Orbits of the left-right equivalence of maps in arbitrary characteristic

Dmitry Kerner

ABSTRACT. The germs of maps \((\mathbb{k}^n, o) \xrightarrow{f} (\mathbb{k}^p, o)\) are traditionally studied up to the right \((\mathbb{R})\), left-right \((\mathfrak{A})\) or contact \((\mathcal{K})\) equivalence. Various questions about the group-orbits \(\mathcal{G}f\) (for \(\mathcal{G}\) one of \(\mathbb{R}, \mathcal{K}, \mathfrak{A}\)) are reduced to their tangent spaces, \(T_{\mathcal{G}}f\). Classically the passage \(T_{\mathcal{G}}f \rightsquigarrow \mathcal{G}f\) was done by vector fields integration, hence it was bound to the \(\mathbb{R}/\mathbb{C}\)-analytic or \(C^r\)-category.

The purely-algebraic (characteristic-free) approach to the group-orbits of \(\mathbb{R}, \mathcal{K}\) has been developed during the last decades. But those methods could not address the (essentially more complicated) \(\mathfrak{A}\)-equivalence. Moreover, the characteristic-free results (for \(\mathbb{R}, \mathcal{K}\)) were weaker than those in characteristic zero, because of the (inevitable) pathologies of positive characteristic.

In this paper we close these omissions.

• We establish the general (characteristic-free) passage \(T_{\mathcal{G}}f \rightsquigarrow \mathcal{G}f\) for the groups \(\mathbb{R}, \mathcal{K}, \mathfrak{A}\). Submodules of \(T_{\mathcal{G}}f\) ensure (shifted) submodules of \(\mathcal{G}f\). For the \(\mathfrak{A}\)-equivalence this extends (and strengthens) various classical results of Mather, Gaffney, du Plessis, and others.

• Given a filtration on the space of maps one has the filtration on the group, \(\mathcal{G}^{(\bullet)}\), and on the tangent space, \(T_{\mathcal{G}}(\bullet)\). We establish the criteria of type \(T_{\mathcal{G}}^{(\bullet)}f \mapsto \mathcal{G}^{(\bullet)}f\) in their strongest form, for arbitrary base field/ring, provided the characteristic is zero or high for a given \(f\). This brings the “inevitably weaker” results of \(char > 0\) to the level of \(char = 0\).

As an auxiliary step, important on its own, we develop the mixed-module structure of the tangent space \(T_{\mathfrak{A}}f\) and establish various properties of the annihilator ideal \(a_{\mathfrak{A}}\), defining the instability locus of the map.

Contents

1. Introduction 1
2. Notations, conventions, preliminaries 5
3. Group actions \(\mathcal{G} \cap \text{Maps}(X,Y)\) and the corresponding tangent spaces \(T_{\mathcal{G}}f\) 9
4. Criteria for right orbits. “\(\mathfrak{A}f\) vs \(T_{\mathfrak{A}}f\)” 15
5. Criteria for contact orbits. “\(\mathcal{K}f\) vs \(T_{\mathcal{K}}f\)” 17
6. Properties of the (extended) tangent space \(T_{\mathcal{G}}f\) 20
7. Criteria for left-right orbits. “\(\mathfrak{A}f\) vs \(T_{\mathfrak{A}}f\)” 24

References 31

1. Introduction

1.1. Consider \(\mathbb{k}\)-analytic map-germs, \(f : (\mathbb{k}^n, o) \to (\mathbb{k}^p, o)\), here \(\mathbb{k}\) is \(\mathbb{R}\) or \(\mathbb{C}\). Whitney’s initial results on stable maps were greatly developed and extended by Thom, Mather and many others. (See e.g. [A.G.L.V], [Martinet], [Mon. Nuñ.-Bal..]) The maps are traditionally studied up to the left-right \((\mathfrak{A})\) equivalence, with the auxiliary right \((\mathbb{R})\) and contact \((\mathcal{K})\) equivalences. The first questions are about the group-orbits, \(\mathcal{G}f \subset \text{Maps}(\mathbb{k}^n,o),(\mathbb{k}^p,o)\), in particular the finite determinacy.

The classical methods relied heavily on vector fields integration. This chained the whole theory to the \(\mathbb{R}, \mathbb{C}\)-analytic (or \(C^r\)-differentiable) setting. Even the Nash function-germs, \(\mathbb{R}(x)\), could not be treated.

The study of \(\mathbb{R}, \mathcal{K}\)-equivalences by purely algebraic methods (in zero and positive characteristic) has begun in 80’s, e.g. [Greuel-Kröning.90]. This was motivated by the study of more complicated (e.g. non-isolated) singularities, by the applications in Algebraic Geometry/Commutative Algebra, and by the need to enable calculations in Computer Algebra. By now numerous results are available, [Bon.Gr.Ma.12], [Bon.Gr.Ma.11], [Greuel-Nguyen.16], [Nguyen.19], [Greuel-Pham.19], [B.K.16], [B.G.K.22]. The results in positive characteristic are (non-surprisingly) weaker than those in characteristic zero.

Date: June 18, 2024. Filename: A.equivalence.Orbits.1.17.tex.

Key words and phrases. Singularities of Maps, Left-Right Equivalence of Maps, Stable Maps, Finite Determinacy of Maps, Group-orbits of Left-Right Equivalence and their tangent spaces.

I was supported by the Israel Science Foundation, grants No. 1910/18 and 1405/22.
The $\mathcal{A}$-case (the most essential for the study of maps) is essentially more complicated than the (auxiliary) $\mathcal{R}, \mathcal{K}$-cases. Even the case of characteristic zero was untouched. The $\mathcal{A}$-orbits could not be studied by the (algebraic) methods of the previous papers, e.g. because of the following difficulties.

- The group-action $\mathcal{A} \circ \text{Maps}((k^n,o),(k^p,o))$ is neither additive, nor $k$-multiplicative, unlike the actions $\mathcal{R}, \mathcal{K} \circ \text{Maps}((k^n,o),(k^p,o))$.
- The tangent space $T_\mathcal{A} f$ is not an $O_{k^n,o}$-module, it is only a “mixed $(O_{k^n,o}, O_{k^p,o})$-module”.
- The essential ingredient was the Thom-Levine lemma (“naturalness of vector field integration”) on the solution of a certain differential equation. Its algebraic version was absent.
- The classical Artin approximation (so helpful for $\mathcal{R}, \mathcal{K}$-equivalences) was absent (until recently) for the left-right equivalence. (See §2.3v for more detail.)

1.2. **The results.** (The detailed description is in §1.6) We study maps of germs, $\text{Maps}(X,(k^p,o))$, of arbitrary characteristic. Here $X = \text{Spec}(R_X)$ and $(k^p,o) = \text{Spec}(R_Y)$ are formal/$k$-analytic/$k$-Nash germs of schemes. More precisely, $(R_X, R_Y)$ is one of the pairs $(k[[x]]/j, k[[y]])$, $(k[x]/j, k\{y\})$, $(k[x]/j, k(y))$, see §2.1ii. Here $k$ is any field or an excellent Henselian ring. (The later case is important for deformations.)

The space of maps is an $R_X$-module, $\text{Maps}(X,(k^p,o)) := \text{Hom}(R((k^p,o)), R_X) \cong m \cdot R_X^{\mathbb{Z}^p}$, for the maximal ideal $m \subset R_X$. Fix a map $f \in m \cdot R_X^{\mathbb{Z}^p}$ and let $\mathcal{G}$ be one of the groups $\mathcal{R}, \mathcal{K}, \mathcal{A}$. Our results are of two types: conditions to ensure the large orbit $\mathcal{G} f$ (in terms of the tangent space $T_\mathcal{G} f$), and the structure of the tangent space $T_\mathcal{G} f$.

1.2.1. **The $\mathcal{G}$-implicit function theorems.** Fix an ideal $a \subset R_X$. Theorems 4.1 5.1 7.4 read (roughly):

\[
\text{(1)} \quad \text{If } a \cdot R^{\mathbb{Z}^p}_X \subseteq T_\mathcal{G} f \text{ then } \mathcal{G} f \supseteq \{f\} + a^2 \cdot R^{\mathbb{Z}^p}_X.
\]

This is characteristic-free, and with no restrictions on the critical/singular/instability locus of the map $f$.

In many case $a$ can be chosen as an ideal that defines the non-reduced critical/singular/instability locus (with the natural scheme structure). One can say roughly: $f$ is $\mathcal{G}$-determined by its “2-jet” on the critical/singular/instability locus. (The “2-jet” means the image in $R_X/a^2$.) Some results of this type are known for $a = m^d$ and $R_X = \mathbb{C}\{x\}$, e.g. [Mon. Num.-Bal. Theorem 6.2, pg. 182]. The proofs were heavily analytic. Even the characteristic zero case is totally new.

1.2.2. **The filtration criteria.** “$T_{\mathcal{G}(j)} f$ vs $\mathcal{G}(j)f$”. Fix an ideal $I \subset R_X$, then the space of maps acquires the filtration $I^* \cdot R^{\mathbb{Z}^p}_X$. This induces filtrations on the group, $\mathcal{G}^*$, and on the tangent space, $T_{\mathcal{G}}^*$. Theorems 4.10 5.31 7.8 read (roughly):

\[
\text{(2)} \quad \mathcal{G}(j) f \supseteq \{f\} + I^d \cdot R^{\mathbb{Z}^p}_X \quad \text{if and only if} \quad T_{\mathcal{G}(j)} f \supseteq I^d \cdot R^{\mathbb{Z}^p}_X.
\]

In this case we assume: either $\text{char}(k) = 0$ or $\text{char}(k) > \frac{d-\text{ord}(f)}{j}$.

For $\mathcal{R}, \mathcal{K}$-equivalences this result is known in characteristic zero, [B.G.K.22], while for $\text{char}(k) > \frac{d-\text{ord}(f)}{j}$ it strengthens the known results “by a factor of $2$”. We remark, that the previous proofs of the “only if”-part (in $\text{char} > 0$) needed an additional assumption “$k$ is infinite” and non-trivial deformation/specialization arguments, [Grenel-Pham.19].

For $\mathcal{A}$-equivalence this was known only in the analytic/smooth-cases, $R_X = \mathbb{C}\{x\}, \mathbb{R}\{x\}, C^\infty(\mathbb{R}^n,o)$, with highly analytic proofs, [B.dP.W.87]. Even the zero characteristic case is new.

A remark, the ideals $a, I$ in (1) and (2) are not necessarily $m$-primary. I.e. there is no restriction on the dimension of the critical/singular/instability loci.

For many more results see §1.6.

1.2.3. **The structure of $T_\mathcal{A} f$.** The tangent spaces $T_X f, T_{\mathcal{X}} f \subseteq R_X^{\mathbb{Z}^p}$ are $R_X$-submodules. Their properties can be addressed via the standard commutative/homological algebra. The tangent space $T_\mathcal{A} f$ is not an $R_X$-module, it is only an $R_Y$-module (and not finitely-generated when $\dim(X) > p$ or when $f$ is not $\mathcal{A}$-finite). The classical Nakayama/Artin-Rees lemmas do not hold for $T_\mathcal{A} f$. We establish their substitutions and numerous “little tools” in §6 (See §1.6 for the list of results.)

In particular we prove: if $f$ is a morphism of “finite singularity type”, then the image tangent space $T_\mathcal{A} f$ is a finitely generated module over certain extension of $R((k^p,o))$, namely: the Noetherian, local ring $f^* R((k^p,o)) + a_\mathcal{A}$.
1.3. The methods. The proofs of theorems 4.1, 5.1 are heavily based on the implicit function arguments (hence their name “$\mathcal{G}$-implicit function theorem”). More precisely, we use a version of Newton lemma of the type of [Bourbaki.CA, Tougeron.66]. The proof of (the main) theorem 7.4 is much more complicated and demands additional tools. (Weierstraß division, arc-solution, Kostant-Rosenlicht theorem, and a special argument when the field $\mathbb{k}_{\text{fin}}$ is not algebraically closed.)

The proofs of theorems 4.5, 5.4, 7.8 are based on the algebraic Thom-Levine lemma (§3.1) and the full Baker-Campbell-Hausdorff expansion.

1.4. Establishing these “fundamentals” of $\mathcal{A}$-equivalence opens the way to further results. In this paper I give only the basic/simplest examples, see §1.0. More serious applications contain (in arbitrary characteristic) the theory of unfoldings, the theory of stable maps, Mather-Yau/Gaffney-Hauser theorems, algebraization of (non-finitely-determined) maps, [Kerner.22]. Moreover, the new criteria on the orbit $\mathcal{A}$-function $f$ and on the structure of the tangent space $T_{\mathcal{A}}f$ seem useful for the classification/study of $\mathcal{A}$-simple ($\mathcal{C}$-analytic) maps.

A remark: the proofs do not use the notion of “geometric equivalence” of [Damon.84], the methods seem applicable to the orbit-study of various other groups, e.g. volume preserving/symplectic equivalences, [Domitrz-Rieger.09].

1.5. Acknowledgement. Thanks are to A. F. Boix, G.-M. Greuel, D. Mond, M. A. S. Ruas for important advices. I was heavily influenced by their papers and also by those of T. Gaffney, A. du Plessis, L. Wilson. Special thanks to M. Borovoi and Z. Rosemblit for their help with Kostant-Rosenlicht issues. §7.1.

Finally I thank the referee of [B.G.K.22] who urged me to treat the $\mathcal{A}$-case (and not only $\mathcal{R}, \mathcal{K}$).

1.6. The detailed structure, results, and contents of the paper.

§2 is about the general preliminaries, to make the paper self-contained.

§2.1 fixes the rings we work with, i.e. $k[x]/j$, $k(x)/j$, $k(\mathcal{A})/j$.

§2.2 identifies the space of maps, $\text{Maps}(X, (k^p, o)) = m \cdot R_X^{\mathcal{P}}$, the isomorphism of $R_X$-modules.

§2.3 recalls “the basic tools”, Weierstraß division and finiteness, the full Baker-Campbell-Hausdorff formula, Artin approximation for $k$-analytic/$k$-Nash equations and for the left-right equivalence.

§2.4 is about the “local coordinate changes” (i.e. the ring automorphisms $\text{Aut}_X := \text{Aut}_k(R_X)$) and the “germs of vector fields” (i.e. the derivations $\text{Der}_X := \text{Der}_k(R_X)$). We recall the corresponding filtrations, on the group, $\{\text{Aut}_X\}$, and on the module, $\{\text{Der}_X\}$.

§2.5 recalls the exp/log maps, $\text{Der}_X \cong \text{Aut}_X$ and their truncated versions, $\text{jet}_N \text{Exp}$, $\text{jet}_N \text{Ln}$. In zero or high characteristic they substitute the integration of vector fields.

§3 sets-up the basic notions of $\mathcal{R}, \mathcal{K}, \mathcal{A}$-equivalence over $k$. The cases $\mathcal{R}, \mathcal{K}$ are “known in some sense”, [B.K.16], [B.G.K.22], but the $\mathcal{A}$-case is new.

§3.1 The group-actions on the space of maps, $\mathcal{G} \cup \text{Maps}(X, (k^p, o)) = m \cdot R_X^{\mathcal{P}}$, for $\mathcal{G}$ one of the groups $\mathcal{R}, \mathcal{L}, \mathcal{C}, \mathcal{A} := \mathcal{L} \times \mathcal{R}$, are defined as in the classical case.

§3.2 The (extended) image tangent spaces $T_{\mathcal{A}}f \subset R_X^{\mathcal{P}}$ are defined via the module(s) of derivations, e.g. $T_{\mathcal{A}}f := \text{Der}_X(f)$, $T_{\mathcal{A}}(f) := T_{\mathcal{A}}f + (f) \cdot R_X^{\mathcal{P}}$. As in the classical case, $T_{\mathcal{A}}f$, $T_{\mathcal{A}}(f)$ are $R_X$-submodules of $R_X^{\mathcal{P}}$, while $T_{\mathcal{A}}f$ is only a “mixed” module.

§3.3 Fix an ideal $I \subset R_X$ to get the filtration on the space of maps, $I^{*} \cdot R_X^{\mathcal{P}}$. It induces (natural) filtrations of the group, $\mathcal{G} \triangleright \mathcal{G}^{(\bullet)}$, and of the tangent space, $T_{\mathcal{A}} \supseteq T_{\mathcal{A}}^{(\bullet)}$.

A little twist: the filtrations $\mathcal{A}^{*}, T_{\mathcal{A}}^{\bullet}$ and $\mathcal{K}^{*}, T_{\mathcal{K}}^{\bullet}$ could be also defined in another way. Lemma §3.4 ensures: these distinct definitions are compatible when $k$ is an infinite field.

§3.4 This filtration $I^{*} \cdot R_X^{\mathcal{P}}$ defines the (filtration) topology on $R_X^{\mathcal{P}}$. The tangent space $T_{\mathcal{A}}f$, and the orbit $\mathcal{G}$, are closed in this topology for the groups $\mathcal{G} = \mathcal{R}, \mathcal{K}$. (The standard) proof is based on the Artin-Rees lemma and the classical Artin approximation.

The $\mathcal{A}$-case is more complicated (in the absence of standard methods), see §6.3.

§3.5 gives the algebraic version of the classical Thom-Levine lemma “Naturality of vector fields integration”. That basic lemma was of key importance for $\mathcal{A}$-equivalence in the $\mathbb{R}, \mathbb{C}$-analytic cases. And the algebraic version is used through the paper (when the characteristic is zero or high enough) in the same way.

§3.6 introduces the annihilator ideal $\mathfrak{a}_\mathcal{G} := \text{Ann}[T_{\mathcal{G}}] \subset R_X$. Here $T_{\mathcal{G}}f := R_X^{\mathcal{P}} / T_{\mathcal{G}}f$, this is the tangent space to $\mathcal{G}$-universal unfoldings, when $f$ is $\mathcal{G}$-finite. For functions ($p = 1$) on smooth germs, i.e. $(k^n, o) \rightarrow (k^1, o)$, these are the classical Jacobian/Tjurina ideals, $\mathfrak{a}_\mathcal{A} = \text{Jac}(f)$, $\mathfrak{a}_\mathcal{K} = \text{Jac}(f) + (f)$. When the source $X$ is singular, one gets Bruce-Roberts’ versions of these ideals.
More generally (for \( p \geq 1 \)), the ideal \( \mathfrak{a}_f \) defines the critical locus of a map, \( \mathfrak{a}_X \) defines the singular locus, \( \mathfrak{a}_f \) defines the instability locus. A remark: \( \mathfrak{a}_f \) is defined as the annihilator ideal, not as the Fitting ideal.

The annihilators \( \mathfrak{a}_f, \mathfrak{a}_X \) are well known, but \( \mathfrak{a}_f \) seems to be not studied previously.

Lemma 3.15 extends the classical fact (from \( k = \mathbb{C} \) to local Henselian rings): the restriction \( f|_{\text{Crit}(f)} : \text{Crit}(f) \to (k^p, o) \) is a finite morphism if \( f \) is \( X \)-finite if \( T^k_f \) is a f.g. module over \( R_Y \).

§4 is the first warmup, addressing the group-orbit: \( \mathcal{R} f \supseteq \{ f \} + a \cdot T_X f. \) This statement (and its proof) is characteristic-free.

The proof goes in the style of implicit function theorem of \[ \text{Bourbaki.CA} \] and \[ \text{Tougeron.66}. \]

§4.2 Numerous examples show that this bound is much stronger than the known bounds (even in the \( \mathbb{R}, \mathbb{C} \)-cases). As a trivial corollary we get the Morse lemma over an arbitrary field.

A geometric corollary: the map \( f \) is \( \mathcal{R} \)-determined by its “2-jet” on the critical locus, \( \mathcal{R} f \supseteq \{ f \} + \mathfrak{a}_X^2 \cdot R_X^{\otimes p} \).

§4.3 is the filtration criterion for \( \mathcal{R} \)-orbits via tangent spaces: \( ^m \mathcal{R} f \supseteq \{ f \} + I^d \cdot R_X^{\otimes p} \) if \( T_{\mathcal{R}(j)} f \supseteq I^d \cdot R_X^{\otimes p} \).

(Positive characteristic the integers \( j, d \) are restricted by the condition \( d - \text{ord}(f) < j \cdot \text{char}(k) \), i.e. the characteristic must be large enough for a given \( j, d \).)

A statement of this type has first appeared in \[ \text{BilPaw.Sia} \] for \( k = \mathbb{R}, \mathbb{C} \), and today it is known in characteristic zero. In positive characteristic it strengthens the known criteria, \[ \text{Greuel-Pham.19}, \text{B.G.K.22} \], all being of type “If \( T_{\mathcal{R}(j)} f \supseteq I^d \cdot R_X^{\otimes p} \) then \( \mathcal{R}(j+1) f \supseteq \{ f \} + I^d \cdot \text{ord}(f)+1 \cdot R_X^{\otimes p} \).”

To emphasize, the new statement is of “iff” type.

The proof uses \( \text{the truncated exp/ln operators, jet}_N(\text{Exp}, \text{jet}_N(\text{Ln})) \) the full Baker-Campbell-Hausdorff expansion, \[ \text{2.23} \text{iv.} \] (We bound the prime numbers appearing in the denominators.)

The \( \mathcal{R} \)-equivalence is much more complicated. (Recall that the \( X \)-equivalence was initially introduced as an auxiliary, simpler version of \( \mathcal{R} \)-equivalence.) A difficulty of \( \mathcal{R} \) (besides those mentioned in §1.1) is that the classical arguments were heavily based on Thom-Levine’s lemma, which holds only in zero or high characteristic, and on Mather-Malgrange preparation theorem, which assumes that \( T_\mathcal{R} f \) is an \( R_X \)-module. Without these results one needs special tools.

The next two sections are the core of the paper.

§6 is “The basic theory of \( T_\mathcal{R} f \).”

§6.1 The annihilator ideal \( \mathfrak{a}_\mathcal{R} = \text{Ann}[T^1_\mathcal{R} f] \subset R_X \) (for \( T^1_\mathcal{R} f := R_X^{\otimes p}[T^1_\mathcal{R} f] \)) is much more delicate than the corresponding annihilators \( \mathfrak{a}_\mathcal{R}, \mathfrak{a}_X \). We establish basic properties of \( \mathfrak{a}_\mathcal{R} \).

§6.2 The (extended) tangent space \( T_\mathcal{R} f \) is not an \( R_X \)-module. Considered as an \( R_Y \)-module it is often not finitely-generated (e.g. when \( \text{dim}(X) > p \)). We prove that \( T_\mathcal{R} f \) is a module over the extended ring \( f^\#(R_Y) + \mathfrak{a}_\mathcal{R} \). Moreover, \( T_\mathcal{R} f \) is finitely-generated over \( f^\#(R_Y) + \mathfrak{a}_\mathcal{R} \) in the most important case, “\( f \) is of finite singularity type” (i.e. \( f \) is \( X \)-finite, i.e. the germ \( V(f) \subset X \) has an isolated singularity).

§6.3 As \( T_\mathcal{R} f \) is not an \( R_X \)-module, one does not have the usual Nakayama statement, “\( M \subseteq T_\mathcal{R} f + m \cdot M \) implies \( M \subseteq T_\mathcal{R} f \).” We give weaker versions.

§6.4 Similarly the assumption “\( T^{d+1}_\mathcal{R}(j) f \supseteq I^{d+1} \cdot R_X^{\otimes p} \)” does not imply “\( T^{d+1}_\mathcal{R}(j+1) f \supseteq I^{d+1} \cdot R_X^{\otimes p} \).” Yet, we establish weaker statements of Artin-Rees type.
§6.5 Unlike the tangent spaces $T_{af} f, T_{xf} f$, the space $T_{af} f$ is not “obviously filtration-closed”. One cannot deduce $T_{af} f = T_{xf} f$ directly, just from the faithfulness/exactness of the completion functor $\lim R x / f \cdot \otimes$. And one cannot apply the Artin approximation, as this is an “inverse Artin problem”.

Yet, one has $a_{af} \cdot T_{af} f = a_{af} \cdot T_{xf} f$. For maps of finite singularity type we prove: $T_{af} f = T_{xf} f$.

§7 is about large modules inside $\mathscr{A}$-orbits.

§7.1 recalls the auxiliary result, the Kostant-Rosenlicht theorem (“Orbits of unipotent algebraic groups are closed”) over algebraically-closed fields. I add also an extension for arbitrary fields of characteristic zero (by M. Borovoi) and an example (by Z. Rosegarten) showing the pathology in positive characteristic, even when the field $k$ is perfect.

§7.2 contains the “$\mathscr{A}$-implicit function theorem”, $\mathscr{A} f \supseteq \{ f \} + a^2 \cdot R_x^{\geq 0} + T_{x(1)} f$. This is the main (and hardest) result of the paper. The proof is essentially more complicated than those in the $R, \mathcal{X}$-cases.

i. The condition $\mathscr{A} f \supseteq f + g$ is not an implicit function equation (unlike the conditions $\mathcal{A} f \supseteq f + g$, $\mathcal{X} f \supseteq f + g$). Yet, we convert it to an implicit function equation, modulo terms of high enough order. In this process we pass from the source $X$ to the target $(k^n, o)$, using the Weierstraß division.

ii. We obtain an arc solution of a system, i.e. a solution over the ring $R_1[[f]]$. Then one passes to the finite jets, and finds the ordinary solution for infinite number of $t$-values.

iii. Use Zariski-closedness of the orbits of unipotent algebraic groups (by Kostant-Rosenlicht). If the field $k/m_k$ is not algebraically-closed, then one invokes an argument of Galois-cohomology.

§7.3 The first immediate corollary is for the $\mathscr{A}$-orbits of stable maps: $\mathscr{A} f \supseteq \{ f \} + a^2 \mathcal{X}^{(i)} f + R_x^{\geq 0}$. This bound is completely new even in the classical case $R_X = \mathbb{C}\{x\}, \mathbb{R}\{x\}$, and is much stronger than the known bounds. In fact this bound holds for a broader class of maps, a sufficient assumption is: $T_{af(x)} f = T_{x(1)} f$, i.e. $a_{af(x)} = a_{x(0)}$.

Then come other examples showing how to apply this $\mathscr{A}$-IFT theorem and how to extend numerous classical results to arbitrary characteristic. E.g. any $\mathscr{A}$-finite map is $\mathscr{A}$-finitely determined, various explicit determinacy bounds, and so on.

§7.4 contains the filtration criterion, “$\mathcal{A} f \supseteq \{ f \} + I f \cdot R_x^{\geq 0}$ iff $T_{af(x)} f \supseteq I f \cdot R_x^{\geq 0}$. As in the $R, \mathcal{X}$-cases, for $\text{char}(k) > 0$ the integers $d, j$ are restricted by the characteristic, $2 - 1 - \text{ord}(f) < j < \text{char}(k)$.

This statement is known for $R_X = k\{x\}, k \in \mathbb{R}, \mathbb{C}$, but is completely new in zero/positive characteristic. As in the $R, \mathcal{X}$-case, the proof uses the truncated maps $\text{jet}_N(\text{Exp}), \text{jet}_N(\text{Ln})$. But the additional (essential) ingredient is the algebraic Thom-Levine lemma of §4.5.

§7.5 gives the geometric criterion of determinacy: $f$ is determined by its finite jet along the instability locus in the target, $f(V(a_{af})).$ (Or, along the locus $f^{-1}(f(V(a_{af}))) \cap \text{Crit}(f)$.) For $\mathscr{A}$-finite maps (i.e. $V(a_{af})$ is a point) this extends the classical Mather-Gaffney criterion. Otherwise this seems to be new even in the $\mathbb{R}, \mathbb{C}$-cases.

2. NOTATIONS, CONVENTIONS, PRELIMINARIES

2.1 Local rings/germs of schemes. We study $\text{Maps}(X, (k^n, o))$, where $X = \text{Spec}(R_X), (k^n, o) = \text{Spec}(R_Y)$ are germs of schemes over the base ring $k$. In the simplest case $k$ is a field, classically $k = \mathbb{R}, \mathbb{C}$. The case when $k$ is not a field is important for deformations/unfoldings, e.g. $k = \kbar[t], \kbar\{t\}$. See [Kerner, 22]. Through the paper we use the multi-variables, $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_0)$, $t = (t_1, \ldots, t_r)$.

i. Denote by $k^\times$ the group of invertible elements of $k$. Sometimes we impose the condition “$2, \ldots, d \in k^\times$.” E.g. this holds if $k \supseteq \mathbb{Q}$ or $k$ contains a field of characteristic larger than $d$.

ii. $(R_X, m)$, or just $(R, m)$, is one of the following local $k$-algebras.

* The formal power series $R = k[[x]],$ or the quotient ring $R = k[x]/J$. Here $k$ is either an $\text{any}$ field or a complete local (Noetherian) ring, e.g. $k = \kbar[[t]]/J$. Then $R$ is complete with respect to the filtration $m^d$.

* The $k$-analytic (locally convergent) power series $R = k\{x\},$ or the quotient ring $R = k(x)/J$. In this case $k$ is either a normed ring (e.g. a normed field), complete with respect to its (non-discrete) norm, or $k = \kbar(t)/J$, where $\kbar$ is a normed field (as before). The relevant cases of complete normed fields are $\mathbb{R}, \mathbb{C}, \mathbb{Q}_p$.

* The algebraic power series $R = k(x)$ (i.e. power series in $x$ that satisfy monic polynomial equations over $k[x]$) or the quotient ring $R = k(x)/J$. Here $k$ is either an $\text{any}$ field or a local excellent Henselian
ring. Then \( k(x) \) is the Henselization of the ring \( k[x] \) at the ideal \( (x) \), constructed as the filtered colimit of étale \( k[x] \)-algebras. See \cite{Moret-Bailly:21} for more detail.

The elements of \( \mathbb{R}(x) \) are exactly the Nash germs, see e.g., \cite{Ruiz:82} for the basic introduction. Occasionally we call the elements of \( k(x)/j \) “\( k \)-Nash germs” rather than “algebraic germ”, to avoid any confusion with rings like \( k[x], k[x]_m \).

The assumptions on \( k \) (local complete, resp. normed and complete for its norm, resp. local excellent Henselian) are needed to ensure the implicit function theorem, Weierstraß finiteness, and Artin approximation, see \cite{Hara:22}.

Denote the maximal ideal of \( k \) by \( m_k \).

The elements of \( R \) are represented by power series. Abusing notations we write \( f(x) \), to emphasize the dependence on \( x \).

Geometrically the source of the map \( X = \text{Spec}(R) \subseteq (k^n, o) \) (or \( X = \text{Spec}(R_X) \)) is a formal/\( k \)-analytic/\( k \)-Nash germ, and \( x \) denotes the local coordinates on the ambient space \( (k^n, o) \).

The target of the map is \( Y := (k^p, o) := \text{Spec}(R_Y) \), the formal/\( k \)-analytic/\( k \)-Nash germ (corresponding to the type of \( X \)). Namely, \( R_Y \) is one of the rings \( \mathbb{k}[[y]], \mathbb{k}\{y\}, \mathbb{k}\langle y \rangle \), with \( k \) a field or a ring, as before.

Below we use \( Y \) and \( (k^p, o) \) interchangeably.

Through the paper we have to compose elements \( f(z) \in R[[z]] \) with \( g \in R \). For \( g \in \mathfrak{m} \subset R \) the element \( f \circ g \) is well defined. For \( g \in R \) the element \( f \circ g \) can be not well defined. But in the proof of theorems \([11][15][16][17]\) the series \( f \) are of special form, \( f(z) \in R[[x,z]][[z]] \), i.e., \( f \) is a polynomial in \( z \), whose coefficients are power series in \( m_z \).

In this case the composition \( f \circ g \) is well defined for any \( g \in R \). Sometimes we use the ideal \( \mathfrak{I}^{\text{ord}(f) - 1} \). If \( \mathfrak{I} \leq 2 \) or \( I = R_X \) we put \( \mathfrak{I}^{\text{ord}(f) - 2} := R_X \).

Recall the general fact: any \( R_X \)-submodule \( M \subseteq R_X^{\mathbb{G}_m} \) is finitely generated. Proof: \( R_X^{\mathbb{G}_m} \) is a module over the Noetherian ring \( R_X \), therefore \( R_X^{\mathbb{G}_m} \) is a Noetherian module. Hence \( M \) is f.g.

We consider only local homomorphisms of local \( k \)-algebras, i.e. \( \phi : R_Y \rightarrow R_X \) with \( \phi(m_Y) \subset m_X \). Such homomorphisms are necessarily Krull-continuous, see e.g. Lemma 15.36.2 of \cite{Stacks}.

Therefore they act by substitution, \( \phi(f(y)) = \phi(f(y)) \).

In particular, the homeomorphism \( \phi \) is determined by its action on the generators \( y = (y_1, \ldots, y_p) \).

2.2 The identification Maps\((X,Y) = m \cdot R_X^{\mathbb{G}_m} \) for \( Y = (k^p, o) \). A map of germs \( f : X \rightarrow Y \) is defined algebraically by the (local) homomorphism of \( k \)-algebras, \( f^\#: \text{Hom}(R_Y, R_X) \). Fix generators \( y = (y_1, \ldots, y_p) \) in \( R_Y \). Any homomorphism \( R_Y \rightarrow R_X \) is determined by the image of \( y \) in \( m \subset R_X \), see \([2][1][6][7] \).

Vice-versa, any such map extends to a homomorphism. Accordingly we identify Maps\((X,Y) = \text{Hom}_k(R_Y, R_X) = m \cdot R_X^{\mathbb{G}_m} \).

This identification equips the set Maps\((X,Y) \) with the \( R_X \)-module structure.

If \( k \) is a field then \( m = (x) \subset R_X \) and the maps fix the origin, \( X \ni o \rightarrow o \in (k^p, o) \). If \( k \) is a local ring then \( m = m_k + (x) \), and Maps\((X,Y) \) can be interpreted as families of maps, possibly displacing the origin.

Given a map \( f \in \text{Maps}(X,Y) = m \cdot R_X^{\mathbb{G}_m} \subset R_X^{\mathbb{G}_m} \), the space \( R_X^{\mathbb{G}_m} \) becomes a (not finitely-generated) \( R_Y \)-module via the composition,

\[
q(y) \cdot R_X^{\mathbb{G}_m} := f^\#(q(y)) \cdot R_X^{\mathbb{G}_m} := q(f_1, \ldots, f_p) \cdot R_X^{\mathbb{G}_m}.
\]

We have the \( R_Y \)-submodule \( f^\#(R_X^{\mathbb{G}_m}) \subseteq f^*(R_X^{\mathbb{G}_m}) = R_X^{\mathbb{G}_m} \). Here \( f^\#(y) \subseteq f^*(y) = (f) \subset R_X \), and the first inclusion is usually proper.

Recall the classical notations, \( \omega f(m_p) \) for \( f^\#(y) \), and \( f^*\mathcal{M} \) for \( f^*(y) \).

2.3 The basic tools. (They are used repeatedly in \([1][6] \).

i. Implicit function theorem with unit linear part. IFT\(_X\). We often have to resolve implicit function equations of type \( z = f(z) \) for a vector of (formal/anayltic/algebraic) power series:

\[
(f(z) \in (m + (z^2)) \cdot R[[z]]^{\mathbb{G}_m}, \text{ resp. } f(z) \in (m + (z^2)) \cdot R\{z\}^{\mathbb{G}_m}, \text{ resp. } f(z) \in (m + (z^2)) \cdot R(z)^{\mathbb{G}_m},
\]

Here \( z = (z_1, \ldots, z_p) \) are the unknowns and the equations are non-polynomial in \( z \). We look for the solution \( z \in \mathfrak{m} \subset R_X^{\mathbb{G}_m} \). There exists the unique formal solution \( \hat{z}(x) \in \mathfrak{m} \cdot R^{\mathbb{G}_m} \). It is obtained, e.g. by the order-by-order procedure.
The implicit function theorem holds for our rings and ensures: this formal solution belongs to $R$.

- For the ring $k[[x]]/j$, with $k$ any complete local ring, this statement is trivial.
- If $FT_1$ holds in the rings $k[[x]]/j$, $k(x)/j$, $k(x)/j$, for any (normed) field $k$, see example 2.2 of [B.G.K.22].
- If $FT_1$ holds in the ring $k[x]/j$ for $k$ a normed ring, complete with respect to its norm, [Abhyankar pg.84].
- If $FT_1$ holds in the ring $k[x]/j$, for $k$ a local, excellent, Henselian ring, [Lafon.67], see also page 4 of [Denef-Lipshitz].

Sometimes the system to resolve has the form $z = h(z) + c$, for a vector $c \in R^{\oplus p}$, where the entries of the vector $h(z)$ belong to the subset $m \cdot R[[m \cdot z][z]] \subseteq R[[z]]$. This case is reduced to that of $z = f(z)$ by the substitution $\tilde{z} = z - c$.

ii. Weierstrass division with remainder. Take a local subring $R \subseteq k[[x]]$, here $k$ is a local ring (e.g. a field). The Weierstrass division holds in $R$ if for any $x_n$-regular element $f \in R$ of $x_n$-order $d_n$, and any $g \in R$, one can present $g = q \cdot f + r$, where $q \in R$ and $r \in R \cap k[[x_1, \ldots, x_{n-1}][x_n]]$, with deg$_x r < d_n$. This division with remainder holds in our rings of $\mathbb{Z}$.

iii. Weierstrass finiteness condition holds for a pair of local $k$-algebras $R_X$, $R_Y$ if for any homomorphism $\phi : R_Y \rightarrow R_X$ and any finitely generated $R_X$ module $M$, with the quotient $M/\phi^* (m_Y) \cdot M$ f.g. over $k$, the module $M$ is f.g. over $R_Y$.

This condition is implied by the Weierstrass division condition. In particular it holds for the pairs $(k[[x]]/j, k[[y]])$, $(k(x)/j, k(y))$, $(k(x)/j, k(y))$.

A remark: the $C^\infty$-version of Weierstrass finiteness is a deep preparation theorem of Malgrange.

iv. Baker-Campbell-Hausdorff formula. Suppose $k \supseteq Q$. Let $M$ be a $k$-module (possibly not finitely generated), with a filtration $M_*$. Suppose $M$ is $M_*$-complete. Suppose some operators (not necessarily $k$-linear) $\xi, \eta \in M$ are filtration-nilpotent, i.e. $\xi, \eta (M_*) \subseteq M_{l+1}$. Then the operators $exp(\xi)$, $exp(\eta)$, $exp(\xi + \eta)$ are defined on $M$. One has the classical relation $exp(\xi) exp(\eta) = exp(\sum_{i=1}^\infty p_i(\xi, \eta))$. Here $p_1(\xi, \eta)$ is a homogeneous polynomial of degree 1 in non-commuting variables $\xi, \eta$. The expansion begins as follows: $exp(\xi) \cdot exp(\eta) = exp(\xi + \eta + \frac{[\xi, \eta]}{2} + \frac{[\xi, [\xi, \eta]] - n[\xi, \eta]}{12} + \cdots)$. Below we use the full expansion, see, e.g. [Jacobson §5 of chapter 5] or [Serre pg.29] or [BourbakiLie pg.160]:

\[
\sum_{l=1}^\infty p_l(\xi, \eta) = \sum_{i=1}^{\infty} \frac{(-1)^i}{i} \sum_{r_1 + s_1 > 0} \sum_{r_i + s_i > 0} \frac{[\xi_1 \eta^{s_1} \cdots \xi_i \eta^{s_i}]}{(r_1 + s_1) \cdots (r_i + s_i)}.
\]

Here $[\xi_1 \eta^{s_1} \cdots \xi_i \eta^{s_i}]$ is a certain combination of commutators with $Z$ coefficients. Comparing the two sides we get $\sum_{l=1}^l (r_1 + s_1) = l$ and therefore:

\[
p_l(\xi, \eta) = \frac{1}{l} \sum_{i=1}^{\infty} \frac{(-1)^i}{i} \sum_{\sum_{l=1}^l (r_1 + s_1) = l} \frac{[\xi_1 \eta^{s_1} \cdots \xi_i \eta^{s_i}]}{(r_1 + s_1) \cdots (s_i + s_i)}.
\]

In particular, the rescaled polynomial $(1!^4 \cdot p_l(\xi, \eta))$ has only integer coefficients.

We emphasize that the relation $exp(\xi) \cdot exp(\eta) = \cdots$ is purely formal, and no linearity/algbericity is assumed on the action $\xi, \eta \in M$.

Recall that in the analytic case, i.e. $R = k[[x]]/j$, $k \supseteq Q$, the series $\sum_{l=1}^\infty p_l(\xi, \eta)$ can be nowhere convergent for some derivations $\xi, \eta \in Der_k(R)$. However this series is locally convergent, i.e. the resulting derivation is analytic, $\sum_{l=1}^\infty p_l(\xi, \eta) \in Der_k(R)$, if one assumes the filtration-nilpotence, $\xi, \eta (M_*) \subseteq M_{l+1}$. Indeed, in this case we get the (filtration-unipotent) analytic automorphisms, $exp(\xi), exp(\eta) \in Aut_k(R)$. (See e.g. the proof of part 2 of Lemma 3.17 in [B.K.16].) Their product is a filtration-unipotent automorphism as well, $exp(\sum_{l=1}^\infty p_l(\xi, \eta)) \in Aut_k(R)$. And therefore $\sum_{l=1}^\infty p_l(\xi, \eta) \in Der_k(R)$.

v. Artin approximation. Let $R_X$ be one of $k[[x]]/j$, $k(x)/j$, $k(x)/j$, with $k$ a local ring, see §2.1ii. Take a (finite) system of implicit function equations, $F(y) = 0$, here $F(y) \in R_X[[y]]^{\oplus N}$, resp. $R_X\{y\}^{\oplus N}$, resp. $R_X\{y\}^{\oplus N}$. Suppose we have an order-by-order solution, i.e. a sequence $\tilde{y}_x \in R_X^{\oplus q}$ satisfying:

[Denef-Lipshitz] assume $k$ a field or a DVR (and denote $k$ by $R$). But their proof uses only the Weierstrass preparation, which holds for $k$-local, excellent, Henselian.
\[ F(y_\ast(x)) \in m^* \cdot R_X^{\oplus N}. \] The Pfister-Popescu theorem, Pfister-Popescu.75 (see also §2.4.1 of B.G.K.22), ensures a formal solution, i.e. \( \hat{y}(x) \in \check{R}_X^{\oplus q} \) satisfying \( F(\hat{y}(x)) = 0. \)

Suppose \( R_X \) is one of \( k[x]/J, k(x)/J, \) see 2.1.ii. By the Artin approximation theorem, Artin.68, Artin.69, this formal solution is approximated by \( R_X \)-solutions. Namely, for each \( d \geq 1 \) there exists \( y_d(x) \in \check{R}_X^{\oplus q} \) satisfying: \( F(y_d(x)) = 0 \) and \( \hat{y}(x) - y_d(x) \in (x)^d \cdot \check{R}_X^{\oplus q}. \)

A remark: the general Artin approximation holds for polynomial equations over any Henselian local ring. In our case the equations \( F(y) = 0 \) are non-polynomial, they are (algebraic/analytic) power series. Therefore we restrict to the particular rings \( k[x]/J, k(x)/J. \) See [B.G.K.22 §2.4.2], Rond.18, Popescu.00 for more detail.

The equation of \( \mathcal{A} \)-equivalence is \( \Phi_Y \circ f \circ \Phi_X^{-1} = \hat{f}. \) This is not an implicit function equation. And the Artin approximation does not work in the analytic case, \( k\{x\}, \) due to the example of Gabrielov.73, see Shiota.98 Fact.1.4. The left-right version of the Artin approximation has been established:

- \( R_X = \mathbb{R}\langle x \rangle, \) for any \( f, \) Shiota.98 Fact.1.3, Shiota.10 Theorem.4.
- \( R_X = \mathbb{R}\{x\}, \) for \( f \) of finite singularity type, Shiota.98 Fact.1.7.

The proof for \( \mathbb{R}\langle x \rangle \) is characteristic-free, and is valid over any field. But the proof for \( \mathbb{R}\{x\} \) is heavily based on the \( C^\infty \)-topology. These statements are extended in Kerner.23 to the rings \( k[x]/J, k(x)/J. \)

2.4. Coordinate changes (ring automorphisms, \( Aut_X \)) and vector fields (ring derivations, \( Der_X \)).

i. Denote by \( Aut_k(R) \) the group of (local) \( k \)-linear automorphisms of \( R. \) The elements of \( R \) are presentable by power series, the automorphisms act by substitution, \( \phi(f(x)) = f(\phi(x)) \) for \( \phi(x) \in m. \) See 2.1.

Geometrically (for \( J = 0 \)) these are “the coordinate changes” of the germ \( X = \text{Spec}(R). \) See [B.G.K.22 §2.3.2] for more detail. Because of this identification we abbreviate \( Aut_k(R) \) to \( Aut_X. \)

ii. Denote by \( Der_k(R) \subset End_k(R) \) the set of \( k \)-linear derivations. This is an \( R \)-module. Geometrically this is the module of germs of vector fields on \( X = \text{Spec}(R), \) classically denoted by \( \Theta_X. \)

For \( R = k[x], k\{x\}, k\langle x \rangle \) this module is generated by the partials \( Der_k(R) = R(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}), \) see e.g. Matsumura Theorem 30.6 More generally, for \( R = \mathbb{S}/J \) with \( S = k[x], k\{x\}, k\langle x \rangle, \) we have the presentation

\[ J \cdot Der_k(S) \to Der Log(J) \to Der_k(R) \to 0. \]

Here \( Der Log(J) = \{ \sum c_j \frac{\partial}{\partial x_j} | \sum c_j \frac{\partial}{\partial x_j} (J) \subset J \} \subset Der_k(S) \) is the module of log-derivations. Its elements correspond to the vector fields tangent to the subgerm \( V(J) \subset (k^n, o). \)

Denoting \( X = \text{Spec}(R) \) we abbreviate \( Der_k(R) \) to \( Der_X \) and \( End_k(R) \) to \( End_X. \)

Applying derivations to an element \( f \in R^{\oplus p} \) we get the (finitely generated) submodule \( Der_X(f) \subset R^{\oplus p}. \)

iii. In \( \mathbb{R}/C \)-analytic (or \( C^\infty \)) geometry one integrates vector fields to flows. Taking a flow at time \( t = 1 \) one gets the map \( \Theta_X \to Aut_X. \) Over an arbitrary ring \( k \) (without any topology/integration) the maps \( Der_X \cong Aut_X \) are constructed in §2.5 using the following filtration-topology. (For more detail see B.G.K.22 §2.2.3.)

Fix a descending filtration by ideals, \( R = I_0 \supset I_1 \supset \cdots, \) with \( \cap I_j = (0). \) This defines the filtration topology on \( R. \) Accordingly, for each \( j \geq 0, \) one considers the coordinate changes that are identity modulo higher order terms:

\[ Aut_X^{(j)} := \{ \Phi \in Aut_X | \Phi(I_d) = I_d \quad \text{and} \quad \Phi|_{I_d/I_{d+j}} = I_d|_{I_d/I_{d+j}}, \forall d \geq 1 \}. \]

Thus \( Aut_X^{(0)} \) consists of automorphisms that preserve the filtration, while \( Aut_X^{(1)} \) consists of “topologically unipotent” automorphism. For the particular filtration \( m^* \subset R \) one has \( Aut_X = Aut_X^{(0)}. \) The sets \( Aut_X^{(j)} \) are normal subgroups, and we get the filtration \( Aut_X^{(0)} \supset Aut_X^{(1)} \supset \cdots. \)

Similarly we consider (for every \( j \in \mathbb{Z} \)) “topologically nilpotent” \( k \)-linear endomorphisms and derivations.

- \( \text{End}_k^{(j)}(R) := \{ \phi \in End_k(R) | \phi(I_d) \subset I_{d+j}, \forall d \geq 1 \} \)

Below we often abbreviate \( End_X := End_k(R) \) and \( End_X^{(j)} := End_k^{(j)}(R). \)

- \( \text{Der}_X^{(j)} := \{ \xi \in Der_X | \xi(I_d) \subset I_{d+j}, \forall d \geq 1 \} \subset Der_X \cap End_X^{(j)}(R). \)

We get the filtration by \( R \)-submodules, \( Der_X \supset \cdots \supset Der_X^{(1)} \supset Der_X^{(0)} \supset \cdots \)

E.g. for the filtration by powers of ideals, \( I^* \subset R, \) we have \( Der_X = Der_X^{(1)} \) and \( Der_X^{(j)} \supset I^{j+1}\cdot Der_X. \)
2.5. The maps \( \exp, \ln : \text{Der}_X \cong \text{Aut}_X \) and their approximations. Vector field integration. Let \((R, m, J^*)\) be a local filtered \(\mathbb{k}\)-algebra of \([2.1]\)ii. Assume \(\mathbb{k} \supseteq \mathbb{Q}\) and \(R\) is complete with respect to the filtration. Then the derivations and automorphisms of \(R\) admit the exponential and logarithmic maps, see Theorem 3.6 of \([B.G.K.22]\):

\[ \forall j \geq 1 : \quad \text{Der}_X^{(j)} \exp \text{Aut}_X^{(j)}, \quad \exp(\xi) := e^\xi = \sum_{i=0}^{\infty} \frac{\xi^i}{i!}, \]

\[ \forall j \geq 1 : \quad \text{Der}_X^{(j)} \ln \text{Der}_X^{(j)}, \quad \ln(\Phi) := -\sum_{i=1}^{\infty} \frac{(1 - \Phi)^i}{i}. \]

When \(\mathbb{k} \supseteq \mathbb{R}\) or the ring \(R\) is not \(J^*\)-complete, one cannot use the classical exponential/logarithmic maps. But in many cases one can use the following “\(N\)th jet approximations” to these maps. Fix \(1 \leq N \leq \infty\).

For our filtration criteria, \([L.3, L.5, L.7]\) we impose the conditions (for all \(j \geq 1\)):

- **jet\(_N(\exp)\)**: every derivation \(\xi \in \text{Der}_X^{(j)}\) defines an automorphism \(\text{jet}_N(\exp(\xi)) := \sum_{i=0}^{N} \frac{\xi^i}{i!} + \phi_\xi \in \text{Aut}_X^{(j)}\).
  
  (Here \(\phi_\xi \in \text{End}_{(\mathbb{J}N+1)}\) is an auxiliary \(\mathbb{k}\)-linear endomorphism, see \([L.3]\)ii.)

- **jet\(_N(\ln)\)**: every automorphism \(\phi \in \text{Aut}_X^{(j)}\) defines a derivation \(\text{jet}_N(\ln(\phi)) := -\sum_{i=1}^{\infty} \frac{(1 - \phi)^i}{i} + \xi_\phi \in \text{Der}_X^{(j)}\).
  
  (Here \(\xi_\phi \in \text{End}_{(\mathbb{J}N+1)}\) is an auxiliary \(\mathbb{k}\)-linear endomorphism.)

These conditions imply, in particular, \(2, \ldots, N \in \mathbb{k}^\times\), see \([2.1]\)ii. Thus \(\text{char}(\mathbb{k}) > 0\) or \(\text{char}(\mathbb{k}) = 0\).

For our “\(f\)-implicit function theorems”, \([L.1, L.3, L.7, L.22]\) we impose a variation of condition \(\text{jet}_1\). Take a derivation \(\xi \in \text{Der}_X\) with \(\xi(\mathbb{m}) \subseteq \mathbb{m}^2\). If \(J = 0\) then the map \(f(x) \to f(x + \xi(x))\) does not exist, under very weak assumptions on the filtration: \(\text{Der}_X^{(1)}(\mathbb{m}) \subseteq \mathbb{m}^2\). For more detail see \([B.G.K.22]\) Theorem 3.6.

- **\(R = \mathbb{k}[x]/(x^N)\), \(R = \mathbb{k}[x]/(x)\)**, and \(\mathbb{k}^\times\), \(N \in \mathbb{N}\).

ii. The condition \(\text{jet}_N(\exp)\) (resp. \(\text{jet}_N(\ln)\)) obviously implies \(\text{jet}_1(\exp)\) (resp. \(\text{jet}_1(\ln)\)) for \(l < N\).

iii. The condition \(\text{jet}_\infty(\exp)\) means: every vector field \(\xi \in \mathbb{T}_X\) integrates to the flow \(\Phi_t \circ X\).

iv. For \(\text{char}(\mathbb{k}) > 0\) and \(J \neq 0\) already the condition \(\text{jet}_1\) is non-trivial, see \(\S 3.2\) of \([B.G.K.22]\). For example, let \(\mathbb{k}\) be a field of characteristic \(p\) and take \(f(x) = x^p + y^p \in \mathbb{k}[x, y]\). Take the ring \(R = \mathbb{k}[x, y]/(f)\) filtered by the ideals \((x, y, p)\). Take \(\xi := y^p \partial_{x^p} \in \text{Der}_X^{(1)}\). The conditions \(\text{jet}_0\) and \(\text{jet}_1(\exp)\) imply: there exist \(\phi_x, \phi_y \in (x, y)\) such that the map \(\langle x, y \rangle \to (x + y^p + \phi_x, y + \phi_y)\) defines an automorphism of \(R\). But then we must have

\[ f(x, y) \to f(x + y^p + \phi_x, y + \phi_y) = x^p + y^{2p} + (\phi_x + y^p)^p + (\phi_y + y^p)^p = (f(x, y)) = (x^p + y^p). \]

And this condition is clearly non-resolvable because of the term \(y^{2p}\).

3. Group actions \(\mathscr{G} \circ \text{Maps}(X, Y)\) and the corresponding tangent spaces \(T_{\varphi} f\)

Classically one studies maps of germs \((\mathbb{k}^n, o) \to (\mathbb{k}^p, o)\), for \(\mathbb{k} = \mathbb{R}, \mathbb{C}\), up to the right \((\mathscr{G})\), left \((\mathscr{L})\), contact \((\mathscr{C})\) or left-right \((\mathscr{S})\) equivalences. We set up the corresponding general notions.

Let \(X = (\mathbb{k}^p, o)\) be the formal/\(\mathbb{k}\)-analytic/\(\mathbb{k}\)-Nash germs, i.e. 
\((\text{Aut}_X \circ \text{Maps}(X, Y)) = \mathbb{m} \cdot R_X^{\mathbb{Z}^p}\) by composition on the right, \([\text{Mon. Num. Bal.}]\) \(\S 3.1\]

\[ \Phi_X(f) = \Phi(f_1, \ldots, f_p) = f \circ \Phi_X^{-1} = (f_1 \circ \Phi_X^{-1}, \ldots, f_p \circ \Phi_X^{-1}), \]

\[ f = m \cdot R_X^{\mathbb{Z}^p}. \]

Traditionally one denotes this group action by \(\mathscr{G} := \text{Aut}_X\). The action \(\mathscr{G} \circ m \cdot R_X^{\mathbb{Z}^p}\) is \(\mathbb{k}\)-linear.
ii. Present the automorphisms $Aut_Y \circ R_Y$ explicitly, $(y_1, \ldots, y_p) = y \rightarrow \Phi_Y(y) = (\Phi_1(y), \ldots, \Phi_p(y))$. Accordingly define the left action $Aut_Y \circ Maps(X,Y) = m \cdot R^\oplus_X$, by $f \rightarrow \Phi_Y(y)|f := (\Phi_1(f), \ldots, \Phi_p(f))$. See [Mon. Nuñ.-Bal.] §3.2. Traditionally one denotes this group action by $\mathcal{L}' := Aut_Y$.

Note that the group $\mathcal{L}'$ does not act on the whole module $R^\oplus_X$, as the composition $\Phi_Y(y)|f$ is not defined for $f \not\in m \cdot R^\oplus_X$. (See §2.1 iv.)

The action $Aut_Y \circ R_Y$ is k-linear, but the action $\mathcal{L} \circ m \cdot R^\oplus_X$ is neither additive nor multiplicative. Occasionally we write also $Aut_X, Aut_Y \circ Maps(X,Y)$.

iii. The contact transformations $X \times Y \rightarrow X \times Y$ are defined by $(x, y) \rightarrow (\Phi(x), \Psi(x,y))$, see e.g. [Mon. Nuñ.-Bal. pg. 108]. Here $\Phi \in Aut_X$ and $\Psi \in Maps(X \times Y, \Psi(o,-) \in Aut_Y$. Algebraically, in the formal case: $\Psi(m_X, m_Y) \in m_Y \cdot R_X[[y]]^\oplus$, with $\Psi(o, m_Y) \in k[[y]]^\oplus$. (And similarly in the k-analytic/k-Nash cases.)

Accordingly we define the contact group action $\mathcal{K} \circ Maps(X,Y) = m \cdot R^\oplus_X$ by $f(x) \rightarrow \Psi(x, f^{-1}(x))$.

Taking $\Phi = Id_X \in Aut_X$ we get the subgroup $\mathcal{K} \subset \mathcal{K}$. It acts on the fibres of the projection $X \times Y \rightarrow X$. This presents the contact group as the semi-direct product $\mathcal{K} = G \rtimes \mathbb{R}$, see pg. 156 of [A.G.L.V.] or page 109 of [Mon. Nuñ.-Bal].

The action $\mathcal{K} \circ Maps(X,Y) = m \cdot R^\oplus_X$ is neither additive nor k-multiplicative. But the $\mathcal{K}$-orbits coincide with the orbits of the much smaller group $\mathcal{K}^{lin} := GL(p, R_X) \rtimes Aut_X \circ R^\oplus_X$. For $R = \mathbb{k}\{x\}, \mathbb{k} \in \mathbb{R}, \mathbb{C}$ this is well known, see e.g. [Mon. Nuñ.-Bal. pg.110]. For the general case see [B.K.16 pg.123]. This action $\mathcal{K}^{lin} \circ Maps(X,Y)$ is k-linear.

Two maps $f, \tilde{f} \in m \cdot R^\oplus_X$ are $\mathcal{K}^{lin}$-equivalent if the corresponding local k-algebras are isomorphic, $R_X(f) \cong R_X(\tilde{f})$. And the later is an isomorphism of scheme-germs, $Spec(R_X(f)) \cong Spec(R_X(\tilde{f}))$. Therefore the $\mathcal{K}^{lin}$ equivalence coincides with the $V$-equivalence of [Martinet.76].

iv. The action of the left-right group, $\mathcal{A} := \mathcal{L} \times \mathbb{R} \circ Maps(X,Y) = m \cdot R^\oplus_X$, is defined by

$$\left(\Phi_Y, \Phi_X\right)(f) := \Phi_Y \circ f \circ \Phi_X^{-1} := \Phi_Y(y)|f \circ \Phi_X^{-1}.$$  

This action is compatible with the product structure: $(\Phi_Y, \Phi_X) \circ (\tilde{\Phi}_Y, \tilde{\Phi}_X) = (\Phi_Y \circ \tilde{\Phi}_Y, \Phi_X \circ \tilde{\Phi}_X)$. In particular, $(\Phi_Y, \Phi_X) = (\Phi_Y, Id_X) \circ (Id_y, \Phi_X) = (Id_Y, \Phi_X) \circ (\Phi_Y, Id_X)$.

While the actions $\mathcal{A}, \mathcal{K}^{lin} \circ m \cdot R^\oplus_X$ are k-linear, the actions $\mathcal{L}, \mathcal{A} \circ m \cdot R^\oplus_X$ are neither additive nor k-multiplicative. To my knowledge it is not known whether/how the $\mathcal{A}$-action can be “reduced” to a k-linear one. See [Damon.91], [Mond-Montaldi.91], [Houston-Wik Atique.13] for the related works.

3.2. The (extended) tangent spaces $T_{\mathcal{A}}, T_{\mathcal{L}}, T_{\mathcal{K}}, T_{\mathcal{A}}$. The natural (extended) tangent spaces to the groups $\mathcal{A}, \mathcal{L}, \mathcal{K}, \mathcal{A}$ are defined via the modules of derivations and endomorphisms, see §2.3 and [Mon. Nuñ.-Bal].

$$T_{\mathcal{A}} := Der_X, \quad T_{\mathcal{L}} := Der_Y, \quad T_{\mathcal{K}} := End_{R_X}(R^\oplus_X) \oplus T_{\mathcal{A}}, \quad T_{\mathcal{A}} := T_{\mathcal{L}} \oplus T_{\mathcal{A}}.$$  

As $R_Y = k[[y]], k\{y\}, k(y)$, one can write explicitly $Der_Y = R_Y \langle \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_p} \rangle$, i.e. $\xi_y = \sum \xi_i \frac{\partial}{\partial y_i}$ for $\xi_i \in R_Y$.

The name “tangent space” is justified by the relation $T_{\mathcal{A}} \cong \mathcal{A}$, see [2.5] and [B.G.K.22 §3].

These tangent spaces act on the (extended) space of maps as in the classical case:

$$T_{\mathcal{A}} \circ R^\oplus_X \quad \text{by} \quad \xi_X(f) := \xi_X(f_1, \ldots, f_p) = (\xi_X(f_1), \ldots, \xi_X(f_1)) \in R^\oplus_X,$$

$$T_{\mathcal{L}} : m \cdot R^\oplus_X \rightarrow R^\oplus_X \quad \text{by} \quad \xi_Y(f) := \xi_Y(y)|f = (\xi_Y(1), \ldots, \xi_Y(|f|)) \in R^\oplus_X.$$  

The action $T_{\mathcal{A}} \circ R^\oplus_X$ is k-linear. This embeds $T_{\mathcal{A}} = Der_X \subset End_k(R^\oplus_X)$ as an $R_X$-submodule. The map $T_{\mathcal{L}} : m \cdot R^\oplus_X \rightarrow R^\oplus_X$ by power series, it is neither additive nor k-multiplicative.

Given a map $f \in m \cdot R^\oplus_X$, one gets the (extended) image tangent space $T_{\mathcal{A}}f \subset R^\oplus_X$. For our groups:

$$T_{\mathcal{A}}f = Der_X(f), \quad T_{\mathcal{L}}f = f^\#(R^\oplus_X), \quad T_{\mathcal{K}}f = T_{\mathcal{A}}f + T_{\mathcal{L}}f, \quad T_{\mathcal{A}}f = T_{\mathcal{K}}f + (f) \cdot R^\oplus_X.$$  

Remark 3.1. i. Classically these $T_{\mathcal{A}}f$ are called “the extended tangent spaces”, with the notations from differential geometry, [Mon. Nuñ.-Bal] pg.72]:

$$R^\oplus_X = \theta(f), \quad T_{\mathcal{A}}f = t f(\theta_n), \quad T_{\mathcal{L}}f = \omega f(\theta_p), \quad T_{\mathcal{K}}f = t f(\theta_n) + f^* m_p \cdot \theta(f).$$  

We avoid the notation $T_{\mathcal{A}}f$ to prevent any confusion with the filtered tangent spaces, $T_{\mathcal{A}}(j)$, §3.3 ii. In zero characteristic, $k \supseteq \mathbb{Q}$, these “image tangent spaces” coincide with the “tangent spaces to the orbit”, $T_{\mathcal{A}}f = T_{\mathcal{A}}f$. In positive characteristic the two notions differ, see §1.5.2 of [B.G.K.22].
iii. Note that $T_{\mathcal{G},f}, T_{\mathcal{Y},f} \subset R^{\mathcal{G}}_X$ are $R_X$-submodules, while $T_{\mathcal{Z},f}, T_{\mathcal{Z},f} \subset R^{\mathcal{Z}}_X$ are only $R_Y$-submodules.

Here $R_Y$ acts by substitution, $q(y_1, \ldots, y_p) \cdot T_{\mathcal{Z},f} := q(f_1, \ldots, f_p) \cdot T_{\mathcal{Z},f} \subset R^{\mathcal{Z}}_X$, see (13).

iv. The actions $\Phi_X, \Phi_Y \circ \text{Maps}(X,Y)$ commute, see (12). The derivations do not commute:

$$
(17) \quad \xi_X(\xi_Y(y)|_f) = \xi_X(\xi_Y(y_1)|_f, \ldots, \xi_Y(y_p)|_f) = \left( \sum_i \partial_{y_i,\xi_Y(y)|_f} \cdot \xi_X(f_l), \ldots, \sum_i \partial_{y_i,\xi_Y(y)|_f} \cdot \xi_X(f_l) \right) = \sum_i \partial_{y_i,\xi_Y(y)|_f} \cdot \xi_X(f_l) = \xi_Y(y)|_f \cdot \xi_X(f_l) = \xi_Y(y)|_f \cdot \xi_X(f_l).
$$

To pass from $\xi_X \cdot \xi_Y$ to $\xi_Y \cdot \xi_X$ we apply lemma 3.3. This is used in 3.3.

### 3.3. Filtrations on the group $\mathcal{G}$ and on the tangent space $T_{\mathcal{G}}$

Fix an ideal $I \subseteq \mathfrak{m} \subset R_X$ and take the filtration by powers of this ideal, $M_r := I^r \cdot R^{\mathcal{G}}_X \subseteq R^{\mathcal{G}}_X$. Thus $M_r \subset \text{Maps}(X,Y)$ consists of maps that ‘vanish to a given order’ on the locus $V(I) \subset X$.

#### 3.3.1. For $\mathcal{G} = \mathcal{R}$ or $\mathcal{L}$ we get the corresponding filtrations on the groups/tangent spaces, as in (18):

$$
(18) \quad \mathcal{G}^{(j)} := \{ g \in \mathcal{G} : g \cdot M_d = M_d \text{ and } g|_{M_d/M_{d+1}} = \text{Id}|_{M_d/M_{d+1}}, \forall d \geq 1 \}.
$$

Then:

$$
T_{\mathcal{G}^{(j)}} := \{ \xi \in T_{\mathcal{G}} : \xi(M_d) \subseteq M_{d+1}, \forall d \geq 1 \}.
$$

For the right equivalence we have $T_{\mathcal{G}^{(j)}} = T_{\mathcal{G}} \cap \text{End}_{\mathfrak{k}}^{(j)}(R^{\mathcal{G}}_X)$ and $T_{\mathcal{G}^{(j-1)}} \supseteq T_{\mathcal{G}^{(j)}}$, cf. (18).

For the left equivalence we have $L^{(j)} := \{ \Phi_Y \in L : \Phi_Y(y) = y|_{I^d \cdot R^{\mathcal{G}}_X} = \Phi_Y(y) \in I^{d+j} \cdot R^{\mathcal{G}}_X \}$, i.e. $\Phi_Y(y) = y + \phi(y)$, where $\phi(I^d) \subseteq I^{d+j}$. In particular $L^{(0)} = \mathcal{L}$.

The tangent space filtration is $T_{\mathcal{G}^{(j)}} = \{ \xi_Y \in \text{Der}_Y[y] \mid \xi_Y(y)|_{I^d} \subseteq I^{d+j} \cdot R^{\mathcal{G}}_X, \forall d \geq 1 \}$. In particular:

$$
T_{\mathcal{G}} \supseteq T_{\mathcal{G}^{(j)}} = (y) \cdot T_{\mathcal{G}} \supseteq T_{\mathcal{G}^{(j+1)}} = (y)^2 \cdot T_{\mathcal{G}}.
$$

More generally, $T_{\mathcal{G}^{(j)}} = b_j \cdot T_{\mathcal{G}}$, for the largest ideal $b_j \subset R_Y$ satisfying: $f^#(b_j) \subseteq I^{d+j}$ for all $d \in \mathbb{N}$ and all $f \in I^d \cdot R^{\mathcal{G}}_X$. In particular, the tangent image is $T_{\mathcal{G}^{(j)}} = f = f^#(b_j) \subseteq I^{d+j} \cdot R^{\mathcal{G}}_X$.

We have the obvious inclusions: $(y)^{d+1} \subseteq b_j \subset R_Y$, and $(f)^{d+1} \subseteq f^#(b_j) \cdot R_X$ when $(f) \subseteq I$. These are equalities when $k \supseteq \mathbb{Q}$ is a field, but can be proper if $k$ is a finite field or a ring.

**Example 3.2.** i. Suppose $R_X$ is one of $k[[x]], k\{x\}, k(x)$, where $k$ is an infinite field. Suppose $I = m = (x) \subset R_X$. Take a map $f \in (x) \cdot R^{\mathcal{G}}_X$. Then:

$$
T_{\mathcal{G}^{(j)}} = (x)^{j+1} \cdot T_{\mathcal{G}}, \quad T_{\mathcal{G}^{(j)}} = (x)^{j+1} \cdot \text{Der}_X(f), \quad T_{\mathcal{G}^{(j)}} = (y)^{j+1} \cdot T_{\mathcal{G}}, \quad T_{\mathcal{G}^{(j)}} = (y)^{j+1} \cdot T_{\mathcal{G}} \subseteq R^{\mathcal{G}}_X.
$$

In the classical notations, cf. (16): $T_{\mathcal{G}^{(j)}} = f^#(m^{|I^j}_{F_0})$, $T_{\mathcal{G}^{(j)}} = f^#(m^{|I^j}_{F_0})$.

ii. Consider $\mathcal{G} = k[3], (k^2, o)$, where $k$ is a field with two elements, and the rings are complete. Take the filtration $(x)^{\bullet} \cdot R^{\mathcal{G}}_X$. Take the derivation $\xi_Y = q(y) \partial_y + 0 \partial_y \in \text{Der}_Y$, where $q(y) = y_1y_2(y_1 + y_2) \in k[[y_1, y_2]]$. Then $q(x^i, (x^i)^i) \subseteq (x)^{3i+1}$. Thus $\xi_Y \in T_{\mathcal{G}^{(j)}}$, even though $q(y) \notin (y)^i \subseteq k[[y_1, y_2]]$.

iii. Let $k = \mathbb{Q}[\{t\}]$ for a field $k \supseteq \mathbb{Q}$. Take $f = (t, m)$, then $T_{\mathcal{G}^{(j)}} = (t, m)^{j+1} \cdot T_{\mathcal{G}} \supseteq (y)^{j+1} \cdot T_{\mathcal{G}}.

### Remark 3.3.

i. In the definition (18) it is important to assume $d \geq 1$, rather than $d \geq 0$. In fact the group $\mathcal{L}$ does not act on the whole $R^{\mathcal{G}}_X$, thus the condition “$g|_{M_0/M_1} = \text{Id}|_{M_0/M_1}$” makes no sense.

One could restrict to $m \cdot M_0/M_1$, but then, for $\sqrt{I} \subseteq m$, the condition $g|_{m \cdot M_0/M_1} = \text{Id}|_{m \cdot M_0/M_1}$ would imply $\mathcal{L} = \{ e \}$. Similarly, the condition $\xi(M \cdot M_0) \subseteq M_1$ would imply $T_{\mathcal{G}^{(j)}} = (0)$.

ii. The relation $(y) \cdot T_{\mathcal{G}} = T_{\mathcal{G}^{(0)}}$ motivates the notation $T_{\mathcal{G}^{(j-1)}} := T_{\mathcal{G}}$. This should be used with caution, as $T_{\mathcal{G}^{(d)}} \not\subseteq I$ for any $d$.

#### 3.3.2. For the groups $\mathcal{G}, \mathcal{X}$ we define the filtration in a special way:

- $\mathcal{G}^{(j)} := \mathcal{B}(i) \times \mathcal{X}$ and $T_{\mathcal{G}^{(j)}} := T_{\mathcal{G}} \oplus T_{\mathcal{G}^{(j)}}$.
- $\mathcal{X}^{(j)} := \mathcal{G}(j) \times \mathcal{X}$ and $T_{\mathcal{X}^{(j)}} := T_{\mathcal{G}^{(j)}} \oplus T_{\mathcal{X}^{(j)}}$. Here $T_{\mathcal{G}^{(j)}} := \text{Mat}_{p \times p}(I^j)$.

In particular $T_{\mathcal{G}^{(j)}} = T_{\mathcal{G}^{(j-1)}}$. As in remark 3.3 ii we denote $T_{\mathcal{G}^{(j-1)}} := T_{\mathcal{G}}$. Again care is needed, as $T_{\mathcal{G}}(I^d \cdot R^{\mathcal{G}}_X) \not\subseteq I \cdot R^{\mathcal{G}}_X$ for any $d$.

One would like to use the general definition (18), for the groups $\mathcal{G}, \mathcal{X}$, i.e. to define

$$
(20) \quad \mathcal{G}^{(j)} := (\mathcal{L} \times \mathcal{S}(j)) \quad \text{and} \quad T_{\mathcal{G}^{(j)}} := (T_{\mathcal{G}} \oplus T_{\mathcal{X}^{(j)})}, \quad \text{and so on}.
$$

In this definition $\mathcal{G}^{(j)}, T_{\mathcal{G}^{(j)}}$ become non-explicit/complicated. But the two versions often coincide.
Lemma 3.4. Suppose $j \geq 1$, and $k$ is an infinite field, and $I^\bullet \neq (0)$ for each $\bullet \in \mathbb{N}$. Then:

1. $(\mathcal{L} \times \mathcal{D})^{(j)} = \mathcal{L}^{(j)} \times \mathcal{D}^{(j)}$, i.e. $(\text{Aut}_Y \times \text{Aut}_X)^{(j)} = \text{Aut}_Y^{(j)} \times \text{Aut}_X^{(j)}$.

2. $(T_{\mathcal{D}} \oplus \mathcal{D}^{(j)}) = T_{\mathcal{D}(j)} \oplus T_{\mathcal{D}(j)}^{(j)}$, i.e. $(\text{Der}_Y \oplus \text{Der}_X)^{(j)} = \text{Der}_Y^{(j)} \oplus \text{Der}_X^{(j)}$.

Proof. The inclusions $(\mathcal{L} \times \mathcal{D})^{(j)} \supseteq \mathcal{L}^{(j)} \times \mathcal{D}^{(j)}$ and $(T_{\mathcal{D}} \oplus \mathcal{D}^{(j)}) \supseteq T_{\mathcal{D}(j)} \oplus T_{\mathcal{D}(j)}^{(j)}$ are obvious. We prove the parts $\subseteq$ and $\supseteq$.

Below we use the property: there exists $f \in I$ such that $f^d \in I^d \setminus I^{d+1}$ for each $d \geq 1$. Indeed, otherwise we would get $I^d \subseteq I^{d+1}$ for $d \gg 1$, as $I$ is finitely generated. But then $I^d = 0$ by Nakayama.

1. Suppose $(\Phi_Y, \Phi_X) \in (\mathcal{L} \times \mathcal{D})^{(j)}$. As $\Phi_Y \in \mathcal{L}^{(j)}$, we get: $\Phi_X \in \mathcal{D}^{(j)}$. Suppose $\Phi_Y \in \mathcal{L}^{(j)} \setminus \mathcal{L}^{(l+1)}$, then, after a $k$-linear reshuffling of $y$-coordinates, we can assume: $\Phi_Y(y_1,\ldots,0,\ldots) = (q(y_1),\ldots)$. Here:

\[
q(y_1) = y_1 + a \cdot y_1^{l+1} + \text{h.o.t.} \quad \text{for some } \tilde{l} \leq l, \quad a \in k^\times, \quad \text{h.o.t.} \in (y_1)^{l+2} \quad \text{and } a \cdot I^{l+1} \subseteq I^{l+1}.
\]

As $I^\bullet \neq 0$ for each $\bullet$, we get $\tilde{l} = l$.

For each $f \in I$ define the difference $\Delta_f := [\Phi_Y(y_1,\ldots,0)] - \Phi_X(f)$. By our assumption $\Delta_f \in I^{\text{ord}(f)+j}$. As $\Phi_X \in \mathcal{D}^{(j)}$, we get: $\Delta_f \in I^{\text{ord}(f)+j}$.

- The case $l = 0$. Then $\Phi_Y \in \mathcal{L}^{(0)} \setminus \mathcal{L}^{(1)}$ and $\Phi_X \in \mathcal{D}^{(0)} \setminus \mathcal{D}^{(1)}$. For each $f \in I$ and $d \in \mathbb{N}$ we get:

\[
\Delta_f \subseteq I^{d-\text{ord}(f)+j} = f^{d-1} \cdot \Phi_X(f) - \Phi_X(f^d) + (\text{h.o.t.}), \quad \text{where } (\text{h.o.t.}) \in I^{(d+1)-\text{ord}(f)}.
\]

This holds for each $f \in I$, and the field $k$ is infinite, we get: $\Phi_X \in \mathcal{D}^{(1)}$, i.e. a contradiction.

- The case $l \geq 1$. For any $f \in I$, $c \in k^\times$ we have:

\[
\Delta_{cf} - c \cdot \Delta_f = q(c \cdot f) - c \cdot q(f) = a \cdot (c^{l+1} - c) \cdot f^{l+1} + \text{h.o.t.} \in I^{\text{ord}(f)+j}.
\]

As $k$ is infinite there exists $c \in k^\times$ for which $c^{l+1} - c \in k^\times$. Thus for each $f \in I$ and $l > 0$ we get: $f^{l+1} \in I^{\text{ord}(f)+j}$. Therefore $l \geq j$, hence $\Phi_Y \in \mathcal{L}^{(j)}$ and thus $\Phi_X \in \mathcal{D}^{(j)}$.

2. Suppose $(\xi_Y, \xi_X) \in (T_{\mathcal{D}} \oplus \mathcal{D}^{(j)})$, thus $(\xi_Y, \xi_X)(I \cdot R^{\text{fp}}_X) \subseteq I^{l+1} \cdot R^{\text{fp}}_X$. If $\xi_Y \in T_{\mathcal{D}} \setminus T_{\mathcal{D}(j)}$ then $\xi_Y(I) \subseteq m$. Then for $f \in (I \cap m^2) \cdot R^{\text{fp}}_X$ we get: $(\xi_Y, \xi_X)(f) \notin m \cdot R^{\text{fp}}_X$. Therefore necessarily $\xi_Y \in T_{\mathcal{D}(j)}$, and then $\xi_X \in T_{\mathcal{D}(j)}^{(j)}$.

Suppose $\xi_Y \in T_{\mathcal{D}(j)} \setminus T_{\mathcal{D}(j+1)}$, then after a $k$-linear reshuffling of $y$-coordinates we can assume $\xi_Y(y_1,\ldots,0) = (q(y_1),\ldots)$. Here $q(y_1) = a \cdot y_1^{l+1} + \text{h.o.t.}$ (for some $\tilde{l} \leq l$), $a \in k^\times$, h.o.t. $\in (y_1)^{l+2}$ and $a \cdot I^{l+1} \subseteq I^{l+1}$. Therefore $\tilde{l} = l$.

- The case $l = 0$. For any $f \in I$ and $d \in \mathbb{N}$ we have:

\[
[(\xi_Y, \xi_X)(f^{0,\ldots,0}) - d \cdot f^{d-1} \cdot [(\xi_Y, \xi_X)(f^{0,\ldots,0})] = \Phi_Y(f^d) - d \cdot f^{d-1} \cdot \Phi_X(f) = (a(1 - d)) \cdot f^d + \text{h.o.t.} \in I^{\text{ord}(f)+j}.
\]

Choose $f \in I \setminus T^2$ satisfying $f^2 \in I^2 \setminus I^3$ to get the contradiction.

- The case $l \geq 1$. For any $f \in I$ and $c \in k$ we have:

\[
[(\xi_Y, \xi_X)(c \cdot f^{0,\ldots,0}) - c \cdot [(\xi_Y, \xi_X)(c \cdot f^{0,\ldots,0})] = q(c \cdot f) - c \cdot q(f) = a(c^{l+1} - c) \cdot f^{l+1} + \text{h.o.t.} \in I^{\text{ord}(f)+j}.
\]

As $k$ is infinite there exists $c \in k$ such that $c^{l+1} - c \neq 0$. Therefore we get $l \geq j$, hence $\xi_Y \in T_{\mathcal{D}(j)}$, and thus $\xi_X \in T_{\mathcal{D}(j)}^{(j)}$.

\[\blacksquare\]

Remark 3.5. By similar arguments one can prove, for the contact group $H = G \times \mathcal{D}$, and an infinite field $k$:

\[\langle G \times \mathcal{D} \rangle = G^{(j)} \times \mathcal{D}^{(j)}, \quad (GL(p, R_X) \times \text{Aut}_X)^{(j)} = GL^{(j)}(p, R_X) \times \text{Aut}_X^{(j)}, \quad (T_{\mathcal{D}} \oplus \mathcal{D}^{(j)}) = T_{\mathcal{D}(j)} \oplus T_{\mathcal{D}(j)}^{(j)}.
\]

3.4. The closures in the filtration topology. For our groups $H = \mathcal{D}, \mathcal{L}, \mathcal{D}$ of $\text{GL}(j)$ we have the closures of the orbits in the filtration topology (for $M_* = I^* \cdot R^{\text{fp}}_X$):

\[
\overline{T_{\mathcal{D}(j)} f} = \cap_{\geq 1}(T_{\mathcal{D}(j)} f + M_*), \quad \overline{\mathcal{D}(j) f} = \cap_{\geq 1}(\mathcal{D}(j) f + M_*).
\]

Lemma 3.6. 1. Any $R_X$-submodule of $R^{\text{fp}}_X$ is closed in $I^* \cdot R^{\text{fp}}_X$.

In particular, the tangential spaces $T_{\mathcal{D}(j)} f, T_{\mathcal{D}(j)} f \subseteq R^{\text{fp}}_X$ are closed.

2. The orbits $\mathcal{D}(j) f, \mathcal{L}(j) f$ are closed.

Proof. Let $M \subseteq R^{\text{fp}}_X$ then $M + I^d \cdot R^{\text{fp}}_X \supseteq \overline{M}$ for any $d \geq 1$. Therefore $M + (I^d \cdot R^{\text{fp}}_X) = \overline{M}$. Applying the Artin-Rees lemma to the (finitely generated, $\text{GL}(j)$) submodule $\overline{M} \subseteq R^{\text{fp}}_X$ we get: $M + I^{d} \cdot \overline{M} \supseteq \overline{M}$. Then Nakayama gives $M = \overline{M}$.
2. (The \( \mathcal{H} \)-case.) Suppose \( f + g \in \mathcal{H}^{(j)} \) for a perturbation \( g \in \mathfrak{m} \cdot R_X^{\mathbb{P}} \). Thus \( f + g \in \mathcal{H}^{(\text{fin})} \). Therefore, the system of implicit function equations \( f(\Phi(x)) = U \cdot (f + g) \) has an order-by-order solution. By the theorem of Pfister-Popescu, there is a formal solutions, \( (\Phi, U) \in \mathcal{H}^{(\text{fin})} \). For the rings \( k(x)/f, k(x)/f \) apply the Artin approximation to get an ordinary solution, \( (\Phi, U) \in \mathcal{H}^{(\text{fin})} \).

**Remark 3.7.** i. Part 1 is a pure commutative algebra, it holds for Noetherian local rings, provided all the objects are defined.

ii. Below we use the simple observation: to verify the inclusion \( \mathcal{H}^{(j)} f + M_{\bullet+1} \supseteq \{ f \} + M_\bullet \) it is enough to verify \( \mathcal{H}^{(j)} f + M_{\bullet+1} \supseteq \{ f \} + M_\bullet \) for every \( \bullet \geq d \). And similarly for \( \mathcal{T} \mathcal{H}^{(j)} f \supseteq M_\bullet \).

iii. The case of \( \mathcal{T} \mathcal{H}^{(j)} f \) is more delicate, see Lemma 6.14.

### 3.5. Related derivations and related automorphisms.

The “naturality” of vector fields integration. Given a map of \((C^\infty \text{ or real/complex-analytic})\) manifolds, \( f : X \to Y \), take some vector fields \( \xi_X \in \Theta_X, \, \xi_Y \in \Theta_Y \), and the corresponding time-one flows, i.e. the automorphisms \( \Phi_X \circ X, \, \Phi_Y \circ Y \).

**Lemma 3.8.** [Thom-Levine, e.g. [Mon. Nuñ.-Bal.], Lemma 2.6] If the vector fields are related, i.e. \( \xi_Y \circ f = df \circ \xi_X \), then so are the corresponding automorphisms, \( \Phi_Y \circ f = f \circ \Phi_X \).

We need the algebraic version of this and other results on vector fields. Below we use the exponential and logarithmic maps of \( \mathbb{R}^n \).

Take a local filtered \( k \)-algebra (\( R_X, \mathfrak{m}, I^* \)). Assume \( k \supseteq \mathbb{Q} \) and \( R_X \) is complete with respect to the filtration. Take the filtration of \( R_X^{\mathbb{P}} \) by \( M_\bullet := I^* : R_X^{\mathbb{P}} \). Put \( M_{\infty} := 0 \). The map \( \exp \) of (9) is well defined.

**Lemma 3.9.** Fix some elements \( (\xi_Y, \xi_X) \in T_{\mathcal{H}^{(j)} f}, \, f \in \mathfrak{m} : R_X^{\mathbb{P}} \), for some \( \ell \geq 1, \, 1 \leq d \leq \infty \), and \( w_d \in M_\ell \).

1. \( \xi_Y(e^{\xi_X} f) = e^{\xi_X} \cdot \xi_Y(y) | f \).
2. If \( \xi_X(f) \equiv M_{\ell} \xi_Y(y) | f \) then \( \xi_X(e^{\xi_X} f) \equiv M_{\ell} \xi_Y(y) | f \) for any \( i, j \geq 2 \).
3. (algebraic “naturality”) \( \xi_Y(f) | f - \xi_X(f) \equiv M_{\ell} w_d \) if and only if \( e^{\xi_Y} f - e^{\xi_X} f \equiv M_{\ell} w_d \).

**Proof.**

1. The direct check: \( \xi_Y(e^{\xi_X} f) = \xi_Y(y) |_{e^{\xi_X} f} = \xi_Y(y) |_{f e^{\xi_X}} = e^{\xi_X} \cdot \xi_Y(y) | f \).
2. Induction on \( i \). The case \( i = 0 \) is trivial. Assuming the statement for \( i \) we verify it for \( i+1 \):

\[
(27) \quad \xi_X^{i+1} (\xi_Y(y) | f) = \xi_X \left( \xi_X^{i} (\xi_Y(y) | f) \right) = \xi_X \left( \xi_X^{i+1} (y) | f \right) = \sum_q \partial_{y_q} \xi_X^{i+1} (y) | f \cdot \xi_X(f_q) \equiv \sum_q \partial_{y_q} \xi_X^{i+1} (y) | f \cdot \xi_Y(y) | f = \xi_Y^{i+1} (y) | f.
\]

The statement \( e^{\xi_X} f - e^{\xi_Y} f \in M_\ell \) is now obtained by Taylor expansion of the exponentials.

3. The part \( \Rightarrow \) : \( e^{\xi_Y} f - e^{\xi_X} f = w_d + \sum_{j \geq 2} \frac{\xi_Y(y) - \xi_X(f)}{j} \equiv w_d \mod M_{\ell+1} \). For the last equality we use part 2.

The part \( \Leftarrow \) : \( \xi_Y(y) | f - \xi_X(f) = \ln(e^{\xi_Y} y) | f - \ln(e^{\xi_X} y) | f = \ln(e^{\xi_Y} y) | f = -\sum_{j \geq 1} \frac{(1 - e^{-\xi_X} e^{\xi_Y})^j}{j} = \sum_{j \geq 1} \frac{1 - e^{-\xi_X} e^{\xi_Y}}{j} e^{-\xi_X} w_d \mod M_{\ell+1} \). For the transition \( \Rightarrow \) we use the commutativity of \( \beta \).

**Remark 3.10.** For the subsequent applications \((\delta_{\delta}, \delta_{\delta}) \) one needs the strongest Thom-Levine statement. In Part 3 the conclusion cannot be strengthened to the statement \( (\mod M_{\ell+1}) \). For example, define the map \((k^n, \alpha) \to (k, \alpha)\) by \( f(x) = x + x^{d-1} \in \mathbb{Q}[x] \). Here \( k \supseteq \mathbb{Q} \), the ring \( R_X = \mathbb{Q}[x] \) is filtered by \( (x)^* \), the ring \( R_Y = \mathbb{Q}[y] \) is filtered by \( (y)^* \).

Take the derivations \( \xi_X = x^d \partial_x, \, \xi_Y = x^d \partial_y \). One has \( \xi_X(f) - \xi_Y(y) | f \equiv w_{d-1} (d-3)x^d \). But \( e^{\xi_X} (x) = \frac{x}{1-x} \) and therefore \( e^{\xi_X} (f) - e^{\xi_Y} (f) = f(\frac{x}{1-x}) - \frac{(x)}{1-f(x)} = (d-3)x^d \left( 1 + 3 \frac{d-2}{2} x + \ldots \right) \). Thus \( e^{\xi_X} (f) - e^{\xi_Y} (f) \equiv (d-3)x^d \mod (x)^{d+1} \).

**Remark 3.11.** In the proof of part 3 we needed the exponential/logarithmic expansions only up to order \( \left[ \frac{d-\text{ord}(f)}{2} \right] + 1 \). Therefore this lemma can be used even when the full exp/log maps do not exist, e.g.
when \( k \supseteq \mathbb{Q} \) or \( R_X^{bp} \) is not \( M_s \)-complete. Namely, fix some \((\xi_Y, \xi_X) \in T_{s(o)}, w_d \in M_d\), and suppose the assumptions \( jet_N \exp, jet_N Ln \) of \((2.3)\) hold for \( N = \left[ \frac{d-\ord(f)}{e} \right] + 1 \). Then we get the statement:

\[
(28) \quad \xi_Y(y)|_f - \xi_X(f) \mod M_{k+i} w_d \quad \text{if and only if} \quad \text{jet}_N(e^{-\xi_Y})j_N(e^{\xi_Y})f \mod M_{k+i} f + w_d.
\]

3.6. Critical/singular/instability loci and the annihilator of \( T^1_{d} f \). Fix a group \( G = \mathcal{G}, \mathcal{A}, \mathcal{X} \) and a map \( f \in Maps(X, Y) = m \cdot R_X^{bp} \). Take the image tangent space \( T_G f \subseteq R_X^{bp} \). Take the quotient module \( T^0_G f := R_X^{bp}/T_G f \). (In \cite{Kerner22} we show that this is the tangent space to the \( \mathcal{G} \)-universal unfolding, when \( f \in \mathcal{G} \)-finite.) The standard way to measure how large is this tangent space goes via the support of \( T^0_G f \), i.e. the annihilator of this quotient module

\[
(29) \quad \mathfrak{a}_G := \text{Ann}(T^0_G f) = \{ q \in R_X | q \cdot R_X^{bp} \subseteq T^0_G f \} \subseteq R_X.
\]

This is the largest ideal satisfying: \( T_G f \supseteq \mathfrak{a}_G \cdot R_X^{bp} \).

Fix a filtration \( I^* \subseteq R_X \) and the corresponding filtrations \( \mathcal{G}(\bullet), T_G(\bullet) \), see \cite{Kerner22}. One gets the filtered annihilators \( \mathfrak{a}_{G(j)} := \text{Ann}(T^0_{G(j)} f) \subseteq R_X \) for \( j \geq 0 \).

**Definition 3.12.**

1. For \( \mathfrak{a}_G \subseteq R_X \) the scheme-germ \( V(\mathfrak{a}_G) \subseteq X \) is called the critical \((\text{Crit}(f))\), resp. \( \text{the singular (Sing}(f))\), resp. \( \text{the instability locus of } f \).
2. The map \( f \) is called \( G \)-finite if \( \sqrt{\mathfrak{a}_G} \supseteq m \), i.e. geometrically \( V(\mathfrak{a}_G) = o \in X \) or \( V(\mathfrak{a}_G) = \emptyset \).

If \( f \) is smooth, \( k = \bar{k} \), and \( f_1, \ldots, f_p \in R_X \) is a regular sequence then (set-theoretically) \( \text{Sing}(f) = \text{Sing}(V(f)) \), the singular locus of the complete intersection. Otherwise the two objects can differ.

**Remark 3.13.** Classically the Fitting (determinantal) ideals of the modules \( T^1_{d} f, T^1_{\mathcal{X}} f \) were used. We use the annihilator ideals in \((29)\) for two reasons:

- These annihilators appear naturally in the criteria for group-orbits, \cite{Eisenbud, Mon.-Montaldi, Bruce-Robinson}.
- The ideals \( \text{Fitt}(T^1_{d} f) \) are not immediately defined, as \( T^1_{d} f \) is not an \( R_X \)-module, while as \( R_Y \)-module it is not finitely generated.

3.6.1. Below we discuss the annihilators \( \mathfrak{a}_{\mathcal{G}}, \mathfrak{a}_{\mathcal{X}} \). The annihilator \( \mathfrak{a}_{\mathcal{G}} \) is much more delicate, see \cite{Eisenbud}.

**Example 3.14.** Let \( R \) be one of \( k[\{x\}] / J, k(\{x\}) / J, k(x) / J \), see \((2.1)\).ii.

i. For \( p = 1 \) and \( R \) one of \( k[\{x\}] / J, k(\{x\}) / J, k(x) / J \), \( \mathfrak{a}_{\mathcal{G}} \) are just the Jacobian and the Tjurina ideals,

\[
T^1_{\mathcal{X}} f = R/Jac(f), \quad \mathfrak{a}_{\mathcal{G}} = Jac(f), \quad T^1_{\mathcal{X}} f = R/Jac(f) + Jac(f), \quad \mathfrak{a}_{\mathcal{X}} = Jac(f) + (f).
\]

Here (when \( k \) is not a field) the derivatives are taken with respect to \( x \)-variables only.

For \( J \neq 0 \) one gets the Bruce-Roberts versions of the Jacobian/Tjurina ideals, \cite{Bruce-Robinson}.

ii. For \( p \geq 2 \) the annihilators are not computed easily, but one has immediate bounds via the determinantal ideals of the generating matrix \( [\xi(f_j)] \) of \( T^1_{\mathcal{G}} f \). (For the regular rings \( k[\{x\}], k(\{x\}), k(x) \), this is just the matrix of partials, \( \{\partial_{x_i} f_j\} \).) One has (see e.g. \cite{Eisenbud} [\$20]\>):

\[
(30) \quad I_p[\xi(f_j)] \subseteq \mathfrak{a}_{\mathcal{G}} \subseteq \sqrt{I_p[\xi(f_j)]}, \quad I_p[\xi(f_j)] + (f) \subseteq \mathfrak{a}_{\mathcal{X}} \subseteq \sqrt{I_p[\xi(f_j)]} + (f).
\]

iii. In particular one gets: \( \mathfrak{a}_{\mathcal{G}} + (f) \subseteq \mathfrak{a}_{\mathcal{X}} \subseteq \sqrt{\mathfrak{a}_{\mathcal{G}} + (f)} \). Hence \( \text{Sing}(f) = \text{Crit}(f) \cap V(f) \).

iv. Let \( p \geq 2 \). In many cases one has \( \mathfrak{a}_{\mathcal{G}} \cdot R_X^{bp} \subseteq m \cdot T^1_{\mathcal{G}} f \). For example, this holds if the determinant ideal \( I_p[\xi(f_j)] \subseteq R_X \) is radical, as then \( \mathfrak{a}_{\mathcal{G}} = I_p[\xi(f_j)] \).

v. Take the filtration \( I^* \subseteq R_X \). One has \( I_p^{\bullet} \cdot \mathfrak{a}_{\mathcal{G}} \subseteq \mathfrak{a}_{\mathcal{G}(i)} \subseteq \mathfrak{a}_{\mathcal{G}} \).

vi. (The case of unfolding) Take a map \( f : (X \times k^r, o) \to (k^p \times k^r, o), (x, u) \to (f_0(x) + g(x, u), u) \), where \( (g(x, u)) \subseteq (x) \cdot (u) \). Then \( \mathfrak{a}_{\mathcal{X}}(f) = \mathfrak{a}_{\mathcal{X}}(f_0) + (u) \) and \( \mathfrak{a}_{\mathcal{X}(0)}(f) = \mathfrak{a}_{\mathcal{X}(0)}(f_0) + (u) \).

3.6.2. Maps of finite singularity type. The following criterion is well known for \( k = \mathbb{C} \) and \( X \)-ICIS, see e.g. \cite{Mond-Montaldi, Mon.-Nuñez-Balcazar} pg.224.

**Lemma 3.15.** Let the pair \( (R_X, R_Y) \) be one of \((k[\{x\}] / J, k[\{y\}]), (k(\{x\}) / J, k(\{y\})), (k(x) / J, k(y))\), see \((2.1)\).ii. The following conditions are equivalent for a map \( f : X \to (k^p, o) \):

1. The restriction of \( f \) to \( \text{Crit}(f) \to (k^p, o) \) is a finite morphism, i.e. the \( R_X \)-module \( R_X/\mathfrak{a}_{\mathcal{G}} \) is a f.g. \( R_Y \)-module;
2. The \( R_X \)-module \( T^1_{d} f \) is a f.g. \( R_Y \)-module;
3. \( T^1_{d} f \) is a f.g. \( k \)-module; \( (\text{If } k \text{ is a field then } f \text{ is } \mathcal{X} \text{-finite.}) \)
4. \( f^{-1}(o) \cap \text{Crit}(f) \) is a (fat) point over \( \text{Spec}(k) \) i.e. \( \sqrt{\mathfrak{a}_{\mathcal{G}} + (f)} \supseteq (x) \).
Moreover, if $T^1_X f \in \text{mod-}k$ is generated by $\{v_\bullet\} \subset R^\text{gp}_X$, then $T^1_X f \in \text{mod}(R_Y)$ is generated by $\{v_\bullet\}$.

In this case $f$ is called a map of finite singularity type.

**Proof.** 1 \implies 2. $T^1_X f$ is a f.g. module over $R_X/\mathfrak{a}_f$, and thus a f.g. module over $R_Y$.

2 \implies 3. If $T^1_X f$ is f.g. over $R_Y$, then $T^1_X f \otimes R_Y(y) \in \text{f.g. over } R_Y(y) = k$. Now observe: $T^1_X f \otimes R_Y(y) = T^1_X f$. $3 \implies 4$. As $T^1_X f$ is f.g. over $R_Y$, it is a module over $R_X(x)^d$ for some $d \gg 1$. Therefore $\sqrt{\mathfrak{a}_f} = \sqrt{\mathfrak{a}_f + (f)} \subseteq (x)$. Thus $f^{-1}(a) \cap \text{Crit}(f)$ is a fat point over Spec($k$).

4 \implies 1. By the assumption $\sqrt{\mathfrak{a}_f + (f)} \subseteq (x)$. Hence the map $\text{Crit}(f) \to (k^n, o)$ is quasi-finite. The finiteness follows by the Weierstraß-finiteness, 2.3 iii.

4. CRITERIA FOR RIGHT ORBITS, “$\mathfrak{a} f \text{ vs } T^1_X f$”

Let $R_X$ be one of the rings $k[[x]]/j$, $k(x)/j$, $k(x)/j$, with $k$ a local ring, see 2.1 ii.

4.1. The $\mathfrak{a}$-implicit function theorem. Take two ideals $a \subseteq I \subseteq R_X$, with $a \subseteq m^2$, and a map-germ $f \in I \cdot R^\text{gp}_X$. (The usual choice is: $I = R_X$ or $I = m$ or $I = \sqrt{f}$.) For the notation $I^{\text{ord}(f)}$ see 2.1 iv.

**Theorem 4.1.** Suppose the ideal $a$ satisfies $a^2 \cdot I^{\text{ord}(f)} - 2 \cdot R^\text{gp}_X \subseteq m \cdot a \cdot T^1_X f$.

1. (The case $J = 0$) Then $\{f\} + a \cdot T^1_X f \subseteq \mathfrak{a} f$.

2. (The case $J \neq 0$) Assume the jet-condition of (2.2) Then $\{f\} + a^2 \cdot T^1_X f \subseteq \mathfrak{a} f$.

A remark: the assumption $a^2 \cdot I^{\text{ord}(f)} - 2 \cdot R^\text{gp}_X \subseteq m \cdot a \cdot T^1_X f$ is preserved by $\mathfrak{a}$-equivalence. The automorphism $\Phi_X \in \text{Aut}_X$ (constructed in the proof) satisfies: $\Phi(x) - x \in a$, resp. (for part 2) $\Phi(x) - x \in a^2$.

**Proof.** 1. Take a perturbation $g \in a \cdot T^1_X f$. We want to resolve the condition $\Phi_X(f) = f + g$, where $\Phi_X \in \mathfrak{a} = \text{Aut}_X$. Fix some (finite sets of) generators $\{\xi_i(f)\}$ of the $R_X$-module $a \cdot T^1_X f = a \cdot \text{Der}_X(f)$. Expand $g = \sum c_i^j \xi_i(f)$, here $c_i^j \in R_X$.

We look for the coordinate change $\Phi_X \in \text{Aut}_X$ in the form $x \to x + \sum c_i^j x_i(x)$. Thus we want to resolve the equation

$$f(x + \sum c_i^j x_i(x)) = f(x) + \sum c_i^j \cdot \xi_i(f), \quad \text{with the unknowns } \{c_i^j\}.$$ (31)

Taylor-expand the left hand side: $f(x) + \sum c_i^j \xi_i(f) + H_{\geq 2}((c_i^j \xi_i(f)))$. Here $H_{\geq 2}(z) \in \sum_{i \geq 2} I^{\text{ord}(f) - 1}(x) 2 R^\text{gp}_X$. As $a \subseteq I$ one has: $H_{\geq 2}(a \cdot z) \in I^{\text{ord}(f) - 2} \cdot (a \cdot z) 2 R_X [a \cdot z] [z]$. These are polynomials in $z$ whose coefficients are (formal/analytic/algebraic) power series in $a \cdot z$, see 2.1 iv. Therefore:

$$H_{\geq 2}((c_i^j \xi_i(x))) \subseteq \{(c_i^j)\} 2 \cdot a^2 \cdot I^{\text{ord}(f) - 2} 2 R^\text{gp}_X \subseteq \{(c_i^j)\} 2 \cdot m \cdot a \cdot T^1_X f.$$ (32)

Thus we can present $H_{\geq 2}(c_i^j \xi_i(x)) = \sum h_i(c_i^j) \xi_i(f)$, for some (fixed) power series $h_i(z) \in (z)^2 \cdot m \cdot R_X [a \cdot z] [z]$. These (formal/analytic/algebraic) series $h_i$ satisfy: $h_i(R_X) \subseteq m$.

Thus to resolve the equation (31) it is enough to resolve the (finite) system of equations: $\{c_i^j + h_i(c_i^j) = c_i^j\}_{i}$. And now apply the IFT1, see 2.2 i, to get the solutions $c_i(x) \in R_X$.

Finally we take the coordinate change $\Phi_X : x \to x + \sum c_i^j x_i(x)$. It is invertible (as $a \subseteq m^2$), and defines the needed automorphism $\Phi_X \in \text{Aut}_X$.

2. Take a perturbation $g \in a^2 \cdot T^1_X f$. By part 1 we have a map $\Phi_X : x \to x + \xi(x)$, with $\xi \in a^2 \cdot T^1_X$, satisfying $\Phi_X^{-1}(f + g) = f$. As $J \neq 0$, this $\Phi_X$ is not necessarily a “coordinate change” on $X$, i.e. not an automorphism of $R_X$. By the jet-quality assumption we can extend $\Phi_X$ to an automorphism $\Psi_X \in \text{Aut}_X$, satisfying $\Psi_X(x) - x - \xi(x) \in a^3$. Then $\Psi_X^{-1}(f + g) - f \in a^3 \cdot I^{\text{ord}(f) - 1} 2 R^\text{gp}_X \subseteq a^2 \cdot m \cdot T^1_X f$. (Note that we cannot assume $\Psi_X^{-1}(f + g) - f \in a^3 \cdot T^1_X f$, because $\Psi_X - I_d - x$ is not a derivation of $R_X$.)

Iterate this argument to get: $\Phi_X^{-1}(f + g) - f \notin a^3 \cdot m^d \cdot R^\text{gp}_X$ for each $d \geq 1$. Hence $f + g \notin \mathfrak{a} f$, the orbit-closure in the $m^*\text{-filtration topology}$. The statement follows now by lemma 3.6.

**Example 4.2.** Let $p \geq 2$ and $R_X$ be one of $k[[x]], k\{x\}, k(x)$, see 2.1 ii. Take $I = m$ and $(k^n, o) \to (k^p, o)$.

i. Suppose $f$ is a submersion, i.e. $T^1_X f = R^\text{gp}_X$. (Thus $p \leq n$.) Then for $a = m^2$ one gets $\mathfrak{a} f \subseteq \{f\} + \mathfrak{a} \cdot T^1_X f$. In particular one gets the normal form of submersion, $\mathfrak{a} f \ni (x_1, \ldots, x_p)$.

ii. More generally, the theorem gives: If $a^2 \subseteq m \cdot a \cdot T^1_X f$, then $\mathfrak{a} f \ni \{f\} + a \cdot T^1_X f$.

(Here $a_{\mathfrak{a}}$ is the annihilator of the critical locus, 3.5) E.g. one has $\mathfrak{a} f \ni \{f\} + m \cdot a_{\mathfrak{a}}^2 \cdot R^\text{gp}_X$.}
iii. In many cases one has $a_\omega \cdot R_X^{\otimes 2} \subseteq m \cdot T_\omega f$, see example 3.14iv. Then theorem 4.1 gives: $\mathcal{R} f \supseteq \{f\} + a_\omega \cdot R_X^{\otimes 2}$. Geometrically: $f$ is determined up to $\mathcal{R}$-equivalence by its 2-jet on the critical locus $Crit(f) = V(a_\omega f)$, taken with its annihilator structure.

4.2. More examples for the case of one power series, i.e. $p = 1$. Let $R_X$ be one of $\mathbb{k}[[x]]$, $\mathbb{k}\{x\}$, $\mathbb{k}\langle x \rangle$. Take $f \in m^2$, $I = m$ and $a \subseteq m^2$. Then theorem 4.1 reads:

$$(33) \quad \text{If } a_2 \cdot m^{ord(f)-2} \subseteq m \cdot a \cdot Jac(f) \quad \text{then } \{f\} + a \cdot Jac(f) \subseteq \mathcal{R} f.$$ 

Below we use a simpler though more restrictive condition: $a_2 \cdot m^{ord(f)-2} \subseteq m \cdot Jac(f)$. Then we get:

**Corollary 4.3.** 1. $\mathcal{R} f \supseteq \{f\} + Jac(f) \cdot \left( (m \cdotJac(f)) : m^{ord(f)} \right)^{-2}$. 

2. Take the smallest $d \in \mathbb{N}$ satisfying $m^{2d+ord(f)-2} \subseteq m^{d+1} \cdot Jac(f)$. Then $\mathcal{R} f \supseteq \{f\} + m^d \cdot Jac(f)$.

**Example 4.4.** i. Obviously $(m \cdot Jac(f)) : m^{ord(f)-2} \subseteq m \cdot Jac(f)$. Therefore: $\{f\} + m \cdot Jac(f)^2 \subseteq \mathcal{R} f$.

This is well known for $R_X = \mathbb{C}\{x\}$, e.g. [Ruiz, lemma 2.2, pg.91], or for $\text{char}(k) = 0$, [Kucharz86].

In particular, if $\sqrt{Jac(f)} = m$ then $f$ is finitely-determined.

More generally, in this way one can extend many other results of [Kucharz86]. We omit the details.

ii. (Morse lemma) Let $k$ be a field, $f \in m^2$, and $\text{rank}[f'_{[a]}] = r$. Then $f \not\in \text{Q}_2(x) + \tilde{f}(w)$, where $x = (x_1, \ldots, x_r)$, $w = (w_1, \ldots, w_{r-n})$, $\tilde{f}(w) \in (w)^3$ (is independent of $x$) and $Q_2(x)$ is a homogeneous quadratic polynomial.

Moreover, for $k = \bar{k}$ one can diagonalize the quadratic form $Q_2$ to get: $f \sim \sum x_i^2 + \tilde{f}(w)$. 

**Proof.** Apply a $GL(n, k)$ transformation to get $f - Q_2(x) \in (x, w)^3$. Then $Jac(f) + (x, w)^2 \supseteq (x_1, \ldots, x_r)$. And thus we have $f - Q_2(x) \in (x, w) \cdot Jac(f)^2 + (w)^3$. The statement follows now by part i.

iii. Assuming $ord(f) \geq 3$ one has $(m \cdot Jac(f)) : m^{ord(f)-2} \supseteq Jac(f)$. Then we get: $\{f\} + Jac(f)^2 \subseteq \mathcal{R} f$.

For $R_X = \mathbb{C}\{x\}$ this is Lemma 3.20 in [Ebeling]. Its proof is based on vector field integration, thus cannot be used when $k$ is an arbitrary field.

iv. If $ord(f) = 2$ then by Morse lemma we split the variables, $f = Q_2(x) + \tilde{f}(w)$ with $\tilde{f}(w) \in (w)^3$.

Then one has: $\mathcal{R} f \supseteq \{f\} + (x)^2 \cdot (x, w) + (x) \cdot Jac(\tilde{f}(w)) + Jac(\tilde{f}(w))^2$.

v. To show how this corollary strengthens the classical bounds, let $k$ be a field of characteristic zero, and take $f$ with an ordinary multiple point at $o \in (k^n, o)$. This means: the partials $\partial_t f, \ldots, \partial_n f$ form a regular sequence, and their leading terms (of order $ord(f) - 1$) form a regular sequence as well. Let $q$ be the smallest integer satisfying $m^q \subseteq m^2 \cdot Jac(f)$.

Then $q = n \cdot (ord(f) - 2) + 1$, see exercise 2.27 in [Gr.Lo.Sh]. The known determinancy bound is: $\{f\} + m^q \not\in \mathcal{R} f$, see §5 of [B.G.K22].

**We claim:** $\mathcal{R} f \supseteq \{f\} + m^{\frac{q-ord(f)}{2}} \cdot Jac(f)$. (This bound is better, even asymptotically.)

**Proof.** As $f$ has an ordinary multiple point we have: $m^q = Jac(f) \cdot m^{q-ord(f)+1}$. Assume $ord(f) \geq 2 + \frac{3}{n-1}$ and denote $d := \frac{\frac{q-ord(f)}{2} + 1}{2}$. Take $m^d \supseteq I = m$. To use part 2 of the corollary we verify:

$$(34) \quad a_2 \cdot m^{ord(f)-2} \subseteq m \cdot a \cdot Jac(f), \quad \text{i.e. } m^{2d+ord(f)-2} \subseteq m^{d+1} \cdot Jac(f).$$ 

Indeed, by our assumption:

$$(35) \quad 2d + ord(f) - 2 \geq q \quad \text{and } (d + 1) + (ord(f) - 1) = \left\lfloor \frac{q-ord(f)}{2} \right\rfloor + 1 \leq q.$$ 

vi. Take the $E_6$ singularity, $f(x, y) = x^3 + y^4$. Assume $k$ is a field, $\text{char}(k) \neq 2, 3$. Denote $m = (x, y)$, $a = m^2 \subseteq I = m$. We get: $a_2 \cdot m^{ord(f)-1} \subseteq m \cdot a \cdot Jac(f)$. Therefore $\mathcal{R} f \supseteq \{f\} + m^2 \cdot (x^2, y^3)$. The ideal $m^2 \cdot (x^2, y^3)$ contains all the monomials lying above the Newton diagram of $x^3 + y^4$. Compare this result to the standard determinancy statement: $\mathcal{R} f \supseteq \{f\} + m^3$.

4.3. The filtration criterion for $\mathcal{R}$-orbits. Take the filtration $M_d : = I^d \cdot R_X^{\otimes p}$. Fix some integers $1 \leq j < d$.

**Theorem 4.5.** Suppose the conditions $jet_N(Exp)$, $jet_N(Ln)$ of $\mathcal{R}$ hold for $N = \left\lfloor \frac{q-ord(f)}{2} \right\rfloor$. Then:

$\mathcal{R}^{(j)} f \supseteq \{f\} + I^d \cdot R_X^{\otimes p}$ 

if and only if $\mathcal{T}_{\mathcal{R}^{(j)}} f \supseteq I^d \cdot R_X^{\otimes p}$.

**Proof.** The part "⇒". First we prove: $\mathcal{T}_{\mathcal{R}^{(j)}} f + M_{d+1} \supseteq M_d$. Let $w \in M_d$, then $gf \equiv f + w \mod M_{d+1}$ for some $g \in \mathcal{R}^{(j)}$. Use the assumption $jet_N(Ln)$ to get the derivation $- \sum_{i=1}^N \frac{\xi_i}{y_i} e_i + \xi_d \in \mathcal{T}_{\mathcal{R}^{(j)}}$, for some
\[ \xi_\varnothing \in \text{End}_k^{(Nj+1)}(R). \] And then

\[ T_{\xi}(f) \ni \left( - \sum_{i=1}^N \frac{(I - g)_i}{i} \right) + \xi_\varnothing f \equiv w \mod(M_{d+1} + M_{Nj+1+\text{ord}(f)}) \equiv w \mod M_{d+1}. \]

Here \( Nj + \text{ord}(f) \geq d \) because of the assumption \( N = \left\lceil \frac{d - \text{ord}(f)}{j} \right\rceil \).

Therefore we have \( T_{\xi}(f) \ni f + M_{d+1} \supseteq M_d. \) As \( M_{d+1} = I \cdot M_d \), we get (by Nakayama) \( T_{\xi}(f) \ni M_d. \)

The part “\( \Leftarrow \)”. First we prove \( \mathcal{R}(j+k)f + M_{d+k} \supseteq \{ f \} + M_{d+k} \) for every \( k \geq 0 \). Indeed, \( T_{\xi}(f) \ni M_d \) implies \( T_{\xi}(j+k)f + M_{d+k} \supseteq \{ f \} + M_{d+k} \) for every \( k \geq 0 \). (Using \( M_\ast = I^d \cdot R^\mathcal{D}_p \).) Accordingly we present an element \( w \in M_{d+k} \) as \( w = \xi(f) \), for some \( \xi \in T_{\xi}(j+k) \). The condition \( \text{jet}_N(\text{Exp}) \) ensures an element \( \Phi := \sum_{i=0}^N \frac{\xi_i}{i} + \phi_\xi \in \mathcal{R}(j+k) \), where \( \phi_\xi \in \text{End}_k^{(N(j+k)+1)}(R_X) \). And then \( \Phi(f) - f - w \in M_{d+k+1} + M_{(j+k)N + \text{ord}(f)+1} \).

Finally use \( (j+k)N + \text{ord}(f) \geq d + k \) to get \( \Phi(f) - f - w \in M_{d+k+1} \).

Iterating \( \mathcal{R}(j+k)f + M_{d+k+1} \supseteq \{ f \} + M_{d+k} \) for \( k \geq 0 \) we get: \( \mathcal{R}(j) \ni \{ f \} + M_d \). Now apply part 2 of Lemma 3.6. \( \square \)

**Corollary 4.6.** \( R_X \) is one of \( k][\{ x \}, k\{ x \}, k \{ x \} \), with \( k \) a local ring, see \( \{ 2 \} \) ii. Take \( I \subseteq (x) \) and \( f \in (x) \cdot R^\mathcal{D}_p \).

1. Suppose \( 2, 3, \ldots, N \in k^{\times} \) for \( N = \left\lceil \frac{d - \text{ord}(f)}{j} \right\rceil \). Then: \( \mathcal{R}(f) \ni \{ f \} + I^d \cdot R^\mathcal{D}_p \) if and only if \( T_{\xi}(f) \ni I^d \cdot R^\mathcal{D}_p \).
2. Suppose \( 2 \in k^{\times} \). If \( T_{\xi}(f) \ni I^d \cdot R^\mathcal{D}_p \) then \( \mathcal{R}(d-j-\text{ord}(f)) \ni \{ f \} + I^{2d-2j-\text{ord}(f)} \cdot R^\mathcal{D}_p \).

**Proof.** 1. Combine theorem \( \{ 4 \} \) with lemma \( \{ 3 \} \). The \( \text{jet}_N \)-condition holds by example \( \{ 2 \} \).

2. We get \( T_{\xi}(d-j-\text{ord}(f)) \ni \{ f \} + I^{2d-2j-\text{ord}(f)} \cdot R^\mathcal{D}_p \). Now apply part one with \( N := \left\lceil \frac{2d-2j-\text{ord}(f)}{d-j-\text{ord}(f)} \right\rceil = 2 \).

**Example 4.7.**

1. Take \( p = 1 \), \( I = \mathfrak{m} \), \( j = 1 \) and suppose \( k \) is a field, with \( \text{char}(k) = 0 \) or \( \text{char}(k) > d - \text{ord}(f) \). Then: \( \mathcal{R}(1) \ni \{ f \} + \mathfrak{m}^d \) if and only if \( \mathfrak{m}^2 \cdot J acid(f) \geq \mathfrak{m}^d \).

In particular, the \( \mathcal{R} \)-order of determinacy of \( f \) is \( \leq \mu(f) + 1 \). In \( \text{char}(k) = 0 \) this is well known, e.g. [Gr.Lo.Sh. Corollary 2.24], [Greuel-Pham.19]: the \( \mathcal{R} \)-order of determinacy of \( f \) is \( \leq 2\mu(f) - \text{ord}(f) + 2 \).

ii. In part two of corollary \( \{ 4 \} \) the conclusion is weaker, but also the assumption is weaker. For \( j = 1 \) this part two is known, e.g. see corollary 5.2 of [B.G.K.22].

5. **Criteria for contact orbits, \( \mathcal{X}f \) vs \( T_{\xi}f \)**

Let \( R_X \) be one of the rings \( k][\{ x \}, k\{ x \}, k\{ x \} ]/f, with \( k \) a local ring, see \( \{ 2 \} \) ii.

5.1. **The \( \mathcal{X} \)-implicit function theorem.** Take two ideals \( \mathfrak{a} \subseteq I \subseteq R_X, \) with \( \mathfrak{a} \subseteq \mathfrak{m}^2 \), and a map-germ \( f \in I \cdot R^\mathcal{D}_p \). (The usual choice is \( I = R_X, \mathfrak{m} = \mathfrak{m} \) or \( I = \sqrt{(f)} \).) For the notation \( I^{\text{ord}(f)} \) see \( \{ 2 \} \).

**Theorem 5.1.** Suppose \( \mathfrak{a} \cdot I^{\text{ord}(f)-2} \cdot R^\mathcal{D}_p \subseteq \mathfrak{m} \cdot \mathfrak{a} \cdot T_{\xi}f + \mathfrak{m} \cdot (f) \cdot R^\mathcal{D}_p \).

1. (The case \( J = 0 \).) Then \( \{ f \} + \left( \mathfrak{a}^2 \cdot I^{\text{ord}(f)-2} + \mathfrak{m} \cdot (f) \right) \cdot R^\mathcal{D}_p \subseteq \mathcal{X}f. \)
2. (The case \( \neq 0 \). Assume the \( \text{jet}_0 \)-condition of \( (x, 0) \) Then \( \{ f \} + \left( \mathfrak{a}^3 \cdot I^{\text{ord}(f)-2} + \mathfrak{m} \cdot (f) \right) \cdot R^\mathcal{D}_p \subseteq \mathcal{X}f. \)

A remark: the assumption \( \mathfrak{a} \cdot I^{\text{ord}(f)-2} \cdot R^\mathcal{D}_p \subseteq \mathfrak{m} \cdot \mathfrak{a} \cdot T_{\xi}f + \mathfrak{m} \cdot (f) \cdot R^\mathcal{D}_p \) is preserved by \( \mathcal{X} \)-equivalence.

**Proof.**

1. Take a perturbation \( g \in (\mathfrak{a} \cdot I^{\text{ord}(f)-2} + \mathfrak{m} \cdot (f)) \cdot R^\mathcal{D}_p \). We want to resolve the condition \( \Phi_X(f) = (\mathfrak{I} + U) \cdot f + g \), where \( \Phi_X \in \text{Aut}_X \) and \( U \in \text{Mat}_{p \times p}(\mathfrak{m}) \). As in the proof of theorem \( \{ 4 \} \) we fix some (finite) set of generators \( \{ \xi_i(f) \} \) of \( \mathfrak{a} \cdot T_{\xi}f \subseteq R^\mathcal{D}_p \).

Present the coordinate change in the form \( \Phi_X : x \rightarrow x + \sum c_i \xi_i(x) \), here \( \{ c_i \} \) are unknowns.

- As in the \( \mathcal{R} \)-case we get: \( \Phi_X(f) - f - \sum c_i \xi_i(f) \in \{ \{ c_i \} \} \cdot \mathfrak{a}^2 \cdot I^{\text{ord}(f)-2} - \mathcal{X}f. \)

Therefore, as in the \( \mathcal{R} \)-case, we can present

\[ \Phi_X(f) = f + \sum \left( c_i + \mathcal{H}_{i}^{(1)}(\{ c_i \}) \right) \xi_i(f) + \mathcal{H}_{i}^{(2)}(\{ c_i \}) \cdot f. \]

Here \( \mathcal{H}_{i}^{(1)}(\{ z \}) \subseteq \mathfrak{m} \cdot R_X[[m \cdot z]][[z]] \) and \( \mathcal{H}_{i}^{(1)}(\{ m \}) \subseteq \mathfrak{m} \subseteq R_X \). Similarly, \( \mathcal{H}_{i}^{(2)}(\{ z \}) \subseteq \text{Mat}_{p \times p}(\mathfrak{m} \cdot R_X[[m \cdot z]][[z]]) \) and \( \mathcal{H}_{i}^{(2)}(\{ m \}) \subseteq \text{Mat}_{p \times p}(\mathfrak{m}). \)
Suppose a submodule \( C \). It is enough to prove:

\[
\begin{align*}
  f + \sum \left( c_i + H_i^{(1)}(\{c_i\}) \right) \xi(f) + H^{(2)}(\{c_i\}) \cdot f &= f + \sum l_i(U) \cdot \xi_i(f) + (U + U^g + U \cdot U^g + U^g) \cdot f.
\end{align*}
\]

Therefore we have transformed the condition \( \Phi_X(f) = (I + U) \cdot (f + g) \) into the system of equations:

\[
\begin{align*}
  c_i + H_i^{(1)}(\{c_i\}) &= l_i(U), \quad H^{(2)}(\{c_i\}) = U + U^g + U \cdot U^g + U^g, \quad \text{with the unknowns } \{c_i\}, U.
\end{align*}
\]

Apply the IFT, to determine \( c_i(x) \in m \) and \( U \in \text{Mat}_{p \times p}(m) \). We get the transformation \( \Phi_X : (I + U) \in X \times GL(p, R_X) \) that satisfies the condition \( \Phi_X(f) = (I + U) \cdot (f + g) \).

2. It is enough to prove: \( \mathcal{X} f \supseteq \{ f \} + a^3 \cdot I^{\text{ord}(f)-2} \cdot R^{\text{sp}}_{X} \) (using the GL\( (p, R_X) \)-transformation). Take a perturbation \( g \in a^3 \cdot I^{\text{ord}(f)-2} \cdot R^{\text{sp}}_{X} \). The previous argument gives a map \( \Phi_X : x \to x + \xi(x) \), with \( \xi(m) \subseteq m \cdot a^3 \), and a matrix \( U \), satisfying \( \Phi_X(f) = (I + U)(f + g) \). As in the proof of theorem this \( \Phi_X \) is not necessarily "a coordinate change" on \( X \). By the \( \psi \)-assumption we can extend \( \Phi_X \) to a coordinate change \( \Psi_X \in \text{Aut}_X \), satisfying \( \Psi_X(x) = x - \sum c_i \xi_i(x) \in m \cdot a^3 \). Then \( \Psi_X^{-1}(I + U)(f + g) \in \{ f \} + a^3 \cdot I^{\text{ord}(f)-2} R^{\text{sp}}_{X} \).

(As in the \( \Psi \)-case we remark: one cannot assume \( \Psi_X^{-1}(I + U)(f + g) \in \{ f \} + a^3 \cdot m \cdot T_{\bar{X}} \) )

Iterate this argument to get \( f + g \not\sim \{ f \} + m^d \cdot a^3 \cdot I^{\text{ord}(f)-2} R^{\text{sp}}_{X} \) for each \( d \geq 1 \). Thus \( f + g \in \mathcal{X} f \), the orbit-closure in the \( m^\bullet \)-filtration topology. The statement follows now by part 2 of lemma 3.6.

Example 2.5. Let \( R_X \) be one of the rings \( k[[x]], k(x), k[x] \), with \( k \) a local ring, see 2.1.ii. Take \( I = m \) and assume \( f \subseteq m^2 \). We get the following bounds on the orbit \( \mathcal{X} f \).

i. \( \mathcal{X} f \supseteq \{ f \} + (m^2 \cdot a^2 \cdot m + f) \cdot R^{\text{sp}}_{X} \). For \( p = 1 \) this reads: \( \mathcal{X} f \supseteq m^2 \cdot Jac(f)^2 + f \cdot m \).

Suppose \( f \) has an isolated singularity, i.e. \( \text{Crit}(f) \cap V(f) = \{ 0 \} \subseteq k \), i.e. \( \sqrt{a^2 + f} = m \). Then \( f \) is finitely-\( \mathcal{X} \)-determined.

The inclusion \( \mathcal{X} f \supseteq \{ f \} + (m^2 \cdot a^2 + f) \cdot R^{\text{sp}}_{X} \) strengthens Theorem 2.1 of [Cut.Sri.97], which reads: If \( R_X = k[[x]] \) and \( a = (f) \in \text{Fitt}(T^1_{\mathcal{X}} f) \) then \( \mathcal{X} f \supseteq \{ f \} + a^3 \cdot R^{\text{sp}}_{X} \).

Note that \( a \not\subseteq \text{Fitt}(T^1_{\mathcal{X}} f) \), and for \( p \geq 2 \) one often has \( m \cdot a \not\subseteq \text{Fitt}(T^1_{\mathcal{X}} f) \), see example 3.1.4.

ii. If \( f \in m^3 \) then \( \mathcal{X} f \supseteq \{ f \} + (a^2 + m^3 \cdot f) \cdot R^{\text{sp}}_{X} \). For \( p = 1 \) this reads: \( \mathcal{X} f \supseteq \{ f \} + Jac(f)^2 + f \cdot m \).

iii. For \( p = 1 \) and \( f \in m^2 \) we get by the Morse lemma, example 3.4.ii, to present \( f = Q_2(x) + f(w) \), with \( f(w) \in (w)^3 \). Then we get: \( \mathcal{X} f \supseteq \{ f \} + m^2 \cdot (x) \cdot (x + Jac(w) f(w)) + (Jac(w) f(w))^3 \cdot (f(w)) \).

iv. For \( p \geq 2 \) the assumption on the modules \( a^2 \cdot I^{\text{ord}(f)-2} \cdot R^{\text{sp}}_{X} \subseteq m \cdot a \cdot T_{\bar{X}} f + m \cdot f \cdot R^{\text{sp}}_{X} \) is weaker than the assumption on the ideals \( a^2 \cdot I^{\text{ord}(f)-2} \subseteq m \cdot a \cdot a^2 + m \cdot f \). But the latter assumption is often simpler to verify.

v. Below we assume \( p = 1 \).

Suppose \( a^2 \cdot I^{\text{ord}(f)-2} \subseteq m \cdot a \cdot Jac(f) + f \cdot m \). Then \( \mathcal{X} f \supseteq \{ f \} + a \cdot Jac(f) + f \cdot m \).

Suppose \( m^d \cdot Jac(f) + (f) m \supseteq m^d \) for \( e = \lfloor \frac{d}{2} \rfloor + 1 \).

Take \( a = m^e \) to get: \( \mathcal{X} f \supseteq \{ f \} + (m^e \cdot Jac(f) + f) \cdot R^{\text{sp}}_{X} \).

Take the smallest \( d \in \mathbb{N} \) satisfying \( m^{2d+\text{ord}(f)-2} \subseteq m^{d+1} \cdot Jac(f) + f \cdot m \).

Then \( \mathcal{X} f \supseteq \{ f \} + m^d \cdot Jac(f) + f \cdot m \).

5.2. An application: the \( \mathcal{X} \)-orbit of a reduced complete intersection curve germ via the semigroup of values. Let \( R_X \) be one of \( k[[x]], k(x), k[x] \), where \( k = \mathbb{A} \) is a field, see 2.1.ii. Suppose \( (C, o) := V(f) \subseteq k^n \), a reduced complete intersection curve germ. (Here \( f = (f_1, \ldots, f_{n-1}) \).) Take its branch decomposition, \( (C, o) = \bigsqcup_{i=1}^r (C_i, o) \). We have the normalizing morphism \( O_{(C, o)} \to \prod O_{(C_i, o)} \).

Here each ring \( O_{(C_i, o)} \) is a DVR. It has the valuation map \( val : O_{(C_i, o)} \to \bar{N} := \mathbb{N} \cup \{ \infty \} \). The "full" valuation map is the product of the morphisms, \( val : R_X \to \prod O_{(C_i, o)} \to \prod \mathbb{N} \).

Accordingly we have two sub-semigroups: \( val(O_{(C, o)}) \subseteq \oplus \mathbb{N} \) and \( val(a^2_{\mathcal{X}}) \subseteq \oplus \mathbb{N} \).

Note that \( val(a^2_{\mathcal{X}}) = val(a^2_{\mathcal{X}}) + f \).

Below we consider the "N-submodules" of \( \mathcal{N} \), i.e. subsets \( S \subseteq \mathcal{N} \) satisfying: \( S + \mathcal{N} = S \).

Proposition 5.3. Suppose a submodule \( S \subseteq \mathcal{N} \) satisfies: \( S \cap val(R_X) = S \cap val(m \cdot a^2_{\mathcal{X}}) \). Then \( \mathcal{X} f \supseteq \{ f \} + m \cdot val^{-1}(S) \cdot R^{\text{sp}}_{X} \).
If \((C, o)\) is not a plane curve germ, i.e. \(p \geq 2\), then in many cases one has \(a_2 \cdot R_X^{\geq p} \subseteq m \cdot T_X f\), rather than just \(a_2 \cdot T_X f \subseteq T_X f\). Then the bound is strengthened to: \(\mathcal{H} f \subseteq \{f\} + \text{val}^{-1}(S) \cdot R_X^{\geq p}\).

**Proof.** Let \(g \in \text{val}^{-1}(S) \subseteq R_X\). Then \(\text{val}(g) = \text{val}(q)\) for an element \(q \in m \cdot a_2^2\). Thus \(\text{val}(g - c \cdot q) > \text{val}(g)\) for some constant \(c \in k\). Namely, \(\text{val}_i(g - c \cdot q) \geq \text{val}_i(g)\) for each \(i = 1, \ldots, r\), and this inequality is strict for at least one \(i\).

Apply this procedure for each coordinate of the vector \(\text{val}(g) \in \mathbb{N}^r\) to get an element \(q \in a_2^2\) that satisfies: \(\text{val}(g - c \cdot q) \geq \text{val}(g) + (1, \ldots, 1)\). Iterating this we get: \(\text{val}^{-1}(S) \subseteq a_2^2 + (f) + \text{val}^{-1}(S + d(1, \ldots, 1))\) for each \(d \geq 1\).

As \((C, o)\) is a reduced germ, the ideal \(a_2^2 + (f)\) is \(m\)-primary. Therefore we get: \(\text{val}^{-1}(S) \subseteq m \cdot a_2^2 + (f)\). And hence \(\{f\} + m \cdot \text{val}^{-1}(S) \cdot R_X^{\geq p} \subseteq \{f\} + m \cdot (m \cdot a_2^2 + (f)) \cdot R_X^{\geq p}\). Finally, apply Example 5.2.i. \(\blacksquare\)

### 5.3. The filtration criterion for \(\mathcal{H}\)-orbits.

Take the filtration \(M_* = I^* \cdot R_X^{\geq p}\). Fix some integers \(1 \leq j < d\).

**Theorem 5.4.** Suppose the conditions \(\text{jet}_N(\text{Exp}), \text{jet}_N(\text{Ln})\) of (2.3) hold for \(N = \lfloor \frac{d - \text{ord}(f)}{j} \rfloor\). Then:
\[
\mathcal{H}^{(j)} f \subseteq \{f\} + I^d \cdot R_X^{\geq p}
\]
if and only if
\[
\text{jet}_{\mathcal{H}^{(j)}} f \subseteq I^d \cdot R_X^{\geq p}.
\]

**Proof.**

**Step 1.** In the proof of \(\mathcal{B}\)-case, Theorem 4.5, we have used the transitions \(T_{\mathcal{H}^{(j)}} \equiv \mathcal{B}^{(j)}\), ensured by the \(\text{jet}_N\) assumptions. Now we need the transitions \(T_{\mathcal{H}^{(j)}} \equiv \mathcal{H}^{(j)}\). These are ensured by lemma 3.22 of [B.G.K.22]:

- For any element \((u, \xi) \in T_{\mathcal{H}^{(j)}}\) there exists an element \((U, \Phi) \in \mathcal{H}^{(j)}\) satisfying:
  \[
  \text{ord}_I[[U, \Phi] - f] > \text{ord}_I[[u, \xi] - f].
  \]
- For any element \((U, \Phi) \in \mathcal{H}^{(j)}\) there exists an element \((u, \xi) \in T_{\mathcal{H}^{(j)}}\) satisfying:
  \[
  \text{ord}_I[[U, \Phi] - f] > \text{ord}_I[[u, \xi] - f].
  \]

The statement of that lemma in [B.G.K.22] assumes the conditions \(\text{jet}_N(\text{Exp}), \text{jet}_N(\text{Ln})\) for each \(N\). In particular this implies: \(k \supsetneq \mathbb{Q}\). However, the proof in [B.G.K.22] uses these conditions only up to \(N = \lfloor \frac{d - \text{ord}(f)}{j} \rfloor\), as we explain now.

The only part of the proof involving the conditions \(\text{jet}_N(\text{Exp}), \text{jet}_N(\text{Ln})\) is the Baker-Campbell-Hausdorff formula, (2.3) iv. For the transitions \(T_{\mathcal{H}^{(j)}} \equiv \mathcal{H}^{(j)}\) we need the expansion only up to order \(l\) that satisfies: \(l \cdot j + \text{ord}(f) \geq d\). Thus it suffices to expand up to order \(l = \lfloor \frac{d - \text{ord}(f)}{j} \rfloor\), which coincides with \(N\). Our \(\text{jet}_{\mathcal{H}^{(j)}}\) assumptions imply \(2, \ldots, N \in k^*\). Therefore the factor \((l!)^j\) is invertible. Therefore the Baker-Campbell-Hausdorff formula, (5), holds in \(R_X\) up to order \(l = N\). Thus lemma 3.22 of [B.G.K.22] is applicable to our case.

**Step 2.** We restate lemma 3.22 of [B.G.K.22] in the form:
\[
(40) \quad w_d \in T_{\mathcal{H}^{(j)}} f + M_{d + 1} \quad \text{iff} \quad w_d \in (\mathcal{H}^{(j)} f - f) + M_{d + 1}.
\]
Using this equivalence the proof goes as follows.

- If \(\mathcal{H}^{(j)} f \subseteq \{f\} + M_d\) then \(T_{\mathcal{H}^{(j)}} f + M_{d + 1} \supseteq M_d\). As \(M_{d + 1} = I \cdot M_d\) we get (by Nakayama):
  \(T_{\mathcal{H}^{(j)}} f \subseteq M_d\).
- If \(T_{\mathcal{H}^{(j)}} f \supseteq M_d\) then \(T_{\mathcal{H}^{(j)} + 1} f \supseteq M_{d + 1}\) for every \(l \geq 0\). For each \(l\) we use (40) with \(N = \frac{d + 1 - \text{ord}(f)}{j} \leq \frac{d - \text{ord}(f)}{j}\). And then \(\mathcal{H}^{(j + 1)} f \supseteq \{f\} + M_{d + 1}\) for every \(l \geq 0\). Thus \(\mathcal{H}^{(j + 1)} f \subseteq \{f\} + M_{d + 1}\). Finally, apply part 2 of Lemma 3.6. \(\blacksquare\)

**Corollary 5.5.** \((R_X \text{ is one of } k[[x]], k[x], k(x), \text{with } k \text{ a local ring, see (2.1) ii.) Take } I \subseteq (x) \text{ and } f \in (x) R_X^{\geq p}\). 

1. Suppose \(2, \ldots, N \in k^\times \text{ for } N = \lfloor \frac{d - \text{ord}(f)}{j} \rfloor\). Then: \(\mathcal{H}^{(j)} f \supseteq \{f\} + I^d \cdot R_X^{\geq p}\) if and only if \(T_{\mathcal{H}^{(j)}} f \supseteq I^d \cdot R_X^{\geq p}\).

2. Suppose \(\text{ord}(f) \geq d\). Then: \(\mathcal{H}^{(j)} f \supseteq \{f\} + I^d \cdot R_X^{\geq p}\) if and only if \(\text{ord}(f) \geq d\).

The proofs of both parts are the same as in Corollary 4.6.

**Example 5.6.** i. Take \(p = 1, I = m, j = 1\) and suppose \(k\) is a field, \(\text{char}(k) = 0\) or \(\text{char}(k) > d - \text{ord}(f)\).
\[(41) \quad \text{Then: } \mathcal{H}^{(1)} f \supseteq \{f\} + m^d \quad \text{if and only if } \ m^2 \cdot \text{Jac}(f) + m \cdot (f) \supseteq m^d.\]
In particular, the $\mathcal{H}$-order of determinacy of $f$ is $\leq \tau(f) + 1$. For $\text{char}(k) = 0$ this is well known, e.g. [Gr.Lo.Sh, Corollary 2.24]. For $\text{char}(k) > d - \text{ord}_f(0)$ this strengthens the bound $(2\tau(f) - \text{ord}_f(0) + 2)$ of [Bon.Gr.Ma.11, Greuel-Pham.19].

ii. For $j = 1$ (and $k$ a field) part two of Corollary 5.5 is known, e.g. see Corollary 5.10 of [B.C.K.22].

6. Properties of the (extended) tangent space $T_{\mathcal{A}}f$

The tangent space $T_{\mathcal{A}}f$ is not an $R_X$-module. And $T_{\mathcal{A}}f$ is not finitely generated as a module over $R_Y$. As is mentioned in 1.9 numerous simple/immediate properties of $T_{\mathcal{A}}f, T_{\mathcal{A}}f$ become cumbersome (and non-trivial) for $T_{\mathcal{A}}f$. E.g. even the simplest properties of $T_{\mathcal{A}}f$ in the classical $C^\infty$-case rely heavily on the deep Mather-Malgrange preparation theorem.

Below we study the annihilator $a_\mathcal{A}$, the extended module structure of $T_{\mathcal{A}}f$, the Nakayama&Artin-Rees properties, and the topological closure $\overline{T_{\mathcal{A}}f} \subseteq T_X^{\mathcal{A}}$.

Let the pair $(R_X, R_Y)$ be one of $(k[[x]]/j, k[[y]])$, $(k(x)/j, k(y))$, $(k(x)/j, k(y))$, $\mathcal{A}$. Fix the filtration $I^\bullet \cdot R_X^{\mathcal{A}}$ of $R_X^{\mathcal{A}}$, we always assume $(f) \subseteq I$. Accordingly one has the filtrations $\mathcal{A}^\bullet$ and $T_{\mathcal{A}}(0)$, with $T_{\mathcal{A}}(0) := T_{\mathcal{A}}$, see 3.3.2 Fix a map $f : X \to (k^p, o)$, i.e. $f \in m \cdot R_X^{\mathcal{A}}$. Take the annihilator ideals $a_\mathcal{A}, a_\mathcal{A}(0) \subseteq R_X$ defining the instability locus, see 3.6. In general $a_\mathcal{A}, a_\mathcal{A}(0) \subseteq f^\#(R_Y)$.

The map $f$ is called infinitesimally $\mathcal{A}$-stable if $T_{\mathcal{A}}f = R_X^{\mathcal{A}}$, i.e. $a_\mathcal{A} = R_X$.

6.1. The annihilator $a_\mathcal{A}$.

6.1.1. The case $p = 1$, i.e. $f : X \to (k^1, o)$, is elementary. (Probably well-known.)

Lemma 6.1. $(p = 1)$ Assuming $f \in m^2$, one has $a_\mathcal{A} \subseteq a_\mathcal{A} \subseteq a_\mathcal{A} : m$ and $a_\mathcal{A} = a_\mathcal{A} + f^\#(R_Y)$.

In particular, for $R_X$ one of $k[[x]], k\{x\}, k(x)$, one has: $\text{Jac}(f) \subseteq a_\mathcal{A} \subseteq \text{Jac}(f) : m$.

Proof. (The part $a_\mathcal{A} \subseteq a_\mathcal{A}$ is obvious.) Take any $g \in a_\mathcal{A}$, then $g \in a_\mathcal{A} + f^\#(g)^d$. For any $g \in m$ one has $q \cdot g \in a_\mathcal{A} + f^\#(g)^d$. (Here we chose $d, d' \leq \infty$ as the largest possible.)

- If $d' \leq d$ then $(1 - q \cdot q) g \in a_\mathcal{A}$ for some $q \in R_Y \subset R_X$. And therefore $g \in a_\mathcal{A}$.
- Suppose $d' > d$ for each $q \in m$. Then $m \cdot g \subseteq a_\mathcal{A} + g \cdot (f) \subseteq a_\mathcal{A} + g \cdot m^2$. Nakayama gives: $m \cdot g \subseteq a_\mathcal{A}$.

In both cases we get $m \cdot a_\mathcal{A} \subseteq a_\mathcal{A}$. Hence $a_\mathcal{A} \subseteq a_\mathcal{A} : m$. The statement $a_\mathcal{A} = a_\mathcal{A} + f^\#(R_Y)$ follows.

6.1.2. For $p \geq 2$ the annihilator $a_\mathcal{A}$ is much more complicated. We list several basic properties. Most of them are immediate/well known. Let the pair $(R_X, R_Y)$ be as in 2.2. Take the filtration $I^\bullet \cdot R_X^{\mathcal{A}}$, with $(f) \subseteq I$.

Properties 6.2. i. $a_\mathcal{A} \cdot R_X^{\mathcal{A}} \subseteq a_\mathcal{A} \cdot R_X^{\mathcal{A}} + k \cdot \mathcal{A}$.

ii. If $(f) \subseteq a_\mathcal{A}$ then $a_\mathcal{A} \subseteq a_\mathcal{A}$.

If $f$ is infinitesimally stable, i.e. $a_\mathcal{A} = R_X$, then $T_{\mathcal{A}}(0) f = T_{\mathcal{A}}(0) f$.

(In Lemma 7.1 of [Mon. Num. Bal] this is stated for $I = m$, but the proof is the same.)

More generally, $T_{\mathcal{A}}(0) f = T_{\mathcal{A}}(0) f$ iff $(f) \cdot R_X^{\mathcal{A}} \subseteq T_{\mathcal{A}}(0) f$ iff $a_\mathcal{A}(0) = a_\mathcal{A}(0)$.

iii. If $(f) \subseteq m^2$ and $a_\mathcal{A} \subseteq m$ then $a_\mathcal{A} \subseteq m$.

iv. If $a_\mathcal{A} \subseteq m$ then $\sqrt{a_\mathcal{A} + (f)} = \sqrt{a_\mathcal{A}} \subseteq R_X$.

Moreover, $\sqrt{a_\mathcal{A} + (f)} = \sqrt{a_\mathcal{A}}$ for $j \geq 0$.

(Geometrically: if $f$ is $\mathcal{A}$-unstable then the singular locus $\text{Sing}(f)$ coincides with the instability locus restricted to the germ $V(f)$.)

Proof. The part $\subseteq$. For $d > 1$ we have: $(a_\mathcal{A} + (f)) d \cdot R_X^{\mathcal{A}} \subseteq T_{\mathcal{A}}f + T_{\mathcal{A}}(0) f \subseteq T_{\mathcal{A}}f + (f) \cdot R_X^{\mathcal{A}} = T_{\mathcal{A}}f$.

The case $j \geq 0$ is immediate.

The part $\supseteq$. Take the filtration $I^\bullet \subseteq R_X$. Then $\sqrt{a_\mathcal{A} + (f)} \supseteq \sqrt{a_\mathcal{A}(0) + (f)} \supseteq \sqrt{a_\mathcal{A}(0) + (f)} = \sqrt{a_\mathcal{A} + (f)} = \sqrt{a_\mathcal{A}}$. The last transition is by example 3.14. The statement with $j \geq 0$ is similar: $\sqrt{a_\mathcal{A}(j) + (f)} \supseteq \sqrt{a_\mathcal{A}(j) + (f)} = \sqrt{a_\mathcal{A}(j)}$.

v. If $a_\mathcal{A} \subseteq \sqrt{I}$ and $a_\mathcal{A} \subseteq m$ then $\sqrt{a_\mathcal{A}(j) + (f)} = \sqrt{a_\mathcal{A} + (f)}$ for each $j \geq 0$.

Proof. $\sqrt{a_\mathcal{A}(j) + (f)} = \sqrt{I} \cdot \sqrt{a_\mathcal{A}(j) + (f)} = \sqrt{a_\mathcal{A} + (f)}$.

vi. If $(f) \subseteq m^2$ and $a_\mathcal{A} \subseteq m$ then $a_\mathcal{A} \cdot R_X^{\mathcal{A}} \subseteq (a_\mathcal{A} + (f)) \cdot T_{\mathcal{A}}f + T_{\mathcal{A}}(0) f$.

If $(f) \subseteq m^2$ and $a_\mathcal{A} \subseteq m$ then $a_\mathcal{A} \cdot R_X^{\mathcal{A}} \subseteq (a_\mathcal{A} + (f)) \cdot T_{\mathcal{A}}f + T_{\mathcal{A}}(0) f$.

vii. $a_\mathcal{A}(0) \cdot a_\mathcal{A} \cdot R_X^{\mathcal{A}} \subseteq (a_\mathcal{A}(0) \cdot T_{\mathcal{A}}f + f) \cdot T_{\mathcal{A}}(0) f + f \cdot T_{\mathcal{A}}(1) f$.

$a_\mathcal{A}(0) \cdot a_\mathcal{A}(0) \cdot R_X^{\mathcal{A}} \subseteq (a_\mathcal{A}(0) + (f)) \cdot T_{\mathcal{A}}(0) f + f \cdot T_{\mathcal{A}}(1) f$.

$a_\mathcal{A} \cdot a_\mathcal{A} \cdot R_X^{\mathcal{A}} \subseteq (a_\mathcal{A} + (f))^2 \cdot T_{\mathcal{A}}f + f \cdot T_{\mathcal{A}}(1) f$. 

Lemma 6.3. Let $\mathfrak{a}_{j+1} R \subseteq I$.
Therefore (by Artin-Rees or via the primary decomposition) $\mathfrak{a}_{j+1} R \subseteq I \setminus \mathfrak{a}_{d_j}$.

The quotient $T_{|j+1|}/T_{|j|}$ is a finitely-generated module over $R_Y$. Thus the submodule $\mathfrak{a}_{j+1} R X_{\mu} + T_{|j+1|}/T_{|j|} f \subseteq T_{|j+1|}/T_{|j|}$ is finitely generated, see (4.6). Take the filtration $(I : \mathfrak{a}_{j+1})^d : \mathfrak{a}_{j+1} R X_{\mu} + T_{|j+1|}/T_{|j|} f$.
It is strictly decreasing. Therefore for each $i \geq 1$ there exists $d_i < \infty$ satisfying:

$$(I : \mathfrak{a}_{j+1})^{d_i} \cdot \mathfrak{a}_{j+1} R X_{\mu} + T_{|j+1|}/T_{|j|} f \subseteq f^g(y)^{i+1} \cdot T_{|j+1|}/T_{|j|} f.$$

Hence $(I : \mathfrak{a}_{j+1})^{d_i+1} R X_{\mu} \subseteq T_{|j+1|} f + T_{|j+1|}/T_{|j|} f$. Then $(I : \mathfrak{a}_{j+1})^{d_i+1} R X_{\mu} \subseteq T_{|j+1|} f \cap f^{i+1} R X_{\mu} + T_{|j+1|}/T_{|j|} f$, because $(f) \subseteq I$. Artin-Rees over $R_X$ gives: $(I : \mathfrak{a}_{j+1})^{d_i} R X_{\mu} \subseteq T_{|j+1|} f + T_{|j+1|}/T_{|j|} f$ for each $j \geq 0$ and $d_j \gg j$. Hence

$\sqrt{\mathfrak{a}_{j+1}} \supseteq I : \mathfrak{a}_{j+1}$.

6.1.3. The following technical lemma is used to apply theorem [73] in particular cases.

Lemma 6.3. Let $M \subseteq T_{|j+1|}/T_{|j|} f$ such that $M \subseteq m : R X_{\mu}$ is an $R_X$-submodule. (Thus $M$ is uniquely defined.) In particular $M \supseteq T_{|j+1|}/T_{|j|} f + \mathfrak{a}_{j+1} R X_{\mu}$. Therefore we can present $M = T_{|j+1|}/T_{|j|} f + \Lambda$ for some $R_Y$-submodule $\Lambda \subseteq T_{|j+1|}/T_{|j|} f$. We take $\Lambda$ the largest possible, thus $\Lambda \supseteq \mathfrak{a}_{j+1} R X_{\mu} \cap T_{|j+1|}/T_{|j|} f$. Therefore Ann$(R_Y/\Lambda) \supseteq \mathfrak{a}_{j+1} R X_{\mu} \cap f^g(y)$.

We claim: Ann$(R_Y/\Lambda) = \mathfrak{a}_{j+1} R X_{\mu} \cap f^g(y)$. Indeed, suppose $\Lambda \supseteq b_Y R Y_{\mu}$ for an ideal $b_Y \subseteq f^g(Y_Y)$. Then $M \supseteq b_Y R X_{\mu} \cdot \mathfrak{a}_{j+1} R X_{\mu}$. And thus $\mathfrak{a}_{j+1} R X_{\mu} \supseteq b_Y R X_{\mu}$. Hence $b_Y \subseteq \mathfrak{a}_{j+1} R X_{\mu} \cap f^g(Y_Y)$.

Altogether: $(\mathfrak{a}_{j+1} R X_{\mu} \cap f^g(y)) R X_{\mu} \subseteq M = T_{|j+1|}/T_{|j|} f + \Lambda$. Therefore

$$(\mathfrak{a}_{j+1} R X_{\mu} \cap f^g(y))^2 \cdot R X_{\mu} = (\mathfrak{a}_{j+1} R X_{\mu} \cap f^g(y)) \mathfrak{a}_{j+1} R X_{\mu} R X_{\mu} \cap (\mathfrak{a}_{j+1} R X_{\mu} \cap f^g(y))^2 \cdot R X_{\mu} \subseteq (\mathfrak{a}_{j+1} R X_{\mu} \cap f^g(y)) \cdot T_{|j+1|}/T_{|j|} f + (\mathfrak{a}_{j+1} R X_{\mu} \cap f^g(y)) \cdot \Lambda.$$

6.2. Extending the module structure of $T_{|j+1|}/T_{|j|} f$. The tangent spaces $T_{|j+1|}/T_{|j|} f \subset R X_{\mu}$ are $R_Y$-submodules. In fact they are modules over a larger ring, and are often finitely generated.

Lemma 6.4. Let $\mathfrak{a}_{j+1} R X_{\mu} \subseteq T_{|j+1|}/T_{|j|} f$.

2. $\mathfrak{a}_{j+1} R X_{\mu} \vartriangleleft T_{|j|}/T_{|j|} f$.

Therefore all the spaces $T_{|j|}/T_{|j|} f$ are modules over the subring $f^g(Y_Y) + \mathfrak{a}_{j+1} R X_{\mu}$.

3. The map $f^g$ is of finite singularity type (i.e. the $k$-module $T_{|j|}/T_{|j|} f$ is finitely generated, (3.6.2) if and only if the $k$-module $R X_{\mu}/(f^g Y_Y) + \mathfrak{a}_{j+1} R X_{\mu}$ is finitely generated.

4. If $f$ is of finite singularity type, then any $(f^g(Y_Y) + \mathfrak{a}_{j+1})$-submodule of $R X_{\mu}$ is finitely generated.

In particular, $R X_{\mu}$ and $T_{|j|}/T_{|j|} f$ are f.g. over the local Noetherian ring $f^g(Y_Y) + \mathfrak{a}_{j+1}$.

Proof.

1. Immediate verification: $\mathfrak{a}_{j+1} R X_{\mu} \vartriangleleft T_{|j|}/T_{|j|} f$.

2. We have: $\mathfrak{a}_{j+1} R X_{\mu} \subseteq T_{|j+1|}/T_{|j|} f + f^g(Y_Y) \cdot T_{|j+1|}/T_{|j|} f$. Note that $f^g(Y_Y) \subseteq I^{j+1}$, as $(f) \subseteq I$. Thus $f^g(Y_Y) \cdot T_{|j+1|}/T_{|j|} f \subseteq T_{|j+1|}/T_{|j|} f$. In addition $f^g(Y_Y) \cdot T_{|j|}/T_{|j|} f \subseteq T_{|j+1|}/T_{|j|} f$. Therefore $f^g(Y_Y) \cdot T_{|j|}/T_{|j|} f \subseteq T_{|j+1|}/T_{|j|} f$.

3. The subring $f^g(Y_Y) + \mathfrak{a}_{j+1} R X_{\mu}$ is local as the rings $R X_{\mu}$, $R X_{\mu}$ are local and $\mathfrak{a}_{j+1} R X_{\mu}$ is an ideal.

We can assume $\mathfrak{a}_{j+1} \subseteq m$. Then, by property (6.2) we have $\sqrt{(f) + \mathfrak{a}_{j+1}} = \mathfrak{a}_{j+1}$. Hence $R X_{\mu}/(f) + \mathfrak{a}_{j+1}$ is $k$-finite if $\sqrt{(f) + \mathfrak{a}_{j+1}} \subseteq m$ if $T_{|j|}/T_{|j|} f$ is $k$-finite.

4. By part 3: $R X_{\mu}/(f) + \mathfrak{a}_{j+1}$ is finitely generated over $k$.

Applying Weierstraß finiteness to $R X_{\mu}/(f) \otimes R X_{\mu}/(f)/(Y_Y) = R X_{\mu}/(f) + (f)$ we get: $R X_{\mu}/(f)$ is f.g. over $R Y$. And then $R X_{\mu}$ is a f.g. module over $f^g(Y_Y) + \mathfrak{a}_{j+1}$. (Fix a finite set of $R X_{\mu}$-preimages of the generators of $R X_{\mu}/(f) \in mod-f^g(Y_Y)$. They generate $R X_{\mu}$ over $f^g(Y_Y) + \mathfrak{a}_{j+1}$.)

Therefore the subring $f^g(Y_Y) + \mathfrak{a}_{j+1} \subseteq R X_{\mu}$ is Noetherian. (This is the Eakin-Nagata theorem, (Eisenbud, Exercise A.3.7, pg.625).)

Finally, apply the remark of (2.3) viii to $R X_{\mu}$ as a f.g. module over $f^g(Y_Y) + \mathfrak{a}_{j+1}$. ■
6.3. Nakayama-type results for $T_{\delta}f$. Suppose a submodule $M \subseteq R_X^{\mathbb{P}}$ satisfies $T_{\delta}g + m \cdot M \supseteq M$. For $G = \mathcal{S}$, $\mathcal{K}$ one concludes (by Nakayama): $T_{\delta}g \supseteq M$. But $T_{\delta}f$ is not an $R_X$-module. If considered as $R_Y$-modules, $T_{\delta}f, M$ are not always finitely generated. Recall the “adjusted Nakayama lemma”.

**Lemma 6.5.** [Gaffney.79, Wall.81, lemma 1.6] Take a homomorphism of (local, Noetherian) rings $f : (R_Y, m_Y) \to (R_X, m)$, with Weierstrass finiteness condition. Take f.g. modules $M_X \in \text{mod-} R_X, M_Y \in \text{mod-} R_Y$.

1. If $M_X \subseteq M_Y + (f) \cdot M_X$ then $M_X \subseteq M_Y$.
2. Suppose the inclusion $M_X \subseteq M_Y + b \cdot M_X$ holds for an ideal $b \subseteq m$ satisfying: $b \cdot M_Y \subseteq (f) \cdot M_X$. Then $M_X \subseteq M_Y$.

**Proof.**
1. One has $R_Y/(y) \otimes M_X \cong M_Y/(f)M_X \subseteq M_Y + (f)M_X/(f)M_Y$. Here $M_Y + (f)M_X/(f)M_Y$ is f.g. over $R_Y$. Thus $R_Y/(y) \otimes M_X$ is f.g. over $R_Y$, see [26, vi]. By Weierstrass finiteness $M_X$ is f.g. over $R_Y$. Finally, apply Nakayama over $R_Y$ to the inclusion $M_X \subseteq M_Y + (f) \cdot M_X$.
2. One gets $b \cdot M_X \subseteq (f) \cdot M_X + b^2 \cdot M_X$. Nakayama (over $R_X$) gives $b \cdot M_X \subseteq (f) \cdot M_X$. The initial assumption gives $M_X \subseteq M_Y + (f) \cdot M_X$. Apply part one.

**Corollary 6.6.** Fix some $j \geq -1$, and $M \subseteq R_X^{\mathbb{P}}$, and $(f) \subseteq I$. 

1. If $T_{\delta}(j) + (f) \cdot M \supseteq M$ then $T_{\delta}(j) + (f) \cdot M \supseteq M$.
2. Suppose $a_{\delta} \subseteq m$. If $T_{\delta}(j) + (a_{\delta} + (f)) \cdot M \supseteq M$ then $T_{\delta}(j) + (f) \cdot M \supseteq M$.

**Proof.** As we need an upper bound on $M$, we can assume $M \supseteq T_{\delta}(j) + (f)$. 
1. Pass to the quotients, $T_{\delta}(j)/T_{\delta}(j) + (f) \cdot M/T_{\delta}(j) \supseteq M/T_{\delta}(j)$. Here $M/T_{\delta}(j) \in \text{mod-} R_X$ and $T_{\delta}(j)/T_{\delta}(j) = \text{mod-} R_X$. Therefore (by part one of lemma 6.5), $T_{\delta}(j)/T_{\delta}(j) \supseteq M/T_{\delta}(j)$.
2. Multiply the assumption by $a_{\delta}$ to get $a_{\delta} \cdot T_{\delta}(j) + (a_{\delta} + (f)) \cdot M \supseteq a_{\delta} \cdot M$. Nakayama over $R_X$ gives $a_{\delta} \cdot T_{\delta}(j) \supseteq a_{\delta} \cdot M$. Part two of lemma 6.5 gives: $T_{\delta}(j) \supseteq a_{\delta} \cdot M$. The initial assumption becomes: $T_{\delta}(j) + (f) \cdot M \supseteq M$. Now invoke part 1.

**Example 6.7.** Let $(R_X, R_Y)$ be one of the pairs $(\mathbb{k}[x]/f, \mathbb{k}[y]), (\mathbb{k}[x]/f, \mathbb{k}[y]), (\mathbb{k}[x]/f, \mathbb{k}[y])$, see [26, ii]. Let $I = m$ and suppose $a_{\delta} \subseteq m$. Take $M = m^j \cdot (f) \cdot R_X^{\mathbb{P}}$ for some $j \geq 0$. Part 2 of corollary 6.6 gives:

$$\text{(43)} \quad \text{If } T_{\delta}(j) + (a_{\delta} + (f)) \cdot m^j \cdot (f) \cdot R_X^{\mathbb{P}} \supseteq m^j \cdot (f) \cdot R_X^{\mathbb{P}} \text{ then } T_{\delta}(j) = T_X(j).$$

For $j = 0$ this is well known, e.g. theorem 5.1 of [Ruas.83].

**Remark 6.8.** One would like a stronger bound: “If $T_{\delta}(j) + (\sqrt{a_{\delta} + (f)}) \cdot m \supseteq M$ then $T_{\delta}(j) \supseteq M$.” This fails already for the case $\sqrt{a_{\delta}} = m$. For example, one has $T_{\delta}(j) + m \cdot R_X^{\mathbb{P}} = R_X^{\mathbb{P}}$, though of course $T_{\delta}(j) \neq R_X^{\mathbb{P}}$.

**Corollary 6.9.** Suppose $b \cdot R_X^{\mathbb{P}} \subseteq T_{\delta}(j) + b \cdot (f) \cdot (f)^{j+1} \cdot R_X^{\mathbb{P}}$, for an ideal $b \subseteq R_X$. Then $b \cdot R_X^{\mathbb{P}} \subseteq T_{\delta}(j)$.

**Proof.** Pass to the quotients, $b \cdot R_X^{\mathbb{P}}/T_{\delta}(j) \subseteq T_{\delta}(j) + b \cdot (f) \cdot (f)^{j+1} \cdot R_X^{\mathbb{P}}/T_{\delta}(j)$. Note: $(b) : (f)^{j+1}) \cdot T_{\delta}(j)/T_{\delta}(j) \subseteq (b) \cdot (f) \cdot R_X^{\mathbb{P}}/T_{\delta}(j)$. Then (by part 2 of lemma 6.5): $b \cdot R_X^{\mathbb{P}}/T_{\delta}(j) \subseteq T_{\delta}(j)$.

**Example 6.10.** i. Take $b = I^r$ and $(f) \subseteq I$, for some $0 \leq j < r$. One gets the well-known bound: if $T_{\delta}(j) + I^{r-j} \cdot R_X^{\mathbb{P}} \supseteq I^r \cdot R_X^{\mathbb{P}}$ then $T_{\delta}(j) \supseteq I^r \cdot R_X^{\mathbb{P}}$. (See e.g. corollary 1.23, pg. 17 of [Ruas.83].)

ii. Moreover, for any map $g \in m \cdot R_X^{\mathbb{P}}$ satisfying $g \cdot f \in I^{2r-2j} \cdot R_X^{\mathbb{P}}$ one gets (by the direct check) $T_{\delta}(j) \supseteq I^r \cdot R_X^{\mathbb{P}}$. For $I = m$ this is [du Plessis.80, corollary 2.4].

6.4. Artin-Rees type properties of $T_{\delta}f$. The image tangent spaces $T_{\delta}g, T_{\delta}f \subseteq R_X^{\mathbb{P}}$ are $R_X$-submodules. For them the condition “$T_{\delta}(j) \supseteq I^d \cdot R_X^{\mathbb{P}}$” implies “$T_{\delta}(j+1) \supseteq I^{d+j} \cdot R_X^{\mathbb{P}}$”, for every $d \geq 0$. This is heavily used in the study of $\mathcal{S}, \mathcal{K}$-orbits, theorems 6.3, 6.4.

The tangent space $T_{\delta}f$ is a module over $R_X$, not over $R_Y$. The condition $T_{\delta}(j) \supseteq I^d \cdot R_X^{\mathbb{P}}$ does not imply “$T_{\delta}(j+1) \supseteq I^{d+j+1} \cdot R_X^{\mathbb{P}}$” for every $d \geq 0$. (For example, in the infinitesimally stable case, i.e. $T_{\delta}f = R_X^{\mathbb{P}}$, one does not have $T_{\delta}(j) \supseteq m \cdot R_X^{\mathbb{P}}$. Yet, some properties of Artin-Rees type do hold.

Take a map $X \to Y = (k^p, a)$, the corresponding map $R_Y \to R_X$, and a pair $mod- R_Y \supset M_X \subset M_X \in mod- R_X$.

**Lemma 6.11.** Then $M_Y \cap m^d \cdot M_X \subseteq m^d \cdot M_Y \cap m^d \cdot M_X$ for every $j$ and a corresponding $d_j < \infty$. 

Proof. The filtration $m_X^* \cdot M_X$ is strictly decreasing (by Nakayama over $R_X$). Take the induced filtration $M_Y/m_Y^d \cdot M_Y \supseteq M_Y \cap m_Y^d \cdot M_X + m_Y^d \cdot M_Y/m_Y^d \cdot M_Y$. Here $M_Y/m_Y^d \cdot M_Y$ is an Artinian module over the ring $R_Y/m_Y^d$. Thus the induced filtration stabilizes,

\[
\bigcap_{d \geq 1} M_Y \cap m_Y^d \cdot M_X + m_Y^d \cdot M_Y/m_Y^d \cdot M_Y = 0 \quad \text{for} \quad d \gg 1.
\]

Finally, $\bigcap \{ M_Y \cap m_Y^d \cdot M_X + m_Y^d \cdot M_Y/m_Y^d \cdot M_Y \mid d \geq 1 \} = 0$. Hence the statement. 

Lemma 6.12. Assume $(f) \subseteq I$.

1. Suppose $T_{g(f)}(l) \supseteq c_1 \cdot R_{X}^{\varepsilon_p}$ for $l = 1, 2$ and some ideals $c_1 \subseteq I^{b+1}$, with $j_1 \geq 0$. Then $T_{g(f)l+1}^{(j_1+1)} f \supseteq c_1 \cdot c_2 \cdot R_{X}^{\varepsilon_p}$.

2. Suppose $T_{g(f)}(l) \supseteq I_d \cdot R_{X}^{\varepsilon_p}$ and $T_{g(l)}(l) \supseteq I_d \cdot R_{X}^{\varepsilon_p}$ for some $d_j \geq d \geq 0$ and $d_j > j \geq 1$. Then $T_{g(l)}(l) \supseteq I_d \cdot R_{X}^{\varepsilon_p}$.

By iterating this we get: $T_{g(l+k)} f \supseteq I_{d+k(d+1)} \cdot R_{X}^{\varepsilon_p}$ for each $k \geq 0$.

3. Suppose an ideal $a \subseteq R_X$ and a subset $b_Y \subseteq f \cap \alpha \subseteq R_X$ satisfy: $a^2 \subseteq a \cdot b_Y$. Then $a \cdot T_{g(f)} + T_{g(l)}(l) \supseteq b_Y \cdot a \cdot R_{X}^{\varepsilon_p} \supseteq a^2 \cdot R_{X}^{\varepsilon_p}$.

For a subset $b_Y \subseteq f \cap \alpha \subseteq R_X$ satisfying $a^2 \subseteq a \cdot C$, one has: $a \cdot m \cdot T_{g(f)} + T_{g(l)}(l) \supseteq b_Y \cdot a \cdot R_{X}^{\varepsilon_p} \supseteq a^2 \cdot R_{X}^{\varepsilon_p}$.

4. Take a subset $C \subseteq (y) \subseteq R_Y$. Then $C \cdot m \cdot T_{g(f)} + T_{g(l)}(l) \supseteq C \cdot a \cdot R_{X}^{\varepsilon_p}$.

In particular, $a \cdot f \cdot T_{g(l)}(l) \supseteq (a \cdot f \cdot R_Y) + a \cdot T_{g(l)}(l) \supseteq (a \cdot f \cdot R_Y) + a \cdot f \cdot R_{X}^{\varepsilon_p}$.

And also: $(a \cdot f \cdot R_Y) + a \cdot T_{g(l)}(l) \supseteq (a \cdot f \cdot R_Y) + a \cdot f \cdot R_{X}^{\varepsilon_p}$.

For any subset $b_Y \subseteq f \cap \alpha$, one has: $a \cdot m \cdot f \cdot T_{g(l)}(l) \supseteq b_Y \cdot a \cdot f \cdot R_{X}^{\varepsilon_p}$.

5. Suppose $(f) \subseteq I$ then $a \cdot f \cdot T_{g(l)}(l) \cap I_{j+2} \subseteq a \cdot f \cdot T_{g(l)}(l)$ for each $j \geq 1$.

If $(f) \subseteq I$ and $a \cdot f \cdot T_{g(l)}(l) \subseteq a \cdot f \cdot T_{g(l)}(l)$ then $a \cdot f \cdot T_{g(l)}(l) \subseteq a \cdot f \cdot T_{g(l)}(l)$ for each $j \geq 1$.

We remark that in part 2 the sequence $(d_i - i)$ is non-increasing and $d_i - i \geq \text{ord}(f)$.

Proof. All the proofs are direct verifications.

1. $c_1 \cdot c_2 \cdot R_{X}^{\varepsilon_p} \supseteq c_1 \cdot \left( T_{g(l)+1} f + f \cdot (b_{j_2}) \cdot T_{g(f)} \right) \subseteq T_{g(l)+1} f + f \cdot (b_{j_2}) \cdot \left( T_{g(l)+1} f + f \cdot (b_{j_2}) \cdot \left( T_{g(l)+1} f + f \cdot (b_{j_2}) \cdot T_{g(l)+1} f \right) \right)$.

2. $I_d \cdot I_{d+1} \cdot R_{X}^{\varepsilon_p} \subseteq I_d \cdot I_{d+1} \cdot \left( T_{g(l)+1} f + f \cdot (f) R_{X}^{\varepsilon_p} \right) \subseteq T_{g(l)+1} f + f \cdot (f) R_{X}^{\varepsilon_p} \subseteq T_{g(l)+1} f + f \cdot (f) T_{g(l)+1} f$.

Example 6.13. i. Part 1 gives: $a_{f}^2 \subseteq (a_{f}^2 \cdot (a_{f} \cdot I_{j+2} \cap I_{j+1})) \subseteq (a_{f} \cdot I_{j+1})$.

ii. Let $I = m$ and $a \subseteq T_{g(f)}(l)$ and $(f) \subseteq a$. Then $a^2 \cdot R_{X}^{\varepsilon_p} \subseteq I \cdot m \cdot f \cdot T_{g(f)}(l)$.

6.5. Filtration closures, $T_{g(f)}(l)$ vs $T_{g(f)}$. The tangent spaces $T_{g(f)} f, T_{g(f)}$ are filtration-closed, by lemma 5.6. As was explained in 5.6.1, the methods of that lemma are not directly applicable to $g$-equivalence. We give the $\omega$-version. Fix some $j \geq -1$.

Proposition 6.14. 1. $a_{f} \cdot T_{g(f)}(l) \supseteq a_{f} \cdot T_{g(f)}(l)$.

2. Suppose either $R_X = \mathbb{k}[x]/f$ or $f$ is of finite singularity type, 3.6.2. Then $T_{g(f)}(l) f = \overline{T_{g(f)}(l) f}$.

Proof. 1. Observe the obvious inclusions: $a_{f} \cdot T_{g(f)}(l) f \subseteq a_{f} \cdot T_{g(f)}(l) f \subseteq a_{f} \cdot T_{g(f)}(l) f$. Note that $a_{f} \cdot T_{g(f)}(l) f \subseteq R_{X}^{\varepsilon_p}$ is an $R_X$-module. Therefore it is closed (lemma 5.6), $a_{f} \cdot T_{g(f)}(l) f = a_{f} \cdot T_{g(f)}(l) f$. Hence $a_{f} \cdot T_{g(f)}(l) f = a_{f} \cdot T_{g(f)}(l) f$.

2. The case $R_X = \mathbb{k}[x]/f$ is trivial. Suppose $f$ is of finite singularity type.

We have the $(f \cdot (R_Y) + a_{f})$-submodule $T_{g(f)} f \subseteq R_{X}^{\varepsilon_p}$, by lemma 5.6. It satisfies: $T_{g(f)} + f \cdot R_{X}^{\varepsilon_p} \supseteq T_{g(f)} f$ for any $d \gg 1$. Therefore $T_{g(f)} + f \cdot R_{X}^{\varepsilon_p} \supseteq T_{g(f)} f$. 

The ring \( \mathfrak{f}(R_Y) + \mathfrak{a}_\mathfrak{f} \) is local Noetherian (lemma 6.4), therefore \( T_{\mathfrak{f}}f \) is f.g. over this ring (2.1 iv).

As \( f \) is of finite singularity type, \( \sqrt{\mathfrak{a}_\mathfrak{f}} = \mathfrak{m} \supset I \). For \( d \gg 1 \) one has \( I^d \subseteq (f) + \mathfrak{a}_\mathfrak{f} \), by property 6.2 iv. By lemma 6.11 one has: \( \{(f\#(y) + \mathfrak{a}_\mathfrak{f})^d \cdot R_{X_Y}^{2p} \} \cap T_{\mathfrak{f}}f \subseteq (f\#(y) + \mathfrak{a}_\mathfrak{f}) \cdot T_{\mathfrak{f}}f \). Thus we get \( T_{\mathfrak{f}}f + (f\#(y) + \mathfrak{a}_\mathfrak{f}) \cdot T_{\mathfrak{f}}f \supseteq T_{\mathfrak{f}}f \). Apply Nakayama to \( T_{\mathfrak{f}}f \) as a f.g. module over \( f\#(R_Y) + \mathfrak{a}_\mathfrak{f} \).

**Example 7.2.** Take a map \( f : X \to (k^p, o) \). If its critical locus, \( \text{Crit}(f) \), see 3.3.6 is a point, i.e. \( \sqrt{\mathfrak{a}_\mathfrak{f}} = \mathfrak{m} \), then already theorem 1.1 gives a good control on the orbit, e.g. \( \mathcal{R}f \supset \{f\} + \mathfrak{a}_\mathfrak{f}^2 \cdot R_{X_Y}^{2p} \). For \( p \geq 2 \) and \( f \) not a submersion the critical locus is always of positive dimension. And “in most cases” the instability locus, \( V(\mathfrak{a}_\mathfrak{f}) \), is of dimension smaller than \( \text{Crit}(f) \). We adopt the following assumptions.

**Assumptions 7.3.**

i. Take an ideal \( \mathfrak{m} \cdot \mathfrak{a}_\mathfrak{f} \subseteq \mathfrak{a} \subseteq \mathfrak{m} \), so that \( V(\mathfrak{a}) \subseteq \text{Crit}(f) \). Suppose either \( \text{Crit}(f) = V(\mathfrak{m}) \subset X \) or the locus \( V(\mathfrak{a}) \subset \text{Crit}(f) \) contains no irreducible component of \( \text{Crit}(f) \subset X \).

ii. If \( J \neq 0 \) and \( \mathfrak{Z} \supset \mathfrak{J} \) then we assume the jet_0-condition of (2.5).

Moreover, for the ring \( R_X = k[x]/J \) (with \( J \neq 0 \)) the statements are for the \( \mathfrak{m} \)-adic closure, \( \mathcal{J} \mathfrak{f} \supset \{f\} \cdots \).

Recall the tangent space of the left equivalence, \( T_{\mathcal{J}f}f = f^\ast((y)^2 \cdot R_{Y_Y}^{2p}) \), see 3.3.2.

**Theorem 7.4.** Suppose \( \mathfrak{a}_\mathfrak{f}^2 \cdot R_{X_Y}^{2p} \subseteq \mathfrak{a} \cdot \mathfrak{m} \cdot T_{\mathfrak{f}}f + T_{\mathfrak{f}}f \).

a. Then \( \mathcal{J}f \supset \{f\} \cdot (\mathfrak{m}_k + \mathfrak{a}_\mathfrak{f}^2 \cdot (f)) \cdot \mathfrak{a}_\mathfrak{f} \cdot T_{\mathfrak{f}}f + T_{\mathfrak{f}}f \).

b. Suppose the field \( k_{f_{\mathfrak{m}_k}} \) is of characteristic zero or is algebraically closed. Then \( \mathcal{J}f \supset \{f\} + \mathfrak{a}_\mathfrak{f}^2 \cdot R_{X_Y}^{2p} + T_{\mathfrak{f}}f \).
Step 1. (Preparations)

i. Consider the $R_X$-module $M_i := \mathfrak{a}^2 \cdot R_X^p + \mathfrak{a} \cdot T_{\mathfrak{a}} f + T_{\mathfrak{a}'}(1)f$. Take its annihilator ideal $\mathfrak{a}_M := \text{Ann}(M) \subset R_X$. We have $\mathfrak{a}_M \cdot M = 0$, thus $\mathfrak{a}_M \supseteq \mathfrak{a}$. Geometrically: $\text{Supp}(\mathfrak{a}_M) \subseteq \text{Crit}(f) = V(\mathfrak{a}_M) \subset X$.

We claim: $\sqrt{\mathfrak{a}_M} = \sqrt{\mathfrak{a}}$. This is obvious for $\mathfrak{a}_M = \mathfrak{a}$ (i.e. $\text{Crit}(f) = V(\mathfrak{a}_M) = o \subset X$). Thus we assume: dim $\text{Crit}(f) > 0$ and $V(\mathfrak{a})$ contains no irreducible component of $\text{Crit}(f)$. Obviously $\text{Supp}(\mathfrak{a}_M) \cap V(\mathfrak{a}) = V(v(\mathfrak{a}_M) \setminus V(\mathfrak{a})$. (In fact, over $X \setminus V(\mathfrak{a})$ one has: $\mathfrak{a} \equiv T_{\mathfrak{a}}^d f$.) And $\text{Supp}(\mathfrak{a}_M) \subset X$ is closed. Therefore $\text{Supp}(\mathfrak{a}_M) = \text{Crit}(f)$, i.e. $\sqrt{\mathfrak{a}_M} = \sqrt{\mathfrak{a}}$.

ii. Consider $M$ as an $R_Y$-module. By our assumption $M$ is a submodule of $T_{\mathfrak{a}'}(1)f + \mathfrak{a} \cdot T_{\mathfrak{a}} f$. The later module is finitely generated over $R_Y$. Therefore $M$ is f.g. over $R_Y$, $\mathfrak{a}_M$. Each $x_i \subset R_Y$ acts on $M$ as an $R_Y$-linear operator. And its action is filtration-nilpotent, i.e. $x_i^d \cdot M \subseteq f^d y \cdot M$ for $d \gg 1$. Therefore the characteristic polynomial of $x_i$ is of the form

$$\chi_{x_i}(t) := \det [tI - x_i] = t^{d_i} + y(\ldots) \in R_Y[t].$$

One has $\chi_{x_i}(x_i) = 0 \subset M$, by Cayley-Hamilton theorem. (Hence $x_i^d \in (f) + \mathfrak{a}_M$.) Therefore we can apply the Weierstraß division, Theorem 3.1 vii, and present $R_X = \mathfrak{a}_M + \text{Span}_{R_Y}[v_\bullet]$. Here $\{v_\bullet\}$ is a finite tuple whose image generates the $R_Y$-module $R_X/\mathfrak{a}_M$.

We take $v_0 = 1$ and $v_{d_i} \in \mathfrak{m}$.

iii. We claim: $\{f\} + \mathfrak{c} \cdot R_X^p + T_{\mathfrak{a}'}(1) f \subseteq \mathcal{L} \{\{f\} + c \cdot R_X^p\}$, for any ideal $\mathfrak{c} \subset R_X$. Indeed, given an element $g(y) \in T_{\mathfrak{a}'}(1)$, define the automorphism $\Phi_Y(\mathfrak{c}) \in L(1)$ by $\Phi_Y(\mathfrak{c}) = y + q(y)$. Then $\Phi_Y^1(\{f\} + \mathfrak{c} \cdot R_X^p + q(f)) \subseteq \{f\} + \mathfrak{c} \cdot R_X^p$. In addition $\mathcal{R}_f \supseteq \{f\} + \mathfrak{m}^2 \cdot \mathfrak{a}_M \cdot R_X^p$, by Theorem 4.1. In view of the previous observations, it is enough to prove (for part b.):}

$$\{f\} + \mathfrak{a}^2 \cdot R_X^p \subseteq \mathcal{A} \{\{f\} + \mathfrak{a}_M \cdot R_X^p + T_{\mathfrak{a}'}(1)f\} \quad \text{for} \quad d \gg 1.$$

From now and until Step 5, we assume $J = 0$, i.e. $R_X$ is one of the rings $k[[x]], k\{x\}, k(x)$.

Step 2. (Reduction of the condition 17 to a system of implicit function equations)

As in the $\mathcal{R}, \mathcal{K}$-cases we fix some generators $\{a_{M}^{I}\}$ of $\mathfrak{a}_M$ and $\{\xi_i(f)\}$ of $\mathfrak{a} \cdot T_{\mathfrak{a}} f$ (as $R_X$-modules). Recall the expansion $R_X = \mathfrak{a}_M + \text{Span}_{R_Y}[v_\bullet]$ of Step 1 ii.

i. (The first attempt.) Take a perturbation $g \in \mathfrak{a}^2 \cdot R_X^p$. Expand it, $g - \sum c_i \xi_i(f) \in T_{\mathfrak{a}'}(1)$, here $c_i^d \in \mathfrak{m}$. Accordingly define the coordinate change $\Phi_X \in \mathcal{R}_f$ by $x \to x + \sum c_i \xi_i(x)$, here $\{c_i\} \in R_X$ are unknowns. As in the proofs of the $\mathcal{R}, \mathcal{K}$-cases we have:

$$\Phi_X(f) = f - \sum c_i \cdot \xi_i(f) \in \mathfrak{a}^2 \cdot \{c_i\}^2 \cdot R_X^p, \quad \Phi_X(g) = g - \{a \cdot \{c_s\} \cdot f^2 \cdot a^2 \cdot (a^2)^n \{c_s\}^2\} \cdot R_X^p.$$

We cannot use the assumption $\mathfrak{a}^2 R_X^p \subseteq \mathfrak{m} \cdot \mathfrak{a} \cdot T_{\mathfrak{a}} f + T_{\mathfrak{a}'}(1)f$ directly, as $\mathfrak{m} \cdot \mathfrak{a} \cdot T_{\mathfrak{a}} f + T_{\mathfrak{a}'}(1)f$ is not an $R_X$-module. Instead we expand $c_i = \sum c_{i,l} t_l + \sum_m c_{i,m} a_{M,m}$, with the unknowns $c_{i,l} \in R_Y$, $c_{i,m} \in R_X$. (Using $R_X = \mathfrak{a}_M + \text{Span}_{R_Y}[v_\bullet]$.) Similarly one expands $\{c_i\}$. The total expression is:

$$\Phi_X(f + g) = f + f^\#(y)^2 \cdot H(\{c_s\}) +$$

$$+ \sum l \left[ \sum v_l(c_{i,l} + c_{i,d}) + \sum_{m} a_{M,m}(c_{i,m} + c_{i,m}^2) + Q_i(c_s) + \mathcal{H}_i(\{c_s\}, \{c_s\}) \right] \xi_i(f).$$
Here $Q_i(z) \in (z)^2 \cdot R_Y[[z]]$, $H_i(z,w) \in (z,w)^2 \cdot R_Y[[z,w]]$ and $H \in T_{\mathcal{Y}}f$.

We would like to get rid of the term $\ldots$, i.e. the coefficient of $\xi_j(f)$. This cannot be done directly. Instead we will force this coefficient to belong to $a_M^f$ for $d \gg 1$. This will ensure the condition \cite{17}.

ii. \textit{(The actual reduction.)} Iterate the expansion $R_X = a_M + \text{Span}_{R_Y}\{v_i\}$ to get $R_X = a_M^f + \sum_{d=0}^{d-1} a_M^{f_d} \cdot \text{Span}_{R_Y}\{v_i\}$, for any $d \geq 1$. Thus $R_X/a_M^f$ is a f.g. $R_Y$-module. Fix some generators, $R_X = a_M^f + \text{Span}_{R_Y}\{v_i\}$. As before, we take $v_0 = 1$ and $v_{i>0} \in M$. Moreover, we choose the generators in a filtered way: $v_0, \ldots, v_{i_j}$ generate $R_X/a_M^f$, then $v_0, \ldots, v_{i_j}, \ldots, v_{i_j}$ generate $R_X/a_M^f$, and so on.

As in Step 1.ii., the action $\mathfrak{m} \circ R_X/a_M^f$ is filtration-nilpotent, i.e. $\mathfrak{m}^N \subseteq (y) + a_M^f$ for $N \gg 1$.

Expand $c_i = \sum_{i,l} v_i c_{i,l}$, where $c_{i,l} \in \mathcal{Y}$ are the new unknowns.

Repeat the procedure of Step 2.i, then equation \cite{49} becomes:

$$\Phi_X(f + g) - f - \sum_{i,l} v_i \cdot [c_{i,l} + c_{i,l}^g + Q_{i,l}(\{c_{ss}\})] \cdot \xi_j(f) \in a_M^f \cdot R_X^{\mathcal{Y}} + T_{\mathcal{Y}(1)}f.$$  \hfill (50)

Here $Q_{i,l}(z) \in (z) \cdot R_Y[y \cdot z][z]$, resp. $(z) \cdot R_Y[y \cdot z][z]$, resp. $(z) \cdot R_Y[y \cdot z][z]$, rather than just $Q_{i,l}(z) \in (z) \cdot R_Y[[z]]$.

As $v_0 = 1$ and $v_{i>0} \in \mathfrak{m}$ one has here: $c_{i,0}^g \in (y)$ and $Q_{i,0}(z) \in (y) \cdot (z)$.

We get the (finite) system of implicit function equations over the ring $R_Y$:

$$c_{i,l} + c_{i,l}^g + Q_{i,l}(\{c_{ss}\}) = 0, \quad \forall i,l.$$  \hfill (51)

Once they are satisfied, equation \cite{51} will imply equation \cite{47}.

Unlike the $\mathfrak{R}, \mathcal{X}$-cases we cannot immediately apply the $IFT_1$ of \cite{23} ii. In fact the coefficients $c_{i,l}^g$ do not necessarily belong to the maximal ideal $(y) + \mathfrak{m}_k \subseteq \mathcal{Y}$. Neither can one assume that $Q_{i,l}(z)$ is nilpotent in some sense, e.g. $Q_{i,l}(z) \notin (f^R(y) + \mathfrak{m}_k) \cdot (z)$.

In Step 3 we prove part a., there the system \cite{51} is replaced by a simpler system. To prove part b. (in Step 4) we introduce the indeterminate $t$, and get an “arc-solution”. Then we extend this arc-solution to an ordinary solution.

Step 3. (The proof of part a. in the case $J = 0$.) The system \cite{51} was obtained for the perturbation $g \in a^2 \cdot R_X^{\mathcal{Y}}$. To prove part a. we start from the perturbation $g \in (\mathfrak{m}_k + (f) + a^2) \cdot a^2 \cdot R_X^{\mathcal{Y}}$. Observe:

$$a_M^f = (\mathfrak{m}_k + (f) + a^2) \cdot a^2 \cdot R_X^{\mathcal{Y}} \subseteq (\mathfrak{m}_k + (f) + a^2) \cdot a \cdot \mathfrak{m} \cdot T_{\mathfrak{R}}f + T_{\mathcal{Y}(1)}f.$$  \hfill (52)

Thus we fix some generators $\{\xi_i\}$ of the $R_X$-module $(\mathfrak{m}_k + (f) + a^2) \cdot a \cdot T_{\mathfrak{R}}f$. Define $\Phi_X \in \mathfrak{R}$ by $x \mapsto x + \sum c_l\xi_l(x)$. We get:

$$\Phi_X(f + g) - f - \sum_{i,l} c_l\xi_l(f) \in (\mathfrak{m}_k + (f) + a^2)^2 \cdot a^2 \cdot \{c_{ss}\}^2 +$$

$$+ (\mathfrak{m}_k + (f) + a^2) \cdot a^2 \cdot ((\mathfrak{m}_k + (f) + a^2) a^2') \{c_{ss}\} \subseteq (\mathfrak{m}_k + (f) + a^4) \cdot a \cdot \mathfrak{m} \cdot T_{\mathfrak{R}}f + T_{\mathcal{Y}(1)}f.$$  \hfill (53)

Expand $c_i = \sum_{i,l} v_i c_{i,l}$ and note that the ideal $(\mathfrak{m}_k + (f) + a^4)$ acts nilpotently on the filtration of $\{v_i\}$.

We get again equation \cite{51}. However, now $Q_{i,l}(z) \in (\mathfrak{m}_k + (y)) + \text{Nilp}$, where the (non-linear) operator $\text{Nilp}$ is filtration-nilpotent on $\{v_i\}$. And this system is indeed resolvable by the $IFT_1$.

Once it is resolved, we have the transition $\{f + a^2 \cdot R_X^{\mathcal{Y}} \cdot (f) + a_M^f \cdot R_X^{\mathcal{Y}} + T_{\mathcal{Y}(1)}f$. By Step 1.iii. this implies the statement a.

Step 4. (The proof of part b. in the case $J = 0$.)

i. Extend the base ring $k$ to $k[[t]]$, resp. $k\{t\}$, resp. $k\{t\}$, for an indeterminate $t$. Instead of $f + g$ take $f + t g$. Accordingly define the coordinate change $\Phi_X$ by $x \mapsto x + t \sum c_{i,l}\xi_l(x)$. Repeat Step 2.ii up to equation \cite{50}:

$$\Phi_X(f + g) - f - t \sum_{i,l} v_i \cdot [c_{i,l} + c_{i,l}^g + t \cdot Q_{i,l}(\{c_{ss}\})] \xi_j(f) \in a_M^f \cdot R_X^{\mathcal{Y}} + t \cdot \{c_{ss}\}^2 \cdot T_{\mathcal{Y}(1)}f.$$  \hfill (54)

Then the system \cite{51} becomes $c_{i,l} + c_{i,l}^g + t \cdot Q_{i,l}(\{c_{ss}\}) = 0$.

Applying $IFT_1$ over the ring $k[[y, t]]$, resp. $k\{y, t\}$, resp. $k\{y, t\}$, we get the solution $\{c_{i,l}(y, t)\}$. As $c_{i,0}^g \in (y)$ and $Q_{i,0}(z) \in (y) \cdot (z)$, we get: $c_{i,0}(y, t) \in (y)$. 


For this solution we define the coordinate change $\Phi_X : \mathcal{X} \to \mathbb{A}$ by $x \to x + t \sum c_{i,t} \cdot v_i \cdot \xi_i(x)$. Observe: this coordinate change is unipotent. Indeed, $\xi(x) \in \mathbb{A} \cdot T_{\mathcal{X}}(x) = a \subseteq m$ and $c_{i,0}(y) \in (y)$ and $v_{i,t} > 0 \in m$.

Altogether, we have proved: $\Phi_X(f + tg) \in \{f\} + a_M^2 \cdot R_{\mathcal{X}}^{\geq p} + T_{\mathcal{X}}(1)f$, for an indeterminate $t$ and any $d \gg 1$. Now part iii. of Step 1 ensures: $f + t \cdot g \not\in f$.

An aside remark, taking $t = m_k + (y)$ gives $\{f\} + a^2 \cdot (m_k + (y)) \cdot R_{\mathcal{X}}^{\geq p} \subseteq \mathcal{X}f$, which is a weaker version of part a.

From now on we assume: the field $k/m_k$ is of characteristic zero or algebraically closed. Moreover, we assume $\{Q_{*}(z)\} \not\in (m_k + (y)) \cdot (z)$. (Otherwise the system [31] is directly resolvable.)

ii. We prove: $f + tg \not\in f$ for an infinite set of values of $t \in k$. We still have to resolve the system $c_{i,t} + c_{i,t}^2 + t \cdot Q_{i,t}(c_{*}) = 0$, now for a fixed $t \in k$. Here $Q_{i,t}(z)$ are power series in $z$, not polynomials. However, it is enough to resolve this system over the quotient ring $R_{\mathcal{X}}(y) + m_k$. Indeed, $Q_{i,t} \in R_{\mathcal{X}}[y,z][z]$ (resp. analytic, resp. algebraic). And if $\{c_{*}\}$ is a solution $mod(y + m_k)$, then by shifting the variables, $c_{*} \sim c_{*} - c_{**}$, we get the system of $\text{IFT}\_1$-type, $c_{i,t} = Q_{i,t}(c_{*})$, where now $Q_{i,t}(z) \in ((y) + m_k) \cdot (z)$. (This later system is readily resolvable.)

Over the ring $R_{\mathcal{X}}(y) + m_k$ we get a polynomial system in variables $\{c_{i,t}\}$, over the (infinite) field $k/m_k$. This system defines a closed algebraic subscheme of a finite dimensional affine space $\mathcal{X}^{\mathcal{X}}_{k/m_k}$. In Step 4i we have constructed an arc on this subscheme, $\{c_{i,t}\}$. This arc is non-trivial (i.e. not a point) as $\{Q_{*}(z)\} \not\in m_k + (y)$. Thus the set of $k/m_k$-points of this (closed, affine, algebraic) subscheme is of positive dimension.

Moreover, the projection of this scheme onto the line $\mathcal{X}^1$ has image of positive dimension. Hence, as the field $k/m_k$ is infinite, the system is solvable for an infinite number of $t$-values. Altogether, $f + tg \not\in f$ for an infinite set of values of $t \in k$.

iii. Finally we deduce: $f + tg \not\in f$ for any $t \in k$. As in Step 4.ii, it is enough to resolve the system over the ring $R_{\mathcal{X}}(y) + m_k$.

Through the whole proof we have used only the unipotent subgroup $\mathcal{X}^{\mathcal{X}} < \mathcal{X}$, for the filtration $m^* \subset R_{\mathcal{X}}$. Passing to the quotient $R_{\mathcal{X}}/m_k + m^d$ for $d \gg 1$, we get the action on the finite dimensional vector space, $\mathcal{X}^{\mathcal{X}} \cap (m \cdot R_{\mathcal{X}}/m_k + m^d)^{\geq p}$. This action is algebraic and unipotent.

If the field $k/m_k$ is algebraically closed then the group-orbits are Zariski-closed, [77]. The intersection of such an orbit with a line, $\mathcal{X}^{\mathcal{X}}(f) \cap \{f + tg\}_{t}$, is either finite, or contains the whole line. In our case this intersection is infinite, thus $\mathcal{X}^{\mathcal{X}}(f) \supseteq f + tg$ for each $t \in k$.

The case “$k/m_k$ is of characteristic zero” is done similarly, using lemma [71].

Step 5. (The case $J \neq 0$. We prove part b., as part a. is simpler.) In Steps 3,4 we have constructed the map $\Phi_X : x \to x + \sum c_i \xi_i(x)$ satisfying $\Phi_Y \circ f \circ \Phi_X = f + g$. For $J \neq 0$ this map is not a ring-automorphism. As in the $\mathcal{X}$-cases we adjust it by higher order terms. To use the jet-zero- assumption we expand further, $c_{i,t} = \sum y^m \cdot c_{i,mt}$, now $c_{i,mt} \in k$ are unknowns. Then we extend $\Phi_X$ to the automorphism $\Psi : x \to x + t \sum y^m \cdot c_{i,mt} \xi_i(x) + t^2 \varphi(\{c_{**}\})^2$. Here the entries of $\varphi(\{c_{**}\})^2$ are power series in $\{c_{**}\}$. Resolve the corresponding system with $t$-indeterminate. This is an arc solution on the variety $A^N_k$. Arguing as in Step 4.ii we get: $f + tg \not\in \mathcal{X}^{\mathcal{X}}(f)$. Proceed as in Step 4.iii to get: $f + g \in \mathcal{X}^{\mathcal{X}}(f) + a_M^2 \cdot R_{\mathcal{X}}^{\geq p}$. This holds for any $d \geq 1$. Therefore $f + g \in \mathcal{X}^{\mathcal{X}}(f)$. This proves the statement for the ring $k[[x]]/J$. For the ring $k(x)/J$ one uses the left-right Artin approximation, §2.2.3v.

Remark 7.5. The factor $m$ in the assumption $a^2 \cdot R_{\mathcal{X}}^{\geq p} \subseteq a \cdot m \cdot T_{\mathcal{X}}f + y^2 \cdot T_{\mathcal{X}}f$ is used only once in this whole proof, in Step 4.ii. It is needed to ensure that the coordinate change $\Phi_X : x \to x + t \sum c_i \xi_i(x)$ is unipotent. But this unipotence holds trivially if one assumes $a \subseteq m^2$. Thus (almost) the same proof leads to the modified part b.: If $a \subseteq m^2$ and $a^2 \cdot R_{\mathcal{X}}^{\geq p} \subseteq a \cdot T_{\mathcal{X}}f + T_{\mathcal{X}}(f)$ then $\{f\} + a^2 \cdot R_{\mathcal{X}}^{\geq p} + T_{\mathcal{X}}(1)f \subseteq \mathcal{X}f$.

7.3. Examples and Corollaries. Let $R_{\mathcal{X}}$ be one of $k[[x]]/J$, $k[x]/J$, $k(x)/J$, where $k$ is a local ring (e.g. any field), §2.11.ii. Suppose the field $k/m_k$ is algebraically closed or of characteristic zero. Take the assumptions
7.3. We use the filtration $\mathfrak{m}^* \cdot R_X^\mathbb{Q}$ and the corresponding tangent spaces $T_{f_\mathcal{H}(j)}, T_{f_{\mathcal{K}(j)}}, T_{f_{\mathfrak{A}(j)}},$ see \[333\]

7.3.1. The case $p = 1$, i.e. the scalar valued functions. Let $f : X \to \mathbb{k}^1, o$ and suppose $a_{f} \subseteq m^2$. Then $a_{f} \not\supseteq \{f\} + a_{f}^2 + \text{Span}_{\mathbb{k}}(f^2, f^3, f^4, \ldots)$.

Proof. Combine lema \[6.1\] with remark \[7.5\] ■

7.3.2. Maps whose $\mathcal{A}$ and $\mathcal{K}$ orbits are “close”.

i. Suppose $a_{f_{\mathcal{O}(o)}} \supseteq \mathfrak{m} \cdot \{f\}$, equivalently $T_{f_{\mathcal{O}(o)}} f = T_{\mathcal{K}(o)} f$, equivalently $a_{f_{\mathcal{O}(o)}} = a_{f_{\mathcal{O}(o)}}$. (This holds, e.g. for infinitesimally-stable maps, see Property \[6.2\] ii.) Then

$$a_{f_{\mathcal{O}(o)}}^2 R_X^\mathbb{Q} \subseteq a_{f_{\mathcal{O}(o)}} \cdot T_{f_{\mathcal{O}(o)}} f + a_{f_{\mathcal{O}(o)}} \cdot T_{f_{\mathcal{K}(o)}} f \subseteq a_{f_{\mathcal{O}(o)}} \cdot T_{f_{\mathcal{O}(o)}} f + T_{f_{\mathcal{K}(o)}} f.$$  \[55\]

Part b. of Theorem \[7.4\] gives (for $a = a_{f_{\mathcal{O}(o)}}$): $\{f\} + a_{f_{\mathcal{O}(o)}}^2 \cdot R_X^\mathbb{Q} \subseteq a_{f}$. This is close to Mather’s statement “Stable maps are determined by their local algebras”.

Note: it is much simpler to compute/to control the ideal $a_{f_{\mathcal{O}(o)}}$ than $a_{f_{\mathcal{O}(o)}}$.

ii. As a particular case, suppose $J = 0$ and take a stable map. It is a versal unfolding of its genotype, i.e. $f : (\mathbb{k}^{n+1}, o) \to (\mathbb{k}^{p+1}, o)$ with $f = (f_o + \sum v_j u_j, u)$, where $f_o : (\mathbb{k}^n, o) \to (\mathbb{k}^p, o)$ and $\text{Span}_{\mathbb{k}}\{v_\bullet\} = \mathfrak{m} \cdot T_{f_o} f_o$. (Thus $\tau = \dim T_{f_o} f_o$.) For $k = \mathbb{R}, \mathbb{C}$ this is the classical Mather’s theorem, [Mon. Nuñ.-Bal., Theorem 7.2], for an arbitrary field see \[Kerne\], [22], in most cases the ideal $a_{f_{\mathcal{O}(o)}}$ is larger than $\mathfrak{m}^{l+1}$. (Even the asymptotics is better.)

iii. More generally, for $J = 0$, suppose $a_{f_{\mathcal{O}(o)}} \supseteq \mathfrak{m} \cdot \{f\}^j$ for some $j \geq 0$. (Thus $a_{f_{\mathcal{O}(o)}} = a_{f_{\mathcal{O}(o)}}$.) Then $a_{f_{\mathcal{O}(o)}}^2 \cdot R_X^\mathbb{Q} \subseteq a_{f_{\mathcal{O}(o)}}^2 \cdot T_{f_{\mathcal{O}(o)}} f + T_{f_{\mathcal{O}(o)}} f$. \[56\]

Therefore $\{f\} + a_{f_{\mathcal{O}(o)}}^2 \cdot R_X^\mathbb{Q} \subseteq a_{f}$. Yet more generally, it is enough to assume $(f)_{j+1} \cdot a_{f_{\mathcal{O}(o)}} \cdot R_X^\mathbb{Q} \subseteq a_{f_{\mathcal{O}(o)}} \cdot T_{f_{\mathcal{O}(o)}} f + T_{f_{\mathcal{O}(o)}} f$ for some $j \geq 0$.

7.3.3. The main assumption of theorem \[7.4\] is: $a_{f}^2 \cdot R_X^\mathbb{Q} \subseteq \mathfrak{a} \cdot \mathfrak{m} \cdot T_{f} f + T_{f} f$. This can be difficult to verify.

Corollary 7.6. (A weaker but simpler version)

1. Suppose $\mathfrak{b}_Y^2 \subseteq \mathfrak{b}_Y \cdot a_{f_{\mathcal{O}(o)}}$ for an ideal $\mathfrak{b}_Y \subseteq f^\#(y)$. Then $\{f\} + (\mathfrak{b}_Y + a_{f_{\mathcal{O}(o)}}) \cdot R_X^\mathbb{Q} \subseteq a_{f}$. Moreover, if the field $\mathbb{k}/\mathfrak{m}$ is algebraically closed or of characteristic zero then $\{f\} + (\mathfrak{b}_Y + a_{f_{\mathcal{O}(o)}}) \cdot R_X^\mathbb{Q} + T_{f} f \subseteq a_{f}$.

2. Suppose $\mathfrak{b}_Y \subseteq \mathfrak{b}_Y \cdot a_{f}$ for an ideal $\mathfrak{b}_Y \subseteq f^\#(y)^2$. Suppose $a_{f} \subseteq m^2$. Then $\{f\} + (\mathfrak{b}_Y + a_{f})^2 \cdot R_X^\mathbb{Q} + T_{f} f \subseteq a_{f}$. Moreover, if the field $\mathbb{k}/\mathfrak{m}$ is algebraically closed or of characteristic zero then $\{f\} + (\mathfrak{b}_Y + a_{f})^2 \cdot R_X^\mathbb{Q} + T_{f} f \subseteq a_{f}$.

Proof.

1. It is enough to observe: $(\mathfrak{b}_Y + a_{f_{\mathcal{O}(o)}})^2 \cdot R_X^\mathbb{Q} \subseteq (\mathfrak{b}_Y + a_{f_{\mathcal{O}(o)}}) \cdot T_{f_{\mathcal{O}(o)}} f + \mathfrak{b}_Y^2 \cdot R_X^\mathbb{Q} \subseteq (\mathfrak{b}_Y + a_{f_{\mathcal{O}(o)}}) \cdot T_{f_{\mathcal{O}(o)}} f + \mathfrak{b}_Y \cdot T_{f} f$. Then we apply theorem \[7.4\]

2. The same proof, just use remark \[7.5\] ■

Example 7.7. (Assuming the field $\mathbb{k}/\mathfrak{m}$ is algebraically closed or of characteristic zero)

i. Take $\mathfrak{b}_Y = a_{f_{\mathcal{O}(o)}} \cap f^\#(y)$ to get $a_{f} \subseteq \{f\} + (a_{f_{\mathcal{O}(o)}} \cap f^\#(y)) \cdot R_X^\mathbb{Q} + T_{f} f$.

Suppose $a_{f} \subseteq m^2$. Then part two gives: $a_{f} \subseteq \{f\} + (a_{f} \cap f^\#(y)) \cdot R_X^\mathbb{Q} + T_{f} f$.

More generally, if $\mathfrak{b}_Y^2 \subseteq \mathfrak{b}_Y \cdot a_{f_{\mathcal{O}(o)}}$ then the ideal $\mathfrak{b}_Y$ satisfies this condition as well.

ii. Suppose $a_{f} \supseteq \mathfrak{m} \cdot \{f\}$, then we have $a_{f} \cap f^\#(y) \supseteq f^\#(y)^d$. Assuming $a_{f} \subset m^2$ we get $\{f\} + (f^d + a_{f})^2 \cdot R_X^\mathbb{Q} + T_{f} f \subseteq a_{f}$.

We remark: $(f^d + a_{f})^2 = \sqrt{a_{f}}$, thus in this case the $(a_{f})$ instability and the $(\mathcal{K})$ singularity loci coincide.

See \[7.5\] for the geometric interpretation.
7.3.4. Computational cases.

i. Define the map \( f : (k^2, o) \to (k^3, o) \) by \( f(x_1, x_2) = (x_1, x_2^2, x_2^3 + x_2^{k+1} x_2) \). Suppose 2, 3, \( k + 1 \in \mathbb{K} \).

By the direct check: \( a_{dr} = (x_1^2, x_2^2) \subset R_X \) and \( a_{dr}^2 \cdot R_X^{\ast p} \subset m \cdot a_{dr} \cdot T_{\ast f} + T_{\ast f} \). Thus \( a_{dr} \) holds for \( \{ f \} + (x_1^2, x_2^2)^2 \cdot R_X^{\ast p} + T_{\ast f (l)} f \). Compare this to the classical criterion: \( m^{k+1} \subset a_{dr} \) and therefore \( a_{dr} \) to \( \{ f \} + m^{2k+2} \cdot R_X^{\ast p} \). [Mon. Num.-Bal] Proposition 6.l, pg. 197).

ii. Define the map \( f : (k^2, o) \to (k^3, o) \) by \( f(x_1, x_2) = (x_1, x_2^2, x_1 x_2^{k}) \), with \( k \)-odd. Suppose 2, \( k \in \mathbb{K} \). Note that \( f \# (y) \supset (x_1 x_2^{-k}) \subset R_X \). By the direct check: \( a_{dr} = (x_2^2, x_1 x_2^{k-1}) \subset R_X \). This is a non-isolated instability.

By the direct check: \( a_{dr}^2 \cdot R_X^{\ast p} \subset m \cdot a_{dr} \cdot T_{\ast f} + T_{\ast f} \). Thus \( a_{dr} \) holds for \( \{ f \} + (x_2^2, x_2^{k-1})^2 \cdot R_X^{\ast p} + T_{\ast f (l)} f \).

7.4. The filtration criterion for \( \mathcal{A} \)-orbits. Take an ideal \( I \subset R_X \) and the filtration \( M_n := I^n \cdot R_X^{\ast p} \).

**Theorem 7.8.** Fix some integers \( 1 \leq j < d \). Assume one of the following:

i. either the conditions \( jet_N(Exp), jet_N(Ln) \) of (2.3) hold for all \( N \geq 1 \). (Thus in particular \( k \geq Q \).)

ii. or \( \{ f \} \subset I \) and the conditions \( jet_N(Exp), jet_N(Ln) \) hold for \( N = \lceil \frac{2d - 1 - ord(f)}{j} \rceil + 1 \). Then: \( a_{dr} \) holds for \( \{ f \} + I^d \cdot R_X^{\ast p} \) if and only if \( T_{a_{dr} (l)} f \subset I^d \cdot R_X^{\ast p} \).

**Proof.** The part \( \Leftarrow \).

**The case i.** Let \( w_d \in M_d \) then \( w_d \equiv \xi_Y (y) | f + \xi_X (f) \) mod \( M_d \) for some \( (\xi_Y, \xi_X) \in T_{a_{dr} (l)} f \). We can assume \( d' \gg d \). Using the \( jet_N(Exp) \)-assumption, for \( N = \lceil \frac{d - ord(f)}{j} \rceil + 1 \), we take the corresponding automorphisms \( jet_N(e^{-X}) \subset \mathcal{A}^{(l)}, jet_N(e^{-\xi}) \subset \mathcal{A}^{(l)} \).

By Remark 3.11, applied to \( M_d \) and \( a_{dr} \), we have: \( jet_N(e^{-X}) \cdot jet_N(e^{-\xi}) f - f - w_d \subset M_{d+t} \). Therefore \( a_{dr} f + M_{d+t} \supset \{ f \} + M_d \). Now apply the same argument to the pair \( (M_{d+j}, a_{dr} f) \), with \( jet_N(Exp) \)-assumption for \( N = \lceil \frac{j + 1 - ord(f)}{j} \rceil + 1 \). Iterate this to get \( a_{dr} f + M_{d+t} \supset \{ f \} + M_d \) for each \( d' \gg 1 \). Hence \( \{ f \} + M_d \subset a_{dr} f \).

**The case ii.** It is enough to prove: \( a_{dr} f + M_{d+1} \supset \{ f \} + M_k \) for each \( k \geq d \). For this we prove:

\[ a_{dr}^{(k+1)} f + M_{kd+l+1} \supset \{ f \} + M_{kd+l+1} \quad \text{for each} \quad k \geq 1 \quad \text{and each} \quad l = 0, \ldots, d - 1. \]

At each step work modulo \( M_d \) with \( d' \gg 1 \), thus we can replace \( T_{a_{dr} (l)} f \) by \( T_{a_{dr} (l)} f \).

By part one of Lemma 6.12 we get \( T_{a_{dr} (k+1)} f \subset M_{kd} \) for each \( k \geq 1 \).

Take any element \( w \in M_{kd+l+1} \) and present it as \( w = \xi_X (f) + \xi_Y (y) | f \in T_{a_{dr} (k+1)} f \).

Take the corresponding automorphisms \( jet_N(e^{-X}), jet_N(e^{-\xi}) \), ensured by the condition \( jet_N(Exp) \).

Apply Remark 3.11 to \( M_{kd+l+1} \) and \( a_{dr}^{(k+1)} f \) to get: \( jet_N(e^{-X}) \cdot jet_N(e^{-\xi}) f - f - w \subset M_{kd+l+1} + (k+1) \). This verifies the condition (57).

We should only justify the use of Remark 3.11 i.e. to verify the condition:

\[ \frac{k d + l - ord(f)}{k j + k - 1} + 1 \leq \lceil \frac{2d - 1 - ord(f)}{j} \rceil + 1 \quad \text{for all} \quad l = 0, \ldots, d - 1, \quad \text{and all} \quad k, j \geq 1. \]

Here the maximum of the left hand side is achieved for \( l = d - 1 \) and \( k = 1 \). And then we get the equality.

The part \( \Rightarrow \).

**The case i.** Take \( w_d \in M_d \) then \( f + w_d \equiv (\Phi_Y, \Phi_X) f \) mod \( M_d \) for some \( (\Phi_Y, \Phi_X) \in \mathcal{A}^{(l)} \). We can assume \( d' \gg d \). Using the \( jet_N(Exp) \)-assumptions we approximate \( (\Phi_Y, \Phi_X) \) by the elements \( jet_N(e^{-X}) \subset Aut_X^{(l)}, jet_N(e^{-\xi}) \subset Aut^{(l)} \) satisfying:

\[ \Phi_X - jet_N(e^{-X}) \subset End^{(l+1)}(R_X), \quad \Phi_Y - jet_N(e^{-\xi}) \subset End^{(l+1)}(R_Y). \]

Therefore \( jet_N(e^{-X}), jet_N(e^{-\xi}) f - f - w_d \subset M_d + M_{d+1} + ord(f) \). Here we can assume \( d' \geq j N + 1 + ord(f) \geq d + j \). Apply Remark 3.11 to the pair \( (M_d, a_{dr} (l)) \) to get: \( (\xi_X + \xi_Y) f - w_d \subset M_{d+j} \). Therefore \( T_{a_{dr} (l)} f + M_{d+j} \supset M_d \).

Now apply the same argument to the pair \( (M_{d+j}, a_{dr} (l)) \), and so on. One gets \( T_{a_{dr} (l)} f + M_d \supset M_d \) for each \( d' \). Hence the statement.
The case ii. Apply the argument of case i. to the pair \((M_d, \mathcal{A}(j))\) and \(N = \lceil \frac{d-\text{ord}(f)}{j} \rceil + 1\) to get: \(T_{\mathcal{A}(j)} f + M_{d+j} \supseteq M_d\). Iterate this argument for all pairs \((M_{d+i}, \mathcal{A}(j))\) and \(N = \lceil \frac{d+i-\text{ord}(f)}{j} \rceil + 1\), with \(i = 1, \ldots, d-1\). We reach \(T_{\mathcal{A}(j)} f + M_{2d} \supseteq M_d\), and for this we have used the \(j\ell\) assumptions with \(N = \lceil \frac{d-\text{ord}(f)}{j} \rceil + 1, \ldots, \lceil \frac{d-2}{j} \rceil + 1\).

And now observe:

\[
M_{2d} = I^d \cdot M_d \subseteq I^d(T_{\mathcal{A}(j)} f + M_{2d}) \subseteq I^d \cdot T_{\mathcal{A}(j)} f + f^\#(b_j)T_{\mathcal{A}(j)} f + M_{3d} \quad (j) \subseteq T_{\mathcal{A}(j)} f + M_{3d}.
\]

Iterate this to get \(T_{\mathcal{A}(j)} f + M_{d'} \supseteq M_d\) for all \(d'\). Hence the statement.

7.4.1. The conclusion of theorem [7.8] is for filtration closures, \(\mathcal{A}(j) f \subseteq T_{\mathcal{A}(j)} f\). With additional assumptions we get the statement \(\mathcal{A}(j) f = T_{\mathcal{A}(j)} f\).

Let \(R_X\) be one of \(k[[x]], k\{x\}, k(x)\), where the local ring \(k\) contains a field, \(2\) if \(j < d\). Suppose either \(\text{char}(k) = 0\) or \(\text{char}(k) > 2^{d-1-\text{ord}(f)}\). Take an ideal \((f) \subseteq I \subseteq m\).

Corollary 7.9. 1. Suppose \(f\) is \(\mathcal{K}\)-finite. If \(\mathcal{A}(j) f \supseteq \{f\} + I^d \cdot R_X^p\) then \(T_{\mathcal{A}(j)} f \supseteq I^d \cdot R_X^p\).

2. Suppose \(I \subseteq \sqrt{a_{\mathcal{A}} + (a_{\mathcal{A}} \cap f^\#(y))X}\). Then \(T_{\mathcal{A}(j)} f \supseteq \{f\} + I^d \cdot R_X^p\).

Geometrically the condition \(I \subseteq \sqrt{a_{\mathcal{A}} + (a_{\mathcal{A}} \cap f^\#(y))X}\) reads: \(V(I) \cap f^{-1}(f(V(a_{\mathcal{A}}))) \cap \text{Crit}(f) = \emptyset\).

For \(k(x)\) with \(k = \mathbb{R}, \mathbb{C}, f = m, j = 1\), this is Theorem 2.5 of [4,5,5,2,2,2,2,2].

Proof. W.l.o.g. we assume \(I \neq 0\), thus \(V(a_{\mathcal{A}}) \subseteq (k^n, o)\).

1. By theorem [7.8] we have \(T_{\mathcal{A}(j)} f \supseteq I^d \cdot R_X^p\). Thus \(T_{\mathcal{A}(j)} f \supseteq I^d \cdot R_X^p\), by part two of lemma 6.12.

2. By theorem [7.8] it is enough to show: \(\mathcal{A}(j) f \supseteq \{f\} + I^n \cdot R_X^p\) for some \(N > 1\). Part 2 of corollary 7.6 gives (for \(b_j = a_{\mathcal{A}} \cap f^\#(y)^j\)): \(f \supseteq I^n \cdot R_X^p \subseteq \{f\} + (a_{\mathcal{A}} + a_{\mathcal{A}} \cap f^\#(y)^j) \cdot R_X^p \subseteq a_{\mathcal{A}} f\).

Moreover, \(f \supseteq I^n \cdot R_X^p \subseteq \{f\} + (a_{\mathcal{A}} + a_{\mathcal{A}} \cap f^\#(y)^j) \cdot R_X^p \subseteq a_{\mathcal{A}} f\).

Example 7.10. Let \(R_X\) be one of \(k[[x]], k\{x\}, k(x)\). Take the filtration \(I^* \cdot R_X^p\) and the corresponding pairs \((T_{\mathcal{A}(j)}, \mathcal{A}(j))\) of \(N\). Assume \(I \subseteq I^* \cdot R_X^p\). Assume either \(\text{char}(k) = 0\) or \(\text{char}(k) > 2(d_{\mathcal{A}} + d_{\mathcal{A}}) - \text{ord}(f)\). Then \(\mathcal{A}(j) f \supseteq \{f\} + I^d \cdot R_X^p\).

Proof. By part 2 of lemma 6.12 we get: \(T_{\mathcal{A}(j)} f \supseteq I^d \cdot R_X^p\). Thus \(a_{\mathcal{A}} \supseteq I \supseteq (f)\). Hence \(a_{\mathcal{A}} + (a_{\mathcal{A}} \cap f^\#(y))X = a_{\mathcal{A}} + (f) = a_{\mathcal{A}} + (f) \supseteq I^d \cdot R_X^p\).

ii. We compare the conclusion \(\mathcal{A}(j) f \supseteq \{f\} + I^d \cdot R_X^p\) to the classical results (for \(k = \mathbb{R}, \mathbb{C}\)):

- [Gaffney-du Plessis, 82] Theorems 0.2 and 2.1: Assume \(f\) is \(\mathcal{K}\)-finite. If \(m \cdot T_{\mathcal{A}} f + T_{\mathcal{A}(0)} f \supseteq \{f\} + m^d \cdot R_X^p\), then \(\mathcal{A}(j) f \supseteq \{f\} + I^{d+2} \cdot R_X^p\).

- [Mon. Nuñ.-Bal, Theorem 6.2]: If \(m \cdot T_{\mathcal{A}} f + T_{\mathcal{A}(0)} f \supseteq m^d \cdot R_X^p\) then \(\mathcal{A}(j) f \supseteq \{f\} + m^d \cdot R_X^p\).

- [Mon. Nuñ.-Bal, Corollary 6.3]: If \(T_{\mathcal{A}} f \supseteq m^d \cdot R_X^p\) and \(T_{\mathcal{A}} f \supseteq m^d \cdot R_X^p\) then \(\mathcal{A}(j) f \supseteq \{f\} + m^d \cdot R_X^p\)

For most types of maps (if one is far from a stable map) one has: if \(I^d \cdot R_X^p \subseteq T_{\mathcal{A}} f\), then \(m^d \cdot R_X^p \subseteq T_{\mathcal{A}(j)} f \subseteq T_{\mathcal{A}(j)} f \supseteq \{f\} + I^d \cdot R_X^p\).

Finally, apply Part 2 of corollary 7.9.

7.5. Geometric characterization of \(\mathcal{A}\)-determinacy. Given a map \(f \in \text{Maps}(X,Y)\), the support of \(T_{\mathcal{A}} f\) is the instability locus, \(V(a_{\mathcal{A}}) \subseteq \text{Crit}(f) \subseteq X\). The ideal \(a_{\mathcal{A}} \cap f^\#(y) \subseteq R_Y\) defines the image \(f(V(a_{\mathcal{A}})) \subseteq (k^p, o)\). The ideal \(R_X(a_{\mathcal{A}} \cap f^\#(y)) + a_{\mathcal{A}}\) defines the preimage, \(f^{-1}(f(V(a_{\mathcal{A}}))) \cap \text{Crit}(f)\).

This preimage contains the instability locus. If \(f\) is of finite singularity type (i.e. the restriction \(f : \text{Crit} \rightarrow (k^p, o)\) is finite), then the loci \(V(a_{\mathcal{A}}), f^{-1}(f(V(a_{\mathcal{A}}))) \cap \text{Crit}(f)\) have the same dimension. Moreover, the two loci often coincide.

Corollary 7.6 extends the classical determinacy criterion to the generality of \(k[[x]][J], k\{x\}[J], k\{x\}/J\). Let \(a \subseteq R_X\) be the defining ideal of the locus \(f^{-1}(f(V(a_{\mathcal{A}}))) \cap \text{Crit}(f)\).

Corollary 7.11. Assume \(a_{\mathcal{A}} \subseteq m\).

1. The map \(f : X \rightarrow (k^p, o)\) is \(\mathcal{A}\)-determined by its finite jet on the locus \(f^{-1}(f(V(a_{\mathcal{A}}))) \cap \text{Crit}(f)\).

Namely, \(\mathcal{A} f \supseteq \{f\} + a_{\mathcal{A}} \cdot R_X^p\), for \(d > 1\), where \(a \subseteq R_X\) is the defining ideal of \(f^{-1}(f(V(a_{\mathcal{A}}))) \cap \text{Crit}(f)\).

2. If \(f\) is \(\mathcal{A}\)-finite (i.e. \(\sqrt{a} = m\)), then \(f\) is \(\mathcal{A}\)-finitely determined.
3. Suppose \( f \) is \( \mathcal{X} \)-finite. Take the filtration \( m^* \cdot R^\mathcal{X}_X \). Suppose the local ring \( k \) contains a field, and \( \text{char}(k) = 0 \) or \( \text{char}(k) > 1 \). If \( \mathcal{A}^{(1)} f \supseteq \{ f \} + \mathcal{A}^d \cdot R^\mathcal{X}_X \) then \( T_{\mathcal{A}^d} f \supseteq (\mathcal{A}^d + \mathcal{A}^d) \cdot R^\mathcal{X}_X \). In particular, \( \text{Supp}(T_{\mathcal{A}^d} f) \subseteq V(\mathcal{A}^d) \subset X \), i.e. \( f \) is infinitesimally stable over \( X \setminus V(\mathcal{A}^d) \).

For complex analytic maps, \( f : (\mathbb{C}^n, o) \to (\mathbb{C}^p, o) \), with isolated instabilities (i.e. \( V(\mathcal{A}^d) = o \in (\mathbb{C}^n, o) \)) this gives the classical finite determinacy criterion of Mathe-Gaffney.

In the case \( \sqrt{\mathcal{A}^d} \subseteq m \), i.e. \( f \) has a non-isolated instability, this result seems to be new even for \( \mathbb{C}[x] \).

**Proof.** Parts 1 and 2 follow by Corollary \( \text{Corollary 7.6} \) Part 3 follows by Corollary \( \text{Corollary 7.9} \). ■

**References**

[Ahryankar] Sh.-Sh. Abhyankar, Local analytic geometry. Singapore: World Scientific. xv, 488 p. (2001).

[A.G.L.V.] V. I. Arnold, V.V. Goryunov, O.V. Lyashko, V.A. Vasil’ev, Singularity theory. I. Reprint of the original English edition from the series Encyclopaedia of Mathematical Sciences. Springer-Verlag, Berlin, 1998. iv+245 pp

[Artin.68] M. Artin, On the solutions of analytic equations. Invent. Math. 5 (1968), 277–291.

[Artin.69] M. Artin, Algebraic approximation of structures over complete local rings, Publ. Math. IHES, 36, (1969), 23-58.

[B.K.16] G. Belitski, D. Kerim, Group actions on filtered modules and finite determinacy. Finding large submodules in the orbit by linearization, C. R. Math. Acad. Sci. Soc. R. Can. 38 (2016), no. 4, 113–153.

[B.G.K.22] A.-F. Boix, G.-M. Greuel, D. Kerim, Pairs of Lie-type and large orbits of group actions on filtered modules. (A characteristic-free approach to finiteness determination.), Math. Z. 301 (2022), no. 3, 2415–2463.

[Bo.Da.Re.92] M. Borovoi, A.-M. Greuel, T. Haas, Actions and invariants of algebraic groups. Monographs and Research Notes in Mathematics. CRC Press, Boca Raton, FL, 2017. xx+459 pp.

[Bou.19] G.-M. Greuel, T. H. Pham, Finite determinacy of matrices and ideals in arbitrary characteristics.

[Bo.Da.Re.92] M. Borovoi, C. Daw, and J. Ren, Local analytic geometry. Parts 1 and 2 follow by Corollary 7.6. Part 3 follows by Corollary 7.9.

[B.G.K.22] A.-F. Boix, G.-M. Greuel, D. Kerim, Pairs of Lie-type and large orbits of group actions on filtered modules. (A characteristic-free approach to finiteness determination.), Math. Z. 301 (2022), no. 3, 2415–2463.

[Bo.Da.Re.92] M. Borovoi, A.-M. Greuel, T. Haas, Actions and invariants of algebraic groups. Monographs and Research Notes in Mathematics. CRC Press, Boca Raton, FL, 2017. xx+459 pp.

[Bou.19] G.-M. Greuel, T. H. Pham, Finite determinacy of matrices and ideals in arbitrary characteristics.

[Bou.19] G.-M. Greuel, T. H. Pham, Finite determinacy of matrices and ideals in arbitrary characteristics.

[Br.Rob.88] J. W. Bruce, R. M. Roberts, Critical points of functions on analytic varieties. J. Algebra 188 (1997), no. 1, 16–57.

[B.D.P.87] J. W. Bruce, A. A. du Plessis, C. T. C. Wall, Determinacy and unipoitency. Invent. Math. 88 (1987), no. 3, 521–554.

[Bou.19] G.-M. Greuel, T. H. Pham, Finite determinacy of matrices and ideals in arbitrary characteristics.

[Br.Rob.88] J. W. Bruce, R. M. Roberts, Critical points of functions on analytic varieties. Topology 27 (1988), no. 1, 57–90.

[Bru.Rua.Sai.92] J. W. Bruce, M. A. S. Ruas, M. J. Saia, A note on determinacy. Proc. Amer. Math. Soc. 115 (1992), no. 3, 865–871.

[Cut.Sri.97] S. D. Cutkosky, H. Srinivasan, Equivalence and finite determinacy of mappings. J. Algebra 188 (1997), no. 1, 16–57.

[Damon.88] J. Damon, The unfolding and determinacy theorems for subgroups of \( \mathcal{A}^d \) and \( \mathcal{X} \). Mem. Amer. Math. Soc. 50 (1984), no. 306, x+88 pp.

[Damon.88] J. Damon, Topological triviality and versality for subgroups of \( \mathcal{A}^d \) and \( \mathcal{X} \). Mem. Amer. Math. Soc. 75 (1988), no. 389, x+106 pp.

[Damon.91] J. Damon, \( \mathcal{A} \)-equivalence and the equivalence of sections of images and discriminants. Singularity theory and its applications. Pt. I: Geometric aspects of singularities. Proc. Symp., Warwick 1988-89, Lect. Notes Math. 1462, 93–121 (1991).

[Damon.92] J. Damon, Topological triviality and versality for subgroups of \( \mathcal{A}^d \) and \( \mathcal{X} \). II. Sufficient conditions and applications. Nonlinearity 5 (1992), no. 2, 372–417.

[Denef-Pham.90] M. Domitrz-Rieger, J. H. Rieger, Volume preserving subgroups of \( \mathcal{A}^d \) and \( \mathcal{X} \) and singularities in unimodular geometry. Math. Ann. 345 (2009), no. 4, 783–817.

[Du Plessis.80] A. du Plessis, On the determinacy of smooth map-germs. Invent. Math. 58 (1980), no. 2, 107–160.

[Ebeling] W. Ebeling, Functions of several complex variables and their singularities. Translated from the 2001 German original by Philip G. Spain. Graduate Studies in Mathematics, 83. AMS, Providence, RI, 2007. xviii+312 pp.

[Eisenbud] D. Eisenbud, Commutative algebra. With a view toward algebraic geometry. Graduate Texts in Mathematics, 150. Springer-Verlag, New York, 1995.

[For.95] R. W. P. Ferrers Santos, A. Rittatore, Actions and invariants of algebraic groups. Second edition. Monographs and Research Notes in Mathematics. CRC Press, Boca Raton, FL, 2017. xx+459 pp.

[For.95] R. W. P. Ferrers Santos, A. Rittatore, Actions and invariants of algebraic groups. Second edition. Monographs and Research Notes in Mathematics. CRC Press, Boca Raton, FL, 2017. xx+459 pp.

[For.95] R. W. P. Ferrers Santos, A. Rittatore, Actions and invariants of algebraic groups. Second edition. Monographs and Research Notes in Mathematics. CRC Press, Boca Raton, FL, 2017. xx+459 pp.

[For.95] R. W. P. Ferrers Santos, A. Rittatore, Actions and invariants of algebraic groups. Second edition. Monographs and Research Notes in Mathematics. CRC Press, Boca Raton, FL, 2017. xx+459 pp.

[For.95] R. W. P. Ferrers Santos, A. Rittatore, Actions and invariants of algebraic groups. Second edition. Monographs and Research Notes in Mathematics. CRC Press, Boca Raton, FL, 2017. xx+459 pp.

[For.95] R. W. P. Ferrers Santos, A. Rittatore, Actions and invariants of algebraic groups. Second edition. Monographs and Research Notes in Mathematics. CRC Press, Boca Raton, FL, 2017. xx+459 pp.

[For.95] R. W. P. Ferrers Santos, A. Rittatore, Actions and invariants of algebraic groups. Second edition. Monographs and Research Notes in Mathematics. CRC Press, Boca Raton, FL, 2017. xx+459 pp.
