Hermite spectral method for Fokker-Planck-Landau equation modeling collisional plasma

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Abstract

We propose an efficient Hermite spectral method for the spatially non-homogeneous Fokker-Planck-Landau (FPL) equation. We split the FPL equation into three parts, which are the convection step, the acceleration step and the collision step. Two different expansion centers are employed in the Hermite spectral method, among which the standard one used in [29] is selected during the convection and collision step in order to utilize the expansion coefficients of the quadratic collision term calculated therein and the one constituted by the local macroscopic velocity and temperature is selected for the acceleration step, by which the effect of the external force can be simplified into a system of ODE. The transformations between distribution functions with different expansion centers are achieved by a highly efficient algorithm [23]. In order to further reduce the complexity of the quadratic FPL collision term, a novel collision model is designed with a combination of the quadratic collision term and the simplified collision term. Several numerical examples are studied to test and validate our new approach.

keyword: quadratic collision operator, Hermite spectral method, Strang splitting

1 Introduction

The evolution of a collisional plasma constituted of different species of particles is commonly described by the Fokker-Planck-Landau (FPL) equation at the kinetic level [19, 12]. The FPL equation, derived by Landau [26] and Fokker-Planck [34] independently, describes binary collisions between charged particles with long-range Coulomb interactions, which is the limit of the Boltzmann transport equation when all binary collisions are grazing [11].

Due to the complexity of the quadratic collision operator, it suffers great difficulty to solve the FPL equation numerically. The Fokker-Planck collision operator is a nonlinear, integro-differential operator in the velocity space, which describes the infinite-range potential interactions in the plasma, causing most of statistical methods such as the DSMC method relatively fettered [30], despite great success in the simulation of the Boltzmann equation [2]. Several

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deterministic methods have also been suggested to solve the FPL equation and several related simplifications. In [31], a spectral method is brought up by truncating Fourier series and extending solutions by periodicity, which is then developed for the non-homogeneous FPL equation in [16]. The entropic scheme which guarantees a non-decreasing entropy is studied in [28, 4, 11, 12]. To deal with the stiffness of the collision operator, an asymptotic-preserving (AP) strategy is applied in [14], while the conservative spectral method is adopted in [17]. The FPL equation with stochasticity is also studied in [15]. A positivity-preserving scheme for the linearized 1D FPL equation is proposed in [7], modified to preserve energy later in [25] and then extended to the 2D FPL equation in cylindrical geometry in [44]. Several other numerical methods such as the multipole expansions [27] and multigrid techniques [5] are also deployed. In [39], a fully implicit nonlinearly converged algorithm for the multidimensional Rosenbluth-Fokker-Planck form of the FPL equation is proposed. Moreover, there are several kinds of numerical methods to solve the Vlasov equations. For example, the finite element methods are utilized in [46, 45], and the semi-Lagrangian schemes [10, 38, 37, 33, 43] and the reference therein are also used to solve the Vlasov equations. The spectral methods are also widely used, for example in [17, 21]. The Hermite spectral method is proposed in [3, 32, 18] to discretize the microscopic velocity space.

Recently, an Hermite spectral method is proposed to solve the Boltzmann equation, where an elaborate algorithm is designed to handle the complicated quadratic collision term [23]. In this paper, a similar numerical method is proposed for the non-homogeneous FPL equation. Distribution functions of the FPL equation are approximated with a series of basis functions derived from Hermite polynomials. For the quadratic collision term, the expansion coefficients can be calculated accurately with the method introduced in [29]. Moreover, the new collision model is constructed by a novel modeling technique which combines the quadratic collision model with the simplified collision model to reduce computational cost. With Strang splitting method adopted, the convection, external force and collision parts can be solved separately and successively. During the external force step, the expansion centers are chosen as the local macroscopic velocity and temperature, under which the effect of the external force can be reduced into an ODE system of the macroscopic velocity. Moreover, the fast projection algorithm introduced in [23] is utilized to handle the transform between distribution functions with different expansion centers. Numerical simulations will be conducted for both the linear and nonlinear Landau damping problems, with the decay rate of the electric energy studied and the effect of the collisional frequency tested. Moreover, the two-stream instability problem and the bump-on-tail instability are also simulated to show the effectiveness, efficiency and accuracy of our novel method.

The rest of this paper is organized as follows: Section 2 will introduce the FPL equation and some related properties. The detailed spectral method used to approximate distribution functions and the quadratic FPL operator will be introduced in Section 3. The numerical algorithm and our novel new collision model will be proposed in Section 4. Several numerical examples will be exhibited in Section 5. The conclusion and future work will be stated in Section 6.

2 Preliminaries

The Fokker-Planck-Landau (FPL) equation is used to model long-range Coulomb interactions between charged particles. In this section, we will give a brief review of the FPL equation and some related properties.
2.1 Fokker-Planck-Landau equation

The Fokker-Planck-Landau equation describes binary collisions between charged particles with long-range Coulomb interactions. The evolution of particles $\alpha$ is described by the distribution function $f_\alpha(t, x, v)$, which depends on time $t$, position $x \in \Omega \subset \mathbb{R}^3$ and microscopic velocity $v \in \mathbb{R}^3$. The governing equation or the FPL equation reads

$$\frac{\partial f_\alpha}{\partial t} + v \cdot \nabla_x f_\alpha + F \cdot \nabla_v f_\alpha = \nu \sum_\beta Q_{\alpha,\beta}[f_\alpha, f_\beta],$$

(2.1)

where the force field $F(t, x)$ only depends on time and space position, produced externally or self-consistently. Here we only consider the self-consistent case, in which $F(t, x)$ corresponds to the electrostatic force $qE(t, x)$, where $q$ is the electric charge and $E(t, x)$ is the self-consistent electrostatic field coupled with the distribution function through Poisson’s equation

$$E(t, x) = -\nabla x \psi(t, x), \quad -\Delta x \psi = \sum_\beta \int_{\mathbb{R}^3} f_\beta(v) \, dv,$$

(2.2)

where $f_\beta$ is the distribution function of particles $\beta$. The non-negative constant $\nu$ is related to the collision frequency and $Q_{\alpha,\beta}[f_\alpha, f_\beta]$ describes collisions between particles $\alpha$ and particles $\beta$, which has the form

$$Q_{\alpha,\beta}[f_\alpha, f_\beta] = \nabla_v \cdot \left( \int_{\mathbb{R}^3} A(v - v') \left( \nabla_v f_\alpha(v) f_\beta(v') - \nabla_v' f_\beta(v') f_\alpha(v) \right) \, dv' \right).$$

(2.3)

The collision kernel $A(\cdot)$ reflects the interaction between particles with the form of a $3 \times 3$ negative and symmetric matrix

$$A(v) = \Psi(|v|) \Pi(v),$$

(2.4)

where $\Psi(|v|)$ is a non-negative radial function, and $\Pi(v)$ is the orthogonal projection upon the space orthogonal to $v$, as $\Pi_{ij}(v) = \delta_{ij} - \frac{v_i v_j}{|v|^2}$. The inverse-power-law(IPL) model, with which we are primarily concerned, supposes that the function $\Psi(v)$ holds the form

$$\Psi(v) = \Lambda |v|^\gamma + 2$$

(2.5)

with $\Lambda$ being a positive constant and $\gamma$ being the index of the power of distance. For the inverse-power law model, we have $\gamma \leq -3$ [16]. Similar to the Boltzmann equation, we obtain the hard potential model when $\gamma > 0$ and the soft potential model when $\gamma < 0$. There are two special cases, the first of which is the model of Maxwell molecules when $\gamma = 0$ and the other is the model with Coulomb interactions when $\gamma = -3$ [16].

2.2 Collision operator

The Fokker-Planck-Landau operator is used to describe binary collisions between charged particles, with the potential of long-range Coulomb interactions. In this paper, the operator $Q_{\alpha,\alpha}$ is used to describe the electron-electron collisions, which has the form

$$Q_{\alpha,\alpha}[f_\alpha, f_\alpha] = \nabla_v \cdot \left( \int_{\mathbb{R}^3} A(v - v') \left( \nabla_v f_\alpha(v) f_\alpha(v') - \nabla_v' f_\alpha(v') f_\alpha(v) \right) \, dv' \right),$$

(2.6)
with $A(\cdot)$ defined in (2.13). Certainly the equilibrium state of the FPL operator is attained when the distribution function $f_\alpha$ satisfies $Q_{\alpha,\alpha}[f_\alpha, f_\alpha] = 0$, which has the form

$$\mathcal{M}_{[\alpha]}^{[u_\alpha, T_\alpha]}(v) = \rho_\alpha \frac{1}{(2\pi T_\alpha)^{3/2}} \exp\left(-\frac{|v - u_\alpha|^2}{2T_\alpha}\right),$$  \hspace{1cm} (2.7)

where $\rho_\alpha$ is the density, $u_\alpha$ is the macroscopic velocity and $T_\alpha$ is the temperature of the electron. Their relations with the distribution function $f_\alpha$ are

$$\rho_\alpha = \int_{\mathbb{R}^3} f_\alpha(t, x, v) \, dv, \quad \rho_\alpha u_\alpha = \int_{\mathbb{R}^3} vf_\alpha(t, x, v) \, dv, \quad \frac{3}{2}\rho_\alpha T_\alpha = \frac{1}{2} \int_{\mathbb{R}^3} |v - u_\alpha|^2 f_\alpha(t, x, v) \, dv. \hspace{1cm} (2.8)$$

Moreover, the FPL operator (2.6) also preserves mass, momentum and energy as

$$\int_{\mathbb{R}^4} Q_{\alpha,\alpha}[f_\alpha, f_\alpha] \begin{pmatrix} 1 \\ |v|^2 \end{pmatrix} = 0. \hspace{1cm} (2.9)$$

Due to the complicated form of the FPL operator, several simplified operators are introduced to approximate the original quadratic operator, for example the linearized collision operator

$$\mathcal{L}_{\alpha}[f_{\alpha}] = Q_{\alpha,\alpha}[f_{\alpha}, \mathcal{M}_{[\alpha]}^{[u_\alpha, T_\alpha]}] + Q_{\alpha,\alpha}[\mathcal{M}_{[\alpha]}^{[u_\alpha, T_\alpha]}, f_{\alpha}]$$  \hspace{1cm} (2.10)

and the diffusive Fokker-Planck (FP) operator [24]

$$\mathcal{P}_{\text{FP}}[f_{\alpha}] = \nabla_v \cdot \left( \mathcal{M}_{[\alpha]}^{[u_\alpha, T_\alpha]} \nabla_v \left( \frac{f_{\alpha}}{\mathcal{M}_{[\alpha]}^{[u_\alpha, T_\alpha]}} \right) \right) \hspace{1cm} (2.11)$$

The collisions between two different particle species are described by the operator $Q_{\alpha,\beta}$, which can be derived from the full form of the Landau operator (2.3). In the instance of the two species being ions and electrons, the elements have much smaller mass, and the ions may be assumed to be stationary. If we assume that the temperature of ions $T_\beta$ is negligible compared to that of electrons $T_\alpha$, this will allow us to assume the distribution function of particles $\beta$ to be given by a Dirac measure in velocity [47]

$$f_\beta(t, x, v) = \rho_\beta(t, x)\delta_0(v - u_\beta(t, x)), \hspace{1cm} (2.12)$$

where $\rho_\beta$ and $u_\beta$ are the density and the macroscopic velocity of particles $\beta$. Consequently, the collision operator $Q_{\alpha,\beta}$ is reduced into

$$Q_{\alpha,\beta}[f_{\alpha}] = \rho_\beta \nabla_v \cdot (A(v - u_\beta) \nabla_v f_{\alpha}). \hspace{1cm} (2.13)$$

The collision operator (2.13) still preserves mass and energy as

$$\int_{\mathbb{R}^3} Q_{\alpha,\beta}[f_{\alpha}] \, dv = 0, \quad \int_{\mathbb{R}^3} |v - u_\beta|^2 Q_{\alpha,\beta}[f_{\alpha}] \, dv = 0. \hspace{1cm} (2.14)$$

We refer [9, 14] for more details of this reduced collision operator.

The high-dimensionality of the distribution function and the complicated form of collision operators in the FPL equation inhibit us from solving it analytically while several numerical methods are proposed. The simplified collision operators such as the linearized FPL operator and the FP operator are also adopted to approximate the original quadratic collision operators. In this paper, we will use the simplified collision operators and propose a numerical method to solve the FPL equation efficiently. In our method, distribution functions are approximated by the series expansion of the Hermite polynomial and a novel collision model is built based on the complicated quadratic collision and simplified collision models. In the next section, we will first introduce the approximation to the distribution function and the FPL operator.
### 3 Series approximation of the FPL equation

In this paper, the Hermite polynomials are adopted to approximate distribution functions. The complicated FPL operator (2.3) is also expanded in terms of these basis functions. In this section, we will introduce the specific approximation method and properties of this series expansion in detail.

#### 3.1 Distribution functions

Due to the similarity between the FPL equation and the Boltzmann equation, Hermite spectral method proposed in [23] is adopted for the FPL equation. Similarly, the distribution function is discretized as

\[
\tilde{f}(\tilde{\alpha}) = \sum_{(i,j,k) \in N^3} f_{(i,j,k)}(\tilde{\alpha}) \tilde{H}_{(i,j,k)}^{(\tilde{\alpha})}(\tilde{\mathbf{v}}),
\]

where \(\tilde{\mathbf{v}} \in \mathbb{R}^3\) and \(\tilde{T} \in \mathbb{R}_+\) possess the same dimensions as \(\mathbf{v}\) and \(T_\alpha\), respectively. There are several kinds of methods to select these two parameters. For example, these two parameters are chosen as the local macroscopic velocity \(\mathbf{u}_\alpha(t, \mathbf{x})\) and temperature \(T_\alpha(t, \mathbf{x})\), respectively in [6], while constants are chosen in [23]. The basis function \(\tilde{H}_{(i,j,k)}^{(\tilde{\alpha})}(\tilde{\mathbf{v}})\) is defined as

\[
\tilde{H}_{(i,j,k)}^{(\tilde{\alpha})}(\tilde{\mathbf{v}}) = (-1)^{i+j+k} \exp \left( \frac{|\tilde{\mathbf{v}}|^2}{2} \right) \frac{\partial^{i+j+k}}{\partial v_1^i \partial v_2^j \partial v_3^k} \left[ \exp \left( \frac{|\tilde{\mathbf{v}}|^2}{2} \right) \right].
\]

From the expansion (3.1), we can derive several relations between expansion coefficients \(f_{(i,j,k)}^{\alpha,\tilde{\alpha}}\) and macroscopic variables, among which some most frequently used are

\[
\begin{align*}
\rho_\alpha &= f_{(0,0,0)}^{\alpha,\tilde{\alpha}}, & \rho_\alpha \cdot \mathbf{u}_\alpha &= \rho \tilde{\mathbf{u}} + \left( f_{(1,0,0)}^{\alpha,\tilde{\alpha}}, f_{(0,1,0)}^{\alpha,\tilde{\alpha}}, f_{(0,0,1)}^{\alpha,\tilde{\alpha}} \right)^T, \\
\frac{1}{2} \rho_\alpha |\mathbf{u}_\alpha|^2 + \frac{3}{2} \rho_\alpha T_\alpha &= \rho_\alpha \cdot \tilde{\mathbf{u}} - \frac{1}{2} \rho_\alpha |\tilde{\mathbf{u}}|^2 + \frac{3}{2} \rho_\alpha \tilde{T} + f_{(1,0,0)}^{\alpha,\tilde{\alpha}} + f_{(2,0,0)}^{\alpha,\tilde{\alpha}} + f_{(0,0,1)}^{\alpha,\tilde{\alpha}} + f_{(0,0,2)}^{\alpha,\tilde{\alpha}}. 
\end{align*}
\]

Some other related macroscopic variables such as the shear stress and the heat flux can also be expressed in terms of the expansion coefficients \(f_{(i,j,k)}^{\alpha,\tilde{\alpha}}\), which will not be articulated in this paper. According to (3.4), if the expansion parameters are chosen as the local macroscopic velocity and the temperature of the particles, or precisely \(\tilde{\mathbf{u}} = \mathbf{u}_\alpha\) and \(\tilde{T} = T_\alpha\), it holds that

\[
f_{e_i}^{\alpha,\tilde{\alpha}}(\mathbf{u}_\alpha, T_\alpha) = 0, \quad \sum_{i=1}^{3} f_{e_i}^{\alpha,\tilde{\alpha}}(\mathbf{u}_\alpha, T_\alpha) = 0, \quad i = 1, 2, 3
\]

where \(e_i, i = 1, 2, 3\) denote the multi-indices \((1,0,0), (0,1,0), (0,0,1)\), respectively.
The parameters \( \tilde{u} \) and \( \tilde{T} \) are supposed to be chosen to accelerate the convergence of the series (3.1), based on a prior understanding of the problem. In [29], a special case when \( \tilde{u} = 0 \) and \( \tilde{T} = 1 \) is studied, where an algorithm has been proposed to calculate the expansion coefficients for the FPL collision operators (2.3). In order to utilize the results in [29], we will expand the collision terms under this particular case in this paper, which will be discussed in detail in the next section.

### 3.2 Collision operators

For the moment, we have obtained the discretization of the distribution function and we will study the discretization of the collision terms (2.3) and (2.13) in the following, where the major difficulty is the quadratic FPL collision operator (2.3), which will be discussed first.

In order to utilize the algorithm proposed in [29], the expansion parameters in (3.1) are chosen as \( \tilde{u} = 0 \) and \( \tilde{T} = 1 \) and the FPL collision term is expanded accordingly as

\[
Q_{\alpha,\alpha}[f_\alpha,f_\alpha](t,x,v) = \sum_{(i,j,k) \in \mathbb{N}^3} Q_{(i,j,k)}^{(\alpha,\alpha),(0,1)}(t,x) H_{(i,j,k)}^{(0,1)}(v),
\]

the coefficients in which can be evaluated by

\[
Q_{(i,j,k)}^{(\alpha,\alpha),(0,1)} = \frac{1}{i!j!k!} \int_{\mathbb{R}^3} H_{(i,j,k)}(v) Q_{\alpha,\alpha}[f_\alpha,f_\alpha](t,x,v) \, dv.
\]

Relatively high computational cost of the calculation of these coefficients demands efficient algorithm. The corresponding algorithm has been proposed in [23] for the quadratic Boltzmann operator, where the method to calculate the coefficients for the dimensionless Boltzmann collision operator is first brought and later generalized to normal case. Moreover, in [29], we have also proposed an algorithm to evaluate these coefficients for the dimensionless FPL operator, the result of which will be utilized here to get the expansion coefficients (3.7).

Substituting (3.1) into (3.7), we can derive that

\[
Q_{(i,j,k)}^{(\alpha,\alpha),(0,1)} = \sum_{(l_1,l_2,l_3) \in \mathbb{N}^3} \sum_{(n_1,n_2,n_3) \in \mathbb{N}^3} A_{(i,j,k)}^{(l_1,l_2,l_3,n_1,n_2,n_3)} f_{(l_1,l_2,l_3)}^{\alpha,\alpha} f_{(n_1,n_2,n_3)}^{\alpha,\alpha},
\]

where \( A_{(i,j,k)}^{(l_1,l_2,l_3,n_1,n_2,n_3)} \) is defined in [29]. Following the same method, it holds that the linearized collision operator (2.10) can also be expanded as

\[
L_\alpha[f_\alpha] = \sum_{(i,j,k) \in \mathbb{N}^3} L_{(i,j,k)}^{\alpha,\alpha}(t,x) H_{(i,j,k)}^{(0,1)}(v)
\]

with the coefficients expressed as

\[
L_{(i,j,k)}^{\alpha,\alpha}(t,x) = \sum_{(l_1,l_2,l_3) \in \mathbb{N}^3} \left( A_{(i,j,k)}^{(l_1,l_2,l_3,0,0,0)} + A_{(i,j,k)}^{(0,0,0,l_1,l_2,l_3)} \right) f_{(l_1,l_2,l_3)}^{\alpha,\alpha}.
\]

By utilizing several properties of the Hermite polynomial

\[
\left( H_i(v) \exp \left( -\frac{v^2}{2} \right) \right)' = -H_{i+1}(v) \exp \left( -\frac{v^2}{2} \right),
\]

\[
H_{i+1}(v) = vH_i(v) - iH_{i-1}(v), \quad H_i(v)' = iH_{i-1}(v),
\]

for \( i = 0, 1, 2, \ldots \), we have
the Fokker-Planck operator (2.11) is expanded as
\[ \mathcal{P}_{FP}[f_\alpha] = \sum_{(i,j,k) \in \mathbb{N}^3} \mathcal{P}^{\alpha,\beta}_{(i,j,k)} H^{[\alpha,\beta]}_{(i,j,k)}(v), \]
\[ \mathcal{P}^{\alpha,\beta}_{(i,j,k)} = \sum_{m=1}^3 \left( 1 - \frac{1}{T_\alpha} \right) \alpha_{(i,j,k),m} + \frac{um}{T_\alpha} \beta_{(i,j,k),m} \right) - \frac{i + j + k}{T_\alpha} \alpha_{(i,j,k)}. \] (3.12)

Next, we will derive the series expansion of the linearized collision operator (2.13) between different species. Without loss of generality, we set \( \rho_\beta = 1 \) and the expansion center as \([u_\beta,1]\) based on the form of the collision operator (2.13). The collision operator (2.13) is expanded accordingly as
\[ Q_{(i,j,k)}[f_\alpha](t,x,v) = \sum_{(i,j,k) \in \mathbb{N}^3} Q^{\alpha,\beta}_{(i,j,k)}[u_\beta,1](t,x,v) H^{[\alpha,\beta]}_{(i,j,k)}(v), \] (3.13)
with the coefficients being
\[ Q^{\alpha,\beta}_{(i,j,k)}[u_\beta,1](t,x,v) = \frac{1}{i!j!k!} \int_{\mathbb{R}^3} H_{(i,j,k)}(v-u_\beta) Q_{\alpha,\beta}[f_\alpha](t,x,v) dv. \] (3.14)

By means of substituting (3.1) and (2.13) into (3.14) and changing variables, the coefficients can be expressed as
\[ Q^{\alpha,\beta}_{(k_1,k_2,k_3)}[u_\beta,1] = \frac{1}{i!j!k!} \sum_{(l_1,l_2,l_3) \in \mathbb{N}^3} \alpha_{(i,j,k)}^{[l_1,l_2,l_3]} \sum_{m,n=1}^3 k_m \left[ \delta_{m,n} \sum_{s=1}^3 G_{ss}(\gamma, (k_1, k_2, k_3) - e_i, (l_1, l_2, l_3) + e_j) - G_{mn}(\gamma, (k_1, k_2, k_3) - e_i, (l_1, l_2, l_3) + e_j) \right], \] (3.15)
where \( G_{mn}(\gamma, (k_1, k_2, k_3), (l_1, l_2, l_3)) \) is defined in [29, Eq.(3.14)]. The detailed calculation of (3.15) is similar as \( A_{(i,j,k)}^{[l_1,l_2,l_3,n_1,n_2,n_3]} \) in (3.8), and we refer [29] for more details. It is obvious that the expansion center in (3.14) is different from that of (3.6), which may bring difficulty to the calculation of the expansion coefficients between the collision \( Q_{\alpha,\alpha} \) and \( Q_{\alpha,\beta} \). The difficulty is overcome by the theorem in [23] explained as below

**Theorem 1.** Suppose the function \( \psi(v) \) satisfies
\[ \int_{\mathbb{R}^3} (1 + |v|^M)|\psi(v)| \, dv < \infty, \] (3.16)
for some positive integer \( M \). Given \( w, w^* \in \mathbb{R}^3 \) and \( \eta, \eta^* > 0 \), for any \((i,j,k) \in \mathbb{N}^3 \) satisfying \(|(i,j,k)| \leq M\), define
\[ \tilde{\psi}^{(i,j,k)} = \frac{1}{i!j!k!} \frac{1}{2} \left( \int_{\mathbb{R}^3} H_{(i,j,k)} \left( \frac{v-w}{\sqrt{\eta}} \right) \psi(v) \, dv, \right. \] (3.17)
\[ \tilde{\psi}^{(i,j,k)*} = \frac{1}{i!j!k!} \frac{1}{2} \left( \int_{\mathbb{R}^3} H_{(i,j,k)} \left( \frac{v-w^*}{\sqrt{\eta}} \right) \psi(v) \, dv. \right. \]
Then
\[ \tilde{\psi}^{(i,j,k)*} = \sum_{l=0}^{(|i,j,k|)} \tilde{\psi}^{(l)}_{(i,j,k)}. \] (3.18)
where $\tilde{\psi}^{(l)}_{(i,j,k)}$ is recursively defined by

$$
\tilde{\psi}^{(l)}_{(i,j,k)} = \begin{cases} 
\tilde{\psi}^{(l-1)}_{(i,j,k)} & \text{if } l = 0, \\
\frac{1}{T} \sum_{m=1}^{M} \left( (w^*_{d} - w_{d}) \tilde{\psi}^{(l-1)}_{(i,j,k)} + \frac{1}{2}(\eta^* - \eta) \tilde{\psi}^{(m-1)}_{(i,j,k)} \right) & \text{if } 1 \leq m \leq \left| (i,j,k) \right|.
\end{cases}
$$

(3.19)

Here terms with negative value in the subscript index are regarded to vanish.

We refer [23] for more details of Theorem 1. Theorem 1 provides an algorithm to obtain $f_{\alpha, [\tilde{u}_1, \tilde{T}_1]}^{(i,j,k)}$ from $f_{\alpha, [\tilde{u}_2, \tilde{T}_2]}^{(i,j,k)}$ for any two different expansion centers, based on which we can also derive the expansion coefficients for the collision models.

For the moment, all the series expansions of the collision terms have been derived. Although the expansions are always complicated, the expansion coefficients can be calculated accurately and most importantly, most of the calculation can be accomplished by pre-computation offline. The numerical algorithm to solve the FPL equation (2.1) will be discussed in the next section.

4 Hermite spectral method

In the last section, the expansion of distribution functions and the collision term have been discussed. In this section, we will introduce the specific numerical method to solve the FPL equation, which is a natural extension of the method in [24]. By the standard Strang splitting method, the FPL equation is split into the following three parts:

- the convection step:
  \[ \frac{\partial f_{\alpha}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{x} f_{\alpha} = 0, \]  
  (4.1)

- the acceleration step:
  \[ \frac{\partial f_{\alpha}}{\partial t} + \mathbf{F}(t, \mathbf{x}) \cdot \nabla \mathbf{v} f_{\alpha} = 0, \]
  \[ \mathbf{F}(t, \mathbf{x}) = q \mathbf{E}(t, \mathbf{x}), \quad \mathbf{E}(t, \mathbf{x}) = -\nabla \mathbf{x} \psi(t, \mathbf{x}), \quad \Delta_{\mathbf{x}} \psi = \sum_{\beta} \int_{\mathbb{R}^3} f_{\beta}(\mathbf{v}) \, d\mathbf{v}, \]  
  (4.2)

- the collision step:
  \[ \frac{\partial f_{\alpha}}{\partial t} = \nu \sum_{\beta} Q_{\alpha, \beta}(f_{\alpha}, f_{\beta}). \]  
  (4.3)

The discretization of distribution functions is simply a truncation of the series

$$
f_{\alpha}(t, \mathbf{x}, \mathbf{v}) \approx \sum_{\left| (i,j,k) \right| \leq M} f_{\alpha, [\tilde{u}, \tilde{T}]}^{(i,j,k)}(t, \mathbf{x}) \mathcal{H}_{(i,j,k)}^{[\tilde{u}, \tilde{T}]}(\mathbf{v}), \quad M \in \mathbb{N}^+. \tag{4.4}
$$

For simplicity, coefficients $f_{\alpha, [\tilde{u}, \tilde{T}]}^{(i,j,k)}$ will be referred as $f_{(i,j,k)}$ and the basis function $\mathcal{H}_{(i,j,k)}^{[\tilde{u}, \tilde{T}]}(\mathbf{v})$ as $\mathcal{H}_{(i,j,k)}(\mathbf{v})$ in below. Parameters, which are different from $[\tilde{u}, \tilde{T}]$ or the distribution function
of which is not \( f_a(t, x, v) \), will be written out explicitly. Here we restrict our study in the 1D spatial space. The numerical scheme adopted in the \( x \)-direction is the standard finite volume discretization. Suppose \( \Gamma_h \) be a uniform mesh in \( \mathbb{R} \) and each cell is identified by an index \( j \). For a fixed \( x_0 \in \mathbb{R} \) and \( \Delta x > 0 \\
\Gamma_h = \{ \Gamma_i = x_0 + (j\Delta x, (j+1)\Delta x) : j \in \mathbb{Z} \}. \quad (4.5) \\
In the following sections, we will propose the numerical scheme to update the distribution function.

### 4.1 Convection step

In this section, the numerical scheme to solve the convection step (4.1) is proposed. We will first derive the equations for the expansion coefficients of the distribution function. Substituting (4.4) into (4.1), and matching the coefficients of the same basis, we can derive the equations for \( f(i,j,k) \) as

\[
\frac{\partial}{\partial t} f(i,j,k) + \frac{\partial}{\partial x} \left( (i+1)f(i,j,k) + \hat{u}_1 f(i,j,k) + \hat{T} f(i,j,k) - e_1 \right) = 0, \quad |(i,j,k)| \leq M. \quad (4.6)
\]

Let \( f = (f(0,0,0), f(1,0,0), \cdots, f(i,j,k), \cdots)^T \), then (4.6) can be rewritten as

\[
\frac{\partial f}{\partial t} + A \frac{\partial f}{\partial x} = 0, \quad (4.7)
\]

where \( A \) is an \( N \times N \) matrix, whose entries are determined by (4.7) and \( N \) is the number of \( f(i,j,k) \), \( |(i,j,k)| \leq M \) or

\[
N = \frac{(M+1)(M+2)(M+3)}{6}. \quad (4.8)
\]

The system (4.7) is solved by Euler’s method with time step \( \Delta t \) as following:

\[
f_j^{n+1} = f_j^n - \frac{\Delta t}{\Delta x} [F_{j+1/2}^n - F_{j-1/2}^n], \quad (4.9)
\]

where \( f_j^n \) is used to approximate the average of \( f \) over the \( j \)-th grid cell at time \( t^n \), and \( F_{j+1/2}^n \) is the numerical flux at the boundary between the cells \( \Gamma_j \) and \( \Gamma_{j+1} \). In this paper, HLL flux [12] is utilized similarly as [23]. Precisely, the HLL flux has the form as below

\[
F_{j+1/2}^n = \begin{cases} \\
A f_{j+1/2}^{n,R} & \text{for } \lambda \geq 0, \\
\frac{\lambda \mathbf{R} \mathbf{A} f_{j+1/2}^{n,L} - \lambda \mathbf{L} \mathbf{A} f_{j+1/2}^{n,R} + \lambda \mathbf{R} \mathbf{A} \left( f_{j+1/2}^{n,R} - f_{j+1/2}^{n,L} \right)}{\lambda \mathbf{R} - \lambda \mathbf{L}} & \text{for } 0 < \lambda < \lambda \mathbf{R}, \\
A f_{j+1/2}^{n,L} & \text{for } \lambda \leq 0 \end{cases}, \quad (4.10)
\]

where the \( \lambda \mathbf{L} \) and \( \lambda \mathbf{R} \) are the fastest signal velocities as \( \lambda \mathbf{L} = \hat{u}_1 - C_{M+1} \sqrt{T} \) and \( \lambda \mathbf{R} = \hat{u}_1 + C_{M+1} \sqrt{T} \), where \( C_{M+1} \) is the maximum root of the Hermite polynomial of degree \( M + 1 \). In order to get higher-order temporal schemes, the approximated solutions on the cell boundary \( f_{j+1/2}^{n,R} \) and \( f_{j+1/2}^{n,L} \) are computed by the linear reconstruction [24]. In addition, the time step is decided by the CFL condition

\[
\Delta t \frac{\hat{u}_1 + C_{M+1} \sqrt{T}}{\Delta x} < \text{CFL}. \quad (4.11)
\]
4.2 Acceleration step

In this section, the numerical scheme to solve the acceleration step will be proposed, which has also been studied meticulously in [42]. In our case, the distribution function is expanded under the center of local macroscopic velocity \( u(t, x) \) and temperature \( T(t, x) \), or precisely \( \tilde{u} = u(t, x) \) and \( \tilde{T} = T(t, x) \) in (3.1), respectively. By this expansion, the acceleration step can be simply reduced into solving the ODE system of the macroscopic velocity \( u_1 \) as

\[
\frac{\partial u_1}{\partial t} - F_1 = 0, \quad F_1(t, x, v) = qE_1(t, x),
\]

\[
E_1(t, x) = -\frac{\partial \psi(t, x)}{\partial x}, \quad -\partial_{xx} \psi = \sum_\beta \int_{\mathbb{R}^3} f_\beta(v) \, dv,
\]

(4.12)

In this paper, we will adopt the scheme listed as below to update during the acceleration step:

1. Find \( \left(f^{n+1,\ast}_{\alpha,[u,T]}(i,j,k)\right) \) from \( f_n^{n+1,\ast}(i,j,k) \) based on Theorem 1, where \( f_n^{n+1,\ast} \) is the numerical solution after the convection step at \( t = t^n \).

2. Solve (4.12) by the forward Euler scheme

\[
u_{i,j}^{n+1} = u_{i,j}^{n+1,\ast} + \Delta t F_{1,i,j}^{n,\ast},
\]

(4.13)

where \( F_{1,i,j}^{n,\ast} \) is the electric force in the \( j \)-th cell after the convection step at time \( t = t^n \). Moreover, the three-point central difference scheme and central difference scheme are adopted to solve the potential equation of \( \psi(t, x) \) and the electric field \( E_1(t, x) \) [42].

3. Update distribution functions \( \left(f^{n+1,\ast}_{\alpha,[u,T]}(i,j,k)\right) \) using \( u_{i}^{n+1} \) and obtain \( f_{i,j}^{n+1,\ast\ast} \).

4. Find \( f_{i,j}^{n+1,\ast\ast} \) from \( f_{i,j}^{n+1,\ast\ast}(i,j,k) \) based on Theorem 1. \( f_{i,j}^{n+1,\ast\ast} \) will be used in the collision step.

4.3 Collision step

In this section, we will introduce the numerical method to update the collision step. Since it is always quite expensive to numerically solve the quadratic collision term (2.3), certain simplifications have to be performed to make the numerical method computable. A strategy to reduce the computational cost has been proposed in [29] for the homogeneous FPL collision term, featuring a combination of the quadratic collision term and a simplified collision term. In this paper, the same strategy is adopted to approximate the complicated quadratic collision term for the non-homogeneous FPL collision term. Precisely, the novel collision term is constructed as

\[
Q_{\alpha,\alpha}^{\text{new}}\left[f_{\alpha}, f_{\alpha}\right](t, x, v) = \sum_{(i,j,k) \in \mathbb{N}^3} Q_{(i,j,k)}^{\text{new}(\alpha,\alpha),[0,1]}(t, x) \mathcal{H}_{(i,j,k)}^{[0,1]}(v)
\]

(4.14)

with the coefficients being

\[
Q_{(i,j,k)}^{\text{new}(\alpha,\alpha),[0,1]}(t, x) = \begin{cases} 
Q_{(i,j,k)}^{(\alpha,\alpha),[0,1]}(t, x) & |(i, j, k)| \leq M_0 \\
\mu_0 F_{\alpha,\alpha}^{(i,j,k)}(t, x) & |(i, j, k)| > M_0
\end{cases}
\]

(4.15)
where the Fokker-Planck operator (2.11) is chosen as the simplified collision operator. The linearized collision operator (2.10) can also be utilized here to construct the new collision model. Here, \( M_0 \) is an arbitrarily positive integer and \( \mu_0 \) is the decay rate of higher-order coefficients. In our numerical computation, \( M_0 \) is chosen based on the specific problem with no general guiding principles. The damping rate \( \mu_0 \) is chosen as \( \mu_0 = \text{DIM} - 1 \) according to the isotropic model derived for the Fokker-Planck equation in [41], where DIM is the dimension number of the microscopic velocity space.

For the moment, we have obtained the reduced collision model \( Q_{\alpha,\alpha}^{\text{new}} \) based on the original quadratic FPL collision term and the linearized collision model. Similarly, the novel reduced collision term (4.14) and (2.13) are all truncated as

\[
Q_{\alpha,\alpha}^{\text{new}}(f_{\alpha}, f_{\alpha})(t, x, v) \approx \sum_{|(i,j,k)| \leq M} Q_{(i,j,k)}^{\text{new}(\alpha,\alpha)}(t, x) H_{(i,j,k)}(v),
\]

\[
Q_{\alpha,\beta}(f_{\alpha})(t, x, v) \approx \sum_{|(i,j,k)| \leq M} Q_{(i,j,k)}^{\alpha,\beta}(t, x) H_{(i,j,k)}(v).
\]

Thus, the governing equations of \( f_{\alpha} \) are changed into

\[
\frac{\partial f_{\alpha}}{\partial t} = Q, \quad Q = Q_{\alpha,\alpha}^{\text{new}} + Q_{\alpha,\beta},
\]

where \( Q_{\alpha,\alpha}^{\text{new}} \) and \( Q_{\alpha,\beta} \) are vectors with the same length of \( f \), formed by the expansion coefficients. Here, (4.17) is solved by the forward Euler scheme. Finally, the numerical scheme to solve the numerical scheme is summarized as below

1. Find \( (f_{\alpha,(0,1)}^{(i,j,k)})^{n+1,*} \) and \( (f_{\alpha,(u,1)}^{(i,j,k)})^{n+1,*} \) from \( f^{n+1,*} \) respectively based on Theorem 1
2. Build the collision model \( Q_{\alpha,\alpha}^{\text{new}} \) and \( Q_{\alpha,\beta} \), and calculate the corresponding expansion coefficients \( (Q_{(i,j,k)}^{\text{new}(\alpha,\alpha),(0,1)})^{n+1,*} \) and \( (Q_{(i,j,k)}^{\alpha,\beta,(u,1)})^{n+1,*} \).
3. Find \( (Q_{(i,j,k)}^{\alpha,\beta})^{n+1,*} \) based on Theorem 1
4. Solve (4.17) using forward Euler scheme as

\[
f_{j}^{n+1} = f_{j}^{n+1,*} + \Delta Q_{j}^{n+1,*}.
\]

Remark 1. The high order Runge-Kutta numerical scheme in (4.18) can also be adopted to update the collision term.

For now, we have proposed the numerical algorithm to solve the collision step. In the numerical scheme, the computational cost to obtain expansion coefficients is \( \mathcal{O}(M_0)^9 \) and the computational cost for the linear part is \( \mathcal{O}(M)^3 \). As is stated in [23], the computational cost for projection is \( \mathcal{O}(M)^4 \). Therefore, the total computational cost to get the collision term is \( \mathcal{O}(M_0^9 + M^4) \). In our numerical computation, since \( M_0 \) is always much smaller than \( M \), the novel collision model can tremendously reduce the computational cost.
5 Numerical algorithms and experiments

In this section, several numerical examples are tested to show the efficiency of the new algorithm. In all the tests, the CFL is set as 0.45, and $[\tilde{u}, \tilde{T}]$ is set as $[0, 1]$. The Landau damping problems are studied first to show the capability of the new algorithm to simulate the FPL equations quantitatively. The examples of two-stream instability and bump-on-tail instability are also tested to show that our method can detect the evolution in the microscopic velocity space rather satisfactorily.

5.1 Linear Landau damping problems

The Landau damping problems are one of the most popular problems in plasma physics, which is caused by the strong interactions between the electromagnetic wave and particles with velocities comparable to the phase velocity which tend to synchronize with the wave. Particles having velocities slightly lower than the phase velocity are accelerated and thus gain energy from the wave while those with slightly higher velocities are decelerated and thus lose energy to the wave, which results in an exponential decrease of the wave. It has already been widely studied and simulated in [16].

The initial data of Landau damping problems are adopted from [47] with $\rho_\beta = 1$, $u_\beta = 0$ and the initial condition as

$$f_\alpha = \frac{1}{(2\pi)^{3/2}} \exp\left(-\frac{|v|^2}{2}\right)(1 + A \cos(kx)), \quad (x, v) \in [0, 2\pi/k] \times \mathbb{R}^3,$$

where $A$ is the amplitude of the perturbation. Periodic boundary condition is implemented in this example.

In Landau damping problem, our interest lies in the evolution of the square root of the electrostatic energy which is defined as

$$\mathcal{E}(t) = \left(\sum_j \Delta x E_{1,j}(t)^2\right)^{1/2}.$$  

(5.2)

According to Landau’s theory, $\mathcal{E}(t)$ is expected to decrease exponentially with a fixed rate $\omega_i$, which may be regarded as the imaginary part of the frequency $\omega$. The theoretical damping rate is often estimated as [47] [36]

$$\gamma = \gamma_L + \gamma_C,$$

(5.3)

where the damping rate of collisionless plasma $\gamma_L$ being

$$\gamma_L = \begin{cases} -\sqrt{\frac{\pi}{8k^3}} \exp\left(-\frac{1}{2k^2} - \frac{3}{2}\right), & k \text{ is large,} \\ -\sqrt{\frac{\pi}{8}} \left(\frac{1}{k^3} - 6k\right) \exp\left(-\frac{1}{2k^2} - \frac{3}{2} - 3k^2 - 12k^4\right), & k \text{ is small.} \end{cases}$$

(5.4)

and a “correction”

$$\gamma_C = -\frac{1}{3} \nu \sqrt{2/\pi},$$

(5.5)
which only depends on the collisional frequency and reflects the effect of the collision, where $\nu$ refers to the collisional frequency.

In this numerical experiment of linear Landau damping, the amplitude of the perturbation $A$ should be sufficiently small and is chosen as $10^{-5}$ here. In addition, the expansion length $M$ is set as $M = 20$, and the grid size as $N = 800$. First, we will study the quality of our novel collision model. Figure 1 and 2 shows the time evolution of the electric energy $E(t)$ with wave number $k$ set as 0.3 and 0.5, respectively. In both, the Coulomb case $\gamma = -3$ is studied and the collision frequency is set as $\nu = 0$ and 0.01 to provide a comparison in order to demonstrate the effect of collision. The quadratic length $M_0$ is chosen as 5 and 10, respectively.

The results show that our method successfully simulates the linear Landau damping and the numerical damping rate of the electric energy is almost identical to the theoretical result in (5.4) for both wave numbers. When collision is exerted, the electrostatic energy shows a faster decay accounted by the effect of collision, which is reflected in larger damping rates with the increments compared to that of the collisionless case exactly matching the theoretical results in (5.5). This proves the accuracy of both our spectral method and our collision model. Most importantly, the numerical solution with small quadratic length $M_0 = 5$ is almost the same as that of $M_0 = 10$, which indicates that even with a small quadratic length $M_0 = 5$, which requires much less storage and computational cost, our collisional model could also capture the Landau damping phenomenon satisfactorily. This exactly validates the efficiency of our collision model with a small quadratic length. For that reason, the quadratic length is set as $M_0 = 5$ in the following numerical experiments.

![Figure 1](image1.png)

Figure 1: Time evolution of $\ln(E(t))$ with $N = 800$, $M = 20$ for different $M_0$. The blue line is that with $M_0 = 5$ while the red dashed line is that with $M_0 = 10$. The wave number is $k = 0.3$.

The cases of different potential indices $\gamma$, the index of the power of distance in the IPL model (2.5) aforementioned, have also been tested. The time evolution of the electric energy $E(t)$ in both the Maxwell case $\gamma = 0$ and the Coulomb case $\gamma = -3$ is tested and compared in Figure 3. Here we set the collisional frequency $\nu$ as 0.01 and the wave number $k$ as 0.3 and 0.5, respectively. This illustrates that our novel collision model is capable of simulating the Landau damping for different $\gamma$ and from our experiment, we can conclude that the collision model with softer potential imposes smaller damping rate.
Figure 2: Time evolution of $\ln(\mathcal{E}(t))$ with $N = 800$, $M = 20$ for different $M_0$. The blue line is that with $M_0 = 5$ while the red dashed line is that with $M_0 = 10$. The wave number is $k = 0.5$.

Figure 3: Time evolution of $\ln(\mathcal{E}(t))$ with $N = 800$, $M_0 = 5$ for different $\gamma$. The blue line is that with $\gamma = 0$ while the red dashed line is that with $\gamma = -3$.

5.2 Nonlinear Landau damping

As is shown in the last section, when the wave amplitude $A$ is sufficiently small, the linear regime is valid, which yields exponentially decreasing electrostatic energy. However, the Landau damping problem with larger amplitude, which divorces from the linear theory and hence also known as nonlinear Landau damping, remains enigmatic and even intractable by analytic methods. Typically one finds that the amplitude decays, grows and oscillates before settling down to a relatively steady state. In this section, we will study the nonlinear Landau damping problem numerically. Despite other operative effects which may also diminish the validity of the linear regime, the nonlinear Landau damping is primarily attributed to the ”trapping” phenomenon, or the phenomenon that a particle is caught in the potential wall of a wave, shuttles back and forth and ends up with gaining and losing energy to the wave.
Figure 4: Time evolution of $\ln(\mathcal{E}(t))$ with $N = 800$, $M_0 = 5$ for different collisional frequencies $\nu = 0$, $\nu = 0.01$, $0.05$ and $0.1$. The wave number is $k = 0.3$.

In this numerical experiment of nonlinear Landau damping, the form of the initial data is the same as that in the last section with $A$ augmented to 0.2 and the electrostatic energy is again studied. The nonlinear Landau damping problem with this particular initial data has also been studied in [9, 47], to which we refer for the comparison of numerical results. The case of Maxwell molecules $\gamma = 0$ is studied and the spatial grid size, the moment expansion order and the quadratic order are set as $N = 800$, $M = 20$ and $M_0 = 5$, respectively. Moreover, in order to avoid recurrence [13], the moment number is chosen as $M = 200$ for the collisionless case.

Figure 4 and 5 show the time evolution of electrostatic energy for $k = 0.3$ and $k = 0.5$ with collisional frequency $\nu = 0$, $\nu = 0.01, 0.05$ and 0.1. We can conclude that for the nonlinear collisionless problem, instead of exponential damping as in the linear case, the electrostatic energy decreases exponentially at the beginning and then growing exponentially with a smaller rate, which is consistent with the results achieved by [8, 47]. In the instance of collision, we find that the electrostatic energy exhibits an exponential-like damping for both wave numbers $k = 0.3$ and $0.5$ and the damping rate increases with collisional frequency, which is reasonable because stronger collision means more frequent energy exchange between particles and results in less “trapping” phenomena happening and faster damping rates. This numerical result also
Figure 5: Time evolution of \( \ln(\mathcal{E}(t)) \) with \( N = 800, M_0 = 5 \) for different collisional frequencies \( \nu = 0, \nu = 0.01, 0.05 \) and 0.1. The wave number is \( k = 0.5 \).

accords with that in [47, 9].

The cases of different potential indices are also studied, where the model of Maxwell molecules \( \gamma = 0 \) and the model with Coulomb interactions \( \gamma = -3 \) are tested. Figure 6 shows the time evolution of the electrostatic energy for wave number \( k = 0.3 \) and 0.5 with different collisional frequencies \( \nu = 0.05 \) and 0.1, from which we find that the damping rate for the Maxwell case \( \gamma = 0 \) is much larger than that of the Coulomb case \( \gamma = -3 \). This result is compatible with a similar conclusion in the linear case.

5.3 Two-stream instability

Two-stream instability is a very common instability in plasma physics and of primary importance for studying nonlinear effects of plasmas in the future. The mechanism of two-stream instability is similar to that of Landau damping to some extent with particles with different velocities transferring energy with each other.

In this numerical experiment of two-stream instability, we only consider a plasma with a
Figure 6: Time evolution of $\ln(\mathcal{L}(t))$ with $N = 800, M_0 = 5$ for different potential indices $\gamma$, where the blue line is that for $\gamma = 0$ and red line is that for $\gamma = -3$. The first column is that for the wave number $k = 0.3$ and the bottom column is that for the wave number $k = 0.5$.

(a) Initial MDF $g(0, x, v_x)$
(b) Contours of $g(0, x, v_x)$
(c) Initial MDF $g(0, \frac{\pi}{4}, x, v_x)$
(d) $k = 0.5, \nu = 0.1$

Figure 7: Initial marginal distribution functions. In (b) and (c), the blue solid lines correspond to the exact solution, and the red dashed lines correspond to the numerical approximation. Figure (a) shows only the numerical approximation. Figure (c) show the numerical approximation and the exact solution at the position $x = \frac{\pi}{4}$.
Figure 8: Evolution of the marginal distribution function $g(t, x, v_x)$ with different collisional frequencies $\nu$. The left row is $\nu = 0$, and the middle row is $\nu = 0.001$, with the right row $\nu = 0.01$.

fixed ion background. The initial data is given with a non-isotropic two-stream flow

$$f_\alpha = \frac{1 + A \cos(kx)}{\sqrt{2\pi T_\alpha}} \left[ 0.5 \exp \left( -\frac{|v - (u_x, 0, 0)|^2}{2T_\alpha} \right) + 0.5 \exp \left( -\frac{|v + (u_x, 0, 0)|^2}{2T_\alpha} \right) \right], \quad (5.6)$$

with $A = 0.01$, $T_\alpha = 0.25$. Similar assumptions and initial data can be found in [47]. The time evolution of the particles with the collisional model of Coulomb interactions $\gamma = -3$ is studied and the wave number $k$ is chosen as $k = 0.5$. The grid size and the expansion moment order is chosen as $N = 400$ and $M = 40$, respectively. Here the collisional frequencies are set as $\nu = 0$, 0.001 and 0.01.

In order to capture the electron “trapping” phenomenon, the marginal distribution function

$$g(t, x, v_x) = \int_{\mathbb{R}^2} f(t, x, v_x, v_y, v_z) \, dv_y \, dv_z, \quad (5.7)$$

is plotted. Clearly our chosen parameters can approximate the initial distribution function very satisfactorily (see Figure 7).
In order to suppress the recurrence and the non-physical oscillations, the filter developed in [22, 13] is applied here. Figure 8 shows the time evolution of the marginal distribution function (5.7) in the $x - v_x$ plane. From these, we can find that for the collisionless case, the linear two-stream instability grows exponentially at first and the nonlinearity becomes important and “trapping” emerges. At the same time, the original distribution begins to twist and curve until an electron hole-like structure finally forms, which is consistent with the results in [20]. For the collisional case, smaller electron hole-like structure forms with the increase of the collisional frequency $\nu$ and there is even no visible hole-like structure occurring in the case of the collisional frequency $\nu = 0.1$. This again substantiates the effect of collision to reduce the “trapping” phenomenon.

The time evolution of the total energy is also studied in order to test the conservation property of this numerical scheme. The total energy $E_t(t)$ is defined as

$$E_t(t) = \frac{1}{2} \Delta x \sum_j \int_{\mathbb{R}^3} f(t, x_j, v) |v|^2 \, dv + \frac{1}{2} E(0)^2.$$  (5.8)

The evolution of the total energy $E_t(t)$ for different collisional frequencies are plotted in Figure 9, from which we can see that although the numerical scheme cannot exactly preserve the total energy, the variation of the total energy is minute, especially for the linear instability stage where the variation is almost negligible. This demonstrates the robustness of our numerical method and the superior quality of our collision model.

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The evolution of the total energy $E_t(t)$ for different collisional frequencies are plotted in Figure 9, from which we can see that although the numerical scheme cannot exactly preserve the total energy, the variation of the total energy is minute, especially for the linear instability stage where the variation is almost negligible. This demonstrates the robustness of our numerical method and the superior quality of our collision model.

Bump-on-tail instability is another important micro-instability where the electron velocity distribution function is multi- and hetero-peaked. The distribution function is unstable which will lead to a growth of the initial perturbation followed by saturation and oscillation of the particles trapped in the potential through the wave [35, 40].

In this numerical experiment of bump-on-tail instability, we also consider a plasma with a fixed ion background and only the electron-electron collisions. Similar to that of two-stream
instability, we also initialize with a non-isotropic hetero-peaked flow

\[ f_\alpha = \frac{(1 + A \sin(kx))}{\sqrt{2\pi T_\alpha}} \left[ n_m \exp \left( -\frac{|v - (u_x, 0, 0)|^2}{2T_\alpha} \right) \right] + n_b \exp \left( -\frac{|v + (u_x, 0, 0)|^2}{2T_\alpha} \right) \],

(5.9)

where \( A = 0.01, T_\alpha = 0.25, u_x = 1. \) \( n_m = 0.7 \) which represents the magnitude of the “main stream” and \( n_b = 0.3 \) representing the magnitude of the “bump” on the tail of the “main stream”. The wave number \( k \) is chosen as \( k = 0.3 \). In this example, we also focus on the Coulomb model. The grid size is chosen as \( N = 400 \) and the expansion moment order as \( M = 40 \), which gives a satisfying approximation of the initial distribution function (see Figure 10).

![Initial MDF](image)

Figure 10: Initial marginal distribution functions. In (b) and (c), the blue solid lines correspond to the exact solution, and the red dashed lines correspond to the numerical approximation. Figure (a) shows only the numerical approximation. Figure (c) shows the numerical approximation and the exact solution at the position \( x = 0 \).

The time evolution of the particles with the collision frequencies \( \nu = 0, 0.001 \) and 0.01 are studied. Figure 11 shows the time evolution of the marginal distribution function (5.7) in the \( x - v_x \) plane. We can observe that for the collisionless case, the bump is trapped by the electric field and gradually forms a crawling vortex-like structure. For the collisional case, the trap of the bump is much weaker and the distribution of the “mainstream” is less affected. In the case of the collisional frequency \( \nu = 0.1 \), no vortex-like structure is perceptible.

The evolution of the total energy defined in (5.8) is also studied. Figure 12 shows the evolution of the total energy for different collisional frequencies. Although the total energy is not perfectly preserved, the variation in the total energy is rather small, especially at the beginning of the evolution and decreases with the increase of the collisional frequency.

6 Conclusion

In this paper, we have developed a numerical algorithm for the FPL equation, based on the Hermite spectral method. A new reduced collision model is built by combining the quadratic FPL collision operator with the linearized collision operator. A fast algorithm to change the basis functions is adopted here and the total time complexity of the numerical algorithm is \( \mathcal{O}(M_0^3 + M^4) \). Both collisions within the same species and between different species are considered to simulate the time evolution of plasmas. Several numerical experiments show that our numerical algorithm can capture the movement of and the interactions between the particles accurately and efficiently, even with a small number of the quadratic length in our collision model.
(a) $t = 20, \nu = 0.0$

(b) $t = 20, \nu = 0.001$

(c) $t = 20, \nu = 0.01$

(d) $t = 30, \nu = 0.0$

(e) $t = 30, \nu = 0.001$

(f) $t = 30, \nu = 0.01$

(g) $t = 40, \nu = 0.0$

(h) $t = 40, \nu = 0.001$

(i) $t = 40, \nu = 0.01$

Figure 11: Evolution of the marginal distribution function $g(t, x, v_x)$ with different collisional frequency $\nu$. The left row is $\nu = 0$, and the middle row is $\nu = 0.001$, with the right row $\nu = 0.01$.

Figure 12: Time evolution of the variation of the total energy $\mathcal{E}_t(t)$ for different collisional frequencies. The variation is defined as $(\mathcal{E}_t(t) - \mathcal{E}_t(0))/\mathcal{E}_t(0)$.

The virtue of low computational cost makes our algorithm highly promising when applied to more complicated problems. However, our method is not capable of dealing with some problems...
during which the state of the plasma may diverge greatly from the equilibrium or in which the plasma consists of particles with different temperatures, to which we will apply ourselves in the future. Research works on more multi-dimensional problems with the magnetic field exerted and coupled with the existing electric field are also ongoing.

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