ACTION INTEGRALS AND INFINITESIMAL CHARACTERS

ANDRÉS VIÑA

Abstract. Let $G$ be a reductive Lie group and $\mathcal{O}$ the coadjoint orbit of a hyperbolic element of $\mathfrak{g}^*$. By $\pi$ is denoted the unitary irreducible representation of $G$ associated with $\mathcal{O}$ by the orbit method. We give geometric interpretations in terms of concepts related to $\mathcal{O}$ of the constant $\pi(g)$, for $g \in \mathbb{Z}(G)$. We also offer a description of the invariant $\pi(g)$ in terms of action integrals and Berry phases. In the spirit of the orbit method we interpret geometrically the infinitesimal character of the differential representation of $\pi$.

MSC 2000: Primary: 53D50, Secondary: 22E45

1. Introduction

Roughly speaking, the orbit method [8], [14] suggests that the unitary dual of a Lie group $G$ (i.e. the set of equivalence classes of unitary irreducible representations of $G$) is in bijective correspondence with the space of coadjoint orbits of $G$. Moreover the orbit method relates geometric properties of the coadjoint orbit with properties of the corresponding irreducible representation. This bijective correspondence exists if $G$ is a connected simply connected nilpotent group; in other cases where the correspondence is not a perfect bijection this method gives valuable suggestions about the geometric meaning of some facts of representation theory.

In this paper $G$ will be a reductive group and $\mathcal{O}$ will be a coadjoint orbit of a hyperbolic element $\eta \in \mathfrak{g}^*$, where $\mathfrak{g}^*$ is the dual of the Lie algebra $\mathfrak{g}$ of $G$. In the spirit of the orbit method we will give geometric interpretations of some invariants of the representation associated with $\mathcal{O}$. This will allow us, in turn, to offer physical interpretations of those invariants in terms of action integrals and Berry phases along curves generated in physical systems by the action of symmetry groups. This is valid for groups relevant in Physics, such as: $SO(p,q)$, $Sp(2n)$, $SL(n,\mathbb{R})$, etc.

For the construction of a representation of $G$ from the orbit we will assume that $\mathcal{O}$ admits an integral datum (see Section 2). By means of an integral datum one defines a unitary irreducible representation $\pi$ of $G$ by induction from a parabolic subgroup of $G$. According to Schur’s lemma, if $g_1$ belongs to the center of $G$, $\pi(g_1)$ is a scalar operator defined by a constant $\kappa$,

$$\pi(g_1) = \kappa \text{Id}.$$  

(1.1)

We will give an interpretation in geometric terms of:

1) The constant $\kappa$.

Key words and phrases. Orbit method, geometric quantization, coadjoint orbits, representation theory.

This work has been partially supported by Ministerio de Educación y Ciencia, grant MAT2007-65097-C02-02.
2) The infinitesimal character $\pi'$ of $\pi$, the differential representation of $\pi$, considered as a representation of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$.

3) Some values of the character $\chi_\tau$ of $\pi$, where $\tau$ is any irreducible representation of any maximal compact subgroup $K$ of $G$, which occurs in $\pi|_K$.

The orbit $O$ is the homogeneous space $G/L$, where $L$ is the stabilizer of $\eta$. An integral datum is a unitary irreducible representation of $L$ on a Hilbert space $H$, satisfying an additional condition (see (2.5)). Given an integral datum $\Lambda$, by $\Phi$ we denote the representation of $L$, tensor product of $\Lambda$ and the character of $L$ on half-densities (2.3).

As $\eta$ is a hyperbolic element its orbit $O$ possesses a real polarization defined by a subalgebra $\mathfrak{u}$ of $\mathfrak{g}$ (see (2.2)). $B_1$ will be the space of smooth $\Phi$-equivariant maps $s : G \to H$, with compact support modulo $L$, and such that $L_C(s) = 0$, for $C \in \mathfrak{u}$, where $L_A$ is the left invariant vector field on $G$ determined by $A \in \mathfrak{g}$. The representation $\pi$ is the left regular representation of $G$ defined on the completion of the pre-Hilbert space $B_1$. Thus the operator associated to $A \in \mathfrak{g}$ in the differential representation $\pi'$ of $\pi$ is $-RA$, the opposite of the right invariant vector field on $G$ defined by $A$.

As a first step we will define the representation $\pi'$ in the context of fibre bundles. We will consider the $GL(H)$-principal bundle $\mathcal{F} := G \times _\Phi GL(H)$ over $\mathcal{O}$, defined by means of the representation $\Phi$ of $L$. On $\mathcal{F}$ there is a natural left $G$-action and an obvious $C^*$-action induced by the multiplication by nonzero scalars of elements of $GL(H)$. In particular each $A \in \mathfrak{g}$ defines a vector field $Y_A$ on $\mathcal{F}$ by the $G$-action. Furthermore, on $\mathcal{F}$ one can define a $G$-invariant connection in a natural way, whose curvature is denoted by $K$. The $G$-action on $\mathcal{F}$ has a moment map $\mu : \mathcal{F} \to \mathfrak{g}(H) \otimes \mathfrak{g}^*$, relative to the 2-form $K$; that is, $D(\mu, A) = -K(Y_A, \cdot)$, where $D$ is the covariant derivative. Moreover $\langle \mu, A \rangle = \mathfrak{h}_A$ induces a map $h_A$ from $\mathcal{O}$ to $\mathfrak{g}(H)$.

We will denote by $\mathcal{V}$ the vector bundle with fibre $H$ associated to $\mathcal{F}$. We write $B_2$ for the space of smooth sections $\sigma$ of $\mathcal{V}$ which can be identified with the elements of $B_1$, and we put $B_3$ for the space of maps $f : \mathcal{F} \to H$ associated with the sections of $\mathcal{V}$ that belong to $B_2$.

Given $A \in \mathfrak{g}$ we will denote by $X_A$ the vector field on $\mathcal{O}$ defined by the coadjoint action of $G$. On the space $B_2$ we consider the following operator

$$\mathcal{P}_A := -D_{X_A} + h_A.$$ 

In Section 3 we prove the following theorem, that gives the representation $\pi'$ on the spaces $B_i$, for $i = 2, 3$.

**Theorem 1.** The representations of $\mathfrak{g}$

$$A \in \mathfrak{g} \mapsto \mathcal{P}_A \in \text{End}(B_2)$$

and

$$A \in \mathfrak{g} \mapsto -Y_A \in \text{End}(B_3)$$

are equivalent to $\pi'$, the differential representation of $\pi$.

Theorem 1 gives the representation $\pi'$ on geometric objects. To determine a geometric description of $\pi(g_1)$ we will “integrate” $\pi'$ along a curve in $G$ with final point at $g_1$. To abbreviate, the smooth curves in $G$ with initial point at $e$ will be called *paths* in $G$. Let $\{g_t\}$ be a path in $G$ with $g_1$ in $Z(G)$, the center of $G$. This curve determines its velocity curve; that is, the family $\{A_t\} \subset \mathfrak{g}$ given by the
relation $\dot{g}_t g_t^{-1} = A_t$. The corresponding time-dependent vector field $Y_{A_t}$ defines a Hamiltonian flow $F_t$ on $\mathcal{F}$. We will prove that the time-1 map of this flow is precisely the multiplication by $\kappa$ in $\mathcal{F}$ (see item (d) of Theorem 2).

Given $s \in \mathcal{B}_1$, we define a family of maps $\{s_t : G \to H\}_t$ by the equations

$$\frac{ds_t}{dt} = -R_{A_t} s_t, \quad s_0 = s.$$  

(1.2)

Given a section $\sigma \in \mathcal{B}_2$ we consider the family of sections $\sigma_t$ of $V$ determined by the following equations

$$\frac{d\sigma_t}{dt} = P_{A_t} \sigma_t, \quad \sigma_0 = \sigma.$$  

(1.3)

Similarly, given $f \in \mathcal{B}_3$, let $\{f_t\}$ be the set of maps $f_t : \mathcal{F} \to H$ such that

$$\frac{df_t}{dt}(p) = -Y_{A_t}(p)(f_t), \quad f_0 = f.$$  

(1.4)

The following theorem relates the constant $\kappa$ with the solutions of the “evolution” equations (1.2), (1.3), (1.4) and with the time-1 map $F_1$.

**Theorem 2.** Let $g_t$ be an arbitrary path on $G$ with $g_1 \in Z(G)$, and $A_t$ the corresponding velocity curve. If $\kappa$ is the constant given by (1.1), the following statements hold

(a) If $s_t$ is the solution of (1.2), then $s_1 = \kappa s$.

(b) If $\sigma_t$ is the solution to (1.3), then $\sigma_1 = \kappa \sigma$.

(c) If $f_t$ is the solution to (1.4), then $f_1 = \kappa f$.

(d) $F_1$ is the multiplication by $\kappa$; that is, $F_1[g, \alpha] = \kappa[g, \alpha]$.

For the above path $\{g_t\}$ with endpoint at $g_1 \in Z(G)$, we denote by $\psi_t$ the closed isotopy on $O$ determined by the time-dependent vector field $X_{A_t}$, that is,

$$\frac{d\psi_t}{dt} = X_{A_t} \circ \psi_t, \quad \psi_0 = \text{id}.$$  

(1.5)

When $\dim H = 1$ the curvature $K$ projects a 2-form $-\omega$ on the orbit $O$, and $F_1$ is the action integral ([15], [9]) around $\psi_t$ (see Section 5). This fact is the statement of the following theorem, which will be proved in Section 4.

**Theorem 3.** If $\dim H = 1$

$$\kappa = \exp \left( \int_S \omega + \int_0^1 h_{A_t}(q_t) dt \right),$$

where $q$ is an arbitrary point of $O$ and $S$ any 2-chain in $O$ whose boundary is the curve $\{q_t := \psi_t(q)\}_t$.

Let $K$ be a maximal compact subgroup of $G$, and let $\tilde{g}$ an element of $K$, such that its conjugacy class meets $L_0$, the connected component of the identity of $L$; that is, there exists $a \in G$ such that $a^{-1} \tilde{g} a \in L_0$. Let $g_t$ be a path in $G$ with $g_1 = \tilde{g}$ and $a^{-1} g_t a \in L$, $A_t$ the corresponding velocity path and $h_{A_t}$ the map on $O$ induced by $\langle \mu, A_t \rangle$. Let $\tau$ be an irreducible representation of $K$ which occurs in the representation $\pi|_K$ with nonzero multiplicity. We will prove the following theorem, that gives an expression for the value of character $\chi_\tau$ at $\tilde{g}$ in terms of the “Hamiltonian” functions $h_{A_t}$.
Theorem 4. Let \( \tilde{g} \) be an element of \( K \), such that there exists \( a \in G \) with \( a^{-1}ga \) in \( L_0 \). Let \( C_t \) be the velocity curve of a path in \( L \) with endpoint at \( a^{-1}ga \). If \( \dim \sigma = m \) and \( \dim H = 1 \), then
\[
\chi_{\sigma}(\tilde{g}) = m \exp \left( \int_{t_0}^1 h_{A_t}(x_0)dt \right),
\]
where \( A_t = \text{Ad}_a C_t \) and \( x_0 = a \cdot \eta \in \mathcal{O} \).

An important invariant of the representation \( \pi' \) is its infinitesimal character \( \chi \) defined on \( \mathcal{Z}(\mathfrak{g}_C) \), the center of the universal enveloping algebra \( U(\mathfrak{g}_C) \) of \( \mathfrak{g}_C = \mathfrak{g} \otimes \mathbb{C} \). If \( \mathfrak{h} \) is a Cartan subalgebra of \( \mathfrak{g}_C \) contained in \( \mathfrak{h}_C \) and \( Z \in \mathcal{Z}(\mathfrak{g}_C) \), we will denote by \( \tilde{Z} \) the projection of \( Z \) into \( U(\mathfrak{h}) \). Let \( Y_1, \ldots, Y_r \) be a basis of \( \mathfrak{h} \). For each \( Y_i \) we define the map \( h_i : \mathcal{F} \to \mathfrak{gl}(H) \), by \( h_i = \langle \mu, Y_i \rangle \); that is, \( h_i \) is the Hamiltonian function associated with \( Y_i \). If \( q(Y_1, \ldots, Y_r) \) is a polynomial
\[
q(Y_1, \ldots, Y_r) = a + \sum_k a_k Y_k + \sum_{i,j} a_{ij} Y_i Y_j + \sum_{k,i,j} a_{kij} Y_k Y_i Y_j + \ldots \quad \text{(finite sum)}
\]
such that \( \tilde{Z} = q(Y_1, \ldots, Y_r) \in U(\mathfrak{h}) \), a geometric interpretation of \( \chi(Z) \) is given in the following theorem.

Theorem 5. If \( Z \) is an element of \( \mathcal{Z}(\mathfrak{g}_C) \) such that \( \tilde{Z} \) is defined by the polynomial \( q(Y_1, \ldots, Y_r) \). Then
\[
q(h_1, \ldots, h_r) : \mathcal{F} \to \mathfrak{gl}(H)
\]
is a constant map on the fiber over \( \eta \), and its value on this fiber is \( \chi(Z)\text{Id} \).

This article is organized as follows. In Section 2 we introduce the definitions and notations which will be used. Following Vogan ([13], [14]) we define the representation \( \pi \) associated to the orbit \( \mathcal{O} \) of a hyperbolic element.

In Section 3 we describe the differential representation \( \pi' \) on the spaces \( B_i \), for \( i = 2, 3 \), proving Theorem 1.

In Section 4 we give geometric interpretations of \( \pi(g_1) \), for \( g_1 \in Z(G) \). We will prove Theorem 2 and Theorem 3. In Subsection 4.2 we prove Theorem 4 about the character of \( \pi \). Subsection 4.3 concerns with the geometric interpretation of the infinitesimal character of \( \pi' \); in this subsection we will prove Theorem 5.

Section 5 provides an interpretation of Theorem 3 in terms of physical concepts. We will show that the invariant \( \kappa \) can be considered as the exponential of the action integral around the closed curve \( \psi_1 \), and also as the Berry phase of a loop of Lagrangian submanifolds of \( \mathcal{O} \). In a worked example we will consider a hyperbolic orbit of the restricted Lorentz group \( SO^+(1, 3) \). Using Theorem 5 we will calculate the value of the corresponding infinitesimal character on the Casimir element \( C \), and we will interpret this value in terms of the “quantum” operator that Geometric Quantization associates with \( C \).

2. Definitions and notations.

Here we review the construction of the representation associated to the coadjoint orbit of a hyperbolic element (see [13], [14] for details).

By \( G \) we denote a reductive group. As definition of reductive group we adopt the one given by Vogan in [14]. We recall this definition. A linear group is reductive if it has finitely many connected components and is preserved by the Cartan involution. A reductive group \( G \) is a Lie group endowed with a homomorphism
from $G$ onto $G_1$ of finite kernel, $G_1$ being a linear reductive group. In particular,
$GL(n, \mathbb{R})$, $SL(n, \mathbb{R})$, $SO(p, q)$, $O(p, q)$, $Sp(2n)$ and all compact Lie groups are re-
ductive groups. If $g \in G$ and $A \in \mathfrak{g}$, we put $g \cdot A = \text{Ad}_g(A)$, and if $\xi \in \mathfrak{g}^*$ we write
$g \cdot \xi$ for $\text{Ad}_g^* (\xi)$.

Let $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$, with $\mathfrak{t}$ the Lie algebra of a maximal
compact subgroup $K$ of $G$.

Given $\eta \in \mathfrak{g}^*$, its stabilizer for the coadjoint adjoint of $G$ will be denoted by $L$. On the other hand, $\eta$
determine an element $X_0 \in \mathfrak{g}$ by the equality
\[(2.1) \quad \eta(Y) = \text{Re } \text{Tr} (X_0 Y), \quad \text{for all } Y \in \mathfrak{g},\]
where $\mathfrak{g}$ is identified with a Lie algebra of matrices.

Let us assume that $\eta$ is a hyperbolic element of $\mathfrak{g}^*$. We can suppose that $X_0 \in \mathfrak{p}$, after replacing
$\eta$ by an $\text{Ad}^*(G)$-equivariant element. As $\text{ad}(X_0)$ is a diagonalizable endomorphism of $\mathfrak{g}$ with real eigenvalues
\[
\mathfrak{g} = \bigoplus_{r \in \mathbb{R}} \mathfrak{g}_r, \quad \mathfrak{g}_r = \{ Y \in \mathfrak{g} \mid \text{ad}(X_0)(Y) = rY \},
\]
and $\mathfrak{g}_0$ is the Lie algebra $l$ of the subgroup $L$. Moreover the adjoint action of $l \in L$
preserves each $\mathfrak{g}_r$.

We put
\[(2.2) \quad u = \bigoplus_{r > 0} \mathfrak{g}_r, \quad u^- = \bigoplus_{r < 0} \mathfrak{g}_r, \quad U = \exp u.
\]

Then $U$ is a simply connected nilpotent subgroup of $G$ and $L$ normalizes $U$. So $Q := LU$ is a Levi decomposition of the subgroup $Q$.

We define the following positive character on $Q$
\[(2.3) \quad \Delta(q) = | \det (\text{Ad}(q)|_u) |^{1/2}.
\]
The derivative of $\Delta$ will be denoted by $\delta$. Since $[\mathfrak{g}_r, \mathfrak{g}_s] \subseteq \mathfrak{g}_{r+s}$, $\delta(B) = 0$ for all $B \in \mathfrak{u}$. On the other hand $\Delta(l_1 l^{-1}) = \Delta(l_1)$, for all $l, l_1 \in L$, so $\delta(l \cdot A) = \delta(A)$, for $l \in L$ and $A \in l$. We extend $\delta$ to a linear map on $\mathfrak{g}$ by setting $\delta|_{\mathfrak{u}^-} = 0$. As the
action of $L$ preserves the $\mathfrak{g}_r$ then
\[(2.4) \quad \delta(l \cdot Y) = \delta(Y), \quad \text{for all } l \in L \text{ and } Y \in \mathfrak{g}.
\]

Let $\Lambda$ be an integral datum at $\eta$ [13]; that is, $\Lambda$ is an irreducible unitary repre-
sentation of $L$ in a Hilbert space $H$, such that
\[(2.5) \quad \Lambda(\exp A) = e^{i\eta(A)} \text{Id}, \quad \text{for all } A \in l.
\]
So $\Lambda(l \exp(A) l^{-1}) = \Lambda(\exp A)$. We extend $\Lambda$ to $Q$ by $\Lambda(lu) = \Lambda(l)$, and write $\lambda$
for the derivative of $\Lambda$. In turn, $\lambda$ can be extended to a linear map on $\mathfrak{g}$ by putting $\lambda|_{\mathfrak{u}^-} = 0$. As in the preceding case
\[(2.6) \quad \lambda(l \cdot Y) = \lambda(Y), \quad \text{for all } l \in L \text{ and } Y \in \mathfrak{g}.
\]

$\text{GL}(H)$ will denote the group of continuous linear automorphisms of $H$. We put
$\Phi : Q \rightarrow \text{GL}(H)$ for the representation of $Q$ tensor product $\Lambda \otimes \Delta$. By $\phi$ we denote the linear map
\[(2.7) \quad \phi : A \in \mathfrak{g} \mapsto \lambda(A) + \delta(A) \text{Id} \in \mathfrak{gl}(H).
\]
From (2.4) and (2.6) it follows
\[(2.8) \quad l \cdot \phi = \phi.
\]
By \( \pi \) we denote the representation \( \text{Ind}^G_O(\Phi) \); that is, the irreducible unitary representation of \( G \) induced by \( \Phi \). The space of \( C^\infty \) vectors of \( \pi \) is the space of smooth functions \( s : G \to H \), with compact support modulo \( L \), such that
\[
(2.9) \quad s(gl) = \Phi(l^{-1})s(g), \quad \text{for all } g \in G, \ l \in L; \quad \text{and } L_As = 0, \ \text{for all } A \in u,
\]
where \( L_A \) is the left invariant vector field on \( G \) defined by \( A \). The representation \( \pi \) on this space is given by
\[
(2.10) \quad \pi(g)(s) = s \circ L_g^{-1},
\]
\( L_g^{-1} \) being the left multiplication in \( G \) by \( g \). Therefore the differential representation \( \pi' \) of \( \pi \) on the smooth function \( s \), with compact support modulo \( L \), which satisfy (2.9) is given by
\[
(2.11) \quad \pi'(C)(s) = -R_C(s),
\]
\( R_C \) being the right invariant vector filed on \( G \) defined by \( C \in g^* \).

3. THE DIFFERENTIAL REPRESENTATION

Henceforth \( G \) will be a reductive group, \( \eta \) a hyperbolic element of \( g^* \) and \( \Lambda \) an integral datum at \( \eta \). The coadjoint orbit of \( \eta \) will be denoted by \( O \). If \( B \in \mathfrak{g}^* \), \( X_B(g,\eta) \) will be the tangent vector to \( O \) at \( g,\eta \) defined by the curve \( \{\exp(tB)\cdot(g,\eta)\}_t \).

By (2.8) the map
\[
(3.1) \quad h_B : g \in G \mapsto \phi(g^{-1} \cdot B) \in \mathfrak{g}(H)
\]
induces a mapping on \( O \), that will be also denoted by \( h_B \). For any \( A, B \in \mathfrak{g}^* \)
\[
(3.2) \quad X_A(h_B) = -h_{[A,B]}.
\]

Next we define the following \( GL(H) \)-principal bundle over \( O \simeq G/L \).
\[
\mathcal{F} = G \times_L GL(H) = \{(g, \alpha) \mid g \in G, \ \alpha \in GL(H)\}/\sim,
\]
where \( (g, \alpha) \sim (gl, \Phi(l^{-1})\alpha) \), with \( l \in L \).

On \( \mathcal{F} \) there is a natural left \( G \)-action \( \mathcal{L} \), so each \( B \in \mathfrak{g}^* \) determines a vector field \( Y_B \) on \( \mathcal{F} \) and
\[
(\mathcal{L}_g)_*(Y_B) = Y_{g \cdot B}.
\]

On the other hand, each \( y \in \mathfrak{g}(H) \) determines a vertical vector field \( W_y \) on \( \mathcal{F} \) by means of right \( GL(H) \)-action \( \mathcal{R} \).

Given \( A \in \mathfrak{l} \), the trivial curve \( \{[g \cdot e^{tA}, \Phi(e^{-tA})\alpha]\}_t \) in \( \mathcal{F} \) defines the vector
\[
Y_{g,A}[g, \alpha] = W_y[g, \alpha], \quad \text{with } y = \text{Ad}\alpha^{-1}(\phi(A)), \quad \text{where Ad denotes the adjoint action of } GL(H) \text{ on } \mathfrak{g}(H)
\]
So the tangent space to \( \mathcal{F} \) at \( [g, \alpha] \) is
\[
(3.3) \quad T_{[g,\alpha]}\mathcal{F} = \{Y_B[g, \alpha] \mid B \in \mathfrak{g} \} \oplus \{W_y[g, \alpha] \mid y \in \mathfrak{g}(H)\}/\{Y_{g,A}[g, \alpha] - W_{\text{Ad}\alpha^{-1}(\phi(A))[g, \alpha] \mid A \in \mathfrak{l}]\}
\]

On \( \mathcal{F} \) we define the following \( \mathfrak{g}(H) \)-valued 1-form
\[
(3.4) \quad \Omega(Y_B[g, \alpha] + W_y[g, \alpha]) = \text{Ad}\alpha^{-1}(\phi(g^{-1} \cdot B)) + y.
\]

\( \Omega \) is, in fact, well-defined on the quotient \( \mathcal{L}_g \), and it is easy to check that
\[
\mathcal{L}_g^*\Omega = \Omega \quad \text{and} \quad \mathcal{R}_g^*\Omega = \text{Ad}_{g^{-1}} \circ \Omega,
\]
for \( g \in G \) and \( \alpha \in GL(H) \). That is, one has the proposition

**Proposition 6.** The 1-form \( \Omega \) defined in (3.4) is a \( G \)-invariant connection on the \( GL(H) \)-principal bundle \( \mathcal{F} \).
We can lift \( h_A \) to a well-defined function \( h_A : \mathcal{F} \to \mathfrak{gl}(H) \) by setting
\[
(3.5) \quad h_A[g, \alpha] = \text{Ad}_{\alpha^{-1}}h_A(g).
\]
Then
\[
(3.6) \quad \Omega(Y_B + W_y) = h_B + y
\]

**Lemma 7.** Given \( A \in \mathfrak{g} \) and \( y \in \mathfrak{gl}(H) \), then
\[
Y_A(h_B) = -h_{[A, B]} \quad \text{and} \quad W_y[g, \alpha](h_B) = -[y, h_B([g, \alpha])],
\]
where \([ , ]_{\mathfrak{gl}}\) is the bracket in the Lie algebra \( \mathfrak{gl}(H) \).

**Proof.** From (3.2) it follows
\[
Y_A[g, \alpha](h_B) = \frac{d}{dt} \bigg|_{t=0} h_B[\exp(tA) \cdot g, \alpha] = \text{Ad}_{\alpha^{-1}}(X_A(g)(h_B)) = -h_{[A, B]}[g, \alpha].
\]

The second formula can be directly deduced from
\[
Y_A[g, \alpha](h_B) = \text{Ad}_{\exp(-ty)}h_B[g, \alpha].
\]

Next we will calculate the value of the curvature \( K \) of the connection \( \Omega \) on the pair \((Y_B, Y_C)\) of vector fields.

**Proposition 8.** The curvature \( K \) of the connection \( \Omega \) satisfies
\[
(3.7) \quad K(Y_B, Y_C) = -h_{[B, C]} + [h_B, h_C]_{\mathfrak{gl}},
\]
for all \( B, C \in \mathfrak{g} \).

**Proof.** By the structure equation
\[
K(Y_B, Y_C) = d\Omega(Y_B, Y_C) + [\Omega(Y_B), \Omega(Y_C)]_{\mathfrak{gl}}.
\]
From Lemma 7 and (3.6) it follows
\[
Y_B[g, \alpha](\Omega(Y_C)) = h_{[C, B]}[g, \alpha].
\]
Similarly \( Y_C(\Omega(Y_B)) = h_{[B, C]} \). Hence \( d\Omega(Y_B, Y_C) = -h_{[B, C]} \) and (3.7) follows.

By \( D \) we denote the covariant derivative determined by \( \Omega \). Since the horizontal component of \( Y_A[g, \alpha] \) is \( Y_A[g, \alpha] + W_y[g, \alpha] \), with \( y = -h_A[g, \alpha] \), by Lemma 7
\[
D h_B(Y_A[g, \alpha]) = -h_{[A, B]} + [h_A, h_B]_{\mathfrak{gl}}.
\]
It follows from (3.7) that
\[
D h_B(Y_A) = -K(Y_B, Y_A), \quad \text{for all } A \in \mathfrak{g}.
\]
Thus we have
\[
(3.8) \quad D h_B = -\iota_{Y_B} K, \quad \text{for all } B \in \mathfrak{g}.
\]

Equation (3.8) can be interpreted saying that the \( \mathfrak{gl}(H) \)-valued function \( h_B \) is a “Hamiltonian” for the vector field \( Y_B \), with respect to the covariantly closed \( \mathfrak{gl}(H) \)-valued 2-form \( K \). That is, we define
\[
\mu : \mathcal{F} \to \mathfrak{gl}(H) \otimes \mathfrak{g}^*,
\]
by \( \langle \mu, A \rangle = h_A \), with \( A \in \mathfrak{g} \). This map is \( G \)-equivariant, that is, \( \langle \mu(gp), A \rangle = \langle \mu(p), g^{-1} \cdot A \rangle \), for all \( p \in \mathcal{F} \) and all \( g \in G \) and \( D(\mu, A) = -\iota_{Y_B} K \). We call \( \mu \) the moment map for the \( G \)-action on \((\mathcal{F}, K)\).
We denote by $\mathcal{V}$ the vector bundle on $\mathcal{O}$ with fibre $H$

$$G \times_L H = \{ (g, v) \mid g \in G, v \in H \},$$

with $(g, v) = (gl, \Phi(l^{-1})v)$. The vector bundle $\mathcal{V}$ can also be considered as associated to $\mathcal{F}$ by the natural representation of $GL(H)$. That is, $\mathcal{V}$ is

$$\{ \{ p, v \} \mid p \in \mathcal{F}, v \in H \},$$

where $\{ p, v \} = \{ p \beta, \beta^{-1}v \}$ for all $\beta \in GL(H)$. The correspondence $(g, v) \mapsto \{ [g, \text{Id}], v \}$ gives the isomorphism between those definitions of $\mathcal{V}$.

Now we consider the following three vector spaces of smooth maps

$$\mathcal{B}'_1 = \{ s : G \to H \mid s(gl) = \Phi(l^{-1})s(g), \forall g \in G, \forall l \in L, \text{supp}(s) \text{ compact modulo } L \},$$

$$\mathcal{B}'_2 = \{ \tau \mid \tau \text{ section of } \mathcal{V}, \text{supp}(\tau) \text{ compact} \},$$

$$\mathcal{B}'_3 = \{ f : \mathcal{F} \to H \mid f(p \beta) = \beta^{-1}f(p), \forall p \in \mathcal{F}, \forall \beta \in GL(H), \text{pr} (\text{supp}(f)) \text{ compact} \},$$

where $\text{pr}$ is the projection $\text{pr} : \mathcal{F} \to G/L$.

Given $s \in \mathcal{B}'_1$, determines a section $\sigma \in \mathcal{B}'_2$ by the relation

$$(3.9) \quad \sigma(gL) = (g, s(g)).$$

Moreover $s$ defines $\sigma^2 \in \mathcal{B}'_3$ by

$$\sigma^2[g, \alpha] = \alpha^{-1}s(g).$$

With the above notations

$$(3.11) \quad \sigma(x) = \{ p, \sigma^2(p) \},$$

for any $p \in \mathcal{F}$ in the fibre of $x \in \mathcal{O}$.

It is well-known that the correspondences $s \mapsto \sigma$ and $s \mapsto \sigma^2$ allow us to identify the $\mathcal{B}'_j$'s. We denote also by $D$ the covariant derivative on sections of $\mathcal{V}$ defined by the connection $\Omega$. It is a known fact that the map of $\mathcal{B}'_3$ associated with the section $D_{X_A} \sigma$ of $\mathcal{V}$ is $X_A^2(\sigma^2)$, where $X_A^2$ is the horizontal lifting of the vector field $X_A$.

We set

$$(3.12) \quad \mathcal{B}_1 = \{ s \in \mathcal{B}'_1 \mid L_A s = 0, \forall A \in u \}.$$

Since $Q$ is connected the condition defining $\mathcal{B}_1$ is equivalent to $s(qg) = \Phi(q^{-1})s(g)$ for all $q \in Q$ and all $g \in G$. In order to interpret $\mathcal{B}_1$ in terms of sections of $\mathcal{V}$ and equivariant functions on $\mathcal{F}$ we need an additional fibre bundle.

$\mathcal{F}_Q$ is the $GL(H)$-principal fibre bundle over $G/Q$ defined by $\Phi$; that is,

$$\mathcal{F}_Q := G \times_Q GL(H).$$

One has a natural fibre map $\Xi : \mathcal{F} \to \mathcal{F}_Q$ over the canonical projection

$$(3.13) \quad G/L \to G/Q.$$

We put

$$(3.14) \quad \mathcal{B}_3 = \{ f \in \mathcal{B}'_3 \mid f \text{ factors through } \Xi \}.$$

Analogously we defines $\mathcal{V}_Q := G \times_Q H$, and the natural fibre map $\Xi : \mathcal{V} \to \mathcal{V}_Q$ will be also denoted by $\Xi$. We set

$$(3.15) \quad \mathcal{B}_2 := \{ \tau \in \mathcal{B}'_2 \mid \Xi \circ \sigma \text{ is constant along the fibers of } \mathcal{F}_Q \}.$$
Proposition 9. The correspondences $s \mapsto \sigma$ and $s \mapsto \sigma^{\sharp}$ define bijective maps between the $B_i$'s.

Proof of Theorem 1. From (3.10) one deduces

\[ Y_A[g, 1](\sigma^{\sharp}) = R_A(g)(s), \quad \text{and} \quad W_y[g, 1](\sigma^{\sharp}) = -y s(g). \]

On the other hand, by (3.3)

\[ X_B^{\sharp}[g, \alpha] = Y_B[g, \alpha] + W_y[g, \alpha], \]

with $y + h_B[g, \alpha] = 0$.

From (3.10) together with (3.17) and (3.10) it follows that the $\Phi$-equivariant function on $G$ associated with $D_X A \sigma$ is $R_A(s)$ and $h_2(s)$. So the section $-D_X A \sigma + h_2(s)$ of $\mathcal{V}$ has as associated $\Phi$-equivariant function to $-R_A(s)$. Therefore if we put

\[ P_A(s) := -D_X A \sigma + h_2(s), \]

then the family $\{P_A\}$ of endomorphisms is a representation of $g$ on $B_2$ equivalent to $\pi'_{\Phi}$ defined in (2.11).

From (3.10) it follows

\[ Y_A[g, \alpha](\sigma^{\sharp}) = \alpha s^{-1} R_A(g)(s). \]

That is, $Y_A \sigma^{\sharp}$ is the function of $B_3$ associated to $R_A(s) \in B_1$. Hence the algebra representation $\pi'$ defined in (2.11) is equivalent to the representation of $g$ on the space $B_3$ given by the operators $\{-Y_A\}$.

\[ \square \]

4. Schur’s Lemma

Let $\{B_t\}_{t \in [0, 1]}$ be a family of elements in $g$. This family generates time-dependent vector fields on $G$, $O$ and $F$, which give rise to evolution equations for several sorts of objects. In Propositions 10, 11 and 12 we state properties of the solutions to these equations that we use later.

By $\varphi_t$ we denote the isotopy on $O$ determined by the time-dependent vector field $X_{B_t}$; that is,

\[ \frac{d\varphi_t}{dt} = X_{B_t} \circ \varphi_t, \quad \varphi_0 = \text{id}. \]

On the other hand the time-dependent vector field $Y_{B_t}$ on $F$ defines a flow $H_t$; that is, the family of diffeomorphisms of $F$ determined by

\[ \frac{dH_t(p)}{dt} = Y_{B_t}(H_t(p)), \quad H_0 = \text{Id}. \]

Given $s \in B_1$, we define a family of maps $\{s_t : G \to H\}_t$ by the equations

\[ \frac{ds_t}{dt} = -R_{B_t}s_t, \quad s_0 = s. \]

Proposition 10. If the family $\{s_t\}$ is solution of (4.3), then $s_t \in B_1$ for all $t$.

Proof.

\[ \frac{d}{dt}(L_Z s_t) = L_Z \frac{ds_t}{dt} = -L_Z R_{B_t}(s_t) = -R_{B_t} L_Z (s_t). \]

If $Z \in \mathfrak{u}$, then $L_Z s_0 = 0$. The uniqueness of solutions of the first order differential equation (4.3) implies $L_Z s_t = 0$, for all $t$. 
On the other hand, given \( l \in L \), on the space of smooth maps \( h : G \to H \) we define the operator \( \alpha \) by

\[
\alpha(h) = h \circ R_l - \Phi(l^{-1})h,
\]

\( R_l \) being the right multiplication in \( G \) by \( l \). If \( A \in g \), it is straightforward to check (4.5)

\[
\alpha R_A = R_A \alpha.
\]

If \( s_t \) is solution of (4.3), it follows from (4.5)

\[
\frac{d}{dt} \alpha(s_t) = \frac{d}{dt} R_A \alpha = R_A \alpha(s_t) = -R_B s_t.
\]

As \( \alpha(s) = 0 \) since \( s \in B_1 \), so we conclude \( \alpha s_t = 0 \) for all \( t \). Thus \( s_t \in B_1 \).

\[
\square
\]

Given a section \( \sigma \in B_2 \) we can consider the family of sections \( \sigma_t \) of \( V \) defined by the following equations

\[
\frac{d}{dt} \sigma_t = P_B \sigma_t, \quad \sigma_0 = \sigma.
\]

Similarly, given \( f \in B_3 \), let \( f_t \) be the set of functions \( f_t : F \to H \) such that

\[
\frac{d}{dt} f_t(p) = -Y_B(p)(f_t), \quad f_0 = f.
\]

By Theorem 1 together with the preceding Proposition one has

**Proposition 11.** Let \( \sigma_t \) be the solution of (4.6) and \( f_t \) the solution of (4.7). If \( \sigma_t \) and \( f_t \) are associated with \( s \in B_1 \), then \( \sigma_t \) and \( f_t \) are associated with \( s_t \), solution of (4.3). In particular \( \sigma_t \in B_2 \) and \( f_t \in B_3 \).

Given a \( f \in B_3 \), we put

\[
\hat{f}_t = f \circ H_t^{-1}.
\]

We have the following proposition.

**Proposition 12.** The set of functions \( \hat{f}_t \) defined by (4.8) satisfies

\[
\frac{d}{dt} \hat{f}_t(p) = -Y_B(p)(\hat{f}_t), \quad \hat{f}_0 = f.
\]

Hence \( \hat{f}_t \) is the solution of (4.7).

**Proof.** By (4.2)

\[
\frac{d}{dt} \bigg|_{u=t} H_u(H_t^{-1}(p)) = Y_B(p);
\]

that is, \( Y_B(p) \) is the vector defined by the curve \( \{H_u(H_t^{-1}(p))\}_u \) at \( u = t \). On the other hand, by (4.3)

\[
\frac{d}{dt} \hat{f}_t(p) = W(f),
\]

where \( W \) is the tangent vector to \( F \) at \( H_t^{-1}(p) \) defined by the curve \( \{H_u^{-1}(p)\}_u \). Since \( H_u(H_u^{-1}(p)) = p \) for all \( u \), it turns out that \( Y_B(p) = -(H_t)_*(W) \). So

\[
-Y_B(p)(\hat{f}_t) = W(\hat{f}_t \circ H_t) = W(f) = \frac{d}{dt} \hat{f}_t(p).
\]

\[
\square
\]
If we integrate the family \( \{ B_t \} \) we will obtain the solutions of differential equations (4.1) and (4.2). That is, we define the curve \( b_t \) in \( G \) by the conditions
\[
(4.10) \quad b_t b_t^{-1} = B_t, \quad b_0 = e.
\]
Then the isotopy \( \varphi_t \) determined by (4.1) is the multiplication by \( b_t \); that is,
\[
(4.11) \quad \varphi_t(g \cdot \eta) = b_t \cdot (g \cdot \eta).
\]
Analogously, the bundle diffeomorphism \( H_t \) defined in (4.2) is the left multiplication by \( b_t \) in \( F \),
\[
(4.12) \quad H_t = L b_t.
\]

Proposition 13. The solution of (4.3) is
\[
(4.13) \quad \Phi(g_1) = \kappa \text{Id},
\]
that is, \( s \circ L b_t^{-1} \) satisfies (4.3). □

Proof of Theorem 2. Now \( g_1 \in Z(G) \) and the isotopy \( \psi_t \) defined in (1.5) is closed; that is, \( \psi_1 = \text{Id} \). By Proposition 13 (2.10) and (1.1) one has
\[
(4.14) \quad s_1 = s \circ L g_1^{-1} = \pi_1(g_1)(s) = \kappa s,
\]
for any \( s \in B_1 \), which proves item (a).

By (3.9) and Proposition 11 the result stated in (a) expressed in terms of the solutions to (1.3) gives (b).

Moreover
\[
\kappa s(g) = s_1(g) = s(g_1^{-1} g) = s(gg_1^{-1}) = \Phi(g_1) s(g);
\]
that is,
\[
(4.15) \quad \Phi(g_1) = \kappa \text{Id}.
\]

It follows from (4.12) and (4.13) that
\[
(4.16) \quad F_1[g, \alpha] = [g_1 g, \alpha] = [gg_1, \alpha] = [g, \Phi(g_1) \alpha] = \kappa [g, \alpha],
\]
and (d) is proved.

From Proposition 12 together with (4.12) and (4.14), it follows
\[
f_1[g, \alpha] = (f \circ F_1^{-1})[g, \alpha] = f[g, \kappa^{-1} \alpha] = \kappa f[g, \alpha],
\]
which proves (c). □
Let us assume that there is a fixed point $x_0 \in \mathcal{O}$ for the isotopy $\{\psi_t\}$ defined in (1.5) that is, $\psi_t(x_0) = x_0$, for all $t$. So $X_{A_t}(x_0) = 0$ and (1.3) evaluated at $x_0$ reduces to
\[
\frac{d\sigma_t(x_0)}{dt} = h_{A_t}(x_0)\sigma_t(x_0), \quad \sigma_0(x_0) = \sigma(x_0).
\]
This is a differential linear equation for $\nu_t := \sigma_t(x_0) \in H$. Let $M(t) \in \mathfrak{gl}(H)$ be the “fundamental matrix” of this linear equation, in other words
\[
\frac{dM(t)}{dt} = h_{A_t}(x_0)M(t), \quad M(0) = \text{Id}.
\]
By Theorem 2 it follows
\[
M(1) = \kappa \text{Id}.
\]
**Corollary to Theorem 2.** If $A_t = A$ for all $t$, and $x_0$ is a fixed point of the isotopy $\{\psi_t\}_t$, then
\[
\kappa \text{Id} = \exp(h_A(x_0)).
\]

4.1. **Case when** $\dim H = 1$. Now the bracket and the adjoint action in $\mathfrak{gl}(H)$ are trivial. It follows from (3.7)
\[
K_{[g,\alpha]}(Y_B, Y_C) = -\phi(g^{-1}[B, C]) = -\eta_{g[B, C]}(g),
\]
and $K$ projects a closed 2-form $K$ on $\mathcal{O}$. We denote by $\omega := -K$; that is,
\[
\omega(X_{A_t}, X_B)(g \cdot \eta) = h_{[A_t, B]}(g).
\]
In this case (3.8) reduces to
\[
dh_B = \iota_{X_B}\omega.
\]
In this context $\psi_t$ defined in (1.5) is an isotopy which determines the time-dependent Hamiltonian $h_{A_t}$ through the form $\omega$ [9].

**Proof of Theorem 3.** Let $\mu$ be a local frame for the line bundle $\mathcal{V}$. The solution $\sigma_t$ to (1.3) can be written $\sigma_t = m_t \mu$, where $m_t$ is a complex function defined on an open set of $\mathcal{O}$. Then (1.3) gives rise to
\[
\frac{d m_t}{dt} = -\gamma(X_{A_t})m_t - X_{A_t}(m_t) + h_{A_t}m_t,
\]
where $\gamma$ is the connection form of $\mathcal{V}$ in the frame $\mu$. Given an arbitrary point $q$ of $\mathcal{O}$, then $\{q_t := \psi_t(q)\}_t$ is a closed curve on $\mathcal{O}$. If $q$ belongs to the domain of $\mu$, we define $m_t := m_t(q_t)$. If we evaluate (4.18) at the point $q_t$ we obtain
\[
\frac{d m_t'}{dt} = -\gamma_{q_t}(X_{A_t}) + h_{A_t}(q_t))m_t'.
\]
So
\[
m_t' = m_0' \exp\left(\int_0^t (-\gamma_{q_u}(X_{A_{u}}) + h_{A_{u}}(q_u))du\right).
\]
On the other hand, we can consider on $\mathcal{O}$ the Kirillov symplectic structure [2], then $\psi_t$ is a Hamiltonian isotopy with respect to this structure, and consequently the evaluation closed curve $\{q_t\}$ is nullhomologous (Lemma 10.31 in [9]), that is, it is the boundary of a 2-chain. By Stokes’ theorem
\[
m_t' = m_0' \exp\left(\int_0^1 h_{A_t}(q_t)dt\right),
\]
where $S$ a 2-chain whose boundary is the curve is $\{q_t\}_t$. By Theorem 2

\begin{equation}
\kappa = \exp\left(\int_S \omega + \int_0^1 h_{A_t}(q_t) dt\right).
\end{equation}

\[\Box\]

Remarks. If $q$ is a fixed point for $\psi_t$ and $A_t = A$ for all $t$, then from Theorem 3 it follows $\kappa = \exp(h(0))$; this agrees with Corollary to Theorem 2.

By (1.1) the exponential in the statement of Theorem 3 depends only on the final point of the curve $g_t$ and it is independent of the family $A_t$ defined by $g_t$.

4.2. The character. A slight modification of the preceding developments allows us to prove the formula for the character given in Theorem 4.

Proof of Theorem 4. Let $c_t$ denote a path in $L$ with $c_1 = a^{-1} \tilde{g} a$. Then $g_t := ac_t a^{-1}$ is a path in $G$ with $g_1 = \tilde{g}$. The point $x_0 = a \cdot \eta \in O$ is a fixed point for the isotopy on $O$ defined by multiplication by $g_t$. We put $A_t \in g$ and $C_t \in l$ for the velocity paths associated with $g_t$ and $c_t$, respectively. So $A_t = \text{Ad}_a C_t$.

We denote by $V_2$ the subspace of $B_2$ on which $\tau$ is defined. The action of $\tau(\tilde{g})$ on a section $\sigma \in V_2$ is the section $\sigma(1)$ determined by the equations

\begin{equation}
\frac{d\sigma(t)}{dt} = P_{A_t}(\sigma(t)), \quad \sigma(0) = \sigma.
\end{equation}

Evaluating these equations at the point $x_0$, and taking into account that $X_{A_t}(x_0) = 0$, one obtains

\begin{equation}
\frac{d\sigma(t)(x_0)}{dt} = h_{A_t}(x_0)\left(\sigma(t)(x_0)\right), \quad \sigma(0)(x_0) = \sigma(x_0).
\end{equation}

As $\text{dim} \ H = 1$, it follows from (4.20) that

\begin{equation}
\sigma(1)(x_0) = \exp\left(\int_0^1 h_{A_t}(x_0) dt\right) \sigma(x_0),
\end{equation}

for any $\sigma \in V_2$.

If $\text{dim} \ \tau = m$, let $\sigma_1, \ldots, \sigma_m$ be a basis of $V_2$, then

\[\sigma_j(1) = \tau(\tilde{g})\sigma_j = \sum_i M_{ij} \sigma_i,\]

with $M_{ij} \in \mathbb{C}$. By (4.21)

\[M_{ji} = \delta_{ji} \exp\left(\int_0^1 h_{A_t}(x_0) dt\right),\]

and the proof is complete.

Remark. As $h_{A_t}(x_0) = \phi(a^{-1} \cdot A_t) = \phi(C_t)$ the formula for the character can be written

\[\chi_\tau(\tilde{g}) = m \exp\left(\phi\left(\int_0^1 C_t dt\right)\right).\]

If $\tilde{c} = a^{-1} \tilde{g} a$ equals $e^C$, with $C \in l$, then we can take $C_t = C$ for all $t$ and

\begin{equation}
\chi_\tau(\tilde{g}) = m \Phi(\tilde{c}).
\end{equation}

(4.22) can also be deduced by considering $\tau$ as a representation on a subspace $V_1$ of $B_1$. Given $s \in V_1$, \[\tau(\tilde{g})s(a) = s(a \tilde{c}^{-1}) = \Phi(\tilde{c})s(a).\]
From this formula it follows \([122]\).

4.3. The infinitesimal character. Now we consider the “representation” of the associative algebra \(U(\mathfrak{g}_C)\) induced by \(\pi'\) on the space \((B_2)_K\) of \(K\)-finite vectors in \(B_1\). Since \(\pi\) is unitary and irreducible, each element of \(Z(\mathfrak{g}_C)\) acts as a scalar operator (see [4], Corollary 8.14). By \(\chi : Z(\mathfrak{g}_C) \to \mathbb{C}\) we denote the corresponding infinitesimal character.

Let \(\mathfrak{h}\) be a Cartan subalgebra of \(\mathfrak{g}_C\) contained in \(\mathfrak{l}_C\). We denote by \(\Delta^+\) a set of positive roots for the pair \((\mathfrak{g}_C, \mathfrak{h})\). For \(\alpha \in \Delta^+\), \(E_{\alpha}\) will be a basis for the corresponding root space. According to Lemma 8.17 in [4], if \(Z \in Z(\mathfrak{g}_C)\) then \(Z \in U(\mathfrak{h}) \oplus \mathcal{P}\), where
\[
\mathcal{P} = \sum_{\alpha \in \Delta^+} U(\mathfrak{g}_C)E_{\alpha}.
\]
The projection of \(Z\) into \(U(\mathfrak{h})\) will be denoted \(\hat{Z}\).

Let \(V \subset (B_2)_K\) be an irreducible representation of the maximal compact subgroup \(K\) which occurs in \(\pi_K\). Now we consider \(s_0 \in V\) a highest weight vector of the representation \(V\), so \(\pi'(E_{\alpha})s_0 = 0\) for all \(\alpha \in \Delta^+\). Thus, if \(Z \in Z(\mathfrak{g}_C) \cap \mathcal{P}\) then the action of \(Z\) on \(s_0\) vanishes. As \(\pi'(Z)s_0 = \chi(Z)s_0\), it follows \(\chi(Z) = 0\). We have the following proposition

**Proposition 14.** If \(Z \in Z(\mathfrak{g}_C) \cap \mathcal{P}\), then \(\chi(Z) = 0\).

To prove Theorem \([3]\) we need the following Lemma

**Lemma 15.** With \(\phi\) denoting the extension of the map \([2,7]\) to \(\mathfrak{g}_C\) and \(1\) the identity element of \(G\), we have
(i) If \(Y, W \in \mathfrak{h}\), then \(\pi'(Y)s = \phi(Y)s\).
(ii) If \(Y, W \in \mathfrak{h}\), then
\[
(\pi'(Y)\pi'(W)s)(1) = \phi(Y)\phi(W)s(1)
\]

**Proof.** Since \(\mathfrak{h} \subset \mathfrak{l}_C\) we can assume that \(Y, W\) are elements of \(\mathfrak{l}\). The item (i) follows from \([2,11]\) together with the fact that \(s\) is \(\Phi\)-equivariant.

If \(Y, W \in \mathfrak{l}\), then \(s(e^{uW}e^{Y}) = \Phi(e^{-uY})\Phi(e^{-uW})s(1)\). Hence
\[
(R_YR_Ws)(1) = \frac{d}{dt}\bigg|_{t=0} \frac{d}{du}\bigg|_{u=0} \Phi(e^{-uY})\Phi(e^{-uW})s(1) = \phi(Y)\phi(W)s(1),
\]
and (ii) follows. \(\square\)

Given \(\{Y_1, \ldots, Y_r\}\) a basis of \(\mathfrak{h}\), and \(Z \in Z(\mathfrak{g}_C)\), then there exist a polynomial \(q(Y_1, \ldots, Y_r)\) as in \([1,6]\) such that \(\hat{Z} = q(Y_1, \ldots, Y_r)\).

**Proposition 16.** With the above notations
\[
q(\phi(Y_1), \ldots, \phi(Y_r)) = \chi(Z)\text{ Id}.
\]

**Proof.** By Proposition \([14]\) the operator \(\pi'(Z)\) associated to \(Z\) is \(q(-R_{Y_1}, \ldots, -R_{Y_r})\).

By Lemma \([15]\) if \(s \in B_1\) then
\[
(q(-R_{Y_1}, \ldots, -R_{Y_r})s)(1) = q(\phi(Y_1), \ldots, \phi(Y_r))s(1).
\]
As \(q(-R_{Y_1}, \ldots, -R_{Y_r}) = \chi(Z)\text{ Id}\), we obtain the proposition. \(\square\)
Theorem 17. Given $Z \in \mathcal{Z}(\mathfrak{g}_C)$, if $\hat{Z} = q(Y_1, \ldots, Y_r)$ and $h_k := h_{Y_k}$, the Hamiltonian map associated with $Y_k$, then the function

$$q(h_1, \ldots, h_r) : O_\eta \to \mathfrak{g}(H)$$

takes at the point $\eta$ the value $\chi(Z) \text{Id}$.

Proof. It follows from Proposition [13] and (3.1).

Proof of Theorem [5] If $\tilde{q}$ is any polynomial in the variables $Y_1, \ldots, Y_r$, from (3.3) one deduces that

$$\tilde{q}(h_1, \ldots, h_r)[g, \alpha] = \alpha^{-1}(\tilde{q}(h_1, \ldots, h_r)(g \cdot \eta))\alpha.$$  

On the other hand, $q(h_1, \ldots, h_r)(\eta)$ is a multiple of identity, by Theorem [17] From this fact together with (4.23) we deduce

$$q(h_1, \ldots, h_r)[1, \alpha] = q(h_1, \ldots, h_r)(\eta),$$

for any $[1, \alpha] \in \mathcal{F}$. The theorem follows from Theorem [17].

Let $x_0 = g \cdot \eta$ be a point of $\mathcal{O}$ and let $\mathfrak{h}'$ be a Cartan subalgebra of $\mathfrak{g}_C$ such that, $g^{-1} \cdot \mathfrak{h}' \subset \mathfrak{c}$. A generalization of Theorem [5] is the following proposition

Proposition 18. If $Y_{r}'$, $\ldots$, $Y_1'$ is a basis of $\mathfrak{h}'$ and the polynomial $q(Y_1', \ldots, Y_r')$ is the projection of $\tilde{Z} \in \mathcal{Z}(\mathfrak{g}_C)$ into $U(\mathfrak{h}')$, then the function $q(h_{Y_1'}, \ldots, h_{Y_r'})$ is constant on the fiber of $\mathcal{F}$ over $x_0$, and its value on this fiber is $\chi(Z)\text{Id}$.

5. Physical interpretations

As it is well known Geometric Quantization [17] is a mathematical procedure for understanding the relation between a classical physical system and its “quantization”. From the mathematical point of view the classical phase space is a symplectic manifold $(M, \alpha)$, and the set of rays of a Hilbert space $\mathcal{H}$ is the mathematical model for the space of states of the quantum system. The manifold $(M, \alpha)$ is said to be quantizable if the cohomology class of $\alpha/(2\pi)$ is integral. In this case there exists a Hermitian line bundle $\mathcal{L}$ on $M$ equipped with a connection whose curvature is $-i\alpha$. $\mathcal{L}$ is called a “prequantum bundle”. For the construction of the Hilbert space $\mathcal{H}$ from the quantizable manifold $M$ one fixes a polarization $\mathfrak{P}$ on $M$, then $\mathcal{H}$ is a subset of the space of sections of $\mathcal{L}$ polarized with respect to $\mathfrak{P}$ (see [17] and [10] for the details omitted in this schematic summary).

The coadjoint orbit $\mathcal{O}$ of $\eta \in \mathfrak{g}^*$ supports a canonical symplectic structure $\tilde{\omega}$, the Kirillov form [2]. Denoting by $L$ the stabilizer of $\eta$, the orbit $\mathcal{O}$ admits a $G$-invariant prequantization iff the linear operator $i\eta : I = \text{Lie}(L) \to i\mathbb{R}$ is integral, in the sense that there exists a character $\Lambda : L \to U(1)$ whose derivative is $i\eta$ [8]. In this case the corresponding prequantum bundle is $\mathcal{L} = G \times_{\Lambda} \mathbb{C}$. Since the group $G$ acts by translation on the orbit, it is reasonable to impose the quantization to have a $G$-invariant Hilbert space structure. In general it is not possible to integrate the absolute value of sections of $\mathcal{L}$ in a translation-invariant way, since $\mathcal{O} = G/L$ does not admit a measure invariant under the action of $G$ (see p. 537 [5]). To define such an integration it is necessary to consider a prequantum bundle different from $\mathcal{L}$; specifically, one takes the bundle $\mathcal{V}$ determined by the character $\Phi = \Lambda \cdot \Delta$, where $\Delta^2$ is the modular function on $G/L$. If $\sigma_1, \sigma_2$ are compactly supported
sections of $\mathcal{V}$, and $s_1, s_2 : G \to \mathbb{C}$ are the corresponding $\Phi$-equivariant functions, then $m(g) := s_1(g)s_2(g)$ satisfies
\begin{equation}
\tag{5.1}
m(gl) = \Delta^2(l^{-1})m(g), \quad \text{for all } l \in L.
\end{equation}
(That is, $m$ defines a section of the bundle of densities on $G/L$.) The functions on $G$ which satisfies \((5.1)\) have a translation invariant integral “over $G/L$” (see p. 65 [12], p. 41 [6]). Then $\langle \sigma_1, \sigma_2 \rangle := \int_{G/L} m$ defines a $G$-invariant product of compactly supported sections of $\mathcal{V}$.

On the other hand, if $\eta$ is a hyperbolic element then the orbit $\mathcal{O}$ possesses the polarization determined by the subalgebra $\mathfrak{u}$ defined in \((2.2)\). So our space $\mathcal{B}_2$, defined in \((3.16)\), is a $G$-invariant quantization of the orbit. By Proposition \((9)\) the spaces $\mathcal{B}_j$, $j = 1, 2, 3$, can be considered as equivalent $G$-invariant quantizations of $\mathcal{O}$.

The Kirillov symplectic form $\hat{\omega}$ is defined by
\[
\hat{\omega}_{g, \eta}(X_A, X_B) = \eta(g^{-1} : [A, B]),
\]
and the Hamiltonian function associated to $A$ is $\hat{h}_A(g \cdot \eta) = \eta(g^{-1} : A)$. From \((5.1)\), \((4.10)\) and \((2.7)\) one obtains
\[
\omega = i\hat{\omega} + \hat{\omega}, \quad h_A = i\hat{h}_A + \hat{h}_A,
\]
where
\[
\hat{\omega}_{g, \eta}(X_A, X_B) = \delta(g^{-1} : [A, B]), \quad \hat{h}_A(g \cdot \eta) = \delta(g^{-1} : A).
\]
$\hat{\omega}$ is not a symplectic form (because it is degenerate), but the analogous relations to \((1.17)\) with tildes and with hats are also valid.

The cotangent bundle $M = T^*P$ to a manifold carries a canonical 1-form $\beta_0$ [9], and $\alpha_0 := -d\beta_0$ defines a symplectic structure on $M$. If $q : t \in [0, 1] \to M$ is a curve and $h_t : M \to \mathbb{R}$ is a time dependent Hamiltonian on $M$, the action integral along the curve $q(t)$ is defined by the following formula [1], [9]
\[
\int_0^1 ( - \beta_0(\dot{q}(t)) + h_t(q(t)))dt.
\]

For a general symplectic manifold $(M, \alpha)$ the time dependent Hamiltonian $h_t$, with $t \in [0, 1]$, determines a time dependent Hamiltonian vector field $X_t$, which in turn defines an isotopy of symplectomorphisms $\xi_t$. If $\xi_1 = \text{id}$ (that is, $\{\xi_t\}_{t \in [0, 1]}$ is a loop in $\text{Ham}(M)$, the Hamiltonian group of $M$ [9]), then evaluation curve $\{\xi_t(p)\}_t$ is nullhomotopic, for all point $p \in M$ [7]. Hence the action integral around this curve can be written
\begin{equation}
\tag{5.2}
\hat{A}(\xi) := \int_S \alpha + \int_0^1 h_t(\xi_t(p))dt,
\end{equation}
$S$ being a 2-chain whose boundary is the curve $\{\xi_t(p)\}_t$. It is known that the value of \((5.2)\) is independent of the point $p$ [11].

In the case when the manifold is a coadjoint orbit $\mathcal{O}$ and the loop $\{\psi_t\}$ in $\text{Ham}(\mathcal{O})$ is defined as in \((15)\), one can also consider the “action integral” $\hat{A}(\psi)$ defined by means of the 2-form $\hat{\omega}$ and the “Hamiltonian” $\hat{h}_A$,
\[
\hat{A}(\psi) := \int_S \hat{\omega} + \int_0^1 \hat{h}_A(\psi_t(p))dt.
\]
Thus the result stated in Theorem 3 can be written as \( \kappa = \exp(i\hat{A}(\psi)) \times \exp(\hat{A}(\psi)) \).

Since the representation \( \pi \) is unitary, \( \kappa \in U(1) \). So
\[
\kappa = \exp(i\hat{A}(\psi)).
\]

That is, the invariant of the representation \( \pi \) associated to \( g_1 \in Z(G) \), by Schur’s lemma, equals the exponential of \( i \) times the action integral around the loop in \( \Ham(O) \) generated by any path in \( G \) with endpoint at \( g_1 \).

In view of item (d) of Theorem 2 and the above facts, we may consider the flow \( H_t \) (defined in (4.2)) as a generalized action integral along the isotopy \( \varphi_t \), relative to the integral datum \( \Lambda \).

The Berry phase is a general phenomenon which may appear when a quantum system undergoes a cyclic evolution. We summarize the geometric definition of Berry phase given in [16], where general references can be found. Let \( \{ (N_t, \epsilon_t) \}_{t \in [0,1]} \) be a family of weighted Lagrangian submanifolds of \( (\mathbb{R}^2, \omega) \) determined by a time dependent Hamiltonian \( H_t \). Let us assume that \( (M, \alpha) \) is quantizable and \( \mathcal{L} \) is a prequantum bundle. \( \mathcal{L}^\times = \mathcal{L} \setminus \{ \text{zero section} \} \) is the corresponding principal bundle. We denote by \( F_t \) the flow on \( \mathcal{L}^\times \) generated by the vector field \( X^f_t - W_{h_t} \), where \( X^f_t \) is the horizontal lift of the respective Hamiltonian vector field \( X_t \), and \( W_{h_t} \) is \( h_t \) times the fundamental vertical vector field on \( \mathcal{L}^\times \). If \( \sigma \) is a section of \( \mathcal{L}^\times_0 \), then \( F_t(\sigma(N)) \) differs from \( \sigma(N) \) by a phase \( \theta \). If the Hamiltonian \( h_t \) are normalized so that \( \int_{N_t} h_t \epsilon_t = 0 \), then \( \theta \) is the Berry phase of the loop \( \{ (N_t, \epsilon_t) \}_{t \in [0,1]} \) (p.142 [16]).

By (3.17) the statement (d) in Theorem 2 when \( \dim H = 1 \), can be interpreted by saying that the invariant \( \kappa \) is the Berry phase of any loop \( \{ (N_t, \epsilon_t) \}_{t \in [0,1]} \) generated by the Hamiltonian functions \( h_{A_t} \), where \( A_t \) is the velocity curve of any path in \( G \) with endpoint at \( g_1 \), and \( h_{A_t} \) is given by (3.1).

**Example.** Let \( G \) be the restricted Lorentz group \( SO^+(1, 3) \). A basis for the Lie algebra \( so^+(1, 3) \) is \( X_1, \ldots, X_6 \), where \( X_1, X_2, X_3 \) are the generators of boosts along the axes, and \( X_4, X_5, X_6 \) are the generators of rotations around those axes. The matrix of Killing metric in the basis \( X_i \) is
\[
(g_{ij}) = \left( \Tr (\text{ad} X_i \circ \text{ad} X_j) \right) = \text{diag}(1, 1, 1, 1, 1, 1),
\]
and the Casimir element \( C \) of \( U(g_{\mathbb{C}}) \) is
\[
C = \sum g_{ij} X^i Y^j, \quad \text{where} \quad X^i = \sum_k g^{ia} X_k.
\]
That is,
\[
(5.3) \quad C = \frac{1}{4} \left( \sum_{i=1}^{3} X_i^2 - \sum_{i=4}^{6} X_i^2 \right).
\]

Let \( Y = (Y_{ab}) \) be a matrix in \( so^+(1, 3) \), with \( a, b = 0, 1, 2, 3 \), and let \( \eta \) denote the element in \( so^+(1, 3)^* \) defined by \( \eta(Y) = kY_{01} \), with \( k \in \mathbb{R} \setminus \{0\} \) (equivalently \( \eta(X_j) = k\delta_{1j} \)). The matrix associated with \( \eta \) according to (2.1) is precisely \( kX_1 \), which has real eigenvalues; i.e. the coadjoint orbit \( O \) of \( \eta \) is hyperbolic. Furthermore \( O \) is \( G/L \), with \( L = SO^+(1, 1) \times SO(2) \). If \( (A, B) \in L \), then \( A \) will have the form \( A = \exp a X_1 \) and \( \eta \) can be extended to a character \( \Lambda \) on \( L \) by putting \( \Lambda(A, B) = e^{ika} \).

One has the following relations
\[
[X_1, X_2] = X_6, \quad [X_1, X_3] = X_5, \quad [X_1, X_4] = 0, \quad [X_1, X_5] = X_3, \quad [X_1, X_6] = X_2.
\]
So a basis for the subalgebra $u$ defined in (2.2) is $X_2 + X_6$, $X_3 + X_5$, and for the operators $\delta$ and $\phi$ introduced in Section 2 we have

\begin{equation}
\delta(X_1) = \text{Tr} (\text{ad}(X_1|_u)) = 2, \quad \phi(X_1) = ik + 2.
\end{equation}

Analogously

\begin{equation}
\delta(X_4) = 0, \quad \phi(X_4) = 0.
\end{equation}

$\mathfrak{h} = l_C$ is a Cartan subalgebra of $\mathfrak{g}_C$, and it is generated by $X_1, X_4$. By (5.3), the projection $\hat{C}$ of $C$ on $U(\mathfrak{h})$ is $\hat{C} = \frac{1}{4}(X_1^2 - X_4^2)$. By Theorem 5 it follows from (5.4), (5.5) and (5.1) that the value $\chi(C)$ of the infinitesimal character of the representation $\pi'$ associated with the orbit $\mathcal{O}$ is $(1/4)(ik + 2)^2$.

For $A \in \mathfrak{g}$, the operator $\mathcal{P}_A = -D_{X_A} + h_A$ acting on polarized sections of the prequantum bundle is the “quantization” of the vector field on $\mathcal{O}$ determined by $A$. From the above result it turns out that the operator

\begin{equation}
\frac{1}{4} \left( \sum_{i=1}^{3} (P_{X_i})^2 - \sum_{i=4}^{6} (P_{X_i})^2 \right),
\end{equation}

associated with the Casimir element $C \in Z(\mathfrak{g}_C)$ is simply the multiplication by the constant $(1/4)(ik + 2)^2$.

REFERENCES

[1] Abraham, R. Marsden, J.E.: Foundations of Mechanics. Benjamin/Cummings Publishing Co. London (1985)
[2] Kirillov, A. A.: Elements of the theory of representations. Springer-Verlag, Berlin (1976)
[3] Kirillov, A. A.: Lectures on the orbit method. American Mathematical Society, Providence (2004)
[4] Knapp, A. W.: Representation theory of semisimple groups: An overview based on examples. Princeton University Press, Princeton (2001)
[5] Knapp, A. W.: Lie groups beyond an introduction. Progr. Math., 140, Birkhuser, Boston, MA, (2005)
[6] Knapp, W.A., Trapa, P.E.: Representations of semisimple Lie groups. In Representation theory of Lie groups. (J. Adams and D. Vogan editors). IAS/Park City Mathematics Series, vol 8, pp. 177-238. AMS, Providence, RI (1999)
[7] Lalonde, T, McDuff, D., Polterovich, L.: On flux conjectures. In CRM Proceedings and Lecture Notes 15, pp. 69-85 AMS, Providence, RI (1998)
[8] Kostant, B.: Quantization and unitary representations. In Modern analysis and applications. Lecture Notes in Mathematics, Vol 170, pp 87-207, Springer-Verlag, Berlin (1970).
[9] McDuff, D., Salamon, D.: Introduction to symplectic topology. Clarenton Press, Oxford (1998)
[10] Śniatycki, J. E.: Geometric quantization and quantum mechanics. Springer-Verlag, New-York (1980)
[11] Viña, A.: Symplectic action around loops in Ham($M$). Geom. Dedicata 109, 31-49, (2004).
[12] Vogan, D.: Unitary representations of reductive Lie groups. Princeton University Press, Princeton, 1987.
[13] Vogan, D.: The orbit method and unitary representations. Algebraic and Analytic Methods in Representation Theory (Sonderborg, 1994), Perspectives in Mathematics, vol 17, pp. 243-339. Academic Press, San Diego (1997)
[14] Vogan, D.: The orbit method of coadjoint orbits for real reductive groups. In Representation theory of Lie groups. (J. Adams and D. Vogan editors). IAS/Park City Mathematics Series, vol 8, pp. 177-238. AMS, Providence, RI (1999)
[15] Weinstein, A.: Cohomology of symplectomorphism groups and critical values of Hamiltonians. Math. Z. 201, 75-82 (1989)
[16] Weinstein, A.: Connections of Berry and Hannay type for moving Lagrangian submanifolds. Adv. Math. 82, 133-159 (1990)
[17] Woodhouse, N.M.J.: Geometric quantization. Clarenton Press, Oxford (1992)

Departamento de Física. Universidad de Oviedo. Avda Calvo Sotelo. 33007 Oviedo. Spain.

E-mail address: vina@uniovi.es