STABILITY ANALYSIS IN SOME STRONGLY PRESTRESSED RECTANGULAR PLATES

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Abstract. We consider an evolution plate equation aiming to model the motion of the deck of a periodically forced strongly prestressed suspension bridge. Using the prestress assumption, we show the appearance of multiple time-periodic uni-modal longitudinal solutions and we discuss their stability. Then, we investigate how these solutions exchange energy with a torsional mode. Although the problem is forced, we find a portrait where stability and instability regions alternate. The techniques used rely on ODE analysis of stability and are complemented with numerical simulations.

1. Introduction. A long narrow rectangular thin plate hinged at two opposite edges and free on the remaining two edges was used in [14] to model the deck of a suspension bridge which, at the short edges, is supported by the ground. If \( L \) denotes its length and \( 2\ell \) denotes its width, a realistic assumption is that \( 2\ell \approx \frac{L}{75} \).

After scaling, we may take \( L = \pi \) so that, in the sequel, the plate we focus on will be given by

\[
\Omega = (0, \pi) \times (-\ell, \ell) \subset \mathbb{R}^2.
\]

A suspension bridge is subject to external actions such as the vortex shedding induced by the wind, the traffic load, the synchronized step of pedestrians. These somehow periodic loads generate oscillations of the deck that can be of different kinds. In turn, the oscillations are characterized by means of the eigenfunctions of a suitable eigenvalue problem, see Section 2, where we classify the oscillations in longitudinal and torsional. The latter, that consist in a rotation of the deck around its main axis, are the most dangerous ones, see [33]: they were seen in several suspension bridges, see [20, §1.3,1.4] for a survey, and may lead to collapses as occurred for the Tacoma Narrows Bridge (TNB) [2, 32, 33]. There is a great confusion on possible explanations of the TNB collapse: the words flutter, parametric resonance, self-excited oscillations are recurrent and are used in most attempts of explanation. Basically, flutter is a self-feeding and potentially destructive vibration where aerodynamic forces on an object couple with a natural mode of vibration of the structure to produce a rapid periodic motion. Flutter may occur in any object within a strong
fluid flow, under the conditions that a positive feedback occurs between the natural vibration of the structure and the aerodynamic forces. That is, the vibrational movement of the object increases an aerodynamic load, which in turn drives the object to move further. If the energy input by the aerodynamic excitation in a cycle is larger than that dissipated by the damping in the system, the amplitude of vibration will increase, resulting in self-excitation oscillations. This definition implicitly assumes the appearance of an external resonance or of a parametric resonance. In fact, all these phenomena are somehow related, see [9, 23, 25] and also [20] for the adaptation to several different bridge models. Scanlan [31, p.194] writes that the term “flutter” is used broadly in the context of any oscillatory dynamic instabilities that typically exhibit divergent character with increasing wind velocity. We refer to [12] for the study of flutter in a buckled plate and to [10, 24] for the analysis of the asymptotic behavior of time-dependent plate equations. Billah-Scanlan [6, p.122] believe that the failure of the TNB was related to an aerodynamically induced condition of self-excitation in a torsional degree of freedom: from [30, Section 1.2] we quote: the truth about self-excited oscillations is that they are not truly self-excited. Hence, not only there is no commonly shared explanation of the TNB collapse but there is also disagreement on the terminology.

One of the main issues in suspension bridges is thus their stability, another general word that may have several meanings, strictly related to flutter. In this paper, we analyze the stability through the study of the PDE

$$\begin{align*}
    u_{tt} + \delta u_t + \Delta^2 u + [P - S \int_{\Omega} u_x^2] u_{xx} &= f & \text{in } \Omega \times (0, T) \\
    u &= u_{xx} = 0 & \text{on } \{0, \pi\} \times [-\ell, \ell] \\
    u_{yy} + \sigma u_{xx} &= u_{yyy} + (2 - \sigma)u_{xxy} = 0 & \text{on } [0, \pi] \times \{-\ell, \ell\} \\
    u(x, y, 0) &= u_0(x, y), \quad u_t(x, y, 0) = v_0(x, y) & \text{in } \Omega,
\end{align*}$$

where $\delta > 0$ is a damping parameter, $\sigma > 0$ is the Poisson ratio, $P > 0$ is the prestressing constant, and $S \int_{\Omega} u_x^2$ measures the geometric nonlinearity of the plate due to its longitudinal stretching. We are interested in both the asymptotic stability of periodic motions and the possibility that a longitudinal oscillation suddenly transforms into a torsional oscillation. In some recent papers [4, 7, 13], this topic was tackled for the plate equation introduced in [14]. These papers only consider the case of weakly prestressed plates (small $P$), for which the associated energy is convex, while for strongly prestressed ones (large $P$) this is no longer true, although the energy remains bounded from below, see (15). Weak prestressing yields a horizontal equilibrium position, while for strong prestressing the equilibrium position is no longer horizontal and the so called buckled states appear. Several suspension bridges, such as the Deer Isle Bridge (see Figure 1), are strongly prestressed.

The purpose of the present paper is to analyze the stability of these positions in strongly prestressed plates under different points of view, in particular we show that

in presence of strong prestressing, the effect of the wind depends on the initial position of the deck. (2)

In Section 2 we give some details on the physical model used to derive (1). In Section 3 we prove the existence of periodic (in time) solutions of (1), providing some bounds for the ones concentrated on the first longitudinal mode, which we also
determine explicitly in some particular cases. In Section 4, we first analyze the stability properties of these periodic solutions and then we study the torsional stability of bi-modal solutions of (1) having one longitudinal and one torsional component; for the latter, we use a notion of stability introduced in [17, 18], see Definition 2. Throughout the paper we mainly use ODE techniques, taking advantage of the results in [11, 21, 22, 28], and numerical simulations.

2. Strongly prestressed plates. In this section we explain the physical meaning of the parameters in (1). For the full dimensional form of (1) we refer to [7], here we mainly focus our attention on the parameter $P$.

The boundary conditions in (1) describe a rectangular plate which is hinged on the short edges and free on the long edges, as the deck of a suspension bridge. The constant $\sigma$ is the Poisson ratio, which is the negative ratio of transverse to axial strain: when a material is compressed in one direction, it tends to expand in the other two directions. The parameter $\sigma$ is a measure of this effect, representing the fraction of expansion divided by the fraction of compression for small values of these changes. For metals, one has $\sigma \approx 0.3$, while for concrete $\sigma$ is approximately between 0.1 and 0.2; since the deck of a bridge is a mixture of metal and concrete, we take $\sigma = 0.2$.

The length of the deck of a bridge (in the $x$-direction) is about 1km while the width is about 13m (4 lanes of 3m each plus 1m of separation). In order to maintain this proportion, we take

$$\ell = \frac{\pi}{150}. \quad (4)$$

For a partially hinged plate such as $\Omega$, the buckling load only acts in the $x$-direction and therefore one obtains the term $\int_{\Omega} u_x^2$ as for a one-dimensional beam; see [26]. The nonlinear nonlocal term $\left(\int_{\Omega} u_x^2\right) u_{xx}$ in (1) models the fact that the stiffness of the plate increases with the stretching energy: if the plate is stretched somewhere in the $x$-direction, then in all the other points of the plate this increases the resistance to further stretching. Prestressed models were introduced for beams by Woinowsky-Krieger [35] and, independently, by Burgreen [8]. A few years later they were extended to plates by Berger [5]. The parameter $P$ is the prestressing...
This inner product defines a norm in $H^2_2(\Omega)$; see [14, Lemma 4.1]. We also denote by $(H^2_2(\Omega))^*$ the dual space of $H^2_2(\Omega)$, with duality $\langle \cdot, \cdot \rangle$.

The vibrating modes of the plate $\Omega$ are the eigenfunctions of the problem

$$\Delta^2 u = \lambda u \quad \text{in} \quad \Omega,$$

$$u = u_{xx} = 0 \quad \text{on} \quad \{0, \pi\} \times [-\ell, \ell],$$

$$u_{yy} + \sigma u_{xx} = u_{yy} + (2 - \sigma)u_{xx} = 0 \quad \text{on} \quad \{0, \pi\} \times [-\ell, \ell].$$

The eigenvalues and eigenfunctions of (5) were fully determined in [14] and their properties were analyzed in several subsequent works [4, 7, 13]. We refer to these papers for all the details, here we only recall the following facts which will be useful for our purposes.

**Proposition 1.** The set of eigenvalues of (5) may be ordered in an increasing sequence of strictly positive numbers diverging to $\infty$ and any eigenfunction belongs to $C^\infty(\Omega)$. The set of eigenfunctions of (5) is a complete system in $H^2_2(\Omega)$. Moreover:

(i) for any $m \geq 1$, there exists a unique eigenvalue $\lambda = \mu_{m,1} \in ((1 - \sigma^2)m^4, m^4)$ with corresponding eigenfunction

$$\begin{bmatrix} \mu_{m,1}^{1/4} - (1 - \sigma)m^2 \end{bmatrix} \cosh \left( \frac{y}{\sqrt{m^2 + \mu_{m,1}^{1/2}}} \right) + \left[ \mu_{m,1}^{1/2} + (1 - \sigma)m^2 \right] \cosh \left( \frac{y}{\sqrt{m^2 - \mu_{m,1}^{1/2}}} \right) \sin(m\alpha);$$

(ii) for any $m \geq 1$ and any $k \geq 2$ there exists a unique eigenvalue $\lambda = \mu_{m,k} > m^4$ satisfying

$$\left( m^2 + \frac{2}{\mu_{m,k}} (k - \frac{2}{\mu_{m,k}}) \right)^2 < \mu_{m,k} < \left( m^2 + \frac{2}{\mu_{m,k}} (k - 1)^2 \right)^2$$

and with corresponding eigenfunction

$$\begin{bmatrix} \mu_{m,k}^{1/4} - (1 - \sigma)m^2 \end{bmatrix} \cosh \left( \frac{y}{\sqrt{m^2 + \mu_{m,k}^{1/2}}} \right) + \left[ \mu_{m,k}^{1/2} + (1 - \sigma)m^2 \right] \cosh \left( \frac{y}{\sqrt{m^2 - \mu_{m,k}^{1/2}}} \right) \sin(m\alpha);$$

(iii) for any $m \geq 1$ and any $k \geq 2$ there exists a unique eigenvalue $\lambda = \nu_{m,k} > m^4$ with corresponding eigenfunction

$$\begin{bmatrix} \nu_{m,k}^{1/4} - (1 - \sigma)m^2 \end{bmatrix} \sinh \left( \frac{y}{\sqrt{m^2 + \nu_{m,k}^{1/2}}} \right) + \left[ \nu_{m,k}^{1/2} + (1 - \sigma)m^2 \right] \sinh \left( \frac{y}{\sqrt{m^2 - \nu_{m,k}^{1/2}}} \right) \sin(m\alpha);$$

(iv) for any $m \geq 1$ satisfying $\tanh(\sqrt{2}m\ell) < \left( \frac{\sigma}{\sqrt{2} - \sigma} \right)^2 \sqrt{2}m\ell$ there exists a unique eigenvalue $\lambda = \nu_{m,1} \in (\mu_{m,1}, m^4)$ with corresponding eigenfunction

$$\begin{bmatrix} \nu_{m,1}^{1/4} - (1 - \sigma)m^2 \end{bmatrix} \sinh \left( \frac{y}{\sqrt{m^2 + \nu_{m,1}^{1/2}}} \right) + \left[ \nu_{m,1}^{1/2} + (1 - \sigma)m^2 \right] \sinh \left( \frac{y}{\sqrt{m^2 - \nu_{m,1}^{1/2}}} \right) \sin(m\alpha).$$
Notice that, in principle, there could be a further eigenvalue, with an associated eigenfunction: in fact, this occurs if the unique positive solution $s > 0$ of the equation 
\[
\tanh(\sqrt{2}s\ell) = \left(\frac{\sigma}{2-s}\right)^2 \sqrt{2}s\ell
\]
is an integer, see [14]. Clearly, this condition has probability 0 to occur in general plates and, when (3) and (4) hold (as in our case), is not satisfied. Thus, no eigenvalues other than (i) – (ii) – (iii) – (iv), provided by Proposition 1, exist.

Proposition 1 classifies the eigenfunctions in even and odd with respect to $y$: the former (cases (i) – (ii)) are called longitudinal, while the latter (cases (iii) – (iv)) are called torsional. Hence, as noticed in [7], longitudinal and torsional displacements have a simple characterization after introducing the subspaces of even and odd functions with respect to $y$:

\[
H^2_E(\Omega) := \{ u \in H^2_\ast(\Omega) : u(x, y) = u(x, -y) \forall (x, y) \in \Omega \},
\]

\[
H^2_O(\Omega) := \{ u \in H^2_\ast(\Omega) : u(x, y) = -u(x, y) \forall (x, y) \in \Omega \}.
\]
The space $H^2_E(\Omega)$ is spanned by the longitudinal eigenfunctions (classes (i) and (ii) in Proposition 1) whereas the space $H^2_O(\Omega)$ is spanned by the torsional eigenfunctions (classes (iii) and (iv)). We have

\[
H^2_E(\Omega) \perp H^2_O(\Omega), \quad H^2_\ast(\Omega) = H^2_E(\Omega) \oplus H^2_O(\Omega),
\]

and these spaces allow to split the solutions of (1) in their longitudinal and torsional components.

If we assume (3) and (4), then the eigenvalues of (5) can be computed numerically, see [4, Table 1]. One finds that the least two eigenvalues are $\mu_{1,1}$ and $\mu_{2,1}$, which are both longitudinal and

\[
\mu_{1,1} \approx 0.96, \quad \mu_{2,1} \approx 15.37.
\]

Moreover, the least 10 eigenvalues are all longitudinal, while the least torsional eigenvalues are the 11th and the 16th and are given by

\[
\nu_{1,2} \approx 10943.25, \quad \nu_{2,2} \approx 43785.56.
\]

A weakly prestressed plate is obtained for $P \leq \mu_{1,1}$, in which case the equilibrium position of the plate is horizontal. In strongly prestressed plates one has $P > \mu_{1,1}$ and the equilibrium position is no longer horizontal; see, e.g., Section 12.1 in [3]. Although prestressing reinforces a plate, one should not abuse of it since complicated unstable equilibria may appear, see [3, Proposition 29]. For these reasons, throughout this paper we assume that

\[
\mu_{1,1} < P < \mu_{2,1},
\]

where $P$ is the prestressing constant and $\mu_{m,1}$ are the longitudinal eigenvalues, see Proposition 1. This situation is called “softening regime” in [21], to be compared with the “hardening regime” in which $P < \mu_{1,1}$. As mentioned therein, the case (9) is by far less explored.

3. Existence of periodic solutions for strongly prestressed plates. In order to state our results, we adopt the following conventions:

(C1) All the eigenfunctions in Proposition 1 are multiplied by a normalization constant so as to have $L^2(\Omega)$-norm equal to 1.

(C2) The $L^2$-normalized eigenfunctions will be denoted by $L_{m,k}$ (longitudinal, associated to $\mu_{m,k}$) and $T_{m,k}$ (torsional, associated to $\nu_{m,k}$).
(C3) The solutions of (1) are always intended in $C^0(\mathbb{R}_+, (H^2_x(\Omega))')$, see (10) below.

To simplify the notation, henceforth and throughout the paper we set
$$\xi := (x, y) \in \Omega.$$ We first recall that (1) is well-posed, as proved in [13, Theorem 3].

**Proposition 2.** Given $\delta, S, P, T > 0$, $u_0 \in H^2_x(\Omega)$, $v_0 \in L^2(\Omega)$ and $f \in C^0([0, T], L^2(\Omega))$, there exists a unique weak solution $u$ of (1), that is,
$$u \in C^0([0, T], H^2_x(\Omega)) \cap C^1([0, T], L^2(\Omega)) \cap C^2([0, T], (H^2_x(\Omega))'),$$
such that $u(\xi, 0) = u_0(\xi)$, $u_t(\xi, 0) = v_0(\xi)$ and
$$\langle u_{tt}, w \rangle_{L^2} + \delta\langle u_t, w \rangle_{L^2} + \langle u, w \rangle_{H^2} + \left[ P - S \int_\Omega u_x^2 \right] \langle u_x, w_x \rangle_{L^2} = \langle f, w \rangle_{L^2}, \quad (10)$$
for all $t \in [0, T]$ and all $w \in H^2_x(\Omega)$.

We also sort the eigenvalues provided by Proposition 1 into a unique sequence $\{\lambda_j\}$, accordingly ordering the corresponding eigenfunctions into the sequence $\{e_j\}$, and shortening their explicit form into $e_j(\xi) = \varphi_j(y)\sin(m_jx)$.

### 3.1. One-mode solutions

We denote by $X_T \subset C^0(\mathbb{R}_+, L^2(\Omega))$ the subspace of functions $f = f(\xi, t)$ which are $T$-periodic in the $t$-variable and we define
$$Y_T = X_T \cap C^0(\mathbb{R}_+, H^2_x(\Omega)) \cap C^1(\mathbb{R}_+, L^2(\Omega)) \cap C^2(\mathbb{R}_+, (H^2_x(\Omega))').$$

Then we state our first result about the existence of periodic solutions.

**Theorem 1.** Fix $T > 0$. Then, for every forcing term $f \in X_T$, there exists at least a $T$-periodic solution $u \in Y_T$ of (1). Moreover, if for a suitable set of indexes $J$ it holds $\int_\Omega f(\xi, t)e_j(\xi)\,d\xi = 0$ for every $j \in J$, then there exists a $T$-periodic solution $u$ of (1) with $\int_\Omega u(\xi, t)e_j(\xi)\,d\xi = 0$ for every $j \in J$.

**Proof.** We proceed by following the steps of the proof of [7, Theorem 6], with some changes. We preliminarily notice that (1) is equivalent to the infinite-dimensional dynamical system
$$\ddot{h}_k(t) + \delta \dot{h}_k(t) + \lambda_k h_k(t) + m_k^2 \left[ -P + S \sum_{j=1}^{\infty} m_j^2 h_j(t)^2 \right] h_k(t) = f_k(t) \quad (11)$$
for $k = 1, 2, \ldots$, where $u(\xi, t) = \sum_k h_k(t)e_k(\xi)$ and $f_k(t) = \int_\Omega f(\xi, t)e_k(\xi)\,d\xi$. We aim first to show the existence of a periodic solution for the approximation of (11) obtained by considering only the first $n$ components of the forcing term; namely, we deal with system (11) after having set $f_k = 0$ for $k \geq n + 1$. Hence, we consider (1) where $f$ has been replaced by
$$f^n(\xi, t) = \sum_{k=1}^{n} f_k(t)e_k(\xi).$$

Then we claim that a $T$-periodic solution $u^n$ of (1) can be found in the form
$$u^n(\xi, t) = \sum_{k=1}^{n} h_k(t)e_k(\xi), \quad (12)$$
thereby solving the following \textit{finite-dimensional} system
\[ \ddot{h}_k(t) + \delta \dot{h}_k(t) + \lambda_k h_k(t) + m_k^2 \left[ -P + S \sum_{j=1}^{\infty} m_j^2 h_j(t)^2 \right] h_k(t) = f_k(t) \quad \text{for } k = 1, \ldots, n. \]  

(13)

The Fourier coefficients \( h_k \) also depend on \( n \) but we simply write \( h_k \).

We consider the spaces \( C^2_T(\mathbb{R}) \) and \( C^0_T(\mathbb{R}) \) of \( T \)-periodic scalar functions and we define both the linear diagonal operator \( L_n : (C^2_T(\mathbb{R}))^n \rightarrow (C^0_T(\mathbb{R}))^n \), whose \( k \)-th component reads
\[ L_n^k(h_1, \ldots, h_n) = \ddot{h}_k(t) + \delta \dot{h}_k(t) + (\lambda_k - m_k^2 P) h_k(t) \quad (k = 1, \ldots, n), \]

and the potential
\[ G_n(h_1, \ldots, h_n) = \frac{S}{4} \sum_{j,k=1}^{n} m_j^2 m_k^2 h_j^2 h_k^2. \]

Note that \( L_1 \) is somehow different from the other operators since \( \lambda_1 < P \). Denoting by \( s = (s_1, \ldots, s_n) \) any \( n \)-tuple, (13) becomes
\[ L_n(h(t)) + \nabla G_n(h(t)) = f(t). \]

As \( \delta > 0 \), for any \( q \in (C^0_T(\mathbb{R}))^n \) there exists a unique \( h \in (C^2_T(\mathbb{R}))^n \) satisfying \( L_n(h) = q \), which may be determined by solving the system of ODEs. By the compact embedding \( (C^2_T(\mathbb{R}))^n \subset (C^0_T(\mathbb{R}))^n \), the inverse \( L_n^{-1} : (C^0_T(\mathbb{R}))^n \rightarrow (C^2_T(\mathbb{R}))^n \) is compact. The nonlinear map \( \Gamma_n : (C^0_T(\mathbb{R}))^n \times [0,1] \rightarrow (C^0_T(\mathbb{R}))^n \) defined by
\[ \Gamma_n(h, \nu) = L_n^{-1}(f - \nu \nabla G_n(h)) \quad \forall (h, \nu) \in (C^0_T(\mathbb{R}))^n \times [0,1] \]
is also compact and we claim that there exists \( H_n > 0 \) (independent of \( \nu \)) such that, if \( h \in (C^0_T(\mathbb{R}))^n \) satisfies \( h = \Gamma_n(h, \nu) \), then
\[ \|h\|_{(C^0_T(\mathbb{R}))^n} \leq H_n. \]  

(14)

Indeed, we introduce the energy
\[ E(t) = \frac{1}{2} \|u^n(t)\|_{L^2}^2 + \frac{1}{2} \|u^n(t)\|_{H^2}^2 - \frac{P}{2} \|u^n(t)\|_{L^2}^2 + \frac{S}{4} \|u^n(t)\|_{L^2}^2 + \frac{\delta P}{4} \|u^n(t)\|_{L^2}^2 + \frac{3S\delta}{8} \|u^n(t)\|_{H^2}^2 \]
and we observe that, using (1), one has
\[ \dot{E}(t) + \frac{\delta}{2} E(t) = -\frac{\delta}{4} \|u^n(t)\|_{L^2}^2 - \frac{\delta}{4} \|u^n(t)\|_{H^2}^2 + \frac{\delta P}{4} \|u^n(t)\|_{L^2}^2 - \frac{3S\delta}{8} \|u^n(t)\|_{H^2}^2 \]

(15)

\[ -\frac{\delta^2}{4} \int_{\Omega} u^n(\xi, t) u^n(\xi, t) d\xi + \int_{\Omega} f^n(\xi, t) \left( u^n(\xi, t) + \frac{\delta}{2} u^n(\xi, t) \right) d\xi. \]

The embedding inequalities \( \|v\|_{L^2}^2 \leq \|v\|_{H^2}^2 \) and \( \lambda_1 \|v\|_{L^2}^2 \leq \|v\|_{H^2}^2 \), together with the Young inequality, imply that
\[ \dot{E}(t) + \frac{\delta}{2} E(t) \leq \frac{\delta}{2} \left( P - \lambda_1 + \frac{\delta^2}{4} \right) \|u^n(t)\|_{L^2}^2 - \frac{3S\delta}{8} \|u^n(t)\|_{L^2}^2 + \frac{1}{\delta} \|f^n(t)\|_{L^2}^2 \]

\[ \leq \frac{\delta C}{2} + \frac{1}{\delta} \|f^n(t)\|_{L^2}^2, \]
where \( C = (P - \lambda_1 + \frac{\delta^2}{4})^2/(12S) \). For every \( t > 0 \), it then follows that
\[ E(t) \leq e^{-\frac{\delta t}{2}} E(0) + \left( C + \frac{2f^\infty}{\delta^2} \right) (1 - e^{-\frac{\delta t}{2}}), \]
where \( f_\infty = \max_{0 \leq t \leq T} \| f(t) \|_{L^2}^2 \) (so that the upper bound on \( E \) is independent of \( n \)). Therefore, by letting \( t \to \infty \), we deduce

\[
\limsup_{t \to \infty} E(t) \leq C + \frac{2f_\infty^2}{\delta^2}.
\]

By the \( T \)-periodicity of \( f \) and \( u^n \), this is also a global bound:

\[
E(t) \leq C + \frac{2f_\infty^2}{\delta^2} \quad \forall t \in [0, T].
\]

Back to the finite dimensional Hamiltonian system (13), this proves the claimed \((C^0_2(\mathbb{R}))^n\)-bound (14). Hence, since the equation \( h = \Gamma_n(h,0) \) admits a unique solution, the Leray-Schauder principle guarantees the existence of a solution \( h \in (C^0_2(\mathbb{R}))^n \) of \( h = \Gamma_n(h,1) \). This proves the existence of a \( T \)-periodic solution \( u^n \) of the finite system (13). Using similar arguments as in [7, Lemmas 19 and 20] and the expression in (15), we deduce that \( \| u^n(t) \|_{H^2} \) and \( \| u^n(t) \|_{L^2} \) are bounded in \([0, T]\), independently of \( n \). The equation

\[
\langle u^n_t, v \rangle + \delta(u^n, v)_{L^2} + \langle u^n, v \rangle_{H^2} + [S\| u^n \|_{L^2} - P](u^n, v)_{L^2} = (f^n, v)_{L^2},
\]

for all \( t \in [0, T] \) and all \( v \in H^2_2(\Omega) \), then yields a uniform \((H^2_2)'\)-bound on \( u^n_{tt} \). Up to a subsequence, we can therefore pass to the limit in (16):

\[
\begin{align*}
&u^n \to u \text{ weakly* in } L^\infty([0, T], H^2_2(\Omega)), \\
u^n_t \to u_t \text{ weakly* in } L^\infty([0, T], L^2(\Omega)), \\
u^n_{tt} \to u_{tt} \text{ weakly* in } L^\infty([0, T], (H^2_2(\Omega))').
\end{align*}
\]

Hence, there exists a \( T \)-periodic solution \( u \) of equation (1) in the sense of \( L^\infty([0, T], (H^2_2(\Omega))') \). To conclude the proof of the existence part of the statement, observe that the continuity properties of \( u \) follow from [34, Lemma 4.1] and from the fact that (1) is satisfied in \( C^0(\mathbb{R}_+, (H^2_2(\Omega))') \).

The second part of the statement follows by noticing that, in view of the assumption on \( f \), projecting (1) onto the eigenspace generated by \( \{ e_j \}_{j \in J} \) yields the ODE system

\[
\ddot{U}_j(t) + \delta \dot{U}_j(t) + (\lambda_j - P)U_j(t) + S \left( \sum_{k} k^2 U_k^2(t) \right) U_j(t) = 0, \quad j \in J,
\]

where \( u(\xi, t) = \sum_{k=1}^{\infty} U_k(t)e_k(\xi) \). System (17) has the solution \( U_j \equiv 0 \), \( j \in J \); the other components \( U_k \) of the solution \( u(\xi, t) \), \( k \notin J \), can now be recovered by solving the infinite-dimensional system of ODEs satisfied by their Fourier coefficients, which has a \( T \)-periodic solution (this can be shown similarly as in the previous step of the proof).

We now consider the case where the deck at rest is excited by a periodic external force acting only on the first longitudinal mode \( L_{1,1} = e_1 \), that is,

\[
f(\xi, t) = A(t)L_{1,1}(\xi) \quad \text{with} \quad A \in C(\mathbb{R}/T, \mathbb{R})
\]

for some minimal period \( T > 0 \). From Theorem 1 we immediately infer:

**Corollary 1.** Fix \( T > 0 \) and let \( f \in X_T \) be as in (18). Then there exists at least a \( T \)-periodic solution \( u \in Y_T \) of (1) satisfying \( \int_{\Omega} u(\xi, t)e_j(\xi) \, d\xi \equiv 0 \) for every \( j \geq 2 \).
In fact, more can be said. Since \( P > \mu_{1,1} \), assuming (18) we may have multiplicity of periodic solutions, as the next statement shows. By “subharmonic solution proportional to \( L_{1,1} \)” we mean a solution \( u \) of (1) in the form \( u(\xi, t) = z(t)L_{1,1}(\xi) \) where, for some integer \( k > 1 \), the time coefficient \( z \) is \( kT \)-periodic, \( kT \) being its minimal period.

**Theorem 2.** Assume (9). Let \( f \) be as in (18) and assume that

\[
\delta \geq \sqrt{8(P - \mu_{1,1})}, \quad \|f\|_{L^\infty} < \frac{\delta L_{1,1}^\infty}{32} \sqrt{\frac{(P - \mu_{1,1})^3}{S(\delta^2 + P - \mu_{1,1})}},
\]

where \( L_{1,1}^\infty = \|L_{1,1}\|_{\infty} \). Then (1) admits exactly three \( T \)-periodic solutions \( \psi^1, \psi^2, \psi^3 \) having the form

\[
\psi^j(\xi, t) = z_j(t)L_{1,1}(\xi), \quad j = 1, 2, 3,
\]

and has no subharmonic solutions proportional to \( L_{1,1} \).

Moreover, one of the three periodic solutions found (which we name \( \psi^1 \)) satisfies the bound

\[
\max_{t \in [0,T]} |z_1(t)| < \sqrt{\frac{P - \mu_{1,1}}{3S}},
\]

while the other two fulfill the estimates

\[
\max_{t \in [0,T]} \left| z_2(t) - \sqrt{\frac{P - \mu_{1,1}}{S}} \right| < \left[ \sqrt{\frac{3}{4}} - 1 \right] \sqrt{\frac{P - \mu_{1,1}}{S}},
\]

\[
\max_{t \in [0,T]} \left| z_3(t) + \sqrt{\frac{P - \mu_{1,1}}{S}} \right| < \left[ \sqrt{\frac{5}{4}} - 1 \right] \sqrt{\frac{P - \mu_{1,1}}{S}}.
\]

In particular, \( z_2 \) and \( z_3 \) have constant sign.

**Proof.** Solutions of (1) which are \( T \)-periodic in time and are proportional to \( L_{1,1} \) have the form \( \psi(\xi, t) = z(t)L_{1,1}(\xi) \) for some \( z \in C^2(\mathbb{R}/T \mathbb{Z}, \mathbb{R}) \). By replacing such an expression into (1), we find that \( z \) has to solve the ODE

\[
\ddot{z}(t) + \delta \dot{z}(t) + (\mu_{1,1} - P)z(t) + \dot{S}z(t)^3 = A(t),
\]

where \( A \in C(\mathbb{R}/T \mathbb{Z}, \mathbb{R}) \) is as in (18). We perform the change of variables

\[
z(t) = \sqrt{\frac{P - \mu_{1,1}}{S}} \psi \left( \sqrt{\frac{P - \mu_{1,1}}{S}} t \right) \iff \psi(t) = \sqrt{\frac{S}{P - \mu_{1,1}}} z \left( \sqrt{\frac{P - \mu_{1,1}}{S}} t \right)
\]

so that \( \psi \) satisfies

\[
\ddot{\psi}(t) + \varepsilon \dot{\psi}(t) - \psi(t) + \psi(t)^3 = g(t),
\]

where

\[
\varepsilon = \frac{\delta}{\sqrt{P - \mu_{1,1}}}, \quad g(t) = \sqrt{\frac{S}{(P - \mu_{1,1})^3}} A \left( \frac{t}{\sqrt{P - \mu_{1,1}}} \right).
\]

In this setting, the assumptions in (19) become

\[
\varepsilon \geq 2\sqrt{2}, \quad \|g\|_{L^\infty} < \frac{\varepsilon}{32 \sqrt{1 + \varepsilon^2}}.
\]

Equation (25) has the form of (1) in [22] so that, in view of (27), we may apply the statements therein. By combining Propositions 12 and 13 in the Appendix we infer that:

- equation (25) admits exactly three periodic solutions \( \psi_1, \psi_2, \psi_3 \) having the same period as \( g \);
- the same equation has no subharmonic solutions.
After undoing the change of variables (24) and after recalling (20), these two facts prove the first statement of the theorem.

Next, by combining Propositions 10 and 11 in the Appendix, we infer that
\[
\|\psi_1\|_{L^\infty} < \frac{1}{\sqrt{3}}, \quad \|\psi_2 - 1\|_{L^\infty} < \frac{\sqrt{5}}{3} - 1, \quad \|\psi_3 + 1\|_{L^\infty} < \frac{\sqrt{5}}{3} - 1.
\]
Undoing the change of variables (24) then yields the bounds in the statement. □

We point out that the lower bound (19) may be dropped, provided less explicit bounds on \(f\) hold, see [21].

3.2. Two explicit examples of periodic solutions. In this section, we provide two examples where one periodic solution of (1) can be written explicitly. To this end, we intensively exploit the Jacobi elliptic functions \(\text{sn}, \text{cn}, \text{dn}\) and their properties [1], see the Appendix for further details.

We start with a result in case (19) is satisfied.

**Proposition 3.** Assume that \(\delta \geq \sqrt{8(P - \mu_{1,1})}\) and that
\[
0 < k^2 < \frac{2\sqrt{P - \mu_{1,1}}}{\sqrt{P - \mu_{1,1} + 16\sqrt{2}(\delta^4 + P - \mu_{1,1})}}.
\]
Moreover, take
\[
f(\xi, t) = -\frac{P - \mu_{1,1}}{\sqrt{S}} \frac{\delta \sqrt{2} k^2}{2 - k^2} \text{sn} \left(\sqrt{\frac{P - \mu_{1,1}}{2 - k^2}} t, k \right) \text{cn} \left(\sqrt{\frac{P - \mu_{1,1}}{2 - k^2}} t, k \right) L_{1,1}(\xi).
\]
Then the assumptions of Theorem 2 are fulfilled and the function
\[
\psi^2(\xi, t) = \sqrt{\frac{P - \mu_{1,1}}{S}} \sqrt{\frac{2}{2 - k^2}} \text{dn} \left(\sqrt{\frac{P - \mu_{1,1}}{2 - k^2}} t, k \right) L_{1,1}(\xi)
\]
is a periodic solution of (1) satisfying the bound (22).

**Proof.** By using the properties of the Jacobi functions [1], one can check that for all \(k \in (0, 1)\) and all \(\varepsilon > 0\) the function
\[
\psi(t) = \sqrt{\frac{2}{2 - k^2}} \text{dn} \left(\frac{t}{\sqrt{2 - k^2}}, k \right)
\]
solves equation (25) with
\[
g(t) = -\varepsilon \sqrt{2} k^2 \text{sn} \left(\frac{t}{\sqrt{2 - k^2}}, k \right) \text{cn} \left(\frac{t}{\sqrt{2 - k^2}}, k \right).
\]
We thus have to show that the assumptions of Theorem 2 are fulfilled. To this end, we notice that the first bound in (27) follows from the lower bound on \(\delta\). On the other hand, since \(\max_s |\text{sn}(s, k)\text{cn}(s, k)| = \frac{1}{2}\) for any \(k \in (0, 1)\) (see again [1]), we have
\[
\|g\|_{L^\infty} = \frac{\varepsilon k^2}{(2 - k^2)\sqrt{2}}.
\]
Thus, \(g\) fulfills the second bound in (27) provided that
\[
0 < k^2 < \frac{2}{1 + 16\sqrt{2}(1 + \varepsilon^2)}.
\]
which is equivalent to (28). Theorem 2 thus applies and the statement is proved.

Notice that from [1] we also know that
\[ \sqrt{1 - \varepsilon^2} \leq \text{dn}(s, k) \leq 1 \]
for all \( s \geq 0 \) so that, from the constraints on \( k \) and \( \varepsilon \),
\[ \psi(t) \geq \sqrt{2(k^2 - 1)} \geq \sqrt{1 + \frac{1}{48\sqrt{2}}} \approx 0.993 \]
and
\[ \psi(t) \leq \sqrt{2(k^2 - 1)} \leq \sqrt{1 + \frac{1}{48\sqrt{2}}} \approx 1.007 \]
for all \( t > 0 \), improving here the bounds on the solution \( \psi \) (since \( \sqrt{5/3} - 1 \approx 0.29 \)).

We now turn to a situation where (19) is not satisfied.

Proposition 4. Assume that \( k \in (1/\sqrt{2}, 1) \) and take
\[ f(\xi, t) = -\frac{P - \mu_{1,1}}{\sqrt{S}} \frac{\delta\sqrt{2} k}{2k^2 - 1} \text{sn}\left( \sqrt{\frac{P - \mu_{1,1}}{2k^2 - 1}}, t, k \right) \text{dn}\left( \sqrt{\frac{P - \mu_{1,1}}{2k^2 - 1}}, t, k \right) L_{1,1}(\xi) \]
(which does not satisfy (19)). Then the function
\[ \varphi(\xi, t) = \sqrt{\frac{P - \mu_{1,1}}{S}} \sqrt{\frac{2k^2}{2k^2 - 1}} \text{cn}\left( \sqrt{\frac{P - \mu_{1,1}}{2k^2 - 1}}, t, k \right) L_{1,1}(\xi) \]
is a periodic solution of (1).

Proof. By using the properties of the Jacobi functions [1], one can check that for all \( k \in (1/\sqrt{2}, 1) \) and all \( \varepsilon > 0 \) the function
\[ \psi(t) = \sqrt{\frac{2k^2}{2k^2 - 1}} \text{cn}\left( \frac{t}{\sqrt{2k^2 - 1}}, k \right) \]
solves equation (25) with
\[ g(t) = -\frac{\varepsilon\sqrt{2} k}{2k^2 - 1} \text{sn}\left( \frac{t}{\sqrt{2k^2 - 1}}, k \right) \text{dn}\left( \frac{t}{\sqrt{2k^2 - 1}}, k \right) \]
for any \( k \in (1/\sqrt{2}, 1) \) (see again [1]), we have
\[ \|g\|_{L^\infty} = \frac{\varepsilon}{\sqrt{2(2k^2 - 1)}}. \]
It can now be checked that with no choice of \( k \) we can fulfill the second bound in (27).

4. Stability of periodic solutions for strongly prestressed plates. In this section, we study the stability of periodic solutions of (1), both from a theoretical and from a numerical point of view.

4.1. Stability of 1-mode solutions. To state our next result, we denote by
\[ P = \{ \varphi^1, \varphi^2, \varphi^3 \} \]
the set of periodic solutions of (1) given by Theorem 2.

Theorem 3. Under the assumptions of Theorem 2, if \( u = u(\xi, t) \) is any solution of (1) with initial conditions proportional to \( L_{1,1} \), that is,
\[ u(\xi, 0) = \alpha L_{1,1}(\xi) \quad \text{and} \quad u_t(\xi, 0) = \beta L_{1,1}(\xi) \quad \text{in} \ \Omega, \quad (29) \]
for some (possibly null) $\alpha, \beta \in \mathbb{R}$, then there exists $\varphi \in \mathbb{P}$ such that
\[
\lim_{t \to \infty} \left( \|u_t(t) - \varphi_t(t)\|_{L^2} + \|u(t) - \varphi(t)\|_{H^2_*} \right) = 0.
\] (30)

Proof. For a given couple $(\alpha, \beta) \in \mathbb{R}^2$, consider the solution $u = u(\xi, t)$ of (1) with initial conditions (29). By exploiting the uniqueness of the solution of (1) (see Proposition 2), one sees that $u$ has the form
\[
u(\xi, t) = U(t) L_1, 1(\xi),
\]
with $U \in C^2(\mathbb{R}_+)$ solving
\[
\ddot{U}(t) + \delta \dot{U}(t) - (P - \mu_{1,1}) U(t) + S U(t)^3 = A(t), \quad U(0) = \alpha, \quad \dot{U}(0) = \beta.
\]

We then perform the same change of variables as (24) by setting
\[
U(t) = \sqrt{P - \mu_{1,1}} V(\sqrt{P - \mu_{1,1}} t) \iff V(t) = \sqrt{\frac{S}{P - \mu_{1,1}}} U\left(\frac{t}{\sqrt{P - \mu_{1,1}}}\right)
\]
so that $V$ satisfies (25), that is,
\[
\ddot{V}(t) + \varepsilon \dot{V}(t) - V(t) + V(t)^3 = g(t).
\] (31)

Since all the assumptions of Proposition 11 in the Appendix are satisfied (see above), we infer that any solution of (31), thereby also $V$, satisfies (for some $j = 1, 2, 3$)
\[
\lim_{t \to \infty} \left( |\dot{V}(t) - \dot{z}_j(t)| + |V(t) - z_j(t)| \right) = 0.
\]

After undoing the change of variables, this proves the desired statement. \qed

Theorem 3 states that if $f$ has the form (18) and if the phase space is reduced to the first longitudinal mode, then the attractor for (1) is given by the set $\mathbb{P}$ of the three periodic solutions of (1) having the form (20). If in this framework it is nice to learn that there are no subharmonic solutions (called a "bad phenomenon" by Haraux [22]), on the other hand the multiplicity of periodic solutions yields an uncertainty of the behavior of all the other solutions. In fact, in [21] it is shown that Theorem 3 holds also in the whole phase space; nevertheless, since we are mainly interested in uni-modal solutions and in quantitative results, we preferred to state it in this precise form.

Theorem 3 also states that, if $P > \mu_{1,1}$, one cannot predict the long-time behavior of the solutions of (1) even for simple forces such as (18). Translated to the bridge model, this shows (2).

A fundamental problem is then to study the asymptotic stability of the three periodic solutions in $\mathbb{P}$, according to the following definition.

**Definition 1.** We say that $\varphi \in \mathbb{P}$ is (Lyapunov) **stable** if for every $\zeta > 0$ there exists $\eta > 0$ such that
\[
\|u_t(0) - \varphi_t(0)\|_{L^2} + \|u(0) - \varphi(0)\|_{H^2} < \eta \implies \|u_t(t) - \varphi_t(t)\|_{L^2} + \|u(t) - \varphi(t)\|_{H^2} < \zeta
\]
for every $t > 0$, where $u$ solves (1) with initial conditions such as (29); we say that $\varphi \in \mathbb{P}$ is **unstable** if it is not stable.
We say that $\phi \in \mathbb{P}$ is **asymptotically stable** if it is stable and there exists $\eta > 0$ such that

$$\|u_t(0) - \phi_t(0)\|_{L^2} + \|u(0) - \phi(0)\|_{H^2} < \eta \implies \lim_{t\to \infty} \left(\|u_t(t) - \phi_t(t)\|_{L^2} + \|u(t) - \phi(t)\|_{H^2}\right) = 0,$$

where $u$ solves (1) with initial conditions such as (29).

To characterize the stability of a periodic solution, it will be useful to remark that if $|c| < \frac{2}{3\sqrt{3}}$ then $\rho^3 - \rho = c$ has three solutions $-\rho_3(-c) < \rho_0(c) < \rho_3(c)$, with $\rho_3 > 0$ and increasing, $\rho_0(0) = 0$ and $c \rho_0(c) < 0$ if $c \neq 0$. We also set

$$T(\delta, f) = \pi \sqrt{\frac{12(P - \mu_{1,1})}{12(P - \mu_{1,1})\rho_* \left(\sqrt{\frac{S}{(P - \mu_{1,1})^3} \|f\|_{L^\infty}}\right)^2}} - 4(P - \mu_{1,1}),$$

$$R_0 = \rho_0 \left(-\sqrt{\frac{S}{(P - \mu_{1,1})^3} \|f\|_{L^\infty}}\right), \quad R_- = -\rho_* \left(-\sqrt{\frac{S}{(P - \mu_{1,1})^3} \|f\|_{L^\infty}}\right),$$

$$R_+ = \rho_* \sqrt{\frac{S}{(P - \mu_{1,1})^3} \|f\|_{L^\infty}},$$

where the notation $L^\infty_{1,1}$ has been introduced in the statement of Theorem 2. Then the following stability results hold.

**Theorem 4.** Let $f$ be as in (18) and assume (9) and (19). Then the following properties hold for the periodic solutions $\phi^j \in \mathbb{P}$ ($j = 1, 2, 3$) defined in (20):

1. $z_1(t)$ fulfills the bound

   $$\|z_1\|_{L^\infty} < R_0$$

   and $\phi^1$ is unstable;

2. if one of the two following facts occurs:

   $$\sqrt{\frac{\delta^2}{12(P - \mu_{1,1})}} + \frac{1}{3} < \tilde{\rho}_{\delta} \quad \text{and} \quad T(\delta, f)$$

   or

   $$\sqrt{\frac{\delta^2}{12(P - \mu_{1,1})}} + \frac{1}{3} \geq \tilde{\rho}_{\delta},$$

   then for every $t \in [0, T]$ one has

   $$0 < -R_- < z_2(t) < R_+ \quad \text{and} \quad -R_+ < z_3(t) < R_- < 0$$

   and $\phi^2$ and $\phi^3$ are asymptotically stable.

Before proving this result, we briefly make some comments and state a more readable corollary. Conditions (35) should be read as assumptions of “smallness” and “largeness” of the damping, respectively; thus, Theorem 4 essentially says that the asymptotic stability of the two periodic solutions $\phi^2, \phi^3$ is obtained either for “small” damping (fulfilling however (19)) provided that the period of $f$ is “small”, or for sufficiently “large” damping independently of the period of $f$. Notice that $R_0 \to 0$ and $R_\pm \to \pm 1$ for $\|f\|_{L^\infty} \to 0$, so that the intervals in (34) and (36) shrink to a constant for $f \to 0$; moreover, as functions of $\|f\|_{L^\infty}$, $R_-$ and $R_+$ are increasing, while $R_0$ is decreasing. Since by (19) we have $\|f\|_{L^\infty} < \frac{L^\infty_{1,1}}{32} \sqrt{\frac{(P - \mu_{1,1})^3}{S}}$, we derive the following statement.
Corollary 2. Under the assumptions of Theorem 4, it is
\[ \|z_1\|_{L^\infty} < \rho_0 \left( \frac{-1}{32} \right) \approx 0.0312806. \]

Moreover, if either
\[ \left( \sqrt{8(P - \mu_{1,1})} \leq \delta < \sqrt{8.18234(P - \mu_{1,1})} \right) \quad \text{and} \quad T < 25.4232 \]

or \[ \delta \geq \sqrt{8.18234(P - \mu_{1,1})} \]

then, for every \( t \in [0, T] \),
\[ 0.983993 \approx \rho_* \left( \frac{-1}{32} \right) < z_2(t) < \rho_* \left( \frac{1}{32} \right) \approx 1.01527, \]
\[ -1.01527 \approx -\rho_* \left( \frac{1}{32} \right) < z_2(t) < -\rho_* \left( \frac{-1}{32} \right) \approx -0.983993. \]

The last part of this statement is obtained by taking the worst cases \( \delta = \sqrt{8(P - \mu_{1,1})} \) and \( \|f\|_{L^\infty} = \frac{L_{d_1}}{2} \sqrt{(P - \mu_{1,1})} \) to compute \( T(\delta, f) \). To cover all the possible cases for the values of \( \delta \), it would remain to prove the asymptotic stability of \( \phi^2 \) and \( \phi^3 \) in case \( \delta < \sqrt{8.18234(P - \mu_{1,1})} \) and \( T \geq 25.4232 \), range in which the technique which we here exploit does not work. We believe, also in view of numerical simulations, that stability is reached also herein, but filling in this gap seems nontrivial from a theoretical point of view.

To prove Theorem 4, we rely on lower and upper solutions techniques. We recall that by “lower solution” (resp., “upper solution”) of (25) one means a function \( \alpha_- \) (resp., \( \alpha_+ \)) such that
\[ \bar{\alpha}_- + \epsilon \bar{\alpha}_- + \alpha_-^3 - \alpha_- \geq g(t), \quad \text{(resp., } \bar{\alpha}_+ + \epsilon \bar{\alpha}_+ + \alpha_+^3 - \alpha_+ \leq g(t)\text{).} \]

Here the symbol \( \geq \) means: \( \geq \) for all \( t \) and \( > \) for some \( t \) (similarly for \( \leq \)). Since we will merely use constant lower-upper solutions, we further say that constant functions \( \alpha_-(t) \equiv \alpha_- \) and \( \alpha_+(t) \equiv \alpha_+ \) are, respectively, “lower solution” and “upper solution” of (25) if
\[ \alpha_-^3 - \alpha_- \leq g(t) \leq \alpha_+^3 - \alpha_+ \quad \forall t. \quad (37) \]

In the proof we will exploit two results in [11] and [28], which we briefly recall here for the reader’s convenience, in versions adapted to equation (25) and for constant lower and upper solutions.

Proposition 5. [11, Proposition 3.1] Let \( \epsilon \geq 0 \), let \( g \) be \( T \)-periodic \((T > 0)\), and assume that there exist constants \( \alpha_- < \alpha_+ \) such that (37) holds. Then (25) has at least one unstable \( T \)-periodic solution \( \psi \) such that
\[ \alpha_- \leq \psi(t) \leq \alpha_+ \quad \forall t \in [0, T], \quad (38) \]
provided that the number of periodic solutions \( \psi \) of (25) fulfilling (38) is finite.

Proposition 6. [28, Theorem 1.2] Let \( \epsilon > 0 \), let \( g \) be \( T \)-periodic \((T > 0)\), and assume that there exist constants \( \alpha_- > \alpha_+ \) such that (37) holds. Moreover, assume that
\[ \max\{\alpha_-^2, \alpha_+^2\} \leq \frac{\pi^2}{32^2} + \frac{1}{2} + \frac{\epsilon^2}{12}. \quad (39) \]

Then (25) has at least one asymptotically stable \( T \)-periodic solution \( \psi \) such that
\[ \alpha_+ \leq \psi(t) \leq \alpha_- \quad \forall t \in [0, T], \quad (40) \]
provided that the number of periodic solutions \( \psi \) of (25) fulfilling (40) is finite.
We emphasize the different assumptions $\alpha_- \leq \alpha_+$ in these two statements. In fact, Proposition 5 is a variant of [11, Proposition 3.1] combined with [29, Lemma 3.2], see also [28, Theorem 1.1]. We are now ready to give the proof.

Proof of Theorem 4. After performing the change of variables (24), we are led to deal with equation (25), for which there exist exactly three periodic solutions $\psi_j$ $(j = 1, 2, 3)$ in view of Theorem 2. We thus know that the number of periodic solutions of (25) is finite.

Let us first prove Item 1). Recalling (19) and (26), we have that $\|g\|_{L^\infty} < 1/32$ and thus, by (33), there exist three solutions of the equations $\rho^3 - \rho = \pm\|g\|_{L^\infty}$. Then, also with the help of the left picture in Figure 2, we know that $\alpha_0 \equiv \rho_0(\|g\|_{L^\infty}) < 0$ and $\alpha^0 \equiv \rho_0(-\|g\|_{L^\infty}) = -\alpha_0$ are constant lower and upper solutions of (25) satisfying the assumptions of Proposition 5. Thus, there exists at least one unstable periodic solution $\psi$ of (25) such that $\alpha_0 \leq \psi(t) \leq \alpha^0$. By undoing the change of variables (24), this corresponds to one of the three $z_j$’s found in Theorem 2. Taking into account (21), it is necessarily $z_1$, which is therefore unstable (and thus $\psi^1$ is unstable, recall (20)) and satisfies the bound (34), completing the proof of Item 1).

As for Item 2), we notice that $\alpha^- \equiv \rho_*(-\|g\|_{L^\infty})$ and $\alpha^+ \equiv \rho_*(\|g\|_{L^\infty})$ are constant lower and upper solutions of (25) such that $\alpha^- > \alpha^+$, see the right picture in Figure 2. The same is fulfilled by $\alpha^- \equiv -\rho_*(-\|g\|_{L^\infty})$ and $\alpha^+ \equiv -\rho_*(\|g\|_{L^\infty})$.

In view of (39), for $\psi^2$ to be asymptotically stable it is sufficient that

$$\rho_*(\|g\|_{L^\infty})^2 \leq \frac{\pi^2}{T^2} + \frac{1}{3} + \frac{\varepsilon^2}{12}$$

This reads as

$$T \leq \frac{\pi}{\sqrt{\rho_*(\|g\|_{L^\infty})^2 - \frac{1}{3} - \frac{\varepsilon^2}{12}}}$$

as long as $\rho_*(\|g\|_{L^\infty})^2 > \frac{1}{3} + \frac{\varepsilon^2}{12}$, corresponding to (35)1 in view of (26). If instead $\rho_*(\|g\|_{L^\infty})^2 \leq \frac{1}{3} + \frac{\varepsilon^2}{12}$, then $T$ can be arbitrary and this is the situation described in (35)2, yielding the asymptotic stability of $\psi^2$. One can reason analogously on $\psi^3$. Again by undoing the change of variables (24), we finally obtain the bounds in (36), concluding the proof.

\[\text{Figure 2. The choice of the lower and upper solutions in the proof of the bounds and of the stability properties of } \psi^1 \text{ (left) and } \psi^2, \psi^3 \text{ (right).}\]
\( \alpha_- \leq \alpha_+ \), there exists a periodic solution \( \psi(t) \) such that \( \alpha_- \leq \psi(t) \leq \alpha_+ \), the same is not guaranteed in presence of reversed-order lower and upper solutions. The stability issue is even more complicated. On the other hand, according to [28, Remark 3.2], there is a significant number of situations in which the periodic solutions of (25) satisfying (38) or (40) are in finite number; in particular, we can partially extend Theorem 4 also to cases where (19) is not fulfilled. Roughly speaking, the local maximum and the local minimum of \( h(\rho) = \rho^3 - \rho \), found in correspondence of \( \rho = \mp \sqrt{\frac{3}{5}} \), play here a central role.

**Proposition 7.** Assume that \( \|f\|_{L^\infty} < \frac{2\delta L_{\mu_1}}{3\sqrt{3}} \sqrt{\frac{(P-\mu_{1,1})^3}{S(P+\mu_{1,1})}} \). Then, there exists a unique unstable periodic solution of (23) which has the same period as \( f \) and satisfies the bound (34).

This follows from [28, Remark 3.2], by noticing that we can find constant lower and upper solutions \( \alpha_0, \alpha^0 \) such that \( \alpha_0 = -\alpha^0 \) and \( 3\alpha_0^2 < 1 \), provided that \( \|g\|_{L^\infty} < \frac{2}{3\sqrt{3}} \). Indeed, we can take as lower and upper solutions the local maximum and the local minimum of \( h \), respectively. Notice that, except for the stability part, the statement corresponds to the first part of Proposition 10 in the Appendix.

Also for the two asymptotically stable solutions it is possible to prove a similar statement; we here limit ourselves to give a “generalization” of Corollary 2, obtained taking a slightly larger value of the damping \( \delta \).

**Proposition 8.** Assume that \( \|f\|_{L^\infty} < \frac{2\delta L_{\mu_1}}{3\sqrt{3}} \sqrt{\frac{(P-\mu_{1,1})^3}{S(P+\mu_{1,1})}} \) and \( \delta \geq \sqrt{12(P-\mu_{1,1})} \). Then, there exist two asymptotically stable periodic solutions of (23) having the same period as \( f \) and satisfying the bounds in (36).

The assumptions of Proposition 8 guarantee, in fact, that it is possible to choose (constant) lower and upper solutions \( \alpha_- \), \( \alpha_+ \) both greater than \( 1/\sqrt{3} \) (or less than \( -1/\sqrt{3} \)) and such that \( \alpha_+ \leq \alpha_- \), so that \( 3s^2 - 1 \geq 0 \) for every \( s \in [\alpha_+, \alpha_-] \), entering the condition given in [28, Remark 3.2]. In this setting, what is not guaranteed is that only three periodic solutions exist; in principle, more complicated pictures for the solvability of (25) may appear. Concerning the existence and the localization of the periodic solutions, the reader may compare Proposition 8 with Proposition 10 in the Appendix, where in presence of a slightly stronger bound on \( f \) it is possible to relax the assumption on \( \delta \) by taking it as in (19).

Finally, using once more the results in [22] it is possible to determine a subset of the basin of attraction for \( \varphi_2, \varphi_3 \), as the following statement shows. The proof comes directly from Proposition 14 in the Appendix.

**Proposition 9.** Assume (19). Then, for any \( \gamma < \frac{1}{16} \sqrt{\frac{P-\mu_{1,1}}{S}} \) there exists \( c > 0 \) such that if

\[
|z(t_0) - \sqrt{\frac{P - \mu_{1,1}}{S}}| \leq \gamma, \quad |z'(t_0)| \leq c,
\]

then the solution of (1) in the form (20) converges to \( \varphi_2 \). Under the same assumptions, if the first condition in (41) is replaced by

\[
|z(t_0) + \sqrt{\frac{P - \mu_{1,1}}{S}}| \leq \gamma,
\]

then the solution of (1) in the form (20) converges to \( \varphi_3 \).
4.2. Some numerical considerations on torsional instability. We analyze here the stability of the plate modeled by (1) from a different point of view. In order to introduce our characterization of torsional instability, we recall that the vortex shedding around the deck of a bridge generates a periodic lift force \( f \) which starts the longitudinal oscillations of the structure. The frequency of \( f \) determines which longitudinal mode is prevailing [17, 18] (that is, which mode captures almost all the energy from the vortices). We analyze the situation where the prevailing mode is \( L_{1,1} \). After some transition time \( T_W \), related to the so-called Wagner effect, the longitudinal oscillation is maintained in amplitude by a somehow perfect equilibrium between the input of energy from the wind and the structural dissipation. This corresponds to one of the periodic solutions in \( \mathbb{P} \) found in Theorem 2. We are thus led to investigate whether “nearly periodic” longitudinal oscillations suddenly transform into (wide) torsional oscillations as in several bridges; to this end, we focus on the behavior of the special class of solutions of (1) having the form

\[
w(\xi, t) = U(t)L_{1,1}(\xi) + V(t)T_{m,2}(\xi),
\]

namely possessing only two nontrivial modes, one longitudinal and one torsional. To obtain such solutions we associate to (1) the initial conditions

\[
w(\xi, 0) = U_0L_{1,1}(\xi) + V_0T_{m,2}(\xi), \quad w_t(\xi, 0) = U_1L_{1,1}(\xi) + V_1T_{m,2}(\xi).
\]

Assuming again that \( f \) has the form (18), if we replace (42) into (1) we find that \((U, V)\) solves the nonlinear system

\[
\begin{aligned}
\dot{U}(t) + \delta U(t) + (\mu_{1,1} - P)U(t) + S[U(t)^2 + m^2V(t)^2]U(t) &= A(t), \\
\dot{V}(t) + \delta V(t) + (\nu_{m,2} - P)V(t) + S[U(t)^2 + m^2V(t)^2]V(t) &= 0,
\end{aligned}
\]

while the initial conditions (43) become

\[
U(0) = U_0, \quad \dot{U}(0) = U_1, \quad V(0) = V_0, \quad \dot{V}(0) = V_1.
\]

If \( V_0 = V_1 = 0 \) then \( w(\xi, t) \) is proportional to \( L_{1,1} \), see Corollary 1. Moreover, harking back to Theorem 2, if we assume (19) we know that \( w \) will converge to some \( \varphi \in \mathbb{P} \) as \( t \to \infty \). The ideal situation would be that \( w \) coincides with one \( \varphi \in \mathbb{P} \), with the explicit form of \( \varphi \) being available; however, this is the case only in very particular situations, see Section 3.2. The natural question is then to investigate whether a suitable “smallness condition” on \( V_0 \) and \( V_1 \) implies that the \( V \)-component of \( w \) in (42) remains small for all \( t \), possibly converging to 0 at infinity: this would mean that the three periodic solutions in \( \mathbb{P} \) also attract solutions of (1) with two modes such as (42). A full answer to this question appears far from being trivial, since already at a linear level the available stability criteria are very few and fairly involved; cf. [36]. Furthermore, a linear analysis may not reflect the real behavior of structures, since linear instability does not necessarily correspond to the onset of large torsional oscillations. Therefore, we limit ourselves to the numerical simulations shown in Figures 3-5, for which we fix \( m = 2 \) (thus considering only the second torsional mode) and we use the approximations (7) and (8) of the eigenvalues \( \mu_{1,1} \) and \( \nu_{2,2} \). Moreover, we set

\[
\delta = 6, \quad P = 5, \quad S = 100, \quad f(\xi, t) = 0.005 \cos(9t),
\]

integrating (44) on the time interval \([0, 20]\). Notice that (19) is here fulfilled. For Figure 3, we take \( U_0 = 0.2 \) (in view of (22)) and \( V_0 = 0.01 \); we see that \( V(t) \)
converges very rapidly to 0 and $U(t)$ converges to $z_2$ (using the notation of Theorem 2), suggesting that $\wp^2$ is asymptotically stable.

In Figure 3, we depict the solution of (44) with the same choices as in (46) and $U_0 = 0.2, V_0 = 0.01$. Here again $V(t) \to 0$ for $t$ large, but, due to the action of the $V$-component, $U$ may converge to $z_3$ even if it starts near $z_2$. On increasing of $V_0$, one can come back to the situation of Figure 3, where the $U$ component converges to $\wp^2$ (Figure 5).

It thus seems that, under the assumptions of Theorem 2 (in particular, in presence of a large damping), the periodic solutions $\wp^2$, $\wp^3$ indeed attract the solutions of
(44), but the law according to which these “choose” to converge to \(\varphi^2\) or to \(\varphi^3\) appears unclear. Our feeling is that there exist alternating intervals of initial data for which the solution converges either to \(\varphi^2\) or \(\varphi^3\) and, as the total energy increases, the number of back-and-forth across a neighborhood of the saddle point of the energy function also increases.

The notion of torsional instability we deal with was introduced in [18] (see also [17]), and we here recall it.

**Definition 2.** Let \(T_W > 0\). We say that a weak solution \(u\) of (1) having the form (42) is **unstable** before time \(T > 2T_W\) if there exists a time instant \(\tau\) with \(2T_W < \tau < T\) such that

\[
\frac{\|V\|_{L^\infty(0,\tau)}}{\|V\|_{L^\infty(0,\tau/2)}} > 10. \tag{47}
\]

We say that \(u\) is **stable** until time \(T\) if (47) is not fulfilled for any \(\tau \in (2T_W, T)\).

This definition extends in a quantitative way the classical linear instability to a nonlinear context, since it requires an “exponential-like” behavior of \(V(t)\) in order to fulfill condition (47); for more details, see [17, 18]. Such a growth condition thus allows to highlight abrupt changes in the nature of the oscillations, from longitudinal to torsional, being more in line with the behavior of real structures. Our aim is to check it in relation to the values of the parameters appearing in (1). We proceed numerically and we first notice that if we set \((U(t), V(t)) = \alpha(U(t), V(t))\), for \(\alpha > 0\), we obtain the system

\[
\begin{align*}
\dot{U}(t) + \delta U(t) + (\mu_{1,1} - P)U(t) + S[U(t)^2 + m^2V(t)^2]U(t) &= \hat{A}(t) \\
\dot{V}(t) + \delta V(t) + (\mu_{2,2} - P)V(t) + S[U(t)^2 + m^2V(t)^2]V(t) &= 0,
\end{align*}
\]

where

\[
\hat{S} = S\alpha^2, \quad \|\hat{A}\|_{L^\infty} = \frac{\|A\|_{L^\infty}}{\alpha}.
\]

Hence, we may scale (44) fixing either \(S\) or \(\|A\|_{L^\infty}\). We fix \(S = 100\) and we check condition (47) on varying of the parameters \(\delta\) and \(\|A\|_{L^\infty}\). Moreover, wishing to highlight the relationships between the time-frequency of the forcing term \(f\) and instability as well (cf. [7]), we fix \(f(\xi, t) = M \cos(\nu t)L_{1,1}(\xi)\), so that \(\|A\|_{L^\infty} = M\) and variations in the time-frequency of \(f\) correspond to variations of the parameter \(\nu\). Finally, as we want to reproduce the dynamics of the plate giving more emphasis to the action of the external forcing (in line with the previous discussion), we fix small initial conditions both on the longitudinal and on the torsional component, by setting \(U_0 = V_0 = 0.01\) and \(U_1 = V_1 = 0\) in (45). This position, which agrees with the fact that torsional oscillations are initially very small, allows not to undergo the influence of the initial width of longitudinal oscillations in spotting instability, enabling to better identify the role of the parameters in the dynamics of the structure.

We show some numerical results whose response in terms of stability is analyzed with the use of the same algorithm as in [18, Section 4.5]. In Figure 6, we show the plot of the \(V\)-component of the solutions of (44) on the time interval \([0, T] := [0, 16]\) for

\[
\delta = 0.01, \quad P = 5, \quad S = 100, \quad \nu = 9, \quad M = 5 \cdot 10^5 \div 7 \cdot 10^5, \tag{48}
\]

where the increase step in the amplitude \(M\) of the forcing term is equal to 2000. Such components are all plotted within the range \([-6, 6]\) as for the \(y\)-axis, in order to
highlight the huge differences between solutions with different initial amplitudes. By use of the above mentioned algorithm, we checked that the corresponding solution $u$ of (1) in the form (42) is unstable for

$$M \in \mathcal{M}_1 = [508000, 558000], \; M = 604000, \; M \in \mathcal{M}_2 = [608000, 682000]$$

(the corresponding pictures being framed by black rectangle-like polygons in Figure 6), while it is stable otherwise. Moreover, the time $\tau$ in correspondence of which instability is observed has a quite neat trend: it starts being close to $T = 16$ in correspondence of the left endpoints of $\mathcal{M}_1$ and $\mathcal{M}_2$, then it reaches a (small) minimum while proceeding inside such intervals, before growing again until stability is acquired once more.

In Figure 7, we let the frequency vary, taking $\delta = 0.01$, $P = 5$, $S = 100$, $M = 5 \cdot 10^5$, $\nu = 0 \div 19.8$, (49) with $\nu$ growing each time with step 0.2. Again, the situations where instability is spotted are framed by black boxes. Notice that for $\nu = 0$, corresponding to the time-constant forcing term $f(\xi, t) = ML_{1,1}(\xi)$, we are in an instability situation.

Finally, as for the damping $\delta$, we proceed with the choices

$$P = 5, \; S = 100, \; M = 5.5 \cdot 10^5, \; \nu = 9, \; \delta = 0 \div 0.495,$$

(50) increasing $\delta$ with step 0.005 each time. Here we still find unstable solutions alternating with stable ones at the beginning (Figure 8), but then we are always in a stable regime.

According to all these figures, we conclude that, on varying of the parameters $M$ and $\nu$,

the portrait for system (44) highlights an alternation of stability and instability regions.

This pattern - which is also typical within other equations such as, e.g., the Mathieu one [27] - has somehow already been spotted (on growing of the initial datum) in
Figures 3-5, where $\psi^2$ and $\psi^3$ seem to alternatively attract the solutions of (44) on varying of $V_0$. On the other hand, focusing on the role of the damping $\delta$, we infer that

there exists a threshold $\delta_0 > 0$ such that

if $\delta \geq \delta_0$, then $V(t) \to 0$ for $t \to \infty$.

An analytic proof of such claims appears tough and particularly challenging and will be the object of a future research.

Appendix. In this section, we collect some of the tools used along the paper and we recall some terminology for the reader’s ease. First, we briefly list some of the results in [22] regarding the existence of periodic solutions for the Duffing equation with damping

$$\ddot{\psi} + \varepsilon \dot{\psi} - \psi + \psi^3 = g(t),$$

with $\varepsilon > 0$ and $g \in L^\infty(\mathbb{R})$.

**Proposition 10.** [22, Theorem 1.1] Let $\|g\|_{L^\infty} < 2/3 \sqrt{3}$. Then, equation (51) has a unique solution $\psi_0 \in W^{2,\infty}(\mathbb{R})$ such that $\|\psi_0\|_{L^\infty} < 1/\sqrt{3}$. If moreover

$$\varepsilon \geq 2 \sqrt{2} \quad \text{and} \quad \|g\|_{L^\infty} < 2 \left( \frac{5}{3} \sqrt{\frac{5}{3} - 2} \right),$$

then (51) has a unique solution $\psi_+$ and a unique solution $\psi_-$ in $W^{2,\infty}(\mathbb{R})$ such that

$$\|\psi_+ - 1\|_{L^\infty} < \sqrt{\frac{5}{3} - 1}, \quad \|\psi_- + 1\|_{L^\infty} < \sqrt{\frac{5}{3} - 1}.$$

**Proposition 11.** [22, Theorem 1.2] Let $g \in C_b(\mathbb{R})$. If

$$\varepsilon \geq 2 \sqrt{2} \quad \text{and} \quad \|g\|_{L^\infty} < \frac{\varepsilon}{32 \sqrt{\varepsilon + 1}},$$

(52)
then any solution $\psi$ of (51) on some half-line $J = (t_0, +\infty)$ is asymptotic to one of the three solutions $\psi_0, \psi_+, \psi_-$ given by Proposition 10.

Proposition 12. [22, Corollary 1.3] Under the assumptions of Proposition 11, if $g$ is almost periodic then (51) has exactly three almost periodic solutions $\psi_0, \psi_+, \psi_-$. If $g$ is $T$-periodic for some $T > 0$, then $\psi_0, \psi_+, \psi_-$ are $T$-periodic.

Proposition 13. [22, Corollary 1.4] Under the assumptions of Proposition 11, if $g$ is $T$-periodic then (51) has no subharmonic solutions.

Proposition 14. [22, Lemma 6.1] Let the second inequality in (52) hold. Then, for any $\delta < 1/16$, there exists $\eta > 0$ such that the conditions

$$|\psi(t_0) - 1| \leq \delta \quad \text{and} \quad |\psi'(t_0)| \leq \eta$$

imply

$$\forall t \geq t_0, \quad |\psi(t) - 1| \leq \frac{1}{4}$$

and

$$\limsup_{t \to +\infty} |\psi(t) - 1| \leq \frac{4\sqrt{2}\sqrt{1 + \varepsilon^2}}{\varepsilon} \|g\|_{L^\infty}.$$

If also the first inequality in (52) holds, then under the same assumptions it holds

$$\lim_{t \to +\infty} (|\psi(t) - \psi_+(t)| + |\psi'(t) - \psi_+'(t)|) = 0.$$

When studying the undamped Duffing equation (namely (51) with $\varepsilon = 0$), a tool which has been largely exploited throughout the paper arises: the Jacobi functions $\text{sn}, \text{cn}, \text{dn}$. We briefly recall their expressions and their properties, referring the reader to [1] for further details.

After having defined

$$u = \int_0^\varphi \frac{d\theta}{\sqrt{1 - m \sin^2 \theta}}$$

Figure 8. The $V$-component of the solution of (44) for values of the parameters given by (50).
for $0 \leq m \leq 1$, the elliptic sine $sn\,u$, the elliptic cosine $cn\,u$ and the delta-amplitude $dn\,u$ are respectively given by

$$sn\,u = \sin \varphi, \quad cn\,u = \cos \varphi, \quad dn\,u = \sqrt{1 - m \sin^2 \varphi}.$$  

By definition, such functions satisfy the equalities

$$sn^2 u + cn^2 u = 1, \quad cn^2 u + (1 - m) sn^2 u = dn^2 u; \quad (53)$$

moreover, it can be seen that they also satisfy the three differential equations

$$\frac{d}{du} sn\,u = cn\,u \, dn\,u, \quad \frac{d}{du} cn\,u = -sn\,u \, dn\,u, \quad \frac{d}{du} dn\,u = -m \, sn\,u \, cn\,u. \quad (54)$$

Finally, the Jacobi functions solve some particular Duffing equations: using (53) and (54), one sees that $sn\,t$, $cn\,t$ and $dn\,t$ respectively satisfy the differential equations

$$\ddot{\psi} + (1 + m)\psi - 2m\psi^3 = 0, \quad \ddot{\psi} + (1 - 2m)\psi + 2m\psi^3 = 0, \quad \ddot{\psi} - (2 - m)\psi + 2\psi^3 = 0.$$  

An alternative way of indicating $sn\,u$, $cn\,u$, $dn\,u$ is to write $sn(u, k)$, $cn(u, k)$, $dn(u, k)$, highlighting their dependence on the so-called elliptic modulus $k = \sqrt{m}$; this is also the notation used in Section 3.2, where the mentioned properties of these elliptic functions have been exploited.

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