Elliptic Curves with Supersingular Reduction over $\Gamma$-extensions

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Let $p$ be a prime number, $k_0$ a finite extension of the rationals $\mathbb{Q}$, $k_\infty/k_0$ a Galois extension with [Galois] group $\Gamma$ isomorphic to the group of $p$-adic integers $\mathbb{Z}_p$. Put $\Gamma_n := \Gamma^{p^n}, k_n := k_\infty^{p^n}$. Let $E$ be an elliptic curve over $\mathbb{Q}$ with supersingular reduction at $p$, $E(k_n)$ the $k_n$-rational points of $E$, and $\ker_n^{(p)}$ the $p$-component of the Shafarevich-Tate group of the curve $E \otimes k_n$.

**Theorem 1.1.** We assume the following conditions:

(a) $p$ is not 2 and does not divide the number of the rational connected components of bad reduction of the curve $E \otimes k_0$.

(b) For all places $v$ of $k_0$ dividing $p$, the completion $k_{0,v}$ is unramified over the field of the $p$-adic numbers $\mathbb{Q}_p$, and its degree over $\mathbb{Q}_p$ is not divisible by 4.

(c) The $\Gamma$-extension $k_\infty/k_0$ is cyclotomic, i.e.

$$k_\infty \subset \bigcup_{n=1}^{\infty} k_0(\sqrt[p^n]{1})$$

Then, if $E(k_0)$ is finite and $\ker_0^{(p)} = 0$, the groups $E(k_n), E(k_\infty)$ and $\ker_n^{(p)}$ are finite and

$$\log_p[\ker_n^{(p)}] = [k_0 : \mathbb{Q}][\left(\frac{p^{n+1}}{p^2 - 1} - \frac{n+1}{2}\right)].$$

We denote by $a_p$ the trace of the Frobenius automorphism of the reduction of $E \mod p$. Note that $E \mod p$ is supersingular if and only if it is non-singular and $p$ divides $a_p$. Consequently, $a_p = 0$ for $p > 3$ and $a_p = 0, \pm p$ when $p = 2, 3$.

**Theorem 1.2.** Suppose that $k_0/\mathbb{Q}$ is abelian and $k_\infty/k_0$ is cyclotomic. Then:

(a) There are integers $\rho(0), \rho(1) \geq 0$, equal for $a_p \neq 0$, such that for all sufficiently large $n \equiv s \mod 2$ ($s = 0, 1$),

$$\text{rk } E(k_n) + \text{cork } \ker_n^{(p)} - \text{rk } E(k_{n-1}) - \text{cork } \ker_{n-1}^{(p)} = \rho(s)(p^n - p^{n-1}),$$

where $\text{rk } E(k_n)$ is the rank of $E(k_n)$ and $\text{cork } \ker_n^{(p)}$ is the corank of $\ker_n^{(p)}$;

(b) if $a_p \neq 0$ and the degree $[k_0 : \mathbb{Q}]$ divides a number of the form $(p^l + 1)p^n$, then $\text{rk } E(k_n)$ stabilizes, and consequently $E(k_\infty)$ is finitely generated;

(c) if $E(k_0)$ and $\ker_0^{(p)}$ are finite, and for $a_p = 0$ we have the condition (b) of Theorem 1.1, then $\rho(0) = \rho(1) = 0$, i.e. $\text{rk } E(k_n)$ and $\text{cork } \ker_n^{(p)}$ stabilize;

(d) if $\rho(0) = \rho(1) = 0$, then there are integers $\mu(s), \delta(s) \geq 0, \lambda(s)(s = 0, 1)$ such that $\delta(0) = \delta(1) = 0$ for $a_p = 0$ and for all sufficiently large $n \equiv s \mod 2$,

$$\log_p[\ker_n^{(p)}] - \log_p[\ker_{n-1}^{(p)}] = \mu(s)(p^n - p^{n-1}) + ([k_0 : \mathbb{Q}] - \delta(s))\left[\frac{p^n}{p+1}\right] + \delta(s)\left[\frac{p^{n-1}}{p+1}\right] + \lambda(s),$$

where $\ker_n^{(p)}$ is the cotorsion of $\ker_n^{(p)}$. 
Denote by $T_n$ the set of places of the field $k_n$ dividing $p$ and ramified in $k_\infty$.

**Theorem 1.3.** Suppose that $a_p = 0$ and that for all $n$ and $v \in T_n$, the extensions $k_{n,v}/\mathbb{Q}_p$ are abelian. Then there are integers $\rho^{(s)}, \nu^{(s)}, \mu^{(s)}, \lambda^{(s)}$ ($s = 0, 1$), $\mu_i (i = 1, 2, \cdots)$, satisfying the relations

\[
\rho^{(s)} \geq r^{(s)} \geq \nu^{(s)} \geq 0, \quad \rho^{(0)} - r^{(0)} = \rho^{(1)} - r^{(1)},
\]

\[
\mu_1 \geq \mu_2 \geq \cdots \geq 0, \mu_i = 0 \text{ for } i > \min(\nu^{(0)}, \nu^{(1)}),
\]

\[
\mu^{(s)} \geq 0, \quad r^{(0)} + r^{(1)} \leq r, \quad \text{where } r = \sum_{v \in T_n} [k_{0,v} : \mathbb{Q}_p] \leq [k_0 : \mathbb{Q}],
\]

and such that for sufficiently large $n \equiv s \mod 2$ the following assertions hold:

(a) $\text{rk} E(k_n) + \text{cork} \, \text{III}_n^{(p)} = \text{rk} E(k_{n-1}) - \text{cork} \, \text{III}_{n-1}^{(p)} = \rho^{(s)}(p^n - p^{n-1});$

(b) $\log_p |\text{III}_n^{(p)}| - \log_p |\text{III}_{n-1}^{(p)}| =

\[
[\mu^{(s)}] \left( \frac{p^n}{p+1} \right) + \left( r^{(s)} + \nu^{(s)} \right) \left( \frac{p^{n-\mu_1}}{p+1} \right) = \text{rk} E(k_n) = \text{rk} E(k_{n-1});
\]

(c) if $r^{(s)} = 0$, then $\text{rk} E(k_n) = \text{rk} E(k_{n-1});$

(d) if $\text{cork} \, \text{III}_n^{(p)}$ stabilizes, then $\text{rk} B_n - \text{rk} B_{n-1} = r^{(s)}(p^n - p^{n-1})$, where $B_n$ is the image of $E(k_n) \otimes \mathbb{Z}_p \to \sum_{v \in T_n} E(k_{n,v})^{(p)}$ and $\text{rk} B_n$ is the rank of $B_n$ over $\mathbb{Z}_p$.

In the case of nonsupersingular reduction, the behavior of the groups $E(k_n)$ and $\text{III}_n^{(p)}$ has been investigated by B. Mazur (see [1], [2]). One of the main points of his research is the description of the $\Gamma$-modules $E(k_{n,v})^{(p)}$ for $v \in T_n$. Analogously, the proofs of Theorems 1.1, 1.2, and 1.3 are based on the theorem in the following paragraph.

## 2 The Local Group of Points

Let $E$ be an elliptic curve over $\mathbb{Q}_p$, $E \mod p$ be supersingular, $a_p$ the trace of the Frobenius automorphism on the reduction $E \mod p$. For any abelian extension $K/\mathbb{Q}_p$, set $K_n := K \cap \mathbb{Q}_p^{nr}(\zeta_n)$, where $n = -1, 0, 1, \cdots; \mathbb{Q}_p^{nr}$ denotes the maximal unramified extension of $\mathbb{Q}_p$, and $\zeta_n$ a primitive root of unity of degree $p^{n+1}$ if $p \neq 2$, and of degree $p^{n+2}$ if $p = 2$. We will denote by $m(K)$ the smallest $n$ for which $K_n = K$.

**Theorem 2.1.** Let $K/\mathbb{Q}_p$ be a finite abelian extension with [Galois] group $G = \text{Gal}(K/\mathbb{Q}_p)$. Then the $\mathbb{Z}_p[G]$-module $E(K)^{(p)}$ is free of $p$-torsion and has a system of generators $\{e_n | n = -1, -2, \cdots, m(K)\}$, all of whose relations can be derived from the following:

\[
e_n \in E(K_n),
\]

\[
\text{Nor}_{n/n-1} e_n = a_p e_{n-1} - e_{n-2} \quad (n \geq 2),
\]

\[
\text{Nor}_{1/0} e_1 = \begin{cases} a_p e_0 - [K_0(\zeta_0) : K_0] e_{-1}, & (p \neq 2), \\
 a_p e_0 - [K_0(\zeta_0) : K_0](a_p - F - F^{-1}) e_{-1}, & (p = 2), \end{cases}
\]

\[
\text{Nor}_{0/1} e_0 = \begin{cases} (a_p - F - F^{-1}) e_{-1}, & (p \neq 2), \\
 (a_p^2 - a_p F - a_p F^{-1} - 1) e_{-1}, & (p = 2), \end{cases}
\]

where $\text{Nor}_{n/n-1} : E(K_n) \to E(K_{n-1})$ is the norm homomorphism and $F \in \text{Gal}(K_{-1}/\mathbb{Q}_p)$ is the Frobenius automorphism.

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References

[1] Yu. Manin: *Cyclotomic Fields and Modular Curves*, Uspehi Matematičeskikh Nauk 26:6 (1971), 7-71.

[2] B. Mazur: *Rational Points of Abelian Varieties with Values in Towers of Number Fields*, Inventiones Mathematicae 18 (1972), 183-266.

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