EXISTENCE AND REGULARITY THEOREMS
OF ONE-DIMENSIONAL BRAKKE FLOWS

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Abstract. Given a closed countably 1-rectifiable set in $\mathbb{R}^2$ with locally finite 1-dimensional Hausdorff measure, we prove that there exists a Brakke flow starting from the given set with the following regularity property. For almost all time, the flow locally consists of a finite number of embedded curves of class $W^{2,2}$ whose endpoints meet at junctions with angles of either 0, 60 or 120 degrees.

1. Introduction

A family of $n$-dimensional surfaces $\{\Gamma_t\}_{t\geq 0}$ in $\mathbb{R}^{n+1}$ is called the mean curvature flow (abbreviated hereafter as MCF) if the velocity is equal to its mean curvature at each point and time. Given a smooth compact surface $\Gamma_0$, there exists a smoothly evolving MCF starting from $\Gamma_0$ until some singularities such as vanishing or pinching occur. There are numerous notions of generalized MCF past singularities and the Brakke flow is the earliest one conceived by Brakke [5] within the framework of geometric measure theory.

In [15], by reworking the proof of Brakke’s general existence theorem [5, Chapter 4], the following time-global existence theorem was established: Let $\Gamma_0$ be a closed countably $n$-rectifiable set whose complement $\mathbb{R}^{n+1}\setminus \Gamma_0$ is not connected and whose $n$-dimensional Hausdorff measure $\mathcal{H}^n(\Gamma_0)$ is finite or at most of exponential growth near infinity. Then there exists a non-trivial Brakke flow starting from $\Gamma_0$. The similar existence problem was considered with fixed boundary conditions [22], where the Brakke flow converges subsequentially to a solution for the Plateau problem as $t \to \infty$. The Brakke flow is a family of varifolds which satisfies the motion law of MCF in a distributional sense, and the flow may not be smooth in general. In fact, for $n = 1$, a typical Brakke flow is expected to look like an evolving network of curves joined by junctions, and the stable junctions are the ones with three curves meeting with equal angles of 120 degrees. Let us call such junction simply as the triple junction. Physically, one can associate the motion of grain boundaries in polycrystalline materials to this mathematical model. As the network goes through various topological changes, junctions of more than three curves are formed when two or more triple junctions collide. They then should instantaneously break up into triple junctions connected by short curves since the latter has shorter curve length. With this formal intuition, it is reasonable to speculate that, for
general initial data, there should exist a 1-dimensional Brakke flow which locally consists of finite
number of smooth curves meeting only at triple junctions for almost all time, or even better, for
all co-countable time.

The present paper edges towards the positive resolution of this speculation. We prove that the
1-dimensional Brakke flow constructed in [15, 22] has this property, albeit degenerately. Roughly
speaking, we prove that the support of Brakke flow in [15, 22] locally consists of embedded \( W^{2,2} \)
curves whose endpoints meet at junctions with either 0, 60 or 120 degrees for almost all time
(Theorem 2.2 and 2.3). Here, \( W^{2,2} \) curve means that it is locally represented as the graph of a
function whose first and second weak derivatives are square-integrable. In particular, these curves
are \( C^{1,1/2} \) by the standard embedding theorem and the angle condition says that they cannot
intersect each other with angles other than 0, 60 or 120 degrees (see Figure 1 for some examples).
The configuration (a) is the typical picture that three curves meet at a triple junction. Note that
the configurations like (b)-(d) can arise as limits of curves connected only by triple junctions while
keeping the curvatures bounded. In that sense, they may be seen as degenerate triple junctions.
For example, (b) can arise as a limit of two triple junctions connected by an infinitesimally short
curve, and (c) can be a limit of infinitesimally small regular hexagon with 6 curves leaving from
the each vertex. The configuration like (d) can also arise as a limit of combination of hexagon and
triple junctions. In addition, we prove that the same regularity property holds for any tangent
flow (Theorem 2.4). The regularity statement for the support itself is a substantial improvement

\[
\begin{array}{cccc}
(a) & (b) & (c) & (d)
\end{array}
\]

**Figure 1.**

compared to the result deduced from the standard regularity theorem in geometric measure theory.
The Brakke flow by definition has locally square-integrable generalized mean curvature and integer-
multiplicity for almost all time, and with the Allard regularity theorem [1], there exists a dense
open set in which the flow consists of \( W^{2,2} \) curves. But it is not known that the complement of
such “regular part” has null 1-dimensional measure in general when the Allard theorem is applied.
As for the regularity near the junctions, we are not aware of any general theorem on the uniqueness
of tangent cone or regularity theorem other than the one for 1-dimensional stationary varifolds due
to Allard-Almgren [2].
To avoid a possible confusion, one should note that the claim of the present paper is pertinent only to the Brakke flows constructed in [15, 22], and not necessarily to arbitrary 1-dimensional Brakke flows as defined in [5, 24]. Just to clarify this point, here is a simple illustration: take a union of two lines crossing at 90 degrees as the initial data. Then, the Brakke flow starting from it in [15] cannot stay time-independent due to the result of the present paper, since such crossing, even though it is stationary and a Brakke flow itself, does not satisfy the angle condition as above. The implication is that the method of construction in [15, 22] selects a special subclass of Brakke flows with this additional regularity property. This class is also compact with respect to the weak convergence (Lemma 6.1) and because of this, it is a natural category to study measure-theoretically.

There are numerous closely related results even if we focus on the 1-dimensional problem and we mention some of the most relevant works. The problem is also called curvature flow, curve-shortening flow or network flow in the literature. For an embedded closed curve as initial data, the well-known theorem due to Gage-Hamilton [9] and Grayson [11] says that the flow stays embedded and becomes convex, and eventually shrinks to a round circle. For a flow with a triple junction, Bronsard-Reitich [6] proved the short-time existence and uniqueness for $C^{2+\alpha}$ initial data. The long-time behavior of the flow was studied in [18, 16] for fixed boundary condition. For less regular initial data, Gösswein-Menzel-Pluda [10] studied the short-time existence and uniqueness within a $W^{2,p}$ class. For initial data with general junctions (non-120 degree triple junction or junction with more edges), Ilmanen-Neves-Schulze [13] showed the short-time existence of the flow. For more information on the existence and uniqueness for various related works, see [17] and the references therein. All of the aforementioned flows may be called classical network flows in the sense that the junctions are all triple junctions with equal angles as they evolve. For more general flow which allows topological changes, in [15, 22], the long-time existence was studied within the framework of geometric measure theory. It is expected that the solution is unique for “regular enough” initial network, but the precise condition is not known (see [8] for a related uniqueness question). The present work establishes an interesting connection between the solution of [15, 22] and classical network flow in the sense that the former is found to be within the measure-theoretic closure of the class of classical network flows.

Next we briefly describe the idea of the proof. In [15] (see Section 7 for comments on [22]), the first task is to construct a time-discrete approximate mean curvature flow. For $n = 1$, it is proper to say simply as “curvature” in place of “mean curvature”, but we may be referring to both 1-dimensional and general $n$-dimensional cases so that we continue to use “mean curvature”. Let $\Delta t_j$ be a time step size which converges to 0 as $j \to \infty$ and suppose that we inductively have $\Gamma_t$ at $t = (k-1)\Delta t_j$ for an integer $k$. To obtain $\Gamma_t$ at $t = k\Delta t_j$, a restricted Lipschitz deformation is first applied to $\Gamma_{(k-1)\Delta t_j}$ so that the resulting intermediate $\tilde{\Gamma}_{k\Delta t_j}$ is almost measure-minimized within a length scale of $O(1/j^2)$. Then one computes a smooth analogue of mean curvature vector $h_{\epsilon_j}$ of
\( \tilde{\Gamma}_{k\Delta t_j} \), and moves \( \tilde{\Gamma}_{k\Delta t_j} \) by \( h_{\varepsilon_j} \Delta t_j \) to obtain \( \Gamma_{k\Delta t_j} \). The parameter \( \varepsilon_j \) is much smaller than \( 1/j^2 \), and in view of the length scale of measure-minimization, \( \varepsilon_j \) is so small that \( h_{\varepsilon_j} \) behaves like the real mean curvature vector of \( \tilde{\Gamma}_{k\Delta t_j} \). For each \( j \), we continue the induction for \( k = 1, \ldots, [j/\Delta t_j] \) and define \( \Gamma_j(t) \) as a piecewise constant approximate flow for \( t \in [0,j] \). The limit of \( \{\Gamma_j(t)\}_{t \geq 0} \) as \( j \to \infty \) corresponds roughly to the desired Brakke flow. Because of the accompanying estimates, for almost all \( t \in \mathbb{R}^+ \), we can make sure that \( \Gamma_j(t) \) is almost measure-minimized within any ball of radius \( o(1/j^2) \) and that we have a control of square-integral of approximate mean curvature vector. This minimality in the 1-dimensional situation gives the result that \( \Gamma_j(t) \) is very close to either a line or a triple junction within any ball of radius \( o(1/j^2) \). We patch these short lines (or triple junctions) globally. The variation of these line or triple junction can be controlled by the square-integral of approximate mean curvature and this gives us \( C^{1,1/2} \) control of the curves in a length scale of \( O(1) \) independent of \( j \). Then we can make sure that \( \Gamma_j(t) \) behaves more or less like a regular network of triple junctions, and the limit with the \( L^2 \) curvature bound is expected to have the same type of regularity, except that some triple junctions may collapse and create junctions with multiple edges. On the other hand, the angle condition is preserved. To justify this argument, we need to evaluate various errors of approximations. We actually need to take a subsequence so that we have \( \{j_\ell\}_{\ell=1}^\infty \) in place of \( \{j\}_{j=1}^\infty \) in the above argument.

The organization of the paper is as follows. In Section 2 the existence theorem of [15] is recalled and the main regularity theorems are stated. Section 3 gathers relevant definitions, lemmas and estimates for approximate MCF from [15]. In Section 4 the main result is Theorem 4.1 which shows that the converging sets are asymptotically close to some measure-minimizing hypersurfaces in any ball of radius \( o(1/j^2) \) as \( \ell \to \infty \). In Section 5 specializing for the case of 1-dimension, we prove Theorem 5.7 which is a regularity theorem for the “flat” portion of the varifold with multiplicity. Up to this point, we are left with the analysis of isolated singularities, and Section 6 shows Theorem 2.2 and 2.3 as well as the compactness property. The last Section 7 gives some further comments.

2. Main results

We use the same notation stated in [15, Section 2]. Since some of the results obtained in this paper are for arbitrary dimensions, we recall the notation for general \( n \)-dimensional case in the following. The only minor notational difference from [15] is that we use \( z \) for a point in \( \mathbb{R}^{n+1} \), and when we specialize in \( n = 1 \), we use \( z = (x,y) \in \mathbb{R}^2 \), reserving \( x \) and \( y \) for the coordinates on \( \mathbb{R}^2 \). Moreover, for the consistency of presentation, we state in this section the results for the Brakke flow obtained in [15], even though Theorem 2.2, 2.4 hold true for the Brakke flow in [22] (see Section 7.1).
Let $\Omega \in C^2(\mathbb{R}^{n+1})$ be a weight function satisfying
\begin{equation}
0 < \Omega(z) \leq 1, \quad |\nabla \Omega(z)| \leq c_1 \Omega(z), \quad \|\nabla^2 \Omega(z)\| \leq c_1 \Omega(z)
\end{equation}
for all $z \in \mathbb{R}^{n+1}$ where $c_1$ is a constant. The function $\Omega$ is introduced to handle initial data $\Gamma_0$ with infinite $\mathcal{H}^n$ measure. If $\Gamma_0$ has finite $\mathcal{H}^n$ measure, we may take $\Omega(z) \equiv 1$ and $c_1 = 0$. If there exists $c > 0$ such that $\mathcal{H}^n(B_R \cap \Gamma_0) \leq \exp(cR)$ for all large $R$, for example, we may choose $\Omega(z) = \exp(-2c\sqrt{1+|z|^2})$ so that (2.1) is satisfied for a suitable $c_1$ depending on $c$ and $n$. With such choice, we have the following (2.2). We excerpt the main existence theorem of Brakke flow from [15, Theorem 3.2 and 3.5]:

**Theorem 2.1.** Suppose that $\Gamma_0 \subset \mathbb{R}^{n+1}$ is a closed countably $n$-rectifiable set whose complement $\mathbb{R}^{n+1} \setminus \Gamma_0$ is not connected and suppose
\begin{equation}
\mathcal{H}^n \mathbf{L}_\Omega (\Gamma_0) := \int_{\Gamma_0} \Omega(z) \, d\mathcal{H}^n(z) < \infty.
\end{equation}
For some $N \geq 2$, choose a finite collection of non-empty open sets $\{E_i\}_{i=1}^N$ in $\mathbb{R}^{n+1}$ such that they are mutually disjoint and $\bigcup_{i=1}^N E_i = \mathbb{R}^{n+1} \setminus \Gamma_0$. Then there exist a family of $n$-dimensional varifolds $\{V_t\}_{t \in \mathbb{R}^+} \subset \mathbf{V}_n(\mathbb{R}^{n+1})$ and a family of open sets $\{E_i(t)\}_{t \in \mathbb{R}^+}$ in $\mathbb{R}^{n+1}$ for each $i = 1, \ldots, N$ with the following property.

1. $V_0 = |\Gamma_0|$ and $E_i(0) = E_{0,i}$ for $i = 1, \ldots, N$.
2. For $\mathcal{L}^1$ a.e. $t \in \mathbb{R}^+$, $V_t \in \mathbf{IV}_n(\mathbb{R}^{n+1})$ and $h(\cdot, V_t) \in L^2(\|V_t\| \Omega)$.
3. For all $0 \leq t_1 < t_2 < \infty$ and $\phi \in C^1(\mathbb{R}^{n+1} \times \mathbb{R}^+; \mathbb{R}^+)$, we have
\begin{equation}
\|V_{t_1}\|(\phi(\cdot, t_1)) \leq \int_{t_1}^{t_2} \delta(V_t, (\phi(\cdot, t))) + \|V_t\| \|\phi_t(\cdot, t)\| dt,
\end{equation}
where $\|V_{t_1}\|(\phi(\cdot, t_1)) := \|V_{t_1}\|(\phi(\cdot, t_2)) - \|V_{t_1}\|(\phi(\cdot, t_1))$.
4. $E_1(t), \ldots, E_N(t)$ are mutually disjoint open sets for each $t \in \mathbb{R}^+$.
5. Let $d\mu := d\|V_t\| dt$. Then $\{z : (z, t) \in \text{spt} \mu\} = \mathbb{R}^{n+1} \setminus \bigcup_{i=1}^N E_i(t) = \bigcup_{i=1}^N \partial E_i(t)$ for each $t > 0$.
6. $\|V_t\| \geq \|\nabla \chi_{E_i(t)}\|$ for each $t \in \mathbb{R}^+$ and $i = 1, \ldots, N$.
7. For each $i = 1, \ldots, N$, $z \in \mathbb{R}^{n+1}$ and $R > 0$, $\chi_{E_i(t)} \in C([0, \infty); L^1(B_R(z)))$.
8. If $\mathcal{H}^n(\Gamma_0 \setminus \bigcup_{i=1}^N \partial^* E_{0,i}) = 0$, then $\lim_{t \to 0^+} \|V_t\| = \mathcal{H}^n \mathbf{L}_{\partial \Gamma_0}$.

Here, $|\Gamma_0|$ is the $n$-dimensional varifold naturally induced from the countably $n$-rectifiable set $\Gamma_0$, $\|V_t\|$ is the weight measure of $V_t$, $h(x, V_t)$ is the generalized mean curvature vector of $V_t$, and
\begin{equation}
\delta(V_t, \phi(\cdot, t))(h(\cdot, V_t)) := \int_{\mathbb{R}^{n+1}} \nabla \phi(z, t) \cdot h(z, V_t) - \phi(z, t)|h(z, V_t)|^2 \, d\|V_t\|(z).
\end{equation}

See more complete description of the properties of $V_t$ and $E_i(t)$ in [15]. The claim (8) is not stated in [15], but the same argument in [22, Proposition 6.10] shows this continuity property.

Specializing in the case of $n = 1$, the main theorem of the present paper in technical terms is the following.
Theorem 2.2. Suppose \( n = 1 \) and let \( \{V_t\}_{t \in \mathbb{R}^+} \) be the Brakke flow obtained in [15] as in Theorem 2.1. Then for almost all \( t \in \mathbb{R}^+ \), \( V_t \) has the following description at each \( z \in \text{spt} \|V_t\| \). There exist \( \rho > 0, k_R, k_L \in \mathbb{N} \), functions \( f_{1,R}, \ldots, f_{k_R,R} \in W^{2,2}([0,\rho]) \) and \( f_{1,L}, \ldots, f_{k_L,L} \in W^{2,2}([-\rho,0]) \) such that \( f_{1,R}(x) \leq \ldots \leq f_{k_R,R}(x) \) for \( x \in [0,\rho] \) and \( f_{1,L}(x) \leq \ldots \leq f_{k_L,L}(x) \) for \( x \in [-\rho,0] \),

\[
f_{1,R}(0) = \ldots = f_{k_R,R}(0) = f_{1,L}(0) = \ldots = f_{k_L,L}(0) = 0, \tag{2.5}
\]

\[
f'_{1,R}(0), \ldots, f'_{k_R,R}(0), f'_{1,L}(0), \ldots, f'_{k_L,L}(0) \in \{0, \pm \sqrt{3}\}, \tag{2.6}
\]

and after translation of \( z \) to the origin and an orthogonal rotation, we have

\[
\|V_t\|_{B_\rho} = \sum_{i=1}^{k_R} \|H^1_{L_{B_\rho} \cap \{(x,f_{i,R}(x)) : x \in [0,\rho]\}}\| + \sum_{i=1}^{k_L} \|H^1_{L_{B_\rho} \cap \{(x,f_{i,L}(x)) : x \in [-\rho,0]\}}\|. \tag{2.8}
\]

Here, \( f \in W^{2,2}([a,b]) \) means that \( f, f', f'' \) are in \( L^2([a,b]) \). It is well-known that \( W^{2,2}([a,b]) \subset C^{1,1/2}([a,b]) \) so that \( f' \) is well-defined as a \( 1/2 \)-Hölder continuous function on \([a,b]\). Note that each term of (2.7) is the inward-pointing unit tangent vector to the curve at the origin and the claim is that their vector sum is 0. To see the content of the claim clearly, consider the special case of \( k_R = k_L = 1 \). Then, we have \( f_{1,R} \) on \([0,\rho]\) and \( f_{1,L} \) on \([-\rho,0]\) with \( f_{1,R}(0) = f_{1,L}(0) = 0, f'_{1,R}(0) = f'_{1,L}(0) \in \{0, \pm \sqrt{3}\} \) by (2.5)-(2.7). In this case, \( f_1 : [-\rho,\rho] \to \mathbb{R} \) defined by \( f_1(x) := f_{1,R}(x) \) for \( x \in [0,\rho] \) and \( f_1(x) := f_{1,L}(x) \) for \( x \in [-\rho,0] \) is in \( W^{2,2}([-\rho,\rho]) \) and we have \( \|V_t\|_{B_\rho} = H^1_{L_{B_\rho} \cap \{(x,f_1(x)) : x \in [-\rho,\rho]\}} \). Hence, this is the case that \( \text{spt} \|V_t\| \) is represented locally as the graph of \( f_1 \). If \( k_R = 1 \) and \( k_L = 2 \) (and similarly for \( k_R = 2 \) and \( k_L = 1 \)), because of (2.6) and (2.7), we see that \( f'_{1,R}(0) = 0, f'_{1,L}(0) = \sqrt{3} \) and \( f'_{2,L}(0) = -\sqrt{3} \) have to be true. This case corresponds to the triple junction with multiplicity 1, that is, three curves meet at the origin with equal angles of 120 degrees. In other cases of \( k_R, k_L \geq 2 \), we have \( k_R \) curves coming from the right-hand side and \( k_L \) curves from the left-hand side, and they meet at the origin with angles of either 0, 60 or 120 degrees, and so that (2.7) holds true. If the derivatives of all the functions are equal at the origin, then \( k := k_R = k_L \) by (2.7), and this case corresponds to the situation that \( \text{spt} \|V_t\| \) is locally represented by \( W^{2,2} \) functions \( f_1 \leq \ldots \leq f_k \) which are tangent at the origin. It is also clear from this description that genuine junctions (with some non-zero angles between curves meeting at the junction) are isolated, and away from them, \( \text{spt} \|V_t\| \) is a union of embedded \( W^{2,2} \) curves which may be tangent to each other but which do not cross transversally.

If \( N = 2 \) (i.e., the “two-phase” case of \( \mathbb{R}^2 \setminus \Gamma_0 = E_{0,1} \cup E_{0,2} \)), we can conclude in the next Theorem 2.3 that there are no genuine junctions and the worst possible irregularities are curves being tangent.
Theorem 2.3. Assume in addition that \( N = 2 \) in Theorem 2.2. Then we have \( k_R = k_L (=: k) \) and \( f'_{1,R}(0) = \ldots = f'_{k,R}(0) = f'_{1,L}(0) = \ldots = f'_{k,L}(0) \). In particular, for almost all \( t \), \( \text{spt} \| V_t \| \) locally consists of a finite number of embedded \( W^{2,2} \) curves which are tangent if they intersect.

Next, we note that the class of Brakke flows with the regularity property in Theorem 2.2 (and 2.3 for \( N = 2 \)) is compact with respect to the natural weak convergence of measures (see Lemma 6.1). In particular, we have the following:

Theorem 2.4. Any tangent flow of Brakke flow obtained in [15] has the same regularity property as in Theorem 2.2 (and 2.3 for \( N = 2 \)).

See [24, 26] for the definition and properties of tangent flow. One of the corollaries is that the support of any static tangent flow (the one which is homogeneous and independent of time) is either a line, a triple junction, two lines crossing with 60 degrees, or three lines crossing with 60 degrees, all of them with possible integer multiplicities. If \( N = 2 \), then the static tangent flow is a line (with a possible integer multiplicity). We remark that the Brakke flow is \( C^\infty \) in a space-time neighborhood of a point if there exists a static tangent flow at that point which is a line with multiplicity 1. This is due to Brakke’s partial regularity theorem [5], the proof of which has been given in [14, 23]. If there exists a static tangent flow which is a triple junction with multiplicity 1, then the Brakke flow is \( C^{1,\alpha} \) in the parabolic sense in a space-time neighborhood of that point for all \( \alpha \in (0, 1) \) (see [25] for the detail). For further discussion on the main results, see Section 7.

3. Preliminaries

3.1. Basic definitions and lemmas. We recall some essential definitions and lemmas from [15] for the subsequent proofs. See [15, Section 4 & 5] for a more comprehensive treatment of these concepts.

Definition 3.1. An ordered collection \( \mathcal{E} = \{E_1, \ldots, E_N\} \) of subsets in \( \mathbb{R}^{n+1} \) is called an \( \Omega \)-finite open partition of \( N \) elements if

(a) \( E_1, \ldots, E_N \) are open and mutually disjoint;

(b) \( \mathcal{H}^n (\mathbb{R}^{n+1} \setminus \bigcup_{i=1}^N E_i) < \infty \);

(c) \( \bigcup_{i=1}^N \partial E_i \) is countably \( n \)-rectifiable.

We do not exclude the possibility that some of \( E_i \)’s are empty set \( \emptyset \). The set of all \( \Omega \)-finite open partitions of \( N \) elements is denoted by \( \mathcal{OP}^N_{\Omega} \). By Definition 3.1(b) and (c) as well as \( \Omega > 0 \), we have \( \mathcal{H}^n (B_R \setminus \bigcup_{i=1}^N E_i) < \infty \) for all \( R > 0 \) and

\[
\bigcup_{i=1}^N \partial E_i = \mathbb{R}^{n+1} \setminus \bigcup_{i=1}^N E_i. \tag{3.1}
\]

Given \( \mathcal{E} = \{E_1, \ldots, E_N\} \in \mathcal{OP}^N_{\Omega} \), we define (with a slight abuse of notation)

\[
\partial \mathcal{E} := \big| \bigcup_{i=1}^N \partial E_i \big| \in \mathbf{IV}_n (\mathbb{R}^{n+1}), \tag{3.2}
\]
which is a unit density varifold naturally induced from the countably $n$-rectifiable $\bigcup_{i=1}^{N} \partial E_i$. We also regard $\partial E$ as a set $\bigcup_{i=1}^{N} \partial E_i$ with no fear of confusion. The weight measure of $\partial E$ satisfies

$$\|\partial E\| = \mathcal{H}^n \Big|_{\bigcup_{i=1}^{N} \partial E_i}. \quad (3.3)$$

By Definition 3.1(b) and (3.1), we have $\|\partial E\|(\Omega) < \infty$.

**Definition 3.2.** Given $E = \{E_1, \ldots, E_N\} \in \mathcal{OP}_\Omega^N$, a function $f : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is called $E$-admissible if it is Lipschitz continuous and satisfies the following. Define $\tilde{E}_i := \text{int} (f(E_i))$ for $i = 1, \ldots, N$. Then:

(a) $\tilde{E}_1, \ldots, \tilde{E}_N$ are mutually disjoint;
(b) $\mathbb{R}^{n+1} \setminus \bigcup_{i=1}^{N} \tilde{E}_i \subset f(\bigcup_{i=1}^{N} \partial E_i);
(c) \sup_{z \in \mathbb{R}^{n+1}} |f(z) - z| < \infty.$

From the definition, one can prove (see [15, Lemma 4.4] for the proof)

**Lemma 3.3.** For $E = \{E_1, \ldots, E_N\} \in \mathcal{OP}_\Omega^N$ and a $E$-admissible function $f$, define $\tilde{E} := \{\tilde{E}_1, \ldots, \tilde{E}_N\}$ with $\tilde{E}_i := \text{int} (f(E_i))$. Then we have $\tilde{E} \in \mathcal{OP}_\Omega^N$.

In the following, $f_*E$ denotes the above $\tilde{E}$, that is, $f_*E := \tilde{E}$.

**Definition 3.4.** For every $j \in \mathbb{N}$, the class $A_j$ is defined as follows:

$$A_j := \{\phi \in C^2(\mathbb{R}^{n+1}; \mathbb{R}^+) : \phi(z) \leq \Omega(z), |\nabla \phi(z)| \leq j \phi(z), \|\nabla^2 \phi(z)\| \leq j \phi(z) \text{ for every } z \in \mathbb{R}^{n+1}\}. \quad (3.4)$$

**Definition 3.5.** For $E = \{E_1, \ldots, E_N\} \in \mathcal{OP}_\Omega^N$ and $j \in \mathbb{N}$, define $E(E, j)$ to be the set of all $E$-admissible functions $f$ such that:

(a) $|f(z) - z| \leq 1/j^2$ for every $z \in \mathbb{R}^{n+1}$;
(b) $\mathcal{L}^{n+1}(E_i \Delta \tilde{E}_i) \leq 1/j$ for all $i = 1, \ldots, N$, where $\tilde{E}_i := \text{int} (f(E_i))$;
(c) $\|\partial f_*E\| (\phi) \leq \|\partial E\| (\phi)$ for all $\phi \in A_j$. Here, $f_*E = \{\tilde{E}_1, \ldots, \tilde{E}_N\}$.

Since the identity map $f(z) = z$ is in $E(E, j)$, $E(E, j)$ is not empty.

**Definition 3.6.** Given $E \in \mathcal{OP}_\Omega^N$ and $j \in \mathbb{N}$, we define the quantity

$$\Delta_j \|\partial E\| (\Omega) := \inf_{f \in E(E, j)} \{\|\partial f_*E\| (\Omega) - \|\partial E\| (\Omega)\}. \quad (3.5)$$

Since the identity map is in $E(E, j)$, we have $\Delta_j \|\partial E\| (\Omega) \leq 0$. We also define a localized version of $E(E, j)$ and $\Delta_j \|\partial E\| (\Omega)$ as follows.

**Definition 3.7.** For $E \in \mathcal{OP}_\Omega^N$, $j \in \mathbb{N}$ and a compact set $C \subset \mathbb{R}^{n+1}$ we define

$$E(E, C, j) := \{f \in E(E, j) : \{z : f(z) \neq z\} \cup \{f(z) : f(z) \neq z\} \subset C\}, \quad (3.6)$$

$$\Delta_j \|\partial E\| (C) := \inf_{f \in E(E, C, j)} (\|\partial f_*E\| (C) - \|\partial E\| (C)). \quad (3.7)$$
We use the following (see [15, Lemma 4.12] for the proof):

**Lemma 3.8.** Suppose $\mathcal{E} = \{E_1, \ldots, E_N\} \in \mathcal{OP}_\Omega^N$, $j \in \mathbb{N}$, $C$ is a compact set, $f$ is $\mathcal{E}$-admissible such that

(a) $\{z : f(z) \neq z\} \cup \{f(z) : f(z) \neq z\} \subset C$;
(b) $|f(z) - z| \leq 1/j^2$ for all $z \in \mathbb{R}^{n+1}$;
(c) $\mathcal{L}^{n+1}(E_i \triangle E_j) \leq 1/j$ for all $i, j, N$ and where $E_i = \text{int}(f(E_i))$;
(d) $\|\partial f, \mathcal{E}\|(C) \leq \exp(-j \text{diam } C)\|\partial \mathcal{E}\|(C)$.

Then we have $f \in \mathcal{E}(\mathcal{E}, C, j)$.

Let $\psi \in C^\infty(\mathbb{R}^{n+1})$ be a radially symmetric function such that

$$
\psi(z) = 1 \text{ for } |z| \leq 1/2, \quad \psi(z) = 0 \text{ for } |z| \geq 1,
$$

$$
0 \leq \psi(z) \leq 1, \quad |\nabla \psi(z)| \leq 3, \quad \|\nabla^2 \psi(z)\| \leq 9 \text{ for all } z \in \mathbb{R}^{n+1}.
$$

Define for each $\varepsilon > 0$

$$
\hat{\Phi}_\varepsilon(z) := \frac{1}{(2\pi \varepsilon^2)^{n+1/2}} \exp \left(-\frac{|z|^2}{2\varepsilon^2}\right), \quad \Phi_\varepsilon(z) := c(\varepsilon)\psi(z)\hat{\Phi}_\varepsilon(z),
$$

where the constant $c(\varepsilon)$ is chosen so that $\int_{\mathbb{R}^{n+1}} \Phi_\varepsilon(z) \, dz = 1$.

**Definition 3.9.** For $V \in \mathcal{V}_n(\mathbb{R}^{n+1})$, we define $\Phi_\varepsilon * V \in \mathcal{V}_n(\mathbb{R}^{n+1})$ through

$$
(\Phi_\varepsilon * V)(\phi) := V(\Phi_\varepsilon * \phi) := \int_{\mathcal{G}_n(\mathbb{R}^{n+1})} \int_{\mathbb{R}^{n+1}} \phi(z - \hat{z}, S) \Phi_\varepsilon(\hat{z}) \, d\hat{z} \, dV(z, S)
$$

for $\phi \in C_c(\mathcal{G}_n(\mathbb{R}^{n+1}))$. For a Radon measure $\mu$ on $\mathbb{R}^{n+1}$, we define a Radon measure $\Phi_\varepsilon * \mu$ on $\mathbb{R}^{n+1}$ through

$$
(\Phi_\varepsilon * \mu)(\phi) := \mu(\Phi_\varepsilon * \phi) := \int_{\mathbb{R}^{n+1}} \int_{\mathbb{R}^{n+1}} \phi(z - \hat{z}) \Phi_\varepsilon(\hat{z}) \, d\hat{z} \, d\mu(z)
$$

for $\phi \in C_c(\mathbb{R}^{n+1})$.

One may prove the following by the Fubini theorem (see [15] Section 4 (4.28))):

**Lemma 3.10.** The Radon measure $\Phi_\varepsilon * \mu$ may be identified with the $C^\infty$ function

$$
(\Phi_\varepsilon * \mu)(z) := \int_{\mathbb{R}^{n+1}} \Phi_\varepsilon(\hat{z} - z) \, d\mu(\hat{z})
$$

since $(\Phi_\varepsilon * \mu)(\phi) = \int_{\mathbb{R}^{n+1}} (\Phi_\varepsilon * \mu)(z) \phi(z) \, dz$ holds for $\phi \in C_c(\mathbb{R}^{n+1})$.

We next define the “smoothed first variation” of $V$ as follows:

**Definition 3.11.** For $V \in \mathcal{V}_n(\mathbb{R}^{n+1})$, define the following $C^\infty$ vector field

$$
(\Phi_\varepsilon * \delta V)(z) := \int_{\mathcal{G}_n(\mathbb{R}^{n+1})} S(\nabla \Phi_\varepsilon(\hat{z} - z)) \, dV(\hat{z}, S).
$$

For any vector field $g \in C_c(\mathbb{R}^{n+1}; \mathbb{R}^{n+1})$, we define

$$
(\Phi_\varepsilon * \delta V)(g) := \int_{\mathbb{R}^{n+1}} (\Phi_\varepsilon * \delta V)(z) \cdot g(z) \, dz.
$$
The following can be verified (see [15, Lemma 4.16] for the proof):

**Lemma 3.12.** For \( V \in V_n(\mathbb{R}^{n+1}) \), we have
\[
\Phi_\varepsilon \ast \|V\| = \|\Phi_\varepsilon \ast V\|, \tag{3.15}
\]
\[
\int_{\mathbb{R}^{n+1}} (\Phi_\varepsilon \ast \delta V)(z) \cdot g(z) \, dz = \delta V(\Phi_\varepsilon \ast g) \quad \text{for} \quad g \in C^1_c(\mathbb{R}^{n+1};\mathbb{R}^{n+1}), \tag{3.16}
\]
\[
\Phi_\varepsilon \ast \delta V = \delta(\Phi_\varepsilon \ast V). \tag{3.17}
\]

The following is the “smoothed mean curvature vector” of \( V \):

**Definition 3.13.** For \( V \in V_n(\mathbb{R}^{n+1}) \) and \( \varepsilon > 0 \), define
\[
h_\varepsilon(\cdot, V) := -\Phi_\varepsilon \ast \left( \frac{\Phi_\varepsilon \ast \delta V}{\Phi_\varepsilon \ast \|V\| + \varepsilon \Omega^{-1}} \right). \tag{3.18}
\]

We use the following quantity as a proxy for a weighted “\( L^2 \)-norm of smoothed mean curvature vector” (see [15, Lemma 5.2]):

**Lemma 3.14.** For \( V \in V_n(\mathbb{R}^{n+1}) \) with \( \|V\|(\Omega) \leq M \) and \( \varepsilon \in (0, \epsilon_1) \) (where \( \epsilon_1 \) depends only on \( n, c_1 \) and \( M \)), we have
\[
\int_{\mathbb{R}^{n+1}} \frac{|\Phi_\varepsilon \ast \delta V|^2 \Omega}{\Phi_\varepsilon \ast \|V\| + \varepsilon \Omega^{-1}} \, dz < \infty. \tag{3.19}
\]

3.2. **Construction of approximate MCF.** In this subsection, we summarize the relevant results of approximate MCF established in [15, Section 6]. The following is [15, Proposition 6.1] which proves the existence of a time-discrete approximate MCF starting from \( \mathcal{E}_0 \). Here, given \( \Gamma_0 \) and \( \{E_{0,i}\}_{i=1}^N \) as in Theorem 2.1, we set \( \mathcal{E}_0 := \{E_{0,1}, \ldots, E_{0,N}\} \). Since \( \Gamma_0 \) is a closed countably n-rectifiable set satisfying (2.2), one can see that \( \mathcal{E}_0 \in \mathcal{OP}_\Omega^N \) and \( \partial \mathcal{E}_0 = |\Gamma_0| \).

**Proposition 3.15.** Given \( \mathcal{E}_0 \in \mathcal{OP}_\Omega^N \) and \( j \in \mathbb{N} \) with \( j \geq c_1 \), there exist \( \varepsilon_j \in (0, j^{-6}) \), \( p_j \in \mathbb{N} \), a family of open partitions \( \mathcal{E}_{j,\ell} \in \mathcal{OP}_\Omega^N \) (\( \ell = 0, 1, \ldots, j^{2p_j} \)) with the following property:
\[
\mathcal{E}_{j,0} = \mathcal{E}_0 \quad \text{for all} \quad j \in \mathbb{N} \tag{3.20}
\]
and with the notation of
\[
\Delta t_j := \frac{1}{2^p_j}, \tag{3.21}
\]
we have
\[
\|\partial \mathcal{E}_{j,\ell}\|(\Omega) \leq \|\partial \mathcal{E}_0\|(\Omega) \exp\left( \frac{c_1^2}{2} \ell \Delta t_j \right) + \frac{2 \varepsilon_j^\frac{1}{8}}{c_1^2} \left( \exp\left( \frac{c_1^2}{2} \ell \Delta t_j \right) - 1 \right), \tag{3.22}
\]
\[
\frac{\|\partial \mathcal{E}_{j,\ell}\|(\Omega) - \|\partial \mathcal{E}_{j,\ell-1}\|(\Omega)}{\Delta t_j} + \frac{1}{4} \int_{\mathbb{R}^{n+1}} \frac{|\Phi_\varepsilon \ast \delta(\partial \mathcal{E}_{j,\ell}\|)(\Omega)}{\Phi_\varepsilon \ast \|\partial \mathcal{E}_{j,\ell}\| + \varepsilon_j \Omega^{-1}} - (1 - j^{-5}) \Delta t_j \|\partial \mathcal{E}_{j,\ell-1}\|(\Omega) \leq \varepsilon_j^\frac{1}{8} + \frac{c_1^2}{2} \|\partial \mathcal{E}_{j,\ell-1}\|(\Omega), \tag{3.23}
\]
\[
\frac{\|\partial \mathcal{E}_{j,\ell}\|(\phi) - \|\partial \mathcal{E}_{j,\ell-1}\|(\phi)}{\Delta t_j} \leq \delta(\partial \mathcal{E}_{j,\ell}, \phi)(h_{\varepsilon_j}(\cdot, \partial \mathcal{E}_{j,\ell})) + \varepsilon_j^\frac{1}{8} \tag{3.24}
\]
Theorem 3.18. The limit of subsequence (denoted by the same index) \( \{ \partial \mathcal{E}_j \} \) should be understood as the limit when \( c_1 \to 0^+ \) and is equal to \( \mathcal{H}^n(\Gamma_0) + \varepsilon_j \frac{t}{\ell} \Delta t_j \).

We remark on the relation of \( \varepsilon_j \) and \( \Delta t_j \) in the proof of Proposition 3.17. As is explained in [15, p.83],

\[
\Delta t_j = \frac{1}{2p_j} \in (2^{-\frac{1}{3} j}, 2^{\frac{n+20}{3} j}].
\]

Definition 3.16. We define for each \( j \in \mathbb{N} \) with \( j \geq \max\{1, c_1\} \) a family \( \mathcal{E}_j(t) \in \mathcal{P}^N_{\Omega} \) for \( t \in [0, j] \) by

\[
\mathcal{E}_j(t) := \mathcal{E}_{j, \ell} \quad \text{if} \quad t \in ((\ell - 1)\Delta t_j, \ell \Delta t_j].
\]

The next is [15, Proposition 6.4]:

Proposition 3.17. There exist a subsequence \( \{ j_\ell \}_{\ell=1}^{\infty} \) and a family of Radon measures \( \{ \mu_t \}_{t \in \mathbb{R}^+} \) on \( \mathbb{R}^{n+1} \) such that

\[
\lim_{\ell \to \infty} \| \partial \mathcal{E}_{j_\ell}(t) \|((\phi)) = \mu_t(\phi)
\]

for all \( \phi \in C_c(\mathbb{R}^{n+1}) \) and for all \( t \in \mathbb{R}^+ \). For all \( T < \infty \), we have

\[
\limsup_{\ell \to \infty} \int_0^T \left( \int_{\mathbb{R}^{n+1}} \left| \frac{1}{\Phi_{\varepsilon_{j_\ell}}(t)} \right|^2 \Omega \right) dt < \infty.
\]

Because of (3.28) and Fatou’s lemma, for a.e. \( t \in \mathbb{R}^+ \), we may choose a time-dependent further subsequence (denoted by the same index) \( \{ j_\ell \}_{\ell=1}^{\infty} \) such that

\[
\sup_{\ell \in \mathbb{N}} \left( \int_{\mathbb{R}^{n+1}} \left| \frac{1}{\Phi_{\varepsilon_{j_\ell}}(t)} \right|^2 \Omega \right) dt < \infty.
\]

Theorem 3.18. The limit of \( \{ \partial \mathcal{E}_{j_\ell}(t) \}_{\ell=1}^{\infty} \) satisfying (3.29) is necessarily an integral varifold \( V_t \) with \( \| V_t \| = \mu_t \), and \( V_t \) has a locally square integrable generalized mean curvature. Moreover, \( \{ V_t \}_{t \in \mathbb{R}^+} \) is a Brakke flow.

The main purpose of the present paper is to prove that spt \( \| V_t \| \) for such \( t \) consists of locally finite number of \( W^{2,2} \) curves with junctions of specific type, namely, these curves at junctions meet either at 0, 60 or 120 degrees. This proves Theorem 2.2 since the Brakke flow obtained in [15] is precisely the limit of sequence satisfying (3.29) for a.e. \( t \).

For the rest of the paper, dropping the variable \( t \), we let \( \{ \partial \mathcal{E}_{j_\ell} \}_{\ell=1}^{\infty} \) be the subsequence with the uniform bound (3.29).
4. Small scale behavior of approximate sequence

The purpose of this section is to prove that \( \partial E_{j\ell} \) is almost measure-minimizing within a length scale smaller than \( 1/j_{\ell}^2 \). This is an expected result due to Definition 3.5(a) and the uniform bound \( -\Delta_{j\ell} \|\partial E_{j\ell}\|((\Omega)) \leq c_2 \Delta t_{j\ell} \) as in (3.29). Since \( \Delta t_{j\ell} \leq \varepsilon_{j\ell}^{3n+20} < j_{\ell}^{-6(3n+20)} \) by (3.25) and \( \varepsilon_{j\ell} \in (0, j_{\ell}^{-6}) \) in Proposition 3.15, we have

\[
-\Delta_{j\ell} \|\partial E_{j\ell}\|((\Omega)) < c_2 j_{\ell}^{-6(3n+20)},
\]

and even after rescaling \( \partial E_{j\ell} \) by \( 1/j_{\ell}^2 \), it is still very close to being measure-minimizing under Lipschitz deformations of admissible class, \( E(E_{j\ell}, j_{\ell}) \).

Suppose throughout this section that \( \{z^{(\ell)}\}_{\ell=1}^{\infty} \subset \mathbb{R}^{n+1} \) is a bounded sequence and suppose that \( \{r_{\ell}\}_{\ell=1}^{\infty} \) is a sequence of positive numbers such that

\[
\lim_{\ell \to \infty} r_{\ell}(j_{\ell})^2 = 0 \quad (4.2)
\]

and

\[
\lim_{\ell \to \infty} r_{\ell}(j_{\ell})^3 = \infty. \quad (4.3)
\]

It is not necessary but to fix the idea, we set \( r_{\ell} := 1/(j_{\ell})^{2.5} \) so that (4.2) and (4.3) are satisfied. The motivation for the choice of \( r_{\ell} \) is that we want \( r_{\ell} = o(1/j_{\ell}^2) \) but not too small so that \( \varepsilon_{j\ell} \ll r_{\ell} \).

For each \( \ell \in \mathbb{N} \), we define \( F_{\ell} : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1} \) by

\[
F_{\ell}(z) := \frac{z - z^{(\ell)}}{r_{\ell}} \quad (4.4)
\]

and define

\[
V_{\ell} := (F_{\ell})_#(\partial E_{j\ell}). \quad (4.5)
\]

Using Lemma 4.2 below, for each \( R > 0 \), we can prove

\[
\limsup_{\ell \to \infty} \|V_{\ell}\|(B_R) \leq \mathcal{H}^n(\partial B_1) R^n. \quad (4.6)
\]

Once this is proved, by the standard compactness theorem of Radon measures, we have a converging subsequence (denoted by the same index) and a limit \( V \in \mathbf{V}_n(\mathbb{R}^{n+1}) \), namely,

\[
\lim_{\ell \to \infty} V_{\ell}(\phi) = V(\phi)
\]

for all \( \phi \in C_c(G_n(\mathbb{R}^{n+1})) \). The main result in this section is the following characterization of this limit \( V \).

**Theorem 4.1.** Let \( V \) be obtained as above and suppose that \( V \neq 0 \). Then \( V \) is measure-minimizing with respect to any compact diffeomorphism and belongs to \( \mathbf{IV}_n(\mathbb{R}^{n+1}) \) with unit density. Moreover, \( V \) satisfies the following.

1. For \( N \geq 3 \) and \( n = 1 \), \( \text{spt} \|V\| \) is either a line or a triple junction (with three half-lines) of 120 degrees.
(2) For $N \geq 3$ and $n \geq 2$, $\text{spt} \|V\|$ consists of three mutually disjoint sets, $\text{reg} V$, $\text{sing}_1 V$ and $\text{sing}_2 V$. The set $\text{reg} V$ is relatively open in $\text{spt} \|V\|$ and is a real-analytic minimal hypersurface. For any point in $\text{sing}_1 V$, there exists a neighborhood in which $\text{spt} \|V\|$ consists of three real-analytic minimal hypersurfaces with boundaries which meet along an $n-1$-dimensional real-analytic surface. The set $\text{sing}_2 V$ is a closed set of Hausdorff dimension $\leq n-2$. In the case of $n = 2$, $\text{sing}_2 V$ is a set of isolated points in $\mathbb{R}^3$.

(3) For $N = 2$ and $1 \leq n \leq 6$, $\text{spt} \|V\|$ is a hyperplane.

(4) For $N = 2$ and $n \geq 7$, $\text{spt} \|V\|$ is a real-analytic minimal hypersurface away from a closed set of Hausdorff dimension $\leq n-7$ and a set of isolated points if $n = 7$.

One can expect that this should be true due to the “almost measure-minimizing property” (4.1), and Theorem 4.4 is known in a variety of different settings in area-minimizing problems. In fact, the present setting of the small scale is close to that of Almgren’s $(F,\varepsilon,\delta)$ minimal sets [3]. On the other hand, since it is not precisely the same with the use of open partitions and the admissible class, we give a self-contained proof (except that we cite results from [15] and well-known results in geometric measure theory) and the rest of this section is devoted to the proof of Theorem 4.4.

We use results for $n = 1$ in the subsequent sections.

We start with the following upper density ratio bound.

**Lemma 4.2.** For any $R > 0$,

$$\limsup_{\ell \to \infty} \frac{1}{(r_{tR})^n} \|\partial E_{j_{\ell}}\|(B_{\ell R}(z^{(\ell)})) \leq \mathcal{H}^n(\partial B_1).$$

**Proof.** Suppose that $\ell$ is large so that $r_{tR} < \frac{1}{2(j_{\ell})^2}$. Assume for a contradiction that there exists $\beta > 0$ such that

$$\|\partial E_{j_{\ell}}\|(B_{\ell R}(z^{(\ell)})) \geq (\beta + \mathcal{H}^n(\partial B_1))(r_{tR})^n$$

for some large $\ell$. Define a Lipschitz map $\hat{F} : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ as follows (though $\hat{F}$ depends on $\ell$, we drop the dependence for simplicity). Choose a ball $B_{\ell}(z') \subset U_{tR}(z^{(\ell)}) \setminus \partial E_{j_{\ell}}$. Let $\hat{F}$ be a retraction map such that $U_{tR}(z^{(\ell)}) \setminus U_{\ell}(z')$ is projected onto $\partial B_{r_{tR}}(z^{(\ell)})$. The ball $B_{\ell}(z')$ is expanded onto $B_{r_{tR}}(z^{(\ell)})$ bijectively. For $z \in \mathbb{R}^{n+1} \setminus U_{tR}(z^{(\ell)})$, define $\hat{F}(z) = z$. Note this $\hat{F}$ is $E_{j_{\ell}}$-admissible (see Definition 3.2 as noted in [15] 4.3.4). We also claim that $\hat{F} \in E(E_{j_{\ell}},j_{\ell})$ (see Definition 3.5).

To prove this, we use Lemma 3.8 (note that $E(E, C, j) \subset E(E, j)$ by definition). We take $C$ in the statement as $B_{r_{tR}}(z^{(\ell)})$. The first three conditions (a)-(c) (with $j$ there replaced by $j_{\ell}$) are satisfied for all large $j_{\ell}$. Thus we only need to check (d) of Lemma 3.8

$$\|\partial \hat{F}_E E_{j_{\ell}}\|(B_{r_{tR}}(z^{(\ell)})) \leq \exp(-2j_{\ell}r_{tR})\|\partial E_{j_{\ell}}\|(B_{r_{tR}}(z^{(\ell)})).$$

(4.8)

Since $\partial \hat{F}_E E_{j_{\ell}} \cap U_{tR}(z^{(\ell)}) = \emptyset$, we have

$$\|\partial \hat{F}_E E_{j_{\ell}}\|(B_{r_{tR}}(z^{(\ell)})) = \|\partial \hat{F}_E E_{j_{\ell}}\|(\partial B_{r_{tR}}(z^{(\ell)})) \leq \mathcal{H}^n(\partial B_{r_{tR}}(z^{(\ell)})) = (r_{tR})^n \mathcal{H}^n(\partial B_1).$$

(4.9)
By (4.2), (4.7) and (4.9), we have (4.8) for all large $\ell$. Thus we proved that $\hat{F} \in E(\mathcal{E}_{j\ell}, j\ell)$ by Lemma 3.8. Recall that

$$\Delta_{j\ell} \|\partial \mathcal{E}_{j\ell}\| (\Omega) = \inf_{f \in E(\mathcal{E}_{j\ell}, j\ell)} (\|\partial f_*\mathcal{E}_{j\ell}\| (\Omega) - \|\partial \mathcal{E}_{j\ell}\| (\Omega))$$

(4.10)

which is bounded from below by $-c_2 \Delta t_{j\ell}$ due to (3.29). Since $\partial \hat{F}_*\mathcal{E}_{j\ell}$ and $\partial \mathcal{E}_{j\ell}$ coincide outside $B_{r_\ell R}(z^{(\ell)})$, we have by (4.9) and (4.7) that

$$\|\partial \hat{F}_*\mathcal{E}_{j\ell}\| (\Omega) - \|\partial \mathcal{E}_{j\ell}\| (\Omega)$$

$$\leq \left( \left( \max_{B_{r_\ell R}(z^{(\ell)})} \Omega \right) \mathcal{H}^n(\partial B_1) - \left( \min_{B_{r_\ell R}(z^{(\ell)})} \Omega \right) (\beta + \mathcal{H}^n(\partial B_1)) \right) (r_\ell R)^n$$

$$\leq \left( \min_{B_{r_\ell R}(z^{(\ell)})} \Omega \right) \left( \mathcal{H}^n(\partial B_1) \exp(2c_1 r_\ell R) - (\beta + \mathcal{H}^n(\partial B_1)) \right) (r_\ell R)^n$$

(4.11)

for all large $\ell$, where we also used $\Omega(z) \leq \Omega(z') \exp(c_1|z - z'|)$ which follows from (2.1). By (3.29), (4.10) and (4.11), for all large $\ell$, we have

$$\left( \min_{B_{r_\ell R}(z^{(\ell)})} \Omega \right) \beta (r_\ell R)^n \leq 2c_2 \Delta t_{j\ell}.$$ 

Since $\{z^{(\ell)}\}_{\ell=1}^\infty$ is a bounded sequence and $\Omega > 0$, we have $\inf_{\ell} \min_{B_{r_\ell R}(z^{(\ell)})} \Omega > 0$. We now obtain a contradiction since $\Delta t_{j\ell} \ll (j\ell)^{-3n}$ and $(j\ell)^{-3n} \ll (r_\ell)^n$ as $\ell \to \infty$ by (4.11) and (4.3). \(\square\)

This proves (4.6) and the existence of a limit $n$-dimensional varifold $V$. It is also useful to consider the convergence of partitions. For this purpose, for each $\mathcal{E}_{j\ell}$, write

$$\{E_{j\ell,k}\}_{k=1}^N = \mathcal{E}_{j\ell}.$$ 

(4.12)

Note that $\bigcup_{k=1}^N \partial E_{j\ell,k} = \partial \mathcal{E}_{j\ell}$ and each $E_{j\ell,k}$ satisfies $\|\nabla \chi_{F_{\ell}(E_{j\ell,k})}\| \leq \|V\|$ by \([4]\) Proposition 3.62. Thus by the compactness theorem of BV functions, there exist a further subsequence (denoted by the same index) and Caccioppoli sets $E_1, \ldots, E_N \subset \mathbb{R}^{n+1}$ such that

$$\chi_{F_{\ell}(E_{j\ell,k})} \to \chi_{E_k}$$

(4.13)

for each $k = 1, \ldots, N$ in $L^1_{loc}(\mathbb{R}^{n+1})$ and a.e. pointwise $z \in \mathbb{R}^{n+1}$ as $\ell \to \infty$. Since $\{F_{\ell}(E_{j\ell,k})\}_{k=1}^N$ are open partitions of $\mathbb{R}^{n+1}$, one can also prove that $E_1, \ldots, E_N$ satisfy

$$\mathcal{L}^{n+1}(E_i \cap \bar{E}_k) = 0$$

for $i \neq k$ and $\sum_{k=1}^N \chi_{E_k} = 1$ a.e. on $\mathbb{R}^{n+1}$. (4.14)

By the lower-semicontinuity of the BV semi-norm, we also have

$$\|\nabla \chi_{E_k}\| \leq \|V\|$$

(4.15)

for all $k = 1, \ldots, N$.

Next we prove that $V$ is measure-minimizing in the following sense, proving the first claim of Theorem 4.1.
Lemma 4.3. For any diffeomorphism $g \in C^1(\mathbb{R}^{n+1}; \mathbb{R}^{n+1})$ with $g|_{\mathbb{R}^{n+1}\setminus U_R}(z) = z$ for some $R > 0$, we have
\[
\|g_2V\|(B_R) \geq \|V\|(B_R). \tag{4.16}
\]

Proof. Suppose on the contrary that there exists a diffeomorphism $g$ as above such that $\|g_2V\|(B_R) - \|V\|(B_R) < -\beta$ for some $\beta > 0$. Then, by the definition of the push-forward of varifold, we have
\[
\int_{G_n(B_R)} (|A_n \nabla g \circ S| - 1) dV(z, S) < -\beta.
\]
Since $|A_n \nabla g(z) \circ S| - 1 = 0$ for $z \in \mathbb{R}^{n+1} \setminus U_R$, $|A_n \nabla g(z) \circ S| - 1$ is an element of $C_c(G_n(\mathbb{R}^{n+1}))$. Thus, by the varifold convergence, for all sufficiently large $\ell$, we have
\[
\|g_\ell V\|(B_R) - \|V\|(B_R) = \int_{G_n(B_R)} (|A_n \nabla g(z) \circ S| - 1) dV_\ell(z, S) < -\beta. \tag{4.17}
\]
We want to interpret this inequality in terms of $\partial \mathcal{E}_{\ell \ell}$. Consider the map $\hat{F}_\ell = (F_\ell)^{-1} \circ g \circ F_\ell : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$, where $F_\ell$ is defined as in (4.4). This is a diffeomorphism which is the identity map on $\mathbb{R}^{n+1} \setminus U_{r_\ell R}(z(\ell))$ and maps $B_{r_\ell R}(z(\ell))$ to itself bijectively. Since $g_\ell V_\ell = (g \circ F_\ell)_\ell(\partial \mathcal{E}_{\ell \ell})$, we have from (4.17)
\[
\|\partial(\hat{F}_\ell)_\ell^* \mathcal{E}_{\ell \ell}\|(B_{r_\ell R}(z(\ell))) - \|\partial \mathcal{E}_{\ell \ell}\|(B_{r_\ell R}(z(\ell))) < -\beta r^n_\ell. \tag{4.18}
\]
We claim that $\hat{F}_\ell \in \mathcal{E}(\mathcal{E}_{\ell \ell}, j_\ell)$ for all sufficiently large $\ell$. If this holds, note that we would obtain a contradiction just as in the proof of Lemma 4.2 by the same argument following (4.10), which would conclude the proof. Thus we only need to check $\hat{F}_\ell \in \mathcal{E}(\mathcal{E}_{\ell \ell}, j_\ell)$. The $\mathcal{E}_{\ell \ell}$-admissibility is fine since it is a diffeomorphism. Just as before, we use Lemma 3.8 with $C = B_{r_\ell R}(z(\ell))$ there. The first three conditions (a)-(c) are satisfied since $r_\ell R < \frac{1}{2(\ell \ell)}$. To check (d), that is,
\[
\|\partial(\hat{F}_\ell)_\ell^* \mathcal{E}_{\ell \ell}\|(B_{r_\ell R}(z(\ell))) \leq \exp(-2j_\ell r_\ell R)\|\partial \mathcal{E}_{\ell \ell}\|(B_{r_\ell R}(z(\ell))),
\]
we use (4.18). For above to be true, we only need to see from (4.18) that
\[
\|\partial \mathcal{E}_{\ell \ell}\|(B_{r_\ell R}(z(\ell))) - \beta r^n_\ell \leq \exp(-2j_\ell r_\ell R)\|\partial \mathcal{E}_{\ell \ell}\|(B_{r_\ell R}(z(\ell)))
\]
but this holds true since
\[
\|\partial \mathcal{E}_{\ell \ell}\|(B_{r_\ell R}(z(\ell)))(1 - \exp(-2j_\ell r_\ell R)) \leq 2(r_\ell R)^n \mathcal{H}^n(\partial B_1)(1 - \exp(-2j_\ell r_\ell R)) \leq \beta r^n_\ell
\]
for all sufficiently large $\ell$, where we used $j_\ell r_\ell \to 0$ and Lemma 4.2. This proves that $\hat{F}_\ell \in \mathcal{E}(\mathcal{E}_{\ell \ell}, j_\ell)$. 

In particular, we obtain

Proposition 4.4. The limit varifold $V$ is stable and stationary.

Next, we prove the following lower density ratio bound of the limit varifold $V$ using [15] Proposition 7.2].
Lemma 4.5. There exists a constant $c_3 \in (0, 1)$ depending only on $n$ with the following property. For any $x \in \text{spt} \|V\|$ and $R > 0$, we have $\|V\|(B_R(x)) \geq c_3 R^n$.

Proof. Let $c_3$ be the constant appearing in [15, Proposition 7.2] also as $c_3$ and assume for a contradiction that we have some $\hat{z} \in \text{spt} \|V\|$ and $R > 0$ such that $\|V\|(B_R(\hat{z})) < c_3 R^n$. Then, since $\|V\| \to \|V\|$ and $\hat{z} \in \text{spt} \|V\|$, we have a sequence $\hat{z}(\ell) \to \hat{z}$ such that $\hat{z}(\ell) \in \text{spt} \|V_\ell\|$ and $\|V_\ell\|(B_R(\hat{z}(\ell))) < c_3 R^n$ for all sufficiently large $\ell$. In terms of $\partial E_{j_\ell}$, this means

$$\|\partial E_{j_\ell}\|(B_{\ell R}(\hat{z}(\ell) + r_\ell \hat{z}(\ell))) < c_3 (\ell R)^n$$

for all sufficiently large $\ell$ with $\hat{z}(\ell) + r_\ell \hat{z}(\ell) \in \text{spt} \|\partial E_{j_\ell}\|$. We may apply [15, Proposition 7.2] to $E_{j_\ell}$ in $B_{\ell R}(\hat{z}(\ell) + r_\ell \hat{z}(\ell))$, which gives a $E_{j_\ell}$-admissible function $g_\ell$ and a radius $r_\ell \in [r_\ell R/2, r_\ell R]$ such that, writing $B(\ell) := B_{r_\ell}(\hat{z}(\ell) + r_\ell \hat{z}(\ell))$,

1. $g_\ell(z) = z$ for $z \in \mathbb{R}^{n+1} \setminus B(\ell)$,
2. $g_\ell(z) \in B(\ell)$ for $z \in B(\ell)$,
3. $\|\partial (g_\ell)_* E_{j_\ell}\|(B(\ell)) \leq \frac{1}{2} \|\partial E_{j_\ell}\|(B(\ell))$.

By Lemma 3.8, we also have $g_\ell \in E(E_{j_\ell}, j_\ell)$ for all large $\ell$. Then we have with (2.1) and (3)

$$\|\partial (g_\ell)_* E_{j_\ell}\|(\Omega) - \|\partial E_{j_\ell}\|(\Omega) \leq \max_{B(\ell)} \|\partial (g_\ell)_* E_{j_\ell}\|(B(\ell)) - \min_{B(\ell)} \|\partial E_{j_\ell}\|(B(\ell))$$

$$\leq \min_{B(\ell)} \left( \exp(2c_1 r_\ell R) - 1 \right) \|\partial E_{j_\ell}\|(B(\ell)).$$

Since the left-hand side of (4.19) is bounded below by $-c_2 \Delta t_{j_\ell}$, we have

$$\left( \inf_{\ell} \min_{B(\ell)} \left( 1 - \exp(2c_1 r_\ell R) \right) \right) \|\partial E_{j_\ell}\|(B(\ell)) \leq c_2 \Delta t_{j_\ell}. \quad (4.20)$$

Since $\Delta t_{j_\ell} \ll r_\ell^n$ and $B_{r_\ell R}(\hat{z}(\ell) + r_\ell \hat{z}(\ell)) \subset B(\ell)$, (4.20) implies $\|V_\ell\|(B_{R/2}(\hat{z}(\ell))) \leq r_\ell^n \|\partial E_{j_\ell}\|(B(\ell)) \to 0$ as $\ell \to \infty$. Since $\hat{z}(\ell) \to \hat{z}$, this implies that $\|V\|(U_{R/2}(\hat{z})) = 0$, contradicting $\hat{z} \in \text{spt} \|V\|$. \qed

Proposition 4.4, Lemma 4.5, and Allard’s rectifiability theorem [1, 5.5(1)] show that $V$ is rectifiable in particular. We next see that

Lemma 4.6. The limit varifold $V$ is integral.

Proof. Since $V$ is rectifiable, for $\|V\|$ a.e. $\hat{z} \in \mathbb{R}^{n+1}$, $V$ has the approximate tangent space $\text{Tan}^n(\|V\|, \hat{z})$ and the blow-up of $V$ at $\hat{z}$ converges to $\theta^n(\|V\|, \hat{z})|\text{Tan}^n(\|V\|, \hat{z})|$ as varifolds. We prove that $\theta(\|V\|, \hat{z}) \in \mathbb{N}$ in the following, which proves that $V$ is integral. For simplicity, we write

$$\beta := \theta^n(\|V\|, \hat{z}) \text{ and } T := \text{Tan}^n(\|V\|, \hat{z}). \quad (4.21)$$

For each $\ell \in \mathbb{N}$, define $g_\ell(z) := \ell(z - \hat{z})$. Because of the above, we have $\lim_{\ell \to \infty} (g_\ell)_* V = \beta |T|$. Since $V_\ell \to V$, we may choose a further subsequence (denoted by the same index) such that $\lim_{\ell \to \infty} (g_\ell)_* V_\ell = \beta |T|$. Since $V_\ell = (F_\ell)_2 \partial E_{j_\ell}$ and

$$(g_\ell \circ F_\ell)(z) = \frac{z - \hat{z}(\ell) - r_\ell \hat{z}}{r_\ell/\ell},$$

we have

$$(g_\ell \circ F_\ell)_* (\partial E_{j_\ell}) = (g_\ell \circ F_\ell)^{-1} \circ \partial (g_\ell)_* E_{j_\ell}.$$(4.22)
we have \((g_\ell)\partial V_\ell = (g_\ell \circ F_\ell)\partial E_{j_\ell}\). We set
\[
\bar{r}_\ell := r_\ell / \ell, \quad \bar{z}^{(\ell)} := z^{(\ell)} + r_\ell \hat{z} \quad \text{and} \quad \tilde{F}_\ell(z) := (g_\ell \circ F_\ell)(z) = \frac{z - z^{(\ell)}}{\bar{r}_\ell}.
\] (4.22)

In the following, we assume that \(\bar{z}^{(\ell)} = 0\) for simplicity. The general case can be handled with suitable parallel translations and no difficulties arise. Writing
\[
\tilde{V}_\ell := (g_\ell)\partial V_\ell = (\tilde{F}_\ell)\partial E_{j_\ell},
\] (4.23)

and by the above discussion and notation, we have
\[
\lim_{\ell \to \infty} \tilde{V}_\ell = \beta |T|.
\] (4.24)

By choosing a further subsequence if necessary, we have (4.3) and (4.2) since \(\bar{r}_\ell < r_\ell\) with \(r_\ell\) replaced by \(\bar{r}_\ell\), i.e.,
\[
\lim_{\ell \to \infty} \bar{r}_\ell(j_\ell)^2 = 0 \quad \text{and} \quad \lim_{\ell \to \infty} \bar{r}_\ell(j_\ell)^3 = \infty.
\] (4.25)

Suppose that \(\nu\) is the smallest positive integer strictly greater than \(\beta\), namely,
\[
\nu \in \mathbb{N} \quad \text{and} \quad \nu \in (\beta, \beta + 1).
\] (4.26)

We use [15, Lemma 8.1]. In the assumption of [15] Lemma 8.1, we fix \(\alpha = 1/2\), and choose \(\zeta \in (0,1)\) so that
\[
\nu - \zeta > \beta.
\] (4.27)

This choice is possible due to (4.26). Let \(\gamma \in (0,1)\) and \(j_0 \in \mathbb{N}\) be constants given as the result of [15] Lemma 8.1, which have the following properties. We write
\[
E^\ast(r) := \{z \in \mathbb{R}^{n+1} : |T(z)| \leq r, \text{dist}(T^\perp(z), Y) \leq (1 + R^{-1}r)\rho\}
\] (4.28)

for \(r, R, \rho \in (0, \infty), Y \subset \mathbb{R}^{n+1}\) and assume

(1) \(E = \{E_j\}_{j=1}^N \in \mathcal{OP}_{\mathbb{N}}^N, j \in \mathbb{N}\) with \(j \geq j_0, R \in (0, \frac{1}{2}j^{-2}), \rho \in (0, \frac{1}{2}j^{-2})\),
(2) \(\rho \geq \alpha R\),
(3) \(Y \subset T^\perp\) satisfies \(\mathcal{H}^0(Y) \leq \nu\) with \(\text{diam} \ Y \leq j^{-2}\) and \(\theta^n(\|\partial E\|, \zeta') = 1\) for all \(\zeta' \in Y\),
(4) \(\int_{G_n(E^\ast(r))} \|S - T\| \|\partial E\|(z, S) \leq \gamma \|\partial E\|(E^\ast(r))\) for all \(r \in (0, R)\),
(5) \(\Delta_j \|\partial E\|(E^\ast(r)) \geq -\gamma \|\partial E\|(E^\ast(r))\) for all \(r \in (0, R)\).

The conclusion under these assumptions (1)-(5) is that
\[
\|\partial E\|(E^\ast(R)) \geq (\mathcal{H}^0(Y) - \zeta)\omega_n R^n.
\] (4.29)

We will apply this result for \(\partial E_{j_\ell}\) with \(j = j_\ell \geq j_0, R = \bar{r}_\ell\) and \(\rho = 2\bar{r}_\ell\). Note that \(R, \rho \in (0, \frac{1}{2}j_{\ell}^{-2})\) is satisfied due to (1.25). We consider a cylinder
\[
C_\ell := \{z \in \mathbb{R}^{n+1} : |T(z)| \leq \bar{r}_\ell, |T^\perp(z)| \leq \bar{r}_\ell\}
\]
for \(\ell \in \mathbb{N}\) and consider the behavior of \(\partial E_{j_\ell}\) in \(C_\ell\). Note that
\[
\tilde{F}_\ell(C_\ell) = \{z \in \mathbb{R}^{n+1} : |T(z)| \leq 1, |T^\perp(z)| \leq 1\}.
\]
Let
\[ G_\ell := \{ z \in \partial \mathcal{E}_{j\ell} \cap C_\ell : \theta^n(\|\partial \mathcal{E}_{j\ell}\|, z) = 1 \} \]
and note that \( \|\partial \mathcal{E}_{j\ell}\|(C_\ell \setminus G_\ell) = 0 \) due to the rectifiability of \( \partial \mathcal{E}_{j\ell} \). We then set
\[ G_\ell^* := \{ z \in T : \mathcal{H}^0(T^{-1}(z) \cap G_\ell) \geq \nu \}. \]

We next prove that for all \( z \in G_\ell^* \), there exist a set \( Y \subset T^{-1}(z) \cap G_\ell \) with \( \mathcal{H}^0(Y) = \nu \) and \( r_z \in (0, \tilde{r}_\ell) \) such that, writing \( E^*(z, r) := E^*(r) + z \) where \( E^*(r) \) is defined with respect to \( Y \) and \( T \), either
\[ \int_{G_n(E^*(z, r_z))} \| S - T \| d(\partial \mathcal{E}_{j\ell}) > \gamma \|\partial \mathcal{E}_{j\ell}\|(E^*(z, r_z)) \tag{4.30} \]
or
\[ \Delta_{j\ell}\|\partial \mathcal{E}_{j\ell}\|(E^*(z, r_z)) < -\gamma \|\partial \mathcal{E}_{j\ell}\|(E^*(z, r_z)). \tag{4.31} \]

In fact, let \( Y \subset T^{-1}(z) \cap G_\ell \) be any subset with \( \mathcal{H}^0(Y) = \nu \). By the definition of \( G_\ell^* \), there exists such a \( Y \). If there were no \( r_z \in (0, \tilde{r}_\ell) \) with both (4.30) and (4.31), then we have all the assumptions (1)-(5) satisfied and (4.29) shows
\[ \|\partial \mathcal{E}_{j\ell}\|(E^*(z, \tilde{r}_\ell)) \geq (\nu - \zeta)\omega_n \tilde{r}_\ell^n. \tag{4.32} \]

On the other hand, \( E^*(z, \tilde{r}_\ell) = \{ z' \in \mathbb{R}^{n+1} : |T(z' - z)| \leq \tilde{r}_\ell, \text{dist}(T^\perp(z'), Y) \leq 4\tilde{r}_\ell \} \), and since \( Y \) is not empty and \( Y \subset C_\ell \), we have
\[ E^*(z, \tilde{r}_\ell) \subset \{ z' \in \mathbb{R}^{n+1} : |T(z' - z)| \leq \tilde{r}_\ell, |T^\perp(z')| \leq 5\tilde{r}_\ell \} =: \hat{E}(z, \tilde{r}_\ell). \tag{4.33} \]

Since \( \|(\tilde{F}_\ell)z\partial \mathcal{E}_{j\ell}\| \to \|\beta|T|\| = \beta \mathcal{H}^n \mathcal{L}_T \), we have
\[ \limsup_{\ell \to \infty} \tilde{r}_\ell^{-n}\|\partial \mathcal{E}_{j\ell}\|((\hat{E}(z, \tilde{r}_\ell))) = \beta \omega_n. \tag{4.34} \]

This shows with (4.32) and (4.33) that \( \nu - \zeta \leq \beta \), which is a contradiction to (4.27). The convergence (4.34) may be made uniform in \( z \). Thus for all sufficiently large \( \ell \) and for all \( z \in G_\ell^* \) and \( Y \) as above, we proved that either (4.30) or (4.31) hold for some \( r_z \in (0, \tilde{r}_\ell) \). Now we use the Besicovitch covering theorem to the family of \( n \)-dimensional closed balls \( \{ B^n_{r_z}(z) : z \in G_\ell^* \} \) in \( T \). Then we have a subfamilies \( C_1, \ldots, C_{B(n)} \) each of which consists of a family of disjoint balls and that
\[ G_\ell^* \subset \bigcup_{l=1}^{B(n)} \bigcup_{B^n_{r_z}(z) \in C_l} B^n_{r_z}(z). \]

Because of the definitions of \( E^*(z, r) \) and \( E^*(r) \), we have for any \( z \in G_\ell^* \)
\[ T^{-1}(B^n_{r_z}(z)) \cap C_\ell \subset E^*(z, r_z) \subset T^{-1}(B^n_{r_z}(z)) \cap \{ z' \in \mathbb{R}^{n+1} : |T^\perp(z')| \leq 5\tilde{r}_\ell \}. \]
Thus, we have
\[
\|\partial \mathcal{E}_{j_\ell}\|(T^{-1}(G^*_\ell) \cap C_\ell) \\
\leq \|\partial \mathcal{E}_{j_\ell}\|(\cup_{i=1}^{B(n)} \cup B^a_{r_\ell}(z) \cap C_\ell) \\
\leq \sum_{i=1}^{B(n)} \sum_{B^a_{r_\ell}(z) \cap C_i} \|\partial \mathcal{E}_{j_\ell}\|(E^a(z, r_\ell)) \\
= \sum_{i=1}^{B(n)} \sum_{B^a_{r_\ell}(z) \cap C_i} \gamma^{-1}(\int_{G_n(E^a(z, r_\ell))} \|S - T\| d(\partial \mathcal{E}_{j_\ell}) - \Delta_{j_\ell}\|\partial \mathcal{E}_{j_\ell}\|(E^a(z, r_\ell))) \\
\leq \mathcal{B}(n) \gamma^{-1}(\int_{\mathcal{G}_n(\hat{C}_\ell)} \|S - T\| d(\partial \mathcal{E}_{j_\ell}) - \Delta_{j_\ell}\|\partial \mathcal{E}_{j_\ell}\|(\hat{C}_\ell)),
\]
where \(\hat{C}_\ell := \{ z \in \mathbb{R}^{n+1} : |T(z)| \geq 2\sqrt{\ell}, |T^{\perp}(z)| \leq 5\sqrt{\ell} \}. \) Since \((\hat{F}_\ell)_{z} \partial \mathcal{E}_{j_\ell}\) converges to \(\beta|T|\) as varifolds, we have
\[
\lim_{\ell \to \infty} \tilde{r}^{-n}_\ell \int_{\mathcal{G}_n(\hat{C}_\ell)} \|S - T\| d(\partial \mathcal{E}_{j_\ell})(z, S) = 0
\]
and we conclude that
\[
\lim_{\ell \to \infty} \tilde{r}^{-n}_\ell \|\partial \mathcal{E}_{j_\ell}\|(T^{-1}(G^*_\ell) \cap C_\ell) = 0.
\]
(4.36)
We have
\[
\|T_{z} \partial \mathcal{E}_{j_\ell}\|(C_\ell) = \int_{T \cap C_\ell} \mathcal{H}^0(T^{-1}(z) \cap G_\ell) d\mathcal{H}^n(z).
\]
Since \(\mathcal{H}^0(T^{-1}(z) \cap G_\ell) \leq \nu - 1\) on \(T \setminus G^*_\ell\), and because of (4.36), we obtain
\[
\lim_{\ell \to \infty} \tilde{r}^{-n}_\ell \|T_{z} \partial \mathcal{E}_{j_\ell}\|(C_\ell) \leq \limsup_{\ell \to \infty} \tilde{r}^{-n}_\ell \int_{C_\ell \cap T \setminus G^*_\ell} \mathcal{H}^0(T^{-1}(z) \cap G_\ell) d\mathcal{H}^n(z)
\leq (\nu - 1) \lim_{\ell \to \infty} \tilde{r}^{-n}_\ell \mathcal{H}^n(T \cap C_\ell) = (\nu - 1)\omega_n.
\]
(4.37)
On the other hand, we have
\[
\lim_{\ell \to \infty} \tilde{r}^{-n}_\ell \|T_{z} \partial \mathcal{E}_{j_\ell}\|(C_\ell) = \lim_{\ell \to \infty} \|T_{\hat{z}} \hat{V}_{\ell}\|(\hat{F}_\ell(C_\ell)) = \beta\|T\|(B^a_{r_\ell}) = \beta\omega_n.
\]
This proves that \(\beta \leq \nu - 1\). By (4.26), we have \(\beta = \nu - 1\) and \(\beta \in \mathbb{N}\).

We next show

**Lemma 4.7.** The limit varifold \(V\) is a unit density varifold, that is, \(\theta^n(\|V\|, z) = 1\) for \(\|V\|\) a.e. \(z\). Moreover, any tangent cone and any blow-down limit of \(V\) are also unit density varifolds.

**Proof.** Suppose the contrary. Then there exists \(z \in \mathbb{R}^{n+1}\) such that the blowup of \(V\) at \(z\) converges to \(\nu|T|\) for some \(\nu \in \mathbb{N}\) with \(\nu \geq 2\) and \(T \in \mathcal{G}(n+1, n)\). As we saw in the proof of Lemma 4.6, let \(\hat{F}_\ell\) be chosen so that \((\hat{F}_\ell)_{z} \partial \mathcal{E}_{j_\ell} \to \nu|T|\). Without loss of generality, we may assume that \(T = \mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}\). In the following, we define a Lipschitz map which reduces the measure in the cylinder of radius 1. Namely, in such a cylinder, we have a measure of \((\hat{F}_\ell)_{z} \partial \mathcal{E}_{j_\ell}\) close to that
of \( \nu \) parallel discs and we will reduce it to that of one disc. Let \( \delta > 0 \) be a small number to be chosen. Write \( z = (z_1, \ldots, z_n, z_{n+1}) = (\hat{z}, z_{n+1}) \). In the cylinder \( |\hat{z}| \leq 1 \), define

\[
g(\hat{z}, z_{n+1}) = \begin{cases} 
(\hat{z}, z_{n+1}) & \text{if } |z_{n+1}| \geq \delta, \\
(\hat{z}, 0) & \text{if } |z_{n+1}| \leq \frac{\delta}{2}, \\
(\hat{z}, 2z_{n+1} - \delta) & \text{if } \frac{\delta}{2} \leq z_{n+1} \leq \delta, \\
(\hat{z}, 2z_{n+1} + \delta) & \text{if } -\delta \leq z_{n+1} \leq -\frac{\delta}{2},
\end{cases}
\]

and in the annular region \( 1 \leq |\hat{z}| \leq 1 + \delta \), define

\[
g(\hat{z}, z_{n+1}) = \begin{cases} 
(\hat{z}, z_{n+1}) & \text{if } |z_{n+1}| \geq \delta \text{ or } |z_{n+1}| \leq |\hat{z}| - 1, \\
(\hat{z}, |\hat{z}| - 1) & \text{if } |\hat{z}| - 1 \leq z_{n+1} \leq \frac{|\hat{z}| - 1}{2} + \frac{\delta}{2}, \\
(\hat{z}, 2z_{n+1} - \delta) & \text{if } \frac{|\hat{z}| - 1}{2} + \frac{\delta}{2} \leq z_{n+1} \leq \delta, \\
(\hat{z}, 1 - |\hat{z}|) & \text{if } \frac{1 - |\hat{z}|}{2} - \frac{\delta}{2} \leq z_{n+1} \leq 1 - |\hat{z}|, \\
(\hat{z}, 2z_{n+1} + \delta) & \text{if } -\delta \leq z_{n+1} \leq 1 - \frac{|\hat{z}|}{2} - \frac{\delta}{2}.
\end{cases}
\]

For \( z \) with \( |\hat{z}| > 1 + \delta \), define \( g(z) = z \). The map \( g \) is defined so that it crushes the region \( \{ z : |\hat{z}| \leq 1, |z_{n+1}| \leq \frac{\delta}{2} \} \) to the disc \( B^\rho_0 \subset T \) and stretches the top and the bottom. In the annular region, a small triangular region is also crushed to an \( n \)-dimensional set (Figure 2). Since \( g \) is a retraction, one can check that it is an admissible map for any open partition. Now consider \( \partial(g \circ \tilde{F}_\ell)_*, \mathcal{E}_{j_\ell} \) for all large \( \ell \) and compare its measure to \( (\tilde{F}_\ell)_* \partial \mathcal{E}_{j_\ell} \). Since \( (\tilde{F}_\ell)_* \partial \mathcal{E}_{j_\ell} \) converges to \( \nu|T| \), we have

\[
\|(\tilde{F}_\ell)_* \partial \mathcal{E}_{j_\ell}\|(\{ z : |\hat{z}| \leq 1 + \delta, 1 \geq |z_{n+1}| \geq \delta/2 \}) \rightarrow 0
\]

as \( \ell \rightarrow \infty \). Let \( A \subset \mathbb{R}^{n+1} \) be open set defined by

\[
A = \{ |\hat{z}| \leq 1, \frac{\delta}{2} < |z_{n+1}| < \delta \} \cup \{ 1 \leq |\hat{z}| < 1 + \delta, \frac{|\hat{z}| - 1}{2} + \frac{\delta}{2} < |z_{n+1}| < \delta \}.
\]

The map \( g \) stretches the set \( A \) by twice, and \( g \) maps \( A \) bijectively to its image \( g(A) \). Thus, using (4.40), we obtain

\[
\|\partial(g \circ \tilde{F}_\ell)_*, \mathcal{E}_{j_\ell}\|(g(A)) = \|(g \circ \tilde{F}_\ell)_* \partial \mathcal{E}_{j_\ell}\|(g(A)) \leq 2^n \|(\tilde{F}_\ell)_* \partial \mathcal{E}_{j_\ell}\|(A) \rightarrow 0
\]

as \( \ell \rightarrow \infty \). Because of the property of \( g \), in the region \( \{ |\hat{z}| < 1 + \delta, |z_{n+1}| < \delta \} \setminus g(A) \), we have

\[
\partial(g \circ \tilde{F}_\ell)_*, \mathcal{E}_{j_\ell} \subset \partial(g(A)) \cup ((\tilde{F}_\ell)_* \partial \mathcal{E}_{j_\ell} \setminus g(A)).
\]

Note that \( \mathcal{H}^n(\partial(g(A))) \cap \{ |\hat{z}| < 1 + \delta, |z_{n+1}| < \delta \} \leq \omega_n + c(n)\delta \). Combining (4.41) and (4.42), we have

\[
\limsup_{\ell \rightarrow \infty} \|\partial(g \circ \tilde{F}_\ell)_*, \mathcal{E}_{j_\ell}\|(g(A)) \leq \omega_n + c(n)\delta.
\]
Since \( \|(\tilde{F}_\ell)_\sharp \partial E_{j_\ell}\|((\{ |\hat{z}| < 1 + \delta, |z_{n+1}| < \delta \}) \to (1 + \delta)^n \omega_n \nu \) with \( \nu \geq 2 \) as \( \ell \to \infty \), for suitably small \( \delta \) depending only on \( n \), we see that \( g \) reduces the measure by definite amount for all large \( \ell \). By the similar argument as in the previous proofs, we may obtain a contradiction to the almost measure-minimizing property. The argument up to this point is also valid for any limit varifold obtained from \( V \) under dilations centered at any point. Since \( V \) is stationary, the tangent cone is integral stationary varifold and we can prove that it has unit density for almost everywhere. By (4.6), we have \( \|V\|(B_R) \leq \mathcal{H}^n(\partial B_1)R^n \) for all \( R > 0 \), so that the blow-down limit centered at arbitrary point exists and is integral stationary varifold which is also a cone (the latter can be proved by the same argument for the proof of tangent cone being a cone). Thus any tangent cone or blow-down limit of \( V \) has the same density property.

Next we see that \( V \) is “two-sided” in the following sense. Here recall the definition of \( E_1, \ldots, E_N \) as in (4.13). The stationarity of \( V \) shows that \( \|V\| = \|V|_{spt\|V\|} \) (see [20, 17.9(1)]). By (4.15), each \( \chi_{E_k} \) is constant on each connected component of \( \mathbb{R}^{n+1} \setminus spt\|V\| \). By (4.14), there is only one \( k \) such that \( \chi_{E_k} = 1 \) there. Thus, we may regard each \( E_k \) to be open and \( \cup_{k=1}^N E_k = \mathbb{R}^{n+1} \setminus spt\|V\| \).

**Lemma 4.8.** For any point \( z \) with \( \theta^n(\|V\|, z) = 1 \), there are two different indices \( k_1, k_2 \in \{1, \ldots, N\} \) and some \( r > 0 \) such that \( L^{n+1}(U_r(z) \cap E_{k_1}) > 0 \) for \( i = 1, 2 \) and \( U_r(z) \cap spt\|V\| = U_r(z) \cup (E_{k_1} \cup E_{k_2}) \).

**Proof.** Since \( V \) is stationary and integral, by the Allard regularity theorem [1], \( spt\|V\| \) is a real-analytic minimal hypersurface in some neighborhood of such \( z \). Thus for sufficiently small neighborhood, \( U_r(z) \setminus spt\|V\| \) consists of two connected non-empty open sets. To prove the claim, we only need to prove that this set is \( (E_{k_1} \cup E_{k_2}) \cap U_r(z) \) with \( k_1 \neq k_2 \). For a contradiction, assume \( k_1 = k_2 \) and we may assume \( k_1 = k_2 = 1 \) without loss of generality. We proceed just as in the proof of Lemma 4.7. Since \( \chi_{\tilde{F}_\ell(E_{j_{\ell,1}})} \to \chi_{E_1} = 1 \) in \( L^1(U_r(z)) \) as \( \ell \to \infty \), we may choose \( \{\tilde{F}_\ell\} \) so that \( (\tilde{F}_\ell)_\sharp \partial E_{j_{\ell,1}} \to |T| \) (note that the multiplicity is 1) and additionally so that

\[
\chi_{\tilde{F}_\ell(E_{j_{\ell,1}})} \to 1 \text{ in } L^1_{loc}(\mathbb{R}^{n+1}).
\]  

(4.44)

We use the same Lipschitz map \( g \) as in Lemma 4.7 to reduce the measure as follows. Because of (4.44) and Fubini theorem, for a.e. \( \delta > 0 \) (as in the proof of Lemma 4.7), we have

\[
\lim_{\ell \to \infty} \mathcal{H}^n(\{z : |\hat{z}| \leq 1, |z_{n+1}| = \delta/2 \} \setminus \tilde{F}_\ell(E_{j_{\ell,1}}) = 0.
\]  

(4.45)

With such \( \delta \), let \( g \) be defined as in (4.38) and (4.39). Consider \((g \circ \tilde{F}_\ell)_\sharp \partial E_{j_{\ell,1}} =: \{E_{j_{\ell,1}, k}\}_{k=1}^N \). Recall that \( E_{j_{\ell,1}, k} \) is the set of interior points of \((g \circ \tilde{F}_\ell)(E_{j_{\ell,1}, k}) \) for each \( k = 1, \ldots, N \). We pay special attention to \( E_{j_{\ell,1},1} \). Suppose that \((\hat{z}, \delta/2) \in \tilde{F}_\ell(E_{j_{\ell,1}}) \) and \((\hat{z}, -\delta/2) \in \tilde{F}_\ell(E_{j_{\ell,1}}) \) for \( \hat{z} \) with \( |\hat{z}| < 1 \). Since \( \tilde{F}_\ell(E_{j_{\ell,1}}) \) is open, there are some neighborhoods of \((\hat{z}, \pm \delta/2) \) which also belong to \( \tilde{F}_\ell(E_{j_{\ell,1}}) \). The map \( g \) sends both \((\hat{z}, \pm \delta/2) \) to the same point \((\hat{z}, 0) \) and there will be some neighborhood of \((\hat{z}, 0) \) which is included in \((g \circ \tilde{F}_\ell)(E_{j_{\ell,1}}) \). Thus it is an interior point of \((g \circ \tilde{F}_\ell)(E_{j_{\ell,1}}) \) and \((\hat{z}, 0) \in E_{j_{\ell,1}} \).
Lemma 4.9. Suppose that $k \rightarrow \infty$ as $\ell \rightarrow \infty$. Thus we proved $\omega_n \rightarrow 0$. Hence we again see that $\omega_n \rightarrow 0$. Since $\omega_n \rightarrow 0$, we may obtain $\lim \inf \| g \circ \tilde{\ell} \| (g(A)) \leq c(n)\delta$ instead, without $\omega_n$ on the right-hand side. Since $(\tilde{\ell})_J^\ell \| g \circ \tilde{\ell} \| (\{ |\tilde{z}| < 1+\delta, |z_{n+1}| < \delta \}) \rightarrow (1+\delta)^n\omega_n$ as $\ell \rightarrow \infty$, we again see that $g$ reduces the measure by definite amount and we may argue similarly to obtain a contradiction. Thus we proved $k_1 \neq k_2$, ending the proof. 

Lemma 4.9. Suppose that $V$ has a tangent cone or blow-down limit which is given as an orthogonal rotation of $|S \times \mathbb{R}^{n-1}| \in \mathbf{IV}_n(\mathbb{R}^{n+1})$, where $S \subset \mathbb{R}^2$ is a finite union of half-lines emanating from the origin. Then, (i) $S$ consists of three half-lines with equal 120 degree angles or, (ii) $S$ is a line through the origin.

Proof. Consider the case $n = 1$. We need to exclude the possibility that $S$ consists of more than three half-lines. For a contradiction, assume the contrary. If there are four or more half-lines, then there would be at least one pair of half-lines intersecting with an angle $\leq 90^\circ$. Since one can reduce the length of such pair of half-lines by a Lipschitz map, we may follow the similar procedure as 

in Lemma 4.7. Thus let $\{ \tilde{\ell} \}$ be chosen so that $(\tilde{\ell})_J^\ell \| g \circ \tilde{\ell} \| (\partial S) \rightarrow |S|$ as $\ell \rightarrow \infty$. For the Lipschitz map, we simply give the schematic picture in Figure 3 which describes the map on the upper half part of the bisected region between the two half-lines. On the lower half, the map is symmetrically defined. Let $A$ and $B$ be closed sets indicated in Figure 3. The Lipschitz map $g$ is defined so that the region $A$ is piece-wise smoothly expanded to cover the union of $A$ and $B$ bijectively while the region $B$ is crushed to the solid line segments in the figure on the right-hand side. Except for the neighborhood of endpoints, the region $A$ is away from $S$ so that $\lim_{\ell \rightarrow \infty} \| (\tilde{\ell})_J^\ell \| (g(A)) = 0$. Consider $\| \partial (g \circ \tilde{\ell}) \| (\partial S) \rightarrow |S|$ as $\ell \rightarrow \infty$. Thus most of the measure of $\| \partial (g \circ \tilde{\ell}) \| (A \cup B)$ lies on $B \backslash (A \cup \text{int} B)$. Now let $C$ be the union of $A \cup B$ and its reflection with respect to the bisecting line, and let $g$ be defined symmetrically on $C \backslash (A \cup B)$. We have the most of the measure of $\| \partial (g \circ \tilde{\ell}) \| (C)$ on
the solid line segments, and the $H^1$ measure is strictly less than $H^1(S \cap C)$. Thus for all large $\ell$, 
\[ \| \partial (g \circ \tilde{F}_{\ell}) \|_2 \| E_\ell \|(C) \text{ is strictly smaller than } \| \tilde{F}_{\ell} \|_2 \| \partial E_\ell \|(C), \]
which would lead to a contradiction to the almost measure-minimizing property just as in Lemma \ref{lem:4.7}. For $n > 1$, the picture is similar. If $S$ has more than three half-lines, again we have a pair of two half-hyperplanes intersecting with an angle $\leq 90^\circ$. Then one can construct a Lipschitz map $g$ which is the same on $\mathbb{R}^2 \times \{0\}$ as $g$ for $n = 1$ and which is homogeneously extended in the $\mathbb{R}^{n-1}$ direction on $\{(z', \hat{z}) \in \mathbb{R}^2 \times \mathbb{R}^{n-1} : |\hat{z}| \leq R\}$ for a large $R$. On $\{(z', \hat{z}) \in \mathbb{R}^2 \times \mathbb{R}^{n-1} : R \leq |\hat{z}| \leq R + 1\}$, as $|\hat{z}|$ changes from $R$ to $R + 1$, one can change $g$ on $\mathbb{R}^2 \times \{\hat{z}\}$ piece-wise smoothly to the identity map. Then, using the same idea, the reduction of measure in $\{(z', \hat{z}) : |\hat{z}| \leq R\}$ due to the map $g$ is proportional to $R^{n-1}$ for all large $\ell$. The possible increase of mass in $\{(z', \hat{z}) : R \leq |\hat{z}| \leq R + 1\}$ can be estimated by a constant multiple of $R^{n-2}$ and we again see a definite amount of reduction for sufficiently large $R$, from which we may derive a contradiction as in Lemma \ref{lem:4.7} \hfill \Box

Finally we give a proof of Theorem \ref{thm:4.1}

**Proof.** The claim that the limit $V$ is measure-minimizing is proved in Lemma \ref{lem:4.3} and the claim of unit density is proved in Lemma \ref{lem:4.7}. We prove (1)-(4) next. For $n = 1$, since $V$ is a 1-dimensional stationary integral varifold, \cite{2} shows that spt $\|V\|$ consists of locally finite line segments with discrete junctions. At these junctions, Lemma \ref{lem:4.9} shows that they have to be triple junctions. We claim that there can be only one triple junction at most. If there is no triple junction, then it is a line, so assume that we have at least one junction. To see that there cannot be more than one junction, we may consider a blow-down limit centered at one of the junctions. We shift the junction to the origin. By the monotonicity formula, we know that such limit $\tilde{V}$ is a cone, and again by Lemma \ref{lem:4.9} the limit is also a triple junction (it cannot be a line). This implies that there exists a sequence $r \to \infty$ such that $\lim_{r \to \infty} \|V\|(B_{2r})/(2r) = 3/2$. Since the origin is a triple junction, we have $\|V\|(B_r)/(2r) = 3/2$ for sufficiently small $r$. But then we have $\|V\|(B_r)/(2r) = 3/2$ for all $r > 0$ by the monotonicity formula and $V$ itself has to be a cone. Thus $V$ can have at most one triple junction. This proves (1). The claim (2) follows from Federer’s dimension reducing argument \cite{7}, Lemma \ref{lem:4.9} and by the well-known free-boundary regularity theorem (see for example \cite{21} and the references therein). More precisely, at the top $(n - 1)$-dimensional stratum $\text{sing}_1 V$ of singularities, the tangent cone has to be some orthogonal rotation of $|S \times \mathbb{R}^{n-1}|$, and Lemma \ref{lem:4.9} specifies that $S$ has to be a triple junction. Then by \cite{21}, $\text{sing}_1 V$ has the desired regularity. Federer’s dimension reducing argument shows that the next dimensional stratum $\text{sing}_2 V$ has Hausdorff dimension $\leq n - 2$. For the case of $N = 2$, by Lemma \ref{lem:4.8} note that there cannot be triple junctions for all dimensions. Thus, for $n = 1$, spt $\|V\|$ is a line, and for $n \geq 2$, $\text{sing}_1 V$ is empty. Since the tangent cone of $V$ is stable, again by the dimension reducing argument, the Hausdorff dimension of $\text{sing}_2 V$ is $\leq n - 7$. For $2 \leq n \leq 6$, since the blow-down limit of $V$ is a stable minimal cone with unit density, it is
5. Behavior at larger scales

In this section, we specialize in the case of \( n = 1 \). The idea of the line of proof is the following. We know asymptotically how \( \partial \mathcal{E}_{j\ell} \) looks like within the length scale of \( O(\ell) \) due to Theorem \([1.1]\). Namely, they are very close to either a line segment or a triple junction with three half-lines. We would like to patch this local picture together globally. To do so, we take advantage of the \( L^2 \) bound of smoothed curvature \([5.29]\) which is a good enough quantity to control the variation of tangent lines of curves. Smoothing parameter \( \varepsilon_{j\ell} \) is much smaller than \( \ell \) so that \([5.29]\) serves like a real \( L^2 \) curvature bound for \( \partial \mathcal{E}_{j\ell} \). In the following, we first single out a “good portion” of \( \partial \mathcal{E}_{j\ell} \), denoted by \( Z_{\ell} \). We show that \( Z_{\ell} \) looks more or less like a network of \( C^{1,1/2} \) curves.

**Definition 5.1.** Let \( \{r_\ell\}_{\ell=1}^\infty \) be a sequence satisfying \([1.2]\) and \([1.3]\). For each \( \ell \in \mathbb{N} \), define

\[
Z_{\ell} := \{ z \in \partial \mathcal{E}_{j\ell} : \inf_{r \in (0,r_\ell)} \mathcal{H}^1(\partial \mathcal{E}_{j\ell} \cap B_r(z))/2r \geq 1 \}, \quad Z_{\ell}^c := \partial \mathcal{E}_{j\ell} \setminus Z_{\ell}.
\]  

(5.1)

**Lemma 5.2.** The set \( Z_{\ell} \) is closed and for any \( R > 0 \) we have

\[
r^2 \mathcal{H}^1(Z_{\ell}^c \cap B_R) \leq -\frac{c_4 R^2}{\text{min}_{B_{R+1}}} \Delta_{j\ell} \| \partial \mathcal{E}_{j\ell} \|_2(\Omega),
\]

(5.2)

where \( c_4 > 0 \) is an absolute constant.

**Remark 5.3.** Once we have \([5.2]\), combined with \([1.1]\) and \( r_\ell = 1/(j\ell)^{2.5} \), we have

\[
\mathcal{H}^1(Z_{\ell}^c \cap B_R) < \frac{c_4 R^2}{\text{min}_{B_{R+1}}} e^{2j_{138} \ell^{-2}} = \frac{R^2 c_4 c_2}{\text{min}_{B_{R+1}}} \Omega j_{133}^{138},
\]

(5.3)

which is negligibly small even if we rescale \( Z_{\ell} \) by \( 1/r_\ell \). Also the limits of \( \partial \mathcal{E}_{j\ell} \) and \( Z_{\ell} \) as measures are equal.

*Proof.* If \( z \in Z_{\ell}^c \), there exists some \( r \in (0,r_\ell) \) such that \( \mathcal{H}^1(\partial \mathcal{E}_{j\ell} \cap B_r(z)) < 2r \). Then, there exists some \( \epsilon > 0 \) such that \( \mathcal{H}^1(\partial \mathcal{E}_{j\ell} \cap B_{\varepsilon}(\tilde{z})) < 2r \) for all \( \tilde{z} \in U_\epsilon(z) \) and thus \( Z_{\ell} \cap U_\epsilon(z) = \emptyset \). This shows that \( Z_{\ell}^c \) is relatively open in \( \partial \mathcal{E}_{j\ell} \) and since \( \partial \mathcal{E}_{j\ell} \) itself is closed, \( Z_{\ell} \) is closed in \( \mathbb{R}^2 \). Let \( B_R \) be covered by a union of balls of radius \( r_\ell \) so that the number of balls is bounded by \( 16R^2(r_\ell)^{-2} \), which can be done easily. Let \( B_{r_\ell}(\tilde{z}) \) be any of such balls. If we prove that

\[
\mathcal{H}^1(Z_{\ell}^c \cap B_{r_\ell}(\tilde{z})) \leq -\frac{c}{\text{min}_{B_{R+1}}} \Omega \Delta_{j\ell} \| \partial \mathcal{E}_{j\ell} \|_2(\Omega)
\]

(5.4)

for an absolute constant \( c \), we will be done. For simplicity, rewrite \( Z_{\ell}^c \cap B_{r_\ell}(\tilde{z}) \) as \( Z_{\ell} \) and we prove \([5.4]\) for this \( Z_{\ell} \). For each point \( z \in Z_{\ell}^c \), by \([5.1]\), there exists \( r \in (0,r_\ell) \) such that \( \mathcal{H}^1(\partial \mathcal{E}_{j\ell} \cap B_r(z)) < 2r \). Then, since

\[
\int_0^r \mathcal{H}^0(\partial \mathcal{E}_{j\ell} \cap \partial B_s(z)) \, ds \leq \mathcal{H}^1(\partial \mathcal{E}_{j\ell} \cap B_r(z)) < 2r,
\]
we must have some $s_z \in (0, r)$ such that $\mathcal{H}^0(\partial \mathcal{E}_{j^*} \cap \partial B_{s_z}(z)) = 1$ or $= 0$. Consider a covering of $Z^c_{\ell}$ by $\{B_{s_z}(z)\}_{z \in Z^c_{\ell}}$. By the Besicovitch covering theorem, there exists a subfamily which consists of mutually disjoint balls (and at most countable) $\{B_{s_{z_k}}(z_k)\}_k$ such that

$$\mathcal{H}^1(Z^c_{\ell}) \leq B_2 \sum_k \mathcal{H}^1(Z^c_{\ell} \cap B_{s_{z_k}}(z_k)),$$

where $B_2$ is the constant appearing in the Besicovitch covering theorem. Let $k_0 \in \mathbb{N}$ be chosen so that

$$\sum_k \mathcal{H}^1(Z^c_{\ell} \cap B_{s_{z_k}}(z_k)) \leq 2 \sum_{k=1}^{k_0} \mathcal{H}^1(Z^c_{\ell} \cap B_{s_{z_k}}(z_k)).$$

Since $Z^c_{\ell} \subset \partial \mathcal{E}_{j^*}$, (5.5) and (5.6) show

$$\mathcal{H}^1(Z^c_{\ell}) \leq 2B_2 \|\partial \mathcal{E}_{j^*}\|((\cup_{k=1}^{k_0} B_{s_{z_k}}(z_k))).$$

We next fix a Lipschitz map $f$ which is the identity map on $\mathbb{R}^2 \setminus \cup_{k=1}^{k_0} U_{s_{z_k}}(z_k)$. On each $U_{s_{z_k}}(z_k)$, we define $f$ as follows. For each $k = 1, \ldots, k_0$, $\partial \mathcal{E}_{j^*} \cap \partial B_{s_k}(z_k)$ consists of at most one point and $\partial B_{s_k}(z_k) \setminus \partial \mathcal{E}_{j^*}$ is connected. Thus for each $k$, there is one component of $\mathcal{E}_{j^*} = \{E_{j^*i}\}_{i=1}^N$, say $E_{j^*k}$, such that $\partial B_{s_k}(z_k) \setminus \partial \mathcal{E}_{j^*} \subset E_{j^*k}$. Choose a ball $B_{s_k}(z_k')$ such that $B_{s_k}(z_k') \subset U_{s_{z_k}}(z_k) \cap E_{j^*k}$ and consider a Lipschitz retraction map $f$ which expands $B_{s_k}(z_k')$ bijectively to $B_{s_k}(z_k)$ and maps $B_{s_k}(z_k) \setminus B_{s_k}(z_k')$ onto $\partial B_{s_k}(z_k)$. Since $f(B_{s_k}(z_k')) = B_{s_k}(z_k)$ and $B_{s_k}(z_k') \subset E_{j^*k}$, $f$ has the property that

$$B_{s_k}(z_k) \setminus (\partial \mathcal{E}_{j^*} \cap \partial B_{s_k}(z_k)) \subset \text{int } f(E_{j^*k}).$$

Now one can check that this $f$ is $\mathcal{E}_{j^*}$-admissible since it is a retraction map on each disjoint balls. By writing $E_{j^*} := f \mathcal{E}_{j^*}$ and $\{E_{j^*k}\}_{i=1}^N := E_{j^*}$, note that $B_{s_k}(z_k) \setminus (\partial \mathcal{E}_{j^*} \cap \partial B_{s_k}(z_k)) \subset \text{int } E_{j^*k}$ for each $k = 1, \ldots, k_0$. Since $\partial \mathcal{E}_{j^*} \cap \partial B_{s_k}(z_k)$ is a point or empty set, we have

$$\|\partial \mathcal{E}_{j^*}\|((\cup_{k=1}^{k_0} B_{s_k}(z_k))) = 0.$$  

It follows from (5.9) and Definition 3.5 that $f \in \mathcal{E}(\mathcal{E}_{j^*}, j^*)$. It follows also from Definition 3.6 that

$$\Delta_{j^*}\|\partial \mathcal{E}_{j^*}\|(\Omega) \leq -\|\partial \mathcal{E}_{j^*}\| \sum_{k=1}^{k_0} \mathcal{H}^1(B_{s_k}(z_k)) \leq -(\min \Omega) \|\partial \mathcal{E}_{j^*}\|((\cup_{k=1}^{k_0} B_{s_k}(z_k))).$$

Combining (5.7) and (5.10) and setting $c = 2B_2$, we obtain (5.4), and subsequently (5.2) with $c_4 = 16c$.

\[\square\]

Lemma 5.4. Depending only on $c_2$, $\min_{B_{\ell+1}} \Omega$ and $\sup_\ell \|\partial \mathcal{E}_{j^*}\|(B_{\ell+1})$, there exists a positive constant $c_5 \in (0, 1)$ such that the following property holds for all sufficiently large $\ell$. For $z \in Z^c_{\ell} \cap B_{\ell}$ and $r \in (0, c_5]$, we have

$$\|\partial \mathcal{E}_{j^*}\|(B_r(z)) \geq r/8.$$  

\[\text{Proof.}\] By Definition 5.1 (5.11) is satisfied for $r \in (0, r_\ell]$, already. Thus we need to prove the case for $r \in (r_\ell, c_5]$ for a suitable $c_5$. By [1] 5.1(1), for general varifold $V \in V_1(\mathbb{R}^2)$ with locally bounded
first variation \(\|\delta V\|\) and \(z \in \mathbb{R}^2\), we have
\[
s_2^{-1} \|V\|(B_{s_2}(z)) \exp \left( \int_{s_1}^{s_2} \delta V(\kappa(V, z, r)) \, dr \right) - s_1^{-1} \|V\|(B_{s_1}(z)) \geq 0. \tag{5.12}
\]
Here, dist \((z, \text{spt} \|V\|) < s_1 < s_2 < \infty\) and \(\kappa(V, z, r)\) is a vector field defined by
\[
\kappa(V, z, r)(\bar{z}) = \begin{cases} (r \|V\|(B_r(z)))^{-1}(\bar{z} - z) & \text{if } \bar{z} \in B_r(z), \\ 0 & \text{if } \bar{z} \notin B_r(z). \end{cases}
\]
We use (5.12) with \(V = \Phi_{\varepsilon_{r\ell}} \ast \partial E_{j\ell}, z \in Z_{\ell} \cap B_R\) and \(r_{\ell} \leq s_1 < s_2 \leq 1\). By definition, \(\|\kappa(V, z, r)\| \leq (\|V\|(B_r(z)))^{-1}\) and we may estimate
\[
|\delta(\Phi_{\varepsilon_{r\ell}} \ast \partial E_{j\ell})(\kappa(\Phi_{\varepsilon_{r\ell}} \ast \partial E_{j\ell}, z, r))| = |(\Phi_{\varepsilon_{r\ell}} \ast \delta(\partial E_{j\ell}))(\kappa(\Phi_{\varepsilon_{r\ell}} \ast \partial E_{j\ell}, z, r))|
\leq ((\Phi_{\varepsilon_{r\ell}} \ast \|\partial E_{j\ell}\|)(B_r(z)))^{-1} \int_{B_r(z)} |\Phi_{\varepsilon_{r\ell}} \ast \delta(\partial E_{j\ell})|. \tag{5.13}
\]
We used (3.17), (3.15) and (3.14) here. Next, by the Cauchy-Schwarz inequality and (3.29),
\[
\int_{B_r(z)} |\Phi_{\varepsilon_{r\ell}} \ast \delta(\partial E_{j\ell})| \leq \left( \int_{B_r(z)} \left| \frac{\Phi_{\varepsilon_{r\ell}} \ast \delta(\partial E_{j\ell})}{\|\partial E_{j\ell}\| + \varepsilon_{r\ell} \Omega^{-1}} \right|^2 \, d\omega \right)^{1/2} \left( \int_{B_r(z)} \frac{\|\partial E_{j\ell}\| + \varepsilon_{r\ell} \Omega^{-1}}{\Omega} \, d\omega \right)^{1/2}
\leq \varepsilon_{r\ell}^{1/2} (\min_{B_{r+1}} \Omega)^{-1/2} ((\Phi_{\varepsilon_{r\ell}} \ast \|\partial E_{j\ell}\|)(B_r(z)) + \varepsilon_{r\ell} (\min_{B_{r+1}} \Omega)^{-1/2} \Omega^{-1/2}). \tag{5.14}
\]
The last term can be controlled as follows. We have
\[
(\Phi_{\varepsilon_{r\ell}} \ast \|\partial E_{j\ell}\|)(B_r(z)) = \|\partial E_{j\ell}\| \Phi_{\varepsilon_{r\ell}} \ast \chi_{B_r(z)} \geq \frac{1}{2} \|\partial E_{j\ell}\| \geq \frac{1}{2} r_{\ell} \tag{5.15}
\]
for all large \(\ell\) using \(r \geq r_{\ell}\) and \(\varepsilon_{r\ell} \ll r_{\ell}\). The last inequality follows from \(\|\partial E_{j\ell}\|(B_{r/2}(z)) \geq r_{\ell}\) for \(r \geq r_{\ell}\) due to (5.1) and \(z \in Z_{\ell}\). This combined with (5.15) shows that for any \(r \in [r_{\ell}, 1]\) and sufficiently large \(\ell\),
\[
\varepsilon_{r\ell} (\min_{B_{r+1}} \Omega)^{-1/2} 2\pi r^2 \leq (\Phi_{\varepsilon_{r\ell}} \ast \|\partial E_{j\ell}\|)(B_r(z)). \tag{5.16}
\]
By (5.13), (5.14) and (5.16), for \(r \in [r_{\ell}, 1]\), we obtain
\[
|\delta(\Phi_{\varepsilon_{r\ell}} \ast \partial E_{j\ell})(\kappa(\Phi_{\varepsilon_{r\ell}} \ast \partial E_{j\ell}, z, r))| \leq (2c_2)^{1/2} (\min_{B_{r+1}} \Omega)^{-1/2} ((\Phi_{\varepsilon_{r\ell}} \ast \|\partial E_{j\ell}\|)(B_r(z)))^{-1/2}. \tag{5.17}
\]
Substituting (5.17) into (5.12) and writing \(\xi(r) := (\Phi_{\varepsilon_{r\ell}} \ast \|\partial E_{j\ell}\|)(B_r(z))\) and \(\xi := 2c_2 (\min_{B_{r+1}} \Omega)^{-1}\), we obtain for \(r_{\ell} \leq s_1 < s_2 \leq 1\)
\[
s_2^{-1} \xi(s_2) \exp(c \int_{s_1}^{s_2} \xi(r) \, dr) \geq s_1^{-1} \xi(s_1). \tag{5.18}
\]
Note that \(\xi\) is a smooth positive function, and (5.18) shows that \(s^{-1}(\xi(s)) \exp(c \int_s \xi^{-1/2}(r) \, dr)\) is a monotone increasing function. After setting \(\tilde{\xi}(s) := s^{-1} \xi(s)\) and by differentiation, we obtain \((\tilde{\xi}^{1/2} + cs^{1/2})' \geq 0\). Since \(\tilde{\xi}(r_{\ell}) \geq \frac{1}{2}\) by (5.15), for any \(r \in [r_{\ell}, 1]\) we obtain
\[
\begin{align*}
r^{-1}(\Phi_{\varepsilon_{r\ell}} \ast \|\partial E_{j\ell}\|)(B_r(z)) &= \tilde{\xi}(r)^{1/2} \geq -cr^{1/2} + \tilde{\xi}(r_{\ell})^{1/2} + cr_{\ell}^{1/2} \geq -cr^{1/2} + 1/2^{1/2}. \tag{5.19}
\end{align*}
\]
Thus, we restrict \(r\) so that \(cr^{1/2} < 1/4\), for example, we obtain a positive lower bound for the density ratio of \(\Phi_{\varepsilon_{r\ell}} \ast \|\partial E_{j\ell}\|\). Finally, we can estimate as
\[
\Phi_{\varepsilon_{r\ell}} \ast \chi_{B_{r/2}(z)} \leq e^{-\varepsilon_{r\ell}^{-1}} \chi_{B_{r+1}}
\]
for all large $\ell$, and we use inequality $\chi_{B_r(z)} + (\Phi_{\epsilon \ell} * \chi_{B_{r/2}(z)}) \mathbb{L} \mathbb{R}^3 \mathcal{B}_r(z) \geq \Phi_{\epsilon \ell} * \chi_{B_{r/2}(z)}$ to derive
\[
\|\partial E_{\ell}\|(B_r(z)) \geq (\Phi_{\epsilon \ell} * \|\partial E_{\ell}\|)(B_{r/2}(z)) - e^{-\epsilon \ell^{-1}} \|\partial E_{\ell}\|(B_{r+1}).
\] (5.20)
The last term converges to 0 even after dividing by $r \geq r_\ell$. Thus, using (5.19) with $r/2$ in place of $r$ and (5.20), we obtain the desired estimate.

**Lemma 5.5.** Let $R, s > 0$ be fixed and let $\epsilon > 0$ be arbitrary. Then there exists $\ell_0 \in \mathbb{N}$ such that the following holds for all $\ell \geq \ell_0$. If $N \geq 3$, for $z \in Z_\ell \cap B_R$, at least one of the following (a) or (b) holds (they are not mutually exclusive). If $N = 2$, (a) holds for $z \in Z_\ell \cap B_R$.

(a) There exists a line denoted by $S$ such that $z \in S$ and
\[
dist H(Z_\ell \cap B_{r_\ell}(z), S \cap B_{r_\ell}(z)) \leq \epsilon r_\ell.
\] (5.21)

(b) There exists a triple junction with three half-lines denoted by $S$ such that $z \in S$ and with the junction in $U_{r_\ell}(z)$ and such that
\[
dist H(Z_\ell \cap B_{2r_\ell}(z), S \cap B_{2r_\ell}(z)) \leq \epsilon r_\ell.
\] (5.22)

(c) Furthermore, in the case of (a), with the same $S$ and for any $\phi \in C^1_c(U_{r_\ell}(z))$ with $\sup (|\phi| + r_\ell |\nabla \phi|) \leq s$, we have
\[
\left| \int_{B_{r_\ell}(z)} \phi(\tilde{z}) d\|\partial E_{\ell}\|((\tilde{z}) - \int_{B_{r_\ell}(z) \cap S} \phi(\tilde{z}) dH^1(\tilde{z}) \right| \leq \epsilon r_\ell
\] (5.23)
and
\[
\left| \int_{G_1(B_{r_\ell}(z))} \tilde{S}\phi(\tilde{z}) d(\partial E_{\ell})(\tilde{z}, \tilde{S}) - \int_{G_1(B_{r_\ell}(z))} \tilde{S}\phi(\tilde{z}) d(|S|)(\tilde{z}, \tilde{S}) \right| \leq \epsilon r_\ell.
\] (5.24)

**Remark 5.6.** Note that the integrand of (5.24) is a $2 \times 2$ matrix and the absolute value is the Euclidean norm of the matrix. One can check that the estimate (5.24) is independent of the coordinate system under orthogonal rotation.

**Proof.** Consider the case $N \geq 3$. If the first claim were not true, we have some $\epsilon_0 > 0$ and subsequences (denoted by the same index) $j_\ell$ and $z^{(\ell)} \in B_R \cap Z_\ell$ such that (5.21) and (5.22) are not true. Since $\partial E_{j_\ell} = Z_\ell \cup Z_\ell^c$ and $(r_\ell)^{-1}H^1(Z_\ell \cap B_{R+1}) \to 0$ by (5.3), the limit of $V_\ell$ (see the definition (1.5)) and that of $\|F_\ell\|_2 Z_\ell$ coincide. By Theorem 4.11, we know that the limit of $V_\ell$ is either a line or a triple junction, denoted by $\hat{S}$. For any $0 < t < 1/2$, $H^1(B_{r_\ell}(z^{(\ell)}) \cap \partial E_{j_\ell}) \geq 2t r_\ell$ by (5.1), hence we have $H^1(B_\ell \cap \hat{S}) \geq 2t$ and $\hat{S}$ has to include the origin in particular. By the contradiction argument, we have $\text{dist}_H(Z_\ell \cap B_{r_\ell}(z^{(\ell)}), S \cap B_{r_\ell}(z^{(\ell)})) > \epsilon_0 r_\ell$ for any line $S$ with $z^{(\ell)} \in S$ and the same inequality with $B_{2r_\ell}(z^{(\ell)})$ in place of $B_{r_\ell}(z^{(\ell)})$ for any triple junction $S$ with the junction in $U_{r_\ell}(z^{(\ell)})$ and $z^{(\ell)} \in S$. After stretching by $F_\ell$, we have $\text{dist}_H((F_\ell)_2 Z_\ell \cap B_1, S \cap B_1) > \epsilon_0$, again for any line $S$ with $0 \in S$ and the same inequality with $B_2$ in place of $B_1$ for any triple junction $S$ with $0 \in S$ and the junction in $U_1$. With $S = \hat{S}$, on the other hand, we would have a contradiction since $(F_\ell)_2 Z_\ell$ converges to $\hat{S}$ in the Hausdorff distance, which follows from (5.1). Note that, if $\hat{S}$ is a triple.
juncture with the junction in $U_1$, we would have a contradiction to $\text{dist}_H((F_0)_{\ell}Z_{\ell} \cap B_2, \hat{S} \cap B_2) > \epsilon_0$
for all large $\ell$. If $\hat{S}$ is either a line or a triple junction with the junction outside of $U_1$, we have a contradiction to $\text{dist}_H((F_0)_{\ell}Z_{\ell} \cap B_1, \hat{S} \cap B_1) > \epsilon_0$. This proves the first claim for $N \geq 3$. If $N = 2$, the above argument using Theorem 4.1 proves the claim.

If (c) were not true, we have subsequences (again denoted by the same index) $z_{\ell} \in Z_{\ell} \cap B_R$, lines $S_{\ell}$ with $z_{\ell} \in S_{\ell}$ and $\phi_{\ell} \in C^1(U_{\ell},(z_{\ell}))$ with $\sup(|\phi_{\ell}(\cdot) + r_{\ell} |\nabla \phi_{\ell}(\cdot)|) \leq s$ such that $r_{\ell}^{-1}\text{dist}_H(Z_{\ell} \cap B_{2}, S_{\ell} \cap B_{2}) \to 0$ and either (5.23) or (5.24) fails for these with $\epsilon_0$. As before, the limit of $V_{\ell}$ and $|(F_{\ell})_{\ell}Z_{\ell}|$ coincide, and since $\text{dist}_H((F_{\ell})_{\ell}Z_{\ell} \cap B_1, (F_{\ell})_{\ell}S_{\ell} \cap B_1) \to 0$, $V_{\ell}$ subsequentially converges to a line $\hat{S} = \lim_{\ell \to \infty}(F_{\ell})_{\ell}S_{\ell}$ on $U_1$ as varifolds. If (5.24) is violated, in terms of $V_{\ell}$, we have

$$\left| \int_{G_1(B_1)} \hat{S}\phi_{\ell}(z) dV_{\ell}(z, \hat{S}) - \int_{G_1(B_1)} \hat{S}\phi_{\ell}(z) d|(F_{\ell})_{\ell}S_{\ell}(\cdot)|(z, \hat{S}) \right| > \epsilon_0. \quad (5.25)$$

Here, $\hat{\phi}_{\ell}(z) := \phi_{\ell}(r_{\ell}z + z_{\ell})$ and $\sup(|\hat{\phi}_{\ell}(\cdot) + |\nabla \hat{\phi}_{\ell}(\cdot)|) \leq s$. Because of the latter uniform bound for $\hat{\phi}_{\ell}$, there exists a subsequence which converges uniformly to a Lipschitz function, say, $\phi$, with support in $B_1$. Thus, with the varifold convergence, we have (for a not-relabeled subsequence)

$$\lim_{\ell \to \infty} \int_{G_1(B_1)} \hat{S}\hat{\phi}_{\ell}(z) dV_{\ell} = \int_{G_1(B_1)} \hat{S}\phi d(|\hat{S}|) = \lim_{\ell \to \infty} \int_{G_1(B_1)} \hat{S}\hat{\phi}_{\ell}(z) d|(F_{\ell})_{\ell}S_{\ell}(\cdot)|.$$ 

This cannot be compatible with (5.25) for all large $\ell$. Thus we obtain a desired contradiction. The case that (5.23) does not hold can be similarly handled.

\[\Box\]

**Theorem 5.7.** Given $\nu \in \mathbb{N}$, there exist $0 < c_6, c_7 < 1$ with the following property. Suppose $\{\partial E_{\ell}\}$ is a sequence satisfying (3.29) and $\mu = \lim_{\ell \to \infty} \|\partial E_{\ell}\|$ on $\mathbb{R}^2$. Assume that for $B_r(a) \subset \mathbb{R}^2$, we have

$$\limsup_{\ell \to \infty} \int_{B_{2\ell}(a)} r \left| \Phi_{\ell} * \delta(\partial E_{\ell}) \right|^2 \leq c_6, \quad (5.26)$$

$$\text{spt} \mu \cap B_{2\ell}(a) \subset \{a + (x, y) \in \mathbb{R}^2 : |y| \leq c_7 \ell \}, \quad (5.27)$$

$$r(\nu - \frac{1}{2}) \leq \mu(\{a + (x, y) \in \mathbb{R}^2 : |y| \leq c_7 \ell, |x| \leq \ell/2\}), \quad (5.28)$$

$$\mu(\{a + (x, y) \in \mathbb{R}^2 : |y| \leq c_7 \ell, |x| \leq \ell \}) \leq 2r(\nu + \frac{1}{2}). \quad (5.29)$$

Then there exist functions $f_i : [-r, r] \to [-c_7 \ell, c_7 \ell]$ $(i = 1, \ldots, \nu)$ with

$$f_i \in W^{2,2}([-r, r]), \quad f_1(x) \leq f_2(x) \leq \ldots \leq f_\nu(x) \text{ for } x \in [-r, r] \quad (5.30)$$

and such that, writing graph $f_i := \{a + (x, f_i(x)) \in \mathbb{R}^2 : x \in [-r, r]\}$, we have

$$\mu(L_{B_r(a)}(f_i) = \sum_{i=1}^{\nu} H^1(L_{B_r(a)} \cap \text{graph } f_i). \quad (5.31)$$

**Proof.** Choose and fix a large $R > 1$ so that $B_{2\ell}(a) \subset B_{R/2}$, and let $c_5$ be the corresponding constant obtained in Lemma 5.4 with $c_2$ in (3.29), $\min_{B_{R+1}} \Omega$ and sup $\|\partial E_{\ell}\|(B_{R+1})$ with the fixed $R$. We will fix $c_6$ and $c_7$ later as absolute constants. Let $\{\epsilon_{\ell}\}_{\ell \in \mathbb{N}}$ be a sequence of positive numbers converging to 0, and let $\{s_{\ell}\}_{\ell \in \mathbb{N}}$ be a sequence of positive numbers diverging to $\infty$. Using Lemma
we pick a subsequence (denoted by the same index) such that $Z_\ell$ satisfies the properties listed in Lemma 5.5 with $\epsilon = \epsilon_\ell$ and $s = s_\ell$ and so that $r_\ell \ll \epsilon_\ell$ as $\ell \to \infty$. Define

$$Z_\ell^* := \{ z \in Z_\ell \cap B_{2r}(a) : \text{there is a line } S \text{ satisfying } z \in S \text{ and } (5.21) \}.$$  \hfill (5.32)

By Lemma 5.5 for any point $z \in B_{2r}(a) \cap Z_\ell \setminus Z_\ell^*$, there exists a triple junction $S$ with the junction in $U_{r_\ell}(z)$ satisfying $z \in S$ and (5.22). First in Step 1, we show that the set $Z_\ell^*$ is composed of a finite set of “almost $C^{1,\frac{\delta}{2}}$ curves” for sufficiently small $c_6$.

**Step 1.** Pick an arbitrary point $z_0 \in Z_\ell^* \cap B_{5r/4}(a)$. Then there exists a line $S$ with the stated properties in (5.21). Call the line segment $S \cap B_{r_\ell}(z_0)$ as $L_0$. For notational convenience, we momentarily consider a coordinate system so that $L_0$ is parallel to the $x$-axis and suppose that $z_0 = (x_0, y_0)$. Write the coordinates of the endpoints of $L_0$ as $z_0^- := (x_0 - r_\ell, y_0)$ and $z_0^+ := (x_0 + r_\ell, y_0)$. Then, we inductively choose $z_k \in Z_\ell$ and a line segment $L_k$ for $k \geq 1$ as long as $L_k \subset B_{3r/2}(a)$ or $z_k$ is not in $r_\ell$-neighborhood of $Z_\ell \setminus Z_\ell^*$, as detailed in the following (see Figure 4). Write the right-hand side endpoint of $L_k$ as $z_k^+$. By (5.21), we have $B_{\epsilon r_\ell}(z_k^+) \cap Z_\ell \neq \emptyset$.

![Figure 4](image)

There are two possibilities: (i) $Z_\ell^* \cap B_{\epsilon r_\ell}(z_k^+) \neq \emptyset$ and (ii) $Z_\ell^* \cap B_{\epsilon r_\ell}(z_k^+) = \emptyset$.

(i) We pick $z_{k+1} \in Z_\ell^* \cap B_{\epsilon r_\ell}(z_k^+)$. Then there exists a line $S$ with $z_{k+1} \in S$ and (5.21) in $B_{r_\ell}(z_{k+1})$. We let $L_{k+1} := S \cap B_{r_\ell}(z_{k+1})$. We define $z_{k+1}$ as before and continue with the inductive process. By (5.21) in particular, we have

$$Z_\ell \cap B_{r_\ell}(z_{k+1}) \subset (L_{k+1})_{\epsilon r_\ell}.$$ \hfill (5.33)

Here, for shorthand, $(A)_t$ is the $t$-neighborhood of the set $A$.

(ii) Since $B_{\epsilon r_\ell}(z_k^+) \cap Z_\ell \neq \emptyset$, there exists $z_{k+1} \in B_{\epsilon r_\ell}(z_k^+) \cap Z_\ell \setminus Z_\ell^*$. By Lemma 5.5, there exists a triple junction $S$ with $z_{k+1} \in S$ and with the junction in $U_{r_\ell}(z_{k+1})$ such that (5.22) holds with $z = z_{k+1}$. In the following, the occurrence of (ii) is casually called “encounter with a triple junction”, since this is a place where $Z_\ell$ looks like a triple junction.

When (ii) occurs for the first time, we end the induction, and let this $k$ be $k_0$ (so that $z_{k_0+1}$ is in $r_\ell$-neighborhood of $Z_\ell \setminus Z_\ell^*$. Also, if (ii) does not occur but if $L_k \setminus B_{3r/2}(a) \neq \emptyset$ for the first time, we stop the induction and name this $k$ as $k_0$. In the following, we prove that the angle between the $x$-axis and $L_k$ ($1 \leq k \leq k_0$) remains small if $c_6$ is appropriately small, which effectively tells that $\bigcup_{k=0}^{k_0} L_k$ can be approximated well by a graph over the $x$-axis with small slope. For that purpose,
we prove
\[
|(y_k^+ - y_k)/(x_k^+ - x_k)| \leq c_8(\epsilon_\ell + \left( \int_{\cup z_i=0}^{L} 1 + \frac{|\Phi_{r_k} \ast \delta(\partial E_{\ell j})|^2}{\Phi_{r_k} \ast \|\partial E_{\ell j}\| + \epsilon_\ell \Omega^{-1}} \right)^{1/2}(k r_\ell + \epsilon_\ell)^{1/2}).
\] (5.34)

Here, \(z_k^+ = (x_k^+, y_k^+)\) and \(z_k = (x_k, y_k)\), thus the left-hand side is the slope of \(L_k\). The constant \(c_8\) is an absolute constant. By assuming \(c_6 \leq (600c_8)^{-1}\), for example, we can make sure that the slope is \(\leq 1/10\) for all sufficiently small \(\epsilon_\ell\). For the moment, assume that the slopes of \(L_1, \ldots, L_k\) are smaller than 1/9, for example. By definition, \(|z_{i+1} - z_i| \leq \epsilon_\ell r_\ell\), and (5.21) and a simple geometric argument show that the angle between the neighboring line segments \(L_i\) and \(L_{i+1}\) is at most a fixed constant multiple of \(\epsilon_\ell\). With this in mind, though we do not give the explicit formula, we may construct a function \(\psi_1\) satisfying the following properties.

- \(\psi_1 = 1\) on the \(r_\ell/8\)-neighborhood of \(\cup_{i=0}^{k} L_i\).
- \(\psi_1 = 0\) on the complement of \(r_\ell/6\)-neighborhood of \(\cup_{i=0}^{k} L_i\).
- \(0 \leq \psi_1 \leq 1\) and \(\sup(r_\ell|\nabla \psi_1| + r_\ell^2\|\nabla^2 \psi_1||) \leq c\), where \(c\) is an absolute constant.

Next, we define a smooth function \(\psi_2\) which depends only on the \(x\)-variable and such that

- \(\psi_2(x) = 0\) for \(x < x_0 - r_\ell/4\) or \(x > x_k + r_\ell/4\).
- \(\psi_2(x) = 1\) for \(x_0 - r_\ell/8 < x < x_k + r_\ell/8\).
- \(\sup(r_\ell|\psi_2| + r_\ell^2\|\psi_2''\|) \leq c\), where \(c\) is an absolute constant.

Finally, we set \(\psi(x, y) = \psi_1(x, y)\psi_2(x)\). Because of the properties of \(\psi_1\) and \(\psi_2\), one can check that

\[
\psi = 1 \text{ on } (\cup_{i=1}^{k-1} L_i)_{r_\ell/10} \text{ and } \text{spt } \psi \subset (\cup_{i=1}^{k-1} L_i)_{r_\ell/2}
\] (5.35)

for small \(\epsilon_\ell\). Another property we use is

\[
\frac{\partial \psi}{\partial y} = \psi_2 \frac{\partial \psi_1}{\partial x} = 0 \text{ on } Z_\ell.
\] (5.36)

To check this, since \(\psi_2\) is independent of \(y\), we only need to check that the set of points with \(\nabla \psi_1 \neq 0\) and \(\psi_2 > 0\) does not intersect \(Z_\ell\). The set satisfying these conditions is included in \(\{(x, y) : x_0 - r_\ell/4 \leq x \leq x_k + r_\ell/4\} \cap (\cup_{i=0}^{k} L_i)_{r_\ell/6} \setminus (\cup_{i=0}^{k} L_i)_{r_\ell/8}\). But having a point \(\tilde{z}\) of \(Z_\ell\) in this set implies that \(\tilde{z} \in \cup_{i=0}^{k} U_{r_\ell}(z_i) \setminus (\cup_{i=0}^{k} L_i)_{r_\ell/8}\), which is a contradiction to (5.33) for small \(\epsilon_\ell\). This proves (5.30). We next estimate

\[
\delta(\partial E_{\ell j})(0, \psi) = \int_{G_i(\mathbb{R}^2)} \tilde{S} \cdot \nabla(0, \psi(\tilde{z})) \, d(\partial E_{\ell j})(\tilde{z}, \tilde{S})
\]

\[
= \int_{G_i(\mathbb{R}^2)} \tilde{S}_1 \frac{\partial \psi}{\partial x} + \tilde{S}_2 \frac{\partial \psi}{\partial y} \, d(\partial E_{\ell j})(\tilde{z}, \tilde{S}).
\] (5.37)
Here $\tilde{S}_{21}$ is the $(2,1)$ component of $\tilde{S} \in G(2,1)$ and similarly for $\tilde{S}_{22}$. For the second term of (5.37), by (5.36), we have
\[
\left| \int_{G_1(R^2)} \tilde{S}_{22} \frac{\partial \psi}{\partial y} d(\partial \mathcal{E}_{j\ell}) \right| \leq \int_{\text{spt} \psi \cap Z_{\ell}^c} \left| \frac{\partial \psi}{\partial y} \right| d\|\partial \mathcal{E}_{j\ell}\| \leq c\varepsilon^{-1} H^1(\text{spt} \psi \cap Z_{\ell}^c) \leq \varepsilon_{\ell}.
\] (5.38)
The last inequality follows since $H^1(Z_{\ell}^c \cap B_R) \ll r_{\ell}^3$ by (5.3), and by $r_{\ell} \ll \varepsilon_{\ell}$. For the first term of (5.37), for $\frac{\partial \psi}{\partial x} = \psi'_y \psi_1 + \psi \frac{\partial \psi_1}{\partial x}$, the integral of the second term can be handled similarly as above using (5.36). The term $\psi'_y \psi_1$ is nonzero only on $U_{r_{\ell}/2}(z_0)$ and $U_{r_{\ell}/2}(z_k)$. We use (5.24) with $\phi = r_{\ell} \psi'_y \psi_1$ on these balls. For all large $\ell$, we have $s_{\ell} \geq \sup(|\phi| + r_{\ell} \nabla \phi|$). Thus we obtain
\[
\left| \int_{G_1(U_{r_{\ell}/2}(z_i))} \tilde{S}_{21} \psi'_y \psi_1 d(\partial \mathcal{E}_{j\ell}) - \int_{G_1(U_{r_{\ell}/2}(z_i))} \tilde{S}_{21} \psi'_y \psi_1 d(|L_i|) \right| \leq \varepsilon_{\ell}
\] (5.39)
for $i = 0$ and $i = k$. When $i = 0$, since $L_0$ is a line segment parallel to the $x$-axis, we have $\tilde{S}_{21} = 0$ and
\[
\int_{G_1(U_{r_{\ell}/2}(z_0))} \tilde{S}_{21} \psi'_y \psi_1 d(|L_0|) = 0.
\] (5.40)
On the other hand, when $i = k$ and writing the slope of $L_k$ as $\alpha := (y_k^+ - y_k^-)/(x_k^+ - x_k)$, $\tilde{S}_{21}$ on $L_k$ is $\alpha/(1 + \alpha^2)$ and
\[
\int_{G_1(U_{r_{\ell}/2}(z_k))} \tilde{S}_{21} \psi'_y \psi_1 d(|L_k|) = -\frac{\alpha}{\sqrt{1 + \alpha^2}}.
\] (5.41)
Combining (5.37)-(5.41), we obtain the estimate
\[
\left| \delta(\partial \mathcal{E}_{j\ell})((0, \psi)) + \frac{\alpha}{\sqrt{1 + \alpha^2}} \right| \leq 4\varepsilon_{\ell}.
\] (5.42)
We next estimate
\[
\left| \delta(\partial \mathcal{E}_{j\ell})((0, \psi)) - (\Phi_{\varepsilon_{j\ell}} * \delta(\partial \mathcal{E}_{j\ell}))((0, \psi)) \right| \leq \int_{\mathbb{R}^2} |\nabla \psi - \Phi_{\varepsilon_{j\ell}} * \nabla \psi| d\|\partial \mathcal{E}_{j\ell}\|.
\] (5.43)
Since $\psi = 0$ outside of $B_R$ in particular, and since $\Phi_{\varepsilon_{j\ell}}$ has support on $B_1$, the integrand of the above vanishes outside of $B_{R+1}$. Furthermore, one can estimate
\[
|\nabla \psi(z) - (\Phi_{\varepsilon_{j\ell}} * \nabla \psi)(z)| \leq \int_{B_1(z)} |\nabla \psi(z) - \nabla \psi(\hat{z})| \Phi_{\varepsilon_{j\ell}}(z - \hat{z}) d\hat{z}
\] (5.44)
\[
\leq c\varepsilon^{-2} \int_{B_1(z)} |z - \hat{z}| \Phi_{\varepsilon_{j\ell}}(z - \hat{z}) d\hat{z} \leq c\varepsilon^{-2} (\varepsilon_{j\ell}^{9/10} + \varepsilon_{j\ell}^{-2} \exp(-1/(2\varepsilon_{j\ell}^{1/5}))).
\]
The last inequality may be obtained by splitting the domain of integration to $\{|z - \hat{z}| \leq \varepsilon_{j\ell}^{9/10}\}$ and the complement, where $\Phi_{\varepsilon_{j\ell}}$ is exponentially small (see the analogous estimate (15) (5.6)). Thus, by (5.43) and (5.44), we have
\[
\left| \delta(\partial \mathcal{E}_{j\ell})((0, \psi)) - (\Phi_{\varepsilon_{j\ell}} * \delta(\partial \mathcal{E}_{j\ell}))((0, \psi)) \right| \leq c\varepsilon^{-2} \varepsilon_{j\ell}^{9/10}(\min_{B_{R+1}} \Omega)^{-1}\|\partial \mathcal{E}_{j\ell}\|.(\Omega).
\] (5.45)
Since $\varepsilon_{j\ell} \ll c_{\ell}^{-6}$ and $r_{\ell} = c_{\ell}^{-2.5}$, we have $r_{\ell}^{-2} \varepsilon_{j\ell}^{9/10} \ll 1$ and this is estimated by $\varepsilon_{\ell}$ for all large $\ell$. Next, since $0 \leq \psi \leq 1$, we have (writing $U = \text{spt} \psi$)
\[
|(\Phi_{\varepsilon_{j\ell}} * \delta(\partial \mathcal{E}_{j\ell}))((0, \psi))| \leq \left( \int_U \left( \frac{|\Phi_{\varepsilon_{j\ell}} * \delta(\partial \mathcal{E}_{j\ell})|^2}{\Phi_{\varepsilon_{j\ell}} * \|\partial \mathcal{E}_{j\ell}\| + \varepsilon_{j\ell} \Omega^{-1}} \right)^{1/2} \right)^{1/2} \left( \int_U \Phi_{\varepsilon_{j\ell}} * \|\partial \mathcal{E}_{j\ell}\| + \varepsilon_{j\ell} \Omega^{-1} \right)^{1/2}.
\] (5.46)
To estimate \( \int_U \Phi_{\varepsilon_{jk}} \ast \|\partial \mathcal{E}_{jk}\| \, dz \), consider a partition of unity \( \{\zeta_i\}_{i=0}^k \) subordinate to \( \{U_{3r_{\ell}/4}(z_i)\}_{i=0}^k \) such that \( \zeta_i \in C_c^\infty(U_{3r_{\ell}/4}(z_i)) \), \( 0 \leq \zeta_i \leq 1 \), \( \sup (\zeta_i + r_\ell |\nabla \zeta_i|) \leq s_\ell \) (which is true for all large \( \ell \)) and \( \sum_{i=0}^k \zeta_i = 1 \) on \( (U_{i=1}^{k-1} L_i)_{\ell/2} \). Note that the latter includes \( \text{spt } \psi \) by (5.35). Thus we have
\[
\int_U \Phi_{\varepsilon_{jk}} \ast \|\partial \mathcal{E}_{jk}\| \, dz \leq \sum_{i=0}^k \int_{U_{3r_{\ell}/4}(z_i)} \Phi_{\varepsilon_{jk}} \ast \|\partial \mathcal{E}_{jk}\| \zeta_i \, dz.
\] (5.47)

We may estimate
\[
|\int \Phi_{\varepsilon_{jk}} \ast \|\partial \mathcal{E}_{jk}\| \zeta_i - \int \zeta_i \, d\|\partial \mathcal{E}_{jk}\| | \leq \int_{\mathbb{R}^2} |\zeta_i - \Phi_{\varepsilon_{jk}} \ast \zeta_i| \, d\|\partial \mathcal{E}_{jk}\|
\] which can be estimated just like (5.43)-(5.45) (with \( r_\ell^{-1} \) in place of \( r_\ell^{-2} \)) and we may assume that this is less than \( \epsilon_\ell r_\ell \). By (5.23),
\[
\int \zeta_i \, d\|\partial \mathcal{E}_{jk}\| \leq \int \zeta_i \, d(|L_i|) + \epsilon_\ell r_\ell \leq r_\ell (2 + \epsilon).
\] (5.49)

By combining (5.47)-(5.49), we obtain
\[
\int_U \Phi_{\varepsilon_{jk}} \ast \|\partial \mathcal{E}_{jk}\| \, dz \leq 2(k+1) r_\ell (1 + \epsilon).
\] (5.50)

By (5.35), we have \( L^2(U) \leq kr_\ell^2 \) and \( \int_U \varepsilon_{jk} \Omega^{-1} \leq (\min_Bb) r_\ell^{-1} kr_\ell^2 \varepsilon_{jk} \) and with (5.50), we obtain (for all large \( \ell \) so that \( (\min_Bb) r_\ell^{-1} r_\ell \varepsilon_{jk} < 2\epsilon \))
\[
\int_U (\Phi_{\varepsilon_{jk}} \ast \|\partial \mathcal{E}_{jk}\| + \varepsilon_{jk} \Omega^{-1}) \leq 2(k+1) r_\ell (1 + 2\epsilon).
\] (5.51)

Now, (5.42), (5.45), (5.46) and (5.51) prove (5.34) with a suitable choice of absolute constant.

Using (5.34), we can make sure that this induction does not continue indefinitely and after a finite number of steps, we have either \( L_k \) exits from \( B_{3r_{\ell}/2}(a) \) or there is an encounter with a triple junction. We can similarly proceed to choose line segments starting again from \( L_0 \) in the opposite direction. Let these points and line segments be \( z_{-1}, \ldots, z_{-k_0} \) and \( L_{-1}, \ldots, L_{-k_0} \). In this opposite direction, the same estimate for the slope of line segments holds. The similar slope estimate holds starting from any \( L_{\pm k} \). This gives an analogue of \( 1/2 \)-Hölder estimate for the slopes of segments.

From \( \bigcup_{k=-k_0}^{k_0} L_k \), we can construct an approximate curve as follows. Define \( I_k := \{x \in \mathbb{R} : (x, y) \in L_k\} \), i.e. the projection of \( L_k \) to the x-axis. They are intervals of length approximately \( 2r_\ell \).

Recall \( z_k = (x_k, y_k) \) and \( z_k^\pm = (x_k^\pm, y_k^\pm) \), and due to the construction, we have \( |x_k^\pm - x_{k+1}| \leq \epsilon_\ell r_\ell \) for \( k = 0, \ldots, k_0 \) and \( |x_k^\pm - x_{k-1}| \leq \epsilon_\ell r_\ell \) for \( k = 0, \ldots, -k_0 \). With the slope \( \leq 1/10 \) (for example) and for small \( \epsilon_\ell \), we can guarantee that at most three \( I_k \)'s can intersect each other. Let \( \{\tilde{\zeta}_k\}_{k=-k_0}^{k_0} \) be a partition of unity subordinate to \( \{\text{int } I_k\}_{k=-k_0}^{k_0} \) such that \( \tilde{\zeta}_k \in C_c^\infty(\text{int } I_k) \), \( 0 \leq \tilde{\zeta}_k \leq 1 \) and \( \sum_{k=-k_0}^{k_0} \tilde{\zeta}_k = 1 \) on \( \bigcup_{k=-k_0}^{k_0-1} I_k \). We may in addition ask that \( \sup (r_\ell |\tilde{\zeta}_k'(x)| + r_\ell^2 |\tilde{\zeta}_k''(x)|) \leq c \) for some absolute constant \( c \). On each \( I_k \), let \( g_k : I_k \to \mathbb{R} \) be the function so that \( \{ (x, g_k(x)) : x \in I_k \} = L_k \) holds. For \( x \in \bigcup_{k=-k_0}^{k_0-1} I_k \), define
\[
f(x) := \sum_{k=-k_0}^{k_0} g_k(x) \tilde{\zeta}_k(x).
\] (5.52)
Note that \( f'(x) \) involves at most three neighboring terms on \( I_i \), say, \( k = i - 1, i, i + 1 \) (or \( k = i - 2, i - 1, i \), or \( k = i, i + 1, i + 2 \)). Consider the first case and the other two cases are similar. Due to (5.21), \( (\epsilon r_\ell)^{-1}|g_i(x) - g_{i\pm 1}(x)| + \epsilon_\ell^{-1}|g'_i(x) - g'_{i\pm 1}(x)| \) is bounded by an absolute constant. Since we have \( \sum_{k=i-1}^{i+1} \zeta_k = 1 \),

\[
    f'(x) - g'_i(x) = \sum_{k=i-1}^{i+1} \{(g_k'(x) - g'_i(x))\zeta_k + (g_k(x) - g_i(x))\zeta_k\}. \tag{5.53}
\]

The right-hand side may be estimated by a constant multiple of \( \epsilon_\ell \) using the bounds on \( \zeta_k \). The variation of the slopes of \( L_k \) (i.e. \( g_k'(x) \)) are estimated by (5.31), so we obtain for any \( x, \tilde{x} \in \cup_{k=-k_0+1}^{k_0-1} \mathcal{I}_k \)

\[
    |f'(x) - f'(\tilde{x})| \leq c_8 \left( \epsilon_\ell + \left( \int_{k_1=1}^{k_2} |\Phi_{\varepsilon_j} * \delta(\varepsilon \partial \mathcal{E}_j)|^2 \right)^{1/2} (|x - \tilde{x}| + \epsilon_\ell)^{1/2} \right). \tag{5.54}
\]

Here, \( k_1, k_2 \in \mathbb{Z} \) are chosen so that \( |x - x_k| \leq r_\ell \) and \( |\tilde{x} - x_k| \leq r_\ell \), and \( c_8 \) here may be different from \( c_8 \) in (5.31) by a factor of absolute constant. From the construction, it is also clear that \( \sup_{x \in \mathcal{I}_k} |g_k(x) - f(x)| \) is estimated by a constant multiple of \( r_\ell \epsilon_\ell \), and due also to (5.33), there exists an absolute constant \( c_9 \) such that

\[
    Z_\ell \cap \{(x, y) : |y - f(x)| \leq r_\ell/4, x \in \cup_{k=-k_0+1}^{k_0-1} \mathcal{I}_k \}
\]

\[
    \subset \{(x, y) : |y - f(x)| \leq c_9 \epsilon_\ell r_\ell, x \in \cup_{k=-k_0+1}^{k_0-1} \mathcal{I}_k \}. \tag{5.55}
\]

**Step 2.** We next prove using the assumption (5.27) with small \( c_7 \) that there is no triple junction in \( B_{9r/8}(a) \) for all large \( \ell \), or more precisely, there is no point of \( Z_\ell \setminus Z_\ell^* \). First note that, because of Lemma 5.4, for all sufficiently large \( \ell \), we have

\[
    B_{3\ell/2}(a) \cap Z_\ell \cap \{a + (x, y) : |y| \geq 2c_7 r \} = \emptyset. \tag{5.56}
\]

Otherwise, we would have a converging subsequence (with the same index) \( \{z^{(i)}\} \in B_{3\ell/2}(a) \cap Z_\ell \cap \{a + (x, y) : |y| \geq 2c_7 r \} \) which converges to \( \hat{z} \in B_{3\ell/2}(a) \cap \{a + (x, y) : |y| \geq 2c_7 r \} \). By Lemma 5.4, we have \( \|\partial \mathcal{E}_j\|(B_s(z^{(i)})) \geq s/8 \) for all large \( \ell \) and \( s \in (0, c_9] \). Then we have \( \mu(B_s(\hat{z})) \geq s/8 \) for \( s \in (0, c_9] \), and in particular \( \hat{z} \in \text{spt } \mu \). This is a contradiction to (5.27).

Suppose for a contradiction that we had some \( z_0 \in B_{9r/8}(a) \cap Z_\ell \setminus Z_\ell^* \), thus there exists a triple junction \( S \) with the junction in \( U_{r_\ell}(z_0) \) satisfying \( z_0 \in S \) and (5.22). We may assume that \( z_0 \in \{a + (x, y) : |y| < 2c_7 r \} \) due to (5.56). Out of the three half-lines of \( S \), there is a half-line which goes upwards (i.e. towards the positive \( y \)-direction) and which has at least 30 degrees with positive or negative \( x \)-axis (or the absolute value of slope \( \geq 1/\sqrt{3} \)). Let \( z_0^+ \) be the intersection of this half-line and \( \mathcal{S}B_{2r_\ell}(z_0) \). By (5.22), we know that there exists \( z_1 \in Z_\ell \cap B_{4r_\ell}(z_0^+) \). Starting from this \( z_1 \), we may start the previous inductive process to select line segments until the encounter with a triple junction or the exit from \( B_{3\ell/2}(a) \) occur. Suppose that the encounter with a triple junction does not occur. Since one can estimate \( |y_k - y_0| / |x_k - x_0| \geq 3/(4\sqrt{3}) \) (where \( z_k = (x_k, y_k) \)) due to (5.34) with sufficiently small \( c_6 \), the \( y \)-coordinate of \( z_k \) keeps going upwards. Then by selecting
a sufficiently small absolute constant $c_7$, the union of line segments (which is close to a straight line due to (5.54) with small $c_6$) would enter $B_{5r/4}(a) \cap \{a + (x, y) : |y| \geq 2c_7 r\}$ after a finite number of induction. Since $L_k$ and $Z_\ell$ in $B_{r/\kappa}(Z_k)$ are close in the Hausdorff distance, this would be a contradiction to (5.56). Thus, after a finite number of induction, there must be another triple junction which is close to $Z_\ell$. This triple junction has one half-line whose angle with the $x$-axis is at least 30 degrees and which is going upwards. Along the direction of this half-line, we can select line segments as before, which again has a definite slope going upwards. By the similar reasoning, there must be another triple junction, and then we choose a half-line going upwards just as before. In this process, we can make sure that the slope of line segments have lower bound (say, $3/(4\sqrt{3})$, for example) and the line segments inevitably enter into $B_{5r/4}(a) \cap \{a + (x, y) : |y| \geq 2c_7 r\}$ after a finite number of induction. Thus, we inevitably have a contradiction to (5.56). This proves that there is no point of $Z_\ell \setminus Z^*_\ell$ inside $B_{9r/8}(a)$.

**Step 3.** Consider $\tilde{Z}_\ell := Z_\ell \cap \{a + (x, y) : |x| \leq r\} \cap B_{9r/8}(a)$. The set $\tilde{Z}_\ell$ is included in $\{a + (x, y) : |y| \geq 2c_7 r\}$ for all large $\ell$ due to (5.56), and by Step 2, we have $B_{9r/8}(a) \cap Z_\ell \setminus Z^*_\ell = \emptyset$. Thus, starting from any $z_0 \in \tilde{Z}_\ell$, the inductive step to choose $L_k$ (and $L_{-k}$) in Step 1 does encounter a triple junction and will continue until $L_k$ (resp. $L_{-k}$) exits from $\{a + (x, y) : |x| \leq r\}$. Namely, with the same notation in Step 1, we have some $k_0 \geq 1$ and $k_0' \geq 1$ such that $r \in I_{k_0-1}, -r \in I_{-k_0'+1}$ and $[-r, r] \subset \bigcup_{k=k_0-1}^{k_0+1} I_k$. The function $f$ defined as in (5.52) over $x \in [-r, r]$ has a small slope for small $c_0$ and $c_7$. By (5.55), note that

\[
\tilde{Z}_\ell \cap \{a + (x, y) : |y - f(x)| \leq r/4, |x| \leq r\} \subset \{a + (x, y) : |y - f(x)| \leq c_0r/\ell, |x| \leq r\}. \tag{5.57}
\]

Let this $f$ be renamed as $f_{(1, \ell)}$. If $\tilde{Z}_\ell \setminus \{a + (x, y) : |y - f_{(1, \ell)}(x)| \leq r/4, |x| \leq r\} \neq \emptyset$, then, we can pick a point $z_0'$ from this set and start the construction of Step 1. Let $f_{(2, \ell)}$ be the resulting function. Note that

\[
|f_{(1, \ell)}(x) - f_{(2, \ell)}(x)| \geq r_\ell/8 \quad \text{for } x \in [-r, r], \tag{5.58}
\]

which can be seen as follows. Because $z_0' = (x_0', y_0') \notin \{x, y) : |y - f_{(1, \ell)}(x)| \leq r/4, |x| \leq r\}$ (which implies $|f_{(1, \ell)}(x_0') - y_0'| > r/4$ and $|y_0 - f_{(2, \ell)}(x_0')| \leq c_0r/\ell$ due to (5.57) satisfied by $f_{(2, \ell)}$, we have $|f_{(1, \ell)}(x_0') - f_{(2, \ell)}(x_0')| > 3r_\ell/16$ for large $\ell$. If there exists $x \in [-r, r]$ such that $|f_{(1, \ell)}(x) - f_{(2, \ell)}(x)| < r_\ell/8$, then by the continuity of $f_{(1, \ell)}$ and $f_{(2, \ell)}$, there must exist some $\hat{x} \in [-r, r]$ such that $|f_{(1, \ell)}(\hat{x}) - f_{(2, \ell)}(\hat{x})| = 3r_\ell/16$. Then there must be a point $\hat{z} \in Z_\ell \cap B_{c_\ell r/\ell}(\hat{x})$ with $f = f_{(1, \ell)}$ and sufficiently small $c_\ell$. Thus, we have (5.58), and if $\tilde{Z}_\ell \setminus \bigcup_{i=1}^{j} \{a + (x, y) : |y - f_{(i, \ell)}(x)| \leq r_\ell/4, |x| \leq r\} \neq \emptyset$, then we can pick a point from this set and construct another function, $f_{(3, \ell)}$. By repeating this construction and due to the above disjointedness property of these graphs, we can exhaust all points of $\tilde{Z}_\ell$ after finite steps and we have a sequence of graphs $f_{(1, \ell)}, \ldots, f_{(n_\ell, \ell)}$, that is,

\[
\tilde{Z}_\ell \setminus \bigcup_{i=1}^{n_\ell} \{a + (x, y) : |y - f_{(i, \ell)}(x)| \leq r_\ell/4, |x| \leq r\} = \emptyset. \tag{5.59}
\]
In the following, we will see that \( \nu' \) has to be necessarily equal to \( \nu \) given in \((5.28)\) for all large \( \ell \) if \( c_6 \) and \( c_7 \) are sufficiently small. Set

\[
\mu_\ell := \mathcal{H}^1 \mathbf{1}_{\{a+(x, f_{(i, \ell)}(x)) : |x| \leq r\}},
\]

that is, \( \mu_\ell \) is the 1-dimensional measure obtained from the union of above graphs. We claim that

\[
\mu = \lim_{\ell \to \infty} \| \partial \mathcal{E}_{f_{\nu'}} \| = \lim_{\ell \to \infty} \mu_\ell
\]
in \( U_{9r/8}(a) \cap \{a + (x, y) : |x| \leq r\} \) as measures. The difference as measures between \( \| \partial \mathcal{E}_{f_{\nu'}} \| \) and \( \mathcal{H}^1 \mathbf{1}_{\{a+(x, f_{(i, \ell)}(x)) : |x| \leq r\}} \) is negligible in the limit due to Remark \( 5.3 \), and within the domain under consideration, \( Z_\ell = \tilde{Z}_\ell \), thus we need to prove \( \lim_{\ell \to \infty} \mathcal{H}^1 \mathbf{1}_{Z_\ell} = \lim_{\ell \to \infty} \mu_\ell \). For this, fix a smooth function \( \phi \in C_c^\infty(U_{9r/8}(a) \cap \{a + (x, y) : |x| \leq r\}) \) with \( |\phi| \leq 1 \). We know that \( \tilde{Z}_\ell \) is within \( c_9 \epsilon r r_{\ell} \)-neighborhood of graphs of \( f_{(1, \ell)}, \ldots, f_{(\nu'_{\ell}, \ell)} \) by \((5.57)\), and these graphs are separated by \( r_{\ell}/8 \) as in \((5.58)\). Now, take a neighborhood of the graph of \( f_{(1, \ell)} \). Recall that \( f_{(1, \ell)} \) is defined as in \((5.52)\), with the partition of unity \( \{\tilde{\xi}_k\}_{k=-k_0}^{k_0} \) subordinate to the intervals \( \{I_k\}_{k=-k_0}^{k_0} \) such that \( -r \in I_{-k_0+1}, \) \( r \in I_{k_0-1} \) and \( [-r, r] \subset \bigcup_{k=-k_0+1}^{k_0-1} I_k \) (these objects depend also on the indices of the functions but we drop the dependence). Let \( \psi(1, \ell) \) be a function such that

\[
\begin{align*}
\psi(1, \ell) = 1 & \quad \text{on} \ \{a + (x, y) : |y - f_{(1, \ell)}(x)| < r_{\ell}/32, |x| \leq r\}, \\
\psi(1, \ell) = 0 & \quad \text{on} \ \{a + (x, y) : |y - f_{(1, \ell)}(x)| > r_{\ell}/16, |x| \leq r\},
\end{align*}
\]

\( 0 \leq \psi(1, \ell) \leq 1 \) and \( \sup(\epsilon \ell r_{\ell}) |\nabla \psi(1, \ell)| \) \( \leq c \) where \( c \) is an absolute constant. Such \( \psi(1, \ell) \) can be constructed due to the definition of \( f_{(1, \ell)} \). We then use \((5.23)\) with \( \tilde{\xi}_k \psi(1, \ell) \phi \) in place of \( \phi \) there. Note that the projection of \( U_{r_{\ell}}(z_k) \) to the \( x \)-axis is \( I_k \) in this case, and we have \( \tilde{\xi}_k \psi(1, \ell) \phi \in C^1_c(U_{r_{\ell}}(z_k)) \) for each \( k = -k_0+1, \ldots, k_0-1 \) and for sufficiently large \( \ell \)

\[
|\int_{B_{r_{\ell}}(z_k)} \tilde{\xi}_k \psi(1, \ell) \phi \, d\|\partial \mathcal{E}_{f_{\nu'}}\| - \int_{B_{r_{\ell}}(z_k) \cap L_k} \tilde{\xi}_k \psi(1, \ell) \phi \, d\mathcal{H}^1| \leq \epsilon \ell r_{\ell}.
\]

By the definition \((5.52)\) and the subsequent discussion, within \( U_{r_{\ell}}(z_k) \), the distance between \( L_k \) and the graph of \( f_{(1, \ell)} \) is within \( c \epsilon \ell r_{\ell} \) and the difference of the derivatives is within \( c \epsilon \ell \) (with absolute constant \( c \)), so that we may estimate as

\[
|\int_{B_{r_{\ell}}(z_k) \cap L_k} \tilde{\xi}_k \psi(1, \ell) \phi \, d\mathcal{H}^1 - \int_{B_{r_{\ell}}(z_k) \cap \text{graph } f_{(1, \ell)}} \tilde{\xi}_k \psi(1, \ell) \phi \, d\mathcal{H}^1| \leq c(\|\phi\|_{C^1}) \epsilon \ell r_{\ell}.
\]

Now, summing over \( k \) and using the fact that \( \sum \tilde{\xi}_k = 1 \) and \( \psi(1, \ell) = 1 \) on graph \( f_{(1, \ell)} \), we obtain from \((5.63)\) and \((5.64)\) that

\[
|\int \psi(1, \ell) \phi \, d\|\partial \mathcal{E}_{f_{\nu'}}\| - \int_{\text{graph } f_{(1, \ell)}} \phi \, d\mathcal{H}^1| \leq c(\|\phi\|_{C^1}) \epsilon \ell r,
\]

where we used \((k_0 + k'_0) r_{\ell} \leq cr \) with an absolute constant (for example, we may take \( c = 5 \)). Now, suppose that \( \nu' \geq \nu + 1 \) for large \( \ell \). Then, we can obtain \((5.64)\) for each \( f_{(1, \ell)}, \ldots, f_{(\nu+1, \ell)} \), and summing over \( i = 1, \ldots, \nu + 1 \), we have (with \( \psi(2, \ell), \ldots, \psi(\nu+1, \ell) \) being the corresponding cut-off
functions for graphs of \( f_{(2, \ell)}, \ldots, f_{(\nu+1, \ell)} \)
\[
| \int \sum_{i=1}^{\nu+1} \psi_{(i, \ell)} \phi d\|\partial \mathcal{E}_{j_k}\| - \sum_{i=1}^{\nu+1} \int_{\text{graph } f_{(i, \ell)}} \phi d\mathcal{H}^1 | \leq c(\nu, \|\phi\|_{C^1}) \epsilon r. \tag{5.66}
\]

We choose \( \phi \) which is a smooth approximation to the characteristic function of the set \( \{ a + (x, y) : |y| < 2c_7r, |x| < r \} \) but which has a compact support within the same set. We may assume that the graphs of \( f_{(1, \ell)}, \ldots, f_{(\nu+1, \ell)} \) are within \( \{ a + (x, y) : |y| < 3c_7r/2 \} \) and we can make sure that the following inequality holds:
\[
\sum_{i=1}^{\nu+1} \int_{\text{graph } f_{(i, \ell)}} \phi d\mathcal{H}^1 \geq 2r(\nu + 3/4). \tag{5.67}
\]

On the other hand, since the supports of \( \psi_{(1, \ell)}, \ldots, \psi_{(\nu+1, \ell)} \) are mutually disjoint, \( \psi_{(\nu+1+\ell), \ell} \) implies
\[
\limsup_{\ell \to \infty} \int \sum_{i=1}^{\nu+1} \psi_{(i, \ell)} \phi d\|\partial \mathcal{E}_{j_k}\| \leq \int \phi d\|\partial \mathcal{E}_{j_k}\| \leq 2r(\nu + 1/2). \tag{5.68}
\]

Then (5.66)–(5.68) lead to a contradiction as \( \ell \to \infty \). Thus we proved \( \nu' \leq \nu \). To see that \( \nu' \leq \nu - 1 \). This time, we choose \( \phi \) which is a smooth approximation to the characteristic function of the set \( \{ a + (x, y) : |y| < 2c_7, |x| < r/2 \} \) but which has a compact support within the same set. We may assume that the slopes of \( f_{(1, \ell)}, \ldots, f_{(\nu', \ell)} \) are small by restricting \( c_6 \) and \( c_7 \) and we can make sure that
\[
\sum_{i=1}^{\nu'} \int_{\text{graph } f_{(i, \ell)}} \phi d\mathcal{H}^1 \leq r(\nu' + 1/4) \leq r(\nu - 3/4). \tag{5.69}
\]

By (5.59), on the other hand, we have
\[
\liminf_{\ell \to \infty} \int \sum_{i=1}^{\nu'} \psi_{(i, \ell)} \phi d\|\partial \mathcal{E}_{j_k}\| = \lim_{\ell \to \infty} \int_{Z_{\ell}} \phi d\mathcal{H}^1 = \int \phi d\mu. \tag{5.70}
\]

By (5.28) and choosing an appropriate \( \phi \) to begin with, we may assume
\[
\int \phi d\mu \geq r(\nu - 2/3). \tag{5.71}
\]

Since we have (5.66) with \( \nu + 1 \) there replaced by \( \nu' \), (5.69)–(5.71) lead to a contradiction, which proves \( \nu' = \nu \) at last.

Thus, for sufficiently large \( \ell \), we have (5.66) with \( \nu + 1 \) there replaced by \( \nu \), and we may assume that \( f_{(1, \ell)} < f_{(2, \ell)} < \ldots < f_{(\nu, \ell)} \) for \( |x| \leq r \). These functions satisfy (5.54), and by Arzelà-Ascoli theorem (with a slight modification of the proof due to the vanishing error \( \epsilon \ell \)), there exists a subsequence denoted by the same index converging in the \( C^1 \) norm and the limit \( C^{1,1/2} \) functions \( f_1 = \lim_{\ell \to \infty} f_{(1, \ell)} \leq \ldots \leq f_\nu = \lim_{\ell \to \infty} f_{(\nu, \ell)} \) defined on \( |x| \leq r \). Because of the uniform \( C^1 \) convergence and by (5.66) (with \( \nu \) in place of \( \nu + 1 \)), we proved (5.61) as well as (5.31). Lastly, we prove that the limit functions are in \( W^{2,2} \). We prove that for any \( \phi \in C^2_c((-r, r)) \) and for any
We next use (5.24) to obtain
\[ \int \phi' \frac{f'_i}{\sqrt{1 + (f'_i)^2}} \, dx \leq \left( \frac{\epsilon}{r} \right) \int \phi^2 \frac{\sqrt{1 + (f'_i)^2}}{\sqrt{1 + (f'_i)^2}} \, dx. \] (5.72)

This proves that \( f'_i/\sqrt{1 + (f'_i)^2} \) has the weak derivative in \( L^2 \), and which shows that \( f_i \in W^{2,2} \).

Assume \( i = 1 \) and the other cases are similar. By the \( C^1 \) convergence, we have
\[ \int \phi' \frac{f'_i}{\sqrt{1 + (f'_i)^2}} \, dx = \lim_{\ell \to \infty} \int \phi' \frac{f'_{i(\ell)}}{\sqrt{1 + (f'_{i(\ell)})^2}} \, dx. \] (5.73)

Fix a large \( \ell \). We go back to the argument and notation following (5.61). Since \( \sum \tilde{\zeta}_k = 1 \), we have
\[ \int \phi' \frac{f'_{i(\ell)}}{\sqrt{1 + (f'_{i(\ell)})^2}} \, dx = \sum_{k=-k_0}^{k_0} \int_{I_k} \tilde{\zeta}_k \phi' \frac{g'_k}{\sqrt{1 + (g'_k)^2}} \, dx. \] (5.74)

On each \( I_k \), by (5.53),
\[ \left| \int_{I_k} \tilde{\zeta}_k \phi' \frac{f'_{i(\ell)}}{\sqrt{1 + (f'_{i(\ell)})^2}} \, dx - \int_{I_k} \tilde{\zeta}_k \phi' \frac{g'_k}{\sqrt{1 + (g'_k)^2}} \, dx \right| \leq c (\sup |\phi'|) \epsilon_{r \ell}. \] (5.75)

Recall that graph \( g_k \) represents \( L_k \), and the \( (1,2) \)-component of the orthogonal projection to the tangent space of \( L_k \) is given by \( g'_k/(1 + (g'_k)^2) \). Thus, in terms of the language of varifold,
\[ \int_{I_k} \tilde{\zeta}_k \phi' \frac{g'_k}{\sqrt{1 + (g'_k)^2}} \, dx = \int_{G_1(U_{r \ell}(z_k))} \tilde{\zeta}_k(x) \phi'(x) \tilde{S}_{12} \, d(|L_k|)(z, \bar{S}), \] (5.76)

where \( z = (x, y) \). Since \( \psi_{(1,\ell)} = 1 \) on \( L_k \), we have
\[ \int_{G_1(U_{r \ell}(z_k))} \tilde{\zeta}_k(x) \phi'(x) \tilde{S}_{12} \, d(|L_k|)(z, \bar{S}) = \int_{G_1(U_{r \ell}(z_k))} \tilde{\zeta}_k(x) \phi' \psi_{(1,\ell)}(x) \tilde{S}_{12} \, d(|L_k|)(z, \bar{S}). \] (5.77)

We next use (5.21) to obtain
\[ \left| \int_{G_1(U_{r \ell}(z_k))} \tilde{\zeta}_k \phi' \psi_{(1,\ell)} \tilde{S}_{12} \, d(|L_k|) - \int_{G_1(U_{r \ell}(z_k))} \tilde{\zeta}_k \phi' \psi_{(1,\ell)} \tilde{S}_{12} \, d(|\partial E_{\ell}|) \right| \leq \epsilon_{r \ell}. \] (5.78)

For this to be true, we note that \( \tilde{\zeta}_k \phi' \psi_{(1,\ell)} \in C^1_c(U_{r \ell}(z_k)) \) and \( \epsilon_{r \ell} |\nabla (\tilde{\zeta}_k \phi' \psi_{(1,\ell)})| \) can be bounded by a constant depending only on \( \|\phi\|_{C^2} \), thus (5.21) is valid with this function for sufficiently large \( \ell \).

By (5.74)-(5.78), and using that \( (k_0 + k_0) r \ell \leq 5 r \), we obtain
\[ \left| \int \phi' \frac{f'_{i(\ell)}}{\sqrt{1 + (f'_{i(\ell)})^2}} \, dx - \int_{G_1(\mathbb{R}^2)} \phi' \psi_{(1,\ell)} \tilde{S}_{12} \, d(|\partial E_{\ell}|) \right| \leq 5 (\sup |\phi'| + 1) \epsilon_{r \ell}. \] (5.79)

In particular, (5.73) and (5.79) show
\[ \int \phi' \frac{f'_{i}}{\sqrt{1 + (f'_{i})^2}} \, dx = \lim_{\ell \to \infty} \int_{G_1(\mathbb{R}^2)} \phi' \psi_{(1,\ell)} \tilde{S}_{12} \, d(|\partial E_{\ell}|). \] (5.80)

Now \( \varphi(x) \psi_{(1,\ell)}(z) = \frac{\partial}{\partial x} (\phi \psi_{(1,\ell)}) - \phi \frac{\partial}{\partial y} \psi_{(1,\ell)} \) and \( \nabla \psi_{(1,\ell)} \) is non-zero only on \( \{ a + (x, y) : r \ell/32 \leq |y - f_{(1,\ell)}(x)| \leq r \ell/16 \} \) by (5.62). By (5.57), on this set, there is no point of \( \tilde{Z}_{\ell} \). Since \( \partial E_{\ell} = \tilde{Z}_{\ell} \cup \tilde{Z}_{\ell}^c \)
and the measure of $\tilde{Z}_f^\ell$ is negligible as noted in Remark 5.3, we have
\[
\lim_{\ell \to \infty} \int_{\mathbf{G}_1(\mathbb{R}^2)} \phi \psi (1, \ell) \tilde{S}_{12} d(\partial \mathcal{E}_{j_\ell}) = \lim_{\ell \to \infty} \int_{\mathbf{G}_1(\mathbb{R}^2)} \left\{ \frac{\partial}{\partial x} (\phi \psi (1, \ell)) \tilde{S}_{12} + \frac{\partial}{\partial y} (\phi \psi (1, \ell)) \tilde{S}_{22} \right\} d(\partial \mathcal{E}_{j_\ell}).
\] (5.81)

By the definition of the first variation, we have
\[
\int_{\mathbf{G}_1(\mathbb{R}^2)} \left\{ \frac{\partial}{\partial x} (\phi \psi (1, \ell)) \tilde{S}_{12} + \frac{\partial}{\partial y} (\phi \psi (1, \ell)) \tilde{S}_{22} \right\} d(\partial \mathcal{E}_{j_\ell}) = \delta (\partial \mathcal{E}_{j_\ell}) ((0, \phi \psi (1, \ell))).
\] (5.82)

We estimate as in (5.43)-(5.45) with $\psi$ there replaced by $\phi \psi (1, \ell)$, which gives
\[
\lim_{\ell \to \infty} |\delta (\partial \mathcal{E}_{j_\ell}) ((0, \phi \psi (1, \ell))) - (\Phi_{\varepsilon_j_\ell} * \delta (\partial \mathcal{E}_{j_\ell})) ((0, \phi \psi (1, \ell)))| = 0.
\] (5.83)

In place of (5.46), we obtain (with $U = \text{spt} \phi \psi (1, \ell)$ and (5.26))
\[
|\Phi_{\varepsilon_j_\ell} * \delta (\partial \mathcal{E}_{j_\ell}) ((0, \phi \psi (1, \ell)))| \leq \left( \int_U \left| \frac{\Phi_{\varepsilon_j_\ell} * \delta (\partial \mathcal{E}_{j_\ell})^2}{\Phi_{\varepsilon_j_\ell} * \| \partial \mathcal{E}_{j_\ell} \| + \varepsilon_j_\ell \Omega^{-1}} \right| \right)^{1/2} \left( \int (\phi \psi (1, \ell))^2 (\Phi_{\varepsilon_j_\ell} * \| \partial \mathcal{E}_{j_\ell} \| + \varepsilon_j_\ell \Omega^{-1}) \right)^{1/2}.
\] (5.84)

Since $\phi \psi (1, \ell)$ is bounded, it follows
\[
\lim_{\ell \to \infty} \int (\phi \psi (1, \ell))^2 \varepsilon_j_\ell \Omega^{-1} = 0.
\] (5.85)

We can also estimate $|\Phi_{\varepsilon_j_\ell} * (\phi \psi (1, \ell)) - \phi \psi (1, \ell)|$ as in (5.44) so that
\[
\lim_{\ell \to \infty} \left| \int (\phi \psi (1, \ell))^2 (\Phi_{\varepsilon_j_\ell} * \| \partial \mathcal{E}_{j_\ell} \|) - \int (\phi \psi (1, \ell))^2 d\| \partial \mathcal{E}_{j_\ell} \| \right| = 0.
\] (5.86)

By the similar argument leading to (5.65) and the $C^1$ convergence of $f_{(1, \ell)}$ to $f_1$, one can prove that
\[
\lim_{\ell \to \infty} \int (\phi \psi (1, \ell))^2 d\| \partial \mathcal{E}_{j_\ell} \| = \int_{\text{graph} f_1} \phi^2 d\mathcal{H}^1.
\] (5.87)

Combining (5.80)-(5.87), we obtain (5.72). This concludes the proof of Theorem 5.7.

6. PROOF OF MAIN THEOREMS

Finally, we give a proof of Theorem 5.22 and 5.23.

Proof. As stated in Theorem 5.18, for almost all $t \in [0, \infty)$, the limit varifold $V_t$ is integral with locally square-integrable generalized mean curvature, and we have Theorem 5.7 available for this $V_t$, where $\mu = \|V_t\|$ there. We omit $t$ in the following. By the monotonicity formula (see [20, 17.7] for the precise form we need), the density function $\theta^1(\|V\|, \cdot)$ is an upper semicontinuous function on $\mathbb{R}^2$ and $\theta^1(\|V\|, x) \geq 1$ for $x \in \text{spt} \|V\|$. Moreover, for any sequence $R_t \to 0+$ and $z_0 \in \text{spt} \|V\|$, define $F_i(z) := (z - z_0)/R_i$, and consider the sequence $(F_i)_t V$. By the well-known argument on the existence of tangent cone (see for example [20, Sec. 42]), there exists a subsequence (denoted by the same index) and the limit $\tilde{V} = \lim_{i \to \infty} (F_i)_t V$ which is stationary integral varifold and which is homogeneous degree 0. That means that there exist distinct half-lines emanating from the origin $M_1, \ldots, M_m$ and integer multiplicities $\theta_1, \ldots, \theta_m$ such that $\tilde{V} = \sum_{k=1}^m \theta_k |M_k|$. If $m = 2$, the
stationarity of \( \tilde{V} \) implies \( \theta_1 = \theta_2 \) and \( M_1 \) and \( M_2 \) are parallel, that is, \( \text{spt} \| \tilde{V} \| \) is a line through the origin. If \( \theta_1 = 1 \), we can apply the Allard regularity theorem [11] and conclude that \( \text{spt} \| V \| \) is a \( W^{2,2} \) curve in a neighborhood of \( z_0 \). If \( \theta_1 \geq 2 \), then, we apply Theorem 5.7 to \( V \) with \( \nu = \theta_1, a = z_0 \) and \( B_{R_i}(a) \). The condition (5.26) is satisfied for all sufficiently large \( i \) since \( R_i \rightarrow 0^+ \). We may assume after rotation that \( M_1 \) is parallel to the \( x \)-axis, and \( \text{spt} \| (F_i)_x V \| \) converges to the \( x \)-axis locally in the Hausdorff distance due to the monotonicity formula. Thus (5.27) is satisfied for all large \( i \), and the convergence of \( \| (F_i)_y V \| \) to \( \nu \mathcal{H}^1 \{ y = 0 \} \) shows that (5.28) and (5.29) are satisfied for all large \( i \). Thus, there exists a neighborhood of \( z_0 \) having the description of (5.31), that is, \( \text{spt} \| V \| \) is a union of \( W^{2,2} \) curves tangent at \( z_0 \). Let \( \text{reg} V \) be the set of points where the tangent cone is a line with multiplicity as above, and let \( \text{sing} V \) be \( \text{spt} \| V \| \setminus \text{reg} V \). This is the set of points where there exists a tangent cone which is not a line, that is, there exists a tangent cone of the form \( \tilde{V} = \sum_{k=1}^m \theta_k |M_k| \) with \( m \geq 3 \). Note that \( \text{sing} V \) is a closed set. Moreover, by the following well-known argument, it is a discrete set: Otherwise, there is a sequence \( \{ z_i \}_{i=1}^\infty \subset \text{sing} V \) converging to \( z_0 \in \text{sing} V \). We may consider a map \( F_i(z) = (z - z_0)/|z_i - z_0| \) and \( (F_i)_y V \). By the same argument, a subsequence converges to a tangent cone, and we may assume after choosing a further subsequence that \( (z_i - z_0)/|z_i - z_0| \) converges to \( \tilde{z} \) with \( |	ilde{z}| = 1 \). But then, \( (F_i)_y V \) approaches to a line with possible multiplicity near \( \tilde{z} \) as \( i \rightarrow \infty \), which implies from the preceding argument that \( (F_i)_y V \) is regular in a neighborhood of \( \tilde{z} \), and that means that \( z_i \) is in \( \text{reg} V \), a contradiction.

Next, fix \( z_0 \in \text{sing} V \) and after a change of variables, assume \( z_0 = 0 \). Suppose \( \lim_{i \rightarrow \infty} (F_i)_y V = \sum_{k=1}^m \theta_k |M_k| \), where \( F_i(z) = z/R_i \) and \( R_i \rightarrow 0^+ \). Assume that \( M_1, \ldots, M_m \) are ordered counter-clockwise. Let us denote the annulus \( \{ z : R_i \leq |z| \leq 2R_i \} \) as \( A_i \). Since the convergence is also in the Hausdorff distance for \( (F_i)_y V \), we have \( \lim_{i \rightarrow \infty} (R_i)^{-1} d_H(\text{spt} \| V \| \cap A_i, \bigcup_{k=1}^m M_k \cap A_i) = 0 \). Depending only on the smallest angle between the half lines \( M_1, \ldots, M_m \), we choose a sufficiently small \( \beta > 0 \) so that for each \( k = 1, \ldots, m \) and \( \tilde{z} \in M_k \cap \{ z : 1 \leq |z| \leq 2 \} \), we have \( B_{4\beta}(\tilde{z}) \cap M_{k'} = \emptyset \) for all \( k' \neq k \). By the stated convergence, we may apply Theorem 5.7 to each \( B_{2\beta R_i}(\tilde{z}) \) for \( \tilde{z} \in M_k \cap A_i \) and conclude that \( \| V \| \) in \( A_i \) is represented by \( \theta_k \) graphs over \( M_k \) denoted by \( f^{(k)}_1 \leq \ldots \leq f^{(k)}_{\theta_k} \) of \( W^{2,2} \) functions near \( A_i \cap M_k \). Here, the index is chosen so that graph \( f_1^{(k)} \), \ldots, graph \( f_{\theta_k}^{(k)} \) are ordered counter-clockwise for each \( k = 1, \ldots, m \) (the "graph over \( M_k \)" means that \( M_k \) is identified with the positive \( x \)-axis and the graph is considered as \( (x, f^{(k)}_1(x)) \) and similarly for others with this coordinate). For all large \( i \), these graphs are \( C^1 \) close to \( M_k \), so their slopes may be arbitrarily close to that of \( M_k \) in \( A_i \) by choosing a large \( i \). Recall that we have a sequence \( \{ \partial E_{j\ell} \} \) converging to \( V \) and that we may consider \( Z_\ell \) in place of \( \{ \partial E_{j\ell} \} \) due to Remark 5.3. Since \( \text{sing} V \) is a discrete set, we may assume \( B_{4R_i} \cap \text{sing} V = \{ 0 \} \). We repeat the argument in the proof of Theorem 5.7 with the same notation and with \( B_{4R_i} \) in place of \( B_{2\nu}(a) \) there. For each \( z \in \text{spt} \| V \| \setminus \{ 0 \} \subset \text{reg} V \), we saw in the proof of Theorem 5.7 that there exists a neighborhood \( B_{\varepsilon}(z) \) and a subsequence \( \{ Z_{\ell'} \} \subset \{ Z_\ell \} \) such that \( B_{\varepsilon'}(z) \cap (Z_{\ell'} \setminus Z_{\ell' \nu}) = \emptyset \) for all sufficiently large \( \ell' \), or, there is no triple
we can make sure that the subsequence (denoted by the same index) such that
\[(Z_\ell \setminus Z^*_{\ell}) \cap (B_{4R_i} \setminus B_\ell) = \emptyset \quad \text{for all sufficiently large } \ell \text{ for each fixed } \epsilon \in (0, R_i) \quad (6.1)\]
and we consider this subsequence in the following. For the construction of the approximate $C^{1,1/2}$ curves with junctions, we start in the neighborhood of $A_i \cap M_k$. As we saw in the Step 3 of the proof of Theorem 5.7 in $A_i$, we may construct $C^{1,1/2}$ functions $f^{(k)}_{(1, \ell)} < \ldots < f^{(k)}_{(\theta_k, \ell)}$ over $M_k \cap A_i$ each of which converges to $W^{2,2}$ functions $f^{(k)}_1, \ldots, f^{(k)}_{\theta_k}$ in $C^1$ topology as $\ell \to \infty$. Thus we have
\[
\|V\|_{A_i} = \sum_{k=1}^{\theta_k} \sum_{j=1}^{\ell} \mathcal{H}^1_{A_i \cap \text{graph } f^{(k)}_j} \quad (6.2)
\]
Consider in particular $M_1$ and $f^*_\ell := f^{(1)}_{(\theta_1, \ell)}$ and for convenience, consider the coordinate system so that $M_1$ is the positive $x$-axis. We may continue choosing line segments to extend the graph $f^*_\ell$ in the negative direction until (a) the exit from $B_{2R_i}$ occurs or (b) the encounter with a triple junction occurs. Note that (a) is not possible: for a contradiction, this would mean that we can construct $f^*_\ell$ for $x \in [-R_i, 2R_i]$ with the property that (see (5.55))
\[
Z_\ell \cap \{(x, y) : |y - f^*_\ell(x)| \leq r_\ell/4, x \in [-R_i, 2R_i]\} \subseteq \{(x, y) : |y - f^*_\ell(x)| \leq 4\epsilon r_\ell, x \in [-R_i, 2R_i]\},
\]
which means that there is an empty horizontal strip devoid of $Z_\ell$ just above graph $f^*_\ell$ over $[-R_i, R_i]$ in particular. But then, as one construct a graph of $f^{(2)}_{(1, \ell)}$ starting near $M_2 \cap A_i$ towards the origin, we would have the following contradiction. Since the angle between graph $f^{(2)}_{(1, \ell)}$ and graph $f^*_\ell$ is strictly smaller than $\pi$ (since the angle between $M_1$ and $M_2$ is), we cannot extend graph $f^{(2)}_{(1, \ell)}$ past this horizontal strip without having a triple junction. The argument of Step 3 shows that the resulting network of line segments starting from the triple junction encountered by graph $f^{(2)}_{(1, \ell)}$ has to eventually needs to approach to this horizontal strip non-tangentially, but that would again be a contradiction. Thus (a) is not possible, and we have the case (b). This implies that there exists $x_\ell := x^{(1)}_{(\theta_1, \ell)} \in [-R_i, R_i]$ such that $f^*_\ell$ is defined on $[x_\ell, 2R_i]$, and that $(Z_\ell \setminus Z^*_{\ell}) \cap B_{\epsilon r_\ell}(x_\ell, f^*_\ell(x_\ell))) \neq \emptyset$. By (6.1), $\lim_{\ell \to \infty} (x_\ell, f^*_\ell(x_\ell))) = 0$. We can similarly argue that each $f^{(1)}_{(j, \ell)}$ ($j = 1, \ldots, \theta_1 - 1$) can be defined on $[x^{(1)}_{(j, \ell)}, 2R_i]$ with $\lim_{\ell \to \infty} (x^{(1)}_{(j, \ell)}, f^{(1)}_{(j, \ell)}(x^{(1)}_{(j, \ell)})) = 0$. Since we have $C^{1,1/2}$ estimates (5.54) for these functions, there exists a subsequence such that $f^{(1)}_{(j, \ell)}$ converges as $\ell \to \infty$ to $f^{(1)}_j \in W^{2,2}$ defined on $[0, 2R_i]$ with respect to the $C^1([\epsilon, 2R_i])$-norm for fixed but arbitrary $\epsilon > 0$ for each $j = 1, \ldots, \theta_1$. We can make sure that $\sup_{x \in [0, 2R_i]} \|f^{(1)}_{(j, \ell)}(x)\|$ is small due to (5.54) for each $j = 1, \ldots, \theta_1$. Moreover, we claim that $(f^{(1)}_j)'(0) = 0$. Otherwise, note that $\lim_{\ell \to \infty} (F_{f^{(1)}_j})_2 V$ would have to have a half-line with slope given by $(f^{(1)}_j)'(0) \neq 0$ with respect to the $x$-axis, which is different from any of $M_1, \ldots, M_{m_1}$, a contradiction. In summary, we may conclude the following: for arbitrary $\epsilon \in (0, R_i)$, $C^{1,1/2}$ functions $f^{(1)}_{(1, \ell)} < \ldots < f^{(1)}_{(\theta_1, \ell)}$ satisfy
\[
\lim_{\ell \to \infty} \|f^{(1)}_{(j, \ell)} - f^{(1)}_j\|_{C^1([\epsilon, 2R_i])} = 0, \quad (6.3)
\]
\[ Z_\ell \cap \{(x, y) : |y - f_{j,\ell}^{(1)}(x)| \leq r_\ell / 4, x \in [\epsilon, 2R_\ell]\} \subset \{(x, y) : |y - f_{j,\ell}^{(1)}(x)| \leq c_9 \epsilon r_\ell, x \in [\epsilon, 2R_\ell]\}, \]

and

\[ (f_j^{(1)})'(0) = 0 \]  \hspace{1cm}  \text{(6.5)}

for \( j = 1, \ldots, \theta_1 \). The similar conclusion follows near \( M_2, \ldots, M_m \) as well. We next claim that \( Z_\ell \cap \{z : \epsilon \leq |z| \leq R_\ell\} \) is all included in the \( c_9 \epsilon r_\ell \)-neighborhood of \( \cup_{k=1}^m \cup_{j=1}^{\theta_k} \text{graph} \ f_j^{(k)} \). Otherwise, we have a sequence \( z_\ell \in Z_\ell \cap \{z : \epsilon \leq |z| \leq R_\ell\} \) outside of the neighborhood. We can construct a \( C^{1,1/2} \) curve with small slope variation starting from \( z_\ell \), and since there is no point of \( Z_\ell \setminus Z_\ell^* \) within \( B_{4R_\ell} \setminus B_\ell \) by \textcolor{blue}{[6.1]} and it has to be disjoint from \( \cup_{j,k} \text{graph} \ f_j^{(k)} \), one can argue that this curve denoted by \( Q_\ell \) is close to a straight line intersecting \( \partial B_\ell \) and \( \partial B_{2R_\ell} \). In particular, \( \lim_{\ell \to \infty} \mathcal{H}^1 \mathbf{1}_{A_\ell \cap Q_\ell} \) contributes positively to \( \|V\| \) on \( A_\ell \), and we can argue that

\[ \|V\| \mathbf{1}_{A_\ell} > \sum_{k=1}^m \sum_{j=1}^{\theta_k} \mathcal{H}^1 \mathbf{1}_{A_\ell \cap \text{graph} f_j^{(k)}}. \]

But this is a contradiction to \textcolor{red}{[6.2]}. Thus, we have the conclusion. Since \( Z_\ell \) is covered by \( c_9 \epsilon r_\ell \)-neighborhood of \( \cup_{j,k} \text{graph} \ f_j^{(k)} \), the same argument proving \textcolor{blue}{(5.61)} shows

\[ \|V\| = \sum_{k=1}^m \sum_{j=1}^{\theta_k} \mathcal{H}^1 \mathbf{1}_{\text{graph} f_j^{(k)}} \text{ on } B_{2R_\ell}. \]  \hspace{1cm}  \text{(6.6)}

In fact, one shows \textcolor{red}{(6.6)} on \( B_{2R_\ell} \setminus B_\ell \) for arbitrary \( \epsilon > 0 \) first, which is enough to conclude the claim on \( B_{2R_\ell} \).

Finally we prove that the angle between the neighboring lines \( M_k \) and \( M_{k+1} \) is either 60 or 120 degrees. It is enough to consider the case of \( M_1 \) and \( M_2 \), and let us identify \( M_1 \) with the positive \( x \)-axis. As we saw already, we have a sequence of \( C^{1,1/2} \) functions \( f^*_{\ell} := f_{j,\ell}^{(1)} \) defined on \( [x_\ell, 2R_\ell] \) (where \( x_\ell := x_{(\theta_1, \ell)}^{(1)} \) with \( (Z_\ell \setminus Z_\ell^*) \cap B_{r_\ell \epsilon'} ((x_\ell, f^*_{\ell}(x_\ell))) \neq \emptyset \)). Moreover, since \( (f^{(1)}_{\ell \theta_1})'(0) = 0 \), one can show that \( \lim_{\ell \to \infty} (f^*_{\ell})'(x_\ell) = 0 \). Starting from \( (x_\ell, f^*_{\ell}(x_\ell)) \), one can construct two curves approximating \( Z_\ell \) again. Follow the curve going up in the direction of \( (-1, \sqrt{3}) \) denoted by \( C_\ell \), and note that the angle between \( C_\ell \) and \( M_1 \) (\( x \)-axis) at \( (x_\ell, f^*_{\ell}(x_\ell)) \) approaches to 120 degrees as \( \ell \to \infty \). We consider the following three cases, (a), (b), (c).

(a) Suppose that for each \( \ell \), \( C_\ell \) extends without encountering a triple junction until it exits from \( B_{2R_\ell} \) for all large \( \ell \). By the estimate of \textcolor{blue}{(5.54)}, there is a limit curve \( C_\infty \) of \( C_\ell \) which contributes to \( \|V\| \), and it has to be one of graph \( f^{(k)}_{\ell} \). In fact, \( C_\infty \) has to be graph \( f^{(2)}_{\ell} \). This is because of the following. Consider the connected component of

\[ B_{R_\ell} \cap \left( \{z : c_9 \epsilon r_\ell < d(z, C_\ell) < r_\ell / 4\} \cup \{(x, y) : c_9 \epsilon r_\ell < y - f^*_{\ell}(x) < r_\ell / 4, x \in [x_\ell, R_\ell]\} \right). \]

This looks like a wedge-shaped strip of width \( (1/4 - c_9 \epsilon) r_\ell \) just above graph \( f^*_{\ell} \cup C_\ell \), and due to the construction of \( f^*_{\ell} \) and \( C_\ell \) (see \textcolor{red}{(6.4)}), it does not contain any point of \( Z_\ell \). This implies that there is no point of \( Z_\ell \) in the set bounded by this wedge-shaped strip and \( \partial B_{R_\ell} \) (and between \( M_1 \) and \( M_2 \)),
since otherwise we should be able to construct a curve (and possibly a network) approximating $Z_\ell$. Then, arguing as in Step 3 in the proof of Theorem 5.7, this has to cross this wedge-shaped strip, but that is not possible. Thus there cannot be any point of $\text{spt} \|V\|$ in the region bounded by graph $f_{\theta_1}^{(1)}$, $C_\infty$ and $\partial B_{R_i}$, and $C_\infty$ has to be graph $f_1^{(2)}$. This also shows that the angle between $M_2$ and $M_1$ has to be 120 degrees.

(b) Suppose that for each $\ell$, as one extends $C_\ell$ by choosing line segments, $C_\ell$ meets some point of $Z_\ell \setminus Z_\ell^*$ (an encounter with a triple junction). Because of (6.1), this point has to converge to the origin as $\ell \to \infty$. We can again construct two curves starting from this point for each $\ell$, and let $\tilde{C}_\ell$ be the one going in the direction of (approximately) $(1, \sqrt{3})$. Assume that $\tilde{C}_\ell$ can be extended without encountering a triple junction and exits from $B_{2R_i}$. In this case, note that the length of $C_\ell$ approaches to 0 as $\ell \to \infty$ and the slope of $\tilde{C}_\ell$ at the junction approaches to $\sqrt{3}$. By arguing as in the case (a), we can prove that the limit curve $\tilde{C}_\infty$ has to be graph $f_1^{(2)}$, proving that the angle between $M_1$ and $M_2$ is 60 degrees.

(c) Continuing from (b), we consider the possibility that $\tilde{C}_\ell$ encounters another triple junction. There, we can again construct two curves starting from this point, and we follow the curve $\tilde{C}_\ell$ going to the right. Just to visualize the setting, imagine that we “walk” on graph $f_\ast^{(1)} \ell$ in the negative $x$ direction, then, at the triple junction $(x_\ell, f_\ast^{(1)} \ell(x_\ell))$, turn 60 degrees to the right, and walk along $C_\ell$. Then at another triple junction, we turn 60 degrees to the right again and walk along $\tilde{C}_\ell$. In the present case, we assume that we encounter another triple junction, and turn right by 60 degrees, and walk along $\tilde{C}_\ell$. All these triple junctions converge to the origin as $\ell \to \infty$ by (6.1). Note that $\tilde{C}_\ell$ has to be almost parallel to the $x$-axis and we would be walking in the positive $x$ direction. If $\tilde{C}_\ell$ can be extended without triple junction until it exits from $B_{2R_i}$, this means that we have another curve above graph $f_{\theta_1}^{(1)} \ell$ of $Z_\ell$, and the limit $\hat{C}_\infty$ of $\tilde{C}_\ell$ will be a curve tangent to $M_1$ at the origin which contribute to $\|V\|$ in addition to $f_1^{(1)}, \ldots, f_{\theta_1}^{(1)}$. This would be a contradiction to (6.6). Thus, $\tilde{C}_\ell$ has to encounter a triple junction. But then, just as in the argument of Step 3, the resulting curve (or network) starting from this triple junction has to go down in the negative $y$ direction until it intersects graph $f_{\theta_1}^{(1)} \ell$, but that is not possible. To sum up, the case (c) actually does not occur, and we have either (a) or (b), proving the desired angle condition.

The proof of Theorem 2.2 is now complete, by rotating the coordinate so that $M_1$ is the positive $x$-axis. The equality (2.7) follows immediately from the fact that the tangent cone is stationary. The proof of Theorem 2.3 ($N = 2$ case) can be completed by reviewing the present proof. The point is that, if $N = 2$ we have $Z_\ell = Z_{\ast \ell}$ due Lemma 5.5 with $N = 2$, that is, there is no triple junction in $Z_\ell$. Then, the above analysis around sing $V$ shows that the tangent cone has to be a line without junction and sing $V$ is empty. \hfill $\square$

**Lemma 6.1.** Suppose that $\{V_i^\ell\}_{t \in I}$ ($i \in \mathbb{N}$) is a sequence of 1-dimensional Brakke flows in $U_R(a) \times I$ ($I$ is an interval) such that $\sup_{t \in I} \sup_{i \in \mathbb{N}} \|V_i^\ell(U_R(a))\| < \infty$ and such that the property stated in
Thus, for a.e. $t \in I$. Then there exist a subsequence (denoted by the same index) and a Brakke flow $\{V_t\}_{t \in I}$ such that $\lim_{t \to \infty} ||V_t^i|| = ||V_i||$ in $U_R(a)$ for all $t \in I$ and $V_t$ satisfies Theorem 2.2 (resp. 2.3) holds true in $U_R(a)$. In fact, it is much easier since the converging objects are regular curves $V_t$. Theorem 2.2 holds, which is true for a.e. $V$ we already know that $W$ with uniform $F$ or any non-negative function $h$ is uniformly bounded on a compactly supported domain in $[12, 24]$ so we only discuss the limit satisfying the regularity property. Proof. One can find the proof of compactness (the existence of convergent subsequence and the limit being a Brakke flow) in [12, 24] so we only discuss the limit satisfying the regularity property. For any non-negative function $\phi \in C^2_c(U_R(a))$ and $t_1 < t_2$ in $I$, by (2.3) and (2.4),

$$
\left. \frac{||h^i||}{t^2} \right|_{t=t_1} \leq \int_{t_1}^{t_2} \int_{U_R(a)} |\nabla \phi||h^i| - \phi|h^i|^2 d\|V_t^i\| dt \leq \int_{t_1}^{t_2} \int_{U_R(a)} \frac{|\nabla \phi|^2}{2\phi} - \frac{1}{2}\phi|h^i|^2 d\|V_t^i\| dt, 
$$

(6.7)

where $h^i = h(\cdot, V_t^i)$ and we used $ab \leq \frac{1}{2}(a^2 + b^2)$ and $|\nabla \phi|^2/\phi \leq 2\sup ||\nabla^2 \phi||$. By Fatou’s Lemma and the given uniform bound, we have

$$
\int_{t_1}^{t_2} \liminf_{i \to \infty} \left( \int_{U_R(a)} \phi|h^i|^2 d\|V_t^i\| \right) dt < \infty. 
$$

(6.8)

Thus, for a.e. $t \in I$, we can choose a subsequence (denoted by the same index) such that the $L^2$ norm of $h^i$ is uniformly bounded on a compactly supported domain in $U_R(a)$. Consider $t \in I$ such that such a subsequence exists and additionally assume that the regularity property described in Theorem 2.2 holds, which is true for a.e. $t$. By the lower semicontinuity argument, the limit $V_t$ has also $h(\cdot, V_t) \in L^2_{loc}(||V_t||)$ for each $t$ and we may also assume that $V_t$ is integral for each $t$ (since we already know that $V_t$ is integral for a.e. $t$). For any $z_0 \in \text{spt}(||V_t||)$, a tangent cone exists, which consists of a finite number of half-lines with integer multiplicities. Let $F_i(z) := (z - z_0)/r_i$, where $(F_i)_t V_t$ converges to a tangent cone. We can choose a further subsequence $\{V_t^i\}_{i \in N}$ (with the same index) so that $(F_i)_t V_t^i$ also converges to the same tangent cone as $i \to \infty$. Since $\int_{B_{r_i}(z_0)} |h^i|^2 d\|V_t^i\|$ is uniformly bounded and due to the regularity property of $V_t^i$, the variation of the slope of tangent line along each curve of $\text{spt}(||V_t^i||)$ in $B_{r_i}(z_0)$ is $O((r_i)^{1/2})$, and any of their junctions are of the type described in Theorem 2.2. The rest of the argument is similar to the one given in the proof of Theorem 2.2. In fact, it is much easier since the converging objects are regular curves $V_t^i$ instead of “approximately regular” curves $Z_t$. If the tangent cone of $V_t$ at $z_0$ is a line with multiplicity $\nu$, then, for all sufficiently large $i$, $||(F_i)_t V_t^i||$ will be close to it in measure in $B_1$, and we may conclude that $\text{spt}(||V_t^i||)$ cannot have a junction and it is a union of stacked $\nu$ curves of class $W^{2,2}$ with uniform $W^{2,2}$ bound. With the radius fixed at this point and letting $i \to \infty$, we conclude that $V_t$ is also made up by a stacked $W^{2,2}$ curves near $z_0$. If the tangent cone is not a line but a union of half-lines with multiplicities, we know that such point is isolated from the same argument as before. Also $(F_i)_t V_t^i$ is locally close to half-lines away from the origin for large $i$, and arguing as before, we can conclude that angles of the tangent cone have to be either 60 or 120 degrees depending on how many triple junctions there are along the curve. We then prove that the limit $||V_t||$ near each
half-line is obtained as the limit of graphs of $W^{2,2}$ functions. We omit the detail since the idea is similar. The case of $N = 2$ is simpler since $V_t^1$ is a union of embedded $W^{2,2}$ curves which may be tangent to each other at some point. Since there is a uniform $W^{2,2}$ bound, one can check that $V_t$ cannot have a tangent cone which is not a line. Since any tangent cone is a line with multiplicity, one can show the desired local property.

Theorem 2.4 is a direct corollary of Lemma 6.1.

Proof. Any tangent flow is obtained as a limit of parabolically rescaled Brakke flow, and the regularity property in Theorem 2.2 and 2.3 is not affected under the rescaling. Thus Lemma 6.1 shows that the tangent flow has the same property.

□

7. Concluding remarks

7.1. Results for Brakke flows in [22]. In Section 2, mainly to avoid confusion, we state results for Brakke flows obtained in [15]. However, since the results are local in nature, the same regularity properties hold true for Brakke flows obtained in [22], where the existence of Brakke flow with fixed boundary condition in a strictly convex domain $U \subset \mathbb{R}^{n+1}$ was established. The proof of the present paper relies on the estimate (3.28), and the same but localized version [22, Proposition 4.13] is available and the proof proceeds by the same argument in the interior of $U$ (with the weight function $\Omega = 1$). Additionally, we recall the following.

Theorem 7.1. ([22, Theorem B])

Let $\{V_t\}_{t \in \mathbb{R}^+}$ be a Brakke flow obtained in [22]. Then there exists a sequence of times $\{t_k\}_{k=1}^\infty$ with $\lim_{k \to \infty} t_k = \infty$ such that the corresponding varifolds $V_k := V_{t_k}$ converge to a stationary integral varifold $V_\infty$ in $U$ such that $(\text{clos (spt } \|V_\infty\|)) \setminus U = \partial \Gamma_0$.

Here $\partial \Gamma_0$ is the given boundary condition. Since $V_\infty$ is stationary and we are concerned with $n = 1$, by [2], spt $\|V_\infty\|$ consists of finite line segments with junctions. A corollary of the present paper is the following.

Theorem 7.2. For $n = 1$, the limit $V_\infty$ of Theorem 7.1 has to satisfy the same angle condition at junctions in $U$, namely, angles at junctions are either 60 or 120 degrees. For $N = 2$, spt $\|V_\infty\|$ consists of lines with no junction in $U$.

The proof is by observing that we may choose the sequence $\{t_k\}_{k=1}^\infty$ so that the $L^2$ curvatures of $V_k$ converge to 0 and so that $V_k$ has the regularity property stated in Theorem 2.2 (or 2.3 if $N = 2$). See [22, Section 7] for the detail of the choice of the sequence. Then the same argument for the proof of Lemma 6.1 shows the claim for $V_\infty$. 
7.2. The higher dimensional case. The present paper does not give any definite conclusion for \( n \geq 2 \), except for Theorem 4.1 that tells that the approximate MCF \( \partial E_{j_\ell}(t) \) is close to a measure-minimizing regular cluster in a small length scale of \( o(1/j_\ell^2) \) for a.e. \( t \). Even though we have the \( L^2 \) bound of smoothed mean curvature vector of \( \partial E_{j_\ell} \), unlike the case of \( n = 1 \), it does not provide enough control of the converging \( n \)-dimensional surfaces. It still raises a natural question as to what is the additional regularity property of the Brakke flow obtained in [15, 22]. As a closely related result, the measure-theoretic closure of smooth MCF of clusters is studied in [19], where it is proved that such class is compact under a set of natural assumptions. Formally, a major difficulty of pursuing the similar line of idea for the present case is that \( \partial E_{j_\ell}(t) \) satisfies the approximate Brakke’s inequality for test functions which cannot vary within the length scale of \( o(1/j_\ell) \) (see (3.24) which is an approximate Brakke’s inequality and the definition of the class of test functions \( A_j \)). Because of this restriction, we cannot fully utilize the aspect that \( \partial E_{j_\ell}(t) \) is an approximate MCF. On the other hand, it seems reasonable to expect that certain “unstable” singularity cannot persist in time in general.

7.3. Further regularity results. The present paper narrows down the possibility of angles of junctions to 0, 60 and 120 degrees, but this does not necessarily mean that they all actually appear for a set of times with positive Lebesgue measure. In fact, except for triple junctions of 120 degrees with unit density, we expect that the other types of junctions should not persist in time and are likely to break up into triple junctions immediately. Also, higher multiplicity should not persist as well, since setting the multiplicity equal to 1 simply reduces the total mass of varifold and Brakke’s formulation should not pose serious difficulty doing so. It may be the case that certain restarting procedure is necessary to obtain a “better” Brakke flow which is a network with triple junctions with unit density for a.e. \( t \).

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