An \( \epsilon \)-regularity criterion and estimates of the regular set for Navier-Stokes flows in terms of initial data

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Abstract

We prove an \( \epsilon \)-regularity criterion for the 3D Navier-Stokes equations in terms of initial data. It shows that if a scaled local \( L^2 \) norm of initial data is sufficiently small around the origin, a suitable weak solution is regular in a set enclosed by a paraboloid started from the origin. The result is applied to the estimate of the regular set for local energy solutions with initial data in weighted \( L^2 \) spaces. We also apply this result to studying energy concentration near a possible blow-up time and regularity of forward discretely self-similar solutions.

Keywords: Navier-Stokes equations, \( \epsilon \)-regularity, regular set

1 Introduction

1.1 Regular set for suitable weak solutions

We consider the regularity of weak solutions for the incompressible Navier-Stokes equations

\[
\partial_t v - \Delta v + v \cdot \nabla v + \nabla p = 0, \quad \text{div} \, v = 0
\]

associated with the initial value \( v|_{t=0} = v_0 \) with \( \text{div} \, v_0 = 0 \). The global in time existence of weak solutions for finite energy initial data was proved by Leray [32] and Hopf [18]. Despite a lot of effort since their foundational work, the global regularity of the weak solutions remains a longstanding open problem. After the pioneering work by Scheffer [39, 40], Caffarelli, Kohn, and Nirenberg [11] established local regularity theory for suitable weak solutions which are weak solutions satisfying the local energy inequality; see Section 2 for details. As an application of their celebrated \( \epsilon \)-regularity criterion, they showed the following result on the regular set:

Theorem D [11]. There exists \( \epsilon_0 > 0 \) such that if \( v_0 \in L^2(\mathbb{R}^3) \) satisfies

\[
|||x|^{-\frac{1}{2}} v_0||^2_{L^2} = \epsilon < \epsilon_0,
\]

then there exists a suitable weak solution which is regular in the set \( \Pi_{\epsilon_0 - \epsilon} \), where

\[
\Pi_\delta := \left\{ (x, t) : t > \frac{|x|^2}{\delta} \right\} \quad \text{for} \ \delta > 0.
\]

This theorem shows that smallness of initial data implies regularity of the solution above a paraboloid

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with vertex at the origin. There are at least two interesting features in this result: No regularity condition (better than $L^2$) is assumed away from the origin and the regularity around the origin is propagated globally in time. We also note that if the size of $v_0$ tends to 0, the regular set increases and invades a limit set $\Pi_{\varepsilon_0}$. This observation leads to the following questions:

(a) Can the size of regular set $\Pi_\delta$ be enlarged?

(b) Can the condition (1.1) of initial data be relaxed in terms of regularity and smallness?

The question (a) is addressed by D’Ancona and Luca [13], where it is shown that there exists $\delta_0 > 0$ such that if $v_0 \in L^2(\mathbb{R}^3)$ satisfies

$$\| |x|^{-\frac{2}{3}} v_0 \|_{L^2} < \delta_0 e^{-4L^2} \tag{1.2}$$

for some $L > 1$, (NS) has a suitable weak solution which is regular in the set $\Pi_{L\delta_0} = \{(x, t) : t > \frac{|x|^2}{L\delta_0}\}$. In particular, the regular set $\Pi_{L\delta_0}$ invades the whole half space $\mathbb{R}^3 \times (0, \infty)$ when $v_0$ tends to zero, though (1.2) still assumes smallness of the data. One of the goals of this paper is trying to answer questions (a) and (b) by employing approach based on a framework of scaled local energy explained below.

### 1.2 Main result on local-in-space regularity near initial time

It is known from works [14, 44, 23, 16] that for $v_0 \in L^q(\mathbb{R}^3)$ with $q \geq 3$, (NS) has a (unique) mild solution defined on some short time interval. Motivated by the problem for constructing large forward self-similar solutions to (NS), Jia and Šverák [20] asked which condition this result can be localized in space. Let $B_r(x) = \{y \in \mathbb{R}^3 : |x - y| < r\}$ and $B_r = B_r(0)$. Then their question can be stated as follows:

(c) If $v_0$ is a general initial data for which suitable weak solutions $v$ is defined and $v_0|_{B_2} \in L^q(B_2)$, can we conclude that $v$ is regular in $B_1 \times [0, t_1)$ for some time $t_1 > 0$?

Although non-local effect of the pressure might prevent the solution from having the same amount of the regularity as the one for the heat equation, such effect is expected to be handled at least for a short time and $q \geq 3$. Indeed this question is settled affirmatively for the subcritical case $q > 3$ in [20] and for the critical case $q = 3$ in [1, 22]. Notice that the results for the critical case have some similarities with Theorem D in [11]. Namely, the assumptions for the initial data ensure critical regularity at the origin and they lead to local-in-space regularity. As the first main result of this paper, we present a new type of local-in-space regularity estimate, which guarantees regularity in the set like $\Pi_\delta$. In order to formulate it, define the scaled local energy of the initial data by

$$N_0 = N_0(v_0) := \sup_{r \in (0, 1]} \frac{1}{r} \int_{B_r} |v_0(x)|^2 \, dx,$$

which plays a central role in this paper.\(^1\)

\(^1\)See also [41] for the condition on the initial enstrophy and [1] for further extension to the $L^{3, \infty}$ space and the critical Besov spaces.
Theorem 1.1. Let \((v, p)\) be a suitable weak solution in \(B_2 \times (0, 4)\) with the initial data \(v_0 \in L^2(B_2)\) in the sense that \(\lim_{t \to 0^+} \|v(t) - v_0\|_{L^2(B_2)} = 0\). Assume that
\[
M := \|v\|^2_{L^\infty(0,4;L^3(B_2))} + \|\nabla v\|^2_{L^2(B_2 \times (0,4))} + \|p\|^2_{L^2(B_2 \times (0,4))} < \infty
\]
and that
\[
N_0 \leq \epsilon_*.
\]
Then there exists \(T = T(M) \geq \frac{\epsilon_*}{1 + M^{1/3}}\) such that \(v\) is regular in the set
\[
\Gamma = \{(x, t) \in B_1 \times (0, 1) : cN_0^2 |x|^2 \leq t < T\}
\]
and satisfies
\[
|v(x, t)| \leq \frac{C}{t^{2}} \quad \text{for} \quad (x, t) \in \Gamma,
\]
where \(\epsilon_*, c, \text{and } C\) are positive absolute constants.

Remark 1.2. (1) This theorem shows that smallness of the scaled energy implies regularity above a paraboloid for a short time. It may be viewed as an \(\epsilon\)-regularity criterion in terms of the initial data.

(2) One can relate Theorem 1.1 to results in [11, 1, 22] by noting that
\[
N_0 \leq C \min\{\|v_0\|_{L^3(B_1)}^2, \|v_0\|_{L^{2,1}(B_1)}^2\},
\]
where
\[
\|v_0\|_{L^{2,\alpha}(\Omega)} := \|\|x\|^{\alpha} v_0\|_{L^2(\Omega)}
\]
for \(\alpha \in \mathbb{R}\) and \(\Omega \subset \mathbb{R}^3\). Thus (1.4) holds if either \(L^3\) norm or \(L^{2,1}\) norm is small in \(B_1\). Hence our theorem can be regarded as a local version of Theorem D in [11].

(3) By rescaling and translation, it is easy to see that if the data satisfies \(\sup_{r \in (0, R]} \frac{1}{r} \int_{B_r(x_0)} |v_0|^2 \leq \epsilon_*\) for some \(x_0 \in B_2\) and \(R > 0\), then \(v\) is regular in the set enclosed by a paraboloid with vertex at \((x_0, 0)\) and the plane \(t = TR^2\). In particular, if \(\|v_0\|_{m^{2,1}(B_2)} \leq \epsilon_*\), there exists \(T = T(M) > 0\) such that the suitable weak solution is regular in \(B_1 \times (0, T)\) and satisfies \(|v(x, t)| \leq \frac{C}{\sqrt{t}}\) in \(B_1 \times (0, T)\).

Here \(\|\cdot\|_{m^{2,1}}\) denotes the local Morrey norm:
\[
\|f\|_{m^{2,1}(\Omega)} := \sup_{x_0 \in \Omega, r \in (0, 1]} \left( \frac{1}{r} \int_{B_r(x_0) \cap \Omega} |f(x)|^2 \, dx \right)^{\frac{1}{2}}.
\]

By well-known relations \(m^{2,1} \supset L^{3,\infty} \supset L^3\), we see that this extends results of [22] \((L^3\) case) as well as [1] \((L^{3,\infty}\) case). Recently it is shown in [9, Theorem 1.2] that the local energy solution is regular in time interval \((0, cR^2]\), provided \(\sup_{x_0 \in \mathbb{R}^3, r \in (0, R]} \frac{1}{r} \int_{B_r(x_0)} |v_0|^2 \) is small. It should be pointed out that the Morrey space is introduced in the context of the Navier-Stokes equations by Giga and Miyakawa [17] for the study of vortex filaments in \(\mathbb{R}^3\). Well-posedness of (NS) in the Morrey space \(m^{2,1}\) (and its homogeneous version \(M^{2,1}\)) is studied by Kato [24] in \(\mathbb{R}^n\) and Taylor [42] in compact Riemannian manifolds. Recently, solutions from vortex filaments with arbitrary circulation are constructed in [2]; see references therein for development of this direction.
The proof of Theorem 1.1 is based on a local-in-space a priori estimate for the scaled energy of \((v, p)\) defined by

\[
E_r(t) := \sup_{0 < s < t} \frac{1}{r} \int_{B_r} |v(s)|^2 + \frac{1}{r} \int_0^t \int_{B_r} |\nabla v|^2 + \frac{1}{r^2} \int_0^t \int_{B_r} |p|^2.
\] (1.5)

Our strategy is partially inspired by a uniformly local \(L^2\) estimate established in the fundamental work [29] of Lemarié-Rieusset. However, in contrast to his estimate, our a priori estimate guarantees scale-critical regularity at the origin so that the \(\epsilon\)-regularity criterion of [11] can apply. Furthermore, since we make assumptions only locally in space, nontrivial modification is required to deal with the non-local effect of the pressure. It should be noted that our strategy is also different from previous works, the proof relied on decomposition of the solution in \(B_2\) into two parts; one is the (regular) solution to (NS) for the data \(B(v_0|B_1) \in L^q\) where \(B\) is the Bogovskii extension operator to \(\mathbb{R}^3\), and the other is the solution to the perturbed Navier-Stokes equation for the data \(v_0 - B(v_0|B_1)\). The main task in previous works is to show the regularity of the perturbed part, and hence they developed local regularity theory for the perturbed Navier-Stokes equations. In contrast, we estimate the solution of (NS) directly without using such decomposition.

1.3 Applications
1.3.1 Regular set

We now return to the questions (a) and (b) concerning the regular set. In order to state our result, it is natural and convenient to use the notion of local energy solutions introduced by Lemarié-Rieusset [29] and later slightly modified in [26, 20, 9]. The local energy solution is a suitable weak solution of (NS) defined in \(\mathbb{R}^3\) which satisfies certain uniformly local energy bound and pressure representation; see Definition 2.1 for the details. In this context, let us recall the uniformly local \(L^q\) spaces for \(1 < q < \infty\). We say \(f \in L^q_{uloc}\) if \(f \in L^q_{loc}(\mathbb{R}^3)\) and

\[
\|f\|_{L^q_{uloc}} = \sup_{x \in \mathbb{R}^3} \|f\|_{L^q(B_1(x))} < \infty.
\] (1.6)

Local in time existence of local energy solutions for initial data in \(L^2_{uloc}\) and also global existence for initial data in \(E^2 := L^2_{uloc} \subset L^2_{loc}\) are established in [29]. One of the advantage of the local energy solution is it can be defined even for infinite energy data; see [28, 31, 21] and references therein for further developments and its applications, and [35] for the local energy solutions in the half space. We also define the global version of the scaled energy by

\[
\tilde{\mathcal{N}}_0 := \sup_{r > 0} \frac{1}{r} \int_{B_r} |v_0(x)|^2 \, dx
\]

Here note that \(\tilde{\mathcal{N}}_0\) is invariant under the Navier-Stokes scaling: \(u(x, t) \mapsto u_\lambda(x, t) := \lambda u(\lambda x, \lambda^2 t)\). The following result shows the estimates of the regular set for the local energy solution for initial data with small scaled energy and also about that for large data in \(L^{2,1}(\mathbb{R}^3)\):

**Theorem 1.3.** Let \((v, p)\) be a local energy solution in \(\mathbb{R}^3 \times (0, \infty)\) for the initial data \(v_0 \in L^2_{uloc}(\mathbb{R}^3)\).

(i) There exist absolute constants \(\epsilon_*\) and \(c\) such that if \(v_0\) satisfies

\[
\sup_{x_0 \in \mathbb{R}^3} \sup_{r \geq 1} \frac{1}{r} \int_{B_r(x_0)} |v_0(x)|^2 \, dx < \infty,
\] (1.7)
and if
\[ \hat{N}_0 \leq \epsilon_*, \]  \hspace{1cm} (1.8)
then \( v \) is regular in the set
\[ \left\{ (x, t) \in \mathbb{R}^3 \times (0, \infty) : c\hat{N}_0^2 |x|^2 \leq t \right\}. \]

(ii) For any \( v_0 \in L^{2,-1}(\mathbb{R}^3) \) there exist positive constants \( T(v_0) \) and \( c(v_0) \) such that \( v \) is regular in the set
\[ \left\{ (x, t) \in \mathbb{R}^3 \times (0, \infty) : c(v_0)|x|^2 \leq t < T(v_0) \right\}. \]

Note that \( \hat{N}_0 \leq \|v_0\|^2_{L^{2,-1}(\mathbb{R}^3)} \) holds and that the condition (1.7) only assumes some mild decay of the data at infinity, and it is satisfied for the data in \( L^2 \) or even for infinite energy data in the homogeneous Morrey space \( M^{2,1} \) (see (1.12)). Therefore (i) relaxes the assumptions for the initial data given in [11, 13]. It also refines the convergence rate of the regular set to \( \mathbb{R}^3 \times (0, \infty) \) in [13] as \( \|v_0\|_{L^{2,-1}} \) tends to zero. Thus Theorem 1.3 extends [11, Theorem D] and [13]. Moreover, [11, Theorem D] and [13] are existence results for initial data satisfying the conditions, while Theorem 1.3 is a regularity result for any solution for such data.

Global existence of local energy solutions for data satisfying (1.7) is proved in [9]. See Theorem 4.1 for further results including eventual regularity for \( v_0 \in L^{2,-1}(\mathbb{R}^3) \) also satisfying (1.7).

The following corollary concerns estimates of regular set for the data in the weighted space \( L^{2,\alpha} \) with \( \alpha > -1 \), which generalize the classical result [32] of Leray and [11, Theorem C] for the cases \( \alpha = 0 \) and \( \alpha = 1 \), respectively.

**Corollary 1.4.** Let \((v, p)\) be a local energy solution for the initial data in \( L^2_{uloc}(\mathbb{R}^3) \).

(i) Assume that \( v_0 \in L^{2,\alpha}(\mathbb{R}^3) \) for some \( \alpha \geq 0 \). Then \( v_0 \in L^2(\mathbb{R}^3) \) and there is \( K = K(\|v_0\|_{L^2}, \|v_0\|_{L^{2,\alpha}}) \) such that \( v \) is regular in the set
\[ \left\{ (x, t) \in \mathbb{R}^3 \times (0, \infty) : |x|^{-2\alpha} \leq t \leq \text{min}\{K,v_0\|v_0\|_{L^2}^4\} \right\}. \]

(ii) Assume that \( v_0 \in L^{2,\alpha}(\mathbb{R}^3) \) for some \( \alpha \in (-1, 0) \) and that (1.7) holds. Then there exist \( K = K(\|v_0\|_{L^{2,\alpha}}) \) and \( T = T(\|v_0\|_{L^{2,\alpha}}) \) such that \( v \) is regular in the set
\[ \left\{ (x, t) \in \mathbb{R}^3 \times (0, \infty) : t \geq \text{max}\{K|x|^{-2\alpha}, T\} \right\}. \]

**Remark 1.5.** (1) Regularity near the initial time is not guaranteed in this case. Note that the condition \( v_0 \in L^{2,\alpha} \) for \( \alpha > -1 \) assumes faster decay at infinity than the case for \( \alpha = -1 \), but less regularity at the origin.

(2) Global existence of local energy weak solutions for data in \( L^2_{uloc} \cap L^{2,\beta} \) for \( \beta \in (-2, 0) \) is proved by [15]. See also more recent work [4] for further generalization.

### 1.3.2 Energy concentration near a possible blow up time

Regularity theory developed in this paper has applications to other problems as well. The following theorem shows concentration of the scaled local energy in a shrinking ball near a possible blow-up time.
Theorem 1.6. Let \( v \) be a local energy solution and \( T_* \in (0, \infty) \) the maximal time so that \( v \in C((0, T_*); L^\infty(\mathbb{R}^3)) \). There exists \( \epsilon_* > 0 \) such that the following holds:

(i) There exist \( S_0 > 0 \) and \( x(t) \in \mathbb{R}^3 \) for \( t \in (0, T_*) \) such that

\[
\frac{1}{\sqrt{T_* - t}} \int_B \frac{|v(x, t)|^2}{\sqrt{\frac{T_* - t}{S_0}}} dx > \epsilon_*.
\] (1.9)

(ii) Suppose that \( (x, t) = (0, T_*) \) is a singular point and that the type I condition

\[
\sup_{x_0 \in \mathbb{R}^3} \sup_{r \in (0, r_0]} \frac{1}{r} \int_{B_r(x_0)} |v(x, t)|^2 dx =: M_* < \infty
\] (1.10)

holds for some \( r_0 > 0 \). Then, there exist \( S = S(M_*) > 0 \) and \( \delta_* > 0 \) such that

\[
\frac{1}{\sqrt{T_* - t}} \int_B \frac{|v(x, t)|^2}{\sqrt{\frac{T_* - t}{S}}} dx > \epsilon_* \quad \text{for } t \in (T_* - \delta_*, T_*).
\] (1.11)

Remark 1.7. (1) Part (i) of Theorem 1.6 is a restatement of [10, Theorem 8.2] with slightly different conditions on \( v \). Our new proof is based on Theorem 3.4.

(2) There is a lot of literature on the behavior of the critical norm near the blow up time. However, to our knowledge, the behavior of the critical norm in shrinking balls has not been studied until recently. Li, Ozawa, and Wang [33] showed concentration of \( L^3 \) norm at the blow up time in a ball \( B(c\sqrt{T_* - t}) \) along some sequence., i.e, they showed that there exist \( t_n \uparrow T_* \) and \( x_n \in \mathbb{R}^3 \) such that

\[
\|v(t_n)\|_{L^3(B_{c\sqrt{T_* - t_n}}(x_n))} > \gamma_0
\]

holds with some absolute constants \( c, \gamma_0 > 0 \). In [36, Corollary 1.1], \( L^3 \) concentration in \( B_{c(T_* - t)}(x(t)) \) for all \( t < T_* \) is shown with some trajectory \( \{x(t)\}_{t < T_*} \). Statement (i) improves [36, Corollary 1.1] since the scaled energy is bounded by \( L^3 \) norm in \( B_{\sqrt{\frac{T_* - T}{S}}} \). We note that similar concentration phenomena are studied extensively for nonlinear dispersive equations with \( L^2 \)-critical nonlinearities; see [37, 3, 25].

(3) This version of type I condition (1.10) is introduced in [1]. It is implied from classical type I conditions such as

\[
|v(x, t)| \leq C(T_* - t)^{-\frac{1}{2}}, \quad |v(x, t)| \leq C|x|^{-1},
\]

or the uniform Morrey bound \( \sup_{t \in (0, T_*)} \|v(t)\|_{m, 1} \leq C < \infty \); see [45, 1] for details. In [1], a concentration of the \( L^3 \) norm at the singular point; i.e., \( \|v(t)\|_{L^3(B_{\sqrt{\frac{T_* - T}{S}}})} > \gamma_* \) with some \( \gamma_* > 0 \), \( S' = S'(M_*) > 0 \) is shown under the condition (1.10). From (1.11) and the Hölder inequality, we can also deduce the similar concentration estimate:

\[
\int_B \frac{|v(x, t)|^3}{\sqrt{\frac{T_* - t}{S}}} dx \geq \sqrt{\frac{S}{T_* - t}} \int_B \frac{|v(x, t)|^2}{\sqrt{\frac{T_* - t}{S}}} dx > \sqrt{S} \epsilon_*.
\]

### 1.3.3 Regularity of discretely self-similar solutions

As another application of our result, we study regularity of discretely self-similar solutions. A solution \( v \) defined in \( \mathbb{R}^3 \times (0, \infty) \) is called (forward) self-similar (SS) if \( v^\lambda(x, t) = v(x, t) \) for all \( \lambda > 0 \) and is discretely self-similar with factor \( \lambda \) (i.e. \( v \) is \( \lambda \)-DSS) if this scaling invariance holds for
a given \( \lambda > 1 \). Similarly, \( v_0 \) is self-similar (or \((-1)\)-homogeneous) if \( v_0(x) = \lambda v_0(\lambda x) \) for all \( \lambda > 0 \) or \( \lambda \)-DSS if this holds for a given \( \lambda > 1 \). In [20], Jia and Šverák constructed self-similar solutions for Hölder continuous (away from the origin) initial data. This result has been generalized in a number of directions [5, 6, 7, 8, 12, 27, 31, 43]; see also the survey [21]. Among other results, Chae and Wolf [12] constructed DSS solutions for any DSS data in \( L^2_{\text{loc}} \). Another construction was given by Bradshaw and Tsai [8], where the constructed solutions satisfy the local energy inequality. The following result shows regularity of DSS solutions for data in \( E^2 = C^\infty_0 \mathcal{L}^2_{\text{uloc}} \) provided \( \lambda \) is close to 1.

**Theorem 1.8.** (i) Let \( \lambda > 1 \) and \( v \) be a \( \lambda \)-DSS local energy solution of the Navier-Stokes equations (NS) with \( \lambda \)-DSS data \( v_0 \in E^2 \). Then

\[
\|v_0\|_{M^2,1} := \sup_{x_0 \in \mathbb{R}^3, r > 0} \left( \frac{1}{r} \int_{B_r(x_0)} |v_0(x)|^2 \, dx \right)^{\frac{1}{2}} < \infty
\]  

(1.12)

holds and there exists \( \mu \in (0, 1) \) such that

\[
\sup_{x_0 \in \mathbb{R}^3, r \in (0, |x_0|]} \frac{1}{r} \int_{B_r(x_0)} |v_0(x)|^2 \, dx \leq \epsilon_*. \tag{1.13}
\]

Moreover \( v \) is regular in the set

\[
\left\{ (x, t) \in \mathbb{R}^3 \times (0, \infty) : 0 < t \leq T_2(\|v_0\|_{M^2,1}) \mu^2 |x|^2 \right\}.
\]

(ii) For any \( \mu \in (0, 1) \), there exists \( \lambda_* = \lambda_*(\mu) \in (1, 2) \) such that if any \( \lambda \)-DSS data \( v_0 \in E^2 \) with factor \( \lambda \in (1, \lambda_*) \) satisfies (1.13), then the \( \lambda \)-DSS local energy solution \( v \) is regular in \( \mathbb{R}^3 \times (0, \infty) \) and

\[
|v(x, t)| \leq \frac{C}{\sqrt{t}} \quad \text{in } \mathbb{R}^3 \times (0, \infty)
\]

holds for a positive constant \( C = C(v_0) \).

**Remark 1.9.** The statement (ii) extends [22, Corollary 1.5 (ii)] on regularity of DSS solutions for data in \( L^{3,\infty}_\text{loc} \). We note that (1.12) was already proved in [9, Lemma 6.1], where it was shown that for \( \lambda \)-DSS data, (1.12) holds if and only if \( v_0 \in L^2_{\text{uloc}}(\mathbb{R}^3) \). On the other hand, (1.13) does not hold for general \( v_0 \in L^2_{\text{uloc}} \). See Remark 4.4 for details.

### 1.4 Outline of the paper and notation

In section 2, we introduce the notion of local energy weak solutions and recall the local regularity criterion due to [11] as well as some technical lemmas. Section 3 is devoted to stating and proving our main results including Theorem 1.1. In section 4, as applications, local regularity, regular set and a blow-up criterion are analyzed in terms of scaled local \( L^2 \) energy. Regularity of discretely self-similar solutions is also studied for locally \( L^2 \) data.

Throughout this paper, \( C \in (0, \infty) \) denotes an absolute constant which may change line by line.

### 2 Preliminaries

In this section, we recall some notions about the weak solution to (NS) and some results such as the \( \epsilon \)-regularity theorems and a priori estimates for the solutions.
For any domain \( \Omega \subset \mathbb{R}^3 \) and open interval \( I \subset (0, \infty) \), we say \((v, p)\) is a suitable weak solution in \( \Omega \times I \) if it satisfies (NS) in the sense of distributions in \( \Omega \times I \),

\[
v \in L^\infty(I; L^2(\Omega)) \cap L^2(I; H^1(\Omega)), \quad p \in L^{3/2}(\Omega \times I),
\]

and the local energy inequality:

\[
\int_\Omega |v(t)|^2 \phi(t) \, dx + 2 \int_0^t \int_\Omega |\nabla v|^2 \phi \, dx \, dt \\
\leq \int_0^t \int_\Omega |v|^2 (\partial_t \phi + \Delta \phi) \, dx \, dt + \int_0^t \int_\Omega (|v|^2 + 2p)(v \cdot \nabla \phi) \, dx \, dt
\]  

(2.1)

for all non-negative \( \phi \in C_0^\infty(\Omega \times I) \). Note that no boundary condition is required.

We next define the notion of local energy solutions. The following definition is formulated in [9], which is slightly revised from the notions of the local Leray solution defined in [29], the local energy solution in [26] and the Leray solution in [20]. We refer to [22, Section 2] for discussion of their relation.

**Definition 2.1** (Local energy solutions [9]). A vector field \( v \in L^2_{\text{loc}}(\mathbb{R}^3 \times [0, T)) \) is a local energy solution to (NS) with divergence free free initial data \( v_0 \in L^2_{\text{loc}} \) if

1. for some \( p \in L^3/2_{\text{loc}}(\mathbb{R}^3 \times [0, T)) \), the pair \((v, p)\) is a distributional solution to (NS),
2. for any \( R > 0 \),

\[
\operatorname{ess} \sup_{0 \leq t \leq \min\{R^2, T\}} \sup_{x_0 \in \mathbb{R}^3} \int_{B_R(x_0)} |v|^2 \, dx + \sup_{x_0 \in \mathbb{R}^3} \int_0^{\min\{R^2, T\}} \int_{B_R(x_0)} |\nabla v|^2 \, dx \, dt < \infty,
\]  

(2.2)

3. for all compact subsets \( K \) of \( \mathbb{R}^3 \) we have \( v(t) \to v_0 \) in \( L^2(K) \) as \( t \to 0^+ \),
4. \((v, p)\) satisfies the local energy inequality (2.1) for all non-negative functions \( \phi \in C_0^\infty(Q) \) with all cylinder \( Q \) compactly supported in \( \mathbb{R}^3 \times (0, T) \),
5. for every \( x_0 \in \mathbb{R}^3 \) and \( r > 0 \), there exists \( c_{x_0, r} \in L^3/2(0, T) \) such that

\[
p(x, t) - c_{x_0, r}(t) = \frac{1}{3} |v(x, t)|^2 + \int_{B_{2r}(x_0)} K(x - y) : v(y, t) \otimes v(y, t) \, dy
\]

\[
+ \int_{\mathbb{R}^3 \setminus B_{3r}(x_0)} (K(x - y) - K(x_0 - y)) : v(y, t) \otimes v(y, t) \, dy
\]  

(2.3)

in \( L^3/2(B_{2r}(x_0) \times (0, T)) \), where \( K(x) = \nabla^2 (\frac{1}{4\pi |x|}) \), and
6. for any compact supported functions \( w \in L^2(\mathbb{R}^3)^3 \),

\[
\text{the function } t \mapsto \int_{\mathbb{R}^3} v(x, t) \cdot w(x) \, dx \text{ is continuous on } [0, T).
\]  

(2.4)

The next lemma shows a priori bounds for the local energy solution, where the crucial part is proved by Lemarié-Rieusset [29] and later revised in [26, 19]. The following version is deduced from [22, 9].
Lemma 2.2 (a priori bound of local energy solutions). Suppose \((v, p)\) is a local energy solution to (NS) with divergence free initial data \(v_0 \in L^2_{uloc}\). For any \(s, q > 1\) with \(\frac{2}{s} + \frac{3}{q} = 3\), there exist \(C(s, q) > 0\) and \(c_{x_0}(t) \in \mathbb{R}^3\) such that

\[
E_{uloc}(t) := \text{ess sup}_{0 \leq s \leq t} \|v(t)\|^2_{L^2_{uloc}} + \sup_{x_0 \in \mathbb{R}^3} \int_0^t \int_{B_r(x_0)} |\nabla v|^2 \, dx \, dt \leq 2 \|v_0\|^2_{L^2_{uloc}},
\]

for \(t \leq T_{uloc} := \frac{c_0}{1 + \|v_0\|^2_{L^2_{uloc}}}\) with a universal constant \(c_0 > 0\). Similar estimates with \(B_1\) replaced by \(B_r\) are valid.

We now recall the scaled version of the \(\epsilon\)-regularity theorem of Caffarelli-Kohn-Nirenberg [11, Proposition 1]. It is formulated in the present form in [38, 34].

Lemma 2.3. There are absolute constants \(\epsilon_{CKN}\) and \(C_{CKN} > 0\) with the following property. Suppose \((v, p)\) is a suitable weak solution of (NS) in \(B_r(x_0) \times (t_0 - r^2, t_0)\), \(r > 0\), with

\[
\frac{1}{r^2} \int_{t_0 - r^2}^{t_0} \int_{B_r(x_0)} |v|^3 + |p|^{3/2} \, dx \, dt \leq \epsilon_{CKN},
\]

then \(v \in L^\infty(B_{2r}(x_0) \times (t_0 - \frac{r^2}{4}, t_0))\) and

\[
\|v\|_{L^\infty(B_{2r}(x_0) \times (t_0 - \frac{r^2}{4}, t_0))} \leq \frac{C_{CKN}}{r}.
\]

We recall a useful Gronwall-type inequality from Bradshaw and Tsai [9, Lemma 2.2].

Lemma 2.4. Suppose \(f \in L^\infty_{loc}([0, T_0); [0, \infty))\) (which may be discontinuous) satisfies, for some \(a, b > 0\), and \(m \geq 1\),

\[
f(t) \leq a + b \int_0^t (f(s) + f(s)^m) \, ds \quad \text{for } t \in (0, T_0),
\]

then we have \(f(t) \leq 2a\) for \(t \in (0, T)\) with

\[
T = \min \left( T_0, \frac{C}{b(1 + a^{m-1})} \right),
\]

where \(C\) is a universal constant.

Finally we also show the following elementary bound for the scaled energy.

Lemma 2.5. Assume that \(f \in L^2_{loc}(\mathbb{R}^3)\) and let \(N_R = N_R(f) := \sup_{R \leq r \leq 1} \frac{1}{r} \int_{B_r} |f|^2\) for some \(R \in (0, \frac{1}{2})\). If \(\delta \geq 2N_R\), then for any \(x_0 \in B_{\frac{1}{2}}\) we have

\[
\sup_{R(x_0) \leq r \leq 1 - |x_0|} \frac{1}{r} \int_{B_r(x_0)} |f|^2 \leq \delta,
\]

with \(R(x_0) = \max\{\frac{R}{2}, \frac{2N_R|x_0|}{8}\}\).
Proof. Assume that \( r \in [R(x_0), 1 - |x_0]| \). If \( 1/2 < r \leq 1 - |x_0| \),
\[
\frac{1}{r} \int_{B_r(x_0)} |f|^2 \leq \frac{1}{r} \int_{B_1(0)} |f|^2 \leq \frac{1}{r} N_R \leq \delta.
\]
If \( |x_0| \leq r \leq 1/2 \), (2.8) clearly holds by the following estimate:
\[
\frac{1}{r} \int_{B_r(x_0)} |f|^2 \leq \frac{1}{r} \int_{B_{2r}(0)} |f|^2 \leq 2N_R \leq \delta;
\]
where we have used \( |x_0| + r \leq 2r \). used \( |x_0| + r \leq 2r \). Finally, if \( R(x_0) \leq r < |x_0| \) (this case is empty if \( |x_0| < R/2 \leq R(x_0) \)), we have \( |x_0| + r < 2|x_0| \), and hence
\[
\frac{1}{r} \int_{B_r(x_0)} |f|^2 \leq \frac{1}{r} \int_{B_{2|x_0|}(0)} |f|^2 \leq \frac{2|x_0|}{r} N_R.
\]
The right hand side is bounded by \( \delta \) since \( r \geq \frac{2N_R|x_0|}{\delta} \). Therefore we have verified (2.8). \( \square \)

3 Main results

In this section we prove Theorem 1.1 regarding local regularity of suitable weak solutions. It is obtained as a consequence of the following theorem; see Remark 3.2 below.

**Theorem 3.1.** Let \((v, p)\) be a suitable weak solution in \(B_2 \times (0, T_0)\), \( T_0 > 0 \), associated with the initial data \( v_0 \) in the sense that \( \lim_{t \to 0^+} \|v(t) - v_0\|_{L^2(B_2)} = 0 \). There are absolute constants \( c, C \in (1, \infty) \) such that the following holds true.

(i) Let \( N_R = \sup_{R < \lambda \leq 1} \frac{1}{r} \int_{B_r} |v_0|^2 < \infty, \ R \geq 0 \). For any \( M \in (0, \infty) \) and \( \delta \in [5N_R, \infty) \), there exists \( T = T(M, \delta, T_0) \in (0, T_0]\) such that if
\[
|v|_{L^\infty(0,T;L^2(B_2))} + \|
abla v\|_{L^2(0,T;L^2(\mathbb{R}^N \setminus B_2))} + \|p\|_{L^2(0,T;L^2(\mathbb{R}^N \setminus B_2))} \leq M,
\]
then \( E_r(t) \) defined by (1.5) and \((v, p)\) satisfy
\[
E_r(t) \leq \delta \quad \text{for } t \in (0, \min\{\lambda r^2, T\}], \quad \text{for all } r \in [R, r_1]
\]
if \( R \leq r_1 \), and
\[
\frac{1}{r^2} \int_0^{\lambda r^2} \int_{B_r} |v|^3 + |p|^\frac{4}{3} \, dx \, dt \leq C(\delta + \delta^\frac{2}{3}) \quad \text{for all } r \in [R, r_2]
\]
if \( R \leq r_2 \), where \( T, \lambda, r_1, \) and \( r_2 \) are given by
\[
T = \min \left( T_0, \frac{c \min\{1, \delta^{12}\}}{1 + M^{18}} \right), \quad \lambda = \frac{c}{1 + \delta^2}, \quad r_1 = \min \left\{ \frac{c \delta}{M^2}, \frac{1}{3} \right\}, \quad r_2 := \min \left\{ \sqrt{\frac{T}{\lambda}}, r_1 \right\}.
\]
(ii) There exists \( \epsilon_* > 0 \) such that if \( N_R \leq \epsilon_* \) and \( R^2 \leq \tilde{T} \), then \( v \) is regular in the set
\[
\Pi = \left\{(x, t) \in B_{\frac{3}{2}} \times [R^2, \tilde{T}] : t \geq cN_R^2 |x|^2 \right\}
\]
and satisfies
\[
|v(x, t)| \leq \frac{C}{\sqrt{t}} \quad \text{for } (x, t) \in \Pi
\]
with \( \tilde{T} = \min\{T_0, \frac{c}{1 + M^{18}}\} \) and absolute constants \( c, C > 0 \).
Remark 3.2. (1) Theorem 3.1 sharpens and generalizes Theorem 1.1 in the following sense: (a) The assumption (3.1) is weaker than (1.3). More importantly, (b) it treats the case of general \( R \geq 0 \) so that the regular set \( \Pi \) is more specifically characterized. It should be emphasized that if \( R > 0 \) the assumption \( N_R \leq \epsilon \) in (ii) does not require scale critical regularity at the origin.

(2) In Part (i) (and also Theorem 3.4 (i)), \( \delta \) can be large, although it is small in the rest of the paper in particular (ii). We hope it might be useful for some other applications.

Proof. (i) For the convenience, we let

\[
\mathcal{E}_{R,r_1}(t) := \sup_{R \leq r \leq r_1} E_r(t)
\]

with the constant \( r_1 \) to be specified later. The local energy inequality (2.1) with a test function \( \varphi \in C^\infty_0(B_{2r}) \) such that \( 0 \leq \varphi \leq 1 \) in \( B_{2r} \) with \( \varphi = 1 \) in \( B_r \) and \( \| \nabla^k \varphi \|_{L^\infty} \leq C_k r^{-k} \) leads to\(^2\).

\[
\int |v(t)|^2 \varphi^2 dx + 2 \int_0^t \int |\nabla v|^2 \varphi^2 dx \leq \int |v_0|^2 \varphi^2 dx + \int_0^t \int |v|^2 \Delta(\varphi^2) + (|v|^2 + p) \cdot \nabla \varphi^2 dx ds
\]

\[
\leq \int_{B_{2r}} |v_0|^2 dx + C \int_0^t \int_{B_{2r}} |v|^2 + \frac{C}{r^2} \int_0^t \int_{B_{2r}} |v|^3 + |p|^\frac{3}{2} dx ds.
\]

This implies

\[
E_r(t) \leq 2N_R + \frac{C}{r^3} \int_0^t \int_{B_{2r}} |v|^2 + \frac{C}{r^2} \int_0^t \int_{B_{2r}} |v|^3 + \frac{C}{r^2} \int_0^t \int_{B_{2r}} |p|^\frac{3}{2}
\]

\[
=: 2N_R + I_{lin} + I_{nonlin} + I_{pr}.
\]

(3.4)

For any \( \rho \in (3r, 1] \) we decompose the pressure as \( p = \tilde{p} + p_h \) with

\[
\tilde{p}(x) := \text{p.v.} \int K(x - y) \xi(y)(v \otimes v)(y) dy - \frac{1}{3}(\xi|v|^2)(x),
\]

where \( \xi \) is a smooth cut-off function with \( \xi = 1 \) in \( B_\rho \) and supported in \( B_{2\rho} \). Since \( p_h \) is harmonic in \( B_\rho \),

\[
\int_{B_{2\rho}} |p|^{\frac{3}{2}} \leq C \int_{B_{2\rho}} |\tilde{p}|^{\frac{3}{2}} + C \int_{B_{2\rho}} |p_h|^{\frac{3}{2}}
\]

\[
\leq C \int_{B_{2\rho}} |\tilde{p}|^{\frac{3}{2}} + C \frac{r^3}{\rho^3} \int_{B_\rho} |p_h|^{\frac{3}{2}}
\]

\[
\leq C \int_{B_{2\rho}} |\tilde{p}|^{\frac{3}{2}} + C \frac{r^3}{\rho^3} \int_{B_\rho} |p|^{\frac{3}{2}} + C \frac{r^3}{\rho^3} \int_{B_\rho} |p|^{\frac{3}{2}}
\]

By the Calderón-Zygmund estimate we see

\[
\int_{B_{2\rho}} |p|^{\frac{3}{2}} \leq C \int_{B_{2\rho}} |v|^3 + C \frac{r^3}{\rho^3} \int_{B_\rho} |p|^{\frac{3}{2}}.
\]

(3.5)

We now divide the proof into two cases. Let \( r_0 \) be a constant satisfying \( 0 \leq r_0 \leq r_1/3 \) to be fixed later.

\(^2\)It is not difficult to see that we may take time-independent test functions in the local energy inequality provided the solution is continuous at \( t = 0 \) in \( L^2_{loc} \). See, e.g., [35, Remark 1.2].
**Case I**: $R \leq r \leq r_0$. If $r_0 < R < r_1$, this case is empty, which is fine. Noting that $2r \in [R, r_1]$, we easily observe that

$$I_{\text{lin}} \leq \frac{C}{r^2} \int_0^t \mathcal{E}_{R, r_1}(s) ds,$$

and also by the interpolation inequality,

$$I_{\text{nolin}} \leq \frac{C}{r^2} \int_0^t \left( \int_{B_{2r}} |\nabla v|^2 \right)^\frac{3}{4} \left( \int_{B_{2r}} |v|^2 \right)^\frac{1}{4} ds + \frac{C}{r^2} \int_0^t \left( \frac{1}{r} \int_{B_{2r}} |v|^2 \right)^\frac{3}{2} ds
\leq \frac{\epsilon}{r} \int_0^t \int_{B_{2r}} |\nabla v|^2 ds + \frac{C}{r^2} \int_0^t \left( \frac{1}{r} \int_{B_{2r}} |v|^2 \right)^\frac{3}{2} ds
\leq \epsilon \mathcal{E}_{R, r_1}(t) + \frac{C}{r^2} \int_0^t \mathcal{E}_{R, r_1}^3(s) + \mathcal{E}_{R, r_1} \frac{3}{2} ds + \frac{1}{10} \mathcal{E}_{R, r_1}(t).$$

(3.7)

with some constant $\epsilon \in (0, 1)$. For the pressure term, integrating (3.5) in time yields

$$I_{\text{pr}} \leq \frac{C}{r^2} \int_0^t \int_{B_{2r}} |v|^3 + \frac{C'}{\rho} \int_0^t \int_{B_{2r}} |p| \frac{3}{2} ds$$

with an absolute constant $C' > 1$. Choose $\rho = 10C'$ so that $\frac{C'}{\rho} \leq \frac{1}{10}$ and also let $r_0 = \frac{r}{20C'}$. Noting that $2\rho = 20C' \leq r_1$ and (3.7), we see that

$$I_{\text{pr}} \leq \frac{C}{r^2} \int_0^t \int_{B_{20C'}} |v|^3 + \frac{1}{10} \mathcal{E}_{R, r_1}(t)
\leq \epsilon \mathcal{E}_{R, r_1}(t) + \frac{C}{r^2} \int_0^t \mathcal{E}_{R, r_1}^3(s) + \mathcal{E}_{R, r_1} \frac{3}{2} ds + \frac{1}{10} \mathcal{E}_{R, r_1}(t).$$

(3.8)

Hence applying (3.6)-(3.8) in (3.4) with $\epsilon = \frac{1}{20}$, we obtain

$$\mathcal{E}_{R, r_0}(t) \leq \frac{2\delta}{5} + \frac{C}{r^2} \int_0^t \mathcal{E}_{R, r_1}^3(s) + \mathcal{E}_{R, r_1}(s) ds + \frac{1}{5} \mathcal{E}_{R, r_1}(t).$$

(3.9)

**Case II**: $r_0 \leq r \leq r_1$. For $t \leq T$ with $T$ specified later, we estimate the right hand side of (3.4) with the aid of $M$. A straightforward estimate yields

$$I_{\text{lin}} \leq \frac{C}{r_0^2} \int_0^t \int_{B_2} |u|^2 \leq \frac{Ct}{r_0^2} M \leq \frac{\delta}{10},$$

provided $t \leq \frac{c^2 r_0^3}{M}$ with a small absolute constant $c \in (0, 1)$. In the similar way to (3.7) we have

$$I_{\text{nolin}} \leq \frac{C}{r_0^2} \int_0^t \int_{B_2} |v|^3
\leq \frac{C}{r_0^2} \int_0^t \left( \int_{B_2} |\nabla u|^2 \right)^\frac{3}{2} \left( \int_{B_2} |u|^2 \right)^\frac{1}{2} ds + \frac{C}{r_0^2} \int_0^t \left( \frac{1}{r_0} \int_{B_2} |v|^2 \right)^\frac{3}{2} ds
\leq \frac{Ct M^\frac{3}{2}}{r_0^2} + \frac{Ct M^\frac{3}{2}}{r_0^2}.$$

(3.10)
Thus $I_{nonlin}$ is bounded by $\frac{\delta}{M}$ if $t \leq \min\{\frac{c\delta r^8}{M^6}, \frac{c\delta r^7}{M^7}\}$ with a suitable absolute constant $c \in (0, 1)$. Concerning the pressure, we choose $\rho = 1$ (using $3r_1 \leq 1 = \rho$) in (3.5) and apply (3.10) to get

$$I_{pr} \leq \frac{C}{r^2} \int_0^t \int_{B_{2r}} \left| \tilde{p} \right|^\frac{3}{2} + Cr \int_0^t \int_{B_1} |p|^{\frac{7}{2}}$$

$$\leq \frac{C}{r^2} \int_0^t \int_{B_1} \left| \tilde{p} \right|^\frac{3}{2} + Cr_1 M^{\frac{7}{2}}$$

$$\leq \frac{C}{r^2} \int_0^t \int_{B_2} \left| v \right|^3 + Cr_1 M^{\frac{7}{2}}$$

$$\leq \frac{C t^{\frac{3}{2}} M^{\frac{7}{2}}}{r^2} + \frac{C t M^{\frac{5}{2}}}{r^2} + Cr_1 M^{\frac{7}{2}} \leq \frac{\delta}{10},$$

(3.11)

provided $t \leq \min\{\frac{c\delta r^8}{M^6}, \frac{c\delta r^7}{M^7}\}$ and $r_1 \leq \min\{\frac{c\delta}{M^\frac{7}{2}}, \frac{1}{3}\}$ with an absolute constant $c > 0$. Making use of these estimates in (3.4), we obtain that

$$\mathcal{E}_{r_0, r_1}(t) \leq \frac{\delta}{2} \quad \text{if} \quad t \leq \min\left\{\frac{c\delta r_0^3}{M}, \frac{c\delta r_0^8}{M^6}, \frac{c\delta r_1^7}{M^7}\right\} \quad \text{and} \quad r_1 \leq \min\left\{\frac{c\delta}{M^\frac{7}{2}}, \frac{1}{3}\right\}. \quad (3.12)$$

Combining (3.9) and (3.12), we see

$$\mathcal{E}_{r, r_1}(t) \leq \frac{\delta}{2} + \frac{C}{R^2} \int_0^t \mathcal{E}_{r, r_1}(s) + \mathcal{E}_{R, r_1}(s) ds, \quad 0 < t < T,$$

(3.13)

where

$$T = \min\left(T_0, \frac{c \min\{1, \delta^{12}\}}{1 + M^{18}}\right). \quad (3.14)$$

Note that $T \leq \min\{\frac{c\delta r_0^3}{M}, \frac{c\delta r_0^8}{M^6}, \frac{c\delta r_1^7}{M^7}\}$ with a suitable small constant $c > 0$ upon the choice of $r_0 = \frac{\sqrt{c}}{20C}$ with $r_1 = \min\{\frac{c\delta}{M^\frac{7}{2}}, \frac{1}{3}\}$. The inequality (3.13) is also true if $R$ is replaced by any $r \in [R, r_1]$ with the same $\delta$. Therefore we may invoke Gronwall Lemma 2.4 with $f(t) = \mathcal{E}_{r, r_1}(t)$ with $a = \frac{\delta}{2}$, $b = \frac{C}{R^2}$, and $m = 3$ to see that

$$\mathcal{E}_{r, r_1}(t) \leq \delta \quad \text{for} \quad t \in (0, \min\{T, \lambda r^2\}] \quad \text{and} \quad r \in [R, r_1], \quad \lambda = \frac{c}{1 + \delta^2}. \quad (3.15)$$

This proves (3.2).

In the similar way to (3.7), we see that if $\lambda r^2 \in (0, T]$ and $r \in [R, r_1],$

$$\frac{1}{r^2} \int_0^{\lambda r^2} \int_{B_r} \left| v \right|^3 \leq \frac{C}{r^2} \int_0^{\lambda r^2} \left( \int_{B_r} |\nabla v|^2 \right)^{\frac{3}{4}} \left( \int_{B_r} \left| v \right|^2 \right)^{\frac{3}{4}} ds + \frac{C}{r^2} \int_0^{\lambda r^2} \left( \frac{1}{r} \int_{B_r} \left| u \right|^2 \right)^{\frac{3}{2}} ds$$

$$\leq C\left(\lambda^{\frac{3}{4}} + \lambda\right) E_r(\lambda r^2)^{\frac{3}{2}}.$$

Hence taking $r_2 := \min\{\sqrt{\frac{T}{\lambda}}, r_1\}$, we have from (3.2) and $\lambda \leq 1$ that

$$\frac{1}{r^2} \int_0^{\lambda r^2} \int_{B_r} \left| v \right|^3 \leq C\delta^{\frac{3}{2}} \quad \text{for} \quad r \in [R, r_2].$$
This together with the pressure bound

\[ \frac{1}{r^2} \int_0^{\lambda r'^2} \int_{B_r} |p|^{\frac{3}{2}} \leq E_r(\lambda r'^2) \leq C\delta \]

leads to (3.3) as desired.

(ii) In order to show the statement (ii), we claim that there exist \( \epsilon_* \) and \( c > 0 \) such that if \( N_R < \epsilon_* \), then for any \( x_0 \in B_{\frac{3}{2}} \) and \( r \in \max\{R, cN_R|x_0|, c(1 - |x_0|)r_2\} \),

\[ \frac{1}{r^2} \int_0^{\lambda r'^2} \int_{B_r(x_0)} |v|^3 + |p|^{\frac{3}{2}} \, dx \, dt \leq \epsilon_{CKN} \tag{3.16} \]

holds. Here \( \epsilon_{CKN} \) is the small constant in Lemma 2.3. To this end, we first note that for each \( \eta \geq 2N_R \) and \( x_0 \in B_{1/2} \), Lemma 2.5 implies

\[ \sup_{R(x_0) \leq r \leq \rho} \frac{1}{r} \int_{B_r(x_0)} |v_0|^2 \, dx \leq \eta, \quad \rho = 1 - |x_0|, \]

with \( R(x_0) = \max\{R/2, \frac{2N_R}{\eta} |x_0|\} \). Let \( v_{x_0}(x, t) = \rho v(x_0 + \rho x, \rho^2 t) \), \( p_{x_0}(x, t) = \rho^2 p(x_0 + \rho x, \rho^2 t) \) and \( \delta = 5\eta \). Since \( (v_{x_0}, p_{x_0}) \) solves (NS) in \( B_2(0) \times (0, \rho^{-2}T_0) \), corresponding to \((v, p)\) in \( B_{2\rho}(x_0) \times (0, T_0) \), and \( 1/2 \leq \rho \leq 1 \),

\[ \|v_{x_0}\|_L^\infty(0, \rho^{-2}T_0; L^2(B_2)) + \|v_{x_0}\|_L^2(B_2 \times (0, \rho^{-2}T_0)) + \|p_{x_0}\|_L^\frac{2}{3}(B_2 \times (0, \rho^{-2}T_0)) \leq CM, \]

\[ \sup_{\rho^{-1}R(x_0) \leq r \leq 1} \frac{1}{r} \int_{B_r(0)} |v_{x_0}(x, 0)|^2 \, dx = \sup_{R(x_0) \leq r \leq \rho} \frac{1}{r} \int_{B_r(x_0)} |v_0|^2 \, dx \leq \frac{\delta^3}{\lambda}, \]

by (3.3) and (3.14) we get

\[ \sup_{\rho^{-1}R(x_0) \leq r \leq r_2} \frac{1}{r^2} \int_0^{\lambda r'^2} \int_{B_r(0)} |v_{x_0}|^3 + |p_{x_0}|^{\frac{3}{2}} \leq C(\delta + \delta^\frac{3}{2}). \]

Here \( r_2 = \min\left(\sqrt{\frac{T'}{\lambda}}, r_1\right) \), with

\[ T' = \min\left(\rho^{-2}T_0, \frac{c \min\{1, \delta^{12}\}}{1 + M^{18}}\right), \quad r_1 = \min\left(\frac{c\delta}{M^{3}}, \frac{1}{3}\right), \]

with a smaller constant \( c \). Note \( T' \) differs from \( T \) in (3.14) by the factor \( \rho^{-2} \) for \( T_0 \). This implies

\[ \sup_{R(x_0) \leq r \leq \rho r_2} \frac{1}{r^2} \int_0^{\lambda r'^2} \int_{B_{\sqrt{T'}}(x_0)} |v|^3 + |p|^{\frac{3}{2}} \leq \frac{C(\delta + \delta^\frac{3}{2})}{\lambda} \leq C(1 + \delta^2)(\delta + \delta^\frac{3}{2}). \tag{3.17} \]

Take a constant \( \delta_0 > 0 \) so small that \( C(1 + \delta_0^2)(\delta_0 + \delta_0^\frac{3}{2}) \leq \epsilon_{CKN} \). We now assume that \( v_0 \) satisfies \( N_R \leq \delta_0/10 \). Then we may choose \( \delta = \delta_0 \) since \( \delta_0 \geq 10N_R \). With this choice and with \( \lambda_0 = \lambda(\delta_0) \), (3.17) shows (3.3) holds for \( \lambda_0 \frac{3}{2}R(x_0) \leq r \leq \lambda_0 \frac{3}{2}r_2 \). This enables us to apply Lemma 2.3 for \( x_0 \in B_{\frac{3}{2}} \) and \( t_0 = r^2 \in [\lambda_0 \max\{R^2, \frac{CN_R^2}{\delta_0^2}\}, \lambda_0 \rho^2 r_2^2] \) to see

\[ |v(x_0, t_0)| \leq \frac{C_{CKN}}{r} = \frac{C_{CKN}}{\sqrt{t_0}}. \]
and hence $v$ is regular at $(x_0, t_0)$. Since $r_2 = \min\{\sqrt{\frac{L}{\lambda_0}}, r_1\}$ and $1/2 \leq \rho \leq 1$,

$$\rho^2 \lambda_0 r_2^2 = \rho^2 \min(T', \lambda_0 r_1^2) = \rho^2 \min\left(\rho^{-2} T_0, \frac{c \min\{1, \delta_0^2\}}{1 + M^{18}}, \lambda_0 r_1^2\right) \geq \min\left(T_0, \frac{c}{1 + M^{18}}\right).$$

Thus $|v(x_0, t_0)| \leq C_{\text{CKN}t_0^{-1/2}}$ for $x_0 \in B_{1/2}$ and

$$\max\{R^2, c N_R^2 |x_0|^2\} \leq t_0 \leq \min\left(T_0, \frac{c}{1 + M^{18}}\right).$$

This shows Part (ii) of Theorem 3.1.

**Remark 3.3.** If we assume slightly stronger regularity for the pressure as $p \in L^2(0, 4; L^2(B_2))$ and

$$\|v\|_{L^\infty(0, T; L^2(B_2))} \cap L^2(0, T; H^1(B_2)) + \|p\|_{L^2(0, T; L^2(B_2))} \leq L$$

for some $L > 0$ instead of (3.1), then we have a bound for $T > 0$ in terms of $L$: Inserting the estimate

$$\int_0^t \int_{B_1} |p| d^t L^\frac{2}{3}$$

in (3.11) we have

$$I_{pr} \leq \frac{\delta}{10}$$

for $t \leq c \delta L^6$ and $r_1 = \frac{1}{3}$ with a suitable constant $c > 0$. Inspecting the proof we easily verify we can replace $T = T(M, \delta)$ by $T' = T'(L, \delta) = \frac{\min\{1, \delta^2\}}{1 + L^2}$. Note that $T'(L, \delta)$ is bigger than $T(M, \delta)$ for sufficiently large $L = M$. This will be used in the next theorem.

One implication of Theorem 3.1 is that for $L^2_{\text{uloc}}$ initial data in $\mathbb{R}^3$, similar conclusions hold for the local energy solutions:

**Theorem 3.4.** Let $v$ be a local energy solution of (NS) in $\mathbb{R}^3 \times (0, T_0)$ associated with initial data $v_0 \in L^2_{\text{uloc}}, N_R = \sup_{R < r_1} \frac{1}{r} \int_{B_r(0)} |v_0|^2 < \infty$. The following hold true.

(i) For $R \in [0, \frac{1}{3}]$, let $\delta \geq 5N_R$. Then

$$E_r(t) \leq \delta$$

for $t \leq \min\{\lambda r^2, T_1\}$, for all $r \in [R, \frac{1}{3}]$, \hspace{1cm} (3.19)

holds with constants given by

$$\lambda = \frac{c}{1 + \delta^2} \leq 1, \hspace{1cm} T_1 = \min\left(T_0, \frac{c \min\{1, \delta^2\}}{1 + \|v_0\|_{L^2_{\text{uloc}}}^2}\right).$$

Moreover for any $r \in [R, R_1]$ with $R_1 := \min\{\sqrt{\frac{r_2}{\lambda}}, \frac{1}{3}\}$, there exists $c_2(t)$ such that

$$\frac{1}{r^2} \int_0^{r^2} \int_{B_r} |v|^2 + |p - c_2(t)||^2 dx dt \leq C(\delta + \delta^2).$$

(3.20)

Here $c$ and $C > 0$ are absolute constants.

(ii) There are absolute constants $\epsilon_*$, $c_0$, $c_1$, and $C_2 > 0$ such that the following holds. If $N_R \leq \epsilon_*$ for some $R \leq \min\{\sqrt{\frac{r_2}{\lambda}}, \frac{1}{3}\}$ with $T_2 = \min\{T_0, \frac{c_0}{1 + \|v_0\|_{L^2_{\text{uloc}}}^2}\}$, then $v$ is regular in the set

$$\Pi := \left\{(x, t) \in B_{1/2} \times [R^2, T_2]; \ c_0 N_R^2 |x|^2 \leq t\right\}$$

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and satisfies

\[ |v(x, t)| \leq \frac{C_2}{\sqrt{t}} \quad \text{for} \quad (x, t) \in \Pi. \]  

(3.21)

**Proof.** By Lemma 2.2 with \((s, q) = (2, 3/2)\),

\[
\begin{aligned}
\sup_{0 \leq t \leq T_{uloc}} \int_{B_2} |v|^2 \, dx + \int_0^{T_{uloc}} \int_{B_2} |\nabla v|^2 \, dx \, dt &\leq 2 \|v_0\|^2_{L_{uloc}^2}, \\
\|p - c_2(t)\|_{L^2(0, T_{uloc}; L^2(B_2))} &\leq C(2, 3/2)\|v_0\|^2_{L_{uloc}^2}.
\end{aligned}
\]

The pair \((v, p - c_2(t))\) is a suitable weak solution in \(B_2 \times (0, \min(T_0, T_{uloc}))\) and it satisfies the assumptions in (3.18) with \(L = (2 + C(2, 3/2))\|v_0\|^2_{L_{uloc}^2}\). Hence from Remark 3.3, we see (3.2) holds with \(T\) replaced by

\[
T' = \min \left\{ T_0, \frac{c \min\{1, \delta^4\}}{1 + \|v_0\|^2_{L_{uloc}^2}} \right\} \leq \min \left\{ T_0, T_{uloc}, \frac{c \min\{1, \delta^4\}}{1 + L^3} \right\},
\]

with \(c > 0\) sufficiently small. Thus we obtain (3.19) as desired. We omit the proof of the remaining claim, since it can be proved in exactly the same way as in Theorem 3.1. \(\square\)

**Remark 3.5.** In the appendix, we provide an alternative proof of Theorem 3.4 based on the pressure decomposition (2.3) instead of the observation in Remark 3.3, which seems to be of independent interest.

### 4 Applications

#### 4.1 Regular set

We now show Theorem 4.1, which contains Theorem 1.3 as a special case and is useful for further applications. To this end, define \(\hat{N}_R\) by

\[
\hat{N}_R := \sup_{r > R} \frac{1}{r} \int_{B_r(0)} |v_0|^2 \quad (R \geq 0).
\]

**Theorem 4.1.** Let \((v, p)\) be a local energy solution in \(\mathbb{R}^3 \times (0, \infty)\) for the initial data \(v_0 \in L^2_{uloc}(\mathbb{R}^3)\). Let \(\epsilon_*\) and \(c_0\) be the absolute constants in Theorem 3.4 (ii). The following statements hold:

(i) If \(v_0\) satisfies (1.7), i.e.,

\[
M_1 := \sup_{x_0 \in \mathbb{R}^3} \sup_{r \geq 1} \frac{1}{r} \int_{B_r(x_0)} |v_0|^2 \, dx < \infty,
\]

and if

\[
\hat{N}_R \leq \epsilon_* \quad \text{for some} \quad R \geq 0, \tag{4.1}
\]

then \(v\) is regular in the set \(\left\{ (x, t) \in \mathbb{R}^3 \times (0, \infty) : \max\{R^2, c_0 \hat{N}_R^2 |x|^2\} \leq t \right\}\).

(ii) Suppose \(v_0 \in L^2(\mathbb{R}^3)\). For any \(0 < \delta \leq \epsilon_*\), there exist positive constant \(T(v_0, \delta)\) such that \(v\) is regular in the set

\[
\left\{ (x, t) \in \mathbb{R}^3 \times (0, \infty) : c_0 \delta^2 |x|^2 \leq t \leq T(v_0, \delta) \right\}. \tag{4.2}
\]
If \( v_0 \in L^{2,-1}(\mathbb{R}^3) \) also satisfies (1.7), then for any \( \delta \in (0, \epsilon_*] \), there is \( T'(v_0, \delta) \) such that \( v \) is regular in
\[
\{ (x, t) \in \mathbb{R}^3 \times (0, \infty) : \max(c_0 \delta^2|x|^2, T'(v_0, \delta)) \leq t \}.
\] (4.3)
It is also regular in
\[
\{ (x, t) \in \mathbb{R}^3 \times (0, \infty) : c_0 \epsilon_*^2|x|^2 \leq t < r_*^2 T_2(M_2) \},
\] (4.4)
with \( r_* := \sup\{ r > 0 \mid \|v_0\|_{L^{2,-1}(B_r)} \leq \epsilon_* \}, \) \( 0 < r_* \leq \infty, \) and \( M_2 = \left[ \max(1, \frac{1}{r})M_1 \right]^{1/2} \). When \( r_* = \infty, \) i.e., \( \|v_0\|_{L^{2,-1}(\mathbb{R}^3)} \leq \epsilon_* \), the regular set (4.4) has no time upper bound.

Theorem 4.1 is more general than Theorem 1.3 since it assumes \( \hat{N}_R < \epsilon_* \) for some \( R \geq 0 \) while Theorem 1.3 assumes \( \hat{N}_0 < \epsilon_* \). The regular set in Part (i) does not depend on the value of \( M_1 \), although it needs \( M_1 < \infty \). If \( v_0 \in L^{2,-1}(\mathbb{R}^3) \) also satisfies (1.7), \( v \) is regular in the union of the two sets (4.2) and (4.3). These sets are inside the parabola \( c_0 \delta^2|x|^2 \leq t \) with arbitrarily small \( \delta > 0 \). The set (4.3) is similar to the eventual regularity result given in [4, Theorem 1.4], which contains an additional a priori bound assumption on the solution, available for solutions constructed in [4] for \( v_0 \) in certain Morrey type space. That bound plays a similar role as the assumption (1.7) for \( v_0 \).

**Proof.** (i) We first consider the case \( R > 0 \). Let \( u_0(x) = \lambda v_0(\lambda x), u(x,t) = \lambda v(\lambda x, \lambda^2 t) \) for \( \lambda > 2R \). By the assumption, we have
\[
\sup_{\frac{R}{2} \leq r < 1} \frac{1}{r} \int_{B_r} |u_0|^2 \leq \hat{N}_R \leq \epsilon_*,
\]
and
\[
\|u_0\|_{L^2_{\text{uloc}}} = \sup_{x_0 \in \mathbb{R}^3} \left( \frac{1}{\lambda} \int_{|x-x_0|<\lambda} |v_0(x)|^2 dx \right)^{1/2} \leq \max \left( \frac{1}{R} \|v_0\|_{L^2_{\text{uloc}}}^2, M_1 \right)^{1/2} =: C_R,
\] (4.5)
with \( C_R > 0 \) independent of \( \lambda \). By Theorem 3.4 (ii), there exists \( T'_2 = T'_2(C_R) \) independent of \( \lambda \) such that \( u \) is regular if \( \max\{R^2/\lambda^2, c_0 \hat{N}_{R}^2 |x|^2 \} \leq t \leq T'_2 \) for \( \lambda \geq \frac{R}{\sqrt{T'_2}} \). Scaling back we see that \( v \) is regular if \( \max\{R^2, c_0 \hat{N}_{R}^2 |x|^2 \} \leq t \leq \lambda^2 T'_2 \). Since \( \lambda > \max\{2R, \frac{R}{\sqrt{T'_2}} \} \) is arbitrary, \( v \) is regular in the set \( \{ (x,t) : \max\{R^2, c_0 \hat{N}_{R}^2 |x|^2 \} \leq t \} \). This proves the case \( R > 0 \). For the case \( R = 0 \), by \( \hat{N}_R \leq \hat{N}_0 \), the above argument shows \( v \) is regular in \( \{ (x,t) : \max\{r^2, c_0 \hat{N}_{R}^2 |x|^2 \} \leq t \} \). Since \( r > 0 \) is arbitrary, we conclude the proof.

(ii) Suppose now \( v_0 \in L^{2,-1}(\mathbb{R}^3) \). For any \( \delta \in (0, \epsilon_*] \), there exists \( R_0 > 0 \) such that
\[
\sup_{0 < r \leq R_0} \frac{1}{r} \int_{B_r} |v_0|^2 dx \leq \int_{B_{R_0}} \frac{|v_0|^2}{|x|} dx \leq \delta.
\]
Let \( u_0(x) = \lambda v_0(\lambda x) \) and \( u(x,t) = \lambda v(\lambda x, \lambda^2 t) \) with \( \lambda = R_0 \). We easily see \( u_0 \in L^2_{\text{uloc}} \) and
\[
\sup_{0 < r \leq \frac{1}{\lambda}} \frac{1}{r} \int_{B_r} |u_0|^2 dx \leq \delta \leq \epsilon_*.
\]
By Theorem 3.4 (ii), \( u \) is regular if \( c_0 \delta^2|x|^2 \leq t \leq T_2(u_0) \). Hence \( v \) is regular in the set \( \{ (x,t) : \max\{r^2, c_0 \hat{N}_{R}^2 |x|^2 \} \leq t \leq R^2_0 T_2(u_0) \} \). Note \( T_2(u_0) \) depends on \( R_0 \) and \( \|v_0\|_{L^2_{\text{uloc}}} \) and goes to zero rapidly as \( R_0 \to 0 \).

Suppose now \( v_0 \in L^{2,-1} \) also satisfies (1.7). There is \( \rho > 0 \) such that \( \int_{\mathbb{R}^3 \setminus B_\rho} \frac{1}{|x|} |v_0|^2 \leq \delta/2 \). Let \( R_1 = \max(\rho, \frac{2}{3} \int_{B_\rho} |v_0|^2) \). For any \( r \geq R_1 \), we have
\[
\frac{1}{r} \int_{B_r} |v_0|^2 \leq \frac{1}{r} \int_{B_\rho} |v_0|^2 + \int_{B_r \setminus B_\rho} \frac{1}{|x|} |v_0|^2 \leq \frac{\delta}{2} + \frac{\delta}{2}.
\]
Thus $\tilde{N}_{R_t} \leq \delta$. By Part (i), $v$ is regular in the set \(\{\max(R_t^2, c_0\delta^2|x|^2) \leq t\}\).

The remaining statement follows by choosing \(\delta = \epsilon_*, \lambda = R_0 \to r_*, \) and noting \(|u_0|_{L^2_{uloc}} \leq M_2\) using (4.5).

**Remark 4.2.** By a standard localization argument and using the a priori estimate derived in this theorem, it is not difficult to construct a suitable weak solution (but not necessarily a local energy solution) satisfying the conclusion in (i) for the data satisfying (4.1) without assuming neither conditions \(v_0 \in L^2_{uloc}\) nor (1.7). However, since such a result is already studied in [4] by a slightly different approach, we do not discuss this fact in details here.

**Remark 4.3** (Initial regular set for general data). Following [22, Comment 4 after Corollary 1.2], define
\[
\rho(x) = \rho(x; v_0) := \sup \left\{ r > 0 : v_0 \in L^2(B_r(x)), \frac{1}{r} \int_{B_r(x)} |v_0|^2 \leq \epsilon_* \right\}.
\]

Let \(\rho(x) = 0\) if such \(r\) does not exist, and let \(\rho(x) = \infty\) if \(\sup_{r>0} \frac{1}{r} \int_{B_r(x)} |v_0|^2 \leq \epsilon_*\). We also set
\[
T(x) = \sup_{\rho < \rho(x)} \frac{c_2 \rho^2}{1 + \rho^{-6} \|v_0\|_{L^2_{uloc}}^2} \in [0, \infty] \quad \text{with} \quad \|v_0\|_{L^2_{uloc}} := \sup_{x_0 \in \mathbb{R}^3} \|v_0\|_{L^2(B_0(x_0))}.
\]

For each \(x_0 \in \mathbb{R}^3\), applying Theorem 3.4 (ii) to
\[
u(y, t) = \lambda v(\lambda y + x_0, \lambda^2 t)
\]
with \(\lambda = \rho(x_0)\) we see that any local energy solution \(v\) is regular in the set
\[
\{(x, t) \in \mathbb{R}^3 \times (0, \infty) : 0 < t < T(x)\}.
\]

This improves [22, Comment 4 after Corollary 1.2] because we replace local \(L^3\) norm by local Morrey-type norm in the definition of \(\rho(x)\).

We next apply Theorem 4.1 to prove Corollary 1.4.

**Proof of Corollary 1.4.** (i) By the assumptions, we have \(v_0 \in L^2(\mathbb{R}^3)\), and hence it satisfies (1.7) since
\[
\sup_{x_0 \in \mathbb{R}^3} \frac{1}{r} \int_{B_r(x_0)} |v_0|^2 \leq \frac{1}{r} \|v_0\|_{L^2}^2.
\]

This estimate also implies
\[
\sup_{x_0 \in \mathbb{R}^3, r \geq R_*} \frac{1}{r} \int_{B_r(x_0)} |v_0|^2 \leq \epsilon_* \quad \text{with} \quad R_* := \frac{\|v_0\|_{L^2}}{\epsilon_*}.
\]

Therefore applying Theorem 4.1 (i) to \(v_{x_0}(x, t) := v(x + x_0, t)\) for each \(x_0 \in \mathbb{R}^3\), we see
\[
\{(x, t) \in \mathbb{R}^3 \times (0, \infty) : \max\{R_*^2, c_0 \tilde{N}_{R_*}^2 \} \leq t\}
\]
is the regular set of \(v\). Since \(x_0 \in \mathbb{R}^3\) is arbitrary, this shows that \(v\) is regular for \(t \geq R_*^2 = \frac{\|v_0\|_{L^2}}{\epsilon_*}^2\).

We next show that there exists \(M = M(\|v_0\|_{L^2})\) such that if \(|x_0| \geq 2R_*\),
\[
\sup_{M|x_0|^{-\alpha} \leq r \leq R_*} \frac{1}{r} \int_{B_r(x_0)} |v_0|^2 \leq \epsilon_*.
\]

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Indeed since $r \leq R_* \leq |x_0|/2$, we see
\[
\frac{1}{r} \int_{B_r(x_0)} |v_0|^2 = \frac{1}{r} \int_{B_r(x_0)} \frac{|x|^\alpha |v_0|^2}{|x|^\alpha} \leq \frac{2 \alpha}{r|x_0|^\alpha} \|v_0\|_{L^{2,\alpha}}^2,
\]
from which (4.7) follows with $M = \frac{2 \alpha \|v_0\|_{L^{2,\alpha}}^2}{\epsilon_*}$. Combining this with (4.6) implies
\[
\sup_{M|x_0|^{-\alpha} \leq r \leq \epsilon_* r} \int_{B_r(x_0)} |v_0|^2 \leq \epsilon_* \quad \text{provided } |x_0| \geq 2 R_*. 
\]
Then if $|x_0| \geq 2 R_*$, we may apply Theorem 4.1 (i) for $v_{x_0}$ with $R = M|x_0|^{-\alpha}$, to see that $v$ is regular at $(x_0, t)$ for $t \geq M^2|x_0|^{-2\alpha}$. Since $v$ is also regular for $t \geq R_*^2 = \|v_0\|_{L^{2,\alpha}}^4/\epsilon_*^2$, this finishes the proof of (i) of Corollary 1.4, with $K = \max(M^2, 4^\alpha R_*^{2+2\alpha})$. Note that $K|x|^{-2\alpha} \geq R_*^2$ when $|x| \leq 2 R_*$.

(ii) Now $\alpha \in (-1, 0)$. We use similar approach as above: If $r \geq |x_0|/2$,
\[
\frac{1}{r} \int_{B_r(x_0)} |v_0|^2 \leq \frac{1}{r} \int_{B_{3r}} |v_0|^2 = \frac{1}{r} \int_{B_{3r}} \frac{|x|^\alpha |v_0|^2}{|x|^\alpha} \leq \frac{C_1}{r^{1+\alpha}} \|v_0\|_{L^{2,\alpha}}^2.
\]
Hence we have
\[
\frac{1}{r} \int_{B_r(x_0)} |v_0|^2 \leq \epsilon_* \quad \text{if } r \geq \max \left\{ \frac{|x_0|}{2}, \left( \frac{C_1 \|v_0\|_{L^{2,\alpha}}^2}{\epsilon_*} \right)^{\frac{1}{1+\alpha}} \right\}. 
\tag{4.8}
\]
Therefore, by virtue of (1.7), we may use Theorem 4.1 (i) to see that
\[
v \text{ is regular at } (x_0, t) \quad \text{if } t \geq \max \left\{ \frac{|x_0|^2}{4}, \left( \frac{C_1 \|v_0\|_{L^{2,\alpha}}^2}{\epsilon_*} \right)^{\frac{2}{1+\alpha}} \right\}. 
\tag{4.9}
\]
For $r \leq |x_0|/2$, we have
\[
\frac{1}{r} \int_{B_r(x_0)} |v_0|^2 \leq \frac{C_2}{r|x_0|^\alpha} \int_{B_r(x_0)} |x|^\alpha |v_0|^2 \leq \frac{C_2}{r|x_0|^\alpha} \|v_0\|_{L^{2,\alpha}}^2,
\]
and hence
\[
\frac{1}{r} \int_{B_r(x_0)} |v_0|^2 \leq \epsilon_* \quad \text{if } \frac{C_2 \|v_0\|_{L^{2,\alpha}}^2}{|x_0|^\alpha \epsilon_*} \leq r \leq \frac{|x_0|}{2}. 
\tag{4.10}
\]
We may increase $C_1$ so that when $\frac{|x_0|}{2} \geq \left( \frac{C_1 \|v_0\|_{L^{2,\alpha}}^2}{\epsilon_*} \right)^{\frac{1}{1+\alpha}}$, then $\frac{|x_0|}{2} \geq \frac{C_2 \|v_0\|_{L^{2,\alpha}}^2}{|x_0|^\alpha \epsilon_*}$. For such $x_0$, (4.8) and (4.10) imply
\[
\frac{1}{r} \int_{B_r(x_0)} |v_0|^2 \leq \epsilon_* \quad \text{if } r \geq \frac{C_2 \|v_0\|_{L^{2,\alpha}}^2}{|x_0|^\alpha \epsilon_*}.
\]
Thus Theorem 4.1 (i) shows $v$ is regular for $t \geq \frac{C_2 \|v_0\|_{L^{2,\alpha}}^4}{|x_0|^{4\alpha} \epsilon_*^2}$. This and (4.9) show Part (ii).
### 4.2 Energy concentration

We now prove Theorem 1.6 on the energy concentration at a blow up time by using Theorem 3.4.

**Proof of Theorem 1.6.** (i) Let $u$ be a local energy solution, and $\epsilon_*$ the small constant of Theorem 3.4 (ii). If $N^*_R := \sup_{x \in \mathbb{R}^3, R \leq 1} \int_{B_1(x)} |u_0|^2 \leq \epsilon_*$, then $u$ is regular in $\mathbb{R}^3 \times \{R^2, T\}$ with $T := \frac{c_1}{1 + c_2}$. Let $S_0 = \frac{T}{2} \leq \frac{1}{T}$.

Suppose that (1.9) is false. Then there exists $t_1 < T_*$ such that

$$\sup_{x_0 \in \mathbb{R}^3} \frac{1}{\sqrt{T_0 - t_1}} \int_{B_1(x_0)} |v(x, t_1)|^2 dx \leq \epsilon_*.$$  

Let $\lambda = \sqrt{\frac{T_0 - t_1}{S_0}}$ and let $u(x, t) := \lambda v(\lambda x, \lambda^2 t + t_1)$ and $u_0(x) = u(x, 0)$. Then $u$ satisfies

$$N^*_S = \sup_{x \in \mathbb{R}^3, \sqrt{S_0} \leq r \leq 1} \frac{1}{r} \int_{B_r(x)} |u_0|^2 \leq \sup_{x \in \mathbb{R}^3} \frac{1}{\sqrt{S_0}} \int_{B_1(x)} |u_0|^2 \leq \epsilon_*,$$

which implies that $u$ is regular in $\mathbb{R}^3 \times \{T_*, 2T_* - t_1\}$. Rescaling back, $v$ is regular in $\mathbb{R}^3 \times \{T_*, 2T_* - t_1\}$. This shows $v$ can be smoothly extended beyond $t = T_*$ and shows the desired contradiction.

(ii) Although the proof mostly follow the outlines of those of (i) and [1, Theorem 2], we will give the details, since some modification is needed.

Let $T_2 = T_2(M^*_1) < 1$ be the constant in Theorem 3.4 (ii). Let $S = T_2/2$ and $\delta_* = \min(Sr_0^2, T_*)$. Assume the contrary so that there exists a local energy solution $v$ which is singular at the point $(0, T_*)$, satisfies the type I condition (1.10), and

$$\frac{1}{\sqrt{T_* - t_0}} \int_{B_{\sqrt{T_* - t_0}}} |v(x, t_0)|^2 \leq \epsilon_* \quad (4.11)$$

for some $t_0 \in (T_* - \delta, T_*)$. Let $\lambda = \sqrt{\frac{T_0 - t_0}{S}}$ and $u(x, t) = \lambda v(\lambda x, \lambda^2 t + t_0)$. We see $\frac{1}{\sqrt{S}} \int_{B_1} |u(x, 0)|^2 dx \leq \epsilon_*$ since

$$\int_{B_1} |u(x, 0)|^2 dx = \lambda^{-1} \int_{B_\lambda} |v(y, t_0)|^2 dy = \sqrt{\frac{S}{T_* - t_0}} \int_{B_{\sqrt{T_* - t_0}}} |v(y, t_0)|^2 dy.$$ 

Therefore

$$\sup_{\sqrt{T_*} \leq r \leq 1} \frac{1}{r} \int_{B_r} |u(x, 0)|^2 \leq \epsilon_*.$$ 

On the other hand, we observe that

$$\sup_{x_0 \in \mathbb{R}^3} \int_{B_1(x_0)} |u(x, 0)|^2 = \sup_{y_0 \in \mathbb{R}^3} \lambda^{-1} \int_{B_\lambda(y_0)} |v(y, t_0)|^2 dy \leq M_*.$$ 

In the last inequality, (1.10) can be used since $r = \lambda \leq r_0$ and $T_* - r^2 < t_0 < T_*$.

Hence we may apply Theorem 3.4 (ii) to see that $u$ is regular at $(0, t)$ for $t \in [T_2/2, T_2)$. Hence $v$ is regular at $(0, t)$ for $t \in [T_*, 2T_* - t_0)$. This contradicts the assumption that $v$ is singular at $(0, T_*)$.  

\[\square\]
4.3 Regularity of discretely self-similar solutions

Based on Theorem 3.4 and the proof of [22, Corollary 1.5], we now prove Theorem 1.8.

**Proof of Theorem 1.8.** (i) For any \( r > 0 \), let \( k \) be the unique integer such that \( \lambda^{k-1} \leq r < \lambda^k \), then by virtue of the discretely self-similarity,

\[
\frac{1}{r} \int_{B_r(x_0)} |v_0|^2 \leq \frac{1}{\lambda^{k-1}} \int_{B_{\lambda^k}(x_0)} |v_0|^2 = \lambda \int_{B_1(\frac{x_0}{\lambda k})} |v_0(y)|^2 dy
\]

\[
\leq \lambda \|v_0\|^2_{L^2_{\text{local}}}. \tag{4.12}
\]

Since \( x_0 \) and \( r \) are arbitrary, this proves (1.12) with \( M := \|v_0\|_{H^2} \leq \sqrt{\lambda} \|v_0\|_{L^2_{\text{local}}} \).

Since \( v_0 \in E^2 \), there exists \( R_* = R_*(v_0, \lambda) > 0 \) such that

\[
\|v_0\|^2_{L^2(B_1(\lambda x_0))} \leq \frac{\epsilon_*}{\lambda^k} \quad \text{if } |x| \geq R_*. \tag{4.13}
\]

If \( 0 < r \leq \frac{|x_0|}{\lambda R_*} \) and \( \lambda^{k-1} \leq r < \lambda^k \), then

\[
\frac{|x_0|}{\lambda^k} \geq \frac{R_* r}{\lambda^{k-1}} \geq R_*.
\]

By (4.12) and (4.13), we obtain

\[
\sup_{r \in (0, \frac{|x_0|}{\lambda R_*})} \frac{1}{r} \int_{B_r(x_0)} |v_0|^2 \leq \epsilon_*
\]

Since \( x_0 \) is arbitrary in \( \mathbb{R}^3 \setminus \{0\} \) and \( R_* \) is independent of \( x_0 \), this proves (1.13) with \( \mu = \frac{1}{M R_*} \).

Applying Theorem 3.4 (ii) to \( u_0(x) := \mu |x_0| v_0(x_0 + \mu |x_0| x) \) for each \( x_0 \), and noting that \( \|u_0\|_{L^2_{\text{local}}} \leq \|v_0\|_{H^2} \), we see \( u \) is regular at \( x = 0 \) for \( t \in (0, T_2(M)) \), that is, \( v \) is regular at \( (x_0, t) \) for \( t \in (0, T_2(M) \mu^2 |x_0|^2) \). This concludes the proof of (i).

(ii) As the result of (i), it is direct that

\[
|v(x, t)| \leq \frac{C}{\sqrt{t}} \quad 0 < t \leq C |x|^2.
\]

As in the proof of Corollary 1.5 in [22], it suffices to show that \( v \) is regular in the set

\[
t \geq C |x|^2, \quad 1 \leq t \leq \lambda^2.
\]

Localizing the solution in the region above, it turns out that the localized solution \( \tilde{v} \) is a weak solution of the Navier-Stokes equations with a regular source term. Since the weak solutions satisfy the classical energy inequality, there is time \( t_* \in [1, \lambda^2) \) such that the solution is in \( H^1 \) at the time. Then due to the local existence of strong solution and the weak-strong uniqueness, there is a constant \( \delta > 0 \) independent of \( \mu \) such that \( \tilde{v} \) is regular in \([t_*, t_* + \delta]\). Therefore, if \( \lambda_* \) is taken to be sufficiently small close to 1, the solution is regular globally in time. For any \( t \), there is an integer \( k \) such that \( \lambda^{2k} t \in [1, \lambda^2] \), and we then have, due to scaling invariance,

\[
|v(x, t)| = |\lambda^k v(\lambda^k x, \lambda^{2k} t)| \leq C \lambda^k \leq C \frac{\lambda_*}{\sqrt{t}} \leq \frac{2C}{\sqrt{t}}.
\]
Remark 4.4. If $v_0 \in L^2_{uloc} \setminus E^2$, (1.13) does not hold in general. To see this we show that there exists a DSS function $F \in L^2_{uloc} \setminus E^2$ such that

$$\sup_{x \in \mathbb{R}^3, r \in (0, \mu|x|]} \frac{1}{r} \int_{B_r(x)} |F|^2 > \epsilon_* \quad \text{for any } \mu \in (0, 1).$$

Indeed for $K > \sqrt{\epsilon_*/4\pi}$, define

$$F(x) := \sum_{k \in \mathbb{Z}} \lambda^k f(\lambda^k x) \quad \text{with } f(x) = K|x - \lambda^1 e_1|^{-1} \chi(x) \quad (4.14)$$

where $\chi$ is the characteristic function supported in $B_{\lambda} \setminus B_1$. This function is given in [5, Comments on Theorem 1.2] and [9, Lemma 6.3] as an example in $M^{2,1} \setminus E^2$ while $L^{3,\infty} \subset E^2$. It is also shown in [9, Lemma 6.1] that DSS $\cap L^2_{uloc} = \text{DSS} \cap M^{2,1}$. For any $\mu \in (0, 1 - \lambda^{-1/2})$ we have

$$\sup_{x \in \mathbb{R}^3, r \in (0, \mu|x|]} \frac{1}{r} \int_{B_r(x)} |F|^2 \geq \frac{1}{\mu \lambda^2} \int_B |\lambda^1 e_1| \frac{|F|^2}{\mu \lambda^2} = 4\pi K^2 > \epsilon_*.$$

Hence (1.13) does not hold for $F$ for any $\mu > 0$.

5 Appendix: alternative proof of Theorem 3.4

By the definition of the local energy solution, there exists $c_r = c_r(t)$ such that the pressure admits the following decomposition (2.3):

$$p - c_r + \frac{|v|^2}{3} = p_{loc} + p_{nonloc}$$

$$:= \text{p.v.} \int_{B_{3r}} K(x - y)(v \otimes v)(y)dy + \int_{\mathbb{R}^3 \setminus B_{3r}} (K(x - y) - K(-y))(v \otimes v)(y)dy. \quad (5.1)$$

Since $(v, p - c_r)$ is a suitable weak solution to (NS) in $B_{2r}$, the local energy inequality with the test function given in the proof of Theorem 3.1 readily yields

$$E_r(t) \leq \frac{2}{2r} \int_{B_{2r}} |v_0|^2 + \frac{C}{r^3} \int_0^t \int_{B_{2r}} |v|^2 + \frac{C}{r^2} \int_0^t \int_{B_{2r}} |v|^3 + |p_{loc}|^2 + |p_{nonloc}|^2$$

$$=: 2N_{2r} + I_{lin} + I_{nonlin} + I_{loc} + I_{nonloc} \quad (5.2)$$

We divide the estimate into two cases.

**Case I:** $R \leq r \leq \frac{1}{6}$. For the simplicity of notation Let $E_r(t) := \sup_{r \leq \rho \leq 1/2} M_\rho(t)$. In the same way as (3.6) and (3.7) we have

$$I_{lin} \leq \frac{C}{R^2} \int_0^t E_R(s)ds,$$

$$I_{nonlin} \leq \epsilon E_R(t) + \frac{C}{r^2} \int_0^t E_R(s)^3 + E_R(s)^2 ds. \quad (5.4)$$
For the local pressure term, the Calderón-Zygmund estimate and (5.4) give
\[ I_{ploc} \leq \frac{C}{r^2} \int_0^t \int_{B_{2r}} |v|^3 \leq \epsilon \mathcal{E}_R(t) + \frac{C_r}{R^2} \int_0^t \mathcal{E}_R(s)^3 + \mathcal{E}_R(s)^{3/2} ds. \]  
(5.5)

On the other hand, since \( |x - y| \geq |y|/3 \) for \( x \in B_{2r} \) and \( y \in \mathbb{R}^3 \setminus B_{3r} \), we see
\[ |p_{nonloc}(x)| \leq \int_{\mathbb{R}^3 \setminus B_{3r}} |K(x - y) - K(-y)||v(y)|^2 dy \]
\[ \leq C r \int_{\mathbb{R}^3 \setminus B_{3r}} \frac{1}{|x - y|^4} |v(y)|^2 dy \]
\[ \leq C r \int_{\mathbb{R}^3 \setminus B_{3r}} \frac{|v(y)|^2}{|y|^4} dy \]
\[ \leq C r \sum_{k=2}^{[-\log_2 r - 1]} \int_{B_{2k^4} \setminus B_{2k^4-1}} |v(y)|^2 dy + C r \int_{\mathbb{R}^3 \setminus B_{14}} \frac{|v(y)|^2}{|y|^4} dy \]
\[ \leq C r^2 \sum_{k=2}^{[-\log_2 r - 1]} \left( \frac{1}{2^{3k}} \int_{B_{2k^4}} |v(y)|^2 dy \right) + C \mathcal{E}_{uloc} \]
\[ \leq C r^2 \mathcal{E}_R + C \|v_0\|^2_{L_{\text{uloc}}}, \]  
(5.6)

provided \( t \leq T_{uloc} \), where we used Lemma 2.2 in the last line. We then obtain
\[ \frac{C}{r^2} \int_0^t \int_{B_{2r}} |p_{nonloc}|^3 \leq \frac{C}{r^2} \int_0^t \mathcal{E}_R(s)^3 ds + t \|v_0\|^3_{L_{\text{uloc}}} \]
\[ \leq C t \|v_0\|^3_{L_{\text{uloc}}} + \delta \frac{10}{10}, \]  
(5.7)

provided \( t \leq \min\{ \frac{c \delta}{C \|v_0\|^3_{L_{\text{uloc}}}}, T_{uloc} \} \) with a small absolute constant \( c > 0 \). Hence applying (5.3), (5.4), (5.5), and (5.7) to (5.2), we obtain
\[ \sup_{R \leq r < 1/6} \mathcal{E}_R(t) \leq \frac{\delta}{2} + \frac{C}{R^2} \int_0^t \mathcal{E}_R(s)^3 ds \]  
for \( \delta \geq 5N_R \).

**Case II:** \( \frac{1}{6} \leq r \leq \frac{1}{2} \). In order to bound the right hand side of (5.2), we observe from Lemma 2.2 that
\[ \sup_{1/3 \leq r \leq 2} \mathcal{E}_R(t) \leq C \mathcal{E}_{uloc}(t) \leq C \|v_0\|^2_{L_{\text{uloc}}} \]  
holds for \( t \leq T_{uloc} \).  
(5.8)

This shows
\[ I_{lin} \leq C \int_0^t \sup_{1/3 \leq r \leq 1} \mathcal{M}_r(s) ds \]
\[ \leq Ct \|v_0\|^2_{L_{\text{uloc}}}, \]
and hence if \( t \leq \min\{T_{uloc}, \frac{c\delta}{\|v_0\|_{L^2_{uloc}}}^2\} \) with a suitable small constant \( c > 0 \), we have

\[
I_{lin} \leq \frac{\delta}{10}.
\]  

(5.9)

For the nonlinear term in (5.2), arguing as (3.10) we have

\[
I_{nonlin} \leq C \left( \int_0^t \int_{B_2} |\nabla v|^2 \right)^{\frac{3}{4}} \left( \int_0^t \left( \int_{B_2} |v|^2 \right)^{\frac{3}{4}} \right)^{\frac{1}{4}} + C \int_0^t \left( \int_{B_2} |v|^2 \right)^{\frac{3}{4}} ds
\]

\[
\leq C(t^{\frac{3}{4}} + t)\|v_0\|_{L^3_{uloc}}^3,
\]

(5.10)

from which the Calderón-Zygmund estimate also gives

\[
I_{ploc} \leq C \int_0^t \int_{B_{3r}} |v|^3 \leq C(t^{\frac{3}{4}} + t)\|v_0\|_{L^3_{uloc}}^3.
\]

(5.11)

Thus the right hand sides in (5.10) and (5.11) are bounded by \( \frac{\delta}{10} \) provided

\[
t \leq \min\left\{ T_{uloc}, \frac{c\delta^4}{\|v_0\|_{L^2_{uloc}}^{12}}, \frac{c\delta}{\|v_0\|_{L^2_{uloc}}^{3}} \right\}
\]

with some small absolute constant \( c > 0 \). On the other hand, in the same way as in (5.6), we have

\[
|p_{nonloc}(x)| \leq C \int_{\mathbb{R}^3\setminus B_{\frac{3}{4}}} \frac{|v(y)|^2}{|y|^4} dy \leq C\|v_0\|_{L^2_{uloc}}^2,
\]

which implies

\[
I_{pnonloc} \leq \frac{\delta}{10} \quad \text{for} \quad t \leq \frac{c\delta}{\|v_0\|_{L^2_{uloc}}^3}.
\]

Making use of these estimates in (5.2), we obtain

\[
\sup_{1/6 \leq r \leq 1/2} E_r(t) \leq \frac{\delta}{2}.
\]

Note here that by choosing \( c_1 > 0 \) sufficiently small, we may take

\[
T_1 := \frac{c_1 \min\{1, \delta^4\}}{1 + \|v_0\|_{L^2_{uloc}}^{12}} \leq \min\left\{ T_{uloc}, \frac{c\delta}{\|v_0\|_{L^2_{uloc}}^2}, \frac{c\delta^4}{\|v_0\|_{L^2_{uloc}}^{12}}, \frac{c\delta}{\|v_0\|_{L^2_{uloc}}^3} \right\}.
\]

Therefore combining the conclusions of the cases I and II, we have

\[
\mathcal{E}_R(t) \leq \frac{\delta}{2} + C \int_R^t \mathcal{E}_R(s) + \mathcal{E}_R(s)^3 ds
\]

for \( t \leq T_1 \). Applying Lemma 2.4 we obtain

\[
\mathcal{E}_R(t) \leq \delta \quad \text{for} \quad t \leq \min\{\lambda R^2, T_1\}.
\]

(5.12)

Since \( \delta \geq 5N_R \geq 5N_r \) for any \( r \in [R, 1/2] \), we may replace \( R \) by \( r \) in (5.12), and thus we have verified (3.19) for \( R \leq r \leq \frac{1}{2} \).

Since the remaining proof is the same as Theorem 3.1, we omit the details. \( \square \)
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