Multiple finite Riemann zeta functions

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Abstract

Observing a multiple version of the divisor function we introduce a new zeta function which we call a multiple finite Riemann zeta function. We utilize some $q$-series identity for proving the zeta function has an Euler product and then, describe the location of zeros. We study further multi-variable and multi-parameter versions of the multiple finite Riemann zeta functions and their infinite counterparts in connection with symmetric polynomials and some arithmetic quantities called powerful numbers.

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1 Introduction

The divisor function $\sigma_k(N) := \sum_{d \mid N} d^k$ is a basic multiplicative function and plays an important role from the beginning of the modern arithmetic study, and in particular, $\sigma_k(N)$ appears in the Fourier coefficients of the (holomorphic) Eisenstein series $E_{k+1}(\tau)$. Usually perhaps, it is not so common to regard $\sigma_k(N)$ as a sort of zeta function. However, in the present paper we treat the divisor function $\sigma_k(N)$ as a function of a complex variable $k = -s$. Still, it is clear that when $N \to \infty$ through factorial (or 0) we have $\sigma_{-s}(N) \to \zeta(s)$, the Riemann zeta function, and there are at least two interpretations of

$$Z_N^1(s) := \sum_{n \mid N} n^{-s} = \sigma_{-s}(N)$$

as a zeta function in number theory:

- Fourier coefficients of real analytic Eisenstein series,
- Igusa zeta functions.

Concerning the first, we refer to Bump et al [BCKV] where the so-called “local Riemann hypothesis” is studied. In case of the real analytic Eisenstein series $E(s, \tau)$ for the modular group $SL(2, \mathbb{Z})$, the $N$-th Fourier coefficient is essentially given by

$$c_N(s, \tau) := Z_N^1(2s - 1)K_{s - \frac{1}{2}}(2\pi N \Im(\tau))e^{2\pi i \Re(\tau)}$$
and hence it satisfies the local Riemann hypothesis: if $c_N(s, \tau) = 0$ then $\text{Re}(s) = \frac{1}{2}$. On the other hand, for the second, the interpretation is coming from the (global) Igusa zeta function $\zeta^{\text{Igusa}}(s, R)$ of a ring $R$ defined as

$$\zeta^{\text{Igusa}}(s, R) := \sum_{m=1}^{\infty} \# \text{Hom}_{\text{ring}}(R, \mathbb{Z}/(m))m^{-s}.$$ 

Then, in fact, it is easy to see that

$$Z^1_N(s) = \zeta^{\text{Igusa}}(s, \mathbb{Z}/(N)).$$

Thus, the purpose of the present paper is initially to study the function defined by the series

(1.1) $$Z^m_N(s) := \sum_{n_1|n_2|\cdots|n_m|N} (n_1n_2\cdots n_m)^{-s}.$$ 

We call $Z^m_N(s)$ the multiple finite Riemann zeta function of type $N$. We then study several basic properties of $Z^m_N(s)$ such as an Euler product, a functional equation and an analogue of the Riemann hypothesis in an elementary way by the help of some $q$-series identity. We also study the limit case $Z^m_\infty(s) := \lim_{N \to \infty} Z^m_N(s)$, $Z^m_0(s) = \sum_{n_1|n_2|\cdots|n_m}(n_1n_2\cdots n_m)^{-s}$. Here and throughout the paper we consider the limit $N \to \infty$ as the one through factorials.

Moreover, we generalize this zeta function in two directions. The first one is to increase the number of variables. We prove that the Euler product of a multi-variable version of the zeta function is expressed in terms of the complete symmetric polynomials with a remarkable specialization of variables (Theorem 3.2). The second one is further to add parameters indexed by a set of positive integers. For general parameters it seems difficult to calculate an explicit expression of the Euler product using symmetric functions with some meaningful specialization of variables. However, when we restrict ourselves to the one variable case, under a special but non-trivial specialization of parameters, we show that the corresponding multiple zeta functions are written as a product of Riemann’s zeta functions $\zeta(cs)$ with several constants $c$’s determined by the given parameters and the Dirichlet series associated with generalized powerful numbers (see Section 4 for the definition). Moreover, we determine the condition whether the Dirichlet series associated with such generalized powerful numbers can be extended as a meromorphic function to the entire complex plane $\mathbb{C}$ or not (see Theorem 4.8 and its corollary). As a consequence, the most of such Dirichlet series are shown to have the imaginary axis as a natural boundary. The result is a generalization of the one in [IS].

In the final position of the paper, we make a remark on the relation between the multiple finite Riemann zeta functions and the number of isomorphism classes of abelian groups. In particular, using the Tauberian theorem, for a given positive integer $m$, we prove that the asymptotic average of the isomorphism classes of abelian groups of order $n$ which are given by the direct sum of $p$-groups $A_p$ such that $p^mA_p = 0$, equals $\zeta(2)\zeta(3)\cdots \zeta(m)$ (when $n$ tends to $\infty$). We make also a small
discussion on an analogous notion of (holomorphic) multiple Eisenstein series which are defined via these multiple finite Riemann zeta functions as coefficients of its Fourier series (or related to the so-called Lambert type series).

Throughout the paper, we denote the sets of all integers, positive integers, non-negative integers, real numbers and complex numbers by \( \mathbb{Z} \), \( \mathbb{Z}_{>0} \), \( \mathbb{Z}_{\geq0} \), \( \mathbb{R} \) and \( \mathbb{C} \) respectively.

## 2 Multiple finite Riemann zeta function

In this section we prove the fundamental properties of the multiple finite Riemann zeta function \( Z^m_N(s) \) defined by (1.1) and make a discussion on some related Dirichlet series. Namely we first show the following theorem.

**Theorem 2.1.** Let \( N \) be a positive integer.

1. **Euler product** :

\[
Z^m_N(s) = \prod_{p \text{ prime}} \prod_{k=1}^{m} \frac{1 - p^{-s(\ord_p N + k)}}{1 - p^{-sk}},
\]

where \( \ord_p N \) denotes the order of \( p \)-factor in the prime decomposition of \( N \).

2. **Functional equation** :

\[
Z^m_N(-s) = N^{-ms} Z^m_N(s).
\]

3. **Analogue of the Riemann hypothesis** : All zeros of \( Z^m_N(s) \) lie on the imaginary axis \( \Re s = 0 \).

More precisely, the zeros of \( Z^m_N(s) \) are of the form \( s = \frac{2\pi in}{(\ord_p N + k) \log p} \) for \( k = 1, \ldots, m, p|N \) and \( n \in \mathbb{Z} \setminus \{0\} \). Consequently, the order \( \text{Mult}^m(n, p, k) \) of the zero at \( s = \frac{2\pi in}{(\ord_p N + k) \log p} \) is given by \( \text{Mult}^m(n, p, k) = \#\{(l, j) : 1 \leq l \leq m, \ j \in \mathbb{Z} \setminus \{0\}, \ (\ord_p N + k)j = (\ord_p N + l)n\} \).

4. **Special value** : When \( n \) is a positive integer one has \( Z^m_N(-n) \in \mathbb{Z} \).

For the proof of the theorem the following lemma is crucial.

**Lemma 2.2.** Let \( m \) be a positive integer. Then

1. For any integers \( l \geq 0 \) it holds that

\[
\sum_{d=0}^{l} \left\lfloor \frac{m-1+d}{m-1} \right\rfloor q^d = \left\lfloor \frac{m+l}{m} \right\rfloor q^l,
\]

where \( \left\lfloor \frac{n}{k} \right\rfloor_q \) is the \( q \)-binomial coefficient defined by \( \left\lfloor \frac{n}{k} \right\rfloor_q = \prod_{j=1}^{k} (1 - q^{n+1-j})/(1 - q^j) \).
2. For $|x| < 1$, $|q| < 1$ it holds that

$$\sum_{d=0}^{\infty} \sum_{m} \binom{m+d}{m} x^d = \prod_{k=0}^{m} \frac{1}{1-q^k x}.$$  

Proof. We prove the formula (2.3) by induction on $l$. When $l = 0$ the formula (2.3) clearly holds. Suppose that it holds for $l$. Then we see that

$$\sum_{d=0}^{l+1} \binom{m-1+d}{m-1} q^d = \binom{m+l}{m} q^{l+1}$$

$$= \prod_{k=1}^{m} (1-q^k)^{-1} \prod_{k=1}^{m-1} (1-q^{l+1+k}) \left\{ (1-q^{l+1}) + (1-q^m)q^{l+1} \right\}$$

$$= \prod_{k=1}^{m} (1-q^k)^{-1} \prod_{k=1}^{m} (1-q^{l+1+k}) = \binom{m+l}{m} q^{l+1},$$

whence (2.3) is also true for $l + 1$.

The second formula (2.4) can be proved in the same manner by induction, but on $m$. (It is also obtained from the so-called $q$-binomial theorem. See, e.g. [AAR]). This completes the proof of the lemma.

Proof of Theorem 2.1. The functional equation (2.2) is easily seen from the definition. Actually, we have

$$Z_N^m(-s) = \sum_{n_1|n_2|\cdots|n_m|N} (n_1 n_2 \cdots n_m)^s = N^{ms} \sum_{n_1|n_2|\cdots|n_m|N} \left( \frac{n_1}{N} \frac{n_2}{N} \cdots \frac{n_m}{N} \right)^s = N^{ms} Z_N^m(s).$$

Next we prove the Euler product (2.1). We first show that $Z_N^m(s)$ is multiplicative with respect to $N$. In fact, if we suppose $N$ and $M$ are co-prime, then we observe

$$Z_{NM}^m(s) = \sum_{n_1|n_2|\cdots|n_m|NM} (n_1 n_2 \cdots n_m)^{-s}$$

$$= \sum_{c_1|c_2|\cdots|c_m|N} (c_1 c_2 \cdots c_m)^{-s} \sum_{d_1|d_2|\cdots|d_m|M} (d_1 d_2 \cdots d_m)^{-s} = Z_N^m(s)Z_M^m(s),$$

because every divisor $n$ of $NM$ is uniquely written as $n = cd$ where $c|N$ and $d|M$. By means of this fact, in order to get the Euler product expression (2.1) of $Z_N^m(s)$ it suffices to calculate the case where $N$ is a power of prime $p$. In this case, one proves the expression

$$(2.5) \quad Z_p^m(s) = \prod_{k=1}^{m} \frac{1 - p^{-(l+k)s}}{1 - p^{-sk}}$$
by induction on \( m \) as follows: It is clear that (2.5) holds for \( m = 1 \). Now, suppose (2.5) is true for \( m-1 \). Then we see that

\[
Z^m_{p^l}(s) = \sum_{j_1 \leq \cdots \leq j_m \leq l} p^{-(j_1+\cdots+j_m)s} \prod_{d=0}^{l-1} \frac{1 - p^{-(d+k)s}}{1 - p^{-sk}} p^{-ds} = \sum_{j_m=0}^{l} Z^{m-1}_{p^{l}}(s) p^{-j_m s} = \sum_{d=0}^{l} \left[ m - 1 + \frac{d}{d} \right] p^{-ds}.
\]

Therefore the assertion follows immediately from the formula (2.3) in the lemma above. This proves (2.5), whence the Euler product for \( Z^m_N(s) \) follows.

Using the Euler product (2.1), we observe that the meromorphic function \( Z^m_N(s) \) may have zeros at each \( s = \frac{2\pi in}{(\text{ord}_p N + k) \log p} \) for \( k = 1, \ldots, m, p|N \) and \( n \in \mathbb{Z} \). Note however, since

\[
Z^m_N(0) = \prod_{p: \text{prime}} \left( \frac{\text{ord}_p N + m}{m} \right) \neq 0,
\]

\( s = 0 \) is not a zero of \( Z^m_N(s) \). Suppose now

\[
\frac{2\pi in}{(\text{ord}_p N + k) \log p} = \frac{2\pi im}{(\text{ord}_q N + l) \log q}
\]

holds for some \( 1 \leq k, l \leq m, p, q|N \) and \( n, j \in \mathbb{Z} \setminus \{0\} \). Then it becomes \( p^{(\text{ord}_p N + k)j} = q^{(\text{ord}_q N + l)n} \). This immediately shows that \( p = q \) and \( (\text{ord}_p N + k)j = (\text{ord}_p N + l)n \). Hence the order of zero at \( s = \frac{2\pi in}{(\text{ord}_p N + k) \log p} \) is actually given by \( \text{Mult}^m(n, p, k) \). Obviously, one has \( \text{Mult}^1(n, p, 1) = 1 \).

The last claim about the special value of \( Z^m_N(s) \) is clear from the definition. This completes the proof of the theorem.

**Remark 2.1.** Note that \( Z^m_p(s) = Z^l_{p^m}(s) \) from (2.5).

Letting \( N \to \infty \) (or \( N \to 0 \)) in Theorem 2.1, we have the following

**Corollary 2.3.** Define a function \( Z^m_{\infty}(s) \) by

\[
Z^m_{\infty}(s) := \sum_{n_1 n_2 \cdots n_m \in \mathbb{Z} > 0} (n_1 n_2 \cdots n_m)^{-s} \quad (\text{Re } s > 1).
\]

Then we have \( Z^m_{\infty}(s) = \prod_{k=1}^{m} \zeta(ks) \). Here \( \zeta(s) \) is the Riemann zeta function.

We next consider a Dirichlet series defined via the multiple finite Riemann zeta functions.

**Corollary 2.4.** Retain the notation in the corollary above. Define a Dirichlet series \( \zeta^m(s) \) by

\[
\zeta^m(s) := \sum_{n=1}^{\infty} Z^m_n(s)n^{-s}.
\]
Then we have

\[ \zeta^m(s) = \prod_{k=1}^{m+1} \zeta(sk) = Z_{\infty}^{m+1}(s). \]

**Proof.** Since \( Z_n^m(s) \) is multiplicative relative to \( n \), the Dirichlet series \( \zeta^m(s) \) has an Euler product of the form

\[ \zeta^m(s) = \prod_{p: \text{prime}} \left( \sum_{l=0}^{\infty} Z_p^m(s)p^{-sl} \right). \]

Hence the first equality follows from Lemma 2.2. In fact,

\[ \sum_{l=0}^{\infty} Z_p^m(s)p^{-sl} = \sum_{l=0}^{m} \prod_{k=1}^{m+1} \frac{1 - p^{-s(k+l)}}{1 - p^{-sk}} p^{-sl} = \prod_{k=1}^{m+1} \frac{1}{1 - p^{-sk}} \]

when \( \Re s > 0 \). The second equality is obvious from Corollary 2.3. This proves the corollary. \( \square \)

**Remark 2.2.** Let \( g(n) \) be the number of the isomorphism classes of the abelian groups of order \( n \). Then we have (see, e.g. [Z, p.16])

\[ \lim_{m \to \infty} \zeta^m(s) = \prod_{k=1}^{\infty} \zeta(k) = \sum_{n=0}^{\infty} g(n)n^{-s} \quad (\Re s > 1). \]

Observing the analytic property of \( Z_n^m(s) \), we have much precise information about the isomorphism classes and their asymptotic averages. For details, see Section 5. \( \square \)

We now slightly generalize the discussion above. Put

\[ \zeta^m(s, t) := \sum_{n=1}^{\infty} Z_n^m(s)n^{-t}. \]

It is clear that \( \zeta^m(s) = \zeta^m(s, s) \). Then, by the same reasoning, \( \zeta^m(s, t) \) has the Euler product

\[ \zeta^m(s, t) = \prod_{p: \text{prime}} \left( \sum_{l=0}^{\infty} Z_p^m(s)p^{-lt} \right) = \prod_{k=0}^{m} \zeta(sk + t). \]

Actually, it follows from the formula (2.4) that

\[ \sum_{l=0}^{\infty} \left( \prod_{k=1}^{m} \frac{1 - p^{-s(k+l)}}{1 - p^{-sk}} p^{-lt} \right) = \prod_{k=0}^{m} \frac{1}{1 - p^{-sk-t}} \]

for \( \Re s > 0 \) and \( \Re t > 0 \). This is a generalization of the formula (2.7). One may consider \( \zeta^m(s, t) \) as a zeta function for two variables. We give here a few examples:

**Example 2.1.** The following is quite well-known. For \( \Re t > k + 1 \), we have

\[ \zeta^1(-k, t) = \sum_{n=1}^{\infty} Z_n^1(-k)n^{-t} = \sum_{n=1}^{\infty} \sigma_k(n)n^{-t} = \zeta(t)\zeta(t - k). \]
Example 2.2. From (2.6), for Re \( t > 1 \), it follows that
\[
\zeta^m(0, t) = \sum_{n=1}^{\infty} Z^m_n(0)n^{-t} = \sum_{n=1}^{\infty} \prod_{p \text{ prime}} \left( m + \text{ord}_p n \right)^{-t} = \zeta(t)^{m+1}.
\]

Example 2.3. Let \( l \) be a positive integer. For Re \( t > 1 - l \), we have
\[
\zeta^m(l, t + l) = \prod_{k=1}^{m+1} \zeta(t + lk).
\]
As \( m \to \infty \), \( \lim_{m \to \infty} \zeta^m(l, t + l) = \prod_{k=1}^{\infty} \zeta(s + lk) \) is the higher Riemann zeta function studied in [KMW] (see also [KW]). It should be noted that the higher Riemann zeta function \( \zeta_{\infty}(s) := \prod_{k=1}^{\infty} \zeta(s + lk) \) possesses a functional equation.

3 Multivariable version

We generalize the definition (1.1) of the multiple finite Riemann zeta functions \( Z^m_N(s) \). For \( \gamma = (\gamma_1, \ldots, \gamma_m) \in \mathbb{Z}^m_{>0} \) and \( N \in \mathbb{Z}_{>0} \), we define
\[
Z^\gamma_N(t_1, \ldots, t_m) := \sum_{n_1, \ldots, n_m > 0, \text{ such that } \sum_{j=1}^{m} \gamma_j n_j = N} n_1^{-\gamma_1 t_1} \cdots n_m^{-\gamma_m t_m}.
\]
This is multiplicative with respect to \( N \). We also notice that \( Z^m_N(s) = Z^\gamma_N(s, \ldots, s) \) when \( \gamma = (1^m) = (1, \ldots, 1) \). We can prove the following lemma in a similar way to Theorem 2.1.

Lemma 3.1. For \( l \geq 0 \), define a function \( G^\gamma_l(q_1, q_2, \ldots, q_m) \) by
\[
G^\gamma_l(q_1, \ldots, q_m) = \sum_{l \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 0} q_1^{\lambda_1} \cdots q_m^{\lambda_m},
\]
where the sum is taken over all partitions \( \lambda = (\lambda_1, \lambda_2, \ldots) \) of length \( \leq m \) such that \( \lambda_1 \leq l \) and \( \gamma_j|\lambda_j \) for \( 1 \leq j \leq m \). Then the Euler product of \( Z^\gamma_N(t_1, \ldots, t_m) \) is given as
\[
Z^\gamma_N(t_1, \ldots, t_m) = \prod_{p \text{ prime}} G^\gamma_{\text{ord}_p N}(p^{-t_1}, \ldots, p^{-t_m}).
\]

Remark 3.1. Let \( f(n_1, \ldots, n_m) \) be a function defined on \( \in \mathbb{Z}^m_{>0} \) which is multiplicative with respect to each variable \( n_j \). Then, in general, we can define a multiple zeta function by
\[
Z^\gamma_N(f) = \sum_{n_1, \ldots, n_m > 0, \text{ such that } \sum_{j=1}^{m} \gamma_j n_j = N} f(n_1, \ldots, n_m).
\]
Actually, one can show that $Z_N^\gamma(f)$ is multiplicative with respect to $N$, whence has the Euler product as $Z_N^\gamma(f) = \prod_{p: \text{prime}} Z_{p^{|\gamma|}}^\gamma \cdot n(f)$.

We first look at the simplest case where $\gamma = (1^m)$. We abbreviate respectively $Z_N^{(1^m)}(t_1, \ldots, t_m)$ and $G_l^{(1^m)}(q_1, \ldots, q_m)$ to $Z_N(t_1, \ldots, t_m)$ and $G_l(q_1, \ldots, q_m)$.

**Theorem 3.2.** Let $h_j(x_1, \ldots, x_m)$ be the $j$-th complete symmetric polynomial defined by

$$h_j(x_1, \ldots, x_m) = \sum_{i_1 + \cdots + i_m = j, i_k \in \mathbb{Z}_{\geq 0}} x_1^{i_1} \cdots x_m^{i_m}.$$

Then we have

$$(3.3) \quad G_l^m(q_1, \ldots, q_m) = \sum_{j=0}^l h_j(q_1 q_2, \ldots, q_1 q_2 \cdots q_m).$$

In particular, we have

$$(3.4) \quad G_\infty^m(q_1, \ldots, q_m) := \lim_{l \to \infty} G_l^m(q_1, \ldots, q_m) = \prod_{k=1}^m \frac{1}{1 - q_1 q_2 \cdots q_k}.$$

**Proof.** Since

$$h_j(q_1 q_2, \ldots, q_1 q_2 \cdots q_m) = \sum_{i_1 + i_2 + \cdots + i_m = j} q_1^{i_1+i_2+\cdots+i_m} q_2^{i_2+i_3+\cdots+i_m} \cdots q_m^{i_m},$$

the first formula (3.3) is clear from the definition (3.2). The second formula (3.4) follows from (3.3) together with the fact that the generating function of complete symmetric polynomials is given by $\sum_{j=0}^\infty h_j(x_1, \ldots, x_m)z^j = \prod_{k=1}^m (1 - x_k z)^{-1}$ (see [M]).

As a corollary of the theorem, we obtain the Euler product of $Z_N^\gamma(t_1, \ldots, t_m)$.

**Corollary 3.3.** Let $m$ and $N$ be positive integers. Then we have

$$Z_N^m(t_1, \ldots, t_m) = \sum_{n_m | \cdots | n_1 | N} n_1^{-t_1} \cdots n_m^{-t_m}$$

$$= \prod_{p: \text{prime}} \left( \sum_{j=0}^{\ord_p N} h_j(p^{-t_1}, p^{-t_1-t_2}, \ldots, p^{-t_1-t_2-\cdots-t_m}) \right).$$

Further, for $\Re t_j > 1$ ($1 \leq j \leq m$), it follows that

$$Z_\infty^m(t_1, \ldots, t_m) := \sum_{n_m | \cdots | n_1} n_1^{-t_1} \cdots n_m^{-t_m} = \prod_{k=1}^m \zeta(t_1 + t_2 + \cdots + t_k).$$
The following lemma gives a recurrence equation among $G^<_1(q_1, \ldots, q_m)$’s.

**Lemma 3.4.** For $\gamma = (\gamma_1, \ldots, \gamma_m) \in \mathbb{Z}_{>0}^m$, we have

$$G^<_1(q_1, \ldots, q_m) = \sum_{n=0}^{\lfloor l/\gamma_1 \rfloor} q_1^{\gamma_1n} G_{\gamma_1n, \ldots, \gamma_m}(q_2, \ldots, q_m).$$

Here $\lfloor x \rfloor$ is the largest integer not exceeding $x$.

**Proof.** Observe that

$$G^<_1(q_1, \ldots, q_m) = \sum_{0 \leq n \leq l/\gamma_1} q_1^{\gamma_1n} \sum_{\gamma_1n \geq \gamma_2 \geq \cdots \geq \gamma_m \geq 0, \gamma_j | \lambda_j (2 \leq j \leq m)} q_2^{\lambda_2} \cdots q_m^{\lambda_m} = \sum_{n=0}^{\lfloor l/\gamma_1 \rfloor} q_1^{\gamma_1n} G_{\gamma_1n, \ldots, \gamma_m}(q_2, \ldots, q_m).$$

The following lemma shows that it is enough to study the case where $\gamma_1, \ldots, \gamma_m$ are relatively prime.

**Lemma 3.5.** For $\gamma = (dc_1, dc_2, \ldots, dc_m)$, we have

$$G^<_1(q_1, \ldots, q_m) = G_{[l/d]}(q_{d_1}, \ldots, q_{d_m}).$$

**Proof.** It is straightforward.

Let us calculate several examples of $G^<_1(q_1, \ldots, q_m)$ for special parameters $\gamma$ which are rather non-trivial.

**Example 3.1.** Let $\gamma = (c, 1)$ and calculate $G^{(c, 1)}_1(q_1, q_2)$. Putting $d = \lfloor l/c \rfloor$, we see by Lemma 3.4 that

$$G^{(c, 1)}_l(q_1, q_2) = \sum_{n=0}^{d} q_1^{cn} G^{1}_{cn}(q_2)$$

$$= \sum_{n=0}^{d} q_1^{cn} \frac{1 - q_2^{cn+1}}{1 - q_2} = \frac{1}{1 - q_2} \left( \frac{1 - q_1^{c(d+1)}}{1 - q_1^c} - q_2 \frac{1 - (q_1 q_2)^{c(d+1)}}{1 - (q_1 q_2)^c} \right)$$

$$= \frac{(1 - q_1^{c(d+1)})(1 - (q_1 q_2)^c) - q_2(1 - q_1^c)(1 - (q_1 q_2)^{c(d+1)})}{(1 - q_2)(1 - q_1^c)(1 - (q_1 q_2)^c)}.$$  \hspace{1cm} (3.5)

If we take a limit $l \to \infty$, we obtain

$$G^{(c, 1)}_{\infty}(q_1, q_2) = \frac{1 - q_2 + q_1 q_2 - q_1^c q_2^c}{(1 - q_2)(1 - q_1^c)(1 - q_1 q_2^c)}.$$
Therefore we have
\[
Z^{(c,1)}_{\infty}(t_1, t_2) = \sum_{n_1 \mid n_2} n_1^{-ct_1}n_2^{-t_2} = \zeta(t_2)\zeta(ct_1)\zeta(c(t_1 + t_2)) \prod_{p: \text{prime}} (1 - p^{-t_2} + p^{-ct_1-t_2} - p^{-ct_1-t_2})
\]
for Re\(t_j > 1\) \((j = 1, 2)\). Thus we may have various possibility of Euler products of the form \(\prod_{p: \text{prime}} H(p^{-s}, p^{-t})\), where \(H(S, T) \in 1 + S \cdot C[S, T] + T \cdot C[S, T]\) is arising from \(Z^{(c,1)}_{\infty}(t_1, t_2)\).

**Example 3.2.** We calculate \(G^{(cd,c,1)}(q, q, q)\). If we set \(q_1 = q_2 = q\) in (3.5), then we have
\[
G^{(c,1)}_1(q, q) = \frac{(1 - q^{(d+1)})(1 - q + q^c - q^{cd+1})}{(1 - q)(1 - q^{2c})}.
\]
It follows from Lemma 3.4 that
\[
G^{(cd,c,1)}(q, q, q) = \sum_{n=0}^{\infty} q^{cdn} G^{(c,1)}_{cdn}(q, q) = \sum_{n=0}^{\infty} q^{cdn} \frac{(1 - q^{c(dn+1)})(1 - q + q^c - q^{cdn+c+1})}{(1 - q)(1 - q^{2c})}
\]
\[
= \frac{1}{(1 - q)(1 - q^{2c})} \left\{ \frac{1 - q + q^c}{1 - q^{cd}} - \frac{q^c(1 + q^c)}{1 - q^{2cd}} + \frac{q^{2c+1}}{1 - q^{3cd}} \right\}.
\]
In particular, if we put \(d = 1\) then
\[
G^{(c,c,1)}_{\infty}(q, q) = \frac{1 - q + q^c - q^{c+1} + q^{2c+1} - q^{3c} + q^{3c+1} - q^{4c}}{(1 - q)(1 - q^{2c})(1 - q^{3c})} = \frac{1 - q + q^c - q^{c+1} + q^{2c}}{(1 - q)(1 - q^{2c})(1 - q^{3c})}.
\]
Hence we obtain
\[
Z^{(c,c,1)}_{\infty}(s) = \sum_{n_3 | n_2 | n_1^2} (n_1^n n_2^n n_3)^{-s} = \zeta(s)\zeta(2cs)\zeta(3cs) \prod_{p: \text{prime}} (1 - p^{-s} + p^{-cs} - p^{-(c+1)s} + p^{-2cs}).
\]

**4 Multiple zeta functions and powerful numbers**

In this section, we study \(Z_{\infty}^{(\gamma)}(s)\) for \(\gamma = (k, k, \ldots, k, 1)\) in connection with a certain generalized notion of **powerful numbers**.

Let us first recall the definition of powerful numbers. A positive number \(n\) is called a \(k\)-powerful number if \(\text{ord}_p n \geq k\) for any prime number \(p\) unless \(\text{ord}_p n = 0\) (see, e.g., [IS], [I]). Extending this we arrive at a new notion, **\(l\)-step \(k\)-powerful numbers**: a positive integer \(n\) is said to be an \(l\)-step \(k\)-powerful number if \(n\) satisfies the condition that \(\text{ord}_p n = 0, k, 2k, \ldots, (l-1)k\) or \(\text{ord}_p n \geq lk\) for any prime number \(p\). Clearly, if \(n\) is an \(l\)-step \(k\)-powerful number, then \(n\) is again a \(j\)-step \(k\)-powerful number for each \(j\) \((1 \leq j \leq l)\). In particular, \(1\)-step \(k\)-powerful numbers are nothing
but the usual $k$-powerful numbers. Note also that every natural number is an $l$-step 1-powerful number for any $l$; this agrees with the claim for $k = 1$ in Theorem 4.8 below.

As an example of $l$-step $k$-powerful numbers, for instance, the first few of 2-step 2-powerful numbers are $1, 4, 9, 16, 25, 32, 36, 49, 64, 81, 100, 121, 128, 144, 169, 196, 225, 243, \ldots$. We note that in general, an $l$-step $k$-powerful number $n$ has the following canonical representation:

$$n = a_1^k a_2^{2k} \cdots a_l^{(l-1)k} \times m,$$

where $a_1, \cdots, a_l$ are square-free, $m$ is $lk$-powerful and these satisfy $\gcd(a_1, \cdots, a_l, m) = 1$. Note that here a $k$-powerful number $m$ is uniquely expressed as $m = b_1^k b_2^{k+1} \cdots b_k^{2k-1}$ if we stipulate that $b_2, \cdots, b_k$ are all square-free.

Let us put

$$f_{k,l}(n) := \begin{cases} 1 & \text{if } n \text{ is an } l\text{-step } k\text{-powerful number,} \\ 0 & \text{otherwise} \end{cases}$$

for a positive integer $n$. We define also $F_{k,l}(s) := \sum_{n=1}^{\infty} f_{k,l}(n)n^{-s}$. This arithmetic function $f_{k,l}(n)$ is multiplicative with respect to $n$. Note that $F_{1,l}(s) = \zeta(s)$ for any $l$. We show that $Z_{\infty}(s)$ is represented by the product of the Riemann zeta functions times $F_{k,l}(s)$. Namely, we have the

**Theorem 4.1.** Let $k, l$ be positive integers and put $\gamma = (k, k, \ldots, k, 1)$. Then we have

$$Z_{\infty}(s) = \sum_{n_{l+1}|n_1^k | \cdots | n_l^k} (n_1^k \cdots n_{l+1}^k)^{-s} = F_{k,l}(s) \prod_{j=2}^{l+1} \zeta(jks) \quad (\text{Re } s > 1).$$

**Remark 4.1.** When $k = 1$, this theorem gives Corollary 2.3. \qed

To prove the theorem, we prepare the following two lemmas.

**Lemma 4.2.** Let $\gamma = (k, k, \ldots, k, 1)$. Then we have

$$G_{\infty}^\gamma(q) = G_{\infty}^\gamma(q, q, \ldots, q) = \prod_{j=1}^{l+1} \frac{1}{1 - q^{jk}} \cdot \frac{1 - q + q^{k+1} - q^{k(l+1)}}{1 - q}.$$
Proof. By definition, it follows that

\[
G_{\infty}^{(k,\ldots,k,1)}(q) = \sum_{\lambda_1 \geq \cdots \geq \lambda_l \geq \lambda_{l+1} \geq 0} q^{\lambda_1 + \cdots + \lambda_l + \lambda_{l+1}} = \sum_{n=0}^{\infty} q^n \sum_{\lambda_1 \geq \cdots \geq \lambda_l \geq n} q^{\lambda_1 + \cdots + \lambda_l}
\]

\[
= \sum_{\lambda_1 \geq \cdots \geq \lambda_l \geq 0} q^{\lambda_1 + \cdots + \lambda_l} + \sum_{a=0}^{\infty} q^{ak+b} \sum_{\lambda_1 \geq \cdots \geq \lambda_l \geq ak+b} q^{\lambda_1 + \cdots + \lambda_l}
\]

\[
= \sum_{\mu_1 \geq \cdots \geq \mu_l \geq 0} q^{k(\mu_1 + \cdots + \mu_l)} + \sum_{a=0}^{\infty} q^{ak+b} \sum_{\mu_1 \geq \cdots \geq \mu_l \geq a+1} q^{k(\mu_1 + \cdots + \mu_l)}
\]

\[
= G_{\infty}^l(q^k) + \sum_{a=0}^{\infty} q^{ak} \sum_{b=1}^{k} q^{b} \sum_{\nu_1 \geq \cdots \geq \nu_l \geq 0} q^{k(\nu_1 + \cdots + \nu_l)}
\]

\[
= G_{\infty}^l(q^k) + q^{lk} \left( \sum_{a=0}^{\infty} q^{a(k+1)} \right) \left( \sum_{b=1}^{k} q^{b} \right) \left( \sum_{\nu_1 \geq \cdots \geq \nu_l \geq 0} q^{k(\nu_1 + \cdots + \nu_l)} \right).
\]

Since \( G_{\infty}^l(q) = \prod_{j=1}^{l} (1 - q^j)^{-1} \), we have

\[
G_{\infty}^{(k,\ldots,k,1)}(q) = \prod_{j=1}^{l} \frac{1}{1 - q^k} \left( 1 + \frac{q^k}{1 - q^{k(l+1)}} \frac{1 - q^k}{1 - q} \right)
\]

\[
= \prod_{j=1}^{l+1} \frac{1}{1 - q^k} \frac{1 - q)(1 - q^{k(l+1)}) + q^{lk+1}(1 - q^k)}{1 - q}
\]

\[
= \prod_{j=1}^{l+1} \frac{1}{1 - q^k} \frac{1 - q + q^{lk+1} - q^{k(l+1)}}{1 - q}.
\]

This proves the assertion. □

Lemma 4.3. We have

\[
\frac{1 - q + q^{lk+1} - q^{k(l+1)}}{1 - q} = (1 - q^k) \left( 1 + q^k + q^{2k} + \cdots + q^{(l-1)k} + q^{lk} \sum_{j=0}^{\infty} q^j \right).
\]

Proof. The calculation is straightforward. Actually, we have

\[
(1 - q^k) \left( 1 + q^k + q^{2k} + \cdots + q^{(l-1)k} + q^{lk} \sum_{j=0}^{\infty} q^j \right)
\]

\[
= (1 - q^k) \left( \frac{1 - q^{lk}}{1 - q^k} + \frac{q^k}{1 - q} \right)
\]

\[
= \frac{1}{1 - q} \left( (1 - q^{lk})(1 - q) + q^k(1 - q^k) \right) = \frac{1 - q + q^{lk+1} - q^{k(l+1)}}{1 - q}.
\]

□
Proof of Theorem 4.1. It follows from Lemma 4.2 and Lemma 4.3 that

\[
Z_{\infty}^{(k,\ldots,k,1)}(s) = \prod_{p: \text{prime}} G_{\infty}^{(k,\ldots,k,1)}(p^{-s})
\]

\[
= \prod_{p: \text{prime}} \left( \prod_{j=2}^{\infty} \frac{1}{1 - p^{-jks}} \right) \times \prod_{p: \text{prime}} \left( 1 + p^{-ks} + p^{-2ks} + \cdots + p^{-(l-1)ks} + p^{-lks} + p^{-(l+1)ks} + \cdots \right)
\]

\[
= \prod_{j=2}^{l+1} \zeta(jks) \cdot F_{k,l}(s).
\]

This completes the proof of the theorem.

Now we determine the condition if the Dirichlet series \( F_{k,l}(s) \) can be meromorphically extended to the whole complex plane \( \mathbb{C} \). We recall the following Estermann’s result \([E]\) (see \([K]\) for a generalization). A polynomial \( f(T) \in 1 + T \cdot \mathbb{C}[T] \) is said to be unitary if and only if there is a unitary matrix \( M \) such that \( f(T) = \det(1 - MT) \).

Lemma 4.4. For a polynomial \( f(T) \in 1 + T \cdot \mathbb{C}[T] \), put \( L(s, f) = \prod_{p: \text{prime}} f(p^{-s}) \). Then

1. \( f(T) \) is unitary if and only if \( L(s, f) \) can be extended as a meromorphic function on \( \mathbb{C} \).

2. \( f(T) \) is not unitary if and only if \( L(s, f) \) can be extended as a meromorphic function in \( \text{Re } s > 0 \) with the natural boundary \( \text{Re } s = 0 \); each point on \( \text{Re } s = 0 \) is a limit-point of poles of \( L(s, f) \) in \( \text{Re } s > 0 \).

Since

\[
F_{k,l}(s) = \prod_{p: \text{prime}} \left( 1 + p^{-ks} + p^{-2ks} + \cdots + p^{-(l-1)ks} + p^{-lks} \sum_{j=0}^{\infty} p^{-js} \right)
\]

\[
= \zeta(s) \zeta(ks) \prod_{p: \text{prime}} \left( 1 - p^{-s} + p^{-(l+1)s} - p^{-(k+1)s} \right)
\]

by Lemma 4.3, we have only to see whether the polynomial \( G_{k,l}(T) := 1 - T + T^{lk+1} - T^{(l+1)k+1} \) is unitary or not. The polynomial \( G_{k,l}(T) \) can be expressed as \( G_{k,l}(T) = (1 - T^k)H_{k,l}(T) \) with \( H_{k,l}(T) := 1 + (T^k - T) \sum_{j=0}^{l-1} T^{kj} \).

Proposition 4.5. Let \( k \) and \( l \) be positive integers. The polynomial \( G_{k,l}(T) \) is unitary if and only if \( k = 1, 2 \).
In order to prove this proposition, we need the following two lemmas.

**Lemma 4.6.** Let $k$ be a positive integer such that $k \geq 3$. Then the unitary root $\alpha$ (i.e. $|\alpha| = 1$) of the polynomial $G_{k,l}(T)$ must satisfy $\alpha^k = 1$ or $\alpha^{k-2} = 1$.

**Proof.** Let $\alpha = e^{2\pi i \theta} \neq 1$ ($\theta \in \mathbb{R}$) be a unitary root of $G_{k,l}(T)$. Since $G_{k,l}(T)/(1-T) = 1+T^{k+1}(1-T^{k-1})/(1-T)$, we have $1+\alpha^{k+1}(1-\alpha^{-1})/(1-\alpha) = 0$ so that $|1-\alpha^{-1}|/(1-\alpha) = |\alpha^{-(k+1)}| = 1$. Hence we see that $\Re \alpha^{k-1} = \Re \alpha$, that is, $\cos 2\pi(k-1)\theta - \cos 2\pi \theta = -2 \sin \pi k \theta \sin \pi (k-2)\theta = 0$. Thus we conclude that either $k\theta \in \mathbb{Z}$ or $(k-2)\theta \in \mathbb{Z}$. This proves the lemma. \qed

**Lemma 4.7.** Let $k$ be a positive integer such that $k \geq 3$. Suppose that a complex number $\alpha$ satisfies $\alpha^{k-2} = 1$. Then $G_{k,l}''(\alpha) \neq 0$.

**Proof.** Since $G_{k,l}''(T) = (lk+1)(kl+1)^2-kl(k+1)(kl-1)T^{(l+1)(kl+1)+2l}$, if we assume that $\alpha$ satisfies $G_{k,l}''(\alpha) = 0$ and $\alpha^{k-2} = 1$, we have $G_{k,l}''(\alpha) = (lk+1)\alpha^{2l-1}-(kl+k)(kl+k-1)\alpha^{2l} = 0$. This immediately shows that $\alpha = \frac{l(l+1)}{l+1(kl+k-1)}$, which contradicts $\alpha^{k-2} = 1$. Hence the assertion follows. \qed

**Proof of Proposition 4.5.** Let $l$ be a positive integer. Since the unitarity of $G_{k,l}(T)$ and that of $H_{k,l}(T)$ are equivalent, it suffices to check the unitarity of $H_{k,l}(T)$. It is clear that $H_{k,l}(T)$ is a unitary polynomial when $k = 1, 2$. Actually we have $H_{1,l}(T) = 1$ and $H_{2,l}(T) = 1 - T + T^{2l+1} - T^{2l+2} = (1-T)(1+T^{2l+1})$, which are indeed unitary.

Suppose $k \geq 3$. If $G_{k,l}(T)$ is unitary, then every root of $H_{k,l}(T)$ satisfies $\alpha^k = 1$ or $\alpha^{k-2} = 1$ by Lemma 4.6. However, if $\alpha^k = 1$ we immediately see that $H_{k,l}(\alpha) = 1 + (1-\alpha)l$. Thus, $H_{k,l}(\alpha)$ can not be 0 because of the unitarity of $\alpha$. Thus any root of $H_{k,l}(T)$ must satisfy $\alpha^{k-2} = 1$ and $\alpha^k \neq 1$. By Lemma 4.7, the multiplicity of these roots of $H_{k,l}(T)$ is at most 2. Since $H_{k,l}(T)$ is assumed to be unitary and $H_{k,l}(1) \neq 0$, it follows that $2(k-3) \geq \deg H_{k,l}(T) = lk$. This is possible only when $l = 1$.

Therefore it is enough to prove that $H_{k,1}(T)$ is not unitary for $k \geq 3$. We put $H_k(T) = H_{k,1}(T) = 1 - T + T^k$ for simplicity.

Suppose that $k$ is an odd integer such that $k \geq 3$. Then $H_k(T)$ has a real root in the interval $(-2, -1)$ since $H_k(-1) = 1 > 0$ and $H_k(-2) = 3 - 2^k < 0$. This implies the polynomial $H_k(T)$ is not unitary.

Thus, only we have to consider is the case where $k$ is even and $k \geq 4$. Suppose that $H_k(T)$ is unitary and let $e^{i\theta}$ ($-\pi < \theta \leq \pi$) be its unitary root. Then we see that $\theta$ satisfies the equations $\cos k\theta = \cos \theta - 1$ and $\sin k\theta = \sin \theta$. Since $1 = \sin^2 k\theta + \cos^2 k\theta = 2 - 2\cos \theta$, we have $\theta = \pm \pi/3$. Further, since $1 = \cos^2 \theta + \sin^2 \theta = (\cos k\theta + 1)^2 + \sin^2 k\theta = 2\cos k\theta + 2$, we have $\cos(k\pi/3) = -1/2$. Hence we see that either $k \equiv 2$ or $4$ (mod 6). On the other hand, since
1 = (\cos \theta - \cos k\theta)^2 + (\sin \theta - \sin k\theta)^2 = 2 - 2 \cos((k - 1)\theta), we have \cos((k - 1)\pi/3) = 1/2. It follows that either \(k \equiv 0 \text{ or } 2 \pmod{6}\) holds. Thus we have \(k \equiv 2 \pmod{6}\).

Now we show that every unitary root of \(H_k(T)\) is simple. If we assume that \(\beta\) is a multiple root of \(H_k(T)\), it follows that \(\beta^k - \beta + 1 = 0\) and \(k\beta^{k-1} - 1 = 0\). Then we have \(|\beta| = k^{-1/(k-1)}\) and \(\beta = k\beta^k\) by the second equation. On the other hand, by the first equation, we obtain \(1 = \beta - \beta^k = (k - 1)\beta^k\) so that \(|\beta| = (k - 1)^{-1/k}\). Therefore we have \(k^k = |\beta|^{-k(k-1)} = (k - 1)^{k-1}\), but this contradicts to the assumption of the unitarity of \(H_k(T)\). This completes the proof of the proposition.

Finally we obtain the following theorem, which gives actually a generalization of the result in [IS] concerning the powerful numbers.

**Theorem 4.8.** Let \(k\) and \(l\) be positive integers. When \(k = 1, 2\) we have

\[
F_{1,l}(s) = \zeta(s) \quad \text{and} \quad F_{2,l}(s) = \frac{\zeta(2s)\zeta((2l + 1)s)}{\zeta(2(2l + 1)s)}.
\]

When \(k \geq 3\), \(F_{k,l}(s)\) can be extended as a meromorphic function in \(\text{Re } s > 0\) and has a natural boundary \(\text{Re } s = 0\).

**Proof.** The assertion is clear from Lemma 4.4 and Proposition 4.5.

As a consequence we have the following result.

**Corollary 4.9.** Let \(Z_{\infty}^{(k, \ldots, k, 1)}(s)\) be as in Theorem 4.1. Then we have for \(k = 1, 2\)

\[
Z_{\infty}^{(1, \ldots, 1, 1)}(s) = \prod_{j=1}^{l+1} \zeta(js), \quad Z_{\infty}^{(2, \ldots, 2, 1)}(s) = \frac{\zeta((2l + 1)s)}{\zeta(2(2l + 1)s)} \prod_{j=1}^{l+1} \zeta(2js),
\]

and for \(k \geq 3\) the function \(Z_{\infty}^{(k, \ldots, k, 1)}(s)\) can be meromorphically extended to the half plane \(\text{Re } s > 0\) with a natural boundary \(\text{Re } s = 0\).

**5 Closing remarks**

We make two small remarks; the first one is to describe the relation between a multiple finite Riemann zeta function and the theory of elementary divisors, and the second one is to examine a multiple Eisenstein series defined via the multiple finite Riemann zeta function.

- The isomorphism classes of abelian groups \(A\) of order \(n\) are parametrized by the map \(\lambda\) from the set of all prime numbers to that of partitions such that \(n = \prod_{p \text{ prime}}^\lambda p^{\lambda(p)}\) and

\[
A \cong \bigoplus_{p \text{ prime}} \bigoplus_{j=1}^{\ell(\lambda(p))} \mathbb{Z}/p^{\lambda_j(p)}\mathbb{Z},
\]
where $|\lambda(p)|$ and $\ell(\lambda(p))$ are the size and the length of the partition $\lambda(p) = (\lambda_j(p))_{j \geq 1}$ respectively. The multiple finite Riemann zeta function $Z_N^m(s)$ is expressed also as $Z_N^m(s) = \sum_{n \mid N^m} g_N^m(n)n^{-s}$. Here $g_N^m(n)$ denotes the number of isomorphism classes of abelian groups of order $n$, parametrized by $\lambda$ such that $\lambda_1(p) \leq m$ and $\ell(\lambda(p)) \leq \text{ord}_p N$ for all $p$. It is clear that $g_N^m(n)$ is multiplicative with respect to $n$ and $N$. If we put $g_N^\infty(n) := \lim_{N \to \infty} g_N^m(n)$, then $g_N^\infty(n)$ equals the number of the isomorphism classes of abelian groups $A$ of order $n$, which is the direct sum of $p$-groups $A_p$ such that $p^m A_p = 0$ for $p|n$.

We now just mention about the asymptotic average for $g_N^\infty(n)$ and $Z_N^m(\sigma)$ ($\sigma \in \mathbb{R}$) with respect to $n$. Thus we need the Tauberian theorem below (see, e.g. [MM]).

**Lemma 5.1.** Let $F(t) = \sum_{n=1}^{\infty} a_n n^{-t}$ be a Dirichlet series with non-negative real coefficients which converges absolutely for $\text{Re}(t) > \beta$. Suppose that $F(t)$ has a meromorphic continuation to the region $\text{Re}(t) \geq \beta$ with a pole of order $\alpha + 1$ at $t = \beta$ for some $\alpha \geq 0$. Put

$$c := \frac{1}{\alpha!} \lim_{t \to \beta} (t - \beta)^{\alpha+1} F(t).$$

Then we have

$$\sum_{n \leq x} a_n = (c + o(1))x^\beta (\log x)^\alpha$$

as $x \to \infty$. \hfill $\Box$

Using the lemma above, we have easily the following facts: Let $m$ be a positive integer.

1. \hspace{1cm} (5.1) \hspace{1cm} $\sum_{n \leq x} g_N^m(n) = (\zeta(2)\zeta(3)\cdots\zeta(m) + o(1))x$

as $x \to \infty$. In other words, the asymptotic average of $g_N^m(n)$ with respect to $n$ is given by $\zeta(2)\zeta(3)\cdots\zeta(m)$.

2. For a fixed $\sigma > 0$,

$$\sum_{n \leq x} Z_N^m(\sigma) = (\zeta(\sigma + 1)\zeta(2\sigma + 1)\cdots\zeta(m\sigma + 1) + o(1))x,$$

$$\sum_{n \leq x} Z_N^m(-\sigma) = (\zeta(\sigma + 1)\zeta(2\sigma + 1)\cdots\zeta(m\sigma + 1) + o(1))x^{1+m\sigma},$$

$$\sum_{n \leq x} Z_N^m(0) = \sum_{n \leq x \text{ prime}} \prod_{p \mid n} \left( \frac{\text{ord}_p n + m}{m} \right) = \left( \frac{1}{m!} + o(1) \right) x (\log x)^m$$

as $x \to \infty$. \hfill $\Box$
Actually, since $\zeta(s)$ has a single pole at $s = 1$ and $\text{Res}_{s=1} \zeta(s) = 1$, it follows the first formula (5.1) from (2.7) and Lemma 5.1. Next, fix $\sigma \in \mathbb{R}$. By (2.8), we have $\zeta^m(\sigma, t) = \sum_{n=1}^{\infty} Z_n^m(\sigma)n^{-t} = \prod_{k=0}^{m} \zeta(t + k\sigma)$. This shows that the abscissa of absolute convergence of the Dirichlet series $\zeta^m(\sigma, t)$ is given by $t = \max\{1, 1 - m\sigma\}$. Hence the remaining formulas follow similarly.

Remark 5.1. Put $g(n) := \lim_{m \to \infty} g_m(n)$. It is well-known that $\sum_{n \leq x} g(n) = \left( \prod_{k=2}^{\infty} \zeta(k) \right)x + O(\sqrt{x})$.

Remark 5.2. Since $Z_n^1(0) = d(n) := \sum_{d|n} 1$, it follows from (5.4) that $\sum_{n \leq x} d(n) \sim x \log x$. It is also well-known [Z] that there exists a constant $C$ such that $\sum_{n \leq x} d(n) = x \log x + Cx + O(\sqrt{x})$ in an elementary way. See, e.g. [A].

• We define a multiple Eisenstein series with parameter $s$ of type $m$ by

$$E^m_s(q) = \sum_{n=1}^{\infty} Z_n^m(1-s)q^n.$$  

We sometimes write $E^m_s(\tau)$ instead of $E^m_s(q)$ when $q = e^{2\pi i \tau}$ with $\tau \in \mathbb{C}$, $\text{Im}(\tau) > 0$. It is obvious that $E^1_s(q)$ is (essentially) the usual holomorphic Eisenstein series of weight $k$. In this remark we make an experimental study of this multiple Eisenstein series. Here we assume that $m = 2$; we shall make a detailed study of these series in somewhere else.

First we observe the following simple relation.

Lemma 5.2. We have

$$E^2_{s+1}(q) = \sum_{l=1}^{\infty} \sum_{N=1}^{\infty} \sigma_s(N) N^s q^{Nl}.$$  

Proof. The calculation is straightforward. Actually, one has

$$\sum_{l=1}^{\infty} \sum_{N=1}^{\infty} \sigma_s(N) N^s q^{Nl} = \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (nm)^s n^s q^{nm} = \sum_{l=1}^{\infty} \sum_{j=1}^{\infty} \sum_{n|j} j^s n^s q^{jl}$$

$$= \sum_{N=1}^{\infty} \sum_{n|j} n^s j^s q^N = \sum_{N=1}^{\infty} Z_N^2(-s)q^N = E^2_{s+1}(q).$$

This proves the assertion. 

Recall now the Fourier expansion of the holomorphic Eisenstein series $E_{k+1}(\tau)$ of weight $k + 1$ with $k$ being odd;

$$E_{k+1}(\tau) = \frac{1}{2} \sum_{(c,d)=1} (ct + d)^{-k-1} = 1 + \frac{1}{\zeta(k+1)} \sum_{m=1}^{\infty} \sum_{n \in \mathbb{Z}} (m\tau + n)^{-k}$$

$$= 1 + \frac{1}{\zeta(k+1)} \frac{(2\pi i)^{k+1}}{k!} \sum_{n=1}^{\infty} \sigma_k(n) q^n \left( = 1 + \frac{1}{\zeta(k+1)} \frac{(2\pi i)^{k+1}}{k!} E^1_{k+1}(\tau) \right).$$
Taking $k$-times derivative of $E_{k+1}(\tau)$, we have easily
\[
\frac{d^k}{d\tau^k}E_{k+1}(\tau) = \frac{(2\pi i)^{2k+1}}{\zeta(k+1)k!} \sum_{n=1}^{\infty} \sigma_k(n) n^k q^n.
\]
Hence by Lemma 5.2, we immediately obtain the expression of $E_{k+1}^2(\tau)$ when $k$ is odd:
\[
E_{k+1}^2(\tau) = \frac{\zeta(k+1)k!}{(2\pi i)^{2k+1}} \sum_{l=1}^{\infty} \left( \frac{d^k}{d\tau^k}E_{k+1} \right)(l\tau).
\]

**Remark 5.3.** There is another expression of $E_{k+1}^2(\tau)$ for $k$ being odd as follows.
\[
E_{k+1}^2(\tau) = -\frac{(2k)!}{(2\pi i)^{2k+1}} \sum_{l=1}^{\infty} \sum_{(c,d)\neq 1}^{(c,d)=1} \sigma_k(cl) l^{-2k-1}(c\tau + d)^{-2k-1}.
\]

- It is clearly interesting to make an extensive study of the multi-variable and multi-parameter version of multiple finite Riemann zeta functions from the symmetric functions point of view.

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Multiple finite Riemann zeta functions

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