A Formal Separation Between Strategic and Nonstrategic Behavior

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Abstract

It is common to make a distinction between “strategic” behavior and other forms of intentional but “nonstrategic” behavior: typically, that strategic agents model other agents while nonstrategic agents do not. However, a crisp boundary between these concepts has proven elusive. This problem is pervasive throughout the game theoretic literature on bounded rationality and particularly critical in parts of the behavioral game theory literature that make an explicit distinction between the behavior of “nonstrategic” level-0 agents and “strategic” higher-level agents (e.g., the level-k and cognitive hierarchy models). Overall, work discussing bounded rationality rarely gives clear guidance on how the rationality of nonstrategic agents must be bounded, instead typically just singling out specific decision rules (e.g., randomizing uniformly, playing toward the best case, optimizing the worst case) and informally asserting that they are nonstrategic. In this work, we propose a new, formal characterization of nonstrategic behavior. Our main contribution is to show that it satisfies two properties: (1) it is general enough to capture all purportedly “nonstrategic” decision rules of which we are aware in the behavioral game theory literature; (2) behavior that obeys our characterization is distinct from strategic behavior in a precise sense.

Keywords: game theory; behavioral game theory; bounded rationality; cognitive models; cognitive hierarchy; level-k

1 Introduction

A common assumption in the game theoretic literature is that agents are perfect optimizers who form correct, explicitly probabilistic beliefs and best respond to those beliefs. The behavior of such agents is commonly said to be strategic; indeed, in the early days of game theory, the term “strategic” was used as a synonym for perfect rationality (e.g., Bernheim, 1984; Pearce, 1984). At the other extreme, we might assume that agents neglect to model other agents at all, following some fixed rule like playing a specific default action or uniformly randomizing across all available actions. The behavior of such agents is commonly said to be nonstrategic. Things get muddier in between these extremes. Human players are clearly not perfect optimizers; e.g., nobody knows the Nash equilibrium strategy for chess. However, at least some of us surely do reason about the

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behavior and beliefs of other agents with whom we interact. The literature generally also calls such “boundedly rational” behavior strategic, even when the behavior is inconsistent with perfect optimization; intuitively, the dividing line is generally taken to be the question of whether agents model other agents and their incentives when deciding how to act. More formally, the term “strategic” is generally used to describe agents who act to maximize their own utilities based on explicit probabilistic beliefs about the actions of other agents (Roth and Ockenfels, 2002; Li and Tesauro, 2003; Babaioff et al., 2004; Lee, 2014; Gerding et al., 2011; Ghosh and Hummel, 2012; Grabisch et al., 2017), and the term “nonstrategic” is generally used to describe agents who follow some fixed, known decision rule (Sandholm and Lesser, 2001; Airiau and Sen, 2003; Li and Tesauro, 2003; Lee, 2014; Gerding et al., 2011; Grabisch et al., 2017).

Being able to make a sharp distinction between strategic and nonstrategic behavior matters particularly in the context of a prominent family of predictive models from behavioral game theory that describe iterative strategic reasoning, including the level-k (Nagel, 1995; Costa-Gomes et al., 2001; Crawford et al., 2010), cognitive hierarchy (Camerer et al., 2004), and quantal cognitive hierarchy models (e.g., Wright and Leyton-Brown, 2017). In all of these models, some agents are strategic in a boundedly rational sense, performing a finite number of recursive steps of reasoning about the behavior of other agents, ultimately terminating in reasoning about so-called level-0 agents, who are assumed to be nonstrategic. Level-0 behavior is frequently defined simply as uniform randomization. However, the predictive performance of such models can often be substantially improved by allowing for richer level-0 specifications (Wright and Leyton-Brown, 2014, 2019). For example, one could specify that level-0 agents act to maximize the utility of their worst case (maxmin) or of their best case (maxmax). Because these rules require only the acting agent’s utilities as inputs (and not those of any of the other agents), it seems clear that they are nonstrategic. But just because a proposed level-0 behavior can be written without reference to beliefs about other agents’ strategies, we cannot conclude that there does not exist another, equivalent way of writing it that does depend on such beliefs. For example, the maxmax rule just described can also be expressed as a best response to the belief that the opposing agents will play actions that make it possible for the acting agent to achieve their best-case outcome. Things get even worse if one aspires to learn the level-0 specification directly from data, effectively optimizing over a space of specifications (Wright and Leyton-Brown, 2014, 2019): the task now becomes reassuring a skeptic that no point in this space corresponds to behavior that could somehow be rewritten in strategic terms.

This paper defines minimal conditions that we argue must be satisfied by any strategic behavioral model. We first define several standard solution concepts, and prove that they are strategic according to our definition. We then leverage this definition to characterize a broad family of “nonstrategic” decision rules—called the elementary behavioral models—and show that they deserve the name: i.e., that no rule in this class can represent strategic reasoning. Our proposed characterization is a structural notion: it restricts the information that agents are permitted to use by restricting them to summarize all outcomes into a single number before performing their reasoning. Finally, we consider the effects of combining elementary models to construct more complicated behavioral models. Convex combinations of elementary models are also nonstrategic. Applying a sequence of elementary models one after the other and returning the first prediction that satisfies a stopping criterion (such
as non-uniform prediction) can be used to construct a minimally strategic model; however, we also demonstrate that it cannot be used to construct any of the standard solution concepts that we use as exemplar models of strategic behavior. Overall, our results are important because they distinguish strategic from nonstrategic behavioral models via formal mathematical criteria, rather than relying on the intuitive sense that a model “depends on” an explicit model of an opponent’s behavior. This makes it possible, for example, to introduce a rich, highly parameterized level-0 specification into an iterative strategic model while guaranteeing that there is no way of instantiating the level-0 specification to produce strategic behavior.

2 Background

We begin by briefly defining our formal framework and notation, discussing normal-form games, solution concepts, and behavioral models.

2.1 Normal-Form Games

A normal-form game \( G = (N, A, u) \) is defined by a tuple \( (N, A, u) \), where \( N = \{1, \ldots, n\} \) is a finite set of agents; \( A = A_1 \times \ldots \times A_n \) is the set of possible action profiles; \( A_i \) is the finite set of actions available to agent \( i \); and \( u = \{u_i\}_{i \in N} \) is a set of utility functions \( u_i : A \to \mathbb{R} \), each of which maps from an action profile to a utility for agent \( i \). Agents may also randomize over their actions. It is standard in the literature to call such randomization a mixed strategy; however, for our purposes this terminology will be confusing, since it would lead us to discuss the strategies of nonstrategic agents. We thus instead adopt the somewhat nonstandard terminology behavior for this concept. We denote the set of agent \( i \)'s possible behaviors by \( S_i = \Delta_{|A_i|} \), and the set of possible behavior profiles by \( S = S_1 \times \ldots \times S_n \), where \( \Delta^k \) denotes the standard \( k \)-simplex (the set \( \{\theta_0 + \cdots + \theta_k \mid \sum_{i=0}^{k} \theta_i = 1 \text{ and } \theta_i \geq 0 \text{ for all } i\} \)), and hence \( \Delta^{|X|} \) is the set of probability distributions over a finite set \( X \). Overloading notation, we represent the expected utility to agent \( i \) of a behavior profile \( s \in S \) by \( u_i(s) \). We use the notation \( s_{-i} \) to refer to the behavior profile of all agents except \( i \), and \((s_i, s_{-i})\) to represent a full behavior profile.

2.2 Solution Concepts

A solution concept is a mapping from a game \( G \) to a behavior profile (or set of behavior profiles) that satisfies some criteria. We will primarily be concerned with these solution concepts as formalizations of strategic behavior in games.

The foundational solution concept in game theory, and the most commonly used, is the Nash equilibrium.

Definition 1 (Nash equilibrium). Let \( BR_i(s_{-i}) = \arg \max_{a_i \in A_i} u_i(a_i, s_{-i}) \) denote the set of agent \( i \)'s best responses to a behavior profile \( s_{-i} \in S_{-i} \). A Nash equilibrium is a behavior profile in which every agent simultaneously best responds to all the other agents. Formally, \( s^* \in S \) is a Nash equilibrium if \( \forall i \in N, a_i \in A_i : s^*_i(a_i) > 0 \implies a_i \in BR_i(s^*_{-i}). \)
When agents play a Nash equilibrium, they must randomize independently. A correlated equilibrium relaxes this requirement, and allows for joint distributions of actions that are correlated.

**Definition 2** (Correlated equilibrium). A correlated equilibrium is a distribution $\sigma \in \Delta^{|A|}$ over action profiles which satisfies the following for every agent $i \in N$ and every mapping $\xi_i : A_i \to A_i$:

$$\sum_{a \in A} \sigma(a) u_i(a_i, a_{-i}) \geq \sum_{a \in A} \sigma(a) u_i(\xi_i(a_i), a_{-i}).$$

Note that every Nash equilibrium $s^*$ corresponds to a correlated equilibrium $\sigma^*(a) = \Pi_{i \in N} s^*_i(a_i)$.

One important idea is that people become more likely to make errors as the cost of making those errors decreases. This can be modeled by assuming that agents best respond quantally, rather than via strict maximization. A quantal response plays actions with high expected utility with high probability, and actions with low expected utility with lower probability. An equilibrium in which agents quantally respond to each other, rather than best responding to each other, is called a quantal response equilibrium (McKelvey and Palfrey, 1995).

**Definition 3** (Quantal best response). A (logit) quantal best response $QBR_i(s_{-i}; \lambda, G)$ by agent $i$ to $s_{-i}$ in game $G$ is a behavior $s_i$ such that

$$s_i(a_i) = \frac{\exp[\lambda \cdot u_i(a_i, s_{-i})]}{\sum_{a'_i} \exp[\lambda \cdot u_i(a'_i, s_{-i})]}, \quad (1)$$

where $\lambda$ (the precision parameter) indicates how sensitive agents are to utility differences. When $\lambda = 0$, quantal best response is equivalent to uniform randomization. As $\lambda \to \infty$, quantal best response corresponds to best response in the sense that actions that are not best responses are played with probability that approaches zero; i.e., for all $a_i \notin BR_i(s_{-i})$, it is that case that $\lim_{\lambda \to \infty} QBR_i(s_{-i}; \lambda)(a_i) = 0$.

**Definition 4** (QRE). A quantal response equilibrium with precision $\lambda$ is a behavior profile $s^*$ in which every agent’s behavior is a quantal best response to the behaviors of the other agents; i.e., for all agents $i$, $s^*_i = QBR_i^G(s^*_{-i}; \lambda)$.

In general, a game can have multiple QREs with a given precision. However, when using QRE for predictions, it is common to select a particular equilibrium for a given $\lambda$ that lies on a one-dimensional manifold in the joint space of strategies and precisions that starts from the uniform strategy at $\lambda = 0$. Note that, although quantal best response approaches best response as precision goes to infinity, there nevertheless exist Nash equilibria that are not the limits of any sequence of QREs.

### 2.3 Models of Agent Behavior

We now turn to what we term behavioral models, functions that return a probability distribution over a single agent’s action space for every given game. We will often refer to this...
distribution informally as the model’s “behavior”. Unlike a solution concept, which represents a set of joint strategies that are consistent with some criterion, a behavioral model represents a prediction of a single agent’s actions. Profiles of behavioral models can thus be seen as solution concepts that always encode a single product distribution over a given set of individual behaviors. In a later section we consider what can be said about the profiles of behavioral models induced by existing solution concepts.

Before we can define behavioral models, we must introduce some basic notation. Let:

- \( \mathcal{G} \) denote the space of all finite normal-form games;
- \( \mathbb{R}^* = \bigcup_{k=1}^{\infty} \mathbb{R}^k \) denote the space of all finite vectors;
- \( (\mathbb{R}^*)^* = \bigcup_{k=1}^{\infty} (\mathbb{R}^*)^k \) denote the space of all finite-sized, finite-dimensional tensors;
- \( \Delta^* = \bigcup_{k=1}^{\infty} \Delta^k \) denote the space of all finite standard simplices; and
- \( A_{G,i} \) denote player \( i \)'s action space in game \( G \).

**Definition 5** (Behavioral models). A behavioral model is a function \( f_i : \mathcal{G} \to \Delta^* \); \( f_i(G) \in \Delta^{\left| A_{G,i} \right|} \) for all games \( G \). We use a function name with no agent subscript, such as \( f \), to denote a profile of behavioral models with one function \( f_i \) for each agent. We write \( f(G) \) to denote the behavior profile that results from applying each \( f_i \) to \( G \).

Much work in behavioral game theory proposes behavioral models rather than solution concepts (though the formal definition of a behavioral model is our own). One key idea from that literature is that humans can only perform a limited number of steps of strategic reasoning, or equivalently that they only reason about higher-order beliefs up to some fixed, maximum order.

We begin with the so-called level-\( k \) model (Nagel, 1995; Costa-Gomes et al., 2001). Unlike Nash equilibrium, correlated equilibrium, and quantal response equilibrium, all of which describe fixed points, the level-\( k \) model is computed via a finite number of best response calculations. Each agent \( i \) is associated with a level \( k_i \in \mathbb{N} \), corresponding to the number of steps of reasoning the agent is able to perform. A level-0 agent plays nonstrategically (i.e., without reasoning about its opponent); a level-\( k \) agent (for \( k \geq 1 \)) best responds to the belief that all other agents are level-\((k - 1)\). The level-\( k \) model implies a distribution over play for all agents when combined with a distribution \( D \in \Delta^* \) over levels.

**Definition 6** (Level-\( k \) prediction). Fix a distribution \( D \in \Delta^* \) over levels and a level-0 behavior \( s^0 \in S \). Then the level-\( k \) behavior for an agent \( i \) is defined as

\[
s^k_i(a_i) \propto I\left[ a_i \in BR_i(s^k_{-i}) \right],
\]

where \( I[\cdot] \) is the indicator function that returns 1 when its argument is true and 0 otherwise. The level-\( k \) prediction \( \pi^{Lk} \in S \) for a game \( G \) is the average of the behavior of the level-\( k \) strategies weighted by the frequency of the levels, \( \pi^{Lk}_i(a_i) = \sum_{k=0}^{\infty} D(k)s^k_i(a_i) \).

Cognitive hierarchy (Camerer et al., 2004) is a very similar model in which agents respond to the distribution of lower-level agents, rather than believing that every agent performs exactly one step less of reasoning.
Definition 7 (Cognitive hierarchy prediction). Fix a distribution \( D \in \Delta^* \) over levels and a level-0 behavior \( s^0 \in S \). Then the level-\( k \) hierarchical behavior for an agent \( i \) is

\[
\pi^k_i(a_i) \propto \mathbb{I} \left[ a_i \in BR_i(\pi^{0:k-1}_i) \right],
\]

where \( \pi^0 = s^0 \) and \( \pi^{0:k-1}_i(a_i) = \sum_{m=0}^{k-1} D(m) \pi^m_i(a_i) \). The cognitive hierarchy prediction is again the average of the level-\( k \) hierarchical strategies weighted by the frequencies of the levels, \( \pi^{CH}_i(a_i) = \sum_{k=0}^{\infty} D(k) \pi^k_i(a_i) \).

As we did with quantal response equilibrium, it is possible to generalize these iterative solution concepts by basing agents’ behavior on quantal best responses rather than best responses. The resulting models are called quantal level-\( k \) and quantal cognitive hierarchy (e.g., Stahl and Wilson, 1994; Wright and Leyton-Brown, 2017).

All of the iterative solution concepts described above rely on the specification of a nonstrategic level-0 behavior. This need is a crucial motivation for the current paper, in which we explore what behaviors can be candidates for this specification.

3 Strategic Behavioral Models

As discussed in the introduction, there is general qualitative agreement in the literature that strategic agents act to maximize their own utilities based on explicit probabilistic beliefs about the actions of other agents. Our ultimate goal is to characterize behavioral models that are unambiguously nonstrategic; thus, to strengthen our results, we adopt a somewhat more expansive notion of strategic behavior. Specifically, we define an agent as weakly strategic if it satisfies two conditions, which we call (1) other responsiveness and (2) domination aversion. These conditions require that the agent chooses actions both (1) with at least some dependence on others’ payoffs; and (2) with at least some concern for its own payoffs.

The key feature of strategic agents is that they take account of the incentives of other agents when choosing their own actions. To capture this intuition via the weakest possible necessary condition, we say that an agent is other responsive if its behavior is ever influenced by changes (only) to the utilities of other agents.

Definition 8 (Other responsiveness). A behavioral model \( f_i \) is other responsive if there exists a pair of games \( G = (N, A, u) \) and \( G' = (N, A, u') \) such that \( u_i(a) = u'_i(a) \) for all \( a \in A \), but \( f_i(G) \neq f_i(G') \).

It is also traditional to assume that agents always act to maximize their expected utilities. This assumption is too strong for our purposes; for example, we want to allow for deviations from perfect utility maximization such as quantal best response. However, it does not seem reasonable to call an agent strategic if they pay no attention whatsoever to their own payoffs. We thus introduce a concept that we call domination aversion, which is a considerably weaker sense in which an agent might show concern for their own payoffs. The condition requires only that gross changes in agents’ own incentives will cause them to change their behavior. Specifically, we say that a behavioral model is domination averse if the player can be induced to play any action with probability arbitrarily close 0, simply by
ensuring that it is sufficiently dominated by another action, while leaving the rest of the game unchanged.

**Definition 9** (Domination aversion). A behavioral model $f_i : G \rightarrow \Delta^*$ is domination averse if for every $\delta > 0$, finite set $V \subset \mathbb{R}$, there is a number $\zeta(f_i, V, \delta)$ such that $f_i(G)(a_i) < \delta$ in all games $G = (N, A, u)$ that satisfy:

1. There exists $a^+_i \in A_i$ such that $u_i(a_i, a_{-i}) \leq u_i(a^+_i, a_{-i}) - \zeta(f_i, V, \delta)$ for every $a_{-i} \in A_{-i}$; and

2. $u_i(a_i, a_{-i}) \in V$ for every $a_i \in A_i \setminus \{a^+_i\}$ and $a_{-i} \in A_{-i}$.

We will sometimes write $\zeta(f_i, G, \delta)$ as a shorthand for $\zeta(f_i, \{u_i(a) | a \in A\}, \delta)$.

Where other responsiveness requires only that a behavioral model sometimes change its behavior, domination aversion requires its behavior to change in a specific way. We impose this stronger requirement as a way to strike a balance between two objectives. On the one hand, we want to require that a strategic model in some way aims to achieve high utility outcomes. A weaker condition that required only that a model sometimes change its behavior in response to changes in its own payoffs would allow for pathological behavior such as playing toward low utilities, particular favorite numbers to the exclusion of all else, preferring dominated actions and so forth. However, we do not want to assume a specific mechanism for seeking higher utilities; in particular, we do not want to require optimization with respect to probabilistic beliefs.

Notice that we do not require that a domination averse model respect domination in the sense of assigning low or zero probability to every strictly dominated strategy. However, we do require that strategies that are dominated by a sufficient margin are played with low probability. This implies a symmetric property for dominant strategies: a strategy that dominates all other strategies by a sufficient margin will be played with probability close to 1.

Behavioral models that are both other responsive and domination averse satisfy a very low bar for strategic behavior. That means that models that fail to satisfy either one or both condition are unambiguously nonstrategic; we refer to such models as strongly nonstrategic.

**Definition 10** (Strongly nonstrategic behavioral model). A behavioral model $f_i$ is weakly strategic if it is both other responsive and domination averse. Conversely, a behavioral model that is not weakly strategic is strongly nonstrategic.

### 4 Existing Solution Concepts are Weakly Strategic

We now demonstrate that our definition of weakly strategic behavioral models does more than describe qualitative patterns of behavior that have been called “strategic” in the past: it also formally captures the predictions of various solution concepts both from classical game theory and from behavioral game theory (Nash equilibrium, correlated equilibrium, quantal response equilibrium, level-$k$, cognitive hierarchy, and quantal cognitive hierarchy).

We first show that any model defined as either a best response or a quantal best response to a profile of domination averse behavioral models is itself domination averse.
Lemma 1. For any $\lambda > 0$ and profile of behavioral models $f_{-i}$, the behavioral model $q_i(G) = QBR_i(f_{-i}(G); \lambda, G)$ is domination averse.

Proof. We directly derive $\zeta(q_i, G, \delta)$ from Definition 9. Fix arbitrary $G = (N, A, u)$ with $|A_i| = m$ and $a_i^+ \neq a_i \in A_i$, and let $z = \min_{a'_{-i} \in A} u_i(a_i^+, a'_{-i}) - u_i(a_i, a_{-i})$. Then

$$q_i(G)(a_i) = \frac{\exp(\lambda u_i(a_i, f_{-i}(G)))}{\sum_{a'_{-i} \in A} \exp(\lambda u_i(a'_{-i}, f_{-i}(G)))} < \frac{\exp(\lambda u_i(a_i, f_{-i}(G)))}{\exp(\lambda|u_i(a_i^+, f_{-i}(G)) + z|)} \leq \frac{\exp(\lambda u_i(a_i, f_{-i}(G)))}{\exp(\lambda u_i(a_i, f_{-i}(G)) + \exp(\lambda z)} = \exp(-\lambda z).$$

Thus, to guarantee $f_i(G)(a_i) < \delta$, it is sufficient to ensure that

$$\exp(-\lambda z) < \delta \iff -\lambda z < \log \delta \iff z > -\frac{\log \delta}{\lambda}.$$ 

Therefore $q_i$ is domination averse, with $\zeta(q_i, V, \delta) = (-\log \delta)/\lambda$ regardless of $V$. \hfill \Box

Lemma 2. For any profile of behavioral models $f_{-i}$, any behavioral model $b_i$ that satisfies

$$a_i \notin BR_i(f_{-i}(G)) \implies b_i(G)(a_i) = 0$$

is domination averse.

Proof. For any actions $a_i^+, a_i \in A_i$, if $u_i(a_i^+, a_{-i}) > u_i(a_i, a_{-i})$ for all $a_{-i} \in A_{-i}$, then by assumption $b_i(G)(a_i) = 0$. Therefore, it is sufficient to define $\zeta(b_i, V, \delta) = \epsilon$ for any constant $\epsilon > 0$, regardless of $V$. \hfill \Box

We now show that quantal best response to any profile of domination averse behavioral models is strategic. From this result, it will immediately follow that Nash equilibrium, correlated equilibrium, quantal response equilibrium, level-$k$, cognitive hierarchy, and quantal cognitive hierarchy are weakly strategic.

Lemma 3. For any $\lambda > 0$ and domination averse profile of behavioral models $f_{-i}$, the behavioral models $q_i$ and $b_i$ are both weakly strategic, where

$$q_i(G) = QBR_i(f_{-i}(G); \lambda, G)$$

and $b_i$ satisfies

$$a_i \notin BR_i(f_{-i}(G)) \implies b_i(G)(a_i) = 0.$$
Proof. Both models are domination averse by Lemmas 1 and 2 respectively. It remains only to show that both models are other responsive.

Consider the following games.

\[
\begin{array}{c|c|c}
 & L & R \\
\hline
U & 1, \zeta(f_i, \{0, 1\}, 0.5) & 0, 0 \\
D & 0, \zeta(f_i, \{0, 1\}, 0.5) & 1, 0 \\
\end{array}
\]

\[
\begin{array}{c|c|c}
 & L & R \\
\hline
U & 1, 0 & 0, \zeta(f_j, \{0, 1\}, 0.5) \\
D & 0, 0 & 1, \zeta(f_j, \{0, 1\}, 0.5) \\
\end{array}
\]

Let \( i \) be the row player and \( j \) be the column player. Since \( f_j \) is domination averse, \( \zeta(f_j, \{0, 1\}, 0.5) \) is well defined, and thus \( f_j(G_1)(L) > f_j(G_1)(R) \), and \( f_j(G_2)(L) < f_j(G_2)(R) \). So then \( q_i(G_1)(U) > q_i(G_1)(D) \), whereas \( q_i(G_2)(U) < q_i(G_2)(D) \). So we can change \( q_i \)'s behavior by changing only \( j \)'s payoffs, and therefore \( q_i \) is other responsive.

Similarly, \( b_i(G_1)(U) > b_i(G_1)(D) = 0 \), whereas \( 0 = b_i(G_2)(U) < b_i(G_2)(D) \), so \( b_i \) is also other responsive.

\[\]

**Theorem 1.** All of QRE, Nash equilibrium, and correlated equilibrium are (profiles of) weakly strategic behavioral models.

Proof. QRE is immediate from Lemma 3 by letting \( f_i(G) = QBR_i(f_{-i}(G); \lambda_i, G) \) for some set \( \{\lambda_i > 0\}_{i=1}^N \). Nash equilibrium and correlated equilibrium are immediate from Lemma 3 by letting \( f \) return an arbitrary Nash (correlated) equilibrium.

\[\]

**Theorem 2.** All of level-k, cognitive hierarchy, and quantal cognitive hierarchy are weakly strategic behavioral models for agents of level 2 and higher. Level 1 is weakly strategic or not depending on level 0.

Proof. Let \( f^0 \) be a profile of behavioral models and \( \lambda_1^1, \lambda_2^2 > 0 \) for all \( i \in N \). To prove the result for quantal cognitive hierarchy, choose behavioral model profiles \( f^1, f^2 \) satisfying \( f^1_i(G) = QBR_i(f_{-i}^0(G); \lambda_1^1, G) \) and \( f^2_i(G) = QBR_i(f_{-i}^1(G); \lambda_2^2, G) \) for all games \( G \) and players \( i \).

If all of the behavioral models in \( f^0 \) are domination averse, then by the argument in the proof of Theorem 3 all of the behavioral models in \( f^1 \) are weakly strategic.

Otherwise, all of the behavioral models in \( f^1 \) are domination averse by Lemma 1 and thus by the argument of the proof of Theorem 3 all of the behavioral models in \( f^2 \) are weakly strategic.

To prove the result for level-k and cognitive hierarchy, instead choose

\[
f^1_i(G)(a_i) = \frac{\mathbb{I} \left[ a_i \in BR_i(f_{-i}^0(G)) \right]}{\sum_{a_i' \in A_i} \mathbb{I} \left[ a_i' \in BR_i(f_{-i}^0(G)) \right]} \quad \text{and} \quad f^2_i(G)(a_i) = \frac{\mathbb{I} \left[ a_i \in BR_i(f_{-i}^1(G)) \right]}{\sum_{a_i' \in A_i} \mathbb{I} \left[ a_i' \in BR_i(f_{-i}^1(G)) \right]},
\]

and follow an identical argument. \( \square \)
For example, when level-0 is uniform, level-1 is strongly nonstrategic, because no change to \( j \)'s payoffs can change \( i \)'s level-1 behavior. But when level-0 is maxmax, level-1 is weakly strategic.

5 Elementary Behavioral Models

Our main task in this paper is to separate nonstrategic behavior from strategic behavior. Now that we have formally defined the latter, we can introduce a class of behavioral models, called elementary models, that we will ultimately show are always strongly nonstrategic. Observe that an agent reasoning strategically needs to account both for its own payoffs (in order to be domination averse) and for others' payoffs (in order to be other responsive); thus, it must evaluate each outcome in multiple terms. Our key idea is thus to require that nonstrategic behavior independently “scores” each outcome using a single number. In this section, we formalize such a notion and illustrate its generality via examples of how it can be used to encode previously proposed “nonstrategic” behaviors.

5.1 Defining Elementary Behavioral Models

The formal definition of elementary behavioral models is unfortunately more complex than the intuition we just gave. The reason is that any tuple of \( k \) real values can be encoded into a single real number; in information economics this is referred to as dimension smuggling (e.g., Nisan and Segal, 2006). Without ruling out dimension smuggling, therefore, a restriction that nonstrategic agents rely on only a single number would lack any force. We thus restrict the class of functions that an elementary model can use to those that are either dictatorial or non-encoding; that is, functions that either ignore all inputs but one, or combine their inputs in a way that makes it impossible to determine what any single input must have been.

**Definition 11 (Dictatorial function).** A function \( \varphi : \mathbb{R}^m \to \mathbb{R}^n \) is dictatorial if its value is completely determined by a single input: \( \exists i \in \{1, \ldots, m\} \) such that \( \forall x, x' \in \mathbb{R}^{m-1}, \forall c \in \mathbb{R}, \varphi(x_1, \ldots, x_{i-1}, c, x_i, \ldots, x_{m-1}) = \varphi(x'_{i}, \ldots, x'_{i-1}, c, x'_{i}, \ldots, x'_{m-1}). \)

This class of functions takes its name from the social choice condition from which it is inspired; one input to the function \( \varphi \) acts as a dictator over \( \varphi \)'s output.

**Definition 12 (Non-encoding function).** A function \( \varphi : \mathbb{R}^N \to \mathbb{R} \) for \( N \geq 2 \) is non-encoding iff for every \( y \in \mathbb{R} \) and \( 1 \leq i \leq N \), the set consisting of the \( i \)-th element of every vector in the preimage \( \varphi^{-1}\{\{y\}\} \) has infinite diameter. That is, for every \( y, b \in \mathbb{R} \), there exist \( x, x' \in \mathbb{R}^N \) such that

1. \( \varphi(x) = \varphi(x') = y \), and
2. \( |x_i - x'_i| > b \).

The reason we want such a condition is to restrict our attention to functions that combine utility values for multiple players into a single value in a meaningfully nonreversable way. (Simple examples include summing the values, taking their max, taking the first value if it
is greater than some constant and otherwise taking the second, taking a convex combination of different values, etc.)

Our condition is stronger than simply requiring the function to be non-invertible. For a function to be non-invertible, it is sufficient that there exist a single value in its range that is mapped to by multiple inputs. In contrast, we require every value in the range to be mapped to by infinitely many inputs. Furthermore, every output must be mapped to by a pair of inputs that are arbitrarily far apart in a specific dimension. In the context of utilities that have been summarized by some non-encoding function, this means that there is no way to recover the utility for a specific player based on the player, to any degree of approximation, unless the output is always computed using only that dimension.

For example, the linear combination \( \psi^{(1)}(x) = x_1 + x_2 \) is non-encoding; knowing that \( \psi^{(1)}(1)(x) = 7 \) gives no information about what value \( x_1 \) must be, nor even a non-trivial neighborhood that must contain \( x_1 \). In contrast, the function \( \psi^{(2)}(x) = 10|x_1| + \exp(x_2)/(1 + \exp(x_2)) \) does not satisfy non-encoding, even though each output is mapped to by infinitely many inputs, because given \( \psi^{(2)}(x) \), it is possible to approximate the value of \( x_1 \) to within \( \pm 1 \), and it is possible to recover \( x_2 \) exactly.

We are now ready to formally define elementary behavioral models. Intuitively, an elementary behavioral model is one which first summarizes the outcome of each action profile as a single number computed only from the profile of utilities it induces. The model then computes its behavior based only on these “potentials”.

**Definition 13** (Elementary behavioral model). A behavioral model \( f_i : G \rightarrow \Delta^* \) is elementary if it can be represented as \( h(\Phi(G)) \), where

1. \( \Phi(G) \) maps a game \( G = (N, A, u) \) to a vector with one entry containing \( \varphi(u(a)) \) for each action profile \( a \in A \), and
2. \( \varphi \) is either dictatorial or non-encoding, and
3. \( h : (\mathbb{R}^*)^* \rightarrow \Delta^{\lvert A_i \rvert} \) is an arbitrary function; we use it to map from \( \mathbb{R}^A \) to \( \Delta^{\lvert A_i \rvert} \).

For convenience, when condition \( \Box \) holds we refer to \( \Phi \) as the potential map for \( \varphi \).

That is, an elementary behavioral model works as follows. First, given an arbitrary game \( G = (N, A, u) \), and for each action profile \( a \in A \), we apply the same (either dictatorial or non-encoding) function \( \varphi \) to the \( \lvert N \rvert \)-tuple of real values \( \langle u_1(a), \ldots, u_{\lvert N \rvert}(a) \rangle \), producing in each case a single real value. We represent all of these real values in a mapping we call \( \Phi \); this potential map is a function of the same size as each of the utility functions. We then apply an arbitrary function \( h \) to \( \Phi(G) \), producing a probability distribution over \( A_i \).

### 5.2 Examples of Elementary Behavioral Models

To demonstrate the generality of elementary behavioral models, we show how to encode each of the candidate level-0 behavioral models that we proposed in our past work (Wright and Levton-Brown, 2014). (Thus, although that work only appealed to intuition, we can now conclude that these behavioral models are indeed all strongly nonstrategic.)

We begin with the simplest behavioral models: those that depend only on a given agent \( i \)'s utilities \( u_i \).
Example 1 (Maxmax behavioral model). A maxmax action for agent $i$ is an action whose best case-utility for $i$ is greater than the best case-utility of any of $i$'s other actions. An agent that wishes to maximize its possible payoff will play a maxmax action. The maxmax behavioral model $f_i^\text{maxmax}(G)$ uniformly randomizes over all of $i$’s maxmax actions in $G$: $f_i^\text{maxmax}(G)(a_i) \propto I \left[ a_i \in \arg \max_{a'_i \in A_i} \max_{a_{-i} \in A_{-i}} u_i(a'_i, a_{-i}) \right]$. 

Example 2 (Maxmin behavioral model). A maxmin action for agent $i$ is the action with the best worst-case guarantee. This is the safest action to play against hostile agents. The maxmin behavioral model $f_i^\text{maxmin}(G)$ uniformly randomizes over all of $i$’s maxmin actions in $G$: $f_i^\text{maxmin}(G)(a_i) \propto I \left[ a_i \in \arg \max_{a'_i \in A_i} \min_{a_{-i} \in A_{-i}} u_i(a'_i, a_{-i}) \right]$. 

Example 3 (Minimax regret behavioral model). Following [Savage 1951], for each action profile, an agent has a possible regret: how much more utility could the agent have gained by playing the best response to the other agents’ actions? Each of the agent’s actions is therefore associated with a vector of possible regrets, one for each possible profile of the other agents’ actions. A minimax regret action is an action whose maximum regret (in the vector of possible regrets) is minimal. The minimax regret behavioral model $f_i^\text{mmr}(G)$ uniformly randomizes over all of $i$’s minimax regret actions in $G$. That is, if $r(a_i, a_{-i}) = u_i(a_i, a_{-i}) - \max_{a'_i \in A_i} u_i(a'_i, a_{-i})$ is the regret of agent $i$ in action profile $(a_i, a_{-i})$, then $f_i^\text{mmr}(G)(a_i) \propto I \left[ a_i \in \arg \min_{a'_i \in A_i} \max_{a_{-i} \in A_{-i}} r(a_i, a_{-i}) \right]$. 

Because each of the maxmax, maxmin, and minimax regret behavioral models depends only on agent $i$’s payoffs, we can set $\varphi(u(a)) = u_i(a)$ in each case; this $\varphi$ is dictatorial. The encodings differ only in their choice of $h$. These vary in their complexity (e.g., maxmax: simply uniformly randomize over all actions that tie for corresponding to the largest potential value; for minimax regret it is necessary to compute a best response for each action profile). However, recall that $h$ is an arbitrary function; thus, this is not a problem for our encoding.

Other behavioral models depend on both agents’ utilities, and so require different $\varphi$ functions.

Example 4 (Max welfare behavioral model). An max welfare action is part of some action profile that maximizes the sum of agents’ utilities. The max welfare behavioral model $f_i^W(G)$ uniformly randomizes over max welfare actions in $G$: $f_i^W(G)(a_i) \propto I \left[ a_i \in \arg \max_{a'_i \in A_i} \max_{a_{-i} \in A_{-i}} \sum_{j \in N} u_j(a'_i, a_{-i}) \right]$. 

We can encode the efficient behavioral model as elementary by setting $\varphi(u(a)) = \sum_j u_j(a)$. This $\varphi$ satisfies non-encoding; in fact, all linear combinations do. We then define $h$ to uniformly randomize over all actions maximal potential value.
Proposition 1. Any linear function $\varphi(x) = w_0 + \sum_{j=1}^{N} w_j x_j$ is either dictatorial or non-encoding.

Proof. If there are zero or one weights $w_j \neq 0$ with $1 \leq j \neq N$, then the function is dictatorial and we are done. Otherwise, fix arbitrary $x \in \mathbb{R}^N$, $b, \epsilon > 0$, and $1 \leq i, j \leq N$ with $w_j \neq 0$ and $j \neq i$. Construct $x' \in \mathbb{R}^N$ by setting $x'_i = x_i + (1 + \epsilon)b$, setting $x'_j = x_j - \frac{w_i}{w_j}(1 + \epsilon)b$, and setting $x'_l = x_l$ for all $l \neq i$ with $1 \leq l \leq N$. Observe that $\varphi(x) = \varphi(x')$ and $|x_i - x'_i| > b$, as required. Since $x$ was arbitrary, this pair can be found for any $y = \varphi(x)$.

Example 5 (Fair behavioral model). Let the unfairness of an action profile be the difference between the maximum and minimum payoffs among the agents under that action profile: $d(a) = \max_{i,j \in N} u_i(a) - u_j(a)$.

Then a “fair” outcome minimizes this difference in utilities. A fair action is part of a minimally unfair action profile. The fair behavioral model $f_i^\text{fair}(G)$ uniformly randomizes over fair actions: $f_i^\text{fair}(G)(a_i) \propto I\left[a_i \in \arg \min_{a_i' \in A_i} \min_{a_{-i} \in A_{-i}} d(a'_i, a_{-i})\right]$.

We can encode the fair behavioral model as elementary by setting $\varphi(u(a)) = \max_{j,k}(u_j(a) - u_k(a))$; it is again straightforward to demonstrate that this potential function is non-encoding. We then define $h$ to uniformly randomize over all actions with minimal potential value.

Proposition 2. Let $\varphi : \mathbb{R}^N \to \mathbb{R}$ for some $N \geq 2$ be defined by $\varphi(x) = \max_{1 \leq j,k \leq N}(x_j - x_k)$. Then $\varphi$ is non-encoding.

Proof. Fix arbitrary $x \in \mathbb{R}^N$, $b, \epsilon > 0$. Let $x'_i = x_i + (1 - \epsilon)b$ for every $1 \leq i \leq N$. Clearly $|x_i - x'_i| > b$ for every $1 \leq i \leq N$. Since $x'_j - x'_i = x_j - x_i$ for every $1 \leq i, j \leq N$, we also have $\varphi(x) = \varphi(x')$.

Finally, we note that all of the examples just given are binary: actions are either fair/maxmin/etc or they are not. By changing only $h$, we could similarly construct continuous variants of each concept in which actions that achieve nearly maximal potentials are played nearly as often by the behavioral model, etc.

6 Elementary Behavioral Models are Strongly Nonstrategic

We are now ready to show that elementary behavioral models are always strongly nonstrategic. This result is important because it achieves our key goal of distinguishing strategic from nonstrategic behavioral models via a formal mathematical criterion, rather than relying on the intuitive sense that a model “depends on” an explicit model of an opponent’s behavior. In fact, we do a bit better than simply showing that elementary models are nonstrategic: we show that the space of domination averse behavioral models is exactly partitioned into elementary models and weakly strategic models.

Theorem 3. Every elementary behavioral model is strongly nonstrategic.

Proof. Suppose for contradiction that elementary behavioral model $f_i(G) = h_i(\Phi(G))$ is weakly strategic, where $\Phi$ is the potential map for $\varphi$. By the definition of elementary behavioral models, $\varphi$ is either dictatorial or non-encoding.
1. \(i\) is a dictator for \(\varphi\). Because \(f_i\) is other responsive, there exist \(G = (N, A, u)\) and \(G' = (N, A, u')\) with \(u_i(a) = u'_i(a)\) for all \(a \in A\), such that \(f_i(G) \neq f_i(G')\). But since \(i\) is a dictator for \(\varphi\), \(\Phi(G) = \Phi(G')\), and hence \(f_i(G) = f_i(G')\), a contradiction.

2. \(\varphi\) is non-encoding. Let \(x, x' \in \mathbb{R}^2\) be two vectors such that \(x'_i - x_i \geq \zeta(f_i, \{x_i\}, 0.5)\) and \(\varphi(x) = \varphi(x')\). These vectors are guaranteed to exist by the definition of non-encoding and the assumption that \(f_i\) is domination averse. Now we use these utility vectors to construct 2-player games \(G_3\) and \(G_4\) in which \(i\) has two actions \(\{U, D\}\) and the other agent has actions \(\{L, R\}\).

|   | L | R |
|---|---|---|
| U | \(x\) | \(x'\) |
| D | \(x'\) | \(x\) |

\(G_3\)

|   | L | R |
|---|---|---|
| U | \(x\) | \(x\) |
| D | \(x'\) | \(x'\) |

\(G_4\)

Note that \(i\)'s utility in \(G_3\) for playing \(U\) exceeds that of playing \(D\) by \(\zeta(f_i, \{x_i\}, 0.5)\), and thus by domination aversion \(f_i(G_3)(U) > f_i(G_3)(D)\). By the same argument, \(f_i(G_4)(U) < f_i(G_4)(D)\). But since \(\varphi(x) = \varphi(x')\), and since \(x\) and \(x'\) are the only payoff tuples that occur in either \(G_3\) or \(G_4\), \(\Phi(G_3) = \Phi(G_4)\) and hence \(f_i(G_3) = f_i(G_4)\), a contradiction.

\[\square\]

The set of elementary behavioral models includes every strongly nonstrategic domination averse model.

**Theorem 4.** A domination averse model is other responsive iff it is not elementary.

**Proof.** Only-if direction: Domination averse and not elementary implies not other responsive. If a model is both domination averse and elementary, then by Theorem 3 it is not other responsive.

If direction: If a model is domination averse and not other responsive, then it is elementary. Suppose that a behavioral model \(f_i\) is domination averse, but not other responsive. Therefore, for every pair of games \(G = (N, A, u)\) and \(G' = (N, A, u')\) with \(u_i(a) = u'_i(a)\) for all \(a \in A\), \(f_i(G) = f_i(G')\). We show how to represent \(f_i\) as an elementary function by constructing appropriate \(\varphi\), \(\Phi\), and \(h_i\) functions. Define \(\varphi(x) = x_i\). Let \(h_i(\Phi(G)) = f_i(z(\Phi(G)))\), where \(z: \mathbb{R}^A \rightarrow \mathcal{G}\) is a function that returns a game with the utilities of \(i\) set to its argument and the utilities of the other players set to 0. Since differences in the other agents’ utilities never change the output of \(f_i\), \(h_i(\Phi(G)) = f_i(z(\Phi(G))) = f_i(G)\) for all \(G\).

\[\square\]

## 7 Combinations of Elementary Models

We now consider behavioral models that are constructed by drawing together the predictions of multiple elementary models. We first consider convex combinations of elementary models, followed by convex combination of elementary models with a pre-processing step to remove
“uninformative” predictions. Our key result in these two subsections is that combining elementary models in this way preserves the strong nonstrategic property.

We then move on to consider an alternative scheme for combining elementary models into a class of models that we call the sequentially elementary models. After first demonstrating with an example that this class of models includes weakly strategic models, we show that it is nevertheless not possible to represent any of our exemplar strategic models (Nash equilibrium, correlated equilibrium, level-k, cognitive hierarchy, quantal cognitive hierarchy, or QRE) as a sequentially elementary model.

7.1 Convex Combinations of Elementary Models

Convex combinations preserve the strong nonstrategic property of elementary behavioral models.

Theorem 5. Every convex combination of elementary models is strongly nonstrategic.

Proof. Suppose not. Then there exists some set of $K$ elementary models $f^k_i$ and $K$ weights $w_k \in [0, 1]$ with $\sum_{k=1}^{K} w_k = 1$ such that $g_i(G) = \sum_{k=1}^{K} w_k f^k_i(G)$ is both domination averse and other-responsive. Since $g_i$ is other-responsive, there must be at least one $k$ such that $f^k_i$ is other-responsive. Now construct $G_3$ and $G_4$ again as in the proof of Theorem 3, this time using $x, x'$ values that satisfy $x_i = 0, \varphi^k_i(x) = \varphi^k_i(x')$, and $x'_i - x_i \geq \zeta(f^k_i, \{0\}, w_k/2)$.

Since $g_i$ is domination averse, it must be the case that $g_i(G_3)(D) < w_k/2$ and $g_i(G_4)(U) < w_k/2$. But since $\varphi^k_i(x) = \varphi^k_i(x')$, it must also be that $f^k_i(G_3) = f^k_i(G_4)$, and thus either $g_i(G_3)(D) \geq w_k/2$ or $g_i(G_4)(U) \geq w_k/2$, a contradiction. \qed

7.2 Convex Combinations with Filtering

In earlier work (Wright and Leyton-Brown, 2014, 2019), we proposed a purportedly non-strategic level-0 model that consisted of a convex combination of elementary behavioral models, with an additional informativeness filtering step. Models of this kind discard uniform predictions by sub-models as “uninformative”; instead, for a given game, the model’s prediction is a convex combination of those sub-models that made an “informative”, non-uniform prediction. In this section, we formally define a generalization of these models, and demonstrate that all models of this kind are indeed strongly nonstrategic.

Definition 14 (Filtered convex combination of elementary models). A behavioral model $g_i$ is a filtered convex combination of elementary models when there exist a set of $K$ elementary models $f^k_i$, $K$ weights $w_k \in [0, 1]$ with $\sum_{k=1}^{K} w_k = 1$, and a filtering function $h : \Delta^* \rightarrow \{0, 1\}$ such that for all games $G = (N, A, u)$ and actions $a_i \in A_i$,

$$g_i(G)(a_i) = \begin{cases} 1/|A_i| & \text{if } h(f^k_i(G)) = 0 \ \forall 1 \leq k \leq K, \\ \frac{1}{\sum_{k=1}^{K} h(f^k_i(G)) w_k f^k_i(G)(a_i)} & \sum_{k=1}^{K} h(f^k_i(G)) w_k \\ & \text{otherwise.} \end{cases}$$ \hspace{1cm} (2)

A natural choice of filtering function is to exclude uniform predictions, but this definition allows for any filter that is a function only of a sub-model’s prediction.

The addition of a filtering function turns out to be insufficient to allow convex combinations of elementary models to represent weakly strategic behavioral models.

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Theorem 6. Every filtered convex combination of elementary models is strongly nonstrategic.

Proof. Suppose not. Then there exists some filtering function $h$, some set of $K$ elementary models $f_i^{(k)}$ and $K$ weights $w_k ∈ [0, 1]$ with $\sum_{k=1}^{K} w_k = 1$ such that $g_i(G)$ defined by equation (??) is both domination averse and other-responsive.

Since $g_i$ is other-responsive, there must be at least one $k$ such that $f_i^{(k)}$ is other-responsive. Fix an arbitrary game $G = (N, A, u)$ such that $h(f_i^{(k)}(G)) = 1$. If no such game exists, then $g_i$ is not other-responsive, and we are done.

Choose two actions $a_i^+, a_i^- ∈ A_i$ with $a_i^+ \neq a_i^-$. Let $M = \max_{a_i ∈ A} u_i(a)$. For every $a_i^{(l)} ∈ A_i$, let $x^{(l, -)}(a_i^+, a_i^{(l)}) = u(a_i^+, a_i^{(l)})$ and $x^{(l, +)}(a_i^+, a_i^{(l)}) = u(a_i^+, a_i^{(l)}).$ Choose $y^{(l, -)}(a_i^+, a_i^{(l)})$ such that $φ_i^{(k)}(y^{(l, -)}) = φ_i^{(k)}(x^{(l, -)})$ and $y^{(l, -)}(a_i^+, a_i^{(l)}) > M + ζ_i(g_i, G, w_k/|A_i|)$. Similarly, choose $y^{(l, +)}(a_i^+, a_i^{(l)})$ such that $φ_i^{(k)}(y^{(l, +)}) = φ_i^{(k)}(x^{(l, +)})$ and $y^{(l, +)}(a_i^+, a_i^{(l)}) > M + ζ_i(g_i, G, w_k/|A_i|).$ Now construct two new games: in $G_5$, replace the utility vector for each $(a_i^+, a_i^{(l)})$ with $y^{(l, -)}$ instead of $x^{(l, -)}$. Similarly, in $G_6$, replace the utility vector for each $(a_i^+, a_i^{(l)})$ with $y^{(l, +)}$ instead of $x^{(l, +)}$.

Observe that $Φ_i^{(k)}(G_5) = Φ_i^{(k)}(G_6) = Φ_i^{(k)}(G)$, and hence $f_i^{(k)}(G_5) = f_i^{(k)}(G_6) = f_i^{(k)}(G)$.

Since $g_i(G)(a_i') < w_k/|A_i|$ for every $a_i' \neq a_i^-$, it must be that $f_i^{(k)}(G_5)(a_i') < 1/|A_i|$, and so $f_i^{(k)}(G_5)(a_i^-) > 1 - 1/|A_i|$. By a similar argument, $f_i^{(k)}(G_6)(a_i^+) > 1 - 1/|A_i|$. But $a_i^+ \neq a_i^-$ and $f_i^{(k)}(G_6)(a_i^+) = f_i^{(k)}(G_5)(a_i^+)$, so $f_i^{(k)}(G_6)(a_i^+) < 1/|A_i|$, a contradiction. □

7.3 Sequentially Elementary Models

In sections 4.4 and 7.2 we considered ways to construct more complicated behavioral models from a set of elementary behavioral models by combining their predictions into a single prediction. Another natural approach is to apply each individual model in some order, and return the first prediction that satisfies some criteria such as informativeness, in the same manner as the priority heuristic (Brandstätter et al., 2006). We call a behavioral model that combines elementary models in this way a sequentially elementary model.

Definition 15 (Sequentially elementary behavioral models). A behavioral model $g_i$ is sequentially elementary if there exists an arbitrary function $h : Δ^* → \{0, 1\}$ and a finite set of $K$ elementary behavioral models $\{f_i^{(k)}\}_{k=1}^{K}$ such that $g_i$ can be represented as

$$g_i(G) = f_i^{(k^*(G))}(G),$$

where $k^*(G) = \min\{1 ≤ k ≤ K \mid h(f_i^{(k)}(G)) = 1\} \cup \{K\}$ is the index of the model that catches $G$; that is, the first model to return a prediction that $h$ maps to 1, or the last model if no such model exists. Equivalently,

$$g_i(G) = \begin{cases} f_i^{(1)}(G) & \text{if } h(f_i^{(1)}(G)) = 1, \\ f_i^{(2)}(G) & \text{if } h(f_i^{(2)}(G)) = 1, \\ \vdots & \\ f_i^{(K)}(G) & \text{otherwise.} \end{cases}$$
Sequentially elementary models are strictly more powerful than elementary models. In fact, it is straightforward to construct a sequentially elementary model that is weakly strategic:

**Proposition 3.** Let \( h(s_i) = \mathbb{I}[s_i \text{ is a uniform distribution}] \) and \( f_i^{\text{und}} \) be a behavioral model that randomizes uniformly over all of \( i \)'s undominated actions. Then the sequentially elementary model \( g_i \) with \( f_i^{(1)} = f_i^{\text{und}} \), \( f_i^{(2)} = f_i^{W} \), and filter function \( h \), is weakly strategic.

**Proof.** Since \( f_i^{\text{und}} \) will catch any game with a dominated strategy, \( g_i \) is domination averse, with \( \zeta(g_i, V, \delta) = \epsilon \) for any \( \delta, \epsilon > 0 \).

The two following games differ only in the column player \( j \)'s payoffs.

\[
\begin{array}{c|cc}
 & L & R \\
\hline
U & 1,1 & 0,0 \\
D & 0,1 & 1,0 \\
\end{array}
\]

\[
\begin{array}{c|cc}
 & L & R \\
\hline
U & 1,0 & 0,1 \\
D & 0,0 & 1,1 \\
\end{array}
\]

The row player \( i \) has no dominated strategy in either game, and so \( h(f_i^{\text{und}}(G_7)) = h(f_i^{\text{und}}(G_8)) = 0 \); thus, \( g_i(G_7) = f_i^{W}(G_7) \) and \( g_i(G_8) = f_i^{W}(G_8) \). But \( f_i^{W}(G_7)(U) = 1 \), whereas \( f_i^{W}(G_8)(U) = 0 \), so \( g_i \) is other responsive. \( \square \)

Although sequentially elementary models are strictly more powerful than elementary models, they are still unable to represent any of our exemplar strategic models.

**Theorem 7.** Fix a profile \( f \) of domination averse behavioral models and a precision \( \lambda > 0 \). Then the behavioral model \( q_i(G) = \text{QBR}_i(f_{-i}(G); \lambda, G) \) cannot be represented by a sequentially elementary behavioral model.

**Proof.** Suppose not. Then there exists a function \( h : \Delta^* \rightarrow \{0, 1\} \) and a finite set of \( K \) elementary behavioral models \( \{f_i^{(k)}\}_{k=1}^K \) such that \( g_i(G) = f_i^{(k^*(G))}(G) = q_i(G) \) for all games \( G \).

We will prove the result by constructing a sequence of games, and demonstrating that \( g_i \) and \( q_i \) make different predictions for at least one of them. Let \( m \) be the number of elementary models \( f_i^{(k)} \) that are other responsive.

First, for each non-domination averse submodel \( k \), choose a sequence of payoff vectors \( o^{(k,1)}, x^{(k,1)}, o^{(k,2)}, \ldots, o^{(k,K)}, x^{(k,K)} \) and

1. \( x_i^{(k,l)} > o_i^{(k,l)} \),
2. \( o_i^{(k,l+1)} > x_i^{(k',l)} \) for all non-domination averse \( k' \), and
3. \( \phi^{(k)}(x^{(k,l)}) = \phi^{(k)}(o^{(k,l)}) = \phi^{(k)}(x^{(k,l+1)}) = \phi^{(k)}(o^{(k,l+1)}) \) for all \( 1 \leq l < K \).
These sequences are straightforward to construct since every non-dictatorial elementary model must have a non-encoding potential.

First, pick an arbitrary integer \( Z > 1 \). For the first non-domination averse model \( k \), construct a sequence \( z(k,1), z(k,2), \ldots, z(k,Z) \) satisfying \( z(k,l+1) - z(k,l) > 1 \) and \( \varphi(k)(z(k,l)) = \varphi(k)(z(k,l+1)) \) for all \( 1 \leq l \leq Z \). This sequence is guaranteed to exist by non-encoding. For the every other non-domination averse model \( k' \), construct a sequence satisfying the same conditions, with the additional condition that either \( z(k',1) = z(k,1) \), or \( z(k',Z) = z(k,Z) \). Set \( o(k,1) = z(k,1) \) and \( x(k,1) = z(k,2) \) for each \( k \). Then, for each \( 1 < l \leq K \), iteratively set \( o(k,l) = z(k,p) \) and \( x(k,l) = z(k,p+1) \) where \( p = \min\{r \mid z(k,r) > x(k',l-1) \forall \text{non-domination averse } k' \} \). This will be possible whenever \( Z \) is large enough.

Now we construct a \( ((K - m)(m + 1)) \times (m + 1) \) game \( H_1 = (N, A, u^1) \) as follows. Let \( A_i = \{a_{i,k,l} \mid 1 \leq l \leq m + 1, f_{i}^{(k)} \text{ is not domination averse.}\} \), and \( A_j = \{a_{j,l} \mid 1 \leq l \leq m + 1\} \). Specify the normal form such that \( u^1(a_{i,k,l}, a_{j,l}) = x(k,1) \) and \( u^1(a_{i,k,l}, a_{j,l'}) = o(k,1) \) for \( l \neq l' \):

\[
\begin{array}{cccc}
  & a_{i,1} & a_{i,2} & \ldots & a_{i,m+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_{i,k,1} & x(k,1) & o(k,1) & \ldots & o(k,1) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_{i,k,2} & o(k,1) & x(k,1) & \ldots & o(k,1) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_{i,k,m+1} & o(k,1) & o(k,1) & \ldots & x(k,1) \\
\end{array}
\]

\( H_1 \)

Starting with \( p = 1 \), perform the following case analysis on \( H_p \): Let \( f_{i}^{*} = f_{i}^{k^*(H_p)} \) be the submodel that “catches” \( H_p \). Let \( a_{i,k,l} \in \arg\max_{a_{i} \in A_i} f_{i}^{*}(H_p)(a_i) \) be an action which \( f_{i}^{*} \) plays with maximal probability in \( H_p \). There are three possible cases:

1. \( f_{i}^{*} \) is domination averse. Let \( z = \min_{a \in A} u^p(a) - \zeta(f_j, H_p, \delta) \) for some \( \delta \approx 0 \). Construct \( H_{p+1} \) such that \( u^{p+1}_i(a) = u^p(a) \) for all \( a \in A_i \), and \( u^{p+1}(a_{i,k,l}) = z \) for all \( a_i \in A_i \), and \( u^{p+1}_j(a_{i,k,l'}, a_{j,l'}) = u^p(a_{i,k,l'}) \) for all \( a_i \in A_i \) and \( l' \neq l \). This has the effect of making action \( a_{j,l} \) strictly dominated for player \( j \) in \( H_{p+1} \), by a sufficient margin that \( f_{j}(H_{p+1})(a_{j,1}) \approx 0 \). If \( k^*(H_{p+1}) = k^*(H_p) \), then \( g_i(H_{p+1}) = g_i(H_p) \), because \( f_{i}^{*} \) cannot tell the two games apart, as all of player \( i \)’s utilities are identical. Hence \( g_i(H_{p+1})(a_{i,k,l}) = g_i(H_{p+1})(a_{i,k,l}) \in \arg\max_{a_{i} \in A_i} g_i(H_{p+1})(a_i) \). But since \( f_j \) is domination averse and action \( a_{j,l} \) is strictly dominated by a large margin, \( f_j(H_{p+1})(a_{j,1,l}) \approx 0 \). But then \( u_i(a_{i,k,l}, f_j(H_{p+1})) > u_i(a_{i,k,l}, f_j(H_{p+1})) \) for some \( l' \neq l \) (specifically, \( l' \in \arg\max_{1 \leq l' \leq m+1} f_j(H_{p+1})(a_{i,l'}) \)), so \( g_i(H_{p+1})(a_{i,k,l'}) > g_i(H_{p+1})(a_{i,k,l}) \), a contradiction.

If \( k^*(H_{p+1}) \neq k^*(H_p) \), then repeat this case analysis for \( p + 1 \).

2. \( k^*(H_p) \neq k \) and \( f_{i}^{*} \) is not domination averse. Construct \( H_{p+1} \) by setting

\[
u^{p+1}(a_i, a_j) = \begin{cases} 
  x(k^*(H_p), p+1), & \text{if } a_i = a_{i,k^*(H_p),l} \text{ and } a_j = a_{j,l} \text{ for some } 1 \leq l \leq m + 1, \\
  o(k^*(H_p), p+1), & \text{if } a_i = a_{i,k^*(H_p),l'} \text{ and } a_j = a_{j,l'} \text{ for } l' \neq l, \\
  u^p(a_i, a_j), & \text{otherwise.}
\end{cases}
\]

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This has the effect of making \( a_{i,k',l} \) strictly dominated for all \( k' \neq k^*(H_p) \).

Repeat this case analysis for \( p+1 \). Notice that by construction \( a_{i,k^*(H_p)} \in \arg\max_{a_i \in A_i} q_i(H_{p+1})(a_i) \) for some \( 1 \leq l \leq m+1 \), since \( x_i^{(k^*(H_p),p+1)} > o_i^{(k^*(H_p),p+1)} > x_i^{(k^*(H_p),p')} > o_i^{(k^*(H_p),p')} \) for all \( p' < p + 1 \), so this case can occur at most once for any submodel \( f_i^* \).

3. \( k^*(H_p) = k \) and \( f_i^* \) is not domination averse. Construct \( H_{p+1} \) by setting

\[
    w^{p+1}(a_i, a_j) = \begin{cases} 
        o^{(k,p+1)} & \text{if } a_i = a_{i,k,l}, \\
        x^{(k,p+1)} & \text{if } a_i = a_{i,k,l'} \text{ for } l' \neq l, \\
        w^p(a_i, a_j) & \text{otherwise.}
    \end{cases}
\]

If \( k^*(H_{p+1}) = k \), then \( g_i(H_{p+1}) = g_i(H_p) \), because \( f_i^* \) cannot tell the two games apart. Hence \( g_i(H_p)(a_{i,k,l}) = g_i(H_{p+1})(a_{i,k,l}) \in \arg\max_{a_i \in A_i} g_i(H_{p+1})(a_i) \). But \( a_{i,k,l} \) is strictly dominated in \( H_{p+1} \), and so we have a contradiction.

If \( k^*(H_{p+1}) \neq k \), then repeat this case analysis for \( p + 1 \).

Whenever the same model catches two consecutive games in case 1 or case 3, we derive a contradiction. Crucially, \( k^*(H_{p+1}) \leq k^*(H_p) \) for every new game \( H_{p+1} \), because at each iteration we either derive a contradiction or construct a new game \( H_{p+1} \) that is not distinguishable from \( H_p \) by submodel \( k^*(H_p) \); so either an earlier submodel catches \( H_{p+1} \), or it is caught by \( k^*(H_p) \), since \( f_i^{k^*(H_p)}(H_{p+1}) = f_i^{k^*(H_p)}(H_p) \), and thus \( h(f_i^{k^*(H_p)}(H_{p+1})) = h(f_i^{k^*(H_p)}(H_p)) = 1 \). Therefore, a contradiction will be derived after a finite number of iterations. \( \square \)

Thus, sequentially elementary models are an interesting example of a class of models that are in some sense “less strategic” than the standard models of Sections 2.2 and 2.3 while still failing to satisfy our strongly nonstrategic condition.

8 Discussion and Future Work

In this work, we proposed elementary behavioral models (and their convex combinations) as mathematical characterizations of classes of strongly nonstrategic decision rules. These classes are constructively defined, in the sense that membership of a rule is verified by demonstrating how to represent the rule in a specific form—as a function of the output of a non-encoding potential map—rather than by proving that it cannot be represented as a response to probabilistic beliefs.

It is interesting to note that various special cases of strategic solution concepts are nonstrategic under our definition. For example, the equilibrium of a two-player zero-sum game can be computed by considering only the utility of a single agent, and hence the behavior for an equilibrium-playing player in such a game can be computed by an elementary behavioral model that computes the agent’s maxmin strategy.\(^1\) Similarly, an equilibrium

\(^1\)However, note that such a behavioral model would act in every game as though the game was zero sum; it is in this sense that we would still say that the model is nonstrategic. An analogous caveat applies to the other examples we give here.
for a potential game can of course be computed in terms of outcome values computed by a potential function \(^{1}\) (Monderer and Shapley, 1996).

One thing that these exceptions all have in common is that they are also \textit{computationally} easy, unlike general \(\epsilon\)-equilibrium, which is known to be hard in a precise computational sense (Daskalakis et al., 2009; Chen and Deng, 2006). The equilibrium of a zero-sum game can be solved in polynomial time by a linear program; the equilibrium of a potential game can be found simply by finding the maximum of the potential function over all pure outcomes.\(^2\)

However, the connection between ease of computation and strategic simplicity is not an equivalence. For example, correlated equilibrium in general games can be computed in polynomial time by a linear program, but cannot in general be computed by an elementary behavioral model (see Theorem 1). An attractive future direction with potential applications in the design and analysis of multiagent environments is to shed further light on the connection between computational and strategic simplicity.

We also observe that our characterization of nonstrategic behavior in this paper is a binary distinction: in the view we have advanced, a behavioral model is either nonstrategic or it is not. An intriguing question for future work is whether such a distinction can be made more quantitative: i.e., is there a sense in which agents are nonstrategic to a greater or lesser degree that is distinct from the number of steps of strategic reasoning that they perform?

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\(^{2}\) Note that although this procedure is tractable in terms of the normal form, congestion games (the most important representation of potential games) can often be represented in an asymptotically more compact form than the normal form; finding the equilibrium of such games is intractable relative to the compact representation (Fabrikant et al., 2004; Babichenko and Rubinstein, 2021).
References

Airiau, S. and Sen, S. (2003). Strategic bidding for multiple units in simultaneous and sequential auctions. *Group Decision and Negotiation*, 12(5):397–413.

Babaioff, M., Nisan, N., and Pavlov, E. (2004). Mechanisms for a spatially distributed market. In *Proceedings of the 5th ACM Conference on Electronic Commerce*, pages 9–20.

Babichenko, Y. and Rubinstein, A. (2021). Settling the complexity of nash equilibrium in congestion games. In *Proceedings of the 53rd Annual ACM SIGACT Symposium on Theory of Computing*, pages 1426–1437.

Bernheim, B. (1984). Rationalizable Strategic Behavior. *Econometrica*, 52(4):1007–1028.

Brandstätter, E., Gigerenzer, G., and Hertwig, R. (2006). The priority heuristic: making choices without trade-offs. *Psychological Review*, 113(2):409.

Camerer, C., Ho, T., and Chong, J. (2004). A cognitive hierarchy model of games. *Quarterly Journal of Economics*, 119(3):861–898.

Chen, X. and Deng, X. (2006). Settling the complexity of two-player nash equilibrium. In *Foundations of Computer Science, 2006. FOCS’06. 47th Annual IEEE Symposium on*, pages 261–272.

Costa-Gomes, M., Crawford, V., and Brosseta, B. (2001). Cognition and behavior in normal-form games: An experimental study. *Econometrica*, 69(5):1193–1235.

Crawford, V. P., Costa-Gomes, M. A., Iriberri, N., et al. (2010). Strategic thinking. Working paper.

Daskalakis, C., Goldberg, P. W., and Papadimitriou, C. H. (2009). The complexity of computing a Nash equilibrium. *SIAM Journal on Computing*, 39(1):195–259.

Fabrikant, A., Papadimitriou, C., and Talwar, K. (2004). The complexity of pure nash equilibria. In *Proceedings of the Thirty-Sixth Annual ACM Symposium on Theory of Computing*, pages 604–612.

Gerding, E. H., Robu, V., Stein, S., Parkes, D. C., Rogers, A., and Jennings, N. R. (2011). Online mechanism design for electric vehicle charging. In *The 10th International Conference on Autonomous Agents and Multiagent Systems-Volume 2*, pages 811–818.

Ghosh, A. and Hummel, P. (2012). Implementing optimal outcomes in social computing: a game-theoretic approach. In *Proceedings of the 21st International Conference on World Wide Web*, pages 539–548.

Grabisch, M., Mandel, A., Rusinowska, A., and Tanimura, E. (2017). Strategic influence in social networks. *Mathematics of Operations Research*, 43(1):29–50.

Hartford, J. S., Wright, J. R., and Leyton-Brown, K. (2016). Deep learning for predicting human strategic behavior. In *Advances in Neural Information Processing Systems*, pages 2424–2432.
Lee, H. (2014). Algorithmic and game-theoretic approaches to group scheduling. In *Proceedings of the 2014 International Conference on Autonomous Agents and Multi-Agent Systems*, pages 1709–1710.

Li, C. and Tesauro, G. (2003). A strategic decision model for multi-attribute bilateral negotiation with alternating. In *Proceedings of the 4th ACM Conference on Electronic Commerce*, pages 208–209.

McKelvey, R. and Palfrey, T. (1995). Quantal response equilibria for normal form games. *Games and Economic Behavior*, 10(1):6–38.

Monderer, D. and Shapley, L. S. (1996). Potential games. *Games and Economic Behavior*, 14(1):124–143.

Nagel, R. (1995). Unraveling in guessing games: An experimental study. *American Economic Review*, 85(5):1313–1326.

Nisan, N. and Segal, I. (2006). The communication requirements of efficient allocations and supporting prices. *Journal of Economic Theory*, 129(1):192–224.

Pearce, D. (1984). Rationalizable Strategic Behavior and the Problem of Perfection. *Econometrica*, 52(4):1029–1050.

Roth, A. E. and Ockenfels, A. (2002). Last-minute bidding and the rules for ending second-price auctions: Evidence from ebay and amazon auctions on the internet. *American Economic Review*, 92(4):1093–1103.

Sandholm, T. W. and Lesser, V. R. (2001). Leveled commitment contracts and strategic breach. *Games and Economic Behavior*, 35(1-2):212–270.

Savage, L. (1951). The Theory of Statistical Decision. *Journal of the American Statistical Association*, 46(253):55–67.

Stahl, D. and Wilson, P. (1994). Experimental evidence on players’ models of other players. *Journal of Economic Behavior and Organization*, 25(3):309–327.

Wright, J. R. and Leyton-Brown, K. (2014). Level-0 meta-models for predicting human behavior in games. In *Proceedings of the ACM Conference on Economics and Computation (EC’14)*, pages 857–874.

Wright, J. R. and Leyton-Brown, K. (2017). Predicting human behavior in unrepeated, simultaneous-move games. *Games and Economic Behavior*, 106:16–37.

Wright, J. R. and Leyton-Brown, K. (2019). Level-0 models for predicting human behavior in games. *Journal of Artificial Intelligence Research*, 64:357–383.