Differential subordination for certain generalized operator

M. H. Al-Abbadi and M. Darus
DIFFERENTIAL SUBORDINATION FOR A CERTAIN
GENERALIZED OPERATOR

M. H. AL-ABBADI AND M. DARUS

Received 16 February, 2011

Abstract. The authors have recently introduced a new generalized derivative operator \( \mu_{\lambda_1, \lambda_2}^{n,m} \),
that generalized many well-known operators. The trend of finding new differential or integral
operators has attracted widespread interest. The aim of this paper is to use the relation
\[
(1 + n) \mu_{\lambda_1, \lambda_2}^{n+1,m} f(z) = \left( \mu_{\lambda_1, \lambda_2}^{n,m} f(z) \right) + n \left( \mu_{\lambda_1, \lambda_2}^{n,m} f(z) \right)
\]
to discuss some interesting results by using the technique of differential subordination. The
results include both subordination and inclusion. In the case of \( n = 0, \lambda_2 = 0 \), we obtain the
results of Oros [11].

2000 Mathematics Subject Classification: 30C45

Keywords: analytic function, Hadamard product (or convolution), univalent function; convex
function, derivative operator, differential subordination, dominant, best dominant

1. INTRODUCTION AND DEFINITIONS

Let \( A \) denote the class of functions of the form
\[
f(z) = z + \sum_{k=2}^{\infty} a_k z^k,
\]
which are analytic in the open unit disk \( U = \{ z \in \mathbb{C} : |z| < 1 \} \) on the complex plane \( \mathbb{C} \). Let \( S, S^*(\alpha), C(\alpha) (0 \leq \alpha < 1) \) denote the subclasses of \( A \) consisting of functions
that are univalent, starlike of order \( \alpha \) and convex of order \( \alpha \) in \( U \), respectively. In
particular, the classes \( S^*(0) = S^* \) and \( C(0) = C \) are the familiar classes of starlike
and convex functions in \( U \), respectively. And a function \( f \in C(\alpha) \) if \( \text{Re}(1 + \frac{z}{f''}) > \alpha \). Furthermore a function \( f \) analytic in \( U \) is said to be convex if it is univalent and
\( f(U) \) is convex.

Let \( H(U) \) be the class of holomorphic functions in unit disk
\( U = \{ z \in \mathbb{C} : |z| < 1 \} \). Consider
\[
A_n = \{ f \in H(U) : f(z) = z + a_{n+1} z^{n+1} + \ldots, \quad (z \in U) \}, \quad \text{with } A_1 = A.
\]
For $a \in \mathbb{C}$ and $n \in \mathbb{N} = \{1, 2, 3, \ldots, \}$ we let

$\mathcal{H}[a, n] = \{ f \in \mathcal{H}(U) : f(z) = z + a_n z^n + a_{n+1} z^{n+1} + \ldots, \ (z \in U) \}.$

Given two functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$ analytic in the unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$, the Hadamard product (or convolution) $f \ast g$ is defined by

$$f(z) \ast g(z) = (f \ast g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k.$$ 

Next, we state the basic ideas on subordination. If $f$ and $g$ are analytic in $U$, then the function $f$ is said to be subordinate to $g$, written as

$$f < g \quad \text{or} \quad f(z) < g(z) \quad (z \in U),$$

if and only if there exists the Schwarz function $w$, analytic in $U$, with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z)) \ (z \in U).$

Furthermore if $g$ is univalent in $U$, then $f < g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U).$ [see [14], p.36].

Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and let $h$ be univalent in $U$. If $p$ is analytic in $U$ and satisfies the (second-order) differential subordination

$$\psi(p(z),zp'(z),z^2p''(z);z) < h(z), \quad (z \in U),$$

then $p$ is called a solution of the differential subordination.

The univalent function $q$ is called a dominant of the solutions of the differential subordination, or simply a dominant, if $p < q$ for all $p$ satisfying (1.2).

A dominant $\tilde{q}$ that satisfies $\tilde{q} < q$ for all dominants $q$ of (1.2) is said to be the best dominant of (1.2). (Note that the best dominant is unique up to a rotation of $U$).

Now, $(x)_k$ denotes the Pochhammer symbol (or the shifted factorial) defined by

$$(x)_k = \begin{cases} 1 & \text{for } k = 0, x \in \mathbb{C} \setminus \{0\}, \\ x(x+1)(x+2)\ldots(x+k-1) & \text{for } k \in \mathbb{N} = \{1, 2, 3, \ldots\} \text{ and } x \in \mathbb{C}. \end{cases}$$

In [1], the authors introduced and studied the generalized derivative operator $\mu_{\lambda_1, \lambda_2}^{n,m} f(z)$ given by the following definition.

**Definition 1.** For $f \in \mathcal{A}$ the generalized derivative operator $\mu_{\lambda_1, \lambda_2}^{n,m}$ is defined by

$$\mu_{\lambda_1, \lambda_2}^{n,m} f(z) = z + \sum_{k=2}^{\infty} \frac{(1 + \lambda_1 (k-1))^m}{(1 + \lambda_2 (k-1))^{m-1}} c(n,k) a_k z^k, \quad (z \in U),$$

where $n, m \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}, \lambda_2 \geq \lambda_1 \geq 0$ and $c(n,k) = \binom{n+k-1}{n} = \frac{(n+1)_{k-1}}{(1)_{k-1}}.$
Special cases of this operator include the Ruscheweyh derivative operator in two cases when \( \mu_{0,0}^{n,1} \equiv R^n \) and \( \mu_{n,0}^{n,0} \equiv R^n \) [16], the Salagean derivative operator for \( \mu_{0,1}^{0,m} \equiv S^n \) [17], the generalized Ruscheweyh derivative operator in the cases \( \mu_{1,1}^{n,0} \equiv R^n \) and \( \mu_{n,0}^{n,0} \equiv R^n \) [3], the generalized Salagean derivative operator introduced by Al-Oboudi \( \mu_{0,0}^{0,m} \equiv S^n \) [2], and the generalized Al-Shaqsi and Darus derivative operator \( \mu_{n,0}^{n,m} \equiv D^n_{\lambda,\beta} \) [5]. Now, let us recall the well known Carlson-Shaffer operator \( L(a, c) \) [4] associated to the incomplete beta function \( \beta(a, c) \), defined by

\[
L(a, c) : \mathcal{A} \to \mathcal{A},
\]

\[
L(a, c) f(z) := \phi(a, c; z) * f(z) \quad (z \in U),
\]

where \( \phi(a, c; z) = z + \sum_{k=2}^{\infty} \frac{(a-1)k-1}{(c-1)k-1} z^k \).

It can be easily seen that

\[
\mu_{0,1}^{0,0} f(z) = \mu_{0,0}^{1,0} f(z) = f(z)
\]

and

\[
\mu_{1,0}^{1,0} f(z) = \mu_{0,0}^{1,1} f(z) = z f'(z).
\]

Also \( \mu_{1,0}^{a-1,0} f(z) = \mu_{0,0}^{a-1,1} f(z) \) where \( a = 1, 2, 3, \ldots \).

To prove our results, we need the following equality:

\[
(1 + n) \mu_{1,2}^{n+1,m} f(z) = z \left( \mu_{1,2}^{n,m} f(z) \right) + n \left( \mu_{1,2}^{n,m} f(z) \right), \quad (z \in U)
\]

where \( n, m \in \mathbb{N}_0 = \{0, 1, 2, \ldots\} \) and \( \lambda_2 \geq 1 \geq \lambda_1 \geq 0 \).

In addition, we need the following lemmas to prove our main results:

**Lemma 1** ([9], p.71). Let \( h \) be analytic, univalent, convex in \( U \), with \( h(0) = a, \gamma \neq 0 \) and \( \Re \gamma \geq 0 \). If \( p \in \mathcal{H}[a,n] \) and

\[
p(z) + \frac{zp'(z)}{\gamma} < h(z), \quad (z \in U),
\]

then

\[
p(z) < q(z) < h(z), \quad (z \in U),
\]

where

\[
q(z) = \frac{\gamma}{nz\pi} \int_0^z h(t) t^{(\frac{n}{2})-1} dt, \quad (z \in U).
\]

The function \( q \) is convex and is the best \((a,n)\)-dominant.

**Lemma 2** ([8]). Let \( g \) be a convex function in \( U \) and let

\[
h(z) = g(z) + n\alpha z g'(z),
\]

where \( \alpha > 0 \) and \( n \) is a positive integer.

If

\[
p(z) = g(0) + pnz^n + pn+1z^{n+1} + \ldots, \quad (z \in U),
\]
is holomorphic in $U$ and
\[ p(z) + \alpha z p'(z) < h(z), \quad (z \in U), \]
then
\[ p(z) < g(z) \]
and this result is sharp.

**Lemma 3 ([10]).** Let $f \in \mathcal{A}$, if
\[ \text{Re} \left( 1 + \frac{zf'''(z)}{f''(z)} \right) > -\frac{1}{2}, \]
then
\[ \frac{2z}{\pi} \int_{0}^{z} f(t) \, dt, \quad (z \in U \text{ and } z \neq 0), \]
is a convex function.

In the present paper, we shall use the method of differential subordination to derive certain properties of the generalized derivative operator $\mu_{\lambda_1, \lambda_2}^{n, m} f(z)$. Note that, differential subordination has been studied by various authors, and we follow the similar work of Oros [12] and Oros and Oros [13].

2. **Main Results**

Before we state our first theorem, we give another definition.

**Definition 2.** For $n, m \in \mathbb{N}_0$, $\lambda_2 \geq \lambda_1 \geq 0$ and $0 \leq \alpha < 1$, we let $R_{\lambda_1, \lambda_2}^{n, m} (\alpha)$ denote the class of functions $f \in \mathcal{A}$ which satisfy the condition
\[ \text{Re} \left( \mu_{\lambda_1, \lambda_2}^{n, m} f(z) \right)' > \alpha, \quad (z \in U). \quad (2.1) \]

It is clear that the class $R_{\lambda_1, 0}^{0, 1} (\alpha) \equiv R(\lambda_1, \alpha)$ consists of functions $f \in \mathcal{A}$ satisfying
\[ \text{Re}(\lambda_1 zf''(z) + f'(z)) > \alpha, \quad (z \in U), \]
studied by Ponnusamy [15] and many others.

Now we begin with our first result.

**Theorem 1.** Let
\[ h(z) = \frac{1 + (2\alpha - 1)z}{1 + z}, \quad (z \in U), \]
be convex in $U$, with $h(0) = 1$ and $0 \leq \alpha < 1$. If $n, m \in \mathbb{N}_0$, $\lambda_2 \geq \lambda_1 \geq 0$, and the differential subordination
\[ (\mu_{\lambda_1, \lambda_2}^{n+1, m} f(z))' < h(z), \quad (z \in U), \quad (2.2) \]
\[ \left( \mu_{\lambda_1, \lambda_2}^{n,m} f(z) \right)' < q(z) = 2\alpha - 1 + \frac{2(n + 1)(1 - \alpha)\sigma(n)}{z^{n+1}}, \]

where \( \sigma \) is given by

\[ \sigma(x) = \int_0^x \frac{1}{1 + t} \, dt, \quad (z \in U). \quad (2.3) \]

The function \( q \) is convex and is the best dominant.

**Proof.** By differentiating (1.3), with respect to \( z \), we obtain

\[ \left( \mu_{\lambda_1, \lambda_2}^{n+1,m} f(z) \right)' = \frac{(1 + n)\left( \mu_{\lambda_1, \lambda_2}^{n,m} f(z) \right)' + z\left( \mu_{\lambda_1, \lambda_2}^{n,m} f(z) \right)''}{1 + n}. \quad (2.4) \]

Using (2.4) in (2.2), differential subordination (2.2) becomes

\[ \frac{(1 + n)\left( \mu_{\lambda_1, \lambda_2}^{n,m} f(z) \right)' + z\left( \mu_{\lambda_1, \lambda_2}^{n,m} f(z) \right)''}{1 + n} < h(z) \]

\[ = 1 + (2\alpha - 1)z \]

\[ = 1 + z. \quad (2.5) \]

Let

\[ p(z) = \left[ \mu_{\lambda_1, \lambda_2}^{n,m} f(z) \right]' = \left[ z + \sum_{k=2}^{\infty} \frac{(1 + \lambda_1(k-1))^{m}}{(1 + \lambda_2(k-1))^{m-1}} c(n,k) a_k z^k \right]' \]

\[ = 1 + p_1 z + p_2 z^2 + \ldots, \quad (p \in \mathcal{K}[1, 1], \, z \in U). \quad (2.6) \]

Using (2.6) in (2.5), the differential subordination becomes:

\[ p(z) + \frac{zp'(z)}{1 + n} < h(z) = \frac{1 + (2\alpha - 1)z}{1 + z}. \]
By using Lemma 1, we have
\[
p(z) < q(z) = \frac{(n+1) \int_0^z h(t)t^n dt}{z^{n+1}}.
\]
\[
\begin{align*}
&= \frac{(n+1) \int_0^z \left( \frac{1+(2\alpha-1)t}{1+t} \right) t^n dt}{z^{n+1}}, \\
&= \frac{(n+1)}{z^{n+1}} \left[ \sigma(n) + (2\alpha-1) \int_0^z \frac{t^{n+1}}{1+t} dt \right], \\
&= 2\alpha - 1 + \frac{2(n+1)(1-\alpha)\sigma(n)}{z^{n+1}},
\end{align*}
\]
where \(\sigma\) is given by (2.3), so we get
\[
\left[ \mu_{\lambda_1,\lambda_2}^{n,m} f(z) \right]' < \frac{2(n+1)(1-\alpha)\sigma(n)}{z^{n+1}}.
\]
The functions \(q\) is convex and is the best dominant. The proof is complete. \(\square\)

**Theorem 2.** If \(n,m \in \mathbb{N}_0, \lambda_2 \geq \lambda_1 \geq 0\) and \(0 \leq \alpha < 1\), then we have
\[
R_{\lambda_1,\lambda_2}^{n+1,m} (\alpha) \subset R_{\lambda_1}^{n,m} (\delta)
\]
where
\[
\delta = 2\alpha - 1 + 2(n+1)(1-\alpha)\sigma(n),
\]
where \(\sigma\) is given by (2.3).

**Proof.** Let \(f \in R_{\lambda_1,\lambda_2}^{n+1,m} (\alpha)\), then from (2.1) we have
\[
\text{Re}(\mu_{\lambda_1,\lambda_2}^{n+1,m} f(z))' > \alpha, \quad (z \in U),
\]
which is equivalent to
\[
(\mu_{\lambda_1,\lambda_2}^{n+1,m} f(z))' < h(z) = \frac{1 + (2\alpha-1)z}{1+z}.
\]
Using Theorem 1, we have
\[
\left[ \mu_{\lambda_1,\lambda_2}^{n,m} f(z) \right]' < \frac{2(n+1)(1-\alpha)\sigma(n)}{z^{n+1}}.
\]
Since \(q\) is convex and \(q(U)\) is symmetric with respect to the real axis, we deduce
\[
\text{Re}(\mu_{\lambda_1,\lambda_2}^{n,m} f(z))' > \text{Re} q(1) = \delta = \delta(\alpha, \lambda_1) = 2\alpha - 1 + 2(n+1)(1-\alpha)\sigma(n).
\]
From that we deduce $R_{\lambda_1, \lambda_2}^{n+1, m}(\alpha) \subset R_{\lambda_1, \lambda_2}^{n, m}(\delta)$. This completes the proof of Theorem 2.

**Theorem 3.** Let $q$ be a convex function in $U$, with $q(0) = 1$ and let

$$h(z) = q(z) + \lambda_1 z q'(z), \quad (z \in U).$$

If $n, m \in \mathbb{N}_0, \lambda_2 \geq \lambda_1 \geq 0, f \in \mathcal{A}$ and it satisfies the differential subordination

$$(\mu_{n+1, m}^{n, m} f(z))' \prec h(z), \quad (z \in U),$$

then

$$\left[\mu_{n, m}^{n, m} f(z)\right]' \prec q(z), \quad (z \in U),$$

and this result is sharp.

**Proof.** Let

$$p(z) = \left(\mu_{n, m}^{n, m} f(z)\right)'.$$

Using (2.4), the differential subordination (2.7) becomes

$$p(z) + z p'(z) \frac{1}{1+n} \prec h(z) = q(z) + \lambda_1 z q'(z), \quad (z \in U).$$

Using Lemma 2, we obtain

$$p(z) \prec q(z), \quad (z \in U).$$

Hence

$$\left[\mu_{n, m}^{n, m} f(z)\right]' \prec q(z), \quad (z \in U).$$

The result is sharp. This completes the proof of the theorem.

We give a simple application for Theorem 3.

**Example 1.** For $n = 1, m = 0, \lambda_2 \geq \lambda_1 \geq 0, q(z) = \frac{1+z}{1-z}, f \in \mathcal{A}$ and $z \in U$ and applying Theorem 3, we have

$$h(z) = \frac{1+z}{1-z} + \lambda_1 z \left(\frac{1+z}{1-z}\right)' = \frac{1+2\lambda_1 z - z^2}{(1-z)^2}.$$
By using (2.4) we find

\[
\left( \mu_{\lambda_1,\lambda_2}^{1,0} f(z) \right)' = \left( \mu_{\lambda_1,\lambda_2}^{0,0} f(z) \right)' + z \left( \mu_{\lambda_1,\lambda_2}^{0,0} f(z) \right)'',
\]

\[
= 1 + \sum_{k=2}^{\infty} (1 + \lambda_2 (k - 1)) k a_k z^{k-1}
+ \sum_{k=2}^{\infty} (1 + \lambda_2 (k - 1)) k^2 a_k z^{k-1},
\]

\[
= 1 + \sum_{k=2}^{\infty} (1 + \lambda_2 (k - 1)) k^2 a_k z^{k-1},
\]

(2.8)

Similarly we compute \( \left( \mu_{\lambda_1,\lambda_2}^{2,0} f(z) \right)' \). By using (2.4), we find

\[
\left( \mu_{\lambda_1,\lambda_2}^{2,0} f(z) \right)' = \left( \mu_{\lambda_1,\lambda_2}^{1,0} f(z) \right)' + \frac{z}{2} \left( \mu_{\lambda_1,\lambda_2}^{1,0} f(z) \right)''.
\]

Then, by using (2.8) we have

\[
\left( \mu_{\lambda_1,\lambda_2}^{1,0} f(z) \right)'' = \sum_{k=2}^{\infty} (1 + \lambda_2 (k - 1)) k^2 (k - 1) a_k z^{k-2}.
\]

(2.9)

(2.10)

After that, by (2.8) and (2.10), (2.9) becomes

\[
\left( \mu_{\lambda_1,\lambda_2}^{2,0} f(z) \right)' = 1 + \sum_{k=2}^{\infty} (1 + \lambda_2 (k - 1)) k^2 a_k z^{k-1}
+ \frac{1}{2} \sum_{k=2}^{\infty} (1 + \lambda_2 (k - 1)) k^2 (k - 1) a_k z^{k-1},
\]

\[
= 1 + \sum_{k=2}^{\infty} (1 + \lambda_2 (k - 1)) k^2 (k + 1) a_k z^{k-1},
\]

\[
f(z) * \left[ z + \sum_{k=2}^{\infty} \frac{1}{2} (1 + \lambda_2 (k - 1)) (1 + k) k^2 z^k \right] = \frac{z}{z}.
\]
From Theorem 3 we deduce that

\[
 f(z) \left[ z + \sum_{k=2}^{\infty} \frac{1}{2} (1 + \lambda_2 (k-1)) (1 + k) k^2 z^k \right] < \frac{1 + 2\lambda_1 z - z^2}{(1-z)^2}
\]

implies

\[
 f(z) \left[ \frac{z + \sum_{k=2}^{\infty} k^2 (1 + \lambda_2 (k-1)) z^k}{z} \right] < \frac{1 + z}{1-z}, \quad (z \in U).
\]

**Theorem 4.** Let \( q \) be a convex function in \( U \), with \( q(0) = 1 \) and let

\[ h(z) = q(z) + zq'(z), \quad (z \in U). \]

If \( n, m \in \mathbb{N}_0, \lambda_2 \geq \lambda_1 \geq 0, f \in \mathcal{A} \) and satisfies the differential subordination

\[
 (\mu_{\lambda_1, \lambda_2}^{n,m} f(z))' < h(z), \quad (2.11)
\]

then

\[
 \frac{\mu_{\lambda_1, \lambda_2}^{n,m} f(z)}{z} < q(z), \quad (z \in U).
\]

The result is sharp.

**Proof.**

\[
 p(z) = \frac{\mu_{\lambda_1, \lambda_2}^{n,m} f(z)}{z}, \quad (2.12)
\]

Differentiating (2.12), with respect to \( z \), we obtain

\[
 \left( \mu_{\lambda_1, \lambda_2}^{n,m} f(z) \right)' = p(z) + z p'(z), \quad (z \in U). \quad (2.13)
\]

Using (2.13), the differential subordination (2.11) becomes

\[ p(z) + z p'(z) < h(z) = q(z) + zq'(z), \]

and by using Lemma 2, we deduce

\[ p(z) < q(z), \quad (z \in U). \]

Next using (2.12), we have

\[
 \frac{\mu_{\lambda_1, \lambda_2}^{n,m} f(z)}{z} < q(z), \quad (z \in U).
\]
This proves Theorem 4.

We give a simple application of Theorem 4.

**Example 2.** For $n = 1, m = 0$, $\lambda_2 \geq \lambda_1 \geq 0$, $q(z) = \frac{1}{1-z}$, $f \in A$ and $z \in U$, by using Theorem 4, we obtain

$$h(z) = \frac{1}{1-z} + z \left( \frac{1}{1-z} \right)' = \frac{1}{(1-z)^2}.$$

From (1.3), we have

$$\left( \mu_{\lambda_1, \lambda_2}^{1,0} f(z) \right) = z \left( \mu_{\lambda_1, \lambda_2}^{0,0} f(z) \right)' = z + \sum_{k=2}^{\infty} (1 + \lambda_2 (k-1)) k a_k z^k = f(z) * \left[ z + \sum_{k=2}^{\infty} (1 + \lambda_2 (k-1)) k z^k \right].$$

From example 1, we have

$$\left( \mu_{\lambda_1, \lambda_2}^{1,0} f(z) \right)' = \frac{f(z) * \left[ z + \sum_{k=2}^{\infty} (1 + \lambda_2 (k-1)) k z^k \right]}{z}.$$

Now, applying Theorem 4, we deduce that

$$\frac{f(z) * \left[ z + \sum_{k=2}^{\infty} (1 + \lambda_2 (k-1)) k z^k \right]}{z} < \frac{1}{(1-z)^2}$$

implies

$$\frac{f(z) * \left[ z + \sum_{k=2}^{\infty} (1 + \lambda_2 (k-1)) k z^k \right]}{z} < \frac{1}{1-z}.$$

**Theorem 5.** Let

$$h(z) = \frac{1 + (2\alpha - 1) z}{1 + z}, \ (z \in U)$$

be convex in $U$, with $h(0) = 1$ and $0 \leq \alpha < 1$. If $n, m \in \mathbb{N}_0$, $\lambda_2 \geq \lambda_1 \geq 0$, $f \in A$ and the differential subordination

$$(\mu_{\lambda_1, \lambda_2}^{n,m} f(z))' < h(z) \quad (2.14)$$
is satisfied, then
\[ \frac{\mu_{\lambda_1, \lambda_2}^{n,m} f(z)}{z} < q(z) = 2\alpha - 1 + \frac{2(1-\alpha)\ln(1+z)}{z}. \]

The function q is convex and is the best dominant.

Proof. Let
\[ p(z) = \frac{\mu_{\lambda_1, \lambda_2}^{n,m} f(z)}{z}, \quad (2.15) \]

Differentiating (2.15), with respect to z, we obtain
\[ \left( \frac{\mu_{\lambda_1, \lambda_2}^{n,m} f(z)}{z} \right)' = p(z) + z p'(z). \quad (z \in U). \]

Using (2.16), the differential subordination (2.14) becomes
\[ p(z) + z p'(z) < h(z) = \frac{1 + (2\alpha - 1)z}{1 + z}, \quad (z \in U). \]

From Lemma 1, we deduce
\[ p(z) < q(z) = \frac{1}{z} \int_{0}^{z} h(t) \, dt, \]
\[ = \frac{1}{z} \int_{0}^{z} \left( \frac{1 + (2\alpha - 1)t}{1 + t} \right) \, dt, \]
\[ = \frac{1}{z} \left[ \int_{0}^{z} \frac{1}{1 + t} \, dt + (2\alpha - 1) \int_{0}^{z} \frac{t}{1 + t} \, dt \right], \]
\[ = 2\alpha - 1 + \frac{2(1-\alpha)\ln(1+z)}{z}. \]

Using (2.15), we have
\[ \frac{\mu_{\lambda_1, \lambda_2}^{n,m} f(z)}{z} < q(z) = 2\alpha - 1 + \frac{2(1-\alpha)\ln(1+z)}{z}. \]

The proof is complete. \qed

From Theorem 5, we deduce the following Corollary:
Corollary 1. If $f \in R_{\lambda_1, \lambda_2}^{n,m}$, then

$$\text{Re}\left(\frac{\mu_{n,m}}{\lambda_1, \lambda_2} f(z) \right) > (2\alpha - 1) + 2(1 - \alpha) \ln 2, \quad (z \in U).$$

Proof. Since $f \in R_{\lambda_1, \lambda_2}^{n,m}$, and from Definition 2 we have

$$\text{Re}\left(\frac{\mu_{n,m}}{\lambda_1, \lambda_2} f(z) \right) > \alpha, \quad (z \in U),$$

which is equivalent to

$$(\mu_{n,m} f(z))' = h(z) = \frac{1 + (2\alpha - 1)z}{1 + z}.$$

Using Theorem 5, we have

$$\frac{\mu_{n,m} f(z)}{z} < q(z) = (2\alpha - 1) + 2(1 - \alpha) \frac{\ln(1 + z)}{z}.$$

Since $q$ is convex and $q(U)$ is symmetric with respect to the real axis, we deduce

$$\text{Re}\left(\frac{\mu_{n,m}}{\lambda_1, \lambda_2} f(z) \right) > \text{Re} q(1) = (2\alpha - 1) + 2(1 - \alpha) \ln 2, \quad (z \in U).$$

□

Theorem 6. Let $h \in \mathcal{H}(U)$, with $h(0) = 1$, $h'(0) \neq 0$ and assume that it satisfies the inequality

$$\text{Re}\left(1 + \frac{zh''(z)}{h'(z)}\right) > -\frac{1}{2}, \quad (z \in U).$$

If $n, m \in \mathbb{N}_0$, $\lambda_2 \geq \lambda_1 \geq 0$, $f \in \mathcal{A}$ and it satisfies the differential subordination

$$(\mu_{n,m} f(z))' < h(z), \quad (z \in U),$$

then

$$\frac{\mu_{n,m} f(z)}{z} < q(z) = \frac{1}{z} \int_0^z h(t)dt.$$

Proof. Let

$$p(z) = \frac{\mu_{n,m} f(z)}{z},$$

$$\frac{z + \sum_{k=2}^{\infty} \frac{(1 + \lambda_1 k - 1)^m}{(1 + \lambda_2(k - 1))^{m-1}} c(n,k) a_k z^k}{z}, \quad (p \in \mathcal{H}[1, 1], \ z \in U).$$
Differentiating (2.18), with respect to $z$, we have
\[
\left( \mu_{\lambda_1, \lambda_2}^{n,m} f(z) \right)' = p(z) + z p'(z), \quad (z \in U). \tag{2.19}
\]
Using (2.19), the differential subordination (2.17) becomes
\[
p(z) + z p'(z) < h(z), \quad (z \in U).
\]
From Lemma 1, we deduce
\[
p(z) \prec q(z) = \frac{1}{z} \int_0^z h(t) \, dt.
\]
With (2.18), we obtain
\[
\mu_{\lambda_1, \lambda_2}^{n,m} f(z) \prec z q(z) = \frac{1}{z} \int_0^z h(t) dt.
\]
From Lemma 3, we obtain that the function $q$ is convex, and from Lemma 1, $q$ is the best dominant for the subordination (2.17). This completes the proof of Theorem 6.

\[\Box\]

3. Conclusion

We remark that several subclasses of analytic univalent functions can be derived using the operator $\mu_{\lambda_1, \lambda_2}^{n,m}$. Several of their properties can be studied with this method, for example properties related to the ones that were studied in [7] and [6].

Acknowledgement

This work was supported by UKM-ST-06-FRGS0244-2010, Malaysia. The authors would like to thank the referee for giving some suggestions to improve the content of the article.

References

[1] M. H. Al-Abbadi and M. Darus, “Differential subordination defined by new generalised derivative operator for analytic functions,” Int. J. Math. Math. Sci., vol. 2010, p. 15, 2010.
[2] F. M. Al-Oboudi, “On univalent functions defined by a generalized Sălăgean operator,” Int. J. Math. Math. Sci., vol. 2004, no. 25–28, pp. 1429–1436, 2004.
[3] K. Al-Shaqsi and M. Darus, “On univalent functions with respect to $k$-symmetric points defined by a generalized Ruscheweyh derivatives operator,” J. Anal. Appl., vol. 7, no. 1, pp. 53–61, 2009.
[4] B. Carlson and D. B. Shaffer, “Starlike and prestarlike hypergeometric functions,” SIAM J. Math. Anal., vol. 15, pp. 737–745, 1984.
[5] M. Darus and K. Al-Shaqsi, “Differential sandwich theorems with generalised derivative operator,” Int. J. Comput. Math. Sci., vol. 2, no. 2, pp. 75–78, 2008.
[6] M. Darus and I. Faisal, “A study of Pescar’s univalence criteria for space of analytic functions,” Journal of Inequalities and Applications, vol. 2011, no. 109, p. 7, 2011.
[7] M. Darus and I. Faisal, “A study on Becker’s univalence criteria,” Abstr. Appl. Anal., vol. 2011, p. 13, 2011.
[8] S. S. Miller and P. T. Mocanu, “On some classes of first-order differential subordinations,” Mich. Math. J., vol. 32, pp. 185–195, 1985.
[9] S. S. Miller and P. T. Mocanu, Differential subordinations: theory and applications, ser. Series on monographs and textbooks in pure and applied mathematics. New York: Marcel Dekker, 2000, vol. 225.
[10] P. T. Mocanu, T. Bulboaca, and G. S. Salagean, Teoria geometrica a functiilor univalente. Cluj-Napoca: Casa Cartii de Stiinta, 1999.
[11] G. I. Oros, “On a class of holomorphic functions defined by the Ruscheweyh derivative,” Int. J. Math. Math. Sci., vol. 2003, no. 65, pp. 4139–4144, 2003.
[12] G. I. Oros, “A class of holomorphic functions defined using a differential operator,” Gen. Math., vol. 13, no. 4, pp. 13–18, 2005.
[13] G. Oros and O. G. Irina, “Differential superordination defined by Sălăgean operator,” Gen. Math., vol. 12, no. 4, pp. 3–10, 2004.
[14] K. S. Padmanabhan and R. Manjini, “Certain applications of differential subordination,” Publ. Inst. Math., Nouv. Sér., vol. 39, no. 53, pp. 107–118, 1986.
[15] S. Ponnusamy, “Differential subordination and starlike functions,” Complex Variables, Theory Appl., vol. 19, no. 3, pp. 185–194, 1992.
[16] S. Ruscheweyh, “New criteria for univalent functions,” Proc. Am. Math. Soc., vol. 49, pp. 109–115, 1975.
[17] G. S. Sălăgean, “Subclasses of univalent functions,” in Complex analysis - Proc. 5th Rom.-Finn. Semin., Bucharest 1981, ser. Lect. Notes Math., vol. 1013. Springer-Verlag, 1983, pp. 362–372.

**Authors’ addresses**

**M. H. Al-Abbadi**  
Universiti Kebangsaan Malaysia, School of Mathematical Sciences, Faculty of Science and Technology, Bangi 43600, Selangor, Malaysia  
*E-mail address:* mamoun.nn@yahoo.com

**M. Darus**  
Universiti Kebangsaan Malaysia, School of Mathematical Sciences, Faculty of Science and Technology, Bangi 43600, Selangor, Malaysia  
*E-mail address:* maslina@ukm.my