SCALING LIMIT FOR A LONG-RANGE DIVISIBLE SANDPILE

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Abstract. We study the scaling limit of a divisible sandpile model associated to a truncated α-stable random walk. We prove that the limiting distribution is related to an obstacle problem for a truncated fractional Laplacian. We also provide, as a fundamental tool, precise asymptotic expansions for the corresponding rescaled discrete Green’s functions. In particular, the convergence rate of these Green’s functions to its continuous counterpart is derived.

1. Introduction

In this paper we study the divisible sandpile model, which is a continuous version of the Abelian sandpile introduced in 1987 by Bak, Tang and Weisenfeld [1] as an example of a dynamical system displaying self-organized criticality.

Up to our knowledge, the divisible sandpile was introduced by Levine and Peres in [10]. The main difference between the divisible sandpile and the Abelian sandpile is that instead of discrete particles of unit mass, each site can contain a continuous amount of mass. At the start there is an initial density of mass distributed on the lattice \( \mathbb{Z}^d \). A lattice site is said to be a full site if it has mass at least 1. At each time step, each full site is toppled by keeping mass 1 for itself and distributing the excess mass among the lattice proportionally to the step distribution of a certain transition probability. As time goes to infinity the mass approaches a final distribution in which each site has mass less than or equal to 1.

In [10] and [11], Levine and Peres studied the scaling limit of this final distribution for the divisible sandpile in which the excess mass is split in every toppling equally among neighbors. In other words, they study the divisible sandpile model associated to the transition probability of a symmetric, simple random walk on \( \mathbb{Z}^d \). The proof is based within the framework of simple random walks. The corresponding scaling limit is related to the obstacle problem for the classical Laplacian operator. In [12] Lucas studied a divisible sandpile model that he calls the unfair divisible sandpile, in which the toppling procedure distributes mass to neighbors proportionally to a nearest neighbors, drifted random walk. The corresponding scaling limit of the final distribution is related to a so-called true heat ball.

The so-called internal Diffusion Limited Aggregation model (iDLA) is a stochastic version of the Abelian sandpile introduced by Lawler, Bramson and Griffeath in [8]. In the mentioned paper, the authors studied the scaling limit of the random set of final occupied sites starting with \( m \) particles at the origin, which happens to be a ball with volume \( m \) in \( \mathbb{R}^d \). In [11] Levine and Peres also studied the scaling limit of the iDLA but starting from an arbitrary distribution of mass, they proved that...
considering the same initial distribution in the divisible sandpile and in the iDLA, the scaling limit of the final set of occupied sites in both models happens to be the same. In [12], a drifted version of the iDLA was also studied and the scaling limit was also proved to coincide with the scaling limit of the drifted divisible sandpile.

Regarding the Abelian sandpile model some recent advances correspond to Pegden and Smart, they proved in [13] that the Abelian sandpile converges in the weak-* topology to a limit characterized by an elliptic obstacle problem. Later in [9], Levine, Pegden and Smart identified Apollonian structures in the scaling limit of the Abelian sandpile. Structures of the limit of the Abelian sandpile were previously observed in [14].

Our aim is to study a family of divisible sandpiles, which we call truncated, $\alpha$-stable divisible sandpiles, that distribute mass not only to nearest neighbors. We present a scaling limit for the final distribution of a sequence of divisible sandpiles on which the excess mass is distributed proportionally to the transition probability of a truncated $\alpha$-stable random walk. Figure 1 represents a simulation of the truncated $\alpha$-stable sandpile starting with mass in two different lattice sites.

Green’s function estimates for the truncated $\alpha$-stable random walk are a key tool in this work. In the context of simple random walks, such estimates are well known. A comprehensive reference, which also shows the power of these estimates as a tool in a variety of contexts, is Lawler’s book [6]. In [5], Le Gall and Rosen obtained powerful estimates for Green’s functions of random walks in the domain of attraction of stable laws.

In the specific context of truncated $\alpha$-stable random walks with $\alpha \in (1, 2)$ in dimension $d = 2$, we present in Theorem 1.3 a precise asymptotic expansion for the corresponding rescaled discrete Green’s functions. In particular, the convergence rate of these Green’s functions to its continuous counterpart is derived. This asymptotic expansion is the most technical part of this work, and it is of independent interest. In particular, the optimal convergence rate $n^{-\alpha}$ is obtained only after the introduction of a non-trivial, nearest-neighbor correction to the transition probability of the truncated $\alpha$-stable sandpile. Without this correction term, the convergence rate can be checked to be $n^{\alpha-2}$.

Figure 1. Truncated $\alpha$-stable sandpile starting with mass 700 and 1300 in two different sites. In each picture we use a scale of color that indicates the amount of mass at each site.

(a) 3400 iterations

(b) 26500 iterations
Let \( \rho \) be a non-negative function on \( \mathbb{R}^2 \) with compact support. We will use this function \( \rho \) to choose the initial mass density of the truncated \( \alpha \)-stable sandpile. As the lattice spacing goes to zero, we run the sandpile with initial distribution given by \( \rho \) restricted to the lattice. Our main result, Theorem 1.2 states that the final distribution of the rescaled sandpile converges and that its limit is related to an obstacle problem for the truncated fractional Laplacian. Obstacle problems are a current subject of interest in the literature, see for example [2], [4]. However, up to our knowledge the obstacle problem we need to deal with in this work has not been considered in the literature.

1.1. Notations and results. The next few paragraphs are devoted to describe the random walk we use in the toppling procedure of the truncated \( \alpha \)-stable divisible sandpile model.

Let us introduce the following notations:

For \( R > 0 \), we denote \( B_R = \{ x \in \mathbb{R}^2 ; |x| < R \} \). For a domain \( A \) in \( \mathbb{R}^2 \), we denote \( A^\circ = A \cap \mathbb{Z}^2 \). We define the discrete boundary of the domain \( A \), as

\[
\partial A^\circ = \{ x \in \mathbb{Z}^2 ; |x - \frac{1}{2}, x + \frac{1}{2}|^2 \notin A \text{ and } |x - \frac{1}{2}, x + \frac{1}{2}|^2 \cap A \neq \emptyset \}.
\]

We fix a parameter \( \alpha \in (1, 2) \), and two positive parameters \( r < M \). We define \( A_{r,M} := B_M \setminus B_r \). Now we will consider a function \( F_1 = F_1_{r,M} : \mathbb{Z}^2 \rightarrow \mathbb{R} \), which is essentially the indicator of the set \( A_{r,M}^\circ \), but for computational convenience, our function \( F_1 \) takes specific values in \( \partial A_{r,M}^\circ \). We define \( F_1(y) \) as the proportion of the square \( |y - \frac{1}{2}, y + \frac{1}{2}|^2 \) that is inside the set \( A_{r,M} \), that is

\[
F_1(y) = \mu(|y - \frac{1}{2}, y + \frac{1}{2}|^2 \cap A_{r,M}), \text{ for } y \in \mathbb{Z}^2,
\]

where \( \mu \) represents the Lebesgue measure in \( \mathbb{R}^2 \). Notice that \( F_1 \) is equal to 1 in \( A_{r,M}^\circ \setminus \partial A_{r,M}^\circ \), and equal to 0 outside \( A_{r,M}^\circ \cup \partial A_{r,M}^\circ \).

We define the probability jump \( p_1 : \mathbb{Z}^2 \rightarrow [0, 1] \) as

\[
p_1(y) = c_1 \left( k_1 1_{\{|y|=1\}} + \frac{F_1(y)}{|y|^{2+\alpha}} \right) \text{ for } y \in \mathbb{Z}^2 \setminus \{0\}
\]  

(1.1)

and \( p_1(0) = 0 \), where \( c_1 \) is the normalizing constant and \( k_1 \) is a positive constant, which depends on \( \alpha, r \) and \( M \), defined as

\[
k_1 = \frac{1}{4} \left( \int_{B_M} \frac{1}{|y|^{\alpha}} dy - \sum_{y \in A_{r,M}^\circ \cup \partial A_{r,M}^\circ} \frac{F_1(y)}{|y|^\alpha} \right).
\]  

(1.2)

We choose \( r \) and \( M \) in order to have \( k_1 > 0 \), see Lemma 1.1. Then the probability \( 1.1 \) is well defined.

In order to obtain the desired convergence of the Green’s function, instead of working with a pure truncated \( \alpha \)-stable random walk, we add a simple random walk multiplied by a carefully chosen constant. This is the constant \( k_1 \) defined in (1.2). This strategy leads us to cancellations that allows us to obtain convergence of order \( n^{-\alpha} \) as Theorem 1.3 states. In particular, the convergence stated in Theorem 1.3 does not hold without this correction.

Now we run in \( \mathbb{Z}^2 \) our sandpile model starting with initial distribution \( \rho \). When time goes to infinity, in Proposition 2.1 bellow we will prove that if every full site is toppled infinitely often, then the mass converges to a limiting distribution on which each site has mass taking values between 0 and 1.
Two individual topplings in the truncated $\alpha$-stable sandpile do not commute, but the sandpile is *Abelian* in the sense that the limiting distribution does not depend on the ordering of topplings. This is the subject of Proposition 2.2.

In the truncated $\alpha$-stable divisible sandpile, as in the classical sandpile model, we use the *odometer function* as the fundamental tool to identify the limiting mass distribution. The odometer function measures the total amount of mass emitted from the point $x \in \mathbb{Z}^2$, counted with repetitions:

$$u(x) = \text{total mass emitted from } x.$$ (1.3)

Also as a consequence of the Abelian property, the odometer function does not depend on the order of the topplings. The total mass received by site $x$ from other lattice points throughout the toppling procedure, is equal to

$$\sum_{y \in B_M \cup \partial B_M} p_1(y)u(x+y),$$

hence

$$L_{M,1}u(x) = \nu(x) - \rho(x),$$ (1.4)

for a function $f$ in $\mathbb{Z}^2$.

We say that a function $f$ is superharmonic at a point $x \in \mathbb{Z}^2$ with respect to $L_{M,1}$, if

$$L_{M,1}f(x) \leq 0 \text{ on } \mathbb{Z}^2 \text{ and } f \geq \gamma.$$ (1.5)

The following lemma gives us an expression for the odometer.

**Lemma 1.1.** Let $\rho$ be a nonnegative bounded function on $\mathbb{Z}^2$ with finite support. Then the odometer function for the truncated $\alpha$-stable sandpile started with mass $\rho$ satisfies

$$u = s - \gamma$$

where $\gamma$ satisfies $L_{M,1}\gamma = \rho - 1$ and $s$ given by (1.6) is the least superharmonic majorant of $\gamma$.

**Proof.** The odometer function is nonnegative, thus $u + \gamma \geq \gamma$. Since $\nu \leq 1$, by (1.4) we have $L_{M,1}(u + \gamma) \leq 0$, hence $s \leq u + \gamma$ by (1.6).

For the converse inequality let $f \geq \gamma$ be any superharmonic function lying above $\gamma$, then $L_{M,1}(f - \gamma - u) \leq 0$ in the domain $D = \{x \in \mathbb{Z}^2; \nu(x) = 1\}$ of fully occupied sites. Outside $D$, since $u$ vanishes $f - \gamma - u \geq 0$ and hence, using a maximum principle for $L_{M,1}$, we conclude that it is nonnegative everywhere. \qed

The expression obtained for $u$ in Lemma 1.1 is called *odometer’s equation*. Lemma 1.1 gives us a way to find the odometer function once we have a function $\gamma$ which satisfies $L_{M,1}\gamma = \rho - 1$. This function is called the obstacle function.
Consider the random walk \( \{S_m\}_{m \geq 0} \) starting at 0 whose jump probability is given by \( p_1 \), we define its Green’s function \( G_{M,1} \) as

\[
G_{M,1}(x) = \lim_{N \to \infty} \sum_{m=0}^{N} [p_1^m(0) - p_1^m(x)]
\]  

(1.7)

where \( p_1^m(x) = P[S_m = x] \) and

\[
\sigma_{M,1}^2 = L_{M,1}|x|^2 = \sum_{y \in B_M \cup \partial B_M} p_1(y)|y|^2.
\]

(1.8)

Notice that \( \sigma_{M,1}^2 \) is the variance of the random walk with transition probability \( p_1 \). Sometimes in literature the function \( G_{M,1} \) defined in (1.7) is called potential kernel, see for instance [6] or [7]. Using the Markov property we can check that \( L_{M,1}G_{M,1}(x) = 0 \) for \( x \neq 0 \) and \( L_{M,1}G_{M,1}(0) = -1 \).

The obstacle function \( \gamma \) can be, for instance, be equal to

\[
\gamma(x) = -\frac{|x|^2}{\sigma_{M,1}^2} - \sum_{y \in \mathbb{Z}^2} G_{M,1}(x - y) \rho(y) \quad \text{for all } x \in \mathbb{Z}^2.
\]

(1.9)

1.2. Scaling procedure. We will introduce a notation for the transition from the Euclidean space to the rescaled lattice \( \frac{1}{n}\mathbb{Z}^2 \).

For \( x \in \mathbb{R}^2 \) we write \( x^{n::} \) for the nearest point in \( \frac{1}{n}\mathbb{Z}^2 \) breaking ties to the right, that is

\[
x^{n::} = (x - \frac{1}{2n}, x + \frac{1}{2n}) \cap \frac{1}{n}\mathbb{Z}^2.
\]

For \( x \in \frac{1}{n}\mathbb{Z}^2 \), we write \( x^{n:\square} = x + [-\frac{1}{2n}, \frac{1}{2n}]^2 \).

For a function \( f \) in \( \mathbb{R}^2 \), we write \( f^{n::} \) for its restriction to the lattice, that is

\[
f^{n::} = f |_{\frac{1}{n}\mathbb{Z}^2}.
\]

For a function \( f \) in \( \frac{1}{n}\mathbb{Z}^2 \), we write \( f^{n:\square} \) for its extension to the Euclidean space, defined as \( f^{n:\square}(x) = f(x^{n::}) \).

For a domain \( A \subset \mathbb{R}^2 \), we write \( A^{n::} = A \cap \frac{1}{n}\mathbb{Z}^2 \).

If \( A \) is a domain in \( \frac{1}{n}\mathbb{Z}^2 \), we write \( A^{n:\square} = A + [-\frac{1}{2n}, \frac{1}{2n}]^2 \).

For a domain \( A \) in \( \mathbb{R}^2 \), we define its discrete boundary with respect to the rescaled lattice \( \frac{1}{n}\mathbb{Z}^2 \) as

\[
\partial A^{n::} = \{ x \in \frac{1}{n}\mathbb{Z}^2 : x^{n::} \notin A \text{ and } x^{n::} \cap A \neq \emptyset \}.
\]

In the above notations, the index \( n \) will be omitted whenever is clear from the context that we are making reference to rescaled lattice \( \frac{1}{n}\mathbb{Z}^2 \).

Recall the notation \( A^{\frac{1}{n}}_M = B_M \setminus B_{\frac{1}{n}} \). We run our sandpile model in \( \frac{1}{n}\mathbb{Z}^2 \) distributing the excess mass at each toppling, proportionally to the step distribution of the random walk with jump probability \( p_n : \frac{1}{n}\mathbb{Z}^2 \to [0,1] \), given by

\[
p_n(y) = c_n \left( k_n \mathbf{1}_{\{y \in \frac{1}{n}\mathbb{Z}^2, |y| = \frac{1}{n}\}} + \frac{F_n(y)}{n^{2+\alpha} |y|^{2+\alpha}} \right) \quad \text{for } y \in \frac{1}{n}\mathbb{Z}^2 \setminus \{0\}
\]

(1.10)

and \( p_n(0) = 0 \), where \( c_n \) is the normalizing constant and the function \( F_n \) is defined as

\[
F_n(y) = n^2 \times \mu(\{y\} \cap A^{\frac{1}{n}}_M) \quad \text{for } y \in \frac{1}{n}\mathbb{Z}^2.
\]

(1.11)

Recall that \( \mu \) denotes the Lebesgue measure in \( \mathbb{R}^2 \). Notice that \( F_n \) is equal to 1 in \( A^{\frac{1}{n}}_M \setminus \partial A^{\frac{1}{n}}_M \), and equal to 0 outside \( A^{\frac{1}{n}}_M \cup \partial A^{\frac{1}{n}}_M \).
The constant $c_n$ is the normalizing constant, and the constant $k_n$ is equal to

$$k_n = \frac{n^{2-\alpha}}{4} \left( \int_{B_M} \frac{1}{|y|^\alpha} dy - \frac{1}{n^2} \sum_{y \in B_M \cup \partial B_M} \frac{F_n(y)}{|y|^\alpha} \right).$$

(1.12)

As we mentioned before, we choose $r < M$ to ensure that $k_1$ defined in (1.2) is positive. The existence of such constants is the subject of Lemma 4.1. Moreover, as a consequence of the proof of Lemma 4.1 we have that $k_n$ is also positive for the same choice of the parameters $r$ and $M$. Note that the truncation remains macroscopically constant.

For a function $f$ defined in $\frac{1}{n}\mathbb{Z}^2$, define the discrete, truncated fractional Laplacian as

$$L_{M,n} f(x) = n^{\alpha} \sum_{y \in B_M \cup \partial B_M} p_n(y)(f(x + y) - f(x)).$$

(1.13)

Suppose we start the sandpile in $\frac{1}{n}\mathbb{Z}^2$ with initial distribution given by the function $\rho_n = \rho^{n,\cdot}$. Let us define the obstacle $\gamma_n$ as a function satisfying

$$L_{M,n} \gamma_n(x) = \rho_n(x) - 1.$$  

(1.14)

The obstacle function can be defined for instance as

$$\gamma_n(x) = -\frac{|x|^2}{\sigma_{M,n}^2} - \frac{1}{n^2} \sum_{y \in \frac{1}{n}\mathbb{Z}^2} G_{M,n}(x - y) \rho_n(y),$$

(1.15)

where $\sigma_{M,n}^2 = L_{M,n}|x|^2$ and $G_{M,n}$ is a function which satisfies $L_{M,n}G_{M,n} = -n^2 \delta_0$. We call $G_{M,n}$ the discrete Green’s function.

In Section 3 we discuss how we define the Green’s function $G_{M,n}$ and give an integral expression for it, that we anticipate here without proofs. All the details can be found in Section 3. For a fixed $x_0 \in \mathbb{R}^2$ with $|x_0| = 1$, define

$$G_{M,n}(x) = \frac{1}{(2\pi)^2} \int_{[-n\pi,n\pi]^2} \frac{\cos(\theta \cdot x) - \cos(\theta \cdot x_0)}{\psi_{M,n}(\theta)} d\theta,$$

(1.16)

where

$$\psi_{M,n}(\theta) = n^{\alpha} \sum_{y \in B_M \cup \partial B_M} p_n(y)(1 - \cos(\theta \cdot y)).$$

(1.17)

The least superharmonic majorant for the obstacle $\gamma_n$ is a function $s_n$ in $\frac{1}{n}\mathbb{Z}^2$ defined as

$$s_n(x) = \inf \{ f(x); L_{M,n} f \leq 0 \text{ on } \frac{1}{n}\mathbb{Z}^2 \text{ and } f \geq \gamma_n \}.$$  

(1.18)

By analogy with (1.3) we define the odometer function starting with mass $\rho_n$ in $\frac{1}{n}\mathbb{Z}^2$ as

$$u_n(x) := \frac{1}{n^2} \times \text{total amount of mass emitted from the site } x.$$  

(1.19)

It is easy to see that the odometer’s equation also holds, that is

$$u_n = s_n - \gamma_n.$$  

(1.20)

and the final distribution of mass $\nu_n$ starting the sandpile with initial distribution of mass $\rho_n$ is given by

$$\nu_n = \rho_n + L_{M,n} u_n.$$  

(1.21)
1.3. **Main results.** The odometer’s equation \((1.20)\) allows us to formulate our problem in a way that translates naturally to the continuum. Let \(c\) be defined as

\[
\lim_{n \to \infty} c_n,
\]

where \(c_n\) is the normalizing constant in \((1.10)\), in Section \(4\) we will prove that this limit exists and the convergence is of the order of \(n^{-\alpha}\).

The continuous counterpart of the discrete, truncated fractional Laplacian \(L_{M,n}\) defined in \((1.13)\) is the truncated fractional Laplacian whose integral form is given by

\[
L_M f(x) = \frac{c}{2} \int_{B_M} f(x + y) + f(x - y) - 2f(x) \frac{1}{|y|^{2+\alpha}} dy,
\]

(1.22)
defined for a twice differentiable, bounded function \(f\).

In Proposition \(4.3\) below, it becomes clear why \(L_M\) is the natural extension of \(L_{M,n}\) to the continuum.

Let us fix a function \(\rho\) in \(\mathbb{R}^2\) with compact support, which represents the initial mass density. We define the continuous obstacle function as

\[
\gamma(x) = -\frac{|x|^2}{\sigma_M^2} - \int_{\mathbb{R}^2} G_M(x - y)\rho(y)dy,
\]

(1.23)

where \(\sigma_M^2 = c \int_{B_M} \frac{1}{|y|^{2+\alpha}} dy\), and \(G_M\) is the Green’s function of the truncated fractional Laplacian defined as

\[
G_M(x) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{\cos(\theta \cdot x) - \cos(\theta \cdot x_0)}{\psi_M(\theta)} d\theta,
\]

(1.24)

where

\[
\psi_M(\theta) = c \int_{B_M} \frac{1 - \cos(\theta \cdot y)}{|y|^{2+\alpha}} dy
\]

(1.25)

and \(x_0 \in \mathbb{R}^2\) is arbitrary but fixed.

The idea behind the definition of the function \(G_M\) as the Green’s function of the truncated fractional Laplacian \(L_M\), is that \(G_M\) is, in some sense, the inverse of the operator \(L_M\). Such property will be discussed in Lemma \(4.4\).

We say that a function \(f\) on \(C^2\) is **superharmonic** with respect to \(L_M\) if \(L_M f \leq 0\). Since we will often be working with functions that are not twice differentiable, it is convenient to define the truncated fractional Laplacian in a more general setting.

We extend \(L_M\) by duality to a large class of distributions. A function \(f \in L^1_{\text{loc}}\) is a linear functional in the dual of the space of infinitely differentiable functions with compact support \(C_c^\infty\). The symmetry of the operator \(L_M\) allows us to define, for \(f \in L^1_{\text{loc}}\),

\[
\langle L_M f, \phi \rangle = \langle f, L_M \phi \rangle,
\]

where \(\phi \in C_c^\infty\). This definition coincides with the previous one in the case \(f \in C^2\).

We say a function \(f \in L^1_{\text{loc}}(\mathbb{R}^2)\) is superharmonic in an open set \(\Omega\) if for every nonnegative test function \(\phi \in C_c^\infty(\mathbb{R}^2)\) whose support is inside \(\Omega\), we have

\[
\langle f, L_M \phi \rangle \leq 0.
\]

(1.26)

Throughout the text, superharmonic function will always mean superharmonic with respect to \(L_M\).

Now we consider the least superharmonic majorant

\[
s(x) = \inf\{f(x); f \text{ is continuous, superharmonic and } f \geq \gamma\}.
\]

(1.27)

The majorant \(s\) is the solution of the obstacle problem for the truncated fractional Laplacian with obstacle \(\gamma\).
The odometer function for $\rho$ is given by
\[ u = s - \gamma. \] (1.28)

We say that a sequence of functions $f_n \in L^1_{loc}$ converges to a distribution $T$ in the weak-$*$ topology if for any test function $\phi \in C_c^\infty$ we have
\[ \lim_{n \to \infty} \langle f_n, \phi \rangle = \langle T, \phi \rangle. \]

Our main result states the convergence of the final distribution of the truncated $\alpha$-stable sandpile with respect to the weak-$*$ topology.

We denote by $C^2_c(\mathbb{R}^2)$ the space of the twice differentiable functions with compact support.

**Theorem 1.2.** Let $\alpha \in (1, 2)$. Consider $\rho \in C^2_c(\mathbb{R}^2)$ as the initial density of the truncated $\alpha$-stable sandpile. Let $\nu_n$ the final distribution of the sandpile in $\frac{1}{n}\mathbb{Z}^2$ starting with initial distribution $\rho_n = \rho^\circ$. Then
\[ \nu_n^{\Box} \to \rho + L_M u \]
in the weak-$*$ topology, where $u$ is the limiting odometer function defined in (1.28).

As we mentioned before, we obtained the next following convergence result for the Green’s functions.

**Theorem 1.3.** Let $G_{M,n}$ and $G_M$ be respectively the rescaled discrete Green’s functions and the continuous Green’s function defined in (1.16) and in (1.24). Then there exists a constant $K > 0$ which only depends on $\alpha$, such that for all $x \neq 0$ and all $n$, holds
\[ \left| G_M(x) - G_{M,n}(x) + \beta_n \log |x| \right| \leq \frac{K}{n^\alpha} \left(1 + \frac{1}{|x|^{2-\alpha}} + \frac{1}{|x|^2}\right) \] (1.29)

where $\beta_n$ is an explicitly computable constant (given in terms of the probability jumps) which converges to zero at rate $n^{-\alpha}$.

In particular $G_{M,n}$ converges towards $G_M$ uniformly on compacts of $\mathbb{R}^2 \setminus \{0\}$.

1.4. **Organization of the paper.** In this work we will follow the next structure

convergence of Green’s functions $\Rightarrow$ convergence of obstacles $\Rightarrow$ convergence of majorants $\Rightarrow$ convergence of odometers $\Rightarrow$ convergence of final distribution.

In Section 6 we treat the convergence of the Green’s functions, which is the subject of Theorem 1.3. In Section 7 we prove the convergence of obstacles as a consequence of Theorem 1.3. Although we do not use all the information that Theorem 1.3 gives in the proof of the convergence of obstacles, we do need such a strong result for the convergence of majorants. In Section 8 we prove the convergence of majorants and the convergence of the odometer functions. A fundamental tool for Section 8 is the continuity of the majorant (1.27) which is the subject of Section 5. In Section 9 we obtain our main result, Theorem 1.2. In Section 4 we put together some important analytical estimates that we will use throughout this work. Proof of various auxiliary results are in Sections 10 and 11.
2. Existence of the odometer function and Abelian property

We start with an initial distribution of mass $\rho$ in $\mathbb{Z}^2$ with finite support. At each time step we topple a full site, recall that a toppling in the site $x$ consists in leaving mass 1 to $x$ and distribute the excess mass among the lattice proportionally to the step distribution of the random walk (1.1).

In this section we prove that if every full site is toppled infinitely often, the mass distribution converges to a limiting distribution on which each site has mass less than or equal to 1. We also prove the Abelian property which states that the final distribution does not depend on the order of the topplings.

We define a toppling scheme $T$ as an infinite sequence of indexes in $\mathbb{Z}^2$ in which each full site that is initially full or becomes full after the realization of the previous toppling appears in the sequence infinitely often. We also say that a toppling scheme is legal if it only topple full sites.

Let $\nu_k$ be the mass distribution after the toppling of the first $k$ sites listed in $T$, and $u_k(x)$ the mass emitted from $x$ up to the $k$-th toppling.

Proposition 2.1. Suppose we start with an initial configuration $\rho$ with finite total amount of mass and bounded support. Consider a legal toppling scheme $T$ for this initial configuration. Then as $k$ goes to infinity, $\nu_k$ and $u_k$ tend to limits $\nu$ and $u$.
Moreover the limiting configuration $\nu$ satisfies $\nu \leq 1$ in $\mathbb{Z}^2$ and $\nu = \rho + L_{M,1}u$, for $L_{M,1}$ defined in (1.5).

Proof. We claim that there exists a bounded set $K$ which contains all the possible sites with positive mass throughout the realization of the topplings. To see that denote by $S$ the support of the initial distribution $\rho$ and let $m$ be the total amount of mass. If a lattice site $y$ outside $S$ becomes full at certain instant of the toppling scheme $T$, then there exists a sequence $y_1, y_2, \ldots, y_\ell$ of full sites such that $y_1 = y$, $y_\ell \in S$ and $y_{i+1} \in \{y_i\} + B_{M+1}$. Note that, since the mass is preserved, $\ell \leq m$.
Then all possible full sites are inside the set $\tilde{K} := S + B_{M+1}$. This allows us to conclude that the final set of sites with positive mass is contained in the set $K := \tilde{K} + B_{M+1}$.

Let us consider the function
$$W_k = \sum_{x \in \mathbb{Z}^2} \nu_k(x) \frac{|x|^2}{\sigma_{M,1}^2},$$
where $\sigma_{M,1}^2$ was defined in (1.8).

Note that $W_k$ is uniformly bounded.

Assume that $x$ is the $k$-th site to be toppled, then the mass at point $x$ is modified by $\beta_k(x) = \nu_{k-1}(x) - \nu_k(x)$, and this amount will be transfered to the lattice according to the toppling rule. Hence,
$$W_k(x) - W_{k-1}(x) = \sum_{y \in \mathbb{Z}^2} \left( \nu_k(y) - \nu_{k-1}(y) \right) \frac{|y|^2}{\sigma_{M,1}^2}$$
$$= -\beta_k(x) \frac{|x|^2}{\sigma_{M,1}^2} + \sum_{y \neq x} \beta_k(x) p_1(y - x) \frac{|y|^2}{\sigma_{M,1}^2}$$
$$= \beta_k(x) L_{M,1} \frac{|x|^2}{\sigma_{M,1}^2}$$
$$= \beta_k(x).$$
Since $u_k$ is the sum up to $k$ of all the relevant $\beta_i(x)$, we get:

$$W_k = W_0 + \sum_{x \in \mathbb{Z}^2} u_k(x).$$

Then for every $x$, $u_k(x)$ is a bounded increasing sequence, then it converges to certain a function $u(x)$.

The relation $\nu_k(x) = \rho(x) + L_{M,1} u_k(x)$ holds for all finite time $k$, so the convergence of $\nu_k$ is a consequence of the convergence of $u_k$. Moreover, its limit $\nu$ satisfies $\nu = \rho + L_{M,1} u$.

Finally a point $x$ is either never toppled, in which case we have $\nu_k(x) \leq 1$ for all $k$, or it is toppled infinitely often, and then when a toppling occurs at time $k$ at the point $x$ we have $\nu_k(x) \leq 1$. In both cases we conclude that the limit $\nu$ satisfies $\nu(x) \leq 1$.

The next proposition states the Abelian property for the truncated $\alpha$-stable sandpile.

**Proposition 2.2.** Consider two legal toppling schemes $T_1$ and $T_2$ for an initial configuration $\rho$. For $i = 1, 2$, let $\nu^{T_i}$ and $u^{T_i}$ be respectively the final distribution and the odometer function for the toppling scheme $T_i$.

Then $\nu^{T_1} = \nu^{T_2}$ and $u^{T_1} = u^{T_2}$.

**Proof.** Assume that at time $k$ we topple the site $x_k$ in the toppling scheme $T_1$. We will prove by induction that $u^{T_2}(x_k) \geq u^{T_1}(x_k)$.

This property is trivially true for $k = 1$. Suppose that this holds for all $i < k$. For $x \neq x_k$ we have two possibilities: either $x$ is not toppled before time $k$ in the scheme $T_1$, in which case $u^{T_1}(x) = 0$, or $x$ is toppled before time $k$. In this case we consider the last index $i$ in which the site $x$ is toppled. Then $u^{T_2}(x) \geq u^{T_1}(x) = u^{T_1}(x)$. In both cases

$$u^{T_2}(x) \geq u^{T_1}(x). \quad (2.1)$$

Since $T_1$ is a legal toppling scheme,

$$\nu^{T_2}(x_k) \leq 1 \leq \nu^{T_1}(x_k). \quad (2.2)$$

The initial configuration is the same for both toppling schemes, so equation (2.2) says:

$$L_{M,1} u^{T_2}(x_k) \leq L_{M,1} u^{T_1}(x_k).$$

That is,

$$u^{T_2}(x_k) - u^{T_1}(x_k) \geq \sum_{x \neq x_k} p_1(x - x_k) (u^{T_2}(x) - u^{T_1}(x)).$$

Each term in the sum above is nonnegative by (2.1), then we conclude that $u^{T_2}(x_k) \geq u^{T_1}(x_k)$. Since the site $x_k$ is toppled infinitely many times, it follows that $u^{T_2} \geq u^{T_1}$. The same argument shows the reversed inequality, which concludes the proof. \qed

### 3. The obstacle function

Equation (1.20) gives us a way to find the odometer function for the truncated $\alpha$-stable sandpile, provided that we have a function $\gamma_n$ that satisfies condition (1.14). This section is devoted to the construction of our obstacle function $\gamma_n$. 

Recall the definition of $p_n$ given in (1.10) and define the jump probability $	ilde{p}_n$ on $\mathbb{Z}^2$ as:

$$\tilde{p}_n(y) = p_n\left(\frac{y}{n}\right).$$  (3.1)

For a function $f$ on $\mathbb{Z}^2$, we define the generator $\tilde{L}_{M,n}$ of this random walk, as:

$$\tilde{L}_{M,n}f(x) = \sum_{y \in B_{M,n}^i} \tilde{p}_n(y) (f(x + y) - f(x)).$$  (3.2)

For each $n$ we consider in $\mathbb{Z}^2$ the random walk $\{S^m_n\}_{m \geq 0}$ starting at the origin, with law given by $\tilde{p}_n$. Define $\tilde{p}^{m}_n(x) = \mathbb{P}[S^m_n = x]$. The function $\tilde{G}_{M,n}$ given by

$$\tilde{G}_{M,n}(x) = \lim_{N \to \infty} \sum_{m=0}^{N} \left[ \tilde{p}^{m}_n(0) - \tilde{p}^{m}_n(x) \right]$$  (3.3)

verifies $\tilde{L}_{M,n}\tilde{G}_{M,n}(x) = 0$ for $x \neq 0$ and $\tilde{L}_{M,n}\tilde{G}_{M,n}(0) = -1.$

The function (3.3) can be written as (see [7]):

$$\tilde{G}_{M,n}(x) = \frac{1}{(2\pi)^2} \int_{[-\pi,\pi]^2} \frac{\cos(\theta \cdot x) - 1}{\psi_{M,n}(\theta)} d\theta,$$  (3.4)

where $\tilde{\psi}_{M,n}(\theta)$ is one minus the characteristic function of the probability of (3.1), that is

$$\tilde{\psi}_{M,n}(\theta) = \sum_{y \in B_{M,n}^i \cup \partial B_{M,n}^i} \tilde{p}_n(y)(1 - \cos(\theta \cdot y)).$$  (3.5)

We are not actually interested in $\tilde{G}_{M,n}$, but in the value of its Laplacian, then we can add constants in a convenient way in order to obtain convergence when the function is properly rescaled. So we fix some $x_0$ in $\mathbb{R}^2$ such that $|x_0| = 1$ and notice that the function

$$\tilde{G}_{M,n}(x) = \frac{1}{(2\pi)^2} \int_{[-\pi,\pi]^2} \frac{\cos(\theta \cdot x) - \cos(\theta \cdot nx_0)}{\psi_{M,n}(\theta)} d\theta$$  (3.6)

also satisfies $\tilde{L}_{M,n}\tilde{G}_{M,n}(x) = 0$ for $x \neq 0$ and $\tilde{L}_{M,n}\tilde{G}_{M,n}(0) = -1.$

We define the function $G_{M,n} : \frac{1}{n} \mathbb{Z}^2 \to \mathbb{R}$ given by

$$G_{M,n}(x) = n^{2-n} \tilde{G}_{M,n}(nx)$$  (3.7)

This function satisfies $L_{M,n}G_{M,n}(x) = 0$ for $x \neq 0$ and $L_{M,n}G_{M,n}(0) = -n^2.$ A change of variables shows that $G_{M,n}$ has the integral form given in (1.16). We call $G_{M,n}$ the Green’s function associated to the operator $L_{M,n}.$

Define

$$G_{M,n} \ast \rho_n(x) = \frac{1}{n^2} \sum_{y \in \frac{1}{n} \mathbb{Z}^2} G_{M,n}(x-y) \rho_n(y).$$

The operator $L_{M,n}$ commutes with the sum, therefore

$$L_{M,n}(G_{M,n} \ast \rho_n)(x) = (L_{M,n}G_{M,n}) \ast \rho_n(x) = -\rho_n(x).$$

Now we are ready to define the obstacle function $\gamma_n$ on $\frac{1}{n} \mathbb{Z}^2$ as

$$\gamma_n(x) = -\frac{|x|^2}{\sigma_{M,n}^2} - G_{M,n} \ast \rho_n(x),$$  (3.8)

where $\sigma_{M,n}^2 = L_{M,n}|x|^2$, as defined in (1.15).
4. Analytical estimates

In this section we put together some of the analytical estimates that we use throughout this paper. We also state some qualitative properties of the Green’s function, the obstacle function, and the odometer function. The proofs of these propositions can be found in Section 10.

Recall from (1.2) the definition of $k_1$. The next lemma states that we can choose the parameters in the definition of $k_1$ in order to ensure that $k_1$ is positive and consequently the probability jump $p_1$ is well defined.

**Lemma 4.1.** There exist constants $r < M$ such that $k_1$ is positive.

The next lemma tells us a simple fact concerning the normalizing constants $c_n$ of the jump probability $p_n$ (recall from (1.10)), and the constant $c = \lim_{n \to \infty} c_n$.

**Lemma 4.2.** There exists a positive constant $C$ which depends on $\alpha$, $r$ and $M$ such that

$$|c_n - c| \leq \frac{C}{n^\alpha} \text{ for all } n.$$

In the next lemma we show that for a sufficiently smooth function $f$ defined in $\mathbb{R}^2$, its discrete fractional Laplacian $L_{M,n}$ defined in (1.13) approximates its fractional truncated Laplacian $L_M$ defined in (1.22).

**Lemma 4.3.** If $f: \mathbb{R}^2 \to \mathbb{R}$ is twice differentiable in $\mathbb{R}^2$ and $K \subset \mathbb{R}^2$ is a compact set. Then

$$|L_M f(x) - L_{M,n} f(x)| \leq \frac{C}{n^{2-\alpha}},$$

for $x \in K$, where the constant $C$ depends on $\alpha$ and on the first and second partial derivatives of the function $f$ in $K + B_M$.

As we mentioned before, the Green’s function $G_M$ (recall the definition from (1.24)), is in some sense the inverse of the truncated fractional Laplacian $L_M$. This is the subject of the next lemma.

**Lemma 4.4.** Let $\phi: \mathbb{R}^2 \to \mathbb{R}$ a infinitely differentiable function with compact support. Then

$$L_M (G_M * \phi) = -\phi.$$

The next two propositions give us a better understanding of the behavior of the Green’s function $G_M$ defined in (1.24).

A change of variables shows that

$$G_M(x) = \frac{1}{M^{2-\alpha}} \left( G_1\left( \frac{x}{M} \right) - G_1\left( \frac{x_0}{M} \right) \right). \quad (4.1)$$

We have the following estimates for the function $G_1$

**Proposition 4.5.** There exists a bounded function $g$ in $B_1$, and a bounded function $h$ in $B_1$, so that

$$G_1(x) = \frac{\alpha^2}{c(2\pi)^2} \frac{1}{|x|^{2-\alpha}} + g(x), \text{ for all } |x| < 1. \quad (4.2)$$

and

$$G_1(x) = -\frac{2}{\pi \sigma_1^2} \log |x| + h_1(x) + \frac{h(x)}{|x|^{2-\alpha}}, \text{ for all } |x| \geq 1, \quad (4.3)$$
where $h_1$ is the bounded function in $\mathcal{B}^c_1$, given by $h_1(x) = \frac{1}{\pi \sigma^2} \int_{B(|x|) \setminus B_1} \frac{\cos(\theta \cdot \omega)}{|\theta|^2} d\theta$, where $\omega$ is any unitary vector.

We have the following estimates for $G_M$.

**Proposition 4.6.** There exists a positive constant $C$ independent of $x$ and $M$, such that

$$
\left| G_M(x) - \frac{\alpha^2}{c(2\pi)^2} \frac{1}{|x|^{2-\alpha}} + \frac{\alpha^2}{c(2\pi)^2} \right| < \frac{C}{M^{2-\alpha}}, \text{ for all } x \in B_M
$$

and

$$
\left| G_M(x) + \frac{2}{\pi \sigma^2_M} \log |x| - h_M(x) - \delta_M \right| < \frac{C}{|x|^{2-\alpha}} \text{ for all } x \in B_M^c.
$$

where $\delta_M$ is an explicitly computable constant, converging to $-\frac{\alpha^2}{(2\pi)^2 c}$ as $M$ goes to infinity, and $h_M$ is a bounded function in $\mathcal{B}^c_M$, given by the formula $h_M(x) = \frac{1}{\pi \sigma^2_M} \int_{B(|x|) \setminus B_1} \frac{\cos(\theta \cdot \omega)}{|\theta|^2} d\theta$, where $\omega$ is any unitary vector.

The function $G(x) := \frac{\alpha^2}{c(2\pi)^2} \frac{1}{|x|^{2-\alpha}}$ is the Green’s function of the fractional Laplacian whose integral form is given by (see [13])

$$
Lf(x) = \int_{\mathbb{R}^2} \frac{f(x+y) + f(x-y) - 2f(x)}{|y|^{2+\alpha}} dy,
$$

for a twice differentiable, bounded function $f$.

Also note that the function $-\frac{2}{\pi \sigma^2_M} \log |x|$ is the Green’s function of a Brownian motion (see [1]), then Proposition 4.6 states that the Green’s function $G_M$ is an interpolation between the Green’s functions of a Levy $\alpha$-stable process and a Brownian motion.

The corollary bellow is a direct consequence of (4.4).

**Corollary 4.7.** The function $G_M(x)$ converges to $G(x) - \frac{\alpha^2}{(2\pi)^2 c}$ uniformly on compacts sets of $\mathbb{R}^2$.

In some situations we will need to integrate the first partial derivatives of the Green’s function $G_M$ over finite sets. In the following proposition we justify such operations.

**Proposition 4.8.** The first partial derivatives of the Green’s function $G_M$ are locally integrable.

Later on we will use that the odometer function $u$ defined in (1.28) has compact support. The next lemma will be used to prove this fact.

**Lemma 4.9.** There exists a ball $\Omega$ such that the continuous obstacle function $\gamma$ defined in (1.23) is concave outside $\Omega$.

**Proposition 4.10.** The continuous odometer function $u$ defined in (1.28) has compact support.
5. Continuity of the majorant

In this section we prove the continuity of the majorant \( s \) defined in (1.27).

The continuity of the majorant, in the previous related works (see for example [11]) is a simple consequence of the mean value inequality for the classical Laplacian operator. We can translate the same technique to our context and for that, in this section we present a mean value inequality for the truncated fractional Laplacian. We follow an idea from Caffarelli and Silvestre [15], where they proved similar results for the fractional Laplacian. However, in our case, an extra care must be taken since we are dealing with a truncation parameter that may change when we do simple operations like change of variables on integration.

Recall from (1.22) and (1.24), the definition of the truncated fractional Laplacian \( L_M \) and its respective Green’s function \( G_M \). For \( \lambda > 0 \), consider the Green’s function \( \hat{G}_M \).

Next we rescale \( \hat{G}_M \) in the following way,

\[
\hat{\Gamma}_\lambda(x) = \frac{1}{\lambda^{\frac{2-\alpha}{\alpha}}} \Gamma_\lambda(\frac{x}{\lambda}).
\]  

(5.1)

A change of variables shows that

\[
G_M(x) = \lambda^{2-\alpha} G_M(\lambda x) - \lambda^{2-\alpha} G_M(\lambda x_0),
\]  

(5.2)

so the function \( \hat{\Gamma}_\lambda \) coincides with \( G_M(x) - G_M(\lambda x_0) \) when \( x \) is outside \( B_\lambda \).

Let us consider then the family of functions

\[
\hat{\Gamma}_\lambda = \hat{\Gamma}_\lambda + G_M(\lambda x_0)
\]  

(5.3)

and note that \( \hat{\Gamma}_\lambda \) coincides with \( G_M \) outside \( B_\lambda \). Moreover \( \hat{\Gamma}_{\lambda_1} \leq \hat{\Gamma}_{\lambda_2} \) if \( \lambda_1 \geq \lambda_2 \).

**Proposition 5.1.** For all \( \lambda > 0 \), \( -L_M \hat{\Gamma}_\lambda \) is a positive continuous function in \( L^1 \). Moreover

\[
\int_{\mathbb{R}^2} -L_M \hat{\Gamma}_\lambda(x) dx = 1.
\]

**Proof.** We have the following rescaling property

\[
L_M \hat{\Gamma}_\lambda(x) = L_M \hat{\Gamma}_\lambda(x) = \frac{1}{\lambda^2} L_M \hat{\Gamma}_\lambda(\frac{x}{\lambda}).
\]  

(5.4)

Then a change of variables shows that it is suffices to prove that for all \( \lambda > 0 \), \( -L_M \hat{\Gamma}_\lambda \) is a positive continuous function in \( L^1 \) and \( \int_{\mathbb{R}^2} -L_M \hat{\Gamma}_\lambda(x) dx = 1 \).

We prove first the \( -L_M \hat{\Gamma}_\lambda \) is positive.
If \( x \notin B_1 \), then \( \Gamma_\lambda(x) = G_\lambda(x) \) and for every other \( y \), \( \Gamma_\lambda(y) \leq G_\lambda(y) \), then:
\[
-L \bar{M}_\lambda \Gamma_\lambda(x) = \frac{c}{2} \int_{B_{2\delta}} \frac{2 \Gamma_\lambda(x) - \Gamma_\lambda(x - y) - \Gamma_\lambda(x + y)}{|y|^{2+\alpha}} dy
\]
\[
\geq \frac{c}{2} \int_{B_{2\delta}} \frac{2 G_\lambda(x - x_1) - G_\lambda(x - y - x_1) - G_\lambda(x + y - x_1)}{|y|^{2+\alpha}} dy
\]
\[
= 0
\]

since \( G_\lambda \) is the Green’s function of \( L \bar{M}_\lambda \).

If \( x \in B_1 \setminus \{0\} \), there exist a \( x_1 \neq x \) and a positive \( \delta \) such that \( G_\lambda(\cdot - x_1) + \delta \) touches \( \Gamma_\lambda \) from above at the point \( x \). Then
\[
-L \bar{M}_\lambda \Gamma_\lambda(x) = \frac{c}{2} \int_{B_{2\delta}} \frac{2 \Gamma_\lambda(x) - \Gamma_\lambda(x - y) - \Gamma_\lambda(x + y)}{|y|^{2+\alpha}} dy
\]
\[
\geq \frac{c}{2} \int_{B_{2\delta}} \frac{2 G_\lambda(x - x_1) - G_\lambda(x - y - x_1) - G_\lambda(x + y - x_1)}{|y|^{2+\alpha}} dy
\]
\[
= 0
\]

since \( G_\lambda \) is the Green’s function of \( L \bar{M}_\lambda \). In the second line above we cancelled the parameter \( \delta \).

If \( x = 0 \), then \( \Gamma_\lambda \) attains its maximum at \( x \):
\[
-L \bar{M}_\lambda \Gamma_\lambda(x) = \frac{c}{2} \int_{B_{2\delta}} \frac{2 \Gamma_\lambda(x) - \Gamma_\lambda(x - y) - \Gamma_\lambda(x + y)}{|y|^{2+\alpha}} dy > 0,
\]
because we are integrating a positive function.

To show that \( \int_{\mathbb{R}^2} -L \bar{M}_\lambda \Gamma_\lambda(x) dx = 1 \) we consider a smooth function \( \omega \) such that \( \omega(x) \leq 1 \) for every \( x \in \mathbb{R}^2 \), \( \omega(x) = 1 \) for every \( x \in B_1 \) and \( \text{supp} \omega \subset B_2 \). Let \( \omega_R(x) = \omega(x/R) \),
\[
\int_{\mathbb{R}^2} -L \bar{M}_\lambda \Gamma_\lambda(x) dx - 1 = \lim_{R \to 0} \left( -L \bar{M}_\lambda (\Gamma_\lambda - G_\lambda) , \omega_R \right) = \lim_{R \to 0} \left( \Gamma_\lambda - G_\lambda, -L \bar{M}_\lambda \omega_R \right) = 0
\]

since \( G_\lambda \omega_R \) goes to zero uniformly on compacts sets, and \( G_\lambda - \Gamma_\lambda \) is an \( L^1 \) function with compact support. \( \square \)

Now let us define \( \eta_\lambda = -L \bar{M}_\lambda \widehat{\Gamma_\lambda} \).

**Proposition 5.2.** The family \( \eta_\lambda \) is an approximation of the identity as \( \lambda \to 0 \) in the sense that, if \( f \) is a continuous function in \( \mathbb{R}^2 \), then
\[
\eta_\lambda * f \to f \text{ when } \lambda \to 0 \text{ uniformly on compacts.}
\]

**Proof.** We will prove first that the collection of functions \( \{ L \bar{M}_\lambda \Gamma_\lambda, \lambda > 0 \} \) is uniformly integrable for \( \lambda \) sufficiently small. That is, for all \( \epsilon > 0 \) there exists \( R > 0 \) such that
\[
\int_{\mathbb{R}^2 \setminus B_R} |L \bar{M}_\lambda \Gamma_\lambda(y)| dy < \epsilon
\]
for \( \lambda \) sufficiently small.

Fix \( y \) outside \( B_2 \). Hence \( \Gamma_\lambda(y) = G_\frac{\lambda}{4}(y) \), we have

\[
-L_M \Gamma_\lambda(y) = \frac{c}{2} \int_{B_M} \frac{2G_\frac{\lambda}{4}(y) - \Gamma_\lambda(y - z) - \Gamma_\lambda(y + z)}{|z|^{2+\alpha}} dz.
\]

Since for \( y \neq 0 \), \( L_M G_\frac{\lambda}{4}(y) = 0 \), the equation above can be written as

\[
-L_M \Gamma_\lambda(y) = \frac{c}{2} \int_{B_M} \frac{G_\frac{\lambda}{4}(y + z) - \Gamma_\lambda(y + z)}{|z|^{2+\alpha}} dz + \frac{c}{2} \int_{B_M} \frac{G_\frac{\lambda}{4}(y - z) - \Gamma_\lambda(y - z)}{|z|^{2+\alpha}} dz
\]

\[= I_1 + I_2.\]

Since the support of \( G_\frac{\lambda}{4} - \Gamma_\lambda \) is contained in \( B_1 \), the integral \( I_1 \) above can be taken over the set \( B_M \cap B(y, 1) \), and the integral \( I_2 \) can be taken in \( B_M \cap B(-y, 1) \).

Since \( G_\frac{\lambda}{4} \) is bigger than \( \Gamma_\lambda \) in \( B_1 \), for a positive constant \( C \) independent of \( y \), we write

\[
I_1 \leq C \int_{B(-y,1)} \frac{G_\frac{\lambda}{4}(y + z)}{|z|^{2+\alpha}} dz \leq \frac{C}{|y|^{2+\alpha}} \int_{B_1} G_\frac{\lambda}{4}(z) dz,
\]

where the constant \( C \) have changed from one line to the other.

By Corollary 4.7, \( G_\frac{\lambda}{4} \) converges uniformly on \( B_1 \) to \( G - \frac{\alpha^2}{c(2\pi)^2} \) (where \( G \) is the Green’s function of the fractional Laplacian) when \( \lambda \) goes to 0.

Then we can pass the limit inside the integral:

\[
\lim_{\lambda \to 0} \int_{B_1} G_\frac{\lambda}{4}(y) dy = \int_{B_1} (G(y) - \frac{\alpha^2}{c(2\pi)^2}) dy.
\]

Since \( G \) is integrable in \( B_1 \), for \( \lambda \) sufficiently small, there exists a constant \( C > 0 \) independent of \( \lambda \) and \( y \) such that

\[
I_1 \leq \frac{C}{|y|^{2+\alpha}} \quad \text{for } y \in B_2^c. \tag{5.5}
\]

The exactly same argument proves that (5.5) also holds for \( I_2 \). Then we have

\[
-L_M \Gamma_\lambda(y) \leq \frac{C}{|y|^{2+\alpha}}, \quad \text{for all } y \in B_2^c.
\]

Take \( R > 2 \), then

\[
\int_{\mathbb{R}^2 \setminus B_R} -L_M \Gamma_\lambda(y) dy \leq \frac{C}{|y|^{2+\alpha}} dy = \frac{2\pi C}{\alpha R^\alpha},
\]

which proves the uniform integrability.

Now fix a compact set \( K \), we will prove that \( \eta_\lambda * f \to f \) as \( \lambda \) goes to zero, uniformly on \( K \), assuming that the function \( f \) is continuous.
Recall that $\eta_\lambda(x) = -L_M \hat{\Gamma}_\lambda(x) = -\frac{1}{N^2} L_M \Gamma_\lambda(\frac{x}{\lambda})$. We have, for $x \in K$

$$\eta_\lambda * f(x) - f(x) = \int_{\mathbb{R}^2} \eta_\lambda(x - y)(f(y) - f(x))dy$$
$$= \int_{\mathbb{R}^2} -L_M \frac{1}{\lambda} \Gamma_\lambda(y)(f(x - \lambda y) - f(x))dy$$
$$= \int_{B(\frac{M}{\lambda} + 1)} -L_M \frac{1}{\lambda} \Gamma_\lambda(y)(f(x - \lambda y) - f(x))dy, \quad (5.6)$$

where in (5.6) we used that the support of $L_M \Gamma_\lambda$ is contained in $B(\frac{M}{\lambda} + 1)$, this can be easily seen from the fact that $\Gamma_\lambda$ coincides with $G_M$ outside $B_1$.

By the uniform integrability of $\{L_M \Gamma_\lambda, \lambda > 0\}$, we fix $\epsilon > 0$ and take $R > 0$ so that

$$\int_{B_R^c} -L_M \frac{1}{\lambda} \Gamma_\lambda(y)dy < \epsilon, \text{ for all } \lambda \text{ sufficiently small.} \quad (5.7)$$

Then we write, for $x \in K$

$$\eta_\lambda * f(x) - f(x) = \int_{B_R \cap B(\frac{M}{\lambda} + 1)} -L_M \frac{1}{\lambda} \Gamma_\lambda(y)(f(x - \lambda y) - f(x))dy$$
$$+ \int_{B_R \cap B(\frac{M}{\lambda} + 1)} -L_M \frac{1}{\lambda} \Gamma_\lambda(y)(f(x - \lambda y) - f(x))dy$$
$$=: J_1 + J_2$$

**Estimate of $J_1$:** Since $f$ is uniformly continuous on compact sets, we choose $\lambda$ small enough, so that

$$|f(x - \lambda y) - f(x)| < \epsilon, \text{ for all } x \in K, y \in B_R.$$ 

Then

$$|J_1| \leq \epsilon \int_{\mathbb{R}^2} -L_M \frac{1}{\lambda} \Gamma_\lambda(y)dy = \epsilon.$$

**Estimate of $J_2$:** For $y \in B(\frac{M}{\lambda} + 1)$, we have $\lambda y \in B_{M + 1} \subset B_{M+1}$. Then in the integral $J_2$, we are only considering $f$ taking values in the set $K + B(\frac{M}{\lambda} + 1)$. Since $f$ is continuous, in particular is bounded in $K + B(M+1)$, let say by $A$.

Then, by (5.7), we have

$$|J_2| \leq 2A \int_{B_R^c} -L_M \frac{1}{\lambda} \Gamma_\lambda(y)dy < 2A \epsilon.$$ 

Since $\epsilon$ is arbitrary, we conclude our proof. \qed

In the next proposition we prove a mean value inequality for superharmonic functions with respect to the operator $L_M$. Recall from (1.26) the definition of superharmonicity for functions that are not twice differentiable.

**Proposition 5.3.** Let $f$ be a continuous function. Then $f$ is superharmonic in an open set $U$ if and only if

$$f(x) \geq \eta_\lambda * f(x) \quad (5.8)$$

for any $x$ in $U$ and any $\lambda$ satisfying $B(x, \lambda) \subset U$. 


Proof. Suppose $L_M f \leq 0$ in $U$.

Let $r > 0$ such that, $r > \lambda_1 > \lambda_2$, then $\hat{\Gamma}_{\lambda_2} - \hat{\Gamma}_{\lambda_1}$ is a nonnegative smooth function supported in $B_r$. If $L_M f \leq 0$ in $B(x, r)$ then:

$$\langle L_M f, \hat{\Gamma}_{\lambda_2}(\cdot - x) - \hat{\Gamma}_{\lambda_1}(\cdot - x) \rangle \leq 0.$$ 

Using the self-adjointness of $L_M$:

$$\langle f, (-L_M)\hat{\Gamma}_{\lambda_2}(\cdot - x) - (-L_M)\hat{\Gamma}_{\lambda_1}(\cdot - x) \rangle \leq 0.$$ 

Therefore

$$\langle f, \eta_{\lambda_1}(\cdot - x) \rangle \leq \langle f, \eta_{\lambda_2}(\cdot - x) \rangle \Rightarrow f * \eta_{\lambda_1}(x) \leq f * \eta_{\lambda_2}(x).$$  \ (5.9)$$

By Proposition 5.2, since $f$ is continuous, we have $f * \eta_{\lambda} \to f$ uniformly on compact sets, as $\lambda \to 0$.

We take $\lambda_2 \to 0$ in (5.9), and this concludes our proof.

The conversely part follows easily. □

In Proposition 5.3 we obtained the desired mean value inequality for the truncated fractional Laplacian. Before proving the continuity of the majorant $s$ defined in (1.27) we need a few lemmas.

Given a bounded open set $U$, define the function

$$s^U(x) = \inf\{f(x) : f \text{ is continuous, superharmonic on } U, \text{ and } f \geq \gamma\}. \ (5.10)$$

By Proposition 4.10 there exists a ball $\Omega$ which contains the support of the odometer function $u$. We fix this ball $\Omega$ and assume that $U$ satisfies

$$\Omega + B_{M+1} \subset U. \ (5.11)$$

Lemma 5.4. Let $U$ be a bounded open set satisfying (5.11). Then the function $s^U$ defined in (5.10) satisfies

$$s^U(x) \geq \eta_{\lambda} * s^U(x),$$

for all $x \in \mathbb{R}^2$ and $\lambda$ sufficiently small.

Proof. Note that since the infimum in (5.10) is taken over a larger set than the infimum in the definition of $s$, we have that $s^U \leq s$.

By Proposition 4.10 $s$ coincides with $\gamma$ outside the ball $\Omega$. Then $\gamma \leq s^U \leq s$, and hence, $s^U$ also coincides with $\gamma$ in $\Omega^c$.

Consider a continuous function $f$ bigger or equal than $\gamma$, and superharmonic in $U$. Then $f \geq s^U$ and moreover by Proposition 5.3 for $x \in U$ and $\lambda$ sufficiently small we have

$$f(x) \geq \eta_{\lambda} * f(x) \geq \eta_{\lambda} * s^U(x),$$

taking infimum over $f$, we have:

$$s^U(x) \geq \eta_{\lambda} * s^U(x) \text{ for } x \in U. \ (5.12)$$

By our choice of $U$, the function $\gamma$ is superharmonic in $U^c$. Then, using Proposition 5.3 for $x \in U^c$ and $\lambda$ sufficiently small

$$s^U(x) = \gamma(x) \geq \eta_{\lambda} * \gamma(x)$$

holds.
Notice that \( \text{supp}(\eta_\lambda) \subset B_{M+1} \) for \( \lambda < 1 \). Since \( U \) satisfies (5.11), we can replace \( \gamma \) by \( s^U \) in the equation above, and obtain
\[
s^U(x) \geq \eta_\lambda \ast s^U(x) \text{ for } x \in U^c. \tag{5.13}
\]
By (5.12) and (5.13) we conclude our proof. \( \square \)

The next lemma states the continuity of \( s^U \).

**Lemma 5.5.** Let \( U \) be a bounded open set satisfying (5.11). Then the function \( s^U \) defined in (5.10) is continuous.

**Proof.** We have noticed before that \( s^U \) coincides with \( \gamma \) in \( \Omega^c \); then it is continuous in \( \Omega^c \). We will prove that it is continuous in \( U \).

By Lemma 5.4 we have
\[
s^U \geq \eta_\lambda \ast s^U,
\]
for \( \lambda \) sufficiently small.

Consider now a bounded, open set \( U_1 \), such that, \( U_1 \supset U + B_M \). Since \( \gamma \) is continuous, by Proposition 5.2 we have that \( \eta_\lambda \ast \gamma \to \gamma \) uniformly in \( U_1 \) as \( \lambda \to 0 \), then for \( \epsilon > 0 \) we take \( \lambda \) small, so that
\[
\eta_\lambda \ast s^U(x) \geq \eta_\lambda \ast \gamma(x) \geq \gamma(x) - \epsilon, \text{ for } x \in U_1. \tag{5.14}
\]

Note that \( \eta_\lambda \ast s^U \) is a continuous function. By Lemma 5.4 and Proposition 5.3 we conclude that it is also superharmonic.

Consider the function \( g = \max(\eta_\lambda \ast s^U + \epsilon, \gamma) \). By (5.14), \( g \) coincides with \( \eta_\lambda \ast s^U + \epsilon \) in \( U_1 \), and hence it is superharmonic in \( U \). Since \( g \) is above \( \gamma \), we have \( g \geq s^U \). We have proved
\[
\eta_\lambda \ast s^U(x) \leq s^U(x) \leq \eta_\lambda \ast s^U(x) + \epsilon, \text{ for } x \in U.
\]
Thus \( s^U \) is an increasing limit of continuous functions in \( U \), and hence lower semicontinuous. Since \( s^U \) is defined as an infimum of continuous functions, it is also upper semicontinuous. \( \square \)

By Lemmas 5.4 and 5.5 and Proposition 5.3 we have that \( s^U \) is continuous and superharmonic. Since \( s^U \) is above the obstacle \( \gamma \), we obtain \( s = s^U \).

We have just proved the following proposition which states the continuity of \( s \).

**Proposition 5.6.** The majorant \( s \) defined in (1.27) is a continuous function. Moreover, if \( U \) is a bounded open set satisfying (5.11), then \( s = s^U \).

6. **Convergence of Green’s functions**

In this section we prove Theorem 1.3 which tells us about the convergence of the Green’s functions. We refer to Section 11 for the proofs of the auxiliary results presented here.

Recall from (1.16) and (1.24) the definition of \( G_{M,n} \) and \( G_M \). The first step in order to prove Theorem 1.3 is to investigate the convergence of \( \psi_{M,n} \) to \( \psi_M \) defined in (1.17) and (1.25) respectively.

A simple change of variables shows that
\[
\psi_M(\theta) = \frac{1}{M^\alpha} \psi_1(\theta M). \tag{6.1}
\]
For $\theta \in B_{\frac{1}{n}}$, by (10.7) we have that
\[
\psi_M(\theta) = \frac{\sigma_2}{4} |\theta|^2 + |\theta|^2 g_M(\theta),
\] (6.2)
for a bounded function $g_M : B_{\frac{1}{n}} \to \mathbb{R}$.

In the same way, for $\theta \in B_{\frac{1}{n}}^c$, by (10.9)
\[
\psi_M(\theta) = c_n |\theta|^\alpha + h_M(\theta),
\] (6.3)
for a bounded function $h_M : B_{\frac{1}{n}}^c \to \mathbb{R}$.

This suggests that we must split the integrals that define $G_M$ and $G_{M,n}$ in three different regions: $B_{\frac{1}{n}}$, $[-n\pi, n\pi]^2 \setminus B_{\frac{1}{n}}$, and $\mathbb{R}^2 \setminus [-n\pi, n\pi]^2$. Recall from (1.11) the definition of $F_n$.

For $\theta \in [-n\pi, n\pi]^2$, denote $\hat{\psi}_{M,n}(\theta) = \frac{1}{n^2} \sum_{y \in B_{\frac{1}{n}} \cup \partial B_{\frac{1}{n}}} F_n(y) \left(1 - \cos(\theta \cdot y)\right)$. Then we have
\[
\psi_{M,n}(\theta) = n^\alpha c_n k_n \sum_{|y| = \frac{1}{n}} (1 - \cos(\theta \cdot y)) + c_n \hat{\psi}_{M,n}(\theta).
\]

Note that for $\theta \in [-n\pi, n\pi]^2$, we have $\frac{\partial}{\partial x}$ is bounded, then we use Taylor’s Theorem to ensure that there exists a positive constant $C$, such that
\[
\left| 2(1 - \cos(\frac{1}{n} \theta)) + 2(1 - \cos(\frac{2}{n} \theta)) - \frac{|\theta|^2}{n^2} \right| \leq \frac{C|\theta|^4}{n^4},
\]
then
\[
\left| \psi_{M,n}(\theta) - c_n \hat{\psi}_{M,n}(\theta) - \frac{c_n k_n |\theta|^2}{n^{2-\alpha}} \right| \leq \frac{C|\theta|^4}{n^4}. \quad (6.4)
\]

The next two lemmas describe the convergence of $\psi_{M,n}$ to $\psi_M$ when $n$ goes to infinity.

**Lemma 6.1.** For $\theta \in B_{\frac{1}{n}}$, we have
\[
|\psi_{M,n}(\theta) - c_n \psi_M(\theta)| \leq \frac{C|\theta|^4}{n^2}, \quad (6.5)
\]
where the constant $C$ only depends on $\alpha$.

**Lemma 6.2.** For $\theta \in [-n\pi, n\pi]^2 \setminus B_{\frac{1}{n}}$, there exists an uniformly bounded family of functions $r_n$ such that
\[
\psi_{M,n}(\theta) = c_n \psi_M(\theta) + \frac{r_n(\theta)|\theta|^{2+\alpha}}{n^2}. \quad (6.6)
\]
Moreover there exists a positive constant $C$ depending only on $\alpha$ such that sequence $r_n$ satisfies
(i) $\frac{d}{dx}(r_n(\theta)|\theta|^{2+\alpha}) \leq C|\theta|^{1+\alpha}$ for $i = 1, 2$
(ii) $\Delta(r_n(\theta)|\theta|^{2+\alpha}) \leq C|\theta|^\alpha$.

Let us introduce the following notations:
\[
\hat{G}_{M,n}(x) = \int_{B_{\frac{1}{n}}} \frac{\cos(\theta \cdot x) - 1}{\psi_{M,n}(\theta)} d\theta + \int_{[-n\pi, n\pi]^2 \setminus B_{\frac{1}{n}}} \frac{\cos(\theta \cdot x)}{\psi_{M,n}(\theta)} d\theta \quad (6.7)
\]
\[
\hat{G}_M(x) = \int_{B_{\frac{1}{n}}} \frac{\cos(\theta \cdot x) - 1}{\psi_M(\theta)} d\theta + \int_{\mathbb{R}^2 \setminus B_{\frac{1}{n}}} \frac{\cos(\theta \cdot x)}{\psi_M(\theta)} d\theta. \quad (6.8)
\]
The next lemma states the convergence of the auxiliary functions defined in (6.7) and (6.8). In the sequence we will conclude the proof of Theorem 1.3 which is the subject of this section.

**Lemma 6.3.** There exists a constant $C$ which only depends on $\alpha$, such that
\[
\left| \frac{c}{c_n} \tilde{G}_M(x) - \tilde{G}_{M,n}(x) \right| \leq \frac{C}{n^\alpha} \left( 1 + \frac{1}{|x|} + \frac{1}{|x|^2} \right).
\]

**Proof of Theorem 1.3.** Note that
\[
G_{M,n}(x) = \frac{1}{(2\pi)^2} (\tilde{G}_{M,n}(x) - \tilde{G}_{M,n}(x_0))
\]
and
\[
G_M(x) = \frac{1}{(2\pi)^2} (\tilde{G}_M(x) - \tilde{G}_M(x_0)).
\]

By Lemma 6.3
\[
\left| \frac{c}{c_n} G_M(x) - G_{M,n}(x) \right| \leq \frac{C}{n^\alpha} \left( 1 + \frac{1}{|x|} + \frac{1}{|x|^2} \right).
\]

Let us define $\beta_n = \frac{2}{\pi \sigma M} \left( \frac{c}{c_n} - 1 \right)$, and note that $\beta_n$ converges to zero with rate $\frac{1}{n^\alpha}$ due to Lemma 4.2. Finally, using Proposition 4.6 we conclude the proof of Theorem 1.3. □

### 7. Convergence of obstacles

Fix $x \in \mathbb{R}^2$ and $y \in \frac{1}{n} \mathbb{Z}^2$ define
\[
\alpha_n^x(y) := \frac{1}{n^2} G_{M,n}(x^i - y) - \int_{y^\Box} G_M(x - z)dz.
\]

**Lemma 7.1.** The function $\alpha_n^x$ satisfies
\[
|\alpha_n^x(y)| \leq C \frac{f(x^i - y)}{n^{1+\alpha}},
\]
where $f$ is a positive function in $L^1_{\text{loc}}$ and $C$ is a positive constant which depends on $\alpha$.

**Proof.** Write
\[
\alpha_n^x(y) = \frac{1}{n^2} (G_{M,n}(x^i - y) - G_M(x^i - y)) + \int_{y^\Box} (G_M(x^i - y) - G_M(x - z))dz.
\]

For $z \in y^\Box$, by Taylor’s Theorem, there exists a positive constant $C$ such that
\[
|G_M(x^i - y) - G_M(x - z)| \leq C |\nabla G_M(x^i - y)|,
\]
Integrating we obtain
\[
\int_{y^\Box} |G_M(x^i - y) - G_M(x - z)|dz \leq C \frac{\nabla G_M(x^i - y)}{n^3}.
\]

By Proposition 4.8, the function $|\nabla G_M(x)|$ is in $L^1_{\text{loc}}(\mathbb{R}^2)$.

On the other hand by Theorem 1.3 there exist a constant $C > 0$ depending only on $\alpha$, such that
\[
|G_M(x^i - y) - G_{M,n}(x^i - y)| \leq C \frac{\tilde{f}(x^i - y) + \frac{1}{|x^i - y|^2}}{n^\alpha},
\]
where $\tilde{f}$ is the function in $L^1_{\text{loc}}(\mathbb{R}^2)$.
where \( \tilde{f} \) is a function in \( L^1_{\text{loc}}(\mathbb{R}^2) \).

The function \( \frac{1}{|x|} \) is not locally integrable. To overcome this difficulty, since \( x \) and \( y \) are both in \( \frac{1}{n}\mathbb{Z}^2 \), we write

\[
\frac{1}{n|x| - y|} \leq \frac{1}{|x - y|}.
\]

This last inequality, together with (7.2), (7.3) and (7.4) concludes the proof of the lemma. \( \square \)

Proposition 7.2. Let \( \gamma_n \) and \( \gamma \) be the obstacles defined in (3.8) and (1.23). Then as \( n \) goes to infinity

\[
\gamma_n \to \gamma \text{ uniformly on compact sets of } \mathbb{R}^2.
\]

Proof. We have \( \sigma_{M,n}^2 \to \sigma_M^2 \). We only need to prove that \( (G_{M,n} * \rho_n)^\square \to G_M * \rho \) uniformly on compact sets of \( \mathbb{R}^2 \).

Since \( \rho_n = \rho^\square \), we write

\[
(G_{M,n} * \rho)^\square(x) - G_M * \rho(x) = G_{M,n} * \rho(x) - G_M * \rho(x)
\]

\[
= \frac{1}{n} \sum_{y \in \frac{1}{n}\mathbb{Z}^2} \rho(y)G_{M,n}(x - y) - \int_{\mathbb{R}^2} \rho(y)G_M(x - y)dy
\]

\[
= \sum_{y \in \frac{1}{n}\mathbb{Z}^2} \rho(y)\alpha_n^\square(y),
\]

where \( \alpha_n^\square(y) \) is given by (7.1).

Recall that \( \rho \) has compact support, let us say \( E \). By Lemma 7.1 there exists a positive locally integrable function \( f \) which satisfies

\[
|G_{M,n} * \rho(x) - G_M * \rho(x)| \leq \frac{C}{n^{1+\alpha}} \sum_{y \in E^\square} f(x - y)
\]

\[
\leq \frac{C}{n^{\alpha-1}} \int_E f(x - y)dy,
\]

where the constant \( C \) has changed from one line to the other. Since \( f \in L^1_{\text{loc}} \) and \( \alpha > 1 \), we conclude the proof. \( \square \)

8. Convergence of odometers

In this section we prove the convergence of majorants, and we obtain as a consequence the convergence of odometers. Before that we need a few lemmas.

Recall from (1.15), (1.18) and (1.19) the definition of the discrete obstacle function \( \gamma_n \), the discrete majorant \( s_n \) and the discrete odometer function \( u_n \), and recall the definition of its respective continuous counterparts \( \gamma, s \) and \( u \) from (1.23), (1.27) and (1.28) respectively.

Lemma 8.1. There exists a ball \( \Omega \) which contains the support of the odometer functions. That is

\[
\text{supp}(u) \subset \Omega \text{ and } \bigcup_n \text{supp}(u_n) \subset \Omega.
\]
Proof. By Lemma 4.9, there exists a ball \( \Omega \) such that the obstacle function \( \gamma \) is concave outside \( \Omega \). In Proposition 4.10 we proved that this ball \( \Omega \) also contains the support of the odometer function \( u \).

We can write the discrete obstacle function \( \gamma_n \) as

\[
\gamma_n(x) = \frac{|x|^2}{\sigma_{M,n}^2} - (G_{M,n} - G_M^r) * \rho_n(x) - G_M^r * \rho_n(x).
\]

We use Theorem 1.3 and repeat the argument used in the proof of Lemma 4.9 to conclude that there exists a ball \( \Omega \) (bigger if necessary) such that the obstacles functions \( \gamma_n \) are concave outside \( \Omega \).

Then, the concave envelope \( \Gamma_n \) of the obstacle \( \gamma_n \), defined as the least concave function lying above \( \gamma_n \), satisfies that \( \Gamma_n(x) = \gamma_n(x) \) for all \( x \) outside \( \Omega \) and for all \( n \). The same argument used in the proof of Proposition 4.10 concludes our proof.

\[ \square \]

Lemma 8.2. Let \( \gamma_n \) and \( s_n \) be the obstacle function and the majorant given by (1.15) and (1.18). If \( \Omega \subset \mathbb{R}^2 \) is a bounded set like the one in Lemma 8.1, then

\[
s_n(x) = \inf \{ f(x); L_{M,n}f \leq 0 \text{ on } \Omega^{-} \text{ and } f \geq \gamma_n \} \quad (8.1)
\]

Proof. We observe first that \( L_{M,n} s_n = 0 \) on the set \( D_n = \{ \nu_n = 1 \} \), since

\[
L_{M,n} s_n = L_{M,n}(u_n + \gamma_n) = \nu_n - \sigma_n + \sigma_n - 1 = \nu_n - 1.
\]

Let us call

\[
s_n^\Omega(x) = \inf \{ f(x); L_{M,n}f \leq 0 \text{ on } \Omega^{-} \text{ and } f \geq \gamma_n \}.
\]

We want to prove that \( s_n^\Omega = s_n \). Since the infimum is taken over a strictly larger set, the inequality \( s_n^\Omega \leq s_n \) is trivial.

For the converse inequality consider a function \( f \geq \gamma_n \) such that \( L_{M,n}f \leq 0 \) on \( \Omega^{-} \). Since \( L_{M,n} s_n = 0 \) on \( D_n \), the difference \( f - s_n \) satisfies \( L_{M,n}(f - s_n) \leq 0 \) on \( D_n \), then by the maximum principle \( f - s_n \) attains its minimum at a point \( x \) outside \( D_n \), where \( s_n(x) = \gamma_n(x) \). Since \( f \geq \gamma_n \) we conclude \( f \geq s_n \) on \( \Omega^{-} \) and hence everywhere.

\[ \square \]

Recall that Lemma 4.4 states

\[
L_M(G_M * \phi) = -\phi, \text{ for a function } \phi \in C_c^\infty. \quad (8.2)
\]

The next lemma studies the equation above removing the smoothness assumption.

Lemma 8.3. Let \( f \) be a bounded, measurable function, with compact support. Then \( L_M(G_M * f) = -f \) in the sense of distributions.

Proof. For a function \( \phi \in C_c^\infty \), holds \( L_M(G_M * \phi) = G_M * (L_M \phi) \). We have

\[
\langle L_M(G_M * f), \phi \rangle = \langle f, L_M(G_M * \phi) \rangle = \langle f, -\phi \rangle = \langle -f, \phi \rangle,
\]

where in the second line we used equation (8.2).

\[ \square \]

Recall from (1.26) the definition of superharmonicity for functions that are not twice differentiable. The next lemma is a consequence of the previous one.
Lemma 8.4. Let $f$ be a bounded measurable function, with compact support. Suppose that $f$ is positive in an open set $U$. Then $G_M * f$ is continuous, and super-harmonic in $U$.

Proof. By Lemma 4.8 and since $f$ is bounded with compact support and the first partial derivatives of $G_M$ are locally integrable, it is easy to see that $G_M * f$ is continuous. Now, fix a positive function $\phi \in C_\infty^\infty$ supported in $U$. Then, by Lemma 8.3, we have

$\langle L_M (G_M * f), \phi \rangle = \langle -f, \phi \rangle \leq 0,$

since $f$ and $\phi$ are both positive in $U$. \hfill $\square$

The next proposition states the convergence of the majorants.

Proposition 8.5. Let $s_n$ and $s$ the majorants defined on (1.18) and (1.27) respectively. Then

$s_n ^\square \rightarrow s$

uniformly on compact sets of $\mathbb{R}^2$.

By Lemma 8.1 there exists a ball $\Omega \subset \mathbb{R}^2$ containing $\text{supp}(u)$ and $\text{supp}(u_n)$ for all $n$. Outside $\Omega$, we have

$s_n ^\square = \gamma_n ^\square \rightarrow \gamma = s$

uniformly on compact sets by Proposition 7.2. We only need to prove convergence in $\Omega$.

Let us consider four compact sets $K_1, K_2, K_3$ and $K_4$, such that $\text{supp}(\rho) \subset K_1$, $\Omega + B_{M+1} \subset K_1$, and

$K_j + B_{M+1} \subset K_{j+1}$, for $j = 1, 2, 3$. \hfill (8.3)

By our assumption about $K_1$, Lemma 8.2 and Proposition 5.6, we have that the discrete and continuous majorants can be written as

$s_n = \inf \{ f(x); L_{M,n} f \leq 0 \text{ on } K_1 \text{ and } f \geq \gamma_n \}$ \hfill (8.4)

$s = \inf \{ f(x); f \text{ is continuous, } L_M f \leq 0 \text{ on } K_1 \text{ and } f \geq \gamma \}$. \hfill (8.5)

Consider the function $\omega(x)$ such that $0 \leq \omega \leq 1$, $\omega(x) = 1$ for $x \in K_3$, and $\text{supp} \omega \subset K_3 + B_1$.

Define the following function on $\frac{1}{n} \mathbb{Z}^2$:

$\phi_n = -L_{M,n}(s_n \omega)$. \hfill (8.6)

Lemma 8.6. The sequence of functions $(\phi_n)_n$ is uniformly bounded.

Proof. Since $\text{supp}(\omega) \subset K_3 + B_1$ we have, $\text{supp}(\phi_n) \subset (K_3 + B_{M+1}) \subset K_4$.

If $x \in K_2$, $\phi_n$ coincides with $-L_{M,n}s_n$ which is uniformly bounded by 1, as can be easily seen as a consequence of the odometer’s equation.

Consider then $x \in K_4 \setminus K_2 =: E$. In the set $E$ we can replace $s_n$ by $\gamma_n$, then in (8.6), we have

$\phi_n(x) = -L_{M,n}(\gamma_n \omega)(x)$ for $x \in E^\circ$. \hfill (8.7)

We write

$L_{M,n}(\gamma_n \omega)(x) = L_{M,n}(\gamma_n - \gamma) \omega(x) + L_{M,n}(\gamma \omega)(x)$. \hfill (8.8)

By Lemma 4.3, the last term on the right-hand side of (8.8) converges to $L_M(\gamma \omega)(x)$ uniformly on $E$, once we have that $\gamma \omega$ is $C^2$. We conclude that $L_{M,n}(\gamma \omega)(x)$ is uniformly bounded in $E$. 
We only need to check that the first term on the right-hand side of (8.8) is also uniformly bounded on $E$.

Note that

$$L_{M,n} \left( \left( \frac{|x|^2}{\sigma_{M,n}^2} - \frac{|y|^2}{\sigma_{M}^2} \right) \omega \right)(x) = \left( \frac{1}{\sigma_{M,n}^2} - \frac{1}{\sigma_{M}^2} \right) L_{M,n}(|x| \omega)(x)$$

is uniformly bounded on $E$, also as a consequence of Lemma 4.3.

Then, it suffices to prove that

$$L_{M,n}((G_M * \rho - G_{M,n} * \rho_n) \omega)$$

is uniformly bounded on $E$.

We write

$$L_{M,n} = \frac{4c_n k_n \Delta_n}{n^{2-\alpha}} + c_n \tilde{L}_{M,n},$$

where

$$\Delta_n f(x) = \frac{n^2}{4} \sum_{y \in \frac{1}{n} \mathbb{Z}^2, |y| = \frac{1}{n}} f(x + y) - f(x)$$

and

$$\tilde{L}_{M,n} f(x) = \frac{1}{n^2} \sum_{y \in B_M \cup \partial B_M} \frac{F_n(y)(f(x + y) - f(x)))}{|y|^{2+\alpha}},$$

for a function $f$ defined on $\frac{1}{n} \mathbb{Z}^2$.

Let us see that $\tilde{L}_{M,n}((G_M * \rho - G_{M,n} * \rho_n) \omega)$ is uniformly bounded on $E$. Recall that for $x \in \frac{1}{n} \mathbb{Z}^2$, we defined $\rho_n(x) = \rho^\perp(x)$.

Denote $E_M := E + B_{M+1}$. For $x \in E^*_M$ we have

$$G_M * \rho(x) - G_{M,n} * \rho(x) = \sum_{y \in \frac{1}{n} \mathbb{Z}^2} \int_{y \cap [0,1)^d} (G_M(z) \rho(x - z) - G_{M,n}(y) \rho(x - y)) dz$$

$$= I_1(x) + I_2(x),$$

where

$$I_1(x) := \sum_{y \in \frac{1}{n} \mathbb{Z}^2} \int_{y \cap [0,1)^d} (G_M(z) \rho(x - z) - G_M(y) \rho(x - y)) dz,$$

$$I_2(x) := \frac{1}{n^2} \sum_{y \in \frac{1}{n} \mathbb{Z}^2} (G_M(y) \rho(x - y) - G_{M,n}(y) \rho(x - y)).$$

Since $K_1$ contains the support of the initial distribution $\rho$, the sums on the definition of $I_1(x)$ and $I_2(x)$ are taken over a set which is safely away from the origin. Let us say $\tilde{E}$.

By symmetry, we write $I_1(x)$ as

$$I_1(x) = \frac{1}{2} \sum_{y \in \tilde{E} : |y| = \frac{1}{n}} G_M(y + z) \rho(x - y - z) + G_M(y - z) \rho(x - y + z) - 2G_M(y) \rho(x - y) dz.$$

Since the sum above has been taken over a set $\tilde{E}$ that does not intersects a neighborhood of the origin, the function $G_M$, and consequently the function $G_M(\cdot) \rho(x - \cdot)$ is $C^2$, that means that using second order Taylor’s Theorem we have
where \( \mu \) denotes the Lebesgue measure in \( \mathbb{R}^2 \) and \( C \) is a positive constant depending only on \( \alpha \), which has changed from one line to the other.

Let us now estimate \( I_2 \). As noticed before, the sum \( I_2 \) (as the sum \( I_1 \)) is taken over a set \( \tilde{E} \) that does not intersects a neighborhood of the origin. By Theorem 1.3, we have

\[
|I_2(x)| \leq \frac{C}{n^2}, \quad (8.11)
\]

where \( C \) is a positive constant which only depends on \( \alpha \) and on the set \( \tilde{E} \).

We have proved that, for all \( x \in E_M \)

\[
|G_M * \rho(x) - G_{M,n} * \rho(x)| \leq \frac{C}{n^\alpha}, \quad (8.14)
\]

As a consequence of (8.14), for \( x \in E \) we have

\[
\left| \hat{\mathcal{L}}_{M,n}((G_M * \rho - G_{M,n} * \rho) \omega)(x) \right| \leq \sum_{y \in \mathbb{Z}^2 \setminus B_1^+} \frac{1}{|y|^{2+\alpha}},
\]

which proves that \( \hat{\mathcal{L}}_{M,n}((G_M * \rho - G_{M,n} * \rho) \omega)(x) \) is uniformly bounded in \( E \).

The same argument proves that \( \frac{\Delta_n((G_M * \rho - G_{M,n} * \rho) \omega)}{n^{\alpha}} \) is uniformly bounded on \( E \), which concludes the proof. \( \square \)

**Lemma 8.7.** The function \( \phi_n \) defined in (8.6) satisfies

\[
|s_n - G * (\phi_n)\Box| \to 0
\]

uniformly on \( K_2 \).

**Proof.** In the beginning of the proof of Lemma 8.6 we observed that \( \text{supp}(\phi_n) \subset K_4 \). Note that

\[
G_M * (\phi_n)\Box(x) = \sum_{y \in K_4} \phi_n(y) \int_{S^\Box} G_M(x - z)dz,
\]

and

\[
s_n(x) = G_{M,n} * \phi_n(x) \text{ for } x \in K_2^2.
\]

Hence for \( x \in K_2 \)

\[
s_n(x) - G_M * (\phi_n)\Box(x) = G_{M,n} * \phi_n(x^+) - G_M * (\phi_n)\Box(x)
= \sum_{y \in K_4} \phi_n(y) \alpha_n^x(y),
\]

for \( \alpha_n^x(y) \) defined in (7.4).
By Lemmas 8.6 and 7.1 there exists a positive constant $C$ independent of $n$ and $x$, and a positive function $f$ in $L^1_{loc}$ such that

$$|s_n(x) - G_M * (\phi_n)(x)| \leq \frac{C}{n^{1+\alpha}} \sum_{y \in K_q} f(y - x^\circ)$$

$$\leq \frac{C}{n^{\alpha-1}} \int_{K_q} f(y - x^\circ) dy \quad (8.15)$$

$$\leq \frac{C}{n^{\alpha-1}} \int_{(K_2 + K_4)} f(y) dy \quad (8.16)$$

$$\leq \frac{C}{n^{\alpha-1}}, \quad (8.17)$$

where the constant $C$ has changed from one line to the other. In (8.15) we used the approximation of the integral by Riemann’s sums, in (8.16) we used that $x \in K_2$ and $y \in K_4$, and in (8.17) we used that $f$ is $L^1_{loc}$. This finishes the proof. \(\square\)

**Proof of Proposition 8.5.** As we pointed out before, we only need to prove that $s_n$ converges to $s$ uniformly in a ball $\Omega$ as the one considered in Proposition 8.1. Recall from (8.4) and (8.5) the expression we are using for $s_n$ and $s$.

Recall from (8.3), our assumptions about the sets $K_j$, for $j = 1, 2, 3, 4$. We will prove that $s_n$ converges to $s$ uniformly on $K_1$.

We write

$$\tilde{s}(x) = \int_{\mathbb{R}^2} s(y) \lambda^{-2} \eta(\frac{x - y}{\lambda}) dy, \quad (8.18)$$

where $\eta$ is the standard smooth mollifier

$$\eta(x) = \begin{cases} \beta e^{1/|x|^2-1} & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases}$$

normalized so that $\int_{\mathbb{R}^2} \eta dx = 1$ (see [3]). Then $\tilde{s}$ is smooth and superharmonic.

By Proposition 5.6, $\tilde{s}$ is continuous, and by compactness, $s$ is uniformly continuous on $K_2$, so taking $\lambda$ sufficiently small in (8.18), we have $|s - \tilde{s}| < \epsilon$ in $K_2$.

By Proposition 4.3, there exists a positive constant $C_1$ such that the function

$$q_n(x) = \tilde{s}^\circ - C_1 n^{2-\alpha} |x|^2$$

satisfies $L_{M,n}q_n \leq 0$ in $K_2^\circ$. By Proposition 7.2, we have $\gamma_n^\circ \to \gamma$ uniformly in $K_2$. Taking $n$ large enough to ensure that $\frac{C_1}{n^{2-\alpha}} |x|^2 \leq \epsilon$ in $K_2$ and $|\gamma_n - \gamma^\circ| < \epsilon$ in $K_2^\circ$, we obtain

$$q_n > \tilde{s}^\circ - \epsilon > s^\circ - 2\epsilon \geq \gamma^\circ - 2\epsilon > \gamma_n - 3\epsilon$$

in $K_2^\circ$.

Now, the function $f_n = \max(q_n + 3\epsilon, \gamma_n)$ coincides with $q_n + 3\epsilon$ in $K_2^\circ$, then $f_n$ satisfies $L_{M,n}f_n \leq 0$ in $K_1^\circ$. It follows that $f_n \geq s_n$, hence, in $K_1^\circ$, we have

$$s_n \leq q_n + 3\epsilon < \tilde{s}^\circ + 3\epsilon < s^\circ + 4\epsilon.$$

By the uniform continuity of $s$ on $K_1$, taking $n$ larger if necessary we have $|s - s^\circ| < \epsilon$ in $K_1$, and hence $s_n^\circ < s + 5\epsilon$ in $K_1$.

For the reverse inequality, recall from (8.6) the definition of the function $\phi_n$. By Lemma 8.7 we have

$$|s_n^\circ - G_M * (\phi_n)^\circ| < \epsilon.$$
in $K_2$, and hence
\[ G_M \ast (\phi_n) \square > \gamma_n - \epsilon > \gamma - 2\epsilon \]
in $K_2$ for $n$ sufficiently large. Since $\phi_n$ is nonnegative on $K_2$, by Lemma 8.4 the function $G_M \ast (\phi_n) \square$ is superharmonic on $K_2$, so the function
\[ g_n = \max(G_M \ast (\phi_n) \square + 2\epsilon, \gamma) \]
coincides with $G_M \ast (\phi_n) \square + 2\epsilon$ in $K_2$, then $g_n$ is superharmonic on $K_1$ for sufficiently large $n$. By the definition of $s$ it follows that $g_n \geq s$, hence
\[ s \square > G_M \ast (\phi_n) \square - \epsilon \geq s - 3\epsilon \]
on $K_1$ for sufficiently large $n$.

**Proposition 8.8.** Let $u_n$ and $u$ respectively be the discrete and continuous odometer function defined in (1.19) and (1.28). Then
\[ u_n \square \to u \]
uniformly as $n$ goes to infinity.

**Proof.** Let $\Omega$ be a ball as in Proposition 8.1. By Propositions 7.2 and 8.5 we have
\[ \gamma_n \square \to \gamma \text{ and } s \square \to s \text{ uniformly on } \Omega. \]
By Lemma 1.20 we have $u_n = s_n - \gamma_n$. Since $u_n = 0 = u$ outside $\Omega$, we conclude $u_n \square \to s - \gamma = u$ uniformly.

9. Convergence of the final distribution

Now we are in position to prove the main theorem of this paper, which tells us about the weak-* convergence of the final distributions.

**Proof of Theorem 1.2.** Recall from (1.21) that the final distribution $\nu_n$ of the truncated $\alpha$-stable divisible sandpile in $1/n\mathbb{Z}^2$ starting with initial density of mass $\rho_n = \rho^i$ satisfies the equation
\[ \nu_n = \rho_n + L_{M,n} u_n. \]
Let $\phi \in C_c^\infty(\mathbb{R}^2)$ be a test function. We have
\[ \langle \nu_n \square, \phi \rangle = \langle (L_{M,n}(u_n - G_{M,n} \ast \rho_n)) \square, \phi \rangle = \langle u_n \square - (G_{M,n} \ast \rho_n) \square, L_{M,n} \phi \rangle. \]
Lemma 1.3 ensures that $L_{M,n} \phi$ converges to $L_M \phi$ uniformly. By Proposition 8.8 and 7.2 we have the convergence $(u_n - G_{M,n} \ast \rho^i) \square \to u - G_M \ast \rho$ uniformly on compact sets of $\mathbb{R}^2$. Since $\phi$ has compact support, there exists a ball which contains the support of $L_{M,n} \phi$ for all $n$. From (9.1) we have
\[ \lim_{n \to \infty} \langle \nu_n \square, \phi \rangle = \langle u - G_M \ast \rho, L_M \phi \rangle = \langle \rho + L_M u, \phi \rangle. \]
Let us denote $\nu = \rho + L_M u$, we have just proved that $\nu_n$ converges towards $\nu$ in the weak-* topology. We must show that the distribution $\nu$ is actually a function.

By Lemma 8.1 we can also say that there exists a ball $U$ which contains the support of the final distribution $\nu_n \square$ for all $n$. Then we have
\[ \langle \nu_n \square, \phi \rangle = \int_U \nu_n \square(x) \phi(x) dx. \]
Since $0 \leq \nu_n \square \leq 1$ for all $n$, by Holder’s inequality we have
\[ |\langle \nu, \phi \rangle| = \lim_{n \to \infty} |\langle \nu_n \square, \phi \rangle| \leq \int_U |\phi(x)| dx \leq \mu(U \cap \text{supp}(\phi)) \| \phi \|_\infty \]
(9.2)
where $\mu$ denotes the Lebesgue measure. This shows that $\nu$ is a continuous linear functional, then by the Riesz-Markov Theorem, there exists a Radon measure $\lambda$ such that

$$\langle \nu, \phi \rangle = \int \phi(x) d\lambda(x).$$

Moreover, the measure $\lambda$ is non-negative, since the functional $\nu$ is positive.

Let $A$ be a measurable set, consider a family $(A_\epsilon)$ of measurable sets such that $A_\epsilon \downarrow A$ when $\epsilon \downarrow 0$. Now consider a family $(g_\epsilon)$ of infinitely differentiable functions such that $0 \leq g_\epsilon \leq 1$ and $\text{supp}(g_\epsilon) \subset A$. In particular, from (9.2), for all $\epsilon$ we have

$$\lambda(A) \leq \langle \nu, g_\epsilon \rangle \leq \mu(A_\epsilon) \| g_\epsilon \|_\infty.$$

Taking limits as $\epsilon \downarrow 0$ we obtain

$$\lambda(A) \leq \mu(A).$$

Then $\lambda$ is absolutely continuous with respect to the Lebesgue measure, therefore the linear functional $\nu$ is indeed a function. □

10. Proofs of analytical estimates

In this section we will proof the propositions and lemmas of Section 4. We will often use in our estimates an universal constant $C$ which may change form line to line and depends only on the parameter $\alpha$, except when we specify another possible dependency.

Proof of Lemma 4.1. For $R > 0$, consider the function $F_R$ in $\mathbb{Z}^2$, defined as

$$F_R(y) = \mu( [y - \frac{1}{2}, y + \frac{1}{2}]^2 \cap B_R).$$

For a domain $A$ in $\mathbb{R}^2$, we denote $A^{n,:,:}$ for its restriction to $\frac{1}{n} \mathbb{Z}^2$, that is $A^{n,:,:} = A \cap \frac{1}{n} \mathbb{Z}^2$. And denote the discrete boundary as

$$\partial A^{n,:,:} = \{ x \in \frac{1}{n} \mathbb{Z}^2 : [x - \frac{1}{2n}, x + \frac{1}{2n}]^2 \not\subseteq A \text{ and } [x - \frac{1}{2n}, x + \frac{1}{2n}]^2 \cap A \neq \emptyset \}.$$ 

Notice that $\hat{F}_R$ is equal to 1 in $B_{1,:}^{1,:,:} \setminus \partial B_{1,:}^{1,:,:}$, and equal to 0 outside $B_{1,:}^{1,:,:} \cup \partial B_{1,:}^{1,:,:}$. Let us denote

$$k(R) = \int_{B_R} \frac{1}{|y|^{\alpha}} dy - \sum_{y \in B_{1,:}^{1,:,:} \cup \partial B_{1,:}^{1,:,:} \setminus \{0\}} \hat{F}(y) |y|^{\alpha},$$

Consider $R' > R$. A change of variables allows us to write

$$k(R') - k(R) = \frac{n^{2-\alpha}}{2} \sum_{y \in A^{n,:,:} \cap \partial A^{n,:,:} \setminus \{0\}} \int_{[-\frac{1}{2n}, \frac{1}{2n}]^2} \left( \frac{1}{|y + z|^{\alpha}} + \frac{1}{|y - z|^{\alpha}} - 2 \frac{1}{|y|^{\alpha}} \right) dz.$$

We use second order Taylor’s Theorem and obtain

$$|k(R') - k(R)| \leq \frac{C}{n^{2+\alpha}} \sum_{y \in A^{n,:,:} \cap \partial A^{n,:,:} \setminus \{0\}} \frac{1}{|y|^{2+\alpha}}$$

$$\leq \frac{C}{n^\alpha} \int_{\mathbb{R}^2 \setminus B_R} \frac{1}{|y|^{2+\alpha}} dy$$

$$\leq \frac{C}{R^\alpha}.$$
We have proved that the sequence \( \{k(R)\}_R \) is Cauchy, so it does converge. We can ensure then that there exists \( S > 0 \) such that

\[
|k(R)| \leq S \text{ for all } R. \tag{10.1}
\]

Although the sequence \( k(R) \) does converge, neither the integral nor the sum on its definition converge. Then there exists \( r > 0 \) so that

\[
\sum_{y \in B^1_{Mn}} \frac{1}{|y|^{2+\alpha}} > S. \tag{10.2}
\]

Taking \( r \) as in (10.2) and \( M \) bigger that \( r \), by equation (10.1) we have that \( k_1 = k_1(r, M) \) as in (1.2) is positive. □

**Proof of Lemma 4.2.** Denote by \( k := \lim_{n \to \infty} k_n \). In particular, from the proof of Lemma 4.1, we have

\[
|k - k_n| \leq C n^{2-\alpha},
\]

for a positive constant \( C \) which only depends on \( \alpha \).

On the other hand, we have

\[
\sum_{y \in \Z^2} \frac{1}{|y|^{2+\alpha}} - \sum_{y \in B^1_{Mn} \cup \partial B^1_{Mn}} \frac{F_n(y)}{|y|^{2+\alpha}} \leq \sum_{y \in \Z^2 \setminus B_{Mn}} \frac{1}{|y|^{2+\alpha}} \leq \frac{C}{n^\alpha},
\]

also for a constant \( C \) depending only on \( \alpha \). This concludes the proof. □

**Proof of Lemma 4.3.** Let us write

\[
L_{M,n} f(x) = \frac{4c_n k_n \Delta_n f(x)}{n^{2-\alpha}} + \frac{c_n}{n^2} \sum_{y \in A^1_{\frac{1}{M},M} \cup \partial A^1_{\frac{1}{M},M}} \frac{F_n(y)(f(x+y) - f(x))}{|y|^{2+\alpha}},
\]

where \( \Delta_n f(x) = \frac{n^2}{2} \sum_{y \in \Z^2, |y|=\frac{1}{M}} (f(x+y) - f(x)) \).

Let us suppose \( x \in K \), for a compact set \( K \). Since \( f \in C^2 \), there exists a constant \( C \), which depends on the first and second derivatives of \( f \) in \( K \), such that

\[
|4c_n k_n \Delta_n f(x)| \leq \frac{C \Delta f(x)}{n^{2-\alpha}} \leq \frac{C}{n^{2-\alpha}}.
\]

Denote

\[
g_x(y) = \frac{f(x+y) + f(x-y) - 2f(x)}{2|x|^{2+\alpha}}.
\]

Since, by Lemma 4.2 \( |c_n - c| \leq \frac{C}{n^{\alpha}} \), for \( C > 0 \) depending only on \( \alpha \), it suffices to prove that

\[
\left| \int_{B_M} g_x(y) dy - \frac{1}{n^2} \sum_{y \in A^1_{\frac{1}{M},M} \cup \partial A^1_{\frac{1}{M},M}} F_n(y) g_x(y) \right| \leq \frac{C}{n^{2-\alpha}}, \tag{10.3}
\]

for a constant \( C \) depending on the first and second derivatives of \( f \) in \( K + B_M \).

In order to prove the inequality (10.3) we write the left-hand side as:

\[
\int_{B_{\frac{1}{M}}} g_x(y) dy + \frac{1}{2} \sum_{y \in A^1_{\frac{1}{M},M} \cup \partial A^1_{\frac{1}{M},M}} \int_{\frac{1}{M} \setminus \frac{1}{M} |z|^2} \tilde{g}_{x,y}(z) dz,
\]
where $\tilde{g}_{x,y}(z) = g_x(y+z) + g_x(y-z) - 2g_x(y)$. For $y$ small, we have $|g_x(y)| \leq \frac{C}{|y|}$, for $x \in K$, with $C$ depending on the second partial derivatives of $f$ in $K$. Then

$$\left| \int_{B_{\frac{r}{2}}} g_x(y) dy \right| \leq \frac{C}{n^{2-\alpha}}.$$ We need to estimate

$$I := \sum_{y \in A_{\frac{r}{2},M} \cup \partial A_{\frac{r}{2},M}} \int_{[-\frac{1}{2n}, \frac{1}{2n}]^2} \tilde{g}_{x,y}(z) dz.$$

(10.4)

We use second order Taylor's Theorem in every argument of the sum and write

$$\tilde{g}_{x,y}(z) = \sum_{|\beta| = 2} z^\beta R_\beta(z),$$

where $R_\beta(z) = \frac{|\beta|}{|\beta|!} \int_0^1 (1-t)^{|\beta|} g_x(y) (tz) dt$. Since $f$ is twice differentiable, there exists a positive constant $C$ which only depends on the first and second partial derivatives of $f$ in $K + B_M$ and in the parameter $\alpha$, such that for all $z \in [-\frac{1}{2n}, \frac{1}{2n}]^2$, holds:

$$|R_\beta(z)| \leq C \left( \frac{1}{|y+z|^{2+\alpha}} + \frac{1}{|y-z|^{2+\alpha}} \right).$$

Notice that $|z| \leq \frac{\sqrt{2}}{2n}$ if $z \in [-\frac{1}{2n}, \frac{1}{2n}]^2$, since $y \in \frac{1}{n} \mathbb{Z}^2 \setminus \{0\}$, we have (for a bigger constant $C$)

$$|R_\beta(z)| \leq C \left( \frac{1}{|y| - \frac{\sqrt{2}}{2n}} \right)^{2+\alpha}. $$

We conclude

$$|I| \leq \frac{C}{n^4} \sum_{y \in A_{\frac{r}{2},M} \cup \partial A_{\frac{r}{2},M}} \frac{1}{(|y| - \frac{\sqrt{2}}{2n})^{2+\alpha}} \leq \frac{C}{n^{2-\alpha}} \sum_{y \in \mathbb{Z}^2 \setminus B_r} \frac{1}{(|y| - \frac{\sqrt{2}}{2n})^{2+\alpha}} \leq \frac{C}{n^{2-\alpha}}.$$ We conclude

This finishes our proof. \hfill \square

**Proof of lemma 4.4.** Note that

$$L_M(G_M \ast \phi)(x) = \int_{\mathbb{R}^2} G_M(y) L_M \phi(x-y) dy.$$ By selfadjointness of $L_M$, we have

$$L_M(G_M \ast \phi)(x) = \int_{\mathbb{R}^2} L_M G_M(x-y) \phi(y) dy = \int_{\mathbb{R}^2} F_x(y) dy,$$

where

$$F_x(y) = \frac{c \phi(y)}{2(2\pi)^2} \int_{B_M} \int_{\mathbb{R}^2} e^{i(\theta \cdot x - y + z)} + e^{i(\theta \cdot x - y - z)} - 2e^{i(\theta \cdot x - y)} \frac{z^{2+\alpha} \psi_M(\theta)}{|z|^{2+\alpha}} d\theta dz.$$ (10.5)
Since the integrand in \([10.5]\) is not absolutely integrable, we are not able to apply Fubini’s theorem directly. To overcome this difficulty we make the following trick,

\[
F_x(y) = \lim_{t \to 0} \frac{c \phi(y)}{2(2\pi)^2} \int_{\mathbb{R}^2} e^{i(\theta-x-y)z} + e^{i(\theta-x-y)z} - 2e^{i(\theta-x-y)}e^{-t|\theta|^2} d\theta dz.
\]

(10.6)

Now we use Fubini and integrate first on \(z\). By the definition of \(\psi_M\) we have

\[
F_x(y) = \lim_{t \to 0} \frac{c \phi(y)}{2(2\pi)^2} \int_{\mathbb{R}^2} e^{i(\theta-x-y)}e^{-t|\theta|^2} d\theta.
\]

Hence, using Fubini again

\[
\int_{\mathbb{R}^2} \phi(y)L_M G_M(x - y)dy = \lim_{t \to 0} \frac{-1}{(2\pi)^2} \int_{\mathbb{R}^2} \phi(y)e^{i(\theta-x-y)}e^{-t|\theta|^2} dy d\theta
\]

\[= \lim_{t \to 0} \frac{-1}{(2\pi)^2} \int_{\mathbb{R}^2} \phi(\theta)e^{i(\theta-x)}e^{-t|\theta|^2} d\theta
\]

\[= \frac{-1}{(2\pi)^2} \int_{\mathbb{R}^2} \phi(\theta)e^{i(\theta-x)} d\theta
\]

\[= -\phi(x).
\]

Since \(\phi \in C_c^\infty\), in particular \(\hat{\phi} \in L^1\). This fact justifies the last two lines above. \(\Box\)

**Proof of Lemma 4.5.** Before starting the proof, a little note about the constants may be useful.

**Remark 10.1.** Recall that the constant \(c\) is defined by \(c = \lim c_n\). Let us denote by \(c_\alpha\) and \(\tilde{c}_\alpha\), the constant satisfying

\[c_\alpha|\theta|^\alpha = c \int_{\mathbb{R}^2} \frac{1 - \cos(\theta \cdot x)}{|x|^{2+\alpha}} dx
\]

and

\[
\tilde{c}_\alpha = \int_{\mathbb{R}^2} \frac{\cos(\theta \cdot x)}{|x|^{2-\alpha}} dx.
\]

Standard computations shows that \(\tilde{c}_\alpha = \frac{\alpha^2 c_\alpha}{c}\).

If \(|\theta| < 1\) we use Taylor’s Theorem and see that

\[
\psi_1(\theta) = c \int_{B_1} \frac{1 - \cos(\theta \cdot y)}{|y|^{2+\alpha}} dy = c \int_{B_1} \frac{(\theta \cdot y)^2}{2|y|^{2+\alpha}} dy + |\theta|^4 g_1(\theta),
\]

for a smooth function \(g\) in \(B_1\).

Note that

\[
\frac{c}{2} \int_{B_1} \frac{(\theta \cdot y)^2}{|y|^{2+\alpha}} dy = \frac{|\theta|^2}{4} \int_{B_1} \frac{1}{|y|^{\alpha}} dy = \frac{\sigma_1^2 |\theta|^2}{4}.
\]

Then for \(\theta \in B_1\)

\[
\psi_1(\theta) = \frac{\sigma_1^2}{4} |\theta|^2 + |\theta|^4 g_1(\theta).
\]

(10.7)

Since \(\psi_1(\theta) = 0\) only if \(\theta = 0\), we easily see that there exists a smooth function \(g_2\) in \(B_1\) such that

\[
\frac{1}{\psi_1(\theta)} = \frac{4}{\sigma_1^2 |\theta|^2} + g_2(\theta).
\]

(10.8)
Now let us suppose $|\theta| \geq 1$. Recalling from (10.1) the definition of the constant $c_\alpha$, we write

$$
\psi_1(\theta) = c_\alpha |\theta|^{\alpha} - c \int_{B_1} \frac{1 - \cos(\theta \cdot y)}{|y|^{2+\alpha}} dy.
$$

(10.9)

The function $h_1$ defined by $h_1(\theta) := c \int_{B_1} \frac{1 - \cos(\theta \cdot y)}{|y|^{2+\alpha}} dy$ is clearly bounded. We can easily see that for $|\theta| > 1$ there exists a bounded function $h_2$, such that:

$$
\frac{1}{\psi_1(\theta)} = \frac{1}{c_\alpha |\theta|^{\alpha}} + \frac{h_2(\theta)}{|\theta|^{2\alpha}}.
$$

(10.10)

Furthermore the function $h_2$ satisfies $\frac{d}{d\theta} \frac{h_2(\theta)}{|\theta|^{2\alpha}} = \frac{h_2(\theta)}{|\theta|^{2\alpha}}$ for $i = 1, 2$, and $\Delta \left( \frac{h_2(\theta)}{|\theta|^{2\alpha}} \right) = \frac{h_3(\theta)}{|\theta|^{2\alpha}}$ for bounded functions $h_2, h_3$ in $B_1^c$. The proof of this assertion lies in the computation of the first to derivatives of $\frac{1}{\psi_1(\theta)} - \frac{1}{c_\alpha |\theta|^{\alpha}}$ outside $B_1$.

We can write $G_1$ as

$$
G_1(x) = \frac{1}{(2\pi)^2} \tilde{G}_1(x) - \frac{1}{(2\pi)^2} \tilde{G}_1(x_0),
$$

(10.11)

where

$$
\tilde{G}_1(x) = \int_{B_1} \frac{\cos(\theta \cdot x) - 1}{\psi_1(\theta)} d\theta + \int_{B_1^c} \frac{\cos(\theta \cdot x)}{\psi_1(\theta)} d\theta.
$$

(10.12)

Let us denote

$$
f_1(x) = \int_{B_1} \frac{\cos(\theta \cdot x) - 1}{|\theta|^2} d\theta,
$$

(10.13)

$$
f_2(x) = \int_{B_1^c} \frac{\cos(\theta \cdot x)}{|\theta|^\alpha} d\theta,
$$

(10.14)

$$
f_3(x) = \int_{B_1} (\cos(\theta \cdot x) - 1) g_2(\theta) d\theta,
$$

(10.15)

$$
f_4(x) = \int_{B_1^c} \frac{h_2(\theta) \cos(\theta \cdot x)}{|\theta|^{2\alpha}} d\theta.
$$

(10.16)

By (10.8) and (10.10) we have

$$
\tilde{G}_1(x) = \frac{4}{\sigma_1^2} f_1(x) + \frac{1}{c_\alpha} f_2(x) + f_3(x) + f_4(x).
$$

(10.17)

We split the estimation of $G_1$ in four steps, consisting in the estimation for $|x| < 1$ and $|x| \geq 1$ of the functions $f_1, f_2, f_3$ and $f_4$.

Estimate of $f_1$: Suppose first $|x| < 1$, since in (10.13) we are integrating a bounded function, $f_1$ is also bounded in the unitary ball. That is, there exists a constant $C > 0$ such that

$$
|f_1(x)| < C, \text{ for all } |x| < 1.
$$

(10.18)

Now let us assume $|x| \geq 1$, we denote $\hat{x} = \frac{x}{|x|}$ and perform a change of variables to see that:

$$
f_1(x) = \int_{B_1 \setminus B_1^c} \frac{\cos(\theta \cdot \hat{x}) - 1}{|\theta|^2} d\theta
$$

$$
= \int_{B_1 \setminus B_1^c \setminus B_1} \frac{1}{|\theta|^2} d\theta + \int_{B_1 \setminus B_1^c \setminus B_1} \frac{\cos(\theta \cdot \hat{x})}{|\theta|^2} d\theta + \int_{B_1} \frac{\cos(\theta \cdot \hat{x}) - 1}{|\theta|^2} d\theta.
$$
The first term of the sum above can be explicitly computed and give us a logarithmic term:
\[- \int_{B|x| \setminus B_1} \frac{1}{|\theta|^2} d\theta = -2\pi \log |x|.
\]
Notice that the last term is a constant, let us call it \(C_1\). Let us investigate the second term, we call it \(h(x)\), that is
\[h(x) = \int_{B|x| \setminus B_1} \frac{\cos(\theta \cdot \hat{x})}{|\theta|^2} d\theta.
\]
(10.19)

We integrate by parts in the following way
\[
h(x) = \int_{B|x| \setminus B_1} \frac{\Delta(1 - \cos(\theta \cdot \hat{x}))}{|\theta|^2} d\theta
\]
\[= \int_{B|x| \setminus B_1} \frac{4(1 - \cos(\theta \cdot \hat{x}))}{|\theta|^4} d\theta - \int_{\partial(B|x| \setminus B_1)} \frac{1}{|\theta|^2} \frac{\partial}{\partial \nu} \cos(\theta \cdot \hat{x}) dS
\]
\[- \int_{\partial(B|x| \setminus B_1)} (1 - \cos(\theta \cdot \hat{x})) \frac{1}{|\theta|^2} \frac{\partial}{\partial \nu} dS,
\]
where \(\nu\) represents the inward pointing unit normal along \(\partial(B|x| \setminus B_1)\) (see [3]). The integration by parts procedure allows us to conclude that \(\tilde{h}(x)\) is bounded.

We have proved that there exists a explicitly computable constant \(C_1\) and a bounded function \(\tilde{h}(x)\) given by (10.19), such that
\[f_1(x) = -2\pi \log |x| + \tilde{h}(x) + C_1, \text{ for all } |x| \geq 1.
\]
(10.20)

**Estimate of \(f_2\):** Recall from (10.1) the definition of the constant \(\tilde{c}_\alpha\). Let us write
\[f_2(x) = \int_{B_1^+} \frac{\cos(\theta \cdot x)}{|\theta|^\alpha} d\theta
\]
\[= \tilde{c}_\alpha \frac{|x|^{2-\alpha}}{|x|^{2-\alpha}} - \int_{B_1} \frac{\cos(\theta \cdot x)}{|\theta|^\alpha} d\theta.
\]

Let us investigate the behavior of the function
\[
\tilde{f}_2(x) := \int_{B_1} \frac{\cos(\theta \cdot x)}{|\theta|^\alpha} d\theta.
\]

We easily see that the function \(\tilde{f}_2\) is bounded for \(|x| < 1\). Then there exists a constant \(C > 0\) independent of \(x\) such that
\[|f_2(x) - \tilde{c}_\alpha| < C, \text{ for all } |x| < 1.
\]
(10.21)

Let us study its asymptotic behavior for \(x\) large. Suppose \(|x| \geq 1\), we write
\[
\tilde{f}_2(x) = \frac{1}{|x|^2} \int_{B_1} \frac{\Delta(1 - \cos(\theta \cdot x))}{|\theta|^\alpha} d\theta.
\]

We perform an integration by parts below. We again denote by \(\nu\) the inward pointing unit normal vector along the surface \(\partial B_1\).
\[
\tilde{f}_2(x) = \frac{\alpha^2}{|x|^2} \int_{B_1} \frac{1 - \cos(\theta \cdot x)}{|\theta|^{2+\alpha}} d\theta - \frac{1}{|x|^2} \int_{\partial B_1} (1 - \cos(\theta \cdot x)) \frac{1}{|\theta|^\alpha} \frac{\partial}{\partial \nu} dS
\]
\[+ \frac{1}{|x|^2} \int_{\partial B_1} \frac{1}{|\theta|^\alpha} \frac{\partial}{\partial \nu} (1 - \cos(\theta \cdot x)) dS.
\]
The first two integrals above decay as \( \frac{1}{|x|^2} \), and the last one decays as \( \frac{1}{|x|} \). Since we are considering \(|x| \geq 1\) we obtain that there exists a constant \( C > 0 \) which only depends on \( \alpha \), such that \( |f_2(x)| < \frac{C}{|x|^2} \).

Since \( f_2(x) = \frac{r^{\alpha}}{|x|^2 - \alpha} - f_2(x) \), and since \( 2 - \alpha < 1 \), we obtain that there exists a positive constant \( C \) which depends on \( \alpha \), such that

\[
|f_2(x)| < \frac{C}{|x|^{2-\alpha}}, \text{ for all } |x| \geq 1.
\]  

(10.22)

**Estimate of \( f_3 \):** It is easy to see that \( f_3(x) \) is bounded for \(|x| < 1\), that is, there exists a positive constant \( C \) depending on \( \alpha \), such that

\[
|f_3(x)| < C, \text{ for all } |x| < 1.
\]

(10.23)

If \(|x| \geq 1\), we have

\[
f_3(x) = \int_{B_1} g_2(\theta) \cos(\theta \cdot x) d\theta + C_2,
\]

for a constant \( C_2 \). We perform an integration by parts in the first term on the right of the above formula, just like we did in the estimation of \( f_2 \) and conclude that there exists a positive constant \( C \) depending only on \( \alpha \), such that

\[
|f_3(x) - C_2| < \frac{C}{|x|^2}, \text{ for all } |x| \geq 1.
\]

(10.24)

**Estimate of \( f_4 \):** Since \( \alpha > 1 \) and consequently \( 2\alpha > 2 \), we easily see that \( f_4 \) is bounded. Then there exists a positive constant \( C \) depending on \( \alpha \), such that

\[
|f_4(x)| < C, \text{ for all } |x| < 1.
\]

(10.25)

Consider \(|x| \geq 1\). Recall from the paragraph below equation (10.10) that there exists bounded functions \( h_3 \) and \( h_{2,i} \) for \( i = 1, 2 \) such that \( \Delta \frac{h_3(\theta)}{|\theta|^2} = \frac{h_3(\theta)}{|\theta|^{2+\frac{\alpha}{\alpha}}} \) and \( \frac{d h_3(\theta)}{d\theta} = \frac{h_3(\theta)}{|\theta|^{2+\frac{\alpha}{\alpha}}} \).

We write

\[
f_4(x) = \frac{1}{|x|^2} \int_{B_1} h_2(\theta) \Delta(1 - \cos(\theta \cdot x)) \frac{d\theta}{|\theta|^{2\alpha}}.
\]

Denoting by \( \nu \) the inward pointing unit normal vector along \( \partial B_1 \), we integrate by parts and write

\[
f_4(x) = \frac{1}{|x|^2} \int_{B_1} h_3(\theta)(1 - \cos(\theta \cdot x)) \frac{d\theta}{|\theta|^{2\alpha+2}} - \frac{1}{|x|^2} \int_{\partial B_1} (1 - \cos(\theta \cdot x)) \frac{\partial}{\partial\nu} \frac{h_2(\theta)}{|\theta|^{2\alpha}} dS
\]

\[+ \frac{1}{|x|^2} \int_{\partial B_1} \frac{h_3(\theta)}{|\theta|^{2\alpha}} \frac{\partial}{\partial\nu}(1 - \cos(\theta \cdot x)) dS.
\]

The first two integrals above decay as \( \frac{1}{|x|^2} \), and the last one decays as \( \frac{1}{|x|} \). Then there exists a positive constant \( C \) independent of \( x \) such that

\[
|f_4(x)| < \frac{C}{|x|}, \text{ for all } |x| \geq 1.
\]

(10.26)

We have proved that the functions \( f_1 \), \( f_2 \) and \( f_4 \) are bounded in the unitary ball \( B_1 \) (see (10.18), (10.23) and (10.25)). In (10.21) we proved that the function \( f_2 \)
has a special behavior in $B_1$. By (10.17), we conclude that there exists a positive constant $C$ which only depends on $\alpha$, such that
\[
\left| \tilde{G}_1(x) - \frac{\alpha^2}{c} \frac{1}{|x|^{2-\alpha}} \right| < C.
\] (10.27)
See Remark 10.1 to justify the constants in the formula above.

Now, in (10.22), (10.24) and (10.26) we estimate the asymptotic behavior of $f_2$, $f_3$ and $f_4$. Since the lowest decay of the three functions is $1/|x|^{2-\alpha}$, which is the decay of $f_2$, we can say that there exists a constant $C > 0$ depending on $\alpha$, such that
\[
|f_2(x)|, |f_3(x)|, |f_4(x)| < \frac{C}{|x|^{2-\alpha}}.
\]
Finally notice that the function $f_1$ has a special behavior in $B_1$ (see (10.20)). Then, by (10.17) we conclude that, there exists an explicitly computable constant $\tilde{C}$, and a positive constant $C$ independent of $x$, such that
\[
\left| \tilde{G}_1(x) + \frac{8\pi}{\sigma_1^2} \log |x| - \frac{4}{\sigma_1^2} \tilde{h}(x) - \tilde{C} \right| < \frac{C}{|x|^{2-\alpha}}, \text{ for all } |x| \geq 1,
\] (10.28)
where the function $\tilde{h}$ is given by (10.19).

By (10.11) and since we fixed $x_0$ so that $|x_0| = 1$, we conclude the proof.

**Proof of Proposition 4.6.** Recall from (4.2) and (4.3) the definitions of the functions $g$, $h$ and $h_1$. By Proposition 4.5 we have that, for $x \in B_M$, holds
\[
G_1(x) = \frac{\alpha^2 M^{2-\alpha}}{(2\pi)^2 c} \frac{1}{|x|^{2-\alpha}} + g\left(\frac{x}{M}\right),
\] (10.29)
and for $x \in B_M^c$ holds
\[
G_1\left(\frac{x}{M}\right) = -\frac{2}{\pi \sigma_1^2} \log |x| + \frac{2}{\pi \sigma_1^2} \log M \cdot h_1\left(\frac{x}{M}\right) + \frac{M^{2-\alpha} h\left(\frac{x}{M}\right)}{|x|^{2-\alpha}}.
\] (10.30)
Since we fixed $x_0$, so that $|x_0| = 1$, we will always have $\frac{|x_0|}{M} < 1$, then
\[
G_1\left(\frac{x_0}{M}\right) = \frac{\alpha^2 M^{2-\alpha}}{(2\pi)^2 c} + g\left(\frac{x_0}{M}\right).
\] (10.31)
By equation (4.1) we have
\[
G_M(x) = \frac{\alpha^2}{(2\pi)^2 c} \frac{1}{|x|^{2-\alpha}} - \frac{\alpha^2}{(2\pi)^2 c} + g\left(\frac{x}{M}\right) - g\left(\frac{x_0}{M}\right), \text{ for } x \in B_M
\]
and
\[
G_M(x) = -\frac{2}{\pi \sigma_1^2} \log |x| + \delta_M + \frac{h\left(\frac{x}{M}\right)}{|x|^{2-\alpha}} + h_M(x), \text{ for } x \in B_M^c
\]
where $\delta_M = -\frac{\alpha^2}{(2\pi)^2 c} + \frac{2}{\pi \sigma_1^2} \log M - \frac{g\left(\frac{x_0}{M}\right)}{M^{2-\alpha}}$, and $h_M$ is given by
\[
h_M(x) = \frac{1}{\pi^2 \sigma_1^2} \int_{B_{|x|} \setminus B_1} \cos(\theta \cdot \omega) \frac{1}{|\theta|^2} d\theta,
\]
for any unitary vector $\omega$.

In the equations above we use that $\sigma_1^2 M = M^{2-\alpha} \sigma_1^2$, this can be checked by a simple change of variables. \qed
Proof of Proposition 4.8. By equation (4.1), it suffices to prove that the first partial derivatives of $G_1$ are in $L^1_{loc}$.

We recall from (10.11) that $G_1(x) = \frac{1}{(2\pi)^2} G(x) - \frac{1}{(2\pi)^2} G_1(x_0)$, for $G_1$ defined in (10.12). Then we only need to prove that the first partial derivatives of $G_1$ are in $L^1_{loc}$.

By equation (10.17), it suffices that the first partial derivatives of the functions $f_1, f_2, f_3$ and $f_4$ defined in (10.13), (10.14), (10.15) and (10.16) are in $L^1_{loc}$. Standard arguments show this fact. □

Proof of Lemma 4.9. Let us write

$$G_M \ast \rho(x) = \int_{\mathbb{R}^2} G_M(y) \rho(x-y) dy$$

$$= \int_{B_M} G_M(y) \rho(x-y) dy + \int_{\mathbb{R}^2 \setminus B_M} G_M(y) \rho(x-y) dy.$$ We consider $x$ big enough so that $B_M \cap B(x, \text{supp}(\rho)) = \emptyset$, then we have

$$G_M \ast \rho(x) = \int_{\mathbb{R}^2 \setminus B_M} G_M(y) \rho(x-y) dy.$$ Now we use Proposition 4.6 to ensure that for $|y| > M$, there exists a constant $\delta_M$ (explicitly computable), and a bounded function $h_M$ in $B_M$ given by

$$h_M(y) = \frac{1}{\pi^2 \sigma_M^2} \int_{\mathbb{R}^2 \setminus B_1} \frac{\cos(\theta \cdot \omega)}{|\theta|^2} d\theta,$$ where $\omega$ is any unitary vector, (10.32) and a function $S(y)$ converging to zero as $y$ goes to infinity, such that

$$G_M(y) = -\frac{2}{\pi \sigma_M} \log |y| + h_M(y) + \delta_M + S(y).$$

Hence, we can write

$$G_M \ast \rho(x) = \int_{\mathbb{R}^2 \setminus B_M} S(y) \rho(x-y) dy + \int_{\text{supp}(\rho)} (-\frac{2}{\pi \sigma_M} \log |x-y| + h_M(x-y) + \delta_M) \rho(y) dy.$$ Now we differentiate twice, with respect to the variable $x$, under the integral sign. We note that all the second order derivatives of the logarithm function goes to zero at infinity, we have to check that the function $h_M$ has the same property, we write $h_M$ in polar coordinates:

$$h_M(x) = \frac{1}{\pi^2 \sigma_M^2} \int_0^{2\pi} \int_1^{\frac{M}{|x|}} \frac{\cos(r \cos \beta)}{r} dr d\beta.$$ This change of variables facilitates the computation of the second order derivatives of $h_M$, so we can easily check that they converge to zero at infinity.

We have proved that the second order derivatives of $G_M \ast \rho(x)$ converge to zero as $x$ goes to infinity.

Notice that the Hessian matrix of the function $-\frac{|x|^2}{\sigma_M}$ is equal to $-\frac{2}{\sigma_M} I d$ for all $x$. Then we can choose a ball $\Omega$ big enough, such that the Hessian matrix of the obstacle $\gamma$ is negative definite in $\mathbb{R}^2 \setminus \Omega$. As a consequence, $\gamma$ is concave outside the set $\Omega$. □
Proof of Proposition 4.10. We consider the concave envelope of the obstacle function $\gamma$, that is, the least concave function bigger than or equal to $\gamma$. We call this function $\Gamma$.

By Lemma 4.9 there exists a ball $\Omega$ such that the function $\Gamma$ coincides with $\gamma$ outside $\Omega$. We want to prove that $\Gamma$ is continuous and superharmonic.

The concave envelope is always continuous. Since the obstacle $\gamma$ is twice differentiable, then $\Gamma$ is twice differentiable almost everywhere.

Fix $x \in \mathbb{R}^2$ such that $\Gamma$ is twice differentiable in $x$. Consider a tangent line $\ell_x$ to the graph of the function $\Gamma$ at the point $x$. By concavity, $\ell_x$ lies above the function $\Gamma$, then $\Gamma - \ell_x$ is nonpositive and attains its maximum at $x$ (where it is equal to zero). Hence $L_M(\Gamma - \ell_x)(x) \leq 0$. Since the truncated fractional Laplacian of a linear function is equal to zero, we obtain that $L_M\Gamma(x) \leq 0$.

We repeat the same argument for all $x$ such that $\Gamma$ is twice differentiable and conclude that $L_M\Gamma(x) \leq 0$ almost everywhere. This implies that for every smooth function $\phi$ with compact support and positive, holds
\[
\langle \Gamma, L_M\phi \rangle \leq 0.
\]
We have proved that $\Gamma$ is continuous and superharmonic (recall Definition 1.26). Hence we can ensure that $\Gamma \geq s$, where $s$ is the least superharmonic majorant defined in (1.27), and since $\Gamma$ coincides with $\gamma$ in $\Omega^c$, so it does $s$. Since $u = s - \gamma$, we finish our proof. $\square$

11. Proof of Propositions of Section 6

In this section we use an universal positive constant $C$ that only depends on $\alpha$, which may change from line to line.

Proof of Lemma 6.1. We write
\[
\frac{1}{c} \psi_M(\theta) - \hat{\psi}_{M,n}(\theta) = \int_{B_M} \frac{1 - \cos(\theta \cdot y)}{|y|^{2+\alpha}} dy - \frac{1}{n^2} \sum_{y \in B_M \cup \partial B_M} \frac{F_n(y)(1 - \cos(\theta \cdot y))}{|y|^{2+\alpha}}. \tag{11.1}
\]

Since for $\theta \in B_{\frac{1}{2}}$ and $y \in B_M$ we have $|\langle \theta, y \rangle| \leq 1$, by Taylor’s Theorem there exists a smooth function $g$ defined in the unitary ball that satisfies
\[
1 - \cos(\theta \cdot y) = \frac{(\theta \cdot y)^2}{2} + (\theta \cdot y)^4 g(\theta \cdot y). \tag{11.2}
\]

An easy computation shows that
\[
\frac{1}{2} \int_{B_M} \frac{(\theta \cdot y)^2}{|y|^{2+\alpha}} dy = \frac{|\theta|^2}{4} \int_{B_M} \frac{1}{|y|^\alpha} dy.
\]

Analogously
\[
\frac{1}{2} \sum_{y \in B_M \cup \partial B_M} \frac{F_n(y)(\theta \cdot y)^2}{|y|^{2+\alpha}} = \frac{|\theta|^2}{4} \sum_{y \in B_M \cup \partial B_M} \frac{F_n(y)}{|y|^\alpha}.
\]

Then using the definition of $k_n$ (recall from (1.12)), in (11.1) we have
\[
\frac{1}{c} \psi_M(\theta) - \hat{\psi}_{M,n}(\theta) = \frac{k_n |\theta|^2}{n^{2-\alpha}} + A_1. \tag{11.3}
\]
where $A_1 := \int_{B_M} \frac{(\theta \cdot y)^4 g(\theta \cdot y)}{|y|^{2+\alpha}} \, dy - \frac{1}{n^2} \sum_{y \in B_M \cup \partial B_M} F_n(y)(\theta \cdot y)^4 g(\theta \cdot y) |y|^{2+\alpha}$ (11.4)

Let us denote $f_{y,\theta}(z) = \frac{(\theta \cdot y + z)^4 g(\theta \cdot y + z)}{|y + z|^{2+\alpha}} + \frac{(\theta \cdot y - z)^4 g(\theta \cdot y - z)}{|y - z|^{2+\alpha}} - 2(\theta \cdot y)^4 g(\theta \cdot y)$. We write

$$A_1 = \int_{B_1} \frac{(\theta \cdot z)^4 g(\theta \cdot z)}{|z|^{2+\alpha}} \, dz + \frac{1}{2} \sum_{y \in A_{1/8,1/8} \cup \partial A_{1/8,1/8}} \int_{(-\frac{1}{2n}, \frac{1}{2n})^2} f_{y,\theta}(z) \, dz.$$ (11.5)

We use second order Taylor’s Theorem with remainder on $f_{y,\theta}(z)$, since $z \in [-\frac{1}{2n}, \frac{1}{2n}]^2$ we have

$$f_{y,\theta}(z) = \sum_{|\beta| = 2} z^\beta R_{f_{y,\theta}}^\beta(z),$$

where $R_{f_{y,\theta}}^\beta(z) = \int_0^1 (1 - t)^{|\beta| - 1} D^\beta f_{y,\theta}(tz) \, dt$. We easily see that there exists a constant $C$ which depends on $\alpha$ such that

$$|R_{f_{y,\theta}}^\beta(z)| \leq C|\theta|^4 \left( \frac{1}{|y + z|^\alpha} + \frac{1}{|y - z|^\alpha} \right).$$

Since $z \in [-\frac{1}{2n}, \frac{1}{2n}]^2$ we have $|z| \leq \frac{\sqrt{2}}{2n}$, then

$$|R_{f_{y,\theta}}^\beta(z)| \leq \frac{C|\theta|^4}{(|y| - \frac{\sqrt{2}}{2n})^\alpha}$$

for a constant $C$ which only depends on $\alpha$. Then

$$\frac{1}{2} \sum_{y \in A_{1/8,1/8} \cup \partial A_{1/8,1/8}} \int_{(-\frac{1}{2n}, \frac{1}{2n})^2} |f_{y}(z)| \, dz \leq \frac{C|\theta|^4}{n^4} \sum_{y \in A_{1/8,1/8} \cup \partial A_{1/8,1/8}} \frac{1}{(|y| - \frac{\sqrt{2}}{2n})^\alpha}$$

$$\leq \frac{C|\theta|^4}{n^{4-\alpha}} \sum_{B_{1/8}} \frac{1}{(|y| - \frac{\sqrt{2}}{2n})^\alpha}$$

$$\leq \frac{C|\theta|^4}{n^2},$$

where the constant $C$ have changed from line to line.

The first term in the right hand side of (11.5) is easily seen to be $\leq \frac{C|\theta|^4}{n^{4-\alpha}}$ for a constant $C$ depending only on $\alpha$. Since $4 - \alpha > 2$ we conclude

$$|A_1| \leq \frac{C|\theta|^4}{n^2}.$$ 

We have just proved that there exists a positive constant $C$ which depends on $\alpha$ such that

$$\left| \frac{1}{c} \psi_M(\theta) - \hat{\psi}_{M,n}(\theta) - \frac{k_n|\theta|^2}{n^{2-\alpha}} \right| \leq \frac{C|\theta|^4}{n^2},$$

which together with (6.4) concludes the proof of the lemma.

\[ \square \]

**Proof of Lemma 6.2** We consider the function $\tilde{F}$ in $\frac{1}{n} \mathbb{Z}^2$ defined as

$$\tilde{F}(y) = n^2 \times \mu(\square \cap B_{\frac{1}{n}}).$$
We split the difference \( \frac{1}{c} \psi_M(\theta) - \tilde{\psi}_{M,n}(\theta) \) in two parts and estimate each one separately. We write

\[
\frac{1}{c} \psi_M(\theta) - \tilde{\psi}_{M,n}(\theta) = B_1 + B_2.
\] (11.6)

where

\[
B_1 := \int_{B_{1,M}} \frac{1 - \cos(\theta \cdot y)}{|y|^{2+\alpha}} \, dy - \frac{1}{n^2} \sum_{y \in B_n^M \cup \partial B_n \setminus \bar{M}} \tilde{F}(y)(1 - \cos(\theta \cdot y))
\] (11.7)

and

\[
B_2 := \int_{A_{1,M}} \frac{1 - \cos(\theta \cdot y)}{|y|^{2+\alpha}} \, dy - \frac{1}{n^2} \sum_{y \in A_n^M \cup \partial A_n \setminus \bar{M}} (F_n(y) - \tilde{F}(y))(1 - \cos(\theta \cdot y))
\] (11.8)

recalling that \( A_{1,M} = B_M \setminus B_{1,M} \).

**Estimate of \( B_1 \):** For \( y \in B_{2,M} \), we can write

\[
1 - \cos(\theta \cdot y) = \frac{(\theta \cdot y)^2}{2} + (\theta \cdot y)^4 g(\theta \cdot y)
\]

for a smooth function \( g \) defined in the unitary ball. Hence, we write

\[
B_1 = B_{1,1} + B_{1,2}
\] (11.9)

where

\[
B_{1,1} = \frac{\theta^2}{4} \left( \int_{B_{1,M}} \frac{1}{|y|^\alpha} \, dy - \frac{1}{n^2} \sum_{y \in B_n^M \cup \partial B_n \setminus \bar{M}} \tilde{F}(y) \right)
\] (11.10)

and

\[
B_{1,2} = \int_{B_{1,M}} \frac{(\theta \cdot y)^4 g(\theta \cdot y)}{|y|^{2+\alpha}} \, dy - \frac{1}{n^2} \sum_{y \in B_n^M \cup \partial B_n \setminus \bar{M}} \tilde{F}(y)(\theta \cdot y)^4 g(\theta \cdot y)
\] (11.11)

We estimate first \( B_{1,1} \). Note from the definition of \( k_n \) (see (11.12)) that

\[
B_{1,1} = k_n \frac{\theta^2}{n^{2-\alpha}} - \frac{\theta^2}{4} \left( \int_{A_{1,M}} \frac{1}{|y|^\alpha} \, dy - \frac{1}{n^2} \sum_{y \in A_n^M \cup \partial A_n \setminus \bar{M}} \frac{F_n(y) - \tilde{F}(y)}{|y|^\alpha} \right).
\] (11.12)

Let us denote \( f_y(z) := \frac{1}{c} \left( \frac{1}{|y+z|^\alpha} + \frac{1}{|y-z|^\alpha} - \frac{2}{|y|^\alpha} \right) \). Then

\[
B_{1,1} = k_n \frac{\theta^2}{n^{2-\alpha}} - \frac{\theta^2}{4} \sum_{y \in A_n^M \cup \partial A_n \setminus \bar{M}} \left| \int \left[ -\frac{1}{c} \frac{1}{|y|^\alpha} \right]^2 f_y(z) \, dz \right|
\]

We use Taylor’s Theorem and bound \( f_y(z) \) by its second order derivatives just like we did in the proof of Lemma 6.1. We obtain that there exists a constant \( C \) which
only depends on $\alpha$ such that

$$\int_{[-\frac{1}{2n}, \frac{1}{2n}]^2} |f_y(z)|dz \leq C|z|^4 \sup_{z \in [-\frac{1}{2n}, \frac{1}{2n}]^2} \left( \frac{1}{|y + z|^{2+\alpha}} + \frac{1}{|y - z|^{2+\alpha}} \right)$$

$$\leq \frac{C|z|^4}{(|y| - \sqrt{2^2n})^{2+\alpha}},$$

where the constant $C$ have changed from one line to the other.

As we noticed before, for $z \in [-\frac{1}{2n}, \frac{1}{2n}]^2$ we have that $|z| \leq \sqrt{2^2n}$. Then

$$\left| \sum_{y \in \mathbb{Z}^2} \int_{[-\frac{1}{2n}, \frac{1}{2n}]^2} |f_y(z)|dz \right| \leq C|\theta|^2 \sum_{y \in \mathbb{Z}^2} \frac{1}{(|y| - \sqrt{2^2n})^{2+\alpha}}$$

$$\leq C|\theta|^{2+\alpha} \int_{B_{1/n}} \frac{1}{|y|^{2+\alpha}} dy$$

$$\leq C|\theta|^{2+\alpha} \frac{n^2}{n^2}.$$

We have proved that there exists a constant $C$ which only depends on $\alpha$, such that

$$\left| B_{1,1} - \frac{k_n |\theta|^2}{n^{2-\alpha}} \right| \leq C|\theta|^{2+\alpha} \frac{n^2}{n^2}. \quad (11.13)$$

Now let us estimate $B_{1,2}$. We proceed in the same way as in the estimation of $B_{1,1}$. Using second order Taylor’s Theorem we obtain that there exists a constant $C$ depending only on $\alpha$, such that

$$|B_{1,2}| \leq C|\theta|^4 \sum_{y \in \mathbb{Z}^2, \frac{1}{n^{2-\alpha}}} \frac{1}{(|y| - \sqrt{2^2n})^{2+\alpha}}$$

$$\leq C|\theta|^4 \int_{B_{1/n}} \frac{1}{|y|^{2+\alpha}} dy,$$

where the constant $C$ have changed form one line to the other. Therefore

$$|B_{1,2}| \leq C|\theta|^{2+\alpha} \frac{n^2}{n^2}. \quad (11.14)$$

Putting (11.13) and (11.14) in (11.9), we obtain

$$\left| B_{1} - \frac{k_n |\theta|^2}{n^{2-\alpha}} \right| \leq C|\theta|^{2+\alpha} \frac{n^2}{n^2}. \quad (11.15)$$

**Estimate of $B_2$:** We repeat the same argument used in the estimate of $B_1$. That is, first we write the difference between the integral and the sum, as a sum of integrals over small squares. Then we symmetrize the arguments in the integrals and use second order Taylor’s Theorem with remainder. We conclude that there
exists a constant $C$ which only depends on $\alpha$, such that
\[
|B_2| \leq \frac{C|\theta|^2}{n^4} \sum_{y \in A_{|y|}^{\alpha} \cap \theta A_{|y|}^{\alpha}} \frac{1}{(|y| - \frac{\sqrt{2}}{2})^{2+\alpha}}
\leq \frac{C|\theta|^2}{n^2} \int_{B_{\frac{\sqrt{2}}{2}}} \frac{1}{|y|^{2+\alpha}} dy,
\]
where the constant $C$ have changed from one line to the other. We conclude
\[
|B_2| \leq \frac{C|\theta|^{2+\alpha}}{n^2}. \tag{11.16}
\]
The estimations of $B_1$ and $B_2$ together give us
\[
\frac{1}{c} \psi(\theta) - \bar{\psi}_n(\theta) - \frac{k_n|\theta|^2}{n^{2-\alpha}} = \bar{r}_n(\theta)|\theta|^{2+\alpha}, \tag{11.17}
\]
where $\bar{r}_n$ is an uniformly bounded sequence of functions defined in $[-n\pi, n\pi] \setminus B_{\frac{\sqrt{2}}{2}}$.

Finally, by (6.4) we conclude the proof of the first part of Lemma 6.2. The proof of (i) and (ii) follows that same idea used in the first part of the proof. \hfill \Box

\textbf{Proof of Lemma 6.3} Let us introduce the notation
\[
\begin{align*}
f_1(x) &= \int_{B_{\frac{\sqrt{2}}{2}}} \frac{\cos(\theta \cdot x) - 1}{\psi_M(\theta)} d\theta \\
f_{1,n}(x) &= \int_{B_{\frac{\sqrt{2}}{2}}} \frac{\cos(\theta \cdot x) - 1}{\psi_{M,n}(\theta)} d\theta \\
f_2(x) &= \int_{[-n\pi,n\pi] \setminus B_{\frac{\sqrt{2}}{2}}} \frac{\cos(\theta \cdot x)}{\psi_M(\theta)} d\theta \\
f_{2,n}(x) &= \int_{[-n\pi,n\pi] \setminus B_{\frac{\sqrt{2}}{2}}} \frac{\cos(\theta \cdot x)}{\psi_{M,n}(\theta)} d\theta \\
f_3(x) &= \int_{B^2 \setminus [-n\pi,n\pi]^2} \frac{\cos(\theta \cdot x)}{\psi_M(\theta)} d\theta.
\end{align*}
\]
Then
\[
\frac{c}{c_n} \tilde{G}_M(x) - \tilde{G}_{M,n}(x) = \left( \frac{c}{c_n} f_1(x) - f_{1,n}(x) \right) + \left( \frac{c}{c_n} f_2(x) - f_{2,n}(x) \right) + \frac{c}{c_n} f_3(x). \tag{11.18}
\]
We split the estimation of $\frac{c}{c_n} \tilde{G}_M - \tilde{G}_{M,n}$ in three different steps, consisting in the estimation of $\frac{c}{c_n} f_1(x) - f_{1,n}(x)$, $\frac{c}{c_n} f_2(x) - f_{2,n}(x)$ and $\frac{c}{c_n} f_3(x)$ separately.

\textbf{Estimate of $\frac{c}{c_n} f_1(x) - f_{1,n}(x)$:} By Lemma 6.1 we have that there exists a constant $C$ which only depends on $\alpha$, such that
\[
\left| \frac{c}{c_n} f_1(x) - f_{1,n}(x) \right| \leq \frac{C}{n^2} \int_{B_{\frac{\sqrt{2}}{2}}} \frac{1 - \cos(\theta \cdot x)}{|\theta|^4} d\theta.
\]
Since $\psi_{M,n}$ and $\psi_M$ are both of the order of $|\theta|^2$ in $B_{\frac{\sqrt{2}}{2}}$, we conclude
\[
\left| \frac{c}{c_n} f_1(x) - f_{1,n}(x) \right| \leq \frac{C}{n^2}. \tag{11.19}
\]
Integrating by parts, we obtain
\[ \partial_n \psi \]
where then have
\[ \psi \]
and
\[ \psi \]
Now let us estimate the boundary terms. Since
\[ \int \left[ -n \pi, n \pi \right]^2 \setminus B_\frac{1}{\pi} \]
we have
\[ \phi_n(\theta) = \frac{c}{n^2 c_n} r_n(\theta) |\theta|^{2+\alpha} \psi_{M,n}(\theta), \]
for a uniformly bounded sequence of function \( r_n \) defined in \([-n \pi, n \pi]^2 \setminus B_\frac{1}{\pi} \). We have
\[
\frac{c}{c_n} f_2(x) - f_{2,n}(x) = \int_{[-n \pi, n \pi]^2 \setminus B_\frac{1}{\pi}} \cos(\theta \cdot x) \frac{\partial n}{\partial \nu} \phi_n(\theta) d\theta
\]
\[
= -\frac{1}{|x|^2} \int_{[-n \pi, n \pi]^2 \setminus B_\frac{1}{\pi}} \Delta(\cos(\theta \cdot x)) \phi_n(\theta) d\theta.
\]
Integrating by parts, we obtain
\[
\frac{c}{c_n} f_2(x) - f_{2,n}(x) = -\frac{1}{|x|^2} \int_{[-n \pi, n \pi]^2 \setminus B_\frac{1}{\pi}} \cos(\theta \cdot x) \Delta \phi_n(\theta) d\theta
\]
\[
+ \frac{1}{|x|^2} \int_{\partial([-n \pi, n \pi]^2 \setminus B_\frac{1}{\pi})} \cos(\theta \cdot x) \frac{\partial}{\partial \nu} \phi_n(\theta) dS
\]
\[
- \frac{1}{|x|^2} \int_{\partial([-n \pi, n \pi]^2 \setminus B_\frac{1}{\pi})} \phi_n(\theta) \frac{\partial}{\partial \nu} \cos(\theta \cdot x) dS,
\]
where \( \nu \) represents the inward pointing unit normal vector along the surface \( \partial([-n \pi, n \pi]^2 \setminus B_\frac{1}{\pi}) \). By items (i) and (ii) of Lemma 6.2, we have \( |\Delta \phi_n(\theta)| \leq \frac{C}{n^{2+|\theta|^\alpha}} \), then
\[
\frac{1}{|x|^2} \int_{[-n \pi, n \pi]^2 \setminus B_\frac{1}{\pi}} \cos(\theta \cdot x) \Delta \phi_n(\theta) d\theta \leq \frac{C}{|x|^2} \int_{[-n \pi, n \pi]^2 \setminus B_\frac{1}{\pi}} \frac{1}{n^2 |\theta|^\alpha} d\theta
\]
\[
\leq \frac{C}{n^\alpha |x|^2}.
\]
Now let us estimate the boundary terms. Since \( \psi_{M,n} \) is periodic with period \([-n \pi, n \pi]^2 \) we have
\[
\int_{\partial([-n \pi, n \pi]^2)} \cos(\theta \cdot x) \frac{\partial}{\partial \nu} \frac{1}{\psi_{M,n}(\theta)} dS = \int_{\partial([-n \pi, n \pi]^2)} \frac{1}{\psi_{M,n}(\theta)} \frac{\partial}{\partial \nu} \cos(\theta \cdot x) dS = 0.
\]
On the other hand it is easy to see that
\[
\frac{1}{|x|^2} \int_{\partial B_\frac{1}{\pi}} \cos(\theta \cdot x) \frac{\partial}{\partial \nu} \phi_n(\theta) dS \leq \frac{C}{n^2 |x|^2}.
\]
and
\[
\frac{1}{|x|^2} \int_{\partial B_\frac{1}{\pi}} \phi_n(\theta) \frac{\partial}{\partial \nu} \cos(\theta \cdot x) dS \leq \frac{C}{n^2 |x|^2}.
\]
Define 

\[ T_n(x) := \frac{c}{c_n |x|^2} \int_{\partial[-n\pi,n\pi]^2} \cos(\theta \cdot x) \frac{\partial}{\partial \nu} \frac{1}{\psi_M(\theta)} \, dS - \frac{1}{\psi_M(\theta)} \frac{\partial}{\partial \nu} \cos(\theta \cdot x) \, dS. \]  

(11.24)

Then, from (11.20), (11.21), (11.22) and (11.23), we conclude

\[ \left| \frac{c}{c_n} f_2(x) - f_{2,n}(x) - T_n(x) \right| \leq \frac{C}{n^\alpha} \left( \frac{1}{|x|^2} + \frac{1}{|x|} \right). \]  

(11.25)

Estimate of \( \frac{c}{c_n} f_3(x) \): Note that

\[ \frac{c}{c_n} f_3(x) = \frac{c}{c_n} \int_{\mathbb{R}^2 \setminus [-n\pi,n\pi]^2} \cos(\theta \cdot x) \frac{\Delta(\psi_M(\theta))}{\psi_M(\theta)} \, d\theta. \]

Performing again an integration by parts, we obtain

\[ \frac{c}{c_n} f_3(x) = -\frac{c}{c_n |x|^2} \int_{\mathbb{R}^2 \setminus [-n\pi,n\pi]^2} \cos(\theta \cdot x) \Delta(\frac{1}{\psi_M(\theta)}) \, d\theta - T_n(x), \]

where \( T_n \) was defined in (11.24).

Since \( |\Delta(\frac{1}{\psi_M(\theta)})| \leq \frac{C}{|\theta|^2} \) on \( \mathbb{R}^2 \setminus B_{\frac{1}{\pi}} \), and in particular on \( \mathbb{R}^2 \setminus [-n\pi,n\pi]^2 \), we have that

\[ \frac{1}{|x|^2} \int_{\mathbb{R}^2 \setminus [-n\pi,n\pi]^2} \cos(\theta \cdot x) \Delta(\frac{1}{\psi_M(\theta)}) \, d\theta \leq \frac{C}{n^\alpha |x|^2}. \]

Then

\[ \left| \frac{c}{c_n} f_3(x) + T_n(x) \right| \leq \frac{C}{n^\alpha |x|^2}. \]  

(11.26)

By (11.18), (11.19), (11.25) and (11.26) we conclude the proof of Lemma 6.3 \( \square \)

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