INJECTIVE COGENERATORS AMONG OPERATOR BIMODULES

BOJAN MAGAJNA

Abstract. Given C*-algebras $A$ and $B$ acting cyclically on Hilbert spaces $H$ and $K$, respectively, we characterize completely isometric $A, B$-bimodule maps from $B(K, H)$ into operator $A, B$-bimodules. We determine cogenerators in some classes of operator bimodules. For an injective cogenerator $X$ in a suitable category of operator $A, B$-bimodules we show: if $A$, regarded as a C*-subalgebra of $A_ℓ(X)$ (adjointable left multipliers on $X$), is equal to its relative double commutant in $A_ℓ(X)$, then $A$ must be a W*-algebra.

1. Introduction

An operator space $Z$ is called injective provided that for each inclusion of operator spaces $X ⊆ Y$ every completely bounded map $ϕ$ from $X$ to $Z$ extends to a completely bounded map $̃ϕ$ from $Y$ to $Z$ with the same completely bounded norm. If, in addition, $X, Y$ and $Z$ are operator bimodules over a pair of C*-algebras $A$ and $B$ and $ϕ$ is a bimodule map, then $̃ϕ$ can be achieved to be an $A, B$-bimodule map too, and $Z$ is then an injective operator $A, B$-bimodule. This property, shared by $Z = B(H)$ (by the well known extension theorem [19], [29]), enables one to treat completely bounded bimodule maps into $Z$, to a certain extent, as linear functionals. But there is another, equally important, property of linear functionals that is not contained in the concept of injectivity, namely linear functionals separate points of the space. An operator $A, B$-bimodule $Z$ is called a cogenerator if for every operator $A, B$-bimodule $X$, every $x ∈ X$ and $ε > 0$ there exists an $A, B$-bimodule complete contraction $ϕ : X → Z$ such that $∥ϕ(x)∥ > ∥x∥ − ε$. If (in addition) $ϕ$ can be found such that $∥ϕ(x)∥ = ∥x∥$, then $Z$ is called a strict cogenerator. Not all injective operator bimodules are cogenerators: for example, it is not hard to show (but will not be needed here) that a continuous von Neumann algebra $A$ is not a cogenerator as an operator $A$-bimodule.

Before describing the content of the paper, let us mention briefly a wider context for motivation. In pure algebra duality for modules has been intensively studied at least since the appearance of paper [17] by Morita (see also [13] Chapter 19 for a more modern account). While the related notion of Morita equivalence has been vigorously studied also outside pure algebra, in particular in operator algebra theory (see [24], [6], [3] and the references there), the operator module duality itself has been considered so far only for the range bimodules of special nature ([18], [22], [16]).

(We note at this point that, unlike in pure algebra, there is no natural reduction of operator $A, B$-bimodules to, say, left modules over $A ⊗ B^{op}$, since there is in general no operator algebra norm structure known on $A ⊗ B^{op}$ turning all operator

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A, B-bimodules into operator left $A \otimes B^{op}$-modules.) However, the recent theory of multipliers of operator spaces ([2], [1], [4], [23]) enables us a natural definition of duality. Namely, according to Blecher [2, 5.4] for each operator space X there exist $C^*$-algebras $A_t(X)$ and $A_r(X)$ such that X is an operator $A_t(X), A_r(X)$-bimodule and each operator $A, B$-bimodule structure on X induces by some $*$-homomorphisms $A \rightarrow A_t(X)$ and $B \rightarrow A_r(X)$. This suggests the following definition.

Let $\alpha : A \rightarrow A_t(X)$ and $\beta : B \rightarrow A_r(X)$ be $*$-homomorphisms and denote by $A^c$ and $B^c$ the commutants of $\alpha(A)$ and $\beta(B)$ in $A_t(X)$ and $A_r(X)$, respectively. Then the dual of an operator $A, B$-bimodule $Y$ with respect to $X$ is the operator $A^c, B^c$-bimodule $CB_4(Y, X)_B$ (completely bounded $A, B$-bimodule maps from $Y$ to $X$), where $(c\phi)(y) := c\phi(y)$ and $(\phi d)(y) := \phi(y)d$ $(y \in Y, c \in A^c, d \in B^c)$.

Restricting in this definition $X$ to be an (injective) cogenerator will guarantee the existence of enough ‘functionals’. Then the bidual is an operator bimodule over $C^*$-algebras $A^{cc}$ and $B^{cc}$, which contain $A$ and $B$, respectively. We shall say that the pair $(A, B)$ admits a duality if for some injective cogenerator $X$ we have $A^{cc} = A$ in $A_t(X)$ and $B^{cc} = B$ in $A_r(X)$. (Here we have identified $A$ with $\alpha(A)$ and $B$ with $\beta(B)$.) In Section 5 we prove that $(A, B)$-admits a duality if and only if $A$ and $B$ are von Neumann algebras. For this we shall need some preparation.

Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert modules over $C^*$-algebras $A$ and $B$ (respectively), $X$ an operator $A, B$-bimodule and $\phi : B(\mathcal{K}, \mathcal{H}) \rightarrow X$ a completely contractive $A, B$-bimodule map. In Section 3 we will prove that, if $\mathcal{H}$ and $\mathcal{K}$ are cyclic with unit cyclic vectors $\xi \in \mathcal{H}$ and $\eta \in \mathcal{K}$ and $\|\phi(\xi \otimes \eta^*)\| = 1$, then $\phi$ is automatically completely isometric. In Section 4 we will show that if $X$ is injective, $\phi$ induces a completely contractive and completely positive $A$-bimodule map $\phi : B(\mathcal{H}) \rightarrow A_t(X)$, which is completely isometric if $\phi$ is completely isometric and $\mathcal{H}$ is cyclic. This will imply that $A_t(X)$ is a cogenerator for operator $A$-bimodules if $X$ is a cogenerator for operator $A, B$-bimodules. Further, if $\phi$ is injective, then the kernel of $\phi_t$ does not contain positive (nonzero) operators.

Now assume that $A \subseteq B(\mathcal{H})$, $\pi$ is a $*$-homomorphism from $A$ into an injective $C^*$-algebra $C$ (say, $C = A_t(X)$ when $X$ is injective) making $C$ an $A$-bimodule, and $\psi : K(\mathcal{H}) \rightarrow C$ is a bounded left $A$-module map, the kernel of which does not contain nonzero positive elements. In this situation we will prove in Section 6 that, if $\pi(A)$ is equal to its relative double commutant in $C$, $A$ must be a von Neumann algebra. A consequence of this (and some other results from Sections 5 and 6) is characterization of pairs $(A, B)$ admitting duality, deduced in Section 6.

In Section 5 we introduce a suitable class of categories of operator bimodules and characterize cogenerators in such categories. The most important cases are of course the categories of all and (if $A$ and $B$ are von Neumann algebras) of all normal operator $A, B$-bimodules.

It is natural to try to find the ‘smallest’ possible among cogenerators. Let us call a strict cogenerator $Z_0$ initial if $Z_0$ is contained completely isometrically as an $A, B$-bimodule in every strict cogenerator. If all families of disjoint cyclic representations of $A$ and $B$ are countable, then the category of all operator $A, B$-bimodules has an initial strict cogenerator (Section 7). This condition is quite restrictive since separable $C^*$-algebras satisfying it turn out to be a special subclass of type I $C^*$-algebras. On the other hand, an analogous condition for a von Neumann algebra
such that restriction to $X$ the infimum of $\|H\|_\infty$ be a Hilbert space multipliers obtained by Blecher, Effros and Zarikian [4] and Blecher and Paulsen countably initial if for each cogenerator $\mathbf{Z}$ there is a completely isometric $M,N$-bimodule map from $\mathbf{Z}$ into $X$ ($= \ell^\infty$-direct sum of countably many copies of $X$). If the centers of $M$ and $N$ are $\sigma$-finite, we shall specify in Section 7 a countably initial cogenerator in $M\text{NOM}_N$. If the center of $M$ is not $\sigma$-finite, we shall show that the category $M\text{NOM}_M$ has no countably initial cogenerators.

The results here show that bimodules of the form $B(K,H)$ (with $H,K$ cyclic) are in some sense minimal among (injective) cogenerators, and the duality for such range bimodules is developed in [22], [16]. Apart from cogenerators, bicommutation is the only aspect of duality studied in the present paper.

Besides the basic operator space theory (which can be found in the initial chapters of any of the books [5], [10], [19], [21]), we shall only need the notions of an injective envelope and of a multiplier of an operator space, which we now recall (more details are in [5] Chapter 4 and [19] Chapter 15).

2. Preliminaries and notation

Throughout the paper $A$ and $B$ are $C^*$-algebras. $A\text{OM}_B$ denotes the category of all operator $A,B$-bimodules and $\text{CB}_A(X,Y)_B$ the space of all completely bounded $A,B$-bimodule maps from $X$ to $Y$. The category of Hilbert $A$-modules (that is, Hilbert spaces with the column operator space structure on which $A$ acts as a $C^*$-algebra) is denoted by $A\text{HM}$, while $B_A(K,H)$ means the space of all (completely) bounded $A$-module maps from $K$ to $H$. $B(K,H)$ and $K(K,H)$ are the spaces of all bounded and all compact operators, respectively, from $K$ to $H$. For an operator $T \in B(K,H)$ and a cardinal $\mathbb{I}$ we denote by $T^{(\mathbb{I})}$ the direct sum of $\mathbb{I}$ copies of $T$ and, for a subset $S \subseteq B(K,H)$, $S^{(\mathbb{I})}$ is the set $\{ T^{(\mathbb{I})} : T \in S \}$. By $R_I(X), C_I(X)$ and $M_I(X)$ we mean the set of all rows, columns and $\mathbb{I} \times \mathbb{I}$ matrices, respectively, indexed by a set $\mathbb{I}$, with the entries in an operator space $X$, that represent bounded operators.

The injective envelope ([11], [25], [4], [10], [19]) of an operator space $X$ is an injective operator space $Y = I(X)$ containing $X$ such that $Y$ is the only injective subspace of $Y$ containing $X$. An injective operator space $Y$ containing $X$ is the injective envelope of $X$ if and only if $Y$ has one (hence both) of the following two equivalent properties (see e.g. [5] 4.2.4 or [19] 15.8) for a proof).

(i) (Rigidity) If $\phi : Y \to Y$ is a complete contraction such that $\phi|X$ is the identity, then $\phi$ is the identity.

(ii) (Essentiality) If $\phi : Y \to Z$ is a complete contraction, where $Z$ is any operator space, such that $\phi|X$ is a complete isometry, then $\phi$ is a complete isometry.

A left multiplier of an operator space $X$ is a map $\phi : X \to X$ such that there exist a Hilbert space $H$, a complete isometry $\iota : X \to B(H)$ and an operator $T \in B(H)$ such that $\phi(x) = T\iota(x)$ for all $x \in X$. (That is, identifying $X$ with $\iota(X)$, $\phi$ is the restriction to $X$ of a left multiplication by $T$). If, in addition, $T^*\iota(X) \subseteq \iota(X)$, then $\phi$ is called an adjointable left multiplier on $X$. The multiplier norm $\|\phi\|_m$ of $\phi$ is the infimum of $\|T\|$ over all possible such representations of $\phi$.

In the theorem quoted below we recall two beautiful canonical descriptions of multipliers obtained by Blecher, Effros and Zarikian [4] and Blecher and Paulsen...
Since every representation of $K$ is a multiple of the identity $I$, we may assume that $\mathcal{H}_\sigma = \mathcal{G}^I$ and $\sigma(x) = x^{(l)} (x \in K)$ for an index set $I$. Further, since $\phi$ is an $A$-module map, the equality $[K^{(l)}Vl] = \mathcal{G}^I$ implies (by a standard computation) that $U$ is an $A$-module map. Similarly $V \in B_B(l, \mathcal{G}^I)$. Finally, with $P : \mathcal{G} \to \mathcal{H}$ and $Q : \mathcal{G} \to K$ the orthogonal projections, we have $x = PxQ$ if $x \in K(\mathcal{K}, \mathcal{H})$. 

\[ (2.1) \quad \phi(x) = U^*x^{(l)}V = \sum_{i \in I} u_i^*xv_i \quad (x \in \mathcal{K}(\mathcal{H}, \mathcal{H})), \]

where $I$ is an index set, $U \in B_A(l, \mathcal{H}^I)$, $V \in B_B(l, \mathcal{K}^2)$ are such that $\|U\||V\| = \|\phi\|_{cb}$ and $u_i \in B_A(l, \mathcal{H})$ and $v_i \in B_B(l, \mathcal{K})$ are the components of $U$ and $V$, respectively, when $U$ and $V$ are regarded as columns of operators using the identification $B_A(l, \mathcal{H}^I) = C_2(B_A(l, \mathcal{H}))$.

The proof of (2.1) can be reduced to the special case $\mathcal{K} = \mathcal{H}$ by putting $\mathcal{G} = \mathcal{H} \oplus \mathcal{K}$ (an $A \oplus B$-module), regarding $K(\mathcal{K}, \mathcal{H})$ as the $(1,2)$ corner of $K := K(\mathcal{G})$ in the usual way and extending $\phi$ by $0$ to the other three corners of $K$ to get a bimodule map $\phi : K \to B(l)$. Now we can refer to more general results ([14], 2.1), [26] or reason as follows. By the representation theorem $\phi$ is of the form $\tilde{\phi}(x) = U^*\sigma(x)V \quad (x \in K)$ for a representation $\sigma$ of $K$ on a Hilbert space $\mathcal{H}_\sigma$ and operators $U, V \in B(l, \mathcal{H}_\sigma)$ with $\|U\||V\| \leq \|\tilde{\phi}\|_{cb} = \|\phi\|_{cb}$, whereby we may assume in addition that $[\sigma(K)Vl] = \mathcal{H}_\sigma = [\sigma(K)Ul]$. Since every representation of $K$ is a multiple of the identity $I$, we may assume that $\mathcal{H}_\sigma = \mathcal{G}^I$ and $\sigma(x) = x^{(l)} (x \in K)$ for an index set $I$. Further, since $\phi$ is an $A$-module map, the equality $[K^{(l)}Vl] = \mathcal{G}^I$ implies (by a standard computation) that $U$ is an $A$-module map. Similarly $V \in B_B(l, \mathcal{G}^I)$. Finally, with $P : \mathcal{G} \to \mathcal{H}$ and $Q : \mathcal{G} \to K$ the orthogonal projections, we have $x = PxQ$ if $x \in K(\mathcal{K}, \mathcal{H})$. 

\[ (2.1) \quad \phi(x) = U^*x^{(l)}V = \sum_{i \in I} u_i^*xv_i \quad (x \in \mathcal{K}(\mathcal{H}, \mathcal{H})), \]
3. Embeddings of $B(K,\mathcal{H})$ into operator bimodules

**Theorem 3.1.** Let $\mathcal{H} \in \mathcal{A}HM$ and $\mathcal{K} \in \mathcal{B}HM$ be cyclic with unit cyclic vectors $\xi \in \mathcal{H}$ and $\eta \in \mathcal{K}$ and let $\phi$ be a completely contractive $A, B$-bimodule map from $B(K,\mathcal{H})$ (or $K(\mathcal{K},\mathcal{H})$) into an operator $A, B$-bimodule $X$. Then $\phi$ is completely isometric if and only if $\|\phi(\xi \otimes \eta^*)\| = 1$.

The Theorem is an immediate consequence of the following Lemma (in fact, of the identity [5, 3.3.1], proved in the third paragraph of the proof of the Lemma). In later sections only the Theorem will be used, but the Lemma is a more complete result with a consequence (Corollary 3.3) that can not be deduced from the Theorem.

**Lemma 3.2.** In the situation of Theorem 3.1 there exists a Hilbert module $l \in \mathcal{A}HM \cap \mathcal{B}HM$ such that:

(i) $X \subseteq B(l)$ completely isometrically as an operator bimodule.

(ii) If $\phi|K(\mathcal{K},\mathcal{H})$, regarded as a map into $B(l)$, is represented in the form [24], where $U \in B_\mathcal{A}(l,\mathcal{H})$ and $V \in B_\mathcal{B}(l,\mathcal{K})$ are contractions, then $\|\phi(\xi \otimes \eta^*)\| = 1$ if and only if there exists an $\alpha \in C_1(\mathbb{C})$, with $\|\alpha\| = 1$, such that

\[
UU^*\alpha = \alpha \quad \text{and} \quad VV^*\alpha = \alpha.
\]

In this case

\[
(3.1) \quad UU^*\alpha = \alpha \quad \text{and} \quad VV^*\alpha = \alpha.
\]

are isometries and

\[
(3.2) \quad u := U^*\alpha \in B_\mathcal{A}(\mathcal{H}, l) \quad \text{and} \quad v := V^*\alpha \in B_\mathcal{B}(\mathcal{K}, l)
\]

\[
(3.3) \quad \phi(x) = uxv^* + e^\perp \phi(x)f^\perp \quad (x \in B(\mathcal{K}, \mathcal{H})),
\]

where $e = uu^*$ and $f = vv^*$.

**Proof.** If $\alpha$, as stated in the Lemma, does exist, then using (3.1) and the relations $\|\alpha\| = 1 = \|\alpha\|$, $\|U\| \leq 1$ and $\|V\| \leq 1$, we have for each $x \in K(\mathcal{K}, \mathcal{H})$

\[
\|\phi(x)\| = \|UU^*x^{(1)}V\| \geq \langleUU^*x^{(1)}VV^*\alpha\eta, \alpha\xi\rangle = \langle x^{(1)}\alpha\eta, \alpha\xi\rangle = \langle x\eta, \xi\rangle.
\]

This implies in particular that $\|\phi(\xi \otimes \eta^*)\| = 1$.

For the converse, first recall that by the CES representation theorem for operator bimodules [3 3.3.1] there exists a $C^*$-algebra $D$ containing $X$ completely isometrically as an operator $A, B$-bimodule (where the $A, B$-bimodule structure on $D$ is induced by a pair of $\ast$-homomorphisms $A \to D$ and $B \to D$). Let $l$ be the Hilbert space of the universal representation of $D$, regard $D$ as contained in $B(l)$ and $\phi$ as a map into $B(l)$. Set $T = \phi(\xi \otimes \eta^*)$ and suppose that $\|T\| = 1$. Since each norm 1 linear functional on $D$ is induced by a pair of unit vectors in $l$ (see [12 10.1.3]), there exists unit vectors $\zeta, \tau \in l$ such that $\langle T\tau, \zeta \rangle = \|T\| = 1$, hence from (2.4)

\[
(3.4) \quad 1 = \langle\phi(\xi \otimes \eta^*)\tau, \zeta\rangle = \langle(\xi \otimes \eta^*)^{(1)}V\tau, U\zeta\rangle \leq \|V\tau\|\|U\zeta\| \leq 1.
\]

Since equality occurs in the Cauchy - Schwarz inequality only for colinear vectors, [5.3.4] implies that

\[
(3.5) \quad (\xi \otimes \eta^*)^{(1)}V\tau = U\zeta \quad \text{and similarly} \quad (\eta \otimes \xi^*)^{(1)}U\zeta = V\tau.
\]
Let $u_i$ and $v_i$ be the components of $U \in C_1(B_A(l, \mathcal{H}))$ and $V \in C_0(B_B(l, \mathcal{K}))$, respectively, and set $\alpha_i = (u_i, \zeta_i, \xi_i)$. Then, written in components, the second identity in (3.5) is

$$v_i \tau = \alpha_i \eta.$$  

Inserting (3.6) in the first identity of (3.5) shows that

$$u_i \zeta = \alpha_i \xi.$$  

From (3.4) we also conclude that $\|U\zeta\| = 1 = \|V\tau\|$, which together with (3.6) (or (3.7)) implies that $\sum_{i \in I} |\alpha_i|^2 = 1$. Thus, the vector

$$\alpha := (\alpha_i) \in C_1(\mathbb{C})$$

has norm 1, so $\alpha \xi \in \mathcal{H}^\perp$ and $\alpha \eta \in \mathcal{K}^\perp$ are also unit vectors and $U \zeta = \alpha \zeta$, $V \tau = \alpha \eta$ by (3.7) and (3.6). Since $U$ and $V$ are contractions, this implies that $U^*U \zeta = \zeta$ and $V^*V \tau = \tau$, which can be written as $U^*(\alpha \zeta) = \zeta$ and $V^*(\alpha \eta) = \tau$, hence

$$UU^*\alpha \xi = \alpha \xi \quad \text{and} \quad VV^*\alpha \eta = \alpha \eta.$$  

Since $UU^*\alpha$ is an $A$-module map and $\xi$ is cyclic for $A$, the first of the above two identities implies that $UU^*\alpha = \alpha$. Similarly $VV^*\alpha = \alpha$, which proves (3.1).

With $u := U^*\alpha$ and $v := V^*\alpha$, we have from (3.1) that $u^*u = \alpha^*UU^*\alpha = \alpha^*\alpha = 1$ and similarly $v^*v = 1$. So $u$ and $v$ are isometries; let $e := uu^*$ and $f := vv^*$ be their range projections. Using (2.1), (3.2) and (3.1) we compute for $x \in K(K, \mathcal{H})$:

$$e\phi(x)f = uu^*U^*x^{(i)}Vvv^* = uu^*U^*x^{(i)}VV^*\alpha v^* = \alpha^*x^{(i)}\alpha v^* = \alpha^*\alpha uu^*.$$  

Thus

$$e\phi(x)f = uu^*.$$  

To simplify the notation, we shall now regard $\mathcal{H}$ and $\mathcal{K}$ as subspaces of $l$ using the isometries $u$ and $v$. Then, relative to the decompositions $l = \mathcal{H} \oplus \mathcal{H}^\perp$ and $l = \mathcal{K} \oplus \mathcal{K}^\perp$, the map $\phi$ can be represented as

$$\phi(x) = \begin{bmatrix} \phi_{11}(x) & \phi_{12}(x) \\ \phi_{21}(x) & \phi_{22}(x) \end{bmatrix} \quad (x \in B(\mathcal{K}, \mathcal{H})), $$

where $\phi_{ij}$ are complete contractions and $\phi_{11}(x) = x$ for all $x \in K(\mathcal{K}, \mathcal{H})$ by (3.8). Since $B(\mathcal{K}, \mathcal{H})$ is the injective envelope of $K(\mathcal{K}, \mathcal{H})$ [4.6.12], $\phi_{11}$ must be the identity by rigidity.

To show that $\theta := \phi_{21} = 0$, note that

$$\|x^*x + \theta(x)^*\theta(x)\| \leq \|x^*x\| \quad (x \in B(\mathcal{K}, \mathcal{H}))$$

since the first column of $\phi$ represents a contraction and $\phi_{11} = \text{id}$. Choosing for $x$ any partial isometry and denoting by $p_x$ its initial projection $x^*x$, it follows that $\|p_x + p_x\theta(x)^*\theta(x)p_x\| \leq 1$, hence $\theta(x)p_x = 0$ and therefore

$$\theta(x)x^* = 0.$$  

If $\dim \mathcal{K} \leq \dim \mathcal{H}$, this implies in particular that $\theta(x) = 0$ for each isometry $x \in B(\mathcal{K}, \mathcal{H})$, hence $\theta = 0$ since the unit ball of $B(\mathcal{K}, \mathcal{H})$ is the closure of the convex hull of isometries (by a polar decomposition argument and the fact that the open unit ball of a unital $C^*$-algebra is the convex hull of unitaries [12 10.5.91]). If $\dim \mathcal{K} > \dim \mathcal{H}$, let $\mathcal{K}_1$ and $\mathcal{K}_2$ be any orthogonal subspaces of $\mathcal{K}$ such that there exist partial isometries $x_i \in B(\mathcal{K}, \mathcal{H}) (i = 1, 2)$ with orthogonal ranges and the initial spaces equal to $\mathcal{K}_i$. Then $x_1 + x_2$ is a partial isometry (with the initial
space $K_1 \oplus K_2$), hence $\theta(x_1 + x_2)(x_1 + x_2)^* = 0$ by (3.11). Using (3.10) again, this implies $\theta(x_1)x_1^* + \theta(x_2)x_2^* = 0$ and then, replacing $x_2$ by $ix_2$, it follows that $\theta(x_2)x_2^* = 0$, hence $\theta(x_2)|K_1 = 0$. Since $K_2$ is spanned by subspaces of the type $K_1$, $\theta(x_2)|K_2 = 0$. But the equality $\theta(x_2)x_2^* = 0$ tells us that $\theta(x_2)|K_2 = 0$, hence $\theta(x_2) = 0$. We conclude that $\theta(x) = 0$ for each non-surjective partial isometry in $B(K, H)$. Since each coisometry is a sum of two such partial isometries, this implies $\theta(x) = 0$ for each coisometry $x \in B(K, H)$. Since $B(K, H)$ is the closure of the span of coisometries, $\phi_{21} = 0$, which proves that the above matrix of $\phi(x)$ is diagonal for each $x \in B(K, H)$. Returning to the original notation, this proves the decomposition (3.3) of $\phi$.

Remark 3.3. In the situation of Lemma 3.2(ii) there exist two families $\{u_\alpha : \alpha \in \mathbb{A}\}$ and $\{v_\alpha : \alpha \in \mathbb{A}\}$ of isometries $u_\alpha \in B_A(H, l)$ and $v_\alpha \in B_B(K, l)$ such that:

1. $u_\alpha^*u_\beta = 0$ and $v_\alpha^* v_\beta = 0$ if $\beta \neq \alpha$;
2. $\phi(x) = \sum_{\alpha \in \mathbb{A}} u_\alpha x_\alpha^* + p^\perp \phi(x)q^\perp$, with $p := \sum_{\alpha \in \mathbb{A}} u_\alpha u_\alpha^*$ and $q := \sum_{\alpha \in \mathbb{A}} v_\alpha v_\alpha^*$;
3. $||p^\perp \phi(\xi_1 \otimes \eta_1^*)q^\perp|| < 1$ for all unit cyclic vectors $\xi_1 \in H$ and $\eta_1 \in K$ and all unit vectors $\tau \in l$.

To prove this, let $\mathbb{A} \subseteq C_1(\mathbb{C})$ be a maximal orthogonal family of vectors $\alpha$ satisfying the two equalities in (3.14). For each $\alpha \in \mathbb{A}$, $u_\alpha := U^* \alpha$ and $v_\alpha := V^* \alpha$ are isometries by what we have already proved in Lemma 3.2, let $e_\alpha = u_\alpha u_\alpha^*$ and $f_\alpha = v_\alpha v_\alpha^*$ be their range projections. If $\beta \neq \alpha$ are in $\mathbb{A}$ then, using (3.14) and the orthogonality of the set $\mathbb{A}$,

$$u_\alpha^*u_\beta = \beta^* U U^* \alpha = \beta^* \alpha = 0.$$ 

This shows that the family of projections $\{e_\alpha : \alpha \in \mathbb{A}\}$ is orthogonal and the same holds also for $\{f_\alpha : \alpha \in \mathbb{A}\}$. Let

$$p = \sum_{\alpha \in \mathbb{A}} e_\alpha \quad \text{and} \quad q = \sum_{\alpha \in \mathbb{A}} f_\alpha.$$ 

Multiplying the identity $\phi(x) = u_\alpha x_\alpha^* + e_\alpha^* \phi(x)f_\alpha^*$ (which is just (3.14) stated for each index $\alpha$) from the left by $e_\beta$ and from the right by $f_\beta$, where $\beta \neq \alpha$, we get $e_\alpha \phi(x)f_\beta = 0$. Similarly $e_\alpha \phi(x)q^\perp = 0$, $p^\perp \phi(x)f_\alpha = 0$ and it follows that

$$\phi(x) = \sum_{\alpha \in \mathbb{A}} e_\alpha \phi(x)f_\alpha + p^\perp \phi(x)q^\perp = \sum_{\alpha \in \mathbb{A}} u_\alpha x_\alpha^* + p^\perp \phi(x)q^\perp \quad (x \in B(K, H)).$$

Suppose that there existed unit cyclic vectors $\xi_1 \in H$ and $\eta_1 \in K$ and a unit vector $\tau \in l$ such that $||p^\perp \phi(\xi_1 \otimes \eta_1^*)q^\perp \tau|| = 1$. Then $\tau \in q^\perp l$ and, denoting by $\zeta \in p^\perp l$ the unit vector such that $\langle p^\perp \phi(\xi_1 \otimes \eta_1^*)q^\perp \tau, \zeta \rangle = 1$, we may apply the arguments from the proof of Lemma 3.2 (from (3.4) on) to the map $x \mapsto p^\perp \phi(x)q^\perp = (U p^\perp)^* x(\zeta) (Vq^\perp)$. It follows that there exists $\beta \in C_1(\mathbb{C})$ of norm 1 such that

$$(3.10) \quad U p^\perp U^* \beta = \beta \quad \text{and} \quad Vq^\perp V^* \beta = \beta.$$

The first of these two equalities implies that $\beta^*(1 - U p^\perp U^*) \beta = 0$. Since $0 \leq 1 - U U^* \leq 1 - U p^\perp U^*$, it follows that $\beta^*(1 - U U^*) \beta = 0$, hence $(1 - U U^*) \beta = 0$ (by positivity) and $U U^* \beta = \beta$. Similarly $V V^* \beta = \beta$, hence $\beta$ satisfies (3.1). With $u_\beta$ and $v_\beta$ defined by

$$u_\beta = p^\perp U^* \beta \quad \text{and} \quad v_\beta = q^\perp V^* \beta,$$
it follows from (3.10) that \( u_\beta \) and \( v_\beta \) are isometries (since \( \|\beta\| = 1 \)). Moreover, since \( u_\alpha = e_\alpha u_\alpha \) for each \( \alpha \in \Lambda \), we have

\[
u_\alpha^* u_\beta = u_\alpha^* e_\alpha p \bot U^* \beta = 0.
\]

But, on the other hand, \( u_\alpha^* u_\beta = \alpha^* U^* [U^* \beta = \alpha^* \beta \) by (3.10). It follows that \( \alpha^* \beta = 0 \). This means that \( \beta \perp \alpha \) for all \( \alpha \in \Lambda \), which contradicts the maximality of the set \( \Lambda \).

Since each compact operator on a Hilbert space achieves its norm, the arguments from Remark 3.3 and the proof of Lemma 3.2 imply the following corollary.

**Corollary 3.4.** If \( \mathcal{H} \in \mathcal{A}_{HM} \) and \( K \in \mathcal{B}_{HM} \) are cyclic with unit cyclic vectors \( \xi \in \mathcal{H} \) and \( \eta \in K \), and \( \phi \in CB_4(K(K, \mathcal{H}), K(l))_B \) is completely contractive (for an \( l \in \mathcal{A}_{HM} \cap \mathcal{B}_{HM} \)) such that \( \|\phi(\xi \otimes \eta^*)\| = 1 \), then for some (necessarily finite) \( n \) there are orthogonal decompositions \( l \cong \mathcal{H}^n \oplus \mathcal{H}_1 \) (as \( \mathcal{A}\)-modules) and \( l \cong K^n \oplus K_1 \) (as \( \mathcal{B}\)-modules) relative to which \( \phi \) is of the form \( \phi(x) = x^{(n)} \oplus \psi(x) \), where \( \psi \) satisfies \( \|\psi(\xi_1 \otimes \eta_1)\| < 1 \) for all unit cyclic vectors \( \xi_1 \in \mathcal{H} \) and \( \eta_1 \in K \).

4. **Induced Maps on Multiplier \( C^*\)-Algebras**

In this section we show how complete contractions induce maps between the corresponding multiplier \( C^*\)-algebras, but we shall only consider a special situation needed later in the paper.

**Theorem 4.1.** Suppose that \( \mathcal{H} \in \mathcal{A}_{HM} \), \( \mathcal{K} \) is a Hilbert space, let \( \eta \in K \) be a unit vector, \( (\varepsilon_i)_{i \in I} \) an orthonormal basis of \( \mathcal{H} \) and \( x_i = \varepsilon_i \otimes \eta^* \). Further, let \( X \) be an injective operator space and operator left \( \mathcal{A}\)-module, and let \( \phi : K(K, \mathcal{H}) \rightarrow X \) be a completely contractive \( \mathcal{A}\)-module map. Then the formula

\[
\phi_\xi (T) = \sum_{i \in I} \phi(Tx_i) \phi(x_i)^* \quad (T \in K(\mathcal{H}))
\]

defines a completely contractive, completely positive \( \mathcal{A}\)-bimodule map \( \phi_\xi : K(\mathcal{H}) \rightarrow \mathcal{A}_{\ell}(X) \). If \( \phi \) is injective, the kernel of \( \phi_\xi \) does not contain any positive (nonzero) elements. Moreover, if \( \phi \) is completely isometric and \( \mathcal{H} \) is cyclic, then \( \phi_\xi \) is completely isometric. Finally, there exists an extension of \( \phi_\xi \) to a map \( \phi_\xi : B(\mathcal{H}) \rightarrow \mathcal{A}_{\ell}(X) \) with the same properties.

**Proof.** Since \( \sum_{i \in I} x_i x_i^* = 1_{\mathcal{H}} \) in the strong operator topology, for each \( x \in K(K, \mathcal{H}) \) we have that

\[
x = \sum_{i \in I} x_i x_i^* x,
\]

where the sum is norm convergent. For each finite subset \( \mathbb{F} \) of \( I \) denote by \( [x_i]_\mathbb{F} \) the row matrix in \( R_\mathbb{F}(K(K, \mathcal{H})) \) with the entries \( x_i \). Since \( \phi \) is a complete contraction and \( \|[x_i]_\mathbb{F}\| \leq 1 \), for each \( T \in K(\mathcal{H}) \) we have

\[
\| \sum_{i \in \mathbb{F}} \phi(Tx_i) \phi(x_i)^* \| = \| \phi(T[x_i]_\mathbb{F}) \phi(x_i)^* \|_2 \leq \| T[x_i]_\mathbb{F} \| \| [x_i]_\mathbb{F} \| \leq \| T \|.
\]

Since the sum \( \sum_{i \in \mathbb{F}} Tx_i x_i^* T^* \) is norm convergent, this estimate implies that also the sum \( \sum_{i \in \mathbb{F}} T x_i x_i^* T^* \) is norm convergent and defines a contraction \( \phi_\xi : K(\mathcal{H}) \rightarrow K_\ell(X) := [X X^*] \). A similar argument (available also in a more general context of Remark 4.2(ii) below) shows that \( \phi_\xi \) is completely contractive. But here an even simpler argument is possible: if \( [t_{ij}] \) is the matrix of \( T \) relative to the orthonormal basis
(ε_i)i∈I, then by a short computation φ_ℓ(T) is just a product of three operator matrices

\[ \phi_ℓ(T) = [\phi(x_i)][r_{ij}][\phi(x_j)]^*, \]

from which one can see that φ_ℓ is completely contractive and completely positive. Since X is injective, it is known that [XX^∗] is a C∗-subalgebra of A_ℓ(X) and that A_ℓ(X) is injective. (See Theorem 2.1(ii) and the paragraph above Theorem 2.1 or [5] 4.4.3. A_ℓ(X) is known to be just the multiplier C∗-algebra of [XX^∗].) Since φ is a left A-module map, φ_ℓ is clearly a left A-module map by (4.1). To show that φ is also a right A-module map, we use the specific form of the operators x_i. For each T ∈ K(H) of the form T = ξ ⊗ ζ^* (ξ,ζ ∈ H) we have

\[ \sum_{i∈I} φ(T x_i) φ(x_i)^* = \sum_{i∈I} φ(⟨ε_i,ζ⟩ ξ ⊗ η^*) φ(ε_i ⊗ η^*)^* = φ(ξ ⊗ η^*) φ(∑_{i∈I} ⟨ε_i,ε_i⟩ ε_i ⊗ η^*)^*, \]

hence, by (4.1) and since (ε_i)i∈I is an orthonormal basis of H,

\[ φ_ℓ(ξ ⊗ ζ^*) = φ(ξ ⊗ η^*) φ(ζ ⊗ η^*)^*. \]

Therefore, for T of the form ξ ⊗ ζ^* and a ∈ A we have

\[ φ_ℓ(Ta) = φ_ℓ(ξ ⊗ (a^* ζ^*)) = φ(ξ ⊗ η^*) φ(a^* ζ ⊗ η^*)^* = φ(ξ ⊗ η^*) φ(ζ ⊗ η^*)^* a = φ_ℓ(T)a \]

since φ is a left A-module map. Since operators of the form ξ ⊗ ζ^* densely span K(H), this shows that φ_ℓ is a right A-module map. If φ is completely isometric, then it follows from (4.3) that \|φ_ℓ(ξ ⊗ ζ^*)\| = 1 for all unit vectors ξ ∈ H, hence, if H is cyclic, φ_ℓ is completely isometric by Theorem 3.1. If φ is injective, then (4.4) with ζ = ξ (together with positivity) implies that the kernel of φ_ℓ does not contain any nonzero positive operator.

We may assume that A_ℓ(X) ⊆ B(l) for some Hilbert space l. Since A_ℓ(X) is injective, there exists a completely contractive projection E : B(l) → A_ℓ(X), which is automatically an A_ℓ(X)-bimodule map by a well known result of Tomiyama [12] 10.5.86. We can now extend φ_ℓ to a (normal) completely positive A-bimodule complete contraction from B(H) to B(l) and then compose this extension with E to get the completely positive complete contraction φ_ℓ : B(H) → A_ℓ(X) with required properties.

**Remark 4.2.** (i) Let H be a Hilbert space, X a ternary ring of operators (see [5]) and ψ : H → X a completely bounded map. Denote by ψ^* : H^* → X^* the map ψ^*(ξ^*) = ψ(ξ)^* and by τ_φ : K(H) → A_ℓ(X) the composition of the maps in the diagram

\[ K(H) = H ⊗ H^* \xrightarrow{ψ^* ⊗ ψ^*} X ⊗ X^* \xrightarrow{μ} [XX^*] ⊆ A_ℓ(X), \]

where μ is the multiplication: μ(x ⊗ y^*) = xy^*. For a Hilbert space K and a fixed unit vector η ∈ K let τ_η : H → H ⊗ K^* be the embedding τ_η(ξ) = ξ ⊗ η^*. Given a completely bounded map φ : K(K, H) = H ⊗ K^* → X, let ψ_η : H → X be ψ_η = φ_η. Then it can be verified that τ_φ = just the map φ_ℓ defined by (4.1) in Theorem 1.1 which explains some of the properties of φ_ℓ and shows that φ_ℓ is independent of the choice of orthonormal basis. It depends, however, on the choice of η.

(ii) Using formula (4.1) one can define a map in the situation when K(K, H) is replaced by a ternary ring of operators W such that W has a left frame. Here a **left frame** in W is a row [x_i] ∈ R_l(W) (for some index set I) with \| [x_i] \| ≤ 1 such
that \( w = \sum_{i \in I} x_i^* w \) for all \( w \in W \), where the sum is norm convergent. It follows from a result of Brown ([9, 2.2], [23, 5.53]) that \( W \) has a left frame if \( [WW^*] \) has a countable approximate identity. We do not know if \( \phi_{\ell} \) can be constructed in a more general situation, without assuming the existence of a left frame in \( W \).

5. On cogenerators

In the definition below we extract some properties which are common to certain natural categories of operator bimodules, such as the category \( {\mathcal{A}}{\mathcal{O}}{\mathcal{M}} \_B \) of all operator \( A, B \)-bimodules and, if \( M \) and \( N \) are von Neumann algebras, the category \( {\mathcal{M}} \_N \) of all normal operator \( M, N \)-bimodules (as well as some other categories defined in [16]). The main feature of such categories is that they have enough bimodules of the form \( B(\mathcal{K}, \mathcal{H}) \). A reason for introducing them in the definition below is that their cogenerators are suitable for studying a bicommutation problem (Theorem 6.6 below). But, if not interested in general categories of operator bimodules, the reader may skip a few paragraphs and continue reading at Definition 5.5 with the concrete categories \( {\mathcal{A}}{\mathcal{O}}{\mathcal{M}} \_B \) or \( {\mathcal{M}} \_N \) in mind.

**Remark 5.1.** In (vii) of the following definition we will use the elementary fact (which may be proved by using matrix units) that each Hilbert \( M_n(A) \)-module \( \mathcal{H} \) is (up to a unitary equivalence) of the form \( G^n \) for a Hilbert \( A \)-module \( G \), and that an \( M_n(A), M_n(B) \)-bimodule map \( \psi \) from \( M_n(X) \) into \( B(l^n, G^n) = M_n(B(l, G)) \) is necessarily of the form \( [x_{ij}] \mapsto [\phi(x_{ij})] \) for a (unique) \( A, B \)-bimodule map \( \phi : X \to B(l, G) \). We shall write this as \( \phi = \phi_n \).

**Definition 5.2.** A subcategory \( \mathcal{O} \) of \( {\mathcal{A}}{\mathcal{O}}{\mathcal{M}} \_B \) is called an ample category if it satisfies the following conditions:

(i) If \( a \in A \) and \( aX = 0 \) for all \( X \in \mathcal{O} \) then \( a = 0 \) and similarly for \( b \in B \).

(ii) \( M_1(X) \in \mathcal{O} \) for each \( X \in \mathcal{O} \) and each index set \( I \).

(iii) For each index sets \( I, J \) and all \( a \in M_{i,j}(\mathbb{C}), b \in M_{i,j}(\mathbb{C}) \) the map \( \mu_{a,b} \) defined by \( \mu_{a,b}(x) = axb \) is a morphism in \( {\mathcal{O}}(M_1(X), M_1(X)) \).

(iv) Given a map \( \phi = [\phi_{ij}] \in CB(X, M_1(Y)) \) such that \( \phi_{ij} \in \mathcal{O}(X, Y) \) for all \( i, j \in I \), we have that \( \phi \in \mathcal{O}(X, M_2(Y)) \).

(v) For any collection \( (X_i)_{i \in I} \) of modules in \( \mathcal{O} \) the \( \ell^\infty \)-direct sum \( X := \bigoplus_{i \in I} X_i \) is in \( \mathcal{O} \) and the coordinate inclusions \( X_i \to X \) and projections \( X \to X_i \) are morphisms in \( \mathcal{O} \). Moreover, for each \( Y \in \mathcal{O} \) and bounded family of morphisms \( \phi_i \in \mathcal{O}(Y, X_i) \) the map \( \phi \) defined by \( \phi(y) = \bigoplus_{i \in I} \phi_i(y) \) is in \( \mathcal{O}(Y, X) \).

(vi) If \( B(K, \mathcal{H}) \in \mathcal{O} \) for some Hilbert modules \( \mathcal{H} \in \mathcal{A}{\mathcal{H}}M \) and \( K \in \mathcal{B}{\mathcal{H}}M \) and if \( \mathcal{H}_0 \) and \( K_0 \) are (isometrically) isomorphic to Hilbert submodules of \( \mathcal{H} \) and \( K \), respectively, then \( B(K_0, \mathcal{H}_0) \in \mathcal{O} \) and the induced ‘inclusion’ \( B(K_0, \mathcal{H}_0) \to B(K, \mathcal{H}) \) and the compression \( B(K, \mathcal{H}) \to B(K_0, \mathcal{H}_0) \) are morphisms in \( \mathcal{O} \).

(vii) For each \( X \in \mathcal{O}, n \in \mathbb{N}, x \in M_n(X) \) and \( \varepsilon > 0 \) there exist Hilbert modules \( \mathcal{H}^n \in M_n(\mathcal{A}) \mathcal{H}M \) and \( \mathcal{K}^n \in M_n(\mathcal{B}) \mathcal{H}M \) such that \( B(K, \mathcal{H}) \in \mathcal{O} \), and there exists a completely contractive \( M_n(A), M_n(B) \)-bimodule map \( \phi_n : M_n(X) \to B(\mathcal{K}^n, \mathcal{H}^n) \) such that \( \phi \in \mathcal{O}(X, B(K, \mathcal{H})) \) and \( \|\phi_n(x)\| > \|x\| - \varepsilon \).

The conditions (vii) and (v) together imply that each bimodule \( X \) in an ample category can be completely isometrically embedded into an \( \ell^\infty \)-direct sum of bimodules of the form \( B(K, \mathcal{H}) \) with a morphism within the category.
Remark 5.3. (i) Using for $a$ and $b$ suitable matrices (with the entries 1 and 0 only), 

(iii) implies in particular that if $I \subseteq I$, the canonical inclusion $M_2(X) \to M_1(X)$ and the compression $M_1(X) \to M_1(X)$ are morphisms in $\mathcal{O}$.

(ii) Since a composition of morphisms in a category is a morphism, it follows from (iii) and (iv) of Definition 5.2 that $O(X,Y)$ is a vector subspace of $CB_A(X,Y)_B$.

To see, for example, that the sum $\phi + \psi$ of two morphisms in $O(X,Y)$ is in $O(X,Y)$, note that the diagonal matrix $\delta := \phi \oplus \psi$ is in $O(X,M_2(Y))$ by (v). Then, with $\mu_{a,b} \in O(M_2(Y), Y)$ defined as in (iii), where $a = [1,1]$ and $b = [1,1]^T$, we have that $\phi + \psi = \mu_{a,b} \circ \delta$ in $O(X,Y)$.

(iii) For each $\phi \in O(X,Y)$ the amplification $\phi_1 : [x_{ij}] \to [\phi(x_{ij})]$ is a morphism in $O(M_1(X), M_1(Y))$.

To show this, let $E_i \in C_2(\mathbb{C})$ have 1 on the $i$-th position and 0 elsewhere and let $E_i^T$ be its transposition. Then the map $\phi_1(y) := \phi(E_i^T y E_j)$ is in $O(M_1(X), X)$ by (iii). Since the map $[\phi_{ij}] : M_1(X) \to M_1(X)$ is just the amplification of $\phi$, it follows from (iv) that $\phi_1 \in O(M_1(X))$.

(iv) If $\{\psi_i : i \in I\}$ is a bounded set of morphisms $\psi_i \in O(Y_i, X_i)$, then $\psi := \oplus_{\iota \in I} \psi_i$ is a morphism in $O(Y, X)$, where $X = \oplus_{\iota \in I} X_i$ and $Y = \oplus_{\iota \in I} Y_i$. This follows by applying (v) to morphisms $\phi_i := \psi_i \eta_i$, where $\eta_i \in O(Y_i, Y_i)$ is the coordinate projection.

Injectivity in an ample category is defined as usual, considering only completely contractive morphisms in the category.

Remark 5.4. If $O$ is ample and $X$ is injective in $O$, then $X$ is injective as an operator space, hence injective in $\mathcal{AOM}_B$.

To see this, let $\iota$ be a completely isometric embedding of $X$ into an $\ell^\infty$-direct sum $Y \in O$ of bimodules of the form $B(\mathcal{K}, H) \in O$ such that $\iota \in O(X,Y)$ and (v)). By injectivity of $X$ in $O$ there exists a completely contractive extension of the identity $1_X$ to a morphism $E \in O(Y, X)$ (that is, $E\iota = 1_X$). Hence $X$ must be injective as an operator space since $Y$ is.

Definition 5.5. If $O$ is a subcategory in $\mathcal{AOM}_B$, a bimodule $Z \in O$ is called a cogenerator if for each $Y \in O$, $y \in Y$ and $\varepsilon > 0$ there exists a completely contractive morphism $\phi \in O(Y, Z)$ such that $\|\phi(y)\| > \|y\| - \varepsilon$. If (in addition) $\phi$ can be chosen so that $\|\phi(y)\| = \|y\|$, then $Z$ is called a strict cogenerator.

For example, if $\mathcal{H} \in \mathcal{AHM}$ and $\mathcal{K} \in \mathcal{BHM}$ contain (up to a unitary equivalence) all cyclic Hilbert modules over $A$ and $B$, respectively, then $B(\mathcal{K}, \mathcal{H})$ is an injective strict cogenerator in $\mathcal{AOM}_B$. (That it is a cogenerator can be deduced from the well known CES theorem [5, 3.3.1], or from the operator bipolar theorem [15, 1.1]). That such a $B(\mathcal{K}, \mathcal{H})$ is in fact a strict cogenerator follows from [22, 4.1], but we can also present an alternative argument. Given $y$ in an operator bimodule $Y$, first choose a norm one functional $\rho$ on $Y$ such that $\rho(y) = \|y\|$. Then apply the well known CSPS factorization theorem to the map $\tilde{\rho} : A \times Y \times B \to \mathbb{C}$, $\tilde{\rho}(a, x, b) := \rho(AXB)$, to show that $\rho$ is of the form $\rho(AXB) = (\pi(a)\phi(x)a)\sigma(a)\eta, \xi$ for some cyclic representations $\pi : A \to B(\mathcal{H}_B)$ and $\sigma : B \to B(\mathcal{K}_A)$, with unit cyclic vectors $\xi$ and $\eta$, respectively, and a completely contractive $A,B$-bimodule map $\phi : Y \to B(\mathcal{K}_A, \mathcal{H}_B)$. Then $(\phi(\eta)\xi, \xi) = \rho(y) = \|y\|$ implies that $\|\phi(y)\| = \|y\|$. By cyclicity we can regard $\mathcal{H}_B$ and $\mathcal{K}_A$ as contained in $\mathcal{H}$ and $\mathcal{K}$, respectively.

Similarly, if $A$ and $B$ are von Neumann algebras and $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces of the universal normal representations of $A$ and $B$, respectively, then $B(\mathcal{K}, \mathcal{H})$ is
an injective cogenerator in the category of all normal operator $A, B$-bimodules (and normal dual operator $A, B$-bimodules).

**Example 5.6.** As an example of a cogenerator in $AOM_B$ that is not injective and not strict, consider $A = B = c_0$ acting on $\ell^2$ in the usual way and let $Z = K(\ell^2)$.

To see that $Z$ is a cogenerator, first note that each cyclic representation of $c_0$ is contained in the representation of $c_0$ on $\ell^2$ as diagonal matrices (namely, $\ell^\infty$ is the universal von Neumann envelope of $c_0$). This implies that $B(\ell^2)$ is a (strict) cogenerator. Now, let $p_n$ be the projection onto the first $n$ coordinates in $\ell^2$. Since $p_n \in c_0 \subseteq c_0'$, the compression $x \mapsto p_n x p_n$ ($x \in B(\ell^2)$) is a $c_0$-bimodule map and the map $x \mapsto \oplus_{n=1}^\infty p_n x p_n$ is a completely isometric $c_0$-bimodule map from $B(\ell^2)$ into $K(\ell^2)^\infty$. This implies that $K(\ell^2)$ is a cogenerator for operator $c_0$-bimodules.

To show that $K(\ell^2)$ is not a strict cogenerator, first note that each $c_0$-bimodule complete contraction $\phi : B(\ell^2) \to K(\ell^2)$ is weak* continuous (when considered as a map into $B((\ell^2))$, hence of the form $\phi(x) = U^* x (I) V$ for suitable contractions $U$ and $V$ (as in [22 Chapter 10, 10.5.15]). For each $x \in B(\ell^2)$ we have $0 = \phi(x) e_i e_j = e_i \phi(x) e_j$, hence $\phi(x) = 0$ since $\sum e_i = 1$. Now choose an operator $y \in B(\ell^2)$ which does not achieve its norm. Since $\phi(y)$ is compact, it achieves its norm on a unit vector $\eta \in \ell^2$, so that $||\phi(y)|| = ||\phi(y)\eta|| = ||U^* y (I) V \eta|| \leq ||y||$. If $||\phi(\eta)|| = ||\eta||$, then it follows that $||U^* y (I) V \eta|| = ||y||$, hence $||y (I) V \eta|| = ||y||$ or
\begin{equation}
\sum_{i \in I} ||y v_i \eta||^2 = ||y||^2,
\end{equation}
where $v_i$ are the components of $V$. But, since $||V|| \leq 1$,
\begin{equation}
\sum_{i \in I} ||y v_i \eta||^2 \leq ||y||^2 \sum_{i \in I} ||v_i \eta||^2 \leq ||y||^2
\end{equation}
and therefore the equality $[5.1]$ is possible only if $||y v_i \eta|| = ||y|| ||v_i \eta||$, which means that $y$ achieves its norm at the unit vector $||v_i \eta||^{-1} v_i \eta$ for some $i$. But this contradicts the choice of $y$.

**Proposition 5.7.** A bimodule $Z$ in an ample category $O$ is a cogenerator (is a strict cogenerator in $O = AOM_B$) if and only if for each $B(K, H) \in O$, with $H \in AHM$ and $K \in BHM$ cyclic, there exists a complete isometry in $O(B(K, H), Z^{(\infty)})$ (a complete isometry in $CB_A(B(K, H), Z)^B$, respectively).

**Proof.** If $Z$ is a cogenerator in $O$ and $H \in AHM$, $K \in BHM$ are cyclic, with unit cyclic vectors $\xi \in H$ and $\eta \in K$ and such that $B(K, H) \in O$, then for each $n = 1, 2, \ldots$ there exists a complete contraction $\phi_n \in O(B(K, H), Z)$ such that $||\phi_n (\xi \otimes \eta^* )|| > 1 - \frac{1}{n}$. Then by Theorem 5.1 the direct sum $\phi$ of the maps $\phi_n$ embeds $B(K, H)$ into $Z^{(\infty)}$ completely isometrically and $\phi \in O(B(K, H), Z^{(\infty)})$ by $5.2(v)$.

For the converse, let $Y \in O$, $y \in Y$, with $||y|| = 1$, and $\varepsilon > 0$. By $5.2(vii)$ there exist $B(K, H) \in O$ and a complete contraction $\phi \in O(Y, B(K, H))$ such that $||\phi(y)|| > 1 - \varepsilon$. Let $\xi \in H$ and $\eta \in K$ be unit vectors such that $\langle \phi(y) \eta, \xi \rangle > 1 - \varepsilon$, let $H_0 := [A\xi]$ and $K_0 := [B\eta]$ be the corresponding cyclic submodules and $p : H \to H_0$ the orthogonal projection. Then $B(K_0, H_0) \in O$, the map $\phi_0(x) := p\phi(x)|K_0$ from
Y into $B(K_0, H_0)$ is a morphism in $\mathcal{O}$ by [5.2 vi) and satisfies $\|\phi_0(y)\| > 1 - \varepsilon$. By hypothesis there exists a complete isometry $\psi \in \mathcal{O}(B(K_0, H_0), Z^N)$. Let $\theta = \psi \phi_0$, so that $\theta \in \mathcal{O}(Y, Z^N)$ is a complete contraction. Composing $\theta$ with a suitable coordinate projection $Z^N \to Z$ (which is in $\mathcal{O}(Z^N, Z)$ by [5.2 vi)), we find a map in $\mathcal{O}(Y, Z)$ satisfying the requirement in the definition of a cogenerator.

The proof for strict cogenerators in $AOM_B$ is similar (see the comment following Definition 5.5).

\begin{corollary}
If $Z$ in an injective strict cogenerator in $AOM_B$, then $\Lambda_i(Z)$ is an injective strict cogenerator in $AOM_A$.
\end{corollary}

\begin{proof}
Let $H$ be any cyclic Hilbert module over $A$ and choose any such module $K$ over $B$. By Proposition 5.7 there exists a completely isometric $A, B$-bimodule map $\phi : B(K, H) \to Z$, which by Theorem 1.1 induces a completely isometric $A$-bimodule map $\phi_\ell : B(H) \to \Lambda_i(Z)$. By Proposition 5.7 again this implies that $\Lambda_i(Z)$ is a strict cogenerator in $AOM_A$. It is known that $\Lambda_i(Z)$ is injective if $Z$ is injective.
\end{proof}

\begin{theorem}
If $Z$ is a cogenerator in an ample category $O$, then for each $X \in O$ there exist an index set $\mathcal{K}$ and a complete isometry in $\mathcal{O}(X, M_{\mathcal{K}}(Z))$.
\end{theorem}

\begin{proof}
From [5.2 vi)(v) there exist two families of Hilbert modules $H_m \in A_{HM}$ and $K_m \in B_{HM}$ ($m \in M$), such that $B(K_m, H_m) \in \mathcal{O}$, $Y := \oplus_{m \in M} B(K_m, H_m) \in \mathcal{O}$, and there exists a complete isometry $\iota \in \mathcal{O}(X, Y)$. Suppose that for each $m$ we have found a complete isometry $\phi_m \in \mathcal{O}(B(K_m, H_m), M_{K_m}(Z))$ for some $K_m$. Then, let $K$ be the disjoint union of the sets $K_m$, $W = \oplus_{m \in M} M_{K_m}(Z)$, $\eta_m : W \to M_{K_m}(Z)$ the coordinate projection (a morphism in $\mathcal{O}$ by [5.2 vi)), $\iota_m : M_{K_m}(Z) \to M_{K_m}(Z)$ the canonical inclusion and $\tilde{\eta}_m := \iota_m \eta_m$. Since $\iota_m$ is a morphism in $\mathcal{O}$ by Remark 5.3(i), the same holds for $\tilde{\eta}_m$ and then it follows from [5.2 iv) that the diagonal matrix $\oplus_m \tilde{\eta}_m \in CB(W, M_{\mathcal{M}}(M_{\mathcal{K}}(Z)))$ represents a morphism $\delta$ in $\mathcal{O}(W, M_{\mathcal{M}}(M_{\mathcal{K}}(Z)))$. For each $k \in \mathcal{K}$ denote by $m_k$ the unique element of $M$ such that $k \in K_{m_k}$ and regard $K$ as a subset in $M \times K$ by $k \mapsto (m_k, k)$. The compression

$$
\gamma : M_{\mathcal{M}}(M_{\mathcal{K}}(Z)) = M_{M \times K}(Z) \to M_{\mathcal{K}}(Z)
$$

is a morphism in $\mathcal{O}$ by Remark 5.3(i), hence so is $\gamma \delta$. But this map $\gamma \delta$ is just the canonical inclusion $\kappa$ of $W$ into (the block-diagonal matrices in) $M_{\mathcal{K}}(Z)$. Then the composition $\kappa \circ (\oplus_{m \in M} \phi_m) \mu$ is a morphism in $\mathcal{O}$ (Remark 5.3 iv)) and embeds $X$ completely isometrically into $M_{\mathcal{K}}(Z)$. Thus, it suffices to find the appropriate maps $\phi_m$, which reduces the proof to showing that each bimodule of the form $B(K, H) \in \mathcal{O}$ can be embedded completely isometrically into $M_{\mathcal{K}}(Z)$ (for some $K$) by a morphism in $\mathcal{O}$.

Denote by $\pi : A \to B(H)$ and $\sigma : B \to B(K)$ the representations that induce the module structures on $H$ and $K$. Let $\{H_i : i \in I\}$ be a maximal set of disjoint cyclic Hilbert submodules in $H$. Here ‘disjoint’ means that $H_i$ and $H_j$ have no isomorphic non-zero submodules if $i \neq j$. Denoting by $e'_i : H \to H_i$ the orthogonal projections (thus $e'_i \in \pi(A)'$), disjointness means that the central carriers $p_i \in \pi(A)'$ of projections $e'_i$ are mutually orthogonal [12 10.3.3]. Let $e' = \sum_{i \in I} e'_i$. Since the central carrier of $e'$ is $1_H$ by maximality, the map $\pi(A) \to \pi(A)e'$, $a \mapsto ae'$ is a $*$-isomorphism of von Neumann algebras by [12 5.5.5], hence $H := \oplus_{i \in I} H_i$ is a faithful $\pi(A)$-module. Therefore $H$ is (isometrically isomorphic to) a submodule in $\mathcal{N}$ for
some $L$ (as a module over $\pi(A)$, hence also over $A$). Denoting by $\tilde{K}$ a submodule of $K$ that is constructed in the same way as $H$ (and enlarging $L$ if necessary), it follows that $B(K, H)$ is ‘contained’ in $B(\tilde{K}, \tilde{H}) = M_L(B(\tilde{K}, \tilde{H}))$, where the ‘inclusion’ is a morphism in $O$ by [5.21 vi]. Hence (simplifying the notation), the proof reduces to

modules of the form $B(K, H)$, where $H = \oplus_{i \in I} H_i$ and $K = \oplus_{j \in J} K_j$ are now direct sums of disjoint cyclic submodules.

Let $p_i' : H \to H_i$ and $q_j' : K \to K_j$ be the projections. Let $S$ be the set of all pairs $(G, l)$, where $G \subseteq H$ and $l \subseteq K$ are cyclic Hilbert modules and for each $s = (G, l) \in S$ let $\psi_s$ be the compression $x \mapsto P_x l$, where $P : H \to G$ is the orthogonal projection. Note that $\psi_s \in O(B(K, H), B(l, G))$ by [5.2 vi]. By Proposition 5.7 for each $s = (G, l) \in S$ there is a complete isometry $\phi_s \in O(B(l, G), Z^H)$. Define

$$\phi : B(K, H) \to (Z^H)^S, \quad \phi(x) = \oplus_{s \in S} \phi_s(\psi_s(x)) \quad (x \in B(K, H)).$$

Then $\phi \in O(B(K, H), (Z^H)^S)$ by [5.2 v]. To show that $\phi$ is isometric, note (by finite rank approximation) that for a compact $x$ there exist countable subsets $I_x \subseteq I$ and $J_x \subseteq J$ such that $p_{i'} x = 0$ if $i \notin I_x$ and $x q_j' = 0$ if $j \notin J_x$. By disjointness the modules $H_x := \oplus_{i \in I_x} H_i$ and $K_x := \oplus_{j \in J_x} K_j$ are cyclic [5 5.10], so $s := (K_x, H_x) \in S$. For this particular $s$ we have that $\|\phi_s(\psi_s(x))\| = \|x\|$. This shows that $\phi(H(K, H)$ is isometric and a similar argument shows that $\phi(K(K, H)$ is completely isometric. Since $B(K, H)$ is the injective envelope of $K(K, H)$, $\phi$ must be a complete isometry.

6. On relative bicommutants

We shall now turn to the characterization of pairs $(A, B)$ admitting duality, but first we need a general result concerning bicommutants and two short lemmas.

**Theorem 6.1.** Let $\pi : A \to C$ be a $*$-homomorphism between $C^*$-algebras (making $C$ an $A$-bimodule). Assume that $C$ is injective and that $A \subseteq B(\mathbb{H})$ for some Hilbert space $\mathbb{H}$ such that there exists a bounded left $A$-module map $\phi : K(\mathbb{H}) \to C$ the kernel of which does not contain any nontrivial right ideals. If $\pi(A)$ is equal to its relative double commutant $\pi(A)^{cc}$ in $C$, then $A$ is a von Neumann algebra.

**Proof.** We may assume that $C \subseteq B(l)$ for some Hilbert space $l$ and denote by $\iota$ the inclusion of $C$ into $B(l)$ and by $E$ the conditional expectation from $B(l)$ to $C$. Let $\tilde{A}$ be the universal von Neumann envelope of $A$, $\Phi : A \to \tilde{A}$ the universal representation and let $\alpha : A \to \tilde{A}$ be the weak* continuous extension of $\Phi^{-1}$ (see [12] Section 10.1) if necessary). Let $\tilde{\pi} : A \to B(l)$ be the weak* continuous extension of $\iota \pi \Phi^{-1}$ from $\Phi(A)$ to $\tilde{A}$. Since $\tilde{\pi}(T) \in \pi(A)$ ($= \text{the weak* closure of } \pi(A)$ in $B(l)$) and $E$ is a $C$-bimodule map, for each $c \in \pi(A)^c$ we have

$$E(\tilde{\pi}(T)) c = E(\tilde{\pi}(T) c) = E(c \tilde{\pi}(T)) = c E(\tilde{\pi}(T)),$$

hence $E(\tilde{\pi}(T)) \in \pi(A)^{cc}$. By hypothesis there exists an $a_T \in A$ such that

$$E(\tilde{\pi}(T)) = \pi(a_T).$$

For each $x \in K(H)$ the map $\phi_x : \tilde{A} \to B(l), \phi_x(T) := \iota \phi(\alpha(T) x)$, is weak* continuous, since $\phi_x$ is the composition of $\alpha$, the right multiplication by $x$ and $\iota \phi$. (Note that $\iota \phi$, as any bounded linear map on $K(H)$, is weak* continuous since this holds for bounded linear functionals). Further, the map $\psi_x : \tilde{A} \to B(l), \psi_x(T) = \tilde{\pi}(T) \phi(x)$ is also weak* continuous since this holds for $\tilde{\pi}$. If $T = \Phi(a)$ for some $a \in A$, then $\phi_x(T) = \phi(ax) = \pi(a) \phi(x) = \tilde{\pi}(T) \phi(x) = \psi_x(T)$ (since $\phi$...
is a left \(a\)-module map), hence by weak* continuity \(\phi_\pi(T) = \psi_\pi(T)\) for all \(T \in \tilde{A}\). This means that \(\tilde{\pi}(T)\phi(x) = \phi(\alpha(T)x)\), hence, since \(E\) is a \(C\)-bimodule map fixing elements of \(C\),

\[ E(\tilde{\pi}(T))\phi(x) = \phi(\alpha(T)x) \quad \text{for all } T \in \tilde{A} \text{ and } x \in K(H). \]

Using (6.1) it follows now that

\[ \phi(\alpha_T x) = \pi(\alpha_T)\phi(x) = E(\tilde{\pi}(T))\phi(x) = \phi(\alpha(T)x), \]

hence \(\phi((\alpha_T - \alpha(T))K(H)) = 0\) and \(\alpha_T = \alpha(T)\) since the kernel of \(\phi\) does not contain any nontrivial right ideal. This shows that \(\alpha(A) \subseteq A\). But, since \(\alpha\) is weak* continuous, it is a well known consequence of the Kaplansky density theorem (together with Alaoglu’s theorem) that \(\alpha(\tilde{A}) = \tilde{A}\). Thus, \(A = \tilde{A}\).

**Example 6.2.** We note that the assumption in Theorem 6.1 that \(C\) is injective is not redundant. To see this, let \(H = \ell^2\), \(A = c_0\) (the sequences, converging to 0 identified as diagonal operators relative to the standard orthonormal basis of \(\ell^2\)), \(C = K(H)\), \(\pi : A \to C\) the inclusion and \(\phi : K(H) \to C\) the identity mapping. Then the relative bicommutant of \(A\) in \(C\) is \(A\), but \(A\) is not a von Neumann algebra.

Clearly, requiring merely that a \(C^*\)-algebra \(A\) is equal to its relative bicommutant in an injective \(C^*\)-algebra containing \(A\), does not imply that \(A\) is a von Neumann algebra (since any \(C^*\)-algebra \(A\) is equal to its relative bicommutant in \(A\)). But presently the author does not know if each monotone complete \(C^*\)-algebra \(A\) satisfies this bicommutation condition.

For \(C^*\)-algebras \(A, B\) admitting faithful cyclic Hilbert modules, one could now deduce from Theorems 6.1, 4.1 and 3.1 that the pair \((A, B)\) is a duality (if and) only if \(A\) and \(B\) are von Neumann algebras. But we shall prove this conclusion without assuming the existence of faithful cyclic Hilbert modules.

**Lemma 6.3.** For any operator space \(X\) and index set \(\mathcal{J}\) the multiplier algebra \(A_\mathcal{J}(M_2(X))\) can naturally be identified (completely isometrically) as a subspace in \(M_2(A_\mathcal{J}(X))\).

**Proof.** First observe, from the fact that each left multiplier \(\theta\) on \(M_2(X)\) is a module map over \(M_2(C) \subseteq A_\mathcal{J}(M_2(X))\), that \(\theta\) must act on a matrix \([x_{ij}] \in M_2(X)\), decomposed into columns \(x_j \in C(X)\), as \(\theta([x_{ij}]) = [\phi(x_{ij})]\), where \(\phi \in CB(C(X))\). Moreover, using Theorem 2.1(i) it follows that \(\phi\) must be a left multiplier on \(C(X)\).

Thus, a left multiplier on \(M_2(X)\) is just a left multiplier on \(C(X)\) applied to all columns of a matrix. The inclusion \(A_\mathcal{J}(C(X)) \subseteq M_2(A_\mathcal{J}(X))\) is proved in [5]. Remark following 5.10.1. To explain a slightly different approach, let \(Y = I(X)\) be the injective envelope of \(X\). Then \(C_\mathcal{J}(Y)\) is the injective envelope of \(C_\mathcal{J}(X)\) by [5, 4.6.12]. Assuming that \([YY^*]\) is contained in a von Neumann algebra \(R\) (which may be taken of the form \(R = B(H)\)), we have that the \(C^*\)-algebra \([C_\mathcal{J}(Y)R_\mathcal{J}(Y^*)]\) is contained in \(M_1(R)\), hence the same holds for its multiplier \(C^*\)-algebra \(A_\mathcal{J}(C_\mathcal{J}(Y))\).

But it follows from Theorem 2.1(ii) that \(A_\mathcal{J}(C_\mathcal{J}(X)) \subseteq A_\mathcal{J}(C_\mathcal{J}(Y))\) since \(C_\mathcal{J}(Y)\) is the injective envelope of \(C_\mathcal{J}(X)\), hence \(A_\mathcal{J}(C_\mathcal{J}(X)) \subseteq M_1(R)\). Finally, a matrix \([T_{ij}] \in M_1(R)\) multiplies \(C_\mathcal{J}(X)\) into itself only if \(T_{ij}X \subseteq X\) for all \(i, j \in \mathcal{J}\), hence \(A_\mathcal{J}(C_\mathcal{J}(X)) \subseteq M_1(A_\mathcal{J}(X))\).
Lemma 6.4. Let $A$ be a $C^*$-subalgebra of $A_\ell(X)$. Then for any index set $\mathbb{J}$ the bicommutant of $A^{(\mathbb{J})}$ in $A_\ell(M_2(X))$ satisfies

$$(A^{(\mathbb{J})})^{cc} = (A^{cc})^{(\mathbb{J})},$$

where $A^{cc}$ is the bicommutant of $A$ in $A_\ell(X)$.

Proof. By Lemma 6.3 $A_\ell(M_2(X)) \subseteq M_2(A_\ell(X))$. Moreover, $A_\ell(M_2(X))$ contains the algebra $\mathcal{F}$ of all finitely supported matrices in $M_2(A_\ell(X))$. It follows that for a $C^*$-subalgebra $A$ of $A_\ell(X)$, the commutant $(A^{(\mathbb{J})})^c$ of $A^{(\mathbb{J})}$ in $A_\ell(M_2(X))$ contains the set

$$C := \{ \phi = [\phi_{ij}] \in \mathcal{F} : \phi a^{(\mathbb{J})} = a^{(\mathbb{J})} \phi \forall a \in A \},$$

This implies that $(A^{(\mathbb{J})})^{cc} \subseteq C^c$. But a standard computation shows that $C^c$ consists of diagonal matrices $x^{(\mathbb{J})}$, where $x \in A^{cc}$, hence $(A^{(\mathbb{J})})^{cc} \subseteq (A^{cc})^{(\mathbb{J})}$. For the reverse inclusion, note that, given $x \in A^{cc}$, $x^{(\mathbb{J})}$ commutes with all matrices $[\phi_{ij}] \in (A^{(\mathbb{J})})^c = M_2(A^c) \cap A_\ell(M_2(X))$.

Remark 6.5. If $Z$ is a cogenerator of an ample category $\mathcal{O}$ of operator $A,B$-bimodules, then the $*$-homomorphism $A \to A_\ell(Z)$, by which the left $A$-module structure is introduced to $Z$, is injective. This can be deduced from Definition 6.2(i) and Theorem 6.6.

Theorem 6.6. Let $\mathcal{O}$ be an ample subcategory of $A_{OM_B}$ and $Z$ an injective cogenerator in $\mathcal{O}$. Suppose that $A$, regarded as a $C^*$-subalgebra of $A_\ell(Z)$, is equal to its bicommutant $A^{cc}$ in $A_\ell(Z)$. Then $A$ is a $W^*$-algebra.

Proof. Let $\mathcal{H} \in A_{HM}$ and $K \in B_{HM}$ be faithful Hilbert modules. By Theorem 6.3 there is a completely isometric $A,B$-bimodule map $\phi : B(K,\mathcal{H}) \to M_2(Z)$ for some index set $K$. By Theorem 6.1 $\phi$ induces an $A$-bimodule map $\phi_\ell : B(\mathcal{H}) \to A_\ell(M_2(Z))$ such that the kernel of $\phi_\ell$ does not contain any nontrivial right ideals. From the hypothesis and Lemma 6.3 the image of $A$ in $A_\ell(M_2(Z))$ is equal to its relative bicommutant, hence by Theorem 6.1 $A$ is a von Neumann algebra on $\mathcal{H}$.

By Theorem 6.6 a pair of $C^*$-algebras $(A,B)$ admits a duality (in the sense defined in the Introduction) only if $A$ and $B$ are von Neumann algebras. Conversely, each pair $(A,B)$ of von Neumann algebras admits a duality by the von Neumann bicommutation theorem, since $B(K,\mathcal{H})$ is an injective cogenerator for normal operator $A,B$-bimodules if $\mathcal{H}$ and $K$ are the universal normal Hilbert modules over $A$ and $B$, respectively.

7. INITIAL COGENERATORS

In this Section $M$ and $N$ are von Neumann algebras, $M_{NHM}$ the category of normal Hilbert $M$-modules and $M_{NONM_N}$ the category of all normal operator $M,N$-bimodules. (We do not require that a normal bimodule $X$ is a dual space, only that there exists a completely isometric $M,N$-bimodule map from $X$ into $B(K,\mathcal{H})$ for some normal Hilbert modules $\mathcal{H} \in M_{NHM}$ and $K \in N_{NHM}$.) Recall from the Introduction that a cogenerator $Z$ in $M_{NONM_N}$ is countably initial if for each cogenerator $X$ there is a completely isometric $M,N$-bimodule map from $Z$ into $X^{\mathbb{J}}$.

If $M$ and $N$ admit cyclic modules $\mathcal{H} \in M_{NHM}$ and $K \in N_{NHM}$ such that all normal states on $M$ and $N$ are vector states coming from vectors in $\mathcal{H}$ and $K$, then
B(K, ℋ) is a cogenerator in _M_ ω _N_ (see the paragraph following Definition 1).

Moreover, it follows from Theorem 3.1 that B(K, ℋ) is countably initial. We shall show that the condition on _M_ and _N_ is satisfied if (and only if) the centers of _M_ and _N_ are σ-finite. This will imply the first part of the following Theorem.

**Theorem 7.1.** If the centers of _M_ and _N_ are σ-finite, then _M_ ω _N_ has a countably initial cogenerator of the form B(K, ℋ), where ℋ ∈ _N_ ω _M_ and _K_ ∈ _N_ ω _M_ are cyclic and such that all normal states on _M_ and _N_ are vector states coming from vectors in _ℋ_ and _Κ_ respectively. If the center of _M_ is not σ-finite, then _M_ ω _N_ has no countably initial cogenerators.

By the observation above, one direction of the Theorem is an immediate consequence of Theorem 3.1 and the following lemma.

**Lemma 7.2.** If the center of _M_ is σ-finite, then there exists a cyclic _ℋ_ ∈ _M_ ω _N_ such that all normal states on _M_ are vector states coming from vectors in _ℋ_.

Since we have not found a reference for the lemma, we include a proof.

**Proof of Lemma 7.2.** We shall use repeatedly the fact that a sum of a countably family of centrally orthogonal cyclic projections in a von Neumann algebra is cyclic [12, 5.5.10]. Decomposing _M_ into the direct sum of the finite, the properly infinite (and semifinite) and the purely infinite part, we may consider each part separately.

If _M_ is finite, then _M_ is σ-finite (since the center of _M_ is) by [12, 8.2.9] or [27, V.2.9], hence it can be represented on a Hilbert space _ℋ_ so that it has a cyclic and separating (trace) vector and then all normal states on _M_ are vector states coming from vectors on _ℋ_ by [12, 7.2.3] (or [27, V.1.12]).

If _M_ is properly infinite and semifinite, then by [27, V.1.40] _M_ is a direct sum of a countable family of algebras of the form _N_ ω _B(_ _G_ _)_ , where _N_ is finite and _G_ is infinite-dimensional. Representing _N_ on a space _ℋ_ _N_ where it has a cyclic and separating vector and taking the tensor product of this representation with the countable multiple of the identity representation of _B(_ _G_ _)_ (cyclic), gives a cyclic representation of _N_ ω _B(_ _G_ _)_ on _ℋ_ _N_ ⊗ _G_ ⊗ ℓ_ 2 _N_ such that all normal states are vector states.

Finally, if _M_ is purely infinite, acting on a Hilbert space _G_ , let _{e_j^′ : j ∈ _J_ }_ be a maximal family of cyclic projections in _M_ ′ with mutually orthogonal central carriers _p_j_. Then _J_ is countable and, since the central carrier of the projection _e′ := ∑_j∈_J_ _e_j^′_ is 1 by maximality, the representation of _M_ on _e′_ _G_ ( _a → a_e′_ ) is faithful and cyclic. In particular ( _M_e′_ ) ′ has a separating vector and is therefore σ-finite. However, since _M_ is of type III with σ-finite center, by [27, V.3.2] _M_ has a unique faithful normal representation with σ-finite commutant, up to a unitary equivalence. So, the countable multiple of the representation of _M_ on _e′_ _G_ is unitarily equivalent to the same representation and therefore each normal state on _M_ must be a vector state coming from a vector in _ℋ_ := _e′_ _G_.

We remark that the Hilbert modules _ℋ_ and _Κ_ in Theorem 3.1 are not necessarily the ones on which _M_ and _N_ are in the standard form. To see this, we may consider _M_ = _N_ = B(ℓ_ 2 _N_ ) = _M_ (C), where _I_ is an uncountable set. Then _X_ := B(ℓ_ 2 _I_ ⊗ ℓ_ 2 _N_ ) = _M_ (B(ℓ_ 2 _N_ )) is an injective cogenerator for operator _M_ -bimodules, but can not be embedded into B(ℓ_ 2 _I_ ⊗ ℓ_ 2 _N_ ) _I_ = _M_ (B(ℓ_ 2 _N_ ) _I_ ) = _M_ (B(ℓ_ 2 _N_ ) _I_ ) as an _M_ -bimodule. Namely, such an embedding would be the amplification of an embedding of B(ℓ_ 2 _I_ ) into B(ℓ_ 2 _N_ ) _I_ , which can not exist if _I_ is large enough.
Since a representation (state) of a C*-algebra $A$ is just a normal representation (state) of its universal von Neumann envelope $\bar{A}$ and the center of $\bar{A}$ is $\sigma$-finite if and only if each family of disjoint cyclic representations of $A$ is countable (this follows using [12 10.3.3]), the method of Theorem 7.1 proves also the first part of the following Proposition.

**Proposition 7.3.** If all families of disjoint cyclic representations of C*-algebras $A$ and $B$ are countable, then the category $\text{AOM}_B$ has an initial strict cogenerator of the form $B(K, \mathcal{H})$, where $\mathcal{H} \in \text{AOM}_K$ and $K \in \langle A \rangle$ are cyclic. If $A$ has an uncountable family of disjoint (cyclic) representations, then $\text{AOM}_A$ has no initial strict cogenerators.

It is known that separable non type I C*-algebras have uncountably many disjoint (cyclic) representations [24 6.8.5]. Thus, separable C*-algebras satisfying the condition of the above Proposition, are of type I and have only countably many inequivalent irreducible representations.

We still have to prove the last parts of Theorem 7.1 and Proposition 7.3. Let $\pi : \mathbb{I} \to \mathbb{I}$ be a (fixed through the rest of the section) maximal orthogonal family of central projections in $M$ that are $\sigma$-finite in the center. Hence $\sum_{i \in \mathbb{I}} \pi_i = 1$ by maximality. We shall assume that $\mathbb{I}$ is uncountable (otherwise the center of $M$ is $\sigma$-finite) and denote by $S$ the family of all countable subsets of $\mathbb{I}$. For each $s \in S$ let $p_s := \sum_{i \in s} \pi_i$. By Lemma 7.2 we may choose for each $i \in \mathbb{I}$ a normal cyclic Hilbert $p_i M$-module $\mathcal{H}_i$ such that all normal states on $p_i M$ are vector states from vectors in $\mathcal{H}_i$. Let $\mathcal{H}_M = \bigoplus_i \mathcal{H}_i$, so that $\mathcal{H}_i = p_i \mathcal{H}_M$. Let $K$ be any fixed Hilbert space ($K = \mathbb{C}$ will be sufficient for our application here).

**Definition 7.4.** Let $Z_M$ be the $M$-submodule of $B(K^G, H_M) = R_S(B(K, H_M))$, consisting of all $z = [z_s]_{s \in S}$, where $z_s \in B(K, \mathcal{H}_M)$ are such that:

(i) $z_s = p_s z_s$ for all $s \in S$ and

(ii) for each $i \in \mathbb{I}$ the set of all $s \in S$ such that $p_s z_s \neq 0$ is countable.

Each normal cyclic Hilbert $M$-module $\mathcal{H}$ is the orthogonal sum of submodules $p_i \mathcal{H}$, where the set $s$ of all $i \in \mathbb{I}$ such that $p_i \mathcal{H} \neq 0$ is countable (by cyclicity). Since each $p_i \mathcal{H}$ is cyclic over $p_i M$ and all normal states on $p_i M$ are vector states coming from vectors in $p_i \mathcal{H}_M$, $p_i \mathcal{H}$ is contained in $p_i \mathcal{H}_M$ (up to a unitary equivalence), hence $\mathcal{H}$ is contained in $\bigoplus_{i \in \mathbb{I}} p_i \mathcal{H}_M = p_\mathcal{H} \mathcal{H}_M$. Since $p_\mathcal{H} \mathcal{H}_M$ is contained in $\mathcal{H}_M$ and in $Z_M$ (if $\mathcal{K} = \mathbb{C}$ then $p_\mathcal{H} \mathcal{H}_M$ is just the $s$-th column of $Z_M$), it follows now from Proposition 5.7 that $\mathcal{H}_M$ and $Z_M$ are cogenerators in $\text{OM}_M$.\n
**Lemma 7.5.** For each $j \in \mathbb{I}$ let $y_j = p_j y_j$ be a norm 1 element in $Z_M^J$ and let $i \in \mathbb{I}$ be fixed. Then the set $J$ of all $j \in \mathbb{I}$ such that

$$\| \begin{bmatrix} y_i & y_j \end{bmatrix} \| > 1$$

is countable.

**Proof.** Let $y_{j,n} \in Z_M$ be the components of $y_j$. If $J$ is uncountable, then for some $n$ the set $J_0$ of all $j \in J$ such that

$$\| \begin{bmatrix} y_{i,n} & y_{j,n} \end{bmatrix} \| > 1$$

(7.1)
is uncountable. Put \( z(j) = y_{j,n} \) and let \( z(j)_s \in B(\mathcal{K}, \mathcal{H}_M) \) be the components of \( z(j) = p_j z(j) \in R_p B(\mathcal{K}, \mathcal{H}_M) \). Let \( S_i = \{ s \in S : z(i)_s \neq 0 \} \) (a countable set by Definition 7.4). Since each \( p_j \mathcal{H}_M \) is cyclic, by Proposition 5.7 \( Z_0 \) contains \( p_j \mathcal{H}_M \) completely isometrically as an \( M \)-module for each \( j \in J \), hence we can choose an element \( y_j = p_j y_j \in Z_0^N \) of norm 1. Since \( \mathcal{H}_M \) is a cogenerator and \( Z_0 \) is countably initial, we may regard \( Z_0 \) as an \( M \)-submodule in \( \mathcal{H}_M^N \), hence \( Z_0^N \) as a submodule in \( \mathcal{H}_M^{N \times N} \cong \mathcal{H}_M^N \). Let \( y_{j,n} \in \mathcal{H}_M \) be the components of \( y_j \) regarded as an element in \( \mathcal{H}_M^N \). Then for each \( j \in J \) there exists \( n \in N \) such that \( \|y_{j,n}\| > 1 - \varepsilon/2 \). Since \( Z_0 \) is uncountable, there exists a \( n \) such that \( \|y_{j,n}\| > 1 - \varepsilon/2 \) for all \( j \) in an uncountable subset \( J \) of \( J \). If \( i, j \in J \), then, since \( y_{i,n} \) and \( y_{j,n} \) are vectors in the Hilbert space \( \mathcal{H}_M \), this implies that
\[
\left\| \begin{bmatrix} y_{i,n} \\ y_{j,n} \end{bmatrix} \right\| > \sqrt{2} - \varepsilon \text{ if } i,j \in J.
\]
It follows that
\[
(7.2) \quad \left\| \begin{bmatrix} y_i \\ y_j \end{bmatrix} \right\| > \sqrt{2} - \varepsilon \text{ if } i,j \in J.
\]

Since \( Z_0 \) is countably initial and \( Z_M \) is a cogenerator, \( Z_0 \) is contained as an \( M \)-submodule in \( Z_M^N \) completely isometrically, hence \( Z_0^N \) is contained in \( Z_M^{N \times N} \cong Z_M^N \). But then by Lemma 7.4 each set of elements \( y_j \) in \( Z_0^N \) satisfying \( (7.2) \) (for a fixed \( i \)) must be countable, which contradicts the fact that \( J \) in \( (7.1) \) is uncountable. □

Proof of Proposition 7.3. We still have to prove that \( A \text{OM}_C \) has no initial strict cogenerators if \( A \) has uncountably many disjoint cyclic representations. The condition on \( A \) means that the center of the universal von Neumann envelope \( \hat{A} \) of \( A \) is not \( \sigma \)-finite. If we put \( M = \hat{A} \) and construct \( \mathcal{H}_M \) and \( Z_M \) as above, then \( \mathcal{H}_M \) and \( Z_M \) are strict cogenerators in \( A \text{OM}_C \) by Proposition 5.7. Thus, if there exists an initial strict cogenerator \( Z_0 \), then both, \( \mathcal{H}_M \) and \( Z_M \) contain a copy of \( Z_0 \). Similarly as in the above proof of Theorem 7.4 (but easier), the fact that \( Z_0 \) is a strict cogenerator implies the existence of an uncountable set \( \{ y_j : j \in J \} \) satisfying \( (7.2) \), while the inclusion \( Z_0 \subseteq Z_M \) together with Lemma 7.4 shows that this is impossible. □
The same technique shows that if the center of at least one of the algebras $M$ or $N$ is not $\sigma$-finite the category $M_{\text{NOM}}N$ has no countably initial cogenerators of the form $B(K, \mathcal{H})$. It seems natural to conjecture that in this case $M_{\text{NOM}}N$ has no countably initial cogenerators at all. The above technique can be upgraded to prove this in the case one of the two algebras has a separable predual (while the center of the other is not $\sigma$-finite), but the general case remains open.

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Department of Mathematics, University of Ljubljana, Jadranska 19, Ljubljana 1000, Slovenia

E-mail address: Bojan.Magajna@fmf.uni-lj.si