1 Introduction

The physics of weakly disordered systems has been the subject of considerable interest over the past years. If interactions effects can be ignored, it is known that quantum interference effects lead to the localization of all electronic states in two-dimensions, even for arbitrarily weak disorder \[1\]. That is, the ohmic conductivity of such systems vanishes. It is well understood how the localization is affected by external fields that destroy the phase coherence underlying the quantum interference effects, e.g., magnetic fields \[1\]. However, there still are open questions about the impact of an electric field, which does not break time reversal invariance, and thus has no direct impact on the interference effects. This is an important question, since transport experiments invariably involve electric fields. In the theoretical literature, one can find arguments that a weak electric field has no impact on the localization corrections at all \[2\], to arguments that predict a strong impact on the localization corrections already at very small fields \[3\], to arguments that predict immediate delocalization \[4\], to arguments that predict delocalization beyond a critical field strength \[5\]. Here we revisit this question. In agreement with Ref. \[4\], we find that an arbitrarily weak field destroys the localization. However, for weak disorder this effect is not observable at realistic temperatures.

2 The model

We consider a Hamilton operator

\[
H = \frac{\hat{p}^2}{2m} + F \cdot x + V(x). \tag{1}
\]

Here \(\hat{p}\) is the momentum operator, \(m\) is the electron mass, \(x = (x, y)\) is the position in real space, \(F = (F, 0)\) is the electric field, and \(V(x)\) is a random potential. For the latter we assume a Gaussian distribution with zero mean and a second moment given by

\[
\langle V(x)V(y) \rangle = \frac{\hbar}{2\pi \nu \tau} \delta(x - y), \tag{2}
\]

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where $\nu$ is the density of states and $\tau$ is the relaxation time in the self-consistent Born approximation.

To investigate the impact of the field on the electron localization, we focus on the density relaxation, as described by the integral equation

$$n(x, E|\Omega) = \int dy \, P(x, y|E, \Omega) n_0(y, E).$$

(3)

Here $n_0(x, E)$ is the initial distribution of the number of particles with energy $E$ at position $x$, the propagator $P$ describes the evolution of this distribution, and $n(x, E|\Omega)$ is the Laplace transform of the time dependent particle number density $n(x, E|t)$.

To calculate the density propagator $P$ we generalize the nonlinear $\sigma$-model of Ref. [6] to allow for the presence of a weak electric field. The action takes the form

$$S = -\frac{\pi \nu \hbar}{4} \int dx \sum_{\alpha=-n_r+1}^{n_r} \Omega Q_{\alpha\alpha}(x) \Lambda_{\alpha} - \frac{\pi \nu \hbar}{8} \int dx \sum_{\alpha,\alpha'=-n_r+1}^{n_r} Q_{\alpha\alpha'}(x) [\nabla \cdot D(\mu_x) \nabla] Q_{\alpha'\alpha}(x).$$

(4)

Here $\alpha$ numbers $2n_r$ replicas, $\Lambda_{\alpha} = +1$ if $\alpha > 0$, and $\Lambda_{\alpha} = -1$ if $\alpha \leq 0$. The matrix field $Q$ is defined on a $O(n_r, n_r)/O(n_r) \times O(n_r)$ Grassmann manifold and has the properties $Q^2 = 1$ and $\text{tr} Q = 0$. The integration over real space is understood to be restricted to the classically accessible region, defined by the requirement $\mu_x = E - F \cdot x > 0$. The generalized diffusion coefficient is given by

$$D(\mu_x) = \tau \mu_x/m.$$  

(5)

In deriving Eq. (4) we have restricted the consideration to small fields, which satisfy the relationship $Fl/\mu_x \ll 1$, where $l$ is the mean free path. It can be shown that the two-point propagator of this field theory determines $P$.

To investigate the action (4) we parameterize the $2n_r \times 2n_r$ matrix $Q$ in terms of real $n_r \times n_r$ matrices $q$, according to

$$Q = \begin{pmatrix} \sqrt{1+q^T q} & q \\ -q^T & -\sqrt{1+q^T q} \end{pmatrix},$$

(6)

and expand the action (4) in powers of $q$. For a one-loop calculation it suffices to keep terms up to $O(q^2)$.

3 The Gaussian fluctuations We first consider the Gaussian approximation. The action (4) then yields a generalized diffusion equation for $P$.

$$\left(\Omega - [\nabla \cdot D(\mu_x) \nabla]\right) P(x, y|E, \Omega) = \delta(x - y).$$

(7)

This differential equation must be supplemented by boundary conditions. We require that the propagator vanishes at infinity in the classically accessible region, and that the current in the direction of the field vanishes at the classical turning point. The structure of the equation is a consequence of the symmetries of the action. It reflects the fact that the configuration averaged Green functions are symmetric with respect to exchange of $x$ and $y$, and invariant against generalized real-space translations $x \rightarrow x + a, E \rightarrow E + F \cdot a$.

The general solution of Eq. (7) can be expressed in terms of hypergeometric functions. For our purposes, it is more illuminating to consider a special initial condition, namely, a homogeneous density $n$ of charge carriers, all with energy $\mu$. An electric field is then suddenly switched on at time $t = 0$, so

$$n_0(x, E) = n \delta(E - (\mu + F \cdot x)).$$

(8)

For such an initial charge carrier density the solution of Eq. (7) gives the probability for finding the charge carriers at time $t$ with energy $\mu'$ if they had the energy $\mu$ at time $t = 0$,

$$\mathcal{P}(\mu', \mu|t) = \frac{\theta(\mu)\theta(\mu')}{D'F^2 t} \exp(-\frac{\mu + \mu'}{D'F^2 t}) I_0(\frac{2}{D'F^2 t} \sqrt{\mu\mu'}).$$

(9)
where $I_0$ is the modified Bessel function, and $D' = \tau/m$ is the Drude mobility. The first moment of this distribution, that is, the mean energy, increases with time according to

$$
\epsilon_\mu(t) \equiv \int_0^\infty d\mu' \mu' \mathcal{P}(\mu', \mu|t) = \mu + D'F^2t.
$$

This heating is accompanied by an ohmic current,

$$
\mathbf{j} = -D'nF.
$$

The generalized diffusion equation (7) thus describes heating of the charge carriers due to the work done on the system by the electric field. Eq. (11) also shows that $D'$ is indeed the mobility.

Fluctuations about the mean energy, calculated from the equation

$$
\sigma^2_\mu(t) = \int_0^\infty d\mu' (\mu' - \epsilon_\mu(t))^2 \mathcal{P}(\mu', \mu|t),
$$

increase with time according to

$$
\sigma^2_\mu(t) = (D'F^2t)^2 + 2D(\mu)F^2t.
$$

Therefore, deviations from the mean energy are only negligible for small times, $t \ll t^*$, where

$$
t^* = 2\mu^2/D(\mu)F^2.
$$

For $t \gg t^*$ the fluctuations about the mean energy are as large as the mean energy itself, so that the mean energy does no longer describe the state of the system adequately.

A more detailed analysis of the solution shows that $t^*$ also sets the time scale for a change of the structure of a particle packet. At $t = 0$ the packet is a delta pulse in energy space, and for $t \ll t^*$ its spread is Gaussian. In this limit, the width of the particle packet increases with time in the same way as in the absence of the field. Therefore, the diffusion volume is not affected by the field for $t \ll t^*$. However, at $t \approx t^*$ the particle packet undergoes a restructuring. For $t \gg t^*$, the mean square deviation in the direction of the electric field increases with time according to

$$
\langle (x - \langle x \rangle)^2 \rangle \approx (D'Ft)^2.
$$

Here $\langle \ldots \rangle$ denotes an average with respect to the distribution $\mathcal{P}$. For $t \gg t^*$, the diffusion volume thus increases much faster with time than in the absence of the field.

### 4 Scaling, and weak-localization corrections

We now perform a scaling analysis of the action (4), using the technique of Ref. [7]. To this end we extend the dimensionality of the system from 2 to $d = 2 + \epsilon$, and first consider the Gaussian fixed point that describes a diffusive phase [7]. At this fixed point, the Gaussian action is invariant against changes of scale of the form $q \rightarrow q' = b^\epsilon q$, $\Omega \rightarrow \Omega' = b^2\Omega$ and $x' = x/b$, if the electric field is scaled according to $F \rightarrow F' = Fb$. Accordingly, $F$ is a relevant operator with respect to the diffusive fixed point, with a scale dimension of 1. If we assign a coupling constant $u$ to the terms quartic in $q$, we find that $u$ decreases according to $u \rightarrow u' = b^{-(d-2)}u$, so $u$ is an irrelevant operator with respect to the diffusive fix point for $d > 2$, and so are all terms of higher order in $q$. Since the scale dimension of the diffusion coefficient $D$ at the diffusive fix point is zero, we have the scaling equation

$$
D(\Omega, F, u) = D(\Omega b^2, Fb, ub^{-(d-2)}).
$$
Putting $\Omega = 0$, and $b = 1/F$, we find
\[ D(0, F, u) \propto \text{const} + F^{d-2}. \] (17)

Equation (17) suggests that in two-dimensions the weak-localization correction to the diffusion coefficient are logarithmic with respect to $F$. We have verified this by an explicit calculation of the one-loop corrections. The corrections obtained in this way are the same as those obtained in Ref. [3].

For $\Omega \neq 0$ there are two scaling regimes. For small $\Omega$ the scaling of the diffusion coefficient is governed by the field, and for large $\Omega$, by the frequency. The scaling analysis shows that the crossover between these scaling regimes occurs at $\Omega \approx \Omega^*$, where $\Omega^* \sim F^{-2}$. As one would expect from these arguments, the explicit calculation of the one-loop corrections yields $\Omega^* = \kappa/\tau^*$, where $\kappa = O(1)$.

The critical fixed point, which describes the Anderson localization transition in the absence of an electric field, is more complicated. It cannot be found by power counting alone, but rather requires an explicit calculation within the framework of a loop expansion [1, 7]. In particular, the scale dimension of $\tilde{F}$ is determined by the loop expansion. However, in $d = 2 + \epsilon$ the scale dimensions at the diffusive and the critical fixed points, respectively, differ only by terms of $O(\epsilon)$. The leading contribution to the scale dimension $[\tilde{F}]$ of $\tilde{F}$ is therefore still given by power counting, and we have
\[ [\tilde{F}] = 1 + O(\epsilon). \] (18)

The electric field is thus a relevant operator with respect to the Anderson localization fixed point. Strictly speaking, this discussion shows only that the localization fixed point is unstable in the presence of an electric field, and it does not tell what happens instead. However, explicit perturbative calculations to one-loop order suggest that there is a metallic phase in $d = 2$ for $F \neq 0$.

5 Conclusions The arguments presented above show that in a weak electric field the scaling of the dc-conductivity at asymptotically low temperatures is governed by the electric field. However, any experiment effectively measures the conductivity at a frequency or temperature given by the inverse phase relaxation time, $\tau_{\phi}^{-1}$. Therefore, the impact of the field on the weak-localization corrections can only be observed if $\Omega^* \tau_{\phi} \gg 1$. Quantitative estimates show $\Omega^* < 10^4 \text{Hz}$, while a typical phase relaxation rate at dilution refrigerator temperatures is on the order of $10^{11} \text{Hz}$. The field scaling is therefore not observable in current experiments.

These considerations solve the following paradox. While it is true that an arbitrarily small electric field destroys localization, as was found by Kirkpatrick [4], this effect manifests itself only at unobservably low temperatures. This explains why the experimental results are consistent with the zero-field theory, even though electric fields are present in the transport experiments.

We finally note that in our model inelastic collisions are not taken into account. Therefore, our results apply only to samples that are shorter than the energy relaxation length. The consideration of inelastic collisions is important in order to establish a nonequilibrium steady state characterized by an effective electron temperature. In our present treatment such a steady state is absent; the charge carriers are continuously heated up. We therefore plan further investigations that will focus on energy relaxation processes.

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