On the sharp constant in “magnetic” 1D embedding theorem

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1 Introduction

We consider the problem of finding the sharp (exact) constant in the “magnetic” embedding theorem

\[ \min_u \frac{\|u' + iAu\|_{L^2}}{\|u\|_{L^q}} =: \mu_q(A), \]

where \( A \in L_1(0, 2\pi) \), and minimum is taken over all \( 2\pi \)-periodic absolutely continuous functions.

It is easy to see that \( \mu_q(A) \) is attained and does not change if we change \( A \mapsto A + k \), \( k \in \mathbb{Z} \). Moreover, the substitution \( u(x) \mapsto u(x) \exp \left( i \int_0^x (A(t) - \alpha) \, dt \right), \quad \alpha = \frac{1}{2\pi} \int_0^{2\pi} A(t) \, dt \)

shows that we can assume without loss of generality \( A \equiv \alpha \) and \(|\alpha| \leq \frac{1}{2}\).

Trivially the value \( \mu_q(0) \equiv 0 \) is attained by any constant function. Further, if \( q \leq 2 \) then due to the evident estimate \( \|u\|_{L^q} \leq (2\pi)^{\frac{1}{q} - \frac{1}{2}} \|u\|_{L^2} \) the constant function also is a minimizer of \( \mu_q(\alpha) \), and \( \mu_q(\alpha) = (2\pi)^{\frac{1}{2} - \frac{1}{q}} \cdot |\alpha| \). Thus, the constant function is a natural candidate to the minimizers of \( \mu_q(\alpha) \). In this paper we show that in fact for \( \alpha \neq 0 \) it is minimizer only for sufficiently small \( q > 2 \), namely, for \((q + 2)\alpha^2 \leq 1 \). In particular, for \( \alpha = \pm \frac{1}{2} \) and \( q > 2 \) the minimizer is always non-constant.

Remark 1. For \( q = \infty \) the sharp constant in (1) was found in [2], see also [3].

In what follows we assume \( 2 < q < \infty \). It is convenient to normalize \( u \) by \( \|u\|_{L^q}^q = 2\pi \), and we arrive at the problem

\[ \mu_q(\alpha)^2 = (2\pi)^{-\frac{2}{q}} \min_u \int_0^{2\pi} |u'|^2 + i\alpha |u|^2 \, dx, \quad \int_0^{2\pi} |u|^q \, dx = 2\pi. \]
To study the problem \( \text{(2)} \) we use the phase plane method. In a similar way in \([4, 5]\) the problem
\[
\min_u \int_{-T}^{T} (u'^2 + u^2) \, dx, \quad \int_{-T}^{T} |u|^q \, dx = 1.
\]
was studied, and the sharp condition of symmetry breaking in this problem was found. See also \([1, \text{Lemma 5}]\).

2 The constant and non-constant minimizers of \( \text{(2)} \)

Denote \( r = |u| \) and \( \varphi = \text{arg}(u) + \alpha x \). Then \( \text{(2)} \) can be rewritten as follows:
\[
J(r, \varphi) = \int_0^{2\pi} |r' + ir\varphi'|^2 \, dx = \int_0^{2\pi} (r'^2 + r^2\varphi'^2) \, dx \rightarrow \min, \quad \int_0^{2\pi} r^q \, dx = 2\pi. \quad (3)
\]
Here \( r \) and \( \varphi' \) are \( 2\pi \)-periodic functions, and
\[
\int_0^{2\pi} \varphi' \, dx = 2\pi \alpha. \quad (4)
\]

The Euler equation with respect to \( \varphi \) reads:
\[
0 \equiv \frac{1}{2} D_{\varphi} J(r, \varphi)(\psi) = r^2 \varphi' \psi \bigg|_0^{2\pi} - \int_0^{2\pi} (r^2 \varphi')' \psi \, dx.
\]
The first term vanishes due to \( 2\pi \)-periodicity, and we obtain
\[
r^2 \varphi' = a = \text{const}. \quad (5)
\]

The Euler–Lagrange equation with respect to \( r \) reads:
\[
r' h \bigg|_0^{2\pi} + \int_0^{2\pi} (r \varphi'^2 - \lambda r'^{-1} - r^q) h \, dx \equiv 0, \quad (\text{6})
\]
and we obtain
\[
-r'' + r \varphi'^2 = \lambda r'^{-1}.
\]
Taking into account \( \text{(5)} \) we arrive at
\[
-r'' + \frac{a^2}{r^3} = \lambda r'^{-1}. \quad (6)
\]
It is easy to see that the function \( r \equiv 1 \) is a solution of \( \text{(6)} \). Moreover, in this case relations \( \text{(5)} \) and \( \text{(4)} \) give \( a = \alpha \), and thus \( \lambda = \alpha^2 \).

**Theorem 2.1.** Let \((q + 2)\alpha^2 > 1\). Then the function \( r \equiv 1 \) cannot provide minimal value in the problem \( \text{(3)} \), and thus we have \( \mu_q(\alpha) < (2\pi)^{\frac{1}{2} - \frac{1}{q}} \cdot |\alpha| \).
Proof. Taking into account (5) we conclude that the second order necessary condition of minimum is positivity of the quadratic form

\[ \int_{0}^{2\pi} \left( h'^2 - \frac{3a^2h^2}{r^4} - \lambda(q-1)r^{q-2}h^2 \right) dx \]

on the space of $2\pi$-periodic function with zero mean value. Substituting $r \equiv 1$, $a = \alpha$, and $\lambda = \alpha^2$ we obtain

\[ \int_{0}^{2\pi} \left( h'^2 - \alpha^2(q+2)h^2 \right) dx \geq 0. \]

For $(q+2)\alpha^2 > 1$ this inequality fails for $h = \sin(x)$.

Theorem 2.2. Let $(q+2)\alpha^2 \leq 1$. Then the function $r \equiv 1$ provides minimal value in the problem (4), and thus we have $\mu_q(\alpha) = (2\pi)^{\frac{1}{2}} \cdot |\alpha|$. 

Proof. Integrating ODE (6) we obtain

\[ \frac{r'^2}{2} = -\frac{a^2}{2r^2} - \frac{\lambda}{q}r^q + c. \] (7)

On the other hand, we can multiply (6) by $r$ and integrate over $[0, 2\pi]$. This gives in view of the normalization condition

\[ \int_{0}^{2\pi} \left( r'^2 + \frac{a^2}{r^2} \right) dx = \lambda \int_{0}^{2\pi} r^q dx = 2\pi \lambda, \]

and (7) implies $c = \frac{1}{2} + \frac{1}{q}$.

If $r$ is not a constant then the right-hand side of (7) has two zeros corresponding to minimal and maximal values of $r$ at the period. Denote these values by $r_1$ and $r_2$ respectively. By the normalization condition we have

\[ r_1 < 1 < r_2. \] (8)

Thus, any non-constant periodic positive solution of ODE (6) corresponds to the motion along an oval given by equation (7) in the phase plane $(r, r')$. Since this oval is symmetric w.r.t. $r'$ axis, without loss of generality we can assume that $r(0) = r(2\pi) = r_1$ and $r(\pi) = r_2$.

Consider a half of the oval corresponding to $r' > 0$. Then we have from (7)

\[ r' = \sqrt{2c - \frac{a^2}{r^2} - \frac{2\lambda}{q}r^q} = \frac{\sqrt{\lambda(1 + \frac{2}{q})r^2 - a^2 - \frac{2\lambda}{q}r^{q+2}}}{r}. \]

By (4) and (5) we obtain

\[ 2\pi \alpha = a \int_{0}^{2\pi} \frac{dx}{r^2} = 2a \int_{0}^{\pi} \frac{dx}{r^2} = \int_{r_1}^{r_2} \frac{2dr}{\sqrt{\frac{a^2}{r^2} \left[ \left( 1 + \frac{2}{q} \right) r^2 - \frac{2\lambda}{q}r^{q+2} \right] - 1}}. \] (9)
By (8) we have $\frac{1}{a^2} > 1$. Changing the variable $t = \frac{1}{a^2} \left(1 + \frac{2}{q}\right) r^2$ we rewrite (9) as follows:

$$M_q(\gamma) := \int_{t_1}^{t_2} \frac{dt}{t\sqrt{t - \gamma t^{q+1}} - 1} = 2\pi\alpha. \quad (10)$$

Here $t_1, t_2$ are the roots of the equation $t - \gamma t^{q+1} - 1 = 0$, and

$$0 < \gamma < \gamma_{\text{max}} = \frac{2}{q + 2} \left(1 + \frac{2}{q}\right)^{-\frac{2}{q}}.$$

The statement of Theorem follows from Lemma which will be proved in Section 3.

Lemma 2.1. For all $\gamma \in (0; \gamma_{\text{max}})$ we have

$$M'_q(\gamma) < 0. \quad (11)$$

Moreover,

$$\lim_{\gamma \uparrow \gamma_{\text{max}}} M_q(\gamma) = \frac{2\pi}{\sqrt{q + 2}}. \quad (12)$$

Namely, it follows from (11) and (12) that if $(q + 2)\alpha^2 \leq 1$ then $M_q(\gamma) > 2\pi\alpha$ for all $\gamma \in (0; \gamma_{\text{max}})$. Therefore, the equation (10) has no solutions, and the constant function is a unique stationary point of the problem (3). This completes the proof.

Remark 2. If $(q + 2)\alpha^2 > 1$ then the equation (10) has a unique solution. Evidently, the motion along corresponding oval in the phase plane just provides the minimum in (3).

3 Proof of Lemma 2.1

We introduce the notation

$$f(t) = t - \gamma t^{\frac{q+1}{2}} - 1. \quad (13)$$

Then

$$f'(t) = 1 - \frac{\gamma(q + 2)}{2} t^{\frac{q}{2}}; \quad f''(t) = -\frac{\gamma q(q + 2)}{4} t^{\frac{q-1}{2}}; \quad f'''(t) = -\frac{\gamma q(q + 2)(q - 2)}{8} t^{\frac{q-2}{2}}. \quad (14)$$

It is easy to see that $f'(t_1) > 0$ and $f'(t_2) < 0$. Denote by $t_0$ a unique root of $f'$.

To prove (12) we observe that by the Rolle Theorem for any $t \in (t_1, t_2)$ there exists $\overline{t}(t) \in (t_1, t_2)$ such that

$$f(t) = -\frac{f''(\overline{t})}{2} (t_2 - t)(t - t_1).$$

Hence

$$M_q(\gamma) = \sqrt{-2 \overline{t}} \int_{t_1}^{t_2} \frac{dt}{t\sqrt{(t_2 - t)(t - t_1)}} = \pi \sqrt{-2 \overline{t}} \int_{t_1}^{t_2} \frac{dt}{t\sqrt{(t_2 - t)(t - t_1)}} = \frac{\pi}{\overline{t}} \sqrt{-2 \overline{t} \overline{t}} \int_{t_1}^{t_2} \frac{dt}{f''(t)} \frac{1}{t\sqrt{(t_2 - t)(t - t_1)}}$$

where $\overline{t}$ and $\tilde{t}$ are some points in $(t_1, t_2)$.

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Notice that $t_1 \uparrow \frac{q+2}{q}$ and $t_2 \downarrow \frac{q+2}{q}$ as $\gamma \uparrow \gamma_{\text{max}}$. Therefore, $\hat{t}$ and $\tilde{t}$ also tend to $\frac{q+2}{q}$, and

$$\lim_{\gamma \uparrow \gamma_{\text{max}}} M_q(\gamma) = \frac{2\pi}{\sqrt{q+2}}$$

and (12) follows.

To prove (11) we proceed similarly to [4, Sec. 2] and [5].

Lemma 3.1. For $\gamma \in (0, \gamma_{\text{max}})$ the following identity holds:

$$M'_{q}(\gamma) = \int_{t_1}^{t_2} \frac{\sqrt{f'}}{\Psi} \cdot H_{\beta} \, dt,$$

where

$$\Psi = f'^2 - 2ff'', \quad H_{\beta} = \beta(3f'^2f'' + 2ff'f''' - 6ff''') - \frac{q(q-2)t^2 - 2}{2}.$$ 

and $\beta$ is an arbitrary number.

Proof. We have

$$M^{(c)}_q(\gamma) := \int_{t_1+\epsilon}^{t_2} \frac{dt}{t\sqrt{f}} \to M_q(\gamma),$$

and convergence is uniform in any compact subset of the interval $(0, \gamma_{\text{max}})$.

Furthermore,

$$\frac{dM^{(c)}_q}{d\gamma} = \frac{d t_2}{d\gamma} \cdot \frac{1}{t\sqrt{f}} \bigg|_{t_2-\epsilon}^{t_2} - \frac{d t_1}{d\gamma} \cdot \frac{1}{t\sqrt{f}} \bigg|_{t_1+\epsilon}^{t_1} + \frac{1}{2} \int_{t_1+\epsilon}^{t_2-\epsilon} \frac{t^2}{\sqrt{f}} \, dt.$$ 

However, $f(t_1) = f(t_2) = 0$ implies

$$\partial f |_{t_k}^{t_1} + f' |_{t_k}^{t_1} \cdot \frac{dt_k}{d\gamma} = 0, \quad k = 1, 2.$$ 

Therefore,

$$\frac{dt_k}{d\gamma} = \frac{t_k^{\frac{q+2}{q} + 1}}{f'} \bigg|_{t_1}^{t_k} = \frac{t_k^{\frac{q+2}{q} + 1} f' - q t_k^{\frac{q+2}{q} f} + 2\beta f^2 f''}{f'^2 - 2ff''} \bigg|_{t_1}^{t_k}, \quad k = 1, 2,$$

and thus

$$\frac{dM^{(c)}_q}{d\gamma} = \frac{1}{t\sqrt{f}} \bigg|_{t_1+\epsilon}^{t_2-\epsilon} \frac{t^2 (t f' - q f) + 2\beta f^2 f''}{\Psi} + O(\epsilon^{\frac{1}{q}}) + \frac{1}{2} \int_{t_1+\epsilon}^{t_2-\epsilon} \frac{t^2}{f^{\frac{2}{q}}} \, dt.$$ 

Note that $\Psi(t_1) = f'^2(t_1) > 0$, and

$$\Psi' = -2f \cdot f''' > 0 \quad \text{in} \quad (t_1, t_2).$$
Hence $\Psi > 0$ in $[t_1, t_2]$, and we can write
\[
\frac{dM_\gamma^{(c)}}{d\gamma} = \int_{t_1+\epsilon}^{t_2-\epsilon} \left[ \frac{d}{dt} \left( \frac{1}{t\sqrt{f}} \cdot \frac{t^2\left(f' - qf\right) + 2\beta f^2 f''}{\Psi} \right) + \frac{t^2}{2f^2} \right] dt + O(\varepsilon^2).
\]

The expression in square brackets is equal to $\frac{\sqrt{f}f'}{\Psi^2} \cdot H_\beta$. Therefore $dM_\gamma^{(c)}/d\gamma$ converges to the right-hand side of (15) as $\varepsilon \to 0$. Moreover, convergence is uniform in any compact subset of the interval $(0, \gamma_{\max})$. This completes the proof.

Using the relations (13)–(14) we calculate
\[
H_\beta(t) = qt^{\frac{q}{2} - 2} \left( \frac{\beta q(q + 2)}{16} h(t) - \frac{q - 2}{2} \right),
\]
where
\[
h(t) = 4(q - 2) - 4(q + 1)t - 4\gamma(q + 1)(q - 2)t^{\frac{q}{2} + 1} + 4\gamma(q + 2)(q + 1)t^{\frac{q}{2}} + \gamma^2(q + 2)(q - 2)t^{q + 1}.
\]

Direct calculation shows that
\[
h''(t) = -\gamma q(q + 1)(q - 2)(q + 2)t^{\frac{q}{2} - 2} f(t).
\]
Thus, $h''(t) < 0$ for $t \in (t_1, t_2)$. Therefore, $h'$ decreases on $[t_1, t_2]$.

Next, the relation (13) implies $\gamma t_1^{\frac{q}{2}} = \frac{t_1 - 1}{t_1}$. Therefore,
\[
h'(t_1) = -(q + 1) \left( 4 + 2\gamma(q - 2)(q + 2)t_1^{\frac{q}{2}} - 2\gamma q(q + 2)t_1^{\frac{q}{2} - 1} - \gamma^2(q + 2)(q - 2)t_1^2 \right)
= -(q + 1) \left[ 4 + (q + 2) \left( 2(q - 2) \frac{t_1 - 1}{t_1} - 2q \frac{t_1 - 1}{t_1^2} - (q - 2) \left( \frac{t_1 - 1}{t_1} \right)^2 \right) \right]
= -\frac{q + 1}{t_1^2} \left( qt_1 - (q + 2) \right)^2 < 0.
\]
Thus, $h'(t) < 0$ on $[t_1, t_2]$. It follows that for $\beta < 0$ the function $H_\beta$ increases on $[t_1, t_2]$.

Now we choose
\[
\beta = -\frac{q(q - 2)t_0^{\frac{q}{2} - 2}}{12 f(t_0) f''(t_0)} < 0
\]
(we recall that $f'(t_0) = 0$). Then $H_\beta(t_0) = 0$. By monotonicity we have $H_\beta < 0$ on $[t_1; t_0)$ and $H_\beta > 0$ on $(t_0; t_2]$. Therefore, $\frac{\sqrt{f}f'}{\Psi^2} \cdot H_\beta \leq 0$ on $[t_1, t_2]$, and (15) implies (11). ■

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