Improved Worst-Case Regret Bounds for Randomized Least-Squares Value Iteration

Priyank Agrawal, Jinglin Chen, Nan Jiang
University of Illinois at Urbana-Champaign

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Abstract

This paper studies regret minimization with randomized value functions in reinforcement learning. In tabular finite-horizon Markov Decision Processes, we introduce a clipping variant of one classical Thompson Sampling (TS)-like algorithm, randomized least-squares value iteration (RLSVI). Our $\tilde{O}(H^2S\sqrt{AT})$ high-probability worst-case regret bound improves the previous sharpest worst-case regret bounds for RLSVI and matches the existing state-of-the-art worst-case TS-based regret bounds.

1 Introduction

We study systematic exploration in reinforcement learning (RL) and the exploration-exploitation tradeoff therein. Exploration in RL (Sutton and Barto, 2018) has predominantly focused on Optimism in the face of Uncertainty (OFU) based algorithms. Since the seminal work of Jaksch, Ortner, and Auer (2010), many provably efficient methods have been proposed but most of them are restricted to either tabular or linear setting (Azar, Osband, and Munos, 2017; Jin et al., 2020). A few paper study a more general framework but subjected to computational intractability (Jiang et al., 2017; Sun et al., 2018; Henaff, 2019). Another broad category is Thompson Sampling (TS)-based methods (Osband, Russo, and Van Roy, 2013; Agrawal and Jia, 2017). They are believed to have more appealing empirical results (Chapelle and Li, 2011; Osband and Van Roy, 2017).

In this work, we investigate a TS-like algorithm, RLSVI (Osband, Van Roy, and Wen, 2016; Osband et al., 2019; Russo, 2019; Zanette et al., 2020). In RLSVI, the exploration is induced by injecting randomness into the value function. The algorithm generates a randomized value function by carefully selecting the variance of Gaussian noise, which is used in perturbations of the history data (the trajectory of the algorithm till the current episode) and then applies the least square policy iteration algorithm of Lagoudakis and Parr (2003). Thanks to the model-free nature, RLSVI is flexible enough to be extended to general function approximation setting, as shown by Osband et al. (2016); Osband, Aslanides, and Cassirer (2018); Osband et al. (2019), and at the same time has less burden on the computational side.

We propose C-RLSVI algorithm, which additionally considers an initial burn-in or warm-up phase on top of the core structure of RLSVI. Theoretically, we prove that C-RLSVI achieves $\tilde{O}(H^2S\sqrt{AT})$ high-probability regret bound.

Significance of Our Results

*These two authors contributed equally.

1 $\tilde{O}(...)$ hides dependence on logarithmic factors.
• Our high-probability bound improves upon previous $\tilde{O}(H^{3/2}S^{3/2}/\sqrt{AT})$ worst-case expected regret bound of RLSVI in [Russo (2019)].

• Our high-probability regret bound matches the sharpest $\tilde{O}(H^2 S\sqrt{AT})$ worst-case regret bound among all TS-based methods (Agrawal and Jia, 2017).²

**Related Works**

Taking inspirations from [Azar, Osband, and Munos (2017); Dann, Lattimore, and Brunskill (2017); Zanette and Brunskill (2019); Yang and Wang (2020)], we introduce clipping to avoid propagation of unreasonable estimates of the value function. Clipping creates a warm-up effect that only affects the regret bound with constant factors (i.e. independent of the total number of steps $T$). With the help of clipping, we prove that the randomized value functions are bounded with high probability.

In the context of using perturbation or random noise methods to obtain provable exploration guarantees, there have been recent works (Osband et al., 2016; Fortunato et al., 2018; Pacchiano et al., 2020; Xu and Tewari 2019; Kyten et al., 2019) in both theoretical RL and bandit literature. A common theme has been to develop a TS-like algorithm that is suitable for complex models where exact posterior sampling is impossible. RLSVI also enjoys such conceptual connections with Thompson sampling (Osband et al., 2019; Osband, Van Roy, and Wen, 2016). Related to this theme, the worst-case analysis of Agrawal and Jia (2017) should be highlighted, where the authors do not solve for a pure TS algorithm but have proposed an algorithm that samples many times from posterior distribution to obtain an optimistic model. In comparison, C-RLSVI does not require such strong optimistic guarantee.

Our results are not optimal as compared with $\Omega(H\sqrt{SAT})$ lower bounds in [Jaksch, Ortner, and Auer (2010)]³. The gap of $\sqrt{SH}$ is sometimes attributed to the additional cost of exploration in TS-like approaches (Abeille, Lazaric et al., 2017). Whether this gap can be closed, at least for RLSVI, is still an interesting open question. We hope our analysis serves as a building block towards a deeper understanding of TS-based methods.

## 2 Preliminaries

**Markov Decision Processes**

We consider the episodic Markov Decision Process (MDP) $M = (H, S, A, P, R, s_1)$ described by [Puterman (2014)], where $H$ is the length of the episode, $S = \{1, 2, \ldots, S\}$ is the finite state space, $A = \{1, 2, \ldots, A\}$ is the finite action space, $P = [P_1, \ldots, P_H]$ with $P_h : S \times A \to \Delta(S)$ is the transition function, $R = [R_1, \ldots, R_H]$ with $R_h : S \times A \to [0, 1]$ is the reward function, and $s_1$ is the deterministic initial state.

A deterministic (and non-stationary) policy $\pi = (\pi_1, \ldots, \pi_H)$ is a sequence of functions, where each $\pi_h : S \to A$ defines the action to take at each state. The RL agent interacts with the environment across $K$ episodes giving us $T = KH$ steps in total. In episode $k$, the agent start with initial state $s_k^1 = s_1$ and then follows policy $\pi_k$, thus inducing trajectory $s_k^1, a_k^1, r_k^1, s_k^2, a_k^2, r_k^2, \ldots, s_k^H, a_k^H$.

For any timestep $h$ and state-action pair $(s, a) \in S \times A$, the Q-value function of policy $\pi$ is defined as $Q_h^\pi(s, a) = R_h(s, a) + \mathbb{E}_{s' \sim P_h(s|s, a)}[\sum_{k=h}^{H} R_k(s_k, \pi_k(s_k)|s, a)]$ and the state-value function is defined as $V_h^\pi(s) = Q_h^\pi(s, \pi_h(s))$. We use $\pi^*$ to denote the optimal policy. The optimal state-value function is defined as $V_h^{\pi^*}(s) := V_h^{\pi^*}(s) = \max_{\pi} V_h^\pi(s)$ and the optimal Q-value function is defined as $Q_h^{\pi^*}(s, a) := Q_h^{\pi^*}(s, a) = \max_{\pi} Q_h^\pi(s, a)$. Both $Q^\pi$ and $Q^{\pi^*}$ satisfy Bellman equations

\[
Q_h^{\pi}(s, a) = R_h(s, a) + \mathbb{E}_{s' \sim P_h(s|s, a)}[V_h^{\pi}(s')]
\]

\[
Q_h^{\pi^*}(s, a) = R_h(s, a) + \mathbb{E}_{s' \sim P_h(s|s, a)}[V_{h+1}^{\pi^*}(s')]
\]

²Agrawal and Jia (2017) studies weakly communicating MDPs with diameter $D$. Bounds comparable to our setting (time in-homogeneous) are obtained by augmenting their state space as $S' \to 2S$ and noticing $D \geq H$.

³The lower bound is translated to time-inhomogeneous setting.
where $V_{H+1}^k(s) = V_{H+1}^*(s) = 0 \forall s$. Notice that by the bounded nature of the reward function, for any $(h, s, a)$, all functions $Q_h^*, V_h^*, Q_h^k, V_h^k$ are within the range $[0, H - h + 1]$. Since we consider the time-inhomogeneous setting (reward and transition change with timestep $h$), we have subscript $h$ on policy and value functions, and later traverse over $(h, s, a)$ instead of $(s, a)$.

**Regret** An RL algorithm is a random mapping from the history until the end of episode $k - 1$ to policy $\pi^k$ at episode $k$. We use regret to evaluate the performance of the algorithm:

$$\text{Reg}(K) = \sum_{k=1}^{K} V^*_1(s_1) - V^*_{1}^{\pi^k}(s_1).$$

Regret $\text{Reg}(K)$ is a random variable, and we bound it with high probability $1 - \delta$. We emphasize that high-probability regret bound provides a stronger guarantee on each roll-out [Seldin et al., 2013; Lattimore and Szepesvári, 2020] and can be converted to the same order of expected regret bound

$$\text{E-Reg}(K) = E \left[ \sum_{k=1}^{K} V^*_1(s_1) - V^*_{1}^{\pi^k}(s_1) \right]$$

by setting $\delta = 1/T$. However, expected regret bound does not imply small variance for each run. Therefore it can violate the same order of high-probability regret bound. We also point out that both bounds hold for all MDP instances $M$ that have $S$ states, $A$ actions, horizon $H$, and bounded reward $R \in [0, 1]$. In other words, we consider worst-case (frequentist) regret bound.

**Empirical MDP** We define the number of visitation of $(s, a)$ pair at timestep $h$ until the end of episode $k - 1$ as $n_k(h, s, a) = \sum_{l=1}^{k-1} \mathbf{1}\{ (s_l^l, a_l^l) = (s, a) \}$. We also construct empirical reward and empirical transition function as $\hat{R}_{k, s, a} = \frac{1}{n_k(h, s, a) + 1} \sum_{l=1}^{k-1} \mathbf{1}\{ (s_l^l, a_l^l) = (s, a) \} r_h^l$ and $\hat{P}_{k, s, a, s'} = \frac{1}{n_k(h, s, a) + 1} \sum_{l=1}^{k-1} \mathbf{1}\{ (s_l^l, a_l^l, s_{l+1}^l) = (s, a, s') \}$. Finally, we use $\hat{M}^k = (H, S, A, \hat{P}^k, \hat{R}^k, s^k_1)$ to denote the empirical MDP. Notice that we have $n_k(h, s, a) + 1$ in the denominator, and it is not standard. The reason we have that is due to the analysis between model-free view and model-based view in Section 3. In the current form, $\hat{P}^k_{h, s, a}$ is no longer a valid probability function, and it is for ease of presentation. More formally, we can slightly augment the state space by adding one absorbing state for each level $h$ and let all $(h, s, a)$ transit to the absorbing states with remaining probability.

### 3 C-RLSVI Algorithm

The major goal of this paper is to improve the regret bound of TS-based algorithms in the tabular setting. Different from using fixed bonus term in the optimism-in-face-of-uncertainty (OFU) approach, TS methods [Agrawal and Goyal, 2013; Abeille, Lazaric et al., 2017; Russo, 2019; Zanette et al., 2020] facilitate exploration by making large enough random perturbation so that optimism is obtained with at least a constant probability. However, the range of induced value function can easily grow unbounded and this forms a key obstacle in previous analysis [Russo, 2019]. To address this issue, we apply a common clipping technique in RL literature [Azar, Osband, and Munos, 2017; Zanette et al., 2020; Yang and Wang, 2020].

We now formally introduce our algorithm C-RLSVI as shown in Algorithm 1. C-RLSVI follows a similar approach as RLSVI in [Russo, 2019]. The algorithm proceeds in episodes. In episode $k$, the agent first samples $Q_h^{\text{pri}}$ from prior $\mathcal{N}(0, \frac{1}{2}I)$ and adds random perturbation on the data (lines 3-10), where $\mathcal{D}_h = \{(s_h^l, a_h^l, r_h^l, s_{h+1}^l) : l < k \}$ for $h < H$ and $\mathcal{D}_H = \{(s_H^l, a_H^l, r_H^l, \emptyset) : l < k \}$. The injection of Gaussian perturbation (noise) is essential for the purpose of exploration and we set $\beta_k = H^3 S \log(2HSAk)$. Later we will see the magnitude of $\beta_k$ plays a crucial role in the regret bound and it is tuned to satisfy the optimism...
with a constant probability in Lemma 4. Given history data, the agent further performs the following procedure from timestep $H$ back to timestep 1: (i) conduct regularized least square regression (lines 13), where $L(Q \mid Q', D) = \sum_{(s,a,r,s') \in D} Q(s, a) - r - \max_{a' \in A} Q'(s', a'))^2$, and (ii) clips the Q-value function to obtain $\hat{Q}_k$ (lines 14-19). Finally, the clipped Q-value function $\hat{Q}_k$ is used to extract the greedy policy $\pi^k$ and the agent rolls out a trajectory with $\pi^k$ (lines 21-22).

**Algorithm 1 C-RLSVI**

1: **input:** variance $\beta_k$ and clipping threshold $\alpha_k$;
2: **for** episode $k = 1, 2, \ldots, K$ **do**
3: **for** timestep $h = 1, 2, \ldots, H$ **do**
4: Sample prior $Q^\text{pri}_h \sim \mathcal{N}(0, \frac{\beta_k}{4} I)$;
5: $\hat{D}_h \leftarrow \{\}$;
6: **for** $(s, a, r, s') \in D_h$ **do**
7: Sample $w \sim \mathcal{N}(0, \beta_k/2)$;
8: $\hat{D}_h \leftarrow \hat{D}_h \cup \{(s, a, r + w, s')\}$;
9: **end for**
10: **end for**
11: Define terminal value $\overline{Q}_{H+1,k}(s, a) \leftarrow 0 \ \forall s, a$;
12: **for** timestep $h = H, H - 1, \ldots, 1$ **do**
13: $\hat{Q}_h^k \leftarrow \arg\min_{Q \in \mathcal{R}^S A} \left[ L(Q \mid \overline{Q}_{h+1,k}, \hat{D}_h) + \|Q - Q^\text{pri}_h\|_2^2 \right]$
14: (Clipping) $\forall (s, a)$
15: if $n^k(h, s, a) > \alpha_k$ then
16: $\overline{Q}_{h,k}(s, a) = \hat{Q}_h^k(s, a)$;
17: else
18: $\overline{Q}_{h,k}(s, a) = H - h + 1$;
19: **end if**
20: **end for**
21: Apply greedy policy $(\pi^k)$ with respect to $(\overline{Q}_{1,k}, \ldots, \overline{Q}_{H,k})$ throughout episode;
22: Obtain trajectory $s_1^k, a_1^k, r_1^k, \ldots, s_H^k, a_H^k, r_H^k$;
23: **end for**

C-RLSVI as presented is a model-free algorithm, which can be easily extended to more general setting and achieve computational efficiency [Osband et al. 2018; Zanette et al. 2020]. When the clipping does not happen, it also has an equivalent model-based interpretation [Russo 2019] by leveraging the equivalence between running Fitted Q-Iteration [Geurts, Ernst, and Wohlen 2006; Chen and Jiang 2019] with batch data and using batch data to first build empirical MDP and then conducting planning. In our later analysis, we will utilize the following property (Eq 1) of Bayesian linear regression [Russo 2019; Osband et al. 2019] for line 13

$$
\hat{Q}_h^k(s, a) \overline{Q}_{h+1,k} \sim \mathcal{N}(\hat{R}_h^k, a, + \sum_{s' \in S} \hat{P}_{h,s,a}^k(s') \max_{a' \in A} \overline{Q}_{h+1,k}(s', a'), \frac{\beta_k}{2(n^k(h, s, a) + 1)})$
$$

$$
\sim \hat{R}_h^k, a, + \sum_{s' \in S} \hat{P}_{h,s,a}(s') \max_{a' \in A} \overline{Q}_{h+1,k}(s', a') + w_h^k, a, \sigma_k(h, s, a) = \frac{\beta_k}{2(n^k(h, s, a) + 1)}.
$$

where the noise term $w_k \in \mathbb{R}^{HS A}$ satisfies $w_k(h, s, a) \sim \mathcal{N}(0, \sigma_k^2(h, s, a))$ and $\sigma_k(h, s, a) = \frac{\beta_k}{2(n^k(h, s, a) + 1)}$. In terms of notation, we denote $\nabla_{h,k}(s) = \max_{a} \overline{Q}_{h,k}(s, a)$.

Compared with RLSVI in [Russo 2019], we introduce a clipping technique to handle the abnormal case in the Q-value function. C-RLSVI has simple one-phase clipping and the threshold $\alpha_k = 4H^3 S \log (2HS Ak) \log (40SAT/\delta)$
is designed to guarantee the boundness of the value function. Clipping is the key step that allows us to introduce new analysis as compared to Russo (2019) and therefore obtain a high-probability regret bound. Similar to as discussed in Zanette et al. (2020), we want to emphasize that clipping also hurts the optimism obtained by simply adding Gaussian noise. However, clipping only happens at an early stage of visiting every \((h, s, a)\) tuple. Intuitively, once \((h, s, a)\) is visited for a large number of times, its estimated Q-value will be rather accurate and concentrates around the true value (within \([0, H - h + 1]\)), which means clipping will not take place. Another effect caused by clipping is we have an optimistic Q-value function at the initial phase of exploration since \(Q_h^* \leq H - h + 1\). However, this is not the crucial property that we gain from enforcing clipping. Although clipping slightly breaks the Bayesian interpretation of RLSVI (Russo, 2019; Osband et al., 2019), it is easy to implement empirically and we will show it does not introduce a major term on regret bound.

### 4 Main Result

In this section, we present our main result: high-probability regret bound in Theorem 1.

**Theorem 1.** For \(0 < \delta < 4\Phi(-\sqrt{2})\), C-RLSVI enjoys the following high-probability regret upper bound, with probability \(1 - \delta\),

\[
\text{Reg}(K) = \tilde{O} \left( H^2 S \sqrt{AT} \right).
\]

Theorem 1 shows that C-RLSVI matches the state-of-the-art TS-based method (Agrawal and Jia, 2017). Compared to the lower bound (Jaksch, Ortner, and Auer, 2010), the result is at least off by \(\sqrt{HS}\) factor. This additional factor of \(\sqrt{HS}\) has eluded all the worst-case analyses of TS-based algorithms known to us in the tabular setting. This is similar to an extra \(\sqrt{d}\) factor that appears in the worst-case upper bound analysis of TS for \(d\)-dimensional linear bandits (Abeille, Lazaric et al., 2017).

It is useful to compare our work with the following contemporaries in related directions.

**Comparison with Russo (2019)** Other than the notion of clipping (which only contributed to warm-up or burn-in term), the core of C-RLSVI is the same as RLSVI considered by Russo (2019). Their work presents significant insights about randomized value functions but the analysis does not extend to give high-probability regret bounds, and the latter requires a fresh analysis. Theorem 1 improves his worst-case expected regret bound \(\tilde{O}(H^{3/2}S^{3/2}\sqrt{AT})\) by \(\sqrt{HS}\).

**Comparison with Zanette et al. (2020)** Very recently, Zanette et al. (2020) proposed frequentist regret analysis for a variant of RLSVI with linear function approximation and obtained high-probability regret bound of \(\tilde{O}(H^2d^2\sqrt{T})\), where \(d\) is the dimension of the low rank embedding of the MDP. While they present some interesting analytical insights which we use (see Section 5), directly converting their bound to tabular setting \((d \rightarrow SA)\) gives us quite loose bound \(\tilde{O}(H^2SA^2\sqrt{T})\).

**Comparison with Azar, Osband, and Munos (2017); Jin et al. (2018)** These OFU works guaranteeing optimism almost surely all the time are fundamentally different from RLSVI. However, they develop key technical ideas which are useful to our analysis, e.g. clipping estimated value functions and estimation error propagation techniques. Specifically, in Azar, Osband, and Munos (2017); Dann, Lattimore, and Brunskill (2017); Jin et al. (2018), the estimation error is decomposed as a recurrence. Since RLSVI is only optimistic with a constant probability (see Section 4 for details), their techniques need to be substantially modified to be used in our analysis.
5 Proof Outline

In this section, we outline the proof of our main results, and the details are deferred to the appendix. The major technical flow is presented from Section 5.1 onward. Before that, we present three technical prerequisites: (i) the total probability for the unperturbed estimated $\hat{M}_k$ to fall outside a confidence set is bounded; (ii) $\nabla V_{1,k}$ is an upper bound of the optimal value function $V^*_h$ with at least a constant probability at every episode; (iii) the clipping procedure ensures that $\nabla h,k$ is bounded with high probability.4

Notations To avoid cluttering of mathematical expressions, we abridge our notations to exclude the reference to $(s, a)$ when it is clear from the context. Concise notations are used in the later analysis:

\[
R_{h,s}^k \rightarrow R_k^h, \hat{R}_{h,s}^k \rightarrow \hat{R}_k^h, P_{h,s}^k \rightarrow P_k^h, \hat{P}_{h,s}^k \rightarrow \hat{P}_k^h, n^k(h, s, a_h^k) \rightarrow n^k(h), w_{h,s}^k \rightarrow w_h^k.
\]

High probability confidence set In Definition 1, $M^k$ represents a set of MDPs, such that the total estimation error with respect to the true MDP is bounded.

Definition 1 (Confidence set).

\[
M^k = \{(H, S, A, P', R', s_1) : \forall (h, s, a), |R'_{h,s,a} - R_{h,s,a} + \langle P'_{h,s,a} - P_{h,s,a}, V^*_h \rangle| \leq \sqrt{e_k(h, s, a)} \} \]

where we set

\[
\sqrt{e_k(h, s, a)} = H \sqrt{\frac{\log (2HSAk)}{n^k(h, s, a) + 1}} \]  \hspace{1cm} (2)

Through an application of Hoeffding’s inequality (Jaksch, Ortner, and Auer, 2010; Osband, Russo, and Van Roy, 2013), it is shown via Lemma 2 that the empirical MDP does not often fall outside confidence set $M^k$. This ensures exploitation, i.e., the algorithm’s confidence in the estimates for a certain $(h, s, a)$ tuple grows as it visits that tuple many numbers of times.

Lemma 2. $\sum_{k=1}^{\infty} P \left( \hat{M}_k \notin M^k \right) \leq 2006HSA.$

Bounded Q-function estimates It is important to note the pseudo-noise used by C-RLSVI has both exploratory (optimism) behavior and corrupting effect on the estimated value function. Since the Gaussian noise is unbounded, the clipping procedure (lines 14-19 in Algorithm 1) avoids propagation of unreasonable estimates of the value function, especially for the tuples $(h, s, a)$ which have low visit counts. This saves from low rewarding states to be misidentified as high rewarding ones (or vice-versa). Intuitively, the clipping threshold $\alpha_k$ is set such that the noise variance ($\sigma_k(h, s, a) = \beta_k n^k(h, s, a) + 1$) drops below a numerical constant and hence limiting the effect of noise on the estimated value functions. This idea is stated in Lemma 3, where we claim the estimated Q-value function is bounded for all $(h, s, a)$.

Lemma 3 ((Informal) Bound on the estimated Q-value function). Under some good event, for the clipped Q-value function $\widetilde{Q}_h$ defined in Algorithm 1 we have $|\widetilde{Q}_{h,k} - Q^*_h(s, a)| \leq H - h + 1$.

See Appendix C for the precise definition of good event and a full proof. Lemma 3 is striking since it suggests that randomized value function needs to be clipped only for constant (i.e. independent of $T$) number of times to be well-behaved.

4We drop/hide constants by appropriate use of $\gtrsim, \lesssim, \simeq$ in our mathematical relations. All the detailed analyses can be found in our appendix.
Optimism The event when none of the rounds in episode \( k \) need to be clipped is denoted by \( E_k^\text{th} := \{ \cap_{h \in H}(h^k(h) \geq \alpha_k) \} \). Due to the randomness in the environment, there is a possibility that a learning algorithm may get stuck on “bad” states, i.e. not visiting the “good” \((h, s, a)\) enough or it grossly underestimates the value function of some states and as result avoid transitioning to those state. Effective exploration is required to avoid these scenarios. To enable correction of faulty estimates, most RL exploration algorithms maintain optimistic estimates almost surely. However, when using randomized value functions, C-RLSVI does not always guarantee optimism. In Lemma 4 we show that C-RLSVI samples an optimistic value function estimate with at least a constant probability for any \( k \). We emphasize that such difference is fundamental.

Lemma 4. When conditioned on \( H_{k-1}^k \),
\[
P\left( V_{1,k}(s_1^k) \geq V^*(s_1^k) | G_k \right) \geq \Phi(-\sqrt{2})/2.
\]
Here \( \Phi(\cdot) \) is the CDF of \( N(0,1) \) distribution and \( H_{k-1}^k \) is all the history of the past observations made by C-RLSVI till the end of the episode \( k - 1 \). We use \( G_k \) to denote the good intersection event of \( \hat{M}^k \in M^k \) and bounded noises and values (the specific definition of good event \( G_k \) can be found in Appendix A).

Lemma 4 is adapted from Zanette et al. (2020); Russo (2019), and we reproduce the proof in Appendix B for completeness.

Now, we are in a position to simplify the regret expression with high probability as
\[
\text{Reg}(K) \leq \sum_{k=1}^{K} 1\{G_k\} \left( V^*_1 - V_{1,k}(s_1^k) \right) (s_1^k) + H \mathbb{P}(\hat{M}^k \notin \mathcal{M}^k),
\]
where we use Lemma 2 to show that for any \((h, s, a)\), the edge case that the estimated MDP lies outside the confidence set is a transient term (independent of \( T \)). The proof of Eq (3) is deferred to Appendix D.

We also define \( \tilde{w}^k(h, s, a) \) as an independent sample from the same distribution \( N(0, \sigma^2(h, s, a)) \) conditioned on the history of the algorithm till the last episode. Armed with the necessary tools, over the next few subsections we sketch the proof outline of our main results. All subsequent discussions are under good event \( G_k \).

5.1 Regret as Sum of Estimation and Pessimism

Now the regret over \( K \) episodes of the algorithm decomposes as
\[
\sum_{k=1}^{K} \left( \frac{V^*_1 - V_{1,k}(s_1^k)}{\text{Pessimism}} + \frac{V_{1,k}(s_1^k) - V_{1}(s_1^k)}{\text{Estimation}} \right).
\]
In OFU-style analysis, the pessimism term is non-positive and insignificant [Azar, Osband, and Munos (2017); Jin et al. (2018)]. In TS-based analysis, the pessimism term usually has zero expectation or can be upper bounded by the estimation term [Osband, Russo, and Van Roy (2013); Osband and Van Roy (2017); Agrawal and Jia (2017); Russo (2019)]. Therefore, the pessimism term is usually relaxed to zero or reduced to the estimation term, and the estimation term can be bounded separately. Our analysis proceeds quite differently. In Section 5.2 we show how the pessimism term is decomposed to terms that are related to the algorithm’s trajectory (estimation term and pessimism correction term). In Section 5.3 and Section 5.4 we show how to bound these two terms through two independent recurrences. Finally, in Section 5.5 we reorganize the regret expression whose individual terms can be bounded easily by known concentration results.
5.2 Pessimism in Terms of Estimation

In this section we present Lemma 5 where the pessimism term is bounded in terms of the estimation term and a correction term \((V_1 - \bar{V}_{1,k}) (s^k_{1})\) that will be defined later. This correction term is further handled in Section 5.4. While the essence of Lemma 5 is similar to that given by Zanette et al. (2020), there are key differences: we need to additionally bound the correction term; the nature of the recurrence relations for the pessimism and estimation terms necessitates a distinct solution, hence leading to different order dependence in regret bound. In all, this allows us to obtain stronger regret bounds as compared to Zanette et al. (2020).

**Lemma 5.** Under the event \(G_k\),
\[
(V_1^* - \bar{V}_{1,k})(s^k_{1}) \leq (\bar{V}_{1,k} - V_1^{\pi^k})(s^k_{1}) + (V_1^{\pi^k} - \bar{V}_{1,k})(s^k_{1}) + M_{1,k}^w,
\]
where \(M_{1,k}^w\) is a martingale difference sequence (MDS).

The detailed proof can be found in Appendix D.2 while we present an informal proof sketch here. The general strategy in bounding \(V_1^* - \bar{V}_{1,k}(s^k_{1})\) is that we find an upper bounding estimate of \(V_1^* (s^k_{1})\) and a lower bounding estimate of \(\bar{V}_{1,k}(s^k_{1})\), and show that the difference of these two estimates converge.

We define \(\bar{V}_{1,k}\) to be the value function obtained when running Algorithm 1 with random noise \(\tilde{w}\) and \(\tilde{\pi}_{pb}\). This correction term is further handled in place of \(w^k\). We use \(w^k\) to denote the solution of the optimization program and \(V_1\) to be the minimum. This ensures \(\bar{V}_{1,k} \leq \bar{V}_{1,k}\) and \(V_1 \leq V_1\). Thus the pessimism term is now given by
\[
(V_1^* - \bar{V}_{1,k})(s^k_{1}) \leq (V_1^* - \bar{V}_{1,k})(s^k_{1}).
\]

Define event \(\tilde{G}_{1,k} := \{\tilde{V}_{1,k}(s^k_{1}) \geq V_1^*(s^k_{1})\}\), \(\tilde{G}_k\) to be a similar event as \(G_k\), and use \(\mathbb{E}_{\tilde{w}}[\cdot]\) to denote the expectation over the pseudo-noise \(\tilde{w}\). Since \(V_1^*(s^k_{1})\) does not depend on \(\tilde{w}\), we get \(V_1^*(s^k_{1}) = \mathbb{E}_{\tilde{w}[\tilde{G}_{1,k} \tilde{G}_k]}[\tilde{V}_{1,k}(s^k_{1})]\).

We can further upper bound Eq (6) by
\[
(V_1^* - \bar{V}_{1,k})(s^k_{1}) \leq \mathbb{E}_{\tilde{w}[\tilde{G}_{1,k} \tilde{G}_k]}[(\tilde{V}_{1,k} - \bar{V}_{1,k})(s^k_{1})].
\]

Thus, we are able to relate pessimism to quantities which only depend on the algorithm’s trajectory. Further we upper bound the expectation over marginal distribution \(\mathbb{E}_{\tilde{w}[\tilde{G}_{1,k} \tilde{G}_k]}[\cdot]\) by \(\mathbb{E}_{\tilde{w}[\tilde{G}_k]}[\cdot]\). This is possible because we are taking expectation of non-negative entities. Moreover, we can show:
\[
\mathbb{E}_{\tilde{w}[\tilde{G}_k]}[(\tilde{V}_{1,k} - \bar{V}_{1,k})(s^k_{1})] \simeq M_{1,k}^w + \bar{V}_{1,k}(s^k_{1}) - \bar{V}_{1,k}(s^k_{1}).
\]

Now consider
\[
(\bar{V}_{1,k} - \bar{V}_{1,k})(s^k_{1}) = (\bar{V}_{1,k} - V_1^{\pi^k})(s^k_{1}) + (V_1^{\pi^k} - \bar{V}_{1,k})(s^k_{1}).
\]

In Eq. 9, the estimation term is decomposed further in Section 5.3. The correction term is simplified in Section 5.4.
5.3 Bounds on Estimation Term

In this section we show the bound on the estimation term. Under the high probability good event $\mathcal{G}_k$, we show decomposition for the estimation term $(\nabla_{h,k} - V^*_h)(s_h^k)$ holds with high probability. By the property of Bayesian linear regression and the Bellman equation, we get

$$
(\nabla_{h,k} - V^*_h)(s_h^k) = 1{\{\epsilon_{h,k}^\text{th}\}}(\langle \hat{P}_h^k - P_h^k, \nabla_{h+1,k} \rangle + \langle P_h^k, \nabla_{h+1,k} - V_{h+1}^* \rangle + R_h^k - R_h^* + w_h^k) + H 1{\{\epsilon_{h,k}^\text{th}\}}.
$$

We first decompose Term (1) as

$$
(1) = (\hat{P}_h^k - P_h^k, V_{h+1}^*) + (\hat{P}_h^k - P_h^k, \nabla_{h+1,k} - V_{h+1}^*).
$$

Term (2) represents the error in estimating the transition probability for the optimal value function $V^*_h$, while Term (3) is an offset term. The total estimation error, $\epsilon_{h,k}^\text{est} := |\text{Term } (2) + \hat{R}_h^k - R_h^*|$ is easy to bound since the empirical MDP $\hat{M}_k$ lies in the confidence set (Eq 23). Then we discuss how to bound Term (3). Unlike OFU-styled analysis, here we do not have optimism almost surely. Therefore we cannot simply relax $\nabla_{h+1,k} - V_{h+1}^*$ to $\nabla_{h+1,k} - V_{h+1}^k$ and form the recurrence. Instead, we will apply $(L_1, L_\infty)$ Cauchy-Schwarz inequality to separate the deviation of transition function estimation and the deviation of value function estimation, and then further bound these two deviation terms. Noticing that $\nabla_{h+1,k} - V_{h+1}^*$ might be unbounded, we use Lemma 3 to assert that $||V_{h+1}^* - \nabla_{h+1}||_\infty \leq H$ under event $\mathcal{G}_k$. With the boundedness of the deviation of value function estimation, it suffices to bound the remaining $||\hat{P}_h.s_h.a_h - P_h.s_h.a_h||_1$ term. Proving an $L_1$ concentration bound for multinomial distribution with careful application of the Hoeffding’s inequality shows

$$
||\hat{P}_h^k - P_h^k||_1 \leq 4 \sqrt{\frac{SL}{n^k(h) + 1}}.
$$

where $L = \log (40 SAT/\delta)$. In Eq (11), we also decomposes Term (1’) to a sum of the next-state estimation and a MDS.

Clubbing all the terms starting from Eq (10), with high probability, the upper bound on estimation is given by

$$
(\nabla_{h,k} - V^*_h)(s_h^k) \lesssim 1{\{\epsilon_{h,k}^\text{th}\}}(\langle \nabla_{h+1,k} - V_{h+1}^*(s_{h+1}^k) \rangle + c_{h,k}^{\text{err}} + w_h^k + M_{\delta_{h,k}^k(s_h^k)} + 4H \sqrt{\frac{SL}{n^k(h) + 1}} + H 1{\{\epsilon_{h,k}^\text{th}\}},
$$

where $M_{\delta_{h,k}^k(s_h^k)}$ is a Martingale difference sequence (MDS). Thus, via Eq (12) we are able decompose estimation term in terms of total estimation error, next-step estimation, pseudo-noise, a MDS term, and a $\tilde{O}\left(\sqrt{1/n^k(h)}\right)$ term. From the form of Eq (12), we can see that it forms a recurrence. Due to this style of proof, our Theorem 1 is $\sqrt{HS}$ superior than the previous state-of-art result [Russel, 2019], and we are able to provide a high probability regret bound instead of just the expected regret bound.
5.4 Bounds on Pessimism Correction

In this section, we give the decomposition of the pessimism correction term \((V_{h}^{\pi k} - V_{h,k})(s_{h}^{k})\). Shifting from \(\overrightarrow{v}_{k}\) to \(\overrightarrow{v}_{h}\) and re-tracing the steps of Section 5.3 with high probability, it follows

\[
(V_{h}^{\pi k} - V_{h,k})(s_{h}^{k}) \lesssim \prod_{h' = 1}^{h} \left\{ \epsilon_{h,h'}^{th} \right\} \left( |\epsilon_{h,k}^{err}| + \left| w_{h}^{k} \right| + \mathcal{M}_{g_{h,k}}^{k} + 4H \sqrt{\frac{S L}{n^{k}(h + 1)}} + H1\{(\epsilon_{k}^{th} \mathcal{C})\} \right).
\]

The decomposition Eq (13) also forms a recurrence. The recurrences due to Eq (12) and Eq (13) are later solved in Section 5.5.

5.5 Final High-Probability Regret Bound

To solve the recurrences of Eq (12) and Eq (13), we keep unrolling these two inequalities from \(h = 1\) to \(h = H\). Then with high probability, we get

\[
\text{Reg}(K) \lesssim \sum_{k=1}^{K} \sum_{h=1}^{H} \left( \prod_{h' = 1}^{h} \left\{ \epsilon_{h,h'}^{th} \right\} \left( |\epsilon_{h,k}^{err}| + \left| w_{h}^{k} \right| + \mathcal{M}_{g_{h,k}}^{k} + \mathcal{M}_{\delta_{h,k}}^{k} + \mathcal{M}_{\psi_{h,k}}^{k} + 4H \sqrt{\frac{S L}{n^{k}(h + 1)}} + H1\{(\epsilon_{h}^{th} \mathcal{C})\} \right) + \sum_{k=1}^{K} \mathcal{M}_{1,k}^{w} \right).
\]

Bounds of individual terms in the above equation are given in Appendix E, and here we only show the order dependence.

The maximum estimation error that can occur at any round is limited by the size of the confidence set Eq (2). Lemma 18 sums up the confidence set sizes across the \(h\) and \(k\) to obtain \(\sum_{k=1}^{K} \sum_{h=1}^{H} |\epsilon_{h,k}^{err}| = \mathcal{O}(\sqrt{H S})\). In Lemma 19, we use Azuma-Hoeffding inequality to bound the summations of the martingale difference sequences with high probability by \(\mathcal{O}(\sqrt{H S \sqrt{T}})\). The pseudo-noise \(\sum_{k=1}^{K} \sum_{h=1}^{H} w_{h}^{k}\) and the related term \(\sum_{k=1}^{K} \sum_{h=1}^{H} \mathcal{M}_{g_{h,k}}^{k}\) are bounded in Lemma 17 with high probability by \(\mathcal{O}(H^{2} S \sqrt{T})\). Similarly, we have \(\sum_{k=1}^{K} \sum_{h=1}^{H} \sqrt{\frac{S L}{n^{k}(h + 1)}} = \mathcal{O}(H S \sqrt{T})\) from Lemma 17. Finally, Lemma 20 shows that the warm-up term due to clipping is independent on \(T\). Putting all these together yields the high-probability regret bound of Theorem 1.

6 Discussions and Conclusions

In this work, we provide a sharper regret analysis for a variant of RLSVI and advance our understanding of TS-based algorithms. Compared with the lower bound, the looseness mainly comes from the magnitude of the noise term in random perturbation, which is delicately tuned for obtaining optimism with constant probability. Specifically, the magnitude of \(\beta_{k}\) is \(\mathcal{O}(\sqrt{H S})\) larger than sharpest bonus term \(\mathcal{O}(\sqrt{H S})\), which leads to an additional \(\mathcal{O}(\sqrt{H S})\) dependence. Naively using a smaller noise term will affect optimism, thus breaking the analysis. Another obstacle to obtaining \(\mathcal{O}(\sqrt{S})\) results is attributed to the bound on Term (3) of Eq (11). Regarding the dependence on the horizon, one \(\mathcal{O}(\sqrt{T})\) improvement may be achieved by applying the law of total variance type of analysis in [Azar, Osband, and Munos, 2017]. The future direction of this work includes bridging the gap in the regret bounds and the extension of our results to the time-homogeneous setting.
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### A Notations, Constants and Definition

#### A.1 Notation Table

| Symbol | Explanation |
|--------|-------------|
| $S$    | The state space |
| $A$    | The action space |
| $S$    | Size of state space |
| $A$    | Size of action space |
| $H$    | The length of horizon |
| $K$    | The total number of episodes |
| $T$    | The total number of steps across all episodes |
| $\pi_k$ | The greedy policy obtained in the Algorithm 1 at episode $k$, $\pi^k = \{ \pi_1^k, \ldots, \pi_H^k \}$ |
| $\pi^*$ | The optimal policy of the true MDP |
| $(s_h^k, a_h^k)$ | The state-action pair at timestep $h$ in episode $k$ |
| $(s_h^k, a_h^k, r_h^k)$ | The tuple representing state-action pair and the corresponding reward at timestep $h$ in episode $k$ |
| $\mathcal{H}_h^k$ | $\{(s_j^l, a_j^l, r_j^l) : j < k \text{ then } h \leq H, \text{ else if } j = k \text{ then } l \leq h\}$ |
| $\overline{\mathcal{H}}_h^k$ | The history (algorithm trajectory) till timestep $h$ of the episode $k$. |
| $n^k(h, s, a)$ | The number of visits to state-action pair $(s, a)$ in timestep $h$ upto episode $k$ |
| $P_{h,s_h^k,a_h^k}$ | The transition distribution for the state action pair $(s_h^k, a_h^k)$ |
| $R_{h,s_h^k,a_h^k}$ | The reward distribution for the state action pair $(s_h^k, a_h^k)$ |
| $P_{h,s,a}$ | The transition distribution for the state action pair $(s, a)$ at timestep $h$ |
| $R_{h,s,a}$ | The reward distribution for the state action pair $(s, a)$ at timestep $h$ |
| $\hat{P}_{h,s_h^k,a_h^k}$ | The estimated transition distribution for the state action pair $(s_h^k, a_h^k)$ |
| $\hat{R}_{h,s_h^k,a_h^k}$ | The estimated reward distribution for the state action pair $(s_h^k, a_h^k)$ |
| $\mathcal{M}^k$ | The confidence set around the true MDP |
| $w_h^k$ | The pseudo-noise used for exploration |
| $\tilde{w}_{h,s_h^k,a_h^k}$ | The independently pseudo-noise sample, conditioned on history till episode $k - 1$ |
| $\hat{\mathcal{M}}^k$ | The estimated MDP without perturbation in data in episode $k$ |
| $\overline{\mathcal{M}}^k$ | $(H, S, A, \hat{P}^k, \hat{R}^k + w^k, s_1^k)$ |
| $V_h^*$ | The optimal value function under true MDP on the sub-episodes consisting of the timesteps $\{h, \ldots, H\}$ |
| $V_{h,k}^*$ | The state-value function of $\pi^k$ evaluated on the true MDP on the sub-episodes consisting of the timesteps $\{h, \ldots, H\}$ |
| $\nabla_{h,k}$ | The state-value function calculated in Algorithm \ref{algo:policy-iteration} with noise $w^k$ |
| $Q_{h,k}$ | The Q-value function calculated in Algorithm \ref{algo:policy-iteration} with noise $w^k$ |
| $\hat{\mathcal{M}}^k$, $\hat{\nabla}_{h,k}$, $\hat{Q}_{h,k}$ | Refer to Definition \ref{def:mpd} |
| $\hat{V}_{h,k}^k$, $\hat{V}_{h,k}^k$, $\hat{Q}_{h,k}^k$ | Refer to Definition \ref{def:mpd} |
| $\tilde{V}_{h,k}^k$, $\tilde{Q}_{h,k}^k$ | Refer to Definition \ref{def:mpd} |
| $\nabla_{h,k}^k(s_h^k)$ | $V_h^k(s_h^k) - \nabla_{h,k}(s_h^k)$ |
\[
\begin{align*}
\delta_{h,k}(s^k_h) & \quad V^*_h(s^k_h) - V_{h,k}(s^k_h) \\
\sigma_{h,k}(s^k_h) & \quad \nabla_{h,k}(s^k_h) - V^\Pi_{h,k}(s^k_h) \\
\gamma_{h,k}(s^k_h) & \quad V^\gamma_{h,k}(s^k_h) - \nabla_{h,k}(s^k_h) \\
R^k_{h,s^k_h,a^k_h} & \quad R^k_{h,s^k_h,a^k_h} - R_{h,s^k_h,a^k_h} \\
\mathcal{P}^k_{h,s^k_h,a^k_h} & \quad (\hat{P}^k_{h,s^k_h,a^k_h} - P_{h,s^k_h,a^k_h}, V^*_{h+1}) \\
C & \quad \sqrt{\frac{\Phi_{1-\gamma(2/2)^2}}{\log(40\text{SAT}/\delta)}} \\
L & \quad 2\sqrt{H^3 S \log(2HSAk) \log(40\text{SAT}/\delta)} \\
\sigma^2_h(h,s,a) & \quad \frac{\beta_g}{2(n^s(h,s,a)+1)} = \frac{H^3 S \log(2HSAk)}{2(n^s(h,s,a)+1)} \\
\gamma_h(h,s,a) & \quad \sqrt{\sigma^2_h(h,s,a) \bar{L}} \\
\beta_k & \quad H^3 S \log(2HSAk) \\
M_{h^k} & \quad \text{Refer to Appendix A.3} \\
\mathcal{M}^k_{h^k} & \quad \text{Refer to Appendix A.3} \\
\mathcal{C}_k & \quad \{M^k \in \mathcal{M}^k\} \\
\mathcal{E}_{h^k} & \quad \{\mathcal{E}_{h^k}(h,s,a)\} \leq \gamma_k(h,s,a), \forall(s,a) \}
\end{align*}
\]
A.2 Definitions of Synthetic Quantities

In this section we define some synthetic quantities required for analysis.

**Definition 2** ($\tilde{V}_{h,k}$). Given history $H_{H}^{h-1}$, and $\hat{P}^k$ and $\hat{R}^k$ defined in empirical MDP $M^k = (H, S, A, \hat{P}^k, \hat{R}^k, s^k_1)$, we define independent Gaussian noise term $w^k(h, s, a)|H_{H}^{h-1} \sim N(0, \sigma^2_k(h, s, a))$, perturbed MDP $\tilde{M}^k = (H, S, A, \tilde{P}^k, \tilde{R}^k + w^k, s^k_1)$, and $\tilde{V}_{h,k}$ to be the value function obtained by running Algorithm 1 with random noise $w^k$.

Notice that $\tilde{w}^k$ can be different from the realized noise term $w^k$ sampled in the Algorithm 1. They are two independent samples from the same Gaussian distribution. Therefore, conditioned on the history $H_{H}^{h-1}$, $\tilde{M}^k$ has the same marginal distribution as $M^k$, but is statistically independent of the policy $\pi^k$ selected by C-RLSVI.

**Definition 3** ($V_{1,k}$). Similar as in Definition 2, given history $H_{H}^{h-1}$ and any fixed noise $w^k_{ptb} \in \mathbb{R}^{HSA}$, we define a perturbed MDP $\tilde{M}^k_{ptb} = (H, S, A, \tilde{P}^k, \tilde{R}^k + w^k_{ptb}, s^k_1)$ and $V^k_{h,k}$ to be the value function obtained by running Algorithm 1 with random noise $w^k_{ptb}$.

Let $w^k$ be the solution of following optimization program

$$\min_{w^k_{ptb} \in \mathbb{R}^{HSA}} V_{1,k}^{w^k_{ptb}}(s^k_1)$$

s.t. $|w^k_{ptb}(h, s, a)| \leq \gamma_k(h, s, a) \ \forall h, s, a.$

We also use $V_{h,k}$ to denote the minimum of the optimization program (i.e., value function $V^k_{h,k}$) and define MDP $\tilde{M}^k = (H, S, A, \tilde{P}^k, \tilde{R}^k + w^k, s^k_1)$. Then we get that $V_{1,k} \leq V^k_{h,k}$ for any $|w^k_{ptb}| \leq \gamma_k$.

**Definition 4** (Confidence set, restatement of Definition 1).

$$\mathcal{M}^k = \left\{ (H, S, A, P', R', s_1) : \forall (h, s, a), \left| R'_{h,s,a} - R_{h,s,a} + (P'_{h,s,a} - P_{h,s,a}, V^*_h) \right| \leq \sqrt{c_k(h, s, a)} \right\},$$

where we set

$$\sqrt{c_k(h, s, a)} = H \sqrt{\frac{\log(2HSAk)}{h^k(h, s, a) + 1}}. \quad (14)$$

A.3 Martingale Difference Sequences

In this section, we give the filtration sets that consists of the history of the algorithm. Later we enumerate the martingale difference sequences needed for the analysis based on these filtration sets. We use the following to denote the history trajectory:

$$H_h^k := \{(s_l^j, a_l^j, r_l^j) : \text{if } j < k \text{ then } l \in [H], \text{ else if } j = k \text{ then } l \in [h]\}$$

$$\overline{H}_h^k := H_h^k \cup \{ w^k(l, s, a) : l \in [H], s \in S, a \in A \}.$$

With $a_h^k = \pi_h^k(s_h^k)$ as the action taken by C-RLSVI following the policy $\pi_h^k$ and conditioned on the history of the algorithm, the randomness exists only on the next-step transitions. Specifically, with filtration sets $\{ \overline{H}_h^k \}_{h,k}$, we define the following notations that is related to the martingale difference sequences (MDS) appeared in the final regret bound:

$$M_{\overline{H}_h^k(s_h^k)} = 1\{ g_k \} \left[ \mathbb{E} \left[ \delta_{h+1,k}(s') - \delta_{h+1,k}(s_{h+1}) \right] \right],$$

15
$$\mathcal{M}_{\delta^k_H}(s^k_h) = 1\{\mathcal{G}_k\} \left[ \mathbb{E} \left[ \delta^k_{h+1,k}(s') - \delta^k_{h+1,k}(s_{h+1}) \right] \right],$$

where the expectation is over next state $s'$ due to the transition distribution: $P_{h,s^k_h,a^k_h}$.

We use with the filtration sets $\{\mathcal{H}_{H-1}^k\}_k$ for the following martingale difference sequence

$$\mathcal{M}^w = 1\{\mathcal{G}_k\} \left[ \mathbb{E}_{\tilde{w}|\mathcal{G}_k} \left[ \tilde{V}_{1,k}(s^k_h) \right] - \tilde{V}_{1,k}(s^k_1) \right].$$

The detailed proof related to martingale difference sequences is presented in Lemma [19].

### A.4 Events

For reference, we list the useful events in Table [1].

## B Proof of Optimism

Optimism is required since it is used for bounding the pessimism term in the regret bound calculation. We only care about the probability of a timestep in an episode $k$ being optimistic. The following proof is adapted from Zanette et al. (2020); Russo (2019).

**Lemma 6** (Optimism with a constant probability, restatement of Lemma 4). *Conditioned on history $\mathcal{H}_{H-1}^k$, we have*

$$\mathbb{P} \left( \tilde{V}_{1,k}(s^k_1) \geq V^*_1(s^k_1) | \mathcal{G}_k \right) \geq \Phi(-\sqrt{2}),$$

*where $\mathcal{G}_k$ refers to the event that $\hat{M}^k \in M^k$, where $M^k$ is the confidence set defined in Eq (14). In addition, when $0 < \delta < 4\Phi(-\sqrt{2})$, we have

$$\mathbb{P} \left( \tilde{V}_{1,k}(s^k_1) \geq V^*_1(s^k_1) | \mathcal{G}_k \right) \geq \Phi(-\sqrt{2})/2.$$

*Proof. The analysis is valid for any episode $k$ and hence we skip $k$ from the subsequent notations in this proof.

For the first result, we conditioned all the discussions on event $\mathcal{G}_k$. Let $(s_1, \cdots, s_H)$ be the random sequence of states drawn by the policy $\pi^*$ (optimal policy under true MDP $M$) in the estimated MDP $\hat{M}$ and $a_h = \pi^*_h(s_h)$. By the property of Bayesian linear regression and Bellman equation, for any $s_h$ (or more specifically $(h, s_h, a_h)$) that is not clipped, we have

$$\tilde{V}_h(s_h) - V^*_h(s_h)$$

$$\geq \tilde{Q}_h(s_h, \pi^*_h(s_h)) - Q^*_h(s_h, \pi^*_h(s_h))$$

$$= \hat{Q}_h(s_h, \pi^*_h(s_h)) - Q^*_h(s_h, \pi^*_h(s_h))$$

$$= \hat{R}_{h,s_h,a_h} + w_{h,s_h,a_h} + \langle \hat{P}_{h,s_h,a_h}, \tilde{V}_{h+1} \rangle - R_{h,s_h,a_h} - \langle P_{h,s_h,a_h}, V^*_{h+1} \rangle$$

$$= \hat{R}_{h,s_h,a_h} - R_{h,s_h,a_h} + \langle \hat{P}_{h,s_h,a_h}, \tilde{V}_{h+1} - V^*_{h+1} \rangle + \langle \hat{P}_{h,s_h,a_h} - P_{h,s_h,a_h}, V^*_{h+1} \rangle + w_{h,s_h,a_h},$$

where step (a) is from the definition of the confidence sets (Definition 4).

For any $s_h$ that is clipped, we have

$$\tilde{V}_h(s_h) - V^*_h(s_h) = H - h + 1 - V^*_h(s_h) \geq 0.$$ 

From the above one-step expansion, we know that we will keep accumulating $w_{h,s_h,a_h} - \sqrt{e(h,s_h,a_h)}$ when unrolling an trajectory until clipping happens. Define $d(h,s)$ as the probability of the random sequence
$(s_1, \ldots, s_H)$ that satisfies $s_h = s$ and no clipping happens at $s_1, \ldots, s_h$. Unrolling from timestep $1$ to timestep $H$ and noticing $a_h = \pi_h^*(s_h)$ gives us

$$\frac{1}{H} \left( \nabla_1(s_1) - V_1^*(s_1) \right)$$

$$\geq \frac{1}{H} \sum_{s \in S, 1 \leq h \leq H} d(h, s) \left[ w_{h,s, \pi_h^*(s)} - \sqrt{e(h, s, \pi_h^*(s))} \right]$$

$$\geq \left( \sum_{s \in S, 1 \leq h \leq H} (d(h, s)/H) w_{h,s, \pi_h^*(s)} \right) - \sqrt{H} \sum_{s \in S, 1 \leq h \leq H} (d(h, s)/H)^2 e(h, s, \pi_h^*(s))$$

$$:= X(w).$$

The first inequality is due to the definition of $d(h, s)$, and the second inequality is due to Cauchy-Schwarz. Since

$$(d(h, s)/H) w_{h,s, \pi_h^*(s)} \sim N \left( 0, (d(h, s)/H)^2 HS e(h, s, \pi_h^*(s))/2 \right),$$

we get

$$X(w) \sim N \left( -\sqrt{HS} \sum_{s \in S, 1 \leq h \leq H} (d(h, s)/H)^2 e(h, s, \pi_h^*(s)), HS \sum_{s \in S, 1 \leq h \leq H} (d(h, s)/H)^2 e(h, s, \pi_h^*(s))/2 \right).$$

Upon converting to standard Gaussian distribution it follows that

$$\Pr(X(W) \geq 0) = \Phi(-\sqrt{2}).$$

Therefore $\Pr \left( \nabla_1(s_1) \geq V_1^*(s_1) \mid G_k \right) \geq \Phi(-\sqrt{2}).$

For the second part, Lemma 7 tells us that $P(G_k \mid C_k) \geq 1 - \delta/8$. Applying the law of total iteration yields

$$\Pr \left( \nabla_1(s_1) \geq V_1^*(s_1) \mid G_k \right) = \Pr(G_k \mid C_k) \Pr(\nabla_1(s_1) \geq V_1^*(s_1) \mid G_k, C_k) + \Pr(G_k \mid C_k) \Pr(\nabla_1(s_1) \geq V_1^*(s_1) \mid G_k^C, C_k)$$

$$\leq \Pr(\nabla_1(s_1) \geq V_1^*(s_1) \mid G_k) + \delta/8.$$

Therefore, we get

$$\Pr(\nabla_1(s_1) \geq V_1^*(s_1) \mid G_k) \geq \Phi(-\sqrt{2}) - \delta/8 \geq \Phi(-\sqrt{2})/2.$$

This completes the proof.

### C Concentration of Events

**Lemma 7** (Bound on the confident set, restatement of Lemma 2). $\sum_{k=1}^{\infty} \Pr(C_k) = \sum_{k=1}^{\infty} \Pr(\hat{M}^k \notin M^k) \leq 2006HSA.$

**Proof.** Similar as [Russu (2019)], we construct “stack of rewards” as in Lattimore and Szepesvári (2020). For every tuple $z = (h, s, a)$, we generate two i.i.d sequences of random variables $r_{z,n} \sim R_{h,s,a}$ and $s_{z,n} \sim P_{h,s,a}(\cdot)$. Here $r_{(h,s,a),n}$ and $s_{(h,s,a),n}$ denote the reward and state transition generated from the $n$th time action $a$ is played in state $s$, timestep $h$. Set

$$Y_{z,n} = r_{z,n} + V_{h+1}^*(s_{z,n}) \quad n \in \mathbb{N}.$$
They are i.i.d, with $Y_{z,n} \in [0, H]$ since $\|V_{h+1}^*\|_\infty \leq H - 1$, and satisfies 

$$E[Y_{z,n}] = R_{h,s,a} + \langle P_{h,s,a}, V_{h+1}^* \rangle.$$ 

Now let $n = n^k(h, s, a)$. First consider the case $n \geq 0$. From the definition of empirical MDP, we have 

$$\hat{R}_{h,s,a}^k + \langle \hat{P}_{h,s,a}^k, V_{h+1}^* \rangle = \frac{1}{n} \sum_{i=1}^n Y_{h,s,a,i} - \frac{1}{n(n+1)} \sum_{i=1}^n Y_{h,s,a,i}.$$ 

Applying triangle inequality gives us 

$$P \left( \left| \hat{R}_{h,s,a}^k - R_{h,s,a} + \langle \hat{P}_{h,s,a}^k - P_{h,s,a}, V_{h+1}^* \rangle \right| \geq H \sqrt{\frac{\log(2/\delta_n)}{2n}} \right)$$ 

$$= P \left( \left| \frac{1}{n} \sum_{i=1}^n Y_{h,s,a,i} - R_{h,s,a} - \langle P_{h,s,a}, V_{h+1}^* \rangle \right| \geq H \sqrt{\frac{\log(2/\delta_n)}{2n}} \right)$$ 

$$\leq P \left( \left| \frac{1}{n} \sum_{i=1}^n Y_{h,s,a,i} - R_{h,s,a} - \langle P_{h,s,a}, V_{h+1}^* \rangle \right| + \frac{1}{n(n+1)} \sum_{i=1}^n Y_{h,s,a,i} \right) \geq H \sqrt{\frac{\log(2/\delta_n)}{2n}}$$ 

$$\leq P \left( \left| \frac{1}{n} \sum_{i=1}^n Y_{h,s,a,i} - R_{h,s,a} - \langle P_{h,s,a}, V_{h+1}^* \rangle \right| + \frac{1}{n+1} H \right) \geq H \sqrt{\frac{\log(2/\delta_n)}{2n}} - \frac{1}{n+1} H.$$ 

When $n \geq 126$, we have 

$$P \left( \left| \frac{1}{n} \sum_{i=1}^n Y_{h,s,a,i} - R_{h,s,a} - \langle P_{h,s,a}, V_{h+1}^* \rangle \right| \geq H \sqrt{\frac{\log(2/\delta_n)}{2n}} - \frac{1}{n+1} H \right)$$ 

$$\leq P \left( \left| \frac{1}{n} \sum_{i=1}^n Y_{h,s,a,i} - R_{h,s,a} - \langle P_{h,s,a}, V_{h+1}^* \rangle \right| \geq H \sqrt{\frac{\log(2/\delta_n)}{2n}} - H \sqrt{\frac{\log(2/\delta_n)}{2n}} \right)$$ 

$$= P \left( \left| \frac{1}{n} \sum_{i=1}^n Y_{h,s,a,i} - R_{h,s,a} - \langle P_{h,s,a}, V_{h+1}^* \rangle \right| \geq \frac{7H}{8} \sqrt{\frac{\log(2/\delta_n)}{2n}} \right).$$ 

By Hoeffding’s inequality, for any $\delta_n \in (0, 1)$, 

$$P \left( \left| \frac{1}{n} \sum_{i=1}^n Y_{h,s,a,i} - R_{h,s,a} - \langle P_{h,s,a}, V_{h+1}^* \rangle \right| \geq \frac{7H}{8} \sqrt{\frac{\log(2/\delta_n)}{2n}} \right) \leq \frac{1}{2^{15}} \delta_n^{3\beta}. $$ 

For $\delta_n = \frac{1}{M^2 H^2 n^3}$, a union bound over HSA values of $z = (h, s, a)$ and all possible $n \geq 127$ yields 

$$P \left( \bigcup_{h \in [H], s \in [S], a \in [A], n \geq 126} \left\{ \left| \frac{1}{n} \sum_{i=1}^n Y_{h,s,a,i} - R_{h,s,a} - \langle P_{h,s,a}, V_{h+1}^* \rangle \right| \geq H \sqrt{\frac{\log(2/\delta_n)}{2n}} \right\} \right).$$
By Hoeffding’s inequality, for any $\delta < 1$, we instead have

$$
\Pr \left( \left| \frac{1}{n} \sum_{i=1}^{n} Y_{(h,s,a),i} - R_{h,s,a} - \langle P_{h,s,a} , V^*_{h+1} \rangle \right| \geq H \sqrt{\frac{\log(2/\delta_n)}{2n}} - \frac{1}{n+1} H \right) 
$$

$$
\leq 2(HSA)^{15/64} \sum_{n=1}^{\infty} \left( \frac{1}{n} \right)^{49/32} 
$$

$$
\leq 2(HSA)^{15/64} \int_{x=1}^{\infty} \left( \frac{1}{x} \right)^{49/32} dx + 1 
$$

$$
\leq 6(HSA)^{15/64}. 
$$

For $1 \leq n \leq 125$, we instead have

$$
\Pr \left( \left| \frac{1}{n} \sum_{i=1}^{n} Y_{(h,s,a),i} - R_{h,s,a} - \langle P_{h,s,a} , V^*_{h+1} \rangle \right| \geq H \sqrt{\frac{\log(2/\delta_n)}{2n}} - \frac{1}{n+1} H \right) 
$$

$$
\leq \Pr \left( \left| \frac{1}{n} \sum_{i=1}^{n} Y_{(h,s,a),i} - R_{h,s,a} - \langle P_{h,s,a} , V^*_{h+1} \rangle \right| \geq H \sqrt{\frac{\log(2/\delta_n)}{2n}} - \frac{H}{2} \sqrt{\frac{\log(2/\delta_n)}{2n}} \right) 
$$

$$
= \Pr \left( \left| \frac{1}{n} \sum_{i=1}^{n} Y_{(h,s,a),i} - R_{h,s,a} - \langle P_{h,s,a} , V^*_{h+1} \rangle \right| \geq \frac{H}{2} \sqrt{\frac{\log(2/\delta_n)}{2n}} \right). 
$$

By Hoeffding’s inequality, for any $\delta_n \in (0, 1)$, we have

$$
\Pr \left( \left| \frac{1}{n} \sum_{i=1}^{n} Y_{(h,s,a),i} - R_{h,s,a} - \langle P_{h,s,a} , V^*_{h+1} \rangle \right| \geq \frac{H}{2} \sqrt{\frac{\log(2/\delta_n)}{2n}} \right) \leq 8^n \delta_n. 
$$

For $\delta_n = \frac{1}{HSA^2}$, a union bound over $HSA$ values of $z = (h, s, a)$ and all possible $1 \leq n \leq 125$ gives

$$
\Pr \left( \left| \frac{1}{n} \sum_{i=1}^{n} Y_{(h,s,a),i} - R_{h,s,a} - \langle P_{h,s,a} , V^*_{h+1} \rangle \right| \geq \frac{H}{2} \sqrt{\frac{\log(2/\delta_n)}{2n}} \right) 
$$

$$
\leq \Pr \left( \left| \frac{1}{n} \sum_{i=1}^{n} Y_{(h,s,a),i} - R_{h,s,a} - \langle P_{h,s,a} , V^*_{h+1} \rangle \right| \geq \frac{H}{2} \sqrt{\frac{\log(2/\delta_n)}{2n}} \right) 
$$

$$
\leq \sum_{s=1}^{S} \sum_{a=1}^{A} \sum_{h=1}^{H} \sum_{n=1}^{125} \sqrt{8} \frac{1}{HSA^2} 
$$

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\[(HSA)^{3/4} \sum_{n=1}^{125} \sqrt{\frac{1}{n^2}} \leq 2000(HSA)^{3/4}.
\]

Combining the above two cases, we have

\[P\left(\exists (k, h, s, a) : n > 0, \left| \hat{R}_{h,s,a}^k - R_{h,s,a} + \langle \hat{P}_{h,s,a}^k - P_{h,s,a}, V_{h+1}^* \rangle \right| \geq H \sqrt{\frac{\log (2HSA^k)}{2n}} \right) \leq 3(HSA)^{15/64} + 2000(HSA)^{3/4} \leq 2006HSA.
\]

Note that by definition, when \(n = n^k(h, s, a) > 0\) we have

\[\sqrt{e_k(h, s, a)} \geq H \sqrt{\frac{\log (2HSA^k)}{2n^k(h, s, a)}}
\]

and hence this concentration inequality holds with \(\sqrt{e_k(h, s, a)}\) on the right hand side.

When \(n = n^k(h, s, a) = 0\), we have \(\hat{R}_{h,s,a}^k = 0\) and \(\hat{P}_{h,s,a}^k(\cdot) = 0\) by definition, so we trivially have

\[\left| \hat{R}_{h,s,a}^k - R_{h,s,a} + \langle \hat{P}_{h,s,a}^k - P_{h,s,a}, V_{h+1}^* \rangle \right| = |R_{h,s,a} + \langle P_{h,s,a}, V_{h+1}^* \rangle| \leq H \leq e_k(h, s, a).
\]

\[\textbf{Lemma 8 (Bound on the noise).} \text{ For } w^k(h, s, a) \sim N(0, \sigma_k^2(h, s, a)), \text{ where } \sigma_k^2(h, s, a) = \frac{H^3S\log(2HSAk)}{2(n^k(h, s, a) + 1)}, \text{ we have that for any } k \in [K], \text{ the event } E^w_k \text{ holds with probability at least } 1 - \delta/8.
\]

\[\text{Proof. For any fix } s, a, h, k, \text{ the random variable } w^k(h, s, a) \text{ follows Gaussian distribution } N(0, \sigma_k^2). \text{ Therefore, Chernoff concentration bounds (see e.g. Wainwright (2019)) suggests}
\]

\[P\left[|w^k(h, s, a)| \geq t\right] \leq 2 \exp\left(-\frac{t^2}{2\sigma_k^2}\right).
\]

Substituting the value of \(\sigma_k^2\) and rearranging, with probability at least \(1 - \delta'\), we can write

\[|w^k(h, s, a)| \leq \sqrt{\frac{H^3S\log(2HSAk)\log(2/\delta')}{n^k(h, s, a) + 1}}.
\]

Union bounding over all \(s, a, k, h\) (i.e. over state, action, timestep, and episode) imply that \(\forall s, a, k, h\), the following hold with probability at least \(1 - \delta'\),

\[|w^k(h, s, a)| \leq \sqrt{\frac{H^3S\log(2HSAk)\log(2SAT/\delta')}{n^k(h, s, a) + 1}}.
\]

Setting \(\delta' = \delta/8\), for any \(s \in [S], a \in [A], h \in [H], k \in [K]\), we have that

\[|w^k(h, s, a)| \leq \sqrt{\frac{H^3S\log(2HSAk)\log(16SAT/\delta)}{n^k(h, s, a) + 1}} \leq \gamma_k(h, s, a).
\]

Finally recalling the definition of \(E^w_k\), we complete the proof.
Lemma 9 (Bounds on the estimated action-value function, restatement of Lemma 3). When the events \( C_k \) and \( E^w_k \) hold then for all \((h, s, a)\)

\[
\left| (\overline{Q}_{h,k} - Q^*_h)(s,a) \right| \leq H - h + 1.
\]

Proof. For simplicity, we set \( \overline{Q}_{h+1,k}(s,a) = Q^*_{h+1}(s,a) = 0 \) and it is a purely virtual value for the purpose of the proof. The proof goes through by backward induction for \( h = H + 1, H, \ldots, 1 \).

Firstly, consider the base case \( h = H + 1 \). The condition \( |\overline{Q}_{h+1,k}(s,a) - Q^*_{h+1}(s,a)| = 0 \leq H - (H + 1) + 1 \) directly holds from the definition.

Now we do backward induction. Assume the following inductive hypothesis to be true

\[
\left| (\overline{Q}_{h,k} - Q^*_h)(s,a) \right| \leq H - h.
\] (15)

We consider two cases:

Case 1: \( n^k(h, s, a) \leq \alpha_k \)

The Q-function is clipped and hence \( \overline{Q}_{h,k} = H - h + 1 \). By the definition of the optimal Q-function, we have \( 0 \leq Q^*_h \leq H - h + 1 \). Therefore it is trivially satisfied that

\[
\left| (\overline{Q}_{h,k} - Q^*_h)(s,a) \right| \leq H - h + 1.
\]

Case 2: \( n^k(h, s, a) > \alpha_k \)

In this case, we don’t have clipping, so \( \overline{Q}_{h,k}(s,a) = \hat{Q}_{h,k}(s,a) \). From the property of Bayesian linear regression and Bellman equation, we have the following decomposition

\[
\begin{aligned}
\left| \overline{Q}_{h,k}(s,a) - Q^*_h(s,a) \right| &= \left| \hat{R}_{h,s,a}^k + w_{h,s,a}^k + (\hat{P}_{h,s,a}^k, \overline{V}_{h+1,k}) - \hat{R}_{h,s,a}^k - (\hat{P}_{h,s,a}^k, V^*_h) \right| \\
&\leq \left| (\hat{P}_{h,s,a}^k, \overline{V}_{h+1,k} - V^*_h) \right| + \left| \hat{R}_{h,s,a}^k - \hat{R}_{h,s,a}^k + (\hat{P}_{h,s,a}^k, V^*_h) \right| + \left| w_{h,s,a}^k \right|.
\end{aligned}
\]

Term (1) is bounded by \( H - h \) due to the inductive hypothesis in Eq (15). Under the event \( C_k \), term (2) is bounded by \( \sqrt{c_k(h,s,a)} = H \sqrt{\frac{\log(2HS Ak)}{n^k(h,s,a) + 1}} \). Finally, term (3) is bounded by \( \gamma_k(h,s,a) \) as the event \( E^w_k \) holds. With the choice of \( \alpha_k \), it follows that the sum of terms (2) and (3) is bounded by 1 as

\[
\frac{\sqrt{H^2 \log(2HS Ak)} + \sqrt{H^2 S \log(2HS Ak)L}}{\sqrt{n^k(h,s,a)}} < 1.
\]

Thus the sums of all the three terms is upper bounded by \( H - h + 1 \). This completes the proof.

Lemma 10 (Intersection event probability). For any episode \( k \in [K] \), when the event \( C_k \) holds (i.e. \( \hat{M}^k \in \mathcal{M}^k \)), the intersection event \( \mathcal{E}_k = \mathcal{E}^w_k \cap \mathcal{E}^\overline{Q}_k \) holds with probability at least \( 1 - \delta/8 \). In other words, whenever the unperturbed estimated MDP lies in the confidence set (Definition 3), the each pseudo-noise and the estimated Q function are bounded with high probability \( 1 - \delta/8 \). Similarly defined, \( \mathcal{E}_k \) also holds with probability \( 1 - \delta/8 \) when \( C_k \) happens.

Proof. The event, \( \mathcal{E}^w_k \) holds with probability at least \( 1 - \delta/8 \) from Lemma 8. Lemma 9 gives that whenever \( (C_k \cap \mathcal{E}^w_k) \) holds then almost surely \( \mathcal{E}^\overline{Q}_k \) holds. Therefore, \( \mathcal{E}_k \) holds with probability \( 1 - \delta/8 \), whenever \( C_k \) holds.
D Regret Decomposition

In this section we give a full proof of our main result Theorem 11, which is a formal version of Theorem 1. We will give a high-level sketch proof before jumping into the details of individual parts in Sections D.1 and D.2.

Theorem 11. For $0 < \delta < 4\Phi(-\sqrt{2})$, C-RLSVI enjoys the following high probability regret upper bound, with probability at least $1 - \delta$,

$$\text{Reg}(K) = \tilde{O}\left(H^2S\sqrt{A T} + H^5S^2A\right).$$

We first decompose the regret expression into several terms and show bounds for each of the individual terms separately. With probability at least $1 - \delta/4$, we have

$$\text{Reg}(K) = \sum_{k=1}^{K} \{C_k\} \left( V_1^*(s^k_h) - V_1^*(s^k_1) \right) + \sum_{k=1}^{K} \{C_k\} \left( V_1^*(s^k_1) - V_1^*(s^k_1) \right).$$

Step (a) holds with probability at least $1 - \delta/4$ due to Lemma 11.

Term (1) is upper bounded due to Lemma 11 and the fact that $V_1^*(s^k_h) - V_1^*(s^k_1) \leq H$, $\forall k \in [K]$. Term (2), additive inverse of optimism, is called pessimism (Zanette et al., 2020) and is further decomposed in Lemma 13 and Lemma 16. Term (3) is a measure of how well the estimated MDP tracks the true MDP and is called estimation error. It is discussed further by Lemma 12, Lemma 13 and finally decomposed in Lemma 14. We start with the results that decompose the terms in Eq (16) and later aggregate them back to complete the proof of Theorem 11.

D.1 Bound on the Estimation Term

Lemma 12 decomposes the deviation term between the Q-value function and its estimate, and the proof relies on Lemma 13. This result is extensively used in our analysis. For the purpose of the results in this subsection, we assume the episode index $k$ is fixed and hence dropped from the notation in both the lemma statements and their proofs when it is clear.

Lemma 12. With probability at least $1 - \delta/4$, for any $h, k, s_h, a_h$, it follows that

$$1\{G_k\} \left[ Q_h(s_h, a_h) - Q_h^*(s_h, a_h) \right] \leq 1\{G_k\} 1\{E_{h,k}^{th}\} \left( \mathcal{P}_{h,s_h,a_h} + \mathcal{R}_{h,s_h,a_h} + w_{h,s_h,a_h} + \delta_{h+1}(s_{h+1}) + M_{s_{h+1}}^{\pi}(s_h) + 4H \sqrt{\frac{SL}{n(h,s_h,a_h) + 1}} \right) + H1\{E_{h,k}^{th}\}.$$
This is because with bounded range $[-n, n]$ from $n > 0$ in $(h, s, a)$ before step $T$, (i.e. $n^k(h, s, a) = n$, but for simplicity we still use $n^k(h, s, a)$ in the analysis below). Applying Hoeffding’s inequality to the transition probabilities obtained from $n$ observations, with probability at least $1 - \delta'$, we have

$$u^\top \left( \frac{1}{n^k(h, s, a)} \sum_{l=1}^{k-1} \mathbb{1}\{(s^l_h, a^l_h, s^l_{h+1}) = (s, a, \cdot)\} - P_{h, s, a}(\cdot) \right) \leq 2 \sqrt{\frac{\log(2/\delta')}{2n^k(h, s, a)}}.$$  

This is because $u^\top \left( \frac{1}{n^k(h, s, a)} \sum_{l=1}^{k-1} \mathbb{1}\{(s^l_h, a^l_h, s^l_{h+1}) = (s, a, \cdot)\} \right)$ is the average of i.i.d. random variables $u^\top e_{s', a'}$ with bounded range $[-1, 1]$. Notice here we have fixed $n^k(h, s, a) = n$, so $n^k(h, s, a)$ is not a random variable.

By triangle inequality, we have

$$\left| \hat{P}_{h, s, a}(\cdot) - \frac{1}{n^k(h, s, a)} \sum_{l=1}^{k-1} \mathbb{1}\{(s^l_h, a^l_h, s^l_{h+1}) = (s, a, \cdot)\} \right| = \frac{1}{n^k(h, s, a)(n^k(h, s, a) + 1)} \sum_{l=1}^{k-1} \mathbb{1}\{(s^l_h, a^l_h, s^l_{h+1}) = (s, a, \cdot)\}$$

\begin{align*}
\leq 1 & \{g_k\} \{\mathcal{C}_h^{th}\} \left[ \hat{Q}_h(s_h, a_h) - Q^*_h(s_h, a_h) \right] + H1\{\mathcal{E}_h^{th}\} \\
= 1 & \{g_k\} \{\mathcal{C}_h^{th}\} \left[ \hat{P}_{h, s, a, \cdot}(V_{h+1}) - (P_{h, s, a, \cdot}, V^*_{h+1}) + \hat{R}_{h, s, a, \cdot} - R_{h, s, a, \cdot} + w_{h, s, a, \cdot} \right] + H1\{\mathcal{E}_h^{th}\} \\
= 1 & \{g_k\} \{\mathcal{C}_h^{th}\} \left[ \hat{P}_{h, s, a, \cdot}(\nabla_{h+1}) - (P_{h, s, a, \cdot}, \nabla^*_{h+1}) + \hat{R}_{h, s, a, \cdot} - R_{h, s, a, \cdot} + w_{h, s, a, \cdot} \right] + H1\{\mathcal{E}_h^{th}\} \\
\leq 1 & \{g_k\} \{\mathcal{C}_h^{th}\} \left[ \mathcal{P}_{h, s, a, \cdot} + \mathcal{R}_{h, s, a, \cdot} + w_{h, s, a, \cdot} + (P_{h, s, a, \cdot}, \nabla_{h+1} - \nabla^*_{h+1}) \right] \\
+ 1 & \{g_k\} \{\mathcal{C}_h^{th}\} \left[ (P_{h, s, a, \cdot} - P_{h, s, a, \cdot}, \nabla_{h+1} - \nabla^*_{h+1}) \right] + H1\{\mathcal{E}_h^{th}\} \\
\leq 1 & \{g_k\} \{\mathcal{C}_h^{th}\} \left[ \mathcal{P}_{h, s, a, \cdot} + \mathcal{R}_{h, s, a, \cdot} + w_{h, s, a, \cdot} + \nabla^*_{h+1}(s_{h+1}) + H\frac{\sqrt{SL}}{n(h, s, a) + 1} \right] + H1\{\mathcal{E}_h^{th}\},
\end{align*}
where the last step is by noticing visiting \((s_h, a'_h, s_h^{l+1}) = (s, a, \cdot)\) implies visiting \((h, s, a)\).

Therefore, we get

\[
\frac{1}{n^k(h, s, a)}. 
\]

Finally, union bounding over all \(h, s, a, u \in \{-1, +1\}^S\), \(n^k(h, s, a) \in \mathbb{Z}\) and set \(\delta = \delta'/(2^S SAT)\), we get

\[
\frac{1}{n^k(h, s, a)} \leq 3 \sqrt{\log(2/\delta') n^k(h, s, a)}. 
\]

This implies \(\|\hat{P}_{h,s,a} - P_{h,s,a}\| \leq 4 \sqrt{\frac{SL}{n^k(h, s, a)+1}}\), which completes the proof. 

The following Lemma 14 is the \(V\) function version of its \(Q\) function version counterpart in Lemma 12. It is applied in the proof of final regret decomposition in Theorem 11.

**Lemma 14.** With probability at least \(1 - \delta/4\), for any \(h, k, s_h, a_h\), the following decomposition holds

\[
1\{G_k\} \left[ \overline{V}_k(s_h) - V^*_k(s_h) \right] 
\leq 1\{G_k\} \left[ \sum_{h,s,a} \mathcal{P}_{h,s,a} + \mathcal{R}_{h,s,a} + w_{h,s,a} \cdot \delta_h + \mathcal{M}_{\pi_s(s_h)} + 4H \sqrt{\frac{SL}{n(h, s_h, a_h)}} \right] + H1\{\varepsilon^h_{\pi}\}.
\]

**Proof.** With \(a_h\) as the action taken by the algorithm \(\pi(s_h)\), it follows that \(\overline{V}_h(s_h) = \overline{Q}_h(s_h, a_h)\) and \(V^*_h(s_h) = Q^*_h(s_h, a_h)\). Thus, the proof follows by a direction application of Lemma 12. 

### D.2 Bound on the Pessimism Term

In this section, we will upper bound the pessimism term with the help of the probability of being optimistic and the bound on the estimation term. The approach generally follows Lemma G.4 of Zanette et al (2020). The difference here is that we also provide a bound for \(V^*_i(s^k_i) - V_{1,k}(s^k_i)\) and \(V^*_i(s^k_i) - \overline{V}_{1,k}(s^k_i)\). This difference enable us to get stronger bounds in the tabular setting as compared to Zanette et al (2020). The pessimism term will be decomposed to the two estimation terms \(\overline{V}_{1,k}(s^k_i) - V^*_i(s^k_i)\) and \(V^*_i(s^k_i) - \overline{V}_{1,k}(s^k_i)\), and the martingale difference term \(\mathcal{M}^w_{1,k}\).

**Lemma 15** (Restatement of Lemma 3). For any \(k\), the following decomposition holds,

\[
1\{G_k\} \left( V^*_i(s^k_i) - \overline{V}_{1,k}(s^k_i) \right) 
\leq 1\{G_k\} \left( V^*_i(s^k_i) - \overline{V}_{1,k}(s^k_i) \right) 
\leq C1\{G_k\} \left( \overline{V}_{1,k}(s^k_i) - V^*_i(s^k_i) + V^*_i(s^k_i) - \overline{V}_{1,k}(s^k_i) + \mathcal{M}^w_{1,k} \right),
\]

where \(1\{G_k\} \left[ V^*_i(s^k_i) - \overline{V}_{1,k}(s^k_i) \right] \) will be further bounded in Lemma 10.

**Proof.** For the purpose of analysis we use two “virtual” quantities \(\hat{V}_{1,k}(s^k_i)\) and \(\overline{V}_{1,k}(s^k_i)\), which are formally stated in the Definitions 2 and 3 respectively. Thus we can define the event \(\mathcal{O}_{1,k} \overset{def}{=} \{V_{1,k}(s^k_i) \geq V^*_i(s^k_i)\}\). For simplicity of exposition, we skip showing dependence on \(k\) in the following when it is clear.

By Definition 2 we know that \(\overline{V}_1(s_1)\) and \(\overline{V}_1(s_1)\) are identically distributed conditioned on last round history \(H^k_{1-1}\). From Definition 3 under event \(G_k\), it also follows that \(\overline{V}_1(s_1) \leq \overline{V}_1(s_1)\).
Since $V_1(s_1) \leq \overline{V}_1(s_1)$ under event $\mathcal{G}_k$, we get
\begin{equation}
1\{\mathcal{G}_k\} \left[ V_1^*(s_1) - \overline{V}_1(s_1) \right] \leq 1\{\mathcal{G}_k\} \left[ V_1^*(s_1) - \overline{V}_1(s_1) \right].
\end{equation}

We also introduce notation $E_{\tilde{w}}[\cdot]$ to denote the expectation over the pseudo-noise $\tilde{w}$ (recall that $\tilde{w}$ discussed in Definition [3]). Under event $\mathcal{O}_1$, we have $\tilde{V}_1(s_1) \geq V_1^*(s_1)$. Since $V_1^*(s_1)$ does not depend on $\tilde{w}$, we get $V_1^*(s_1) \leq E_{\tilde{w}|\mathcal{O}_1, \mathcal{G}_k} \left[ \tilde{V}_1(s_1) \right]$. Using the similar argument for $V_1(s_1)$, we know that $V_1(s_1) \leq E_{\tilde{w}|\mathcal{O}_1, \mathcal{G}_k} \left[ \tilde{V}_1(s_1) \right]$. Subtracting this equality from the inequality $V_1^*(s_1) \leq E_{\tilde{w}|\mathcal{O}_1, \mathcal{G}_k} \left[ \tilde{V}_1(s_1) \right]$, it follows that
\begin{equation}
V_1^*(s_1) - \overline{V}_1(s_1) \leq E_{\tilde{w}|\mathcal{O}_1, \mathcal{G}_k} \left[ \tilde{V}_1(s_1) - \overline{V}_1(s_1) \right].
\end{equation}

Therefore, we have
\begin{equation}
1\{\mathcal{G}_k\} \left[ V_1^*(s_1) - \overline{V}_1(s_1) \right] \leq 1\{\mathcal{G}_k\} E_{\tilde{w}|\mathcal{O}_1, \mathcal{G}_k} \left[ \tilde{V}_1(s_1) - \overline{V}_1(s_1) \right].
\end{equation}

From the law of total expectation, we can write
\begin{equation}
E_{\tilde{w}|\hat{\mathcal{G}}_k} \left[ \tilde{V}_1(s_1) - \overline{V}_1(s_1) \right] = P(\mathcal{O}_1|\hat{\mathcal{G}}_k) E_{\tilde{w}|\mathcal{O}_1, \hat{\mathcal{G}}_k} \left[ \tilde{V}_1(s_1) - \overline{V}_1(s_1) \right] + P(\mathcal{O}_1|\hat{\mathcal{G}}_k) E_{\tilde{w}|\mathcal{O}_0, \hat{\mathcal{G}}_k} \left[ \tilde{V}_1(s_1) - \overline{V}_1(s_1) \right].
\end{equation}

Since $\tilde{V}_1(s_1) - \overline{V}_1(s_1) \geq 0$ under event $\hat{\mathcal{G}}_k$, multiplying both sides of Eq (20) by $1\{\mathcal{G}_k\}$, relaxing the second term on RHS to 0 and rearranging yields
\begin{equation}
1\{\mathcal{G}_k\} E_{\tilde{w}|\mathcal{O}_1, \hat{\mathcal{G}}_k} \left[ \tilde{V}_1(s_1) - \overline{V}_1(s_1) \right] \leq \frac{1}{P(\mathcal{O}_1|\hat{\mathcal{G}}_k)} 1\{\mathcal{G}_k\} E_{\tilde{w}|\mathcal{O}_1, \hat{\mathcal{G}}_k} \left[ \tilde{V}_1(s_1) - \overline{V}_1(s_1) \right].
\end{equation}

Noticing $\tilde{V}$ is an independent sample of $\overline{V}$, we can invoke Lemma 13 for $\tilde{V}$, and it follows that $P(\mathcal{O}_1|\hat{\mathcal{G}}_k) \geq \Phi(-\sqrt{2})/2$. Set $C = \frac{1}{\Phi(-\sqrt{2})/2}$ and consider
\begin{equation}
1\{\mathcal{G}_k\} E_{\tilde{w}|\hat{\mathcal{G}}_k} \left[ \tilde{V}_1(s_1) - \overline{V}_1(s_1) \right] = 1\{\mathcal{G}_k\} \left( E_{\tilde{w}|\hat{\mathcal{G}}_k} \left[ \tilde{V}_1(s_1) - \overline{V}_1(s_1) \right] \right) + 1\{\mathcal{G}_k\} \left( \overline{V}_1(s_1) - \overline{V}_1(s_1) \right).
\end{equation}

where the equality is due to $\tilde{w}$ is independent of $\overline{V}_1(s_1)$.

Since $\overline{V}_1(s_1)$ and $\tilde{V}_1(s_1)$ are identically distributed from the definition, we will later show term (1)
\begin{equation}
1\{\mathcal{G}_k\} \left[ E_{\tilde{w}|\hat{\mathcal{G}}_k} \left[ \tilde{V}_1(s_1) - \overline{V}_1(s_1) \right] \right] := \mathcal{M}_1^w
\end{equation}

is a martingale difference sequence in Lemma [19] Term (2) can be further decomposed as
\begin{equation}
\overline{V}_1(s_1) - \overline{V}_1(s_1) = \overline{V}_1(s_1) - V^*_1(s_1) + V^*_1(s_1) - \overline{V}_1(s_1).
\end{equation}

Term (3) in Eq (23) is same as estimation term in Lemma [14]. For term (4), to make it clearer, we will show a bound separately in Lemma [10].

Combining Eq (21), (22), and (23) gives us that
\begin{equation}
1\{\mathcal{G}_k\} \left[ E_{\tilde{w}|\hat{\mathcal{G}}_k} \left[ \tilde{V}_1(s_1) - \overline{V}_1(s_1) \right] \right]
\leq C 1\{\mathcal{G}_k\} \left( V^*_1(s_1) - \overline{V}_1(s_1) + \overline{V}_1(s_1) - V^*_1(s_1) + \mathcal{M}_1^w \right).
\end{equation}

This completes the proof. □

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In Lemma 16 we provide a missing piece in Lemma 15. It will be applied when we do the regret decomposition of major term in Theorem 11.

**Lemma 16.** With probability at least $1 - \delta/4$, for any $h, k, s_h^k, a_h^k$, the following decomposition holds with the intersection event $G_k$

$$1\{G_k\} \left[V_{h}^{\pi^k}(s_h^k) - V_{h,k}(s_h^k)\right]$$

$$\leq 1\{G_k\} 1\{\mathcal{E}_{h,k}^{th}\} \left(-\mathcal{D}_{h,s_h^k,a_h^k}^k - \mathcal{D}_{h,s_h^k,a_h^k}^k + w_{h,s_h^k,a_h^k}^k + \Delta_{h+1,k}(s_h^k) + \mathcal{M}_{h,k}(s_h^k) + 4H \left(\frac{SL}{n(h, s_h^k, a_h^k) + 1}\right)\right) + H 1\{\mathcal{E}_{h,k}^{th}\}. \tag{24}$$

**Proof.** We continue to show how to bound term (4) in Lemma 15 and we will also drop the superscript $k$ here.

Noticing that $a_h$ as the action chosen by the algorithm $\pi(s_h)$, we have $V_{h}^{\pi}(s_h) = Q_{h}^{\pi}(s_h, a_h)$. By the definition of value function $V_{h}(s_h) = \max_{a} Q_{h}(s_h, a)$. This gives $Q_{h}(s_h, a_h) \leq V_{h}(s_h)$. Hence,

$$V_{h}^{\pi}(s_h) - V_{h}(s_h) = Q_{h}^{\pi}(s_h, a_h) - V_{h}(s_h) \leq Q_{h}(s_h, a_h) - Q_{h}(s_h, a_h).$$

From the definition of $V_{h}$, we know that its noise satisfies $|w(h, s, a)| \leq \gamma(h, s, a)$. Therefore, we can show a version of Lemma 9 for $V_{h}$ and get $1\{G_k\}(V_{h}^{\pi} - V_{h}) \leq H$. This implies the version of Lemma 13 for $V_{h}$ would hold. Since the decomposition and techniques in Lemma 12 only utilize the property that $Q_{h}$ is the solution of the Bayesian linear regression and the Bellman equation for $Q_{h}^{\pi}$, we can directly get another version for instance $Q_{h}^\pi$. Also noticing that we flip the sign of $V_{h}^{\pi}(s_h) - V_{h}(s_h)$, therefore, we obtain the following decomposition for the term (4) in Lemma 15

$$1\{G_k\} \left[V_{h}^{\pi}(s_h) - V_{h}(s_h)\right]$$

$$\leq 1\{G_k\} 1\{\mathcal{E}_{h,k}^{th}\} \left(-\mathcal{D}_{h,s_h^k,a_h^k}^k - \mathcal{D}_{h,s_h^k,a_h^k}^k + w_{h,s_h^k,a_h^k}^k + \Delta_{h+1,k}(s_h^k) + \mathcal{M}_{h,k}(s_h^k) + 4H \left(\frac{SL}{n(h, s_h^k, a_h^k) + 1}\right)\right) + H 1\{\mathcal{E}_{h,k}^{th}\}. \tag{25}$$

**D.3 Final Bound on Theorem 11**

Armed with all the supporting lemmas, we present the remaining proof of Theorem 11.

**Proof.** Recall that in the regret decomposition Eq (16), it remains to bound

$$\sum_{k=1}^{K} 1\{G_k\} \left(V_{1}^{\pi}(s_1^k) - V_{1,k}(s_1^k) + V_{1,k}(s_1^k) - V_{1}^{\pi}(s_1^k)\right).$$

Again, we would skip notation dependence on $k$ when it is clear. For each episode $k$, it suffices to bound

$$1\{G_k\} \left[V_{1}^{\pi}(s_1) - V_{1}(s_1) + V_{1}(s_1) - V_{1}^{\pi}(s_1)\right]$$

$$\leq 1\{G_k\} \left[V_{1}^{\pi}(s_1) - V_{1}(s_1)\right] + 1\{G_k\} \left[V_{1}(s_1) - V_{1}^{\pi}(s_1)\right]$$

$$= 1\{G_k\} \overline{\delta}_{1}(s_1) + 1\{G_k\} \overline{\delta}_{1}^\pi(s_1). \tag{25}$$

We first use Lemma 15 to relax the first term in Eq (25). Applying Eq (17) in Lemma 15 gives us the following

$$1\{G_k\} \overline{\delta}_{1}(s_1)$$

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\[ = 1\{G_k\} \left[ V_1^*(s_1) - \nabla_1(s_1) \right] \]
\[ \leq C1\{G_k\} \left( V_1^*(s_1) - \nabla_1(s_1) + V_1^*(s_1) - M_1^w \right) \]
\[ = C1\{G_k\} \left( \delta_1^*(s_1) + \delta_1^*(s_1) + M_1^w \right). \]  

(26)

Combining Eq (26) and Eq (25), we get
\[ 1\{G_k\} \left( V_1^*(s_1) - \nabla_1(s_1) + V_1^*(s_1) - V_1^*(s_1) \right) \]
\[ \leq (C + 1)1\{G_k\}\delta_1^*(s_1) + C1\{G_k\} \left( M_1^w + \delta_1^*(s_1) \right). \]  

(27)

We will bound first and second term in Eq (27) correspondingly. In the sequence, we always consider the case that Lemma 10 and Lemma 14 hold. Therefore, the following holds with probability at least \(1 - \delta/4 - \delta/2\).

For the \(\delta_1^*(s_1)\) term in Eq (27), applying Eq (24) in Lemma 16 yields
\[ 1\{G_k\} \delta_1^*(s_1) = 1\{G_k\} \left[ V_1^*(s_1) - \nabla_1(s_1) \right] \]
\[ \leq 1\{G_k\} 1\{\xi_{1,k}^{th} \} \left( P_{1,s_1,a_1} + R_{1,s_1,a_1} + w_{1,s_1,a_1} + \delta_2^*(s_2) + M_{\delta_1^*(s_1)} + 4H \sqrt{\frac{SL}{n(1,s_1,a_1)+1}} \right) + H1\{\xi_k^{th} \xi \}. \]  

(28)

For the \(\nabla_1^*(s_1)\) term Eq (27), applying Lemma 14 yields
\[ 1\{G_k\} \nabla_1^*(s_1) = 1\{G_k\} \left[ \nabla_1(s_1) - V_1^*(s_1) \right] \]
\[ \leq 1\{G_k\} 1\{\xi_{1,k}^{th} \} \left( P_{1,s_1,a_1} + R_{1,s_1,a_1} + w_{1,s_1,a_1} + \delta_2^*(s_2) + M_{\delta_1^*(s_1)} + 4H \sqrt{\frac{SL}{n(1,s_1,a_1)+1}} \right) + H1\{\xi_k^{th} \xi \}. \]  

(29)

Plugging Eq (28) and (29) into Eq (27) gives us, with probability at least \(1 - \delta/2\),
\[ 1\{G_k\} \left( V_1^*(s_1) - \nabla_1(s_1) + V_1^*(s_1) - V_1^*(s_1) \right) \]
\[ \leq (C + 1)1\{G_k\}\delta_1^*(s_1) + C1\{G_k\} \left( M_1^w + \delta_1^*(s_1) \right) \]
\[ \leq C1\{G_k\} 1\{\xi_{1,k}^{th} \} \left( P_{1,s_1,a_1} + R_{1,s_1,a_1} + w_{1,s_1,a_1} + \delta_2^*(s_2) + M_{\delta_1^*(s_1)} + 4H \sqrt{\frac{SL}{n(1,s_1,a_1)+1}} \right) + CH1\{\xi_k^{th} \xi \} \]
\[ + (C + 1)1\{G_k\} 1\{\xi_{1,k}^{th} \} \left( P_{1,s_1,a_1} + R_{1,s_1,a_1} + w_{1,s_1,a_1} + \delta_2^*(s_2) + M_{\delta_1^*(s_1)} + 4H \sqrt{\frac{SL}{n(1,s_1,a_1)+1}} \right) + C1\{G_k\} M_1^w \]
\[ + (C + 1)H1\{\xi_k^{th} \xi \} + C1\{G_k\} M_1^w \]
\[ = C1\{G_k\}\delta_2^*(s_2) + (C + 1)1\{G_k\}\nabla_1^*(s_1) + C1\{G_k\} M_1^w \]
\[ + C1\{G_k\} 1\{\xi_{1,k}^{th} \} \left( P_{1,s_1,a_1} + R_{1,s_1,a_1} + w_{1,s_1,a_1} + \delta_2^*(s_2) + M_{\delta_1^*(s_1)} + 4H \sqrt{\frac{SL}{n(1,s_1,a_1)+1}} \right) + CH1\{\xi_k^{th} \xi \} \]

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\[ + (C + 1) \mathbb{1}\{G_k\} \mathbb{1}\{c_{i,k}^{th}\} \left( |P_{1,s_1,a_1} + R_{1,s_1,a_1}| + w_{1,s_1,a_1} + M\pi_1(s_1) + 4H \sqrt{\frac{SL}{n(1,s_1,a_1) + 1}} \right) + (C + 1) H \mathbb{1}\{c_{i}^{th}\}. \]

(30)

Keep unrolling Eq (30) to timestep \( H \) and noticing \( \delta_{H+1}^*(s_{H+1}) = \delta_{H+1}^*(s_{H+1}) = 0 \) and \( M\pi_{H+1}^* = 0 \) yields that with probability at least \( 1 - \delta/2 \),

\[ 1\{G_k\} \left[ V^*_1(s_1) - V^*_n(s_1) \right] \leq (2C + 1) H^2 \mathbb{1}\{c_{i}^{th}\} + CM\pi
\]

\[ + C \sum_{k=1}^{H} \sum_{h=1}^{H} 1\{G_k\} \prod_{h'=1}^{h} 1\{E_{h',k}^{th}\} \left( |P_{h,s_h,a_h} + R_{h,s_h,a_h}| + w_{h,s_h,a_h} + M\pi_{h}(s_h) + 4H \sqrt{\frac{SL}{n(h,s_h,a_h) + 1}} \right)
\]

\[ + (C + 1) \sum_{k=1}^{H} 1\{G_k\} \prod_{h'=1}^{h} 1\{E_{h',k}^{th}\} \left( |P_{h,s_h,a_h} + R_{h,s_h,a_h}| + w_{h,s_h,a_h} + M\pi_{h}(s_h) + 4H \sqrt{\frac{SL}{n(h,s_h,a_h) + 1}} \right). \]

(31)

It suffices to bound each individual term in Eq (31) and we will take sum over \( k \) outside.

Lemma 18 gives us the bound on transition function and reward function

\[ \sum_{k=1}^{K} \sum_{h=1}^{H} 1\{G_k\} |P_{h,s_h,a_h}^k + R_{h,s_h,a_h}^k| = \tilde{O}(\sqrt{H^3 SAT}). \]

Following the steps in Lemma 18 we also get the bound

\[ \sum_{k=1}^{K} \sum_{h=1}^{H} 1\{G_k\} |P_{h,s_h,a_h}^k + R_{h,s_h,a_h}^k| = \tilde{O}(H^3\sqrt{AT}). \]

(32)

Lemma 19 bounds the martingale difference sequences. Replacing \( \delta \) by \( \delta' \) in Lemma 19 gives us that with probability at least \( 1 - \delta' \),

\[ \sum_{k=1}^{K} \sum_{h=1}^{H} 1\{G_k\} \prod_{h'=1}^{h} 1\{E_{h',k}^{th}\} M\pi_{h}(s_h) = \tilde{O}(H \sqrt{T}) \]

\[ \sum_{k=1}^{K} \sum_{h=1}^{H} 1\{G_k\} \prod_{h'=1}^{h} 1\{E_{h',k}^{th}\} M\pi_{h}(s_h) = \tilde{O}(H \sqrt{T}) \]

\[ \sum_{k=1}^{K} 1\{G_k\} M\pi_{i,k} = \tilde{O}(H \sqrt{T}). \]

For the noise term, we first notice that under event \( G_k \), \( w_{h,s_h,a_h}(s_h) \) can be upper bounded by \( |w_{h,s_h,a_h}(s_h)| \). Applying Lemma 17 and (replacing \( \delta \) by \( \delta' \) in Lemma 17) gives us, with probability at least \( 1 - 2\delta' \)

\[ \sum_{k=1}^{K} \sum_{h=1}^{H} 1\{G_k\} w_{h,s_h,a_h}(s_h) = \tilde{O}(H^2 S \sqrt{AT}) \]

and

\[ \sum_{k=1}^{K} \sum_{h=1}^{H} 1\{G_k\} w_{h,s_h,a_h}(s_h) = \tilde{O}(H^2 S \sqrt{AT}). \]
The warm-up regret term is bounded in Lemma \[20\]

\[H^2 \sum_{k=1}^{K} 1 \{e_k^h \leq \epsilon\} = \tilde{O}(H^5 S^2 A).\]

Putting all these pieces together and setting \(\delta' = \delta/12\), with probability at least \(1 - \delta\), we get

\[
\sum_{k=1}^{K} 1 \{G_k\} \left( V^*_1(s_1) - \nabla_{1,k}(s_1) + \nabla_{h,k}(s_1) - V^*_1(s_1) \right) = \tilde{O} \left( H^2 S \sqrt{AT} + H^5 S^2 A \right).
\] (33)

This completes the proof of Theorem \[11\].

\[\square\]

E  Bounds on Individual Terms

E.1  Bound on the Noise Term

Lemma 17. With \(w^k_{h,s,a} \) as defined in Definition \[2\] and \(a^k_h = \pi^k(s^k_h)\), the following bound holds:

\[
\sum_{k=1}^{K} \sum_{h=1}^{H} 1 \{G_k\} \left| w^k_{h,s,a} \right| = \tilde{O} \left( H^2 S \sqrt{AT} \right).
\]

Proof. We have:

\[
\sum_{k=1}^{K} \sum_{h=1}^{H} \left| w^k_{h,s,a} \right| = \sqrt{\frac{\beta_h L}{2}} \sum_{k=1}^{K} \sum_{h=1}^{H} \sqrt{\frac{1}{n^k(h,s,a) + 1}} = \sqrt{\frac{\beta_h L}{2}} \sum_{h,s,a} \sum_{n=1}^{n^k(h,s,a)} \sqrt{\frac{1}{n}}.
\]

Upper bounding by integration followed by an application of Cauchy-Schwarz inequality gives:

\[
\sum_{h,s,a} n^k(h,s,a) \sum_{n=1}^{n^k(h,s,a)} \frac{1}{n} \leq \sum_{h,s,a} \int_0^{n^k(h,s,a)} \sqrt{\frac{1}{x}} dx = 2 \sum_{h,s,a} \sqrt{n^k(h,s,a)} \leq 2 \sqrt{HSA} \sum_{h,s,a} n^k(h,s,a) = O \left( \sqrt{HSA} \sqrt{SAT} \right).
\]

This leads to the bound of \(O \left( \sqrt{\beta_h L} \sqrt{HSA} \sqrt{SAT} \right) = \tilde{O} \left( H^2 S \sqrt{AT} \right).

\[\square\]

E.2  Bound on Estimation Error

Lemma 18. For \(a^k_h = \pi^k(s^k_h)\), the following bound holds

\[
\sum_{k=1}^{K} \sum_{h=1}^{H} 1 \{G_k\} \left| \hat{P}^k_{h,s,a} - P_{h,s,a} \right| = \tilde{O} \left( H^{3/2} \sqrt{SA} \right).
\]

Proof. Under the event \(G_k\), the estimated MDP \(\hat{M}^k\) lies in the confidence set defined in Appendix \[A\]. Hence

\[
\left| \hat{P}^k_{h,s,a} - P_{h,s,a} \right| = \sqrt{e_k(h,s,a) + \frac{\log(2HSAk)}{n^k(h,s,a) + 1}}.
\]

We bound the denominator as

\[
\sum_{k=1}^{K} \sum_{h=1}^{H} \sqrt{\frac{1}{n^k(h,s,a) + 1}}
\]

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\[
\leq \sum_{h,s,a} \sum_{n=1}^{n^K(h,s,a)} \sqrt{\frac{1}{n}} \\
\leq \sum_{h,s,a} \int_0^{n^K(h,s,a)} \sqrt{\frac{1}{x}} \, dx \\
\leq 2 \sum_{h,s,a} \sqrt{n^K(h,s,a)} \\
\leq 2 \sqrt{HSA \sum_{h,s,a} n^K(h,s,a)} \\
= O(\sqrt{HSAT}),
\]

where step (a) follows Cauchy-Schwarz inequality.

Therefore we get

\[
\sum_{k=1}^{K} \sum_{h=1}^{H} \sqrt{e_k(h, s_k^h, a_k^h)} = H\tilde{O}\left(\sqrt{HSAT}\right) = \tilde{O}\left(H^{3/2}\sqrt{SAT}\right).
\]

E.3 Bounds on Martingale Difference Sequences

Lemma 19. The following martingale difference summations enjoy the specified upper bounds with probability at least \(1 - \delta\),

\[
\sum_{k=1}^{K} \sum_{h=1}^{H} \sqrt{e_k(h, s_k^h, a_k^h)} = H\tilde{O}\left(\sqrt{HSAT}\right) = \tilde{O}\left(H^{3/2}\sqrt{SAT}\right).
\]

Here \(\prod_{h'=1}^{h} 1\{\mathcal{E}^{th}_{h',k}\}M_{\delta_{h,k}(s_k^h)}^{a_k^h}\prod_{h'=1}^{h} 1\{\mathcal{E}^{th}_{h',k}\}M_{\delta_{h,k}(s_k^h)}^{a_k^h}\) are considered under filtration \(\mathcal{H}_k\), while \(\mathcal{M}_{w_{1,k}}\) is considered under filtration \(\mathcal{H}^{k-1}_{H}\). Noticing the definition of martingale difference sequences, we can also drop \(1\{G_k\}\) in the lemma statement.

Proof. This proof has two parts. We show (i) above are summations of martingale difference sequences and (ii) these summations concentrate under the event \(G_k\) due to Azuma-Hoeffding inequality (Wainwright, 2019).

We only present the proof for \(\prod_{h'=1}^{h} 1\{\mathcal{E}^{th}_{h',k}\}M_{\delta_{h,k}(s_k^h)}^{a_k^h}\) and \(\mathcal{M}_{w_{1,k}}\), and another one follow like-wise.

We first consider \(\prod_{h'=1}^{h} 1\{\mathcal{E}^{th}_{h',k}\}M_{\delta_{h,k}(s_k^h)}^{a_k^h}\) term. Given the filtration set \(\mathcal{H}_k\), we observe that

\[
\mathbb{E}\left[1\{G_k\} \prod_{h'=1}^{h} 1\{\mathcal{E}^{th}_{h',k}\}\delta_{h+1,k}(s_{h+1})^k_{k} \mathcal{H}^k_{h} \right] = \mathbb{E}\left[1\{G_k\} \prod_{h'=1}^{h} 1\{\mathcal{E}^{th}_{h',k}\}\delta_{h+1,k}(s')^k_{k} \mathcal{H}^k_{h} \right].
\]
This is because the randomness is due to the random transitions of the algorithms when conditioning on $\overline{H}_h^k$. Thus we have $E \left[ \prod_{h'=1}^h 1 \{ \xi_{h',k}^i \} \mathcal{M}_{s_{h,k}(s_k^{h})}^k \left| \overline{H}_h^k \right. \right] = 0$ and $\left\{ \prod_{h'=1}^h 1 \{ \xi_{h',k}^i \} \mathcal{M}_{s_{h,k}(s_k^{h})}^k \right\}$ is indeed a martingale difference on the filtration set $\overline{H}_h^k$.

Under event $\mathcal{G}_k$, we also have $\overline{\delta}_{h+1,k}(s_{h+1}^k) = V_{h+1,k}(s_{h+1}^k) - V_{h+1,k}(s_{h+1}^k) \leq 2H$. Applying Azuma-Hoeffding inequality (e.g. Azar, Osband, and Munos (2017)), for any fixed $K' \in [K]$ and $H' \in [H]$, we have with probability at least $1 - \delta'$,

$$\sum_{k=1}^{K'} \sum_{h=1}^{H'} \prod_{1}^{h} 1 \{ \xi_{h',k}^i \} \mathcal{M}_{s_{h,k}(s_k^{h})}^k \leq H \sqrt{4T \log \left( \frac{2T}{\delta'} \right)} = \tilde{O} \left( H \sqrt{T} \right).$$

Union bounding over all $K'$ and $H'$, we know the following holds for any $K' \in [K]$ and $H' \in [H]$ with probability at least $1 - \delta'$,

$$\sum_{k=1}^{K'} \sum_{h=1}^{H'} \prod_{1}^{h} 1 \{ \xi_{h',k}^i \} \mathcal{M}_{s_{h,k}(s_k^{h})}^k \leq H \sqrt{4T \log \left( \frac{2T}{\delta'} \right)} = \tilde{O} \left( H \sqrt{T} \right).$$

Then we consider $\mathcal{M}_{i,1,k}^w$ term. Given filtration $\mathcal{H}_{K-1}^k$, we know that $\tilde{V}_{i,k}$ has identical distribution as $\overline{V}_{i,k}$. Therefore, for any state $s$, we have

$$E \left[ 1 \{ \tilde{G}_k \} \tilde{V}_{i,1,k}(s) \right| \mathcal{H}_{K-1}^k] = E \left[ 1 \{ \tilde{G}_k \} \overline{V}_{i,1,k}(s) \right| \mathcal{H}_{K-1}^k].$$

Besides, from the definition of $E_\pi$ and $\tilde{w}$ is the only randomness given $\mathcal{H}_{K-1}^k$, we have that for any state $s$,

$$E \left[ 1 \{ \tilde{G}_k \} E_{w} \tilde{G}_k \left[ \tilde{V}_{i,1,k}(s) \right| \mathcal{H}_{K-1}^k \right] = E \left[ 1 \{ \tilde{G}_k \} E_{\tilde{G}_k} \left[ 1 \{ \tilde{G}_k \} \tilde{V}_{i,1,k}(s) \right| \mathcal{H}_{K-1}^k \right] = E \left[ 1 \{ \tilde{G}_k \} E_{w} \tilde{G}_k \left[ 1 \{ \tilde{G}_k \} \tilde{V}_{i,1,k}(s) \right| \mathcal{H}_{K-1}^k \right] = E \left[ 1 \{ \tilde{G}_k \} E_{\tilde{G}_k} \left[ 1 \{ \tilde{G}_k \} \tilde{V}_{i,1,k}(s) \right| \mathcal{H}_{K-1}^k \right].$$

Combining these two equations and setting $s = s_k^h$, we have $E \left[ \mathcal{M}_{i,1,k}^{w} \right| \mathcal{H}_{K-1}^k]$. Therefore the sequence $\{ \mathcal{M}_{i,1,k}^{w} \}$ is indeed a martingale difference.

Under event $\mathcal{G}_k$, we also have $E_{w} \tilde{G}_k \left[ \overline{V}_{i,1,k}(s_k^h) \right] - E_{w} \tilde{G}_k \left[ \overline{V}_{i,1,k}(s_k^h) \right] \leq 2H$ from Lemma. Applying from Azuma-Hoeffding inequality (e.g. Azar, Osband, and Munos (2017)) and similar union bounding argument above, for any $K' \in [K]$, with probability at least $1 - \delta'$, we have

$$\sum_{k=1}^{K'} \mathcal{M}_{i,1,k}^{w} \leq H \sqrt{4T \log \left( \frac{2T}{\delta'} \right)} = \tilde{O} \left( H \sqrt{T} \right).$$
The remaining results as in the lemma statement is proved like-wise. Finally let $\delta' = \delta/3$ and uniform bounding over these 3 martingale difference sequences completes the proof. \hfill \Box

### E.4 Bound on the Warm-up Term

**Lemma 20** (Bound on the warm-up term).

$$
\sum_{k=1}^{K} \mathbf{1}\{\mathcal{C}^{th}_{k}\} = \tilde{O}(H^3 S^2 A).
$$

**Proof.**

\[
\sum_{k=1}^{K} \mathbf{1}\{\mathcal{C}^{th}_{k}\} \\
= \sum_{k=1}^{K} \mathbf{1}\{ \bigcup_{h \in [H]} n^k(h, s, a) \leq \alpha_k, \forall (h, s, a) = (h, s_h^k, a_h^k) \} \\
\leq \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbf{1}\{ n^k(h, s, a) \leq \alpha_k, \forall (h, s, a) = (h, s_h^k, a_h^k) \} \\
\leq \sum_{a \in A} \sum_{s \in S} \sum_{h=1}^{H} \alpha_k \\
\leq 4H^3S^2A \log (2HSAK) \log \left( \frac{40SAT}{\delta} \right) \\
= \tilde{O}(H^3 S^2 A).
\]

Step (a) is by substituting the value of $\alpha_k$ followed by upper bound for all $4H^3S \log (2HSAK) \log \left( \frac{40SAT}{\delta} \right)$.

\hfill \Box