ELECTRIC-MAGNETIC DUALITY AND EFFECTIVE FIELD THEORIES†

César Gómez

Instituto de Matemáticas y Física Fundamental, CSIC
Serrano 123, 28006 Madrid, Spain

and

Rafael Hernández

Departamento de Física Teórica, C-XI, Universidad Autónoma de Madrid
Cantoblanco, 28049 Madrid, Spain

† Lectures given by C.G. in the Advanced School on Effective Theories, Almuñecar, Granada, 1995.
## Contents

1 Introduction  

2 The Duality Group  
  2.1 The Dirac Monopole  
  2.2 The 't Hooft-Polyakov Monopole  
    2.2.1 Topological Properties of Monopoles  
    2.2.2 BPS States  
    2.2.3 Quantum Mass Formulas  
  2.3 Duality  

3 Duality and Effective Field Theories  
  3.1 Flat Potentials and Classical Moduli  
  3.2 The Quantum Moduli  
  3.3 Effective Field Theory description of the Quantum Moduli  
  3.4 Duality and EFT-Geometry  
    3.4.1 Wilsonian Effective Theories  
    3.4.2 The $N=2$ Prepotential  
    3.4.3 Zamolodchikov’s metric of the Quantum Moduli  
    3.4.4 Dual Coordinates  
    3.4.5 Duality Transformations  
  3.5 $N=2$ Non Renormalization Theorems  
  3.6 The Singularity at Infinity  
  3.7 Strong Coupling Regime: heuristic approach  
  3.8 Montonen-Olive Duality in $N=2$ Theories  

4 Exact Results and Coupling to Gravity  
  4.1 Singularities and Phases  
    4.1.1 Holomorphicity and Abelian Monodromy  
    4.1.2 Duality and Phases  
  4.2 Seiberg-Witten Solution for $N=2\ SU(2)$ Supersymmetric Yang-Mills  
  4.3 Some Comments on Seibeg-Witten Solution  
    4.3.1 The Abelian Confinement Argument  
  4.4 Geometrical Interpretation  
  4.5 The Stringy Approach to the Quantum Moduli
1 Introduction.

The very concept of law of nature reflects a terminology which appears to be the heritage of a normative metaphor rooted in the ancestral image of a universe ruled by God. Maybe, a deeper concept which underlies more basically our way of thinking in physics is that of symmetry, which moves us from the normative or legal metaphor to the belief that beauty is the closest to truth. This was partially Dirac’s philosophy of physics; these lectures will try to present some of the recent and exciting developments in quantum field theory and string theory that his original ideas of duality and magnetic monopoles have made possible.

It was on the basis of the original idea of electric-magnetic duality [1] that the physics of magnetic monopoles begins [2]. The classical mass formulas [1] for the ‘t Hooft-Polyakov monopole [1], together with Witten’s discovery of the dyon nature of the monopole and the role played by the $\theta$-parameter [1], were crucial to promote the electric-magnetic duality transformations into a group of symmetries, namely the modular group $Sl(2, Z)$. The unexpected connection between the BPS mass formula and the $N=2$ supersymmetry algebra discovered by Olive and Witten [3] was already the first indication that a deep relation between duality and supersymmetry existed [7].

In a different context, what might be called a “dual” way of thinking turned to be an extremely powerful tool [8] to get a qualitative understanding of the phenomena of confinement: we only need to think of the dual to the well known Meissner effect in BCS-superconductivity, to interpret confinement as the dual to the Higgs mechanism.

However, the strong constraints imposed by the Montonen-Olive duality conjecture [9] reduce at first sight its field of application to phenomenologically uninteresting theories with vanishing $\beta$-function (for instance, $N=4$ supersymmetric Yang-Mills). It was only very recently, thanks to the seminal work of Seiberg and Witten [10], that we have learned how to extend, in a fruitful way, the ideas of duality to more realistic theories, with non vanishing $\beta$-function. Besides, Seiberg-Witten results are not only interesting from the strict sense of duality, as they deal with a phenomenon specially important in the phenomenology of supersymmetric theories (as well as in string theories): the question of the physics associated with the existence of flat potentials. The natural tool for studying this physics is, of course, effective lagrangians, and that is the main reason why these lectures will appear in this volume. A flat potential forces us to deal, from the beginning, with a manifold of unequivalent field theories; the geometry of this manifold can be described by the effective field theories that result from integrating, at each point in the flat direction, the heavy modes. Now, typical questions in the effective lagrangian physics, as the range of validity of a particular choice of light degrees of freedom, acquires a new geometrical meaning, namely the structure of singularities of the quantum moduli.

All these recent developments in quantum field theory have become possible, in part, thanks to the new way of thinking physics provided by string theory. In fact, Montonen-Olive duality was rediscovered in the context of string theory (interpreted as two dimen-
sional σ-model physics) in what is called T-duality [11]. In this framework, the dynamical origin of this symmetry is located in the extended nature of the fundamental string. Montonen-Olive duality refers, of course, to four dimensional physics, but the natural way in which strings explain the two dimensional analog strongly forces us to look for the meaning of duality in a stringy framework. This was the idea for postulating a new stringy symmetry, that was christened S-duality [12]. Recently [13], many new results indicate that such a dream is close to the truth; the discovery of the so called duality of dualities provides a bridge to connect T and S-duality in pairs of string theories. These fascinating topics dealing with string theory will, however, only be briefly treated at the end of the lectures.

In summary, the concept of duality is becoming an impressively useful tool to work in many areas of theoretical physics, ranging from confinement and effective field theories to extended supersymmetry and string theory.

2 The Duality Group.

2.1 The Dirac Monopole.

Maxwell’s equations, governing the behaviour of the electromagnetic field, offer a compact form when written in relativistic notation; in the absence of sources, they look like

\[
\partial_\nu F^{\mu\nu} = 0 \\
\partial_\nu \ast F^{\mu\nu} = 0
\] (2.1)

if a dual tensor of the electromagnetic tensor \(F_{\mu\nu}\) is introduced:

\[
\ast F^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma},
\] (2.2)

with \(\varepsilon^{\mu\nu\rho\sigma}\) the totally antisymmetric tensor.

These equations in vacua are obviously symmetric under the duality transformation

\[
\mathcal{F}^{\mu\nu} \rightarrow \ast \mathcal{F}^{\mu\nu} \quad \ast \mathcal{F}^{\mu\nu} \rightarrow -\mathcal{F}^{\mu\nu}
\] (2.3)

which amounts to interchanging the role played by electricity and magnetism. This symmetry is immediately broken if a non zero electric current \(j^{\mu}\) enters the theory, unless a magnetic current \(k^{\mu}\) is introduced, leading Maxwell’s equations to the form

\[
\partial_\nu F^{\mu\nu} = -j^{\mu} \\
\partial_\nu \ast F^{\mu\nu} = -k^{\mu}.
\] (2.4)

Duality appears again under the transformation, generalizing (2.3),

\[
\mathcal{F}^{\mu\nu} \rightarrow \ast \mathcal{F}^{\mu\nu} \quad j^{\mu} \rightarrow k^{\mu} \\
\ast \mathcal{F}^{\mu\nu} \rightarrow -\mathcal{F}^{\mu\nu} \quad k^{\mu} \rightarrow -j^{\mu}.
\] (2.5)
The immediate step one should give is wondering about whether this duality is consistent with quantum theory. Quantization relies on the power of the canonical formalism, so we should keep track of the canonical variables for the electromagnetic field, which are not the components of the tensor $F_{\mu\nu}$, but of the potential $A^\mu$, defined through

$$F_{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu. \quad (2.6)$$

However, finding a four-vector satisfying (2.3) is only possible if the magnetic (dual) equation in (2.4) vanishes:

$$\partial^\nu F_{\mu\nu} = 0. \quad (2.7)$$

But this condition, implied by equation (2.6), seems to destroy the duality coming from equations (2.4). Fortunately, there is an approach able to maintain the chance to build an electromagnetic potential; all we have to do is notice that in the neighbourhood of a magnetic charge (a monopole) the electromagnetic potential must be singular. To see this, let us suppose that somewhere in space we have a magnetic monopole of charge $g$, implying the nonvanishing of the magnetic current $k^\mu$. Then, the magnetic flux leaving a sphere surrounding the monopole can be easily calculated if we suppose that the electromagnetic potential is nonsingular everywhere. If we decompose $S^2$ into the two hemispheres $H^+$ and $H^-$, and use Stoke’s theorem, we will have a contribution

$$\int \int_{H^+} F_{\mu\nu} d\Sigma^{\mu\nu} = \oint_{\partial H^+} A_\mu d x^\mu, \quad (2.8)$$

where the line integral, along the equator (the boundary of $H^+$), is taken in the clockwise direction. Integrating over $H^-$,

$$\int \int_{H^-} F_{\mu\nu} d\Sigma^{\mu\nu} = \oint_{\partial H^-} A_\mu d x^\mu, \quad (2.9)$$

where now the line integration is in the counterclockwise direction. Adding the two integrals implies a zero for the total magnetic flux. As this contradicts the assumption that the surface contains a magnetic monopole, in which case the flux must be $4\pi g$, we conclude that $A_\mu$ must have a singularity somewhere on the sphere.

The argument above can be used for any radius of the sphere surrounding the monopole, so by increasing it from zero to infinity we conclude that the monopole has attached a line of singularities. Dirac [1] was the first to notice such a line, which is known as the Dirac string. The magnetic charge introduced to make Maxwell’s equations appear symmetric, and giving rise to the line of singularities described, is the magnetic partner of the electron, and is called a Dirac monopole.

Therefore, in the presence of a monopole the electromagnetic potential can not be defined everywhere; all what can be done is find it everywhere except on a line joining the monopole to infinity. The orientation of the string is, of course, arbitrary, and potential configurations in which the singularity extends along different lines are related by gauge transformations.
The Dirac string should not be thought of as a physical singularity, but as a singularity in the representation of the potential in a particular gauge choice. It has the same meaning as the singularities in a stereographic projection of the sphere: one can not help the appearance of a singular coordinate, say the north or south pole, unless two different coordinate patches are used in the description; in this case there is, in addition, the need to specify which point in one projection corresponds to which point in the other projection in the overlap region of the projections of the two patches. The singularity in the electromagnetic potential coming from the presence of a monopole can, in just the same way, be understood as the need to use more than one coordinate patch to describe the potential, a perfectly licit manipulation if we use gauge symmetry transformations to pass from one patch to another.

The formulation given by Wu and Yang [14] of the above fact leads to the mathematical meaning of magnetic charges. In the mathematical approach to monopoles, the sphere $S^2$ surrounding the monopole becomes the base space of a $U(1)$ (we are dealing with electromagnetism, with gauge group $U(1)$) principal fibre bundle. When a connection satisfying Maxwell’s equations is chosen, we get a description of a magnetic monopole; in local coordinates, the connection 1-form $\omega$ is written as

$$\omega = g^{-1}Ag + g^{-1}dg$$  \hspace{1cm} (2.10)

where $A(x) = A^a(x)\lambda_a/2i d x^\mu$ is the potential 1-form, with $\lambda_a/2i$ an element of the Lie algebra. For a $U(1)$ principal bundle $g = e^{i\psi}$, so the connection is simply

$$\omega = A + d\psi.$$  \hspace{1cm} (2.11)

Dividing $S^2$ into $H^+$ and $H^-$,

$$\omega = \begin{cases} 
A^+ + d\psi^+, & \text{on } H^+ \\
A^- + d\psi^-, & \text{on } H^-
\end{cases}$$  \hspace{1cm} (2.12)

The transition functions must depend on the coordinates describing the overlap region $H^+ \cap H^-$, so they must now be functions of $\varphi$; besides, the transition must take place through elements of the gauge group, so

$$e^{i\Psi^-} = e^{i\varphi} e^{i\Psi^+}.$$  \hspace{1cm} (2.13)

Equivalently, the potential 1-forms in the two hemispheres are related by

$$A^+ = A^- + n d\varphi,$$  \hspace{1cm} (2.14)

that is, $A^+_\mu$ and $A^-_\mu$ are related by a gauge transformation:

$$A^+_\mu = A^-_\mu + \partial_\mu \lambda.$$  \hspace{1cm} (2.15)
An important conclusion follows from equation (1.13): in order to make sure that the resulting structure is a manifold, the fibers must fit together exactly when completing full revolutions around the equator, so \( n \) must be an integer \( \mathbb{Z} \).

The monopole charge can be shown to coincide with the integral of the first Chern class \( c_1 \) for the Dirac monopole \( U(1) \) bundle over \( S^2 \): the first Chern class is, up to a \( 2\pi \) factor, the curvature 2-form \( F \) (the electromagnetic field strength); in fact,

\[
c_1 = -\frac{F}{2\pi} \tag{2.16}
\]

Applying Stoke’s theorem, with \( F = dA \) in mind, and taking into account that \( A \) is separately defined in \( H^+ \) and \( H^- \), we get

\[
\int_{S^2} c_1 = -\int_{S^2} \frac{F}{2\pi} = -\frac{1}{2\pi} \left[ \int_{H^+} dA^+ + \int_{H^-} dA^- \right] = -\frac{1}{2\pi} \int_{S^1} A^+ - A^- \tag{2.17}
\]

From (1.14), we then get

\[
\int_{S^2} c_1 = -\frac{1}{2\pi} \int_{S^1} n \, d\varphi = -n. \tag{2.18}
\]

Comparing now (2.15) and (2.18), we observe that the magnetic charge of the monopole can be directly interpreted as the winding number of the gauge transformation \( \partial \lambda \), which defines a map from the overlap region, the equator, to the gauge group \( U(1) \):

\[
\partial \lambda : S^1 \rightarrow U(1) \sim S^1. \tag{2.19}
\]

These maps are classified (see below) by the first homotopy group, \( \Pi_1(U(1)) \sim \mathbb{Z} \), with the corresponding integer number, given by (2.18), representing the winding number of the map.

In classical electrodynamics, the whole theory is described in terms of the electromagnetic field tensor \( F^\mu\nu \); however, when entering quantum theory the knowledge of \( F^\mu\nu \) does not allow us to determine the phase of the electron wave function, as the Aharanov-Bohm effect shows: the potential appears again as the proper tool in quantum theory, as it bears that information. When paralellly transporting a wave function along a path \( \Gamma \), it picks up a Dirac phase factor

\[
\exp[i \varepsilon \int_{\Gamma} A_\mu(x) \, dx^\mu], \tag{2.20}
\]

\footnote{For \( n = 0 \), we have a trivial bundle: \( S^2 \times S^1 \) \((S^1 \simeq U(1))\). When \( n = 1 \), we have the Hopf fibering of the sphere \( S^3 \), describing a Dirac monopole of charge one, so the Dirac monopole is a non trivial \( U(1) \) principal fibre bundle with base \( S^2 \).}

\footnote{To be precise, the curvature for a \( U(1) \) principal bundle is purely imaginary and can be written as \( \Omega = iF \).}
where $A_\mu$ is the potential due to a monopole. Then,

$$\Psi(P) \rightarrow \Psi(P') = \exp[ie \int \Gamma A_\mu dx^\mu] \Psi(P).$$

(2.21)

As for a closed trajectory (describing a closed trajectory with our electron amounts to looping once around the Dirac string) we must have

$$\Psi(P) = \Psi(P'),$$

(2.22)

the phase factor must become the identity, implying

$$eg = 2\pi n, \ n \in Z.$$  

(2.23)

This is Dirac quantization condition, and contains a deep consequence: the existence of magnetic charge implies the quantization of electric charge. There are many other different ways to get it; a topological version of it is the comment stated above following from equation (1.13).

From the Dirac quantization condition we notice an extremely interesting result: the interchange of the role played by electricity and magnetism (duality), obtained by exchanging the coupling constants, implies the interchange of strong and weak coupling.

It is important to stress that the Dirac magnetic monopole is not part of the spectrum of standard QED. Moreover, we can not define a local field theory possessing, as part of its physical spectrum, both electrons and Dirac monopoles. In order to use Dirac’s duality symmetry in a more fundamental way, we should look for local field theories which contain, as a part of the physical spectrum, both electrically and magnetically charged particles. As it was discovered by ’t Hooft and Polyakov, spontaneously broken non abelian gauge theories satisfying some topological criteria on its vacuum manifold possess classical field configurations, which are solutions to the equations of motion, topologically stable, magnetically charged, and have particle like behaviour.

### 2.2 The ’t Hooft-Polyakov Monopole.

Let us consider the Georgi-Glashow model. It consists of an $SO(3)$ gauge field, interacting with an isovector Higgs field $\phi$:

$$\mathcal{L} = -\frac{1}{4} F_{a\mu\nu} F^{a\mu\nu} + \frac{1}{2} D^{\mu} \phi \cdot D_{\mu} \phi - V(\phi), \ a = 1, 2, 3,$$

(2.24)

where $D^a_{\mu}$ is the covariant derivative, and $V(\phi)$ the Higgs potential:

$$V(\phi) = \frac{1}{4} \lambda (\phi^2 - a^2)^2.$$

(2.25)
In a Yang-Mills-Higgs configuration (a configuration where there is a gauge in which \( \partial_0 A = \partial_0 \phi = 0 \)) with no time component, \( A_0 = 0 \), the energy can be written in the simple form
\[
E = \int \left[ \frac{1}{2} |F|^2 + \frac{1}{2} |D\phi|^2 + V(\phi) \right] d^3 r.
\] (2.26)

Independently, 't Hooft and Polyakov [4] realized that the Georgi-Glashow model contains some remarkable finite energy solutions. As the Higgs potential appears in the energy density, the integral (2.26) can only converge if, for large distances, the Higgs field tends to the constant value \( a \); a configuration satisfying this is
\[
\phi_a = a \delta_{a3}.
\] (2.27)

Besides, the energy is gauge invariant, so any gauge rotation of this configuration is also a finite energy solution.

The set of \( \phi \) which minimizes \( V(\phi) \) constitutes the vacuum; in this case, the vacuum manifold \( V \), as follows from equation (2.24), is a two-dimensional sphere of radius \( a \). The structure of this manifold is determined by the gauge group \( G \), and a subgroup in it known as the little group:

The little group \( H \) consists of the elements of \( G = SO(3) \) leaving a given \( \phi \) invariant, so it is the group of rotations around the \( \phi \) axis (of radius \( a \)), that is, \( SO(2) \); equivalently, \( H \simeq U(1) \). Therefore, finite energy enforces the gauge group \( G \) to break, at large distances, down to \( H = U(1) \); the only field component remaining massless is that associated with the residual \( U(1) \), component we identify with the photon.

Assuming that \( G \) acts transitively on \( V \), that is, given two fields in \( V \) there is an element in \( G \) relating them \((\phi_1, \phi_2) \in V \Rightarrow \exists g_{12}, \phi_1 = D(g_{12})\phi_2 \), with \( D \) a representation of \( G \), the structure of the vacuum \( V \) is determined by \( G \) and \( H \): the vacuum is the space of right cosets of \( H \) in \( G \),
\[
V = G/H.
\] (2.28)

To understand the meaning of this finite energy solution, we must have a closer look around the region where it is located: so far, we have dealt with the asymptotic properties the solution must satisfy. In the regions where the gauge symmetry is unbroken, determining a solution to the field equations is a difficult task, unless some simplifying ansatz is used; the one used by 't Hooft and Polyakov is a spherically symmetric one. With this ansatz, the field strength behaves, at large distances, as
\[
F_{ij}^a \sim \frac{1}{ae r^3} \epsilon_{ijk} r^k \phi_a.
\] (2.29)

The only surviving component of this field is that associated with the neutral vector boson, the photon, and yields a magnetic field
\[
B^i = -\frac{1}{e} \frac{r^i}{r^3}.
\] (2.30)
Using Dirac quantization condition (and the fact that the smallest charge which might enter the theory is not $e$, but $\frac{1}{2}e$) we notice that, at large distances, our solution behaves like one unit of magnetic charge (that is, a monopole).

Therefore, the 't Hooft-Polyakov monopole, at short distances, mantains all fields excited, giving rise to an $SU(2)$ symmetric finite energy solution; at large distances the non-abelian symmetry is broken and all fields but the residual photon are unexcited, giving rise to a solution that resembles a Dirac monopole of unit magnetic charge. The ansatz given by 't Hooft and Polyakov allows to define a Compton wavelength, such that the monopole can be thought of as having a definite size, of the order $1/m_W$; in the inside, the massive fields provide a smooth structure and, in the outside, they vanish, leaving a field configuration indistinguishable from the Dirac monopole.

However, the smooth internal structure satisfying $SO(3)$ gauge theory equations of the 't Hooft-Polyakov monopole implies that there is no need to introduce string singularities, in contrast to the Dirac monopole.

To summarize, the field configurations obtained by 't Hooft and Polyakov satisfy:

- They are finite energy solutions.
- They represent magnetically charged states.

The monopoles contain, besides, a rich property, that of topological stability, as will be pointed in the next paragraph.

### 2.2.1 Topological Properties of Monopoles.

We should first notice that as for large distances $\phi$ does not depend on $r$, it provides a mapping from the sphere at infinity into the vacuum manifold $\mathcal{V}$ (which is also a two-sphere):

$$\phi : S^2 \to \mathcal{V}. \quad (2.31)$$

This map admits a winding number, an integer representing the number of times $\phi$ covers the sphere $\mathcal{V}$ as $(\theta, \varphi)$ covers once the sphere at infinity.

Another interesting remark, that of topological stability, comes from observing that outside the monopole (that is, for a radius larger than its Compton wavelength) the field configuration is close to a Higgs vacuum,

$$\mathcal{D}_\mu \phi = \partial_\mu \phi - eA_\mu \wedge \phi = 0. \quad (2.32)$$

The form of the potential satisfying (2.32) is

$$A_\mu = \frac{1}{a^2} e \phi \wedge \partial_\mu \phi + \frac{1}{a} \phi A_\mu, \quad (2.33)$$
and the field strength can be written

\[ F_{\mu\nu}^a = \frac{1}{a} \phi_a F_{\mu\nu} = \frac{1}{a} \phi_a \frac{1}{a^3 e} \phi (\partial^\mu \phi \wedge \partial^\nu \phi) + \partial^\mu A^\nu - \partial^\nu A^\mu. \] (2.34)

Besides, the equations of motion derived from the Georgi-Glashow lagrangian \((2.24)\), when combined with condition \((2.32)\), reduce to Maxwell’s equations \((2.1)\), a fact remembering us that outside the monopole the \(SO(3)\) gauge theory can not be distinguished from electromagnetic theory.

Let us now calculate the magnetic flux through a closed surface \(S\). Using \((2.34)\),

\[ g = \int_S B \, dS = -\frac{1}{2e a^3} \int_S \epsilon_{ijk} \phi \cdot (\partial^j \phi \wedge \partial^k \phi) \, dS^i. \] (2.35)

This expression is invariant under smooth deformations of the Higgs field: for a field variation \(\phi' = \phi + \delta \phi\), such that

\[ \phi \cdot \delta \phi = 0, \] (2.36)

the variation term

\[ \delta[\phi \cdot (\partial^j \phi \wedge \partial^k \phi)] = 3\delta \phi \cdot (\partial^j \phi \wedge \partial^k \phi) + \partial^j[\phi \cdot (\delta \phi \wedge \partial^k \phi)] - \partial^k[\phi \cdot (\delta \phi \wedge \partial^j \phi)] \] (2.37)

vanishes up since, on integration, the last two terms vanish by Stoke’s theorem, and the first, as \(\partial^j \phi \wedge \partial^k \phi\) is parallel to \(\phi\) (because \(\partial^i \phi \perp \phi\)), is zero due to \((2.38)\). Therefore, small variations in \(\phi\) do not modify the flux \(g\). This result can be extended to all changes which can be built from small deformations; these small deformations are called homotopies\(^3\).

If we write the magnetic flux as \(g = -4\pi N/e\), the integral \(N\), defined through

\[ N = \frac{1}{4\pi a^3} \int dS^i \frac{1}{2} \epsilon_{ijk} \phi \cdot (\partial^j \phi \wedge \partial^k \phi), \] (2.38)

turns out to be the winding number mentioned above characterizing the map \(\phi\): the number of times \(S^2\) is wrapped around \(\mathcal{V}\) by \(\phi\). It must therefore be an integer (a fact in agreement with Dirac quantization condition).

\(^3\)The maps \(\phi\) can be divided into equivalence classes under homotopy, two maps being in the same class if and only if they are continuously deformable into each other (homotopic). \(\phi\) defines a non trivial element of the second homotopy group \(\Pi_2(\mathcal{V})\):

\[ \Pi_2(SU(2)/U(1)) \sim \Pi_1(U(1)) \sim \mathbb{Z}. \]
2.2.2 BPS States.

As the monopole described so far is a smooth structure, a mass can be calculated; the expression for this mass was shown by Bogomolny to satisfy a simple bound:

\[ M \geq a |g| \]  \hspace{1cm} (2.39)

This bound can be saturated so, as was shown by Prasad and Sommerfield, we can obtain a solution with

\[ M = a |g| \]  \hspace{1cm} (2.40)

Monopoles satisfying this bound are called Bogomolny-Prasad-Sommerfield (BPS) monopoles [3].

The original ansatz by 't Hooft and Polyakov, leading to these simple expressions, was electrically neutral; however, the absence of electric charge is not a necessary condition coming from spherical symmetry: it is possible to obtain solutions, called dyons, containing both electric and magnetic charge. Again, a simple bound holds for the mass of these BPS (saturated) states:

\[ M = a(q^2 + g^2)^{1/2} \]  \hspace{1cm} (2.41)

If we use the relation between electric and magnetic charge coming from the asymptotic expression (1.29), \( g = -4\pi/e \), the mass formula for dyons with \( n_e \) units of electric charge, and \( n_m \) units of magnetic charge, can be written

\[ M = |a e(n_e + \tau_0 n_m)| \]  \hspace{1cm} (2.42)

where we have introduced a parameter \( \tau_0 \) containing the coupling:

\[ \tau_0 \equiv \frac{4\pi}{e^2} \]  \hspace{1cm} (2.43)

An important modification of equation (2.42), coming from the CP non invariant \( \theta \)-term in the lagrangian, was discovered by Witten [5]. In non abelian gauge theories, pure gauge vacuum configurations define, if we use the temporal gauge \( A_0 = 0 \), maps from the compactified space three sphere \( S^3 \) into the gauge group \( G \) (maps classified into homotopy classes by \( \Pi_3(G) \), which for \( G = SU(N) \) is the set of integer numbers, \( \mathbb{Z} \)).

Instantons are euclidean field configurations that tunnel between vacua in different topological classes. Denoting, in the temporal gauge \( A_0 = 0 \), by \( |n> \) the pure gauge vacuum corresponding to a pure gauge configuration characterized by the value \( n \) in \( \Pi_3(G) \), the net effect of instantons [13] is to define the \( \theta \)-vacuum as the coherent state \( |\theta> = \sum e^{i\theta} |n> \), with the tunneling amplitude \( <n|n+1> \) given, in semiclassical approximation, by \( e^{-S_{inst}} = e^{-8\pi^2/g^2} \) (\( S_{inst} \) is the classical action for the instanton). The generating function \( <\theta|\theta> \) is then given by

\[ <\theta|\theta> = \int dA \exp[-\int \mathcal{L} + \frac{\theta}{32\pi^2} \mathcal{F} \tilde{\mathcal{F}}] \]  \hspace{1cm} (2.44)
with $\mathcal{F}$ the dual field tensor. Now, we should take into account the effect of the $\theta$-parameter on the classical mass formula for the magnetic monopole. Witten’s result [5] is that the monopole in the presence of a non vanishing value of $\theta$ becomes effectively a dyon with electric charge $\frac{\theta}{2\pi}n_m$. This effect modifies the BPS mass formula (2.42) to

$$M = |ae \left(n_e + \tau n_m\right)|,$$

(2.45)

where now $\tau$ is defined by

$$\tau \equiv \frac{\theta}{2\pi} + i\frac{4\pi}{e^2}.$$

(2.46)

The appearance in such a natural way of the “complexified” coupling constant $\tau$ in the BPS mass formula is already a hint that some supersymmetry is hiddenly governing monopole dynamics. In fact, as supersymmetry practitioners know, the holomorphicity properties underlying non renormalization theorems are intimately connected with the complexification (2.46) of the coupling constant [16] (more on this can be found on next section).

One further comment is the manifest symmetry of (2.44) under the change $\theta \to \theta + 2\pi$. Witten’s dyon effect provides a new physical flavour to this innocent symmetry: the transformation $\theta \to \theta + 2\pi$ changes in a non trivial way the induced electric charge of the monopole:

$$\frac{\theta n_m}{2\pi} \to \frac{\theta n_m}{2\pi} + n_m$$

(2.47)

This transformation will marry, as we will describe at the end of this section, the old duality introduce by Dirac, to define the full duality group.

2.2.3 Quantum Mass Formulas.

The hidden supersymmetry that was trying to show up through the natural appearance of the complexified coupling constant $\tau$ in the classical BPS mass formula becomes manifest after the seminal comment by Olive and Witten [6], that points out that the BPS mass formula can be derived directly from the $N=2$ supersymmetry algebra once we take into account the existence of non vanishing central extensions in the Higgs phase. In fact, the usual supersymmetry algebra

$$\{Q_{\alpha i}, \bar{Q}_{\beta j}\} = \delta_{ij} \gamma^\mu_{\alpha\beta} P_\mu, \ i, j = 1, 2,$$

(2.48)

should be modified to include central terms [17]:

$$\{Q_{\alpha i}, \bar{Q}_{\beta j}\} = \delta_{ij} \gamma^\mu_{\alpha\beta} P_\mu + \delta_{\alpha\beta} U_{ij} + (\gamma_5)_{\alpha\beta} V_{ij}, \ i, j = 1, 2.$$

(2.49)

The central terms in the above expression verify $U_{ij} = -U_{ji}$, $V_{ij} = -V_{ji}$. 

12
For the $N=2$ supersymmetric extension of the Georgi-Glashow model,

$$
\mathcal{L} = -\frac{1}{4} F^\mu_{\nu} F_{\mu\nu} + \frac{1}{2} \bar{\psi} a_i D \psi^i_a + \frac{1}{2} D^\mu A_a D_\mu A_a + \frac{1}{2} D^\mu B_a D_\mu B_a + \\
\frac{1}{2} e^2 Tr[A, B][A, B] + \frac{1}{2} i \epsilon_{ij} Tr([\bar{\psi}^i, \psi^j] A + [\bar{\psi}^i, \gamma_5 \psi^j] B),
$$

(2.50)

and for non zero vacuum expectation value $<A>$, the central extensions become respectively

$$
U = <A> e \quad V = <A> g,
$$

(2.51)

where $g$ is the magnetic charge. Now, the algebra (2.49) can be seen to imply, for the mass of each particle, the relation

$$
M^2 \geq U^2 + V^2,
$$

(2.52)

that is, the Bogomolny bound. With the notation used so far,

$$
M^2 \geq a^2 (q^2 + g^2).
$$

(2.53)

The main interest of the previous result is that now we can claim that the bound (2.52) has not only classical, but also quantum mechanical meaning. In fact, if supersymmetry is not dynamically broken, then we can be sure that a formula like (2.52) will be exact, even after including all quantum, perturbative and non perturbative, corrections. The only thing we need is to use the $N=2$ supersymmetric algebra of the effective theory obtained after taking into account all quantum corrections.

$N=2$ supersymmetry does not only explain from a fundamental point of view the Bogomolny bound, but also clarifies the meaning of BPS saturated states: the question on when the above bound is saturated. Following the reasoning by Olive and Witten, concerns the representations of the $N=2$ supersymmetry algebra. An irreducible representation has $2^N$ helicity states for zero mass, and $2^{2N}$ states for nonzero mass, so we might wonder whether the irreducible representations of the extended supersymmetry algebra should have four or sixteen states; it turns out that representations with four helicity states (which are denoted “small irreps”) are the ones saturating the Bogomolny bound (2.53), while it is not saturated for representations with sixteen states.

Particles getting mass by the Higgs mechanism ($W^\pm$, monopoles, dyons), must be irreducible representations of the extended supersymmetry algebra; in fact, for $<A> \neq 0$ the central terms are non vanishing. Moreover, if they get mass by the Higgs mechanism, which does not change the number of degrees of freedom, they must have 4 helicity states, as massless particles have, and therefore they will transform under $N=2$ supersymmetry as small irreps, so that they will be BPS states.
2.3 Duality.

In this section we will use the symmetries of mass formula for BPS saturated states to define the duality group. To do so, let us introduce some notation to rewrite the mass formula (2.45),

\[ M = |ae(ne + \tau nm)|, \quad \tau = \frac{4\pi}{e^2} + \frac{\theta}{2\pi}, \]

in a more convenient form: if we define

\[ a \equiv a \cdot e, \quad a_D \equiv \tau a, \quad (2.54) \]

then the mass spectrum will be given by

\[ M = |an_e + a_D n_m| . \quad (2.55) \]

Now, we know that shifting the \( \theta \) angle by \( 2\pi \), \( \theta \to \theta + 2\pi \), should have no effect. Such a shift amounts to

\[ \tau \to \tau + 1 \quad (2.56) \]

or, in terms of our new variables,

\[ a \to a \quad a_D \to a + a_D . \quad (2.57) \]

In order to make sure that the mass spectrum is not modified, (2.57) should be accompanied by the change

\[ (n_e, n_m) \to (n_e - n_m, n_m) . \quad (2.58) \]

The shift (2.56) is a fractional linear transformation,

\[ \tau \to \frac{a\tau + b}{c\tau + d} , \quad (2.59) \]

with \( ad - bc \neq 0 \), where now \( a = b = d = 1, c = 0 \). In matrix form,

\[ \begin{pmatrix} a_D \\ a \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_D \\ a \end{pmatrix} = \begin{pmatrix} a + a_D \\ a \end{pmatrix} . \quad (2.60) \]

The matrix appearing in the above expression is known as \( T \):

\[ T \equiv \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} . \quad (2.61) \]

It is an element of \( Sl(2, Z) \), the group ("special linear group") of \( 2 \times 2 \) matrices of unit determinant, with integer entries: \( T \in Sl(2, Z) \). This special group is also known as the (full) modular group.
Notice from (2.58) that the effect of $T$ on the mass spectrum is simply Witten’s effect (2.47).

Now let us consider Dirac’s electric-magnetic transformation (2.5):

\[ n_e \rightarrow n_m \quad n_m \rightarrow -n_e. \]  

(2.62)

No change will appear in the mass formula if we also perform

\[ a \rightarrow a_D \quad a_D \rightarrow -a. \]  

(2.63)

Again, this is a fractional linear transformation, with $a = d = 0, b = -1, c = 1$:

\[ \tau \rightarrow -\frac{1}{\tau}. \]  

(2.64)

In matrix form,

\[ \left( \begin{array}{c} a_D \\ a \end{array} \right) \rightarrow \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) \left( \begin{array}{c} a_D \\ a \end{array} \right) = \left( \begin{array}{c} -a \\ a_D \end{array} \right), \]  

(2.65)

with

\[ S \equiv \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) \in Sl(2, Z). \]  

(2.66)

The electric and magnetic variables exchange generated by $S$ leads to a strong-weak coupling transformation,

\[ S : \tau \rightarrow -\frac{1}{\tau} \]  

(2.67)

(Notice that it is $\tau \equiv i \frac{4\pi}{e^2} + \frac{\theta}{2\pi}$, and not $e$, which is properly inverted.)

Now we can combine Dirac’s duality $S$ with Witten’s effect as defined by $T$. These two transformations generate the modular group $Sl(2, Z) \equiv \Gamma \ddash$, an ubiquitous symmetry group in physics. This group is defined by the relations

\[ S^4 = \mathbb{I} \quad (ST)^3 = \mathbb{I}. \]  

(2.68)

We will call this group the duality group. The complexified coupling constant $\tau$ is a number living in the complex upper half plane, once we impose on it the physical constraint of positivity. The action of the duality group on this plane is defined by (2.58).

The existence of a fundamental domain for $Sl(2, Z)$ shows how difficult will be to find a theory invariant under duality transformations, if such a theory has non vanishing $\beta$-functions (i.e., non trivial running coupling constant). The necessary condition will be that the renormalization group trajectories be concentrated in the fundamental region.

The Montonen-Olive conjecture, that the symmetry under the duality group is an exact symmetry of some quantum theory, implies that strong coupling is equivalent to the weak coupling limit, with particles and solitons exchanged.
When the $\beta$-function is zero, the duality conjecture proposed by Montonen and Olive can be directly expressed as the modular invariance of the partition function. Let us then think of a theory with a lagrangian $\mathcal{L}(A,\psi,\bar{\psi},\phi;e,\theta)$; the Montonen-Olive duality conjecture states that the partition function

$$Z(e,\theta) = \int DA D\psi D\bar{\psi} D\phi e^{-\int \mathcal{L}(A,\psi,\bar{\psi},\phi;e,\theta)} \equiv Z(\tau)$$  \hspace{1cm} (2.69)

remains invariant under the action of the duality group; therefore, the transformations generated by $T$ and $S$ must leave $Z$ invariant:

$$Z(\tau) = Z(\tau + 1)$$

$$Z(\tau) = Z(-1/\tau)$$  \hspace{1cm} (2.70)

Thus, invariance under the duality group $Sl(2,Z)$ means that the partition function $Z$ must be a modular form.

A two dimensional example of duality in a theory is that given by string $T$-duality, where

$$Z(G,B) = \int DX \exp\left[\int (G^{ij}\partial X^i\partial X^j + \epsilon B^{ij}\partial X^i\partial X^j) d^2z\right].$$

As the $\beta$-functions vanish, $\beta_G = \beta_B = 0$, $Z(\tau) = Z(-1/\tau)$, with $\tau \equiv iG + B$.

Candidates to four dimensional dual theories are those with vanishing $\beta$-function: $N=4$ supersymmetric Yang-Mills (the original place where Montonen-Olive duality used to live), and $N=2$ supersymmetric $SU(N_c)$ QCD with $N_f=2N_c$.

In next section we will study the extension of duality given by Seiberg and Witten to the context of $N=2$ supersymmetric theories.

### 3 Duality and Effective Field Theories.

#### 3.1 Flat Potentials and Classical Moduli.

The existence of flat potentials is generic in $N=1$ and $N=2$ supersymmetric gauge theories, as well as in the low energy limit of string theories. The best way to characterize the special features of a gauge theory possessing a flat potential is using the concept of a classical moduli. In the previous section we have introduced the vacuum manifold $\mathcal{V}$, defined as the set of all the vacuum field configurations. Different points in $\mathcal{V}$ can be parametrized by the corresponding vacuum expectation values of the scalar fields entering into the theory. Given a generic point $P \in \mathcal{V}$ we define the gauge group $H_P$ as the part of the gauge symmetry group $G$ of the lagrangian consisting of symmetries of the vacuum
state parametrized by $P$. Different points $P, P'$ in $\mathcal{V}$ will describe the same physics if we can reach $P'$ from $P$ by acting with some element of the gauge symmetry group $G$, i.e., $P' \in G/H_P$ or, in other words, $P$ and $P'$ are related by some Goldstone boson.

Now, we define the *moduli space* $\mathcal{M}(\mathcal{V})$ as the space of equivalence classes of vacua, where two vacua will be in the same equivalence class if they describe the same physics, i.e., if they are related by the action of Goldstone bosons. If the theory we start with possesses a flat potential then the moduli $\mathcal{M}(\mathcal{V})$ will be a connected manifold with dimension bigger or equal to one.

A more geometrical description of the moduli can be done as follows. Given a generic point $P \in \mathcal{V}$ we can decompose the tangent space $T_P$ as:

$$T_P = T^G_P \otimes T^M_P,$$

(3.1)

where the generators of $T^G_P$ are the Goldstone bosons, and the generators of $T^M_P$ are properly speaking the tangents to the moduli directions. The dimension of $T^G_P$ is given by the dimension of the homogeneous space $G/H_P$.

*Singularities* in $\mathcal{M}(\mathcal{V})$ will appear at points $P$ where jumps in the dimension of $T^G_P$ take place. The meaning of these singularities is clear from the physical point of view, namely they correspond to points in $\mathcal{V}$ where the symmetry of the vacuum $H_P$ changes.

It is clear, from the definition of $\mathcal{M}(\mathcal{V})$, that in order to define good coordinates we should use gauge invariant quantities. To fix ideas, let us consider a concrete example:

For $N = 2$ supersymmetric Yang-Mills with gauge group $G = SU(2)$ the potential for the complex scalar field $\phi$ is given by

$$V(\phi) = \frac{1}{g^2} Tr[\phi, \phi^\dagger]^2,$$

(3.2)

with $g$ the coupling constant. The flat direction is defined by

$$\phi = \frac{1}{2} a \sigma^3,$$

(3.3)

where $a$ is a complex parameter, and $\sigma^3$ is the diagonal Pauli matrix. For $a \neq 0$ the gauge symmetry is spontaneously broken to $U(1)$. Vacuum states corresponding to values $a$ and $-a$ are equivalent, since they are related by the action of the Weyl subgroup of $SU(2)$. A gauge invariant parametrization of $\mathcal{M}(\mathcal{V})$ in this case can be defined in terms of the expectation value of the Casimir $Tr \phi^2$:

$$u \equiv Tr \phi^2 = \frac{1}{2} a^2.$$

(3.4)

---

4We follow the conventions of [10], so the Higgs field is normalized so that its kinetic term is multiplied by $1/4\pi\alpha = 1/g^2$.3
For $u = 0$ we find a singularity of $\mathcal{M}(\mathcal{V})$. In fact, at this point there is an enhancement of the gauge symmetry, and $H(u=0) = SU(2)$, while for all the other points, with $u \neq 0$, we have $H(u \neq 0) = U(1)$. In this example, the moduli $\mathcal{M}(\mathcal{V})$ is simply the complex plane punctured at the origin $^5$.

### 3.2 The Quantum Moduli.

Generically we expect that the flat directions of the classical potential disappear once we take into account perturbative and non-perturbative quantum corrections. Notice that different points in the moduli $\mathcal{M}(\mathcal{V})$ correspond to inequivalent physical theories; for instance, for a theory with spontaneously broken gauge symmetry and non trivial moduli space $\mathcal{M}(\mathcal{V})$, the mass of the gauge vector boson will be different for different points in $\mathcal{M}(\mathcal{V})$. From physical grounds we expect that after including the quantum corrections the vacuum degeneracy will be lifted and one particular theory will be selected. If this is not the case, and after including all quantum corrections the flat direction remains flat, then we will be able to define the moduli $\mathcal{M}_q(\mathcal{V})$ of the complete quantum theory; this manifold will be called the *quantum moduli*.

The first question we should consider in such a case is in what way the geometry of $\mathcal{M}_q(\mathcal{V})$ will differ from that of the classical moduli $\mathcal{M}(\mathcal{V})$. Clearly, the differences between $\mathcal{M}(\mathcal{V})$ and $\mathcal{M}_q(\mathcal{V})$ will reside in the location and number of singularities. In fact, as we have already explained, singularities are associated with jumps in the symmetry invariance of the vacuum; more concretely, they appear whenever a charged particle in the spectrum becomes massless. The simplest example was the singularity of $\mathcal{M}(\mathcal{V})$, for $SU(2)$ $N = 2$ SYM, at the origin, where the gauge vector bosons $W^\pm$ become massless and the gauge invariance is classically restored.

When quantum mechanical effects are turned on, a classical singularity can disappear if the associated massless particle is not quantum mechanically stable. In the same way, some new singularities can appear whenever a classical massive particle becomes massless once quantum corrections are taken into account.

But before entering into a detailed study of the singularities of the quantum moduli $\mathcal{M}_q(\mathcal{V})$, we should consider the previous question of the very existence of the quantum moduli $^5$:

**Theorem**  
For $N = 2$ supersymmetric Yang-Mills the classical flat direction $(3.3)$ remains flat after quantum corrections, perturbative and non perturbative.

---

$^5$The concept of moduli in the sense described above should be familiar to practitioners of conformal field theory (CFT). Given a CFT its moduli is generated by the set of truly marginal operators. The singularities of the moduli correspond to those points where some relevant operator becomes truly marginal. The metric of the moduli space is known in CFT as the Zamolodchikov’s metric $^{20}$. In the context of string theory the flat directions of the four dimensional low energy physics are associated with the moduli of the CFT used to characterize the internal space-time.
The proof of this statement goes as follows. First, we observe that the only way to generate a superpotential for the $N=1$ chiral matter superfield $\Phi$ is by breaking the extended $N=2$ to $N=1$. In fact, the most general $N=2$ invariant theory is described by

$$
\mathcal{L}_{\text{eff}} = \text{Im} \left[ \int d^2\theta \, d^2\bar{\theta} \frac{1}{2} K(\Phi^a, \bar{\Phi}^a, V^a) + \int d^2\theta f_{ab}(\Phi) W^a W^b + \text{cc} \right],
$$

(3.5)

where $f_{ab}(\Phi)$ is a holomorphic function, and $K$ is a Kähler potential. Quantum corrections will determine the specific form of $K$ and $f$ in (3.5). The second step in the proof uses the fact that Witten’s index $\text{tr} (-1)^F$, for $N=2$ supersymmetric Yang-Mills, is different from zero \cite{21}, which automatically implies that supersymmetry is not broken dynamically. Combining these two facts, the theorem follows.

### 3.3 Effective Field Theory description of the Quantum Moduli.

Let us consider an $N=2$ supersymmetric gauge theory\footnote{Here we consider the case of a gauge group $SU(N)$.} possessing a non trivial quantum moduli $\mathcal{M}_q(V)$. For each point $P \in \mathcal{M}_q(V)$, let us denote by $S_{\text{pec}}(P)$ the corresponding mass spectrum. Some particles in $S_{\text{pec}}(P)$ will become massive by the Higgs mechanism with respect to the vacuum expectation values which parametrize the point $P$ in the flat direction. To define an effective field theory at $P$ requires:

- To split $S_{\text{pec}}(P)$ into light and heavy particles.
- To integrate the heavy particles.

Using the theorem discussed in the previous paragraph we can conclude that the effective theory at $P$ will be described, up to higher derivative terms, by a Lagrangian of the type (3.5) for a set of $N=2$ hypermultiplets $\Psi(P)$ that describe the light particles. More precisely, the Kähler potential $K$ and the holomorphic function $f$ in (3.5) will be determined by the integration of the heavy modes, and the $N=2$ hypermultiplets entering into the lagrangian will correspond to the light particles. Notice that we should split the spectrum into light and heavy modes in a way consistent with $N=2$ supersymmetry.

Now, we define effective field theory coordinates of the point $P \in \mathcal{M}_q(V)$ by the expectation values of the scalar components of the $N=2$ hypermultiplets describing the light modes. Again, as a concrete example, let us consider the case of $SU(2)$ $N=2$ supersymmetric Yang-Mills: For a point with a value of $u \neq 0$ the spectrum of massive particles is determined by the Higgs mechanism, and the light modes will be described by one $N=2$ hypermultiplet containing the $U(1)$ photon.

The question we should address now is the range of validity of some “EFT-coordinates”. Two are the general criteria we must take into account:
1. The range of validity of the split of the spectrum,

\[ S_{pec}(P) = S_{pec}^{light}(P) \oplus S_{pec}^{heavy}(P) \]  

into light and heavy modes. We should require that when moving continuously in \( \mathcal{M}_q(\mathcal{V}) \) light particles go into light, and heavy into heavy. The effective field theory description around a point \( P \) will break down whenever we reach a point in \( \mathcal{M}_q(\mathcal{V}) \) such that a heavy particle becomes light. This is a similar phenomena to a spectral flow (see Figure 1).

![Spectral flow in the quantum moduli](image)

Figure 1: Spectral flow in the quantum moduli.

The net effect, at the level of the effective field theory, of some heavy particle becoming light is, when this particle is charged with respect to some gauge group \( G \), that the corresponding effective coupling constant \( g \) develops a logarithmic singularity, of the type

\[ \frac{1}{g^2} \sim \ln(m), \]  

with \( m \) the mass of the light particle.

2. The second general criteria for the validity of the effective field theory description is, of course, to reduce ourselves to regions in \( \mathcal{M}_q(\mathcal{V}) \) for which the corresponding effective field theory is weakly coupled.

The two criteria just described give us already some hints on what is going to be the “EFT-geometry” of the quantum moduli space \( \mathcal{M}_q(\mathcal{V}) \). In fact, we are going to need different EFT-coordinates to cover the whole quantum moduli. In the overlapping regions, where one should pass from some EFT-coordinates to others, we will need to use a change of EFT-coordinates satisfying some conditions: namely, to be isometries of the quantum
moduli Zamolodchikov’s metric. It is at this point where the concept of duality will play
an important role, as we will see in the rest of this notes.

Before ending this introductory section on the EFT-geometry of the quantum moduli,
let us briefly come back to the criterion 1, in connection with the structure of singularities
of $\mathcal{M}_q(\mathcal{V})$. Recall that singularities in $\mathcal{M}_q(\mathcal{V})$ were also associated with some massive
particle in the spectrum becoming massless. From our previous discussion, it becomes
clear that the structure of singularities of $\mathcal{M}_q(\mathcal{V})$ is telling us how many different local
EFT-coordinates we will need to use in order to cover the whole quantum moduli.

3.4 Duality and EFT-Geometry.

3.4.1 Wilsonian Effective Theories.

It has become traditional, after the work of the russian school [16], to differentiate between
the effective action interpreted as the 1PI generating functional, and what is known
as the Wilsonian effective action. The Wilsonian effective action, $S_W(\mu)$, is defined by
integrating the vacuum loops with virtual momentum $p > \mu$. Thus, the difference between
the wilsonian effective action and the 1PI generating functional depends on the infrared
region, where $p < \mu$. Denoting by $g_W(\mu)$ the wilsonian effective coupling, and using for
$\mu$ the mass scale determined by the Higgs mechanism, i.e., the vacuum expectation value
of the scalar field, we can define $1/g_W^2(\mu)$ as a function on $\mathcal{M}_q(\mathcal{V})$. In most cases it is
important to differentiate between the wilsonian effective coupling $g_W(\mu)$ and the effective
coupling $g_{eff}(\mu)$, defined as the coefficient of the corresponding 1PI vertex for external
momentum equal to $\mu$. In particular, the physical $\beta$-function, $\beta(g)$, is defined for $g_{eff}(\mu)$,
and not for $g_W(\mu)$.

To see the difference between $g_{eff}(\mu)$ and $g_W(\mu)$ it is convenient to use the so called
Konishi anomaly [22]. To illustrate the phenomena, let us just consider the simpler case
of $N=1$ SQED. The wilsonian effective action is

$$\mathcal{L}_W = \frac{1}{4g_W^2(\mu)} \int d^2\theta WW + \frac{Z(\mu)}{4} \int d^4\theta (\bar{T}e^V T + \bar{U}e^{-V}U),$$

with

$$\frac{8\pi^2}{g_W^2(\mu)} = \frac{8\pi^2}{g_0^2} + 2\ln\left(\frac{\Lambda}{\mu}\right),$$

where $\Lambda$ is the ultraviolet cut-off. The D-term in (3.8) can be written, using standard
superfield notation, as follows:

$$\int d^4\theta(\bar{T}e^V T + \bar{U}e^{-V}U) = -\frac{1}{2} \int d^2\theta D^2(\bar{T}e^V T + \bar{U}e^{-V}U).$$

By using the Konishi anomaly relation,

$$ZD^2(\bar{T}e^V T + \bar{U}e^{-V}U) = \frac{1}{2\pi^2} W^2,$$
we easily get the extra “infrared” contribution to \( g_{\text{eff}}(\mu) \):
\[
\frac{8\pi^2}{g_{\text{eff}}^2(\mu)} = \frac{8\pi^2}{g_0^2} + 2 \ln \left( \frac{\Lambda}{\mu} \right) - 2 \ln Z(\mu) = \frac{8\pi^2}{g_W^2(\mu)} - 2 \ln Z(\mu)
\]
(3.12)
and, in this way, the explicit relation between the wilsonian effective coupling and the effective coupling for \( N=1 \) SQED.

### 3.4.2 The \( N=2 \) Prepotential.

One of the main characteristics of \( N=1 \) supersymmetric theories, which is at the origin of non renormalization theorems, is the holomorphic dependence of the wilsonian effective coupling on the scale \( \mu \). As it can be observed from the previous argument based on the Konishi anomaly, this is not in general true for the effective coupling constant \( g_{\text{eff}}(\mu) \), which contains pieces of an infrared origin coming from \( D \)-terms in the lagrangian which are not holomorphic. This phenomenon is the field theory analog of the string holomorphic anomaly.

For \( N=2 \) supersymmetric theories holomorphic constraints are stronger than for \( N=1 \). In fact, a generic \( N=2 \) effective theory can be completely described, in \( N=2 \) superfield notation, by means of the lagrangian
\[
L = \text{Im} \int d^4 \theta F_{\text{eff}}(\Psi)
\]
(3.13)
with \( \Psi \) representing the set of \( N=2 \) hypermultiplets, and \( F_{\text{eff}}(\Psi) \) the prepotential, which is a holomorphic function. To pass from (3.13) to (3.5), we use
\[
f_{ab}(\Phi) = \frac{\partial^2 F}{\partial \Phi^a \partial \bar{\Phi}^b},
\]
(3.14)
\[
K(\Phi^a, \bar{\Phi}^b, V) = \partial_a F \cdot (e^V)_{ab} \bar{\Phi}^b.
\]
(3.15)
Notice that the whole \( N=2 \) lagrangian (3.13) is, in \( N=2 \) superfield notation, of the same type as the \( N=1 \) \( F \)-term superpotential, and therefore we can extend the \( N=1 \) non renormalization theorems of the superpotential to the whole \( N=2 \) lagrangian.

### 3.4.3 Zamolodchikov’s metric of the Quantum Moduli.

Let us now come back to the quantum moduli \( \mathcal{M}_q(V) \). As we have discussed in Section 3.3, we parametrize points in \( \mathcal{M}_q(V) \) by the expectation value of the scalar component of the hypermultiplet \( \Psi(P) \) used to describe the light modes in \( S_{pec}(P) \). Moreover, the effective lagrangian is given by \( F_{\text{eff}}(\Psi(P)) \), where the effective prepotential is obtained integrating the massive (heavy) modes in \( S_{pec}(P) \).

Now, we can use a \( \sigma \)-model inspired approach to the geometry of \( \mathcal{M}_q(V) \). Based on relation (3.14), and taking now into account that we are using the scalar components of...
the chiral multiplets Φ as the coordinates of \( \mathcal{M}_q(\mathcal{V}) \), we can use the holomorphic function \( f_{ab}(\Phi) \) to define the Zamolodchikov’s metric of \( \mathcal{M}_q(\mathcal{V}) \). The physical meaning of this metric is now specially clear from equation (3.3). Namely, and for the simplest case of a one complex dimensional quantum moduli, we get:

\[
\frac{4\pi}{g_W^2(\mu)} = \text{Im} f(\Phi) \quad \text{and} \quad \frac{\theta_W(\mu)}{2\pi} = \text{Re} f(\Phi),
\]

(3.16)

where, as usual, \( \mu \) here refers to the expectation value of the scalar part in the multiplet \( \Phi \), and we have introduced an effective “wilsonian” \( \theta \)-parameter as the real part of \( f(\Phi) \).

From (3.14), the relation between the Zamolodchikov’s metric on \( \mathcal{M}_q(\mathcal{V}) \) and the renormalization group of the theory is manifest.

### 3.4.4 Dual Coordinates.

We will now come back to the notation used in section 1, besides working, for simplicity, with a one complex dimensional quantum moduli. Denoting then by \( a \) the expectation value of the scalar field, the tree level prepotential for the scalar component is given by

\[
\mathcal{F}^{(0)}(a) = \frac{1}{2} \tau^{(0)} a^2,
\]

(3.17)

with \( \tau^{(0)} = \frac{i4\pi}{g^{(0)}} + \frac{\theta^{(0)}}{2\pi} \). The tree level value \( \theta^{(0)} \) is equal to zero. Now, using the same notation as in section 2.3, we introduce the dual variable \( a_D \) as:

\[
a_D \equiv \tau^{(0)} a = \frac{\partial \mathcal{F}^{(0)}}{\partial a}.
\]

(3.18)

The generalization of (3.18) for the effective field theory defined by \( \mathcal{F}_{\text{eff}}(a) \) is just

\[
a_D \equiv \frac{\partial \mathcal{F}_{\text{eff}}(a)}{\partial a}.
\]

(3.19)

A physical way to check if definition (3.19) of the dual variable is meaningful would be computing the mass of BPS states of the effective theory or, in other words, to find the central extensions of the \( N=2 \) supersymmetric algebra for the effective field theory defined by \( \mathcal{F}_{\text{eff}}(a) \). In fact, the mass formula for the BPS-saturated states of the effective theory is given by

\[
M^2(n_e, n_m) = | a n_e + \frac{\partial \mathcal{F}_{\text{eff}}}{\partial a} n_m |^2,
\]

(3.20)

in agreement with definition (3.19).

Summarizing, at each point \( P \in \mathcal{M}_q(\mathcal{V}) \) we have introduced the following set of geometrical objects:
i) The effective field theory coordinate \( a(P) \), defined by the expectation value of the scalar component of the hypermultiplet \( \Psi(P) \), which describes the light modes in \( S_{pec}(P) \).

ii) The effective prepotential \( F_{eff}(a(P)) \) obtained by integrating the heavy modes in \( S_{pec}(P) \).

iii) The dual coordinate \( a_D = \frac{\partial F_{eff}(a(P))}{\partial a} \), in terms of which we reproduce the BPS mass formula derived from the centrally extended supersymmetric algebra of the effective theory.

iv) The Zamolodchikov’s metric \( g(P) = \frac{\partial^2 F_{eff}(a(P))}{\partial a^2} \) of \( M_q(V) \) at the point \( P \).

### 3.4.5 Duality Transformations.

In terms of the dual coordinate \( a_D(P) \), the Zamolodchikov’s metric can be written as follows:

\[
ds^2 = Im \frac{\partial^2 F_{eff}(a(P))}{\partial a \partial \bar{a}} da \, d\bar{a} = Im \, d\bar{a} \, da_D.
\]

(3.21)

which is manifestly invariant under the \( S \)-duality transformation

\[
S : \left( \begin{array}{c} a_D \\ a \end{array} \right) \rightarrow \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \left( \begin{array}{c} a_D \\ a \end{array} \right) = \left( \begin{array}{c} a \\ -a_D \end{array} \right).
\]

(3.22)

Moreover, in the dual variables the metric \( g(P) \) becomes

\[
g^D(P) = -\frac{1}{g(P)}.
\]

(3.23)

The physical meaning of the dual coordinates \( a_D(P) \) can be easily understood in the general effective field theory framework we have used in our previous discussion. Let us denote by \( S_{pec}^{(l)}(P) \) the light part of the spectrum that we are describing by means of the \( N=2 \) hypermultiplet \( \Psi(P) \), with scalar component \( a(P) \). Now, we can formally define \( S_{pec}^{(l)}(P) \) as the dual of \( S_{pec}^{(l)}(P) \). Particles in \( S_{pec}^{(l)}(P) \) are related to the ones in \( S_{pec}^{(l)}(P) \) by interchanging electric with magnetic charge. Therefore, in the case of \( N=2 \) \( SU(2) \) SYM, \( S_{pec}^{(l)}(P) \) is described by the \( N=2 \) hypermultiplet of the unbroken \( U(1) \) photon, and \( S_{pec}^{(l)}(P) \) will be described by a new hypermultiplet \( \Psi^D(P) \), containing a “dual” photon. The dual coordinate \( a_D(P) \) will represent the scalar component of the \( N=2 \) hypermultiplet \( \Psi^D(P) \).

Equation (3.23) implies that if the effective field theory for \( \Psi(P) \) is weakly (strongly) coupled, then the effective field theory for \( \Psi^D(P) \) will be strongly (weakly) coupled. Using now that the duality transformation (3.22) is an isometry of the Zamolodchikov’s metric, we can try to use duality to extend, beyond the weak coupling regime, the range of validity of a set of EFT-coordinates.
3.5 \( N=2 \) Non Renormalization Theorems.

The popular way to prove non renormalization theorems in supersymmetric theories is using the multiplet of anomalies argument \[23\]. This argument is based on the Adler-Bardeen theorem for the \( U(1) \) axial anomaly, and the fact that by supersymmetry the axial and the conformal currents are in the same hypermultiplet. From these two facts, formally follows that the \( \beta \)-function is saturated by one loop contributions. This argument is known to be wrong for \( N=1 \) supersymmetric theories, the reason being that the supersymmetric partner of the conformal anomaly is not the one to which the Adler-Bardeen theorem applies. The physical origin of this problem is the same already discussed in section 3.4, concerning the differences between the wilsonian and the effective coupling constant. In fact, a “wilsonian \( \beta \)-function” would be saturated by one loop corrections; however, this is not the case for the standard \( \beta \)-function, as can be easily observed from equation (3.12). For \( N=2 \) supersymmetric theories, the situation changes dramatically, and the multiplet of anomalies argument produces the right result \[24\]. Starting from equation (3.13) for the effective lagrangian, the form of \( \mathcal{F}_{\text{eff}}(\Psi) \) can be fixed using:

\[ i) \text{ Holomorphy of the effective prepotential, and} \]
\[ ii) \text{ The } U(1) \text{ axial anomaly.} \]

From the tree level prepotential \( \mathcal{F}^{(0)}(\Psi) = \frac{1}{2} \tau^{(0)} \Psi^2 \), we fix the \( U(1) \) \( R \)-charges of the \( N=2 \) hypermultiplet:
\[ R(\Psi) = 2. \quad (3.24) \]

The \( U(1) \) axial anomaly implies that \( \mathcal{L}_{\text{eff}} \) is transforming under the \( U(1) \) axial transformations as
\[ \delta_\alpha \mathcal{L}_{\text{eff}} = \frac{\alpha}{4\pi^2} \mathcal{F} \tilde{F}. \quad (3.25) \]

Using (3.13), (3.23), and the condition of holomorphy we derive the non renormalization theorem:
\[ \mathcal{F}_{\text{eff}}(\Psi) = \frac{1}{8g^2} \Psi^2 \left[ 1 + \frac{g^2}{4\pi^2} \ln \left( \frac{\Psi^2}{\Lambda^2} \right) \right], \quad (3.26) \]

which implies for the effective coupling defined by
\[ \text{Im} \frac{\partial^2 \mathcal{F}_{\text{eff}}}{\partial \Phi \partial \Phi} \] the following renormalization group relation:
\[ \frac{1}{g_{\text{eff}}^2(a)} = \frac{1}{g_0^2} + \frac{1}{4\pi^2} \ln \left( \frac{a^2}{\Psi^2} \right), \quad (3.28) \]

where we have already denoted by \( a \) the scalar part of the \( N=2 \) superfield \( \Psi \).
For $N=1$ supersymmetric theories, the previous way to prove the non renormalization theorems can be directly applied to the superpotential $F$-term of the lagrangian, which is also constrained by holomorphicity and $R$-symmetries. The difference with $N=2$, concerning the coupling constant, and therefore the non renormalization theorems for the $\beta$-function, is that in $N=1$ theories the effective coupling constant gets contributions through the Konishi anomaly from $D$-terms in the lagrangian, which are not constrained by holomorphicity.

3.6 The Singularity at Infinity.

For $N=2$ supersymmetric $SU(2)$ Yang-Mills we can use $\mathcal{F}_{\text{eff}}(\Psi)$, as given by equation (3.26), to define the effective field theory for the light modes, i.e., for $\Psi$ representing the $N=2$ gauge field of the photon. From equation (3.28), we know that this effective field theory will be “reasonable” in the neighbourhood of $a = \infty$, where the theory is weakly coupled. We can now compactify the moduli space $\mathcal{M}_q(V)$ by adding the point at infinity; therefore, with this compactification, the point at $\infty$ appears as a singularity in $\mathcal{M}_q(V)$.

In fact, this singularity has a quantum origin, as it comes from the logarithmic term in equation (3.28).

To understand the physical meaning of this singularity, we can use the dual coordinate introduced in equation (3.19). From (3.26) and (3.19), we get

$$a_D = \frac{2ia}{\pi} \ln \left( \frac{a}{\Lambda} \right) + \frac{ia}{\pi}, \quad (3.29)$$

which implies that at the point at $\infty$ the mass of the magnetic monopole becomes infinite.

The effective field theory described by (3.26) represents the weak coupling regime of the electric light modes, once we have integrated the particles becoming massive by the Higgs mechanism. The parametrization of $a$ in terms of the gauge invariant coordinate $u$, can be defined by equation (3.4):

$$a(u) = \sqrt{2}u. \quad (3.30)$$

Using (3.29) and (3.30), we can compute the monodromy induced by the singularity at $\infty$, by looping in the physical parameter $u$:

$$a(e^{2\pi i}u) = -a(u)$$
$$a_D(e^{2\pi i}u) = -\frac{2ia}{\pi} \ln \left( \frac{a}{\Lambda} \right) - \frac{ia}{\pi} + 2a = -a_D + 2a. \quad (3.31)$$

With the matrix notation also used in section 1, we get:

$$\begin{pmatrix} a_D \\ a \end{pmatrix} \longrightarrow \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a_D \\ a \end{pmatrix} = M_\infty \begin{pmatrix} a_D \\ a \end{pmatrix}, \quad (3.32)$$

26
which in terms of the $Sl(2, Z)$ generators is given by:

$$M_\infty = PT^{-2},$$  \hspace{1cm} (3.33)

where $P = -I$.

From equation (3.26), and the definition of the Zamolodchikov’s metric, we get the transformation of $g(P)$ under the monodromy $M_\infty$:

$$g(e^{2\pi i P}) = \frac{-g + 2}{-1}$$  \hspace{1cm} (3.34)

which from (3.16) simply means

$$\theta_{eff} \rightarrow \theta_{eff} - 4\pi,$$  \hspace{1cm} (3.35)

a perfectly nice symmetry of the effective theory. This result is in fact general: any monodromy around a singularity of $\mathcal{M}_q(\mathcal{V})$ defines an exact symmetry of the quantum theory.

Before going into a more detailed study of the physical meaning of this symmetries, it would be adequate to introduce the following classification of the different elements in $Sl(2, Z)$:

**Classical symmetries** Diagonal matrices in $Sl(2, Z)$.

**Perturbative symmetries** Matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with $c = 0$.

**Non perturbative symmetries** Matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with $c \neq 0$.

Perturbative symmetries correspond to shifts in $\theta$ ($\theta \rightarrow \theta + 2\pi n$), while non perturbative symmetries connect strong with weak coupling.

### 3.7 Strong Coupling Regime: heuristic approach.

For $u$ close to $\Lambda$, the dynamically generated scale of the theory, we reach the strong coupling regime, so around this point in $\mathcal{M}_q(\mathcal{V})$ we should look for a new set of weakly coupled light modes. From the classical formula for the ‘t Hooft–Polyakov monopole, we can expect that magnetic monopoles are good candidates for defining the relevant light modes, weakly coupled, in terms of which the effective field theory description of the strong coupling region in $\mathcal{M}_q(\mathcal{V})$ can be defined.

Let us assume that the mass of the monopole vanishes at $u = \Lambda$. In this case, we know that the dual coupling constant defining the coupling of the magnetic monopole to the dual photon will become singular at the point $\Lambda$ (see equation (3.7)). The singularity will
go as the logarithm of the monopole mass, which from the BPS mass formula we know is
given by the value of $a_D$. Based on this simple argument, we can guess the form of the
effective prepotential $F_{eff}^D(a_D)$, describing the region of $\mathcal{M}(\mathcal{V})$ around $\Lambda$ in terms of the
dual coordinate $a_D$; namely,

$$F_{eff}^D(a_D) \sim a_D^2 \ln a_D,$$

(3.36)

with $a_D(u) \mid_{u=\Lambda} = 0$. In the neighbourhood of $\Lambda$ we write $a_D(u) \sim (u - \Lambda)$. From (3.36)
we can now define, using equation (3.19), the dual coordinate $a$ by:

$$a = \frac{\partial F_{eff}^D}{\partial a_D} = a_D(u) \ln a_D(u) + a_D(u).$$

(3.37)

Notice that the behaviour of $a$, given by (3.37), is very different to the behaviour (3.30)
defined by the classical Higgs mechanism in the weak coupling regime, already indicating
that a different phase is governing the strong coupling region.

### 3.8 Montonen-Olive Duality in $\mathcal{N}=2$ Theories.

From the previous discussion we can get some insight on the structure of monodromy ma-
trices associated with singularities of the quantum moduli space. For BPS stable particles
with charges $(0, n_e)$, $(n_m, 0)$, the two ingredients we shall use to find the corresponding
monodromy matrix at the point where the particle becomes massless are: i) The logarith-
icmic singularity in the effective coupling constant associated to the charge of the particle
(see equation (3.7)), and ii) The BPS mass formula. In terms of variables $a(u), a_D(u)$ we
find for an “electric” singularity, i. e., a $(0, 1)$ massless particle, the following behaviour:

$$a_D(u) \sim a(u) \ln a(u) + a(u),$$

(3.38)

with $a(u) \to 0$ at the singular point. In the same way, for a “magnetic” singularity
associated with a massless $(1, 0)$ particle we will get:

$$a(u) \sim a_D(u) \ln a_D(u) + a_D(u),$$

(3.39)

with $a_D \to 0$ at the singular point. In a graphical representation (3.38) and (3.39)
correspond to one loop Feynman diagrams in which the particle running inside the loop is
the one that becomes massless at the singular point and the (two) external legs represent
the $U(1)$ photon, in the case of (3.38), or the dual $U(1)$ photon for (3.39).

Originally, Montonen-Olive duality was propposed as a symmetry with respect (in
Georgi-Glashow model) to the interchange of $W^\pm$ charged vector bosons and ’t Hooft-
Polyakov magnetic monopoles. At tree level, this conjecture can be described in terms of the
Feynman diagram identity depicted in Figure 2.

---

7Notice that the contribution of the monopole to the one loop vacuum polarization of the dual theory
should be computed with an ultraviolet cutoff $\Lambda$ of the order $m_W$, since the size of the monopole is $1/m_W$. 

28
It is well known that this conjecture is not true for $N = 0$ theories. In the case of the $N=2$ supersymmetric extension of the Georgi-Glashow model the main problem this conjecture faces is that the $W^\pm$ vector bosons and the magnetic monopoles transform under different representations of $N = 2$ supersymmetry (vector bosons live in a vector multiplet, with spin one, while the magnetic monopole is an hypermultiplet with $s = 1/2$). Independently of this general argument we can try to check the Montonen-Olive conjecture already at the level of the monodromy matrices, just comparing the one associated with massless $W^\pm$ and that generated by massless magnetic monopoles. Denoting this monodromy matrices respectively by $M_W$ and $M_m$, it is easy to convince ourselves, just using (3.38) and (3.39), that both monodromies are certainly different. Moreover, $M_W$ will be generated by the $T$ generator of $Sl(2, Z)$, while $M_m$ will contain the $S$ generator. At this point, the best we can do to generalize Montonen-Olive duality conjecture to $N = 2$ supersymmetric gauge theories with a one dimensional quantum moduli will be to look for some extra monodromy matrix $M$ such that:

$$M_W = M_m M.$$  

(3.40)

As we will see in next section, this is in fact what happens in $SU(2)$ super Yang-Mills with $M$ the monodromy matrix corresponding to the singularity generated by a massless dyon. In fact, the general result for $N = 2$ supersymmetric gauge theories with a one dimensional quantum moduli is:

$$M_W = M_m \prod_i M_{R_i(m)},$$  

(3.41)

where $M_{R_i(m)}$ are the monodromy matrices corresponding to singularities associated with massless $R_i(m)$ particles, where by $R_i(m)$ we denote BPS stable states obtained from the monopole by acting with the set of unbroken global $R$-symmetries of the theory.
Notice that the only way to satisfy the Montonen-Olive duality conjecture, interpreted as the identity $M_W = M_m$, is when either the massless $W^\pm$ or the massless monopole produce a singularity in the one loop diagrams, i.e., when the theory has a vanishing $\beta$-function.

Equations (3.40) and (3.41) will characterize the way Montonen-Olive duality is extended to $N = 2$ supersymmetric theories with non-vanishing $\beta$-function. In the next lecture, we will work out the previous picture in more detail.

4 Exact Results and Coupling to Gravity.

4.1 Singularities and Phases.

In this section we will reduce ourselves to the study of $N=2$ $SU(2)$ super Yang-Mills. As we have seen in the previous section, this theory possess a flat potential which is not lifted by quantum corrections, and therefore a one dimensional quantum moduli $\mathcal{M}_q(\mathcal{V})$. A gauge invariant parametrization of $\mathcal{M}_q(\mathcal{V})$ is defined by means of the Casimir coordinate $u = Tr \, \phi^2$. In the Higgs phase the electrically charged vector bosons $W^\pm$ have a mass given by $a(u) n_e$ ($n_e = 1$), with $a(u)$ the vacuum expectation value of the Higgs field, i.e., $\sqrt{2u}$. If we maintain ourselves in the Higgs phase, the geometry of $\mathcal{M}_q(\mathcal{V})$ can be described by the effective field theory prepotential $\mathcal{F}_{eff}(a(u))$, where we integrate all the massive (heavy) particles, with the mass of these particles obtained by the standard Higgs mechanism. Singularities in the Higgs phase will appear whenever one of these massive particles becomes massless. This singularity will introduce some logarithmic dependence of the dual variable $a_D \equiv \frac{\mathcal{F}_{eff}(a(u))}{\partial_a(u)}$ on $a(u)$. The origin of this logarithmic singularity is, of course, the coupling of the $a(u)$ field to a Higgsed massive state that becomes massless at the singularity.

The monodromy matrices in the Higgs phase, $\{M_i^{Higgs}\}$, coming from

$$
\begin{pmatrix}
a_D(u) \\
a(u)
\end{pmatrix} \rightarrow M_i^{Higgs} \begin{pmatrix}
a_D(u) \\
a(u)
\end{pmatrix} = \begin{pmatrix}
a_D(u e^{2\pi i}) \\
a(u e^{2\pi i}) = -a
\end{pmatrix},
$$

will be of the type

$$
M_i^{Higgs} = \begin{pmatrix}
a & b \\
0 & -1
\end{pmatrix} \in SL(2, \mathbb{Z})
$$

for some integer values of $a$ and $b$ depending on the quantum corrections to the effective prepotential.

From (4.2) we immediately derive the following general result:

$R_1$ On the Higgs phase of $\mathcal{M}_q(\mathcal{V})$, for $N = 2$ $SU(2)$ super Yang-Mills, the monodromy group generated by the singularities is abelian, i.e., it is generated by $T$ and $P$. 

30
The previous result in particular means (see section 2.3) that the exact quantum symmetry defined by the monodromy group reduces to simply the well known symmetry \( \theta \rightarrow \theta + 2\pi n, \ n \in \mathbb{Z} \).

The question we should address now is whether the whole quantum moduli \( \mathcal{M}_q(\mathcal{V}) \) is in the Higgs phase.

4.1.1 Holomorphicity and Abelian Monodromy.

A simple holomorphicity argument can prove to us that the quantum moduli \( \mathcal{M}_q(\mathcal{V}) \) contains more than the Higgs phase. The argument goes as follows: First of all we define the coupling constant, in the way described in the previous Section, as

\[
\text{Im} \ \tau(a(u)),
\]

with

\[
\tau(a(u)) = \frac{\partial^2 \mathcal{F}_{\text{eff}}(a(u))}{\partial a^2}.
\]

(4.4)

Obviously, with respect to the abelian monodromy of type \( (4.2) \), and on the Higgs phase defined by \( a(u) \sim \sqrt{2u} \), the effective coupling constant (4.3) will be single valued or, in other words, globally defined.

From the general structure of \( \mathcal{N}=2 \) supersymmetric theories we know that the prepotential is an holomorphic function; therefore, \( \text{Im} \ \tau(a(u)) \) is harmonic. Now, we only need to remember some basics in complex analysis, namely the well known theorem that states that an harmonic function can only reach the maximum at the boundary of its domain of definition. Therefore, if we impose positivity of the effective coupling constant (4.3), then the Higgs phase can only correspond to a local region of the quantum moduli space. Summarizing, we have obtained the following second result:

\( R_2 \) The quantum moduli of \( \mathcal{N}=2 \) supersymmetric Yang-Mills can not be globally described in terms of the Higgs phase variables.

Geometrically, we are learning that \( a(u) \) and \( a_D(u) \) should be interpreted as sections on \( \mathcal{M}_q(\mathcal{V}) \). It should be already clear, from our discussion in the previous sections on the way the duality group is defined, that these sections are sections of a two dimensional vector bundle on \( \mathcal{M}_q(\mathcal{V}) \) with structure group, acting on the fibre, the duality group \( SL(2, \mathbb{Z}) \).

The simplest (and naive) way to interpret \( R_2 \) would be noticing that at the origin we are out of the Higgs phase, because at that point the gauge group is unbroken; however,

\[ \text{Im} \ \tau \] (equation (2.56):

\[
T : \tau \rightarrow \tau + 1,
\]

which implies that the imaginary part of \( \tau \) is unchanged.

---

8This can be trivially derived from the way the abelian subgroup of \( SL(2, \mathbb{Z}) \) generated by \( T \) is acting on \( \tau \) (equation (2.56):
this comment, as we will see in a moment, is wrong. In fact, at the origin the charged particles that become massless are the gauge vector bosons. The monodromy at that point should be such that the spectrum vector \((n_e = 1, n_m = 0)\) is invariant under its action. This already means that this monodromy should be part of the abelian subgroup generated by \(T\) and \(P\), and therefore will not help us in solving the problem of positivity of \(\text{Im } \tau(a(u))\). The reader should notice that we are generically calling Higgs phase the whole region of \(\mathcal{M}_q(\mathcal{V})\) where the Higgs parametrization \(a(u) \sim \sqrt{2u}\) is correct.

### 4.1.2 Duality and Phases.

Now we would like to have an heuristic and simple minded way to understand the previous phenomena, namely the existence of more than one phase on \(\mathcal{M}_q(\mathcal{V})\). If we describe the Higgs phase by \(a(u) \sim \sqrt{2u}\), we can try to use the dual description to get some insight on what can be the physical origin of the failure of this Higgs relation. To do that we can try to work out, with the dual variable \(a_D(u)\), an effective theory \(\mathcal{F}_{\text{eff}}^D(a_D(u))\), and to define \(a(u)\) by \(\frac{\partial \mathcal{F}_{\text{eff}}^D}{\partial a_D(u)}\). The variable \(a_D(u)\), as we have discussed in section 3.4.3, describes what we can call the “dual” photon; this photon is coupled to magnetically charged particles which are represented under \(N = 2\) supersymmetry by \(N = 2\) matter hypermultiplets (the spin of the magnetic monopole is \(1/2\)). From this picture we can now expect a logarithmic dependence in \(a(u)\) (see equation (3.29)), in contrast to the Higgs dependence \(a(u) \sim \sqrt{2u}\), whenever the dual photon is coupled to massless magnetic hypermultiplets. Therefore, the other phase on which the theory is living can correspond to having massless magnetic monopoles, something that, as we will see, can be described as a dual (magnetic) Higgs phase.

### 4.2 Seiberg-Witten Solution for \(N = 2 SU(2)\) Supersymmetric Yang-Mills.

Based on the previous discussion, we can introduce a set of rules that can be used to derive the exact geometry of the quantum moduli space.

**Rule 1** Monodromy condition. We will assume that the quantum moduli is compactified by adding the point at \(\infty\). In these conditions, we should impose

\[
\prod M_i = \mathbb{1}, \tag{4.5}
\]

where the product in (4.5) is over the whole set of singularities. If the moduli space has dimension bigger than one, we will assume that singularities always define codimension one regions.

---

\(^9\)Notice that monodromies are associated with non contractible paths in the \(u\) plane. The product in (4.5) is determined by combining paths with the same base point.
Rule 2 Positivity of the coupling constant. In the one dimensional case this condition, together with the holomorphy of the prepotential, is enough to prove the existence of more than one Higgs phase.

Rule 3 Global $R$-symmetries. Global $R$-symmetries are generically broken, by non perturbative instanton effects, to some discrete residual subgroup. We will require that singularities of the quantum moduli are mapped into singularities by the action of these global $R$-symmetries.

Rule 4 BPS stability of massless particles. To each singular point $P_i$ we associate a massless charged particle characterized by the charge vector $(n_m^{(i)}, n_e^{(i)})$. This vector should satisfy
\[(n_m^{(i)} n_e^{(i)}) M_i = (n_m^{(i)} n_e^{(i)}), \tag{4.6}\]
and must correspond to a stable BPS particle.

We will now use the previous set of rules to build up the exact quantum moduli for $N=2 \ SU(2)$ super Yang-Mills. The starting point is of course the singularity at $\infty$ that we have already described (see section 3.6). The corresponding monodromy is
\[M_\infty = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}. \tag{4.7}\]
By Rule 2, and the argument in 4.1.1 and 4.1.2, we assume the existence of one point where the magnetic monopole becomes massless; this point can be characterized by the dynamically generated scale $\Lambda$. From (3.38) we derive
\[M_\Lambda = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}. \tag{4.8}\]
(From (4.6) it can be now verified that the particle becoming massless at $u = \Lambda$ is in fact a magnetic monopole of charge one.)

Now we should take into account Rule 3, and to act with the residual global $R$-symmetries on the point $u = \Lambda$. For $N = 2 \ SU(2)$ super Yang-Mills, and when the vacuum expectation value is zero, the instanton induces an effective vertex with eight external gluinos; when the vacuum expectation value is different from zero, the number of zero modes reduces to four, two coming from the fermionic partner of the gauge field, $\lambda$, and two coming from the fermionic partner of the scalar field, $\psi$.

Therefore the $Z_8$ residual $R$-symmetry breaks to $Z_4$ for generic points in $\mathcal{M}_q(\mathcal{V})$ or, in other words, $Z_8$ is spontaneously broken to $Z_4$, and thus the orbit of the $R$-symmetry is $Z_2$. Hence, if we have a singularity at $\Lambda$, we should have another singularity at the point $-\Lambda$ (as $Z_2 : u \rightarrow -u$).
This defines the minimal solution compatible with the previous set of rules, where $M_{-\Lambda}$ is obtained from condition (4.5):

$$
M_{-\Lambda} = \begin{pmatrix}
-1 & 2 \\
-2 & 3
\end{pmatrix}.
$$

(4.9)

Using (4.6) we now observe that the particle that becomes massless at the point $-\Lambda$ is a dyon, which is again a BPS stable particle.

The $\mathbb{Z}_2$ transformation can be implemented in the $u$ plane by a matrix $A$:

$$
M_{-\Lambda} = AM_{\Lambda}A^{-1}.
$$

(4.10)

It is easy to observe that any $A$ of the form

$$
A = TM_{\Lambda}^r
$$

(4.11)

provides a solution to (4.5). However, only for $r = 1$ we get a solution satisfying $A^2 = -\mathbb{1}$, which in particular implies that after the action of $\mathbb{Z}_2^2$ the stable monopole at $u = \Lambda$ will become an antimonopole, a fact related to the existence of $P$ in $M_\infty$ (see (3.33)).

### 4.3 Some Comments on Seiberg-Witten Solution.

The most impressive implication of Seiberg-Witten solution is certainly that the classical singularity at the origin is not there when quantum mechanical effects are taken into account. This, in particular, means that over the whole quantum moduli the gauge symmetry is $U(1)$, with no point where the (full $SU(2)$) gauge symmetry is restored. This fact will be crucial to connect Seiberg-Witten quantum moduli with type II strings.

A way to understand why the origin is not a singular point of the quantum moduli can be motivated by our discussion in section 3.8 on Montonen-Olive duality for $N = 2$ supersymmetric theories. In fact, Seiberg-Witten solution can be directly derived from equation (3.40). In other words, we can start formally with a quantum moduli possessing only the singularities at $\infty$ and at the origin. The singularity at the origin should be generated by massless gauge vector bosons, and therefore will be given by $M_W$. Next, we impose the Montonen-Olive duality relation (3.40) for $N = 2$ theories and we obtain Seiberg-Witten solution or, equivalently, the split of the classical singularity at the origin into the two singularities at the points $\pm \Lambda$.

A different way to approach the meaning of the two singularities at $\pm \Lambda$ is in the framework of the ‘t Hooft, Polyakov and Mandelstam theory of confinement. For asymptotically free theories the confinement phase is expected to correspond to unbroken gauge symmetry (the classical singularity at the origin) but with a vacuum characterized, as a “dual” BCS superconductor, by a non vanishing magnetic order parameter which, very likely, will require massless magnetically charged objects (the two quantum singularities at $\pm \Lambda$).
4.3.1 The Abelian Confinement Argument.

Let us now summarize the main steps of Seiberg-Witten confinement argument.

i) The quantum moduli of $N=1$ supersymmetric $SU(2)$ Yang-Mills is a discrete set of two points, related by a $Z_2$ global $R$-symmetry transformation. This result comes from the exact computation of Witten’s index, $tr (-1)^F$, which in this particular case is two (if the gauge group is taken to be $SU(N)$, the value of the index is $N$).

ii) The two vacua of $N=1$ supersymmetric $SU(2)$ Yang-Mills are characterized by a non vanishing expectation value $<\lambda \lambda>$, a gaugino condensate $[25]$.

iii) Instantons contribute to a non vanishing expectation value $<\lambda \lambda(x) \lambda \lambda(y)> [26]$. As, from supersymmetric Ward identities we know that this expectation value is independent of $|x-y|$, the gaugino condensate can be derived from the instanton contribution to $<\lambda \lambda(x) \lambda \lambda(y)>$ using cluster decomposition or, equivalently, assuming the existence of a mass gap in the $N=1$ theory.

iv) Adding a soft supersymmetry breaking term $m \text{tr} \phi^2$ to the $N=2$ theory, and using the decoupling theorem, we can define $N=1$ $SU(2)$ super Yang-Mills as the corresponding effective low energy field theory. Once we add this soft breaking term, we lift the flat direction of the $N=2$ potential.

v) The two vacua defining the quantum moduli of the low energy effective $N=1$ theory should correspond to two points of the $N=2$ quantum moduli. Moreover, the existence of a mass gap in the $N=1$ theory implies that the massless $U(1)$ photon must become massive by some Higgs mechanism at these two points. The only candidate that can play the role of the Higgs field is the massless monopole at the points $\pm \Lambda$ that will higgs the dual “magnetic” photon.

vi) This dual Higgs mechanism explains, in the dual analog of BCS superconductivity, the confinement and the mass gap of the $N=1$ theory.

More quantitatively, we can define a superpotential $W(M)$ for the monopole field:

$$W(M) = m \text{tr} \phi^2 + a_D M \tilde{M},$$ (4.12)

with the term $a_D M \tilde{M}$ describing the coupling of the monopole to the “dual” photon, as required by $N=2$ supersymmetry. On the quantum moduli we can rewrite $W(M)$ using the fact that $a_D = a_D(u)$ as

$$W(M) = m u(a_D) + a_D M \tilde{M}.$$ (4.13)

The vacuum defined by $dW = 0$ is characterized, if $\frac{\partial u}{\partial a_D} \neq 0$, by the magnetic order parameter $<M> \neq 0$ which will induce, by the dual Higgs mechanism, the mass gap of the $N=1$ theory. Notice that the confinement picture we are presenting here takes only into account the abelian gauge symmetry. The analog in $N=0$ quantum field theory is Polyakov’s compact quantum electrodynamics.
4.4 Geometrical Interpretation.

The monodromy group generated by $M_\Lambda$, $M_{-\Lambda}$ and $M_\infty$ is the group $\Gamma_2$ of unimodular matrices congruent to the identity mod(2):

$$\Gamma_2 \equiv \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, Z), a \equiv b \equiv 1 \mod(2), b \equiv c \equiv 0 \mod(2) \right\}. \quad (4.14)$$

As explained above this group defines the exact quantum symmetry of the theory, which in particular implies that we can reduce the upper half complex plane $H$, parametrizing the coupling constant and the $\theta$-parameter, to the $\Gamma_2$ fundamental domain $H/\Gamma_2$. This fundamental domain has a nice interpretation in algebraic geometry that we will describe now in very qualitative terms: Let us define an elliptic curve $E_u$ as the vanishing locus of a cubic polynomial in $\mathbb{P}^2$,

$$W(x, y, z; u) = 0, \quad (4.15)$$

and let us denote by $\tau(u)$ the corresponding elliptic modulus. Singularities of the curve defined by (4.15) will appear at values of $u$ for which

$$W = \frac{\partial W}{\partial x} = \frac{\partial W}{\partial y} = \frac{\partial W}{\partial z} = 0. \quad (4.16)$$

Let us now denote by $\Gamma_M$ the monodromy group of the map $\tau(u)$ at these singular points. By construction, $\tau(u)$ and $\Gamma_M \tau(u)$ should correspond to the same elliptic curve, and therefore $\Gamma_M \subset SL(2, Z)$, the modular group of an elliptic (genus one) curve. The quotient group

$$SL(2, Z)/\Gamma_M \quad (4.17)$$

will map among themselves the singular points solution to equation (4.16). In fact, all of them should correspond to the boundary of the moduli space of complex structures of the defining elliptic curve. We can now characterize the quotient (4.17) as the set of transformations $x \rightarrow x', y \rightarrow y', z \rightarrow z'$ such that

$$W(x', y', z'; u) = f(u)W(x, y, z; u), \quad (4.18)$$

i. e., as changes of local coordinates that can be compensated by a change of the moduli parameter $u$.

In order to reproduce Seiberg-Witten solution in the previous framework we should find a cubic polynomial $W(x, y, z; u)$ [27] with solutions to (4.13) at the points $u = \infty, \pm \Lambda$, and with monodromy group $\Gamma_M = \Gamma_2$. It is easy to check that

$$W(x, y, z; u) = -zy^2 + x(x^2 - \Lambda^4 z^2) - uz(x^2 - \Lambda^4 z^2) = 0 \quad (4.19)$$

which defines the elliptic curve

$$E_u : \quad y^2 = (x - \Lambda^2)(x + \Lambda^2)(x - u) \quad (4.20)$$
possesses singularities at precisely the points $u = \infty, \pm \Lambda$, and that the group of transformations (4.18) is isomorphic to $Sl(2,\mathbb{Z})/\Gamma_2$, which indirectly means, by the argument above, that the monodromy group of the corresponding map $\tau(u)$ is $\Gamma_2$.

For the curve (4.20) the map $\tau(u)$ can be easily defined as:

$$\tau(u) = \oint_{\gamma_1} \lambda_1 = \oint_{\gamma_2} \lambda_1 = \frac{dx}{y} \in H^{1,0}(E_u, C),$$  

(4.21)

with $\gamma_1, \gamma_2$ an homology basis of $E_u$. From the definition of $\tau(u)$ in terms of the prepotential $\mathcal{F}(a(u))$ we get:

$$\tau(u) = \frac{da_D/du}{da/du}.$$  

(4.22)

To match (4.21) and (4.22) we can try to look for a one form $\lambda$ in $H^1(E_u, C) = H^{0,1} \oplus H^{1,0}$ such that

$$a_D(u) = \oint_{\gamma_1} \lambda, \quad a(u) = \oint_{\gamma_2} \lambda,$$  

(4.23)

and

$$\frac{d\lambda}{du} = f(u)\lambda_1$$  

(4.24)

for some function $f(u)$. To fix this function we can use the asymptotic behaviour of $a_D(u)$ and $a(u)$ at the points $u = \pm \Lambda, \infty$, which we have already derived in the previous sections (see equations (3.29) and (3.30) for the asymptotic behaviour at $\infty$, and (3.37) for the behaviour at the point $+\Lambda$; recall that the behaviour at $-\Lambda$ is derived from the one at $\Lambda$ by acting with the residual global $Z_2$ R-symmetry). We leave as an exercise to check that the correct one form $\lambda$ is given by

$$\lambda = \sqrt{2}(\lambda_2 - u\lambda_1) \frac{1}{2\pi}, \quad \lambda_2 = \frac{xdx}{y}.$$  

(4.25)

The natural connection between $N=2$ supersymmetry and algebraic geometry should be understood in the framework of Picard-Fuchs equation. From the $N=2$ supersymmetry transformations we can derive the following set of relations [28]:

$$d_u V = U,$$

$$D_u U = C_{uuu} G_u^{-1} \bar{U},$$

$$d_u \bar{U} = 0,$$  

(4.26)

where we have introduced

$$V \equiv (a, a_D),$$

$$G_{u\bar{u}} \equiv Im \tau(u),$$

$$C_{uuu} \equiv d\tau \left( \frac{da}{du} \right)^2,$$  

$$D_u \equiv d_u - \Gamma_u,$$

$$\Gamma_u \equiv G_{u\bar{u}}^{-1} (d_u G_{u\bar{u}}),$$  

(4.27)

37
with $\tau(u)$ and $a_D$ defined in the usual way in terms of the prepotential $F(a(u))$. The rigid Kähler relations (4.26) can be organized in the form of a differential equation; using relation (4.22) this differential equation becomes the Picard-Fuchs equation for the periods of the algebraic curve (1.20).

Before finishing this section we would like to add some general comments.

$C1$ Solution (4.25) gives us the exact geometry of the quantum moduli space or, in other words, the exact prepotential. In physical terms this is equivalent to knowing the effective low energy lagrangian up to higher (bigger than two) derivative terms.

$C2$ The exact solution (4.25) contains all the information concerning instanton effects.

### 4.5 The Stringy Approach to the Quantum Moduli.

Once you have the algebraic geometrical description of the quantum moduli space it is difficult to resist (mostly if you have been exposed during the last decade to the stringy way of thinking high-energy physics) the temptation to interpret the results in stringy terms. In this spirit, it would be very natural to put in parallel Seiberg-Witten solution for $N=2$ supersymmetric gauge theories with the effective field theory interpretation of the special Kähler geometry of the moduli space of complex structures of some Calabi-Yau manifold. The analogy is certainly more than formal. If you consider a type II string compactified on some Calabi-Yau manifold $X$ the special Kähler geometry on the moduli $\mathcal{M}(X)$ of complex structures can be interpreted in terms of an effective $N=2$ supergravity with as many $U(1)$ gauge fields as the dimension of $\mathcal{M}(X)$, which is given by the Betti number $b_{2,1}$ of the manifold $X$.

The first similarity with the quantum moduli of $N=2$ supersymmetric gauge theories is the appearance of only $U(1)$ gauge fields, a consequence of choosing a type II superstring. The formal analogy goes on in the sense that $N=2$ local supersymmetry implies the Picard-Fuchs equation for the periods of the top form on the Calabi-Yau manifold. Moreover, the exact quantum duality symmetry of the effective supergravity theory describing the geometry on $\mathcal{M}(X)$ is given by the $T$-duality, in string language, of the Calabi-Yau manifold. The map $\tau(u)$ becomes, in this picture, the mirror map mapping the moduli $\mathcal{M}(X)$ of complex structures into its mirror, the moduli of Kähler structures, which we describe by $\tau$. Singularities in $\mathcal{M}(X)$, which are known as conifold singularities, should be the stringy parallel of the singularities of the quantum moduli. The previous set of analogies can be temptatively summarized in the following set of “translation” rules:
A: QFT Language  

Quantum moduli of $N=2$ supersymmetric gauge theory.  

$\leftrightarrow$  

B: String Language  

Moduli of complex structures of some Calabi-Yau manifold $X$ with respect to which we compactify a type II superstring.  

Singularities (monopoles).  

$\leftrightarrow$  

Conifold singularities.  

$\tau(u)$ map.  

$\leftrightarrow$  

Mirror map.  

$\Gamma_M$ monodromy group.  

$\leftrightarrow$  

$T$-duality.  

$E_u$ curve.  

$\leftrightarrow$  

Calabi-Yau manifold $X$.

The previous set of analogies is very suggestive, but presents severe difficulties of interpretation. First of all we should notice that column B can be interpreted as describing the quantum moduli of some $N=2$ supergravity theory, and therefore in order to pass to column A we need to work out some way to decouple gravitational effects. In second place, the analogy between the quantum moduli of $N=2$ supersymmetric gauge theories and the moduli, in column B, of complex structures of some Calabi-Yau manifold, presents the problem that, as we have discussed in the previous sections, the moduli of $N=2$ $SU(2)$ super Yang-Mills is not the moduli of complex structures of $E_u$, but something bigger containing extra geometrical information on $E_u$, an effect that is related to the way Montonen-Olive duality is realized in $N=2$ supersymmetric theories with non vanishing $\beta$-function. Another problem of our set of translation rules is to unravel the meaning of the conifold singularity as associated with some massless charged hypermultiplet\textsuperscript{10}.

\textsuperscript{10}The meaning of the conifold singularities constituted a while a serious puzzle. Recently, Strominger has proposed to interpret these singularities, in perfect parallel with the approach of Seiberg and Witten, as coming from a BPS stable massless charged hypermultiplet, that in string theory has the interpretation of a charged black hole. This conjecture can be checked once we consider string loop corrections near the conifold point. Using the topological twisted version we can compute the topological amplitudes $F_g$ at genus $g$ using the Kodaira-Spencer theory \cite{Kodaira}. In this approach, $F_g$ is given by:

$$F_g = (V_{mmm})^{2g-2} P^{3g-3},$$

with $P$ the Kodaira-Spencer propagator, and $V_{MMM}$ the vertex for the massive excitations. Using the relations between the structure constants $C_{ijk}$, the propagator $P$ and the vertex,

$$\partial_t C_{ijk} \sim P(V_{ttM})^2,$$

we easily get,

$$F_g \sim \left(\frac{\partial_t^3 C_{ttt}}{\partial_t^2 C_{ttt}}\right)^{2g-2},$$

for $C_{ttt}$ the three point function corresponding to the marginal directions defined by the $t$-direction of the moduli. Now, we observe from the above relation that the tree level information of the special geometry of the moduli of complex structures, which determines $C_{ttt}$ and, therefore, the string tree level conifold singularity, implies, for instance for $F_1$, a logarithmic singularity that admits Strominger’s interpretation.
And a final and most urgent problem is, of course, to figure out what can the classical supergravity theory be whose quantum moduli is precisely described by the moduli of complex structures of some Calabi-Yau manifold \( X \).

Let us comment on this last issue. For a Calabi-Yau manifold \( X \) with \( b_{2,1} = r \) we can naively think that the moduli \( \mathcal{M}(X) \) is the quantum moduli of a supergravity theory with gauge group \( G \) spontaneously broken to \( U(1)^r \), with \( r \) the rank of \( G \). However, this is not the correct answer because of the special role played by the dilaton field in string theory. If, for instance, we consider that \( \mathcal{M}(X) \) is the quantum moduli of some classical moduli, described by \( \mathcal{M}(Y) \), of some different manifold \( Y \), we need to count with a way to control the quantum corrections on the theory compactified on the manifold \( Y \) that produces the quantum moduli \( \mathcal{M}(X) \). This will be impossible if we consider a type II compactification on \( Y \) because in type II strings the dilaton, which is counting the string loop effects, is in a hypermultiplet and therefore does not interfere with the dynamics of vector multiplets, which are the ones describing the geometry of the moduli space of complex structures. The only possible way out is of course compactifying on \( Y \) a heterotic string. In this case the dilaton appears as a vector hypermultiplet and therefore we have some chances that quantum corrections of the heterotic string compactified on \( Y \) sum up into the quantum moduli space \( \mathcal{M}(X) \). In this framework the second Betti number of \( X \) should be equal to \( r + 1 \), the rank of the gauge group of the supergravity theory, \( G \), plus one extra vector field, corresponding to the dilaton of the heterotic string. Pairs of Calabi-Yau manifolds \((X,Y)\) such that \( \mathcal{M}(X) \) is the quantum moduli of the heterotic string compactified on \( Y \) are known as Heterotic-type II dual pairs. Part of the beauty of these dual pairs is that the \( T \)-duality on \( \mathcal{M}(X) \) is inducing, when we read it in the variables of the heterotic string compactified on \( Y \), an \( S \)-duality transformation on the dilaton field, that in fact is part of the non perturbative quantum monodromy for the heterotic string on \( Y \). Continuing with our translation rules, the previous discussion can be summarized in the following diagram:

**A: QFT Language   B: String Language**

| Classical moduli | Heterotic on \( Y \) |
|------------------|---------------------|
| \( \uparrow \)    | \( \uparrow \)       |
| Quantum moduli   | Type II on \( X \)   |
|                  | \( (X,Y) \) dual pair|

To end this lectures, we wish just to mention a \( N=2 \) dual pair, defined by heterotic compactification on \( K3 \times T^2 \) and type II on the weighted projective space \( WP^{(1,1,2,2,6)} \) \[31\]. This dual pair is the natural candidate for recovering from the string the quantum moduli structure of \( N=2 \) \( SU(2) \) super Yang-Mills \[32\], opening the possibility to explore of a massless black hole running in the loop, in perfect analogy with our discussion of the singularities of Seiberg-Witten quantum moduli.
how strings and gravitational effects modify the point particle limit, quantum moduli physics described in these notes.

Acknowledgements.

It is pleasure for C. G. to thank M. J. Herrero, A. Dobado and F. Cornet for their invitation to give these lectures in such a lovely place. The work of C. G. is supported in part by PB 92-1092, ERBCHRXCT920069 and OFES contract number 93.0083. The work of R. H. is supported by a fellowship from UAM.
References

[1] P. A. M. Dirac, Proc. R. Soc. A133 (1931), 60.

[2] For a review, see for instance P. Goddard and D. I. Olive, Rep. Prog. Phys. 41 (1978), 1357.

[3] E. B. Bogomolny, Sov. J. Nucl. Phys. 24 (1976), 449.
   M. K. Prasad and C. M. Sommerfield, Phys. Rev. Lett. 35 (1975), 760.

[4] G. ‘t Hooft, Nucl. Phys. B79 (1974), 276.
   A. M. Polyakov, JETP Lett. 20 (1974), 194.

[5] E. Witten, Phys. Lett. B86 (1979), 283.

[6] E. Witten and D. Olive, Phys. Lett. B78 (1978), 97.

[7] H. Osborn, Phys. Lett. B83 (1979), 321.

[8] G. ‘t Hooft, Nucl. Phys. B190 (1981), 455.

[9] C. Montonen and D. Olive, Phys. Lett. B72 (1977), 117.
   P. Goddard, J. Nuyts and D. Olive, Nucl. Phys. B125 (1977), 1.

[10] N. Seiberg and E. Witten, Nucl. Phys. B426 (1994), 19.

[11] For reviews see A. Giveon, M. Porrati and E. Rabinovici, Phys. Rep. 244 (1994), 77;
   E. Álavez, L. Álvarez-Gaumé and Y. Lozano, hep-th/9410237.

[12] A. Font, L. Ibañez, D. Lüst and F. Quevedo, Phys. Lett. B249 (1990), 35.

[13] A. Sen, Int. J. Mod. Phys. A9 (1994), 3007.
   C. Hull and P. Townsend, Nucl. Phys. B438 (1995), 109.
   M. Duff, Nucl. Phys. B442 (1995), 47.
   P. Townsend, Phys. Lett. B350 (1995), 184.
   E. Witten, Nucl. Phys. B443 (1995), 85.
   A. Sen, “String-String Duality Conjecture in Six Dimensions and Charged Solitonic Strings”, hep-th/9504027.
   J. Harvey and A. Strominger, “The Heterotic String is a Soliton”, hep-th/9504047.
   B. Greene, D. Morrison and A. Strominger, “Black Hole Condensation and the Unification of String Vacua”, hep-th/9504145.
C. Vafa and E. Witten, Nucl. Phys. B447 (1995), 261.

C. Hull and P. Townsend, “Enhanced Gauge Symmetries in Superstring Theories”, hep-th/9505073.

[14] T. T. Wu and C. N. Yang, Phys. Rev. D12 (1975), 3845.

[15] G. ’t Hooft, Phys. Rev. Lett. 37 (1976), 8.

R. Jackiw and C. Rebbi, Phys. Rev. Lett. 37 (1976), 172.

[16] M. A. Shifman and A. I. Vainshtein, Nucl. Phys. B277 (1986), 456; Nucl. Phys. B359 (1991), 571.

[17] R. Haag, J. T. Lopuszanski and M. Sohnius, Nucl. Phys. B88 (1975), 257.

[18] J.P. Serre, Course d’Aritmetique, PUF (1970).

[19] N. Seiberg, Phys. Lett. B318 (1993), 469.

[20] A. B. Zamolodchikov, JETP Lett. 43 (1986), 730.

[21] E. Witten, Nucl. Phys. B202 (1982), 253.

[22] K. Konishi, Phys. Lett. B135 (1984), 439.

[23] M. T. Grisaru and P. C. West, Nucl. Phys. B254 (1985), 249.

[24] N. Seiberg, Phys. Lett. B206 (1988), 75.

[25] E. Cohen and C. Gómez, Phys. Rev. Lett. 52 (1984), 237.

[26] V. A. Novikov, M. A. Shifman, A. I. Vainshtein, V. I. Zakharov, Nucl. Phys. B223 (1983), 445; Nucl. Phys. B260 (1989), 197.

[27] A. Ceresole, R. D’Auria and S. Ferrara, Phys. Lett. B339 (1994), 71.

[28] A. Strominger, Commun. Math. Phys. bf 133 (1990), 163.

B. de Wit and A. van Proeyen, Nucl. Phys. B245 (1984), 89.

[29] A. Strominger, “Massless black holes and conifolds in string theory”, hep-th/9504090.

[30] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, “Holomorphic Anomalies in Topological Field Theories”, hep-th/9302103.

[31] S. Kachru and C. Vafa, “Exact Results for N = 2 Compactifications of Heterotic Strings”, hep-th/9505107.
[32] G. Lopes Cardoso, D. Lust and T. Mohaupt, “Duality Symmetries and Supersymmetry Breaking in String Compactifications”, hep-th/9404095.

C. Gómez and E. López, “A Note on the String Analog of $N=2$ Supersymmetric Yang-Mills”, hep-th/9506024.

M. Billó, A. Ceresole, R. D’Auria, S.Ferrara, P. Fré, T. Regge, P. Soriani and A. van Proeyen, “A Search for Non-Perturbative Dualities of Local $N=2$ Yang-Mills Theories from Calabi-Yau Threefolds”, hep-th/9506075.

S. Kachru, A. Klemm, W. Lerche, P. Mayr and C. Vafa, “Nonperturbative Results on the Point Particle Limit of $N=2$ Heterotic String Compactifications”, hep-th/9508155.