Levy processes with summable Levy measures,
long time behavior

Lev Sakhnovich

99 Cove ave., Milford, CT, 06461, USA
E-mail: lsakhnovich@gmail.com

Mathematics Subject Classification (2010): Primary 60G51;
Secondary 60J45; 45A05

Keywords. Semigroup, generator, convolution form, potential, quasi-potential,
long time behavior.

Abstract

In our previous paper [17] we have proved that a representation
of the infinitesimal generators $L$ for Levy processes $X_t$ can be written
down in a convolution type form. For the case of non-summable Levy
measures we constructed the quasi-potential operators $B$ and investi-
gated the long time behavior of $X_t$. In the present paper we consider
Levy processes $X_t$ with summable Levy measures. In this case the
form of the quasi-potential operators $B$ essentially differs from the
form in the case of non-summable Levy measures. We use this new
form in order to study the long time behavior of $X_t$ for the case of
summable Levy measures.

1 Introduction

Let us introduce the notion of the Levy processes.

Definition 1.1 A stochastic process $\{X_t : t \geq 0\}$ is called Levy process, if the
following conditions are fulfilled:
1. Almost surely $X_0 = 0$, i.e. $P(X_0 = 0) = 1$. 
(One says that an event happens almost surely (a.s.) if it happens with probability one.)

2. For any \(0 \leq t_1 < t_2 \ldots < t_n < \infty\) the random variables
\[X_{t_2} - X_{t_1}, X_{t_3} - X_{t_2}, \ldots, X_{t_n} - X_{t_{n-1}}\]
are independent (independent increments).

(To call the increments of the process \(X_t\) independent means that increments \(X_{t_2} - X_{t_1}, X_{t_3} - X_{t_2}, \ldots, X_{t_n} - X_{t_{n-1}}\) are mutually (not just pairwise) independent.)

3. For any \(s < t\) the distributions of \(X_t - X_s\) and \(X_{t-s}\) are equal (stationary increments).

4. Process \(X_t\) is almost surely right continuous with left limits.

Then Levy-Khinchine formula gives (see [2, 18])
\[\mu(z, t) = E\{\exp[izX_t]\} = \exp[-t\lambda(z)], \quad t \geq 0,\] (1.1)
where
\[\lambda(z) = \frac{1}{2} Az^2 - i\gamma z - \int_{-\infty}^{\infty} (e^{ixz} - 1 - ixz1_{|x|<1})\nu(dx).\] (1.2)

Here \(A \geq 0, \quad \gamma = \bar{\gamma}, \quad z = \bar{z}\) and \(\nu(dx)\) is a measure on the axis \((-\infty, \infty)\) satisfying the conditions
\[\int_{-\infty}^{\infty} \frac{x^2}{1 + x^2} \nu(dx) < \infty.\] (1.3)

The Levy-Khinchine formula is determined by the Levy-Khinchine triplet \((A, \gamma, \nu(dx))\).

By \(P_t(x_0, \Delta)\) we denote the probability \(P(X_t \in \Delta)\) when \(P(X_0 = x_0) = 1\) and \(\Delta \in \mathbb{R}\). The transition operator \(P_t\) is defined by the formula
\[P_t f(x) = \int_{-\infty}^{\infty} P_t(x, dy) f(y).\] (1.4)

Let \(C_0\) be the Banach space of continuous functions \(f(x)\), satisfying the condition \(\lim_{|x| \to \infty} f(x) = 0\) with the norm \(||f|| = \sup_{x} |f(x)|\). We denote by \(C_0^n\) the set of \(f(x) \in C_0\) such that \(f^{(k)}(x) \in C_0, \quad (1 \leq k \leq n)\). It is known that [18]
\[P_t f \in C_0,\] (1.5)
if \(f(x) \in C_0^n\).

Now we formulate the following important result (see [18]).
Theorem 1.2 The family of the operators \( P_t \) \((t \geq 0)\) defined by the Levy process \( X_t \) is a strongly continuous semigroup on \( C_0 \) with the norm \(||P_t|| = 1\). Let \( L \) be its infinitesimal generator. Then

\[
Lf = \frac{1}{2} \frac{d^2 f}{dx^2} + \gamma \frac{df}{dx} + \int_{-\infty}^{\infty} (f(x + y) - f(x) - y \frac{df}{dx} 1_{|y| < 1}) \nu(dy),
\]

where \( f \in C^2_0 \).

Slightly changing the well-known classification (see \[18\]) we introduce the following definition:

Definition 1.3 We say that a Levy process \( X_t \) generated by \((A, \nu, \gamma)\) has type I if

\[
A = 0 \text{ and } \int_{-\infty}^{\infty} \nu(dx) < \infty,
\]

and \( X_t \) has type II if

\[
A \neq 0 \text{ or } \int_{-\infty}^{\infty} \nu(dx) = \infty.
\]

Remark 1.4 The introduced type I coincides with the type A in the usual classification. The introduced type II coincides with the union of the types B and C in the usual classification.

The properties of these two types of the Levy processes are quite different. The paper \[17\] was dedicated to type II.

Remark 1.5 Some results of the paper \[17\] are proved for all Levy processes. We shall use these results.

In the present paper we shall consider the type I. Without loss of generality we assume that

\[
\gamma - \int_{|y| < 1} y \nu(dy) = 0.
\]

According to (1.6) the generator \( L \) of the corresponding process \( X_t \) can be represented in the form (see \[16\],p.13)

\[
Lf = \int_{-\infty}^{\infty} [f(x + y) - f(x)] \nu(dy).
\]
Remark 1.6 If condition (1.9) is valid, then type I coincides with the class of the compound Poisson processes.

As in the case of type II we use the convolution representation of the generator $L$. To do it we introduce the functions

$$
\mu_-(x) = \int_{-\infty}^{x} \nu(dx), \quad x < 0,
$$

(1.11)

$$
\mu_+(x) = -\int_{x}^{\infty} \nu(dx), \quad x > 0,
$$

(1.12)

where the functions $\mu_-(x)$ and $\mu_+(x)$ are monotonically increasing and right continuous on the half-axis $(-\infty, 0]$ and $[0, \infty)$ respectively. We note that

$$
\mu_+(x) \to 0, \quad x \to +\infty; \quad \mu_-(x) \to 0, \quad x \to -\infty,
$$

(1.13)

$$
\mu_-(x) \geq 0, \quad x < 0; \quad \mu_+(x) \leq 0, \quad x > 0.
$$

(1.14)

Now we define the functions

$$
k_-(x) = \int_{-a}^{x} \mu_-(t)dt, \quad -\infty \leq x < 0, \quad a > 0,
$$

(1.15)

$$
k_+(x) = -\int_{x}^{a} \mu_+(t)dt, \quad 0 < x \leq +\infty.
$$

(1.16)

Proposition 1.7 (see [17], p.13). Let the Levy process $X_t$ belong to the type I and let the condition (1.9) be fulfilled. Then formula (1.10) can be written in the following convolution form

$$
Lf = \frac{d}{dx} S \frac{d}{dx} f,
$$

(1.17)

where

$$
Sf = \int_{-\infty}^{\infty} k(y - x)f(y)dy,
$$

(1.18)

$$
k(x) = k_+(x), \quad \text{if} \ x > 0; \quad k(x) = k_-(x), \quad \text{if} \ x < 0.
$$

(1.19)

Using (1.15)-(1.18) we obtain the assertion.
Proposition 1.8 Let conditions of Proposition 1.7 be fulfilled. Then relation (1.17) takes the form

\[ Lf = -\Omega f + \int_{-\infty}^{\infty} f(y)d\nu(y), \quad \Omega = \int_{-\infty}^{\infty} d\nu(y). \]  

(1.20)

Proof. It follows from (1.15)-(1.18) that

\[ Lf = -\int_{-\infty}^{x} \mu_{-}(y-x)f'(y)dy - \int_{x}^{\infty} \mu_{+}(y-x)f'(y)dy. \]  

(1.21)

Integrating by parts (1.21) we have

\[ Lf = -[\mu_{-}(0) - \mu_{+}(0)]f + \int_{-\infty}^{\infty} f(y)d\nu(y - x). \]  

(1.22)

The proposition is proved.

In formulas (1.20) and (1.22) we use the equality \( d\nu(x) = \nu(dx) \).

Definition 1.9 We say that Levy process \( X_t \) belongs to type \( I_c \) if the corresponding Levy measure \( \nu(y) \) is summable and continuous.

We say that Levy process \( X_t \) belongs to type \( I_d \) if the corresponding Levy measure \( \nu(y) \) is summable and discrete.

It is easy to see that the following assertion is true.

Proposition 1.10 If Levy process \( X_t \) belongs to the type \( I \), then \( X_t \) can be represented in the form

\[ X_t = X_t^{(1)} + X_t^{(2)}, \]  

(1.23)

where \( X_t^{(1)} \in I_c \) and \( X_t^{(2)} \in I_d \).

The main part of the present paper is dedicated to investigating the Levy processes from the type \( I_c \).

We denote by \( p(t, \Delta) \) the probability that a sample of the process \( X_t \in I_c \) remains inside the domain \( \Delta \) for \( 0 \leq \tau \leq t \) (ruin problem). With the help of representation (1.17) we find a new formula for \( p(t, \Delta) \). This formula allow us to obtain the long time behavior of \( p(t, \Delta) \). Namely, we have proved the following asymptotic formula

\[ p(t, \Delta) = e^{-t/\lambda_1} [c_1 + o(1)], \quad c_1 > 0, \quad \lambda_1 > 0, \quad t \to +\infty. \]  

(1.24)
Let $T_\Delta$ be the time during which $X_t$ remains in the domain $\Delta$ before it leaves the domain $\Delta$ for the first time. It is easy to see that

$$p(t, \Delta) = P(T_\Delta > t).$$

(1.25)

An essential role in our theory plays the operator

$$L_\Delta f = -\Omega f + \int_\Delta f(y)d_\nu(y - x),$$

(1.26)

which is generated by the operator $L$ (see (1.8)). We note that $\lambda_1$ in formula (1.23) is the greatest eigenvalue of $-L_\Delta^{-1}$.

**Definition 1.11** The measure $\nu(y)$ is unimodal with mode 0 if $\nu(y)$ is concave when $y < 0$ and convex if $y > 0$.

The unimodality and its properties were actively investigated (see [18]). In the paper we found a new important property of unimodal measure:

**Proposition 1.12** If Levy measure $\nu(y)$ is continuous, summable and unimodal with mode 0 then the operator $L_\Delta^{-1}$ has the form

$$L_\Delta^{-1} = -\frac{1}{\Omega}(I + T_1),$$

(1.27)

where the operator $T_1$ is compact in the space of continuous functions.

In the last part of the paper we investigate the operator $L_\Delta$ when $X_t \in I_d$.

## 2 Quasi-potential

1. By domain $\Delta$ we denote the set of segments $[a_k, b_k]$, where $a_1 < b_1 < a_2 < b_2 < \ldots < a_n < b_n$, $1 \leq k \leq n$.

We denote by $D_\Delta$ the space of the continuous functions $g(x)$ on the domain $\Delta$. The norm in $D_\Delta$ is defined by the relation $||f|| = \sup_{x \in \Delta}|f(x)|$. The space $D_\Delta^0$ is defined by the relations:

$f(x) \in D_\Delta$ and $f(a_k) = f(b_k) = 0$, $1 \leq k \leq n$.

We introduce the operator $P_\Delta$ by relation $P_\Delta f(x) = f(x)$ if $x \in \Delta$ and $P_\Delta f(x) = 0$ if $x \notin \Delta$. 


Definition 2.1 The operator

\[ L_\Delta = P_\Delta L P_\Delta = \frac{d}{dx} S_\Delta \frac{d}{dx}, \text{ where } S_\Delta = P_\Delta S P_\Delta, \quad (2.1) \]

is called a truncated generator. (We use here the equality \( P_\Delta \frac{d}{dx} = \frac{d}{dx} P_\Delta, \quad x \in \Delta \).)

Definition 2.2 The operator \( B \) with the definition domain \( D_\Delta \) is called a quasi-potential if the following relation

\[ -BL_\Delta g = g, \quad g \in D^0_\Delta \quad (2.2) \]

is true.

According to Proposition 1.8 the operator \( L_\Delta \) has the form

\[ L_\Delta f = -\Omega f + \int_\Delta f(y) d_\nu(y - x), \quad f(x) \in D^0_\Delta, \quad (2.3) \]

We introduce the operator

\[ T f = \int_\Delta f(y) d_\nu(y - x), \quad f(x) \in D^0_\Delta. \quad (2.4) \]

Inequality (2.4) implies the statement.

Theorem 2.3 Let the Levy process \( X_t \) belong to the type I and let condition (1.9) be fulfilled. Then the operator \( T \) acts from \( D_\Delta \) into \( D_\Delta \) and

\[ ||T|| = \sup \int_\Delta d_\nu(y - x) \leq \Omega, \quad x \in \Delta \quad (2.5) \]

Further we suppose in addition that

\[ ||T|| < \Omega. \quad (2.6) \]

Remark 2.4 Let the support of the Levy measure \( \nu \) is unbounded. Then in view of (2.5) the inequality (2.6) holds.

We consider separately the case when the Levy process \( X_t \) belongs to the type \( I_c \).
Theorem 2.5 Let $X_t$ belong to the type $I_c$ and let the condition (2.6) be true. Then the operator $L^{-1}_\Delta$ exists and has the form

$$L^{-1}_\Delta = -B = -\frac{1}{\Omega}(I + T_1), \quad (2.7)$$

where the operator $T_1$ acts in the space $D_\Delta$ is bounded and is defined by the formula

$$T_1 f = \int_\Delta f(y) d_y \Phi(x, y). \quad (2.8)$$

The function $\Phi(x, y)$ is continuous with respect to $x$ and $y$, monotonically increasing with respect to $y$.

3 Quasi-potential, compactness

1. In this section we consider the following problem:

Under which conditions the operators $T$ and $T_1$ are compact in the space $D_\Delta$?

We remind that the operators $T$ and $T_1$ are defined by (2.4) and (2.8) respectively. We need the following definitions

Definition 3.1 The total variation of a complex-valued function $g$, defined on $\Delta$ is the quantity

$$V_\Delta(g) = \sup_{P} \sum_{i=0}^{n_P-1} |g(x_{i+1}) - g(x_i)|,$$

where the supremum is taken over the set of all partitions $P = (x_0, x_1, ..., x_{n_P})$ of the $\Delta$.

Definition 3.2. A complex-valued function $g$ on the $\Delta$ is said to be of bounded variation (BV function) on the $\Delta$ if its total variation is finite.

By $D^*_\Delta$ we denote the conjugate space to $D_\Delta$. It is well-known that the space $D^*_\Delta$ consists from functions $g(x)$ with a bounded total variation $V_\Delta(g)$. The norm in $D^*_\Delta$ is defined by the relation $||g|| = V_\Delta(g)$, the functional in $D_\Delta$ is defined by the relation

$$(f, g)_\Delta = \int_\Delta f(x) \overline{g(x)}, \quad f \in D_\Delta, \quad g \in D^*_\Delta. \quad (3.1)$$
Hence, the conjugate operator $B^*$ maps the space $D_\Delta^*$ into himself and has the form
\[ B^*g = \int_\Delta \Phi(y, x)dg(y). \] (3.2)

J. Radon [14] proved the following theorem.

**Theorem 3.3** The operator $T_1$ defined by formula (2.8) is compact in the space $D_\Delta$ if and only if
\[ \lim_{x \to \xi} ||\Phi(x, y) - \Phi(\xi, y)||_V = 0, \quad x, y, \xi \in \Delta. \] (3.3)

Hence we have the assertion.

**Proposition 3.4** If measure $\nu(y)$ is summable and has continuous derivative, then the corresponding operators $T$ and $T_1$ are compact in the space $D_\Delta$.

**Proof.** Relation (2.4) takes the form:
\[ Tf = \int_\Delta f(y)\nu'(y - x)dy, \quad x \in \Delta. \] (3.4)

The conditions of the Radon’s theorem are fulfilled, i.e. the operator $T$ is compact. In view of (2.4), (2.5) and (2.8), the operator $T_1$ is compact as well. The Proposition is proved.

2. Let us consider the important case, when the Levy measure is unimodal (see Definition 1.11). We shall use the following convex and concave properties.

**Proposition 3.5** Let the points $x_p$, $(p = 1, 2, 3, 4)$ be such that $x_1 < x_2 \leq x_3 < x_4$.

1. If a function $f(x)$ is convex then
\[ \frac{f(x_2) - f(x_1)}{x_2 - x_1} \geq \frac{f(x_4) - f(x_3)}{x_4 - x_3}. \] (3.5)

2. If a function $f(x)$ is concave then
\[ \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_4) - f(x_3)}{x_4 - x_3}. \] (3.6)
Corollary 3.1 The assertions 1. and 2. of Proposition 3.5 are true if
\( x_1 < x_3 \leq x_2 < x_4 \) and \( x_2 - x_1 = x_4 - x_3 \).

Proof. We consider the convex case and take such integer \( n \) that \( \ell/n < \ell_1 \),
where \( \ell = x_2 - x_1, \ell_1 = x_3 - x_1 \). It follows from (3.5) that
\[
\sum_{k=1}^{n} [f(x_1 + k\ell/n) - f(x_1 + (k-1)\ell/n)] \geq \sum_{k=1}^{n} [f(x_3 + k\ell/n) - f(x_3 + (k-1)\ell/n)].
\] (3.7)
Hence in the convex case the corollary is proved. In the same way the corollary can be proved in the concave case.

Theorem 3.6 If a measure \( \nu(dx) \) on \( \mathbb{R} \) is unimodal with mode \( 0 \) then the corresponding operators \( T \) and \( T_1 \) are compact in the space \( D_\Delta \).

Proof.
Let us consider the case when \( \Delta = [c, d] \) and \( c = y_0 < y_1 < \ldots < y_n = d \). We introduce the variation
\[
V_n(x, \xi) = \sum_{k=1}^{n} |[\mu(y_k - x) - \mu(y_{k-1} - x)] - [\mu(y_k - \xi) - \mu(y_{k-1} - \xi)]|,
\] (3.8)
where \( c \leq \xi < x \leq d \). Without loss of generality we assume that
\[
\max |y_k - y_{k-1}| < x - \xi, \quad 1 \leq k \leq n.
\] (3.9)
We denote by \( y_N \) such point that
\[
y_N - x \geq 0, \quad y_{N-1} - x \leq 0.
\] (3.10)
We represent equality (3.8) in the form
\[
V_n(x, \xi) = \sum_{k=1}^{N-2} |b_k| + \sum_{k=N-1}^{N} |b_k| + \sum_{N+1}^{n} |b_k|,
\] (3.11)
where
\[
b_k = [\mu(y_k - x) - \mu(y_{k-1} - x)] - [\mu(y_k - \xi) - \mu(y_{k-1} - \xi)].
\] (3.12)
Proposition 3.6 implies that
\[
[\mu(y_k - x) - \mu(y_{k-1} - x)] - [\mu(y_k - \xi) - \mu(y_{k-1} - \xi)] \geq 0, \quad k \geq N,
\] (3.13)
\[ [\mu(y_k - x) - \mu(y_{k-1} - x)] - [\mu(y_k - \xi) - \mu(y_{k-1} - \xi)] \leq 0, \ k \leq N - 2. \quad (3.14) \]

It follows from (3.11)-(3.14) that

\[ V_n(x, \xi) = -\sum_{k=1}^{N-2} b_k + \sum_{k=N-1}^{N} |b_k| + \sum_{N+1}^{n} b_k. \quad (3.15) \]

Hence we have

\[ V_n(x, \xi) \leq D_0 + 2(D_{N-2} + D_{N-1} + D_N) + D_n, \quad (3.16) \]

where \( D_k = |\mu(y_k - x) - \mu(y_k - \xi)| \). The function \( \mu(y) \) is continuous. Therefore

\[ \sup V_n(x, \xi) \to 0, \ x \to \xi. \quad (3.17) \]

From the last relation and Radon’s theorem follows that in the case, when \( \Delta = [c, d] \), the assertion of the theorem is true. Then the theorem is true in the case of arbitrary domain \( \Delta \).

**Remark 3.7** The Theorem 3.6 is true in the case when \( \nu(dx) \) is \( n \)-modal \( (1 \leq n < \infty) \).

**Remark 3.8** The unimodality of the Levy measure \( \nu(dx) \) is closely connected with the unimodality of the probability distribution \( F(x, t) \) of the corresponding Levy process (see [18], section 52).

It is easy to obtain the following assertion.

**Proposition 3.9** Let condition (2.6) be fulfilled. Then the spectrum of the operator \( B \) belongs to the right half plane. If in addition the operator \( T \) is compact, then the eigenvalues \( \lambda_j \) of \( B \) are such that

\[ \lambda_j \to 1/\Omega, \ j \to \infty. \quad (3.18) \]

4 The Probability of the Levy process (type \( I_c \)) remaining within the given domain

1. We remind, that the definition of the Levy processes and the definition of the type \( I_c \) are given in section 1.
Conditions 4.1 Further we assume that the following conditions are fulfilled:
1. The Levy process $X_t$ belong to the type $I_c$.
2. The relation (1.9) is valid.

We denote by $F_0(x, t)$ the distribution function of Levy process $X_t$, i.e.

$$F_0(x, t) = P(X_t \leq x).$$  \hfill (4.1)

We need the following statement (see [18], Remark 27.3)

**Theorem 4.1** Let Conditions 4.1 be fulfilled. The distribution function $F_0(x, t)$ is continuous with respect to $x$ if $x \neq 0$.

Let us investigate the behavior of $F_0(x, t)$ in the point $x = 0$.

**Proposition 4.2** Let Conditions 4.1 be fulfilled. Then the relation

$$F_0(+0, t) - F_0(-0, t) = e^{-t\Omega}, \quad \Omega = \int_{-\infty}^{\infty} \nu(dx)$$ \hfill (4.2)

is valid. Here by definition we have

$$F(0, t) = [F_0(+0, t) + F_0(-0, t)]/2.$$ \hfill (4.3)

**Proof.** In our case equality (1.2) takes the form

$$\lambda(z) = -\int_{-\infty}^{\infty} (e^{ixz} - 1)\nu(dx) = \Omega - \omega(z),$$ \hfill (4.4)

where

$$\omega(z) = \int_{-\infty}^{\infty} e^{ixz}\nu(dx).$$ \hfill (4.5)

Using the inverted Fourier-Stieltjes transform we have

$$F_0(x, t) - F_0(0, t) = \frac{1}{2\pi} e^{-t\Omega} \int_{-\infty}^{\infty} \frac{e^{-ixz} - 1}{-iz} \sum_{k=0}^{\infty} \frac{(t\omega(z))^k}{k!} dz.$$ \hfill (4.6)

In view of (4.4) the relation

$$\mu(x) - \mu(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-ixz} - 1}{-iz} \omega(z) dz$$ \hfill (4.7)
holds. The measure $\nu(dx)$ is continuous. Hence formula (4.7) implies, that
\[
\int_{-\infty}^{\infty} \frac{e^{-ixz} - 1}{-iz} \omega(z) dz \bigg|_{x=0} = 0.
\]
(4.8)

In the same way we can prove the formulas
\[
\int_{-\infty}^{\infty} \frac{e^{-ixz} - 1}{-iz} \omega^k(z) dz \bigg|_{x=0} = 0, \quad k \geq 1.
\]
(4.9)

It follows from (4.5) and (4.9) that
\[
F_0(+0, t) - F_0(-0, t) = e^{-t\Omega} \lim_{x \to +0} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{-ixz} - 1}{-iz} dz = e^{-t\Omega}.
\]
(4.10)

The Proposition is proved.

We introduce the sequence of functions
\[
F_{n+1}(x, t) = \int_{0}^{t} \int_{-\infty}^{\infty} F_0(x - \xi, t - \tau)V(\xi)d\xi F_n(\xi, \tau) d\tau, \quad n \geq 0,
\]
(4.11)
where the function $V(x)$ is defined by relations $V(x) = 1$ when $x \notin \Delta$ and $V(x) = 0$ when $x \in \Delta$. In the right side of (4.11) we use Stieltjes integration. According to (4.11) the function $F_1(x, t)$ is continuous with respect to $x$, when $x \neq 0$. The point $x = 0$ we shall consider separately.

**Theorem 4.3** Let Conditions 4.1 be fulfilled. If the point $x = 0$ belongs to $\Delta$ then the functions $F_n(x, t), \quad n > 0$ are continuous with respect to $x$.

**Proof.** Using (4.11) we have
\[
F_1(+0, t) - F_1(-0, t) = \int_{0}^{t} [F_0(+0, t - \tau) - F_0(-0, t - \tau)] V(0) d\tau.
\]
(4.12)

If the point $x = 0$ belongs to $\Delta$ then $V(0) = 0$. Hence, $F_1(+0, t) - F_1(-0, t) = 0$. Thus, the function $F_1(x, t)$ is continuous with respect to $x$. Now the assertion of the Theorem follows directly from (4.11).

It follows from (1.1) that
\[
\mu(z, t) = \mu(z, t - \tau)\mu(z, \tau).
\]
(4.13)
Due to (4.13) and convolution formula for Stieltjes-Fourier transform ([3], Ch.4) the relation

\[ F_0(x, t) = \int_{-\infty}^{\infty} F_0(x - \xi, t - \tau) d\xi F_0(\xi, \tau) \] (4.14)

is true. Using (4.11) and (4.14) we have

\[ 0 \leq F_n(x, t) \leq t^n F_0(x, t)/n!. \] (4.15)

Hence the series

\[ F(x, t, u) = \sum_{n=0}^{\infty} (-1)^n u^n F_n(x, t) \] (4.16)

converges. The probabilistic meaning of \( F(x, t, u) \) is defined by the relation (see [10], Ch.4):

\[ E\{\exp[-u \int_{0}^{t} V(X_\tau)d\tau], c_1 < X_t < c_2\} = F(c_2, t, u) - F(c_1, t, u). \] (4.17)

The inequality \( V(x) \geq 0 \) and relation (4.17) imply that the function \( F(x, t, u) \) monotonically decreases with respect to the variable “\( u \)” and monotonically increases with respect to the variable “\( x \)”. Hence, the following formula

\[ 0 \leq F(x, t, u) \leq F(x, t, 0) = F_0(x, t) \] (4.18)

is true. In view of (4.18) the Laplace transform

\[ \Psi(x, s, u) = \int_{0}^{\infty} e^{-st} F(x, t, u) dt, \quad s > 0 \] (4.19)

is correct. Since the function \( F(x, t, u) \) monotonically decreases with respect to \( u \), this is also true for the function \( \Psi(x, s, u) \). Hence the limits

\[ F_\infty(x, t) = \lim F(x, t, u), \quad \Psi_\infty(x, s) = \lim \Psi(x, s, u), \quad u \to \infty \] (4.20)

exist. It follows from (4.17) that

\[ p(t, \Delta) = P(X_\tau \in \Delta, 0 < \tau < t) = \int_{\Delta} d_x F_\infty(x, t). \] (4.21)

Hence we have

\[ \int_{0}^{\infty} e^{-st} p(t, \Delta) dt = \int_{\Delta} dx F_\infty(x, s). \] (4.22)

2. Relations (4.22) implies the following assertion
**Proposition 4.4** Let Conditions 4.1 be fulfilled. If the point \( x = 0 \) belongs to \( \Delta \) then the function \( \Psi_\infty(x, s) \) for all \( s > 0 \) is monotonically increasing and continuous with respect to \( x \) if \( x \neq 0 \).

Using (4.10) and (4.19) we have the assertion.

**Proposition 4.5** Let Conditions 4.1 be fulfilled. If the point \( x = 0 \) belongs to \( \Delta \) then

\[
\Psi(+0, s, u) - \Psi(-0, s, u) = \Psi_\infty(+0, s) - \Psi_\infty(-0, s) = \frac{1}{s + \Omega}, \quad s > 0. \tag{4.23}
\]

The behavior of \( \Psi_\infty(x, s) \) when \( s = 0 \) we shall consider separately, using the following Hengartner and Thedorescu result ([8], see [18] too):

**Theorem 4.6** Let \( X_t \) be a Levy process. Then for any finite interval \( K \) the estimation

\[
P(X_t \in K) = 0(t^{-1/2}) \quad \text{as} \quad t \to \infty \tag{4.24}
\]

is valid.

Hence, we have the assertion (see [17]).

**Theorem 4.7** Let \( X_t \) be a Levy process. Then for any integer \( n > 0 \) the estimation

\[
p(t, \Delta) = 0(t^{-n/2}) \quad \text{as} \quad t \to \infty \tag{4.25}
\]

is valid.

We need the following partial case of (4.22):

\[
\int_0^\infty p(t, \Delta)dt = \int_\Delta dx \Psi_\infty(x, 0), \tag{4.26}
\]

According to (4.25) the integral in the left side of (4.26) exists.

2. Let us investigate in details the functions \( F(x, t, u) \) and \( \Psi(x, s, u) \).

According to (4.11) and (4.16) the function \( F(x, t, u) \) satisfies the equation

\[
F(x, t, u) + u \int_0^t \int_{-\infty}^\infty F_0(x - \xi, t - \tau)V(\xi)d\xi F(\xi, \tau, u)d\tau = F_0(x, t). \tag{4.27}
\]

Taking from both parts of (4.27) the Laplace transform, using (4.19) and the convolution property (see [3], Ch.4) we obtain

\[
\Psi(x, s, u) + u \int_{-\infty}^\infty \Psi_0(x - \xi, s)V(\xi)d\xi \Psi(\xi, s, u) = \Psi_0(x, s), \tag{4.28}
\]

15
where
\[ \Psi_0(x, s) = \int_0^\infty e^{-st}F_0(x, t)dt. \] (4.29)

It follows from (1.1) and (4.29) that
\[ \int_{-\infty}^\infty e^{ixp}d_x\Psi_0(x, s) = \frac{1}{s + \lambda(p)}. \] (4.30)

According to (4.28) and (4.29) we have
\[ \int_{-\infty}^\infty e^{ixp}[s + \lambda(p) + uV(x)]d_x\Psi(x, s, u) = 1. \] (4.31)

Now we introduce the function
\[ h(p) = \frac{1}{2\pi} \int_\Delta e^{-ixpf(x)}dx, \] (4.32)

where the function \( f(x) \) belongs to \( C_\Delta \).

By \( C_\Delta \) we denote the set of functions \( g(x) \) on \( L^2(\Delta) \) such that
\[ g(a_k) = g(b_k) = g'(a_k) = g'(b_k) = 0, \quad 1 \leq k \leq n, \quad g''(x) \in L^p(\Delta), \quad p > 1. \] (4.33)

Multiplying both parts of (4.31) by \( h(p) \) and integrating them with respect to \( p \) \( (-\infty < p < \infty) \) we deduce the equality
\[ \int_{-\infty}^\infty \int_{-\infty}^\infty e^{ixp}[s + \lambda(p)]h(p)d_x\Psi(x, s, u)dp = f(0). \] (4.34)

We have used the relations
\[ V(x)f(x) = 0, \quad -\infty < x < \infty, \] (4.35)
\[ \frac{1}{2\pi} \lim_{N \to \infty} \int_{-N}^N \int_\Delta e^{-ixpf(x)}dxdp = f(0), \quad N \to \infty. \] (4.36)

Since the function \( F(x, t, u) \) monotonically decreases with respect to "\( u \)" the function \( \Psi(x, s, u) \) (see (4.19)) monotonically decreases with respect to "\( u \)" as well. Hence there exist the limits
\[ F_\infty(x, s) = \lim_{u \to \infty}F(x, s, u), \quad \Psi_\infty(x, s) = \lim_{u \to \infty}\Psi(x, s, u), \quad u \to \infty. \] (4.37)
Using relations (1.2) and (1.6) we deduce that
\[ \lambda(z) \int_{-\infty}^{\infty} e^{-iz\xi} f(\xi) d\xi = -\int_{-\infty}^{\infty} e^{-iz\xi}[Lf(\xi)] d\xi. \] (4.38)

Relations (4.34), (4.37) and (4.38) imply the following assertion.

**Theorem 4.8** Let Conditions 4.1 be fulfilled. If the point \( x = 0 \) belongs to \( \Delta \) then the relation
\[ \int_{\Delta} (sI - L_\Delta)f \Psi_\infty(x, s) = f(0) \] (4.39)
is true.

**Remark 4.9** For the Levy processes of type II equality (4.39) was deduced in the paper [17].

Now we shall prove the following assertion.

**Theorem 4.10** Let Conditions 4.1 and inequality (2.7) be fulfilled. If the point \( x = 0 \) belongs to \( \Delta \) then the function
\[ \Psi(x, s) = (I + sB^*)^{-1} \Phi(0, x), \] (4.40)
satisfies relation (4.39).

**Proof** In view of (2.8) we have
\[ -BL_\Delta f = f, \quad f \in C_\Delta. \] (4.41)

Relations (4.40) and (4.41) imply that
\[ ((sI - L_\Delta)f, \Psi(x, s))_\Delta = -((I + sB)L_\Delta f, \Psi)_\Delta = -(L_\Delta f, \Phi(0, x))_\Delta. \] (4.42)

It is easy to see that
\[ \Phi(0, x) = B^*\sigma(x), \] (4.43)
where \( \sigma(x) = -1/2 \) when \( x < 0 \) and \( \sigma(x) = 1/2 \) when \( x > 0 \). Then according to (4.40) and (4.42) relation (4.39) is true.

The theorem is proved.
5 Long time behavior

1. We apply the following Krein-Rutman theorem ([12], section 6):

**Theorem 5.1** If a linear compact operator $T_1$ leaving invariant a cone $K$, has a point of the spectrum different from zero, then it has a positive eigenvalue $\lambda_1$ not less in modulus than any other eigenvalues $\lambda_k$, $(k > 1)$. To this eigenvalue $\lambda_1$ corresponds at least one eigenvector $g_1 \in K$, $(T_1 g_1 = \lambda_1 g_1)$ of the operator $T_1$ and at least one eigenvector $h_1 \in K^*$, $(T_1^* h_1 = \lambda_1 h_1)$ of the operator $T_1^*$.

We remark that in our case the operator $T_1$ has the form (2.8), the cone $K$ consists of non-negative continuous real functions $g(x) \in D_\Delta$ and the cone $K^*$ consists of monotonically increasing bounded real functions $h(x) \in D^*_\Delta$.

In this section we investigate the asymptotic behavior $p(t, \Delta)$ when $t \to \infty$.

2. The spectrum of the operator $B = (1/\Omega)I + T_1$ is situated in the domain $\Re z > 0$. The eigenvalue $\mu_1 = 1/\Omega + \lambda_1$ of the operator $B$ is greater in modulus than any other eigenvalues $\mu_k$, $k > 1$ of $B$. We introduce the domain $D_\varepsilon$:

$$|z - 1/\Omega| < \varepsilon, \ 0 < \varepsilon < 1/\Omega. \quad (5.1)$$

We denote the boundary of the domain $D_\varepsilon$ by $\Gamma_\varepsilon$. If $z$ belongs to $D_\varepsilon$ then the relations

$$\Re(1/z) > c_\varepsilon > 0 \quad (5.2)$$

holds. We denote

$$\text{rank} \lambda_1 = r. \quad (5.3)$$

Now we formulate the main result of this section.

**Theorem 5.2** Let Levy process $X_t$ have type $I_c$, $0 \in \Delta$ and let the corresponding operator $T_1$ satisfy the following conditions:

1. Operator $T_1$ is compact in the Banach space $D_\Delta$.
2. Operator $T_1$ has a point of the spectrum different from zero.

Then the asymptotic equality

$$p(t, \Delta) = e^{-t/\mu_1}[q + o(1)], \quad t \to + \infty, \quad q \geq 0 \quad (5.4)$$

is true.
Proof. Using (4.40) we obtain the equality

\[ p(t, \Delta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (e^{iyt}, \Psi_{\infty}(x, iy))_\Delta dy, \quad t > 0. \]  

(5.5)

Changing the variable \( z = i/y \) we rewrite (5.5) in the form

\[ p(t, \Delta) = \frac{1}{2i\pi} \int_{i\infty}^{-i\infty} (e^{-t/z}, (zI - B^*)^{-1}\Phi(0, x))_\Delta \frac{dz}{z}, \quad t > 0. \]  

(5.6)

As the operator \( T_1 \) is compact, only a finite number of eigenvalues \( \lambda_k, \quad 1 < k \leq m \) of this operator does not belong to the domain \( D_\epsilon \). We deduce from formula (5.6) the relation

\[ p(t, \Delta) = \sum_{k=1}^{m} \sum_{j=0}^{n_k-1} e^{-t/\lambda_k} t^j c_{k,j} + J, \]  

(5.7)

where \( n_k \) is the index of the eigenvalue \( \lambda_k \),

\[ J = -\frac{1}{2i\pi} \int_{\Gamma_\epsilon} \frac{1}{z} e^{-t/z} (1, (B^* - zI)^{-1}\Phi(0, x))_\Delta dz. \]  

(5.8)

We remind that the index of the eigenvalue \( \lambda_k \) is defined as the dimension of the largest Jordan block associated to that eigenvalue. We note that

\[ n_1 = 1. \]  

(5.9)

Indeed, if \( n_1 > 1 \) then there exists such a function \( f_1 \) that

\[ Bf_1 = \lambda_1 f_1 + g_1. \]  

(5.10)

In this case the relations

\[ (Bf_1, h_1)_\Delta = \lambda_1 (f_1, h_1)_\Delta + (g_1, h_1)_\Delta = \lambda_1 (f_1, h_1)_\Delta \]  

(5.11)

are true. Hence \( (g_1, h_1)_\Delta = 0 \), this relation contradicts (5.3). It proves equality (5.9).

Relation (4.43) implies that

\[ \Phi(0, x) \in D_\Delta^*. \]  

(5.12)
Among the numbers $\mu_k$ we choose the ones for which $\text{Re}(1/\mu_k)$ has the smallest value $\delta$. Among the obtained numbers we choose $\mu_k$, $(1 \leq k \leq \ell)$ the indexes $n_k$ of which have the largest value $n$. We deduce from (5.7) and (5.8) that
\[
p(t, \Delta) = e^{-t\delta}t^n[Q(t) + o(1)], \quad t \to \infty.
\] (5.13)

We note that the function
\[
Q(t) = \sum_{k=1}^{\ell} e^{it \text{Im}(\mu_k^{-1})} c_k
\] (5.14)
is almost periodic (see [13]). Hence in view of (5.13) and the inequality $p(t, \Delta) > 0$, $(t \geq 0)$ the following relation
\[
Q(t) \geq 0, \quad -\infty < t < \infty
\] (5.15)
is valid.

First we assume that at least one of the inequalities
\[
\delta < \lambda_{1}^{-1}, \quad n > 1
\] (5.16)
is true. Using (5.16) and the inequality
\[
\lambda_1 > |\lambda_k|, \quad k = 2, 3, ...
\] (5.17)
we have
\[
\text{Im}\mu_j^{-1} \neq 0, \quad 1 \leq j \leq \ell.
\] (5.18)

It follows from (5.14) that
\[
c_j = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} Q(t)e^{-it(\text{Im}\mu_j^{-1})} dt, \quad T \to \infty.
\] (5.19)

In view of (5.15) the relations
\[
|c_j| \leq \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} Q(t) dt = 0, \quad T \to \infty,
\] (5.20)
are valid, i.e. $c_j = 0$, $1 \leq j \leq \ell$. This means that relations (5.14) are not true. Hence the equalities
\[
\delta = \lambda_{1}^{-1}, \quad n = 1
\] (5.21)
are true. From (5.21) we get the asymptotic equality

\[ p(t, \Delta) = e^{-t/\lambda_1} [q + o(1)] \quad t \to \infty, \quad q \geq 0, \]

(5.22)

where

\[ q = \sum_{k=1}^{r} g_k(0) \int_{\Delta} dh_k(x). \]  

(5.23)

Here \( g_k(x) \) are the eigenfunctions of the operator \( B \) corresponding to the eigenvalues \( \mu_1 \), and \( h_k(x) \) are the eigenfunctions of the operator \( B^* \) corresponding to the eigenvalues \( \mu_1 \). The following conditions are fulfilled

\[ (g_k, h_k) = \int_{\Delta} g_k(x) dh_k(x) = 1, \]  

(5.24)

\[ (g_k, h_\ell) = \int_{\Delta} g_k(x) dh_\ell(x) = 0, \quad k \neq \ell. \]  

(5.25)

The theorem is proved.

6 Example

1. Let us consider the example, when

\[ \nu'(x) = p^2 e^{-p|x|}, \quad p > 0, \quad \Delta = [0, \omega], \quad A = 0, \quad \gamma = 0. \]  

(6.1)

Using (1.26) we have

\[ L_\Delta f = -2p[f(x)] - (p/2) \int_{0}^{\omega} e^{-p|x-y|} f(y) dy. \]  

(6.2)

Condition (2.6) is fulfilled. Hence the operator \( L_\Delta^{-1} \) has form (2.7), where the operator \( T_1 \) is defined by the relation

\[ T_1 f = \int_{0}^{\omega} \gamma(x, t) f(t) dt. \]  

(6.3)

It follows from (2.7) and (6.2), that

\[ \gamma(x, t) - (p/2)e^{-p|x-t|} - (p/2) \int_{0}^{\omega} e^{-p|x-y|} \gamma(y, t) dy = 0. \]  

(6.4)
According to (6.4) we have
\[
\frac{\partial^2 \gamma}{\partial x^2} = 0, \ x \neq t. \tag{6.5}
\]
From (6.5) we obtain that
\[
\gamma(x, t) = c_1(t) + c_2(t)x, \ x > t, \tag{6.6}
\]
\[
\gamma(x, t) = c_3(t) + c_4(t)x, \ x < t. \tag{6.7}
\]
With the help of (6.6), (6.7) and equality \(\gamma(x, t) = \gamma(t, x)\) we deduce that
\[
\gamma(x, t) = (\alpha_1 + \alpha_2t) + (\beta_1 + \beta_2t)x, \ t < x. \tag{6.8}
\]
Using (6.6) and relations
\[
\gamma(x, 0) - (p/2)e^{-px} - (p/2) \int_0^\omega e^{-p|x-y|} \gamma(y, 0) dy = 0, \tag{6.9}
\]
\[
\left. \frac{\partial \gamma(x, t)}{\partial t} \right|_{t=0} - (p^2/2)e^{-px} - (p/2) \int_0^\omega e^{-p|x-y|} \left. \frac{\partial \gamma(y, t)}{\partial t} \right|_{t=0} dy = 0. \tag{6.10}
\]
we obtain
\[
\alpha_1 = \frac{p(1 + p\omega)}{2 + p\omega}, \ \beta_1 = -\frac{p^2}{2 + p\omega}, \ \alpha_2 = p\alpha_1, \ \beta_2 = p\beta_1. \tag{6.11}
\]
Thus, the corresponding operator \(T_1\) has the form (6.3), where the kernel \(\gamma(x, t)\) is defined by the relations (6.8) and (6.11).

2. Let us find the eigenvalues and eigenfunctions of the operator \(T_1\), when (6.1) is valid. The eigenfunctions \(f(x)\) and eigenvalues \(\lambda\) satisfy the relations
\[
\int_0^\omega \gamma(x, t) f(t) dt = \lambda f(x), \quad \int_0^\omega \frac{\partial}{\partial t} \gamma(x, t) f(t) dt = \lambda f'(x). \tag{6.12}
\]
It follows from (6.12) that
\[
\lambda f''(x) = -p^2 f. \tag{6.13}
\]
Hence,
\[
f(x, \lambda) = c_1(\lambda) \sin(xp/\sqrt{\lambda}) + c_2(\lambda) \cos(xp/\sqrt{\lambda}). \tag{6.14}
\]
According to (6.12) and (6.14) we have
\[ c_1(\lambda)a(\lambda) + c_2(\lambda)b(\lambda) = \lambda c_2(\lambda), \quad (6.15) \]
\[ c_1(\lambda)a(\lambda) + c_2(\lambda)b(\lambda) = \sqrt{\lambda} c_1(\lambda), \quad (6.16) \]
where
\[ a(\lambda) = \frac{1}{2 + p\omega} [-\sqrt{\lambda} \cos(p\omega/\sqrt{\lambda}) - \lambda \sin(p\omega/\sqrt{\lambda}) + (1 + p\omega)\sqrt{\lambda}], \quad (6.17) \]
\[ b(\lambda) = \frac{1}{2 + p\omega} [\sqrt{\lambda} \sin(p\omega/\sqrt{\lambda}) - \lambda \cos(p\omega/\sqrt{\lambda}) + \lambda]. \quad (6.18) \]
The eigenvalues of the operator $B$ are defined by the relations (6.15) and (6.16), i.e.
\[ \sqrt{\lambda} a(\lambda) + b(\lambda) = \lambda. \quad (6.19) \]
Equalities (6.17)-(6.19) imply
\[ \tan(p\omega/\sqrt{\lambda}) = \frac{2\sqrt{\lambda}}{1 - \lambda}. \quad (6.20) \]
The following table gives the maximal positive roots of the equation (6.20).
\[
\begin{pmatrix}
p\omega & \pi/4 & \pi/3 & \pi/2 & 2\pi/3 & \pi \\
\lambda & 0.445 & 0.617 & 0.162 & 1.433 & 2.454
\end{pmatrix}
\quad (6.21)
\]

7 Discrete Levy measure

We consider the case when Levy measure $\nu(u)$ is discrete, where the gap points we denote by $\nu_k (1 \leq k \leq n)$ and the corresponding gaps we denote by $\sigma_k > 0$. Let condition (1.7) and (1.9) be fulfilled. In this case formula (1.26) takes the form
\[ L_{\Delta} f = -\Omega f(x) + \sum_{k=1}^{n} f(\nu_k + x)\sigma_k, \quad \Omega \geq \sum_{k=1}^{n} \sigma_k \quad (7.1) \]
and the operator $T$ is defined by the relation
\[ Tf = \sum_{k=1}^{n} f(\nu_k + x)\sigma_k, \quad x \in \Delta. \quad (7.2) \]
We note that $f(x) = 0$ if $x \notin \Delta$. Relation (7.2) implies the assertion.
Proposition 7.1 If $\nu_k > 0$, $(1 \leq k \leq n)$ then the operator $T$ has the following properties:

1. There exists such integer number $m$ that the equality
   \[ T^m = 0 \]  
   (7.3)

is valid.
2. The operator $T$ maps the bounded functions in the bounded functions.
3. The operator $T$ maps the non-negative functions in the non-negative functions.
4. The operator $T_1$ in equality (2.8) has the form
   \[ T_1 = T/\Omega + (T/\Omega)^2 + \ldots + (T/\Omega)^{m-1}. \]  
   (7.4)

Acknowledgements The author is very grateful to I. Tydniouk for his calculation of the table 6.21.

References

[1] G. Baxter and M. D. Donsker, *On the Distribution of the Supremum Functional for Processes with Stationary Independent Increments*. Trans. Amer. Math. Soc. 8 (1957), 73–87.

[2] J. Bertoin, *Levy Processes*. University Press, Cambridge, 1996.

[3] S. Bochner, *Vorlesungen über Fouriersche Integrale*. Akademische Verlagsgesellschaft, Leipzig, 1932.

[4] A. Erdelyi, W. Magnus, F. Oberhettingen, and F. G. Tricomi, *Higher Transcendental functions*. New York, 1953.

[5] K.E.Gustatson and D.Rao *Numerical Range*, Springer, 1996.

[6] M.Haase, *The Functional Calculus for Sectorial Operators* Operator Theory 169, Birkhauser, 2006.

[7] P.Hartman and A.Wintner *On the Infinitesimal Generators of Integral Convolutions*, Amer.UJ.Math., 64, 273-298, 1942.

[8] Hengartner W. and Theodorescu R. *Concentration Functions* Academic Press, New York, 1973.
[9] K. Ito, *On Stochastic Differential Equations*. Memoirs Amer. Math. Soc. 4, 1951.

[10] M. Kac, *Probability and Related Topics in Physical Sciences*. Colorado, 1957.

[11] M. Kac, *Some stochastic problems in physics and mathematics*. Dallas, 1957.

[12] M. G. Krein and M. A. Rutman, *Linear Operators Leaving Invariant a Cone in a Banach Space*. Amer. Math. Soc., Translation 26, 1950.

[13] B.M. Levitan, *Some Questions of the Theory of Almost Periodic Functions*. Amer. Math. Soc. Transl. 28, 1950.

[14] J. Radon, *Über lineare Functionaltransformationen und Functionalgleichungen*. Sitzber. Akad. Wiss. Wien, 128, 1083-1121, 1919.

[15] L.A. Sakhnovich, *Integral Equations with Difference Kernels*. Operator Theory 84, Birkhäuser, 1996.

[16] L.A. Sakhnovich, *Levy Processes, Integral Equations, Statistical Physics: Connections and Interactions*. Operator Theory 225, Birkhäuser, 2012.

[17] L.A. Sakhnovich, *Levy Processes: long time behavior and convolution-type form of the Ito representation of the infinitesimal generator*. [arXiv:1306.1492](http://arxiv.org/abs/1306.1492)

[18] K. Sato, *Levy Processes and Infinitely Divisible Distributions*. University Press, Cambridge, 1999.

[19] W. Schoutens, *Levy Processes in Finance*. Wiley series in Probability and Statistics, 2003.

[20] M. Sharpe, *Zeroes of Infinitely Divisible Densities*, The Annals of Mathematical Statistics, Vol. 40, No. 4, 1503-1505, 1969.

[21] M. Stone, *Linear Transformation in Hilbert Space*. New York, 1932.

[22] M. Thomas and O. Barndorff (ed.), *Levy Processes: Theory and Applications*. Birkhauser, 2001.
[23] E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals*. Oxford, 1937.

[24] H.G. Tucker, *The Supports of Infinitely Divisible Distribution functions*, Proc. Amer. Math. Soc., 49, 436-440, 1975.

[25] A. Zygmund, *Trigonometric Series*, v.II, Cambridge, 1959.