STRING TOPOLOGY AND A CONJECTURE OF VITERBO

EGOR SHELUKHIN

Abstract. We identify a class of closed smooth manifolds for which there exists a uniform bound on the Lagrangian spectral norm of Hamiltonian deformations of the zero section in a unit cotangent disk bundle. This class of manifolds is characterized in topological terms involving the Chas-Sullivan algebra and the BV-operator on the homology of the free loop space. In particular, it contains spheres and is closed under products. This settles a conjecture of Viterbo from 2007 as the special case of $T^n$. We discuss generalizations and applications.

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1. Introduction

In this paper we prove a generalization of a conjecture of Viterbo from 2007 [41, Conjecture 1] on the spectral norm $\gamma(L', L)$ of Lagrangian submanifolds in the disk cotangent bundle of a large class of closed connected manifolds $L$, including $T^n$. We shall consider all homological invariants, including the spectral norm, with coefficients in a field $K$.

The main tool of the paper is the Viterbo isomorphism [1, 2, 4, 31, 40] of BV-algebras between the symplectic cohomology of the cotangent bundle and the homology of the loop space of the base, combined with the TQFT operations on symplectic cohomology and Lagrangian Floer homology studied by Seidel and Solomon [33].

We start by describing the class of manifolds to which our results apply. Fix a base field $K$ as a coefficient ring for homology and cohomology groups. For a closed connected smooth manifold $L$ of dimension $\dim(L) = n$, consider the constant-loop inclusion map

$$\iota: H_*(L) \to H_*(\mathcal{L}L),$$

and the evaluation map

$$ev: H_*(\mathcal{L}L) \to H_*(L),$$

between its homology and the homology of the free loop space $\mathcal{L}L$ of $L$. Given a homogeneous class $a \in H_*(\mathcal{L}L)$, let

$$m_a : H_*(\mathcal{L}L) \to H_{*+\lfloor a\rfloor-n+1}(\mathcal{L}L)$$
be the right Chas-Sullivan string bracket $[-, a]$ with $a$. We recall that the string bracket is given in terms of the Chas-Sullivan product $\ast$, and the BV-operator
\[
\Delta : H_s(\mathcal{L}L) \to H_{s+1}(\mathcal{L}L).
\]
It is essentially the $\Delta$-differential of the product: for homogeneous elements $a, b \in H_{n-s}(\mathcal{L}L)$,
\[
[b, a] = (-1)^{|b|} (\Delta(b \ast a) - \Delta(b) \ast a - (-1)^{|b|} b \ast \Delta(a)).
\]
This bracket together with the product forms the structure of a Gerstenhaber algebra on $H_{n-s}(\mathcal{L}L)$. In terms of these operations, the main operator that we consider in this paper is
\[
P_a : H_s(L) \to H_{s+|a|-n+1}(L)
\]
\[
P_a = ev \circ m_a \circ \iota.
\]

**Definition 1.** We call a closed connected smooth manifold $L$ string point-invertible over $\mathbb{K}$ if it is $\mathbb{K}$-orientable and there exists a collection of classes $a_1, \ldots, a_N \in H_s(\mathcal{L}L)$ such that the composition $P = P_{a_N} \circ \ldots \circ P_{a_1}$ satisfies $[L] = P([pt])$, where $[L] \in H_n(L)$ is the fundamental class and $[pt] \in H_0(L)$ is the class of the point. Reformulated more abstractly, $[L] \in \mathcal{P}([pt]) = \{P([pt]) \mid P \in \mathcal{P}\}$, where $\mathcal{P}$ is the subalgebra of $Hom(H_s(L), H_s(L))$ generated by $\{P_a \mid a \in H_s(\mathcal{L}L)\}$.

**Remark 2.** We note the following three points regarding Definition 1:

i. Set $H_s(\mathcal{L}L)^+ = \ker(ev : H_s(\mathcal{L}L) \to H_s(L))$. It is easy to see that we may, without loss of generality, restrict $a_1, \ldots, a_N$ in Definition 1 to lie in $H_s(\mathcal{L}L)^+$ and replace $\mathcal{P}$ with its subalgebra $\mathcal{P}^+$ generated by $\{P_a \mid a \in H_s(\mathcal{L}L)^+\}$. Indeed, it is enough to consider homogeneous elements $a$, in which case $P_a : H_s(L) \to H_{s+|a|-n+1}(L)$ is a homogeneous operator. Further, for all $b \in \iota(H_s(L))$, $P_b = 0$ since $\Delta = 0$ on $\iota(H_s(L))$, and $\iota, ev$ are maps of algebras, $H_s(L)$ being endowed with the intersection product. Hence we may correct each homogeneous element $a \in H_s(\mathcal{L}L)$ by $a_0 = ev \circ m_a \circ \iota$ to obtain the homogeneous element $a' = a - a_0 \in H_s(\mathcal{L}L)^+$, with the property that $P_{a'} = P_a$. Hence $[L] = P_{a_N} \circ \ldots \circ P_{a_1}([pt]), a_1, \ldots, a_N \in H_s(\mathcal{L}L)$ if and only if $[L] = P_{a'_N} \circ \ldots \circ P_{a'_1}([pt])$, with $a'_1, \ldots, a'_N \in H_s(\mathcal{L}L)^+$.

ii. For a class $a \in H_s(\mathcal{L}L)$ we may consider the operator $Q_a : H_s(L) \to H_s(L)$, given by $Q_a = ev \circ m'_a \circ \iota$, where $m'_a$ is the Chas-Sullivan product by $a$. The technical arguments in this paper apply to this simpler map, however since $ev$ and $\iota$ are maps of algebras, and $ev \circ \iota = \text{id}$, we observe that for $x \in H_s(L)$, $Q_a = ev \circ m_a \circ \iota(x) = ev(a \ast \iota(x)) = ev(a) \ast ev \circ \iota(x) = ev(a) \ast x$. Therefore $Q_a$ is the multiplication operator by $ev(a) \in H_s(L)$ with respect to the intersection product on $H_s(L)$. In particular it does not increase degree. Therefore, while adding the operations $Q_b, b \in H_s(\mathcal{L}L)$, to Definition 1 may theoretically be useful, in practice it seems to have little effect.

iii. Note that if $P_a : H_s(L) \to H_s(L)$ increases degree, then the homological degree of $a \in H_s(\mathcal{L}L)$ satisfies $|a| \geq n$.

We proceed to discuss the size of the class of string point-invertible manifolds by describing examples and non-examples, based on known calculations of the Chas-Sullivan Gerstenhaber algebra. These calculations turn out to be quite delicate, and to depend on the choice of coefficients, and hence so does the property of string point-invertibility.

By a result of Menichi [27] this class contains spheres of odd dimension $S^{2m+1}, m \geq 0$, with arbitrary coefficients, and $S^2$ with coefficients in $\mathbb{F}_2$. A minor modification of the argument of Menichi for $S^2$ shows that the even-dimensional spheres $S^{2m}, m \geq 1$ are in this class, with $\mathbb{F}_2$ coefficients. By a result of Tamanoi [37], the complex Stiefel manifolds $V_{n+1-k}(\mathbb{C}^{n+1}) \cong$
SU(n+1)/SU(k) of orthonormal \((n+1-k)\)-frames in \(\mathbb{C}^{n+1}\) for all \(n \geq 0, 0 \leq k \leq n\), are string point-invertible over arbitrary coefficients, and by Menichi [28] (see related result of Hepworth [18]), all compact connected Lie groups are string point-invertible, with characteristic zero coefficients. Results of Westerland [42] indicate that over \(\mathbb{F}_2\), certain projective spaces, for example \(\mathbb{C}P^n\) and \(\mathbb{H}P^n\), where \(n\) is odd, are also string point-invertible. Even though the results of [42] pertain to a different BV-operator than the one we use here, since these spaces are simply connected, results of Malm [25] and Felix-Menichi-Thom [13] or Keller [19] imply that the Gerstenhaber algebras induced by the two BV-operators are in fact isomorphic (see also the discussion in [12]). Furthermore, as string point-invertibility depends only on the Gerstenhaber algebra structure on \(H_{n-*}(\mathcal{L}L)\) and the evaluation and inclusion maps, at least for \(L\) simply connected, by the latter results, as well as [14], it depends only on the singular cochain dg-algebra \(C^*(L)\) of \(L\), up to isomorphism.

By Menichi [27], this class does not contain the even-dimensional spheres \(S^{2m}, m \geq 1\) for coefficients of characteristic zero, for instance, and the same is true for \(\mathbb{C}P^n, n \geq 1, \mathbb{H}P^n, n \geq 1\) and \(O^2\) by results of Yang [43], Chataur-Le Borgne [11], Hepworth [17], and Cadek-Moravec [9]. Moreover, by [9, 11, 17] the same is true for \(\mathbb{C}P^n, \mathbb{H}P^n\), where \(n\) is even, and for \(O^2\), with \(\mathbb{F}_2\)-coefficients. By a result of Vaintrob [39], this class does not contain the closed surface \(\Sigma_g\) of genus \(g\), for each \(g > 1\), and any choice of coefficients. The same is true for closed manifolds of strictly negative sectional curvature, again by [39] or by an index argument of Tonkonog [38].

Finally, we have the following general structural result for this class.

**Proposition 3.** The class of string point-invertible manifolds over a fixed field \(\mathbb{K}\) is closed under products.

In particular, the \(n\)-torus \(T^n\) is string point-invertible over any field \(\mathbb{K}\). To verify the definition one can take (the image under coefficient change to \(\mathbb{K}\) of) the sequence \(a_1, \ldots, a_n\) of positive generators of \(H_n(\mathcal{L}e_i; L; \mathbb{Z}) \cong \mathbb{Z}\) for free homotopy classes of loops \(e_1, \ldots, e_n\) corresponding to a positively oriented basis of \(\mathbb{Z}^n\). The main result of this paper is the following.

**Theorem A.** Let \(L\) be string point-invertible over a field \(\mathbb{K}\). Let \(g\) be a Riemannian metric on \(L\). Then there exists a constant \(C(g, L; \mathbb{K})\) such that for all exact Lagrangian submanifolds \(L_0, L_1\) contained in the unit codisk bundle \(D_g^*L \subset T^*L\), the spectral norm of the pair \(L_0, L_1\) satisfies

\[
\gamma(L_0, L_1; \mathbb{K}) \leq C(g, L; \mathbb{K}).
\]

**Remark 4.** By the triangle inequality for the spectral norm, it is enough to prove the above statement for \(L_1 = L\), the zero section in \(D_g^*L\).

This statement was previously known for \(\mathbb{K} = \mathbb{F}_2\), and \(L \in \{\mathbb{R}P^n, \mathbb{C}P^n, \mathbb{H}P^n, S^n : n \geq 1\}\) by [35], essentially in the case when \(L_1\) is Hamiltonianly isotopic to the zero section, and \(L_0 = L\). In particular the case of \(T^n\) for \(n > 1\) has remained completely open. We note that by the above examples, Theorem A is somewhat complementary to the existing result. Furthermore, while quite a few manifolds including surfaces of higher genus are not string point-invertible, and hence Theorem A does not apply as such, there is an ongoing work [7] proving results related to the Viterbo conjecture for bases given by arbitrary closed connected manifolds.

The strategy of the proof of Theorem A differs significantly from that of [35]. The main idea is threefold: first, a homogeneous class \(a \in H_{n-*}(\mathcal{L}L)^+\) corresponds by the Viterbo isomorphism [1, 2, 4, 31, 40] to a class \(\alpha \in SH^*(\mathcal{L}L)\) (the latter computed with suitable
background class) with the property that \( r_{L'}(\alpha) = 0 \in HF^*(L', L') \) for each exact Lagrangian \( L' \subset T^*L \), where \( r_{L'} : SH^*(L L) \to HF^*(L', L') \) is the natural closed-open restriction map (by \([3, 5, 12, 21]\)) each such \( L' \) has vanishing Maslov class, is Spin relatively to the above background class, and endowed with suitable Spin structure is Floer-theoretically equivalent to the zero section \( L \). Second, working up to \( \varepsilon > 0 \), given that \( L_0, L_1 \subset D = D^g_yL \), a Liouville domain with contact boundary \( S = \partial D = S^*_yL \), the work of Seidel and Solomon \([33]\) gives an operation \( HF^*(L_0, L_1) \to HF^{*+|\alpha|-1}(L_0, L_1) \), which raises the action filtration by no more than a symplectic-homological spectral invariant \( c(\alpha, D, S) \) corresponding to the class \( \alpha \) and the domain \( D \). Finally, using further TQFT operations for Lagrangian Floer cohomology and symplectic cohomology \([1, 4]\), we calculate that under the Floer-theoretic equivalence with the zero section, Poincaré duality \( c(\alpha, D, S) \) is given by \( P_{\alpha} : H_*(L) \to H_{*+|\alpha|}(L) \). Therefore, in view of string point-invertibility, assuming for simplicity that all \( P_{\alpha_j} \), \( 1 \leq j \leq N \), increase degree, which tends to happen in practice, by successively writing inequalities that bound the Lagrangian spectral invariants of classes of higher homological degree in terms of those of classes of lower homological degree, we arrive to a uniform upper bound on the spectral distance \( \gamma(L_0, L_1; \mathbb{K}) \), finishing the proof.

**Remark 5.** A few remarks on Theorem \([\mathbb{A}]\) are in order.

i. It is not necessary that our Weinstein domain be \( D^g_yL \) for a Riemannian metric \( g \). In fact the same result holds for any Weinstein domain \( D \) with completion given by \( T^*L \). In this case the upper bound will be given in terms of a constant \( c(D, L; \mathbb{K}) \). For example \( D \) may be given by a Finsler metric, or an optical domain: one that is strictly fiberwise star-shaped, and has a smooth boundary. Finally, approximating general, not necessarily smooth, strictly fiberwise starshaped domains by ones with smooth boundary, we obtain a uniform bound in that case as well.

ii. In fact \( C(g, L; \mathbb{K}) \) in Theorem \([\mathbb{A}]\) can be chosen to be equal to a certain sum of spectral invariants relative to the domain \( D \) with boundary \( S \), corresponding to any \( N \)-tuple \( a_N, \ldots, a_1 \in H_0(L L) \) as in Definition \([1]\) See Equation \([7]\). Furthermore, it is easy to see that the spectral invariants \( c(a, D, S) \) are continuous in the Banach-Mazur distance with respect to the natural \( \mathbb{R}_{*,0} \)-action on \( T^*L \) \([30, 36]\) (see \([11]\)), and hence extend for example to the non-smooth strictly fiberwise star-shaped case. As a consequence we obtain bounds in the non-smooth fiberwise star-shaped case in terms of the extension of the spectral invariants.

iii. Let \( g_0 \) be the standard metric of diameter 1/2 on \( S^1 = \mathbb{R}/\mathbb{Z} \). Let \( D_0 = D^g_0S^1 = [-1, 1] \times S^1 \). From \([7]\), it is evident that \( C(D_0, L_0; \mathbb{K}) = 1 \) in this case. This upper bound is sharp, since for each \( \varepsilon > 0 \) sufficiently small, it is easy to construct a Lagrangian \( L'_0 \subset D_0 \) Hamiltonian isotopic to \( L_0 = S^1 \) in \( D_0 \), with \( \gamma(L'_0, L_0) > 1 - \varepsilon \), and the intersection \( L'_0 \cap L_0 \) is transverse and consists of precisely 2 points \( x, y \) of index 1 and 0 respectively. Consider now the strictly fiberwise star-shaped domain \( D \subset T^*(T^n) \) given by \( D = (D_0)^n = [-1, 1]^n \times T^n \).

It is easy to calculate that the upper bound obtained by continuity from \([7]\) is in this case \( C(D, L; \mathbb{K}) = n \). It is seen to be sharp by noting that \( L' = (L'_0)^n \) satisfies \( \gamma(L', L) = n \cdot \gamma(L'_0, L_0) \), since \( L' \cap L \), and the only intersection point of \( L' \) and \( L \) indices \( n \) and 0 are \((x, \ldots, x)\), and \((y, \ldots, y)\) respectively.

iv. We note that Theorem \([\mathbb{A}]\) fails for general bounded Liouville domains. For example it is false for Lagrangians Hamiltonian isotopic to \( L \) in plumbings of \( D^gL \) with two or more cotangent disk bundles by \([44]\).
1.1. Applications. As observed in [35], an argument of neck-stretching around a divisor in $M \times M^-$ that makes the Lagrangian diagonal $\Delta_M \subset M \times M^-$ exact, where $(M, \omega)$ is a closed symplectic manifold such that Theorem A holds for $L = M$, and $M^-$ denotes the symplectic manifold $(M, -\omega)$, allows one to prove, for example in the symplectically aspherical case, that the Hamiltonian spectral norm on $\text{Ham}(M, \omega)$ is Lipschitz in the $C^0$ norm, in a $C^0$-neighborhood of the identity. We pick one instance of such an application. The $C^0$-distance between two diffeomorphisms $\phi_0, \phi_1$ of $M$ is defined as $d_{C^0}(\phi_0, \phi_1) = \max_{x \in M} d(\phi_0(x), \phi_1(x))$, the distance $d$ being taken with respect to a background Riemannian metric on $M$.

**Corollary 6.** Let $g$ be a Riemannian metric on $T^{2n}$, and $\mathbb{K}$ be a field. The spectral norm

$$\gamma : \text{Ham}(T^{2n}, \omega_{st}) \to \mathbb{R}_{\geq 0}$$

over $\mathbb{K}$ satisfies the following. There exist constants $C, \delta > 0$, such that

$$\min\{\gamma(\phi), \delta C\} \leq C \cdot d_{C^0}(\phi, 1)$$

for all $\phi \in \text{Ham}(T^{2n}, \omega_{st})$.

We refer to [35] for a discussion of results of this kind, such as [8, 20, 24], and a proof of a similar, though more complex, implication [35, Theorem C]. Essentially the same proof allows one to show that the same proof allows one to show that $\gamma_{pt} : \text{Ham}(M, \omega) \to \mathbb{R}_{\geq 0}$ satisfies the same property as in Corollary 6, whenever $(M, \omega)$ is a monotone symplectic manifold such that as a smooth manifold $M$ is string point-invertible over $\mathbb{K}$. Here $\gamma_{pt}(H)$ for a Hamiltonian $H \in C^\infty([0,1] \times \mathbb{R}, \mathbb{R})$ is given in terms of Hamiltonian spectral invariants by

$$\gamma_{pt}(H) = \inf \{c(a, H) - c([pt] \ast a, H)\}$$

the infimum running over $a \in QH_*(M, A_\mathbb{K})$, the quantum homology algebra over the Novikov field $A_\mathbb{K}$, where $\ast$ is the quantum product, and $[pt] \in QH_0(M, A_\mathbb{K})$ is the point class. One quickly verifies that $\gamma_{pt}(H)$ depends only on the time-one map $\phi = \phi^1_0$ of the Hamiltonian flow of $H$, and it is well-known [26, Equation 12.4.6] that $\gamma_{pt}(\phi) > 0$ if and only if $\phi \neq \text{id}$. It does not however define a metric on $\text{Ham}(M, \omega)$, as it does not in general satisfy the triangle inequality. Finally we note that for $\mathbb{K} = \mathbb{F}_2$, monotone products of spheres $S^2 \times \ldots \times S^2$ and their products $V \times T^{2m}$, $m \geq 1$, with tori, provide examples of such manifolds.

Similarly to [35], Theorem A yields the existence of non-trivial homogeneous quasi-morphisms on $\text{Ham}_c(D^*_a L)$ for $L$ string point-invertible, providing new examples of quasi-morphisms on compactly supported Hamiltonian diffeomorphism groups of Weinstein domains. In particular, the quasiisomorphism $\mu : \text{Ham}_c(D^*_a T^n) \to \mathbb{R}$ in the case of $T^n$ is immediately seen to be invariant under finite coverings $T^n \to T^n$, scaled suitably, as defined in [41] (see also [29]). Of course a similar invariance holds for products $T^n \times L$ with $L$ string point-invertible, with the induced coverings, or for finite coverings $L' \to L$ with both $L, L'$ string point-invertible. It is an interesting topological question to determine whether or not the class of string point-invertible manifolds is closed with respect to finite coverings: we expect this to be the case when working with coefficients of characteristic zero. Furthermore, Theorem A provides a different proof, and indeed a strengthening, of the results of [41, Section 7].

We finish with yet another application, that is proved again by a neck-stretching argument (see [35, Theorem F]), which is again somewhat simpler, because of the weakly exact setting.

**Corollary 7.** Let $L$ be string point-invertible. Suppose $L$ is embedded as a weakly exact Lagrangian submanifold in a symplectically aspherical symplectic manifold $M$ that is closed or tame at infinity. Consider the pair $(U, L)$, for $U \subset M$ a Weinstein neighborhood of $L$, that is symplectomorphic to the pair $(D, 0_L)$, for a Weinstein domain $D \subset T^*L$ containing...
the zero-section $0_L \subset T^*L$. Consider $r \in (0,1)$, and let $U'$ be the preimage of $r \cdot D$ by the symplectomorphism. Then there exists a constant $C(D,L;\mathbb{K})$ such that if $L' \subset M$ is another weakly exact Lagrangian submanifold that is contained in $U'$, then

$$\gamma(L',L;\mathbb{K}) \leq C(D,L;\mathbb{K}) \cdot r.$$ 

An example of the situation described in Corollary 7 is the torus $L = T^n$ embedded as $L_1 \times \ldots \times L_n$ inside $\Sigma_1 \times \ldots \times \Sigma_n$, where for all $1 \leq j \leq n$, the submanifold $L_j \subset \Sigma_j$ is an embedded simple closed curve in the closed oriented surface $\Sigma_j$ of genus at least 1, that does not bound a disk. In the cases of Corollaries 6 and 7 arguments following [20, Theorem B] show that the associated Floer-theoretic barcodes, up to shift, are continuous in a suitable $C^0$-sense. Moreover, one can deduce analogues of both corollaries for certain monotone symplectic manifolds, respectively monotone Lagrangian submanifolds, however for reasons of conciseness we defer this discussion to a further publication.

As a closing remark, we mention that it would be very interesting to see if additional algebraic structures on symplectic cohomology and string topology could be applied to extend the class of manifolds $L$ for which Viterbo’s conjecture holds. For instance, introducing local systems on $\mathcal{L}L$ that are trivial when restricted the image of the constant-loop embedding $L \to \mathcal{L}L$, or considering higher operations in the suitable $L_\infty$-algebras or SFT algebras, may yield further such examples.

### 2. Preliminaries

Throughout the paper we follow the definitions and notations of Seidel and Solomon [33], with one distinction: we take the opposite sign for all action functionals. Furthermore, we adopt the following convention: everywhere we argue up to $\epsilon$, and allow arbitrarily small perturbations of all Hamiltonian terms involved. For example, when the Hamiltonian perturbation data has curvature zero, it means that we may achieve regularity by a Hamiltonian term arbitrarily close to the given one, in such a way as to make the curvature arbitrarily small.

We sketch the part of definitions where additional detail is required. In particular we look at exact Lagrangian submanifolds $L$ inside a Weinstein manifold $W$ with Liouville form $\theta$, and symplectic form $\omega = d\theta$. We restrict attention to the case when $W$ is the completion of a Weinstein domain $D$ with compact contact boundary $S$, and we consider $L \subset D$. For the definition of symplectic cohomology we choose a cofinal family of Hamiltonians $H_\alpha$ that are $\epsilon$-small in the $C^2$ norm on $D$, and are in fact non-positive Morse functions there with gradient pointing outward of $D$ at $S$. Furthermore outside of $D \cup C$ for a small collar neighborhood $C = C_\lambda$ of $S$, $H_\alpha = \lambda \cdot r$, where $r$ is the radial coordinate on the infinite end $([1,\infty) \times S, (d\alpha))$, $\alpha = \theta|_S$, of the completion, with the property that $\lambda \notin \text{Spec}(\alpha, S)$, that is, it is not a period of a closed Reeb orbit of $\alpha$. The latter is a smooth loop $\gamma : \mathbb{R}/T\mathbb{Z} \to S$, $T > 0$, such that $\gamma'(t) = R_\alpha \gamma(t)$ for all $t \in \mathbb{R}/T\mathbb{Z}$, and the Reeb vector field $R_\alpha$ on $S$ is defined by the conditions $\nu_{R_\alpha} \alpha = 1$, $L_{R_\alpha} \alpha = \iota_{R_\alpha} d\alpha = 0$. Furthermore, we require that $0 < \epsilon \ll \epsilon_\alpha = \min \text{Spec}(\alpha, S)$, and that $H_\alpha$ be radial increasing and convex in $C$. Furthermore, (an arbitrarily small perturbation in $D \cup C$ of) $H_\lambda$ is non-degenerate at all its 1-periodic orbits, which necessarily lie in $D \cup C$. All closed $H_\lambda$ one-periodic orbits in $C$ are in a $2$ to $1$ correspondence with the Reeb orbits of $\alpha$ of periods in $[\epsilon_\alpha, \lambda)$, and we choose $C, H_\lambda$ so that for a fixed $\delta > 0$ independent of $\lambda$ the $H_\lambda$-actions of these orbits are $\delta$-close to their $\alpha$-periods. We choose $\delta < \epsilon_\alpha$. Furthermore, we require that $H_{\lambda_k} \leq H_{\lambda_{k+1}}$, $k \geq 1$, on $W$ for a strictly increasing sequence $\{\lambda_k\}_{k \geq 1}$, $\lambda_k \xrightarrow{k \to \infty} \infty$, in $\mathbb{R}_{>0} \setminus \text{Spec}(\alpha, S)$, and that $||H_{\lambda_k}||_{C^2(D)} \xrightarrow{k \to \infty} 0$. That these choices can be made is standard.
material on symplectic cohomology (see for example [16, Section 5]). From now on, when we write \( H_\lambda \) we assume that \( \lambda = \lambda_k \). Finally, for two fixed Lagrangian submanifolds \( L_0, L_1 \subset D \) we may choose \( H_\lambda \) on \( D \) so that the intersection \( \phi^t_{H_\lambda}(L_0) \cap L_1 \) is transverse for all \( k \geq 1 \).

We recall that the \( \omega \)-compatible almost complex structures \( J \) that we consider are of convex type: on the infinite end of \( W, J \partial_r = R \), and \( J \) is invariant under translations in \( \rho = \log(r) \).

The action of a periodic orbit \( x \) of \( H \) is defined as

\[
\mathcal{A}_H(x) = -\int_0^1 H(t, x(t)) \, dt + \int_x \theta.
\]

We consider the Floer cohomology groups \( CF^*(H_\lambda) \), that as \( K \)-modules have generators corresponding to 1-periodic orbits of \( H_\lambda \), the coefficient near \( x_- \) of whose differential \( d_{H,J} \) evaluated on \( x_+ \), for \( J \)-generic, counts isolated solutions \( u : \mathbb{R} \times S^1 \to W \) to the Floer equation

\[
\partial_s u + J_t(u)(\partial_t u - X_H^t(u)) = 0,
\]

with asymptotic conditions \( u(s, -) \to x_\pm(-) \), as \( s \to \pm \infty \), for 1-periodic orbits \( x_\pm \) of \( H = H_\lambda \).

Here \( X_H \) is the time-dependent Hamiltonian vector field of \( H \) given by \( tX_H^t \omega = -d(H(t, \cdot)) \).

Note that the critical points of \( \mathcal{A}_H \) on the loop space \( \mathcal{L}W \) are precisely given by time-1 periodic orbits of the isotopy \( \{ \phi^t_H \} \) generated by \( X_H \). Furthermore, if \( d_{H,J}(y) = z \), then \( \mathcal{A}_{H}(y) > \mathcal{A}_{H}(z) \). Finally, \( CF^*(H_\lambda) \) forms a direct system with respect to the natural order on \( \{ \lambda_k \} \), by means of Floer continuation maps: \( CF^*(H_{\lambda_k}) \to CF^*(H_{\lambda_{k'}}) \) for \( \lambda_k \leq \lambda_{k'} \). Here it is important that \( H_{\lambda_k}(t, x) \) is increasing as a function of \( k \). The symplectic cohomology of \( W \) is defined as

\[
SH^*(W) = \lim_{\lambda \to \infty} CH^*(H_\lambda) = \lim_{\lambda \to -\infty} CH^*(H_\lambda).
\]

Its filtered version associated to \((D, S)\) is defined as

\[
SH^*(W)^{<t} = \lim_{\lambda \to \infty} CH^*(H_\lambda)^{<t},
\]

where \( CH^*(H_\lambda)^{<t} \) is the subcomplex generated by 1-periodic orbits of action strictly smaller than \( t \).

Given two exact Lagrangian submanifolds \( L_0, L_1 \subset D \), we choose generic perturbation data \( D = (J^{L_0,L_1}, K^{L_0,L_1}) \) consisting of an almost complex structure \( J^{L_0,L_1} \) that depends on time \( t \in [0, 1] \), and a Hamiltonian \( K^{L_0,L_1} \) that is radial outside of \( D \cup C \) (for example zero there), and define the Floer complex \( CF(L_0, L_1; D) \) with generators corresponding to \( X_{K^{L_0,L_1}} \)-chords from \( L_0 \) to \( L_1 \), the matrix coefficients \( \langle d_{L_0,L_1;D}(x_+), x_- \rangle \) of whose differential \( d_{L_0,L_1;D} \) count isolated solutions \( u : \mathbb{R} \times [0, 1] \to W \) to the Floer equation

\[
\partial_s u + J^t_{L_0,L_1}(u)(\partial_t u - X_H^t(u)) = 0,
\]

with boundary conditions

\[
u(\mathbb{R}, 0) \subset L_0, \quad u(\mathbb{R}, 1) \subset L_1,
\]

and uniform asymptotics

\[
u(s, -) \xrightarrow{s \to \pm \infty} x_\pm(-).
\]

Enhancing \( L_0, L_1 \) to \( L_0' = (L_0, f_0), L_1' = (L_1, f_1) \) by choices of primitives \( f_0 \in C^\infty(L_0, \mathbb{R}), f_1 \in C^\infty(L_1, \mathbb{R}) \), we define the action functional on the space of paths \( \mathcal{P}(L_0, L_1) \) in \( W \) from \( L_0 \) to \( L_1 \),

\[
\mathcal{A}_{L_0,L_1,D} : \mathcal{P}(L_0, L_1) \to \mathbb{R}
\]

\[
\mathcal{A}_{L_0,L_1,D}(x) = -\int_0^1 K^{L_0,L_1}(t, x(t)) + \int_x \theta + f_1(x(1)) - f_0(x(0)).
\]
The critical points of \( A_{L_0, L_1; D} \) correspond to the generators of \( CF(L_0, L_1; D) \), and if \( d_{L_0, L_1; D}(y) = z \) then \( A_{L_0, L_1; D}(y) > A_{L_0, L_1; D}(z) \). Furthermore, as we assume that \( L_0, L_1 \) are connected, \( A_{L_0, L_1; D} \) does not depend on the enhancements \( L_0, L_1 \) up to an additive constant.

For a class \( a \in SH^*(W) \setminus \{0\} \), its symplectic cohomology spectral invariant \( c(a, D, S) \) relative to the domain \( D \) with contact-type boundary \( S \), is defined as

\[
c(a, D, S) = \inf\{ t \in \mathbb{R} | a \in \im (SH^*(W)^{<t} \to SH^*(W)) \},
\]

where \( SH^*(W)^{<t} \to SH^*(W) \) is the natural map induced by the inclusions of complexes \( CF^*(H_\lambda)^{<t} \subset CF^*(H_\lambda) \). These spectral invariants (see [23, 30, 36]) are known to satisfy the following properties. First, \( c(a, D, S) \) is given as \( \int_\gamma \alpha_S \) for a certain \( \alpha_S \)-Reeb orbit \( \gamma \) on \( S \). In particular \( c(a, D, S) > 0 \). Second, \( c(a, D, S) \) is monotone with inclusions of Liouville domains \( D \subset D' \), with completion \( W \) ([36, 16, Section 8]). Finally, for \( t \in \mathbb{R} \),

\[
c(a, \psi^tD, \psi^tS) = e^tc(a, D, S)
\]

where \( \psi^t \) is the flow of the Liouville vector field \( X \) given by \( i_X \omega = \lambda \). In particular if \( \psi^{-t}D \subset D' \subset \psi^tD \) then

\[
| \log c(a, D, S) - \log c(a, D', S') | \leq t.
\]

For a class \( x \in HF^*(L_0, L_1) \setminus \{0\} \) its spectral invariant \( c(x, L_0, L_1; D) \) relative to the enhancements \( L_0, L_1 \) and perturbation data \( D \), is set to be

\[
c(x, L_0, L_1; D) = \inf\{ t \in \mathbb{R} | x \in \im (HF^*(L_0, L_1; D)^{<t} \to HF(L_0, L_1; D)) \},
\]

where \( HF(L_0, L_1; D)^{<t} \) is the homology of the subcomplex \( CF^*(L_0, L_1; D)^{<t} \) of \( CF^*(L_0, L_1; D) \) generated by chords \( z \) of action \( A_{L_0, L_1}(z) < t \). It is well-known (see [22, 35] and references therein) that \( c(x, L_0, L_1; D) \) is given by \( A_{L_0, L_1; D}(z) \) for a generator \( z \) of \( CF^*(L_0, L_1; D) \), and is therefore finite. Furthermore, \( c(x, L_0, L_1; D) \) does not depend on the almost complex structure part \( J_{L_0, L_1} \) of \( D \), and is Lipschitz in the Hofer norm of the Hamiltonian term \( K_{L_0, L_1} \) of \( D \), in the sense that if the Hamiltonian terms \( K, K' \) of \( D, D' \) agree outside a compact set (in our case this means that their slopes at infinity agree), then

\[
| c(x, L_0, L_1; D) - c(x, L_0, L_1; D') | \leq \int_0^1 \left( \max_W (F_t) - \min_W (F_t) \right) dt,
\]

where \( F = K' \# K \) is the Hamiltonian generating the flow \( \phi_K \circ (\phi_K')^{-1} \). This allows us to extend the spectral invariant to arbitrary perturbations (even continuous ones), and in particular we define \( c(x, L_0, L_1) \) as the limit of \( c(x, L_0, L_1; D) \) as the norm of the Hamiltonian term of \( D \) tends to zero. Finally, we remark that if \( \phi_K(L_0) \subset C \), for all \( t \in [0, 1] \), and where \( C \) is the collar neighborhood of \( S \) such that \( K \) is convex radial in \( C \) and has slope \( \lambda \) outside \( D \cup C \), then \( c(x, L_0, L_1; D) \) depends only on \( K(t, x) \) for \( (t, x) \in [0, 1] \times (D \setminus C) \), by a suitable maximum principle. Indeed, in this case the filtered Floer complex \( (CF^*(L_0, L_1; D), A_{L_0, L_1; D}) \) does not depend on \( K(t, x) \) for \( (t, x) \notin [0, 1] \times (D \setminus C) \). In particular, if the Hamiltonian term of \( D_k \) is given by \( H_{\lambda_k} \), then

\[
c(x, L_0, L_1; D_k) \xrightarrow{k \to \infty} c(x, L_0, L_1).
\]

From now on, for each exact Lagrangian \( L \) we fix an enhancement \( L_0 \), and set \( c(x, L_0, L_1; D) := c(x, L_0, L_1; D) \), and \( c(x, L_0, L_1) := c(x, L_0, L_1) \). Our results will not depend on this choice.

Signs in the count of the differentials, as well as gradings, in both kinds of Floer complexes are determined by certain background classes. We summarize these below, and refer
to [33, 5, 3, 21, 32, 1] for details. For grading in the symplectic homology, we assume that
\(2c_1(TW) = 0\), in which case the Grassmannian Lagrangian bundle \(\text{Lag}(M) \rightarrow M\) admits a
cover \(\tilde{\text{Lag}}(M) \rightarrow M\) with fibers given by universal covers of the former fibers, and for signs
we fix a background class \(b \in H^2(W, \mathbb{F}_2)\). In the main case we consider, \(W = T^*L\), for \(L\) a
closed manifold, our assumption holds, and we set \(b = \pi^*w_2(L)\), for \(\pi : T^*L \rightarrow L\) the natural
projection. The existence of the cover can be deduced by considering the section of \(\text{Lag}(M)\)
given by the Lagrangian subspaces tangent to the fibres. We equip each exact Lagrangian
\(L' \subset T^*L\) with the structure of a brane as follows. By [21], the Maslov class of \(L'\) vanishes,
whence \(\tilde{\text{Lag}}(M)|_{L'}\) admits \(H^0(L', \mathbb{Z}) = \mathbb{Z}\)-worth of sections, which we call gradings, of which
we pick one. By [3], \(\pi_1^*w_2(L) = w_2(L')\), hence \(L'\) is relatively Spin with respect to \(b\), and
furthermore out of the \(H^1(L', \mathbb{F}_2) = H^1(L, \mathbb{F}_2)\) choices of a relative Spin structure we fix one,
such that \(L'\) endowed with these choices is Floer-theoretically equivalent to the zero-section \(L\) (see Theorem C) with the standard relative Spin structure and grading. Throughout the
paper, when considering Lagrangians, we keep in mind such an underlying determination of
a brane structure.

Cycles in Deligne-Mumford moduli spaces of disks, considered as Riemann surfaces with
boundary, decorated with interior and boundary punctures, whose universal curves are equipped
with choices of positive or negative (input or output type) cylindrical ends at each puncture,
induce operations on the various Floer homology groups considered. Indeed, we may equip
the universal curves with Floer data compatible with gluing and compactification, wherein
the cylindrical ends allow one to write suitable Floer equations and asymptotic conditions on
the punctures to land in the correct Floer complexes. For more details we refer to [33]. In our
case, as we wish to consider the behavior of actions and energies in our operations, we need
to make further choices. In particular, we use the notion of cylindrical strips introduced and
used in [20]. In fact, the Floer decorations for our main homological operation were already
considered in [20] in the case of closed monotone symplectic manifolds, and their monotone
Lagrangian submanifolds. We note that in contrast to the closed case, in the case of Liouville
manifolds one must ensure that the images of all Floer solutions lie in a compact subset of \(W\).
This is accomplished by the integrated maximum principle (see [6, Lemma 7.2] or [4, Section
5.2.7]).

In particular, consider the moduli space of disks with a unique input interior marked point
and one output boundary marked point, with the cylindrical end at the interior marked point
chosen so that the asymptotic marker points towards the boundary marked point. This moduli
space is a point, and we may equip it with a choice of a cylindrical strip from the input to the
output. We set the Hamiltonian Floer datum to be \(H_\lambda \otimes dt\) on the cylindrical strip.
Furthermore, we choose boundary condition \(L\) for the Floer solutions. This gives us, for a
suitable perturbation datum \(\mathcal{D} = \mathcal{D}^{L,L}\) with Hamiltonian part compactly supported and of
\(C^2\) norm \(o(1)\) as \(\lambda \rightarrow \infty\), an operation
\[
\phi_L^0 : CF^*(H_\lambda) \rightarrow CF^*(L, L; H_\lambda) \rightarrow CF^*(L, L; \mathcal{D}),
\]
which is a chain map. This operation yields the canonical restriction map
\[
r_L : SH^*(W) \rightarrow HF^*(L, L).
\]
As by [20, Section 2.5], arguing up to \(\epsilon\), the Floer data chosen as above has zero curvature, all
perturbation data, in particular \(\mathcal{D}^{L,L}\), can be chosen to have Hamiltonian parts sufficiently
small, so that this operation satisfies \(A_{L,\mathcal{D}}(\phi_L^0(x)) \leq A_{H_\lambda}(x) + 2\epsilon\).
Furthermore, consider the moduli space of disks with two boundary marked points, an input and an output, and one interior marked point with asymptotic marker pointing towards the output. This moduli space is identified with an interval $\mathbb{R}$, and its Deligne-Mumford compactification is identified with a closed interval by adding nodal disks at $-\infty$, and $+\infty$. See [33] for a description of these nodal disks. Choose cylindrical ends accordingly, and choose a cylindrical strip between the interior input and the boundary output. On this cylindrical strip, let the Hamiltonian part of the Floer datum be $H_\lambda \otimes dt$. This yields an operation:

$$\phi^1_{L_0,L_1} : CF^*(H_\lambda) \otimes CF^*(L_0,L_1) \to CF^*(L_0,L_1;[1]) \to CF^*(L_0,L_1)[-1].$$

Considering the above compactification, one obtains [33] that $\phi^1_{L_0,L_1}$ provides a homotopy between the two maps

$$\mu_2(\phi^0_{L_0}(a), x), \quad (-1)^{|a|}|x| \mu_2(x, \phi^0_{L_1}(a)),$$

where $a \otimes x \in CF^*(H_\lambda) \otimes CF^*(L_0,L_1)$. Furthermore, by our choice of Floer data on the cylindrical strip, whose curvature vanishes by definition, choosing the Floer data $D = D^{L_0,L_1}$ to have sufficiently small Hamiltonian part we obtain that for all $a \otimes x \in CF^*(H_\lambda) \otimes CF^*(L_0,L_1)$,

$$A_{L_0,L_1,D}(\phi^1_{L_0,L_1}(a,x)) \leq A_{H_\lambda}(a) + A_{L_0,L_1,D}(x) + 2\epsilon.$$

Finally, let $a \in CF^*(H_\lambda)$ be a cycle, whose cohomology class represents $\alpha \in SH^*(W)$, with

$$r_L(\alpha) = 0 \in HF^*(L,L).$$

Following [33, Definition 4.2], we call $L$ $a$-equivariant with primitive $c_L \in CF^*(L,L)$ if

$$\phi^0_{L,L}(a) = \mu_1(c_L).$$

As in [33, Equation 4.4], given a cycle $a \in CF^k(H_\lambda)$, and two $a$-equivariant Lagrangians $L_0, L_1 \subset D$, with primitives $c_{L_0} \in CF^{k-1}(L_0,L_0)$, $c_{L_1} \in CF^{k-1}(L_1,L_1)$, we can upgrade $\phi^1_{L_0,L_1}(a,-)$ to a chain map

$$\overline{\phi}^1_{L_0,L_1}(a,-) : CF^*(L_0,L_1) \to CF^*(L_0,L_1)[-1 + k],$$

by setting for homogeneous $x \in CF^*(L_0,L_1)$,

$$\overline{\phi}^1_{L_0,L_1}(a,x) = \phi^1_{L_0,L_1}(a,x) - \mu_2(c_{L_0},-\mu_2(-,c_{L_1}).$$

For sufficiently $C^2$ Hamiltonian-small perturbation data $D^{L_0,L_0}, D^{L_0,L_1}$ it is easy to see that all chains in $CF^*(L_0,L_0), CF^*(L_1,L_1)$ are of actions $A_{L_0,L_0,D}, A_{L_1,L_1,D}$ bounded in absolute value by $\epsilon$. Hence, assuming that $A_{H_\lambda}(a) \geq 2\epsilon_0 \geq \epsilon$, we obtain that

$$A_{L_0,L_1,D}(\overline{\phi}^1_{L_0,L_1}(a,x)) \leq A_{L_0,L_1,D}(x) + A_{H_\lambda}(a) + 2\epsilon.$$

By our choices in the construction of symplectic cohomology, the assumption on $A_{H_\lambda}(a)$ is verified unless $a$ lies in the subcomplex formed by generators located in $D$. In case $a$ does lie in this subcomplex, and $a \neq 0$, then $A_{H_\lambda}(a) \geq -\epsilon$, in which case [33] holds still. The case $a = 0$ is trivial. Hence (3) applies to all $a \in CF^*(H_\lambda)$, and $x \in CF^*(L_0,L_1)$.

Finally, we recall two fundamental results on the symplectic topology of cotangent bundles. The first result, proved by Viterbo [40], Abbondandolo-Schwarz [1], and Salamon-Weber [31] in the case of $K = \mathbb{R}_2$, or for $Spin$ manifolds and arbitrary coefficients, and by Abouzaid and Krall in the general case (see [4] and the references therein), asserts a relation between the symplectic cohomology of $(T^*L, \theta_{can})$ considered as a Weinstein manifold, and the homology of the free loop space $\mathcal{L} L$. In general, to compare signs between the two theories, a local system
on $\mathcal{L}T^*L$ should be introduced, as mentioned above. For certain choices of $L$ and $\mathbb{K}$, such as $\mathbb{K} = \mathbb{F}_2$, or $L$ being Spin, this local system is trivial and can therefore be ignored. Finally, we note that it is important for our purposes, that this is an isomorphism of Gerstenhaber algebras over $\mathbb{K}$, rather than simply one of $\mathbb{K}$-algebras. This aspect of the isomorphism is discussed in [4]: in fact it is an isomorphism of BV-algebras.

**Theorem B** (Viterbo isomorphism). There exists an isomorphism
\[ \Phi : SH^*(W) \to H_{n-*}(\mathcal{L}L), \]
of BV-algebras over $\mathbb{K}$, where $SH^*(W)$ is endowed with the pair-of-pants product, and the BV-operator arising from the moduli space of cylinders with free asymptotic markers at infinity, while $H_{n-*}(\mathcal{L}L)$ is endowed with the Chas-Sullivan product, and the BV-operator given by suspending the $S^1$-action by loop-rotation on $\mathcal{L}L$. The map $ev : H_{n-*}(\mathcal{L}L) \to H_{n-*}(L)$ corresponds to the map $r_L : SH^*(W) \to HF^*(L)$ by the isomorphism $\Phi$ and Poincaré duality.

The second result, due to Fukaya-Seidel-Smith [15] in the simply connected case, and Abouzaid [3] and Kragh [21], in the general case, asserts that each exact Lagrangian $L'$ in the cotangent bundle $T^*L$ is isomorphic to $L$ in the Fukaya category of $T^*L$. It is not difficult to observe that this isomorphism is in fact an isomorphism of modules over $SH^*(T^*L)$: indeed, after one knows that the isomorphism is given by multiplication by continuation elements, this is a consequence of the homotopy property of $\phi_1^1_{L_0,L_1}$. We state a simplified version that is sufficient for our purposes, referring to [5].

**Theorem C** (Exact nearby Lagrangians are Floer-theoretically equivalent). Let $L$ and $\mathbb{K}$ be as above, and let $L'$ be an exact Lagrangian in $T^*L$. Then for each exact Lagrangian $K$, the $SH^*(T^*L)$-modules $HF^*(L',K)$, and $HF^{*+i}(L,K)$ are isomorphic, and the same is true for $HF^*(K,L')$, and $HF^{*+i}(K,L)$, for certain $i \in \mathbb{Z}$. The isomorphisms in both directions can be taken to be multiplication operators $\mu_2(-,[x]), \mu_2(-,[y])$, respectively $\mu_2([y],-), \mu_2([x],-)$, for $[x] \in HF^i(L,L'), [y] \in HF^{-i}(L',L)$.

At this point we define the spectral norm $\gamma(L_0,L_1)$ for exact Lagrangians in $T^*L$ as follows. Choose primitives $f_0,f_1$ of the restrictions $\theta_{L_0},\theta_{L_1}$ of the Liouville form $\theta$ to $L_0,L_1$ respectively. This allows us to filter $CF^*(L_0,L_1)$ by an action functional induced by $L_0 = (L_0,f_0), L_1 = (L_1,f_1)$. Since $HF^*(L_0,L_1) \cong HF^*(L,L) \cong H^*(L)$, consider the classes $\mu, e \in HF^*(L_0,L_1)$ that correspond to the generator $\mu_L = PD([pt]) \in H^n(L)$, and the unit $1 = PD([L]) \in H^0(L)$ respectively. Recall that for a class $a \in HF^*(L_0,L_1) \setminus \{0\}$, we defined the Lagrangian spectral invariant as
\[ c(a,L_0,L_1;D) = \inf\{ t \in \mathbb{R} : a \in \text{im}(HF^*(L_0,L_1;D)^t) \to HF^*(L_0,L_1;D) \}. \]
These invariants are finite, and satisfy numerous useful properties, and in particular they are defined for arbitrary exact $L_0,L_1$, by taking the limit as the Hamiltonian term in the perturbation datum $D$ goes to zero. We set the spectral norm to be
\[ \gamma(L_0,L_1) = c(\mu,L_0,L_1) - c(e,L_0,L_1). \]
Note that as a difference of two spectral invariants it does not depend on the choice of enhancements $L_0,L_1$ of $L_0,L_1$. Furthermore, by considering the identity $\mu_L = \mu_L * 1$ in $H^*(L)$, one obtains the identity $\mu = \mu_L * e$, under the isomorphisms $H^*(L) \cong HF^*(L,L) \cong HF^*(L_1,L_1)$ and $H^*(L) \cong HF^*(L,L) \cong HF^*(L_0,L_1)$, from which one obtains that $\gamma(L_0,L_1) \geq 0$, and that the inequality is strict unless $L_0 = L_1$ (see [20]). Further properties of spectral invariants imply that $\gamma(L_0,L_1) = \gamma(L_1,L_0)$ for all $L_0,L_1$ exact, and that $\gamma(L_0,L_1) \leq \gamma(L_0,K) + \gamma(K,L_1)$ for all $L_0,L_1,K$ exact, whence $\gamma$ defines a metric on the
space of exact Lagrangian submanifolds of $T^*L$. Furthermore, this metric is invariant under the action of the group of Hamiltonian diffeomorphisms: for all $H \in C^\infty_c([0,1] \times T^*L, \mathbb{R})$ and $L_0, L_1$ exact, $\gamma(\phi L, \phi L_1) = \gamma(L, L_1)$, where $\phi = \phi^1_H$, is the time-one map of the Hamiltonian isotopy generated by $H$. Finally, note that we shall study the restriction of $\gamma$ to the subspace of exact Lagrangian submanifolds in $D \subset T^*L$, where $D$ is a bounded Liouville domain with completion $T^*L$.

Set $\tilde{\phi}^1_{(L_0, L_1)}(a) := \tilde{\phi}^1_{L_0, L_1}(a, -).$ By means of Theorem [C] if we are merely interested in the operator $P'_{(L_0, L_1), a} = \tilde{\phi}^1_{(L_0, L_1), a}$ on the homological level, and $L_0, L_1$ are exact Lagrangians in $T^*L$, then we may replace both $L_0$, and $L_1$ by $L$ in the definition. In this case, we compute the operation $P'_a = [\tilde{\phi}^1_{(L, L), a}]$ as follows.

**Proposition 8.** The isomorphism $HF^*(L, L) \cong H_{n-s}(L)$ obtained from by the isomorphism $HF^*(L, L) \cong H^*(L)$ followed with the Poincaré duality isomorphism $H^*(L) \cong H_{n-s}(L)$, identifies $P'_a : HF^*(L, L) \to HF^*(L, L)$ and the map $P_a : HF_{n-s}(L) \to HF_{n-s}(L)$.

**Proof of Proposition 8 (sketch).** One way to prove this result involves first showing that $P'_a = r_L \circ m'_a \circ \iota'$, where $\iota'$ is a homological inclusion map $HF^*(L, L) \cong HF_{n-s}(L, L) \to SH^*(T^*L)$, $m'_a : SH^*(T^*L) \to SH^*(T^*L)$ is the right symplectic homology bracket with $a \in SH^*(T^*L)$, given by the pair of pants product and the BV-operator $\Delta' : SH^*(T^*L) \to SH^*(T^*L)[-1]$, and $r_L : SH^*(T^*L) \to HF^*(L, L)$ is the restriction map. Consequently, one shows that each of these maps is identified with $\iota, m_{\Phi(a)}, \Delta$, and $ev$, respectively under the isomorphisms $\Phi$ and $HF^*(L, L) \cong H^*(L) \cong H_{n-s}(L)$. Theorem [E] takes care of identifying $m_{\Phi(a)}$ and $m_{\Phi(a)}$, and $\Delta'$ with $\Delta$. It is left to identify $\iota'$ with $\iota$, and $r_L$ with $ev$. The map $r_L$ in the latter pair is a suitable closed-open map, and the identification is carried out by following the isomorphism $\Phi$ from [4]. The map $\iota'$ defined below can again be seen to correspond to $\iota$ by following the construction in [4]. Alternatively, see [1, 2].

To show that $P'_a = r_L \circ m'_a \circ \iota'$, we first observe that $\iota'$ is given [4] (see also [11]) by the moduli space of disks with one interior marked point, which is an output, and boundary conditions on a cotangent fibre $T^*_x L$ where $x$, that we consider to be an input, is allowed to vary freely in $L$. We choose the perturbation datum to coincide with $H_\lambda \otimes dt$ on a cylindrical end by the output, and with zero near the boundary. This is possible to achieve while keeping the conditions of the integrated maximum principle of Abouzaid and Seidel [6], that keeps the corresponding solutions to the Floer equations within $D \cup C_\lambda$.

Similarly, $r_L$ is given by the moduli space of disks with one interior marked point, which is an input, and one boundary marked point which is an output, with the asymptotic marker at the interior marked point pointing towards the boundary marked point, and with boundary conditions on $L$. It is then easy to see by gluing, that $r_L \circ \iota'$ is given by the moduli space of annuli with boundary conditions on a cotangent fiber on one boundary component and $L$ on the other, with a boundary marked point, an output: that is in a concrete model, if the annulus is given by $[0,1] \times S^1$, the boundary marked point is $(l, \zeta)$ for $\zeta \in S^1$, and the boundary condition at $\{0\} \times S^1$ is on $T^*_x L$ for a varying $x$. Moreover it is easy see, for example by a suitable homotopy of the Hamiltonian perturbation data and a pearly model for this operation, that on the homology level $r_L \circ \iota' = id$. Hence $P'_a = P'_a \circ r_L \circ \iota'$. We claim that it is therefore given by the compactified moduli space of annuli with one boundary component marked by a varying cotangent fibre, the other boundary component marked by $L$ and endowed with an unconstrained boundary marked point, an output, and one interior input marked point. The asymptotic marker at the interior marked point is pointing towards the output, and we use the interior marked point to plug in $a$. 

This last claim requires further elaboration. We proceed as follows. First we glue the moduli space of disks describing $\phi^1_{L,L}(a,-)$ and the moduli space of disks describing $r_L \circ i'$ on the chain level. After fixing parametrization, we can fix the former to be given by the standard disk $D^2 \subset \mathbb{C}$ with two fixed marked points $\zeta_- = -1, \zeta_+ = 1$ on the boundary, and an interior marked point $z = iy$ with $\text{Re}(z) = 0$, and asymptotic marker pointing towards $\zeta_+$ along a hyperbolic geodesic. Hence the glued operation is given by an cylinder $[0, 1] \times S^1$ with boundary conditions as above, and interior marked point constrained to an arc $\gamma_1$ with boundary $\{(l, \zeta_1)\} - \{(l, \zeta_0)\}$ on $l \times S^1$ separating $(l, 0)$ and $\{0\} \times S^1$, with asymptotic marker pointing towards $(l, 0)$. Furthermore, we identify, up to chain-homotopy, the correction term $-\mu_2(c_L, -) + (-1)^{k-1}|x| \mu_2(-, c_L)$ in $\overline{\phi^1_{L,L,a}}$, precomposed with the chain-level map $\psi$ giving $r_L \circ i'$, as the operator given by the same Riemann surface, now considered as an annulus with outer boundary $\{l\} \times S^1$, with the same boundary conditions and boundary marked points, except that the interior marked point, used to plug in $a$, is now constrained to an arc $\gamma_2$ with boundary $\{-\{(l, \zeta_1)\} + \{(l, \zeta_0)\}\}$, which now does not separate $(l, 0)$ and $\{0\} \times S^1$ (this is a result of parametric gluing applied to a one-parametric family of nodal annuli, consisting of an annulus and a disk related by a node $(l, \zeta)$, with the interior marked point in the disk, used to plug in $a$, the asymptotic marker pointing at the node, interpolating between the surface for $(l, \zeta_1)$, and the one for $(l, \zeta_0)$, with $\zeta$ in the spherical arc $[\zeta_1, \zeta_0]$ not containing $0$). The asymptotic marker at the interior marked point still points towards $(l, 0)$. Choosing $\gamma_1$ and $\gamma_2$ suitably, and considering a homotopy of the decorated Riemann surfaces corresponding to the loop $\gamma_1 \# \gamma_2$, we obtain the claim.

Finally, it is easy to see, by gluing again, that $r_L \circ m'_0 \circ i'$ is given by a homotopic moduli space of decorated annuli. This is immediate by gluing from the description of the string bracket [A, Section 2.5.1] as the moduli space of spheres with 3 marked points, where in the model of $S^2 = \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ we choose the marked points $z_0 = \infty, z_1 = 0$, and $z_2$ restricted to the unit circle $z_2 = e^{i\theta}$ for $\theta \in \mathbb{R}/(2\pi)\mathbb{Z}$. The asymptotic markers at $z_0, z_1$ are chosen to be tangent to the imaginary axis, and pointing in the negative, resp. positive, direction. The asymptotic marker at each $z_2 = e^{i\theta}$ is chosen by identifying $S^2 \backslash \{z_0, z_2\}$ biholomorphically with $\mathbb{R} \times S^1$, so that the output asymptotic marker at $z_0$ correspond to the ray $(-\infty, 0) \times \{0\}$ in the negative end $(-\infty, 0) \times S^1$, and choosing the input asymptotic marker at $z_2$ to correspond to the ray $(0, \infty) \times \{0\}$ in the positive end $(0, \infty) \times S^1$.  

3. Proofs

3.1. Proof of Theorem [A] Let $a_1, \ldots, a_N \in H_{n-*}(\mathcal{L}L)^+$ be such that

$$P_{a_N} \circ \ldots \circ P_{a_1}([pt]) = [L].$$

Let $\mu, e \in HF^*(L_0, L_1)$ be such that $\mu$ corresponds to $[pt]$ and $e$ corresponds to $[L]$ under the isomorphism

$$HF^{* + i_0 - i_1}(L_0, L_1) \cong HF^{* + i_0}(L_0, L) \cong HF^*(L, L) \cong H_{n-*}(L),$$

for suitable integers $i_0, i_1 \in \mathbb{Z}$. In this case the spectral norm $\gamma(L_0, L_1)$ is given by

$$\gamma(L_0, L_1) = c(\mu, L_0, L_1) - c(e, L_0, L_1).$$

It is therefore sufficient to prove that there exists a constant $C(g, L; \mathbb{K})$ such that

$$c(\mu, L_0, L_1) \leq c(e, L_0, L_1) + C(g, L; \mathbb{K}).$$

Let $a'_1 = \Phi^{-1}(a_1), \ldots, a'_N = \Phi^{-1}(a_N)$. Since $a_1, \ldots, a_N \in H_{n-*}(\mathcal{L}L)^+$, we obtain that $L_0, L_1$ are $a_j$-equivariant for all $1 \leq j \leq N$. In view of Proposition [K], the identity (5) corresponds to
the identity

\[ P'_{(L_0,L_1),a'_N} \circ \cdots \circ P'_{(L_0,L_1),a'_1} (e) = \mu. \]

Set \( C_j = A_{\mathbf{H}_{\lambda}} (a'_j) \) for representatives \( a'_j \) of \( a'_j \), and \( x_j = P'_{(L_0,L_1),a'_j} \circ \cdots \circ P'_{(L_0,L_1),a'_1} (e), \) \( 1 \leq j \leq N, \) \( x_N = e. \) We lift these elements to the chain level as follows. Let \( \tilde{x}_0 = \tilde{e} \in CF^*(L_0, L_1) \) be a chain representative of \( e \) with

\[ A_{L_0,L_1,\mathcal{D}}(\tilde{e}) \leq c(e, L_0, L_1) + \epsilon \]

(where \( \mathcal{D} \) is chosen to be Hamiltonian-small). Then for \( 1 \leq j \leq N \) we set

\[ \tilde{x}_j = \phi_{1(L_0,L_1)} \circ \cdots \circ \phi_{1(L_0,L_1)}(a'_j). \]

In this situation we obtain by (6) that for all \( 0 \leq j < N, \)

\[ A_{L_0,L_1,\mathcal{D}}(\tilde{x}_{j+1}) \leq A_{L_0,L_1,\mathcal{D}}(\tilde{x}_j) + A_{\mathbf{H}_{\lambda}}(a_{j+1}) + 2\epsilon. \]

Therefore by (6), we obtain that

\[ A_{L_0,L_1,\mathcal{D}}(\tilde{x}_N) \leq c(e, L_0, L_1) + \sum_{j=1}^{N} C_j + 2(N + 1)\epsilon. \]

As \( [\tilde{x}_N] = x_N = \mu, \) by definition of the spectral invariant we have

\[ c(\mu, L_0, L_1) \leq A_{L_0,L_1,\mathcal{D}}(\tilde{x}_N), \]

and this finishes the proof.

Remark 9. The inequality

\[ c(\mu, L_0, L_1) \leq c(e, L_0, L_1) + \sum_{j=1}^{N} C_j + 2(N + 1)\epsilon \]

finishing the proof can be optimized as follows. First we may choose \( C_j = c([a_j], H_{\lambda}) \), to obtain

\[ c(\mu, L_0, L_1) \leq c(e, L_0, L_1) + \sum_{j=1}^{N} c(a'_j, H_{\lambda}) + 3(N + 1)\epsilon. \]

Furthermore, as \( \lambda \to \infty, \) by definition of spectral invariants in filtered symplectic homology, we obtain that \( c(a'_j, H_{\lambda}) \to c(a'_j, D, S) \). Therefore we can write

\[ c(\mu, L_0, L_1) \leq c(e, L_0, L_1) + \sum_{j=1}^{N} c(a'_j, D, S) + 3(N + 1)\epsilon, \]

and sending \( \epsilon \) to 0, we finally get the bound

\[ \gamma(L_0, L_1) \leq \sum_{j=1}^{N} c(a'_j, D, S). \]
3.2. Proof of Proposition 3. Let $L$, $L'$ be string point-invertible. We will show that so is $L \times L'$. Let $a_1, \ldots, a_N$, and $a_1', \ldots, a_{N'}$, be sequences exhibiting the fact that $L$, respectively $L'$, are string point-invertible. Consider $b_k = a_k \otimes \iota([L'])$, for $1 \leq k \leq N$, and $b_k = \iota([L]) \otimes a'_{k-N}$, for $N < k \leq N + N'$. We claim that the sequence $b_1, \ldots, b_{N+N'}$, up to sign, is the required one for $L \times L'$. Since it is indeed enough to consider the question up to signs, we will ignore signs coming from the Koszul rule for tensor products. Note that by Künneth theorem $H_*(L \times L') \cong H_*(L) \otimes H_*(L')$ and $H_*(\mathcal{L}L \times \mathcal{L}L') \cong H_*(\mathcal{L}L) \otimes H_*(\mathcal{L}L')$. Furthermore, under changing the grading to $* - n$ everywhere, these splittings are tensor products of graded algebras. Moreover the maps $\iota$ and $ev$ commute with this tensor product decomposition. The BV-operator behaves in the following way: for homogeneous elements $x \in H_*(\mathcal{L}L)$, $x' \in H_*(\mathcal{L}L')$,

$$\Delta_{L \times L'}(x \otimes x') = \Delta_L(x) \otimes x' + (-1)^{|x|} x \otimes \Delta_{L'}(x').$$

This implies that for $a \in H_*(L)$, $a' \in H_*(L')$,

$$m_a \otimes a'(x \otimes x') = \pm m_a(x) \otimes a' \ast x' \pm a \ast x \otimes m_a'(x').$$

Observe that $[pt_L \times L'] = [pt_L] \otimes [pt_{L'}]$, $[L \times L'] = [L] \otimes [L']$, and furthermore for $a' = \iota([L'])$,

$$m_{a'} = 0.$$ Therefore

$$m_{b_1} \iota([pt_L] \otimes [pt_{L'}]) = m_{a_1} \otimes \iota([L']) \iota([pt_L] \otimes [pt_{L'}]) = \pm m_{a_1} \iota([pt_L]) \otimes \iota([pt_{L'}]).$$

Finally, since $ev$ is an algebra map, and it commutes with the Künneth decompositions, we obtain

$$P_{b_1}([pt_L \times L']) = \pm P_{a_1}([pt_L] \otimes [pt_{L'}]).$$

The same argument shows that for all $1 \leq k \leq N$,

$$P_{b_k} \circ \cdots \circ P_{b_1}([pt_L \times L']) = \pm P_{a_k} \circ \cdots \circ P_{a_1}([pt_L] \otimes [pt_{L'}]).$$

In particular, for $k = N$,

$$(8) \quad P_{b_N} \circ \cdots \circ P_{b_1}([pt_L \times L']) = \pm [L] \otimes [pt_{L'}].$$

Similarly, we obtain that for all $1 \leq k \leq N'$,

$$P_{b_{N+k}} \circ \cdots \circ P_{b_{N+k}}([L] \otimes [pt_{L'}]) = \pm [L] \otimes P_{a'_k} \circ \cdots \circ P_{a'_1}([pt_L]).$$

In particular, for $k = N'$,

$$(9) \quad P_{b_{N+N'}} \circ \cdots \circ P_{b_{N+N'}}([L] \otimes [pt_{L'}]) = \pm [L] \otimes [L'].$$

Hence, by (8) and (9), we obtain

$$P_{b_{N+N'}} \circ \cdots \circ P_{b_1}([pt_L \times L']) = \pm [L \times L'].$$

If necessary, changing the sign of $b_{N+N'}$, we finish the proof.

Acknowledgements

I thank Mohammed Abouzaid for suggesting to look at equivariant Lagrangians in the quantitative context, and Leonid Polterovich for motivating discussions on applying symplectic cohomology. I thank Paul Biran, Lev Buhovsky, Octav Cornea, Michael Entov, Viktor Ginzburg, Helmut Hofer, Dmitry Tonkonog, and Sara Venkatesh, for inspiring conversations. This work was supported by an NSERC Discovery Grant and by the Fonds de recherche du Québec - Nature et technologies.
References

[1] A. Abbondandolo and M. Schwarz. Floer homology of cotangent bundles and the loop product. *Geom. Topol.*, 14(3):1569–1722, 2010.

[2] A. Abbondandolo and M. Schwarz. Corrigendum: On the Floer homology of cotangent bundles. *Comm. Pure Appl. Math.*, 67(4):670–691, 2014.

[3] M. Abouzaid. Nearby Lagrangians with vanishing Maslov class are homotopy equivalent. *Invent. Math.*, 189(2):251–313, 2012.

[4] M. Abouzaid. Symplectic cohomology and Viterbo’s theorem. In *Free loop spaces in geometry and topology*, volume 24 of *IRMA Lect. Math. Theor. Phys.*, pages 271–485. Eur. Math. Soc., Zürich, 2015.

[5] M. Abouzaid and T. Kragh. Simple homotopy equivalence of nearby Lagrangians. *Acta Math.*, 220(2):207–237, 2018.

[6] M. Abouzaid and P. Seidel. An open string analogue of Viterbo functoriality. *Geom. Topol*.

[7] P. Biran and O. Cornea. Private communication, 2018.

[8] L. Buhovsky, V. Humilière, and S. Seyfaddini. The action spectrum and $C^0$ symplectic topology. Preprint arXiv:1808.09790, 2018.

[9] M. Cadek and Z. Moravec. Loop homology of quaternionic projective spaces. Preprint arXiv:1004.1550 [math.AT], 2010.

[10] M. Chas and D. Sullivan. String topology. Preprint, arXiv:math/9911159 [math.GT], 1999.

[11] D. Chataur and J.-F. Le Borgne. On the loop homology of complex projective spaces. *Bull. Soc. Math. France*, 139(4):503–518, 2011.

[12] T. Etgü and Y. Lekili. Koszul duality patterns in Floer theory. *Geom. Topol.*, 21(6):3313–3389, 2017.

[13] Y. Félix, L. Menichi, and J.-C. Thomas. Gerstenhaber duality in Hochschild cohomology. *J. Pure Appl. Algebra*, 199(1-3):43–59, 2005.

[14] Y. Félix and J.-C. Thomas. Rational BV-algebra in string topology. *Bull. Soc. Math. France*, 136(2):311–327, 2008.

[15] K. Fukaya, P. Seidel, and I. Smith. Exact Lagrangian submanifolds in simply-connected cotangent bundles. *Invent. Math.*, 172(1):1–27, 2008.

[16] J. Gutt and M. Hutchings. Symplectic capacities from positive $S^1$-equivariant symplectic homology. *Algebr. Geom. Topol.*, 18(6):3537–3600, 2018.

[17] R. A. Hepworth. String topology for complex projective spaces. Preprint arXiv:0908.1013 [math.AT], 2009.

[18] R. A. Hepworth. String topology for Lie groups. *J. Topol.*, 3(2):424–442, 2010.

[19] B. Keller. Derived invariance of higher of higher structures on the hochschild complex. Preprint, 2004, available at https://webusers.imj-prg.fr/bernhard.keller/publ/dih.pdf.

[20] A. Kislev and E. Shelukhin. Bounds on spectral norms and applications. Preprint arXiv:1810.09865, 2018.

[21] T. Kragh. Parametrized ring-spectra and the nearby Lagrangian conjecture. *Geom. Topol.*, 17(2):639–731, 2013. With an appendix by Mohammed Abouzaid.

[22] R. Leclercq and F. Zapolsky. Spectral invariants for monotone lagrangians. *J. Topol. Anal.* Available at https://doi.org/10.1142/S1793525318500267.

[23] J. Lee. Fiberwise convexity of Hill’s lunar problem. *J. Topol. Anal.*, 9(4):571–630, 2017.

[24] F. Leroux, S. Seyfaddini, and C. Viterbo. Barcodes and area-preserving homeomorphisms. Preprint arXiv:1810.03139 [math.SG], 2018.
[25] E. Malm. *String topology and the based loop space*. PhD thesis, Stanford University, 2010.

[26] D. McDuff and D. Salamon. *J-holomorphic curves and symplectic topology*. 2nd ed., volume 52. Providence, RI: American Mathematical Society (AMS), 2nd ed. edition, 2012.

[27] L. Menichi. String topology for spheres. *Comment. Math. Helv.*, 84(1):135–157, 2009. With an appendix by Gerald Gaudens and Menichi.

[28] L. Menichi. A Batalin-Vilkovisky algebra morphism from double loop spaces to free loops. *Trans. Amer. Math. Soc.*, 363(8):4443–4462, 2011.

[29] A. Monzner, N. Vichery, and F. Zapolsky. Partial quasimorphisms and quasistates on cotangent bundles, and symplectic homogenization. *J. Mod. Dyn.*, 6(2):205–249, 2012.

[30] L. Polterovich, D. Rosen, K. Samvelyan, and J. Zhang. Topological persistence in geometry and analysis. Preprint, arXiv:1904.04044 [math.AT], 2019.

[31] D. A. Salamon and J. Weber. Floer homology and the heat flow. *Geom. Funct. Anal.*, 16(5):1050–1138, 2006.

[32] P. Seidel. *Fukaya categories and Picard-Lefschetz theory*. Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, 2008.

[33] P. Seidel and J. P. Solomon. Symplectic cohomology and q-intersection numbers. *Geom. Funct. Anal.*, 22(2):443–477, 2012.

[34] S. Seyfaddini. Descent and C0-rigidity of spectral invariants on monotone symplectic manifolds. *J. Topol. Anal.*, 4(4):481–498, 2012.

[35] E. Shelukhin. Viterbo conjecture for Zoll symmetric spaces. Preprint arXiv:1811.05552 [math.SG], 2018.

[36] V. Stojisavljević and J. Zhang. Persistence modules, symplectic Banach-Mazur distance and Riemannian metrics. Preprint arXiv:1810.11151 [math.SG], 2018.

[37] H. Tamanoi. Batalin-Vilkovisky Lie algebra structure on the loop homology of complex Stiefel manifolds. *Int. Math. Res. Not.*, pages Art. ID 97193, 23, 2006.

[38] D. Tonkonog. Private communication, 2019.

[39] D. Vaintrob. The string topology BV algebra, Hochschild cohomology and the goldman bracket on surfaces. Preprint arXiv:math/0702859 [math.AT], 2007.

[40] C. Viterbo. Functors and computations in Floer homology with applications. I. *Geom. Funct. Anal.*, 9(5):985–1033, 1999.

[41] C. Viterbo. Symplectic homogenization. 2014. Preprint arXiv:0801.0206 [math.SG].

[42] C. Westerland. String homology of spheres and projective spaces. *Algebr. Geom. Topol.*, 7:309–325, 2007.

[43] T. Yang. A Batalin-Vilkovisky algebra structure on the Hochschild cohomology of truncated polynomials. *Topology Appl.*, 160(13):1633–1651, 2013.

[44] F. Zapolsky. On the Hofer geometry for weakly exact Lagrangian submanifolds. *J. Symplectic Geom.*, 11(3):475–488, 2013.

Egor Shelukhin, Department of Mathematics and Statistics, University of Montreal, C.P. 6128 Succ. Centre-Ville Montreal, QC H3C 3J7, Canada

E-mail address: shelukhin@dms.umontreal.ca