Fermion propagator at finite temperatures on extremely anisotropic lattice.

Vladimir K. Petrov*

*Bogolyubov Institute for Theoretical Physics, 252143 Kiev, Ukraine.

Abstract

Fermion propagator is computed in a simple model on an extremely anisotropic lattice $\xi \gg 1$. Fermion determinant is evaluated up to $\xi^{-4}$ order. Chiral condensate is estimated in mean field approximation.

1 Introduction

Finite-temperature studies of the fermion contribution in lattice gauge theories were on the agenda over a number of years and both MC experiments and analytical models were used to challenge the problem. Calculations with dynamical fermions and small lattice spacing, however, are still highly expensive. Hence the approximations which would hopefully capture some of the essential features of the physics are worth a try. We attempt to study the fermionic part of the action (see e.g. [1])

$$-S_F = n_f a^3 \sum_{x,x'} \left( \overline{\psi}_{x'} D^0_{x,x'} \psi_x + \xi^{-1} \overline{\psi}_{x'} \sum_{n=1}^3 D^n_{x,x'} \psi_x \right)$$

(1)

on an extremely anisotropic lattice ($\xi \gg 1$) in a simple model [2], where terms in (1) proportional to $\xi^{-1}$ are discarded. Here $n_f$ is the number of flavors and $\xi = \xi (g, \xi_0)$ is the anisotropy parameter. The dependance of $\xi$ on coupling $g$ and on 'bare' anisotropy parameter $\xi_0 = a/a_r$ is defined by the

*E-mail address: vkpetrov@yahoo.com
condition of independence of physical values on spatial $a$ and temporal $a_{\tau}$ lattice spacings. Dirac operator is defined by

$$D_{x'x}^\nu = \frac{1 - \gamma^\nu U^\nu(x) \delta_{x,x'-\nu} + \frac{1 + \gamma^\nu}{2} U^\dagger(x') \delta_{x,x'+\nu} - \left(1 + \delta^0 \nu ma_{\tau}\right) \delta_{x',x},$$

(2)

where $U^\nu(x)$ are link $(x, x + \nu)$ variables, $\gamma^\nu$ are Dirac matrices, and $\nu = 0, 1, 2, 3$. We consider the case of infinitely heavy mirror fermions and choose Hamiltonian gauge

$$U_0(x,t)_{\alpha\lambda} = \delta_{\alpha\lambda} \left[1 - \left(\exp \left\{-i \varphi_{\alpha}(x)\right\} + 1\right) \delta_{t',0}\right],$$

(3)

where $\exp \left\{-i \varphi_{\alpha}(x)\right\} \equiv \left(\prod_{t=0}^{N_{\tau}-1} U_0(x,t)\right)_{\alpha\alpha}$ are the eigenvalues of Polyakov matrix and $N_{\tau}$ is the lattice length in the temporal direction and $\alpha, \lambda = 1, ..., N$ are color indices. In particular, for the temporal component of Dirac matrix one may write

$$D^0_{x'x} = \delta_{x',x} \left(\frac{1 - \gamma^0}{2} \Delta_{t'-t} + \frac{1 + \gamma^0}{2} \Delta^\dagger_{t'-t}\right),$$

(4)

with

$$\Delta_{t'-t}(x) \equiv \exp \left\{-i \varphi_{\alpha}(x)\right\} \delta_{t'-t-1} - \left(1 + ma_{\tau}\right) \delta_{t'-t}.$$  

(5)

For antiperiodic border conditions on fermion fields in temporal direction the matrix $\Delta_{t'-t}$ may be diagonalized by discrete Fourier transformation with half-integer variables, so

$$\Phi_{t}(k) = \exp \left\{2\pi i \left(k + 1/2\right) \left(t + 1/2\right) / N_{\tau}\right\}$$

(6)

are the eigenfunctions of the operator $\Delta_{t'-t}$. After Fourier transformation we get

$$\tilde{\lambda}_{k,k'} = \sum_{t,t'=0}^{N_{\tau}-1} \Phi_{t'}(k') \Delta_{t',t} \Phi_{t}(k)^* = \delta_{k,k'} \lambda_k,$$

(7)

where

$$\lambda_k = \exp \left\{-i \left[\varphi_{\alpha} - 2\pi \left(k + 1/2\right)\right] / N_{\tau}\right\} - \left(1 + ma_{\tau}\right)$$

(8)

$^1$Further for convenience we put $a = 1.$
are the eigenvalues of $\Delta_{t't}$, Thereby the inverse matrix is given by

$$
\Delta_{t't}^{-1} = N_r^{-1} \sum_{k=0}^{N_r-1} \lambda_k^{-1} \exp \left\{ 2\pi i (t - t') \left( k + 1/2 \right) / N_r \right\}.
$$

(9)

Common properties of projectors $\frac{1+\gamma_0}{2} = \left( \frac{1+\gamma_0}{2} \right)^2$ and $\frac{1+\gamma_0}{2} \left( \frac{1+\gamma_0}{2} \right)^2 = 0$ allow to write

$$
f \left( \frac{1+\gamma_0}{2} u + \frac{1-\gamma_0}{2} v \right) = \frac{1+\gamma_0}{2} f(u) + \frac{1-\gamma_0}{2} f(v)
$$

(10)

for the arbitrary regular function, in particular the inverse operator $(D_{t't}^0)^{-1}$ may be written as

$$
(D_{t't}^0)^{-1} = \frac{1+\gamma_0}{2} \Delta_{t't}^{-1} + \frac{1-\gamma_0}{2} \left( \Delta_{t't}^{-1} \right)^\dagger.
$$

(11)

2 Ansatz

To sum over the Fourier variables $k$ in (9) we use a standard trick. Let $f(\omega)$ be a rational function of cos $\omega$ and sin $\omega$, which has no poles on lines $\Re \omega = 2\pi n$. The integral over the whole area of the complex variable $\omega$ for the periodic function splits into a set of equal integrals over 'elementary' bands $2n\pi \leq \Re \omega - \frac{\pi}{N_r} < 2\pi (n + 1)$. Due to periodicity, the result of integration over any band boundary is

$$
\Phi = \frac{1}{2} \left\{ \int_{-i\infty}^{2\pi-i\infty} - \int_{i\infty}^{2\pi+i\infty} \right\} f(\omega) \tan \left( \frac{N_r \omega}{2} \right) \frac{d\omega}{2\pi i}.
$$

(12)

The loop integral over each band area of $\omega$ is equal to zero as well, so expressing it as sum of residues one may write

$$
\oint f(\omega) \tan \left( \frac{N_r \omega}{2} \right) \frac{d\omega}{2\pi i} = \sum \text{res} \left\{ f(\omega) \tan \left( \frac{N_r \omega}{2} \right) \right\} + \Phi = 0.
$$

(13)

Taking into account that tan $(N_r \omega/2)$ has poles at $\omega = 2\pi (k + 12) / N_r$, one may finally write

$$
\frac{1}{N_r} \sum_{k=0}^{N_r-1} f \left( 2\pi \left( k + \frac{1}{2} \right) / N_r \right) = -\frac{1}{2} \sum_{\omega_r} \text{res} \left\{ f(\omega) \tan \left( \frac{N_r \omega}{2} \right) \right\} - \Phi.
$$

(14)
where $\omega_r$ poles of the $f(\omega)$ are located in the band $0 \leq \Re \omega - \frac{\pi}{N_\tau} < 2\pi$. Now one may apply the ansatz (14) to compute $\Delta_{t,t'}^{-1}$. In the case considered

$$
f(\omega) = \begin{cases} 
egthinspace\negthinspace\negthinspace\negthinspace\negthinspace -e^{i(t-t')\omega}^{\lambda_k^{-1}} & \text{for} \quad 0 \leq t - t' < N_\tau \\
\negthinspace\negthinspace\negthinspace\negthinspace\negthinspace e^{i(t-t'+N_\tau)\omega}^{\lambda_k^{-1}} & \text{for} \quad -N_\tau < t - t' < 0
\end{cases}
$$

has poles at $\omega_0 = -\varphi_\alpha + ima$, so taking into account that $f(\omega) \to 0$ for $\Im \omega \to \pm\infty$ we find

$$
(\Delta_{t,t'}^{-1})_{\alpha} = \frac{\theta_{t,t'} e^{-i\varphi_\alpha} - (\theta_{t',t} + \delta_{t',t}) e^{\frac{m T}{2}} e^{-m \frac{t-t'-1}{N_\tau}}}{e^{-i\varphi_\alpha} + e^{\frac{m T}{2}}},
$$

where $T = N_\tau^{-1} a^{-1}$ and the periodic in $t_k$ function $\theta_{t_1,t_2}$ is given by (for $t_k = (t_k)_{\text{mod} N_\tau}$)

$$
\theta_{t_1,t_2} = \begin{cases} 
egthinspace\negthinspace\negthinspace\negthinspace\negthinspace 1 & \text{for} \quad t_1 > t_2 \\
\negthinspace\negthinspace\negthinspace\negthinspace\negthinspace 0 & \text{for} \quad t_1 \leq t_2
\end{cases}
$$

Hence, in our approximation the expression for fermion determinant may be found in a closed form from (11) and (16).

### 3 Fermion determinant

Since $(D^0)^{-1}$ is known, for the fermion contribution up to the $O(\xi^{-4})$ one can write

$$
n_f^{-1} \ln \int d\bar{\psi}_x d\psi_x \exp \{-S_F \,(1/\xi)\} = -S_F^{(eff)} \,(1/\xi) = -S_F^{(eff)} \,(0) - \frac{1}{2} S_F^{(eff)''} \,(0) \xi^{-2},
$$

where

$$
-S_F^{(eff)} \,(0) = \ln \det(D^0) = \sum_x \left\{ \sum_{\alpha=1}^{N} \ln \left( \cos \varphi_\alpha(x) + \cosh \frac{m}{T} \right) - S_{\text{mirr}} \right\}
$$

with

$$
-S_{\text{mirr}} = N m / T
$$

and

$$
S_F^{(eff)''} \,(0) = \sum_{n=1}^{3} \text{Sp} \left\{ (D^0)^{-1} D^n (D^0)^{-1} D^n \right\}.
$$
In case of light \((m \ll T)\) fermions we get
\[
-\frac{1}{2} S^{(eff)n}_F (0) = \left( 1 - \frac{2m}{T} + O \left( \frac{m^2}{T^2} \right) \right) \sum_{t_2, t_1=0}^{N_R-1} \sum_{n=1}^{3} \times \nonumber \\
\sum_{\alpha \nu} \sum_{t_1} Re \left\{ U_n (t_2, x)_{\alpha \nu} \Phi_{\alpha \nu} (x, n)_{t_2, t_1} U_n^\dagger (t_1, x)_{\nu \alpha} - \Phi_{\alpha \alpha} (x, n)_{t_2, t_1} \right\}. 
\]

where
\[
\Phi_{\alpha \nu} (x, n)_{t_2, t_1} = -\frac{1}{2} \left( \frac{\theta_{t_1, t_2}}{1 - e^{-i\varphi_{\nu}(x+n)}} + \frac{\theta_{t_2, t_1}}{1 - e^{-i\varphi_{\alpha}(x)}} \right) \times \nonumber \\
\frac{\tan \frac{\varphi_{\alpha} (x)}{2}}{\tan \frac{\varphi_{\nu} (x+n)}{2}} .
\]

It is easy to see that \(S^{(eff)n}_F (0)\) turns into zero in the case of \(U(1)\) gauge group, if spatial link variables \(U_n\) are static.

**4 Chiral condensate**

To estimate \(\langle \bar{\psi} \psi \rangle\) one may, in the spirit of Curie-Weiss method, change all \(\varphi_{\alpha} (x)\) for the mean field \(\bar{\varphi}_{\alpha}\) defined by the mean field equation. For example, in case of SU(2) gauge group \(\varphi_1 (x) = -\varphi_2 (x) \equiv \varphi (x) / 2 \simeq \bar{\varphi} / 2\). Thereby for the chiral condensate we get
\[
\langle \bar{\psi} \psi \rangle = T \frac{\partial}{\partial m} \ln Z = n_f T \left( \frac{\partial}{\partial m} \ln \det D \right)_{\varphi_{\alpha} (x) = \bar{\varphi}_{\alpha}}
\]

or
\[
\langle \bar{\psi} \psi \rangle / 2n_f = 1 + \frac{\xi^{-2}}{2} S^{(eff)n}_F (0) \bigg|_{m=0} + \frac{m/T}{1 + \cos (\bar{\varphi}/2)} + O \left( m^2, \xi^{-4} \right). 
\]

It should be noted that even for infinitely heavy mirror fermion the remnant term \(S_{mirr}\), defined in (20) survives. It is easy to make sure that this very term introduces a constant component \((2n_f)\) into \(\langle \bar{\psi} \psi \rangle\), that doesn’t disappear even at \(m/T \to 0\). The mirror fermion contribution may be totally removed ”by brute force”. If we introduce ”the counterterm” \(S_{ct} = -S_{mirr}\) into the original action (1), but preserve a standard definition for \(\langle \bar{\psi} \psi \rangle\), such ”counterterm” would evidently cancel the undesirable component in the first term of (24). Therefore, we get \(\langle \bar{\psi} \psi \rangle \to n_f \xi^{-2} S^{(eff)n}_F (0) \bigg|_{m=0} \) for \(m/T \to 0\).
5 Discussion

In the parameter area available for modern computers sea quark contribution introduces minor changes in MC data. For example, the comparison of spectroscopy results obtained with dynamical and quenched fermions shows no dramatic difference. Apparently at the parameter area of the MC simulation the sea quarks simply do not affect the spectroscopy above five to ten percent level [3]. Major part of MC data [4] helps to trace the sea quark contribution, rather than locate the area where the sea quarks should obligatory be taken into account.

Suggested model allows to evaluate the fermion contribution analytically for arbitrary small values of lattice spacings and infinitely large lattices, where MC simulations easily become prohibitively costly. Although such contribution can be totally incorporated into the invariant measure and may appear almost trivial, it significantly changes the phase structure of the model. Such changes, especially in the case of heavy quarks, are very similar to those introduced by external magnetic field into spin systems. We hope that suggested model will help not only to estimate the sea quark contribution at lattices with size and spacings unaccessible for MC experiment, but might also assist in qualifying the parameter area where the presence of dynamical fermions leads to particularly appreciable effects.

References

[1] P. Hasenfratz, F. Karsch, Phys.Lett.125B(1983)308.

[2] V.K. Petrov, [hep-lat/9803019].

[3] C. Bernard, et.al., Phys.Rev.Lett.78(1997)598.

[4] S. Collins, et.al., Nucl.Phys.Proc.Suppl.47(1996)3785, A. Spitz, et.al., Nucl.Phys.Proc.Suppl.63(1998)317.