SPIN GRAPHS

K. M. BUGAJSKA

Abstract. We show that on any Riemann surface $\Sigma$ of genus $g > 1$ any non-singular even spin bundle $\xi$ defines an $\epsilon$-foliation of $\Sigma$. When a surface is hyperelliptic then all leaves of this foliation are finite and almost all of them consist of $2g + 2$ points. Moreover, each leaf carries an additional structure which allows us to view it as a concrete graph. We find the properties of these spin-graphs and we describe the classification of surfaces which is given by these properties. The classification is based on a finite number of exceptional graphs which have to be present on any surface $\Sigma$ of genus $g \geq 2$.

1. INTRODUCTION

We will show that for any point $P \in \Sigma$ any non-singular even spin bundle $\xi$ on $\Sigma$ determines the spin-graph $S_P$. For almost all non-Weierstrass points of $\Sigma$ the spin-graphs are isomorphic to each other, that is, they have the same generic form $S(g)$, where $g$ is the genus of a surface $\Sigma$. In addition, there are two spin-graphs through the Weierstrass points and besides we must have "exceptional" spin graphs which involve "exceptional points" (at most $4g$ of them).

In this paper we will consider only hyperelliptic Riemann surfaces. The image of any point $P$ of a surface $\Sigma$ under the hyperelliptic involution is denoted by $\tilde{P}$. The set of the Weierstrass points (i.e. points with the property $P = \tilde{P}$) will be denoted by $W$.

Let $S_P$ be a spin-graph through a point $P \in \Sigma$ and let $\{S_P\}$ denote the set of its vertices. Any spin-graph $S_P$ consists of vertices, edges, faces and n-cells ($n \leq (g - 1)$). For each vertex $Q \in \{S_P\}$ the graph $S_P$ allows us to read the divisor of the unique meromorphic section of $\xi$ with the single, simple pole at $Q$. Besides, it is shown in [1], [2], that $S_P$ carries a structure which allows us to associate to any of its vertex $Q$ a group $G_Q$ of permutations acting transitively on the set $\hat{Q} := \Phi_Q(\Sigma) \cap \Theta$. (Here $\Phi_Q : \Sigma \rightarrow \text{Jac} \Sigma$ denotes the Jacobi mapping with the origin at $Q$ and $\Theta$ is the divisor of the theta function $\Theta(\xi)$ on $\text{Jac} \Sigma$. [3], [4])

A generic spin graph $S(g)$ has $2g + 2$ vertices and all edges between the appropriate points are simple and straight. Each of two graphs through the Weierstrass points has $g + 1$ vertices which are connected to each other by straight, simple edges. The vertices of an exceptional spin graph will be called $\epsilon$-exceptional points. Any exceptional spin graph on a surface of genus $g$ is characterized by an integer $0 \leq \epsilon \leq g - 1$.

We will find all possible isomorphic classes of exceptional spin-graphs on a surface of genus $g$ and we will describe properties of such graphs that allow us to classify hyperelliptic surfaces. More precisely, to each surface $\Sigma$ equipped with a nonsingular even spin structure $\xi$, we will associate an $M$-tuple of integers $m_{\epsilon,s}$ which have to satisfy some condition (see (4.1) and (4.6)). Here, $r = 0, 1, \ldots, g - 1$; $s = 0, 1, \ldots, N(r)$ with $N(r)$ equal to the number $\sigma_{r+1}(g+1)$ of unordered partitions
of $g + 1$ into $r + 1$ positive integers and

$$M = \sum_{r=0}^{g-1} N(r)$$

The general classification of exceptional spin-graps on a surface of genus $g$ will be given. We will illustrate this by giving explicit examples of all classes of isomorphic exceptional graphs for $g$ equal to $2, 3, 4$ and $5$. There are exactly two classes of isomorphic exceptional spin graphs on a surface of genus $2$ and hence exactly three different types of surfaces of genus $2$. There are four isomorphic classes of possible exceptional graphs on a surface of genus $3$ and hence, by the condition (1.1), there are exactly nine different types of surfaces of genus $3$.

When $g = 4$ the number of isomorphic classes of exceptional spin graphs is six. Now, by the condition (4.1), each possible 6-tuple $(m_0, m_1, m_1, m_2, m_2, m_3)$ that satisfies $16 = 8m_0 + 6(m_1 + m_1) + 4(m_2 + m_2) + 2m_3$ determines a concrete type of a hyperelliptic Riemann surface equipped with a nonsingular spin structure $\xi$.

The number $M = \sum_{r=0}^{g-1} N(r)$ is the same for all surfaces of genus $g$ and for all possible nonsingular even spin-structures. It is an open question whether, for a given surface $\Sigma$, the $M$-tuples $(m_r,s)$, with $r = 0, \ldots, g-1; s = 0, \ldots, N(r)$; are the same or they are different for different non-singular even characteristics $[\epsilon]'s$.

More subtle classification of hyperelliptic Riemann surfaces which uses not only exceptional spin graphs but also exceptional spin groups associated to each vertex $Q$ of an exceptional spin-graph $S_p$ is given in the paper [2].

2. PRELIMINARIES

Let $\Sigma$ be a compact, hyperelliptic Riemann surface of genus $g \geq 2$ and let $\xi$ be a nonsingular even spin bundle over $\Sigma$. For any point $P \in \Sigma$ there exists (see [5], [6]) a unique point-like representation of $\xi$, namely:

$$\xi \cong \xi_P^{-1} \otimes \xi_{P_1} \otimes \ldots \otimes \xi_{P_g}; \quad P \notin \{P_1, P_2, \ldots, P_g\}$$

with the property that the index of specialty of the divisor

$$A_P^\epsilon := P_1 P_2 \ldots P_g$$

vanishes. (Here the points $P_1, \ldots, P_g$ are not necessarily distinct.) Equivalently, there exists unique (up to a nonzero multiplicative constant) section $\sigma_P$ of the line bundle $\xi$, with a single, simple pole at $P$ and hence with the divisor given by $(\sigma_P) := div \sigma_P = P^{-1} A_P^\epsilon$.

Let $D$ be any integral divisor on $\Sigma$. By $\{D\}$ we will denote the set of all distinct points of the divisor $D$. Moreover, for any divisor $U$ on a hyperelliptic surface $\Sigma$ the divisor that is obtained from $U$ by replacing all of its points by their conjugate respectively, will be denoted by $\tilde{U}$.

Suppose that we have fixed a point $P \in \Sigma$. Let $P_i \in \{A_P^\epsilon\}$. Now, for each $i = 1, \ldots, g$, the point $P_i$ determines a new set of points $P_{i,j}, j = 1, \ldots, g$, which form the divisor $A_P^{\epsilon_{P_i}}$ respectively. We will continue this process for all such points obtained in the previous step.

**Definition 1.** The set of all points of $\Sigma$ containing a point $P$ and obtained in the way described above starting from the set $\{A_P^\epsilon\}$ will be denoted by $\{S_P^\epsilon\}$. 
Lemma 3. For any Weierstrass point we have
\[ P \in \{ \tilde{P} \} \]
Proof. The property (1) follows immediately when we consider a meromorphic differential given by the product of two sections \( \sigma_p \) and \( \sigma_{\tilde{p}} \). Analogously, by considering a section \( \sigma_p \sigma_{\tilde{p}} \), we obtain the property (2).

Corollary 1.

(1) When a point \( P \in \Sigma \) has \( \epsilon \)-degree equal to \( g \) then we have \( \tilde{D}_k^P = \tilde{D}_k^P \).

(2) When \( \deg_{,} P = g \) then the divisors
\[ P^{-1}A_p = P^{-1}P_1\ldots P_g \quad \text{and} \quad P_k^{-1}P_{\tilde{P}_1}\ldots P_{\tilde{P}_k}\ldots P_g \]
are equivalent to each other. (Here the hat means that the point \( \tilde{P}_k \) is omitted.

Lemma 2. Suppose that point \( P \in \Sigma \) has \( \epsilon \)-degree equal to \( g \).

(1) When \( \tilde{P} \notin \{ A_p \} \) then each point \( Q \in \{ S_p \} \) has \( \epsilon \)-degree equal to \( g \) and we must have \( Q \notin \{ A_Q \} \).

(2) When \( \tilde{P} \in \{ A_p \} \) then there exists \( Q \in \{ S_q \} \) with \( \epsilon \)-degree necessarily less than \( g \); \( \deg, Q < g \).

Proof. Simple.

Corollary 2. When the \( \epsilon \)-degree of a non-Weierstrass point \( P \in \Sigma \) is equal to \( g \) and when \( \tilde{P} \notin \{ A_p \} \) then the cardinality of the set \( \{ S_p \} \) is \( 2g + 2 \). More precisely we have
\[ \{ S_p \} = \{ P, P_k, \tilde{P}, \tilde{P}_k; k = 1 \ldots g \} \]

Lemma 3. For any Weierstrass point \( P \in W \subset \Sigma \) we have \( \deg, P = g \). Moreover, all points \( P_k \in \{ A_p \} \) are also the Weierstrass points.

Proof. Since \( P = \tilde{P} \) we have \( A_p = A_{\tilde{P}} = \tilde{A}_p \) and hence the set \( \{ A_p \} \) is contained in the set \( \mathcal{W} \) of the Weierstrass points of \( \Sigma \). Now, since the \( \epsilon \)-degree of \( P \) less than \( g \) implies that the divisor of the section \( \sigma_p \) is an integral divisor (what is impossible for a non-singular even characteristic \( [\epsilon] \) we immediately obtain \( \deg, P = g = \deg, P_k \).

3. STANDARD AND WEIERSTRASS SPIN-GRAPHS

Let us fix a non-singular spin bundle \( \xi \) on a hyperelliptic surface \( \Sigma \). Let \( Q \) be any point of the set \( \{ S_p \} \subset \Sigma \) introduced by the definition 1 above. (From now on the index \( \epsilon \) may be omitted.) Notice that we must have \( \{ S_p \} = \{ S_Q \} \). Moreover, lemma 1 implies that when \( P' \in A_Q \) for some point \( P' \in \{ S_p \} \) then \( Q \in \{ A_{P'} \} \). We will say that points \( P' \) and \( Q \) are \( \epsilon \)-connected (or mutually connected by \( \xi \)).
Definition 4. Let $P$ be an arbitrary point of $\Sigma$. The spin graph $S_P$ through $P$ has vertices given by all points of the set $\{S_P\}$ and it has edges which connect only vertices that are $\epsilon$-connected. Suppose that vertices $Q$ and $R$ of the graph $S_P$ are $\epsilon$-connected. When $n_1 \geq 1$ is the maximal integer such that the divisor $Q^{n_1} < A_R$ and when $n_2 \geq 1$ is the maximal integer such that $R^{n_2} < A_Q$ and, for example, $n_1 \leq n_2 = n_1 + k; k \geq 0$, then the straight edge between $Q$ and $R$ has the multiplicity equal to $n_1$ and the spin-graph $S_P$ has the additional, oriented arc edge from $R$ to $Q$ labelled by $k = n_2 - n_1$.

Definition 5. A non-Weierstrass point $P$ will be called a standard point of $\Sigma$ when $\deg_{\epsilon} P = g$ and $\tilde{P} \notin \{A_P\}$.

Lemma 2 implies that all vertices of the spin-graph $S_P$ through a standard point $P$ are also standard points of $\Sigma$ as well as that all edges of $S_P$ must be simple straight edges. The spin-graph through a standard point will be called a standard graph.

Corollary 3. All standard spin-graphs on any hyperelliptic Riemann surface $\Sigma$ of genus $g$ equipped with any non-singular spin bundle $\xi_\epsilon$ are isomorphic to each other. In other words, the isomorphic class of standard spin-graphs depends only on the genus $g$ of a surface.

The standard graphs for genus $g = 2$ and $g = 3$ are given by the Pict1a and Pict1b respectively.

Let $P$ be a Weierstrass point. We already know that the $\epsilon$-degree of $P$ must be equal to $g$ and that all points of the divisor $A_P$ are also Weierstrass points. Since we have the property that $P \in \{A_Q\}$ if and only if $Q \in \{A_P\}$ the graph $S_P$ through a point $P$ has $g + 1$ vertices (which all are Weierstrass points) and all of them are mutually connected by straight, simple (i.e. with multiplicity equal to 1) edges. The examples of such graphs for genus $g = 2$ and for genus $g = 3$ are given by the Pict2a and by Pict2b respectively.
4. EXCEPTIONAL SPIN-GRAPHS

4.1. Generalities. Any non Weierstrass point \( Q \in \Sigma \) that is not a standard point will be called an exceptional point of \( \Sigma \). It occurs that on any hyperelliptic Riemann surface we must have points whose \( \epsilon \)-degree is equal to \( \rho < g \) (i.e. exceptional points). To show this let us consider the unique (up to a non zero multiplicative constant) meromorphic function \( f_P \) which connects two sections \( \sigma_P \) and \( \tilde{\sigma}_P \) of the line bundle \( \xi_\epsilon \); here \( P \) is an arbitrary, fixed standard point of \( \Sigma \). From the relation \( \tilde{\sigma}_P = f_P \sigma_P \) we see that the function \( f_P \) has degree equal to \( g + 1 \) and its divisor is

\[
(f_P) = (\frac{\sigma_{\tilde{P}}}{\sigma_P}) = \frac{P\tilde{P}_1\ldots\tilde{P}_g}{PP_1\ldots P_g}
\]

We see immediately that the ramification number of the mapping \( f_P : \Sigma \to \hat{\mathbb{C}} \) at any standard or at any Weierstrass point of \( \Sigma \) is equal to 1.

Let \( Q \) be any point of \( \Sigma \). Let \( A_Q \) denote the integral divisor \( QA_Q \) and let \( \{A_Q\} \) denote, as usually, the set of its distinct points.

**Lemma 4.** Let \( P \) be a standard point and let \( f_P \) be the meromorphic function on \( \Sigma \) introduced above. Then

1. For each point \( Q \in \Sigma \) the function \( f_P \) is constant on the set \( \{A_Q\} \).
2. If \( Q \) is any standard point different than \( P \) than functions \( f_P \) and \( f_Q \) are related by some Moebius transformation.
3. There are points on \( \Sigma \) whose \( \epsilon \)-degree is equal to \( \rho < g \). There are at most \( 4g \) such points.

**Proof.**

1. Suppose that \( f_P(Q) = z_1 \in \mathbb{C}^* \) (i.e. point \( Q \notin \{S_P\} \)). Since the index of specialty \( i(A_P) = 0 \) we see that the zero divisor of the function \( f_P - z_1 \) is \( (f_P - z_1)^0 = QA_Q \). This means that we have \( f_P(R) = z_1 = f_P(Q) \) for each point \( R \in \{A_Q\} \).
2. It is a simple consequence of the fact that whenever \( Q \) is a standard point with \( f_P(Q) = z_1 \neq 0 \) then \( f_P(Q) = z_2 \in \mathbb{C}^* \) and \( z_1 \neq z_2 \). Hence the divisor of the function \( (\frac{f_P}{f_P - z_2}) \) is equal to the divisor \( \frac{A_Q}{\epsilon_Q} = (f_Q) \).
By the Hurwitz-Riemann theorem the total branch number $B$ of the function $f_P$ is $B = 4g$. Since neither the Weierstrass points nor standard points can be ramification points of $f_P$ we must have some other points which have non-zero branching numbers. However, a non-zero branching number at a point $Q \in \Sigma$ means that this point occurs with the multiplicity $m > 1$ in the divisor $A_R$ for some point $R \in \{A_Q\}$. In other words, the $\epsilon$-degree of $R$ must be smaller than $g$. So, on any hyperelliptic surface $\Sigma$ there are exceptional points and there are at most $4g$ of them.

The spin-graph through an exceptional point will be called an exceptional graph. The number of such graphs is restricted by the value of the total branching number $B = 4g$. Contrary to the fact that all standard graphs belong to exactly one isomorphic class of graphs (depending only on the genus $g$ of $\Sigma$) and that the same is true for the Weierstrass spin graphs, the exceptional spin graphs may belong to distinct isomorphic classes.

Suppose that on a surface $\Sigma$ of genus $g$ we have some number $M$ of possible isomorphic classes of exceptional spin graphs. Let $B_i$, $i = 1, \ldots, N$ denote the total branch number carried by the $i$-th class of such graphs. This number is uniquely determined by the multiplicities of edges occurring in the graph. Let $m_i$ denote the number of exceptional graphs on $\Sigma$ that belong to the $i$-th class. We observe that we may classify hyperelliptic Riemann surfaces of genus $g$ equipped with a non-singular even spin structure $\xi$ by an ordered array of $M$ nonnegative integers $(m_1, m_2, \ldots, m_M)$ which satisfy

\begin{equation}
4g = \sum_{i=1}^{M} m_i B_i
\end{equation}

From our general considerations above we may notice that for any point $Q \in \Sigma$ and for any vertex $R$ of the graph $S_Q$ we have

\begin{equation}
\text{either } A_R = A_Q \text{ or } A_R = A_{\tilde{Q}}
\end{equation}

**Lemma 5.** Suppose that the integral divisor $A_Q$ corresponding to an exceptional point $Q \in \Sigma$ is given by

$$A_Q = \tilde{Q}^{k_0-1} Q_1^{k_1} \ldots Q_r^{k_r}; \quad k_0 \geq 1 \quad (k_0 - 1) + k_1 + \ldots + k_r = g$$

with $r < g$. The total branch number carried by the spin graph $S_Q$ is equal to $B(S_Q) = 2(g - r)$.

**Proof.** The form of the divisor $A_Q$ implies that the divisor $\kappa_Q := QA_{\tilde{Q}}$ may be written as follows

\begin{equation}
\kappa_Q = Q^{k_0} \tilde{Q}_1^{k_1} \tilde{Q}_2^{k_2} \ldots \tilde{Q}_r^{k_r} \quad \text{with} \quad k_0 + k_1 + k_2 + \ldots + k_r = g + 1
\end{equation}

The relation (4.2) implies that for each vertex $R \in \{S_Q\}$ the divisor $\kappa_R$ (which has the same form as (4.3)) may have distinct values of $k_i; i = 0, 1, \ldots, r$, but the number $r$ of the remaining points (different than $R$) in the set $\{\kappa_R\}$ is exactly the same for every vertex $R$. This means that we may characterize any exceptional spin graph $S_P$ by an integer $r < g$. Besides, we see that to evaluate the total branch number $B(S_Q)$ we may use an arbitrary vertex of the graph $S_Q$. More precisely we have

\begin{equation}
B(S_Q) = 2[(k_0 - 1) + (k_1 - 1) + \ldots +(k_r - 1)] = 2(g - r)
\end{equation}
Suppose that 

Example 1. for graphs with a given

Conversely, the equality of such graphs that occur on a given hyperbolic surface $\Sigma$. According to (4.3) it is called the order of a graph. However, exceptional graphs with the same order must have an $M$ that we may classify hyperelliptic Riemann surfaces by associating to each of them $\{A^*_r; Q \in \{\Sigma_Q\}\}$

Lemma 6. Let $\Sigma^*_P$ and $\Sigma^*_Q$ be two different exceptional graphs with heads $P$ and $Q$ respectively. These graphs are isomorphic if and only if, after eventually change of the enumerations of their vertices, $\hat{k}(P) = (k_0, k_1, \ldots, k_r)$ coincides with $\hat{k}(Q) = (k'_0, k'_1, \ldots, k'_s)$.

Proof. When the graphs are isomorphic then obviously we have $\hat{k}(P) = \hat{k}(Q)$. Conversely, the equality $\hat{k}_P = \hat{k}_Q$ means that $r = s$ and that $k_l(P) = k_l(Q)$ for $l = 0, 1, \ldots, r$. Equivalently, the divisors of the sections $\sigma_P$ and $\sigma_Q$ are: $\sigma_P = P^{-1}P^{s_0}P^{s_1} \ldots P^{s_r}$ and $\sigma_Q = Q^{-1}Q^{s_0}Q^{s_1} \ldots Q^{s_r}$. Now the isomorphism between the graphs $\Sigma^*_P$ and $\Sigma^*_Q$ follows immediately.

For any vertex $Q$ of any spin graph $\Sigma^*_P$ the integer $r \geq 0$ denotes the number of points of the set $\{A_Q\}$ that are different than the conjugate point $\tilde{Q}$. Although the $\epsilon$-degree $\deg Q \geq r$ may be different for different vertices of a given exceptional graph $\Sigma^*_P$, the integer $r$ is for all vertices exactly the same and, by the definition 6, it is called the order of a graph. However, exceptional graphs with the same order $r$, $0 \leq r < g$, may belong to different classes of isomorphic spin graphs that are possible on a given surface of genus $g$.

This means that the integer $r$ does not define uniquely a class of isomorphic spin graphs. More precisely, the lemma 6 implies that the number of possible different classes of isomorphic spin graphs with a given $0 \leq r < g$ is equal to the number $N(r)$ of representations of $g + 1$ as a sum $g + 1 = k_0 + k_1 + \ldots + k_r$ of non-decreasing integers $1 \leq k_0 \leq k_1 \leq \ldots k_r \leq g - r - 1$ i.e. $N(r) = \sigma_{r+1}(g + 1)$.

Now, $N(0) = 1$ and it corresponds to the unique class of isomorphic graphs with $\hat{k} = (k_0)$ where $k_0 = g + 1$. The value of $N(g - 1)$ is also one and it corresponds to the unique class with $\hat{k} = (k_0, k_1, \ldots, k_{g-1}) = (1, 1, \ldots, 1, 2)$.

Let us fix the genus $g$ and let $\Sigma^*_r = \Sigma^*_r(g)$ denote a class of appropriate exceptional graphs with a given $r$ and with $s \in \{1, 2, \ldots, N(r)\}$. Let $m_{r,s}$ denote the number of such graphs that occur on a given hyperbolic surface $\Sigma$. According to (4.1) we must have

$$4g = m_0B_0 + B_1(m_{1,1} + \ldots + m_{1,N(1)}) + \ldots + B_r(m_{r,1} + \ldots + m_{r,N(r)}) + \ldots + B_{g-1}m_{g-1}$$

Since $M$, given by the formula (1.1), determines the number of possible different classes of isomorphic exceptional spin graphs on a surface of the genus $g$ we see that we may classify hyperelliptic Riemann surfaces by associating to each of them an $M$-tuple of non-negative integers $(m_{r,s}; r = 0, 1, \ldots, g - 1; s = 1, \ldots, N(r))$.

Example 1. Suppose that $g = 2$. Since we must have $r \in \{0, 1\}$ the possible classes $\Sigma^*_r$ of exceptional graphs are $\Sigma^*_0$ and $\Sigma^*_1$. Hence, to any surface of genus
So, we have nine different types of surfaces of genus $g$. We see that we may have exactly three different types of such surfaces corresponding to $(m_0, m_1)$ equal either to $(2, 0)$ or to $(1, 2)$ or to $(0, 4)$ respectively.

**Example 2.** Let $g = 3$. Now $r \in \{0, 1, 2\}$. A possible exceptional graph $S_P$ with a head $P$ may belong to the following classes: either to $S^0$ (when $A_P = \tilde{P}^1$), or to $S^1_1$ (when $A_P = P^1_1$), or to $S^2_1$ (when $A_P = \tilde{P}P^1_1$) or to the class $S^3_1$ (when $A_P = P^2_1P^1_1$). Since the condition (4.6) requires that $12 = 6m_0 + 4(m_{1,1} + m_{1,2}) + 2m_2$ any surface of genus $g = 3$ can be characterized by a quadruple $(m_0, m_{1,1}, m_{1,2}, m_2)_\epsilon$ of non-negative integers which belongs to the set:

$$\{(2, 0, 0, 0), (1, 1, 0, 1), (1, 0, 1, 1), (0, 1, 2, 0), (0, 2, 1, 0), (0, 11, 2), (0, 1, 0, 4), (0, 0, 1, 4), (0, 0, 0, 6)\}$$

So, we have nine different types of surfaces of genus $g = 3$ equipped with an even, non-singular spin structure $\xi_e$.

**Example 3.** Suppose that the genus of a hyperelliptic surface is $g = 4$. Now we may have $r \in \{0, 1, 2, 3\}$. A possible exceptional spin graph $S_P$ with a head $P$ belongs to one of the following classes of isomorphic graphs: to the class $S^0$ (when $A_P = \tilde{P}^1$), or to $S^1_1$ (when $A_P = P^1_1$), or to $S^2_1$ (when $A_P = \tilde{P}P^1_1$) or to the class $S^3_1$ (when $A_P = P^2_1P^1_1$), or to $S^3_2$ (when $A_P = P^2_2P^1_1$), or to $S^3_2$ (when $A_P = P^2_1P^2_2$), or to $S^3_2$ (when $A_P = P^1_1P^2_2$). Thus, to any surface of genus $g = 4$ we will associate a six-tuple $(m_0, m_{1,1}, m_{1,2}, m_{2,1}, m_{2,2}, m_3)_\epsilon$ of non-negative integers which satisfy

$$16 = 8m_0 + 6(m_{1,1} + m_{1,2}) + 4(m_{2,1} + m_{2,2}) + 2m_3$$

Each non-negative integer $m_{r,s}$ indicates the number of exceptional spin graphs that occurs on $\Sigma$ which belong to the isomorphic class $S^r_\epsilon$; where $r = 0, 1, 2, 3$ and $s = 1, \ldots, N(r)$.

**4.2. Exceptional graphs for arbitrary genus $g$.** Let $S_P$ be an exceptional graph on a surface $\Sigma$ of genus $g$ whose head is $P$. Suppose that the divisor of the section $\sigma_P$ of $\xi_e$ is

$$\sigma_P = P^{-1}\tilde{P}^{1}P_1^{k_1} \ldots P_r^{k_r} \quad \text{with} \quad i = k_0 - 1, \quad 0 \leq r \leq g - 1$$

and with $k_0 \leq k_1 \leq \ldots \leq k_r$

We will introduce the following notation:

$$k_0 = 1 + i, \quad k_n = k_{n-1} + p_n; \quad n = 1, 2, \ldots, r$$

Now, the class $S^r_\epsilon = S^r_\epsilon(g)$ of isomorphic exceptional graphs will be denoted by $S_{r, p_1, \ldots, p_r}$. Since from (4.3) we have

$$g + 1 = (r + 1)(1 + i) + rp_1 + (r - 1)p_2 + \ldots + p_r$$

we see that the maximal possible value of $i = k_0 - 1$ is

$$i_{\text{max}} = \left\lfloor \frac{g - r}{r + 1} \right\rfloor$$
4.2.1. Let $P$ be a head of an exceptional spin graph with $r = 1$ i.e. the divisor $k_P = P^{k_0} \tilde{P}^{k_1}$ (equivalently $A_P = \tilde{P}^i P^{k_1}$) and $g = i + k_1$. Since $P$ is a head we must have $k_0 \leq k_1$ and hence

$$0 \leq i \leq \left\lfloor \frac{g - 1}{2} \right\rfloor$$

The class $S_{i,p_1}(g) = S_{i,p_1}$ of isomorphic graphs corresponds to $\hat{k} = (k_0, k_1) = (1 + i, 1 + i + p_1)$. When $i = 0$ then $k_1 = g$ and the graph $S_{0,g-1}$ is given on Pic3a.

When $i \geq 1$ then all conjugate vertices must be connected and the general form of the graph $S_{i,p_1}$ is given by Pic3b.

In particular, when the genus is odd then for $i_{\text{max}} = \frac{g-1}{2}$ we have $p_1 = 0$ i.e. $k_0 = k_1$. For the remaining values of $i \leq \frac{g-1}{2}$ we have $p_1 = 2i_{\text{max}} - 2i + 1$ and hence it is always odd. When the genus $g$ is an even integer then $i_{\text{max}} < \frac{g-1}{2}$ and the corresponding to it $p_1$ must be equal to 1. For the remaining possible values of $i$ we have $p_1 = 2i_{\text{max}} - 2i + 1$ which is always odd.

When $g = 3$ then there are two possible classes of isomorphic exceptional graphs corresponding to the values of $(i, p_1)$ equal to $(0, 2)$ and to $(1, 0)$. When $g = 4$ then the possible graphs $S_{i,p_1}$ correspond to $(i, p_1) \in \{(0, 3), (1, 1)\}$. When $g = 5$ then $(i, p_1) \in \{(04), (1, 2), (2, 0)\}$. For all of these values of the genus $g$ we may use the pictures Pic3a and Pic3b to draw appropriate spin graphs.

For a general $g \geq 2$ the total number of possible isomorphic classes $S_{i,p_1}$ is equal to

$$N(1) = \sigma_2(g + 1)$$

More precisely, when $g$ is even then

$$\hat{k} = (k_0, k_1) \in \{(1, g), (2, g - 1), \ldots, \left(\frac{g}{2}, \frac{g}{2} + 1\right)\}$$
and when \( g \) is odd we have
\[
\hat{k} = (k_0, k_1) \in \{(1, g), (2, g - 1), \ldots, (1 + \left\lfloor \frac{g}{2} \right\rfloor, 1 + \left\lfloor \frac{g}{2} \right\rfloor)\}
\]

4.2.2. \( r=2 \). Suppose that \( S_P \) is an exceptional spin graph with a head \( P \). We have \( \hat{k}(P) = (k_0, k_1, k_2) = (1 + i, 1 + i + p_1, 1 + i + p_1 + p_2) \) with \( i, p_1, p_2 \geq 0 \) and with \( i + k_1 + k_2 = g \). The general form of a spin graph with \( r = 2 \) is given by Pict.4. Since
\[
3i + 2p_1 + p_2 + 2 = g
\]
such exceptional graph is possible only when the genus of a surface is \( g \geq 3 \). The maximal possible value of \( i \) is
\[
i_{\text{max}} = \left\lfloor \frac{g - 2}{3} \right\rfloor
\]

When \( r = 2 \) then, depending from the genus \( g \), we may have one or two possible isomorphic classes of graphs with the value \( i = i_{\text{max}} \). More precisely:

- When \( g \equiv 0 \mod 3 \) then \( i_{\text{max}} = \frac{g}{3} - 1 \) and the property \( (4.10) \) implies that \( p_1 = 0, p_2 = 1 \). Hence there is only one class of exceptional graphs with such \( i_m \equiv i_{\text{max}} \), namely \( S_{i_m,0,1} \).
- When \( g \equiv 1 \mod 3 \) then \( i_{\text{max}} = \left\lfloor \frac{g-2}{3} \right\rfloor = \left\lfloor \frac{g}{3} \right\rfloor - 1 \) and we have \( 2p_1 + p_2 = 2 \). There are two possible classes of isomorphic spin graphs with the maximal value of \( i = i_m \): \( S_{i_m,0,2} \) and \( S_{i_m,1,0} \).
- When \( g \equiv 2 \mod 3 \) then \( i_m = \left\lfloor \frac{g-2}{3} \right\rfloor = \left\lfloor \frac{g}{3} \right\rfloor \). Now we must have \( p_1 = p_2 = 0 \) and hence the unique class \( S_{i_m,0,0} \) of graphs with the maximal value \( i = i_m \).

From Pict.4 we see that when \( i = 0 \) then a head \( P \) of the exceptional graph \( S_P \in S_{0,p_1,p_2} \) is not connected to its conjugate \( \tilde{P} \). We may have a situation where either only one pair of conjugate vertices ( i.e. \( P_2 \) and \( \tilde{P}_2 \)) is connected or two pairs \( \{P_i, \tilde{P}_i\} \) for \( i = 1, 2 \) are connected. In the first case \( S_P \) belongs to the unique class \( S_{0,0,2} \) and in the latter case we have \( \sigma_2^2(g) \) possible classes \( S_{0,p_1,p_2} \), (with \( p_1 \geq 1 \) and \( k_1 + k_2 = g; \ 2 \leq k_1 \leq k_2 \)) of isomorphic graphs. (Here \( \sigma_2^k(m) \) denotes the number of representations of \( m \) as a sum of \( k \) non-decreasing integers that are grater than or equal to \( l \).) Summarizing, the number of possible graphs with \( i = 0 \) is equal to \( \sigma_2(g) = 1 + \sigma_2^2(g) \).

When \( i > 0 \) then all pairs of conjugate vertices are connected. Since we have \( k_0 = 1 + i \geq 2 \) and \( k_0 \leq k_1 \leq k_2; i + k_1 + k_2 = g \), it is easy to see that the number of all possible classes of exceptional graphs is now given by \( \sigma_3^2(g + 1) \).

The graph \( S_P \) is totally symmetric with respect to all of its vertices (see Pict.5) when \( k_0 = k_1 = k_2 = 1 + i \). This is possible only on a surface whose genus is \( g \equiv 2 \mod 3 \) and \( g = 3i + 2 \geq 5 \). In particular, for \( g = 5 \) this occurs when \( i = i_{\text{max}} = 1, \hat{k} = (k_0, k_1, k_2) = (2, 2, 2) \).

We notice that the cardinality of all possible classes of isomorphic exceptional spin graphs with \( r = 2 \) is
\[
N(2) = \sigma_2(g) + \sigma_3^2(g + 1) = \sigma_3(g + 1)
\]
as expected.
4.2.3. $r = 3$. Let $P$ be a head of an exceptional spin graph $S_P$ with $r = 3$. This means that the section $\sigma_P$ of the holomorphic bundle $\xi_\epsilon$ has the divisor

$$(\sigma_P) = P^{-1}A_P = P^{-1}\tilde{P}^iP_1^{k_1}P_2^{k_2}P_3^{k_3} \quad \text{with} \quad i + k_1 + k_2 + k_3 = g$$

Each isomorphic class of graphs with $r = 3$ is uniquely determined by a quadruple $(i, p_1, p_2, p_3)$ where $p_l = k_l - k_{l-1}$ for $l = 1, 2, 3$. Since

$$(4.12) \quad g = 4i + 3p_1 + 2p_2 + p_3 + 3$$

the genus of a surface that carries such graph must be $g \geq 4$ and the value of $i = k_0 - 1$ may vary from 0 to $i_{\text{max}} = i_m = \left\lfloor \frac{g-3}{4} \right\rfloor$. There are two or one (depending on the genus ) isomorphic classes of graphs with $i = i_m$.

- When $g \equiv 3 \text{mod} 4$ then $i_m = \left\lfloor \frac{3}{4} \right\rfloor$ and we have only one class $S_{i_m,0,0,0}$.
- When $g \equiv 2 \text{mod} 4$ then $i_m = \left\lfloor \frac{3}{4} \right\rfloor - 1$ and possible classes with $i = i_m$ are $S_{i_m,1,0,0}$ and $S_{i_m,0,0,3}$.
- When $g \equiv 1 \text{mod} 4$ then $i_m = \left\lfloor \frac{3}{4} \right\rfloor - 1$ again but now we have two possible classes of graphs: $S_{i_m,0,1,0}$ and $S_{i_m,0,0,2}$.
- When $g \equiv 0 \text{mod} 4$ then $i_m = \frac{3}{4} - 1$ and there is unique possible class of exceptional graphs with $i = i_m$, namely $S_{i_m,0,0,1}$. 
We observe that only on a surface of genus $g \equiv 3 \mod 4$, $g \geq 7$ we may have an exceptional spin graph whose all pairs of conjugate vertices are connected and the graph is symmetric with respect to all of its vertices (i.e. $k_0 = k_1 = k_2 = k_3 \geq 2$).

When $i = 0$, i.e. when a head $P$ is not connected with its conjugate, then the number of all possible classes of equivalent graphs is equal to $\sigma_3(g)$. More precisely:

- When only one pair $P_3$ and $\tilde{P}_3$ of conjugate vertices is connected then we have $\tilde{k}(P) = (1,1,1,g-2) = (1,1,1,1+p_3)$ and $S_P \cong S_{0,0,0,g-3}$.
- When two pairs, $P_2, \tilde{P}_2$ and $P_3, \tilde{P}_3$ are connected then there are $\sigma_2(g-1) - 1$ possible isomorphic classes $S_{0,0,p_2,p_3}, p_2 > 0$, of exceptional spin graphs.
- When only a head of an exceptional graph is not connected with its conjugate (i.e. when $p_1 > 0$) then $\tilde{k}(P) = (1,1+p_1,1+p_1+p_2,1+p_1+p_2+p_3) = (k_0,k_1,k_2,k_3)$ with $2 \leq k_1 \leq k_2 \leq k_3$. The number of possible isomorphic classes of such graphs is $\sigma_3^2(g) = \sigma_3(g) - \sigma_2(g-1)$.

When $i > 0$ then all pairs of conjugate vertices are connected. Now, for each $1 \leq i \leq \left[\frac{g-3}{4}\right] = i_{\text{max}}$ there are $\sigma_3^{k_0}(g-i)$ possible classes of isomorphic spin graphs. The set $\{S_{i,p_1,p_2,p_3}\}, i > 0$, of all such classes has $\sigma_3^2(g+1)$ elements. (This number is equal to the number of representations of $g+1$ as a sum $k_0 + k_1 + k_2 + k_3$ of integers that satisfy $2 \leq k_0 \leq k_1 \leq k_2 \leq k_3$.)

Summarizing, the total number $N(3)$ of isomorphic classes of possible exceptional spin graphs with $r = 3$ is equal

\[
N(3) = \sigma_3(g) + \sigma_3^2(g+1) = \sigma_4(g+1)
\]

4.2.4. $r > 3$. Keeping the same notation as above, an exceptional graph $S_P$ on a surface $\Sigma$ of genus $g \geq r + 1$ belongs to an isomorphic class $S_{i,p_1,\ldots,p_r}$ where $i + k_1 + \ldots + k_r = g$, $1 + i \leq k_1 \leq \ldots \leq k_r$ and

\[
g = (r+1)i + rp_1 + \ldots + 2p_{r-1} + p_r + r
\]
where $B_m$ negative integers and $S$ graph which belongs to a class $\mathcal{S}_{r,p_1,...,p_r}$ class isomomorphic hyperelliptic Riemann surfaces of genus $g$ on a surface $\Sigma$ not all types of exceptional graphs have to occur. Thus, we may classify hyperelliptic Riemann surfaces of genus $g$.

$$4 = \sigma_r(g) + \sigma_{r+1}^2(g+1) = \sigma_{r+1}(g+1)$$

For the maximal possible value of $r$, i.e. for $r = g-1$ this formula gives $N(g-1) = \sigma_g(g+1) = 1$ as expected. We see that the number of all isomorphism classes of exceptional graphs that are possible on a surface $\Sigma$ of genus $g$ is

$$M = M(g) = \sum_{r=0}^{g-1} n(r) = 1 + \sum_{r=1}^{g-1} \sigma_r(g+1) = 1 + \sum_{m=2}^{g} \sigma_m(g+1)$$

5. Summary

Let $\Sigma$ be a hyperelliptic Riemann surface and let $\xi_e$ be any even, non singular spin bundle on this surface. The hyperelliptic involution results in the existence of interrelations between some sections of $\xi_e$. These relations always link merely finite number of meromorphic sections of this bundle, each with a single, simple pole.

In almost all cases the number of such related sections is equal to $2g+2$. In two cases this number is $g+1$ and besides of this, we have finite number of cases when the number of interrelated sections vary. In other words, the bundle $\xi_e$ introduces $\epsilon$-foliation of the surface $\Sigma$ whose all leaves consist of finite number of points. These points are vertices of spin graphs. Each graph carries all informations about the sections of $\xi_e$ with the unique simple pole at a given vertex of this graph (i.e. at a given point of the leaf).

Almost all leaves of the $\epsilon$-foliation have $2g+2$ points that are vertices of standard graphs. These graphs are isomorphic to each other i.e. they all belong to the unique, (standard) class of spin graphs $\mathcal{S}(g)$. There are two leaves through the Weierstrass points $\mathcal{W}$, each consisting of $g+1$ points (they are vertices of Weierstrass graphs).

The remaining leaves of the $\epsilon$-foliation are associated with exceptional graphs. The number of non-isomorphic classes of exceptional spin graphs that are possible on a surface of genus $g$ is equal to $M(g)$ (see the formula (4.16)). However, on a given surface $\Sigma$ not all types of exceptional graphs have to occur. Thus, we may classify hyperelliptic Riemann surfaces of genus $g$ by giving an $M(g)$-tuple of non-negative integers $m_{r,s}$, $r = 0, 1, \ldots, g-1; s = 1, 2, \ldots, N(r)$. Each integer $m_{r,s}$ indicates how many exceptional graphs belonging to a given class of isomorphic graphs, are actually present on a surface. These integers must satisfy the conditions

$$4g = B_0m_0 + \ldots + B_r(m_{r,1} + \ldots + m_{r,N(r)}) + \ldots + B_{g-1}m_{g-1}$$

where $B_r = 2(g - r)$ is the total branch number produced by any exceptional spin graph which belongs to a class $\mathcal{S}_{r,p_1,...,p_r}$ with a given $r = 0, 1, \ldots, g-1$.

This $\epsilon$-foliation of a hyperbolic Riemann surface $\Sigma$ together with the spin-graph structure of each of its leaf will allow us (in []) to attach to each point $P$ of $\Sigma$ a concrete spin group $G_P$. 
REFERENCES

[1] Bugajska, K., *Standard and Weierstrass spin groups on hyperelliptic Riemann surfaces*, submitted for publication

[2] Bugajska, K., *Exceptional spin groups on hyperelliptic Riemann surfaces*, submitted for publication

[3] Farkas, H. M., I. Kra, *Theta constants, Riemann Surfaces and the Modular Group*, AMS, GSM vol.37, 2001

[4] Mumford, D., *Tata Lectures on Theta II*, Birkhäuser, Progress in Mathematics vol.43, 1984

[5] Varolin, D., *Riemann surfaces by Way of Complex Analytic Geometry*, AMS, GSM vol.125, 2011

[6] Gunning, R., C., *Lecture on vector bundles over Riemann surfaces*, MNPUP, Princeton, 1967

[7] Miranda, R., *Algebraic curves and Riemann Surfaces*, AMS, GSM vol.5, 1995

[8] Farkas, H., M., I. Kra, *Riemann Surfaces*, Springer-Verlag, GTM vol.71, 1992

Department of Mathematics and Statistics, York University, Toronto, ON, M3J 1P3