INFLATIONS OF IDEAL TRIANGULATIONS

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Abstract. Starting with an ideal triangulation of $\tilde{M}$, the interior of a compact 3–manifold $M$ with boundary, no component of which is a 2–sphere, we provide a construction, called an inflation of the ideal triangulation, to obtain a strongly related triangulation of $M$ itself. Besides a step-by-step algorithm for such a construction, we provide examples of an inflation of the two-tetrahedra ideal triangulation of the complement of the figure-eight knot in $S^3$, giving a minimal triangulation, having ten tetrahedra, of the figure-eight knot exterior. As another example, we provide an inflation of the one-tetrahedron Gieseking manifold giving a minimal triangulation, having seven tetrahedra, of a nonorientable compact 3–manifold with Klein bottle boundary. Several applications of inflations are discussed.

1. Introduction

Triangulations play a central role in the study and understanding of 3–manifolds. They are used directly or indirectly for the major work on a census of 3–manifolds [3, 14, 13, 12] and are fundamental to most of our advances on decision problems, algorithms, and issues of computational complexity. Triangulations naturally give rise to classes of surfaces called normal and almost normal surfaces, these surfaces in turn have been used in constructions of decompositions and recognition algorithms for 3–manifolds. A triangulation of a 3–manifold can be thought of as a combinatorial analog of a metric on the manifold and just as we try to deform metrics to gain geometric and topological information about a 3–manifold, we can similarly hope to gain geometric and topological information about a 3–manifold by deforming a given triangulation to a ‘good’ triangulation of the 3–manifold. This work contributes to constructions that can be used to modify one triangulation to another that exhibits desirable properties.

It is well known that triangulations contain many normal surfaces that are not very interesting topologically but are artifacts of the triangulation; on the other hand, with certain modification, we can often arrive at a triangulation where there are useful connections between the geometry and topology of the manifold and the normal surfaces in the triangulation. In our work on 0–efficient triangulations [5], the aim was to control normal surfaces with positive Euler characteristic; this leads to a very nice algorithm for the connected sum decomposition of a 3–manifold [5] and

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triangulations that lend themselves nicely to the 3–sphere recognition algorithm [15, 17, 5]. For many algorithms and structure problems it is very desirable to control normal surfaces with zero Euler characteristic; our work on such triangulations includes the work presented here. One of its applications is control of normal annuli in 3–manifolds with boundary [7]. Angle structures in ideal triangulations give interesting examples of connections between normal surfaces and the geometry and topology of 3–manifolds. In fact the space of normal surfaces of an ideal triangulation forms a natural dual object of the space of angle structures [10, 11].

In our study of positive Euler characteristic normal surfaces in a 3–manifold [5], we developed a technique for crushing a triangulation of a 3–manifold along a normal surface; in this work and subsequent work [7], we extend these techniques to manifolds with boundary, crushing a triangulation along the boundary (crushing the boundary to a point) and arriving at a related ideal triangulation of the interior of the 3–manifold. In our considerations of surface with zero Euler characteristic, we discovered an operation on ideal triangulations that is dual to the operation of crushing a triangulation of a 3–manifold with boundary along its boundary. We call this operation on an ideal triangulation an inflation of the ideal triangulation. Starting from an ideal triangulation of the interior of a compact 3–manifold with boundary, an inflation gives a strongly related triangulation of the compact 3–manifold itself, which, in turn, admits a crushing along its boundary returning to the original ideal triangulation.

In Section 3 of this paper, we review the construction of crushing a triangulation of a 3–manifold along a normal surface and apply these techniques to this work, which we distinguish by saying we crush a triangulation along a normal boundary. Theorem 3.1 can be considered the Fundamental Theorem for Crushing Triangulations along a normal surface. There can be obstructions to crushing a triangulation along a normal surface; in fact, there are two such obstructions which can be manifested in the natural cell-decomposition coming from splitting a triangulation along a normal surface. We provide examples of the obstructions that can occur in Figure 4: (A) demonstrates what we refer to as “too many product blocks” and (B) demonstrates “a cycle of prisms.” We follow these examples with an example in Figure 5 of crushing a triangulation along a normal surface for which there are no obstructions. Full details and a proof of Theorem 3.1 can be found in [5]. We end Section 3 by introducing the new notion of a combinatorial crushing of a triangulation along a normal surface. By definition a combinatorial crushing has no obstructions and has a very discrete aspect that may not be the situation in more general crushing without obstructions.

Section 4 introduces inflations of ideal triangulations. If $M$ is a compact 3–manifold with boundary, $T$ is a triangulation of $M$ with all of its vertices in $\partial M$, then by “crushing the triangulation $T$ along $\partial M$ means the crushing of the triangulation $T$ along a normal surface that is the frontier of a small regular neighborhood of $\partial M$. For it to be possible to crush a triangulation along $\partial M$ it is necessary that a small regular neighborhood of $\partial M$ be normally isotopic to a normal surface and that the hypothesis of Theorem 3.1 be satisfied. If these conditions are satisfied, then the process takes some proper subcollection of the tetrahedra of $T$ and uses the face identifications of $T$ to give face identifications to the specific subcollection of tetrahedra resulting in an ideal triangulation of $\overset{\circ}{M}$, the interior of $M$, necessarily with fewer tetrahedra than those in $T$. 
Suppose \( M \) is a compact 3–manifold with boundary, no component of which is a 2–sphere, and \( T^* \) is an ideal triangulation of \( \overset{\circ}{M} \). A triangulation \( T \) of \( M \), having all of its vertices in \( \partial M \), is called an inflation of the ideal triangulation \( T^* \), if there is a combinatorial crushing of \( T \) along \( \partial M \) giving the ideal triangulation \( T^* \). While a combinatorial crushing of a triangulation \( T \) along \( \partial M \) gives a unique ideal triangulation \( T^* \) of \( \overset{\circ}{M} \), the operation of inflation, which is dual to crushing, can consist of many choices which lead to possibly inequivalent triangulations of \( M \).

The construction of an inflation \( T \) of an ideal triangulation \( T^* \) uses all of the tetrahedra of \( T^* \) along with some number of new tetrahedra; the precise number of the new tetrahedra necessary to the construction can be determined at the beginning. Furthermore, crushing the inflated triangulation \( T \) along its boundary eliminates precisely the new tetrahedra that were added to the tetrahedra of \( T^* \) in the inflation construction and gives back the triangulation \( T^* \). Finally, we remark that any ideal triangulation \( T^* \) of the interior of a compact 3–manifold with boundary, no component of which is a 2–sphere, admits an inflation. All of the details in the construction of an inflation of any ideal triangulation are given in Section 4. The precise statement is given in Theorem 4.3.

In Section 5, we provide two examples of the inflation construction. The first is an inflation of the two-tetrahedron ideal triangulation of the \textit{figure-eight}–knot complement. The example has a minimal complexity for the inflation and produces a minimal triangulation of the \textit{figure-eight} knot exterior. Recall, that it is necessary that a minimal triangulation of a knot exterior in \( S^3 \) have precisely one vertex and it must be in its boundary. The second example is an inflation of the one-tetrahedron ideal triangulation of the Giesking manifold. This is a non-orientable 3–manifold that is double covered by the ideal triangulation of the \textit{figure-eight} knot complement. The inflation in this example gives a compact, non-orientable 3–manifold with a Klein Bottle boundary; it is a seven-tetrahedron triangulation and is, again, a minimal triangulation. However, inflations, even of minimal ideal triangulations, do not need to be minimal. It is not known if a minimal triangulation can always be constructed as an inflation.

In the Appendix we give the standard ideal triangulation of the Whitehead link complement; we use this to exhibit in Section 4 how certain steps in the construction take care of inflations having multiple ideal vertices.

Applications of the inflation construction are given in [4, 8, 7]. In [4] we provide a relationship between inflations and adding two–handles to the boundary of a 3–manifold. In particular, this construction, called inflation along a curve, when used in the inflation of an ideal triangulation of the interior of a compact 3–manifold with a torus boundary results in a Dehn filling of the compact manifold along the slope of the curve used in the inflation. In [8], we provide a relationship between inflations and the (closed) normal surfaces in an ideal triangulation and the closed normal surfaces in any inflation. In particular, we prove that if \( T^* \) is an ideal triangulation of the interior of the compact 3–manifold \( M \) with boundary, no component of which is a 2–sphere, and \( T \) is an inflation of \( T^* \), then there is a bijective correspondence between the closed normal surfaces in \( T^* \) and those in \( T \). In particular, all inflations of an ideal triangulation have isomorphic collections of closed normal surfaces. In [7], we use the inflation construction as a main tool to show that any triangulation of a compact, orientable, irreducible, \( \partial \)-irreducible, and anannular 3–manifold can be modified to an annular-efficient triangulation; i.e., a 0–efficient triangulation so
that the only normal annuli with essential boundary are edge-linking. A result of this work, also in [7], is that in any annular efficient triangulation of the compact 3-manifold $M$, there are only finitely many boundary slopes for connected normal surfaces in $\mathcal{T}$ of bounded Euler characteristic.

2. Triangulations

We follow the notation and basic results of [5] on (pseudo-) triangulations, ideal triangulations, and normal surface theory.

Suppose $\Delta = \{\tilde{\Delta}_1, \ldots, \tilde{\Delta}_t\}$ is a pairwise-disjoint collection of compact, convex, linear 3-cells and $\Phi$ is a set of face pairings on the faces of the cells in $\Delta$ so that if $\phi \in \Phi$, then $\phi$ is an affine isomorphism from a face $\sigma_i \in \tilde{\Delta}_i$ to a face $\sigma_j \in \tilde{\Delta}_j$, possibly $i = j$. A face appears in at most one face pairing and the natural quotient map $p : \Delta \to \Delta/\Phi$ is injective on the interior of each simplex of each dimension.

Under these conditions, the quotient space $\Delta/\Phi$ is a 3-manifold, except possibly at the image of a vertex or at the image of the midpoint of an edge. We collect all this information into a single symbol $\mathcal{T}$ and call $\mathcal{T}$ a cell-decomposition of $\Delta/\Phi$, if $\Delta/\Phi$ is a manifold, or ideal cell-decomposition of $\Delta/\Phi$, if $\Delta/\Phi$ is a manifold except possible at the image of a vertex. If each cell in $\Delta$ is a tetrahedron, we call $\mathcal{T}$ a triangulation or ideal triangulation of $\Delta/\Phi$. A cell (tetrahedron), face, edge, or vertex in this cell decomposition is, respectively, the image under $p : \Delta \to \Delta/\Phi$ of a cell (tetrahedron), face, edge, or vertex from the collection $\Delta = \{\tilde{\Delta}_1, \ldots, \tilde{\Delta}_t\}$. We will denote the image of the faces by $T^{(2)}$, the image of the edges by $T^{(1)}$ and the image of the vertices by $T^{(0)}$. We call $T^{(i)}$ the $i$-skeleton of $\mathcal{T}$; but, generally, we just refer to these as the faces, edges, or vertices of $\mathcal{T}$. We will denote the image of $\tilde{\Delta}_i$ by $\Delta_i$ and call $\tilde{\Delta}_i$ the lift of $\Delta_i$. A cell is the quotient of a unique cell and a face is the quotient of one or two faces; edges and vertices may be the quotient of a number of edges or vertices, respectively. We define the degree of an edge $e$ of $\mathcal{T}$ to be the number of edges in $p^{-1}(e)$.

The collection of normal triangles made up of precisely one normal triangle of each type forms a normal surface; a component is called a vertex-linking surface. $\Delta/\Phi$ is a 3-manifold if and only if each vertex-linking surface is a 2-sphere or a 2-cell (in the latter case $M$ has boundary and the vertex is in $\partial M$). Typically, for an ideal triangulation, no vertex-linking surface is a 2-sphere and all vertex-linking surfaces are closed; however, such restrictions are not necessary. The index of an ideal vertex is the genus of its vertex-linking surface.

For a triangulation of a 3-manifold with boundary, if the frontier of a small regular neighborhood of the boundary is normally isotopic to a normal surface, then we say the triangulation has normal boundary. In general, for a triangulation of a 3-manifold with boundary, it is not necessary that the frontier of a small regular neighborhood of the boundary be normally isotopic to a normal surface. For example, if we layer a tetrahedron along an edge in the boundary of a triangulation, then the resulting triangulation will not have normal boundary; in particular, layered triangulations of handlebodies [6] do not have normal boundaries.

We recall some well-known results about triangulations of 3-manifolds.

2.1. Theorem. A closed 3-manifold admits a triangulation with precisely one-vertex.
2.2. **Theorem.** A compact 3–manifold with boundary, no component of which is a 2–sphere, admits a triangulation with all vertices in the boundary and then precisely one vertex in each boundary component.

In each such situation, we say the manifold has a *minimal-vertex triangulation*.

2.3. **Theorem.** The interior of a compact 3–manifold with boundary, no component of which is a 2–sphere, admits an ideal triangulation.

### 3. Crushing triangulations

In [5] we introduced the notion of “crushing a triangulation along a normal surface” and stated and proved the fundamental theorem for crushing.

Crushing a triangulation of a 3–manifold along a normal surface provides a global method for modifying the triangulation. It can be used to reduce the number of tetrahedra in a given triangulation [5], construct the prime decomposition of a 3–manifold [5] [9], construct ideal triangulations [5] [7], and gain a better understanding of the normal surfaces in a triangulation [9] [7]. In this section we give definitions and state a special case of the fundamental theorem on crushing triangulations, which is applicable to our needs in this work. This version is the inverse of an inflation of an ideal triangulation. The latter is the main purpose of this paper and is described in Section 4. In fact, understanding crushing in this special case provided the motivation and understanding for developing the inflation construction in the next section.

Suppose \( \mathcal{T} \) is a triangulation of the compact 3–manifold \( M \) or an ideal triangulation of the interior of \( M \). Suppose \( S \) is a closed normal surface embedded in \( M \) and \( X \) is the closure of a component of the complement of \( S \) in \( M \) that does not contain any vertices of \( \mathcal{T} \). For our purposes in this paper, \( X \) will be homeomorphic to \( M \). In this situation, we want to use the tetrahedra of \( \mathcal{T} \) to construct a particularly nice ideal triangulation of \( \overset{\circ}{X} \), the interior of \( X \). Since none of the vertices of \( \mathcal{T} \) are in \( X \), we observe that \( X \) has a nice cell-decomposition, \( \mathcal{C} \), consisting of at most four types of cells: *truncated tetrahedra*, *truncated prisms*, *triangular product blocks*, and *quadrilateral product blocks*. See Figure 1.

**Figure 1.** Cells in induced cell-decomposition of \( X \) and ideal triangulation of \( \overset{\circ}{X} \).
The boundary of each 3–cell in \( C \) has an induced cell decomposition in which some of the cells are in \( S \) and some are not. The edges and faces in the decomposition \( C \) are called horizontal if their interiors are in \( S \) and vertical if their interiors are not in \( S \). The quadrilateral vertical 2–cells are called trapezoids; there are two in a truncated-prism, three in a triangular block, and four in a quadrilateral block. The non-trapezoidal vertical 2–cells are in truncated-prisms and truncated-tetrahedra and are hexagons.

We define \( P(C) \) as the union, \( P(C) = \{ \text{vertical edges of } C \} \cup \{ \text{trapezoids} \} \cup \{ \text{triangular blocks} \} \cup \{ \text{quadrilateral blocks} \} \). \( P(C) \) is called the combinatorial product for \( C \).

Each component of \( P(C) \) is an I–bundle. Suppose \( P(C) \neq X \) and each component is a product I–bundle. Under these assumptions, a component of \( P(C) \) is a product \( P_i = K_i \times [0,1] \), where \( K_i \) is isomorphic to a subcomplex in the induced normal cell structure on \( S \), \( i = 1, \ldots, k \), and \( k \) is the number of components of \( P(C) \). Let \( K^0_i = K_i \times \varepsilon, \varepsilon = 0 \) or 1. Then \( K^0_i \) and \( K^1_i \) are disjoint, isomorphic subcomplexes of the induced normal cell structure on \( S \).

Now, consider the truncated-prisms in \( C \). Each truncated-prism has two hexagonal faces. In \( C \), these hexagonal faces are identified via the face identifications of the given triangulation \( T \) to a hexagonal face of a truncated-tetrahedron or to a hexagonal face of truncated-prism. If we follow a sequence of such identifications through hexagonal faces of truncated-prisms, we trace out a well-defined arc that terminates at an identification with a hexagonal face of a truncated-tetrahedron or possibly does not terminate but forms a complete cycle through hexagonal faces of truncated-prisms. See Figure 2. We call a collection of truncated-prisms identified in this way a chain. If a chain ends in a truncated-tetrahedra, we say the chain terminates; otherwise, we call the chain a cycle of truncated-prisms.

![Figure 2. Chain of truncated prisms.](image)

In general, conditions sufficient for crushing must be established; for example, in a general situation it may not be true that each \( K_i \times I \) is a product I-bundle or that each \( K_i \) is simply connected, or that \( P(C) \neq X \). However, for the purposes of this work, each \( K_i \times I \) is a product I-bundle, each \( K_i \) is a simply connected planar complex and hence, cell-like, and \( P(C) \neq X \). Under all these conditions, we say \( P(C) \) is a trivial combinatorial product. Furthermore, in this work, there
are no cycles of truncated-prisms. In the general situation, we can allow cycles of truncated-prisms but they must be consumed by more general product regions than we are considering here. Again, we refer the reader to [5].

By our assumptions, there are truncated-tetrahedra in the cell decomposition $C$ of $X$ (there are not too many product blocks, $X \neq \mathbb{P}(C)$), and there are not too many truncated-prisms (no cycles of truncated-prisms). To go from the cell decomposition $C$ of $X$ to an ideal triangulation of $\hat{X}$, it is necessary to crush cells (or collections of cells) of $C$, arriving at an ideal triangulation of $\hat{X}$. In particular, each component of $S$ is crushed to a point (distinct points for distinct components), all products $K_i \times I$ are crushed to arcs (edges) so that if $K_i \times I$ is crushed to the edge $e_i$, then the crushing projection coincides with the projection of $K_i \times I$ onto the $I$ factor. Each vertical edge, each trapezoid, and each product block in $C$ becomes an edge. Each truncated-prism becomes a face and each truncated-tetrahedron becomes a tetrahedron. See Figure 1.

Let $\{\Delta_1, \ldots, \Delta_n\}$ be the collection of truncated-tetrahedra in $C$. Notice that each truncated-tetrahedron in $X$ has its triangular faces in $S$. If we crush each such triangular face of a truncated-tetrahedron to a point (for the moment, distinct points for each triangular face), we get a tetrahedron. We use the notation $\hat{\Delta}_*^i$ such triangular face of a truncated-tetrahedron to a point (for the moment, distinct points for distinct components), all products $K_i \times I$ are crushed to arcs (edges) so that if $K_i \times I$ is crushed to the edge $e_i$, then the crushing projection coincides with the projection of $K_i \times I$ onto the $I$ factor. Each vertical edge, each trapezoid, and each product block in $C$ becomes an edge. Each truncated-prism becomes a face and each truncated-tetrahedron becomes a tetrahedron. See Figure 1.

Let $\Delta^* = \{\hat{\Delta}_1^*, \ldots, \hat{\Delta}_n^*\}$ be the tetrahedra obtained from the collection of truncated-tetrahedra $\{\Delta_1, \ldots, \Delta_n\}$ following the crushing of the normal triangles in the surface $S$ to points. It follows that there is a family $\hat{\Phi}^*$ of face-pairings induced on the collection of tetrahedra $\Delta^*$ by the face-pairings of $C$ (coming from the face-pairings of $T$) as follows (see Figure 3):

- if the face $\sigma_i$ of $\Delta_i$ is paired with the face $\sigma_j$ of $\Delta_j$, then this pairing induces the pairing of the face $\hat{\sigma}_i^*$ of $\hat{\Delta}_i^*$ with the face $\hat{\sigma}_j^*$ of $\hat{\Delta}_j^*$;
- if the face $\sigma_i$ of $\Delta_i$ is paired with a face of a truncated-prism in a chain of truncated-prisms and the face $\sigma_j$ of the truncated-tetrahedron $\Delta_j$ is also paired with a face of this chain of truncated-prisms, then the face $\hat{\sigma}_i^*$ of $\hat{\Delta}_i^*$ has an induced pairing with the face $\hat{\sigma}_j^*$ of $\hat{\Delta}_j^*$ through the chain of truncated-prisms.

Hence, we get a 3–complex $\hat{\Delta}^*/\hat{\Phi}^*$, which is a 3–manifold except, possibly, at its vertices. We will denote the associated ideal triangulation by $\hat{T}^*$. We call $\hat{T}^*$ the ideal triangulation obtained by crushing the triangulation $T$ along $S$. We denote the image of a tetrahedron $\hat{\Delta}_i^*$ by $\Delta_i^*$ and, as above, call $\hat{\Delta}_i^*$ the lift of $\Delta_i^*$.

We have the following theorem.

3.1. **Theorem.** Suppose $T$ is a triangulation of a compact 3–manifold or an ideal triangulation of the interior of a compact 3–manifold, $M$. Suppose $S$ is a normal surface embedded in $M$, $X$ is the closure of a component of the complement of $S$, $X$ does not contain any vertices or ideal vertices of $T$, and $C$ is the induced cell-decomposition on $X$. Let $\mathbb{P}(C)$ denote the combinatorial product region for $X$. If

i) $X \neq \mathbb{P}(C)$,

ii) $\mathbb{P}(C)$ is a trivial product region for $X$, and
iii) there are no cycles of truncated-prisms in $X$, then the triangulation $T$ can be crushed along $S$ giving a unique ideal triangulation $T^*$ of $\tilde{X}$.

Proof. Using the above notation, we have that $T$ induces a cell-decomposition $\mathcal{C}$ on $X$. The truncated-tetrahedra in $\mathcal{C}$ (by hypothesis, there must be some) determine a collection of tetrahedra $\Delta^* = \{\Delta^*_1, \ldots, \Delta^*_n\}$ and, as described above, the face-pairings of $T$, along with our hypothesis that there are no cycles of truncated-prisms, determine a family $\Phi^*$ of face-pairings for $\Delta^*$. The underlying point set for the triangulation $T^*$, $\Delta^*/\Phi^*$, is obtained from $X$ by identifying each component of $S$ to a point (distinct points for distinct components), identifying each component, $K_i \times [0, 1]$ of $\mathcal{P}(\mathcal{C})$, of the product region for $X$ to an edge $e_i$ (distinct edges for distinct components; see Figure 1), and identifying each chain of truncated-prisms to a face (see Figure 3). If we look at this identification map we have the inverse image of a point in the interior of a tetrahedron $\Delta^*_i$ is just a point in the interior of the truncated-tetrahedron $\Delta^*_i$; the inverse image of a point in the interior of a face is either a point or an arc, the latter in the case a chain of truncated-prisms is identified to a face; and the inverse image of a point in the interior of an edge is a copy $K_j \times x$ for some $j$ and $x \in [0, 1]$. Notice that in the identification of a chain of truncated-prisms to a face; the associate identification of the edges is through a band of trapezoids and so there are no new identifications not already made in $K_j \times [0, 1]$ for some $j$. Thus the identification map on $\tilde{X}$ is a cell-like map. It follows by [11, 10], that $T^*$ is an ideal triangulation of $\tilde{X}$. Furthermore, there are no choices for the truncated-tetrahedra; they are completely determined by $T$ and $S$. Under our assumptions the truncated-tetrahedra are crushed to tetrahedra and face identifications of $T^*$ are completely (and uniquely) determined by the face-pairings of $T$. We conclude that $T^*$ is uniquely determined by $T$ and $S$.  \[\square\]
Following are three elementary examples exhibiting the construction of crushing a triangulation along a normal 2–sphere. In Figure 4(A) the construction terminates when the induced cell decomposition of $X$ “has too many product blocks”. In this case, we have that $X = P(X)$ is a twisted I-bundle over $\mathbb{R}P^2$ and $M$ is $\mathbb{R}P^3$. In Figure 4(B) the construction terminates when the induced cell decomposition of $X$ “has a cycle of prisms”, giving that $M$ is the 3–manifold $L(3,1)$. In Figure 5 the construction crushes a four-tetrahedron, two-vertex triangulation of $L(4,1)$ to the one-tetrahedron, one-vertex, minimal triangulation of $L(4,1)$. Note when there are no obstructions to crushing along a normal 2–sphere, the ideal triangulation in the conclusion of Theorem 3.1 gives a triangulation.

**Example.** Obstructions when crushing a triangulation along a normal 2–sphere. See Figure 4.

(A) too many product blocks

(B) cycle of prisms

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{example}
\caption{In (A) crushing a two-tetrahedron triangulation of $L(2,1) = \mathbb{R}P^3$ along a normal $S^2$ with obstruction $X = P(X)$. In (B) crushing a two-tetrahedron triangulation of $L(3,1)$ along a normal 2–sphere with obstruction a cycle of prisms.}
\end{figure}

**Example.** Crushing a triangulation along a normal 2–sphere. See Figure 5.

A four-tetrahedron triangulation of $L(4,1)$.

| $\text{tet}$ | (012) | (013) | (023) | (123) |
|-------------|-------|-------|-------|-------|
| (0)         | (3)(023) | (2)(013) | (1)(021) | (2)(123) |
| (1)         | (0)(032) | (2)(013) | (3)(031) | (3)(123) |
| (2)         | (0)(031) | (1)(013) | (3)(021) | (0)(123) |
| (3)         | (2)(032) | (1)(032) | (0)(012) | (1)(123) |

The cell-decomposition of $X$ in this example consists of three truncated-prisms and one truncated-tetrahedron, $\{3\}$. The three truncated-tetrahedra form two chains, one having two truncated-prisms denoted $\{2\}$ and $\{0\}$ and the other having just one truncated-prism denoted $\{1\}$.

The new face identifications after crushing are given as:

\((3^*)(012) \leftrightarrow (2)(032) \leftrightarrow (2)(013) \leftrightarrow (1)(021) \leftrightarrow (2)(123) \leftrightarrow (0)(023) \leftrightarrow (1)(032) \leftrightarrow (0)(012) \leftrightarrow (1)(123)\)
Figure 5. Crushing a four-tetrahedron triangulation of \( L(4,1) \) along a normal \( S^2 \) giving the one-tetrahedron triangulation of \( L(4,1) \).

\[
\begin{align*}
(3^*)(123) & \leftrightarrow (1)(123) & \text{crush} & \leftrightarrow (1)(023) & \leftrightarrow (3^*)(031)
\end{align*}
\]

The new triangulation of \( L(4,1) \) after crushing is the one-tetrahedron, one-vertex triangulation \( T^* \).

\[
\begin{array}{cccc}
\text{tet} & (012) & (013) & (023) & (123) \\
(3^*) & (3^*)(203) & (3^*)(132) & (3^*)(102) & (3^*)(031)
\end{array}
\]

We end this section with a definition and an observation. If \( M \) is a 3–manifold, \( \mathcal{T} \) is a triangulation or ideal triangulation of \( M \), \( S \) is a normal surface, and \( X \) is the closure of a component of the complement of \( S \) meeting no vertices of \( \mathcal{T} \), then under the special conditions \( X \neq \mathbb{P}(X) \), the combinatorial product \( \mathbb{P}(X) \) is trivial, and there are no cycles of truncated-prisms, we have from Theorem 3.1 that the triangulation \( \mathcal{T} \) admits a crushing along \( S \). In this special situation, we say the triangulation \( \mathcal{T} \) admits a \emph{combinatorial crushing along} \( S \). More general conditions for crushing are given in [5]. In the case of a combinatorial crushing along \( S \), the tetrahedra of the ideal triangulation \( \mathcal{T}^* \) are in one-one correspondence with the truncated-tetrahedra of the cell-decomposition \( \mathcal{C} \) of \( X \). Hence, if \( t \) is the number of tetrahedra of \( \mathcal{T} \) and \( t^* \) is the number of tetrahedra of \( \mathcal{T}^* \), then \( t^* \leq t \) with equality if and only if \( S \) is a vertex-linking surface, in which case, \( \mathcal{T} = \mathcal{T}^* \).

4. Inflations of ideal triangulations

Suppose \( X \) is a compact 3–manifold with boundary and \( \mathcal{T} \) is a triangulation of \( X \) with normal boundary. If the triangulation \( \mathcal{T} \) can be crushed along the normal surface that is the frontier of a small regular neighborhood of the boundary, we say \( \mathcal{T} \) \emph{admits a crushing along} \( \partial X \).

4.1. **Definition.** If \( \mathcal{T}^* \) is an ideal triangulation of \( \overset{\circ}{X} \), the interior of the compact 3–manifold \( X \), an \emph{inflation} of \( \mathcal{T}^* \) is a minimal-vertex triangulation \( \mathcal{T} \) of \( X \) with a normal boundary that admits a combinatorial crushing along \( \partial X \) giving the ideal triangulation \( \mathcal{T}^* \).
In this section we provide an algorithm for constructing an inflation of any given ideal triangulation of the interior of a compact 3–manifold.

4.1. Frames. Suppose $S$ is a triangulated surface. A graph in the 1–skeleton of the triangulation of $S$ is called a spine if each component of its complement in $S$ is an open disk. We say a spine $\xi$ for $S$ is a frame if it is minimal with respect to set inclusion; i.e., if $\xi'$ is a spine for $S$ and $\xi' \subset \xi$, then $\xi' = \xi$. Note that a frame has only one component of its complement. A vertex on a frame is called a branch point if its index is greater than 2, in which case its index is called its branching index. A component of a frame minus its branch points is an open arc; we call its closure a branch of the frame. For a surface of genus $g$ there are only finitely many configurations, up to graph isomorphism, for branches and branch points making up a frame. In Figure 6 we show the only two possible configurations for the torus and give examples for frames for genus 2 and genus 3 surfaces. In the case of the torus, we refer to the two possible frames as an index 4 frame or a double index 3 frame. In Figure 7, we give explicit examples of frames; one is a double index 3 frame in the vertex-linking Klein bottle of the one-tetrahedron ideal triangulation of the Gieseking manifold and the other is an index 4 frame in the vertex-linking torus of the two-tetrahedron ideal triangulation of the figure-eight knot complement in $S^3$. In the latter, the frame is the standard meridian/longitude frame. The bars on 6 (6) and on 4 (4) in Figure 7 indicate traversing the edges 6 and 4 in the direction opposite that used in the face identifications of the triangulation (see Figure 31).

**Figure 6.** On the left are the only two possible frames for the torus. On the right are two examples of frames: one for the genus two surface and the other for a genus three surface.

**Figure 7.** A double index 3 frame with three branches for the vertex-linking Klein bottle in the ideal triangulation of the Gieseking manifold and an index 4 frame with two branches for the vertex-linking torus in the two-tetrahedron ideal triangulation of $S^3 \setminus \text{(figure-eight)}$. 
4.2. **Existence of inflations.** The construction of an inflation of an ideal triangulation $\mathcal{T}^*$ begins with the choice of frames in the induced triangulations of the various vertex-linking surfaces of $\mathcal{T}^*$. We have organized the construction with notation that, hopefully, may aid in coding the algorithm so that it can be used more effectively to generate and study examples.

Suppose $\xi$ is a frame in the vertex-linking surface $S$, we label the edges of $\xi$. We choose to label the edges by selecting some direction on a branch; the choice of direction is arbitrary. For such a directed branch, we label the edges successively, $e_1, e_2, \ldots, e_J$, beginning at the initial branch point (determined by the chosen direction on the branch) and ending at the terminal branch point. Of course, the initial and terminal branch point might be the same. We include additional information (again, using the direction of the branch) by labeling the initial and terminal vertices of the edge $e_j$ by $e_{0j}$ and $e_{1j}$, respectively; it is possible that these are the same point, for example, if the branch has only one edge. Hence, if $v_0$ and $v_1$ are the initial and terminal branch points for the branch in question, we have: $v_0 = e_{01}^1, e_{11}^1 = e_{02}^1, \ldots, e_{1J-1}^1 = e_{0J}^1, e_{1J}^1 = v_1$. See Figures 7 and 8; the former gives examples of actual frames with labeled branches. We label the frames in all of the vertex-linking surfaces.

In addition to choosing a direction for each branch and labeling its edges, we choose a transverse direction for each branch. We consistently choose the transverse direction for a branch by using the right-hand-rule at its initial vertex; i.e., if at the initial vertex of the branch, the thumb of the right-hand is pointing in the direction of the ideal vertex, then the index finger of the right hand is pointing in the transverse direction to that branch. We then transport the transverse direction, determined at the initial vertex, along the branch inducing a transverse direction on each of its directed edges. For orientable surfaces, the right-hand rule (described above) can be used at any vertex; however, this is not the case for non-orientable surfaces. We indicate the transverse direction by small transverse arrows on two of the branches in Figure 8; also, see Examples of inflations given below in Section 5.

**Figure 8.** Example of labeling the edges of a frame; the edges in each branch are labeled in succession, beginning (and ending) at a branch point. Transverse directions are shown on two branches.

Before we present the inflation construction, we give an overview so the reader will understand our motivation at various steps in the construction. An inflation of an ideal triangulation, the very definition of which involves crushing, is motivated by an attempt to achieve a model crushing. In crushing a triangulation along a normal surface, quadrilaterals in the surface lead to prisms or quad product regions.
in the tetrahedra containing these quads. The model situation is to have no cycles of truncated prisms and for quad products, if there is more than one quad in a tetrahedron, then there are only two giving a single quad product region between them. The crushed prisms become faces in the ideal triangulation and the crushed product regions become edges in the ideal triangulation. If we consider a copy of the normal surface along which we are crushing (a parallel, normally isotopic copy), then its image after crushing becomes a vertex-linking surface; furthermore, each quad in this surface becomes an edge in the induced triangulation on the vertex-linking surface. The inflation construction reverses this model and uses a frame in each of the vertex-linking surfaces as its guide. In particular, in the inflation construction, we inflate the vertex-linking surfaces in the ideal triangulation \( T^* \) getting normal surfaces in the inflation triangulation \( T \). We show that these surfaces admit a combinatorial crushing that returns to the starting ideal triangulation \( T^* \); the quads in the induced triangulation of the surface crush to the edges in the branches of the frame.

So, how do the frames guide the construction? For each edge in a frame we add a tetrahedron into the ideal triangulation by what we call “an inflation at a face of \( T^* \)”. The edge in the frame inflates to a quadrilateral in the inflation of the vertex-linking surface and is in the added tetrahedron, giving a truncated-prism to be crushed back to the face, as described in Section 3. For edges along a branch of the frame, there is, in general, a unique way to make face identifications for two of the four faces of each added tetrahedron. The identifications of the remaining faces are determined at the vertices of the frame, where an edge in the ideal triangulation meets the vertex-linking surface. We refer to this part of the construction as “an inflation at an edge of \( T^* \).” All of the constructions needed for inflating at an edge of \( T^* \) are combinations of three basic constructions. One is called generic and is associated with an edge of \( T^* \) that only meets the frames in a single point of index 2; another is called a crossing and is associated with an edge of \( T^* \) that meets the frames in two distinct points, each of index 2; the third is called a branch and is associated with an edge of \( T^* \) meeting the frames in one point, which is a branch point. In the generic case, there is only one choice for identification and we do not need to add any tetrahedra. For a crossing, we need to add a tetrahedron to make the necessary face identifications of tetrahedra previously added. For a branch, it is necessary to add a cone over a planar polygon to make the necessary face identifications of tetrahedra previously added; then we make some arbitrary choice of subdividing the polygon (without adding vertices) and cone over the subdivided polygon to achieve the desired triangulation for the inflation.

In Figure 9 we show how the frames can be viewed in a face of \( T^* \); and in Figure 10 we show how they can be viewed at an edge of \( T^* \).

An edge \( E \) of \( T^* \) meets the vertex-linking surfaces in two points; we denote these two points by \( E^+ \) and \( E^- \). In determining these labels, we have implicitly given a direction to the edge (say, it is directed from \( E^- \) to \( E^+ \)); there is no preferred direction and the choice is arbitrary for each edge of \( T^* \). However, this direction is important to our construction as we use it below to assure we get a 3–manifold when we inflate at an edge of \( T^* \). Suppose \( D^+_E \) and \( D^-_E \) are small regular neighborhoods of \( E^+, E^- \), respectively, in the vertex-linking surfaces. Both \( D^+_E \) and \( D^-_E \) receive induced subdivisions into triangles from the triangulation on the vertex-linking surface.
If \( \sigma \) is a face of \( T^* \) having \( E \) as an edge there are unique edges, one in \( D^+_E \) and one in \( D^-_E \), lying in \( \sigma \) and meeting \( E \); we say one of these edges is above the other (relative to \( E \)). If the edge \( x \) is above the edge \( y \), then \( y \) is also above \( x \). The combinatorial structures induced on \( D^+_E \) and \( D^-_E \) by the triangulation of the vertex-linking surfaces are isomorphic via the correspondence that takes the edge \( x \) in \( D^+_E \) (\( D^-_E \)) to the edge above it in \( D^-_E \) (\( D^+_E \)). If \( x \) is an edge in a frame \( \xi \) of one of the vertex-linking surfaces and \( x \) has a vertex at \( E^- \) (\( E^+ \)), then we call the edge in the vertex-linking surface above \( x \) a virtual edge of \( \xi \). It is possible that a virtual edge of \( \xi \) over \( x \) is also an edge \( z \) of a frame (possibly \( \xi \)); if this is the case, then \( x \) is a virtual edge over \( z \). Each edge in a frame has two virtual edges associated with it. We remark that the local structure of edges and virtual edges of the frames about an edge can take numerous forms. Below, we catalog all of the possibilities for an inflation of an ideal triangulation having only one ideal vertex being of index one. In Figure 11, we show a local picture of the vertex-linking surfaces at each end of the edge \( E \) along with edges and virtual edges of the frames meeting \( E \). We have presented the figure with the ideal vertices \( v^* \) and \( w^* \), as well as a transverse direction on the edge \( x_j \). Note that the transverse direction for a virtual edge is taken from that induced on the edge and as such follows the rule for the edge (which in Figure 11 looks like a left-hand-rule on the virtual edge at \( D^+_E \)).

**Inflation at a face of \( T^* \).** Given the triangulation \( T^* \); that is, we have the collection of tetrahedra and the associated family of face identifications of \( T^* \). We shall discard some of the face identifications, add tetrahedra and make new face identifications. We consider the vertices of the tetrahedra in \( T^* \), the ideal vertices, as being included.
Suppose $\sigma$ is a face of $T^*$ and the frames meet $\sigma$. Let $(p)$ and $(p')$ denote the tetrahedra in $T^*$ having $\sigma$ as a face and suppose $(p)(abc) = \sigma = (p')(a'b'c')$ is the face identification. See Figure 11.

![Figure 11](image-url)

**Figure 11.** Two tetrahedra of $T^*$ meeting along the face $\sigma$. We use the notation of REGINA to label simplicies and give face identifications.

As mentioned above, we add a tetrahedron to the triangulation $T^*$ for each edge of the frame(s). If $x_j$ is an edge of a frame we denote the tetrahedron to be added by $(x_j)$ and think of it as the join of two edges, one with vertices 0, 1 and the other with vertices 2, 3; hence, the vertices of $(x_j)$ are labeled 0, 1, 2, 3. We follow the notational conventions of REGINA [2]: i.e., the faces are $(x_j)(012), (x_j)(013), (x_j)(023), \ldots$; edges are $(x_j)(01), (x_j)(02), \ldots$; vertices are $(x_j)(0), \ldots$, etc. The choice of the edge with vertices 2 and 3 in $(x_j)$ is arbitrary; however, later, this choice will be significant in our choice of face identification.

![Figure 12](image-url)

**Figure 12.** The added tetrahedron $(x_j)$ is a join of edge (23) with (01) where the vertices 0 and 1 are chosen after 2 and 3, using the “right-hand rule”.

The choice for the vertices 0 and 1 is made after that for 2 and 3 and uses the convention of labeling the edge $(x_j)(01)$ so that a right-hand twist while going along the edge $(x_j)(23)$ from $(x_j)(2)$ to $(x_j)(3)$ moves the vertex 0 to the vertex 1 in the faces $(x_j)(301)$ and $(x_j)(201)$. We indicate this in the tetrahedra in Figure 12.

We now give the construction for an inflation at a face $\sigma$ of $T^*$, which meets the frames.

**One edge in $\sigma$.** Suppose $\sigma$ contains just one edge of the frame(s). As an edge of a frame it has been labeled and been given a direction and a transverse direction. Suppose its label is $x_j$. In this case, we add one tetrahedron, denoted $(x_j)$ with vertices 0, 1, 2, 3. We discard the face identification $(p)(abc) \leftrightarrow (p')(a'b'c')$ and add
Two new face identifications. The rule for the new face identifications is that in the face \((p)(abc)\) the edge of the face opposite the vertex with the edge \(x_j\) of the frame is identified to the edge \((x_j)(23)\) in the same direction as the directed edge \(x_j\); and similarly, in the face \((p')(a'b'c')\) the edge of the face opposite the vertex with the edge \(x_j\) of the frame is identified to the edge \((x_j)(23)\) in the same direction as the directed edge \(x_j\). In this case, we have the edge \((p)(eb)\) identified with \((x_j)(23)\) and the edge \((p')(e'b')\) also identified with \((x_j)(23)\). It then needs to be determined which vertices of \((x_j)\) the vertices \((p)(a)\) and \((p')(a')\) are to be identified with; one to be identified with \((x_j)(0)\) and the other with \((x_j)(1)\). The rule for these last identifications is determined by the transverse direction to the edge \(x_j\). If the transverse direction along \(x_j\) points out of the tetrahedron \((p)\), then the vertex \(p(a)\) is identified with the vertex \((x_j)(0)\), leaving the vertex \(p'(a')\) to be identified with the vertex \((x_j)(1)\) and the face identifications are \((p)(abc) \leftrightarrow (x_j)(032)\) and \((x_j)(132) \leftrightarrow (p')(a'b'c')\). In the exhibited case (Figure 13), the transverse direction to the edge \(x_j\) points out of the tetrahedron \((p)\); hence, the vertex \(p(a)\) is identified with the vertex \((x_j)(0)\), leaving the vertex \(p'(a')\) to be identified with the vertex \((x_j)(1)\) (the transverse direction to the edge \(x_j\) points into the tetrahedron \((p')\)). The face identifications are \((p)(abc) \leftrightarrow (x_j)(032)\) and \((x_j)(132) \leftrightarrow (p')(a'b'c')\). A quadrilateral is added to the vertex-linking surface for \(x_j\) and two triangles are added for the two virtual edges corresponding to \(x_j\). See Figure 13.

\[
(p)(abc) \leftrightarrow (x_j)(032) \quad \text{and} \quad (x_j)(132) \leftrightarrow (p')(a'b'c')
\]

**Figure 13.** The face \(\sigma\) meets the frames in one edge \(x_j\); an inflation at the face \(\sigma\) adds one tetrahedron.

Two edges in \(\sigma\). Suppose \(\sigma\) contains two edges of the frames. As edges of frames, they have labels and directions; suppose their labels are \(x_j\) and \(y_k\). In this case, we add two tetrahedra; one denoted \((x_j)\) with vertices 0, 1, 2, 3 and the other denoted \((y_k)\) with vertices 0, 1, 2, 3. We discard the face identification \((p)(abc) \leftrightarrow (p')(a'b'c')\) and add three new face identifications \((p)(abc) \leftrightarrow (x_j)(032); (x_j)(132) \leftrightarrow (y_k)(312);\) and \((y_k)(302) \leftrightarrow (p')(a'b'c')\).

The rule for the new face identifications is just as that above and is determined by the direction of the edge in the frame and the transverse direction. See Figure 14. In the face \((p)(abc)\) the edge of the face opposite the vertex with the edge \(x_j\) of the frame is \((p)(eb)\); it is identified to the edge \((x_j)(23)\) in the same direction as the directed edge \(x_j\). Since the transverse direction is pointing out of the tetrahedron \((p)\), the vertex \((p)(a)\) is identified with \((x_j)(0)\). This gives the face identification \((p)(abc) \leftrightarrow (x_j)(032)\). The edge \(y_k\) carries its direction and transverse direction to...
the tetrahedron \((x_j)\) and we have \((x_j)(21)\) being the edge opposite \(y_k\) and in the same direction. Hence, \((x_j)(21)\) is identified with \((y_k)(23)\). The transverse direction on \(y_k\) is pointing into \((p)\) (out of \((p')\)); this is carried over to \((x_j)\) and we identify \((x_j)(3)\) with \((y_k)(1)\). This gives the face identification \((x_j)(132) \leftrightarrow (y_k)(312)\). Finally, we have \((y_k)(302) \leftrightarrow (p')(a'b'c')\).

Two quadrilaterals are added to the vertex-linking surfaces for \(x_j\) and \(y_k\) and four triangles are added for the two virtual edges corresponding to each. Again, see Figure 14.

*Inflate face* 
\((p)(abc) \leftrightarrow (x_j)(032); (x_j)(132) \leftrightarrow (y_k)(312); (y_k)(302) \leftrightarrow (p')(a'b'c')*

**Figure 14.** The face \(\sigma\) meets the frames in two edges \(x_j\) and \(y_k\); an inflation at the face \(\sigma\) adds two tetrahedra.

Note that it does not matter in our construction of an inflation whether we add the tetrahedron \((x_j)\) along \((p)\) or add \((y_k)\) along \((p)\). In fact, one can see that this construction is the same as adding a single pyramid (also shown in Figure 14) and then selecting one of the two diagonals in the quadrilateral face to subdivide it into two tetrahedra. Making one choice verses the other at this step comes up later when it may be necessary to add an additional tetrahedron in order to make necessary face identifications in inflating at the edges of \(T^*\). Below we consider economy in the number of tetrahedra added in an inflation and for this the order does matter but we will see a good choice (economic) is dictated by the frame.

**Three edges in \(\sigma\).** Suppose the frames meet the face \(\sigma\) in three edges. The edges in the frames have labels and directions; suppose the labels are \(x_j\), \(y_k\) and \(z_n\). In this case, we add three tetrahedra, denoted \((x_j), (y_k),\) and \((z_n)\); we label the vertices of \((x_j)\) as 0, 1, 2, 3, those of \((y_k)\) as 0, 1, 2, 3, and those of \((z_n)\) as 0, 1, 2, 3. We discard the face identification \((p)(abc) \leftrightarrow (p')(a'b'c')\) and add four new face identifications. Again, the choice for the order we add the tetrahedra is arbitrary. For this demonstration we shall add the new tetrahedra in the order \((x_j), (z_n)\) and then \((y_k)\). Also, we need to assume some direction on the edges \(x_j, z_n,\) and \(y_k\), as well as transverse directions. The new face identifications are: \((p)(abc) \leftrightarrow (x_j)(032); (x_j)(132) \leftrightarrow (z_n)(231); (z_n)(230) \leftrightarrow (y_k)(203);\) and \((y_k)(213) \leftrightarrow (p')(a'b'c')\). See Figure 15. In the tetrahedron \((x_j)\) the edge \((x_j)(13)\) has the direction of \(z_n\) and, hence, is identified to \((z_n)(23)\); similarly, in \((z_n)\) the edge \((z_n)(20)\) has the direction of \((y_k)\) and, hence, is identified with \((y_k)(23)\). Three quadrilaterals are added to the vertex-linking surface, one for each of the edges \(x_j, y_k,\) and \(z_n\), and two triangles are added for the two virtual edges corresponding to each added tetrahedron, making six triangles added to the vertex-linking surface(s).
For a face meeting three edges of the frames, one can see that the construction is the same as adding a single prism (also shown in Figure 15) and then making a choice of diagonals in the quadrilateral faces to subdivide the prism. In this case, however, the choice of diagonals in the quad faces of a prism impact on further subdivision of the prism; some choices can be extended to a subdivision requiring only three tetrahedra and for other choices it is necessary to use four tetrahedra to subdivide the prism. See Figure 16. In adding tetrahedra as we have done, the diagonals are chosen so that we only need three tetrahedra; adjustments may then need to be made when we address necessary face identifications in inflating at the edges of $T^*$. As mentioned above in the case of two edges of the frames in a face, such choices need much more scrutiny when we come to the issue of economy in adding tetrahedra.

**Figure 15.** The face sigma meets the frames in three edges $x_j$, $y_k$, and $z_n$; an inflation at the face $\sigma$ adds three tetrahedra.

**Figure 16.** The configuration of diagonals in the quadrilateral faces of the prism on the left requires only three tetrahedra to triangulate the prism while the configuration on the right requires four tetrahedra to triangulate.
Remark 4.1. When we inflate $T^*$ at a face, we have well-defined face identifications for two of the four faces of each added tetrahedron; at this step, we leave two of the faces in these tetrahedron unidentified. We have chosen notation so that these two unidentified faces in the tetrahedron $(x_j)$ are $(x_j)(012)$ and $(x_j)(013)$; the edge common to these faces is $(x_j)(01)$, which we shall refer to as a free edge. Below these faces will be identified to other faces when we inflate at the edges of $T^*$. After we have completed the inflation construction, we shall see that all of the free edges coming from adding the tetrahedra $(x_1), \ldots, (x_j), \ldots, (x_J)$ to $T^*$ along a fixed branch of of a frame having edges labeled $x_1, \ldots, x_j, \ldots, x_J$ are identified to a single edge $e_x$ in the inflation $T$ and $e_x$ is in $\partial X$.

For later reference, we also note that for each edge in the frame, we can modify the vertex-linking surface in $T^*$ by adding a quadrilateral about the added free edge and two triangles, one at each end of the edge opposite the free edge in the added tetrahedron. Hence, there is an “inflation” of the vertex-linking surface about each ideal vertex. These inflated surfaces will become boundary-linking surfaces in the inflated triangulation; i.e., the frontiers of a small regular neighborhood of the boundary components in the inflated triangulation. These added quads and triangles are shown in Figures 13, 14, and 15.

Finally, we note that at this stage, the vertices of the 3-complex we have constructed are exactly the ideal vertices we started with for $T^*$.

**Inflation at an edge of $T^*$**. Following the inflation at faces of $T^*$ there are a number of unidentified faces of the added tetrahedra. These unidentified faces correspond to the vertices of the frame in the sense that if $x_j$ is an edge in the frame and $(x_j)$ is a tetrahedron added when we inflate the face containing $x_j$, then the faces $(x_j)(012)$ and $(x_j)(013)$ are faces not identified in that step. The former is associated with the vertex $x_j^0$ of $x_j$ and the latter with the vertex $x_j^1$. The vertices of the vertex-linking surfaces are precisely where the edges of the triangulation $T^*$ meet the vertex-linking surface and the vertices we are interested in are where the edges of $T^*$ meet the frames.

Suppose $E$ is an edge of $T^*$ and $E$ meets the frames; then $E^+$ or $E^-$ is a vertex of a frame; it is possible that both are vertices of frames and that they are vertices of different frames. Locally about $E$ there are edges and virtual edges of the frames meeting $E$. The arrangement of edges and virtual edges in $D^+_E$ meeting $E^+$ is isomorphic to the arrangement of those in $D^-_E$ meeting $E^-$ via the isomorphism between the combinatorial structures induced on $D^+_E$ and $D^-_E$ that takes an edge to the edge above it. In this isomorphism, edges go to virtual edges and vice-versa.

If $x_j$ is an edge of a frame in $D^+_E$, then $E^+$ is a vertex of $x_j$. If $E^+ = x_j^i$, then the face $(x_j)(013)$ is unidentified at $E$; and if $E^+ = x_j^0$, then the face $(x_j)(012)$ is unidentified at $E$. If $\tilde{z}_n$ is a virtual edge of a frame in $D^+_E$, then $E^-$ is a vertex of $z_n$. If $E^- = z_n^i$, then the face $(z_n)(013)$ is unidentified at $E$; and if $E^- = z_n^0$, then the face $(z_n)(012)$ is unidentified at $E$. Starting at any point in $D^+_E$ ($D^-_E$) and making a complete cycle about the boundary of $D^+_E$ ($D^-_E$) we can sequentially list the edges and virtual edges of the frames meeting $E$. Suppose $x_j, y_k, w_l, \ldots, v_m, \ldots, z_n$ is such a listing, say at $D^+_E$, where $e$ denotes the virtual edge of $e$. Then the unidentified faces of the tetrahedra, added at an inflation of the faces containing the edges $x_j, y_k, w_l, \ldots, v_m, \ldots, z_n$, form a band of triangles identified as shown in Figure 15.

It is possible, and most likely, that some of the edges involved in such a sequence
are in the same branch; and if \( E^+ \) is an index 2 vertex in a branch, then consecutive edges of that branch appear in the sequence but are not necessarily consecutive in the cyclic order of edges in the frames at \( E \).

We have labeled the unidentified faces by using the label of the associated edge or virtual edge; of course, the role of edge or virtual edge changes depending on being at \( D^+_E \) or \( D^-_E \). An edge in a frame locally about \( E \) may also be a virtual edge; this creates a potential ambiguity but can easily be resolved. If an edge of a frame about \( E \) is also a virtual edge, then there is a face \( \sigma \) of \( T^* \) containing \( E \) along with both of these edges in the frame; so, from our earlier considerations in the inflation of the face \( \sigma \), we made a choice of diagonals in the subdivision of the pyramid or prism we added. This choice of diagonals determines the order about the edge \( E \) of the faces of the tetrahedron added for the edge versus that added for the virtual edge. The possibilities are demonstrated in Part (C) of Figure 18 where the arrangement of faces about the edge \( E \) incorporates the choice of diagonal in our subdivision of a pyramid in the inflation of the face \( \sigma \).

Our algorithm needs information on the arrangement of edges and virtual edges about an edge \( E \). This can be recorded at either end of \( E \) as represented in \( D^+_E \) or \( D^-_E \); we only need to be consistent and stay with one choice or the other at the edge \( E \). This enables us to code the arrangement of edges and virtual edges about the edge \( E \) as follows. Make a choice of either \( D^+_E \) or \( D^-_E \), then using a planar polygon having the number of sides as there are edges and virtual edges of the frames about \( E \), label its boundary edges either “+” or “−”, where we use “+” to correspond to an edge of a frame and “−” to correspond to a virtual edge of a frame. In Figure 17 we show this method of coding the arrangement of edges about \( E \); later, we shall need to add information about the transverse directions which can also be recorded from \( D^+_E \) or \( D^-_E \). These polygons, along with recorded information on transverse directions will be called \textit{configuration polygons}.

In Figure 18 we provide three examples; in Example (A) there are 4 edges of the frames meeting \( E \) (at \( D^-_E \) there are two edges and two virtual edges), in Example
Basic inflations at an edge of $T^*$. There are three basic configurations of unidentified faces about an edge of $T^*$ and hence, three basic constructions for the inflation at an edge. All other configurations are decomposed into a combination of these three; hence, all other inflation constructions at an edge of $T^*$ are a combination of these basic constructions. We give an algorithm in Lemma 4.2 to decompose an arbitrary configuration about an edge to a combination of the basic configurations.

Generic. In the generic configuration the edge $E$ only meets the frames in a single point that is a vertex of index 2 in the frame. Suppose notation is such that $E$ meets the frame in $E^-$, which is the vertex $x_j^1 = x_j^{0,1}$ between the edges $x_j$ and $x_{j+1}$ of the frame $\xi$. Then the two unidentified faces about the edge $E$ are $(x_j)(013)$ and $(x_{j+1})(012)$. In this situation, we make the face identification
In a generic configuration the edge \( E \) meets the frame in only one point, an index 2 vertex. Free faces can be identified with no tetrahedra added.

\[ (x_j)(013) \leftrightarrow (x_{j+1})(012) \]. Note that our convention for labeling the vertices of the tetrahedra \( x_j \) and \( x_{j+1} \) and the identifications when inflating at a face of \( T^* \) now have the free edges \( (x_j)(01) \leftrightarrow (x_{j+1})(01) \) identified.

**Crossing.** In the crossing configuration the edge \( E \) meets the frames in two points, each is a vertex of index 2 in the frames (here the frames may be different). Suppose notation is such that \( E \) meets the frames at \( y_{k-1}^1 = y_k^1 \) between the edges \( y_{k-1} \) and \( y_k \) and \( E \) meets the frames at \( x^+_j = x^0_{j+1} \) between the edges \( x_j \) and \( x_{j+1} \). Then there are four unidentified faces about the edge \( E \): \( (x_j)(013), (y_{k-1})(013), (x_{j+1})(012), \) and \( (y_k)(012) \). In \( D^-_E \), the edge formed by \( y_{k-1} \) and \( y_k \) crosses the edge formed by the virtual edges \( \bar{x}_j \) and \( \bar{x}_{j+1} \); hence, the name crossing. It is possible that the edge \( E \) meets the frames in two points where each is a vertex of index 2 in the frames but the edges and virtual edges in this case do not cross. If this is the situation, we do not have a crossing and will see below that we can decompose this into two distinct generic configurations.

In the case of a crossing, it is necessary that we add a new tetrahedron. We shall denote this tetrahedron by \( c \) and its vertices by 0, 1, 2, 3; we think of it as the join \( (c)(02) * (c)(13) \). If there is a crossing at the edge \( E \), we shall use the convention that the edge \( (c)(02) \) is always associated with the vertex at \( E^+ \) and then the edge \( (c)(13) \) is always associated with the vertex \( E^- \); that is, since \( x_j \) and \( x_{j+1} \) are edges at \( E^+ \), then \( (x_j)(01) = (c)(02) = (x_{j+1})(01) \) and since \( y_{k-1} \) and \( y_k \) are edges at \( E^- \), then \( (y_{k-1})(01) = (c)(13) = (y_k)(01) \). Using these conventions, then the transverse directions in \( D^-_E \) (or \( D^+_E \)) determine the remaining vertices for the face identifications of the free faces about the edge \( E \). Hence, we have the face identification \( (x_j)(013) \leftrightarrow (c)(023); (c)(021) \leftrightarrow (x_{j+1})(012); (y_{k-1})(013) \leftrightarrow (c)(132); \) and \( (c)(130) \leftrightarrow (y_k)(012) \). See Figure 20. The scheme is that the free edge associated with the branch containing \( x_j \) and \( x_{j+1} \) match through the edge \( (c)(02) \) and the free edge associated with the branch containing \( y_{k-1} \) and \( y_k \) match through the edge \( (c)(13) \); of course, it is possible these are all the same branch and the same free edge.

**Remark 4.2.** Again, for later reference, we note that during inflation at a crossing, we add a tetrahedron and can modify the vertex-linking surface in \( T^* \) by adding...
two quadrilaterals, one about each of the added free edges. See Figure 20 which shows the added quads.

\[
\tilde{x}_j \quad \tilde{y}_{k-1} \quad D_E \quad E
\]

Crossing: one tetrahedron added

\[
(x_j)(013) \leftrightarrow (c)(023);
\]
\[
(c)(021) \leftrightarrow (x_{j+1})(012);
\]
\[
(y_{k-1})(013) \leftrightarrow (c)(132);
\]
\[
(c)(130) \leftrightarrow (y_k)(012)
\]

**Figure 20.** At a crossing the edge \( E \) meets the frames in two points, each an index 2 vertex and the edges and virtual edges form a crossing. A new tetrahedron is added.

**Branch.** In the *branching configuration* the edge \( E \) meets a frame in one point that is a branch point of index \( b, b \geq 3 \). Suppose notation is such that \( E \) meets the frame at \( E^- \); hence, \( E^- \) is a branch point of index \( b \) of a frame. In \( D_E^- \) there are \( b \) edges in the frame, which we may assume are ordered cyclically as \( x, y, \ldots, z \).

Note, it is possible that two edges may be from the same branch of the frame and it is possible that \( E^- \) is an initial point for some branches and a terminal point for others. Furthermore, here we need to use the transverse directions given to the branches of the frames. There are \( b \) unidentified faces of tetrahedra, which were added to the triangulation \( T^* \) from the \( b \) branches at \( E^- \), about the edge \( E \); our conventions give labels to these unidentified faces as \( (x)(01\varepsilon_x), (y)(01\varepsilon_y), \ldots, \) and \( (z)(01\varepsilon_z) \), where \( \varepsilon_e = 2 \) or 3, depending on the direction induced on the edge \( e \) by the directed branch of the frame containing \( e \). In this situation, we let \( P_b \) be a planar \( b \)-gon and form the cone \( (b^*) = 0 \circ P_b \) over the \( b \)-gon \( P_b \). We label the vertices of \( (b^*) \) as 0, 1, 2, \ldots, \( b \), where 1, 2, \ldots, \( b \) are the vertices of \( P_b \) ordered counter-clockwise from the view at the cone point. The cone \( (b^*) \) has \( b \) triangular faces: \( (b^*)(120), (b^*)(230), \ldots, (b^*)(b10) \). In making face identifications we must take into consideration the induced transverse directions on the edges meeting \( E^- \). For demonstration, see Figure 20 we assume the edges are cyclically ordered in \( D_E^- \) as \( x_1, y_K, x_j, z_1, \ldots \). Then we have face identifications \( (x_1)(012) \leftrightarrow (b^*)(b10); (y_K)(013) \leftrightarrow (b^*)(210); (x_j)(013) \leftrightarrow (b^*)(320); (z_1)(012) \leftrightarrow (b^*)(340), \ldots \). The cone point of \( (b^*) \) is identified with the ideal vertex at \( E^+ \).

Note that we may start the cyclical ordering in \( D_E^- \) at any edge; we then start the face labeling of \( b^* \) so that we start \( (b^*)(b1) \) at the first edge in our cyclic...
ordering; it is after this, we need to follow the transverse directions in making face identification. If $e = e_1$ is an initial edge of a branch, then the identification is $(e_1)(012) \leftrightarrow (b^*)(n(n+1)0)$ in the case of an orientable surface; if $e_N$ is a terminal edge of a branch, then the identification is $(e_N)(013) \leftrightarrow (b^*)((n+1)n0)$ in the case of an orientable surface.

\[
\begin{align*}
(x_1)(012) & \leftrightarrow (b^*)(b10); \\
(y_K)(013) & \leftrightarrow (b^*)(210); \\
(x_J)(013) & \leftrightarrow (b^*)(320); \\
(z_1)(012) & \leftrightarrow (b^*)(340)
\end{align*}
\]

Figure 21. The edge $E$ meets the frames in one point, an index $b$ branch point ($b \geq 3$). The cone over a planar $b$-gon is added and then is subdivided into $(b - 2)$ tetrahedra, without adding vertices.

**Remark 4.3.** In the case of an inflation at a branch configuration, we have the possibility of subdividing the planar $b$-gon in a number of ways; hence, we see that even for a fixed frame, the inflation construction does not lead to a unique triangulation.

Also, for later reference, we note that during inflation at a branch point, in each added tetrahedron we modify the vertex-linking surface in $T^*$ by adding two parallel triangles, one that is vertex-linking at the vertex opposite the free face and one that is of the same normal triangle type but we refer to it as parallel to the free face. Also, the faces of subdivided polygon make up faces in the boundary of the manifold resulting from inflation.

We now have the various pieces in place to put together our inflation construction. The following lemma provides the algorithm to reduce any configuration of frames about an edge of the ideal triangulation $T^*$ to a composition of the three basic configurations given above. We assume we are given a configuration via a planar polygon with its edges marked by either a “$+$” or a “$-$”. We shall call a maximal sequence of adjacent edges having the same mark a link.

**4.2. Lemma.** Suppose $P$ is a planar polygon with edges marked either “$+$” or “$-$” and there is not just one edge of a given mark. Then $P$ can be subdivided into
sub-polygons where each sub-polygon has edges marked with either a “+” or “−”, extending the markings on the boundary of \( P \), and each sub-polygon has one of the three forms given above; generic, crossing, or branch.

**Proof.** The proof is by induction on the number of edges of the polygon \( P \).

By hypothesis there is not just one edge of a given mark; hence, the induction begins with \( P \) a bi-gon and the configuration is generic.

We assume the conclusion is true for any marked polygon satisfying the hypothesis and having fewer than \( p \) edges, \( p \geq 3 \). Now, suppose \( P \) is a marked \( p \)-gon.

Consider the various links (maximal collections of adjacent edges having the same mark) in the boundary of \( P \).

If there is only one link, then we have a branch configuration.

If there are no links of length at least 2, then the markings, as we proceed about the edges of \( P \), alternate. Since \( P \) must have an even number of edges (by alternating markings) and \( p \geq 3 \), we have that \( P \) must have at least 4 edges. If \( P \) has precisely 4 edges, then its configuration is a crossing. So, we may assume \( P \) has at least 6 edges. Hence, there are edges of the boundary of \( P \) marked with a “+” and “−” and which are separated by at least two other edges; we denote these edges \( e^+ \) and \( e^- \), respectively. Let \( v \) be a point in the interior of \( P \) and consider the two triangles \( v \ast e^+ \) and \( v \ast e^- \) formed by the join of the edges \( e^+ \) and \( e^- \) with \( v \). Mark the edges of \( v \ast e^+ \) with a “+” extending the mark on \( e^+ \) and mark the edges \( v \ast e^- \) with a “−” extending the mark on \( e^- \). This subdivides \( P \) into four sub-polygons each with marked edges that satisfy our hypothesis and each having fewer than \( p \) edges. In fact the triangles \( v \ast e^+ \) and \( v \ast e^- \) are marked with a branch configuration. The other two sub-polygons have alternating markings. See Figure 22.

So, we may assume there are links having length at least two. If there are just two such links, then we can pinch \( P \) along a segment interior to \( P \) that separates the two links in its boundary. This gives two polygons; these polygons have either generic or branch configurations. Hence, we may assume there are links having length at least two and together these do not determine the totality of marks on the boundary of the configuration polygon.

Now, for a link having length at least two, draw a straight line through the interior of \( P \) from one of its end points to the other. Mark this edge with the same mark as the link that determined it. In this case we have no ambiguity since we have more than two links in the boundary. Repeating this for links in the boundary, we subdivide the polygon \( P \) into a number of sub-polygons, all but one having the configuration of a branch and the exceptional marked sub-polygon having alternating marks and it at least 4 edges. Hence, the induction step follows. □

We exhibit some of the steps in the algorithm from Lemma 4.2 in the following three examples.

In Example (A) there are only two links in the boundary of the marked polygon. The link of length 2 in this case leads to a generic configuration. The link of length 4 leads to a branch configuration, requiring the addition of 2 tetrahedra.

In Example (B), there are two links having length at least two. We draw line segments in the polygon separating off these links and mark the new edges. This gives two branches, each of index 3, and a new polygon. The new polygon has a crossing configuration. Each branch of index 3 requires that we add a tetrahedron.
Figure 22. Three examples giving the steps in the algorithm to subdivide a marked planar polygon into marked sub-polygons having generic, crossing, or branch configurations.

and the crossing configuration requires that we add a tetrahedron; hence, we add 3 tetrahedra.

In Example (C), each link in the boundary has length one, the markings are alternating. Hence, we must add a vertex to the interior of the polygon and cone on two edges in the boundary that are length at least two apart and have distinct markings. We mark the new edges, getting two branch configurations of index 3 and two new marked polygons. In this example one of these is a crossing configuration. The other has alternating markings and so we must again add a vertex to its interior. The result is two more branch configurations of index 3 and a crossing. We must add 7 tetrahedra to resolve the configuration of Example (C).

Face identifications for any of the resulting decompositions of the configuration polygons are induced by edges and transverse directions in the boundary of the configuration polygon. A configuration is either generic, a crossing, branched or is decomposed into a collection of generic, branch and crossing configurations. A branch configuration with edges labeled “+” has its cone vertex identified with the ideal vertex of $E$ at the end labeled $E^-$ and all of its other vertices at the ideal vertex of $E$ labeled $E^+$. Each crossing configuration adds a tetrahedron with two of its vertices identified to the ideal vertex at one end of $E$ and the remaining two vertices identified to the ideal vertex at the other end of $E$. If $c = (c)(02) * (c)(13)$ is
a crossing, then our conventions have the vertices \((c)(0)\) and \((c)(2)\) identified to the ideal vertex at the end designated \(E^+\); and, then, \((c)(1)\) and \((c)(3)\) identified to the ideal vertex at the end designated \(E^-\). Every configuration polygon is associated with a unique edge of \(\mathcal{T}^*\).

**Demonstration.** As a demonstration of what we might have for the configuration polygons, we catalog in Figures 23-25 all the possibilities for an ideal triangulation of the interior of a knot-manifold (one ideal vertex of index one).

![Figure 23.](image)

In the previous construction, we start with an ideal triangulation \(\mathcal{T}^*\) of the interior of a compact 3–manifold \(X\). We select a collection \(\Lambda\) of frames in the vertex-linking surfaces of \(\mathcal{T}^*\). Using these frames as guides, we add tetrahedra to the collection of tetrahedra in \(\mathcal{T}^*\); we discard some of the face identifications in \(\mathcal{T}^*\) and add new face identifications. The new collection of tetrahedra with some of the face identifications from \(\mathcal{T}^*\) and some new face identifications give us a triangulation \(\mathcal{T}_\Lambda\) of a 3–complex. Below in Theorem 4.3 we show that the underlying point set of the 3-complex \(\mathcal{T}_\Lambda\) is a compact 3–manifold with boundary, homeomorphic with \(X\), and \(\mathcal{T}_\Lambda\) is an inflation of \(\mathcal{T}^*\). First, we describe two important surfaces constructed during the inflation.

### 4.3. Boundary and normal boundary in \(\mathcal{T}_\Lambda\).

For each ideal vertex \(v_j^*\) of \(\mathcal{T}^*\) we have an associated vertex-linking surface \(V_j^*\) and a frame \(\xi_j\) in the 1–skeleton of the induced triangulation on \(V_j^*\), which is in the collection \(\Lambda\).

**Boundary surfaces.** The inflation along \(\xi_j\) creates what we are calling free edges and free faces, as was pointed out in the remarks above following various steps in the inflation construction. Free edges are introduced during inflations at the faces of \(\mathcal{T}^*\) that contain an edge of \(\xi_j\). Other free edges and free faces are created when subdividing branching polygons. All the free edges created along a branch of \(\xi_j\) are
identified and are identified with boundary edges in the branching polygons. The identification of free faces and free edges is equivalent to pairwise identification of the boundary edges of the branching polygons. For the inflation along the frame $\xi_j$, these identifications give a connected surface, $B_j$, with a one-vertex triangulation and $B_j$ is a subcomplex of $\mathcal{T}_\Lambda$. It follows that $B_j$ is homeomorphic to $V_j^*$. Since a face in $B_j$ is contained in only one tetrahedron, it is a boundary face of $\mathcal{T}_\Lambda$; below we will have that $B_j$ is a component of $\partial M$, where $M$ is the underlying point set of $\mathcal{T}_\Lambda$.

**Boundary-linking normal surfaces.** The inflation along $\xi_j$ also leads to the construction of a boundary-linking normal surface in $\mathcal{T}_\Lambda$, which can be viewed as an inflation of the vertex-linking surface $V_j^*$. While the significate elements of the inflation of the vertex-linking surface $V_j^*$ are related to the inflation along the frame $\xi_j$, it is possible that inflations along other frames in $\Lambda$ also contribute to the inflation of $V_j^*$.

At each edge of $\xi_j$, we have an inflation in a face of $\mathcal{T}^*$ which removes a face identification in the triangulation and adds a tetrahedron and two new face identifications. The affect on the vertex-linking surface $V_j^*$ is to remove edge identifications of $V_j^*$ in the faces containing the edges of $\xi_j$ and for each tetrahedron added in this fashion, we can add to $V_j^*$ a quadrilateral around the added free edge. It also is necessary to add two vertex-linking triangles at each end of the edge that is opposite the free edge in the added tetrahedron; these vertex-linking triangles may or may not be added to the vertex-linking surface $V_j^*$ but do add to a vertex-linking surface as part of its inflation. These additions are exhibited in Figures 13, 14, and 15. At a crossing, we add a tetrahedron and make identifications between its faces and four unidentified faces of previously added tetrahedra. As part of the inflation of the vertex-linking surfaces, we add two quadrilaterals, which might be added to

![Diagram](image-url)
distinct vertex-linking surfaces; these quadrilaterals are added about the free edges of the crossing tetrahedron. See Figure 21. At a branching, we have a cone over a planar polygonal $b$–gon, $P_b$, which we denoted above $(b^*) = 0 * P_b$. The cone faces of $(b^*)$ are then identified with $b$ unidentified faces of previously added tetrahedra. We subdivide $P_b$ into $b - 2$ triangles and extend this to a subdivision of $(b^*)$ into $b - 2$ tetrahedra by coning each triangle from the cone point (which as noted above is one of the ideal vertices of $T^*$). This gives a triangulation of $(b^*)$ with $b - 2$ free faces corresponding to the faces in the subdivided $b$–gon, $P_b$. In this case we add two triangles in each of the $b - 2$ tetrahedra, one is a vertex-linking triangle about the cone point and the other a face-linking triangle about the free face of the tetrahedron, which is in $P_b$. We exhibit this is Figure 26 for a valence 4 branch point.
Hence, starting with a vertex-linking surface $V_j^*$, and by removing various edge identifications in $V_j^*$ that are dictated by the inflation construction and adding quadrilaterals and triangles, we arrive at a closed normal surface in the triangulation $T_\Lambda$. We denote this surface by $V_j$ to indicate its relationship with $V_j^*$. The normal surface $V_j$ is the frontier of a small neighborhood of the boundary subcomplex $B_j$ from above. Further, we observe that the normal surface $V_j$ maps (“crushes”) via a cell-like map to the vertex-linking surface $V_j^*$; and therefore, the boundary-linking normal surface $V_j$ is homeomorphic to the vertex-linking surface $V_j^*$.

**Figure 26.** Triangles added to the boundary-linking surface at a branch point.

In Figure 27 we exhibit local results of the inflation of the vertex-linking surfaces coming from inflations at faces of $T^*$ along with a contiguous inflation along an edge of $T^*$. We show the latter at a generic, a crossing, and a branch point of the frame. An inflation in a face creates an inflation in the vertex-linking surfaces at each vertex of the face and an inflation along an edge $E$ of $T^*$ creates an inflation of the vertex-linking surfaces at both ends of the edge $E$.

**4.3. Theorem.** Suppose $X$ is a compact 3–manifold with boundary, no component of which is a 2–sphere, and $T^*$ is an ideal triangulation of the interior of $X$. Then for any collection of frames $\Lambda$, one frame in each of the vertex-linking surfaces of $T^*$, the underlying point set of $T_\Lambda$ is a compact 3–manifold $M$ homeomorphic to $X$ and the triangulation $T_\Lambda$ of $M$ is an inflation of the ideal triangulation $T^*$.

**Proof.** First we shall show that the construction gives a minimal vertex triangulation of a compact 3-manifold with boundary.

Let $T_\Lambda$ denote the triangulation constructed and let $M$ be the underlying point set. $M$ is a 3-manifold at the image of each interior point of a tetrahedron and at each interior point of a face. Having chosen a fixed direction on each edge of $T^*$ and respecting this orientation with face identifications in both the inflations at the faces and at the edges, the construction gives a 3-manifold at the image of each edge having a complete cycle of face identifications about it. The edges that do not have a complete cycle of face identifications about them correspond to the “free edges” added when we inflate at a face of $T^*$ or the “free edges” added in the subdivision of the $b$-gons, $b \geq 3$, for branches in the configuration polygons. Starting at any one of the free edges and following face identifications in both directions, there are chains of face identifications that end only when we come to a face of a $b$-gon in one of the configuration polygons. Hence, an interior point $p$ of any of these edges has a 3–cell neighborhood with the point $p$ in the boundary of the 3–cell.
It follows that $M$ is a 3–manifold at each point, except possibly at the vertices of $T_\Lambda$. We shall show that $M$ is a 3–manifold by showing that each vertex-linking surface in $T_\Lambda$ is a disk.

For $v_j$ a vertex of $T_\Lambda$, we have $v_j$ in $B_j$ and have denoted the boundary-linking normal surface along $B_j$ by $V_j$. We shall show that we can modify the boundary-linking normal surface $V_j$ to get the vertex-linking surface about $v_j$.

The normal surface $V_j$ is the frontier of a small regular neighborhood of $B_j$. Hence, we can catalog how $V_j$ meets the tetrahedra of $T_\Lambda$.

For $\Delta$ a tetrahedron of $T_\Lambda$, we have $V_j$ meets $\Delta$ in only one of the following:

- a subset of vertex-linking triangles and $\Delta$ meets $B_j$ only in those vertices where there is a vertex-linking triangle in $V_j$,
- an edge-linking quadrilateral about an edge $e$ of $\Delta$ and a subset of vertex-linking triangles about the vertices of the edge of $\Delta$ opposite $e$ and $\Delta$ only meets $B_j$ in the edge $e$ and those vertices where $V_j$ has a vertex-linking triangle,
- two parallel edge-linking quadrilaterals and $\Delta$ only meets $B_j$ in those edges about which $V_j$ has the edge-linking quadrilaterals, or
- a face-linking triangle and possibly a vertex-linking triangle at the opposite vertex and $\Delta$ only meets $B_j$ in the face at which we have the face-linking
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triangle and at the opposite vertex and then only if \( V_j \) has a vertex-linking triangle at that vertex.

We now show how to construct the vertex-linking surface at \( v_j \) from the normal surface \( V_j \). See Figure 28. First, remove each quadrilateral, \( Q \), in \( V_j \) and replace it with two vertex-linking disks, one at each end of the edge the quadrilateral \( Q \) was linking; see Figure 28(A). Then remove each face-linking triangle, \( \sigma \), in \( V_j \) and replace it with three vertex-linking triangles, one at each vertex of the face in \( B_j \) that the triangle \( \sigma \) in \( V_j \) was linking; see 28(B).

As observed above, the surfaces \( V_j^*, B_j, \) and \( V_j \) are all homeomorphic. So, if they have genus \( g \), then each has Euler characteristic, \( \chi = 2 - 2g \). Starting with \( V_j \), removing an edge-linking quadrilateral and adding two vertex-linking disks does not change the Euler characteristic, if we do not count the new vertices added to the edge in \( B_j \). Removing a face-linking triangle and replacing it with three vertex-linking triangles changes the Euler characteristic by \(-\frac{5}{2}\) if we again do not count the new vertices at the edges in \( B_j \). Note in this case, we have included in the count all edges, including those in the face in \( B_j \).

Thus the Euler characteristic of \( V_j \) is modified by \(-\frac{5}{2}\) for each face of \( B_j \) and by +2 for each edge in \( B_j \). Let \( D_j \) denote the surface we get from these modifications of \( V_j \). There are \( 4g - 2 \) faces in \( B_j \) and \( 6g - 3 \) edges. It follows that

\[
\chi(D_j) = 2 - 2g - 5(2g - 1) + 2(6g - 3) = 1
\]

and \( D_j \) is a disk.

It follows that \( M \), the underlying point set of \( T_\Lambda \) is a compact 3–manifold with boundary, \( T_\Lambda \) has normal boundary, and is a minimal vertex-triangulation.

If we crush the triangulation \( T_\Lambda \) along the boundary of \( M \), we get the ideal triangulation \( T^* \) and so the manifold \( M \) is homemorphic to \( X \). Thus we have that \( T_\Lambda \) is an inflation of the ideal triangulation \( T^* \).

We have the following summary to the construction of an inflation.

1. Any ideal triangulation \( T^* \) of the interior of a compact 3–manifold \( M \) with boundary, no component of which is a 2–sphere, admits an inflation to a minimal-vertex triangulation \( T \) of \( M \) and \( T \) has normal boundary. An inflation depends on the choice of frames in the vertex-linking surfaces of
\( T^* \); however, even for fixed frames, there are choices and a finite number of distinct inflations.

2. The ideal vertices “inflate” to one-vertex-triangulations of the boundary components of \( M \), which are the triangulations induced on \( \partial M \) by \( T \).

3. The vertex-linking surfaces in \( T^* \) inflate to boundary-linking normal surfaces in \( T \).

4. The tetrahedra of \( T^* \) become tetrahedra of \( T \) and some of the face identifications of \( T^* \) are taken as face identifications of \( T \).

5. The triangulation \( T \) combinatorially crushes to \( T^* \) along the boundary of \( M \). A combinatorial crushing is unique.

4.4. Inflations with complexity. We have established that any ideal triangulation of the interior of a compact 3–manifold \( X \) has an inflation to a minimal vertex triangulation of \( X \). We now show that we can precisely determine the number of tetrahedra in the inflation triangulation as a function of the number of tetrahedra in the given ideal triangulation and a complexity (number) assigned to the frame of the inflation.

Suppose \( T^* \) is an ideal triangulation of the interior of the compact 3–manifold \( X \) and \( \Lambda = \{ \xi_1, \ldots, \xi_V \} \) is a collection of frames in the vertex-linking surfaces of \( T^* \). Above, in the inflation of \( T^* \) for the collection of frames \( \Lambda \), we did not take particular notice of the number of tetrahedra added; however, without purposely adding extra tetrahedra at the various steps of the inflation, the only choices that effect the number of tetrahedra added are during an inflation at a face of \( T^* \), and then only if there is more than one edge of the frames in the face. At this step and in the case of two edges or three edges in the face, we can arbitrarily choose the order in which we add the tetrahedra associated to these edges. (Recall if we have two or three edges in a face, then the inflation in that face is equivalent to adding a pyramid or a prism, respectively, and our choices then are in choosing diagonals in the quadrilateral faces of these pyramids or prisms when we subdivide them into tetrahedra.) These choices are reflected in the configuration polygons and necessitate the addition of crossings in the configuration polygons.

So, suppose for \( \Lambda \) we consider all possible orders for inflations at the faces of \( T^* \) and then count the number of crossing configurations in the configuration polygons. The minimal such number will be called the crossing number for \( \Lambda \) and is denoted \( \times(\Lambda) \). We let \( b(\Lambda) \) denote the number of branches of \( \Lambda \) (the branches are the components left after removing all branch points of the various frames in \( \Lambda \)); and we let \( v_b(\Lambda) \) denote the number of branching points of \( \Lambda \). Finally, we let \( e(\Lambda) \) denote the number of edges in the frames in \( \Lambda \).

In the above inflation construction, the frames in \( \Lambda \) determine various branch configurations in the configuration polygons. If a branch configuration has branching index \( b \), then we add \( b - 2 \) tetrahedra for this branch configuration. It follows that if \( \xi_j \) is a frame in \( \Lambda \), \( b(\xi_j) \) is the number of branches of \( \xi_j \), and \( v_b(\xi_j) \) is the number of branching points, then the number of tetrahedra needed for the branching configurations coming from \( \xi_j \) is \( 2[b(\xi_j) - v_b(\xi_j)] \). These are additive functions over the various frames in \( \Lambda \); hence, the number of tetrahedra added in the inflation over all branching configurations for the frames in \( \Lambda \) is \( 2[b(\Lambda) - v_b(\Lambda)] \).

We define the complexity of \( \Lambda \) as

\[
\mathcal{C}(\Lambda) = e(\Lambda) + \times(\Lambda) + 2[b(\Lambda) - v_b(\Lambda)].
\]
Example A. $\xi_1 = \langle 1 \rangle \cup \langle 9, 3, 6, 4 \rangle$

Example B. $\xi_2 = \langle 1 \rangle \cup \langle 4, 6, 12, 11 \rangle$

Figure 29. Examples of two frames with different complexities in the vertex-linking torus of the two-tetrahedra ideal triangulation of the figure-eight knot complement in $S^3$. 
In particular, the complexity for a frame $\xi$ in the vertex-linking torus of a cusped 3–manifold with one cusp is $C(\xi) = e(\xi) + \times(\xi) + 2$.

4.4. **Theorem.** Suppose $X$ is a compact 3–manifold with boundary, no component of which is a 2–sphere, and $T^*$ is an ideal triangulation of the interior of $X$. If $\Lambda$ is a collection of frames in the vertex-linking surfaces of $T^*$, one from each vertex-linking surface of $T^*$. An inflation triangulation $T^*_\Lambda$ of $T^*$ has $\text{card}(T^*) + C(\Lambda)$ tetrahedra, where $\text{card}(T^*)$ is the number of tetrahedra in $T^*$ and $C(\Lambda)$ is the complexity of $\Lambda$.

We give some examples of frames and compute their complexities.

**Example C.**

$\xi_{V^*} = \langle 2 \rangle \cup \langle 6, 12 \rangle$

$\xi_{W^*} = \langle 15 \rangle \cup \langle 7, 9, 3, 1 \rangle$

$e(\Lambda) = 8$; $\times(\Lambda) = 2$;

$b(\Lambda) = 4$; $v(\Lambda) = 2$

$C(\Lambda) = 14$

\textbf{Figure 30.} Frames in the two vertex-linking tori for a four tetrahedra ideal triangulation of the complement of the Whitehead Link in $S^3$. The complexity of $\Lambda = \{\xi_{V^*}, \xi_{W^*}\}$ is greater than the sum of the complexities of $\xi_{V^*}$ and $\xi_{W^*}$. 
In the first two examples (Figure 29), we have the two-tetrahedron ideal triangulation of the figure-eight knot complement in the $3$–sphere. This ideal triangulation has one ideal vertex and the vertex-linking surface is a torus. There are two edges. In Example (A) the frame is given as $\xi_1 = (1) \cup (9, 3, 5, 7)$, where $\pi$ indicates the reverse direction to the arrow on the edge $e$, and has two branches; one is the standard meridian, labeled $(1)$, and the other is the standard homological longitude, labeled $(9, 3, 5, 7)$. This frame $\xi_1$ has complexity 8. Hence, the inflation of this two-tetrahedron ideal triangulation of the figure-eight knot complement, using the frame $\xi_1$, gives a one-vertex triangulation of the figure-eight knot exterior having 10 tetrahedra. In this case, we conjecture the inflation triangulation is also a minimal triangulation. In the next section, we construct this inflation. In Example (B) the frame is given as $\xi_2 = (1) \cup (4, 6, 12, 11)$ and again has two branches; one branch is the meridian but the other, while a longitude, is not the homological longitude. The complexity in this example is 9. Hence, we have two frames in the same vertex-linking surface having different complexities.

In the next example (Figure 45) in the Appendix, we have a four-tetrahedron ideal triangulation of the complement of the Whitehead link in the $3$–sphere. This ideal triangulation has two ideal vertices and the vertex-linking surface at each is a torus. There are three edges, $E$, $F$ and $G$. We denote the ideal vertices $v^*$ and $w^*$ and their vertex-linking tori $V^*$ and $W^*$, respectively. In $V^*$ the frame is $\xi_{V^*} = \langle 2 \rangle \cup (6, 12)$ and in $W^*$ the frame is $\xi_{W^*} = \langle 15 \rangle \cup (7, 9, 3, 1)$. We set $\Lambda = \{\xi_{V^*}, \xi_{W^*}\}$. The collection $\Lambda$ has four branches and two vertices; its complexity is 14. Note that the edges $E$ and $F$ meet different frames at their ends. Individually, the frame $\xi_{V^*}$ has complexity 5 and the frame $\xi_{W^*}$ has complexity 7; however, we cannot add these complexities to get the complexity of $\Lambda$. This example displays that we must consider the frames at each end of an edge in an inflation. If we should first inflate along the frame $\xi_{V^*}$, then we change the induced triangulation on the vertex-linking torus at $w^*$ and change the frame and its complexity there.

5. Examples of the Inflation Construction.

In this section we give two examples of the inflation construction. The first is an inflation of the two-tetrahedra ideal triangulation of the complement of the figure-eight knot in $S^3$ and the second is an inflation of the one-tetrahedron triangulation of the Gieseking manifold. As we have remarked elsewhere, we considered using other examples as these manifolds have been extensively studied and are repeatedly used as examples; however, we finally decided that familiarity with these examples may be useful in introducing the ideas of the inflation construction.

Example. Inflation of figure-eight knot complement.

Step 1. Given an ideal triangulation.

For this example, the given ideal triangulation $T^*$ is the two-tetrahedra ideal triangulation of the figure-eight knot complement in $S^3$ given in Figure 31.

We have two tetrahedra, $(p)$ and $(p')$ with face identifications:

\[
\begin{align*}
(p)(012) & \leftrightarrow (p')(012) & (p)(013) & \leftrightarrow (p')(312) \\
(p)(023) & \leftrightarrow (p')(310) & (p)(123) & \leftrightarrow (p')(320)
\end{align*}
\]
Step 2. Construct the vertex-linking surface and choose a frame.

The vertex-linking surface also is shown in Figure 31. We shall use as a frame $\xi = < 1 > \cup < 9, 3, 6, 4 >$, which is given as an example in Figure 7. It is the standard meridian/longitude pair for the figure-eight knot in $S^3$.

Step 3. Direct each branch, successively label edges in the branches, and determine the transverse direction for each branch.

There are two branches for this frame. One branch is the single edge 1, the meridian. In this example, we chose directions on the edges in the induced triangulation of the vertex-linking torus to aid the reader in the face identifications; we shall utilize these labels and directions. So, the branch (1) is given the same direction as the edge 1 and we have indicated a transverse direction for this branch in Figure 32.

The second branch and direction is $< 9, 3, 6, 4 >$, where $\bar{e}$ means the edge $e$ taken in the opposite direction to that used in the face identifications and given in Figure 31. This branch corresponds to the homological longitude slope in $S^3$. The transverse direction is given in Figure 32.

Step 4. Determine the configuration polygons; using the transverse directions, determine the directions on the boundary edges of the configuration polygons.

The configuration polygons are given in Example A in Figure 29; we give them here in Figure 33 with labels and transverse directions. Note we use $\bar{6}$ and $\bar{4}$ and give them the direction induced by that of the branch $(9, 3, 6, 4)$.

Step 5. Add a tetrahedron for each edge in the frame.
Figure 32. A one vertex, index 4 frame with two branches: \( \langle 1 \rangle \) and \( \langle 9, 3, 6, 4 \rangle \), along with transverse directions. This frame has complexity 8.

Figure 33. Labeled configuration polygons for the inflation of the figure eight knot complement.

For this example we have 5 edges, giving 5 tetrahedra: \((1)(0123), (9)(0123), (3)(0123), (5)(0123)\) and \((4)(0123)\).

**Step 6.** Inflation at the faces of \( T \).

Figure 34. Inflation at the face \((p)(012) \leftrightarrow (p')(012)\), which meets the frame in two edges 1 and 3.

- Inflation at the face \((p)(012) \leftrightarrow (p')(012)\): this face contains the two edges 1 and 3. See Figure 34. Notice that we have made a choice of order in adding the tetrahedra (1) and (3) (same as selecting a diagonal if we think of the inflation at the face as adding a prism). Our choice was made by looking at the configuration polygon in Figure 33 and seeing that we avoid a crossing if we make the selection by first attaching the tetrahedron \((p)\) to the tetrahedron (3); then attaching (3) to (1); and,
finally, ending by attaching (1) to \((p')\). This is discussed above where we discussed inflations with complexity.

- Inflation at the face \((p)(013) \leftrightarrow (p')(312)\); this face contains the two edges \(\overline{\text{5}}\) and \(\overline{\text{4}}\). See Figure 35. Again we have used the configuration polygon to make a selection for the order we add the tetrahedra \((\overline{\text{5}})\) and \((\overline{\text{4}})\) to avoid adding additional crossings. Here we avoid a crossing if we make the selection by first attaching the tetrahedron \((p)\) to the tetrahedron \((\overline{\text{4}})\); then attaching \((\overline{\text{4}})\) to \((\overline{\text{5}})\); and, finally, ending by attaching \((\overline{\text{5}})\) to \((p')\).

\[\begin{align*}
(p)(013) & \leftrightarrow (\overline{\text{4}})(132); \\
(\overline{\text{4}})(032) & \leftrightarrow (\overline{\text{6}})(213); \\
(\overline{\text{6}})(203) & \leftrightarrow (p')(312)
\end{align*}\]

**Figure 35.** Inflation at the face \((p)(013) \leftrightarrow (p')(312)\), which meets the frame in two edges \(\overline{\text{5}}\) and \(\overline{\text{4}}\).

- Inflation at the face \((p)(023) \leftrightarrow (p')(310)\); this face contains the single edge 9. See Figure 36.

\[\begin{align*}
(p)(023) & \leftrightarrow (9)(320); \\
(9)(321) & \leftrightarrow (p')(310)
\end{align*}\]

**Figure 36.** The face \((p)(023) \leftrightarrow (p')(310)\) meets the frame in one edge 9.

**Step 7.** Inflation at the edges of \(T\). There are two edges, \(E\) and \(F\). The inflation at an edge is determined by the configuration polygon at that edge. In Step 4, Figure 33 we give the configuration polygons for the edges \(E\) and \(F\).
- Inflation at the edge $E$. In this case the configuration polygon splits into two independent polygons; one is a generic polygon and the other is a branch polygon for a branch of index 4. See Figure 37.

\[
(\overline{6})(012) \leftrightarrow (3)(013) \\
(1)(012) \leftrightarrow (b^*)(120); \\
(9)(012) \leftrightarrow (b^*)(230) \\
(1)(013) \leftrightarrow (b^*)(b30); \\
(\overline{3})(013) \leftrightarrow (b^*)(1b0)
\]

Figure 37. Inflation at the edge $E$ where the configuration polygon splits into a generic bi-gon and a branch pyramid, which is subdivided into $(b^*_2), b = 3$ and $(b^*_1), b = 1$.

- Inflation at the edge $F$. In this case the configuration polygon is a crossing. See Figure 38.

\[
(\overline{5})(013) \leftrightarrow (c)(023) \\
(\overline{4})(012) \leftrightarrow (c)(021) \\
(3)(012) \leftrightarrow (c)(130) \\
(9)(013) \leftrightarrow (c)(132)
\]

Figure 38. Inflation at the edge $F$ where the configuration polygon is a crossing.

**Step 8.** Finish.
We have face identifications from $T^*$ which may remain. In this example, the face identification $(p)(123) \leftrightarrow (p')(320)$ is retained.

We have cones over branch configurations, which may need to be subdivided into tetrahedra. This is done by first triangulating the branch polygon, without adding vertices, and then using the cone structure to extend the triangulation of the polygon to a triangulation of the cone. For $b$-gons we have $(b-2)$ tetrahedra. While the number of tetrahedra for a $b$–gon will remain fixed at $(b-2)$, there are numerous options for subdividing the polygons (the number of options is the $(b-2)$ Catalan number). In this example, the branch polygon is a quadrilateral and we have two choices, depending on the choice of diagonal, for triangulating the pyramid $(b^*)$. For example, if we choose the diagonal from the vertex $b$ to the vertex 2; then we have a triangulation of the pyramid with two tetrahedra, $(b_1^*)(012b)$ and $(b_2^*)(023b)$, and a single face identification $(b_1^*)(02b) \leftrightarrow (b_2^*)(02b)$.

We collect the tetrahedra and face identifications into an array following the notational conventions of REGINA [2]. A tetrahedron (using the abbreviation “tet”) from our collection is given in the first column, its faces are given in the top row, and the face identifications are given at the intersection of a row with a column. For example, to know the face identification of the face (013) of the tetrahedron (9), go down the first column to (9) and then cross to the column under (013) where we find (c)(132); hence, the face identification is $(9)(013) \leftrightarrow (c)(132)$. To include the subdivision of the pyramid $(b^*)$, we arbitrarily choose to subdivide $(b^*)$ into two tetrahedra $(b_1^*)(012b)$ and $(b_2^*)(023b)$; hence, in the array, we indicate the vertex $b$ by 3 in $(b_1^*)$ and indicate the vertex $b$ by 1 in $(b_2^*)$.

Notice that the two faces $(b_1^*)(123)$ and $(b_2^*)(123)$ are not identified and become the boundary of the compact manifold, which is the exterior of the *figure-eight* knot in $S^3$. If one follows the identifications of the edges in the boundary of the quadrilateral, which is the base of the pyramid $(b^*)$, using our notational conventions, we have

$$
(b_1^*)(12) \rightarrow (1)(01) \rightarrow (b_2^*)(13)
$$

and

$$
(b_2^*)(23) \rightarrow (9)(01) \rightarrow (c)(13) \rightarrow (3)(01) \rightarrow (\overline{6})(01) \rightarrow (c)(02) \rightarrow (\overline{7})(01) \rightarrow (b_1^*)(13).
$$

The boundary is the one-vertex, two triangle triangulation of the torus. We see, clearly, here that the free edges added when inflating at a face along a single branch are all identified and form an edge in the boundary. This guided our choice of notation for this edge being $(01)$ along all branches. We also see the crossing where we have free edges, $(c)(13)$ and $(c)(02)$, as opposite edges of the tetrahedron $(c)$; in this example, they are on the same branch.

| tet  | (012) | (013) | (023) | (123) |
|------|-------|-------|-------|-------|
| $(p)$ | (3)(320) | (4)(132) | (9)(320) | (p')(320) |
| $(p')$ | (1)(132) | (9)(123) | (p)(321) | (6)(032) |
| (1) | $(b_1^*)(120)$ | $(b_2^*)(130)$ | (3)(312) | (p')(021) |
| (3) | $(c)(130)$ | $(6)(012)$ | (p)(210) | (1)(230) |
| (4) | $(c)(021)$ | $(b_1^*)(130)$ | (6)(231) | (p)(031) |
| (6) | (3)(013) | $(c)(023)$ | $(p')(132)$ | (4)(302) |
| (9) | $(b_2^*)(230)$ | $(c)(132)$ | (p)(320) | (p')(013) |
| (c) | $(\overline{4})(021)$ | (3)(201) | $(\overline{6})(013)$ | (9)(031) |
We conclude this example by remarking that we suspect this is a minimal triangulation of the figure-eight knot exterior in $S^3$. It has 10 tetrahedra. There are many other non-isomorphic 10-tetrahedra triangulations of the figure-eight knot exterior; some of the examples we know are not inflations of the two-tetrahedron ideal triangulation of the figure-eight knot complement but of a three-tetrahedra ideal triangulation of the figure-eight knot complement, which is formed by a $2 \leftrightarrow 3$ Pachner move on the two-tetrahedra ideal triangulation.

**Example. Inflation of the Gieseking manifold.**

**Step 1.** Given an ideal triangulation.

For this example, the given ideal triangulation $\mathcal{T}^*$ is the one-tetrahedra ideal triangulation of the Gieseking manifold given in Figure 39.

![Figure 39](image)

**Figure 39.** The one-tetrahedron ideal triangulation of the Gieseking manifold (non-orientable), along with the induced triangulation on its vertex-linking Klein bottle.

We have one tetrahedra, $(p)$, with face identifications:

$(p)(012) \leftrightarrow (p)(302)$

$(p)(013) \leftrightarrow (p)(123)$

**Step 2.** Construct the vertex-linking surface and choose a frame.

The vertex-linking surface also is shown in Figure 39; notice that it is a Klein bottle and the Gieseking manifold is non-orientable. We shall use as a frame $\xi = < 5 > \cup < 2 > \cup < 3 >$, which was given in the example in Figure 4.

**Step 3.** Direct each branch, successively label edges in the branches, and determine the transverse direction for each branch.

There are three branches for this frame. Each branch has just one edge. As above, we have directions on the edges in the induced triangulation of the vertex-linking Klein bottle, which were given to aid the reader in the face identifications; we utilize these labels and directions. All branches are given the direction of their
edges. However, here when finding the transverse directions, we see that the orientation reversing edges 3 and 5 change the transverse directions from what we would have in the orientable case. The transverse directions are given in Figure 40.

![Figure 40](image)

Figure 40. A double index three frame with three branches \( \langle 5 \rangle \), \( \langle 2 \rangle \), and \( \langle 3 \rangle \), along with their transverse directions. This frame has complexity 6.

**Step 4.** Determine the configuration polygons; using the transverse directions, determine the directions on the boundary edges of the configuration polygons.

There is only one edge and therefore only one configuration polygon; it is given in Figure 41 with labels and transverse directions.

![Figure 41](image)

Figure 41. Labeled configuration polygons for the inflation of the Gieseking manifold.

**Step 5.** Add a tetrahedron for each edge in the frame.

For this example we have three edges and therefore add three tetrahedra: \((3)(0123)\), \((2)(0123)\) and \((5)(0123)\).

**Step 6.** Inflation at the faces of \( T \).

- Inflation at the face \((p)(013) \leftrightarrow (p)(123)\); this face contains the single edge 5. See Figure 42.

- Inflation at the face \((p)(012) \leftrightarrow (p)(302)\); this face contains the two edges 2 and 3 of the frame \( \xi \). See Figure 43. In this example when we remove the identification \((p)(012) \leftrightarrow (p)(302)\), we have a choice of order in adding the tetrahedra (2) and (3); this is the same as adding a diagonal in the base of the pyramid we add to give a triangulation of the pyramid. We make the selection of first adding the tetrahedron (3) along the face \((p)(012)\) and then adding the tetrahedron (2).

**Step 7.** Inflation at the edges of \( T \). There is only one edge, \( E \). The inflation at an edge is determined by the configuration polygon at that edge. In Step 4, Figure 41 we give the configuration polygon and the transverse directions.
Figure 42. The face \((p)(013) \leftrightarrow (p)(123)\) meets the frame in one edge 5.

Figure 43. Inflation at the face \((p)(012) \leftrightarrow (p)(302)\), which meets the frame in two edges 2 and 3.

- Inflation at the edge \(E\). The configuration polygon decomposes into two branch configurations and a crossing. See Figure 44.

Step 8. Finish.

In this example, there are no face identification from \(T^*\) retained and there are no cones that need subdivided.

We collect the tetrahedra and face identifications into an array (again, following the notational conventions of Regina [2]).

Here the two faces \((b_1^*)(123)\) and \((b_2^*)(123)\) are not identified and become the boundary of the compact manifold given by the inflation triangulation. The identifications of the edges of the triangles \((b_1^*)(123)\) and \((b_2^*)(123)\) are determined by the face identifications and are:

\[
(b_1^*)(12) \rightarrow (3)(10) \rightarrow (c)(20) \rightarrow (b_1^*)(23);
\]
\[
(b_1^*)(13) \rightarrow (2)(10) \rightarrow (c)(31) \rightarrow (b_2^*)(13);
\]

and

\[
(b_2^*)(12) \rightarrow (5)(10) \rightarrow (b_2^*)(23).
\]
The inflations of the one-tetrahedron Gieseking manifold gives a 7-tetrahedron triangulation of a compact, non-orientable 3–manifold with a normal boundary and interior homeomorphic to the Gieseking manifold. The boundary is a Klein bottle.

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APPENDIX

In Figure 45, we give the complement of the Whitehead Link in S^3 as an ideal cell-decomposition with just one 3-cell, an octahedron. There are two ideal vertices, one labeled v^* and the other w^*; hence, two vertex-linking surfaces, each a torus and labeled V and W, respectively. The vertex-linking tori have induced cell-decompositions consisting of quadrilaterals. As is well known, an octahedron can be decomposed into a triangulation having four tetrahedra by choosing one of the three possible diagonals. We consider the triangulations from each of these choices. In (A) the diagonal is between the vertices labeled v^* in the figure; in (B) the diagonal is between the vertices labeled w^* in the figure and in (C) the diagonal is between the two unlabeled vertices in the figure, which are also identified with w^*.

In each case, we give the induced triangulation on the vertex-linking torus. The meridian slope on V is designated μ_V and on W it is designated μ_W. In all subdivisions, the meridian slope μ_V = ⟨2⟩ has length one; however, the meridian slope μ_W has length 1 in (A), length 2 in (B) and we can choose the meridian slope, μ_W, to be either length 1 or 2 in (C). The longitudinal slopes, λ_V = ⟨6, 2, 8, 12⟩ and λ_W = ⟨7, 9, 3, 1⟩, (considered as the longitude, independently, in each component of the link) are circuits in all the induced triangulations of the vertex-linking tori and in all cases each has length 4. The pair μ_W, λ_W forms a frame; however, the pair μ_V, λ_V does not form a frame. We can choose as a frame in the vertex-linking torus V the pair λ' = ⟨6, 12⟩ and μ_V = ⟨2⟩ (see Figure 30).
Example A3. Whitehead Link complement.

Figure 45. An ideal octagonal decomposition of the complement of the Whitehead Link in $S^3$. Shown are the vertex-linking tori at the vertices $A$ and $B$, depending on the choice of diagonal in the octagon which subdivide it into a four-tetrahedra ideal triangulation of the Whitehead Link complement.
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