A new mechanism for gain in time dependent media

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Time dependent systems do not in general conserve energy invalidating much of the theory developed for static systems and turning our intuition on its head. This is particularly acute in luminal space time crystals where the structure moves at or close to the velocity of light. Conventional Bloch wave theory no longer applies, energy grows exponentially with time, and a new perspective is required to understand the phenomenology. In this letter we identify a new mechanism for pulse amplification: the compression of lines of force that are nevertheless conserved in number.

Energy can be added to electromagnetic waves in several different fashions. We identify a mechanism, distinct from conventional ones, in which compression of lines of force is the active ingredient. There are instances of this in other contexts: a superconductor repels magnets because the magnetic lines of forces are compressed as they are rejected by the superconductor. A more dramatic example is the generation of thousand Tesla magnetic fields by explosive collapse of a copper cylinder enclosing magnetic lines of force \cite{1}. Here we invoke the concept in the context of amplifying electromagnetic waves. We show that in some circumstances the number of lines of force, electric and magnetic, in a time dependent system is conserved and amplification occurs when these lines of force are squeezed closer together.

In this letter we use a simple model of a time dependent grating moving with a uniform velocity, \( c_g = \frac{\Omega}{g} \),

\[
\begin{align*}
\varepsilon(x-c_g t) &= \varepsilon_1 + 2\alpha_e \cos(gx - \Omega t), \\
\mu(x-c_g t) &= \mu_1 + 2\alpha_\mu \cos(gx - \Omega t)
\end{align*}
\]  

(1)

where \( g \) and \( \Omega \) are spatial and temporal modulation frequencies, and \( \alpha_e \) and \( \alpha_\mu \) the strength of electric and magnetic modulations, respectively. The model has been extensively deployed in other studies of time dependent systems and a recent review is to be had in \cite{2}. We assume that \( \varepsilon, \mu \) are both real. Complications arising when the medium is dispersive and lacking in periodicity are treated in the accompanying PRB paper. We stress that the medium itself does not move so that there is no restriction from relativity on the magnitude of \( c_g \) which may take any value \( 0 < c_g < \infty \).
A new mechanism for gain in time dependent media

Similar travelling-wave media have been investigated theoretically in the past [1, 2, 3]. More recently, they gained renewed interest thanks to their ability to break Lorentz reciprocity without need for an applied magnetic bias, both by inducing asymmetric band-gaps [4, 5, 6, 7, 8, 9] and, more recently, in the quasistatic regime [10, 11], as well as due to their topological [12] and non-Hermitian [13] features. The luminal modulation regime considered here has recently been proposed for pulse formation [14] and broadband, nonreciprocal amplification [15]. However, a fundamental explanation of the physical mechanism responsible for this amplification has never been developed.

When waves interact with static structures we have many tools not only for calculating but also for understanding the processes and for conceptualizing a problem before we even begin to calculate. In periodic structures the Bloch wave vector is conserved and together with frequency, the other conserved quantity, gives a wealth of understanding. Its dispersion with frequency tells us where the band gaps are and where we are likely to find surface states. Bloch waves are the basis for understanding the topology of the states through such quantities as the Berry phase [19] and Chern number [20]. We stress that these concepts are not merely computational devices but central to our thought processes as tools for understanding and creating such systems.

**FIG. 1.**

A: dispersion relation of light travelling in a uniform dielectric. B: a grating modulated as shown in Eq. (1) where \( c_g < c_0 \sqrt{\varepsilon_1 \mu_1} \). C: a grating modulated as shown in Eq. (1) where \( c_g > c_0 \sqrt{\varepsilon_1 \mu_1} \). D: the case of \( c_g = c_0 \sqrt{\varepsilon_1 \mu_1} \) where all the forward travelling states are degenerate and therefore strongly coupled.
In this paper we seek to provide a set of tools for understanding time dependent structures for which the traditional static concepts fail, by identifying a conserved quantity in the form of the number of lines of force contained in the system.

At first sight Eq. (1) would imply a straightforward generalization of Bloch's theorem mixing together waves differing by a space-time reciprocal lattice vector,

\[ k', \omega' = k + ng, \omega + n\Omega \]

so that as well as a Bloch wave vector, \( k \), there is a Bloch frequency \( \omega \), both conserved modulo \((g, \Omega)\). This is a good description of the problem for \( c_g \ll c_0/\sqrt{\epsilon_1\mu_1} \) where \( c_0/\sqrt{\epsilon_1\mu_1} \) is the velocity of light in the background medium.

FIG. 1A shows dispersion of light in the background medium and FIG. 1B what happens when the grating is turned on: band gaps open in the normal way but now there is asymmetry about \( k = 0 \) due to the breaking of time reversal symmetry. These structures have potential for isolators [21]

In the other extreme, \( c_g \gg c_0/\sqrt{\epsilon_1\mu_1} \), shown in FIG. 1C, we may still cling onto the Bloch wave picture except that the band gaps are regions of complex \( \omega \) rather than of complex \( k \). In these gaps waves can lose or gain energy, a process of parametric amplification [22].

The focus of our interest will be in the middle of these extremes: the luminal region in which the speed of the grating approaches that of light in the medium. FIG. 1D shows the catastrophe that occurs when \( c_g = c_0/\sqrt{\epsilon_1\mu_1} \): all the forward travelling waves become degenerate. Whereas the band gaps formed between forward and backward travelling waves dominate scattering outside this regime, it is forward-forward scattering that dominates here. There is a clearly defined range of \( c_g \) within which the band picture is invalid. In this range light is not scattered by the structure but is captured and localized, carried along with the structure with velocity \( c_g \), amplified, and ejected from the medium as a series of pulses as shown in FIG. 2. Here a picture in terms of freely propagating waves is meaningless. Instead we look to the basic elements of the electromagnetic field: the lines of force embodied in the \( D \) and \( B \) fields. Because their velocity varies with position in the structure there will be a point towards which they migrate, an accumulation point [17,18], and here they are compressed and our new mechanism of amplification comes into play.
A new mechanism for gain in time dependent media

FIG. 2. Schematic figure of the effect of a luminal grating, travelling to the right, on a plane wave incident from the left. The grating has little effect on waves incident from the right. Note the compression of phase which mirrors the compression of lines of force shown schematically as grey lines.

In order to show the complete contrast with parametric amplification which operates at higher grating velocities and is due to the formation of band gaps, we eliminate all band gaps from our model which is done by requiring that the medium has a constant impedance,

$$\mu(x-c_g t)/\epsilon(x-c_g t) = Z^2$$

(3)

This assumption turns off parametric amplification and allows us to identify our new mechanism in its purest form. In essence Eq. (3) eliminates backscattering from the system. The theory will hold good whenever backscattering can be neglected. For example when the time dependent disturbance is weak whether or not (3) is satisfied as we show in the accompanying PRB paper.

When back scattering is negligible the lines of force are conserved. Therefore any gain in energy can only come from compression of the lines into a sharp pulse. If their local density is increased by a factor of $f^2$, then the local energy density increases by $f^2$ and hence there is not only a local increase in energy but also a net increase.

Here we stress that this is an entirely novel insight into an amplification process. We have already identified parametric amplification which occurs when band gaps open and give rise to complex values of $\omega$ and wave fields that grow in time. No compression is at work here, simply a uniform addition of lines of force. Next consider a slab of uniform gain medium characterized by a complex refractive index $n = n_r - in_i$ so that a wave of frequency $\omega$ injected into the system acquires a complex wave vector,

$$k = (n_r - in_i)\omega/c_0$$

(4)

and as a result the wave amplitude increases exponentially with penetration into the medium. Clearly there is no conservation of lines of force here. Many more are ejected from the far side of the medium than enter.

The closest analogy to the luminal mechanism of compression can be seen when a magnet approaches a superconductor. Its lines of force cannot enter the
superconductor and are distorted and compressed but at the same time conserved. This requires energy giving rise to the repulsive force experienced by the magnet. In a similar fashion compressing the field lines of trapped radiation results in a frictional force on the moving structure.

Within the same moving structure both parametric and luminal compression mechanisms are in general at work. To separate the two we consider structures in which back scattering can be neglected. This eliminates the band gaps and thus turns off the parametric mechanism. This model accurately describes weakly modulated structures and structures obeying Eq. (3). As can be seen from FIG. 1 in the luminal regime forward travelling waves crowd close together and are more densely packed than backwards travelling waves and hence the dominant interaction will be between the forward waves. We discuss likely experimental realizations in the accompanying PRB publication.

**Equation of motion for the density of lines of force**

Consider a plane wave travelling in the +x direction:

\[ D(x,t) = D_y \exp(ikx - i\omega t), \]
\[ B(x,t) = ZD_y \exp(ikx - i\omega t) \]

The system is defined by Eq. (3) and Maxwell's equations reduce to a single first order partial differential equation,

\[ + \frac{\partial D_y}{\partial t} = -\frac{\partial H_z}{\partial x} = -\frac{1}{Z} \frac{\partial E_y}{\partial x} \]

If we assume that \( Z, \epsilon \) are both real, and that \( \epsilon \) is independent of frequency and a local operator, then \( D \) and \( E \) have the same phase and there is a further factorization of (5) into real and imaginary parts,

\[ + \frac{\partial D_y}{\partial t} = -\frac{1}{Z} \frac{\partial E_y}{\partial x}, \quad + \frac{D_y}{\partial t} \frac{\partial \phi}{\partial t} = -\frac{1}{Z} \frac{E_y}{\partial x} \frac{\partial \phi}{\partial x} \]

These equations are used in [PRB] to prove two theorems: that the number of lines of force in each period of the grating is conserved and that the change of phase across one period is a constant: results that are independent of the shape of the grating.

Eqs. (7) allow approximate but accurate analytic solutions which provide further insight. From the fields we can calculate the energy density, \( U \), within one period of the grating that is to say at a fixed value of \( gX = gx - \Omega t \). We change coordinates to,

\[ X = x - c_g t, \quad t' = t; \quad x = X + c_g t, \quad t = t' = \tau \sqrt{\epsilon_1 \mu_1} / c_0 \]

and defining \( c_l(X) = c_0 / \sqrt{\epsilon(X) \mu(X)} \), the local speed of waves in the medium, we arrive at our defining equation,
A new mechanism for gain in time dependent media

\[
\frac{\partial U'(X,t')}{\partial t'} = \frac{1}{2} U'(X,t') \left( c_g + c_l(X) \right) \left[ \frac{\partial \ln \varepsilon(X)}{\partial X} + \frac{\partial \ln \mu(X)}{\partial X} \right] + \left[ c_g - c_l(X) \right] \frac{\partial U'(X,t')}{\partial X} \tag{9}
\]

There are two terms on the right hand side. The first tells us that the rate of change of \( \varepsilon \) and \( \mu \) determines how much energy is pumped into the system. However the energy is mobile: because the refractive index varies with position energy flows at a different rate in each part of the grating and the second term accounts for this flow. This redistribution plays a vital role in evolution of energy density.

It is possible to find an approximate solution to Eq. (9). We show below the result where only \( \varepsilon \) is modulated in order to demonstrate that our theory is valid whenever back scattering is weak, which it is for weak modulations.

\[
U'(X,\tau) = \exp \left\{ -\Omega (2+\delta) \alpha \sin (g\chi) \tau \right\}
\]

\[
\left[ -\delta + \alpha \cos (g\chi) \right] ^2 \frac{\Omega}{2g\alpha^2} \left[ -1 - 2\alpha g\tau + e^{2\alpha g\tau} \right]
\]

\[
+ \delta \left[ -\delta + \alpha \cos (g\chi) \right] \frac{2\Omega}{g\alpha^2} \left[ -1 - \alpha g\tau + e^{g\chi} \right]
\]

where \( c_g = (1+\delta)c_0/\sqrt{\varepsilon_1\mu_1} \) and \( \delta \) defines the grating’s deviation from the average velocity of light. A superluminal grating corresponds to \( \delta > 0 \) and a sub luminal grating to \( \delta < 0 \).

The first term in brackets in Eq. (10) arises from the rate of change of \( \varepsilon \) and \( \mu \) and is a maximum when that rate is a maximum. It gives rise to exponential growth in amplitude. There is a point in the grating where gain switches to loss. The other terms arise from the flow of energy and substantially change the shape of the pulse that forms and where the pulse forms. The narrowing of the pulses is also exponential as our theory requires to be the case.

If the grating travels at the average velocity of light, \( \delta = 0 \), our approximations are rather accurate and in fact are exact at \( g\chi = \pi/2 \) and at \( g\chi = 3\pi/2 \), the maximum loss and maximum gain points in the medium, where only the first term matters. When \( \delta \neq 0 \) energy drifts away from the point of maximum creation towards the point where the velocity of light is the same as that of the grating. Peak energy density lies between the two.

Here is the origin of trapping lines of force to which we alluded earlier. If there is no point within the grating where the local velocity of light is the same as that of the grating, then lines of force escape into the next period and the growth mechanism collapses and we revert to a Bloch wave description. The condition for the growth mechanism to operate is [2],

\[
1/\sqrt{1+2\alpha} < \varepsilon_1 c_g / c_0 < 1/\sqrt{1-2\alpha}
\tag{11}
\]
A new mechanism for gain in time dependent media

FIG. 3. Intensity transmitted through a finite luminal medium, $\delta = 0$, at the exit surface, as a function of $gX$, **A**: after a short time propagating along the medium, $\alpha = 0.05$, $\Omega = 0.07$, $\tau = 150$ and **B**: after a longer time, $\tau = 600$. The dashed black line is calculated from the transfer matrix, and the red line from the present approximate theory. **C**: same parameters as for **B** except for $\delta$. **D**: same parameters as for **B** but here we plot the real part of the amplitude which includes a calculation of the phase where the incident wave has frequency $\omega = 1$.

In FIG. 3 we compare the analytic result with transfer matrix based simulations. The first two figures show transmission through slabs of grating of two different thicknesses whilst the grating travels at the average velocity of light, $\delta = 0$. In FIG. 3A the light has been in the grating for a short time, $\tau = 150$, in FIG. 3B for a longer time, $\tau = 600$ resulting in a higher narrower pulse. Our compression of lines of force theory predicts that the pulse height scales inversely as the square of the pulse width, which it does. Note the agreement with the transfer matrix calculations.

In FIG. 3C the calculation in the 3B is extended to other grating velocities. Here the analytic result is less accurate (it depends on linearisation of $\varepsilon(X)$ about $gX = 3\pi/2$). The amplification mechanism is on the point of breakdown at $\delta = \pm 0.05$ beyond which values a Bloch wave picture reasserts itself. Note how the peak lags behind the point of maximum gain when $\delta > 0$ and conversely when $\delta < 0$.

Finally in FIG. 3D we add the phase calculated analytically in [PRB] and compare the real part of the electric field with that calculated using the transfer matrix. Derivation of the phase formula follows closely that for the field amplitude. The formula is accurate where the linearisation assumption holds good.

In conclusion, we have introduced a new theory of luminal materials where light travels in a medium in which disturbances to $\varepsilon$ and $\mu$ propagate at or near the speed of light. A picture emerges in which the medium grabs handfuls of lines of force compressing them at accumulation points resulting in a net increase in energy and...
A new mechanism for gain in time dependent media

generation of a supercontinuum of radiation containing many frequency-momentum harmonics. In the absence of back reflection and of loss, two theorems hold: conservation of lines of force and conservation of phase during the compression process: the field lines hang onto their phase during compression. We believe that this picture of luminal materials provides a new understanding of a process which lacked the insights available for other gain media.

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