ON THE TRANSVERSAL NUMBER AND VC-DIMENSION OF FAMILIES OF POSITIVE HOMOTHETS OF A CONVEX BODY

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Abstract. Let $F$ be a family of positive homothets (or translates) of a given convex body $K$ in $\mathbb{R}^n$. We investigate two approaches to measuring the complexity of $F$. First, we find an upper bound on the transversal number $\tau(F)$ of $F$ in terms of $n$ and the independence number $\nu(F)$. This question is motivated by a problem of Grünbaum [2]. Our bound $\tau(F) \leq 2^n (\frac{2^n}{n}) (n \log n + \log \log n + 5\log \log n)$ is exponential in $n$, an improvement from the previously known bound of Kim, Nakprasit, Pelsmajer and Skokan [10], which was of order $n^n$. By a lower bound, we show that the right order of magnitude is exponential in $n$.

Next, we consider another measure of complexity, the Vapnik–Cervonenkis dimension of $F$. We prove that vcdim($F$) $\leq 3$ if $n = 2$ and is infinite for some $F$ if $n \geq 3$. This settles a conjecture of Grünbaum [6]: Show that the maximum dual VC-dimension of a family of positive homothets of a given convex body $K$ in $\mathbb{R}^n$ is $n + 1$. This conjecture was disproved by Naiman and Wynn [13] who constructed a counterexample of dual VC-dimension $\lfloor \frac{3}{2} n^2 \rfloor$. Our result implies that no upper bound exists.

1. Definitions and Results

A convex body in $\mathbb{R}^n$ is a compact convex set with non-empty interior. A positive homothet of a set $S \subseteq \mathbb{R}^n$ is a set of the form $\lambda S + x$, where $\lambda > 0$ and $x \in \mathbb{R}^n$. The cardinality, closure, convex hull and volume of $S$ are denoted as $\text{card}(S)$, $\text{cl}(S)$, $\text{conv}(S)$ and $\text{vol}(S)$, respectively. The origin of $\mathbb{R}^n$ is denoted $o$.

Let $F$ be a family of positive homothets (or translates) of a given convex body $K$ in $\mathbb{R}^n$. In this note we study two approaches to measuring the complexity of $F$.

First, we bound the transversal number $\tau(F)$ in terms of the dimension $n$ and the independence number $\nu(F)$. The transversal number $\tau(F)$ of a family of sets $\mathcal{F}$ is defined as

$$\tau(F) = \min \{ \text{card}(S) : S \cap F \neq \emptyset \text{ for all } F \in \mathcal{F} \}.$$ 

The independence number $\nu(F)$ of $\mathcal{F}$ is defined as

$$\nu(F) = \max \{ \text{card}(S) : S \subseteq F \text{ and } S \text{ is pairwise disjoint} \}.$$ 

Clearly $\nu(F) \leq \tau(F)$. The problem of finding an inequality in the reverse direction originates in the following question of Grünbaum [2]: Is it true that $\nu(F) = 1$
implies \( \tau(\mathcal{F}) \leq 3 \) for any family \( \mathcal{F} \) of translates of a convex body in \( \mathbb{R}^2 \). Karasev [9] proved the affirmative answer. One of the main results of [10] by Kim, Nakprasit, Pelsmajer and Skokan is that in \( \mathbb{R}^n \) we have \( \tau(\mathcal{F}) \leq 2^{n-1} n^\nu(\mathcal{F}) \). We improve the dependence on \( n \) to exponential.

**Theorem 1.** Let \( K \subseteq \mathbb{R}^n \) be a convex body and \( \mathcal{F} \) a family of positive homothets of \( K \). Then

\[
\nu(\mathcal{F}) \leq \tau(\mathcal{F}) \leq \frac{\text{vol}(2K - K)}{\text{vol}(K)}(n \log n + \log \log n + 5n)\nu(\mathcal{F})
\]

\[
\leq \begin{cases} 
3^n(n \log n + \log \log n + 5n)\nu(\mathcal{F}) & \text{if } K = -K, \\
2^n(\binom{n}{2})n(n \log n + \log \log n + 5n)\nu(\mathcal{F}) & \text{otherwise.}
\end{cases}
\]

The following proposition shows that an exponential bound is the best possible, even when \( \mathcal{F} \) contains only translates of \( K \).

**Proposition 2.** For sufficiently large \( n \), there is a convex body \( K \) in \( \mathbb{R}^n \) and a family \( \mathcal{F} \) of translates of \( K \) such that \( \tau(\mathcal{F}) \geq \frac{1}{2}(1.058)^n\nu(\mathcal{F}) \).

Our second approach is to investigate the VC-dimension of a family \( \mathcal{F} \) of positive homothets (or translates) of a convex body \( K \). This combinatorial measure of complexity was introduced by Vapnik and Červonenkis [19], and is defined as

\[ \text{vcdim}(\mathcal{F}) = \sup \{ \text{card}(X) : \mathcal{F} \text{ shatters } X \} \]

where a set system \( \mathcal{F} \) is said to shatter a set of points \( X \) if for every subset \( X' \subseteq X \), there exists a set \( F \in \mathcal{F} \) such that \( X \cap F = X' \). Note that if there is no upper bound on the sizes of sets shattered by \( \mathcal{F} \), then this definition yields \( \text{vcdim}(\mathcal{F}) = \infty \).

Our main motivation in studying the VC-dimension is its involvement in upper bounds on transversal numbers (see the Epsilon Net Theorem of Haussler and Welzl [7] and Corollary 10.2.7 of [11]) and related phenomena (see [12], for example). We show, however, that \( \text{vcdim}(\mathcal{F}) \) is bounded from above only in dimension two.

**Theorem 3.** If \( K \subseteq \mathbb{R}^2 \) is a convex body and \( \mathcal{F} \) is a family of positive homothets of \( K \), then \( \text{vcdim}(\mathcal{F}) \leq 3 \).

**Example 4.** We construct a convex body \( K \subseteq \mathbb{R}^3 \) and a countable family \( \mathcal{F} \) of translates of \( K \) such that \( \text{vcdim}(\mathcal{F}) = \infty \).

This example can, of course, be embedded in \( \mathbb{R}^n \) for \( n > 3 \) as well.

Example 4 also settles a conjecture of Grünbaum on dual VC-dimension (see Section 10.3 of [11] for this notion). He showed [6] that if \( \mathcal{F} \) is a family of positive homothets of a convex body in \( \mathbb{R}^2 \), then \( \text{vcdim}(\mathcal{F}^*) \leq 3 \), and conjectured (point (7) on p. 21 of [6]) the upper bound \( \text{vcdim}(\mathcal{F}^*) \leq n + 1 \) for such families in \( \mathbb{R}^n \).

(Grünbaum uses a different terminology: instead of dual VC-dimension, he writes “the maximal number of sets in independent families”, where “independence” is not as we defined above.) Naiman and Wynn [13] disproved this conjecture by giving an example with \( \text{vcdim}(\mathcal{F}^*) = \left\lfloor \frac{3n}{2} \right\rfloor \); our example shows that no upper bound exists, since \( \text{vcdim}(\mathcal{F}) < 2^{\text{vcdim}(\mathcal{F}^*)+1} \) (III, Lemma 10.3.4).

**Corollary 5.** There is a convex body \( K \subseteq \mathbb{R}^3 \) and a countable family \( \mathcal{F} \) of translates of \( K \) such that \( \text{vcdim}(\mathcal{F}^*) = \infty \).
The construction of example 4 shares some principles with the constructions given in [8] and in Theorem 2.9 of [4] to show that certain Helly-type and Hadwiger-type theorems for line transversals of families of translates of a convex set in the plane do not generalize to $\mathbb{R}^3$. These examples and ours show that, in some sense, translates of a convex set in $\mathbb{R}^3$ may form set systems of high complexity. They also suggest that finding good bounds for the transversal numbers of such families is a difficult task.

In Section 2, we prove Theorem 1 and Proposition 2. In Section 3, we prove Theorem 3 and construct Example 4.

2. Transversal and Independence Numbers of Positive Homothets

Let $K$ and $L$ be convex bodies in $\mathbb{R}^n$. Let $N(K, L)$ denote the covering number of $K$ by $L$; that is, the smallest number of translates of $L$ required to cover $K$.

**Theorem 6** (Rogers [14], Rogers–Zong [16]). Let $K, L \subset \mathbb{R}^n$ be convex sets. Then

$$N(K, L) \leq \frac{\text{vol}(K - L)}{\text{vol}(L)} (n \log n + \log \log n + 5n).$$

**Proof of Theorem 6**. First, we prove the theorem in the case when $F$ consists of translates of $K$ only. Let $\{K_1, K_2, \ldots, K_\ell\}$ be a maximal set of independent (i.e., pairwise disjoint) elements of $F$. Clearly, $\ell \leq \nu(F)$. Let $F_1 = \{F \in F : F \cap K_1 \neq \emptyset\}$, and for $i = 2, \ldots, \ell$ let

$$F_i = \left\{ F \in F \setminus \bigcup_{j=1}^{i-1} F_j : F \cap K_i \neq \emptyset \right\}.$$

We will construct a transversal $T_i$ for each $F_i$.

It is easy to show that, for any set $S \subseteq \mathbb{R}^n$,

$$S - K = \{x \in \mathbb{R}^n : (K + x) \cap S \neq \emptyset\}.$$

An immediate consequence is that if $K_i - K \subseteq T_i - K$, then $T_i$ is a transversal of $F_i$. By Theorem 6, for each $i$, there is such a set $T_i$ with

$$\text{card}(T_i) \leq \frac{\text{vol}(K_i - K + K)}{\text{vol}(-K)} (n \log n + \log \log n + 5n)$$

$$= \frac{\text{vol}(2K - K)}{\text{vol}(K)} (n \log n + \log \log n + 5n)$$

$$\leq \begin{cases} 3^n (n \log n + \log \log n + 5n) & \text{if } K = -K, \\ 2^n \left(\frac{2^n}{n}\right) (n \log n + \log \log n + 5n) & \text{otherwise.} \end{cases}$$

The last inequality for the non-symmetric case follows from the Rogers–Shephard inequality [15]. Hence, $T = \bigcup_{i=1}^{\ell} T_i$ is a transversal of $F$ of cardinality bounded from above as stated in the theorem.

The proof of the case when $F$ contains finitely many positive homothets of $K$ follows from an argument given in [10], which we repeat here. First, assume that $\inf \{\lambda : \lambda K + x \in F\} > 0$. Let $\varepsilon$ be a positive number, to be specified later. We say that $\lambda K + x$ is a small member of a subset $A \subseteq F$ if

$$\lambda < (1 + \varepsilon) \inf \{\mu : \mu K + x \in A\}.$$
Let $F_1$ be a small element of $\mathcal{F}$, and let $\mathcal{F}_1 = \{F \in \mathcal{F} : F \cap F_1 \neq \emptyset \}$. Next, for each $i = 2, 3, \ldots , \ell$ inductively, let $F_i$ be a small element in $\mathcal{F} \setminus \bigcup_{j=1}^{i-1} \mathcal{F}_j$, and let

$$\mathcal{F}_i = \left\{ F \in \mathcal{F} \setminus \bigcup_{j=1}^{i-1} \mathcal{F}_j : F \cap F_i \neq \emptyset \right\}.$$ 

Let $\lambda_i = \inf \{ \lambda : \lambda K + x \in \mathcal{F}_i \}$. By assumption, $\lambda_i > 0$. Our inductive procedure of defining $F_i, \mathcal{F}_i$ and $\lambda_i$ will terminate with $\ell \leq \nu(\mathcal{F})$.

Now, for each $F \in \mathcal{F}_i$, choose a point $z$ in $F \cap F_i$, and shrink $F$ with center $z$ to obtain a translate of $\lambda_i K$. The shrunk copy of $F$ is clearly contained in $F$. Let $\mathcal{F}'_i$ be the family of these shrunk copies. Now, $\mathcal{F}'_i$ contains only translates of $\lambda_i K$, any transversal of $\mathcal{F}'_i$ is a transversal of $\mathcal{F}_i$, and each member of $\mathcal{F}'_i$ intersects $F_i$. Thus if $F_i - \lambda_i K \subseteq T_i - \lambda_i K$, then $T_i$ is a transversal of $\mathcal{F}_i$. Theorem 6 yields such a set $T_i$ with cardinality

$$\text{card}(T_i) \leq \frac{\text{vol}(1 + \varepsilon) \lambda_i K - \lambda_i K + \lambda_i K)}{\text{vol}(-\lambda_i K)}(n \log n + \log \log n + 5n).$$

Since $\text{card}(T_i)$ is an integer, choosing a sufficiently small $\varepsilon$ provides the right bound.

Finally, we sketch the additions necessary to handle the case when $\inf \{ \lambda : \lambda K + x \in \mathcal{F} \} = 0$, a case not considered in [10]. Let $(\delta_m)_{m=1}^\infty$ be a sequence of positive real numbers with $\delta_m \downarrow 0$. For every $m \in \mathbb{Z}^+$ we define $\mathcal{F}^m = \{ \lambda K + x \in \mathcal{F} : \lambda > \delta_m \}$. Using the previous proof, we obtain a transversal $T^m = \{ t^m_1, \ldots , t^m_k \}$ of $\mathcal{F}^m$ for each $m$, where $k$ is the desired bound. Now, choose some $G_1 \in \mathcal{F}$. By the pigeonhole principle, there is an $i \in \{1, \ldots , k\}$ with $t^m_i \in G_1$ for infinitely many $m$; assume $i = 1$. Passing to a subsequence of $(T^m)_{m=1}^\infty$, we may further assume that $t^m_i \to t_1 \in G_1$. If $\{t_1\}$ is not a transversal of $\mathcal{F}$, choose $G_2 \in \mathcal{F}$ with $t_1 \notin G_2$; passing to a further subsequence of $(T^m)_{m=1}^\infty$, we may assume that $t^m_2 \to t_2 \in G_2$. If $\{t_1, t_2\}$ is not a transversal of $\mathcal{F}$, continue in this manner, obtaining eventually a transversal of $\mathcal{F}$.

For the proof of Proposition 2, we need the following definition. A set $S \subseteq \mathbb{R}^n$ is called strictly antipodal if, for any two points $x_1$ and $x_2$ in $S$, there exists a hyperplane $H$ through $o$ such that $H + x_1$ and $H + x_2$ support $S$ and $(H + x_1) \cap S = \{ x_1 \}$ and $(H + x_2) \cap S = \{ x_2 \}$. For more on this notion, see [5].

Proof of Proposition 2 First, we show that if $S$ is a strictly antipodal set then $\mathcal{F} = \{ K + s : s \in S \}$, where $K = \text{conv}(S)$, is a family of pairwise touching translates of $K$, and no three members of $\mathcal{F}$ have a point in common. We may assume that $o \in K$. Let $x_1, x_2$ be two distinct points in $S$. Clearly, $x_1 + x_2 \in (K + x_1) \cap (K + x_2)$. On the other hand, if $H$ is a hyperplane as in the definition of strict antipodality, then $H' = H + x_1 + x_2$ separates $K + x_1$ and $K + x_2$. Moreover, $(K + x_1) \cap H' = (K + x_2) \cap H' = \{ x_1 + x_2 \}$. So, $K + x_1$ and $K + x_2$ touch each other. We need to show that for any $x_3 \in S \setminus \{ x_1, x_2 \}$, we have that $K + x_3$ does not contain $x_1 + x_2$. Suppose it does. Then $x_1 + x_2$ is a common point of $K + x_1$ and $K + x_2$, hence, by the previous argument, $x_1 + x_2 = x_1 + x_3$, so $x_2 = x_3$, a contradiction.

On the other hand, Füredi, Lagarias and Morgan (Theorem 2.4. in [3]) give a construction, for sufficiently large $n$, of a symmetric strictly convex body $K$ and a finite set $S \subseteq \mathbb{R}^n$ with the property that any two translates of $K$ in the family $\{ s + K : s \in S \}$ touch each other, moreover $\text{card}(S) \geq (1.02)^n$. It follows that $S$
is a strictly antipodal set. Later, Swanepoel observed (Theorem 2 in Section 2.2, [17]) that a better bound, \( \text{card}(S) \geq (1.058)^n \) follows from the proof in [3]. Thus, for the resulting \( F \) we have \( \nu(F) = 1 \) and \( \tau(F) \geq \frac{1}{2} \text{card}(F) = \frac{1}{2}(1.058)^n \). \( \square \)

3. VC-Dimension of Positive Homothets

Proof of Theorem 3. Let \( F \) be a family of positive homothets of a convex body \( K \subseteq \mathbb{R}^2 \). Suppose, for contradiction, that \( F \) shatters some set of four points, say, \( X = \{x_1, x_2, x_3, x_4\} \).

Case 1: One of the points of \( X \) is in the convex hull of the other three, say, \( x_1 \in \text{conv}(\{x_2, x_3, x_4\}) \). By hypothesis, there is an \( F \in F \) such that \( X \cap F = \{x_2, x_3, x_4\} \). But since \( F \) is convex, it follows that \( x_1 \in F \), which is a contradiction.

Case 2: The points of \( X \) are in convex position, forming the vertices of a convex quadrilateral in, say, the order \( x_1 x_2 x_3 x_4 \). (See Figure 1) Without loss of generality, \( X \cap K = \{x_1, x_3\} \) and \( X \cap TK = \{x_2, x_4\} \), where \( T : \mathbb{R}^2 \to \mathbb{R}^2, Tx = \lambda x + t \) is a homothety with ratio \( \lambda \geq 1 \).

First suppose \( \lambda > 1 \). Let

\[
p = \frac{1}{1-\lambda} t
\]

be the centre of the homothety \( T \). If \( p \) is in the (closed) region \( A \) shown in Figure 1, then \( x_2 \in \text{conv}(\{x_1, x_3, p\}) \). On the other hand, \( T^{-1}x_2 \) is a convex combination of \( p \) and \( x_2 \); thus \( x_2 \in \text{conv}(\{x_1, x_3, T^{-1}x_2\}) \). (See Figure 2) But \( \{x_1, x_3, T^{-1}x_2\} \subseteq K \), so by convexity, \( x_2 \in K \), a contradiction.

Similarly, if \( p \in B \) then \( x_4 \in \text{conv}(\{x_1, x_3, T^{-1}x_4\}) \subseteq K \); if \( p \in C \cup D \) then \( x_3 \in \text{conv}(\{x_2, x_4, Tx_3\}) \subseteq TK \); and if \( p \in D \cup E \) then \( x_1 \in \text{conv}(\{x_2, x_4, Tx_1\}) \subseteq TK \). In all cases we obtain a contradiction.

The case \( \lambda = 1 \), when \( T \) is a translation, succumbs to essentially the same argument, with \( p \) an ideal point corresponding to the direction of the translation. We omit the details.

\( \square \)
Figure 3. The paraboloid $z = x^2 + y^2$ and a few sections of it.

Construction of Example 4. To illustrate the ideas of the construction, we first sketch how to construct, for any $M \in \mathbb{N}$, a convex body $K$ whose translates shatter a set of $M$ points.

The sections of the paraboloid $z = x^2 + y^2$ by planes parallel to the $yz$-plane are all translates of the same parabola. (See Figure 3.) Choose some $2^M$ of these sections and some set $X$ of $M$ points on one of them. Each section contains a translated copy of $X$; assign a subset to each section, take that subset of its copy of $X$, and let $K$ be the convex hull of the points in these subsets of copies. The translates of $K$ then shatter $X$, since an appropriate translation will superimpose the section corresponding to any desired subset on the section containing $X$.

Now, we present Example 4. Let $\mathcal{E}$ be the family of all finite subsets of $\mathbb{N}$, and let $E : \mathbb{N} \to \mathcal{E}$ be a bijection. Set $A = \{(m, n) \in \mathbb{N}^2 : m \in E(n)\}$.

For $m, n \in \mathbb{N}$, let $u_m = (\frac{1}{m}, 0, \frac{1}{m^2})$ and $v_n = (0, \frac{1}{n}, \frac{1}{n^2})$, and define $p : \mathbb{N}^2 \to \mathbb{R}^3$, $p(m, n) = u_m + v_n$.

Let $K = \text{conv}(\text{cl}(p(A)))$ and $\mathcal{F} = \{K - v_n : n \in \mathbb{N}\}$. We claim that $\text{vcdim}(\mathcal{F}) = \infty$.

Let $P \subseteq \mathbb{R}^3$ be the paraboloid with equation $z = x^2 + y^2$. Since $P$ is the boundary of a strictly convex set, $P \cap \text{conv}(S) = S$ for any $S \subseteq P$. Since $p(\mathbb{N}^2)$ is a discrete set, $p(\mathbb{N}^2) \cap \text{cl}(S) = S$ for any $S \subseteq p(\mathbb{N}^2)$. So if $T \subseteq p(\mathbb{N}^2)$, then $T \cap K = T \cap p(\mathbb{N}^2) \cap P \cap K = T \cap p(\mathbb{N}^2) \cap \text{cl}(p(A)) = T \cap p(A)$.

Now, let $M \in \mathbb{N}$, $X = \{u_1, \ldots, u_M\}$, and $X' \subseteq X$. Let $n \in \mathbb{N}$ be such that $X' = \{u_m : m \in E(n)\}$. Then

$$(X + v_n) \cap K = (X + v_n) \cap p(A) = X' + v_n,$$

that is, $X \cap (K - v_n) = X'$. Thus $\mathcal{F}$ shatters $X$, so $\text{vcdim}(\mathcal{F}) \geq M$. \qed

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4. A NOTE

After the publication of the paper, Konrad Swanepoel brought the following to our attention: In Lemma 9.11.2 of [1] (proved by I. Talata in [18]) an explicit construction of an $o$-symmetric strictly convex smooth body is given with $\sqrt[3]{3n}/3$ pairwise touching translates. That changes the bound in Proposition 2 to $\tau(F) \geq \frac{2\sqrt{3}n}{9} \nu(F)$.

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