Scattering threshold for radial defocusing-focusing mass-energy
double critical nonlinear Schrödinger equation in $d \geq 5$

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Abstract

We extend the scattering result given by Cheng et al. for the radial defocusing-focusing mass-energy double critical nonlinear Schrödinger equation in $d \leq 4$ to the whole range $d \geq 3$. The main ingredient is a suitable long time perturbation theory which is applicable for $d \geq 5$.

1 Introduction and main results

In this paper, we consider the defocusing-focusing mass-energy double critical nonlinear Schrödinger equation (DFDCNLS)

$$i\partial_t u + \Delta u - |u|^{2^* - 2}u + |u|^{2^*}u = 0 \quad \text{in } \mathbb{R} \times \mathbb{R}^d$$

(1.1)

with $d \geq 5$, $2^* = 2 + \frac{4}{d}$ and $2^* - 2 = 2 + \frac{4}{d}$. (1.1) is a special case of the NLS with combined nonlinearities

$$i\partial_t u + \Delta u + \mu_1|u|^{p_1 - 2}u + \mu_2|u|^{p_2 - 2}u = 0 \quad \text{in } \mathbb{R} \times \mathbb{R}^d$$

(1.2)

with $d \geq 1$, $\mu_1, \mu_2 \in \mathbb{R}$ and $p_1, p_2 \in (2, \infty)$. (1.2) is a prototype model in many applications of quantum physics such as nonlinear optics and Bose-Einstein condensation. For example, in the study of Bose-Einstein condensation, the nonlinearities $|u|^2u$, $|u|^3u$ and $|u|^4u$ model the two-body interaction, quantum fluctuation and three-body interaction respectively. The signs $\mu_i$ can be tuned to be defocusing ($\mu_i < 0$) or focusing ($\mu_i > 0$), indicating the repulsivity or attractivity of the nonlinearity. For a comprehensive introduction on the physical background of (1.2), we refer to [2, 11] and the references therein.

From a mathematical point of view, we are particularly interested in problems with critical nonlinearities due to the following aspects: On the one hand, the nonlinear estimates for non-critical problems can usually be derived from the critical ones by means of interpolation; on the other hand, by dealing with critical problems additional symmetry operator such as dilation or Galilean boosts will also come into play, which makes the problem more challenging and interesting. The above mentioned reasons hence motivate our study on the mass-energy double critical NLS, whose mixed type nature also prevents any potential applications concerning scaling invariance property. At this point, we also refer the readers to the representative papers [12, 1, 6, 8, 5, 3, 10] for scattering results of (1.2), in which at least one of the nonlinearities has critical growth.

We restrict our attention to the radial DFDCNLS (1.1), which was studied by Cheng, Miao, Zhao [6] in the case $d \leq 4$. The precise statement is as follows:

**Theorem 1.1** ([6]). Let $d \in \{3, 4\}$. Define

$$\mathcal{H}(u) := \frac{1}{2}||\nabla u||_2^2 + \frac{1}{2^*}||u||_{2^*}^2 - \frac{1}{2^*}||u||_{2^*}^{2^*},$$

$$\mathcal{K}(u) := ||\nabla u||_2^2 + \frac{d}{d + 2}||u||_{2^*}^{2^*} - ||u||_{2^*}^{2^*},$$

and

$$\mathcal{A} := \{u \in H^{1}_{rad}(\mathbb{R}^d) : \mathcal{H}(u) < d^{-1}S_{d}^{\frac{d}{d-2}}, \mathcal{K}(u) \geq 0\},$$

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where $S$ is the optimal constant of the Sobolev inequality, i.e.

$$S := \inf_{D_1^1(\mathbb{R}^d) \setminus \{0\}} \frac{\|\nabla u\|_2^2}{\|u\|_2^2}.$$ 

Then the unique solution $u$ of (1.1) with $u(0) \in A$ is global and scatters in time.

The main obstacle that prevents Theorem 1.1 to hold in $d \geq 5$ is the absence of a suitable long time perturbation theory. More precisely, since the gradient of the nonlinearity $|u|^4 u$ is merely Hölder continuous for $d \geq 5$, the proof of the long time perturbation result for $d \leq 4$ is no longer valid. By appealing to fractional calculus we show that such a long time perturbation result indeed continues to hold also in the case $d \geq 5$.

**Theorem 1.2.** Let $d \geq 5$ and let $u \in C(I; H^1(\mathbb{R}^d))$ be a solution of (1.1) defined on some interval $I \ni t_0$. Assume also that $w$ is an approximate solution of the following perturbed NLS

$$i\partial_t w + \Delta w = |w|^4 w - |w|^4 \Delta w + e$$

such that

$$\|u\|_{L^\infty_t H^1_x(I)} \leq B_1, \quad \|u(t_0) - w(t_0)\|_{H^1} \leq B_2, \quad \|w\|_{W^{2,\infty}_x W^{2,\infty}_x(I)} \leq B_3$$

for some $B_1, B_2, B_3 > 0$. Then there exists some positive $\alpha = \alpha(B_1, B_2, B_3) \ll 1$ with the following property: if

$$\|e^{i(t-t_0)\Delta} (u(t_0) - w(t_0))\|_{W^{2,\infty}_x(I)} \leq \beta, \quad (1.7)$$

then

$$\|\langle \nabla \rangle e^{i(t-t_0)\Delta} (u(t_0) - w(t_0))\|_{L^2_{t,x}(I)} \leq \beta, \quad (1.8)$$

for some $0 < \beta < \alpha$, then

$$\|\langle \nabla \rangle u\|_{S(I)} \lesssim B_1, B_2, B_3 1.$$  

**Remark 1.3.** By interpolation, (1.7) and (1.8) can be replaced by the stronger condition

$$\|\langle \nabla \rangle e^{i(t-t_0)\Delta} (u(t_0) - w(t_0))\|_{W^{2,\infty}_x(I)} \leq \beta, \quad (1.11)$$

which is the smallness condition given in the long time perturbation theory [6, Prop. 3.2].

For the precise definition of the function spaces defined in Theorem 1.2 we refer to Section 1.1 below for details. The main challenge for proving Theorem 1.2 lies in the fact that both nonlinearities of (1.1) are endpoint critical nonlinearities and there is no chance to estimate one by another using interpolation. This will force us to directly derive suitable estimates for both of the nonlinearities using fractional calculus.

As a direct consequence, we immediately deduce the following generalization of Theorem 1.1. The proof is a straightforward modification of the arguments from [6], thus we omit the details.

**Theorem 1.4.** Theorem 1.1 continues to hold for all $d \geq 5$.

The rest of the paper is organized as follows: In Section 1.1 we introduce the notation and definitions which will be used throughout the paper. In Section 2 we give the proof of Theorem 1.2.
1.1 Notations and definitions

We use the notation $A \lesssim B$ whenever there exists some positive constant $C$ such that $A \leq CB$. Similarly we define $A \gtrsim B$ and we use $A \sim B$ when $A \lesssim B \lesssim A$. For an interval $I \subset \mathbb{R}$, the space $L^q_t L^r_x(I)$ is defined by

$$L^q_t L^r_x(I) := \{ u : I \times \mathbb{R}^2 \to \mathbb{C} : \|u\|_{L^q_t L^r_x(I)} < \infty \},$$

where

$$\|u\|_{L^q_t L^r_x(I)}^q := \int_I \|u\|^q_t \, dt.$$

When $q = r$, we simply write $L^q_{t,x} := L^q_t L^r_x$. A pair $(q,r)$ is said to be $\dot{H}^s$-admissible with $s \in [0,1]$ if $q,r \in [2,\infty]$ and $\frac{2}{q} + \frac{d}{r} = \frac{d}{2} - s$. When $s = 0$, we simply say the pair $(q,r)$ is $L^2$-admissible. For any $L^2$-admissible pairs $(q_1,r_1)$ and $(q_2,r_2)$ we have the following Strichartz estimate: if $u$ is a solution of

$$i\partial_t u + \Delta u = F(u)$$

in $I \subset \mathbb{R}$ with $t_0 \in I$ and $u(t_0) = u_0$, then

$$\|u\|_{L^q_t L^r_x(I)} \lesssim \|u_0\|_2 + \|F(u)\|_{L^q_t L^r_x(I)},$$

where $(q_2',r_2')$ is the Hölder conjugate of $(q_2,r_2)$. For a proof, we refer to [7, 4].

For $s \in \mathbb{R}$, the multipliers $|\nabla|^s$ and $\langle \nabla \rangle^s$ are defined by the symbols

$$|\nabla|^s f(x) = F^{-1}(|\xi|^s \hat{f}(\xi))(x),$$

$$\langle \nabla \rangle^s f(x) = F^{-1}\left((1 + |\xi|^2)^{s/2} \hat{f}(\xi)\right)(x).$$

The following function spaces will be used throughout the paper:

$$W_{2s} := L_{t,x}^{\frac{4d+2}{d-2}}, \quad W_{2s}^* := L_{t,x}^{\frac{4d+2}{d+2}},$$

$$V_{2s} := L_{t,x}^{\frac{2d+4}{d-2}}, \quad V_{2s}^* := L_{t,x}^{\frac{2d+4}{d+2}},$$

$$S := L_{t,x}^{\infty} \cap L_{t,x}^{2s},$$

$$X := L_{t,x}^{\frac{d(2-d)}{d-2}} \cap L_{t,x}^{2d+6d+16},$$

$$Y := L_{t,x}^{\frac{d}{d-2}} \cap L_{t,x}^{2d+4d^2+4d-16},$$

$$Z := L_{t,x}^{\frac{d(2-d)}{d-2}} \cap L_{t,x}^{2d^2+2d^2-8d+16}.$$

One easily verifies using Hölder and Sobolev that

$$\|\langle \nabla \rangle^s |u|^\frac{2}{r} u\|_{L_{t,x}^{\frac{4d+2}{d-2}}(I)} \lesssim \|\langle \nabla \rangle^s u\|_{W_{2s}(I)} \|u\|_{W_{2s}^*(I)}^\frac{2}{3},$$

(1.12)

$$\|\langle \nabla \rangle^s |u|^\frac{2}{r} u\|_{L_{t,x}^{\frac{4d+2}{d-2}}(I)} \lesssim \|\langle \nabla \rangle^s u\|_{W_{2s}(I)} \|u\|_{W_{2s}^*(I)}^\frac{2}{3},$$

(1.13)

for $s \in \{0,1\}$, and

$$\|u\|_{W_{2s}(I)} \lesssim \|\nabla u\|_{V_{2s}(I)} \lesssim \|\nabla u\|_{L_{t,x}^{2s}}^{\frac{2}{r}} \|u\|_{L_{t,x}^{2s}}^{1-\frac{2}{r}}.$$  

(1.14)

We also record here the following useful elementary inequalities: By fundamental calculus we have the following elementary inequality

$$\left| \sum_{j=1}^k z_j \right|^\alpha - \sum_{j=1}^k |z_j|^\alpha \lesssim \sum_{1 \leq i,j \leq k, i \neq j} |z_i|^\alpha |z_j|^{\alpha-1} |z_j|.$$  

(1.15)
for \( z \in \mathbb{C}^k \) and \( \alpha \in (1, \infty) \); For function \( H(z) = |z|^{\frac{1}{k}} z \) we have
\[
|(H(u) - H(v))| \lesssim (|u|^{\frac{1}{k}} + |v|^{\frac{1}{k}})|u - v|
\] (1.16)
for \( d \geq 3 \),
\[
\left| \nabla \left( H(u) - H(v) \right) \right| \lesssim |u|^{\frac{1}{k}}|\nabla u - \nabla v| + |\nabla v|(|u|^{\frac{d-1}{k}} + |v|^{\frac{d-1}{k}})|u - v|
\] (1.17)
for \( d = 3 \), and
\[
\left| \nabla \left( H(u + v) - H(u) - H(v) \right) \right| \lesssim |u|^{\frac{1}{k}}|\nabla v| + |v|^{\frac{1}{k}}|\nabla u|
\] (1.18)
for \( d \geq 4 \).

## 2 Perturbation theory

In this section we prove Theorem 1.2. The proof relies on a modification of the arguments involving fractional calculus given in [9]. To proceed, we first record some auxiliary tools (Lemma 2.1 to Lemma 2.6). For details of their proofs, we refer to [9] and the references therein. We will also restrict the space dimension \( d \) to be larger than four in this section.

**Lemma 2.1.** Let \( s \in (0, 1] \) and \( q, q_1, q_2, q_3, q_4 \in (1, \infty) \) with
\[
\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q_3} + \frac{1}{q_4}.
\]
Then
\[
\|\nabla^\alpha u\|_q \lesssim \|u\|_{q_1} \|\nabla^\alpha v\|_{q_2} + \|\nabla^\alpha u\|_{q_3} \|v\|_{q_4}.
\] (2.1)

**Lemma 2.2.** Let \( G : \mathbb{C} \to \mathbb{C} \) be a \( C^1 \)-function and let \( s \in (0, 1] \). Then for all \( 1 < p, p_1, p_2 < \infty \) with
\[
\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2},
\]
we have
\[
\|\nabla^\alpha G(u)\|_p \lesssim \|G'(u)\|_{p_1} \|\nabla^\alpha u\|_{p_2}.
\] (2.2)

**Lemma 2.3.** Let \( h : \mathbb{C} \to \mathbb{C} \) be a Hölder continuous function of order \( \alpha \in (0, 1) \). Then for any \( s \in (0, \alpha) \), \( q \in (1, \infty) \) and \( \sigma \in (\frac{\alpha}{\alpha}, 1) \) we have
\[
\|\nabla^\sigma h(u)\|_q \lesssim \|h\|_{(\alpha - \frac{\sigma}{\alpha})q_1} \|\nabla^\sigma u\|_{q_2},
\] (2.3)
provided that \( \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} \) and \( (\alpha - \frac{\sigma}{\alpha})q_1 > \alpha \).

**Lemma 2.4.** For any interval \( I \) we have
\[
\|\nabla^\frac{1}{\alpha} u\|_{X(I)} \lesssim \|u\|_{W^\frac{1}{\alpha^2}H^\frac{1}{\alpha^2}(I)} \|\nabla u\|_{S^\frac{1}{\alpha^2}(I)} \lesssim \|\nabla u\|_{S(I)},
\] (2.4)
\[
\|u\|_{W^\frac{1}{\alpha}H^\frac{1}{\alpha}(I)} \lesssim \|\nabla^\frac{1}{\alpha} u\|_{X(I)} \|\nabla u\|_{S(I)}^{-c},
\] (2.5)
for some \( c = c(d) \in (0, 1) \).

**Lemma 2.5.** For any interval \( I \) we have
\[
\left\| \int_{I_0}^t |\nabla|^{\frac{1}{\alpha^2}} e^{(t-s)i}\Delta f(s) \, ds \right\|_{X(I)} \lesssim \|\nabla|^{\frac{1}{\alpha}} f\|_{Y(I)}.
\] (2.6)

**Lemma 2.6.** For any interval \( I \) we have
\[
\|\nabla|^{\frac{1}{\alpha}} (|u|^{\frac{1}{\alpha^2}} u)\|_{Y(I)} \lesssim \|\nabla|^{\frac{1}{\alpha}} u\|_{X(I)}
\] (2.7)
and
\[
\|\nabla|^{\frac{1}{\alpha}} (u + w)\|_{Y(I)} \lesssim \left( \|\nabla|^{\frac{1}{\alpha}} u\|_{X(I)} \|\nabla u\|_{S(I)} + \|\nabla|^{\frac{1}{\alpha}} w\|_{X(I)} \|\nabla w\|_{S(I)} \right) \|\nabla|^{\frac{1}{\alpha}} v\|_{X(I)}.
\] (2.8)
Next, we prove an exotic dual Strichartz estimate for the mass-critical term:

**Lemma 2.7.** For any interval $I$ we have

$$\|\nabla |^{\frac{4}{d+2}}\left(|u|^{\frac{2}{d}}u\right)\|_{Y(I)} \lesssim \|u\|_{S(I)}^{\frac{4(d+1)\left(\frac{d+1}{d}\right)}{4+2d}} \|\nabla |^{\frac{4}{d+2}} u\|_{X(I)}$$

(2.9)

and

$$\|\nabla |^{\frac{4}{d+2}}\left(|u + w|^{\frac{2}{d}}v\right)\|_{Y(I)} \lesssim \left(\|u\|_{S(I)}^{\frac{4(d+1)\left(\frac{d+1}{d}\right)}{4+2d}} + \|w\|_{S(I)}^{\frac{4(d+1)\left(\frac{d+1}{d}\right)}{4+2d}}\right) \|\nabla |^{\frac{4}{d+2}} v\|_{X(I)}$$

(2.10)

with $c = c(d) \in (0,1)$.

**Proof.** To simplify notations we omit the symbol $I$ in the following proof. First notice that using interpolation, for any $L^2$-admissible pair $(q, r)$ with $q \in (2, 2')$ we have

$$\|u\|_{L^q_t L^r_x} \lesssim \|u\|_{S(I)}^{1-c} \|u\|_{S(I)}^c$$

(2.11)

for some $c = c(q, r) \in (0,1)$. For different $L^2$-admissible pairs $(q_1, r_1)$ and $(q_2, r_2)$ with $c_1 = c(q_1, r_1) > c(q_2, r_2) = c_2$, we also have

$$\|u\|_{S(I)}^{1-c_1} \|u\|_{S(I)}^{c_1} \|u\|_{S(I)}^{c_2} \lesssim \|u\|_{S(I)}^{1-c_2} \|u\|_{S(I)}^{c_2}.$$  

(2.12)

We will thus refer to a unified (and possibly small) $c \in (0,1)$ by the application of (2.11) for different $L^2$-admissible pairs. Using Lemma 2.3, we infer that

$$\|\nabla |^{\frac{4}{d+2}}\left(|u|^{\frac{2}{d}}u\right)\|_Y \lesssim \|u\|_{L^q_t L^r_x}^{\frac{4(d+1)\left(\frac{d+1}{d}\right)}{4+2d}} \|\nabla |^{\frac{4}{d+2}} u\|_X$$

$$= \|u\|_{L^q_t L^r_x}^{\frac{4(d+1)\left(\frac{d+1}{d}\right)}{4+2d}} \|\nabla |^{\frac{4}{d+2}} u\|_X$$

$$\lesssim \|u\|_{S(I)}^{\frac{4(d+1)\left(\frac{d+1}{d}\right)}{4+2d}} \|u\|_{S(I)}^{c} \|\nabla |^{\frac{4}{d+2}} u\|_X,$$

(2.13)

which gives (2.9). By Lemma 2.1 and (2.11) we have

$$\|\nabla |^{\frac{4}{d+2}}\left(|u + w|^{\frac{2}{d}}v\right)\|_{L^q_t L^r_x} \lesssim \|u + w\|_{L^q_t L^r_x}^{\frac{2d(d+1)\left(\frac{d+1}{d}\right)}{4+2d}} \|\nabla |^{\frac{4}{d+2}} v\|_X$$

$$\lesssim \left(\|u\|_{S(I)}^{\frac{4(d+1)\left(\frac{d+1}{d}\right)}{4+2d}} + \|w\|_{S(I)}^{\frac{4(d+1)\left(\frac{d+1}{d}\right)}{4+2d}}\right) \|\nabla |^{\frac{4}{d+2}} v\|_X$$

(2.14)
It is left to estimate the second product in (2.14). Using Hölder and Sobolev we obtain that
\[
\|v\|_{L^1_{d+1+\alpha}(\mathbb{R}^d)} \leq \|v\|_{L^1_{d+1+\alpha}(\mathbb{R}^d)} \leq \|v\|_{L^1_{d+1+\alpha}(\mathbb{R}^d)} \leq \|v\|_{L^1_{d+1+\alpha}(\mathbb{R}^d)}.
\]

On the other hand, by Lemma 2.3 we know that
\[
\|\nabla \|^{d+1}_{d+1+\alpha}(u + w)\|_{L^1_{d+1+\alpha}(\mathbb{R}^d)} \leq \|\nabla \|^{d+1}_{d+1+\alpha}(u + w)\|_{L^1_{d+1+\alpha}(\mathbb{R}^d)} \leq \|\nabla \|^{d+1}_{d+1+\alpha}(u + w)\|_{L^1_{d+1+\alpha}(\mathbb{R}^d)}.
\]

where we set
\[
\sigma = \frac{d + 1}{d + 2}, \quad q_1 = \frac{d^2(d + 1)(d + 2)}{2(d^2 + 2d - 10)}, \quad q_2 = \frac{d(d + 1)(d + 2)}{2(d^2 + 2d - 10)}
\]

therein. Using Hölder, Sobolev and interpolation we finally conclude that
\[
\|\nabla \|^{d+1}_{d+1+\alpha}(u + w)\|_{L^1_{d+1+\alpha}(\mathbb{R}^d)} \leq \|\nabla \|^{d+1}_{d+1+\alpha}(u + w)\|_{L^1_{d+1+\alpha}(\mathbb{R}^d)} \leq \left(\|\nabla \|^{d+1}_{d+1+\alpha}(u + w)\|_{L^1_{d+1+\alpha}(\mathbb{R}^d)} + \|\nabla \|^{d+1}_{d+1+\alpha}(u + w)\|_{L^1_{d+1+\alpha}(\mathbb{R}^d)}\right)
\]

for some \(c \in (0, 1)\). (2.14), (2.15) and (2.17) then imply (2.10).

Next, we formulate a small data well-posedness result for (1.1), which is slightly different from the standard one and is suitable for the proof of Lemma 2.9 given below.

**Lemma 2.8.** For any \(A > 0\) there exists some \(\beta > 0\) such that the following is true: Suppose that \(t_0 \in I\) for some interval \(I\). Suppose also that \(u_0 = u(t_0) \in H^1(\mathbb{R}^d)\) with
\[
\|u_0\|_{H^1} \leq A
\]

and
\[
\|e^{it(t-\tau_0)}u\|_{W_{2,2}(I)} + \|\nabla \|^{d+1}_{d+1+\alpha}e^{it(t-\tau_0)}u\|_{X(I)} \leq \beta.
\]

Then (1.1) has a unique solution \(u \in C(I; H^1(\mathbb{R}^d))\) such that
\[
\|u\|_{S(I)} \leq \|u_0\|_{H^1},
\]

\[
\|u\|_{W_{2,2}(I)} \leq 2\|e^{it(t-\tau_0)}u\|_{W_{2,2}(I)},
\]

\[
|||\nabla \|^{d+1}_{d+1+\alpha}u\|_{X(I)} \leq 2\|||\nabla \|^{d+1}_{d+1+\alpha}e^{it(t-\tau_0)}u\|_{X(I)}
\]

**Proof.** We define the space \(B(I)\) by
\[
B(I) := \left\{ u \in L^\infty_t H^1_x(I) : \|\nabla \|^{d+1}_{d+1+\alpha}\|_{L^d_x(I)} \leq 2C\|u_0\|_{H^1}, \right\}
\]

\[
\|u\|_{W_{2,2}(I)} \leq 2\|e^{it\Delta}u_0\|_{W_{2,2}(I)},
\]

\[
\|\nabla \|^{d+1}_{d+1+\alpha}u\|_{X(I)} \leq 2\|||\nabla \|^{d+1}_{d+1+\alpha}e^{it\Delta}u_0\|_{X(I)}
\]

(2.18)
One easily checks that the set $B(I)$ equipped with the metric $\rho$ defined by

$$\rho(u, v) := \|u - v\|_{S(I)}$$

is a complete metric space. Now define the operator $\Phi$ by

$$\Phi(u) := e^{i(t-t_0)\Delta}u_0 - i\int_{t_0}^t e^{i(t-s)\Delta}(\|\hat{u}\|^2 u - |u|^2 u)\,ds.$$  

(2.19)

We show that $\Phi$ is a contraction on $B(I)$. Using (2.25) we infer that there exists some $c \in (0, 1)$ such that

$$\|u\|_{W^{2,1}(I)} \leq (2CA)^{1-c}\|\nabla|^{\frac{4}{7}} u\|_{X(I)} \leq (2CA)^{1-c}2^{\beta}c^c.$$  

(2.20)

Combining with Strichartz, (1.12) and (1.13) we deduce that

$$\sum \leq \|\nabla\Phi\|_{S(I)} + C\sum_{s=1}^2 \left(\|\nabla|^{\frac{4}{7}} u\|_{\frac{5}{2}L^{\infty}_{t,x}(I)}\right)$$

$$\leq \|\nabla\Phi\|_{S(I)} + C\|\nabla\Phi\|_{W^{2,1}(I)}|u|_{\frac{5}{2}L^{\infty}_{t,x}(I)} + C\|\nabla u\|_{S(I)}|u|_{\frac{5}{2}L^{\infty}_{t,x}(I)}$$

$$\leq \|\nabla\Phi\|_{S(I)} + C\|\nabla u\|_{S(I)}|u|_{\frac{5}{2}L^{\infty}_{t,x}(I)} + C\|\nabla u\|_{S(I)}|u|_{\frac{5}{2}L^{\infty}_{t,x}(I)}$$

$$\leq \sum \leq C\|u\|_{H^1} + C((2\beta)\frac{2}{5} + ((2CA)^{1-c}2^{\beta}c)^{\frac{4}{7}})\|u_0\|_{H^1}.$$  

(2.21)

Analogously, we have

$$\|\Phi(u)\|_{W^{2,1}(I)} \leq \|\nabla u_0\|_{W^{2,1}(I)} + C\|\nabla u\|_{L^\infty_{t,x}} + C\|\nabla\Phi\|_{W^{2,1}(I)}$$

$$\leq \|\nabla u_0\|_{W^{2,1}(I)} + C\|\nabla (2\beta)\frac{2}{5} + ((2CA)^{1-c}2^{\beta}c)^{\frac{4}{7}}\|\nabla u_0\|_{W^{2,1}(I)}.$$

(2.22)

Using (2.7) and (2.23) we see that

$$\|\nabla\Phi\|_{X(I)} \leq \|\nabla e^{i\Delta u_0}\|_{X(I)} + C\|\nabla \phi\|_{Y(I)} + C\|\nabla e^{i\Delta u_0}\|_{X(I)}$$

$$\|\phi\|_{X(I)} \leq \|\nabla\Phi\|_{X(I)} + C\|\nabla\Phi\|_{Y(I)} + C\|\nabla\Phi\|_{X(I)}.$$  

(2.23)

Since $Z$ corresponds to an $L^2$-admissible pair, we know that there exists some $\kappa \in (0, 1)$ such that

$$\|u\|_{Z(I)} \leq \|u\|_{S(I)}^{1-\kappa}.$$  

(2.24)

Summing up we have

$$\|\nabla\Phi\|_{X(I)} \leq \|\nabla e^{i\Delta u_0}\|_{X(I)} + C(2\beta)^{\frac{2}{5}} + C(2CA)^{\kappa(1-c)}(2\beta)^{\frac{4}{3}}\|\nabla e^{i\Delta u_0}\|_{X(I)}.$$  

(2.25)
Hence by choosing \( \beta \) sufficiently small we see that \( \Phi \) maps \( B(I) \) into itself. In a similar way, using (1.1) followed by Strichartz, Hölder and (2.20) we obtain that

\[
\| \Phi(u) - \Phi(v) \|_{S(I)} \\
\leq C(\| u \|_{W^2_x(I)} + \| v \|_{W^2_x(I)} + \| u \|_{W^2_x(I)} + \| v \|_{W^2_x(I)}) \| u - v \|_{W^2_x(I)} \\
\leq C\left(\| u \|_{W^2_x(I)} + \| v \|_{W^2_x(I)} \right) \\
+ (2CA)^{\frac{4(1-\gamma)}{2}} \| \nabla^\gamma u \|_{X(I)} + (2CA)^{\frac{4(1-\gamma)}{2}} \| \nabla^\gamma v \|_{X(I)} \| u - v \|_{S(I)} \\
\leq 2C(2\beta)^{\frac{4}{5} + (2CA)^{\frac{4(1-\gamma)}{2}} (2\beta)^{\frac{4}{5}}) \| u - v \|_{S(I)}.
\]

(2.26)

Thus choosing even smaller \( \beta \) if necessary we infer that \( \Phi \) is a contraction on \( B(I) \). Now the existence and uniqueness of a solution \( u \) of (1.1) are ensured by the Banach fixed point theorem. The continuity of \( u \) follows immediately from the fact that \( u \) satisfies the integral equation. \( \square \)

**Lemma 2.9** (Short time perturbation). Let \( u \in C(I; H^1(\mathbb{R}^d)) \) be a solution of (1.1) defined on some interval \( I \ni t_0 \). Assume also that \( w \) is an approximate solution of the following perturbed NLS

\[
i\dot{\theta}_t w + \Delta w = |w|^{\frac{4}{d-2}} w - |w|^\frac{2}{d-2} w + \epsilon
\]

(2.27)

such that

\[
\| w \|_{L^\infty_t H^1(I)} \leq B_1, \quad \| u(t_0) - w(t_0) \|_{H^1} \leq B_2
\]

(2.28)

for some \( B_1, B_2 > 0 \). Then there exist some positive \( \beta_0, \beta_1 \ll 1 \), depending on \( B_1 \) and \( B_2 \), with the following property: if

\[
\| \nabla^\gamma u \|_{X(I)} + \| \nabla w \|_{W^2_x(I)} \leq \beta_0
\]

(2.29)

and

\[
\| e^{i(t-t_0)\Delta} (u(t_0) - w(t_0)) \|_{W^2_x(I)} \leq \beta,
\]

(2.30)

\[
\| \nabla^\gamma e^{i(t-t_0)\Delta} (u(t_0) - w(t_0)) \|_{X(I)} \leq \beta,
\]

(2.31)

\[
\| \nabla^\gamma \|_{L^\infty_t L^{\frac{2(d+2)}{d-2}}(I)} \leq \beta
\]

(2.32)

for some \( 0 < \beta \leq \beta_1 \), then

\[
\| u - w \|_{Z^{t} W^2_x(I)} \lesssim \kappa,
\]

(2.33)

\[
\| \nabla^\gamma (u - w) \|_{X(I)} \lesssim \kappa,
\]

(2.34)

\[
\| F(u) - F(w) \|_{L^{\frac{2(d+2)}{d-2}}(I)} \lesssim \kappa,
\]

(2.35)

\[
\| \nabla^\gamma (F(u) - F(w)) \|_{Y(I)} \lesssim \kappa,
\]

(2.36)

\[
\| \nabla \|_{S(I)} + \| \nabla w \|_{S(I)} \lesssim B_1 + B_2
\]

(2.37)

for some \( \kappa \in (0, 1) \). Here \( F(z) = -|z|^{\frac{4}{d-2}} z + |z|^{\frac{2}{d-2}} z \).

**Proof.** First denote by \( \kappa_1 \) the number such that

\[
\| u \|_Z \lesssim \| u \|_{S}^{1-\kappa_1} \| u \|_{W^2_x}^{\kappa_1}.
\]

(2.38)

By definition of \( u \) and \( w \) we have

\[
u(t) = e^{i(t-t_0)\Delta} u(t_0) + \int_{t_0}^{t} e^{i(t-s)\Delta} F(u) \, ds,
\]

(2.39)

\[
w(t) = e^{i(t-t_0)\Delta} w(t_0) + \int_{t_0}^{t} e^{i(t-s)\Delta} (F(w) - \epsilon) \, ds.
\]

(2.40)
Using Strichartz, (1.12), (1.13), (2.29) and (1.14) we obtain that
\[ \|\langle \nabla \rangle w\|_{S(I)} \leq \|w(t_0)\|_{H^\varepsilon_x} + \left(\|w\|_{W^{2,1}_x(I)} + \|\langle \nabla \rangle w\|_{S(I)}\right)\|\langle \nabla \rangle e\|_{L^{2(d+2)}_{t,x}} \]
\[ \leq B_1 + \beta + \left(\frac{\beta}{\beta_0} + B_1^{\frac{1}{d+2} - 1} \right)\|\langle \nabla \rangle w\|_{S(I)}. \]
(2.41)

By choosing \( \beta_0 \) sufficiently small we infer that
\[ \|\langle \nabla \rangle w\|_{S(I)} \lesssim B_1. \]
(2.42)

Now (2.38) and (2.20) also yield
\[ \|\langle \nabla \rangle w\|_{Z(I)} \lesssim \beta_0^{\varepsilon_1}. \]
(2.43)

Using Strichartz, (2.29), (1.12), (1.13), (2.42), (1.14) and (2.32) we infer that
\[ \|e^{i(t-t_0)\Delta} u_0\|_{W^{2,1}_x(I)} \lesssim \|w\|_{W^{2,1}_x(I)} + \|F(w)\|_{L^{2(d+2)}_{t,x}} + \|\langle \nabla \rangle e\|_{L^{2(d+2)}_{t,x}} \]
\[ \leq \beta_0 + \frac{\beta}{\beta_0} + \beta \lesssim \beta_0^{\varepsilon_2}. \]
(2.44)

Similarly, (2.7), (2.9), (2.42), (2.29) and (2.32) yield
\[ \|\|\nabla\|^{\frac{1}{d+2}} e^{i(t-t_0)\Delta} u_0\|_{X(I)} \| \lesssim \|\nabla\|^{\frac{1}{d+2}} w\|_{X(I)} + \|\|\nabla\|^{\frac{1}{d+2}} F(w)\|_{Y(I)} + \|\langle \nabla \rangle e\|_{L^{2(d+2)}_{t,x}} \]
\[ \lesssim \beta_0 + \frac{\beta}{\beta_0} + \beta \lesssim \beta_0. \]
(2.45)

Combining with (2.41) and the triangular inequality we deduce that
\[ \|e^{i(t-t_0)\Delta} u(t_0)\|_{W^{2,1}_x(I)} \lesssim \beta_0^{\varepsilon_2}, \]
\[ \|\|\nabla\|^{\frac{1}{d+2}} e^{i(t-t_0)\Delta} u(t_0)\|_{X(I)} \lesssim \beta_0. \]
(2.46)

(2.47)

Hence, by choosing \( \beta_0 \) sufficiently small, we know from Lemma (2.8) that
\[ \|\langle \nabla \rangle u\|_{S(I)} \lesssim \|u(t_0)\|_{H^\varepsilon_x} \lesssim B_1 + B_2, \]
(2.48)
\[ \|u\|_{W^{2,1}_x(I)} \leq 2\|e^{i(t-t_0)\Delta} u_0\|_{W^{2,1}_x(I)} \lesssim \beta_0^{\varepsilon_2}, \]
(2.49)
\[ \|\|\nabla\|^{\frac{1}{d+2}} u\|_{X(I)} \leq 2\|\|\nabla\|^{\frac{1}{d+2}} e^{i(t-t_0)\Delta} u_0\|_{X(I)} \lesssim \beta_0. \]
(2.50)

Combining with (2.41) we already have (2.37). Using Strichartz, (2.46), (2.48), (1.12), (1.13), (2.30) and (2.5) we obtain
\[ \|\langle \nabla \rangle u\|_{W^{2,1}_x(I)} \lesssim \|e^{i(t-t_0)\Delta} u(t_0)\|_{W^{2,1}_x(I)} + \|F(u)\|_{L^{2(d+2)}_{t,x}} \]
\[ \lesssim \beta_0^{\varepsilon_2} + \|\langle \nabla \rangle u\|_{W^{2,1}_x(I)}^{1+\frac{1}{d+2}} \lesssim \beta_0^{\varepsilon_2} + \|\langle \nabla \rangle u\|_{W^{2,1}_x(I)}. \]
(2.51)

By standard continuity arguments we conclude that
\[ \|\langle \nabla \rangle u\|_{W^{2,1}_x(I)} \lesssim \beta_0^{\varepsilon_2}. \]
(2.52)

Similarly, from Strichartz, (2.47), (2.48), (2.7), (2.9), (2.38) and (2.32) we infer that
\[ \|\|\nabla\|^{\frac{1}{d+2}} u\|_{X(I)} \lesssim \|\|\nabla\|^{\frac{1}{d+2}} e^{i(t-t_0)\Delta} u(t_0)\|_{X(I)} + \|\|\nabla\|^{\frac{1}{d+2}} F(u)\|_{Y(I)} \]
\[ \lesssim \beta_0 + \|\|\nabla\|^{\frac{1}{d+2}} u\|_{X(I)}^{1+\frac{1}{d+2}} \lesssim \beta_0 + \|\|\nabla\|^{\frac{1}{d+2}} u\|_{X(I)}. \]
(2.53)
and using standard continuity arguments we see that
\[ \| |\nabla|^{\frac{1}{1+\epsilon}} u \|_{X(I)} \lesssim \beta_0. \] (2.54)

Next, we define \( v := u - w \). Then
\[ v(t) = e^{i(t-t_0)\Delta}(u(t_0) - w(t_0)) + \int_{t_0}^t e^{i(t-s)\Delta}(F(v + w) - F(w) + e) \, ds \] (2.55)
and
\[ \| (\nabla) v \|_{S(I)} \leq \| (\nabla) u \|_{S(I)} + \| (\nabla) w \|_{S(I)} \lesssim B_1 + B_2. \] (2.56)

From Strichartz, Sobolev, (2.38), (2.48), (2.42) and (2.30) we know that
\[ \| e^{i(t-t_0)\Delta}(u(t_0) - w(t_0)) \|_{Z(I)} \lesssim \beta_0^\epsilon. \] (2.57)

By Strichartz, Sobolev, (2.30), (2.24), (1.33), (1.16), (2.57), (2.29), (1.14), (2.5) and (2.42) we have
\[ \begin{align*}
&\| v \|_{Z_{\gamma}W_{2_\gamma}(I)} \\
&\lesssim \| e^{i(t-t_0)(u(t_0) - w(t_0))} \|_{Z_{\gamma}W_{2_\gamma}(I)} + \| e \|_{\frac{2(4+\epsilon)}{L_{k_\epsilon}}} \\
&+ \| (F(v + w) - F(w)) \|_{\frac{2(4+\epsilon)}{L_{k_\epsilon}}} \\
&\lesssim \beta_0^\epsilon + (\| v \|_{W_2(I)} + \| w \|_{W_2(I)}) \|
u\|_{W_2(I)} \\
&\lesssim \beta_0^\epsilon + \| v \|_{Z_{\gamma}W_{2_\gamma}(I)} + \| (\nabla)\frac{1}{2} \|_X v \|_{\frac{1}{2} X(I)} \|
u\|_{Z_{\gamma}W_{2_\gamma}(I)} + \| \beta_0^\epsilon \|_{Z_{\gamma}W_{2_\gamma}(I)}. \end{align*} \] (2.58)

Now using Strichartz, Sobolev, (2.43), (2.29), (2.31), (2.56), (2.6), (2.8), (2.10) and the identity
\[ F(v + w) - F(w) = v \int_0^1 F_z(v + (1 + \theta)w) \, d\theta + \bar{v} \int_0^1 F_z(v + (1 + \theta)w) \, d\theta \]
we obtain that
\[ \begin{align*}
&\| |\nabla|^{\frac{1}{1+\epsilon}} e^{i(t-t_0)(u(t_0) - w(t_0))} \|_{X(I)} + \| (\nabla) e \|_{\frac{2(4+\epsilon)}{L_{k_\epsilon}}} \\
&+ \| |\nabla|^{\frac{1}{1+\epsilon}} (F(v + w) - F(w)) \|_{Y(I)} \\
&\lesssim \beta + \left( \| |\nabla|^{\frac{1}{1+\epsilon}} e^{i(t-t_0)} \|_{X(I)} \| \nabla\|_{\frac{1}{2} X(I)} \|\nabla\|_{\frac{1}{2} X(I)} \right) \| |\nabla|^{\frac{1}{1+\epsilon}} v \|_{X(I)} \\
&+ \left( \| |\nabla|^{\frac{4(1-\epsilon)}{S(I)}} \| e \|_{\frac{4}{Z(I)}} + \| e \|_{\frac{4(1-\epsilon)}{S(I)}} \| e \|_{\frac{4}{Z(I)}} \right) \| |\nabla|^{\frac{1}{1+\epsilon}} v \|_{X(I)} \\
&+ \left( \| (\nabla v) \|_{\frac{1}{2} S(I)} + \| (\nabla v) \|_{\frac{1}{2} S(I)} \right) \| e \|_{\frac{4(1+\epsilon)}{S(I)}} \| e \|_{\frac{4}{Z(I)}} + \| e \|_{\frac{4(1+\epsilon)}{S(I)}} \| e \|_{\frac{4}{Z(I)}} \right) \\
&\times \| |\nabla|^{\frac{1}{2} X(I)} \| (\nabla|^{\frac{1}{1+\epsilon}} v \|_{X(I)}) \\
&\lesssim \beta + \| |\nabla|^{\frac{1}{1+\epsilon}} v \|_{X(I)} + \| |\nabla|^{\frac{1}{1+\epsilon}} v \|_{X(I)} + \| |\nabla|^{\frac{1}{1+\epsilon}} v \|_{X(I)} + \| |\nabla|^{\frac{1}{1+\epsilon}} v \|_{X(I)} \\
&+ \beta^\epsilon_0 \| |\nabla|^{\frac{1}{1+\epsilon}} v \|_{X(I)} + \| |\nabla|^{\frac{1}{1+\epsilon}} v \|_{X(I)} + \| |\nabla|^{\frac{1}{1+\epsilon}} v \|_{X(I)} \\
&+ \beta^\epsilon_0 \| |\nabla|^{\frac{1}{1+\epsilon}} v \|_{X(I)} + \| |\nabla|^{\frac{1}{1+\epsilon}} v \|_{X(I)} \right) \| |\nabla|^{\frac{1}{1+\epsilon}} v \|_{X(I)} \\
&\lesssim \beta_0 + \| |\nabla|^{\frac{1}{1+\epsilon}} v \|_{X(I)} + \| |\nabla|^{\frac{1}{1+\epsilon}} v \|_{X(I)} + \| |\nabla|^{\frac{1}{1+\epsilon}} v \|_{X(I)} + \| |\nabla|^{\frac{1}{1+\epsilon}} v \|_{X(I)} \] (2.59)

Define
\[ \| v \|_{W(I)} := \| v \|_{Z_\gamma W_{2\gamma}(I)} + \| |\nabla|^{\frac{1}{1+\epsilon}} v \|_{X(I)}. \] (2.60)
We can choose $\delta \beta$ with $\alpha$ and (2.59). Adding (2.59) to (2.58) and absorbing the terms with powers of $\beta_0$ on the r.h.s. to the l.h.s., we obtain
\[
\|v\|_{W(I)} \lesssim \beta^{\kappa_1} + \|v\|_{W(I)}^{1+\frac{\phi}{2}} + \|v\|_{W(I)}^{1+\frac{\theta}{2}} + \|v\|_{W(I)}^{1+\frac{\theta}{3}}.
\] (2.61)

By standard continuity arguments we infer that
\[
\|v\|_{W(I)} \lesssim \beta^{\kappa_1}
\] (2.62)
and we conclude (2.63) and (2.64). Thus we may iterate the previous step over all
\[
0 \leq j \leq \kappa_2 + \frac{1}{\kappa_1}
\]
which yields already from the calculation given in (2.58) and (2.59). This completes the proof.

Having all the preliminaries, we are at the position to prove Theorem 1.2.

Proof of Theorem 1.2. We first show that
\[
\|\langle \nabla \rangle w\|_{S(I)} \leq C
\] (2.63)
for some $C = C(B_1, B_2, B_3) > 0$. By (1.6) we may subdivide $I$ into $J_1 = J_1(B_3)$ subintervals $K_j = [s_j, s_{j+1}], j = 0, \cdots, J_1 - 1$, such that
\[
\|w\|_{W_{2,0}^2(K_j)} \leq \delta_0
\] (2.64)
for some small $\delta_0$ to be chosen later. On $K_0$, using Strichartz, Hölder and (1.9) we obtain that
\[
\|\langle \nabla \rangle w\|_{S(K_0)} \lesssim \|w(0)\|_{H^1} + \|\langle \nabla \rangle e\|_{L^{2+\frac{2}{\beta}}_{t,x}(K_0)} + \|\langle \nabla \rangle F(w)\|_{L^{2+\frac{2}{\beta}}_{t,x}(K_0)}
\lesssim B_1 + B_2 + \beta + \left(\|w\|_{W_{2,0}^2(K_0)} + \|w\|_{W_{2,0}^2(K_0)}\right)\|\langle \nabla \rangle w\|_{S(K_0)}
\lesssim B_1 + B_2 + \beta + \left(\delta_0 + \delta_0^{\frac{2}{\beta}}\right)\|\langle \nabla \rangle w\|_{S(K_0)}.
\] (2.65)

We can choose $\delta_0$ sufficiently small to absorb the term $(\delta_0^{\frac{2}{\beta}} + \delta_0^{\frac{1}{\beta}})$ to the r.h.s. of (2.65). This yields
\[
\|\langle \nabla \rangle w\|_{S(K_0)} \lesssim B_1 + B_2.
\]

In particular,
\[
\|w(0)\|_{H^1} \lesssim B_1 + B_2.
\]

Notice also that $\delta_0$ is only dependent on $B_3$. Thus we may iterate the previous step over all $j$ to infer that
\[
\|\langle \nabla \rangle w\|_{S(K_j)} \lesssim B_1 + B_2
\]
for all $j$. Summing all the estimates on $K_j$ over $j$ up yields (2.63). Using (2.4) and (2.63) we are able to divide $I$ into $J_2 = J_2(B_1, B_2, B_3)$ intervals $L_j = [l_j, l_{j+1}], j = 0, \cdots, J_2 - 1$, such that
\[
\|\langle \nabla \rangle\frac{1}{\sqrt{1+\beta_0}}w\|_{X(L_j)} + \|\langle \nabla \rangle w\|_{W_{2,0}^2(L_j)} \leq \beta_0,
\] (2.66)
with $\beta_0 = \beta_0(C(B_1, B_2, B_3), C(B_1, B_2, B_3) + 1)$ defined by Lemma 2.9. By (1.7) and (1.8) we have
\[
\|e^{(t-t_0)\Delta} (u(t_0) - w(t_0))\|_{W_{2,0}^2(L_0)} \leq \beta,
\] (2.67)
\[
\|\langle \nabla \rangle \frac{1}{\sqrt{1+\beta_0}} e^{(t-t_0)\Delta} (u(t_0) - w(t_0))\|_{X(L_0)} \leq \beta
\] (2.68)
by setting initially $\alpha = \beta_1$ with $\beta_1 = \beta_1(B_1, B_2, B_3)$ from Lemma 2.4. Thus Lemma 2.9 is applicable for $L_0$. In particular, we have for all $j = 1, \cdots, J_2 - 1$ and $\beta_0 \in (0, \alpha)$
\[
\|u - w\|_{L^{\infty}_{t,x}(L_j)} \leq C_0 \beta^{\kappa},
\|\langle \nabla \rangle\frac{1}{\sqrt{1+\beta_0}}(u - w)\|_{X(L_j)} \leq C_0 \beta^{\kappa},
\|F(u) - F(w)\|_{L^{2+\frac{2}{\beta}}_{t,x}(L_j)} \leq C_0 \beta^{\kappa},
\|\langle \nabla \rangle\frac{1}{\sqrt{1+\beta_0}}(F(u) - F(w))\|_{Y(L_j)} \leq C_0 \beta^{\kappa},
\|\langle \nabla \rangle u\|_{S(L_j)} \leq C_0 C(B_1, B_2, B_3),
\|\langle \nabla \rangle F(u)\|_{S(L_j)} \leq C_0 C(B_1, B_2, B_3),
\]...
with $\kappa \in (0, 1)$ and $C_0 = C_0(B_1, B_2, B_3) > 0$, provided that

$$\|e^{i(t-t_j)}\Delta (u(t_j) - w(t_j))\|_{W^{s,2}(L_j)} \leq \beta, \quad (2.69)$$

$$\|\nabla |\nabla|^{\frac{1}{2}} e^{i(t-t_j)}\Delta (u(t_j) - w(t_j))\|_{X(L_j)} \leq \beta \quad (2.70)$$

hold for all $j = 1, \cdots, J_2 - 1$. We prove this using inductive arguments. One checks that

$$\|e^{i(t-t_j)}\Delta (u(t_j) - w(t_j))\|_{W^{s,2}(L_j)}$$

$$\lesssim \|e^{i(t-t_0)}\Delta (u(t_0) - w(t_0))\|_{W^{s,2}(L_j)} + \|e\|_{L_{t,x}^{2s+2}} [t_0, t_j]$$

$$+ \|F(u) - F(w)\|_{L_{t,x}^{2s+2}} [t_0, t_j] \lesssim \beta + \beta + C_0 J \beta^\kappa, \quad (2.71)$$

Choosing $\alpha$ iteratively small completes the proof. $\blacksquare$

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References

[1] Akahori, T., Ibrahim, S., Kikuchi, H., and Nawa, H. Existence of a ground state and scattering for a nonlinear Schrödinger equation with critical growth. Selecta Math. (N.S.) 19, 2 (2013), 545–609.

[2] Barashenkov, I. V., Goecheva, A. D., Makhankov, V. G., and Puzyin, I. V. Stability of the soliton-like “bubbles”. Phys. D 34, 1-2 (1989), 240–254.

[3] Carles, R., and Sparber, C. Orbital stability vs. scattering in the cubic-quintic Schrödinger equation. Rev. Math. Phys. 33, 3 (2021), 2150004, 27.

[4] Cazenave, T. Semilinear Schrödinger equations, vol. 10 of Courant Lecture Notes in Mathematics. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2003.

[5] Cheng, X. Scattering for the mass super-critical perturbations of the mass critical nonlinear Schrödinger equations. Illinois J. Math. 64, 1 (2020), 21–48.

[6] Cheng, X., Miao, C., and Zhao, L. Global well-posedness and scattering for nonlinear Schrödinger equations with combined nonlinearities in the radial case. J. Differential Equations 261, 6 (2016), 2881–2934.

[7] Keel, M., and Tao, T. Endpoint Strichartz estimates. Amer. J. Math. 120, 5 (1998), 955–980.

[8] Killip, R., Oh, T., Pocovnicu, O., and Vişan, M. Solitons and scattering for the cubic-quintic nonlinear Schrödinger equation on $\mathbb{R}^3$. Arch. Ration. Mech. Anal. 225, 1 (2017), 469–548.

[9] Killip, R., and Vişan, M. Nonlinear Schrödinger equations at critical regularity. In Evolution equations, vol. 17 of Clay Math. Proc. Amer. Math. Soc., Providence, RI, 2013, pp. 325–437.
[10] Luo, Y. Sharp scattering threshold for the cubic-quintic NLS in the focusing-focusing regime, 2021, 2105.15091.

[11] Pelinovsky, D. E., Afanasjev, V. V., and Kivshar, Y. S. Nonlinear theory of oscillating, decaying, and collapsing solitons in the generalized nonlinear Schrödinger equation. *Phys. Rev. E* 53 (Feb 1996), 1940–1953.

[12] Tao, T., Visan, M., and Zhang, X. The nonlinear Schrödinger equation with combined power-type nonlinearities. *Comm. Partial Differential Equations* 32, 7-9 (2007), 1281–1343.