Landau degeneracy and black hole entropy

Miguel S. Costa\textsuperscript{1} and Malcolm J. Perry\textsuperscript{2}

\textit{D.A.M.T.P.}
\textit{University of Cambridge}
\textit{Cambridge CB3 9EW}
\textit{UK}

Abstract

We consider the supergravity solution describing a configuration of intersecting D-4-branes with non-vanishing worldvolume gauge fields. The entropy of such a black hole is calculated in terms of the D-branes quantised charges. The non-extreme solution is also considered and the corresponding thermodynamical quantities are calculated in terms of a D-brane/anti-D-brane system. To perform the quantum mechanical D-brane analysis we study open strings with their ends on branes with a magnetic condensate. Applying the results to our D-brane system we managed to have a perfect agreement between the D-brane entropy counting and the corresponding semi-classical result. The Landau degeneracy of the open string states describing the excitations of the D-brane system enters in a crucial way. We also derive the near-extreme results which agree with the semi-classical calculations.

\textsuperscript{1}M.S.Costa@damtp.cam.ac.uk
\textsuperscript{2}malcolm@damtp.cam.ac.uk
1 Introduction

It is now widely accepted that superstring theory is the most promising unifying theory. Among its great achievements is the resolution of a longstanding problem in black hole physics: the statistical origin of the Bekenstein-Hawking entropy at least for certain extreme and near-extreme black holes [1, 2, 3, 4, 5, 6]. The resolution of this problem has been made possible by the realization of Polchinski [7, 8] that D-branes, i.e. branes carrying Ramond-Ramond charge, are extended objects with the property that open strings can end on them. The D-brane excitations are described by such open strings. The microscopic origin of the black hole entropy arises in D-brane physics in two steps: firstly, we identify a certain classical supersymmetric black hole carrying charge in the $R \otimes R$ sector with a corresponding excited D-brane configuration; secondly, we count the number of open string states that reproduce the excited D-brane system.

The worldvolume bosonic field content that describes the low energy excitations of a D-$p$-brane is given by $9 - p$ scalar fields $\phi^m \ (m = p + 1, \ldots, 9)$ and by a 1-form gauge potential $A_\alpha \ (\alpha = 0, \ldots, p)$ [9, 10]. These fields are interchanged by $T$-duality transformations. The scalar fields are associated to the massless excitations of the brane that describe transverse displacements, and the gauge field is associated to worldvolume massless excitations on the brane. Through this paper we will be interested in non-threshold D-brane bound states. These configurations arise whenever the worldvolume gauge field associated with a D-$p$-brane forms a condensate. If this is the case the D-$p$-brane may carry the $R \otimes R$ charge that is usually associated with lower dimensional D-branes. In fact, the D-$p$-brane effective action contains the correct coupling terms to $R \otimes R$ gauge potentials with rank lower than $p + 1$ [11]. Remarkably, the existence of these generalised Wess-Zumino terms (and for the D-9-brane the anomalous Bianchi identity and the G-S coupling) is a consequence of $T$-duality and Lorentz invariance [11, 12].

To be definite consider the $(p|p - 2)$ D-brane bound state. This configuration may be obtained by performing a $T$-duality at an angle on a D-$(p - 1)$-brane, i.e.

$$
p - 1 : \quad X^1, \ldots, X^{p-1} \quad \phi^p = \tan \zeta \ X^{p-1} \quad \rightarrow \quad p : \quad X^1, \ldots, X^p \quad A_p = \tan \zeta \ X^{p-1}
$$

where we start with a D-$(p - 1)$-brane parallel to the $x^1, \ldots, x^{p-2}$-directions and making an angle $\zeta$ with the $x^{p-1}$-direction in the $x^{p-1}x^p$ 2-torus. The $T$-duality
transformation along the $x^p$-direction interchanges the scalar field $\phi^p$ with the
gauge field $A_p$ forming a magnetic condensate (through this paper we shall scale
the gauge potential as $A_\alpha \equiv 2\pi\alpha' A_\alpha$). From the supergravity theory perspective
the existence of these bound states has been deduced in [13, 14].

In the linear approximation the field theory describing the low-lying states on
the D-brane bound state in (1.1) is the super Yang Mills theory with $U(m)$ gauge
group [15]. The integer $m$ is the D-$p$-brane winding number in the $x^{p-1}$-direction.
The brane carries a $U(1)$ flux which induces a t'Hooft twist in the $SU(m)/\mathbb{Z}_m$
part of the theory [16, 17, 18, 19]. As a consequence the gauge potential and the
scalar fields will obey twisted boundary conditions. For an appropriate choice
of this $U(m)$ bundle multiple transition functions the only non-vanishing gauge
potential is in the $U(1)$ center of the group and it is given by (1.1). Had we
considered $N$ D-branes on top of each other, the scalar and gauge fields would
become $U(Nm)$ matrices and equation (1.1) would still refer to the $U(1)$ center
of the group.

The aim of this paper is to study the D-brane entropy counting when the D-
brane system has non-vanishing worldvolume gauge fields. In particular, we will
be considering a D-brane configuration presented in [20] interpreted as the intersection of $N_1$ and $N_2$ D-4-branes with non-vanishing worldvolume gauge fields
and carrying momentum along a common direction. Although this configuration
is $T$-dual to the configuration used by Callan and Maldacena [2], the origin of
the statistical entropy has a remarkable new feature. Namely, we have to take
into account the Landau degeneracy of the open string sector corresponding to
open strings with each end on different branes.

We begin in section two by presenting our D-brane configuration and by con-
sidering the corresponding charges quantisation. We shall then present the super-
gravity solution describing our D-brane system and write the integer valued form
of the Bekenstein-Hawking entropy. The supergravity solution corresponding to
the non-extreme configuration will also be presented and the corresponding thermodynamical formulae in terms of a brane/anti-brane system will be written. In
section three we shall consider open strings with ends on different D-branes which
have a magnetic condensate on its worldvolume theory. We will keep our discus-
sion general and specify for our D-brane system in section four, where we shall
give the D-brane entropy counting. We will also consider the near-extreme black
hole solution, matching the corresponding thermodynamical quantities with the
brane/anti-brane system. We shall give our conclusions in section five.
Figure 1: $3 \perp 3$ D-brane system with each group of D-3-branes wrapped on the $(-q_1,p_1)$- and $(q_2,p_2)$-cycles of the $x^4x^5$ 2-torus. In this example $q_1 = 1$, $p_1 = 5$ and $q_2 = 3$, $p_2 = 4$. The moduli $R_4$ and $R_5$ will then be related by $(R_5/R_4)^2 = 3/20$.

2 D-brane system

Let us start by considering the $3 \perp 3$ D-brane configuration. We place $N_1$ D-3-branes along the $x^1, x^2, x^5$-directions and $N_2$ D-3-branes along the $x^1, x^3, x^4$-directions. These directions are taken to be circles of radius $R_i$. Next, we rotate this configuration in the $x^4x^5$ 2-torus by an angle $\zeta$ as described in figure 1. The resulting D-brane system may be represented by

$$3_1 : \quad X^1, X^2, X^5,$$
$$\phi^i_{(1)} = -\tan \zeta X^5, \quad (2.1)$$

$$3_2 : \quad X^1, X^3, X^5,$$
$$\phi^i_{(2)} = \cot \zeta X^5,$$

where as explained in the introduction we are considering the U(1) center of the worldvolume scalar fields associated with each group of D-3-branes. Wrapping a direction of the D-3-branes around a cycle of the $x^4x^5$ 2-torus yields quantisation conditions on the rotation angle $\zeta$. Start by considering the first D-3-branes. If the corresponding winding number in the $x^4$-direction is $-q_1$, and in the $x^5$-direction is $p_1$, the angle $\zeta$ will obey the quantisation condition

$$\tan \zeta = \frac{q_1 R_4}{p_1 R_5}, \quad (2.2)$$
where $q_1$ and $p_1$ are co-prime. Similarly, if the other D-3-branes are wrapped on the $(q_2, p_2)$-cycle the quantisation condition on the angle $\zeta$ is

$$\cot \zeta = \frac{q_2}{p_2} \frac{R_4}{R_5}. \quad (2.3)$$

Thus, the moduli $R_4$ and $R_5$ are not independent but obey the relation

$$\left( \frac{R_4}{R_5} \right)^2 = \frac{p_1 p_2}{q_1 q_2}. \quad (2.4)$$

The mass formula for $N$ D-$p$-branes wrapped on a $p$-torus is given by \[8\]

$$M_p = N \frac{(\omega^1 R_1) \ldots (\omega^p R_p)}{g \alpha'^2}, \quad (2.5)$$

where $g$ is the string coupling constant, $2\pi \alpha'$ is the inverse string tension and $R_i$ is the radius of the circle along which the D-branes are wrapped with winding number $\omega^i$. It is straightforward to generalise this formula for the case where a given direction of the D-branes wraps around a 2-torus. For the configuration described by (2.1) we have

$$M_{3_1} = N_1 \frac{R_1 R_2}{g \alpha'^2} \sqrt{(q_1 R_4)^2 + (p_1 R_5)^2}, \quad (2.6)$$

$$M_{3_2} = N_2 \frac{R_1 R_3}{g \alpha'^2} \sqrt{(q_2 R_4)^2 + (p_2 R_5)^2}.$$

We proceed by performing a $T$-duality transformation along the $x^4$-direction on the configuration (2.1). The resulting D-brane system is described by (2.1)

$$4_1 : X^1, X^2, X^4, X^5, \quad F_{45}^{(1)} = \tan \zeta,$$

$$4_2 : X^1, X^3, X^4, X^5, \quad F_{45}^{(2)} = - \cot \zeta. \quad (2.7)$$

Consider the first group of D-4-branes. They arise by performing a $T$-duality transformation along the $x^4$-direction on $N_1$ D-3-branes that were wrapped on the $x^5$-direction with winding number $p_1$. Therefore we end up with $N_1$ D-4-branes with winding $p_1$ in the $x^5$-direction. Also, we have seen that the D-3-branes were wrapped on the dualised direction with winding $-q_1$. Therefore the D-4-branes will also carry D-2-brane charge. They carry $-N_1 q_1$ units of such charge. In the
language of [3], the D-2-branes dissolve in the D-4-branes leaving flux. This fact is described by the magnetic field condensate $F^{(1)}$. Similar comments apply to the second group of D-4-branes. The resulting intersecting D-4-brane system does not obey the standard intersecting rules, but supersymmetry of this configuration is guaranteed by the presence of non-vanishing worldvolume gauge fields [21].

In the dual theory the quantisation conditions on the rotation angle $\zeta$ translate into quantisation conditions on the flux carried by the worldvolume gauge field strengths $F^{(1)}$ and $F^{(2)}$. In terms of the dual radius $\left( R_4 \rightarrow \frac{\alpha'}{R_4} \right)$ and dual string coupling $\left( g \rightarrow g \sqrt{\alpha'} \right)$ these read as

$$F^{(1)}_{45} = \tan \zeta = \frac{q_1}{p_1} \frac{\alpha'}{R_4 R_5},$$

$$F^{(2)}_{45} = - \cot \zeta = - \frac{q_2}{p_2} \frac{\alpha'}{R_4 R_5}.$$ (2.8)

Similarly, the mass of the D-4-branes is now given by

$$M_{41} = N_1 \frac{R_1 R_2}{g \alpha'^{5/2}} \sqrt{(q_1 \alpha')^2 + (p_1 R_4 R_5)^2} = p_1 N_1 \frac{R_1 R_2 R_4 R_5}{g \alpha'^{5/2}} \sqrt{1 + \left( F^{(1)}_{45} \right)^2},$$

$$M_{42} = N_2 \frac{R_1 R_3}{g \alpha'^{5/2}} \sqrt{(q_2 \alpha')^2 + (p_2 R_4 R_5)^2} = p_2 N_2 \frac{R_1 R_3 R_4 R_5}{g \alpha'^{5/2}} \sqrt{1 + \left( F^{(2)}_{45} \right)^2}.$$ (2.9)

The factor $\sqrt{1 + F^2}$ multiplying the usual D-4-brane mass formula is due to the presence of the magnetic field condensate. As it will be explained in the subsection 3.4, this term also arises in the vacuum amplitude calculation for a system of two parallel D-branes with such fields.

Finally, we remark that all constituent D-branes in the configuration (2.7), including the D-2-branes that are dissolved in the D-4-branes, have a common $x^1$-direction. As a consequence, the D-branes may carry momentum along this direction while leaving some unbroken supersymmetry. If we introduce $N$ units of (left-moving) momentum, the compactified system will have an extra mass term given by

$$P = M_P = \frac{N}{R_1}.$$ (2.10)

### 2.1 Supergravity solution

The D-brane system presented above has a corresponding supergravity solution. This solution may be found by applying $T$-duality at an angle on the $3 \perp 3$ D-
brane solution or by using the $SL(2,\mathbb{R})$ symmetry of $N = 2$, $D = 8$ supergravity\cite{22, 20}. The corresponding type IIA background fields are

$$ds^2 = (H_1 H_2)^{\frac{1}{2}} \left[ (H_1 H_2)^{-1} \left( -dt^2 + dx_1^2 + (H_3 - 1)(dx_1 - dt)^2 \right) + H_1^{-1} dx_2^2 + H_2^{-1} dx_3^2 + \tilde{H}_{12}^{-1} \left( dx_4^2 + dx_5^2 \right) + ds^2(\mathbb{E}^4) \right] ,$$

$$e^{2(\phi - \phi_\infty)} = \tilde{H}_{12}^{-1} (H_1 H_2)^{\frac{1}{2}} ,$$

$$F = - \sin \zeta \left[ dH_1^{-1} \wedge dt \wedge dx_1 \wedge dx_2 + \star dH_2 \wedge dx_2 \right] - \cos \zeta \left[ \star dH_1 \wedge dx_3 + dH_2^{-1} \wedge dt \wedge dx_1 \wedge dx_3 \right] ,$$

$$\mathcal{H} = - \cos \zeta \sin \zeta \left( H_2 - H_1 \right) \left( \frac{H_1}{H_{12}} \right) \wedge dx_4 \wedge dx_5 ,$$

where

$$H_i = 1 + \frac{\alpha_i}{r^2} , \quad i = 1, 2, 3 ,$$

are one-centered harmonic function on the $\mathbb{E}^4$ Euclidean space and $\star$ is the dual operator with respect to the Euclidean metric on $\mathbb{E}^4$. The harmonic function $\tilde{H}_{12}$ is defined by $\tilde{H}_{12} = \cos^2 \zeta H_1 + \sin^2 \zeta H_2$. Reducing the solution (2.11) to five dimensions we obtain a black hole with a regular horizon. The corresponding Einstein frame metric is

$$ds^2_E = - (H_1 H_2 H_3)^{-\frac{2}{3}} dt^2 + (H_1 H_2 H_3)^{\frac{1}{3}} ds^2(\mathbb{E}^4) .$$

Calculating the horizon area the Bekenstein-Hawking entropy is seen to be

$$S_{BH} = \frac{A_H}{4G_{N}^5} = \frac{A_3}{4G_{N}^5} \sqrt{\alpha_1 \alpha_2 \alpha_3} ,$$

where $G_{N}^5$ is the five-dimensional Newton’s constant and $A_3 = 2\pi^2$ is the volume of the unit 3-sphere.

We want to express the macroscopic entropy formula (2.14) in terms of the quantities describing the D-brane system (2.7). This may be done by matching the mass formulae (2.9) and (2.10) to the corresponding string frame five-dimensional masses\cite{2, 3}. The result is

$$\frac{16\pi G_{N}^5}{A_3} M_i = 2\alpha_i .$$
The five- and ten-dimensional Newton’s constants are related by
\[ G_N^{10} = (2\pi)^5 R_1 \ldots R_5 G_N^5 = 8\pi^6 g^2 \alpha'^4. \]  
(2.16)

After some straightforward algebra the previous relations will give the following entropy formula
\[ S_{BH} = 2\pi \sqrt{NN_1 N_2 n_L}, \]  
(2.17)
where
\[ n_L = p_2 q_1 + p_1 q_2, \]  
(2.18)
is, as we shall see, the Landau degeneracy for open strings with ends on different branes. In the T-dual picture described by (2.1) the same factor arises in the entropy formula. This factor is interpreted as the D-3-branes’ number of intersecting points on the \( x^4 x^5 \) 2-torus [23]. The string theory entropy counting is then clear because on the intersecting points strings with both ends on different D-3-branes become massless.

The D-4-brane charges \( Q_i = N_i \beta_i \) may be expressed as an integral over spacial infinity of the 4-form field strength. The properly normalised expression for these charges is
\[ Q_i = \frac{1}{2A_3(2\pi)g\alpha'^3/2} \int_{\Sigma_i} F = \begin{cases} \frac{R_3}{g^2\alpha'^2} \cos \zeta \alpha_1, & i = 1, \\ \frac{R_2}{g^2\alpha'^2} \sin \zeta \alpha_2, & i = 2, \end{cases} \]  
(2.19)
where \( \Sigma_i = S_3^\infty \times S^1_i \) with \( S_3^\infty \) the asymptotic 3-sphere and \( S^1_i \) the compactified dimension transverse to the \( i \)-th 4-branes. We remark that it is also possible to write the D-2-brane charges carried by the D-4-branes as an integral of the dual of the 4-form field strength. The ‘charge’ associated with the (left-moving) momentum carried by the D-4-branes may be calculated by using equations (2.10) and (2.15)
\[ N = \frac{R_1^2 R_2 R_3 R_4 R_5}{g^2 \alpha'^4} \alpha_3. \]  
(2.20)

2.2 Non-extreme solution

The supergravity solution (2.11) can be easily generalised to the corresponding non-extreme solution [24, 25, 26]. Start with the solution (2.11) without the plane
wave and perform the following redefinitions

\[ dt^2 \rightarrow f \ dt^2, \]
\[ dr^2 \rightarrow f^{-1} dr^2, \]
\[ Q_i \rightarrow Q_i \sqrt{(\mu + \alpha_i)/\alpha_i}, \quad i = 1, 2, \quad (2.21) \]

where \( f = 1 - \frac{\mu}{r^2} \) and \( Q_i \) is either the electric or magnetic charge associated with the \( i \)-th 4-branes. For later convenience we will express our solution in terms of the parameters \( r_0 \) and \( \beta_i \) defined by

\[ \mu = r_0^2, \]
\[ \alpha_i = r_0^2 \sinh^2 \beta_i, \quad i = 1, 2. \quad (2.22) \]

In the limit \( r_0 \rightarrow 0 \) and \( \beta_i \rightarrow \infty \), holding \( \alpha_i \) fixed, we obtain the extreme solution. The charge associated with the momentum along the \( x^1 \)-direction may be obtained by performing the boost transformation

\[ t \rightarrow \cosh \beta_3 \ t - \sinh \beta_3 \ x^1, \]
\[ x^1 \rightarrow - \sinh \beta_3 \ t + \cosh \beta_3 \ x^1. \quad (2.23) \]

The resulting non-extreme solution is described by

\[ ds^2 = (H_1 H_2)^{\frac{1}{2}} \left[ (H_1 H_2)^{-1} \left( -dt^2 + dx_1^2 + \left( \frac{r_0}{r} \right)^2 (\cosh \beta_3 \ dt - \sinh \beta_3 \ dx_1)^2 \right) \right. \]
\[ \left. + H_1^{-1} dx_2^2 + H_2^{-1} dx_3^2 + \tilde{H}_{12}^{-1} \left( dx_4^2 + dx_5^2 \right) + f^{-1} dr^2 + r^2 d\Omega_3^2 \right] , \quad (2.24) \]

\[ F = - \sin \zeta \left[ \coth \beta_1 \ dH_2^{-1} \wedge dt \wedge dx_1 \wedge dx_2 + \coth \beta_2 \ dH_2 \wedge dx_2 \right] \]
\[ - \cos \zeta \left[ \coth \beta_1 \star dH_1 \wedge dx_3 + \coth \beta_2 \ dH_2^{-1} \wedge dt \wedge dx_1 \wedge dx_3 \right] , \]

and the dilaton field and 3-form field strength are still given as in \((2.11)\). The charges \((2.19)\) and \((2.20)\) previously defined are now given by

\[ Q_1 = \frac{R_3}{2g \alpha'^3/2} \cos \zeta \ r_0^2 \ \sinh 2\beta_1, \]
\[ Q_2 = \frac{R_2}{2g \alpha'^3/2} \sin \zeta \ r_0^2 \ \sinh 2\beta_2, \quad (2.25) \]
\[ N = \frac{R_1^2 R_2 R_3 R_4 R_5}{2g^2 \alpha'^4} \ r_0^2 \ \sinh 2\beta_3. \]
The ADM mass, entropy and Hawking temperature may be calculated by reducing our solution to five dimensions. The result is

\[ M = \frac{R_1 R_2 R_3 R_4 R_5}{2 g^2 \alpha'^4} r_0^2 \sum_{i=1}^{3} \cosh 2\beta_i , \]

\[ S_{BH} = 2\pi \frac{R_1 R_2 R_3 R_4 R_5}{g^2 \alpha'^4} r_0^3 \prod_{i=1}^{3} \cosh \beta_i , \]

\[ T_H = \left( 2\pi r_0 \prod_{i=1}^{3} \cosh \beta_i \right)^{-1} = \frac{S_{BH} r_0^2 R_1 R_2 R_3 R_4 R_5}{g^2 \alpha'^4} . \]  

(2.26)

As in the extreme case, we are interested in expressing these quantities in terms of an associated D-brane system. In other words, we want to introduce right-moving momentum and anti-D-4-branes. In the near-extremal limit the resulting system will describe the near-extremal black-hole \[ [27] \]. With that in mind, we define the quantities \((N_1, N_1, N_2, N_2, N_R, N_L)\) by

\[ Q_{4_1} = N_1 p_1 = \frac{R_3 r^2}{4 g \alpha'^2} \cos \zeta \ e^{2\beta_1} , \quad Q_{4_2} = N_2 p_2 = \frac{R_3 r^2}{4 g \alpha'^2} \sin \zeta \ e^{2\beta_2} , \]

\[ N_L = \frac{R_1 R_2 R_3 R_4 R_5 r_0^3}{4 g^2 \alpha'^4} e^{2\beta_3} , \quad N_R = \frac{R_1 R_2 R_3 R_4 R_5 r_0^3}{4 g^2 \alpha'^4} e^{-2\beta_3} . \]

(2.27)

We shall see in section 4 that in the near-extremal limit, these quantities are interpreted as the units of left \((N_L)\) and right \((N_R)\) moving momentum along the \(x^1\)-direction, and as the winding numbers along the \(x^1\)-direction of the D-4-branes \((N_1, N_2)\) and anti-D-4-branes \((N_1, N_2)\) both with the worldvolume gauge field turned on as in \((2.7)\). In terms of these quantities the ADM mass, entropy and Hawking temperature in \((2.26)\) are given by

\[ M = \frac{R_1 R_2 R_3 R_4 R_5}{g \alpha'^{5/2}} p_1 (N_1 + N_1) \sqrt{1 + (F_{45}^{(1)})^2} + \frac{R_1 R_3 R_4 R_5}{g \alpha'^{5/2}} p_2 (N_2 + N_2) \sqrt{1 + (F_{45}^{(2)})^2} + \frac{1}{R_1} (N_R + N_L) , \]

\[ S_{BH} = 2\pi \left( \sqrt{N_1} + \sqrt{N_1} \right) \left( \sqrt{N_2} + \sqrt{N_2} \right) \left( \sqrt{N_R} + \sqrt{N_L} \right) \sqrt{n_L} , \]

\[ T_H = S_{BH}^{-1} \left( \frac{4}{R_1} \right) \sqrt{N_R N_L} , \]  

(2.28)
where $n_L$ is defined in (2.18).

3 D-branes with worldvolume magnetic fields

In this section we will study open superstrings ending on D-branes which have a worldvolume gauge field strength of the form

$$F_{\hat{\alpha}\hat{\beta}} = \begin{pmatrix} 0 & f \\ -f & 0 \end{pmatrix}$$  \hspace{1cm} (3.1)

where $f$ is a constant and the indices $\hat{\alpha}$ and $\hat{\beta}$ label two spatial worldvolume coordinates.\footnote{D-branes with worldvolume field condensates have been earlier considered in [28, 29, 30] using the boundary state formalism.} We will just consider the superstring mode expansion for these two coordinates. This, together with well known facts, will be sufficient to explain the three open string sectors of the D-brane system (2.7), i.e. the two sectors corresponding to open strings with both ends on the same D-branes and the sector corresponding to open strings with each end on different D-branes. In the first case, the field strength (3.1) is equal on both ends of the string while in the second case it is not. Of course there are other possibilities, for example one end of the string may be let to move freely \footnote{D-branes with worldvolume field condensates have been earlier considered in [28, 29, 30] using the boundary state formalism.}. An important remark is that we are just considering the $U(1)$ center of the D-branes’ worldvolume fields. In our D-brane system we have $N_1$ and $N_2$ D-4-branes on top of each other wrapping on the $x^5$-direction with winding numbers $p_1$ and $p_2$, respectively. This means that the corresponding worldvolume fields are in the adjoint representation of $U(N_ip_i)$. As explained in the introduction, the $U(1)$ flux induces a twist on the fields. The gauge potential as well as the scalar fields will then obey twisted boundary conditions \cite{[13, 17, 18, 19]}. We can choose a set of multiple transition functions on this $U(N_ip_i)$ bundle such that the only non-vanishing D-branes’ worldvolume fields are in the $U(1)$ center of the group. This means that we can perform the calculations as if our open strings have each of its ends on a singly wrapped D-brane.

Let us start by writing the boundary term of the open superstring action in the presence of a $U(1)$ background gauge field. Using the orthonormal gauge coordinates $\tau (-\infty < \tau < \infty)$ and $\sigma (0 < \sigma < \pi)$ we have (taking $\alpha' = 1/2$)

$$S_b = \frac{1}{\pi} \int_{-\infty}^{+\infty} d\tau \left[ A_a \partial_\tau X^a - \frac{i}{4} F_{ab} \left( \Psi^a_- \Psi^b_- + \Psi^a_+ \Psi^b_+ \right) \right]^{\sigma=\pi}_{\sigma=0}, \hspace{1cm} (3.2)$$
where \(a, b = 0, \ldots, 9\), and \(\Psi^a_-\) and \(\Psi^a_+\) are respectively the positive and negative chirality spinor components of the worldsheet Majorana spinor \(\Psi^a\). Performing \(T\)-duality transformations along the directions \(m = p + 1, \ldots, 9\) we find the boundary term for the action describing open superstrings with the ends on a D-p-brane with worldvolume fields \(A_\alpha(X^\beta)\) \((\alpha, \beta = 0, \ldots, p)\) and \(\phi^m(X^\sigma)\)

\[
S_b = \frac{1}{\pi} \int_{-\infty}^{+\infty} d\tau \left[ \left( A_\alpha \partial_{\tau} X^\alpha + \phi_m \partial_{\sigma} X^m \right) - \frac{i}{4} F_{\alpha\beta} \left( \Psi^a_- \Psi^\beta_+ + \Psi^a_+ \Psi^\beta_- \right) + \frac{i}{2} \partial_{\alpha} \phi_m \left( \Psi^a_- \Psi^m_- - \Psi^a_+ \Psi^m_+ \right) \right]_{\sigma=0}^{\sigma=\pi},
\]

where we used the \(T\)-duality transformation rules \(\partial_{\tau} \rightarrow \partial_{\sigma}, A_m \rightarrow \phi_m, \Psi^m_- \rightarrow +\Psi^m_+\). If the open string ends on different D-branes the background fields will differ for \(\sigma = \pi\) and \(\sigma = 0\).

As explained above, we will find the superstring mode expansion for the case where the only non-vanishing worldvolume fields are

\[
F^{(1)}_{\hat{\alpha}\hat{\beta}}(\sigma = 0) = \tan \gamma \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\]

\[
F^{(2)}_{\hat{\alpha}\hat{\beta}}(\sigma = \pi) = \tan \gamma' \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\]

where \(\gamma\) and \(\gamma'\) are arbitrary constants and the indices \(\hat{\alpha}\) and \(\hat{\beta}\) label only two spatial worldvolume coordinates. For the D-brane system considered in this paper, and for the open string sector corresponding to open strings ending on different branes, the indices \(\hat{\alpha}\) and \(\hat{\beta}\) label the \(x^4\)- and \(x^5\)-directions and

\[
\tan \gamma = \frac{q_1}{p_1 R_4 R_5} \alpha', \quad \tan \gamma' = -\frac{q_2}{p_2 R_4 R_5} \alpha'.
\]

The variation of the \(X^{\hat{\alpha}}\) bosonic fields in the superstring action with a boundary term like in (3.3) yields the usual wave equation together with the boundary conditions

\[
\partial_{\sigma} X^{\hat{\alpha}} = F^{\hat{\alpha}\hat{\beta}} \partial_{\tau} X^{\hat{\beta}}, \quad \sigma = 0, \pi.
\]

Similarly, to cancel the boundary term that arises on varying the fermionic fields in the action (3.3), the right \((\Psi^\hat{\alpha}_-\) and left \((\Psi^\hat{\alpha}_+\) moving world-sheet fermions
should obey the boundary conditions \cite{33,34}

\begin{equation}
\Psi^{\hat{\alpha}}_+ - \Psi^{\hat{\alpha}}_- = F^{\hat{\alpha}}_\beta \left( \Psi^{\hat{\beta}}_+ + \Psi^{\hat{\beta}}_- \right), \quad \sigma = 0,
\end{equation}

\begin{equation}
\Psi^{\hat{\alpha}}_+ \mp \Psi^{\hat{\alpha}}_- = F^{\hat{\alpha}}_\beta \left( \Psi^{\hat{\beta}}_+ \pm \Psi^{\hat{\beta}}_- \right), \quad \sigma = \pi.
\end{equation}

The upper and lower sign choices correspond to the Ramond and Neveu-Schwarz sectors of the theory, respectively. In fact, the boundary conditions (3.7) are found by requiring the fermionic fields variations to obey \( \delta \Psi^{\hat{\alpha}}_+ = \delta \Psi^{\hat{\alpha}}_- \) at \( \sigma = 0 \), and \( \delta \Psi^{\hat{\alpha}}_+ = \pm \delta \Psi^{\hat{\alpha}}_- \) at \( \sigma = \pi \).

### 3.1 Bosonic mode expansion

In this subsection we will study the mode expansion for the worldsheet bosonic fields satisfying the boundary conditions (3.6). All the results presented here have been obtained in \cite{33} in the context of open strings in a constant magnetic field, and are included for the sake of clarity. Has explained above we will just consider the excitations along the \( x^{\hat{\alpha}} \)-directions. Relabelling these worldsheet fields by \( X \) and \( Y \) we define the complex field

\begin{equation}
Z \equiv \frac{1}{\sqrt{2}} (X + iY) .
\end{equation}

The boundary conditions (3.6) now read

\begin{align}
\partial_\sigma Z &= -i \tan \gamma \partial_\tau Z , \quad \sigma = 0, \\
\partial_\sigma Z &= -i \tan \gamma' \partial_\tau Z , \quad \sigma = \pi.
\end{align}

The mode expansion is then seen to be

\begin{equation}
Z = z + i \left[ \sum_{n=1}^{\infty} a_{n-\epsilon} \phi_{n-\epsilon}(\tau, \sigma) - \sum_{n=0}^{\infty} a_{n+\epsilon}^\dagger \phi_{-n-\epsilon}(\tau, \sigma) \right],
\end{equation}

where

\begin{equation}
\phi_{n-\epsilon} = \frac{1}{\sqrt{|n - \epsilon|}} \cos \left[ (n - \epsilon)\sigma + \gamma \right] e^{-i(n-\epsilon)\tau},
\end{equation}

with \( \epsilon = (\gamma - \gamma')/\pi \), \( n \) an integer and \( a_{n-\epsilon}, a_{n+\epsilon}^\dagger \) real. Note that by an appropriate redefinition of the modes in (3.10) the parameter \( \epsilon \) may be taken to lie between 0 and 1. In fact, by interchanging the \( a_{n-\epsilon} \) and \( a_{n+\epsilon} \) mode operators it may be taken to lie between 0 and 1/2. Since the integrals of the mode functions \( \phi_{n-\epsilon} \)
are non-zero, the constant $z$ can not be interpreted as the string’s center-of-mass. The functions $\phi_{n-\epsilon}$, together with a constant function, form a complete basis of functions on the interval $[0, \pi]$ (the details may be found in [35]).

The canonical momentum is as usual given by
\[ P_Z = \frac{\partial L}{\partial (\partial_\tau Z)} = \frac{1}{\pi} \left[ \partial_\tau Z^\dagger + A^{(2)\dagger} \delta(\pi - \sigma) - A^{(1)\dagger} \delta(\sigma) \right] , \]
(3.12)
where the complex gauge fields $A^{(i)}$ are defined by
\[ A^{(i)} = \frac{1}{\sqrt{2}} \left( A^{(i)}_x + i A^{(i)}_y \right) , \quad i = 1, 2. \]
(3.13)

At this point we have to choose a gauge for the D-branes’ worldvolume gauge fields. A convenient choice is
\[ A^{(1)}_{\tilde{\alpha}} dx^{\tilde{\alpha}} = \tan \gamma \left( X dy - Y dx \right) , \]
\[ A^{(2)}_{\tilde{\alpha}} dx^{\tilde{\alpha}} = \tan \gamma' \left( X dy - Y dx \right) . \]
(3.14)
The canonical momentum is then seen to be
\[ P_Z = \frac{1}{\pi} \left( \partial_\tau Z^\dagger + \frac{i}{2} Z^\dagger \left[ \tan \gamma \delta(\sigma) - \tan \gamma' \delta(\pi - \sigma) \right] \right) . \]
(3.15)
The first quantised string is obtained by introducing the equal time canonical commutation relations. This gives the following non-vanishing commutators for the mode operators in the expansion (3.10)
\[ [a_{n-\epsilon}, a^\dagger_{m-\epsilon}] = \delta_{nm} , \quad [a_{n+\epsilon}, a^\dagger_{m+\epsilon}] = \delta_{nm} , \quad [z, z^\dagger] = \frac{\pi}{\tan \gamma - \tan \gamma'} . \]
(3.16)
Note that these commutation relations do not depend on the gauge choice (3.14). Thus, the spectrum of string states is gauge invariant.

Introducing the light cone coordinates $\sigma^\pm = \tau \pm \sigma$, the Virasoro operators are defined to be
\[ L_n = \frac{1}{\pi} \int_0^\pi d\sigma \left( e^{in\sigma} T_{++} + e^{-in\sigma} T_{--} \right) , \]
(3.17)
where $T_{\pm\pm}$ are the $\pm\pm$ components of the energy-momentum tensor. The corresponding contributions from the $X$ and $Y$ worldsheet fields are
\[ T_{\pm\pm} = \partial_\pm X^{\tilde{\alpha}} \partial_\pm X_{\tilde{\alpha}} = \frac{1}{2} \left( \partial_\tau Z \pm \partial_\sigma Z \right) \left( \partial_\tau Z \pm \partial_\sigma Z \right)^\dagger . \]
(3.18)
After some straightforward algebra we obtain for $n \geq 0$

$$L_n^{(B)} = \sum_{p=1}^{\infty} \sqrt{(p - \epsilon)(n + p - \epsilon)} \ a_{p-\epsilon}^\dagger a_{n+p-\epsilon}$$

$$+ \sum_{p=0}^{\infty} \sqrt{(p + \epsilon)(n + p + \epsilon)} \ a_{p+\epsilon}^\dagger a_{n+p+\epsilon}$$

$$(3.19)$$

$$+ \sum_{p=0}^{n-1} \sqrt{(p + \epsilon)(n - p - \epsilon)} \ a_{p+\epsilon} a_{n-p-\epsilon}.$$

For $n < 0$, $L_n$ is obtained by substituting $n$ by $-n$ and by taking the hermitian conjugate of (3.19). The Virasoro operators satisfy the algebra

$$[L_n^{(B)}, L_m^{(B)}] = (n - m) L_{n+m}^{(B)} + A(n) \delta_{n+m,0},$$

(3.20)

where the c-number $A(n)$ is due to the usual normal ordering ambiguity in the definition of $L_0$. The value of $A(n)$ may be calculated for the bosonic string as it was done in \[35\]. We will calculate this anomalous term in the full superstring theory.

As a final remark, we note that the frequencies of the $a_{n-\epsilon}$ and $a_{n+\epsilon}$ ($n > 0$) oscillators will be shifted with respect to their usual values by $-\epsilon$ and $+\epsilon$, respectively. Also, the operators $a_\epsilon$ and $a_\epsilon^\dagger$ annihilate and create quanta of frequency $\epsilon$. If $\epsilon = 0$ the zero mode operators disappear (we will comment on this case in subsection 3.4). If $\epsilon = \frac{1}{2}$ the frequencies of the mode operators $a_{n-\epsilon}$ and $a_{n+\epsilon}$, and therefore the spectrum, become the same as those for open strings with ND boundary conditions. The only difference is that the zero mode $z$ is not absent.

### 3.2 Fermionic mode expansion

At the beginning of this section we wrote the boundary conditions satisfied by the worldsheet fermionic fields for the case when the D-brane worldvolume gauge field does not vanish. As in the bosonic case we will just consider the two relevant directions corresponding to the ansatz (3.4). Start by defining the complex spinor field

$$\Psi \equiv \frac{1}{\sqrt{2}} (\Psi^x + i \Psi^y) = \frac{1}{\sqrt{2}} \left( \begin{array}{c} \Psi_+^x + i \Psi_+^y \\ \Psi_-^x + i \Psi_-^y \end{array} \right) \equiv \begin{pmatrix} \Psi_- \\ \Psi_+ \end{pmatrix}.$$

(3.21)

In terms of this field the boundary conditions (3.7) read

$$\Psi_+ - \Psi_- = -i \tan \gamma (\Psi_+ + \Psi_-), \quad \sigma = 0, \quad (3.22)$$

$$\Psi_+ \mp \Psi_- = -i \tan \gamma' (\Psi_+ \pm \Psi_-), \quad \sigma = \pi, \quad (3.22)$$
where the upper and lower sign choices correspond to the Ramond and Neveu-Schwarz sectors, respectively.

Let us consider first the Ramond sector. The mode expansion for the field $\Psi$ may be written as

$$
\Psi_{\mp} = \frac{1}{\sqrt{2}} \left[ \sum_{n=1}^{\infty} \Psi_{n-\epsilon}^{(\mp)}(\tau, \sigma) - \sum_{n=0}^{\infty} \Psi_{n+\epsilon}^{(\mp)}(\tau, \sigma) \right], \quad (3.23)
$$

where

$$
\phi_{n-\epsilon}^{(\mp)} = e^{(n-\epsilon)(\tau \mp \sigma) \mp \gamma}, \quad (3.24)
$$

with $\epsilon$ defined as in (3.11) and $n$ an integer. The canonical momentum of the complex spinor field $\Psi$ is

$$
P_{\Psi} = \frac{\partial L}{\partial (\partial_{\tau} \Psi)} = \frac{i}{2\pi} \Psi^\dagger. \quad (3.25)
$$

Introducing the equal time anti-commutation relations we find the following non-vanishing anti-commutators for the mode operators in the expansion (3.23)

$$
\{ \Psi_{n-\epsilon}, \Psi_{m-\epsilon}^\dagger \} = \delta_{nm}, \quad \{ \Psi_{n+\epsilon}, \Psi_{m+\epsilon}^\dagger \} = \delta_{nm}. \quad (3.26)
$$

The contribution of the worldsheet fermionic fields $\Psi^x$ and $\Psi^y$ to the $\pm\pm$ components of the energy-momentum tensor is

$$
T_{\pm\pm} = \frac{i}{2} \Psi^\dagger_{\pm\pm} \partial_{\pm} \Psi_{\pm\pm} = \frac{i}{2} \left( \Psi_{\pm\pm} \partial_{\pm} \Psi_{\pm\pm}^\dagger + \Psi_{\pm\pm}^\dagger \partial_{\pm} \Psi_{\pm\pm} \right). \quad (3.27)
$$

Thus, using the definition (3.17) and after some algebra we find that for $n \geq 0$

$$
L_n^{(R)} = L_n^{(B)} + \sum_{p=1}^{\infty} \left( \frac{n}{2} + p - \epsilon \right) \Psi_{p-\epsilon}^\dagger \Psi_{n+p-\epsilon} + \sum_{p=0}^{\infty} \left( \frac{n}{2} + p + \epsilon \right) \Psi_{p+\epsilon}^\dagger \Psi_{n+p+\epsilon} + \sum_{p=0}^{n-1} \left( \frac{n}{2} - p - \epsilon \right) \Psi_{n-p-\epsilon} \Psi_{p+\epsilon}, \quad (3.28)
$$

where $L_n^{(B)}$ is the bosonic contribution to the Virasoro operators given in (3.19). For $n < 0$ we have to replace $n$ by $-n$ and take the hermitian conjugate. The Virasoro algebra may be seen to be

$$
[L_n^{(R)}, L_m^{(R)}] = (n - m) L_{n+m}^{(R)} + \frac{2}{8} n^3 \delta_{n+m,0}. \quad (3.29)
$$
This is exactly the standard result expected for the Ramond sector. Thus the zero energy is not changed in the presence of the magnetic field condensate (3.4). This is just the consequence of the fact that both the worldsheet bosons and fermions have the same moding. As in the bosonic case the frequencies of the $\Psi_{n-\epsilon}$ and $\Psi_{n+\epsilon}$ ($n > 0$) oscillators are shifted with respect to their usual values by $-\epsilon$ and $\epsilon$, respectively. Also, the operators $\Psi_\epsilon$ and $\Psi_\epsilon^\dagger$ annihilate and create quanta of frequency $\epsilon$. If $\epsilon = 0$ these operators will take ground states into ground states forming the usual Ramond vacua. If $\epsilon = 1/2$ the mode operators, and therefore the spectrum, become similar to the case of open superstrings with ND boundary conditions, i.e. the expansion becomes of Neveu-Schwarz type.

We now turn to the Neveu-Schwarz sector of the theory. In this case the mode expansion takes the form

$$\Psi_{\mp} = \frac{1}{\sqrt{2}} \left[ \sum_{r=\frac{1}{2}}^{\infty} \Psi_{r-\epsilon} \phi_{r-\epsilon}^{(\mp)}(\tau, \sigma) - \sum_{r=\frac{1}{2}}^{\infty} \Psi_{r+\epsilon}^\dagger \phi_{r+\epsilon}^{(\mp)}(\tau, \sigma) \right] , \quad (3.30)$$

where

$$\phi_{r-\epsilon}^{(\mp)} = e^{(r-\epsilon)(\mp\tau)}$$

with $\epsilon$ defined as before and $r$ a half-integer. After quantisation the non-vanishing anti-commutation relations for the mode operators in the expansion (3.30) are

$$\{\Psi_{r-\epsilon}, \Psi_{s-\epsilon}^\dagger\} = \delta_{rs} , \quad \{\Psi_{r+\epsilon}, \Psi_{s+\epsilon}^\dagger\} = \delta_{rs} . \quad (3.32)$$

A similar calculation to the one in the previous case gives the Virasoro operators ($n \geq 0$)

$$L_n^{(NS)} = L_n^{(B)} + \sum_{r=\frac{1}{2}}^{\infty} \left( \frac{n}{2} + r - \epsilon \right) \Psi_{r-\epsilon} \Psi_{n+r-\epsilon}$$

$$+ \sum_{r=\frac{1}{2}}^{\infty} \left( \frac{n}{2} + r + \epsilon \right) \Psi_{r+\epsilon}^\dagger \Psi_{n+r+\epsilon}$$

$$+ \sum_{r=\frac{1}{2}}^{n-\frac{1}{2}} \left( \frac{n}{2} - r - \epsilon \right) \Psi_{n-r-\epsilon} \Psi_{r+\epsilon} . \quad (3.33)$$

For $n < 0$ we have to replace $n$ by $-n$ and take the hermitian conjugate. The Virasoro algebra is now given by

$$[L_n^{(NS)}, L_m^{(NS)}] = (n - m)L_{n+m}^{(NS)} + \delta_{n+m,0} \left[ \frac{2}{8} (n^3 - n) + \epsilon n \right] . \quad (3.34)$$
The first term inside the square brackets is the usual term that would be obtained if the magnetic field was absent. The other term is proportional to $n$ and it may be absorbed by a redefinition $L_0 \to L_0 + \epsilon/2$, which shifts the zero energy by a term $\epsilon/2$ (in units of $(\alpha')^{-1}$).

As in the previous cases the frequencies of the $\Psi_{\tau-\epsilon}$ and $\Psi_{\tau+\epsilon}$ oscillators are shifted with respect to their usual values by $-\epsilon$ and $\epsilon$, respectively. If $\epsilon = 0$ we just obtain the standard spectrum for the worldsheet fermions in the NS sector. If $\epsilon = \frac{1}{2}$ the frequency of the mode operators $\Psi_{1/2-\epsilon}$ and $\Psi_{1/2-\epsilon}^\dagger$ vanishes. In this case the spectrum is similar to open superstrings with ND boundary conditions, i.e. it becomes of Ramond type.

### 3.3 The Landau degeneracy

In the subsection 3.1 we have seen that the zero modes $z$ and $z^\dagger$ are non-commuting variables. In terms of the $x$ and $y$ coordinates the corresponding commutation relation reads

$$[x, y] = i \frac{\pi}{\tan \gamma - \tan \gamma'}.$$  

(3.35)

Thus, $k = y(\tan \gamma - \tan \gamma')/\pi$ may be seen as the conjugate momentum of $x$.

Since both zero modes $x$ and $y$ commute with $L_0$ we can take any string state to be an eigenstate of either $x$ or $y$. In the D-brane system we are interested on the $x$- and $y$-directions are compactified on a torus. As a result, the eigenstates of $x$ or $y$ will have a degeneracy, the Landau degeneracy \[35\].

Assume that the D-branes where the open strings end are wrapped around the $y$-direction with winding numbers $p_1$ and $p_2$ for each D-brane. We consider first the case where $p_1$ and $p_2$ are co-prime. The open strings ending on both branes carry Chan-Paton factors in the fundamental representation of $U(p_1) \times U(\bar{p}_2)$. The worldsheet fields are identified according to

$$(Y(\tau, \sigma))_a^b + 2\pi R_y \equiv (Y(\tau, \sigma))_{a+1,b+1},$$  

(3.36)

where $a = 1, \ldots, p_1$ and $b = 1, \ldots, p_2$. Thus, going around $p_1p_2$ cycles in the $y$-direction we have the identification

$$(Y(\tau, \sigma))_a^b + 2\pi p_1 p_2 R_y \equiv (Y(\tau, \sigma))_{a\bar{b}}.$$  

(3.37)

To deduce the existence of the Landau levels we essentially follow the same steps as in ref. \[35\]. Since the D-branes are singly wrapped in the $x$-direction
the zero mode wave function $\langle x_{ab}|k_{ab}\rangle = \exp(ikx)$ must be single-valued. This implies that the $k$ eigenvalues are quantised as $(k_m)_{ab} = m/R_x$. In terms of the zero mode $y_{a\bar{b}}$ this condition reads

$$(y_m)_{a\bar{b}} = \frac{\pi}{\tan \gamma - \tan \gamma'} \frac{m}{n_L} L ,$$

where $n_L = p_2q_1 + p_1q_2$ and $L = 2\pi p_1p_2R_y$. In the last equality we have used equation (3.36) which applies to the D-brane system considered in this paper. More generally, the fact that $n_L$ has to be an integer follows from the flux quantisation condition on the D-branes' gauge fields. This may also be derived as a Dirac quantisation condition since a constant magnetic field on a torus is a monopole field [35].

In order to interpret equation (3.38) we note that $p_1$ and $p_2$ were taken to be co-prime. This means that for a given $(y_m)_{a\bar{b}}$ all the other zero modes $(y_m)_{a'\bar{b}'}$ are determined by (3.36). Since the system has periodicity $L$ it follows from (3.38) that there are $n_L$ independent $(y_m)$ states. Thus, the degeneracy of any string state is $n_L$, i.e. any string state will be found by acting with creation operators on the degenerated ground state

$$|y_m\rangle , \quad m = 1, ..., n_L .$$

To relax the assumption that $p_1$ and $p_2$ are co-prime, we just have to realise that in the general case the system has periodicity $L' = 2\pi l p_1' p_2' R_y$, where $p_1 = lp_1'$ and $p_2 = lp_2'$ with $p_1'$ and $p_2'$ co-prime and $l$ an integer. Equation (3.38) may be written as

$$(y_m)_{a\bar{b}} = \frac{m}{n_L} L' ,$$

where $n'_L = p_2'q_1 + p_1'q_2$. The point now is that for a given $(y_m)_{a\bar{b}}$ not all the other zero modes $(y_m)_{a'\bar{b}'}$ are determined by (3.36). A minimal set of zero modes that determines all the other zero modes is

$$(y_m)_{a\bar{b}} , \quad (y_m)_{a+1\bar{b}} , \ldots , \quad (y_m)_{a+l-1\bar{b}} .$$

Thus, there are $n'_L$ Landau levels but each level is itself $l$ times degenerated. This means that again any string state will be found by acting with the creation operators on the $n_L$ times degenerated ground state

$$|y_{m,r}\rangle , \quad m = 1, ..., n'_{L} , \quad r = 1, ..., l .$$
3.4 The $(p|p-2)$ bound state

For completeness we will comment on the $(p|p-2)$ D-brane bound state \[1\]. This configuration is interpreted as a D-$p$-brane carrying also the charge of a D-$(p-2)$-brane. As it is well known the excitations of this D-brane bound state are described by open superstrings with the following boundary condition for the worldsheet bosons (at $\sigma = 0$ and $\sigma = \pi$)

\[
\begin{align*}
\partial_\sigma X^\alpha &= 0, & \alpha &= 0, \ldots, p-2, \\
\partial_\sigma X^{\hat{\alpha}} &= F^{\hat{\alpha}}_{\hat{\beta}} \partial_\tau X^{\hat{\beta}}, & \hat{\alpha}, \hat{\beta} &= p-1, p, \\
\partial_\tau X^m &= 0, & m &= p+1, \ldots, 9,
\end{align*}
\]

where $F^{\hat{\alpha}}_{\hat{\beta}}$ is given by (3.1). The boundary conditions for the fermionic fields in the $x^{\hat{\alpha}}$-directions are as in (3.7). The spectrum for directions other than the $x^{\alpha}$-directions is just standard. For the $x^{\hat{\alpha}}$-directions, and using the above results, we have $\epsilon = 0$ and therefore the fermionic worldsheet excitations are also standard. For the bosonic fields the only relevant difference is the corresponding zero modes which become commuting variables and no longer satisfy the commutation relation (3.35). Also, the operators $a_\epsilon$ and $a_\epsilon^\dagger$ disappear and are substituted by a linear term in $\tau - if\sigma$ \[37\]. The resulting spectrum is exactly the same as in the case for vanishing worldvolume gauge field (up to a normalisation factor in the momentum modes that is correctly reproduced for the low-lying excitations if one considers the Born-Infeld action \[19\]).

A crucial difference between the open superstrings obeying the boundary conditions (3.43) and the boundary conditions corresponding to a D-$p$-brane with vanishing worldvolume gauge field is the vacuum amplitude for two parallel of such branes. In the bound state case there is a overall factor $(1+f^2)$ multiplying the corresponding result for two D-$p$-branes. This factor arises in the momentum integration as it was explained in \[35\]. In the effective field theory this factor also appears as it is expected from the agreement of both calculations \[7\]. Consider first the attractive term in the amplitude that is due to the graviton, dilaton and antisymmetric tensor field exchanges. The coupling of a D-brane to this fields will have an extra $\sqrt{1+f^2}$ factor due to the Born-Infeld character of the D-brane

\footnote{Note that taking the limit $\gamma = \gamma'$ the commutation relation (3.35) becomes ill defined. However, in this limit the spectrum of the Landau levels becomes continuum as may be seen from \[33\]. This is the expected result because the zero modes are now continuum variables. We thank Gary Gibbons for bringing this point to our attention.}
action \cite{36}, giving the correct result in the corresponding one-loop calculation. The repulsive RR exchange will now have two contributions as the D-branes carry the charge of the \( A_{p+1} \) and \( A_{p-1} \) form field potentials. To be more precise, the coupling of a D-\( p \)-brane to these fields will be given by

\[
\mu_p \int d^{p+1} \sigma (\hat{A}_{p+1} + F \wedge \hat{A}_{p-1}) = \\
\mu_p \int d^{p+1} \sigma \hat{A}_{p+1} + \mu_p f (2\pi R_{p-1}s) (2\pi R_p p) \int d^{p-1} \sigma \hat{A}_{p-1},
\]

(3.44)

where the hat denotes the pullback to the D-brane’s worldvolume, \( \mu_p \) is the D-\( p \)-brane quantum of charge and this D-brane is wrapped around the \( x^{p-1} \)- and \( x^p \)-directions with winding numbers \( s \) and \( p \), respectively. Both the RR exchanges will give the desired \( 1 + f^2 \) factor. We remark that since just integer quanta of the \( A_{p-1} \) field can be exchanged there has to be a quantisation condition on the worldvolume gauge field. In other words, the usual flux quantisation condition has to be satisfied. This gives the relation

\[
q\mu_{p-2} = \mu_p f (2\pi R_{p-1}s)(2\pi R_p p),
\]

(3.45)

where \( q \) is an integer. Substituting for the known values of \( \mu_p \) and \( \mu_{p-2} \) we find

\[
f = \frac{1}{s p R_{p-1} R_p} \frac{\alpha'}{\alpha}.
\]

(3.46)

This is exactly the quantisation condition \cite{2.8} for the case \( s = 1 \) obtained geometrically by performing \( T \)-duality at an angle. If \( s \neq 1 \) the corresponding \( T \)-dual D-brane system is given by \( s \) parallel branes wrapped on a \( (q, p) \)-cycle on this 2-torus.

### 4 D-brane entropy counting

In this section we will study the D-brane system describing the extreme and near-extreme black hole presented in section 2. This discussion parallels that of \cite{33} \cite{34} with some new ingredients. We start by performing the D-brane entropy counting in the extreme case. Our D-brane system consisted of \( N_1 \) and \( N_2 \) intersecting D-4-branes with the \( U(1) \) center of the corresponding gauge fields turned on. It
was represented as
\[ F_{45}^{(1)} = \tan \zeta, \]
\[ F_{45}^{(2)} = -\cot \zeta, \]
where the branes carry \( N \) units of left-moving momentum in the \( x^1 \)-direction. The D-4-branes are also wrapped on the \( x^5 \)-direction with windings \( p_1 \) and \( p_2 \). The excitations of this D-brane system are described by three different sectors. The first two correspond to open strings with both ends on the same group of D-branes and the third to open strings with each end on a different group of D-branes. The former cases were explained at the end of the previous section and the spectrum is similar to the case with vanishing worldvolume gauge field. The maximum number of massless excitation is \( 8N_i \) (and not \( 8N_i^2 \) because some excitations will give a mass term to others) which is far to few to explain the entropy formula \( (2.17) \) [6]. This excitations represent subleading contributions to the entropy formula. We expect these string states to condense ensuring supersymmetry of the excited D-brane system. In the gauge theory description this fact translates into the conditions for the vanishing of the D-terms of the theory and of the mass terms for the hypermultiplet associated with strings ending on different branes[6]. The relevant excitations come from the sector of open strings with the ends on different branes [6]. Open strings attached to both type of branes will have the following boundary conditions
\[ X^0, X^1 : \quad NN, \]
\[ X^2, X^3 : \quad ND, DN, \]
\[ X^4, X^5 : \quad FF, \]
\[ X^6, ..., X^9 : \quad DD, \]
where the FF boundary conditions represent open strings satisfying the boundary conditions \( (3.6) \) and \( (3.7) \) with the field strength given by \( (3.4) \) and \( (3.5) \). The only non-standard results come from these \( x^4\) - and \( x^5\)-directions. The parameters in the previous section read now in terms of our D-brane system as
\[ \gamma = \zeta, \quad \gamma' = \zeta - \frac{\pi}{2}, \quad \epsilon = \frac{1}{2}, \]
\[ n_L = p_2q_1 + p_1q_2. \]

Consider first the Neveu-Schwarz sector of the theory. Since \( \epsilon = 1/2 \), the worldsheet bosons become half-integer moded and the worldsheet fermions integer
moded. The zero moded operators $\Psi_0, \Psi_0^\dagger$ take ground states into ground states. Thus, the mode expansion is just as in the ND case. The zero energy is then seen to be zero as it is the case of intersecting D-branes with four ND directions

$$E = (6 - 2) \left( -\frac{1}{24} - \frac{1}{48} \right) + 4 \left( \frac{1}{48} + \frac{1}{24} \right) = 0,$$

where the first term is the contribution of the $x^0, x^1, x^6, ..., x^9$-directions and the second term the contribution of the DN, ND and FF directions. There is however a crucial difference between the ND and FF directions since the latter admits a degenerated zero mode in the bosonic expansion. The vacuum state for the NS sector forms a representation of the 4-dimensional algebra

$$\{\Psi_0^I, \Psi_0^J\} = \delta^{IJ}, \quad I, J = 2, ..., 5,$$

and it has a further degeneracy due to the bosonic zero mode. A convenient basis for the vacuum states is

$$|s_1, s_2, y_m\rangle,$$

where $s_i = \pm \frac{1}{2}$ and $y_m$ comes from the Landau degeneracy as explained in subsection 3.3. This vacuum transforms as a spinor under the internal space group $SO(4)_I$ giving therefore bosonic states. The GSO projections leaves half of the states out. We end up with $2n_L$ states, and since there are $2N_1N_2$ of these strings (the factor of 2 is because the strings are oriented) we have $4N_1N_2n_L$ massless bosonic particles.

Next, consider the Ramond sector. The zero energy is zero as it is usual even when we have the FF directions. Since $\epsilon = 1/2$, the worldsheet fermions become half-integer moded as it is the case for ND directions. This vacuum forms a representation of the 6-dimensional Dirac algebra

$$\{\Psi_0^\mu, \Psi_0^\nu\} = \eta^{\mu\nu}, \quad \mu, \nu = 0, 1, 6, ..., 9,$$

with the Landau degeneracy. A basis for the vacuum states is

$$|s_1, s_2, s_3, y_m\rangle,$$

where $s_i = \pm \frac{1}{2}$ and $y_m$ is defined as in (4.6). This vacuum transforms as a spinor under the corresponding $SO(1, 5)$ group. The GSO projection plus the fact that

---

Footnote:
The zero energy in the FF directions is shifted by $\epsilon/2 = 1/4$ with respect to the usual result for two NN or DD directions as it was explained in the previous section.
physical states are annihilated by the zero mode of the supersymmetry generator leaves us with \(2n_L\) states and therefore we have \(4N_1N_2n_L\) massless fermionic excitations.

The argument now is exactly as presented in [2]. We have a gas of strings with \(4N_1N_2n_L\) fermionic and bosonic species carrying all together the left-moving momentum \(P = \frac{N}{R_1}\). This gives exactly the entropy formula (2.17).

The previous counting argument is valid for \(N \gg n_LN_1N_2\) [37]. Outside this regime, we would obtain an entropy much smaller than the Bekenstein-Hawking entropy. The point is that the configurations of branes winding around in the \(x^1\)-direction become the relevant ones to explain the classical geometry limit. Consider the case where \(N_1\) and \(N_2\) are co-prime (this assumption may be easily relaxed [37]). For the system corresponding to two intersecting D-4-branes with winding numbers \(N_1\) and \(N_2\) in the \(x^1\)-direction we will have a gas of \(4n_L\) bosonic and fermionic species. The momentum carried by these states is however quantised in units of \((N_1N_2R_1)^{-1}\) giving the correct entropy formula for \(NN_1N_2 \gg n_L\).

\subsection{4.1 Near-extremal black hole}

In the extremal case we can extrapolate the entropy calculation from the weakly coupled D-brane phase to the strongly coupled classical black hole geometry phase because of supersymmetry. In other words, the BPS nature of our configuration allow us to extrapolate between small and strong coupling regimes while leaving the number (degeneracy) of our BPS states unchanged. For the non-extreme solution case this argument is no longer valid. While the small coupling calculation is still reliable in the non-BPS case, we can not extrapolate it to the strong coupling region. However, it is seen that in the near-extreme case the D-brane picture reproduces the correct semi-classical thermodynamical analysis [4]. As explained in [37], the excitations of our D-brane system above the BPS state that correctly reproduce the near-extreme behaviour correspond to multiply wrapped branes. The corresponding D-brane system is defined by keeping the total momentum and D-brane charges fixed, while introducing some right-moving momentum \(\delta N_R (= \delta N_L)\), and anti-D-4-branes both with magnetic condensates as in (4.1) and with windings along the \(x^1\)-direction \(\delta N_1 (= \delta N_1)\) and \(\delta N_2 (= \delta N_2)\).

The calculation of the variation of the entropy due to the change in momentum is similar to the one presented in [2, 37]. We have \(4n_L\) bosonic and fermionic
species with momentum quantised in units of \((N_1 N_2 R_1)^{-1}\). The change in the left-moving entropy is \(\Delta S = 2\pi \sqrt{N_1 N_2 n_L (\sqrt{N + \delta N_R} - \sqrt{N})}\), while the change in the right-moving entropy is \(\Delta S = 2\pi \sqrt{N_1 N_2 n_L} \sqrt{\delta N_R}\) which dominates. The resulting change in the entropy agrees with the semi-classical value in (2.28).

We proceed by calculating the contribution to the entropy due to the anti-D-4-branes. In order to do that we shall assume that in the near-extreme regime we may perform duality transformations leaving the degeneracy of states unchanged [4]. Consider the case when we have a D-4-brane and an anti-D-4-brane along the \(x^1, x^2, x^4, x^5\)-directions with a magnetic condensate as in (4.1) and wrapped on the \(x^1\)-direction with winding numbers \(N_1 + \delta N_1\) and \(\delta N_1\), respectively. Next, perform the following duality transformations

\[
T_1 \; S \; T_4 \; T_5 \; T_2 \; (Up),
\]

where \(T_i\) means a T-duality transformation along the \(x_i\)-direction, \(S\) the S-duality transformation and \((Up)\) the uplift of the configuration to 11-dimensions. To follow this duality orbit one has to realize that S-duality acts as electromagnetic duality on the worldvolume gauge field of the D-3-branes [38, 39]. The duality transformation (4.9) interchanges the \(N\) quanta of momentum in the \(x^1\)-direction with a M-5-brane parallel to the \(x^1, x^4, x^5, x^2, x^{11}\)-directions with winding number \(N\) along the \(x^{11}\)-direction. The D-4-brane with winding \(N_2\) in the \(x^1\)-direction is transformed into a membrane singly wrapped on the \(x^3\)-direction and wrapping diagonally the \(x^2 x^{11}\) 2-torus on the \((p_2, -q_2)N_2\)-cycle [40]. This membrane makes an angle \(-\zeta\) with the \(x^{11}\)-direction in this squared 2-torus where

\[
cot \zeta = \frac{q_2 R_{11}}{p_2 R_2}.
\]  

(4.10)

The M-5-brane intersects the membrane along the string defined by this direction. The D-4-brane with winding \(N_1 + \delta N_1\) (anti-D-4-brane with winding \(\delta N_1\)) is transformed under the duality operation (1.9) into left(right)-moving momentum modes propagating along the common string direction. There will be \(p_1 (N_1 + \delta N_1)\) left-moving quanta of momentum along the \(x^{11}\)-direction and \(q_1 (N_1 + \delta N_1)\) left-moving quanta of momentum along the \(x^2\)-direction. The condition that these momentum modes propagate along the common string is

\[
\tan \zeta = \frac{q_1 R_{11}}{p_1 R_2}.
\]  

(4.11)
The total left-moving momentum along this direction is then seen to be

\[ P_L = \frac{(N_1 + \delta N_1)n_L}{R N N_2}, \quad R = \sqrt{(q_2 R_{11})^2 + (p_2 R_{2})^2}. \quad (4.12) \]

We now note that the momentum modes along the intersection string are described by a superconformal field theory with central charge \( c = 6 \) \([41, 42]\). In fact, reducing our M-theory configuration along the \( x^4 \)-direction we obtain a D-4-brane intersecting a D-2-brane over the string direction described above. The open strings carrying the momentum along this string direction are described by a theory with the required central charge. Thus, from equation (4.12) we conclude that the left sector of the theory is at level \( NN_2(N_1 + \delta N_1)n_L \) giving a change in the entropy \( \Delta S = 2\pi \sqrt{NN_2n_L(\sqrt{N_1 + \delta N_1} - \sqrt{N_1})} \). Similarly, the change in the entropy arising from the right-moving sector is \( \Delta S = 2\pi \sqrt{NN_2n_L\sqrt{\delta N_1}} \) which dominates. The resulting change in the entropy again agrees with the semi-classical result (2.28). The near-extreme entropy due to the other anti-D-4-brane may be calculated in a similar way.

The calculation of the Hawking temperature goes through as in [2]. The result again agrees with (2.28) and is

\[ T_H = 2T_R = \frac{4}{2\pi R_1} \sqrt{\frac{\delta N_R}{N_1 N_2 n_L}}, \quad (4.13) \]

where \( T_R \) is the right-moving temperature.

## 5 Conclusion

In this paper we have studied a black hole described by a configuration of intersecting D-4-branes with non-vanishing worldvolume gauge fields. We have shown the agreement between the semi-classical and D-brane calculations of the entropy and Hawking temperature in the extreme and near-extreme cases. As a new ingredient, we found that the Landau degeneracy of open string states describing the excitations of the D-brane system enters in a fundamental way in the explanation of the microscopic structure of this black hole. It is amusing that an old result of quantum mechanics that has been brought into string theory in [35] now has its place in the fundamental description of black holes, providing an impressive new matching with the corresponding semi-classical results.
The results presented in this paper should also arise in the worldvolume description of the low-lying states of our D-brane system. The corresponding theory should be dual to an 8-dimensional theory with two vector gauge fields $A_{\alpha}$ ($\alpha = 0, ..., 7$) and two scalar fields $\phi_m$ ($m = 1, 2$) in the adjoint representation of $U(N_1p_1)$ and $U(N_2p_2)$, plus a hypermultiplet transforming in the fundamental representation of $U(N_1p_1) \times U(N_2p_2)$. Naively one should expect from the D-brane intersection rules that such a supersymmetric theory does not exist. However, the coupling to the non-vanishing background gauge fields should render the theory supersymmetric.

As another application of our work there is the possibility of considering the dual configuration of $n$ D-branes intersecting at $SU(2)$ angles [43, 44, 23]. The entropy of such configuration of D-5-branes with a self-dual field strength on a compact $T^4$ also has its origin in the Landau degeneracy of open strings describing the excitations of this system. Further, in this case it is easy to write down the worldvolume field theory governing the dynamics of the D-brane system. It is just the super Yang Mills theory with twisted boundary conditions on $T^4$. The Landau levels arise in this description as torons [45, 19], and the correct matching with the entropy formula arises from non-trivial results on $\Theta$-functions on $T^4$ [46].

Finally, let us note that the intersection of other D-branes with worldvolume gauge fields may be studied by using the results of section 3. An example, are the configurations found in [47]. The corresponding worldvolume theory should also be expressed in terms of bundles with twisted boundary conditions.

Acknowledgements

We would like to thank M. Gutperle and G. Gibbons for helpful comments. One of us (M.S.C.) acknowledges the financial support of JNICT (Portugal) under programme PRAXIS XXI.
References

[1] A. Strominger and C. Vafa, Phys. Lett. B379 (1996) 99.

[2] C. Callan and J.M. Maldacena, Nucl. Phys. B472 (1996) 591.

[3] J.C. Breckenridge, R.C. Myers, A.W. Peet and C. Vafa, Phys. Lett. B391 (1997) 93.

[4] J.M. Maldacena and A. Strominger, Phys. Rev. Lett. 77 (1996) 428.

[5] C.V. Johnson, R.R. Khuri and R.C. Myers, Phys. Lett. B378 (1996) 78.

[6] J.M. Maldacena, Black Holes in String Theory, Ph.D. Thesis, hep-th/9607235.

[7] J. Polchinski, Phys. Rev. Lett. 75 (1995) 4724.

[8] J. Polchinski, TASI Lectures on D-branes, hep-th/9611050.

[9] J. Dai, R.G. Leigh and J. Polchinski, Mod. Phys. Lett. A4 (1989) 2073.

[10] R.G. Leigh, Mod. Phys. Lett. A4 (1989) 2767.

[11] M. Douglas, Branes within Branes, hep-th/9512077.

[12] C. Bachas, Phys. Lett. B374 (1996) 37.

[13] J.C. Breckenridge, G. Michaud and R.C. Myers, Phys. Rev. D55 (1997) 6438.

[14] M.S. Costa and G. Papadopoulos, Superstring dualities and p-brane bound states, hep-th/9612204 (to appear in Nucl. Phys. B).

[15] E. Witten, Nucl. Phys. B460 (1995) 335.

[16] G. ’t Hooft, Nucl. Phys. B153 (1979) 141.

[17] G. ’t Hooft, Commun. Math. Phys. 81 (1981) 267.

[18] Z. Guralnik and S. Ramgoolam, Nucl. Phys. B499 (1997) 241.

[19] A. Hashimoto and W. Taylor, Nucl. Phys. B503 (1997) 193.
[20] M.S. Costa and M. Cvetič, Phys. Rev. D56 (1997) 4834.

[21] V. Balasubramanian and R.G. Leigh, Phys. Rev. D55 (1997) 6415.

[22] J.M. Izquierdo, N.D. Lambert, G. Papadopoulos and P.K. Townsend, Nucl. Phys. B460 (1996) 560.

[23] V. Balasubramanian, F. Larsen and R.G. Leigh, Branes at Angles and Black Holes, [hep-th/9704143].

[24] M.J. Duff, H. Lu and C.N. Pope, Phys. Lett. B382 (1996) 73.

[25] M. Cvetič and A.A. Tseytlin, Nucl. Phys. B478 (1996) 181.

[26] M.S. Costa, Nucl. Phys. B495 (1997) 195.

[27] G. Horowitz, J.M. Maldacena and A. Strominger, Phys. Lett. B383 (1996) 151.

[28] M. Li, Nucl. Phys. B460 (1996) 351.

[29] C.G. Callan, Jr. and R. Klebanov, Nucl. Phys. B465 (1996) 473.

[30] M.B. Green and M. Gutperle, Nucl. Phys. B476 (1996) 484.

[31] G. Lifschytz, Nucl. Phys. B499 (1997) 283.

[32] M.B. Green, J.H. Schwarz and E. Witten, Superstring Theory, Vol. 1, Cambridge University Press, 1987.

[33] C.G. Callan, C. Lovelace, C.R. Nappi and S.A. Yost, Nucl. Phys. B308 (1988) 221.

[34] C. Bachas and M. Porrati, Phys. Lett. B296 (1992) 77.

[35] A. Abouelsaood, C.G. Callan, C.R. Nappi and S.A. Yost, Nucl. Phys. B280 (1987) 599.

[36] M.B. Green, D-branes and related topics, lectures given at the Cargèse summer school on Strings, Branes and Dualities.

[37] J.M. Maldacena and L. Susskind, Nucl. Phys. B475 (1996) 679.

[38] A.A. Tseytlin, Nucl. Phys. B469 (1996) 51.
[39] M.B. Green and M. Gutperle, Phys. Lett. **B377** (1996) 28.

[40] J.G. Russo and A.A. Tseytlin, Nucl. Phys. **B490** (1997) 121.

[41] I.R. Klebanov and A.A. Tseytlin, Nucl. Phys. **B475** (1996) 179.

[42] V. Balasubramanian and F. Larsen, Nucl. Phys. **B478** (1996) 199.

[43] M. Berkooz, M.R. Douglas and R.G. Leigh, Nucl. Phys. **B480** (1996) 265.

[44] J.C. Breckenridge, G. Michaud and R.C. Myers, Phys. Rev. **D56** (1997) 5172.

[45] P. Van Baal, Commun. Math. Phys. **94** (1984) 397.

[46] M.S. Costa and M.J. Perry, *Torons and black hole entropy*, to appear.

[47] M.S. Costa, Nucl. Phys. **B490** (1997) 202.