ON SUM-PRODUCT BASES

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ABSTRACT. Besides various asymptotic results on the concept of sum-product bases in \( \mathbb{N}_0 \), we consider by probabilistic arguments the existence of thin sets \( A, A' \) of integers such that \( AA + A = \mathbb{N}_0 \) and \( A'A + A'A = \mathbb{N}_0 \).

1. Introduction

Additive bases, and in less importance multiplicative bases, have been extensively studied for several centuries. More recently, expanding polynomials (of course with more than one variable) arise in this scope, whose point is to study the expansion of finite sets under polynomials. If \( f \in \mathbb{Z}[x_1, x_2, \ldots, x_d] \) and \( A \) be contained in a given subset \( R \) of a commutative ring, then let \( f(A, A, \ldots, A) \) (with \( k \) arguments) denote the set of all terms \( f(a_1, a_2, \ldots, a_k) \) where the \( a_i \)'s come from \( A \). The polynomial \( f \) is called an expander if there exists \( \delta > 0 \) such that \( |f(A, \ldots, A)| > |A|^{1+\delta} \) for any finite set \( A \), where \( |B| \) denotes the cardinality of a finite set \( B \). If \( R \) is finite, as for instance \( \mathbb{F}_p \) or \( \{1, \ldots, N\} \), we need to restrict the above definition by assuming that \( |R|^\epsilon < |A| < |R|^{1-\epsilon} \), for some \( \epsilon > 0 \). A more restrictive notion is the one of covering polynomial: is there a non trivial minimal size such that if \( A \) attains it then \( f(A, A, \ldots, A) \) entirely covers \( R \)?

We shall use the notation \( AB \) to denote the set of elements \( x \) such that \( x = ab \) for some \( a \in A \) and \( b \in B \). When \( A = B \), we use the notation \( A^2 = AA \) and by extension \( A^k = AA^{k-1} \), for \( k > 1 \) with the convention \( A^1 = A \). We shall focus on \( R = \mathbb{N}_0 \), the set of all nonnegative integers and the two special polynomials \( x + yz \) and \( xy + zt \) which are known to be expanders in different contexts (cf. [1, 3]). They also bring to light the important sum-product phenomenon. It can also be enlightened by their ability to break the natural threshold for the size of a set \( A \) satisfying \( f(A, A, \ldots, A) = R \) that can be deduced from the sum or the product taken separately. More precisely and taking an instance, the set \( A^2 + A \) contains both \( Aa_0 + A \) and \( A^2 + a_0 \) provided that \( a_0 \in A \). But we can expect to find sets \( A \) such that \( A^2 + A = R \) which are much smaller, with respect to their size, than sets satisfying \( Aa_0 + A = R \) or \( A^2 + a_0 = R \).

We call \( A \) to be a \( f \)-sum-product basis for \( R \) if \( f(A, A, \ldots, A) = R \). When \( R \) is finite, the measure of the size \( A \) of a set could be its cardinality. For infinite \( R \), and mainly \( \mathbb{N}_0 \), we can use an appropriate notion of counting function of a set \( A \) or an appropriate notion of its density.

Notation. We let \( \mathbb{N} := \mathbb{N}_0 \setminus \{0\} \) be the set of positive integers. For \( A \subset \mathbb{N}_0 \) and \( X > 0 \), let \( A(X) := |A \cap [1, X]| \) and

\[
\underline{d}(A) := \liminf_{X \to \infty} X^{-1}A(X), \quad \overline{d}(A) := \limsup_{X \to \infty} X^{-1}A(X),
\]

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called respectively the *lower density* and *upper density* of $A$. We let $d(A)$ denote their common value if it is the case and call it the *density* of $A$.

We shall use the symbols $\ll, \gg, \sim$ in the usual way. The notation $g(x) \asymp f(x)$ means $f(x) \ll g(x) \ll f(x)$ for any $x$ large enough. All the implied constants in Vinogradov’s symbol $\ll$ are generally absolute. If they depend upon $\varepsilon$, we write $\ll \varepsilon$.

In this paper we shall study those subsets $A$ of natural numbers such that the set $A^k + A^l$ contains all sufficiently large natural numbers or at least has positive lower density, where $k, l$ are positive integers and $\max(k, l) \geq 2$. Clearly if we want $A^2 + A$ (resp. $A^2 + A^2$) to cover all the positive integers, or at least to have a positive lower density, one needs $A(X) \gg \sqrt{X}$ (resp. $A(X) \gg X^{1/4}$). Since there exist additive bases $B$ of order 2 with counting function $B(X) \ll \sqrt{X}$, one may hope to find a set $A$ such that $A(X) = o(\sqrt{X})$ in both the particular discussed cases. On the other hand, thin multiplicative bases of order 2, that is sets $A$ satisfying $A^2 = \mathbb{N}$, cannot be too small since they must contain all the primes, hence $A(X) \gg X/\log X$ (see the recent [6] for recent progress on the subject). This suggests us that the gain below $\sqrt{X}$ cannot be more than a power of $\log X$. In Section 2, we shall prove the following result.

**Theorem 1.1.** Let $k \geq l$ be positive integers with $k \geq 2$ and $A \subset \mathbb{N}$ such that the set $A^k + A^l$ has a positive lower density. Then for infinitely many positive integers $X$, we have

$$A(X) \gg \frac{\sqrt{X}}{\log^\alpha(k, l) X},$$

where $\alpha(k, l) = \frac{k+l-2}{k+l}$.

In Section 2, we shall also prove the following result.

**Theorem 1.2.** There exists an $A \subset \mathbb{N}_0$ such that $A^k + A = \mathbb{N}_0$ and for all sufficiently large $X$, we have

$$A(X) \ll \frac{\sqrt{X}}{\log^\alpha(k) X},$$

where $\alpha(k) = \frac{k-2}{k+1}$.

The probabilistic method remains an efficient method for proving the existence of thin bases by controlling the asymptotic behaviour in a probabilistic way. Nevertheless it could not provide optimally thin bases by a sufficiently general model.

In Section 3, we study the possible deviation in the behaviour of the counting function $A(X)$ in the family of all sets $A$ such that $A^2 + A = \mathbb{N}_0$.

The existence of a set $A \subset \mathbb{N}_0$ such that $A^2 + A = \mathbb{N}_0$ and $A(X) = o(\sqrt{X})$ is not yet solved. We only mention that the dyadic set

$$T = \{2\} \cup \left\{ \sum_{i=0}^{k} \varepsilon_i 2^{2i}, \ k \geq 0, \ \varepsilon_i \in \{0, 1\} \right\}$$

satisfies $T^2 + T \supseteq 2 \cdot T + T = \mathbb{N}$ and $\limsup_{X \to \infty} \frac{T(X)}{\sqrt{X}} = \sqrt{3}$.

In Section 4, we will show
Theorem 1.3. For any positive increasing function \( \phi(X) \) going to infinity as \( X \to \infty \), there exists a set \( A \subset \mathbb{N} \) such that \( d(A^2 + A) = 1 \) and \( \lim \inf_{X \to \infty} A(X)(X\phi(X))^{-1/3} < \infty \).

In Section 4, we give the necessary tools of probability theory.

In Section 6, we construct a thin set \( A \) such that \( A^2 + A = \mathbb{N}_0 \) and whose counting function satisfies \( A(X) = O(\sqrt{X}) \). More precisely, we prove the following result.

Theorem 1.4. There exists \( A \subset \mathbb{N}_0 \) with \( A^2 + A = \mathbb{N}_0 \) and \( A(X) = O(\sqrt{X} \log 1/4 X) \).

2. General asymptotic bounds

In this section, we shall prove Theorems 1.1 and 1.2. For this we need the following result, which follows by partial summation.

Lemma 2.1. Let \( \alpha, \beta < 1 \) (not necessarily positive) be real numbers. Let \( A, B \subset \mathbb{N} \) such that \( A(X) \ll \sqrt{X} \log \alpha X \) and \( B(X) \ll \sqrt{X} \log \beta X \) for all sufficiently large \( X \). Then for all sufficiently large \( X \), we have

(1) \((AB)(X) \leq \sum \sum_{a \in A, b \in B} ab \leq X^2 \ll \sqrt{X} \log^{1-\alpha-\beta} X.\)

We also have

Lemma 2.2. Let \( \alpha, \beta < 1 \) be real numbers. Let \( A, B \subset \mathbb{N} \) such that \( |A(X)| \gg \sqrt{X} \log \alpha X \) and \( B(X) \gg \sqrt{X} \log \beta X \) for all sufficiently large \( X \). Then we have

\[ \sum \sum_{a \in A, b \in B} ab \leq X^2 \gg \sqrt{X} \log^{1-\alpha-\beta} X. \]

for all sufficiently large \( X \).

Proof of Lemma 2.1. For any real number \( X \geq 2 \), we have \( A(X) \leq c \sqrt{X} \log^\alpha(X) \) and \( B(X) \leq c \sqrt{X} \log^\beta(X) \) for some \( c > 0 \). Therefore

(2) \[ \sum \sum_{a \in A, b \in B} 1 \leq A(2)B(X) + B(2)A(X) + \sum \sum_{2 \leq a \leq \frac{X}{2}, a \in A} B(X/a) \leq c \sqrt{X} \sum_{2 \leq a \leq \frac{X}{2}} \frac{I_A(a)}{\sqrt{a} \log^\beta \left( \frac{X}{a} \right)} + O \left( \frac{\sqrt{X}}{\log \min(\alpha, \beta) X} \right). \]

By partial summation we obtain

(3) \[ \sum_{2 \leq a \leq \frac{X}{2}} \frac{I_A(a)}{\sqrt{a} \log^\beta \left( \frac{X}{a} \right)} = \frac{1}{\log^\alpha(X) \log^2 2} + \int_2^{\frac{X}{2}} A(t) \frac{\log \left( \frac{X}{t} \right) - \beta}{2t^{3/2} \log^{\beta+1}(\frac{X}{t})} dt \ll \int_2^{\frac{X}{2}} \frac{dt}{t \log^\beta \left( \frac{X}{t} \right) \log^\alpha(t)} + O(\log^{-\alpha}(X)). \]
Since $1 - \alpha > 0$, $1 - \beta > 0$, we obtain

$$
\int_{2}^{X} \frac{dt}{t \log^\beta(\frac{X}{t}) \log^\alpha(t)} \ll B(1 - \beta, 1 - \alpha) \log^{1-(\alpha+\beta)}X,
$$

where $B$ is the beta function. We obtain (1), (2), (3), and (4) from (2), (3), and (4).

A similar argument gives Lemma 2.2. We provide the details below.

PROOF OF LEMMA 2.2 There exists $c > 0$ and a real number $X_0$ such that $A(X) \geq c\frac{\sqrt{X}}{\log^\alpha(X)}$ and $B(X) \geq \frac{\sqrt{X}}{\log^\beta(X)}$ for all $X \geq X_0$. Therefore for all sufficiently large $X$, we have

$$
\sum_{\substack{a \in A, b \in B \\ ab \leq X}} \frac{1}{n} \geq \sum_{\substack{X_0 \leq n \leq \sqrt{X} \\ n \in A}} B\left(\frac{X}{n}\right) \geq c\sqrt{X} \int_{X_0}^{\sqrt{X}} A(t) \frac{\log(X/t) - \beta}{2t^{3/2} \log^{\beta+1}(X/t)} dt + O(\sqrt{X} \log^{-\alpha}(X)).
$$

When $t \in [X_0, \sqrt{X}]$, we have $\log(X/t) - \beta \geq \frac{\log(X)}{2}$ for all sufficiently large $X$. Using this we obtain that

$$
\int_{X_0}^{\sqrt{X}} A(t) \frac{\log(X/t) - \beta}{2t^{3/2} \log^{\beta+1}(X/t)} dt \gg \int_{X_0}^{\sqrt{X}} \frac{1}{t \log^\beta(X/t) \log^\alpha(t)} dt \gg \log^{1-(\alpha+\beta)}X \int_{\frac{\log X_0}{\log X}}^{\frac{1}{2}} \frac{du}{(1-u)^{\beta} u^\alpha},
$$

provided $X \gg X_0^4$. Using this and (5) the claim follows.

COROLLARY 2.3. Let $\alpha < 1$ be a real number and $n \geq 2$ be an integer. Let $A \subset \mathbb{N}$ such that $A(X) \ll \frac{\sqrt{X}}{\log^\alpha X}$ for all sufficiently large $X$. Then for all sufficiently large $X$, we have

$$
A^k(X) \ll \sqrt{X} \log^{k-1-k\alpha}X.
$$

Proof. Using induction, this is an immediate corollary of Lemma 2.1.

Theorem 1.2 now follows from Corollary 2.3 and the inequality $AB(X) \leq A(X)B(X)$, which is easy to verify.

For proving Theorem 1.2, we need the following result.

LEMMA 2.4. Let $\alpha > 0$ be a real number and $P$ be the set of primes. Then there exists a set $P_1 \subset P$ such that for any sufficiently large integer $X$, we have

$$
\frac{\sqrt{X}}{\log^\alpha X} \ll P_1(X) \ll \frac{\sqrt{X}}{\log^\alpha X}.
$$

In fact, we also have $P_1 \cap (0.5X, X) \gg \frac{\sqrt{X}}{\log^\alpha X}$.

Proof. For any sufficiently large natural number $n$, we have $|P \cap (n, 2n]| \geq \frac{n}{2 \log n}$. We choose any $P_1 \subset P$ which satisfies that for all sufficiently large natural numbers $l$, $|P_1 \cap (2^l, 2^{l+1})| = \frac{2^{l/2}}{2 \log^\alpha 2^l}$. Then $P_1$ is a set as required.
Corollary 2.5. Let $\alpha < 1$ be a real number. Let $P_1$ be a subset of primes with $P_1(X) \geq \frac{c\sqrt{X}}{\log X}$ for any sufficiently large real number $X$, where $c > 0$ is a constant. Then for any $k \geq 2$ we have

$$P_k^k(X) \geq c_1 \sqrt{X} \log^{k-1-k\alpha} X$$

for any sufficiently large $X$ with $c_1 > 0$ being a constant depending only on $c$ and $k$.

Proof. The claim is trivial for $k = 1$. Suppose it is true for $k = l - 1$ with $l \geq 2$. Let $A = P_{l-1}$ and $B = P_l$. For any natural number if $r(n)$ denotes the number of solutions $(a, b)$ of $n = ab$ with $a \in A$ and $b \in B$, then $r(n) \leq l$. Hence we have

$$\sum_{a \in A, b \in B, ab \leq X} 1 = \sum_{n \in AB, n \leq X} r(n) \leq lP_l^l(X).$$

Using the above inequality and applying Lemma 2.2, the claim follows.

We need the following result due to Lorentz.

Theorem 2.6. [2, page 13, Theorem 6] Let $A \subset \mathbb{N}$ with at least 2 elements. Then there exists an additive complement $B \subset \mathbb{N}$ of $A$, namely such that $\mathbb{N} \setminus (A + B)$ is finite, with

$$B(X) \ll \sum_{n=1}^{X} \frac{\log A(n)}{A(n)}.$$ 

We now give a proof of Theorem 1.2.

Proof of Theorem 1.2. Let $\alpha = \alpha(k)$ be as in Theorem 1.2. For this $\alpha$, let $P_1$ be as in Lemma 2.4. Then using Corollary 2.5, we have $P_1^k(X) \gg \sqrt{X} \log^{k-1-k\alpha} X$. Then using Theorem 2.6 there exists an additive complement $B$ of $P_1^k$ with $B(X) \ll \frac{\sqrt{X}}{\log^{k-1-k\alpha} X}$. We obtain the result by taking $A = P_1 \cup B$ and noticing that our choice of $\alpha$ satisfies $\alpha = k - 2 - k\alpha$.

3. Asymptotic behaviour for sets $A$ such that $A^2 + A = \mathbb{N}_0$

In this section we give an account on the deviation for the counting function (beforehand normalized) of sets $A$ such that $A^2 + A = \mathbb{N}_0$.

Let $A \subset \mathbb{N}_0$ and define

$$\alpha_A = \inf \left\{ t \geq 0 : \liminf_{X \to \infty} \frac{A(X)}{X^t} < \infty \right\}, \quad \beta_A = \inf \left\{ t \geq 0 : \limsup_{X \to \infty} \frac{A(X)}{X^t} < \infty \right\}.$$ 

Proposition 3.1. Let $A$ such that $A^2 + A = \mathbb{N}_0$. Then

(a) $\alpha_A \geq 1/3$,
(b) $\beta_A \geq 1/2$,
(c) $\alpha_A + \beta_A \geq 1$.

Proof. (a) We must have for any positive real number

$$X \leq \sum_{a,b,c\in A, ab+c\leq X} 1 \leq A^2(X)A(X) \leq A(X)^3$$ 

hence $\alpha_A \geq 1/3$. 

(b) Let $\beta > \beta_A$. Then $A(X) \leq X^\beta$ for any $X$ large enough. It follows that

$$A^2(X) \leq \sum_{a, b \in A \atop ab \leq x} 1 \leq \sum_{a \in A \atop a \leq X} A \left( \frac{X}{a} \right) \ll X^\beta \sum_{a \in A \atop a \leq X} a^{-\beta} = X^\beta \left( \frac{A(X)}{X^{\beta}} + \beta \int_1^X \frac{A(t)}{t^{\beta+1}} dt \right) \ll X^\beta \left( 1 + \beta \int_1^X t^{-1} dt \right) \ll X^\beta \log X.$$ 

Thus $X \leq A(X)A^2(X) \leq X^{2\beta} \log X$, yielding $\beta \geq 1/2$, whence $\beta_A \geq 1/2$.

(c) Let $\alpha > \alpha_A$ and $\beta > \beta_A$ and $X$ large enough such that $A(X) \ll X^\alpha$. We also have $A(Y) \ll Y^\beta$ for any $Y$. Thus $A^2(X) \ll X^\beta \log X$ and

$$X^\alpha \gg A(X) \gg \frac{X}{A^2(X)} \gg \frac{X^{1-\beta}}{\log X}.$$ 

It follows that $\alpha + \beta \geq 1$ for any $\alpha > \alpha_A$ and any $\beta > \beta_A$. Hence $\alpha_A + \beta_A \geq 1$. \hfill $\Box$

We now prove the reverse statement:

**Proposition 3.2.** For any pair of real numbers $(\alpha, \beta)$ satisfying

$$0 \leq 1 - \beta \leq \alpha \leq \beta < 1 \text{ and } \alpha \geq 1/3,$$

there exists $A \subset \mathbb{N}_0$ such that $A^2 + A = \mathbb{N}_0$ and

$$A(X) \ll X^\beta \log^{2/3} X,$$

$$A(X) \gg X^\alpha \log^{1/3} X, \quad \text{and}$$

$$A(X) \ll X^\alpha \log^{1/3} X \text{ for infinitely many natural numbers } X.$$

In particular, we have $(\alpha, \beta) = (\alpha_A, \beta_A)$.

**Proof.** Let $x_1 \geq 64$ be a sufficiently large natural number so that for any real number $x \geq x_1$, we have

$$\pi(x, 2x) \gg \frac{x}{\log x},$$

where $\pi(x, 2x)$ denotes the number of primes in the interval $(x, 2x]$. Let $\{x_1, x_2, \ldots\}$ be the sequence of natural numbers defined by $x_{i+1} = x_i^2$, $i \geq 1$. For any $i \geq 1$, let $y_i = x_i^{7/3}$ be a real number. Then $y_i \geq x_i^{1/3} \geq 4x_i^2$. Let $P_1$ be a subset of primes with following properties:

$$P_1 \cap [x_i, x_{i+1}] \subset (x_i, y_i], \quad \forall i,$$

$$\left| P_1 \cap [2^j x_i, 2^{j+1} x_i] \right| \ll (2^{3/2} r_{i+1})^3 \log^{1/3} x_i, \quad 1 \leq j \leq \log_2 \frac{y_i}{x_i} - 1, \quad \forall i.$$ 

By (5), since $\beta < 1$, there exist such $P_1$. Let $A_1 = P_1 \cup \{0, 1\}$. It is easy to verify that we have

$$A_1(t) \gg t^{3/2} \log^{1/3} t, \quad 2x_i \leq t \leq y_i, \quad \forall i,$$

$$A_1(t) \ll t^{3/2} \log^{1/3} t, \quad \forall t,$$

$$A_1(t) \gg t^{3/2} \log^{1/3} t, \quad \forall t,$$

$$A_1(x_i) \ll x_i^{3/2} \log^{1/3} x_i, \quad \forall i.$$
Here all the implied constants in the above inequalities are absolute. Let \( t \geq x_2 \) be a real number. Then \( t \in (x_i, x_{i+1}] \) for some \( i \geq 2 \). Using the above inequalities, we have

\[
A^2_i(t) \gg t^{2\beta} \log^{2/3} t, \text{ if } 4x_i^2 \leq t \leq y_i^2, \\
A^2_i(t) \gg t^{2\alpha} \log^{2/3} t, \text{ if } t \in (x_i, 2x_i] \cup (y_i^2, x_{i+1}], \\
A^2_i(t) \gg t^{\beta} \log^{1/3} t, \text{ if } 2x_i \leq t \leq 4x_i^2.
\]

In particular, we have \( A^2_i(t) \gg t^{\min(2\alpha, \beta)} \log^{1/3} t \). Using Theorem 2.6 there exists \( B \subset \mathbb{N}_0 \) such that \( A^2_i + B = \mathbb{N}_0 \) and \( B(t) \leq t^{\alpha} \log^{2/3} t \). Moreover we have for any \( i \geq 1 \),

\[
B(x_{i+1}) \ll x_i^2 + \sum_{4x_i^2 \leq t \leq x_{i+1}} \frac{\log A^2_i(t)}{A^2_i(t)} \ll x_{i+1}^{1-2\alpha} \log^{1/3} x_i \ll x_{i+1}^{-\alpha} \log^{1/3} x_{i+1}.
\]

Let \( A = B \cup A_1 \). Then we have \( A^2 + A = \mathbb{N}_0 \) and \( A(x_i) \ll x_i^{\alpha} \log^{1/3} x_i \). We have

\[
\max(A_1(t), B(t)) \leq A(t) \leq A_1(t) + B(t).
\]

Using this, the result follows. In case \( \alpha < \beta \), in fact we also have \( A(t) \ll t^{\beta} \log^{1/3} t \) for every \( t \).

In [4] Theorem 1.8] the authors proved that for any \( n \) there exists a finite set \( S \subset \mathbb{N}_0 \) such that \( |S| \ll (n \log n)^{1/3} \) and \( \{0, 1, \ldots, n\} \subset S^2 + S \). We can extend the idea of [4] to show the following:

**Corollary 3.3.** There is an infinite set \( A_0 \subset \mathbb{N}_0 \) such that

\[
A^2_0 + A_0 = \mathbb{N}_0 \quad \text{and} \quad \liminf_{X \to \infty} \frac{A_0(X)}{(X \log X)^{1/3}} < \infty.
\]

**Proof.** Using Proposition 3.2 with \( \alpha = 1/3 \) and \( \beta = 2/3 \), the result follows. Note that we also have

\[
\limsup_{X \to \infty} \frac{A_0(X)}{(X \log X)^{2/3}} < \infty.
\]

\[
\square
\]

4. Some basic results in probability

Let \( Y = \{0, 1\}^\mathbb{N} \). Any set \( A \subset \mathbb{N} \) is in one-one correspondence with its indicator function which is an element of \( Y \). One can show an existence of a set \( A \subset \mathbb{N} \) satisfying certain properties by assigning a suitable probability measure on \( Y \) (that is collection of all subsets of \( \mathbb{N} \)) such that the probability of collection of those subsets of \( \mathbb{N} \) which satisfy the required properties is strictly positive. In Sections 5 and 6 we shall use this method to show an existence of a set with the properties we are interested in.

Now \( \{0, 1\} \) is a discrete topological space and \( Y \) is a product topological space. Let \( \mathcal{B} \subset \mathcal{P}(Y) \) be the Borel \( \sigma \)-algebra on \( Y \). Given any sequence of real numbers \( \{x_a\}_{a \in \mathbb{N}} \) with \( 0 \leq x_a \leq 1 \), let \( p_a : \mathcal{P}(\{0, 1\}) \to [0, 1] \) be a sequence of probability measure such that \( p_a(\{1\}) = x_a \). Then there exists a unique probability measure \( \mathbb{P} : \mathcal{B} \to [0, 1] \) such that \( \mathbb{P} = \prod_{a \in \mathbb{N}} p_a \). One says that we are selecting a random subset \( A \) of \( \mathbb{N} \) by selecting every element \( a \in \mathbb{N} \) with probability \( x_a \) and the elements are selected independently. We shall write \( \mathbb{E}^\mathbb{P}(Z) \) (or simply \( \mathbb{E}(Z) \)) and \( \mathbb{V}^\mathbb{P}(Z) \) (or simply \( \mathbb{V}(Z) \)) respectively for the expectation and the variance of a random variable \( Z \) on this probability space.
For any \( a \in \mathbb{N} \), let \( \xi_a : X \rightarrow \{0, 1\} \) be the projection to the \( a \)-th coordinate and we define

\[
A(n) = \sum_{a \leq n} \xi_a, \quad \text{and} \quad \lambda_n = \sum_{a \leq n} x_a.
\]

Then the following result is an easy corollary of [8, Corollary 1.10].

**Lemma 4.1.** For any \( 0 < \varepsilon < 1/2 \), we have

\[
\mathbb{P} \left( \{ A \subset \mathbb{N} : (1 - \varepsilon)\lambda_n \leq A(n) \leq (1 + \varepsilon)\lambda_n \} \right) \geq 1 - 2 \exp \left( \frac{-\varepsilon^2 \lambda_n}{4} \right).
\]

If there exists a finite set \( C \subset \mathbb{N} \) such that for \( a \notin C \), we have \( y_a = 0 \), then \( \mathbb{P} \) induces a probability measure on \( \{0, 1\}^{\mathbb{N}} \) and for any \( 0 < \varepsilon < 1/2 \), we have

\[
\mathbb{P} \left( \{ A \subset C : (1 - \varepsilon)\lambda \leq |A| \leq (1 + \varepsilon)\lambda \} \right) \geq 1 - 2 \exp \left( \frac{-\varepsilon^2 \lambda}{4} \right),
\]

where \( \lambda = \sum_{a \in C} y_a \).

**Lemma 4.2** (Borel-Cantelli Lemma). Let \( E_n \in \mathcal{B} \) with \( \sum_n \mathbb{P}(E_n) < \infty \). Then we have

\[
\mathbb{P} \left( \{ A \subset \mathbb{N} : A \notin E_n \text{ for all sufficiently large } n \} \right) = 1.
\]

**Corollary 4.3.** Suppose \( \lambda_n \geq \kappa \log^2 n \) then

\[
\mathbb{P} \left( \{ A \subset \mathbb{N} : \lim_{n \to \infty} \lambda_n^{-1} A(n) = 1 \} \right) = 1.
\]

**Proof.** We choose \( \varepsilon = \frac{8}{\sqrt{\kappa \log n}} \) in Lemma 4.1. This implies that the probability that \( |\frac{A(n)}{\lambda_n} - 1| \gg \frac{1}{\log n} \) is \( O(\frac{1}{n^2}) \). We conclude by the Borel-Cantelli Lemma. \( \square \)

Let \( R(n) \) be a sequence of random variables on \( Y \). In our applications, we shall need to show that for almost every set, \( R(n) \neq 0 \) for all sufficiently large \( n \). The following result is an immediate corollary of Lemma 4.2.

**Lemma 4.4.** Let \( R(n) \) be a sequence of random variables on \( Y \). If \( \mathbb{P}(\{ R(n) = 0 \}) \leq \frac{1}{n^{1+\eta}} \) for some fixed \( \eta > 0 \), then we have

\[
\mathbb{P}(\{ A \subset \mathbb{N} : R(n) \neq 0 \text{ for all sufficiently large } n \}) = 1.
\]

We assume that \( R(n) \) depends only upon the first \( n \) coordinates. Then \( R(n) \) may be viewed as a random variable on \( Y_n = \{0, 1\}^n \). Moreover \( \mathbb{P} \) induces a probability measure \( \mathbb{P}_n = \prod_{i=1}^n \mathbb{P}_i \) on \( Y_n \) and

\[
\mathbb{P}(\{ R(n) = 0 \}) = \mathbb{P}_n(\{ R(n) = 0 \}).
\]

In order to obtain an upper bound for the probability of those sets such that \( R(n) = 0 \), we shall use Janson’s inequality. Before stating it, we need some assumptions on \( R(n) \) and some notations.

For any \( n \), we shall assume that there exist a finite index set \( I \) and for every \( i \in I \) a Boolean random variable \( Z_i \) on \( Y_n \) such that

\[
R(n) = \sum_{i \in I} Z_i.
\]
Lemma 4.5 (Janson’s inequality). We have
\[ \mathbb{P}(R(n) = 0) \leq \exp(-\mu_n + \Delta_n). \]

A function \( f : Y_n \to \mathbb{R} \) is said to be monotone increasing function if \( f(x_1, \ldots, x_n) \leq f(y_1, \ldots, y_n) \), whenever \( x_i \leq y_i \ \forall i \). In our applications \( I_{R(n)\neq0} \) will be a monotone increasing function. The following result shall be useful in obtaining an upper bound for \( \mathbb{P}(\{R(n) = 0\}) \).

Lemma 4.6. Let \( f : Y_n \to \mathbb{R} \) be a monotone increasing function. Let \( \mathbb{P}_n = \prod_{i=1}^{n} p_i \) and \( \mathbb{P}_n' = \prod_{i=1}^{n} p_i' \) be two probability measure on \( Y_n \) with \( p_i(\{1\}) \geq p_i'(\{1\}) \ \forall i \). Then
\[ \mathbb{E}^{\mathbb{P}_n} f := \sum_{y \in Y_n} f(y)\mathbb{P}_n(\{y\}) \geq \sum_{y \in Y_n} f(y)\mathbb{P}_n'(\{y\}) := \mathbb{E}^{\mathbb{P}_n'} f. \]

Proof. We first show the result when there exists an \( i_0 \) with \( 1 \leq i_0 \leq n \) such that \( p_i = p_i' \) for every \( i \neq i_0 \). We may assume, without any loss of generality that \( i_0 = 1 \). Then we have
\[ (8) \quad \mathbb{E}^{\mathbb{P}_n} f = \sum_{y \in \{0,1\}^{n-1}} f(y,0) \left( \prod_{i=2}^{n} p_i \right)(\{y\}) \]
\[ + \sum_{y \in \{0,1\}^{n-1}} (f(1, y) - f(0, y))p_1(\{1\}) \left( \prod_{i=2}^{n} p_i \right)(\{y\}). \]
Since \( f \) is monotone increasing, for any \( y \in \{0,1\}^{n-1} \), we have \( f(y, 1) - f(y, 0) \geq 0 \). Hence we have
\[ (9) \quad (f(y, 1) - f(y, 0))p_1(\{1\}) \geq (f(y, 1) - f(y, 0))p_1'(\{1\}). \]
Using (8) and (9), we obtain the result when \( p_i = p_i' \) for any \( i \neq 1 \). Using the induction hypothesis, we may assume that the result holds when the number of \( i \) such that \( p_i \neq p_i' \) is at most \( k \geq 1 \). If \( k = n \), then we have nothing to prove. If \( k < n \), we need to show that the result holds when the number of \( i \) such that \( p_i \neq p_i' \) is equal to \( k + 1 \). Without any loss of generality, we may assume that \( p_i = p_i' \) for every \( i \geq k + 2 \). Let \( \mathbb{P}_n'' = \prod_{i=1}^{k+1} p_i'' \) be the measure on \( Y_n \) with \( p_i'' = p_i' \) for \( i \leq k \) and \( p_i'' = p_i \) for \( i \geq k + 1 \). Using the induction hypothesis, we have
\[ \mathbb{E}^{\mathbb{P}_n} f \geq \mathbb{E}^{\mathbb{P}_n''} f \geq \mathbb{E}^{\mathbb{P}_n'} f. \]
Hence the result follows. \( \square \)

5. Locally extremely thin almost sum-product basis

In Corollary 3.3 it was shown that there exists \( A \subset \mathbb{N}_0 \) with \( A^2 + A = \mathbb{N}_0 \) and \( A(X) \ll (X \log X)^{1/3} \) for infinitely many integers \( X \). To obtain a thinner set in the sense that \( A(X) \ll X^{1/3} \) for infinitely many integers \( X \) is out of reach. Nevertheless it happens that by relaxing the covering condition \( A^2 + A = \mathbb{N}_0 \) into \( d(A^2 + A) > 1 - \varepsilon \), we can obtain such a set \( A \) satisfying \( A(X) \ll_{\varepsilon} X^{1/3} \) for infinitely many integers \( X \) (cf. Theorem 5.2).
We shall use the ideas from an additive complement lemma for finite sets of integers due to Ruzsa (see [7, Lemma 2.1]). We state and prove the needed version.

**Lemma 5.1.** Let $0 < \varepsilon < \frac{1}{2}$ be sufficiently small, $n \in \mathbb{N}$ and $A \subset \mathbb{N}$ such that $n > 10^5\varepsilon^{-9/2}$ and
\begin{equation}
\forall x \in [n^{1/3}, \varepsilon n], \; \forall m \in [2x, 2\varepsilon^{-1} x], \; |A \cap [m - 2x, m - x]| > \varepsilon x^{2/3} \log \left(\frac{n}{x}\right).
\end{equation}
Then there exists $B \subset [n^{1/3}, 2\varepsilon n]$ such that $|B| \leq \varepsilon^{-2/3}n^{1/3}$ and
\begin{equation}
\forall t \in [2n^{1/3}, 2n], \; \left|\left[2n^{1/3}, t\right] \cap (A + B)\right| \leq \varepsilon t.
\end{equation}

**Proof.** Let $C = [n^{1/3}, 2\varepsilon n]$. We define a probability measure $\mathbb{P} = \prod_{a \in C} p_a$ on $Y = \{0, 1\}^C$ by choosing
\[ y_a := p_a(\{1\}) = \frac{10\varepsilon^{-1}}{a^{2/3}}. \]

Our assumption implies that $y_a < 1$ and hence there exists such a probability measure. Then
\[ \lambda := \sum_{a \in C} y_a \leq \varepsilon^{-2/3}n^{1/3}. \]

Using [7], we have
\[ \mathbb{P}(B \subset C : |B| \geq 2\lambda) \leq 2 \exp \left(\frac{-\lambda}{4}\right) \]
which can be made smaller than $1/4$ by choosing $\varepsilon$ small enough. Hence
\begin{equation}
\forall x \in [n^{1/3}, \varepsilon n], \; \forall m \in [2x, 2\varepsilon^{-1} x], \; |A \cap [m - 2x, m - x]| > \varepsilon x^{2/3} \log \left(\frac{n}{x}\right).
\end{equation}

For any $B \subset C$, we denote $B_j = B \cap [\varepsilon^{j+1}n, 2\varepsilon^{j+1}n]$ for any $0 \leq j \leq J_\varepsilon := \left\lceil \frac{\log n^{2/3}}{\log \varepsilon^{-1}} \right\rceil - 1$. Let $m \in [2\varepsilon^{j+1}n, 2\varepsilon^j n]$. Then since
\begin{equation}
\forall a \in [m - 2\varepsilon^{j+1}n, m - \varepsilon^{j+1} n], \; y_{m-a} \geq \frac{10\varepsilon^{-1}}{(2\varepsilon^{j+1} n)^{2/3}} > p := \frac{5\varepsilon^{-1}}{(\varepsilon^{j+1} n)^{2/3}},
\end{equation}
we get
\[ \mathbb{P}(m - a \notin B_j, \; \forall a \in A \cap [m - 2\varepsilon^{j+1} n, m - \varepsilon^{j+1} n]) \leq (1 - p)^{|A \cap [m - 2\varepsilon^{j+1} n, m - \varepsilon^{j+1} n]|} \leq \exp \left(-p|A \cap [m - 2\varepsilon^{j+1} n, m - \varepsilon^{j+1} n]|\right). \]

By (10) and (12) this gives
\[ \mathbb{P}(m - a \notin B_j, \; \forall a \in A \cap [m - 2\varepsilon^{j+1} n, m - \varepsilon^{j+1} n]) \leq \exp \left(-5(j + 1)\log \varepsilon^{-1}\right) \leq \frac{\varepsilon^2}{8(j + 1)^2}. \]

We infer
\[ \mathbb{E}(|2\varepsilon^{j+1} n, 2\varepsilon^j n] \setminus (A + B_j)| \leq \frac{\varepsilon^{j+2} n}{4(j + 1)^2}, \]
hence by Markov’s inequality
\[ \mathbb{P}(|2\varepsilon^{j+1} n, 2\varepsilon^j n] \setminus (A + B_j)| > \varepsilon^{j+2} n) \leq \frac{1}{4(j + 1)^2}, \]
and finally
\[ \mathbb{P}(\exists j, \ 0 \leq j \leq J_\varepsilon : \|2\varepsilon^{j+1}n, 2\varepsilon^j n\| \setminus (A + B_j) > \varepsilon^{j+2}n) < \frac{1}{2}. \]

With (11), we deduce that there exists a set \( B \) such that \( |B| \ll \varepsilon^{-2/3}n^{1/3} \) and
\[ \forall j, \ 0 \leq j \leq J_\varepsilon : \|2\varepsilon^{j+1}n, 2\varepsilon^j n\| \setminus (A + B_j) \leq \varepsilon^{j+2}n. \]

Now let \( t > 2n^{1/3} \). Then there is a \( j \) with \( 0 \leq j \leq J_\varepsilon \) such that \( 2\varepsilon^{j+1}n < t \leq 2\varepsilon^j n \). Hence
\[ \|2n^{1/3}, t\| \setminus (A + B) \leq \sum_{i=j}^{J_\varepsilon} \varepsilon^{i+2}n \leq 2\varepsilon^{j+2}n \leq \varepsilon t. \]

This ends the proof of the lemma. \( \square \)

We deduce the main result of the section.

**Theorem 5.2.** For any \( \varepsilon > 0 \), there exists an infinite sequence \( A_0 \) of integers such that
\[ d(A_0^2 + A_0) \geq 1 - \varepsilon \quad \text{and} \quad \liminf_{n \to \infty} X^{-1/3} A_0(X) \ll \varepsilon^{-5/6} \sqrt{\log \varepsilon^{-1}} \ll \varepsilon^{-1} \]
where the implied constant is absolute.

**Proof.** We can plainly assume \( 0 < \varepsilon < \frac{1}{2} \). Let \( (N_k)_{k \geq 1} \) be a sequence of integers where \( N_1 \varepsilon^{9/2} \)

is big enough and \( N_{k+1} = N_k^3 \). This implies that
\[ N_{2k+1}^{1/9} \varepsilon^{9/2} \]

is big enough for any \( k \geq 1 \).

In order to apply Lemma 5.1, we define our sufficiently big set \( A \) according to hypothesis (10).

Let \( k \geq 1 \) and \( N = N_{2k+1} \). Firstly we define a set of prime numbers \( P_{2k+1} \subset [\sqrt{\varepsilon N^{1/3}}, \sqrt{2N}] \). Let \( j \) be an integer such that
\[ 0 \leq j \leq \left \lfloor \frac{\log N^{2/3}}{\log \varepsilon^{-1}} \right \rfloor - 1. \]

We split the interval \([\varepsilon^{j+1}N, 2\varepsilon^j N]\) into \( O(\varepsilon^{-2}) \) intervals of size \( \frac{\varepsilon^{j+2}N}{2} \). If for some \( 2\varepsilon^{-1} \leq r \leq 4\varepsilon^{-2} \)
\[ I_{j,r} := \left[ \frac{r \varepsilon^{j+2}N}{2}, \frac{(r+1) \varepsilon^{j+2}N}{2} \right] \]
is such an interval, then the interval
\[ \left[ \sqrt{\frac{r \varepsilon^{j+2}N}{2}}, \sqrt{\frac{(r+1) \varepsilon^{j+2}N}{2}} \right] \]
has length \( \gg \frac{\sqrt{r \varepsilon^{j+2}N}}{r} \), hence by the Prime Number Theorem it contains at least \( \frac{\varepsilon^{j+2+\sqrt{N}}}{\sqrt{r \log N}} \gg \frac{\varepsilon^{j+2+\sqrt{N}}}{\log N} \)
many primes.

We observe that the above remains true when \( \varepsilon \) tends to 0 when \( N \) increases to infinity, as for instance \( \varepsilon > (\log \log N)^{-1} \). We shall use this fact in the proof of Theorem 11.3.

We have \( \varepsilon^{-j} \leq N^{2/3} \) hence by condition (13)
\[ \varepsilon^{j+2} \frac{\sqrt{N}}{\log N} \geq 2 \sqrt{\varepsilon(\varepsilon^{j+2}N)^{2/3} \log \varepsilon^{-j(j+1)}}. \]
We thus may assign

\[
2\sqrt{\varepsilon (\varepsilon j^2 N)^{2/3} \log \varepsilon^{-j+1}}
\]

prime numbers into \(P_{2k+1}\). Arguing similarly for each interval \(I_{j,r}\), we obtain the required sequence of primes \(P_{2k+1}\).

Our aim is now to show that hypothesis \((10)\) in Lemma 5.1 holds with \(A = P_{2k+1}^2\) and \(n = N\). Let \(x \in [N^{1/3}, \varepsilon N]\) and \(m \in [2x, \varepsilon^{-1}x] \subset [2N^{1/3}, 2N]\).

- If \(2\varepsilon^{j+1}N \leq m - x < 2\varepsilon^{j}N\) and \(x \geq \varepsilon^{j+1}N\) then either \(m - 2x \geq \varepsilon^{j+1}N\) or \(\frac{m - x}{2} \geq m - 2x\). In both cases \([m - 2x, m - x] \cap [\varepsilon^{j+1}N, 2\varepsilon^{j}N]\) has length \(\geq x/2\). Hence

\[
|P_{2k+1}^2 \cap [m - 2x, m - x]| > 2 \left\lfloor \frac{x}{\varepsilon j + 2 N} \right\rfloor \varepsilon (\varepsilon j^2 N)^{2/3} \log \varepsilon^{-j+1}
\]

\[
> \varepsilon x^{2/3} \log \left(\frac{N}{x}\right).
\]

- If \(2\varepsilon^{j+1}N \leq m - x < 2\varepsilon^{j}N\) and \(x < \varepsilon^{j+1}N\) then

\[
[m - 2x, m - x] \subset [\varepsilon^{j+1}N, 2\varepsilon^{j}N]
\]

hence

\[
|P_{2k+1}^2 \cap [m - 2x, m - x]| > 2 \left\lfloor \frac{x}{\varepsilon j + 2 N} \right\rfloor \varepsilon (\varepsilon j^2 N)^{2/3} \log \varepsilon^{-j+1}
\]

\[
> \varepsilon x^{2/3} \log \left(\frac{N}{x}\right) \times \frac{2(j + 1)}{j + 2} \geq \varepsilon x^{2/3} \log \left(\frac{N}{x}\right)
\]

since \(x \geq \frac{\varepsilon m}{2} > \varepsilon^{j+2}N\).

Applying Lemma 5.1 we obtain a partial additive complement \(B_{2k+1}\) of \(P_{2k+1}^2\) in \([2N^{1/3}, 2N]\) such that

\[
|B_{2k+1}| \ll \varepsilon^{-2/3} N^{1/3}.
\]

Moreover since for any \(j\) there are \(O(\varepsilon^{-2})\) intervals \(I_{j,r}\) we deduce

\[
|P_{2k+1}| \ll \sum_{j \geq 0} \varepsilon^{-2} \sqrt{\varepsilon (\varepsilon j^2 N)^{2/3} \log \varepsilon^{-j+1}} = \varepsilon^{-5/6} \sqrt{\log \varepsilon^{-1} N^{1/3}} \sum_{j \geq 0} \varepsilon^{j/3} \sqrt{j + 1}
\]

\[
\ll \varepsilon^{-5/6} \sqrt{\log \varepsilon^{-1} N^{1/3}}.
\]

We define

\[
A_0 = \{0, 1\} \cup \bigcup_{k \geq 1} (\N_{2k-1} \cup \N_{2k} \cup P_{2k+1} \cup B_{2k+1}).
\]

Notice that \(S_k := \[N_{2k-1}, N_{2k}\] \cup P_{2k+1} \cup B_{2k+1} \subset [N_{2k-1}, N_{2k+1}]\) hence the sets \(S_k\)'s do not overlap. By (14), (15) and since \(N_{2k} = N_{2k+1}^{1/3}\), we infer

\[
A_0(N_{2k+1}) \leq N_{2k} + |P_{2k+1}| + |B_{2k+1}| \ll \varepsilon^{-5/6} \sqrt{\log \varepsilon^{-1} N_{2k+1}^{1/3}}.
\]

By Lemma 5.1 again,

\[
|[2N_{2k}, t] \setminus (A_0^2 + A_0)| \leq \varepsilon t, \quad \text{for any} \ 2N_{2k} \leq t \leq 2N_{2k+1}.
\]

Furthermore we have \([2N_{2k-1}, 2N_{2k}], [2N_{2k+1}, 2N_{2k+2}] \subset A_0 + A_0 \subset A_0^2 + A_0\). Thus

\[
|[1, t] \setminus (A_0^2 + A_0)| \leq \varepsilon t + 2N_{2k-1} = \varepsilon t + O(t^{1/3}), \quad \text{for any} \ 2N_{2k} < t \leq 2N_{2k+2}.
\]
We infer \( d(A_0^2 + A_0) \geq 1 - \varepsilon \).

\[ \text{Proof of Theorem 1.3.} \]

For deriving Theorem 1.3 we slightly modify the proof of Theorem 5.2 by letting \( \varepsilon \) to be a function of \( k \). We may assume that \( \phi(t) < (\log \log t)^3 \). For fixed \( k \), we take \( \varepsilon_k = \phi(N_{2k+1}^{-1/3}) \). We check that \( N_{2k+1}^{1/3}\varepsilon_k^{-9/2} \) is big enough and that \( \varepsilon_k > (\log \log N_{2k+1})^{-1} \), allowing us to construct \( P_{2k+1} \) as in the above proof using the Prime Number Theorem in slightly shorter intervals of the type \([X, X(1 + (\log \log X))^{-1}]\). By (16), (17) with \( \varepsilon = \varepsilon_k \) and letting \( k \) tend to infinity we deduce the required result.

\[ \square \]

6. Probabilistic construction of a thin set \( A \) such that \( A^2 + A^2 = \mathbb{N}_0 \)

We define the probability measure \( \mathbb{P} = \prod_{a \in \mathbb{N}} p_a \) on \( Y = \{0, 1\}^\mathbb{N} \) by choosing

\[
x_a := p_a(\{1\}) = \frac{c}{\sqrt{a(\log(a + 1))^{1/4}}}.
\]

We shall choose \( c < 1 \) so that \( x_a < 1 \) and there exists such a probability measure. The following result is easy to prove by partial summation.

**Lemma 6.1.** With the notations as above, we have

\[
\lambda_n := \sum_{a \leq n} x_a \sim 2c\sqrt{n(\log n)^{-1/4}}.
\]

Hence using Lemma 6.1 and Corollary 4.3, we obtain that

**Corollary 6.2.** For any \( \varepsilon > 0 \), we have

\[
\mathbb{P}(\{ A \subset \mathbb{N} : A(n) \sim 2c\sqrt{n(\log n)^{-1/4}} \text{ as } n \text{ tends to infinity } \}) = 1.
\]

We now define the random variable \( R(n) \) counting the number of representations of \( n \) under the form \( n = ab + cd \) restricted to quadruples of distinct integers \( a, b, c, d \in A \). In order to avoid repetitions, we also assume that \( a < b, c < d, ab \leq cd \):

\[
R(n) = \sum_{a, b, c, d} \xi_a \xi_b \xi_c \xi_d
\]

where the dash indicates the above restrictions. In the rest of this section, we shall prove the following result.

**Proposition 6.3.** For a suitable \( c > 0 \), we have

\[
\mathbb{P}(\{ R(n) = 0 \}) \leq \frac{1}{n^{1+\eta}}
\]

for some \( \eta > 0 \).

Using Corollary 6.2, Proposition 6.3 and Lemma 4.3, we obtain Theorem 1.4.

Let \( \mathbb{P}' = \prod_{a=1}^n p'_a \) be a probability measure on \( Y_n = \{0, 1\}^\mathbb{N} \) with \( p'_a(\{1\}) = \frac{c}{\sqrt{a(\log(a + 1))^{1/4}}} \). It is easy to see that \( I_{R(n)\neq 0} \) is a monotone increasing function on \( Y_n \). Therefore to prove Proposition 6.3 using Lemma 4.6, it is sufficient to prove that for a suitable \( c > 0 \), we have

\[
\mathbb{P}'_n(\{ R(n) = 0 \}) \leq \frac{1}{n^{1+\eta}}
\]

for some \( \eta > 0 \).
Let $I = \{(a, b, c, d) \in \mathbb{N}^4 : ab + cd = n; \ a, b, c, d \text{ distinct}, \ a < b, \ c < d, \ ab < cd\}$ be an index set and for any $(a, b, c, d) \in I$, with

$$Z_{(a,b,c,d)} = \xi_a \xi_b \xi_c \xi_d; \text{ we have } Z = \sum_{(a,b,c,d) \in I} Z_{(a,b,c,d)}.$$ 

Hence $R(n)$ is a sum of Boolean random variables. For $(a, b, c, d), (a', b', c', d') \in I$, the random variables $Z_{(a,b,c,d)}$ and $Z_{(a',b',c',d')}$ are independent if and only if $(a, b, c, d) \cap \{a', b', c', d'\} = \emptyset$.

Note that if $n = ab + cd = a'b' + c'd'$ and $(a, b, c, d) \neq (a', b', c', d')$, then $(a, b, c, d) \neq \{a', b', c', d\}$: indeed if for instance $n = ab + cd = ac + bd$ then $a(b - c) = d(b - c)$ hence $a = d$ since $b \neq c$, a contradiction.

Let $n \geq 1$. For any quadruple of distinct positive integers $a, b, c, d$, we denote by $E_n(a, b, c, d)$ the event

$$n = ab + cd \quad \text{and} \quad \xi_a \xi_b \xi_c \xi_d = 1.$$ 

We observe that the events $E_n(\sigma(a), \sigma(b), \sigma(c), \sigma(d))$, where $\sigma$ runs in the set of all permutations of $\{a, b, c, d\}$, are disjoint. Moreover

$$\mu_n := \mathbb{E}(R(n)) = \sum_{a,b,c,d}^I \mathbb{P}(E_n(a, b, c, d)).$$ 

where the dash in the summation means $a, b, c, d$ are distinct and $a < b, c < d$ and $ab < cd$.

If the events $E_n(a, b, c, d)$ where mutually independent we would have

$$\mathbb{P}(R(n) = 0) = \mathbb{P} \left( \bigcap_{a,b,c,d} E_n(a, b, c, d) \right) = \prod_{a,b,c,d} (1 - \mathbb{P}(E_n(a, b, c, d))) \sim e^{-\mu_n}$$

as $n$ tends to infinity. If $\mu_n \sim c' \log n$ as $n$ tends to infinity, with $c' > 1$, then we could deduce from Borel-Cantelli Lemma (cf. Lemma 4.2) that almost surely $R(n) \neq 0$ for any large enough $n$.

But the events $E_n(a, b, c, d)$ are not mutually independent, hence we need to measure their dependence. We denote $(a, b, c, d) \sim (a', b', c', d')$ if $(a, b, c, d) \cap \{a', b', c', d'\} \neq \emptyset$ and $(a, b, c, d) \neq (a', b', c', d')$. We are going to concentrate on the estimation of

$$\Delta_n := \sum_{(a,b,c,d)\sim(a',b',c',d')} \mathbb{P}(E_n(a, b, c, d) \cap E_n(a', b', c', d')).$$

Our goal is to prove that $\mu_n \sim c' \log n$ and $\Delta_n = o(\log n)$. We will conclude by Janson’s inequality (cf. Lemma 4.5).

Let $\tau$ the divisor function. Our estimates will need the following classical facts:

$$\tau(m) \leq 2 \sum_{\substack{l \leq \sqrt{m} \atop l|n}} 1,$$

Moreover for any $\varepsilon > 0$, we have $\tau(n) \ll n^\varepsilon$. Finally $\sum_{d|n} \frac{1}{d} \ll \log \log n$.

We now come to our problem and start to estimate $\mu_n$ and $\Delta_n$.

Firstly by the next lemma (cf. Lemma 6.4) we have the lower bound

$$\mu_n = \sum_{a,b,c,d}^I p_a p_b p_c p_d \sim \frac{c^4}{8 \log(n + 1)} \sum_{0 < k < n} \frac{\tau(k) \tau(n - k)}{\sqrt{n - k} \sqrt{k}} > \left(\frac{3c^4}{4\pi} + o(1)\right) \log n.$$ (18)
The factor 8 in the denominator compensates for the restrictions on \(a, b, c, d\). We used also the fact that the contribution in the sum over \(a, b, c, d\) in which two variables coincide is \(O(\log \log n)\). Indeed:
- when \(a = b\) in \(n = ab + cd\), the contribution is
  \[
  \ll \sum_{0 < a < \sqrt{n}} \frac{\tau(n - a^2)}{a\sqrt{n - a^2}} \ll n^\varepsilon \sum_{0 < a < \sqrt{n}} \frac{1}{a\sqrt{n - a^2}}.
  \]
In the sum, for \(0 < a \leq \frac{4\sqrt{n}}{2}\), we get \(O\left(\frac{\log n}{\sqrt{n}}\right)\); for \(\frac{4\sqrt{n}}{2} < a < \sqrt{n} - 1\), we get \(O\left(\frac{1}{n^{1/4}}\right)\); for \(a = \lfloor \sqrt{n} \rfloor\) we get \(O\left(\frac{1}{\sqrt{n}}\right)\).
- when \(a = c\) the contribution is \(\sum_{|a|} \frac{1}{a} \sum_{0 < b < \frac{n}{a}} \frac{1}{\sqrt{b\sqrt{\frac{n}{a} - b}}} \ll \log \log n\) by the easy estimate
  \[
  \sum_{0 < k < n} \frac{1}{\sqrt{k(n - k)}} \sim \int_0^n \frac{dt}{\sqrt{t(n - t)}} = \pi + o(1).
  \]

**Lemma 6.4.** One has

\[
T_n := \sum_{0 < k < n} \frac{\tau(k)\tau(n - k)}{\sqrt{n - k}\sqrt{k}} \geq \frac{6 + o(1)}{\pi} (\log n)^2, \quad n \to \infty.
\]

**Proof of the lemma.** We argue by partial summation, using the estimate due to Ingham (cf. \[5\]).

\[
\sum_{0 < k < n} \tau(k)\tau(n - k) = \frac{6}{\pi^2} n (\log n)^2 \sum_{q|n} \frac{1}{q} \geq U(n) := \frac{6}{\pi^2} n (\log n)^2.
\]

We thus have

\[
\frac{T_n}{2} \geq \sum_{0 < k < \frac{n}{2}} \frac{\tau(k)\tau(n - k)}{\sqrt{n - k}\sqrt{k}} \geq \frac{U\left(\frac{n}{2}\right)}{\pi} + \frac{3}{\pi^2} \int_1^{\pi \over 2} (n - 2t)(\log t)^2 dt.
\]

The above integral is equivalent to

\[
(\log n)^2 \int_1^{\pi \over 2} \frac{(n - 2t)}{\sqrt{t(n - t)^{3/2}}} dt = (\log n)^2 \int_1^{\pi \over 2} \frac{dt}{\sqrt{t\sqrt{n - t}}} - (\log n)^2 \int_1^{\pi \over 2} \sqrt{t} dt.
\]

By partial summation

\[
\int_1^{\pi \over 2} \frac{\sqrt{t} dt}{(n - t)^{3/2}} = 2 + o(1) - \int_1^{\pi \over 2} \frac{dt}{\sqrt{t\sqrt{n - t}}},
\]

hence the result since \(\int_0^1 \frac{du}{\sqrt{u\sqrt{1 - u}}} = \pi\).

Secondly we observe that \((a, b, c, d) \sim (a', b', c', d')\) holds for only 5 different types of configurations:

i) \(a = a'\) and \(a, b, c, d, b', c', d'\) are distinct
ii) \(a = a', b = b'\) and \(a, b, c, d, c', d'\) are distinct
iii) \(a = a', c = c'\) and \(a, b, c, d, b', d'\) are distinct
iv) \(a = a', b = d'\) and \(a, b, c, d, c', d'\) are distinct
v) \(a = a', b = d', c = c'\) and \(a, b, c, d, d'\) are distinct
In the sequel we shall treat them separately and show that the corresponding contributions $E_i, i = 1, \ldots, 5$, are negligible.

Contribution (i). The representations of $n$ under the form $n = ab + cd = ab' + c'd'$ contribute for at most

$$E_1 \ll \frac{1}{(\log n)^{7/4}} \sum_{a,b,c,d,b',c',d'} \frac{1}{\sqrt{abcdb'c'd'}} = \frac{1}{(\log n)^{7/4}} \sum_{1 \leq a < n} \frac{1}{\sqrt{a}} \left( \sum_{b} \frac{\tau(n-ab)}{\sqrt{b(n-ab)}} \right)^2.$$

**Lemma 6.5.** Let $a < b$ be real numbers and $a_1, \ldots, a_k \in [a, b]$ with $a_i - a_{i-1} \geq l > 0$. If $f : (a-l, b] \to \mathbb{R}^+$ is a monotonically decreasing function, then

$$f(a_1) + \ldots + f(a_k) \leq \frac{1}{l} \int_{a-l}^{b} f(u) du.$$

If $f : [a, b+1) \to \mathbb{R}^+$ is a monotonically increasing function, then

$$f(a_1) + \ldots + f(a_k) \leq \frac{1}{l} \int_{a}^{b+1} f(u) du.$$

We will readily derive $E_1 \ll (\log n)^{1/4}$ from the following lemma.

**Lemma 6.6.** For any $a$, let $S_a = \sum_{b} \frac{\tau(n-ab)}{\sqrt{b(n-ab)}}$. Then

(19) $$\sum_{a \leq n} \frac{S_a^2}{\sqrt{a}} \ll \log^2 n.$$

*Proof of the lemma.* For any $\varepsilon > 0$ and $a \geq n^\varepsilon$, we have $\tau(n-ab) \ll \varepsilon a^\varepsilon$ with implied constant being independent of $b$ and depending only upon $\varepsilon$. Hence we have

$$\sum_{1 \leq b \leq \frac{a}{2n}-1} \frac{\tau(n-ab)}{\sqrt{b(n-ab)}} \ll \varepsilon \frac{a^\varepsilon}{\sqrt{a}} \left( \sum_{1 \leq b \leq \frac{a}{2n}} \frac{1}{\sqrt{b\sqrt{a-b}}} + \sum_{\frac{a}{2n} \leq b \leq \frac{a}{n}-1} \frac{1}{\sqrt{b\sqrt{\frac{a}{n}-b}}} \right)$$

$$\ll \varepsilon \frac{a^\varepsilon}{\sqrt{a}} \left( \sum_{1 \leq b \leq \frac{a}{2n}} \frac{1}{\sqrt{b}} + \sum_{\frac{a}{2n} \leq b \leq \frac{a}{n}-1} \frac{1}{\sqrt{\frac{a}{n}-b}} \right)$$

$$\ll \varepsilon \frac{a^\varepsilon}{\sqrt{a}}.$$

Hence we have

(20) $$\sum_{n^\varepsilon \leq a \leq n} \frac{1}{\sqrt{a}} S_a^2 \ll \varepsilon 1 + \sum_{n^\varepsilon \leq a \leq n} \frac{1}{\sqrt{a}} \left( \sum_{\frac{a}{n}-1 < b \leq \frac{a}{n}} \frac{\tau(n-ab)}{\sqrt{b(n-ab)}} \right)^2.$$
For any fixed $a$, there exists at most one integer $b_0 \in \left\{ \frac{n}{a} - 1, \frac{n}{a} \right\}$ and for such an integer $b_0$, let $k_a = n - ab_0$. We have that $a$ divides $n - k_a$ and $b_0 \gg \frac{n}{a}$. Hence we get

\begin{equation}
\sum_{a \leq n} \frac{1}{\sqrt{a}} \left( \sum_{\frac{n}{a} - 1 < b \leq \frac{n}{a}} \frac{\tau(n - ab)}{\sqrt{b(n - ab)}} \right)^2 \leq \frac{1}{n} \sum_{a \leq n} \frac{\sqrt{ad^2(k_a)}}{k_a} \leq \frac{1}{n} \sum_{k \leq n} \frac{d^2(k)}{k} \sum_{a|n-k} \sqrt{a} \leq \epsilon \frac{n^\varepsilon}{\sqrt{n}} \log n.
\end{equation}

Using (20), (21) and the inequality $(c + d)^2 \leq 2(c^2 + d^2)$, we obtain

\begin{equation}
\sum_{n^\varepsilon \leq a \leq n} \frac{S_a^2}{\sqrt{a}} \ll \varepsilon 1.
\end{equation}

When $a \leq n^\varepsilon$ and $n$ is sufficiently large, we have

\begin{equation}
\sum_{\sqrt{n} < b \leq \frac{\sqrt{n}}{a} - \sqrt{n}} \frac{\tau(n - ab)}{\sqrt{b(n - ab)}} \leq 2 \sum_{l \leq \sqrt{n}} \sum_{\sqrt{n} < b \leq \frac{\sqrt{n}}{a} - \sqrt{n}} \frac{1}{\sqrt{l(n - ab)}}
\end{equation}

\begin{equation}
\leq 2 \sum_{l_1 \leq \sqrt{n}} \sum_{l_2 \leq \frac{\sqrt{n}}{a} - \sqrt{n}} \frac{1}{\sqrt{l_1(n - ab)}} \ll \frac{d(a)}{\sqrt{a}} \log n.
\end{equation}

Hence by Lemma 6.5

\begin{equation}
\sum_{\sqrt{n} < b \leq \frac{\sqrt{n}}{a} - \sqrt{n}} \frac{\tau(n - ab)}{\sqrt{b(n - ab)}} \leq 2 \sum_{l_1 \leq \sqrt{n}} \sum_{l_2 \leq \frac{\sqrt{n}}{a} - \sqrt{n}} \frac{1}{l_1 \sqrt{\nu(n - au)}}
\end{equation}

\begin{equation}
\ll \frac{\varepsilon}{\sqrt{a}} \log n.
\end{equation}

When $a \leq n^\varepsilon$, we also have

\begin{equation}
\sum_{b \leq \sqrt{n}} \frac{\tau(n - ab)}{\sqrt{b(n - ab)}} + \sum_{\frac{n}{a} - \sqrt{n} \leq b \leq \frac{n}{a}} \frac{\tau(n - ab)}{\sqrt{b(n - ab)}} \ll \varepsilon \frac{n^\varepsilon}{n^{1/4}} + \frac{\sqrt{an^\varepsilon}}{\sqrt{n}} \sum_{m \leq a\sqrt{n}} \frac{1}{\sqrt{m}} \ll \varepsilon \frac{an^\varepsilon}{n^{1/4}}.
\end{equation}

Using (23) and (24), we obtain that

\begin{equation}
\sum_{a \leq n^\varepsilon} \frac{1}{\sqrt{a}} S_a^2 \ll \varepsilon \log^2 n + n^{(9\varepsilon - 1)/2}.
\end{equation}

Using (22) and (25) with $\varepsilon = 1/9$, we obtain the result. \qed
Contribution (ii). The representations of \( n \) under the form \( n = ab + cd = ab + c'd' \) contribute for at most

\[
E_2 \ll \frac{1}{(\log n)^{3/2}} \sum_{a,b,c,d,c',d'} \frac{1}{\sqrt{|abcdc'd'|}}
\]

Letting \( h = ab \), the inner sum becomes

\[
\sum_{0 < h < n} \frac{\tau(h)\tau(n-h)^2}{\sqrt{h(n-h)}} \ll n^{-1/2}.
\]

Contribution (iii). We have \( n = ab + cd = ab' + cd' \). Let \( q = \gcd(a, c) \). Then \( q \mid n \) and

\[
\frac{n}{q} = \alpha b + \gamma d = \alpha b' + \gamma d', \quad \alpha = \frac{a}{q}, \quad \gamma = \frac{c}{q}.
\]

Let \( \alpha, \gamma \) fixed. Since \( \gcd(\alpha, \gamma) = 1 \) we have \( \alpha \mid (d - d') \) and \( \gamma \mid (b - b') \). Let \((b_{\alpha,\gamma}, d_{\alpha,\gamma})\) a fixed solution of the equation \( \frac{n}{q} = \alpha x + \gamma y \). Then there exists \( \lambda \in \mathbb{Z} \) such that \((b, d) = (b_{\alpha,\gamma} - \lambda\gamma, d_{\alpha,\gamma} + \lambda\alpha)\). Similarly \((b', d') = (b_{\alpha,\gamma} - \mu\gamma, d_{\alpha,\gamma} + \mu\alpha)\) for some integer \( \mu \). When \( b, d \) run in \((0, n) \cap \mathbb{N}\) according to the given restrictions, \( \lambda \) runs in some interval \( I_{\alpha,\gamma} \). Further there exists at most a \( \lambda_0 \in I_{\alpha,\gamma} \) such that \( b_{\alpha,\gamma} - \lambda_0 \gamma < \frac{2}{q} \) and at most a \( \lambda_1 \in I_{\alpha,\gamma} \) such that \( d_{\alpha,\gamma} - \lambda_1 \alpha < \frac{2}{q} \). The contribution corresponding to case (3) is

\[
\ll E_3 := \frac{1}{(\log n)^{3/2}} \sum_{q \mid n} \frac{1}{q} \sum_{\alpha, \gamma} \sqrt{\alpha \gamma} \sum_{b, d, b', d'} \frac{1}{\sqrt{b'v'd'v'd'}}
\]

The inner sum can be rewritten and bounded by

\[
\frac{1}{\alpha \gamma} \sum_{\lambda, \mu \in I_{\alpha,\gamma}, \lambda \neq \mu} \left( \frac{b_{\alpha,\gamma}}{\gamma} - \lambda \right)^{-1/2} \left( \frac{d_{\alpha,\gamma}}{\alpha} + \lambda \right)^{-1/2} \left( \frac{b_{\alpha,\gamma}}{\gamma} - \mu \right)^{-1/2} \left( \frac{d_{\alpha,\gamma}}{\alpha} + \mu \right)^{-1/2}.
\]

For brevity let \( F(\lambda, \mu) \) be denote the summand in the above double sum. Observe also that \( \lambda_0 \geq 0 \geq \lambda_1 \) hence \( \lambda_0 = \lambda_1 = 0 \) only if their common value is 0 in which case \( I_{\alpha,\gamma} = \{0\} \). Hence in that case the summation over \( \lambda, \mu \) is empty and the corresponding contribution is zero.

We now assume \( \lambda_0 > \lambda_1 \). By developing the sum over \( \lambda, \mu \) we obtain

\[
\sum_{\lambda, \mu \in I_{\alpha,\gamma}, \lambda \neq \mu} F(\lambda, \mu) + \sum_{i=0}^{1} \sum_{\lambda \in I_{\alpha,\gamma}, \lambda \neq \lambda_0, \lambda_1} F(\lambda, \lambda_0, \lambda_1) + 2F(\lambda_0, \lambda_1).
\]

The first sum involves

(26)

\[
\sum_{\lambda \in I_{\alpha,\gamma}, \lambda \neq \lambda_0, \lambda_1} \left( \frac{b_{\alpha,\gamma}}{\gamma} - \lambda \right)^{-1/2} \left( \frac{d_{\alpha,\gamma}}{\alpha} + \lambda \right)^{-1/2} = O(1)
\]

by the next lemma (with absolute constant). Hence

\[
\sum_{\lambda, \mu \in I_{\alpha,\gamma}, \lambda \neq \lambda_0, \lambda_1} F(\lambda, \mu) \ll 1.
\]
Lemma 6.7. Let \( u, v \) two positive real numbers. Then

\[
\sum_{\frac{1}{2}-v \leq j \leq u-\frac{1}{2}} \frac{1}{\sqrt{(u-j)(v+j)}} \leq 12.
\]

Proof of the lemma. The sum splits into 3 terms according to the range covered by \( j \): for \( \frac{1}{2} - v \leq j \leq 1 \), \( j = 0 \) and \( 1 \leq j \leq u - 1 \). The variable \( j \) takes the value 0 only if \( u, v \geq \frac{1}{2} \) hence \( \frac{1}{\sqrt{uv}} \leq 4 \). The first and the third cases are similar. Letting \( f_{u,v}(j) \) the summand in the considered sum, one has

\[
\sum_{0 \leq j \leq u-\frac{1}{2}} f_{u,v}(j) = \sum_{1 \leq j \leq \frac{u}{2}} f_{u,v}(j) + \sum_{\frac{u}{2} < j \leq u-\frac{1}{2}} f_{u,v}(j)
\]

\[
\leq \frac{1}{\sqrt{u}} \sum_{1 \leq j \leq \frac{u}{2}} \frac{1}{\sqrt{v+j}} + \frac{1}{\sqrt{v+\frac{u}{2}}} \sum_{1 \leq j < \frac{u}{2}} \frac{1}{\sqrt{j}}
\]

\[
\leq \frac{2}{\sqrt{u}} \sum_{1 \leq j \leq \frac{u}{2}} \frac{1}{\sqrt{j}} \leq 2 \frac{\sqrt{\frac{u}{2}}}{\sqrt{0}} \int_{\frac{u}{2}}^{\frac{u}{2}} \frac{dt}{\sqrt{t}} = 4,
\]

and the bound follows. \( \square \)

Since \( \frac{b_{\alpha,\gamma}}{\gamma} - \lambda_0 \geq \frac{1}{\gamma} \) and \( \frac{d_{\alpha,\gamma}}{\alpha} + \lambda_0 \geq \frac{1}{\alpha} \) we get

\[
\sum_{i=0}^1 \sum_{\lambda \in \Gamma_{\alpha,\gamma} \setminus \lambda_0, \lambda_1} F(\lambda, \lambda_i) = O(\sqrt{\gamma} + \sqrt{\alpha})
\]

where we use again (26). Finally \( F(\lambda_0, \lambda_1) = O(\sqrt{\alpha \gamma}) \) since \( \lambda_0 \neq \lambda_1 \).

This readily gives

\[
E_3 \ll \frac{1}{(\log n)^{3/2}} \sum_{q \mid n} \frac{1}{q} \sum_{0 < \alpha, \gamma < \frac{n}{q}} \frac{1}{\alpha^{3/2} \gamma^{3/2}} (1 + \sqrt{\alpha} + \sqrt{\gamma} + \sqrt{\alpha \gamma})
\]

\[
\ll \sqrt{\log n \log \log n}.
\]

Contribution (iv). Here \( n = ab + cd = ab' + c'b \), hence these representations contribute for

\[
E_4 \ll \frac{1}{(\log n)^{3/2}} \sum_{a,b,c,d,b',c' \atop n = ab + cd = ab' + c'b} \frac{1}{\sqrt{abcdb'c'}}.
\]

For any \( a, b \), one has \( q = \gcd(a, b) \mid n \). Further \( q^2 \mid ab < n \) thus \( q < \sqrt{n} \). Hence

\[
E_4 \ll \frac{1}{(\log n)^{3/2}} \sum_{q \mid n} \sum_{a,b \atop q \mid \gcd(a,b) = q} \frac{\tau(n-ab)}{\sqrt{n-ab}} \sum_{a \mid k} \frac{1}{b \sqrt{(n-k)}}.
\]

We fix \( a, b \) and denote by \( K_{a,b} \) the smallest positive integer such that \( a \mid K_{a,b} \) and \( b \mid (n - Kq) \). Let also \( \lambda_{a,b} = (qn - q^2 K_{a,b})/ab \). Then the inner sum in the above inequality is

\[
\frac{1}{q} \sum_{0 \leq \lambda \leq \lambda_{a,b}} \left( K_{a,b} + \lambda \frac{ab}{q^2} \right)^{-1/2} \left( \frac{n}{q} - K_{a,b} - \lambda \frac{ab}{q^2} \right)^{-1/2}.
\]
This sum restricted to $0 < \lambda < \lambda_1$ is bounded from Lemma 6.7 by $O\left(\frac{a^2}{ab}\right)$. Letting $f(\lambda)$ the summand in the above sum we obtain the bound

$$\sum_{\substack{a|k \\ b|(n-k)}} 1 \sqrt{k} \sqrt{n-k} \ll \frac{q}{ab} + \frac{f(0)}{q} + f(\lambda_{a,b}) \frac{1}{q}.$$ 

This yields 2 types of contribution for $E_4$, those given by $f(0)/q$ and $f(\lambda_{a,b})/q$ being treated similarly. The first one is

$$E_4' = \frac{1}{(\log n)^{3/2}} \sum_{\substack{q | n \\ q < \sqrt{n}}} \sum_{a,b} \frac{\tau(n-ab)}{\sqrt{n-ab}} \times \frac{q}{ab} \ll n^\varepsilon \sum_{\substack{q | n \\ q < \sqrt{n}}} \frac{1}{q^2} \sum_{h < \frac{q}{n}} \frac{1}{h} \left( \frac{n}{q^2} - h \right)^{-1/2}.$$ 

Since $q \mid n$ the fractional part of $\frac{n}{q^2} \neq 0$ is $\geq \frac{1}{q}$. Separating the case $h = \lfloor \frac{n}{q^2} \rfloor$ from the rest of the sum over $h$ we find that it is $\ll \frac{n^\varepsilon}{q^2} + q \frac{\log n}{\sqrt{n}}$. It follows that

$$E_4 \ll_{\varepsilon} \frac{n^\varepsilon}{n} \sum_{\substack{q | n \\ q < \sqrt{n}}} q + n^\varepsilon \sum_{\substack{q | n \\ q < \sqrt{n}}} \frac{1}{q} \ll n^\varepsilon \frac{1}{\sqrt{n}}.$$ 

For the remaining contribution and by symmetry we only have to consider that coming from the term $f(0)$. By definition of $K_{a,b}$, the product $\frac{K_{a,b}}{aq^{-1}} \left( \frac{nq^{-1}-K_{a,b}}{bq^{-1}} \right)$ is a positive integer, hence we have

$$(\log n)^{3/2} E_4'' \ll \sum_{\substack{q | n \\ q < \sqrt{n}}} \frac{1}{q} \sum_{a,b} \frac{\tau(n-ab)}{\sqrt{n-ab}} \times \frac{q}{ab} \left( \frac{K_{a,b}}{aq^{-1}} \left( \frac{nq^{-1}-K_{a,b}}{bq^{-1}} \right) \right)^{-1/2} \leq \sum_{\substack{q | n \\ q < \sqrt{n}}} \frac{1}{q} \sum_{a,b} \frac{\tau(n-ab)}{\sqrt{n-ab}} \times \frac{q}{ab} = \sum_{a,b} \frac{\tau(n-ab)}{\sqrt{n-ab} \sqrt{ab}} = \mu_n \log n.$$ 

Hence $E_4 \ll \frac{\mu_n}{\log n}$.

Contribution (v). We have $n = ab + cd = ab' + cb$. Hence $b'$ is uniquely determined by the other variables. This yields the bound for the contribution

$$E_5 \ll \frac{1}{(\log n)^{5/4}} \sum_{\substack{a,b,c,d,b' \\ n = ab + cd = ab' + cb}} \frac{1}{\sqrt{abcdb'}} \leq \frac{1}{(\log n)^{3/2}} \sum_{\substack{a,b,c,d \\ n = ab + cd}} \frac{1}{\sqrt{abcd}} \ll \frac{\mu_n}{(\log n)^{1/4}}.$$ 

We conclude that

$$(27) \quad \Delta_n \ll \sum_{i=1}^{5} E_i \ll \frac{\mu_n}{(\log n)^{1/4}}.$$ 

It thus follows that if $3c^4\pi/4 > 1$, almost surely the random set $A$ has counting function $A(n) \sim 2cn^{1/2}(\log n)^{-1/4}$ and satisfies $\mathbb{N} \setminus (A^2 + A^2)$ is finite. By completing if necessary $A$ by a finite number of nonnegative integers, we get the announced result in Theorem 1.3 we state it under the sharpest following form (the constant is the best possible provided by this probabilistic approach):
Theorem 6.8. Let \( c > (4\pi)^{1/4} \). There exists a set of integers \( A \) such that \( A^2 + A^2 = \mathbb{N}_0 \) and \( A(X) \sim \frac{2c\sqrt{X}}{(\log X)^{1/4}} \) as \( X \to \infty \).

Remark. Let \( l \in \mathbb{N} \). Theorem 1.1 can be extended and Theorem 1.4 can be straightforwardly generalized to the sum-product set

\[
\Sigma_{l,2}(A) := A + \cdots + A + A^2 + A^2.
\]

Namely there exists a set \( A \subset \mathbb{N}_0 \) such that \( A(X) \ll \frac{X^{1/(l+2)}}{\log X} \) and \( \Sigma_{l,2}(A) = \mathbb{N}_0 \). We do not provide the complete proof, we only point out the main points. Since we are no longer concerned with the constant, we may assume that in \( n = x_1 + \cdots + x_l + ab + cd \) satisfy \( x_i, ab, cd \gg n \). The elementary probability for \( x \in \mathbb{N} \) is given by 

\[
p'_x = \frac{c}{x^{(l+1)/(l+2)}(\log n)^{1/(l+4)}}.
\]

Then using plain notation the expectation \( \mu_n = \mathbb{E}(R(n)) \) is

\[
\mu_n \gg \frac{1}{n^{(l+1)/(l+2)} \log n} \sum_{\substack{x_1, \ldots, x_l, h, k \geq n \\ n = x_1 + \cdots + x_l + h + k}} \tau(h) \tau(k) \gg \log n.
\]

The estimation of \( \Delta_n \) concerns variable coincidences inside both representations

\[
n = x_1 + \cdots + x_l + ab + cd = x'_1 + \cdots + x'_l + a'b' + c'd'.
\]

Each collision \( x_i \) with some variable in the second representation induces a lesser degree of freedom in the summation with the counterpart that a factor \( n^{-(l-1)/l} \) is cleared. There could be an additional \( n^\varepsilon \) coming from the divisor function when for instance \( x_i = a \). It gives a contribution to \( \Delta_n \) being \( \ll n^{-1/l+\varepsilon} \mu_n \).

In case of a unique collision among \( a, b, c, d \) and \( a', b', c', d' \), we consider \( n = x_1 + \cdots + x_l + ab + cd = x'_1 + \cdots + x'_l + ab' + c'd' \). Letting \( h = ab, k = cd, h' = a'b' \) and \( k' = c'd' \), the related contribution reduces to

\[
\ll \frac{1}{n^4(\log n)^{-1/(l+4)}} \sum_{h, h', k, k'} \tau(k) \tau(k') \sum_{a | \gcd(h, k)} a^{(l+1)/(l+2)} \ll \frac{(\log n)^{1/(l+4)}}{n^2} \sum_{h, h', k} a^{(l+1)/(l+2)}
\]

Inverting the summations gives \( O(n^2) \) for the triple sum, hence a total contribution \( \ll (\log n)^{1/(l+4)} = o(\mu_n) \). The remaining cases with 2, 3 or 4 collisions are easy to consider and yields smaller contributions. We infer \( \Delta_n = o(\mu_n) \).

Remark. Let \( \delta > 0 \). The arguments used to prove Theorem 1.4 can be used to prove the existence of \( A \subset \mathbb{N} \) such that for any sufficiently large \( n \in \mathbb{N} \) and \( x \in \mathbb{R} \), we have

\[
n = ab + cd, \ a, b, c, d \in A, \quad \text{with} \quad d \leq n^\delta \quad \text{and} \quad A(x) \ll \delta \frac{x^{1/2}}{\log^{1/4} x}.
\]

We do not provide the complete proof and only point out the main points. Since we are no longer concerned with the constant, we may assume that in \( n = ab + cd \) satisfy \( ab, cd \gg n \) and \( d \leq n^\delta \). The elementary probability for \( x \in \mathbb{N} \) is given by 

\[
p'_x = \frac{c}{x^{1/2}(\log n)^{1/4}}.
\]

Let

\[
R(n) = \sum_{ab+cd=n} \xi_a \xi_b \xi_c \xi_d,
\]
where the dash in the above summation indicates the restriction $a, b, c, d$ being distinct and $ab, cd \leq n$ with $d \leq n^\delta$. We have the following lower bound

$$
\mu_n := \mathbb{E}(R(n)) \gg \frac{c^4}{n \log n} \sum_{h+k=n, h, k \leq n} \tau(h) \tau_\delta(k),
$$

where $\tau_\delta(k) = \sum_{d|k, d \leq n^\delta} 1$. Assuming that $\delta \leq 1/2$, using the lower bound $\tau(h) \geq \sum_{a|k, a \leq n^{1/4}} 1$ we obtain that $\mu_n \geq c(\delta) c^4 \log n$, where $c(\delta) > 0$ is a constant depending only upon $\delta$. We choose $c$ such that $c(\delta) c^4 > 1$. For the purpose of obtaining an upper bound for $\Delta_n$, we may ignore the condition that $d \leq n^\delta$ and use directly the bound provided by (27) to obtain that $\Delta_n \ll \log^{3/4} n = o(\mu_n)$.

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