Homogenisation for a stationary Maxwell system

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Abstract

We study homogenization problem for the stationary Maxwell system. It is supposed that the magnetic permeability and the dielectric permittivity locally close to fast-oscillating (with respect to some small parameter) periodic functions which can change the form on rather big distances. An asymptotic behavior to solutions of the Maxwell system outside of its spectrum is obtained. We also describe asymptotic behavior of resolvent with control of the remainder in terms of some appropriate operator norms.

1 Introduction

The work is devoted to study the homogenization problem for the Maxwell operator. Roughly speaking it is supposed that the magnetic permeability and the dielectric permittivity locally close to fast-oscillating periodic functions which can change the form on rather big distances. Problems of this kind for operators with purely periodic coefficients were intensively studied by many authors [1, 2, 3, 6, 7, 8, 10]. The Maxwell operator with purely periodic coefficients was in details studied in a series of works of M.Sh.Birman

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and T.A.Suslina, last results can be found in work \[9\]. However, homogenization problems concerned to operators whose coefficients aren’t purely periodic are investigated less in details \[4, 5\]. Moreover, it seems like, the Maxwell operator with coefficients locally close to fast-oscillating periodic functions remains almost not studied.

Let us denote by \( L^2(\mathbb{R}^3; \mathbb{C}^3) \) the \( L^2 \) class of \( \mathbb{C}^3 \)-valued functions in \( \mathbb{R}^3 \) and by \( H^p(\mathbb{R}^3; \mathbb{C}^3) \) the corresponding Sobolev classes of order \( p \), where \( p \in \mathbb{N} \).

Let us put

\[
J = \{ u : u \in L^2(\mathbb{R}^3; \mathbb{C}^3), \quad \text{div}(u) = 0 \}.
\]

Here equality \( \text{div}(u) = 0 \) is understood in sense of distribution theory

\[
\text{div}(u) = 0 \iff \int_{\mathbb{R}^3} (u, \nabla w) \, dx = 0 \quad \forall \ w \in H^1(\mathbb{R}^3; \mathbb{C}^3),
\]

where \( (\cdot, \cdot) \) is the standard scalar product in \( \mathbb{C}^3 \).

Let us consider the model described by the Maxwell operator

\[
\mathcal{M} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & i \text{rot}(\mu^{-1}) \\ -i \text{rot}(\alpha^{-1}) & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix},
\]

where \( \alpha \) and \( \mu \) are some matrix-valued functions which will be described in detail lately. The Maxwell operator \( \mathcal{M} \) acts in the space \( J \oplus J \) on the domain

\[
\text{Dom} \mathcal{M} = \{ u, v : u, v \in H^1(\mathbb{R}^3, \mathbb{C}^3), \quad \text{div}(u) = 0, \quad \text{div}(v) = 0 \}.
\]

Now let us describe \( \alpha \) and \( \mu \). Let \( \Gamma \) be a lattice in \( \mathbb{R}^3 \) and \( \Omega \) be the elementary cell of the lattice \( \Gamma \). Let coefficients \( \alpha \) and \( \mu \) have the form

\[
\alpha = \alpha \left( x, \frac{x}{\varepsilon} \right), \quad \mu = \mu \left( x, \frac{x}{\varepsilon} \right),
\]

where \( \varepsilon \) is a small positive parameter. Suppose that the matrix-valued functions \( \alpha \) and \( \mu \) satisfy the following assumption.

**Assumption 1.1**

1) \( \alpha \) and \( \mu \) are real and positive-definite \((3 \times 3)\) matrixes;
2) \( \alpha, \mu \in C^\infty(\mathbb{R}^6, M_3) \);
3) \( \alpha(x, y) \) and \( \mu(x, y) \) are \( \Gamma \)-periodic with respect to \( y \);
4) \( \alpha(x, y) \equiv I \) and \( \mu(x, y) \equiv I \) for any \( x \notin B_R = \{ |x| < R \} \) and \( y \in \mathbb{R}^3 \), where \( I \) is the identity matrix and \( R \) is a positive constant.
Denote by $\mathcal{M}(\varepsilon)$ the Maxwell operator with coefficients $\alpha$ and $\mu$ satisfying assumption 1.1. The Maxwell operator $\mathcal{M}(\varepsilon)$ is closed in $J \oplus J$ with the standard scalar product and selfadjoint in $J \oplus J$ with scalar product of the form
\[
\left\langle \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{pmatrix} , \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{pmatrix} \right\rangle = \int_{\mathbb{R}^3} (\alpha^{-1} u_1, u_2) + (\mu^{-1} v_1, v_2) \, dx.
\]
We study an asymptotic behaviour as $\varepsilon \to 0$ of the resolvent $(\mathcal{M}(\varepsilon) - E)^{-1}$ for $\text{Im} \, E > 0$. We also investigate asymptotic behaviour of solutions to the Maxwell system
\[
(\mathcal{M}(\varepsilon) - E) \begin{pmatrix} U(x, \varepsilon) \\ V(x, \varepsilon) \end{pmatrix} = \begin{pmatrix} f^u(x) \\ f^v(x) \end{pmatrix}, \quad \begin{pmatrix} U(x, \varepsilon) \\ V(x, \varepsilon) \end{pmatrix} \in \text{Dom} \mathcal{M}(\varepsilon),
\]
where $\text{Im} \, E > 0$ and
\[
f^u, f^v \in C^\infty(\mathbb{R}^3, \mathbb{C}^3) \cap L^2(\mathbb{R}^3, \mathbb{C}^3), \quad \text{div}(f^u) = 0, \quad \text{div}(f^v) = 0.
\]

The main results of the work are contained in theorem 3.1 and corollary 3.2.

2 Formal asymptotic solutions

This section is devoted to the constructing of formal asymptotic solutions, as $\varepsilon \to 0$, of the system (1). To separate the slow and fast dependencies on the argument we seek the solution of system (1) in the form
\[
\begin{pmatrix} U(x, \varepsilon) \\ V(x, \varepsilon) \end{pmatrix} = \begin{pmatrix} u(x, x/\varepsilon, \varepsilon) \\ v(x, x/\varepsilon, \varepsilon) \end{pmatrix}.
\]
It is easy to see that the following lemma holds.

Lemma 2.1 Let functions $u$ and $v$ satisfy equations
\[
(\varepsilon^{-1} \mathcal{M}_y + \mathcal{M}_x - E) \begin{pmatrix} u(x, y, \varepsilon) \\ v(x, y, \varepsilon) \end{pmatrix} = \begin{pmatrix} f^u(x) \\ f^v(x) \end{pmatrix},
\]
\[
\varepsilon^{-1} \text{div}_y u(x, y, \varepsilon) = \text{div}_x u(x, y, \varepsilon), \quad \varepsilon^{-1} \text{div}_y v(x, y, \varepsilon) = \text{div}_x v(x, y, \varepsilon),
\]
where
\[
\mathcal{M}_x \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix} = \begin{pmatrix} i \text{rot}_x (\mu^{-1}(x, y) v(x, y)) \\ -i \text{rot}_x (\alpha^{-1}(x, y) v(x, y)) \end{pmatrix},
\]
\[ \mathcal{M}_y \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} = \begin{pmatrix} i \text{rot}_y (\mu^{-1}(x,y)v(x,y)) \\ -i \text{rot}_y (\alpha^{-1}(x,y)v(x,y)) \end{pmatrix} \]

and the operators \( \text{rot}_x \), \( \text{div}_x \) and \( \text{rot}_y \), \( \text{div}_y \) act upon a variables \( x \) and \( y \) respectively. Then a function

\[ \begin{pmatrix} U(x, \varepsilon) \\ V(x, \varepsilon) \end{pmatrix} = \begin{pmatrix} u \left(x, \frac{x}{\varepsilon}, \varepsilon\right) \\ v \left(x, \frac{x}{\varepsilon}, \varepsilon\right) \end{pmatrix} \]

satisfies equation (1) and conditions \( \text{div}(U(x, \varepsilon)) = 0 \), \( \text{div}(V(x, \varepsilon)) = 0 \).

**Proof.** It can be easily proved by using direct calculations. □

Thus, the constructing of formal solutions of system (1) is reduced to the constructing of formal solutions of equations (2), (3).

The formal solutions of equations (2), (3) are sought in the form

\[ \begin{pmatrix} u(x,y,\varepsilon) \\ v(x,y,\varepsilon) \end{pmatrix} = \sum_{n \geq 0} \varepsilon^n \begin{pmatrix} u_n(x,y) \\ v_n(x,y) \end{pmatrix}, \quad (4) \]

where functions \( u_n(x,y) \) and \( v_n(x,y) \) are \( \Gamma \)-periodic with respect to \( y \).

**Lemma 2.2** Suppose \( u_n, v_n \) satisfy the following recurrence system of equations

\[ \mathcal{M}_y \begin{pmatrix} u_n(x,y) \\ v_n(x,y) \end{pmatrix} = \begin{pmatrix} f^u(x) \\ f^v(x) \end{pmatrix} \delta_n + (E - \mathcal{M}_x) \begin{pmatrix} u_{n-1}(x,y) \\ v_{n-1}(x,y) \end{pmatrix}, \quad (5) \]

\[ \text{div}_y (u_n(x,y)) = -\text{div}_x (u_{n-1}(x,y)), \quad \text{div}_y (v_n(x,y)) = -\text{div}_x (v_{n-1}(x,y)), \quad (6) \]

where \( u_{-1} \equiv 0 \) and \( v_{-1} \equiv 0 \). Then (1) formally satisfies problem (2), (3).

**Proof.** Substituting expansion (4) in system (2), (3), and comparing the coefficients corresponding to the equal powers of \( \varepsilon \), we obtain recurrence system (5), (6). □

For detailed description of solutions to system (5), (6) we recall the following well known result.

**Lemma 2.3** There exist 3 linearly independent \( \Gamma \)-periodic (with respect to \( y \)) solutions to the system

\[ \begin{cases} \text{rot}_y (\zeta(x,y)) = 0, \\ \text{div}_y (\alpha(x,y)\zeta(x,y)) = 0. \end{cases} \]
\[ \zeta_k(x, y) = e_k + \nabla_y \varphi_k(x, y), \quad k = 1, 2, 3, \]
\[ e_1 = (1, 0, 0)^t, \quad e_2 = (0, 1, 0)^t, \quad e_3 = (0, 0, 1)^t, \]
where \( \varphi_k(x, y) \) is any \( \Gamma \)-periodic (with respect to \( y \)) solutions to the equation
\[ \text{div}_y(\alpha(x, y) \nabla_y \varphi_k(x, y) + \alpha(x, y)e_k) = 0. \]

There exist 3 linearly independent \( \Gamma \)-periodic (with respect to \( y \)) solutions to the system
\[
\begin{cases}
\text{rot}_y(\xi(x, y)) = 0, \\
\text{div}_y(\mu(x, y)\xi(x, y)) = 0.
\end{cases}
\]
\[ \xi_k(x, y) = e_k + \nabla_y \psi_k(x, y), \quad k = 1, 2, 3, \]
where \( \psi_k(x, y) \) is any \( \Gamma \)-periodic (with respect to \( y \)) solutions to the equation
\[ \text{div}_y(\mu(x, y)\nabla_y \psi_k(x, y) + \mu(x, y)e_k) = 0. \]

Moreover, \( \zeta_k, \xi_k \in C^\infty(\mathbb{R}^3, \mathbb{C}^3) \) and \( \zeta_k(x, y) = e_k, \xi_k(x, y) = e_k \) for \( x \notin B_R, \) where \( k = 1, 2, 3. \)

**Proof.** We only need to prove that \( \zeta_k(x, y) = e_k, \xi_k(x, y) = e_k \) for \( x \notin B_R, \) where \( k = 1, 2, 3. \) It is easy to see that \( \alpha(x, y) = \mu(x, y) = I \) for \( x \notin B_R. \) Hence \( \varphi_k, \psi_k \) satisfy the following equations
\[ \Delta_y \varphi_k(x, y) = 0, \quad \Delta_y \psi_k(x, y) = 0, \quad x \notin B_R, \quad k = 1, 2, 3. \]

Since \( \varphi_k(x, y), \psi_k(x, y) \) are \( \Gamma \)-periodic functions (with respect to \( y \)), we see that \( \varphi_k(x, y), \psi_k(x, y) \) are constant for \( x \notin B_R. \) This implies the necessary assertion. \( \Box \)

Consider recurrence system (5), (6) for \( n = 0 \)
\[ \mathfrak{M}_y \begin{pmatrix} u_0(x, y) \\ v_0(x, y) \end{pmatrix} = 0, \quad \text{div}_y(u_0(x, y)) = \text{div}_y(v_0(x, y)) = 0. \tag{7} \]

Lemma 2.3 implies that general solution of (7) has the following form
\[ u_0(x, y) = \alpha(x, y) \sum_{k=1}^{3} a_k^0(x) \zeta_k(x, y), \quad v_0(x, y) = \mu(x, y) \sum_{k=1}^{3} b_k^0(x) \xi_k(x, y), \]
where \( a_k^0(x) \) and \( b_k^0(x) \) are arbitrary smooth functions.
Recurrence system (5), (6) for \( n = 1 \) has the form
\[
\begin{align*}
\mathcal{M}_y \begin{pmatrix} u_1(x, y) \\ v_1(x, y) \end{pmatrix} &= \begin{pmatrix} f^u(x) \\ f^v(x) \end{pmatrix} + (E - \mathcal{M}_x) \begin{pmatrix} u_0(x, y) \\ v_0(x, y) \end{pmatrix}, \\
\text{div}_y(u_1(x, y)) &= -\text{div}_x(u_0(x, y)), \quad \text{div}_y(v_1(x, y)) = -\text{div}_x(v_0(x, y)).
\end{align*}
\]

Let us rewrite it in the following from
\[
\begin{align*}
\mathcal{M}_y \begin{pmatrix} f^u(x) \\ f^v(x) \end{pmatrix} &= \begin{pmatrix} f^u(x) \\ f^v(x) \end{pmatrix} + (E - \mathcal{M}_x) \begin{pmatrix} u_0(x, y) \\ v_0(x, y) \end{pmatrix}, \\
\text{div}_y(u_1(x, y)) &= -\text{div}_x(u_0(x, y)), \quad \text{div}_y(v_1(x, y)) = -\text{div}_x(v_0(x, y)).
\end{align*}
\]

(8)

Suppose that the following conditions hold
\[
\begin{align*}
\int \left( -i f^u - i E u_0 - \text{rot}_x(\mu^{-1} v_0), \xi_k \right) dx = 0, \quad k = 1, 2, 3, \\
\int \text{div}_x(v_0) dx = 0,
\end{align*}
\]

(11)

then problem (9) can be solved. It is easy to check that
\[
\frac{1}{|\Omega|} \int_{\Omega} \begin{pmatrix} \xi_1(x, y), \xi_2(x, y), \xi_3(x, y) \end{pmatrix}^t dx = I,
\]

\[
\begin{pmatrix} f^u(x), \xi_1(x, y) \\ f^u(x), \xi_2(x, y) \\ f^u(x), \xi_3(x, y) \end{pmatrix} = |\Omega| f^u(x).
\]

Similarly,
\[
\begin{pmatrix} u_0(x, y), \xi_1(x, y) \\ u_0(x, y), \xi_2(x, y) \\ u_0(x, y), \xi_3(x, y) \end{pmatrix} = |\Omega| \Lambda^u(x) a_0^0(x),
\]

\[
\begin{pmatrix} a_1^0(x) \\ a_2^0(x) \\ a_3^0(x) \end{pmatrix}.
\]
where
\[
\Lambda_{kp}(x) = \frac{1}{|\Omega|} \left( \alpha(x,y) \zeta_p(x,y), \xi_k(x,y) \right)_y = \frac{1}{|\Omega|} \left( \alpha(x,y) \zeta_p(x,y), \zeta_k(x,y) \right)_y. \tag{12}
\]

From (12) it follows that \(\Lambda^u(x)\) is real and positive-definite \((3 \times 3)\) matrix.

Using direct calculations, we get
\[
\left( \text{rot}_x (\mu^{-1}(x,y)v_0(x,y)), \xi_k(x,y) \right)_y = \sum_{p=1}^{3} \left( \text{rot}_x (b^0_p(x)) \zeta_p(x,y), \xi_k(x,y) \right)_y = \]
\[
= \sum_{p=1}^{3} b^0_p(x) \left( \text{rot}_x (\xi_p(x,y)), \xi_k(x,y) \right)_y + \]
\[
+ \sum_{p=1}^{3} \left( [\nabla_x (b^0_p(x)) \times \xi_p(x,y)], \xi_k(x,y) \right)_y.
\]

It is easy to see that
\[
\left( \text{rot}_x (\xi_p(x,y)), \xi_k(x,y) \right)_y = 0,
\]
\[
\int_\Omega [\xi_p(x,y) \times \xi_k(x,y)] \, dy = |\Omega| [e_p \times e_k].
\]

Therefore,
\[
\left( \text{rot}_x v_0(x,y), \xi_k(x,y) \right)_y = |\Omega| \sum_{p=1}^{3} \left( [e_p \times e_k], \nabla_x (b^0_p(x)) \right),
\]
\[
\begin{pmatrix}
\left( \text{rot}_x v_0(x,y), \xi_1(x,y) \right)_y \\
\left( \text{rot}_x v_0(x,y), \xi_2(x,y) \right)_y \\
\left( \text{rot}_x v_0(x,y), \xi_3(x,y) \right)_y
\end{pmatrix} = |\Omega| \text{rot}_x b^0(x), \quad b^0(x) = \begin{pmatrix} b^0_1(x) \\ b^0_2(x) \\ b^0_3(x) \end{pmatrix}.
\]

Thus, the first three equations from (11) can be rewritten in the following form
\[
i \text{rot}_x b^0(x) - E \Lambda^u(x) a^0(x) = f^u(x).
\]
The last equation from (11) can be reduced to the form
\[ \text{div}_x (\Lambda^v(x)b^0(x)) = 0, \]
where
\[ \Lambda^v_{\mu}(x) = \frac{1}{|\Omega|} (\mu(x, y)\xi_p(x, y), \zeta_k(x, y))_y = \frac{1}{|\Omega|} (\mu(x, y)\xi_p(x, y), \xi_k(x, y)). \]

From (13) it follows that \( \Lambda^v(x) \) is real and positive-definite (3 \times 3) matrix.

Similarly, one can check that solvability conditions for problem (10) can be written in the form
\[ -i \text{rot}_x a^0(x) - E\Lambda^v(x)b^0(x) = f^v(x), \]
\[ \text{div}_x (\Lambda^u(x)a^0(x)) = 0. \]

Let us introduce notations
\[ \hat{u}_0(x) = \Lambda^u(x)a^0(x), \quad \hat{v}_0(x) = \Lambda^v(x)b^0(x). \]

Finally, the solvability conditions for (8) have the following form
\[
\begin{cases}
(\mathcal{M} - E) \begin{pmatrix} \hat{u}_0 \\ \hat{v}_0 \end{pmatrix} = \begin{pmatrix} f^u(x) \\ f^v(x) \end{pmatrix}, & \mathcal{M} = \begin{pmatrix} 0 & i \text{rot}_x(\Lambda^u(x))^{-1} \\ -i \text{rot}_x(\Lambda^u(x))^{-1} & 0 \end{pmatrix}, \\
\text{div}_x(\hat{u}_0) = 0, & \text{div}_x(\hat{v}_0) = 0,
\end{cases}
\]
(14)

General solution of (8) can be represented in the form
\[ u_1(x, y) = \tilde{u}_1(x, y) + \alpha(x, y) \sum_{k=1}^{3} a^1_k(x)\zeta_k(x, y), \]
\[ v_1(x, y) = \tilde{v}_1(x, y) + \mu(x, y) \sum_{k=1}^{3} b^1_k(x)\xi_k(x, y), \]
where \( \tilde{u}_1(x, y), \tilde{v}_1(x, y) \) are partial solutions of (8) that are orthogonal to solutions of homogeneous equations.

Let us check that \( \tilde{u}_1(x, y) \equiv 0 \) and \( \tilde{v}_1(x, y) \equiv 0 \) for \( x \notin B_R \). It is easy to see that
\[ \alpha(x, y) = \mu(x, y) = \Lambda^u(x) = \Lambda^v(x) = I, \quad \hat{u}_0(x) = a^0(x), \quad \hat{v}_0(x) = b^0(x), \]
for $x \notin B_R$. Taking into account (14), we see that for $x \notin B_R$ the following equations hold
\[
\begin{align*}
\text{rot}_y(\tilde{v}_1) &= 0, \\
\text{div}_y(\tilde{v}_1) &= 0,
\end{align*}
\tag{15}
\]
Recalling that $\tilde{u}_1(x, y)$, $\tilde{v}_1(x, y)$ are orthogonal to solutions of homogeneous equations, we get that $\tilde{u}_1(x, y) \equiv 0$ and $\tilde{v}_1(x, y) \equiv 0$ for $x \notin B_R$.

Now let us consider recurrence system (5), (6) for $n \geq 2$. General solution of this system can be represented in the form
\[
u_n(x, y) = \tilde{v}_n(x, y) + \mu(x, y) \sum_{k=1}^{3} b^u_k(x) \xi_k(x, y),
\tag{16}
\]
where $\tilde{u}_n(x, y)$, $\tilde{v}_n(x, y)$ are partial solutions of (5), (6) that are orthogonal to solutions of homogeneous system. Coefficients $a^u_k(x)$ and $b^u_k(x)$ can be defined from the solvability conditions of system (5), (6) with $n$ substituted by $n + 1$, which can be written in the form
\[
\begin{align*}
\begin{pmatrix} \hat{\mathcal{M}} - E \end{pmatrix} \begin{pmatrix} \hat{u}_n \\ \hat{v}_n \end{pmatrix} &= \begin{pmatrix} F^u_n(x) \\ F^v_n(x) \end{pmatrix}, \\
\text{div}_x(\hat{u}_n) &= G^u_n(x), \quad \text{div}_x(\hat{v}_n) = G^v_n(x),
\end{align*}
\tag{17}
\]
where
\[
\begin{align*}
\hat{u}_n(x) &= \Lambda^u(x) \begin{pmatrix} a^u_1(x) \\ a^u_2(x) \\ a^u_3(x) \end{pmatrix}, \quad \hat{v}_n(x) = \Lambda^v(x) \begin{pmatrix} b^u_1(x) \\ b^u_2(x) \\ b^u_3(x) \end{pmatrix}, \\
F^u_n(x) &= \frac{1}{|\Omega|} \begin{pmatrix} (f^u_n(x, y), \xi_1(x, y))_y \\ (f^u_n(x, y), \xi_2(x, y))_y \\ (f^u_n(x, y), \xi_3(x, y))_y \end{pmatrix}, \\
f^u_n(x, y) &= E\tilde{v}_n(x, y) + i \text{rot}_x (\mu^{-1}(x, y)\tilde{v}_n(x, y)), \\
F^v_n(x) &= \frac{1}{|\Omega|} \begin{pmatrix} (f^v_n(x, y), \xi_1(x, y))_y \\ (f^v_n(x, y), \xi_2(x, y))_y \\ (f^v_n(x, y), \xi_3(x, y))_y \end{pmatrix}, \\
f^v_n(x, y) &= E\tilde{u}_n(x, y) + i \text{rot}_x (\alpha^{-1}(x, y)\tilde{u}_n(x, y)),
\end{align*}
\]
\[ G_n^u(x) = \frac{1}{|\Omega|} \int_\Omega \text{div}_x (\tilde{u}_n(x,y)) \, dy, \quad G_n^v(x) = \frac{1}{|\Omega|} \int_\Omega \text{div}_x (\tilde{v}_n(x,y)) \, dy. \quad (18) \]

Let us collect the obtained result in the following theorem.

**Theorem 2.4** Suppose \( \text{Im}(E) > 0, \varepsilon > 0, \) assumption 1.1 and conditions 2 hold. Then Maxwell system (1) has asymptotic solution of the form

\[ \begin{pmatrix} U(x, \varepsilon) \\ V(x, \varepsilon) \end{pmatrix} = \sum_{n \geq 0} \varepsilon^n \begin{pmatrix} u_n(x, \frac{x}{\varepsilon}, \varepsilon) \\ v_n(x, \frac{x}{\varepsilon}, \varepsilon) \end{pmatrix}, \quad (19) \]

where \( u_n \) and \( v_n \) can be represented in form (16). The corresponding coefficients can be founded from recurrence equations (5), (6), and (17).

Moreover, \( \tilde{u}_n(x, y) \equiv 0 \) and \( \tilde{v}_n(x, y) \equiv 0 \) for \( x \not\in B_R \) and \( n \geq 0 \).

The following lemma describe important estimates for the components of formal solution (19).

**Lemma 2.5** Suppose that the statements of theorem 2.4 hold. Then

\[
\max_{0 \leq |k| \leq 2} \max_{y \in \Omega} \left\| \frac{\partial^{|k|}}{\partial y^k} u_n(\cdot, y, \varepsilon) \right\|_{H^{s-n+1}(\mathbb{R}^3, C^1)} \leq C \| f \|_{H^s(\mathbb{R}^3, C^3)},
\]

\[
\max_{0 \leq |k| \leq 2} \max_{y \in \Omega} \left\| \frac{\partial^{|k|}}{\partial y^k} v_n(\cdot, y, \varepsilon) \right\|_{H^{s-n+1}(\mathbb{R}^3, C^1)} \leq C \| f \|_{H^s(\mathbb{R}^3, C^3)},
\]

\[
\| u_n(\cdot, /\varepsilon, \varepsilon) \|_{H^{s-n+1}(\mathbb{R}^3, C^1)} \leq C \varepsilon^{n-s-1} \| f \|_{H^s(\mathbb{R}^3, C^3)},
\]

\[
\| v_n(\cdot, /\varepsilon, \varepsilon) \|_{H^{s-n+1}(\mathbb{R}^3, C^1)} \leq C \varepsilon^{n-s-1} \| f \|_{H^s(\mathbb{R}^3, C^3)},
\]

for \( 0 \leq n \leq s + 1 \), where the constant \( C \) does not depend on \( f \).

**Proof.** The statement of the lemma for \( n = 0 \) follows from (14). Using recurrent system of equations (5), (6), and (17), the statement of the lemma for \( n \geq 1 \) can be easily proved by induction. \( \Box \)
3 Asymptotic expansion of the resolvent

In this section we describe asymptotic behaviour of the resolvent \((M(\varepsilon) - E)^{-1}\) for a small \(\varepsilon\). To do that we consider a partial sum for formal series \(19\) of the form

\[
(U_N(x, x/\varepsilon, \varepsilon)) = \sum_{n=0}^{N} \varepsilon^n \left(\begin{array}{c} u_n(x, x/\varepsilon) \\ v_n(x, x/\varepsilon) \end{array}\right) + \varepsilon^N \delta(x, \varepsilon),
\]

where \(N \geq 1\) and \(\delta = (\delta^u, \delta^v)^t\) is an auxiliary function. Here we also suppose that \(\hat{v}_N(x) \equiv \hat{u}_N(x) \equiv 0\). This implies that \(u_N(x, x/\varepsilon) \equiv v_N(x, x/\varepsilon) \equiv 0\) for \(x \notin B_R\).

It easy to see that

\[
(M(\varepsilon) - E) \left(\begin{array}{c} U_N \\ V_N \end{array}\right) = \left(\begin{array}{c} f^n(x) \\ f^v(x) \end{array}\right) + \varepsilon^N \left[ (\mathcal{M} - E) \left(\begin{array}{c} u_N(x, y) \\ v_N(x, y) \end{array}\right) \right]_{y=x/\varepsilon} + \varepsilon^N (M(\varepsilon) - E) \delta(x, \varepsilon),
\]

\(1\)

\[
\text{div} (U_N) = \varepsilon^N \left[ \text{div}_x (u_N(x, y)) \right]_{y=x/\varepsilon} + \varepsilon^N \text{div}(\delta^u(x, \varepsilon)),
\]

\(2\)

\[
\text{div} (V_N) = \varepsilon^N \left[ \text{div}_x (v_N(x, y)) \right]_{y=x/\varepsilon} + \varepsilon^N \text{div}(\delta^v(x, \varepsilon)).
\]

\(3\)

Now we chose \(\delta\) such that \((U_N, V_N)^t \in \text{Dom } M\). To do that we find an appropriate solution to the following equations \(\text{div} (U_N) = 0\), \(\text{div} (V_N) = 0\).

First equation can be rewritten in the form

\[
\text{div}(\delta^u(x, \varepsilon)) = g(x, \varepsilon) \equiv -[\text{div}_x (u_N(x, y))]_{y=x/\varepsilon}.
\]

\(4\)

Let us consider the following solution of equation \((4)\)

\[
\delta^u(x, \varepsilon) = x \int_{0}^{1} g(tx, \varepsilon)t^2 \, dt.
\]

Recalling that supp \(g \subset B_R\), we obtain

\[
\|\delta^u(\cdot, \varepsilon)\|_{H^1(\mathbb{R}^3, C^2)}^2 \leq (R^2 + 6) \int_{\mathbb{R}^3} \left( \int_{0}^{1} |g(tx, \varepsilon)|t^2 \, dt \right)^2 \, dx + 2R^2 \sum_{j=1}^{3} \int_{\mathbb{R}^3} \left( \int_{0}^{1} |g_j(tx, \varepsilon)|t^3 \, dt \right)^2 \, dx,
\]

11
where \( g_j(x, \varepsilon) = \partial_{x_j} g(x, \varepsilon) \). Using simple calculations, we get
\[
\int_{\mathbb{R}^3} \left( \int_0^1 |g(tx, \varepsilon)| t^2 \, dt \right)^2 \, dx \leq \|g(\cdot, \varepsilon)\|_{L^2(\mathbb{R}^3, \mathbb{C})}^2,
\]
and hence
\[
\int_{\mathbb{R}^3, \mathbb{C}} \left( \int_0^1 |g_j(tx, \varepsilon)| t^3 \, dt \right)^2 \, dx \leq \|g_j(\cdot, \varepsilon)\|_{L^2(\mathbb{R}^3, \mathbb{C})}^2, \quad k = 1, 2, 3,
\]
and hence
\[
\|\delta^u(\cdot, \varepsilon)\|_{H^1(\mathbb{R}^3, \mathbb{C}^3)}^2 \leq (R^2 + 6)\|g(\cdot, \varepsilon)\|_{L^2(\mathbb{R}^3, \mathbb{C})}^2 + 2R^2 \sum_{j=1}^3 \|g_j(\cdot, \varepsilon)\|_{L^2(\mathbb{R}^3, \mathbb{C})}^2 \leq (2R^2 + 6)\|g(\cdot, \varepsilon)\|_{H^1(\mathbb{R}^3, \mathbb{C})}^2.
\]
Using lemma 2.5 and definition (4) of \( g \) we obtain
\[
\|\delta^u(\cdot, \varepsilon)\|_{H^1(\mathbb{R}^3, \mathbb{C}^3)} \leq (2R^2 + 6)\|f\|_{H^{N+1}(\mathbb{R}^3, \mathbb{C}^3)} \leq (2R^2 + 6)\|f\|_{H^{N+1}(\mathbb{R}^3, \mathbb{C}^3)}.
\]
In the same way one can define \( \delta^v \) such that \( \text{div}(V_N) = 0 \) and
\[
\|\delta^v(\cdot, \varepsilon)\|_{H^1(\mathbb{R}^3, \mathbb{C}^3)} \leq (2R^2 + 6)\|f\|_{H^{N+1}(\mathbb{R}^3, \mathbb{C}^3)}.
\]
Since \( (u_N(x, y), v_N(x, y))^t \in \text{Dom} \mathcal{M} \), we can apply the resolvent \( (\mathcal{M}(\varepsilon) - E)^{-1} \) to the both sides of (1) for \( \text{Im}(E) > 0 \)
\[
(\mathcal{M}(\varepsilon) - E)^{-1} \begin{pmatrix} f^u(x) \\ f^v(x) \end{pmatrix} = \begin{pmatrix} U_N(x, x/\varepsilon, \varepsilon) \\ V_N(x, x/\varepsilon, \varepsilon) \end{pmatrix} = -\varepsilon^N (\mathcal{M}(\varepsilon) - E)^{-1} h(x, x/\varepsilon, \varepsilon)
\]
where
\[
h(x, x/\varepsilon, \varepsilon) = \left( (\mathcal{M}_x - E) \begin{pmatrix} u_N(x, y) \\ v_N(x, y) \end{pmatrix} \right)_{y=x/\varepsilon} + (\mathcal{M}(\varepsilon) - E)\delta(x, \varepsilon).
\]
Using lemma 2.5, (5), (6), and (7) we get
\[
\left\| (\mathcal{M}(\varepsilon) - E)^{-1} \begin{pmatrix} f^u(x) \\ f^v(x) \end{pmatrix} - \sum_{n=0}^{N-2} \varepsilon^n \begin{pmatrix} u_n(x, x/\varepsilon) \\ v_n(x, x/\varepsilon) \end{pmatrix} \right\|_{L^2(\mathbb{R}^3, \mathbb{C}^6)} \leq C_1 \varepsilon^{N-1} \left\| (\begin{pmatrix} f^u(x) \\ f^v(x) \end{pmatrix}) \right\|_{H^{N+1}(\mathbb{R}^3, \mathbb{C}^6)},
\]
for $N \geq 2$, where the constant $C_1$ does not depend on $f^u(x)$ and $f^v(x)$.

Finally, we proved the following theorem.

**Theorem 3.1** Suppose $\text{Im}(E) > 0$, $\varepsilon > 0$, assumption 1.1 and conditions (2) hold. Then estimate (8) holds for $N \geq 2$, where the constant $C_1$ does not depend on $f^u(x)$ and $f^v(x)$.

**Corollary 3.2** Suppose $\text{Im}(E) > 0$, $\varepsilon > 0$, assumption 1.1 and conditions (2) hold. Then

$$\left\| (\mathcal{M}(\varepsilon) - E)^{-1} - \Theta(\varepsilon)(\hat{\mathcal{M}} - E)^{-1}; H^3(\mathbb{R}^3, \mathbb{C}^6) \rightarrow L_2(\mathbb{R}^3, \mathbb{C}^3) \right\| \leq C_1 \varepsilon,$$

where constant $C_1$ does not depend on $\varepsilon$ and $\Theta(\varepsilon)$ is a multiplication operator on the matrix-valued function

$$\begin{pmatrix}
\alpha(x, x/\varepsilon)Z^u(x, x/\varepsilon)\Lambda^u(x)^{-1} & 0 \\
0 & \mu(x, x/\varepsilon)Z^v(x, x/\varepsilon)\Lambda^v(x)^{-1}
\end{pmatrix},$$

$$Z^u = (\zeta_1, \zeta_2, \zeta_3), \quad Z^v = (\xi_1, \xi_2, \xi_3).$$

**Proof.** For the proof it is sufficient to apply theorem 3.1 for $N = 2$. □

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