On the dynamics of finite temperature trapped Bose gases

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Abstract

The system that describes the dynamics of a Bose-Einstein Condensate (BEC) and the thermal cloud at finite temperature consists of a nonlinear Schrödinger (NLS) and a quantum Boltzmann (QB) equations. In such a system of trapped Bose gases at finite temperature, the QB equation corresponds to the evolution of the density distribution function of the thermal cloud and the NLS is the equation of the condensate. The quantum Boltzmann collision operator in this temperature regime is the sum of two operators $C_{12}$ and $C_{22}$, which describe collisions of the condensate and the non-condensate atoms and collisions between non-condensate atoms. Above the BEC critical temperature, the system is reduced to an equation containing only $C_{22}$, which possesses a blow-up positive radial solution with respect to the $L^{\infty}$ norm (cf. [38]). On the other hand, at the very low temperature regime, the system becomes an equation of $C_{12}$, with a different (much higher order) transition probability, which has a unique global positive radial solution with weighted $L^{1}$ norm (cf. [3]). In the current temperature regime, we first decouple the QB and NLS equations, then show a global existence and uniqueness result for positive radial solutions to the spatially homogeneous kinetic system. Different from the case considered in [38], due to the presence of the BEC, the collision integrals are associated to sophisticated energy manifolds rather than spheres, since the particle energy is approximated by the Bogoliubov dispersion law. Moreover, the mass of the full system is not conserved while it is conserved for the case considered in [38]. A new theory is then supplied.

Keyword: Quantum kinetic theory; Bose-Einstein condensate; quantum Boltzmann equation; defocusing cubic nonlinear Schrödinger equation; quantum gases.

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1 Introduction

The study of kinetic equations has a very long history, starting with the classical Boltzmann equation, which provides a description of the dynamics of dilute monoatomic gases (cf.

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As an attempt to extend the Boltzmann equation to deal with quantum gases, the Uehling-Uhlenbeck equation was introduced \[80, 95\]. However, the Uehling-Uhlenbeck equation fails to describe a Bose gas at temperatures which are close to and below the Bose-Einstein Condensate (BEC) critical temperature, due to the fact that its steady-state solution is a Bose-Einstein distribution in particle energies. Below the critical temperature, many-body effects modify the equilibrium distribution so that this distribution depends on quasiparticle energies. These are accounted for by mean fields which break the the unperturbed Hamiltonian \(U(1)\) gauge symmetry. Therefore, a new description in terms of quasiparticles is required. The first attempt to obtain a kinetic model for Bose gases below the critical temperature was carried on by Kirkpatrick and Dorfman \[66, 67\]. After the production of the first BECs, that later led Cornell, Wieman, and Ketterle to the 2001 Nobel Prize of Physics \[4, 5, 11\], there has been an explosion of research on the kinetic theory associated to BECs (see \[88, 87, 65, 18, 30, 34, 59, 66, 67, 98, 83, 90, 51, 91, 1, 48, 43, 62, 44, 61, 45, 49\], and references therein). The Kirkpatrick-Dorfman model was revisited by Zaremba, Nikuni, Griffin in \[98, 51\] with a simpler technique. Later, Gardiner, Zoller and collaborators derived a Master Quantum Kinetic Equation for BECs, which at the limit returns to the Kirkpatrick-Dorfman-Zaremba-Nikuni-Griffin (KDZNG) system, which reads model, and introduced the terminology “Quantum Kinetic Theory” in the series of papers \[48, 43, 62, 44, 61, 45, 49\]. Note that in the pioneering BEC experiment \[5\], one can observe the growth of the condensate after fast evaporative cooling, which cools the gas below the BEC transition temperature. The condensate growth is an interesting dynamical process, and a complete theoretical description must include the condensate and the interactions with the cloud of thermal atoms. It was in Gardiner, Zoller, et. al. \[48\] that the first numerical studies of such condensate growth were done, before any available condensate formation data. Their results are, indeed, in qualitative agreement with existing experiments \[4, 29\]. Later on, their theory was improved to include the dynamics of low-lying trap levels, following the MIT-controlled growth experiment \[5\]; that led to the first quantitative results with a good overall agreement with the experiment \[49\]. We refer to the review paper \[6\], the book \[60\] and the series \[82, 46, 94, 47\], for more discussions and a complete list of references on this topic. In \[84, 55, 54\], Reichl and Gust made a breakthrough in discovering a new collision operator, which had been missing in the previous works. More details on the derivation of this new collision operator can be found in the work \[85\].

Notice that the first proof of Bose-Einstein Condensation was given in 2002 by Lieb and Seiringer (cf. \[70\]). Besides the kinetic theory point of view, there are other ways of understanding the dynamics of BECs and their thermal clouds. For instance, the excitation spectrum \[86\], the time-dependent Hartree-Fock-Bogoliubov equations \[10\].

For the last 20 years, as an essential tool in modelling diverse phenomena, kinetic theory has become a very active search field in mathematics (see \[23, 98, 95, 82, 77, 52, 53, 92, 76, 73, 19, 58, 12, 61\] and references therein). Recently, quantum kinetic theory has also emerged as an important topic of strong interest (see \[69, 68, 87, 39, 27, 18, 15, 14, 35, 22\] and references therein). A remarkable mathematical result in quantum kinetics has been
Figure 1: The Bose-Einstein Condensate (BEC) and the excited atoms.

done in [38], where the authors constructed a class of initial data, which leads to finite time blow-up radial solutions of the Uehling-Uhlenbeck equation in the $L^\infty$ norm.

In the current paper, we are interested in the mathematical study of the Kirkpatrick-Dorfman-Zaremba-Nikuni-Griffin (KDZNG) system, which reads

\[
\frac{\partial f}{\partial t} + p \cdot \nabla_r f = Q[f] := C_{12}[f] + C_{22}[f], (t, r, p) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3, \quad (1.1)
\]

\[
f(0, r, p) = f_0(r, p), (r, p) \in \mathbb{R}^3 \times \mathbb{R}^3,
\]

\[
C_{12}[f](t, r, p_1) := \frac{2g^2 N_c}{(2\pi)^2 \hbar} \int \int_{\mathbb{R}^3 \times \mathbb{R}^3} K^{12}(p_1, p_2, p_3) \delta(p_1 - p_2 - p_3) \delta(E_{p_1} - E_{p_2} - E_{p_3}) f(t, r, p_1) \left[(1 + f(t, r, p_1))f(t, r, p_2)f(t, r, p_3) - f(t, r, p_1)(1 + f(t, r, p_2))(1 + f(t, r, p_3))\right] dp_2 dp_3,
\]

\[
C_{22}[f](t, r, p_1) := \frac{2g^2 N_c}{(2\pi)^5 \hbar^4} \int \int \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3} K^{22}(p_1, p_2, p_3, p_4) \delta(p_1 + p_2 - p_3 - p_4) \delta(E_{p_1} + E_{p_2} - E_{p_3} - E_{p_4}) \times
\]

\[
\times [(1 + f(t, r, p_1))(1 + f(t, r, p_2))f(t, r, p_3)f(t, r, p_4) - f(t, r, p_1)f(t, r, p_2)(1 + f(t, r, p_3))(1 + f(t, r, p_4))] dp_2 dp_3 dp_4,
\]

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where \( N_c = |\Phi|^2 \) is the condensate density, \( \Phi \) satisfies

\[
i\hbar \frac{\partial \Phi(r, t)}{\partial t} = \left( -\frac{\hbar \Delta_r}{2m} + g|\Phi(r, t)|^2 \right) \Phi(r, t), \quad \Phi(0, r) = \Phi_0(r), \forall r \in \mathbb{R}^3,
\]

and \( E_p \) is the Bogoliubov dispersion law

\[
E_p = E(p) = \sqrt{\kappa_1 |p|^2 + \kappa_2 |p|^4}, \quad \kappa_1 = \frac{gN_c}{m} > 0, \quad \kappa_2 = \frac{1}{4m^2} > 0,
\]

\( m \) is the mass of the particles, \( g \) is the interaction coupling constant.

Notice that (1.2) describes collisions of the condensate and the non-condensate atoms (condensate growth term), (1.3) describes collisions between non-condensate atoms, and (1.4) is the defocusing nonlinear Schrodinger equation of the condensate (see Figure 1). For the sake of simplicity, we denote \( \lambda_1 = \frac{2g^2N_c}{(2\pi)^2\hbar^2} \) and \( \lambda_2 = \frac{2g^2}{(2\pi)^4\hbar^2} \).

The transition probability kernel

\[
K^{12}(p_1, p_2, p_3) = |A^{12}(|p_1|, |p_2|, |p_3|)|^2
\]

defined by the scattering amplitude

\[
A^{12}(|p_1|, |p_2|, |p_3|) :=
\]

\[
:= (u_{p_3} - v_{p_3})(u_{p_3} u_{p_2} + v_{p_3} v_{p_2}) + (u_{p_2} - v_{p_2})(u_{p_1} u_{p_3} + v_{p_1} v_{p_3}) - (u_{p_1} - v_{p_1})(u_{p_2} v_{p_3} + v_{p_2} u_{p_3}),
\]

where

\[
u^2_p = \frac{\nu^2}{2m} + \frac{gN_c + E_p}{2E_p}, \quad u^2_p - v^2_p = 1.
\]

Define the characteristic momentum for the crossover between the linear and the quadratic part of the spectrum to be \( p_0 = 2mN_c \nu \), in which \( \nu \) is the repulsive point interaction. The following approximations hold true (cf. \[34, 59\]):

- If \(|p_1|, |p_2|, |p_3| > p_0\), then

\[
A^{12}(|p_1|, |p_2|, |p_3|) \approx 1.
\]

- If \(|p_1|, |p_2|, |p_3| < p_0\), then

\[
A^{12}(|p_1|, |p_2|, |p_3|) \approx \frac{3}{2^{7/4}} \left( \frac{|p_1||p_2||p_3|}{p_0^3} \right)^{1/2}.
\]

- If \(|p_1|, |p_2| > p_0, |p_3| < p_0\), then

\[
A^{12}(|p_1|, |p_2|, |p_3|) \approx 2^{3/4} \left( \frac{|p_3|}{p_0} \right)^{3/2}.
\]
Note that, $K^{12}$ also has a simplified form

$$C_K([p_1 \wedge p_0]| [p_2 \wedge p_0], [p_3 \wedge p_0]) = C_K \min\{\min\{p_1, p_0\}, \min\{p_2, p_0\}, \min\{p_3, p_0\}\}, \quad (1.7)$$

where $C_K$ is some physical constant. This form of the transition probability $K^{12}$ is very similar to the one used in \[81, 78, 34\].

The transition probability kernel

$$K^{22}(p_1, p_2, p_3, p_4) = |A^{22}(p_1, p_2, p_3, p_4)|^2$$

of $C^{22}$ is given by the scattering amplitude

$$A^{22}([p_1], [p_2], [p_3], [p_4]) := u_{p_1} u_{p_2} u_{p_3} u_{p_4} + u_{p_1} v_{p_2} u_{p_3} v_{p_4} + v_{p_1} u_{p_2} v_{p_3} u_{p_4} + v_{p_1} v_{p_2} v_{p_3} v_{p_4}. \quad (1.8)$$

Let us point out a difficulty that arises from the form of the transition probability $A^{22}$. If all momenta are much smaller than $p_0$ i.e. $|p_1|, |p_2|, |p_3| << p_0$, we obtain the following unphysical asymptotic behavior (cf. \[34\])

$$|A^{22}([p_1], [p_2], [p_3], [p_4])|^2 \approx |p_1|^{-1} |p_2|^{-1} |p_3|^{-1} |p_4|^{-1}. \quad (1.9)$$

However, phenomenological approaches predict (cf. \[176, 34\]) the following asymptotic behavior for $A^{22}$

$$|A^{22}([p_1], [p_2], [p_3], [p_4])|^2 \approx C_{A^{22}} |p_1| |p_2| |p_3| |p_4|,$$

where $C_{A^{22}}$ is some positive physical constant. This may have a connection with the divergences discussed in \[81, 78, 34\], and the correct transition probability could be very complicated. We do not intend to go through a deep investigation of this sophisticated question in the scope of our paper. As a consequence, to avoid this singular behavior, the following transition probability is chosen

$$K^{22}(p_1, p_2, p_3, p_4) = |A^{22}([p_1], [p_2], [p_3], [p_4])|^2 \chi_{\{|p_1|, |p_2|, |p_3|, |p_4| \geq p_*\}}, \quad (1.10)$$

where $\chi_{\{|p_1|, |p_2|, |p_3|, |p_4| \geq p_*\}}$ is the characteristic function of the set $\{|p_1|, |p_2|, |p_3|, |p_4| \geq p_*\}$, for some positive constant $p_*$. namely, it turns out that we need to consider the collisions between non-condensate atoms only in the high-temperature region with large momenta.

With this truncated transition probability, there exists a positive constant $\Gamma$ depending on $p_*$, such that

$$K^{22}(p_1, p_2, p_3, p_4) < \Gamma. \quad (1.10)$$

Now, for Equation (1.4), putting $\Phi = e^{-it}\Psi$ yields

$$ih\frac{\partial\Psi(r, t)}{\partial t} = \left(-\frac{\hbar^2}{2m} + g|\Psi(r, t)|^2 - \hbar\Psi(r, t)\right)\Psi(r, t), \quad \Psi(0, r) = \Phi_0(r), \forall r \in \mathbb{R}^3. \quad (1.11)$$
We impose the following boundary condition on $\Psi$

$$\lim_{|r| \to \infty} \Psi = C\Psi,$$  \hspace{1cm} (1.12)

where $C\Psi$ is some positive constant. For more physical background of the boundary condition (1.12), we refer to \[42, 64, 63, 17, 93\] and references therein.

Denote

$$\langle x \rangle = \sqrt{2 + |x|^2},$$

we recall the following theorem from \[57\].

**Theorem 1.1** There exists a positive constant $\delta$ such that for any initial condition $\Phi_0 \in H^1(\mathbb{R}^3)$ and

$$\int_{\mathbb{R}^3} \langle r \rangle^2 \left( |\text{Re}\Phi_0(r)|^2 + |\nabla\Phi_0(r)|^2 \right) dr < \delta^2,$$ \hspace{1cm} (1.13)

Equation (1.11)-(1.12) has a unique global solution $\Psi_\delta$. Moreover, there exists a positive constant $n_c$ depending on $m, g, h$, such that

$$\Psi_\delta = \sqrt{n_c} + \Omega_\delta,$$

where $\Omega_\delta = \Omega_{\delta 1} + i\Omega_{\delta 2}$ and

$$\|\Omega_{\delta 1}(t)\|_{L^\infty} \leq O(t^{-1}), \quad \|\Omega_{\delta 2}(t)\|_{L^\infty} \leq O(t^{-9/10}).$$ \hspace{1cm} (1.14)

The above theorem implies that with a suitable choice of $\Phi_0$, the condensate density distribution function $N_c = |\Psi_\delta|^2$ can be considered as a constant $n_c$. By the theorem above, we have decoupled the quantum Boltzmann and the cubic nonlinear Schrödinger equations. In an ongoing work \[89\], we are working on the strong coupling between the nonlinear Schrödinger and the quantum Boltzmann equations.

Imposing the assumption that

$$N_c = n_c,$$

the system (1.1)-(1.4) is valid in the high temperature range. In the lower temperature range, sometimes, we can suppose (cf. \[34, 40\]) that the interaction between bosons, i.e. the $C_{22}$ collision operator, is negligible, and the BEC is very stable. In this case, the system can be reduced to a kinetic equation involving the $C_{12}$ collision operator only:

$$\frac{\partial f}{\partial t} = C_{12}[f], \quad f(0, p) = f_0(p), \forall p \in \mathbb{R}^3.$$ \hspace{1cm} (1.15)

In this low temperature regime, the transition probability takes the form $C_K|p_1||p_2||p_3|$, which is unbounded, while (1.6) is bounded. An attempt to build a mathematical theory for Equation (1.15) has been carried on in the series of work \[3, 79, 28, 41\]. In \[3, 79\] it has been proved that (1.15) has a unique positive radial solution, based on an argument of
propagation of polynomial and exponential moments. We will see later that, unlike (1.15), polynomial and exponential moments of solutions of the system (1.1)-(1.4) are not propagating on the time interval \([0, \infty)\), due to the presence of the collision operator \(C_{22}\). In [79], it is proved that the solution of (1.15) is bounded from below by a Gaussian. In other words, the operator \(C_{12}\) is “strongly” positive.

In the current work, we restrict our attention to spatial homogeneous and radial solutions of (1.1)-(1.4)
\[
f(0, r, p) = f_0(|p|), \quad f(t, r, p) = f(t, |p|).
\]
By the same argument as in [38], \(C_{22}\) could be transformed into
\[
C_{22}[f] = \kappa_3 \int \int \int_{\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+} K^{22}(p_1, p_2, p_3, p_4) \frac{\min\{|p_1|, |p_2|, |p_3|, |p_4|\}}{|p_1|^2} \delta(E_{p_1} + E_{p_2} - E_{p_3} - E_{p_4}) f(p_3) f(p_4) (1 + f(p_1) + f(p_2)) - f(p_1) f(p_2) (1 + f(p_3) + f(p_4)) |d[p_2]| |d[p_3]| |d[p_4]|,
\]
where \(\kappa_3\) is some positive constant.

Equation (1.1) can be simplified as follows
\[
\frac{\partial f}{\partial t} = Q[f] = C_{12}[f] + C_{22}[f], \quad f(0, p) = f_0(|p|), \forall p \in \mathbb{R}^3.
\]

Above the BEC critical temperature, the density of the condensate \(n_c\) is 0, then \(C_{12} = 0\). Equation (1.17) is reduced to the Uehling-Ulenbeck equation
\[
\frac{\partial f}{\partial t} = C_{22}[f], \quad f(0, p) = f_0(|p|), \forall p \in \mathbb{R}^3,
\]
which has a blow-up positive radial solution in the \(L^\infty\) norm if the mass of the initial data is too concentrated around the origin (cf. [38]). Note that in this temperature regime, the transition probability is \(K^{22} = 1\) (cf. [54, 56, 85]), which is different from the regime considered in this paper. The existence of a global weak and measure solution for the equation was treated in [71, 72, 73]. In [21], local existence and uniqueness results, with respect to the \(L^\infty\) norm, were obtained for the Uehling-Ulenbeck equation. Let us mention that when the temperature is above the BEC critical temperature, the energy is of the form \(\frac{p^2}{2m}\). The collision of two microscopic boxes of particles with momenta \(p_1\) and \(p_2\) changes the momenta into \(p_3\) and \(p_4\); and the conservation laws read:
\[
|p_1|^2 + |p_2|^2 = |p_3|^2 + |p_4|^2, \quad p_1 + p_2 = p_3 + p_4.
\]
Since \(p_1, p_2, p_3, p_4\) belong to the sphere centered at \(\frac{p_1 + p_2}{2}\) with radius \(\frac{|p_1 - p_2|}{2}\), the collision operator \(C_{22}\) can be expressed as an integration on a sphere, following the strategy represented in [23, 96] for the classical Boltzmann operator.
In the temperature regime considered in our paper, $E_p$ is approximated by the Bogoliubov dispersion law (1.5), which means that the collision operators are integrals on much more complicated manifolds. New estimates on these energy manifolds are required. Moreover, (1.18) conserves the mass of the solution, while the full equation (1.17) does not. As a consequence, estimating the mass of the solution to (1.17) is a crucial task.

Let us define

$$L^1_m(\mathbb{R}^3) = \left\{ f \left| \| f \|_{L^1_m} := \int_{\mathbb{R}^3} |p|^m |f(p)| dp < \infty \right. \right\}, \quad (1.19)$$

$$L^1_m(\mathbb{R}^3) = \left\{ f \left| \| f \|_{L^1_m} := \int_{\mathbb{R}^3} |f(p)| E_p^{m/2} dp < \infty \right. \right\}, \quad (1.20)$$

$$L^1_m(\mathbb{R}^3) = \left\{ f \left| \| f \|_{L^1_m} := \int_{\mathbb{R}^3} |f(p)| \left(1 + E_p^{m/2}\right) dp < \infty \right. \right\}. \quad (1.21)$$

Our main result is the following theorem.

**Theorem 1.2** Suppose that $f_0(p) = f_0(|p|) \geq 0$, and

$$\int_{\mathbb{R}^3} (1 + E_p)f_0(p) dp < \infty.$$

Define

$$R = \max\left\{1, \int_{\mathbb{R}^3} f_0(p) \right\}.$$

For any time interval $[0, T]$, let $n$, $n^*$ be two positive integers, $n > 1$, $n_*$ is an odd number, $n^* > n + 4$, and $c_{n^*}$ be as in (2.8). We assume that

$$\int_{\mathbb{R}^3} E_p^{n^*} f_0(p) < c_{n^*}.$$

Then there exists a unique classical positive radial solution

$$f(t, p) = f(t, |p|) \in C^0([0, T], L^1_m(\mathbb{R}^3)) \cap C^1((0, T), L^1_{2n}(\mathbb{R}^3))$$

of (1.17). Moreover, there exists a constant $C_{n^*, T}$ depending on $n^*$ and $T$ such that

$$\sup_{t \in [0, T]} \int_{\mathbb{R}^3} E_p^{n^*} f(t, p) dp < C_{n^*, T}.$$

Let us mention the more general open problem for the system (1.1)-(1.4).

**Open Problem:** Let $\Psi_\delta$ be the solution defined in Theorem 1.1, then $\Phi_\delta = e^{-it}\Psi_\delta$ is the solution of (1.1). Suppose that $f_0(p) = f_0(|p|) \geq 0$. A reasonable question is if Equation
(1.1) with \( N_c = |\Phi_\delta|^2 \), has a unique positive solution \( f_\delta \) and if the sequence of solutions \( \{f_\delta\} \) converges to the solution \( f \) found in Theorem (1.2) in some topology as \( \delta \) tends to 0.

One of the key ingredients of the proof of Theorem (1.2) is the following theorem about the existence and unique of solutions to ODEs on Banach spaces. The theorem has an inspiration from \([20, 74, 3, 2]\). Notice that different from the previous cases considered in \([20, 74, 3, 2]\), we do not have the propagation of polynomial and exponential moments of the solution, as a consequence, we introduce new ideas to deal with this difficulty. Those ideas are discussed in Remarks (1.1) (1.2) and (1.3).

**Theorem 1.3** Let \([0, T]\) be a time interval, \( E := (E, \| \cdot \|) \) be a Banach space, \( S \) be a bounded, convex and closed subset of \( E \), and \( Q : S \to E \) be an operator satisfying the following properties:

(\(A\)) Let \( \| \cdot \|_\ast \) be a different norm of \( E \), satisfying \( \| \cdot \|_\ast \leq C_E \| \cdot \| \) for some universal constant \( C_E \), and the function

\[
| \cdot |_\ast : E \to \mathbb{R}, \quad u \to |u|_\ast,
\]

satisfying

\[
|u + v|_\ast \leq |u|_\ast + |v|_\ast, \quad \text{and} \quad |\alpha u|_\ast = \alpha |u|_\ast
\]

for all \( u, v \) in \( E \) and \( \alpha \in \mathbb{R}_+ \).

Moreover,

\[
|u|_\ast = \| u \|_\ast, \forall u \in S,
\]

\[
|u|_\ast \leq \| u \|_\ast \leq C_E \| u \|, \forall u \in E,
\]

and

\[
|Q(u)|_\ast \leq C_\ast (1 + |u|_\ast), \forall u \in S,
\]

then

\[
S \subset B_\ast \left(O, (2R_\ast + 1)e^{(C_\ast + 1)T}\right) := \left\{ u \in E \| u \|_\ast \leq (2R_\ast + 1)e^{(C_\ast + 1)T} \right\},
\]

for some positive constant \( R_\ast \geq 1 \).

(\(B\)) Sub-tangent condition

\[
\liminf_{h \to 0^+} h^{-1} \text{dist}(u + hQ[u], S) = 0, \quad \forall u \in S \cap B_\ast \left(O, (2R_\ast + 1)e^{(C_\ast + 1)T}\right),
\]

(\(C\)) Hölder continuity condition

\[
\|Q[u] - Q[v]\| \leq C\|u - v\|^\beta, \quad \beta \in (0, 1), \quad \forall u, v \in S,
\]
(3) one-side Lipschitz condition

\[ [Q[u] - Q[v], u - v] \leq C\|u - v\|, \quad \forall u, v \in S, \]

where

\[ [\varphi, \phi] := \lim_{h \to 0^-} h^{-1} (\|\phi + h\varphi\| - \|\phi\|). \]

Then the equation

\[ \partial_t u = Q[u] \text{ on } [0,T] \times E, \quad u(0) = u_0 \in S \cap B_*(O, R_*) \quad (1.22) \]

has a unique solution in \( C^1((0,T), E) \cap C([0,T], S) \).

**Remark 1.1** Note that for \((1.17)\), the mass is not conserved. We indeed prove that it grows exponentially in Section 2.1.3. As a consequence, in Theorem 1.3 besides the norm \( \| \cdot \| \) of the Banach space \( E \), we also need the second norm \( \| \cdot \|_* \) and the ball

\[ B_*(O, (2R_* + 1)e^{(C_*+1)T}), \]

which take the crucial role in controlling the mass of the solution on the time interval \([0, T]\). Thanks to the control on the mass, we can later prove that the collision operator \( Q \) in \((1.17)\) is indeed Holder continuous, which means Condition (C) of Theorem 1.3 is satisfied.

**Remark 1.2** In Theorem 1.3, \( \| \cdot \|_* \) is a function from \( E \) to \( \mathbb{R} \), that coincides with the second norm in \( \| \cdot \|_* \) in the set \( S \). This is due to the fact that, we will choose \( S \) to be a subset of the positive cone of \( E = L^{1_{2n}}(\mathbb{R}^3) \).

**Remark 1.3** In Condition (B) of Theorem 1.3 we do not consider the boundary case

\[ \|u\|_* = (2R_* + 1)e^{(C_*+1)T}. \]

Our idea of the proof is to start with an initial condition \( u(0) \) in the intersection of \( S \) and the ball \( B_*(O, R_*) \), and make \( u(t) \) evolve as long as

\[ \|u(t)\|_* < (2R_* + 1)e^{(C_*+1)T}. \]

This idea is realized, in a discrete way, in Part 2 of the proof of Theorem 1.3.

The plan of the paper is as follows:

- Section 2 is devoted to the proof of Theorem 1.2. This proof is divided into several steps:
In Section 2.1, basic properties of Equation (1.17) are presented. We prove that solutions of (1.17) conserve momentum and energy in Section 2.1.1. However, different from the Uehling-Uhlenbeck equation (1.18), the mass is not conserved for the full equation. Therefore, estimating the mass is a crucial task. Notice that different from previous studies (cf. [38]), where the energy is

\[ E_p = \frac{p^2}{2m}, \]

in our case, due to the presence of the condensate, the energy is approximated by the Bogoliubov dispersion law (1.5). This requires new estimates on the energy surfaces. Section 2.1.2 is devoted to such estimates. Based on these estimates, in Section 2.1.3 we provide a bound of the mass of solutions to Equation (1.17) on a finite time interval \([0, T]\).

As a key ingredient of the proof of Theorem 1.2, we show in Section 2.2 that polynomial moments with arbitrary high orders of solutions of (1.17) are bounded on a finite time interval \([0, T]\), which is the content of Proposition 2.4. Note that different from the very low temperature regimes considered in [3], in our regimes, polynomial moments are not propagating an created on \([0, \infty)\). The strategy of the proof of the proposition is to estimate moments of the collision operators \(C_{12}\) and \(C_{22}\), which are done in Sections 2.2.1 and 2.2.2 using results on energy surfaces of Section 2.1.2. Based on these estimates, we obtain a differential inequality for finite time moments of high orders in Section 2.2.3 which leads to the desired results of Proposition 2.4.

In Section 2.3 we prove that the collision operators \(C_{12}\) and \(C_{22}\) are Holder continuous, thanks to Proposition 2.4. In order to do this, we decompose \(C_{22}\) as the sum of two operators \(C_{22}^1\) and \(C_{22}^2\), where the first one is of second order and the second one is of third order. The operators \(C_{12}\), \(C_{22}^1\) and \(C_{22}^2\) are proven to be Holder continuous in Sections 2.3.1, 2.3.2 and 2.3.3 respectively, on any time interval \([0, T]\).

Using Theorem 1.3, we prove in Section 2.4 that Equation (1.17) has a unique positive, radial solution on any time interval \([0, T]\).

• The proof of Theorem 1.3 is given in Section 3

2 The quantum Boltzmann equation

2.1 Mass, momentum and energy of solutions of the kinetic equation

We will make use of the following notation

\[ m_k[f] = \int_{\mathbb{R}^3} \mathcal{E}_k(p_1)f(p_1)dp_1. \]
For convenience, we introduce

\[ C_{12}[f] = C_{12}^{1}[f] + C_{12}^{2}[f] \]  

(2.2)

with

\[ C_{12}^{1}[f] := \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \mathcal{K}^{12}(p_1, p_2, p_3) \left[ f(p_2)f(p_3) - f(p_1)(f(p_2) + f(p_3) + 1) \right] dp_2 dp_3 \]

\[ C_{12}^{2}[f] := -2 \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \mathcal{K}^{12}(p_2, p_1, p_3) \left[ f(p_1)f(p_3) - f(p_2)(f(p_1) + f(p_3) + 1) \right] dp_2 dp_3, \]

where the collision kernel is defined by

\[ \mathcal{K}^{12}(p_1, p_2, p_3) = \lambda_{1} n_{c} K^{12}(p_1, p_2, p_3) \left( \delta(\mathcal{E}(p_1) - \mathcal{E}(p_2) - \mathcal{E}(p_3)) \delta(p_1 - p_2 - p_3) \right). \]

We also define the energy surfaces

\[ S_{p} : = \{ p_{*} \in \mathbb{R}^3 : \mathcal{E}(p - p_{*}) + \mathcal{E}(p_{*}) = \mathcal{E}(p) \} \]

\[ S_{p}^{'} : = \{ p_{*} \in \mathbb{R}^3 : \mathcal{E}(p + p_{*}) = \mathcal{E}(p) + \mathcal{E}(p_{*}) \} \]

\[ S_{p}^{''} : = \{ p_{*} \in \mathbb{R}^3 : \mathcal{E}(p_{*}) = \mathcal{E}(p) + \mathcal{E}(p_{*} - p) \} \]

(2.3)

for all \( p \in \mathbb{R}^3 \setminus \{0\} \). Set

\[ \tilde{K}^{12}(p_1, p_2, p_3) = \lambda_{1} n_{c} K^{12}(p_1, p_2, p_3), \]

by the nature of the Dirac delta function, the collision operators can be expressed under the form of the following surface integrals

\[ C_{12}^{1}[f] := \int_{S_{p_{1}}} \tilde{K}^{12}(p_1, p_2, p_3) \left[ f(p_2)f(p_3) - f(p_1)(f(p_2) + f(p_3) + 1) \right] d\sigma(p_3) \]

\[ C_{12}^{2}[f] := 2 \int_{S_{p_{1}}} \tilde{K}^{12}(p_{1} + p_3, p_2, p_3) \left[ f(p_{1} + p_3)(f(p_1) + f(p_3) + 1) - f(p_1)f(p_3) \right] d\sigma(p_3). \]

We also split \( C_{12}[f] \) as the sum of gain and loss terms:

\[ C_{12}[f] = C_{12}^{\text{gain}}[f] - C_{12}^{\text{loss}}[f] \]

(2.4)

with

\[ C_{12}^{\text{gain}}[f] := \int_{S_{p_{1}}} \tilde{K}^{12}(p_{1} + p_3, p_2, p_3)f(p_1 + p_3)f(p_3) d\sigma(p_3) \]

+ \[ 2 \int_{S_{p_{1}}} \tilde{K}^{12}(p_{1} + p_3, p_1, p_3)f(p_1 + p_3)(f(p_1) + f(p_3) + 1) d\sigma(p_3), \]

\[ C_{12}^{\text{loss}}[f] := fC_{12}^{-}[f], \]

\[ C_{12}^{-}[f] := \int_{S_{p_{1}}} \tilde{K}^{12}(p_{1} + p_3, p_1, p_3) \left( f(p_1 + p_3) + f(p_3) + 1 \right) d\sigma(p_3) \]

+ \[ 2 \int_{S_{p_{1}}} \tilde{K}^{12}(p_{1} + p_3, p_1, p_3)f(p_3) d\sigma(p_3). \]
Similar as for \( C_{12} \), we also split \( C_{22} \) into gain and loss operators, as follows
\[
C_{22}[f] = C_{22}^{\text{gain}}[f] - C_{22}^{\text{loss}}[f],
\]
where
\[
C_{22}^{\text{gain}}[f] := \lambda_2 \iiint_{\mathbb{R}^3} K^{22}(p_1, p_2, p_3, p_4)(1 + f(p_1))(1 + f(p_2))f(p_3)f(p_4)dp_2dp_3dp_4,
\]
\[
C_{22}^{\text{loss}}[f] := fC_{22}^{-}[f],
\]
\[
C_{22}^{-}[f] := \lambda_2 \iiint_{\mathbb{R}^3} K^{22}(p_1, p_2, p_3, p_4)f(p_2)(1 + f(p_3))(1 + f(p_4))dp_2dp_3dp_4,
\]
and
\[
K^{22}(p_1, p_2, p_3, p_4) = \lambda_2 K^{22}(p_1, p_2, p_3, p_4)\delta(p_1 + p_2 - p_3 - p_4)\delta(\mathcal{E}_{p_1} + \mathcal{E}_{p_2} - \mathcal{E}_{p_3} - \mathcal{E}_{p_4}).
\]
We also split \( Q \) into the sum of a gain and a loss operators
\[
Q[f] = Q^{\text{gain}}[f] - Q^{\text{loss}}[f],
\]
where
\[
Q^{\text{gain}}[f] = C_{12}^{\text{gain}}[f] + C_{22}^{\text{gain}}[f],
\]
\[
Q^{\text{loss}}[f] = C_{12}^{\text{loss}}[f] + C_{22}^{\text{loss}}[f],
\]
and
\[
Q^{\text{loss}}[f] = fQ^{-}[f],
\]
with
\[
Q^{-}[f] = C_{12}^{-}[f] + C_{22}^{-}[f].
\]

2.1.1 Conservation of momentum and energy and the H-Theorem
In this section, we obtain the basic properties of smooth solutions of (1.17).

Lemma 2.1 There holds
\[
\int_{\mathbb{R}^3} Q[f](p_1)\varphi(p_1)dp_1 = \iint_{\mathbb{R}^3 \times \mathbb{R}^3} R_{12}[f](p_1, p_2, p_3)\left(\varphi(p_1) - \varphi(p_2) - \varphi(p_3)\right)dp_1dp_2dp_3
\]
\[
+ \frac{1}{2} \iint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3} R_{22}[f](p_1, p_2, p_3, p_4)\left(\varphi(p_1) + \varphi(p_2) - \varphi(p_3) - \varphi(p_4)\right)dp_1dp_2dp_3dp_4,
\]
for any smooth test function \( \varphi \), where
\[
R_{12}[f](p_1, p_2, p_3) = \lambda_1 n_c K^{12}(p_1, p_2, p_3)\delta(p_1 - p_2 - p_3)\delta(\mathcal{E}_{p_1} - \mathcal{E}_{p_2} - \mathcal{E}_{p_3})
\times [(1 + f(p_1))f(p_2)f(p_3) - f(p_1)(1 + f(p_2))(1 + f(p_3))],
\]
\[ R_{22}[f](p_1, p_2, p_3, p_4) = \lambda_2 K_{22}(p_1, p_2, p_3, p_4) \delta(p_1 + p_2 - p_3 - p_4) \delta(\mathcal{E}_{p_1} + \mathcal{E}_{p_2} - \mathcal{E}_{p_3} - \mathcal{E}_{p_4}) \times (1 + f(p_1))(1 + f(p_2))f(p_3)f(p_4) - f(p_1)f(p_2)(1 + f(p_3))(1 + f(p_4)) \times (1 + f(p_1))(1 + f(p_2))f(p_3)f(p_4) - f(p_1)f(p_2)(1 + f(p_3))(1 + f(p_4)). \]

**Proof** By a view of (1.17), we have

\[ \int_{\mathbb{R}^3} C_{12}[f](p_1) \varphi(p_1) dp_1 + \int_{\mathbb{R}^3} C_{22}[f](p_1) \varphi(p_1) dp_1 = I_1 + I_2, \]

where

\[ I_1 := \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3} \left( R_{12}[f](p_1, p_2, p_3) - R_{12}[f](p_2, p_1, p_3) - R_{12}[f](p_3, p_2, p_1) \right) \varphi(p_1) \, dp_1 dp_2 dp_3, \]

\[ I_2 := \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3} R_{22}[f](p_1, p_2, p_3, p_4) \varphi(p_1) \, dp_1 dp_2 dp_3 dp_4. \]

By switching the variables \( p_1 \leftrightarrow p_2, p_1 \leftrightarrow p_3 \) in the integrals of \( I_1 \) and \( (p_1, p_2) \leftrightarrow (p_2, p_1), (p_1, p_2) \leftrightarrow (p_3, p_4) \) in the integrals of \( I_2 \), respectively, as in [79] [3] [38], the lemma follows at once.

As a consequence, we obtain the following two corollaries.

**Corollary 2.1 (Conservation of momentum and energy)** Smooth solutions \( f(t, p) \) of (1.17) satisfy

\[ \int_{\mathbb{R}^3} f(t, p) dp = \int_{\mathbb{R}^3} f_0(p) dp \] (2.7)

\[ \int_{\mathbb{R}^3} f(t, p) \mathcal{E}(p) dp = \int_{\mathbb{R}^3} f_0(p) \mathcal{E}(p) dp \] (2.8)

for all \( t \geq 0 \).

**Proof** This follows from Lemma 2.1 by taking \( \varphi(p) = p \) or \( \mathcal{E}(p) \). \( \blacksquare \)

**Corollary 2.2 (H-Theorem)** Smooth solutions \( f(t, p) \) of (1.17) satisfy

\[ \frac{d}{dt} \int_{\mathbb{R}^3} [f(t, p) \log f(t, p) - (1 + f(t, p)) \log(1 + f(t, p))] \, dp \leq 0 \]

A radial symmetric equilibrium of the equation has the following form

\[ f_{\infty}(p) = \frac{1}{e^{c\mathcal{E}(p)} - 1} \] (2.9)

where \( c \) is some positive constant.
Proof. Observe that

\[ \partial_t \int_{\mathbb{R}^3} [f(t,p) \log f(t,p) - (1 + f(t,p)) \log(1 + f(t,p))] \, dp = \int_{\mathbb{R}^3} \partial_t f(t,p) \log \left( \frac{f(t,p)}{f(t,p) + 1} \right) \, dp, \]

and

\[ \int_{\mathbb{R}^3} Q[f(t,p)] \varphi(t,p) \, dp \]

\[ = \lambda_1 n_c \int \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3} K^{12}(p_1, p_2, p_3) \delta(p_1 - p_2 - p_3) \delta(E_{p_1} - E_{p_2} - E_{p_3}) \]

\[ \times (1 + f(t, p_1))(1 + f(t, p_2))(1 + f(t, p_3)) \left( \frac{f(t, p_2)}{f(t, p_2) + 1} \frac{f(t, p_3)}{f(t, p_3) + 1} - \frac{f(t, p_1)}{f(t, p_1) + 1} \right) \]

\[ \times [\varphi(p_1) - \varphi(p_2) - \varphi(p_3)] dp_1 dp_2 dp_3 \]

\[ + \frac{\lambda_2}{2} \int \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3} K^{22}(p_1, p_2, p_3, p_4) \delta(p_1 + p_2 - p_3 - p_4) \delta(E_{p_1} + E_{p_2} - E_{p_3} - E_{p_4}) \]

\[ \times (1 + f(t, p_1))(1 + f(t, p_2))(1 + f(t, p_3))(1 + f(t, p_4)) \times \]

\[ \times \left( \frac{f(t, p_3)}{f(t, p_3) + 1} \frac{f(t, p_1)}{f(t, p_1) + 1} - \frac{f(t, p_2)}{f(t, p_2) + 1} \right) \]

\[ \times [\varphi(p_1) + \varphi(p_2) - \varphi(p_3) - \varphi(p_4)] dp_1 dp_2 dp_3 dp_4. \]

Notice that

\[ (\alpha - \beta) \log \left( \frac{\alpha}{\beta} \right) \geq 0. \]

In the above inequality, the equality holds if and only if \( \alpha = \beta \). Now suppose that \( f_\infty(p) \) is a radial symmetric equilibrium. By Lemma 2.1 with \( \varphi(p) = \log \left( \frac{f_\infty(p)}{f_\infty(p) + 1} \right) \), we obtain

\[ \int_{\mathbb{R}^3} Q[f_\infty](p) \varphi(p) dp \leq 0. \]

This yields the inequalities in the H-theorem:

\[ \frac{f_\infty(p_2)}{f_\infty(p_2) + 1} \frac{f_\infty(p_3)}{f_\infty(p_3) + 1} - \frac{f_\infty(p_1)}{f_\infty(p_1) + 1} = 0, \]

\[ \frac{f_\infty(p'_2)}{f_\infty(p'_2) + 1} \frac{f_\infty(p'_3)}{f_\infty(p'_3) + 1} - \frac{f_\infty(p'_4)}{f_\infty(p'_4) + 1} = 0. \]

Setting \( h(p) = \log \left( \frac{f_\infty(p)}{f_\infty(p) + 1} \right) \), with the notice that \( h \) is radial symmetric, we get the following set of equations

\[ h(p_2) + h(p_3) = h(p_1), \quad \text{(2.10)} \]

and

\[ h(p'_3) + h(p'_4) = h(p'_2) + h(p'_4). \quad \text{(2.11)} \]
Let us consider (2.10). In particular, by the conservation law

\[ p_1 = p_2 + p_3, \]

the function \( h(p) \) possesses the following property

\[ h(p_2 + p_3) = h(p_2) + h(p_3), \]

for all \((p_2, p_3) \in \mathbb{R}^6\) satisfying

\[ \mathcal{E}(p_2 + p_3) = \mathcal{E}(p_2) + \mathcal{E}(p_3). \]

As a consequence, since \( h \) is radial symmetric,

\[ h \circ \mathcal{E}^{-1}(\alpha + \beta) = h \circ \mathcal{E}^{-1}(\alpha) + h \circ \mathcal{E}^{-1}(\beta), \]

where \( p_2 = \mathcal{E}^{-1}(\alpha) \) and \( p_3 = \mathcal{E}^{-1}(\beta) \). Notice that \( \alpha, \beta \) can take arbitrary values in \( \mathbb{R}_+ \), which implies \( h \circ \mathcal{E}^{-1}(\alpha) = -c\alpha \) for some positive constant \( c \) and for all \( \alpha \geq 0 \). Hence \( h(p) = -c\mathcal{E}(p) \), for all \( p \in \mathbb{R}^3 \). Identity (2.9) is proved. \( \blacksquare \)

### 2.1.2 Energy surfaces

We recall the following two lemmas from [79] about estimates on the energy surface integrals on \( S'_p \) and \( S''_p \).

**Lemma 2.2** Let \( S'_p \) be defined as in (2.3) and \( F : \mathbb{R}^3 \to \mathbb{R} \) be an arbitrary positive radial function

\[ F(u) = F(|u|) \]

satisfying

\[ \int_{\mathbb{R}^+} |u| F(|u|) d|u| < \infty. \]

There are positive constants \( C_0, C_1 \) independent of \( p \) such that

\[ \int_{S'_p} F(|w|) \, d\sigma(w) \leq C_0 \int_{\mathbb{R}^+} |u| F(|u|) d|u| = C_1 \int_{\mathbb{R}^3} \frac{F(u)}{|u|} d|u|. \]

**Lemma 2.3** Let \( S''_p \) be defined as in (2.3) and \( F : \mathbb{R}^3 \to \mathbb{R} \) be an arbitrary positive radial function

\[ F(u) = F(|u|) \]

satisfying

\[ \int_{\mathbb{R}^+} |u| F(|u|) d|u| < \infty. \]

There are positive constants \( C_0, C_1 \) independent of \( p \) such that

\[ \int_{S''_p} F(|w|) \, d\sigma(w) \leq C_0 \int_{\mathbb{R}^+} |u| F(|u|) d|u| = C_1 \int_{\mathbb{R}^3} \frac{F(u)}{|u|} d|u|. \]
For integrals on $S_p$, we need

**Lemma 2.4** Let $S_p$ be defined as in (2.3). The following estimate holds

$$
\int_{S_p} K^{12}(p, w, p - w)|w|^{k_1}|p - w|^{k_2} d\sigma(w) \geq c_1|p|^{k_1+k_2+2} \min\{1, |p|\}^{k_1+k_2+6},
$$

(2.12)

where $k_1, k_2$ is are non-negative constants.
In addition, there are positive constants $c_0, C_0$ independent of $p$ such that

$$
c_0|p|^2 \min\{1, |p|\} \leq \int_{S_p} d\sigma(w) \leq C_0|p|^2 \min\{1, |p|\}.
$$

(2.13)

Moreover, for any function $F(\cdot) : \mathbb{R}^3 \to \mathbb{R}$ which is radial and positive

$$
F(u) = F(|u|),
$$

we have

$$
\int_{S_p} F(|w|) d\sigma(w) \leq c_2(1 + |p|) \int_0^{|p|} |u| F(|u|) \, du,
$$

(2.14)

for some positive constant $c_2$ independent of $p$.

**Proof** By definition $S_p$ is the surface containing all $w$ satisfying

$$
\mathcal{E}(p - w) + \mathcal{E}(w) = \mathcal{E}(p).
$$

For $w = 0$ and $p$, the above identity is automatically satisfied, hence $\{0, p\} \subset S_p$. If we consider $\mathcal{E}(\hat{q})$ as a function of $|\hat{q}|$: $\mathcal{E}(\hat{q}) = \mathcal{E}(|\hat{q}|)$, then

$$
\mathcal{E}'(|\hat{q}|) = \frac{\kappa_1 + 2\kappa_2 |\hat{q}|^2}{\sqrt{\kappa_1 + 2\kappa_2 |\hat{q}|^2}} > 0,
$$

which means that $\mathcal{E}(|\hat{q}|)$ is strictly increasing. Since for all $w \in S_p \setminus \{0, p\}$, $\mathcal{E}(|p - w|) < \mathcal{E}(|p|)$ and $\mathcal{E}(|w|) < \mathcal{E}(|p|)$, by the monotonicity of $\mathcal{E}(|\hat{q}|)$, we have $|w| < |p|$ and $|p - w| < |p|$, for all $w \in S_p \setminus \{0, p\}$. As a consequence, the energy surface $S_p$ is a subset of $B(0, |p|) \cap B(p, |p|)$.

Now, define

$$
H(w) := \mathcal{E}(p - w) + \mathcal{E}(w) - \mathcal{E}(p).
$$

The directional derivative of $H$ in the direction of $w$ can be computed as

$$
\nabla_w H = \frac{w - p}{|p - w|} \mathcal{E}'(|p - w|) + \frac{w}{|w|} \mathcal{E}'(|w|).
$$

(2.15)

For $w$ of the form $w = \gamma p + q e_0, \gamma, q \in \mathbb{R}_+, e_0 \cdot p = 0$, the derivative of $H$ with respect to $q$ is

$$
\partial_q H = \partial_q w \cdot \nabla_w H = e_0 \cdot \nabla_w H = q|e_0|^2 \left[ \frac{\mathcal{E}'(p - w)}{|p - w|} + \frac{\mathcal{E}'(w)}{|w|} \right] > 0,
$$

(2.16)
which means that $H(w)$ is strictly increasing with respect to $q$.
For $q = 0$ and $\gamma \in (0, 1)$, we will show that

$$H(w) = H(\gamma p) < 0.$$  \hfill (2.17)

Let us start by the following true fact

$$\sqrt{(\kappa_1 + \kappa_2 \gamma^2 |p|^2) (\kappa_1 + \kappa_2 (1 - \gamma)^2 |p|^2)} < \kappa_1 + \kappa_2 (\gamma^2 - \gamma + 2) |p|^2 \quad \text{for } p \neq 0.$$  \hfill (2.18)

Multiplying both sides of the above inequality with $2\gamma (1 - \gamma) |p|^2$ yields

$$2 \sqrt{(\kappa_1 \gamma^2 + \kappa_2 \gamma^4 |p|^2) (\kappa_1 (1 - \gamma)^2 + \kappa_2 (1 - \gamma)^4 |p|^2)} < 2 \kappa_1 \gamma (1 - \gamma) |p|^2 + 2 \kappa_2 \gamma (1 - \gamma) (\gamma^2 - \gamma + 2) |p|^4.$$  \hfill (2.19)

Adding $\kappa_1 \gamma^2 |p|^2 + \kappa_2 \gamma^4 |p|^4 + \kappa_1 (1 - \gamma)^2 |p|^2 + \kappa_2 (1 - \gamma)^4 |p|^4$ to both sides of the above inequality, we obtain

$$\kappa_1 \gamma^2 |p|^2 + \kappa_2 \gamma^4 |p|^4 + \kappa_1 (1 - \gamma)^2 |p|^2 + \kappa_2 (1 - \gamma)^4 |p|^4 + 2 \sqrt{(\kappa_1 \gamma^2 + \kappa_2 \gamma^4 |p|^2) (\kappa_1 (1 - \gamma)^2 + \kappa_2 (1 - \gamma)^4 |p|^2)} < \kappa_1 |p|^2 + \kappa_2 |p|^4.$$  \hfill (2.20)

Rearranging the terms in the above inequality and taking the square root gives

$$\sqrt{\kappa_1 \gamma^2 + \kappa_2 \gamma^4 |p|^2} + \sqrt{\kappa_1 (1 - \gamma)^2 + \kappa_2 (1 - \gamma)^4 |p|^2} < \sqrt{\kappa_1 |p|^2 + \kappa_2 |p|^4},$$

and (2.17) is proved.
As a consequence, for a unit vector $e_0$ which is orthogonal to $p$, the surface $S_p$ and the set $P_\gamma = \{\gamma p + q e_0, q \in \mathbb{R}_+\}$ intersect at only one point, for each $\gamma \in (0, 1)$. Define the intersection by $W_\gamma = \gamma p + q_\gamma e_0$. Since

$$E(p - W_\gamma) + E(W_\gamma) = E(p),$$

then $E(W_\gamma) < E(p)$; there holds

$$|W_\gamma| = \sqrt{\gamma^2 |p|^2 + |q_\gamma|^2} < |p|, \quad |W_\gamma - p| = \sqrt{(1 - \gamma)^2 |p|^2 + |q_{1 - \gamma}|^2} < |p|$$

which implies

$$|q_\gamma| < |p|,$$  \hfill (2.18)

and

$$\gamma |p| < |W_\gamma| < |p|, \quad (1 - \gamma) |p| < |p - W_\gamma| < |p|.$$  \hfill (2.19)

Taking the derivative with respect to $\gamma$ of the identity

$$H(W_\gamma) = 0$$

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yields:

\[ 0 = \partial_\gamma W_\gamma \cdot \nabla_w H = p \cdot \left( \frac{W_\gamma - p}{|p - W_\gamma|} \mathcal{E}'(|p - W_\gamma|) + \frac{W_\gamma}{|W_\gamma|} \mathcal{E}'(|W_\gamma|) \right) \]

\[= \frac{1}{2} \partial_\gamma |W_\gamma|^2 \left[ \frac{\mathcal{E}'(|p - W_\gamma|)}{|p - W_\gamma|} + \frac{\mathcal{E}'(|W_\gamma|)}{|W_\gamma|} \right] - |p|^2 \frac{\mathcal{E}'(|p - W_\gamma|)}{|p - W_\gamma|} \]

\[= \frac{1}{2} \partial_\gamma |q_\gamma|^2 \left[ \frac{\mathcal{E}'(|p - W_\gamma|)}{|p - W_\gamma|} + \frac{\mathcal{E}'(|W_\gamma|)}{|W_\gamma|} \right] + \gamma |p|^2 \frac{\mathcal{E}'(|W_\gamma|)}{|W_\gamma|} - (1 - \gamma) |p|^2 \frac{\mathcal{E}'(|p - W_\gamma|)}{|p - W_\gamma|} \]

(2.20)

where the identities \( \partial_\gamma W_\gamma = p \), \(|W_\gamma|^2 = \gamma^2 |p|^2 + |q_\gamma|^2 \) have been used.

With the notice that \( \mathcal{E}'(|W_\gamma|) > 0 \), the above identity yields

\[
\frac{1}{2} \partial_\gamma |q_\gamma|^2 \leq (1 - \gamma) |p|^2
\]  

(2.21)

for all \( p \) and all \( \gamma \in (0, 1) \).

We now provide an estimate on \( q_\gamma \). In order to do this, let us consider two cases \(|p| \geq 1 \) and \(|p| < 1 \).

- **Case 1**: \(|p| \geq 1 \). Observe that at \( \gamma = \frac{1}{2} \), due to the symmetry of the geometry

  \[ |W_{1/2}| = |W_{1/2} - p|, \]

  which implies

  \[ 2\mathcal{E}(W_{1/2}) = \mathcal{E}(p). \]

Noting that \(|W_{1/2}|^2 = \frac{1}{4} |p|^2 + |q_{1/2}|^2 \), yields

\[ 4 \left[ \kappa_1 \left( \frac{1}{4} |p|^2 + |q_{1/2}|^2 \right) + \kappa_2 \left( \frac{1}{4} |p|^2 + |q_{1/2}|^2 \right)^2 \right] = \kappa_1 |p|^2 + \kappa_2 |p|^4, \]

then

\[ \kappa_2 \left( \frac{1}{4} |p|^2 + |q_{1/2}|^2 \right)^2 + \kappa_1 |q_{1/2}|^2 = \frac{\kappa_2}{4} |p|^4, \]

which implies

\[ c_0 |p|^2 = c_0 |p|^2 \min \left\{ 1, |p|^2 \right\} \leq |q_{1/2}|^2 \leq C_0 |p|^2 \min \left\{ 1, |p|^2 \right\} = C_0 |p|^2 \]

(2.22)

for some constants \( c_0, C_0 \), independent of \(|p|\).

Combining (2.21), (2.22) and the fact that

\[ |q_\gamma|^2 = |q_{1/2}|^2 - \int_{\gamma}^{\gamma} \partial_\gamma |q_\gamma|^2 \ d\gamma' \]

yields

\[ |q_\gamma|^2 \geq c_0 |p|^2 - 2 \left| \gamma - \frac{1}{2} \right| |p|^2 \geq \frac{1}{2} c_0 |p|^2 \]

(2.23)

for all \( \gamma \) satisfying \(|\gamma - \frac{1}{2}| \leq \frac{1}{2} \).
\[ \left( \mathcal{E}(w) + \mathcal{E}(p - w) \right)^2 - \mathcal{E}(p)^2 \]
\[ = \kappa_1 (|p - w|^2 + |w|^2 - |p|^2) + \kappa_2 (|p - w|^4 + |w|^4 - |p|^4) + 2\mathcal{E}(w)\mathcal{E}(p - w) \]
\[ = 2\kappa_1 w \cdot (w - p) + 2\kappa_2 w \cdot (w - p) \left( |w|^2 + |w - p|^2 + |p|^2 \right) \]
\[ - 2\kappa_2 |w|^2 |p - w|^2 + 2\mathcal{E}(w)\mathcal{E}(p - w) \]
which leads to
\[ -w \cdot (w - p) \left( \kappa_1 + \kappa_2 |w|^2 + \kappa_2 |w - p|^2 + \kappa_2 |p|^2 \right) = \mathcal{E}(w)\mathcal{E}(p - w) - \kappa_2 |w|^2 |p - w|^2 \]
for all \( w \in S_p \), in which the right hand side can be computed explicitly as
\[ \mathcal{E}(w)\mathcal{E}(p - w) - \kappa_2 |w|^2 |p - w|^2 \]
\[ = |w||p - w| \sqrt{(\kappa_1 + \kappa_2 |w|^2)(\kappa_1 + \kappa_2 |w - p|^2) - \kappa_2 |w|^2 |p - w|^2} \]
\[ = |w||p - w| \frac{\kappa_1 (\kappa_1 + \kappa_2 |w|^2 + \kappa_2 |w - p|^2)}{\sqrt{(\kappa_1 + \kappa_2 |w|^2)(\kappa_1 + \kappa_2 |w - p|^2) + \kappa_2 |w||p - w|}}. \]

We will develop an asymptotic expansion of the above expression in term of \(|p|\). In order to do this, we observe that
\[ \sqrt{\left( 1 + \frac{\kappa_2 |w|^2}{\kappa_1} \right) \left( 1 + \frac{\kappa_2}{\kappa_1} |w - p|^2 \right)} = 1 + \frac{\kappa_2}{2\kappa_1} (|w|^2 + |w - p|^2) + \mathcal{O}(|p|^4), \]
which leads to
\[ \mathcal{E}(w)\mathcal{E}(p - w) - \kappa_2 |w|^2 |p - w|^2 \]
\[ = |w||p - w| \left( \kappa_1 + \kappa_2 |w|^2 + \kappa_2 |w - p|^2 \right) \]
\[ \times \left( 1 - \frac{\kappa_2}{2\kappa_1} (|w|^2 + |w - p|^2) - \frac{\kappa_2}{\kappa_1} |w||w - p| + \mathcal{O}(|p|^4) \right) \]
\[ = |w||p - w| \left( \kappa_1 + \frac{1}{2} \kappa_2 |w|^2 + \frac{1}{2} \kappa_2 |w - p|^2 - \kappa_2 |w||w - p| + \mathcal{O}(|p|^4) \right) \]
\[ = |w||p - w| \left( \kappa_1 + \frac{1}{2} \kappa_2 |w|^2 + \frac{1}{2} \kappa_2 |w - p|^2 + \kappa_2 |p|^2 \right) \]
\[ - \frac{\kappa_2}{2} |w||w - p| \left( |w|^2 + |w - p|^2 + 2|w||w - p| + 2|p|^2 \right) \left( 1 + \mathcal{O}(|p|^2) \right). \]

Define \( \rho_\gamma \) be the angle between \( W_\gamma \) and \( W_\gamma - p \), then \( W_\gamma \cdot (W_\gamma - p) = |W_\gamma||W_\gamma - p| \cos \rho_\gamma \), which, together with (2.25) - (2.26), leads to
\[ 1 + \cos \rho_\gamma = \frac{\kappa_2}{2} \left( \frac{|W_\gamma|^2 + |W_\gamma - p|^2 + 2|W_\gamma||W_\gamma - p| + 2|p|^2}{\kappa_1 + \kappa_2 |w|^2 + \kappa_2 |w - p|^2 + \kappa_2 |p|^2} \right) \left( 1 + \mathcal{O}(|p|^2) \right) = \mathcal{O}(|p|^2). \]
Hence $\sin \rho_\gamma = O(|p|)$. The area of the parallelogram formed by $W_\gamma$ and $W_\gamma - p$ can be computed as

$$2|p||q_\gamma| = |W_\gamma \times (W_\gamma - p)| = |W_\gamma||W_\gamma - p|\sin \rho_\gamma,$$

which, together with (2.19), implies that there exist universal constants $c_2, c_3$ satisfying

$$c_3(1 - \gamma)|p|^2 \leq |q_\gamma| \leq c_2|p|^2 \quad (2.27)$$

for all $\gamma \in (0, 1)$.

The two inequalities (2.23) and (2.27) are the two estimates we need to obtain (2.12).

To continue, we parametrize the surface $S_p$ as follows: We choose $p^\perp$ to be a vector in $P_0 = \{p \cdot q = 0\}$ and $e_\theta$ to be the unit vector in $P_0$ so that the angle between $p^\perp$ and $e_\theta$ is $\theta$. The surface $S_p$ can be represented as

$$S_p = \{W(\gamma, \theta) = \gamma p + |q_\gamma| e_\theta : \theta \in [0, 2\pi], \gamma \in [0, 1]\}.$$

Notice that the vector $\partial_\theta e_\theta$ is orthogonal to both vectors $p$ and $e_\theta$, the surface area can be computed as

$$d\sigma(w) = |\partial_\gamma W_\gamma \times \partial_\theta W_\gamma|d\gamma d\theta = \left|\left(p + \partial_\gamma |q_\gamma| e_\theta\right) \times |q_\gamma| \partial_\theta e_\theta\right|d\gamma d\theta$$

$$= \sqrt{|p|^2|q_\gamma|^2 + \frac{1}{4}|\partial_\gamma(|q_\gamma|^2)|^2}d\gamma d\theta. \quad (2.28)$$

With (2.23) and (2.27), we are now able to estimate the integral

$$Z := \int_{S_p} K^{12}(p, w, p - w)|w|^{k_1}|p - w|^{k_2}d\sigma(w).$$

Notice that

$$K^{12}(p, w, p - w) \geq C(|p| \wedge p_0)(|p - w| \wedge p_0)(|w| \wedge p_0),$$

where $C$ is some positive constant varying from lines to lines. As a result, $Z$ can be bounded from below by $CZ'$, where $Z'$ is defined as

$$Z' := \int_{S_p} (|p| \wedge p_0)(|w| \wedge p_0)(|p - w| \wedge p_0)|w|^{k_1}|p - w|^{k_2}d\sigma(w).$$

By (2.28), $Z'$ can be rewritten as

$$\int_0^{2\pi} \int_0^1 (|p| \wedge p_0)(|W_\gamma| \wedge p_0)(|p - W_\gamma| \wedge p_0)|W_\gamma|^{k_1}|p - W_\gamma|^{k_2} \sqrt{|p|^2|q_\gamma|^2 + \frac{1}{4}|\partial_\gamma(|q_\gamma|^2)|^2}d\gamma d\theta.$$
Due to (2.23), for \( p \) large, on the interval \([2^{-c_0}, 2^{+c_0}]\),

\[
|W_\gamma|^2 \geq |q_\gamma|^2 \geq \frac{1}{2} c_0 |p|^2
\]

and

\[
|p - W_\gamma|^2 \geq |q_\gamma|^2 \geq \frac{1}{2} c_0 |p|^2.
\]

Therefore, \( Z' \) can be estimated as follows

\[
Z' \geq \int_0^{2\pi} \int_{\frac{1+c_0}{2}}^{\frac{c_0}{2}} \left( |p| \wedge p_0 \right) \left( \sqrt{\frac{c_0}{2}} |p| \right)^{k_1+k_2} \left( \sqrt{\frac{c_0}{2}} |p|^2 \right)^{k_1+k_2+2} d\gamma d\theta
\]

\[
\geq C(|p| \wedge 1)^3 |p|^{k_1+k_2+2}
\]

\[
\geq C|p|^{k_1+k_2+2},
\]

where \( C \) is some positive constant varying from line to line.

Thanks to (2.27), for \( p \) small, on the interval \( \gamma \in \left[\frac{1}{3}, \frac{1}{2}\right] \),

\[
|W_\gamma|^2 \geq |q_\gamma|^2 \geq c_1 |p|^4
\]

and

\[
|p - W_\gamma|^2 \geq |q_\gamma|^2 \geq c_1 |p|^4.
\]

Therefore, \( Z' \) can be estimated as follows

\[
Z' \geq \int_0^{2\pi} \int_{\frac{1}{2}}^{\frac{1}{4}} \left( |p| \wedge p_0 \right) \left( \sqrt{c_1} |p|^2 \wedge p_0 \right)^2 \left( \sqrt{c_1} |p|^{2k_1+2k_2} \left( \sqrt{c_1} |p|^3 \right) \right) d\gamma d\theta
\]

\[
\geq C(|p| \wedge 1)^5 |p|^{2k_1+2k_2+3}
\]

\[
\geq C|p|^{2k_1+2k_2+8}.
\]

The above shows that (2.12) holds true.

Inequalities (2.14), (2.13) can be proved following the same path as in [79].

2.1.3 Boundedness of the total mass for the kinetic equation

**Proposition 2.1** Suppose that the positive radial initial condition \( f_0(p) = f_0(|p|) \) satisfies

\[
\int_{\mathbb{R}^3} f_0(p_1) dp_1 < \infty, \int_{\mathbb{R}^3} f_0(p_1) \mathcal{E}(p_1) dp_1 < \infty.
\]

There exist universal positive constants \( C_1, C_2 \) such that the mass of the positive radial solution \( f(t, p) = f(t, |p|) \) of (1.17) could be bounded as

\[
\int_{\mathbb{R}^3} f(t, p_1) dp_1 \leq C_1 e^{C_2 t}.
\]
Proof  First, observe that the constant function 1 can be used as the test function for (1.17), to get
\[
\frac{d}{dt} \int_{\mathbb{R}^3} f(p_1)dp_1 = \int_{\mathbb{R}^3} C_{12}[f](p_1)dp_1 + \int_{\mathbb{R}^3} C_{22}[f](p_1)dp_1, \tag{2.29}
\]
with the notice that
\[
\int_{\mathbb{R}^3} C_{22}[f](p_1)dp_1 = 0,
\]
and
\[
\int_{\mathbb{R}^3} C_{12}[f](p_1)dp_1 = \lambda_1 n_c \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3} K^{12}(p_1, p_2, p_3) \delta(p_1 - p_2 - p_3) \delta(\mathcal{E}_{p_1} - \mathcal{E}_{p_2} - \mathcal{E}_{p_3}) \times [f(p_1) + 2f(p_1)f(p_2) - f(p_2)f(p_3)] dp_1 dp_2 dp_3.
\]
From the above computations, we can see that the control of the total mass really comes from estimating the collision operator $C_{12}$, since the integral of $C_{22}$ is already 0. Set
\[
J_1 = \lambda_1 n_c \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3} K^{12}(p_1, p_2, p_3) \delta(p_1 - p_2 - p_3) \delta(\mathcal{E}_{p_1} - \mathcal{E}_{p_2} - \mathcal{E}_{p_3}) f(p_1) dp_1 dp_2 dp_3
\]
and
\[
J_2 = 2\lambda_1 n_c \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3} K^{12}(p_1, p_2, p_3) \delta(p_1 - p_2 - p_3) \delta(\mathcal{E}_{p_1} - \mathcal{E}_{p_2} - \mathcal{E}_{p_3}) f(p_1) f(p_2) dp_1 dp_2 dp_3,
\]
to get
\[
\frac{d}{dt} \int_{\mathbb{R}^3} f(p_1)dp_1 = \int_{\mathbb{R}^3} Q[f](p_1)dp_1 \leq J_1 + J_2, \tag{2.30}
\]
note that in the above inequality, we have dropped the negative term containing $f(p_2)f(p_3)$. Now, $J_1$ can be estimated the following way, by using the definition of the Dirac functions $\delta(p_1 - p_2 - p_3)$, $\delta(\mathcal{E}_{p_1} - \mathcal{E}_{p_2} - \mathcal{E}_{p_3})$ and the boundedness of $K^{12}(p_1, p_2, p_1 - p_2)$
\[
J_1 = \lambda_1 n_c \int_{\mathbb{R}^3 \times \mathbb{R}^3} K^{12}(p_1, p_2, p_1 - p_2) \delta(\mathcal{E}_{p_1} - \mathcal{E}_{p_2} - \mathcal{E}_{p_1 - p_2}) f(p_1) dp_1 dp_2 \\
\leq C \int_{\mathbb{R}^3} f(p_1) \left( \int_{S_{p_1}} d\sigma(p_2) \right) dp_1,
\]
which, by Lemma 2.4, can be bounded as
\[
J_1 \leq C \int_{\mathbb{R}^3} f(p_1) |p_1|^2 \min\{1, |p_1|\} dp_1.
\]
Using the fact that $|p_1|^2 \min\{1, |p_1|\}$ is dominated by $\mathcal{E}_{p_1}$ up to a constant, yields
\[
J_1 \leq C \int_{\mathbb{R}^3} f(p_1) \mathcal{E}_{p_1} dp_1 \leq C, \tag{2.31}
\]
23
where $C$ is a constant varying from lines to lines and the last inequality follows from the conservation of energy (2.8).

It remains to estimate $J_2$. By a straightforward use of the definition of the Dirac functions $\delta(p_1 - p_2 - p_3)$ and $\delta(E_{p_1} - E_{p_2} - E_{p_3})$

\[
J_2 = 2\lambda n_c \int_{\mathbb{R}^3 \times \mathbb{R}^3} K_{12}^{12}(p_1, p_2, p_1 - p_2) \delta(E_{p_1} - E_{p_2} - E_{p_1 - p_2}) f(p_1) f(p_2) dp_1 dp_2
\]

\[
= 2\lambda n_c \int_{\mathbb{R}^3} f(p_2) \left( \int_{S''_{p_2}} K_{12}^{12}(p_1, p_2, p_1 - p_2) f(p_1) d\sigma(p_1) \right) dp_2,
\]

which, by Lemma 2.3, can be bounded as

\[
J_2 \leq C \int_{\mathbb{R}^3} f(p_2) \left( \int_{S''_{p_2}} K_{12}^{12}(p_1, p_2, p_1 - p_2) f(p_1) d\sigma(p_1) \right) dp_2,
\]

Since $K_{12}^{12}(p_1, p_2, p_1 - p_2)$ is bounded by $|p_2|$, up to a constant, $J_2$ is dominated by

\[
J_2 \leq C \int_{\mathbb{R}^3} |p_2| f(p_2) \left( \int_{\mathbb{R}^3} f(p_1) dp_1 \right) dp_2
\]

\[
\leq C \left( \int_{\mathbb{R}^3} f(p_2) E(p_2) dp_2 \right) \left( \int_{\mathbb{R}^3} f(p_1) dp_1 \right),
\]

notice that $C$ is a positive constant varying from lines to lines and we have just used the fact that $|p|$ is bounded by $E(p)$ up to a constant, which by the conservation of energy (2.8), implies

\[
J_2 \leq C \left( \int_{\mathbb{R}^3} f(p_1) dp_1 \right), \tag{2.32}
\]

Combining (2.30), (2.31) and (2.32) leads to

\[
\frac{d}{dt} \int_{\mathbb{R}^3} f(p_1) dp_1 = \int_{\mathbb{R}^3} Q[f](p_1) dp_1 \leq C^* \left( 1 + \int_{\mathbb{R}^3} f(p_1) dp_1 \right), \tag{2.33}
\]

for some positive constant $C^*$, which implies the conclusion of the Proposition. \hfill \blacksquare

\section*{2.2 Finite time moment estimates of the solution to the kinetic equation}

\subsection*{2.2.1 Estimating $C_{12}$}

\begin{proposition}
For any positive, radial function $f(p) = f(|p|)$, for any $n \in \mathbb{N}$, there exists a universal positive constant $C$ depending on $n$, such that the following bound on the
collision operator $C_{12}$ holds true
\[
\int_{\mathbb{R}^3} C_{12}[f](p_1)\mathcal{E}^n(p_1)dp_1 \leq \sum_{k=1}^{n-1} \mathcal{C}m_k[f](m_{n-k-1}[f] + m_{n-k}[f]) - \mathcal{C}m_{n+1}[f] + \mathcal{C}m_1[f]. \quad (2.34)
\]

**Proof** For the sake of simplicity, we denote $m_k[f]$ by $m_k$. By a view of Lemma 2.1
\[
\int_{\mathbb{R}^3} C_{12}[f](p_1)\mathcal{E}^n(p_1)dp_1 = \quad (2.35)
= n_c\lambda_1 \iiint_{\mathbb{R}^{3\times3}} K^{12}(p_1, p_2, p_3)\delta(p_1 - p_2 - p_3)\delta(\mathcal{E}_{p_1} - \mathcal{E}_{p_2} - \mathcal{E}_{p_3}) \times
\]
\[
\times [f(p_2)f(p_3) - f(p_1) - 2f(p_1)f(p_2)]\mathcal{E}^n_{p_1} + \mathcal{E}^n_{p_2} - \mathcal{E}^n_{p_3}dp_1dp_2dp_3.
\]
By the definition of $\delta(\mathcal{E}_{p_1} - \mathcal{E}_{p_2} - \mathcal{E}_{p_3})$, the term $\mathcal{E}^n_{p_1} - \mathcal{E}^n_{p_2} - \mathcal{E}^n_{p_3}$ could be rewritten as
\[
(\mathcal{E}_{p_2} + \mathcal{E}_{p_1})^n - \mathcal{E}^n_{p_1} + \mathcal{E}^n_{p_2} - \mathcal{E}^n_{p_3} = \sum_{k=1}^{n-1} \binom{n}{k} \mathcal{E}^k_{p_2} \mathcal{E}^{n-k}_{p_3}.
\]
which yields
\[
\int_{\mathbb{R}^3} C_{12}[f](p_1)\mathcal{E}^n(p_1)dp_1 = \quad (2.36)
= n_c\lambda_1 \iiint_{\mathbb{R}^{3\times3}} K^{12}(p_1, p_2, p_3)\delta(p_1 - p_2 - p_3)\delta(\mathcal{E}_{p_1} - \mathcal{E}_{p_2} - \mathcal{E}_{p_3}) \times
\]
\[
\times [f(p_2)f(p_3) - f(p_1) - 2f(p_1)f(p_2)]\sum_{k=1}^{n-1} \binom{n}{k} \mathcal{E}^k_{p_2} \mathcal{E}^{n-k}_{p_3} dp_1dp_2dp_3.
\]
Dropping the term containing $-2f(p_1)f(p_2)$, the above quantity could be bounded as
\[
\int_{\mathbb{R}^3} C_{12}[f](p_1)\mathcal{E}^n(p_1)dp_1 \leq L_1 + L_2,
\]
where
\[
L_1 := n_c\lambda_1 \iiint_{\mathbb{R}^{3\times3}} K^{12}(p_1, p_2, p_3)\delta(p_1 - p_2 - p_3)\delta(\mathcal{E}_{p_1} - \mathcal{E}_{p_2} - \mathcal{E}_{p_3}) \times
\]
\[
\times f(p_2)f(p_3)\sum_{k=1}^{n-1} \binom{n}{k} \mathcal{E}^k_{p_2} \mathcal{E}^{n-k}_{p_3} dp_1dp_2dp_3
\]
\[
L_2 := -n_c\lambda_1 \iiint_{\mathbb{R}^{3\times3}} K^{12}(p_1, p_2, p_3)\delta(p_1 - p_2 - p_3)\delta(\mathcal{E}_{p_1} - \mathcal{E}_{p_2} - \mathcal{E}_{p_3}) \times
\]
\[
\times f(p_1)\sum_{k=1}^{n-1} \binom{n}{k} \mathcal{E}^k_{p_2} \mathcal{E}^{n-k}_{p_3} dp_1dp_2dp_3.
\]
Let us first look at $L_1$. By the definition of $\delta(p_1 - p_2 - p_3)$,

$$L_1 = n_c \lambda_1 \int_{\mathbb{R}^3} K^{12}(p_2 + p_3, p_2, p_3) \delta(\mathcal{E}_{p_2 + p_3} - \mathcal{E}_{p_2} - \mathcal{E}_{p_3}) \times$$

$$\times f(p_2) f(p_3) \left[ \sum_{k=1}^{n-1} \left( \frac{n}{k} \right) \mathcal{E}_{p_2}^{k} \mathcal{E}_{p_3}^{n-k} \right] dp_2 dp_3,$$

which by the boundedness of $K^{12}$, could be bounded as

$$L_1 \leq C \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \delta(\mathcal{E}_{p_2 + p_3} - \mathcal{E}_{p_2} - \mathcal{E}_{p_3}) \times$$

$$\times f(p_2) f(p_3) \left[ \sum_{k=1}^{n-1} \left( \frac{n}{k} \right) \mathcal{E}_{p_2}^{k} \mathcal{E}_{p_3}^{n-k} \right] dp_2 dp_3$$

$$\leq \sum_{k=1}^{n-1} C \int_{\mathbb{R}^3} f(p_2) \mathcal{E}_{p_2}^{k} \left[ \int_{\mathbb{R}^3} f(p_3) \mathcal{E}_{p_3}^{n-k} d\sigma(p_3) \right] dp_2,$$

Applying Lemma 2.2 to the above inequality leads to

$$L_1 \leq \sum_{k=1}^{n-1} C \int_{\mathbb{R}^3} f(p_2) \mathcal{E}_{p_2}^{k} \left[ \int_{\mathbb{R}^3} f(p_3) \frac{\mathcal{E}_{p_3}^{n-k}}{|p_3|} dp_3 \right] dp_2,$$

where $C$ is some constant varying from lines to lines.

Observe that

$$\frac{\mathcal{E}_{p_3}^{n-k}}{|p_3|} \leq C \left( \mathcal{E}_{p_3}^{n-k-1} + \mathcal{E}_{p_3}^{n-k} \right),$$

which implies

$$L_1 \leq \sum_{k=1}^{n-1} C \left[ \int_{\mathbb{R}^3} f(p_1) \mathcal{E}_{p_1}^{k} dp_1 \right] \left[ \int_{\mathbb{R}^3} f(p_1) \mathcal{E}_{p_1}^{n-k-1} dp_1 + \int_{\mathbb{R}^3} f(p_1) \mathcal{E}_{p_1}^{n-k} dp_1 \right]$$

$$\leq \sum_{k=1}^{n-1} C m_k [m_{n-k-1} + m_{n-k}].$$

Now, by the definition of $\delta(p_1 - p_2 - p_3)$ and $\delta(\mathcal{E}_{p_1} - \mathcal{E}_{p_2} - \mathcal{E}_{p_1-p_2})$, the second term $L_2$ can be rewritten as

$$L_2 = -n_c \lambda_1 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} K^{12}(p_1, p_2, p_1 - p_2) \delta(\mathcal{E}_{p_1} - \mathcal{E}_{p_2} - \mathcal{E}_{p_1-p_2}) \times$$

$$\times f(p_1) \left[ \sum_{k=1}^{n-1} \left( \frac{n}{k} \right) \mathcal{E}_{p_2}^{k} \mathcal{E}_{p_1-p_2}^{n-k} \right] dp_1 dp_2$$

$$\leq \sum_{k=1}^{n-1} C \left[ \int_{\mathbb{R}^3} K^{12}(p_1, p_2, p_1 - p_2) \mathcal{E}_{p_2}^{k} \mathcal{E}_{p_1-p_2}^{n-k} d\sigma(p_2) \right] dp_1.$$
Since 
\[ \psi_{p_2} \psi_{p_1-p_2} \geq C \left[ |p_2|^k |p_1-p_2|^n + |p_2|^{2k} |p_1-p_2|^{2(n-k)} \right], \]
where \( C \) is some positive constant varying from lines to lines, \( L_2 \) can be estimated as follows

\[
L_2 \leq \sum_{k=1}^{n-1} C \int_{\mathbb{R}^3} f(p_1) \left[ \int_{S_{p_1}} K^{12}(p_1, p_2, p_1-p_2) \left( |p_2|^k |p_1-p_2|^n + |p_2|^{2k} |p_1-p_2|^{2(n-k)} \right) d\sigma(p_2) \right] dp_1,
\]
which, due to Lemma 2.4, can be bounded by

\[
L_2 \leq -C \int_{\mathbb{R}^3} f(p_1) \left( (|p_1| \wedge 1)^{n+6} |p_1|^{n+2} + (|p_1| \wedge 1)^{2n+6} |p_1|^{2n+2} \right) dp_1.
\]

Splitting the integral on \( \mathbb{R}^3 \) into two integrals on \( |p_1| > 1 \) and \( |p_1| \leq 1 \) yields

\[
L_2 \leq -C \int_{|p_1| > 1} f(p_1) \left( |p_1|^{n+2} + |p_1|^{2n+2} \right) dp_1 \]
\[
- C \int_{|p_1| \leq 1} f(p_1) \left( |p_1|^{2n+6} + |p_1|^{4n+6} \right) dp_1 \]
\[
\leq -C \int_{|p_1| > 1} f(p_1) \left( |p_1|^{n+2} + |p_1|^{2n+2} \right) dp_1,
\]
where \( C \) is some positive constant varying from lines to lines and we have used the inequality \( -|p_1|^{n+1} > -|p_1|^{n+2} \) for \( |p_1| > 1 \). Adding and subtracting the right hand side of the above inequality with an integral on the domain \( |p_1| \leq 1 \), we obtain

\[
L_2 \leq -C \left[ \int_{\mathbb{R}^3} f(p_1) \left( |p_1|^{n+1} + |p_1|^{2n+2} \right) dp_1 - \int_{|p_1| \leq 1} f(p_1) \left( |p_1|^{n+1} + |p_1|^{2n+2} \right) dp_1 \right] \]
\[
\leq -C \left[ \int_{\mathbb{R}^3} f(p_1) \left( |p_1|^{n+1} + |p_1|^{2n+2} \right) dp_1 - \int_{|p_1| \leq 1} |p_1| f(p_1) dp_1 \right],
\]
where the last inequality is due to the fact that we are integrating on \( |p_1| \leq 1 \). Bounding the integral on \( |p_1| \leq 1 \) by the integral on the full space \( \mathbb{R}^3 \), we get

\[
L_2 \leq -C \int_{\mathbb{R}^3} f(p_1) \left( |p_1|^{n+1} + |p_1|^{2n+2} \right) dp_1 + C \int_{\mathbb{R}^3} |p_1| f(p_1) dp_1.
\]

By the inequality
\[ |p_1|^{n+1} + |p_1|^{2n+2} \geq C \psi_{p_1}^{n+1}, \]
we obtain the following estimate on \( L_2 \)

\[
L_1 \leq -Cm_{n+1} + Cm_1. \tag{2.38}
\]
Combining (2.36), (2.37) and (2.38), we get the conclusion of the Proposition.
2.2.2 Estimating \( C_{22} \)

**Proposition 2.3** For any positive, radial function \( f(p) = f(|p|) \), for any \( n \in \mathbb{N} \), \( n > 2 \), \( n \) is odd, there exists a universal positive constant \( C \) depending on \( n \), such that the following bound on the collision operator \( C_{22} \) holds true

\[
\int_{\mathbb{R}^3} C_{22}[f](p_1) \mathcal{E}^n_{p_1} dp_1 \leq C \sum_{0 \leq i, j, k < n; \ i + j + k = n} m_{i+j+k} (m_{j+k-s} + m_{j+k-s+1/2}) + C \sum_{0 \leq i, j, k < n; \ i + j + k = n; \ j, k > 0} m_i (m_{j-1} + m_{j-1/2}) (m_{k-1} + m_{k-1/2}).
\]  

(2.39)

**Proof** For the sake of simplicity, we denote \( m_k[f] \) by \( m_k \). We first observe that, by a spherical change of variables

\[
\int_{\mathbb{R}^3} C_{22}[f](p_1) \mathcal{E}^n_{p_1} dp_1 = C \int_{\mathbb{R}_+} C_{22}[f](p_1)|p_1|^2 \mathcal{E}^n_{p_1} d|p_1|,
\]

where \( C \) is some universal constant varying from lines to lines, and

\[
\int_{\mathbb{R}^3} C_{22}[f](p_1) \mathcal{E}^n_{p_1} dp_1 = \kappa_3 \int_{\mathbb{R}_+^4} K^{22}(p_1, p_2, p_3, p_4) \min\{|p_1|, |p_2|, |p_3|, |p_4|\}|p_1||p_2||p_3||p_4|\delta(\mathcal{E}_{p_1} + \mathcal{E}_{p_2} - \mathcal{E}_{p_3} - \mathcal{E}_{p_4})
\times [f(p_3)f(p_4)(1 + f(p_1) + f(p_2)) - f(p_1)f(p_2)(1 + f(p_3) + f(p_4))] \mathcal{E}^n_{p_2} d|p_1|d|p_2|d|p_3|d|p_4|.
\]

By the classical change of variables \((p_1, p_2) \leftrightarrow (p_2, p_1), (p_1, p_2) \leftrightarrow (p_3, p_4)\) (cf. [96]), the above equation could be expressed in the following way

\[
\int_{\mathbb{R}^3} C_{22}[f](p_1) \mathcal{E}^n_{p_1} dp_1 = C \int_{\mathbb{R}_+^4} K^{22}(p_1, p_2, p_3, p_4) \min\{|p_1|, |p_2|, |p_3|, |p_4|\}|p_1||p_2||p_3||p_4|\delta(\mathcal{E}_{p_1} + \mathcal{E}_{p_2} - \mathcal{E}_{p_3} - \mathcal{E}_{p_4})
\times f(p_1)f(p_2)(1 + f(p_3) + f(p_4)) [\mathcal{E}^n_{p_4} + \mathcal{E}^n_{p_3} - \mathcal{E}^n_{p_2} - \mathcal{E}^n_{p_1}] d|p_1|d|p_2|d|p_3|d|p_4|,
\]

where \( C \) is some universal constant varying from lines to lines.

Taking into account the fact that \( p_3 \) and \( p_4 \) are symmetric, and using the definition of the
Since \( n \) is an odd number, applying Newton formula to the term \((\mathcal{E}_{p_1} + \mathcal{E}_{p_2} - \mathcal{E}_{p_3})^n + \mathcal{E}_{p_2}^n - \mathcal{E}_{p_1}^n\) yields

\[
(\mathcal{E}_{p_1} + \mathcal{E}_{p_2} - \mathcal{E}_{p_3})^n + \mathcal{E}_{p_3}^n - \mathcal{E}_{p_2}^n - \mathcal{E}_{p_1}^n = \sum_{0 \leq i, j, k < n; i + j + k = n} C_{i,j,k,n} \mathcal{E}_{p_1}^i \mathcal{E}_{p_2}^j \mathcal{E}_{p_3}^k. \tag{2.43}
\]
Plugging (2.43) into (2.42), integrating with respect to $d\mathcal{E}_4$ and using the bound (1.10) leads to

$$
\int_{\mathbb{R}^3} C_{22}[f](p_1)\mathcal{E}_{p_1}^n \, dp_1 \leq C \int_{\{\mathcal{E}_{p_1} + \mathcal{E}_{p_2} \geq \mathcal{E}_{p_3}\}} \min\{|p_1|, |p_2|, |p_3|\} |p_1||p_2||p_3| f(p_1) f(p_2) (1 + 2 f(p_3)) \times \sum_{0 \leq i, j, k < n; i + j + k = n} |C_{i,j,k,n}| \mathcal{E}_{p_1}^i \mathcal{E}_{p_2}^j \mathcal{E}_{p_3}^k \, dp_1 \, dp_2 \, dp_3 \tag{2.44}
$$

$$
\leq C \sum_{0 \leq i, j, k < n; i + j + k = n} \int_{\{\mathcal{E}_{p_1} + \mathcal{E}_{p_2} \geq \mathcal{E}_{p_3}\}} \min\{|p_1|, |p_2|\} |p_1||p_2| f(p_1) f(p_2) (1 + 2 f(p_3)) \times \mathcal{E}_{p_1}^i \mathcal{E}_{p_2}^j \mathcal{E}_{p_3}^k \, dp_1 \, dp_2 \, dp_3 \tag{2.45}
$$

In order to estimate the right hand side of (2.44), we estimate each term containing $f(p_1) f(p_2)$ and $2 f(p_1) f(p_2) f(p_3)$ seperately.

Let us first look at the term containing $f(p_1) f(p_2)$

$$
H_1 := C \sum_{0 \leq i, j, k < n; i + j + k = n} \int_{\{\mathcal{E}_{p_1} + \mathcal{E}_{p_2} \geq \mathcal{E}_{p_3}\}} \min\{|p_1|, |p_2|\} |p_1||p_2| f(p_1) f(p_2) \times \mathcal{E}_{p_1}^i \mathcal{E}_{p_2}^j \mathcal{E}_{p_3}^k \, dp_1 \, dp_2 \, dp_3 \tag{2.46}
$$

where we have used (2.44) to get $|p_3| dp_3 \leq C d\mathcal{E}_{p_3}$ and the fact that

$$
\min\{|p_1|, |p_2|, |p_3|\} \leq \min\{|p_1|, |p_2|\}.
$$

In (2.45), integrating with respect to $d\mathcal{E}_{p_3}$ leads to

$$
H_1 \leq C \sum_{0 \leq i, j, k < n; i + j + k = n} \int_{\mathbb{R}^3} \min\{|p_1|, |p_2|\} |p_1||p_2| f(p_1) f(p_2) \times \mathcal{E}_{p_1}^i \mathcal{E}_{p_2}^j \frac{(\mathcal{E}_{p_1} + \mathcal{E}_{p_2})^{k+1}}{k+1} \, dp_1 \, dp_2, \tag{2.47}
$$

where $C$ is some universal constant varying from lines to lines.

Again, by Newton formula

$$
(\mathcal{E}_{p_1} + \mathcal{E}_{p_2})^{k+1} = \sum_{s=0}^{k+1} \binom{k+1}{s} \mathcal{E}_{p_1}^s \mathcal{E}_{p_2}^{k+1-s}, \tag{2.47}
$$

30
which, together with (2.45) leads to

\[ H_1 \leq C \sum_{0 \leq i,j,k < n; \ i+j+k=n} \sum_{s=0}^{k+1} \int_{\mathbb{R}^2_+} \min\{|p_1|, |p_2|\} |p_1||p_2| f(p_1) f(p_2) \mathcal{E}_p^{i+s} \mathcal{E}_p^{j+k+1-s} d|p_1| d|p_2| \]
\[ \leq C \sum_{0 \leq i,j,k < n; \ i+j+k=n} \sum_{s=0}^{k+1} \int_{\mathbb{R}^2_+} |p_1|^2 |p_2| f(p_1) f(p_2) \mathcal{E}_p^{i+s} \mathcal{E}_p^{j+k+1-s} d|p_1| d|p_2|. \]  

Note that integrals of \( d|p_1| \) and \( d|p_2| \) in (2.48) are separated and it is straightforward that the integral of \( d|p_1| \) can be computed, by a spherical coordinate change of variables, as

\[ \int_{\mathbb{R}^3} |p_1|^2 f(p_1) \mathcal{E}_p^{i+s} d|p_1| = \int_{\mathbb{R}^3} f(p_1) \mathcal{E}_p^{i+s} dp_1 = m_{i+s}. \]  

Now, for the second integral concerning \( d|p_2| \), by the inequality

\[ \mathcal{E}_p^{i+s} \leq C(|p_2| + |p_2|^2), \]

for some positive constant \( C \), one gets

\[ \int_{\mathbb{R}^3} |p_2| f(p_2) \mathcal{E}_p^{j+k+1-s} d|p_2| \leq C \int_{\mathbb{R}^3} (|p_2|^2 + |p_2|^3) f(p_2) \mathcal{E}_p^{j+k-s} d|p_2| \]
\[ \leq C \int_{\mathbb{R}^3} (1 + |p_2|) f(p_2) \mathcal{E}_p^{j+k-s} dp_2, \]

which, by the inequality

\[ \mathcal{E}_p^{1/2} \geq C |p_2|, \]

implies that

\[ \int_{\mathbb{R}^3} |p_2| f(p_2) \mathcal{E}_p^{j+k+1-s} d|p_2| \leq C \int_{\mathbb{R}^3} \left( 1 + \mathcal{E}_p^{1/2} \right) f(p_2) \mathcal{E}_p^{j+k-s} dp_2 \]
\[ \leq C (m_{j+k-s} + m_{j+k-s+1/2}). \]  

Combining (2.48), (2.49) and (2.50) lead to

\[ H_1 \leq C \sum_{0 \leq i,j,k < n; \ i+j+k=n} \sum_{s=0}^{k+1} m_{i+s} \left( m_{j+k-s} + m_{j+k-s+1/2} \right). \]  

(2.51)
Now, for the term containing $2f(p_1)f(p_2)f(p_3)$, by bounding the integral on $\{E_{p_1} + E_{p_2} \geq E_{p_3}\}$ by the integral on $\mathbb{R}^3_+$, we get

$$H_2 := C \sum_{0 \leq i,j,k < n; i+j+k=n} \int_{\mathbb{R}^3_+} \min\{ |p_1|, |p_2|, |p_3| \} |p_1| |p_2| |p_3| \cdot$$

$$\times 2f(p_1)f(p_2)f(p_3)E_{p_1}^i E_{p_2}^j E_{p_3}^k dp_1 dp_2 dp_3 \leq C \sum_{0 \leq i,j,k < n; i+j+k=n} \int_{\mathbb{R}^3_+} \min\{ |p_1|, |p_2|, |p_3| \} |p_1| |p_2| |p_3| \cdot$$

$$\times f(p_1)f(p_2)f(p_3)E_{p_1}^i E_{p_2}^j E_{p_3}^k dp_1 dp_2 dp_3,$$

where $C$ is some universal constant varying from lines to lines.

Notice that there are only two cases: $i,j,k > 0$ and one of $i,j,k$ is 0. Indeed, due to the condition that $i + j + k = n$ and $0 \leq i,j,k < n$, the case where two of the index $i,j,k$ are 0 will not happen. Therefore, we can suppose without loss of generality that $i \geq 0$ and $j,k > 0$.

The terms on the right hand side of (2.52) can be estimated as

$$\int_{\mathbb{R}^3_+} \min\{ |p_1|, |p_2|, |p_3| \} |p_1| |p_2| |p_3| f(p_1)f(p_2)f(p_3)E_{p_1}^i E_{p_2}^j E_{p_3}^k dp_1 dp_2 dp_3 \leq$$

$$\int_{\mathbb{R}^3_+} |p_1|^2 E_{p_1}^i f(p_1) dp_1 \int_{\mathbb{R}^3_+} |p_2|^2 E_{p_2}^j f(p_2) dp_2 \int_{\mathbb{R}^3_+} |p_3|^2 E_{p_3}^k f(p_3) dp_3.$$ (2.53)

For each term on the right hand side of (2.53), one can write, by the spherical coordinate change of variables

$$\int_{\mathbb{R}^3_+} |p_1|^2 E_{p_1}^i f(p_1) dp_1 = \int_{\mathbb{R}^3} E_{p_1}^i f(p_1) dp_1 = m_i,$$ (2.54)

$$\int_{\mathbb{R}^3_+} |p_2|^2 E_{p_2}^j f(p_2) dp_2 \leq C \left( m_{j-1} + m_{j-1/2} \right),$$ (2.55)

$$\int_{\mathbb{R}^3_+} |p_3|^2 E_{p_3}^k f(p_3) dp_3 \leq C \left( m_{k-1} + m_{k-1/2} \right),$$ (2.56)

where (2.54) and (2.57) are obtained by exactly the same manner as (2.50).

Combining (2.53), (2.54), (2.55) and (2.56) yields

$$\int_{\mathbb{R}^3_+} \min\{ |p_1|, |p_2|, |p_3| \} |p_1| |p_2| |p_3| f(p_1)f(p_2)f(p_3)E_{p_1}^i E_{p_2}^j E_{p_3}^k dp_1 dp_2 dp_3 \leq$$

$$C m_i \left( m_{j-1} + m_{j-1/2} \right) \left( m_{k-1} + m_{k-1/2} \right).$$ (2.57)

The two inequalities (2.52) and (2.57) yield

$$H_2 \leq C \sum_{0 \leq i,j,k < n; i+j+k=n: j,k>0} m_i \left( m_{j-1} + m_{j-1/2} \right) \left( m_{k-1} + m_{k-1/2} \right),$$ (2.58)
where $C$ is some universal constant varying from lines to lines.

From (2.44), (2.51) and (2.58), we get

$$
\int_{\mathbb{R}^3} C_{22}[f](p_1)\mathcal{E}^n_{p_1} dp_1 \leq
\leq C \sum_{0 \leq i,j,k < n;\ i+j+k=n} \sum_{s=0}^{k+1} m_i (m_{j+k-s} + m_{j+k-s+1/2}) +
+ C \sum_{0 \leq i,j,k < n;\ i+j+k=n;\ j,k>0} m_i (m_{j-1} + m_{j-1/2}) (m_{k-1} + m_{k-1/2}).
$$

\[\square\]

2.2.3 Finite time moment estimates

Proposition 2.4 Suppose that $f_0(p) = f_0(|p|)$ is a positive radial initial condition and

$$
\int_{\mathbb{R}^3} f_0(p)\mathcal{E}_p dp < \infty, \quad \int_{\mathbb{R}^3} f_0(p) dp < \infty,
$$

then for any finite time interval $[0,T]$, and for any $n \geq 1$, the positive radial solution $f(t,p) = f(t,|p|)$ of (1.17) satisfies

$$
\sup_{t \in [\tau,T]} \int_{\mathbb{R}^3} f(t,p)\mathcal{E}^n_p dp < C_\tau, \quad \forall \ 0 < \tau \leq T,
$$

where $C_\tau$ is a constant depending on $\tau$.

If

$$
\int_{\mathbb{R}^3} f_0(p)\mathcal{E}_p dp < \infty,
$$

then

$$
\sup_{t \in [0,T]} \int_{\mathbb{R}^3} f(t,p)\mathcal{E}^n_p dp < \infty.
$$

In order to prove Proposition 2.4 we would need the following Holder inequality.

Lemma 2.5 Let $f$ be a function in $L^1(\mathbb{R}^3) \cap L_n^1(\mathbb{R}^3)$, then

$$
\|f\|_{L^1_1} \leq C\|f\|_{L^n_1}^{\frac{k}{n}},
$$

where $C$ is a constant depending on $\|f\|_{L^1}$, $k$ and $n$.  

\[\text{33}\]
Proof By Holder inequality, we have
\[
\int_{\mathbb{R}^3} |p|^k f(p) dp \leq \left( \int_{\mathbb{R}^3} |f(p)| dp \right)^{\frac{n-k}{n}} \left( \int_{\mathbb{R}^3} |p|^n |f(p)| dp \right)^{\frac{k}{n}}
\]
\[
\leq C \left( \|f\|_{L^1,k,n} \right) \left( \int_{\mathbb{R}^3} |p|^n f(p) dp \right)^{\frac{k}{n}}.
\]

Proof [of Proposition 2.4] Fix a time interval \([0,T]\). It is sufficient to prove Proposition 2.4 for \(n \in \mathbb{N}, n \text{ odd} \). Using \(E_{p_1}^n\) as a test function in \([1,17]\), as a view of Lemma 2.1 we get
\[
\frac{d}{dt} \int_{\mathbb{R}^3} f(p_1) E_{p_1}^n dp_1 = \int_{\mathbb{R}^3} C_{12}[f](p_1) E_{p_1}^n dp_1 + \int_{\mathbb{R}^3} C_{22}[f](p_1) E_{p_1}^n dp_1. \tag{2.59}
\]
For the sake of simplicity, we denote \(m_k(f(t)) = m_k(t)\). First, let us consider the \(C_{12}\) collision operator. By Proposition 2.2
\[
\int_{\mathbb{R}^3} C_{12}[f](p_1) E_{p_1}^n dp_1 \leq \sum_{k=1}^{n-1} C m_k(t)(m_{n-k-1}(t) + m_{n-k}(t)) - C m_{n+1}(t) + C m_1(t).
\]
Since, according to Proposition 2.1 \(m_0(t)\) is bounded by a constant \(C\) on \([0,T]\), we deduce from Lemma 2.5 that
\[
m_k(t) \leq C m_n(t)^{\frac{k}{n}}, \quad m_{n-k-1}(t) \leq C m_n(t)^{\frac{n-k-1}{n}},
\]
\[
m_{n-k}(t) \leq C m_n(t)^{\frac{n-k}{n}}, \quad C m_{n+1}(t) \geq m_n(t)^{\frac{n+1}{n}}, \quad C m_1(t) \leq m_n(t)^{\frac{1}{n}}.
\]
where \(C\) depends on \(n, k\), and the bound of the mass on \([0,T]\) in Proposition 2.1. As a consequence, we obtain the following estimate for \(C_{12}\)
\[
\int_{\mathbb{R}^3} C_{12}[f](p_1) E_{p_1}^n dp_1 \leq C m_n(t) + C m_n(t)^{\frac{n-k}{n}} + C m_n(t)^{\frac{1}{n}} - C m_n(t)^{\frac{n+1}{n}}. \tag{2.60}
\]
Now, for the \(C_{22}\) collision operator, according to Proposition 2.3
\[
\int_{\mathbb{R}^3} C_{22}[f](p_1) E_{p_1}^n dp_1 \leq C \sum_{0 \leq i,j,k<n; i+j+k=n} \sum_{s=0}^{k+1} \left( m_{i+s}(t) + m_{j+k-s}(t) + m_{j+k-s+1/2}(t) \right) +
\]
\[
+ C \sum_{0 \leq i,j,k<n; i+j+k=n; j,k>0} m_i(t) \left( m_{j-1}(t) + m_{j-1/2}(t) \right) \left( m_{k-1}(t) + m_{k-1/2}(t) \right).
\]
Again, by Proposition 2.1 and Lemma 2.5
\[
m_{i+s}(t) \leq C m_n(t)^{\frac{i+s}{n}}, \quad m_{j+k-s}(t) \leq C m_n(t)^{\frac{j+k-s}{n}},
\]

\[
\begin{align*}
m_{j+k-s+1/2}(t) &\leq C m_n(t)^{\frac{j+k-s+1/2}{n}}, \quad m_s(t) \leq C m_n(t)^{\frac{k}{n}}, \\
m_{j-1}(t) &\leq C m_n(t)^{\frac{j-1}{n}}, \quad m_{j-1/2}(t) \leq C m_n(t)^{\frac{j-1/2}{n}}, \\
m_{k-1}(t) &\leq C m_n(t)^{\frac{k-1}{n}}, \quad m_{k-1/2}(t) \leq C m_n(t)^{\frac{k-1/2}{n}},
\end{align*}
\]
we obtain
\[
\int_{\mathbb{R}^3} C_{22}[f](p_1) \mathcal{E}_{p_1}^n dp_1 \leq
\]
\[
\leq C \sum_{0 \leq i,j,k<n; i+j+k=n} \sum_{s=0}^{k+1} m_n(t)^{\frac{i+s}{n}} \left( m_n(t)^{\frac{j+k-s}{n}} + m_n(t)^{\frac{j+k-s+1/2}{n}} \right) +
\]
\[
+ C \sum_{0 \leq i,j,k<n; i+j+k=n; j,k>0} m_n(t)^{\frac{i+k}{n}} \left( m_n(t)^{\frac{j-1}{n}} + m_n(t)^{\frac{j-1/2}{n}} \right) \left( m_n(t)^{\frac{k-1}{n}} + m_n(t)^{\frac{k-1/2}{n}} \right).
\]
Combining (2.59), (2.60) and (2.61) yields
\[
\frac{d}{dt} m_n(t)
\]
\[
\leq C m_n(t) + C m_n(t)^{\frac{n-1}{n}} + C m_n(t)^{\frac{1}{n}} - C m_n^{\frac{n+1}{n}}
\]
\[
+ C \sum_{0 \leq i,j,k<n; i+j+k=n} \sum_{s=0}^{k+1} m_n(t)^{\frac{i+s}{n}} \left( m_n(t)^{\frac{j+k-s}{n}} + m_n(t)^{\frac{j+k-s+1/2}{n}} \right) +
\]
\[
+ C \sum_{0 \leq i,j,k<n; i+j+k=n; j,k>0} m_n(t)^{\frac{i+k}{n}} \left( m_n(t)^{\frac{j-1}{n}} + m_n(t)^{\frac{j-1/2}{n}} \right) \left( m_n(t)^{\frac{k-1}{n}} + m_n(t)^{\frac{k-1/2}{n}} \right),
\]
where \( C \) depends on \( n, k \), and the bound of the mass on \([0, T]\) in Proposition 2.1. Notice that \(-C m_n(t)^{\frac{n+1}{n}}\) has the highest order on the right hand side of (2.62). By the same argument as in [97], the conclusion of the theorem then follows.

2.3 Holder estimates for the collision operators

In this section, we will provide Holder estimates for the two collision operators \( C_{12} \) and \( C_{22} \). For \( C_{22} \), we split it into two operators

\[
C_{22}^1[f](p_1) = \kappa_3 \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3} K^{22}(p_1, p_2, p_3, p_4) \frac{\min\{|p_1|, |p_2|, |p_3|, |p_4|\}|p_1||p_2||p_3||p_4|}{|p_1|^2} \]
\[
\times \delta(\mathcal{E}_{p_1} + \mathcal{E}_{p_2} - \mathcal{E}_{p_3} - \mathcal{E}_{p_4})[f(p_3)f(p_4) - f(p_1)f(p_2)]d|p_2|d|p_3|d|p_4|,
\]

(2.63)
and
\[
C_{22}^2[f](p_2) = \kappa_3 \iint_{\mathbb{R}^3 \times \mathbb{R}^3} K^{22}(p_1, p_2, p_3, p_4) \frac{\min\{|p_1|, |p_2|, |p_3|, |p_4|\}}{|p_1|^2} |p_1| |p_2| |p_3| |p_4| \\
\times \delta(\mathcal{E}_{p_1} + \mathcal{E}_{p_2} - \mathcal{E}_{p_3} - \mathcal{E}_{p_4}) [f(p_3)f(p_4)(f(p_1) + f(p_2)) - f(p_1)f(p_2)(f(p_3) + f(p_4))] dp_1 dp_2 dp_3 dp_4.
\]

We will show in Proposition 2.2, Proposition 2.6 and Proposition 2.7 that $C_{12}$, $C_{22}^1$ and $C_{22}^2$ are Holder continuous.

### 2.3.1 Holder estimates for $C_{12}$

**Proposition 2.5** Let $f$ and $g$ be two functions in $L^1_{n+3}(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$, $n \in \mathbb{R}_+$, $n$ can be 0; then there exists a constant $C$ depending on $\|f\|_{L^1_{n+3}}, \|f\|_{L^1}, \|g\|_{L^1_{n+3}}, \|g\|_{L^1}$ such that
\[
\|C_{12}[f] - C_{12}[g]\|_{L^1_{n+3}} \leq C (\|f - g\|_{L^1_{n+3}} + \|f - g\|_{L^1}).
\]
If $\|f\|_{L^1_{n+3}}, \|g\|_{L^1_{n+3}} \leq C_0$, then
\[
\|C_{12}[f] - C_{12}[g]\|_{L^1_{n+3}} \leq C_1 (\|f - g\|_{L^1_{n+3}}^{1/3} + \|f - g\|_{L^1}),
\]
where $C_1$ is a constant depending on $C_0, C$.

**Proof** First, let us consider the $L^1_{n}$ norm of the difference $C_{12}[f] - C_{12}[g]$. As a view of Lemma 2.1
\[
\|C_{12}[f] - C_{12}[g]\|_{L^1_{n}} = \int_{\mathbb{R}^3} |p_1|^n |C_{12}[f] - C_{12}[g]| dp_1 \\
\leq n_c \lambda_1 \iint_{\mathbb{R}^3} K^{12}(p_1, p_2, p_3) \delta(p_1 - p_2 - p_3) \delta(\mathcal{E}_{p_1} - \mathcal{E}_{p_2} - \mathcal{E}_{p_3}) \\
\times \left| f(p_2)f(p_3) - 2f(p_3)f(p_1) - f(p_1) - g(p_2)g(p_3) + 2g(p_3)g(p_1) + g(p_1) \right| |p_1|^n |p_2|^n |p_3|^n dp_1 dp_2 dp_3.
\]

The above identity implies that $\|C_{12}[f] - C_{12}[g]\|_{L^1_{n}}$ can be bounded by the sum of the following three terms
\[
N_1 = n_c \lambda_1 \iint_{\mathbb{R}^3} K^{12}(p_1, p_2, p_3) \delta(p_1 - p_2 - p_3) \delta(\mathcal{E}_{p_1} - \mathcal{E}_{p_2} - \mathcal{E}_{p_3}) \\
\times \left| f(p_2)f(p_3) - g(p_2)g(p_3) \right| |p_1|^n |p_2|^n |p_3|^n dp_1 dp_2 dp_3,
\]
\[
N_2 = 2n_c \lambda_1 \iint_{\mathbb{R}^3} K^{12}(p_1, p_2, p_3) \delta(p_1 - p_2 - p_3) \delta(\mathcal{E}_{p_1} - \mathcal{E}_{p_2} - \mathcal{E}_{p_3}) \\
\times \left| f(p_3)f(p_1) - g(p_3)g(p_1) \right| |p_1|^n |p_2|^n |p_3|^n dp_1 dp_2 dp_3,
\]

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and

\[
N_3 = n_c \int \int \int_{\mathbb{R}^3} K^{12}(p_1, p_2, p_3) \delta(p_1 - p_2 - p_3) \delta(\mathcal{E}_{p_1} - \mathcal{E}_{p_2} - \mathcal{E}_{p_3})
\times |f(p_1) - g(p_1)| |p_1|^n + |p_2|^n + |p_3|^n \, dp_1 dp_2 dp_3.
\]

In the sequel, we will estimate \( N_1, N_2, N_3 \) in three steps.

**Step 1: Estimating \( N_1 \).**

By the definition of \( \delta(p_1 - p_2 - p_3) \), \( N_1 \) can be rewritten as:

\[
N_1 = n_c \lambda_1 \int \int \int_{\mathbb{R}^3} K^{12}(p_2 + p_3, p_2, p_3) \delta(\mathcal{E}_{p_2+p_3} - \mathcal{E}_{p_2} - \mathcal{E}_{p_3})
\times |f(p_2) f(p_3) - g(p_2) g(p_3)| |p_2 + p_3|^n + |p_2|^n + |p_3|^n \, dp_2 dp_3.
\]

By the triangle inequality,

\[
|f(p_2) f(p_3) - g(p_2) g(p_3)| \leq |f(p_2) - g(p_2)| |f(p_3)| + |f(p_3) - g(p_3)| |g(p_2)|,
\]

the term \( N_1 \) can be bounded as

\[
N_1 \leq n_c \lambda_1 \int \int \int_{\mathbb{R}^3} K^{12}(p_2 + p_3, p_2, p_3) \delta(\mathcal{E}_{p_2+p_3} - \mathcal{E}_{p_2} - \mathcal{E}_{p_3})
\times |f(p_2) - g(p_2)||f(p_3)| |p_2 + p_3|^n + |p_2|^n + |p_3|^n \, dp_2 dp_3
\]

\[
+ n_c \lambda_1 \int \int \int_{\mathbb{R}^3} K^{12}(p_2 + p_3, p_2, p_3) \delta(\mathcal{E}_{p_2+p_3} - \mathcal{E}_{p_2} - \mathcal{E}_{p_3})
\times |f(p_3) - g(p_3)||g(p_2)| |p_2 + p_3|^n + |p_2|^n + |p_3|^n \, dp_2 dp_3.
\]

Again, by the triangle inequality

\[
|p_2 + p_3|^n \leq (|p_2| + |p_3|)^n \leq 2^{n-1}(|p_2|^n + |p_3|^n),
\]

one can estimate \( N_1 \) as

\[
N_1 \leq C \int \int \int_{\mathbb{R}^3} \delta(\mathcal{E}_{p_2+p_3} - \mathcal{E}_{p_2} - \mathcal{E}_{p_3}) K^{12}(p_2 + p_3, p_2, p_3) \times
\times |f(p_2) - g(p_2)||f(p_3)| |p_2|^n + |p_3|^n \, dp_2 dp_3
\]

\[
+ C \int \int \int_{\mathbb{R}^3} \delta(\mathcal{E}_{p_2+p_3} - \mathcal{E}_{p_2} - \mathcal{E}_{p_3}) K^{12}(p_2 + p_3, p_2, p_3) \times
\times |f(p_3) - g(p_3)||g(p_2)| |p_2|^n + |p_3|^n \, dp_2 dp_3,
\]

where \( C \) is a constant varying from lines to lines. The above estimate can be rewritten, taking into account the definition of \( \delta(\mathcal{E}_{p_2+p_3} - \mathcal{E}_{p_2} - \mathcal{E}_{p_3}) \), as

\[
N_1 \leq C \int_{\mathbb{R}^3} \int_{S_{p_3}} K^{12}(p_2 + p_3, p_2, p_3) |f(p_2) - g(p_2)||f(p_3)| |p_2|^n + |p_3|^n \, d\sigma(p_3) dp_2
\]

\[
+ C \int_{\mathbb{R}^3} \int_{S_{p_2}} K^{12}(p_2 + p_3, p_2, p_3) |f(p_3) - g(p_3)||g(p_2)| |p_2|^n + |p_3|^n \, d\sigma(p_2) dp_3.
\]
By Lemma 2.2, one can estimate $N_1$ as follows

$$
N_1 \leq C \int_{R^3} |f(p_2) - g(p_2)||f(p_3)||f(p_3)|K^{12}(p_2 + p_3, p_2, p_3) [p_2]^n + |p_3|^n \, dp_3 dp_2 + C \int_{R^3} |f(p_3) - g(p_3)||g(p_2)||f(p_3)|K^{12}(p_2 + p_3, p_2, p_3) [p_2]^n + |p_3|^n \, dp_2 dp_3.
$$

Since $K^{12}(p_2 + p_3, p_2, p_3)$ and $K^{12}(p_2 + p_3, p_2, p_3)$ are bounded, $N_1$ is bounded as

$$
N_1 \leq C \int_{R^3} |f(p_2) - g(p_2)||f(p_2)||f(p_3)||f(p_3)||p_2|^n + |p_3|^n \, dp_3 dp_2 + C \int_{R^3} |f(p_3) - g(p_3)||g(p_2)||f(p_3)||p_2|^n + |p_3|^n \, dp_2 dp_3,
$$

which leads to the following straightforward estimates on $N_1$

$$
N_1 \leq C \int_{R^3} |f(p_2) - g(p_2)||p_2|^n dp_2 \int_{R^3} |f(p_3)| dp_3 + C \int_{R^3} |f(p_2) - g(p_2)| dp_2 \int_{R^3} |f(p_3)||p_3|^n dp_3 + C \int_{R^3} |f(p_3) - g(p_3)| dp_3 \int_{R^3} |f(p_2)| dp_2 + C \int_{R^3} |f(p_3) - g(p_3)| dp_3 \int_{R^3} |f(p_2)||p_2|^n dp_2 \\
\leq C \int_{R^3} |f(p_1) - g(p_1)||p_1|^n dp_1 + C \int_{R^3} |f(p_1) - g(p_1)| dp_1.
$$

**Step 2: Estimating $N_2$.**

By the definition of $\delta(p_1 - p_2 - p_3)$, $N_2$ can be rewritten as:

$$
N_2 = 2n_4 \lambda_1 \int_{R^3} K^{12}(p_1 - p_1, p_2, p_3) [E_{p_1} - E_{p_1} - E_{p_3}] \\
\times |f(p_3) - g(p_3)||f(p_1)||p_1|^n + |p_1 - p_3|^n + |p_3|^n \, dp_1 dp_3,
$$

which, by the inequality,

$$
|p_1 - p_3|^n \leq (|p_1| + |p_3|)^n \leq 2^{n-1}(|p_1|^n + |p_3|^n),
$$

can be bounded as

$$
N_2 \leq C \int_{R^3} K^{12}(p_1, p_1, p_3, p_3) [E_{p_1} - E_{p_1} - E_{p_3}] \\
\times |f(p_3) - g(p_3)||f(p_1)||p_1|^n + |p_3|^n \, dp_1 dp_3 + C \int_{R^3} K^{12}(p_1, p_1, p_3, p_3) [E_{p_1} - E_{p_1} - E_{p_3}] \\
\times |f(p_1) - g(p_1)||g(p_3)||p_1|^n + |p_3|^n \, dp_1 dp_3.
$$
Employing the definition of $\delta(\mathcal{E}_{p_1} - \mathcal{E}_{p_1-p_3} - \mathcal{E}_{p_3})$, one can estimate $N_2$ as

\[
N_2 \leq C \int_{\mathbb{R}^3} \int_{S_{p_1}} K^{12}(p_1, p_1 - p_3, p_3) |f(p_3) - g(p_3)||f(p_1)| \left[ |p_1|^n + |p_3|^n \right] d\sigma(p_3) dp_1 \\
+ C \int_{\mathbb{R}^3} \int_{S_{p_1}} K^{12}(p_1, p_1 - p_3, p_3) |f(p_1) - g(p_1)||g(p_3)| \left[ |p_1|^n + |p_3|^n \right] d\sigma(p_3) dp_1,
\]

which, by Lemma [2.4], yields

\[
N_2 \leq \int_{\mathbb{R}^3} (1 + |p_1|) \int_{0}^{|p_1|} K^{12}(p_1, p_1 - p_3, p_3) |f(p_1) - g(p_1)||g(p_3)| \left[ |p_1|^n + |p_3|^n \right] |p_3| d|p_3| dp_1 \\
+ \int_{\mathbb{R}^3} (1 + |p_1|) \int_{0}^{\infty} K^{12}(p_1, p_1 - p_3, p_3) |f(p_3) - g(p_3)||f(p_1)| \left[ |p_1|^n + |p_3|^n \right] |p_3| d|p_3| dp_1.
\]

Bounding the integral from 0 to $|p_1|$ by an integral from 0 to $\infty$ implies

\[
N_2 \leq \int_{\mathbb{R}^3} (1 + |p_1|) \int_{0}^{\infty} K^{12}(p_1, p_1 - p_3, p_3) |f(p_1) - g(p_1)||g(p_3)| \left[ |p_1|^n + |p_3|^n \right] |p_3| d|p_3| dp_1 \\
+ \int_{\mathbb{R}^3} (1 + |p_1|) \int_{0}^{\infty} K^{12}(p_1, p_1 - p_3, p_3) |f(p_3) - g(p_3)||f(p_1)| \left[ |p_1|^n + |p_3|^n \right] |p_3| d|p_3| dp_1.
\]

We now switch the integral from $d|p_3|$ to $dp_3$ from the above inequality to obtain

\[
N_2 \leq \int_{\mathbb{R}^3 \times 2} (1 + |p_1|) \frac{K^{12}(p_1, p_1 - p_3, p_3)}{|p_3|} |f(p_3) - g(p_3)||f(p_1)| \left[ |p_1|^n + |p_3|^n \right] dp_3 dp_1 \\
+ \int_{\mathbb{R}^3 \times 2} (1 + |p_1|) \frac{K^{12}(p_1, p_1 - p_3, p_3)}{|p_3|} |f(p_3) - g(p_3)||f(p_1)| \left[ |p_1|^n + |p_3|^n \right] dp_3 dp_1.
\]

Applying the inequality

\[
(1 + |p_1|)(|p_1|^n + |p_3|^n) \leq C(1 + |p_1|^{n+1} + |p_3|^{n+1})
\]

to the above bound on $N_2$, we get

\[
N_2 \leq \int_{\mathbb{R}^3 \times 2} \frac{K^{12}(p_1, p_1 - p_3, p_3)}{|p_3|} |f(p_3) - g(p_3)||f(p_1)| \left[ 1 + |p_1|^{n+1} + |p_3|^{n+1} \right] dp_3 dp_1 \\
+ \int_{\mathbb{R}^3 \times 2} \frac{K^{12}(p_1, p_1 - p_3, p_3)}{|p_3|} |f(p_1) - g(p_1)||g(p_3)| \left[ 1 + |p_1|^{n+1} + |p_3|^{n+1} \right] dp_3 dp_1.
\]

The same argument as for (2.68) yields

\[
N_2 \leq C \int_{\mathbb{R}^3} |f(p_1) - g(p_1)||p_1|^{n+1} dp_1 + C \int_{\mathbb{R}^3} |f(p_1) - g(p_1)| dp_1. \quad (2.69)
\]
Step 3: Estimating $N_3$.

By the definition of $\delta(p_1 - p_2 - p_3)$, $N_3$ can be rewritten as:

$$N_3 = n_c \lambda_1 \int_{\mathbb{R}^3} K^{12}(p_1, p_2, p_1 - p_2) \delta(\xi_{p_1} - \xi_{p_2} - \xi_{p_1 - p_2}) \times |f(p_1) - g(p_1)| [||p_1||^n + ||p_2||^n + ||p_1 - p_2||^n] d\sigma_1 dp_1 dp_2,$$

which, by the inequality,

$$||p_1 - p_2||^n \leq (||p_1|| + ||p_2||)^n \leq 2^{n-1}(||p_1||^n + ||p_2||^n),$$

can be bounded as

$$N_3 \leq C \int_{\mathbb{R}^3} \int_{S_{p_1}} K^{12}(p_1, p_2, p_1 - p_2) |f(p_1) - g(p_1)| [||p_1||^n + ||p_2||^n] d\sigma_2 dp_1.$$

Now, as an application of Lemma 2.4,

$$\int_{S_{p_1}} (||p_1||^n + ||p_2||^n) d\sigma_2 (p_2) \leq C \left( ||p_1||^{n+2} \min\{1, ||p_1||\} + \int_{S_{p_1}} ||p_2||^n d\sigma_2 (p_2) \right) \leq C \left( ||p_1||^{n+2} \min\{1, ||p_1||\} + (1 + ||p_1||) \int_{0}^{||p_1||} ||p_2||^{n+1} d||p_2|| \right) \leq C (1 + ||p_1||^{n+3}),$$

which together with the fact that $K^{12}(p_1, p_2, p_1 - p_2)$ is bounded, implies

$$N_3 \leq C \int_{\mathbb{R}^3} |f(p_1) - g(p_1)| [||p_1||^{n+3} + 1] dp_1. \quad (2.70)$$

Combining (2.68), (2.69), and (2.70) yields

$$\|C_{12}[f] - C_{12}[g]\|_{L^\infty} \leq C \int_{\mathbb{R}^3} |f(p_1) - g(p_1)| [||p_1||^{n+3} + ||p_1||^{n+1} + ||p_1||^n + 1] dp_1. \quad (2.71)$$

Since

$$||p||^n \leq C (||p||^{n+3} + 1), ||p||^{n+1} \leq C (||p||^{n+3} + 1),$$

Inequality (2.65) follows from (2.71). Inequality (2.66) is a consequence of Inequality (2.65), Lemma 2.4 and

$$\|f - g\|_{L^{n+3}_{\infty}} \leq \|f - g\|_{L^1} \left( \|f\|_{L^{n+4}_{n+4}} + \|g\|_{L^{n+4}_{n+4}} \right)^{n+4 \over n+4}. \quad \square$$
2.3.2 Holder estimates for $C_{22}^{1}$

**Proposition 2.6** Let $f$ and $g$ be two functions in $L_{n}^{1}(\mathbb{R}^{3}) \cap L_{n}^{1}(\mathbb{R}^{3})$, $n \in \mathbb{N}$, $n/2$ is an odd number, or $n = 0$, then there exists a constant $C$ depending on $\|f\|_{L_{n+1}^{1}}$, $\|f\|_{L_{n+1}^{1}}$, $\|g\|_{L_{n+1}^{1}}$, $\|g\|_{L_{n+1}^{1}}$ such that

$$\|C_{22}^{1}[f] - C_{22}^{1}[g]\|_{L_{n}^{1}} \leq C \left( \|f - g\|_{L_{n+1}^{1}} + \|f - g\|_{L_{1}^{1}} \right). \tag{2.72}$$

If $\|f\|_{L_{n+2}^{1}}$, $\|g\|_{L_{n+2}^{1}} < C_{0}$, then

$$\|C_{22}^{1}[f] - C_{22}^{1}[g]\|_{L_{n}^{1}} \leq C_{1} \left( \|f - g\|_{L_{n+2}^{1}}^{\frac{1}{n+2}} + \|f - g\|_{L_{1}^{1}} \right), \tag{2.73}$$

where $C_{1}$ is a constant depending on $C_{0}$, $C$.

**Proof** Let us consider the $L_{n}^{1}$ norm of the difference $C_{22}^{1}[f] - C_{22}^{1}[g]$. As a view of Lemma [2.1]

$$\int_{\mathbb{R}^{3}} \left| C_{22}^{1}[f](p_{1}) - C_{22}^{1}[g](p_{1}) \right| \left| p_{1} \right|^{n} dp_{1}$$

$$\leq C \int_{\mathbb{R}_{+}^{3}} K^{22}(p_{1}, p_{2}, p_{3}, p_{4}) \min\{p_{1}, \left| p_{2} \right|, \left| p_{3} \right|, \left| p_{4} \right|, \left| p_{4} \right| p_{1} \left| p_{2} \right| p_{3} \left| p_{4} \right| \delta(\varepsilon_{p_{1}} + \varepsilon_{p_{2}} - \varepsilon_{p_{3}} - \varepsilon_{p_{4}}) \right.$$  

$$\times \left| f(p_{1}) f(p_{2}) - g(p_{1}) g(p_{2}) \right| \left[ \left| p_{4} \right|^{n} + \left| p_{3} \right|^{n} + \left| p_{2} \right|^{n} + \left| p_{1} \right|^{n} \right] \left| p_{1} \right| \left| p_{2} \right| \left| p_{3} \right| \left| p_{4} \right|$$

By the inequality

$$\left| p_{1} \right|^{n} \leq C \varepsilon_{p_{1}}^{n/2},$$

one gets

$$\left| p_{4} \right|^{n} + \left| p_{3} \right|^{n} + \left| p_{2} \right|^{n} + \left| p_{1} \right|^{n} \leq C \varepsilon_{p_{4}}^{n/2} + C \varepsilon_{p_{3}}^{n/2} + C \varepsilon_{p_{2}}^{n/2} + C \varepsilon_{p_{1}}^{n/2},$$

which implies

$$\int_{\mathbb{R}^{3}} \left| C_{22}^{1}[f](p_{1}) - C_{22}^{1}[g](p_{1}) \right| \left| p_{1} \right|^{n} dp_{1}$$

$$\leq C \int_{\mathbb{R}_{+}^{3}} K^{22}(p_{1}, p_{2}, p_{3}, p_{4}) \min\{p_{1}, \left| p_{2} \right|, \left| p_{3} \right|, \left| p_{4} \right|, \left| p_{4} \right| p_{1} \left| p_{2} \right| p_{3} \left| p_{4} \right| \delta(\varepsilon_{p_{1}} + \varepsilon_{p_{2}} - \varepsilon_{p_{3}} - \varepsilon_{p_{4}}) \right.$$  

$$\times \left| f(p_{1}) f(p_{2}) - g(p_{1}) g(p_{2}) \right| \left[ \varepsilon_{p_{4}}^{n/2} + \varepsilon_{p_{3}}^{n/2} + \varepsilon_{p_{2}}^{n/2} + \varepsilon_{p_{1}}^{n/2} \right] \left| p_{1} \right| \left| p_{2} \right| \left| p_{3} \right| \left| p_{4} \right|$$

Now, thanks to the Dirac function $\delta(\varepsilon_{p_{1}} + \varepsilon_{p_{2}} - \varepsilon_{p_{3}} - \varepsilon_{p_{4}})$, one can write $\varepsilon_{p_{4}}$ as $\varepsilon_{p_{1}} + \varepsilon_{p_{2}} - \varepsilon_{p_{3}}$, which implies

$$\int_{\mathbb{R}^{3}} \left| C_{22}^{1}[f](p_{1}) - C_{22}^{1}[g](p_{1}) \right| \left| p_{1} \right|^{n} dp_{1}$$

$$\leq C \int_{\mathbb{R}_{+}^{3}} K^{22}(p_{1}, p_{2}, p_{3}, p_{4}) \min\{p_{1}, \left| p_{2} \right|, \left| p_{3} \right|, \left| p_{4} \right| \left| p_{4} \right| p_{1} \left| p_{2} \right| p_{3} \left| p_{4} \right| \delta(\varepsilon_{p_{1}} + \varepsilon_{p_{2}} - \varepsilon_{p_{3}} - \varepsilon_{p_{4}}) \right.$$  

$$\times \left| f(p_{1}) f(p_{2}) - g(p_{1}) g(p_{2}) \right| \left[ (\varepsilon_{p_{1}} + \varepsilon_{p_{2}} - \varepsilon_{p_{3}})^{n/2} + \varepsilon_{p_{3}}^{n/2} + \varepsilon_{p_{2}}^{n/2} + \varepsilon_{p_{1}}^{n/2} \right] \left| p_{1} \right| \left| p_{2} \right| \left| p_{3} \right| \left| p_{4} \right|.$$
Similar as for (2.44), \(|p_4|d|p_4|\) can be bounded by \(C d \mathcal{E}_{p_4}\) and \(\min\{|p_1|, |p_2|, |p_3|, |p_4|\}\) can be bounded by \(\min\{|p_1|, |p_2|, |p_3|\}\). Moreover, \(K^{22}(p_1, p_2, p_3, p_4)\) is bounded by \(\Gamma\) due to (1.10).

As a consequence,

\[
\int_{\mathbb{R}^3} |C_{22}^1[f](p_1) - C_{22}^1[g](p_1)| |p_1|^n dp_1 
\leq C \int_{\mathbb{R}^3} \min\{|p_1|, |p_2|, |p_3|\}|p_1||p_2||p_3|\delta(\mathcal{E}_{p_1} + \mathcal{E}_{p_2} - \mathcal{E}_{p_3} - \mathcal{E}_{p_4})|f(p_1)f(p_2) - g(p_1)g(p_2)| \left[ (\mathcal{E}_{p_1} + \mathcal{E}_{p_2} - \mathcal{E}_{p_3})^{n/2} + \mathcal{E}_{p_2}^{n/2} + \mathcal{E}_{p_1}^{n/2} \right] \left[ f(p_1)f(p_2) - g(p_1)g(p_2) \right] |p_1||p_2||p_3|d\mathcal{E}_{p_4},
\]

where in the last inequality, we have taken the integration with respect to \(d\mathcal{E}_{p_4}\). Since \(n/2\) is an odd number, by Newton formula

\[
(\mathcal{E}_{p_1} + \mathcal{E}_{p_2} - \mathcal{E}_{p_3})^{n/2} + \mathcal{E}_{p_2}^{n/2} + \mathcal{E}_{p_1}^{n/2} = \sum_{0 \leq i, j, k; i + j + k = n/2; k \neq n/2} B_{i,j,k,n} \mathcal{E}_{p_1}^i \mathcal{E}_{p_2}^j \mathcal{E}_{p_3}^k,
\]

we obtain

\[
\int_{\mathbb{R}^3} |C_{22}^1[f](p_1) - C_{22}^1[g](p_1)| |p_1|^n dp_1 \leq X, \tag{2.74}
\]

where

\[
X := C \int_{\mathcal{E}_{p_3} \leq \mathcal{E}_{p_1} + \mathcal{E}_{p_2}} \min\{|p_1|, |p_2|, |p_3|\}|p_1||p_2||p_3|\left[ f(p_1)f(p_2) - g(p_1)g(p_2) \right] \left[ \sum_{0 \leq i,j,k; i + j + k = n/2; k \neq n/2} B_{i,j,k,n} \mathcal{E}_{p_1}^i \mathcal{E}_{p_2}^j \mathcal{E}_{p_3}^k \right] |p_1||p_2||p_3|.\]

The rest of the proof is devoted to estimates of \(X\).

Similar as for (2.44), \(|p_3|d|p_3|\) can be bounded by \(C d \mathcal{E}_{p_3}\) and \(\min\{|p_1|, |p_2|, |p_3|\}\) can be bounded by \(\min\{|p_1|, |p_2|\}\):
Integrating with respect to \( d\mathcal{E}_{p_3} \) the above integral and using Newton formula yields

\[
X \leq C \int_{\mathbb{R}_+^2} \min\{|p_1|, |p_2|\} |p_1| |p_2| |f(p_1) f(p_2)| - g(p_1) g(p_2) \left[ \sum_{k=0}^{k+1} \sum_{i+j+k=n/2} C \int_{\mathbb{R}_+^2} \min\{|p_1|, |p_2|\} |p_1| |p_2| |f(p_1) f(p_2)| - g(p_1) g(p_2) |\mathcal{E}^{i+s} \mathcal{E}^{k+1-j-s} p_1 d|p_1| d|p_2| \right]

\leq \sum_{k=0}^{k+1} \sum_{i+j+k=n/2} C \int_{\mathbb{R}_+^2} \min\{|p_1|, |p_2|\} |p_1| |p_2| |f(p_1) f(p_2)| - g(p_1) g(p_2) |\mathcal{E}^{i+s} \mathcal{E}^{k+1-j-s} p_1 d|p_1| d|p_2| + C \int_{\mathbb{R}_+^2} \min\{|p_1|, |p_2|\} |p_1| |p_2| |f(p_1) f(p_2)| - g(p_1) g(p_2) |\mathcal{E}^{n/2+1} p_1 d|p_1| d|p_2|.

By the inequalities

\[
\min\{|p_1|, |p_2|\} |p_1| |p_2| \leq |p_1| |p_2|^2,
\]

and

\[
\min\{|p_1|, |p_2|\} |p_1| |p_2| \leq |p_1|^2 |p_2|,
\]

one deduce that

\[
X \leq X_1 + X_2,
\]

(2.75)

where

\[
X_1 := \sum_{k=0}^{k+1} \sum_{i+j+k=n/2} C \int_{\mathbb{R}_+^2} |p_1| |p_2|^2 |f(p_1) f(p_2)| - g(p_1) g(p_2) |\mathcal{E}^{i+s} \mathcal{E}^{k+1-j-s} p_1 d|p_1| d|p_2|.
\]

\[
X_2 := C \int_{\mathbb{R}_+^2} |p_1|^2 |p_2| |f(p_1) f(p_2)| - g(p_1) g(p_2) |\mathcal{E}^{n/2+1} p_1 d|p_1| d|p_2|.
\]

Let us first estimate \( X_1 \) by looking at the terms inside the sum

\[
\int_{\mathbb{R}_+^2} |p_1| |p_2|^2 |f(p_1) f(p_2)| - g(p_1) g(p_2) |\mathcal{E}^{i+s} \mathcal{E}^{k+1-j-s} p_1 d|p_1| d|p_2|
\]

\[
\leq \int_{\mathbb{R}_+^2} |p_1| |p_2|^2 |f(p_1)| |g(p_2)| |\mathcal{E}^{i+s} \mathcal{E}^{k+1-j-s} p_1 d|p_1| d|p_2|
\]

\[
+ \int_{\mathbb{R}_+^2} |p_1| |p_2|^2 |f(p_2)| |g(p_1)| |\mathcal{E}^{i+s} \mathcal{E}^{k+1-j-s} p_1 d|p_1| d|p_2|.
\]
where we have used the triangle inequality
\[ |f(p_1)f(p_2) - g(p_1)g(p_2)| \leq |f(p_1) - g(p_1)||g(p_2)| + |f(p_2) - g(p_2)||f(p_1)|. \]
Since \( 0 < i + s \leq n/2 + 1 \) and \( 0 \leq k + 1 + j - s \leq n/2 + 1 \), we have
\[ \mathcal{E}_{p_1}^{i+s} \leq C \left( |p_1| + |p_1|^{n+2} \right), \]
and
\[ \mathcal{E}_{p_2}^{k+1+j-s} \leq C \left( 1 + |p_2|^{n+2} \right), \]
which yields
\[
\int_{\mathbb{R}^3} |p_1||p_2|^2 |f(p_1)f(p_2) - g(p_1)g(p_2)| \mathcal{E}_{p_1}^{i+s} \mathcal{E}_{p_2}^{k+1+j-s} dp_1 dp_2
\]
\[
\leq C \int_{\mathbb{R}^3} |p_1| \left( |p_1| + |p_1|^{n+2} \right) |f(p_1) - g(p_1)| dp_1 \int_{\mathbb{R}^3} |p_1|^2 \left( 1 + |p_1|^{n+2} \right) |g(p_1)| dp_1
\]
\[
+ C \int_{\mathbb{R}^3} |p_1| \left( |p_1| + |p_1|^{n+2} \right) |f(p_1) - g(p_1)| dp_1 \int_{\mathbb{R}^3} |p_1|^2 \left( 1 + |p_1|^{n+2} \right) |f(p_1)| dp_1,
\]
where in the last inequality, we have switched the integration on \( \mathbb{R}_+ \) to \( \mathbb{R}^3 \), by a spherical change of variables. Now, by the boundedness of \( f \) and \( g \) in \( L^1 \) and \( L^{n+2}_1 \),
\[
\int_{\mathbb{R}^3} |p_1||p_2|^2 |f(p_1)f(p_2) - g(p_1)g(p_2)| \mathcal{E}_{p_1}^{i+s} \mathcal{E}_{p_2}^{k+1+j-s} dp_1 dp_2
\]
\[
\leq C \int_{\mathbb{R}^3} \left( 1 + |p_1|^{n+1} \right) |f(p_1) - g(p_1)| dp_1,
\]
which implies the following estimate on \( X_1 \)
\[
X_1 \leq C \|f - g\|_{L^1} + C \|f - g\|_{L_1^{n+1}}. \tag{2.76}
\]
We now estimate \( X_2 \). As an application of the inequality
\[ \mathcal{E}_{p_2}^{n/2+1} \leq C \left( |p_2| + |p_2|^{n+2} \right), \]
\( X_2 \) can be bounded as follows
\[
X_2 \leq C \int_{\mathbb{R}_+^2} |p_1|^2 |p_2||f(p_1)f(p_2) - g(p_1)g(p_2)| \left( |p_2| + |p_2|^{n+2} \right) dp_1 dp_2
\]
\[
\leq C \int_{\mathbb{R}_+^2} |p_1|^2 |f(p_1) - g(p_1)||g(p_2)| \left( |p_2|^2 + |p_2|^{n+3} \right) dp_1 dp_2
\]
\[
+ C \int_{\mathbb{R}_+^2} |p_1|^2 |f(p_2) - g(p_2)||f(p_1)| \left( |p_2|^2 + |p_2|^{n+3} \right) dp_1 dp_2.
\]
The same argument as for (2.76) leads to

\[ X_2 \leq C\|f - g\|_{L^1} + C\|f - g\|_{L^{n+1}}. \tag{2.77} \]

Combining (2.75), (2.76) and (2.77) yields

\[ X \leq C\left(\|f - g\|_{L^1} + \|f - g\|_{L^{n+1}}\right). \tag{2.78} \]

The two inequalities (2.74) and (2.78) lead to

\[ \int_{\mathbb{R}^3} |C_{22}^1[f](p_1) - C_{22}^1[g](p_1)| |p_1|^n dp_1 \leq C\left(\|f - g\|_{L^1} + \|f - g\|_{L^{n+1}}\right). \tag{2.79} \]

Inequality (2.73) is a consequence of Inequality (2.72), Lemma 2.5 and

\[ \|f - g\|_{L^{n+1}} \leq \|f - g\|_{L^{n+1/2}} \left(\|f\|_{L_{n+2}^1} + \|g\|_{L_{n+2}^1}\right)^{n+1/2}. \]

\[ \blacksquare \]

### 2.3.3 Holder estimates for \( C_{22}^2 \)

**Proposition 2.7** Let \( f \) and \( g \) be two functions in \( L^1_n(\mathbb{R}^3) \cap L^1(\mathbb{R}^3), n/2 \in \mathbb{N}, n \) can be 0, then there exists a constant \( C \) depending on \( \|f\|_{L^1_n}, \|f\|_{L^1}, \|g\|_{L^1_n}, \|g\|_{L^1} \), such that

\[ \|C_{22}^2[f] - C_{22}^2[g]\|_{L^1_n} \leq C\left(\|f - g\|_{L^1_n} + \|f - g\|_{L^1}\right). \tag{2.80} \]

If \( \|f\|_{L^1_n}, \|g\|_{L^1_n} < C_0 \), then

\[ \|C_{22}^2[f] - C_{22}^2[g]\|_{L_n^1} \leq C_1\left(\|f - g\|_{L_n^1}^{1/2} + \|f - g\|_{L^1}\right), \tag{2.81} \]

where \( C_1 \) is a constant depending on \( C_0, C \).

**Proof** As a view of Lemma 2.1 the \( L^1_n \) norm of the difference \( C_{22}^2[f] - C_{22}^2[g] \) can be written as

\[
\int_{\mathbb{R}^3} |C_{22}^2[f](p_1) - C_{22}^2[g](p_1)| |p_1|^n dp_1 \\
\leq C \int_{\mathbb{R}^4} K^{22}(p_1, p_2, p_3, p_4) \min\{|p_1|, |p_2|, |p_3|, |p_4|\}|p_1||p_2||p_3||p_4| \delta(E_{p_1} + E_{p_2} - E_{p_3} - E_{p_4}) \times \nabla \times |f(p_1)f(p_2)f(p_3) - g(p_1)g(p_2)g(p_3)| |p_4|^n + |p_3|^n + |p_2|^n + |p_1|^n |dp_1|dp_2|dp_3|dp_4|,
\]

Similar as for Proposition 2.6 by the inequality

\[ |p_4|^n + |p_3|^n + |p_2|^n + |p_1|^n \leq C\mathcal{E}^{n/2}_{p_4} + C\mathcal{E}^{n/2}_{p_3} + C\mathcal{E}^{n/2}_{p_2} + C\mathcal{E}^{n/2}_{p_1}, \]

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one has

\[
\int_{\mathbb{R}^3} |C_{22}^2[f](p_1) - C_{22}^2[g](p_1)| |p_1|^n dp_1 \\
\leq C \int_{\mathbb{R}^4_+} K^{22}(p_1, p_2, p_3, p_4) \min\{|p_1|, |p_2|, |p_3|, |p_4|\} |p_1||p_2||p_3||p_4| \delta(\mathcal{E}_{p_1} + \mathcal{E}_{p_2} - \mathcal{E}_{p_3} - \mathcal{E}_{p_4}) \times \\
\times |f(p_1)f(p_2)f(p_3) - g(p_1)g(p_2)g(p_3)| \left[ \mathcal{E}_{p_4}^{n/2} + \mathcal{E}_{p_3}^{n/2} + \mathcal{E}_{p_2}^{n/2} + \mathcal{E}_{p_1}^{n/2} \right] d|p_1|d|p_2|d|p_3|d|p_4|,
\]

By the Dirac function \(\delta(\mathcal{E}_{p_1} + \mathcal{E}_{p_2} - \mathcal{E}_{p_3} - \mathcal{E}_{p_4})\), \(\mathcal{E}_{p_4}\) can be written as \(\mathcal{E}_{p_1} + \mathcal{E}_{p_2} - \mathcal{E}_{p_3}\), which implies

\[
\int_{\mathbb{R}^3} |C_{22}^2[f](p_1) - C_{22}^2[g](p_1)| |p_1|^n dp_1 \\
\leq C \int_{\mathbb{R}^4_+} K^{22}(p_1, p_2, p_3, p_4) \min\{|p_1|, |p_2|, |p_3|, |p_4|\} |p_1||p_2||p_3||p_4| \delta(\mathcal{E}_{p_1} + \mathcal{E}_{p_2} - \mathcal{E}_{p_3} - \mathcal{E}_{p_4}) \times \\
\times |f(p_1)f(p_2)f(p_3) - g(p_1)g(p_2)g(p_3)| \left[ (\mathcal{E}_{p_1} + \mathcal{E}_{p_2} - \mathcal{E}_{p_3})^{n/2} + \\
+ \mathcal{E}_{p_1}^{n/2} + \mathcal{E}_{p_2}^{n/2} + \mathcal{E}_{p_3}^{n/2} \right] d|p_1|d|p_2|d|p_3|d|p_4|.
\]

Similar as for (2.44), \(|p_4||d|p_4|\) can be bounded by \(C d\mathcal{E}_{p_4}\) and \(\min\{|p_1|, |p_2|, |p_3|, |p_4|\}\) can be bounded by \(\min\{|p_1|, |p_2|, |p_3|\}\), which leads to

\[
\int_{\mathbb{R}^3} |C_{22}^2[f](p_1) - C_{22}^2[g](p_1)| |p_1|^n dp_1 \\
\leq C \int_{\mathbb{R}^4_+} K^{22}(p_1, p_2, p_3, p_4) \min\{|p_1|, |p_2|, |p_3|\} |p_1||p_2||p_3| \delta(\mathcal{E}_{p_1} + \mathcal{E}_{p_2} - \mathcal{E}_{p_3} - \mathcal{E}_{p_4}) \times \\
\times |f(p_1)f(p_2)f(p_3) - g(p_1)g(p_2)g(p_3)| \left[ (\mathcal{E}_{p_1} + \mathcal{E}_{p_2} - \mathcal{E}_{p_3})^{n/2} + \mathcal{E}_{p_1}^{n/2} + \mathcal{E}_{p_2}^{n/2} + \mathcal{E}_{p_3}^{n/2} \right] d|p_1|d|p_2|d|p_3|d\mathcal{E}_{p_4} \\
\leq C \int_{\mathcal{E}_{p_3} \leq \mathcal{E}_{p_1} + \mathcal{E}_{p_2}} K^{22}(p_1, p_2, p_3, p_4) \min\{|p_1|, |p_2|, |p_3|\} |p_1||p_2||p_3| |f(p_1)f(p_2)f(p_3) - \\
- g(p_1)g(p_2)g(p_3)| \left[ (\mathcal{E}_{p_1} + \mathcal{E}_{p_2} - \mathcal{E}_{p_3})^{n/2} + \mathcal{E}_{p_1}^{n/2} + \mathcal{E}_{p_2}^{n/2} + \mathcal{E}_{p_3}^{n/2} \right] d|p_1|d|p_2|d|p_3|,
\]

where we have taken the integration with respect to \(d\mathcal{E}_{p_4}\).

Since \(n/2\) is a natural number, by Newton formula

\[
(\mathcal{E}_{p_1} + \mathcal{E}_{p_2} - \mathcal{E}_{p_3})^{n/2} + \mathcal{E}_{p_3}^{n/2} + \mathcal{E}_{p_2}^{n/2} + \mathcal{E}_{p_1}^{n/2} = \sum_{0 \leq i,j,k ; i+j+k=n/2} D_{i,j,k,n} \mathcal{E}_{p_1}^i \mathcal{E}_{p_2}^j \mathcal{E}_{p_3}^k,
\]

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where \( D_{i,j,k,n} \) are positive constants. As an application of the above Newton formula, one has

\[
\int_{\mathbb{R}^3} |C_{22}^2[f](p_1) - C_{22}^2[g](p_1)| |p_1|^n dp_1 \\
\leq C \sum_{0 \leq i,j,k ; \; i+j+k=n/2} \int_{\mathcal{E}_{p_3} \leq \mathcal{E}_{p_1} + \mathcal{E}_{p_2}} K^{22}(p_1, p_2, p_3, p_4) \min\{|p_1|, |p_2|, |p_3|\} |p_1||p_2||p_3| \\
\times |f(p_1) f(p_2) f(p_3) - g(p_1) g(p_2) g(p_3)| \mathcal{E}_{p_1}\mathcal{E}_{p_2}\mathcal{E}_{p_3} \, dp_1 \, dp_2 \, dp_3,
\]

where \( C \) is a positive constant varying from lines to lines.

By using the fact that

\[
K^{22}(p_1, p_2, p_3, p_4) \min\{|p_1|, |p_2|, |p_3|\} |p_1||p_2||p_3| \leq C |p_1|^2 |p_2|^2 |p_3|^2,
\]

where \( C \) is a positive constant depending on \( p_* \) defined in (1.9), we get

\[
\int_{\mathbb{R}^3} |C_{22}^2[f](p_1) - C_{22}^2[g](p_1)| |p_1|^n dp_1 \\
\leq C \sum_{0 \leq i,j,k ; \; i+j+k=n/2} \int_{\mathcal{E}_{p_3} \leq \mathcal{E}_{p_1} + \mathcal{E}_{p_2}} |p_1|^2 |p_2|^2 |p_3|^2 |f(p_1) f(p_2) f(p_3) -
\]

\[
- g(p_1) g(p_2) g(p_3)| \mathcal{E}_{p_1}\mathcal{E}_{p_2}\mathcal{E}_{p_3} \, dp_1 \, dp_2 \, dp_3| \\
\leq C \sum_{0 \leq i,j,k ; \; i+j+k=n/2} \int_{\mathbb{R}^3} |p_1|^2 |p_2|^2 |p_3|^2 |f(p_1) f(p_2) f(p_3) -
\]

\[
- g(p_1) g(p_2) g(p_3)| \mathcal{E}_{p_1}\mathcal{E}_{p_2}\mathcal{E}_{p_3} \, dp_1 \, dp_2 \, dp_3|.
\]

Changing from the radial integration on \( \mathbb{R}_+ \) to the integration on \( \mathbb{R}^3 \) in the above inequality, by a spherical coordinate change of variables, yields

\[
\int_{\mathbb{R}^3} |C_{22}^2[f](p_1) - C_{22}^2[g](p_1)| |p_1|^n dp_1 \\
\leq C \sum_{0 \leq i,j,k ; \; i+j+k=n/2} \int_{\mathbb{R}^3} |f(p_1) f(p_2) f(p_3) - g(p_1) g(p_2) g(p_3)| \mathcal{E}_{p_1}\mathcal{E}_{p_2}\mathcal{E}_{p_3} \, dp_1 \, dp_2 \, dp_3.
\]

Applying the triangle inequality

\[
|f(p_1) f(p_2) f(p_3) - g(p_1) g(p_2) g(p_3)| \\
\leq |f(p_1) - g(p_1)| |f(p_2)||f(p_3)| + |f(p_2) - g(p_2)| |g(p_1)||f(p_3)| + |f(p_3) - g(p_3)| |g(p_1)||g(p_3)|,
\]
to the previous inequality gives
\[
\int_{\mathbb{R}^3} |C_{22}^2[f](p_1) - C_{22}^2[g](p_1)| |p_1|^n dp_1 \leq C \sum_{0 \leq i,j,k ; i+j+k=n/2} \int_{\mathbb{R}^3} |f(p_1) - g(p_1)||f(p_2)||f(p_3)||\mathcal{E}_{p_1}^i \mathcal{E}_{p_2}^j \mathcal{E}_{p_3}^k| dp_1 dp_2 dp_3
\]
\[+ C \sum_{0 \leq i,j,k ; i+j+k=n/2} \int_{\mathbb{R}^3} |f(p_2) - g(p_2)||g(p_1)||f(p_3)||\mathcal{E}_{p_1}^i \mathcal{E}_{p_2}^j \mathcal{E}_{p_3}^k| dp_1 dp_2 dp_3
\]
\[+ C \sum_{0 \leq i,j,k ; i+j+k=n/2} \int_{\mathbb{R}^3} |f(p_3) - g(p_3)||g(p_1)||g(p_3)||\mathcal{E}_{p_1}^i \mathcal{E}_{p_2}^j \mathcal{E}_{p_3}^k| dp_1 dp_2 dp_3.\]

Notice that we can estimate $\mathcal{E}_{p_1}^i$, $\mathcal{E}_{p_2}^j$ and $\mathcal{E}_{p_3}^k$ as
\[
|\mathcal{E}_{p_1}^i| \leq C(1 + |p|^n), \quad |\mathcal{E}_{p_2}^j| \leq C(1 + |p|^n), \quad |\mathcal{E}_{p_3}^k| \leq C(1 + |p|^n),
\]
which leads to the following estimate on the norm of $C_{22}^2[f] - C_{22}^2[g]$
\[
\int_{\mathbb{R}^3} |C_{22}^2[f](p_1) - C_{22}^2[g](p_1)| |p_1|^n dp_1 \leq C \sum_{0 \leq i,j,k ; i+j+k=n/2} \int_{\mathbb{R}^3} |f(p_1) - g(p_1)||f(p_2)||f(p_3)|
\]
\[\times (1 + |p_1|^n)(1 + |p_2|^n)(1 + |p_3|^n) dp_1 dp_2 dp_3
\]
\[+ C \sum_{0 \leq i,j,k ; i+j+k=n/2} \int_{\mathbb{R}^3} |f(p_2) - g(p_2)||g(p_1)||f(p_3)|
\]
\[\times (1 + |p_1|^n)(1 + |p_2|^n)(1 + |p_3|^n) dp_1 dp_2 dp_3
\]
\[+ C \sum_{0 \leq i,j,k ; i+j+k=n/2} \int_{\mathbb{R}^3} |f(p_3) - g(p_3)||g(p_1)||g(p_3)|
\]
\[\times (1 + |p_1|^n)(1 + |p_2|^n)(1 + |p_3|^n) dp_1 dp_2 dp_3.\]

Now, since
\[
\int_{\mathbb{R}^3} |f(p)|(1 + |p|^n) = \|f\|_{L^1} + \|f\|_{L^1}, \quad \int_{\mathbb{R}^3} |g(p)|(1 + |p|^n) = \|g\|_{L^1} + \|g\|_{L^1},
\]
we get from the above inequality that
\[
\int_{\mathbb{R}^3} |C_{22}^2[f](p_1) - C_{22}^2[g](p_1)| |p_1|^n dp_1 \leq C \left(\|f - g\|_{L^1} + \|f - g\|_{L^1} \right).
\]
Inequality (2.81) is a consequence of Inequality (2.80), Lemma 2.25 and
\[
\|f - g\|_{L^1} \leq \|f - g\|_{L^1} \left(\|f\|_{L^1} + \|g\|_{L^1} \right)^\frac{n+1}{n+1}.\]
2.4 Proof of Theorem 1.2

In order to prove Theorem 1.2 we will use Theorem 1.3. Choose \( E = L^1_{2n}(\mathbb{R}^3) \). We define the function \(| \cdot |_*\) to be

\[
|f|_* = \int_{\mathbb{R}^3} f(p)dp.
\]

Set

\[
\|f\|_* = \int_{\mathbb{R}^3} |f(p)|dp.
\]

By (2.33), it is clear that for all \( f \geq 0, f \in E \), the following inequality holds true

\[
|Q[f]|_* \leq C^* (1 + \|f\|_*), \quad (2.82)
\]

where \( C^* \) depends on \( \|f\|_{L^1_{2n}(\mathbb{R}^3)} \). We then choose \( C_* \) in Theorem 1.3 as \( C^* \).

The set \( S \) is defined as follows:

\[
S := \left\{ f \in L^1_{2n}(\mathbb{R}^3) \mid (S_1) f \geq 0, \quad f(p) = f(|p|), \quad (S_2) \int_{\mathbb{R}^3} f(|p|)dp \leq \zeta_0,
\right. \]

\[
(S_3) \int_{\mathbb{R}^+} f(|p|)\xi p dp = \zeta_1, \quad (S_4) \int_{\mathbb{R}^+} f(|p|)\xi^*_p dp \leq \zeta^{*}_n \right\}, \quad (2.83)
\]

where

\[
\zeta_0 := (2R + 1)e^{(C^*+1)T}, \quad (2.84)
\]

and

\[
\zeta^{*}_n = \frac{3\rho n^{*}_n}{2}, \quad (2.85)
\]

with \( \rho_n^{*} \) defined in (2.87). It is clear that \( S \) is a bounded, convex and closed subset of \( L^1_{2n}(\mathbb{R}^3) \). Moreover for all \( f \) in \( S \), it is straightforward that \(|f|_* = \|f\|_*\).

In the four Sections 2.4.2, 2.4.1, 2.4.3, 2.4.4 we will verify the four conditions (A), (B), (C) and (D) of Theorem 1.3. Then, Theorem 1.2 follows as an application of Theorem 1.3.

2.4.1 Checking Condition (A)

We choose the constant \( R_* \) to be \( R \), then for all \( u \) in \( S \), \( \|u\|_* \leq (2R_* + 1)e^{(C^*+1)T} \). Condition (A) is satisfied.
2.4.2 Checking Condition (28)

First, the same argument as for (2.62) gives

\[
\int_{\mathbb{R}^3} Q[f] e_p^p \, dp \leq \mathcal{P}[m_0^*(f)] :=
\]

\[C m_0^*(f) + C m_0^*(f) \frac{a^{n-1}}{n} + C m_0^*(f) \frac{a^l}{n} - C m_0^*(f) \frac{a^{n+1}}{n+1} + C \sum_{0 \leq i,j,k < n^*; i+j+k = n^*} \sum_{s=0}^{k+1} m_n^*(f) \frac{i+s}{n} \left( m_n^*(f) \frac{i+k-s}{n} + m_n^*(f) \frac{i+k-s+1/2}{n} \right) +
\]

\[+ C \sum_{0 \leq i,j,k < n^*; i+j+k = n^*; j,k > 0} m_n^*(f) \frac{j+k}{n} \left( m_n^*(f) \frac{j-l}{n} + m_n^*(f) \frac{j-l+1/2}{n} \right) \times \left( m_n^*(f) \frac{k-l}{n} + m_n^*(f) \frac{k-l+1/2}{n} \right), \quad \forall f \in \mathcal{S},
\]

where \( C \) is a positive constant depending on \( \epsilon_0 \).

Let \( \rho_{n^*} \) be the solution of \( \mathcal{P}(\rho) = 0 \): if \( 0 < \rho < \rho_{n^*} \), \( \mathcal{P}(\rho) < 0 \); if \( \rho > \rho_{n^*} \), \( \mathcal{P}(\rho) > 0 \).

Notice that \( \rho_{n^*} \) depends on \( \epsilon_0 \).

Let \( f \) be an arbitrary element of the set \( \mathcal{S} \cap B \left( \Omega, (2R^*_n + 1)e^{(C^*_n + 1)T} \right) \) and consider \( f + hQ[f] \). We will show that for all \( \epsilon > 0 \), there exists \( h_\epsilon \) depending on \( f \) and \( \epsilon \) such that \( B(f + hQ[f], \epsilon) \cap \mathcal{S} \) is not empty for all \( 0 < h < h_\epsilon \). Define \( \chi_R(p) \) to be the characteristic function of the ball \( B(0, R) \) centered at the origin with radius \( R \). Set \( f_R(p) = \chi_R(p)f(p) \) and \( w_R = f + hQ[f_R] \). Since \( Q[f_R] \in L^1_{2n}(\mathbb{R}^3) \), we find that \( w_R \in L^1_{2n}(\mathbb{R}^3) \). We will prove that for \( h_\epsilon \) small enough and \( R \) large enough, \( w_R \) belongs to \( \mathcal{S} \). We now verify the four conditions (S1), (S2), (S3) and (S4).

- **Condition (S1):** Since \( f_R \) is compactly supported, it is clear that \( Q^-[f_R] \), with \( Q^- \) defined in (2.6), is bounded by \( C(f, R, \epsilon_0, \epsilon_{n^*}) \), a positive constant depending on \( f \), \( R \), \( \epsilon_0 \), \( \epsilon_{n^*} \), which implies

\[ w_R \geq f - hQ^-[f_R] \geq f(1 - hQ^{-}[f_R]) \geq 0, \]

for \( h < C(f, R, \epsilon_0, \epsilon_{n^*})^{-1} \).

- **Condition (S2):** Since

\[ \|f\|_* < (2R^*_n + 1)e^{(C^*_n + 1)T}, \]

and

\[ \lim_{h \to 0} \|f - w_R\|_* = 0, \]

we can choose \( h_\epsilon \) small enough such that

\[ \|w_R\|_* < (2R^*_n + 1)e^{(C^*_n + 1)T}. \]
• **Condition (S₃):** By the conservation of energy, we have

\[
\int_{\mathbb{R}^3} w_R \mathcal{E}_p \, dp = \int_{\mathbb{R}^3} (f + hQ[f_R]) \mathcal{E}_p \, dp = \int_{\mathbb{R}^3} f \mathcal{E}_p \, dp = c_1.
\]

• **Condition (S₄):** Now, we claim that \( R \) and \( h_* \) can be chosen, such that

\[
\int_{\mathbb{R}^3} w_R \mathcal{E}_p^{n_*} \, dp < \frac{3\rho_{n_*}}{2}.
\]

In order to see this, we consider two cases:

If \( \int_{\mathbb{R}^3} f \mathcal{E}_p^{n_*} \, dp < \frac{3\rho_{n_*}}{2} \),

we deduce from the fact

\[
\lim_{h \to 0} \int_{\mathbb{R}^3} |w_R - f| \mathcal{E}_p^{n_*} \, dp = 0,
\]

that we can choose \( h_* \) small enough such that

\[
\int_{\mathbb{R}^3} w_R \mathcal{E}_p^{n_*} \, dp < \frac{3\rho_{n_*}}{2}.
\]

If, on the other hand, we have

\[
\int_{\mathbb{R}^3} f \mathcal{E}_p^{n_*} \, dp = \frac{3\rho_{n_*}}{2},
\]

we can choose \( R \) large enough such that

\[
\int_{\mathbb{R}^3} f R \mathcal{E}_p^{n_*} \, dp > \rho_{n_*},
\]

which implies, by (2.87), that

\[
\int_{\mathbb{R}^3} Q[f_R] < 0.
\]

As a consequence,

\[
\int_{\mathbb{R}^3} w_R \mathcal{E}_p^{n_*} \, dp < \int_{\mathbb{R}^3} f \mathcal{E}_p^{n_*} \, dp = \frac{3\rho_{n_*}}{2}.
\]

Finally, we have \( w_R \in S \) for all \( 0 < h < h_* \).

Now since

\[
\lim_{R \to \infty} \frac{1}{h} \|w_R - f - hQ[f_R]\|_{L^{\infty}(\mathbb{R}^3)} = \lim_{R \to \infty} \|Q[f] - Q[f_R]\|_{L^{1}_{\text{loc}}(\mathbb{R}^3)} = 0,
\]

then for \( R \) large enough, \( w_R \in B(f + hQ[f], h\epsilon) \), which implies \( B(f + hQ[f], h\epsilon) \cap S \setminus \{f + hQ[f]\} \). Condition (B) is verified.
2.4.3 Checking Condition (C)
Condition (C) follows from Propositions 2.5, 2.6, and 2.7.

2.4.4 Checking Condition (D)
By the Lebesgue dominated convergence theorem, we have that
\[
\left[ \varphi, \phi \right] \leq \int_{\mathbb{R}^3} \varphi(p) \text{sign}(\phi(p))(1 + \mathcal{E}^n_p)dp,
\]
which means that Condition (D) is satisfied if we have the following inequality
\[
\mathcal{M}_0 := \int_{\mathbb{R}^3} [Q[f](p) - Q[g](p)] \text{sign}((f - g)(p))(1 + \mathcal{E}^n_p)dp \leq C \| f - g \|_{L^1_{2n}}.
\]  
Since \( Q = C_{12} + C_{22} \), let us split
\[
\mathcal{M}_0 = \mathcal{M}_1 + \mathcal{M}_2,
\]
where
\[
\mathcal{M}_1 := \int_{\mathbb{R}^3} [C_{12}[f](p) - C_{12}[g](p)] \text{sign}((f - g)(p))(1 + \mathcal{E}^n_p)dp,
\]
and
\[
\mathcal{M}_2 := \int_{\mathbb{R}^3} [C_{22}[f](p) - C_{22}[g](p)] \text{sign}((f - g)(p))(1 + \mathcal{E}^n_p)dp.
\]

**Step 1: Estimating \( \mathcal{M}_1 \).**
Define \( \varphi_k(p) = \text{sign}((f - g)(p))\mathcal{E}^k_p \), \( k \in \mathbb{Z}, k \geq 0, k \neq 1 \). Let us consider the following generalized term of \( \mathcal{M}_1 \)
\[
\mathcal{N}_0 := \int_{\mathbb{R}^3} [C_{12}[f](p) - C_{12}[g](p)]\varphi_k(p)dp,
\]
which by Lemma 2.1 can be rewritten as
\[
\mathcal{N}_0 := \int_{\mathbb{R}^3} [R_{12}[f](p_1) - R_{12}[g](p_1)][\varphi_k(p_1) - \varphi_k(p_2) - \varphi_k(p_3)]dp_1dp_2dp_3
\]
\[
= \int_{\mathbb{R}^3} \tilde{K}^{12}(p_2 + p_3, p_2, p_3)\delta(\mathcal{E}_{p_2+p_3} - \mathcal{E}_{p_2} - \mathcal{E}_{p_3}) \{ (f(p_2)f(p_3) - g(p_2)g(p_3)) \}
\]
\[
- 2(f(p_2)f(p_2 + p_3) - g(p_2)g(p_2 + p_3)) - (f(p_2 + p_3) - g(p_2 + p_3)) \times
\]
\[
\times [\varphi_k(p_2 + p_3) - \varphi_k(p_2) - \varphi_k(p_3)]dp_2dp_3.
\]
Split \( \mathcal{N}_0 \) into the sum of three terms:
\[
\mathcal{N}_1 := \int_{\mathbb{R}^3} \tilde{K}^{12}(p_2 + p_3, p_2, p_3)\delta(\mathcal{E}_{p_2+p_3} - \mathcal{E}_{p_2} - \mathcal{E}_{p_3}) \{ (f(p_2)f(p_3) - g(p_2)g(p_3)) \}
\]
\[
\times [\varphi_k(p_2 + p_3) - \varphi_k(p_2) - \varphi_k(p_3)]dp_2dp_3,
\]
\[
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\]
\[ N_2 := -2 \int_{\mathbb{R}^3} \tilde{K}^{12}(p_2 + p_3, p_2, p_3) \delta(\mathcal{E}_{p_2 + p_3} - \mathcal{E}_{p_2})[f(p_2)f(p_2 + p_3) - g(p_2)g(p_2 + p_3)] \times [\varphi_k(p_2 + p_3) - \varphi_k(p_2) - \varphi_k(p_3)] dp_2 dp_3, \]

and

\[ N_3 := -\int_{\mathbb{R}^3} \tilde{K}^{12}(p_2 + p_3, p_2, p_3) \delta(\mathcal{E}_{p_2 + p_3} - \mathcal{E}_{p_2})[f(p_2 + p_3) - g(p_2 + p_3)] \times [\varphi_k(p_2 + p_3) - \varphi_k(p_2) - \varphi_k(p_3)] dp_2 dp_3. \]

The same arguments as for (2.68) and (2.69) give

\[ N_1 \leq C \| f - g \|_{L^1_{2k}(\mathbb{R}^3)}, \]

and

\[ N_2 \leq C \| f - g \|_{L^1_{2k+1}(\mathbb{R}^3)}, \]

where \( C \) is a positive constant varying from lines to lines. The third term \( N_3 \) can be estimated as

\[ N_3 = -\int_{\mathbb{R}^3} \tilde{K}^{12}(p_2 + p_3, p_2, p_3) \delta(\mathcal{E}_{p_2 + p_3} - \mathcal{E}_{p_2})[f(p_2 + p_3) - g(p_2 + p_3)] \times \]

\[ \times [\mathcal{E}^{k}_{p_2 + p_3} \text{sign}((f(p_2 + p_3) - g(p_2 + p_3)) - \mathcal{E}^{k}_{p_2} \text{sign}((f(p_2) - g(p_2)) - \]

\[ - \mathcal{E}^{k}_{p_3} \text{sign}((f(p_3) - g(p_3))] dp_2 dp_3 \]

\[ \leq \int_{\mathbb{R}^3} \tilde{K}^{12}(p_2 + p_3, p_2, p_3) \delta(\mathcal{E}_{p_2 + p_3} - \mathcal{E}_{p_2})[f(p_2 + p_3) - g(p_2 + p_3)] \times \]

\[ \times [\mathcal{E}^{k}_{p_2} + \mathcal{E}^{k}_{p_3} - \mathcal{E}^{k}_{p_2 + p_3}] dp_2 dp_3. \]

Now, let us consider the two cases \( k = 0 \) and \( k > 1 \) separately.

- If \( k = 0 \),

\[ N_3 \leq \int_{\mathbb{R}^3} \tilde{K}^{12}(p_2 + p_3, p_2, p_3) \delta(\mathcal{E}_{p_2 + p_3} - \mathcal{E}_{p_2} - \mathcal{E}_{p_3})[f(p_2 + p_3) - g(p_2 + p_3)] dp_2 dp_3, \]

which, by the same arguments that lead to (2.70), can be bounded as

\[ N_3 \leq C \| f - g \|_{L^1(\mathbb{R}^3)}. \]

- If \( k > 1 \), since \( \mathcal{E}_{p_2 + p_3} = \mathcal{E}_{p_2} + \mathcal{E}_{p_3} \), it is straightforward that

\[ \mathcal{E}^{k}_{p_2} + \mathcal{E}^{k}_{p_3} - \mathcal{E}^{k}_{p_2 + p_3} = \mathcal{E}^{k}_{p_2} + \mathcal{E}^{k}_{p_3} - (\mathcal{E}_{p_2} + \mathcal{E}_{p_3})^k \leq -k \mathcal{E}_{p_2} \mathcal{E}_{p_3}^{k-1} \leq 0. \]
As a consequence, we can estimate $N_3$ as

$$N_3 \leq \int_{\mathbb{R}^3} |f(p) - g(p)| \left( |p|^{2k+2} \min \{ 1, |p| \}^{2k+6} \right) dp.$$  \hfill (2.101)

Combining (2.95), (2.96), (2.99) and (2.101) for the two cases $k = 0$ and $k = n$, yields

$$M_1 \leq C \int_{\mathbb{R}^3} |f(p) - g(p)| \left( 1 + |p| + |p|^3 + |p|^{2n} + |p|^{2n+1} \right)$$

$$\leq - |p|^{2n+2} \min \{ 1, |p| \}^{2n+6} \right) dp.$$  \hfill (2.102)

**Step 2: Estimating $M_2$.**
We can estimate $M_2$ in a straightforward manner by employing Propositions 2.6 and 2.7 as follows

$$M_2 \leq C \int_{\mathbb{R}^3} |f(p) - g(p)| \left( 1 + |p| + |p|^{2n} + |p|^{2n+1} \right) dp.$$  \hfill (2.103)

**Step 3: Estimating $M_0$.**
Combining (2.102) and (2.103) yields

$$M_0 \leq C \int_{\mathbb{R}^3} |f(p) - g(p)| \left( 1 + |p| + |p|^3 + |p|^{2n} + |p|^{2n+1} \right)$$

$$\leq - |p|^{2n+2} \min \{ 1, |p| \}^{2n+6} \right) dp.$$  \hfill (2.104)

Since for $|p| \leq 1$,

$$1 + |p| + |p|^3 + |p|^{2n} + |p|^{2n+1} + |p|^{4n+8} \leq 5,$$
and for $|p| > 1$, there exists $C > 0$ independent of $p$ such that

$$1 + |p| + |p|^3 + |p|^{2n} + |p|^{2n+1} + |p|^{4n+2} \leq C,$$
we find that the weight

$$1 + |p| + |p|^3 + |p|^{2n} + |p|^{2n+1} + |p|^{4n+2} \min \{ 1, |p| \}^{2n+6}$$

of (2.104) is bounded uniformly in $p$ by a universal positive constant $C$. As a consequence, Inequality (2.104) implies

$$M_0 \leq C \int_{\mathbb{R}^3} |f(p) - g(p)| dp,$$  \hfill (2.105)

which concludes the proof of (2.89).
3 Proof of Theorem 1.3

Our proof is an extension and generalization of the framework proposed in [20]. The proof is divided into four parts:

Part 1: Fix a element \( v \) of \( \mathcal{S} \), due to the Holder continuity property of \( Q \), we have
\[
\|Q(u)\| \leq \|Q(v)\| + C\|u - v\|^\beta, \quad \forall u \in \mathcal{S}.
\]
According to our assumption, \( \mathcal{S} \) is bounded by a constant \( C_S \). We deduce from the above inequality that
\[
\|Q(u)\| \leq \|Q(v)\| + C_S (\|u\| + \|v\|)^\beta =: C_Q, \quad \forall u \in \mathcal{S}.
\]
For an element \( u \) be in \( \mathcal{S} \), there exists \( \xi_u > 0 \) such that for \( 0 < \xi < \xi_u \), \( u + \xi Q(u) \in \mathcal{S} \), which implies
\[
B(u + \xi Q(u), \delta) \cap \mathcal{S} \setminus \{u + \xi Q(u)\} \neq \emptyset,
\]
for \( \delta \) small enough. Choose \( \epsilon = 2C((C_Q + 1)\xi)^\beta \), then \( \|Q(u) - Q(v)\| \leq \frac{\xi}{2} \) if \( \|u - v\| \leq (C_Q + 1)\xi \), by the Holder continuity of \( Q \). Let \( z \in B\left(u + \xi Q(u), \frac{\epsilon \xi}{2}\right) \cap \mathcal{S} \setminus \{u + \xi Q(u)\} \) and define
\[
t \mapsto \vartheta(t) = u + \frac{t(z - u)}{\xi}, \quad t \in [0, \xi].
\]
Since \( \mathcal{S} \) is convex, \( \vartheta \) maps \([0, \xi]\) into \( \mathcal{S} \). It is straightforward that
\[
\|\vartheta(t) - u\| \leq \xi\|Q(u)\| + \frac{\epsilon \xi}{2} < (C_Q + 1)\xi,
\]
which implies
\[
\|Q(\vartheta(t)) - Q(u)\| \leq \frac{\epsilon}{2}, \quad \forall t \in [0, \xi].
\]
The above inequality and the fact that
\[
\|\dot{\vartheta}(t) - Q(u)\| \leq \frac{\epsilon}{2},
\]
leads to
\[
\|\dot{\vartheta}(t) - Q(\vartheta(t))\| \leq \epsilon, \quad \forall t \in [0, \xi]. \tag{3.1}
\]

Part 2: Let \( \vartheta \) be a solution to (3.1), which is constructed step by step, following the procedure of Part 1 on a sequence of intervals \([0, \tau_1], \ldots, [\tau_1 + \cdots + \tau_{n-1}, \tau_1 + \cdots + \tau_n]\) and \( \tau_1 + \cdots + \tau_n \leq T \). Inequality (3.1) leads to
\[
\left| \frac{\vartheta(\tau_1) - \vartheta(0)}{\tau_1} - Q(\vartheta(0)) \right| \leq C_E \epsilon,
\]
which yields
\[ |\vartheta(\tau_1)|_* \leq \tau_1|\vartheta(0)|_* + \tau_1C_* (|\vartheta(0)|_* + 1) + \tau_1C_E \varepsilon. \]
Since we can assume that \( C_E \varepsilon < 1 \), we obtain
\[ |\vartheta(\tau_1)|_* \leq (C_* + 1)\tau_1|\vartheta(0)|_* + (C_* + 1)\tau_1. \]
As a consequence, we find the following series of inequalities
\[ |\vartheta(\tau_1)|_* \leq (C_* + 1)\tau_1|\vartheta(0)|_* + (C_* + 1)\tau_1, \]
\[ \ldots \]
\[ |\vartheta(\tau_1 + \cdots + \tau_n)|_* \leq (C_* + 1)\tau_n|\vartheta(\tau_1 + \cdots + \tau_{n-1})|_* + (C_* + 1)\tau_n. \]
We obtain from the above inequalities that
\[ |\vartheta(\tau_1 + \cdots + \tau_n)|_* \leq \tau_n \cdots \tau_1|\vartheta(0)|_*(C_* + 1)^n \]
\[ + \tau_n \cdots \tau_2 \tau_1 (C_* + 1)^n \]
\[ \ldots \]
\[ + \tau_n \tau_{n-1}(C_* + 1)^2 \]
\[ + \tau_n(C_* + 1). \]
Applying the inequality
\[ \frac{(x_1 + x_2 + \cdots + x_m)^m}{m!} \geq x_1x_2 \cdots x_m, \forall m \in \mathbb{N}, x_1, \ldots, x_m \in \mathbb{R}_+, \]
to the above estimate, we find
\[ |\vartheta(\tau_1 + \cdots + \tau_n)|_* \leq \frac{(\tau_n + \cdots + \tau_1)^n}{n!}|\vartheta(0)|_*(C_* + 1)^n \]
\[ + \frac{(\tau_n + \cdots + \tau_1)^n}{n!}(C_* + 1)^n \]
\[ \ldots \]
\[ + \frac{(\tau_n + \tau_{n-1})^2}{2!}(C_* + 1)^2 \]
\[ + \tau_n(C_* + 1), \]
which leads to
\[ ||\vartheta(\tau_1 + \cdots + \tau_n)||_* = |\vartheta(\tau_1 + \cdots + \tau_n)|_* \leq (|\vartheta(0)|_* + 1) \left( e^{(C_* + 1)(\tau_1 + \cdots + \tau_n)} - 1 \right) \]
\[ \leq (2R_* + 1)e^{(C_* + 1)T}. \]
(3.2)
Now, suppose that we have a solution \( \bar{\vartheta} \) to (3.1) on the time interval \([0, \tau]\), that satisfies
\[ |\bar{\vartheta}(\tau)|_* \leq (|\bar{\vartheta}(0)|_* + 1) \left( e^{(C_* + 1)\tau} - 1 \right). \]
(3.3)
Using the procedure of Part 1, we assume that \( \dot{\vartheta} \) can be extended to the interval \([\tau, \tau + \tau']\), with \( \tau + \tau' \leq T \), \( \tau' \leq \tau \).

The same arguments that lead to (3.2) imply

\[
|\dot{\vartheta}(\tau + \tau')|_* \leq ((|\dot{\vartheta}(\tau)|_* + 1)e^{(C_* + 1)\tau'} - 1).
\]

Combining the above inequality with (3.3) yields

\[
||\dot{\vartheta}(\tau + \tau')||_* = |\dot{\vartheta}(\tau + \tau')|_* \\
\leq \left( (|\dot{\vartheta}(0)|_* + 1) \left( e^{(C_* + 1)\tau} - 1 \right) + 1 \right) \left( e^{(C_* + 1)\tau'} - 1 \right) \\
\leq \left( (|\dot{\vartheta}(0)|_* + 1) e^{(C_* + 1)\tau} + 1 \right) \left( e^{(C_* + 1)\tau'} - 1 \right) \\
\leq (|\dot{\vartheta}(0)|_* + 1) e^{(C_* + 1)(\tau + \tau')} + e^{(C_* + 1)\tau'} - (|\dot{\vartheta}(0)|_* + 1) e^{(C_* + 1)\tau} - 1 \\
\leq (|\dot{\vartheta}(0)|_* + 1) \left( e^{(C_* + 1)(\tau + \tau')} - 1 \right) \\
\leq (2R_* + 1)e^{(C_* + 1)T},
\]

where the last inequality follows from the fact that \( R_* \geq 1 \).

Notice that in the above argument, we can only extend the solution from \([0, \tau]\) to \([0, \tau + \tau']\) with the restriction \( \tau' \leq \tau \). However, suppose that we can extend the solution from \([0, \tau]\) to \([0, 2\tau]\). Now, from \([0, 2\tau]\) we can extend the solution to \([0, 2\tau + \tau'']\), and the constraint on \( \tau'' \) is much better: \( \tau'' \leq 2\tau \).

**Part 3:** From Part 1, there exists a solution \( \vartheta \) to the equation (3.1) on an interval \([0, h]\).

Now, we have the following procedure.

- **Step 1:** Suppose that we can construct the solution \( \vartheta \) of (3.1) on \([0, \tau]\) \((\tau < T)\). Since \( \vartheta(\tau) \in S \), by the same process as in Part 1 and by (3.2) and (3.4), the solution \( \vartheta \) could be extended to \([\tau, \tau + h_{\tau}]\) where \( \tau + h_{\tau} \leq T, h_{\tau} \leq \tau \).

- **Step 2:** Suppose that we can construct the solution \( \vartheta \) of (3.1) on a series of intervals \([0, \tau_1], [\tau_1, \tau_2], \ldots, [\tau_n, \tau_{n+1}], \ldots \). Observe that the increasing sequence \( \{\tau_n\} \) is bounded by \( T \), the sequence has a limit, defined by \( \tau \). Recall that \( Q(\vartheta) \) is bounded by \( C_Q \) on \([\tau_n, \tau_{n+1}]\) for all \( n \in \mathbb{N} \), then \( \dot{\vartheta} \) is bounded by \( \epsilon + C_Q \) on \([0, \tau]\). As a consequence \( \vartheta(\tau) \) can be defined as

\[
\vartheta(\tau) = \lim_{n \to \infty} \vartheta(\tau_n), \dot{\vartheta}(\tau) = \lim_{n \to \infty} \dot{\vartheta}(\tau_n),
\]

which, together with the fact that \( S \) is closed, implies that \( \vartheta \) is a solution of (3.1) on \([0, \tau]\).

By Step 2, if the solution \( \vartheta \) can be defined on \([0, T_0]\), \( T_0 < T \), it could be extended to \([0, T_0]\). Now, we suppose that \([0, T_0]\) is the maximal closed interval that \( \vartheta \) could be defined, by Step
1, θ could be extended to a larger interval \([T_0, T_0 + T_h]\), which means that \(T = T_0\) and \(θ\) is
defined on the whole interval \([0, T]\).

**Part 4:** Finally, let us consider a sequence of solution \(\{u^ε\}\) to (3.1) on \([0, T]\). We will
prove that this is a Cauchy sequence. Let \(\{u^ε\}\) and \(\{v^ε\}\) be two sequences of solutions
to (3.1) on \([0, T]\). We note that \(u^ε\) and \(v^ε\) are affine functions on \([0, T]\). Moreover by the
one-side Lipschitz condition
\[
\frac{d}{dt}||u^ε(t) - v^ε(t)|| = \left[u^ε(t) - v^ε(t), \dot{u}^ε(t) - \dot{v}^ε(t)\right]
\leq \left[u^ε(t) - v^ε(t), Q[u^ε(t)] - Q[v^ε(t)]\right] + 2ε
\leq C||u^ε(t) - v^ε(t)|| + 2ε,
\]
for a.e. \(t \in [0, T]\), which leads to
\[
||u^ε(t) - v^ε(t)|| \leq 2ε\frac{e^{LT}}{L}.
\]
Let \(ε\) tend to 0, \(u^ε \to u\) uniformly on \([0, T]\). It is straightforward that \(u\) is a solution to
(1.22).

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