A class of bistochastic positive optimal maps in $M_d(\mathbb{C})$

Adam Rutkowski$^{1,2}$, Gniewomir Sarbicki$^3$, and Dariusz Chruściński$^3$

$^1$ Faculty of Mathematics, Physics and Informatics, University of Gdańsk, 80-952 Gdańsk, Poland
$^2$ Quantum Information Centre of Gdańsk, 81-824 Sopot, Poland
$^3$ Institute of Physics, Faculty of Physics, Astronomy and Informatics, Nicolaus Copernicus University, Grudziadzka 5, 87–100 Toruń, Poland

Abstract

We provide a straightforward generalization of a positive map in $M_3(\mathbb{C})$ considered recently by Miller and Olkiewicz [5]. It is proved that these maps are optimal and indecomposable. As a byproduct we provide a class of PPT entangled states in $d \otimes d$.

Positive maps in matrix algebras play important role both in mathematics and theoretical physics [1, 2, 3, 4]. In the recent paper [5] paper Miller and Olkiewicz considered a linear map $\Lambda_3 : M_3(\mathbb{C}) \to M_3(\mathbb{C})$ ($M_d(\mathbb{C})$ denotes a matrix algebra of $d \times d$ complex matrices) defined as follows

$$\Lambda_3 \left( \begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array} \right) = \left( \begin{array}{ccc} \frac{1}{2} (a_{11} + a_{22}) & 0 & \frac{1}{\sqrt{2}} a_{13} \\ 0 & \frac{1}{2} (a_{11} + a_{22}) & \frac{1}{\sqrt{2}} a_{32} \\ \frac{1}{\sqrt{2}} a_{31} & \frac{1}{\sqrt{2}} a_{23} & a_{33} \end{array} \right) \geq 0 \ .$$ (1)

It was proved [5] that $\Lambda_3$ is a bistochastic positive extremal (even exposed) non-decomposable map. In this paper we provide the following generalization $\Lambda_d : M_d(\mathbb{C}) \to M_d(\mathbb{C})$:

$$\Lambda_d (A) = \frac{1}{d-1} \left( \begin{array}{ccccccc} \sum_{i=1}^{d-1} a_{ii} & \cdots & 0 & 0 & \sqrt{d-1} a_{1d} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \cdots & \sum_{i=1}^{d-1} a_{ii} & 0 & \sqrt{d-1} a_{d-2,d} \\ 0 & \cdots & 0 & \sum_{i=1}^{d-1} a_{ii} & \sqrt{d-1} a_{d,d-1} \\ \sqrt{d-1} a_{d1} & \cdots & \sqrt{d-1} a_{d,d-2} & \sqrt{d-1} a_{d-1,d} & (d-1) a_{dd} \end{array} \right) \ ,$$ (2)

where $A = [a_{ij}] \in M_d(\mathbb{C})$.

**Proposition 1.** $\Lambda_d$ is a positive map.
Proof: let \( y = \begin{pmatrix} x \\ x_d \end{pmatrix} \in \mathbb{C}^d, x \in \mathbb{C}^{d-1} \) and \( P_i = |i\rangle\langle i| \) for \( i = 1, \ldots, d-1 \). One has

\[
\Lambda_d(yy^\dagger) = \frac{1}{d-1} \begin{pmatrix} \|x\|^2 \mathbb{I}_{d-1} & \sqrt{d-1} \left(x_d P_1 x + \sum_{i=2}^{d-1} \bar{x}_d P_i \bar{x} \right) \end{pmatrix} \begin{pmatrix} \sqrt{d-1} \left(x_d P_1 x + \sum_{i=2}^{d-1} \bar{x}_d P_i \bar{x} \right)^\dagger & (d-1) |x_d|^2 \end{pmatrix}
\]

Now we use the well known result [2]: a block matrix

\[
\begin{pmatrix} A & B \\ B^\dagger & C \end{pmatrix}
\]

with \( C > 0 \) is positive iff

\[
A \geq BC^{-1}B^\dagger.
\]

Hence, to prove that \( \Lambda_d(yy^\dagger) \geq 0 \) it is necessary and sufficient to show that

\[
|x_d|^2 \|x\|^2 \mathbb{I}_{d-1} - \left(x_d P_1 x + \sum_{i=2}^{d-1} \bar{x}_d P_i \bar{x} \right) \left(x_d P_1 x + \sum_{i=2}^{d-1} \bar{x}_d P_i \bar{x} \right)^\dagger \geq 0.
\]

One has

\[
\left(x_d P_1 x + \sum_{i=2}^{d-1} \bar{x}_d P_i \bar{x} \right) \left(x_d P_1 x + \sum_{i=2}^{d-1} \bar{x}_d P_i \bar{x} \right)^\dagger \leq \left\| \left(x_d P_1 x + \sum_{i=2}^{d-1} \bar{x}_d P_i \bar{x} \right) \right\|_{d-1}^2
\]

\[
= |x_d|^2 \left( \|P_1 x\|^2 + \sum_{i=2}^{d-1} \|P_i \bar{x}\|^2 \right) \mathbb{I}_{d-1} = |x_d|^2 \|x\|^2 \mathbb{I}_{d-1},
\]

which ends the proof. \( \square \)

Remark 1. It is very easy to check that \( \Lambda_d \) is unital and trace-preserving and hence it defines a positive bistochastic map.

Proposition 2. \( \Lambda_d \) is nondecomposable.

Proof: to prove it we construct a PPT state \( \rho_{\text{PPT}} \) such that \( \text{Tr}(W_d \rho) < 0 \), where \( W_d = (\mathbb{I} \otimes \Lambda_d) P_d^+ \) denotes the corresponding entanglement witness ([6] and the recent review [4]). Let us define

\[
\rho = \begin{pmatrix}
\sqrt{d-2} e_{11} + e_{dd} & 0 & \cdots & 0 & -e_{1d} \\
0 & \sqrt{d-2} e_{22} + e_{dd} & \cdots & 0 & -e_{2d} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \sqrt{d-2} e_{d-1,d-1} + e_{dd} & -e_{d-1,d} \\
-e_{d1} & -e_{d2} & \cdots & -e_{d,d-1} & \mathbb{I} - (1 - \sqrt{d-2}) e_{dd}
\end{pmatrix}
\]
where $e_{ij} = |i⟩⟨j|$. Let us observe that $\rho \geq 0$ iff the following $d - 1 \times d - 1$ submatrix

$$
\begin{pmatrix}
\sqrt{d-2} & 0 & \cdots & 0 & -1 \\
0 & \sqrt{d-2} & \cdots & 0 & -1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \sqrt{d-2} & -1 \\
-1 & -1 & \cdots & -1 & \sqrt{d-2}
\end{pmatrix}
\geq 0 ,
$$

(4)

which is the case due to the fact that its eigenvalues read: $\{\lambda_1 = 0, \lambda_2 = 2\sqrt{d-2}, \lambda_3 = 2\sqrt{d-2}\}$, where $\lambda_1, \lambda_3$ are simple and $\lambda_2$ has multiplicity $d - 3$. Consider now the partial transposed

$$
\rho^\Gamma = \begin{pmatrix}
\sqrt{d-2} e_{11} + e_{dd} & 0 & \cdots & 0 & -e_{d1} \\
0 & \sqrt{d-2} e_{22} + e_{dd} & \cdots & 0 & -e_{d2} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \sqrt{d-2} e_{d-1,d-1} + e_{dd} & -e_{d,d-1} \\
-e_{1d} & -e_{2d} & \cdots & -e_{d-1,d} & 1 - (1 - \sqrt{d-2}) e_{dd}
\end{pmatrix}
\text{.}
$$

Its positivity follows from the simple observation that the following $2 \times 2$ submatrices

$$
\begin{pmatrix}
\sqrt{d-2} & -1 \\
-1 & \sqrt{d-2}
\end{pmatrix}
\text{ and }
\begin{pmatrix}
1 & -1 \\
-1 & 1
\end{pmatrix}
$$

(5)

are positive. Now,

$$
\text{Tr} (W_d\rho) = 2(d - 1) \left(\sqrt{d-2} - \sqrt{d-1}\right) < 0 ,
$$

which finally proves that $\Lambda_d$ is nondecomposable. \hfill \square

Now we are ready to show that a map $\Lambda_d$ is optimal [7].

**Proposition 3.** $\Lambda_d$ is optimal.

Proof: to prove optimality we use the following result from [7]: if the entanglement witness $W = (1 \otimes \Lambda)P^+_d$ allows for a set of product vectors $\psi_k \otimes \phi_k$ such that

$$
⟨\psi_k \otimes \phi_k|W|\psi_k \otimes \phi_k⟩ = 0 ,
$$

(6)

then if $\psi_k \otimes \phi_k$ span $\mathbb{C}^d \otimes \mathbb{C}^d$ the map $\Lambda$ is optimal. Now, take arbitrary $x \in \mathbb{C}^d$ and define

$$
W_d(x) = \text{Tr}_1(W_d \cdot |x⟩⟨x| \otimes I_d).
$$

(7)

One finds

$$
W_d(x) = \left[ \begin{array}{c|c}
\frac{zI_{d-1}}{\bar{a}^T} \\
\bar{a}^T \end{array} \right] ,
$$

(8)

where

$$
a_i = \sqrt{d-1} \cdot \left\{ \begin{array}{ll}
x_i x_i^* & \text{for } i < d - 1 \\
x_d x_i^* & \text{for } i = d - 1
\end{array} \right. ,
$$
\[ z = \sum_{i=1}^{d-1} |x_i|^2 \] and \( u = (d - 1)|x_d|^2 \). Note that \( W_d(x) \) is at least of rank \( d - 1 \) and hence its kernel is at most 1-dimensional. To find the corresponding zero-mode of \( W_d(x) \) we consider
\[
\det W_d(x) = -(d - 1)|x_d|^2 \left( \sum_{i=1}^{d-1} |x_i|^2 \right) \cdot z^{d-2} + (d - 1)|x_d|^2 z^{d-1} = 0.
\]

Observing that the last row of \( W_d(x) \) is a combination of the previous ones, we find the vector of the kernel solving the equation
\[
[zI|\tilde{a}] \begin{bmatrix} v \\ w \end{bmatrix} = z\tilde{v} + \tilde{a}w = 0
\]
which implies (up to a scalar), that \( \tilde{v} = \tilde{a} \) and \( w = -z \). Denoting the solution as \( y(x) \), one gets the family \( q(x) = x \otimes y(x) \) of product vectors such that \( \langle x \otimes y(x) | W | x \otimes y(x) \rangle = 0 \). A vector from the family has the following coordinates:
\[
\begin{align*}
&x_1 x_1 x^*_d \ldots x_1 x_{d-2} x^*_d \quad x_1 x^*_1 x^*_d \\
&x_2 x_1 x^*_d \ldots x_2 x_{d-2} x^*_d \quad x_2 x^*_2 x^*_d \\
&\vdots \quad \vdots \\
&x_d x_1 x^*_d \ldots x_d x_{d-2} x^*_d \quad x_d x^*_d x^*_d \\
&x_d x_{d-1} x^*_d \ldots x_d x_{d-2} x^*_d \quad x_d x_{d-1} x^*_d \\
&x_d \sum_{i=1}^{d-1} x_i x^*_i .
\end{align*}
\]

It remains to show that vectors \( q(x) = x \otimes y(x) \) span \( \mathbb{C}^d \otimes \mathbb{C}^d \). Suppose that there exists a vector \( \alpha = \sum_{i,j=1}^d \alpha_{i,j} |e_i \otimes e_j \rangle \) orthogonal to \( q(x) \) for all \( x \), that is,
\[
\sum_{i=1}^d \left( \sum_{j=1}^{d-2} \alpha_{i,j}^* x_i x^*_j x^*_d + \alpha_{i,d-1}^* x_i x^*_d + \alpha_{i,d}^* \left( \sum_{i=1}^{d-1} x_i x^*_i \right) \right) = 0 .
\]

We stress that in the linear space of polynomials of 2\( d \) variables \( x_i \) and \( x^*_i \) are linearly independent. The monomial \( x_i x^*_1 \) appears in the sum only once multiplied by the coefficient \( \alpha_{i,d} \). Hence because different monomials are linearly independent in the space of polynomials one concludes that \( \alpha_{i,d} = 0 \). Next observe, that the monomial \( x_i x^*_d x^*_d \) appears only once multiplied by the coefficient \( \alpha_{i,d-1} \). Thus one concludes that \( \alpha_{i,d-1} = 0 \). Finally, we have to prove, that the sum \( \sum_{i=1}^d \sum_{j=1}^{d-2} \alpha_{i,j}^* x_i x^*_j x^*_d \) is zero iff all coefficients are zero. Indeed, all the coefficients multiply the different monomials. There are no non-zero vectors orthogonal to the subspace spanned by the vectors \( q(x) \), so these vectors span the whole Hilbert space of the system, what implies optimality of the witness.

In conclusion we have shown how to generalize a positive map in \( M_2(\mathbb{C}) \) considered in [5] to a positive map in \( M_d(\mathbb{C}) \). We have proved that this map is optimal and indecomposable. As a byproduct we provide a class of PPT entangled states in \( d \otimes d \). It would be interesting check whether this generalized map is extremal or even exposed.

**Acknowledgements**

A. Rutkowski was supported by a postdoc internship decision number DEC–2012/04/S/ST2/00002, from the Polish National Science Center. D. Chruściński and G. Sarbicki were partially supported by the Polish National Science Center project DEC-2011/03/B/ST2/00136.
References

[1] V. Paulsen, *Completely Bounded Maps and Operator Algebras*, Cambridge: Cambridge University Press 2003.

[2] R. Bhatia, *Positive Definite Matrices*, Princeton, NJ: Princeton University Press 2006.

[3] E. Størmer, *Positive Linear Maps of Operator Algebras*, (Berlin: Springer) 2013.

[4] D. Chruściński and G. Sarbicki, J. Phys. A: Math. Theor. 47, 483001 (2014).

[5] M. Miller and R. Olkiewicz, *Stable subspaces of positive maps of matrix algebras*, arXiv:1412.7469 (to appear in OSID).

[6] B. Terhal, Phys. Lett. A 271, 319 (2000).

[7] M. Lewenstein, B. Kraus, J. I. Cirac, and P. Horodecki, Phys. Rev. A 62, 052310 (2000).