The time-asymptotic expansion for the compressible Euler equations with damping

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Abstract
In 1992, Hsiao and Liu first showed that the solution to the compressible Euler equations with damping time-asymptotically converges to the diffusion wave \((\bar{v}, \bar{u})\) of the porous media equation. Geng et al. proposed a time-asymptotic expansion around the diffusion wave \((\bar{v}, \bar{u})\), which is a better asymptotic profile than \((\bar{v}, \bar{u})\).

In this paper, we rigorously justify the time-asymptotic expansion by the approximate Green function method and the energy estimates. Moreover, the large time behavior of the solution to compressible Euler equations with damping is accurately characterized by the time-asymptotic expansion.

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1 INTRODUCTION

In this paper, we are concerned with the 1-d compressible Euler equations with damping in Lagrangian coordinates, which reads as

\[ \begin{aligned}
  v_t - u_x &= 0, \\
  u_t + P(v)_x &= -\alpha u,
\end{aligned} \tag{1.1} \]
with the initial data

\[(v, u)(x, 0) = (v_0, u_0)(x) \to (v_\pm, u_\pm) \quad \text{as} \quad x \to \pm \infty, \tag{1.2}\]

where \(v = v(x, t)\) denotes the specific volume, \(u = u(x, t)\) is the velocity, and \(P(v)\) is the pressure satisfying \(P(v) > 0, P'(v) < 0\). The damping term \(\alpha u\) is the friction effect with the physical parameter \(\alpha > 0\). Without loss of generality, we assume \(\alpha = 1\) in this paper.

Since the velocity \(u_i\) in (1.1) \(_2\) decays to zero faster than the other terms due to the damping effect, it is commonly conjectured that (1.1) is time-asymptotically equivalent to the porous medium equation (PME):

\[
\begin{cases}
\ddot{v}_t - \ddot{u}_x = 0, \\
P(\dot{v})_x = -\ddot{u},
\end{cases}
\quad \text{equivalently, } \ddot{v}_t = -P(\dot{v})_{xx}, \quad \text{PME,} \tag{1.3}
\]

and the velocity \(\ddot{u}\) obeys the Darcy’s law. This conjecture was first justified by Hsiao and Liu \([9]\) around the diffusion wave \(\ddot{v}(x, t) =: \ddot{v}(\xi), \xi = \frac{x}{\sqrt{1+t}}\) of PME with the boundary condition at the far field

\[
\ddot{v}(x, t) \to v_\pm \quad \text{as} \quad x \to \pm \infty. \tag{1.4}
\]

Since \(\ddot{v}(\xi)\) is the self-similar solution to Cauchy problem of PME (1.3)-(1.4), then \(\ddot{v}(\xi)\) solves the following ordinary differential equation:

\[
\frac{1}{2} \ddot{v}' - (P(\ddot{v}))'' = 0, \quad ' = \frac{d}{d\xi}\tag{1.5}
\]

with

\[
\lim_{\xi \to \pm \infty} \ddot{v}(\xi) = v_\pm. \tag{1.6}
\]

In fact, there exists a unique solution \(\ddot{v}(\xi)\) up to a shift in \([9]\). Once \(\ddot{v}(\xi)\) is determined, \(\ddot{u}(x, t)\) is defined as

\[
\ddot{u}(x, t) = -(1 + t)^{-\frac{1}{2}}(P(\ddot{v}')). \tag{1.7}
\]

Furthermore, it was shown in \([9]\) that the solution \((v, u)(x, t)\) to (1.1)-(1.2) converges to the diffusion wave \((\ddot{v}, \ddot{u})(x, t)\) given in (1.5)-(1.7) in the form of

\[
||v - \ddot{v}, u - \ddot{u}||_{L^\infty} = O(1)\left(1 + t\right)^{-\frac{1}{2}}, \left(1 + t\right)^{-\frac{3}{2}}\).
\]

Later, Nishihara \([24]\) improved the decay rate to

\[
||v - \ddot{v}, u - \ddot{u}||_{L^\infty} = O(1)\left(1 + t\right)^{-\frac{1}{4}}, \left(1 + t\right)^{-\frac{5}{4}}\).
\]

By constructing a fine approximate Green function and elaborate estimates, Nishihara–Wang–Yang \([25]\) further improved to

\[
||v - \ddot{v}, u - \ddot{u}||_{L^\infty} = O(1)\left(1 + t\right)^{-1}, \left(1 + t\right)^{-\frac{3}{2}}\).
Mei [23] chose another asymptotic profile (not self-similar solution \( \bar{\nu} \left( \frac{x}{\sqrt{1+t}} \right) \) of PME) and obtained the corresponding decay rate. For the other interesting works, see [3, 10–21, 26, 27, 29–31] and the references therein.

On the other hand, in [4], we considered the compressible Euler equation with time-dependent damping

\[
\begin{aligned}
\frac{\partial v}{\partial t} - u_x &= 0, \\
\frac{\partial u}{\partial t} + P(u)_x &= -\frac{1}{(1+t)^2} u
\end{aligned}
\]  

(1.8)

and proposed a time-asymptotic expansion

\[
\begin{aligned}
\tilde{v}_k &= \bar{\nu} + \sum_{i=1}^{k} (1+t)^{-i\sigma} v_i(\xi), \\
\tilde{u}_k &= \bar{u} + \sum_{i=1}^{k} (1+t)^{-(i+\frac{1}{2})\sigma} u_i(\xi),
\end{aligned}
\]

(1.9)

If the expansion holds, then the asymptotic behavior of solutions beyond the diffusion wave \( \bar{\nu}(\xi) \) can be accurately characterized. We justified the expansion as \( \lambda \in \left( \frac{1}{7}, 1 \right) \) and further conjectured that the expansion still holds for any \( \lambda \in [0, 1) \) in [4]. For more results on the time-dependent system, we refer to [1, 5–8, 28] and reference therein.

The aim of this paper is to justify the expansion (1.9) as \( \lambda = 0 \), that is, the constant damping system (1.1). We consider the case of \( k = 1 \), namely,

\[
\begin{aligned}
v_* (\xi, t) &= \bar{\nu} + (1+t)^{-1} v_1(\xi), \\
u_* (\xi, t) &= \bar{u} + (1+t)^{-\frac{3}{2}} u_1(\xi),
\end{aligned}
\]

(1.10)

where \( \xi = \frac{x}{\sqrt{1+t}} \), \( (\bar{\nu}, \bar{u}) \) is the diffusion wave of (1.3)–(1.4) and the subsequent term \( (v_1, u_1)(\xi) \) will be determined in Section 2 below.

Without loss of generality, we focus on the case of \( u_+ = u_- = 0 \). The other cases can be treated by introducing a correction function

\[
\begin{aligned}
\hat{v}(x, t) &= -(u_+ - u_-) m_0(x) e^{-t}, \\
\hat{u}(x, t) &= e^{-t} (u_- + (u_+ - u_-) \int_{-\infty}^{x} m_0(y) dy),
\end{aligned}
\]

(1.11)

where \( m_0(x) \) is a smooth function with compact support satisfying \( \int_{\mathbb{R}} m_0(x) dx = 1 \), see [9] for the details. Let

\[
V(x, t) = \int_{-\infty}^{x} \left( v(y, t) - v_* \left( \frac{y}{\sqrt{1+t}}, t \right) \right) dy, \quad V_1(x, t) = u(x, t) - u_* (\xi, t),
\]

(1.12)

and \( V_0(x) = : V(x, 0), V_1(x) = : V_1(x, 0) \). Assume that the initial data

\[
(V_0, V_1)(x) \in H^5(\mathbb{R}) \times H^4(\mathbb{R}), \quad V_0(x), V_1(x) \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})
\]

(1.13)
with \( V_0(x) = \partial_x V_0(x), \ V_1(x) = \partial_x V_1(x) \). Set

\[
N(0) =: \|V_0\|_5 + \|V_1\|_4, \quad \delta =: |v_+ - v_-|, \quad \delta_1 =: \|\widetilde{V}_0\|_{L^1} + \|\widetilde{V}_1\|_{L^1}.
\]

Then, we have the following.

**Theorem 1.1.** Suppose that the initial data \( V_0(x), V_1(x) \) satisfy (1.13). Then, there is a small constant \( \varepsilon_0 > 0 \) such that if

\[
\sqrt{N^2(0)} + \delta + \delta_1 \leq \varepsilon_0,
\]

the Cauchy problem (1.1)–(1.2) admits a unique smooth solution \((v, u)(x, t)\). Moreover, it holds that

\[
\sum_{l+k \leq 5, l \leq 1} (1 + t)^{l + k + \frac{3}{2}} \|\partial_t^l \partial_x^k V(\cdot, t)\|_{L^2} \leq C \left( \sqrt{N^2(0)} + \delta + \delta_1 \right).
\]

(1.14)

Furthermore, noting the relationship between \( v - v_*, u - u_* \) and \( V \) in Section 3 below, we use the Cauchy–Schwartz inequality to get the following corollary.

**Corollary 1.1.** Under the assumptions of Theorem 1.1, it holds that

\[
\|(v - v_*, u - u_*)(t)\|_{L^\infty} \leq C \left( \sqrt{N^2(0)} + \delta + \delta_1 \right) \left( (1 + t)^{-\frac{3}{2}}, (1 + t)^{-2} \right),
\]

(1.15)

which justifies the expansion (1.10).

We now sketch the main strategy. To justify the time-asymptotic expansion (1.10), it remains to show that the remainder \( v - v_* =: V_x \) decays faster than \((1 + t)^{-1}\). To this end, we first reduce the perturbation system into a nonlinear wave equation

\[
V_{tt} + (P'(v_*)V_x)_x + V_t = g_1 + S[v_*],
\]

(1.16)

and establish the basic estimates for \( V_{tt} \) by energy method, where \( g_1 \) is the nonlinear term and \( S[v_*] \) is the error term induced by the expansion (1.10), see (2.7) below. Motivated by the diffusion phenomenon observed in [9] and [25], we then regard (1.16) as a diffusion equation with source term, that is,

\[
V_t + (P'(v_*)V_x)_x = g_1 + S[v_*] - V_{tt},
\]

(1.17)

and use the approximate Green function \( G(x, t; y, s) \) to obtain the integral formula of \( V(x, t) \) through Duhamel principle,

\[
V(x, t) = \int_R G(x, t; y, 0)V_0(y)dy + \int_0^t \int_R G(x, t; y, s)[g_1 + S[v_*] - V_{ss}]dyds
\]

\[
+ \int_0^t \int_R R_G(x, t; y, s)V(y, s)dyds,
\]

(1.18)
where $R_G$ denotes the difference between the Green function and approximate one $G(x, t; y, s)$. Thanks to the expansion (1.10), $S[v_+]$ decays faster than that of [25]. Thus, we can obtain a better decay rate for

$$
\int_0^t \int_{\mathbb{R}} G(x, t; y, s) S[v_+] dy ds.
$$

On the other hand, we have new estimates for

$$
\int_0^t \int_{\mathbb{R}} G(x, t; y, s) g_1 y dy ds \quad \text{and} \quad \int_0^t \int_{\mathbb{R}} R_G(x, t; y, s) V(y, s) dy ds.
$$

Finally, we can obtain the desired decay rate $(1 + t)^{-\frac{3}{2}}$ for $V_x$, as shown in (1.15), with the help of the energy estimates for the wave equation (1.16).

The arrangement of the present paper is as follows. In Section 2, we introduce the time-asymptotic expansion $(v_*, u_*) (x, t)$ for (1.1)–(1.2). In Section 3, we reduce the perturbation system to a nonlinear wave equation and establish the basic estimates by energy method. Section 4 is devoted to proving Theorem 1.1 by the approximate Green function method with the help of the basic energy estimates.

Notations. Throughout this paper, the symbol $c$, $C$ will be used to represent a generic constant that is independent of $x$ and $t$ and may vary from line to line. $\| \cdot \|_{L^p}$ stands for the $L^p(\mathbb{R})$-norm $(1 \leq p \leq \infty)$. The $L^2$-norm on $\mathbb{R}$ is simply denoted by $\| \cdot \|$. Moreover, the domain $\mathbb{R}$ will be often abbreviated without confusions.

2 \quad THE TIME-ASYMPTOTIC EXPANSION

We first list some properties on the diffusion wave $\bar{v}(\xi)$ with $\xi = \frac{x}{\sqrt{1+t}}$ of PME (1.3)–(1.4).

**Lemma 2.1** [2, 10]. For the diffusion wave $\bar{v}(\xi)$ of PME (1.3)–(1.4), it holds that

$$
|\bar{v}(\xi) - v_+|_{\xi > 0} + |\bar{v}(\xi) - v_-|_{\xi < 0} \leq C(|v_+ - v_-|) e^{-c \xi^2}, 
$$

$$
|\partial_x^k \partial_t^l \bar{v}| \leq C(|v_+ - v_-|) (1 + t)^{-\frac{k}{2} - l} e^{-c \xi^2}, \quad k + l \geq 1, k, l \geq 0,
$$

$$
\|\partial_x^k \partial_t^l \bar{v}\| \leq C(|v_+ - v_-|) (1 + t)^{-\frac{k}{2} - l + \frac{1}{2}}, \quad k + l \geq 1.
$$

As in [4], we introduce the time-asymptotic expansion

\[
\begin{cases}
    v_*(\xi, t) = \bar{v} + (1 + t)^{-1} v_1(\xi), \\
    u_*(\xi, t) = \bar{u} + (1 + t)^{-\frac{3}{2}} u_1(\xi),
\end{cases}
\]
where $\bar{u}$ is given in (1.7) and $(v_1, u_1)(\xi)$ will be determined as follows. Note that $\bar{v}_t - \bar{u}_x = 0$, we expect
\[
((1 + t)^{-1}v_1(\xi))_t - \left((1 + t)^{-\frac{3}{2}}u_1(\xi)\right)_x = 0, \quad \text{equivalently,} \quad v_{st} - u_{sx} = 0, \quad (2.5)
\]
which implies that
\[
u_1(\xi) = -\frac{1}{2}\xi v_1 - \frac{1}{2} \int_{-\infty}^{\xi} v_1 d\xi = -\frac{1}{2}(\xi G_1)_\xi, \quad (2.6)
\]
where $G_1 = \int_{-\infty}^{\xi} v_1 d\xi$. Denote the source term
\[
S[v_*] = u_{st} + P(v_*)_x + u_*.
\]
For convenience, we also write $P'(v) = \frac{dP(v)}{dv}$. From Darcy’s law $P(\bar{v})_x + \bar{u} = 0$ and (2.6), a straightforward computation shows that
\[
S[v_*] = \bar{u}_t + (1 + t)^{-\frac{3}{2}}P'(\bar{v})g_2 + g_3 \quad (2.7)
\]
where
\[
S[v_*] = \bar{u}_t + (1 + t)^{-\frac{3}{2}}P'(\bar{v})v_1 + g_2 + g_3
\]
and
\[
S[v_*] = \bar{u}_t + (1 + t)^{-\frac{3}{2}}P'(\bar{v})G_1 + P(\bar{v})_x + O(1)(1 + t)^{-\frac{5}{2}}, \quad (2.8)
\]
where
\[
g_2 = P(v_*) - P(\bar{v}) - P'(\bar{v})(v_* - \bar{v}), \quad (2.9)
\]
and
\[
g_3 = (1 + t)^{-\frac{3}{2}}u_1. \quad (2.10)
\]
To eliminate the term concerning $(1 + t)^{-\frac{3}{2}}$ in (2.8), we seek for $G_1(\xi)$ satisfying
\[
P'(\bar{v})G_1 - \frac{1}{2}\xi G_1 + \frac{1}{2}\xi P(\bar{v})_\xi = 0 \quad (2.11)
\]
and $v_1(\xi) = G_1(\xi)$. Thus, $S[v_*] = g_2 + g_3$. We choose a solution $G_1$ to (2.11) by
\[
G_1(\xi) = -e^{\int_{\bar{v}}^{\xi} \frac{d\xi}{P'(\eta)}} \int_{0}^{\xi} e^{-\int_{\eta}^{\xi} \frac{d\eta}{P'(\theta)}} \frac{d\eta \eta P'(\eta)}{2}. \quad (2.12)
\]
Lemma 2.2. Let $\delta = |v_+ - v_-|$. Then, for $(v_1, u_1)$ given in (2.6) and (2.12), it holds that
\[
S[v_*] = O(1)\delta e^{-c \xi^2} (1 + t)^{-\frac{5}{2}}. \quad (2.13)
\]
Proof. It follows from (2.9)–(2.10) that
\[
g_2 = O(1)(1 + t)^{-2} v_1^2, \quad g_3 = (1 + t)^{-\frac{3}{2}}u_1.
\]
By Lemma 2.1 and (2.6) and (2.12), we have
\[
S[v_*] = g_{2x} + g_{3t} = O(1)\delta e^{-c_{x}^2} (1 + t)^{-\frac{5}{2}}.
\]
Thus the proof is completed. □

Moreover, we have the following.

**Lemma 2.3.** It holds that
\[
\|g_2\|_{L^1} \leq C\delta (1 + t)^{-\frac{3}{2}}, \quad \|\delta_t^n \delta_x^k g_2\|_{L^2} \leq C\delta (1 + t)^{-n-k-\frac{7}{4}},
\]
\[
(2.14)
\]
\[
\|g_3\|_{L^1} \leq C\delta (1 + t)^{-1}, \quad \|\delta_t^n \delta_x^k g_3\|_{L^2} \leq C\delta (1 + t)^{-n-k-\frac{5}{4}}.
\]
\[
(2.15)
\]

*Proof.* Lemma 2.3 can be proved by the straightforward computations on (2.9)–(2.10). □

### 3 Reduced System

From (1.1), (2.5), and (2.7), we get the following perturbation system:

\[
\begin{aligned}
(v - v_*)_t - (u - u_*)_x &= 0, \\
(u - u_*)_t + (P'(v_*)(v - v_*))_x + u - u_* &= g_{1x} + S[v_*]
\end{aligned}
\]

(3.1)

with the initial data
\[
(v - v_*, u - u_*)(x, 0) = (v_0(x) - v_*(x, 0), u_0(x) - u_*(x, 0)),
\]

(3.2)

where
\[
g_1 = -(P(v) - P(v_*) - P'(v_*)(v - v_*)).
\]

(3.3)

Let
\[
V = \int_{-\infty}^{x} (v(y, t) - v_*(y, t)) dy,
\]

which yields that \(V_x = v - v_*\) and \(V_t = u - u_*\). Then, we can rewrite (3.1)–(3.2) as a nonlinear wave equation

\[
V_{tt} + (P'(v_*)V_x)_x + V_t = g_{1x} + S[v_*]
\]

(3.4)

with the initial data
\[
(V, V_t)(x, 0) = (V_0, V_1)(x).
\]

(3.5)

Following the framework of [24], we seek for the solution \(V(x, t)\) of (3.4) in the following solution space:

\[
X_T = \{ V(x, t) \mid V \in C([0, T); H^5(\mathbb{R})), V_t \in C([0, T); H^4(\mathbb{R})) \}.
\]
Since the local existence of the solution of (3.4) can be proved by the standard iteration method, see [22], the main effort in this section is to establish the a priori estimates for the solution.

For any $T \in (0, +\infty)$, define
\[
N(T) = \sup_{0 \leq t \leq T} \sum_{l+k \leq 5, l \leq 1} (1+t)^{l+k/2+3/4} \|\partial_t^l \partial_x^k V(t)\|_{L^2}.
\] (3.6)

Suppose that $N(T) \leq \epsilon$, where $\epsilon$ is sufficiently small and will be determined later. Then, it follows from (2.13) and (3.4) that
\[
\sum_{j+s \leq 5, j \geq 0, s \geq 0} \|\partial_t^j \partial_x^s V(0)\|_{L^2}^2 \leq C(N^2(0) + \delta)
\] (3.7)
for some positive constant $C$.

**Lemma 3.1.** For any $T > 0$, assume that $V(x, t) \in X_T$ is the solution of (3.4). If $\epsilon$ and $\delta = |v_+ - v_-|$ are small, then it holds that for any $0 < t < T$,
\[
\|V(\cdot, t)\|^2 + \sum_{s=0}^4 (1+t)^{s+1} \left( \|\partial_x^s V_t(\cdot, t)\|^2 + \|\partial_x^s V_x(\cdot, t)\|^2 \right) \leq C(N^2(0) + \delta)
\] (3.8)
and
\[
(1+t)^2 \|V_t(\cdot, t)\|^2 + \sum_{s=0}^3 (1+t)^{s+3} \left( \|\partial_x^s Vtt(\cdot, t)\|^2 + \|\partial_x^s Vxt(\cdot, t)\|^2 \right) \leq C(N^2(0) + \delta).
\] (3.9)

*Proof.* The proof can be completed by the same line as in [24] and the details are omitted. \(\square\)

The decay rates of $\partial_x^s \partial_t^l V_{tt}$ can be improved by the following estimates.

**Lemma 3.2.** Under the assumptions of Lemma 3.1, it holds that for any $0 < t < T$,
\[
(1+t)^4 \|V_{tt}(\cdot, t)\|^2 + \sum_{s=0}^2 (1+t)^{s+5} \left( \|\partial_x^s V_{ttt}(\cdot, t)\|^2 + \|\partial_x^s V_{xtt}(\cdot, t)\|^2 \right) \leq C(N^2(0) + \delta).
\] (3.10)

*Proof.* For $s = 0, 1, 2$, taking the procedure as
\[
\int_0^t \int_{\mathbb{R}} \left[ \partial_x^2 \partial_t^2 (3.4) \times (\mu + t)^{s+4} \partial_x^3 \partial_t^2 V + \partial_x^2 \partial_t^2 (3.4) \times (\mu + t)^{s+4} \partial_x^2 \partial_t^2 V \right] dx d\tau
\]
and choosing constant $\mu$ large enough, together with Lemmas 2.1 and 3.1, yield that
\[
(1+t)^4 \|V_{tt}(\cdot, t)\|^2 + (1+t)^{s+4} \left( \|\partial_x^s V_{ttt}(\cdot, t)\|^2 + \|\partial_x^s V_{xtt}(\cdot, t)\|^2 \right)
\]
\[
+ \int_0^t (1+\tau)^{s+4} \left[ \|\partial_x^s V_{xtt}(\cdot, \tau)\|^2 + \|\partial_x^s V_{ttt}(\cdot, \tau)\|^2 \right] d\tau \leq C(N^2(0) + \delta).
\] (3.11)
In the similar way as for
\[
\int_0^t \int_{\mathbb{R}} \partial_x^2 \partial_t^2 (3.4) \times (\mu + t)^{s+5} \partial_x^2 \partial_t^3 V dx d\tau,
\]
we have
\[
(1 + t)^{s+5} \left( \| \partial_x V_{ttt}(\cdot, t) \|^2 + \| \partial_x V_{xttt}(\cdot, t) \|^2 \right) + \int_0^t (1 + \tau)^{s+5} \| \partial_x V_{ttt}(\cdot, \tau) \|^2 d\tau \leq C(N^2(0) + \delta).
\]

(3.12)

Thus, (3.11) and (3.12) lead to (3.10). Therefore, the proof of Lemma 3.2 is completed. □

Similarly, we have the following.

**Lemma 3.3.** Under the assumptions of Lemma 3.1, it holds that for any \(0 < t < T\),
\[
(1 + t)^6 \| V_{ttt}(\cdot, t) \|^2 + \sum_{s=0}^1 (1 + t)^{s+7} \left( \| \partial_x^s V_{tttt}(\cdot, t) \|^2 + \| \partial_x^s V_{xttt}(\cdot, t) \|^2 \right) \leq C(N^2(0) + \delta).
\]

(3.13)

**Lemma 3.4.** Under the assumptions of Lemma 3.1, it holds that for any \(0 < t < T\),
\[
(1 + t)^9 \| \partial_t^4 V(\cdot, t) \|^2 + (1 + t)^9 \left( \| \partial_t^5 V(\cdot, t) \|^2 + \| \partial_t^4 V_x(\cdot, t) \|^2 \right) \leq C(N^2(0) + \delta).
\]

(3.14)

### 4 | GREEN FUNCTION METHOD

Note that the decay rates obtained in Lemmas 3.1–3.4 are not fast enough to close the a priori assumption \(N(T) \leq \epsilon\) in (3.6). In this section, we will use the approximate Green function method to improve the decay rates so that the a priori assumption can be closed. Since Lemmas 3.1–3.4 have provided the desired estimates to close the a priori assumption for local time, we focus on the large time \(t > 1\) in what follows. As in [25], we rewrite (3.4) as
\[
V_t + (a(x, t)V_x)_x = g_{1x} + S[v_*] - V_{tt},
\]
where \(a(x, t) = P'(v_*)\) and construct a minimizing Green function as
\[
G(x, t; y, s) = \left( \frac{-1}{4\pi a(x, t)(t-s)} \right)^{\frac{3}{2}} \exp \left( \frac{(x-y)^2}{4A(y, s; t)(t-s)} \right)
\]
satisfying the basic requirement
\[
G(x, t; y, t) = \delta(y-x),
\]
where \(\delta\) is the Dirac function and \(A(y, s; t) = P'(v_*(\eta, t))\) and
\[
\eta = \begin{cases} 
  y/\sqrt{1 + s}, & s > t/2, \\
  y/\sqrt{1 + t/2}, & s \leq t/2.
\end{cases}
\]
Recall that $S[v_*] = g_{2x} + g_{3y}$, then the solution $V(x, t)$ to (4.1) can be written as the integral form

$$
V(x, t) = \int_{\mathbb{R}} G(x, t; y, 0)V_0(y)dy + \int_0^t \int_{\mathbb{R}} G(x, t; y, s)[g_{1y} + g_{2y} + g_{3y} - V_{ss}]dyds
$$

$$
+ \int_0^t \int_{\mathbb{R}} R_G(x, t; y, s)V(y, s)dyds,
$$

(4.3)

where

$$
R_G(x, t; y, s) = G_3(x, t; y, s) - \{a(y, s)G_3(x, t; y, s)\}_y.
$$

### 4.1 Properties of approximate green function

In this subsection, we recall the properties of $G(x, t; y, s)$ and $R_G(x, t; y, s)$ introduced in [25].

**Lemma 4.1 [25].** For $l, h \leq 1$, it holds that

$$
|\partial_l^\ell \partial_s^h \partial_x^k \partial_y^m G(x, t; y, s)| \leq O(1) \left( \sum_{m_1 + m_2 = m} (t - s)^{-m_1 + m_2} \right) \left( \sum_{k_1 + k_2 = k} (t - s)^{-k_1 + k_2} (1 + t)^{-k_2} \right)
$$

$$
\times \left( (1 + t)^{-1} + (t - s)^{-1} \right)^{l} \left( \partial_1^2 + (t - s)^{-1} \right)^{h} G_D(x - y, t - s),
$$

(4.4)

where

$$
G_D(y, s) = \left( \frac{1}{4\pi s} \right)^{\frac{1}{2}} \exp \left( -\frac{y^2}{D_0 s} \right)
$$

with $\|G_D\|_{L^p} \leq C s^{-\frac{1}{2}(1 - \frac{1}{p})}$ for $p \geq 1$,

and $\vartheta(s) = \vartheta_1(s) + \vartheta_2(s)$ with

$$
\vartheta_1(s) = \begin{cases} 
(1 + s)^{-\frac{1}{2}}, & s > t/2, \\
0, & s \leq t/2,
\end{cases}
$$

and

$$
\vartheta_2(s) = \begin{cases} 
0, & s > t/2, \\
(1 + t)^{-\frac{1}{2}}, & s \leq t/2.
\end{cases}
$$

**Lemma 4.2 [25].** It holds that

$$
R_G = O(1) \delta \Theta(t, s) \tilde{E}(y, t, s) G_D(x - y, t - s),
$$

(4.5)

where

$$
\Theta(t, s) = \begin{cases} 
(1 + s)^{-1} + (t - s)^{-\frac{1}{2}}(1 + s)^{-\frac{1}{2}}, & s > t/2, \\
(1 + t)^{-1} + (t - s)^{-\frac{1}{2}}(1 + s)^{-\frac{1}{2}}, & s \leq t/2,
\end{cases}
$$

$$
\tilde{E}(y, t, s) = \begin{cases} 
E(y, s), & s > t/2, \\
E(y, t), & s \leq t/2,
\end{cases}
$$

with $E(y, \tau) = \exp(-Cy^2/(1 + \tau))$ for some constant $C > 0$. Moreover, it holds for $s < t/2$ that

$$
|\partial_l^\ell \partial_s^h R_G(x, t; y, s)| \leq C \delta(1 + s)^{-\frac{1}{2}} (t - s)^{-l-k+1} E(y, t) G_D(x - y, t - s),
$$

(4.6)
and for $s = t/2$ that

$$\lim_{s \to t/2^\pm} \left| \partial^k_x R_G(x, t; y, s) \right| \leq C \delta t^{-1 - k/2} E(y, t/2) G_D(x - y, t/2). \tag{4.7}$$

**Proof.** Although Lemma 4.2 was obtained in [25], we give more details to obtain (4.5) and some details will be used later. The direct computation shows that

$$R_G = \frac{G}{2(t - s)} \left( 1 - \frac{a(y, s)}{A(y, s; t)} \right) \frac{(x - y)^2 G}{4A(y, s)(t - s)^2} \left( 1 - \frac{a(y, s)}{A(y, s; t)} \right) + G \bar{R}, \tag{4.8}$$

where $\bar{R}$ satisfies

$$\bar{R} = O(1) \delta \begin{cases} (1 + s)^{-1} + (t - s)^{-1/2} (1 + s)^{-1/2} E(y, s), & s > t/2, \\ (t - s)^{-1/2} (1 + s)^{-1/2} E(y, s) + (t - s)^{-1/2} (1 + t)^{-1/2} E(y, t), & s \leq t/2 \end{cases} \tag{4.9}$$

$$= O(1) \delta \Theta(t, s) E(y, t).$$

When $s > t/2$, it follows from $a(y, s) = A(y, s; t)$ that $R_G = G \bar{R}$, which implies that (4.5) holds. It remains to show (4.5) for $s \leq t/2$. The straightforward computations show that

$$\left| \frac{1}{t - s} p'(v_* \left( \frac{y}{\sqrt{1+t/2}} \right)) - p'(v_* \left( \frac{y}{\sqrt{1+s}} \right)) \right| \leq O(1) \delta E(y, t)(t - s)^{-1/2} (1 + s)^{-1/2}, \tag{4.10}$$

which, together with (4.9), leads to (4.5) directly. \qed

Next, we will summarize the properties of $G(x, t; y, s)$ and $R_G(x, t; y, s)$ introduced in [25].

**Lemma 4.3.** For $s > t/2$ and $k_1 \geq 1, k_2 \geq 0$, it holds that

$$\partial_x^{k_1} G = (-1)^{k_1} \partial_x^{k_1} G + \sum_{0 \leq \beta < k_1} C_\beta \partial_y^\beta (G \bar{R}_{1k_1-\beta-1}^1), \tag{4.11}$$

$$\partial_y^{k_2} G = -\partial_y^{k_2} G + \sum_{0 \leq \beta < k_2} C_\beta \partial_x^\beta (G \bar{h}_{1k_2-\beta}^1), \tag{4.12}$$

where $\bar{R}_{1k_1}^1$ and $\bar{h}_{1k_2}^1$ represent some generic functions satisfying

$$\bar{R}_{1k_1}^1 = O(1) \delta (1 + t)^{-1/2} E(y, t) \quad \text{and} \quad \partial_y \bar{R}_{1k_1}^1 = O(1) \delta (1 + t)^{-1/2} \left[ (t - s)^{-1/2} + (1 + t)^{-1/2} \right]. \tag{4.13}$$
and
\[
\tilde{h}_1^l = O(1)\delta(1 + t)^{-1 - \frac{1}{2}} E(y, t) \quad \text{and} \quad \partial_y \tilde{h}_1^l = O(1)\delta(1 + t)^{-1 - \frac{1}{2}} \left[ (t - s)^{-\frac{1}{2}} + (1 + t)^{-\frac{1}{2}} \right], \quad \forall l \geq 0.
\]
Furthermore, for the above mentioned $R_1^l$, it holds that
\[
\partial_x (GR_1^l) = -\partial_y (GR_1^l) + GR_1^{l+1}. \tag{4.15}
\]

**Proof.** Lemma 4.3 can be proved by the straightforward computations and Lemma 4.1. The details are omitted. \( \square \)

**Lemma 4.4.** For $s > \frac{t}{2}$ and $k_1 \geq 1, k_2 \geq 0$, it holds that
\[
\partial_x^{k_1} R_G = (-1)^{k_1} \partial_y^{k_1} R_G + \sum_{\beta < k_1} C_\beta \partial_y^\beta \left( GR_2^{k_1-\beta-1} \right), \tag{4.16}
\]
\[
\partial_t \partial_x^{k_2} R_G = -\partial_s \partial_x^{k_2} R_G + \sum_{\beta \leq k_2} C_\beta \partial_y^\beta \left( G\tilde{h}_2^{k_2-\beta} \right), \tag{4.17}
\]
where $R_2^l$ and $\tilde{h}_2^l$ represent some generic functions satisfying
\[
R_2^l = O(1)\delta \Theta(t, s) E(y, t)(1 + t)^{-1 - \frac{1}{2}} \quad \text{and} \quad \tilde{h}_2^l = O(1)\delta \Theta(t, s) E(y, t)(1 + t)^{-1 - \frac{1}{2}}, \quad \forall l \geq 0. \tag{4.18}
\]

**Proof.** Recall from (4.9) that for $s > \frac{t}{2}, R_G = GR$ and $R = O(1)\delta \Theta(t, s) E(y, t)$ satisfies
\[
\partial_x R = -\partial_y R + \tilde{R} \quad \text{with} \quad \tilde{R} = O(1)\delta \Theta(t, s) E(y, t)(1 + t)^{-\frac{1}{2}},
\]
\[
\partial_t R = -\partial_s R + \tilde{h} \quad \text{with} \quad \tilde{h} = O(1)\delta \Theta(t, s) E(y, t)(1 + t)^{-1}.
\]
Then, by the direct computations and Lemma 4.3, we can verify (4.16)–(4.17). Thus, the proof of Lemma 4.4 is completed. \( \square \)

Note that (4.6) gives
\[
||\partial_t^l \partial_x^k R_G||_{L^1} = O(1)\delta t^{-l-\frac{k+1}{2}} (1 + s)^{-\frac{1}{2}} \quad \text{for} \quad s \leq \frac{t}{2},
\]
whose decay rate is not fast enough in the analysis. We need to derive a better decay rate as follows.

**Lemma 4.5.** For $s < t/2$, it holds that
\[
||\partial_t^l \partial_x^k R_G||_{L^1} = O(1)\delta \left[ t^{-l-\frac{k+1}{2}} \left( 1 + s \right)^{-\frac{1}{2}} + t^{-\frac{k}{2}} \right] \quad \text{for} \quad q_1 \geq 1.
\]
Proof. Different from (4.10), we estimate

\[ 1 - \frac{a(y,s)}{A(y,s;t)} = \frac{p'(v_* \left( \frac{y}{\sqrt{1+s}} \right)) - p'(v_* \left( \frac{y}{\sqrt{1+t/2}} \right))}{p'(v_* \left( \frac{y}{\sqrt{1+t/2}} \right))} \]

\[ = O(1)\delta \left( \frac{t}{2} - s \right) \int_0^1 E(y,s + \theta(t/2 - s))(1 + s + \theta(t/2 - s))^{-1} d\theta = : O(1)\delta \left( \frac{t}{2} - s \right) \hat{F}. \]

Differentiating (4.8), using Lemma 4.1 and (4.9) gives that

\[ \partial_t^l \partial_x^k R_G = O(1)\delta(t - s)^{-l - \frac{k+1}{2}} \left[ (1 + s)^{-\frac{1}{2}} E(y,s) + (1 + t)^{-\frac{1}{2}} E(y,t) + (t - s)^{-\frac{1}{2}} \left( \frac{t}{2} - s \right) \hat{F} \right] G_D, \]

and then

\[ \| \partial_t^l \partial_x^k R_G \|_{L^1} = O(1)\delta(t - s)^{-l - \frac{k+1}{2}} \left[ (1 + s)^{-\frac{1}{2}} \| E(y,s) \|_{L^{q_1}} \| G_D \|_{L^{p_1}} + (1 + t)^{-\frac{1}{2}} \| E(y,t) \|_{L^2} \| G_D \|_{L^2} \right] \]

\[ + O(1)\delta(t - s)^{-l - \frac{k}{2} - 1} (t/2 - s) \| \hat{F} \|_{L^{q_2}} \| G_D \|_{L^{p_2}}, \]

where \( \frac{1}{p_i} + \frac{1}{q_i} = 1, p_i, q_i \geq 1 \). Thanks to the Minkowski inequality, we have

\[ \| \hat{F} \|_{L^q} = \left( \int_{\mathbb{R}} \left( \int_0^1 E(y,s + \theta(t/2 - s))(1 + s + \theta(t/2 - s))^{-1} d\theta \right)^q dy \right)^{\frac{1}{q}} \]

\[ \leq \int_0^1 \left( \int_{\mathbb{R}} \left[ E(y,s + \theta(t/2 - s))(1 + s + \theta(t/2 - s))^{-1} \right]^q dy \right)^{\frac{1}{q}} d\theta \]

\[ \leq \int_0^1 \left( \int_{\mathbb{R}} e^{-\frac{c q y^2}{1+s+\theta(t/2 - s)}} dy \right)^{\frac{1}{q}} (1 + s + \theta(t/2 - s))^{-1} d\theta \]

\[ = O(1) \int_0^1 (1 + s + \theta(t/2 - s))^{-1 + \frac{1}{2q}} d\theta \]

\[ = O(1)(t/2 - s)^{-1} (1 + t)^{\frac{1}{2q}}, \]

which leads to

\[ \| \partial_t^l \partial_x^k R_G \|_{L^1} = O(1)\delta(t - s)^{-l - \frac{k+1}{2} + \frac{1}{2} \left( 1 - \frac{1}{p_1} \right) (1 + s)^{-\frac{1}{2} + \frac{1}{2q_1}} + O(1)\delta(t - s)^{-l - \frac{k+1}{2} - \frac{1}{2} (1 + t)^{-\frac{1}{2}}} \]

\[ + O(1)\delta(t - s)^{-l - \frac{k}{2} - 1 - \frac{1}{2} \left( 1 - \frac{1}{p_2} \right) (1 + t)^{\frac{1}{2q_2}}} \]

\[ = O(1)\delta t^{-l - \frac{k+1}{2} + \frac{1}{2q_1}} (1 + s)^{-\frac{1}{2} + \frac{1}{2q_1}} + O(1)\delta t^{-l - \frac{k}{2} - 1}. \]

Thus, the proof of Lemma 4.5 is completed. \( \square \)
Decay rate

In this subsection, we will derive the sharper decay rate for $V(x, t)$ to close the a priori assumption $N(T) \leq \varepsilon$ in (3.6). From (4.3), we have

$$\partial_l \partial_k^k V(x, t) = \sum_{i=1}^{6} I_{l,k}^{i}, \quad (4.19)$$

where $l \leq 1, k + l \leq 5$ and

$$I_{1}^{l,k} = \int_{\mathbb{R}} \partial_l^l \partial_k^k G(x, t; y, 0) V(y, 0) dy,$$

$$I_{2}^{l,k} = \partial_l \int_{0}^{t} \int_{\mathbb{R}} \partial_k^k G(x, t; y, s) g_{2y} ds dy,$$

$$I_{3}^{l,k} = \partial_l \int_{0}^{t} \int_{\mathbb{R}} \partial_k^k G(x, t; y, s) g_{3y} ds dy,$$

$$I_{4}^{l,k} = \partial_l \int_{0}^{t} \int_{\mathbb{R}} \partial_k^k G(x, t; y, s) g_{1y} ds dy,$$

$$I_{5}^{l,k} = -\partial_l \int_{0}^{t} \int_{\mathbb{R}} \partial_k^k G(x, t; y, s) V_{ss} ds dy,$$

$$I_{6}^{l,k} = \partial_l \int_{0}^{t} \int_{\mathbb{R}} \partial_k^k R_{G}(x, t; y, s) V ds dy.$$

For $I_{1}^{l,k}$, we use Lemma 4.1, the facts $V(y, 0) = \partial_y \tilde{V}(y, 0)$ and $\tilde{V}(y, 0) \in L^2 \cap L^1$ with $\|\tilde{V}(y, 0)\|_{L^1} < \delta_1$ to get

$$\|I_{1}^{l,k}\|_{L^2} = \|\int_{\mathbb{R}} \partial_l^l \partial_k^k G(x, t; y, 0) \tilde{V}(y, 0) dy\|_{L^2} = \|\partial_l^l \partial_k^k G(x, t; y, 0) * \tilde{V}(y, 0)\|_{L^2}$$

$$= \|\tilde{V}(y, 0)\|_{L^1} \|G\|_{L^2} \|g_{2y}\|_{L^1} = O(1) \delta_1 (1 + t)^{-\frac{l-k+3}{4}}, \quad (4.20)$$

where we have used the Young inequality $\|f * g\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}$ with $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ for $1 \leq p, q \leq \infty$.

For $I_{2}^{l,k}$ with $l = 0, k \leq 5$, we have

$$I_{2}^{0,k} = \int_{0}^{t} \int_{\mathbb{R}} \partial_k^k G(x, t; y, s) g_{2y} ds dy = \left( \int_{0}^{\frac{t}{2}} + \int_{\frac{t}{2}}^{t} \right) \int_{\mathbb{R}} \partial_k^k G(x, t; y, s) g_{2y} ds dy =: I_{2,1}^{0,k} + I_{2,2}^{0,k}, \quad (4.21)$$

It follows from (2.14) and (4.4) that

$$\|I_{2,1}^{0,k}\|_{L^2} = \|\int_{0}^{\frac{t}{2}} \int_{\mathbb{R}} \partial_y \partial_k^k G(x, t; y, s) g_{2y} ds dy\|_{L^2} = O(1) \int_{0}^{\frac{t}{2}} t^{-\frac{k}{2}} (1 + t)^{\frac{l}{2}} \|G\|_{L^2} \|g_{2y}\|_{L^1} ds$$

$$= O(1) \delta(1 + t)^{-\frac{k-3}{4}} \int_{0}^{\frac{t}{2}} (1 + s)^{-\frac{3}{2}} ds = O(1) \delta(1 + t)^{-\frac{k-3}{4}}. \quad (4.22)$$
In addition, we use \((2.14), (4.11),\) and \((4.13)\) to get

\[
\|I_{0, k}^{0, k}\|_{L^2} = \| \left[ \int_0^t \int_\mathbb{R} \left( -1 \right)^k \partial_\gamma \mathcal{G} + \sum_{\beta < k} C_\beta \partial_\gamma \left( \mathcal{G} \mathcal{R}_{k - \beta - 1}^1 \right) \right] g_2 y dy ds \|_{L^2} 
\]

\[= O(1) \int_0^t \| G_D \|_{L^1} \| \partial_\gamma^{k+1} g_2 \|_{L^2} ds + O(1) \delta \sum_{\beta < k} (1 + t)^{-\frac{k - \beta}{2}} \int_0^t \| G_D \|_{L^1} \| \partial_\gamma^{\beta+1} g_2 \|_{L^2} ds \]

\[= O(1) \delta (1 + t)^{-k + \frac{5}{2}}. \tag{4.23} \]

Substituting \((4.22)\)–\((4.23)\) into \((4.21)\) yields that for \(k \leq 5\),

\[
\|I_{0, k}^{0, k}\|_{L^2} = O(1) \delta (1 + t)^{-\frac{k}{2} - \frac{3}{4}}. \tag{4.24} \]

When \(l = 1, k \leq 4\), we deduce from \((4.11)\)–\((4.12)\) that

\[
\|I_{l, k}^{1, k}\|_{L^2} = \left[ \int_\mathbb{R} \partial_\gamma \mathcal{G}(x, t; y, \frac{t}{2}) g_2 y \left( \frac{t}{2} \right) dy + \int_0^t \int_\mathbb{R} \partial_\gamma \partial_\gamma^{k} \mathcal{G}(x, t; y, s) g_2 dy ds + \int_0^t \int_\mathbb{R} \mathcal{G} \partial_\gamma \partial_\gamma^{k+1} g_2 dy ds \right] \]

\[+ \sum_{\beta < k} C_\beta \int_\mathbb{R} \int_\mathbb{R} \mathcal{R}_{k - \beta - 1}^1 \partial_\gamma \partial_\gamma^{\beta+1} g_2 dy ds + \sum_{\beta < k} C_\beta \int_\mathbb{R} \int_\mathbb{R} \mathcal{R}_{k - \beta}^1 \partial_\gamma^{\beta+1} g_2 dy ds, \]

which, together with \((2.14), (4.4),\) and \((4.13)\)–\((4.14)\), yields that

\[
\|I_{l, k}^{1, k}\|_{L^2} = O(1) t^{-\frac{k}{2}} \| G_D \|_{L^1} \| g_2 \|_{L^2} ds 
\]

\[+ O(1) \int_0^t \| G_D \|_{L^1} \| \partial_\gamma \partial_\gamma^{k+1} g_2 \|_{L^2} ds + O(1) \delta \sum_{\beta < k} (1 + t)^{-\frac{k - \beta}{2}} \int_0^t \| G_D \|_{L^1} \| \partial_\gamma^{\beta+1} g_2 \|_{L^2} ds \]

\[+ O(1) \delta \sum_{\beta < k} (1 + t)^{-1 - \frac{k - \beta}{2}} \int_0^t \| G_D \|_{L^1} \| \partial_\gamma^{\beta+1} g_2 \|_{L^2} ds = O(1) \delta (1 + t)^{-\frac{k}{2} - \frac{7}{4}}. \tag{4.25} \]

In summary, we have from \((4.24)\) and \((4.25)\) that

\[
\|I_{l, k}^{1, k}\|_{L^2} = O(1) \delta (1 + t)^{-\frac{l - k}{2} - \frac{3}{4}}. \tag{4.26} \]

For \(I_{l, k}^{0, k}\) with \(l = 0, k \leq 5\), we have

\[
I_{l, k}^{0, k} = \int_0^t \int_\mathbb{R} \partial_\gamma^{k} \mathcal{G}(x, t; y, s) g_{3s} dy ds = \left( \int_0^\frac{t}{2} + \int_\frac{t}{2}^t \right) \int_\mathbb{R} \partial_\gamma^{k} \mathcal{G}(x, t; y, s) g_{3s} dy ds = I_{3, 1}^{0, k} + I_{3, 2}^{0, k}. \tag{4.27} \]

Then,

\[
I_{3, 1}^{0, k} = \int_\mathbb{R} \partial_\gamma^{k} \mathcal{G}(x, t; y, \frac{t}{2}) g_3 (y, \frac{t}{2}) dy - \int_\mathbb{R} \partial_\gamma^{k} \mathcal{G}(x, t; y, 0) g_3 (y, 0) dy \]
\begin{equation}
- \int_0^t \int_{\mathbb{R}} \partial_y \partial^k_x G(x, t; y, s) g_3 dy ds.
\end{equation}

(4.28)

It follows from (2.6) and (2.10) that
\begin{equation}
g_3(y, 0) = u_1(y) = -\frac{1}{2} y v_1(y) - \frac{1}{2} \int_{-\infty}^y v_1(\tau) d\tau = -\mathcal{G}_y,
\end{equation}

(4.29)

which yields that
\begin{equation}
\int_{\mathbb{R}} \partial_k x G(x, t; y, 0) g_3(y, 0) dy = \int_{\mathbb{R}} \partial_y \partial^k_x G(x, t; y, 0) \mathcal{G} dy.
\end{equation}

(4.30)

Substituting (4.30) into (4.28), together with (2.15) and (4.4), leads to
\begin{align}
\| I_{3,1}^{0,k} \|_{L^2} &= O(1)t^{-\frac{k}{2}} \| G_D \|_{L^1} \| g_3 \|_{L^2} + O(1)t^{-\frac{k}{2} - \frac{3}{2}} \| G_D \|_{L^2} \| \mathcal{G} \|_{L^1} + O(1)t^{-1 - \frac{k}{2}} \int_0^t \| G_D \|_{L^2} \| g_3 \|_{L^1} \| \partial_s \partial^k y \|_{L^2} ds \\
&= O(1)\delta t^{-\frac{k}{2} - \frac{5}{4}} + O(1)\delta t^{-\frac{k}{2} - \frac{3}{4}} + O(1)\delta t^{-\frac{k}{2} - \frac{5}{4}} \ln(1 + t) = O(1)\delta(1 + t)^{-\frac{k}{2} - \frac{3}{4}},
\end{align}

(4.31)

where we have used \( \| \mathcal{G} \|_{L^1} = O(1)\delta \) due to \( |\mathcal{G}| = O(1)\delta e^{-cy} \) from (2.12). In addition, it follows from (2.15) and (4.11) that
\begin{align}
\| I_{3,2}^{0,k} \|_{L^2} &= \left\| \int_{\frac{t}{2}}^t \int_{\mathbb{R}} \partial_\beta \partial^k_x g_3 dy ds + \sum_{\beta < k} C_\beta \int_{\frac{t}{2}}^t \int_{\mathbb{R}} GR^{1}_{k-\beta-1} \partial_\beta \partial^\beta y g_3 dy ds \right\|_{L^2} \\
&= O(1) \int_{\frac{t}{2}}^t \| G_D \|_{L^1} \| \partial_\beta \partial^k \partial_y g_3 \|_{L^2} ds + O(1)\delta \sum_{\beta < k} (1 + t)^{-\frac{k-\beta}{2}} \int_{\frac{t}{2}}^t \| G_D \|_{L^1} \| \partial_\beta \partial^\beta y g_3 \|_{L^2} ds \\
&= O(1)\delta(1 + t)^{-\frac{k}{2} - \frac{3}{4}}.
\end{align}

(4.32)

Substituting (4.31)–(4.32) into (4.27) yields that for \( k \leq 5 \),
\begin{equation}
\| I_{3,1}^{0,k} \|_{L^2} = O(1)\delta(1 + t)^{-\frac{k}{2} - \frac{3}{4}}.
\end{equation}

(4.33)

When \( l = 1, k \leq 4 \), we deduce from (4.11)–(4.12) that
\begin{align}
I_{3}^{1,k} &= \int_{\mathbb{R}} \partial^k_x G(x, t; y, \frac{t}{2}) g_3 ydy + \int_{\mathbb{R}} \partial_\beta \partial^k_x G(x, t; y, \frac{t}{2}) g_3 ydy \\
&- \int_{\mathbb{R}} \partial_\beta \partial^k_x G(x, t; y, 0) g_3 ydy \\
&+ \int_{0}^{\frac{t}{2}} \int_{\mathbb{R}} \partial_\beta \partial^k_x G(x, t; y, s) g_3 dy ds + \int_{\frac{t}{2}}^t \int_{\mathbb{R}} \partial^2_\beta \partial^k_x g_3 dy ds \\
&+ \sum_{\beta < k} C_\beta \int_{\frac{t}{2}}^t \int_{\mathbb{R}} GR^{1}_{k-\beta-1} \partial_\beta \partial^\beta y g_3 dy ds + \sum_{\beta < k} C_\beta \int_{\frac{t}{2}}^t \int_{\mathbb{R}} \mathcal{G}^{1}_{k-\beta} \partial_\beta \partial^\beta y g_3 dy ds.
\end{align}

(4.34)
It follows from (4.29) that

\[
\int_{\mathbb{R}} \partial_t \partial_x^k G(x, t; y, 0) g_3(y, 0) dy = \int_{\mathbb{R}} \partial_t \partial_y^k \partial_x^k G(x, t; y, 0) \tilde{G} dy.
\]  

(4.35)

Thus using (2.15), (4.4), (4.13)–(4.14), and (4.35) gives that for \( k \leq 4 \),

\[
\| I_{3, k}^{1} \|_{L^2} = O(1) t^{-\frac{k}{2}} \| G_D \|_{L^1} \| g_3 \|_{L^2} + O(1) t^{-1-\frac{k}{2}} \| G_D \|_{L^1} \| g_3 \|_{L^2} + O(1) t^{-\frac{3}{2}-\frac{k}{2}} \| G_D \|_{L^2} \| \tilde{G} \|_{L^1} \\
+ O(1) t^{-2-\frac{k}{2}} \int_0^{\frac{t}{2}} \| G_D \|_{L^2} \| g_3 \|_{L^1} ds + (1+\delta) \sum_{\beta < k} (1+t)^{-\frac{k-\beta}{2}} \int_0^{\frac{t}{2}} \| \partial^\beta \partial_s \partial_x^k G_3 \|_{L^2} ds \\
+ O(1) \delta \sum_{\beta \leq k} (1+t)^{-1-\frac{k-\beta}{2}} \int_\frac{t}{2}^{t} \| G_D \|_{L^1} \| \partial^\beta \partial_s G_3 \|_{L^2} ds = O(1) \delta (1+t)^{-\frac{k-3}{4}}.
\]  

(4.36)

In summary, we have from (4.33) and (4.36) that

\[
\| I_{3, k}^{1} \|_{L^2} = O(1) \delta (1+t)^{-l-\frac{k}{2}-\frac{3}{4}}.
\]  

(4.37)

Therefore, we conclude from (4.20), (4.26), and (4.37) that

**Lemma 4.6.** It holds that for \( l + k \leq 5 \), \( l \leq 1 \),

\[
\sum_{i=1}^{3} \| I_{i, k}^{l} \|_{L^2} = O(1) (\delta + \delta_1) (1+t)^{-l-\frac{k}{2}-\frac{3}{4}}.
\]  

(4.38)

Next, we estimate the nonlinear terms \( I_{i, k}^{l} \) for \( i = 4, 5, 6 \). First, we have the following.

**Lemma 4.7.** It holds that for \( l + k \leq 5 \), \( l \leq 1 \),

\[
\| I_{4, k}^{l} \|_{L^2} = O(1) (N^2(0) + \delta + \varepsilon^2) (1+t)^{-l-\frac{k}{2}-\frac{3}{4}}.
\]  

(4.39)

**Proof.** When \( l = 0, k \leq 4 \), we have

\[
I_{4, k}^{0} = \int_0^t \int_{\mathbb{R}} \partial_x^k G(x, t; y, s) g_{1y} dy ds = \left( \int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^{t} \right) \int_{\mathbb{R}} \partial_x^k G(x, t; y, s) g_{1y} dy ds = : I_{4, 1}^{0} + I_{4, 2}^{0}.
\]  

(4.40)

It follows from (3.3) and the a priori assumption \( N(T) \leq \varepsilon \) that

\[
\int_0^{\frac{t}{2}} \| g_1 \|_{L^1} ds = O(1) \varepsilon^2 \int_0^{\frac{t}{2}} (1+s)^{-\frac{5}{2}} ds = O(1) \varepsilon^2,
\]  

(4.41)
which, together with Lemma 4.1 and the Young inequality, yields that for \( t > 1 \)
\[
\|I_{0,k}^{4,1}\|_{L^2} \leq \| \int_0^t \int_\mathbb{R} \partial_y \partial_x^k G(x, t; y, s) g_1(y) dy ds \|_{L^2} = O(1) \int_0^t \int_\mathbb{R} \partial_y \partial_x^k G(x, t; y, s) g_1(y) dy ds \|_{L^2} ds \\
= O(1) \int_0^t t^{-k} (1 + t)^{-1} \| G \|_{L^2} \| g_1 \|_{L^1} ds \leq O(1) (1 + t)^{-k/2 - 3/4} \int_0^t \| g_1 \|_{L^1} ds.
\]

(4.42)

In addition, it follows from (4.11), (4.13), and (4.15) that
\[
\|I_{0,k}^{4,2}\|_{L^2} = \| \int_0^t \int_\mathbb{R} \partial_y \partial_x^k G(x, t; y, s) g_1(y) dy ds \|_{L^2} \\
\leq \| \int_0^t \int_\mathbb{R} \partial_y \partial_x^k G(x, t; y, s) g_1(y) dy ds \|_{L^2} + \sum_{\beta < k} O(1) \delta \int_0^t (1 + t)^{-k - \beta - 1/2} \| G \|_{L^2} \| \partial_y^{k+1} g_1 \|_{L^2} ds \\
+ \sum_{\beta < k} O(1) \delta \int_0^t (1 + t)^{-k - \beta - 1/2} \| G \|_{L^2} \| \partial_y^{k+1} g_1 \|_{L^2} ds.
\]

(4.43)

From (3.3), \( g_1 = O(1) V^2_x \). Since \( V \in C([0,T); H^5(\mathbb{R})) \), the derivative order of \( g_1 \) is at most 4. This is why we consider \( k \leq 4 \) in (4.43). From (3.8), it holds that
\[
\| \partial_y^k g_1(s) \|_{L^2} = O(1)(N^2(0) + \delta)(1 + s)^{-k/2 - 5/4},
\]
which yields that
\[
\|I_{0,k}^{4,2}\|_{L^2} = O(1) \| N^2(0) + \delta + \epsilon^2 \| (1 + t)^{-k/2 - 3/4}.
\]

(4.45)

Substituting (4.42) and (4.45) into (4.40) gives that for \( k \leq 4 \),
\[
\|I_{0,k}^{4,2}\|_{L^2} = O(1) \left[ N^2(0) + \delta + \epsilon^2 \right] (1 + t)^{-k/2 - 3/4}.
\]

(4.46)

When \( l = 1, k \leq 3 \), we deduce from (4.12) that
\[
I_{0,k}^{1,1} = \int_\mathbb{R} \partial_x^k G(x, t; y, t) g_1(y) dy + \int_0^t \int_\mathbb{R} \partial_x^k \partial_x G(x, t; y, s) g_1(y) dy ds \\
= \int_\mathbb{R} \partial_x^k G(x, t; y, t) g_1(y) dy + \int_0^t \int_\mathbb{R} \partial_x^k \partial_x G(x, t; y, s) g_1(y) dy ds \\
+ \int_0^t \int_\mathbb{R} \partial_x^k G \partial_x g_1 dy ds + \sum_{\beta < k} C_{\beta} \int_0^t \int_\mathbb{R} \partial_x^k \partial_y^{\beta+1} g_1 dy ds.
\]

(4.47)
It follows from Lemma 4.1 and (4.44) that
\[
\| \int_{\mathbb{R}} \frac{\partial^k}{\partial x^k} G(x, t; y, \frac{f}{2}) \, dy \|_{L^2} = \| \frac{\partial^k}{\partial x^k} G(x, t; y, \frac{f}{2}) * g_1(y, \frac{t}{2}) \|_{L^2} = O(1) t^{-\frac{k}{2}} \| G_D \|_{L^1} \| g_1 \|_{L^2}
\]
\[
= O(1) (N^2(0) + \delta + \epsilon^2) (1 + t)^{-\frac{k}{2} - \frac{7}{4}},
\]
(4.48)
and from (4.41) that
\[
\| \int_{0}^{t} \int_{\mathbb{R}} \partial_t \frac{\partial^k}{\partial x^k} G(x, t; y, s) \, dy \, ds \|_{L^2} = \| \int_{0}^{t} \int_{\mathbb{R}} \partial_t \frac{\partial^k}{\partial x^k} \tilde{G}(x, t; y, s) \, dy \, ds \|_{L^2}
\]
\[
= O(1) t^{-\frac{k}{2} - \frac{3}{2}} \int_{0}^{t} \| G_D \|_{L^1} \| g_1 \|_{L^2} \, ds = O(1) \epsilon^2 (1 + t)^{-\frac{k}{2} - \frac{7}{4}}.\]
(4.49)

Similar to (4.43), we use (4.11) and (4.15) to get
\[
\int_{\frac{t}{2}}^{t} \int_{\mathbb{R}} \frac{\partial^k}{\partial x^k} G \partial_s g_1(y, t) \, dy \, ds = \int_{\frac{t}{2}}^{t} \int_{\mathbb{R}} \partial_x \left( (-1)^{k-1} \partial_y^{k-1} G + \sum_{\beta < k-1} C_\beta \partial_y^{\beta} (G^1_{k-\beta-2}) \right) \partial_s g_1(y, t) \, dy \, ds
\]
\[
= \int_{\frac{t}{2}}^{t} \int_{\mathbb{R}} \partial_x G \partial_s \partial^k g_1(y, t) \, dy \, ds + \sum_{\beta < k-1} C_\beta \int_{\frac{t}{2}}^{t} \int_{\mathbb{R}} G^1_{k-\beta-2} \partial_s \partial_y^{\beta+2} g_1(y, t) \, dy \, ds
\]
\[
+ \sum_{\beta < k-1} C_\beta \int_{\frac{t}{2}}^{t} \int_{\mathbb{R}} G^1_{k-\beta-1} \partial_y^{\beta+1} g_1(y, t) \, dy \, ds.
\]
Thus, for the last two terms in the right-hand side of (4.47), we use Lemma 4.1 and Equations (4.13) and (4.44) to get
\[
\| \int_{\frac{t}{2}}^{t} \int_{\mathbb{R}} \partial_x G \partial_s g_1(y, t) \, dy \, ds + \sum_{\beta < k} C_\beta \int_{\frac{t}{2}}^{t} \int_{\mathbb{R}} G^1_{k-\beta} \partial_y^{\beta+1} g_1(y, t) \, dy \, ds \|_{L^2}
\]
\[
= \int_{\frac{t}{2}}^{t} \| \partial_s G \|_{L^1} \| \partial_s \partial_y^k \partial_s g_1 \|_{L^2} \, ds + O(1) \delta \left\{ \sum_{\beta < k-1} (1 + t)^{-\frac{k-\beta-1}{2}} \int_{\frac{t}{2}}^{t} \| G_D \|_{L^1} \| \partial_s \partial_y^{\beta+2} g_1 \|_{L^2} \, ds \right. 
\]
\[
+ \sum_{\beta < k-1} (1 + t)^{-\frac{k-\beta}{2}} \int_{\frac{t}{2}}^{t} \| G_D \|_{L^1} \| \partial_s \partial_y^{\beta+1} g_1 \|_{L^2} \, ds 
\]
\[
\left. + \sum_{\beta < k} (1 + t)^{-1-\frac{k-\beta}{2}} \int_{\frac{t}{2}}^{t} \| G_D \|_{L^1} \| \partial_y^{\beta+1} g_1 \|_{L^2} \, ds \right\}
\]
\[
= O(1) (N^2(0) + \delta)(1 + t)^{-\frac{k}{2} - \frac{7}{4}},
\]
(4.50)
where we have used the fact that
\[
\| \partial_s \partial_y^k \partial_s g_1 \|_{L^2} = O(1) (N^2(0) + \delta)(1 + s)^{-\frac{n}{2} - \frac{9}{4}}.
\]
(4.51)
Substituting (4.48)–(4.50) into (4.47) yields that for \( k \leq 3 \),
\[
\| I_{4}^{1,k} \|_{L^2} = O(1) (N^2(0) + \delta + \epsilon^2)(1 + t)^{-\frac{k}{2} - \frac{7}{4}}.
\]
(4.52)
When \( l = 0, k = 5 \), we have

\[
I_{4,5}^{0.5} = \int_0^t \int_{\mathbb{R}} \delta^5_x G(x, t; y, s) g_1(y) dy ds = \left( \int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^t \right) \int_{\mathbb{R}} \delta^5_x G(x, t; y, s) g_1(y) dy ds =: i_{4,1}^{0.5} + i_{4,2}^{0.5}.
\]

(4.53)

Similar to (4.42), we get

\[
\|I_{4,1}^{0.5}\|_{L^2} = O(1) \varepsilon^2 (1 + t)^{-\frac{5}{2} - \frac{3}{4}}.
\]

(4.54)

Moreover, it follows from (4.11) that

\[
I_{4,2}^{0.5} = \int_{\frac{t}{2}}^t \int_{\mathbb{R}} \left( (1) G^4_y \right) g_1(y) dy ds
\]

(4.55)

The key point to deal with \( I_{4,2}^{0.5} \) is to control the \( \int_{\frac{t}{2}}^t \int_{\mathbb{R}} \delta^2_y G^4_y g_1(y) dy ds \). Recall that

\[
\delta^2_y G(x, t; y, s) = -\frac{1}{a} (R_G + a_y G_y) + \frac{1}{a} G_z,
\]

(4.56)

which, together with (4.5), yields that

\[
\int_{\frac{t}{2}}^t \int_{\mathbb{R}} \delta^2_y G^4_y g_1(y) dy ds = \left[ (1 + s)^{-1} + (t - s)^{-\frac{1}{2}} \right] \|G_D\|_{L^1} \|\delta_y^4 g_1\|_{L^2} ds + O(1) \|\delta_y^4 g_1(t)\|_{L^2}
\]

(4.57)

Thus, substituting (4.57) into (4.55) and using Lemma 4.1, (4.13), (4.44), and (4.51) lead to

\[
\|I_{4,2}^{0.5}\|_{L^2} = O(1) \varepsilon^2 (1 + t)^{-\frac{5}{2} - \frac{3}{4}}.
\]

(4.58)
where we have used the fact from (4.2) that

\[
\| \frac{1}{a} \int g_1 \|_{L^2} = \frac{1}{a} \| \partial_s^4 g_1 \|_{L^2}.
\]

Substituting (4.54) and (4.58) into (4.53) yields that

\[
\| t_4^{0.5} \|_{L^2} = O(1)(N^2(0) + \delta + \epsilon^2)(1 + t)^{\frac{3}{2} - \frac{3}{4}}.
\]  

(4.59)

For the last case with \( l = 1, k = 4 \) for the term \( I_{4}^{1,k} \), we deduce from (4.12) and (4.13) that

\[
I_{4}^{1,4} = \int \partial_s^4 G(x,t; y, t) g_1(t) dy + \int_0^t \int \partial_s^4 G(x,t; y, s) g_1 dy ds
\]

\[
= \int \partial_s^4 G(x,t; y, \frac{t}{2}) g_1 \left( \frac{t}{2} \right) dy + \int_0^t \int \partial_s^4 G(x,t; y, s) g_1 dy ds
\]

\[
+ \int_{\frac{t}{2}}^t \int \partial_s^4 G \partial_s g_1 dy ds + \sum_{\beta < 4} C_\beta \int_{\frac{t}{2}}^t \int (\partial_y G^1_{\beta} + \partial_y G^1_{\beta}) \partial_s^\beta g_1 dy ds.
\]  

(4.60)

Similar to (4.48)–(4.49), we get

\[
\| \int \partial_s^4 G(x,t; y, \frac{t}{2}) g_1 \left( \frac{t}{2} \right) dy \|_{L^2} = O(1)(N^2(0) + \delta)(1 + t)^{\frac{3}{4} - \frac{7}{4}}
\]  

(4.61)

and

\[
\| \int_0^t \int \partial_s^4 G(x,t; y, s) g_1 dy ds \|_{L^2} = O(1)\epsilon^2(1 + t)^{\frac{3}{4} - \frac{7}{4}}.
\]

Using (4.56), we have

\[
\int_{\frac{t}{2}}^t \int \partial_s^4 G \partial_s g_1 dy ds = \int_{\frac{t}{2}}^t \int \partial_s^2 G \partial_s^3 g_1 dy ds + \sum_{\beta < 4} C_\beta \int_{\frac{t}{2}}^t \int (\partial_y G^{1}_{3-\beta} + \partial_y G^{1}_{3-\beta}) \partial_s^\beta g_1 dy ds
\]

\[
= - \int_{\frac{t}{2}}^t \int \left( \frac{1}{a} R_G + a_y G_y \right) \partial_s^3 g_1 dy ds + \int_{\frac{t}{2}}^t \frac{1}{a} G \partial_s^3 g_1 dy ds
\]

\[
+ \int_{\frac{t}{2}}^t \int \left( \frac{1}{a^2} a_s G \partial_s^3 g_1 + \frac{1}{a} G_y \partial_s^2 g_1 - \frac{1}{a^2} a_y G \partial_s^2 g_1 \right) dy ds
\]

\[
+ \sum_{\beta < 4} C_\beta \int_{\frac{t}{2}}^t \int (\partial_y G^{1}_{3-\beta} + \partial_y G^{1}_{3-\beta}) \partial_s^\beta g_1 dy ds.
\]

From (3.8)–(3.10) and (4.2), we have

\[
\| \int \frac{1}{a} G(x,t; y, t) \partial_s^3 g_1 (y, t) dy \|_{L^2} = \frac{1}{a} \| \partial_s^3 g_1 (x, t) \|_{L^2}
\]

and

\[
\| \partial_s^2 \partial_y^n g_1 \|_{L^2} = O(1)(N^2(0) + \delta)(1 + s)^{\frac{n}{2} - \frac{13}{4}}.
\]
which, together with Lemmas 4.1-4.2, (4.13)–(4.14), (4.44), and (4.51), yields that for the last two terms of (4.60),

\[
\| \int_{\frac{t}{2}}^{t} \int_{\mathbb{R}} \partial_{x}^{\delta} g_{1} \, dy \, ds + \sum_{\beta \leq 4} C_{\beta} \int_{\frac{t}{2}}^{t} \int_{\mathbb{R}} (\partial_{y} G_{1}^{1-\delta} + G_{\partial_{y}^{1}}) \partial_{y}^{\delta} g_{1} \, dy \, ds \|_{L^{2}} = O(1) \delta \int_{\frac{t}{2}}^{t} (1+\frac{s}{1+2}) + (t-s)^{-\frac{1}{2}} \| G_{D} \|_{L^{1}} \| \partial_{s}^{3} g_{1} \|_{L^{2}} \, ds + O(1) \| \partial_{y}^{3} g_{1}(t) \|_{L^{2}}
\]

Substituting (4.61)–(4.62) into (4.60) leads to

\[
\| I_{1,4}^{l,k} \|_{L^{2}} = O(1)(N^{2}(0) + \delta + \epsilon^{2})(1+\frac{t}{2})^{-l-k-\frac{1}{4}}.
\]

Thus, we obtain (4.39) from (4.46), (4.52), (4.59), and (4.63). Therefore, the proof is completed. □

Next, for the term \( I_{5}^{l,k} \), we have the following.

**Lemma 4.8.** It holds that for \( l + k \leq 5, l \leq 1 \),

\[
\| I_{5}^{l,k} \|_{L^{2}} = O(1) \left( \sqrt{N^{2}(0) + \delta} + \delta_{1} \right)(1+\frac{t}{2})^{-l-k-\frac{1}{4}}.
\]

**Proof.** When \( l = 0, k \leq 4 \), we have

\[
I_{5}^{0,k} = \left( \int_{0}^{t} + \int_{\frac{t}{2}}^{t} \right) \int_{\mathbb{R}} \partial_{x}^{k} G(x,t;y,s) V_{s} \, dy \, ds = : I_{5,1}^{0,k} + I_{5,2}^{0,k}.
\]

It follows that

\[
I_{5,1}^{0,k} = \int_{\mathbb{R}} \partial_{x}^{k} G(x,t;\frac{t}{2}) V_{s}(\frac{t}{2}) \, dy - \int_{\mathbb{R}} \partial_{x}^{k} G(x,t;y,0) V_{s}(y,0) \, dy
\]

Note that \( V_{s}(y,0) = \partial_{y} V_{s}(y,0) \) and \( V_{s}(y,0) \in L^{2} \cap L^{1} \) with \( \| V_{s}(y,0) \|_{L^{1}} = O(1) \delta_{1} \), thus we have

\[
\| \partial_{x}^{k} G(x,t;y,0) \|_{L^{2}} = \| V_{s}(y,0) \|_{L^{2}} \| \tilde{V}_{s}(y,0) \|_{L^{1}} = O(1) \delta_{1}(1+t)^{-\frac{1}{4} - \frac{3}{4}},
\]
which, together with (3.9) and (4.4), yields that
\[
\|I_{5,1}^{0,k}\|_{L^2} = O(1)t^{-\frac{k}{2}}\|G_D\|_{L^1}\|V_s\left(\frac{t}{2}\right)\|_{L^2} + O(1)\delta_1t^{-\frac{k}{2} - \frac{3}{4}} + O(1)t^{-\frac{k}{2} - 1}\int_0^t \|G_D\|_{L^1}\|V_s\|_{L^2} ds
\]
\[
= O(1)\left(\sqrt{N^2(0) + \delta + \delta_1}\right)(1 + t)^{-\frac{k}{2} - \frac{3}{4}}. \tag{4.66}
\]
Moreover, it follows from (3.10), (4.11), (4.13), and (4.15) that for \(k \leq 3\),
\[
\|I_{5,2}^{0,k}\|_{L^2} = \left\| \int_\frac{t}{2}^t \int_\mathbb{R} (-1)^k \partial^k G + \sum_{\beta < k} C_\beta \partial^\beta (GR_{k-\beta-1}^1) \right\|_{L^2} \|V_s\| dyds
\]
\[
= O(1) \int_\frac{t}{2}^t \|G_D\|_{L^1} \|\partial^2_x \partial^\beta V\|_{L^2} ds + \sum_{\beta < k} O(1)(1 + t)^{-\frac{k-\beta}{2}} \int_\frac{t}{2}^t \|G_D\|_{L^1} \|\partial^2_x \partial^\beta V\|_{L^2} ds
\]
\[
= O(1)\sqrt{N^2(0) + \delta}(1 + t)^{-\frac{k-\beta}{2} - 1}, \tag{4.67}
\]
and
\[
\|I_{5,2}^{0,4}\|_{L^2} = \left\| \int_\frac{t}{2}^t \int_\mathbb{R} -\partial_x \partial^3_y G + \sum_{\beta < 3} C_\beta \partial^\beta \left[ -\partial_y (GR_{2-\beta}^1) + GR_{3-\beta}^1 \right] \right\|_{L^2} \|V_s\| dyds
\]
\[
= O(1) \int_\frac{t}{2}^t \|\partial_x G\|_{L^1} \|\partial^2_x \partial^3_y V\|_{L^2} ds + \sum_{\beta < 3} O(1)(1 + t)^{-\frac{3-\beta}{2} - \frac{1}{2}} \int_\frac{t}{2}^t \|G_D\|_{L^1} \|\partial^2_x \partial^3_y V\|_{L^2} ds
\]
\[
+ \sum_{\beta < 3} O(1)(1 + t)^{-\frac{3-\beta}{2} - \frac{1}{2}} \int_\frac{t}{2}^t \|G_D\|_{L^1} \|\partial^2_x \partial^\beta V\|_{L^2} ds = O(1)\sqrt{N^2(0) + \delta}(1 + t)^{-3}. \tag{4.68}
\]
Substituting (4.66)–(4.68) into (4.65) gives that for \(k \leq 4\),
\[
\|I_{5}^{0,k}\|_{L^2} = O(1)\left(\sqrt{N^2(0) + \delta + \delta_1}\right)(1 + t)^{-\frac{k}{2} - \frac{3}{4}}. \tag{4.69}
\]
When \(l = 1, k \leq 3\), it follows from (4.11)–(4.12) that
\[
I_{5}^{1,k} = \int_\mathbb{R} \partial^k_x G(x, t; y, \frac{t}{2}) V_s\left(\frac{t}{2}\right) dy + \int_\mathbb{R} \partial^k_x G(x, t; y, \frac{t}{2}) V_s\left(\frac{y}{2}\right) dy
\]
\[
- \int_\mathbb{R} \partial^k_x G(x, t; y, 0) V_s(y, 0) dy
\]
\[
+ \sum_{\beta < k} C_\beta \int_\frac{t}{2}^t \int_\mathbb{R} GR_{k-\beta-1}^1 \partial^3_x \partial^\beta V dyds
\]
\[
+ \sum_{\beta \leq k} C_\beta \int_\frac{t}{2}^t \int_\mathbb{R} G\partial^1 \partial^2_x \partial^\beta V dyds.
\]
Since \( \| \tilde{V}_s(y,0) \|_{L^1} = O(1) \delta_1 \), we have
\[
\| \partial_t \partial_t^k G(x,t;0,0) * V_s(y,0) \|_{L^2} = \| \partial_t \partial_t^k \partial_t^j G(x,t;y,0) \|_{L^2} \| V_s(y,0) \|_{L^1} = O(1) \delta_1 t^{-\frac{k}{2} - \frac{7}{4}},
\] (4.70)
which, together with (3.9)–(3.10), (3.13), and (4.13)-(4.14), yields that for \( k \leq 3 \),
\[
\| I_{s,5}^{1,k} \|_{L^2} = O(1) t^{-\frac{k}{2}} \| G_D(x,t;y,\frac{t}{2}) \|_{L^1} | V_{ss} \|_{L^2} \| \tilde{V}_s \|_{L^1} = O(1) \delta_1 t^{-\frac{k}{2} - \frac{7}{4}},
\] (4.71)
When \( l = 0, k = 5 \), we have
\[
I_{s,5}^{0,5} = \left( \int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^{t} \right) \int_{\mathbb{R}} \partial_t^5 G(x,t;y,s) V_{ss} dy ds = : I_{s,5}^{0,5}_{5,1} + I_{s,5}^{0,5}_{5,2}.
\] (4.72)
By the same argument as in (4.66), we get
\[
\| I_{s,5}^{0,5}_{5,1} \|_{L^2} = O(1) \left( \sqrt{N^2(0) + \delta + \delta_1} \right)(1 + t)^{-\frac{5}{2} - \frac{3}{4}}.
\] (4.73)
In addition, it follows from (4.11) that
\[
I_{s,5}^{0,5}_{5,2} = \int_{\frac{t}{2}}^{t} \int_{\mathbb{R}} (\partial_t^5 G + \sum_{\beta < 5} C_\beta \partial_t^\beta (G \tilde{R}_4^{1-\beta})) V_{ss} dy ds
\]
\[
= \int_{\frac{t}{2}}^{t} \int_{\mathbb{R}} \partial_t^5 G \partial_t^2 \partial_t^3 y V dy ds + \sum_{\beta < 4} C_\beta \int_{\frac{t}{2}}^{t} \int_{\mathbb{R}} G \tilde{R}_4^{1-\beta} \partial_t^3 \partial_t^\beta y V dy ds
\]
\[
+ \int_{\frac{t}{2}}^{t} \int_{\mathbb{R}} (\partial_t^5 G \tilde{R}_4^{1-\beta} + G \partial_t^5 \tilde{R}_4^{1-\beta}) \partial_t^3 \partial_t^\beta y V dy ds.
\]
Then, we use (4.56) to get
\[
\int_{\frac{t}{2}}^{t} \int_{\mathbb{R}} \partial_t^5 G \partial_t^2 \partial_t^3 y V dy ds = - \int_{\frac{t}{2}}^{t} \int_{\mathbb{R}} \frac{1}{a} (R_G + a_y G_y) \partial_t^2 \partial_t^3 y V dy ds + \frac{1}{a} \int_{\mathbb{R}} G \partial_t^2 \partial_t^3 y V dy ds
\]
\[
= O(1) \delta \int_{\frac{t}{2}}^{t} \int_{\mathbb{R}} \Theta(t,s) E(y,s) G_D(x-y,t-s) \partial_t^2 \partial_t^3 y V dy ds
\]
\[
+ \left. \int_{\mathbb{R}} \frac{1}{a} G \partial_t^2 \partial_t^3 y V dy \right|_{s=\frac{t}{2}}^{s=t}
\]
\[ \int_\frac{t}{2}^t \int_\mathbb{R} \left( \frac{1}{a^2} a_x G \frac{\partial^3}{\partial y^3} V + \frac{1}{a} G_y \frac{\partial^3}{\partial x^3} V - \frac{1}{a^2} a_y G \frac{\partial^3}{\partial y^3} V \right) dy ds, \]

which, together with (3.10), (3.13), and Lemma 4.1, indicates that

\[ \| \int_\frac{t}{2}^t \int_\mathbb{R} \frac{\partial^2}{\partial y^3} G \frac{\partial^3}{\partial y^3} V dy ds \|_{L^2} = O(1) \delta \int_\frac{t}{2}^t \left( (1+s)^{-\frac{1}{2}} + (t-s)^{-\frac{1}{2}} \right) \| G_D \|_{L^1} \| \frac{\partial^2}{\partial y^3} V \|_{L^2} ds + O(1) \| \frac{\partial^2}{\partial y^3} V(t) \|_{L^2} \]

\[ + O(1) \| G_D \|_{L^1} \| \frac{\partial^2}{\partial y^3} V \|_{L^2} \]

\[ = O(1) \sqrt{N^2(0) + \delta} (1 + t)^{-\frac{7}{2}}, \quad (4.74) \]

where we have used the fact that

\[ \| \int_\mathbb{R} \frac{1}{a} G(x, t; y, t) \frac{\partial^2}{\partial y^3} V(y, t) dy \|_{L^2} = \| \frac{1}{a} \frac{\partial^2}{\partial y^3} V(x, t) \|_{L^2}. \]

Thus, it follows from (3.10), (4.13), (4.15), and (4.74) that

\[ \| I^{0,5}_{5,2} \|_{L^2} = O(1) \sqrt{N^2(0) + \delta} (1 + t)^{-\frac{7}{2}} + O(1) \delta \sum_{\beta < 4} \int_\frac{t}{2}^t (1 + t)^{-\frac{5-\beta}{2}} \| G_D \|_{L^1} \| \frac{\partial^2}{\partial y^3} V \|_{L^2} ds \]

\[ + O(1) \delta \int_\frac{t}{2}^t \left( (1+s)^{-\frac{1}{2}} + (t-s)^{-\frac{1}{2}} \right) (1 + t)^{-\frac{5-\beta}{2}} \| G_D \|_{L^1} \| \frac{\partial^2}{\partial y^3} V \|_{L^2} ds \]

\[ = O(1) \sqrt{N^2(0) + \delta} (1 + t)^{-\frac{7}{2}}. \quad (4.75) \]

Substituting (4.73) and (4.75) into (4.72) leads to

\[ \| I^{0,5}_{5} \|_{L^2} = O(1) \left( \sqrt{N^2(0) + \delta} + \delta_1 \right) (1 + t)^{-\frac{5}{2} - \frac{3}{4}}. \quad (4.76) \]

When \( l = 1, k = 4 \), similar to (4.34), we use (4.11)–(4.12) to get

\[ I^{1,4}_{5} = \int_\mathbb{R} \frac{\partial^4}{\partial x^4} G(x, t; y, \frac{t}{2}) V_{ss} \left( \frac{t}{2} \right) dy + \int_\mathbb{R} \frac{\partial}{\partial t} \frac{\partial^4}{\partial x^4} G(x, t; y, \frac{t}{2}) V_s \left( y, \frac{t}{2} \right) dy \]

\[ - \int_\mathbb{R} \frac{\partial}{\partial t} \frac{\partial^4}{\partial x^4} G(x, t; y, 0) V_s(y, 0) dy \]

\[ - \int_0^t \int_\mathbb{R} \frac{\partial}{\partial s} \frac{\partial}{\partial x} \frac{\partial^4}{\partial x^4} G(x, t; y, s) V_s dy ds + \int_\frac{t}{2}^t \int_\mathbb{R} \frac{\partial^2}{\partial y^2} \frac{\partial^3}{\partial y^3} V dy ds \]

\[ + \sum_{\beta < 3} C_\beta \int_\frac{t}{2}^t \int_\mathbb{R} G \tilde{R}^1_{3-\beta} \frac{\partial^3}{\partial y^3} V dy ds \]
\[
\begin{align*}
&\quad + \int_t^\infty \int_\mathbb{R} (\partial_y GR_1^{1-\beta} + G \partial_y R_1^{1-\beta}) \partial_3^2 \partial_y G dy ds + \sum_{\beta \leq 3} C_\beta \int_t^\infty \int_\mathbb{R} G h_1^{4-\beta} \partial_3^2 \partial_y G dy ds \\
&\quad + \int_t^\infty \int_\mathbb{R} (\partial_y G h_1^{4-\beta} + G \partial_y h_1^{4-\beta}) \partial_3^2 \partial_y G dy ds.
\end{align*}
\] (4.77)

Using (4.56) again, we have

\[
\begin{align*}
&\quad \int_t^\infty \int_\mathbb{R} \partial_2^2 G \partial_3^3 \partial_y^3 V dy ds = - \int_t^\infty \int_\mathbb{R} \frac{1}{a} (\alpha R_G + a_y G_y) \partial_3^3 \partial_y^3 V dy ds + \int_t^\infty \int_\mathbb{R} \frac{1}{a} G_s \partial_3^3 \partial_y^3 V dy ds \\
&\quad = O(1) \int_t^\infty \int_\mathbb{R} \Theta(t,s) E(y,s) G_D(x-y,t-s) \partial_3^3 \partial_y^3 V dy ds \\
&\quad + \int_t^\infty \frac{1}{a} G \partial_3^3 \partial_y^3 V dy \bigg|_{s = \frac{t}{2}} \\
&\quad + \int_t^\infty \int_\mathbb{R} \left( \frac{1}{a^2} a_y G \partial_3^3 \partial_y^3 V + \frac{1}{a} G_y \partial_4^4 \partial_y V - \frac{1}{a^2} a_y G \partial_4^4 \partial_y V \right) dy ds,
\end{align*}
\]

which, together with (3.13)–(3.14) and (4.4)–(4.5), indicate that

\[
\begin{align*}
&\quad \| \int_t^\infty \int_\mathbb{R} \partial_2^2 G \partial_3^3 \partial_y^3 V dy ds \|_{L^2} \\
&\quad = O(1) \delta \int_t^\infty \left( (1 + s)^{-\frac{1}{2}} + (t - s)^{-\frac{1}{2}} (1 + s)^{-\frac{1}{2}} \right) \| G_D \|_{L^1} \| \partial_3^3 \partial_y^3 V \|_{L^2} ds + O(1) \| \partial_3^3 \partial_y^3 V \|_{L^2} \\
&\quad + O(1) \| G_D \left( \frac{t}{2} \right) \|_{L^1} \| \partial_3^3 \partial_y^3 V \left( \frac{t}{2} \right) \|_{L^2} + O(1) \int_t^\infty \left( (1 + s)^{-\frac{1}{2}} + (t - s)^{-\frac{1}{2}} \right) \| G_D \|_{L^1} \| \partial_4^4 \partial_y V \|_{L^2} ds \\
&\quad = O(1) \sqrt{N^2(0) + \delta} (1 + t)^{-4},
\end{align*}
\] (4.78)

where we have used the fact that

\[
\| \int_\mathbb{R} \frac{1}{a} G(x,t;\gamma,y) \partial_3^3 \partial_y^3 V(y,t) dy \|_{L^2} = \| \frac{1}{a} \partial_3^3 \partial_y^3 V(x,t) \|_{L^2}.
\]

Substituting (4.78) into (4.77), together with (3.9), (4.4), (4.13)–(4.15), and (4.70), yields that

\[
\| I_{1,4}^{1,4} \|_{L^2} = O(1) \left( \sqrt{N^2(0) + \delta} + \delta_1 \right) (1 + t)^{-2 - \frac{7}{4}}.
\] (4.79)

Thus, we obtain (4.64) from (4.69), (4.71), (4.76), and (4.79). Therefore, the proof is completed. \(\square\)

Finally, for the term \(I_{6}^{l,k}\), we have the following.

**Lemma 4.9.** It holds that for \(l + k \leq 5\), \(l \leq 1\),

\[
\| I_{6}^{l,k} \|_{L^2} = O(1) \delta \epsilon (1 + t)^{-l - \frac{k}{2} - \frac{3}{4}}.
\] (4.80)
**Proof.** When \( l = 0, k \leq 5 \), we have

\[
I_{6, 1}^{0, k} = \left( \int_{0}^{\frac{l}{2}} + \int_{\frac{l}{2}}^{l} \right) \int_{\mathbb{R}} \partial_{x}^k R_G(x, t; y, s)Vdyds =: I_{6, 1}^{0, k} + I_{6, 2}^{0, k}.
\]

Let \( 1 \leq q_1 < 2 \) in Lemma 4.5, then we use the a priori assumption (3.6) to get

\[
\|I_{6, 1}^{0, k}\|_{L^2} \leq \int_{0}^{\frac{l}{2}} \|\partial_{x}^k R_G\|_{L^1} \|V\|_{L^2} ds = O(1)\delta \varepsilon \int_{0}^{\frac{l}{2}} \left[ t^{-\frac{k+1}{2}} (1 + s)^{-\frac{1}{2} + \frac{1}{2q_1}} + t^{-\frac{k+1}{2} - \frac{1}{2}} \right] \left( 1 + s \right)^{-\frac{3}{2}} ds
\]

\[
= O(1)\delta \varepsilon (1 + t)^{-\frac{k}{2} - \frac{3}{4}}. \tag{4.81}
\]

It follows from the a priori assumption (3.6) that for \( l \leq 1 \),

\[
\|\partial_{s}^l \partial_{y}^n V(s)\|_{L^2} = O(1)\varepsilon (1 + s)^{-l - \frac{n}{2} - \frac{3}{4}},
\]

which implies that

\[
\int_{\frac{l}{2}}^{l} \Theta(t, s)\|\partial_{s}^l \partial_{y}^n V(s)\|_{L^2} ds = O(1)\varepsilon (1 + t)^{-l - \frac{n}{2} - \frac{3}{4}}.
\]

Together with (4.5), (4.16), and (4.18), we have

\[
\|I_{6, 1}^{0, k}\|_{L^2} = \int_{0}^{l} \int_{\mathbb{R}} R_{G} \partial_{x}^k Vdyds + \sum_{\beta < k} C_{\beta} \int_{0}^{l} \int_{\mathbb{R}} G\tilde{R}_{k-\beta-1} \partial_{x}^\beta Vdyds \leq O(1)\delta \int_{0}^{l} \Theta(t, s)\|G_D\|_{L^1} \|\partial_{x}^k V\|_{L^2} ds + \sum_{\beta < k} O(1)\delta (1 + t)^{-\frac{k-\beta}{2}} \int_{0}^{l} \Theta(t, s)\|G_D\|_{L^1} \|\partial_{s}^\beta V\|_{L^2} ds
\]

\[
= O(1)\delta \varepsilon (1 + t)^{-\frac{k}{2} - \frac{3}{4}}. \tag{4.82}
\]

Then, (4.81) and (4.82) imply that for \( k \leq 5 \),

\[
\|I_{6, 1}^{0, k}\|_{L^2} = \int_{0}^{l} \int_{0}^{\frac{l}{2}} \|\partial_{x}^k R_G\|_{L^1} \|V\|_{L^2} ds = O(1)\delta \varepsilon (1 + t)^{-\frac{k}{2} - \frac{3}{4}}. \tag{4.83}
\]

When \( l = 1, k \leq 4 \), it follows from (4.16)–(4.17) that

\[
I_{6, 1}^{1, k} = \int_{\mathbb{R}} \partial_{x}^k R_G(x, t; y, \frac{l}{2}) V(\frac{l}{2}) dy + \int_{0}^{\frac{l}{2}} \int_{\mathbb{R}} \partial_{x}^k \partial_{t} R_G(x, t; y, s)Vdyds + \int_{\frac{l}{2}}^{l} \int_{\mathbb{R}} R_{G} \partial_{s} \partial_{x}^k Vdyds
\]

\[
+ \sum_{\beta < k} C_{\beta} \int_{\frac{l}{2}}^{l} \int_{\mathbb{R}} G\tilde{R}_{k-\beta-1} \partial_{x}^\beta \partial_{s} Vdyds + \sum_{\beta < k} C_{\beta} \int_{\frac{l}{2}}^{l} \int_{\mathbb{R}} G\tilde{R}_{k-\beta-1} \partial_{x}^\beta \partial_{s} \partial_{s} Vdyds.
\]

By the same way as in (4.81)–(4.82), we get that for \( k \leq 4 \),

\[
\|I_{6, 1}^{1, k}\|_{L^2} = O(1)\delta t^{-1 - \frac{k}{2}} \|G_D\|_{L^1} \|V(\frac{l}{2})\|_{L^2} + O(1)\delta \int_{\frac{l}{2}}^{l} \Theta(t, s)\|G_D\|_{L^1} \|\partial_{s}^k \partial_{y} V\|_{L^2} ds
\]
\[ + \int_0^t \| \partial_t^{\frac{k}{2}} \partial_x R_G \|_{L^2} \| V \|_{L^2} ds + O(1) \delta \sum_{\beta < k} (1 + t)^{-\frac{k - \beta}{2}} \int_0^t \Theta(t, s) \| G_D \|_{L^1} \| \partial_s \partial_y^\beta V \|_{L^2} ds \\
+ O(1) \delta \sum_{\beta \leq k} (1 + t)^{-1 - \frac{k - \beta}{2}} \int_0^t \Theta(t, s) \| G_D \|_{L^1} \| \partial_s \partial_y^\beta V \|_{L^2} ds \\
= O(1) \delta (1 + t)^{-\frac{k}{2}}. \tag{4.84} \]

Thus, we obtain (4.80) from (4.83)–(4.84). Therefore, the proof is completed. \( \square \)

According to the above discussion, it is easy to show the proof of Theorem 1.1 as follows.

**Proof of Theorem 1.1.** Combining (4.19), (4.38), (4.39), (4.64), and (4.80), we have

\[
\sup_{0 \leq t \leq T} \sum_{l + k \leq 5, l \leq 1} \| \partial_t^l \partial_x^k V(\cdot, t) \|_{L^2} \leq C_1 \left[ \left( \sqrt{N^2(0) + \delta} + \delta_1 \right) + (\epsilon + \delta) \epsilon \right] (1 + t)^{-\frac{k}{2} - \frac{3}{4}},
\]

which, together with choosing \( \epsilon = 2C_1 \left( \sqrt{N^2(0) + \delta} + \delta_1 \right) \) small enough, yields that

\[
\sup_{0 \leq t \leq T} \sum_{l + k \leq 5, l \leq 1} (1 + t)^{\frac{k}{2} + \frac{3}{4}} \| \partial_t^l \partial_x^k V(\cdot, t) \|_{L^2} \leq 2 \epsilon < \epsilon.
\]

Thus, the proof of Theorem 1.1 is completed. \( \square \)

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