Mass gap from pressure inequalities

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(Dated: March 26, 2022)

We prove that a temperature independent mass distribution is identically zero below a mass threshold (mass gap) value, if the pressure satisfies certain inequalities. This supports the finding of a minimal mass in quark matter equation of state by numerical estimates and by substitution of analytic formulas. We present a few inequalities for the mass distribution based on the Markov inequality.

INTRODUCTION

Quark matter, searched for in relativistic heavy ion collisions, reveals itself in signatures on observed hadron spectra which are (can best be) interpreted in terms of quark level properties. In particular scaling of the elliptic flow component $v_2$ with the constituent quark content of the finally observed mesons and baryons [1] and successful description of $p_T$-spectra of pions and antiprotons using quark coalescence rules for hadron building [2] utilize the fast hadronization concept of quark redistribution. Albeit this simple idea brings also problems with it, e.g. in dealing with energy conservation and entropy non-decreasing, these issues can be resolved by using a distributed mass quasiparticle model for quark matter [3], and are in accord with the quark matter equation of state obtained in lattice QCD calculations [4]. The surmised mass distribution gives rise to specific equation of state (pressure as a function of temperature, $p(T)$), and reversed, a mass distribution may be outlined from knowledge on the $p(T)$ curve.

While traditional, fixed mass quasiparticle models already succeed to describe the equation of state obtained in lattice QCD [5], those mass values are themselves temperature dependent. Furthermore a temperature dependent width is associated to the quasiparticle mass, too [6]. This is unavoidable, as it can be easily seen from the following comparison of pressures obtained from a mass distribution and from perturbative QCD in the high temperature limit:

$$\frac{p(T)}{p_{id}(0,T)} = 1 - \frac{\langle m^2 \rangle}{4T^2} + \frac{3}{4} - \gamma + \frac{\langle m^4 \rangle}{16T^4} + \frac{\langle m^4 \ln(2T/m) \rangle}{16T^4} + \ldots,$$

$$\frac{p_{QCD}(T)}{p_{id}(0,T)} = 1 - a_2g^2 + a_4g^4 + b_4g^4\ln \frac{2\pi T}{\Lambda} + \ldots$$

where the ideal gas pressure for relativistic particles with mass $m$ defines the scaling function

$$\Phi \left( \frac{m}{T} \right) = \frac{p_{id}(m,T)}{p_{id}(0,T)} = \frac{1}{2} \left( \frac{m}{T} \right)^2 K_2 \left( \frac{m}{T} \right).$$

We note that $\Phi(x)$ can also be obtained for Bose or Fermi distributions instead of the Boltzmann one; the numerical difference is overall minor, less than six per cent. The basic assumption, $\langle m^2 \rangle \sim g^2T^2$ sets the scale for a simplified treatment of the quark matter pressure at high temperature. The comparison reveals that $\langle m^4 \rangle \neq \langle m^2 \rangle^2$, whence the necessity of a width in the mass distribution emerges. The temperature dependence of the pressure ratio to the massless ideal gas value is concentrated on the temperature dependence of the coupling constant: $g = g(T)$ in the traditional interpretation. We have recently pursued an alternative approach to the quasiparticle mass distribution in quark matter [2, 3], where a temperature independent $w(m)$ distribution is reconstructed from the pressure ratio $\sigma(T) = p(T)/p_{id}(0,T)$ curve:

$$\sigma(T) = \int_0^\infty w(m) \Phi \left( \frac{m}{T} \right) \, dm.$$  \hfill (3)

This is possible only with a single $w(m)$ curve for each exactly known $\sigma(T)$. Such $w(m)$ distributions show diverging expectation values for positive powers of the mass (like $\langle m^2 \rangle$) signaling a high mass tail not falling faster than $\sim m^{-3}$. Another remarkable property of this approach is that it indicates a temperature independent threshold (smallest mass) in the $w(m)$ spectrum for lattice QCD pressure data[7, 8].

The pressure is, however, not known analytically, the numerical results are smeared with error bars. This problem is more severe in the light of the fact that eq.(3) is an integral transformation (related to the Meijer K-transformation, a generalization of the Laplace transformation). There is no mathematical guarantee that by inverting such a transformation one obtains close results for $w(m)$ from close functions for $\sigma(T)$. In fact this is known as the "inverse imaging problem" [10].
Our goal in the present paper is to support knowledge about a T-independent \( w(m) \) mass distribution when the pressure \( p(T) \) satisfies certain inequalities. In particular we prove that if the pressure \( p(T) \) is below the corresponding ideal gas pressure with a given mass \( M_0 \) at all temperatures, then the mass distribution is exactly zero for all masses below \( M_0 \). For inequalities with other than ideal gas pressure curves as estimators we apply the Markov inequality for probability measures, which directly offers upper bounds on the integrated probability \( P(M) = \int_0^M w(m) \, dm \). It turns out that the appearance and value of the highest possible estimators for \( \sigma - \) and \( \exp(\cdot) \) for lattice gauge theory data (bottom) with scaled pressure data in Figure 1 (top) and to pure SU(3) and QCD simulation of Ref.[7] (2+1 flavor QCD) and of [9] (pure SU(3) gauge theory).

**MARKOV INEQUALITY AND MASS GAP**

In this section we derive the mass gap based on the Markov inequality and use this approach to estimate upper bounds on the integrated probability for the mass being lower than a given value. The general form of the Markov inequality is given by [11]

\[
\mu(\{ x \in X : f(x) \geq t \}) \leq \frac{1}{g(t)} \int_{x \in X} g(f(x)) \, d\mu(x) \quad (4)
\]

with measure \( \mu \), a real valued \( \mu \)-measurable function \( f \), and a monotonic growing non-negative measurable real function \( g \). The proof, based on the monotony of integration, can be presented in a few lines. For a non-negative and monotonic growing function \( g(t) \leq g(f(x)) \) for \( t \leq f(x) \). We obtain

\[
g(t) \int_{f(x) \geq t} d\mu(x) = \int_{f(x) \geq t} g(t) \, d\mu(x) \leq \int_{f(x) \geq t} g(f(x)) \, d\mu(x). \quad (5)
\]

This quantity can be bounded by

\[
\int_{f(x) \geq t} g(f(x)) \, d\mu(x) \leq \int_{x \in X} g(f(x)) \, d\mu(x). \quad (6)
\]

A division by \( g(t) \geq 0 \) delivers the original statement in eq.(4).

In order to apply this inequality to the mass spectrum we choose \( f(m) = tM/m \). In this case

\[
\mu(\{ m \geq t \}) = \int_0^M d\mu(m), \quad (7)
\]

and the Markov inequality reads as

\[
P(M) := \int_0^M d\mu(m) \leq \frac{1}{g(t)} \int_0^\infty g\left(\frac{tM}{m}\right) \, d\mu(m). \quad (8)
\]

For a continuum mass spectrum \( d\mu(m) = w(m) \, dm \) can be chosen with \( w(m) \) being the probability density function. The generalized Markov inequality stated above is valid for general probability measures[12] \( \mu \) possibly including bound state contributions.

**Power law estimate**

In the following we discuss a few examples for monotonic rising functions \( g(z) \), which allow us to draw some conclusions about the integrated probability for masses below \( M \). Applying the special form of \( g(t) = t^n \) we arrive at

\[
P(M) \leq \frac{1}{t^n} \int_0^\infty \left(\frac{tM}{m}\right)^n w(m) \, dm, \quad (9)
\]
whence we obtain:

\[ P(M) \leq M^n \int_0^\infty m^{-n} w(m) \, dm. \tag{10} \]

It is easy to see that the negative integral moments of the mass on the right hand side of the above inequality are connected to the negative integral moments of scaled pressure \( \sigma(T) = p(T)/\rho(d)(0,T) \). The final inequality for the probability of having masses smaller than \( M \) is given by

\[ P(M) \leq M^n \int_0^\infty T^{-n} \sigma(T) \, dT. \tag{11} \]

The most striking inequality is obtained by using \( \sigma(T) \leq \exp(-\lambda/T) \) (cf. the dotted line in Figure 1 for \( \lambda = T_c \)). In this case eq.(11) delivers

\[ P(M) \leq \left( \frac{M}{\lambda} \right)^n \frac{2\Gamma(n)}{\Gamma(2+n/2)\Gamma(n/2)}. \tag{13} \]

The large \( n \) limit of this result is given by

\[ P(M) \leq \left( \frac{M}{\lambda} \right)^{2} \frac{1}{\pi n^{3/2}} \tag{14} \]

to leading order in \( 1/n \). Again the right hand side approaches zero for \( M \leq \lambda \) and diverges for \( M > \lambda \). This points out a mass gap stretching to (and including) \( \lambda \) from zero.

**The pressure itself is a majorant**

The most striking inequality is obtained by using \( g(t) = \Phi(1/t) \). This function is also admissible, its rise from zero to one is strict monotonic. Eq. (11) leads to

\[ P(M) \leq \frac{\sigma(tM)}{\Phi(1/t) \, \Phi(1/t)} \tag{15} \]

For \( t = 1 \) using the numerical value \( \Phi(1) \approx 0.81 \) one arrives at \( P(M) \leq 1.23\sigma(M) \), which can be directly read off from numerical simulation or theoretical predictions of \( \sigma(T) \). Figure 2 presents curves for different \( t \)-values (see legend), all being an upper estimate for the integrated probability \( P(M) \) in the respective cases of 2+1 flavor QCD and pure SU(3) gauge theory. The higher seems to be the starting \( M_0 \) value for the rise of the upper bound on \( P(M) \), the higher also the magnification of the error bars. A secure estimate for the \( P(M) \leq 0.05 \) is given for masses \( M > 1.7T_c = 280 \text{ MeV} \) for the 2+1 flavor QCD case, while for \( M > 7.2T_c = 1.9 \text{ GeV} \) for the pure SU(3) gauge case. While in the first case this can be at best an average between quark and gluon-like quasiparticle masses, in the second case should be close to observed glueball mass. We note that using \( \sigma(tM) \leq \Phi(M_0/tM) \) in the \( t \rightarrow 0 \) limit again a mass gap at \( M_0 \) follows from

**Upper bound by a fixed mass ideal gas**

Let us apply this result to the simplest majorant, that of a fixed mass relativistic ideal gas. In this case \( \sigma(T) \leq \Phi(M_0/T) \) with some \( M_0 \) (cf. dashed line in Figure 1). Equation (11) leads to

\[ P(M) \leq M^n \int_0^\infty T^{-n} \Phi\left(\frac{M_0}{T}\right) \, dT \leq \frac{M^n}{\Phi(M_0)} \tag{12} \]
eq.(15). In this respect the use of different $g(z)$ functions in the Markov inequality does not matter\[13\].

A related version of the inequality (15) is obtained for $tM = T$, $g(z) = \Phi(tM/zT)$. The upper bound is obtained at any fixed $T$ as being

$$P(M) \leq \frac{\sigma(T)}{\Phi(M/T)} \quad (16)$$

Figure 3 plots upper bounds for $P(M)$ obtained using the eq.(16). The most restrictive are the lowest temperature data for $\sigma(T)$, they are, however, also the most contaminated by errors. It is probably safe to conclude that as much as 90 – 95\% of the masses are above $1.5 T_c \approx 440$ MeV according to these data.

Our mathematical treatment of the mass gap leaves the point $m = 0$ in the possible mass distribution as a special case. Assuming that there were such a contribution of finite measure, i.e. $P(0) = a$ were a finite value between zero and one, one concludes from the definition eq.(3) that in this case $\sigma(T) \geq a$ would be. There is no sign of such an indication in lattice QCD data.

Finally we note that there is a potential to use our method presented in this letter in a context wider than quark matter: the quasiparticle test based on the generalized Markov inequality can in principle be done for any system with sufficiently known thermal equation of state. The estimate for a lowest mass can then be checked against knowledge on the mass spectrum obtained from the study of correlation functions.

**Acknowledgment** This work has been supported by the Hungarian National Research Fund, OTKA (T49466, T048483) and by the Bolyai scholarship for Péter Ván. Discussions with Prof. T. Matolcsi, Prof. T. Móri and A. Lukács are gratefully acknowledged.

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[12] a measure normalized to one

[13] From practical viewpoint, however, in the $t \to 0$ limit the error bars on the original $p/T^4$ data are infinitely enlarged.