Thermal stability of the Nagaoka-Thouless theorems

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Abstract

We prove that the Aizenman-Lieb theorem on ferromagnetism in the Hubbard model holds true even if the electron-phonon interactions and the electron-photon interactions are taken into account. Our proof is based on path integral representations of the partition functions.

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1 Introduction

1.1 Background

The Hubbard model of interacting electrons occupies a special place in the study of ferromagnetism; this is because it is the simplest model which can describe the following fundamental properties:

• the Pauli exclusion principle;
• the Coulomb repulsion between electrons;
• itinerancy of the electrons.

It is believed that ferromagnetism arises from the interplay of these properties. However, to reveal the mechanism of ferromagnetism has been mystery, even today. A first rigorous example of the ferromagnetism in the Hubbard model was constructed by Nagaoka and Thouless [30, 38]. They proved that the ground state of the model exhibits ferromagnetism when one electron is fewer than the half-filling and the Coulomb strength $U$ is very large. The Nagaoka-Thouless theorem is restricted to the ground states. It is logical as well as important to ask whether we can extend the theorem to positive temperatures. This problem was solved by Aizenman and Lieb [1]; in the present paper, we call their result the Aizenman-Lieb theorem, see Theorem 1.4 for detail.
In the real world, the electrons are constantly influenced by the surrounding environment, e.g., the lattice vibrations, the radiation field and the thermal fluctuations. Therefore, the following question naturally arises: are the Nagaoka-Thouless theorem and related properties of many-electron system stable under the influences from the environment? This question has been studied by the author, successfully; in [23, 25, 26, 27], the Nagaoka-Thouless theorem and Lieb’s theorem are shown to be stable, even if the electron-phonon and the electron-photon interactions are taken into account; furthermore, a general structure behind these stabilities is explored in [28].

The principle purpose of the present paper is to prove stabilities of the Aizenman-Lieb theorem under the influences of the lattice vibrations and the quantized radiation field.

For later use, we provide more precise statements of the Nagaoka-Thouless theorem and the Aizenman-Lieb theorem. For each \( \ell \in \mathbb{N} \) even, let \( \Lambda = [-\ell/2, \ell/2)^d \cap \mathbb{Z}^d \). The elements of \( \Lambda \) are called vertices and we say that \( x, y \in \Lambda \) are neighbours if \( \|x - y\| = 1 \), where \( \|x\| = \max_{j=1,\ldots,d} |x_j| = 1 \). A pair \( e = \{x, y\} \in \Lambda \times \Lambda \) is called edge if \( x \) and \( y \) are neighbours. We denote by \( E_{\Lambda} \) the set of all edges. Clearly, \( (\Lambda, E_{\Lambda}) \) becomes a graph.

The Hubbard model on \( \Lambda \) is defined by the Hamiltonian

\[
H_{\text{H},0} = \sum_{\sigma = \pm 1} \sum_{x,y \in \Lambda} (-t_{xy}) c_{x\sigma}^* c_{y\sigma} + U \sum_{x \in \Lambda} n_{x,+1} n_{x,-1} + \sum_{x, y \in \Lambda \atop x \neq y} U_{xy} n_x n_y. \tag{1.1}
\]

The self-adjoint operator \( H_{\text{H},0} \) acts in the \( \Lambda \)-electron space \( \delta_{\Lambda}^{(N)} = \Lambda^N (\ell^2(\Lambda) \oplus \ell^2(\Lambda)) \), where \( \Lambda^N \) indicates the \( N \)-fold antisymmetric tensor product. \( c_{x\sigma}^* \) and \( c_{x\sigma} \) are the fermionic creation- and annihilation operators satisfying the usual anticommutation relations:

\[
\{c_{x\sigma}^*, c_{y\tau}\} = \delta_{\sigma\tau} \delta_{xy}, \quad \{c_{x\sigma}, c_{y\tau}\} = 0, \tag{1.2}
\]

where \( \delta_{ab} \) is the Kronecker delta. The number operators are defined by \( n_{x\sigma} = c_{x\sigma}^* c_{x\sigma} \), \( \sigma = \pm 1 \) and \( n_x = n_{x,+1} + n_{x,-1} \). For simplicity, the hopping matrix \( (t_{xy}) \) satisfies the following:

\[
(T) \quad t_{xy} = \begin{cases} 
 t > 0 & \text{if } \{x, y\} \in E_{\Lambda} \\
 0 & \text{otherwise}.
\end{cases}
\]

( Remark that many of the results in the present paper can be extended to general hopping matrices \( (t_{xy}) \) with \( t_{xy} \geq 0 \) and \( t_{xy} = t_{yx} \).) \( U \) and \( U_{xy} \) are the local and non-local Coulomb matrix elements, respectively. In the present paper, we always assume the following:

\[
(U. 1) \quad U > 0;
\]

\[
(U. 2) \quad U_{xy} = U_{yx} \in \mathbb{R} \text{ for all } x, y \in \Lambda \text{ with } x \neq y.
\]

The spin operators at \( x \in \Lambda \) are defined by

\[
S^{(j)}_x = \frac{1}{2} \sum_{\sigma, \sigma' = \pm 1} c_{x\sigma}^* (s^{(j)})_{\sigma\sigma'} c_{x\sigma'}, \quad j = 1, 2, 3, \tag{1.3}
\]
where \( s^{(j)} (j = 1, 2, 3) \) are the \( 2 \times 2 \) Pauli matrices. The total spin operators are defined by

\[
S_{\text{tot}}^{(j)} = \sum_{x \in \Lambda} S_{x}^{(j)}, \quad j = 1, 2, 3
\]

and

\[
S_{\text{tot}}^{2} = \sum_{j=1}^{3} (S_{\text{tot}}^{(j)})^{2}
\]

with eigenvalues \( S(S + 1) \). The Hubbard Hamiltonian in a uniform magnetic field \( h = (0, 0, 2b) \) is given by

\[
H_{H} = H_{H,0} - 2bS_{\text{tot}}^{(3)}, \quad b > 0.
\]

Let us derive an effective Hamiltonian describing the system with very large \( U \). For this purpose, we introduce the Gutzwiller projection \( P_{G} \) by

\[
P_{G} = \prod_{x \in \Lambda} (1 - n_{x,+1}n_{x,-1}).
\]

\( P_{G} \) is the orthogonal projection onto the subspace with no doubly occupied sites.

**Proposition 1.1** ([26]) *Let us consider the Hubbard model. Assume that \( N = |\Lambda| - 1 \). We define an effective Hamiltonian \( H_{H,\infty} \) by \( H_{H,\infty} = P_{G}H_{H}^{U=0}P_{G} \), where \( H_{H}^{U=0} \) is the Hubbard Hamiltonian \( H_{H} \) with \( U = 0 \). Then we have*

\[
\lim_{U \to \infty} (H_{H} - z)^{-1} = (H_{H,\infty} - z)^{-1}P_{G}, \quad z \in \mathbb{C}\backslash\mathbb{R}
\]

*in the operator norm topology.*

We denote the restriction of \( H_{H,\infty} \) to \( \text{ran}(P_{G}) \) by the same symbol. The Nagaoka-Thouless theorem can be stated as follows.

**Theorem 1.2** ([36, 37]) *Let us consider the Hubbard model. Assume that \( N = |\Lambda| - 1 \). The ground state of \( H_{H,\infty} \) has total spin \( S = (|\Lambda| - 1)/2 \) and is unique apart from the trivial \((2S + 1)\)-degeneracy.*

**Remark 1.3** By [17, Corollary 2.2], we have

\[
\lim_{b \to +0} \lim_{|\Lambda| \to \infty} \frac{\langle S_{\text{tot}}^{(3)} \rangle_{H,\infty}(b)}{|\Lambda|} \geq \sqrt{3} \lim_{|\Lambda| \to \infty} \sqrt{\frac{\langle (S_{\text{tot}}^{(3)})^{2} \rangle_{H,\infty}(b=0)}{|\Lambda|^{2}}} = \frac{1}{2},
\]

where \( \langle \cdot \rangle_{H,\infty}(b) \) is the ground state expectation associated with \( H_{H,\infty} \): \( \langle O \rangle_{H,\infty}(b) = \frac{1}{2S+1} \sum_{m=-S}^{S} \langle \psi_{m}|O\psi_{m} \rangle \). Here, \( \psi_{m} \) is the normalized unique ground state of \( H_{\infty} \) in the \( m \)-subspace \( \mathcal{H}_{H,\infty}^{(N)} \) \( m = \ker (S_{\text{tot}}^{(3)} - m), \ m \in \text{spec}(S_{\text{tot}}^{(3)}) \). Because \( \frac{S_{\text{tot}}^{(3)}}{|\Lambda|} \leq \frac{1}{2} \), we arrive at

\[
\lim_{b \to +0} \lim_{|\Lambda| \to \infty} \frac{\langle S_{\text{tot}}^{(3)} \rangle_{H,\infty}(b)}{|\Lambda|} = \frac{1}{2}.
\]

As we will see, this conclusion is a key to understand extensions of this theorem to positive temperatures.
Let
\[ Z_{H,\infty}(\beta) = \text{Tr}_{\mathcal{H}(N)} \left[ e^{-\beta H_{H,\infty}} \right]. \]  

We define the thermal expectation values of operators by
\[ \langle O \rangle_{H,\infty}(b; \beta) = \frac{\text{Tr}_{\mathcal{H}(N)} \left[ O e^{-\beta H_{H,\infty}} \right]}{Z_{H,\infty}(\beta)}. \]  

**Theorem 1.4 (The Aizenman-Lieb theorem)** Let us consider the Hubbard model. Suppose that \( N = |\Lambda| - 1 \). For all \( 0 < \beta < \infty \) and \( 0 < b \), we obtain
\[ \langle S_{\text{tot}}^{(3)} \rangle_{H,\infty}(b; \beta) > \frac{N}{2} \tanh(\beta b). \]  

Thus, we have
\[ \lim_{b \to +0} \lim_{|\Lambda| \to \infty} \lim_{\beta \to \infty} \frac{\langle S_{\text{tot}}^{(3)} \rangle_{H,\infty}(b; \beta)}{|\Lambda|} = \frac{1}{2}. \]  

In this sense, (1.13) is an extension of the Nagaoka-Thouless theorem to positive temperatures.

### 1.2 Models

#### 1.2.1 The Holstein-Hubbard model

We consider the interaction between the electrons and the lattice vibrations. The Holstein-Hubbard model is widely accepted as a standard model describing such a system. The Hamiltonian of the Holstein-Hubbard model is given by
\[ H_{\text{HH}} = H_{H} + \sum_{x,y \in \Lambda} g_{xy} n_{x}(b_{y}^{*} + b_{y}) + \sum_{x \in \Lambda} \omega b_{x}^{*} b_{x}. \]  

\( H_{\text{HH}} \) acts in the Hilbert space \( \mathcal{H}(N) = \mathcal{H}(N) \otimes \mathcal{H}_{\text{HH}}, \) where \( \mathcal{H}_{\text{HH}} = \mathcal{F}(\ell^{2}(\Lambda)) \), the bosonic Fock space over \( \ell^{2}(\Lambda) \); in general, the bosonic Fock space over \( X \) is defined by
\[ \mathcal{F}(X) = \bigoplus_{n=0}^{\infty} \otimes_{s}^{n} X, \]  

where \( \otimes_{s}^{n} X \) is the \( n \)-fold symmetric tensor product of \( X \) with \( \otimes_{s}^{0} X = \mathbb{C} \). \( b_{x}^{*} \) and \( b_{x} \) are the bosonic creation- and annihilation operators at site \( x \) satisfying the standard commutation relations:
\[ [b_{x}, b_{y}^{*}] = \delta_{xy}, \quad [b_{x}, b_{y}] = 0 \]  

on \( \mathcal{F}_{\text{HH,fin}} = \mathcal{F}_{\text{fin}}(\ell^{2}(\Lambda)) \), where \( \mathcal{F}_{\text{fin}}(X) \) is the indirect direct sum of \( \otimes_{s}^{n} X \). \( g_{xy} \) is the strength of the electron-phonon interaction. The phonons are assumed to be dispersionless with energy \( \omega > 0 \). Henceforth, we assume the following:

\( (G) \) \( (g_{xy})_{x,y} \) is a real symmetric matrix.

Note that \( H_{\text{HH}} \) is a self-adjoint operator, bounded from below.
1.2.2 The Hubbard model coupled to the quantized radiation field

We consider an $N$-electron system coupled to the quantized radiation field. Suppose that the lattice $\Lambda$ is embedded into the region $V = [-L/2, L/2]^3 \subset \mathbb{R}^3$ with $L > 0$. (Thus, when we consider this system, we assume that $d \leq 3$.) The system is described by the following Hamiltonian

\begin{equation}
H_{\text{rad}} = \sum_{x,y \in \Lambda} \sum_{\sigma = \pm 1} (-t_{xy}) \exp \left\{ i \int_{C_{xy}} dr \cdot A(r) \right\} c_{x\sigma}^* c_{y\sigma} + \sum_{k \in V^*} \sum_{\lambda = 1, 2} \omega(k) a(k,\lambda)^* a(k,\lambda)
+ U \sum_{x \in \Lambda} n_{x,1} n_{x,-1} + \sum_{x,y \in \Lambda, x \neq y} U_{xy} n_{x} n_{y}.
\end{equation}

$H_{\text{rad}}$ acts in the Hilbert space $\mathcal{H}_{\text{rad}}^{(N)} = \mathcal{H}_{\text{H}}^{(N)} \otimes \mathcal{H}_{\text{rad}}$. $\mathcal{H}_{\text{rad}}$ is the Fock space over $\ell^2(V^* \times \{1, 2\})$ with $V^* = (\mathbb{Z}/2\mathbb{Z})^3$. $a(k,\lambda)^*$ and $a(k,\lambda)$ are the bosonic creation- and annihilation operators, respectively. These operators satisfy the following commutation relations:

\begin{equation}
[a(k,\lambda), a(k',\lambda')] = \delta_{\lambda\lambda'} \delta_{kk'}, ~ [a(k,\lambda), a(k',\lambda')] = 0
\end{equation}
on $\mathcal{H}_{\text{rad}, \text{fin}} := \mathcal{H}_{\text{fin}}(\ell^2(V^* \times \{1, 2\}))$. The quantized vector potential is given by

\begin{equation}
A(x) = |V|^{-1/2} \sum_{k \in V^*, \lambda = 1, 2} \frac{\chi_\kappa(k)}{2\omega(k)} \varepsilon(k,\lambda) \left( e^{ikx} a(k,\lambda) + e^{-ikx} a(k,\lambda)^* \right).
\end{equation}

The form factor $\chi_\kappa$ is the indicator function of the ball of radius $0 < \kappa < \infty$. The dispersion relation $\omega(k)$ is chosen to be $\omega(k) = |k|$ for $k \in V^* \setminus \{0\}$, $\omega(0) = m_0$ with $0 < m_0 < \infty$. For concreteness, the polarization vectors are chosen as

\begin{equation}
\varepsilon(k, 1) = \frac{(k_2, -k_1, 0)}{\sqrt{k_1^2 + k_2^2}}, ~ \varepsilon(k, 2) = \frac{k}{|k|} \wedge \varepsilon(k, 1).
\end{equation}

To avoid ambiguity, we set $\varepsilon(k,\lambda) = 0$ if $k_1 = k_2 = 0$. $A(x)$ is essentially self-adjoint. We denote its closure by the same symbol. $C_{xy}$ is a piecewisely smooth curve from $x$ to $y$. Note that the more precise definition of $\int_{C_{xy}} A(r) \cdot dr$ will be given in Section 3. This model was introduced by Giuliani et al. in [10]. $H_{\text{rad}}$ is essentially self-adjoint and bounded from below. We denote its closure by the same symbol.

1.3 Results

We will display an extension of Theorem [14]. For this purpose, we need the following proposition.

**Proposition 1.5 ([26])** Assume that $N = |\Lambda| - 1$. For $\mathfrak{z} = \text{rad}, \text{HH}$, we define an effective Hamiltonian $H_{\mathfrak{z}, \infty}$ by $H_{\mathfrak{z}, \infty} = P_{\mathfrak{z}} H_{\mathfrak{z}, U=0}^U P_{\mathfrak{z}}$, where $H_{\mathfrak{z}, U=0}$ is the corresponding Hamiltonian $H_{\mathfrak{z}}$ with $U = 0$. Then we have

\begin{equation}
\lim_{U \to \infty} (H_{\mathfrak{z}} - z)^{-1} = (H_{\mathfrak{z}, \infty} - z)^{-1} P_{\mathfrak{z}}, \quad z \in \mathbb{C} \setminus \mathbb{R}
\end{equation}
in the operator norm topology.
We denote the restriction of $H_{\natural,\infty}$ to ran($P_G$) by the same symbol. The following theorem is a generalized version of the Nagaoka-Thouless theorem.

**Theorem 1.6** ([26]) The ground state of $H_{\natural,\infty}$ has total spin $S = (|\Lambda| - 1)/2$ and is unique apart from the trivial $(2S + 1)$-degeneracy for $\natural = \text{rad}, HH$.

**Remark 1.7** As before, we have the following:

$$\lim_{b \to +0} \lim_{|\Lambda| \to \infty} \frac{\langle S_{\text{tot}}^{(3)} \rangle_{\natural,\infty}(b)}{|\Lambda|} = \frac{1}{2}, \quad \natural = \text{rad}, HH,$$

where $\langle \cdot \rangle_{\natural,\infty}(b)$ is the ground state expectation associated with $H_{\natural,\infty}$: $\langle O \rangle_{\natural,\infty}(b) = \frac{1}{Z_{\natural,\infty}(b)}$.

We also remark that there are some other extensions of the Nagaoka-Thouless theorem, see, e.g., [15, 16].

Theorem 1.6 can be extended to positive temperatures as follows. Let

$$Z_{\natural,\infty}(\beta) = \text{Tr}_{\mathcal{H}_{\natural,\infty}}[e^{-\beta H_{\natural,\infty}}], \quad \natural = \text{rad}, HH.$$  

The thermal expectation values of operators are defined by

$$\langle O \rangle_{\natural,\infty}(\beta; b) = \frac{\text{Tr}_{\mathcal{H}_{\natural,\infty}}[O e^{-\beta H_{\natural,\infty}}]}{Z_{\natural,\infty}(\beta)}.$$  

The following theorem is an extension of Theorem 1.4.

**Theorem 1.8** Suppose that $N = |\Lambda| - 1$. For all $0 < \beta < \infty$, $0 < b$ and $\natural = \text{rad}, HH$, we obtain

$$\langle S_{\text{tot}}^{(3)} \rangle_{\natural,\infty}(\beta; b) > \frac{N}{2} \tanh(\beta b).$$

Thus, we have

$$\lim_{b \to +0} \lim_{|\Lambda| \to \infty} \lim_{\beta \to \infty} \frac{\langle S_{\text{tot}}^{(3)} \rangle_{\natural,\infty}(\beta; b)}{|\Lambda|} = \frac{1}{2}. \quad (1.27)$$

In this sense, (1.26) can be regarded as an extension of Theorem 1.4 to positive temperatures.

We will provide a proof of Theorem 1.8 in Section 7.

**Remark 1.9**

- Theorem 1.8 can be extended to models on a general bipartite lattice with nearest neighbour hopping.
- Let us consider the multi-polaron model in a bounded region $[-L, L]^N$ with periodic boundary conditions, and a hard-core repulsion. (As for the definition of this model, see, e.g., [8].) If $N$ is odd, then our method can be applicable to the model. Similar observations hold true for the Pauli-Fierz model in $[-L, L]^N$. For the precise definition of this model, we refer to [5, 11, 21, 35].
1.4 Organization

The organization of the present paper is as follows. In Section 2, we construct Feynman-Kac-Itô formulas for the magnetic Hubbard model. In Section 3, we provide trace formulas for free Bose fields in terms of path integral representations. By combining the formulas in Sections 2 and 3, we construct Feynman-Kac-Itô formulas for the partition functions for $H_{HH}$ and $H_{rad}$. Section 6 is devoted to give random loop representations for the partition functions. Applying these representations, we give a proof of Theorem [in Section 7]. In Section 8, we derive an interesting formula for the partition functions. Appendix A is devoted to prove a useful proposition.

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2 Feynman-Kac-Itô formulas for the Hubbrd models

2.1 A Feynman-Kac-Itô formuals for one-electron Hamiltonians

2.1.1 Discrete magnetic Schrödinger operators

Let us consider a single electron living in $\Lambda$. One is given a hopping matrix $(t_{xy})$ satisfying $(T)$ in Section 1. The kinetic energy of the electron is a self-adjoint operator $h_0$ acting in $\ell^2(\Lambda) \oplus \ell^2(\Lambda) \cong \ell^2(\Omega)$ with $\Omega = \Lambda \times \{-1, +1\}$ given by

$$ (h_0 f)(x, \sigma) = \sum_{\sigma = \pm 1} \sum_{y \in \Lambda} t_{xy} \left( f(x, \sigma) - f(y, \sigma) \right), \quad f \in \ell^2(\Omega). \quad (2.1) $$

Note that the inner product of $\ell^2(\Omega)$ is given by $\langle f | g \rangle_{\ell^2(\Omega)} = \sum_{\sigma = \pm 1} \sum_{x \in \Lambda} f(x, \sigma)^* g(x, \sigma) = \sum_{X \in \Omega} f(X)^* g(X)$. By a magnetic potential on $\Lambda$, we understand a matrix $\alpha = (\alpha_{xy})$ such that

- $\alpha_{xy} \in \mathbb{R}$ for all $\{x, y\} \in E_\Lambda$;
- $\alpha_{xy} = - \alpha_{yx}$ for all $\{x, y\} \in E_\Lambda$.

The kinetic energy of the electron in the magnetic potential is given by

$$ (h_0(\alpha) f)(x, \sigma) = \sum_{\sigma = \pm 1} \sum_{y \in \Lambda} t_{xy} \left( f(x, \sigma) - e^{i\alpha_{xy}} f(y, \sigma) \right), \quad f \in \ell^2(\Omega). \quad (2.2) $$

Let $v$ be a potential, i.e., a multiplication operator by the real-valued function $v$: $(v f)(x, \sigma) = v(x) f(x, \sigma)$ for all $f \in \ell^2(\Omega)$. Then discrete magnetic Schrödinger operators are defined by

$$ h_v(\alpha) = h_0(\alpha) + v. \quad (2.3) $$

Trivially, $h_v(\alpha)$ is self-adjoint.
2.1.2 A Feynman-Kac-Itô formula for $h_v(\alpha)$

As a first step, we will construct a Feynman-Kac-Itô formula for $h_v(\alpha)$. In this study, we employ a useful description by Güneysu, Keller and Schmidt [13].

For notational simplicity, we set $N_0 = \{0\} \cup \mathbb{N}$. Let $(Y_n)_{n \in N_0}$ be a discrete time Markov chain with state-space $\Omega$, which satisfies, for $X = (x, \sigma), Y = (y, \tau) \in \Omega$,

$$P(Y_n = X | Y_{n-1} = Y) = \delta_{\sigma \tau} \frac{t_{xy}}{d(y)} , \quad n \in \mathbb{N},$$

(2.4)

where $d(x) = \sum_{y \in \Lambda} t_{xy}$. Throughout, we work with fixed probability spaces $(M, \mathcal{F}, P)$. Let $(T_n)_{n \in \mathbb{N}}$ be independent exponentially distributed random variables of parameter $1$, independent of $(Y_n)_{n \in N_0}$. Set

$$S_n = \frac{T_n}{d(Y_{n-1})}, \quad J_n = S_1 + \cdots + S_n$$

(2.5)

and

$$X_t = Y_n \quad \text{if} \quad J_n \leq t < J_{n+1} \quad \text{for some} \quad n.$$  

(2.6)

$(X_t)_{t \geq 0}$ becomes a right continuous process. Furthermore, $J_0 := 0, J_1, J_2, \ldots$ are the jump times of $(X_t)_{t \geq 0}$ and $S_1, S_2, \ldots$ are the holding times of $(X_t)_{t \geq 0}$. Let $P_X(\cdot) = P(\cdot|X_0 = X)$ and let $(\mathcal{F}_t)_{t \geq 0}$ be the filtration defined by $\mathcal{F}_t = \sigma(X_s|s \leq t)$. Then $(M, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, (P_X)_{X \in \Omega})$ is a strong Markov process, see, e.g., [31, Theorems 2.8.1 and 6.5.4].

Set $N(t) := \sup\{n \in \mathbb{N}_0 | J_n \leq t\}$, the number of jumps of $(X_t)_{t \geq 0}$ until $t \geq 0$. For $s < t$, we introduce random variables by

$$\int_s^t \alpha(dX_u) = \sum_{n=N(s)+1}^{N(t)} \alpha X_{J_n-1} X_{J_n},$$

(2.7)

$$S_{[s,t]}(v, \alpha|X_\bullet) = i \int_s^t \alpha(dX_u) - \int_s^t v(X_u)du,$$

(2.8)

In the above definition, we understand that $\int_s^t \alpha(dX_u) = 0$, provided that $N(s) = N(t)$.

Let $g$ be a function on $\Omega$. The multiplication operator associated with $g$ is defined by $(M_g f)(X) = g(X) f(X)$ for $f \in \ell^2(\Omega)$. In what follows, $M_g$ is abbreviated as $g$ for notational simplicity. Trivially, we have $\|g\|_{C^*(\Omega)} \leq \|g\|_{\ell^\infty(\Omega)} \|f\|_{C^*(\Omega)}$, where $\|g\|_{\ell^\infty(\Omega)} = \max_{X \in \Omega} |g(X)|$. We denote by $\ell^\infty(\Omega)$ the abelian $C^*$-algebra of multiplication operators on $\ell^2(\Omega)$, equipped with the norm $\|\cdot\|_{\ell^\infty(\Omega)}$.

**Proposition 2.1** Let $\alpha_1, \ldots, \alpha_n$ be magnetic potentials. Let $f_0, f_1, \ldots, f_n \in \ell^\infty(\Omega)$ and $f \in \ell^2(\Omega)$. For each $0 < t_1 < t_2 < \cdots < t_n$ and $X \in \Omega$, we have

$$\left( f_0 e^{-t_1 h_v(\alpha_1)} f_1 e^{-(t_2-t_1) h_v(\alpha_2)} f_2 \cdots f_{n-1} e^{-(t_n-t_{n-1}) h_v(\alpha_n)} f \right)(X)$$

$$= E_X \left[ f_0(X_0) f_1(X_{t_1}) \cdots f_{n-1}(X_{t_{n-1}}) f(X_n) \exp \left\{ \sum_{\ell=1}^n S_{[t_{\ell-1}, t_\ell]}(v, \alpha|X_\bullet) \right\} \right],$$

(2.9)

where $E_X[F] = \int_M dP_X F$ and $t_0 = 0$. 

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Proof. In [13, Theorem 4.1], the following Feynman-Kac-Itô formula has been established:

\[
(e^{-th_v(\alpha)}f)(X) = E_X\left[e^{S_{[0,t]}(v,\alpha)X}f(X_t)\right], \quad f \in \ell^2(\Omega), \quad X \in \Omega.
\]

(2.10)

By this formula and the Markov property of \((X_t)_{t \geq 0}\), we can prove the assertion in Proposition 2.1. \(\Box\)

2.2 A Feynman-Kac-Itô formula for \(N\)-electron Hamiltonians

Let us consider an \(N\)-electron system. The non-interacting Hamiltonian is given by

\[
L(\alpha) = h_v(\alpha) \otimes 1 + \cdots + 1 \otimes h_v(\alpha) \otimes 1 + \cdots + 1 \otimes \cdots \otimes 1 \otimes h_v(\alpha).
\]

(2.11)

Note that \(L(\alpha)\) acts in \(\bigotimes_{j=1}^{N} \ell^2(\Omega) \cong \ell^2(\Omega^N)\). In order to take the Fermi-Dirac statistics into consideration, we introduce the antisymmetrizer \(A_N\) on \(\ell^2(\Omega^N)\) by

\[
(A_N F)(X) = \sum_{\tau \in \mathfrak{S}_N} \frac{\text{sgn}(\tau)}{N!} F(\tau^{-1} X), \quad F \in \ell^2(\Omega^N), \quad X = (X^{(1)}, \ldots, X^{(N)}) \in \Omega^N,
\]

(2.12)

where \(\mathfrak{S}_N\) is the permutation group on \(\{1, \ldots, N\}\), and \(\tau X := (X^{(\tau(1))}, \ldots, X^{(\tau(N))})\), \(\tau \in \mathfrak{S}_N\). As is well-known, \(A_N\) is an orthogonal projection from \(\ell^2(\Omega^N)\) onto \(\ell^2_{as}(\Omega^N)\), the set of all antisymmetric functions on \(\Omega^N\). We define \(L_{as}(\alpha) = A_N L(\alpha) A_N\). Note that we can naturally identify \(L_{as}(\alpha)\) with \(L(\alpha) \upharpoonright \ell^2_{as}(\Omega^N)\).

We wish to construct a Feynman-Kac-Itô formula for \(L_{as}(\alpha)\). For this purpose, let

\[
\Omega^N_\neq = \left\{ X \in \Omega^N \left| X^{(i)} \neq X^{(j)} \text{ for all } i, j \in \{1, \ldots, N\} \right. \right\}.
\]

(2.13)

We introduce an event by

\[
D = D_O \cap D_S
\]

(2.14)

with

\[
D_O = \left\{ m \in (M)^N \left| X_s(m) \in \Omega^N_\neq \text{ for all } s \in [0, \infty) \right. \right\},
\]

(2.15)

\[
D_S = \left\{ m \in (M)^N \left| \sigma_s^{(j)}(m) = \sigma_0^{(j)}(m) \text{ for all } j \in \{1, \ldots, n\} \text{ and } s \in [0, \infty) \right. \right\},
\]

(2.16)

where in the definition of \(D_S\), we used the following notation: \(X_s^{(j)}(m) = (x_s^{(j)}(m), \sigma_s^{(j)}(m))\). For each \(m \in D\), a right-continuous \(\Omega^N\)-valued function \((X_t(m))_{t \geq 0} = (X_t^{(1)}(m), \ldots, X_t^{(N)}(m))\) is simply called a path; the path \((X_t(m))_{t \geq 0}\) represents a trajectory of the \(N\)-electrons.

Let us write \(X_t^{(j)}(m) = (x_t^{(j)}(m), \sigma_t^{(j)}(m))\); \(\sigma_t^{(j)}(m)\) is called the spin component of \(X_t^{(j)}(m)\), and \(x_t^{(j)}(m)\) is called the spatial component of \(X_t^{(j)}(m)\), respectively. We
note that, for \( m \in D \), the path \((X_t(m))_{t \geq 0}\) possesses properties such that \( \sigma^{(j)}_t(m) \) are constant in time; and there are no encounters of electrons of equal spin.

Let \( F \) be a function on \( \Omega^N \). We say that \( F \) is symmetric if \( F(\tau X) = F(X) \) for all \( \tau \in \mathbb{G}_N \) and \( X \in \Omega^N \). We denote by \( \ell^\infty_s(\Omega^N) \) the abelian C\*algebra of multiplication operators by symmetric functions on \( \Omega^N \), equipped with the norm \( \| \cdot \|_{\ell^\infty(\Omega^N)} \).

**Proposition 2.2** For every \( F_0, F_1, \ldots, F_{n-1} \in \ell^\infty_s(\Omega^N), \) \( F \in \ell^2_{as}(\Omega^N), \) \( 0 < t_1 < t_2 < \cdots < t_n \) and \( X = (X^{(1)}, \ldots, X^{(N)}) \in \Omega^N \), we have

\[
\left( F_0 e^{-t_1 L_{as}(\alpha)} F_1 e^{-(t_2-t_1)L_{as}(\alpha_2)} F_2 \cdots F_{n-1} e^{-(t_n-t_{n-1})L_{as}(\alpha_n)} F \right)(X) = E_X \left[ 1_D F_0(X_0) F_1(X_{t_1}) \cdots F_{n-1}(X_{t_{n-1}}) F(X_{t_n}) \exp \left\{ \sum_{j=1}^N \sum_{\ell=1}^n S_{[\ell_{t_1},t_\ell]}(v,\alpha_{\ell} | X^{(j)}_{\ell}) \right\} \right],
\]

(2.17)

where \( E_X[F] = \int_{\Omega^N} \prod_{j=1}^N dP_{X^{(j)}} F \) and \( 1_D \) is the indicator function of the event \( D \).

**Proof.** It is not so difficult to show (2.17) without the term \( 1_D \) from Proposition 2.1. Below, we will explain why the term \( 1_D \) appears in the formula.

Let \( P \) be the multiplication operator by \( 1_{\Omega^N} \). We readily confirm that \( P \) is the orthogonal projection from \( \ell^2_{as}(\Omega^N) \) onto \( \ell^2_{as}(\Omega^N) \). In addition, it holds that

\[ P L_{as}(\alpha) = L_{as}(\alpha) P. \] (2.18)

We denote by \( \sigma_0 = (\sigma_0^{(1)}, \ldots, \sigma_0^{(N)}) \in \{-1, +1\}^N \) the set of spin components of \( X = (X^{(1)}, \ldots, X^{(N)}) \) with \( X^{(j)} = (x_0^{(j)}, \sigma_0^{(j)}) \). Let \( Q \) be the multiplication operator by \( 1_{\{\sigma_0\}^t} \):

\[ (QF)(Y) = \delta_{\sigma_0(\sigma(Y))} F(Y), \] \( F \in \ell^2_{as}(\Omega^N) \), where \( \sigma(Y) \) is the set of spin components of \( Y \in \Omega^N \). Note that

\[ Q L_{as}(\alpha) = L_{as}(\alpha) Q. \] (2.19)

We set \( L_0 = L_{as,v=0}(\alpha = 0) \). By (2.17), we know that

\[ L_0 1_{\{\sigma_0\}} = 0. \] (2.20)

For \( t \geq 0 \), let \( D(t) = \{ m \in (M)^N \mid X_t(m) \in \Omega^N \} \) and \( \sigma_t(m) = \sigma_0 \), where, as before, \( \sigma_t(m) \) is the set of spin components of \( X_t(m) \). By setting \( 1_{\Omega^N \cap \{\sigma_0\}} := 1_{\Omega^N} \times 1_{\{\sigma_0\}} \), we have, for \( 0 < t_1 < t_2 < \cdots < t_n \),

\[
P_X \left( \bigcap_{i=1}^n D(t_i) \right) = E_X \left[ 1_{\Omega^N \cap \{\sigma_0\}}(X_{t_n}) 1_{\Omega^N \cap \{\sigma_0\}}(X_{t_{n-1}}) \cdots 1_{\Omega^N \cap \{\sigma_0\}}(X_{t_1}) X_t \right] = \left( e^{-t_n L_0 1_{\Omega^N \cap \{\sigma_0\}}} \cdot e^{-(t_{n-1}-t_n)L_0 1_{\Omega^N \cap \{\sigma_0\}}} \cdots e^{-(t_2-t_1)L_0 1_{\Omega^N \cap \{\sigma_0\}}} \right)(X) = \left( e^{-t_n L_0 1_{\Omega^N \cap \{\sigma_0\}}} \right)(X) = 1 - \left( e^{-t_n L_0 1_{\Omega^N \cap \{\sigma_0\}}} \right)(X), \]

(2.21)
where \((\Omega^N)^c\) is the complement of \(\Omega^N\). In the third equality, we have used (2.18) and (2.19); in the last equality, we have used (2.20). By using the fact \(\lim_{t \to \infty} e^{-tL_0}1_{(\Omega^N)^c \cap \{\sigma=0\}} = 0\), we have 

\[PX\left(\bigcap_{t \in \mathbb{Q}} D(t)\right) = 1.\]

Because \(D = \bigcap_{t \in \mathbb{Q}} D(t)\) by the right-continuity of \((X_s)_{s \geq 0}\), we arrive at \(PX(D) = 1\). Consequently, Proposition 2.2 follows from Proposition 2.1. □

Now, let \(V\) be an interaction between electrons, which is a multiplication operator on \(\ell^2_{\mathrm{as}}(\Omega^N)\) by a symmetric function \(V\). Our Hamiltonian is given by \(L_V(\alpha) = L_{\mathrm{as}}(\alpha) + V\). Furthermore, by taking an interaction between spins and a uniform field \(h = (0, 0, 2\hbar)\) into consideration, we arrive at the following Hamiltonians:

\[L_{V,h}(\alpha) = L_V(\alpha) - b \sum_{j=1}^N \sigma_j, \quad (2.22)\]

where the linear operator \(\sigma_j\) is defined by

\[(\sigma_j f)(X^{(1)}, \ldots, X^{(N)}) = \sigma(X^{(j)}) f(X^{(1)}, \ldots, X^{(N)}), \quad f \in \ell^2_{\mathrm{as}}(\Omega^N),\]

where \(\sigma(X^{(j)})\) is the spin component of \(X^{(j)}\). In general, \(\alpha\) and \(b\) are independent.

By using (2.17), we obtain the following.

**Proposition 2.3** Let \(\alpha_1, \ldots, \alpha_n\) be magnetic potentials. For every \(F_0, F_1, \ldots, F_{n-1} \in \ell^2_{\mathrm{as}}(\Omega^N)\), \(F \in \ell^2_{\mathrm{as}}(\Omega^N)\), \(0 < t_1 < t_2 < \cdots < t_n\) and \(X \in \Omega^N_{\mathbb{R}}\), we have

\[
\begin{align*}
&\left(F_0 e^{-t_1 L_{V,h}(\alpha_1)} F_1 e^{-(t_2-t_1)L_{V,h}(\alpha_2)} F_2 \cdots F_{n-1} e^{-(t_n-t_{n-1})L_{V,h}(\alpha_n)} F\right)(X) \\
= &E_X \left[1_D F_0(X_0) F_1(X_{t_1}) \cdots F_{n-1}(X_{t_{n-1}}) F(X_{t_n}) \times \right. \\
&\left. \times \exp \left\{\sum_{j=1}^N \sum_{\ell=1}^n S_{[\ell-1,\ell]}(v, \alpha_\ell | X^{(j)}_\bullet) \right. - \int_0^{t_n} V(X_s) ds + b \sum_{j=1}^N \sigma(X^{(j)}_0) \right\} \right]. \quad (2.24)
\end{align*}
\]

### 2.3 A Feynman-Kac-Itô formula for the magnetic Hubbard models

Let \(\alpha = (\alpha_{xy})\) be a magnetic potential. The **magnetic Hubbard Hamiltonian** is given by

\[
H(\alpha) = \sum_{x, y \in \Lambda} \sum_{\sigma = \pm 1} (-t_{xy}) e^{i\alpha_{xy} \ell_{x\sigma} e_{y\sigma}} + U \sum_{x \in \Lambda} n_{x+1} n_{x-1} + \sum_{x \neq y} U_{xy} n_x n_y - 2b \mathcal{S}^{(3)}_{\text{tot}}.
\]

In order to construct a path integral formula for \(H(\alpha)\), we need some preliminaries. Let \(\mathcal{E}\) be the fermionic Fock space over \(\ell^2(\Omega)\):

\[
\mathcal{E} = \bigoplus_{n=0}^{\infty} \ell^2(\Omega).
\]

(2.26)
The $N$-electron space $\mathcal{H}^{(N)}(\cong l^2_{as}(\Omega^N))$ can be regarded as a subspace of $\mathcal{E}$ in the following manner:

$$f_1 \wedge \cdots \wedge f_N = c(f_1)^* \cdots c(f_N)^* \Omega_{el}, \quad f_1, \ldots, f_N \in l^2(\Omega), \quad (2.27)$$

where $c(f)^*$, $f \in l^2(\Omega)$ is defined by $c(f)^* = \sum_{X \in \Omega} f(X) c_X^*$, and $\Omega_{el}$ is the Fock vacuum in $\mathcal{E}$. Let $A = (a_{XY})_{X,Y}$ be a self-adjoint operator on $l^2(\Omega)$. Under the identification (2.27), we have

$$\sum_{X,Y \in \Omega} a_{XY} c_X^* c_Y \mid \mathcal{H}^{(N)} = d\Gamma_{as,N}(A), \quad (2.28)$$

where

$$d\Gamma_{as,N}(A) = \left( \begin{array}{c} A \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes A \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes 1 \otimes A \end{array} \right) \mid \mathcal{H}^{(N)}. \quad (2.29)$$

Let $T(\alpha) = (-t_{xy} e^{i\alpha_{xy}})_{x,y}$. By (2.29), we have

$$\sum_{x,y \in \Lambda} \sum_{\sigma = \pm 1} (-t_{xy} e^{i\alpha_{xy}}) c_{x,\sigma}^* c_{y,\sigma} \mid \mathcal{H}^{(N)} = d\Gamma_{as,N}(T(\alpha)) = d\Gamma_{as,N}(h_\mu(\alpha)), \quad (2.30)$$

where we have used the fact $h_0(\alpha) = -\mu + T(\alpha)$ with $\mu(x) = -\sum_{y \in \Lambda} t_{xy}$. Furthermore, we obtain

$$n_{x,\sigma} = d\Gamma_{as,N}(\delta_{(x,\sigma)}), \quad n_x = \sum_{\sigma = \pm 1} d\Gamma_{as,N}(\delta_{(x,\sigma)}), \quad (2.31)$$

where $\delta_{(x,\sigma)}$ is the multiplication operator by the function $\delta_{(x,\sigma)}(Y) := \delta_{\sigma \tau} \delta_{xy}$, $Y = (y, \tau) \in \Omega$. Thus, the Coulomb interaction term in (2.25) can be identified with the multiplication operator $\tilde{V}$ defined by

$$\tilde{V} = \tilde{V}_0 + \tilde{V}_d \quad (2.32)$$

where

$$\tilde{V}_d = U \sum_{x \in \Lambda} \sum_{i,j=1}^N \delta_{(x+1),i}^{(j)} \delta_{(x-1),i}^{(j)} \quad \tilde{V}_0 = \sum_{\sigma, \tau = \pm 1} \sum_{x \neq y} \sum_{i,j=1}^N U_{xy} \delta_{(x,\sigma)}^{(i)} \delta_{(y,\tau)}^{(j)}, \quad (2.33)$$

and

$$\delta_{(x,\sigma)}^{(j)} = \underbrace{1 \otimes \cdots \otimes 1 \otimes \delta_{(x,\sigma)} \otimes 1 \otimes \cdots \otimes 1}_{j} \quad j = 1, \ldots, N. \quad (2.34)$$

Consequently, we arrive at

$$H(\alpha) = L_{\tilde{V},\mu}(\alpha) \quad (2.35)$$

with $v = \mu$. By Proposition (2.3) and (2.35), we obtain a Feynman-Kac-Itô formula for $H(\alpha)$:
Theorem 2.4 Let $V = \tilde{V} + d\Gamma_{as,N}(\mu)$. Let $\alpha_1, \ldots, \alpha_n$ be magnetic potentials. For every $F_0, F_1, \ldots, F_{n-1} \in \ell_{as}^2(\Omega^N)$, $F \in \ell_{as}^2(\Omega^N)$, $0 < t_1 < t_2 < \cdots < t_n$ and $X \in \Omega_{\mu}^N$, we have
\[
\left( F_0 e^{-t_1 H(\alpha_1)} F_1 e^{-(t_2-t_1) H(\alpha_2)} F_2 \cdots F_{n-1} e^{-(t_n-t_{n-1}) H(\alpha_n)} F \right)(X)
\]
\[= \mathrm{E}_X \left[ 1_{D_\infty} F_0(X_0) F_1(X_{t_1}) \cdots F_{n-1}(X_{t_{n-1}}) F(X_{t_n}) \right. \]
\[\times \exp \left\{ \sum_{j=1}^N \sum_{\ell=1}^n \mathcal{S}_{[\ell-1, \ell]}(0, \alpha_\ell | X^{(j)}_\bullet) - \int_0^{t_n} V(X_s) ds + b \sum_{j=1}^N \sigma(X^{(j)}_0) \right\} \] \hspace{1cm} (2.36)

2.4 A Feynman-Kac-Itô formula for $H(\alpha)$ in the large $U$ limit

Using a manner of proof similar to that applied to [26, Theorem 2.5], we can prove the following proposition.

Proposition 2.5 Let $\alpha$ be a magnetic potential. Assume $N = |\Lambda| - 1$. We define an effective Hamiltonian $H_\infty(\alpha)$ by $H_\infty(\alpha) = P_G H^{U=0}(\alpha) P_G$, where $H^{U=0}(\alpha)$ is the magnetic Hubbard Hamiltonian $H(\alpha)$ with $U = 0$. Then we have
\[
\lim_{U \to \infty} (H(\alpha) - z)^{-1} = (H_\infty(\alpha) - z)^{-1} P_G, \quad z \in \mathbb{C} \setminus \mathbb{R} \] (2.37)
in the operator norm topology.

Let
\[
\Omega_{\mu, \infty}^N = \left\{ X \in \Omega^N \mid x^{(i)} \neq x^{(j)} \text{ for all } i, j \in \{1, \ldots, N\} \text{ with } i \neq j \right\}. \hspace{1cm} (2.38)
\]

Here, we used the following notations: $X = (X^{(1)}, \ldots, X^{(N)})$ with $X^{(j)} = (x^{(j)}, \sigma^{(j)})$. We set $D_\infty = D_{0, \infty} \cap D_S$ with
\[
D_{0, \infty} = \left\{ m \in (M)^N \mid X_s(m) \in \Omega_{\mu, \infty}^N \text{ for all } s \in [0, \beta] \right\}. \hspace{1cm} (2.39)
\]
Note that, for each $m \in D_\infty$, there are no electron encounters in the corresponding path $(X_t(m))_{t \geq 0}$.

Theorem 2.6 Suppose that $N = |\Lambda| - 1$. Let $\alpha_1, \ldots, \alpha_n$ be magnetic potentials. For every $F_0, F_1, \ldots, F_{n-1} \in \ell_{as}^2(\Omega^N)$, $F \in \ell_{as}^2(\Omega^N)$, $0 < t_1 < t_2 < \cdots < t_n$ and $X \in \Omega_{\mu}^N$, we have
\[
\left( F_0 e^{-t_1 H(\alpha_1)} F_1 e^{-(t_2-t_1) H(\alpha_2)} F_2 \cdots F_{n-1} e^{-(t_n-t_{n-1}) H(\alpha_n)} F \right)(X)
\]
\[= \mathrm{E}_X \left[ 1_{D_\infty} F_0(X_0) F_1(X_{t_1}) \cdots F_{n-1}(X_{t_{n-1}}) F(X_{t_n}) \right. \]
\[\times \exp \left\{ \sum_{j=1}^N \sum_{\ell=1}^n \mathcal{S}_{[\ell-1, \ell]}(0, \alpha_\ell | X^{(j)}_\bullet) - \int_0^{t_n} V_0(X_s) ds + b \sum_{j=1}^N \sigma(X^{(j)}_0) \right\} \] \hspace{1cm} (2.40)
where $V_0 = \tilde{V}_0 + d\Gamma_{as,N}(\mu)$. Here, $\tilde{V}_0$ is given by (2.33).
Proof. By Proposition 2.5, we have

\[
\lim_{U \to \infty} \left( F_0 e^{-t_1 H(\alpha_1)} F_1 e^{-t_2 H(\alpha_2)} \cdots F_{n-1} e^{-t_{n-1} H(\alpha_{n-1})} F_n \right) = F_0 e^{-t_1 H(\alpha_1)} F_1 e^{-t_2 H(\alpha_2)} \cdots F_{n-1} e^{-t_{n-1} H(\alpha_{n-1})} H(\alpha_n) F_n. \tag{2.41}
\]

We denote by \(1_D G_U(X_\tau)\) the integrand in the right hand side of (2.36). Then we have

\[
E_X[1_D G_U(X_\tau)] = E_X[1_{D,\infty} G_{U=0}(X_\tau)] + E_X[1_{D,\infty} G_{U=0}(X_\tau)]. \tag{2.42}
\]

Because \(\lim_{U \to \infty} G_U(X_\tau(m)) = 0\) for all \(m \in D \setminus D_{\infty}\), we have, by the dominated convergence theorem,

\[
\lim_{U \to \infty} E_X[1_D G_U(X_\tau)] = E_X[1_{D,\infty} G_{U=0}(X_\tau)]. \tag{2.43}
\]

Combining (2.36), (2.41) and (2.43), we obtain the desired assertion. \(\square\)

2.5 A trace formula for \(H(\alpha)\)

First, let us construct a complete orthonormal system (CONS) for \(\ell^2(\Omega^N)\). For each \(X \in \Omega^N\), we set \(\delta_X = \otimes_{j=1}^N \delta_X^{(j)} \in \ell^2(\Omega^N)\) and \(e_X = A_N \delta_X\), where \(A_N\) is the antisymmetrizer on \(\ell^2(\Omega^N)\). Trivially, \(\{\delta_X \mid X \in \Omega^N\}\) is a CONS for \(\ell^2(\Omega^N)\). To get a CONS for \(\ell^2(\Omega^N)\), we need some preliminaries. For all \(\tau \in \mathcal{S}_N\), we know that \(e_{\tau X} = \text{sign}(\tau) e_X\). Taking this fact into consideration, we introduce an equivalence relation in \(\Omega^N\) as follows: Let \(X, Y \in \Omega^N\). If there exists a \(\tau \in \mathcal{S}_N\) such that \(Y = \tau X\), then we write \(X \equiv Y\). Let \([X]\) be the equivalence class to which \(X\) belongs. We will often abbreviate \([X]\) to \(X\) if no confusion occurs. The quotient set \(\Omega^N/\equiv\) is denoted by \([\Omega^N]\). Then we readily confirm that \(\{e_X \mid X \in [\Omega^N]\}\) is a CONS for \(\ell^2(\Omega^N)\).

Definition 2.7 Let \(P_{\equiv}\) be the orthogonal projection from \(\ell^2(\Omega^N)\) to \(\ell^2(\Omega^N)\). We define a self-adjoint operator on \(\ell^2(\Omega^N)\) by

\[
L = P_{\equiv} L(\alpha = 0)|_{\nu = 0} P_{\equiv}. \tag{2.44}
\]

Here, we note that by recalling (2.11), \(L(\alpha = 0)|_{\nu = 0}\) can be explicitly expressed as \(h_0(0) \otimes 1 \otimes \cdots \otimes 1 + \nu \cdot 1 \otimes \cdots \otimes 1 \otimes h_0(0)\). Fix \(X \in \Omega^N\), arbitrarily. A permutation \(\tau \in \mathcal{S}_N\) is called dynamically allowed associated with \(X\) if there is an \(n \in \mathbb{N}_0\) such that

\[
\langle \delta_X | L^n \delta_{\tau X} \rangle \neq 0. \tag{2.45}
\]

The set of all dynamically allowed permutations associated with \(X\) is denoted by \(\mathcal{S}_N(X)\). As we will see below, the dynamically allowed permutations play important roles.

We give a useful characterization of (2.45). For each \(X = (X^{(j)})_{j=1}^{N}, Y = (Y^{(j)})_{j=1}^{N} \in \Omega^N\), we define a distance between \(X\) and \(Y\) by \(\|X - Y\|_\infty = \max_{j=1,\ldots,N} \|x^{(j)} - y^{(j)}\|_\infty\), where \(x^{(j)}\) (resp. \(y^{(j)}\)) is the spatial component of \(X^{(j)}\) (resp. \(Y^{(j)}\)). We say that \(X\) and \(Y\) are neighbours if \(\|X - Y\|_\infty = 1\). A pair \(\{X, Y\} \in \Omega^N \times \Omega^N\) is called an edge, if \(X\) and \(Y\) are neighbours. We say that a sequence \((X_i)_{i=1}^{m} \subset \Omega^N\) is a path,
if \( \{X_i, X_{i+1}\} \) is an edge for all \( i \). For each edge \( \{X, Y\} \), we define a linear operator acting in \( \ell^2(\Omega^N) \) by
\[
Q(X, Y) = |\delta_X\rangle\langle\delta_Y|.
\] (2.46)

**Lemma 2.8** The following (i) and (ii) are mutually equivalent:

(i) \( \tau \) is dynamically allowed associated with \( X \);

(ii) there is a path \( (X_i)_{i=1}^m \) such that
\[
- X_1 = X \text{ and } X_m = \tau X;
- (\langle\delta_X | Q(X_1, X_2)Q(X_2, X_3) \cdots Q(X_{m-1}, X_m)\delta_X \rangle > 0.
\]

*Proof.* This lemma can be readily confirmed by using the fact that \( L \) is expressed as a linear combination of \( \{Q(X, Y)\} \). \( \square \)

We introduce a subspace of \( \ell^2(\Omega^N) \) by
\[
\ell^2(\Omega^N) = \{ F \in \ell^2(\Omega^N) \mid F \text{ is symmetric} \}.
\] (2.47)

The subspace \( \ell^2(\Omega^N) \) describes wave functions for a system of \( N \) hard-core bosons.

Let
\[
D_P = \{ m \in (M)^N \mid \exists \tau \in S_N(0(m)) \text{ such that } X_\beta(m) = \tau X_0(m) \}.
\] (2.48)

We set
\[
L_\beta = D \cap D_P.
\] (2.49)

Let us introduce a random variable on \( L_\beta \) by
\[
(-1)^{\pi(X_\beta(m))} = \text{sgn(}\tau\text{)},
\] (2.50)

where \( \tau \) is given in (2.48). For all \( m \in L_\beta \), the electron configuration at \( \beta \), i.e., \( X_\beta(m) \) is a permutation of the configuration at \( t = 0 \); \( (-1)^{\pi(X_\beta(m))} \) is the parity of the permutation.

**Theorem 2.9** Let \( \alpha_1, \ldots, \alpha_n \) be magnetic potentials. Suppose that \( F_0, F_1, \ldots, F_{n-1} \in \ell^2(\Omega^N) \). Then, there exists a probability measure \( \nu_\beta \) on \( L_\beta \) such that, for \( 0 < t_1 < t_2 < \cdots < t_{n-1} < \beta \),
\[
\text{Tr}_{\ell^2(\Omega^N)}[F_0 e^{-t_1 H(\alpha_1)} F_1 e^{-t_2 H(\alpha_2)} F_2 \cdots F_{n-1} e^{-t_{n-1} H(\alpha_{n-1})} e^{-\beta L}] = \int_{L_\beta} d\nu_\beta F_0(X_0) F_1(X_{t_1}) \cdots F_{n-1}(X_{t_{n-1}})(-1)^{\pi(X_\beta)} \times \exp \left\{ \sum_{j=1}^N \sum_{\ell=1}^n S_{\ell, t_{\ell-1}, t_\ell}(0, \alpha_\ell | X_{\ell}^{(j)}) - \int_0^{t_n} V(X_s) ds + b \sum_{j=1}^N \sigma(X_0^{(j)}) \right\}
\] (2.51)

with \( t_n = \beta \).
Proof. Let $K_n = F_0 e^{-t_1 H(a_1)} F_1 e^{-(t_2-t_1) H(a_2)} F_2 \cdots F_{n-1} e^{-(\beta-t_{n-1}) H(\alpha_n)}$. Fix $X \in \Omega_\mathbb{P}^N$, arbitrarily. We claim that if $\tau$ is not dynamically allowed associated with $X$, then it holds that, for all $n \in \mathbb{N}$ and $0 < t_1 < t_2 < \cdots < t_{n-1} < \beta$,

$$\langle \delta_X | K_n \delta_X \rangle = 0. \quad (2.52)$$

We shall prove this claim for $n = 2$ only. To prove the claim for general $n$ is similar. Because $F_0$ and $F_1$ are diagonal, we have

$$\langle \delta_X | F_0(-L)^{n_1} F_1(-L)^{n_2} \delta_X \rangle = 0 \quad (2.53)$$

for all $n_1, n_2 \in \mathbb{N}_0$. Indeed, because $L$ is a linear combination of $\{Q(X,Y)\}$, the equation (2.53) follows from

$$\langle \delta_X | Q(X_1, X_2) Q(X_2, X_3) \cdots Q(X_{m-1}, X_m) \delta_X \rangle = 0 \quad (2.54)$$

for any path $(X_i)_{i=1}^m$. But this is obvious from Lemma 2.8. Using (2.53), we can prove (2.52) as follows:

$$\langle \delta_X | K_n \delta_X \rangle = \sum_{n_1=1}^\infty \sum_{n_2=1}^\infty \frac{t_1^{n_1} (t_2 - t_1)^{n_2}}{n_1! n_2!} \langle \delta_X | F_0(-L)^{n_1} F_1(-L)^{n_2} \delta_X \rangle = 0. \quad (2.55)$$

By Theorem 2.4 and (2.52), we get

$$\operatorname{Tr}_{\mathbb{P}_N}[K_n] = \sum_{X \in \Omega_\mathbb{P}^N} \langle e_X | K_n e_X \rangle = \sum_{X \in \Omega_\mathbb{P}^N} \sum_{\tau \in \mathfrak{S}_N} \frac{\text{sgn}(\tau)}{N!} \langle \delta_X | K_n \delta_X \rangle = \sum_{X \in \Omega_\mathbb{P}^N} \sum_{\tau \in \mathfrak{S}_N(X)} \frac{\text{sgn}(\tau)}{N!} \langle \delta_X | K_n \delta_X \rangle = \sum_{X \in \Omega_\mathbb{P}^N} \sum_{\tau \in \mathfrak{S}_N(X)} \frac{\text{sgn}(\tau)}{N!} E_X \left[ K_n(X_{\bullet}) 1_{\{X_{\beta} = \tau X\} \cap D} \right], \quad (2.56)$$

where a random variable $K_n$ is given by

$$K_n(X_{\bullet}) = F_0(X_0) F_1(X_{t_1}) \cdots F_{n-1}(X_{t_{n-1}}) \times e^{\sum_{j=1}^N \sum_{l=1}^n S_{n_{l-1,t_l}}(\tau, \alpha_l | X_{\bullet}^{(j)})} e^{-\int_0^{t_n} V(X_s) ds + b \sum_{j=1}^N \sigma(X_{\bullet}^{(j)}).} \quad (2.57)$$

Let us define a probability measure on $L_\beta$ by

$$\nu_\beta(B) = \sum_{X \in \Omega_\mathbb{P}^N} \sum_{\tau \in \mathfrak{S}_N(X)} \frac{1}{N!} P_X \left( B \cap \{X_{\beta} = \tau X\} \cap D \right) \text{Norm.}, \quad (2.58)$$
where \( \text{Norm.} \) is the normalization constant, which can be computed as follows:

\[
\text{Norm.} = \sum_{X \in [\Omega^N]} \sum_{\tau \in \mathfrak{S}_N(X)} \frac{1}{N!} P_X(\{X_\beta = \tau X\} \cap D) \\
= \sum_{X \in [\Omega^N]} \sum_{\tau \in \mathfrak{S}_N(X)} \frac{1}{N!} \langle \delta_X | e^{-\beta L} \delta_{\tau} X \rangle \\
= \text{Tr}_L(\Omega_N^\alpha) e^{-\beta L}. \tag{2.59}
\]

By combining (2.56) and (2.58), we obtain the desired assertion with \( \text{sgn}(\tau) = (-1)^\pi(X_\beta) \).

\[ \square \]

### 2.6 A trace formula for \( H_\infty(\alpha) \)

First, we note that a natural CONS for \( P_G\ell_2^\alpha(\Omega_N) \) is \( \{e_X | X \in [\Omega_N^{\infty}]\} \), where \( [\Omega_N^{\infty}] \) is the quotient set \( \Omega_N^{\infty}/\equiv \), see Section 2.3. Let \( L_\infty = P_G L P_G \). Even if \( U = \infty \), we can define the dynamically allowed permutations: Fix \( X \in [\Omega_N^\infty] \), arbitrarily. A permutation \( \tau \in \mathfrak{S}_N \) is called dynamically allowed associated with \( X \) if there is an \( n \in \mathbb{N}_0 \) such that

\[
\langle \delta_X | L_\infty^n \delta_{\tau} X \rangle \neq 0. \tag{2.60}
\]

The set of all dynamically allowed permutations associated with \( X \) is denoted by \( \mathfrak{S}_{N,\infty}(X) \). Because there are no encounter of electrons, the trajectories are very simplified; indeed, we can check that if \( \tau \) is dynamically allowed, then \( \tau \) is always even: \( \text{sgn}(\tau) = 1 \). (Reader can readily confirm this fact for one-dimensional chain; in fact, the dynamically allowed permutation can only be the identity in this case.)

Let

\[
D_{\infty,\infty} = \{m \in (M)^N | \exists \tau \in \mathfrak{S}_{N,\infty}(X_0) \text{ such that } X_\beta(m) = \tau X_0(m) \}. \tag{2.61}
\]

We define an event by

\[
L_{\beta,\infty} = D_\infty \cap D_{\infty,\infty}. \tag{2.62}
\]

As before, we can define the random variable \( (-1)^\pi(X_\beta(m)) \) for all \( m \in L_{\beta,\infty} \). But because each dynamically allowed permutation is even in the case where \( U = \infty \), it holds that \( (-1)^\pi(X_\beta(m)) = 1 \). Taking this fact into account and using arguments similar to those in the proof of Theorem 2.9, we can prove the following.

**Theorem 2.10** Suppose that \( N = |\Lambda| - 1 \). Let \( \alpha_1, \ldots, \alpha_n \) be magnetic potentials. Suppose that \( F_0, F_1, \ldots, F_{n-1} \in \ell^\infty_s(\Omega_N) \). Then, there exists a probability measure \( \nu_{\beta,\infty} \) on \( L_{\beta,\infty} \) such that, for \( 0 < t_1 < t_2 < \cdots < t_{n-1} < \beta \),

\[
\text{Tr}_{P_G\ell_2(\Omega_N)} \left[ F_0 e^{-t_1 H_\infty(\alpha_1)} F_1 e^{-(t_2-t_1) H_\infty(\alpha_2)} F_2 \cdots F_{n-1} e^{-(\beta-t_{n-1}) H_\infty(\alpha_n)} \right] \\
= \text{Tr}_{P_G\ell_2(\Omega_N)} \left[ e^{-\beta L_\infty} \right] \int_{L_{\beta,\infty}} d\nu_{\beta,\infty} F_0(X_0) F_1(X_{t_1}) \cdots F_{n-1}(X_{t_{n-1}}) \times \\
\times \exp \left\{ \sum_{j=1}^N \sum_{\ell=1}^n S_{[t_{\ell-1}, t_\ell]}(0, \alpha_j | X_\beta^{(j)}) - \int_0^{t_n} V_0(X_s) ds + b \sum_{j=1}^N \sigma(X_0^{(j)}) \right\} \tag{2.63}
\]

with \( t_n = \beta \), where \( V_0 \) is defined in Theorem 2.6.
3 Trace formulas for free Bose field

3.1 Preliminaries

Let $\mathfrak{X}_r$ be a real separable Hilbert space equipped with the inner product $\langle \cdot | \cdot \rangle_{\mathfrak{X}_r}$. Let $\{ \phi(f) | f \in \mathfrak{X}_r \}$ be the Gaussian random process indexed by $\mathfrak{X}_r$ and let $(Q, \mathcal{F}, \mu)$ be its underlying probability space. Note that

$$\int_Q d\mu \phi(f) \phi(g) = \frac{1}{2} \langle f | g \rangle_{\mathfrak{X}_r}, \quad f, g \in \mathfrak{X}_r. \quad (3.1)$$

Let $A$ be a positive self-adjoint operator acting in $\mathfrak{X}_r$. Suppose that there exists a constant $a_0 > 0$ such that $A \geq a_0$ (i.e., $\langle f | Af \rangle \geq a_0 \| f \|^2$ for all $f \in \text{dom}(A^{1/2})$). For each $s \in \mathbb{R}$, we introduce an inner product $\langle \cdot | \cdot \rangle_s$ on $\text{dom}(A^{s/2})$ by $\langle f | g \rangle_s = \langle A^{s/2} f | A^{s/2} g \rangle$, $f, g \in \text{dom}(A^{s/2})$. For $s > 0$, $\text{dom}(A^{s/2})$ becomes a Hilbert space, which is denoted by $\mathfrak{X}_{r,s}$. For $s < 0$, we denote by $\mathfrak{X}_{r,s}$ the completion of $\mathfrak{X}_{r}(:= \text{dom}(A^{s/2}))$ in the norm $\| \cdot \|_s := \langle \cdot | \cdot \rangle_s^{1/2}$. The dual space of $\mathfrak{X}_{r,s}$ can be identified with $\mathfrak{X}_{r,-s}$ through the bilinear form such that $-s \langle f | g \rangle_s = \langle f | g \rangle_{\mathfrak{X}_r}$, $f \in \mathfrak{X}_{r,-s} \cap \mathfrak{X}_r$, $g \in \mathfrak{X}_{r,s} \cap \mathfrak{X}_r$.

In what follows, we assume the following:

(A) For some $\gamma_0 > 0$, $A^{-\gamma_0}$ is in the trace class.

Choose $\gamma > \gamma_0$, arbitrarily. Clearly, the embedding mapping of $\mathfrak{X}_r$ into $\mathfrak{X}_{r,-\gamma}$ is in the Hilbert-Schmidt class. Thus, by applying [12], Proposition 5.1, we can take $Q = \mathfrak{X}_{r,-\gamma}$ and $\phi(f) = -\gamma \langle \phi(f) \rangle_{\gamma}$, $f \in Q$, $f \in \mathfrak{X}_{r,\gamma}$.

We denote by $\mathfrak{X}$ the complexification of $\mathfrak{X}_r$. Then each element $f$ in $\mathfrak{X}$ can be expressed as $f = f_1 + i f_2$, $f_1, f_2 \in \mathfrak{X}_r$. Now we define $\phi(f)$, $f \in \mathfrak{X}$ by $\phi(f) = \phi(f_1) + i \phi(f_2)$. Trivially, we have $\int_Q d\mu \phi(f) \phi(g) = \frac{1}{2} \langle f | g \rangle_{\mathfrak{X}_r}$, $f, g \in \mathfrak{X}$, where $\overline{f} = f_1 - i f_2$.

For each $f \in \mathfrak{X}$, we define a symmetric operator $\Phi_S(f)$ by

$$\Phi_S(f) = \frac{1}{\sqrt{2}}(a(f)^* + a(f)). \quad (3.2)$$

It is well-known that $\Phi_S(f)$ is essentially self-adjoint on $\mathfrak{F}_\text{fm}(\mathfrak{X})$. We denote its closure by the same symbol. $\Phi_S(f)$ is called the Segal’s field operator.

There is a useful identification between $L^2(Q, d\mu)$ and $\mathfrak{F}(\mathfrak{X})$; namely, there exists a unitary operator $U$ from $\mathfrak{F}(\mathfrak{X})$ onto $L^2(Q, d\mu)$ satisfying the following (i)-(iii) [33]:

(i) $U \Omega_0 = 1$, where $\Omega_0$ is the Fock vacuum in $\mathfrak{F}(\mathfrak{X})$;

(ii) $U(a(f_1)^* \cdots a(f_n)^* \Omega_0 = 2^{n/2} : \phi(f_1) \cdots \phi(f_n) :$, $f_1, \ldots, f_n \in \mathfrak{X}$, where $\phi(f_1) \cdots \phi(f_n)$ indicates the Wick product;

(iii) $U \Phi_S(f) U^{-1} = \phi(f)$, $f \in \mathfrak{X}$.

Let $B$ be a positive self-adjoint operator on $\mathfrak{X}_r$. We define a linear operator $d\Gamma_s(B)$ acting in $\mathfrak{F}(\mathfrak{X})$ by

$$d\Gamma_s(B) \mid \otimes^n_s \mathfrak{X} = B \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes B \otimes 1 \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes 1 \otimes B. \quad (3.3)$$
$d\Gamma_n(B)$ is called the second quantization of $B$. $d\Gamma_n(B)$ is positive and self-adjoint. Let $\{e_n\}_{n=1}^\infty \subset \mathfrak{X}$ be a CONS of $\mathfrak{X}$. Suppose that $B$ is diagonal with respect to $\{e_n\}_{n=1}^\infty$: $Be_n = \lambda_ne_n, \ n \in \mathbb{N}$. Then we have

$$\sum_{n=1}^\infty \lambda_na(e_n)^*a(e_n) = U^{-1}d\Gamma_n(B)U$$

on the dense subspace $\{\Psi = (\Psi_n)_{n=0}^\infty \in \mathfrak{F}_\text{fin}(\mathfrak{X}) | \Psi_n \in \otimes_{\text{alg}}\text{dom}(B)\}$, where $\otimes_{\text{alg}}$ indicates the incompletely extended tensor product.

### 3.2 A trace formula for free Euclidean field I

Let $A$ be the linear operator given in the previous subsection. Note that $A$ can be naturally extended to $\mathfrak{X}$. We denote the extension by the same symbol. By the assumption (A), we know that $e^{-\beta A}$ is in the trace class as an operator on $\mathfrak{X}$ for all $\beta > 0$. Accordingly, $e^{-\beta d\Gamma_n(A)}$ is in the trace class as an operator on $\mathfrak{F} (\mathfrak{X})$ for all $\beta > 0$ satisfying

$$Z_A(\beta) := \text{Tr}_{\mathfrak{F} (\mathfrak{X})}[e^{-\beta d\Gamma_n(A)}] = \frac{1}{\det(1 - e^{-\beta A})},$$

where $\det(\cdots)$ is the determinant.

We set $Q_\beta = C_P([0, \beta]; Q)$, the space of continuous loop of $Q$ with parameter space $[0, \beta]$. For each $\Phi \in Q_\beta$, the value of $\Phi$ at $t$ is denoted by $\Phi_t \in Q$. Let $F_\beta$ be the Borel field on $Q_\beta$ generated by $\Phi_t(f), f \in \mathfrak{X}, t \in [0, \beta]$.

**Proposition 3.1 ([3, 14])** There exists a probability measure $\mu_\beta$ on $(Q_\beta, F_\beta)$ such that $\{\Phi_\beta(f) | f \in \mathfrak{X}, t \in [0, \beta]\}$ is a family of jointly Gaussian random processes on $(Q_\beta, F_\beta, \mu_\beta)$ with covariance

$$\int_{Q_\beta} d\mu_\beta \Phi_\beta(f)\Phi_\beta(g) = \frac{1}{2}(f|(1 - e^{-\beta A})^{-1}(e^{-(\beta - |t-s|)A} + e^{-|t-s|A})g)_{|\mathfrak{X}}, \ f, g \in \mathfrak{X}.$$  

Let $G_0, \ldots, G_n$ be bounded measurable functions on $\mathbb{R}^m$. For $0 < t_1 < t_2 < \cdots < t_n < \beta$, we have

$$\text{Tr}_{\mathfrak{F} (\mathfrak{X})}[G_0^F e^{-t_1d\Gamma_n(A)}G_1^F e^{-(t_2-t_1)d\Gamma_n(A)}G_2^F \cdots G_n^F e^{-(\beta-t_n)d\Gamma_n(A)}] / Z_A(\beta)$$

$$= \int_{Q_\beta} d\mu_\beta G_0 G_1^l \cdots G_n^l,$$  

where

$$G_j^F = G_j(\Phi_S(f_1^{(j)}), \ldots, \Phi_S(f_m^{(j)})),$$

$$G_j^l = G_j(\Phi_l(f_1^{(j)}), \ldots, \Phi_l(f_m^{(j)}))$$

for $f_1^{(j)}, \ldots, f_m^{(j)} \in \mathfrak{X}_{r, \gamma}$. 

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3.3 A trace formula for free Euclidean field II

Let $L^2_r(0, \beta)$ be the real Hilbert space of real-valued measurable functions in $L^2(0, \beta)$ and set $\mathcal{X}_r^\beta = L^2_r(0, \beta) \otimes \mathcal{X}_r$. Let $\Delta_p$ be the periodic Laplacian acting in $L^2(0, \beta)$. We introduce a norm of $\mathcal{X}_r^\beta$ by

$$
\|f\|^2_{-1, \beta} = \frac{1}{2} \left\| \frac{1 \otimes A^2}{(-\Delta_p) \otimes 1 + 1 \otimes A^2} f \right\|^2_{\mathcal{X}_r^\beta}, \quad f \in \mathcal{X}_r^\beta.
$$

(3.10)

We denote by $\mathcal{X}_{-1, r}^\beta$ the completion of $\mathcal{X}_r^\beta$ by the norm $\| \cdot \|_{-1, \beta}$. The following formula will be useful:

$$
\langle \delta_t \otimes f | \delta_t \otimes g \rangle_{-1, \beta} = \frac{1}{2} \left| f \right|^2 (1 - e^{-\beta A})^{-1} (e^{-(-\beta - t s)A} + e^{-|t-s|A}) g \rangle_{\mathcal{X}_r}, \quad f, g \in \mathcal{X}_r,
$$

(3.11)

where $\delta_t$ is the Dirac delta function. Let $f$ be an $\mathcal{X}_r$-valued measurable function on $[0, \beta]$ such that $\int_0^\beta ||f(t)||^2_{-1, \beta} dt < \infty$. We define the smeared random variable by $\Phi(f) = \int_0^\beta \langle \Phi_t | f(t) \rangle_{\gamma} dt$. Using (3.11), we obtain $\int_{Q, \beta} d\mu \Phi(f) \Phi(g) = \langle f|g \rangle_{-1, \beta}$, $f, g \in \mathcal{X}_{-1, r}^\beta$, where the inner product $\langle \cdot \cdot \rangle_{-1, \beta}$ is naturally obtained from (3.10). Therefore, $\{\Phi(f) | f \in \mathcal{X}_{-1, r}^\beta\}$ becomes a Gaussian mean zero random process indexed by $\mathcal{X}_{-1, r}^\beta$; its underlying probability space is $(Q, F, \mathcal{F}, \mu)$.

Let $\beta > 0$. By using the fact that $\coth x > 0$, provided that $x > 0$, we define a self-adjoint operator $B(\beta)$ on $\mathcal{X}_r$ by

$$
B(\beta) = \sqrt{\coth \frac{\beta A}{2}}.
$$

(3.12)

Because $A \geq a_0$, we readily see that $1 \leq B(\beta) \leq \sqrt{\coth a_0}$. Furthermore, by using the elementary fact $\sqrt{\coth x} - 1 \leq D e^{-2x}$ with $D = 2^{-a_0}/(e^{a_0} - e^{-a_0})$ for all $x \in [a_0, \infty)$, we obtain that $0 \leq \text{Tr}_{\mathcal{X}_r} [B(\beta) - 1] \leq D \text{Tr}_{\mathcal{X}_r} [e^{-2\beta A}] < \infty$, which implies that $B(\beta) - 1$ is in the trace class. Especially, $B(\beta) - 1$ is in the Hilbert-Schmidt class as well. By Shale’s theorem [33 Theorem I. 23], there exists a probability measure $\mu_B(\beta)$ on $(Q, F)$ mutually absolutely continuous to $\mu$ such that

$$
\int_Q d\mu_B(\beta) e^{i\phi(f)} = \int_Q d\mu e^{i\phi(B(\beta)f)} = e^{-\|B(\beta) f\|^2_{\mathcal{X}_r} / 4}
$$

(3.13)

and $d\mu_B(\beta) = G_\beta d\mu$ with $G_\beta \in L^p(Q, d\mu)$ for some $p > 1$ and $G_\beta^{-1} \in L^q(Q, d\mu)$ for some $q > 1$.

For $t \in [0, \beta)$, we define a linear operator $j_t$ from $\mathcal{X}_{r, \gamma}$ to $\mathcal{X}_{-1, r}^\beta$ by

$$
j_t f = \delta_t \otimes f, \quad f \in \mathcal{X}_{r, \gamma}.
$$

(3.14)

By (3.11), we have

$$
\|j_t f\|^2_{-1, \beta} = \frac{1}{2} \|B(\beta) f\|^2_{\mathcal{X}_r} =: \|f\|^2_{\beta}.
$$

(3.15)

Let $\mathcal{X}_{r, \gamma}^{(\beta)}$ be the completion of $\mathcal{X}_{r, \gamma}$ by the norm $\| \cdot \|_{\beta}$. From (3.15), it follows that $j_t$ is the isometry from $\mathcal{X}_{r, \gamma}^{(\beta)}$ into $\mathcal{X}_{-1, r}^\beta$. Now we define a linear operator $J_t : L^2(Q, d\mu_B(\beta)) \to$
For each $f \in L^2(Q_\beta, d\mu_\beta)$ by $J_t = \Gamma(j_t)$, where for each contraction operator $C$, $\Gamma(C)$ is defined by $\Gamma(C) : \phi(f_1) \cdots \phi(f_n) := \Phi(C f_1) \cdots \Phi(C f_n) : f_1, \ldots, f_n \in \mathcal{X}_{1,r}^\beta$ and $\Gamma(C)1 = 1$. Note that the mapping $t \rightarrow J_t$ is strongly continuous, and $J_t$ is an isometry. In addition, we have the following \[3, 33\]:

- $J_t$ is positivity preserving;
- $(J_t F)(\Phi) = F(\Phi_t)$, $F \in L^2(Q, d\mu_{B(\beta)})$; thus, the mapping $t \rightarrow F(\Phi_t)$ is continuous in $L^2(Q_\beta, d\mu_\beta)$;
- $J_t$ can be extended to a contraction from $L^p(Q, d\mu_{B(\beta)})$ to $L^p(Q_\beta, d\mu_\beta)$ for all $p \in [1, \infty)$.

**Theorem 3.2** (\[3, 4\]) Let $G_0, \ldots, G_n$ be bounded measurable functions on $\mathbb{R}^m$. We set

$$G_j^F = G_j\left(\Phi_S(f_1^{(j)}), \ldots, \Phi_S(f_n^{(j)})\right)$$

(3.16)

for $f_1^{(j)}, \ldots, f_n^{(j)} \in \mathcal{X}_{1,r}^\beta$. For $0 < t_1 < t_2 < \cdots < t_n < \beta$, we have

$$\text{Tr}_{\mathcal{H}(\mathcal{X})}\left[G_0^F e^{-t_1 d\gamma(A)} G_1^F e^{-t_2-t_1 d\gamma(A)} G_2^F \cdots G_n^F e^{-(\beta-t_n) d\gamma(A)}\right] = \int_{Q_\beta} d\mu_\beta(J_0 G_0^F)(\Phi)(J_1 G_1^F)(\Phi) \cdots (J_n G_n^F)(\Phi).$$

(3.17)

Here, note that

$$\left(J_t G_j^F\right)(\Phi) = G_j\left(\Phi(j_t f_1^{(j)}), \ldots, \Phi(j_t f_n^{(j)})\right).$$

(3.18)

### 3.4 Positivity preservingness of $e^{\Pi(f)}$

For each $f \in \mathcal{X}_{-1,r}^\beta$, we define a linear operator on $\mathcal{H}(\mathcal{X}_{-1}^\beta)$ by

$$\Pi(f) = \frac{i}{\sqrt{2}}(a(f)^* - a(f)),$$

(3.19)

where $\mathcal{X}_{-1}^\beta$ is the complexification of $\mathcal{X}_{-1,r}^\beta$. Then $\Pi(f)$ is essentially self-adjoint. We denote its closure by the same symbol. As before, we have a natural identification $\mathcal{H}(\mathcal{X}_{-1}^\beta) \cong L^2(Q_\beta, d\mu_\beta)$. Under this identification, $\Pi(f)$ can be regarded as a linear operator on $L^2(Q_\beta, d\mu_\beta)$.

The following proposition will play an important role.

**Proposition 3.3** For any $f \in \mathcal{X}_{-1,r}^\beta$, $e^{\Pi(f)}$ is positivity preserving, that is, if $F \in L^2(Q_\beta, d\mu_\beta)$ is a positive function, then $e^{\Pi(f)} F$ is a positive function.

**Proof.** We will apply the idea in \[20, 33\]. First, we note the following equality:

$$e^{\Pi(f)} e^{i\Phi(g)} = e^{-i(f|g|)_-} e^{i\Phi(g)} e^{\Pi(f)}, \quad g \in \mathcal{X}_{-1,r}^\beta.$$
Let \( F(x_1, \ldots, x_n), G(x_1, \ldots, x_n) \in \mathcal{S}(\mathbb{R}^n) \), the functions of rapid decreases. For each \( f_1, \ldots, f_n, g_1, \ldots, g_n \in \mathcal{X}^3_{-1,r} \), we set
\[
\tilde{F} = F(\Phi(f_1), \ldots, \Phi(f_n)), \quad \tilde{G} = G(\Phi(f_1), \ldots, \Phi(g_n)).
\]
(3.21)

By using (3.20), we have
\[
\langle \tilde{F} e^{iHf} \rangle = e^{-i\|f\|^{2,1,\beta}/4} (2\pi)^{-n} \int_{\mathbb{R}^{2n}} ds dt \tilde{F}(s)^* \tilde{G}(t) \exp \left\{ -\frac{1}{4} \left\| \sum_{i=1}^n (s_i f_i - t_i g_i) \right\|_{-1,\beta}^2 \right\} \times
\]
\[
\times \exp \left\{ -\frac{i}{2} \left\langle f \left[ \sum_{i=1}^n (s_i f_i + t_i g_i) \right] \right\rangle_{-1,\beta} \right\},
\]
where \( \tilde{f} \) indicates the Fourier transform of \( f \). Let \( K \) be a bounded linear operator on \( L^2(\mathbb{R}^n) \) defined by
\[
\overline{KG}(s) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} dt \exp \left\{ -\frac{1}{4} \left\| \sum_{i=1}^n (s_i f_i - t_i g_i) \right\|_{-1,\beta}^2 \right\} G(t), \quad G \in L^2(\mathbb{R}^n).
\]
(3.23)

For \( c \in \mathbb{R}^n \), let \( T_c \) be the shift operator on \( L^2(\mathbb{R}^n) \): \( (T_c F)(x) = F(x - c) \). Then
\[
\text{the RHS of (3.22)} = \langle T_a F \rangle \overline{K T_b G},
\]
(3.24)
where \( a = (a_1, \ldots, a_n) \) with \( a_i = -\langle f | f_i \rangle_{-1,\beta}/2 \) and \( b = (b_1, \ldots, b_n) \) with \( b_i = \langle f | g_i \rangle_{-1,\beta}/2 \).

Let \( H_a(t) = \exp \left\{ -\frac{1}{4} \left\| \sum_{i=1}^n (s_i f_i - t_i g_i) \right\|_{-1,\beta}^2 \right\} \). Because \( H_a \) is Gaussian, \( H_a \) has a Fourier transform which is a Gaussian. In particular, \( \hat{H}_a \geq 0 \), where \( \hat{f} \) indicates the inverse Fourier transform of \( f \). Thus, if \( G \) is positive, then \( (KG)(s) = \hat{H}_a * G \) is positive as well, where * indicates the convolution, which implies that the linear operator \( K \) is positivity preserving. Because the shift operator \( T_c \) is positivity preserving, the right hand side of (3.24) is positive. Since any positive \( \Psi \in L^2(Q, d\mu) \) is a limit of such \( F(\Phi(f_1), \ldots, \Phi(f_n)), \quad F \in \mathcal{S}(\mathbb{R}^n) \) [33] Proof of Theorem I.12], we conclude the assertion in Proposition 3.3. \( \square \)

4 Feynman-Kac-Itô formulas for \( H_{\text{rad}} \) and \( H_{\text{rad},\infty} \)

4.1 The Feynman-Schrödinger representation of the radiation field

In this study, we will employ the Feynman-Schrödinger representation of the quantized radiation field, which was first introduced by Feynman [7]. (Some mathematical properties of it were examined in [23, 29].) To explain the representation, we need some preliminaries: Let \( H_f = \sum_{\lambda=1,2} \sum_{k \in V^*} \omega(k) a(k,\lambda)^* a(k,\lambda) \). By (3.3), we have
\[
H_f = d\Gamma_a(\omega \oplus \omega),
\]
(4.1)
where \( d\Gamma_a(A) \) is the second quantization operator defined by (3.3). Furthermore, the vector potentials can be expressed as \( A_j(x) = \Phi_S(\eta_{x,j}^{(1)} \oplus \eta_{x,j}^{(2)}) \), where
\[
\eta_{x,j}^{(\lambda)}(k) = \frac{\bar{v}_{x,j}(k)}{\sqrt{\omega(k)}} \eta(k)e^{-ikx}, \quad \lambda = 1, 2
\]
(4.2)
with \( g(k) = |\Lambda|^{-1/2}\chi_\kappa(k) \). We set
\[
\begin{align*}
\ell^2_{\text{even}}(V^*) &= \{ f \in \ell^2(V^*) \mid f(-k) = f(k) \ \forall k \in V^* \}, \\
\ell^2_{\text{odd}}(V^*) &= \{ f \in \ell^2(V^*) \mid f(-k) = -f(k) \ \forall k \in V^* \}.
\end{align*}
\] (4.3)

Let us introduce subspaces of \( \ell^2(V^*) \) by
\[
\begin{align*}
h_1 &= \{ \epsilon_{1,i}f \mid f \in \ell^2_{\text{even}}(V^*), \ i = 1, 2, 3 \}, \\
h_2 &= \{ \epsilon_{1,i}f \mid f \in \ell^2_{\text{odd}}(V^*), \ i = 1, 2, 3 \}, \\
h_3 &= \{ \epsilon_{2,i}f \mid f \in \ell^2_{\text{even}}(V^*), \ i = 1, 2, 3 \}, \\
h_4 &= \{ \epsilon_{2,i}f \mid f \in \ell^2_{\text{odd}}(V^*), \ i = 1, 2, 3 \}.
\end{align*}
\] (4.5)\-(4.8)

Because
\[
\ell^2(V^*) = \bigcup_{i=1,2,3} \text{ran}(\epsilon_{\lambda,i}), \ \lambda = 1, 2,
\] (4.9)

we have the following identification:
\[
\ell^2(V^*) \oplus \ell^2(V^*) = h_1 \oplus h_2 \oplus h_3 \oplus h_4.
\] (4.10)

Corresponding to (4.10), we have
\[
\begin{align*}
\tilde{\mathcal{F}}_{\text{rad}} &= \tilde{\mathcal{F}}(h_1 \oplus h_2 \oplus h_3 \oplus h_4), \\
H_\ell &= d\Gamma_\kappa(\omega),
\end{align*}
\] (4.11)\-(4.12)

where \( \omega \) is a self-adjoint operator defined by \( \omega = \omega \oplus \omega \oplus \omega \oplus \omega \).

**Lemma 4.1** There is a unitary operator \( W \) satisfying the following (i) and (ii):
\[
\begin{align*}
\text{(i)} \quad & Wd\Gamma_\kappa(\omega)W^{-1} = d\Gamma_\kappa(\omega). \\
\text{(ii)} \quad & WA(x)W^{-1} = \Phi_S(\theta_x), \text{ where } \theta_x = (\theta_{1,x}, \theta_{2,x}, \theta_{3,x}, \theta_{4,x}) \text{ with }
\end{align*}
\] (4.13)\-(4.14)

Proof. Let \( \Pi_S(f) = \frac{i}{\sqrt{2}}(a(f)^* - a(f)) \). By (4.2), we have
\[
A_j(x) = \Phi_S\left(\theta_{1,x,j} \oplus 0 \oplus \theta_{3,x,j} \oplus 0\right) + \Pi_S\left(0 \oplus \theta_{2,x,j} \oplus 0 \oplus \theta_{4,x,j}\right).
\] (4.15)

We set \( W = e^{-i\pi N_2/2}e^{-i\pi N_4/2} \), where
\[
N_2 = d\Gamma_\kappa(0 \oplus 1 \oplus 0 \oplus 0), \quad N_4 = d\Gamma_\kappa(0 \oplus 0 \oplus 0 \oplus 1).
\] (4.16)

Then, because \( W\Pi_S\left(0 \oplus \theta_{2,x,j} \oplus 0 \oplus \theta_{4,x,j}\right)W^{-1} = \Phi_S\left(0 \oplus \theta_{2,x,j} \oplus 0 \oplus \theta_{4,x,j}\right) \), we obtain (ii). To check (i) is easy. \( \square \)

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By the arguments in Section 3.1 and Lemma 4.1, we can regard the vector potentials $A(x)$ as multiplication operators in $L^2(Q, d\mu)$. This representation is called the Feynman-Schrödinger representation, which is useful in constructing Feynman-Kac-Itô formulas in the remainder of this section.

Let $h_{r,\lambda}$ be the real-Hilbert space of real-valued sequences in $h_{\lambda}$. Let us consider the Gaussian random process indexed by $X_{\text{rad}, r} := h_{r, 1} \oplus h_{r, 2} \oplus h_{r, 3} \oplus h_{r, 4} : \{ \phi(f) \mid f \in X_{\text{rad}, r} \}$, and let $(Q, \mathcal{F}, \mu)$ be its underlying probability space. The vector potential $A(x)$ can be expressed as $\phi(\theta_x)$. We readily confirm that $\sum_{k \in V} \omega(k)^{-4} \leq \infty$. Hence, by choosing $A = \omega$, all results in Section 3 hold.

4.2 A Feynman-Kac-Itô formula for $H_{\text{rad}}$

For notational simplicity, we express $X_{\text{rad}, r}$ as $X_r$ in this section. To construct a Feynman-Kac-Itô formula for $H_{\text{rad}}$, we need some preliminaries.

Lemma 4.2 Let

$$\Theta_{XY} = \delta_{\sigma(X)\sigma(Y)} \int_{C_{xy}} \theta_r \cdot dr \in X_r, \quad X = (x, \sigma(X)), Y = (y, \sigma(Y)) \in \Omega,$$

where $\theta_r$ is defined in Lemma 4.1. Then, for all $s \leq u \leq t$, we have

$$\int_s^t j_s \Theta(dX^j_u) \in L^2(L_\beta, d\nu_\beta; X_{1, r}^3), \quad j = 1, \ldots, N.$$

Recall that the probability space $(L_\beta, \nu_\beta)$ and the random variable $\Theta(dX^j_u)$ are defined in Section 3.

Proof. Note that the line integral in (4.17) depends solely on the points $x$ and $y$, and thus independent of the path between them. Accordingly, by choosing $C_{xy} = \{(1 - s)x + sy \in V \mid s \in [0, 1]\}$, we obtain

$$\Theta_{1,XY}(k) = \frac{\rho(k)}{\sqrt{\omega(k)}} \mathbf{e}_1(k) \cdot \frac{x - y \sin(k \cdot x) - \sin(k \cdot y)}{|x - y|},$$

$$\Theta_{2,XY}(k) = \frac{\rho(k)}{\sqrt{\omega(k)}} \mathbf{e}_1(k) \cdot \frac{x - y \cos(k \cdot x) - \cos(k \cdot y)}{|x - y|},$$

where $\mathbf{e}_1(k) = \begin{pmatrix} k \\ (y - x) \end{pmatrix}$. Using these formulas, we obtain

$$|\Theta_{\lambda,XY}(k)| \leq \frac{\rho(k)}{\sqrt{\omega(k)}}.$$  

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We have
\[
\left\| \int_s^t j_u \Theta(dX_u^{(j)}) \right\|^2_{L^2(L_\beta, d\nu_\beta; X^\beta_{-1,r})} 
\leq \int_{L_\beta} \sum_{i=N(j)(s)+1}^{N(j)(t)} \left\langle j_u \Theta_{X_{i-1,i}^{(j)}} \left| j_u \Theta_{X_{i-1,i}^{(j)}} \right. \right\rangle_{X_{-1,r}} \, d\nu_\beta 
\leq 4 \int_{L_\beta} \sum_{i=N(j)(s)+1}^{N(j)(t)} \left\langle \omega 1/2 \left| B(\beta) \omega 1/2 \right. \right\rangle_{L^2(V^*)} \, d\nu_\beta 
= 4 \left\| B(\beta) \omega 1/2 \right\|^2_{L^2(V^*)} \int_{L_\beta} N(j)(t-s) \, d\nu_\beta.
\] (4.21)
The second inequality follows from (4.20). By Proposition A.1 the right hand side of (4.21) is finite. □

**Lemma 4.3** For each \( n \in \mathbb{N} \), set \( t_i^{(n)} = \beta i / 2^n \), \( i = 0, 1, \ldots, 2^n \). We define an \( X^\beta_{-1,r} \)-valued random variable on \( L_\beta \) by
\[
C_n^{(j)} = \sum_{i=1}^{2^n} \int_{t_i^{(n)}}^{t_{i+1}^{(n)}} j_{i}^{(n)} \Theta(dX_u^{(j)}), \quad j = 1, \ldots, N.
\] (4.22)
Then \( (C_n^{(j)})_{n=1}^\infty \) is a Cauchy sequence in \( L^2(L_\beta, d\nu_\beta; X^\beta_{-1,r}) \).

**Proof.** We apply the standard argument in the probability theory, see, e.g., [20, 34]. First, note that
\[
C_{n+1}^{(j)} - C_n^{(j)} = \sum_{i=1}^{2^n} \int_{t_{2i-1}^{(n+1)}}^{t_{2i+1}^{(n+1)}} \left( j_{i}^{(n+1)} - j_{i}^{(n+1)} \right) \Theta(dX_u^{(j)}).
\] (4.23)
In the remainder of this proof, we abbreviate \( t_i^{(n+1)} \) as \( t_i \). For \( s < t \leq \beta \), we set
\[
D_{[s,t]} = \int_s^t \Theta(dX_u^{(j)}), \quad j = 1, \ldots, N.
\] (4.24)
We have
\[
\left\| C_{n+1}^{(j)} - C_n^{(j)} \right\|^2_{L^2(L_\beta, d\nu_\beta; X^\beta_{-1,r})} \leq \sum_{i=1}^{2^n} \left\| \left( j_{i}^{(n+1)} - j_{i}^{(n+1)} \right) D_{[t_{2i-1}^{(n+1)}, t_{2i+1}^{(n+1)}]} \left\|_{L^2(L_\beta, d\nu_\beta; X^\beta_{-1,r})}. \right.
\] (4.25)
For each \( x \geq 0 \), we set
\[
K_{\beta,n}(x) = \coth \left( \frac{\beta x}{2} \right) - e^{-(\beta - 2^{-(n+1)} - \beta)} x + e^{-2^{-(n+1)} - \beta x}.
\] (4.26)
We readily confirm that
\[
K_{\beta,n}(x) \leq \frac{1}{2^{n+1}} R_{\beta,n}(x), \quad R_{\beta,n}(x) = \beta x \frac{1 - e^{-\beta x}}{1 - e^{-\beta}}. \tag{4.27}
\]
By using this, we obtain
\[
\left\| \left( \dot{j}_{t_{2i-1}} - \dot{j}_{t_{2i}} \right) D_{[t_{2i-2},t_{2i-1}]} \right\|_{L^2}^2 (L_{\beta,d\nu};X^{\beta}_{-1,r})
= \int_{L_{\beta}} d\nu_{\beta} \left\langle D_{[t_{2i-2},t_{2i-1}]} K_{\beta,n}(A) D_{[t_{2i-2},t_{2i-1}]} \right\rangle_{X_r}
\leq \frac{1}{2^{n+1}} \int_{L_{\beta}} d\nu_{\beta} \left\langle D_{[t_{2i-2},t_{2i-1}]} R_{\beta,n}(A) D_{[t_{2i-2},t_{2i-1}]} \right\rangle_{X_r}. \tag{4.28}
\]
On the other hand,
\[
\int_{L_{\beta}} d\nu_{\beta} \left\langle D_{[t_{2i-2},t_{2i-1}]} R_{\beta,n}(A) D_{[t_{2i-2},t_{2i-1}]} \right\rangle_{X_r}
\leq 2 \int_{L_{\beta}} d\nu_{\beta} \sum_{k=N(j)(t_{2i-2})+1}^{N(j)(t_{2i-1})} \left\langle \Theta X^{(j)}_{X_{j_k}} X_{j_k} \left| R_{\beta,n}(A) \Theta X^{(j)}_{X_{j_k}} X_{j_k} \right\rangle_{X_r}
\leq 8 \left\| \omega^{-1/2} \sqrt{R_{\beta,n}(\omega)} \theta_{j(V^\ast)} \right\|_{L^2_{T_1^2(V^\ast)}} \int_{L_{\beta}} d\nu_{\beta} N^{(j)}(t_{2i-1} - t_{2i-2})
\leq \frac{C}{2^{n+1}}, \tag{4.29}
\]
where \( C \) is some constant independent of \( n \). In the second inequality, we used (4.20). In the last inequality, we applied Proposition [A.1]. To sum, we arrive at
\[
\left\| C^{(j)}_{n+1} - C^{(j)}_n \right\|_{L^2}^2 (L_{\beta,d\nu};X^{\beta}_{-1,r}) \leq \frac{C'}{2^{n+1}}, \tag{4.30}
\]
where \( C' \) is some constant independent of \( n \). Hence, for \( m < n \), we have
\[
\left\| C_n - C_m \right\|_{L^2}(L_{\beta,d\nu};X^{\beta}_{-1,r}) \leq \sum_{i=m}^{n} \frac{C'}{2^{i+1}}. \tag{4.31}
\]
Thus, we are done. \( \square \)

**Definition 4.4** For each \( j = 1, \ldots, N \), we define an \( X^{\beta}_{-1,r} \)-valued random variable on \( L_{\beta} \) by
\[
a_{\beta}(X^{(j)}_{\bullet}) = \lim_{n \to \infty} C^{(j)}_n, \tag{4.32}
\]
where the right hand side exists in \( L^2 \)-sense.

**Theorem 4.5** Let
\[
Z_{\gamma}(\beta) = \text{Tr}_{\gamma} \left[ e^{-\beta H_{\gamma}} \right]. \tag{4.33}
\]
In addition, let

$$Z_{\text{rad}}(\beta) = \text{Tr}_{\Omega} \left[ e^{-\beta (L + d\Gamma_s(\omega))} \right]. \tag{4.34}$$

We define a probability measure on $\mathcal{Q}_{\beta,\text{rad}} := L_{\beta} \times Q_{\beta,\text{rad}}$ by $P_{\beta,\text{rad}} = \nu_{\beta} \otimes \mu_{\beta,\text{rad}}$. Then we have

$$Z_{\text{rad}}(\beta) / Z_{\text{rad}}(\beta) = \int_{\mathcal{Q}_{\beta,\text{rad}}} dP_{\beta,\text{rad}} \exp \left\{ i \Phi(A_{\beta}(X_\bullet)) - \int_0^\beta V(X_s)ds + \beta b \sum_{j=1}^N \sigma(X_0^{(j)}) \right\} (-1)^\pi(X_{\beta}), \tag{4.35}$$

where $A_{\beta}(X_\bullet) = \sum_{j=1}^N a_{\beta}(X_0^{(j)})$.

Proof. Let $\Theta = (\Theta_{XY})_{X,Y}$ be a matrix defined through (4.17). Note that $\Phi_S(\Theta) := (\Phi_S(\Theta_{XY}))_{X,Y}$ can be regarded as a magnetic potential. By the Trotter-Kato formula, we have

$$Z_{\text{rad}}(\beta) / Z_{\text{rad}}(\beta) = \lim_{n \to \infty} \text{Tr} \left[ \left( e^{-\beta H(\Phi_{t_1}^N(\Theta))} e^{-\beta d\Gamma(\omega)/2} \right)^{2n} \right] / Z_{\text{rad}}(\beta). \tag{4.36}$$

Set $t_i = i\beta / 2^n$, $i = 0, 1, \ldots, 2^n$. By Theorems 2.9 and 3.2, we have

$$\text{the RHS of (4.36)} = \lim_{n \to \infty} \int_{\mathcal{Q}_{\beta,\text{rad}}} dP_{\beta,\text{rad}} \text{Tr}_{\Omega^N} \left[ e^{-t_1 H(\Phi_{t_1}(\Theta))} e^{-\beta H(\Phi_{t_2}(\Theta))} \ldots e^{-\beta (t_2 - t_1) H(\Phi_{t_2}(\Theta))} \ldots e^{-\beta (t_{2^n - t_{2^n - 1}}) H(\Phi_{t_2}(\Theta))} \right] / \text{Tr}_{\Omega^N} \left[ e^{-\beta L} \right].$$

$$= \lim_{n \to \infty} \int_{\mathcal{Q}_{\beta,\text{rad}}} dP_{\beta,\text{rad}} \exp \left\{ \sum_{j=1}^N \sum_{i=1}^{2^n} S_{t_{i-1}, t_i} \left( 0, \Phi_{t_i}(\Theta) \right) X^{(j)} - \int_0^\beta V(X_s)ds + \beta b \sum_{j=1}^N \sigma(X_0^{(j)}) \right\} (-1)^\pi(X_{\beta}). \tag{4.37}$$

By the fact $\Phi_{t_i}(f) = \Phi(f_{t_i})$, we have

$$\sum_{j=1}^N \sum_{i=1}^{2^n} S_{t_{i-1}, t_i} \left( 0, \Phi_{t_i}(\Theta) \right) X^{(j)} = i \Phi \left( \sum_{j=1}^N \sum_{i=1}^{2^n} j_{t_i} \Theta (dX_0^{(j)}) \right). \tag{4.38}$$

By Definition 4.4, we know that the right hand side of (4.38) converges to $i \Phi(A_{\beta}(X_\bullet))$ in $L^2$-sense. Therefore, by the dominated convergence theorem, we arrive at the desired result. □

Remark 4.6 There are some other constructions of the Feynman-Kac-Itô formula for $H_{\text{rad}}$, see, e.g., [9, 24]. Our construction in the present paper has the benefit of usability.
Corollary 4.7 (Diamagnetic inequality) Let
\[
Z_H(\beta) = \text{Tr}_{\beta} \left[ e^{-\beta H} \right].
\] (4.39)
Assume that the dynamically allowed permutations are all even and not restricted to identity. For all \( \beta \geq 0 \), we have
\[
Z_{\text{rad}}(\beta) \leq Z_H(\beta).
\] (4.40)

Theorem 4.8 There is a positive measure \( \rho_{\beta,\text{rad}} \) on \( L_\beta \) such that
\[
Z_{\text{rad}}(\beta) = \int_{L_\beta} d\rho_{\beta,\text{rad}} (-1)^{\pi(X_\beta)} \exp \left\{ \beta b \sum_{j=1}^{N} \sigma(X_j(0)) \right\}.
\] (4.41)

Proof. First, note that the right hand side of (4.35) can be expressed as
\[
\text{the RHS of (4.35)} = \left\{ \int_{L_\beta} d\rho_{\beta,\text{rad}} (-1)^{\pi(X_\beta)} \exp \left\{ \beta b \sum_{j=1}^{N} \sigma(X_j(0)) \right\} \right\}_{L_2(\beta,\text{rad},d\rho_{\beta,\text{rad}})},
\] (4.42)
where \( F = e^{-\int_0^\beta V(X_u)du + \beta \sum_{j=1}^{N} \sigma(X_j(0))} (-1)^{\pi(X_\beta)} \). For each contraction operator \( C \) on \( \mathcal{X}_{-1}^\beta \), we define its second quantization by \( \Gamma(C)1 = 1 \) and \( \Gamma(C) : \Phi(f_1) \cdots \Phi(f_n) := \Phi(Cf_1) \cdots \Phi(Cf_n) \). For each \( \beta \), \( \Phi^{-1} \) is a \( \rho_{\beta,\text{rad}} \)-valued random variable on \( L_\beta \). By setting \( d\rho_{\beta,\text{rad}} = Z_{\text{rad}}(\beta) W(X_\bullet) e^{-\int_0^\beta V(X_u)du} d\nu_\beta \), we obtain
\[
\text{the RHS of (4.42)} = \left\{ \int_{L_\beta} d\nu_\beta \int_{Q_{\beta,\text{rad}}} d\mu_{\beta,\text{rad}} e^{i\Pi(A_\beta(X_\bullet))} F(X_\bullet) \right\}_{L_2(\beta,\text{rad},d\rho_{\beta,\text{rad}})}.
\] (4.43)
Let us define a random variable on \( L_\beta \) by
\[
W(X_\bullet) = \int_{Q_{\beta,\text{rad}}} d\mu_{\beta,\text{rad}} e^{i\Pi(A_\beta(X_\bullet))}.
\] (4.44)
By Proposition 3.3, \( W(X_\bullet) \) is positive \( \nu_\beta \)-a.e.. By setting \( d\rho_{\beta,\text{rad}} = Z_{\text{rad}}(\beta) W(X_\bullet) e^{-\int_0^\beta V(X_u)du} d\nu_\beta \), we obtain the desired result. \( \square \)

4.3 A Fyenman-Kac-Itô formula for \( H_{\text{rad},\infty} \)

Let \( C^{(j)}_n \) be the \( \mathcal{X}_{-1,r}^\beta \)-valued random variable on \( L_{\beta,\infty} \) defined by (4.22). Using arguments similar to those in the proof of Lemma 4.3, we can also prove that \( (C^{(j)}_n)_{n=1}^{\infty} \) is a Cauchy sequence in \( L^2(L_{\beta,\infty},d\nu_\beta;\mathcal{X}_{-1,r}^\beta) \). Thus, we can define an \( \mathcal{X}_{-1,r}^\beta \)-valued random variable on \( L_{\beta,\infty} \) by
\[
a_{\beta,\infty}(X^{(j)}_\bullet) = \lim_{n \to \infty} C^{(j)}_n,
\] (4.45)
where the right hand side exists in \( L^2 \)-sense. In what follows, \( a_{\beta,\infty} \) is simply written as \( a_{\beta} \), if no confusion arises.

Using arguments similar to those in the proof of Theorem 4.5, we can prove the following.
Theorem 4.9 Assume that \( N = |\Lambda| - 1 \). Let \( Z_{\rad,\infty}(\beta) \) be the partition function defined by (1.24). Let

\[
Z_{\rad,\infty}(\beta) = \text{Tr}_{P_\beta(\rad)} \left[ e^{-\beta (L_\infty + d\Gamma_s(\omega))} \right].
\]

(4.46)

We define a probability measure on \( \mathcal{Q}_{\beta,\rad,\infty} := L_{\beta,\infty} \times Q_{\beta,\rad} \) by

\[
P_{\beta,\rad,\infty} = \nu_{\infty} \otimes \mu_{\beta,\rad}.
\]

Then we have

\[
\frac{Z_{\rad,\infty}(\beta)}{Z_{\rad,\infty}(\beta)} = \int_{\mathcal{Q}_{\beta,\rad,\infty}} dP_{\beta,\rad,\infty} \exp \left\{ i\Phi(A_{\beta}(X_{\bullet})) - \int_0^\beta V_o(X_s) ds + \beta b \sum_{j=1}^N \sigma(X^{(j)}_0) \right\},
\]

(4.47)

where \( A_{\beta}(X_{\bullet}) = \sum_{j=1}^N a_{\beta}(X^{(j)}_{\bullet}) \).

We can also prove the following assertions as before.

Corollary 4.10 Assume that \( N = |\Lambda| - 1 \). For all \( \beta \geq 0 \), we have

\[
Z_{\rad,\infty}(\beta) \leq Z_{H,\infty}(\beta),
\]

(4.48)

where \( Z_{H,\infty}(\beta) \) is given by (1.11).

Theorem 4.11 There is a positive measure \( \rho_{\beta,\rad,\infty} \) on \( L_{\beta,\infty} \) such that

\[
Z_{\rad,\infty}(\beta) = \int_{L_{\beta,\infty}} d\rho_{\beta,\rad,\infty} (-1)^{\pi(X_{\beta})} \exp \left\{ \beta b \sum_{j=1}^N \sigma(X^{(j)}(0)) \right\}.
\]

(4.49)

5 Feynman-Kac-Itô formulas for \( H_{\text{HH}} \) and \( H_{\text{HH},\infty} \)

5.1 The Lang-Firsov transformation

Let us introduce a linear operator by

\[
L = \omega^{-1} \sum_{x,y \in \Lambda} g_{xy} n_x (b^*_y - b_y).
\]

(5.1)

\( L \) is essentially anti-self-adjoint. We denote its closure by the same symbol. The unitary operator \( e^L \) is called the Lang-Firsov transformation [18]; in the study of the Holstein-Hubbard model, this transformation is often useful. Observe that

\[
e^{L}c_{x\sigma}c^{-L} = e^{\Pi(\xi)}c_{x\sigma}, \quad e^{L}b_x e^{-L} = b_x - \omega^{-1} \sum_{y \in \Lambda} g_{xy} n_y,
\]

(5.2)

where \( \Pi(\xi) = \frac{i}{\sqrt{2}} (b(\xi)^* - b(\xi))^{**} \) and \( \xi = (\xi_x)_x \in l^2(\Lambda) \) with \( \xi_x = \omega^{-1} \sum_{y \in \Lambda} g_{xy} \). Here, we used the following notation: \( b(\xi) = \sum_{x \in \Lambda} \xi_x b_x \). Let \( N_p = d\Gamma_s(1) \). Using the formula \( e^{-i\pi N_p/2} b_x e^{i\pi N_p/2} = -ib_x \), we arrive at the following.
Lemma 5.1  Let $\mathcal{H} = e^{-i\pi N \rho/2} e^{L}$. Set $\mathcal{H}_{HH} = \mathcal{H} H_{HH} \mathcal{H}^{-1}$. For each $x, y \in \Lambda$, we define a vector $\zeta_{xy} \in \ell^2(\Lambda)$ by $\zeta_{xy} = \xi_x - \xi_y$. Then we have

$$
\mathcal{H}_{HH} = \sum_{\sigma = \pm} \sum_{x, y \in \Lambda} (-t_{xy}) e^{i \Phi_\sigma(\zeta_{xy})} c_x^\sigma c_y^\sigma + \sum_{x \in \Lambda} U_{\text{eff}, xx} n_x + \sum_{x \neq y} U_{\text{eff}, xy} n_x n_y + \omega N_p,
$$

where $U_{\text{eff}, xy} = U_{xy} - \omega^{-1} \sum_{z \in \Lambda} g_{xz} g_{zy}$ if $x \neq y$, and $U_{\text{eff}, xx} = U - 2\omega^{-1} \sum_{z \in \Lambda} g_{xz}^2$.

5.2 Trace formulas for $H_{HH}$ and $H_{HH, \infty}$

By Lemma 5.1, we know that every arguments in Section 4 are applicable to $H_{HH}$ and $H_{HH, \infty} = \mathcal{H} H_{HH, \infty} \mathcal{H}^{-1}$. Below, we exhibit trace formulas for $H_{HH}$ and $H_{HH, \infty}$.

**Theorem 5.2** Let

$$Z_{HH}(\beta) = \text{Tr}_{\mathcal{H}_{HH}} [e^{-\beta H_{HH}}].$$

There is a positive measure $\rho_{\beta, HH}$ on $L_\beta$ such that

$$Z_{HH}(\beta) = \int_{L_\beta} d\rho_{\beta, HH} (-1)^{\pi(X_\beta)} \exp \left\{ \beta b \sum_{j=1}^N \sigma(X_\beta(j))(0) \right\}.$$  

(5.5)

**Theorem 5.3** Assume that $N = |\Lambda| - 1$. Let $Z_{HH, \infty}(\beta)$ be the partition function defined by (1.27). There is a positive measure $\rho_{\beta, HH, \infty}$ on $L_{\beta, \infty}$ such that

$$Z_{HH, \infty}(\beta) = \int_{L_{\beta, \infty}} d\rho_{\beta, HH, \infty} (-1)^{\pi(X_\beta)} \exp \left\{ \beta b \sum_{j=1}^N \sigma(X_\beta(j))(0) \right\}.$$  

(5.6)

We can also prove the following propositions.

**Proposition 5.4** Assume that the dynamically allowed permutations are all even and not restricted to identity. For all $\beta \geq 0$, we have

$$Z_{HH}(\beta) \leq Z_H(\beta).$$

(5.7)

**Proposition 5.5** Assume that $N = |\Lambda| - 1$. For all $\beta \geq 0$, we have

$$Z_{HH, \infty}(\beta) \leq Z_{H, \infty}(\beta).$$

(5.8)

6 Random loops representations

In this section, we derive random loop representations for $Z_\ast(\beta)$ and $Z_{1, \infty}(\beta)$, $\ast = \text{rad, HH}$. Similar representations are known to be a useful tool in quantum spin systems, see, e.g., [2, 39]. As we will see in Sections 7 and 8, the representations also play important roles in the proof of Theorem 1.8.

For each $m \in L_\beta$, the corresponding path $(X_t(m))_{0 \leq t \leq \beta}$ satisfies the following properties:
(i) $\sigma^{(j)}_t(m)$ is constant in $t$ for all $j = 1, \ldots, N$;
(ii) there exists a dynamically allowed permutation $\tau$ such that $X_\beta(m) = \tau X_0(m)$;
(iii) $X_t(m) \in \Omega_N^\tau$ for all $t \in [0, \beta]$, that is, there are no encounters of electrons of equal spin.

Furthermore, the loops are associated with the path $(X_t(m))_{0 \leq t \leq \beta}$; the loops are obtained by the following manner:

- we start drawing the loops from the $t = 0$ location of any of the electrons;
- we trace the electron’s location forward in space-time until its first encounter with another electron;
- at the encounter point, the tracing line switches to the world line of the other electron in the reversed orientation in time;
- such an orientation switch is repeated whenever an electron encounter is reached;
- when trace line reaches the time at $t = 0$ or $t = \beta$, it reemerges at the same location with time treated as periodic;
- the above procedures are continued until a trace line is closed.

Figure 1: The collection of electron world lines. Electron 1 and electron 3 have the spin value +1; electron 2 has the spin value −1.

Figure 2: The loop corresponding to Figure 1. The black colored loop has the winding number 1 and the parity +1.

For each $m \in L_\beta$, let $\Gamma(m)$ be the collection of all loops associated with the path $(X_t(m))_{0 \leq t \leq \beta}$. Each loop in $\Gamma(m)$ is a closed trajectory $\gamma : [0, \ell_\beta] \rightarrow \Lambda \times [0, \beta]$, where $[0, \beta]$ is the interval $[0, \beta]$ with periodic boundary conditions, i.e., the torus of
length $\beta$. For each $\gamma \in \Gamma(m)$, the winding number is defined by the following line integral:

$$w(\gamma) = \left| \int_{\gamma} \frac{d\tau}{\beta} \right|. \quad (6.1)$$

Note that, because $\gamma$ is a piecewise smooth curve, the right hand side of (6.1) is well defined.

Our main result in this section is stated as follows.

**Theorem 6.1 (Random loop representation I)** For $\zeta = \text{rad, HH}$, one obtains

$$Z_\zeta(\beta) = \int_{L_\beta} d\rho(\gamma)(m) (-1)^{\pi(X_\beta(m))} \prod_{\gamma \in \Gamma(m)} \cosh (\beta w(\gamma)). \quad (6.2)$$

For $m \in L_{\beta,\infty}$, we can also associate the path $(X_t(m))_0 \leq t \leq \beta$ with the loops in the same manner as above. Then, we can prove the following.

**Theorem 6.2 (Random loop representation II)** For $\zeta = \text{rad, HH}$, one obtains

$$Z_{\zeta,\infty}(\beta) = \int_{L_{\beta,\infty}} d\rho(\gamma)(m) \prod_{\gamma \in \Gamma(m)} \cosh (\beta w(\gamma)). \quad (6.3)$$

### 6.1 Proof of Theorem 6.1

For each loop $\gamma \in \Gamma(m)$, set $\gamma_t = \gamma \cap X_t(m)$, the cross section of $\gamma$ with the cutting plane described by the equation $\tau = t$ in the $\tau-X$ plane. For notational simplicity, suppose that $\gamma_t = (X^{(i_1)}_t(m), \ldots, X^{(i_n)}_t(m))$ with $X^{(i)}_t(m) = (x^{(i)}_t(m), \sigma^{(i)}_0(m))$. Note that $i_1, \ldots, i_n$ and $n$ could depend on $t$. Then we readily confirm that

$$\sigma^{(i_1)}_t(m) + \cdots + \sigma^{(i_n)}_t(m) = \varepsilon(\gamma)w(\gamma), \quad \varepsilon(\gamma) = \pm 1, \quad (6.4)$$

where we understand that the left hand side of (6.4) equals 0, provided that $\gamma_t = \emptyset$.

The factor $\varepsilon(\gamma)$ is called the parity of $\gamma$, see Figures 1 and 2. The following lemma is an immediate consequence of (6.4).

**Lemma 6.3** For each $m \in L_\beta$, we have

$$\sum_{j=1}^N \varepsilon\sigma^{(j)}_t(m) = \sum_{\gamma \in \Gamma(m)} \varepsilon(\gamma)w(\gamma). \quad (6.5)$$

Let $(X^{(j)}_t(m))_{0 \leq t \leq \beta}$ be a trajectory of the $j$-th electron with $X^{(j)}_t(m) = (x^{(j)}_t(m), \sigma^{(j)}_0(m))$. For each $t \in [0, \beta]$, we set $X^{(j)}_t(m) = (x^{(j)}_t(m), -\sigma^{(j)}_0(m))$, the spin-reversed point corresponding to $X^{(j)}_t(m)$ in space-time. For $\gamma \in \Gamma(m)$ characterized by $\gamma_t = (X^{(i_1)}_t(m), \ldots, X^{(i_n)}_t(m))$, the conjugate loop $\gamma_t$ is defined through the relation $\gamma_t = (X^{(i_1)}_t(m), \ldots, X^{(i_n)}_t(m))$ for all $0 \leq t \leq \beta$.

Let $m \in L_\beta$. We express $\Gamma(m)$ as $\Gamma(m) = \{\gamma_1, \ldots, \gamma_K\}$, provided that $\Gamma(m) \neq \emptyset$. As before, we set $\gamma_{\alpha,t} = \gamma_\alpha \cap X_t(m) = (X^{(i_1)}_{\alpha,t}(m), \ldots, X^{(i_n)}_{\alpha,t}(m))$ with $X^{(i)}_{\alpha,t}(m) = x^{(i)}_{\alpha,t}(m), \sigma^{(i)}_0(m)$.
$(x^{(i)}_{\alpha,t}(m),\sigma^{(i)}_{\alpha,0}(m))$. For each $j \in \{1, \ldots, N\}$, there exists a $k(j) \in \{1, \ldots, K\}$ such that $X_{t=0}^{(j)}(m) \in \gamma_{k(j)}$. With this notation, we define a bijective map $g_j : L_\beta \rightarrow L_\beta$ through the following relation:

$$\Gamma(g_j m) = \{\gamma^b_j, \ldots, \gamma^b_K\}, \quad (6.6)$$

where $\gamma^b_\alpha$ is given by the following manner:

(a) $\gamma^b_{k(j)} = \gamma_{k(j)}$,

(b) remainder loops $\{\gamma^b_\alpha\}_{\alpha \neq k(j)}$ are uniquely determined by the rearrangement of the spin configuration of electron trajectories needed to maintain (i)-(iii).

Roughly speaking, (a) the map $g_j$ flips the spin values along the loop containing the $t = 0$ position of the $j$-th particle; (b) the spin flip induces the rearrangement of the spin configuration of remainder electron trajectories in order to maintain (i)-(iii), see Example 1 below.

**Example 1** For reader’s convenience, let us consider a path given in Figure 3. As shown

![Figure 3](image_url)

in Figure 4, this path has two loops, i.e., the black colored loop and the green colored loop. In this case, the action of $g_2$ induces the change of the spin configuration as follows:

![Figure 4](image_url)
The action of $g_1$ to this path induces the same change of the spin configuration. On the other hand, the action of $g_3$ to this path only flips the spin value along the trajectory of electron 3. □

The composition map $g_i \circ g_j$ will be simply denoted by $g_ig_j$. Trivially, we have the following:

- $g_jg_j = \text{Id}$, the identity map on $L_\beta$, for all $j \in \{1, \ldots, N\}$;
- $g_i g_j = g_j g_i$ for all $i, j \in \{1, \ldots, N\}$.

Let $G$ be the symmetry group generated by $g_1, \ldots, g_N$. Each $g \in G$ can be expressed as $g = g_{\xi}$, where

$$g_{\xi} = g^{\xi_1}_1 g^{\xi_2}_2 \cdots g^{\xi_N}_N, \quad \xi = (\xi_1, \ldots, \xi_N) \in \{0, 1\}^N. \quad (6.7)$$

Here, we understand that $g^0 = \text{Id}$ and $g^1 = g_j$. In particular, $G$ has $2^N$ elements.

**Lemma 6.4** We have the following:

(i) $\rho_\beta, \text{rad}(gE) = \rho_\beta(E)$ for each $E \in \mathcal{F}_\beta^N$, $g \in G$ and $\xi = \text{rad}, \text{HH}$, where $\mathcal{F}_\beta^N := \mathcal{F}_\beta^N \cap L_\beta$. Here, the reader should recall that $\mathcal{F}$ is the $\sigma$-algebra on $M$ introduced in Section 2.1.

(ii) $(-1)^\pi(X_\beta(gm)) = (-1)^\pi(X_\beta(m))$ for all $g \in G$.

**Proof.** (i) First, note that, by the arguments in the proof of Theorem 4.8, we have

$$\rho_\beta, \text{rad}(E) = Z_{\text{rad}}(\beta) \int_E d\nu_\beta W(X_\bullet). \quad (6.8)$$

For all $g \in G$ and $m \in L_\beta$, we readily confirm that

$$W(X_\bullet(gm)) = W(X_\bullet(m)), \quad (6.9)$$

$$\int_0^\beta V(X_u(gm)) du = \int_0^\beta V(X_u(m)) du, \quad (6.10)$$

$$\nu_\beta(gE) = \nu_\beta(E). \quad (6.11)$$

Taking these properties into account, we can show (i).

To check (ii) is easy. □

Let $m \in L_\beta$. Corresponding to $\Gamma(m) = \{\gamma_1, \ldots, \gamma_K\}$, there is a unique partition $(I_\gamma)_{\gamma \in \Gamma(m)}$ of $\{1, \ldots, N\}$ such that $X^{(\cdot)}_{t=0}(m) \in \gamma$ for all $i \in I_\gamma$. For each $\xi = (\xi_i)_{i=1}^N \in \{0, 1\}^N$, we set $\xi_\gamma = (\xi_i)_{i \in I_\gamma} \in \{0, 1\}^{\left|I_\gamma\right|}$. Trivially, $\xi = \bigcup_{\gamma \in \Gamma(m)} \xi_\gamma$. With this notation, we have the following decomposition:

$$g^\xi = \prod_{\gamma \in \Gamma(m)} g^\xi_\gamma, \quad (6.12)$$

where $g^\xi_\gamma = \prod_{i \in I_\gamma} g^\xi_i$.
Lemma 6.5. For each $m \in L_\beta$ and $\xi \in \{0,1\}^N$, we define random variables on $L_\beta$ by
\begin{align*}
B(m) &= \beta b \sum_{\gamma \in \Gamma(m)} \varepsilon(\gamma) w(\gamma), \\
B_\xi(m) &= \beta b \sum_{\gamma \in \Gamma(m)} \varepsilon(\gamma)(-1)^{\sum_{i \in I_\gamma} \xi_i} w(\gamma).
\end{align*}

Then we have
\begin{equation}
B(g^\xi m) = B_\xi(m).
\end{equation}

**Proof.** We will provide a sketch of the proof. Fix $j \in \{1, \ldots, N\}$ arbitrarily. We continue to use the notation (6.6). By the definition of $g^j$, we see that
\begin{equation}
B(g^j m) = \beta b \sum_{i=1}^K \varepsilon(\gamma_i)(-1)^{\delta_{i,k(j)}} \delta_{i,k(j)} w(\gamma_i) = B_\xi(m),
\end{equation}
where $\xi_j = (\delta_{i,k(j)})_{i=1}^N$. Hence, we obtain (6.15) when $\xi = \xi_j$. To extend this argument to general $\xi$ is not so hard. □

**Proof of Theorem 6.1.**

By Lemmas 6.3, 6.4 and 6.5, we observe that
\begin{equation}
Z_\sharp(\beta) = \int_{L_\beta} d\rho_{\beta,\prell}(g^\xi m)(-1)^{\pi(X_\beta(m))} e^{B(m)}
= \int_{L_\beta} d\rho_{\beta,\prell}(m)(-1)^{\pi(X_\beta(m))} e^{B_\xi(m)}
\end{equation}
for all $\xi \in \{0,1\}^N$ and $\prell = \text{rad, HH}$. Consequently, we obtain
\begin{equation}
Z_\sharp(\beta) = \sum_{\xi \in \{0,1\}^N} \frac{1}{2^N} \int_{L_\beta} d\rho_{\beta,\prell}(m)(-1)^{\pi(X_\beta(m))} e^{B_\xi(m)}
= \int_{L_\beta} d\rho_{\beta,\prell}(m)(-1)^{\pi(X_\beta(m))} \left[ \prod_{\gamma \in \Gamma(m)} \frac{1}{2^{|I_\gamma|}} \sum_{\xi \in \{0,1\}^{|I_\gamma|}} \exp \left\{ \beta b \varepsilon(\gamma)(-1)^{\sum_{i \in I_\gamma} \xi_i} w(\gamma) \right\} \right]
= \int_{L_\beta} d\rho_{\beta,\prell}(m)(-1)^{\pi(X_\beta(m))} \prod_{\gamma \in \Gamma(m)} \cosh \left( \beta w(\gamma) \right).
\end{equation}
Thus, we are done. □

**6.2 Proof of Theorem 6.2.**

Even in the case where $U = \infty$, Lemmas 6.3, 6.4 and 6.5 hold true. Therefore, by using arguments similar to those in the proof of Theorem 6.1, we can prove Theorem 6.2. □
7 Proof of Theorem \textbf{1.8}

7.1 A useful expression of $Z_{\natural,\infty}(\beta)$

To prove Theorem \textbf{1.8}, we need the following theorem.

**Theorem 7.1** Suppose that $N = |\Lambda| - 1$. Let $\mathcal{P}_N$ be the set of all partitions of $N$: $\mathcal{P}_N = \bigcup_{i=1}^N \{ n = \{ n_i \}^\ell \mid n_1 + \cdots + n_\ell = N \}$. Suppose that $0 < \beta < \infty$ and $0 < b$. For $\hat{n} \in \mathcal{P}_N$, there is a positive function $D_{n,\hat{n}}(\beta)$ independent of $b$ such that

(i) $Z_{\natural,\infty}(\beta) = \sum_{n \in \mathcal{P}_N} D_{n,\hat{n}}(\beta) \prod_i \cosh(\beta b n_i)$,

(ii) $D_{n,\hat{n}}(\beta)$ is strictly positive for all $0 < \beta$ if, and only if, there is an allowed permutation whose cyclic lengths are $n_1, \ldots, n_\ell$.

**Proof.** In the case where $U = \infty$, there are no encounters of electrons. Therefore, the loops are very much simplified. For each $m \in L_{\beta,\infty}$, there is a dynamically allowed permutation $\tau \in S_N(X_0(m))$ such that $X_\beta(m) = \tau X_0(m)$. Let us write down $\tau$ in cycle notation as

$$\tau = \varepsilon_1 \cdots \varepsilon_\ell, \quad (7.1)$$

where $\varepsilon_i$ is an $n_i$-cycle: $\varepsilon_i = (k_1 \cdots k_{n_i})$ with $k_j \in \{1, \ldots, N\}$. Without loss of generality, we may assume that $\sum_{i=1}^{\ell} n_i = N$. [For example, let us consider a permutation $\tau = (1 2 3)(4 7 9)(5 6 8)$. Thus, $\sum_{i=1}^{\ell} n_i = 9$ holds.] Note that, by the features of the system with $U = \infty$, each $n_i$ must be odd, which implies that $\text{sgn}(\varepsilon_i) = 1$. (Therefore, it holds that $(-1)^{\pi(X_\beta(m))} = 1$ for all $m \in L_{\beta,\infty}$ as we mentioned before.) We can associate the expression (7.1) with the set of loops $\Gamma(m) = \{ \gamma_1, \ldots, \gamma_\ell \}$ such that each $\gamma_i$ has the winding number $n_i$. By combining this and Theorem \textbf{6.2}, we conclude the assertions in Theorem \textbf{7.1}. \hfill \Box

7.2 Proof of Theorem \textbf{1.8}

Let $E$ be an expectation over permutations defined by

$$E[F(n)] = \sum_{n \in \mathcal{P}_N} D_{n,\hat{n}}(\beta) \prod_i \cosh(\beta b n_i) F(n) / Z_{\natural,\infty}(\beta). \quad (7.2)$$

Then we find that

$$\langle S^{(3)}_{\text{lat}} \rangle_{\natural,\infty}(b; \beta) = \frac{1}{2\beta} \frac{\partial}{\partial b} \log Z_{\natural,\infty}(\beta) = \frac{1}{2} E \left[ \sum_i n_i \tanh(\beta b n_i) \right]. \quad (7.3)$$

Because $\tanh(\beta bn) > \tanh(\beta b)$ for all $n \geq 2$, we find that RHS of (7.3) $> \frac{N}{2} \tanh(\beta b)$. \hfill \Box
8 A useful expression for $Z^\natural_\gamma(\beta)$

By using features of the one-dimensional system, we get the following expression for $Z^\natural_\gamma(\beta)$. Note that the theorem below could be useful when we examine finite-temperature extensions of the Lieb-Mattis theorem [19] for $H_{HH}$ and $H_{rad}$. (Remark that a simple extension of [19] to $H_{HH}$ can be found in [22].)

**Theorem 8.1** Let us consider a one-dimensional system. For each $m \in \text{spec}(S^{(3)}_{\text{tot}})$, let

$$Z^\natural_\gamma(\beta; m) = \text{Tr}_{\mathcal{S}^{(N)}_{k, M}[m]}[e^{-\beta H_\natural}], \ \natural = \text{rad}, HH,$$

where $\mathcal{S}^{(N)}_{k, M}[m]$ is the $m$-subspace defined by $\mathcal{S}^{(N)}_{k, M}[m] = \mathcal{S}^{(N)}_{H, M}[m] \otimes \mathcal{S}_2$ with $\mathcal{S}^{(N)}_{H, M}[m] = \ker(S^{(3)} - m)$. We set $Y^{(k)}_M(m) = \dim \mathcal{S}^{(N)}_{H, M}[m]$. For all $\beta > 0$, $b > 0$ and $\natural = \text{rad}, HH$, there are functions $C^{N, \natural}_k(\beta), C^{N-2, \natural}_k(\beta), \ldots, C^0_0(\beta)$ or $C^1_1(\beta)$ which are independent of $m$ such that

$$Z^\natural_\gamma(\beta) = \sum_{k=0}^{N} C^N_k(\beta) \{\cosh(\beta b)\}^k,$$

and

$$Z^\natural_\gamma(\beta; m) = \sum_{k=0}^{N} C^N_k(\beta) Y^{(k)}_M(m),$$

where we understand that $C^N_k(\beta) \equiv 0$ if $N - k$ is odd, and $Y^{(k)}_M(m) \equiv 0$ if $k < 2|m|$.  

**Proof.** In case of the one-dimensional chain, $w(\gamma)$ can take the values 0 and 1. Taking this fact into consideration, we set

$$L_\beta(k) = \left\{ m \in L_\beta \mid \text{the number of loops in } \Gamma(m) \text{ with } w(\gamma) = 1 \text{ is equal to } k \right\}.$$

Because $L_\beta = \bigsqcup_{k=0}^{N} L_\beta(k)$, we have

$$Z^\natural_\gamma(\beta) = \sum_{k=0}^{N} C^N_k(\beta) \{\cosh(\beta b)\}^k, \ C^N_k(\beta) = \int_{L_\beta(k)} d\rho^{\beta, \natural}(-1)^\tau(X_\beta).$$

Next, we will prove (8.3). First, note that

$$Z^\natural_\gamma(\beta) = \sum_{m=-N/2}^{N/2} Z^\natural_\gamma(\beta; m).$$

Let $z = e^{\beta b}$. By Theorem 5.3, we can express $Z^\natural_\gamma(\beta; m)$ as

$$Z^\natural_\gamma(\beta; m) = z^m \int_{\{ \sum_{j=1}^{N} \sigma(X_j^{(j)}) = 2m \}} d\rho^{\beta, \natural}(-1)^X_{\beta}. \quad (8.7)$$

On the other hand, by (8.5), we have

$$Z^\natural_\gamma(\beta) = \sum_{k=0}^{N} C^N_k(\beta) \left( \frac{z + z^{-1}}{2} \right)^k.$$

Comparing (8.7) and (8.8), we obtain (8.3). \(\square\)

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A A useful proposition

The following proposition is needed in Section 4.

**Proposition A.1** For all $j = 1, \ldots, N$, we have the following:

(i) $\int_{L_β} dν_β N(j)(t) = O(t)$ as $t \to +0$.

(ii) $\int_{L_β,∞} dν_β,∞ N(j)(t) = O(t)$ as $t \to +0$.

**Proof.** (i) First, we note that, by (2.58),

$$\int_{L_β} dν_β F(−1)π(X_β) = \sum_{X∈[ΩN]} \sum_{τ∈S_N} \frac{\text{sgn}(τ)}{N!} E_X \left[ F_1{τX} \cap D \right] / \text{Tr}_{P_G^L(Ω_N)} \left[ e^{−βL} \right].$$

(A.1)

Thus, it suffices to show that $E_X[N(j)(t)] = O(t)$ as $t \to +0$. By Lemma A.2 we find that

$$E_X[N(j)(t)] = \sum_{n=0}^{∞} nP_{X(j)}(N(j)(t) = n)
= \sum_{n=0}^{∞} nP_{X(j)}(J_n ≤ t < J_{n+1})
= \sum_{n=0}^{∞} nP_{X(j)}(S_1 + \cdots + S_n ≤ t < S_1 + \cdots + S_{n+1})
≤ d_0 t - 1 + e^{−d_0 t}
= O(t) \quad (A.2)$$

as $t \to +0$.

(ii) To prove (ii), we remark that

$$\int_{L_β,∞} dν_β,∞ F = \sum_{X∈[ΩN,∞]} \sum_{τ∈S_N} \frac{1}{N!} E_X \left[ F_1{τX} \cap D \right] / \text{Tr}_{P_G^L(Ω_N)} \left[ e^{−βT} \right].$$

(A.3)

Therefore, it suffices to prove that $E_X[N(j)(t)] = O(t)$ as $t \to +0$. But this has been already proved in the above. □

**Lemma A.2** Let $d_0 = d(x)$. (Recall that $d(x)$ is given in Section 2.1.2.) One obtains

$$P_X \left( S_1 + \cdots + S_n ≤ t < S_1 + \cdots + S_{n+1} \right) ≤ \frac{(d_0 t)^{n+1}}{(n+1)!} e^{−d_0 t}. \quad (A.4)$$

**Proof.** To prove the lemma, let us consider a single electron on $\tilde{Λ} = [-ℓ/2−1, ℓ/2+1]∩Z$. We impose the periodic boundary conditions on this system as follows: Let $\partial E_Λ$ be the set of pairs $\{x, y\} ∈ \tilde{Λ} × \tilde{Λ}$ satisfying the following:
• there exists an \( i \in \{1, \ldots, d\} \) such that \( x_i - y_i = \ell + 1 \);

• for all \( j \in \{1, \ldots, d\} \setminus \{i\} \), \( x_j - y_j = 0 \) holds.

Let \( \tilde{\Omega} = \hat{\Lambda} \times \{-1, +1\} \). Let \( (t^P_{xy}) \) be the hopping matrix with the periodic boundary conditions defined by \( t^P_{xy} = t_{xy} \) if \( \{x, y\} \in E_{\Lambda} \); \( t^P_{xy} = t \) if \( \{x, y\} \in \partial E_{\Lambda} \). The corresponding kinetic energy of the electron is a self-adjoint operator \( h^P_0 \) acting in \( \ell^2(\tilde{\Omega}) \) defined by

\[
(h^P_0 f)(x, \sigma) = \sum_{\sigma = \pm 1} \sum_{y \in \Lambda} t^P_{xy} (f(x, \sigma) - f(y, \sigma)), \quad f \in \ell^2(\tilde{\Omega}).
\] (A.5)

By replacing (2.1) with

\[
P^P(Y_n = X | Y_{n-1} = Y) = \delta_{\sigma, t} \frac{t^P_{xy}}{d(0)}, \quad X, Y \in \tilde{\Omega},
\] (A.6)

we can construct a Feynman-Kac-Itô formula for \( h^P_0 \). Notice that because we consider the periodic boundary conditions, \( d(x) \) is always constant: \( d(x) = \sum_{y \in \tilde{\Lambda}} t_{xy} = d(0) \).

The probability measures \( P \) and \( P^P \) are related as

\[
P(A) = P^P(A \land \mathcal{R}),
\] (A.7)

where \( \mathcal{R} = \{m \in M | X_t(m) \in \Omega \text{ for all } 0 \leq t\} \). Accordingly, it suffices to prove that

\[
P^P_X\left(S_1 + \cdots + S_n \leq t < S_1 + \cdots + S_{n+1}\right) = \frac{(d_0 t)^{n+1}}{(n+1)!} e^{-d_0 t}.
\] (A.8)

Indeed, taking the definition of \( S_n \) in Section 2 into account, we have

\[
P^P_X\left(S_1 + \cdots + S_n \leq t < S_1 + \cdots + S_{n+1}\right)
= \int_{t_1 + \cdots + t_n \leq d_0 t < t_1 + \cdots + t_{n+1}} e^{-(t_1 + \cdots + t_{n+1})} dt_1 \cdots dt_{n+1}
= \frac{(d_0 t)^{n+1}}{(n+1)!} e^{-d_0 t}.
\] (A.9)

Thus, we are done. \( \square \)

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