Many simple cardinal invariants

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Abstract: For \( g < f \) in \( \omega^\omega \) we define \( c(f, g) \) be the least number of uniform trees with \( g \)-splitting needed to cover a uniform tree with \( f \)-splitting. We show that we can simultaneously force \( \aleph_1 \) many different values for different functions \((f, g)\). In the language of \( [\text{Blass}] \): There may be \( \aleph_1 \) many distinct uniform \( \Pi_0^1 \) characteristics.

0. Introduction

[\text{Blass}] defined a classification of certain cardinal invariants of the continuum, based on the Borel hierarchy. For example, to every \( \Pi_1^0 \) formula \( \varphi(x, y) = \forall n R(x | n, y | n) \) \( (R \) recursive) the cardinal

\[
\kappa_\varphi := \min\{ B \subseteq \omega^\omega : \forall x \in \omega^\omega \exists y \in B : \varphi(x, y) \}
\]

is the “uniform \( \Pi_1^0 \) characteristic” associated to \( \varphi \).

Blass proved structure theorems on simple cardinal invariants, e.g., that there is a smallest \( \Pi_1^0 \) characteristic (namely, \( \text{Cov}(M) \), the smallest number of first category sets needed to cover the reals), and also that the \( \Pi_1^2 \)-characteristics can behave quite chaotically. He asked whether the known uniform \( \Pi_1^0 \) characteristics \((c, d, r, \text{Cov}(M))\) are the only ones or (since that is very unlikely) whether there could be a reasonable classification of the uniform \( \Pi_1^1 \) characteristics — say, a small list that contains all these invariants.

In this paper we give a strong negative answer to this question: For two \( \Pi_1^0 \) formulas \( \varphi_1, \varphi_2 \) we say that \( \varphi_1 \) and \( \varphi_2 \) define “potentially nonequal characteristics” if \( \kappa_{\varphi_1} \neq \kappa_{\varphi_2} \) is consistent. We say that \( \varphi_1 \) and \( \varphi_2 \) define “actually different characteristics”, if \( \kappa_{\varphi_1} \neq \varphi_2 \).

We will find a family of \( \Pi_1^1 \)-formulas indexed by a real parameter \((f, g)\), and we will show not only that there is a perfect set of parameters which defines pairwise potentially nonequal \( \Pi_1^0 \)-characteristics, but we produce a single universe in which (at least) \( \aleph_1 \) many cardinals appear as \( \Pi_1^0 \)-characteristics. (In fact it

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is also possible to produce a universe where there is a perfect set of parameters defining pairwise actually different $\Pi^0_1$-characteristics. See [Shelah 448a]).

If we want more than countably many cardinals, we obviously have to use the boldface pointclass. But the proof also produces many lightface uniform $\Pi^0_1$ characteristics.

For more information on cardinal invariants, see [Blass], [van Douwen], [Vaughan].

From another point of view, this paper is part of the program of finding consistency techniques for a large continuum, i.e., we want $2^{\aleph_0} > \aleph_2$ and have many values for cardinal invariants. We use a countable support product of forcing notions with an axiom A structure.

We will use invariants that were implicitly introduced in [Shelah 326, §2], where it was proved that $c(f, g)$ and $c(f', g')$ (see below) may be distinct.

0.1 Definition: If $f \in \omega^\omega$, we say that $\bar{B} = \langle B_k : k \in \omega \rangle$ is an $f$-slalom if for all $k$, $|B_k| = f(k)$. We write $h \in \bar{B}$ for $h \in \prod_n B_n$, i.e., $\forall n h(n) \in B_n$. (See figure 1) This is a $\Pi^0_1$-formula in the variables $h$ and $\bar{B}$.

Some authors call the set $\{ h : h \in \bar{B} \}$ a “belt”, or “uniform tree”.

For example, $\prod_n f(n)$ is an $f$-slalom, because we identify the number $f(n)$ with the set of predecessors, $\{0, \ldots, f(n) - 1\}$.

Figure 1: A slalom

0.2 Definition: Assume $f, g \in \omega^\omega$. Assume that $\mathcal{B}$ is a family of $g$-slaloms, and $\bar{A} = \langle A_k : k \in \omega \rangle$ is an $f$-slalom.

We say that $\mathcal{B}$ covers $\bar{A}$ iff:

\[ \forall s \in \bar{A} \exists \bar{B} \in \mathcal{B} \text{ such that } s \in \bar{B} \]
0.3 Definition: Assume \( f, g \in \omega^\omega \). Then we define the cardinal invariant \( c(f, g) \) to be the minimal number of \( g \)-slaloms needed to cover an \( f \)-slalom.

(Clearly this makes sense only if \( \forall k \ f(k), g(k) > 0 \), so we will assume that from now on.)

This is a uniform \( \Pi^1_1 \)-characteristic. (Strictly speaking, we are not working in \( \omega^\omega \), but rather in \( \omega (|\omega|^\omega) \), but a trivial coding translates \( c(f, g) \) into a “uniform \( \Pi^0_1 \) characteristic” as defined above.)

Some relations between these cardinal invariants are provable in ZFC: For example, if \( g < g' < f < f' \), then \( c(f', g') \leq c(f, g) \). Also, \( c(f^2, g^2) \leq c(f, g) \).

We will show that if \( (f, g) \) is sufficiently different from \( (f', g') \), then the values of \( c(f, g) \) and \( c(f', g') \) are quite independent, and moreover: if \( ((f_i, g_i) : i < \omega_1) \) are pairwise sufficiently different, then almost any assignment of the form \( c(f_i, g_i) = \kappa_i \) will be consistent.

Similar results are possible for the “dual” version of \( c(f, g) \): \( c^d(f, g) := \) the smallest family of \( g \)-slaloms \( \bar{B} \) such that for every \( h \) bounded by \( f \) there are infinitely many \( k \) with \( h(k) \in B_k \), and for the “tree” version (a \( g \)-tree is a tree where every node in level \( k \) has \( g(k) \) many successors). See [Shelah 448a].

We thank Tomek Bartoszynski for pointing out the following known results about the cardinal characteristics \( c(f, g) \):

For example, lemma 1.11 follows from Theorem 3.17 in [Comfort-Negrepontis]: Taking \( \kappa = \alpha = \omega, \beta = n \), and letting \( S \subseteq n^\omega \) be a family of \( \omega \)-large oscillation, then no family of \( n-1 \)-slaloms of size \( < 2^{\aleph_0} \) can cover \( S \). Indeed, whenever \( F \) is a function on \( S \) such that for each \( s \in S \), \( F(s) \) is a \( n-1 \)-slalom covering \( s \), then \( F \) has to be finite-to-one and in fact at most \( n-1 \)-to-one.

Also, since \( c(f, f-1) \) is the size of the smallest family of functions below \( f \) which does not admit an “infinitely equal” function, i.e.,

\[
c(f, f-1) = \min \{ |G| : G \subseteq \prod_n f(n) \land \forall h \in \prod_n f(n) \exists g \in G \forall \infty_n f(n) \neq g(n) \}
\]

by [Miller] we have that the minimal value of \( c(f, f-1) \) is the smallest size of a set of reals which does not have strong measure zero.

Also, note that if \( r \) is a random real over \( V \) in \( \prod_n f(n) \), and if \( \sum_{n=1}^{\infty} 1/f(n) = \infty \), then \( \prod_n (1-1/f(n)) = 0 \), so \( r \) cannot be covered by any \( f-1 \)-slalom from \( V \).

Conversely, if \( \sum_{n=1}^{\infty} 1/f(n) < \infty \), then for any function \( h \in \prod_n f(n) \cap V \) there is a condition forcing that \( h \) is covered by the \( f-1 \)-slalom \( \{0, \ldots, f(k) - 1\} - \{r(k)\} : k \in \omega \).

Thus, if we add \( \kappa \) many random reals with the measure algebra, a easy density argument shows that in the resulting model we have

\[
c(f, f-1) = \begin{cases} \kappa = 2^{\aleph_0} & \text{if } \sum_{n=1}^{\infty} 1/f(n) = \infty \\ \aleph_1 & \text{otherwise (use any } \aleph_1 \text{ many of the random reals)} \end{cases}
\]

That already shows that we can have at least two distinct values of \( c(f, g) \) and \( c(f', g') \).
1. Results in ZFC

1.1 Notation: Operations and relations on functions are understood to be pointwise, e.g., \( f/g, g^e, g < f \), etc. \([x]\) is the greatest integer \( \leq x \). \( \lim f \) is \( \lim_{k \to \infty} f(k) \).

We write \( f \leq^* g \) for \( \exists n \forall k \geq n \ f(k) \leq g(k) \).

First we state some obvious facts:

1.2 Fact:

1. \( f \leq g \) iff \( c(f, g) = 1 \).

2. \( f \leq^* g \) iff \( c(f, g) \) finite.

3. If \( A := \{ k : g(k) < f(k) \} \) is infinite then \( c(f|A, g|A) = c(f, g) \).

4. If \( \pi \) is a permutation of \( \omega \), then \( c(f \circ \pi, g \circ \pi) = c(f, g) \). \( \circ \) 1.2

(Strictly speaking, we define \( c(f, g) \) only for functions \( f, g \) defined on all of \( \omega \), so (3) should be formally rephrased as \( c(f \circ h, g \circ h) = c(f, g) \), where \( h \) is a 1-1 enumeration of \( A \))

1.3 Convention: We will concentrate on the case where \( c(f, g) \) is infinite, so we will wlog assume that \( g < f \). By (4), we may also wlog assume that \( g \) is nondecreasing.

In these cases we will have that \( c(f, g) \) is infinite, and moreover an easy diagonal argument shows the following fact:

1.4 Fact:

\( c(f, g) \) is uncountable. \( \circ \) 1.4

Furthermore, we have the following properties:

1.5 Fact:

1. (Monotonicity) If \( f \leq^* f', g \geq^* g' \), then \( c(f, g) \leq c(f', g') \).

2. (Multiplicativity) \( c(f \cdot f', g \cdot g') \leq c(f, g) \cdot c(f', g') \).
Lemma: Assume so let \( \prod \) covers \( \{ h \} \) of all \( \{ h \} \). Define \( \text{c}(f, g) \). (where \( f \) is the function defined by \( f^{-1}(n) = f(n + 1) \).)

(4) (Invariance) \( \text{c}(f, g) = \text{c}(f^-, g^-) \) (where \( f^- \) is the function defined by \( f^-(n) = f(n + 1) \)).

(5) (Monotonicity II) If \( A \subseteq \omega \) is infinite, then \( \text{c}(f|A, g|A) \leq \text{c}(f, g) \).

1.6 Remark: (2) implies in particular \( \text{c}(f^n, g^n) \leq \text{c}(f, g) \). See 3.4 for an example of \( \text{c}(f^2, g^2) < \text{c}(f, g) \).

The following inequalities need a little more work.

1.7 Lemma:

(1) \( \text{c}(f \cdot |f|g), f) = \text{c}(f, g) \).

(2) \( \text{c}(f \cdot |f|g), g) = \text{c}(f, g) \).

(3) \( \text{c}(f \cdot |f|g)^m, g) = \text{c}(f, g) \) for all \( m \in \omega \).

Proof: (2) follows from (1) using transitivity, and (3) follows from (2) by induction, so we only have to prove (1).

Proof of (1): By monotonicity we only have to show \( \leq \). So let \( (N, \in) \) be a reasonably closed model of a large fragment of ZFC (say, \( (N, \in) \in (H(\chi^+), \in) \), where \( \chi = 2^\omega \)) of size \( \text{c}(f, g) \) such \( \prod f(n) \) is covered by the set of all \( g \)-slaloms from \( N \).

Define \( h \) by \( h(k) := f(k) \cdot |f(k)/g(k)| \). We can find a family \( \{ B^k_i : i < f(k), k \in \omega \} \) in \( N \) such that for all \( k \), \( \{0, \ldots, h(k) - 1\} = \bigcup_{i < f(k)} B^k_i \), where \( |B^k_i| \leq f(k)/g(k) \). We have to show that the set of \( f \)-slaloms from \( N \) covers \( \prod h(k) \).

So let \( x \) be a function satisfying \( \forall k \ x(k) \in \bigcup_{i < f(k)} B^k_i \). We can define a function \( y \in \prod f(n) \) such that for all \( k \), \( x(k) \in B^y_k \). So there is some \( g \)-slalom \( \bar{C} \subseteq \omega \) such that for all \( k \), \( y(k) \in C_k \).

Define \( \bar{A} = \{ A_k : k \in \omega \} \) by \( A_k := \bigcup_{C_k \subseteq B^y_k} B^k_i \). Then \( |A_k| \leq |C_k| \cdot |B^k_i| \leq g(k) \cdot f(k)/g(k) = f(k) \), so \( \bar{A} \) is an \( f \)-slalom in \( \omega \), and for all \( k \), \( x(k) \in A_k \).

1.8 Lemma: Assume \( f \) \( g \) \( > 0 \). Assume that \( \{ w_i : i \in \omega \} \) is a partition of \( \omega \) into finite sets, and for each \( i \) there are \( H^i = \{ H^i_l : l \in w_i \} \) satisfying (a)-(c). Then \( \text{c}(f^i, g^i) \leq \text{c}(f, g) \).

(a) \( \text{dom} H^i_l = f'(i) = \{0, \ldots, f'(i) - 1\} \)

(b) \( \text{rng} H^i_l \subseteq f(l) = \{0, \ldots, f(l) - 1\} \)

(c) Whenever \( \langle u_i : l \in w_i \rangle \) satisfies

\[
\begin{align*}
  u_i \subseteq f(l) \\
  |u_i| \leq g(l)
\end{align*}
\]

then \( \{ n < f'(i) : \forall l \in w_i H^i_l(n) \in u_i \} \) has cardinality \( \leq g'(i) \)

Proof: To any \( g \)-slalom \( \bar{B} = \{ B_l : l \in \omega \} \) we can associate a \( g' \)-slalom \( \bar{B}^* = \{ B^*_i : i \in \omega \} \) by letting

\[
B^*_i := \{ n < f'(i) : \forall l \in w_i H^i_l(n) \in w_i \}
\]
Conversely, to any function $x \in \prod_i f'(i)$ we can define a function $x^*$ in $\prod_n f(n)$ by

$$\text{if } l \in w_i, \text{ then } x^*(l) = H^i_l(x(i))$$

It is easy to check that if $x^*$ is in $\bar{B}$ then $x$ is in $\bar{B}^*$. The result follows.

1.9 Corollary: Assume $0 = n_0 < n_1 < \cdots$, and let

$$f'(i) := f(n_i) \cdot f(n_i + 1) \cdots f(n_{i+1} - 1)$$

$$g'(i) := g(n_i) \cdot g(n_i + 1) \cdots g(n_{i+1} - 1)$$

Then $c(f', g') \leq c(f, g)$.

Proof: Identify the set of numbers less than $f(n_i) \cdot f(n_i + 1) \cdots f(n_{i+1} - 1)$ with the cartesian product $\prod_{n_i \leq k < n_{i+1}} f(k)$, and let

$$H^i_l : \prod_{n_i \leq k < n_{i+1}} f(k) \to f(l)$$

be the projection onto the $l$-coordinate. We leave the verification of 1.8(c) to the reader.

1.10 Lemma: If $g$ is constant, $f(k) \geq 2^k$, then $c(f, g) = c$.

Proof: Let $\forall k g(k) = n, f(k) = 2^k$. Assume that $\prod_l 2$ can be covered by $< c$ many $g$-slaloms.

For any $\eta \in \{2\}$, the sequence $\bar{\eta} := \langle |\eta| : l \in \omega \rangle$ is in $\prod_l 2$. But any $g$-slalom can contain only $n$ many such $\bar{\eta}$, i.e. for any $g$-slalom $\bar{B} = \langle B_l : l \in \omega \rangle$ we have

$$|\{\eta \in \{2\} : \forall l \eta| l \in B_l\}| \leq m$$

Since there are continuum many $\eta$ we need continuum many $g$-slaloms to cover $\prod_l f(l)$ (or equivalently, $\prod_l 2$).

1.11 Lemma: If $f$ and $g$ are constant with $f > g$, then $c(f, g) = c$.

Proof: Using monotonicity wlog we assume that $f(k) = n + 1, g(k) = n$ for all $k$. We will use 1.8. Let $\omega = \bigcup_{l \in \omega} w_i$ be a partition of $\omega$ where $|w_i| = n^2$.

Let $f'(i) = 2^i, g'(i) = n$, and let $\langle H^i_l : l \in w_i \rangle$ enumerate all functions from $2^i$ to $n$.

We plan to show $c(f, g) \geq c(f', g')$ (so $c(f, g) = c$ by 1.10). We want to apply 1.8, so fix a sequence $\langle u_l : l \in w_i \rangle$, where $u_l \subseteq f(l)$ and $|u_l| \leq g(l)$.

To show that the hypotheses of 1.8 are satisfied, fix $i_0$ and let

$$A := \{x < f'(i_0) : \forall l \in w_{i_0} H^0_l(x) \in u_l\}$$

and assume $A$ has cardinality $> g'(i_0) = n$. So let $x_0, \ldots, x_n$ be distinct elements of $A$. Let $H : f'(i_0) \to n+1$ be a function satisfying

$$\forall j \leq n H(x_j) = j$$

$H$ is one of the functions $\{H^0_l : l \in w_{i_0}\}$, say $H = H^0_{i_0}$. Let $j_0 \notin u_{i_0}$, then also

$$x_{j_0} \notin \{x < f'(i_0) : H^0_{i_0}(x) \in u_{i_0}\} \supseteq A,$$

contradicting $x_{j_0} \in A$.
1.12 Corollary: If $f > g$, and $\liminf_{k \to \infty} g(k) < \infty$, then $c(f, g) = c$.

Proof: This follows from 1.11, using monotonicity and monotonicity II.

We can now extend 1.7 as follows:

1.13 Theorem: If for some $\varepsilon > 0$, $g^{1+\varepsilon} \leq f$, then for all $n$, $c(f^n, g) = c(f, g)$.

Proof: First we consider a special case: Assume that $g^2 \leq f$. Then we get

$$c(f, g) \leq c(f^2, g) \leq c(f^2, f) \cdot c(f, g) \leq c(f^2, g^2) \cdot c(f, g) = c(f, g)$$

Now we use this result on $(f, g)$, then on $(f^2, g)$, etc, to get

$$c(f, g) = c(f^2, g) = c(f^4, g) = c(f^8, g) = \cdots$$

and use monotonicity to get the general result under the assumption $g^2 \leq f$.

Now we consider the general case $g^{1+\varepsilon} \leq f$:

If $g$ does not converge to infinity, we have already (by 1.12) $c(f, g) = c$. Otherwise we can find some $\delta > 0$ such that for almost all $k$,

$$\frac{f(k)}{g(k)} \geq g(k)^\delta + 1,$$

so

$$\left\lfloor \frac{f(k)}{g(k)} \right\rfloor \geq g(k)^\delta$$

Now choose $m$ such that $m \cdot \delta > 1$. Then $\frac{f(k)/g(k)}{m} \geq g$. By 1.7, $c(f \cdot \lfloor f/g \rfloor^m, g) = c(f, g)$ and so by monotonicity also $c(f \cdot g, g) = c(f, g)$. Since $g^2 \leq f \cdot g$, we can apply the result from the special case above to get $c(f, g) = c(f^n \cdot g^n, g)$ so in particular, $c(f^n, g) = c(f, g)$.  

\(1.13\) If $f$ is not much bigger than $g$, the assumption in 1.7 and 1.13 may be false. For these cases, we can prove the following:

1.14 Lemma:

1. $c(2f - g, f) = c(f, g)$.
2. $c(2f - g, g) = c(f, g)$.
3. $c(f + m(f - g), g) = c(f, g)$ for all $m \in \omega$.

Proof: The proof is similar to the proof of 1.7. Again we only have to show (1). Let $(N, \varepsilon)$ be a reasonably closed model of a large fragment of ZFC (say, $(N, \varepsilon) \prec (H(\varepsilon^+), \varepsilon)$, where $\varepsilon = 2^\varepsilon$) of size $c(f, g)$ such $\prod f(n)$ is covered by the set of all $g$-slaloms from $N$.

Define $h$ by $h(k) := f(k) + f(k) - g(k)$. We can find a family $\langle B_k^i : i < f(k), k \in \omega \rangle$ in $N$ such that for all $k$, $\{0, \ldots, h(k) - 1\} = \bigcup_{i < f(k)} B_k^i$, where $|B_k^i| = 2$ for $l < f(k) - g(k)$, and $|B_k^i| = 1$ otherwise. We have to show that the set of $g$-slaloms from $N$ covers $\prod h(k)$.

So let $x$ be a function satisfying $\forall k, x(k) \in \bigcup_i \langle B_k^i \rangle$. We can define a function $y \in \prod f(n)$ such that for all $k, x(k) \in B_k^{p(y, k)}$. So there is some $g$-slalom $C \in N$ such that for all $k, y(k) \in C_k$. Revised version, April 1992
Define $\tilde{A} = \langle A_k : k \in \omega \rangle$ by $A_k := \bigcup_{i \in C_k} B^i_k$. Thus $A_k$ is the union of $g(k)$ many sets, of which at most $f(k) - g(k)$ are pairs, and the others singletons. Thus $|A_k| \leq g(k) + (f(k) - g(k)) = f(k)$, so $\tilde{A}$ is an $f$-slalom in $N$, and for all $k$, $x(k) \in A_k$.

Similar to the proof of 1.13 we now get:

1.15 Lemma:

(1) If $2g \leq f$, then for all $n$, $c(nf, g) = c(f, g)$.

(2) If for some $\varepsilon > 0$, $(1 + \varepsilon)g \leq f$, then for all $n$, $c(nf, g) = c(f, g)$.

2. The forcing notion $Q_{f,g}$

2.1 Definition: We fix sequences $\langle n_k^- : k \in \omega \rangle$ and $\langle n_k^+ : k \in \omega \rangle$ that increase very quickly and satisfy $n_0^- \ll n_0^+ \ll n_1^- \ll n_1^+ \ll \cdots$. In particular, we demand

(1) For all $k$ \( \prod_{j < k} n_j^- \leq n_k^- \)

(2) \( \lim_{k \to \infty} \frac{\log n_k^+}{\log n_k^-} = 0 \).

(3) $n_k^- \cdot n_k^+ < n_{k+1}^-$. 

We will only consider functions $f, g$ satisfying $n_k^- \leq g(k) < f(k) \leq n_k^+$. This is partly justified by 1.9, and it also helps to keep the formulation of the main theorem relatively simple.

2.2 Definition: Let $X \neq \emptyset$ be finite, $c, d \in \omega$. A $(c, d)$-complete norm on $P(X)$ is a map

\[ || \cdot || : P(X) - \{\emptyset\} \to \omega \]

mapping any nonempty $a \subseteq X$ to a number $||a||$ such that

whenever $a = a_1 \cup \cdots \cup a_c \subseteq X$, then for some $i_1, \ldots, i_d \in \{1, \ldots, c\}$, $||a_{i_1} \cup \cdots \cup a_{i_d}|| \geq ||a|| - 1$.

$(||a||$ is the cardinality of the set $a$)

A natural $(c, d)$-complete norm is given by $||a|| := \log_{c/d} |a|$. $c$-complete means $(c, 1)$-complete.

2.3 Definition: We call $(f, g, h)$ progressive, if $f, g, h$ are functions in $^{\omega} \omega$, satisfying

(1) For all $k$, $n_k^- \leq g(k) < f(k) \leq n_k^+$

(2) For all $k$, $n_k^- \leq h(k)$

(3) $\lim_k \frac{\log f(k)}{\log g(k)} / \log h(k) = \infty$.

We call $(f, g)$ progressive, if there is a function $h$ such that $(f, g, h)$ is progressive (or equivalently, if $(f, g, n^-)$ is progressive, where $n^-$ is the function defined by $n^-(k) = n_k^-$).
2.4 Remark: For example, if $f$ and $g$ satisfy (1), then $(f, g, g)$ is progressive iff $\log f / \log g \to \infty$. ☺ 2.4

In 2.6 we will define a forcing notion $Q_{f,g,h}$ for any progressive $(f, g, h)$. First we recall the following notation:

2.5 Notation: $\prec^{\omega} \omega = \bigcup_n n \omega$ is the set of finite sequences of natural numbers. For $s \in \prec^{\omega} \omega$, $|s|$ is the length of $s$.

A tree $p$ is a nonempty subset of $\prec^{\omega} \omega$ with the properties

$$\forall \eta \in p \forall k < |\eta| : \eta | k \in p$$
$$\forall \eta \in p : \text{succ}_p(\eta) \neq \emptyset,$$

where

$$\text{succ}_p(\eta) := \{ \nu \in p : \eta \subset \nu, |\eta| + 1 = |\nu| \}.$$ 

A branch $b$ of $p$ is a maximal linearly $\subseteq$-ordered subset of $p$. Every branch $b$ defines a function $\bar{b} : \omega \to \omega$ by $\bar{b} = \bigcup b$. We usually identify $b$ and $\bar{b}$, so we write $b|k$ (instead of $(\bigcup b)|k$) for the $k$th element of $b$.

The set of all branches of $p$ is written as $[p]$.

For $\eta \in p$, we let

$$p |_{\eta} := \{ \nu \in p : \nu \subseteq \eta \text{ or } \eta \subseteq \nu \}$$

We let

$$\text{split}(p) := \{ \eta \in p : |\text{succ}_p(\eta)| > 1 \} \quad \text{(the splitting nodes of } p)$$
$$\text{split}_n(p) := \{ \eta \in \text{split}(p) : |\{ \nu \subset \eta : \nu \in \text{split}(p) \}| = n \} \quad \text{(the } n\text{-th splitting level)}$$

and we define the stem of $p$ to be the unique element of $\text{split}_0(p)$.

2.6 Definition: Assume $f, g, h$ are as in 2.3. Then we define for all $k$, and for all sets $x$

$$|x|_k := \left\lfloor \frac{\log(|x|/g(k))}{\log h(k)} \right\rfloor$$

and we define the forcing notion $Q_{f,g}$ (or more accurately, $Q_{f,g,h}$) to be the set of all $p$ satisfying

1. $p$ is a perfect tree.
2. $\forall \eta \in p \forall i \in \text{dom}(\eta) \eta(i) < f(i)$.
3. $\forall \eta \in \text{split}_n(p) |\text{succ}_p(\nu)|_{|\nu|} \geq n$.

We let $p \leq q$ ("$q$ extends $p$") iff $q \subseteq p$.

2.7 Remark: If we define

$$p \sqsupseteq_k q \text{ iff } p \leq q \text{ and } \text{split}_k(p) \subseteq q$$

then $Q_{f,g,h}$ satisfies axiom A, and is in fact strongly $\omega$-bounding, i.e., for name of an ordinal, $\varnothing$, for any $p$ and for any $n$ there is a finite set $A$ and a condition $q \sqsupseteq_n p$, $q \Vdash \varnothing \in A$. However, it will be more convenient to use the relation $\leq_n$ that is based on levels rather than splitting levels.
2.8 Definition: For \( p, q \in Q, n \in \omega \) we define
\[
p \leq_n q \text{ iff } p \leq q \text{ and } p \cap \leq_n \omega \subseteq q
\]

2.9 Notation: We will usually write \( \| \eta \|_p \) instead of \( \| \text{succ}_p(\eta) \|_{\| \eta \|} \).

2.10 Remark: This forcing is similar to the forcing in [Shelah 326], but note the following important difference: Whereas in [Shelah 326] all nodes above the stem have to be splitting points, we allow many nodes to have only one successor, as long as there “many” nodes with high norm.

2.11 Remark:
1. The norm \( \| \cdot \|_k \) is \( h(k) \)-complete (hence also \( n_k \)-complete).
2. If \( c/d \leq h(k) \), then the norm is \( (c, d) \)-complete.
3. If \( \| a \|_k > 0 \), then \( |a| > g(k) \).
4. \( \| f(k) \|_k \to \infty \) (so \( Q_{f,g,h} \) is nonempty). \( \odot \) 2.11

We will see in the next section that this forcing (and any countable support product of such forcings) is proper and \( \omega^\omega \)-bounding. For the moment, we only show why this forcing is useful in connection with \( c(f, g) \):

2.12 Fact: Any generic filter \( G \subseteq Q_{f,g} \) defines a “generic branch”
\[
r := \bigcup_{p \in G} \text{stem}(p)
\]
that avoids all \( g \)-slaloms from \( V \).

Proof: Let \( B = \langle B_k : k \in \omega \rangle \) be a \( g \)-slalom in \( V \), and let \( p \in Q_{f,g} \) be a condition. Let \( \eta \in p \) be a node satisfying \( \| \eta \|_p > 0 \). Let \( k := |\eta| \). Then \( |\text{succ}_p(\eta)| > g(k) \) by 2.11(3), so there is \( i \notin B_k, \eta \downarrow i \in p \). So \( p[\eta \downarrow i] \models r(k) = i \notin B_k \). \( \odot \) 2.12

3. The construction
In this section we will prove the following theorem:

3.1 Theorem (CH): Assume that \( (f_\xi, g_\xi : \xi < \omega_1) \) is a sequence of progressive functions, witnessed by functions \( h_\xi \) (see 2.3).

Let \( (\kappa_\xi : \xi < \omega_1) \) be a sequence of cardinals satisfying \( \kappa_\xi^+ = \kappa_\xi \) such that whenever \( \kappa_\xi < \kappa_\zeta \), then
\[
\lim_{k \to \infty} \min \left( \frac{f_\xi(k)}{g_\xi(k)} \cdot \frac{f_\zeta(k)}{g_\zeta(k)} / h_\zeta(k) \right) = 0
\]
(or informally: either \( f_\xi \ll g_\xi \), or \( f_\xi/g_\xi \ll h_\xi \), or a combination of these two condition holds)

Then there is a proper forcing notion \( P \) not collapsing cardinals nor changing cofinalities such that
\[
\models_p \forall \xi : c(f_\xi, g_\xi) = \kappa_\xi
\]
For the proof we use a countable support product of the forcing notions $Q_{f_\xi, g_\xi, h_\xi}$ described in the previous section.

### 3.2 Remark:
The theorem is of course also true (with the same proof) if we have countably or finitely many functions to deal with.

If we are only interested in 2 cardinal invariants $c(f', g')$, $c(f, g)$, then we can phrase the theorem without the auxiliary functions $h$ as follows: If $(f, g)$ and $(f', g')$ are progressive, and satisfy

$$\min \left( \frac{f'}{g} \cdot \frac{\log(f/g)}{\log(f'/g')} \right) \to 0$$

then $c(f, g) < c(f', g')$ is consistent.

In particular, this shows that our result is quite sharp: For example, if for some function $d$ we have $\lim d = \infty$, $f' = f^d$, $g' = g^d$ (and $(f, g)$, $(f', g')$ are progressive with the same $n_k^-$, $n_k^+$), then $c(f, g) < c(f', g')$ is consistent. On the other hand, $c(f^n, g^n) \leq c(f, g)$ for every fixed $n$.

**Proof:** Choose $h'$ such that $\log h' \approx 2\log(f/g)$ whenever $\frac{f'}{g} \geq \frac{\log(f/g)}{\log(f'/g')}$. $(f', g', h')$ is progressive, and the assumptions of the theorem are satisfied. (Recall that $(f, g)$ is progressive, hence $\log f/g \gg \log n^-$, so $h'$ will satisfy $h'(k) \geq n_k^-$.)

A similar simplified formulation of 3.1 is possible when we deal with only countably many functions.

### 3.3 Example:
There is a family $\langle (f_\xi, g_\xi, \xi) : \xi < c \rangle$ of continuum many progressive functions such that for any $\xi \neq \xi$, $\min \left( \frac{f_\xi}{g_\xi}, \frac{f_\xi}{g_\xi} \right) \to 0$. [In particular, under CH we may choose any family $(\kappa_\xi : \xi < \omega_1)$ of cardinals satisfying $\kappa_\xi^\omega = \kappa_\xi$ and get an extension where $c(f_\xi, g_\xi) = \kappa_\xi$.]

**Proof:** Let $\ell_k := \left[ \frac{1}{2} \log \frac{n_k^+}{\log n_k^-} \right]$. (Here, “log” can be the logarithm to any (fixed) base, say 2.) Then $\lim_{k \to \infty} \ell_k = \infty$, and by invariance (1.5(4)) we may assume $\ell_k \geq 1$ for all $k$.

Let $T \subseteq 2^{<\omega}$ be a perfect tree such that for all $k$ we have $\vert T \cap 2^k \vert = \ell_k$, say, $T \cap 2^k = \{s_1(k), \ldots, s_{\ell_k}(k)\}$.

For any $x \in [T]$ (i.e., $x \in 2^\omega$, $\forall k x|k \in T$) we now define functions $f_x$, $g_x$, $h_x$ by:

- If $x|k = s_i(k)$, then
  $$f_x(k) = (n_k^-)^{\ell_k}$$
  $$g_x(k) = (n_k^-)^{\ell_k - 1}$$
  $$h_x(k) = g_x(k)$$

We leave the verification that $(f_x, g_x, h_x)$ is indeed progressive to the reader. [Recall 2.4, and also note that $\log f_x(k) \leq 2\ell_k \log \ell_k + \log \log n_k^- < \log \log n_k^+$. Finally, note that if $x \neq y$, then for almost all $k$ we have $\min \left( \frac{f_x(k)}{g_y(k)} \frac{f_y(k)}{h_x(k)} \right) \ll \frac{1}{n_k^-}$.]

### 3.4 Example: It is consistent to have $c(f^2, g^2) > c(f, g)$ (for certain $f$, $g$).

**Proof:** Let $\ell_k := \left[ \frac{1}{6} \log \frac{n_k^-}{n_k^+} \right]$. Assume $\ell_k > 0$ for all $k$. Then, letting

- $f(k) := (n_k^-)^{3\ell_k}$
- $g(k) := (n_k^-)^{2\ell_k}$
- $h(k) := n_k^-$

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We have that \((f, g, h)\) and \((f^2, g^2, h)\) are progressive, and \(\lim_{x\to a} f = 0\), so we can apply the theorem.

### 3.5 Definition:

Let \(\kappa\) be a disjoint union \(\kappa = \bigcup_{\xi < \omega_1} A_\xi\), where \(|A_\xi| = \kappa_\xi\).

For \(\alpha < \kappa\), let \(Q_\alpha\) be the forcing \(Q_{f_\xi, g_\xi, h_\xi}\) if \(\alpha \in A_\xi\), and let \(P = \prod_{\alpha < \kappa} Q_\alpha\) be the **countable support product** of the forcing notions \(Q_\alpha\), i.e., elements of \(P\) are countable functions \(p\) with \(\text{dom}(p) \subseteq \kappa\), and \(\forall \alpha \in \text{dom}(p) \ p(\alpha) \in Q_\alpha\).

For \(A \subseteq \kappa\), we write \(P|A := \{ p|A : p \in P \}\). Clearly \(P|A \triangleleft P\) for any \(A\). In particular, \(Q_\alpha \triangleleft P\).

We write \(r_\alpha\) for the \(Q_\alpha\)-name (or \(P\)-name) for the generic branch introduced by a generic filter on \(Q_\alpha\).

We say that \(q\) **strictly extends** \(p\), if \(q \geq p\) and \(\text{dom}(q) = \text{dom}(p)\).

### 3.6 Facts:

Assume CH. Then

1. each \(Q_\alpha\) is proper and \(\omega\omega\)-bounding.
2. \(P\) is proper and \(\omega\omega\)-bounding.
3. \(P\) satisfies the \(\aleph_2\)-cc.
4. Neither cardinals nor cofinalities are changed by forcing with \(P\).

Proof of (1), (2): See below (3.23, 3.24)

Proof of (3): A straightforward \(\Delta\)-system argument, using CH.

(4) follows from (2) and (3).

We plan to show that \(\vDash_P c_\xi = \kappa_\xi\) for all \(\xi < \omega_1\).

### 3.7 Definition:

If \(p \in P\), \(k \in \omega\), we let the level \(k\) of \(p\) be

\[
\text{Level}_k(p) := \{ \bar{\eta} : \text{dom}(\bar{\eta}) = \text{dom}(p), \forall \alpha \in \text{dom}(\bar{\eta}) : |\bar{\eta}(\alpha)| = k, \bar{\eta}(\alpha) \in p(\alpha) \}
\]

We define the set of active ordinals at level \(k\) as

\[
\text{active}_k(p) := \{ \alpha \in \text{dom}(p) : |\text{stem}(p(\alpha))| \leq k \}
\]

### 3.8 Remark:

Sometimes we identify the set \(\text{Level}_k(p)\) with the set

\[
\{ \bar{\eta} : \text{dom}(\bar{\eta}) = \text{active}_k(p), \forall \alpha \in \text{dom}(\bar{\eta}) : |\bar{\eta}(\alpha)| = k \}
\]

\[= \{ \bar{\eta} | \text{active}_k(p) : \bar{\eta} \in \text{Level}_k(p) \}\]

### 3.9 Definition:

We say that the \(k\)th level is a splitting level of \(p\) (or “\(k\) is a splitting level of \(p\)”) iff

\[\exists \alpha \in \text{dom}(p) \exists \eta \in \text{split}(p(\alpha)) : |\eta| = k\]

### 3.10 Definition:

If \(\bar{\eta} \in \text{Level}_k(p)\), \(\bar{\eta}' \in \text{Level}_{k'}(p)\), \(k < k'\), then we say that \(\bar{\eta}'\) extends \(\bar{\eta}\) if for all \(\alpha \in \text{dom}(\bar{\eta})\), \(\bar{\eta}'(\alpha)\) extends (i.e., \(\supseteq\)) \(\bar{\eta}(\alpha)\).
3.11 Definition: For \( p, q \in P, \ k \in \omega \), we let

\[
p \leq_k q \quad \text{iff} \quad p \leq q \ \text{and} \ \forall \alpha \in \text{dom}(p) \colon p(\alpha) \leq_k q(\alpha) \ \text{and} \ \text{active}_k(p) = \text{active}_k(q)
\]

That is, we allow \( \text{dom}(q) \) to be bigger than \( \text{dom}(p) \), but for all new \( \alpha \in \text{dom}(q) - \text{dom}(p) \) we require that \( |\text{stem}(q(\alpha))| > k \).

3.12 Definition: Let \( A \subseteq P \). A set \( D \subseteq P \) is

dense in \( A \), if \( \forall p \in A \exists q \in D : p \leq q \)
strictly dense in \( A \), if \( \forall p \in A \exists q \in D : p \leq q \ \text{and} \ \text{dom}(p) = \text{dom}(q) \)
open in \( A \), if \( \forall p \in D \forall q \in A : (p \leq q \ \text{implies} \ q \in D) \)
almost open in \( A \), if \( \forall p \in D \forall q \in A : (p \leq q \ \text{and} \ \text{dom}(p) = \text{dom}(q) \ \text{implies} \ q \in D) \)

These definitions can also be relativized to conditions above a given condition \( p_0 \). If we omit \( A \) we mean \( A = P \).

3.13 Definition: If \( \bar{\eta} \in \text{Level}_k(p) \), we let \( q = p[^{\bar{\eta}}] \) be the condition defined by \( \text{dom}(q) = \text{dom}(p) \), and

\[
\forall \alpha \in \text{dom}(q) \colon q(\alpha) = p(\alpha)^{[\bar{\eta}(\alpha)]}
\]

3.14 Definition: If \( p \models \varphi \in V \), and \( \bar{\eta} \in \text{Level}_k(p) \), we say that \( \bar{\eta} \) decides \( \varphi \) (or more accurately, \( p[^{\bar{\eta}}] \) decides \( \varphi \) if for some \( y \in V \), \( p[^{\bar{\eta}}] \models \varphi = \bar{y} \).

First we simplify the form of our conditions such that all levels are finite.

3.15 Fact: The set of all conditions \( p \) satisfying

\[
\begin{align*}
\text{I} & \quad \forall k |\text{active}_k(p)| < \omega, \ \text{and} \ \text{moreover:} \\
\text{II} & \quad \text{For any splitting level} \ k \ \text{there is exactly one pair} (\eta, \alpha) \ \text{such that} \ |\text{succ}_p(\alpha)(\eta)| > 1.
\end{align*}
\]

is dense in \( P \).

3.16 Fact: If \( p \) is in the dense set given by (I) and (II), then the size of level \( k \) is \( \leq n_k^+ \cdot n_{k-1}^- < n_k^- \).

Proof: By induction. \( \odot \)

From now on we will only work in the dense set of conditions satisfying (I) and (II).

3.17 Notation: For \( p \) satisfying (I)-(II), we let \( k_l = k_l(p) \) be the \( l \)th splitting level. Let \( \eta_l = \eta_l(p) \) and \( \alpha_l = \alpha_l(p) \) be such that \( |\eta_l(p)| = k_l(p) \), \( \eta_l(p) \in \text{split}(p(\alpha_l)) \). We let \( \zeta_l = \zeta_l(p) \) be such that \( \alpha_l \in A_{\zeta_l} \).

We write \( \|p\|_{k_l} \) for \( \|\eta_l\|_{p(\alpha_l)} \) i.e., for \( \|\text{succ}_p(\alpha_l)(\eta_l)\|_{\zeta_l, k_l} \) (See figure 2)

3.18 Definition: If \( p \) is a condition, \( l \in \omega \), \( \alpha^* := \alpha_l(p) \), \( \eta^* := \eta_l(p) \), \( \nu^* \in \text{succ}_p(\alpha^*)(\eta^*) \), we can define a stronger condition \( q \) by letting \( q(\alpha) = p(\alpha) \) for all \( \alpha \neq \alpha^* \), and

\[
q(\alpha^*) := \{\eta \in p(\alpha^*) : \text{If} \ \eta^* \subset \eta, \ \text{then} \ \nu^* \subseteq \eta\}
\]

In this case, we say that \( q \) was obtained from \( p \) by “pruning the splitting node \( \eta^* \).”

To simplify the notation in the fusion arguments below, we will use the following game:
3.19 Definition: For any condition $p \in P$, $G(P, p)$ is the following two person game with perfect information:

There are two players, the spendthrift and the accountant. A play in $G(P, p)$ last $\omega$ many moves (starting with move number 1) The accountant moves first. We let $p_0 := p$, $i_0 := 0$.

In the $n$-th move, the accountant plays a pair $(\eta^n, \alpha^n)$ with $\eta^n \in p_{n-1}(\alpha^n)$, $|\eta^n| = i_{n-1}$, and a number $b_n$. Player spendthrift responds by playing a condition $p_n$ and a finite sequence $\nu^n$ (letting $i_n := |\nu^n| + 1$) satisfying the following: (See Figure 3)

1. $p_n \geq i_{n-1}, p_{n-1}$.
2. $\nu^n \in p_n(\alpha^n)$
3. $\|\nu^n\|_{p_n(\alpha^n)} > b_n$.
4. $\nu^n \supseteq \eta^n$.
5. For all $\alpha \in \text{dom}(p_n) - \text{dom}(p_{n-1})$, $|\text{stem}(p_n(\alpha))| > |\nu^n|$.
6. $|\text{Level}_{\nu^n}(p_n)| = |\text{Level}_{\eta^n}(p_n)| = |\text{Level}_{\eta^n}(p_{n-1})|$

(Remember that all conditions $p_n$ have to be in the dense set given by (I) and (II)) Player accountant wins iff after $\omega$ many moves there is a condition $q$ such that for all $n$, $p_n \leq q$, or equivalently, if the function $q$ with domain $\bigcup_n \text{dom}(p_n)$, defined by

$$q(\alpha) = \bigcup_{\alpha \in \text{dom}(p_n)} p_n(\alpha)$$

is a condition. Note that we have $\eta_l(q) = \nu_l$, $\alpha_l(q) = \alpha_l$, since the only splitting points are the ones chosen by spendthrift.

3.20 Fact: Player accountant has a winning strategy in $G(P, p)$.  

Figure 2: A condition satisfying (I) and (II)
Figure 3: stage $n$

**Proof:** We leave the proof to the reader, after pointing out that a finitary bookkeeping will ensure that the limit of the conditions $p_n$ is in fact a condition.

In particular, this shows that *spendthrift* has no winning strategy. Below we will define various strategies for the *spendthrift*, and use only the fact that there is a play in which the *accountant* wins. \(\circ\) 3.20

The game gives us the following lemma:

**3.21 Lemma:** Assume that $p$ is a condition satisfying (I)--(II). For each $l$ let $\emptyset \neq F_{\eta_l} \subseteq \text{succ}_{p(\alpha_l)}(\eta_l)$ be a set of norm $\|F_{\eta_l}\|_{k_l} \geq \|\text{succ}_{p(\alpha_l)}(\eta_l)\| / 2$.

Then there is a condition $q \geq p$, $\text{dom}(q) = \text{dom}(p)$ such that for all $l$:

\[(\ast) \quad \text{If } \eta_l(p) \in q(\alpha_l(p)), \text{ then } \text{succ}_{q(\alpha_l(p))}(\eta_l(p)) \subseteq F_{\eta_l}\]

**Proof:** The condition $q$ can be constructed by playing the game. In the $n$-th move, *spendthrift* first finds a $\eta^n \supset \nu^n$ satisfying $\eta^n(i) \in F_{\eta_i}$ whenever this is applicable, and $\|\text{succ}_{p_{n-1}}(\eta^n)\| > 2b_n$. Then *spendthrift* obtains $p_n$ by pruning (see 3.18) all splitting nodes of $p_{n-1}$ whose height is between $|\eta^n|$ and $|\nu^n|$ and further thinning out the successors of $\eta^n$ to satisfy $\text{succ}_{p_n}(\eta^n) = F_{\eta^n}$. (Note that $F_{\eta^n} \subseteq \text{succ}_{p_{n-1}}(\eta^n) = \text{succ}_{p_0}(\eta^n)$.)

In the resulting condition $q$ the only splitting nodes will be the nodes $\eta^n$, so $(\ast)$ will be satisfied. \(\circ\) 3.21

(Note that in general $\eta_l(q) \neq \eta_l(p)$, and indeed $k_l(q) \neq k_l(p)$, since many splitting levels of $p$ are not splitting levels in $q$ anymore.)

**3.22 Lemma:** Assume $\tau$ is a $P$-name of a function from $\omega$ to $\omega$, or even from $\omega$ into ordinals. Then the set of conditions satisfying (I)--(III) is dense and almost open.
Whenever $k$ is a splitting level, then every $\bar{\eta}$ in level $k + 1$ decides $\tau↾k$.

Proof of (III): We will use the game from 3.19. We will define a strategy for the spendthrift ensuring that the condition $q$ the accountant produces at the end will satisfy (III).

In the $n$-th move, spendthrift finds a condition $r_n \geq i_{n-1} \cdot p_{n-1}$ such that for every $\bar{\eta} \in \text{Level}_{i_{n-1}}(r_n)$ the condition $(p_n)\bar{\eta}$ decides $\tau↾i_{n-1} + 10$. Then spendthrift finds $\eta^n \in r_n(\alpha^n)$ satisfying the rules and obtains $p_n$ with $\eta^n \in p_n(\alpha^n)$ from $r_n$ by pruning all splitting levels between $i_{n-1}$ and $|\eta_n|$.

Since all levels of $q$ are finite, it is thus possible to find a finite sequence $B = \langle B_k : k \in \omega \rangle$ in the ground model that will cover $\tau$. (I.e. $q \Vdash \exists k (B(k) \in B_k)$). The rest of this section will be devoted to finding “small” such sets $B_k$.

3.23 Corollary: $P$ is $\omega$-bounding and does not collapse $\omega_1$.

3.24 Remark: Although it does not literally follow from 3.22, the reader will have no difficulty in showing that $P$ is actually $\alpha$-proper for any $\alpha < \omega_1$. Indeed, using the partial orders $\sqsubseteq_n$ from 2.7, it is possible to carry out straightforward fusion arguments, without using the game 3.19 at all. However, the orderings $\leq_n$ are more easy to handle, since in induction steps we only have to take care of a single $\eta^n$, instead of a front.

3.25 Fact: $\models_P \forall \tau \in {}^\omega \omega \exists B \subseteq \kappa, B$ countable, $B \in V$, and $\tau \in V[G↾B]$.

Proof: Let $p$ be any condition and let $\tau$ be a name for a real. There is a stronger condition $q$ satisfying (I), (II) and (III). Let $B := \text{dom}(q)$. Clearly $q \Vdash \tau \in V[G↾B]$.

3.26 Corollary: If $\lambda = |A|^\omega$, then $\models_{P↾A} 2^{\aleph_0} \leq \lambda$.

Proof: For each countable subset $B \subseteq A$, $\models_{P↾B} CH$. Since every real in $V[G]$ is in some such $V[G↾B]$, the result follows.

3.27 Fact and Notation: If $p$ satisfies (II), then

1. If $\bar{\eta}(\alpha_l) = \eta_l$, and $\nu \in \text{succ}_{p(\alpha_l)}(\bar{\eta}_{l})$, then the requirement

\[ \bar{\eta}^{+\nu}(\alpha_l) = \nu \]

uniquely defines an extension $\bar{\eta}^{+\nu}$ of $\bar{\eta}$ in Level$_{k+1}(p)$.

2. If $\bar{\eta}(\alpha_l) \neq \eta_l$, $\bar{\eta}$ has a unique extension $\bar{\eta}^+_l \in \text{Level}_{k+1}(p)$. To simplify the notation in 3.33 below, we also define for this case, for any $\nu \in \text{succ}_{p(\alpha_l)}(\bar{\eta}_{l})$, $\bar{\eta}^{+\nu} := \bar{\eta}^+_l$.

3.28 Fact: The set of conditions satisfying (IV) is strictly dense (but not almost open) in the set of conditions satisfying (I)–(II).

IV For all $l$:

\[ |\text{Level}_{k_l}(p)| < \min \left( \frac{\|p\|_{k_r}}{2}, n_{k_l}^{-} \right) \]

For the proof, note that $|\text{Level}_{k_l}(p)| = |\text{Level}_{k_l-1}(p)|$.

\[ \Box \]
3.29 Lemma: Assume \( \mathcal{F} \) is a \( P \)-name of a function \( \in \downarrow \omega \), and \( \models P \forall k \mathcal{F}(k) < n_k^+ \). Then the set of conditions satisfying (V) is strictly dense and almost open in the set give by (I), (II), (III). where

Whenever \( k \) is a splitting level, then every \( \bar{\eta} \) in level \( k \) decides \( \mathcal{F}|k \).

Proof: Fix \( p \) satisfying (I), (II), (III), (IV).

Let \( k_l := k_l(p), \) etc. Let \( m_l := |\text{Level}_{k_l}|. \)

Proof: We will use 3.21. For each \( l \in \omega, F_{\eta_l} \subseteq \text{succ}_{p(\alpha_l)}(\eta_l) \) will be defined as follows: Let \( m_l := |\text{Level}_{k_l}(p)| \), and let \( \eta^0, \ldots, \eta^{m_l-1} \) enumerate \( \text{Level}_{k_l}(p) \). Find a sequence

\[
\text{succ}_{p(\alpha_l)}(\eta_l) = F^0 \supseteq F^1 \supseteq \cdots \supseteq F^{m_l} \quad \forall i \|F^{i+1}\|_k \geq \|F^i\|_k - 1
\]

such that for all \( i \) there exists \( x^i \) such that for all \( \nu \in F^{i+1} \) we have \( p^{(\bar{\eta}^i)_*} \models \mathcal{F}|k = x \). It is possible to find such \( F^{i+1} \) since \( \|\|_k \) is \( n_k^* -(\|\|_k \) complete, and there are only \( n_0^* \cdot n_1^* \cdots n_{k-1}^* < n_k^* \) many possible values of \( \mathcal{F}|k \).

Finally, let \( F_{\eta_l} := F^{m_l} \). Applying 3.21 will yield the desired result. \( \smiley 3.29 \)

3.30 Remark: Note that (V) in particular implies

\( \smiley V_a \)

Whenever \( k \) is not a splitting level, then every \( \bar{\eta} \) in level \( k \) decides \( \mathcal{F}(k) \).

3.31 Proof that \( \models P c(f_\xi, g_\xi) \geq \kappa_\xi \): (This proof is essentially the same as 2.12.)

Recall that \( \mathcal{F}_\alpha \) is the generic real added by the forcing \( Q_\alpha \). Working in \( V[G] \), let \( B \) be a family of less than \( \kappa_\xi \) many \( g_\xi \)-slaloms. We will show that they cannot cover \( \prod f_\xi \), by finding an \( \alpha \) such that \( \mathcal{F}_\alpha \) is forced not to be covered.

There exists a set \( A \in V \) of size < \( \kappa_\xi \) such that \( \mathcal{B} \subseteq V[G\upharpoonright A] \). Since \( |A| < \kappa_\xi \) there is \( \alpha \in A_\xi - A \).

Assume that \( B \) is a \( g_\xi \)-slalom in \( V[G\upharpoonright A] \) covering \( r_\alpha \). So in \( V \) there are a \( P\upharpoonright A \)-name \( \bar{B} \) and a condition \( p \) such that

\( \models P\upharpoonright A \bar{B} \) is a \( g \)-slalom

and

\( p \models P \bar{B} \) covers \( r_\alpha \)

We can find a node \( \eta \) in \( p(\alpha) \) with \( \text{succ}_{p(\alpha)}(\eta) \) having more than \( g(|\eta|) \) elements. Increase \( p\upharpoonright A \) to decide \( \bar{B}_{\upharpoonright |\eta|} \), then increase \( p(\alpha) \) to make \( r_\alpha \) avoid this set. \( \smiley 3.31 \)

3.32 Fact: Fix \( \xi^* \). Then the set of conditions \( p \) satisfying

\( \smiley V_l \)

For all \( l \): If \( \kappa_\xi^* \leq \kappa_{\xi^*}(p) \), then

\[
\min \left( \frac{f_{\xi^*}(k_l)}{g_{\xi^*}(k_l)}, \frac{f_\xi(k_l)}{g_\xi(k_l)} \bigg/ h_{\xi^*}(k_l) \right) < \frac{1}{|\text{Level}_{k_l}(p)|}
\]

is dense almost open.

Proof: Write \( F_\xi \) for the function \( \min \left( \frac{f_\xi}{g_\xi}, \frac{f_{\xi^*}}{g_{\xi^*}} \bigg/ h_{\xi^*} \right) \). Recall that if \( \kappa_\xi < \kappa_\xi^* \), then \( F_\xi \) tends to 0.
Fix a condition $p$. We will use the game $G(P, p)$. *spendthrift* will use the following strategy: Whenever $\alpha_n \in A_\zeta$ and $\kappa_\zeta < \kappa_\zeta^\cdot$, then *spendthrift* first find $m_0$ such that for all $m \geq m_0$ we have $F_\zeta(m) < 1/|\text{Level}_{b_{n-1}}(p_{n-1})|$. Now find $\nu^n \supseteq \eta^n$ of length $> m_0$ with a large enough norm, and play any condition $p_n$ obeying the rules of the game. In particular, we must have $|\text{Level}_{\nu^n}(p^n)| = |\text{Level}_{\eta^n}(p^n)|$.

Clearly the resulting condition from the game satisfies the requirements.

\[3.33 \text{ Proof that } \Vdash_P c(f_\xi, g_\xi) \subseteq \kappa_\xi: \text{ Fix } \xi. \text{ We will write } f \text{ for } f_\xi, \text{ etc.} \]

Let

$$A := \bigcup \{ A_\zeta : \kappa_\zeta \leq \kappa_\xi \}. $$

We will show that the $g$-slaloms from $V^{P/A}$ already cover $\prod f$. This is sufficient, because $\Vdash_P (2^{\kappa_0})^{V^{P/A}} \leq |A| = \kappa_\xi$.

Let $p_0$ be an arbitrary condition. Let $\zeta$ be a name of a function $< f$. Find a condition $p \geq p_0$ satisfying (I)–(VI).

For each $l$ we now define sets $F_m \subseteq \text{succ}_{p(\alpha_1)}(\eta_l)$ as follows:

1. If $\alpha_1 \in A$, then $F_m = \text{succ}_{p(\alpha_1)}(\eta_l)$.
2. If $f_\zeta(k_l) \leq g_\zeta(k_l)/|\text{Level}_{k_l}(p)|$, then again $F_m = \text{succ}_{p(\alpha_1)}(\eta_l)$.
3. Otherwise, we thin out the set $\text{succ}_{p(\alpha_1)}(\eta_l)$ such that each $\eta_l$ in $\text{Level}_{k_l}(p)$ decides $\zeta(k_l)$ up to at most $g(k_l)/|\text{Level}_{k_l}(p)|$ many values.

Here is a more detailed description of case (3): Let $k = k_l$, $\zeta = \zeta_l$.

Note that if neither (1) nor (2) holds, then letting $c := f_\zeta(k)$, $d := g_\zeta(k)/|\text{Level}_{k_l}(p)|$, we have $c/d \leq h_\zeta(k)$.

Using $(c, d)$-completeness of the norm $\| : \|_{\zeta, k}$, we define a sequence

$$\text{succ}_{p(\alpha_1)}(\eta_l) = L(0) \supseteq L(1) \supseteq \cdots \supseteq L(|\text{Level}_{k_l}(p)|)$$

as follows. Let $\eta_0, \ldots, \eta_{|\text{Level}_{k_l}(p)|-1}$ be an enumeration of $\text{Level}_{k_l}(p)$.

Given $L(i)$, we know that for each $\nu \in L(i)$ the sequence $\eta_0^{+\nu}$, (i.e., the condition $p^{[\eta_0^{+\nu}]}$) decides $\zeta(k)$. (See 3.27.) since there only $\leq c$ many possible values for $\zeta(k)$, we can use $(c, d)$-completeness to find a set $L(i + 1) \subseteq L(i)$ and a set $C(i)$ such that

(a) $|L(i + 1)| \geq |L(i)| - 1$

(b) $|C(i)| \leq d$.

(c) For every $\nu \in L(i + 1)$, $p^{[\eta_0^{+\nu}]} \Vdash \zeta(k) \in C(i)$.

Now let $F_m$ be $L(|\text{Level}_{k_l}(p)|)$, and let

$$B_k := \bigcup_i C(i).$$

So $|B_k| \leq |\text{Level}_{k_l}(p)| \cdot d \leq g(k)$.

Clearly $\|F_m\|_{\zeta, k_l} \geq \|p\|_{k_l} - |\text{Level}_{k_l}(p)| > \frac{1}{2} \|p\|_{k_l}$.
This completes the definition of the sets $F_\eta$.

Let $q \geq p$ be the condition defined from $p$ using the $F_\eta$ (see 3.21). We will find a $P|A$-name for a $g$-slalom

\[ \mathcal{B} = \langle B_k : k \in \omega \rangle \]

such that

\[ q \models \mathcal{B} \text{ covers } \tau. \]

If $k$ is not a splitting level, then every $\bar{\eta}$ in level $k$ decides $\tau(k)$ by (Va). So in this case we can let

\[ B_k := \{ i : \exists \bar{\eta} \in \text{Level}_k(p), p[\bar{\eta}] \models \tau(k) = i \} \]

This set is of size $\leq |\text{Level}_k(p)| < g(k)$, and clearly $q \models \tau(k) \in B_k$.

If $k$ is a splitting level, $k = k_l$, then there are three cases.

Case 1: $\alpha_l \in A$: We define $B_k$ to be a $P|A$-name satisfying the following:

\[ \models_{P|A} B_k = \{ i : \exists \bar{\eta} \in \text{Level}_{k+1}(p), V \models p[\bar{\eta}] \models \tau(k) = i, \bar{\eta}(\alpha_l) \subseteq r_{\alpha_l} \} \]

Thus, we only admit those $\bar{\eta}$ which agree with the generic real added by the forcing $Q_{\alpha_l}$. Clearly $\models_{P|A} |B_k| \leq \text{Level}_k(p) < g(k)$, and $p \models_{P} \tau(k) \in B_k$.

Case 2: $f_{\xi}(k) \leq g_{\xi}(k)/|\text{Level}_k(p)|$.

So we have $|\text{Level}_{k+1}(p)| \leq f_{\xi}(k) \cdot |\text{Level}_k(p)| \leq g(k)$, so we can let

\[ B_k := \{ i : \exists \bar{\eta} \in \text{Level}_{k+1}(p), p[\bar{\eta}] \models \tau(k) = i \} \]

This set is of size $\leq |\text{Level}_{k+1}(p)| \leq g(k)$, and again $p \models \tau(k) \in B_k$.

Case 3: Otherwise. We have already defined $B_{k_l}$ in $(\oplus)$. By condition (c) above, $q \models \tau(k) \in B_k$.

So indeed $q \models \text{“} \mathcal{B} = \langle B_k : k \in \omega \rangle \text{ is a } g\text{-slalom covering } \tau\text{”}$.  

\[ \odot 3.33 \odot 3.1 \odot |\text{GS}h 448| \]

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