THE GOLOMB SPACE IS TOPOLOGICALLY RIGID

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Abstract. The Golomb space \( N_\tau \) is the set \( N \) of positive integers endowed with the topology \( \tau \) generated by the base consisting of arithmetic progressions \( \{a + bn : n \geq 0\} \) with coprime \( a, b \). We prove that the Golomb space \( N_\tau \) is topologically rigid in the sense that its homeomorphism group is trivial. This resolves a problem posed by the first author at Mathoverflow in 2017.

1. Introduction

In the AMS Meeting announcement \[3\] M. Brown introduced an amusing topology \( \tau \) on the set \( N \) of positive integers turning it into a connected Hausdorff space. The topology \( \tau \) is generated by the base consisting of arithmetic progressions \( a + bN_0 := \{a + bn : n \in N_0\} \) with coprime parameters \( a, b \in N \). Here by \( N_0 = \{0\} \cup N \) we denote the set of non-negative integer numbers.

In \[14\] the topology \( \tau \) is called the relatively prime integer topology. This topology was popularized by Solomon Golomb \[6\], \[7\] who observed that the classical Dirichlet theorem on primes in arithmetic progressions is equivalent to the density of the set II of prime numbers in the topological space \( (\mathbb{N}, \tau) \). As a by-product of such popularization efforts, the topological space \( N_\tau := (\mathbb{N}, \tau) \) is known in General Topology as the Golomb space, see \[15\], \[16\].

The topological structure of the Golomb space \( N_\tau \) was studied by Banakh, Mioduszewski and Turek \[2\] who proved that the space \( N_\tau \) is not topologically homogeneous (by showing that 1 is a fixed point of any homeomorphism of \( N_\tau \)). Motivated by this results, the authors of \[2\] posed a problem of the topological rigidity of the Golomb space. This problem was also repeated by the first author at Mathoverflow \[1\]. A topological space \( X \) is defined to be topologically rigid if its homeomorphism group is trivial.

The main result of this note is the following theorem answering the above problem.

**Theorem 1.** The Golomb space \( N_\tau \) is topologically rigid.

The proof of this theorem will be presented in Section 5 after some preparatory work made in Sections 3, 4. The idea of the proof belongs to the second author who studied in \[12\] the rigidity properties of the Golomb topology on a Dedekind ring with removed zero, and established in \[12\] Theorem 6.7 that the homeomorphism group of the Golomb topology on \( \mathbb{Z} \setminus \{0\} \) consists of two homeomorphisms. The proof of Theorem 1 is a modified (and simplified) version of the proof of Theorem 6.7 given in \[12\]. It should be mentioned that the Golomb topology on Dedekind rings with removed zero was studied by Knopfmacher, Porubský \[10\], Clark, Lebowitz-Lockard, Pollack \[4\], and Spirito \[12, 13\].

2. Preliminaries and notations

In this section we fix some notation and recall some known results on the Golomb topology. For a subset \( A \) of a topological space \( X \) by \( \overline{A} \) we denote the closure of \( A \) in \( X \).

A poset is a set \( X \) endowed with a partial order \( \leq \). A subset \( L \) of a partially ordered set \( (X, \leq) \) is called

- linearly ordered (or else a chain) if any points \( x, y \in L \) are comparable in the sense that \( x \leq y \) or \( y \leq x \);
- an antichain if any two distinct elements \( x, y \in A \) are not comparable.

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By \( \Pi \) we denote the set of prime numbers. For a number \( x \in \mathbb{N} \) we denote by \( \Pi_x \) the set of all prime divisors of \( x \). Two numbers \( x, y \in \mathbb{N} \) are coprime iff \( \Pi_x \cap \Pi_y = \emptyset \). For a number \( x \in \mathbb{N} \) let \( x^k := \{x^n : n \in \mathbb{N}\} \) be the set of all powers of \( x \).

For a number \( x \in \mathbb{N} \) and a prime number \( p \) let \( l_p(x) \) be the largest integer number such that \( p^{l_p(x)} \) divides \( x \). The function \( l_p(x) \) plays the role of logarithm with base \( p \).

The following formula for the closures of basic open sets in the Golomb topology was established in \cite{BanakhMioduszewskiTurek2000}.

**Lemma 2** (Banakh, Mioduszewski, Turek). For any \( a, b \in \mathbb{N} \)

\[
a + b\mathbb{N}_0 = \mathbb{N} \cap \bigcap_{p \in \Pi_b} (p\mathbb{N} \cup (a + p^{l_p(b)}\mathbb{Z})).
\]

Also we shall heavily exploit the following lemma, proved in \cite{BanakhMioduszewskiTurek2000}.

**Lemma 3** (Banakh, Mioduszewski, Turek). Each homeomorphism \( h : \mathbb{N}_r \to \mathbb{N}_r \) of the Golomb space has the following properties:

\begin{enumerate}
  \item \( h(1) = 1; \)
  \item \( h(\Pi) = \Pi; \)
  \item \( \Pi h(x) = h(\Pi x) \) for every \( x \in \mathbb{N}; \)
  \item \( h(x^k) = h(x)^n \) for every \( x \in \mathbb{N} \).
\end{enumerate}

Let \( p \) be a prime number and \( k \in \mathbb{N} \). Let \( \mathbb{Z} \) be the ring of integer numbers, \( \mathbb{Z}_{p^k} \) be the residue ring \( \mathbb{Z}/p^k\mathbb{Z} \), and \( \mathbb{Z}_{p^k}^\times \) be the multiplicative group of invertible elements of the ring \( \mathbb{Z}_{p^k} \). It is well-known that \( |\mathbb{Z}_{p^k}^\times| = \phi(p^k) = p^{k-1}(p-1) \). The structure of the group \( \mathbb{Z}_{p^k}^\times \) was described by Gauss in \cite{Gauss1801} art.52–56 (see also Theorems 2 and 2’ in Chapter 4 of \cite{Gauss1801}).

**Lemma 4** (Gauss). Let \( p \) be a prime number and \( k \in \mathbb{N} \).

\begin{enumerate}
  \item If \( p \) is odd, then the group \( \mathbb{Z}_{p^k}^\times \) is cyclic;
  \item If \( p = 2 \) and \( k \geq 2 \), then the element \( -1 + 2^k\mathbb{Z} \) generates a two-element cyclic group \( C_2 \) in \( \mathbb{Z}_{p^k}^\times \), the element \( 5 + 2^k\mathbb{Z} \) generates a cyclic subgroup \( C_{2^{k-2}} \) of order \( 2^{k-2} \) in \( \mathbb{Z}_{2^k}^\times \) such that \( \mathbb{Z}_{2^k}^\times = C_2 \oplus C_{2^{k-2}} \).
\end{enumerate}

3. Golomb topology versus the \( p \)-adic topologies on \( \mathbb{N} \)

Let \( p \) be any prime number. Let us recall that the \( p \)-adic topology on \( \mathbb{Z} \) is generated by the base consisting of the sets \( x + p^n\mathbb{Z} \), where \( x \in \mathbb{Z} \) and \( n \in \mathbb{N} \). This topology induces the \( p \)-adic topology on the subset \( \mathbb{N} \) of \( \mathbb{Z} \). It is generated by the base consisting of the sets \( x + p^n\mathbb{N}_0 \) where \( x, n \in \mathbb{N} \). The following lemma is a special case of Proposition 3.1 in \cite{GelfandShilov1964}.

**Lemma 5.** For any clopen subset \( \Omega \) of \( \mathbb{N}_r \setminus p\mathbb{N} \), and any \( x \in \Omega \), there exists \( n \in \mathbb{N} \) such that \( x + p^n\mathbb{N}_0 \subset \Omega \).

**Proof.** Since the set \( p\mathbb{N} \) is closed in \( \mathbb{N}_r \), the set \( \Omega \) is open in \( \mathbb{N}_r \) and hence \( x + p^n\mathbb{N}_0 \subset \Omega \) for some \( k \in \mathbb{N} \) and \( b \in \mathbb{N} \), which is coprime with \( px \). We claim that \( x + p^n\mathbb{N}_0 \subset \Omega \). To derive a contradiction, assume that \( x + p^n\mathbb{N}_0 \setminus \Omega \) contains some number \( y \). Since \( \Omega \) is closed in \( \mathbb{N}_r \setminus p\mathbb{N} \), there exist \( m \geq n \) and \( d \in \mathbb{N} \) such that \( d \) is coprime with \( p \) and \( y \), and \( y + p^m\mathbb{N}_0 \cap \Omega = \emptyset \). It follows that \( y + p^m\mathbb{N}_0 \subset (x + p^n\mathbb{N}_0) + p^m\mathbb{N}_0 \subset x + p^n\mathbb{N}_0 \). Since \( p \notin \Pi_b \cup \Pi_d \), we can apply the Chinese Remainder Theorem \cite{Mordell1953} 3.12 and conclude that \( \emptyset \neq (y + p^m\mathbb{N}) \cap \bigcap_{q \in \Pi_b \cup \Pi_d} q\mathbb{N} \). Applying Lemma 2 and taking into account that the set \( \Omega \) is clopen in \( \mathbb{N}_r \setminus p\mathbb{N} \), we conclude that

\[
\emptyset \neq (y + p^m\mathbb{N}_0) \cap \left( \bigcap_{q \in \Pi_b \cup \Pi_d} q\mathbb{N} \right) = (x + p^n\mathbb{N}_0) \cap \left( \bigcap_{q \in \Pi_b} q\mathbb{N} \right) \cap (y + p^m\mathbb{N}_0) \cap \left( \bigcap_{q \in \Pi_d} q\mathbb{N} \right) \subseteq
\]

\[
\frac{x + p^n\mathbb{N}_0 \cap y + p^m\mathbb{N}_0}{\mathbb{N} \setminus \mathbb{N} \setminus \emptyset} \subset \mathbb{N} \cap (\mathbb{N} \setminus p\mathbb{N} \setminus \Omega) \subset p\mathbb{N},
\]

which is not possible as the sets \( x + p^n\mathbb{N}_0 \) and \( p\mathbb{N} \) are disjoint. This contradiction shows that \( x + p^n\mathbb{N}_0 \subset \Omega \). \( \square \)
A subset of a topological space is clopen if it is closed and open. By the zero-dimensional reflection of a topological space $X$ we understand the space $X$ endowed with the topology generated by the base consisting of clopen subsets of the space $X$.

**Lemma 6.** The $p$-adic topology on $\mathbb{N} \setminus p\mathbb{N}$ coincides with the zero-dimensional reflection of the subspace $\mathbb{N}_r \setminus p\mathbb{N}$ of the Golomb space $\mathbb{N}_r$.

**Proof.** Lemma implies that the $p$-adic topology $\tau_p$ on $\mathbb{N} \setminus p\mathbb{N}$ is stronger than the topology $\zeta$ of zero-dimensional reflection on $\mathbb{N}_r \setminus p\mathbb{N}$. To see that the $\tau_p$ coincides with $\zeta$, it suffices to show that for every $x \in \mathbb{N} \setminus p\mathbb{N}$ and $n \in \mathbb{N}$ the basic open set $\mathbb{N} \cap (x + p^n\mathbb{Z})$ in the $p$-adic topology is clopen in the subspace topology of $\mathbb{N}_r \setminus p\mathbb{N} \subset \mathbb{N}_r$. By the definition, the set $\mathbb{N} \cap (x + p^n\mathbb{Z})$ is open in the Golomb topology. To see that it is closed in $\mathbb{N}_r \setminus p\mathbb{N}$, take any point $y \in (\mathbb{N} \setminus p\mathbb{N}) \setminus (x + p^n\mathbb{Z})$ and observe that the Golomb-open neighborhood $y + p^n\mathbb{N}_0$ of $y$ is disjoint with the set $\mathbb{N} \cap (x + p^n\mathbb{Z})$.

For every prime number $p$, consider the countable family

$$X_p = \{a^N : a \in \mathbb{N} \setminus p\mathbb{N}, \ a \neq 1\},$$

where the closure $\overline{a^N}$ is taken in the $p$-adic topology on $\mathbb{N} \setminus p\mathbb{N}$, which coincides with the topology of zero-dimensional reflection of the Golomb topology on $\mathbb{N} \setminus p\mathbb{N}$ according to Lemma.

The family $X_p$ is endowed with the partial order $\leq$ defined by $X \leq Y$ iff $Y \subseteq X$. So, $X_p$ is a poset carrying the partial order of reverse inclusion.

**Lemma 7.** For any prime number $p$, any homeomorphism $h$ of the Golomb space $\mathbb{N}_r$ induces an order isomorphism

$$h : X_p \to X_{h(p)}, \ h : a^N \mapsto h(a^N) = \overline{h(a)^N}$$

of the posets $X_p$ and $X_{h(p)}$.

**Proof.** By Lemma $h(1) = 1$ and $(h(p)$ is a prime number. First we show that $h(p\mathbb{N}) = h(p)\mathbb{N}$. Indeed, for any $x \in p\mathbb{N}$ we have $p \in \Pi(x)$ and by Lemma $h(p) \in h(\Pi(x)) = \Pi(h(x))$ and hence $h(x) \in h(p)\mathbb{N}$ and $h(p\mathbb{N}) \subset h(p)\mathbb{N}$. Applying the same argument to the homeomorphism $h^{-1}$, we obtain $h^{-1}(h(p)\mathbb{N}) \subset p\mathbb{N}$, which implies the desired equality $h(p\mathbb{N}) = h(p)\mathbb{N}$. The bijectivity of $h$ ensures that $h$ maps homeomorphically the space $\mathbb{N}_r \setminus p\mathbb{N}$ onto the space $\mathbb{N}_r \setminus h(p)\mathbb{N}$.

Then $h$ also is a homeomorphism of the spaces $\mathbb{N} \setminus p\mathbb{N}$ and $\mathbb{N}_r \setminus h(p)\mathbb{N}$ endowed with the zero-dimensional reflections of their subspace topologies inherited from the Golomb topology of $\mathbb{N}_r$. By Lemma these reflection topologies on $\mathbb{N} \setminus p\mathbb{N}$ and $\mathbb{N} \setminus h(p)\mathbb{N}$ coincide with the $p$-adic and $(h(p))$-adic topologies on $\mathbb{N} \setminus p\mathbb{N}$ and $\mathbb{N} \setminus h(p)\mathbb{N}$, respectively.

By Lemma for any $a \in \mathbb{N} \setminus \{1\} \cup p\mathbb{N}$ we have

$$h(a)^N = h(a^N) \subseteq h(\mathbb{N} \setminus p\mathbb{N}) = h(p\mathbb{N})$$

and by the continuity of $h$ in the topologies of zero-dimensional reflections, we get $h(a^N) = \overline{h(a)^N} = \overline{h(a)^N}$. The same argument applies to the homeomorphism $h^{-1}$. This implies that

$$h : X_p \to X_{h(p)}, \ h : a^N \mapsto h(a^N) = \overline{h(a)^N},$$

is a well-defined bijection. It is clear that this bijection preserves the inclusion order and hence it is an order isomorphism between the posets $X_p$ and $X_{h(p)}$.

4. The order structure of the posets $X_p$

In this section, given a prime number $p$, we investigate the order-theoretic structure of the poset $X_p$.

For every $n \in \mathbb{N}$ denote by $\pi_n : \mathbb{N} \to \mathbb{Z}_{p^n}$ the homomorphism assigning to each number $x \in \mathbb{N}$ the residue class $x + p^n\mathbb{Z}$. Also for $n \leq m$ let

$$\pi_{m,n} : \mathbb{Z}_{p^m} \to \mathbb{Z}_{p^n}$$

be the ring homomorphism assigning to each residue class $x + p^n\mathbb{Z}$ the residue class $x + p^n\mathbb{Z}$. It is easy to see that $\pi_n = \pi_{m,n} \circ \pi_m$. Observe that the multiplicative group $\mathbb{Z}_{p^n}^\times$ of invertible elements of the ring $\mathbb{Z}_{p^n}$ coincides with the set $\mathbb{Z}_{p^n} \setminus p\mathbb{N}$ and hence has cardinality $p^n - p^{n-1} = p^{n-1}(p-1)$.

First we establish the structure of the elements $a^N$ of the family $X_p$. 

□
Lemma 8. If for some \( a \in \mathbb{N} \setminus p\mathbb{Z} \) and \( n \in \mathbb{N} \) the element \( \pi_n(a) \) has order \( \geq \max\{p,3\} \) in the multiplicative group \( \mathbb{Z}_p^\times \), then \( a^n = \pi_n^{-1}(\pi_n(a)^n) \).

Proof. Let \( B = B^n \) be the cyclic group generated by the element \( b = \pi_n(a) \) in the multiplicative group \( \mathbb{Z}_p^\times \). Since \( |\mathbb{Z}_p^\times| = p^{n-1}(p-1) \), and \( b \) has order \( \geq \max\{p,3\} \), the cardinality of the group \( B \) is equal to \( p^kd \) for some \( k \in [1,n-1] \) and some divisor \( d \) of the number \( p-1 \). Moreover, if \( p = 2 \), then \( 2^k \geq 3 \) and hence \( k \geq 2 \) and \( n \geq 3 \).

For any number \( m \geq n \), consider the quotient homomorphism \( \pi_{m,n} : \mathbb{Z}_{pm}^\times \rightarrow \mathbb{Z}_{pn}^\times, \pi_{m,n} : x + pm\mathbb{Z} \mapsto x + pn\mathbb{Z} \). We claim that the subgroup \( H = \pi_{m,n}^{-1}(B) \) of the multiplicative group \( \mathbb{Z}_{pm}^\times \) is cyclic. For odd \( p \) this follows from the cyclicity of the group \( \mathbb{Z}_{p^n}^\times \), see Lemma [1].

For \( p = 2 \), by Lemma [1] the multiplicative group \( \mathbb{Z}_{2m}^\times \) is isomorphic to the additive group \( \mathbb{Z}_2 \times \mathbb{Z}_{2m-2} \). Assuming that \( H \) is not cyclic, we conclude that \( H \) contains the 4-element Boolean subgroup

\[ V = \{1 + 2^m \mathbb{Z}, -1 + 2^m \mathbb{Z}, 1 + 2^{m-1} + 2^m \mathbb{Z}, -1 + 2^{m-1} + 2^m \mathbb{Z}\} \]

of \( \mathbb{Z}_{2m}^\times \). Then \( B = \pi_{m,n}(H) \supset \pi_{m,n}(V) \ni -1 + 2^n \mathbb{Z} \). Taking into account that \(-1 + 2^n \mathbb{Z} \) has order 2 in the cyclic group \( B \), we conclude that \(-1 + 2^n \mathbb{Z} = a^{2^n-1} + 2^n \mathbb{Z} \). Since \( k \geq 2 \), the odd number \( c = a^{2^{k-2}} \) is well-defined and \( c^3 + 42 = a^{2^{k-2}} + 4Z = -1 + 4Z \) which is not possible (as squares of odd numbers are equal 1 modulo 4). This contradiction shows that the group \( H \) is cyclic.

By [1] 1.5.5, the number of generators of the cyclic group \( H \) can be calculated using the Euler totient function as

\[ \phi(|H|) = \phi(p^{m-n} |B|) = \phi(p^{m-n} p^kd) = \phi(p^{m-n+k} |d| (p-1)) = p^{m-n} \phi(p^k) \phi(d) = p^{m-n} \phi(p^k) \phi(|B|), \]

which implies that for every generator \( g \) of the group \( B \), every element of the set \( \pi_{m,n}(g) \) is a generator of the group \( H \). In particular, the element \( \pi_{m,n}(a) \in \pi_{m,n}^{-1}(\pi_n(a)) \) is a generator of the group \( H \). By the definition of \( p \)-adic topology,

\[ a^n = \bigcap_{m \geq n} \pi_{m,n}^{-1}(\pi_{m,n}(a)^n) = \bigcap_{m \geq n} \pi_{m,n}^{-1}(\pi_{m,n}(B)) = \bigcap_{m \geq n} \pi_{m,n}^{-1}(B) = \pi_{m,n}^{-1}(B) = \pi_{n}^{-1}(\pi_n(a)^n). \]

Lemma 9. (1) For any \( X \in \mathcal{X}_p \), there exists \( n \in \mathbb{N} \) and a cyclic subgroup \( H \) of \( \mathbb{Z}_p^\times \) of order \( \geq \max\{p,3\} \) such that \( X = \pi_n^{-1}(H) \).

(2) For any \( n \in \mathbb{N} \) and cyclic subgroup \( H \) of \( \mathbb{Z}_p^\times \) of order \( |H| \geq \max\{p,3\} \), there exists a number \( a \in \mathbb{N} \setminus p\mathbb{N} \) such that \( \pi_n^{-1}(H) = a^n \in \mathcal{X}_p \).

Proof. 1. Given any \( X \in \mathcal{X}_p \), find a number \( a \in \mathbb{N} \setminus (\{1\} \cup p\mathbb{N}) \) such that \( X = a^n \). Choose any \( n \in \mathbb{N} \) with \( p^n > a^p \) and observe that the cyclic subgroup \( H \subset \mathbb{Z}_p^\times \), generated by the element \( \pi_n(a) = a + p^n \mathbb{Z} \) has order \( |H| \geq p + 1 \geq \max\{p,3\} \). By Lemma [1] \( X = a^n = \pi_n^{-1}(H) \).

2. Fix \( n \in \mathbb{N} \) and a cyclic subgroup \( H \) of \( \mathbb{Z}_p^\times \) of order \( |H| \geq \max\{p,3\} \). Find a number \( a \in \mathbb{N} \) such that the residue class \( \pi_n(a) = a + p^n \mathbb{Z} \) is a generator of the cyclic group \( H \). Then \( \pi_n(a) \) has order \( |H| \geq \max\{p,3\} \), Lemma [1] ensures that \( \pi_n^{-1}(H) = \pi_n^{-1}(\pi_n(a)^n) = a^n \in \mathcal{X}_p \).

For any \( X \in \mathcal{X}_p \), let

\[ n(X) = \min \{ n \in \mathbb{N} : X = \pi_n^{-1}(\pi_n(X)), |\pi_n(X)| \geq \max\{p,3\} \}. \]

Lemma [1] implies that the number \( n(X) \) is well-defined and \( \pi_n(X) \) is a cyclic subgroup of order \( \geq \max\{p,3\} \) in the multiplicative group \( \mathbb{Z}_p^\times \). Let \( i(X) \) be the index of the subgroup \( \pi_n(X) \) in \( \mathbb{Z}_p^\times \).

Lemma 10. For any odd prime number \( p \) and two sets \( X,Y \in \mathcal{X}_p \) the inclusion \( X \subseteq Y \) holds if and only if \( i(Y) \) divides \( i(X) \).
For any odd prime number $p$, any $n \in \mathbb{N}$, and the number $a = 1 + p^n$ we have $\overline{a^n} = 1 + p^n \mathbb{N}_0$ and $i(\overline{a^n}) = p^n(p - 1)$.

**Proof.** Observe that for any $k < p$ we have $a^k = (1 + p^n)^k \in 1 + kp^n + p^{n+1} \mathbb{Z} \neq 1 + p^{n+1} \mathbb{Z}$ and $a^n = (1 + p^n)p \in 1 + p^{n+1} \mathbb{Z}$, which means that the element $\pi_{n+1}(a)$ has order $p$ in the group $\mathbb{Z}^{\times}_{p^{n+1}}$. By Lemma 11

$$\overline{a^n} = \pi_{n+1}^{-1}(\{a^k + p^{n+1} \mathbb{Z} : 0 \leq k < p\}) = \bigcup_{k=0}^{p-1}(a^k + p^{n+1} \mathbb{N}_0) = 1 + p^n \mathbb{N}_0.$$  

Also $i(\overline{a^n}) = |\mathbb{Z}^{\times}_{p^{n+1}}|/p = p^n(p - 1)$. □

Lemmas 8, 9, and 11 imply that for an odd $p$, the poset $\mathcal{X}_p$ is order isomorphic to the set 

$$D_p = \{d \in \mathbb{N} : d \text{ divides } p^n(p - 1) \text{ for some } n \in \mathbb{N}\},$$

described with the divisibility relation.

An element $t$ of a partially ordered set $(X, \leq)$ is called a **chain** if its upper set $\uparrow t = \{x \in X : x \geq t\}$ is a chain. It is easy to see that the set of **chain** elements of the poset $D_p$ coincides with the set $\{p^n(p - 1) : n \in \mathbb{N}_0\}$ and hence is a well-ordered chain with the smallest element $(p - 1)$.

Below on the Hasse diagrams of the posets $D_3$ and $D_5$ (showing that these posets are not order isomorphic) the **chain** elements are drawn with the bold font.

![Hasse diagrams](image)

Lemmas 9, 10, and the isomorphism of the posets $\mathcal{X}_p$ and $D_p$ imply the following lemma.

**Lemma 12.** For an odd prime number $p$, the family $\{1 + p^n \mathbb{N}_0 : n \in \mathbb{N}\}$ coincides with the linearly ordered set of **chain** elements of the poset $\mathcal{X}_p$.

Now we reveal the order structure of the poset $\mathcal{X}_2$. This poset consists of the closures $\overline{a^n}$ in the 2-adic topology of the sets $a^n$ for non-zero odd numbers $a > 1$.

**Lemma 13.** Let $a \in \mathbb{N}$ and $X = a^n$.

1. If $a \in 1 + 4 \mathbb{N}$, then $a^n = 1 + 2^{n(x)-2} \mathbb{N}_0$. 

![Hasse diagram](image)
Lemma 14. If $a \in 3 + 4\mathbb{N}$, then $a^{16} = (1 + 2^{n(X)-1}N_0) \cup (-1 + 2^{n(X)-2} + 2^{n(X)-1}N_0)$.

In both cases, $i(X) = 2^{n(X)-3}$.

Proof. Lemma 8 and the definition of the number $n(X)$ imply that the projection $C_X := \pi_{n(X)}(X) = \pi_{n(X)}(\mathbb{N})$ is a cyclic subgroup of order 4 of the group $\mathbb{Z}_{2^n(X)}$, and that $X = \pi_{n(X)}^{-1}(C_X)$.

By the Gauss Lemma 4, $M_X = \{1 + 4k + 2^{n(X)} \mathbb{Z} \mid 0 \leq k < 2^{n(X)}-2\}$ is a maximal cyclic subgroup of $\mathbb{Z}_{2^n(X)}$. If $a \in 1 + 4\mathbb{N}$, the subgroup generated by $\pi_{n(X)}(a)$ is contained in $M_X$. Then $C_X = \{1 + k \cdot 2^{n(X)-2} + 2^{n(X)} \mathbb{Z} \mid 0 \leq k < 4\}$ and $X = \pi_{n(X)}^{-1}(C_X) = 1 + 2^{n(X)-2} \mathbb{Z}$.

If $a \in 3 + 4\mathbb{N}$, then $C_X$ is not contained in $M_X$. By Gauss Lemma 4 again, the unique cyclic subgroup of $\mathbb{Z}_{2^n(X)}$ of order 4 not contained in $M_X$ is generated by an element $g$ of $\mathbb{Z}_{2^n(X)}$ such that $-g$ generates the cyclic subgroup of $M_X$ of order 4. Therefore,

$C_X = \{1 + 2^{n(X)} \mathbb{Z}, 1 + 2^{n(X)-1} + 2^{n(X)} \mathbb{Z}, -1 + 2^{n(X)-2} + 2^{n(X)} \mathbb{Z}, -1 + 2^{n(X)-2} + 2^{n(X)-1} + 2^{n(X)} \mathbb{Z}\}$.

The first two elements, lifted to $\mathbb{Z}$, give the sequence $1 + 2^{n(X)-1} \mathbb{N}$, while the last two give $-1 + 2^{n(X)-2} + 2^{n(X)-1} \mathbb{N}$. Hence, $X$ is their union. $\square$

Lemma 14. For every $n \geq 2$

1. the set $X = (1 + 2^n)^N \in X_2$ coincides with $1 + 2^n \mathbb{N}_0$ and has $i(X) = 2^{n-1}$;
2. the set $Y = (-1 + 2^n)^N \in X_2$ coincides with $(1 + 2^{n+1} \mathbb{N}_0) \cup (2^n - 1 + 2^{n+1} \mathbb{N}_0)$ and has $i(Y) = 2^n$.

Proof. 1. Observe that for every $k < 4$ we have $(1 + 2^n)^k \in 1 + k2^n + 2^{n+2} \mathbb{Z} \neq 1 + 2^{n+2} \mathbb{Z}$ and $(1 + 2^n)^4 \in 1 + 2^{n+2} \mathbb{Z}$, which means that the element $(1 + 2^n)^k + 2^{n+2} \mathbb{Z}$ has order 4 in the group $\mathbb{Z}_{2^{n+2}}^\times$. Then the element $X = (1 + 2^n)^N \in X_2$ has $n(X) = n + 2$ and hence $X = 1 + 2^n \mathbb{N}_0$ and $i(X) = 2^{n(X)-3} = 2^{n-1}$ by Lemma 13.

2. Also for every $k < 4$ we have $(-1 + 2^n)^k \in (-1)^k + k2^n + 2^{n+2} \mathbb{Z} \neq 1 + 2^{n+2} \mathbb{Z}$ and $(-1 + 2^n)^4 \in 1 + 2^{n+2} \mathbb{Z}$, which means that the element $(-1 + 2^n)^k + 2^{n+2} \mathbb{Z}$ has order 4 in the group $\mathbb{Z}_{2^{n+2}}^\times$. Then the element $Y = (-1 + 2^n)^N \in X_2$ has $n(Y) = n + 2$ and hence $Y = (1 + 2^{n+1} \mathbb{N}_0) \cup (2^n - 1 + 2^{n+1} \mathbb{N}_0)$ and $i(Y) = 2^n(Y)-3 = 2^n$ by Lemma 13. $\square$

Lemma 15. For distinct sets $X, Y \in X_2$, the inclusion $X \subset Y$ holds if and only if $X \subseteq 1 + 4\mathbb{N}_0$ and $i(Y) < i(X)$.

Proof. If $X \subseteq 1 + 4\mathbb{N}_0$, then by Lemma 13, $X = 1 + 2^{n(X)-1} \mathbb{N}_0$. If $i(Y) < i(X)$, then $n(Y) < n(X)$, and thus by Lemma 13 we get $Y \supset 1 + 2^{n(X)-1} \mathbb{N}_0$ (i.e., $Y \supset X$) both if $Y$ is contained in $1 + 4\mathbb{N}_0$ or if it is not.

Conversely, if $X \subset Y$, the claim follows by writing explicitly $X$ and $Y$ through Lemma 13 $\square$

Lemmas 14 and 15 imply:

Lemma 16. The family $\min X_2 = \{X \in X_2 : X \not\subseteq 1 + 8\mathbb{N}_0\}$ coincides with the set of minimal elements of the poset $X_2$ and the set $X_2 \setminus \min X_2 = \{X \in X_2 : X \subseteq 1 + 8\mathbb{N}_0\}$ is linearly ordered and coincides with the set $\{1 + 2^n \mathbb{N}_0 : n \geq 3\}$. 
The Hasse diagram of the poset $X_2$

**Lemma 17.** For any homeomorphism $h$ of the Golomb space $\mathbb{N}_\tau$ and any $n \in \{1, 2, 3\}$ we have $h(n) = n$.

**Proof.**
1. The equality $h(1) = 1$ follows from Lemma 8(1).

2. By Lemma 7, $h$ induces an order isomorphism of the posets $X_2$ and $X_{h(2)}$. By Lemma 14 the set $\{(−1 + 2^n)\mathbb{N} : n \geq 2\}$ is an infinite antichain in the poset $X_2$. Consequently, the poset $X_{h(2)}$ also contains an infinite antichain. On the other hand, for any odd prime number $p$ the poset $X_p$ is order-isomorphic to the poset $D_p$, which contain no infinite antichains. Consequently, $X_{h(2)}$ cannot be order isomorphic to $X_p$, and hence $h(2) = 2$.

3. By Lemma 8(2), $h(3)$ is a prime number, not equal to $h(2) = 2$. By Lemma 7, $h$ induces an order isomorphism of the posets $X_3$ and $X_{h(3)}$. Then the posets $D_3$ and $D_{h(3)}$ also are order isomorphic. The smallest $\uparrow$-chain element of the poset $D_3$ is 2 and the set $\downarrow 2 = \{d \in D_3 : d \text{ divides } 2\}$ has cardinality 2. On the other hand, the smallest $\uparrow$-chain element of the poset $D_{h(3)}$ is $h(3) − 1$. Since the sets $D_3$ and $D_{h(3)}$ are order-isomorphic, the set $\downarrow (h(3) − 1) = \{d \in D_p : d \text{ divides } h(3) − 1\}$ has cardinality 2, which means that the number $h(3) − 1$ is prime. Observing that 3 is a unique odd prime number $p$ such that $p − 1$ is prime, we conclude that $h(3) = 3$. □

**Lemma 18.** For any homeomorphism $h$ of the Golomb space $\mathbb{N}_\tau$, and any prime number $p$ we have $h(1 + p^n\mathbb{N}_0) = 1 + h(p)^n\mathbb{N}_0$ for all $n \in \mathbb{N}$.

**Proof.** By Lemma 7 the homeomorphism $h$ induces an order isomorphism of the posets $X_p$ and $X_{h(p)}$.

If $p = 2$, then $h(p) = 2$ by Lemma 17. Consequently, $h$ induces an order automorphism of the poset $X_2$ and hence $h$ is identity on the well-ordered set $\{1 + 2^n\mathbb{N}_0 : n \geq 3\}$ of non-minimal elements of $X_2$. Consequently, $h(1 + 2^n\mathbb{N}_0) = 1 + 2^n\mathbb{N}_0$ for all $n \geq 3$.

Next, we show that $h(1 + 4\mathbb{N}_0) = 1 + 4\mathbb{N}_0$. Observe that for the smallest non-minimal element $9\mathbb{N} = 1 + 8\mathbb{N}_0$ of $X_2$ there are only two elements $5\mathbb{N} = 1 + 4\mathbb{N}_0$ and $3\mathbb{N} = (1 + 8\mathbb{N}_0) \cup (3 + 8\mathbb{N}_0)$, which are strictly smaller than $9\mathbb{N}$ in the poset $X_2$. Then $h(5\mathbb{N}) \in \{3\mathbb{N}, 5\mathbb{N}\}$. By Lemma 17, $h(3) = 3$ and hence $h(3\mathbb{N}) = 3\mathbb{N}$, which implies that $h(1 + 4\mathbb{N}_0) = h(5\mathbb{N}) = 5\mathbb{N} = 1 + 4\mathbb{N}_0$.

Now assume that $p$ is an odd prime number. Since $h(2) = 2$, the prime number $h(p) ≠ h(2) = 2$ is odd. By Lemma 12 the well-ordered sets $\{1 + p^n\mathbb{N}_0 : n \in \mathbb{N}\}$ and $\{1 + h(p)^n\mathbb{N}_0 : n \in \mathbb{N}\}$ coincide with the sets of $\uparrow$-chain elements of the posets $X_p$ and $X_{h(p)}$, respectively. Taking into account that $h$ is an order isomorphism, we conclude that $h(1 + p^n\mathbb{N}_0) = 1 + h(p)^n\mathbb{N}_0$ for every $n \in \mathbb{N}$. □
In this section we present the proof of Theorem 1. Given any homeomorphism \( h \) of the Golomb space \( \mathbb{N}_r \), we need to prove that \( h(n) = n \) for all \( n \in \mathbb{N} \). This equality will be proved by induction.

For \( n \leq 3 \) the equality \( h(n) = n \) is proved in Lemma 17. Assume that for some number \( n \geq 4 \) we have proved that \( h(k) = k \) for all \( k < n \). For every prime number \( p \) let \( \alpha_p \) be the largest integer number such that \( p^{\alpha_p} \) divides \( n - 1 \) (so, \( \alpha_p = t_p(n-1) \)). For every \( p \in \Pi_{n-1} \) we have \( p \leq n - 1 \) and hence \( h(p) = p \) (by the inductive hypothesis). Then \( h(\Pi_{n-1}) = \Pi_{n-1} \) and \( h(\Pi \setminus \Pi_{n-1}) = \Pi \setminus \Pi_{n-1} \).

Observe that \( n \) is the unique element of the set
\[
\bigcap_{p \in \Pi} (1 + p^{\alpha_p} N_0) \setminus (1 + p^{\alpha_p+1} N_0).
\]

By Lemma 18, \( h(n) \) coincides with the unique element of the set
\[
\bigcap_{p \in \Pi} (1 + h(p)^{\alpha_p} N_0) \setminus (1 + h(p)^{\alpha_p+1} N_0) =
\left( \bigcap_{p \in \Pi_{n-1}} (1 + h(p)^{\alpha_p} N_0) \setminus (1 + h(p)^{\alpha_p+1} N_0) \right) \cap \left( \bigcap_{p \in \Pi \setminus \Pi_{n-1}} N \setminus (1 + h(p) N_0) \right) =
\left( \bigcap_{p \in \Pi_{n-1}} (1 + p^{\alpha_p} N_0) \setminus (1 + p^{\alpha_p+1} N_0) \right) \cap \left( \bigcap_{p \in \Pi \setminus \Pi_{n-1}} N \setminus (1 + p N_0) \right) = \{ n \}
\]
and hence \( h(n) = n \).

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