Self-testing nonlocality without entanglement

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(Dated: March 25, 2022)

Quantum theory allows for nonlocality without entanglement. Notably, there exist bipartite quantum measurements consisting of only product eigenstates, yet they cannot be implemented via local quantum operations and classical communication. In the present work, we show that a measurement exhibiting nonlocality without entanglement can be certified in a device-independent manner. Specifically, we consider a simple quantum network and construct a self-testing procedure. This result also demonstrates that genuine network quantum nonlocality can be obtained using only non-entangled measurements. From a more general perspective, our work establishes a connection between the effect of nonlocality without entanglement and the area of Bell nonlocality.

I. INTRODUCTION

The (un)ability to distinguish certain quantum states via measurements is a central aspect of quantum theory, and central to applications such as quantum key distribution [1, 2] and data-hiding [3]. This question is of particular interest when considering composite systems. One may consider for instance two remote observers, Alice and Bob, sharing a state chosen from a certain set. Alice and Bob should try to identify which state they received. In this context, a natural limitation is that Alice and Bob must restrict to local measurements (or operations) assisted by classical communication (LOCC). Surprisingly in this case, there exist sets of product states forming a basis, which nevertheless cannot be perfectly distinguished by any LOCC measurement. While Ref. [4] provided preliminary results in this direction, the first examples were constructed by Bennett et al. [5], coining the effect “nonlocality without entanglement”. This shows that separable quantum measurements (where all eigenstates are non-entangled) are strictly stronger than LOCC measurements. These ideas have been generalized in many different directions see e.g. [6–12], with connections to the notion of unextendible product basis and bound entanglement [6, 13]. In recent years, renewed interest has been devoted to these ideas, with the discovery of stronger forms of this effect in particular in multipartite systems, see e.g. [14–16].

In this work, we discuss the question of certifying the effect of “nonlocality without entanglement” (NLWE) in a black-box setting. Specifically, we consider the quantum measurement featuring NLWE introduced in [5] (for a two-qudit system), and show that it can be certified in a device-independent manner. For this we consider the simple quantum network of entanglement swapping [17] (also known as “bilocality” network [18]), where the middle party performs the NLWE measurement (see Fig. 1b). Based on the assumption that the two quantum sources present in the network are independent, the standard assumption in network nonlocality [18, 19], we can show that NLWE can be certified using concepts and tools from self-testing [20, 21], a framework for the device-independent certification of quantum resources. In fact, the full quantum setup can be self-tested, including also the shared entangled states and the local measurements of the side parties. Finally, we discuss how our self-testing scheme can be generalized.

Our result shows that a strong form of nonlocal quantum correlations in networks, known as genuine network quantum nonlocality [22], can be obtained without the need for entangled measurements (as traditionally used in network nonlocality). From a more general point of view, our work also finally connects the effect of “nonlocality without entanglement” with the area of Bell nonlocality [23, 24].

II. PROBLEM

We consider the entanglement swapping experiment [17] consisting of two sources and three parties (nodes), as shown in Fig. 1a. We call the central party Bob, and two lateral ones Alice and Charlie. The center of our interest are the correlations obtained in this setting, the so-called bilocality network [18, 25], where the two sources are assumed to be independent from each other. Note that the characterisation of local and quantum correlations in networks featuring independent sources has attracted growing attention in recent years, see e.g. [18, 19, 26–28] and [29] for a review.

In this work, our main goal is to certify in a device-independent manner that Bob performs a quantum measurement featuring NLWE. That is, assuming only the independence of the two sources, we will show, that from observed data alone, the presence of such a measurement can be demonstrated, up to irrelevant local transformations. We use the tools and concepts of self-testing [20, 21], in particular the results of Ref. [30]. While self-tests have been developed for specific joint entangled measurements (such as the well-known Bell-state measurement) [31, 32], we show here that a similar construction is possible for relevant measurements with only separable eigenstates.

Self-testing is a procedure which establishes a form of equivalence between two experiments. First, we have the physical experiment, which corresponds to the actual experiment performed in the laboratory, featuring a priori unknown
FIG. 1: We consider the entanglement swapping (or bilocality) network represented in (a), where the middle party Charlie, prepares the same state. Hence, the reference states are given by

\[ |\psi^t\rangle = |\psi\rangle = |\phi_+\rangle \]

where \( |\phi_+\rangle = (|00\rangle + |11\rangle + |22\rangle)/\sqrt{3} \). Note that we use superscripts to identify the various systems, and that we distinguish the two subsystems of Bob by \( B_1 \) and \( B_2 \). The reference measurements for Alice are given by three ternary measurements:

\[ M_0 = \{ M_{0|0} = |0\rangle\langle 0|, M_{0|1} = |1\rangle\langle 1|, M_{0|2} = |2\rangle\langle 2| \}, \]
\[ M_1 = \{ M_{0|1} = |+\rangle\langle +|, M_{1|1} = |\pm\rangle\langle \pm|, M_{2|1} = |\mp\rangle\langle \mp| \}, \]
\[ M_2 = \{ M_{0|2} = |0\rangle\langle 0|, M_{2|2} = |2\rangle\langle 2| \}, \]

where \(|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2} \). Here \( M_{0|x} \) denotes the projective measurement for input \( x = 0, 1, 2 \). Each measurement produces a ternary output \( a \), with POVM elements denoted \( M_{a|x} \). The reference measurements for Charlie are the same as for Alice, and we will denote them by \( M_{b|y} \) with elements \( M_{b|y} \).

The reference measurements of Bob are of two types. First, we have four measurements with a clear subsystem separation. On the subsystem \( B_1 \), these are used to self-test the state shared with Alice, as well as Alice’s measurements. On the subsystem \( B_2 \), these are used to self-test the state shared with Charlie, and Charlie’s measurements. More precisely, these measurements take the form \( M_{b_1,b_2|y} = M_{b_1|y} \otimes M_{b_2|y} \) for \( b_1, b_2 = 0, 1, 2 \) and \( y = 0, 1, 2, 3 \), where the POVM elements \( M_{b_1|y} \) and \( M_{b_2|y} \) correspond to the eigenvectors of the following four operators:

\[ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & \pm 1 & 0 \\ \pm 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}, \quad \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & \pm 1 \\ 0 & \pm 1 & 1 \end{bmatrix}. \]

The second type of measurement for Bob is our main object of interest. This corresponds to a single extra measurement, which corresponds to the input \( y = \diamond \). This measurement exhibits the property of nonlocality without entanglement, and is denoted \( M_{\diamond} = M_{b_1,b_2|\diamond} \). It consists of nine eigenstates, which are all product with respect to the partition \( B_1 \) vs \( B_2 \), given by

\[
\begin{align*}
M_{0,0|\diamond} &= |1\rangle\langle 1| \otimes |1\rangle\langle 1|, & M_{0,1|\diamond} &= |0\rangle\langle 0| \otimes |+\rangle\langle +|_{0,1}, & M_{0,2|\diamond} &= |0\rangle\langle 0| \otimes |-\rangle\langle -|_{0,1}, \\
M_{1,0|\diamond} &= |2\rangle\langle 2| \otimes |+\rangle\langle +|_{1,2}, & M_{1,1|\diamond} &= |2\rangle\langle 2| \otimes |-\rangle\langle -|_{1,2}, & M_{1,2|\diamond} &= |+\rangle\langle +|_{1,2} \otimes |0\rangle\langle 0|, \\
M_{2,0|\diamond} &= |-\rangle\langle -|_{1,2} \otimes |0\rangle\langle 0|, & M_{2,1|\diamond} &= |+\rangle\langle +|_{0,1} \otimes |2\rangle\langle 2|, & M_{2,2|\diamond} &= |-\rangle\langle -|_{0,1} \otimes |2\rangle\langle 2|. 
\end{align*}
\]
sisted with classical communication (so-called LOCC measurements) [5]. Therefore, the measurement \( \mathcal{M}'_a \) cannot be implemented via LOCC, illustrating the fact that separable measurements are strictly stronger than LOCC measurements.

Combining the above states and measurements, we obtain the statistics of the reference experiment, given by the joint conditional probability distribution

\[
p'(a, b_1, b_2, c|x, y, z) = \text{Tr} \left[ \left( M_{a|x}^A \otimes M_{b_1, b_2|y}^B \otimes M_{c|z}^C \right) \psi_+^{A'B'_1} \otimes \psi_+^{B_2C'} \right]. \tag{3}
\]

where \( x, z = 0, 1, 2 \), \( y = 0, 1, 2, 3, \diamond \) and \( a, b_1, b_2, c = 0, 1, 2 \). Note that when computing the above equation, one should be careful about the order of the subsystems.

III. MAIN RESULT

We now present our main result, namely that the reference experiment is a self-test. Consider a physical experiment with a priori unknown states \( \psi_+^{A'B_1} \) and \( \psi_+^{B_2C} \) and measurements \( \{M_a|x\} \), \( \{M_{b_1, b_2|y}\} \) and \( \{M_{c|z}\} \), resulting in observed correlations

\[
p(a, b_1, b_2, c|x, y, z) = \text{Tr} \left[ \left( M_{a|x}^A \otimes M_{b_1, b_2|y}^B \otimes M_{c|z}^C \right) \psi_+^{A'B_1} \otimes \psi_+^{B_2C} \right]. \tag{4}
\]

Below we show that if these statistics correspond to those of the reference experiment, as given in Eq. (3), then all states and measurements of the physical experiment are equivalent (up to irrelevant local transformation) to the reference states and measurements. In particular, this implies that the measurement \( y = \diamond \) for Bob must feature NLWE. More formally, we have the following theorem.

**Theorem 1.** Consider a physical experiment such that its statistics, as given in Eq. (4), match exactly the statistics of the reference experiment given in Eq. (3). Then there exists a local isometry \( \Phi \) mapping

- Bob’s measurement \( M_\diamond \) to the NLWE measurement \( \mathcal{M}'_a \):

\[
\Phi \left( M_{b_1, b_2|y}^B \otimes |00^{A'B'_1} \otimes |00^{B_2C'} \right) = \left( |\xi\rangle^{A'B_1} \otimes |\xi\rangle^{B_2C} \otimes |00\rangle^{A'B'_1} \right), \tag{5}
\]

where \( |\xi\rangle^{A'B_1} \) is a valid quantum state.

- the states \( |\psi\rangle^{A'B_1} \) and \( |\psi\rangle^{B_2C} \) to the reference states (i.e. pairs of maximally entangled qutrits), and Alice’s and Charlie’s measurements to the corresponding reference measurements \( \mathcal{M}'_y \):

\[
\Phi \left( M_{a|x}^A \otimes M_{c|z}^C \otimes |\psi\rangle^{B_2C} \otimes |00^{A'B'_1} \otimes |00\rangle^{B_2C'} \right) = \left( |\xi\rangle^{A'B_1} \otimes M_\diamond |\phi_+\rangle^{A'B'_1} \otimes |\phi_+\rangle^{B_2C'} \right). \tag{6}
\]

\[\text{Input set } (x, z) \quad \text{Matching output set } (a, b_1, b_2, c)\]

| (0, 0) | (1, 0, 0, 1) |
| --- | --- |
| (0, 1) | (0, 0, 1, 0) |
| (0, 2) | (2, 1, 0, 1) |
| (2, 0) | (1, 1, 2, 0) |
| (1, 0) | (0, 2, 1, 2) |

**Proof.** The theorem consists of two self-testing results, and we start by proving the second one. The idea is to first prove that in order to reproduce some of the marginals of the reference correlations, specifically \( \{p(a, b_1|x, y)\} \), implies the equivalence between \( |\psi\rangle^{A'B_1} \) and \( |\phi_+\rangle \). The formal statement is given in the following lemma.

**Lemma 1.** Let the state \( |\psi\rangle^{A'B_1} \otimes |\psi\rangle^{B_2C} \) and measurements \( \{M_{a|x}\} \) and \( \{M_{b_1, b_2|y}\} \) such that \( M_{b_1, b_2|y} = \sum_b M_{b_1, b_2|y} \) produce correlations \( p(a, b_1|x, y) \). If those correlations are such that

\[
p(a, b_1|x, y) = \text{Tr} \left[ \left( M_{a|x}^A \otimes M_{b_1|y}^B \right)(\phi_+^{A'B'_1}) \right], \tag{7}
\]

for every \( a, b_1, x, y \) then there exists isometry \( \Phi \) such that

\[
\Phi \left( M_{a|x}^A |\psi\rangle^{A'B_1} \otimes |\psi\rangle^{B_2C} \otimes |00\rangle^{A'B'_1} \right) = \left( |\xi\rangle^{A'B_1} \otimes (M_\diamond |\phi_+\rangle^{A'B'_1}) \right), \tag{8}
\]

where \( |\xi\rangle^{A'B_1} \) is a valid quantum state.

The proof can be found in Appendix A and it is directly inspired by the self-testing of maximally entangled pair of qutrits presented in [30]. Combined with methods for self-testing joint measurements introduced in [33] allows us to get the self-testing result for both states, as well as all measurements performed by Alice and Charlie, as stated in eq. (6). The proof of this equation is given in Appendix B.

Based on these self-testing results, we move on to the second part of the proof, for certifying the additional measurement of Bob, namely \( M_\diamond \) from Eq. (5). First, note that simulation of the reference correlations involves

\[
p(b|\diamond) = \text{Tr} \left[ M_{b_1|0}^B (|\psi\rangle^{A'B_1} \otimes |\psi\rangle^{B_2C}) \right] = \frac{1}{9} \forall b.
\]

This implies that the norm of vectors \( M_{b_1|0} (|\psi\rangle^{A'B_1} \otimes |\psi\rangle^{B_2C}) \) for all outcomes \( b \) equals 1/3. Similarly, from the simulation of the reference correlations we have:

\[
p(a, c|x, z) = \frac{1}{9} \forall a, c, x, z, \tag{9}
\]

\[\text{TABLE I: For Bob’s input } y = \diamond, \text{ certain sets of inputs } (x, z) \text{ for Alice and Charlie imply specific outputs patterns.}\]
implying that the norm of vectors $M_{a|x|} \ket{\psi} A_1 B_1 C_1 \otimes M_{c|x} \ket{\psi} B_2 C_2$ for all $a, c, x, z$ equals $1/3$.

Let us define for certain input sets $(x, \hat{0}, z)$ their matching outputs sets $(a, b, c)$, as given in Table I. The simulation of the reference correlations imposes that for every set of inputs from Table I, the matching set of outputs happens with probability $1/9$. Let us now concentrate on the set of inputs $(0, \hat{0}, 0)$ and its matching set of outputs $(1, 0, 1)$, i.e., eq. $p(1, 0, 0, 1|0, \hat{0}, 0) = 1/9$. Given eq. (8) we obtain the following set of relations:

$$p(1, 0, 0, 1|0, \hat{0}, 0) = \frac{1}{9}.$$ 

The second relation is the consequence of the fact that isometries do not change the scalar product. Given that both vectors in the scalar product in the third equality have norm $1/3$ the saturation of Cauchy-Schwarz inequality implies

$$\Phi \left( M_{0,0,0} \ket{\psi} A^B_1 \otimes \ket{0}_1 \otimes \ket{0}_2 \otimes \ket{0}_3 \right) = \frac{1}{9} \Phi \left( M_{0,0,0} \ket{\psi} A^B_1 \right).$$

With similar argumentation we obtain the following set of relations for all $b = b_1, b_2$:

$$\Phi \left( M_{b_1, b_2} \ket{\psi} A^B_1 \otimes \ket{0}_1 \otimes \ket{0}_2 \otimes \ket{0}_3 \right) = \frac{1}{9} \Phi \left( M_{b_1, b_2} \ket{\psi} A^B_1 \right).$$

where states $\ket{\psi_0}$ and $\ket{\phi_0'}$ are such that $M_{b_1, b_2} \ket{\psi} A^B_1 \otimes \ket{\psi_0} \otimes \ket{\phi_0'}$. All these equations together with eq. (B10) imply the self-testing result we needed as given in eq. (5).

IV. GENERAL CONSTRUCTION

The above construction can be generalized to other measurements featuring NLWE, for higher dimensions (still in the bipartite case) and to the multipartite case. The general idea is that the above procedure allows one to basically self-test any measurement with product rank-one eigenstates. When the later involve only real parameters, the construction is rather straightforward, while the general case with complex coefficients is more challenging, as usual in self-testing [33].

For bipartite NLWE measurements, the bilocality network can be readily used. If the measurement $M_{b_1, b_2}$ acts now on $C^{d_A} \otimes C^{d_B}$, the local dimensions of two maximally entangled states distributed in the network must be adapted (to $d_A$ and $d_B$ respectively). These states, and the local measurements of Alice and Charlie, can be self-tested using any of the available methods (see for example [30, 33]). In turn, Alice and Charlie can remotely prepare for Bob (via their certified local measurements acting on half of the shared maximally entangled pairs) the pair of input states in order to match any of the eigenstates of the measurement $M_{b_1, b_2}$.

Moving to the multipartite case will involve a star network, where the central node will perform the measurement with NLWE. For an N-party measurement with qubits, we consider a star network with $N$ branches. On each branch a maximally entangled state of two qubits must first be self-tested, as well as local measurements of the lateral nodes. Second the central NLWE is self-tested as above.

V. DISCUSSION

We discussed the device-independent certification of the effect of “nonlocality without entanglement”. Specifically, we showed that a quantum measurement featuring only separable eigenstates, but which cannot be implemented via an LOCC procedure, can be certified in a quantum network with independent sources, based only on observed statistics.

A point worth noting is that our self-test construction has interesting consequences from the perspective of network nonlocality. In particular, this example shows that genuine quantum network nonlocality [22], a form of quantum nonlocality that can only arise in networks, is in fact possible without involving any entangled measurement.

Let us point out that previous works have discussed the certification of non-classical quantum measurements in a partially device-independent setting, considering prepared quantum states of limited dimension [34, 35], also with an experimental demonstration [35]. A key difference with our work (besides the stronger assumptions), is that these previous results could only certify that a measurement is not achievable via LOCC, but could not certify the property of NLWE, as we do here.
From a more general perspective, our work establishes a connection between two forms of nonlocality in quantum theory, namely Bell nonlocality and the effect of nonlocality without entanglement.

Appendix A: Proof of Lemma 1

In this section we prove Lemma 1. The proof is largely inspired by the proof offered in [30]. The most important part of our contribution is self-testing of measurements as well, and not just the state as in [30]. It will be convenient to define the marginal physical measurements operators for Bob $M_{b_{i,j} | y}^B = \sum_{b_{i,j}} M_{b_{i,j} | y}^B$. Not that this physical coarse-grained measurement can, in principle, act on both Hilbert spaces $\mathcal{H}^{B_1}$ and $\mathcal{H}^{B_2}$. Let us further introduce the following notation:

$$
Z_{0,1}^A = M_{0,0}^A - M_{1,1}^A, \quad Z_{1,2}^A = M_{1,0}^A - M_{2,0}^A, \quad X_{0,1}^A = M_{0,1}^A - M_{1,1}^A, \quad X_{1,2}^A = M_{1,1}^A - M_{2,2}^A, \quad (A1)
$$

$$
D_{0,1}^B = M_{0,0}^B - M_{1,1}^B, \quad D_{1,2}^B = M_{1,2}^B - M_{2,2}^B, \quad E_{0,1}^B = M_{0,1}^B - M_{1,1}^B, \quad E_{1,2}^B = M_{1,3}^B - M_{2,3}^B, \quad (A2)
$$

$$
\tilde{Z}_{0,1}^B = \frac{D_{0,1}^B + E_{0,1}^B}{\sqrt{2}}, \quad \tilde{X}_{0,1}^B = \frac{D_{0,1}^B - E_{0,1}^B}{\sqrt{2}}, \quad \tilde{Z}_{1,2}^B = \frac{D_{1,2}^B + E_{1,2}^B}{\sqrt{2}}, \quad \tilde{X}_{1,2}^B = \frac{D_{1,2}^B - E_{1,2}^B}{\sqrt{2}}. \quad (A3)
$$

Physical operators $Z_{i,j}^A$ and $X_{i,j}^A$ are supposed to act as Pauli’s $\sigma_z$ and $\sigma_x$ respectively on qubit subspaces spanned by vector basis $\{|i\rangle^A, |j\rangle^A\}$, hence the notation. Of course, these operators are uncharacterized and only after self-testing result is proven such notation is justified. Operators $\tilde{Z}_{0,1}^B$ and $\tilde{X}_{0,1}^B$ are supposed to act as Pauli’s $\sigma_z$ and $\sigma_x$ respectively on qubit subspaces spanned by vector basis $\{|i\rangle^B, |j\rangle^B\}$, but for the beginning we note that these operators unlike $Z_{i,j}^A$ and $X_{i,j}^A$ do not even need to be unitary. Let us now define operators which in ideal case on designated qubit subspaces act as identities:

$$
1_{0,1,z}^A = M_{0,0}^A + M_{1,1}^A, \quad 1_{1,2,z}^A = M_{1,0}^A + M_{2,0}^A, \quad 1_{0,1,x}^A = M_{0,1}^A + M_{1,1}^A, \quad 1_{1,2,x}^A = M_{1,1}^A + M_{2,2}^A, \quad (A4)
$$

$$
1_{0,1,d}^B = M_{0,0}^B + M_{1,1}^B, \quad 1_{1,2,d}^B = M_{1,2}^B + M_{2,2}^B, \quad 1_{0,1,e}^B = M_{0,1}^B + M_{1,1}^B, \quad 1_{1,2,e}^B = M_{1,3}^B + M_{2,3}^B \quad (A5)
$$

The operators $1_{0,1,d}^B, 1_{0,1,e}^B, 1_{1,2,d}^B$, and $1_{1,2,e}^B$ are projectors to the subspaces spanned by eigenvectors of $D_{0,1}^B, E_{0,1}^B, D_{1,2}^B$ and $E_{1,2}^B$. In a similar fashion we want to define projectors to the subspaces spanned by $\tilde{Z}_{0,1}^B$, $\tilde{X}_{0,1}^B$, $\tilde{Z}_{0,1}^B$, and $\tilde{X}_{0,1}^B$. However, as we said earlier those operators are not necessarily unitary. However, we define the regularized version of this operators in the following way:

$$
Z_{0,1}^B = \frac{\tilde{Z}_{0,1}^B}{|\tilde{Z}_{0,1}^B|}, \quad X_{0,1}^B = \frac{\tilde{X}_{0,1}^B}{|\tilde{X}_{0,1}^B|}, \quad Z_{1,2}^B = \frac{\tilde{Z}_{1,2}^B}{|\tilde{Z}_{1,2}^B|}, \quad X_{1,2}^B = \frac{\tilde{X}_{1,2}^B}{|\tilde{X}_{1,2}^B|}. \quad (A6)
$$

Such renormalization of eigenvalues is not possible if any of the operators figuring in the numerators have eigenvectors with a corresponding eigenvalue equal to zero. If such case appears we just change all such eigenvalues from 0 to 1. In this way we obtain unitary operators $Z_{i,j}^B$ and $X_{i,j}^B$. Now we define subspace $B_{i,j}$ which comprises the range of operators $D_{i,j}^B$ and $E_{i,j}^B$. Let $1_{i,j}^B$ be the projector to the subspace $B_{i,j}$. The definition of $Z_{i,j}^B$ and $X_{i,j}^B$ implies that the their range is exactly the subspace $B_{i,j}$, and since all their eigenvalues are either 1 or −1 the following equations hold

$$
Z_{i,j}^B = 1_{i,j}^B, \quad X_{i,j}^B = 1_{i,j}^B. \quad (A7)
$$

Let us now define the following subnormalized states:

$$
|\psi_{i,j,z,A}| = 1_{i,j, z}^A |\psi\rangle, \quad |\psi_{i,j,x,A}\rangle = 1_{i,j, x}^A |\psi\rangle \quad (A8)
$$

$$
|\psi_{i,j,d,B}| = 1_{i,j, d}^B |\psi\rangle, \quad |\psi_{i,j,e,B}\rangle = 1_{i,j, e}^B |\psi\rangle, \quad |\psi_{i,j,e,B}\rangle = 1_{i,j, b}^B |\psi\rangle, \quad (A9)
$$

where we used notation $|\psi\rangle \equiv |\psi\rangle^{ABC} = |\psi\rangle^{AB_1} \otimes |\psi\rangle^{B_2C}$. Whenever state is written without any superscript it means that we are taking into account the whole state distributed in the network. Let us now analyse the consequences of the fact that physical

ACKNOWLEDGMENTS

The authors acknowledge financial support from the Starting ERC grant QUSCO and the Swiss National Science Foundation (project 2000021_192244/1 and NCCR SwissMAP).
experiment simulates the reference one. In the first step we concentrate on the following set of correlations:

\[ \langle \psi | M_{A|1}^A \otimes I_B^B | \psi \rangle = \frac{1}{3} \]

(A10)

\[ \langle \psi | M_{A|0}^A \otimes I_B^B | \psi \rangle = \frac{1}{3} \]

(A11)

\[ \langle \psi | M_{2|0}^A \otimes M_{2|0}^B | \psi \rangle = \frac{1}{3} \]

(A12)

\[ \langle \psi | M_{0|1}^A \otimes M_{0|1}^B | \psi \rangle = \frac{1}{3} \]

(A13)

Given that in DI scenario we can consider all measurement operators as projectors, eqs. (A10),(A11) imply that the norms of vectors \( M_{A|1}^A \otimes I_B^B | \psi \), \( M_{2|1}^A \otimes I_B^B | \psi \), \( M_{0|0}^A \otimes I_B^B | \psi \), \( M_{0|2}^A \otimes I_B^B | \psi \), \( I^A \otimes M_{2|1}^B | \psi \), \( I^A \otimes M_{0|2}^B | \psi \) and \( I^A \otimes M_{0|3}^B | \psi \) is equal to \( 1/\sqrt{3} \) as determined from eqs. (A10),(A11) is equal to \( 1/3 \), which by saturation of the Cauchy–Bunyakovsky–Schwarz inequality implies that vectors figuring in the inner product are parallel:

\[ M_{A|1}^A \otimes I_B^B | \psi \rangle = I^A \otimes M_{2|0}^B | \psi \rangle, \quad M_{2|1}^A \otimes I_B^B | \psi \rangle = I^A \otimes M_{2|1}^B | \psi \rangle, \quad M_{0|0}^A \otimes I_B^B | \psi \rangle = I^A \otimes M_{0|3}^B | \psi \rangle, \quad M_{0|2}^A \otimes I_B^B | \psi \rangle = I^A \otimes M_{0|3}^B | \psi \rangle \]

(A14)

\[ M_{A|1}^A \otimes I_B^B | \psi \rangle = I^A \otimes M_{2|1}^B | \psi \rangle, \quad M_{2|1}^A \otimes I_B^B | \psi \rangle = I^A \otimes M_{2|1}^B | \psi \rangle, \quad M_{0|0}^A \otimes I_B^B | \psi \rangle = I^A \otimes M_{0|3}^B | \psi \rangle, \quad M_{0|2}^A \otimes I_B^B | \psi \rangle = I^A \otimes M_{0|3}^B | \psi \rangle \]

(A15)

\[ M_{A|0}^A \otimes I_B^B | \psi \rangle = I^A \otimes M_{0|2}^B | \psi \rangle, \quad M_{A|2}^A \otimes I_B^B | \psi \rangle = I^A \otimes M_{0|3}^B | \psi \rangle \]

(A16)

\[ M_{A|0}^A \otimes I_B^B | \psi \rangle = I^A \otimes M_{0|2}^B | \psi \rangle, \quad M_{A|2}^A \otimes I_B^B | \psi \rangle = I^A \otimes M_{0|3}^B | \psi \rangle \]

(A17)

The tensor product form of the state \( | \psi \rangle = | \psi \rangle_{AB1} \otimes | \psi \rangle_{BC} \) and eqs. (A14)-(A17) imply that measurements \( M_{2|0}^A, M_{2|1}^A, M_{0|2}^A \) and \( M_{0|3}^A \) act nontrivially only on Hilbert space \( \mathcal{H}_{B|1} \). Given that for all measurements it holds \( \sum_i M_{ij} = I \), eqs. (A14)-(A17) imply the following relations:

\[ | \psi_{0,1,2,A} \rangle = | \psi_{0,1,D,B} \rangle = | \psi_{0,1,D,B} \rangle = | \psi_{0,1,2,2,A} \rangle = \frac{1}{\sqrt{3}} | \psi_{0,1} \rangle \]

(A18)

\[ | \psi_{1,2,A} \rangle = | \psi_{1,2,D,B} \rangle = | \psi_{1,2,D,B} \rangle = | \psi_{1,2,2,A} \rangle = \frac{1}{\sqrt{3}} | \psi_{1,2} \rangle \]

(A19)

and furthermore the norm of \( | \psi_{0,1} \rangle \) and \( | \psi_{1,2} \rangle \) is equal to \( \sqrt{3} \). Also since \( | \psi_{0,1,D,B} \rangle = | \psi_{0,1,E,B} \rangle \) and \( | \psi_{1,2,D,B} \rangle = | \psi_{1,2,E,B} \rangle \), given the definition of \( I_{B|1}^A \) and \( I_{B|2}^A \), it must be

\[ | \psi_{0,1,B} \rangle = | \psi_{0,1} \rangle, \quad | \psi_{1,2,B} \rangle = | \psi_{1,2} \rangle \]

(A20)

The simulation of reference correlations implies:

\[ \langle \psi | Z_{0|1}^A \otimes D_{0|1}^E + Z_{0|1}^A \otimes E_{0|1}^B + X_{0|1}^A \otimes D_{0|1}^B - X_{0|1}^A \otimes E_{0|1}^B | \psi \rangle = \frac{2}{3} 2\sqrt{2} \]

(A21)

A variant of sum-of-squares (SOS) decomposition of generalized shifted Bell operator reads:

\[ \sqrt{2} \left[ \frac{Z_{0|1}^A + X_{0|1}^A + 2D_{0|1}^B + 2E_{0|1}^B}{\sqrt{2}} \right] - (Z_{0|1}^A \otimes (D_{0|1}^B + E_{0|1}^B) + X_{0|1}^A \otimes (D_{0|1}^B - E_{0|1}^B)) = \]

\[ = \left( Z_{0|1}^A - \frac{D_{0|1}^B + E_{0|1}^B}{\sqrt{2}} \right)^2 + \left( X_{0|1}^A - \frac{D_{0|1}^B - E_{0|1}^B}{\sqrt{2}} \right)^2 \],

(A22)

where we used the fact that \( Z_{0|1}^A = I_{0|1,Z}^A, X_{0|1}^A = I_{0|1,X}^A, D_{0|1}^B = I_{0|1,D}^B \) and \( E_{0|1}^B = I_{0|1,E}^B \). Given that

\[ \langle \psi | I_{0|1,Z}^A + I_{0|1,X}^A + I_{0|1,D}^B + I_{0|1,E}^B | \psi \rangle = \frac{8}{3} \]

(A23)

the l.h.s. of eq. (A22) is equal to 0, which means that both sums on the r.h.s. must be equal to 0 as well, as they are squares of operators implying they must be nonnegative. Hence, recalling notation introduced in (A3) this implies:

\[ Z_{0|1}^A | \psi \rangle = \tilde{Z}_{0|1}^A | \psi \rangle, \quad X_{0|1}^A | \psi \rangle = \tilde{X}_{0|1}^A | \psi \rangle \]

(A24)
Since \( |\psi\rangle = |\psi\rangle^{AB_1} \otimes |\psi\rangle^{B_2C} \), and on the l.h.s. of eqs. (A24) identity operators act on \( |\psi\rangle^{B_2C} \) we can conclude that operators \( \hat{Z}_{0,1}^B \) and \( \hat{X}_{0,1}^B \) act nontrivially only on Hilbert space \( \mathcal{H}^{B_1} \).

As operators \( \hat{Z}_{0,1}^B \) and \( \hat{X}_{0,1}^B \) anticommute by construction, operators \( Z_{0,1}^A \) and \( X_{0,1}^A \) anticommute on the support of \( \text{Tr}_B[|\psi\rangle\langle\psi|] \):

\[
\{X_{0,1}^A, Z_{0,1}^A\} |\psi\rangle = X_{0,1}^A Z_{0,1}^A |\psi\rangle + Z_{0,1}^A X_{0,1}^A |\psi\rangle \\
= X_{0,1}^A \hat{Z}_{0,1}^B |\psi\rangle + Z_{0,1}^A \hat{X}_{0,1}^B |\psi\rangle \\
= \hat{Z}_{0,1}^B X_{0,1}^A |\psi\rangle + \hat{X}_{0,1}^B Z_{0,1}^A |\psi\rangle \\
= \hat{Z}_{0,1}^B \hat{X}_{0,1}^B |\psi\rangle + \hat{X}_{0,1}^B \hat{Z}_{0,1}^A |\psi\rangle \\
= 0. \tag{A25}
\]

The same procedure can be repeated for the correlations among \( Z_{1,2}^A, X_{1,2}^A, \hat{D}_{1,2}^A \) and \( \hat{E}_{1,2}^A \) to obtain:

\[
Z_{1,2}^A |\psi\rangle = \hat{Z}_{1,2}^B |\psi\rangle, \quad X_{1,2}^A |\psi\rangle = \hat{X}_{1,2}^B |\psi\rangle \\
\{X_{1,2}^A, Z_{1,2}^A\} |\psi\rangle = 0, \quad \{\hat{X}_{1,2}^A, \hat{Z}_{1,2}^B\} |\psi\rangle = 0 \tag{A26, A27}
\]

Again, as is the case for \( \hat{Z}_{0,1}^B \) and \( \hat{X}_{0,1}^B \), operators \( \hat{Z}_{1,2}^B \) and \( \hat{X}_{1,2}^B \) act nontrivially only on the Hilbert space \( \mathcal{H}^{B_2} \). As noted earlier, hatted operators do not necessarily have eigenvalues \(-1, 1\) and in prospect of using unitary operators to build self-testing isometry we defined the regularized operators \( \hat{Z}_{0,1}^B, \hat{X}_{0,1}^B, \hat{Z}_{1,2}^B \) and \( \hat{X}_{1,2}^B \) which are unitary by construction and they act on \( |\psi\rangle \) in the same way as \( \hat{Z}_{0,1}^B, \hat{X}_{0,1}^B, \hat{Z}_{1,2}^B \) and \( \hat{X}_{1,2}^B \) respectfully. The proof of this is described in details in Appendix A2 in [21]. Let us now introduce projective operators

\[
P_0^B = \frac{1 - Z_{0,1}^B + Z_{0,1}^B}{2}, \tag{A28}
\]

\[
P_1^B = \frac{1 - Z_{0,1}^B - Z_{0,1}^B}{2} = \frac{1 - Z_{0,1}^B + Z_{0,1}^B}{2}, \tag{A29}
\]

\[
P_2^B = \frac{1 - Z_{0,1}^B - Z_{0,1}^B}{2} \tag{A30}
\]

which together with \( \omega = \exp \frac{i\pi}{3} \) form the operators used to build the self-testing isometry

\[
Z^A = \sum_{j=0}^{2} \omega^j M_{j,0}^A, \tag{A31}
\]

\[
Z^B = \sum_{j=0}^{2} \omega^j P_j^B, \tag{A32}
\]

\[
X_{(1)}^A = X_{0,1}^A + I^A - I_{0,1}^A, \quad X_{(1)}^B = X_{0,1}^B + I^B - I_{0,1}^B, \tag{A33}
\]

\[
X_{(2)}^A = X_{(1)}^A (I^A - I_{1,2}^A + X_{1,2}^A), \quad X_{(2)}^B = X_{(1)}^B (I^B - I_{1,2}^B + X_{1,2}^B). \tag{A34}
\]

Operators \( Z^A/B, X_{(1)}^A/B \) and \( X_{(2)}^A/B \) are unitary. Indeed, operators \( Z^A/B \) are constructed as a sum of projectors with eigenvalues squaring to 1. For operators \( X_{(1)}^A \) and \( X_{(2)}^A \) unitarity can be proven in the following way

\[
X_{(1)}^A X_{(1)}^A = (X_{0,1}^A + I^A - I_{0,1}^A, X_{0,1}^A + I^A - I_{0,1}^A) = I \tag{A35}
\]

\[
X_{(2)}^A X_{(2)}^A = X_{(1)}^A (I^A - I_{1,2}^A + X_{1,2}^A) (I^A - I_{1,2}^A + X_{1,2}^A) = X_{(1)}^A X_{(1)}^A = I. \tag{A37}
\]

The same unitarity proof holds for \( X_{(1)}^B \) and \( X_{(2)}^B \). Note that \( X_{(1)}^A \) can alternatively be written as

\[
X_{(1)}^A = X_{0,1}^A + M_{2,0}^A. \tag{A40}
\]
Eqs. (A24) and (A26) imply
\[ Z \text{ where } \]

Similarly we obtain the following relations:
\[ F \text{ transform acting in the following way: } \]
\[ X \text{ while for } 1 \text{ that act nontrivially only on Hilbert space } \]
\[ E_{A/B} \]
\[ \Phi = \Phi \text{ is simply } \]
\[ R \text{ is the Fourier transform acting in the following way: } \]
\[ F \text{ are defined as: } \]
\[ \text{ where } Z^j \text{ is simply } j\text{-th power of } Z, X^{(j)} = 1 \text{ and } X^{(j)} \text{ are given in eqs. (A33), (A34). The output state of the isometry is: } \]
\[ \Phi(\psi)^{AB} \otimes |00\rangle^{A'B'} = \sum_{j,k=0}^{2} X^{(j)}A M_{j0}^A \otimes X^{(k)}B P_k^B |\psi\rangle^{ABC} \otimes |jk\rangle^{A'B'}. \]
as per definition of $Z^{A/B}$ given in eqs. (A31),(A32). Eqs. (A47) imply that only surviving elements of the sum are those containing $|j\rangle^{A'B'}$ such that $j = k$. Hence, the explicitly written full output state $\Phi(|\psi⟩^{ABC} \otimes |00⟩^{A'B'})$ reads

$$\Phi(|\psi⟩^{ABC} \otimes |00⟩^{A'B'}) = M_{001}^A |\psi⟩^{ABC} \otimes |00⟩^{A'B'} + X(1)^A M_{110}^A \otimes X(1)^B |\psi⟩^{ABC} \otimes |1⟩^{A'B'} + X(2)^A M_{220}^A \otimes X(2)^B |\psi⟩^{ABC} \otimes |22⟩^{A'B'}.$$  

(A53)

We simplify the second term on the r.h.s. of the last eq.:

$$X(1)^A M_{110}^A \otimes X(1)^B |\psi⟩ = \left( X_{01}^A + X_{22}^A \right) \frac{I_{012} + Z_{01}^A}{2} \otimes \frac{X_{01}^B + M_{22}^B}{2} |\psi⟩ = \frac{I_{012} + Z_{01}^A}{2} \left( X_{01}^A + M_{22}^A \right) \otimes \frac{X_{01}^B + M_{22}^B}{2} |\psi⟩ = M_{001}^A |\psi⟩$$  

(A54)

In the first line we used eqs. (A40) and (A41). To get the second line we used the anticommutation between $X_{01}^A$ and $Z_{01}^A$. The third eq. was obtained from (A24) and the fact that $X_{01}^A |\psi⟩ = |0⟩_{X} |\psi⟩ = |0⟩_{X1}$. In the last equation we used the fact that $M_{001}^A = \frac{I_{012} + Z_{01}^A}{2}$ and $|\psi⟩_{X1} = (M_{001}^A + M_{110}^A) |\psi⟩$. Hence, we get simplified output state

$$\Phi(|\psi⟩^{ABC} \otimes |00⟩^{A'B'}) = M_{001}^A |\psi⟩^{ABC} \otimes \left( |00⟩^{A'B'} + |11⟩^{A'B'} \right) + X(2)^A M_{220}^A \otimes X(2)^B |\psi⟩^{ABC} \otimes |22⟩^{A'B'}.$$  

(A57)

Now we take care of the last term

$$X(2)^A M_{220}^A \otimes X(2)^B |\psi⟩ = \left( X_{01}^A + M_{22}^A \right) \frac{I_{12} + Z_{12}^A}{2} \otimes \frac{X_{01}^B + M_{22}^B}{2} |\psi⟩ = \frac{I_{12} + Z_{12}^A}{2} \left( X_{01}^A + M_{22}^A \right) \otimes \frac{X_{01}^B + M_{22}^B}{2} |\psi⟩ = M_{001}^A |\psi⟩.$$  

(A58)

In the first line we used eqs. (A42). In the second line we used the anticommutation relation between $Z_{12}^A$ and $X_{12}^A$ (cf. (A27)). To get the third line we used the fact that $\left( M_{012}^A + X_{12}^A \right) \otimes \left( M_{012}^B + X_{12}^B \right) |\psi⟩ = |\psi⟩$ and also equality $I_{12} + Z_{12}^A = I_{012} + Z_{01}^A$. The last two lines are the consequence of the anticommutation relation between $Z_{12}^A$ and $X_{12}^A$ (cf. (A25)) and relation $\left( M_{12}^A + X_{01}^A \right) \otimes \left( M_{12}^B + X_{01}^B \right) |\psi⟩ = |\psi⟩$. Finally, we obtain the self-testing remark for the state:

$$\Phi(|\psi⟩^{ABC} \otimes |00⟩^{A'B'}) = \sqrt{3} M_{001}^A |\psi⟩^{ABC} \otimes |\phi+⟩^{A'B'} \equiv |ξ⟩^{ABC} \otimes |ϕ+⟩^{A'B'}.$$  

(A63)

where we introduced notation $|ξ⟩^{ABC} = \left( \sqrt{3} M_{001}^A |\psi⟩^{AB1} \right) \otimes |ϕ⟩^{B2C}$, which is a valid quantum state, because the norm of $M_{001}^A |\psi⟩^{AB1}$ is equal to $1/\sqrt{3}$ (cf. (A16)). Now we move to the self-testing of measurements. Let us start with measurement $M_{10}^A$, and see how the self-testing isometry maps the state $M_{10}^A |\psi⟩^{ABC}$:

$$\Phi(M_{10}^A |\psi⟩^{ABC} \otimes |00⟩^{A'B'}) = \sum_{j,k=0}^{2} X(1)^A M_{j10}^A M_{lj0}^A \otimes X(1)^B P_{k}^B |\psi⟩^{ABC} \otimes |j⟩^{A'B'} = \sum_{j,k=0}^{2} X(1)^A \delta_{jk} M_{lj0}^A \otimes X(1)^B P_{k}^B |\psi⟩^{ABC} \otimes |j⟩^{A'B'} = |ξ⟩^{ABC} \otimes \frac{1}{\sqrt{3}} |\phi+⟩^{A'B'}.$$  

(A64)
which is exactly the self-testing statement for $M_{ij0}$. To get the second line we used orthogonality of projectors \{$M_{ij0}$\}, and the following two lines just reproduce the proof of the self-testing the state. Exactly the same proof holds for measurement operators $M^A_{2|1}$ and $M^A_{0|2}$, as they act on $|\psi\rangle$ in the same way as $M^A_{2|0}$ and $M^A_{0|0}$ respectively. Concerning self-testing of $M^A_{0|1}$ and $M^A_{1|1}$, given that $X^A_{0|1} = M^A_{0|1} - M^A_{1|1}$ and $I^A_{0|1,X} = M^A_{0|1} + M^A_{1|1}$, self-testing $X^A_{0|1}$ and $I^A_{0|1,X}$ is equivalent to self-testing $M^A_{0|1}$ and $M^A_{1|1}$. Given that $I^A_{0|1,X}|\psi\rangle = I^A_{0|1,2}|\psi\rangle$ and $I^A_{0|1,z} = M^A_{0|0} + M^A_{1|0}$ the self-testing statement for \{$M^A_{0|j}$\} allows to conclude

$$
\Phi(\langle I^A_{0|1,X}\rangle^ABC \otimes (00)^{AB'i}) = |\xi\rangle^{ABC} \otimes (0\langle 0|A'\rangle + |1\rangle\langle 1|A')\phi_+^{A'B'i}.
$$

(A65)

We now turn to self-testing od $X^A_{0|1}$:

$$
\Phi(X^A_{0|1}|\psi\rangle^{ABC} \otimes (00)^{AB'i}) = \sum_{j,k=0}^{2} X^{(j)A}M^A_{j|0}X^A_{0|1} \otimes X^{(k)B}P^B_k|\psi\rangle^{ABC} \otimes |j\rangle^{AB'i}.
$$

(A66)

$$
= \sum_{j,k=0}^{2} X^{(j)A}M^A_{j|0}X^A_{0|1} \otimes X^{(k)B}P^B_k|\psi\rangle^{ABC} \otimes |j\rangle^{AB'i}.
$$

(A67)

Note that in the second line the sum has different upper bounds. Let us first prove that all elements of the sum corresponding to $k = 2$ vanish:

$$
\sum_{j=0}^{2} X^{(j)A}M^A_{j|0}X^A_{0|1} \otimes X^{(2)B}P^B_2|\psi\rangle^{ABC} \otimes |2\rangle^{AB'i} = \sum_{j=0}^{2} X^{(j)A}M^A_{j|0}X^A_{0|1} \otimes X^{(2)B}P^B_2|\psi\rangle^{ABC} \otimes |2\rangle^{AB'i}
$$

(A69)

$$
= \sum_{j=0}^{2} X^{(j)A}M^A_{j|0} \left( M^A_{0|1} - M^A_{1|1} \right)M^A_{2|1} \otimes X^{(2)B}P^B_2|\psi\rangle^{ABC} \otimes |2\rangle^{AB'i}
$$

(A70)

$$
= 0,
$$

(A71)

where in the second line we used eq. (A47), and in the second we used eq. (A18) and definition of $X^A_{0|1}$. Finally, to get the last line we used orthogonality of projectors corresponding to the same measurement. Now we consider elements of the sum corresponding to $k = 2$:

$$
\sum_{k=0}^{1} X^{(2)A}M^A_{2|0}X^A_{0|1} \otimes X^{(k)B}P^B_k|\psi\rangle^{ABC} \otimes |2k\rangle^{AB'i} = \sum_{k=0}^{1} X^{(2)A}M^A_{2|0} \otimes X^{(k)B}P^B_kX^A_{0|1}|\psi\rangle^{ABC} \otimes |2k\rangle^{AB'i}
$$

(A72)

$$
= \sum_{k=0}^{1} X^{(2)A} \otimes X^{(k)B}P^B_kX^A_{0|1}|\psi\rangle^{ABC} \otimes |2k\rangle^{AB'i}
$$

(A73)

$$
= 0.
$$

(A74)

In the first line we used eq. (24) and the fact that haded and unhated operators of Bob act on $|\psi\rangle$ in the same way. In the second eq. we used eq. (A44), and to get the last eq. we used the fact that ranges of $P^B_2$ and $X^A_{0|1}$ are orthogonal. Let us write the whole remaining state

$$
\Phi(X^A_{0|1}|\psi\rangle^{ABC} \otimes (00)^{AB'i}) = M^A_{0|0}X^A_{0|1} \otimes P^B_0|\psi\rangle^{ABC} \otimes (00)^{AB'i} + M^A_{0|0}X^A_{0|1} \otimes X^{(1)B}P^B_1|\psi\rangle^{ABC} \otimes (01)^{AB'i} +
$$

(A75)
The second equality (the third and fourth lines) is obtained by using anticommutation relation between $X_{0,1}^A$ and $Z_{0,1}^A$ (cf. (A25)). In the fifth line we used the fact that $M_{1|0}^A \otimes P_{1}^B |\psi\rangle = 0$ and $M_{0|0}^A \otimes P_{1}^B |\psi\rangle = 0$ (cf. (A47)). To get the sixth and seventh lines we used again the anticommutation between $Z_{0,1}^A$ and $X_{0,1}^A$ and relation $X_{0,1}^A X_{1}^A |\psi\rangle = 1_{0,1} \cdot |\psi\rangle$. Given the definition of $M_{0,1}$ and $M_{1|1}$, eqs. (A65) and (A75) imply

$$\Phi(M_{j|1}^A |\psi\rangle ABC \otimes |00\rangle A'B'_{1}) = |\xi\rangle ABC \otimes M_{j|1}^{A'} |\phi_{+}\rangle A'B'_{1},$$  
(A76)

for $j = 0, 1, 2$. Completely analogous proof holds for self-testing the third Alice’s measurement:

$$\Phi(M_{j|2}^A |\psi\rangle ABC \otimes |00\rangle A'B'_{1}) = |\xi\rangle ABC \otimes M_{j|2}^{A'} |\phi_{+}\rangle A'B'_{1},$$  
(A77)

for $j = 0, 1, 2$.

Eqs. (A64), (A76) and (A77) together imply:

$$\Phi(M_{a|x|}^A |\psi\rangle ABC \otimes |00\rangle A'B'_{1}) = |\xi\rangle ABC \otimes M_{a|x|}^{A'} |\phi_{+}\rangle A'B'_{1},$$  
(A78)

The set of eqs. (A14)-(A17) implies that operators $M_{2|0}^B$, $M_{2|1}^B$, $M_{0|2}^B$ and $M_{0|3}^B$ act on $|\psi\rangle$ in the same way as Alice’s measurements self-tested in (A78) which implies:

$$\Phi(M_{2|0}^B |\psi\rangle ABC \otimes |00\rangle A'B'_{1}) = |\xi\rangle ABC \otimes M_{2|0}^{B} |\phi_{+}\rangle A'B'_{1},$$  
(A79)

$$\Phi(M_{2|1}^B |\psi\rangle ABC \otimes |00\rangle A'B'_{1}) = |\xi\rangle ABC \otimes M_{2|1}^{B} |\phi_{+}\rangle A'B'_{1},$$  
(A80)

$$\Phi(M_{0|2}^B |\psi\rangle ABC \otimes |00\rangle A'B'_{1}) = |\xi\rangle ABC \otimes M_{0|2}^{B} |\phi_{+}\rangle A'B'_{1},$$  
(A81)

$$\Phi(M_{0|3}^B |\psi\rangle ABC \otimes |00\rangle A'B'_{1}) = |\xi\rangle ABC \otimes M_{0|3}^{B} |\phi_{+}\rangle A'B'_{1}.$$  
(A82)

Let us now define reference operators:

$$X_{0,1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad Z_{0,1}' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad X_{1,2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad Z_{1,2} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$  
(A83)

Eqs. (A78) implies:

$$\Phi(X_{0,1}^A |\psi\rangle ABC \otimes |00\rangle A'B'_{1}) = |\xi\rangle ABC \otimes X_{0,1}^{A'} |\phi_{+}\rangle A'B'_{1},$$  
(A84)

$$\Phi(Z_{0,1}^A |\psi\rangle ABC \otimes |00\rangle A'B'_{1}) = |\xi\rangle ABC \otimes Z_{0,1}^{A'} |\phi_{+}\rangle A'B'_{1},$$  
(A85)

$$\Phi(X_{1,2}^A |\psi\rangle ABC \otimes |00\rangle A'B'_{1}) = |\xi\rangle ABC \otimes X_{1,2}^{A'} |\phi_{+}\rangle A'B'_{1},$$  
(A86)

$$\Phi(Z_{1,2}^A |\psi\rangle ABC \otimes |00\rangle A'B'_{1}) = |\xi\rangle ABC \otimes Z_{1,2}^{A'} |\phi_{+}\rangle A'B'_{1}.$$  
(A87)

Now eqs. (A24) and (A26) together with the set of eqs. (A84)-(A87) lead to:

$$\Phi(X_{0,1}^B |\psi\rangle ABC \otimes |00\rangle A'B'_{1}) = |\xi\rangle ABC \otimes X_{0,1}^{B'} |\phi_{+}\rangle A'B'_{1},$$  
(A88)

$$\Phi(Z_{0,1}^B |\psi\rangle ABC \otimes |00\rangle A'B'_{1}) = |\xi\rangle ABC \otimes Z_{0,1}^{B'} |\phi_{+}\rangle A'B'_{1},$$  
(A89)

$$\Phi(X_{1,2}^B |\psi\rangle ABC \otimes |00\rangle A'B'_{1}) = |\xi\rangle ABC \otimes X_{1,2}^{B'} |\phi_{+}\rangle A'B'_{1},$$  
(A90)

$$\Phi(Z_{1,2}^B |\psi\rangle ABC \otimes |00\rangle A'B'_{1}) = |\xi\rangle ABC \otimes Z_{1,2}^{B'} |\phi_{+}\rangle A'B'_{1},$$  
(A91)

and equivalently

$$\Phi(D_{0,1}^B |\psi\rangle ABC \otimes |00\rangle A'B'_{1}) = |\xi\rangle ABC \otimes \frac{X_{0,1}^{B'} + Z_{0,1}^{B'} |\phi_{+}\rangle A'B'_{1}}{\sqrt{2}},$$  
(A92)

$$\Phi(E_{0,1}^B |\psi\rangle ABC \otimes |00\rangle A'B'_{1}) = |\xi\rangle ABC \otimes \frac{Z_{0,1}^{B'} - X_{0,1}^{B'} |\phi_{+}\rangle A'B'_{1}}{\sqrt{2}},$$  
(A93)

$$\Phi(D_{1,2}^B |\psi\rangle ABC \otimes |00\rangle A'B'_{1}) = |\xi\rangle ABC \otimes \frac{X_{1,2}^{B'} + Z_{1,2}^{B'} |\phi_{+}\rangle A'B'_{1}}{\sqrt{2}},$$  
(A94)

$$\Phi(E_{1,2}^B |\psi\rangle ABC \otimes |00\rangle A'B'_{1}) = |\xi\rangle ABC \otimes \frac{Z_{1,2}^{B'} - X_{1,2}^{B'} |\phi_{+}\rangle A'B'_{1}}{\sqrt{2}}.$$  
(A95)
The last set of equations together with definitions of $D_{j,k}$ and $E_{j,k}$ and eqs. (A79)-(A82) imply:

$$
\Phi (M^{B}_{b_{1}|y}|\psi\rangle^{ABC}\otimes|00\rangle^{A'B'_{i}}) = |\xi\rangle^{ABC} \otimes M^{B'_{i}}_{b_{1}|y} |\phi_{+}\rangle^{A'B'_{i}}.
$$

(A96)

Eqs. (A78) and (A96) together give

$$
\Phi (M^{A}_{a|x} \otimes M^{B}_{b_{1}|y}|\psi\rangle^{ABC}\otimes|00\rangle^{A'B'_{i}}) = |\xi\rangle^{ABC} \otimes M^{A'}_{a|x} \otimes M^{B'_{i}}_{b_{1}|y} |\phi_{+}\rangle^{A'B'_{i}},
$$

(A97)

where

$$
|\xi\rangle^{ABC} = \left( \sqrt{3}M^{A}_{0|y}|\psi\rangle^{AB_{1}} \right) \otimes |\psi\rangle^{B_{2}C} \\
\equiv |\xi_{1}\rangle^{AB_{1}} \otimes |\psi\rangle^{B_{2}C}.
$$

(A98)

(A99)

**Appendix B: Proof of Eq. (6)**

Eq. (A63) together with (A96) leads to

$$
\Phi^{t}_{B_{1}} \left( M^{B}_{b_{1}|y} \otimes I^{B'_{i}} \right) = I^{B} \otimes M^{B'_{i}}_{b_{1}|y}.
$$

(B1)

Since $M^{B'_{i}}_{b_{1}|y}$ are projective it must be $\Phi^{t}_{B_{1}} \left( M^{B}_{b_{1},b_{2}|y} \otimes I^{B'_{i}} \right) = K^{B}_{b_{1},b_{2}|y} \otimes M^{B'_{i}}_{b_{1}|y}$, where $K^{B}_{b_{1},b_{2}|y}$ is positive semidefinite and $\sum_{b_{2}} K^{B}_{b_{1},b_{2}|y} = I$. With this insight, and eqs. (A78) and (A96) for every collection of inputs $x,y,z$ and outputs $a,b,c$ the equivalence between reference correlations given in eq. (3) and physical correlations given in (4) implies:

$$
\text{Tr} \left[ \left( M^{A'}_{a|x} \otimes M^{B'_{i}}_{b_{1}|y} \right) |\phi_{+}\rangle^{A'B'_{i}} \right] \text{Tr} \left[ \left( M^{B'_{i}}_{b_{2}|y} \otimes M^{C'}_{c|z} \right) |\phi_{+}\rangle^{B'_{i}C'} \right] = \langle \psi | M^{A}_{a|x} \otimes M^{B}_{b_{1},b_{2}|y} \otimes M^{C}_{c|z} |\psi\rangle^{ABC} \otimes |\psi\rangle^{B_{2}C},
$$

(B2)

$$
= \langle \xi |^{ABC} I^{A} \otimes K^{B}_{b_{1},b_{2}|y} \otimes M^{C}_{c|z} |\xi\rangle^{ABC} \langle \phi_{+} |^{A'B'_{i}} M^{A'}_{a|x} \otimes M^{B'}_{b_{1}|y} |\phi_{+}\rangle^{A'B'_{i}}.
$$

(B3)

By cancelling the same terms on two sides of equality we obtain:

$$
\text{Tr} \left[ \left( M^{B'_{i}}_{b_{2}|y} \otimes M^{C'}_{c|z} \right) |\phi_{+}\rangle^{B'_{i}C'} \right] = \langle \xi |^{ABC} I^{A} \otimes K^{B}_{b_{1},b_{2}|y} \otimes M^{C}_{c|z} |\xi\rangle^{ABC}
$$

(B4)

Since for every fixed $b_{1}$ the set of operators $\{K^{B}_{b_{1},b_{2}|y}\}_{b_{2}}$ represents a valid measurement, together with Charlie’s measurements and state $|\xi\rangle$ we have a physical experiment which satisfies conditions given in Lemma 1. Hence, we can repeat exactly the same procedure as in Appendix A to establish existence of the $\Phi' = \Phi_{B_{2}} \otimes \Phi_{C}$ such that $\Phi_{B_{2}}$ acts nontrivially only on Hilbert space $H^{B_{2}}$ and the whole isometry transforms state $|\xi\rangle^{ABC}$ as follows

$$
\Phi' \left( M^{B}_{b_{1}|y} \otimes M^{C}_{c|z}|\xi\rangle^{ABC} \otimes|00\rangle^{B'_{i}C'} \right) = \tilde{\xi}^{ABC} \otimes \left( M^{B}_{b_{2}|y} \otimes M^{C}_{c|z} |\phi_{+}\rangle^{B'_{i}C'} \right),
$$

(B5)

which is the analogue of (A97) and where $|\tilde{\xi}\rangle$ is defined as

$$
|\tilde{\xi}\rangle^{ABC} = |\xi_{1}\rangle^{AB_{1}} \otimes \left( \sqrt{3}M^{C}_{0|z}|\psi\rangle^{B_{2}C} \right)
$$

(B6)

$$
\equiv |\xi_{1}\rangle^{AB_{1}} \otimes |\xi_{2}\rangle^{B_{2}C}.
$$

(B7)

Let us now rewrite eq. (A97)

$$
\Phi \left( M^{A}_{a|x} \otimes M^{B}_{b_{2}|y} \otimes M^{C}_{c|z}|\psi\rangle^{AB_{1}} \otimes |\psi\rangle^{B_{2}C} \otimes|00\rangle^{A'B'_{i}} \right) = M^{C}_{c|z}|\xi\rangle^{ABC} \otimes M^{A'}_{a|x} \otimes M^{B'_{i}}_{b_{1}|y} |\phi_{+}\rangle^{A'B'_{i}}.
$$

(B8)

Insights from the beginning of this Appendix allow us to write further

$$
\Phi \left( M^{A}_{a|x} \otimes M^{B}_{b_{2}|y} \otimes M^{C}_{c|z}|\psi\rangle^{AB_{1}} \otimes |\psi\rangle^{B_{2}C} \otimes|00\rangle^{A'B'_{i}} \right) = K^{B}_{b_{1},b_{2}|y} \otimes M^{C}_{c|z} |\xi\rangle^{ABC} \otimes M^{A'}_{a|x} \otimes M^{B'_{i}}_{b_{1}|y} |\phi_{+}\rangle^{A'B'_{i}}.
$$

(B9)
By acting on the r.h.s. of this equation with $\Phi'$, use eq. (B5), taking into account that $\Phi_{B_1}$ acts nontrivially only on $\mathcal{H}^{B_1}$, while $\Phi_{B_2}$ acts nontrivially only on $\mathcal{H}^{B_2}$, and summing over $b_1$ and $b_2$ we obtain:

$$
\Phi_A \otimes \Phi_{B_1} \circ \Phi_{B_2} \otimes \Phi_C \left( M_{a|x} |\psi\rangle^{A_1 B_1} \otimes M_{c|z} |\psi\rangle^{B_2 C} \otimes |0000\rangle^{A'B_1'B_2'C'} \right) = |\tilde{\xi}\rangle^{AB_1B_2C} \otimes \mathcal{M}'_{a|x} |\phi_u\rangle^{A'B_1'} \otimes \mathcal{M}'_{c|z} |\phi_u\rangle^{B_2'C'},
$$

which completes the proof of eq. (6) if we denote $\tilde{\Phi} = \Phi_A \otimes \Phi_{B_1} \circ \Phi_{B_2} \otimes \Phi_C$.

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