Estimation of the invariant density for discretely observed diffusion processes: impact of the sampling and of the asynchronicity

Chiara Amorino\textsuperscript{a} and Arnaud Gloter\textsuperscript{b}

\textsuperscript{a}Department of Mathematics, Université du Luxembourg, Esch-sur-Alzette, Luxembourg; \textsuperscript{b}Laboratoire de Mathématiques et Modélisation d’Evry, CNRS, Univ Evry, Université Paris-Saclay, Evry, France

\textbf{ABSTRACT}

We aim at estimating in a non-parametric way the density $\pi$ of the stationary distribution of a $d$-dimensional stochastic differential equation $(X_t)_{t \in [0,T]}$, for $d \geq 2$, from the discrete observations of a finite sample $X_{t_0}, \ldots, X_{t_n}$ with $0 = t_0 < t_1 < \cdots < t_n =: T_n$. We propose a kernel density estimator and we study its convergence rates for the pointwise estimation of the invariant density under anisotropic Hölder smoothness constraints. First of all, we find some conditions on the discretization step that ensures it is possible to recover the same rates as if the continuous trajectory of the process was available. As proven in the recent work [Amorino C, Gloter A. Minimax rate of estimation for invariant densities associated to continuous stochastic differential equations over anisotropic Holder classes; 2021. arXiv preprint arXiv:2110.02774], such rates are optimal and new in the context of density estimator. Then we deal with the case where such a condition on the discretization step is not satisfied, which we refer to as the intermediate regime. In this new regime we identify the convergence rate for the estimation of the invariant density over anisotropic Hölder classes, which is the same convergence rate as for the estimation of a probability density belonging to an anisotropic Hölder class, associated to $n$ iid random variables $X_1, \ldots, X_n$. After that we focus on the asynchronous case, in which each component can be observed at different time points. Even if the asynchronicity of the observations complexifies the computation of the variance of the estimator, we are able to find conditions ensuring that this variance is comparable to the one of the continuous case. We also exhibit that the non-synchronicity of the data introduces additional bias terms in the study of the estimator.

\textbf{1. Introduction}

In this paper, we aim at estimating the invariant density belonging to an anisotropic Hölder class starting from the discrete observation of the $d$-dimensional...
process

\[ X_t = X_0 + \int_0^t b(X_s) \, ds + \int_0^t a(X_s) \, dW_s, \quad t \in [0, T], \]

(1)

for \( d \geq 2 \); with \( b : \mathbb{R}^d \to \mathbb{R}^d \), \( a : \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d \) and \( W = (W_t, t \geq 0) \) a \( d \)-dimensional Brownian motion. In this paper we start considering the synchronous case. It means that the observations over the different directions are all recorded at the same instants \( 0 = t_0 \leq t_1 \leq \cdots \leq t_n =: T_n \). After that we consider the asynchronous case, where the components of the process \( X \) are observed at a different point in time. We assume in this second case that we dispose of the discrete observations \( X^{l, t_i} \) for any \( l \in \{1, \ldots, d\} \), with \( 0 \leq t_1^l < \cdots < t_n^l \leq T_n \) for any \( l \in \{1, \ldots, d\} \). We also define the discretization step \( \Delta_n := \sup_{i=0, \ldots, n-1} (t_{i+1} - t_i) \) and \( \Delta_n := \sup_{l=1, \ldots, d} \sup_{i=0, \ldots, n} (t_{i+1}^l - t_i^l) \) with \( t_{n+1}^l = T_n \) in the synchronous and asynchronous framework, respectively. The regime considered is \( T_n \to \infty \) and \( \Delta_n \to 0 \) for \( n \to \infty \).

The model presented in (1) is interesting from a theoretical point of view and because of its applications in many fields. For example, it has application in biology [1] and epidemiology [1] as well as in physics [3] and mechanics [4]. Some other classical examples are neurology [5], mathematical finance [6] and economics [7].

Because of the importance of the model, statistical inference for stochastic differential equations has been widely investigated in a lot of different context: disposing of continuous or discrete observations; on a fixed time interval or on long time intervals; in the parametric or in the non parametric frameworks.

Hence, there have been a big amount of papers on the topic. Among them, we quote Comte et al. [8], Dalalyan and Reiss [9], Genon-Catalot [10], Gobet et al. [11], Hoffmann [12], Kessler [13], Larédo [14], Marie and Rosier [15] and Yoshida [16].

In this paper, we focus on the non-parametric estimation of the invariant density starting from the discrete observations of the stochastic differential equation in (1). In particular, we will consider at the beginning the case where the data is synchronously available and we will study, then, the case where the observations are given asynchronously. The asynchronous case is extremely important for the applications. Asset prices, indeed, are generally measured when markets close, even if the closing times may be different across markets. In some cases, such as the United States and Japan, the markets do not have any common open hours, while in some other countries there is a partial overlap in the trading hours. As a consequence, the value of portfolios, value at risk measures and hedge strategies appear distort. Even if the prices are quoted at only slightly different times, the observations are still biased. This has been proven to be more evident in analysis of individual markets where closing prices may be stale (see e.g [17,18]).

In today’s global market these problems are relevant. Asynchronous data, indeed, makes more complicated the tasks of financial management as the value of the portfolio is never known at a particular time and so the measures such as value at risks and hedge strategies may be misleading.

The treatment of non-synchronous trading effects dates back to Fisher [19]. For several years researchers focussed mainly on the effects that stale quotes have on daily closing prices. Campbell et al. (Chapter 3 of [20]) provides a survey of this literature. There are several models applied to asynchronous financial market closing prices, see for example
Chapters 3 and 4 of [21,22] for respectively autoregressive and log-Gaussian statistical approaches, whilst [23] proposes a recurrent neural network approach. Some other examples in which the authors deal with asynchronicity are [24,25]. In [26] the authors propose to synchronize prices by computing estimates of the values of assets even when markets are closed, starting from the information given from markets which are open. Correlation and volatility of asynchronous data are also studied in [27], where the authors consider the problem of estimating the covariance of two diffusion processes when they are observed at discrete times in a non-synchronous manner. They propose a new estimator which allows them to correct the Epps effect.

In this context, we aim at proposing a kernel density estimator based on the discrete observations of (1) and we aim at finding the convergence rates of estimation for the stationary measure \( \pi \) associated to it, assuming that it belongs to an anisotropic Hölder class. As the smoothness properties of elements of a function space may depend on the chosen direction of \( \mathbb{R}^d \), the notion of anisotropy plays an important role. We will present some conditions that the discretization step needs to satisfy in order to recover the same (optimal) convergence rate achievable when a continuous trajectory of the process \( X \) is available and we will discuss the convergence rates we find in the intermediate regime, which is when the discretization step is not small enough and so the discretization error is not negligible.

With regard to the literature already existing about the estimation of the invariant measure, it is important to say that it is a problem already widely faced in many different frameworks by many authors (see for example [28–36]). The reason why such a problem results very attractive is the huge amount of physical applications and numerical methods connected to it, such as the Markov Chain Monte Carlo method. For example, stability theory is used to study properties of the invariant distribution and mixing properties of the diffusion in [29,37]. In [38,39], instead, it is possible to find an approximation algorithm for the computation of the invariant density. The non-parametric estimation of the invariant density can also be used in order to estimate the drift coefficient in a non-parametric way (see [15,40]).

Kernel estimators are widely employed as powerful tools: in [29,41] some kernel estimators are used to estimate the density of a continuous time process. They are used also in more complicated models, such as in jump-diffusion framework (see [42–45]).

Some references in a context closer to ours are [9,46,47]. In all three papers kernel density estimator have been used for the study of the convergence rate for the estimation of the invariant density associated with a reversible diffusion process with a unit diffusion part (in the first two works) or to the same stochastic differential equation as in (1) (in the third one). They are all based on the continuous observation of the process considered.

In particular in [47] the invariant density \( \pi \) has been estimated by means of the kernel estimator \( \hat{\pi}_{h,T} \) assuming to have the continuous record of the process \( X \) solution to (1) up to time \( T \) and the following upper bound for the mean squared error has been shown, for \( d \geq 3 \):

\[
\sup_{(a,b) \in \Sigma} \mathbb{E}[(\hat{\pi}_{h,T}(x) - \pi(x))^2] \leq \begin{cases} 
\left( \frac{\log T}{T} \right)^{\frac{2\beta_3}{2\beta_3 + d - 2}} & \text{for } \beta_2 < \beta_3, \\
\left( \frac{1}{T} \right)^{\frac{2\beta_3}{2\beta_3 + d - 2}} & \text{for } \beta_2 = \beta_3,
\end{cases}
\]
where $\Sigma$ is a class of coefficients for which the stationarity density has some prescribed regularity, $\beta_1 \leq \beta_2 \leq \cdots \leq \beta_d$ and $\bar{\beta}_3$ is the harmonic mean over the smoothness after having removed the two smallest. In particular, it is $\frac{1}{\bar{\beta}_3} := \frac{1}{d-2} \sum_{i \geq 3} \frac{1}{\beta_i}$ and so it clearly follows that $\bar{\beta}_3 \geq \bar{\beta}_1$, where $\bar{\beta}_1 := \frac{1}{d} \sum_{j=1}^{d} \frac{1}{\beta_j}$. It has also been proven that the convergence rates here above are optimal.

In this paper, we propose to estimate the invariant density $\pi$ starting from the discrete observation of the process $X$ by means of the kernel estimator $\hat{\pi}_{h, n}$, which is the discretized version of $\hat{\pi}_{h, T}$ (see Section 4.1). Then, we prove an upper bound on the variance which is composed of two terms. The first is the same as when the continuous trajectory of the process is available, while the second is the discretization error. If the second is negligible we get a condition the discretization step has to satisfy to recover the continuous convergence rate. On the other side, if the first is negligible compared to the second, we obtain the new convergence rate in the intermediate regime. In particular, for $d \geq 3$, we show the following:

$$\sup_{(a, b) \in \Sigma} \mathbb{E}[|\hat{\pi}_{h, n}(x) - \pi(x)|^2] \leq \begin{cases} \frac{(\log T_n)^{2\bar{\beta}_3}}{T_n^{2\bar{\beta}_3+d-2}} & \text{if } \beta_2 < \beta_3 \text{ and } \Delta_n \leq h_1^* h_2^* \sum_{j=1}^{d} |\log h_j^*|, \\ \left( \frac{1}{T_n} \right)^{2\bar{\beta}_3+d-2} \bar{\beta}_3 & \text{for } \beta_2 = \beta_3 \text{ and } \Delta_n \leq h_1^* h_2^* \end{cases}$$

where $\Delta_n$ is the discretization step and $h^*(T_n) = (h_1^*(T_n), \ldots, h_d^*(T_n))$ is the rate optimal choice for the bandwidth $h$ (see Theorem 4.1 and the discussion below for details about the dependence of $h^*$ in $T_n$).

On the other side, when the condition above on the discretization step are not respected, we obtain

$$\sup_{(a, b) \in \Sigma} \mathbb{E}[|\hat{\pi}_{h, n}(x) - \pi(x)|^2] \leq n \frac{\bar{\beta}_3}{2\bar{\beta}_3+d}.$$ 

We remark that the convergence rate in the intermediate regime is the same as for the estimation of a probability density belonging to an anisotropic Hölder class, associated to $n$ iid random variables $X_1, \ldots, X_n$. An analogous result is shown also in the bi-dimensional case.

After that, we focus on the asynchronous frameworks, in which each component can be observed at a different moment. Such a context implies many difficulties, as in this way even the choice of the estimator to propose appears challenging. The idea is to introduce $d$ functions $\varphi_{n,l} : [0, T] \to \mathbb{R}$ such that

$$\varphi_{n,l}(t) = \sup \left\{ t' \mid t' \leq t \right\},$$

where $(t'_l)_i$ are the instants of time in which the component $X'_l$ is observed, for $l \in \{1, \ldots, d\}$. In this way it is possible to write the sums as integrals and to propose an estimator which is the natural adaptation of $\hat{\pi}_{h, T}$, the one considered in the continuous framework.
in [47]. Moreover, the non-synchronicity involves some other challenges in the computation of the upper bound on the variance, most importantly the fact that the observation times of different components are intertwined complexifies the use of the transition density. We are able to overcome such issues by considering some combinatorics (see the proof of Proposition 5.1 below for details) and by proving some technical sharp bounds which allows us to show that, in the asynchronous context, the condition \( \Delta_n \leq h_1^n h_2^n \) is enough to recover the variance obtained in the continuous case, for \( \beta_2 < \beta_3 \).

We analyse then the bias term. Here, a condition of asynchronicity naturally appears. In particular, having defined

\[
\Delta_n' := \sup_{t \in [0,T_n]} \sup_{i,j=1,...,d} |\varphi_{n,i}(t) - \varphi_{n,j}(t)|,
\]

we obtain the continuous convergence rate when additionally the condition \( \Delta_n' \leq \left( \frac{\log T_n}{T_n} \right)^{2/\beta_3 + d - 2} \) holds.

The outline of the paper is the following. In Section 2, we introduce the model and we list the assumptions we will need in the sequel, while in Section 3 we recall the results in the case where the continuous trajectory of the process \( X \) is available. In Section 4, we consider the synchronous framework. We propose the kernel estimator and we state the upper bounds for the variance which will result in conditions on the discretization step to obtain the continuous regime and in the convergence rates in the intermediate regime. Section 5 is devoted to the statement of our results in the asynchronous framework. In Section 6, we prove the results stated in Section 4 while Section 7 is devoted to the proof of results under asynchronicity.

### 2. Model assumptions

We aim at proposing a non-parametric estimator for the invariant density associated to a \( d \)-dimensional diffusion process \( X \) discretely observed. In Section 3, we will recall what happens when a continuous record of the process \( X^T = \{X_t, 0 \leq t \leq T\} \) up to time \( T \) is available. After that, in Section 4, we will be working in a high-frequency setting and we will wonder which conditions on the discretization step will ensure the achievement of the same results as in the continuous case. The diffusion is a strong solution of the following stochastic differential equation:

\[
X_t = X_0 + \int_0^t b(X_s) \, ds + \int_0^t a(X_s) \, dW_s, \quad t \in [0, T],
\]

where \( b : \mathbb{R}^d \to \mathbb{R}^d \), \( a : \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d \) and \( W = (W_t, t \geq 0) \) is a \( d \)-dimensional Brownian motion. The initial condition \( X_0 \) and \( W \) are independent. We denote \( \tilde{a} := a \cdot a^T \).

We denote with \( \| \cdot \| \) and \( \langle \cdot, \cdot \rangle \) respectively the Euclidian norm and the scalar product in \( \mathbb{R}^d \), and for a matrix in \( \mathbb{R}^{d \times d} \) we denote its operator norm by \( \| \cdot \| \).

**A1:** The functions \( b(x) \) and \( a(x) \) are bounded globally Lipschitz functions of class \( C^1 \), such that for all \( x \in \mathbb{R}^d \),

\[
|a(x)| \leq a_0, \quad |b(x)| \leq b_0, \quad \left| \frac{\partial}{\partial x_i} b(x) \right| \leq b_1, \quad \left| \frac{\partial}{\partial x_i} a(x) \right| \leq a_1, \quad \text{for } i \in \{1, \ldots, d\},
\]
where \( a_0 > 0, b_0 > 0, a_1 > 0, b_1 > 0 \) are some constants. Moreover, for some \( a_{\min} > 0, \)
\[
a_{\min}^2 \mathbb{I}_{d \times d} \leq \tilde{a}(x)
\]
where \( \mathbb{I}_{d \times d} \) denotes the \( d \times d \) identity matrix.

As the inequality here above is between two matrices, it is worth explaining it is intended in the scalar product sense.

**A2 (Drift condition):** There exist \( \tilde{C}_b > 0 \) and \( \tilde{\rho}_b > 0 \) such that \( < x, b(x) > \leq -\tilde{C}_b |x|, \forall x : |x| \geq \tilde{\rho}_b. \)

We suppose that the invariant probability measure \( \mu \) of \( X \) is absolutely continuous with respect to the Lebesgue measure and from now on we will denote its density as \( \pi \):
\[
\mu(dx) = \pi(x) \, dx.
\]
Under the assumptions A1 – A2 the process \( X \) admits a unique invariant distribution \( \mu \) and the ergodic theorem holds. In particular, it implies the exponential ergodicity of the process \( X \). For the exponential mixing property of general multidimensional diffusions, the reader may consult Theorem 3 of Kusuoka and Yoshida [48] (\( \alpha \)–mixing); Meyn and Tweedie [49], Stramer and Tweedie [50] and Veretennikov [51] (\( \beta \)–mixing).

Under our assumptions the process \( X \) is exponentially \( \beta \) mixing and exponentially ergodic. In particular, the following inequality holds true:
\[
\| P_t f \|_{L^1(\mu)} \leq c e^{-\rho t} \| f \|_\infty,
\]
where \( P_t f(x) := \mathbb{E}[f(X_t)|X_0 = x] \) is the transition semigroup of the process \( X \).

The transition density is denoted by \( p_t \) and it is such that \( P_t f(x) = \int_{\mathbb{R}^d} f(y) p_t(x,y) \, dy \).

We want to estimate the invariant density \( \pi \) belonging to the anisotropic Hölder class \( \mathcal{H}_d(\beta, \mathcal{L}) \) defined below.

**Definition 2.1:** Let \( \beta = (\beta_1, \ldots, \beta_d), \beta_i > 0, \mathcal{L} = (\mathcal{L}_1, \ldots, \mathcal{L}_d), \mathcal{L}_i > 0. \) A function \( g : \mathbb{R}^d \to \mathbb{R} \) is said to belong to the anisotropic Hölder class \( \mathcal{H}_d(\beta, \mathcal{L}) \) of functions if, for all \( i \in \{1, \ldots, d\}, \)
\[
\| D^k g \|_\infty \leq \mathcal{L}_i \quad \forall k = 0, 1, \ldots, \lfloor \beta_i \rfloor, \\
\| D^{|\beta_i|} g(. + te_i) - D^{|\beta_i|} g(.) \|_\infty \leq \mathcal{L}_i \| t \|^{\beta_i - \lfloor \beta_i \rfloor} \quad \forall t \in \mathbb{R},
\]
for \( D^k g \) denoting the \( k \)th order partial derivative of \( g \) with respect to the \( i \)th component, \( \lfloor \beta_i \rfloor \) denoting the largest integer strictly smaller than \( \beta_i \) and \( e_1, \ldots, e_d \) denoting the canonical basis in \( \mathbb{R}^d \).

This leads us to consider a class of coefficients \((a, b)\) for which the stationary density \( \pi = \pi_{(a,b)} \) has some prescribed Hölder regularity.

**Definition 2.2:** Let \( \beta = (\beta_1, \ldots, \beta_d), 0 < \beta_1 \leq \cdots \leq \beta_d \) and \( \mathcal{L} = (\mathcal{L}_1, \ldots, \mathcal{L}_d), \mathcal{L}_i > 0, \)
\[
0 < a_{\min} \leq a_0 \text{ and } a_1 > 0, b_0 > 0, b_1 > 0, \bar{C} > 0, \bar{\rho} > 0.
\]
We define $\Sigma(\beta, L, a_{\min}, a_0, a_1, b_0, b_1, \hat{C}, \hat{\rho})$ the set of couple of functions $(a, b)$ where $a : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ and $b : \mathbb{R}^d \to \mathbb{R}^d$ are such that

- $a$ and $b$ satisfy A1 with the constants $(a_{\min}, a_0, a_1, b_0, b_1)$,
- $b$ satisfies A2 with the constants $(\hat{C}, \hat{\rho})$,
- the density $\pi_{(a,b)}$ of the invariant measure associated to the stochastic differential Equation (2) belongs to $\mathcal{H}_d(\beta, L)$.

To have an idea about the link between the coefficients and the invariant density, one can think about a reversible diffusion. Indeed, if the diffusion coefficient is the identical matrix and the drift is such that $b(x) = -\nabla V(x)$, where $V$ is a function we refer to as a potential, then assuming $V \in \mathcal{H}_d(\beta, L)$ implies $\pi = C e^{-V} \in \mathcal{H}_d(\beta + 1, L) = \mathcal{H}_d(\beta_1 + 1, \ldots, \beta_d + 1, L)$.

We aim at estimating the invariant density $\pi$ starting from discrete observations of the process $X$. In particular, we want to find some conditions that the discretization step has to satisfy in order to recover the same convergence rates we had when a continuous record of the process was available. Moreover, one may wonder which are the convergence rates in the intermediate regime, i.e., when the discretization step goes to zero but the associated error is not negligible. In order to answer these questions we recall what happens when the whole trajectory of the process $X$ is available, as detailed discussed in [47]. This is the purpose of the next section.

3. Continuous observations

It is natural to estimate the invariant density $\pi \in \mathcal{H}_d(\beta, L)$ by means of a kernel estimator. We therefore introduce some kernel function $K : \mathbb{R} \to \mathbb{R}$ satisfying

$$\int_{\mathbb{R}} K(x) \, dx = 1, \quad \|K\|_\infty < \infty, \quad \text{supp}(K) \subset [-1,1], \quad \int_{\mathbb{R}} K(x)x^l \, dx = 0, \quad (3)$$

for all $l \in \{0, \ldots, M\}$ with $M \geq \max_i \beta_i$.

For $j \in \{1, \ldots, d\}$, we denote by $X_j^T$ the $j$th component of $X_t$. A natural estimator of $\pi$ at $x = (x_1, \ldots, x_d)^T \in \mathbb{R}^d$ in the anisotropic context is given by

$$\hat{\pi}_{h,T}(x) = \frac{1}{T \prod_{l=1}^d h_l} \int_0^T \prod_{m=1}^d K \left( \frac{x_m - X_m^u}{h_m} \right) \, du. \quad (4)$$

The multi-index $h = (h_1, \ldots, h_d)$ is small. In particular, we assume $h_i < 1$ for any $i \in \{1, \ldots, d\}$.

For $d \geq 3$, from Theorem 1 of [47] we have the following convergence rate for the kernel bandwidth given below, in (6):

$$\sup_{(a,b) \in \Sigma} \mathbb{E}[|\hat{\pi}_{h,T}(x) - \pi(x)|^2] \lesssim \begin{cases} \left( \frac{\log T}{T} \right)^{\frac{2\beta_3}{2\beta_3 + d - 2}} & \text{if } \beta_2 < \beta_3 \\ \left( \frac{1}{T} \right)^{\frac{2\beta_3}{2\beta_3 + d - 2}} & \text{if } \beta_2 = \beta_3, \end{cases}$$
where $\Sigma$ is the set defined in Definition 2.2 and $\bar{\beta}_3$ is such that

$$\frac{1}{\bar{\beta}_3} := \frac{1}{d - 2} \sum_{j=3}^{d} \frac{1}{\beta_j}.$$ 

Moreover, from Theorems 3 and 4 of [47] we know they are optimal in the minimax sense.

Regarding the bi-dimensional case we know, from Theorem 2 in [47] that the following holds true

$$\sup_{(a,b) \in \Sigma} \mathbb{E}[|\hat{\pi}_{h,T}(x) - \pi(x)|^2] \leq \frac{\log T}{T},$$

and the convergence rate here above is optimal in the minimax sense (see Theorem 5 of [47]).

We now suppose that the continuous record of the process, up to time $T$, is no longer available. We dispose instead of the discretization of the process at the instants $0 = t_0 \leq t_1 \leq \cdots \leq t_n = T$, given by $X_{t_0}, \ldots, X_{t_n}$. The first goal of Section 4 is to find some conditions the discretization step has to satisfy in order to recover the same convergence rates as in this section.

### 4. Discrete observations, synchronous framework

In this section we suppose that we observe a finite sample $X_{t_0}, \ldots, X_{t_n}$, with $0 = t_0 \leq t_1 \leq \cdots \leq t_n =: T_n$. The process $X$ is the solution of the stochastic differential Equation (2). Every observation time point depends also on $n$ but, in order to simplify the notation, we suppress this index. We assume the discretization scheme to be uniform which means that, for any $i \in \{0, \ldots, n - 1\}$, it is $t_{i+1} - t_i := \Delta_n$. We will be working in a high-frequency setting i.e., the discretization step $\Delta_n \to 0$ for $n \to \infty$. We assume moreover that $T_n = n\Delta_n \to \infty$ for $n \to \infty$ and that $\Delta_n > n^{-k}$ for some $k \in (0, 1)$.

#### 4.1. Construction estimator

As in Section 3, we propose to estimate the invariant density $\pi \in \mathcal{H}_d(\beta, L)$ associated to the process $X$, solution to (2). To do that, we propose a kernel estimator which is the discretized version of the one introduced in (4). For $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, we define

$$\hat{\pi}_{h,n}(x) := \frac{1}{n\Delta_n} \frac{1}{\prod_{l=1}^{d} h_l} \sum_{i=0}^{n-1} \prod_{l=1}^{d} K \left( \frac{x_l - X_{t_i}^l}{h_l} \right) (t_{i+1} - t_i)$$

$$= \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{K}_h(x - X_{t_i}), \tag{5}$$

with $K$ a kernel function as in (3) and $\mathbb{K}_h = \prod_{l=1}^{d} K_{h_l}$ where $K_{h_l}(\cdot) = \frac{1}{h_l} K(\cdot h_l)$.

In this context we have two objectives:

(1) Find some conditions on $\Delta_n$ to get the same convergence rates we had when a continuous record of the process was available.
Find the convergence rates in the intermediate regime ($\Delta_n$ tends to zero but the discretization error is not negligible).

To achieve them, we have to study the behaviour of the mean squared error $E[|\hat{\pi}_{n,h}(x) - \pi(x)|^2]$ when $d \geq 3$ and when $d = 2$, and we have deal with two asymptotic regimes. The idea mainly consists of some upper bounds for the variance of the estimator. The difference, with respect to the continuous case, is that now we get an extra term which derives from the discretization.

If the new discretization term is negligible compared to the others, then the convergence rates are the same they were in the continuous case; otherwise we will find new convergence rates.

### 4.2. Main results, synchronous framework

The asymptotic behaviour of the estimator proposed in (5) is based on the bias-variance decomposition. To find the convergence rates it achieves we need a bound on the variance, as stated in the following propositions. We recall that the estimator performs differently depending on the dimension $d$. In this paper, we provide the main results for $d \geq 2$.

**Proposition 4.1**: Suppose that $A1–A2$ hold with some constant $0 < a_{\min} \leq a_0$ and $a_1 > 0$, $b_0 > 0$, $b_1 > 0$, $C_b$, $\tilde{b}$ and that $d \geq 3$. Let $\beta = (\beta_1, \ldots, \beta_d)$, $\beta_1 = \beta_2 = \cdots = \beta_{k_0} < \beta_{k_0+1} \leq \cdots \leq \beta_d$, for some $k_0 \in \{1, \ldots, d\}$ and $L = (L_1, \ldots, L_d)$, $L_i > 0$. If $\hat{\pi}_{h,n}$ is the estimator proposed in (5), then there exist $c > 0$ and $T_0 > 0$ such that, for $T_n \geq T_0$, the following holds true.

- If $k_0 = 1$ and $\beta_2 < \beta_3$ or $k_0 = 2$, then
  
  \[
  \text{Var}(\hat{\pi}_{h,n}(x)) \leq \frac{c}{T_n} \left( \frac{\sum_{j=1}^d |\log(h_j)|}{\prod_{l=3}^d h_l} \right) + \frac{\Delta_n}{T_n \prod_{l=1}^d h_l}.
  \]

- If $k_0 \geq 3$, then
  
  \[
  \text{Var}(\hat{\pi}_{h,n}(x)) \leq \frac{c}{T_n} \left( \frac{1}{(\prod_{l=1}^{k_0} h_l)^{1-\frac{2}{k_0}}} \frac{1}{(\prod_{l=k_0+1}^d h_l)} \right) + \frac{\Delta_n}{T_n \prod_{l=1}^d h_l}.
  \]

- If otherwise $k_0 = 1$ and $\beta_2 = \beta_3$, then
  
  \[
  \text{Var}(\hat{\pi}_{h,T}(x)) \leq \frac{c}{T_n} \left( \frac{1}{\prod_{l=4}^d h_l h_4 h_3} + \frac{\Delta_n}{T_n \prod_{l=1}^d h_l} \right).
  \]

Moreover, the constant $c$ is uniform over the set of coefficients $(a, b) \in \Sigma$.

Comparing the result here above with Proposition 2 of [47], it is easy to see that we have an extra term coming from the discretization: $\frac{\Delta_n}{T_n \prod_{l=1}^d h_l}$.

One can remark that, as in the high-frequency framework the estimators can be understood as numerical approximators of the full observation estimators, a natural choice could
be to derive the results in the discrete case from the continuous one. However, following
the same route as Theorem 4.1 in [46] to control the error coming from the approxima-
tion of the estimator, one can check that this approach leads us to a larger discretization
error, which yields a worse condition on the discretization step. This justifies our choice
of proving some upper bounds directly on the variance of the discrete estimator, as in
Proposition 4.1. Such bounds leads us to the first main result of this section.

**Theorem 4.1 (Discretization term negligible):** Suppose that A1–A2 hold and that \( d \geq 3 \).
Let \( \beta = (\beta_1, \ldots, \beta_d) \), \( 0 < \beta_1 \leq \cdots \leq \beta_d \), \( \mathcal{L} = (\mathcal{L}_1, \ldots, \mathcal{L}_d) \), \( \mathcal{L}_i > 0 \), \( 0 < a_{\min} \leq a_0 \leq a_1 > 0 \), \( b_0 > 0 \), \( b_1 > 0 \), \( \bar{C} > 0 \), \( \bar{\rho} > 0 \) and set \( \Sigma = \Sigma(\beta, \mathcal{L}, a_{\min}, a_0, a_1, b_0, b_1, \bar{C}, \bar{\rho}) \), using the
notation of Definition 2.2. Let \( h^* = (h^*_1, \ldots, h^*_d) \) be the rate optimal choice for the bandwidth
as given in (6). Then, there exist \( c > 0 \) and \( T_0 > 0 \) such that, for \( T_n \geq T_0 \), the following hold
true.

- If \( \beta_2 < \beta_3 \) and \( \Delta_n \lesssim h^*_1 h^*_2 \sum_{j=1}^d |\log h^*_j| \), then

\[
\sup_{(a, b) \in \Sigma} \mathbb{E}[|\hat{\pi}_{h,n}(x) - \pi(x)|^2] \leq c \left( \log \frac{T_n}{T_n} \right)^{\frac{2\beta_3}{2\beta_3 + d - 2}}.
\]

- If otherwise \( \beta_2 = \beta_3 \) and \( \Delta_n \lesssim h^*_1 h^*_2 \), then

\[
\sup_{(a, b) \in \Sigma} \mathbb{E}[|\hat{\pi}_{h^*,n}(x) - \pi(x)|^2] \leq c T_n^{-\frac{2\beta_3}{2\beta_3 + d - 2}},
\]

where \( \bar{\beta}_3 \) is such that \( \frac{1}{\bar{\beta}_3} = \frac{1}{d-2} \sum_{l=3}^d \frac{1}{\beta_l} \).

Comparing the results here above with the ones included in Section 3 of [47] we deduce
that, when the discretization step satisfies the constraint \( \Delta_n \lesssim h^*_1 h^*_2 \sum_{j=1}^d |\log h^*_j| \) (or
\( \Delta_n \lesssim h^*_1 h^*_2 \) respectively) it is possible to recover the same optimal convergence rates we
had when the trajectory of the process was observed continuously.

As seen in the proof of Theorem 1 of [47], for \( \beta_2 < \beta_3 \) the rate optimal choice for the
bandwidth \( h \) is given by \( h^*_j = \left( \frac{\log T_n}{T_n} \right)^{a_j} \), while for \( \beta = \beta_3 \) it is \( h^*_j = \left( \frac{1}{T_n} \right)^{a_j} \) with

\[
a_j = \frac{\bar{\beta}_3}{\beta_j(2\bar{\beta}_3 + d - 2)} \quad \text{for any } j \in \{1, \ldots, d\}. \tag{6}
\]

We remark that, according to [47], it is also possible choose smaller bandwidths \( h^*_1 \) and \( h^*_2 \)
in the case \( \beta_2 < \beta_3 \). However, in order to make the condition on the discretization step as
weak as possible, it is convenient to choose \( h^*_1 h^*_2 \) as large as possible, which leads us to the
choice gathered in (6).

Hence, replacing the optimal choice for the bandwidth as in (6) one can recover the
same upper bound for the mean squared error as in Theorem 1 of [47] when the following
conditions hold:

\[ \Delta_n \leq \left( \frac{\log T_n}{T_n} \right)^{\frac{\bar{\beta}_3}{\beta_3+d-2}} \left( \frac{1}{\beta_1} + \frac{1}{\beta_2} \right) \log T_n \quad \text{for } \beta_2 < \beta_3, \tag{7} \]

\[ \Delta_n \leq \left( \frac{1}{T_n} \right)^{\frac{\bar{\beta}_3}{\beta_3+d-2}} \left( \frac{1}{\beta_1} + \frac{1}{\beta_2} \right) \quad \text{for } \beta_2 = \beta_3. \tag{8} \]

When such conditions are not respected, instead, we get a different convergence rate. It is achieved by choosing the optimal bandwidth as

\[ h_j(n) = \left( \frac{1}{n} \right)^{\frac{\bar{\beta}_j}{\beta_j+(\beta+d)}} \],

where \( \bar{\beta} \) is the harmonic mean over the \( d \) different smoothness:

\[ \frac{1}{\bar{\beta}_j} = \sum_{j=1}^{d} \frac{1}{\beta_j}. \]

It leads us to the following result which can be applied when the conditions in the discretization step gathered in Theorem 4.1 are not respected.

**Theorem 4.2 (Discretization term non-negligible):** Suppose that \( d \geq 3 \). Let \( \beta = (\beta_1, \ldots, \beta_d) \), \( 0 < \beta_1 \leq \cdots \leq \beta_d \), \( \mathcal{L} = (\mathcal{L}_1, \ldots, \mathcal{L}_d) \), \( \mathcal{L}_i > 0 \), \( 0 < a_{\min} \leq a_0 a_1 > 0 \), \( b_0 > 0 \), \( b_1 > 0 \), \( \tilde{C} > 0 \), \( \tilde{\rho} > 0 \) and set \( \Sigma = \Sigma(\beta, \mathcal{L}, a_{\min}, a_0, a_1, b_0, b_1, \tilde{C}, \tilde{\rho}) \), using the notation of Definition 2.2. Assume that one of the following holds

- \( \Delta_n > \left( \frac{\log T_n}{T_n} \right)^{\frac{\bar{\beta}_3}{\beta_3+d-2}} \left( \frac{1}{\beta_1} + \frac{1}{\beta_2} \right) \log T_n \) and \( \beta_2 < \beta_3 \).
- \( \Delta_n > \left( \frac{1}{T_n} \right)^{\frac{\bar{\beta}_3}{\beta_3+d-2}} \left( \frac{1}{\beta_1} + \frac{1}{\beta_2} \right) \) and \( \beta_2 = \beta_3 \).

Then, there exist \( c > 0 \) and \( T_0 > 0 \) such that, for \( T_n \geq T_0 \),

\[ \sup_{(a,b) \in \Sigma} \mathbb{E}[|\hat{\pi}_{h,n}(x) - \pi(x)|^2] \leq c n^{-\frac{2\tilde{\beta}}{2\tilde{\beta}+d}}, \]

where \( \bar{\beta} \) is the harmonic mean of the smoothness over the \( d \) direction, defined as

\[ \frac{1}{\bar{\beta}_j} = \sum_{j=1}^{d} \frac{1}{\beta_j}. \]

It is interesting to remark that it is also the convergence rate for the estimation of a probability density belonging to an Hölder class, associated to \( n \) independent and identically distributed random variables \( X_1, \ldots, X_n \).

One may wonder if the rate obtained in the intermediate regime is optimal. The question is addressed for the case \( d = 1 \) in [52]. Therein it is shown that the convergence rate here above is optimal in a minimax sense. However, the approach used in [52] relies on local time and on some bounds on the occupation time which are not easily extended to a higher dimension. Hence, the question of optimality of the rate obtained when the discretization step is not negligible is still an open question for \( d \geq 2 \).
We remark that, as \( T_n = n \Delta_n \), the conditions (7), (8) can be written as function of \( n \). Using that \( \frac{d}{\beta} = \frac{d-2}{\beta_3} + \left( \frac{1}{\beta_1} + \frac{1}{\beta_2} \right) \) and \( \Delta_n > n^{-k} \) with \( k \in (0, 1) \), it is possible to check they are equivalent to the following:

\[
\Delta_n \leq \left( \frac{1}{n} \right)^{\frac{\beta_3}{2 \beta_3 + d - 2}} \log n \quad \text{for } \beta_2 < \beta_3,
\]

\[
\Delta_n \leq \left( \frac{1}{n} \right)^{\frac{\beta_2}{2 \beta_2 + d - 2}} \left( \frac{\beta_3}{\beta_3 + d - 2} \right) \leq \left( \frac{1}{n} \right)^{\frac{\beta_2}{2 \beta_2 + d - 2}} \leq \left( \frac{1}{n} \right)^\frac{\beta_2}{2 \beta_2 + d - 2}.
\]

We are interested in studying the extreme case, for which the discretization step is equal to the right-hand side of the equations here above. Then, the continuous convergence rate always becomes \( \left( \frac{1}{n} \right)^{\frac{\beta_3}{2 \beta_3 + d}} \). Indeed, if \( \beta_2 = \beta_3 \) we have

\[
\left( \frac{1}{T_n} \right)^{\frac{2 \beta_3}{2 \beta_3 + d - 2}} = \left( \frac{1}{n T_n} \right)^{\frac{2 \beta_3}{2 \beta_3 + d - 2}} = \left( \left( \frac{1}{n} \right)^{\frac{\beta_3}{2 \beta_3 + d - 2}} \right)^{\frac{2 \beta_3}{2 \beta_3 + d - 2}} = \left( \frac{\log n}{\Delta_n} \right)^{\frac{2 \beta_3}{2 \beta_3 + d - 2}} = \left( \frac{1}{n} \right)^{\frac{\beta_2}{2 \beta_2 + d - 2}} = \left( \frac{1}{n} \right)^{\frac{2 \beta_3}{2 \beta_3 + d - 2}}.
\]

The very same computation holds for \( \beta_2 < \beta_3 \) as

\[
\frac{\log T_n}{T_n} = \left( \frac{\log n}{\Delta_n} \right) \frac{2 \beta_3}{2 \beta_3 + d - 2} \left( \left( \frac{1}{n} \right)^{\frac{\beta_3}{2 \beta_3 + d - 2}} \right)^{\frac{2 \beta_3}{2 \beta_3 + d - 2}} = \left( \frac{1}{n} \right)^{\frac{\beta_2}{2 \beta_2 + d - 2}}.
\]

It follows that, when the discretization step reaches the threshold, the continuous convergence rate is equal to the convergence rate in the intermediate regime.

For \( d = 2 \), analogous results hold. In particular, we have the following proposition.

**Proposition 4.2:** Suppose that A1–A2 hold and that \( d = 2 \). Let \( \beta = (\beta_1, \ldots, \beta_d) \), \( 0 < \beta_1 \leq \cdots \leq \beta_d \) and \( \mathcal{L} = (\mathcal{L}_1, \ldots, \mathcal{L}_d) \), \( \mathcal{L}_i > 0 \). Let \( \hat{\pi}_{h,n} \) be the estimator proposed in (5), then there exist \( c > 0 \) and \( T_0 > 0 \) such that, for \( T_n \geq T_0 \),

\[
\text{Var}(\hat{\pi}_{h,n}(x)) \leq \frac{c}{T_n} \sum_{j=1}^d |\log(h_j)| + \frac{1}{T_n h_1 h_2} \Delta_n,
\]

where the constant \( c \) is uniform over the set of coefficients \( (a, b) \in \Sigma \).

As before, it leads us to a condition on \( \Delta_n \) that allows us to recover the continuous convergence rate gathered in Theorem 2 of [47], in the continuous case.

**Theorem 4.3:** Suppose that \( d = 2 \). Let \( \beta = (\beta_1, \beta_2) \), \( 0 < \beta_1 \leq \beta_2 \), \( \mathcal{L} = (\mathcal{L}_1, \mathcal{L}_2) \), \( \mathcal{L}_i > 0 \), \( 0 < a_{\min} \leq a_0 a_1 > 0, b_0 > 0, b_1 > 0, \mathcal{C} > 0, \tilde{\rho} > 0 \) and set \( \Sigma = \Sigma(\beta, \mathcal{L}, a_{\min}, a_0, a_1, b_0, b_1, \mathcal{C}, \tilde{\rho}) \), using the notation of Definition 2.2.
Let \( h^* = (h^*_1, h^*_2) \) be the rate optimal choice (6) for the bandwidth \( h \). Then, there exist \( c > 0 \) and \( T_0 > 0 \) such that, for \( T_n \geq T_0 \), the following hold true.

- If \( \Delta_n \leq h^*_1 h^*_2 \sum_{j=1}^{2} \log h^*_j \| \log T_n \|^{1-\beta} \log T_n \), then
  
  \[
  \sup_{(a,b) \in \Sigma} \mathbb{E}[|\hat{\pi}_{h,n}(x) - \pi(x)|^2] \leq c \frac{\log T_n}{T_n}
  \]

- If otherwise \( \Delta_n > (\frac{\log T_n}{T_n})^{1-\beta} \log T_n \), then
  
  \[
  \sup_{(a,b) \in \Sigma} \mathbb{E}[|\hat{\pi}_{h,n}(x) - \pi(x)|^2] \leq c \left( \frac{1}{n} \right)^{\frac{2\beta}{\beta+2}}.
  \]

As below Theorem 4.2, we can write the conditions on \( \Delta \) in function of \( n \). In particular, the condition \( \Delta_n \leq (\frac{\log T_n}{T_n})^{1-\beta} \log T_n \) is equivalent to \( \Delta_n \leq (\frac{1}{n})^{\frac{1}{\beta+1}} \log(n \Delta_n) \). Replacing \( \Delta_n = (\frac{1}{n})^{\frac{1}{\beta+1}} \log(n \Delta_n) \) in the continuous convergence rate we get

\[
\frac{\log T_n}{T_n} = \log(n \Delta_n) \left( \frac{1}{n} \right)^{\frac{1}{\beta+1}} \frac{1}{\log(n \Delta_n)} = \left( \frac{1}{n} \right)^{\frac{\beta}{\beta+1}},
\]

which is the convergence rate obtained in the intermediate regime, as \( \frac{2\beta}{\beta+2} = \frac{\beta}{\beta+1} \).

In this section, we have found the convergence rates for the estimation of the invariant density starting from the discrete observation of the process \( X \). Such observations are, in this section, all taken at the same instant. One may wonder if it is possible to recover the same results when the process \( X \) is observed asynchronously. The goal of the next section is to answer this question.

### 5. Main results, asynchronous framework

In this section, we assume \( d \geq 3 \) and we suppose that the components of the process \( X \) are observed asynchronously, i.e., in different instants. We will see that, up to require conditions on the discretization step and on the asynchronicity of the observations, it is possible to obtain the continuous convergence rate \( (\frac{\log T_n}{T_n})^{\frac{2\beta}{\beta+2d-2}} \).

We assume that we dispose of the discrete observations \( X^l_{i_0^l}, \ldots, X^l_{i_n^l} \) for any \( l \in \{1, \ldots, d\} \), with \( 0 = i_0^l < i_1^l < \cdots < i_n^l \leq T_n \) and \( T_n \to \infty \) for \( n \to \infty \) for any \( l \in \{1, \ldots, d\} \). We also define

\[
\Delta_n := \sup_{l=1, \ldots, d} \sup_{i=0, \ldots, n} (i^l_{i+1} - i^l_i), \text{ where } t^l_{i+1} = T_n \text{ for all } l \in \{1, \ldots, d\}.
\]

We remark it would have been possible to consider a different number of points in different directions, having in particular \( n_l \) observations for the coordinate \( X^l \). We have decided to take \( n_1 = \cdots = n_d \) in order to lighten the notation.
Before we proceed with the statements of our results, we introduce some functions. First of all we observe that we have defined $d$ partitions of $[0, T_n]$ and so, for any $u \in [0, T_n]$, there exist some indexes $i_1, \ldots, i_d$ such that $u \in [t_{i_l}^j, t_{i_{l+1}}^j)$, depending on the considered direction. We introduce then the following $d$ functions $\varphi_{n,l}: [0, T_n] \to \mathbb{R}$ such that $\varphi_{n,l}(u) := t_{i_l}^l$, for any $l \in \{1, \ldots, d\}$. Thanks to these functions we can write the sums in the form of integrals. It leads us to the following estimator, which is the natural adaptation of the one in (4):

$$
\hat{\pi}_{h,T_n}^a(x) = \frac{1}{T_n} \prod_{l=1}^d h_l \int_0^{T_n} \prod_{l=1}^d K \frac{x_l - X_{\varphi_{n,l}(u)}^l}{h_l} \, du \\
=: \frac{1}{T_n} \int_0^{T_n} \prod_{l=1}^d K_{h_l}(x_l - X_{\varphi_{n,l}(u)}^l) \, du.
$$

In this context we introduce the class of coefficient $\tilde{\Sigma} = \tilde{\Sigma}(\beta, \mathcal{L}, a_{\min}, a_0, a_1, b_0, b_1, \tilde{C}, \tilde{\rho})$ as

$$
\tilde{\Sigma} := \{(a, b) \in \Sigma(\beta, \mathcal{L}, a_{\min}, a_0, a_1, b_0, b_1, \tilde{C}, \tilde{\rho}) : a, b \text{ are } \mathcal{C}^3\}.
$$

The regularity requested is needed in order to obtain uniform mixing inequalities on the class of coefficients and control on the bias of the estimator.

We provide two different bounds for the variance, depending on the regime considered. When the discretization error is negligible we have the following result.

**Proposition 5.1:** Suppose that A1–A2 hold with some constant $0 < a_{\min} \leq a_0$ and $a_1 > 0$, $b_0 > 0$, $b_1 > 0$, $\tilde{C}_b$, $\tilde{\rho}_b$ and that $d \geq 3$. Let $\beta = (\beta_1, \ldots, \beta_d)$, $0 < \beta_1 \leq \cdots \leq \beta_d$ and $\mathcal{L} = (\mathcal{L}_1, \ldots, \mathcal{L}_d)$, $\mathcal{L}_i > 0$. Let $h^* = (h_i^*, \ldots, h_d^*)$ be the rate optimal choice for the bandwidth $h$ given in (6). We suppose moreover that $\Delta_n \leq \frac{1}{4} h_i^* h_2^*$, then there exist $c > 0$ and $T_0 > 0$ such that, for $T_n \geq T_0$,

$$
\text{Var}(\hat{\pi}_{h,n}^a(x)) \leq \frac{c}{T_n} \sum_{j=1}^d |\log(h_j^*)| \prod_{l=1}^d h_l^*.
$$

Moreover, the constant $c$ is uniform over the set of coefficients $(a, b) \in \tilde{\Sigma}$.

Comparing the bound here above with the results gathered in Proposition 4.1 it appears clearly that asking the condition $\Delta_n \leq \frac{1}{4} h_i^* h_2^*$ is enough both in the synchronous and asynchronous frameworks to recover the same bound on the variance as in the continuous case, which is optimal for $\beta_2 < \beta_3$.

In the intermediate regime, instead, the following proposition holds true.

**Proposition 5.2:** Suppose that A1–A2 hold with some constant $0 < a_{\min} \leq a_0$ and $a_1 > 0$, $b_0 > 0$, $b_1 > 0$, $\tilde{C}_b$, $\tilde{\rho}_b$ and that $d \geq 3$. Let $\beta = (\beta_1, \ldots, \beta_d)$, $0 < \beta_1 \leq \cdots \leq \beta_d$ and $\mathcal{L} = (\mathcal{L}_1, \ldots, \mathcal{L}_d)$, $\mathcal{L}_i > 0$. Let $h^* = (h_i^*, \ldots, h_d^*)$ be the rate optimal choice for the bandwidth $h$ in the intermediate regime, given by $h_j^*(n) = \left(\frac{1}{n}\right)^{\tilde{\beta}/(2\tilde{\beta}+d)}$ for any $j \in \{1, \ldots, d\}$.
Suppose moreover that $\Delta_n \geq (\prod_{i=1}^d \tilde{h}^i_{T_n})^{\frac{1}{2}}$, then there exist $c > 0$ and $T_0 > 0$ such that, for $T_n \geq T_0$,

$$\text{Var}(\hat{\pi}^a_{h^*,n}(x)) \leq \frac{c \Delta_n}{T_n \prod_{i=1}^d \tilde{h}^i_{T_n}}.$$ 

Moreover, the constant $c$ is uniform over the set of coefficients $(a, b) \in \tilde{\Sigma}$.

If $\frac{n \Delta_n}{T_n}$ is bounded it is possible to compare the conditions on the sampling step required by Propositions 5.1 and 5.2. Indeed, under the condition $\frac{n \Delta_n}{T_n} \leq c$, exactly as we obtain (9), we see that the condition on $\Delta_n$ in Proposition 5.1 is equivalent to $\Delta_n \leq n^{-\frac{2}{2d+2}} (\frac{1}{\beta_1} + \frac{1}{\beta_2})$. On the other hand, the condition on the sampling step in Proposition 5.2 is always equivalent to $\Delta_n \leq n^{-\frac{2}{2d+2}}$. In the isotropic case, the two exponents are the same, and thus Proposition 5.1 and 5.2 covers all the cases for $\Delta_n$. In the anisotropic case, there is a possible gap corresponding to $n^{-\frac{2}{2d+2}} < \Delta_n < n^{-\frac{2}{2d+2}}$, where the sampling step is not small enough to recover the continuous rate and conditions required for Proposition 5.2 are not satisfied either.

In order to obtain the convergence rate in the asynchronous framework we have to provide a bound on the bias term. While in the synchronous context this object has already been deeply studied, this is no longer the case when the process is observed in different instants of time. Hence, in the asynchronous case we introduce a quantity whose goal is to measure the asynchronicity of our data:

$$\Delta_n' := \sup_{t \in [0, T_n]} \sup_{i,j=1,\ldots,d} |\varphi_{n,i}(t) - \varphi_{n,j}(t)|.$$ (12)

By definition we have $\Delta_n' \leq \Delta_n$, and $\Delta_n' = 0$ for synchronous data. Then, the following proposition holds true.

**Proposition 5.3:** Suppose that A1 holds and that $d \geq 3$. Let $\beta = (\beta_1, \ldots, \beta_d)$, $0 < \beta_1 \leq \cdots \leq \beta_d$ and $\mathcal{L} = (\mathcal{L}_1, \ldots, \mathcal{L}_d)$, $\mathcal{L}_i > 0$. Then, there exists $c > 0$ such that for all $T_n > 0$, $0 < h_i < 1$,

$$\left| \mathbb{E}[\hat{\pi}^a_{h^*,n}(x)] - \pi(x) \right| \leq c \sum_{i=1}^d h_i^{\beta_i} + c \sqrt{\Delta_n}.$$ 

Moreover, the constant $c$ is uniform over the set of coefficients $(a, b) \in \tilde{\Sigma}$.

We see that asynchronicity introduces an additional term in the control of the bias. From Propositions 5.1 and 5.3 the next theorem easily follows.

**Theorem 5.1:** Suppose that $d \geq 3$. Let $\beta = (\beta_1, \ldots, \beta_d)$, $0 < \beta_1 \leq \cdots \leq \beta_d$, $\mathcal{L} = (\mathcal{L}_1, \ldots, \mathcal{L}_d)$, $\mathcal{L}_i > 0$, $0 < a_{\min} \leq a_0$, $a_1 > 0$, $b_0 > 0$, $b_1 > 0$, $\tilde{C} > 0$, $\tilde{\rho} > 0$ and $\tilde{\Sigma}$ given by (11). If $\Delta_n \leq \left( \frac{\log T_n}{T_n} \right)^{\frac{2\beta_3}{2d+2}} (\frac{1}{\beta_1} + \frac{1}{\beta_2})$ and $\Delta_n' \leq \left( \frac{\log T_n}{T_n} \right)^{\frac{2\beta_3}{2d+2}}$, then there exist $c > 0$ and
$T_0 > 0$ such that, for $T_n \geq T_0$,

$$\sup_{(a,b) \in \tilde{\Sigma}} \mathbb{E}[|\hat{\pi}_{h,T_n}^a(x) - \pi(x)|^2] \leq c \left( \frac{\log T_n}{T_n} \right)^{\frac{2\tilde{\beta}_3}{2\tilde{\beta}_3 + d - 2}}.$$ 

Regarding the intermediate regime, instead, Proposition 5.2 and 5.3 yields the following

**Theorem 5.2:** Suppose that $d \geq 3$. Let $\beta = (\beta_1, \ldots, \beta_d)$, $0 < \beta_1 \leq \cdots \leq \beta_d$, $\mathcal{L} = (\mathcal{L}_1, \ldots, \mathcal{L}_d)$, $\mathcal{L}_i > 0$, $0 < a_{\min} \leq a_0$, $a_1 > 0$, $b_0 > 0$, $b_1 > 0$, $\tilde{C} > 0$, $\tilde{\rho} > 0$ and $\tilde{\Sigma}$ given by (11). If $\Delta_n \geq (\frac{2}{n})^{\frac{2\tilde{\beta}}{2\tilde{\beta} + d}}$ and $\Delta'_n \leq (\frac{1}{n})^{\frac{2\tilde{\beta}}{2\tilde{\beta} + d}}$, then there exist $c > 0$ and $T_0 > 0$ such that, for $T_n \geq T_0$,

$$\sup_{(a,b) \in \tilde{\Sigma}} \mathbb{E}[|\hat{\pi}_{h,T_n}^a(x) - \pi(x)|^2] \leq c \left( \frac{1}{n} \right)^{\frac{2\tilde{\beta}}{2\tilde{\beta} + d}}.$$ 

Theorems 5.1 and 5.2 extend to the asynchronous case Theorems 4.1 and 4.2, respectively. One can see that the bounds on the variance appearing in Proposition 5.1 are significantly lighter than the condition on the discretization step needed to recover the bound on the variance as in Proposition 5.1. In this case, the continuous trajectories of $X^1$ and $X^2$ are available, as well as the discrete observations $X^l_{t_1}, \ldots, X^l_{t_n}$ for any $l \in \{3, \ldots, d\}$, with $0 \leq t_1^l \leq \cdots \leq t_n^l \leq T_n$.

The discretization step is defined as

$$\Delta_n := \sup_{l=3,\ldots,d} \sup_{i=0,\ldots,n} (t_{i+1}^l - t_i^l), \quad \text{where } t_{n+1}^l = T_n \text{ for all } l \in \{3, \ldots, d\}.$$ 

The kernel estimator in this context is the following:

$$\hat{\pi}_{h,T_n}(x) = \frac{1}{T_n} \prod_{l=1}^d \int_0^{T_n} \prod_{m=1,2} K \left( \frac{x_m - X^m_{\star u}}{h_m} \right) \prod_{i=3}^d K \left( \frac{x_l - X^l_{\star u,\star i(u)}}{h_l} \right) \, du$$

$$= \frac{1}{T_n} \int_0^{T_n} \prod_{m=1,2} K_{h_m}(x_m - X^m_{\star u}) \prod_{l=3}^d K_{h_l}(x_l - X^l_{\star u,\star i(u)}) \, du. \quad (13)$$

Then, it is possible to recover the same upper bound on the variance as in Proposition 2 of [47], where all the components of the process $X$ were continuously available.

**Proposition 5.4:** Suppose that A1–A2 hold and that $d \geq 3$. Let $\beta = (\beta_1, \ldots, \beta_d)$, $0 < \beta_1 \leq \cdots \leq \beta_d$ and $\mathcal{L} = (\mathcal{L}_1, \ldots, \mathcal{L}_d)$, $\mathcal{L}_i > 0$. Let $h^* = (h^*_1, \ldots, h^*_d)$ be the rate optimal choice
for the bandwidth \( h \) given in (6) and suppose that \( \Delta_n \leq \frac{1}{2}(\prod_{l \geq 3} h_1^*)^{\frac{2}{d-2}} = \frac{1}{2}(\frac{\log T_n}{T_n})^{\frac{2}{2\beta_3 + d - 2}} \), then there exist \( c > 0 \) and \( T_0 > 0 \) such that, for \( T_n \geq T_0 \),

\[
\operatorname{Var}(\pi_n h^*, T_n(x)) \leq \frac{c}{T_n} \sum_{j=1}^{d} |\log(h_j^*)| \prod_{l=3}^{d} h_l^*.
\]

Moreover, the constant \( c \) is uniform over the set of coefficients \((a, b) \in \tilde{\Sigma}\).

Comparing the proposition here above with Proposition 2 of [47] one can see that, when \( k_0 = 1, 2 \), not having the continuous record of the last \((d - 2)\) components does not interfere in the computations of the upper bound of the variance. The condition appearing here above is less restrictive than the one in Proposition 4.1 and Theorem 4.1, as \( h_1^* h_2^* \sum_{j=1}^{d} |\log h_j^*| \leq (\prod_{l \geq 3} h_l^*)^{\frac{2}{d-2}} \). Indeed, it is equivalent to ask

\[
(\frac{\log T_n}{T_n})^{\frac{2}{2\beta_3 + d - 2}} (\frac{1}{\beta_1^*} + \frac{1}{\beta_2^*}) \log T_n \leq (\frac{\log T_n}{T_n})^{\frac{2}{2\beta_3 + d - 2}},
\]

which holds true in a pure anisotropic context.

However, it is easy to see that Proposition 5.3 still holds true when \( \varphi_{n,l}(t) = t \) for \( l = 1, 2 \) and so with \( \tilde{\pi}_{h,T_n}(x) \) instead of \( \hat{\pi}_{h,T_n}(x) \). Then, the condition on \( \Delta_n' \) turns out being in this case a condition on \( \Delta_n \), as two components are observed continuously. This constraint is stronger than both the conditions in Propositions 5.1 and 5.4. It implies that one can propose an estimator based on the continuous observations of two components in order to improve the bound on the variance, but this implies a big deterioration in the condition of the bias. In particular, the resulting estimation is not better than the one based on the discrete asynchronous observation of all the components.

One may observe that, even if in the paper it is always assumed that the target stationary distribution \( \pi \) is such that the regularities are ordered, to build our estimators as in (5) and (10) it is not necessary to know in advance which coordinates are less regular. It is no longer the case for the estimator proposed in (13), where the two continuously observed coordinates are the less regular ones. This is the biggest limitation of Proposition 5.4, which is only a theoretical result. Its purpose is to remark that having finer observations in some coordinates yields a worse condition on the discretization step, which is surprising. This suggests it would be better to synchronize the continuous observations of the first two components with the closest discrete observation, to decrease the asynchronicity.

The proofs of all the theorems and propositions stated in this section can be found in Section 7, but for Proposition 5.4, whose proof can be found in the appendix.

6. Proof main results, synchronous framework

This section is devoted to the proof of our main results in the case where all the components are observed at the same time. In the sequel the constant \( c \) may change from line to line.

6.1. Proof of Proposition 4.1

**Proof:** The proof of Proposition 4.1 heavily relies on the proof of the upper bound on the variance of (4), in the continuous case. Intuitively, the integrals in Proposition 2 of [47]
will be now replaced by sums, that we will split in order to use some different bounds on each of them. The main tools are the exponential ergodicity of the process as gathered in Proposition 1 of [47] and a bound on the transition density as in Proposition 5.1 of [53].

From the definition of our estimator \( \hat{\pi}_{h,n} \), using also the fact that we are considering a uniform discretization step, it follows

\[
\text{Var}(\hat{\pi}_{h,n}(x)) = \text{Var}\left( \frac{1}{n\Delta_n} \sum_{j=0}^{n-1} K_h(x - X_{t_j}) \right) \Delta_n
\]

\[
= \frac{\Delta_n^2}{T_n^2} \sum_{j=0}^{n-1} (n - j) \text{Cov}(K_h(x - X_0), K_h(x - X_{t_j}))
\]

\[
= \frac{\Delta_n^2}{T_n^2} \left( \sum_{j=0}^{j_{\delta_1}} + \sum_{j=j_{\delta_1}+1}^{j_{\delta_2}} + \sum_{j=j_{\delta_2}+1}^{j_D} + \sum_{j=j_D+1}^{n-1} \right) (n - j) k(t_j)
\]

\[
=: I_1 + I_2 + I_3 + I_4,
\]

having introduced \( 0 \leq j_{\delta_1} \leq j_{\delta_2} \leq j_D \leq n - 1 \) and set \( \delta_1 = \Delta_n j_{\delta_1}, \delta_2 = \Delta_n j_{\delta_2}, D = \Delta_n j_D \) and

\[
k(t) := \text{Cov}(K_h(x - X_0), K_h(x - X_t)).
\]

Recall also that \( t_j = \Delta_n j \). The quantities \( j_{\delta_1}, j_{\delta_2}, j_D \) (and consequently \( \delta_1, \delta_2 \) and \( D \)) will be chosen later, in order to get an upper bound on the variance as sharp as possible. We will provide some bounds for \( I_1, I_4 \) which do not depend on \( k_0 \), while we will bound differently \( I_2 \) and \( I_3 \) depending on whether or not \( k_0 \) is larger than 3. For \( j \) small we use Cauchy-Schwarz inequality, the stationarity of the process, the boundedness of \( \pi \) and the definition of the kernel function to obtain

\[
|k(t_j)| \leq \text{Var}(K_h(x - X_0))^\frac{1}{2} \text{Var}(K_h(x - X_{t_j}))^\frac{1}{2} \leq \int_{\mathbb{R}^d} (K_h(x - y))^2 \pi(y) \, dy \leq \frac{c}{\prod_{l=1}^{d} h_l}.
\]

It follows

\[
|I_1| \leq \frac{\Delta_n^2 n}{T_n^2} \sum_{j=0}^{j_{\delta_1}} \frac{c}{\prod_{l=1}^{d} h_l} = \frac{\Delta_n^2 n}{T_n^2} \frac{1}{\prod_{l=1}^{d} h_l} (j_{\delta_1} + 1).
\]

(14)

For \( j \in [j_{\delta_1} + 1, j_{\delta_2}] \) we act differently depending on \( k_0 \) as done for \( s \in (\delta_1, \delta_2) \) in Proposition 2 of [47]. For \( k_0 = 1 \) and \( \beta_2 < \beta_3 \) or \( k_0 = 2 \) it provides (see Equation (15) in [47])

\[
|k(s)| \leq \frac{c}{\prod_{j\geq3} h_j s}
\]

that, for \( s = t_j \), becomes

\[
|k(t_j)| \leq \frac{c}{\prod_{l\geq3} h_l t_j}.
\]
It yields
\[ |I_2| \leq \frac{\Delta_n^2 n}{T_n^2} \sum_{j=j_1+1}^{j_{s_2}} \frac{c}{l_{ \geq 3} h_l} \frac{1}{t_j} = \frac{c \Delta_n}{T_n} \prod_{l_{ \geq 3}} \frac{1}{h_l} \sum_{j=j_1+1}^{j_{s_2}} \frac{1}{t_j}. \]

We recall that \( t_j \) can be seen as \( \Delta_n j \). Therefore,
\[ \sum_{j=j_1+1}^{j_{s_2}} \frac{1}{t_j} \leq \frac{c}{\Delta_n} \log \left( \frac{j_{s_2}}{j_{s_1}} \right) = \frac{c}{\Delta_n} \log \left( \frac{\Delta_n j_{s_2}}{\Delta_n j_{s_1}} \right) = \frac{c}{\Delta_n} \log \left( \frac{\delta_2}{\delta_1} \right). \]

It follows
\[ |I_2| \leq \frac{c}{T_n} \prod_{l_{ \geq 3}} \frac{1}{h_l} \log \left( \frac{\delta_2}{\delta_1} \right). \quad (16) \]

When \( k_0 \geq 3 \) instead, acting as to get \((17)\) in [47] and taking \( s = t_j \), we obtain
\[ |k(t_j)| \leq \frac{c}{\prod_{l_{ \geq k_0+1}} \frac{1}{h_l}} t_j^{-\frac{k_0}{2}}. \]

Therefore,
\[ |I_2| \leq \frac{\Delta_n^2 n}{T_n^2} \sum_{j=j_1+1}^{j_{s_2}} \frac{c}{l_{ \geq k_0+1} h_l} t_j^{-\frac{k_0}{2}} \]
\[ = \frac{c \Delta_n}{T_n} \prod_{l_{ \geq k_0+1}} \frac{1}{h_l} \sum_{j=j_1+1}^{j_{s_2}} \Delta_n^{-\frac{k_0}{2}} j^{-\frac{k_0}{2}} \]
\[ \leq \frac{c \Delta_n^{1-\frac{k_0}{2}}}{T_n} \prod_{l_{ \geq k_0+1}} \frac{1}{h_l} j_{s_1}^{1-\frac{k_0}{2}} \]
\[ = \frac{c}{T_n} \prod_{l_{ \geq k_0+1}} \frac{1}{h_l} \delta_1^{1-\frac{k_0}{2}} \]
\[ (17) \]

where we have used that, as \( k_0 \geq 3 \), \( 1 - \frac{k_0}{2} \) is negative.

To conclude the analysis of \( I_2 \) we assume that \( k_0 = 1 \) and \( \beta_2 = \beta_3 \). In this case the estimation here above still holds but, as \( 1 - \frac{k_0}{2} = \frac{1}{2} \) is now positive, it provides
\[ |I_2| \leq \frac{c}{T_n} \prod_{l_{ \geq 2}} \frac{1}{h_l}. \quad (18) \]

We now deal with \( I_3 \). With the same bound on the covariance as in \((20)\) of Proposition 2 in [47] we get in any case, but for \( k_0 = 1 \) and \( \beta_2 = \beta_3 \),
\[ |k(s)| \leq c(s^{-\frac{d}{2}} + 1). \]
Therefore, taking \( s = t_j \),
\[
|I_3| \leq \frac{\Delta^2_n \, n}{T^2_n} \sum_{j=\delta_2+1}^{j_D} c(t_j^{-\frac{d}{2}} + 1)
\leq c \frac{\Delta_n}{T_n} \left( \sum_{j_l \leq 1, j = \delta_2+1}^{j_D} \Delta_n^{-\frac{d}{2}} j_l^{-\frac{d}{2}} + \sum_{j_l > 1, j = \delta_2+1}^{j_D} 1 \right)
\leq c \frac{\Delta_n}{T_n} (\Delta_n^{-\frac{d}{2}} 1^{-\frac{d}{2}} + j_D)
= \frac{c}{T_n} (\delta_2^{1-\frac{d}{2}} + D).
\] (19)

For \( k_0 = 1 \) and \( \beta_2 = \beta_3 \), instead, (22) of [47] provides
\[
|k(s)| \leq c \left( s^{-\frac{3}{2}} \frac{1}{\prod_{l \geq 4} h_l} + 1 \right).
\]

It follows
\[
|I_3| \leq \frac{c \Delta^2_n \, n}{T^2_n} \frac{1}{\prod_{l \geq 4} h_l} \sum_{j=\delta_2+1}^{j_D} (t_j^{-\frac{3}{2}} + 1).
\]

Acting as above we obtain
\[
|I_3| \leq \frac{c}{T_n} \left( \frac{1}{\prod_{l \geq 4} h_l} \frac{1}{\delta_2^{\frac{3}{2}}} + D \right).
\] (20)

To conclude, we need to evaluate the case where \( j \in [\delta_D + 1, n - 1] \). In this interval, we use the exponential ergodicity of the process, as in Proposition 1 of [47]. It follows
\[
|k(t_j)| \leq c \| K_{\eta}(x - \cdot) \|_{L^2}^2 e^{-\rho t_j} \leq \frac{c}{(\prod_{l=1}^d h_l)^2} e^{-\rho t_j},
\]
for \( c \) and \( \rho \) positive constant uniform over the set of coefficients \((a, b) \in \Sigma \). It implies
\[
|I_4| \leq \frac{c \Delta_n}{T_n} \frac{1}{(\prod_{l=1}^d h_l)^2} \sum_{j=\delta_D+1}^{n-1} e^{-\rho \Delta_n j}
\leq \frac{c \Delta_n}{T_n} \frac{1}{(\prod_{l=1}^d h_l)^2} e^{-\rho \Delta_n (j_D+1)}
\leq \frac{c \Delta_n}{T_n} \frac{1}{(\prod_{l=1}^d h_l)^2} e^{-\rho D}.
\] (21)
From (15), (16), (19) and (21) we obtain the following bound for the case $k_0 = 1$ and $\beta_2 < \beta_3$ or $k_0 = 2$:

\[
\text{Var}(\hat{\pi}_{h,n}(x)) \\
\leq \frac{c}{T_n} \left[ \frac{1}{\prod_{l=1}^{d} h_l} (\delta_1 + \Delta_n) + \frac{1}{\prod_{l \geq 3} h_l} \log \left( \frac{\delta_2}{\delta_1} \right) + \delta_2^{1 - \frac{1}{\Delta_1}} + D + \frac{1}{\left( \prod_{l=1}^{d} h_l \right)^2} e^{-\rho D} \right].
\]

It is easy to see that, except for the term $\frac{c}{T_n} \frac{1}{\prod_{l=1}^{d} h_l} \Delta_n$, the bound is the same as in the continuous case (see (22) in [47]). Hence, also the optimal choice for the parameters $\delta_1$, $\delta_2$ and $D$ should be the same as in the continuous case, for which $\delta_1 = h_1 h_2$, $\delta_2 := (\prod_{j \geq 3} h_j)^{2 - \frac{2}{\rho}}$ and $D := \left[ \max(-\frac{2}{\rho} \log(\prod_{j=1}^{d} h_j), 1) \wedge T \right]$. Recalling that $j_{\delta_1}$, $j_{\delta_2}$ and $j_D$ have to be some integers, we can not propose exactly the same choice as above but we can take

\[
j_{\delta_1} := \left\lfloor \frac{h_1 h_2}{\Delta_n} \right\rfloor, \quad j_{\delta_2} := \left\lfloor \frac{(\prod_{j \geq 3} h_j)^{2 - \frac{2}{\rho}}}{\Delta_n} \right\rfloor,
\]

\[
j_{D} := \left\lfloor \frac{\max(-\frac{2}{\rho} \log(\prod_{j=1}^{d} h_j), 1) \wedge T}{\Delta_n} \right\rfloor.
\]

It yields

\[
\text{Var}(\hat{\pi}_{h,n}(x)) \leq \frac{c}{T_n} \frac{\sum_{l=1}^{d} |\log(h)|}{\prod_{l \geq 3} h_l} + \frac{c}{T_n} \frac{1}{\prod_{l=1}^{d} h_l} \Delta_n,
\]
as we wanted.

When $k_0 \geq 3$, instead, we replace the bound gathered in (16) with the one in (17). It follows

\[
\text{Var}(\hat{\pi}_{h,n}(x)) \\
\leq \frac{c}{T_n} \left[ \frac{1}{\prod_{l=1}^{d} h_l} (\delta_1 + \Delta_n) + \frac{1}{\prod_{l \geq k_0+1} h_l} \delta_1^{1 - \frac{k_0}{\Delta_1}} + \delta_2^{1 - \frac{1}{\Delta_1}} + D + \frac{1}{\left( \prod_{l=1}^{d} h_l \right)^2} e^{-\rho D} \right].
\]

Again, every term but $\frac{c}{T_n} \frac{1}{\prod_{l=1}^{d} h_l} \Delta_n$ was already present in the proof of Proposition 2 of [47] (see (23)) and so the best choice would be to take the parameters as before. With this purpose in mind we choose

\[
j_{\delta_1} := \left\lfloor \frac{(\prod_{l=1}^{k_0} h_l)^{k_0}}{\Delta_n} \right\rfloor, \quad j_{\delta_2} := \left\lfloor \frac{1}{\Delta_n} \right\rfloor.
\]
\[ j_D := \left\lceil \max \left( - \frac{2}{\rho} \log \left( \prod_{j=1}^{d} h_j \right), 1 \right) \wedge T \right\rceil \Delta_n. \]

It follows
\[
\text{Var}(\hat{\pi}_{h,n}(x)) \leq \frac{c}{T_n} \frac{1}{\left( \prod_{l=1}^{k_0} h_l \right)^{\frac{1}{2} - \frac{1}{k_0}}} \left( \prod_{l \geq k_0 + 1} h_l \right) + c \frac{1}{T_n} \left( \prod_{l=1}^{d} h_l \right) \Delta_n. 
\]

We are left to study the case where \( k_0 = 1 \) and \( \beta_2 = \beta_3 \). Here, from (15), (18), (20) and (21) we obtain
\[
\text{Var}(\hat{\pi}_{h,n}(x)) \leq \frac{c}{T_n} \frac{1}{\prod_{l=1}^{d} h_l} (\delta_1 + \Delta_n) + \frac{\delta_2}{\prod_{l \geq 4} h_l} + \frac{1}{\prod_{l \geq 4} h_l \delta_2} + D + \frac{1}{\left( \prod_{l=1}^{d} h_l \right)^{\frac{1}{2}}} e^{-\rho D}. 
\]

We take
\[
j_{\delta_1} := 1, \quad j_{\delta_2} := \left\lceil \frac{h_3 h_4}{\Delta_n} \right\rceil, \quad j_D := \left\lceil \max \left( - \frac{2}{\rho} \log \left( \prod_{j=1}^{d} h_j \right), 1 \right) \wedge T \right\rceil \Delta_n \]

to get
\[
\text{Var}(\hat{\pi}_{h,n}(x)) \leq \frac{c}{T_n} \left( \frac{1}{\sqrt{h_2 h_3}} \right) \left( \frac{1}{\prod_{l \geq 4} h_l} \right) + \frac{c}{T_n} \frac{1}{\prod_{l=1}^{d} h_l} \Delta_n. 
\]

All the constants are uniform over the set of coefficients \((a, b) \in \Sigma\). The proof of Proposition 4.1 is then complete.

\[■\]

6.2. Proof of Theorem 4.1

**Proof:** We split the proof according to the choice of \( k_0 \).

- For \( k_0 = 1 \) and \( \beta_2 < \beta_3 \) or \( k_0 = 2 \), when

\[
\Delta_n \leq h_1^* h_2^* \sum_{j=1}^{d} | \log h_j^* | = \left( \frac{\log T_n}{T_n} \right)^{\frac{1}{2} \rho_3 + \rho_2} \left( \frac{1}{\rho_1 + \rho_2} \right) \log T_n
\]

the result is a straightforward consequence of the bias variance decomposition and of the bound on the variance gathered in Proposition 4.1. The rate optimal choice of the
bandwidth $h^*$ as in (6) provides

$$
\text{Var}(\hat{\pi}_{h,n}(x)) \leq \frac{c}{T_n} \frac{\sum_{l=1}^{d} \log(h^*_{l})}{\prod_{l \geq 3} h^*_l} + \frac{c}{T_n} \frac{1}{\prod_{l=1}^{d} h^*_l} \Delta_n
$$

$$
\leq \frac{c}{T_n} \frac{\sum_{l=1}^{d} \log(h^*_{l})}{\prod_{l \geq 3} h^*_l}.
$$

Hence,

$$
\sup_{(a,b) \in \Sigma} \mathbb{E}[|\hat{\pi}_{h,n}(x) - \pi(x)|^2] \leq \frac{c}{T_n} \frac{\sum_{j=1}^{d} h_j^{2\beta_j}}{\prod_{l \geq 3} h^*_l} + \frac{c}{T_n} \frac{\sum_{l=1}^{d} \log(h^*_l)}{\prod_{l \geq 3} h^*_l}
$$

$$
= \left( \frac{\log T_n}{T_n} \right)^{(2\beta_3+d-2)}
$$

- For $k_0 \geq 3$, when $\Delta_n \leq (h^*_1 h^*_2) = (h^*_1)^2 = \left( \frac{1}{T_n} \right)^{\frac{2\beta_3}{\beta_1(2\beta_3+d-2)}}$, we get the continuous convergence rate by the bias-variance decomposition and the second point of Proposition 4.1, choosing $h^*_1 = \cdots = h^*_k$ and $h^*_l(T_n) : \left( \frac{1}{T_n} \right)^{\frac{\beta_3}{\beta_1(2\beta_3+d-2)}}$ for any $l \in \{1, \ldots, d\}$.

The reasoning is the same for $k_0 = 1$ and $\beta_2 = \beta_3$, recalling that we no longer have $h^*_1 = h^*_2$ (as $\beta_1 < \beta_2$) and so the condition on the discretization step becomes $\Delta_n \leq h^*_1 h^*_2 =: \left( \frac{1}{T_n} \right)^{\frac{\beta_3}{2\beta_3+d-2}}$. 

$$
= \left( \frac{\log T_n}{T_n} \right)^{(2\beta_3+d-2)} \left( \frac{1}{\beta_1} + \frac{1}{\beta_2} \right).
$$

### 6.3. Proof of Theorem 4.2

**Proof:** Even if the final result is the same for any $k_0$, the proof of Theorem 4.2 is substantially different depending on the ordering of the smoothness and so on the value of $k_0$.

- We start assuming $k_0 = 1$ and $\beta_2 < \beta_3$ or $k_0 = 2$. When $\Delta_n > \left( \frac{\log T_n}{T_n} \right)^{\frac{\beta_3}{2\beta_3 + d - 2}} \left( \frac{1}{\beta_1} + \frac{1}{\beta_2} \right)$, log $T_n$, the form of the rate optimal bandwidth is no longer as in the continuous case. We observe that $T_n = n \Delta_n$ and

$$
\frac{1}{\beta_1} + \frac{1}{\beta_2} = \frac{d}{\bar{\beta}} - \frac{d - 2}{\beta_3},
$$

and so the condition here above is equivalent to ask $\Delta_n^{1+\alpha} > (\log T_n)^{1+\alpha} \left( \frac{1}{n} \right)^{\alpha}$. It holds true if and only if

$$
\Delta_n > (\log T_n) \left( \frac{1}{n} \right)^{\frac{\alpha}{1+\alpha}},
$$

with

$$
\alpha := \frac{\beta_3}{2\beta_3 + d - 2} \left( \frac{1}{\beta_1} + \frac{1}{\beta_2} \right) = \frac{\beta_3}{2\beta_3 + d - 2} \left( \frac{d}{\bar{\beta}} - \frac{d - 2}{\beta_3} \right) = \frac{\bar{\beta} d - (d - 2) \bar{\beta}}{\bar{\beta} (2\beta_3 + d - 2)}.
$$
From the bias variance decomposition together with Proposition 4.1 we now obtain
\[ E[|\hat{\pi}_{h,n}(x) - \pi(x)|^2] \leq c \sum_{j=1}^{d} h_j^{2\beta_j} + \frac{c}{T_n} \sum_{l=1}^{d} \frac{1}{\prod_{l \geq 3} h_l^\beta} + \frac{c}{T_n} \frac{1}{\prod_{l=1}^{d} h_l^\Delta_n}. \]

We look for the rate optimal choice of the bandwidth by choosing \( a_1, \ldots, a_d \) such that
\[ h_l = (\frac{1}{n})^{a_l}. \]

We get the following bound
\[ E[|\hat{\pi}_{h,n}(x) - \pi(x)|^2] \leq c \sum_{j=1}^{d} \left( \frac{1}{n} \right)^{2\beta_j a_j} + \frac{c}{n \Delta_n} \frac{1}{\prod_{l \geq 3} (\frac{1}{n})^a_l} + \frac{c}{n} \frac{1}{\prod_{l=1}^{d} (\frac{1}{n})^a_l}, \]

having used the condition on \( \Delta_n \) gathered in (22). We recall we have assumed \( n\Delta_n \to \infty \) for \( n \to \infty \) and that \( \Delta_n > n^{-k} \) for some \( k \in (0, 1) \). It follows that \( \log(n\Delta_n) < (1 - k) \log n \), which implies \( \frac{\log n}{\log(n\Delta_n)} \leq c \). Then, we observe that the balance between the three terms in (24) is achieved for \( h_l(n) = (\frac{1}{n})^{\frac{\beta_l}{2\beta_l + d}} \). In this way \( \prod_{l \geq 3} h_l = (\frac{1}{n})^{\frac{\beta_3}{2\beta_3 + d}} \), which implies in particular that the second term in the right-hand side of (24) is upper bounded by
\[
\left( \frac{1}{n} \right)^{\frac{\beta_3}{2\beta_3 + d}} \cdot \frac{1}{n} \frac{1}{n} \cdot \frac{\beta_3}{2\beta_3 + d} \cdot \frac{1}{n} \frac{1}{n} \cdot \frac{1}{n} \frac{1}{n},
\]
which is clearly the size of the other terms as well, after having replaced the rate optimal choice for \( h_l(n) \).

- We now consider the case where \( \Delta_n > (\frac{1}{n})^{\frac{\beta_3}{2\beta_3 + d}} (\frac{1}{n})^{\frac{\beta_1 + \beta_2}{2\beta_1 + \beta_2}} \) and \( \beta_2 = \beta_3 \). We start assuming that \( k_0 = 1 \). We can write
\[ \Delta_n > (\frac{1}{n})^{\frac{\alpha}{1 + \alpha}}, \]

with \( \alpha \) as in (23). From the bound on the variance gathered in Proposition 4.1 and the bias-variance decomposition easily follows the wanted result, acting as above.
When \( k_0 \geq 3 \) it is \( \beta_1 = \beta_2 \) and so \( \Delta_n > \left( \frac{1}{T_n} \right)^{2\beta_3 + d - 2} \left( \frac{1}{\beta_1} + \frac{1}{\beta_2} \right) = \left( \frac{1}{T_n} \right)^{2\tilde{\beta}_3}. \) As \( T_n = n\Delta_n \), the previous condition is equivalent to ask \( \Delta_n > \left( \frac{1}{n} \right)^{\alpha}, \) with

\[
\alpha = \frac{2\tilde{\beta}_3}{\beta_1 (2\beta_3 + d - 2)}.
\]

We underline that \( \alpha \) is exactly the same as in (23), having now \( \beta_1 = \beta_2 \). The previous remark, together with the second point of Proposition 4.1 leads to the following bound for the mean squared error, for \( k_0 \geq 3 \):

\[
\mathbb{E} |\hat{\pi}_{h,n}(x) - \pi(x)|^2 \leq c \sum_{j=1}^{d} h_j^{2\beta_j} + \frac{c}{n} \left( \frac{1}{\prod_{l=1}^{d} h_l} \right)^{1-\frac{2}{k_0}} \prod_{l=k_0+1}^{d} h_l + \frac{c}{n} \left( \frac{\Delta_n}{\prod_{l=1}^{d} h_l} \right)^{1-\frac{2}{k_0}} \prod_{l=k_0+1}^{d} h_l.
\]

As before, we choose the rate optimal bandwidth as \( h_l(n) := \left( \frac{1}{n} \right)^{\frac{\beta}{\beta_l(2\beta_3 + d)}}. \) When \( \beta_1 = \cdots = \beta_{k_0} \) it follows in particular that \( h_1(n) = \cdots = h_{k_0}(n). \) Replacing the value of \( h_l(n) \) in the bound of the mean squared error we get

\[
\mathbb{E} |\hat{\pi}_{h,n}(x) - \pi(x)|^2 \leq c \left( \frac{1}{n} \right)^{\frac{2\tilde{\beta}}{2\tilde{\beta} + d}} + c \left( \frac{1}{n} \right)^{1-\frac{\alpha}{1-\alpha}} \left( \frac{n^{\tilde{\beta}} (k_0-2)}{\beta_1^{\tilde{\beta}}} \right) + c \left( \frac{1}{n} \right)^{1-\frac{\beta}{2\tilde{\beta} + d}} \left( \frac{d}{\tilde{\beta}} \right),
\]

recalling that \( \tilde{\beta}_k \) is the mean smoothness over \( \beta_{k_0+1}, \ldots, \beta_d \) and it is such that \( \frac{1}{\tilde{\beta}_k} = \frac{k_0 - 2}{\beta_1} + \frac{d - k_0}{\beta_k} = \frac{d - 2}{\beta_3}. \)

We remark that

\[
\frac{k_0 - 2}{\beta_1} + \frac{d - k_0}{\beta_k} = \frac{d - 2}{\beta_3}.
\]

Then, using also (25), we have that the exponent of \( \frac{1}{n} \) in the second term here above is

\[
\frac{1}{1 + \alpha} - \frac{\tilde{\beta}}{2\tilde{\beta} + d} \left( \frac{d - 2}{\tilde{\beta}} \right) = \frac{2\tilde{\beta}}{2\tilde{\beta} + d}.
\]

Remarking that the constant \( c \) does not depend on \((a, b) \in \Sigma\), it follows

\[
\sup_{(a,b)\in\Sigma} \mathbb{E} |\hat{\pi}_{h,n}(x) - \pi(x)|^2 \leq c \left( \frac{1}{n} \right)^{\frac{2\tilde{\beta}}{2\tilde{\beta} + d}},
\]

as we wanted.
6.4. Proof of Proposition 4.2

Proof: The proof follows the procedure to bound the variance for $d = 2$ proposed in the continuous case (see Theorem 2 of [47]). We split the sum in three terms:

$$
\text{Var}(\hat{\pi}_{h,n}(x)) = \text{Var} \left( \sum_{j=0}^{n-1} \mathbb{K}_h(x - X_{t_j}) \Delta_n \right)
= \Delta_n^2 \left( \sum_{j=0}^{j_6} + \sum_{j=j_6+1}^{j_D} + \sum_{j=j_D+1}^{n-1} \right) (n-j) k(t_j) =: \tilde{I}_1 + \tilde{I}_2 + \tilde{I}_3.
$$

Repeating the arguments given for the bounds on $\tilde{I}_1$ and $\tilde{I}_4$ (defined in the proof of Proposition 4.1) we find the following bounds for $\tilde{I}_1$ and $\tilde{I}_3$, respectively.

$$
|\tilde{I}_1| \leq c \frac{1}{T_n h_1 h_2} (\delta + \Delta_n),
|\tilde{I}_3| \leq c \frac{1}{T_n (h_1 h_2)^2} e^{-\rho D}.
$$

Regarding $\tilde{I}_2$, we act here as we did on $I_3$ in Proposition 4.1. Recalling that here $d = 2$ we get

$$
|k(t_j)| \leq c (t_j^{-\frac{d}{2}} + 1) \leq c \left( \frac{1}{t_j} + 1 \right).
$$

We need to consider separately what happens when $t_j$ is larger or smaller than 1.

$$
|\tilde{I}_2| \leq c \frac{\Delta_n}{T_n} \sum_{j=j_6+1}^{j_D} \left( \frac{1}{t_j} + 1 \right)
\leq c \frac{\Delta_n}{T_n} \left( \sum_{t_j \leq 1, j=j_6+1}^{j_D} \frac{1}{t_j} + \sum_{t_j > 1, j=j_6+1}^{j_D} 1 \right)
\leq c \frac{1}{T_n} (\log D + \log \delta + D).
$$

Putting all the pieces together, one can see that the choice $j_6 := \lceil \frac{\log h_2}{\Delta_n} \rceil$ and $j_D := \lceil \text{max}(\frac{\log (h_1 h_2) + 1}{T_n}) \rceil$ leads to the wanted result. 

6.5. Proof of Theorem 4.3

Proof: The scheme we follow to prove Theorem 4.3 is the one provided in the proof of Theorem 4.1. We start considering the case where

$$
\Delta_n \leq h_1^* h_2^* \sum_{j=1}^{2} | \log h_j^* | = \left( \frac{\log T_n}{T_n} \right)^{\frac{1}{2p_1} + \frac{1}{2p_2}} \log T_n = \left( \frac{\log T_n}{T_n} \right)^{\frac{1}{p}} \log T_n,
$$
where we have used that, from the proof of Theorem 2 of [47], \( h_i^n(T_n) = \left( \frac{\log T_n}{T_n} \right)^{a_i} \) with \( a_i \geq \frac{1}{2\beta_i} \) and that, for \( d = 2 \), \( \frac{1}{\beta_1} + \frac{1}{\beta_2} = \frac{1}{\beta} \).

From the bias variance decomposition together with Proposition 4.2, taking the rate optimal choice \( h_i^n(T_n) \) as above directly follows

\[
\mathbb{E}[|\hat{\pi}_{h,n}(x) - \pi(x)|^2] \leq c \frac{\log T_n}{T_n} + c \frac{\log(\log T_n)}{T_n} = c \frac{\log T_n}{T_n}.
\]

If \( \Delta_n > \left( \frac{\log T_n}{T_n} \right)^{\frac{\beta}{\beta + d}} \log T_n \), instead, it is also \( \Delta_n > \left( \frac{1}{n} \right)^{\frac{1}{\beta + 1}} (\log(n\Delta_n)) \). Using the bias variance decomposition and Proposition 4.2 we obtain

\[
\mathbb{E}[|\hat{\pi}_{h,n}(x) - \pi(x)|^2] \leq c(h_1^{2\beta_1} + h_2^{2\beta_2}) + c \frac{2}{T_n} \sum_{j=1}^2 |\log h_j| + c \frac{\Delta_n}{T_n h_1 h_2}
\]

\[
\leq c(h_1^{2\beta_1} + h_2^{2\beta_2}) + c \left( \frac{1}{n} \right)^{1 - \frac{1}{\beta + 1}} \sum_{j=1}^2 |\log h_j| + c \frac{\Delta_n}{n h_1 h_2}.
\]

We choose the rate optimal bandwidth \( h_l(n) := \left( \frac{1}{n} \right)^{\frac{\beta_l}{\beta_l(2\beta_l + d)}} \), for \( l = 1, 2 \). We have already discussed the behaviour of \( n\Delta_n \), saying in particular that \( \frac{\log n}{\log n\Delta_n} \leq c \) in the proof of Theorem 4.2. It yields

\[
\mathbb{E}[|\hat{\pi}_{h,n}(x) - \pi(x)|^2] \leq c \left( \frac{1}{n} \right)^{\frac{2\beta}{2\beta + d}} + c \left( \frac{1}{n} \right)^{1 - \frac{1}{\beta + 1}} + c \left( \frac{1}{n} \right)^{\frac{\beta}{\beta + 1}} \left( \frac{1}{\beta_1} + \frac{1}{\beta_2} \right)^{\frac{2\beta + d - 2}{2\beta + d}}
\]

which is what we wanted, as \( d = 2 \). \( \blacksquare \)

7. Proof main results, asynchronous framework

In Section 5, we assume to observe the components of the process \( X \) in different moments.

We start by proving the results gathered in Proposition 5.1. The proof of Proposition 5.4, which follows a similar route, can be found in the appendix.

7.1. Proof of Proposition 5.1

Proof: In analogy to the proofs in the synchronous case we introduce

\[
k(t, s) := \text{Cov} \left( \prod_{l=1}^d K_{h_i^n}(x_l - X_{\phi_{n,t}}^l), \prod_{l=1}^d K_{h_i^n}(x_l - X_{\phi_{n,s}}^l) \right),
\]

such that

\[
\text{Var} (\hat{\pi}_{h,n,T_n}(x)) = \frac{2}{T_n} \int_0^{T_n} \int_0^t k(t, s) 1_{s < t} \, ds \, dt.
\]
We write
\[
\text{Var}(\hat{\pi}_{h_n}^T(x)) = \frac{2}{T_n^2} \int_0^T \int_0^t k(t,s)1_{s \leq t} \left( 1_{|t-s| \leq h_n^* h_n^*} + 1_{h_n^* h_n^* \leq |t-s| \leq (\prod_{j \geq 3} h_j^*)^{\frac{2}{d-2}} + 1_{(\prod_{j \geq 3} h_j^*)^{\frac{2}{d-2}} \leq |t-s| \leq D} + 1_{D \leq |t-s| \leq T_n} \right) d\sigma \, ds
\]
\[
= \sum_{j=1}^4 \bar{I}_j
\]
with \( h^* \) the rate optimal choice of the bandwidth given by (6) and \( D \) will be specified latter. Regarding \( \bar{I}_1 \), acting as in Proposition 4.1 in order to get (14), we have
\[
|k(t,s)| \leq \mathbb{E} \left( \prod_{l=1}^d K_{h_l^*}^2(x_l - X_{\varphi_{\mathcal{N}},l(t)}) \right)^{1/2} \mathbb{E} \left( \prod_{l=1}^d K_{h_l^*}^2(x_l - X_{\varphi_{\mathcal{N}},l(s)}) \right)^{1/2}.
\]
We now state a lemma which will be useful in the sequel and whose proof is postponed to the appendix.

**Lemma 7.1:**

1. We have \( \mathbb{E}(\prod_{l=1}^d |K_{h_l}(x_l - X_{\varphi_{\mathcal{N}},l(t)})|) \leq c \), for some constant \( c \) independent of \( t \) and \( (h_l)_{l=1,...,d} \).

2. We have \( \mathbb{E}(\prod_{l=1}^d K_{h_l}^2(x_l - X_{\varphi_{\mathcal{N}},l(t)}) \leq \frac{c}{\prod_{l=1}^d h_l} \), for some constant \( c \) independent of \( t \) and \( (h_l)_{l=1,...,d} \).

From application of the second point of Lemma 7.1, we obtain \( |k(t,s)| \leq \frac{c}{\prod_{l=1}^d h_l} \).

Therefore, after the change of variable \( t \to t' := t - s \), we have
\[
\bar{I}_1 \leq \frac{c}{T_n^2} \int_0^T \int_0^t h_n^* h_n^* \sum_{j=1}^d |\log h_j^*| \frac{1}{\prod_{l=1}^d h_l^*} \, dt' \, ds = \frac{c}{T_n} \sum_{j=1}^d \frac{|\log h_j^*|}{\prod_{l=3}^d h_l^*}, \tag{26}
\]
where we recall that the constant \( c \) may change from line to line. We now study \( \bar{I}_2 \), which is the most complicated term we have to deal with. Intuitively, we would like to bound the variance as in the interval \([\delta_1, \delta_2]\) in the proof of Proposition 2 of [47]. It relies on a bound for the transition density (as in Proposition 5.1 of [53] or Lemma 1 of [47]). In order to use it we need to know the ordering between the quantities \( s, t, \varphi_{n,1}(s),..., \varphi_{n,d}(s), \varphi_{n,1}(t),..., \varphi_{n,d}(t) \). We know that \( s < t, \varphi_{n,l}(s) \leq \varphi_{n,1}(t) \), and \( \varphi_{n,1}(t) \leq s \) for any \( l \in \{1,...,d\} \). Then, we can consider a permutation \( w_1,..., w_d \) of \( \varphi_{n,1}(s),..., \varphi_{n,d}(s) \) which is such that \( w_1 \leq \cdots \leq w_d \). In particular, we denote by \( w_1 \leq \cdots \leq w_d \) a reordering of \( \varphi_{n,l}(s) \) and \( \sigma \) an element of the permutation group on \( \{1,...,d\} \) such that \( w_i = \varphi_{n,\sigma(i)}(s) \) for all \( i \in \{1,...,d\} \). In the same way we introduce the permutation \( \tilde{w}_1,.., \tilde{w}_d \) of \( \varphi_{n,1}(t),..., \varphi_{n,d}(t) \) which is properly ordered, i.e., \( \tilde{w}_1 \leq \cdots \leq \tilde{w}_d \). In particular, we introduce \( \tilde{\sigma} \) which is an element of
the permutation group such that \( \tilde{w}_i = \varphi_{n, \tilde{\sigma}(i)}(t) \) for all \( i \in \{1, \ldots, d\} \). As \( \Delta_n \leq \frac{1}{4} h^*_1 h^*_2 \) and \( |t - s| > h^*_1 h^*_2 \), we have

\[
w_1 \leq \ldots \leq w_d \leq s \leq \tilde{w}_1 \leq \ldots \leq \tilde{w}_d \leq t.
\]

We also introduce the vectors, which represent the positions in the instants \( w_j \) and \( \tilde{w}_j \). At the instant \( w_j \) we have the vector \( \tilde{y}^j \), for \( j \in \{1, \ldots, d\} \), while the time \( \tilde{w}_j \) are associated to the vectors \( \tilde{y}^j \), for \( j \in \{1, \ldots, d\} \). We observe we can write

\[
|k(t, s)| \leq |\tilde{k}(t, s)| + \left| \mathbb{E} \left[ \prod_{l=1}^d K_{h^*_1}(x_l - X^l_{\varphi_{n,l}(t)}) \prod_{l=1}^d K_{h^*_2}(x_l - X^l_{\varphi_{n,l}(s)}) \right] \right|
\]

\[
\leq |\tilde{k}(t, s)| + c,
\]

where

\[
\tilde{k}(t, s) := \mathbb{E} \left[ \prod_{l=1}^d K_{h^*_1}(x_l - X^l_{\varphi_{n,l}(s)}) \prod_{l=1}^d K_{h^*_2}(x_l - X^l_{\varphi_{n,l}(t)}) \right]
\]

\[
= \mathbb{E} \left[ \prod_{l=1}^d K_{h^*_1}(x_{\sigma(l)} - X^{\sigma(l)}_{w_l}) \prod_{l=1}^d K_{h^*_2}(x_{\sigma(l)} - X^{\tilde{\sigma}(l)}_{\tilde{w}_l}) \right]
\]

and in the equation above we have used the first point of Lemma 7.1. We write the expectation in the definition of \( \tilde{k}(t, s) \) using the law of the random vector \( (X_{w_1}, \ldots, X_{w_d}, X_{\tilde{w}_1}, \ldots, X_{\tilde{w}_d}) \). For simplicity, we assume that all the instants appearing in this vector are different \( w_1 < \ldots < w_d < \tilde{w}_1 < \ldots < \tilde{w}_d \) and in turn the law of this vector admits a density as product of the transition density of the process \( X \). If we are not in the situation where all the instants are distinct, it is possible to slightly move some values of \( w_1, \ldots, w_d, \tilde{w}_1, \ldots, \tilde{w}_d \) in order to get different instants, and then conclude by a density argument in order to get the upper bound on \( |\tilde{k}(s, t)| \). With these considerations, we can write

\[
|\tilde{k}(t, s)| \leq \int_{\mathbb{R}^d} \prod_{l=1}^d \left| K_{h^*_1}(x_{\sigma(l)} - Y^j_{\sigma(l)}) \right| \times \\prod_{l=1}^d \left| K_{h^*_2}(x_{\sigma(l)} - Y^{\tilde{\sigma}(l)}_{\tilde{w}_l}) \right| \times \ldots \times \left| p_{\tilde{w}_d - w_{d-1}}(y^{d-1}, y^d) p_{\tilde{w}_{d-1} - \ldots - w_1}(y^1, \ldots, y^{d-1}) \times \ldots \right| \times \left| \right.
\]

\[
\left. p_{\tilde{w}_d - w_{d-1}}(\tilde{y}^{d-1}, \tilde{y}^d) \right| \right| \text{dy}^1 \, \text{dy}^2 \cdot \ldots \cdot \text{dy}^d \, \text{dy}^{d-1} \ldots \text{dy}^1,
\]

(27)

where we denote \( Y^j_m \) the \( m \)th component of the vector \( Y^j \in \mathbb{R}^d \). As it is important to remove the contribution of the two smallest bandwidths, we need to reorder the components to \( \tilde{y} \).
To do that we introduce $\tilde{\sigma}^{-1}$, the inverse of the permutation $\tilde{\sigma}$, and we write

$$|\tilde{k}(t, s)| \leq \int_{\mathbb{R}^d} \prod_{l=1}^{d} |K_{h_l^*(i)}(x_{\sigma(l)} - y_{\sigma(l)}^l)|$$

$$\times \int_{\mathbb{R}^d} \prod_{l=1}^{d} |K_{h_l^*(i)}(x_l - y_l)^{-1}(l)| p_{w_2-w_1}(y^1, y^2) p_{w_3-w_2}(y^2, y^3)$$

$$\times \cdots \times p_{w_d-w_{d-1}}(y^{d-1}, y^d) p_{\bar{w}_1-w_d}(y^d, y^1) \times \cdots$$

$$\times p_{\bar{w}_d-w_{d-1}}(\tilde{y}^{d-1}, \tilde{y}^d) \pi(y^1) dy^1 dy^2 \cdots dy^d d\tilde{y}^{d-1} \cdots d\tilde{y}^d.$$

We use the upper bound $\prod_{l=3}^{d} |K_{h_l^*(i)}(x_l - y_l)^{-1}(l)| \leq C/ \prod_{l=3}^{d} h_l^*$ and we integrate with respect to the variables $\tilde{y}_l^{\tilde{\sigma}^{-1}(l)}$, $l = 3, \ldots, d$ to get

$$|\tilde{k}(t, s)| \leq \frac{C}{\prod_{l=3}^{d} h_l^*} \int_{\mathbb{R}^d} \prod_{l=1}^{d} |K_{h_l^*(i)}(x_{\sigma(l)} - y_{\sigma(l)}^l)|$$

$$\times \int_{\mathbb{R}^d} \left|K_{h_l^*(i)}(x_1 - y_1^{\tilde{\sigma}^{-1}(1)}) K_{h_2^*(i)}(x_2 - y_2^{\tilde{\sigma}^{-1}(2)})\right|$$

$$\times p_{w_2-w_1}(y^1, y^2) p_{w_3-w_2}(y^2, y^3) \times \cdots \times p_{w_d-w_{d-1}}(y^{d-1}, y^d)$$

$$\times p_{\bar{w}_1-w_d}(y^d, \tilde{y}^d) p_{\bar{w}_1-w_d}(\tilde{y}^d, \tilde{y}^d) \pi(y^1) dy^1 dy^2 \cdots dy^d d\tilde{y}^{d-1} \cdots d\tilde{y}^d,$$

where $i_* = \min(\tilde{\sigma}^{-1}(1), \tilde{\sigma}^{-1}(2))$ and $i^* = \max(\tilde{\sigma}^{-1}(1), \tilde{\sigma}^{-1}(2))$. Using a Gaussian bound on the transition density

$$p_{\bar{w}_1-w_d}(y^d, \tilde{y}^d) p_{\bar{w}_1-w_d}(\tilde{y}^d, \tilde{y}^d)$$

$$\leq \frac{C}{\tilde{w}_d} q\left((\tilde{y}_j^{\tilde{\sigma}^{-1}(1)})_{j\neq 1}, (\tilde{y}_j^{\tilde{\sigma}^{-1}(1)})_{j\neq 2} \mid y^d, \tilde{y}_1^{\tilde{\sigma}^{-1}(1)}, \tilde{y}_2^{\tilde{\sigma}^{-1}(2)}\right)$$

where

$$q\left((\tilde{y}_j^{\tilde{\sigma}^{-1}(1)})_{j\neq 1}, (\tilde{y}_j^{\tilde{\sigma}^{-1}(1)})_{j\neq 2} \mid y^d, \tilde{y}_1^{\tilde{\sigma}^{-1}(1)}, \tilde{y}_2^{\tilde{\sigma}^{-1}(2)}\right)$$

$$= \sqrt{\tilde{w}_d - w_d} \prod_{j=1}^{d} \frac{e^{-c \frac{(y_j^d - \tilde{y}_j^d)^2}{\tilde{w}_d - w_d}}}{\sqrt{\tilde{w}_d - w_d}} \prod_{j=1}^{d} \frac{e^{-c \frac{(y_j^d - \tilde{y}_j^d)^2}{\tilde{w}_d - w_d}}}{\sqrt{\tilde{w}_d - w_d}}.$$

We now prove that

$$\sup_{(y^d, \tilde{y}_1^{\tilde{\sigma}^{-1}(1)}, \tilde{y}_2^{\tilde{\sigma}^{-1}(2)}) \in \mathbb{R}^{d+2}} \int_{\mathbb{R}^{2d-1}} q\left((\tilde{y}_j^{\tilde{\sigma}^{-1}(1)})_{j\neq 1}, (\tilde{y}_j^{\tilde{\sigma}^{-1}(1)})_{j\neq 2} \mid y^d, \tilde{y}_1^{\tilde{\sigma}^{-1}(1)}, \tilde{y}_2^{\tilde{\sigma}^{-1}(2)}\right)$$

$$\prod_{j=2}^{d} d\tilde{y}_j^{\tilde{\sigma}^{-1}(1)} \prod_{j=2}^{d} d\tilde{y}_j^{\tilde{\sigma}^{-1}(2)} \leq C.$$ (29)
To prove (29), assume, in order the simplify the notations, that $i_\ast = \tilde{\sigma}^{-1}(1)$ and $i^* = \tilde{\sigma}^{-2}(2)$, as the other case can be proved symmetrically. Then, the integral in the left-hand side of (29) is

$$
\int_{R^{2(d-1)}} \sqrt{\tilde{w}_{\tilde{\sigma}}^{-1}(1) - w_d} \prod_{j=2}^{d} e^{-\frac{c\tilde{v}_{j}^{-1}(1) - \tilde{v}_{j}^{-1}(2)}{w_{\tilde{\sigma}}^{-1}(1) - w_d}} \prod_{j=1}^{d} e^{-\frac{c\tilde{v}_{j}^{-1}(1) - \tilde{v}_{j}^{-1}(2)}{w_{\tilde{\sigma}}^{-1}(2) - w_{\tilde{\sigma}}^{-1}(1)}} \times \prod_{j=2}^{d} d(\tilde{v}_{j}^{-1}(1)) \prod_{j=1}^{d} d(\tilde{v}_{j}^{-1}(2)).
$$

Integrating with respect to the measures $\prod_{j=3}^{d} d(\tilde{v}_{j}^{-1}(1)) \prod_{j=3}^{d} d(\tilde{v}_{j}^{-1}(2))$, we get that the last integral is upper bounded by

$$
\int_{R^{2}} \sqrt{\tilde{w}_{\tilde{\sigma}}^{-1}(1) - w_d} e^{-\frac{c\tilde{v}_{j}^{-1}(1) - \tilde{v}_{j}^{-1}(2)}{w_{\tilde{\sigma}}^{-1}(1) - w_d}} \prod_{j=1}^{2} e^{-\frac{c\tilde{v}_{j}^{-1}(1) - \tilde{v}_{j}^{-1}(2)}{w_{\tilde{\sigma}}^{-1}(2) - w_{\tilde{\sigma}}^{-1}(1)}} d\tilde{v}_{j}^{-1}(1) d\tilde{v}_{j}^{-1}(2).
$$

Then, integrating with respect to $d\tilde{v}_{2}^{-1}(1)$, the convolution of Gaussian kernels yields to the following upper bound for the LHS of (29),

$$
\int_{R} \sqrt{\tilde{w}_{\tilde{\sigma}}^{-1}(1) - w_d} e^{-\frac{c\tilde{v}_{1}^{-1}(1) - \tilde{v}_{1}^{-1}(2)}{w_{\tilde{\sigma}}^{-1}(1) - w_d}} e^{-\frac{c\tilde{v}_{1}^{-1}(1) - \tilde{v}_{1}^{-1}(2)}{w_{\tilde{\sigma}}^{-1}(2) - w_{\tilde{\sigma}}^{-1}(1)}} d\tilde{v}_{1}^{-1}(2).
$$

Using that $\sqrt{\tilde{w}_{\tilde{\sigma}}^{-1}(1) - w_d} \leq \sqrt{\tilde{w}_{\tilde{\sigma}}^{-1}(2) - w_d}$ and that the first exponential inside the integral above is smaller than 1, we deduce that (29) holds true. Using (28)–(29) we deduce

$$
|\tilde{k}(t,s)| \leq \frac{C}{\prod_{l=3}^{d} \tilde{w}_{l} - w_d} \int_{R^{d}} \prod_{l=1}^{d} |K_{l}^{*}(x_{l} - y_{l}^{1})| \times \int_{R^{2}} \left|K_{l}^{*}(x_{1} - \tilde{y}_{1}^{-1}(1))K_{l}^{*}(x_{2} - \tilde{y}_{2}^{-1}(2))\right| \times p_{w_{2} - w_{1}}(y^{1}, y^{2})p_{w_{3} - w_{2}}(y^{2}, y^{3}) \times \cdots \times p_{w_{d} - w_{d-1}}(y^{d-1}, y^{d})\pi(y^{1}) d^{d} \cdots d^{d} dy^{1} d^{2} \cdots dy^{d} d\tilde{y}_{1}^{-1}(1) d\tilde{y}_{2}^{-1}(2).
$$

We now state a lemma which will be useful in the sequel. Its proof can be found in the appendix.

**Lemma 7.2:** (1) Let $r \geq 1$ be an integer and $q_1, \ldots, q_r \in \{1, \ldots, d\}$. Then, we have

$$
\int_{R^{d}} \prod_{i=1}^{d} |K_{i}(x_{i} - u_{i}^{q_{i}})| \prod_{j=1}^{r-1} p_{w_{j+1} - w_{j}}(u^{j}, u^{j+1}) du^{1} \cdots du^{r} \leq C,
$$

for some constant $C$ independent of $(h_{i})_{i}$ and $(w_{j})_{j}$.
The same upper bound holds true if we replace the transition densities \( p_{w_{j+1} - w_j} \) by Gaussian kernels \( g(w_{j+1} - w_j) \).

Using that \( \pi \) is bounded, and the first point of Lemma 7.2 with \( r = d \), \( q_i = \tilde{\sigma}^{-1}(i) \) for \( i \in \{1, \ldots, d\} \) we deduce

\[
|k(t, s)| \leq \frac{c}{\tilde{w}_1 - w_d} \prod_{l \geq 3} h_l^* \int_{\mathbb{R}^2} \left| K_{h_1^*}^* (x_1 - y_1^*(1)) K_{h_2^*}^* (x_2 - y_2^{*-1}(2)) \right| dy_1^{*-1}(1) dy_2^{*-1}(2)
\]

\[
\leq \frac{c}{\tilde{w}_1 - w_d} \prod_{l \geq 3} h_l^* \leq \frac{c}{\tilde{w}_1 - w_d} \prod_{l \geq 3} h_l^*
\]

which implies

\[
|k(t, s)| \leq \frac{c}{\tilde{w}_1 - w_d} \prod_{l \geq 3} h_l^* + c.
\]

In order to bound \( \tilde{I}_2 \) we also observe that

\[
|t - s| \leq |t - \tilde{w}_1| + |\tilde{w}_1 - w_d| + |w_d - s|
\]

\[
\leq |\tilde{w}_1 - w_d| + 2\Delta_n
\]

\[
\leq |\tilde{w}_1 - w_d| + \frac{1}{2} h_1^* h_2^*.
\]

It implies

\[
1 \prod_{l \geq 3} h_l^* \leq |t - s| - \frac{1}{2} h_1^* h_2^*.
\]

As a consequence

\[
|\tilde{I}_2| \leq \frac{c}{T_n^2} \int_0^T \int_0^t \int_{t - s}^{t} \prod_{l \geq 3} h_l^* \frac{1}{|t - s| - \frac{1}{2} h_1^* h_2^*} \prod_{l \geq 3} h_l^* 1_{t - s| \leq (\prod_{l \geq 3} h_l^*)^{2} ds dt.
\]

By the change of variable \( t - s =: t' \), we obtain

\[
|\tilde{I}_2| \leq \frac{c}{T_n} \int_{h_1^* h_2^*}^{(\prod_{l \geq 3} h_l^*)^{\frac{3}{2}}} \frac{c}{t' - \frac{1}{2} h_1^* h_2^* \prod_{l \geq 3} h_l^*} dt'
\]

\[
= \frac{c}{T_n} \sum_{j=1}^2 |\log h_j^*| \prod_{l \geq 3} h_l^*.
\]

Regarding \( \tilde{I}_3 \), it is

\[
\tilde{I}_3 := \frac{1}{T_n^2} \int_0^T \int_0^t k(t, s) 1_{|t - s| \leq (\prod_{l \geq 3} h_l^*)^{\frac{3}{2}} \leq |t - s| \leq D} ds dt.
\]

We can write \( \tilde{k}(t, s) \) as in (27). We use the rough estimation

\[
p_{\tilde{w}_1 - w_d}(y^1, \tilde{y}^1) \leq \frac{c}{(\tilde{w}_1 - w_d)^{\frac{3}{2}}}.
\]
We replace it in (27), it follows

\[
|\tilde{k}(t, s)| \leq \frac{c}{(\tilde{w}_1 - w_d)} \int_{\mathbb{R}^d} \prod_{l=1}^{d} \left| K_{h_{\sigma(l)}}^*(x_{\sigma(l)} - y_l^j) \right| \times \int_{\mathbb{R}^d} \prod_{l=1}^{d} \left| K_{h_{\tilde{\sigma}(l)}}^*(x_{\tilde{\sigma}(l)} - \tilde{y}_l^j) \right| \left| \prod_{l=1}^{d-1} p_{w_{l+1} - w_l}(y_l^j, y_l^{j+1}) \right|
\]

\[
\times \prod_{l=1}^{d-1} p_{\tilde{w}_{l+1} - \tilde{w}_l}(\tilde{y}_l^j, \tilde{y}_l^{j+1}) \pi(y_1^j) \, dy_1^j \, dy_2^j \ldots dy_d \, d\tilde{y}_1^j \ldots d\tilde{y}_d.
\]

We now apply twice the first point of Lemma 7.2, having on each integral \( r = d \). It provides

\[
|k(t, s)| \leq \frac{c}{(\tilde{w}_1 - w_d)} + c. \quad (31)
\]

We now observe it is

\[
|t - s| \leq |t - \tilde{w}_1| + |\tilde{w}_1 - w_d| + |w_d - s| \leq |\tilde{w}_1 - w_d| + 2\Delta_n.
\]

Hence,

\[
|\tilde{w}_1 - w_d| \geq |t - s| - 2\Delta_n \geq |t - s| - \frac{1}{2} h_1^* h_2^* \geq \frac{1}{2} |t - s|.
\]

From the change of coordinates \( t' := t - s \) we obtain

\[
\tilde{I}_3 \leq \frac{c}{T_n} \int_0^{T_n} \int_{[\prod_{l \geq 3} h_l^*]} (\frac{c}{t^{\frac{d}{2}}} + c) \, dt' \, ds \leq \frac{c}{T_n} \left( \prod_{l \geq 3} h_l^* \right) \left( \frac{2}{d + 2} (1 - \frac{d}{2}) + \frac{1}{2} \right) = \frac{c}{T_n} \left( \prod_{l \geq 3} h_l^* \right)^{-1} + D,
\]

which is the order we wanted.

We are left to study \( \tilde{I}_4 \), which is the case where \( D \leq |t - s| \leq T_n \). Here we want to use the fact that the process is exponential \( \beta \)-mixing. To do that, we use the definition of the covariance. We introduce the notation \( K_{h_l^*}(t) := K_{h_l^*}(x - X_t) \). Then we need to study, up to reorder the components,

\[
\text{Cov}(K_{h_1^*}(w_1) \cdots K_{h_d^*}(w_d), K_{h_1^*}(\tilde{w}_1) \cdots K_{h_d^*}(\tilde{w}_d)),
\]

where \( w_1 \leq \cdots \leq w_d \leq s \leq \tilde{w}_1 \leq \cdots \leq \tilde{w}_d \leq t, D \leq |t - s| \leq T \). We define

\[
g(X_{\tilde{w}_1}) := \mathbb{E}[K_{h_1^*}(\tilde{w}_1) \cdots K_{h_d^*}(\tilde{w}_d) | X_{\tilde{w}_1}].
\]
It follows we can write the covariance as
\[
\mathbb{E}[K_{h_1^*}(w_1) \cdots K_{h_d^*}(w_d)K_{h_1^*}(\tilde{w}_1) \cdots K_{h_d^*}(\tilde{w}_d)] + \\
- \mathbb{E}[K_{h_1^*}(w_1) \cdots K_{h_d^*}(w_d)]\mathbb{E}[K_{h_1^*}(\tilde{w}_1) \cdots K_{h_d^*}(\tilde{w}_d)] \\
= \mathbb{E}[K_{h_1^*}(w_1) \cdots K_{h_d^*}(w_d)g(X_{\tilde{w}_1})] - \mathbb{E}[K_{h_1^*}(\tilde{w}_1) \cdots K_{h_d^*}(w_d)]\mathbb{E}[g(X_{\tilde{w}_1})] \\
= \mathbb{E}[K_{h_1^*}(w_1) \cdots K_{h_d^*}(w_d)(g(X_{\tilde{w}_1}) - \pi(g))] \\
= \mathbb{E}[K_{h_1^*}(w_1) \cdots K_{h_d^*}(w_d)(P_{\tilde{w}_1-w_d}g(X_{\tilde{w}_1}) - \pi(g))]
\]
where we recall that \( P_t f(x) := \mathbb{E}[f(X_t) | X_0 = x] = \int_{\mathbb{R}^d} f(y) p_t(x,y) \, dy \) is the transition semigroup of the process \( X \). From Lemma 7 of [47] we get
\[
|k(t,s)| \leq \prod_{l=1}^{d} \left\| K_{h_l^*} \right\|_{\infty} \left\| P_{\tilde{w}_1-w_d}g(X_{\tilde{w}_1}) - \pi(g) \right\|_{L^1} \\
\leq \frac{c}{\prod_{l=1}^{d} h_l^*} e^{-\rho(\tilde{w}_1-w_d)} \left\| g \right\|_{\infty} \leq \frac{c}{(\prod_{l=1}^{d} h_l^*)^2} e^{-\rho(\tilde{w}_1-w_d)},
\]
As before, it is clearly \(|\tilde{w}_1 - w_d| \geq |t-s| - 2\Delta_n\). Hence,
\[
\overline{t}_4 \leq \frac{c}{(\prod_{l=1}^{d} h_l^*)^2} \frac{1}{T_n^2} \int_{0}^{T_n} \int_{D}^{T_n} e^{-\rho s'} e^{\rho |\Delta_n|} \, dt \, ds' \\
\leq \frac{c}{T_n \left( \prod_{l=1}^{d} h_l^* \right)^2} e^{-\rho D}
\]
Putting all the pieces together, it yields
\[
\text{Var}(\hat{\pi}_{h_*}^a, T_n(x)) \leq \frac{c}{T_n} \sum_{j=1}^{d} \left| \log h_j^* \right| \prod_{l=3}^{d} h_l^* + \frac{c}{T_n} \sum_{j=1}^{d} \left| \log h_j^* \right| \prod_{l=3}^{d} h_l^* \\
+ \frac{c}{T_n} \frac{1}{\prod_{l=3}^{d} h_l^*} + \frac{D}{T_n} + \frac{c}{T_n \left( \prod_{l=1}^{d} h_l^* \right)^2} e^{-\rho D}.
\]
By choosing \( D := [\max(-\frac{2}{\rho} \log(\prod_{l=1}^{d} h_l^*), 1) \wedge T_n] \) we obtain the wanted result. 

### 7.2. Proof of Proposition 5.2

**Proof:** Following the route given by the previous proof we split the variance.
\[
\text{Var}(\hat{\pi}_{h_*}^a, T_n(x)) = \frac{2}{T_n^2} \int_{0}^{T_n} \int_{0}^{T_n} k(t,s) 1_{s< t} \left( (1_{|t-s| \leq 3\Delta_n} + 1_{3\Delta_n \leq |t-s| \leq D} + 1_{D \leq |t-s| \leq T_n}) \right) \, ds \, dt \\
= \sum_{j=1}^{3} \overline{t}_j.
\]
We act as on \( \tilde{I}_1 \) in the proof of Proposition 5.1, obtaining \( |k(t, s)| \leq \frac{c}{\prod_{j=1}^{d} \tilde{h}_j} \) and so \( \tilde{I}_1 \leq \frac{c}{T_n} \sum_{j=1}^{d} \Delta_n \). We bound then \( \tilde{I}_2 \) as \( \tilde{I}_3 \) in the proof of Proposition 5.1. We observe that, in this case, we have \( |\tilde{w}_1 - w_d| \geq |t - s| - 2\Delta_n \). As \( |t - s| \geq 3\Delta_n \), this implies \( |\tilde{w}_1 - w_d| \geq \frac{1}{3} |t - s| \). The change of coordinates \( t' := t - s \), together with (31) yields

\[
\tilde{I}_2 \leq \frac{c}{T_n} \int_0^T \int_{\Delta_n} D \left( \frac{c}{t'} + c \right) dt' ds \leq \frac{c}{T_n} \left( \Delta_n^{1 - \frac{d}{2}} + D \right).
\]

We act then on \( \tilde{I}_3 \) as for \( \tilde{I}_4 \) above. It follows

\[
\tilde{I}_3 \leq \frac{c}{T_n} \left( \prod_{j=1}^{d} \tilde{h}_j \right)^{\frac{1}{2}} e^{-\rho D}.
\]

We choose \( D := \left[ \max \left( -\frac{2}{\rho} \log(\prod_{j=1}^{d} \tilde{h}_j \Delta_n), 1 \right) \right] \). The proof is concluded once one observe that, as \( \Delta_n \geq \left( \prod_{j=1}^{d} \tilde{h}_j \right)^{\frac{1}{3}} \), it is \( \Delta_n^{1 - \frac{d}{2}} \leq \frac{\Delta_n}{\prod_{j=1}^{d} \tilde{h}_j} \).

### 7.3. Proof of Proposition 5.3

**Proof:** From the expression of \( \hat{x}_{h, T_n}^a(x) \) given in Section 4 we have

\[
\mathbb{E}[\hat{x}_{h, T_n}^a(x)] = \frac{1}{T_n} \int_0^T \mathbb{E} \left[ \prod_{i=1}^{d} K_{h_i}(x_{i} - X_{\varphi_{n,l}(t)}) \right] dt.
\] (32)

Hence, we focus on \( \mathbb{E}[\prod_{i=1}^{d} K_{h_i}(x_i - X_{\varphi_{n,l}(t)})] \) for \( t \in [0, T_n] \). We denote by \( w_1 \leq w_2 \leq \cdots \leq w_d \) a reordering of \( (\varphi_{n,l}(t))_{l=1, \ldots, d} \), and let \( \sigma \) an element of the permutation group such that \( w_i = \varphi_{n,\sigma(i)}(t) \) for all \( i \in \{1, \ldots, d\} \). With this notations, \( \mathbb{E}[\prod_{i=1}^{d} K_{h_i}(x_i - X_{\varphi_{n,l}(t)})] = \mathbb{E}[\prod_{i=1}^{d} K_{h_i}(x_{\sigma(i)} - X_{w_i}(i))] \) and we now show the following control

\[
\left| \mathbb{E} \left[ \prod_{i=1}^{d} K_{h_{\sigma(i)}}(x_{\sigma(i)} - X_{w_i}(i)) \right] - \int_{\mathbb{R}^d} \pi(y) \prod_{i=1}^{d} K_{h_i}(x_i - z_i) dz_1 \ldots dz_d \right| \leq c\sqrt{\Delta_n},
\] (33)

for some constant \( c \) independent of \( (w_i)_i \) and \( (h_i)_i \), and where \( \Delta_n \) is defined in (12).

In the proof of (33) we can assume by a density argument that \( w_i < w_{i+1} \) for \( i = 1, \ldots, d - 1 \). In this case, we write

\[
\mathbb{E} \left[ \prod_{i=1}^{d} K_{h_{\sigma(i)}}(x_{\sigma(i)} - X_{w_i}(i)) \right] = \int_{\mathbb{R}^d} \pi(y) \prod_{i=1}^{d} K_{h_{\sigma(i)}}(x_{\sigma(i)} - y_{\sigma(i)}) \prod_{i=1}^{d} p_{w_{i+1} - w_i}(y_i, y_i + 1) dy_1 \ldots dy_d
\]

\[
= \int_{\mathbb{R}^d} \left( \prod_{i=1}^{d} K_{h_{\sigma(i)}}(x_{\sigma(i)} - y_{\sigma(i)}) \right) \xi(w_i, \pi(y_1^{\sigma(1)}, \ldots, y_d^{\sigma(d)})) dy_1^{\sigma(1)} dy_2^{\sigma(2)} \ldots dy_d^{\sigma(d)},
\] (34)
where for any function \( \phi \) we have set

\[
\xi(w_i, \phi(y^{1}_{\sigma(1)}, \ldots, y^{d}_{\sigma(d)}) = \int_{\mathbb{R}^{d(d-1)}} \phi(y^1) \prod_{i=1}^{d-1} p_{w_{i+1} - w_i}(y^i, y^{i+1}) \, dy^1 \ldots dy^d,
\]

(35)

with \( \hat{y}^i = (y^i_j)_{j \in \{1, \ldots, d\} \setminus \{\sigma(i)\}} \). We now state the following lemma, whose proof can be found in the appendix.

**Lemma 7.3:** Suppose that A1 holds and that \( a \) and \( b \) are \( C^3 \) with bounded derivatives. Let \( \phi : \mathbb{R}^d \to \mathbb{R} \) be a bounded \( C^1 \) function with bounded derivatives and let us denote

\[
d_{(w_i), \phi}(y^{1}_{\sigma(1)}, \ldots, y^{d}_{\sigma(d)}) = \xi(w_i, \phi(y^{1}_{\sigma(1)}, \ldots, y^{d}_{\sigma(d)}) - \phi(y^{1}_{\sigma(1)}, \ldots, y^{d}_{\sigma(d)}).
\]

Then, there exists some constant \( C \) such that we have

\[
\int_{\mathbb{R}^d} \left( \prod_{i=1}^{d} |K_{h_{\sigma(i)}}(x_{\sigma(i)} - y^{i}_{\sigma(i)})| \right) |d_{(w_i), \pi}(y^{1}_{\sigma(1)}, \ldots, y^{d}_{\sigma(d)})| \, dy^1_{\sigma(1)} \ldots dy^d_{\sigma(d)} \leq C\sqrt{\Delta'_n}.
\]

The constant \( C \) is independent of \( (h_i)_i \).

Applying Lemma 7.3 with \( \phi = \pi \), and (34), we deduce

\[
\mathbb{E} \left[ \prod_{i=1}^{d} K_{h_{\sigma(i)}}(x_{\sigma(i)} - X^\sigma_{w_i}) \right]
\]

\[
= \int_{\mathbb{R}^d} \left( \prod_{i=1}^{d} K_{h_{\sigma(i)}}(x_{\sigma(i)} - y^{i}_{\sigma(i)}) \right) \pi(y^{1}_{\sigma(1)}, \ldots, y^{d}_{\sigma(d)}) \, dy^1_{\sigma(1)} \ldots dy^d_{\sigma(d)} + O(\sqrt{\Delta'_n}).
\]

Changing the notation \( z_i = y^{\sigma^{-1}(i)}_i \), which is such that \( y^{\sigma(i)}_i = z_{\sigma(i)} \), we get

\[
\mathbb{E} \left[ \prod_{i=1}^{d} K_{h_{\sigma(i)}}(x_{\sigma(i)} - X^\sigma_{w_i}) \right]
\]

\[
= \int_{\mathbb{R}^d} \left( \prod_{i=1}^{d} K_{h_{\sigma(i)}}(x_{\sigma(i)} - z_{\sigma(i)}) \right) \pi(z_1, \ldots, z_d) \, dz_{\sigma(1)} \ldots dz_{\sigma(d)} + O(\sqrt{\Delta'_n})
\]

\[
= \int_{\mathbb{R}^d} \left( \prod_{i=1}^{d} K_{h_i}(x_i - z_i) \right) \pi(z_1, \ldots, z_d) \, dz_1 \ldots dz_d + O(\sqrt{\Delta'_n}).
\]

This implies (33). Then, recalling (32), and \( \mathbb{E}[\prod_{i=1}^{d} K_{h_i}(x_i - X^l_{\varphi_{n,1}(i)})] = \mathbb{E}[\prod_{i=1}^{d} K_{h_{\sigma(i)}}(x_{\sigma(i)} - X^\sigma_{w_i})] \) we deduce that

\[
\left| \mathbb{E}[\hat{\pi}^a_{h, T_n}(x)] - \int_{\mathbb{R}^d} \left( \prod_{i=1}^{d} K_{h_i}(x_i - z_i) \right) \pi(z_1, \ldots, z_d) \, dz_1 \ldots dz_d \right| \leq c\sqrt{\Delta'_n}.
\]
The proposition follows from the following upper bound on the bias of the synchronous case (see [44])

$$\left| \int_{\mathbb{R}^d} \left( \prod_{i=1}^{d} K_{h_i}(x_i - z_i) \right) \pi(z_1, \ldots, z_d) \, dz_1 \ldots dz_d - \pi(x_1, \ldots, x_d) \right| \leq c \sum_{i=1}^{d} h_i^{\beta_i}. $$

7.4. Proof of Theorem 5.1

**Proof:** The proof is a straightforward consequence of the bias-variance decomposition, Propositions 5.1 and 5.3. It is also based on the discussion on the conditions of the discretization step located after the statement of Theorem 5.1.

7.5. Proof of Theorem 5.2

**Proof:** As for the previous theorem, the proof is a direct consequence of the bias-variance decomposition, Propositions 5.2 and 5.3.

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Appendix

A.1 Proof of Proposition 5.4

Proof: In analogy to the previous proofs, we introduce

\[ k(t, s) := \text{Cov} \left( \prod_{m=1,2} K \left( \frac{x_m - X^m_t}{h^m_{m(t)}} \right) \prod_{l=3}^d K \left( \frac{x_l - X^l_{\psi_n(l)(t)}}{h^l_{\psi_n(l)(t)}} \right) \right), \]

\[ \prod_{m=1,2} K \left( \frac{x_m - X^m_s}{h^m_{m(s)}} \right) \prod_{l=3}^d K \left( \frac{x_l - X^l_{\psi_n(l)(s)}}{h^l_{\psi_n(l)(s)}} \right), \]

such that

\[ \text{Var} (\widehat{\pi}_{h^\nu, T_n}(x)) \]

\[ = \frac{2}{T_n^2} \int_0^{T_n} \int_0^t k(t, s) 1_{s < t} \left( 1 |t-s| \leq h^\nu_{h^\nu} \sum_{j=1}^d |\log h^\nu_j| \right) \]
with \( h^* \) the rate optimal choice of the bandwidth as in the proof of Theorem 1 of [47], given by (6).

We start considering \( I_1 \). Acting exactly as in (26) we have

\[
\tilde{I}_1 \leq \frac{c}{T_n} \sum_{j=1}^d |\log h_j^*| \frac{1}{\prod_{l=3}^d h_l^*}.
\]

To study \( I_2 \) we introduce a similar notation as for the analysis of \( \tilde{I}_2 \) for the ordering of \( s, t, \varphi_{n,3}(s), \ldots, \varphi_{n,d}(s), \varphi_{n,3}(t), \ldots, \varphi_{n,d}(t) \). In particular, we denote by \( w_3 \leq \cdots \leq w_d \) a reordering of \( \varphi_{n,l}(t) \) and \( \sigma \) an element of the permutation group on \( \{3, \ldots, d\} \) such that \( w_l = \varphi_{n,\sigma(i)}(t) \) for all \( i \in \{3, \ldots, d\} \). In the same way, we introduce \( \tilde{\sigma} \) which is an element of the permutation group such that \( \tilde{w}_l = \varphi_{n,\tilde{\sigma}(i)}(s) \) for all \( i \in \{3, \ldots, d\} \). In order to admit the possibility that \( \varphi_{n,l}(t) \leq s \), and then \( \varphi_{n,l}(t) = \varphi_{n,l}(s) \) for some index \( l \), we say that \( \tilde{w}_j \in \{w_3, \ldots, w_d\} \) for \( j \leq k \) and \( \tilde{w}_{k+1} \geq s \). It follows that

\[
w_3 \leq \cdots \leq w_d \leq s \leq \tilde{w}_{k+1} \leq \cdots \leq \tilde{w}_d \leq t.
\]

Hence we can write, for \( l \leq k, \tilde{w}_j = w_{\tau(l)} \) for some \( \tau(l) \in \{3, \ldots, d\} \). We now introduce the following vectors, which represent the positions in the instants previously discussed. At the instant \( w_j \) we have the vector \( y_j^l \), for \( j \in \{3, \ldots, d\} \). At the time \( \tilde{w}_j \) we have the vectors \( \tilde{y}_j^l \), for \( j \in \{k+1, \ldots, d\} \). We remark that \( y_j^l \) is the l-component of the vector \( y_j^l \), which gives the position at the instant \( w_j \). We observe that, as \( \tilde{w}_l = w_{\tau(l)} \) for \( l \leq k \) and so we can write, for any \( l \leq k \), \( y_{\sigma(l)}^l = y_{\sigma(\tau(l))}^l \). We have

\[
|k(t, s)| \leq |\tilde{k}(t, s)|
\]

\[
= |\mathbb{E} \left[ \prod_{m=1,2} K \left( \frac{x_m - X^m_t}{h_m} \right) \prod_{l=3}^d K \left( \frac{x_l - \varphi_{n,l}(t)}{h_l} \right) \right]| + |\mathbb{E} \left[ \prod_{m=1,2} K \left( \frac{x_m - X^m_s}{h_m} \right) \prod_{l=3}^d K \left( \frac{x_l - \varphi_{n,l}(s)}{h_l} \right) \right]|
\]

\[\times |\mathbb{E} \left[ \prod_{m=1,2} K \left( \frac{x_m - X^m_t}{h_m} \right) \prod_{l=3}^d K \left( \frac{x_l - \varphi_{n,l}(t)}{h_l} \right) \right]|, \tag{A1}\]

where

\[
\tilde{k}(t, s) := \mathbb{E} \left[ \prod_{m=1,2} K \left( \frac{x_m - X^m_t}{h_m} \right) \prod_{l=3}^d K \left( \frac{x_l - \varphi_{n,l}(t)}{h_l} \right) \right]
\]

\[\times \prod_{m=1,2} K \left( \frac{x_m - X^m_s}{h_m} \right) \prod_{l=3}^d K \left( \frac{x_l - \varphi_{n,l}(s)}{h_l} \right) \].

We aim at proving that $|\tilde{k}(t, s)| \leq \frac{c}{(t-s)^\alpha} \prod_{m=1}^d h_m$. We have

$$
|\tilde{k}(t, s)| \leq \int_{\mathbb{R}^d} \prod_{m=1}^2 |K_{h_m^*}(x_m - z_m)| \int_{\mathbb{R}^d(d-2)} |\prod_{l=3}^d K_{h_m^*}(x_\sigma(l) - y^l_{\sigma(l)})| \int_{\mathbb{R}^d} \prod_{m=1}^2 |K_{h_m^*}(x_m - z_m)|
$$

\[
\times \prod_{l=3}^d |K_{h_m^*}(x_\sigma(l) - y^l_{\sigma(l)})| \int_{\mathbb{R}^d} \prod_{m=1}^2 |K_{h_m^*}(x_m - z_m)|
\]

\[
\times p_{w_4-w_3(y^3, y^4)p_{w_5-w_4(y^4, y^5)} \times \cdots \times p_{w_{d-w_{d-1}}(y^{d-1}, y^d)p_{s-w_d(y^d, z)}}
\]

\[
\times p_{\tilde{w}_{d-w_{d-1}}(y^{d-1}, \tilde{y}^d)p_{t-\tilde{w}_d(\tilde{y}^d, \tilde{z})}(y^3)\, dz\, dy^3 \cdots dy^d\, d\tilde{z}^{k+1} \cdots d\tilde{y}^d.
\]

We bound the above expression and obtain:

\[
|\tilde{k}(t, s)| \leq \frac{1}{\prod_{l=3}^d h_l^*} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d(d-2)} \prod_{m=1}^2 |K_{h_m^*}(x_m - z_m)|
\]

\[
\times \prod_{l=3}^d |K_{h_m^*}(x_\sigma(l) - y^l_{\sigma(l)})| \int_{\mathbb{R}^d} \prod_{m=1}^2 |K_{h_m^*}(x_m - z_m)|
\]

\[
\times p_{w_4-w_3(y^3, y^4)p_{w_5-w_4(y^4, y^5)} \times \cdots \times p_{w_{d-w_{d-1}}(y^{d-1}, y^d)p_{s-w_d(y^d, z)}}
\]

\[
\times p_{\tilde{w}_{d-w_{d-1}}(y^{d-1}, \tilde{y}^d)p_{t-\tilde{w}_d(\tilde{y}^d, \tilde{z})}(y^3)\, dz\, dy^3 \cdots dy^d\, d\tilde{z}^{k+1} \cdots d\tilde{y}^d.
\]

Using Gaussian upper bounds on the transition density (Proposition 5.1 in [53]), we have $p_{t-s}(y^d, \tilde{z}) \leq e^{-(\frac{t-s}{2})^2 \frac{1}{\prod_{l=3}^d h_l^*}}$, with

\[
q_{t-s}(\tilde{z}_3, \ldots, \tilde{z}_d|\tilde{z}_1, \tilde{z}_2, y^d) = e^{-\lambda_0 \frac{(\tilde{z}_3 - y^d)^2}{t-s}} e^{-\lambda_0 \frac{(\tilde{z}_2 - y^d)^2}{t-s}} \cdots \frac{1}{\sqrt{t-s}} e^{-\lambda_0 \frac{(\tilde{z}_d - y^d)^2}{t-s}}.
\]

We observe that

$$
\sup_{t-s \in (0, 1)} \sup_{y^d, \tilde{z}_1, \tilde{z}_2} \int_{\mathbb{R}^{d+2}} q_{t-s}(\tilde{z}_3, \ldots, \tilde{z}_d|\tilde{z}_1, \tilde{z}_2, y^d) \, d\tilde{z}_3 \cdots d\tilde{z}_d < c,
\]

noting that $t-s \in (0, 1)$ as $0 \leq t-s \leq (\prod_{l=3}^d h_l^*)^{-\alpha} < 1$. Moreover, we easily bound

$$
\int_{\mathbb{R}^d} \prod_{m=1}^2 |K_{h_m^*}(x_m - z_m)| \, d\tilde{z}_1 \, d\tilde{z}_2 < c.
\]

Replacing everything in (A3) we obtain

$$
|\tilde{k}(t, s)| \leq \frac{c}{t-s} \prod_{l=3}^d h_l^* \int_{\mathbb{R}^d} \prod_{m=1}^2 |K_{h_m^*}(x_m - z_m)| \prod_{l=3}^d K_{h_m^*}(x_\sigma(l) - y^l_{\sigma(l)})
\]

\[
\times \int_{\mathbb{R}^d(d-2)} \prod_{l=4}^d p_{w_{l-w_{l-1}}(y^{l-1}, y^l)p_{s-w_d(y^d, z)}(y^3)\, dz\, dy^3 \cdots dy^d.
\]
We use the first point of Lemma 7.2 for \( r = d - 1 \). In particular, \( u_1, \ldots, u_{r-1}, u_r \) are in this case \( y^3, \ldots, y^d, z \) while \( q_1 = q_2 = r \) and, for \( l \geq 3 \), \( q_l = \sigma^{-1}(l) \). We obtain

\[
|\tilde{k}(t, s)| \leq \frac{c}{t - s} \frac{1}{\prod_{l \geq 3} h_l^*}.
\]

Using again the first point of Lemma 7.2, there exists a constant \( c \) such that

\[
|\mathbb{E} \left[ \prod_{m=1,2} K \left( \frac{x_m - X_m^u}{h_m} \right) \prod_{l=3}^{d} K \left( \frac{x_l - X_{\varphi_m(l)}^{l}}{h_l} \right) \right] | \leq c, \quad \forall u.
\]

Recalling (A1), it implies \(|k(t, s)| \leq \frac{c}{t - s} \frac{1}{\prod_{l \geq 3} h_l^*} + c\). It yields, applying the change of variable \( t - s =: t' \),

\[
|I_2| \leq \frac{c}{T_n} \frac{1}{\prod_{l \geq 3} h_l^*} \int_{l \geq 3} \left( \prod_{h_l^*} h_l^* \right)^{\frac{2}{d-2}} 1 \left( \prod_{h_l^*} h_l^* \right)^{\frac{1}{d-2}} \left( \prod_{l \geq 3} |\log h_l^*| \right) \frac{1}{t'} \leq \frac{c}{T_n} \frac{\sum_{j=1}^{2} |\log h_j^*|}{\prod_{l \geq 3} h_l^*}.
\]

(A4)

We study now \( I_3 \). We can write \( \tilde{k}(t, s) \) as in (A2). Now \( s \) and \( t \) are distant to each other, which implies that \( \varphi_{n, l}(s) < \varphi_{n, l}(t) \) for any \( l \in \{3, \ldots, d\} \) and so, in particular, the ordering of the quantities previously introduced is the following:

\[
|t - s| \geq \left( \prod_{j \geq 3} h_j^\ast \right)^{\frac{2}{d-2}} \text{ and } \Delta_n \leq \left( \prod_{j \geq 3} h_j^\ast \right)^{\frac{2}{d-2}}.
\]

The study of \( I_3 \) follows the route of the analysis of \( I_3 \) in the proof of Proposition 5.1. We have

\[
|\tilde{k}(t, s)| \leq \int_{\mathbb{R}^d} \prod_{m=1}^2 |K_{h_m^\ast}(x_m - z_m)| \left( \prod_{l=3}^{d} \prod_{l=3}^{d} |K_{h_{\sigma(l)}^\ast}(x_{\sigma(l)} - y_{\sigma(l)}^{\prime})| \right) \int_{\mathbb{R}^d} \prod_{m=1}^2 |K_{h_m^\ast}(x_m - z_m)|
\]

\[
\times \int_{\mathbb{R}^d} \prod_{l=3}^{d} |K_{h_{\sigma(l)}^\ast}(x_{\sigma(l)} - y_{\sigma(l)}^{\prime})| p_{w_4 - w_3}(y^{3}, y^{4}) p_{w_5 - w_4}(y^{4}, y^{5}) \times \cdots
\]

\[
\times p_{w_d - w_{d-1}}(y^{d-1}, y^{d}) p_{w_d - w_{d-1}}(y^{d}, z) x p_{w_d - w_{d-1}}(z, y^{3}) \times \cdots
\]

\[
\times p_{w_d - w_{d-1}}(y^{d-1}, z) p_{w_d - w_{d-1}}(z, \tilde{y}^{3}) \pi(y^{3}) dz dy^{3} \cdots dz dy^{3} \cdots dz dy^{d}.
\]

We remark that the largest interval of time above is \( \tilde{w}_3 - s \). We use on it the rough estimation

\[
p_{w_d - w_3}(z, \tilde{y}^{3}) \leq \frac{c}{(\tilde{w}_3 - s)^{\frac{d}{2}}} \prod_{l=1}^{d} \left( \frac{e^{\lambda_{\Delta_0}}}{{\tilde{w}_3} - {w_d}} \right)^{\frac{2}{d}} \leq \frac{c}{(\tilde{w}_3 - s)^{\frac{d}{2}}}.
\]

We apply twice the first point of Lemma 7.2 (having on each integral \( r = (d - 1) \) as we are considering the integrals in \( y^3, \ldots, y^d, z \) and in \( y^{3}, \ldots, y^{d}, \tilde{z} \)). It implies

\[
|k(t, s)| \leq \frac{c}{(\tilde{w}_3 - s)^{\frac{d}{2}}} + c.
\]

(A5)

We now observe that it is

\[
|t - s| \leq |t - \tilde{w}_3| + |\tilde{w}_3 - s| \leq \Delta_n + |\tilde{w}_3 - s| \leq \frac{1}{2} \left( \prod_{l \geq 3} h_l^* \right)^{\frac{2}{d-2}} + |\tilde{w}_3 - s|.
\]
It follows
\[ |\tilde{w}_3 - s| \geq |t - s| - \frac{1}{2} \left( \prod_{l \geq 3} h^*_l \right)^{\frac{1}{2}} \geq \frac{1}{2} |t - s|. \]

Moreover, \( |\tilde{w}_3 - s| \leq |t - s| \leq D \). From the change of coordinates \( s \to s' := t - s \) we obtain
\[
I_3 \leq \frac{c}{T_n^2} \int_0^{T_n} \int_0^D \left( \frac{c}{s'^2} + c \right) ds' \, dt = \frac{c}{T_n} \left( \prod_{l \geq 3} h^*_l \right)^{-1} + D, \tag{A6}
\]
which is the order we wanted.

We are left to study \( I_4 \), where \( D \leq |t - s| \leq T_n \). Here we act as on \( I_4 \) in Proposition 5.1. We need to study, up to reorder the components,
\[
\text{Cov}(K_{h^*_l}(s)K_{h^*_l}(w_3) \cdots K_{h^*_l}(w_d), K_{h^*_l}(t)K_{h^*_l}(\tilde{w}_3) \cdots K_{h^*_l}(\tilde{w}_d)),
\]
where \( w_3 \leq \cdots \leq w_d \leq s \leq \tilde{w}_3 \leq \cdots \leq \tilde{w}_d \leq t, D \leq |t - s| \leq T \). We define
\[
g(X_{\tilde{w}_3}) := \mathbb{E}[K_{h^*_l}(t)K_{h^*_l}(\tilde{w}_3) \cdots K_{h^*_l}(\tilde{w}_d) | X_{\tilde{w}_3}].
\]
Acting as in the proof of Proposition 5.3 we can clearly write the covariance as
\[
\mathbb{E}[K_{h^*_l}(s)K_{h^*_l}(w_3) \cdots K_{h^*_l}(w_d) (P_{\tilde{w}_3 - s}g(X_s) - \pi(g))].
\]
From Lemma 7 of [47] we easily obtain \( \|P_{\tilde{w}_3 - s}g(X_s) - \pi(g)\|_{L^1} \leq c e^{-\rho(\tilde{w}_3 - s)} \|g\|_{\infty}. \) Therefore,
\[
|k(t,s)| \leq \prod_{l=1}^d \left| K_{h^*_l} \right|_{\infty} \|P_{\tilde{w}_3 - s}g(X_s) - \pi(g)\|_{L^1}
\]
\[ \leq \frac{c}{\left( \prod_{l=1}^d h^*_l \right)^{\frac{1}{2}}} e^{-\rho(\tilde{w}_3 - s)}, \]
where we have also used that, from the definition of \( g \), it is \( \|g\|_{\infty} \leq \frac{c}{\prod_{l=1}^d h^*_l} \). Moreover, acting as in (A5) and remarking that \( \left( \prod_{l \geq 3} h^*_l \right)^{\frac{1}{2}} \leq D \), we easily get
\[
|\tilde{w}_3 - s| \geq |t - s| - \Delta_n.
\]

With the change of variable \( s \to s' := t - s \) we obtain
\[
I_4 \leq \frac{c}{\left( \prod_{l=1}^d h^*_l \right)^{\frac{1}{2}}} \frac{1}{T_n} \int_0^{T_n} \int_0^D e^{-\rho s'} e^{\rho \Delta_n} \, ds' \, dt \leq \frac{c}{T_n} \left( \prod_{l=1}^d h^*_l \right)^{\frac{1}{2}} e^{-\rho D}, \tag{A7}
\]
Putting all the pieces together, using in particular (26), (A4), (A6) and (A7), it yields
\[
\text{Var}(\bar{h}_{T_n}, T_n(x)) \leq \frac{c}{T_n} \sum_{l=1}^d \left| \log h^*_l \right| + \frac{c}{T_n} \sum_{l=3}^d \left| \log h^*_l \right| + \frac{c}{T_n} \frac{1}{\prod_{l=3}^d h^*_l} + \frac{D}{T_n} + \frac{c}{T_n} \left( \prod_{l=1}^d h^*_l \right)^{\frac{1}{2}} e^{-\rho D}.
\]
By choosing \( D := \left[ \max(-\frac{2}{\rho} \log(\prod_{l=1}^d h^*_l), 1) \right] \land T_n \) we obtain the wanted result. \( \blacksquare \)

### A.2 Proof of Lemma 7.3

**Proof:** Let us denote by \( g_\alpha(z) \) the density of a centred Gaussian variable with covariance matrix \( \alpha \).
We recall the approximation of the diffusion transition density by the Gaussian kernel given in [54].
Specifying \( s = 1/N \) and \( T = 1 \) in the notations of the statement of Theorem 3 [54], we have for all \( s \leq 1 \),
\[
|p_s(z, z') - g_\theta(z, z')| \leq C\sqrt{s}\lambda_0ld(z - z'), \tag{A8}
\]
where \( \lambda_0 > 0 \) and \( C > 0 \) are some constant and \( \tilde{a} = a \cdot a^T \). This leads us to introduce a Gaussian approximation of (35)
\[
\xi^G_{(w_i),\phi}(y^{(1)}_{\sigma(1)}, \ldots, y^{d}_{\sigma(d)}) = \int_{\mathbb{R}^{d(d-1)}} \phi(y^{(1)}) \prod_{i=1}^{d-1} g(w_{i+1} - w_i)\tilde{a}(y_i)(y^{i+1} - y^i)\,dy^1 \ldots dy^d. \tag{A9}
\]
With this notation, we split \( d_{(w_i),\phi}(y^{(1)}_{\sigma(1)}, \ldots, y^{d}_{\sigma(d)}) \) as \( \sum_{l=1}^{2} d^{(l)}_{(w_i),\phi}(y^{(1)}_{\sigma(1)}, \ldots, y^{d}_{\sigma(d)}) \), with
\[
d^{(1)}_{(w_i),\phi}(y^{(1)}_{\sigma(1)}, \ldots, y^{d}_{\sigma(d)}) = \xi_{(w_i),\phi}(y^{(1)}_{\sigma(1)}, \ldots, y^{d}_{\sigma(d)}),
\]
\[
d^{(2)}_{(w_i),\phi}(y^{(1)}_{\sigma(1)}, \ldots, y^{d}_{\sigma(d)}) = \xi^G_{(w_i),\phi}(y^{(1)}_{\sigma(1)}, \ldots, y^{d}_{\sigma(d)}) - \phi(y^{(1)}_{\sigma(1)}, \ldots, y^{d}_{\sigma(d)}). \tag{A10}
\]
The lemma is a consequence of the following upper bound for \( l \in \{1, 2\} \),
\[
\int_{\mathbb{R}^d} \left| \prod_{i=1}^{d} |K_{h_{\sigma(i)}}(x_{\sigma(i)} - y^{(1)}_{\sigma(i)})| \right| d^{(l)}_{(w_i),\phi}(y^{(1)}_{\sigma(1)}, \ldots, y^{d}_{\sigma(d)}) \, dy^{(1)}_{\sigma(1)} \ldots dy^{d}_{\sigma(d)} \leq C\sqrt{\Delta_n}. \tag{A11}
\]
- We first prove (A11) with \( l = 1 \). Comparing (35) with (A9), we can write
\[
d^{(1)}_{(w_i),\phi}(y^{(1)}_{\sigma(1)}, \ldots, y^{d}_{\sigma(d)}) = \int_{\mathbb{R}^{d(d-1)}} \phi(y^{(1)}) \prod_{k=1}^{d-1} \prod_{1 \leq i < k} p_{w_{i+1} - w_i}(y^i, y^{i+1})
\]
\[
\times \left[ p_{w_{k+1} - w_{k}}(y^k, y^{k+1}) - g(w_{k+1} - w_{k})\tilde{a}(y^k)(y^{k+1} - y^k) \right]
\]
\[
\times \prod_{k<i \leq d-1} g(w_{i+1} - w_i)\tilde{a}(y^i)(y^{i+1} - y^i)\,dy^1 \ldots dy^d.
\]
Using (A8) and a Gaussian upper bound of the transition density, we deduce
\[
\left| d^{(1)}_{(w_i),\phi}(y^{(1)}_{\sigma(1)}, \ldots, y^{d}_{\sigma(d)}) \right| \leq C \sup_{i=1, \ldots, d-1} \sqrt{w_{i+1} - w_i}
\]
\[
\times \int_{\mathbb{R}^d} |\phi(y^{(1)})| \prod_{1 \leq i \leq d-1} g(w_{i+1} - w_i)\lambda_0ld(y^{i+1} - y^i)\,dy^1 \ldots dy^d,
\]
\[
\leq C\sqrt{\Delta_n} \int_{\mathbb{R}^d} \prod_{1 \leq i \leq d-1} g(w_{i+1} - w_i)\lambda_0ld(y^{i+1} - y^i)\,dy^1 \ldots dy^d,
\]
for some constant \( \lambda_0 > 0 \), and where we used that by (12), \( w_{i+1} - w_i \leq \Delta_n \). It yields,
\[
\int_{\mathbb{R}^d} \left( \prod_{i=1}^{d} K_{h_{\sigma(i)}}(x_{\sigma(i)} - y^{(1)}_{\sigma(i)}) \right) \left| d^{(1)}_{(w_i),\phi}(y^{(1)}_{\sigma(1)}, \ldots, y^{d}_{\sigma(d)}) \right| dy^{(1)}_{\sigma(1)} \ldots dy^{d}_{\sigma(d)}
\]
\[
\leq C\sqrt{\Delta_n} \int_{\mathbb{R}^d} \prod_{i=1}^{d} K_{h_{\sigma(i)}}(x_{\sigma(i)} - y^{(1)}_{\sigma(i)}) \prod_{1 \leq i \leq d-1} g(w_{i+1} - w_i)\lambda_0ld(y^{i+1} - y^i)dy^1 \ldots dy^d.
\]
From the second point of Lemma 7.2 with \( r = d, u^j = y^j \) for \( i = 1, \ldots, d \), and \( q_i = \sigma^{-1}(i) \) we deduce that the last integral is upper bounded by some constant, and in turn that (A11) holds true for \( l = 1 \).
- We now prove (A11) with \( l = 2 \). In the integral defined by the right-hand side of (A9) we make a change of variables, replacing the variables \( (y^{(1)}, \ldots, y^{d}) = (y^j)_{j \in \{1, \ldots, d\}, i \in \{1, \ldots, d\} \setminus \{\sigma(j)\}} \) by new integration variables \( (z^{j}_{1 \leq i \leq d, 2 \leq j \leq d}) \) defined in the following way. For \( j = d \), we define \( z^{d} = (z^{d}_{1 \leq i \leq d}) \).
through the change of variable

\[ y_l^d \to z_l^d := \frac{y_l^d - y_l^{d-1}}{\sqrt{w_d - w_{d-1}}} \quad \text{for } l \in \{1, \ldots, d\} \setminus \{\sigma(d)\}, \quad y_{\sigma(d)}^{d-1} \to z_{\sigma(d)}^d := \frac{y_{\sigma(d)}^d - y_{\sigma(d)}^{d-1}}{\sqrt{w_d - w_{d-1}}}, \]

and more generally for \( 2 \leq j \leq d \) we define \( z_j \) through the formulae

\[
y_j^l \to z_j^l := \frac{y_j^l - y_j^{l-1}}{\sqrt{w_j - w_{j-1}}} \quad \text{for } l \in \{1, \ldots, d\} \setminus \{\sigma(d), \sigma(d-1), \ldots, \sigma(j)\},
\]

\[
y_j^{l-1} \to z_j^l := \frac{y_j^l - y_j^{l-1}}{\sqrt{w_j - w_{j-1}}} \quad \text{for } l \in \{\sigma(d), \sigma(d-1), \ldots, \sigma(j)\}.
\]

From these definitions we have

\[
d_j^1, \ldots, d_j^d = \prod_{j=1}^d \prod_{l=1, \ldots, d} (w_j - w_{j-1})^{d/2}.
\]

Moreover by construction \( z_j^l = \frac{y_j^l - y_j^{l-1}}{\sqrt{w_j - w_{j-1}}} \) for all \( j \in \{2, \ldots, d\} \). We deduce that (A9) can be written after change of variables as

\[
\xi_{(w_j),\sigma}^{G}(y_1^1, \ldots, y_{\sigma(d)}^d) = \int_{\mathbb{R}^{(d-1)}} \phi(\tilde{y}^1) \prod_{j=1}^{d-1} \mathcal{G}_{\tilde{y}^j}(z_j^{j+1}) \, dz^2 \ldots dz^d,
\]

where \( \tilde{y}^j = (z_1^j, \ldots, z_j^j, y_1^1, \ldots, y_{\sigma(d)}^d) \) is a notation for the expression of \( (y_1^1, \ldots, y_{\sigma(d)}^d) \) as a function of the new variables \( z_1^2, \ldots, z_j^j \). They are given by the explicit expression

\[
\tilde{y}_j = \begin{cases} 
  y_j^{\sigma^{-1}(l)-1} + \sum_{u=0}^{j-\sigma^{-1}(l)-1} \sqrt{w_u + \sigma^{-1}(l)+1 - w_u + \sigma^{-1}(l)} z_u^{\mu+1+\sigma^{-1}(l)} & \text{if } j > \sigma^{-1}(l), \\
  y_j^{\sigma^{-1}(l)} & \text{if } j = \sigma^{-1}(l), \\
  y_j^{\sigma^{-1}(l)-j-1} - \sum_{u=0}^{\sigma^{-1}(l)-j-1} \sqrt{w_u + j + 1 - w_u + j} z_u^{\mu+1+j} & \text{if } j < \sigma^{-1}(l).
\end{cases}
\]

This leads us to introduce the following notation which stresses the dependence upon the time intervals \( w_{u+1} - w_u \). We set for \( s_1, \ldots, s_{d-1} > 0 \),

\[
\tilde{y}_j^{(s_1, \ldots, s_{d-1})} = \begin{cases} 
  y_j^{\sigma^{-1}(l)-1} + \sum_{u=0}^{j-\sigma^{-1}(l)-1} s_{u+\sigma^{-1}(l)+1} z_u^{\mu+1+\sigma^{-1}(l)} & \text{if } j > \sigma^{-1}(l), \\
  y_j^{\sigma^{-1}(l)} & \text{if } j = \sigma^{-1}(l), \\
  y_j^{\sigma^{-1}(l)-j-1} - \sum_{u=0}^{\sigma^{-1}(l)-j-1} s_{u+j} z_u^{\mu+1+j} & \text{if } j < \sigma^{-1}(l),
\end{cases}
\]

(A12)

and we define

\[
\tilde{\xi}_{(w_j),\sigma}^{G}(s_1, \ldots, s_{d-1}) = \int_{\mathbb{R}^{(d-1)}} \phi(\tilde{y}^1(s_1, \ldots, s_{d-1})) \prod_{j=1}^{d-1} \mathcal{G}_{\tilde{y}^j(s_1, \ldots, s_{d-1})}(z_j^{j+1}) \, dz^2 \ldots dz^d.
\]

(A13)

With these notations, \( \tilde{y}_j^l = \tilde{y}_j^{(\sqrt{w_2 - w_1}, \ldots, \sqrt{w_d - w_{d-1}})} \) for all \( 1 \leq j, l \leq d \), and

\[
\xi_{(w_j),\sigma}^{G}(y_{\sigma(1)}, \ldots, y_{\sigma(d)}) = \tilde{\xi}_{(w_j),\sigma}^{G}(\sqrt{w_2 - w_1}, \ldots, \sqrt{w_d - w_{d-1}}).
\]

(A14)
For \((s_1, \ldots, s_{d-1}) = (0, \ldots, 0)\) these quantities have simpler expressions. Let us denote \(y^* = (y_{\sigma^{-1}(1)}^1, \ldots, y_{\sigma^{-1}(d)}^1)\), and remark that from (A12), we have \(\tilde{y}(0, \ldots, 0) = y^*\), for all \(1 \leq j \leq d\). It follows
\[
\frac{d\tilde{G}G}{d\tilde{G}}(0, \ldots, 0) = \int_{\mathbb{R}^{(d-1)}} \phi(y^*) \prod_{j=1}^{d-1} g_{\tilde{a}(y^*)}(z^{j+1}) \, dz^{j+1} \ldots \, dz^d = \phi(y^*)
\] (A15)
where we have used that \(y^*\) does not depend on the integration variable and that the Gaussian kernel integrates to one. We deduce from (A14)–(A15),
\[
\frac{\partial}{\partial s_j} \tilde{G}(s_1, \ldots, s_{d-1}) = 0,
\]
for \(1 \leq j \leq d-1\) and \(s_1, \ldots, s_{d-1} > 0\). We successively integrate in \(s_2, \ldots, s_{d-1}\), and remark that from (A12), we have
\[
\frac{\partial}{\partial s_j} \tilde{G}(s_1, \ldots, s_{d-1}) \leq \sqrt{\Delta_n} \sum_{j=2}^{d} \sup_{0 \leq s_1, \ldots, s_{d-1} \leq \sqrt{\Delta_n}} \left| \frac{\partial}{\partial s_j} \tilde{G}(s_1, \ldots, s_{d-1}) \right|
\] (A16)
where we used \(\sqrt{w_j - w_{j-1}} \leq \sqrt{\Delta_n}\) for all \(2 \leq j \leq d\). From the definition (A12), we have \(\frac{\partial^2}{\partial s_j^2} \tilde{G} \leq C \sum_{j=2}^{d} \|z^j\|^\frac{3}{2} \sup_{s_1, \ldots, s_{d-1} \leq \sqrt{\Delta_n}} \left| \frac{\partial}{\partial s_j} \tilde{G}(s_1, \ldots, s_{d-1}) \right|
\]
Using that \(\phi\) and \(\tilde{a}\) are \(C^1\) functions, bounded with bounded derivative, and that \(\tilde{a} \geq \alpha_{\min} I_d\), we deduce from (A13) that
\[
\left| \frac{\partial^2}{\partial s_j^2} \tilde{G}(s_1, \ldots, s_{d}) \right| \leq C \int_{\mathbb{R}^{(d-1)}} \sum_{i=2}^{d} (1 + \|z^j\|^3) \prod_{j=1}^{d-1} g_{\tilde{a}(\tilde{G}(s_1, \ldots, s_{d-1})))}(z^{j+1}) \, dz^{j+1} \ldots \, dz^d.
\]
Used that \(\tilde{a}\) is a bounded function under Assumption A1, we deduce that the last integral is bounded independently of \(s_1, \ldots, s_{d-1}\), and thus
\[
\left| \sup_{s_1, \ldots, s_{d-1} \leq \sqrt{\Delta_n}} \frac{\partial^2}{\partial s_j^2} \tilde{G}(s_1, \ldots, s_{d}) \right| \leq C \sqrt{\Delta_n},
\]
for a constant \(C\) independent of \((w_i)\) and \(y_{\sigma^{-1}(1)}, \ldots, y_{\sigma^{-1}(d)}\). Recalling (A10) and the notation \(y^* = (y_{\sigma^{-1}(1)}^1, \ldots, y_{\sigma^{-1}(d)}^1)\), this is the upper bound
\[
\left| \frac{\partial^2}{\partial (w_i)\partial \theta_{\sigma^{-1}(1)}} \tilde{G}(y_{\sigma^{-1}(1)}, \ldots, y_{\sigma^{-1}(d)}) \right| \leq C \sqrt{\Delta_n},
\]
and we deduce (A11) with \(l = 2\) by integration.

### A.3 Proof of Lemma 7.2

**Proof:** We only prove the first point as the proof of the second is similar. Using the Gaussian upper bound on the transition density (e.g., see Proposition 5.1 in [53])
\[
p_{w_{j+1} - w_j}(u^i, u^{i+1}) \leq C \frac{1}{(w_{j+1} - w_j)^{d/2}} \, e^{-\lambda_0 \frac{|u^{i+1} - u^i|^2}{w_{j+1} - w_j}},
\]
we deduce that the left-hand side of (30) is smaller than
\[
C \int_{\mathbb{R}^d} \prod_{i=1}^{d} \left| K_{h_i}(x_i - u_i^{q_i}) \right| \prod_{j=1}^{r-1} \frac{1}{(w_{j+1} - w_j)^{1/2}} e^{-\lambda_0 \frac{|u^{i+1} - u^i|^2}{w_{j+1} - w_j}} \prod_{i=1}^{d} \left( \int_{\mathbb{R}^d} du_i \right)
\]
which is equal to
\[
C \prod_{i=1}^{d} \left( \int_{\mathbb{R}^d} \left| K_{h_i}(x_i - u_i^{q_i}) \right| \prod_{j=1}^{r-1} \frac{1}{(w_{j+1} - w_j)^{1/2}} e^{-\lambda_0 \frac{|u^{i+1} - u^i|^2}{w_{j+1} - w_j}} du_i \right).
\] (A17)
It is sufficient to show that for all \(i \in \{1, \ldots, d\}\) the integrals in the product (A17) are smaller than some constant independent of \((h_i)\) and \((w_j)\). We successively integrate in \(u_i^1, u_i^2, \ldots, u_i^{q_i+1}\),...
using the change of variables $u_i^j \rightarrow z_i^j := \frac{u_i^j - u_i^{i-1}}{\sqrt{w_{j+1} - w_j}}$ for $j = r, r - 1, \ldots, q_i + 1$. Next, we successively integrate in $u_1^1, \ldots, u_{q_i}^{q_i-1}$, using the change of variable $u_i^j \rightarrow z_i^j := \frac{u_i^{j+1} - u_i^j}{\sqrt{w_{j+1} - w_j}}$. We deduce that the integrals appearing in (A17) are upper bounded by $C \int_{\mathbb{R}} K_{h_i}(x_i - u_i^{q_i}) du_i^{q_i} \leq C$. This proves the lemma.

\section*{A.4 Proof of Lemma 7.1}

\textbf{Proof:} We start by the proof of the first point. We reorder $(\varphi_{n,l}(t))_{l=1,\ldots,d}$ as $w_1 \leq \cdots \leq w_d$ and denote by $\sigma$ a permutation of $\{1, \ldots, d\}$ such that $\varphi_{n,\sigma(l)}(t) = w_l$ for all $l \in \{1, \ldots, d\}$. By a density argument we can assume that the $w_l$ are all distinct. Then, we can write

\begin{equation*}
\mathbb{E} \left[ \prod_{l=1}^d |K_{h_l}(x_l - X_{\varphi_{n,l}(t)})| \right] = \int_{\mathbb{R}^d} \prod_{l=1}^d |K_{h_l}(x_l - y_{\sigma^{-1}(l)})| \prod_{l=1}^{d-1} p_{w_{l+1} - w_l}(y_l, y_{l+1}) \, dy_1 \ldots dy_d.
\end{equation*}

Now the first part of the lemma is a consequence of the first point of Lemma 7.2, with $r = d$ and $q_l = \sigma^{-1}(l)$ for all $l \in \{1, \ldots, d\}$.

The second point of the lemma is obtained as a consequence of the first point, after remarking that we can write $K_{h_l}(\cdot)^2 = \frac{1}{h_l} \frac{1}{h_l} K^2(\frac{\cdot}{h_l})$, and applying the first point with the function $K^2$ instead of $|K|$.