Cosmic Billiards with Painted Walls
in Non Maximal Supergravities.

A worked out example†

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Abstract

The derivation of smooth cosmic billiard solutions by means of the compensator method, introduced by us sometimes ago, is extended to the case of supergravity with non maximal supersymmetry. Here a new key feature is provided by the non-maximal split nature of the scalar coset manifold. To deal with this, one has to consider the theory of Tits Satake projections leading to maximal split projected algebras, where the compensator method can be successfully applied and interesting solutions that display several smooth bounces can be derived. The generic bouncing feature of all exact solutions can thus be checked. From the analysis of the Tits Satake projection emerges a regular scheme applicable to all non maximal supergravity models and in particular a challenging so far unobserved structure, that of the paint group \( G_{paint} \). This latter, which is preserved through dimensional reduction, provides a powerful tool to codify solutions of the exact supergravity theories in terms of solutions of their Tits Satake projected partners, which are much simpler and manageable. It appears that the dynamical walls on which the cosmic ball bounces come actually in \textit{painted copies} rotated into each other by the paint group. So the effective cosmic dynamics is that dictated by the maximal split Tits Satake manifold \textit{plus paint}. In the present paper we work out in all minor details the example provided by \( N = 6, D = 4 \) supergravity, whose scalar manifold is the special Kählerian \( SO^*(12)/SU(6) \times U(1) \) c-mapping in \( D = 3 \) to the quaternionic \( E_{7(-5)}/SO(12) \times SO(3) \). This choice was not random. It is the next one after maximal supergravity and at the same time can be reinterpreted in the context of \( N = 2 \) supergravity. We plan indeed, in a future publication, to apply the results we obtained here, to the discussion of the Tits Satake projection within the context of generic special Kähler manifolds. We also comment on the merging of the Tits-Satake projection with the affine Kač–Moody extension originating in dimensional reduction to \( D = 2 \) and relying on a general field–theoretical mechanism illustrated by us in a separate paper.

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1 Introduction

The cosmological implications of superstring theory have been under attentive consideration in the last few years from various viewpoints \[1\]. This involves the classification and the study of possible time-evolving string backgrounds which amounts to the construction, classification and analysis of supergravity solutions depending only on time or, more generally, on a low number of coordinates including time. In this context a quite challenging and potentially highly relevant phenomenon for the overall interpretation of extra–dimensions and string dynamics is provided by the so named \textit{cosmic billiard} phenomenon \[2\],\[3\],\[4\],\[5\], \[6\],\[7\]. This is based on a profound link between the features of time evolution of the cosmological scale factors and the algebraic structure of string theory duality groups. As it is well known, the dualities that unify the various perturbative quantum string models into a unique M–theory are elements of a unified group \(U(\mathbb{Z})\) which is the suitable restriction to integers of a corresponding Lie group \(U(\mathbb{R})\) encoded in the low energy limit of superstrings, namely supergravity. The group \(U \equiv U(\mathbb{R})\) appears as isometry group of the scalar manifold \(\mathcal{M}_{\text{scalar}}\) emerging in compactifications of 10–dimensional supergravity to lower dimensions and crucially depends on the geometry of the compact dimensions and on the number of preserved supersymmetries \(N_Q \leq 32\). For \(N_Q > 8\) the scalar manifold is always a homogeneous space \(U/H\) and what actually happens is that the cosmological scale factors \(a_i(t)\) associated with the various dimensions of space–time can be interpreted as exponentials of those scalar fields \(h_i(t)\) which lie in the Cartan subalgebra of \(U\), while the other scalar fields in \(U/H\) correspond to positive roots \(\alpha > 0\) of the Lie algebra \(U\). In this way the cosmological evolution is described by the motion of a \textit{fictitious ball} in the CSA of \(U\). This space is actually a billiard table whose walls are the hyperplanes orthogonal to the various roots. The fictitious ball bounces on the billiard walls and this means that there are inversions in the time evolution of scale factors. Certain dimensions that were expanding almost suddenly begin to contract and others do the reverse. Such a scenario was introduced by Damour, Henneaux, Julia and Nicolai in \[2\], and in a series of papers with collaborators \[4\], \[5\],\[6\], which generalize classical results obtained in the context of pure General Relativity \[7\],\[3\]. In this approach the cosmic billiard phenomenon is analyzed as an asymptotic regime in the neighborhood of space-like singularities and the billiard walls are seen as delta function potentials provided by the various \(p\)-forms of supergravity localized at sharp instants of time.

It was observed in \[8\] that the fundamental mathematical setup underlying the appearance of the billiard phenomenon is the so named \textit{Solvable Lie algebra parametrization} of supergravity scalar manifolds, pioneered in \[9\] and later applied to the solution of a large variety of superstring/supergravity problems, including the structure of supersymmetric black-hole solutions \[10\],\[11\], the construction of gauged SUGRA potentials\[12\], \[13\] and several other issues (for a comprehensive review see \[14\]). Indeed we pointed out in those papers that, thanks to the solvable parametrization, one can establish a precise algorithm to implement the following programme:

1. Reduce the original supergravity in higher dimensions \(D \geq 4\) (for instance \(D = 10, 11\)) to a gravity-coupled \(\sigma\)–model in \(D \leq 3\) where gravity is non–dynamical and where the original higher dimensional bosonic field equations reduce to geodesic equations for a solvable group-manifold metrically equivalent to a non compact coset manifold \(\exp \{\text{Solv} (U/H)\} \cong U/H\).

2. Utilize the algebraic structure of the solvable Lie algebra \(\text{Solv} (U/H)\) in order to integrate analytically the geodesic equations. In particular we introduced in \[8\] a general method...
of integration, named the *H-compensator method* which reduces the geodesic differential equations to a triangular form and hence to quadratures when \( U/H \) is maximally split, which is always the case when supersymmetry is maximal \( (N_Q = 32) \).

3. Dimensionally oxidize the solutions obtained in this way to exact time dependent solutions of \( D \geq 4 \) supergravity. In particular we showed in \([\text{I}3]\) that the oxidation process is not unique but is algebraically classified by the embedding of Weyl orbits of subalgebras \( G \subset U \). Indeed the analytic structure of the solution is fully determined only by the algebraic structure of \( G \). Its physical interpretation varies and depends on the explicit embedding \( G \rightarrow U \).

In this way each solution in \( D \leq 3 \) corresponds to an entire orbit of higher dimensional backgrounds, very different from one another, but dual to each other under transformations of the Weyl group \( W \equiv \text{Weyl}(U) \).

Within this approach it was proved in \([\text{I}3]\) that the *cosmic billiard phenomenon* is indeed a general feature of exact time dependent solutions of supergravity and has *smooth realizations*. Calling \( h(t) \) the \( r \)-component vector of Cartan fields (where \( r \) is the rank of \( U \)) and \( h_{\alpha}(t) \equiv \alpha \cdot h(t) \) its projection along any positive root \( \alpha \), a *bounce* occurs at those instant of times \( t_i \) such that:

\[
\exists \alpha \in \Delta_+ \quad \text{such that} \quad h_{\alpha}(t) \big|_{t=t_i} = 0
\]

namely when the Cartan field in the direction of some root \( \alpha \) inverts its behaviour and begins to shrink if it was growing or viceversa begins to grow if it was shrinking. Since all higher dimensional bosonic fields (off-diagonal components of the metric \( g_{\mu\nu} \) or \( p \)-forms \( A[p] \)) are, via the solvable parametrization of \( U/H \), in one-to-one correspondence with roots \( \phi_{\alpha} \leftrightarrow \alpha \), it follows that the bounce on a *wall* (hyperplane orthogonal to the root \( \alpha \)) is caused by the sudden growing of that particular field \( \phi_{\alpha} \). Indeed we showed in \([\text{I}3]\) that in exact smooth solutions which we were able to obtain by means of the compensator method, each bounce is associated with a typical bell-shaped behaviour of the root field \( \phi_{\alpha} \) and that the whole process can be interpreted as a temporary localization of the Universe energy density in a *lump* on a spatial brane associated with the field \( \phi_{\alpha} \).

Although very much encouraging the analysis of \([\text{I}3]\) was still limited in three respects:

- **a** The dimensional reduction process which is responsible for making manifest the duality algebra \( U \) and hence for creating the whole algebraic machinery utilized in deriving the *smooth cosmic billiard solutions* was stopped at \( D = 3 \), namely at the first point where all the bosonic degrees of freedom can be represented by scalars. In \( D = 3 \), \( U \) is still a finite dimensional Lie algebra and the whole richness of the underlying algebraic structure is not yet displayed. As it is well known \([15]\), in \( D = 2 \) and \( D = 1 \), the algebra \( U \) becomes a Kać–Moody algebra, affine or hyperbolic, respectively. The smooth billiard dynamics has to be reconsidered and extended in view of this.

- **b** The constructions of \([\text{I}3]\) depend, in some crucial points, on the assumption that the coset \( U/H \) corresponds to a pair \( \{U,H \subset U\} \) of Lie algebra and Lie subalgebra which is *maximally split*. This is always the case for maximal supersymmetry \( N_Q = 32 \) but it is not true for \( N_Q \leq 32 \). Extending the *H*-compensator method to *non maximally split pairs* \( \{U,H \subset U\} \) is necessary in order to discuss billiard dynamics in lower SUSY theories and hence in compactifications of string theory on internal manifolds \( M_{\text{internal}} \) with restricted holonomies and \( G \)-structures, with or without fluxes.
The solutions considered in [8] were solutions of the pure $\sigma$-model, namely of pure, ungauged supergravity. The extension also to gauged supergravities is mandatory in order to make contact with potentially realistic models, in particular with currently considered flux compactifications. [16]

In a recent paper [17] we have begun to address point a) of the above list. There we have shown that the mechanism outlined several years ago by Nicolai [18] as the origin of the Kač–Moody extension of the duality algebra which appears in $D = 3$ when you step down to $D = 2$, namely the existence of two non-locally related dimensional reduction schemes $D = 4 \mapsto D = 2$, the Ehlers reduction and the Matzner Missner reduction, can be formulated in a general set up which provides a regular scheme of analysis both at the algebraic and at the field theoretical level and which applies to all supergravity theories. In particular we have shown that the $U_{D=3}$ algebra emerges from the Ehlers reduction and has the following general decomposition with respect to the $U_{D=4}$ algebra:

$$\text{adj}(U_{D=3}) = \text{adj}(U_{D=4}) \oplus \text{adj} (SL(2, \mathbb{R})_E) \oplus W(2, \mathbb{W})$$  

(1.2)

where $W$ is a symplectic representation of $U_{D=4}$ determined by the vector fields in the parent $D = 4$ supergravity and $SL(2, \mathbb{R})_E/O(2)$ is the target space for a $\sigma$-model which encodes the degrees of freedom of pure Einstein gravity. Continuing the Ehlers reduction from $D = 3$ to $D = 2$ we obtain a Lagrangian with the same symmetry

$$U_{D=2}^{[E]} = U_{D=3}$$  

(1.3)

Alternatively, following the Matzner Missner reduction scheme we obtain a twisted $\sigma$-model with symmetry

$$U_{D=2}^{[MM]} = U_{D=4} \otimes SL(2, \mathbb{R})_{MM}$$  

(1.4)

where $SL(2, \mathbb{R})_{MM}/O(2)$ is the target space for a $\sigma$-model also encoding the degrees of freedom of pure Einstein gravity. The Matzner Missner $SL(2, \mathbb{R})_{[MM]}$ group, however, is not the same as the Ehlers one $SL(2, \mathbb{R})_{[E]}$ and $U_{D=2}^{[MM]}$, $U_{D=2}^{[E]}$ are just two different finite dimensional subalgebras of the same infinite dimensional one $U_{D=2}$, which is nothing else but the affine Kač–Moody extension of $U_{D=3}$:

$$\begin{align*}
U_{D=2}^{[E]} & \subset U_{D=2} \equiv U_{D=3} \\
U_{D=2}^{[MM]} & \subset \end{align*}$$  

(1.5)

Understanding the general pattern for the Kač–Moody extension and mastering its field theoretical realization provides the necessary basis for the construction of smooth billiard solutions which rely on the full fledged Lorentzian signature CSA, lying behind supergravity. This we emphasized and begun to exploit in [17].

In the present paper we address point b) of the list mentioned above.

Our starting point is ungauged supergravity in $D = 4$, whose bosonic lagrangian takes the following general form:

$$\mathcal{L}^{(4)} = \sqrt{\text{det} g} \left[ -2R[g] - \frac{1}{6} \partial_\alpha \phi^a \partial^\mu \phi^b \partial^\nu h_{ab}(\phi) + \text{Im} N_{\Lambda \Sigma} F^\Lambda_{\mu \dot{\nu}} F^{\Sigma}_{\mu \dot{\nu}} \right] + \frac{1}{2} \text{Re} N_{\Lambda \Sigma} F^\Lambda_{\mu \dot{\nu}} F^\Sigma_{\mu \dot{\nu}} \epsilon^{\mu \dot{\nu} \rho \dot{\sigma}}$$  

(1.6)
In eq. (1.6) $\phi^a$ denotes the whole set of $n_S$ scalar fields parametrizing the scalar manifold $\mathcal{M}_{\text{scalar}}^{D=4}$ which, for $N_Q \geq 8$, is necessarily a coset manifold:

$$\mathcal{M}_{\text{scalar}}^{D=4} = \frac{U_{D=4}}{H} \quad (1.7)$$

For $N_Q \leq 8$, eq. (1.7) is not obligatory but it is possible. Particularly in the $\mathcal{N} = 2$ case, i.e. for $N_Q = 8$, a large variety of homogeneous special Kähler or quaternionic manifolds [19] fall into the set up of the present general discussion. The fields $\phi^a$ have $\sigma$–model interactions dictated by the metric $h_{ab}(\phi)$ of $\mathcal{M}_{\text{scalar}}^{D=4}$. The theory includes also $n$ vector fields $A^\Lambda_{\hat{\mu}}$ for which

$$F_{\hat{\mu}\hat{\nu}}^{\pm|\Lambda} \equiv \frac{1}{2} \left[ F^\Lambda_{\hat{\mu}\hat{\nu}} \mp i \frac{\sqrt{\det g}}{2} \epsilon_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} F^\hat{\rho}\hat{\sigma} \right] \quad (1.8)$$

denote the self-dual (respectively antiself-dual) parts of the field-strengths. As displayed in eq. (1.6) they are non minimally coupled to the scalars via the symmetric complex matrix

$$\mathcal{N}_{\Lambda\Sigma}(\phi) = i \text{Im}\mathcal{N}_{\Lambda\Sigma} + \text{Re}\mathcal{N}_{\Lambda\Sigma} \quad (1.9)$$

which transforms projectively under $U_{D=4}$. Indeed the field strengths $F_{\mu\nu}^\Lambda$ plus their magnetic duals fill up a $2n$–dimensional symplectic representation of $U_{D=4}$ which we call by the name of $W$.

The main point in the analysis of billiard dynamics for the lower SUSY cases is that the pair $\{U_{D=4}, \mathbb{H} \subset U_{D=4}\}$ is generically not maximally split. This implies that $U_{D=3}$, whose decomposition with respect to $U_{D=4}$ is always given by eq. (1.2) is also not maximally split. This happens since, in these cases, $U_{D=4,3}$ is a real section of the corresponding complex Lie algebra $\mathbb{U}(\mathbb{C})$ different from the maximally non compact one. Indeed it is only for the maximally non-compact real section that:

1. All Cartan generators $\mathcal{H}_i$ are non compact and belong to the Solvable Lie algebra: $\forall i \mathcal{H}_i \in \text{Solv}(U/H)$.

2. All step operators $E^\alpha$ associated with positive roots belong to the solvable algebra: $\forall \alpha > 0$, $E^\alpha \in \text{Solv}(U/H)$.

3. The maximal compact subalgebra $\mathbb{H} \subset U$ is the span of all generators $E^\alpha - \bar{E}^{-\alpha}$, for all positive roots $\forall \alpha > 0$.

Since items 1-3 in the above list are essential ingredients in the algorithm to derive exact solutions developed by us in [9], it is evident that our set-up has to be reconsidered carefully in the more general case.

In this paper we make an in depth analysis of a specific example of a non maximally split manifold $U_{D=4}/H$, that of $\mathcal{N} = 6$ supergravity, from which we extrapolate a general elegant result which reduces the non-maximally split cases to associated maximally split ones allowing, in this way, the extension of the compensator method to all values of $N_Q$ and hence the derivation of exact solutions in all instances.

As we are going to see our present results concerning point b are quite relevant also for the appropriate discussion of point a as well. Indeed the concept of painted walls that will emerge and that of paint group $G_{\text{paint}}$ are invariant by dimensional reduction and apply also to the Kač–Moody extensions.

In the next subsection we summarize the main result of our paper.
1.1 Tits Satake subalgebras and painted walls

In the case of non maximally non-compact manifolds $U/H$ the Lie algebra $U$ of the numerator group is some appropriate real form

$$ U = G_R $$

(1.10)

of a complex Lie algebra $G(\mathbb{C})$ of rank $r = \text{rank}(G)$. The Lie algebra $\mathbb{H}$ of the denominator $H$ is the maximal compact subalgebra $\mathbb{H} \subset U$ which has typically rank $r_{\text{compact}} > r$. Denoting, as usual, by $\mathbb{K}$ the orthogonal complement of $\mathbb{H}$ in $G_R$:

$$ G_R = \mathbb{H} \oplus \mathbb{K} $$

(1.11)

and defining as non compact rank or rank of the coset $U/H$ the dimension of the non compact Cartan subalgebra:

$$ r_{\text{nc}} = \text{rank}(U/H) \equiv \dim \mathcal{H}^{n.c.} \quad ; \quad \mathcal{H}^{n.c.} \equiv \text{CSA}_{G(\mathbb{C})} \cap \mathbb{K} $$

(1.12)

we obtain that $r_{\text{nc}} < r$. The manifold $U/H$ is still metrically equivalent to a solvable group manifold $M_{\text{Solv}} \equiv \exp[Solv(U/H)]$ and the field equations of supergravity still reduce to geodesic equations in $M_{\text{Solv}}$, which can be reformulated as first order equations by using the constant Nomizu connection (see [8]):

$$ \dot{Y}^A + \Gamma^A_{BC} \ Y^B \ Y^C = 0 $$

(1.13)

but it is the form of the Solvable Lie algebra $Solv(U/H)$, whose structure constants define the Nomizu connection, which is now more complicated and apparently does not allow the immediate use of the compensator method for the solution of equations (1.13). Yet the system (1.13) can be reduced to an equivalent one which is maximally split and can be solved with the methods of [8]. This is a consequence of Tits-Satake theory of non compact cosets and split subalgebras and, within such a mathematical framework of a peculiar universal structure of the solvable algebra $Solv(U/H)$ that, up to our knowledge, had not been observed before. Explicitly we have the following scheme. Splitting the Cartan subalgebra into its compact and non compact subalgebras:

$$ \text{CSA}_{G_R} = i\mathcal{H}^{\text{comp}} \oplus \mathcal{H}^{n.c.} \quad \upharpoonleft \quad \upharpoonright $$

(1.14)

$$ \text{CSA}_{G(\mathbb{C})} = \mathcal{H}^{\text{comp}} \oplus \mathcal{H}^{n.c.} $$

every vector in the dual of the full Cartan subalgebra, in particular every root $\alpha$ can be decomposed into its parallel and transverse part to $\mathcal{H}^{n.c.}$:

$$ \alpha = \alpha_{\parallel} \oplus \alpha_{\perp} $$

(1.15)

Setting all $\alpha_{\perp} = 0$ corresponds to a projection:

$$ \Pi : \Delta_G \mapsto \bar{\Delta} $$

(1.16)

of the original root system $\Delta_G$ onto a new system of vectors living in an euclidean space of dimension equal to the non compact rank $r_{\text{nc}}$. A priori this is not obvious, but it is nonetheless
true that $\Delta$ is by itself the root system of a simple Lie algebra $G_{TS}$, the Tits-Satake subalgebra of $G_R$:

$$\Delta = \text{root system of } G_{TS} \subset G_R$$ (1.17)

The Tits-Satake subalgebra $G_{TS} \subset G_R$ is always the maximally non compact real section of its own complexification. For this reason, considering its maximal compact subalgebra $H_{TS} \subset G_{TS}$ we have a new smaller coset $G_{TS}/H_{TS}$ which is maximally split and whose associated solvable algebra $Solv(G_{TS}/H_{TS})$ has the standard structure utilized in [8] to solve the differential equations (1.13). What is the relation between the two solvable Lie algebras $Solv(G_R/H)$ and $Solv(G_{TS}/H_{TS})$? The explicit answer to this question and the illustration of its relevance for the solution of the geodesic equations (1.13) is the key result of the present paper. It leads to the concept of billiards with painted walls and can be formulated through the following statements.

- A) In a projection it can occur that more than one higher dimensional vector maps to the same lower dimensional one. This means that in general there will be several roots of $\Delta_G$ which have the same image in $\Delta$. Calling $\Delta^+_G$ and $\Delta^+_T$ the sets of positive roots of the two root systems, it happens that both of them split in two subsets with the following properties.

| $G_R$ | $G_{TS}$ |
|-------|-------|
| $\Delta^+_G = \Delta^+ \cup \Delta^+$ | $\Delta^+_T = \Delta^+ \cup \Delta^+$ |
| $\forall \eta_1, \eta_2 \in \Delta^+; \eta_1 + \eta_2 \in \Delta^+$ | $\forall \alpha_1^\ell, \alpha_2^\ell \in \Delta^\ell; \alpha_1^\ell + \alpha_2^\ell \in \Delta^\ell$ |
| $\forall \eta \in \Delta^+, \forall \delta \in \Delta^\delta; \eta + \delta \in \Delta^\delta$ | $\forall \alpha^\delta \in \Delta^\delta, \forall \alpha^s \in \Delta^s; \alpha^\delta + \alpha^s \in \Delta^s$ |
| $\forall \delta_1, \delta_2 \in \Delta^\delta; \delta_1 + \delta_2 \in \{\Delta^\delta \cup \Delta^s\}$ | $\forall \alpha^\delta_1, \alpha^\delta_2 \in \Delta^\delta; \alpha^\delta_1 + \alpha^\delta_2 \in \{\Delta^\delta \cup \Delta^s\}$ |

(1.18)

The projection acts on the two different sets in the following way:

$$\Pi[\Delta^\eta] = \Delta^\ell$$
$$\Pi[\Delta^\delta] = \Delta^s$$

$\forall \alpha^\ell \in \Delta^\ell; \text{ card } \Pi^{-1}[\alpha^\ell] = 1$
$$\forall \alpha^s \in \Delta^s; \text{ card } \Pi^{-1}[\alpha^s] = m$$

$$\text{card } \Delta^+_G = \text{card } \Delta^+_T = m \times \text{card } \Delta^s$$ (1.19)

It means that there are two type of roots those which have a distinct image in the projected root system and those which arrange into multiplets with the same projection. The possible multiplicities, however, are only two, either 1 or $m$. Because of that we can enumerate the generators of the solvable algebra $Solv(G_R/H)$ in the following way:

$$H_i \Rightarrow \text{Cartan generators}$$
$$\Phi_{\alpha^\ell} \Rightarrow \eta - \text{roots}$$
$$\Omega_{\alpha^s_I} \Rightarrow \delta - \text{roots} \quad (I = 1, \ldots, m)$$ (1.20)
The index $I$ enumerating the $m$–roots of $\Delta_{G_R}$ that have the same projection in $\overline{\Delta}$ is named the paint index.

- B] There exists a compact subalgebra $G_{\text{paint}} \subset G_R$ which acts as an algebra of outer automorphisms (i.e. outer derivatives) on the solvable algebra $Solv_{G_R} \equiv Solv(G_R/H) \subset G_R$, namely:

$$[G_{\text{paint}}, Solv_{G_R}] = Solv_{G_R}$$

(1.21)

- C] The Cartan generators $H_i$ and the generators $\Phi_{\alpha^s}$ are singlets under the action of $G_{\text{paint}}$, i.e. each of them commutes with the whole of $G_{\text{paint}}$:

$$[H_i, G_{\text{paint}}] = [\Phi_{\alpha^s}, G_{\text{paint}}] = 0$$

(1.22)

On the other hand, each of the $m$-multiplets of generators $\Omega_{\alpha^s|I}$ constitutes an orbit under the action of the paint group $G_{\text{paint}}$, i.e. a linear representation $D[\alpha^s]$ which, for different roots $\alpha^s$ can be different, but has always the same dimension $m$:

$$\forall X \in G_{\text{paint}} : [X, \Omega_{\alpha^s|I}] = (D[\alpha^s][X])^J_I \Omega_{\alpha^s|J}$$

(1.23)

- D] The paint algebra $G_{\text{paint}}$ contains a subalgebra

$$G^0_{\text{paint}} \subset G_{\text{paint}}$$

(1.24)

such that with respect to $G^0_{\text{paint}},$ each $m$–dimensional representation $D[\alpha^s]$ branches in the same way as follows:

$$D[\alpha^s] \xrightarrow{G^0_{\text{paint}}} 1 \oplus (m-1)-\text{dimensional}$$

(1.25)

Accordingly we can split the range of the paint index $I$ as follows:

$$I = \begin{cases} 0, x, & \text{singlet} \\ 1, \ldots, m-1 & \text{($m-1$)-dimensional} \end{cases}$$

(1.26)

the index 0 corresponding to the singlet, while $x$ ranges over the representation $J$.

- E] The tensor product $J \otimes J$ contains both the identity representation $1$ and the representation $J$ itself. Furthermore, there exists, in the representation $\bigwedge^3 J$ a $G^0_{\text{paint}}$-invariant tensor $a^{xyz}$ such that the two solvable Lie algebras $Solv_{G_R}$ and $Solv_{G_{TS}}$ can be written as
follows

| \( \text{Solv}_{G_{TS}} \) | \( \text{Solv}_{G_{R}} \) |
|---|---|
| \([H_i, H_j] = 0\) | \([H_i, H_j] = 0\) |
| \([H_i, E^{a\ell}] = \alpha_i^\ell E^{a\ell}\) | \([H_i, \Phi_{a\ell}] = \alpha_i^\ell \Phi_{a\ell}\) |
| \([H_i, E^{a\ast}] = \alpha_i^a E^{a\ast}\) | \([H_i, \Omega_{a^\ast|I}] = \alpha_i^a \Omega_{a^\ast|I}\) |
| \(\alpha^\ell + \beta^\ell \notin \Delta\) | \(\Phi_{a\ell}, \Phi_{\beta\ell} = 0\) |
| \([E^{a\ell}, E^{b\ell}] = 0\) | \([E^{a\ell}, \Phi_{b\ell}] = 0\) |
| \(\alpha^\ell + \beta^\ell \in \Delta\) | \([\Phi_{a\ell}, \Phi_{b\ell}] = N_{a\ell\beta\ell} E^{a\ell+\beta\ell}\) |
| \([\Phi_{a\ell}, \Omega_{b^\ast|I}] = 0\) | \([\Phi_{a\ell}, \Omega_{b^\ast|I}] = 0\) |
| \(\Omega_{a^\ast|I} = 0\) | \([\Phi_{a\ell}, \Omega_{b^\ast|I}] = 0\) |
| \(\alpha^s + \beta^s \notin \Delta\) | \([\Omega_{a^\ast|I}, \Omega_{b^\ast|J}] = 0\) |
| \([E^{a\ast}, E^{b\ast}] = 0\) | \([\Omega_{a^\ast|I}, \Omega_{b^\ast|J}] = \delta^{IJ} N_{a^\ast b^\ast} (\Phi_{a^\ast b^\ast} + \Phi_{a^\ast b^\ast})\) |
| \(\alpha^s + \beta^s \in \Delta\) | \([\Omega_{a^\ast|I}, \Omega_{b^\ast|J}] = 0\) |
| \([E^{a\ast}, E^{b\ast}] = N_{a^\ast b^\ast} E^{a\ast+\beta\ast}\) | \([\Omega_{a^\ast|I}, \Omega_{b^\ast|J}] = \delta^{IJ} N_{a^\ast b^\ast} (\Phi_{a^\ast b^\ast} + \Phi_{a^\ast b^\ast})\) |

(1.27)

The existence of the paint group \(G_{paint}\) and the structure of the solvable Lie algebra displayed in eq. (1.27) imply that we can reduce the geodesic problem on \(G_R/H\) and hence the supergravity field equations to the geodesic problem on \(G_{TS}/H_{TS}\) which is maximally split and can be solved with the compensator method introduced in [8]. It suffices to observe that by setting all the components of the tangent vectors in the directions of the generators \(\Omega_{a^\ast|I}\) to zero we simply reproduce a copy of the solvable Lie algebra of the Tits Satake manifold. Once we have found a solution for this latter, we can extend it to a full fledged solution of the original system by applying rotations of the paint group \(G_{paint}\) with constant parameters. Physically this means that indeed the billiard table is just the Weyl chamber of the Tits Satake algebra as observed by Damour et al [3], yet, in the smooth billiard realization the raising and lowering of the walls occurs in \(paints\) which specify the precise correspondence with the supergravity fields and hence with the oxidation to higher dimensions.
1.2 Content of the paper

In the sequel of this paper we illustrate these general structures by working out in all details a specific example, that arising from $D = 4, \mathcal{N} = 6$ supergravity. Our choice is motivated as follows. On one hand, the case $N_Q = 24$ is the next simplest apart from that of maximal supersymmetry $N_Q = 32$. Indeed there is just the graviton multiplet, the number of fields is completely fixed and so is the geometric structure of the lagrangian. On the other hand the scalar manifold of $\mathcal{N} = 6$ supergravity is an instance of a special Kähler manifold and the bosonic lagrangian can be reinterpreted as the lagrangian of a particular $\mathcal{N} = 2$ model. In other words we could also reconsider our constructions from an $\mathcal{N} = 2$ viewpoint and interpret the scalar fields we deal with as moduli of an abstract Calabi-Yau compactification. Indeed in a subsequent paper we shall extrapolate from the present example general considerations on billiard dynamics and painted walls in the context of special geometries.

Our paper is organized as follows.

In section 2 we present the in depth analysis of the $E_7(-5)$ real section: how generators are constructed, how they are subdivided into compact and non compact ones, how the Tits Satake projection works in this case, what is the structure of the solvable Lie algebra generating the coset manifold $E_7(-5)/SO(12) \times SO(3)$ and what is the structure of the paint group. Then in section 3 we derive the Nomizu connection for both the original manifold and its Tits Satake projection and we compare the structure of the two systems of first order equations for the tangent vectors. In section 4 we derive explicit smooth solutions for the maximally split $F_4$ system and we show that they display several bounces: smooth cosmic billiards. In section 5 we uplift the previously found solutions to the original $E_7(-5)$ system by means of the paint group. Then we discuss the general features of the Tits Satake projection, how it commutes with dimensional reduction and how the paint group is preserved in the reduction. We illustrate these concepts on the specific example. Section 6 contains our conclusions. Then we have two appendices. The first, appendix A contains the listing and ordering of $E_7$ roots utilized throughout the paper. The second, appendix B is devoted to the explicit construction of the fundamental 26–dimensional representation of $F_{4(4)}$ which we used in the paper to calculate the needed $N_{\alpha\beta}$ matrix.

2 The example of $\mathcal{N}=6$ supergravity

In $\mathcal{N} = 6, D = 4$ supergravity there are 30 scalars which span the special Kähler manifold:

$$\mathcal{M}_{\text{scalar}}^{D=4,\mathcal{N}=6} = \frac{SO^\ast(12)}{SU(6) \times U(1)}$$

and the relevant duality algebra is therefore:

$$\mathcal{G}_{D=4} = SO^\ast(12)$$

The 16 graviphotons give rise to 16 electric plus 16 magnetic field strengths that organize into the 32 spinor representation of $SO^\ast(12)$ which is symplectic as it should be.

After reduction to $D = 3$ dimensions and dualization of all the vector fields to scalars we obtain a 3D-gravity coupled $\sigma$–model based on the quaternionic symmetric space:

$$\mathcal{M}_{\text{scalar}}^{D=3,\mathcal{N}=12} = \frac{E_7(-5)}{SO(3)R \times SO(12)}$$
which is the c-map of the special Kähler manifold \( \mathcal{M}_{\text{Tits Satake}}^{\text{scalar}} \)\(^{(2.4)}\). In this section we study the structure of the solvable Lie algebra describing the non maximally split non-compact manifold \( (2.3) \) and how it is related to its Tits Satake submanifold:

\[
\mathcal{M}_{\text{Tits Satake}}^{\text{scalar}} = \frac{F_4(4)}{SU(2)_R \times USp(6)}
\]

which is instead maximally split and it is the relevant submanifold defining the cosmic billiard. Our main goal is to show how the solution of the first order equations for the system \( (2.4) \) can be used to obtain solutions for the system \( (2.3) \). In particular we shall appropriately study how the dynamic walls of the billiard \( (2.3) \) are painted copies of the walls associated with the billiard \( (2.4) \).

To this effect we have to develop all the algebraic machinery associated with the real form \( E_7(-5) \) of the \( E_7 \) complex Lie algebra. We begin by spelling out the particular form of the decomposition \( (1.2) \)

\[
\text{adj}(E_7(-5)) = \text{adj}(SO^*(12)) \oplus \text{adj}(\text{SL}(2, \mathbb{R})) \oplus (2, 32_s)
\]

where \( 32_s \) denotes the spinor representation of \( SO^*(12) \), while \( 2 \) denotes the fundamental representation of \( \text{SL}(2, \mathbb{R}) \). The subgroup \( SO^*(12) \times \text{SL}(2, \mathbb{R}) \) is regularly embedded and non compact. There is another similar decomposition of the adjoint of \( E_7(-5) \) with respect to its maximal compact subgroup:

\[
\text{adj}(E_7(-5)) = \text{adj}(SO(12)) \oplus \text{adj}(SO(3)_R) \oplus (2, 32_s)
\]

where, once again \( 32_s \) denotes the spinor representation of the compact \( SO(12) \), this time.

The non compact symmetric space \( (2.3) \) has rank \( 4 \). This means that of the seven Cartan generators of \( E_7(-5) \), four are non compact and belong to the coset, while three are compact and belong to the compact subalgebra. We proceed to the explicit construction of the involutive automorphism of the complex \( \mathbb{E}_7 \) algebra

\[
\sigma : \mathbb{E}_7^C \rightarrow \mathbb{E}_7^C
\]

which defines the real form \( E_7(-5) \). This given we obtain also the compact subalgebra \( \mathbb{H} \), the complementary non compact subspace \( \mathbb{K} \) and the solvable Lie algebra \( \text{Solv}_{E_7(-5)} \) whose corresponding solvable group manifold is isometrical to the coset manifold \( (2.3) \).

### 2.1 The \( E_7 \) root system, and its projection onto the \( F_4 \) root system

In order to realize the programme we have just outlined, we begin by choosing an explicit basis of simple roots for \( E_7 \). In an Euclidean orthonormal basis they are the following ones:

\[
\begin{align*}
\alpha_1 &= \{1, -1, 0, 0, 0, 0, 0\} \\
\alpha_2 &= \{0, 1, -1, 0, 0, 0, 0\} \\
\alpha_3 &= \{0, 0, 1, -1, 0, 0, 0\} \\
\alpha_4 &= \{0, 0, 0, 1, -1, 0, 0\}, \\
\alpha_5 &= \{0, 0, 0, 0, 1, -1, 0\}, \\
\alpha_6 &= \{0, 0, 0, 0, 1, 1, 0\}, \\
\alpha_7 &= \{-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{\sqrt{2}}\}
\end{align*}
\]
Figure 1: The Dynkin diagram of $E_7$ and the labeling of simple roots. The three orthogonal roots $\alpha_1$, $\alpha_3$ and $\alpha_5$ are marked black, since they are used to define the part of the Cartan subalgebra which is compact in the $E_7(-5)$ real form.

and they are associated with the $E_7$ Dynkin diagram labeled as it is displayed in fig. 1. Next we list all the positive roots of $E_7$ arranged according to their height. They are 63 and they are listed in Appendix A.

The real section $E_{7(-5)}$ of the complex Lie algebra $E_7^C$ is identified by the Tits Satake diagram depicted in fig. 1 where the simple roots $\alpha_1$, $\alpha_3$, $\alpha_5$ are black. This means that in the chosen real form the Cartan generators dual to these three roots $H_{\alpha_1,3,5}$ are compact, while non compact are the Cartan generators in the complementary 4–dimensional subspace. It is fairly easy to describe the space of non-compact Cartan generators $H^{n.c.}$. It is the span of the four weight vectors $\lambda_{2,4,5,7}$ which, by construction, are orthogonal to the roots $\alpha_{1,3,5}$. Thus in the chosen euclidean basis we obtain:

$$H^{n.c.} = \text{span} \{\lambda_2, \lambda_4, \lambda_6, \lambda_7\}$$

$$\lambda_2 = \{1, 1, 0, 0, 0, 0, \sqrt{2}\}$$

$$\lambda_4 = \{1, 1, 1, 1, 0, 0, 2\sqrt{2}\}$$

$$\lambda_6 = \{1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 3/\sqrt{2}\}$$

$$\lambda_7 = \{0, 0, 0, 0, 0, 0, \sqrt{2}\}$$

(2.9)

It is now easy to construct an orthogonal basis of four length-two 7-vectors for the space $H^{n.c.}$ defined by eq. (2.9). It is given by:

$$e_1 = \{1, 1, 0, 0, 0, 0, 0\}$$

$$e_2 = \{0, 0, 1, 1, 0, 0, 0\}$$

$$e_3 = \{0, 0, 0, 0, 1, 1, 0\}$$

$$e_4 = \{0, 0, 0, 0, 0, 0, \sqrt{2}\}$$

(2.10)
Indeed, equivalently to eq. (2.9) we can also write:

$$H^{n.c.} = \text{span}\ \{e_1, e_2, e_3, e_4\} \quad (2.11)$$

We can complete the basis (2.10) with other three vectors also of length 2, which are orthogonal to $e_1, e_2, e_3, e_4$ and also orthogonal among themselves:

$$
e_5 = \{1, -1, 0, 0, 0, 0, 0\}
\ne_6 = \{0, 0, 1, -1, 0, 0, 0\}
\ne_7 = \{0, 0, 0, 0, 1, -1, 0\} \quad (2.12)$$

The compact Cartan subalgebra is provided by the span of these three vectors:

$$H^c = \text{span}\ \{e_5, e_6, e_7\} \quad (2.13)$$

The $E_7$ roots are vectors in the dual of the 7-dimensional space which is the direct sum of the four dimensional space $H^{n.c.}$ plus the three dimensional space $H^c$:

$$H = H^{n.c.} \oplus H^c \quad (2.14)$$

Hence every root $\alpha \in \Delta_{E_7}$ can be decomposed as follows:

$$\alpha = \alpha \| + \alpha \perp \quad (2.15)$$

where $\alpha \|$ lies in $H^{n.c.}$ and $\alpha \perp$ is orthogonal to it. The essential point in Tits Satake theory of real forms is that the parallel projections of the roots, namely $\alpha \|$, are not just arbitrary vectors, rather they are roots of a Lie algebra of rank equal to the dimension of the non compact Cartan Lie algebra which is actually a subalgebra of the original algebra. In our case we have rank $= 4$ and the relevant subalgebra (Tits Satake) is $F_{4(4)} \subset E_7(-5)$. Indeed the parallel projections $\alpha \|$ fill the cardinality 24 root-system $\Delta_{F_4}$.

The actual construction of the real form $E_7(-5)$ involves the careful analysis of the onto projection:

$$\Delta_{E_7} \xrightarrow{\pi} \Delta_{F_4} \quad (2.16)$$

Explicitly, if we decompose the 63 positive roots of $E_7$ along the new orthogonal basis $e_{1,2,3,4,5,7}$ we discover the following:

1. There are just three roots that are orthogonal to the subspace spanned by $e_{1,2,3,4}$, namely such that $\alpha \| = 0$. They are precisely the simple roots $\alpha_1, \alpha_3$ and $\alpha_5$.

2. The remaining 60 roots have a projection onto the space spanned by $e_{1,2,3,4}$ which takes the form of one of the 24 roots of $F_4$ and all such 24 roots are reproduced in the projection. Namely $\alpha \| \in \Delta_{F_4}$.

3. The set of 24 roots of $F_4$ is subdivided in two subsets of 12 roots each. The long and the short roots. Each long root appears only once in the projection of $E_7$ roots. Each of the 12 short roots, instead, appears exactly four times as image of four distinct $E_7$ roots. So that we count $4 \times 12 + 12 = 60$
To understand this pattern we have to introduce the $F_4$ root system. The Dynkin diagram of $F_4$ is given in fig. 2 and calling $y_{1,2,3,4}$ a basis of orthonormal vectors:

$$y_i \cdot y_j = \delta_{ij}$$  \hspace{1cm} (2.17)

a possible choice of simple roots $\varpi_i$ which reproduces the Cartan matrix encoded in the Dynkin diagram (2) is the following:

$$\begin{align*}
\varpi_1 &= -y_1 - y_2 - y_3 + y_4 \\
\varpi_2 &= 2y_3 \\
\varpi_3 &= y_2 - y_3 \\
\varpi_4 &= y_1 - y_2
\end{align*}$$  \hspace{1cm} (2.18)

With this basis of simple roots the full root system composed of 48 vectors is given by:

$$\Delta_{F_4} \equiv \pm y_i \pm y_j ; \pm y_i ; \pm y_1 \pm y_2 \pm y_3 \pm y_4$$  \hspace{1cm} (2.19)

and one can list the positive roots by height as displayed in table 1. If we identify the $E_7$ roots with their progressive number as it is defined by their listing in Appendix A we can reorganize them into the following three subsets according to their projection onto the $F_4$ root space.

1. First we have the $\Delta_\beta$ set:

$$\begin{align*}
\beta_1 &= \alpha_1 \\
\beta_2 &= \alpha_3 \\
\beta_3 &= \alpha_5
\end{align*}$$  \hspace{1cm} (2.20)

which contains the three roots with vanishing projection onto the $F_4$ root space. As we are going to see, together with their negative and with the compact Cartan generators, these roots define a compact subalgebra $SO(3)_1 \times SO(3)_2 \times SO(3)_3$ with respect to which the generators of the solvable Lie algebra of $E_7(-5)/SO(12) \times SU(2)$ transform covariantly and arrange into representations. Indeed this $SO(3)^3$ is, for the present case, the paint group $G_{\text{paint}}$ mentioned in eq. (1.21). The subgroup $G^0_{\text{paint}} \subset G_{\text{paint}}$ mentioned in eqs (1.24) and (1.25) is actually the diagonal subgroup $SO(3)_{\text{diag}} \subset SO(3)^3$.

2. Secondly we have the $\Delta_\eta$ set containing those twelve roots whose projection onto the $F_4$ root space is unique. We organize them according to the height of the $F_4$ root on which they project. The result is displayed in table 2.
Table 1: Listing of all positive roots of $F_4$. The second column gives the Dynkin labels, while the second column gives the form of the root in an euclidean basis.

3 Thirdly we have the $\Delta_8$ set of those 48 $E_7$ roots which arrange into quadruplets having the same projection onto the $F_4$ system. We denote these roots by $\delta^I_i$ where $I = 1, \ldots, 12$ and $i = 1, 2, 3, 4$. They are displayed in table 3.
Table 2: The $\Delta_\eta$ set of those twelve $E_7$ roots whose projection on the $F_4$ root system is unique. As it is evident from the table, from the point of view of $F_4$ the $\Delta_\eta$ set is composed by the long roots. The first column gives the name by means of which these roots will be referred to within the $F_4(4)$ algebra. The last column gives the name of the corresponding root in $E_7$ calculations.

| $\alpha^\ell_i$ | $F_4$ root Dynkin labels | $F_4$ root in eucl. basis | corresp. root of $E_7$ |
|-----------------|--------------------------|--------------------------|-----------------|
| $\alpha_1^\ell$ | $\{0, 1, 0, 0\}$         | $2\ y_3$                | $\eta_1 = \alpha_6$ |
| $\alpha_2^\ell$ | $\{1, 0, 0, 0\}$         | $-y_1 - y_2 - y_3 + y_4$| $\eta_2 = \alpha_7$ |
| $\alpha_3^\ell$ | $\{1, 1, 0, 0\}$         | $-y_1 - y_2 + y_3 + y_4$| $\eta_3 = \alpha_{13}$ |
| $\alpha_4^\ell$ | $\{0, 1, 2, 0\}$         | $2\ y_2$                | $\eta_4 = \alpha_{31}$ |
| $\alpha_5^\ell$ | $\{1, 1, 2, 0\}$         | $-y_1 + y_2 - y_3 + y_4$| $\eta_5 = \alpha_{36}$ |
| $\alpha_6^\ell$ | $\{1, 2, 2, 0\}$         | $-y_1 + y_2 + y_3 + y_4$| $\eta_6 = \alpha_{41}$ |
| $\alpha_7^\ell$ | $\{0, 1, 2, 2\}$         | $2\ y_1$                | $\eta_7 = \alpha_{48}$ |
| $\alpha_8^\ell$ | $\{1, 1, 2, 2\}$         | $y_1 - y_2 - y_3 + y_4$ | $\eta_8 = \alpha_{51}$ |
| $\alpha_9^\ell$ | $\{1, 2, 2, 2\}$         | $y_1 - y_2 + y_3 + y_4$ | $\eta_9 = \alpha_{54}$ |
| $\alpha_{10}^\ell$ | $\{1, 2, 4, 2\}$         | $y_1 + y_2 - y_3 + y_4$ | $\eta_{10} = \alpha_{61}$ |
| $\alpha_{11}^\ell$ | $\{1, 3, 4, 2\}$         | $y_1 + y_2 + y_3 + y_4$ | $\eta_{11} = \alpha_{62}$ |
| $\alpha_{12}^\ell$ | $\{2, 3, 4, 2\}$         | $2\ y_4$                | $\eta_{12} = \alpha_{63}$ |

2.2 The real form $E_7(-5)$ and its associated solvable Lie algebra

Given these preliminaries we can now introduce the real form $E_7(-5)$ which follows from the action of a suitable involutive automorphism (2.7) of the complex Lie algebra $E_7^C$.

Following the general definitions presented in most textbooks on Lie algebra theory (see for instance [20]), the real form $G_R$ is defined as the subspace of eigenvalue 1 of the relevant automorphism $\sigma$, namely we have:

$$\sigma (G_R) = G_R$$

On the other hand, $\sigma$ is completely identified by the Tits Satake diagram depicted in fig. Indeed the action of $\sigma$ is originally defined on the Cartan subalgebra and corresponds to changing the signs of all vectors lying in the compact part while keeping unchanged those lying in the non compact part:

$$\sigma : H^{n.c.} \to H^{n.c.} ; \quad \sigma : H^c \to -H^c$$

From the Cartan algebra, the action of $\sigma$ is canonically extended to the root space. Decomposing each root in its parallel and transverse parts we have:

$$\sigma (\alpha) = \sigma (\alpha_{||} + \alpha_{\perp}) = \alpha_{||} - \alpha_{\perp}$$

If we rewrite all the sixty three $E_7$ roots in the $e_i$ basis defined by eqs (2.10) and (2.12) we unveil the meaning of our regrouping from the point of view of the automorphism $\sigma$. The set $\Delta_\eta$
Finally from the root space the automorphism \( \sigma \) the whole algebra. This last step involves the introduction of a set of sign factors. To see this, let \( H \) be the Cartan and the step operators, respectively, realized in the maximally non compact, split, real section \( G_{\text{split}} \) of the complex Lie algebra \( G^C \). If regarded as matrices in any of its irreducible representations both \( H \) and \( E^\alpha \) are real matrices. In our case the complex Lie algebra is \( E_7^C \) and the maximally non compact split real section is \( E_{7(7)} \). The representation we can focus on is the fundamental 56-dimensional representation and the explicit form of the matrices \( H \) and \( E^\alpha \) we shall utilize was constructed by us in 1997 and it is described in [10]. This fixes the conventions, which is a necessary step, since the definition of the step operators

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
I &= 1 & \alpha_1^s & \{0, 0, 0, 1\} & y_1 - y_2 & \alpha_2 & \alpha_8 & \alpha_9 & \alpha_{14} \\
I &= 2 & \alpha_2^s & \{0, 0, 1, 0\} & y_2 - y_3 & \alpha_4 & \alpha_{10} & \alpha_{11} & \alpha_{16} \\
I &= 3 & \alpha_3^s & \{0, 1, 1, 0\} & y_2 + y_3 & \alpha_{12} & \alpha_{17} & \alpha_{19} & \alpha_{24} \\
I &= 4 & \alpha_4^s & \{0, 0, 1, 1\} & y_1 - y_3 & \alpha_{20} & \alpha_{15} & \alpha_{26} & \alpha_{21} \\
I &= 5 & \alpha_5^s & \{1, 1, 1, 0\} & -y_1 + y_4 & \alpha_{18} & \alpha_{23} & \alpha_{25} & \alpha_{30} \\
I &= 6 & \alpha_6^s & \{0, 1, 1, 1\} & y_1 + y_3 & \alpha_{27} & \alpha_{22} & \alpha_{33} & \alpha_{29} \\
I &= 7 & \alpha_7^s & \{1, 1, 1, 1\} & -y_2 + y_4 & \alpha_{32} & \alpha_{28} & \alpha_{37} & \alpha_{34} \\
I &= 8 & \alpha_8^s & \{0, 1, 2, 1\} & y_1 + y_2 & \alpha_{35} & \alpha_{38} & \alpha_{40} & \alpha_{43} \\
I &= 9 & \alpha_9^s & \{1, 1, 2, 1\} & -y_3 + y_4 & \alpha_{39} & \alpha_{42} & \alpha_{45} & \alpha_{47} \\
I &= 10 & \alpha_{10}^s & \{1, 2, 2, 1\} & y_3 + y_4 & \alpha_{44} & \alpha_{46} & \alpha_{49} & \alpha_{50} \\
I &= 11 & \alpha_{11}^s & \{1, 2, 3, 1\} & y_2 + y_4 & \alpha_{53} & \alpha_{52} & \alpha_{56} & \alpha_{55} \\
I &= 12 & \alpha_{12}^s & \{1, 2, 3, 2\} & y_1 + y_4 & \alpha_{58} & \alpha_{57} & \alpha_{60} & \alpha_{59} \\
\hline
\end{array}
\]

Table 3: The \( \Delta_\delta \) set of those 48 \( E_7 \) roots which arrange into 12 quadruplets having the same \( F_4 \) projection. As it is evident from the table, from the point of view of \( F_4 \) the \( \Delta_\delta \) set is composed by the short roots. The second column gives the name of the projected root within the \( F_{4(4)} \) algebra context. In the last four columns, \( \alpha_i \) denotes an \( E_7 \)-root numbered according to the order listed in Appendix A.

is composed by all those roots whose transverse part vanishes, namely the components of each \( \eta \)-root along \( e_{5,6,7} \) are zero:

\[
\forall \eta \in \Delta_\eta : \eta_\perp = 0
\]

which implies

\[
\sigma (\eta_I) = \eta_I
\]

On the other hand the roots in the set \( \Delta_\delta \) arrange into pairs such that the transverse part of \( \delta^1_I \) is the opposite of that of \( \delta^4_I \) and similarly that of \( \delta^2_I \) is the opposite of that of \( \delta^3_I \). Hence we have:

\[
\sigma (\delta^1_I) = \delta^4_I ; \quad \sigma (\delta^4_I) = \delta^1_I ; \quad \sigma (\delta^2_I) = \delta^3_I ; \quad \sigma (\delta^3_I) = \delta^2_I
\]

Finally from the root space the automorphism \( \sigma \) can be lifted to the step operators and hence to the whole algebra. This last step involves the introduction of a set of sign factors. To see this, let \( H_i \) and \( E^\alpha \) be the Cartan and the step operators, respectively, realized in the maximally non compact, split, real section \( G_{\text{split}} \) of the complex Lie algebra \( G^C \). If regarded as matrices in any of its irreducible representations both \( H_i \) and \( E^\alpha \) are real matrices. In our case the complex Lie algebra is \( E_7^C \) and the maximally non compact split real section is \( E_{7(7)} \). The representation we can focus on is the fundamental 56-dimensional representation and the explicit form of the matrices \( H_i \) and \( E^\alpha \) we shall utilize was constructed by us in 1997 and it is described in [10].
is up to choices of some arbitrary signs. This being set, the lifting of the automorphism $\sigma$ from
the root space to the complex Lie algebra is defined in the following way. Firstly the complex
Lie algebra is defined as the complex span (linear combinations with complex coefficients) of the
generators $H_i$ and $E^\alpha$:

$$G^C = \text{complex span} \{H_i, E^\alpha\} \quad (2.27)$$

Secondly, for each element $g \in G^C$, we require:

$$\sigma (i g) = -i \sigma (g) \quad (2.28)$$

where $i = \sqrt{-1}$ denotes the imaginary unit. Thirdly the automorphism is fixed by writing its
action on the generators:

$$\sigma (H_i^\parallel) = H_i^\parallel; \quad \sigma (H_i^\perp) = -H_i^\perp; \quad \sigma (E^\alpha) = a_\alpha E^{\sigma(\alpha)} \quad (2.29)$$

In the above equation $a_\alpha$ is a real number whose absolute is immediately fixed to one by consis-
tency with Jacobi identities. Hence $a_\alpha = \pm 1$. Yet the choice of these signs is not immediately
obvious. Indeed it follows from the original choice of normalizations of the step operators for the
split algebra $G_{\text{split}}$ and therefore it is convention dependent. In a moment we shall resolve this
ambiguity relative to the already mentioned choice of conventions, namely those of [10]. First
let us observe that once the $a_\alpha$ are fixed, the complex linear combinations of
split generators
forming a complete basis for the real Lie algebra $G_R$ are also fixed. As an example let us consider
the maximally compact real section $G_{\text{compact}}$ for which, as it is well known, we always have:

$$G_{\text{compact}} = \text{real span} \{i H_i, \frac{1}{2} (E^\alpha + E^{-\alpha}), \frac{1}{2} (E^\alpha - E^{-\alpha})\} \quad (2.30)$$

In view of the previous theory this is easily explained as follows. In this case the whole Cartan
subalgebra is compact and hence $\sigma (H_i) = -H_i$ for all Cartan generators. From this it follows
that $a_\alpha = 0$ for all roots and therefore $\sigma (\alpha) = -\alpha$. The actual linear combinations displayed in
eq (2.30) follow from the choice $a_\alpha = -1, \forall \alpha$ which implies:

$$\sigma (i \frac{1}{2} (E^\alpha + E^{-\alpha})) = i \frac{1}{2} (E^\alpha + E^{-\alpha}); \quad \sigma (\frac{1}{2} (E^\alpha - E^{-\alpha})) = \frac{1}{2} (E^\alpha - E^{-\alpha}) \quad (2.31)$$

Had we chosen $a_\alpha = 1$ we would have obtained the same linear combinations but with the $i$-factors
interchanged: $\frac{1}{2} (E^\alpha + E^{-\alpha}), i \frac{1}{2} (E^\alpha - E^{-\alpha})$. Such a choice, however, would be wrong since it
does not define an algebra. Indeed the commutator of two generators of type $i \frac{1}{2} (E^\alpha - E^{-\alpha})$
produces a generator of the same type, but without the $i$-factor in front. On the contrary the
opposite choice of $a_\alpha$, which amounts to the well known choice [2.30] of $i$-prefactors consistently
defines a subalgebra. This discussion shows that:

1. The choice of the $a_\alpha$ factors which completely determines the action the automorphism $\sigma$
is fully equivalent to deciding the position of the $i$-factors, namely to deciding whether, for
each pair $\alpha$ and $\sigma (\alpha)$ of roots mapped into each other by the automorphism it is

$$\frac{1}{2} (E^\alpha - E^{\sigma(\alpha)}); \quad i \frac{1}{2} (E^\alpha + E^{\sigma(\alpha)}) \quad (2.32)$$
or

$$\frac{1}{2} (E^\alpha + E^{\sigma(\alpha)}); \quad i \frac{1}{2} (E^\alpha - E^{\sigma(\alpha)}) \quad (2.33)$$

which appear as generators of the algebra $G^C_R$. 

17
Table 4: Explicit enumeration of the generators of the real Lie algebra E\(_7(-5)\)

2. The decision whether (2.32) or (2.33) is the right choice is determined by the commutation relations and the closure of the algebra \(G_{\mathbb{R}}\) and can be different for different pairs of related roots.

In the case of the E\(_7(-5)\) real section of the E\(_7\) complex Lie algebra, using the normalization of step operators derived in [10] we have carefully inspected by computer calculations all the commutation relations and we have derived the assignment of \(i\)-factors displayed in the explicit enumeration of generators of E\(_7(-5)\) displayed in table 4.

### 2.3 The maximal compact subalgebra SO(3)\(_R\) \(\times\) SO(12) and the basis of coset generators

Having explicitly constructed the real Lie algebra \(G_{\mathbb{R}} = E_7(-5)\) we can now consider its decomposition with respect to its maximal compact subalgebra \(H \equiv SO(3)_{\mathbb{R}} \times SO(12)\) which is to us the most relevant issue, since the final goal of our study is the construction of geodesic motions in the manifold (2.3). Being interested in the splitting:

\[
G_{\mathbb{R}} = H \oplus K
\]  
(2.34)
we proceed to establishing a canonical basis of generators for $G_R$ organized as it follows:

\[
(A = 1, \ldots, 133) \quad T_A = \begin{cases} 
T_i = \mathbb{H}_i & (i = 1, \ldots, 69) \\
T_{i+69} = \mathbb{K}_i & (i = 1, \ldots, 64)
\end{cases}
\]  

(2.35)

where $\mathbb{H}_i$ is a basis of generators for the maximal compact subalgebra $SO(3)_R \times SO(12)$ and $\mathbb{K}_i$ is a basis of generators for its orthogonal complement, namely for the tangent space to the manifold (2.3). With reference to table 4 our choice and ordering of the basis $\mathbb{H}_i$ is the following:

\[
\begin{align*}
\mathbb{H}_{3i-2} &= i \frac{1}{\sqrt{2}} H_{\beta_i} & (i = 1, 2, 3) \\
\mathbb{H}_{3i-1} &= E^+_{\beta_i} & (i = 1, 2, 3) \\
\mathbb{H}_{3i} &= E^-_{\beta_i} & (i = 1, 2, 3) \\
\mathbb{H}_{9+i} &= E^-_I & (I = 1, \ldots, 12) \\
\mathbb{H}_{21+A} &= \frac{1}{\sqrt{2}} \left( E^+_A - (E^+_A)^\dagger \right) & (A = 1, \ldots, 24) \\
\mathbb{H}_{45+A} &= \frac{1}{\sqrt{2}} \left( E^-_A - (E^-_A)^\dagger \right) & (A = 1, \ldots, 24) \\
& & (A = 1, \ldots, 24)
\end{align*}
\]  

(2.36)

Correspondingly, our choice and ordering for the coset generators $\mathbb{K}_i$ is displayed below:

\[
\begin{align*}
\mathbb{K}_i &= \mathcal{H}_i^{n.c.} & (1, 2, 3) \\
\mathbb{K}_4 &= \mathcal{H}_4 \\
\mathbb{K}_{4+I} &= E^+_I & (I = 1, \ldots, 12) \\
\mathbb{K}_{16+A} &= \frac{1}{\sqrt{2}} \left( E^+_A + (E^+_A)^\dagger \right) & (A = 1, \ldots, 24) \\
\mathbb{K}_{40+A} &= \frac{1}{\sqrt{2}} \left( E^-_A + (E^-_A)^\dagger \right) & (A = 1, \ldots, 24)
\end{align*}
\]  

(2.37)

Let us make a few comments. In our ordering of the compact generators $\mathbb{H}_i$, the first nine generate a special subgroup, which we have already identified as the paint group:

\[
G_{\text{paint}} = SO(3)^3_\beta \equiv SO(3)_{\beta_1} \times SO(3)_{\beta_2} \times SO(3)_{\beta_3} \subset SO(3)_R \times SO(12)
\]  

(2.38)

This latter is associated with the three “compact roots” defining the real section and will play an important role as automorphism algebra of the Solvable Lie Algebra $Solv_{7(-5)}$ associated with the coset (2.3). It is appropriate to stress that the subgroup $SO(3)_R$ is none of these three $SO(3)_{\beta_i}$. On the contrary the subgroup $SO(3)_R$ sits inside $SO(12)$. The subgroup $SO(3)_R$, which commutes with the whole $SO(12)$, is instead generated by the following uniquely determined linear combinations of the generators $\mathbb{H}_i$:

\[
J^R_1 = \frac{1}{2\sqrt{2}} \left( \mathbb{H}_{10} - \mathbb{H}_{13} + \mathbb{H}_{16} - \mathbb{H}_{21} \right)
\]
\[ J_R^2 = -\frac{1}{2\sqrt{2}} (H_{12} - H_{14} + H_{17} + H_{20}) \]
\[ J_R^3 = \frac{1}{2\sqrt{2}} (H_{11} - H_{15} + H_{18} - H_{19}) \]

which close the standard commutation relations:
\[ [J^R_i, J^R_j] = \epsilon_{ijk} J^R_k \] (2.40)

The ordering of the coset generators is obvious from equation (2.37). First we have have listed the four non-compact Cartans, then non-compact combinations associated with the 12 roots that project onto the long roots of \( F_4 \) with multiplicity one. Finally we have listed the non-compact combinations associated with the roots that project onto the short roots of \( F_4 \) in exactly the same order as their compact analogues appear in the listing of \( \mathbb{H} \)-generators. From the point of view of representation theory we know that the \( \mathbb{K} \)-space transforms as follows under \( \text{SO}(3)_R \times \text{SO}(12) \):
\[ \mathbb{K} = (2, 32) \] (2.41)

and we could arrange the generators into linear combinations corresponding to the weights of the representation (2.41), yet this is not essential for our present purposes.

### 2.4 Structure of the Solvable Lie algebra

We can now come to the main point of our construction which relates to the solvable Lie algebra \( \text{Solv}_{E_7(-5)} \) whose corresponding group manifold is metrically equivalent to the coset manifold (2.3) and to its relation with the solvable Lie algebra \( \text{Solv}_{F_4(4)} \) whose corresponding group manifold is metrically equivalent to the coset manifold (2.4).

First we define the solvable Lie algebra \( \text{Solv}_{E_7(-5)} \). This is easily done. Following the general theory recalled, for instance in [9], [10], we know that \( \text{Solv}_{E_7(-5)} \) is the linear span of the non-compact Cartan generators plus those linear combinations of the positive root step operators which pertain to the considered real section. In practice this means:

\[
\text{Solv}_{E_7(-5)} = \text{real span} \{ \mathcal{H}_i^{n.c.}, E^{n_{12}}, \mathcal{E}_A^+, \mathcal{E}_A^- \}
\]

\[ (i = 1, \ldots, 4; I = 1, \ldots, 12; A = 1, \ldots, 24) \] (2.42)

As we know the solvable algebra \( \text{Solv}_{G/H} \) associated with any non-compact coset \( G/H \) has the great advantage that by exponentiation it provides a polynomial parametrization of the coset representative and hence of the scalar fields of supergravity. With respect to the traditional parametrization of cosets in terms of \( \exp(\mathbb{K}) \) the advantages of the solvable parametrization are obtained at one price: while \( \mathbb{K} \) is a representation of \( \mathbb{H} \), the solvable algebra \( \text{Solv}_{G/H} \) is not. In the non maximally split case something very useful, however, occurs. Although \( \text{Solv}_{E_7(-5)} \) is not a representation of the full compact group \( \text{SO}(3)_R \times \text{SO}(12) \) yet it transforms covariantly under the action of a proper compact subgroup, the \( \text{paint group} \), specifically \( G_{\text{paint}} = \text{SO}(3)_3 \), defined in eq. (2.38). Indeed the decomposition of \( \text{Solv}_{E_7(-5)} \) with respect to \( \text{SO}(3)_3 \) is:

\[
\text{Solv}_{E_7(-5)} = 16 \times (1, 1, 1) \oplus 4 \times (2, 2, 0) \oplus 4 \times (2, 0, 2) \oplus 4 \times (0, 2, 2) \] (2.43)
and \( \text{SO}(3) \) works as an automorphism group of the solvable Lie algebra. The sixteen singlets are the four Cartan generators plus the twelve \( E^\alpha \) step operators associated with long roots of \( F_4 \). The forty-eight non-singlets, distributed in irreps as described in eq. (2.43) are instead the generators associated with the \( \delta \)-roots that project onto the short ones of \( F_4 \).

This covariant structure of the solvable Lie algebra \( \text{Solv}_{E_7(-5)} \) is responsible for its relation with \( \text{Solv}_{F_4(4)} \) and for the painted billiard phenomenon. Let us see how.

We are interested in the structure constants of the Solvable Lie algebra which in turn determine the Nomizu connection and hence the 1st order equations for the tangent vector to the geodesic [8]. Calling \( T_\Lambda \) a set of generators in the adjoint representation of the algebra we read off the structure constants from the equation:

\[
[T_\Sigma , T_\Pi] = C^\Lambda_{\Sigma \Pi} T_\Lambda
\]  

Let us first consider the solvable Lie algebra \( \text{Solv}_{F_4(4)} \) associated with the coset (2.4). This is a maximally split case and the structure of \( \text{Solv}_{F_4(4)} \) is the canonical one discussed in [8]. We have 4 Cartan generators \( H_a \) and 24 positive roots that split into two subsets of 12 long roots \( \alpha^\ell \) and 12 short roots \( \alpha^s \). Calling \( \Delta^\ell \) and \( \Delta^s \) the two subsets we have the following structure:

\[\forall \alpha^\ell, \beta^\ell \in \Delta^\ell : \alpha^\ell + \beta^\ell = \begin{cases} \text{not a root or} \\ \gamma^\ell \in \Delta^\ell \end{cases}\]

\[\forall \alpha^\ell \in \Delta^\ell \text{ and } \forall \beta^s \in \Delta^s : \alpha^\ell + \beta^s = \begin{cases} \text{not a root or} \\ \gamma^s \in \Delta^s \end{cases}\]

\[\forall \alpha^s, \beta^s \in \Delta^s : \alpha^s + \beta^s = \begin{cases} \text{not a root or} \\ \gamma^s \in \Delta^s \text{ or} \\ \gamma^\ell \in \Delta^\ell \end{cases}\]

Consequently, let \( H_i \) be the Cartan generators and \( E^\alpha^\ell, E^\alpha^s \) be the step operators respectively associated with positive long and short roots. This set of operators completes a basis of 28 generators for the solvable Lie algebra \( \text{Solv}_{F_4(4)} \). In view of eq.s (2.45) the possible structure constants are:

\[C^\Lambda_{\Sigma \Pi} \equiv \{ C_{\alpha^\ell, \beta^\ell, \gamma^\ell}^\alpha, C_{\alpha^\ell, \beta^s, \gamma^s}^\alpha, C_{\alpha^s, \beta^\ell, \gamma^s}^\alpha, C_{\alpha^s, \beta^s, \gamma^s}^\alpha \}\]  

and we further have:

\[C_{\alpha^\ell, \beta^\ell}^\alpha = \delta_{\beta^\ell, \alpha^\ell} \delta_{\beta^\ell, \alpha^\ell}^\ell \]

\[C_{\alpha^s, \beta^s}^\alpha = \delta_{\beta^s, \alpha^s} \delta_{\beta^s, \alpha^s}^s \]

\[C_{\alpha^\ell, \beta^s, \gamma^s}^\alpha = \delta_{\beta^s, \gamma^s}^s N_{\beta^s, \gamma^s}^s \]

\[C_{\alpha^s, \beta^\ell, \gamma^s}^\alpha = \delta_{\beta^\ell, \gamma^s}^s N_{\beta^\ell, \gamma^s}^s \]

\[C_{\alpha^\ell, \beta^s, \gamma^s}^\alpha = \delta_{\beta^s, \gamma^s}^s N_{\beta^s, \gamma^s}^s \]

\[C_{\alpha^s, \beta^s, \gamma^s}^\alpha = \delta_{\beta^s, \gamma^s}^s N_{\beta^s, \gamma^s}^s \]

where the matrix \( N_{\beta^s, \gamma^s} \), defined by the standard Cartan-Weyl commutation relations as given in eq. (B.23) of the appendix, or in table 1.27, differs from zero only when the sum of the two roots
\( \beta \) and \( \gamma \) is a root. Hence it suffices to know \( N_{\beta \gamma} \) and the solvable Lie algebra structure constants are completely determined. In the following three tables (2.48), (2.49), (2.50) we exhibit the values of \( N_{\beta \gamma} \) for the \( \text{F}_{4(4)} \) Lie algebra.

\[
\begin{array}{cccccccccccc}
\hline
\alpha^\ell_1 & \alpha^\ell_2 & \alpha^\ell_3 & \alpha^\ell_4 & \alpha^\ell_5 & \alpha^\ell_6 & \alpha^\ell_7 & \alpha^\ell_8 & \alpha^\ell_9 & \alpha^\ell_{10} & \alpha^\ell_{11} & \alpha^\ell_{12} \\
0 & -\sqrt{2} & 0 & 0 & -\sqrt{2} & 0 & 0 & -\sqrt{2} & 0 & \sqrt{2} & 0 & 0 \\
\sqrt{2} & 0 & 0 & -\sqrt{2} & 0 & 0 & -\sqrt{2} & 0 & 0 & 0 & -\sqrt{2} & 0 \\
0 & 0 & 0 & -\sqrt{2} & 0 & 0 & -\sqrt{2} & 0 & 0 & -\sqrt{2} & 0 & 0 \\
0 & \sqrt{2} & \sqrt{2} & 0 & 0 & 0 & 0 & -\sqrt{2} & \sqrt{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} & 0 & \sqrt{2} & 0 & 0 & 0 \\
\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{2} & -\sqrt{2} & 0 & 0 & 0 \\
0 & 0 & 0 & \sqrt{2} & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
\sqrt{2} & 0 & 0 & \sqrt{2} & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\sqrt{2} & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\sqrt{2} & 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
\end{array}
\]

\( (2.48) \)

\[
\begin{array}{cccccccccccc}
\hline
\alpha^s_1 & \alpha^s_2 & \alpha^s_3 & \alpha^s_4 & \alpha^s_5 & \alpha^s_6 & \alpha^s_7 & \alpha^s_8 & \alpha^s_9 & \alpha^s_{10} & \alpha^s_{11} & \alpha^s_{12} \\
0 & \sqrt{2} & 0 & -\sqrt{2} & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\
0 & 0 & -\sqrt{2} & 0 & 0 & -\sqrt{2} & 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\
0 & -\sqrt{2} & 0 & \sqrt{2} & 0 & 0 & 0 & -\sqrt{2} & 0 & 0 & 0 & 0 \\
\sqrt{2} & 0 & 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\
-\sqrt{2} & 0 & 0 & 0 & 0 & -\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
\sqrt{2} & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
\end{array}
\]

\( (2.49) \)
\[
H[1] = \frac{1}{\sqrt{2}} \mathcal{H}_{1}^{n.c.}; \quad H[2] = \frac{1}{\sqrt{2}} \mathcal{H}_{2}^{n.c.}; \quad H[3] = \frac{1}{\sqrt{2}} \mathcal{H}_{3}^{n.c.}; \quad H[4] = \frac{1}{\sqrt{2}} \mathcal{H}_{4}^{n.c.}
\]

Table 5: Listing of Cartan generators of \(Solv_{E_7(-5)}\) which exactly correspond to the Cartan generators of \(\mathbf{F}_4(4)\)

| \(\alpha_1^s\) | \(\alpha_2^s\) | \(\alpha_3^s\) | \(\alpha_4^s\) | \(\alpha_5^s\) | \(\alpha_6^s\) | \(\alpha_7^s\) | \(\alpha_8^s\) | \(\alpha_9^s\) | \(\alpha_{10}^s\) | \(\alpha_{11}^s\) | \(\alpha_{12}^s\) |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | -1 | 0 | -1 | 0 | 0 | \(\sqrt{2}\) | -\(\sqrt{2}\) | \(\sqrt{2}\) | 0 | 0 | \(\alpha_1^s\) |
| -1 | 0 | \(\sqrt{2}\) | 0 | \(\sqrt{2}\) | 1 | -1 | 0 | 0 | -1 | 0 | \(\sqrt{2}\) | \(\alpha_2^s\) |
| 1 | -\(\sqrt{2}\) | 0 | 1 | -\(\sqrt{2}\) | 0 | -1 | 0 | 1 | 0 | 0 | \(\sqrt{2}\) | \(\alpha_3^s\) |
| 0 | 0 | -1 | 0 | 1 | \(\sqrt{2}\) | \(\sqrt{2}\) | 0 | 0 | 1 | -\(\sqrt{2}\) | 0 | \(\alpha_4^s\) |
| 1 | -\(\sqrt{2}\) | \(\sqrt{2}\) | -1 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | \(\sqrt{2}\) | \(\alpha_5^s\) |
| 0 | -1 | 0 | -\(\sqrt{2}\) | -1 | 0 | \(\sqrt{2}\) | 0 | 1 | 0 | \(\sqrt{2}\) | 0 | \(\alpha_6^s\) |
| 0 | 1 | 1 | -\(\sqrt{2}\) | 0 | -\(\sqrt{2}\) | 0 | 1 | 0 | 0 | \(\sqrt{2}\) | 0 | \(\alpha_7^s\) |
| -\(\sqrt{2}\) | 0 | 0 | 0 | -1 | 0 | -1 | 0 | \(\sqrt{2}\) | \(\sqrt{2}\) | 0 | 0 | \(\alpha_8^s\) |
| \(\sqrt{2}\) | 0 | -1 | 0 | 0 | -1 | 0 | -\(\sqrt{2}\) | 0 | -\(\sqrt{2}\) | 0 | 0 | \(\alpha_9^s\) |
| -\(\sqrt{2}\) | 1 | 0 | -1 | 0 | 0 | 0 | -\(\sqrt{2}\) | \(\sqrt{2}\) | 0 | 0 | 0 | \(\alpha_{10}^s\) |
| -1 | 0 | 0 | \(\sqrt{2}\) | 0 | -\(\sqrt{2}\) | -\(\sqrt{2}\) | 0 | 0 | 0 | 0 | 0 | \(\alpha_{11}^s\) |
| 0 | -\(\sqrt{2}\) | -\(\sqrt{2}\) | 0 | -\(\sqrt{2}\) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | \(\alpha_{12}^s\) |

The ordering of long and short roots of the \(\mathbf{F}_4\) system is that used in tables: 2 and 3. On the other hand the explicit determination of the tensor \(N_{\alpha\beta}\) which appears in the standard Cartan-Weyle commutation relations was performed via the explicit construction of the fundamental 26-dimensional representation of this Lie algebra. This construction is described in appendix B.

Now the exciting point about the solvable Lie algebra of the full non split coset (2.3) which contains all the degrees of freedom of supergravity is that it can be exhibited in terms of the structure constants of its split Tits Satake submanifold (2.4) by utilizing also the covariance with respect to the compact paint subgroup \(G_{\text{paint}} = \text{SO}(3)_{(3)}^3\). This is the result of an essential interplay of impressive elegance between the graded structure of the split Tits Satake algebra, which is non simply laced and for that reason contains a distinction between short and long roots, and the rearrangement of those roots of \(E_7\) which project onto the short ones of \(\mathbf{F}_4\) into representations of the compact paint group \(\text{SO}(3)_{(3)}^3\).

We already emphasized that, under the action of the paint group (2.38), the generators of the Solvable Lie algebra \(Solv_{E_7(-5)}\) decompose into the irreducible representations mentioned in eq. (2.43). Let us define the diagonal subgroup of the three \(\text{SO}(3)_{(3)}\):

\[
G^0_{\text{paint}} \equiv \text{SO}(3)_{(3)}^{diag} = \text{diagonal} \left[ \text{SO}(3)_{(3)} \times \text{SO}(3)_{(3)} \times \text{SO}(3)_{(3)} \right]
\]
\[ \begin{align*}
\Phi[a_1^\ell] &= \frac{1}{\sqrt{2}} E^{\eta_1} ; \\
\Phi[a_2^\ell] &= \frac{1}{\sqrt{2}} E^{\eta_2} ; \\
\Phi[a_3^\ell] &= -\frac{1}{\sqrt{2}} E^{\eta_3} \\
\Phi[a_4^\ell] &= \frac{1}{\sqrt{2}} E^{\eta_4} ; \\
\Phi[a_5^\ell] &= \frac{1}{\sqrt{2}} E^{\eta_5} ; \\
\Phi[a_6^\ell] &= \frac{1}{\sqrt{2}} E^{\eta_6} \\
\Phi[a_7^\ell] &= \frac{1}{\sqrt{2}} E^{\eta_7} ; \\
\Phi[a_8^\ell] &= \frac{1}{\sqrt{2}} E^{\eta_8} ; \\
\Phi[a_9^\ell] &= \frac{1}{\sqrt{2}} E^{\eta_9} \\
\Phi[a_{10}^\ell] &= \frac{1}{\sqrt{2}} E^{\eta_{10}} ; \\
\Phi[a_{11}^\ell] &= -\frac{1}{\sqrt{2}} E^{\eta_{11}} ; \\
\Phi[a_{12}^\ell] &= -\frac{1}{\sqrt{2}} E^{\eta_{12}}
\end{align*} \]

Table 6: Listing of generators of \( Solv_{E7(-5)} \) which correspond to long roots of \( F_4 \) and appear in one copy. The order of \( \eta \)-roots is that listed in table 2.

| \( i \) | \( \Omega_0[a_i^\ell] = \) | \( \Omega_X[a_i^\ell] = \) | \( \Omega_Y[a_i^\ell] = \) | \( \Omega_Z[a_i^\ell] = \) |
|---|---|---|---|---|
| 1 | \( \frac{1}{2} (E^{\delta_2(1)} + E^{\delta_3(1)}) \) | \( \frac{1}{2} (E^{\delta_1(1)} - E^{\delta_4(1)}) \) | \( -\frac{1}{2} (E^{\delta_1(1)} + E^{\delta_4(1)}) \) | \( \frac{1}{2} (E^{\delta_1(1)} - E^{\delta_3(1)}) \)
| 2 | \( \frac{1}{2} (E^{\delta_2(2)} + E^{\delta_3(2)}) \) | \( -\frac{1}{2} (E^{\delta_1(2)} - E^{\delta_4(2)}) \) | \( \frac{1}{2} (E^{\delta_1(2)} + E^{\delta_4(2)}) \) | \( \frac{1}{2} (-E^{\delta_2(2)} + E^{\delta_3(2)}) \)
| 3 | \( -\frac{1}{2} (E^{\delta_2(3)} + E^{\delta_3(3)}) \) | \( \frac{1}{2} (E^{\delta_1(3)} - E^{\delta_4(3)}) \) | \( -\frac{1}{2} (E^{\delta_1(3)} + E^{\delta_4(3)}) \) | \( \frac{1}{2} (E^{\delta_2(3)} - E^{\delta_3(3)}) \)
| 4 | \( -\frac{1}{2} (E^{\delta_1(4)} + E^{\delta_4(4)}) \) | \( \frac{1}{2} E^{\delta_2(4)} - E^{\delta_3(4)} \) | \( \frac{1}{2} (E^{\delta_4(4)} + E^{\delta_3(4)}) \) | \( -\frac{1}{2} (E^{\delta_1(4)} - E^{\delta_4(4)}) \)
| 5 | \( -\frac{1}{2} (E^{\delta_2(5)} + E^{\delta_3(5)}) \) | \( \frac{1}{2} (E^{\delta_1(5)} - E^{\delta_4(5)}) \) | \( -\frac{1}{2} (E^{\delta_1(5)} + E^{\delta_4(5)}) \) | \( -\frac{1}{2} (-E^{\delta_2(5)} + E^{\delta_4(5)}) \)
| 6 | \( -\frac{1}{2} (E^{\delta_1(6)} + E^{\delta_4(6)}) \) | \( \frac{1}{2} E^{\delta_2(6)} - E^{\delta_3(6)} \) | \( \frac{1}{2} E^{\delta_4(6)} + E^{\delta_3(6)} \) | \( -\frac{1}{2} (E^{\delta_1(6)} - E^{\delta_4(6)}) \)
| 7 | \( -\frac{1}{2} (E^{\delta_1(7)} + E^{\delta_4(7)}) \) | \( \frac{1}{2} E^{\delta_2(7)} - E^{\delta_3(7)} \) | \( \frac{1}{2} E^{\delta_4(7)} + E^{\delta_3(7)} \) | \( -\frac{1}{2} (E^{\delta_1(7)} - E^{\delta_4(7)}) \)
| 8 | \( -\frac{1}{2} (E^{\delta_2(8)} + E^{\delta_3(8)}) \) | \( -\frac{1}{2} (E^{\delta_1(8)} - E^{\delta_4(8)}) \) | \( \frac{1}{2} E^{\delta_2(8)} + E^{\delta_4(8)} \) | \( -\frac{1}{2} (E^{\delta_1(8)} + E^{\delta_4(8)}) \)
| 9 | \( -\frac{1}{2} (E^{\delta_2(9)} + E^{\delta_3(9)}) \) | \( \frac{1}{2} (E^{\delta_1(9)} - E^{\delta_4(9)}) \) | \( -\frac{1}{2} (E^{\delta_1(9)} + E^{\delta_4(9)}) \) | \( \frac{1}{2} (E^{\delta_4(9)} - E^{\delta_3(9)}) \)
| 10 | \( -\frac{1}{2} (E^{\delta_2(10)} + E^{\delta_3(10)}) \) | \( -\frac{1}{2} (E^{\delta_1(10)} - E^{\delta_4(10)}) \) | \( \frac{1}{2} (E^{\delta_1(10)} + E^{\delta_4(10)}) \) | \( -\frac{1}{2} (E^{\delta_2(10)} - E^{\delta_4(10)}) \)
| 11 | \( \frac{1}{2} (E^{\delta_1(11)} + E^{\delta_4(11)}) \) | \( -\frac{1}{2} (E^{\delta_2(11)} - E^{\delta_3(11)}) \) | \( \frac{1}{2} (E^{\delta_2(11)} + E^{\delta_3(11)}) \) | \( 1/2 (E^{\delta_1(11)} - E^{\delta_4(11)}) \)
| 12 | \( \frac{1}{2} (E^{\delta_1(12)} - E^{\delta_4(12)}) \) | \( -\frac{1}{2} (E^{\delta_2(12)} + E^{\delta_3(12)}) \) | \( \frac{1}{2} (E^{\delta_2(12)} - E^{\delta_3(12)}) \) | \( 1/2 (E^{\delta_1(12)} + E^{\delta_4(12)}) \)

Table 7: Listing of generators of \( Solv_{E7(-5)} \) which correspond to short roots of \( F_4 \) and appear in four copies. The order of \( \delta \)-roots is that displayed in table 3.

\[
\begin{align*}
\text{since} \\
\text{2} \otimes \text{2} &= \text{1} \oplus \text{3} \\
\text{(2.52)}
\end{align*}
\]

holds true for SO(3) representations, it follows that under \( \text{SO(3)}^{\text{diag}}_\beta \) the Lie algebra \( Solv_{E7(-5)} \) decomposes as follows:

\[
\begin{align*}
Solv_{E7(-5)} \xrightarrow{\text{SO(3)}^{\text{diag}}_\beta} & \left( \begin{array}{c}
\text{4} \\
\text{Cartan} \\
\text{long roots}
\end{array} \right) + \\
\text{12} \\
\text{short roots}
\end{align*}
\]

\[
\times \text{1} \oplus \left( \begin{array}{c}
\text{12} \\
\text{short roots}
\end{array} \right) \times \text{3} \\
\text{(2.53)}
\]

Hence the representation \( \mathbf{J} \) mentioned in eqs (1,25) and (1,26) is \( \mathbf{J} = \mathbf{3} \), the triplet of \( \text{SO(3)}^{\text{diag}} \). The decomposition (2.53) is explicitly exhibited in tables 5, 6 and 7. In table 5 modulo some changes in normalization we list the non compact Cartan generators of \( E_{7(7)} \) which correspond
to the full set of Cartan generators of \( F_{4(4)} \). In table 6 we define a set of generators \( \Phi[\alpha^s] \) associated with the long roots of \( F_{4(4)} \), which are obviously given by the step operators \( E_\eta \) of \( E_{7(-5)} \) since the \( \eta \)-roots project on such short roots of \( F_4 \). There are just some suitable \( \pm \sqrt{2} \) factors in the normalization which are purposely chosen in order to make the relation between the two solvable Lie algebras clean. All these generators are singlets under \( SO(3)_\beta \) and therefore also under \( SO(3)_{\beta diag} \). Finally in table 7 we list a set of four \( E_{7(-5)} \) generators \( \Omega_\alpha[\alpha^s] \), \( (A=0,X,Y,Z) \), associated with each of the short roots \( \alpha^s \) of \( F_{4(4)} \). Indeed each such root is the image, in the projection, of four different \( E_{7(-5)} \) roots, namely, the \( \delta^\ell \) roots, displayed in table 6. Hence the four \( \Omega_\alpha[\alpha^s] \) operators are, with convenient normalization factors, step operators of \( E_{7(-5)} \) corresponding to the \( \delta \)-roots. The normalization factors and the precise correspondence is chosen in such a way that the \( \Omega_0 \) are singlets under \( SO(3)_{\beta diag} \), while \( \Omega_{x=X,Y,Z} \) form a triplet.

If we use these generators and, in order to avoid proliferation of symbols, we denote by the same letter the generator of the algebra and its dual one-form appearing in the Maurer Cartan equations:

\[
d T^\Lambda = \frac{1}{2} C_\Sigma^\Lambda T^\Sigma \wedge T^\Pi \tag{2.54}
\]

the structure constants of \( Solv_{E_{7(-5)}} \) can be exhibited by writing the following Maurer Cartan equations, which just contain the structure constants of the \( F_{4(4)} \) solvable algebra, discussed before, plus the quaternionic structure anticipated in table (1.27) of the introduction.

\[
\begin{align*}
    d H_\lambda &= 0 \\
    d \Phi[\alpha^s] &= C^{\alpha^s}_{\mu^s \nu^s} H_\mu \wedge \Phi[\nu^s] + \frac{1}{2} C^{\alpha^s}_{\beta^s \gamma^s} \Phi[\beta^s] \wedge \Phi[\gamma^s] + \\
    &\quad \frac{1}{2} C^{\alpha^s}_{\beta^s \gamma^s} (\Omega_0[\beta^s] \wedge \Omega_0[\gamma^s] + \Omega_x[\beta^s] \wedge \Omega_x[\gamma^s]) \\
    d \Omega_0[\alpha^s] &= C^{\alpha^s}_{\mu^s \nu^s} H_\mu \wedge \Omega_0[\nu^s] + \frac{1}{2} C^{\alpha^s}_{\beta^s \gamma^s} (\Omega_0[\beta^s] \wedge \Omega_0[\gamma^s] + \Omega_x[\beta^s] \wedge \Omega_x[\gamma^s]) + C^{\alpha^s}_{\beta^s \gamma^s} \Phi[\beta^s] \wedge \Omega_0[\gamma^s] \\
    d \Omega_x[\alpha^s] &= C^{\alpha^s}_{\mu^s \nu^s} H_\mu \wedge \Omega_x[\nu^s] + \frac{1}{2} C^{\alpha^s}_{\beta^s \gamma^s} (\Omega_0[\beta^s] \wedge \Omega_x[\gamma^s] - \Omega_x[\beta^s] \wedge \Omega_0[\gamma^s] - e^{xyz} \Omega_y[\beta^s] \wedge \Omega_z[\gamma^s]) + C^{\alpha^s}_{\beta^s \gamma^s} \Phi[\beta^s] \wedge \Omega_x[\gamma^s]
\end{align*}
\tag{2.55}
\]

Equations (2.55) are just a short way of writing all commutation relations and exhibit the interplay between the graded structure of \( Solv_{F_{4(4)}} \) and the structure of the paint group representation. Indeed we see the announced quaternionic structure! What actually happens is that the Cartan and long root generators are isomorphic in the two algebras, while the short root generators are promoted to quaternions while going from \( F_{4(4)} \) to \( E_{7(-5)} \). We can write

\[
F_{4(4)} \ni E^{\alpha^s} \implies \Omega_0[\alpha^s] + \Omega_X[\alpha^s] j^X + \Omega_Y[\alpha^s] j^Y + \Omega_Z[\alpha^s] j^Z \in E_{7(-5)}
\tag{2.56}
\]

where \( j^X, j^Y, j^Z \) are the three quaternionic imaginary units.

This structure has very relevant consequences for the solution of the differential equations and for the billiard phenomenon. Since the Nomizu connection determining the first order equations for tangent vectors is completely determined by the structure constants of the solvable Lie algebra we can just adopt the following strategy:
1. Rather than considering the original problem associated with the non split manifold (2.3) we consider the problem associated with the split Tits Satake manifold (2.4), which can be solved along the lines of paper [8] using, in particular, the compensator method to integrate the 1st order differential equations.

2. Once we have obtained a solution for the system described by the structure constants (2.46-2.47) we also possess a particular solution for the system (2.55). It just correspond to setting the fields associated with $\Omega_{X,Y,Z}$ to zero.

3. A large class of solutions of the system (2.55) can be obtained from the general solution of the $F_4$ system with structure constants (2.46,2.47) by means of global rotations of the paint group $G_{\text{paint}} = \text{SO}(3)^3$.

From the point of view of the billiard picture we know that switching on root fields correspond to the introduction of dynamical walls on which the fictitious cosmic ball will bounce. The structure of the solvable algebra implies that the billiard chamber is just the Weyl chamber of $F_4(4)$, yet certain dynamical walls are painted, namely occur in four copies constituting a quaternion. In explicit solutions we see also the color of the actual wall which is excited.

### 3 The first order equations for the tangent vectors

As we showed in [8], the field equations of the purely time dependent $\sigma$-model, which is what we are supposed to solve in our quest for time dependent solutions of supergravity, can be written as follows:

$$\dot{Y}^A + \Gamma^A_{BC} Y^B Y^C = 0 \quad (3.1)$$

where $Y^A$ denotes the purely time dependent tangent vectors to the geodesic in an anholonomic basis:

$$Y^A = \begin{cases} Y^i = V^j_i (\phi) \dot{\phi}^j & i \in \text{CSA} \\ Y^\alpha = \sqrt{2} V^i_\alpha (\phi) \dot{\phi}^i & i \in \text{positive root system } \Delta_+ \end{cases} \quad (3.2)$$

$V^A_I (\phi) d\phi^I$ being the vielbein of the target manifold we are considering. In eq. (3.1) the symbol $\Gamma^A_{BC}$ denotes the components of the Levi-Civita connection in the chosen anholonomic basis. Explicitly they are related to the components of the Levi-Civita connection in an arbitrary holonomic basis by:

$$\Gamma^A_{BC} = \Gamma^I_{JK} V^A_I V^B_J V^C_K - \partial_K (V^A_J) V^B_J V^C_K \quad (3.3)$$

where the inverse vielbein is defined in the usual way:

$$V^A_I V^I_B = \delta^A_B \quad (3.4)$$

The basic idea of [8], which was exploited together with the compensator method in order to construct explicit solutions, is the following. As already recalled in eq. (1.13), the connection $\Gamma^A_{BC}$ can be identified with the Nomizu connection defined on a solvable Lie algebra, if the coset representative $L$ from which we construct the vielbein is solvable, namely if it is represented as
the exponential of the associated solvable Lie algebra $\text{Solv}(U/H)$. In fact, as we can read in [21]
once we have defined over $\text{Solv}$ a non degenerate, positive definite symmetric form:

$$\langle , \rangle : \text{Solv} \otimes \text{Solv} \rightarrow \mathbb{R}$$

$$\langle X, Y \rangle = \langle Y, X \rangle$$ (3.5)

whose lifting to the manifold produces the metric, the covariant derivative is defined through the Nomizu operator:

$$\forall X \in \text{Solv} : \mathbb{L}_X : \text{Solv} \rightarrow \text{Solv}$$ (3.6)

so that

$$\forall X, Y, Z \in \text{Solv} : 2\langle Z, \mathbb{L}_X Y \rangle = \langle Z, [X, Y] \rangle - \langle X, [Y, Z] \rangle - \langle Y, [X, Z] \rangle$$ (3.7)

while the Riemann curvature $2$-form is given by the commutator of two Nomizu operators:

$$R^W Z (X, Y) = \langle W, \{ [L_X, L_Y] - L_{[X, Y]} \} Z \rangle$$ (3.8)

This implies that the covariant derivative explicitly reads:

$$\mathbb{L}_X Y = \Gamma^Z_{XY} Z$$ (3.9)

where

$$\Gamma^Z_{XY} = \frac{1}{2} (\langle Z, [X, Y] \rangle - \langle X, [Y, Z] \rangle - \langle Y, [X, Z] \rangle) \frac{1}{< Z, Z >} \quad \forall X, Y, Z \in \text{Solv}$$ (3.10)

Eq. (3.10) is true for any solvable Lie algebra. In the case of maximally non-compact, split algebras we can write a general form for $\Gamma^Z_{XY}$, namely:

$$\Gamma^i_{jk} = 0$$

$$\Gamma^i_{\alpha \beta} = \frac{1}{2} \left( -\langle E_\alpha, [E_\beta, H^i] \rangle - \langle E_\beta, [E_\alpha, H^i] \rangle \right) = \frac{1}{2} \alpha^i \delta_{\alpha \beta}$$

$$\Gamma^\alpha_{ij} = \Gamma^i_{\alpha j} = \Gamma^i_{j \alpha} = 0$$

$$\Gamma^\alpha_{\beta i} = \frac{1}{2} \left( \langle E_\alpha, [E_\beta, H_i] \rangle - \langle E_\beta, [H_i, E_\alpha] \rangle \right) = -\alpha_i \delta^\alpha_{\beta}$$

$$\Gamma^{\alpha+\beta}_{\alpha} = -\Gamma^{\alpha+\beta}_{\beta} = \frac{1}{2} N^{\alpha \beta}$$

$$\Gamma^{\alpha}_{\alpha+\beta} = \frac{1}{2} N^{\alpha \beta}$$ (3.11)

where $N^{\alpha \beta}$ is defined by the commutator $[E_\alpha, E_\beta] = N^{\alpha \beta} E_{\alpha+\beta}$, as usual. In the case of $F_4(4)$, the coefficients $N^{\alpha \beta}$ are read–off from the eq.s 2.49, 2.48, 2.50. The explicit form (3.11) follows from the choice of the non degenerate metric:

$$\langle \mathcal{H}_i, \mathcal{H}_j \rangle = 2 \delta_{ij}$$

$$\langle \mathcal{H}_i, E_\alpha \rangle = 0$$

$$\langle E_\alpha, E_\beta \rangle = \delta_{\alpha \beta}$$ (3.12)

$\forall \mathcal{H}_i, \mathcal{H}_j \in \text{CSA}$ and $\forall E_\alpha$, step operator associated with a positive root $\alpha \in \Delta_+$. For any other non split case, as that of $\text{Solv}(E_{7(-5)})$, the Nomizu connection exists nonetheless although it
does not take the form (3.11). It follows from eq. (3.10) upon the choice of an invariant positive metric on \(\text{Solv}\) and the use of the structure constants of \(\text{Solv}\). Given the list of generators in tables 4 and 5 the positive metric on \(\text{Solv}\) is easily defined in full analogy with the definition of \(\mathbb{S}\). The metric is diagonal and normalized as it follows:

\[
\begin{align*}
\langle H_i, H_j \rangle &= 2 \delta_{ij} \\
\langle \Phi[\alpha]^\ell, H_i \rangle &= 0 \\
\langle \Omega_i[\alpha^s], H_i \rangle &= 0
\end{align*}
\]

The Nomizu connection can be explicitly calculated from eq. (3.10) reading the structure constants from the Maurer Cartan equations (2.55). In the case of all split algebras, the first order equations take the general form:

\[
\begin{align*}
\dot{Y}^i &+ \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha^i Y^2 = 0 \\
\dot{Y}^\alpha &+ \sum_{\beta \in \Delta^+} N_{\alpha \beta} Y^\beta Y^{\alpha + \beta} - \alpha_i Y^i Y^\alpha = 0
\end{align*}
\]

which follows from eq. (3.16). For the solvable Lie algebra of \(F_{4(4)}\) eq. (3.14) takes the form:

\[
\begin{align*}
\dot{H}^i &+ \frac{1}{2} \sum_{\alpha \in \Delta^\ell} \alpha^i \Phi[\alpha]^2 + \frac{1}{2} \sum_{\alpha \in \Delta^s} \alpha^i \Omega[\alpha]^2 = 0 \\
\dot{\Phi}[\alpha] &- \alpha_i \cdot H \Phi[\alpha] + \sum_{\beta \in \Delta^\ell} N_{\alpha \beta} \Phi[\beta] \Phi[\alpha + \beta] \\
&+ \sum_{\beta \in \Delta^s} N_{\alpha \beta} \Omega[\beta] \Omega[\alpha + \beta] = 0 \\
\dot{\Omega}[\alpha] &- \alpha_s \cdot H \Omega[\alpha] + \sum_{\beta, \gamma \in \Delta^\ell} N_{\alpha \beta} \Omega[\beta] \Omega[\alpha + \beta] \\
&+ \sum_{\beta, \gamma \in \Delta^s} N_{\alpha \beta} \Omega[\beta] \Omega[\alpha + \beta] = 0
\end{align*}
\]

where for notation simplicity we have given to the component \(Y^A\) of the tangent vector \(\vec{Y}\) along a generator \(T_A\) of the solvable Lie algebra the same name as the generator itself.

In the case of the \(E_{7(-5)}\) Lie algebra the first order equations for the tangent vector take the following form:

\[
\begin{align*}
\dot{H}^i &+ \frac{1}{2} \sum_{\alpha \in \Delta^\ell} \alpha^i \Phi[\alpha]^2 + \frac{1}{2} \sum_{\alpha \in \Delta^s} \alpha^i \left( \Omega_0[\alpha]^2 + \sum_{i=1}^3 \Omega_i[\alpha]^2 \right) = 0 \\
\dot{\Phi}[\alpha] &- \alpha_i \cdot H \Phi[\alpha] + \sum_{\beta \in \Delta^\ell} N_{\alpha \beta} \Phi[\beta] \Phi[\alpha + \beta] \\
&+ \sum_{\beta \in \Delta^s} N_{\alpha \beta} \left( \Omega_0[\beta] \Omega_0[\alpha + \beta] + \sum_{i=1}^3 \Omega_i[\beta] \Omega_i[\alpha + \beta] \right) = 0
\end{align*}
\]
As one sees from the above equations the E\(_7\)(−5) differential system (3.16) is consistently truncated to the F\(_4\)(4) system (3.15) by setting \(\Omega_i[\alpha_s] = 0\), (\(i = 1, 2, 3\)) and identifying \(\Omega[\alpha_s] = \Omega_0[\alpha_s]\). Hence any solution of the F\(_4\)(4) equations is also a particular solution of the E\(_7\)(−5) ones. On the other hand eq.s (3.16) are invariant under the action of the paint group \(G_{\text{paint}} = SO(3)^3\).

In the next section, we utilize the compen-sator method to solve the first order equations (3.14) in the case of \(\text{Solv}_F \ F_4(4)\) and then we use the paint group to rotate these solutions to general solutions of the first order equations of \(\text{Solv}_E E_7(−5)\).

4 Solutions of the F\(_4\)(4) system by means of the compen-sator method

As we showed in [8] in the split case the first order equations for the tangent vectors can be solved in the following way. First one considers the decomposition (2.34) of the full algebra and recalls that, in this case, the compact subalgebra is generated by \(E^\alpha - E^{-\alpha}\) for all \(\alpha \in \Delta_+\). Secondly, one writes the decomposition of the left-invariant one–form on the coset manifold \(U/H\) along the compact and non-compact generators:

\[
\Omega = L^{-1}dL = V^A K_A + \omega^\alpha t_\alpha. \tag{4.1}
\]

where \(V = V^A K_A\) corresponds to the coset manifold vielbein, while \(\omega = \omega^\alpha t_\alpha\) corresponds to the coset manifold H–connection. One notes that the condition for the coset representative \(L\) to be solvable (namely to be the exponential of the solvable algebra) is expressed very simply by:

\[
V^\alpha = \sqrt{2} \omega^\alpha. \tag{4.2}
\]

Thirdly one derives the condition to be fulfilled by an H-gauge transformation:

\[
L \mapsto L h = \bar{L} \quad h = \exp [\theta^\alpha t_\alpha] \tag{4.3}
\]

in order for the solvable gauge (4.2) to be preserved. This latter reads as follows:

\[
\frac{\sqrt{2}}{\text{tr}(t_\alpha^2)} \text{tr} (h^{-1}(\theta) dh(\theta) t_\alpha) = V^\beta \left( -A(\theta)^\alpha_\beta + D(\theta)^\alpha_\beta \right) + V^i D(\theta)_i^\alpha \tag{4.4}
\]
In the above equation the matrix $A(\theta)$ is the adjoint representation of $h \in H$ and $D(\theta)$ is the $D$-representation of the same group element which acts on the complementary space $K$ and which depends case to case:

$$h^{-1} t_{\alpha} h = A(\theta)_{\alpha}^\beta t_{\beta}$$
$$h^{-1} K_A h = D(\theta)_{A}^B K_B$$ \hspace{2cm} (4.5)

In our example of $F_{4(4)}$ the compact group is $H = SU(2)_R \times USp(6)$ and the representation $D$ is the $(14, 2)$

A simple solution of the first order equations (3.14) is easily obtained by setting $Y^\alpha = 0$ and $Y^i = c^i = \text{const}$, namely we can begin with a constant vector in the direction of the CSA. Such a solution is named the normal form of the tangent vector. In the language of billiard dynamics it corresponds to a fictitious ball that moves on a straight line with a constant velocity. All other solutions of eqs. (3.14) can be obtained from the normal form solution by means of successive rotations of the compact group, with parameters $\theta[t]$ satisfying the differential equation (4.4). The advantage of this method, emphasized in [8] where we introduced it, is that at each successive rotation we obtain an equation which is fully integrable in terms of the integral of the previous ones.

In this paper we just present one solution of the $F_{4(4)}$ system which is fully analytical and already sufficiently complicated to display the billiard dynamics with several bounces. Our solution is obtained by applying 5 successive rotations to a normal form vector that we parametrize in terms of 4 constants. We use an intelligent parametrization which is the following one:

$$Y_{nf} = \{ -\frac{\omega_5}{2} - \omega_6, \frac{\omega_5}{2} - \omega_6 + \omega_7, \frac{-\omega_{24}}{4} \}$$ \hspace{2cm} (4.6)

The way $Y_{nf}$ is parametrized and the name given to the constants $\omega_{24, 7, 6, 5}$ anticipate their physical interpretation in the solution we are going to derive. Indeed we obtain our solution by writing:

$$Y(t) = Y_{nf} \exp[H^{24} \theta_{24}(t)] \cdot \exp[H^7 \theta_7(t)] \cdot \exp[H^6 \theta_6(t)] \cdot \exp[H^5 \theta_5(t)] \cdot \exp[H^4 \theta_4(t)]$$ \hspace{2cm} (4.7)

where

$$H^n = \frac{1}{2} (E^{\alpha[n]} - E^{-\alpha[n]})$$ \hspace{2cm} (4.8)

are the compact generators associated with the $F_4$ roots numbered as in table I. The explicit form of the solution of the differential equations (4.4) for the five rotation angles is given below:

$$\theta_4(t) = \arccos \left[ \frac{e^{(t - \tau_4)} (\omega_5 + 2 \omega_6 - \omega_7) \sqrt{1 + e^{2(t - \tau_7) \omega_7}}}{\sqrt{1 + e^{2(t - \tau_6) \omega_6} + e^{2(t - \tau_7) \omega_7} + e^{2(t - \tau_4) (\omega_5 + 2 \omega_6 - \omega_7)} (1 + e^{2(t - \tau_7) \omega_7})}} \right]$$

$$\theta_5(t) = \arccsc \left[ \sqrt{1 + e^{(t - \tau_5) \omega_5}} \cosh \left( \frac{(t - \tau_4) \omega_7}{2} \right) \right] \cdot \sqrt{2}$$

$$\theta_6(t) = \arccot \left[ \frac{e^{(t - \tau_6) \omega_6}}{\sqrt{1 + e^{2(t - \tau_7) \omega_7}}} \right]$$

$$\theta_7(t) = \arccos \left[ \frac{e^{(t - \tau_7) \omega_7}}{\sqrt{1 + e^{2(t - \tau_7) \omega_7}}} \right]$$
\[ \theta_{24}(t) \rightarrow \arccsc \left[ \frac{\sqrt{1 + e^{(t-t_{24})}\omega_{24}}}{\sqrt{2}} \right] \]  

(4.9)

Let us briefly mention the explicit form of the eq.s \[ \text{(4.4)} \] from which we obtained the above result. As specified in eq.\[ \text{(4.7)} \] we perform the compact rotations in the order \( 24 \rightarrow 7 \rightarrow 6 \rightarrow 5 \rightarrow 4 \). This is not a random choice but it is motivated by the fact that in this way the differential equations \[ \text{(4.4)} \] come up triangular: in other words at each step we just obtain a differential equation for the angle \( \theta_i(t) \) that depends only on the previously determined angles \( \theta_j(t) \). The systematic study of triangulization of the differential system \[ \text{(4.4)} \] for general algebras is postponed to a later publication. We just note that one typically has to perform rotations along roots arranged in reverse order with respect to their height but this criterion, although necessary is not yet sufficient in full generality. A complete solution requires a more systematic analysis. It is however fairly easy, by computer calculations, to obtain ordered lists of root-rotations that have the triangular property and hence lead to exact analytic solutions by quadratures. In the case of the present algebra we have already found lists of up to eight successive such rotations and the solution we present with five rotations has just been chosen as an illustrative example of the physical and analytical mechanisms occurring in the differential system \[ \text{(4.4)} \].

This being clarified, we present the differential equations obtained for the angles \( \theta_i(t) \) in succession. Performing the first rotation around the highest root \( \varpi_{24} \) we obtain:

\[ \frac{\sin \left[ 2 \sqrt{2} \theta_{24}(t) \right]}{4 \sqrt{2}} \omega_{24} + \dot{\theta}_{24}(t) = 0 \]  

(4.10)

Performing the second rotation around the root \( \varpi_7 \) we get:

\[ \frac{\sin \left[ 2 \theta_7(t) \right]}{2} \omega_7 + \dot{\theta}_7(t) \]  

(4.11)

which is still an independent equation. Performing the third rotation around the root \( \varpi_6 \) we get a differential equation that depends on the solution of the previous two:

\[ \frac{\sin[2 \theta_6(t)] \omega_6}{2} - \frac{\cos^2 [\theta_7(t)] \sin[2 \theta_6(t)] \omega_7}{2} + \dot{\theta}_6(t) \]  

(4.12)

The same happens when we introduce the fourth rotation around \( \varpi_5 \). We obtain:

\[ 2 \sqrt{2} \sin \left[ 2 \sqrt{2} \theta_5(t) \right] \omega_5 - 4 \sqrt{2} \cos \left[ 2 \theta_7(t) \right] \sin \left[ 2 \sqrt{2} \theta_5(t) \right] \omega_7 + \sqrt{2} \cos \left[ 2 \sqrt{2} \theta_{24}(t) \right] \sin \left[ 2 \sqrt{2} \theta_5(t) \right] \omega_{24} + 16 \theta_5(t) = 0 \]  

(4.13)

Finally, when we perform the 5-th rotation, we get:

\[ 8 \sin \left[ 2 \theta_4(t) \right] \omega_5 - 4 \left( -3 + \cos \left[ 2 \theta_6(t) \right] \right) \sin \left[ 2 \theta_4(t) \right] \omega_6 - 6 \sin \left[ 2 \theta_4(t) \right] \omega_7 + 2 \cos \left[ 2 \theta_6(t) \right] \sin \left[ 2 \theta_4(t) \right] \omega_7 + \cos \left[ 2 \theta_6(t) - 2 \theta_7(t) \right] \sin \left[ 2 \theta_4(t) \right] \omega_7 + 2 \cos \left[ 2 \theta_7(t) \right] \sin \left[ 2 \theta_4(t) \right] \omega_7 + 16 \dot{\theta}_4(t) = 0 \]  

(4.14)
The explicit functions $\theta_i(t)$ displayed in eqs \((1.9)\) are the general integral of the system of eqs \((4.10), (4.14)\) where the integration constants are represented by the fixed instants of time $\tau_i$. The physical interpretation of these constants becomes clear when we investigate the properties of the scalar fields $h_i(t)$ lying in the Cartan subalgebra of $F_{4(4)}$ and eventually representing, after dimensional oxidation the logarithms of the scale factors in the various available dimensions. Following the discussion of \[8\] we can write

$$h_i(t) = \int H_i(t')dt'$$

$$H_i(t) \equiv Y^i(t) \text{ in CSA}$$

(4.15)

where $H_i(t)$ are obtained by inserting the explicit solutions \((4.9)\) into eq. \((4.7)\) and then extracting the first four components of such a vector. We have an analytic although cumbersome expression for the $H_i(t)$ in the case of all the considered rotations, yet the next integration to $h_i(t)$ can no longer be analytically done if we include all the thetas $\theta_{24}(t), \theta_7(t), \theta_6(t), \theta_5(t)$ and $\theta_4(t)$. For this reason we prefer to discuss the features of billiard dynamics by considering the simpler solution obtained by including only the first three rotations $\theta_{24}(t), \theta_7(t)$ and $\theta_6(t)$. This solution is already complicated enough to display the phenomena we want to illustrate yet it still leads to manageable analytic formulae. Explicitly we obtain:

$$H_1(t) = \frac{-\omega_5}{2} - \omega_6 + \sin^2 [\theta_7(t)] \omega_7$$

$$= \frac{-\omega_5}{2} - \omega_6 + \frac{\omega_7}{1 + e^{2(t-\tau_7)\omega_7}}$$

$$H_2(t) = \frac{1}{1} \left[ \omega_5 + 2 \sin^2 [\theta_6(t)] \left( \omega_6 - \cos^2 [\theta_7(t)] \omega_7 \right) \right]$$

$$H_3(t) = \frac{-\omega_5}{2} + \cos^2 [\theta_6(t)] \left( -\omega_6 + \cos^2 [\theta_7(t)] \omega_7 \right)$$

$$= \frac{-\omega_5}{2} + \frac{e^{2(t-\tau_6)\omega_6} \left( -e^{2t\omega_7} - e^{2\tau_7\omega_7} \right) \omega_6 + e^{2t\omega_7} \omega_7}{1 + e^{2(t-\tau_6)\omega_6} + e^{2(t-\tau_7)\omega_7} \left( e^{2t\omega_7} + e^{2\tau_7\omega_7} \right)}$$

$$H_4(t) = \frac{1}{4} \left( \cos \left[ 2\sqrt{2} \theta_{24}(t) \right] \omega_{24} \right) = \frac{1}{4} \left( -1 + \frac{2}{1 + e^{(t-\tau_{24})\omega_{24}}} \right) \omega_{24}$$

(4.16)

Considering now the five roots involved in this calculation:

$$\omega_{24} = \{0, 0, 0, 2\} \quad \omega_7 = \{1, 0, -1, 0\}$$

$$\omega_6 = \{0, 1, 1, 0\} \quad \omega_5 = \{-1, -1, 1, 1\}$$

$$\omega_4 = \{1, -1, 0, 0\}$$

(4.17)
we can evaluate the five projections of the Cartan fields in the direction of the five relevant roots and we get:

\[
\begin{align*}
    h_{\omega_{24}}(t) & \equiv \int \overrightarrow{\omega_{24}} \cdot \vec{H}(t) = - \log(1 + e^{(t-\tau_{24})\omega_{24}}) + \frac{t \omega_{24}}{2} \\
    h_{\omega_{7}}(t) & \equiv \int \overrightarrow{\omega_{7}} \cdot \vec{H}(t) = - \log(1 + e^{2(t-\tau_{7})\omega_{7}}) + \log(1 + e^{2(t-\tau_{7})\omega_{7}} + e^{2(t-\tau_{7})\omega_{7}}) \\
    h_{\omega_{6}}(t) & \equiv \int \overrightarrow{\omega_{6}} \cdot \vec{H}(t) = \frac{\log(1 + e^{2(t-\tau_{7})\omega_{7}}) - \log(1 + e^{2(t-\tau_{6})\omega_{6}} + e^{2(t-\tau_{7})\omega_{7}} + t \omega_{6}}{2} \\
    h_{\omega_{5}}(t) & \equiv \int \overrightarrow{\omega_{5}} \cdot \vec{H}(t) = \frac{1}{4} \left[ 4 \log(1 + e^{2(t-\tau_{7})\omega_{7}}) - 2 \log(1 + e^{2(t-\tau_{24})\omega_{24}}) - 2 t \omega_{5} - 4 t \omega_{7} + t \omega_{24} \right] \\
    h_{\omega_{4}}(t) & \equiv \int \overrightarrow{\omega_{4}} \cdot \vec{H}(t) = \frac{1}{2} \left[ - \log(1 + e^{2(t-\tau_{7})\omega_{7}}) + \log(1 + e^{2(t-\tau_{5})\omega_{5}} + e^{2(t-\tau_{7})\omega_{7}}) - 2 t \omega_{5} - 4 t \omega_{6} + 2 t \omega_{7} \right]
\end{align*}
\]

(4.18)

In the first of eq.(4.18) we can observe the basic building block for the smooth realization of the cosmic billiard behaviour. It is given by the function:

\[
G(t|\omega, \tau) \equiv - \log(1 + e^{(t-\tau)\omega}) + \frac{t \omega}{2}
\]

(4.19)

For \( t - \tau < < 0 \), that is for asymptotically early times the behaviour of \( G(t|\omega, \tau) \) is the following one:

\[
G(t|\omega, \tau) \simeq \frac{t \omega}{2}
\]

(4.20)

which corresponds to the motion of a fictitious ball with constant velocity \( v = \omega/2 \). For asymptotically late times, namely for \( t - \tau > > 0 \), we have instead:

\[
G(t|\omega, \tau) \simeq -\frac{t \omega}{2}
\]

(4.21)

which corresponds to the motion of a fictitious ball with inverted constant velocity \( v = -\omega/2 \).

The inversion, namely the bounce occurs in the region \( t - \tau \sim 0 \). Hence it appears that the integration constants \( \tau_i \) introduced in our solution have precisely the meaning of instant of times at which bounces occur. Furthermore each bounce occurs precisely on the wall orthogonal to each root around which we have made compact rotations while using the compensator algorithm. On the other hand, the components of the normal form solution in the CSA direction have the interpretation of components of the velocity vector of the fictitious cosmic ball in the asymptotically early times prior to the first cosmic bounce. Each new rotation introduces a new bounce. This is illustrated in fig. 3 where the Cartan fields along the five relevant roots are plotted for the solution with three rotations namely in the case of eq.(4.18). Here we clearly see three bounces, due to the three rotations introduced.
Figure 3: Plots of the Cartan fields $h_i(t) \equiv \vec{\omega}_i \cdot \vec{h}(t)$, as functions of the time in the case of three rotations $\theta_{24}(t), \theta_7(t), \theta_6(t)$ and with the following choice of parameters: $\omega_{24} = 4, \omega_7 = 3.5, \omega_6 = 2.5, \omega_5 = 0, \tau_6 = -2, \tau_7 = 0, \tau_{24} = 2$. It is evident from the plots that there are three bounces, exactly at $t = -2$, $t = 0$ and $t = 2$. 
5 Uplifting of $F_{4(4)}$ solutions to $E_{7(-5)}$ and painted walls

Now that we have obtained explicit solutions of the first order system \((3.15)\) by means of the compensator method, we can appreciate the role of the paint group, \(G_{\text{paint}} = SO(3)^3\) since rotations of this latter applied to the \(F_{4(4)}\) solution generate non trivial solutions of the differential system \((3.16)\).

To illustrate the mechanism with an explicit and manageable example we consider the \(F_{4(4)}\) solution based on the three rotations angles \(\theta_{24}(t), \theta_7(t), \theta_6(t)\) for which we have already written the time dependence of the Cartan fields in eq.\((4.16)\) and we complete it by writing also the time dependence of the root components of the tangent vector. In this case the only non vanishing root fields are \(\Phi^{\alpha_{12}}\) and \(\Omega_{3,4,8}\) respectively associated with the long root \(\alpha_{12}^\ell = 2y_4\) and with the short roots \(\alpha_{3,4,8}^s = y_2 + y_3, y_1 - y_3, y_1 + y_2\). The time dependence of these fields in the considered solution is given by:

\[
\begin{align*}
\Phi[\alpha_{12}^\ell](t) &= -\left(\frac{e^{(t^{-\tau_2})\omega_24} \omega_24}{\sqrt{2} (1 + e^{(t^{-\tau_2})\omega_24})}\right) \\
\Omega[\alpha_3^s](t) &= -2 e^{(t^{-\tau_6})\omega_6} \sqrt{1 + e^{2(t^{-\tau_7})\omega_7}} \left((e^{2t\omega_7} + e^{2\tau_7\omega_7}) \omega_6 - e^{2t\omega_7} \omega_7\right) \\
\Omega[\alpha_4^s](t) &= -2 e^{(t^{-\tau_7})\omega_7} \omega_7 \\
\Omega[\alpha_5^s](t) &= \frac{2 e^{(t^{-\tau_7})\omega_7} (1 + e^{2(t^{-\tau_7})\omega_7}) (1 + e^{2(t^{-\tau_9})\omega_9})}{\sqrt{1 + e^{2(t^{-\tau_7})\omega_7}} (1 + e^{2(t^{-\tau_9})\omega_9} (1 + e^{2(t^{-\tau_9})\omega_9}))}
\end{align*}
\]

and it is displayed in fig.\[4]\]

Uplifting this solution to an \(E_{7(-5)}\) solution is done by identifying the Cartan and the long root fields of the two systems and then by identifying:

\[
\begin{align*}
\Omega[\alpha_3^s](t) &= \Omega_6[\alpha_3^s](t) \\
\Omega[\alpha_4^s](t) &= \Omega_6[\alpha_4^s](t) \\
\Omega[\alpha_5^s](t) &= \Omega_6[\alpha_5^s](t)
\end{align*}
\]

Next we can rotate the so obtained solution with any element of the nine parameter paint group \(SO(3)^3\) whose generators are the first nine operators \([5.2]\) described in eq.s\((2.36)\). For instance we can apply a rotation of a constant angle \(\psi_5\) along the 5-th generator, namely along \(E_{32}^+\). The result putting all-together is given by the Cartan fields in eq.\((4.16)\) and by the following root fields:

\[
\begin{align*}
\Phi[\alpha_{12}^\ell](t) &= -\left(\frac{e^{(t^{-\tau_2})\omega_24} \omega_24}{\sqrt{2} (1 + e^{(t^{-\tau_2})\omega_24})}\right) \\
\Omega_0[\alpha_3^s](t) &= -2 e^{(t^{-\tau_6})\omega_6} \sqrt{1 + e^{2(t^{-\tau_7})\omega_7}} \frac{\cos\left(\frac{\psi_5}{2}\right)}{\sqrt{2}} \left((e^{2t\omega_7} + e^{2\tau_7\omega_7}) \omega_6 - e^{2t\omega_7} \omega_7\right) \\
&\quad \left(1 + e^{2(t^{-\tau_6})\omega_6} + e^{2(t^{-\tau_7})\omega_7}\right) \left(e^{2t\omega_7} + e^{2\tau_7\omega_7}\right)
\end{align*}
\]
Figure 4: Plots of the root fields $\Phi[\alpha_{12}^\ell](t)$ and $\Omega[\alpha_{3,4,8}^s](t)$ as functions of the time in the case of three rotations $\theta_{24}(t), \theta_{7}(t), \theta_{6}(t)$ and with the following choice of parameters: $\omega_{24} = 4$, $\omega_{7} = 3.5$, $\omega_{6} = 2.5$, $\omega_{5} = 0$, $\tau_{6} = -2$, $\tau_{7} = 0$, $\tau_{24} = 2$. It is evident from the plots that the dynamical wall causing the bounce at $t = 2$ is $\Phi[\alpha_{12}^\ell](t)$, while the walls for the $t = 0$ bounce are provided by $\Omega[\alpha_{3}^s](t)$ and $\Omega[\alpha_{8}^s](t)$. Finally the wall for the $t = -2$ bounce is provided by $\Omega[\alpha_{4}^s](t)$.

\begin{align*}
\Omega_{Z}[\alpha_{3}^s](t) &= -2e^{(t-\tau_6)}\omega_6 \omega_7 \sqrt{1 + e^{2(t-\tau_7)}\omega_7 \sin\left(\frac{\pi}{2}\sqrt{\frac{\omega_7}{e^{2(t-\tau_6)}\omega_6}}\right)} \left((e^{2t\omega_7} + e^{2\tau_7\omega_7})\omega_6 - e^{2t\omega_7}\omega_7\right) \\
\Omega_{0}[\alpha_{4}^s](t) &= -2e^{(t-\tau_7)}\omega_7 \omega_7 \\
&= \frac{(-2e^{(t-\tau_7)}\omega_7 \omega_7)}{(1 + e^{2(t-\tau_7)}\omega_7)(1 + e^{2(t-\tau_6)}\omega_6 + e^{2(t-\tau_7)}\omega_7)} \left(e^{2t\omega_7} + e^{2\tau_7\omega_7}\right) \\
&= \frac{-2e^{(t-\tau_7)}\omega_7 \omega_7}{(1 + e^{2(t-\tau_7)}\omega_7) \sqrt{1 + \frac{1 + e^{2(t-\tau_7)}\omega_7}{e^{2(t-\tau_6)}\omega_6}}} 
\end{align*}
Inserting eqs (4.16) and eqs(5.3) into the differential equations (3.16) one can patiently verify that they are all satisfied for any value of the angle $\psi_5$. We could continue with more complicated rotations, but the lesson taught by this example should already be sufficiently clear. In this solution the time dependence of $\Omega_Z[\alpha_s^8](t)$ and $\Omega_0[\alpha_s^8](t)$ is exactly the same and the ratio of these two fields is the constant factor $\tan \left[ \frac{\psi_5}{2\sqrt{2}} \right]$. Similarly for the fields $\Omega_Z[\alpha_s^3](t)$ and $\Omega_0[\alpha_s^3](t)$. Hence it appears that the dynamical walls which raise and lower and cause the bounces of the cosmological factors are just those displayed by the Tits Satake projection of the supergravity scalar manifold, namely, the quaternionic manifold $\mathbb{F}_4(4)/Usp(6) \times SU(2)$, rather than $E_7(-5)/SO(12) \times SO(3)$ in $D = 3$, and after dynamical oxidation to $D = 4$, the special Kähler manifold $Sp(6, \mathbb{R})/SU(3) \times U(1)$ rather than $SO^*(12)/SU(6) \times U(1)$. Indeed as we have pointed out in [17] and recalled in table 8 taken from [17], the Tits Satake projection commutes with the c-map produced by the dimensional reduction à la Ehlers and we have the correspondence:

$$\text{adj}(U_{D=3}) = \text{adj}(U_{D=4}) \oplus \text{adj}(SL(2, \mathbb{R})_E) \oplus W_{(2,2W)}$$

$$\downarrow$$

$$\text{adj}(U_{TS_{D=3}}) = \text{adj}(U_{TS_{D=4}}) \oplus \text{adj}(SL(2, \mathbb{R})_E) \oplus W_{(2,W_{TS})}$$

(5.4)

where $SL(2, \mathbb{R})_E$ is the Ehlers group coming from the dimensional reduction of pure gravity and $W$ denotes the symplectic representation to which vector fields are assigned in $D = 4$. What is actually preserved in the c-map is the paint group $G_{\text{paint}}$.

Hence the dynamical walls are those associated with the Tits Satake projected model but they come, in the true supergravity theory, in painted copies, for instance, within the context of our example, the copy $\Omega_0$ and the copy $\Omega_Z$. The paint group rotates these copies into each other. The explicit form taken by the diagram (5.4) in the worked out example studied by the present paper is:

$$\text{adj}(E_7(-5)) = \text{adj}(SO^*(12)) \oplus \text{adj}(SL(2, \mathbb{R})_E) \oplus (2, 32_s)$$

$$\downarrow$$

$$\text{adj}(F_{4(4)}) = \text{adj}(Sp(6, \mathbb{R}) \oplus \text{adj}(SL(2, \mathbb{R})_E) \oplus (2, 14)$$

(5.5)

The representation 14 of $Sp(6, \mathbb{R})$ is that of an antisymmetric symplectic traceless tensor:

$$\dim_{Sp(6, \mathbb{R})} 14 = 14$$

(5.6)

On the other hand the invariance of the paint group through dimensional reduction and oxidation can be easily checked as follows. First of all we note that $G_{\text{paint}} = SO(3)^3$ is both a subgroup of
| # Q.s | D=4 | D=3 | D=2 | D=1 |
|-------|------|------|------|------|
| \( N = 8 \) | \( U \) | \( E_7(7) \) | \( E(8) \) | \( E_9 \) | \( E_10 \) |
| \( H \) | \( SU(8) \) | \( SO(16) \) | KE_9 | KE_10 |
| \( N = 6 \) | \( U \) | \( SO^*(12) \) | \( E_7(-5) \) | \( E_7(-5) \) | \( E_7(-5) \) |
| \( H \) | \( SU(6) \times U(1) \) | \( SO(12) \times SO(3) \) | KE_7(-5) | KE^\land_7(-5) | KE^\land_7(-5) |
| \( N = 5 \) | \( U \) | \( SU(5,1) \) | \( U(1) \) | \( b = 2 \) | \( A_4^\land(2) \) | \( A_4^\land(2) \) |
| \( H \) | \( SU(5) \times U(1) \) | \( SO(10) \times SO(2) \) | SO(10) × SO(2) | SO(10) × SO(2) | SO(10) × SO(2) |
| \( N = 4 \) | \( U \) | \( SO(6, n) \times SU(1,1) \) | \( SO(8, n + 2) \) | SO(8, n + 2) | SO(8, n + 2) |
| \( H \) | \( SO(6) \times SO(n) \times U(1) \) | \( SO(8) \times SO(n + 2) \) | SO(8) × SO(n + 2) | SO(8) × SO(n + 2) | SO(8) × SO(n + 2) |
| \( N = 5 \) | \( U \) | \( SO(n, n) \times SU(1,1) \) | \( SO(n + 2, n + 2) \) | SO(n + 2, n + 2) | SO(n + 2, n + 2) |
| \( H \) | \( SO(n) \times SO(n) \times U(1) \) | \( SO(n + 2) \times SO(n + 2) \) | SO(n + 2) × SO(n + 2) | SO(n + 2) × SO(n + 2) | SO(n + 2) × SO(n + 2) |
| \( N = 3 \) | \( U \) | \( SU(3,n) \) | \( SU(4,n + 1) \) | \( SU(4,n + 1) \) | \( SU(4,n + 1) \) |
| \( H \) | \( SU(3) \times SU(n) \times U(1) \) | \( SU(4) \times SU(n + 1) \times U(1) \) | SU(4) × SU(n + 1) × U(1) | SU(4) × SU(n + 1) × U(1) | SU(4) × SU(n + 1) × U(1) |
| \( N = 2 \) | \( \text{geom.} \) | \( SK \) | \( Q \) | Q^\land | Q^\land |
| \( \text{TS}[\text{geom.}] \) | \( \text{TS}[SK] \) | \( \text{TS}[Q] \) | \( \text{TS}[Q\land] \) | \( \text{TS}[Q^\land] \) |

Table 8: In this table we present the duality algebras \( U_D \) in \( D = 4, 3, 2, 1 \), for various values of the number of supersymmetry charges. We also mention the corresponding Tits Satake projected algebras (where they are well defined) that are relevant for the discussion of the cosmic billiard dynamics.

SO(12) and of SU(6) as it is easily verified through the subgroup chain:

\[
\begin{align*}
\text{SO}(12) & \supset \text{SU}(6) \supset \text{SU}(4) \times \text{SU}(2) \\
\downarrow & \downarrow & \simeq & \simeq \\
\text{SO}(12) & \supset \text{SU}(6) \supset \text{SO}(6) \times \text{SO}(3) \\
\downarrow & \downarrow & \cup & \downarrow \\
\text{SO}(12) & \supset \text{SU}(6) \supset \text{SO}(4) \times \text{SO}(2) \times \text{SO}(3) \\
\downarrow & \downarrow & \simeq & \downarrow \\
\text{SO}(12) & \supset \text{SU}(6) \supset \text{SO}(3) \times \text{SO}(3) \times \text{SO}(2) \times \text{SO}(3)
\end{align*}
\]

Secondly we note that the non maximally split coset manifold \( 2\mathbb{A}_1 \) appearing in \( D = 4 \) has dimension 30 and rank 3. This means that out of the 30 positive roots there are three, \( \beta_1, \beta_2 \) and \( \beta_3 \) that are orthogonal to the 3 non–compact Cartan generators. Together with the three compact Cartan generators they make up the same SO(3)^3 paint Lie algebra as in the \( D = 3 \) case. Furthermore the remaining 27 non compact roots which together with the 3 non compact Cartans span the solvable Lie algebra of \( \text{Solv}(\text{SO}^*(12)/\text{SU}(6) \times \text{U}(1)) \) are accounted for in the
following way. The Tits Satake projection of $SO^*(12)$ is the maximally split Lie algebra $Sp(6, \mathbb{R})$. This latter is non simply laced and has 9 positive roots which distribute into 3 long ones ($\alpha^i = 2\epsilon_i (i = 1, \ldots, 3)$) and 6 short ones ($\alpha^i = \epsilon_i \pm \epsilon_j, i < j, i, j = 1, 2, 3$). Just as in $D = 3$ the long roots of $Sp(6, \mathbb{R})$ correspond to roots of $SO^*(12)$ that are singlets under the paint group, while the short ones correspond to roots of $SO^*(12)$ which arrange into the following 12 dimensional representation

$$12_{\text{paint}} = (2, 2, 0) \oplus (2, 0, 2) \oplus (0, 2, 2) \tag{5.8}$$

In $D = 3$ we have 4-copies of the representation $12_{\text{paint}}$ while in $D = 4$ we just have 2–copies of the same. It is instructive to compare how the total number of roots is retrieved in the two cases:

$$\# \text{ of } E_7 \text{ roots} = 63 = \underbrace{3}_{\text{compact}} + \underbrace{12}_{\text{long}} + 4 \times 12_{\text{paint}}$$
$$\# \text{ of } SO^*(12) \text{ roots} = 30 = \underbrace{3}_{\text{compact}} + \underbrace{3}_{\text{long}} + 2 \times 12_{\text{paint}} \tag{5.9}$$

In eq. (5.9) the second and fourth lines recall that each of the short roots of either $F_4(4)$ or $Sp(6, \mathbb{R})$ has 4 preimages in the $D = 4$ algebra which arrange into a triplet plus a singlet with respect to the diagonal subgroup $SO(3)_{\text{diag}} = G^0_{\text{paint}}$. This shows how the structure of the paint group filters through the dimensional reduction. We can analyze this phenomenon also at the level of the symplectic representation $\mathbf{W}$ to which the vector fields are assigned. For the full $\mathcal{N} = 6$ supergravity model, this representation is the spinorial $32_s$ of $SO^*(12)$. Following the general discussion given in our recent paper [17], the 32 weights of this representation are in one to one correspondence with the roots of $E_7$, which have non vanishing grading with respect to the highest root $\psi = \alpha[63]$ in the numbering of appendix A. This root set subdivides into $32 = 8 + 24$ where 8 roots are Tits-Satake projected into 8 long roots of $F_4(4)$, while 24 are Tits-Satake projected into 6 short roots of the same. The 14-dimensional representation of $Sp(6, \mathbb{R})$ is just made by these $8 + 6$ roots of $F_4(4)$ which have non vanishing grading with respect to its own highest root $\psi_{TS}$. Indeed, as we have noted in [17] the Tits Satake projection of the highest root is the highest root of the target algebra.

The above discussion provides the essential tools to perform the dimensional oxidation of the solutions we have found to full fledged solutions of supergravity models in $D = 4$ or even in higher dimensions. We do not address this issue in the present paper leaving it for further publications where we also plan to provide a systematic analysis of the Tits Satake projection for all supergravity theories linking it to the properties of the compactification manifolds.

We deem that the present detailed case-study has illustrated the role of the **dimensional reduction invariant paint group** in reducing the study of billiard dynamics to simpler maximally split cosets.
6 Conclusions

In this paper we have considered one of the two necessary extensions of the analysis of *smooth cosmic billiards* initiated by us in \[8\]: that to supergravity theories with lesser supersymmetry than the maximal one. The other necessary extension is the further reduction to \(D < 3\) dimensions, which we have recently addressed in \[17\] by studying the universal field–theoretical mechanism of the affine extension. As displayed in the systematic analysis presented by us in \[17\], lesser supersymmetry involves a general new feature: cosets that are not maximally split and correspond to non maximally non compact real sections of their isometry algebra. For these cosets the *compensator method* devised by us in \[8\] cannot be directly applied. Yet we have shown in this paper that the dynamical problem can be reduced, also in these cases, to a problem which can be solved with the compensator method. In fact the original system can be reduced to a maximally split one, performing the Tits–Satake projection of the original Lie algebra. The solutions of the projected system (that can be easily found with the compensator method) are also solutions of the complete one. Moreover, we also showed that many other solutions can be obtained from these by global rotations of a suitable compact group that we named *paint group*. Although we do not have the general integral for these cases, we showed how to obtain a large class of solutions that, probably, are the most relevant from the physical point of view.

Tits Satake projection of the original Lie algebra has emerged as a central token in discussing cosmic billiards for lesser supersymmetry. We have illustrated its role by an in depth analysis of a specific example that of \(N = 6\) supergravity. Through this case-study we were able to extrapolate the main general features that apply to all supergravity models and which we plan to study systematically in a future publication. In particular we have elucidated the key role of the \(G_{\text{paint}}\) group, a notion not yet introduced in the literature and leading to the idea of *painted billiard walls*. The main property of \(G_{\text{paint}}\) is that it commutes with the \(c\)-map, namely with dimensional reduction. Hence it filters through dimensional oxidation and can be retrieved in higher dimensional supergravity.

The main research line that streams from our results is the analysis of the Tits Satake projection and of its kernel (the paint group) in more general contexts, in particular in the context of generic special Kähler geometry, of which our case study is also an example (see for instance \[14\] for a review). Furthermore keeping in mind the generic interpretation of the scalar manifold \(\mathcal{M}_{\text{scalar}}\) as moduli space for the geometry of the compactification manifold, a Calabi Yau \(\mathcal{M}_{\text{CY}}\) in the case where \(\mathcal{M}_{\text{scalar}}\) is special Kählerian, it is challenging to obtain the interpretation of the Tits Satake projection at the level of the compact manifold geometry. This, as already stressed, we plan to do in the immediate future.

It is at the same time quite interesting to consider the interplay between the Tits Satake projection and the gauging of supergravity models which is also on agenda.

As we have illustrated in this paper we can easily obtain smooth realizations of the cosmic billiard with several bounces. The number of these bounces, however, is finite, as long as we deal with finite algebras, namely as long as we discuss higher dimensional configurations from a \(D = 3\) perspective. This is so because bounces are created, as we have shown, by compact group rotations along different generators and there is a finite number of them if the number of roots is finite. In order to see infinite bounces and may be chaos we have to have infinitely many roots, namely we have to look at higher dimensional supergravity from a \(D = 2\) or \(D = 1\) perspective. This requires to consider the affine or hyperbolic Kač–Moody extensions which we have addressed.
in the recent paper [17]. Yet, as touched upon there and readdressed in the present example here, the Tits Satake projection commutes with the affine extension and in general with dimensional reduction, which preserves the structure of the paint group $G_{paint}$. Hence a door has been open how to paint walls and roots also in Kač-Moody algebras.
A Listing the positive roots of $E_7$

Listing of all positive roots of $E_7$. The first column gives the Dynkin label, the last gives the euclidean components of the root vectors

\[
\begin{align*}
\alpha[1] &= \{1,0,0,0,0,0,0\} &= \{1,-1,0,0,0,0,0\} \\
\alpha[2] &= \{0,1,0,0,0,0,0\} &= \{0,1,-1,0,0,0,0\} \\
\alpha[3] &= \{0,0,1,0,0,0,0\} &= \{0,0,1,-1,0,0,0\} \\
\alpha[4] &= \{0,0,0,1,0,0,0\} &= \{0,0,0,1,-1,0,0\} \\
\alpha[5] &= \{0,0,0,0,1,0,0\} &= \{0,0,0,0,1,-1,0\} \\
\alpha[6] &= \{0,0,0,0,0,1,0\} &= \{0,0,0,0,0,1,0\} \\
\alpha[7] &= \{0,0,0,0,0,0,1\} &= \{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\} \\
\alpha[8] &= \{1,1,0,0,0,0,0\} &= \{1,0,-1,0,0,0,0\} \\
\alpha[9] &= \{0,1,1,0,0,0,0\} &= \{0,1,0,-1,0,0,0\} \\
\alpha[10] &= \{0,0,1,1,0,0,0\} &= \{0,0,1,0,-1,0,0\} \\
\alpha[11] &= \{0,0,0,1,1,0,0\} &= \{0,0,0,1,0,-1,0\} \\
\alpha[12] &= \{0,0,0,1,0,1,0\} &= \{0,0,0,1,0,1,0\} \\
\alpha[13] &= \{0,0,0,0,0,1,1\} &= \{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\} \\
\alpha[14] &= \{1,1,1,0,0,0,0\} &= \{1,0,0,-1,0,0,0\} \\
\alpha[15] &= \{0,1,1,1,0,0,0\} &= \{0,1,0,0,-1,0,0\} \\
\alpha[16] &= \{0,0,1,1,1,0,0\} &= \{0,0,1,0,0,-1,0\} \\
\alpha[17] &= \{0,0,1,1,0,1,0\} &= \{0,0,1,0,0,1,0\} \\
\alpha[18] &= \{0,0,0,1,0,1,1\} &= \{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\} \\
\alpha[19] &= \{0,0,0,1,1,1,0\} &= \{0,0,0,1,1,0,0\} \\
\alpha[20] &= \{1,1,1,1,0,0,0\} &= \{1,0,0,0,-1,0,0\} \\
\alpha[21] &= \{0,1,1,1,1,0,0\} &= \{0,1,0,0,0,-1,0\} \\
\alpha[22] &= \{0,1,1,1,0,1,0\} &= \{0,1,0,0,0,1,0\} \\
\alpha[23] &= \{0,0,1,1,0,1,1\} &= \{-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\} \\
\alpha[24] &= \{0,0,0,1,1,1,0\} &= \{0,0,1,0,1,0,0\} \\
\alpha[25] &= \{0,0,0,1,1,1,1\} &= \{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\} \\
\alpha[26] &= \{1,1,1,1,1,0,0\} &= \{1,0,0,0,0,-1,0\} \\
\alpha[27] &= \{1,1,1,1,0,1,0\} &= \{1,0,0,0,0,1,0\} \\
\alpha[28] &= \{0,1,1,1,0,1,1\} &= \{-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\} \\
\alpha[29] &= \{0,1,1,1,1,1,0\} &= \{0,1,0,0,1,0,0\} \\
\alpha[30] &= \{0,0,1,1,1,1,1\} &= \{-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\} \\
\alpha[31] &= \{0,0,1,2,1,1,0\} &= \{0,0,1,1,0,0,0\}\end{align*}
\]
\( \alpha[32] = \{1, 1, 1, 0, 1, 1\} = \left\{ \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right\} \)

\( \alpha[33] = \{1, 1, 1, 1, 1, 1\} = \{1, 0, 0, 0, 1, 0, 0\} \)

\( \alpha[34] = \{0, 1, 1, 1, 1, 1\} = \left\{ -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right\} \)

\( \alpha[35] = \{0, 1, 1, 2, 1, 1\} = \{0, 1, 0, 1, 0, 0, 0\} \)

\( \alpha[36] = \{0, 0, 1, 2, 1, 1\} = \left\{ -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right\} \)

\( \alpha[37] = \{1, 1, 1, 1, 1, 1\} = \left\{ \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right\} \)

\( \alpha[38] = \{1, 1, 1, 2, 1, 1\} = \{1, 0, 0, 1, 0, 0, 0\} \)

\( \alpha[39] = \{0, 1, 1, 2, 1, 1\} = \left\{ -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2} \right\} \)

\( \alpha[40] = \{0, 1, 2, 2, 1, 1\} = \{0, 1, 1, 0, 0, 0, 0\} \)

\( \alpha[41] = \{0, 0, 1, 2, 1, 2\} = \left\{ -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right\} \)

\( \alpha[42] = \{1, 1, 1, 2, 1, 1\} = \left\{ \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right\} \)

\( \alpha[43] = \{1, 1, 2, 2, 1, 1\} = \{1, 0, 1, 0, 0, 0, 0\} \)

\( \alpha[44] = \{0, 1, 1, 2, 1, 2\} = \left\{ -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right\} \)

\( \alpha[45] = \{0, 1, 2, 2, 1, 1\} = \left\{ -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2} \right\} \)

\( \alpha[46] = \{1, 1, 1, 2, 1, 2\} = \left\{ \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2} \right\} \)

\( \alpha[47] = \{1, 1, 2, 2, 1, 1\} = \left\{ -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2} \right\} \)

\( \alpha[48] = \{1, 2, 2, 2, 1, 1\} = \{1, 1, 0, 0, 0, 0, 0\} \)

\( \alpha[49] = \{0, 1, 2, 2, 1, 2\} = \left\{ -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right\} \)

\( \alpha[50] = \{1, 1, 2, 2, 1, 2\} = \left\{ \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right\} \)

\( \alpha[51] = \{1, 2, 2, 2, 1, 1\} = \left\{ \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2} \right\} \)

\( \alpha[52] = \{0, 1, 2, 3, 1, 2\} = \left\{ \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right\} \)

\( \alpha[53] = \{1, 1, 2, 3, 1, 2\} = \left\{ \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right\} \)

\( \alpha[54] = \{1, 2, 2, 2, 1, 2\} = \left\{ \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2} \right\} \)

\( \alpha[55] = \{0, 1, 2, 3, 2, 1\} = \left\{ -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \right\} \)

\( \alpha[56] = \{1, 1, 2, 3, 2, 1\} = \left\{ \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2} \right\} \)

\( \alpha[57] = \{1, 2, 2, 3, 1, 2\} = \left\{ \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2} \right\} \)

\( \alpha[58] = \{1, 2, 2, 3, 2, 1\} = \left\{ \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \right\} \)

\( \alpha[59] = \{1, 2, 3, 3, 1, 2\} = \left\{ \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right\} \)

\( \alpha[60] = \{1, 2, 3, 3, 2, 1\} = \left\{ \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2} \right\} \)

\( \alpha[61] = \{1, 2, 3, 4, 2, 1\} = \left\{ \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \right\} \)

\( \alpha[62] = \{1, 2, 3, 4, 2, 3\} = \left\{ \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right\} \)

\( \alpha[63] = \{1, 2, 3, 4, 2, 3\} = \{0, 0, 0, 0, 0, \sqrt{2}\} \)
B  Explicit construction of the fundamental and adjoint representation of $F_{4(4)}$

The semisimple complex Lie algebra $F_4$ is defined by the Dynkin diagram in figure 2 and a set of simple roots corresponding to such diagram was provided in eq. (2.18). A complete list of the 24 positive roots was given in table 1. The roots were further subdivided into the set of 12 long roots and 12 short roots respectively listed in table 2 and 3 where their correspondence with $E_7$ roots was spelled out. The adjoint representation of $F_4$ is 52–dimensional, while its fundamental representation is 26–dimensional. This dimensionality is true for all real sections of the Lie algebra but the explicit structure of the representation is quite different in each real section. Here we are interested in the maximally split real section $F_{4(4)}$. For such a section we have a maximal, regularly embedded, subgroup $SO(5,4) \subset F_{4(4)}$. The decomposition of the representations with respect to this particular subgroup is the essential instrument for their actual construction. For the adjoint representation we have the decomposition:

$$52 \quad \text{adj}_{F_{4(4)}} \quad \rightarrow \quad 36 \oplus 16 \quad \text{adj}_{SO(5,4)} \oplus \text{spinor of SO}(5,4) \quad (B.1)$$

while for the fundamental one we have:

$$26 \quad \text{fundamental}_{F_{4(4)}} \quad \rightarrow \quad 9 \oplus 16 \oplus 1 \quad \text{vector of SO}(5,4) \oplus \text{spinor of SO}(5,4) \oplus \text{singlet of SO}(5,4) \quad (B.2)$$

In view of this, we fix our conventions for the $SO(5,4)$ invariant metric as it follows

$$\eta_{AB} = \text{diag} \{+,-,+,-,+,-,-,-\} \quad (B.3)$$

and we perform an explicit construction of the $16 \times 16$ dimensional gamma matrices which satisfy the Clifford algebra

$$\{\Gamma_A, \Gamma_B\} = \eta_{AB} \mathbf{1} \quad (B.4)$$

and are all completely real. This construction is provided by the following tensor products:

$$\begin{align*}
\Gamma_1 &= \sigma_1 \otimes \sigma_3 \otimes 1 \otimes 1 \\
\Gamma_2 &= \sigma_3 \otimes \sigma_3 \otimes 1 \otimes 1 \\
\Gamma_3 &= 1 \otimes \sigma_1 \otimes 1 \otimes \sigma_1 \\
\Gamma_4 &= 1 \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_3 \\
\Gamma_5 &= 1 \otimes \sigma_1 \otimes \sigma_3 \otimes \sigma_3 \\
\Gamma_6 &= 1 \otimes i\sigma_2 \otimes 1 \otimes 1 \\
\Gamma_7 &= 1 \otimes \sigma_1 \otimes i\sigma_2 \otimes \sigma_3 \\
\Gamma_8 &= 1 \otimes \sigma_1 \otimes 1 \otimes i\sigma_2 \\
\Gamma_9 &= i\sigma_2 \otimes \sigma_3 \otimes 1 \otimes 1
\end{align*} \quad (B.5)$$

where by $\sigma_i$ we have denoted the standard Pauli matrices:

$$\begin{align*}
\sigma_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad ; \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad ; \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\end{align*} \quad (B.6)$$

44
Moreover we introduce the $C_+$ charge conjugation matrix, such that:

\[ C_+ = (C_+)^T ; \quad C_+^2 = 1 \]

\[ C_+ \Gamma_A C_+ = (\Gamma_A)^T \]  \hspace{1cm} (B.7)

In the basis of eq. (B.5) the explicit form of $C_+$ is given by:

\[ C_+ = i \sigma_2 \otimes \sigma_1 \otimes i \sigma_2 \otimes \sigma_1 \]  \hspace{1cm} (B.8)

Then we define the usual generators $J_{AB} = -J_{BA}$ of the pseudorthogonal algebra $SO(5,4)$ satisfying the commutation relations:

\[ [J_{AB}, J_{CD}] = \eta_{BC} J_{AD} - \eta_{AC} J_{BD} - \eta_{BD} J_{AC} + \eta_{AD} J_{BC} \]  \hspace{1cm} (B.9)

and we construct the spinor and the vector representations by respectively setting:

\[ J_{CD}^s = \frac{1}{4} [\Gamma_C, \Gamma_D] ; \quad (J_{CD}^v)_A^B = \eta_{CA} \delta_D^B - \eta_{DA} \delta_C^B \]  \hspace{1cm} (B.10)

In this way if $v_A$ denote the components of a vector, $\xi$ those of a real spinor and $\epsilon^{AB} = -\epsilon^{BA}$ are the parameters of an infinitesimal $SO(5,4)$ rotation we can write the $SO(5,4)$ transformation as follows:

\[ \delta_{SO(5,4)} v_A = 2 \epsilon_{AB} v^B ; \quad \delta_{SO(5,4)} \xi = \frac{1}{2} \epsilon^{AB} \Gamma_{AB} \xi \]  \hspace{1cm} (B.11)

where indices are raised and lowered with the metric (B.3). Furthermore we introduce the conjugate spinors via the position:

\[ \bar{\xi} \equiv \xi^T C_+ \]  \hspace{1cm} (B.12)

With these preliminaries, we are now a position to write the explicit form of the 26-dimensional fundamental representation of $F_{4(4)}$ and in this way to construct also its structure constants and hence its adjoint representation, which is our main goal.

According to eq. (B.1) the parameters of an $F_{4(4)}$ representation are given by an anti-symmetric tensor $\epsilon_{AB}$ and a spinor $q$. On the other hand a vector in the 26-dimensional representation is specified by a collection of three objects, namely a scalar $\phi$, a vector $v_A$ and a spinor $\xi$. The representation is constructed if we specify the $F_{4(4)}$ transformation of these objects. This is done by writing:

\[
\delta_{F_{4(4)}} \begin{pmatrix}
\phi \\
v_A \\
\xi
\end{pmatrix} \equiv \begin{bmatrix} \epsilon^{AB} T_{AB} & + & \bar{q} Q \end{bmatrix} \begin{pmatrix}
\phi \\
v_A \\
\xi
\end{pmatrix} = \begin{pmatrix}
\bar{q} \xi \\
2 \epsilon_{AB} v^B + a \bar{q} \Gamma_A \xi \\
\frac{1}{2} \epsilon^{AB} \Gamma_{AB} - 3 \phi q - \frac{1}{a} v^A \Gamma_A \xi
\end{pmatrix}
\]  \hspace{1cm} (B.13)

where $a$ is a numerical real arbitrary but non-null parameter. Eq. (B.13) defines the generators $T_{AB}$ and $Q$ as 26 $\times$ 26 matrices and therefore completely specifies the fundamental representation of the Lie algebra $F_{4(4)}$. Explicitly we have:

\[
T_{AB} = \begin{pmatrix}
0 & 0 & 0 \\
0 & J^v_{AB} & 0 \\
0 & 0 & J^s_{\bar{A}B}
\end{pmatrix}
\]  \hspace{1cm} (B.14)
and

\[
Q_\alpha = \begin{pmatrix}
0 & 0 & \delta^\beta_\alpha \\
0 & 0 & a \Gamma^\beta_A \\
-3 \delta^\beta_\alpha & -\frac{1}{a} \Gamma^\beta_B & 0
\end{pmatrix}
\]  

(B.15)

and the Lie algebra commutation relations are evaluated to be the following ones:

\[
[T_{AB}, T_{CD}] = \eta_{BC} T_{AD} - \eta_{AC} T_{BD} - \eta_{BD} T_{AC} + \eta_{AD} T_{BC}
\]

\[
[T_{AB}, Q] = \frac{1}{2} \Gamma_{AB} Q
\]

\[
[Q_\alpha, Q_\beta] = -\frac{1}{12} (C_+ \Gamma^{AB})_{\alpha\beta} T_{AB}
\]  

(B.16)

Eq. (B.16), together with eqs. (B.5) and eq. (B.7) provides an explicit numerical construction of the structure constants of the maximally split F_{4(4)} Lie algebra. What we still have to do is to identify the relation between the tensorial basis of generators in eq. (B.16) and the Cartan-Weyl basis in terms of Cartan generators and step operators. To this effect let us enumerate the 52 generators of F_{4(4)} in the tensorial representation according to the following table:

| \Omega_1 = T_{12} | \Omega_2 = T_{13} | \Omega_3 = T_{14} | \Omega_4 = T_{15} |
| \Omega_5 = T_{16} | \Omega_6 = T_{17} | \Omega_7 = T_{18} | \Omega_8 = T_{19} |
| \Omega_9 = T_{23} | \Omega_{10} = T_{24} | \Omega_{11} = T_{25} | \Omega_{12} = T_{26} |
| \Omega_{13} = T_{27} | \Omega_{14} = T_{28} | \Omega_{15} = T_{29} | \Omega_{16} = T_{34} |
| \Omega_{17} = T_{35} | \Omega_{18} = T_{36} | \Omega_{19} = T_{37} | \Omega_{20} = T_{38} |
| \Omega_{21} = T_{39} | \Omega_{22} = T_{45} | \Omega_{23} = T_{46} | \Omega_{24} = T_{47} |
| \Omega_{25} = T_{48} | \Omega_{26} = T_{49} | \Omega_{27} = T_{56} | \Omega_{28} = T_{57} |
| \Omega_{29} = T_{58} | \Omega_{30} = T_{59} | \Omega_{31} = T_{67} | \Omega_{32} = T_{68} |
| \Omega_{33} = T_{69} | \Omega_{34} = T_{78} | \Omega_{35} = T_{79} | \Omega_{36} = T_{89} |
| \Omega_{37} = Q_1 | \Omega_{38} = Q_2 | \Omega_{39} = Q_3 | \Omega_{40} = Q_4 |
| \Omega_{41} = Q_5 | \Omega_{42} = Q_6 | \Omega_{43} = Q_7 | \Omega_{44} = Q_8 |
| \Omega_{45} = Q_9 | \Omega_{46} = Q_{10} | \Omega_{47} = Q_{11} | \Omega_{48} = Q_{12} |
| \Omega_{49} = Q_{13} | \Omega_{50} = Q_{14} | \Omega_{51} = Q_{15} | \Omega_{52} = Q_{16} |

(B.17)

Then, as Cartan subalgebra we take the linear span of the following generators:

\[
CSA \equiv \text{span} (\Omega_5, \Omega_{13}, \Omega_{20}, \Omega_{26})
\]  

(B.18)

and furthermore we specify the following basis:

\[
\mathcal{H}_1 = \Omega_5 + \Omega_{13}; \quad \mathcal{H}_2 = \Omega_5 - \Omega_{13}
\]

\[
\mathcal{H}_3 = \Omega_{20} + \Omega_{26}; \quad \mathcal{H}_4 = \Omega_{20} - \Omega_{26}
\]  

(B.19)

With respect to this basis the step operators corresponding to the positive roots of F_{4(4)} as ordered and displayed in table 1 are those enumerated in table 4. The steps operators corresponding to
negative roots are obtained from those associate with positive ones via the following relation:

$$E^{-\alpha} = -C E^\alpha C$$ \hfill (B.20)

where the $26 \times 26$ symmetric matrix $C$ is defined in the following way:

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \eta & 0 \\ 0 & 0 & C_+ \end{pmatrix}$$ \hfill (B.21)

A further comment is necessary about the normalizations of the step operators $E^\alpha$ which are displayed in table 9. They have been fixed with the following criterion. Once we have constructed the algebra, via the generators \((B.14), (B.15)\), we have the Lie structure constants encoded in eq. \((B.16)\) and hence we can diagonalize the adjoint action of the Cartan generators \((B.13)\) finding which linear combinations of the remaining generators correspond to which root. Each root space is one-dimensional and therefore we are left with the task of choosing an absolute normalization for what we want to call the step operators:

$$E^\alpha = \lambda_\alpha \text{ (linear combination of } \Omega\text{.s)}$$ \hfill (B.22)
The values of $\lambda_\varpi$ are now determined by the following non trivial conditions:

1. The differences $H_i = (E^{\varpi_i} - E^{-\varpi_i})$ should close a subalgebra $\mathbb{H} \subset F_{4(4)}$, the maximal compact subalgebra $SU(2)_R \times Usp(6)$

2. The sums $K_i = \frac{1}{\sqrt{2}} (E^{\varpi_i} + E^{-\varpi_i})$ should span a 28-dimensional representation of $\mathbb{H}$, namely the aforementioned $(2, 14)$ of $SU(2)_R \times Usp(6)$

We arbitrarily choose the first four $\lambda_\varpi$ associated with simple roots and then all the others are determined. The result is that displayed in table $\text{9}$. Using the Cartan generators defined by eq.s (B.19) and the step operators enumerated in table $\text{9}$ one can calculate the structure constants of $F_{4(4)}$ in the Cartan-Weyl basis, namely:

\[
\begin{align*}
[H_i, H_j] &= 0 \\
[H_i, E^{\varpi}] &= \varpi^i E^{\varpi} \\
[E^{\varpi}, E^{-\varpi}] &= \varpi \cdot H \\
[E^{\varpi_i}, E^{\varpi_j}] &= \mathcal{N}_{\varpi_i, \varpi_j} E^{\varpi_i+\varpi_j}
\end{align*}
\] (B.23)

in particular one obtains the explicit numerical value of the coefficients $\mathcal{N}_{\varpi_i, \varpi_j}$, which, as it is well known, are the only ones not completely specified by the components of the root vectors in the root system. The result of this computation, following from eq. (B.16) is that encoded in eq.s (2.48, 2.49, 2.50) of the main text.

As a last point we can investigate the properties of the maximal compact subalgebra $SU(2)_R \times Usp(6) \subset F_{4(4)}$. As we know a basis of generators for this subalgebras is provided by:

\[
H_i = (E^{\varpi_i} - E^{-\varpi_i}) \quad ; \quad (i = 1, \ldots, 24)
\] (B.24)

but it is not a priori clear which are the generators of $SU(2)_R$ and which of $Usp(6)$. By choosing a basis of Cartan generators of the compact algebra and diagonalizing their adjoint action this distinction can be established. The generators of $SU(2)_R$ are the following linear combinations:

\[
\begin{align*}
J_X &= \frac{1}{4\sqrt{2}} (H_1 - H_{14} + H_{20} - H_{22}) \\
J_Y &= \frac{1}{4\sqrt{2}} (H_5 + H_{11} - H_{18} + H_{23}) \\
J_Z &= \frac{1}{4\sqrt{2}} (-H_2 + H_9 - H_{16} - H_{24})
\end{align*}
\] (B.25)

close the standard commutation relations:

\[
[J_i, J_j] = \epsilon_{ijk} J_k
\] (B.26)

and commute with all the generators of $Usp(6)$. These latter are displayed as follows.

\[
\begin{align*}
\mathcal{H}_1^{(Usp6)} &= -\frac{H_2}{2} - \frac{H_6}{2} + \frac{H_{16}}{2} - \frac{H_{24}}{2} \\
\mathcal{H}_2^{(Usp6)} &= -\frac{H_2}{2} + \frac{H_6}{2} + \frac{H_{16}}{2} + \frac{H_{24}}{2} \\
\mathcal{H}_3^{(Usp6)} &= \frac{H_2}{2} + \frac{H_6}{2} + \frac{H_{16}}{2} - \frac{H_{24}}{2}
\end{align*}
\] (B.27)
are the Cartan generators. On the other hand the nine pairs of generators which are rotated one into the other by the Cartans with eigenvalues equal to the roots of the compact algebra are the following ones

\[
\begin{align*}
W_1 &= H_{10} & Z_1 &= H_7 \\
W_2 &= H_4 & Z_2 &= -H_{13} \\
W_3 &= H_6 & Z_3 &= -H_3 \\
W_4 &= -H_1 + H_{14} + H_{20} - H_{22} & Z_4 &= -H_5 - H_{11} - H_{18} + H_{23} \\
W_5 &= H_{21} & Z_5 &= -H_8 \\
W_6 &= H_1 + H_{14} + H_{20} + H_{22} & Z_6 &= H_5 - H_{11} - H_{18} - H_{23} \\
W_7 &= -H_1 - H_{14} + H_{20} + H_{22} & Z_7 &= H_5 - H_{11} + H_{18} + H_{23} \\
W_8 &= H_{17} & Z_8 &= H_{15} \\
W_9 &= H_{12} & Z_9 &= H_{19}
\end{align*}
\]

(B.28)

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