Linear continuous surjections of $C_p$-spaces over compacta

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Abstract

Let $X$ and $Y$ be compact Hausdorff spaces and suppose that there exists a linear continuous surjection $T : C_p(X) \to C_p(Y)$, where $C_p(X)$ denotes the space of all real-valued continuous functions on $X$ endowed with the pointwise convergence topology. We prove that $\dim X = 0$ implies $\dim Y = 0$. This generalizes a previous theorem [7, Theorem 3.4] for compact metrizable spaces. Also we point out that the function space $C_p(P)$ over the pseudo-arc $P$ admits no densely defined linear continuous operator $C_p(P) \to C_p([0,1])$ with a dense image.

1 Introduction and Results

For a Tychonoff space $X$, $C_p(X)$ denotes the space of all continuous real-valued functions on $X$ endowed with the pointwise convergence topology. The relationship between the topology of $X$ and linear topological properties of $C_p(X)$ is a subject of extensive research. A theorem of Pestov [13] plays the fundamental role in this study: if $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic for Tychonoff spaces $X$ and $Y$, then we have the equality $\dim X = \dim Y$. The theorem was first proved by Pavlovskii for compact metrizable spaces [12]. A natural question arises whether the inequality

The authors are supported by JSPS KAKENHI Grant Number 26400080. The visit of the second author to University of Tsukuba in August, 2015 was supported by the grant above.

Keywords: $C_p$-theory, linear operators, dimension, hereditarily indecomposable continua

MSC. 54C35, 46E10, 54F45
\[ \dim Y \leq \dim X \] holds for Tychonoff spaces \( X \) and \( Y \) whenever there exists a linear continuous surjection \( C_p(X) \to C_p(Y) \) \[2\] Problem 1046,1047]. This was answered negatively in \[3\],\[7\] even for compact metrizable spaces and the results were recently refined in \[9\]. The exception is the zero-dimensional case \[\ref{7},\text{Theorem }3.4\]: if there exists a linear continuous surjection \( C_p(X) \to C_p(Y) \) for compact metrizable spaces \( X \) and \( Y \), then \( \dim X = 0 \) implies \( \dim Y = 0 \). The present paper extends the above theorem to all compact Hausdorff spaces.

**Theorem 1.1** Let \( X \) and \( Y \) be compact Hausdorff spaces and suppose that there exists a linear continuous surjection \( T : C_p(X) \to C_p(Y) \). If \( \dim X = 0 \), then we have \( \dim Y = 0 \).

Our proof is based on the spectral theorem of Shchepin \[\ref{14},\ref{3}\] which allows us to reduce our consideration to that on compact metrizable spaces (Proposition 2.2). Then a slight modification of \[\ref{7},\text{Theorem }3.4\] supplies the desired result (Proposition 2.1).

The proof of Proposition 2.1 is applied to obtain more information on the existence of linear continuous surjections. It is known that for each finite-dimensional compact metrizable space \( X \), there exists a linear continuous surjection \( C_p([0,1]) \to C_p(X) \) \[8\] and the map may be constructed to be open \[9\]. The assumption of finite-dimensionality cannot be dropped since there exists no linear continuous surjection \( C_p([0,1]) \to C_p([0,1]^\omega) \) \[8\,Remark 4.6\]. We see from the next proposition that \([0,1]\) in these results cannot be replaced by the pseudo-arc \( P \), the topologically unique hereditarily indecomposable continuum which is the limit of an inverse sequence of \([0,1]\) (see a survey article \[10\]). Here a continuum means a compact connected metrizable space and a continuum \( X \) is said to be hereditarily indecomposable if each subcontinuum is not the union of two proper subcontinua. A compact metrizable space is called a Bing compactum if each connected component is either hereditarily indecomposable or is a singleton. A compact metrizable space \( Y \) is said to be hereditarily locally connected if each subcontinuum of \( Y \) is locally connected. Examples are \([0,1]\) and more generally dendrites \([11]\).

**Proposition 1.2** Let \( X \) be a Bing compactum, \( Y \) be a hereditarily locally connected compact metrizable space and let \( T : C_p(X) \to C_p(Y) \) be a densely defined linear continuous operator with a dense image. Then we
have \( \dim Y = 0 \). In particular there exists no linear continuous surjection \( C_p(P) \to C_p([0, 1]) \).

For a linear operator \( T : E \to F \), \( D(T) \) and \( R(T) \) denote the domain of \( T \) and the image of \( T \) respectively. An operator \( T : E \to F \) is said to be densely defined if \( D(T) \) is dense in \( E \). For a continuous map \( \varphi : X \to Y \), the induced operator \( \varphi^\# : C_p(Y) \to C_p(X) \) is defined by

\[
\varphi^\#(f) = f \circ \varphi, \quad f \in C_p(Y).
\]

2 Proofs

**Proposition 2.1** Let \( X \) and \( Y \) be compact metrizable spaces and let \( T : E \to F \) be a linear continuous surjection defined on a dense subspace \( E \) of \( C_p(X) \) onto a dense subspace \( F \) of \( C_p(Y) \). If \( \dim X = 0 \), then we have \( \dim Y = 0 \).

The compactness assumption cannot be dropped in the above theorem as is demonstrated by the following example.

**Example 2.2** There exists a densely defined surjective linear operator \( T : C_p(C') \to C_p([0, 1]) \), where \( C' \) is a \( G_\delta \) subset of the Cantor set \( C \) and hence is a Polish space.

Let \( \varphi : C \to [0, 1] \) be a continuous 2-to-1 map of the Cantor set \( C \) onto \([0, 1]\) such that the set \( \{ y \in [0, 1] \mid |\varphi^{-1}(y)| = 2 \} \) is at most countable. We then obtain a subset \( C' \) of \( C \), by removing a countable subset from \( C \), such that \( \varphi' = \varphi|C' \to [0, 1] \) is a continuous bijection. The map induces a dense embedding \( (\varphi')^\# : C_p([0, 1]) \to C_p(C') \) \((\mathbb{I} \text{0.4.8})\) which naturally defines a densely defined operator \( T : C_p(C') \to C_p([0, 1]) \) with domain \( R((\varphi')^\#) := (\varphi')^\#(C_p([0, 1])) \) onto \( C_p([0, 1]) \).

Proof of Proposition 2.1. Our proof is a modification of \( \mathbb{I} \text{ Theorem 3.4} \) and is based on an analysis of the dual spaces. We first recall basics of the dual space notions \( \mathbb{I} \text{ Chap.0} \). For a Tychonoff space \( X \), \( L_p(X) \) denotes the dual space, that is, the space of all continuous linear functionals on
$C_p(X)$ endowed with the pointwise convergence topology. The space $L_p(X)$ is linearly homeomorphic to the space

$$\{ \sum_{i=1}^{n} \alpha_i x_i \mid \alpha_i \in \mathbb{R}, x_i \in X, n \in \mathbb{Z}_{\geq 0} \},$$

where the topology is described below. Each non-zero point $x$ of $L_p(X)$ is uniquely written as

$$x = \sum_{i=1}^{n} \alpha_i x_i \quad (2.1)$$

where $\alpha_i \in \mathbb{R}^* := \mathbb{R} \setminus \{0\}$ and $x_1, \ldots, x_n$ are mutually distinct points of $X$. The number $n$ above is called the length of $x$ and is denoted by $\ell(x)$. Let $A_n(X) = \{ x \in L_p(X) \mid \ell(x) = n \}$ and $B_n(X) = \{ x \in L_p(X) \mid \ell(x) \leq n \}$. By the definition we have $A_n(X) = B_n(X) \setminus B_{n-1}(X)$ and also we have the equality

$$L_p(X) = \bigcup_{n=1}^{\infty} A_n(X). \quad (2.2)$$

It is known that each $B_n(X)$ is closed in $L_p(X)$ and each point $x = \sum_{i=1}^{n} \alpha_i x_i$ as in (2.1) has a neighborhood basis consisting of the sets

$$\sum_{i=1}^{n} O_i U_i$$

where $O_i$’s are open neighborhoods of $\alpha_i$’s in $\mathbb{R}^*$ and $U_i$’s are open neighborhoods of $x_i$’s respectively such that $U_i \cap U_j = \emptyset$ whenever $i \neq j$. It follows from this that the map

$$\sigma_n : (\mathbb{R}^*)^n \times (X^n \setminus \Delta_n) \to A_n(X), \quad (\alpha_i, (x_i)) \mapsto \sum_{i=1}^{n} \alpha_i x_i, \quad (2.3)$$

where $\Delta_n = \{ (x_1, \ldots, x_n) \mid x_i = x_j \text{ for some } i \neq j \}$, is a homeomorphism.

Starting the proof of Proposition 2.1, let $\xi : E \to C_p(X)$ and $\eta : F \to C_p(Y)$ be the dense inclusions and consider the dual maps $\xi^* : L_p(X) \to E^*$ and $\eta^* : L_p(Y) \to F^*$. These maps $\xi^*$ and $\eta^*$ are continuous bijections because of the denseness of $E$ and $F$. It follows from this that for each compact set $K$ of $L_p(X)$, the restriction $\xi^*|K : K \to \xi^*(K)$ is a homeomorphism. The same holds for compact sets of $L_p(Y)$. The subset $\Delta_n$ above is closed and thus $G_\delta$ in a metrizable compact space $X^n$. Then by (2.3), we see
that $A_n(X)$ is $\sigma$-compact and is represented as the union $A_n(X) = \bigcup_{i=1}^{\infty} A_{n,i}$, where each $A_{n,i}$ is homeomorphic to a compact subset of $(\mathbb{R}^*)^n \times X^n$. By the above remark, we have

$$\xi^*(A_n(X)) = \bigcup_{i=1}^{\infty} \xi^*(A_{n,i})$$

and each $\xi^*(A_{n,i})$ is homeomorphic to $A_{n,i}$.

The composition $T^* \circ \eta^* : L_\mu(Y) \to E^*$ embeds $Y$ into $E^*$ and hence $Y$ is homeomorphic to the subspace

$$T^*\eta^*(Y) \subset \bigcup_{i=1}^{\infty} \xi^*(A_{n,i}).$$

Thus we have $Y = \bigcup_{k=1}^{\infty} Y_k$ where $Y_k$ is a compact set such that the restriction

$$(\xi^*|A_{n(k),i(k)})^{-1} \circ T^* \circ \eta^*|Y_k : Y_k \to Y'_k$$

is a homeomorphism of $Y_k$ onto a compact subset $Y'_k$ of $A_{n(k),i(k)}$ for some $n(k)$ and $i(k)$. Below we show that $\dim Y_k = 0$ from which the desired conclusion follows by the countable sum theorem [5, Theorem 3.1.8].

Let $\hat{Y}_k = \sigma_{n(k)}^{-1}(Y'_k)$ where $\sigma_{n(k)}$ is the homeomorphism of (2.3). Let $p : (\mathbb{R}^*)^{n(k)} \times X^{n(k)} \to X^{n(k)}$ be the projection and consider the restriction $p_k := p|\hat{Y}_k$. As a map defined on the compact set $\hat{Y}_k$, the map $p_k$ is a closed map onto a zero-dimensional subspace $p_k(\hat{Y}_k)$ of $X^{n(k)}$. We show that $p_k$ is at-most-$n(k)$-to-1 map, that is,

$$|p_k^{-1}(x)| \leq n(k)$$

for each $x \in X^n$. Once (2.4) is verified, we obtain the equality $\dim \hat{Y}_k = 0$ from the zero-dimensionality of $p_k(\hat{Y}_k)$ by the dimension lowering theorem [5, Theorem 1.12.4], and therefore we conclude $\dim Y_k = 0$, as desired.

In what follows, $n(k)$ and $p_k$ are simply denoted by $n$ and $p$ respectively. In order to verify (2.4), take a point $x = (x_1, \ldots, x_n)$ of $X^n$ and suppose on the contrary that the fiber $p^{-1}((x_1, \ldots, x_n))$ contains $(n + 1)$ points $y^1, \ldots, y^{n+1}$ of $\hat{Y}_k$. For each $j = 1, \ldots, n + 1$, we may find $\lambda_{ij} \in \mathbb{R}^*$ such that

$$y^j = \sum_{i=1}^{n} \lambda_{ij} x_i.$$  

(2.5)

By the definition of $\hat{Y}_k$, the definitions of $\sigma_{n(k)}$ and the duality, the above (2.5) is rephrased as follows: for each $f \in E$ we have

$$(Tf)(y^j) = \sum_{i=1}^{n} \lambda_{ij} f(x_i), \quad j = 1, \ldots, n + 1.$$  

(2.6)
Consider the matrix $\Lambda = (\lambda_{ij})_{1 \leq i,j \leq n}$ of size $n \times n$. The denseness of $F$ with (2.6) implies that the linear map

$$v \mapsto \Lambda v, \quad \mathbb{R}^n \to \mathbb{R}^n$$

has the dense image. This implies that $\det \Lambda \neq 0$. In particular we may find $a_{ij} \in \mathbb{R}$ such that

$$(Tf)(y^{n+1}) = \sum_{i=1}^{n} a_{ij} Tf(y^i).$$

Then the image of the evaluation map

$$e : F \to \mathbb{R}^{n+1}; \quad e(g) = (g(y^j))_{1 \leq j \leq n+1}$$

is contained in an $n$-dimensional subspace of $\mathbb{R}^{n+1}$. However since $F$ is dense in $C_p(Y)$, the set $e(F)$ must be dense in $\mathbb{R}^{n+1}$. This contradiction finishes the proof of (2.4) and thus finishes the proof of Proposition 2.1.

Here we recall some basics on inverse spectra from [3, Chap.1]. Let $\mathcal{S}_X = \{X_\alpha, p_{\alpha\beta}; A\}$ be an inverse system of topological spaces $X_\alpha$, indexed by a directed set $A$ with the limit space $X = \lim \leftarrow \mathcal{S}_X$. The canonical projection of $X$ to $X_\alpha$ is denoted by $p_\alpha : X \to X_\alpha$. An inverse system $\mathcal{S}_X = \{X_\alpha, p_{\alpha\beta}; A\}$ is called a factorizing $\omega$-spectrum if

(O1) each countable chain $C$ of $A$ has the supremum $\sup C \in A$,

(O2) for each countable chain $B$ of $A$ with $\beta = \sup B$, the canonical map

$$\triangle_{\alpha \in B p_{\alpha\beta}} : X_\beta \to \lim \leftarrow \{X_\alpha, p_{\alpha_1\alpha_2}; B\}$$

is a topological embedding, and

(F) each continuous function $f : X = \lim \leftarrow \mathcal{S}_X \to \mathbb{R}$ admits an $\alpha \in A$ and $f_\alpha : X_\alpha \to \mathbb{R}$ such that $f = f_\alpha \circ p_\alpha$.

For a compact Hausdorff space $X$, there exists a factorizing $\omega$-spectrum $\mathcal{S}_X = \{X_\alpha, p_{\alpha\beta}; A\}$ with $|A| \leq w(X)$ such that $X = \lim \leftarrow \mathcal{S}_X$ and

(C1) each $X_\alpha$ is a compact metrizable space,

(C2) each limit projection $p_\alpha$ as well as each bonding map $p_{\alpha\beta}$ is surjective.

(C3) the canonical map $\triangle_{\alpha \in B p_{\alpha\beta}}$ of (O2) is a surjective homeomorphism.
If \( \dim X \leq n \), then we may choose above spectrum so that \( \dim X_\alpha \leq n \) for each \( \alpha \in A \) \[3\] Propositions 1.3.5, 1.3.2, and 1.3.10]. For a compact metrizable space \( X \), the above spectrum is reduced to the trivial system \( \{ X, \text{id}_X \} \).

**Proposition 2.3** Let \( X = \lim_\leftarrow S_X \) and \( Y = \lim_\leftarrow S_Y \) be compact Hausdorff spaces which are the limits of factorizing \( \omega \)-spectra \( S_X = \{ X_\alpha, p_\alpha; A \} \) and \( S_Y = \{ Y_\beta, p_\beta; B \} \) satisfying the conditions (C1)-(C3) with the projections \( p_\alpha : X \to X_\alpha \) and \( q_\beta : Y \to Y_\beta \) respectively.

Let \( T : C_p(X) \to C_p(Y) \) be a linear continuous operator.

1. For each \( \alpha \), there exist \( \beta = \beta(\alpha) \) and a densely defined operator \( T_{\alpha,\beta} : C_p(X_\alpha) \to C_p(Y_\beta) \) such that

\[
T \circ p^\sharp_\alpha = q^\sharp_\beta \circ T_{\alpha,\beta}.
\]

For each \( \beta_0 \in B \), we may choose the above \( \beta(\alpha) \) so that \( \beta(\alpha) \geq \beta_0 \).

2. If moreover \( T \) is surjective, then for each \( \alpha_0 \in A \) and for each \( \beta_0 \in B \), we may choose \( T_{\alpha,\beta} \) so that \( \alpha \geq \alpha_0, \beta \geq \beta_0 \) and \( T_{\alpha,\beta} \) has a dense image.

The following diagram illustrates the operator \( T_{\alpha,\beta} \).

\[
\begin{array}{ccc}
C_p(X) & \xrightarrow{T} & C_p(Y) \\
\downarrow{p^\sharp_\alpha} & & \uparrow{q^\sharp_\beta} \\
C_p(X_\alpha) & \xrightarrow{T_{\alpha,\beta}} & C_p(Y_\beta)
\end{array}
\]

Proof. Let \( f_\alpha \in C_p(X_\alpha) \) and consider the composition \( f_\alpha \circ p_\alpha \) whose image by \( T \) is factorized as

\[
T(f_\alpha \circ p_\alpha) = g_\beta \circ q_\beta \tag{2.7}
\]

for some \( \beta \) and \( g_\beta \in C_p(Y_\beta) \). Observe that we may choose the above \( \beta \) as large as we wish. An important observation here is that, because \( q_\beta \) is surjective,

\[
g_\beta \quad \text{is uniquely determined by} \quad f_\alpha \quad \text{and} \quad \beta. \tag{2.8}
\]
Fix an arbitrary $\alpha$ and notice that $C_p(X_\alpha)$ is separable ([I.1.5]). Take a countable dense set $D = \{f_{\alpha,i}\}$ of $C_p(X_\alpha)$. For each $i$ take a $\beta_i$ and $g_i \in C_p(Y_{\beta_i})$ such that

$$T(f_{\alpha,i} \circ p_\alpha) = g_i \circ q_{\beta_i}.$$ 

Let $\beta = \sup_i \beta_i \in B$. By the $\omega$-continuity of the spectrum (C3) we have

$$Y_{\beta} = \lim \left\{ Y_{\beta_i}, q_{\beta_i} \right\}$$

with the projection $q_{\beta_i,\beta} : Y_{\beta} \to Y_{\beta_i}$. For each $i$, the function $g_{\beta,i} = q_{\beta_i,\beta}(g_i) \in C_p(Y_{\beta})$ satisfies

$$T(f_{\alpha,i} \circ p_\alpha) = g_{\beta,i} \circ q_{\beta_i}.$$ 

Define $T_{\alpha,\beta} : D \to C_p(Y_{\beta})$ by

$$T_{\alpha,\beta}(f_{\alpha,i}) = g_{\beta,i}$$

and let $D(T_{\alpha,\beta}) = \text{span} D$ which is a dense subspace of $C_p(X_\alpha)$. We make use of the uniqueness (2.8) to extend $T_{\alpha,\beta}$ to a linear operator $T_{\alpha,\beta} : D(T_{\alpha,\beta}) \to C_p(Y_{\beta})$ as follows.

For $f = \sum_i \lambda_i f_{\alpha,i} \in D(T_{\alpha,\beta})$, all but finitely many $\lambda_i$’s being zero, let $T_{\alpha,\beta}(f) = \sum_i \lambda_i g_{\beta,i}$. Then $T_{\alpha,\beta}$ is well-defined and a linear map. Indeed, suppose that $\sum_i \lambda_i f_{\alpha,i} = \sum_i \mu_i f_{\alpha,i}$. Then we have the equality $\sum_i \lambda_i f_{\alpha,i} \circ p_\alpha = \sum_i \mu_i f_{\alpha,i} \circ p_\alpha$ and hence we have that $T(\sum_i \lambda_i f_{\alpha,i} \circ p_\alpha) = T(\sum_i \mu_i f_{\alpha,i} \circ p_\alpha)$. By linearity of $T$ we see that

$$\sum_i \lambda_i g_{\beta,i} \circ q_{\beta} = \sum_i \lambda_i T(f_{\alpha,i} \circ p_\alpha) = T(\sum_i \lambda_i f_{\alpha,i} \circ p_\alpha)$$

$$= T(\sum_i \mu_i f_{\alpha,i} \circ p_\alpha) = \sum_i \mu_i T(f_{\alpha,i} \circ p_\alpha)$$

$$= \sum_i \mu_i g_{\beta,i} \circ q_{\beta},$$

from which we see that $\sum_i \lambda_i g_{\beta,i} = \sum_i \mu_i g_{\beta,i}$ by the surjectivity of $q_{\beta}$. This proves that $T_{\alpha,\beta}$ is well-defined. The same argument is applied to prove that $T_{\alpha,\beta}$ is a linear map.

Finally we verify the continuity of $T_{\alpha,\beta}$. For a finite set $F$ of a space $Z$, for an $\epsilon > 0$ and for a function $h \in C_p(Z)$, let

$$< h, F, \epsilon > = \{ u \in C(Z) \mid |u(p) - h(p)| < \epsilon \text{ for each } p \in F\}.$$
Fix an $\epsilon > 0$ and a finite subset $F_\beta$ of $Y_\beta$. For each $y_\beta \in F_\beta$, take $y \in Y$ such that $q_\beta(y) = y_\beta$ by the surjectivity of $q_\beta$. Let $F$ be the resulting finite subset of $Y$. By the continuity of $T$, we may find a finite subset $E$ of $X$ and $\delta > 0$ such that

$$T(< f_\alpha \circ p_\alpha, E, \delta >) \subset < T(f_\alpha \circ p_\alpha), F, \epsilon >.$$ 

Let $E_\alpha = p_\alpha(E)$. We verify the inclusion

$$T,\beta(< f_\alpha, E_\alpha, \delta > \cap D(T,\alpha,\beta)) \subset < T,\alpha,\beta(f_\alpha), F_\beta, \epsilon >.$$ 

Indeed for $f'_\alpha \in < f_\alpha, E_\alpha, \delta >$, we have $|f'_\alpha(p_\alpha(x)) - f_\alpha(p_\alpha(x))| < \delta$ for each $x \in E$ and hence $f'_\alpha \circ p_\alpha \in < f_\alpha \circ p_\alpha, E, \delta >$. For each $y \in F$ we have

$$|T(f'_\alpha \circ p_\alpha)(y) - T(f_\alpha \circ p_\alpha)(y)| < \epsilon,$$

which implies

$$|T,\alpha,\beta(f'_\alpha)(y_\beta) - T,\alpha,\beta(f_\alpha)(y_\beta)| < \epsilon$$

for each $y_\beta \in F_\beta$ by the equality $T,\alpha,\beta(f_\alpha) \circ q_\beta = T(f_\alpha \circ p_\alpha)$. Thus we see that $T,\alpha,\beta(f'_\alpha) \in < T,\alpha,\beta(f_\alpha), F_\beta, \epsilon >$. This proves the continuity of $T,\alpha,\beta$ and hence completes the proof of (1).

(2) Now assume that $T$ is surjective. For an arbitrary $\alpha$ with $\alpha \geq \alpha_0$, take $\beta = \beta(\alpha) \geq \beta_0$ and $T,\alpha,\beta : C_p(X_\alpha) \to C_p(Y_\beta)$ as in (1). The operator $T,\alpha,\beta$ is defined on a dense subspace $D(T,\alpha,\beta)$ of $C_p(X_\alpha)$ and satisfies

$$T \circ p^*_{\alpha_0} = q^*_\beta \circ T,\alpha,\beta.$$  \hspace{1cm} (2.9)

Take a countable dense subset $E = \{g_i\}$ of $C_p(Y_\beta)$. Since $T$ is surjective, for each $i$ there exists $f_i \in C_p(X)$ such that $T(f_i) = g_i \circ q_\beta$. The function $f_i$ factorizes as

$$f_i = f_{\alpha_i} \circ p_{\alpha_i}$$

for some $\alpha_i \in A$. Let $\alpha(1) = \sup_i \alpha_i$. The function $f_{\alpha(1),i} = f_{\alpha_i} \circ p_{\alpha(1)}$ satisfies $f_i = f_{\alpha(1),i} \circ p_{\alpha(1)}$ and hence

$$T(f_{\alpha(1),i} \circ p_{\alpha(1)}) = g_i \circ q_\beta.$$  \hspace{1cm} (2.10)

Choose a countable dense subset $D_1$ of $C_p(X_{\alpha(1)})$ such that $D_1 \supset p^*_{\alpha,\alpha(1)}(D) \cup \{f_{\alpha(1),i}\}$. Repeat the procedure of (1) with the use of $D_1$ to find $\beta(1) > \beta$. 

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and a linear map \( T_{\alpha(1),\beta(1)} : C_p(X_{\alpha(1)}) \rightarrow C_p(Y_{\beta(1)}) \) densely defined on the linear span \( D(T_{\alpha(1),\beta(1)}) \) of \( D_1 \) such that
\[
T(f_{\alpha(1)} \circ p_{\alpha(1)}) = T_{\alpha(1),\beta(1)} \circ q_{\beta(1)}
\] (2.11)

for each \( f_{\alpha(1)} \in D(T_{\alpha(1),\beta(1)}) \). By (2.10), (2.11), and the surjectivity of \( q_{\beta(1)} \) we see that \( T_{\alpha(1),\beta(1)}(f_{\alpha(1),i}) = g_i \) for each \( i \). Thus we have the inclusion \( R(T_{\alpha(1),\beta(1)}) \supseteq q_{\beta(1)}^*(E) \).

Let us summarize the above properties of \( T_{\alpha(1),\beta(1)} \):

(1.1) \( D(T_{\alpha(1),\beta(1)}) \supseteq p_{\alpha(1)}^*(D(T_{\alpha,\beta}))) \) and

(1.2) \( R(T_{\alpha(1),\beta(1)}) \supseteq q_{\beta(1)}^*(E) \).

We continue this process to obtain increasing sequences of indices \( \alpha(1) < \cdots < \alpha(n) < \cdots, \beta(1) < \cdots < \beta(n) < \cdots \), countable dense subsets \( D_n \) of \( C_p(X_{\alpha(n)}), E_n \) of \( C_p(Y_{\beta(n)}) \), and a sequence of linear continuous operators \( \{T_{\alpha(n),\beta(n)} : C_p(X_{\alpha(n)}) \rightarrow C_p(Y_{\beta(n)})\} \), each \( T_{\alpha(n),\beta(n)} \) being defined on \( D(T_{\alpha(n),\beta(n)}) \), such that

(n.1) \( D(T_{\alpha(n+1),\beta(n+1)}) \supseteq p_{\alpha(n)\alpha(n+1)}^*(D(T_{\alpha(n),\beta(n)})) \),

(n.2) \( R(T_{\alpha(n+1),\beta(n+1)}) \supseteq q_{\beta(n)\beta(n+1)}^*(E_n) \), and

(n.3) \( T_{\alpha(n+1),\beta(n+1)}(p_{\alpha(n)\alpha(n+1)} \circ T_{\alpha(n),\beta(n)}) \) on \( D(T_{\alpha(n),\beta(n)}) \).

Let \( \alpha_\infty = \sup_n \alpha(n) \) and \( \beta_\infty = \sup_n \beta(n) \) and let \( D_\infty = \cup_n p_{\alpha(n)\alpha_\infty}^*(D(T_{\alpha(n),\beta(n)})) \).

Then
\[
D_\infty \text{ is dense in } C_p(X_{\alpha_\infty}).
\] (2.12)

Indeed \( X_{\alpha_\infty} = \lim_\omega X_{\alpha(n)} \) by the \( \omega \)-continuity (C3). Hence for each \( f \in C_p(X_{\alpha_\infty}) \) and for each \( \epsilon > 0 \), there exist \( \alpha(n) \) and \( f_{\alpha(n)} \in C_p(X_{\alpha(n)}) \) such that
\[
\sup_{x_\infty \in X_{\alpha_\infty}} |f(x_\infty) - f_{\alpha(n)}(p_{\alpha(n)}(x_\infty))| < \epsilon.
\]

The function \( f_{\alpha(n)} \) is approximated arbitrarily closely by functions of \( D(T_{\alpha(n),\beta(n)}) \) with respect to the pointwise convergence topology on \( C_p(X_{\alpha(n)}) \). Hence the function \( f \) is approximated arbitrarily closely by functions from the set \( \cup_n p_{\alpha(n)\alpha_\infty}^*(D(T_{\alpha(n),\beta(n)})) = D_\infty \).
An operator $T_{\alpha,\beta}$ is defined by

$$T_{\alpha,\beta}(p_{\alpha,\alpha}(f_{\alpha})) = T_{\alpha,\beta}(f_{\alpha}) \circ q_{\beta}(E)$$

for $f_{\alpha} \in D(T_{\alpha,\beta})$. The equality (n.3) guarantees that $T_{\alpha,\beta}$ is well defined. Now we have the inclusion $T_{\alpha,\beta}(p_{\alpha,\alpha}(D(T_{\alpha,\beta}(n)))) \supset q_{\beta}(E_n)$. Hence $R(T_{\alpha,\beta}) \supset E_\infty := \cup_n q_{\beta}(E_n)$. Using the same argument as in (2.12) we see that $E_\infty$ is dense in $C_p(Y_{\beta})$.

This proves (2).

Proof of Theorem 1.1. Let $T : C_p(X) \to C_p(Y)$ be a linear continuous surjection where $X$ and $Y$ are compact Hausdorff spaces. Let $S_X = \{X_\alpha, p_{\alpha_1,\alpha_2}; A\}$ and $S_Y = \{Y_\alpha, p_{\beta_1,\beta_2}; B\}$ be factorizing $\omega$-spectra such that $X = \varinjlim S_X$ and $Y = \varinjlim S_Y$. By the assumption dim $X = 0$, we may assume that dim $X_\alpha = 0$ for each $\alpha \in A$. For each $\beta \in B$, we apply Proposition 2.2 to find a densely defined operator $T_{\alpha,\beta} : C_p(X_\alpha) \to C_p(Y_{\beta(\alpha)})$ such that $\beta(\alpha) \geq \beta$ and $R(T_{\alpha,\beta})$ is dense in $C_p(Y_{\beta(\alpha)})$. By Proposition 2.1, we have dim $Y_{\beta(\alpha)} = 0$ and hence the set $\{\beta \in B \mid \dim Y_{\beta} = 0\}$ forms a cofinal subset of $B$. We therefore obtain dim $Y = 0$. This completes the proof of Theorem 1.1.

Remark 2.4 Theorem 1.1 holds under a weaker hypothesis that $X$ is a zero-dimensional pseudocompact Tychonoff space and $Y$ is a compact Hausdorff space.

To see the above, assume that $X$ is such a space and $T : C_p(X) \to C_p(Y)$ is a linear continuous surjection onto $C_p(Y)$ where $Y$ is compact. The inclusion $h : X \to \beta X$ of $X$ into the Stone-Čech compactification $\beta X$ of $X$ induces a linear continuous surjection $h^\sharp : C_p(\beta X) \to C_p(X)$. Since dim $\beta X = 0$, we may apply Theorem 1.1 to the composition $T \circ h^\sharp$ to conclude that dim $Y = 0$.

Proof of Proposition 1.2. Let $X$ be a Bing compactum, let $Y$ be a hereditarily locally connected compact metrizable space, and let $T : C_p(X) \to C_p(Y)$ be a densely defined linear continuous operator with a dense image. First recall that each continuous map $\varphi : Z \to B$ of a locally connected continuum $Z$ to a Bing compactum $B$ must be a constant map. Proceeding as in the proof of Proposition 2.1, we see that the space $Y$ is the countable union $Y = \cup_{i=1}^\infty Y_i$. 

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such that each $Y_i$ is homeomorphic to a subspace $\tilde{Y}_i$ of $(\mathbb{R}^*)^n \times X^n$, and such that the projection $p_i : \tilde{Y}_i \to X^n$ is a finite-to-one map.

By the assumption, each component $C$ of $Y_i$ is locally connected. Applying the above remark to the composition of the map $p_i|C : C \to X^n$ with the projection $X^n \to X$ onto any factor, we see that $p_i|C$ must be a constant map. This implies that $C$ is contained in a fiber of $p_i$, hence $C$ is a finite set and thus $C$ is a singleton. This proves that $Y_i$ is totally disconnected which is equivalent to $\dim Y_i = 0$ by the compact metrizability of $Y_i$. By the countable sum theorem [5, Theorem 3.1.8], we conclude that $\dim Y = 0$.

3 Remarks and Problems

The proof of Proposition 2.1 relies on both the compactness and the metrizability of the spaces involved, while Remark 2.4 naturally raises the following problem.

**Problem 3.1** Does Theorem 1.1 hold for Tychonoff spaces $X$ and $Y$?

Recently Krupski and Marciszewski [6] proved that there exists a metrizable space $X$ such that $C_p(X)$ is not homeomorphic to $C_p(X) \times C_p(X) \simeq C_p(X \oplus X)$, negatively answering a long standing problem of Arkhangel’skii. In the paper above they also showed that there exists no linear continuous surjection $C_p(M) \to C_p(M) \times C_p(M)$ for the Cook continuum $M$. Recall that the Cook continuum $M$ is a hereditarily indecomposable continuum such that, for each non-degenerate subcontinuum $C$ of $M$, every continuous map $f : C \to M$ is either the identity map id or a constant map [4]. Since $M \oplus M$ is a subspace of $M \times M$, we see that there exists no linear continuous surjection $C_p(M) \to C_p(M \times M)$. On the other hand as has been pointed out by Marciszewski (a private communication), we have the following.

**Remark 3.2** For the pseudo-arc $P$, there exists a topological linear isomorphism $C_p(P) \simeq C_p(P \oplus P)$.

Here we give a sketch of the above. Take a non-degenerate subcontinuum $Q$ of $P$ and let $P/Q$ be the quotient space obtained from $P$ by shrinking $Q$ into a point. The space $P/Q$ is a metrizable continuum as well. The projection $P \to P/Q$ is a monotone map, and monotone maps preserve the hereditary indecomposability and the arc-likeness (that is, being represented
by the limit of an inverse sequence of [0,1]), from the characterization of the pseudo-arc, we see that $P/Q$ is homeomorphic to $P$. Then we obtain, by [15] Corollary 6.6.13, the following linear topological isomorphisms

\[ C_p(P) \simeq C_p(P/Q) \times C_p(Q) \simeq C_p(P \oplus P). \]

This proves the desired result.

The following problem naturally arises.

**Problem 3.3** Does there exist a linear continuous surjection $C_p(P) \to C_p(P \times P)$?

Here we notice that, if a finite-dimensional compact metrizable space $X$ contains a homeomorphic copy of $[0,1]$, then there does exist a linear continuous surjection $C_p(X) \to C_p(X \times X)$. In order to see this, first observe that an arbitrary embedding $h : [0,1] \to X$ induces a linear continuous surjection $h^\sharp : C_p(X) \to C_p([0,1])$. Since $X$ is finite-dimensional, compact and metrizable, there exists a linear continuous surjection $T : C_p([0,1]) \to C_p(X)$ [8]. Then the composition $T \circ h^\sharp$ is the desired linear continuous surjection.

**Remark 3.4** Theorem 1.1 does not hold when we drop the assumption of linearity of the operator $T$.

In fact we can prove the following: Let $X$ be a compact metrizable space and let $S$ be the convergent sequence, that is, the space homeomorphic to $\{0\} \cup \{\frac{1}{n} \mid n \in \omega\}$. Then there exists a continuous surjection $H : C_p(S) \to C_p(X)$.

Proof. The argument below is extracted from [6, Proposition 5.4]. The space $C_p(S)$ contains a closed homeomorphic copy of $J$, the space of irrationals. Since the Banach space $C(X)$ of all real-valued continuous functions with sup norm is separable, we obtain a continuous surjection $J \to C(X)$ which extends to a continuous surjection $H : C_p(S) \to C(X)$. The map $H$, regarded as a map to $C_p(X)$, is the desired continuous surjection $H : C_p(S) \to C_p(X)$.
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