P-Wave Holographic Insulator/Superconductor Phase Transition

Amin Akhavan\textsuperscript{a,b} and Mohsen Alishahiha\textsuperscript{a}

\textsuperscript{a} School of physics, Institute for Research in Fundamental Sciences (IPM)  
P.O. Box 19395-5531, Tehran, Iran

\textsuperscript{b} Department of Physics, Sharif University of Technology  
P.O. Box 11365-9161, Tehran, Iran

email: amin.akhavan@mehr.sharif.ir, and alishah@ipm.ir

Abstract

Using a five dimensional AdS soliton in an Einstein-Yang-Mills theory with $SU(2)$ gauge group we study p-wave holographic insulator/superconductor phase transition. To explore the phase structure of the model we consider the system in the probe limit as well as fully back reacted solutions. We will also study zero temperature limit of the p-wave holographic superconductor in four dimensions.

Dedicated to the memory of Mohammad Hossein Sarmadi
1 Introduction

The application of AdS/CFT correspondence \[1\] in condensed matter physics has been attracted lots of attentions in recent years. In particular it has provided gravitational descriptions for systems which exhibit superconductor/superfluid \[2-4\] phases. Since in condensed matter physics we are typically dealing with systems at finite charge and temperature, in the context of the AdS/CFT correspondence the dual gravity descriptions should be given by gravitational models which admit charged black holes as vacuum solutions.

Indeed the simplest model may be provided by an Einstein-Maxwell theory coupled to a charged scalar field. For this model, it has been shown that the charged black holes become unstable to develop scalar hair for sufficiently low temperature \[2\] and moreover the $U(1)$ symmetry is broken near the black holes horizon. Within the framework of the AdS/CFT correspondence, the charged scalar field corresponds to an operator which carries the charge of the global $U(1)$ symmetry. Having non-zero hair corresponds to the fact that the dual operator has non-zero expectation value and the global $U(1)$ symmetry is broken as well. This phenomena may be interpreted as a second order phase transition between conductor and superconductor phases. This interpretation has been supported by making use of the behavior of AC conductivity in these phases.

Using a five dimensional AdS soliton in an Einstein-Maxwell-charged scalar field the authors of \[5\] have constructed a model describing an insulator/superconductor phase transition at zero temperature. Actually since in this model the normal phase is described by an AdS soliton where the system exhibits mass gap \[6\] the dual field theory is in an insulator phase. On the other hand for sufficiently large chemical potential the AdS soliton becomes unstable to forming scalar hair which corresponds to the fact that the dual theory is in a superconductor phase. Holographic insulator/superconductor system with back reactions has also been studied in \[7\].

The aim of the present article is to extend the holographic insulator/superconductor for the case where the AdS soliton develops a vector hair. In fact holographic superconductors with vector hair, known as p-wave holographic superconductors, have been first studied in \[8\] and further explored in \[9,10\]. The simplest example of p-wave holographic superconductors may be provided by an Einstein-Yang-Mills theory with $SU(2)$ gauge group \[11\]. In this model the electromagnetic $U(1)$ gauge symmetry is identified with the abelian $U(1)$ subgroup of the $SU(2)$. The other components of the $SU(2)$ gauge field play the role of charged fields whose non-zero expectation values break the $U(1)$ symmetry leading to a phase transition in the dual field theory.

In this paper following \[8\] we will consider a five dimensional $SU(2)$ Einstein-Yang-Mills theory with a negative cosmological constant whose action is given by

\[ S = \int d^5 x \sqrt{-g} \left[ \frac{1}{2} (R - \Lambda) - \frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} \right], \tag{1.1} \]

where $F_{\mu\nu}$ is the field strength of the $SU(2)$ gauge field. In our notation the negative cosmological constant is given by $-12/L^2$ with $L = 1$. The equations of motion coming from

\[1\] See also \[11\] for p-wave holographic superconductors in the context of five dimensional gauged supergravity.

\[2\] In our notation we have set the five dimensional Newton constant to one, $\kappa_5 = 1$. 

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the above action are

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - 6 g_{\mu\nu} = T_{\mu\nu}, \quad \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} F^a{}^{\mu\nu}) + q f^{abc} A^b_\mu F^c{}^{\mu\nu} = 0, \]  

(1.2)

where

\[ T_{\mu\nu} = F^a_{\mu\rho} F^a{}^{\rho\nu} - \frac{1}{4} g_{\mu\nu} F^a_{\rho\sigma} F^{a}{}^{\rho\sigma}, \]  

(1.3)

with \( F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + q f^{abc} A^b_\mu A^c_\nu. \)

The model has only one free parameter which in our notation is given by the gauge coupling \( q \). Whether or not the back reactions of the gauge field on the metric are important is controlled by \( q \). When \( q \) is large the effects of the gauge field on the geometry are negligible, while for finite \( q \) the gauge field back reactions are important and affect the geometry.

The equations of motion (1.2) support an AdS solitonic solution with zero gauge field and the metric which is given by

\[ ds^2 = \frac{1}{r^2 g(r)} dr^2 + r^2 (-dt^2 + dx^2 + dy^2) + r^2 g(r) d\chi^2, \quad g = 1 - \frac{r_0^4}{r^4}. \]  

(1.4)

Essentially this solution can be obtained from a five dimensional AdS black hole solution by making use of two Wick rotations. Physically, this solution corresponds to a five dimensional solution compactified on a circle with anti-periodically boundary condition for the fermions along the compact direction. Since the geometry ends at \( r = r_0 \) where \( g_\mu \) is non-zero, the background provides the gravity description of a three dimensional field theory with a mass gap.

It is worth mentioning that unlike the AdS black hole that finiteness of the gauge potential at the horizon prevents to have a non-zero \( A_t \) at the horizon, in the present case one could still have the above solution with a constant non-zero gauge potential \( A_t = \mu \).

The theory given by the action (1.1) admits another analytic solution with non-zero gauge field. The corresponding solution is, indeed, an AdS Reissner-Nordström black hole which carries the charge of the \( U(1) \) abelian subgroup of the \( SU(2) \) gauge group. The solution is given by

\[ ds^2 = \frac{dr^2}{g} - g dt^2 + r^2 (dx^2 + dy^2 + dz^2), \quad A = \rho \left( 1 - \frac{1}{r^2} \right) \sigma^3 dt, \]  

(1.5)

where \( g = r^2 - \frac{1+\rho^3/3}{r^2} + \frac{\rho^2}{3r^2} \), and \( \sigma^3 \) is the generator of \( U(1) \) subgroup. Here we have normalized the coordinates such that the horizon is located at \( r = 1 \). In this notation the Hawking temperature of the black hole is \( T = \frac{2-\rho^2/3}{2\pi} \).

The organization of this paper is as follows. In the next section we study p-wave holographic insulator/superconductor in the probe limit where we will also compute the AC conductivity of the model. In section three we first consider fully back reacted solutions and then we study the phase structure of the theory. In section four, for completeness, we will study zero temperature limit of p-wave holographic superconductor in four dimensions. The last section is devoted to conclusions.

\footnote{We would like to thank referee for his/her comment on this point}

\footnote{We denote the generators of \( SU(2) \) by \( \sigma^i \) for \( i = 1, 2, 3 \).}
2 Insulator/Superconductor transition at probe limit

In this section we will consider dynamics of the $SU(2)$ gauge field on the background (1.4). In general turning on a gauge field, the background metric (1.4) gets corrections due to back reactions of the gauge field back. Nevertheless at first order we will consider the case where the back reactions are negligible so that the gauge field may be treated as a probe. This can be done with the assumption that $q$ is sufficiently large $[9,10]$.

2.1 Phase transition

To study the insulator/superconductor phase transition, following $[9]$, we will consider the following ansatz for the gauge field:

$$A = \phi(r) \sigma^3 dt + \psi(r) \sigma^1 dx.$$  \hspace{1cm} (2.1)

Since we are in the probe limit, the back reactions of this gauge field on the metric (1.4) are negligible. Note that in this ansatz the $t$-component of the gauge field represents the $U(1)$ gauge field, while the second term plays the role of the charged field whose condensation breaks the $U(1)$ gauge symmetry.

Plugging this ansatz into the equations of motion (1.2) one arrives at

$$\phi'' + \left(\frac{3}{r} + \frac{g'}{g}\right)\phi' - \frac{\psi^2}{r^4 g} \phi = 0, \quad \psi'' + \left(\frac{3}{r} + \frac{g'}{g}\right)\psi' + \frac{\phi^2}{r^4 g} \psi = 0.$$  \hspace{1cm} (2.2)

The aim is to solve the above equations, though in general it is difficult to do it analytically and therefore we will utilize the numerical method. To proceed we note that the above equations are invariant under the rescaling $r \rightarrow r_0 r, \phi \rightarrow r_0 \phi$ and $\psi \rightarrow r_0 \psi$, so that $r_0$ may be dropped from the equations and leaves us to work with dimensionless quantities.

Near the boundary where $r \rightarrow \infty$ the solutions of the equations behave as

$$\phi = \mu - \frac{\rho}{r^2}, \quad \psi = \psi_0 + \frac{\psi_1}{r^2}. \hspace{1cm} (2.3)$$

From dictionary of the AdS/CFT correspondence $[12,13]$, it is known that $\mu$ and $\rho$ correspond to the chemical potential and charge density in the boundary theory, while $\psi_0$ and $\psi_1$ may be identified as a source and the expectation value of the dual operator, respectively. Since we are interested in the case where the dual operator is not sourced, we set $\psi_0 = 0$. Also up to a normalization one has $\langle O \rangle \sim \psi_1$.

On the other hand imposing the finiteness condition near the tip where $r \rightarrow 1$, leads to the following series expansions for the $\psi$ and $\phi$ fields (see also $[5]$)

$$\psi = \alpha_0 + \alpha_1 \left(1 - \frac{1}{r}\right) + \alpha_2 \left(1 - \frac{1}{r}\right)^2 \cdots,$$

$$\phi = \beta_0 + \beta_1 \left(1 - \frac{1}{r}\right) + \beta_2 \left(1 - \frac{1}{r}\right)^2 \cdots. \hspace{1cm} (2.4)$$

See also $[14,15]$. 

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5See also $[14,15]$. 

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With these boundary conditions one may solve the equations (2.2) to find the expectation value of the dual operator as a function of the chemical potential. In fact given the initial values for the gauge field near the tip, one may find the boundary values of the gauge field by making use of the shooting method. Indeed doing so, one finds that the solution is unstable to develop a hair for the chemical potential bigger than a critical value, $\mu > \mu_c$.

More precisely using the Mathematica we have numerically solved the equations leading to the plot depicted in figure 1. In our numerical method we have chosen $\phi$ at the tip as the shooting parameter and fixed $\psi$ at the tip by making use of the asymptotic condition $\psi_0 = 0$.

From the figure 1 (right) we observe that the expectation value of the dual operator is non-zero for $\mu > \mu_c = 2.26$. Having non-zero expectation value spontaneously breaks the $U(1)$ symmetry leading to a phase transition. To find the nature of the phase transition, it is illustrative to find the behavior of the charge density in terms of the chemical potential. Indeed since the charge density may be identified with the derivative of free energy, $\rho = \partial F/\partial \mu$, its behavior around the critical chemical potential can indicate the nature of the phase transition. Indeed, as it is evident from figure 1 (right), it turns out that the phase transition is second order. In this sense the behavior of the system is similar to that for the s-wave one [5].

Since the geometry (1.4) is dual to a three dimensional theory with a mass gap, it is natural to treat the phase of the system before condensation as an insulator phase. Therefore the obtained phase transition may be interpreted as an insulator/superconductor phase transition. To see whether the new phase behaves as a superconductor it is useful to study the response of the theory to an external magnetic field. In other words one may compute the AC conductivity of the system.

### 2.2 AC conductivity

To study the conductivity of the theory we will consider an extra magnetic field along $y$ direction. From gravity description point of view this can be done by turning on a non-zero gauge field in $y$ direction. To do so, one may consider the following ansatz for the gauge
field along $y$ direction

$$A_y = A(r)e^{i\omega t}\sigma^3. \tag{2.5}$$

Since we are working in the probe limit, we will consider the back reactions of the new component of the gauge field neither on the metric nor on the other components of the gauge field. As the result it is sufficient to consider the equation of motion for $A_y$ in the background generated by the metric (1.4) together with the solutions of $\psi$ and $\phi$ given by the equations (2.2).

The corresponding equation of motion for $A_y$ is given by

$$A'' + \left( \frac{3}{r} + \frac{g'}{g} \right) A' + \frac{\phi^2}{r^4g} \left( \omega^2 - \psi^2 \right) A = 0. \tag{2.6}$$

For large $r$ where we approach the boundary the behavior of the gauge field is

$$A = A_0 + \frac{A_1}{r^2}. \tag{2.7}$$

With this notation the conductivity in $y$ direction is given by (see for example [10])

$$\sigma_{yy} = \frac{-iA_1}{\omega A_0}. \tag{2.8}$$

Using the numerical solution we had found for $\psi$ and $\phi$ in the previous subsection we can find the behavior of the AC conductivity in terms of the energy $\omega$. Actually we expect that the imaginary part of the conductivity to have a non trivial behavior at $\omega = 0$. More precisely we would expect to get a delta function support for the real part of the conductivity at $\omega = 0$. Indeed this is what we get from our numerical solution.

In the figure 2 we have plotted the behavior of the conductivity in terms of the energy when the condensation is non-zero. At $\omega \to 0$ we find a pole showing that we have an infinite conductivity as expected for the superconductor phase. Note that the pole structure shown in figure 2 is repeated periodically. This is due to an infinite tower of vector modes.

So far we have seen that the model based on the action (1.1) exhibits a p-wave insulator/superconductor phase transition as we vary the chemical potential. On the other hand as we have already mentioned in the introduction the equations of motion (1.2) admit another analytic solution; the RN black hole solution. Therefore one may expect that the AdS solitonic solution (1.4) could decay to an AdS black hole leading to another phase transition in the model. This is indeed the confinement/deconfinement first order phase transition considered in [6]. In our context from three dimensional theory point of view this corresponds to an insulator/conductor phase transition. It is then natural to look for another conductor/superconductor phase transition in the model.

We note, however, that to explore the whole picture of the phase structure of the model one has to consider the back reactions of the gauge field on the metric as well. In other words we will have to go beyond the probe limit. This is, in fact, what we will do in the next section.\footnote{The complete phase diagrams for s-wave holographic insulator/superconductor system has been studied\footnote{where the authors have considered the back reactions of the gauge and scalar fields.}}
As a final remark, before going to the back reactions analysis, it is an instructive exercise to study the insulator/superconductor phase transition in the presence of a constant DC current (see for example \cite{16}). To do so we will consider the following ansatz for the gauge field

\[ A = \phi(r)\sigma^3 dt + \psi(r)\sigma^1 dx + A(r)\sigma^3 dy. \] (2.9)

Note that although we are still working in the probe limit where the back reactions of the gauge field on the metric are negligible, all components of the gauge fields are treating in the same footing. This means that one should take in to account the effects of \( A_y \) component on the other components of the gauge field. Indeed it turns out that the equation of motion for \( \psi \) is modified as follows

\[ \psi'' + \left( \frac{3}{r} + \frac{g'}{g} \right) \psi' + \frac{\phi^2 - A^2}{r^4 g} \psi = 0, \] (2.10)

while for \( \phi \) it remains unchanged. Finally the last equation is given by (2.6) with \( \omega = 0 \).

It is then the aim to solve these equations to find the behavior of the system. Actually we find that the model is again unstable against developing hair leading to the symmetry breaking when the chemical potential is bigger than a critical value. We note, however, that in the present case the critical value of the chemical increases as we increase the background DC current (see figure 3). In other words having a non-zero background field makes it harder to have insulator/superconductor phase transition.

3 Beyond probe limit

So far we have been considering the model in the probe limit where the back reactions of the gauge field were negligible. The aim of this section is to study the effects of the gauge field on the background metric. We, note, that back reacted solutions of the five dimensional gravity
Figure 3: The behavior of the expectation value of the dual operator in terms of the chemical potential in the present of a constant background field $A_0$. The most left curve is for the case when $A_0 = 0$, the others appear as we increase the field. Therefore as we turn on the background field, the condensation is harder to be formed.

(1.1) have been studied in [17] (see also [18]) where the authors have numerically constructed asymptotically AdS charged black holes of the model. It was also shown that, for sufficiently low temperature, the model develops vector hair. Generically this corresponds to a second order phase transition in the dual field theory. The aim of this section is to further study the back reacted solutions of the theory given by the action (1.1).

3.1 AdS soliton

In this subsection we will study the back reactions of the gauge field on the AdS solitonic solution considered in the previous section. To proceed, we start with the following ansatz for the metric and gauge field

$$ds^2 = \frac{dr^2}{g(r)} + r^2 \left( - f(r) dt^2 + h(r) dx^2 + dy^2 \right) + g(r) e^{-\chi(r)} d\eta^2,$$

$$A = \phi(r) \sigma^2 dt + \psi(r) \sigma^1 dx.$$  \hspace{1cm} (3.1)

Using this ansatz the Einstein equations of motion read

$$\frac{f''}{f} + \frac{f'}{f} \left( \frac{3}{r} - \frac{f'}{2f} + \frac{h'}{2h} + \frac{g'}{g} - \frac{\chi'}{2} \right) = \frac{2\phi'^2}{r^2 f} + \frac{2q^2 \psi^2 \phi^2}{r^4 fgh},$$

$$\frac{h''}{h} + \frac{h'}{h} \left( \frac{3}{r} + \frac{f'}{2f} - \frac{h'}{2h} + \frac{g'}{g} - \frac{\chi'}{2} \right) = -\frac{2\psi'^2}{r^2 h} + \frac{2q^2 \psi^2 \phi^2}{r^4 fgh},$$

$$\chi'' + \chi' \left( \frac{2}{r} - \frac{3g'}{2g} - \frac{\chi'}{2} \right) - \frac{1}{gr^2} (r^2 g - r^4)'' = -\frac{\psi'^2}{r^2 h} + \frac{\phi'^2}{r^2 f} + \frac{3q^2 \psi^2 \phi^2}{r^4 fgh},$$

$$\frac{f'h'}{fh} + \frac{(fh)'}{fh} \left( \frac{1}{r} + \frac{g'}{g} - \frac{\chi'}{r} \right) = \frac{3\chi'}{r} = \frac{4q^2 \psi^2 \phi^2}{r^4 fgh}. \hspace{1cm} (3.2)$$
while from the equations of motion of gauge field one finds

\[
\phi'' + \phi' \left( \frac{1}{r^2} \frac{f'}{2f} + \frac{h'}{2h} + \frac{g'}{g} - \frac{\chi'}{2} \right) - \frac{q^2 \psi^2 \phi}{r^2 fg} = 0,
\]
\[
\psi'' + \psi' \left( \frac{1}{r^2} \frac{f'}{2f} - \frac{h'}{2h} + \frac{g'}{g} - \frac{\chi'}{2} \right) + \frac{q^2 \phi^2 \psi}{r^2 hg} = 0.
\] (3.3)

Now the aim is to solve the above equations though in general it requires numerical analysis. It is evident that the AdS solitonic solution considered in the previous section is, indeed, a solution of the above equation for \( \psi = \phi = 0 \).

Near the boundary where \( r \to \infty \) the behavior of the gauge field is as follows

\[
\phi = \mu - \frac{\rho}{r^2}, \quad \psi = \psi_0 + \frac{\psi_1}{r^2}.
\] (3.4)

Of course we set \( \psi_0 = 0 \) to avoid having source for the dual operator in the field theory. The main goal of this subsection is to find the behavior of \( \psi_1 \) as a function of the chemical potential \( \mu \) for given electric charge \( q \). For this purpose we utilize the numerical method using Mathematica. Essentially the procedure is very similar to what we have done in the previous section, though here we have to deal with more equations.

To proceed we assume that at the tip, \( r = r_0 \), the function \( g \) goes to zero, i.e. \( g(r_0) = 0 \).

On the other hand one may set \( r_0 = 1 \) by making use of an evident scaling symmetry of the equations

\[
r \to \lambda r, \quad (t, x, y, \eta) \to \lambda^{-1}(t, x, y, \eta), \quad g \to \lambda^2 g, \quad (\phi, \psi) \to \lambda(\psi, \psi).
\] (3.5)

Therefore by considering a series solution near the tip we are left with five parameters given \( f(1), h(1), \chi(1) \) and \( \phi(1), \psi(1) \). We note, however, that since the equations are symmetric under a constant shift in \( \chi \), we may set \( \chi = 0 \) at tip. On the other hand due to the scaling symmetries

\[
f \to a^2 f, \quad \phi \to a\phi,
\] (3.6)

and

\[
h \to b^2 h, \quad \psi \to b\psi,
\] (3.7)

one may fix \( f(1) \) and \( h(1) \) by setting, for example, \( f = h = 1 \) at the boundary. Altogether we are left with two parameters \( \psi(1) \) and \( q \). Therefore for a given \( q \) we have a one parameter family of solutions parametrized by the value of \( \psi \) at the tip.

To summarize we can solve the equations by treating \( \phi(1) \) as the shooting parameter for a fixed \( q \) and given \( \psi(1) \). Then we can scan the moduli space of parameters by changing \( q \). Doing so we find the following picture.

For generic \( q \) the behaviors of \( \psi \) and \( \rho \) as the functions of the chemical potential are indeed the same as those we had in the probe limit (see figures[1]). More precisely there is a critical chemical potential above which the system becomes unstable to develop vector hair leading to a second order phase transition in the dual field theory.

We, note, however that in the present case the critical value of the chemical potential depends on the value of \( q \). Indeed as we increase \( q \) the value of the chemical potential, where the phase transition occurs, decreases. An interesting phenomena we encounter is that there
is a critical value for \( q \) (\( q \approx 1.94 \)) below which the system exhibits qualitatively different behaviors. This phenomena is also seen in the AdS charged black hole which we will discuss in the next subsection. Actually the situation is very similar to the s-wave holographic insulator/ superconductor system studied in [7]. We will back to this point latter.

### 3.2 AdS charged black hole

The model given by the action (1.1) admits AdS charged black holes when the back reactions of the gauge field are taken into account. These solutions have been numerically studied in [17] where the authors have also shown that the solutions are unstable to develop a vector hair for sufficiently low temperature. In this subsection, for completeness of our study, we will rederive the results of [17]. Of course our ansatz are slightly different from that in [17], though our final results are similar them, up to a normalization.

Motivated by [24] we consider the following ansatz

\[
\begin{align*}
    ds^2 &= \frac{dr^2}{g(r)} + r^2 (h(r)dx^2 + dy^2 + dz^2) - g(r)e^{-\chi(r)} dt^2,
    \\
    A &= \phi(r) \sigma^3 dt + \psi(r) \sigma^1 dx.
\end{align*}
\]

Plugging the above ansatz into the equations of motion (1.2) for thr metric one finds

\[
\begin{align*}
    \frac{g'}{g} \left( \frac{3}{2r} + \frac{h'}{2h} \right) - \left( \frac{3}{r^2} + \frac{4h'}{rh} + \frac{h''}{h} \right) + \frac{6}{g} = \frac{e^x}{4g} \phi'^2 + \frac{1}{4r^2 h^2} \psi'^2 + \frac{q^2 e^x}{4r^2 g^2 h^2} \phi^2 \psi^2,
    \\
    \frac{h''}{h} + \frac{h'}{h} \left( \frac{3}{r} + \frac{g'}{g} - \frac{\chi'}{2} \right) = - \frac{\psi'^2}{2r^2 h^2} + \frac{q^2 e^x}{2r^2 g^2 h^2} \phi^2 \psi^2,
    \\
    \frac{h'}{h} \left( \frac{1}{r} + \frac{g'}{g} - \chi' \right) - \frac{3\chi'}{2r} = \frac{q^2 e^x}{r^2 g^2 h^2} \phi^2 \psi^2,
\end{align*}
\]

while for the gauge field we get

\[
\begin{align*}
    \phi'' + \phi' \left( \frac{3}{r} + \frac{\chi'}{2} + \frac{h'}{h} \right) - \frac{q^2 \psi'^2}{r^2 g^2 h^2} \phi = 0, \\
    \psi'' + \psi' \left( \frac{1}{r} + \frac{g'}{g} - \frac{h'}{h} - \frac{\chi'}{2} \right) + \frac{e^x q^2 \phi^2}{g^2} \psi = 0.
\end{align*}
\]

For finite \( q \) where the back reactions of the gauge field are important the model admits AdS charge black holes. In particular when we set \( \psi = 0 \) the equations admit an analytic solution which is indeed the Reissner-Nordstrom AdS black hole given by (1.5).

For generic values of the gauge field, beside the RN black hole, there is no other well known analytic solution, though asymptotically AdS charged black holes have been numerically constructed in [17] (see also [18]). Using the same procedure as that in the previous sections we may solve the equations numerically. To proceed we note that for large \( r \) as we approach the boundary the behavior of the gauge field is as follows

\[
\phi = \mu - \frac{\rho}{r^2}, \quad \psi = \psi_0 + \frac{\psi_1}{r^2}.
\]

Again we will set \( \psi_0 = 0 \). On the other hand near the horizon where \( g \) vanishes we assume \( \phi \) is zero too. The horizon may be set to be located at \( r = 1 \). By making use of the
Figure 4: The behavior of the expectation value in terms of temperature for \( q = 3.5 \) where we get \( T_c = 0.226 \).

time reparamerization we can set \( \chi \) to zero at the horizon. On the other hand utilizing the following symmetries

\[
r \rightarrow \lambda r, \quad g \rightarrow \lambda^2 g, \quad (\phi, \psi) \rightarrow \lambda (\phi, \psi),
\]

and

\[
h \rightarrow \xi h, \quad \psi \rightarrow \xi \psi,
\]

one may fix the values of \( g \) and \( h \) at the horizon by setting, for example, \( g = r^2 \) and \( h = 1 \) at the boundary where we recove the AdS solution.

Taking into account that the value of \( \phi \) at the horizon is treated as the shooting parameter we are left with two parameters given by \( q \) and \( \psi \) at the horizon. Therefore for a fixed \( q \) we get one parameter family of solutions parametrized by the value of \( \psi \) at the horizon.

For generic \( q \) one finds that the solution is unstable to develop a vector hair for sufficiently low temperature where the \( U(1) \) gauge symmetry is also broken. This corresponds to a second order conductor/superconductor phase transition from field theory point of view (see figure \[4\] \[17\]).

We note that the critical temperature depends on the value of \( q \). Indeed as we increase \( q \) the critical temperature increases too. It is worth mentioning that there is a critical value for \( q \) where the system shows qualitatively different behaviors. In the normalization of our numerical computations it happens at \( q \approx 1.94 \). This is exactly the behavior obtained in \[17\] where the authors have found that there is a critical \( q \) where the transition becomes first order. This has to be compared with behavior we have found in the previous section. In the following section we will discuss about this point.

### 3.3 Phase structure

In this subsection we would like to explore the phase structure of the model using the results we have found so far. To proceed we note that the equations of motion support two distinctive solutions; AdS soliton and AdS charged black hole. In each case the solution becomes unstable to develop a vector hair as we change the parameters of the model. In particular we have seen that when we change the chemical potential there is a critical point
Figure 5: The qualitative phase structure of the model. As we change $q$ the positions of points A and B are changed and therefore the shape of different regions get modified. There is a critical point where A and B are on top of each other. For sufficiently small $q$ there is a possibility to have a first order phase transition where a superconductor becomes and insulator as we decrease temperature as shown in the left picture. The figure “a” is the phase structure in the probe limit, while the others are with back reactions.

above which the AdS soliton develops a vector hair. From field theory point of view this phenomena corresponds to the insulator/superconductor second order phase transition. On the other hand when we have a charged black hole the solution is unstable to generate a vector hair for sufficiently low temperature which is indeed the holographic realization of the second order conductor/superconductor phase transition.

On top of these we note that using the Euclidean solitonic solution one may associate a temperature to the solution which could be any value for a given chemical potential. Therefore as we draw the phase structure in the $T - \mu$ space the curve which separates insulator phase from the superconductor phase should be a line parallel to the $T$ axes. On the other hand when the period of the Euclidean time of the AdS soliton becomes the same as that in the AdS charged black hole the free energy of the system shows that the favored solution is the AdS charged black hole and thus we have a first order phase transition from AdS soliton to AdS charged black hole [6]. So altogether we get four different phases as shown in figure 5-a.

It should be mentioned that in order to find the above picture for the phase structure of the model the results we have gathered from probe limit considerations were sufficient. We note, however, that in order to fully explore different phases of the model in more detail one needs to go beyond the probe limit. In particular taking into account the back reactions we observe two new features in the phase diagram of the model.

The first observation is that for large chemical potenial as we decrease temperature the favored phase is soliton superconductor. In other words the phase diagram gets modifed as figure 5-b. In particular we cannot have a phase describing hairy charged black hole at zero
temperature which could have been the case if the phase diagram had been given by figure 5a. We will back to this point in the next section. It is important to note that the figure 5 should be treated as a qualitative description and the interfaces between different phases have been plotted schematically. To explore the precise structure of the phase diagram higher numerical precision is needed, specially for small $q$. We hope to back to this point in our future work.

On the other hand there are two special points, named by A and B in figure 5 which in order to understand their physical significant, it requires to study the system beyond the probe limit.

Actually our numerical computations show that the positions of these two points labeled by $\mu_A$ and $\mu_B$ change as we are changing $q$. More precisely, when we decrease $q$, $\mu_A$ and $\mu_B$ increase as well and eventually they meet each other as we approach the critical point $q = 1.94$ where one gets $\mu_B \approx \mu_A \approx 1.87$ (see figure 6).

As the result when we change the parameter $q$, different regions in the phase diagram of the model get modified and in particular we reach a situation where two points A and B are on top of each other as shown in figure 5-c.

This is indeed the critical point we have found in our numerical calculations in the previous subsections where the model shows qualitatively different behavior. This is also the situation which has been observed in [17] (and also in [7] for s-wave) where it was shown that the phase transition becomes first order at the critical point.

As we further decrease $q$ we observe that while both $\mu_A$ and $\mu_B$ increase, the point A passes though the point B (see figure 5-d). In this case we encounter a new phase transition. Actually as we decrease temperature there is a range of $\mu$ between which the superconductor becomes an insulator via a first order phase transition. This phase transition has been also seen in the s-wave consideration in [7]. Going further it seems that the phase where we have soliton superconductor becomes smaller and smaller and eventually disappears from the phase diagram, though due to the uncertainty of our numerical results, we have not been able to explore the situation exactly. In particular for low temperature (for small enough $q$, i.e. $q \approx 0.86$) the numerical solution develops a singularity and one has to study $T \to 0$ limit of hairy charged black hole more carefully. This is indeed what we will do in the next section.
4 Zero temperature limit

So far we have been considering the phase transition between AdS soliton and AdS charged black hole with non-zero temperature. It is then natural to pose the question what happens when we send the temperature of the AdS charged black hole to zero? In general for a charged black hole sending temperature to zero we will end up with an extremal black hole whose near horizon geometry develops an \( AdS_2 \) throat with non-zero entropy. Therefore if we would like to study holographic superconductors at zero temperature the extremal black hole cannot provide the gravity dual descriptions. In fact to get an eligible background one must have a geometry with zero size horizon ensuring that the ground state is a single state (entropy is zero).

Indeed zero temperature limit of s-wave holographic superconductors in three dimensions has been studied in [19] (see also [20–23]) where it was shown that the corresponding hairy solution develops an \( AdS_4 \) geometry at the core of the space time. More precisely they have found numerical solutions which interpolate between hairy \( AdS_4 \) and \( AdS_4 \) geometries. The Zero temperature limit of the holographic p-wave superconductors in three dimensions has also been studied in [24].

To complete our discussions of p-wave holographic superconductors in four dimensions, in this section we will consider zero temperature limit of the four dimensional p-wave holographic superconductor. Actually to do this one needs to solve the equations of motion (3.9) with a particular boundary condition ensuring that the resultant geometry would have zero size horizon. To proceed we consider the following behaviors for the parameters of the ansatz (3.9) in \( r \to 0^+ \) limit.

\[
\phi \sim \phi_0(r), \quad \psi \sim \psi_0 - \psi_1(r), \quad \chi \sim \chi_0 - \chi_1(r), \quad g \sim r^2 + g_1(r), \quad h \sim h_0 + h_1(r),
\]

(4.1)

with the assumption that \( \phi_0, \psi_1, \chi_1, g_1 \) and \( h_1 \) go to zero sufficiently fast in the limit of \( r \to 0^+ \). Plugging these behaviors into the corresponding equations of motion, at leading order, one finds

\[
\phi = \phi_0 e^{-\frac{\alpha}{r^2}}, \quad \chi = \chi_0 - \frac{e^{\chi_0} \alpha \phi_0^2}{6r^2} e^{-\frac{2\alpha}{r^2}}, \quad g = r^2 - \frac{e^{\chi_0} \alpha \phi_0^2}{6r^2} e^{-\frac{2\alpha}{r^2}},
\]

\[
\psi = \psi_0 \left( 1 - \frac{e^{\chi_0} q \phi_0^2}{4r \alpha^2} e^{-\frac{2\alpha}{r^2}} \right), \quad h = h_0 \left( 1 + \frac{e^{\chi_0} \phi_0^2}{8r} e^{-\frac{2\alpha}{r^2}} \right).
\]

(4.2)

where \( \alpha = q \psi_0 / h_0 \). On the other hand for large \( r \) one impose the following asymptotic conditions for the gauge field components

\[
\phi = \mu - \frac{\rho}{r^2}, \quad \psi = \psi_0 + \frac{\psi_1}{r^2}.
\]

(4.3)

Now the aim is to solve numerically the equations of motion with the above boundary conditions. To do so, using the symmetries of the model we can set \( \phi_0 = h_0 = 1 \) and \( \chi_0 = 0 \).

\footnote{Note that the insulator/superconductor phase transition we have considered in the previous section was also occurred at zero temperature. We note, however, that this has not to be compared with the present case. This is because in the previous case to get the solitonic solution we have put anti-periodic boundary condition for the fermions along the compact direction which is not thr case here.}
Figure 7: The behavior of $\phi, g/r^2, \psi, \chi$ and $h$ as functions of $r$. The function is plotted for $q = 0.93, 1, 1.12, 1.22, 1.32, 1.41$ which have been shown by thick, dashed, dotted, red, blue and green curves respectively.

On the other hand we would like to have a solution which is asymptotically $AdS_5$ without hair. Therefore we are interested in the case where $\psi$ vanishes asymptotically. This can be done by imposing a relation between $\alpha$ and $q$. For example for $3 < q < 7$ one finds $\alpha = 0.8 + 0.49q$.

It is straightforward to solve the equations numerically using Mathematica. The final results are shown in figure 7. As we see the solutions approach $AdS_5$ geometries at IR with non-zero hair while they are asymptotically $AdS_5$ without hair. Therefore from field theory point of view in IR we get an emergent CFT broken by the expectation value of the operator that is dual to $\psi$.

It is worth noting that as we approach $q \to \sqrt{3}/2$ limit we lose the accuracy of our numerical solutions. This is the limit where we recover the extremal limit of the charged AdS black hole. Therefore effectively we will have to solve the equations on the $AdS_2$ geometry and $\sqrt{3}/2$ is, indeed, the Breitenlohner-Freedman bound for mass in $AdS_2$ space time.

Having found hairy solutions it is natural to compute the conductivity for the model. To study the conductivity we consider small fluctuations for the metric and $A_y$ component of the gauge field as follows

$$g_{ty} = \epsilon f(r)e^{-\omega t}, \quad A_y^3 = \epsilon \varphi(r)e^{-\omega t}. \quad (4.4)$$

where $\epsilon$ is a small number controls the effects of the perturbations. As far as the equations of motion for gauge field and metric are concerned, since we are working in the probe limit, the effects of the perturbations on the equations of motion of the other components are negligible and thus one can still use the equations (3.9). On the other hand for the perturbations, at
first order in $\epsilon$, the equations of motion are given by

$$\varphi'' + \varphi' \left( \frac{1}{r} + \frac{g'}{g} - \frac{\chi'}{2} + \frac{h'}{h} \right) + \varphi \left( \frac{e\chi \omega^2}{g^2} - \frac{q^2 \psi^2}{r^2 gh^2} \right) + \frac{e \chi f}{g} \left( \varphi'' + \varphi' \left( \frac{1}{r} + \frac{f'}{f} + \frac{\chi'}{2} + \frac{h'}{h} \right) \right) \right) - \frac{g^2 \psi^2}{r^2 gh^2} \varphi = 0,$$

(4.5)

By making use of the equation of motion for $\phi$, (3.9), one arrives at

$$\varphi'' + \varphi' \left( \frac{1}{r} + \frac{g'}{g} - \frac{\chi'}{2} + \frac{h'}{h} \right) + \varphi \left( \frac{e\chi \omega^2}{g^2} - \frac{q^2 \psi^2}{r^2 gh^2} - e \chi \phi'^2 \right) = 0.$$

(4.6)

To find the conductivity one needs to find the asymptotic behavior of gauge field $\varphi$. In fact for large $r$ where we find $\varphi = \varphi_0 + \frac{\varphi_1}{r^2}$ the conductivity is given by

$$\sigma = -\frac{i \varphi_1}{\omega \varphi_0}.$$

(4.7)

On the other hand at $r \to 0$, using the results of the last section and imposing in coming boundary condition at $r \to 0$ we find

$$\varphi(r \to 0) = \tilde{\varphi}_0 e^{i \tilde{\omega} / \sqrt{r}}$$

(4.8)

where $\tilde{\omega} = \sqrt{\omega^2 - \alpha^2}$.

Taking into account that the solutions we have found for the parameters of the anstaz (3.8) are real and using the equation (4.6) together with its complex conjugate we can show that the following expression is a constant of motion in the sense that it is $r$ independent

$$\text{Im} \left( r h g e^{-\chi/2} \phi^* \varphi' \right).$$

(4.9)

Therefore its value at the horizon is the same as that at the boundary. Evaluating the above expression at the boundary and near $r = 0$ one finds $2 \text{Im}(\varphi_0^* \varphi_1) = i |\tilde{\varphi}_0|^2 \text{Re}(\tilde{\omega})$. Plugging this result into (4.7) we arrive at

$$\text{Re}(\sigma) = \frac{\text{Re}(\tilde{\omega}) |\tilde{\varphi}_0|^2}{2 \omega |\varphi_0|^2},$$

(4.10)

that is zero for $\omega < \alpha$ leading to a gap. Indeed, this is exactly (up to a factor of 1/2) the same expression which found in [24] for three dimensional model.

5 Conclusions

In this paper we have studied holographic insulator/superconductor by making use of a five dimensional Einstein-Yang-Mills theory with an $SU(2)$ gauge field. The equations of motion supports both AdS soliton as well as AdS charge black hole solutions. Both solutions may
become unstable to forming a vector hair as we change the parameters of the solutions, including temperature and chemical potential.

Therefore altogether the model exhibits four different phases. From gravity point of view they correspond to AdS soliton, hairy AdS soliton, AdS charged black hole and hairy AdS charged black hole and from field theory point of view they are insulator, superconductor, conductor and another superconductor, respectively. As we see there are two phases where the system is in the superconductor phase. Actually these two phases can be distinguished by the behavior of the conductivity \[7\]; while in the solitonic case the real part of the conductivity has a series of delta functions, in the charged black hole case we just have a gap at low frequency.

Using the probe limit we have been able to study qualitatively the general features of the phase structure of the system, though to explore the phase diagram in more detail it was curtail to take into account the back reactions. Doing so, we have found that the model has rather a rich phase structure. In particular when \( q < 1.94 \) there is a range of \( \mu \) between which the system can show a first order phase transition from superconductor to insulator as we decrease temperature. Indeed this is a new phenomena in which has been first observed in the context of s-wave insulator/superconductor transition in \[7\]. It would be interesting to understand this phase transition better.

We have also studied zero-temperature limit of four dimensional p-wave holographic superconductor. We have numerically solved the corresponding equations of motion have shown that equations of motion admit smooth solutions which interpolate between hairy \( AdS_5 \) solution at IR and \( AdS_5 \) solutions without hair at UV. Having found a hairy \( AdS_5 \) solution at IR shows that there is an emergent CFT at IR such that the conformal symmetry is broken by the expectation value of the dual operator. Evaluating the conductivity for the solutions one finds that the real part of the conductivity vanishes for \( \omega < \alpha \) leading to a gap in the model.

Note:

All results of the present paper are based on several Mathematica codes we have written for each part. We should admit that to develop our Mathematica codes the one prepared by C. P. Herzog which is available at his homepage \((\text{http://wwwphy.princeton.edu/~cpherzog/})\) was illustrative.

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