Eigenvalue problem for radial potentials
in space with SU(2) fuzziness

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Abstract
The eigenvalue problem for radial potentials is considered in a space whose spatial coordinates satisfy the SU(2) Lie algebra. As the consequence, the space has a lattice nature and the maximum value of momentum is bounded from above. The model shows interesting features due to the bound, namely, a repulsive potential can develop bound-states, or an attractive region may be forbidden for particles to propagate with higher energies. The exact radial eigen-functions in momentum space are given by means of the associated Chebyshev functions. For the radial stepwise potentials the exact energy condition and the eigen-functions are presented. For a general radial potential it is shown that the discrete energy spectrum can be obtained in desired accuracy by means of given forms of continued fractions.

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1 Introduction

The noncommutative spaces have been the subject of a great number of studies in recent years [1, 2]. The natural appearance of these spaces in some areas of physics, for example in the string theory, is a part of the motivation. In particular, the canonical relation
\[ [\hat{x}_a, \hat{x}_b] = i \theta_{a,b} \mathbf{1}, \tag{1} \]
is shown to describe the algebra between the coordinates of the longitudinal directions of $D$-branes in presence of a constant $B$-field background [3–7].

The natural extension of the above algebra is to take the commutators of the coordinates non-constant. Examples of this kind are, the noncommutative cylinder and the $q$-deformed plane [8], the $\kappa$-Poincaré algebra [9–12], and linear noncommutativity of the Lie algebra type [13,14]. In the latter the dimensionless spatial position operators satisfy:
\[ [\hat{x}_a, \hat{x}_b] = f_{c,a,b} \hat{x}_c, \tag{2} \]
in which $f_{c,a,b}$’s are the structure constants of a Lie algebra, for example the algebra by SO(3) or SU(2) groups. A special case, the so-called fuzzy sphere, is when an irreducible representation of the position operators is taken, by which the Casimir of the algebra is constant, hence the name sphere [15,16].

The other possibility is to not restrict the representation to an irreducible one, but all of the irreducible representations would be taken [17–20]; see also [21]. In particular, the regular representation of the group would be considered, and as the consequence, the model is built on the whole space, not on a subspace, as the case with fuzzy sphere.

In [17–20] basic ingredients for calculus on a linear Lie type fuzzy space and the field theoretic aspects on such a space were studied in details. The most remarkable features of the field theories on such a space happen to be: 1) They are free from any ultraviolet divergences if the group is compact; 2) The momentum conservation is modified, in the sense that the vector addition is replaced by a non-Abelian operation [17, 22]; 3) In the transition amplitudes only the so-called planar graphs contribute.

The classical motion on noncommutative space has attracted interests as well [23,24]. In particular, the central force problems on space-times with canonical and linear noncommutativity and their observational consequences have been the subject of different research works [25–30]. In [31] the classical mechanics defined on a space with SU(2) algebra was studied. In particular, the Poisson structure induced by noncommutativity of SU(2) type was investigated, for either the Cartesian or Euler parametrization of SU(2) group. In [32] it was shown that on a SU(2) type space it is only the Kepler potential, as a single-term power-law one, for which all of nearly circular orbits are closed. Further, it was proved for the Kepler potential all of bounded orbits, no matter how far from circle, is closed [32].
The commutation relations of the position and momentum operators on a space with SU(2) algebra was studied in [33]. The thermodynamical aspects of these models were explored in [34,35].

The purpose of the present work is to continue the study the quantum mechanics on space with SU(2) algebra. In particular, the eigenvalue problem for radial potentials is considered in a space whose spatial coordinates satisfy the SU(2) Lie algebra. The interesting feature by the SU(2) algebra is, the space happens to have a lattice nature, however a rotationally symmetric one. Also, due to the lattice structure, the maximum value of momentum is bound from above. The bound on momentum would appear as the basis for surprising features for the model. In particular, on such a space a repulsive potential can develop bound-states, or an attractive region may be forbidden for particles to propagate with higher energies. As definition of the position eigenstates of the form $|x_1⟩|x_2⟩|x_3⟩$ is not possible due to the algebra (2), the momentum space is commutative, and hence all the necessary ingredients for the model can be defined in this space.

The scheme of the rest of this paper is as following. In Sec. 2, the basic notions to formulate quantum theory on a space with Lie type noncommutativity are presented. Also in this section the construction is specialized for the case of the SU(2) group. In Sec. 3 the exact radial eigen-functions are constructed by means of associated Chebyshev functions. In Sec. 4 the radial stepwise potentials are considered, and the exact expression for the energy quantization condition as well as the eigen-functions are presented. In Sec. 5 the case with a general radial potential is discussed. In particular it is shown that the discrete energy spectrum can be obtained in desired accuracy by means of given forms of continued fractions.

2 Basic notions

Consider a Lie group G. Denote the members of a basis for the left-invariant vector fields corresponding to this group by $\hat{x}_a$’s. These fields (which are sections of the tangent bundle TG) satisfy (2), with the structure constants of the Lie algebra corresponding to G. The coordinates $k^a$ are defined such that

$$U(k) := \exp(k^a \hat{x}_a) U(0),$$

where $U(k)$ is the group element corresponding to the coordinates $k$, $U(0)$ is the identity, and $\exp(\hat{x})$ is the flux corresponding to the vector field $\hat{x}$. The Hilbert space to be considered is the space of functions defined on G, which are square integrable with respect to the Haar measure of the group. The action of the functions of the group and the vector fields defined on the group, on the functions of the group are defined through multiplication and Lie derivation, respectively. The commutators of the operator forms of the coordinate functions and the left
invariant vector fields are (2) and

\[ [\hat{k}^a, \hat{k}^b] = 0, \]
\[ [\hat{x}_a, \hat{k}^b] = \hat{x}_a^b, \]

where \( \hat{x}_a^b \)'s are functions of \( G \). These satisfy

\[ \hat{x}_a^b(\hat{k} = 0) = \delta_a^b. \]

Next consider the right-invariant vector fields \( \hat{x}_a^R \), so that they coincide with their left-invariant analogues at the identity of the group [33]:

\[ \hat{x}_a^R(\hat{k} = 0) = \hat{x}_a(\hat{k} = 0). \]

These field satisfy the commutation relations

\[ [\hat{x}_a^R, \hat{x}_b^R] = -f^{c}_{ab} \hat{x}_c^R, \]
\[ [\hat{x}_a^R, \hat{x}_b] = 0. \]

Using these, one defines the new vector field \( \hat{J}_a \) through

\[ \hat{J}_a := \hat{x}_a - \hat{x}_a^R. \]

These are the generators of the adjoint action, and satisfy the commutation relations [33]

\[ [\hat{J}_a, \hat{J}_b] = f^{c}_{ab} \hat{J}_c, \]
\[ [\hat{J}_a, \hat{x}_b] = f^{c}_{ab} \hat{x}_c, \]
\[ [\hat{J}_a, \hat{x}_a^R] = f^{c}_{ab} \hat{x}_c^R, \]
\[ [\hat{k}^c \hat{J}_a] = f^{c}_{ab} \hat{k}^b. \]

For the group SU(2), taking \( \hat{k}^a \)'s and \( \hat{x}_a \)'s as momenta and spatial coordinates respectively, \( \hat{J}_a \)'s are the natural candidates for the orbital angular momenta, as suggested by the algebra they satisfy.

Using the dimensionless operators introduced in the above, one can easily construct the corresponding dimensionful ones, simply by multiplication of these operators by suitable factors to make them Hermitian with proper dimension:

\[ p^a := (\hbar/\ell) \hat{k}^a, \]
\[ x_a := i\ell \hat{x}_a, \]
\[ x_a^R := i\ell \hat{x}_a^R, \]
\[ x_a^b(\hbar) := \hat{x}_a^b[(\ell/\hbar) \hbar], \]
\[ J_a := i\hbar \hat{J}_a, \]
where $\ell$ is a constant of dimension length. One then arrives at the following commutation relations [33]

$$
[p^a, p^b] = 0, \quad (20)
$$
$$
[x_a, p^b] = i\hbar x_a^b, \quad (21)
$$
$$
[x_a, x_b] = i\ell f^c_{ab} x_c, \quad (22)
$$
$$
[J_a, x_b] = i\hbar f^c_{ab} x_c, \quad (23)
$$
$$
[p^c, J_a] = i\hbar f^c_{ab} p^b, \quad (24)
$$
$$
[J_a, J_b] = i\hbar f^c_{ab} J_c, \quad (25)
$$

It is seen that in the limit $\ell \to 0$ the ordinary commutation relations are retrieved.

### 2.1 SU(2) setup and the Euler parameters

For the group SU(2), the commutation relations (20), (21), and (22) make in fact the algebra of a rigid rotator, in which the angular momentum and the rotation vector have been replaced by $x$ and $p$, respectively, that is, the roles of position and momenta have been interchanged. As the consequence, the position operators do not have simultaneous eigenstates and the space has a lattice structure. As the momentum space is commutative with well-defined eigenstates, we switch to this space. As usual it is convenient to use the Euler parametrization of SU(2), defined through

$$
\exp(\phi T_3) \exp(\theta T_2) \exp(\psi T_3) := \exp(k^a T_a), \quad (26)
$$

where $T_a$’a are the generators of SU(2) satisfying the relations

$$
[T_a, T_b] = \epsilon^{c}_{ab} T_c, \quad (27)
$$
$$
[T_a, T_b]_+ = -\frac{1}{2} \delta_{ab}, \quad (28)
$$

for which the second is valid for the defining representation of SU(2). It can be seen that the range of the Euler parameters so that each point of the group is covered one and only one time is [33]

$$
0 \leq \phi + \psi \leq 2\pi,
$$
$$
-2\pi \leq \phi - \psi \leq 2\pi,
$$
$$
0 \leq \theta \leq 2\pi. \quad (29)
$$

One also has [36]

$$
\cos \frac{k}{2} = \cos \frac{\theta}{2} \cos \frac{\phi + \psi}{2}, \quad (30)
$$

where $k := \sqrt{\delta_{ab} k^a k^b}$. In the Euler momentum basis the inner-product of wave-functions is defined using the so-called Haar measure $d\mu$, given by:

$$
d\mu = c |\sin \theta| d\phi d\theta d\psi, \quad (31)
$$

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in which $c$ is a constant, and is fixed once the normalization prescription is fixed.

The operators $\hat{x}$ and $\hat{J}$ in the momentum basis with proper dimension are given in [33]:

\begin{align*}
x_1 &\to i \ell \left( -\frac{\cos \psi}{\sin \theta} \frac{\partial}{\partial \phi} + \sin \psi \frac{\partial}{\partial \theta} + \frac{\cos \psi \cos \theta}{\sin \theta} \frac{\partial}{\partial \psi} \right), \quad (32) \\
x_2 &\to i \ell \left( \frac{\sin \psi}{\sin \theta} \frac{\partial}{\partial \phi} + \cos \psi \frac{\partial}{\partial \theta} - \frac{\sin \psi \cos \theta}{\sin \theta} \frac{\partial}{\partial \psi} \right), \quad (33) \\
x_3 &\to i \ell \frac{\partial}{\partial \psi}, \quad (34)
\end{align*}

\begin{align*}
J_1 &\to i \hbar \left[ \frac{\cos \phi \cos \theta - \cos \psi}{\sin \theta} \frac{\partial}{\partial \phi} + (\sin \phi + \sin \psi) \frac{\partial}{\partial \theta} \\
&\quad + \frac{\cos \phi + \cos \psi \cos \theta}{\sin \theta} \frac{\partial}{\partial \psi} \right], \quad (35) \\
J_2 &\to i \hbar \left[ \frac{\sin \phi \cos \theta + \sin \psi}{\sin \theta} \frac{\partial}{\partial \phi} + (-\cos \phi + \cos \psi) \frac{\partial}{\partial \theta} \\
&\quad + \frac{\sin \phi - \sin \psi \cos \theta}{\sin \theta} \frac{\partial}{\partial \psi} \right], \quad (36) \\
J_3 &\to i \hbar \left( -\frac{\partial}{\partial \phi} + \frac{\partial}{\partial \psi} \right). \quad (37)
\end{align*}

It can be shown that the above operators are Hermitian with respect to the inner-product defined by the Haar measure (31).

Introducing the new parameters:

\begin{align*}
\chi &:= \frac{\phi - \psi}{2}, \quad \xi := \frac{\phi + \psi}{2}, \\
v &:= \cos \theta \frac{\phi}{2} \cos \xi, \quad \tau := (1 - v^2)^{-1/2} \cos \theta \frac{\phi}{2} \sin \xi, \quad (38)
\end{align*}

one arrives at ($J_{\pm} = J_1 \pm i J_2$)

\begin{align*}
J_{\pm} &= i \hbar \exp(\pm i \chi) \left( -\sqrt{1 - \tau^2} \frac{\partial}{\partial \tau} \pm i \frac{\tau}{\sqrt{1 - \tau^2}} \frac{\partial}{\partial \chi} \right), \quad (39) \\
J_3 &= -i \hbar \frac{\partial}{\partial \chi}, \quad (40)
\end{align*}

resulting in

\begin{align*}
\mathbf{J} \cdot \mathbf{J} &= -\hbar^2 \left[ (1 - \tau^2) \frac{\partial^2}{\partial \tau^2} - 2 \tau \frac{\partial}{\partial \tau} + \frac{1}{1 - \tau^2} \frac{\partial^2}{\partial \chi^2} \right], \quad (41)
\end{align*}

and subsequently [33]:

\begin{align*}
\mathbf{x} \cdot \mathbf{x} &= -\frac{\ell^2}{4} \left[ -\frac{\hbar^2}{1 - v^2} \mathbf{J} \cdot \mathbf{J} + (1 - v^2) \frac{\partial^2}{\partial \tau^2} - 3v \frac{\partial}{\partial v} \right]. \quad (42)
\end{align*}
Using (40) and (41), it is seen that the angular momentum eigenfunctions \((Y^m_l)'s\) satisfying

\[ J_3 Y^m_l = m \hbar Y^m_l, \tag{43} \]
\[ J \cdot J Y^m_l = l (l + 1) \hbar^2 Y^m_l, \tag{44} \]
are products of an arbitrary function \(f(v)\), and \(Y^m_l)'s \) (the usual spherical harmonics) with the cosine of the colatitude equal to \(\tau\) and the longitude equal to \(\chi\), that is

\[ Y^m_l = f(v) Y^m_l(\cos^{-1} \tau, \chi). \tag{45} \]

Hereafter we consider SU(2)-invariant systems, that is they are rotationally invariant and the Hamiltonian \(H\) and \(J_a\)'s commute. As \(J_a\)'s generate rotations of both \(x\) and \(k\), for a SU(2)-invariant system \(H\) is a function of only \(p \cdot p\) and \(x \cdot x\), namely:

\[ H = K(\sqrt{p \cdot p}) + V(\sqrt{x \cdot x}) \tag{46} \]

in which \(K\) and \(V\) are representing the kinetic and the potential terms, respectively. An example for \(K\) is [18,19,31,33]

\[ K = \frac{4 \hbar^2}{M \ell^2} \left( 1 - \cos \frac{\ell \rho}{2 \hbar} \right), \]
\[ = \frac{4 \hbar^2}{M \ell^2} (1 - v). \tag{47} \]

By the above choice, originated from the characteristics of spin-half irreducible representations of the group, the kinetic term happens to be monotonic with respect to \(k\) (for \(0 < k < 2 \pi\)) [18,19,31,33]. In the commutative limit \(\ell \to 0\) this kinetic term is reduced to the commutative case \(p \cdot p/(2 M)\).

For such a SU(2)-invariant system, \(H, J \cdot J\) and one of the components of \(J\) (say \(J_3\)) can be taken to have common eigen-functions. Now, the aim is to exploit the SU(2)-symmetry of such a Hamiltonian to write down an eigenvalue equation for the Hamiltonian so that that equation contains only one variable, out of the the three variables corresponding to the momentum. It is in fact an easy task by the expressions obtained so far. By the given form of \(x \cdot x\) by (42), and the relation (44), one finds [33]

\[ x \cdot x Y^m_l = Y^m_l \frac{\ell^2}{4} \left[ -(1 - v^2) \frac{d^2}{dv^2} + 3v \frac{d}{dv} + l (l + 1) \frac{1}{1 - v^2} \right] f(v), \tag{48} \]

by which the radial part of the Schrodinger equation in momentum space takes the form

\[ \frac{4 \hbar^2}{M \ell^2} (1 - v) \psi_{Elm}(v) + V(\sqrt{x \cdot x}) \psi_{Elm}(v) = E \psi_{Elm}(v). \tag{49} \]
3 Radial eigenfunctions

As seen in Sec. 2, for the systems with rotational invariance, as in case on ordinary space, the eigenvalue problem is reduced to a one-dimensional one. However, it is reminded that in the present case the one-dimensional problem, in contrary to ordinary space is not the length of the position vector \( r = \sqrt{\mathbf{x} \cdot \mathbf{x}} \), but it is the length of momentum vector, \( p = \sqrt{\mathbf{p} \cdot \mathbf{p}} \), or alternatively \( v = \cos(\ell p / 2\hbar) \) with \(-1 \leq v \leq 1\). So it is natural to define the basis \(|v⟩_l\) as the eigenvector of operator \( \hat{v} \) acting on subspace with orbital angular momentum \( l \):

\[
\hat{v} |v⟩_l = v |v⟩_l,
\]

for which using the Haar measure (31) and the variables (38), we have

\[
l⟨v|v'⟩_l = \delta(v - v') \sqrt{1 - v'^2}.
\]

In the present section the aim is to find the eigen-functions of the operator \( \mathbf{x} \cdot \mathbf{x} \), for which in the \( v \)-space we earlier found:

\[
\mathbf{x} \cdot \mathbf{x} = -\frac{\ell^2}{4} \left[ (1 - v^2) \frac{d^2}{dv^2} - 3v \frac{d}{dv} - \frac{l(l+1)}{1 - v^2} \right].
\]

Fortunately the operator in the bracket for \( l = 0 \) is known, with the Chebyshev polynomials of Type II as eigen-functions, satisfying:

\[
(1 - v^2) U''_n(v) - 3v U'_n(v) = -n(n + 2) U_n(v).
\]

Constructing the eigenfunctions for cases with \( l \neq 0 \) is rather straightforward, just like the method by which the associated Legendre polynomials are constructed [37]. In general, we will find for the associated Chebyshev functions the following

\[
(1 - v^2) U''_n(v) - 3v U'_n(v) - \frac{l(l+1)}{1 - v^2} U_n^l(v) = -n(n + 2) U_n^l(v)
\]

in which \( n = l, l+1, \cdots \), and

\[
U_n^l(v) = \sqrt{\frac{2}{\pi}} \frac{(n-l)!}{(n+l+1)!} (1 - v^2)^{l/2} \frac{d^l}{dv^l} U_n(v),
\]

in which the pre-factor is set in the way that the eigen-functions are normalized, satisfying

\[
\int_{-1}^{1} \sqrt{1 - v^2} U_n^l(v) U_{n'}^l(v) = \delta_{nn'}.
\]

It is reminded that the original \( U_n(v) \) is not normalized to one, and in fact \( U_0^0 = \sqrt{2} U_n \). Also, as \( U_n \) is a polynomial of degree \( n \), by construction \( U_n^l \equiv 0 \), for \( n < l \). As the associated Chebyshev functions are rather less available, in the
Appendix A explicit expressions for them together with the plots are presented. Readily, by the basis constructed by $U^l_n$’s, a representation of δ-function in $v$-space is given
\[
\sum_{n=l}^{\infty} U^l_n(v)U^l_n(v') = \frac{\delta(v - v')}{\sqrt{1 - v'^2}}. \tag{57}
\]
By these all, the eigenvalues of the operator $x \cdot x$ happen to be $n^2(n^2 + 1)\ell^2$, for $l = 0, 1, \cdots, n$, leading to the degeneracy $2(n^2) + 1$. Reminding that the coordinates $\hat{x}_\alpha$’s satisfy the algebra (22) for SU(2), this result is the one to be expected. So every wave-function with orbital angular momentum $l$ can be expanded in terms of $U^l_n$’s, namely
\[
\psi_l(v) = \sum_{n=l}^{\infty} a_n U^l_n(v). \tag{58}
\]
By the above expansion, the probability that the particle would be found at the radial site $r_n = \sqrt{\frac{n^2(n^2 + 1)}{2}}\ell$ is proportional to $|a_n|^2$.

Eq. (54) for $l = 0$, as a second order differential equation, also has another linearly independent solution, usually denoted by $W_n$ [37]. This solution is normalizable, but diverging as $v \to \pm 1$. Similar the construction for $U^l_n$, one can generate the solutions for $l \neq 0$, denoted by $W^l_n$:
\[
W^l_n(v) = \sqrt{\frac{2}{\pi}} \frac{(n - l)! (n + 1)!}{(n + l + 1)!} (1 - v^2)^{l/2} \frac{d^l}{dv^l} W_n(v). \tag{59}
\]
It is seen that these associated solutions are neither normalizable nor finite within the interval $[-1, 1]$. Further, as the original $W_n$’s are not in polynomial form, it can be seen that $W^l_n \neq 0$ for $n < l$. It will be seen later that due to this property these functions can not appear as coefficients in expansion (58) in regions containing sites with radial sites $n < l$. The behavior of the two solutions and their associates best can be obtained by the trigonometric function representation of them [37], namely:
\[
U_n(x) = \frac{\sin(n + 1)\alpha}{\sin \alpha}, \tag{60}
\]
\[
W_n(x) = \frac{\cos(n + 1)\alpha}{\sin \alpha}, \tag{61}
\]
with $x = \cos \alpha$. To express the solutions in the exponential form one can define the linear combinations:
\[
V^l_{\pm n} := W^l_n \pm iU^l_n. \tag{62}
\]
This exponential representations appear useful to express the wave-functions in terms of oppositely oscillating radial waves. Also, the behavior of the above functions at large-$n$ beyond their defining interval $-1 \leq v \leq 1$ would come in forms of exponentially decreasing and growing functions of $n$. As we will see in next section, these functions might appear as the coefficients $a_n$’s in the
expansion (58) in the regions with constant potential; for example in tails of the bound-state solutions of the radial stepwise potentials. In fact, by the above form the behaviors of the linear combinations are summarized at large-$n$ as:

\[ n \to \infty : \begin{cases} |V_{\pm n}(x)| \propto \exp(\mp n \eta), & x > 1, \\ |V_{\pm n}(x)| \propto \exp(\pm n \eta), & x < -1, \end{cases} \tag{63} \]

in which $\cosh^{-1}|x| = \eta > 0$.

It is helpful to remind the recurrence relation [38]

\[ 2v F_n^l = \alpha_{+n}^l F_{n+1}^l + \alpha_{-n}^l F_{n-1}^l \tag{64} \]

with $F$’s are either $U$, $W$, or $V_{\pm n}$ types, and

\[ \alpha_{+n}^l = \sqrt{(n + l + 2)(n - l + 1)}(n + 1)(n + 2), \tag{65} \]

\[ \alpha_{-n}^l = \sqrt{(n + l + 1)(n - l)}n(n + 1). \tag{66} \]

We mention

\[ \alpha_{+n}^l = \alpha_{-(n+1)}^l, \quad \alpha_{-n}^l = \alpha_{+(n-1)}^l, \tag{67} \]

It is easy to check that the above introduced functions not only satisfy the above identity in their defining domain $-1 \leq v \leq 1$, but also formally on the whole real axes, $-\infty < v < \infty$. As the kinetic term is linear in $v$, the above identity comes extremely helpful to obtain the recurrence relations between the coefficients of the trial expansions. As an illustration, let us consider the case of a free particle. By (45), the energy eigen-function in momentum space takes the form:

\[ i(v, \cos^{-1} \tau, \chi|E) = \psi_{E_l}(v) Y^m_l(\cos^{-1} \tau, \chi) \tag{68} \]

in which $(\cos^{-1} \tau, \chi)$ specify the direction of the momentum, and $v$ is related to the momentum by $v = \cos(\ell p/(2\hbar))$. As for a free particle momentum commutes with the Hamiltonian, its energy eigen-function is proportional to $\delta$-function in $v$-space. Using the representation (57), for a free particle with momentum $p_0$, with $v_0 = \cos(\ell p_0/(2\hbar))$, we have:

\[ \psi_{E_l}(v) = c \delta(v - v_0) \propto \sum_{n=l}^{\infty} U_n^l(v) U_n^l(v_0). \tag{69} \]

It would be instructive to check the above result by the use of the expansion (58). By the Hamiltonian of free particle,

\[ H = \frac{\hbar^2}{M\ell^2} (1 - v), \tag{70} \]
and the identity (64), the equation $H\psi = E\psi$ would lead to the recurrence relation for the coefficients in the expansion (58):

$$2 \left(1 - \frac{M\ell^2 E}{4\hbar^2}\right) a_n = \alpha_{-n}^l a_{n-1} + \alpha_{+n}^l a_{n+1},$$

with the boundary condition $a_{l-1} = 0$. Defining

$$v_0 := 1 - \frac{M\ell^2 E}{4\hbar^2},$$

and a fresh use of the identity (64), we find $a_n \propto U_n^l(v_0)$, as confirmation of the result (69). In an alternative way, one may choose the linear combination

$$a_n = C^+ V_{+n}^l(v_0) + C^- V_{-n}^l(v_0), \quad n = l, l+1, \cdots,$$

for which by the condition $a_{l-1} = 0$, we find $C^+ = -C^-$, leading to the previous result. By the condition $-1 \leq v_0 \leq 1$ for detectable particles, we find

$$0 \leq E \leq \frac{8\hbar^2 M\ell^2}{L^2},$$

expressing that the energy of a free particle with its momentum taking values on a compact space would be bounded from above.

### 4 Radial stepwise potential

By the quantized radial distance as $r_n = \sqrt{\frac{n\hbar}{\frac{\pi}{2} + 1}} \ell$, the radial stepwise potentials may be defined by

$$V(r_n) = \begin{cases} 
\pm V_0, & n \leq n_0 \quad \text{(region I)} \\
0, & n > n_0 \quad \text{(region II)} 
\end{cases}$$

in which $+V_0$ and $-V_0$ correspond to the radial barrier and the radial square well potentials, respectively. As mentioned for the free particle, the total energy eigen-function in momentum space has the form (68), for which the dependence on $v$ can be expanded as (58). For the regions I and II of the potential, the recurrence relations for the coefficients are found as below:

$$n \leq n_0 : \quad 2 v_1 a_n = \alpha_{-n}^l a_{n-1} + \alpha_{+n}^l a_{n+1}$$

$$n > n_0 : \quad 2 v_{II} a_n = \alpha_{-n}^l a_{n-1} + \alpha_{+n}^l a_{n+1},$$

in which

$$v_1 := 1 - \frac{M\ell^2}{4\hbar^2} (E \mp V_0),$$

$$v_{II} := 1 - \frac{M\ell^2 E}{4\hbar^2},$$

...
accompanied by the boundary condition $a_{l-1} = 0$. By the condition $n \geq l$ mentioned in the previous section, the more interesting cases happen when $l < n_0$, for which the recurrence relation in region I has nonzero solution. By the properties of $U_n^l$'s mentioned before and the boundary condition, for the region I the acceptable solution comes in the form

$$a_n = C_I U_n^l(v_I), \quad n = l, \cdots, n_0 + 1 \quad (80)$$

For the region II, based on the behavior of the eigen-function for $n \to \infty$, either $V_n^l \pm$ with the argument inside the interval $[-1, 1]$, or exponentially decreasing $V_n^l$ type outside the interval by (63), are acceptable solutions.

Based on the condition $|v| \leq 1$ for directly detectable particles, two different situation should be studied separately, which are: 1) $8\hbar^2/(M\ell^2) > V_0$, and 2) $8\hbar^2/(M\ell^2) < V_0$.

### 4.1 Case with $8\hbar^2/(M\ell^2) > V_0$

Here we consider the barrier and square well cases separately.

**Barrier case:**

In this case three domains for $E$ are recognized, for each one the corresponding $v_{I & II}$ are mentioned:

$$0 \leq E \leq V_0: \quad v_I \geq 1 \quad \text{and} \quad |v_{II}| \leq 1$$

$$V_0 \leq E \leq \frac{8\hbar^2}{M\ell^2}: \quad |v_I| \leq 1 \quad \text{and} \quad |v_{II}| \leq 1$$

$$\frac{8\hbar^2}{M\ell^2} \leq E \leq \frac{8\hbar^2}{M\ell^2} + V_0: \quad |v_I| \leq 1 \quad \text{and} \quad v_{II} \leq -1 \quad (81)$$

Out of three domains mentioned in above we have $|v_{I & II}| > 1$, which are not acceptable for a particle detectable in either region I or II. For the first two domains in above, $v_{II}$ takes the values for which in region II the particle can make oppositely oscillating waves. So

$$0 \leq E \leq \frac{8\hbar^2}{M\ell^2}: \quad a_n = C_{II}^+ V_{+n}^l(v_{II}) + C_{II}^- V_{-n}^l(v_{II}), \quad n = n_0, \cdots, \infty \quad (82)$$

The continuity condition between two regions at $n_0$ and $n_0 + 1$ would give the relations between three pre-factors $C_I$ and $C_{II}^\pm$, and no condition on energy would be required. As the consequence, in the domains the energy spectrum is continuous. We mention that this domain of energy reaches the upper bound for the energy of a free particle obtained in the previous section.

For the third domain for energy, however, the particle can not have a propagating nature in the region II, and so the wave-function should vanish exponentially as $n \to \infty$. So, by $v_{II} < -1$ in third domain and (63), we have

$$\frac{8\hbar^2}{M\ell^2} \leq E \leq \frac{8\hbar^2}{M\ell^2} + V_0: \quad a_n = C_{II}^- V_{-n}^l(v_{II}), \quad n = n_0, \cdots, \infty \quad (83)$$
In this case the two continuity equations, namely

\begin{align}
C_I U_{n_0}^l (v_I) &= C_{II}^- V_{-n_0}^l (v_{II}) \\
C_I U_{n_0+1}^l (v_I) &= C_{II}^- V_{-(n_0+1)}^l (v_{II}),
\end{align}

are sufficient to fix the relation between two pre-factors $C_I$ and $C_{II}^+$, provided that the determinant of the equations would vanish, leading to

$$U_{n_0}^l (v_I) V_{-(n_0+1)}^l (v_{II}) - V_{-n_0}^l (v_{II}) U_{n_0+1}^l (v_I) = 0.$$  \(86\)

The last expression is in fact the quantization condition, by which a discrete set of the energies in the third interval is obtained. As for these kinds of solutions $V_{-n}^l (v_{II}) \rightarrow 0$ by $n \rightarrow \infty$, these states with discrete energies are bound-ones. The surprising feature of these bound-states is that they are obtained with an initially supposed repulsive potential of a barrier. It is in fact the result of the bound on the momentum. In particular, although it is expected that outside the repulsive region I the momentum would grow, but due to the bound, outside the repulsive region would appear forbidden for particle to propagate.

**Square well case:**

Also in this case three domains for $E$ are recognized:

\begin{align*}
- V_0 &\leq E \leq 0 : \quad |v_I| \leq 1 \text{ and } v_{II} \geq 1 \\
0 &\leq E \leq - V_0 + \frac{8 \hbar^2}{M \ell^2} : \quad |v_I| \leq 1 \text{ and } |v_{II}| \leq 1 \\
- V_0 + \frac{8 \hbar^2}{M \ell^2} &\leq E \leq - \frac{8 \hbar^2}{M \ell^2} : \quad v_I \leq -1 \text{ and } |v_{II}| \leq 1
\end{align*}

For the first interval, the acceptable solutions in region II are of the form of (83), but with $V_{-n}$ replaced by $V_{+n}$. So the quantization condition comes to the form

$$U_{n_0}^l (v_I) V_{+(n_0+1)}^l (v_{II}) - V_{+n_0}^l (v_{II}) U_{n_0+1}^l (v_I) = 0.$$  \(88\)

In this case the energy spectrum is discrete, and the eigen-functions, as expected, are bound-states.

For the second and third energy intervals the acceptable solutions are of the form of (82), and we encounter with asymptotically free states. The surprising feature in this case is with the third interval, for which, although the energy is higher, but due to the bound on the maximum momentum, it is forbidden for the particle to propagate in the attractive region I.

### 4.2 Case with $8 \hbar^2/(M \ell^2) < V_0$

Here also we consider the barrier and square well cases separately.

**Barrier case:**

In this case two domains for $E$ are recognized, for each one the corresponding
Table 1: The numerical values for the bound-states energy values by the quantization condition (86) for the barrier potential. The presented values are for \(4\hbar^2/(M\ell^2) = 2\), \(n_o = 6\), \(V_0 = 10\).

\[
\begin{array}{cccccccc}
\ell = 0 & E_0 & E_1 & E_2 & E_3 & E_4 & E_5 & E_6 \\
\ell = 1 & -13.696 & 13.144 & 12.419 & 11.628 & 10.889 & 10.318 &  \\
\ell = 2 & - & -13.497 & 12.823 & 12.025 & 11.218 & 10.525 &  \\
\ell = 3 & - & - & -13.250 & 12.455 & 11.592 & 10.781 &  \\
\end{array}
\]

\(\nu_{I \& II}\) are mentioned:

\[
0 \leq E \leq \frac{8\hbar^2}{M\ell^2} : \quad \nu_I > 1 \quad \text{and} \quad |\nu_{II}| \leq 1
\]

\[
V_0 \leq E \leq \frac{8\hbar^2}{M\ell^2} + V_0 : \quad |\nu_I| \leq 1 \quad \text{and} \quad \nu_{II} \leq -1
\]  

By the same reasonings of the previous part, in the first interval the energy spectrum is continuous and in the second one is discrete (by condition (86)). Also, again due to the bound on momentum, in the second interval region II is forbidden for particle to propagate, and so states are surprisingly bound-ones. Also in this case there is a gap in the interval \([-\frac{8\hbar^2}{M\ell^2}, V_0]\) between the continuous (lower) and the discrete (higher) parts of the spectrum.

**Square well case:**

Also in this case two domains for \(E\) are recognized:

\[
-V_0 \leq E \leq -V_0 + \frac{8\hbar^2}{M\ell^2} : \quad |\nu_I| \leq 1 \quad \text{and} \quad \nu_{II} > 1
\]

\[
0 \leq E \leq \frac{8\hbar^2}{M\ell^2} : \quad \nu_I \leq -1 \quad \text{and} \quad |\nu_{II}| \leq 1
\]  

It is easy to see that in the first interval the energy spectrum is discrete (by condition (88)), and in the second one is continuous. Also, due to the bound on momentum, in the second interval the particle can not propagate in the region I, although this region is attractive. Also in this case there is a gap in the interval \([-V_0 + \frac{8\hbar^2}{M\ell^2}, 0]\) between the discrete (lower) and the continuous (higher) parts of the spectrum.

### 4.3 Numerical samples

The spectrum of energy for stepwise potentials obtained in the previous subsections can be checked in both discrete and continuous parts by approximate methods; for example by the basis \(\{U_n^I\}\) in the Rayleigh-Ritz perturbation method [41]. Here for the discrete part of spectrum we give the samples of the numerical solutions for (86) or (88), for both barrier and square well potentials, presented in Tables 1 & 2, respectively.
| $l = 0$ | $E_0$ | $E_1$ | $E_2$ | $E_3$ | $E_4$ | $E_5$ | $E_6$ |
|-------|-------|-------|-------|-------|-------|-------|-------|
|       | -9.8510 | -9.4258 | -8.7858 | -8.0251 | -7.2571 | -6.5995 | -6.1564 |
| $l = 1$ | -9.6960 | -9.1448 | -8.4199 | -7.6283 | -6.8896 | -6.3185 |
| $l = 2$ | -9.4972 | -8.8239 | -8.0251 | -7.2183 | -6.5254 |
| $l = 3$ | -9.2502 | -8.4558 | -7.5920 | -6.7810 |

Table 2: The numerical values for the bound-states energy values by the quantization condition (88) for the square-well potential. The presented values are for $4\hbar^2/(M\ell^2) = 2$, $n_0 = 6$, $V_0 = 10$.

5 General potential

Here we consider the case with a general rotationally invariant potential. As the result, it is shown that in general the eigenvalue problem would end to solve a 3-term recurrence relation, or equivalently to find the stable points of a corresponding continued fraction. Using

$$ (x \cdot x) |n\rangle_l = \frac{\ell^2}{4} n(n + 2) |n\rangle_l $$

we have

$$ l \langle n | V(\sqrt{x \cdot x}) |\psi\rangle = V \left( \sqrt{\frac{n}{2}} \left( \frac{n}{2} + 1 \right) \ell \right) \psi_l(n). $$

By using the identity (64), the equation $H\psi_E = E\psi_E$ would lead to the recurrence relation for the coefficients of expansion (58):

$$ 2 \left( 1 - \frac{M\ell^2}{4\hbar^2} (E - V_n) \right) a_n = \alpha_{-n} a_{n-1} + \alpha_{l+n} a_{n+1} $$

in which

$$ V_n := V \left( \sqrt{\frac{n}{2}} \left( \frac{n}{2} + 1 \right) \ell \right), $$

accompanied by the boundary condition $a_{l-1} = 0$. The treatment of these kinds of recurrence relations is rather standard [39, 40]. Defining

$$ N_n := \frac{a_{n+1}}{a_n} $$

the above 3-term recurrence relation can be transformed to

$$ N_{n-1} = \frac{\alpha_{-n}}{2 \left( 1 - \frac{M\ell^2}{4\hbar^2} (E - V_n) \right) - \alpha_{l+n} N_n} $$

with $n = l, l+1, \cdots$. We mention, following the condition $a_{l-1} = 0$, $N_{l-1} \to \infty$. As consequence, by setting $n = l$ in above the denominator should vanish, leading to

$$ N_l = \sqrt{2(l + 2)} \left( 1 - \frac{M\ell^2}{4\hbar^2} (E - V_l) \right). $$
On the other hand, one can express $N_l$ by means of continued fractions, namely

$$N_l = \frac{\alpha_{l+1}^l}{\beta_{l+1} - \frac{\alpha_{l+1}^l}{\beta_{l+2} - \frac{\alpha_{l+2}^l}{\beta_{l+3} - \ldots}}}$$

in which we have used the relation $\alpha_{l+1}^l = \alpha_{l_n}$, and defining

$$\beta_n := 2 \left(1 - \frac{M\ell^2}{2\hbar^2} (E - V_n)\right).$$

In practice, firstly one should determine the limiting value of $N_n$ for $n \to \infty$. Then by equating the two values for $N_l$ by (97) and a truncated form of (98) by the limiting value, one can get an equation by which the energy eigenvalues could be evaluated. The accuracy as well as the number of obtained eigenvalues would be determined by the level of truncation of the continued fraction (98). Hence, the desired accuracy could be reached by sufficiently large level of truncation [40], say $n_{\infty}$, by which (98) takes the form

$$N_l = \frac{\alpha_{l+1}^l}{\beta_{l+1} - \frac{\alpha_{l+1}^l}{\beta_{l+2} - \frac{\alpha_{l+2}^l}{\beta_{l+3} - \ldots} \frac{\alpha_{l+n_{\infty}-1}^l}{\beta_{l+n_{\infty}} - \alpha_{l+n_{\infty}}^l} N_{n_{\infty}}}}. \quad (100)$$

Once the energy eigenvalues are determined by the desired accuracy, the eigenfunctions can be constructed by solving the recurrence relations (93) for $a_n$’s, accompanied by appropriate boundary and normalization conditions.

In the following we apply this method to the cases with the harmonic oscillator and the coulomb potentials.

### 5.1 Harmonic oscillator

The harmonic oscillator potential is taken as $\frac{1}{2} M \omega^2 (\mathbf{x} \cdot \mathbf{x})$, by which we have

$$N_l = \sqrt{2(l+2) \left(1 - \frac{M\ell^2}{4\hbar^2} \left(E - \frac{1}{8} M \omega^2 \ell^2(l + 2)\right)\right)} . \quad (101)$$

At large $n$ we assume $N_n \propto c n^\gamma$, by which after inserting in (96), we find

$$\gamma = -2, \quad c = \frac{16 \hbar^2}{M^2 \ell^4 \omega^2} , \quad (102)$$

leading to $N_n \to 0$ as $n \to \infty$. So truncation at level $n_{\infty}$ would be hold by $N_{n_{\infty}} \approx 0$ in (100). Equating (101) and the truncated form of (100) leads to the equation for eigenvalues.

In Table 3 samples of the numerical solutions by the method as the energy eigenvalues are given. All the given numbers can be checked also by the approximation methods, for example the Rayleigh-Ritz method.
| $l = 0$ | $E_0$ | $E_1$ | $E_2$ | $E_3$ | $E_4$ | $E_5$ | $E_6$ |
|---|---|---|---|---|---|---|---|
| 0 | -∞ | -16.614 | -9.5004 | -6.5087 | -4.8372 | -3.7717 | -3.0371 |
| $l = 1$ | - | -16.568 | -9.4921 | -6.5056 | -4.8357 | -3.7708 | -3.0366 |
| $l = 2$ | - | - | -9.4766 | -6.4995 | -4.8326 | -3.7690 | -3.0355 |
| $l = 3$ | - | - | - | -6.4908 | -4.8283 | -3.7666 | -3.0340 |

Table 3: The energy eigenvalues for the harmonic oscillator, setting $\hbar^2/(M\ell^2) = 2$, $\omega = 2$, level of truncation: 14.

| $l = 0$ | $E_0$ | $E_1$ | $E_2$ | $E_3$ | $E_4$ | $E_5$ | $E_6$ |
|---|---|---|---|---|---|---|---|
| 0 | -∞ | -16.614 | -9.5004 | -6.5087 | -4.8372 | -3.7717 | -3.0371 |
| $l = 1$ | - | -16.568 | -9.4921 | -6.5056 | -4.8357 | -3.7708 | -3.0366 |
| $l = 2$ | - | - | -9.4766 | -6.4995 | -4.8326 | -3.7690 | -3.0355 |
| $l = 3$ | - | - | - | -6.4908 | -4.8283 | -3.7666 | -3.0340 |

Table 4: The energy eigenvalues for the Coulomb potential, setting $\hbar^2/(M\ell^2) = 2$, $e^2/\ell = 16$, level of truncation: 14.

5.2 Coulomb potential

The Coulomb potential is taken as $V(\sqrt{x \cdot x}) = -e^2/\sqrt{x \cdot x}$, by which we have

$$N_l = \sqrt{2(l + 2)} \left( 1 - \frac{M\ell^2}{4\hbar^2} \left( E + \frac{2e^2}{\ell \sqrt{l(l + 2)}} \right) \right).$$

In this case we have both the bound-states for $E < 0$, as well as the asymptotically free states with $E > 0$. Here we consider only the case $E < 0$. At large $n$ we assume $N_n \propto c n^2$, by which after inserting in (96), it would be found

$$\gamma = 0, \quad c = 1 - \frac{M\ell^2}{4\hbar^2} \left( E \pm \sqrt{E^2 - \frac{8\hbar^2}{M\ell^2} E} \right) =: c_\pm.$$ 

As far as the aim is to find the energy eigenvalues, both $c_\pm$ can be used as the limiting value in (100). So, equating (103) and truncated form of (100) by $N_n \approx c_\pm$, would yield the required equation for the discrete eigenvalues of bound-states. However, to obtain the coefficients in the expansion the situation is different. It is easy to check

$$c_+ + c_- = 1, \quad c_+ > c_-,$$

by which we have $c_+ > 1$ and $c_- < 1$. As $N_n$ is giving the ratio $a_{n+1}/a_n$, to have normalizable bound-states of the form (58) only $c_-$ can be accepted to solve the recurrence relation (93).

In Table 4 samples of the numerical solutions by the method as the energy eigenvalues are given. Also in this case the given numbers can be checked by the approximation methods, for example the Rayleigh-Ritz method.

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A  Samples of $U_n^l$’s and their plots

Here some explicit expressions of the associated Chebyshev polynomials $U_n^l$’s, and their plots are presented.

Associated Chebyshev functions $U_n^l(x)$

\[
\begin{align*}
U_1^1(x) &= 2\sqrt{\frac{2}{3\pi}} \sqrt{1-x^2} \\
U_1^2(x) &= \frac{1}{2\sqrt{4}} \frac{3\pi}{\sqrt{1-x^2}} \\
U_1^3(x) &= \sqrt{\frac{2}{15\pi}} \sqrt{1-x^2}(24x^2 - 4) \\
U_1^4(x) &= \frac{1}{2\sqrt{3\pi}} \sqrt{1-x^2}(64x^3 - 24x) \\
U_1^5(x) &= \sqrt{\frac{2}{35\pi}} \sqrt{1-x^2}(160x^4 - 96x^2 + 6) \\
U_1^6(x) &= \frac{1}{2\sqrt{6\pi}} \sqrt{1-x^2}(384x^5 - 320x^3 + 48x)
\end{align*}
\]

![Figure 1: Plots $U_n^l(x)$, $n = 1, 2, 3, 4.$](image)

Associated Chebyshev functions $U_n^2(x)$

\[
\begin{align*}
U_2^2(x) &= \frac{4}{\sqrt{5\pi}} (1-x^2) \\
U_2^3(x) &= \frac{1}{\sqrt{3\pi}} (1-x^2) \left(192x^2 - 24\right) \\
U_2^4(x) &= \frac{1}{6\sqrt{4\pi}} (1-x^2) \left(640x^3 - 192x\right) \\
U_2^5(x) &= \frac{1}{8\sqrt{5\pi}} (1-x^2) \left(1920x^4 - 960x^2 + 48\right) \\
U_2^6(x) &= \frac{1}{10\sqrt{6\pi}} (1-x^2) \left(5376x^5 - 3840x^3 + 480x\right)
\end{align*}
\]
Figure 2: Plots of $U_n^2(x)$, $n = 1, 2, 3, 4$.

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