Abstract. The work is devoted to old and recent investigations of the classical M. A. Buhl problem of describing compatible linear vector field equations, its general M.G. Pfeiffer and modern Lax-Sato type special solutions. Especially we analyze the related Lie-algebraic structures and integrability properties of a very interesting class of nonlinear dynamical systems called the dispersionless heavenly type equations, which were initiated by Plebański and later analyzed in a series of articles. The AKS-algebraic and related $R$-structure schemes are used to study the orbits of the corresponding co-adjoint actions, which are intimately related to the classical Lie–Poisson structures on them. It is demonstrated that their compatibility condition coincides with the corresponding heavenly type equations under consideration. It is shown that all these equations originate in this way and can be represented as a Lax compatibility condition for specially constructed loop vector fields on the torus. The infinite hierarchy of conservation laws related to the heavenly equations is described, and its analytical structure connected with the Casimir invariants, is mentioned. In addition, typical examples of such equations, demonstrating in detail their integrability via the scheme devised herein, are presented. The relationship of the very interesting Lagrange–d’Alembert type mechanical interpretation of the devised integrability scheme with the Lax–Sato equations is also discussed.

1. Introduction

In the classical works \cite{11, 12, 13} still in 1928 the French mathematician M.A. Buhl posed the problem of classifying all infinitesimal symmetries of a given linear vector field equation

\[(1.1) \quad A\psi = 0,\]

where function $\psi \in C^2(\mathbb{R}^n; \mathbb{R})$, and

\[(1.2) \quad A := \sum_{j=1}^{n} a_j(x) \frac{\partial}{\partial x_j}\]

is a vector field operator on $\mathbb{R}^n$ with coefficients $a_j \in C^1(\mathbb{R}^n; \mathbb{R}), j = 1, n$. It is easy to show that the problem under regard is reduced \cite{13} to describing all possible vector fields

\[(1.3) \quad A^{(k)} := \sum_{j=1}^{n} a_j^{(k)}(x) \frac{\partial}{\partial x_j}\]

with coefficients $a_j^{(k)} \in C^1(\mathbb{R}^n; \mathbb{R}), j, k = 1, n$, satisfying the Lax type commutator condition

\[(1.4) \quad [A, A^{(k)}] = 0\]

for all $x \in \mathbb{R}^n$ and $k = 1, n$. The M.A. Buhl problem above was completely solved in 1931 by the Ukrainian mathematician G. Pfeiffer in the works \cite{10, 11, 12, 13, 14, 15}, where he has constructed explicitly the searched set of independent vector fields \cite{13}, having made use effectively of the full set of invariants for the vector field \cite{12} and the related solution set structure of the Jacobi-Mayer system of equations, naturally following from \cite{13}. Some results, yet not complete, were also obtained by C. Popovici in \cite{17}.

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\textbf{Key words and phrases.} Lax–Sato equations, heavenly equations, Lax integrability, Hamiltonian system, torus diffeomorphisms, loop Lie algebra, Lie-algebraic scheme, Casimir invariants, R-structure, Lie-Poisson structure, bi-Hamiltonicity, Lagrange–d’Alembert principle.
Some years ago the M.A. Buhl type equivalent problem was independently reanalyzed once more by Japanese mathematicians K. Takasaki and T. Takebe [61, 62] and later by L.V. Bogdanov, V.S. Dryuma and S.V. Manakov [8], for a very special case when the vector field operator (1.2) depends upon Japanese mathematicians K. Takasaki and T. Takebe [61, 62] and later by L.V. Bogdanov, V.S. Dryuma and S.V. Manakov [8] for a very special case when the vector field operator (1.2) depends upon a chosen root element ̃G∗ and some Casimir invariants, we have successively demonstrated that their compatibility condition coincides exactly with the corresponding heavenly equations and the related bi-Hamiltonian structure as well as the Bäcklund transformations. As a matter of fact, there are only a few examples of multi-dimensional integrable systems for which such a detailed description of their mathematical structure has been given. As was aptly mentioned in [59], the heavenly equations comprise an important class of such integrable systems. This is due in part to the fact that some of them are obtained by a reduction of the Einstein equations with Euclidean (and neutral) signature for (anti-) self-dual gravity, which includes the theory of gravitational instantons. This and other cases of important applications of multi-dimensional integrable equations strongly motivated us to study this class of equations and the related mathematical structures. As a very interesting aspect of our approach to describing integrability of the heavenly dynamical systems, there is a very interesting Lagrange–d’Alembert type mechanical interpretation. We need to underline here that the main motivating idea behind this work was based both on the paper by Kulish [27], devoted to studying the super-conformal Korteweg–de-Vries equation and the orbits of the corresponding Lie–Poisson type structures, were reanalyzed and studied in detail. By constructing two commuting flows on the coadjoint space ̃G∗, generated by a chosen root element ̃l ∈ ̃G∗ and some Casimir invariants, we have successively demonstrated that their compatibility condition coincides exactly with the corresponding heavenly equations and the related bi-Hamiltonian structure as well as the Bäcklund transformations. As a matter of fact, there are only a few examples of multi-dimensional integrable systems for which such a detailed description of their mathematical structure has been given. As was aptly mentioned in [59], the heavenly equations comprise an important class of such integrable systems. This is due in part to the fact that some of them are obtained by a reduction of the Einstein equations with Euclidean (and neutral) signature for (anti-) self-dual gravity, which includes the theory of gravitational instantons. This and other cases of important applications of multi-dimensional integrable equations strongly motivated us to study this class of equations and the related mathematical structures. As a very interesting aspect of our approach to describing integrability of the heavenly dynamical systems, there is a very interesting Lagrange–d’Alembert type mechanical interpretation. We need to underline here that the main motivating idea behind this work was based both on the paper by Kulish [27], devoted to studying the super-conformal Korteweg–de-Vries equation.
as an integrable Hamiltonian flow on the adjoint space to the holomorphic loop Lie superalgebra of super-conformal vector fields on the circle, and on the insightful investigation by Mikhailov [32], which studied Hamiltonian structures on the adjoint space to the holomorphic loop Lie algebra of smooth vector fields on the circle. We were also impressed by deep technical results [61, 62] of Takasaki and Takebe, who fully realized the vector field scheme of the Lax–Sato theory. Additionally, we were strongly influenced by the works of Pavlov, Bogdanov, Dryuma, Konopelchenko and Manakov [10, 8, 9, 21], as well as by the work of Ferapontov and Moss [20], in which they devised new effective differential-geometric and analytical methods for studying an integrable degenerate multi-dimensional dispersionless heavenly type hierarchy of equations, the mathematical importance of which is still far from being properly appreciated. Concerning other Lie-algebraic approaches to constructing integrable heavenly equations, we mention work by Szablowski and Sergyeyev [60, 57], Ovsienko [35, 36] and by Kruglikov and Morozov [26].

2. The Lax–Sato type compatible systems of linear vector field equations

2.1. A vector field on the torus and its invariants. Consider a simple vector field $X : \mathbb{R} \times \mathbb{T}^n \to T(\mathbb{R} \times \mathbb{T}^n)$ on the $(n + 1)$-dimensional toroidal cylinder $\mathbb{R} \times \mathbb{T}^n$ for arbitrary $n \in \mathbb{Z}_+$, which we will write in the slightly formal form

$$A = \frac{\partial}{\partial t} + \langle a(t, x), \frac{\partial}{\partial x} \rangle,$$

where $(t, x) \in \mathbb{R} \times \mathbb{T}^n, a(t, x) \in \mathbb{R}^n, \frac{\partial}{\partial x} := (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n})^T$ and $\langle \cdot, \cdot \rangle$ is the standard scalar product on the Euclidean space $\mathbb{R}^n$. With the vector field (2.1), one can associate the linear equation

$$A\psi = 0$$

for some function $\psi \in C^2(\mathbb{R} \times \mathbb{T}^n; \mathbb{R})$, which we will call an “invariant” of the vector field.

Next, we study the existence and number of such functionally-independent invariants to the equation (2.2). For this let us pose the following Cauchy problem for equation (2.2): Find a function $\Psi \in C^2(\mathbb{R} \times \mathbb{T}^n; \mathbb{R})$, which at point $t(0) \in \mathbb{R}$ satisfies the condition $\psi(t, x)|_{t=t(0)} = \psi(0)(x)$, $x \in \mathbb{T}^n$, for a given function $\psi(0) \in C^2(\mathbb{T}^n; \mathbb{R})$. For the equation (2.2) there is a naturally related parametric vector field on the torus $\mathbb{T}^n$ in the form of the ordinary vector differential equation

$$\frac{dx}{dt} = a(t, x),$$

to which there corresponds the following Cauchy problem: find a function $x : \mathbb{R} \to \mathbb{T}^n$ satisfying

$$x(t)|_{t=t(0)} = z$$

for an arbitrary constant vector $z \in \mathbb{T}^n$. Assuming that the vector-function $a \in C^1(\mathbb{R} \times \mathbb{T}^n; \mathbb{R}^n)$, it follows from the classical Cauchy theorem [14] on the existence and unicity of the solution to (2.4) that we can obtain a unique solution to the vector equation (2.3) as some function $\Phi \in C^1(\mathbb{R} \times \mathbb{T}^n; \mathbb{T}^n), x = \Phi(t, z)$, such that the matrix $\partial\Phi(t, z)/\partial z$ is nondegenerate for all $t \in \mathbb{R}$ sufficiently close to $t(0) \in \mathbb{R}$. Hence, the Implicit Function Theorem [14, 15] implies that there exists a mapping $\Psi : \mathbb{R} \times \mathbb{T}^n \to \mathbb{T}^n$, such that

$$\Psi(t, x) = z$$

for every $z \in \mathbb{T}^n$ and all $t \in \mathbb{R}$ sufficiently enough to $t(0) \in \mathbb{R}$. Supposing now that the functional vector $\Psi(t, x) = (\psi(t, x), \psi(2)(t, x), \ldots, \psi(n)(t, x))^T, (t, x) \in \mathbb{R} \times \mathbb{T}^n$, is constructed, from the arbitrariness of the parameter $z \in \mathbb{T}^n$ one can deduce that all functions $\psi(j) : \mathbb{R} \times \mathbb{T}^n \to \mathbb{T}^1, j = 1, n$, are functionally independent invariants of the vector field equation (2.2), that is $A\psi(j) = 0, j = 1, n$. Thus, the vector field equation (2.2) has exactly $n \in \mathbb{Z}_+$ functionally independent invariants, which make it possible, in particular, to solve the Cauchy problem posed above. Namely, let a mapping $\alpha : \mathbb{T}^n \to \mathbb{R}$ be chosen such that $\alpha(\Psi(t, x))|_{t=t(0)} = \psi(0)(x)$ for all $x \in \mathbb{T}^n$ and a fixed $t(0) \in \mathbb{R}$. Inasmuch as the superposition of functions $\alpha \circ \Psi : \mathbb{R} \times \mathbb{T}^n \to \mathbb{T}^1$ is, evidently, also an invariant for the equation (2.2), it provides the solution to this Cauchy problem, which we can formulate as the following result.

**Proposition 2.1.** The linear equation (2.2), generated by the vector field (2.3) on the torus $\mathbb{T}^n$, has exactly $n \in \mathbb{Z}_+$ functionally independent invariants.
Consider now a Plucker type \(2.4\) differential form \(\chi^{(n)} \in \Lambda^n(T^n)\) on the torus \(T^n\) as

\[
\chi^{(n)} := d\psi^{(1)} \land d\psi^{(2)} \land \ldots \land d\psi^{(n)},
\]

generated by the vector \(\Psi : \mathbb{R}^n \times T^n \to T^n\) of independent invariants \(2.5\), depending additionally on \(n \in \mathbb{Z}_+\) parameters \(t \in \mathbb{R}^n\), where by definition, for any \(k = 1, n\)

\[
d\psi^{(k)} := \sum_{j=1,n} \frac{\partial \psi^{(k)}}{\partial x_j} dx_j
\]
on the manifold \(T^n\). As follows from the Frobenius theorem \([3, 14, 22, 24]\), the Plucker type differential form \(2.6\) is for all fixed parameters \(t \in \mathbb{R}^n\) nonzero on the manifold \(T^n\) owing to the functional independence of the invariants \(2.5\). It is easy to see that at the fixed parameters \(t \in \mathbb{R}^n\) the following \([45]\) Jacobi-Mayer type relationship

\[
\frac{\partial \Psi}{\partial x}^{-1} d\psi^{(1)} \land d\psi^{(2)} \land \ldots \land d\psi^{(n)} = dx_1 \land dx_2 \land \ldots \land dx_n
\]

holds for \(k = 1, n\) on the manifold \(T^n\), where \(\left|\frac{\partial \Psi}{\partial x}\right|\) is the determinant of the Jacobi mapping \(\frac{\partial \Psi}{\partial x} : T(T^n) \to T(T^n)\) of the mapping \(2.5\) subject to the torus variables \(x \in T^n\). On the righthand side of \(2.8\) one has the volume measure on the torus \(T^n\), which is naturally dependent on \(t \in \mathbb{R}^n\) owing to the general vector field relationships \(2.8\). Taking into account that for all \(k = 1, n\) the full differentials

\[
d\psi^{(k)} = \sum_{s=1,n} \frac{\partial \psi^{(k)}}{\partial t_s} dt_s + d\psi^{(k)}
\]

vanish on \(\mathbb{R}^n \times T^n\), the corresponding substitution of the reduced differentials \(d\psi^{(k)} \in C^2(\mathbb{R}^n; \Lambda^1(T^n)), k = 1, n\) into \(2.8\) easily gives rise, in particular, to the following set of the compatible vector field relationships

\[
\frac{\partial \Psi}{\partial t_s} - \sum_{j,k=1,n} \left(\frac{\partial \Psi}{\partial x}\right)^{-1}_{jk} \frac{\partial \psi^{(k)}}{\partial t_s} \frac{\partial \Psi}{\partial x_j} = 0,
\]

for all \(s = 1, n\). The latter property, as it was demonstrated by M.G. Pfeiffer in \([45]\), makes it possible to solve effectively the M.A. Buhl problem and has interesting applications \([8, 24]\) in the theory of completely integrable dynamical systems of heavenly type, which are considered in the next section.

2.2. Vector field hierarchies on the torus with “spectral” parameter and the Lax–Sato integrable heavenly dynamical systems. Consider some naturally ordered infinite set of parametric vector fields \(2.1\) on the torus \(T^n\) in the form

\[
A^{(k)} = \frac{\partial}{\partial t_k} + a^{(k)}(t, x; \lambda) \frac{\partial}{\partial x}, \quad \frac{\partial}{\partial \lambda} + a_0^{(k)}(t, x; \lambda) \frac{\partial}{\partial \lambda} := \frac{\partial}{\partial t_k} + A^{(k)}
\]

where \(t_k \in \mathbb{R}, k \in \mathbb{Z}_+, (t, x; \lambda) \in (\mathbb{R}^n_+ \times T^n) \times \mathbb{C}\) are the evolution parameters, and the dependence of smooth vectors \((a^{(k)}, a_0^{(k)})^T \in \mathbb{R} \times \mathbb{R}^n, k \in \mathbb{Z}_+\) on the “spectral” parameter \(\lambda \in \mathbb{C}\) is assumed to be holomorphic. Suppose now that the infinite hierarchy of the linear equations

\[
A^{(k)} \psi = 0
\]

for \(k \in \mathbb{Z}_+\) possesses exactly \(n + 1 \in \mathbb{Z}_+\) common functionally independent invariants \(\psi^{(j)}(\lambda) \in C^2(\mathbb{R}^n_+ \times T^n; \mathbb{C}), j = 0, n\), for \(\lambda \in \mathbb{C}\). Then, owing to the existence theory \([14, 15, 24]\) for ordinary differential equations depending on the “spectral” parameter \(\lambda \in \mathbb{C}\), the solutions may be assumed to be such that allow analytical continuation in the parameter \(\lambda \in \mathbb{C}\) both inside \(S^1_+ \subset \mathbb{C}\) of some circle \(S^1 \subset \mathbb{C}\) and subject to the parameter \(\lambda^{-1} \in \mathbb{C}, |\lambda| \to \infty\), outside \(S^1_+ \subset \mathbb{C}\) of this circle.
S^1 \subset \mathbb{C}. This means that as |\lambda| \to \infty we have the following expansions:

\begin{align*}
\psi^{(0)}(\lambda) &\sim \lambda + \sum_{k=0}^{\infty} \psi^{(0)}_k(t,x)\lambda^{-k}, \\
\psi^{(1)}(\lambda) &\sim \sum_{k=0}^{\infty} \tau^{(1)}_k(t,x)\psi_0(\lambda)^k + \sum_{k=1}^{\infty} \psi^{(1)}_k(t,x)\psi_0(\lambda)^{-k}, \\
\psi^{(2)}(\lambda) &\sim \sum_{k=0}^{\infty} \tau^{(2)}_k(t,x)\psi_0(\lambda)^k + \sum_{k=1}^{\infty} \psi^{(2)}_k(t,x)\psi_0(\lambda)^{-k}, \\
&\vdots \\
\psi^{(n)}(\lambda) &\sim \sum_{k=0}^{\infty} \tau^{(n)}_k(t,x)\psi_0(\lambda)^k + \sum_{k=1}^{\infty} \psi^{(n)}_k(t,x)\psi_0(\lambda)^{-k},
\end{align*}

(2.13)

where we took into account that \psi^{(0)}(\lambda) \in C^2(\mathbb{R}^{Z^+} \times \mathbb{T}^n; \mathbb{C}), \lambda \in \mathbb{S}^1 \subset \mathbb{C}, is the basic invariant solution to the equations (2.12), functions \tau^{(s)}_k(t,x) \in C^2(\mathbb{R}^{Z^+} \times \mathbb{T}^n; \mathbb{C}) for all \lambda \in [1, \infty) and suitable \lambda \in \mathbb{S}^1 \subset \mathbb{C}, and \psi^{(j)}_k(t,x) \in C^2(\mathbb{R}^{Z^+} \times \mathbb{T}^n; \mathbb{C}) for all \lambda \in [1, \infty), j = 0, n. Write down now the condition (2.8) on the manifold \mathbb{C} \times \mathbb{T}^n in the equivalent form

\begin{equation}
|\frac{\partial \Psi}{\partial x}|^{-1} d\psi^{(0)}(t,x) \wedge d\psi^{(1)}(t,x) \wedge \ldots \wedge d\psi^{(n)}(t,x) = d\lambda \wedge dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n
\end{equation}

at fixed parameters \ t \in \mathbb{R}^{Z^+}, where x := (\lambda, x) \in \mathbb{C} \times \mathbb{T}^n, |\frac{\partial \Psi}{\partial x}| is the Jacobi determinant of the mapping \Psi := (\psi^{(0)}(t,x), \psi^{(1)}(t,x), \ldots, \psi^{(n)}(t,x))^T \in C^2(\mathbb{C} \times (\mathbb{R}^{Z^+} \times \mathbb{T}^n); \mathbb{C}^{n+1}) subject to the variables x \in \mathbb{C} \times \mathbb{T}^n\text{. Inasmuch as this mapping subject to the parameter} \lambda \in \mathbb{C}\text{ has analytical continuation inside} \mathbb{S}^1 \subset \mathbb{C}\text{ of the circle} \mathbb{S}^1 \subset \mathbb{C}\text{ and subject to the parameter} \lambda^{-1} \in \mathbb{C}\text{ as} |\lambda| \to \infty\text{ outside} \mathbb{S}^1 \subset \mathbb{C}\text{ of this circle} \mathbb{S}^1 \subset \mathbb{C}, \text{one can easily obtain from the holomorphic structure of the vector fields} (2.11)\text{ subject to the complex variable} \lambda \in \mathbb{C}\text{ and the relationship} (2.14),\text{ reduced on the manifold} \mathbb{S}^1 \times \mathbb{T}^n, \text{the following analytical criterion:}

\begin{equation}
\left( |\frac{\partial \Psi}{\partial x}|^{-1} d\psi^{(0)}(t,x) \wedge d\psi^{(1)}(t,x) \wedge \ldots \wedge d\psi^{(n)}(t,x) \right)_- = 0,
\end{equation}

(2.15)

where (...)_− means the asymptotic part of the expression in brackets, depending on the negative degree parameter \lambda^{-1} \in \mathbb{S}^1 \subset \mathbb{C} of \lambda \to \infty. Now, since the full differentials d\psi^{(j)}(t,x) \in \Lambda^1(\mathbb{C} \times (\mathbb{R}^{n \times Z^+} \times \mathbb{T}^n), j = 0, n, vanish on \mathbb{C} \times (\mathbb{R}^{n \times Z^+} \times \mathbb{T}^n), we have

\begin{equation}
d\psi^{(j)}(t,x) = -\sum_{k=0}^{\infty} \frac{\partial \psi^{(j)}_k(t,x)}{\partial t^{(j)}_k(t,x)} dT_k^{(j)}(t,x),
\end{equation}

(2.16)

whose substitution into (2.15) yields

\begin{equation}
\frac{\partial \Psi}{\partial T_k^{(j)}} = \left[ \left( \frac{\partial \Psi}{\partial x} \right)_0^{-1} \psi^{(0)}(\lambda)^k \right] + \frac{\partial \Psi}{\partial \lambda} + \sum_{s=1}^{n} \left[ \left( \frac{\partial \Psi}{\partial x} \right)_s^{-1} \psi^{(0)}(\lambda)^k \right] \bigg| _{s_j} + \frac{\partial \Psi}{\partial x},
\end{equation}

(2.17)

for all \ k \in \mathbb{Z}^+, j = 1, n; these expressions comprise an infinite hierarchy of Lax–Sato compatible linear equations, where (...)_− denotes the asymptotic part of the expression in brackets, depending on positive powers of the parameter \lambda \in \mathbb{C}. As for the functional parameters \tau^{(j)}_k(t,x) \in C^1(\mathbb{R}^{Z^+} \times \mathbb{T}^n; \mathbb{C}) for all \ k \in \mathbb{Z}^+, j = 1, n, one can prove their functional independence by taking into account their a priori linear dependence on the corresponding independent evolution parameters \ t_k \in \mathbb{R}, \ k \in \mathbb{Z}^+. On the other hand, taking into account the explicit form of the hierarchy of equations (2.17), following [8], it is not hard to show that the corresponding vector fields

\begin{equation}
A_k^{(j)} := \left[ \left( \frac{\partial \Psi}{\partial x} \right)_0^{-1} \psi^{(0)}(\lambda)^k \right] + \frac{\partial \Psi}{\partial \lambda} + \sum_{s=1}^{n} \left[ \left( \frac{\partial \Psi}{\partial x} \right)_s^{-1} \psi^{(0)}(\lambda)^k \right] \bigg| _{s_j} + \frac{\partial \Psi}{\partial x},
\end{equation}

(2.18)
on the manifold $\mathbb{C} \times \mathbb{T}^n$ satisfy for all $k, m \in \mathbb{Z}_+, j, l = 1, n$, the Lax compatibility conditions

$$\frac{\partial A_{m}^{(l)}}{\partial \tau_{k}^{(j)}} - \frac{\partial A_{k}^{(j)}}{\partial \tau_{m}^{(l)}} = [A_{k}^{(j)}, A_{m}^{(l)}],$$

which are equivalent to the independence of the all functional parameters $\tau_{k}^{(j)} \in C^1(\mathbb{R}^{2+} \times \mathbb{T}^n; \mathbb{C})$, $k \in \mathbb{Z}_+, j = 1, n$. As a corollary of the analysis above, one can show that the infinite hierarchy of vector fields (2.18) is a linear combination of the basic vector fields (2.11) and also satisfies the Lax type compatibility condition (2.19). Inasmuch the coefficients of vector fields (2.11) are suitably smooth functions on the manifold $\mathbb{R}^{2+} \times \mathbb{T}^n$, the compatibility conditions (2.19) yield the corresponding sets of differential-algebraic relationships on their coefficients, which have the common infinite set of invariants, thereby comprising an infinite hierarchy of completely integrable so called "heavenly" nonlinear dynamical systems on the corresponding multidimensional functional manifolds. That is, all of the above can be considered as an introduction to a recently devised constructive algorithm for generating completely integrable nonlinear dynamical systems of heavenly type on functional manifolds of arbitrary dimension. It is worthwhile to stress here that the above constructive algorithm for generating completely integrable nonlinear multidimensional dynamical systems still does not make it possible to directly show they are Hamiltonian and construct other related mathematical structures. This important problem is solved by employing other mathematical theories; for example, the analytical properties of the related loop diffeomorphisms groups generated by the hierarchy of vector fields (2.11).

Remark 2.2. The compatibility condition (2.19) allows an alternative differential-geometric description based on the Lie-algebraic properties of the basic vector fields (2.11). Namely, consider the manifold $\mathbb{R}^{n \times 2+}$, as the base manifold of the vector bundle $E(\mathbb{R}^{n \times 2+}, G)$, $E = \bigcup_{\tau} \mathbb{R}^{n \times 2+} \{ (G^{*} \rtimes \tau) / \rho \}$, $G^{*} := \{ \phi^{*} : \phi^{*} \beta^{(1)} := \alpha^{(1)} \circ \phi, \beta^{(1)} \in \Lambda^{1}(\mathbb{C} \times \mathbb{T}^n; \mathbb{C}), \phi \in G \}$ for an equivalence relation $\rho$ and the (holomorphic in $\lambda \in \mathbb{S}_1 \cup \mathbb{S}_1 \subset \mathbb{C}$) structure group $G = Diff_{hol}(\mathbb{C} \times \mathbb{T}^n)$, naturally acting on the vector space $E$. The structure group can be endowed with a connection $\Upsilon$ by means of a mapping $d_{h} : \Gamma(E) \rightarrow \Gamma(T^{*}(\mathbb{R}^{n \times 2+}) \otimes E) \cong \Gamma(\text{Hom}(T(\mathbb{R}^{n \times 2+}); E))$, where

$$d_{h}\varphi^{*}_{\tau} := \sum_{j \in \mathbb{Z}_+} d_{\tau}^{(k)} \otimes \frac{\partial}{\partial \tau^{(k)}} \circ \varphi^{*}_{\tau} + \varphi^{*}_{\tau} \circ \alpha^{(1)} \frac{\partial}{\partial \lambda},$$

\(\alpha^{(1)} := \sum_{j \in \mathbb{Z}_+} \alpha_{j}^{(1)} \partial_{\tau_{j}}^{(k)} \in \Lambda(\mathbb{R}^{n \times 2+}) \otimes \Gamma(E)\), which is defined for any cotangent diffeomorphism $\varphi^{*}_{\tau} \in E, \tau \in \mathbb{R}^{n \times 2+}$, generated by the set of parametric vector fields (2.11), and naturally acting on any mapping $\psi \in C^{2}(\mathbb{R}^{n \times 2+} \times (\mathbb{C} \times \mathbb{T}^n); \mathbb{C})$ as $\varphi^{*}_{\tau} \circ \psi(\tau, x) := \psi(\tau, \varphi(\tau, x))$, $(\tau, x) \in \mathbb{R}^{n \times 2+} \times \mathbb{T}^n$. It is easy now to see that the corresponding to (2.20) zero curvature condition $d_{h}^{2} = 0$ is equivalent to the set of compatibility equations (2.19). Moreover, the parallel transport equation

$$d_{h}\varphi^{*}_{\tau} \circ \psi = 0$$

coincides exactly with the infinite hierarchy of linear vector field equations (2.17), where $\psi \in C^{2}(\mathbb{R}^{n \times 2+} \times \mathbb{T}^n; \mathbb{R})$ is their invariant. Conversely, the Cartan integrable ideal of differential forms $h(o) \in \Lambda(\mathbb{R}^{n \times 2+} \times \mathbb{T}^n) \otimes \Gamma(T^{*}(\mathbb{R}^{n \times 2+}))$, which is equivalent to the zero curvature condition $d_{h}^{2} = 0$, makes it possible to retrieve [4, 48] the corresponding connection $\Upsilon$ by constructing a mapping $d_{h} : \Gamma(E) \rightarrow \Gamma(T^{*}(\mathbb{R}^{n \times 2+}) \otimes E) \cong \Gamma(\text{Hom}(T(\mathbb{R}^{n \times 2+}); E))$ in the form (2.20). These and other interesting related aspects of the integrable heavenly dynamical systems shall be investigated separately elsewhere.

2.3. Example: the vector field representation for the Mikhalev–Pavlov equation. The Mikhalev–Pavlov equation was first constructed in [32] and has the form

$$u_{xt} + u_{yy} = u_{y}u_{xx} - u_{x}u_{xy},$$

where $u \in C^{\infty}(\mathbb{R}^{2} \times \mathbb{T}^1; \mathbb{R})$ and $(t, y, x) \in \mathbb{R}^{2} \times \mathbb{T}^1$. Assume now [8] that the following two functions

$$\psi(0) = \lambda, \quad \psi^{(1)} \sim \sum_{k=3}^{\infty} \lambda^{k} \tau_{k} - \lambda^{2} t + \lambda y + x + \sum_{j=1}^{\infty} \psi_{j}^{(1)}(t, y, \tau; x) \lambda^{-j},$$
where $\psi_1^{(1)}(t, y, \tau; x) = u$, (t, y, \tau; x) $\in \mathbb{R}^2 \times \mathbb{R}^\infty \times \mathbb{T}^1$, are invariants of the set of vector fields \((2.12)\) for an infinite set of constant parameters $\tau_k \in \mathbb{R}$, $k = 3, \infty$, as the complex parameter $\lambda \to \infty$. By applying to the invariants \((2.23)\) the criterion \((2.15), (2.16)\) in the form
\begin{equation}
(\partial \psi^{(1)}/\partial x)^{-1} \text{d}\psi^{(1)}_{\tau} = 0,
\end{equation}
one can easily obtain the following compatible linear vector field equations
\begin{equation}
\frac{\partial \psi}{\partial t} + (\lambda^2 + \lambda u_x - u_y) \frac{\partial \psi}{\partial x} = 0,
\end{equation}
\begin{equation}
\frac{\partial \psi}{\partial y} + (\lambda + u_x) \frac{\partial \psi}{\partial x} = 0,
\end{equation}
\begin{equation}
\frac{\partial \psi}{\partial \tau_k} + P_k(u; \lambda) \frac{\partial \psi}{\partial x} = 0,
\end{equation}
where $P_k(u; \lambda), k = 3, \infty$, are independent differential-algebraic polynomials in the variable $u \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^\infty \times \mathbb{T}^1)$ and algebraic polynomials in the spectral parameter $\lambda \in \mathbb{C}$, calculated from the expressions \((2.17)\). Moreover, as one can check, the compatibility condition \((2.25)\) for the first two vector field equations of \((2.26)\) yields exactly the Mikhalev–Pavlov equation \((2.22)\).

2.4. Example: The Dunajski metric nonlinear equation. The equations for the Dunajski metric \((2.17)\) are
\begin{equation}
u_{x1t} + u_{y12} + u_{x11}u_{x22} - u_{x1x2} - v = 0,
\end{equation}
\begin{equation}
u_{x1t} + v_{x2y} + u_{x1x1}v_{x2x2} - 2u_{x1x2}v_{x1x2} = 0,
\end{equation}
where $(u, v) \in C^\infty(\mathbb{R}^2 \times \mathbb{T}^2; \mathbb{R}^2)$, $(y, t; x_1, x_2) \in \mathbb{R}^2 \times \mathbb{T}^2$. One can construct now, by definition, the following asymptotic expansions
\begin{equation}
\psi^{(0)} \sim \lambda + \sum_{j=1}^{\infty} \psi_j^{(0)}(t, y; x) \lambda^{-j},
\end{equation}
\begin{equation}
\psi^{(1)} \sim \sum_{k=2}^{\infty} (\psi_0^{(1)})^k \tau_k^{(1)} - \psi_0^{(0)} y + x_1 + \sum_{j=1}^{\infty} \psi_j^{(1)}(t, y; x) (\psi_0^{(0)})^{-j},
\end{equation}
\begin{equation}
\psi^{(2)} \sim \sum_{k=2}^{\infty} (\psi_0^{(2)})^k \tau_k^{(2)} + \psi_0^{(0)} t + x_2 + \sum_{j=1}^{\infty} \psi_j^{(2)}(t, y; x) (\psi_0^{(0)})^{-j},
\end{equation}
where $\partial u/\partial x_1 := \psi_1^{(1)}, \partial u/\partial x_2 := \psi_1^{(1)}, v := \psi_1^{(0)}$ and $\tau_k^{(s)} \in \mathbb{R}$, $s = 1, 2, k = 2, \infty$, are constant parameters. Then the Lax–Sato conditions \((2.15), (2.16)\)
\begin{equation}
(\partial \psi^{(0)}, \psi^{(1)}, \psi^{(2)}) \frac{-1}{\partial (\lambda, x_1, x_2)} \text{d}\psi^{(0)} \wedge \text{d}\psi^{(1)} \wedge \text{d}\psi^{(2)} = 0
\end{equation}
yield a compatible hierarchy of the following linear vector field equations:
\begin{equation}
X^{(t_0)} \psi := \frac{\partial \psi}{\partial t} + X^{(t_0)} \psi = 0, \quad X^{(t_0)} := u_{x2x2} \frac{\partial}{\partial x_1} - (\lambda + u_{x1x2}) \frac{\partial}{\partial x_2} + v_{x2} \frac{\partial}{\partial x_3} = 0,
\end{equation}
\begin{equation}
X^{(t_1)} \psi := \frac{\partial \psi}{\partial y} + X^{(t_1)} \psi = 0, \quad X^{(t_1)} := (\lambda - u_{x1x2}) \frac{\partial}{\partial x_1} + u_{x1x1} \frac{\partial}{\partial x_2} - v_{x1} \frac{\partial}{\partial x_3} = 0,
\end{equation}
\begin{equation}
X^{(t_k^{(s)})} \psi := \frac{\partial \psi}{\partial \tau_k^{(s)}} + P_{k}^s(u; \lambda) \frac{\partial \psi}{\partial x} = 0,
\end{equation}
where $P_{k}^s(u; v; \lambda) \in \mathbb{E}^3, s = 1, 2, k = 2, \infty$, are analytic in $\lambda \in \mathbb{C}$ independent differential-algebraic vector polynomials \((10)\) in the variables $(u, v) \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^\infty \times \mathbb{T}^2; \mathbb{R}^2)$ and algebraic polynomials in the spectral parameter $\lambda \in \mathbb{C}$, calculated from the expressions \((2.17)\). In particular, the
compatibility condition \((2.14)\) for the first two equations of \((2.25)\) is equivalent to the Dunajski metric nonlinear equations \((2.26)\).

The description of the Lax–Sato equations presented above, especially their alternative differential-geometric interpretation \((2.20)\) and \((2.21)\), makes it possible to realize that the structure group \(Diff_{\text{hol}}(\mathbb{C} \times \mathbb{T}^n)\) should play an important role in unveiling the hidden Lie-algebraic nature of the integrable heavenly dynamical systems. This is actually the case, and a detailed analysis is presented in the sequel.

3. HEAVENLY EQUATIONS: THE LIE-ALGEBRAIC INTEGRABILITY SCHEME

Let \(\hat{G}_\pm := \hat{Diff}_\pm(\mathbb{T}^n), n \in \mathbb{Z}_+,\) be subgroups of the loop diffeomorphisms group \(\hat{Diff}(\mathbb{T}^n) := \{ \mathbb{C} \supset \mathbb{S}^1 \to Diff(\mathbb{T}^n) \}\), holomorphically extended in the interior \(\mathbb{S}^1_+ \subset \mathbb{C}\) and in the exterior \(\mathbb{S}^1_- \subset \mathbb{C}\) regions of the unit circle \(\mathbb{S}^1 \subset \mathbb{C}\), such that for any \(g(\lambda) \in \hat{G}_\pm, \lambda \in \mathbb{S}^1_, g(\infty) = 1 \in Diff(\mathbb{T}^n)\). The corresponding Lie subalgebras \(\hat{G}_\pm := \hat{diff}_\pm(\mathbb{T}^n)\) of the loop subgroups \(\hat{G}_\pm\) are vector fields on \(\mathbb{T}^n\) holomorphic, respectively, on \(\mathbb{S}^1_\pm \subset \mathbb{C}\), where for any \(\check{a}(\lambda) \in \hat{G}_\check{-}\) the value \(\check{a}(\infty) = 0\). The split loop Lie algebra \(\hat{G} = \hat{G}_+ + \hat{G}_-\) can be naturally identified with a dense subspace of the dual space \(\hat{G}^*\) through the pairing

\[
\langle \tilde{l}, \check{a} \rangle := \frac{1}{2\pi i} \int_{\mathbb{S}^1} \langle \tilde{l}(x, \lambda), a(x, \lambda) \rangle_{H^q} d\lambda, 
\]

for some fixed \(p, q \in \mathbb{Z}_+\). We took above, by definition \([44, 49]\), a loop vector field \(\check{a} \in \hat{G}(\check{T}(\mathbb{T}^n))\) and a loop differential 1-form \(\tilde{l} \in \hat{\Lambda}^1(\mathbb{T}^n)\) given as

\[
\check{a} = \sum_{j=1}^{n} a^{(j)}(x, \lambda) \frac{\partial}{\partial x_j} := \left\langle a(x, \lambda), \frac{\partial}{\partial x} \right\rangle, 
\]

\[
\tilde{l} = \sum_{j=1}^{n} l_j(x, \lambda) dx_j := \left\langle l(x, \lambda), dx \right\rangle, 
\]

introduced for brevity the gradient operator \(\frac{\partial}{\partial x} := \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n} \right)^T\) in the Euclidean space \(\mathbb{E}^n\) and chose the Sobolev type metric \((\cdot, \cdot)_{H^q}\) on the space \(C^\infty(\mathbb{T}^n; \mathbb{R}^n) \subset H^q(\mathbb{T}^n; \mathbb{R}^n)\) for some \(q \in \mathbb{Z}_+\) as

\[
\langle l(x; \lambda), a(x; \lambda) \rangle_{H^q} := \sum_{j=1}^{n} \sum_{|\alpha| = 0}^{q} \int_{\mathbb{T}^n} dx \left( \frac{\partial^{|\alpha|} l_j(x; \lambda)}{\partial x^\alpha} \frac{\partial^{\alpha} a^{(j)}(x; \lambda)}{\partial x^\alpha} \right), 
\]

where \(\partial x^\alpha := \partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}\), \(|\alpha| = \sum_{j=1}^{n} \alpha_j\) for \(\alpha \in \mathbb{Z}_+^n\), generalizing the metric used before in \([33]\). The Lie commutator of vector fields \(\tilde{a}, \tilde{b} \in \hat{G}\) is calculated the standard way and equals

\[
[\tilde{a}, \tilde{b}] = \tilde{a} \tilde{b} - \tilde{b} \tilde{a} = \left\langle a(x; \lambda), \frac{\partial}{\partial x} \right\rangle b(x; \lambda), \frac{\partial}{\partial x} \right\rangle - 
\]

\[
- \left\langle b(x; \lambda), \frac{\partial}{\partial x} \right\rangle a(x; \lambda), \frac{\partial}{\partial x} \right\rangle. 
\]

The Lie algebra \(\hat{G}\) naturally splits into the direct sum of two Lie subalgebras

\[
\hat{G} = \hat{G}_+ \oplus \hat{G}_-, 
\]

for which one can identify the dual spaces

\[
\hat{G}_+^* \simeq \lambda^{p-1} \hat{G}_-, \quad \hat{G}_-^* \simeq \lambda^{p-1} \hat{G}_+^*, 
\]

where for any \(l(\lambda) \in \hat{G}_-^*\) one has the constraint \(\tilde{l}(0) = 0\). Having defined now the projections

\[
P_{\pm} \hat{G} := \hat{G}_{\pm} \subset \hat{G}, 
\]

one can construct a classical \(R\)-structure \([63, 51, 56]\) on the Lie algebra \(\hat{G}\) as the endomorphism \(R : \hat{G} \to \hat{G}\), where

\[
R := (P_+ - P_-)/2, 
\]
which allows to determine on the vector space $\tilde{G}$ the new Lie algebra structure

\begin{equation}
[\tilde{a}, \tilde{b}]_R := [\mathcal{R}\tilde{a}, \tilde{b}] + [\tilde{a}, \mathcal{R}\tilde{b}]
\end{equation}

for any $\tilde{a}, \tilde{b} \in \tilde{G}$, satisfying the standard Jacobi identity.

Let $D(\tilde{G}^*)$ denote the space of smooth functions on $\tilde{G}^*$. Then for any $f, g \in D(\tilde{G}^*)$ one can write the canonical \cite{[63, 51, 48, 4]} Lie–Poisson bracket

\begin{equation}
\{ f, g \} := (\tilde{l}, [\nabla f(\tilde{l}), \nabla g(\tilde{l})]),
\end{equation}

where $\tilde{l} \in \tilde{G}^*$ is a seed element and $\nabla f, \nabla g \in \tilde{G}$ are the usual functional gradients at $\tilde{l} \in \tilde{G}^*$ with respect to the metric \cite{[51, 48, 4]}. The related space $I(\tilde{G}^*)$ of Casimir invariants is defined as the set $I(\tilde{G}^*) \subset D(\tilde{G}^*)$ of smooth independent functions $\gamma_j \in D(\tilde{G}^*), j = 1, n$, for which

\begin{equation}
ad^*_{\nabla \gamma_j(\tilde{l})} \tilde{l} = 0,
\end{equation}

where for any seed element

\begin{equation}
\tilde{l} = \langle l, dx >
\end{equation}

the gradients

\begin{equation}
\nabla \gamma_j(\tilde{l}) := \left( \nabla \gamma_j(l), \frac{\partial}{\partial x} \right)
\end{equation}

and the coadjoint action \cite{[3.10]} can be equivalently rewritten, for instance, in the case $q = 0$, as

\begin{equation}
\left( \frac{\partial}{\partial x}, \nabla \gamma_j(l) \right) l + \left( l, \frac{\partial}{\partial x} \nabla \gamma_j(l) \right) = 0
\end{equation}

for any $j = 1, n$. If one takes two smooth functions $\gamma^{(y)}, \gamma^{(t)} \in I(\tilde{G}^*) \subset D(\tilde{G}^*)$, their second Poisson bracket

\begin{equation}
\{ \gamma^{(y)}, \gamma^{(t)} \}_R := (\tilde{l}, [\nabla \gamma^{(y)}, \nabla \gamma^{(t)}]|_R)
\end{equation}

on the space $\tilde{G}^*$ vanishes; that is,

\begin{equation}
\{ \gamma^{(y)}, \gamma^{(t)} \}_R = 0
\end{equation}

at any seed element $\tilde{l} \in \tilde{G}^*$. Since the functions $\gamma^{(y)}, \gamma^{(t)} \in I(\tilde{G}^*)$, the following coadjoint action relationships hold:

\begin{equation}
ad^*_{\nabla \gamma^{(y)}(\tilde{l})} \tilde{l} = 0, \quad ad^*_{\nabla \gamma^{(t)}(\tilde{l})} \tilde{l} = 0,
\end{equation}

which can be equivalently rewritten (as above in the case $q = 0$) as

\begin{equation}
\left( \frac{\partial}{\partial x}, \nabla \gamma^{(y)}(l) \right) l + \left( l, \frac{\partial}{\partial x} \nabla \gamma^{(y)}(l) \right) =
\end{equation}

\begin{equation}
\left( \nabla \gamma^{(y)}(l), \frac{\partial}{\partial x} \right) l + \left( l, \frac{\partial}{\partial x} \nabla \gamma^{(y)}(l) \right) + \left( l, \frac{\partial}{\partial x} \nabla \gamma^{(t)}(l) \right) :=
\end{equation}

\begin{equation}
= (A_{\nabla \gamma^{(y)}} + B_{\nabla \gamma^{(y)}})l
\end{equation}

and similarly

\begin{equation}
\left( \frac{\partial}{\partial x}, \nabla \gamma^{(t)}(l) \right) l + \left( l, \frac{\partial}{\partial x} \nabla \gamma^{(t)}(l) \right) := (A_{\nabla \gamma^{(t)}} + B_{\nabla \gamma^{(t)}})l,
\end{equation}

where the expressions

\begin{equation}
A_{\nabla \gamma^{(y)}} := \left( \nabla \gamma^{(y)}(l), \frac{\partial}{\partial x} \right), \quad A_{\nabla \gamma^{(t)}} := \left( \nabla \gamma^{(t)}(l), \frac{\partial}{\partial x} \right)
\end{equation}

are true vector fields on $\mathbb{T}^n$, yet the expressions

\begin{equation}
B_{\nabla \gamma^{(y)}} := \left( \frac{\partial}{\partial x}, \nabla \gamma^{(y)}(l) \right) + \left( \frac{\partial}{\partial x} \nabla \gamma^{(y)}(l) \right),
\end{equation}

\begin{equation}
B_{\nabla \gamma^{(t)}} := \left( \frac{\partial}{\partial x}, \nabla \gamma^{(t)}(l) \right) + \left( \frac{\partial}{\partial x} \nabla \gamma^{(t)}(l) \right),
\end{equation}

are the usual matrix homomorphisms of the Euclidean space $\mathbb{E}^n$. 


Consider now the following Hamiltonian flows on the space $\tilde{G}^*$:

$$\frac{\partial \tilde{l}}{\partial y} := \{h^{(y)}, \tilde{l}\}_R = -ad^*_{\nabla h^{(y)}(\tilde{l})^+} \tilde{l},$$

$$\frac{\partial \tilde{l}}{\partial t} := \{h^{(t)}, \tilde{l}\}_R = -ad^*_{\nabla h^{(t)}(\tilde{l})^+} \tilde{l},$$

where $h^{(y)}, h^{(t)} \in I(\tilde{G}^*)$ and $y, t \in \mathbb{R}$ are the corresponding evolution parameters. Since $h^{(y)}, h^{(t)} \in I(\tilde{G}^*)$ are Casimirs, the flows (3.21) commute. Thus, taking into account the representations (3.17), one can recast the flows (3.21) as

$$\frac{\partial \tilde{l}}{\partial t} = -(\tilde{A}_{\nabla h^{(y)}} + B_{\nabla h^{(t)}})\tilde{l}, \quad \frac{\partial \tilde{l}}{\partial y} = -(\tilde{A}_{\nabla h^{(y)}} + B_{\nabla h^{(y)}})\tilde{l},$$

where

$$\tilde{A}_{\nabla h^{(y)}} := \left\langle \nabla h^{(y)}(l^+), \frac{\partial}{\partial x} \right\rangle, \quad \tilde{A}_{\nabla h^{(t)}} := \left\langle \nabla h^{(t)}(l^+), \frac{\partial}{\partial x} \right\rangle.$$

**Lemma 3.1.** The compatibility of commuting flows (3.22) is equivalent to the Lax type vector fields relationship

$$\partial \tilde{A}_{\nabla h^{(y)}} / \partial y - \partial \tilde{A}_{\nabla h^{(t)}} / \partial t + [\tilde{A}_{\nabla h^{(y)}}, \tilde{A}_{\nabla h^{(t)}}] = 0,$$

which holds for all $y, t \in \mathbb{R}$ and arbitrary $\lambda \in \mathbb{C}$.

**Proof.** The compatibility of commuting flows (3.22) implies that $\partial^2 l / \partial t \partial y - \partial^2 l / \partial y \partial t = 0$ for all $y, t \in \mathbb{R}$ and arbitrary $\lambda \in \mathbb{C}$. Taking into account the expressions (3.21), one has for any vector field $Z = \langle Z, \frac{\partial}{\partial x} \rangle \in \tilde{G}$

$$0 = (\partial^2 l / \partial t \partial y - \partial^2 l / \partial y \partial t, Z) = -\frac{\partial}{\partial y} (ad^*_{\nabla h^{(y)}(\tilde{l})^+} \tilde{l}, Z) + \frac{\partial}{\partial y} (ad^*_{\nabla h^{(t)}(\tilde{l})^+} \tilde{l}, Z) =$$

$$= -(\frac{\partial}{\partial y}, [\nabla h^{(y)}(\tilde{l})^+, Z]) + \frac{\partial}{\partial y} (\tilde{l}, [\nabla h^{(y)}(\tilde{l})^+, Z]) =$$

$$= -(\frac{\partial}{\partial y}, [\nabla h^{(y)}(\tilde{l})^+, Z]) - (\tilde{l}, [\frac{\partial}{\partial y} \nabla h^{(y)}(\tilde{l})^+, Z]) +$$

$$+ (\frac{\partial}{\partial y}, [\nabla h^{(t)}(\tilde{l})^+, Z]) + (\tilde{l}, [\frac{\partial}{\partial y} \nabla h^{(y)}(\tilde{l})^+, Z]) =$$

$$= (ad^*_{\nabla h^{(y)}(\tilde{l})^+} \tilde{l}, [\nabla h^{(y)}(\tilde{l})^+, Z]) - (\tilde{l}, [\frac{\partial}{\partial y} \nabla h^{(y)}(\tilde{l})^+, Z]) -$$

$$- (ad^*_{\nabla h^{(t)}(\tilde{l})^+} \tilde{l}, [\nabla h^{(t)}(\tilde{l})^+, Z]) + (\tilde{l}, [\frac{\partial}{\partial y} \nabla h^{(y)}(\tilde{l})^+, Z]) =$$

$$= (\tilde{l}, [\nabla h^{(y)}(\tilde{l})^+, [\nabla h^{(y)}(\tilde{l})^+, Z]]) - (\tilde{l}, [\frac{\partial}{\partial y} \nabla h^{(y)}(\tilde{l})^+, Z]) -$$

$$- (\tilde{l}, [\nabla h^{(y)}(\tilde{l})^+, [\nabla h^{(t)}(\tilde{l})^+, Z]]) + (\tilde{l}, [\frac{\partial}{\partial y} \nabla h^{(t)}(\tilde{l})^+, Z]) =$$

$$= (\tilde{l}, [\nabla h^{(y)}(\tilde{l})^+, [\nabla h^{(y)}(\tilde{l})^+, Z]] - [\nabla h^{(y)}(\tilde{l})^+, [\nabla h^{(t)}(\tilde{l})^+, Z]]) +$$

$$+ (\tilde{l}, [\frac{\partial}{\partial y} \nabla h^{(t)}(\tilde{l})^+, -\frac{\partial}{\partial y} \nabla h^{(y)}(\tilde{l})^+, Z]) =$$

$$= (\tilde{l}, [\nabla h^{(y)}(\tilde{l})^+, \nabla h^{(t)}(\tilde{l})^+]) + \frac{\partial}{\partial y} \nabla h^{(y)}(\tilde{l})^+ - \frac{\partial}{\partial y} \nabla h^{(y)}(\tilde{l})^+, Z) =$$

$$= (\tilde{l}, [\nabla h^{(y)}(\tilde{l})^+, \nabla h^{(t)}(\tilde{l})^+], Z] = (ad^*_{\nabla h^{(y)}(\tilde{l})^+} \tilde{l}, Z),$$

where

$$\nabla H(\tilde{l}) := \partial \tilde{A}_{\nabla h^{(y)}} / \partial y - \partial \tilde{A}_{\nabla h^{(t)}} / \partial t + [\tilde{A}_{\nabla h^{(y)}}, \tilde{A}_{\nabla h^{(t)}}], Z] = (ad^*_{\nabla h^{(y)}(\tilde{l})^+} \tilde{l}, Z).$$

From (3.24) we obtain that $ad^*_{\nabla H(\tilde{l})} \tilde{l} = 0$ for all $y, t \in \mathbb{R}$ and arbitrary $\lambda \in \mathbb{C}$; that is, the vector field (3.26) is the gradient of an analytical Casimir functional $H \in I(\tilde{G}^*)$. Now based on the
analyticity of the vector field expression (3.26), one easily shows that \( \nabla H(\tilde{l}) = 0 \), thus finishing the proof.

For the exact representatives of the functions \( h^{(y)}, h^{(t)} \in I(\hat{G}^*) \), it is necessary to solve the determining equation (3.13), taking into account that if the chosen element \( \tilde{l} \in \hat{G}^* \) is singular as \( |\lambda| \to \infty \), the related expansion

\[
\nabla \gamma^{(p)}(l) \sim \lambda^p \sum_{j \in \mathbb{Z}_+} \nabla \gamma_j(l) \lambda^{-j},
\]

where the degree \( p \in \mathbb{Z}_+ \) can be taken as arbitrary. Upon substituting (3.27) into (3.13) one can find recurrently all the coefficients \( \nabla \gamma_j(l) \), \( j \in \mathbb{Z}_+ \), and then construct gradients of the Casimir functions \( h^{(y)}, h^{(t)} \in I(\hat{G}^*) \) reduced on \( \tilde{G}_+ \) as

\[
\nabla h^{(y)}_+(l) = (\lambda^p \nabla \gamma(l))|_+, \quad \nabla h^{(t)}_+(l) = (\lambda^p \nabla \gamma(l))|_+
\]

for some positive integers \( p_y, p_t \in \mathbb{Z}_+ \). Then the corresponding flows are, respectively, written as

\[
\partial \tilde{l}/\partial t = a d^*_{\nabla h^{(y)}(l)} \tilde{l}, \quad \partial \tilde{l}/\partial y = a d^*_{\nabla h^{(t)}(l)} \tilde{l}.
\]

The above results, owing to (3.11), can be formulated as the following main proposition.

**Proposition 3.3.** Let a seed vector field \( \tilde{l} \in \hat{G}^* \) and \( h^{(y)}, h^{(t)} \in I(\hat{G}^*) \) be Casimir functions subject to the metric \((\cdot, \cdot)\) on the loop Lie algebra \( \hat{G} \) and the natural coadjoint action on the loop co-algebra \( \hat{G}^* \). Then the following dynamical systems

\[
\partial \tilde{l}/\partial y = -a d^*_{\nabla h^{(y)}(l)} \tilde{l}, \quad \partial \tilde{l}/\partial t = -a d^*_{\nabla h^{(t)}(l)} \tilde{l}
\]

are commuting Hamiltonian flows for all \( y, t \in \mathbb{R} \). Moreover, the compatibility condition of these flows is equivalent to the so called vector fields representation

\[
(\partial/\partial t + \hat{A}_{\nabla h^{(t)}}) \psi = 0, \quad (\partial/\partial y + \hat{A}_{\nabla h^{(y)}}) \psi = 0,
\]

where \( \psi \in C^\infty(\mathbb{R}^2 \times T^n; \mathbb{C}) \) and the vector fields \( \hat{A}_{\nabla h^{(y)}}^{(y)}, \hat{A}_{\nabla h^{(t)}}^{(t)} \in \hat{G} \), given by the expressions (3.25) and (3.26), satisfy the Lax relationship (3.24).

The proposition above makes it possible to describe in a very effective way the Bäcklund transformations between two solution sets to the dispersionless heavenly type equations resulting from the Lax compatibility condition (3.24). Namely, let a diffeomorphism \( \xi \in Diff(T^n) \), depending parametrically on \( \lambda, \mu \in \mathbb{C} \) and evolution variables \( (y, t) \in \mathbb{R}^2 \), be such that a seed loop differential form \( \tilde{l}(x; \lambda, \mu) \in \hat{G}^* \approx \hat{L}^1(T^n) \) satisfies the invariance condition

\[
\tilde{l}(\xi(x; \lambda, \mu); \lambda) = k \tilde{l}(x; \mu)
\]

for some non-zero constant \( k \in \mathbb{C} \setminus \{0\} \), any \( x \in T^n \) and arbitrarily chosen \( \lambda \in \mathbb{C} \). As the seed element \( \tilde{l}(\xi(x; \lambda, \mu); \lambda) \in \hat{L}^1(T^n) \), by the construction, simultaneously satisfies the system of compatible equations following from (3.32), the loop diffeomorphism \( \xi \in Diff(T^n) \), found analytically from the invariance condition (3.34), should satisfy the relationships

\[
\frac{\partial}{\partial y} \xi = \nabla h^{(y)}_+(l), \quad \frac{\partial}{\partial t} \xi = \nabla h^{(t)}_+(l),
\]

giving rise exactly to the Bäcklund type relationships for coefficients of the seed loop differential form \( \tilde{l} \in \hat{G}^* \approx \hat{L}^1(T^n) \).
4. Integrable heavenly equations: Examples

4.1. The Mikhalev–Pavlov heavenly equation. This equation \[ (4.1) \]
\[
\frac{\partial^{2} u}{\partial t \partial y} - \frac{\partial^{2} u}{\partial x \partial t} - \frac{\partial u}{\partial x} \frac{\partial u}{\partial t} = \frac{\partial u}{\partial y} \frac{\partial u}{\partial x},
\]
where \( u \in C^\infty(\mathbb{R}^2 \times \mathbb{T}^1; \mathbb{R}) \) and \( (t, y; x) \in \mathbb{R}^2 \times \mathbb{T}^1 \). Set \( \hat{G}^* := \hat{D} \hat{f}^* (\mathbb{T}^1) \) and take the corresponding seed element \( \hat{I} \in \hat{G}^* \) as
\[
\hat{I} = (\lambda - 2u_x)dx.
\]
It generates a Casimir invariant \( h \in I(\hat{G}^*) \) for which the expansion \[ (3.27) \] as \( |\lambda| \to \infty \) is given by the asymptotic series
\[
\nabla h(l) \sim 1 + u_x/\lambda - u_y/\lambda^2 + O(1/\lambda^3)
\]
and so on. If further one defines
\[
\nabla h(l)^{(1)}(l)_+ := (\lambda^2 \nabla h)_+ = \lambda^2 + \lambda u_x - u_y,
\]
\[
\nabla h(l)^{(1)}(l)_+ := (\lambda^2 \nabla h)_+ = \lambda^2 + \lambda u_x - u_y,
\]
it is easy to verify that
\[
\hat{A}_{\nabla h(l)^{(1)}} := \langle \nabla h(l)_+ , \frac{\partial}{\partial x} \rangle = (\lambda^2 + \lambda u_x - u_y) \frac{\partial}{\partial x},
\]
\[
\hat{A}_{\nabla h(l)^{(2)}} := \langle \nabla h(l)_+ , \frac{\partial}{\partial x} \rangle = (\lambda + u_x) \frac{\partial}{\partial x}.
\]
As a result of \[ (4.5) \] and the commuting flows \[ (3.32) \] on \( \hat{G}^* \) we retrieve (the equivalent to the Mikhalev–Pavlov \[ (3.7) \] ) equation \[ (4.1) \] vector field compatibility relationships
\[
\frac{\partial \psi}{\partial t} + (\lambda^2 + \lambda u_x - u_y) \frac{\partial \psi}{\partial x} = 0 = \frac{\partial \psi}{\partial y} + (\lambda + u_x) \frac{\partial \psi}{\partial x},
\]
satisfied for \( \psi \in C^\infty(\mathbb{R}^2 \times \mathbb{T}^1; \mathbb{C}) \), any \( (y, t; x) \in \mathbb{R}^2 \times \mathbb{T}^1 \) and all \( \lambda \in \mathbb{C} \).

We now study the Bäcklund transformation for two special solutions \( u, \tilde{u} \in C^2(\mathbb{T}^1 \times \mathbb{R}^2; \mathbb{R}) \) to the Mikhalev–Pavlov equation \[ (4.1) \]. Let us consider a loop diffeomorphism \( \xi \in \hat{D}iff(\mathbb{T}^1) \) that is the mapping \( \mathbb{T}^1 \ni x \to \tilde{x} = \xi(x; y, t, \lambda) \in \mathbb{T}^1 \), which parametrically depends on \( \lambda \in \mathbb{C} \) and the evolution variables \( (y, t) \in \mathbb{R}^2 \), satisfying the invariance condition \[ (3.34) \] for the seed loop differential form \[ (4.2) \].
\[
(\lambda - 2\tilde{u}_x(x; t, y))d\tilde{x} = (\lambda - 2u_x(x; t, y))dx,
\]
where for simplicity, we define \( \mu = \lambda \in \mathbb{C} \) and the constant parameter \( k = 1 \). From \[ (4.7) \] one easily finds that
\[
\lambda \xi_\lambda(x; t, y) = 2[\tilde{u}(\tilde{x}; t, y) - u(x; t, y)]_x + \lambda,
\]
or, equivalently,
\[
\xi(x; \lambda) = x + 2(\tilde{u} - u)/\lambda + \alpha(y, t; \lambda),
\]
where \( \lambda \in \mathbb{C} \setminus \{0\} \) and \( \alpha \in C^2(\mathbb{R}^2; \mathbb{R}) \) is some mapping. As the loop diffeomorphism \[ (4.8) \] should simultaneously satisfy the vector field equations \[ (3.35) \], giving rise (at \( \alpha(y, t; \lambda) = 0 \)) to the corresponding Bäcklund type differential relationships
\[
2(\tilde{u}_y - u_y)/\lambda = \lambda + \tilde{u}_x, \quad 2(\tilde{u}_t - u_t)/\lambda = \lambda^2 + \lambda \tilde{u}_x - \tilde{u}_y,
\]
which hold for any \( \lambda \in \mathbb{C} \) and two special solutions \( u, \tilde{u} \in C^2(\mathbb{T}^1 \times \mathbb{R}^2; \mathbb{R}) \) to the Mikhalev–Pavlov equation \[ (4.1) \].
4.2. Example: The Witham heavenly type equation. Consider the following heavenly type equation:

\( u_{ty} = u_x u_{xy} - u_y u_{xx}, \)  

where \( u \in C^2(\mathbb{R}^2 \times \mathbb{T}^1; \mathbb{R}) \) and \( (t, y, x) \in \mathbb{R}^2 \times \mathbb{T}^1. \) To prove the Lax-Sato type integrability of \( (4.9) \), let us consider a seed element \( \tilde{l} \in \mathfrak{G}^*, \) defined as

\( \tilde{l} = (u_y - 2\lambda^{-1} + 2u_x + \lambda)dx, \)

where \( \lambda \in \mathbb{C} \setminus \{0\} \) is a complex parameter. The following asymptotic expressions are gradients of the Casimir functionals \( h^{(t)}, h^{(y)} \in I(\mathfrak{G}^*), \) related with the holomorphic loop Lie algebra \( \hat{\mathfrak{g}} = \text{diff}(\mathbb{T}^1) : \)

\( \nabla h^{(t)} \sim \lambda[(u_x\lambda^{-1} - 1) + O(1/\lambda)], \)

as \( \lambda \to \infty, \) and

\( \nabla h^{(y)} \sim u_y\lambda^{-1} + O(\lambda^2), \)

as \( \lambda \to 0. \) Based on the expressions \( (4.11) \) and \( (4.12), \) one can construct the following commuting to each other Hamiltonian flows

\( \frac{\partial}{\partial y} \tilde{l} = -\text{ad}_{\nabla h^{(y)}} \tilde{l}, \quad \frac{\partial}{\partial t} \tilde{l} = -\text{ad}_{\nabla h^{(y)}} \tilde{l} \)

with respect to the evolution parameters \( y, t \in \mathbb{R}, \) which give rise, in part, to the following equations:

\( u_{yt} = u_x u_{xy} - u_y u_{xx}, \)
\( u_t = -u_y^{-2}/2 + 3u_x^2/2, \)
\( u_{yy} = -u_y^2[(u_x u_y)_x + u_x u_{xy}], \)

where the projected gradients \( \nabla h^{(t)}, \nabla h^{(y)} \in \hat{\mathfrak{g}} \) are equal to the loop vector fields

\( \nabla h^{(t)} = (u_x - \lambda) \frac{\partial}{\partial x}, \quad \nabla h^{(y)} = \frac{u_y}{\lambda} \frac{\partial}{\partial x}; \)

satisfying for evolution parameters \( y, t \in \mathbb{R}^2 \) the Lax-Sato vector field compatibility condition:

\( \frac{\partial}{\partial y} \nabla h^{(t)} - \frac{\partial}{\partial t} \nabla h^{(y)} + [\nabla h^{(t)}, \nabla h^{(y)}] = 0. \)

As a simple consequence of the condition one finds exactly the first equation of the \( (4.14), \) coinciding with the heavenly type equation \( (4.9). \) Thereby, we have stated that this equation is a completely integrable heavenly type dynamical system with respect to both evolution parameters.

**Remark 4.1.** It is worth to observe that the third equation of \( (4.14) \) entails the interesting relationship

\( \frac{\partial}{\partial y}(1/u_y) = \frac{\partial}{\partial x}(u_x u_y^2), \)

whose compatibility makes it possible to introduce a new function \( v \in C^2(\mathbb{S}^1; \mathbb{R}), \) satisfying the next differential expressions:

\( u_x = 1/u_y, \quad v_x = u_x u_y^2, \)

which hold for all \( (x, y) \in \mathbb{S}^1 \times \mathbb{R}. \) Based on \( (4.18) \) the seed element \( (4.10) \) is rewritten as

\( \tilde{l} = (v_x^2\lambda^{-1} + 2u_x + \lambda)dx, \)

and the vector fields \( (4.15) \) are rewritten as

\( \nabla h^{(t)} = (u_x - \lambda) \frac{\partial}{\partial x}, \quad \nabla h^{(y)} = \frac{1}{v_x \lambda} \frac{\partial}{\partial x}. \)
whose compatibility condition \( (4.10) \) gives rise to the following system of heavenly type nonlinear integrable flows:

\[
(4.21) \quad v_y = u_x v_y^2, \quad v_x = u_x v_y + u_x v_x,
\]

\[
u_y = 1/v_x, \quad u_t = -v_x^2/2 + 3u_x^2/2,
\]

compatible for arbitrary evolution parameters \( y, t \in \mathbb{R} \).

4.3. Plebański heavenly equation. This equation \( (4.10) \) is

\[
(4.22) \quad u_{tx_1} - u_{yx_2} + u_{x_1 x_1} u_{x_2 x_2} - u_{x_1 x_2}^2 = 0
\]

for a function \( u \in C^\infty(\mathbb{R}^2; \mathbb{T}^2) \), where \( (y, t; x_1, x_2) \in \mathbb{R}^2 \times \mathbb{T}^2 \). We set \( \tilde{G}^* := \text{diff}^* (\mathbb{T}^2) \) and take the corresponding seed element \( \tilde{l} \in \tilde{G}^* \) as

\[
(4.23) \quad \tilde{l} = (\lambda - u_{x_1 x_2} + u_{x_1 x_1}) dx_1 + (\lambda - u_{x_2 x_2} + u_{x_2 x_2}) dx_2.
\]

This generates two independent Casimir functionals \( h^{(1)}, h^{(2)} \in I(\tilde{G}^*) \), whose gradient expansions \( (3.24) \) as \( |\lambda| \to \infty \) are given by the expressions

\[
(4.24) \quad \nabla h^{(1)}(l) \sim (0, 1)^T + (u_{x_2 x_2}, -u_{x_1 x_2})^T \lambda^{-1} + O(\lambda^{-2}),
\]

\[
\nabla h^{(2)}(l) \sim (1, 0)^T + (u_{x_1 x_2}, -u_{x_1 x_2})^T \lambda^{-1} + O(\lambda^{-2}),
\]

and so on. Now, by defining

\[
(4.25) \quad \nabla h^{(1)}(l)_+ := (\lambda \nabla h^{(1)}(l))_+ = (u_{x_2 x_2}, \lambda - u_{x_1 x_1})^T,
\]

\[
\nabla h^{(2)}(l)_+ := (\lambda \nabla h^{(2)}(l))_+ = (\lambda + u_{x_1 x_2}, -u_{x_1 x_1})^T,
\]

one obtains for \( (4.22) \) the following \( (4.16) \) vector field representation

\[
(4.26) \quad \frac{\partial \psi}{\partial t} + u_{x_1 x_1} \frac{\partial \psi}{\partial x_1} + (\lambda - u_{x_1 x_2}) \frac{\partial \psi}{\partial x_2} = 0,
\]

\[
\frac{\partial \psi}{\partial z} + (\lambda + u_{x_1 x_2}) \frac{\partial \psi}{\partial x_1} - u_{x_1 x_1} \frac{\partial \psi}{\partial x_2} = 0,
\]

satisfied for \( \psi \in C^\infty(\mathbb{R}^2 \times \mathbb{T}^2; \mathbb{C}) \), any \( (t, y; x_1, x_2) \in \mathbb{R}^2 \times \mathbb{T}^2 \) and all \( \lambda \in \mathbb{C} \).

4.4. General heavenly equation. This equation was first suggested and analyzed by Schief in \( [51, 53] \), where it was shown to be equivalent to the first Plebański heavenly equation, and later studied by Doubrov and Ferapontov \( [16] \); it has the form

\[
(4.27) \quad \alpha u_{yt} u_{x_1 x_2} + \beta u_{tx_2} u_{yx_1} + \gamma u_{x_1} u_{yx_2} = 0,
\]

where \( \alpha, \beta, \gamma \in \mathbb{R} \) are arbitrary constants, satisfying the constraint

\[
(4.28) \quad \alpha + \beta + \gamma = 0,
\]

and \( t, y \in \mathbb{R}, (x_1, x_2) \in \mathbb{T}^2 \). To demonstrate the Lax integrability of the equation \( (4.27) \) we choose now a seed vector field \( \tilde{l} \in \tilde{G}^* := \text{diff}^* (\mathbb{T}^2) \) in the following rational form

\[
(4.29) \quad \tilde{l} = \left( \frac{\mu u_{x_1 x_2}^2}{\gamma (\mu + \beta)} + \frac{u_{x_1 x_1}^2}{\alpha} - \frac{\mu u_{x_1 x_2}}{\beta (\mu - \gamma)} \right) dx_1 + \left( \frac{u_{x_1 x_2} u_{x_2 x_2}}{\gamma (\mu + \beta)} + \frac{u_{x_1 x_2} u_{x_2 x_2}}{\alpha} - \frac{\mu u_{x_1 x_2} u_{x_2 x_2}}{\beta (\mu - \gamma)} \right) dx_2,
\]

where \( a_j, b_j \in C^\infty(\mathbb{T}^2; \mathbb{R}) \), \( j = 0, 1 \), are smooth functions and \( \mu \in \mathbb{C} \) is a complex parameter. The corresponding equations for independent Casimir invariants \( \gamma^{(j)} \in I(\tilde{G}^*), j = 1, 2 \), are given with respect to the standard metric \((\cdot, \cdot)\) by the following asymptotic expansions:

\[
(4.30) \quad \nabla \gamma^{(1)}(l) \sim \sum_{j \in \mathbb{Z}_+} \nabla \gamma^{(1)}_j(l) \lambda^j,
\]
as \( \mu + \beta = \lambda \rightarrow 0 \) and
\begin{equation}
\nabla_{\gamma^{(2)}}(l) \sim \sum_{j \in \mathbb{Z}^+} \nabla_{\gamma_j^{(2)}}(l) \lambda^j,
\end{equation}
as \( \mu - \gamma = \lambda \rightarrow 0 \). For the first case \([4.30]\) one obtains that
\begin{equation}
\nabla_{\gamma^{(1)}}(l) \sim \left( \frac{\beta u_{tx_x}}{u_{x_1 x_2}} + \frac{u_{tx_x}}{u_{x_1 x_2}} \lambda, - \frac{\beta u_{tx_x}}{u_{x_1 x_2}} \right) + O(\lambda^2)
\end{equation}
and for the second one \([4.31]\) one finds that
\begin{equation}
\nabla_{\gamma^{(2)}}(l) \sim \left( \frac{\gamma u_{yx_x}}{u_{x_1 x_2}} + \frac{u_{yx_x}}{u_{x_1 x_2}} \lambda, - \frac{\gamma u_{yx_x}}{u_{x_1 x_2}} \right) + O(\lambda^2).
\end{equation}
Here we took into account that the following two Hamiltonian flows on
\begin{equation}
\nabla \rightarrow \infty
\end{equation}
\begin{equation}
\nabla
\end{equation}
\begin{equation}
\partial \tilde{l}/\partial y = ad^{*}_{\nabla_{h^{(y)}(1)}} \tilde{l}, \quad \partial \tilde{l}/\partial t = ad^{*}_{\nabla_{h^{(t)}(1)}} \tilde{l}
\end{equation}
with respect to the evolution parameters \( y, t \in \mathbb{R} \) hold for the following conservation laws gradients:
\begin{equation}
\nabla h_{-}^{(t)}(l) := \lambda(\lambda^{2} \nabla_{\gamma^{(1)}}(l)) \bigg|_{\lambda=\mu+\beta} = \left( \frac{\mu u_{tx_x}}{u_{x_1 x_2}(\mu + \beta)}, \frac{\beta u_{tx_x}}{u_{x_1 x_2}(\mu + \beta)} \right),
\end{equation}
\begin{equation}
\nabla h_{-}^{(y)}(l) := \lambda(\lambda^{2} \nabla_{\gamma^{(2)}}(l)) \bigg|_{\lambda=\mu-\gamma} = \left( \frac{\mu u_{yx_x}}{u_{x_1 x_2}(\mu - \gamma)}, \frac{\gamma u_{yx_x}}{u_{x_1 x_2}(\mu - \gamma)} \right).
\end{equation}
Owing to the compatibility condition of two commuting flows \([4.35]\), one can easily rewrite it as
\begin{equation}
\partial \tilde{A}^{(y)}/\partial t - \partial \tilde{A}^{(t)}/\partial y = [\tilde{A}^{(y)}, \tilde{A}^{(t)}],
\end{equation}
where
\begin{equation}
\tilde{A}^{(t)} := \bigg\langle \nabla h_{-}^{(t)}(l), \frac{\partial}{\partial x} \bigg\rangle = \frac{\mu u_{tx_x}}{u_{x_1 x_2}(\mu + \beta)} \frac{\partial}{\partial x_1} + \frac{\beta u_{tx_x}}{u_{x_1 x_2}(\mu + \beta)} \frac{\partial}{\partial x_2},
\end{equation}
\begin{equation}
\tilde{A}^{(y)} := \bigg\langle h_{-}^{(y)}(l), \frac{\partial}{\partial x} \bigg\rangle = \frac{\mu u_{yx_x}}{u_{x_1 x_2}(\mu - \gamma)} \frac{\partial}{\partial x_1} - \frac{\gamma u_{yx_x}}{u_{x_1 x_2}(\mu - \gamma)} \frac{\partial}{\partial x_2}.
\end{equation}
An easy calculation shows that the general heavenly equation \([4.24]\) follows from the compatibility condition \([4.35]\), whose equivalent vector field representation is given as
\begin{equation}
\frac{\mu u_{tx_x}}{u_{x_1 x_2}(\mu + \beta)} \frac{\partial}{\partial x_1} + \frac{\beta u_{tx_x}}{u_{x_1 x_2}(\mu + \beta)} \frac{\partial}{\partial x_2} + \frac{\mu u_{yx_x}}{u_{x_1 x_2}(\mu - \gamma)} \frac{\partial}{\partial x_1} - \frac{\gamma u_{yx_x}}{u_{x_1 x_2}(\mu - \gamma)} \frac{\partial}{\partial x_2} + \frac{\partial}{\partial t} = 0,
\end{equation}
for a function \( \psi \in C^\infty(\mathbb{R}^2 \times T^2; \mathbb{C}) \) for all \((y, t; x_1, x_2) \in \mathbb{R}^2 \times T^2\).

We mention here that the related Backlund transformation for the general heavenly equation \([4.24]\) was recently constructed both in \([68]\) and in \([69]\), and can be retrieved from the differential one-form \([4.29]\).

4.5. The Alonso–Shabat heavenly equation. This equation \([2]\) has the form
\begin{equation}
u_{yx_x} - u u_{yx_x} + u_{yx} u_{tx_x} = 0,
\end{equation}
where \( u \in C^\infty(\mathbb{R}^2 \times T^2; \mathbb{R}) \), \((y, t) \in \mathbb{R}^2 \) and \((x_1, x_2) \in T^2\). To prove its Lax integrability, we define a seed element \( \tilde{l} \in \mathcal{G}^* := \text{diff}^* (T^2) \) of the form
\begin{equation}
\tilde{l} = z_{x_1}^2 (\lambda + 1) dx_1 + z_{x_1} z_{x_2} (\lambda + 1) dx_2,
\end{equation}
for a fixed function \( z \in C^\infty(T^2; \mathbb{R}) \). Then one easily obtains asymptotic expansions as \( |\lambda| \to \infty \) for coefficients of the two independent Casimir functionals \( \gamma^{(j)} \in I(\hat{\mathcal{G}}^*) \), \( j = 1, 2 \), gradients:

\[
\nabla \gamma^{(1)}(l) \sim (1/z_{x_1} + k z_{x_2}/z_{x_1}, -k)^T + O(1/\lambda^2),
\]

\[
\nabla \gamma^{(2)}(l) \sim (z_{x_2}/z_{x_1}, -1)^T + O(1/\lambda^2),
\]

where \( k \neq 1 \) is a constant. Using the Casimir functionals \( (4.41) \), one can construct the simplest two commuting flows

\[
\partial \tilde{l}/\partial y = -\text{ad}_{\nabla h(\psi)(l)}^* \tilde{l}, \quad \partial \tilde{l}/\partial t = -\text{ad}_{\nabla h(\psi)(l)}^* \tilde{l}
\]

with respect to the evolution parameters \( y, t \in \mathbb{R} \), where

\[
\nabla h^{(y)}(l)_+ := (\lambda \nabla \gamma^{(1)}(l))_+ = (\lambda/z_{x_1} + \lambda k z_{x_2}/z_{x_1}, -\lambda k)^T := (\lambda u_y, -\lambda k)^T,
\]

\[
\nabla h^{(t)}(l)_+ := (\lambda \nabla \gamma^{(2)}(l))_+ = (z_{x_2}/z_{x_1}, -\lambda)^T := (\lambda u_t, -\lambda)^T
\]

for some function \( u \in C^\infty(\mathbb{R}^2 \times \mathbb{T}^2; \mathbb{R}) \). From relationships \( (4.43) \), as a result of the commutativity of the flows \( (4.42) \), one derives the equivalent Lax type relationship \( (3.24) \) for the vector fields,

\[
\tilde{A}_{\nabla h^{(\psi)}} := \lambda u_y \partial/\partial x_1 - k \lambda \partial/\partial x_2, \quad \tilde{A}_{\nabla h^{(t)}} := \lambda u_t \partial/\partial x_1 - \lambda \partial/\partial x_2,
\]

which can be rewritten as the compatibility condition for the following vector field equations:

\[
\frac{\partial \psi}{\partial t} + \lambda u_t \frac{\partial \psi}{\partial x_1} - \lambda \frac{\partial \psi}{\partial x_2} = 0, \quad \frac{\partial \psi}{\partial y} + \lambda u_y \frac{\partial \psi}{\partial x_1} - k \lambda \frac{\partial \psi}{\partial x_2} = 0,
\]

satisfied for \( \psi \in C^\infty(\mathbb{R}^2 \times \mathbb{T}^2; \mathbb{C}) \), any \( (t, y; x_1, x_2) \in \mathbb{R}^2 \times \mathbb{T}^2 \) and all \( \lambda \in \mathbb{C} \). The resulting equation is then

\[
u_{yy} - u_t u_{yx_1} + u_y u_{tx_1} + k u_{tx_2} = 0,
\]

which reduces at \( k = 0 \) to the Alonso–Shabat heavenly equation \( (4.39) \).

**Remark 4.2.** It is interesting to observe that the seed elements \( \tilde{l} \in \hat{\mathcal{G}}^* \) of the examples presented above have the differential geometric structure:

\[
\tilde{l} = \eta \, dp,
\]

where \( \eta \) and \( \rho \in C^\infty(\mathbb{R}^2 \times (\mathbb{C} \times \mathbb{T}^2); \mathbb{C}) \) are some smooth functions. For instance,

\[
\tilde{l} = d(\lambda x - 2u) \quad - \text{Mikhaev–Pavlov equation},
\]

\[
\tilde{l} = d(\lambda x_1 + \lambda x_2 - u_{x_2} + u_{x_1}) \quad - \text{Plebański heavenly equation},
\]

\[
\tilde{l} = u_{x_1 x_2} \xi du_{x_2} \xi := (\mu [\gamma(\mu + \beta)]^{-1} + \alpha^{-1} - \mu[\beta(\mu - \gamma)]^{-1}) \quad - \text{general heavenly equation},
\]

\[
\tilde{l} = (\lambda + 1)z_{x_1} dz \quad - \text{Alonso–Shabat heavenly equation}.
\]

5. **The generalized heavenly type Lie-algebraic structures**

It is well known that the loop Lie algebra \( \hat{\mathcal{G}} := \hat{\text{diff}}(\mathbb{T}^n) \) can be centrally extended as \( \hat{\mathcal{G}} := (\hat{\text{diff}}(\mathbb{T}^n); \mathbb{R}^1) \) only \( [21] \) for the case \( n = 1 \), where for any two elements \( (\bar{a}; \alpha) \) and \( (\bar{b}; \beta) \in \hat{\mathcal{G}} \) the commutator

\[
[[\bar{a}; \alpha], (\bar{b}; \beta)] = ([\bar{a}, \bar{b}]; \omega_2(\bar{a}, \bar{b})) \in \hat{\mathcal{G}}
\]

and the 2-cocycle \( \omega_2 : \hat{\mathcal{G}} \times \hat{\mathcal{G}} \to \mathbb{R}^1 \) satisfies the condition

\[
\omega_2(\bar{a}, \bar{b}, \bar{c}) + \omega_2(\bar{b}, \bar{c}, \bar{a}) + \omega_2(\bar{c}, \bar{a}, \bar{b}) = 0
\]
for any \( \tilde{a}, \tilde{b} \) and \( \tilde{c} \in \check{G} \). For the case \( n = 1 \), the Gelfand–Fuchs 2-cocycle \([21]\) on the loop Lie algebra \( \check{G} \) equals the expression

\[
\omega_2(\tilde{a}, \tilde{b}) = \frac{1}{2\pi i} \oint_{\mathbb{S}^1} \left( \frac{\partial^2 a(x; \lambda)}{\partial x^2}, \frac{\partial b(x; \lambda)}{\partial x} \right)_{H_0} d\lambda
\]

for any vector fields \( \tilde{a} = a(x; \lambda) \frac{\partial}{\partial x}, \tilde{b} = b(x; \lambda) \frac{\partial}{\partial x} \in \check{G} \) on \( T^1 \) and a fixed integer \( p \in \mathbb{Z} \).

The integrable dynamical systems related to this central extension were described in detail in \([33]\). Concerning a further generalization of the multi-dimensional case related to the loop group \( \check{G} \) for \( n \in \mathbb{Z}_+ \) one can proceed in the following natural way: as the Lie algebra \( \check{G} = \text{diff}(T^n) \) consists of the elements formally depending additionally on the “spectral” variable \( \lambda \in \mathbb{C}^1 \), one can extend the basic Lie structure on \( \check{G} = \text{diff}(T^n) \) to that on the adjacent holomorphic in \( \lambda \in \mathbb{S}^1 \) Lie algebra \( \check{G} := \text{diff}_{\text{hol}}(\mathbb{C} \times T^n) \subset \text{diff}(\mathbb{C} \times T^n) \) of vector fields on \( \mathbb{C} \times T^n \). This has elements representable as \( \tilde{a}(x; \lambda) := < a(x; \lambda), \frac{\partial}{\partial x} > = \sum_{j=1}^{n} a_j(x; \lambda) \frac{\partial}{\partial x_j} + a_0(x; \lambda) \frac{\partial}{\partial x} \in \check{G} \) for some holomorphic in \( \lambda \in \mathbb{S}^1 \) vectors \( a(x; \lambda) \in \mathbb{E} \times \mathbb{E}^n \) for all \( x \in T^n \), where \( \frac{\partial}{\partial x} := (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, ..., \frac{\partial}{\partial x_n}) \) is the generalized Euclidean vector gradient with respect to the vector variable \( x := (\lambda, x) \in \mathbb{C} \times T^n \).

It is now important to mention that the Lie algebra \( \check{G} \subset \text{diff}(\mathbb{C} \times T^n) \) also splits into the direct sum of two subalgebras:

\[
\check{G} = \check{G}_+ \oplus \check{G}_-,
\]

allowing to introduce on it the classical \( \mathcal{R} \)-structure:

\[
[\check{a}, \check{b}]_{\mathcal{R}} := [\mathcal{R} \check{a}, \check{b}] + [\check{a}, \mathcal{R} \check{b}]
\]

for any \( \check{a}, \check{b} \in \check{G} \), where

\[
[\mathcal{R}]_{ij} := (P_+ - P_-)/2,
\]

and

\[
P_{\pm} \check{G} := \check{G}_\pm \subset \check{G}.
\]

The space \( \check{G}^* \simeq \Lambda^1(\mathbb{C} \times T^n) \), adjoint to the Lie algebra \( \check{G} \) of vector fields on \( \mathbb{C} \times T^n \), can be functionally identified with \( \check{G} \) subject to the Sobolev type metric

\[
(l, \check{a}) = \frac{1}{2\pi i} \oint_{\mathbb{S}^1} \lambda^{-p} d\lambda (l, a)_{H^p},
\]

where \( p \in \mathbb{Z}, q \in \mathbb{Z}_+ \) and for arbitrary \( \bar{l} := < l(x; \lambda), dx > = \sum_{j=0,n} l_j(x; \lambda) dx_j \in \check{G}^*, \check{a} = \sum_{j=0,n} a_j(x; \lambda) \frac{\partial}{\partial x_j} \in \check{G} \) one defines

\[
(l, a)_{H^0} = \sum_{j=0}^{n} \sum_{|\alpha| = 0}^{q} \int_{T^n} dx \frac{\partial^{|\alpha|} l_j}{\partial x^\alpha} \frac{\partial^{|\alpha|} a_j}{\partial x^\alpha}.
\]

In particular, for \( q = 0 \) one has \( (l, a)_{H^0} = \int_{T^n} dx \sum_{j=0}^{n} l_j \ a_j \), the case which will be mainly chosen. Then for arbitrary \( f, g \in D(\check{G}^*) \), one can determine two Lie–Poisson brackets

\[
\{ f, g \} := (\bar{l}, [\nabla f(\bar{l}), \nabla g(\bar{l})])
\]

and

\[
\{ f, g \}_{\mathcal{R}} := (\bar{l}, [\nabla f(\bar{l}), \nabla g(\bar{l})]_{\mathcal{R}}),
\]

where at any seed element \( \bar{l} \in \check{G}^* \) the gradient element \( \nabla f(\bar{l}) \) and \( \nabla g(\bar{l}) \in \check{G} \) are calculated with respect to the metric \([58]\).

Now let us assume that a smooth function \( \gamma \in I(\check{G}^*) \) is a Casimir invariant, that is

\[
ad^\gamma_{\nabla(\bar{l})} \bar{l} = 0
\]
for a chosen seed element \( \bar{I} \in \bar{G}^* \). As the adjoint mapping \( \text{ad}^*_{\nabla f(l)} \bar{I} \) for any \( f \in D(\bar{G}^*) \) can be rewritten in the reduced form as

\[
\text{ad}^*_{\nabla f(l)}(\bar{I}) = \left< \frac{\partial}{\partial x}, \nabla f(l) \right> \bar{I} + \sum_{j=1}^{n} \left< l, \frac{\partial}{\partial x} \nabla f(l) \right> d\bar{x},
\]

where \( \nabla f(l) := \frac{\partial f(l)}{\partial x} \). For the Casimir function \( \gamma \in D(\bar{G}^*) \), the condition \( 5.12 \) is then equivalent to the equation

\[
l \left< \frac{\partial}{\partial x}, \nabla \gamma(l) \right> + \left< \nabla \gamma(l), \frac{\partial}{\partial x} \right> l + \left< l, \frac{\partial}{\partial x} \nabla \gamma(l) \right> = 0,
\]

which should be solved analytically. In the case when an element \( \bar{I} \in \bar{G}^* \) is singular as \( |\lambda| \to \infty \), one can consider the general asymptotic expansion

\[
\nabla := \nabla \gamma(p) \sim \lambda^p \sum_{j \in \mathbb{Z}^+} \nabla \gamma(p) \lambda^{-j}
\]

for some suitably chosen \( p \in \mathbb{Z}^+ \), and upon substituting \( 5.15 \) into the equation \( 5.14 \), one can solve it recurrently.

Now let \( h^{(y)}, h^{(t)} \in I(\bar{G}^*) \) be such Casimir functions for which the Hamiltonian vector field generators

\[
\nabla h^{(y)}_+ (l) := (\nabla \gamma^{(p_y)}(l))_+, \quad \nabla h^{(t)}_+ (l) := (\nabla h^{(p_t)}(l))_+
\]

are, respectively, defined for special integers \( p_y, p_t \in \mathbb{Z}^+ \). These invariants generate, owing to the Lie–Poisson bracket \( 5.11 \), for the case \( q = 0 \) the following commuting flows

\[
\frac{\partial l}{\partial t} = - \left< \frac{\partial}{\partial x}, \nabla h^{(t)}_+(l) \right> l - \left< l, \frac{\partial}{\partial x} \nabla h^{(t)}_+(l) \right>,
\]

\[
\frac{\partial l}{\partial y} = - \left< \frac{\partial}{\partial x}, \nabla h^{(y)}_+(l) \right> l - \left< l, \frac{\partial}{\partial x} \nabla h^{(y)}_+(l) \right> >,
\]

where \( y, t \in \mathbb{R} \) are the corresponding evolution parameters. Since the invariants \( h^{(y)}, h^{(t)} \in I(\bar{G}^*) \) commute with respect to the Lie–Poisson bracket \( 5.11 \), the flows \( 5.17 \) also commute, implying that the corresponding Hamiltonian vector field generators

\[
\hat{A}_{\nabla h^{(t)}_+} := \left< \frac{\partial}{\partial x}, \nabla h^{(t)}_+(l) \right>, \quad \hat{A}_{\nabla h^{(y)}_+} := \left< \frac{\partial}{\partial x}, \nabla h^{(y)}_+(l) \right>
\]

satisfy the Lax compatibility condition

\[
\frac{\partial }{\partial y} \hat{A}_{\nabla h^{(t)}_+} - \frac{\partial }{\partial t} \hat{A}_{\nabla h^{(y)}_+} = [\hat{A}_{\nabla h^{(t)}_+}, \hat{A}_{\nabla h^{(y)}_+}]
\]

for all \( y, t \in \mathbb{R} \). On the other hand, the condition \( 5.19 \) is equivalent to the compatibility condition of two linear equations

\[
\left( \frac{\partial}{\partial t} + \hat{A}_{\nabla h^{(y)}_+} \right) \psi = 0, \quad \left( \frac{\partial}{\partial y} + \hat{A}_{\nabla h^{(t)}_+} \right) \psi = 0
\]

for a function \( \psi \in C^\infty(\mathbb{R} \times \mathbb{T}^n; \mathbb{C}) \) for all \( y, t \in \mathbb{R} \) and any \( \lambda \in \mathbb{C} \).

The above can be formulated as the following key result:

**Proposition 5.1.** Let a seed vector field be \( \bar{I} \in \bar{G}^* \) and \( h^{(y)}, h^{(t)} \in I(\bar{G}^*) \) be Casimir functions subject to the metric \( \langle \cdot, \cdot \rangle \) on the loop Lie algebra \( \bar{G} \) and the natural coadjoint action on the loop co-algebra \( \bar{G}^* \). Then the following dynamical systems

\[
\frac{\partial l}{\partial y} = - \text{ad}^*_{\nabla h^{(y)}_+(l)} \bar{I}, \quad \frac{\partial l}{\partial t} = - \text{ad}^*_{\nabla h^{(t)}_+(l)} \bar{I}
\]

are commuting Hamiltonian flows for all \( y, t \in \mathbb{R} \). Moreover, the compatibility condition of these flows is equivalent to the vector fields representation

\[
\left( \frac{\partial}{\partial t} + \hat{A}_{\nabla h^{(y)}_+} \right) \psi = 0, \quad \left( \frac{\partial}{\partial y} + \hat{A}_{\nabla h^{(t)}_+} \right) \psi = 0,
\]

where \( \psi \in C^\infty(\mathbb{R}^2 \times \mathbb{C} \times \mathbb{T}^n; \mathbb{C}) \) and the vector fields \( \hat{A}_{\nabla h^{(y)}_+}, \hat{A}_{\nabla h^{(t)}_+} \in \bar{G} \) are given by the expressions \( 5.18 \) and \( 5.19 \).
Remark 5.2. As mentioned above, the expansion (5.19) is effective if a chosen seed element \( \bar{I} \in \bar{G}^* \) is singular as \( |\lambda| \to \infty \). In the case when it is singular as \( |\lambda| \to 0 \), the expression (5.19) should be replaced by the expansion

\[
\nabla \gamma^{(p)}(l) \sim \lambda^{-p} \sum_{j \in \mathbb{Z}_+} \nabla \gamma_j^{(p)}(l) \lambda^j
\]

for suitably chosen integers \( p \in \mathbb{Z}_+ \), and the reduced Casimir function gradients then are given by the Hamiltonian vector field generators

\[
\nabla h^{(y)}_+(l) := \lambda(\lambda^{-p_1} - \nabla \gamma^{(p_1)}(l))_-, \quad \nabla h^{(l)}_-(l) := \lambda(\lambda^{-p_2} - \nabla \gamma^{(p_2)}(l))_-
\]

for suitably chosen positive integers \( p_1, p_2 \in \mathbb{Z}_+ \) and the corresponding Hamiltonian flows are, respectively, written as

\[
\partial \bar{I}/\partial t = \text{ad}^*_{\nabla h^{(y)}_+}(l) \bar{I}, \quad \partial \bar{I}/\partial y = \text{ad}^*_{\nabla h^{(y)}_+}(l) \bar{I}.
\]

As in Section 3 the Proposition 5.1 above makes it possible to describe the Bäcklund transformations between two special solution sets for the dispersionless heavenly equations resulting from the Lax compatibility condition (5.21). Let a diffeomorphism \( \xi \in Diff(\mathbb{C} \times \mathbb{T}^n) \) be such that a seed loop differential form \( \bar{l}(\lambda, x) \in \bar{G}^* \simeq \Lambda^1(\mathbb{C} \times \mathbb{T}^n) \) satisfies the invariance condition

\[
\bar{l}(\xi(x; \mu)) = k\bar{l}(\bar{x})
\]

for some non-zero constant \( k \in \mathbb{C} \setminus \{0\} \), any \( x = (\lambda, x) \) and \( \bar{x} = (\mu, x) \in \mathbb{C} \times \mathbb{T}^n \) and arbitrarily an chosen parameter \( \mu \in \mathbb{C} \). As the seed element \( \bar{l}(\xi(x; \mu)) \) \( \in \Lambda^1(\mathbb{C} \times \mathbb{T}^n) \) satisfies simultaneously the system of compatible equations (5.21), the loop diffeomorphism \( \xi \in Diff(\mathbb{C} \times \mathbb{T}^n) \), found analytically from the invariance condition (5.20), satisfies the compatible system of vector field equations

\[
\frac{\partial}{\partial t} \xi = \nabla h^{(l)}_+(l), \quad \frac{\partial}{\partial y} \xi = \nabla h^{(y)}_+(l),
\]

giving rise to the Bäcklund type relationships for the coefficients of the seed loop differential form \( \bar{l} \in \mathcal{G}^* \simeq \Lambda^1_{\text{hol}}(\mathbb{C} \times \mathbb{T}^n) \).

It worth mentioning that, following Ovsienko’s scheme [35, 36], one can consider a wider class of integrable heavenly equations, realized as compatible Hamiltonian flows on the semidirect product of the holomorphic loop Lie algebra \( \mathcal{G} \) of vector fields on the torus \( \mathbb{T}^n \) and its regular co-adjoint space \( \mathcal{G}^* \), supplemented with naturally related cocycles. We plan to analyze this aspect of the construction, devised in the present work, in a paper now in preparation (3.21).

5.1. Example: Einstein–Weyl metric equation. Define \( \mathcal{G}^* = \text{diff}_{\text{hol}}(\mathbb{T}^1 \times \mathbb{C}) \) and take the seed element

\[
\bar{l} = (u_x - 2u_xv_x - u_y) \, dx + (\lambda^2 - v_x \lambda + v_y + v_x^2) \, d\lambda,
\]

which generates with respect to the metric (5.8) (as before for \( q = 0 \)) the gradient of the Casimir invariants \( h^{(p,)} \in \mathcal{I}(\mathcal{G}^*) \) in the form

\[
\nabla h^{(p,)}(l) \sim \lambda^2(0, 1) + (-u_x, v_x)^T \lambda + (u_y, u - v_y)^T + O(\lambda^{-1}),
\]

\[
\nabla h^{(p,)}(l) \sim \lambda(0, 1) + (-u_x, v_x)^T + (u_y, -v_y)^T \lambda^{-1} + O(\lambda^{-2})
\]

as \( |\lambda| \to \infty \) at \( p_t = 2, p_y = 1 \). For the gradients of the Casimir functions \( h^{(l)}(l), h^{(y)}(l) \in \mathcal{I}(\mathcal{G}^*) \), determined by (5.10) one can easily obtain the corresponding Hamiltonian vector field generators

\[
\tilde{A}_{\nabla h^{(l)}_+} = \left< \nabla h^{(l)}_+(l), \frac{\partial}{\partial x} \right> = (\lambda^2 + \lambda v_x + u - v_y) \frac{\partial}{\partial x} + (-\lambda u_x + u_y) \frac{\partial}{\partial \lambda},
\]

\[
\tilde{A}_{\nabla h^{(y)}_+} = \left< \nabla h^{(y)}_+(l), \frac{\partial}{\partial x} \right> = (\lambda + v_x) \frac{\partial}{\partial x} - u_x \frac{\partial}{\partial \lambda},
\]

satisfying the compatibility condition (5.19), which is equivalent to the set of equations

\[
\begin{cases}
    u_{xt} + u_{yy} + (u_{ux})_x + v_x u_{xy} - v_y u_{xx} = 0, \\
v_{xt} + v_{yy} + u v_{xx} + v_x v_{xy} - v_y v_{xx} = 0,
\end{cases}
\]
describing general integrable Einstein–Weyl metric equations [13].

As is well known [28], the invariant reduction of (5.21) at \( v = 0 \) gives rise to the famous dispersionless Kadomtsev–Petviashvili equation

\[
(5.30) \quad (u_t + uu_x)_x + u_yy = 0,
\]

for which the reduced vector field representation (5.20) follows from (5.28) and is given by the vector fields

\[
(5.31) \quad \tilde{A}_{\psi x}^{(t)} = (\lambda^2 + u) \frac{\partial}{\partial x} + (\lambda u_x + u_y) \frac{\partial}{\partial \lambda},
\]

\[
\tilde{A}_{\psi y}^{(y)} = \lambda \frac{\partial}{\partial x} - u_x \frac{\partial}{\partial \lambda},
\]

satisfying the compatibility condition (5.19), equivalent to the equation (5.30). In particular, one derives from (5.20) and (5.31) the vector field compatibility relationships

\[
(5.32) \quad \frac{\partial \tilde{A}_{\psi x}^{(t)}}{\partial y} + (\lambda^2 + u) \frac{\partial \tilde{A}_{\psi y}^{(y)}}{\partial x} + (\lambda u_x + u_y) \frac{\partial \tilde{A}_{\psi y}^{(y)}}{\partial \lambda} = 0
\]

\[
\frac{\partial \tilde{A}_{\psi y}^{(y)}}{\partial x} + \lambda \frac{\partial \tilde{A}_{\psi x}^{(t)}}{\partial \lambda} - u_x \frac{\partial \tilde{A}_{\psi y}^{(y)}}{\partial \lambda} = 0,
\]

satisfied for \( \psi \in C^\infty(\mathbb{R}^2 \times T^1 \times \mathbb{C}; \mathbb{C}) \) and any \( y, t \in \mathbb{R}, (x, \lambda) \in T^1 \times \mathbb{C} \).

5.2. The modified Einstein–Weyl metric equation. This equation system is

\[
(5.33) \quad u_{xt} = u_{yy} + u_x u_y + u_x^2 w_x + uu_yw_x + u_x w_x a,
\]

\[
w_{xt} = uu_yw_x + uy w_x + u_x^2 w_y + aw_x - a_y,
\]

where \( a_x := u_x w_x - w_{xy} \), and was recently derived in [60]. In this case we take also \( \tilde{G}^* = df_{hol}(T^1 \times \mathbb{C}) \), yet for a seed element \( \tilde{I} \in \tilde{G} \) we choose the form

\[
(5.34) \quad \tilde{I} = [\lambda^2 u_x + (2u_x w_x + u_y + 3uu_y) \lambda + 2u_x \partial^{-1} u_x w_x + 2u_x \partial^{-1} u_y +
\]

\[
+ 3u_x w_x^2 + 2u_y w_x + 6uu_x w_x + 2uu_y + 3u_y^2 - 2a u_x dx + ]
\]

\[
+ [\lambda^2 + (w_x + 25 u_x) \lambda + 2 \partial^{-1} u_x w_x + 2 \partial^{-1} u_y + w_x^2 + 3uw_x + 3u^2 - a] d\lambda,
\]

which with respect to the metric (5.8) (as before for \( q = 0 \)) generates two Casimir invariants \( \gamma^{(j)} \in I(\tilde{G}^*), j = 1, 2 \), whose gradients are

\[
(5.35) \quad \nabla \gamma^{(2)}(I) \sim \lambda^2 [(ux_x - 1) + (ux_y + uy) - u + w_x] \lambda^{-1} +
\]

\[
+ (0, uw_x - a)^T \lambda^{-2} \} + O(\lambda^{-1}),
\]

\[
\nabla \gamma^{(1)}(I) \sim \lambda [(ux_x - 1) + (0, w_x)^T \lambda^{-1} + O(\lambda^{-1}),
\]

as \( |\lambda| \rightarrow \infty \) at \( p_y = 1, p_t = 2 \). The corresponding gradients of the Casimir functions \( h^{(t)}, h^{(y)} \in I(\tilde{G}^*), \) determined by (5.10), generate the Hamiltonian vector field expressions

\[
(5.36) \quad \nabla h^{(y)}_+ := \nabla \gamma^{(1)}(I)_+ = (ux_x, -\lambda + w_x)^T,
\]

\[
\nabla h^{(t)}_+ = \nabla \gamma^{(2)}(I)_+ = (ux_x^2 + (ux_x + uy) \lambda, -\lambda^2 + (w_x - u) \lambda + uw_x - a)^T.
\]

Now one easily obtains from (5.36) the compatible Lax system of linear equations
(5.37) \[ \frac{\partial \psi}{\partial y} + (-\lambda + w_x) \frac{\partial \psi}{\partial x} + u_x \lambda \frac{\partial \psi}{\partial \lambda} = 0, \]

\[ \frac{\partial \psi}{\partial t} + (-\lambda^2 + (w_x - u) \lambda + uw_x - a) \frac{\partial \psi}{\partial x} + (u_x \lambda^2 + (uw_x + u_y) \lambda) \frac{\partial \psi}{\partial \lambda} = 0, \]
satisfied for \( \psi \in C^\infty(\mathbb{R}^2 \times \mathbb{T}^1 \times \mathbb{C}; \mathbb{C}) \) and any \( y, t \in \mathbb{R}, (x, \lambda) \in \mathbb{T}^1 \times \mathbb{C}. \)

5.3. Example: The Dunajski heavenly equations. This equation, suggested in [17], generalizes the corresponding anti-self-dual vacuum Einstein equation, which is related to the Plebański [46] second heavenly equation [12,43]. To study the integrability of the Dunajski equations

(5.38) \[ u_{x_1 t} + u_{y x_2} + u_{x_1 x_1} u_{x_2 x_2} - u_{x_1 x_2}^2 - v = 0, \]

\[ v_{x_1 t} + v_{x_2 y} + u_{x_1 x_1} u_{x_2 x_2} - 2 u_{x_1 x_2} v_{x_2 x_2} = 0, \]

where \((u, v) \in C^\infty(\mathbb{R}^2 \times \mathbb{T}^2 \times \mathbb{R}^2), (y, t; x_1, x_2) \in \mathbb{R}^2 \times \mathbb{T}^2, \) we define \( \mathcal{G}^* := dif f h^- (\mathbb{C} \times \mathbb{T}^1) \) and take the following as a seed element \( \tilde{l} \in \mathcal{G}^* \)

(5.39) \[ \tilde{l} = (\lambda + v_{x_1}, -u_{x_1 x_1}, -u_{x_1 x_2}) dx_1 + (\lambda + v_{x_2}, u_{x_1 x_1} u_{x_2 x_2} - u_{x_1 x_2}) dx_2 + (\lambda - x_1 - x_2) d\lambda. \]

With respect to the metric \( [8] \) (as before for \( q = 0 \)), the gradients of two functionally independent Casimir invariants \( h^{(p_1)} h^{(p_2)} \in I(\mathcal{G}^*) \) can be obtained as \( |\lambda| \to \infty \) in the asymptotic form as

(5.40) \[ \nabla h^{(p_1)} (l) \sim \lambda(0, 1, 0)^T + (-v_{x_1}, -u_{x_1 x_2}, u_{x_1 x_1})^T + O(\lambda^{-1}), \]

\[ \nabla h^{(p_2)} (l) \sim \lambda(0, 0, -1)^T + (v_{x_2}, u_{x_2 x_2}, -u_{x_1 x_2})^T + O(\lambda^{-1}) \]
at \( p_1 = 1 = p_2. \) Upon calculating the Hamiltonian vector field generators

(5.41) \[ \nabla h^{(p_1)} := \nabla h^{(p_1)} (l)|_+ = (-v_{x_1}, \lambda - u_{x_1 x_2}, u_{x_1 x_1})^T, \]

\[ \nabla h^{(p_2)} := \nabla h^{(p_2)} (l)|_+ = (v_{x_2}, u_{x_2 x_2}, -\lambda - u_{x_1 x_2})^T, \]

following from the Casimir functions gradients \( [5,10] \), one easily obtains the following vector fields

(5.42) \[ \hat{A}_{\nabla h^{(p_1)}} = < \nabla h^{(p_1)}, \frac{\partial}{\partial \lambda} > = u_{x_2 x_2} \frac{\partial}{\partial x_1} - (\lambda + u_{x_1 x_2}) \frac{\partial}{\partial x_2} + v_{x_2} \frac{\partial}{\partial \lambda}, \]

\[ \hat{A}_{\nabla h^{(p_2)}} = < \nabla h^{(p_2)}, \frac{\partial}{\partial \lambda} > = (\lambda - u_{x_1 x_2}) \frac{\partial}{\partial x_1} + u_{x_1 x_1} \frac{\partial}{\partial x_2} - v_{x_1} \frac{\partial}{\partial \lambda}, \]
satisfying the Lax compatibility condition \( [5,19] \), which is equivalent to the the Dunajski [17] vector field compatibility relationships \( [4,41] \)

(5.43) \[ \frac{\partial \psi}{\partial t} + u_{x_2 x_2} \frac{\partial \psi}{\partial x_1} - (\lambda + u_{x_1 x_2}) \frac{\partial \psi}{\partial x_2} + v_{x_2} \frac{\partial \psi}{\partial \lambda} = 0, \]

\[ \frac{\partial \psi}{\partial y} + (\lambda - u_{x_1 x_2}) \frac{\partial \psi}{\partial x_1} + u_{x_1 x_1} \frac{\partial \psi}{\partial x_2} - v_{x_1} \frac{\partial \psi}{\partial \lambda} = 0, \]
satisfied for \( \psi \in C^\infty(\mathbb{R}^2 \times \mathbb{C} \times \mathbb{T}^2; \mathbb{C}), \) any \((y, t; x_1, x_2) \in \mathbb{R}^2 \times \mathbb{T}^2 \) and all \( \lambda \in \mathbb{C}. \) As was mentioned in [8], the Dunajski equations \( (5.38) \) generalize both the dispersionless Kadomtsev–Petviashvili and Plebański second heavenly equations, and is also a Lax integrable Hamiltonian system.
6. Integrability, bi-Hamiltonian structures and the classical Lagrange-d’Alembert principle

It is evident that all evolution flows like \((\ref{eq:6.2})\) or \((\ref{eq:5.17})\) are Hamiltonian with respect to the second Lie–Poisson bracket \((\ref{eq:6.11})\) on the adjoint loop space \(\tilde{G}^* = \text{diff}^* \mathbb{T}^n\) or on the holomorphic subspace \(\tilde{G}^* = \text{diff}_{\mathbb{C}}^* (\mathbb{C} \times \mathbb{T}^n)\), respectively. Moreover, they are poly-Hamiltonian on the corresponding functional manifolds, as the related bilinear forms \((\ref{eq:6.11})\) and \((\ref{eq:5.8})\) are marked by integers \(p \in \mathbb{Z}\). This leads to \((\ref{eq:63})\) an infinite hierarchy of compatible Poisson structures on the phase spaces, isomorphic, respectively, to the orbits of a chosen seed element \(\tilde{l} \in \tilde{G}^*\) or of a seed element \(l \in \tilde{G}^*\). Taking also into account that all these Hamiltonian flows possess an infinite hierarchy of commuting nontrivial conservation laws, one can prove their formal complete integrability under some naturally formulated constraints. The corresponding analytical expressions for the infinite hierarchy of conservation laws can be retrieved from the asymptotic expansion \((\ref{eq:3.22})\) for Casimir functional gradients by employing the well-known \([\ref{eq:4}, \ref{eq:34}, \ref{eq:63}]\) formal homotopy technique.

As an arbitrary heavenly type equation is a Hamiltonian system with respect to both evolution parameters \(t, y \in \mathbb{R}^2\), one can construct \([\ref{eq:34}, \ref{eq:38}, \ref{eq:4}]\) its suitable Lagrangian representation under some natural constraints. Thus, it is possible to retrieve the corresponding Poisson structures related to both these evolution parameters \(t, y \in \mathbb{R}^2\), which, as follows from the Lie-algebraic analysis in Section \([\ref{eq:6}]\) are compatible with each other. In this way, one can show that any heavenly type equation is a bi-Hamiltonian integrable system on the corresponding functional manifold. It should be mentioned here that this property was introduced by Sergyeyev in \((\text{arXiv:1501.01955})\), published in \([\ref{eq:58}]\), and rediscovered and applied in detail in \([\ref{eq:7}, \ref{eq:9}, \ref{eq:10}]\) for investigating the integrability properties of the general heavenly equation \((\ref{eq:4.27})\) first suggested by Schief in \([\ref{eq:54}]\) and later studied by Dubrov and Ferapontov in \([\ref{eq:16}]\).

Using our approach in the case of the basic loop Lie algebra \(\tilde{G} = \text{diff}^* \mathbb{T}^n\) one needs to recall that a seed element \(\tilde{l}(\lambda) \in \tilde{G}^*, \lambda \in \mathbb{C}\), generates the commuting Hamiltonian flows \((\ref{eq:6.1})\)

\[
\partial_t \tilde{l}(\lambda)/\partial t_j := -ad^*_{\nabla h(\tilde{l}(\lambda))} \tilde{l}(\lambda)
\]

for any \(j \in \mathbb{Z}_+\). Taking into account that the element \(\tilde{l}(\lambda) \in \tilde{G}^*\), the hierarchy of flows \((\ref{eq:6.1})\) can be equivalently rewritten differential geometrically as the generating vector field

\[
\frac{\partial}{\partial t} \tilde{l}(\lambda)(\tilde{Y}) = -\frac{\mu}{\mu - \lambda} \tilde{l}(\lambda)(\nabla h(\tilde{l}(\mu)), \tilde{Y}) = -i_{\tilde{Y}} \tilde{l}(\lambda)(\tilde{Y}) - \frac{d}{dx} \tilde{l}(\lambda)(\mu - x \nabla h(\tilde{l}(\mu))), \tilde{Y}) = \frac{\partial}{\partial t} \tilde{l}(\lambda)(\tilde{Y})
\]

for any \(\tilde{Y} \in \tilde{G}\), where \(\tilde{Y} := \sum_{j \in \mathbb{Z}_+} \tilde{l}(\lambda)(\mu - x \nabla h(\tilde{l}(\mu))), \tilde{Y})
\]

Recall now that for the space \(\text{diff}^* \mathbb{T}^n \simeq \Gamma(T\mathbb{T}^n)\) and space \(\text{diff}^* \mathbb{T}^n_{\mu} \simeq \Gamma(T\mathbb{T}^n)\), the generating relationship \((\ref{eq:6.2})\) can be easily rewritten as the following evolution equation

\[
\frac{d}{dt} \tilde{l}(\lambda) = \frac{\mu}{\mu - \lambda} \nabla h(\tilde{l}(\lambda)) = \frac{\mu}{\mu - \lambda} < \nabla h(\tilde{l}(\lambda)), \partial_{\lambda} >
\]

on the seed element \(\tilde{l}(\lambda) \in \tilde{\Lambda}^1(\mathbb{T}^n)\). Here \(d/dt := \partial/\partial t + L_{\tilde{k}(\lambda)}\) and \(L_{\tilde{k}(\lambda)} = i_{\tilde{k}(\lambda)} d + d i_{\tilde{k}(\lambda)}\) denotes here the well-known \([\ref{eq:11}, \ref{eq:4}, \ref{eq:22}]\) Cartan expression for the derivation along the vector field \((\ref{eq:6.3})\)

\[
\tilde{k}(\lambda) := \frac{\mu}{\mu - \lambda} \nabla h(\tilde{l}(\lambda)) = \frac{\mu}{\mu - \lambda} < \nabla h(\tilde{l}(\lambda)), \partial_{\lambda} >
\]

which holds asymptotically as \(\mu, \lambda \to \infty, |\lambda/\mu| < 1\), and is equivalent to the so called Lax–Sato hierarchy of equations, studied in \([\ref{eq:61}, \ref{eq:52}, \ref{eq:7}, \ref{eq:9}, \ref{eq:10}]\) for the generating vector fields function \((\ref{eq:6.3})\).

We plan to study this and other related algebraic aspects of these equations in more detail in a work under preparation.

The expression \((\ref{eq:6.3})\) allows the following interesting mechanical Lagrange-d’Alembert type principle \([\ref{eq:17}]\) interpretation. Namely, the seed differential form \(\tilde{l}(\lambda) := (x; \lambda), dx > \in \tilde{\Lambda}^1(\mathbb{T}^n)\), can be considered as a virtual infinitesimal work, performed by the “virtual force” \((x; \lambda) \in \tilde{T}^*(\mathbb{T}^n)\) at point \(x \in \mathbb{T}^n\) for any \(\lambda \in \mathbb{C}\) along the infinitesimal path \(dx \in \mathbb{T}^n\). The whole infinitesimal
“virtual work” \(\delta W(t)\), performed by this force within a moving 1-connected arbitrary open domain \(\Omega_t \subset \mathbb{T}^n\) with smooth boundary \(\partial \Omega_t, t \in \mathbb{R}\), equals
\[
\delta W(t) := \int_{\Omega_t} < l(x(t); \lambda), \delta x(t) > d^n x(t),
\]
where the evolution of points \(x(t) \in \Omega_t\) is naturally determined by the vector field \(L\):
\[
\frac{dx(t)}{dt} = \frac{\mu}{\mu - \lambda} \nabla h( l(\mu))(t; x(t))
\]
and the Cauchy data
\[
x(t)|_{t=0} = x_0 \in \Omega_0
\]
for an arbitrarily chosen open 1-connected domain \(\Omega_0 \subset \mathbb{T}^n\) with the smooth boundary \(\partial \Omega_0\). Then the Lagrange–d’Alembert principle in mechanics says that the infinitesimal virtual work \(\delta W\) equals zero for all moments of time, that is \(\delta W(t) = 0 = \delta W(0)\) for all \(t \in \mathbb{R}\). To check that it is really true, let us calculate the temporal derivative of the expression \(\delta W\):
\[
\frac{d}{dt} \delta W(t) = \frac{d}{dt} \int_{\Omega_t} < l(x(t); \lambda), \delta x(t) > d^n x(t) =
\]
\[
= \frac{d}{dt} \int_{\Omega_0} < l(x(t); \lambda), \delta x(t) > |\frac{\partial (x(t))}{\partial x_0}| d^n x_0 = \int_{\Omega_0} \frac{d}{dt} |\frac{\partial (x(t))}{\partial x_0}| d^n x_0 =
\]
\[
= \int_{\Omega_0} |\frac{\partial}{\partial t} L_{K(\mu)}(x(t); \lambda), \delta x(t) > + < l(x(t); \lambda), \delta x(t) > \text{div } \tilde{K}(\mu) |\frac{\partial (x(t))}{\partial x_0}| d^n x_0 =
\]
\[
= \int_{\Omega_1} |\frac{\partial}{\partial t} L_{K(\mu)} + \text{div } \tilde{K}(\mu), \delta x(t) > d^n x(t) = 0,
\]
owing to the equation \(6.3\). Thus, if at \(t = 0\) one has \(\delta W(0) = 0\), the infinitesimal work \(\delta W(t) = 0\) for all \(t \in \mathbb{R}\), proving the Lagrange–d’Alembert principle for the generating evolution equation \(6.3\).

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