Descriptional Complexity of Bounded Context-Free Languages

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Abstract

Finite-turn pushdown automata (PDA) are investigated concerning their descriptional complexity. It is known that they accept exactly the class of ultralinear context-free languages. Furthermore, the increase in size when converting arbitrary PDAs accepting ultralinear languages to finite-turn PDAs cannot be bounded by any recursive function. The latter phenomenon is known as non-recursive trade-off. In this paper, finite-turn PDAs accepting bounded languages are considered. First, letter-bounded languages are studied. We prove that in this case the non-recursive trade-off is reduced to a recursive trade-off, more precisely, to an exponential trade-off. A conversion algorithm is presented and the optimality of the construction is shown by proving tight lower bounds. Furthermore, the question of reducing the number of turns of a given finite-turn PDA is studied. Again, a conversion algorithm is provided which shows that in this case the trade-off is at most polynomial. Finally, the more general case of word-bounded languages is investigated. We show how the results obtained for letter-bounded languages can be extended to word-bounded languages.

Key words: automata and formal languages, descriptional complexity, finite-turn pushdown automata recursive trade-offs, bounded languages

1 Introduction

Finite-turn pushdown automata (PDAs) were introduced in [5] by Ginsburg and Spanier. They are defined by fixing a constant bound on the number of switches between push and

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pop operations in accepting computation paths of PDAs. The class of languages defined by these models is called the class of ultralinear languages and is a proper subclass of the class of context-free languages. It can be also characterized in terms of ultralinear and non-terminal bounded grammars. (In the special case of 1-turn PDAs, i.e., devices making at most one switch between push and pop operations, we get the class of linear context-free languages).

In [12], descriptional complexity questions concerning finite-turn PDAs were investigated, by showing, among other results, the existence of non-recursive trade-offs between PDAs and finite-turn PDAs. Roughly speaking, this means that for any recursive function \( f(n) \) and for arbitrarily large integers \( n \), there exists a PDA of size \( n \) accepting an ultralinear language such that any equivalent finite-turn PDA must have at least \( f(n) \) states. Thus, a PDA with arbitrary many turns may represent an ultralinear language more succinctly than any finite-turn PDA and the savings in size cannot be bounded by any recursive function.

This phenomenon of non-recursive trade-offs was first observed between context-free grammars and deterministic finite automata (DFAs) in the fundamental paper by Meyer and Fischer [13]. Nowadays, many non-recursive trade-offs are known which are summarized, e.g., in [2] and [11]. In the context of context-free languages non-recursive trade-offs are known to exist between PDAs and deterministic PDAs (DPDAs), between unambiguous PDAs (UPDAs) and DPDAs, and between PDAs and UPDAs. Recursive trade-offs are known, e.g., between nondeterministic/alternating finite automata and DFAs and between DPDAs and DFAs.

Interestingly, the witness languages used in [13] were defined over an alphabet of two symbols and leave open the unary case which was recently solved in [14] by proving an exponential trade-off. Thus, the non-recursive trade-off in the binary case turns into a recursive trade-off in the unary case. More generally, a careful investigation of the known cases of non-recursive trade-offs reveals that the used witness languages are not bounded resp. word-bounded, i.e., they are not included in some subset of \( w_1^*w_2^*\ldots w_m^* \) for some fixed words \( w_1, w_2, \ldots, w_m \). So, the question arises whether the above non-recursive trade-offs can be translated to the bounded case or whether the structural limitation on boundedness is one that will allow only recursive trade-offs.

In this paper, we tackle this question and restrict ourselves initially to the case of letter-bounded languages, namely, subsets of \( a_1^*\ldots a_m^* \), where \( a_1, \ldots, a_m \) are pairwise distinct symbols. Our main result shows that for these languages the trade-off between PDAs (or context-free grammars) and finite-turn PDAs becomes recursive. More precisely, in Section 3 we first show that each context-free grammar in Chomsky normal form with \( h \) variables generating a letter-bounded set can be converted to an equivalent finite-turn PDA whose size is \( 2^{O(h)} \). Furthermore, the resulting PDA makes at most \( m - 1 \) turns where \( m \) is the number of letters in the terminal alphabet. In a second step, an exponential trade-off is also shown for arbitrary context-free grammars.

We prove in Section 5 that this result is tight by showing that the size of the resulting PDA and the number of turns cannot be reduced. Note that this result is a generalization of the above-mentioned transformation of unary context-free grammars into finite automata which is presented in [14]. In Section 4 the investigation is further deepened by studying
how to reduce the number of turns in a PDA. In particular, given a $k$-turn PDA accepting a subset of $a_1^*a_2^*\ldots a_m^*$, where $k > m - 1$, we show how to build an equivalent $(m - 1)$-turn PDA. It turns out that in this case the trade-off is polynomial. This result is also used to prove the optimality of our simulation of PDAs accepting letter-bounded languages by finite-turn PDAs. Finally, in Section 6 we consider word-bounded languages. Based on the constructions for letter-bounded languages in the previous sections, we are able to give similar constructions for word-bounded languages. Thus, similar upper and lower bounds can be obtained for the general situation of word-bounded languages.

We would like to remark that bounded context-free languages have very appealing properties concerning their decidability questions. It is known [3] that equivalence and inclusion problems are decidable whereas both problems are undecidable for context-free languages and inclusion is an undecidable problem for deterministic context-free languages. Furthermore, it is decidable whether a given context-free grammar generates a bounded language. In the positive case, the words $w_1, w_2, \ldots, w_m$ can be effectively calculated. For the membership problem we know the Cocke-Younger-Kasami algorithm which solves the problem in cubic time. It is shown in [9] that letter-bounded context-free languages can be accepted by a certain massively parallel computational model. This result implies that the membership problem for letter-bounded context-free languages can be solved in quadratic time and linear space. Since the membership problem for word-bounded context-free languages can be reduced to the membership problem for letter-bounded context-free languages by using suitable inverse homomorphisms, we obtain identical time and space bounds also in the word-bounded case.

2 Preliminaries and Definitions

Given a set $S$, $\#S$ denotes its cardinality. Let $\Sigma^*$ denote the set of all words over the finite alphabet $\Sigma$, with the empty string denoted by $\epsilon$, and $\Sigma^+ = \Sigma^* \setminus \{\epsilon\}$. Given a string $x \in \Sigma^*$, $|x|$ denotes its length. For the sake of simplicity, we will consider languages without the empty word $\epsilon$. However, our results can be easily extended to languages containing $\epsilon$. Let REG denote the family of regular languages. We assume that the reader is familiar with the common notions of formal language theory as presented in [8].

A **context-free grammar** (CFG, for short), is a 4-tuple $G = (V, \Sigma, P, S)$, where $V$ is the set of variables, $\Sigma$ is the set of terminals, $V$ and $\Sigma$ are disjoint sets, $S \in V$ is the initial symbol and $P \subseteq V \times (V \cup \Sigma)^*$ is the finite set of productions. A production $(A, \alpha) \in P$ is denoted by $A \rightarrow \alpha$. The relations $\Rightarrow$, $\Rightarrow^*$, and $\overset{*}{\Rightarrow}$ are defined in the usual way. Given $\alpha, \beta \in (V \cup \Sigma)^*$, if $\theta$ is a derivation of $\beta$ from $\alpha$, then we write $\theta : \alpha \Rightarrow^* \beta$. A useful representation of derivations of context-free grammars can be obtained using parse trees.

A **parse tree** (or **tree**, for short) for a context-free grammar $G$ is a labeled tree satisfying the following conditions:

1. Each internal node is labeled by a variable in $V$.
2. Each leaf is labeled by either a variable, a terminal, or $\epsilon$. However, if the leaf is labeled $\epsilon$, then it must be the only child of its parent.
If an internal node is labeled with a variable $A$, and its children, from left to right, are labeled with $X_1, X_2, \ldots, X_k \in V \cup \Sigma$, then $A \rightarrow X_1X_2\ldots X_k$ is a production of $G$.

If $T$ is a parse tree whose root is labeled with a variable $A \in V$ and such that the labels of the leaves, from left to right, form a string $\alpha \in (V \cup \Sigma)^*$, then we write $T : A \Rightarrow \alpha$. Furthermore, we indicate as $\nu(T)$ the set of variables which appear as labels of some nodes in $T$.

The language generated by the grammar $G$, i.e., the set $\{x \in \Sigma^* | S \Rightarrow x\}$, is denoted by $L(G)$.

The class of languages generated by CFGs is called the class of context-free languages. It is well-known that the class of context-free languages properly contains the class of regular languages (i.e., the languages accepted by finite automata), but in the unary case, these two classes collapse [4].

A grammar $G = (V, \Sigma, P, S)$ is said to be in Chomsky normal form if and only if its productions have the form $A \rightarrow BC$ or the form $A \rightarrow a$, with $A, B, C \in V$ and $a \in \Sigma$. It is well-known that each context-free language not containing the empty word can be generated by a context-free grammar in Chomsky normal form.

A production $A \rightarrow \alpha$ of a context-free grammar $G = (V, \Sigma, S, P)$ is said to be right-linear (left-linear, linear) if $\alpha \in \Sigma^*V$ ($\alpha \in V\Sigma^*$, $\alpha \in \Sigma^*V\Sigma^*$).

A context-free grammar $G = (V, \Sigma, S, P)$ is said to be ultralinear [5] if $V$ is a union of pairwise disjoint (possibly empty) subsets $V_0, \ldots, V_n$ of $V$ with the following property. For each $V_i$ and each $A \in V_i$, each production with left side $A$ is of the form $A \rightarrow w$, where either $w \in \Sigma^*V_i\Sigma^*$ or $w \in (\Sigma \cup V_0 \cup \ldots \cup V_{i-1})^*$. The family of languages generated by ultralinear grammars is called ultralinear languages and denoted by ULTRALIN.

A context-free grammar $G = (V, \Sigma, S, P)$ is said to be non-terminal bounded [5] if there exists an integer $k$ with the following property: If $A \Rightarrow w$, $w \in (V \cup \Sigma)^*$, $A \in V$, then $w$ has at most $k$ occurrences of variables. The rank $r_G(w)$ of a word $w \in (V \cup \Sigma)^*$ is defined to be the largest integer $r$ such that there is a word $u \in (V \cup \Sigma)^*$, with $r$ occurrences of variables, such that $w \Rightarrow u$. It is known [5] that a context-free grammar $G$ is ultralinear if and only if $G$ is non-terminal bounded.

For each ultralinear grammar $G$, the rank of $G$ is defined as the largest integer which is the rank of one of the variables. Let $L$ be an ultralinear language. The rank of $L$, $r(L)$, is defined as zero, if $L$ is regular. If $L$ is nonregular, then $r(L)$ is defined as the smallest integer which is the rank of some ultralinear grammar generating it.

Let $M = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ be a pushdown automaton [5]. A configuration of a pushdown automaton is a triple $(q, w, \gamma)$ where $q$ is the current state, $w$ the unread part of the input, and $\gamma$ the current content of the pushdown store. The leftmost symbol of $\gamma$ is the topmost stack symbol. We write $(q, aw, Z\gamma) \vdash (p, w, \beta\gamma)$, if $\delta(q, a, Z) \ni (p, \beta)$ for $p, q \in Q$, $a \in \Sigma \cup \{\epsilon\}$, $w \in \Sigma^*$, $\gamma, \beta \in \Gamma^*$, and $Z \in \Gamma$. The reflexive and transitive closure of $\vdash$ is denoted by $\vdash^*$. The language accepted by $M$ with accepting states is

$$T(M) = \{w \in \Sigma^* | (q_0, w, Z_0) \vdash^* (q, \epsilon, \gamma) \text{ with } q \in F \text{ and } \gamma \in \Gamma^*\}.$$
A sequence of configurations of $M (q_1, w_1, \gamma_1) \vdash \ldots \vdash (q_k, w_k, \gamma_k)$ is called one-turn if there exist $1 < i \leq j < k$ such that

$$|\gamma_1| \leq \cdots \leq |\gamma_{i-1}| < |\gamma_i| \leq |\gamma_{i+1}| \leq \cdots \leq |\gamma_j| > |\gamma_{j+1}| \geq \cdots \geq |\gamma_k|.$$ 

A sequence of configurations $c_0 \vdash \ldots \vdash c_m$ is called $k$-turn if there are integers $0 = i_0, \ldots, i_l = m$ with $l \leq k$ such that for $j = 0, \ldots, l - 1$ the subsequences $c_{i_j} \vdash \ldots \vdash c_{i_{j+1}}$ are one-turn, respectively. $M$ is a $k$-turn pushdown automaton if every word $w \in T(M)$ is accepted by a sequence of configurations which is $k$-turn.

By $L(k\text{-turn PDA})$ we denote the family of languages accepted by $k$-turn PDAs. The union of all $k$-turn PDAs with fixed $k \geq 1$ is the set of finite-turn PDAs. The family of languages accepted is defined as $L(\text{finite-turn PDA}) = \bigcup_{k \geq 1} L(k\text{-turn PDA})$.

Thus, $k$-turn PDAs are allowed to make new turns not depending on the stack height. The following characterization of ultralinear languages by finite-turn PDAs may be found in [1] and [5], respectively.

**Theorem 1** A language $L$ belongs to ULTRALIN if and only if there is a $k \in \mathbb{N}$ such that $L$ is accepted by a $k$-turn PDA.

We want to consider in this paper PDAs in a certain normal form. Thus, we make, without loss of generality, the following assumptions about PDAs (cf. [14]).

1. At the start of the computation the pushdown store contains only the start symbol $Z_0$; this symbol is never pushed or popped on the stack;
2. the input is accepted if and only if the automaton reaches a final state, the pushdown store only contains $Z_0$ and all the input has been scanned;
3. if the automaton moves the input head, then no operations are performed on the stack;
4. every push adds exactly one symbol on the stack.

The transition function $\delta$ of a PDA $M$ then can be written as

$$\delta : Q \times (\Sigma \cup \{\varepsilon\}) \times \Gamma \rightarrow 2^{Q \times \{\text{pop} \cup \{\text{push}(A) | A \in \Gamma\}\}},$$

In particular, for $q, p \in Q, A, B \in \Gamma, \sigma \in \Sigma, (p, \varepsilon) \in \delta(q, \sigma, A)$ means that the PDA $M$, in the state $q$, with $A$ at the top of the stack, by consuming the input $\sigma$, can reach the state $p$ without changing the stack contents. $(p, \text{pop}) \in \delta(q, \varepsilon, A)$, $(p, \text{push}(B)) \in \delta(q, \varepsilon, A)$, respectively) means that $M$, in the state $q$, with $A$ at the top of the stack, without reading any input symbol, can reach the state $p$ by popping off the stack the symbol $A$ on the top (by pushing the symbol $B$ on the top of the stack, without changing the stack, respectively).

A **descriptional system** $D$ is a recursive set of finite descriptors (e.g. automata or grammars) relating each $A \in D$ to a language $T(A)$. It is additionally required that each descriptor $A \in D$ can be effectively converted to a Turing machine $M_A$ such that $T(M_A) =$
result shows that given a grammar \( G \). In this section, we study the transformation of CFGs into finite-turn PDAs. Our main general information on descriptional complexity may be found in [2].

As a measure for the size of a context-free grammar \( G \) = \((V, \Sigma, P, S)\) we consider the number of symbols of \( G \), defined as \( \text{Symb}(G) = \sum_{(A \rightarrow \alpha) \in P} (2^{|\alpha|} \cdot |\alpha|) \) (cf. [10]). Furthermore, in the paper it will be useful also to consider the number of variables of \( G \), defined as \( \text{Var}(G) = \#V \) (note that this function in general is not a measure for the size). Some general information on descriptional complexity may be found in [2].

3 From Grammars to Finite-Turn PDAs

In this section, we study the transformation of CFGs into finite-turn PDAs. Our main result shows that given a grammar \( G \) of size \( h \), we can build an equivalent finite-turn PDA \( M \) of size \( 2^{O(h)} \). Furthermore, if the terminal alphabet of \( G \) contains \( m \) letters, then \( M \) is an \((m - 1)\)-turn PDA. The tightness of the bounds will be shown in Section 5. For the sake of simplicity, we start by considering CFGs in Chomsky normal form with the measure \( \text{Var} \). At the end of the section, we will discuss the generalization to arbitrary context-free grammars, taking into consideration the size defined by the measure \( \text{Symb} \).

In the following we consider an alphabet \( \Sigma = \{a_1, \ldots, a_m\} \) and a CFG \( G = (V, \Sigma, P, S) \) in Chomsky normal form with \( h \) variables, generating a subset of \( a_1^r \cdots a_m^r \). Without loss of generality, we can suppose that each variable of \( G \) is useful, i.e., for each \( A \in V \), there exist terminal strings \( u, v, w \), such that \( S \stackrel{*}{\Rightarrow} uAw \Rightarrow uvw \).

Lemma 1 For each variable \( A \in V \) there exists an index \( l, 1 \leq l \leq m \) \((r, 1 \leq r \leq m, \)
resp.) such that if \( A \Rightarrow uAv \), with \( u, v \in \Sigma^* \), then \( u \in a^*_l \) (\( v \in a^*_r \), resp.). Furthermore, if there exists at least one derivation \( A \Rightarrow uAv \) with \( u \neq \epsilon \), \( (v \neq \epsilon \), resp.) then such an \( l \) (\( r \), resp.) is unique.

**Proof:** It is easy to see that if \( A \Rightarrow uAv \) and \( u \) contains at least two letters \( a_p, a_q \), with \( l' \neq l'' \), then, because \( A \Rightarrow uuAvv \), the language generated by \( G \) should contain a string not belonging to \( a^*_1 \ldots a^*_m \). Hence, \( u \in a^*_l \), for some \( 1 \leq l \leq m \).

Now suppose that \( u \neq \epsilon \) and \( A \Rightarrow u'Av' \). Because \( A \Rightarrow uuAvv \), using the previous argument it is easy to conclude that \( u' \in a^*_l \).

A similar argument can be given for the right part. \( \square \)

For each variable \( A \) such that \( A \Rightarrow uAv \), with \( u, v \in \Sigma^+ \), we denote by \( \text{border}(A) \) the unique pair \((l, r)\) of indices, given in Lemma 1. On the other hand, if for any derivation \( A \Rightarrow uAv \), the string \( v \) is empty, then we define \( \text{border}(A) \) as the pair \((l, l)\), and if for any derivation \( A \Rightarrow uAv \) the string \( u \) is empty, then we define \( \text{border}(A) \) as the pair \((r, r)\). If there are no derivations of the form \( A \Rightarrow uAv \), then we leave \( \text{border}(A) \) undefined.

Formally,

\[
\text{border}(A) = \begin{cases} 
(l, r) & \text{if } A \Rightarrow uAv \text{ for some } u \in a^*_l, v \in a^*_r \\
(l, l) & \text{if } u \in a^*_l \text{ and } v = \epsilon \text{ for any } A \Rightarrow uAv \\
(r, r) & \text{if } u = \epsilon \text{ and } v \in a^*_r \text{ for any } A \Rightarrow uAv \\
\text{undefined} & \text{otherwise.}
\end{cases}
\]

For the sake of brevity, \( \text{border}(A) \) will be denoted also as \((l_A, r_A)\), if defined.

We now consider the relation \( \leq \) on the set of possible borders defined as \((l, r) \leq (l', r')\) if and only if \( l \leq l' \) and if \( l = l' \) then \( r \geq r' \), for all \((l, r), (l', r') \in \{1, \ldots, m\}^2\), with \( l \leq r \) and \( l' \leq r' \). It is not difficult to verify that \( \leq \) is a total order on the set of pairs of indices \( l, r \) from \( \{1, \ldots, m\} \), such that \( l \leq r \).

Actually, we are interested in computing borders of variables belonging to the same derivation tree. In this case, either a variable is a descendant of the other in the tree, and then the interval defined by its border is inside the interval defined by the border of the other variable, or one variable is to the right of the other one, and then the corresponding interval is to the right of the other one. More formally, we can prove the following:

**Lemma 2** Let \( T \) be a derivation tree and \( A, B \in \nu(T) \) be two variables. If \( \text{border}(A) \leq \text{border}(B) \), then either:

\( (a) \ l_A \leq l_B \leq r_B \leq r_A, \) or
\( (b) \ l_A < l_B, r_A < r_B, \) and \( r_A \leq l_B. \)

**Proof:** The case \( \text{border}(A) = \text{border}(B) \) is trivial. Thus, for the rest of the proof we suppose that \( \text{border}(A) \neq \text{border}(B) \).
If $A$ and $B$ lie in $T$ on the same path from the root, then $A$ must be closer to the root than $B$ (otherwise $\text{border}(B) \leq \text{border}(A)$). It is immediate to conclude that in this case $l_A \leq l_B \leq r_B \leq r_A$.

Otherwise, $A$ should appear in a path to the left of the path containing $B$. This easily implies that $l_A \leq r_A \leq l_B \leq r_B$. We consider the two subcases $r_A = r_B$ and $r_A < r_B$.

If $r_A = r_B$ then we get $l_A < l_B = r_B = r_A$, so (a) holds. On the other hand, if $r_A < r_B$ then it is not possible that $l_A = l_B$ because this should imply, with $\text{border}(A) \leq \text{border}(B)$, the contradiction $r_B \leq r_A$. Hence, $l_A < l_B$ that, with $r_A < r_B$ and $r_A \leq l_B$, gives (b). □

A partial derivation tree (or partial tree, for short) $U : A \Rightarrow vAx$ is a parse tree whose root is labeled with a variable $A$ and all the leaves, with the exception of one whose label is the same variable $A$, are labeled with terminal symbols.

Given a partial tree $U : A \Rightarrow vAx$, any derivation tree $T : S \Rightarrow z$ with $A \in \nu(T)$ can be “pumped” using $U$, by replacing a node labeled $A$ in $T$ with the subtree $U$. In this way, a new tree $T' : S \Rightarrow z'$ is obtained, where $z' = uvwxy$, such that $z = uwy$, $S \Rightarrow uAy$, and $A \Rightarrow w$. Moreover, $\nu(T') = \nu(T) \cup \nu(U)$.

On the other hand, any derivation tree producing a sufficiently long terminal string can be obtained by pumping a derivation tree of a shorter string with a partial tree. This fact, which is essentially the pumping lemma of context–free languages (see, e.g., [8]), is recalled in the following:

**Lemma 3** Let $T : S \Rightarrow z$ be a derivation tree of a string $z \in \Sigma^*$. If $|z| > 2^{h-1}$ then we can write $z = uvwxy$, such that $0 < |vx| < 2^h$, there is a tree $T' : S \Rightarrow uwy$, a variable $A \in \nu(T')$ and a tree $T'' : A \Rightarrow vAx$ such that $\nu(T) = \nu(T') \cup \nu(T'')$ and $A \Rightarrow w$, where $h$ is the number of the variables of the grammar $G$.

**Proof:** First of all, we recall that given a parse tree of a string $\alpha$ according to a context-free grammar $G$ in Chomsky normal form, if the longest path from the root to a leaf in the tree has length $k$ (measured by the number of edges), then $|\alpha| \leq 2^{k-1}$ (see, e.g., [8]). Using this property, we get that if $|z| > 2^{h-1}$ then the tree $T : S \Rightarrow z$ must contain a path of $h + 1$ edges from a node $n$ to a leaf. Hence, this path must contain two nodes labeled with the same variable $A$. This defines the decomposition of $T$ in $T'$ and $T''$. Again by the above property, the terminal string $vwxy$ generated by the subtree rooted at $n$ has length bounded by $2^h$. Furthermore, $|w| \geq 1$, because the grammar is in Chomsky normal form. Hence $|vx| < 2^h$.

By applying the pumping lemma several times, we can prove that any derivation tree can be obtained by starting from a derivation tree of a “short” string (namely, a string of length at most $2^{h-1}$) and iteratively pumping it with “small” partial trees. Furthermore, a sequence of partial trees can be considered such that the sequence of borders of their roots is not decreasing. This fact will be crucial to get the main simulation presented in this section. More precisely, we are able to prove the following result:

**Lemma 4** Given a derivation tree $T : S \Rightarrow z$ of a string $z \in \Sigma^*$, with $|z| > 2^{h-1}$, for some integer $k > 0$ there are:
• $k + 1$ derivation trees $T_0, T_1, \ldots, T_k$, where $T_i : S \Rightarrow z_i$, $i = 0, \ldots, k$, $0 < |z_0| \leq 2^{h-1}$, $T_k = T$, $z_k = z$;

• $k$ partial trees $U_1, \ldots, U_k$, where, for $i = 1, \ldots, k$, $U_i : A_i \Rightarrow v_i A_i x_i$, $0 < |v_i x_i| < 2^h$, and $T_i$ is obtained by pumping $T_{i-1}$ with $U_i$.

Furthermore, $\text{border}(A_1) \leq \text{border}(A_2) \leq \ldots \leq \text{border}(A_k)$.

**Proof:** We can build the sequence from the end starting from $T$ and decomposing it in a tree $T'$ and a partial tree $U$, according to Lemma [3]. This process can be iterated until a derivation tree $T_0$ producing a string of length bounded by $2^{h-1}$ is obtained.

In order to get a sequence of partial trees such that the sequence of the borders of the variables labeling their roots is not decreasing, at each step a partial tree is selected, among all possible candidates, in such a way that the border of its root is maximum. In other words, for $i = 1, \ldots, k$, the variable $A_i$ labeling the root of the tree $U_i : A_i \Rightarrow v_i A_i x_i$ satisfies:

$$\text{border}(A_i) = \max \{ \text{border}(A) \mid \text{there exists a partial tree } U : A \Rightarrow vAx \text{ of } T_i \text{ with } 0 < |vx| < 2^h \}.$$ 

We prove that with this choice $\text{border}(A_i) \leq \text{border}(A_{i+1})$, for $i = 1, \ldots, k - 1$. This is obvious if the partial tree $U_i : A_i \Rightarrow v_i A_i x_i$ of $T_i$ is also a partial tree of $T_{i+1}$, namely, it belongs to the set of candidates when $U_{i+1}$ is chosen to reduce $T_{i+1}$. If this is not the case, then $T_{i+1}$ should contain a partial tree $U' : A_i \Rightarrow v' A_i x'$, with $|v' x'| \geq 2^h$, such that, after removing $U_{i+1}$, $U'$ is reduced in $T_i$ to $U_i$. It can be observed that in $T_{i+1}$ such a reduction is possible only if the root $A_{i+1}$ of $U_{i+1}$ is a descendant of the root $A_i$ of $U'$. Hence, in $T_{i+1}$ the terminal string generated by the subtree whose root, labeled $A_{i+1}$, coincides with the root of $U_{i+1}$, must be a factor of the terminal string generated by the subtree whose root, labeled $A_i$, coincides with the root of $U'$. This implies that $l_{A_i} \leq l_{A_{i+1}} \leq r_{A_{i+1}} \leq r_{A_i}$.

Hence, $\text{border}(A_i) \leq \text{border}(A_{i+1})$.

**Example:** The language $L = \{ a_1^{n+k} a_2^{k+p} a_3^{p+n} \mid n, k, p > 0 \}$ can be generated by a grammar in Chomsky normal form with the following productions:

- $S \rightarrow A_1 E$
- $S' \rightarrow AB$
- $A \rightarrow A_1 F$
- $B \rightarrow A_2 G$
- $A_1 \rightarrow a_1$
- $E \rightarrow S' A_3$
- $A \rightarrow A_1 A_2$
- $F \rightarrow A A_2$
- $G \rightarrow B A_3$
- $A_2 \rightarrow a_2$
- $E \rightarrow S' A_3$
- $B \rightarrow A_2 A_3$
- $A_3 \rightarrow a_3$

Note that $S \Rightarrow a_1 S a_3$, $A \Rightarrow a_1 A a_2$, and $B \Rightarrow a_2 B a_3$. It is easy to get a tree $T_0 : S \Rightarrow a_1^2 a_2 a_3^2$ and three partial trees $U' : S \Rightarrow a_1 S a_3$, $U'' : A \Rightarrow a_1 A a_2$, and $U''' : B \Rightarrow a_2 B a_3$.

Given integers $n, k, p > 0$, a derivation tree for the string $a_1^{n+k} a_2^{k+p} a_3^{p+n}$ can be obtained considering $T_0$, and pumping it $n - 1$ times with the tree $U'$, $k - 1$ times with the tree $U''$, and $p - 1$ with the tree $U'''$. Note that $\text{border}(S) = (1, 3) \leq \text{border}(A) = (1, 2) \leq \text{border}(B) = (2, 3)$.

Lemma [4] suggests a nondeterministic procedure which can be used to generate all the strings $a_1^{k_1} \cdots a_m^{k_m}$ belonging to the language $L(G)$: at the beginning a derivation tree
T : S ® a₁ⁿ₁ . . . aₙₘ is selected. Then the procedure enters a loop which is repeated a nondeterministically chosen number of times. At each iteration, the tree T so far considered is pumped with a nondeterministically chosen partial tree U : A ® vAx (note that, by Lemma 1, v ∈ aₙₐ, x ∈ aₙₓ) such that A is a variable occurring in T. Note that, to implement this strategy, the procedure does not need to remember the whole tree T but only the set of variables occurring in it.

The procedure is the following:

nondeterministically select a tree T : S ® a₁ⁿ₁a₂ⁿ₂ . . . aₙₘ,  
with n₁ + n₂ + . . . + nₘ ≤ 2h⁻¹
k₁ ← n₁, k₂ ← n₂, . . ., kₘ ← nₘ
enabled ← ν(T)
iterate ← nondeterministically choose true or false
while iterate do
  nondeterministically select a tree U : A ® vAx, with 0 < |vx| < 2h,  
  and A ∈ enabled // border(A) = (lₐ, rₐ)
kₐₐ ← kₐₐ + |v|
kₐₓ ← kₐₓ + |x|
enabled ← enabled ∪ ν(U)
iterate ← nondeterministically choose true or false
endwhile
output a₁ᵏ₁a₂ᵏ₂ . . . aₙₘ

Now, we will convert the above procedure into an (m − 1)-turn PDA recognizing the language generated by the grammar G. For the sake of simplicity, let us start by describing the case m = 2 with v ∈ a₁ⁿ and x ∈ a₂ⁿ, for each partial tree U : A ® vAx. The PDA uses two bounded counters n₁, n₂ in order to remember the string a₁ⁿa₂ⁿ generated by the initial “small” tree. In a preliminary phase, the PDA consumes n₁ occurrences of a₁ from the input tape, in order to verify that a₁ⁿ is a prefix of the input (otherwise it stops and rejects). Subsequently, the automaton starts the simulation of the loop above described where, at each iteration, a partial tree U : A ® vAx, with A ∈ enabled is used to pump the generated string. To this aim, the automaton reads v ∈ a₁ⁿ from the input tape and pushes x ∈ a₂ⁿ on the stack (if v is not a prefix of the remaining part of the input tape, then the automaton stops and rejects). At the end of the loop, the pushdown store will contain a string a₂ᵖ, for some p ≥ 0. Finally, the automaton accepts if and only if the remaining part of the input is a₂ⁿ⁺²⁺p. We can observe that the automaton so described simulates the derivation of a string and verifies its matching with the input string. For the occurrences of the letter a₁, the matching is verified immediately, by comparing the generated factors with the input string; for the occurrences of the letter a₂, the verification of the matching is postponed: the generated factors are kept on the stack and compared with the input in the final phase.

This strategy can be extended to the general case by pumping, according to Lemma 4 with partial trees such that the sequence of the borders of their roots is not decreasing. More precisely, the PDA implements the following nondeterministic procedure, whose correctness is proved in Lemma 5 and Theorem 2.
nondeterministically select a tree $T : S \Rightarrow a_1^{n_1}a_2^{n_2}\ldots a_m^{n_m}$, with $n_1 + n_2 + \ldots + n_m \leq 2^{h-1}$.

read $a_1^{n_1}$ from the input

$enabled \leftarrow \nu(T)$

$(l, r) \leftarrow (1, m)$ // the “work context”

$iterate \leftarrow$ nondeterministically choose true or false

while $iterate$ do

nondeterministically select a tree $U : A \Rightarrow vAx$, with $0 < |vx| < 2^h$, $A \in enabled$, and $(l, r) \leq border(A) = (l_A, r_A)$

if $r < r_A$ then //new context to the right of the previous one

for $j \leftarrow l + 1$ to $r - 1$ do

consumeInputAndCounter($j$)

endfor

for $j \leftarrow r$ to $l_A$ do

consumeInputAndCounter($j$)

consumeInputAndStack($j$)

endfor

else // $r_A \leq r$: new context inside the previous one

for $j \leftarrow l + 1$ to $l_A$ do

consumeInputAndCounter($j$)

endfor

endif

$(l, r) \leftarrow (l_A, r_A)$

read $v$ from the input

if $r \neq l$ then push $x$ on the stack

else read $x$ from the input

endif

$enabled \leftarrow enabled \cup \nu(U)$

$iterate \leftarrow$ nondeterministically choose true or false

endwhile

for $j \leftarrow l + 1$ to $r - 1$ do

consumeInputAndCounter($j$)

endfor

for $j \leftarrow r$ to $m$ do

consumeInputAndCounter($j$)

consumeInputAndStack($j$)

endfor

if the end of the input has been reached then accept

else reject

endif

In the previous procedure and in the following macros, the instruction “read $x$ from the input tape,” for $x \in \Sigma^*$, actually means that the automaton verifies whether or not $x$ is a prefix of the next part of the input. If the outcome of this test is positive, then the input head is moved immediately to the right of $x$, namely, $x$ is “consumed,” otherwise the machine stops and rejects.

The macros are defined as follows:

ConsumesInputAndCounter($j$):

while $n_j \geq 0$ do

read $a_j$ from the input tape

$n_j \leftarrow n_j - 1$

endwhile
In order to prove that the pushdown automaton described in the previous procedure accepts the language $L(G)$ generated by the given grammar $G$, it is useful to state the following lemma:

**Lemma 5** Consider one execution of the previous procedure. Let $T_0 : S \Rightarrow a_1^{n_1} \ldots a_m^{n_m}$ be the tree selected at the beginning of such an execution. At every evaluation of the condition of the while loop, there exists a tree $T : S \Rightarrow a_1^{k_1} \ldots a_m^{k_m}$, for some $k_1, \ldots, k_m \geq 0$, such that

- the scanned input prefix is $z = a_1^{k_1} \ldots a_l^{k_l}$;
- the pushdown store contains the string $\gamma = a_r^{p_r} \ldots a_m^{p_m}$, where, for $j = r, \ldots, m$, $p_j \geq 0$ and $k_j = p_j + n_j$;
- for $l < j < r$, $k_j = n_j$;
- enabled = $\nu(T)$.

**Proof:** It is easy to see that at the first evaluation of the condition it holds that $l = 1$, $r = m$, $k_1 = n_1, \ldots, k_m = n_m$, $p_m = 0$, the scanned input prefix is $a_1^{k_1}$ and the pushdown store is empty, namely, it contains $a_m^{p_m}$.

We now suppose the statement to be true before the execution of one iteration and we show that it still holds true at the end of the iteration. Let $U : A \Rightarrow vAx$ be the partial tree selected in the while loop. Because $A \in \nu(T)$, the derivation tree $T : S \Rightarrow a_1^{k_1} \ldots a_m^{k_m}$ can be pumped with the partial tree $U$, obtaining a new tree $T' : S \Rightarrow a_1^{k_1'} \ldots a_m^{k_m'}$, with $\nu(T') = \nu(T) \cup \nu(U)$, where (in the case $l_A \neq r_A$) $k_A' = k_A + |v|$, $k_{r_A}' = k_{r_A} + |x|$ and $k_j' = k_j$ for each $j \neq l_A$ and $j \neq r_A$ (in the special case $l_A = r_A$ we have that $k_A' = k_{r_A}' = k_A + |vx|$).

We now consider two subcases, corresponding to the selection in the while loop.

**Case $r < r_A$.**

By Lemma 2, this implies that $l < l_A$ and $r \leq l_A$.

First, we prove that for each $j$, with $l_A < j < r_A$, the stack cannot contain the symbol $a_j$, i.e., $p_j = 0$. Suppose, by contradiction, that the string $\gamma$ contains at least one occurrence of $a_j$. This symbol must have been pushed on the stack in a previous iteration, with “work context” $(\tilde{l}, \tilde{r})$, for some $\tilde{l} \leq \tilde{r} = j$. Since the procedure never removes variables from the tree, the variable used to pump the tree in such a previous iteration is also in the tree $T'$. Moreover, the procedure chooses contexts in a nondecreasing order, so $(\tilde{l}, \tilde{r}) \leq (l_A, r_A)$. By Lemma 2 it turns out that either $\tilde{l} \leq l_A \leq r_A \leq \tilde{r} = j$, or $\tilde{l} < l_A$, implying the result.
\( \tilde{r} < r_A \), and \( j = \tilde{r} \leq l_A \). It is easy to observe that in both the cases we get a contradiction. Hence, for any \( j \) with \( l_A < j < r_A \), the stack does not contain the symbol \( a_j \), i.e., \( p_{l_A+1} = p_{l_A+2} = \ldots = p_{r_A-1} = 0 \). This implies that \( \gamma = \gamma''\gamma' \) with \( \gamma'' = a_{p_{r_A}} \ldots a_{p_{l_A}} \) and \( \gamma' = a_{p_{r_A}} \ldots a_{p_m} \).

Now, we observe the operations on the input and on the stack that are performed during the execution of the body of the loop:

- the input factor \( a_{p_{l+1}+1} \ldots a_{p_{r-1}+n} \ldots a_{p_{l_A}+n} = a_{l_A+1} \ldots a_{l_A} \) is consumed;
- the string \( \gamma'' \) is popped off the stack;
- the input factor \( v \in a^{*}_e \) is consumed;
- if \( l_A < r_A \) then the string \( x \in a^{*}_e \) is saved on the stack (to be consumed later), otherwise it is consumed immediately.

By summarizing, in the case \( l_A < r_A \), at the end of the iteration the scanned input prefix is \( a_{p_{l+1}} \ldots a_{p_{r-1}+n} \ldots a_{p_{l_A}+n} = a_{l_A+1} \ldots a_{l_A} \) and the pushdown store contains the string \( x\gamma = x\gamma'' \ldots a_{p_m} \), for each \( j, l_A < j < r_A, k_j = k_j = n_j \), and \( enabled = \nu(T') \). With small changes we can deal with the case \( l_A = r_A \).

Case \( r_A \leq l \).

By Lemma 2, this implies that \( l \leq l_A \leq r_A \leq r \).

If \( l_A < r_A \) then the consumed input prefix is \( a_{p_{l_A}} \ldots a_{p_{r_A}} \ldots a_{p_m} \) and the pushdown store contains the string \( x\gamma = x\gamma'' \ldots a_{p_m} \), where \( p_{r_A} = p_{r_A+1} = \ldots = p_{r-1} = 0 \). At this point it is not difficult to verify that the statement of the Lemma is true. The subcase \( l_A = r_A \) can be managed with easy changes.

As a consequence:

**Theorem 2** The pushdown automaton \( M \) described by the previous procedure is an \( (m-1) \)-turn PDA accepting the language \( L(G) \).

**Proof:** First, we show that the number of turns of the PDA \( M \) defined in the above procedure is at most \( m-1 \). To this aim we count how many times the automaton can switch from push operations to pop operations.

At each iteration of the while loop, the automaton can perform push operations. Pop operations are possible only by calling the macro consumeInputAndStack. This happens first in the while loop, when the condition \( r < r_A \) holds true, i.e., when the new context \((l_A, r_A)\) is to the right of the previous context \((l, r)\), and secondly after the end of the loop.

Let \((l_1, r_1), (l_2, r_2), \ldots, (l_k, r_k)\) be the sequence of the contexts which in the computation make the above-mentioned condition hold true. Hence, \( 1 < l_1 < \ldots < l_k \leq m \), that implies \( k \leq m-1 \). If \( k < m-1 \), then the PDA \( M \) makes at most \( k \leq m-2 \) turns in
the simulation of the while loop and one more turn after the loop. So the total number of
turns is bounded by $m-1$.

Now, suppose that $k = m - 1$. This implies that $l_k = m = r_k$. Before reaching the
context $(l_k, r_k)$, at most $m - 2$ turns can be performed. When the automaton switches
to the new context $(l_k, r_k) = (m, m)$, it can make pop operations, by calling the macro
consumeInputAndStack($m$). This requires one more turn. After that, the automaton
can execute further iterations, using the same context $(m, m)$. By reading the procedure
carefully, we can observe that it never executes further push operations. Finally, at the
exit of the loop, further pop operations can be executed (consumeInputAndStack). Hence,
the total number of turns is bounded by $m - 1$.

To prove that the language $L(G)$ and the language accepted by the automaton defined
in the above procedure coincide it is enough to observe that given a string $z \in L(G)$,
the procedure is able to guess the tree $T_0$ and the partial trees $U_1, \ldots, U_k$ of Lemma 4
recognizing in this way $z$. Conversely, using Lemma 5 it is easy to show that each string
accepted by the procedure should belong to $L(G)$.

**Corollary:** Given an alphabet $\Sigma = \{a_1, \ldots, a_m\}$, for any context–free grammar $G$
in Chomsky normal form with $h$ variables generating a letter-bounded language $L \subseteq
a_1^* \ldots a_m^*$, there exists an equivalent $(m-1)$-turn PDA $M$ with $2^{O(h)}$ states and $O(1)$ stack
symbols.

**Proof:** The most expensive information that the automaton defined in the previous
procedure has to remember in its state are the $m - 1$ counters bounded by $2^{h-1}$, and the
set enabled, which is a subset of $V$. For the pushdown store an alphabet with $m + 1$
symbols can be used. With a small modification, the pushdown store can be implemented
using only two symbols (one symbol to keep a counter $p_j$ and another one to separate two
consecutive counters), and increasing the number of states by a factor $m$, to remember
what input symbol $a_j$ the stack symbol $A$ is representing.

Using standard techniques, a PDA of size $n$ can be converted to an equivalent CFG in
Chomsky normal form with $O(n^2)$ variables. Hence, we easily get:

**Corollary:** Each PDA of size $n$ accepting a subset of $a_1^* \ldots a_m^*$ can be simulated by an
equivalent $(m - 1)$-turn PDA of size $2^{O(n^2)}$.

We now consider the situation when the given CFG is not necessarily in Chomsky normal
form.

**Lemma 6** Given a context-free grammar $G = (V, \Sigma, P, S)$, there exists an equivalent context–free grammar
$G' = (V', \Sigma, P', S)$ such that the length of the right hand side of any
production belonging to $P'$ is at most 2, $\text{Var}(G') \leq \text{Symb}(G)$, and the rank of $G'$ coincides
with the rank of $G$.

**Proof:** Without loss of generality, we suppose that for each variable $A$ in the set $V$ there
is a production with $A$ on the left hand side.
The set $V'$ of variables of $G'$ is defined by considering all variables in the set $V$, plus some extra variables as defined below. The set of productions $P'$ is defined as follows: We consider each production $A \to X_1 X_2 \ldots X_m$ belonging to $P$, with $X_i \in V \cup \Sigma$, $i = 1, \ldots, m$:

- If $m \leq 2$, then the production $A \to X_1 X_2 \ldots X_m$ belongs to $P'$.
- If $m > 2$, then $m - 2$ new extra variables $D_1, D_2, \ldots, D_{m-2}$ are introduced in the set $V'$, and the following productions are added to $P'$:
  
  $A \to X_1 D_1, D_1 \to X_2 D_2, \ldots, D_{m-3} \to X_{m-2} D_{m-2}, D_{m-2} \to X_{m-1} X_m$.

- No other productions are in $P'$.

Note that the construction of $P'$ is similar to the last step in the classical reduction of general context-free grammars to Chomsky normal form.

It is easy to verify that $G$ and $G'$ generate the same language. Furthermore, by construction, the right hand side of each production of $G'$ has length at most 2.

We recall that each variable of $V$ appears on the left hand side of some production. Furthermore, for each production $A \to \alpha$, with $|\alpha| > 2$, $|\alpha| - 2$ extra variables have been introduced in $V'$. Hence:

$$
\text{Symb}(G) = \sum_{(A \to \alpha) \in P} (2 + |\alpha|) \\
\geq \#V + \sum_{(A \to \alpha) \in P} (1 + |\alpha|) \\
\geq \#V + \sum_{(A \to \alpha) \in P \text{ s.t. } |\alpha| > 2} (1 + |\alpha|) \\
\geq \#V + \sum_{(A \to \alpha) \in P \text{ s.t. } |\alpha| > 2} (|\alpha| - 2) \\
= \text{Var}(G').
$$

Finally, it is immediate to observe that the definition of $G'$ preserves the rank. \hfill \square

**Corollary:** Given an alphabet $\Sigma = \{a_1, \ldots, a_m\}$, for any context–free grammar $G$ with $\text{Symb}(G) = h$ and generating a letter-bounded language $L \subseteq a_1^{*} \ldots a_m^{*}$, there exists an equivalent $(m - 1)$-turn PDA $M$ with $2^{O(h)}$ states and $O(1)$ stack symbols.

**Proof:** At first, it can be observed that Lemma 4 is true not only for CFGs in Chomsky normal form but also for CFGs whose productions have right hand sides of length at most 2. Thus, all arguments in Section 3 are also true for such “normalized” CFGs. Owing to Lemma 6 we then observe that any CFG $G$ can be converted to an equivalent CFG $G'$ such that the length of the right hand side of any production belonging to $G'$ is at most 2 and $\text{Var}(G') \leq \text{Symb}(G)$. With similar arguments as in Corollary 3 we obtain the claim. \hfill \square

We will discuss and present the extension of Corollary 3 to the word-bounded case in Section 6.
4 Reducing the Number of Turns

By the results presented in Section 3, each context-free subset of $a_1^* \ldots a_m^*$ can be accepted by an $(m - 1)$-turn PDA. In particular, Corollary 3 shows that the size of an $(m - 1)$-turn PDA equivalent to a given PDA of size $n$ accepting a subset of $a_1^* \ldots a_m^*$, is at most exponential in the square of $n$.

In this section, we further deepen this kind of investigation by studying how to convert an arbitrary $k$-turn PDA accepting a letter-bounded language $L \subseteq a_1^* a_2^* \ldots a_m^*$ to an equivalent $(m - 1)$-turn PDA. It turns out that the increase in size is at most polynomial.

All PDAs we consider are in normal form.

Let us start by considering the unary case, i.e., $m = 1$, which turns out to be crucial to get the simulation in the general case. In the following we consider a $k$-turn PDA $M = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ in normal form. Then we know that at most one symbol is pushed on the stack in every transition.

Lemma 7 Let $M$ be a PDA accepting a unary language $L$. Let $L(q_1, A, q_2)$ be the set of all words which are processed by 1-turn sequences $\pi$ of configurations starting with some stack height $h$ in a state $q_1$ and having $A$ as topmost stack symbol and ending with the same stack height $h$ in some state $q_2$ and having $A$ as topmost stack symbol. Then, $L(q_1, A, q_2)$ can be recognized by an NFA $M'$ such that $\text{size}(M') \leq n^2$ and $n = \text{size}(M)$.

Proof: Consider the following CFG $G$ with start symbol $[q_1, A, q_2]$ having the following productions. Let $p, p', q, q' \in Q$, $Z \in \Gamma$, and $\sigma \in \{a, \epsilon\}$.

1. $[p, Z, q] \rightarrow \sigma[p', Z, q]$, if $\delta(p, Z, \sigma) \ni (p', -)$,
2. $[p, Z, q] \rightarrow \sigma[p, Z, q']$, if $\delta(q', Z, \sigma) \ni (q, -)$,
3. $[p, Z, q] \rightarrow [p', Z', q']$, if $\delta(p, Z, \epsilon) \ni (p', \text{push}(Z'))$ and $\delta(q', Z', \epsilon) \ni (q, \text{pop})$,
4. $[p, Z, q] \rightarrow \epsilon$, if $p = q$.

We want to describe how $M'$ simulates a 1-turn sequence $\pi$. We simulate the parts of $\pi$ with $A$ as topmost stack symbol and stack height $h$ with productions (1) and (2). The first part from the beginning up to the first push operation is simulated using productions (1). The second part starting at the end of the computation and going backwards up to the last pop operation is simulated with productions (2). We may change nondeterministically between productions (1) and (2). This is possible, since the input is unary. Having simulated the parts of $\pi$ with stack height $h$ it is decided nondeterministically to proceed with simulating the parts of $\pi$ with stack height $h + 1$. Productions (3) simulate a push operation and the corresponding pop operation. Then, productions (1) and (2) can be again used to simulate the parts of $\pi$ with stack height $h + 1$. Now, we iterate this behavior and simulate all computational steps in $\pi$ while the stack height simulated is growing. Finally, we can terminate the derivation with productions (4) when the stack height has reached its highest level and all computational steps have been simulated.
Now, we construct an equivalent NFA $M' = (Q', \Sigma, \delta', (q_1, A, q_2), F')$ as follows: $Q' = Q \times \Gamma \times Q$ and $F' = \{(q, Z, q) \mid q \in Q, Z \in \Gamma\}$. For $\sigma \in \{a, \epsilon\}$ the transition function $\delta'$ is defined as:

1. $\delta'((p, Z, q), \sigma) \ni (p', Z, q)$, if $\delta(p, Z, \sigma) \ni (p', \epsilon)$,
2. $\delta'((p, Z, q), \sigma) \ni (p, Z, q')$, if $\delta(q', Z, \sigma) \ni (q, \epsilon)$,
3. $\delta'((p, Z, q), \sigma) \ni (p', Z', q')$, if $\delta(p, Z, \epsilon) \ni (p', \text{push}(Z'))$ and $\delta(q', Z', \epsilon) \ni (q, \text{pop})$.

It is not difficult to observe that $T(M') = L(G)$. □

**Corollary:** Let $M$ be some 1-turn PDA accepting a unary language $L$. Then, an equivalent NFA $M'$ can be constructed such that $\text{size}(M') \leq n^2 + 1$ and $n = \text{size}(M)$.

**Proof:** We can use the above construction, but additionally have to guess in a first step in which state a computation ends. Therefore, we add a new initial state $q_0'$ and the following rules $\delta'(q_0', \epsilon) \ni (q_0, Z_0, q_f)$, for all $q_f \in F$ to $M'$.

It is easy to observe that the parts of $\pi$ with stack height one can be again simulated with productions (1) and (2). The remaining part of the simulation is identical to the above described construction. □

A subcomputation $\pi'$ is called **strong** of level $A$ if it starts with some stack height $h$ and topmost stack symbol $A$, ends with the same stack height $h$ and topmost stack symbol $A$, and in all other configurations of $\pi'$ the stack height is greater than $h$.

**Lemma 8** Let $M$ be some $k$-turn PDA accepting a unary language $L$. Let $L(q_1, A, q_2)$ be the set of all words which are processed by sequences $\pi$ of strong computations of level $A$ which, additionally, start in some state $q_1$ and end in some state $q_2$. It can be observed that all words in $L(q_1, A, q_2)$ are accepted with $j \leq k$ turns. Then, $L(q_1, A, q_2)$ can be accepted by an NFA $M'$ such that $\text{size}(M') \in O(n^{2[\log_2 j] + 2})$ and $n = \text{size}(M)$.

**Proof:** The construction is very similar to the above described construction. Additionally, we store the number of turns, which have to be simulated, in the fourth component of the variables. There are two cases how $\pi$ may look like. In the first case (type I, cf. Fig. 4 left) $\pi$ consists of at least two strong computations of level $A$. We introduce a new production type (5) which is used to decompose a sequence of strong computations with $i$ turns into two subsequences with $i_1$ and $i_2$ turns, respectively. A resulting subsequence is then either again of type I and can be again decomposed with the new productions (5), or it is of type II, i.e., it consists of one strong computation of level $A$ (cf. Fig. 4 right). If this computation is 1-turn, it can be simulated with the productions (1) to (3) and finished with productions (4). If it is not 1-turn, we can reduce it to a sequence of strong computations of level $B$ by using the productions (1) to (3). Then, the same analysis can be made for strong computations of level $B$.

The formal construction of the CFG $G$ is as follows. We consider the start symbol $[q_1, A, q_2, j]$ and the following productions. Let $p, p', q, q' \in Q$, $Z \in \Gamma$, and $\sigma \in \{a, \epsilon\}$. 17
(1) \([p, Z, q, i] \rightarrow \sigma[p', Z, q, i]\), if \(\delta(p, Z, \sigma) \ni (p', -)\),
(2) \([p, Z, q, i] \rightarrow \sigma[p, Z, q', i]\), if \(\delta(q', Z, \sigma) \ni (q, -)\),
(3) \([p, Z, q, i] \rightarrow [p', Z', q', i]\), if \(\delta(p, Z, \epsilon) \ni (p', \text{push}(Z'))\) and \(\delta(q', Z', \epsilon) \ni (q, \text{pop})\),
(4) \([p, Z, q, 1] \rightarrow \epsilon\), if \(p = q\),
(5) \([p, Z, q, i] \rightarrow [p, Z, r, i_1][r, Z, q, i_2]\), for all \(r \in Q\) and \(i_1, i_2 \geq 1\) such that \(i_1 + i_2 \leq i\).

It can be shown by an induction on the number of turns that \(G\) generates \(L(q_1, A, q_2)\).

We can observe that all productions are right-linear except for productions (5). Since the last component of a variable \([p, Z, r, i]\) is reduced in every application of a production (5), we can conclude that (5) is applied at most \(j - 1\) times. Thus, every sentential form contains at most \(j\) variables. Thus, we can construct some NFA simulating the single derivation steps by representing all variables of a sentential form in its state. A rough estimation of the number of states is then

\[
\sum_{i=1}^{j} (|Q|^2|\Gamma|)^i = \frac{(|Q|^2|\Gamma|)^{j+1} - 1}{j|Q|^2|\Gamma| - 1} - 1 < \frac{2(|Q|^2|\Gamma|)^{j+1}}{j|Q|^2|\Gamma| - 1} \leq 2(|Q|^2|\Gamma|)^j \in O(n^{2j}).
\]

We now want to do some finer estimation and will obtain \(2(|Q|^2|\Gamma|)^{\lfloor \log_2 j \rfloor + 1}\) as upper bound. To this end, we observe that a simulation of a production (5) increases the number of variables in the current state of the NFA by one and that a simulation of a production (4) at the end of some 1-turn computation decreases the number of variables by one. Thus, our strategy is to apply productions of type (4) as soon as possible. Now, whenever an application of a production (5) has replaced a variable \([p, Z, q, i]\) by two variables \([p, Z, r, i_1]\) and \([r, Z, q, i_2]\), then the derivation of the variable with the lower number of remaining turns is simulated. This makes sure that the total number of variables in a state is as small as possible. The worst case which can occur in this context is that in every application of a production (5) the number of turns is divided into two equal parts. This may happen at most \(\lfloor \log_2 j \rfloor + 1\) many times. Thus, the size of the NFA can be estimated as follows

\[
\sum_{i=1}^{\lfloor \log_2 j \rfloor + 1} (|Q|^2|\Gamma|)^i = \frac{(|Q|^2|\Gamma|)^{\lfloor \log_2 j \rfloor + 2} - 1}{j|Q|^2|\Gamma| - 1} - 1 < \frac{(|Q|^2|\Gamma|)^{\lfloor \log_2 j \rfloor + 2}}{j|Q|^2|\Gamma| - 1} = \frac{j|Q|^2|\Gamma|}{j|Q|^2|\Gamma| - 1} (|Q|^2|\Gamma|)^{\lfloor \log_2 j \rfloor + 1} \leq 2(|Q|^2|\Gamma|)^{\lfloor \log_2 j \rfloor + 1} \in O(n^{2\lfloor \log_2 j \rfloor + 2})
\]

Finally, it can be observed that the last component in a tuple \([p, Z, q, i]\) may be removed, since the maximum number of possible turns only depends on \(p, Z, q\). This may save the constant factor \(j\) in the above estimation.

\(\square\)

**Corollary:** Let \(M\) be some \(k\)-turn PDA accepting a unary language \(L\). Then, an equivalent NFA \(M'\) can be constructed with \(\text{size}(M') \in O(n^{2\lfloor \log_2 k \rfloor + 2})\) and \(n = \text{size}(M)\).
If we introduce a fifth component of the variables in which some element \( s \) is stored, this means that the variable can only produce terminals \( a_i \). If \( s = a_i a_j \) (\( 1 \leq i < j \leq m \)), then such a variable can only produce sentential forms which start with terminals \( a_i \) and end with terminals \( a_j \).

For the construction we first consider a context-free grammar \( G \) with start symbol \( S \) and having the following productions. Let \( p, p', q, q' \in Q, Z \in \Gamma, a, b \in \{ a_1, \ldots, a_m \} \) such that \( ab \in \Pi(m) \), \( \overline{a} \in \{ a, \epsilon \} \), and \( \overline{b} \in \{ b, \epsilon \} \).

1. \( [p, Z, q, i, ab] \rightarrow \overline{a}[p', Z, q, i, ab], \) if \( \delta(p, Z, \overline{a}) \ni (p', -) \),
2. \( [p, Z, q, i, ab] \rightarrow [p, Z, q', i, ab]\overline{b}, \) if \( \delta(q', Z, \overline{b}) \ni (q, -) \),
3. \( [p, Z, q, i, ab] \rightarrow [p', Z', q', i, ab], \) if \( \delta(p, Z, \epsilon) \ni (p', \text{push}(Z')) \) as well as \( \delta(q', Z', \epsilon) \ni (q, \text{pop}) \).

Now, we are able to consider the general case, i.e., \( m \geq 1 \) and start with some definitions. Given the alphabet \( \Sigma = \{ a_1, \ldots, a_m \} \), we define the set \( \Pi(m) \) as follows

\[
\Pi(m) = \{ a_i \mid 1 \leq i \leq m \} \cup \{ a_i a_j \mid 1 \leq i < j \leq m \}
\]

It is easy to show that the cardinality of \( \Pi(m) \) is \( \frac{m^2 + m}{2} \). Let \( w \in a_i a_j \ldots a_m \) be some string. Then \( \pi(w) \) denotes the projection to the first symbol of \( w \) and \( \pi_r(w) \) denotes the projection to the last symbol of \( w \). For example, let \( w = a_2 a_3 a_4 \). Then, \( \pi_1(w) = a_2 \) and \( \pi_r(w) = a_4 \).

**Theorem 3** Let \( M \) be some \( k \)-turn PDA accepting a letter-bounded language \( L \subseteq a_1 a_2 \ldots a_m \). Then, an equivalent \((m - 1)\)-turn PDA \( M' \) can be constructed such that \( \text{size}(M') \in O(m^6 n^{4\lfloor \log_2 k \rfloor + 8}) \) and \( n = \text{size}(M) \).

**Proof:** It has been shown in the previous section that any \( L \subseteq a_1 a_2 \ldots a_m \) can be accepted by an \((m - 1)\)-turn PDA. If \( L \) is accepted by a \( k \)-turn PDA such that \( k > m - 1 \), then some turns are in a way “not necessary.” We will show in this proof that this finite number of additional turns takes place within unary parts of the input, i.e., while reading some input \( a_i \) with \( 1 \leq i \leq m \). Then, with the help of the construction of Lemma these parts can be accepted by NFAs and hence do not affect the stack height in the construction of an \((m - 1)\)-turn PDA accepting \( L \).

The construction is similar to the constructions of the previous two lemmas. Additionally, we introduce a fifth component of the variables in which some element \( s \in \Pi(m) \) is stored.

Figure 1: The two cases arising in the construction in Lemma.
(4) \([p, Z, q, 1, u] \rightarrow \epsilon\), if \(p = q\), for all \(u \in \Pi(m)\),

(5) \([p, Z, q, i, ab] \rightarrow [p, Z, r, i_1, u][r, Z, q, i_2, v]\), for all \(r \in Q\), \(i_1, i_2 \geq 1\) such that \(i_1 + i_2 \leq i\), 
\(|u| > 1, |v| > 1\), \(\pi_l(u) = a\), and \(\pi_r(v) = b\),

(6) \([p, Z, q, i, ab] \rightarrow ([p, Z, r, i_1, a][r, Z, q, i_2, v])\), for all \(r \in Q\), \(i_1, i_2 \geq 1\) such that \(i_1 + i_2 \leq i\), \(\pi_r(v) = b\),

(7) \([p, Z, q, i, ab] \rightarrow ([p, Z, r, i_1, u][r, Z, q, i_2, b])\), for all \(r \in Q\), \(i_1, i_2 \geq 1\) such that \(i_1 + i_2 \leq i\), \(\pi_l(u) = a\).

(8) \(S \rightarrow [q_0, Z_0, q_f, i, u]\) for all \(q_f \in F\), \(1 \leq i \leq k\), and \(u \in \Pi(m)\),

The productions (1) to (4), and (8) are defined similarly to the previous constructions. In the productions (5) to (7), a computation with \(i\) turns is decomposed into two subcomputations with \(i_1\) and \(i_2\) turns, respectively. Additionally, we differentiate whether we obtain subcomputations producing only one type of terminals or not. The former case is handled with productions (6) and (7), for the latter case we have the productions (5).

Moreover, we have to define productions for variables of the form \([p, Z, q, i, a]\) with \(a \in \{a_1, \ldots, a_m\}\). Such variables are from now on called “unary” variables. Owing to Lemma\(^8\) we know that the language \(L(p, Z, q)\) can be accepted by some NFA \(A\) having at most \(2(|Q|^2|\Gamma|)^{\lfloor \log_2 i \rfloor + 1}\) states. This NFA can be converted to some right-linear grammar \(G_A\) with at most \(2(|Q|^2|\Gamma|)^{\lceil \log_2 i \rceil + 1}\) variables. Now, the productions for a unary variable \([p, Z, q, i, a]\) with \(a \in \{a_1, \ldots, a_m\}\) are defined to be the productions of the corresponding right-linear grammar \(G_A\).

In order to finally get a PDA making at most \(m - 1\) turns, we have introduced in the productions (6) and (7) some special variables of the form \(([p, Z, q, i, a], [p', Z', q', i', v])\) which, at a first glance, are not natural and intuitive.

To derive such a variable \(([p, Z, q, i, a], [p', Z', q', i', v])\), we first derive its first component \([p, Z, q, i, a]\) with the above defined unary productions. Observe that the resulting productions are right-linear. For variables having the form \((\epsilon, [p', Z', q', i', v])\) we add productions \((\epsilon, [p', Z', q', i', v]) \rightarrow [p', Z', q', i', v])\). The remaining second component \([p', Z', q', i', v]\) is then derived with the productions (1) to (7) if \(|v| > 1\) and with the above defined productions otherwise.

To derive a variable \(([p, Z, q, i, u], [p', Z', q', i', b])\), we first derive its second component \([p', Z', q', i', b]\) with the above defined unary productions. Observe that the resulting productions are left-linear. We add productions of the form \(([p, Z, q, i, u], \epsilon) \rightarrow [p, Z, q, i, u]\) and the remaining first component \([p, Z, q, i, u]\) is then derived with the productions (1) to (7) if \(|u| > 1\) and with the above defined productions otherwise. We would like to remark that variables of the form \(([p, Z, q, i, a], [p', Z', q', i', b])\) are treated as in the first case, i.e., we start to derive the first component with unary productions and then derive the second component.

It can be observed that \(G\) generates \(T(M)\). Since all productions in \(G\) are linear except those of type (5), the number of variables occurring in a sentential form can only be increased by applications of productions of type (5). It can be observed that the maximum
number of variables introduced by productions of type (5) is bounded by the maximum number of decompositions of the string \( a_1 a_2 \ldots a_m \) into substrings \( w_1, w_2, \ldots, w_l \) such that, for \( 1 \leq t \leq l, \ w_t = a_i a_j \) with \( 1 \leq i < j \leq m \) and \( w_1 w_2 \ldots w_l \in a_1^* a_2^* \ldots a_m^* \). It is easy to show that \( l \leq m - 1 \). Thus, every sentential form contains at most \( m - 1 \) variables which implies that \( G \) is ultralinear of rank \( m - 1 \).

We next convert \( G \) to some equivalent one-state PDA \( M' \) using the standard conversion algorithm as given for example in \([8]\). Since the rank of \( G \) is \( m - 1 \), it can be observed that the maximum number of variables on the stack is bounded by \( m - 1 \). Furthermore, any decreasing of the stack starts by deleting some variable from the stack. This action corresponds to an application of some production of type (4) in the grammar. Thus, the number of turns in \( M' \) is bounded by the number of possible applications of productions of type (4) which is in turn bounded by the number of variables on the stack. Thus, the number of turns in \( M' \) is bounded by \( m - 1 \).

We now want to estimate the size of \( M' \) which is bounded by the number of variables of \( G \) and the size of the alphabet \( \Sigma \). The number of variables of type \([p, Z, q, i, u]\) is bounded by \( O(k m^2 |Q|^2 |\Gamma|) \) and the number of variables resulting from the unary productions is bounded by \( m(2k|Q|^2 |\Gamma|)^{|\log_2 k|+1} \). We now want to estimate the number of variables resulting from variables of type \([p, Z, r, i, u], [r, Z, q, j, v]\). Observe that in each such variable \(|u| = 1\) or \(|v| = 1\), respectively. Since these unary parts are derived first, the number of variables resulting is bounded by the product of the number of variables resulting from unary productions and of the number of variables of type \([p, Z, q, i, u]\). Thus, the number of variables resulting is bounded by \( O(m^3(2k|Q|^2 |\Gamma|)^{|\log_2 k|+2}) \) which implies \(|M'| \in O(m^6(2k|Q|^2 |\Gamma|)^{2|\log_2 k|+4})\) as an upper bound.

Finally, we convert \( M' \) to a PDA in normal form. This may cause at most an additional quadratic blow-up. Thus, we obtain \( O(m^6(2k|Q|^2 |\Gamma|)^{2|\log_2 k|+4}) = O(m^6 n^4 |\log_2 k|+8)\) as an upper bound.

**Corollary:** The trade-offs between finite-turn pushdown automata that accept letter- bounded languages are at most polynomial.

## 5 Lower Bounds

In this section we show the optimality of the simulation of grammars generating letter- bounded languages by finite-turn PDAs (Corollary [3]), and of some other simulation results presented in the paper. Even in this case, the preliminary investigation of the unary case will be useful to afford the general case.

**Theorem 4** For any integer \( n \geq 1 \), consider the language \( L_n = \{a^{2^n}\} \).

1. \( L_n \) can be generated by some CFG in Chomsky normal form with \( n + 1 \) variables.
2. Every NFA accepting \( L_n \) needs at least \( 2^n \) states.
(3) For each $k > 0$, every $k$-turn PDA accepting $L_n$ is at least of size $2^{cn}$ for some constant $c > 0$ and any sufficiently large $n$.

Proof: To prove (1) it is enough to observe that $L_n$ can be generated by the grammar $G$ with the following productions:

$$
S \rightarrow A_1A_1 \\
A_1 \rightarrow A_2A_2 \\
\vdots \\
A_{n-1} \rightarrow A_nA_n \\
A_n \rightarrow a
$$

The proof of (2) is trivial.

Finally, to prove (3) consider a $k$-turn PDA $M$ of size $s(n)$ accepting $L_n$. Due to Corollary 4 we can construct an equivalent NFA of size $s'(n) \leq Hs^K(n)$ for suitable constants $H, K$. Since $s'(n) \geq 2^{n}$, we obtain $s(n) \geq H'2^{n}/K$ for some other constant $H'$.

From Theorem 3, it turns out that for each integer $m$ the simulation result stated in Corollary 3 is optimal. The witness languages are unary. Hence, they can be also accepted by “simpler” devices, i.e., finite automata or PDAs with less than $m - 1$ turns. We now show the optimality in a stronger form, by exhibiting, for each integer $m$, a family of witness languages that cannot be accepted with less than $m - 1$ turns.

**Theorem 5** Given the alphabet $\Sigma = \{a_1, \ldots, a_m\}$, for any integer $n \geq 1$ consider the language

$$
\tilde{L}_n = \{a_1^{n_0+n_1}a_2^{n_1+n_2}\ldots a_{m-1}^{n_{m-2}+n_{m-1}}a_m^{n_{m-1}} \mid n_0 = 2^n, n_1 \geq 1, \ldots, n_{m-1} \geq 1\}.
$$

(1) $\tilde{L}_n$ is generated by some CFG in Chomsky normal form with $n + 4m - 3$ variables.

(2) $\tilde{L}_n$ is accepted by an $(m - 1)$-turn PDA of size $2^{O(n)}$.

(3) For each integer $k \geq m - 1$, every $k$-turn PDA accepting $\tilde{L}_n$ is at least of size $2^{cn}$ for some constant $c > 0$ and any sufficiently large $n$.

(4) $\tilde{L}_n$ cannot be accepted by any PDA which makes less than $m - 1$ turns.

**Proof:** Consider the grammar $G$ with the following productions:

- $S \rightarrow A_0B_1$,
- $B_1 \rightarrow C_1B_2, \ldots, B_{m-3} \rightarrow C_{m-3}B_{m-2}, B_{m-2} \rightarrow C_{m-2}C_{m-1}$
- $A_0 \rightarrow A_1A_1, A_1 \rightarrow A_2A_2, \ldots, A_{n-1} \rightarrow A_nA_n, A_n \rightarrow a_1$
- $C_i \rightarrow D_iE_i \mid D_iD_{i+1}, E_i \rightarrow C_iD_{i+1}$, for $i = 1, \ldots, m - 1$
• $D_i \rightarrow a_i$, for $i = 1, \ldots, m$.

It is possible to verify that this grammar $G$ generates the language $\tilde{L}_n$. In particular, observe that from each $C_i$ we can derive terminal strings only of the form $a_1^n a_{i+1}^t$, with $t \geq 1$, and from $A_0$ we can derive only the string $a_1^{2^n}$. By observing that we can use the same variable for $A_0$ and $D_1$, we easily conclude that the total number of variables is $n + 4m - 3$. This proves (1). Furthermore, as an easy consequence, applying Corollary 3 we get (2).

Now, given $k \geq m - 1$, suppose to have a $k$-turn PDA of size $s$ accepting $\tilde{L}_n$. By replacing each move consuming a symbol $a_i$, where $i > 1$, with an $\epsilon$-move, we get another $k$-turn PDA with $s$ states accepting the language $\tilde{L}_n = \{a_1^{n_1} | n_1 > 2^n\}$. Using a slight modification of Theorem 4 we can get that $s \geq 2^c$ for some constant number $c > 0$ and any sufficiently large $n$. This proves (3).

We finally prove (4). To this aim, we first prove that each context-free grammar which generates the following language $L$ must have rank at least $m - 1$:

$$L = \{a_1^{n_1} a_2^{n_1+n_2} \ldots a_m^{n_m-2+n_m-1} a_m^{n_m-1} | n_1 \geq 1, \ldots, n_m \geq 1\}.$$ 

Let $G$ be a grammar with $h$ variables which generates $L$. We can suppose that the right hand side of each production of $G$ has length at most 2 (by Lemma 6 this restriction preserves the rank, furthermore it can be easily seen that Lemma 7 used in the following, holds even for there grammars). Let $H = a_1^h a_2^{2H} \ldots a_m^{2H} a_m^H \in L$.

Given a derivation tree $T : S \Rightarrow z$, consider an integer $k > 0$, derivation trees $T_0, \ldots, T_k$ of strings $z_0, \ldots, z_k$, partial derivation trees $U_i : A_i \Rightarrow v_i A_i x_i$, $i = 1, \ldots, k$, according to Lemma 4. For $i = 1, \ldots, k$, let $\text{border}(A_i) = (l_i, r_i)$. Note that $r_i = l_i + 1$, otherwise, by pumping $T_i$ (which generates the string $z_i \in L$) with the partial tree $U_i$, the resulting string $z_i$ should not belong to $L$. Considering the definition of the relation $\leq$ between borders and Lemma 2 this easily implies that for each pair of variables $A_i, A_j \in \{A_1, \ldots, A_k\}$, either $A_i$ and $A_j$ have the same border or they lie on two different paths from the root of the tree $T$. We now prove that for each $j = 1, \ldots, m - 1$, there is a variable $A_{i_j} \in \{A_1, \ldots, A_k\}$ such that $\text{border}(A_{i_j}) = (j, j+1)$, obtaining in this way $m - 1$ variables belonging to different paths from the root of $T$.

Suppose, by contradiction, that there is an index $j$ such that $(j, j+1) \notin \{\text{border}(A_i) | i = 1, \ldots, k\}$. Hence, there is an index $r$, $1 \leq r \leq k$, such that $0 < l_1 \leq \ldots \leq l_r \leq j - 1$ and $l_{r+1} > j$ ($r = 1$ in the case $j = 1$). The pumping process described in Lemma 4 starts from the tree $T_0$, which generates a string $z_0 = a_1^{n_1} a_2^{n_2+n_2} \ldots a_m^{n_m-2+n_m-1} a_m^{n_m-1}$. The number of occurrences of the letters $a_1, \ldots, a_j$ can be incremented only by pumping with the partial trees $U_1, \ldots, U_r$, while the number of occurrences of the letters $a_{j+1}, \ldots, a_m$ only by pumping with the partial trees $U_{r+1}, \ldots, U_k$. Hence, the terminal string generated at the $r$th step should be

$$z_r = a_1^{H} a_2^{2H} \ldots a_j^{2H} a_{j+1}^{n_j+n_{j+1}} \ldots a_m^{n_m-2+n_m-1} a_m^{n_m}.$$ 

For $i = 1, \ldots, j$, let $a_i$ be the number of occurrences of letters $a_i$ and $a_{i+1}$ added during the pumping process, which leads from $z_0$ to $z_r$, by those partial trees among $U_1, \ldots, U_r$ such
that the borders of their roots coincide with \((i, i + 1)\). It is easy to verify that \(\alpha_i = H - n_i\), for \(i = 1, \ldots, \tilde{j}\). Furthermore, by our choice of \(\tilde{j}\), it turns out that \(\alpha_{\tilde{j}} = 0\). This implies that \(n_{\tilde{j}} = H\). This is a contradiction, because \(n_{\tilde{j}} < |z_0| < H\). Hence, we finally get that the tree \(T\) contains \(m - 1\) variables \(A_{i_1}, \ldots, A_{i_{m-1}} \in \{A_1, \ldots, A_k\}\) which lie on different paths from the root, i.e., there is a derivation of the form 
\[S \Rightarrow \alpha_1 A_{i_1} \alpha_2 \ldots \alpha_{m-1} A_{i_{m-1}} \alpha_m \Rightarrow z.\]
Thus, we conclude that the rank of the grammar \(G\) is at least \(m - 1\).

Using a slight modification of the construction given in the proof of Theorem 3, we can show that from a \(k\)-turn PDA it is possible to get an equivalent grammar of rank \(k\). This implies that if a \(k\)-turn PDA accepts \(L\) then \(k \geq m - 1\).

To complete the proof, we observe that given a PDA \(\tilde{M}\) accepting the language \(\tilde{L}\), we can build a PDA \(M\) which accepts \(L\) by working in two phases: In the first phase \(M\) simulates the moves of \(\tilde{M}\) from the initial configuration, as long as \(\tilde{M}\) consumes the input \(a_1^{2^n}\). In this phase, each move consuming the symbol \(a_1\) is replaced by an \(\epsilon\)-move (an internal variable counts, up to \(2^n\), the number of these moves). In this way, \(M\) is able to reach, without consuming any input symbol, every configuration reachable by \(\tilde{M}\) by consuming the input prefix \(a_1^{2^n}\). At this point, the second phase can start. In this phase \(M\) makes exactly the same moves as \(\tilde{M}\). It is easy to see that \(M\) accepts the language \(L\). Furthermore, if the given PDA \(\tilde{M}\) is \(k\)-turn, \(M\) is \(k\)-turn, too.

In conclusion, having proved that \(L\) cannot be accepted by \(k\)-turn PDAs with \(k < m - 1\), we can conclude that \(\tilde{L}\) cannot be accepted by \(k\)-turn PDAs with \(k < m - 1\), too. \(\square\)

Remark that we have considered so far only CFGs in Chomsky normal form and the measure \(\text{Var}\). It is easy to observe that we also obtain exponential trade-offs when considering the measure \(\text{Symb}\). This shows that the result of Corollary 3 is also optimal. Since \(\tilde{L}_n\) can be accepted by a PDA of size \(O(n)\), we obtain that the result of Corollary 3 is nearly optimal.

We complete this section by considering again the unary case. In particular, we prove that the upper bound stated in Corollary 4 is tight.

**Theorem 6** Consider the language family

\[L'_n = \{a^t \mid t \geq 0 \land t \equiv 0 \mod n \land t \equiv 0 \mod n + 1\}\]
for natural numbers \(n \geq 2\). Then each \(L'_n\) can be accepted by some 1-turn PDA of size \(2n + 1\), but every NFA accepting \(L'_n\) needs at least \(n^2 + n\) states.

**Proof:** A 1-turn PDA accepting \(L'_n\) starts with checking whether the length of the input is divisible by \(n\) in its states. At the same time, the input is stored in the stack. Then the PDA guesses that the whole input is read and checks whether the length of the input, which is stored on the stack, is divisible by \(n + 1\). Finally, the PDA accepts if the whole input is read, divisible by \(n\) and \(n + 1\), and the stack is empty. Otherwise, the input is rejected. It can be observed that such a PDA is 1-turn, has one stack symbol (apart from \(Z_0\)) and has \(2n + 1\) states. Since \(\gcd(n, n + 1) = 1\), we can apply a result from [7] and obtain the latter claim. \(\square\)
6 Word-Bounded Languages

In this section we study how to extend our results from the letter-bounded case to the word-bounded case. The idea is that of reducing the latter case to the former one. To this aim, a large part of the section is devoted to prove that for fixed $m$ words $w_1, w_2, \ldots, w_m \in \Sigma^*$, $m$ symbols $a_1, \ldots, a_m$ and the homomorphism $\phi$ associating with each symbol $a_i$ the string $w_i$, $i = 1, \ldots, m$, and for each context-free grammar $G = (V, \Sigma, P, S)$ in Chomsky normal form generating a subset of $w_1^* w_2^* \ldots w_m^*$, we can get another context-free grammar $\hat{G}$ whose number of symbols is linear in the number of symbols of $G$, namely $\text{Symb}(\hat{G}) = O(\text{Symb}(G))$, and such that $L(\hat{G}) = \phi^{-1}(L(G))$, i.e., for all integers $k_1, \ldots, k_m \geq 0$: $a_1^{k_1} \ldots a_m^{k_m} \in L(\hat{G})$ if and only if $w_1^{k_1} \ldots w_m^{k_m} \in L(G)$.

The construction is given in two steps: first we introduce a new grammar $G' = (V', \Sigma, P', S')$ equivalent to $G$. In such a grammar, the variables of $G$ are marked with some indices which are useful to recognize where, in a derivation, a variable can produce the first symbol of one of the $w_i$'s. This will be useful to get from $G'$, in a second step, the required grammar $\hat{G}$.

We start by considering the following set of variables:

$$V' = \{[A, i, l, r, j] \mid A \in V, 1 \leq l \leq r \leq m, 0 \leq i \leq |w_l|, 0 \leq j \leq |w_r|\}.$$ 

The definition is given in such a way that a variable $[A, i, l, r, j]$ can generate all terminal strings of the form $\alpha \beta \gamma$ generated by $A$, such that $\beta \in w_1^* \ldots w_r^*$, $\alpha$ is the suffix of $w_l$ which starts in position $i + 1$ and $\gamma$ is the prefix of $w_r$ which ends in position $j$. If $l = r$, furthermore, the variable $[A, i, l, l, j]$ will be able to generate the factor of $w_l$ from position $i + 1$ to position $j$ if $A$ is able to do this.

To this aim, we define the following productions:

1. $[A, j - 1, l, j, j] \rightarrow a$, for all $A \in V$ such that $A \rightarrow a$ is a production in $P$, $1 \leq l \leq m$, $1 \leq j \leq |w_l|$, and $a = w_l j$, i.e., $a$ is the symbol in position $j$ of $w_l$.

2. $[A, i, l, r, j] \rightarrow [B, i, l, h, k][C, k, h, r, j]$, for all $A, B, C \in V$ such that $A \rightarrow BC$ is a production in $P$, $1 \leq l \leq h \leq r \leq m$, $0 \leq i \leq |w_l|$, $0 \leq k \leq |w_h|$, and $0 \leq j \leq |w_r|$.

3. $[A, i, l, r, 0] \rightarrow [A, i, l, r, |w_r|]$, for all $1 \leq l \leq r \leq m$, $0 \leq i \leq |w_l|$.

4. $[A, |w_l|, l, r, j] \rightarrow [A, 0, l, r, j]$, for all $1 \leq l \leq r \leq m$, $0 \leq j \leq |w_r|$.

5. $[A, i, l, r, 0] \rightarrow [A, i, l, h, 0]$, for all $1 \leq l \leq h < r \leq m$, $0 \leq i \leq |w_l|$.

6. $[A, |w_h|, l, r, j] \rightarrow [A, i, |w_h|, h, r, j]$, for all $1 \leq l < h \leq r \leq m$, $0 \leq j \leq |w_r|$.
The initial symbol \( S' \) of \( G' \) is \([S,|w_1|,1,m,0]\).

The following lemma states the main property of the variables of the grammar \( G' \) and it is crucial in order to prove that \( G' \) is equivalent to \( G \):

**Lemma 9** For each variable \([A,i,l,r,j]\) of the grammar \( G' \) above defined and for each string \( w \in \Sigma^* \), it holds that \( [A,i,l,r,j] \xrightarrow{\ast} G' w \) if and only if \( A \xrightarrow{\ast} G w \), and

1. \( w = w_{l,i+1} \ldots w_{l,|w_1|} \beta_1 \ldots w_{r,j}, \) with \( \beta \in w_i^* \ldots w_r^* \),
2. or \( l = r, \ i < j, \) and \( w = w_{l,i+1} \ldots w_{l,j} \).

**Proof:** We first prove the *only if* part. Let \( s \geq 1 \) be an integer such that \( [A,i,l,r,j] \xrightarrow{s} G' w \). The proof is by induction on \( s \).

If \( s = 1 \) then the derivation consists only of the production \( [A,j-1,l,l,j] \rightarrow w_{l,j} \). This implies that \( i = j - 1, \ l = r, \) and \( A \xrightarrow{\ast} G w_{l,j} \).

If \( s > 1 \) then we have to consider several possibilities depending on which kind of production is used in the first step of the derivation. (The following list refers to the above defined productions (2), \ldots, (6):

1. \( [A,i,l,r,j] \rightarrow [B,i,l,h,k] [C,k,h,r,j] \):
   
   Hence \( [B,i,l,h,k] \xrightarrow{\ast} G' w', [C,k,h,r,j] \xrightarrow{\ast} G' w'' \) where \( w = w'w'' \). We observe that, by induction hypothesis, \( B \xrightarrow{\ast} G w' \), \( C \xrightarrow{\ast} G w'' \), and then \( A \xrightarrow{\ast} G w \). The proof can be easily completed by considering the following subcases:
   
   - \( w' = w_{l,i+1} \ldots w_{l,|w_1|} \beta' w_{h,1} \ldots w_{h,k} \), \( w'' = w_{h,k+1} \ldots w_{h,|w_1|} \beta'' w_{r,1} \ldots w_{r,j} \), where \( \beta' \in w_i^* \ldots w_h^* \) and \( \beta'' \in w_h^* \ldots w_r^* \). By choosing \( \beta = \beta' \beta'' \), it turns out that \( \beta \in w_i^* \ldots w_r^* \) and \( w = w_{l,i+1} \ldots w_{l,|w_1|} \beta w_{r,1} \ldots w_{r,j} \).
   
   - \( l = h = r, \ i < k < j, \) \( w' = w_{l,i+1} \ldots w_{l,k} \), and \( w'' = w_{l,k+1} \ldots w_{l,j} \). Then \( w = w_{l,i+1} \ldots w_{l,j} \).
   
   - \( l = h, \ i < k, \) \( w' = w_{l,i+1} \ldots w_{l,k} \), and \( w'' = w_{l,k+1} \ldots w_{l,|w_1|} \beta w_{r,1} \ldots w_{r,j} \), or \( h = r, \ k < j, \) \( w' = w_{l,i+1} \ldots w_{l,|w_1|} \beta w_{r,1} \ldots w_{r,j} \), and \( w'' = w_{r,k+1} \ldots w_{r,j} \). In both these cases, we get that \( w = w_{l,i+1} \ldots w_{l,|w_1|} \beta w_{r,1} \ldots w_{r,j} \).

2. \( [A,i,l,0,0] \rightarrow [A,i,l,r,|w_r|] \):
   
   Hence \( [A,i,l,r,|w_r|] \xrightarrow{s-1} G' w \). By induction hypothesis we get that \( A \xrightarrow{\ast} G w \). If \( w = w_{l,i+1} \ldots w_{l,|w_1|} \beta' w_{r,1} \ldots w_{r,r}, \) with \( \beta' \in w_i^* \ldots w_r^* \), then \( w = w_{l,i+1} \ldots w_{l,|w_1|} \beta \), with \( \beta = \beta' w_r \in w_i^* \ldots w_r^* \). Otherwise, \( l = r \) and then \( w = w_{l,i+1} \ldots w_{l,|w_1|} \beta \), with \( \beta = \epsilon \).

3. \( [A,i,l,r,0] \rightarrow [A,0,l,r,j] \):
   
   This case is similar to the previous one.

4. \( [A,i,l,l,j] \rightarrow [A,0,l,r,j] \):
   
   Hence \( [A,i,l,l,j] \xrightarrow{s-1} G w \) and, by induction hypothesis, \( A \xrightarrow{\ast} G w \) and \( w = w_{l,i+1} \ldots w_{l,|w_1|} \beta \) for some \( \beta \in w_i^* \ldots w_r^* \). From \( h < r \), it turns out that \( \beta \in w_i^* \ldots w_r^* \).
(6) \([A, |w_l|, l, r, j] \rightarrow [A, |w_h|, h, r, j]\):

Hence \([A, |w_h|, h, r, j] \xrightarrow{s+1} \) \(G\) \(w\) and, by induction hypothesis, \(A \xrightarrow{G} w\) and \(w = \beta w_{r,1} \ldots w_{r,j}\) for some \(\beta \in w_h^* \ldots w_r^*\). From \(l < r\), it turns out that \(\beta \in w_i^* \ldots w_r^*\).

We now prove the if part. Even in this case the proof is by induction on the number \(s\) of the steps of the derivation \(A \xrightarrow{*} \) \(w\) under consideration.

If \(s = 1\) then \(|w| = 1\) and \(A \xrightarrow{} w\) must be a production in \(P\). We have to consider two cases:

- \(w = w_{l,i+1} \ldots w_{l,|w_l|} \beta w_{r,1} \ldots w_{r,j}\), with \(\beta \in w_i^* \ldots w_r^*\):
  
  Since \(|w| = 1\), it turns out that:
  
  - either \(w = w_{l,|w_l|}, \beta = \epsilon, i = |w_l| - 1, \text{ and } j = 0,\) or
  
  - \(w = w_{r,1}, \beta = \epsilon, i = |w_l|, \text{ and } j = 1,\) or
  
  - \(\beta = w = w_h, \text{ for some } l \leq h \leq r, i = |w_l|, \text{ and } j = 0.\)

In the first case, the grammar \(G'\) should contain the production \([A, |w_l| - 1, l, l, |w_l|] \rightarrow w\). Hence, using productions (5), (3), and (1), we get:

\[
[A, i, l, r, j] = [A, |w_l| - 1, l, r, 0] \Rightarrow [A, |w_l| - 1, l, l, 0] \Rightarrow [A, |w_l| - 1, l, |w_l|] \Rightarrow G' w.
\]

The second case is similar. In the third case, the grammar \(G'\) contains, by construction, the production \([A, 0, h, h, 1] \rightarrow w\). Hence, using productions (6), (5), (3), (4), we get the following derivation:

\[
[A, i, l, r, j] = [A, |w_l|, l, r, 0] \Rightarrow [A, |w_h|, h, r, 0] \Rightarrow [A, |w_h|, h, h, 0] \Rightarrow G' [A, |w_h|, h, h, 1] \Rightarrow G' [A, 0, h, h, 1] \Rightarrow G' w.
\]

- \(l = r, i < j, \text{ and } w = w_{l,i+1} \ldots w_{l,j}\):
  
  This case is trivial.

Now suppose \(s > 1\). The first production applied in the derivation of \(w\) should be \(A \xrightarrow{} BC\), for two variables \(B\) and \(C\), such that \(B \xrightarrow{G} w', C \xrightarrow{G} w''\), and \(w = w' w''\). As before, the proof is divided into two cases:

- \(w = w_{l,i+1} \ldots w_{l,|w_l|} \beta w_{r,1} \ldots w_{r,j}\), with \(\beta \in w_i^* \ldots w_r^*\):
  
  We have to consider the following three possibilities:

  - \(w' = w_{l,i+1} \ldots w_{l,|w_l|} \beta' w_{h,1} \ldots w_{h,k}, w'' = w_{h,k+1} \ldots w_{h,|w_h|} \beta'' w_{r,1} \ldots w_{r,j}\), for some \(l \leq h \leq r, 0 \leq k \leq |w_h|, \beta' \in w_i^* \ldots w_k^*\), and \(\beta'' \in w_k^* \ldots w_r^*\).

  By the inductive hypothesis, we obtain \([B, i, l, h, k] \Rightarrow G' w', [C, k, h, r, j] \Rightarrow G' w''\). Hence, \([A, i, l, r, j] \Rightarrow G' w\).
Given $G$ generating a word bounded language $L$.

Hence, we get the following result: $S \Rightarrow^* w'$. By the inductive hypothesis, we obtain $[B, i, l, k] \Rightarrow^* w'$, $[C, k, l, j] \Rightarrow^* w''$. Hence, $[A, i, l, r, j] \Rightarrow^* w$.

Theorem 8

Given $m$ strings $w_1, \ldots, w_m$, $m$ symbols $a_1, \ldots, a_m$, and the homomorphism $\phi$ associating with each symbol $a_i$ the string $w_i$, $i = 1, \ldots, m$, for each context-free grammar $G$ generating a word bounded language $L \subseteq w_1^* \ldots w_m^*$, there exists another context-free grammar $\hat{G}$ such that $L(\hat{G}) = \phi^{-1}(L(G))$, and $\text{Symb}(\hat{G}) = O(\text{Symb}(G))$.

We are now ready to extend Corollary 8 to word-bounded languages:

Theorem 8

Given $m$ strings $w_1, \ldots, w_m$, for any context-free grammar $G$ with $\text{Symb}(G) = h$ and generating a word bounded language $L \subseteq w_1^* \ldots w_m^*$, there exists an equivalent $(m - 1)$-turn PDA $M$ with $2^{O(h)}$ states and $O(1)$ stack symbols.
Proof: By Corollary 6, given \( m \) symbols \( a_1, \ldots, a_m \), we can get a context-free grammar \( \hat{G} \) with \( \text{Symb}(\hat{G}) = O(\text{Symb}(G)) \), which generates the language \( \phi^{-1}(L(G)) \). Using Corollary 3, we are able to find an \( (m-1) \)-turn PDA \( \hat{M} \) with \( 2^{O(\text{Symb}(\hat{G}))} \) states and \( O(1) \) stack symbols, recognizing \( \phi^{-1}(L(G)) \).

From \( \hat{M} \) we now define a PDA \( M \) accepting \( L \): \( M \) should simulate \( \hat{M} \) with the difference that a move of \( M \) consuming the input symbols \( a_i \) is simulated by a sequence of moves of \( M \) consuming the input factor \( w_i \). It is easy to implement this with \( 2^{O(h)} \) states and without increasing the number of stack symbols.

Because the letter-bounded case is a special case of the word-bounded case, by the results presented in Section 5 the upper bound presented in Theorem 8 is tight.

Also Corollary 4 can be extended to the word-bounded case.

Theorem 9 The trade-offs between finite-turn pushdown automata that accept word-bounded languages are at most polynomial.

Proof: Given \( m \) strings \( w_1, \ldots, w_m \), let \( M \) be some \( k \)-turn PDA accepting a word-bounded language \( L \subseteq w_1^*w_2^*\ldots w_m^* \). We consider \( m \) letters \( a_1, \ldots, a_m \) and the morphism \( \phi \) associating with each \( a_i \) the string \( w_i, i = 1, \ldots, m \). From the given PDA \( M \), a \( k \)-turn PDA \( \hat{M} \) accepting \( \phi^{-1}(L) \) can be defined by simulating \( M \) step by step, with the only difference that the input tape of \( M \) is replaced by a buffer of length \( \max\{|w_i| \mid i = 1, \ldots, m\} \). When \( \hat{M} \) has to simulate a move of \( M \) that reads an input symbol, it uses the next symbol from the buffer. However, if the buffer is empty, then \( \hat{M} \) reads the next symbol \( a_i \) from its input tape and puts the corresponding string \( w_i \) in the buffer. Note that the size of \( \hat{M} \) is polynomial in the size of \( M \). This is essentially the construction for inverse homomorphism as is given, e.g., in [8].

Now, using Theorem 3 from \( \hat{M} \) it is possible to get an equivalent \( (m - 1) \)-turn PDA \( \hat{M}' \), which is still polynomial in the size of \( M \). Finally, using the same construction outlined in the proof of Theorem 8 this last PDA can be converted in another \( (m-1) \)-turn PDA \( M' \), accepting the original language \( L \), whose size is polynomial in the size of \( M \). \( \square \)

7 Conclusion

In this paper, we have considered context-free grammars generating letter-bounded as well as word-bounded languages. We have shown that such languages can be accepted by finite-turn pushdown automata. Furthermore, we have given constructions for converting context-free grammars which generate bounded languages to equivalent finite-turn pushdown automata as well as minimizing the number of turns of a given pushdown automaton accepting a bounded language. The resulting trade-offs concerning the size of description of the corresponding context-free grammars and pushdown automata have been shown to be exponential when starting with an arbitrary context-free grammar or an arbitrary pushdown automaton. When starting with a finite-turn pushdown automaton, a polynomial trade-off has been obtained. Both trade-offs are in strong contrast to the non-bounded
case where non-recursive trade-offs are known to exist. Moreover, the existence of non-recursive trade-offs implies that such conversion algorithms and minimization algorithms cannot exist in general. Additionally, equivalence or inclusion problems are undecidable for arbitrary context-free languages.

We have shown that boundedness is a structural limitation on context-free languages which reduces non-recursive trade-offs to recursive trade-offs. Together with the known positively decidable questions such as equivalence or inclusion of bounded context-free languages, we obtain that context-free grammars and pushdown automata for bounded languages are much more manageable from a practical point of view than in the general case.

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