Extreme Current Fluctuations in Lattice Gases: Beyond Nonequilibrium Steady States

Baruch Meerson\(^1\) and Pavel V. Sasorov\(^2\)

\(^1\) Racah Institute of Physics, Hebrew University of Jerusalem, Jerusalem 91904, Israel
\(^2\) Keldysh Institute of Applied Mathematics, Moscow 125047, Russia

We use the macroscopic fluctuation theory (MFT) to study large current fluctuations in non-stationary diffusive lattice gases. We identify two universality classes of these fluctuations which we call elliptic and hyperbolic. They emerge in the limit when the deterministic mass flux is small compared to the mass flux due to the shot noise. The two classes are determined by the sign of compressibility of effective fluid, obtained by mapping the MFT into an inviscid hydrodynamics. An example of the elliptic class is the Symmetric Simple Exclusion Process where, for some initial conditions, we can solve the effective hydrodynamics exactly. This leads to a super-Gaussian extreme current statistics conjectured by Derrida and Gershenfeld (2009) and yields the optimal path of the system. For models of the hyperbolic class the deterministic mass flux cannot be neglected, leading to a different extreme current statistics.

PACS numbers: 05.40.-a, 05.70.Ln, 02.50.-r

Large fluctuations of currents of matter or energy in systems away from thermodynamic equilibrium are at the forefront of statistical physics. Over the last decade a major progress has been achieved in the study of large fluctuations of the density profile and of the current in nonequilibrium steady states (NESS) of stochastic diffusive lattice gases driven from the boundaries, see [1] and references therein. Diffusive lattice gases [2–4] constitute a broad family of simple transport models which capture different aspects of transport in extended many-body systems. One extensively studied model is the Symmetric Simple Exclusion Process (SSEP) [2–9], where each particle can randomly hop to a neighboring lattice site if that site is empty. If it is occupied by another particle, the move is disallowed. Applications of this model range from cell biology and biophysics [14] to a host of transport problems in materials science, and they will be in the focus of our attention here. Following Refs. [19–22], we will consider a diffusive lattice gas on an infinite line, and study fluctuations of integrated current of NESS of diffusive lattice gases exhibit qualitatively new features compared to the free energy of equilibrium states [11–18], and these discoveries have attracted great interest. Non-stationary fluctuations of diffusive lattice gases are still poorly understood [19–24], and they will be in the focus of our attention here. Following Refs. [19–22], we will consider a diffusive lattice gas on an infinite line, and study fluctuations of integrated current. At large scales these fluctuations can be described by the Langevin equation

\[ \partial_t n = \partial_x [D(n) \partial_x n] + \partial_x \left[ \sqrt{\sigma(n)} \eta(x,t) \right], \]

where \( \eta(x,t) \) is a zero-average Gaussian noise, delta-correlated both in space and in time [3, 4]. As one can see, a fluctuating lattice gas is fully characterized by \( D(n) \) and the coefficient \( \sigma(n) \), which comes from the shot noise and is equal to twice the mobility of the gas [3, 16].

Starting from Eq. (2), one can arrive at macroscopic fluctuation theory (MFT), which employs \( 1/\sqrt{N} \) \((N\) is the typical number of particles in the relevant region of space) as a small parameter, and is appropriate for dealing with large deviations. The MFT was originally developed for the NESS [23] and more recently extended to non-stationary settings, such as the step-like initial density profile [20, 22]. The MFT can be formulated as a classical Hamiltonian field theory [21, 22, 25, 26], and we will adopt this formulation here.

The MFT formulation of the problem of statistics of integrated current was obtained in Ref. [20]. Until now the problem has defied analytic solution except (i) for non-interacting random walkers (RWs) [20], and (ii) for small fluctuations around the mean. For the RWs, the \( J \rightarrow \infty \) asymptote of the current probability density \( P(J,T) \) is super-Gaussian in \( J \) [20]:

\[ \ln P(J \rightarrow \infty, T) \sim -\frac{J^3}{12n^2 T}. \]

Derrida and Gershenfeld [20] (DG) conjectured that the \( J^3/T \) decay holds for a whole class of interacting gases, and proved their conjecture for \( D(n) = 1 \) and \( \sigma(n) \leq n + \text{const} \) for \( 0 \leq n \leq R \), and \( \sigma(n) = 0 \) otherwise [27].
This Letter reports a major progress in the analysis of extreme current fluctuations. Here is an outline. A natural first step in the analysis of unusually large currents is to neglect, in the MFT equations, the deterministic mass flux compared to the mass flux due to the shot noise. The noise-dominated MFT equations can then be mapped into an effective inviscid hydrodynamics. This mapping uncovers two universality classes (which we call elliptic and hyperbolic) of the diffusive lattice gases with respect to the extreme current statistics. These are determined by the sign of $\sigma''(n)$. For the elliptic class $\sigma''(n) < 0$ for all relevant $n$. Here the DG conjecture holds, as

$$\ln \mathcal{P}(J \to \infty, T) \simeq -\frac{f(n_-, n_+)}{T} J^3. \quad (4)$$

Furthermore, for the SSEP, with $\sigma(n) = 2n(1-n)$ \cite{3, 4}, the effective hydrodynamics can be solved exactly. The solution yields a systematic way of calculating the function $f(n_-, n_+)$ and gives the optimal path of the system, responsible for the specified current.

The hyperbolic case, when $\sigma''(n) > 0$, is more complicated. Here a singularity, present in the noise-dominated equations, has to be regularized by diffusion. The resulting $\ln P$ differs from Eq. \cite{11} \cite{28}. Now we expose our results in some detail.

MFT. A specified current is described by the equation

$$\int_0^\infty dx [n(x, 1) - n_+] = J/\sqrt{T} \equiv j. \quad (5)$$

Here and in the following $t$ and $x$ are rescaled by $T$ and $\sqrt{T}$, respectively. The optimal path $q(x, t)$ in the space of $\{n(x, t)\}$ obeys the equations

$$\begin{align*}
\partial_t q &= \partial_x [D(q) \partial_x q] - \partial_x [\sigma(q) \partial_x p], \quad (6) \\
\partial_t p &= -D(q) \partial_x^2 p - \frac{1}{2} \sigma'(q) (\partial_x p)^2, \quad (7)
\end{align*}$$

where $p(x, t)$ is the “momentum” field conjugate to $q \cite{20}$. These are Hamiltonian equations, with the Hamiltonian $H = \int_{-\infty}^{\infty} dx \mathcal{H}$, where $\mathcal{H}(q, p) = -D(q) \partial_q p \partial_x q + (1/2)\sigma(q)(\partial_x p)^2$. Once $q(x, t)$ and $p(x, t)$ are known, the action $\int \int dt dx \{p \partial_q q - \mathcal{H}\}$ can be written as

$$s = \frac{1}{2} \int_0^1 dt \int_{-\infty}^{\infty} dx \sigma(q)(\partial_x p)^2. \quad (8)$$

The boundary condition for $q(x, 0)$ is given by $n(x, t = 0)$ from Eq. \cite{11}. By varying $q(x, 1)$ to minimize the action under the constraint \cite{5}, DG \cite{20} obtained the second boundary condition:

$$q(x, t = 1) = \lambda \theta(x), \quad (9)$$

where the Lagrange multiplier $\lambda = \lambda(j, n_-, n_+) > 0$ is set by Eq. \cite{5}. Once $s$ is found, $\mathcal{P}(J, T)$ is given by $\ln \mathcal{P}(J, T) \simeq -\sqrt{T} s \cite{20, 22}$.\n
\textbf{INVISCID LIMIT.} When $j \to \infty$, it seems natural to neglect the deterministic diffusion terms in Eqs. \cite{6} and \cite{7}. This yields the first-order Hamiltonian equations

$$\begin{align*}
\partial_t q &= -\partial_x [\sigma(q) \partial_x p], \quad (a) \\
\partial_t p &= -\frac{1}{2} \sigma'(q)(\partial_x p)^2, \quad (b)
\end{align*}$$

stemming from the \textit{noise-dominated} Hamiltonian

$$H_0 = \int_{-\infty}^{\infty} dx \rho(q, p), \quad \rho = \frac{1}{2} \sigma(q)(\partial_x p)^2. \quad (11)$$

We will call this theory inviscid. Comparing Eqs. \cite{8} and \cite{11} and using the fact that $H_0$ is now a constant of motion, we can rewrite Eq. \cite{8} as

$$s = \int_0^1 dt H_0. \quad (12)$$

where we do not use here the rescaling with $\lambda$. Equation \cite{12} includes a shock at $x_s(t) = -2\sqrt{\lambda(1-t)}$. Now, Eq. \cite{10}a is a continuity equation for the density $q(x, t)$ with a known velocity field $2\partial_x p$. The characteristics of this equation are $x = C(1-t)$, where $C = \text{const}$. Consider the region of the $x, t$ plane where $\partial_t p \neq 0$, see Fig. \cite{11}. The characteristics with $-2\sqrt{\lambda} \leq C \leq 0$ cross the boundary $t = 0$, where $q(x, t = 0) = n_-$. This yields a simple solution, $q(x, t) = n_- (1-t)^{-1}$, to the right of the characteristic $x = -2\sqrt{\lambda}(1-t)$.

The characteristics with $C < -2\sqrt{\lambda}$ cross the velocity shock at $x = x_s(t)$. To ensure mass conservation at the shock (where the velocity drops from a positive value to zero), $q$ at the shock must vanish. As a result, $q(x, t) = 0$ on the interval $-2\sqrt{\lambda} < x < -2\sqrt{\lambda}(1-t)$ which expands in time. To summarize,

$$q(x, t) = \begin{cases} 
 n_-, & x < -2\sqrt{\lambda} \\
 0, & -2\sqrt{\lambda} < x < -2\sqrt{\lambda}(1-t), \\
n_- (1-t)^{-1}, & -2\sqrt{\lambda}(1-t) < x < 0, \\
n_+, & x > 0.
\end{cases}$$

Importantly, at $t = 0$ the flow already includes a point-like void, $q = 0$, at the point $x = -2\sqrt{\lambda} \cite{29}$. At $t >$
is happens for the SSEP, where the effective fluid pressure \( \sigma \) where it is assumed that whether of this flow – elliptic or hyperbolic – is determined by dynamic flow of an effective fluid. The general character \( V \) Eq. (15) can be written as equal to the (rescaled) integrated current \( \lambda \) and we obtain \( \lambda = j^2/(4n_\perp^2) \). The rescaled action \( S \) is
\[
s = \int_0^1 dt \int_{-2\sqrt{\lambda}(1-t)} dx \frac{n_-}{1-t} \frac{x^2}{4(1-t)^2} = \frac{2n_- \lambda^{3/2}}{3} = \frac{j^3}{12n^2}.
\]
which, in view of the relation \( \ln P(J,T) \simeq -\sqrt{T} s \), yields Eq. [39]. As we see, for the RWs, the inviscid MFT does yield the correct leading-order result for the extreme current statistics and for the optimal path of the system.

Effective hydrodynamics and two universality classes. Now let us consider some general properties of the inviscid theory as applied to interacting particles. A key observation is that the Hamiltonian density \( \rho \) of the inviscid MFT, see Eq. (11), is conserved locally, as it evolves according to the continuity equation
\[
\partial_t \rho + \partial_x (\rho V) = 0
\]
with the effective velocity \( V = \sigma'(q) \partial_x p \). A direct calculation shows that \( V \) obeys
\[
\partial_t V + V \partial_x V = -\sigma''(q) \partial_x \rho,
\]
where it is assumed that \( \sigma''(q) \) is expressed via \( \rho \) and \( V \). Equations (14) and (15) describe an inviscid hydrodynamic flow of an effective fluid. The general character of this flow – elliptic or hyperbolic – is determined by whether \( \sigma''(q) \) is negative or positive, respectively. The analogy with hydrodynamics becomes complete when \( \sigma(\rho) \) is a quadratic polynomial, and so the right side of Eq. (15) can be written as \(-1/\rho^2 \partial_x P(\rho)\). This is what happens for the SSEP, where the effective fluid pressure is \( P(\rho) = -2\rho^2 < 0 \), exemplifying elliptic flow. In an initial-value problem such a fluid would be intrinsically unstable. Moreover, when starting from generic smooth fields \( \rho \) and \( V \) at \( t = 0 \), a finite-time singularity develops, see Ref. [32] for a detailed review. In our boundary-value problem, \( \rho \) and \( V \) must blow up, in view of Eq. (19), at time \( t = 1 \) at \( x = 0 \). However, they are bounded and smooth at earlier times, \( 0 \leq t < 1 \), in the whole region where \( \rho(x,t) > 0 \). Therefore, the inviscid MFT is well defined for elliptic flows, and Eq. (1) holds [33].

Models where \( \sigma''(n) > 0 \) exhibit a hyperbolic flow. In view of the boundary condition \( \partial_x \rho \) and \( V \) must diverge at \( t = 1 \) and \( x = 0 \). In a hyperbolic flow this implies that the singularities \( \rho = \infty \) and \( V = \infty \) are present at all times \( 0 \leq t \leq 1 \). Here one needs to return to the full MFT equations (8) and (9), where the singularities are regularized by diffusion. This leads to a different extreme current statistics [28].

The RWs, with \( \sigma(n) = 2n \), belong to the marginal class \( \sigma''(0) = 0 \) where the effective fluid pressure vanishes, leading to the Hopf equation as we already observed.

The SSEP. Fortunately, Eqs. (14) and (15) become linear upon the hodograph transformation, where \( \rho \) and \( V \) are treated as the independent variables, and \( t(\rho,V) \) and \( x(\rho,V) \) as the dependent ones [51]. For the SSEP we obtain the elliptic linear second-order equation
\[
\rho^{-1} \partial_{\rho} (\rho^2 \partial_{\rho} t) + 4\partial_x^2 t = 0.
\]
Once \( t(\rho,V) \) is found, \( x(\rho,V) \) can be determined from any of the relations
\[
\partial_y x = V \partial_t x - \rho \partial_{\rho} t, \quad \partial_y x = V \partial_t x + 4\partial_v t.
\]
In the new variables \( X = V/2 \) and \( Y = 2\rho^{1/2} \) Eq. (16) becomes
\[
\partial_X^2 t + \partial_Y^2 t + (3/Y) \partial_Y t = 0.
\]
As we will see shortly, the boundary conditions for \( q \) and \( p \) at \( t = 0 \) and \( 1 \), respectively, define a Dirichlet problem for \( t(X,Y) \). Moreover, Eq. (18) can be transformed into the Laplace’s equation in an extended space [32], which opens the way to exact analytic solution. The full solution also includes non-hodograph regions: (i) static regions where \( q = n_- \) or \( n_+ \) and \( \partial_x q = 0 \), (ii) a void, \( q = 0 \), and (iii) a close-packed cluster, \( q = 1 \). The dynamics of \( \rho \) in the void and cluster regions is described by the Hopf equation \( \partial_t \rho \pm (\partial_x \rho)^2 = 0 \). As \( \sigma(0) = \sigma(1) = 0 \), the non-hodograph regions do not contribute to the action [39]. Importantly, (point-like) void and cluster are already present at \( t = 0 \); they expand at \( t > 0 \).

Here we only consider the simple case of a flat initial density profile with \( q(x,t) = 0 \). In this case the solution is symmetric, \( q(-x,t) = 1 - q(x,t) \) and \( \partial_x q(-x,t) = \partial_x q(x,t) \) and, remarkably, can be obtained in elementary functions. We start from Eq. (18) which should be solved in the upper half-plane \( X < \infty \),
$0 \leq Y < \infty$. To match the non-hodograph part of the full solution, the hodograph solution must be regular at $Y = 0$. The second boundary condition comes from $t = 0$. The value of $\partial_x p(x, t = 0)$ changes, as a function of $x$, from an an a priori unknown finite maximum value $v_0 > 0$. Exploiting the invariance of the inviscid MFT equations under the transformation $x/\sqrt{X} \to x$ and $p/\lambda \to p$, we can first solve the problem for $v_0 = 1$ and then restore the $\lambda$-scalings in the final solution. By virtue of the conditions $q(x, t = 0) = 1/2$ and $0 \leq \partial_x p(x, t = 0) \leq 1$, $t$ must vanish on the segment $X = 0$, $0 \leq Y \leq 1$. The last boundary condition comes from Eq. (9) at $t = 1$. As $\partial_x p(x, t = 1)$ is a delta-function, $t$ must approach 1 as $|X|$ or $Y$ go to infinity. The solution of this Dirichlet problem can be obtained in the elliptic coordinates $s, r$: $X = sr$, $Y = [(1 + s^2)(1 - r^2)]^{1/2}$, $s \geq 0$, $|r| \leq 1$. After some algebra, Eq. (18) becomes

$$
\partial_s [(1 + s^2)\partial_s t] + \frac{1 + s^2}{1 - r^2} \partial_r [(1 - r^2)\partial_r t] = 0. \quad (19)
$$

The new boundary conditions, $t(s = 0, |r| \leq 1) = 0$ and $t(s \to \infty) = 1$, are independent of $r$, and so is the solution: $\pi t(s)/2 = s(1 + s^2)^{-1} + \text{arctan} s$. In the original variables $q$ and $v = \partial_x p$, we obtain

$$
\frac{\pi}{2} t(q, v) = \frac{\sqrt{2(v^2 + R - 1)}}{v^2 + R + 1} + \arctan \frac{\sqrt{v^2 + R - 1}}{2}. \quad (20)
$$

where $R^2(q, v) = (v^2 - 1)^2 + 4v^2(2q - 1)^2$. Then Eqs. (17) yield

$$
x(q, v) = \frac{[v^2 - R - 8q(1 - q) + 1] \sqrt{v^2 + R - 1}}{2\sqrt{2}\pi q(1 - q)(2q - 1)v}. \quad (21)
$$

Functions $t(q, v)$ and $x(q, v)$ are analytic in the whole hodograph region $0 \leq q \leq 1, 0 \leq v < \infty$ except at the branch cut at $q = 1/2, 0 \leq v \leq 1$, see Fig. 2.

The hodograph asymptotics at $v \to \infty$ are $t \approx 1 - 4/(3\pi v^3)$ and $x \approx 4(2q - 1)/(\pi v^2)$. In physical variables, we obtain self-similar asymptotics at $t \to 1$: $2q(x, t - 1) = x/\ell(t)$ and $v(x, t) = (4/3\pi)^{1/3}(1 - t)^{-1/3}$, where $|x| \leq \ell(t)$ and $\ell(t) = 3(4/3\pi)^{1/3}(1 - t)^{2/3}$. Using Eq. (8) or (11), we obtain $s = H_0 = \int_{-t}^{\ell(t)} dx q(1 - q)v^2|_{t=1} = 4/(3\pi)$.

In its turn, the $q \to 1/2$ asymptotic of Eq. (21) corresponds to $t \to 0$, and we obtain $v(x, t = 0) = (1 - x^2/x_0^2)^{1/2}$, for $|x| \leq x_0$, and 0 otherwise, where $x_0 = 4/\pi$. Notably, $x = -x_0$ and $x = x_0$ are the positions of the point-like void and point-like cluster, respectively, at $t = 0$. At $t > 0$ the void and cluster expand, see Fig. 3 and by $t = 1$ all of the material from the interval $-x_0 < x < 0$ is transferred to the interval $0 < x < x_0$. Therefore, the integrated current is $j_0 = (1/2) \times 2\pi = 2/\pi$, whereas $\lambda_0 = \int_{-x_0}^{x_0} dx v(x, t = 0) = 2$. Restoring the $\lambda$- and $j$-scalings, we obtain $\lambda = \pi^2 j^3/2$ and $s = \pi^2 j^3/6$, which leads to Eq. (4) with $f(1/2, 1/2) = \pi^2/6$. (See Supplemental Material below for an alternative derivation of this result.) This completes the evaluation of $\ln P(J, T)$ and justifies the inviscid MFT for the SSEP.

The inviscid MFT can be extended to other non-stationary settings. One of them is the noise-driven void formation, at a specified time $T$, in an initially uniform gas, in the limit of $L \gg \sqrt{T}$, where $L$ is the characteristic void size. As it is evident from Fig. 3, the void formation problem is closely related to the extreme current problem. It also extends to higher dimensions.

We thank Arkady Vilenkin, whose numerical insight directed us in the initial stage of this work, and P.L. Krapivsky for useful discussions. B.M. was supported by the Israel Science Foundation (Grant No. 408/08) and by the US-Israel Binational Science Foundation (Grant No. 2008075). P.V.S. was supported by the Russian Foundation for Basic Research, grant No 13-01-00314.

Supplemental Material: Alternative derivation of the result $f(1/2, 1/2) = \pi^2/6$ for the SSEP

The same result $f(1/2, 1/2) = \pi^2/6$ for the SSEP can be extracted from the expressions obtained by Derrida and Gerschenfeld (DG) [19, 20]. DG employed the moment generating function of $J$:

$$
\langle e^{\lambda J} \rangle = \sum_{J \geq 0} e^{\lambda J} P(J). \quad (22)
$$

For diffusive lattice gases with a step-like initial condition one has, at long times, $\ln P(J) \simeq -\sqrt{T} s(j, n_-, n_+)$,
where \( j = J/\sqrt{T} \), see the main text. Therefore,
\[
\langle e^{\lambda j} \rangle \sim \int_0^\infty dj \ e^{\sqrt{T} [\lambda j - s(j, n_-, n_+)]} \sim e^{\sqrt{T} \mu(\lambda, n_-, n_+)},
\]
where \( \mu(\lambda, n_-, n_+) = \max_j [\lambda j - s(j, n_-, n_+)] \). That is, \( \mu(\lambda) \) is the Legendre transformation of \( s(j) \). In reverse, \( s(j, n_-, n_+) = \max_\lambda [j \lambda - \mu(\lambda, n_-, n_+)] \).

Using the microscopic model of SSEP, DG \[19\] calculated \( \mu(\lambda) \) in the \textit{annealed} setting, that is when the initial densities at \( x < 0 \) and \( x > 0 \) are allowed to fluctuate around their respective mean values \( n_- \) and \( n_+ \), and one averages the result over these fluctuations. DG obtained
\[
\mu_{\text{annealed}}(\lambda, n_-, n_+) = \frac{1}{\pi} \int_{-\infty}^{\infty} dk \ln \left( 1 + \Lambda e^{-k^2} \right),
\]
where
\[
\Lambda = n_- (\lambda^2 - 1) + n_+ (e^{-\lambda} - 1) + n_- n_+ (\lambda^4 - 1) (e^{-\lambda} - 1).
\]
In this paper we only deal with the \textit{deterministic} (also called \textit{quenched}) setting, when no density fluctuations are allowed at \( t = 0 \). Still, in the special case of \( n_- = n_+ = 1/2 \), one can obtain \( \mu_{\text{quenched}} \) from Eq. (25). This is because DG proved, in the framework of MFT formalism that, in this special case, \( \mu_{\text{quenched}} \) is simply related to \( \mu_{\text{annealed}} \):
\[
\mu_{\text{quenched}}(\lambda, 1/2, 1/2) = \frac{1}{\sqrt{2}} \mu_{\text{annealed}}(\lambda, 1/2, 1/2)
\]
for any \( \lambda \). Our inviscid theory is only valid for large currents, \( \lambda \gg 1 \). Calculating the \( \lambda \gg 1 \) asymptotic of Eq. (25) (which is actually independent of \( n_- \) and \( n_+ \)) and using Eq. (26), we obtain
\[
\mu_{\text{quenched}}(\lambda \gg 1, 1/2, 1/2) \simeq \frac{2\sqrt{T}}{3\pi} \lambda^{3/2}.
\]
Now we can determine \( s(j, n_-, n_+) \) from Eq. (24). The maximum is achieved at \( \lambda = \pi^2 j^2 / 6 \), and so \( s = \pi^2 j^3 / 6 \) which yields \( f(1/2, 1/2) = \pi^2 / 6 \) as expected.

---

\[1\] meerson@mail.huji.ac.il
\[2\] pavel.sasorov@gmail.com

[1] B. Derrida, J. Stat. Mech. P07023 (2007).
[2] T. M. Liggett, \textit{Stochastic Interacting Systems: Contact, Voter, and Exclusion Processes} (Springer, New York, 1999).
[3] H. Spohn, \textit{Large Scale Dynamics of Interacting Particles} (New York: Springer-Verlag, 1991).
[4] C. Kipnis and C. Landim, \textit{Scaling Limits of Interacting Particle Systems} (Springer, New York, 1999).
[5] B. Schmittmann and R. K. P. Zia, \textit{Statistical Mechanics of Driven Diffusive Systems}, in: \textit{Phase Transitions and Critical Phenomena}, Vol. 17, eds. C. Domb and J. L. Lebowitz (Academic Press, London, 1995).
[6] B. Derrida, Phys. Rep. \textbf{301}, 65 (1998).
[7] G. Schütz, \textit{Exactly Solvable Models for Many-Body Systems Far From Equilibrium}, in \textit{Phase Transitions and Critical Phenomena}, Vol. 19, eds. C. Domb and J. L. Lebowitz (Academic Press, London, 2000).
[8] R. A. Blythe and M. R. Evans, J. Phys. A \textbf{40}, R333 (2007).
[9] P. L. Krapivsky, S. Redner, and E. Ben-Naim, \textit{A Kinetic View of Statistical Physics} (Cambridge University Press, Cambridge, 2010).
[10] H. Lee, L.S. Levitov, and A. Yu. Yakovets, Phys. Rev. B \textbf{51}, 4079 (1995).
[11] Y.M. Blanter and M. Büttiker, Phys. Rep. \textbf{336}, 1 (2000).
[12] A.N. Jordan, E.V. Sukhorukov and M. Büttiker, Phys. Rev. Lett. \textbf{90}, 206801 (2003).
[13] A.N. Jordan, E.V. Sukhorukov, and S. Pilgram, J. Math. Phys. \textbf{45}, 4386 (2004).
[14] T. Chou, K. Mallick, and R. K. P. Zia, Rep. Prog. Phys. \textbf{74}, 116601 (2011).
[15] H. Touchette, Phys. Rep. \textbf{478}, 1 (2009).
[16] G. Jona-Lasinio, Prog. Theor. Phys. Suppl. \textbf{184}, 262 (2010).
[17] P.I. Hurtado and P.L. Garrido, Phys. Rev. Lett. \textbf{107}, 186001 (2011).
[18] G. Bunin, Y. Kafri, and D. Podolsky, J. Stat. Mech. L10001 (2012); J. Stat. Phys. \textbf{152}, 112 (2013).
[19] B. Derrida and A. Gerschenfeld, J. Stat. Phys. \textbf{136}, 1 (2009).
[20] B. Derrida and A. Gerschenfeld, J. Stat. Phys. \textbf{137}, 978 (2009).
[21] S. Sethuraman and S.R.S. Varadhan, Ann. Prob. \textbf{41}, 1461 (2013).
[22] P. L. Krapivsky and B. Meerson, Phys. Rev. E \textbf{86}, 031106 (2012).
[23] V. Lecomte, J. P. Garrahan, and F. van Wijland, J. Phys. A: Math. Theor. \textbf{45}, 175001 (2012).
[24] P. L. Krapivsky, B. Meerson, and P.V. Sasorov, J. Stat. Mech. P12014 (2012).
[25] L. Bertini, A. De Sole, D. Gabrielli, G. Jona-Lasinio, and C. Landim, Phys. Rev. Lett. \textbf{87}, 040601 (2001); ibid \textbf{94}, 030601 (2005); J. Stat. Phys. \textbf{123}, 237 (2006); ibid \textbf{135}, 857 (2009); J. Stat. Mech. P07014 (2007).
[26] J. Tailleur, J. Kurchan, and V. Lecomte, Phys. Rev. Lett. \textbf{99}, 15006 (2007); J. Phys. A \textbf{41}, 505001 (2008).
[27] Recently the \( \sim J^3/T \) decay was proved for the SSEP for more general initial density profiles \[21\].
[28] B. Meerson and P.V. Sasorov, arXiv:1309.5241.
[29] The presence of a point-like void at \( t = 0 \) in the inviscid Eqs. (10) implies a rapid formation of a deep and narrow density minimum in the complete MFT equations (6) and (7). Other discontinuities, exhibited by the inviscid solutions, will be also regularized by diffusion, except for the true singularity \( \delta_+ p(x, 1) = \lambda \delta(x) \) imposed by Eq. (4).
[30] G.B. Whitham, \textit{Linear and Nonlinear Waves} (Wiley, New York, 1974).
[31] L. D. Landau and E. M. Lifshitz, \textit{Fluid Mechanics} (Reed, Oxford, 2000).
[32] B.A. Trubnikov and S.K. Zhdanov, Phys. Rep. \textbf{155}, 137 (1987).
[33] Interestingly, the rescaled action from Eq. (5) is equal to
the (a priori unknown) finite total mass of the effective fluid which collapses into the origin at $t = 1$. 