Odd Order Pandiagonal Latin and Magic Cubes in Three and Four Dimensions

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I Introduction

By a magic square of order $n$ is here meant an arrangement, without repeats, of the integers $\{0, 1, 2, \ldots, n^2 - 1\}$ into the $n^2$ cells of an $n \times n$ square in a way that the sum of the elements of each row, of each column and of each of the two diagonals is the same. Since the sum of $1 + 2 + \ldots + (n^2 - 1)$ is $n^2(n^2-1)/2$ and since for a magic square this must be equal to the sum of the sum of the $n$-integers in each of the $n$-rows (or columns) it follows that the common sum, $\sigma_2$ of these integers must be

$$\sigma_2 = \frac{n(n^2 - 1)}{2} \quad (1.1)$$

Figure I shows two magic squares for $n = 4$ and $n = 5$

$$
\begin{array}{cccc}
0 & 10 & 15 & 5 \\
7 & 13 & 8 & 2 \\
9 & 3 & 6 & 12 \\
14 & 4 & 1 & 11 \\
\end{array} \quad \begin{array}{cccc}
13 & 19 & 20 & 1 \\
21 & 2 & 8 & 14 \\
9 & 10 & 16 & 22 \\
17 & 23 & 4 & 5 \\
\end{array}
$$

Figure I
with sum values 30 and 60, respectively. Of particular interest is the magic square in Ib. Not only do the sums along the rows, the columns and the two diagonals have the value 60, but so do the eight broken diagonals that are obtained if all partial diagonals are completed by imagining the square bent around into a cylinder. Examples of broken diagonals in Figure Ib are: 21, 10, 4, 18, 7; 9, 23, 12, 15, 1; 17, 6, 3, 14, 20; 11, 22, 8, 19, 0. Magic \( n \times n \) squares, such as this one, where the sums along the \( n \)-rows and columns, the two diagonals and the \( 2n - 2 \) broken diagonals all have the same value are called pandiagonal or "diabolic" magic squares \[9, \text{ ch. 10}], [11]. To simplify, we shall use the term ROW in all of the following to mean any one of a row, a column, a diagonal or a broken diagonal. Thus in a pandiagonal magic square the sum along any ROW is the same and given by (1.1).

By a magic pandiagonal cube of order \( n \) we shall mean an arrangement of the integers 0, 1, 2, \ldots n^3 - 1 on the lattice points of an \( n \times n \times n \) cube so that the 3\( n \) squares parallel to the faces of the cube as well as the six "diagonal" squares which bisect the cube and contain its 4 body-diagonals are all pandiagonal magic squares. Thus in each of the \( 3n + 6 \) (overlapping) squares contained in the cube, the sums along all of the the ROWS—which here includes also the files—are the same. It is easily seen that in this case the common sum \( \sigma_3 \) must be

\[
\sigma_3 = \frac{n(n^3 - 1)}{2}
\]  \hspace{1cm} (1.2)

By extension, we define a four dimensional (4-D) pandiagonal magic \( n \times n \times n \times n \) cube as one in which the integers 0, 1, 2, \ldots n^4 - 1 are placed, without repeats, at the \( n^4 \) lattice sites of a 4-D hypercube, so that the three
dimensional (overlapping) $n \times n \times n$ cubes that can be formed within it, are pandiagonal magic cubes. The sum along each ROW of a 4-D cube is easily shown to be

$$\sigma_4 = n(n^4 - 1)/2$$  \hspace{1cm} (1.3)

In constructing pandiagonal magic squares, cubes, and hypercubes, we follow the idea of Euler [6]—who was concerned exclusively with magic squares—and used latin squares (LS) in their construction. As defined by Denes and Keedwell [4] and by Laywine and Mullen [9] a latin square of order $n$ is an $n \times n$ array each of whose lattice points is occupied by one of $n$ given symbols, in a way so that no row or column contains any one of these symbols more than once. The number of LS’s grows rapidly with $n$ and for example for $n = 10$ and 15 this number is $\sim 10^{36}$ and $\sim 1.5 \times 10^{86}$, respectively [9, p. 5]. Here we are interested in the relatively small subset of the LS’s, the pandiagonal latin squares, for which in addition each diagonal as well as each of the $2n - 2$ broken diagonals—in short it’s ROWS—also contains each of the $n$ elements precisely once.

For a general algebraic theory of pandiagonal (diabolic) magic squares see the analysis by Rosser and Walker [11]. The books by Andrews [2], Kraitchik [5] and Rause Ball and Coxeter [3] contain a more empirical approach to magic squares and include various examples and practical rules for constructing magic squares. Pasles [10] describes some unusual magic squares constructed by Benjamin Franklin.

Some $5 \times 5$ pandiagonal magic squares and some of their important features are described by Gardner [7]. In a companion column [8] he details
a magic cube of order seven and notes its *non-pandiagonal nature*. This is reenforced by the results of Wynne [12] who studied magic cubes of order seven and showed that even if every square of the cube that is parallel to a cube-face is pandiagonal, not all of the six, diagonal squares of the cube can be pandiagonal. This is consistent with the present analysis, according to which only for $n \geq 11$ do magic pandiagonal cubes exist. (Similarly we find that only for $n \geq 17$ can 4-D pandiagonal magic cubes exist.) That *non*-pandiagonal magic cubes of order 7 do exist, however, is established by Andrews [2] and by Alspach and Heinrich [1], the latter, incidental to their discussion of cubes of order 4m.

II. Odd-$n$ Pandiagonal Latin Squares and Cubes in 3 and 4 Dimensions

A. Pandiagonal Latin Squares

To set the stage for our discussion of magic pandiagonal cubes in 3 and 4 dimensions, in this section we collect—and in some instances amplify on—properties of pandiagonal latin squares [4],[9].

A *latin square* of odd order $n$, $LS$, is an $n \times n$ array involving $n$ distinct symbols with each symbol appearing once and only once in each row and column. We shall invariably take the $n$ symbols to be the integers comprising the set $S$ defined by

$$S = \{0, 1, 2, \ldots, n - 1\}$$  \hspace{1cm} (2.1)

Also useful will be the set $\bar{S}$ defined by

$$\bar{S} = \{1, 2, \ldots, n - 1\}$$  \hspace{1cm} (2.2)
A **diagonal LS** has the additional property that each of its two diagonals also contains the \( n \) chosen symbols exactly once. Finally a **pandiagonal LS** is a diagonal LS in which, in addition, each of the \( n \) symbols appears once and only once in each of the \( 2n - 2 \) “broken diagonals”, as defined in the preceding section.

Figure II shows 4 LS’s of order 3,4,5, and 5, respectively. The \( 3 \times 3 \) LS in part (a) is not a diagonal LS by virtue of the three zeros

\[
\begin{array}{ccc}
0 & 1 & 2 \\
2 & 0 & 1 \\
1 & 2 & 0 \\
\end{array}
\]

\[
\begin{array}{ccc}
0 & 1 & 2 & 3 \\
2 & 3 & 0 & 1 \\
3 & 2 & 1 & 0 \\
1 & 0 & 3 & 2 \\
\end{array}
\]

\( (a) \)

\[
\begin{array}{cccc}
3 & 4 & 0 & 1 & 2 \\
1 & 2 & 3 & 4 & 0 \\
4 & 0 & 1 & 2 & 3 \\
2 & 3 & 4 & 0 & 1 \\
0 & 1 & 2 & 3 & 4 \\
\end{array}
\]

\[
\begin{array}{cccc}
4 & 0 & 1 & 2 & 3 \\
2 & 3 & 4 & 0 & 1 \\
0 & 1 & 2 & 3 & 4 \\
3 & 4 & 0 & 1 & 2 \\
1 & 2 & 3 & 4 & 0 \\
\end{array}
\]

\( (b) \)

\[
\begin{array}{cccc}
3 & 4 & 0 & 1 & 2 \\
1 & 2 & 3 & 4 & 0 \\
4 & 0 & 1 & 2 & 3 \\
2 & 3 & 4 & 0 & 1 \\
0 & 1 & 2 & 3 & 4 \\
\end{array}
\]

\[
\begin{array}{cccc}
4 & 0 & 1 & 2 & 3 \\
2 & 3 & 4 & 0 & 1 \\
0 & 1 & 2 & 3 & 4 \\
3 & 4 & 0 & 1 & 2 \\
1 & 2 & 3 & 4 & 0 \\
\end{array}
\]

\( (c) \)

\( (d) \)

**Figure II**

appearing in the upper left to lower right diagonal. By contrast the \( 4 \times 4 \) LS in part (b) is diagonal but not pandiagonal while the \( 5 \times 5 \), LS in part (c) is pandiagonal. Note that, given a pandiagonal LS, if we permute its symbols \( \{0,1,2,...,n-1\} \) it remains pandiagonal. For example,
if in IIc, we carry out the permutation 0 → 1; 1 → 2; 2 → 3, 3 → 4; 4 → 0, the resultant LS remains pandiagonal and is given by Figure IId. Thus a given $n \times n$ pandiagonal LS is the basis for $n!$ different ones that result from the $n!$ possible permutations of the $n$-symbols.

Consider for odd $n$, the $n \times n$ array $L$ with elements $L_{ij}$

$$L_{ij} \equiv \alpha_1 i + \alpha_2 j \pmod{n} \quad (2.3)$$

where $i, j$ run over the elements of $S$ as do the elements $L_{ij}$ themselves and $\alpha_1$ and $\alpha_2$ are non-zero positive integer parameters and thus are elements of $\bar{S}$ in (2.2). When we wish to stress the dependence of $L$ on $\alpha_1$ and $\alpha_2$, we shall also use the notation $L = < \alpha_1, \alpha_2 >$.

We now establish:

**Theorem (2.1):** If $L = < \alpha_1, \alpha_2 >$ is the array in (2.3) and the greatest common divisor with $n$ of each of $\alpha_1, \alpha_2, \alpha_1 \pm \alpha_2$ is 1, that is

$$\alpha_i, n = 1; \quad i = 1, 2 \quad (2.4a)$$
$$\alpha_1 + \alpha_2, n = 1 \quad (2.4b)$$
$$\alpha_1 - \alpha_2, n = 1 \quad (2.4c)$$

then $L$ is a pandiagonal LS for $n \geq 5$. (Note that (2.4) cannot be satisfied for even $n$.)

**Proof:** Firstly, since for $n = 3$, the possible $\alpha$-values are 1 and 2 and since these violate (2.4b), it follows that (2.3) is not a pandiagonal LS for $n = 3$. On the other hand, for $n = 5$, the pairs $[\alpha_1, \alpha_2] = [1, 2]$ and $[1, 3]$, for example, do satisfy each of (2.4).
Secondly to establish first that $L$ in (2.3) for $n \geq 5$ is a diagonal LS, consider it for fixed $j$, say, as $i$ ranges over the $j$th row. As $i$ thus varies from 0 to $n-1$, the $j$th row of $L$ varies over the same set by virtue of the hypothesis $(\alpha_1, n) = 1$. Similarly for fixed $i$, as $j$ varies over the $i$th column, since $(\alpha_2, n) = 1$, $L_{ij}$ varies over the same set $S$. Thus the rows and columns of $L$ satisfy the condition that $L$ be a latin square. Along the diagonal from $(i, j) = (0, 0)$ to $(n-1, n-1)$, $i = j$, so that along here $L_{ij} \equiv (\alpha_1 + \alpha_2)i \pmod{n}$. Thus again as $i$ varies over the set $S$ the $n$ diagonal elements of $L$ must be some permutation of $S$ since $(\alpha_1 + \alpha_2, n) = 1$ according to (2.4b). Finally along the other the diagonal from $(i, j) = (0, n-1)$ to $(n-1, 0)$, $i+j = n-1$, so that $L_{ij} \equiv (\alpha_1 - \alpha_2)i + \alpha_2(n-1) \pmod{n}$ which has the same property by virtue of (2.4c), $(\alpha_1 - \alpha_2, n) = 1$. Thus we have established that $L$ in (2.3) under the constraints in (2.4) is a diagonal LS; it remains only to establish that it is also pandiagonal.

To this end, consider, for example, the “split” diagonal just above—and parallel to—the lower left to upper right diagonal of $L$ and its appendage in the lower right hand corner, $(i, j) = (n-1, 0)$. (We assume $i$ increases from 0 to the right and $j$ increases upward from 0.) Its entries are defined by $j = i+1$, $i = 0, 1, 2, \ldots n-2$ and $(i, j) = (n-1, 0)$. Substituting $j = i+1$ into (2.3) we find

$$L_{i,i+1} \equiv (\alpha_1 + \alpha_2)i + \alpha_2 \pmod{n}$$

If we now let $i$ run over the complete set $S$, it is easy to see that since $L_{in} \equiv L_{io} \pmod{n}$ we obtain both parts of the broken diagonal and that no two of these $n$-elements are the same since they are simply a permutation of the elements of $S$ by virtue of (2.4b). This argument is easily repeated for
all broken diagonals and we conclude that each broken diagonal consists of a permutation of the elements of $S$. We have thus established the theorem.

**Remark 2.1:** If $\alpha_1, \alpha_2$ satisfy each of (2.4), then so do $k\alpha_1, k\alpha_2$ for any $k \epsilon \bar{S}$ for which $(k, n) = 1$.

**Remark 2.2:** Reference to (2.3) shows that if $L =< \alpha_1, \alpha_2 >$ is a pandiagonal LS, then so is $< k\alpha_1, k\alpha_2 >$ for any positive integer $k \epsilon \bar{S}$ provided $(k, n) = 1$. For according to (2.3), $< k\alpha_1, k\alpha_2 >$, is simply $< \alpha_1, \alpha_2 >$ with its elements permuted in some way.

**Remark 2.3:** If we add an integer $x \epsilon \bar{S}$ to the right hand side of (2.3), we obtain $< \alpha_1, \alpha_2 >$ with its elements permuted in some way.

An important notion relating to LS’s is that of the **orthogonality** [9, ch. 2]. Two $n \times n$, LS’s are said to be orthogonal, if when they are superposed, none of the $n^2$ ordered pairs of elements that result occurs more than once. Thus if $L_{ij}^{(1)}$ and $L_{ij}^{(2)} \epsilon S$ are any corresponding elements of the two LS’s, $L^{(1)}$ and $L^{(2)}$ then they are orthogonal if and only if the ordered pairs $[L_{ij}^{(1)}, L_{ij}^{(2)}]$ and $[L_{\alpha\beta}^{(1)}, L_{\alpha\beta}^{(2)}]$ differ $(mod \ n)$ for any fixed values of $i, j \epsilon S$ but for all choices of $\alpha, \beta, \epsilon S$.

**Remark 2.4** If two LS’s are orthogonal, they remain so when the elements of either or both undergo arbitrary permutations.

We now establish:

**Theorem (2.2)** Let $L^{(1)} =< \alpha_1, \alpha_2 >$ and $L^{(2)} = < \beta_1, \beta_2 >$ be two pandiagonal LS’s with both pairs $\alpha_1, \alpha_2$ and $\beta_1, \beta_2$ each satisfying (2.4). If the determinant $d_2$ defined by

$$d_2 = \begin{vmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{vmatrix}$$

\( (2.5) \)
is relatively prime to \(n\), i.e.

\[ (d_2, n) = 1 \]  \hspace{1cm} (2.6)

then \(L^{(1)}\) and \(L^{(2)}\) are orthogonal pandiagonal LS’s.

**Proof:** Suppose for \(i, j \in S\) there is a second pair \(k, \ell \in S\) for which the ordered pairs \([L_{ij}^{(1)}, L_{ij}^{(2)}] \equiv [L_{k\ell}^{(1)}, L_{k\ell}^{(2)}] \mod (n)\). That is, suppose

\[
\begin{align*}
\alpha_1 i + \alpha_2 j &\equiv \alpha_1 k + \alpha_2 \ell \pmod n \\
\beta_1 i + \beta_2 j &\equiv \beta_1 k + \beta_2 \ell
\end{align*}
\]

which for convenience we express in matrix notation

\[
\begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix} \begin{pmatrix} i-k \\ j-\ell \end{pmatrix} \equiv 0 \pmod n
\]

Now since by hypothesis \((d_2, n) = 1\), so that in particular \(d_2 \not\equiv 0 \pmod n\), we can multiply this relation by

\[
d_2 \begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix}^{-1} = \begin{pmatrix} \beta_2 & -\alpha_2 \\ -\beta_1 & \alpha_1 \end{pmatrix}
\]

so that it becomes

\[
d_2 \begin{pmatrix} i-k \\ j-\ell \end{pmatrix} \equiv 0 \pmod n
\]

Finally, since \((d_2, n) = 1\) it follows that \(i = k\) and \(j = \ell\). Thus \(L^{(1)}\) and \(L^{(2)}\) are orthogonal pandiagonal LS’s.

The orthogonality criterion of the two LS’s in (2.6) is very convenient and as we shall see is extendable to higher dimensions.

Let us consider the question as to the number of distinct pairs \([\alpha_1, \alpha_2]\) there are for given \(n\) for which \(L =< \alpha_1, \alpha_2 >\) is a pandiagonal LS. To
simplify let us assume in the following that \( n \) is a prime \( p \). For a given prime \( p \geq 5 \), consider the \( p - 3, [\alpha_1, \alpha_2] \), pairs

\[
[\alpha_1, \alpha_2] = [1, 2], [1, 3], [1, 4], ... [1, p - 2]
\]

\textbf{(2.7)}

Obviously each of these satisfies (2.4) and no other pair for which \( \alpha_1 = 1 \) does so.

**Theorem (2.3):** The pandiagonal latin squares associated with the \([\alpha_1, \alpha_2]\) pairs in (2.7) are:

1. **pairwise mutually orthogonal**

2. **any other pandiagonal LS can be obtained from one associated with**

   \(< 1, \ell >, \ell = 2, 3, ...p - 2, \) in (2.7) by a permutation of symbols.

**Proof:**

1. **Consider the two pandiagonal LS's \(< 1, \ell >, < 1, m >; \ell, m = 2, 3, ...p - 2, (\ell \neq m) \). According to (2.5) the associated determinant \( d_2 \) is**

   \[
d_2 = \begin{vmatrix}
   1 & \ell \\
   1 & m.
   \end{vmatrix} = m - \ell
   \]

   and obviously satisfies (2.6) since we assume \( m \neq \ell \).

2. **Let \( x, y \) be any two unequal positive integers \( \bar{S} = \{1, 2, ...p - 1\} \) that**

   **satisfy each of (2.4) so that \(< x, y > \) is a pandiagonal LS. If \( x^{-1} \) is the**

   **inverse of \( x \) (mod \( p \)) then making use of (2.3) and Remark (2.2), we**

   **find that \(< 1, x^{-1}y > \) is also pandiagonal and is obtained from \(< x, y > \)**

   **by a permutation of its symbols. Finally since \( x^{-1}y \) (mod \( p \)) must be**

   **one of (2.3, ...p - 2) in (2.7) it follows from Remarks 2.1 and 2.2 that**
< x, y > can be obtained from < 1, x^{-1}y > by a permutation of its symbols.

For example, for p = 7, successive multiplication (mod p) by use of k = 2, 3, ... 6, leads to < 1, 2 > → < 2, 4 >, < 3, 6 >, < 4, 1 >, < 5, 3 >, < 6, 5 >. Similarly < 1, 3 > → < 2, 6 >, < 3, 2 >, < 4, 5 >, < 5, 1 >, < 6, 4 >; and < 1, 4 > → < 2, 1 >, < 3, 5 >, < 4, 2 >, < 5, 6 >, < 6, 3 >.

**Remark 2.5:** If we allow for permutation of symbols then all pandiagonal LS’s in (2.3) can be obtained, for given p by use only of the [α₁, α₂] pairs in (2.7).

Finally, as shown for pandiagonal magic squares, by Ball and Coxeter [3, p. 203] and by Martin Gardner [7] for a 5 × 5 pandiagonal magic square, we find that pandiagonal LS’s have an analogous unusual property. If in an n × n pandiagonal LS we move the left hand column so that it becomes the right hand column (or vice versa) or similarly move the top row to the bottom, the resultant array is again a pandiagonal LS. This has the consequence that if we “tile” the plane with a given n × n pandiagonal LS we can outline any n × n square on this infinite pattern and obtain a pandiagonal LS.

The underlying result is contained in:

**Theorem 2.4:** If the left column of a pandiagonal latin square is moved so it becomes the right column, the resulting LS is a pandiagonal LS obtained from the original one by a permutation of symbols.

**Proof:** Let $L = < \alpha_1, \alpha_2 >$ be the original pandiagonal LS and define a second one $L'$ with elements given by

$$L'_{ij} \equiv L_{ij} + \alpha_1 \equiv (\alpha_1 + 1)i + \alpha_2 j \quad (mod \ n)$$
It follows from (2.3) that $L'$ is simply $L$ with its elements permuted. Further we have

$$L'_{o_j} \equiv L_{1j}; L'_{1j} \equiv L_{2j}; \ldots L'_{n-1,j} \equiv L_{o_j}; \ (mod \ n); \ j = 0, 1, \ldots n - 1$$

which shows that $L'$ is simply $L$ with its left column moved so it becomes the right column.

For example, if the left hand column of the LS in Figure IIc, which incidentally is simply $< 1, 3 >$, (with $i$ increasing to the right and $j$ increasing upwards) is moved to the right side, the original LS, but with permuted symbols, in IId results! Of considerable interest perhaps is that a form of this property, as will be shown below, has an analogue in higher dimensions.

**B. Pandiagonal Latin Cubes**

We define a **pandiagonal latin cube** as an $n \times n \times n$ cube each of whose $n^3$ lattice points contains one of the members of $S = \{0, 1, 2, \ldots n - 1\}$ and in a way so that each of its $3n + 6$ constituent squares is a pandiagonal latin square. Recall in this connection that an $n \times n \times n$ cube has $3n$ squares parallel to a cube face plus 6 “diagonal” squares which contain its 4 body diagonals. Note that, as here defined, in a pandiagonal latin cube each row, column, file, diagonal and broken diagonal of each of its squares, i.e., its ROWS, contain each element of $S$ once and only once.

In an obvious generalization of (2.3) to three dimensions, we define an $n \times n \times n$ array $C$ by the formula

$$C_{ijk} \equiv \alpha_1 i + \alpha_2 j + \alpha_3 k \ (mod \ n) \ (2.8)$$

where each element of $C_{ijk} \in S, \{0, 1, \ldots n - 1\}$. Here $i, j, k$ are integer variables each running over $\{0, 1, \ldots n - 1\}$, and $\alpha_1, \alpha_2, \alpha_3$ are elements of
\( S = \{1, 2, \ldots, n - 1\} \). We shall use the notation \( C = \langle \alpha_1, \alpha_2, \alpha_3 \rangle \) when we wish to focus on the dependence of \( C \) on \( \alpha_1, \alpha_2, \alpha_3 \). We now establish the following:

**Theorem (2.5):** The cube defined in (2.8) is a pandiagonal latin cube—in that each of its constituent \( 3n + 6 \) squares is pandiagonal—provided \( \alpha_1, \alpha_2, \alpha_3 \) satisfy the constraints:

\[
\begin{align*}
(\alpha_{\ell}, n) &= 1; \quad \ell = 1, 2, 3 \\
(\alpha_{\ell} + \alpha_{\ell'}, n) &= 1; \quad \ell, \ell' = 1, 2, 3; \; \ell \neq \ell' \\
(\alpha_{\ell} - \alpha_{\ell'}, n) &= 1; \quad \ell, \ell' = 1, 2, 3; \; \ell \neq \ell' \\
(A, n) &= 1; \quad A = \alpha_1 + \alpha_2 + \alpha_3 \\
(A - 2\alpha_{\ell}, n) &= 1; \quad \ell = 1, 2, 3
\end{align*}
\]  

(2.9)

**Proof:** Consider first the \( 3n \) squares parallel to the faces of the cube. For fixed \( k \), say, \( 0 \leq k \leq n - 1 \), consider the square of \( C_{ijk} \) that is parallel to the \( i - j \) plane as \( i \) and \( j \) vary over the elements of \( S \). Then \( C_{ijk} \) has essentially the same structure as does \( L_{ij} \) in (2.3) since the constant \( \alpha_3 k \) is of no consequence. Making use of the restrictions on the \( \alpha \)'s in (2.9 a,b,c) and comparing with those in (2.4 a, b, c) we conclude that for any fixed \( k \), the square \( C_{ijk} \) is a pandiagonal latin square. Repeating this argument for fixed \( i \), with \( j \) and \( k \) variable and for fixed \( j \) with \( i \) and \( k \) variable we conclude that all \( 3n \) squares in the cube defined by (2.8) that are parallel to a cube face are pandiagonal latin squares.
With regard to the six squares not parallel to a cube face we proceed as follows. For that diagonal square, with vertices at \((i, j, k) = (0, 0, 0), (0, 0, n - 1), (n - 1, n - 1, n - 1), (n - 1, n - 1, 0)\), we have \(i = j\) so that (2.8) becomes

\[ C_{iik} \equiv (\alpha_1 + \alpha_2)i + \alpha_3 k \pmod{n} \]

But this is again of the form in (2.3) if we make the replacements in (2.4a, b, c) \(\alpha_1 \rightarrow \alpha_1 + \alpha_2; \alpha_2 \rightarrow \alpha_3\). The first of (2.4) is satisfied because of (2.9a, b) while (2.4b) and (2.4c) become respectively \((\alpha_1 + \alpha_2 + \alpha_3, n) = 1\) and \((\alpha_1 + \alpha_2 - \alpha_3, n) = 1\) and these are the same constraints as in (2.9d) and (2.9e), respectively. Thus \(C_{iik}\), the given square, is a pandiagonal latin square.

Similarly for the diagonal square perpendicular to \(C_{iik}\) whose vertices have the coordinates \((i, j, k) = (n - 1, 0, 0), (0, n - 1, 0), (0, n - 1, n - 1), (n - 1, 0, n - 1)\), we have \(i + j = n - 1\) and

\[ C_{i,(n-1-i),k} \equiv (\alpha_1 - \alpha_2)i + \alpha_3 k + \alpha_2(n - 1) \pmod{n} \]

which on comparison with (2.3) and (2.4) with the replacements \(\alpha_1 \rightarrow \alpha_1 - \alpha_2; \alpha_2 \rightarrow \alpha_3\) leads to the conditions \((\alpha_1 - \alpha_2 + \alpha_3, n) = 1\), and \((\alpha_1 - \alpha_2 - \alpha_3, n) = 1\) The first of these is the same as (2.9e) with \(\ell = 2\) and the second is the same as (2.9e) with \(\ell = 1\) if we make use of the fact that \((-x, y) = 1\) is equivalent to \((x, y) = 1\).

A similar argument shows that for the constraints in (2.9) the remaining four “diagonal squares” of the cube are also pandiagonal latin squares. The theorem is thus established.

Just as for the 2-D case, we require a 3-D analogue of orthogonality of latin cubes. For our purposes, we shall say that three latin cubes are **orthogonal**.
if when they are superposed none of the \( n^3 \) ordered triplets of elements that result occurs more than once. There are other definitions of orthogonal latin cubes, [9, ch. 3], but for purposes of producing *magic* cubes this definition is essential.

We now establish:

**Theorem (2.6):** Consider the 3 pandiagonal latin cubes

\[
C_{ijk}^{(q)} \equiv \alpha_{1q}i + \alpha_{2q}j + \alpha_{3q}k \pmod{n}; \quad q = 1, 2, 3 \tag{2.10}
\]

where \( \alpha_{pq} \) \((p, q = 1, 2, 3)\) are elements of \( \bar{S} \) and for each value for \( q \) satisfy the conditions in (2.9) and let \( d_3 \) be the determinant \( d_3 = |\alpha_{pq}| \). Then if \( d_3 \) is relatively prime to \( n \), that is

\[
(d_3, n) = 1 \tag{2.11}
\]

then the three cubes \( C^{(q)}, q = 1, 2, 3 \) are orthogonal.

**Proof:** Suppose on the contrary that for a given \((i, j, k)\) there existed an integer triplet \((u, v, w)\) each \( \epsilon S \) not equal to \((i, j, k)\) for which \( C_{ijk}^{(q)} \equiv C_{uvw}^{(q)} \pmod{n} \) for each \( q \). Then, as for the 2-D case, we could express this in matrix notation

\[
\begin{pmatrix}
\alpha_{11} & \alpha_{21} & \alpha_{31} \\
\alpha_{12} & \alpha_{22} & \alpha_{32} \\
\alpha_{13} & \alpha_{23} & \alpha_{33}
\end{pmatrix}
\begin{pmatrix}
i - u \\
j - v \\
k - w
\end{pmatrix} \equiv 0 \pmod{n} \tag{2.12}
\]

Since by (2.11), the determinant \( d_3 \) is relatively prime to \( n \), so that in particular \( d_3 \neq 0 \pmod{n} \), we may multiply both sides of (2.12) by \( d_3 \) times the
inverse of the matrix on the left. The result is

\[
d_3 \begin{pmatrix} i - u \\ j - v \\ k - w \end{pmatrix} \equiv 0 \mod n.
\]

Finally since \((d_3, n) = 1\) it follows that \(i = u; j = v; k = w\) and the theorem is proved.

An empirical study of the constraints in (2.9) shows that no triplets \([\alpha_1, \alpha_2, \alpha_3]\) satisfying (2.9) exists for \(n = 5, 7, 9\). (The latter clearly since \(n\) would be divisible by 3.) Such an analysis is most easily carried out by recognizing that—as in the 2-D case—without loss of generality we can take \(\alpha_1 = 1\) and consider simply the triplets \([1, \alpha_2, \alpha_3]\). For \(n = 7\), it is easily confirmed that no values, such \([1, 2, 3], [1, 2, 4], [1, 2, 5]\) would satisfy all of (2.9). For \(n = 11\) however we find, among others, the possibilities \([1, 2, 4], [1, 2, 5], [1, 2, 6], [1, 2, 7], [1, 5, 8], [1, 6, 8]\) as well as these with \(\alpha_2\) and \(\alpha_3\) interchanged. As an example of orthogonal pandiagonal cubes we note that for appropriate integers, \(\ell, m, p\) for the three LS’s \(<1, 2, \ell>, <1, m, 2>\) and \(<1, 2, p>\), for \(n = 11\), \(d_3\) is given by

\[
d_3 = \begin{vmatrix} 1 & 2 & \ell \\ 1 & m & 2 \\ 1 & 2 & p \end{vmatrix} = (m - 2)(p - \ell),
\]

and will for values of \(\ell, m, p\) with \(\ell \neq p\) and \(4 \leq \ell, m, p \leq 9\) lead to \((d_3, n) = 1\).

Figure III shows two planar sections through the cube \(<1, 2, 7>\) for \(n = 11\):
(a) corresponds to the square \(k = 2\) in (2.8) and (b) to the diagonal square.
\[ i = j. \]

Note, however, that for the triplet \(<1, 2, \ell>, <1, 2, m>, <1, 2, p>\), \(d_3 = 0\) so that these three do not constitute orthogonal cubes as we have defined them even though each cube is itself pandiagonal.

As for the analogous 2-D case (Theorem 2.4) we can easily establish the fact that if we move, say, a face of a pandiagonal latin cube to its opposite
Consider $C = \langle \alpha_1, \alpha_2, \alpha_3 \rangle$ in (2.8) and define $C'$ with elements given by

$$C'_{ijk} = C_{ijk} + \alpha_1$$

so that

### Figure III

side, the resultant cube remains a pandiagonal latin cube. To see this,
\[ C'_{ijk} \equiv \alpha_1(i + 1) + \alpha_2j + \alpha_3k \pmod{n}. \]

Obviously,
\[ C'_{ojk} = C_{1jk}; C'_{1jk} \equiv C_{2jk}; \ldots C'_{n-1,jk} \equiv C_{o,jk} \pmod{n}; \quad 0 \leq j, k \leq n - 1 \]

Thus \( C' \) is obtained from \( C \) by transporting its \( i = 0 \) face to \( i = n - 1 \). This implies that if we “tile” all of 3-D space with a given pandiagonal latin cube and any \( n \times n \times n \) cube selected out of this infinite array will be a pandiagonal latin cube but with its elements permuted in some way.

**C. Four dimensional pandiagonal latin cubes**

A 4-D pandiagonal latin cube is an arrangement of the integers \( \{0, 1, 2, \ldots, n - 1\} \) among the cells of an \( n \times n \times n \times n \) cube in a way so that each of its \( 4n + 12 \) constituent 3-D cubes is a pandiagonal latin cube.

In an obvious generalization of (2.8), we define an \( n \times n \times n \times n \) array of integers by the formula
\[ H_{ijkl} \equiv \alpha_1i + \alpha_2j + \alpha_3k + \alpha_4\ell \pmod{n} \tag{2.13} \]

where each of \( H_{ijkl} \) is \( \epsilon S = \{0, 1, 2, \ldots, n - 1\} \) and where \( i, j, k, \ell \) are integer variables each with the range \( 0, 1, 2, \ldots, n - 1 \). The four quantities \( \alpha_1, \alpha_2, \alpha_3, \alpha_4 \) are integer parameters \( \epsilon S = \{1, 2, \ldots, n - 1\} \). As before we use the notation \( < \alpha_1, \alpha_2, \alpha_3, \alpha_4 > \) when the dependence of \( H \) on the \( \alpha \)'s is of interest. We now establish the following:

**Theorem (2.7):** The hypercube defined in (2.13) is a 4-D pandiagonal latin cube provided the four integer parameters \( \alpha_1, \alpha_2, \alpha_3, \alpha_4 \) satisfy the con-
Proof: Consider first the 3-D cube that results for fixed $\ell$ from (2.13) as $i, j, k$ vary over S. It will be a pandiagonal latin cube provided the constraints in (2.9a)-(2.9e) are satisfied. Now (2.9a)-(2.9e) are contained within (by appropriate choice of the subscripts) (2.14a), (2.14b), (2.14c), (2.14d), (2.14e), (2.14g), respectively. Thus since the added constant $\alpha_4\ell$ plays no role the cube is a pandiagonal latin cube. Similarly, the cubes that result for fixed $i$, as $j, k, \ell$ vary, and for fixed $j$ as $i, k, \ell$ vary and for fixed $k$ as $i, j, \ell$ vary over S are all pandiagonal latin cubes. There are altogether $4n$ pandiagonal 3-D latin cubes of this type.

Similarly the “diagonal” 3-D cube that results for $i = j$, as $i, k, \ell$ vary over S has the form

$$H_{iik\ell} \equiv (\alpha_1 + \alpha_2)i + \alpha_3k + \alpha_4\ell \mod n$$

On comparison with (2.8) and (2.9a) - (2.9e) we see that this is also a pandiagonal latin cube since we assumed (2.14a), (2.14b), (2.14c), (2.14d), (2.14e), (2.14g) to be satisfied. And similarly for the other five pairs: $i = k, i = \ell, j = k, j = \ell, k = \ell$. 

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Finally for the “diagonal” latin 3-D cube that results for $i + j = n - 1$, as $i, k$ and $\ell$ vary over $S$

$$H_{i,n-1-i,k,\ell} \equiv (\alpha_1 - \alpha_2)i + \alpha_3k + \alpha_4\ell - \alpha_2(n - 1) \pmod{n}$$

which on comparison with (2.9a) - (2.9c) is a pandiagonal latin cube by virtue of (2.14a), (2.14b), (2.14c), (2.14f), (2.14g), (2.14h). And similarly for the remaining five cubes $i + k = n - 1, i + \ell = n - 1, j + k = n - 1, j + \ell = n - 1$ and $k + \ell = n - 1$. This proves the theorem.

Turning to the question of orthogonality, we define four, 4-D pandiagonal cubes to be orthogonal, if when they are superposed no two of the $n^4$ ordered quartets of elements that result are the same. As in the lower dimensional cases there are other definitions of this orthogonality [9], but for our purposes this one is essential.

By analogy to Theorems (2.2) and (2.6) we have:

**Theorem (2.8):** Consider the four, 4-D pandiagonal latin cubes

$$H^{(q)}_{ijkl} \equiv \alpha_1q_i + \alpha_2q_j + d\alpha_3q_k + \alpha_4q\ell \pmod{n}; \quad q = 1, 2, 3, 4 \quad (2.15)$$

where $\alpha_{pq}$ ($p, q = 1, 2, 3, 4$) are elements of $\bar{S}$ which for each value of $q$ satisfy the conditions in (2.14) and the variables $i, j, k, \ell$ range over $S = \{0, 1, 2, \ldots n - 1\}$. Further, let $d_4 = |\alpha_{pq}|$ be the determinant of the $\alpha$’s. Then if $d_4$ is relatively prime to $n$

$$(d_4, n) = 1 \quad (2.16)$$

then the four 4-D cubes in (2.15) are orthogonal.

**Proof:** Suppose to the contrary there existed a quartet of integers $(u, v, w, x)$ each $\epsilon S$, not equal to any given quartet $(i, j, k, \ell)$ for which $H^{(q)}_{ijkl} \equiv H^{(q)}_{uvw}$
\[(mod\ n)\ for\ each\ q.\ Then\ just\ as\ in\ deriving\ (2.12)\ we\ would\ find\]
\[
\begin{bmatrix}
\alpha_{11} & \alpha_{21} & \alpha_{31} & \alpha_{41} \\
\alpha_{12} & \alpha_{22} & \alpha_{32} & \alpha_{42} \\
\alpha_{13} & \alpha_{23} & \alpha_{33} & \alpha_{43} \\
\alpha_{14} & \alpha_{24} & \alpha_{34} & \alpha_{44}
\end{bmatrix} \begin{bmatrix}
i - u \\
j - v \\
k - w \\
\ell - x
\end{bmatrix} \equiv 0 \mod n \quad (2.17)
\]

and conclude following essentially the same steps as before that \(i, j, k, \ell\) must be equal to \(u, v, w, x\) respectively. Thus concluding the proof of orthogonality.

It is easily confirmed by enumerating the various possibilities that only for \(n \geq 17\) is it possible to find integers \([\alpha_1, \alpha_2, \alpha_3, \alpha_4]\) among \(\{1, 2, \ldots, n - 1\}\) for which \((2.14)\) can be satisfied. In particular for \(n = 17\), possible 4-D latin hypercubes are given by \(<1, 2, 4, 8>, <1, 2, 4, 9>, <1, 2, 13, 8>, <1, 2, 13, 9>\) as is readily confirmed. Thus a possible form for the determinant of the matrix in \((2.17)\) is

\[
d_4 = \begin{vmatrix}
1 & 2 & 4 & 8 \\
1 & 2 & 4 & 9 \\
1 & 2 & 8 & 4 \\
1 & 4 & 9 & 2
\end{vmatrix} = -8
\]

and since \((-8, 17) = 1\), the four 4-D pandiagonal latin cubes \(<1, 2, 4, 8>, <1, 2, 4, 9>, <1, 2, 13, 9>, <1, 4, 9, 2>\) constitute an orthogonal set of such hypercubes, for \(n = 17\).

Figure IV, shows a planar section through the hypercube \(<1, 2, 4, 9>\) corresponding to \(i = 2, j + k = 16\) in \((2.13)\).
\[
\begin{bmatrix}
0 & 9 & 1 & 10 & 2 & 11 & 3 & 12 & 4 & 13 & 5 & 14 & 6 & 15 & 7 & 16 & 8 \\
2 & 11 & 3 & 12 & 4 & 13 & 5 & 14 & 6 & 15 & 7 & 16 & 8 & 0 & 9 & 1 & 10 \\
4 & 13 & 5 & 14 & 6 & 15 & 7 & 16 & 8 & 0 & 9 & 1 & 10 & 2 & 11 & 3 & 12 \\
6 & 15 & 7 & 16 & 8 & 0 & 9 & 1 & 10 & 2 & 11 & 3 & 12 & 4 & 13 & 5 & 14 \\
8 & 0 & 9 & 1 & 10 & 2 & 11 & 3 & 12 & 4 & 13 & 5 & 14 & 6 & 15 & 7 & 16 \\
10 & 2 & 11 & 3 & 12 & 4 & 13 & 5 & 14 & 6 & 15 & 7 & 16 & 8 & 0 & 9 & 1 \\
12 & 4 & 13 & 5 & 14 & 6 & 15 & 7 & 16 & 8 & 0 & 9 & 1 & 10 & 2 & 11 & 3 \\
14 & 6 & 15 & 7 & 16 & 8 & 0 & 9 & 1 & 10 & 2 & 11 & 3 & 12 & 4 & 13 & 5 \\
16 & 8 & 0 & 9 & 1 & 10 & 2 & 11 & 3 & 12 & 4 & 13 & 5 & 14 & 6 & 15 & 7 \\
1 & 10 & 2 & 11 & 3 & 12 & 4 & 13 & 5 & 14 & 6 & 15 & 7 & 16 & 8 & 0 & 9 \\
3 & 12 & 4 & 13 & 5 & 14 & 6 & 15 & 7 & 16 & 8 & 0 & 9 & 1 & 10 & 2 & 11 \\
5 & 14 & 6 & 15 & 7 & 16 & 8 & 0 & 9 & 1 & 10 & 2 & 11 & 3 & 12 & 4 & 13 \\
7 & 16 & 8 & 0 & 9 & 1 & 10 & 2 & 11 & 3 & 12 & 4 & 13 & 5 & 14 & 6 & 15 \\
9 & 1 & 10 & 2 & 11 & 3 & 12 & 4 & 13 & 5 & 14 & 6 & 15 & 7 & 16 & 8 & 0 \\
11 & 3 & 12 & 4 & 13 & 5 & 14 & 6 & 15 & 7 & 16 & 8 & 0 & 9 & 1 & 10 & 2 \\
13 & 5 & 14 & 6 & 15 & 7 & 16 & 8 & 0 & 9 & 1 & 10 & 2 & 11 & 3 & 12 & 4 \\
15 & 7 & 16 & 8 & 0 & 9 & 1 & 10 & 2 & 11 & 3 & 12 & 4 & 13 & 5 & 14 & 6 
\end{bmatrix}
\]

**Figure IV**

Just as for the 2 and 3 dimensional cases, if \( H = < \alpha_1, \alpha_2, \alpha_3, \alpha_4 > \) is a 4-D pandiagonal latin cube, then so is \( H' \) with elements given by

\[
H'_{ijkl} = H_{ijkl} + \alpha_1 \equiv (\alpha_1 + 1)i + \alpha_2 j + \alpha_3 k + \alpha_4 \ell \pmod n.
\]

Again we can imagine “tiling” all of 4-D space with \( H \) and be assured that any \( n \times n \times n \times n \) subcube in this space will be a 4-D pandiagonal latin cube, with its elements a permutation of the original elements of \( H \).
III. Magic, Pandiagonal Squares and Cubes in Two and Three Dimensions

With the results of the preceding section available, it is now straightforward \([3], [6]\) to generate magic pandiagonal cubes in three and four dimensions. To set the stage we first illustrate the matter in two dimensions.

A. Magic Pandiagonal Squares

Recall that a magic pandiagonal square of order \(n\) is an arrangement, without repeats, of the integers \((0, 1, 2, \ldots n^2 - 1)\) on the \(n \times n\) lattice points of an \(n \times n\) array so that the sum of the elements in each of the \(n\) rows, \(n\) columns, \(n\) diagonals (including the \(n - 2\) broken diagonals) has the same value. (As above let us use the generic ROW to represent any one of these rows, columns, diagonals, etc.) This common sum of the ROWS has been given in (1.1).

**Theorem 3.1** [9, p. 178] Let \(L^{(1)} = < \alpha_1, \alpha_2 >, L^{(2)} = < \beta_1, \beta_2 >\) be two orthogonal pandiagonal latin squares that separately satisfy the conditions of theorems (2.1) and (2.2) and define an \(n \times n\) array \(M^{(2)}\) with elements \(M_{ij}^{(2)}\) by

\[
M^{(2)} = nL^{(1)} + L^{(2)}
\]  

(3.1)

Then \(M^{(2)}\) is a magic pandiagonal square.

**Proof:** Since the elements of \(L^{(1)}\) and \(L^{(2)}\) range over \(S = \{0, 1, 2, \ldots n - 1\}\) it follows from (3.1) that each of the elements \(M_{ij}^{(2)}\) must be one of the integers \(0, 1, 2, \ldots n^2 - 1\). Further, since the sum of the elements in each ROW of the pandiagonal \(L^{(1)}\) and \(L^{(2)}\), is \(0 + 1 + 2 + \ldots + n - 1 = n(n - 1)/2\) it follows that the sum of each of the \(4n\) ROWS of \(M^{(2)}\) is

\[
n[n(n - 1)/2] + n(n - 1)/2 = n(n^2 - 1)/2 = \sigma_2
\]

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with \( \sigma_2 \) defined in (1.1). Finally since \( L^{(1)} \) and \( L^{(2)} \) are orthogonal, no two elements of \( M^{(2)} \) can be the same and the theorem is established.

Since \( L^{(1)} \) and \( L^{(2)} \) are pandiagonal LS’s for which we know \( n \geq 5 \), only for \( n \geq 5 \) can \( M^{(2)} \) in (3.1) be a magic pandiagonal square. Further if we tile the plane with \( M^{(2)} \) any \( n \times n \) subsquare will also be a pandiagonal magic square since \( L^{(1)} \) and \( L^{(2)} \) have this same tiling property [7]. (See the discussion at the end of IIA.)

Since the elements of the orthogonal pandiagonal latin squares \( L^{(1)} \) and \( L^{(2)} \) may be permuted among themselves without changing their essential properties, it follows that from a given pandiagonal LS we may generate \( n! \) different versions. This leads to the number \( N_2 \) of pandiagonal magic squares obtainable by our method to be

\[
N_2 = \ell(n)(n!)^2
\]

(3.2)

where \( \ell(n) \) is a low order polynomial in \( n \). For \( n = 5 \), since the only independent LS’s are \(<1,2>\) and \(<1,3>\) (Theorem 2.3), and since we may interchange their roles in (3.1) \( \ell(5) = 2 \) and we obtain consistent with the results of Rosser and Walker [11]

\[
N_2 = 2880
\]

Clearly because of the \((n!)^2\) factor \( N_2 \) rises very rapidly with \( n \). For \( n = 11 \) for example, \( N_2 \) is \( \sim 9.0 \times 10^{16} \).

**B. Pandiagonal Magic Cubes**

A magic pandiagonal \( n \times n \times n \) cube is an arrangement, without repeats, of the integers \( 0, 1, 2, \ldots n^3 - 1 \) onto the \( n^3 \) lattice sites of a cube so that the sums along each ROW (i.e. along each of the \( n^2 \)-rows, \( n^2 \)-columns, \( n^2 \)-files
and the \( n(3n + 6) \) diagonals (including the broken diagonals) in each of its \((3n + 6)\) squares are the same. The common sum along the ROWS is given by \( \sigma_3 \) in (1.2).

**Theorem (3.2):** Let \( C^{(q)} = < \alpha_{1q}, \alpha_{2q}, \alpha_{3q} >, q = 1, 2, 3 \), be three orthogonal pandiagonal \( n \times n \times n \) latin cubes that satisfy the conditions of theorems (2.5), (2.6) and define an \( n \times n \times n \) array \( M^{(3)} \) with elements \( M_{ijk} \) by

\[
M^{(3)} = n^2 C^{(1)} + nC^{(2)} + C^{(3)}. \tag{3.3}
\]

Then \( M^{(3)} \) is a magic pandiagonal cube.

**Proof:** Since the elements of \( C^{(1)}, C^{(2)} \) and \( C^{(3)} \) range over \( S = \{0, 1, 2, \ldots n - 1\} \) it follows from (3.3) that each element of \( M^{(3)} \) must be one of the integers \( \{0, 1, 2, \ldots n^3 - 1\} \). Consider now any ROW of \( M^{(3)} \). The sum of the elements of this ROW is, according to (3.3), given by

\[
n^2\left[n(n - 1)/2\right] + n\left[n(n - 1)/2\right] + (n - 1)/2 = n(n^3 - 1)/2
\]

and yields the value \( \sigma_3 \) in (1.2). Thus the sum of the elements in each ROW of the cube is the same and since \( C^{(1)}, C^{(2)} \) and \( C^{(3)} \) are orthogonal, it follows from (3.3) that no two elements of \( M^{(3)} \) can be the same. The theorem is thus established.

According to the discussion in IIB, since the C’s are pandiagonal latin cubes, for which we know \( n \geq 11 \), \( M^{(3)} \) will exist only for \( n \geq 11 \).

As for the two dimensional case, if we allow for the interchange in \( M^{(3)} \), of \( \alpha_{1q}, \alpha_{2q}, \alpha_{3q} \) \( (q = 1, 2, 3) \) with each other, and of permuting the symbols in each of \( C^{(1)}, C^{(2)} \) and \( C^{(3)} \) independently we can conclude that \( N_3 \), the number of cubes \( M^{(3)} \) in (3.3) is given by

\[
N_3 = \ell_3(n)(n!)^3 \tag{3.4}
\]
where $\ell_3$ is an appropriate polynomial in $n$. For $n = 11, 13$ and 17, $(n!)^3$ assumes the approximate values $6.4 \times 10^{22}$, $2.4 \times 10^{29}$ and $4.5 \times 10^{43}$ respectively. Because of this rapid rise of $N_3$ with $n$, we can anticipate that the factor $\ell_3(n)$ will not affect this variation, qualitatively.

As for the analogous 2-D case, it follows that by virtue of the “tiling” properties of pandiagonal latin cubes (see II.B), we can also tile 3-D space with any magic pandiagonal cube, and be assured that any $n \times n \times n$ cube selected out of this infinite array will also be a magic pandiagonal cube. It will differ from the original cube in that the elements of its underlying latin cubes will have been permuted.

**Four Dimensional Pandiagonal Magic Cubes**

By analogy to the above, we define a magic four dimensional pandiagonal cube, as an arrangement, without repeats, of the integers $0, 1, 2, \ldots n^4 - 1$ among the $n^4$ lattice sites of an $n \times n \times n \times n$ cube so that the sum of the elements in each ROW of the 4-D cube, has the same value $\sigma_4$ as given in (1.3).

**Theorem (3.3):** Let $Q^{(q)} = < \alpha_{1q}, \alpha_{2q}, \alpha_{3q}, \alpha_{4q} >$, $q = 1, 2, 3, 4$ be four orthogonal pandiagonal $n \times n \times n \times n$ latin cubes that satisfy the conditions of theorems (2.6) and (2.7) and define an $n \times n \times n \times n$ array $M^{(4)}$ with elements $M^{(4)}_{ijkl}$ by

$$M^{(4)} = n^3Q^{(1)} + n^2Q^{(2)} + nQ^{(3)} + Q^{(4)}$$

Then $M^{(4)}$ is a magic, pandiagonal four dimensional cube.

**Proof:** Since the elements of each of $Q^{(1)}, Q^{(2)}, Q^{(3)}$ and $Q^{(4)}$ range over $S = \{0, 1, 2, \ldots n - 1\}$, it follows from (3.5) that the elements of $M^{(4)}_{ijkl}$ assume values from the set $\{0, 1, 2, \ldots n^4 - 1\}$. Further, since the $Q$’s are pandiagonal
it follows that the sum of the elements in any ROW of $M^{(4)}$ is given according to (3.5) by

$$n^3 \left[ \frac{n(n-1)}{2} \right] + n^2 \left[ \frac{n(n-1)}{2} \right] + n\left[ \frac{n(n-1)}{2} \right] + (n-1)/2 = \frac{n(n^4 - 1)}{2}$$

and this is $\sigma_4$ in (1.3). Thus the sum of the elements of any ROW of $M^{(4)}$ is given by (1.3). Finally because of the assumed orthogonality of the $Q$’s, it follows that no two elements of $M^{(4)}$ are the same and $M^{(4)}$ is a magic pandiagonal 4-D cube without repeats. The theorem is established.

According to the discussion in IIC, since the $Q$’s are 4-D pandiagonal latin cubes, for which we found $n \geq 17$, it follows that $M^{(4)}$ will exist only for $n \geq 17$.

The number $N_4$ of different $M^{(4)}$’s in (3.5) can be estimated as above to vary for large values of $n$ as

$$N_4 \cong (n!)^4$$

so that for $n = 17, 19$ and $23$, $N_4$ assumes the approximate and rapidly growing values of $n$ of $1.6 \times 10^{58}$, $2.19 \times 10^{68}$ and $4.5 \times 10^{89}$ respectively.

It is also possible, using the tiling properties of the $Q$’s to “tile” $M^{(4)}$ throughout four dimensional space and obtain a pandiagonal magic 4-D cube by selecting any $n \times n \times n \times n$ cube in this space. Such a cube will be the same as the original cube but with the elements of its underlying latin cubes permuted.

Evidently, these arguments are extendable to dimensions higher than 4, but the resulting constraints on the $\alpha$ parameters, analogous to those in (2.9) in three dimensions and those in (2.14) in four, can be expected to become increasingly complex.
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