Same pattern, different visualization: visual support does matter in pre-algebra

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Abstract. This paper is a part of a more significant study in design research aimed to create a learning trajectory in pre-algebra. The study was following the steps of the Preliminary Study, Teaching Implementation and Retrospective Analysis. In the current discussion, we will focus on the importance of visualization employed to represent the specific number pattern. The study was conducted in a state elementary school in Palembang with eight fifth grade students who participate during the Teaching Implementation phase. We gathered data from students’ written work, interviews, and observation during the lesson. The data were analyzed qualitatively using the constant comparative method. From the analysis, it was found that even though the problems were using the same number pattern, the students’ interpretation that lead to a correct generalization were highly influenced by the visual support. Hence, it is recommended to the future learning materials designers and teachers to consider the visual support matters to enhance the students’ mastery.

1. Introduction
According to the study of Pramesti and Retnawati [1], there are three major difficulties encountered by junior high school students to solve algebra problems, including understanding the problem and variables, and working with the operation. Furthermore, it also found that the students perform several misconceptions in pre-algebra due to their incomplete reasoning factor and the lack of understanding in pre-requisite concepts, e.g. arithmetic operation [2]. These findings indicate the need to introduce algebra at an earlier age to provide experience in working with the number structures.

The algebra lesson at an early age is usually called by pre-algebra. Instead of bringing the higher-level school curriculum to the younger students, the lesson promotes the establishment of algebraic thinking ability, such as the generalization and reasoning within an algebraic structure, develop mental models, and formulate pattern to solve the algebra problem [3]. The pre-algebra lesson can take any type of activity that facilitate the students to reinvent the notion of generalization. One of the recommendations is using pattern investigation [4] because it has dynamic representation [5] and it provides the chance to do investigation, identification, and the relation between the structures algebraically [6].

One strategy to use pattern activities is to combine the number pattern with visual representations. Several previous studies which use this approach ([7], [8], [9] & [10]) pointed out that pattern investigation can be helpful for the algebraic activity and therefore recommend this to start algebraic
lessons in earlier grades. Despite of the benefits of employing pattern investigation in algebra class, it was revealed that the strategies implemented to encourage the students to make a conjecture and communicating ideas were still lacking [11].

To support the students in learning by using pattern investigation, we embodied the pattern into geometry representation. Generally, image or visual assistant helps the students to identify the problem [12]. Specifically, in pre-algebra class, the representation of number pattern into geometry objects can be helpful to endorse the structure sense of the students [13]. By that, the students will be able to establish personal inferences to see the general form of a particular pattern.

The study of how visual supports will benefit the students in learning pre-algebra had conducted by several preliminary studies (see [14] & [15]). However, not all geometry representations provided meaningful promotion for students’ generalization ability. Reflected on the reasons mentioned earlier, in the following study, the research question formulated is how different visualization of the same pattern can lead to different thinking process in pre-algebra?

2. Methods
The present study is a part of Design Research to design a Local Instructional Theory in Pre-Algebra by using number patterns. It followed the steps of Design Research, i.e., Preliminary Study, Teaching Experiment, and Retrospective Analysis. The Teaching Experiment Phase conducted twice in a state elementary school in Palembang. The first cycle was involving four students as a pilot study and the second cycle was involving 32 students—four students were chosen as the focus group. Hence, the total number of the participants were eight students. In both cycles, we tried the lesson with fifth-grade students.

The learning trajectories were designed based on the tenets of Realistic Mathematics Education (RME) approach. According to RME point of view, mathematics is a human activity, and therefore it should be experienced meaningfully by the students [16]. Hence, we introduced the idea of pre-algebra by using pattern investigation activities. The pattern was an arithmetic sequence embodied in geometric visualization, as recommended by the previous study [17]. The geometrical shape that visualizes the number pattern was used as the vertical instrument to connect the reality to the mathematical concepts [18]. The current paper will discuss the importance of visual support. We will compare the lesson of triangular pattern we designed and implemented in the first and second cycles.

The data were gathered from students’ written work, interviews, and observation during group and classroom discussions. We analyzed the data qualitatively using the constant comparative method. To increase the reliability of the data, we discussed some of the findings with colleagues to see whether they interpret the similar tendency from the observation.

3. Results and Discussion
In the present study, we focused on the triangular number pattern. We compared the students’ activity to investigate the triangular number pattern in the First and the Second Cycles. Even though the pattern is the same, but the visualizations are different. From the retrospective analysis on both cycles, we argue that different visual support leads to a different interpretation of the pattern.

In the First Cycle, triangular number pattern was introduced in the dance formation context. The triangular number pattern was introduced after the square number pattern. However, there was no relationship between those patterns. The triangular number assignment was organized by providing the first three formation (see Figure 1), where the students were asked to draw the fourth formation. The next question is to figure out the number of dancers in the 7th and 10th formations. Next, they established a strategy to find the general formula that can be used to find the number of dancers in the 100th formation.

![Figure 1. The first three formations of triangular number pattern in first cycle](image.png)
To address the first requirement, the students confused the term “triangle” with the structure given in the worksheet. The first responses of the students can be seen in Figure 2.

![Figure 2](image)

**Figure 2.** Incorrect representation of triangular number pattern formation

One of the participants finally realized that unlike the examples in the worksheet, the triangle drawn in Figure 2 has the “hole” in the middle. It means the triangle in Figure 2 should not be in a sequence. They have to put one more in the third line. She continued her argument by saying that in the next pattern, there will be the additional dots as in the number of the formation. Hence, the fifth formation will have five dots more from the fourth formation. Her additional illustration for the fifth formation can be seen in Figure 3.

![Figure 3](image)

**Figure 3.** The fifth triangular number pattern formation

The students moved to the second problem is asking about the number of dancers in the 7th and the 10th formations. The students used the addition method to solve the problem. There are two additional structured in the students’ solution, horizontal (Figure 4) and slant (Figure 5).

![Figure 4](image)

**Figure 4.** Adding in horizontal

**Translation:**
- 28 people
- 36 people
- 45 people
- 55 people

![Figure 5](image)

**Figure 5.** Adding in slant position
Even though the students see the structure of the pattern differently, they find the sum with the same method by adding the numbers one by one. Without further drawing, the students found that the number of dancers will be the same as they add from one until the index of formation. The strategy is transcribed in the following Fragment 1 (S1&S2 = students, R = Researcher).

**Fragment 1: The Addition of One to Seven**

1. S1: Add four with three with two with one.
2. S2: Wait.
3. S1: Then, seven … add it with three …
4. S2: Please wait me in counting…
5. R: S1, can you explain why you add the numbers?
6. S1: I count this (pointing the picture he made [Figure 5])
7. S2: Oh, I understand (marked the dots in slant position as in [Figure 5]).
8. R: Okay it works for the fourth formation. What is the plan for the 7th formation?
9. S2: To find the number of dancers in the seventh formation? [Adding] one to seven.
10. S1: What is the result?
11. S2: As the addition of one, two, and three resulted on seven; seven plus four is equal to eleven.
12. S1: But, one plus two is three, three and three is six …
13. R: Track your calculation, so you won’t be confused.

The students concluded that no matter the index of formation is, to find the number of dancers on it, they just need to sum all numbers from one to the index number. Hence, for the 100th formation, the number of dancers is equal to 1+2+3+…+100. If the number getting big, let say the unknown formation, they argued that need to add from 1 until the unknown number. Even though the students were able to recognize the relation between the number of formation and the dancers on it, the students did not develop a more efficient strategy to count as they keep adding the numbers based on its order.

Mathematically speaking, the students see the structure of the triangular number in recursive pattern only. It means, to figure out the n number, they need to count the n-1 number first. This addition method also emerged in the study of Amit and Neria [19] that the students solve the generalization problem by using recursive or operational-local strategy. In fact, the goal of the lesson was to enable the students to find the general structure or explicit formula (as is referred by Yeşildere and Akkoç [20]) of a triangular number pattern. By means, without looking for the n-1, the students are expected to notice the relation between any number of the formation and its number of dancers.

Reflected to the result of the First Cycle, we revised the overall design of the pattern investigation sequence. In the Second Cycle, we introduced the triangular number pattern as half of the rectangular number pattern, which was introduced after the square number pattern. Contrary to the design in the First Cycle, in the present design, those three number patterns are related to each other. The square pattern was used as the opening problem, modified to the rectangular number pattern, and altered again to the triangular pattern. By that, we expected the students to employ the general ideas they built on the square and rectangular number patterns to see the structure of the triangular number pattern.

The lesson about rectangular number pattern as closed by the general ideas that the number of dancers is equal to the index number of formation times itself added by one. From that, we asked the students to divide the dancers in rectangular configuration fairly into two groups, to determine the color of their costume: black and white. This task was given as homework.

The students’ responses were varied. Not only considering the matter of “fair-sharing” color costume, the students argue about the aesthetic value of a stage performance- some examples of students work provided in Figure 6.
During the classroom discussion, some patterns were discussed, and the teacher announced that the unique pattern, as in Figure 7, in which two triangles can be constructed in a rectangular formation had chosen.

**Figure 7. Two triangles in a rectangle number pattern**

In the worksheet, the students can observe the idea of the pattern (Figure 7). From the first four formations, the students were asked to draw the position of the dancers with white costume in the fifth formation. Two methods emerged in the problem, the addition, and the halving method.

The first method is by using addition. Here, the students checked the additional dots in every pattern growth, i.e., there are one more dot in every next row. The students argued that the first formation only has one dot, the second formation has one-two dots, and the third formation has one-two-three dots. Similarly, in the fifth formation, there will be one dot in the first row, two dots in the second row, three dots in the third row, four dots in four rows, and five dots in the fifth row. This strategy leads to the recursive formula construction as the finding in the First Cycle of the study.

The second method performed by the students were using the halving method. In halving method, the students were not looking for previous term to determine the present, but by looking the functional relations among the variables involved. Hence, instead of recursive formula, the students will obtain a global structure of the pattern [19].

In this study, to perform the halving method, the students firstly visualize the number of dancers in the complete fifth formation then divide them into two groups. Finally, the students erased the half part of the formation and left the triangle alone. Hence, they come to the idea that a triangular number pattern can be made from halving a rectangular number pattern. Compare the students’ answer in following Figure 8 (a) the addition method and (b) the halving method.
As can be observed in Figure 8 (b), there were several pen corrections in half of the students’ answers. It refers to the students’ idea of halving the rectangle, not by adding one more dot in every next row as in Figure 8 (a). During the discussion, the students combine the idea of (a) and (b) by halving the formation but recursively check the number of dancers. Consider the discussion in the following Transcript 1 (S3, S4 = students, R = Researcher).

Fragment 2: Check the Result
1. S3: We have to fairly divide it.
2. S4: See, there are thirty (refers to 30 dancers in fourth rectangular number pattern formation)
3. R: Half of them will be fifteen.
4. S3: The question is asking about the white costume.
5. S4: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15 (counting the dots one by one to check)
6. S4: See, it is 15!

Fragment 2 illustrates how the students (S3 and S4) try to prove their halving method by count it one by one. They discuss their strategy to the students (S5 and S6) who used the addition method. This result is in line with the study of Tanisli [21] which stated that most of the students were seeing the number pattern in recursive relation. The study found that the functional relationship among two variables only is noticed by few students in middle and high achievement levels.

Looking at the counting result and the illustration made by S3 and S4 (Figure 8(b)), S5 and S6 were convinced to use halving method. To ensure the consistency of the method, we gathered the data further by checking their strategy to solve a similar problem but with a bigger number. The students were easily solving the second and the third task about the number of dancers in tenth and hundredth formation by implementing a similar halving idea.

We added the last question related to the inverse problem in triangular number pattern formation to evaluate students’ understanding of the relation between rectangular and triangular number patterns. The last problem is considered as a reflective generalization of a mathematical activity which consists of analysis, synthesis, and reflective abstraction [22]. For this question, the students have to determine the rectangular formation, which has 210 dancers wearing white costume. The following Figure 3 showed the explanation of the students’ representative (S5) to the Researcher (R) related to their attempt in solving the problem.

Fragment 3: From Triangular to Rectangular Number Pattern
1. R: So, who wants to tell me how you guys are getting the answer?
2. S5: This, two hundred and ten plus two hundred and ten, is equal to four hundred and twenty.
3. R: Okay, then ….
4. S5: Then, it is the result.
5. You know, twenty times twenty resulting in four hundred (refer to square number pattern).
6. As you need a rectangle form, you add twenty more. It makes four hundred and twenty.
7. Therefore, forty-two dancers are in the twentieth formation of rectangular number pattern.
According to the S5 explanation, we can infer the students’ steps to find the unknown number. Here, even though the problem was not employing the alphabet variables, we agree with Britt & Irwin [23] that number itself is a valuable support to develop students’ awareness in seeing the structure of the pattern. First, the students were using the structure of the triangular formation. Second, they doubled the numbers as they want to find the number of two triangles. By means, the students consistently saw the rectangular configuration as two triangular formations. Third, the students applied the intelligent guessing strategy [24] to find the formation number of rectangular number pattern.

The result supports the study of Apsari [25] which found that geometry representation is an essential help to encourage the students to shift from recursive to general formula in pre-algebra. Not just any type of representations, this study highlighted how different representation leads to different method. Reflected from the given task, we can see how the illustration given in First Cycle inspires the students to see the number of additional dancers for every next pattern. Hence, the students came up with the recursive method. Meanwhile, the Second Cycle provides a chance for the students to see the relation between rectangular to triangular formation and its relation to the number of dancers in each formation. Therefore, the students were guided to establish the general formula.

4. Conclusion
The results from the study confirmed that even though the number pattern is same, different visualization may lead the students to build different strategies. It can be seen from the addition strategy emerged in First Cycle compared to addition and halving strategies emerged in Second Cycle. Therefore, to enhance the students’ understanding in certain topics and help them to master certain skills, the learning designers and educators need to pay attention to the kind of visual support employed in the mathematics classroom. Specifically, in the present study, as the goal is to design a set of learning trajectories in pre-algebra which aim is to develop students’ algebraic thinking, a powerful visualization as in Second Cycle had better support to the process of seeing the general structure of the pattern.

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