Knot invariants and higher representation theory I: diagrammatic and geometric categorification of tensor products

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Abstract. In this paper, we study 2-representations of 2-quantum groups (in the sense of Rouquier and Khovanov-Lauda) categorifying tensor products of irreducible representations. Our aim is to construct knot homologies categorifying Reshetikhin-Turaev invariants of knots for arbitrary representations, which will be done in a follow-up paper.

We consider an algebraic construction of these categories, via an explicit diagrammatic presentation, generalizing the cyclotomic quotient of the quiver Hecke algebra and a geometric construction given by Zheng. One of our primary results is that these categories coincide when both are defined.

We also investigate finer structure of these categories. Like many similar representation-theoretic categories, they are standardly stratified and satisfy a double centralizer property with respect to their self-dual modules. The standard modules of the stratification play an important role, as Vermas do in more classical representation theory, as test objects for functors.

The existence of these representations has consequences for the structure of previously studied categorifications; it allows us to prove the non-degeneracy of Khovanov and Lauda’s 2-category (that its Hom spaces have the expected dimension) in all symmetrizable types, and that the cyclotomic quiver Hecke algebras are symmetric Frobenius.

The program of “higher representation theory,” begun (at least as an explicit program) by Chuang and Rouquier in [CR08] and continued by Rouquier [Roub] and Khovanov-Lauda [KLc] is aimed at studying “2-analogues” of the universal enveloping algebras of simple Lie algebras $U(g)$, and their quantizations $U_q(g)$. In this paper, we study certain representations of these analogues. Our objects of study are certain explicitly given categories, of both algebraic and geometric nature, which are categorifications of tensor products of simple integrable modules for $U_q(g)$ (in the sense that their Grothendieck groups are integral forms of these representations). Our interest in these categories has arisen because of their applicability to the construction of knot invariants, which we address in a sequel to this paper [Web]; however, we believe they are also of independent interest.

These algebras also have connections in the type A case to classical representation theory, as has been explored by Brundan and Kleshchev [BK09]. We will build on

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their work in Section 5 by showing that our categories appear in the context of category $O$ in type A. Our categories should be viewed as a generalization of the type A category $O$ orthogonal to that of category $O$ for other groups, just as quiver varieties are a generalization of the type A flag variety orthogonal to the flag varieties of other types.

Our primary construction of these categories is algebraic; the underlying category $\mathcal{V}_\lambda$ is the representations of an algebra $T^\lambda$ defined in this paper. The algebra $T^\lambda$ is a generalization of the cyclotomic quiver Hecke algebra introduced by Khovanov and Lauda. This categorification is well defined for any symmetrizable Kac-Moody algebra, and it depends on a choice of base field $\mathbb{k}$ and polynomial $Q_{ij} \in \mathbb{k}[u, v]$ for all $i, j$ in the Dynkin diagram.

We will also consider a geometric construction; the underlying category is the category of sheaves $\mathcal{Q}_\lambda$ defined by Zheng in [Zheb]. We note that the latter category is defined for a much more restrictive set of situations: only if the Cartan matrix is symmetric, and only for a specific choice of $Q_{ij}$. Our main theorem is as follows:

**Theorem A** The category $\mathcal{V}_\lambda$ is a categorification in the sense of Khovanov-Lauda, i.e. they carry actions of the 2-category $\mathcal{U}$ defined in [CL, §2] and its Grothendieck group is canonically isomorphic to the tensor product

$$
\mathcal{V}_\lambda \cong \mathcal{V}_{\lambda_1} \otimes \cdots \otimes \mathcal{V}_{\lambda_r}.
$$

Furthermore, whenever $\mathcal{Q}_\lambda$ is also defined, these categories are derived equivalent, in the sense that $\mathcal{Q}_\lambda$ is equivalent to the category of bounded below dg-modules over $T^\lambda$ (considered as a dg-algebra with trivial differential) that are finite dimensional in each degree.

We should note that even in the case of $\underline{\lambda} = (\lambda)$, where the algebra $T^\lambda$ is a cyclotomic quotient in the sense of [KLc, §3.4], this is a new theorem, which in particular implies that the induction and restriction functors on these categories are biadjoint. This was proved independently by Kang and Kashiwara [KK] by completely different methods after this paper had already appeared.

A version of this theorem was previously conjectured by Zheng [Zhea], based on geometric considerations; also, the relationship of Zheng’s categories $\mathcal{Q}_\lambda$ to the categorification program has been discussed in lectures and personally communicated to the author by Rouquier [Roua]; we cannot claim credit for recognizing the importance of these categories as representations of 2-quantum groups. However, we believe that this is the first explicit presentation of Zheng’s categories in terms of generators and relations.

We show that these categories have many properties that would be expected by analogy with similar representation-theoretic categories:
**Theorem B** The projectives-injective objects of $\mathfrak{B}^\Delta$ form a categorification of the subrepresentation $V_{\lambda_1+\cdots+\lambda_n} \subseteq V^\Delta$. In particular, if $\Delta = (\lambda)$, then all projectives are injective; in fact, the algebra $T^{(\lambda)}$ is symmetric Frobenius.

The sum of all indecomposable projective-injectives has the double centralizer property; this realizes $T^\Delta$ as the endomorphisms of a natural collection of modules over the algebra for the corresponding simple module $T^{(\lambda_1+\cdots+\lambda_n)}$.

The algebra $T^\Delta$ is standardly stratified; the semi-orthogonal decomposition for this stratification categorifies the decomposition of $V^\Delta$ as the sum of tensor products of weight spaces.

This double centralizer result allows us to generalize a theorem of Brundan and Kleshchev [BK09, Main Theorem], and show that in type $A$, the algebras $T^\Delta$ are endomorphism algebras of certain projectives in parabolic category $O$, while in type $\widehat{A}$, they are related to the representations of the cyclotomic $q$-Schur algebra. This relationship will be explored more fully in forthcoming work of the author and Stroppel [SW].

We see no reason to think that our category has a similar description in terms of classical representation theory when $g \not\simeq \mathfrak{sl}_n$, $\widehat{\mathfrak{sl}}_n$, though we would be quite pleased to be proven wrong in this speculation.

The action on these categories plays a similar role to the actions of equivariant cohomology studied by Lauda in [Laua, Laub] and Khovanov-Lauda in [KLa]; it shows by direct construction that the set of diagrams conjectured by Khovanov and Lauda to give a basis of 2-morphisms indeed does (because there is no linear combination of them that acts trivially on all categories $V^\Delta$).

**Theorem C** The 2-category $\mathcal{U}$ is nondegenerate (in the sense of [KLa, Definition 3.15]) over any field.

Let us now summarize the structure of the paper.

- In Section 1, we discuss the basics of the 2-category $\mathcal{U}$, and prove it acts on $\mathfrak{B}^\Delta$. This is accomplished by the construction of categorifications $\mathcal{U}_{(b)}$ for the minimal non-solvable parabolics $U(b)$. These categories carry a mixture of the characteristics of $U(b)$ and $U(\mathfrak{sl}_2)$; an appropriate non-degeneracy result is already known for both of these algebras separately. By modifying the proofs of these previous results, we can show that $\mathcal{U}_{(b)}$ acts on $\mathfrak{B}^\Delta$. It is an easy consequence of this that the full $\mathcal{U}$ acts, which proves Theorem C.
- In Section 2, we define the algebras $T^\Delta$. As far as we know, these algebras are new to the literature, but are constructed using the familiar tool of Khovanov-Lauda’s graphical calculus. This graphical calculus gives an easy description...
of the action of the category $\mathcal{U}$. We also study the relationship of this category to $T^{(\lambda_1 + \cdots + \lambda_\ell)}$.

- In Section 3, we develop a special class of modules which we term standard modules, which categorify pure tensors. These are typically not the standard modules of a quasi-hereditary structure, but rather of a weaker standardly stratified structure. Amongst other things, these modules will prove crucial as “test” objects for understanding how functors decategorify.

- In Section 4, we discuss Zheng’s geometric construction and its analytic analogue, and show how in the cases where this makes sense (most importantly, $g$ must be of symmetric type), Zheng’s construction essentially coincides with ours.

- In Section 5, we consider the case $g = sl_n$ or $\hat{sl}_n$. In this case, we employ results of Brundan and Kleshchev to show that $T^\lambda$ is in fact the endomorphism algebra of a projective in a parabolic category $O$ in finite type and in the representation category of the cyclotomic $q$-Schur algebra in affine type. This result will be important for comparing our construction of knot homology in the sequel to versions previously defined using category $O$.

**Notation.** We let $g$ be a symmetrizable Kac-Moody algebra, which we will assume is fixed for the remainder of the paper. Let $\Gamma$ denote the Dynkin diagram of this algebra, considered as an unoriented graph. We denote the weight lattice $Y(g)$ and root lattice $X(g)$, the simple roots $\alpha_i$ and coroots $\alpha_i^\vee$. Let $c_{ij} = \alpha_i^\vee(\alpha_j)$ be the entries of the Cartan matrix.

We let $\langle -, - \rangle$ denote the symmetrized inner product on $Y(g)$, fixed by the fact that the shortest root has length $\sqrt{2}$ and

$$2 \frac{\langle \alpha_i, \lambda \rangle}{\langle \alpha_i, \alpha_i \rangle} = \alpha_i^\vee(\lambda).$$

As usual, we let $2d_i = \langle \alpha_i, \alpha_i \rangle$, and for $\lambda \in Y(g)$, we let

$$\lambda^i = \alpha_i^\vee(\lambda) = \frac{\langle \alpha_i, \lambda \rangle}{d_i}.$$

We note that we have $d_i c_{ij} = d_j c_{ij}$ for all $i, j$.

We let $U_q(g)$ denote the deformed universal enveloping algebra of $g$; that is, the associative $\mathbb{C}(q)$-algebra given by generators $E_i, F_i, K_\mu$ for $i \in \Gamma$ and $\mu \in Y(g)$, subject to the relations:

1. $K_0 = 1$, $K_\mu K_{\mu'} = K_{\mu + \mu'}$ for all $\mu, \mu' \in Y(g)$,
2. $K_\mu E_i = q^{\alpha_i^\vee(\mu)} E_i K_\mu$ for all $\mu \in Y(g)$,
3. $K_\mu F_i = q^{-\alpha_i^\vee(\mu)} F_i K_\mu$ for all $\mu \in Y(g)$,
4. $E_i F_j - F_j E_i = \delta_{ij} \kappa_{ij}^{K_{i} - K_{j}}$, where $\kappa_{ij} = K_{\pm d_i \alpha_i}$.
v) For all $i \neq j$

$$\sum_{a+b=-c_{ij}+1} (-1)^{a} E_i^{(a)} E_j^{(b)} = 0$$

$$\sum_{a+b=-c_{ij}+1} (-1)^{a} F_i^{(a)} F_j^{(b)} = 0.$$

This is a Hopf algebra with coproduct on Chevalley generators given by

$$\Delta(E_i) = E_i \otimes 1 + \bar{K}_i \otimes E_i$$

$$\Delta(F_i) = F_i \otimes \bar{K}_{-i} + 1 \otimes F_i$$

We let $U^Z_q(\mathfrak{g})$ denote the Lusztig (divided powers) integral form generated over $\mathbb{Z}[q,q^{-1}]$ by $E_n^i, F_n^i$ for all integers $n$ of this quantum group. The integral form of the representation of highest weight $\lambda$ over this quantum group will be denoted by $V^Z_{\lambda}$; for a sequence $\underline{\lambda}$, we will be interested in the tensor product

$$V^Z_{\underline{\lambda}} = V^Z_{\lambda_1} \otimes_{\mathbb{Z}[q,q^{-1}]} \cdots \otimes_{\mathbb{Z}[q,q^{-1}]} V^Z_{\lambda_{\ell}}.$$

We will also consider the completion of these modules in the $q$-adic topology $V_{\underline{\lambda}} = V^Z_{\underline{\lambda}} \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Z}((q))$.

We will always use $K_0(R)$ for a graded ring $R$ to denote the Grothendieck group of finitely generated graded projective $R$-modules. This group carries an action of $\mathbb{Z}[q,q^{-1}]$ by grading shift $[A(i)] = q^i[A]$, where $A(i)$ is the graded module whose $i$th grade is the $i + j$th of $A$. The careful reader should note that this is opposite to the grading convention of Khovanov and Lauda.

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1. Categorification of quantum groups

1.1. 2-Categories. In this paper, our notation builds on that of Khovanov and Lauda, who give a graphical version of the 2-quantum group, which we denote $\mathcal{U}$ (leaving $\mathfrak{g}$ understood). These constructions could also be rephrased in terms of Rouquier’s description and we have striven to make the paper readable following either [KLc] or [Roub]; however, it is most sensible for us to follow the 2-category defined by Cautis and Lauda [CL] which is a variation on both of these. The difference between this category and the categories defined by Rouquier in [Roub] is quite subtle; it concerns precisely whether the inverse to a particular map is formally added, or imposed to be a particular composition of other generators in the category. Most important for our purposes, the 2-category $\mathcal{U}$ receives a canonical map from each of Rouquier’s categories $\mathfrak{A}$ and $\mathfrak{A}'$, so a representation of it is a representation in Rouquier’s sense as well. Furthermore, Cautis and Lauda have shown that any representation in Rouquier’s sense satisfying very mild technical conditions will be a representation of $\mathcal{U}$.

Since the construction of these categories is rather complex, we give a somewhat abbreviated description. The most important points are these:

- an object of this category is a weight $\lambda \in \mathcal{Y}$.
- a 1-morphism $\lambda \to \mu$ is a formal sum of words in the symbols $E_i$ and $F_i$ where $i$ ranges over $\Gamma$ of weight $\lambda - \mu$, $E_i$ and $F_i$ having weights $\pm \alpha_i$. In [Roub], the corresponding 1-morphisms are denoted $E_i, F_i$, but we use these for elements of $U_q(\mathfrak{g})$. Composition is simply concatenation of words. In fact, we will take idempotent completion, and thus add a new 1-morphism for every projection from a 1-morphism to itself (once we have added 2-morphisms).

By convention, $F_i = F_{i_1} \cdots F_{i_n}$ if $i = (i_1, \ldots, i_n)$ (this somewhat dyslexic convention is designed to match previous work on cyclotomic quotients by Khovanov-Lauda and others). In Khovanov and Lauda’s graphical calculus, this 1-morphism is represented by a sequence of dots on a horizontal line labeled with the sequence $i$.

We should warn the reader, this convention requires us to read our diagrams differently from the conventions of [Laua, KLc, CL]; in our diagrammatic calculus, 1-morphisms point from the left to the right, not from the right to the left as indicated in [Laua, §4]. Technically, the 2-category $\mathcal{U}$ we define is the 1-morphism dual of Cautis and Lauda’s 2-category: the objects are the same, but the 1-morphisms are all reversed. The practical implication will be that our relations are the reflection through a vertical line of Cautis and Lauda’s (without changing the labeling of regions).

- 2-morphisms are a certain quotient of the $k$-span of certain immersed oriented 1-manifolds carrying an arbitrary number of dots whose boundary is
given by the domain sequence on the line $y = 1$ and the target sequence on $y = 0$. We require that any component begin and end at like-colored elements of the 2 sequences, and that they be oriented upward at an $E_i$ and downward at an $F_i$. We will describe their relations momentarily. We require that these 1-manifolds satisfy the same genericity assumptions as projections of tangles (no triple points or tangencies), but intersections are not over- or under-crossings; our diagrams are genuinely planar. We consider these up to isotopy which preserves this genericity.

We draw these 2-morphisms in the style of Khovanov-Lauda, by labeling the regions of the plane by the weights (objects) that the 1-morphisms are acting on.

By Morse theory, we can see that these are generated by

* a cup $\epsilon : E_iF_i \to \emptyset$ or $\epsilon' : F_iE_i \to \emptyset$

\[
\epsilon = \begin{array}{c}
\lambda + \alpha_i \\
\lambda \\
\lambda - \alpha_i
\end{array} \quad \quad \epsilon' = \begin{array}{c}
\lambda + \alpha_i \\
\lambda - \alpha_i
\end{array}
\]

* a cap $\iota' : \emptyset \to E_iF_i$ or $\iota : \emptyset \to F_iE_i$

\[
\iota' = \begin{array}{c}
\lambda \\
\lambda - \alpha_i \\
\lambda + \alpha_i
\end{array} \quad \quad \iota = \begin{array}{c}
\lambda \\
\lambda + \alpha_i
\end{array}
\]

* a crossing $\psi : F_iF_j \to F_jF_i$

\[
\psi = \begin{array}{c}
\lambda \\
\lambda \\
\lambda \\
\lambda
\end{array}
\]

* a dot $y : F_i \to F_i$

\[
y = \begin{array}{c}
\lambda \\
\lambda
\end{array}
\]

Once and for all, fix a matrix of polynomials $Q_{ij}(u, v)$ for $i \neq j \in \Gamma$ (by convention $Q_{ii} = 0$) valued in $\mathbb{K}$. We assume each polynomial is homogeneous of degree $\langle \alpha_i, \alpha_j \rangle = -2d_i c_{ij} = -2d_j c_{ji}$ when $u$ is given degree $2d_i$ and $v$ degree $2d_j$. We will always assume that the leading order of $Q_{ij}$ in $u$ is $-c_{ji}$, and that $Q_{ij}(u, v) = Q_{ji}(v, u)$.
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We let $t_{ij} = Q_{ij}(1, 0)$; by convention $t_{ii} = 1$. Khovanov and Lauda’s original category is the choice $Q_{ij} = u^{-e_{ij}} + v^{-e_{ij}}$.

Before writing the relations, let us remind the reader that these 2-morphism spaces are actually graded; the degrees are given by

$$\deg \begin{array}{c} \alpha_i \\ \alpha_j \end{array} = -\langle \alpha_i, \alpha_j \rangle$$
$$\deg \begin{array}{c} \alpha_i \\ \alpha_i \end{array} = \langle \alpha_i, \alpha_i \rangle$$
$$\deg \begin{array}{c} \alpha_i \\ \alpha_j \end{array} = -\langle \alpha_i, \alpha_j \rangle$$
$$\deg \begin{array}{c} \alpha_i \\ \alpha_i \end{array} = \langle \alpha_i, \alpha_i \rangle$$

The relations satisfied by the 2-morphisms include:

- the cups and caps are the units and counits of a biadjunction. The morphism $y$ is cyclic, whereas the morphism $\psi$ is double right dual to $t_{ij}/t_{ji} \cdot \psi$ (see [CL] for more details).
- Any bubble of negative degree is zero, any bubble of degree 0 is equal to 1. We must add formal symbols called “fake bubbles” which are bubbles labelled with a negative number of dots (these are explained in [KLc, §3.1.1]); given these, we have the inversion formula for bubbles, shown in Figure 1.

- 2 relations connecting the crossing with cups and caps, shown in Figure 2
- Oppositely oriented crossings of differently colored strands simply cancel, shown in Figure 3
- the endomorphisms of words only using $F_i$ (or by duality only $E_i$’s) satisfy the relations of the quiver Hecke algebra $R$, shown in Figure 4

This categorification has analogues of the positive and negative Borels given by the representations of quiver Hecke algebras, the algebra given by diagrams where all strands are oriented downwards, modulo the relations in Figure 4, which is discussed in [Roub, §4] and an earlier paper of Khovanov and Lauda [KLa]. We denote these 2-categories $\mathcal{U}^+$ and $\mathcal{U}^-$. 

1.2. **Categorifications for parabolics.** For our purposes, it will be crucial to have a nondegeneracy result for $\mathcal{U}$; the most important consequence of this will be that the quiver Hecke algebra injects into $\text{End}_U(\oplus_i \mathcal{F}_i \mu)$ for any weight $\mu$. Luckily, we know such results for $\mathfrak{sl}_2$, and for the Borel $\mathfrak{b}_-$ by work of Lauda [Laua] and Khovanov-Lauda [KLa], with independent proofs given by Rouquier [Roub, Proposition 5.15 & Proposition 3.12]. Since a Kac-Moody algebra is essentially a bunch of $\mathfrak{sl}_2$’s with their interactions described by a Borel, we can hope that these cases can lead us to the more general case.

In order to achieve this, we consider a new category categorifying the parabolic generated by $\mathfrak{b}_-$ and $E_i$, for a fixed index $i$ (which we leave fixed for the remainder of this section).

**Definition 1.1** We let $\mathcal{U}_i^{-}$ be the category whose

- objects are weights of $\mathfrak{g}_i$,
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\[ \gamma_i j = t_{ij} \]

\[ \gamma_i j = t_{ji} \]

FIGURE 3. The cancellation of oppositely oriented crossings with different labels.

- **1-morphisms** are compositions of 1-morphisms in \( \mathcal{U}^- \) and the single 1-morphism \( E_i \) from \( \mathcal{U}^+ \),
- **2-morphisms** are a quotient of the \( \mathbb{K} \)-span of string diagrams of the form used in \( \mathcal{U} \) in which only \( i \)-colored strands are allowed to go downwards. The relations killed are exactly those from \( \mathcal{U} \) that relate such diagrams.

In Rouquier's language, we would construct this category by adjoining \( E_i \) to the lower half categorification as a formal left adjoint to \( F_i \), and impose the relations that

- the map \( \rho_{s,\lambda} \) is an isomorphism whose inverse is described by the lower relation in Figure 2 (in the "style" of Rouquier, one would not impose this equation, but simply adjoin an inverse to \( \rho_{s,\lambda} \)).
- the right adjunction between \( F_i \) and \( E_i \) is given by the upper relation of Figure 2.

There an obvious functor \( \mathcal{U}_i^- \to \mathcal{U} \), which is not obviously faithful, since new relations could appear when the other objects are added. We note that the 2-morphisms in this category have a spanning set analogous to \( B_{i,j,\lambda} \) defined in [KLc, §3.2.3], which we will denote \( B_i \).

Just as Lauda’s categorification of \( sl_2 \) acts on a “flag category,” this parabolic categorification acts on a “quiver flag category,” which can be thought of as arising from Zheng’s construction [Zheb] if one only quotients out by the thick subcategory for the vertex \( i \). While this geometric perspective can be made precise for symmetric Kac-Moody algebras (as we discuss in Section 4), we wish to give a proof for all symmetrizable types, and thus will give a completely algebraic construction.
Figure 4. The relations of the quiver Hecke algebra. These relations are insensitive to labeling of the plane.

For ease, we let $m_{ij}^\mu = \omega_i^{\gamma}(\lambda - \mu)$, where $\omega_i^{\gamma} : X \to \mathbb{Z}$ is the linear function sending $\alpha_i$ to 1, and all other simple roots to 0. As usual, we let $\Lambda(\mathbf{p})$ be the algebra of symmetric polynomials on an alphabet $\mathbf{p}$, and let $e_i(\mathbf{p}), h_i(\mathbf{p})$ denote the elementary
and complete symmetric polynomials of degree $i$. Let

$$\tilde{\Lambda}_\mu \cong \bigotimes_{j \in \Gamma} \Lambda(p_{j,1}, \ldots, p_{j,m_j}).$$

Now consider the polynomial in $\tilde{\Lambda}_\mu$ given by

$$\Xi_\mu(p, t) = \left( \sum_{k=0}^{\infty} h_k(p)(-t)^k \right) \prod_{j \neq i} \prod_{k=0}^{m_j} t_{ij}^{-1} \cdot t^{-c_{ij}} Q_{ij}(p_{j,k}, -t),$$

where $p_i$ denotes the alphabet of variables $p_{i,*}$.

We let $\Lambda_\mu$ be the quotient of $\tilde{\Lambda}_\mu$ by the relations:

$$\Xi_\mu(t^g) = 0 \quad \text{for all } g > \mu^i + m_j^i.$$ 

Here $f(t)[t^g]$ denotes the $t^g$ coefficient of a polynomial. We note that these are quite reminiscent of the relations in a Grassmannian $\operatorname{Gr}(n, m)$, which are simply that $h_k(p) = 0$ for all $k > n - m$. In the symmetric case, for a specific choice of $Q_{ij}$, the ring $\Lambda_\mu$ is the cohomology ring of a Grassmannian bundle over a module space of quiver representations, and these constructions can be interpreted geometrically, as we will cover in more detail in Section 4.

**Definition 1.2** The “quiver flag category” $G_\lambda$ is a 2-category that sends each weight $\mu$ to the category of modules over $\Lambda_\mu$ with 1-morphisms given by the categories of bimodules between these algebras, and 2-morphisms given by bimodule morphisms.

**Theorem 1.3** There is an action of the category $\mathcal{U}_i^-$ on $G_\lambda$, and every non-trivial linear combination of elements of $B_i$ in $\mathcal{U}_i^-$ acts non-trivially in one of these categories. That is, $\mathcal{U}_i^-$ is non-degenerate in the sense of Khovanov-Lauda.

**Proof.** First, we describe the action on the level of 1-morphisms.

- The functors $\mathcal{F}_j$ for $j \neq i$ act by tensoring with the $\Lambda_\mu \cdot \Lambda_{\mu-a_i}$ bimodule $\Lambda_\mu[p_{j,m_j+1}]$. The left-module structure over $\Lambda_\mu$ is the obvious one, and right-module over $\Lambda_{\mu-a_i}$ is a slight tweak of this: $e_k(p'_j)$ acts by $e_k(p_j, p_{j,m_j+1})$, $e_k(p'_m)$ by $e_k(p_m)$ for $m \neq j$.

- The functor $\mathcal{F}_i$ acts by an analogue of the action in Lauda’s paper [Laua]; tensor product with a natural $\Lambda_\mu \cdot \Lambda_{\mu-a_i}$ bimodule $\Lambda_{\mu,i}$ which is a quotient of $\Lambda_\mu[p_{i,m_i+1}]$ by the relation

$$\sum_{c=0}^{\infty} (-p_{i,m_i+1})^c \Xi_\mu(t^g) = 0 \quad \text{for all } g > \mu^i + m_i^i - 1$$

with the same left and right actions as above.
• Similarly, the functor $\mathcal{E}_i$ acts by tensor product with $\Lambda_{\mu + \alpha_i \iota}$, the bimodule defined above with the actions above reversed. This can also be presented as a quotient of $\Lambda_{\mu}[p_{i,m_{\mu}^j}]$ by the relation

$$(1 + p_{i,m_{\mu}^j + 1}t) \Xi_{\mu}(t^g) = 0 \quad \text{for all } g > \mu^i + m_{\mu}^j.$$ 

If we only consider $\mathcal{E}_i$’s and $\mathcal{F}_i$’s, then we obtain a sum of specializations of Lauda’s construction of a representation of $\mathcal{U}_{\text{sl}_2}$ on the equivariant cohomology of Grassmannians. That is, for each fixed choice of $m_{\mu}^i$ for $i \neq j$, we realize the functors along the $\text{sl}_2$ weight-string of $\eta = \lambda - \sum m_{\mu}^i \alpha_j$ by extending scalars from Lauda’s construction by the map $H^*_{\text{GL}_{\infty}}(\text{Gr}(m_{\mu}^i, \infty); \mathbb{k}) \to \Lambda$ given by sending

$$x_k \mapsto e_k(p_i) \quad y_k \mapsto \Xi_{\mu}(t^k).$$

Clearly, we have

$$\Lambda_{\mu} \cong H^*_{\text{GL}_{\infty}}(\text{Gr}(m_{\mu}^i, \infty); \mathbb{k}) \otimes_{H^*_{\text{GL}_{\infty}}(\text{Gr}(m_{\mu}^i, \eta); \mathbb{k})} \Lambda.$$ 

This allows to define all necessary 2-morphisms between $\mathcal{F}_i$’s and $\mathcal{E}_i$’s, which automatically satisfy all the appropriate relations by [Laub] Theorem 4.13.

On the other hand, 2-morphisms between $\mathcal{F}_i$’s other than $i$ act as in Khovanov and Lauda [KLb] or Rouquier [Roub, Proposition 3.12]. Similarly, the proof of relations follows over immediately. Thus, the only issue is the interaction between these 2 classes of functors.

In particular, it remains to show the maps corresponding to elements of $R(\nu)$ are well defined (the relations between them then automatically hold, since quotienting out by relations will not cause two things to become unequal).

Now, consider the bimodules $\Lambda_{\mu} \otimes_{\Lambda_{\mu - \alpha_i}} \Lambda_{\mu - \alpha_j}$ and $\Lambda_{\mu} \otimes_{\Lambda_{\mu - \alpha_j}} \Lambda_{\mu - \alpha_i}$. The former is just $\Lambda_{\mu}[p_{j,m_{\mu}^i + 1}]$, so the relations are just (1.1).

The latter is a quotient of $\Lambda_{\mu}[p_{j,m_{\mu}^i + 1}, p_{i,m_{\mu}^j + 1}]$ by

$$t^{-c_{ij}}Q_{ji}(p_{j,m_{\mu}^i + 1}, t^{-1}) \left( \sum_{c=0}^{\infty} (-p_{i,m_{\mu}^j + 1}t)^{c} \right) \Xi_{\mu}(t^g) = 0 \quad \text{for all } g > \mu^i + m_{\mu}^j - 1 - c_{ij}.$$ 

Modulo the relations (1.2) of $\Lambda_{\mu}$ this polynomial is congruent to

$$t^{-c_{ij}}Q_{ji}(p_{j,m_{\mu}^i + 1}, p_{i,m_{\mu}^j + 1}) \left( \sum_{c=0}^{\infty} (-p_{i,m_{\mu}^j + 1}t)^{c} \right) \Xi_{\mu},$$

so the new relations introduced are exactly $Q_{ji}(p_{j,m_{\mu}^i + 1}, p_{i,m_{\mu}^j + 1})$ times those of $\Lambda_{\mu}[p_{j,m_{\mu}^i + 1}]$.

Thus, the usual definition of $\psi$ from Khovanov and Lauda indeed induces a map of modules, as long as we are careful to use the convention that $\epsilon(j, i)\psi$ corresponds to the identity map (in [KLb]), this is the switch map for two variables, since they
do not index the variables for different colors separately) and \( e(i, j) \psi \) corresponds to multiplication by \( Q_{ij}(p_{i,m_i_j+1}, p_{i,m_i_j+1}) \).

Let us illustrate this point in the simplest case, when \( \mu = \lambda \).

\[
\Lambda_\lambda = \mathbb{k}, \quad \Lambda_{\lambda-i} = \mathbb{k}[p_i]/(p_i^{\alpha_{\lambda}(\lambda)})
\]

The only one of these requiring any appreciable computation is the last. In this case, we have the relation \( p_i^{\psi} Q_{ij}(p_j, p_i) = 0 \) by relating the \( t^{(\lambda-i-\alpha_{\lambda}i)} + 2d_i \) term of \((1 - p_it + \cdots) t^{-c_{ij}Q_{ij}(p_j, -t^{-1})} \).

Finally, we must prove the relation shown in Figure 3. This is simply a calculation, given that we have already defined the morphisms for all the diagrams which appear. The composition

\[
\mathcal{F} \mathcal{E}_i \xrightarrow{\iota_1} \mathcal{E}_i \mathcal{F} \mathcal{E}_i \xrightarrow{\psi_2} \mathcal{E}_i \mathcal{F} \mathcal{E}_i \xrightarrow{\epsilon_3} \mathcal{E}_i \mathcal{F}
\]

is given by

\[
e_3 \psi_2 \iota_1 (p_i^{a} \otimes p_i^{b}) = e_3 \psi_2 \left( \sum_{k=0}^{m_i-1} p_i^{a} \otimes p_i^{b} \otimes p_i^{m_i-k+1} \otimes h_k(p_i^{\prime}) \right)
\]

\[
= e_3 \sum_{k=0}^{m_i-1} p_i^{a} \otimes p_i^{m_i-k+1} \otimes p_i^{b} \otimes h_k(p_i^{\prime})
\]

\[
= \sum_{k=0}^{a} (-1)^k p_i^{b} \otimes e_{a-k}(p_i^{\prime}) h_k(p_i^{\prime})
\]

\[
= p_i^{b} \otimes e_{a}(p_i^{\prime})
\]

Now, note that by our assumptions on \( Q_{ij} \), the power series \( \Xi(t) \) has a non-zero constant term, and thus has a formal inverse in \( \Lambda(p)[[t]] \), which we denote \( \Xi^{-1}(t) \). By the usual Cauchy formula, we have

\[
\Xi^{-1}(t) = \left( \sum_{k=0}^{\infty} e_k(p_i^{\prime}) t^k \right) \prod_{j \neq i} \prod_{k=0}^{m_i} t_{ij} \cdot t_{ji}^{-1} / Q_{ij}(p_j, t_{ij}^{-1} - t),
\]

and by [Laua, Definition 3.1],

\[
X_k \mapsto \Xi^{-1}(t) \{t^k\} \quad Y_k \mapsto (-1)^k h_k(p_i).
\]

The composition

\[
\mathcal{E}_i \mathcal{F} \xrightarrow{\iota_2} \mathcal{E}_i \mathcal{F} \mathcal{F} \mathcal{E}_i \xrightarrow{\psi_2} \mathcal{E}_i \mathcal{F} \mathcal{F} \mathcal{E}_i \xrightarrow{\epsilon_3} \mathcal{F} \mathcal{E}_i
\]
is given by

\[ e'_1 \psi_2 \iota'_3 (p^{b}_{j, m'_\mu + 1} \otimes p^{a}_{i, m'_\nu}) = e'_1 \psi_2 \left( \sum_{k=0}^{m'_\mu - 1} (-1)^k \Xi(p'_i, t) \{t^k\} \otimes p^{m'_\mu - k + 1}_{i, m'_\mu} \otimes p^{b}_{j, m'_\mu + 1} \otimes p^{a}_{i, m'_\nu} \right) \]

\[ = e'_1 \left( \sum_{k=0}^{m'_\mu - 1} (-1)^k \Xi(p'_i, t) \{t^k\} \otimes p^{b}_{j, m'_\mu + 1} Q_{ij} (p^{m'_\mu + 1}_{i, m'_\mu}, p_{i, m'_\nu}) \otimes p^{m'_\mu - k + 1}_{i, m'_\mu} \otimes p^{a}_{i, m'_\nu} \right) \]

\[ = \sum_{k=0}^{a} (-1)^k \Xi(p'_i, t) \{t^k\} \cdot \Xi(p_i, t)^{-1} Q_{ij} (p^{m'_\mu + 1}_{i, m'_\mu}, -t) \{t^{a-k-c_{ij}}\} \otimes p^{b}_{j, m'_\mu + 1} \]

\[ = \frac{t_{ij}}{1 - p_{j, m'_\mu} t} \{t^a\} \otimes p^{b}_{j, m'_\mu + 1} \]

\[ = t_{ij} \cdot p^{a}_{i, m'_\mu} \otimes p^{b}_{j, m'_\mu + 1} \]

This shows that the action is well defined. Obviously, if \( \lambda \) is sufficiently large, then we can assure that all relations in \( \Lambda_\mu \) are of arbitrarily large degree, so any linear combination of diagrams in Khovanov and Lauda’s spanning set can be made non-zero for degree reasons.

\[ \square \]

1.3. **Cyclotomic quotients.** Now that we understand how to add the adjoint of one of the \( \mathbb{F}_i \)'s to \( \mathcal{U}^- \), we move towards considering all of them. Just as with \( \mathcal{U}^- \) and \( \mathcal{U}_i^- \), we prove non-degeneracy by constructing a family of actions which are jointly faithful. As in the previous section, \( i \) will denote a fixed element of \( \Gamma \), and we will use \( j \) for an arbitrary index.

**Definition 1.4** The cyclotomic quiver Hecke algebra \( R^\lambda \) for a weight \( \lambda \) is the quotient of \( R \) by the cyclotomic ideal, the 2-sided ideal generated by the elements \( y_1^{13} \cdot e(i) \) for all sequences \( i \).

We let \( \mathfrak{B}^\lambda \) denote the category of finite dimensional \( R^\lambda \)-modules.

This algebra has attracted great interest recently in the work of Brundan-Kleshchev [BK09], Kleshchev-Ram [KR], Hoffmung-Lauda and Lauda-Vazirani [LV, HL], and Hill-Melvin-Mondragon [HMM]. It has a very rich structure and representation theory, and some surprising connections to classical representation theory. More importantly for our purposes, \( \mathfrak{B}^\lambda \) is a module category over \( \mathcal{U} \), as we will show below.

Consider the map \( v_j: R^\lambda \rightarrow R^\lambda \) that adds a strand labeled with \( j \) at the right.

**Definition 1.5** We let \( \mathcal{Y}_j = - \otimes v_j, R^\lambda \) denote the functor of extension of scalars by this map; we will refer to this as an induction functor.
We let $E_j = \text{Hom}_{R^i}(R^j, -)((\mu, \alpha_j) - d_i)$ denote restriction of scalars by this map (with a grading shift), the functors left adjoint to the $\tilde{\mathcal{F}}_i$'s; we call these restriction functors.

It's worth noting that these are graded differently from the most obvious restriction functors; the presence of a cup (see Figure 10) shifts the grading.

The first step to understanding this relation is to realize the cyclotomic quotient in terms of the category $\mathcal{U}_i^-$. Given any object $\lambda$ in the 2-category $\mathcal{U}_i^-$, we have a representation $\mathcal{U}_i^-(\lambda)$ of this 2-category (i.e. a 2-functor to $\text{Cat}$), given by $\mu \mapsto \text{Hom}(\lambda, \mu)$ with 1-morphisms giving functors between these categories and 2-morphisms natural transformations by composition. Given any collection $J$ of 2-morphisms closed under both vertical composition and horizontal composition on the right with arbitrary morphisms (a “ideal” which is 2-sided for the vertical composition, and 1-sided for horizontal composition), we can consider the quotient representation $\mathcal{U}_i^-(\lambda)/J$ by these 2-morphisms; this is again a 2-functor from $\mathcal{U}_i^-$ to $\text{Cat}$. It sends $\mu$ to the quotient of $\text{Hom}(\lambda, \mu)/J$, the category whose objects coincide with those of $\text{Hom}(\lambda, \mu)$, but where all morphisms in $J$ are identified with 0 (of course, if the identity morphism of an object is in $J$, that object is isomorphic to 0 in the quotient category).

**Proposition 1.6** Let $J$ be the smallest set of morphisms containing

$$id: \mathcal{E}_i \lambda \to \mathcal{E}_i \lambda \quad \text{and} \quad y^\lambda: \mathcal{F}_j \lambda \to \mathcal{F}_j \lambda$$

which is closed under both vertical composition and horizontal composition on the right with arbitrary morphisms. The idempotent completion of $\mathcal{U}_i^-(\lambda)/J$ is equivalent to the category of projective $R^i$-modules, and this equivalence intertwines the actions of $\mathcal{F}_i$ with that of with $\mathcal{F}_j$ and $\mathcal{E}_i$ with that of with $\mathcal{E}_i$.

That is, there is a 2-functor $\mathcal{U}_i^- \to \text{Cat}$ given by

$$\mu \mapsto R^\lambda_{\mu^{-} \text{pmod}}$$
$$\mathcal{F}_j \mapsto \mathcal{F}_j$$
$$\mathcal{E}_i \mapsto \mathcal{E}_i.$$

In particular, the functors $\mathcal{E}_i$ and $\mathcal{F}_i$ are biadjoint (up to grading shift) since $\mathcal{F}_i$ and $\mathcal{E}_i$ are in $\mathcal{U}_i^-$. 

**Proof.** First, we show that $R^\lambda$ can also be written as a quotient of the larger algebra $\tilde{R}^\lambda = \text{End}_{\mathcal{U}_i}(\bigoplus_i \mathcal{F}_i)$, again by the 2-sided ideal generated by $id_A \cdot y^\lambda: A\mathcal{F}_j \lambda \to A\mathcal{F}_j \lambda$ for all 1-morphisms $A$; we call this ideal “the cyclotomic ideal of $\tilde{R}^\lambda$.” This ideal contains all positive degree clockwise bubbles at the left of the diagram (since all of these carry at least $\lambda^i$ dots), so the quiver Hecke algebra surjects onto the quotient. On the other hand, if a diagram in $\text{End}_{\mathcal{U}_i}(\bigoplus_i \mathcal{F}_i)$ contains a positive degree bubble,
it cannot be rewritten by the relations to be an element of the quiver Hecke algebra. Thus, the intersection of the cyclotomic ideal in $\tilde{R}^1$ with the included copy of $R$ is the cyclotomic ideal of that smaller algebra.

We also note that in $U^-_i(\lambda)/J$, every object is a summand of one of the form $\oplus \mathcal{F}_i \lambda$ for some set of $i$’s. Since such objects generate under the action of $U^-_i$ (after all, $\lambda$ alone generates), it suffices to show such objects are closed under the action of $\mathcal{E}_i$.

We induct on the length of $i$. If $i = \emptyset$, then $\mathcal{E}_i \lambda = 0$ and we are done. In general, we have that $\mathcal{E}_i \mathcal{F}_i \mathcal{F}_i \lambda$ is a sum of $\mathcal{F}_i \lambda$ plus some number of copies of $\mathcal{F}_i \lambda$, by the relations in Figures 2 and 3 (this is discussed in more detail in [Laua §5.7]). Thus, by induction, we are done.

Combining these results, we see that the statement of the theorem is equivalent to the statement that in $U^-_i(\lambda)/J$, the morphism space $\text{Hom}_{U^-_i/J}(\mathcal{F}_i, \mathcal{F}_j)$ is isomorphic to $e_1 R^1 e_1$ with composition sent to multiplication. We have a multiplicative map $e_1 R^1 e_1 \to \text{Hom}_{U^-_i/J}(\mathcal{F}_i, \mathcal{F}_j)$, and this map sends the cyclotomic ideal to the indicated subcategory, so it induces a map $e_1 R^1 e_1 \to \text{Hom}_{U^-_i/J}(\mathcal{F}_i, \mathcal{F}_j)$. If an element of $R^1$ is in the kernel of this map, its image in $\text{Hom}_{U^-_i}(\mathcal{F}_i, \mathcal{F}_j)$ is in $J$. Since $R^1$ injects into $\text{Hom}_{U^-_i}(\mathcal{F}_i, \mathcal{F}_j)$ by Theorem \[1.3\], this element can be rewritten as a sum of diagrams that factor through $A \mathcal{E}_i \lambda$ for some 1-morphism $A$ plus elements of the cyclotomic ideal. We can assume without loss of generality that it is a sum of elements of the former form.

Said differently, this 2-morphism can be obtained by starting with a 2-morphism $a: \mathcal{F}_i \mathcal{F}_i (\lambda + \alpha_i) \to \mathcal{F}_j \mathcal{F}_i (\lambda + \alpha_i)$, and “capping off” the $\mathcal{F}_i$. We rewrite $a$ in terms of Khovanov and Lauda’s spanning set, where we choose reduced expressions for our permutations so that the left-most simple reflection only happens once.

“Capping off,” we obtain an element where every diagram appearing has either a clockwise bubble at the far left, or a loop-de-loop turning leftward. We can apply the relation of Figure 2 to see that it is a sum of elements in the cyclotomic ideal plus diagrams with a clockwise bubble at the left. By the relation of Figure 1, every positive degree clockwise bubble can be written as a a polynomial in positive degree counter-clockwise bubbles. A positive degree counter-clockwise bubble must carry at least $\lambda^i$ dots and thus lies in the cyclotomic ideal of $\tilde{R}^1$.

This shows that $U^-_i$ acts on the category of projective modules of $R^1$ and clearly $\mathcal{F}_i$ is sent to $\mathcal{F}_i$. Since $\mathcal{E}_i (\langle \mu, \alpha_i \rangle - d_i)$ (resp. $\mathcal{E}_i (\langle \langle \mu, \alpha_i \rangle + d_i)$) is left (resp. right) biadjoint to $\mathcal{F}_i$ in $U^-_i$ (up to shift), $\mathcal{E}_i$ is sent to $\mathcal{E}_i$ by the uniqueness of left adjoints. This also shows that $\mathcal{E}_i (\langle \langle \mu, \alpha_i \rangle + d_i)$ is right adjoint to $\mathcal{E}_i$.

\[\square\]

In particular, this shows that every inclusion $U^- \hookrightarrow U^-_j$ induces the same action of $U^-$ on $\mathbb{R}^1$.

Using these biadjunctions, we can interpret any picture of the type Khovanov and Lauda draw where all strands begin and end pointing downward as an element
of the cyclotomic quotient. We note that it is not immediately obvious that this assignment satisfies all of Cautis and Lauda’s relations.

Still, this equips \( R^\lambda \) with a map \( \tau_\lambda : R^\lambda \to \mathbb{k} \) given by closing a diagram at the right (if top and bottom strands match) and interpreting this as an element of \( R^\lambda(0) \cong \mathbb{k} \), as shown in Figure 5. The biadjunction implies that this functional makes \( R^\lambda \) into a Frobenius algebra.

Recall that a Frobenius structure on a \( \mathbb{k} \)-algebra \( A \) is a linear map \( \text{tr} : A \to \mathbb{k} \) which kills no left ideal.

![Figure 5. Closing a diagram](image)

**Theorem 1.7** The assignment \( E_j \mapsto \mathcal{E}_j, F_j \mapsto \mathcal{F}_j \) gives an action of \( \mathcal{U} \) on \( R^\lambda \) and thus on \( \mathcal{B}^\lambda \). Any non-trivial linear combination of Khovanov and Lauda’s spanning set acts non-trivially on some \( \mathcal{B}^\lambda \). In particular, the functors \( \mathcal{E}_j \) and \( \mathcal{F}_j \) are biadjoint and \( \tau_\lambda \) is a Frobenius structure on \( R^\lambda \).

As a \( \mathcal{U}_q(g) \)-representation, \( K_0(R^\lambda) \) is naturally isomorphic to \( V_\lambda^Z \).

We should note that this theorem has been independently proven by Cautis and Lauda [CL, 8.1] based on work of Kang and Kashiwara [KK].

**Remark 1.8** This Frobenius trace can be easily adjusted to become symmetric. One fixes one reference sequence \( i_\mu \) for each weight \( \mu \); for each other sequence \( i \), we pick a diagram connecting it to \( i_\mu \) and for each crossing with and consider the scalar \( t(i) \) which is the product over all crossings in the diagram of \( t_{ji}/t_{ij} \) where the NE/SW strand of the crossing is labeled with \( i \) and the NW/SE strand is labeled \( j \). If we multiply the trace on \( e(i)R^\lambda e(i) \) by \( t(i) \), the result will still be Frobenius and be cyclic.

The reader may sensibly ask why we use the trace above instead; it is in large part so we may match the conventions of [CL] and use their results. That said, their choice arises very naturally from a coherent principle: that degree 0 bubbles should be 1. Trying to recover cyclicity in \( \mathcal{U} \) will definitely break this condition.

**Proof of Theorem 1.7** We have already established that we have actions of the categorification of \( \mathfrak{sl}_2 \) for each simple root and of \( \mathcal{U}^\mathfrak{g} \), so any relation only involving
these subcategories must be satisfied. In fact, we already know that any relation only involving one $E_i$ is satisfied. This leaves exactly one from Khovanov and Lauda’s relations: fixing the double duals of morphisms.

This is actually equivalent to $\text{tr}$ satisfying the condition that $\text{tr}(ab) = \text{tr}(ba)$ if $a$ is a diagram only involving dots and crossings in one color (which we already know from the action of $\mathcal{U}$) and $\text{tr}(\psi \cdot a) = t_{ij}/t_{ji} \cdot \text{tr}(a \cdot \psi)$ if $\psi$ is crossing with the NE/SW strand labeled with $i$ and the NW/SE strand labeled $j$, and the latter condition is somewhat simpler to prove (primarily as a matter of organizing induction). We prove that $\tau_{\lambda}$ is symmetric by induction on the number of strands, noting that we already know that $\tau_{\lambda}(ab) = \tau_{\lambda}(ba)$ if $b$ is a diagram where all dots and crossing only occur in one color. This establishes the base case of one strand.

We can always use relations in $a$ to assure that the strands at the far right at the top and bottom (if different) cross each other before any other strands. Thus, if $b$ doesn’t cross the rightmost strand, then we can collapse the loop formed when closing $ab$ by crushing the rightmost bubble in $a$. We thus can obtain a diagram $a'$ with fewer strands such that if $b'$ is $b$ with the rightmost strand removed, then $\text{tr}(ab) = \text{tr}(a'b')$ and $\text{tr}(ba) = \text{tr}(b'a')$. Thus, by induction, we have $\text{tr}(ab) = \text{tr}(ba)$.

This reduces us to the case where $b$ is a single crossing of the two rightmost strands, which may assume are of a different color. This separates into 3 cases, grouped by how many the 2 rightmost terminals at top are connected to the the 2 rightmost terminals at the bottom; this is either 0, 1, or 2. Each of these individual cases is an easy calculation, which we show in Figure 6. This establishes that the correct duals hold, and thus that $\mathcal{U}$ acts on $R_{\lambda}^{\mu} - \text{pmod}$.

We know that the functors $E_i$ and $F_i$ extend to all modules as do the natural transformations defined by 2-morphisms in $\mathcal{U}$. Since every object in $\mathcal{B}^\lambda$ has a presentation by projectives, it is enough to check relations between natural transformations on the subcategory of projectives. Thus, these functors also define an action of $\mathcal{U}$ on $\mathcal{B}^\lambda$.

To show that any non-trivial linear combination of Khovanov and Lauda’s spanning set acts non-trivially, it is enough to show that any polynomial in the dots acts non-trivially for some $\lambda$ (since no element of $R^\lambda$ kills the polynomial representation). This, in turn, reduces to the case of a polynomial in positive degree bubbles (we can simply multiply our polynomial in dots by a monomial to assure that each bubble obtain upon closing is positive degree).

Consider the highest degree monomial in the bubbles, and let $\alpha_i$ be a simple root such that a positive degree bubble colored with $\alpha_i$ appears in this term. Let $j$ be the sum of the degrees of the $i$-colored bubbles in this term. Let $k = \max(1, 1 - \mu^j)$, and surround this polynomial in bubbles with $k$ bubbles colored with $i$, with the outer one carrying $\mu^j - 1$ dots. This is a non-zero polynomial in bubbles with lower degree. By induction, we get a non-zero polynomial of 0 degree, i.e. a scalar map $\text{id}_{\lambda'} \to \text{id}_{\lambda'}$.
Figure 6. Establishing the cyclicity of $\sigma_{ij}$. In each case, the proof of cyclicity is to “pull” the indicated strand in the direction of the thin dashed line.
for some weight $\lambda'$. Thus, we need only choose $\lambda$ such that the $\lambda'$-weight space of $\mathfrak{g}^\lambda$ is non-trivial.

Finally, we must check that $K_0(R^\lambda) \cong V_\lambda$. For this, we need only note that

- $K_0(R^\lambda)$ is generated by a single highest weight vector of weight $\lambda$. Thus it is a quotient of the Verma module of highest weight $\lambda$.
- On the other hand, $\mathfrak{g}^\lambda$ is an integrable categorification in the sense of Rouquier: acting by $\mathfrak{g}_i$ or $\mathfrak{g}_i$ a sufficiently large number of times kills any $R^\lambda$-module, so $K_0(R^\lambda)$ is integrable.
- $V^Z_\lambda$ is the only integrable quotient of the the Verma module which is free as a $\mathbb{Z}[q, q^{-1}]$ module. \hfill \Box

Since no element of $\check{U}$ kills all finite dimensional representations, an immediate consequence of this is that

**Corollary 1.9** The map $\gamma: \check{U} \to K(U)$ defined by Khovanov and Lauda in [KLc, §3.6] is an isomorphism.

Recall that the $q$-Shapovalov form $\langle -,- \rangle$ is the unique bilinear form on $V^Z_\lambda$ such that

- $\langle v_h, v_h \rangle = 1$ for a fixed highest weight vector $v_h$.
- $\langle u \cdot v, v' \rangle = \langle v, \tau(u) \cdot v' \rangle$ for any $v,v' \in V_\lambda$ and $u \in U_q(\mathfrak{g})$, where $\tau$ is the $q$-antilinear anti-automorphism defined by
  
  \[
  \tau(E_i) = q_i^{-1} \check{K}_i F_i \quad \tau(F_i) = q_i \check{K}_i E_i \quad \tau(K_\mu) = K_{-\mu}
  \]
- $f \langle v,v' \rangle = \langle f v, v' \rangle = \langle v, f v' \rangle$ for any $v,v' \in V^Z_\lambda$ and $f \in \mathbb{Z}[q, q^{-1}]$.

**Corollary 1.10** The isomorphism $K_0(R^\lambda) \cong V^Z_\lambda$ intertwines the Euler form

$$\langle [P_1],[P_2] \rangle = \dim_q \text{Hom}(P_1, P_2)$$

with the $q$-Shapovalov form described above. In particular,

$$\dim_q e(i)R^\lambda e(j) = \langle F_i v_h, F_j v_h \rangle$$

We let $\langle -, - \rangle_1$ denote the specialization of this form at $q = 1$, which is thus the ungraded Euler form.

**1.4. Universal categorifications.** In [Roub, §5.1.2], Rouquier discusses universal categorifications of simple integrable modules. He proves that there is a unique $\mathcal{U}$-module category $\mathfrak{g}^\lambda$ (he uses the notation $\mathcal{L}(\lambda)$) with generating highest weight object $P$ with the universal property that

(*) for any additive, idempotent-complete $\mathcal{U}$-module category $C$ and any object $C \in \text{Ob} C_\lambda$ with $\mathcal{C}_i(C) = 0$ for all $i$, there is a unique (up to unique isomorphism) functor $\phi_C: \mathfrak{g}^\lambda \to C$ sending $P$ to $C$. 21
On purely formal grounds, such a category must exist for any version of the 2-category categorifying $U_q(\mathfrak{g})$; thus we will study the corresponding module for the 2-category $\mathcal{U}$ we have been using, which is different from Rouquier’s.

In any case, this is a higher categorical analogue of the universal property of a Verma module, but somewhat surprisingly, $\check{\mathcal{R}}^\lambda$ does not categorify a Verma module, but rather an integrable module. We recall that $\text{End}_{\mathcal{U}}(\oplus \mathcal{F}_i \lambda) \cong R \otimes \Lambda$ where $\Lambda \cong (\otimes_{j \in \mathcal{I}} \Lambda(p_j))$ and $p_j$ is an infinite alphabet attached to each node, with the clockwise bubble of degree $2n$ corresponding to $(-1)^n e_n(p_j)$, and the counterclockwise one of degree $2n$ corresponding to $h_n(p_j)$.

**Definition 1.11** Let $\check{\mathcal{R}}^\lambda$ be the quotient of $\text{End}_{\mathcal{U}}(\oplus \mathcal{F}_i \lambda)$ by the relations

$$
\begin{align*}
\begin{array}{cccccc}
\bigcirc & \cdots & = & 0 & \lambda & j \\
\bigcirc & \cdots & = & \lambda^{-1} & -\lambda & j \\
\bigcirc & \cdots & = & \lambda & -1 & j \\
\bigcirc & \cdots & = & 0 & & j \\
\end{array}
\end{align*}
$$

where in both pictures, the ellipses indicate that the portion of the diagram shown is at the far left. More algebraically, these relations can be written in the form

$$
e(i)(y_1^{\lambda_1} - e_1(p_i)y_1^{\lambda_1-1} + \cdots + (-1)^{\lambda_1} e_{\lambda_1}(p_i)) = 0$$

$$e_n(p_j) = 0 \quad (n > \lambda^j)$$

Note that if we specialize $e_n(p_j) = 0$ for every $n > 0$, then we recover the usual cyclotomic quotient $R^\lambda$.

If we extend scalars to polynomials in the $p_\ast$, and form the algebra $\check{\mathcal{R}}^\lambda \otimes k[p_1, \ldots,]$ then we can rewrite these equations as

$$
e(i)(y_1 - p_{i_1,1})(y_1 - p_{i_1,2}) \cdots (y_1 - p_{i_1,\lambda_1}) = 0$$

$$p_{i,n} = 0 \quad (n > \lambda^j)$$

In terms of the geometry of quiver varieties discussed later in this paper (see Section 4), $\check{\mathcal{R}}^\lambda$ arises from considering equivariant sheaves for the action of the group $\prod \text{GL}(W_i)$, and its extension to polynomials from equivariant sheaves for a maximal torus of this group.

**Proposition 1.12** For any additive, idempotent-complete $\mathcal{U}$-module category $C$ and any object $C \in \text{Ob} \mathcal{C}_\lambda$ with $E_i(C) = 0$ for all $i$, there is a unique (up to unique isomorphism) functor $\phi_C : \check{\mathcal{R}}^\lambda \text{-pmod} \to C$ sending $P$ to $C$. 

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Proof. The proof that $\mathcal{U}$ acts on $\mathcal{R}^\lambda$ is precisely the same as that for $R^\lambda$ given in Proposition 1.6 and Theorem 1.7. The algebra $\mathcal{R}^\lambda$ is a Frobenius extension of the algebra $\bigotimes_{j \in \Gamma} \Lambda(p_{j,1}, \ldots, p_{j,\lambda})$.

Thus, we need only construct the functor $\phi_C$. We have a natural functor $\mathcal{U}(\lambda) \to C$ sending $\lambda \mapsto C$. We wish to show that this factors through the functor from $\mathcal{U}(\lambda) \to \mathcal{R}^\lambda$-pmod. On the level of objects, we simply send $e_i \mathcal{R}^\lambda \mapsto S_i C$; on the level of morphisms, we need only note that the relations of $\mathcal{R}^\lambda$ obviously act trivially on any 1-morphism applied to $C$, since they factor through a 1-morphism applied to $S_i C$.

These algebras are quite interesting; though they are infinite dimensional (unlike $R^\lambda$), they have the advantage of being of finite global dimension (unlike $R^\lambda$). We will explore these algebras and their tensor product analogues in future work.

2. The tensor product algebras

2.1. Definition and basic properties. We now proceed to the algebraic construction mentioned in the introduction. This is structured around certain algebras which are pictorial in definition, and similar in flavor to the algebras $R^\lambda$ we have already defined.

The generators of our algebra are pictures in $\mathbb{R}^2$ consisting of red and black oriented embedded smooth curves decorated with a number (possibly 0) of dots such that:

- each curve begins on the line $y = 0$ and ends on the line $y = 1$
- each curve is never tangent to a horizontal line
- locally around each point, our diagram is either a single line or one of the pictures:

\[
\begin{array}{c}
\times \\
\end{array}
\begin{array}{c}
\times \\
\end{array}
\]

In particular, red lines are never allowed to cross, and no pair of lines are allowed to meet the lines $y = 0$ or $y = 1$ at the same point.

We will only ever be interested in these pictures up to isotopy preserving the conditions above.

Consider the algebra $T$ over $\mathbb{K}$ whose generators are pictures as above, with each black line labeled by a simple root of $\mathfrak{g}$, and each red line labeled with a dominant weight. Multiplication is given by the stacking of diagrams if the pattern of red and black lines with their labels can be isotoped to match up at $y = 1$ in the first diagram and $y = 0$ in the second and is defined to be 0 otherwise. Of course, this stacking must be followed by smoothing any kinks at the joins of the lines (which is unique up to isotopy) and vertical scaling to match the ends up with the correct horizontal
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lines. By convention the product $ab$ means stacking the diagram $b$ on top of the diagram $a$.

The black strands satisfy the quiver Hecke relations from Figure 4, which again we apply as local relations (i.e. any time a small portion of a larger diagram matches one side of the relation, we equate it to the diagram with the small portion changed to match the other side of the relation).

We must also include new relations involving red lines which are:

- All black crossings and dots can pass through red lines, with a correction term similar to Khovanov and Lauda’s (for the latter two relations, we also include their mirror images):
  \[
  \begin{align*}
  &\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 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with a black strand), then the diagram is 0. We will refer to such a strand as **violating**.

We also let $\tilde{T}$ denote the algebra without the last relation above. While $T$ is the algebra of primary importance for us, $\tilde{T}$ will be of great technical utility, since we can construct a basis for it, whereas for $T$, this seems to be quite out of reach. Furthermore, the algebra $\tilde{T}$ has a more simple geometric description, as we will discuss in Section 4.

Following Brundan and Kleshchev, we will sometimes use $y_i$ to represent multiplication by a dot on the $i$th black strand, and $\psi_i$ to denote the crossing of the $i$th and $i+1$st black strands and $\epsilon(i)$ to denote the sum of all pictures where there are no crossings or dots, and the black strands are labeled with $i = (i_1, \ldots, i_n)$ in that order.

**Grading.** This algebra is graded with degrees given by

- a black/black crossing: $-\langle \alpha_i, \alpha_j \rangle$,
- a black dot: $\langle \alpha_i, \alpha_i \rangle = 2d_i$
- a red/black crossing: $\langle \alpha_i, \lambda \rangle = d_i \lambda_i$.

This algebra is endowed with a natural anti-automorphism $a \mapsto \hat{a}$ given by reflecting diagrams in the horizontal axis. If $M$ is a right module over this algebra, we let $\hat{M}$ be the left module given by twisting the action by this anti-automorphism.

**Definition 2.1** For a finite-dimensional right module $M$, we define the **dual module** by $M^* = \hat{M}^*$, where $(\cdot)^*$ denotes usual vector space duality interchanging left and right modules.

This is a right module since both vector space dual and the anti-automorphism interchange left and right modules.

**Definition 2.2** For a sequence of weights $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, we let $T_\lambda$ be the subalgebra of $T$ where the red lines are labeled, in order, with the elements of $\lambda$. We let $\mathcal{O}_\lambda = T_\lambda / \text{mod}$ be the category of finite dimensional representations of $T_\lambda$ graded by $\mathbb{Z}$.

We let $T^\lambda_\alpha$ for $\alpha \in \mathcal{Y}(\mathfrak{g})$ be the subalgebra of $T_\lambda$ where the sum of the roots associated to the black strands is $\sum_i \lambda_i - \alpha$. 
We also let $\bar{T}_\lambda$ denote the corresponding subalgebra of $\bar{T}$, and $K_\lambda$ denote the kernel of the natural map $\bar{T}_\lambda \to T_\lambda$. By definition, $K_\lambda$ is the span of the diagrams in $\bar{T}_\lambda$ with a violating strand, since these elements are generators of the kernel and their span is closed under left and right multiplication.

Consider a sequence of simple roots $\iota = (i_1, \ldots, i_n)$, and a weakly increasing map $\kappa: [1, \ell] \to [0, n]$. We can define an idempotent $e(\iota, \kappa)$ as the crossingless diagram where the strands are labeled by the roots in the order given by $\iota$, with the $j$th red line immediately right of the $\kappa(j)$th black line, except that if $\kappa(j)$’s agree, the original order of red lines is preserved. By convention, if $\kappa(i) = 0$, then the $i$th red strand is left of all black strands. Note that if $e(\iota, \kappa)$ is not trivial, we must have $\kappa(1) = 0$.

**Definition 2.3** We consider the projective modules $P_\kappa \iota = e(\iota, \kappa)T_\lambda$ and $\bar{P}_\kappa \iota = e(\iota, \kappa)\bar{T}_\lambda$ and let $K_\kappa \iota$ be the kernel of the natural map $\bar{P}_\kappa \iota \to P_\kappa \iota$.

Note that the kernel $K_\lambda$ can also be described as the span of the elements that factor through $\bar{P}_\iota$ where $\kappa(1) \neq 0$, that is, the trace of these projectives. In categorical terms, the projective modules over $T_\lambda$ are the quotient of the category of projective modules over $\bar{T}_\lambda$ by this collection of projectives.

We can generalize this notion a bit by allowing multiplicities $\vartheta_j$; we associate a projective to the sequence $(i_1^{(\vartheta_1)}, \ldots, i_n^{(\vartheta_n)})$ which is a submodule of the projective for the sequence where $i_j^{(\vartheta_j)}$ has been expanded to $\vartheta_j$ instances of $i_j$. This is the projective given by multiplying each block of strands in the expanded projective on the bottom by the idempotent denoted $e_{\vartheta_j}$ in [KLb, §2], which we illustrate in Figure 8.

![Figure 8. The idempotent $e_4$.](image)

Usually, we will not require these multiplicities, and will thus exclude them from the notation. Unless they are indicated explicitly, the reader should assume that they are 1.

Under decategorification, the projective $P_\kappa \iota$ is sent to the vector

$$F_{i_a}^{(\theta_a)} \cdots F_{i_{(\ell)}}^{(\theta_{(\ell)})} \left( \cdots (F_{i_{(3)}}^{(\theta_{(3)})} \cdots F_{i_{(2)+1}}^{(\theta_{(2)+1})}) (F_{i_{(2)}}^{(\theta_{(2)})} \cdots F_{i_{1}+1}^{(\theta_{1}+1)}) v_1 \otimes v_2 \otimes \cdots \otimes v_\ell, \right)$$

where $v_i \in V_{\lambda_i}$ is a fixed highest weight vector, as we prove in Section 3.2.
2.2. **Examples.** To give a simple illustration of the behavior of our algebra, let us consider $\mathfrak{g} = \mathfrak{sl}_2$, and $\underline{\lambda} = (1, 1)$. Thus, our diagrams have 2 red lines, both labeled with 1’s.

In this case, the algebras $T^\underline{\lambda}_n$ are easily described as follows:

- $T^{(1,1)}_2 \cong \mathbb{K}$: it is just multiples of the diagram which is just a pair of red lines.
- $T^{(1,1)}_0$ is spanned by

\[
\begin{array}{c}
| & | & | & | & | & | & | \\
| & | & \times & \times & | & | & | \\
| & | & \times & | & | & \times & | \\
\end{array}
\]

One can easily check that this is the standard presentation of a regular block of category $\mathcal{O}$ for $\mathfrak{sl}_2$ as a quotient of the path algebra of a quiver (see, for example, [Str03]).

- $T^{(1,1)}_{-2} \cong \text{End}(\mathbb{K}^3)$: quotienting out by the left ideal generated by all diagrams with crossings gives the unique irreducible representation. The algebra is spanned by the diagrams, which one can easily check multiply (up to sign) as the elementary generators of $\text{End}(\mathbb{K}^3)$.

\[
\begin{array}{c}
| | | | \times \times \times \\
\times \times | | \times \times \\
\times \times \times \times | | \\
\times \times \times \times \times \\
\end{array}
\]

2.3. **A basis and spanning set.** Recall that a **reduced word** in the symmetric group is a list of $k$ adjacent transpositions $(i, i+1)$ whose product cannot be written as a shorter product of adjacent transpositions. For each choice of a reduced word $w$ for a permutation of $n + \ell$ letters which doesn’t invert any pair of red strands, we have an element $\psi_w$ of $P^\kappa_i$ given by replacing the simple reflection $(i, i+1)$ with the crossing of the $i$ and $i + 1$st strands (red or black) and multiplying out the result.

**Proposition 2.4** For any fixed choice of reduced word for each permutation, the algebra $\tilde{T}^\underline{\lambda}$ has a basis given $e(i, \kappa)\psi_w y_1^{a_1} \cdots y_n^{a_n}$ for all permutations which preserve the relative order of the red strands and any $n$-tuple $\{a_i \in \mathbb{Z}_{\geq 0}\}$.

This proposition is crucial in that it not only gives us a basis, but an ordered basis; permutations have a natural partial order, the strong Bruhat order.

We will always refer to the process of rewriting an element in terms of this basis as “straightening” since visually, it is akin to pulling all the strands taut until they are straight, though this image is slightly misleading, as we will explain momentarily.

**Proof.** The proof is directly analogous to that of [KLa, Theorem 2.5].
First we show is that this set spans, for which is suffices to show that $\psi_w$ for any word can be rewritten in terms of $y_i$’s times $\psi_w'$ for our fixed choice of reduced words and shorter diagrams.

If $w$ is not a reduced word in the symmetric group, then by applying braid relations (which hold modulo shorter words), we can assume that there are two consecutive crossings of the same strands, which can be simplified using the relations and written in terms of $\psi_w'$ for shorter words $w'$.

If $w$ is a reduced word, then the fixed reduced word corresponding to the same permutation $w'$ differs from $w$ by Tits moves, so the difference between $\psi_w - \psi_w'$ can thus be written in terms of shorter diagrams.

The difficult part is to show that the elements are linearly independent. First, we note that $\tilde{T}^\lambda$ has a version of Khovanov and Lauda’s polynomial representation, where $\tilde{T}^\lambda$ acts on a direct sum of polynomial rings $\mathbb{Z}[y_1, \ldots, y_n]$ over all choices of $i$ and $\kappa$ by the rule (where in each case, there are $k - 1$ black strands to the left of the portion of the diagram shown) shown in Figure 9.

The action of black diagrams is that of Khovanov-Lauda (in original signs, this is [KLa, Theorem 2.3], and is discussed with sign modifications in the final section of
The most general version for arbitrary $Q_{*,*}$ is covered in [Roub Proposition 3.12]), so the only relations we need check are our additional relations (2.1) and (2.2). The only one of these which is interesting is the first line of (2.1). The LHS is

$$f \mapsto \frac{y_k^\lambda f - y_{k+1}^\lambda (k, k+1) \cdot f}{y_k - y_{k+1}}$$

and the RHS is

$$f \mapsto y_{k+1}^\lambda f - \frac{(k, k+1) \cdot f}{y_k - y_{k+1}} + \frac{y_k^\lambda - y_{k+1}^\lambda f}{y_k - y_{k+1}}$$

and the relation is verified.

The most important consequence of this is that Khovanov and Lauda’s algebra $R$ injects into $\tilde{T}^\lambda$, since any element of the kernel acts trivially on the polynomial representation, and thus is trivial.

Now, we show that we have a basis in general by reducing to this case. Assume that there is a non-trivial linear relation between vectors of the form in the statement. Then we can compose on the bottom with the element $\theta_{k, r}$ which pulls all black strands to the right and red to the left, and on the top with $\dot{\theta}_r$. Pulling all black strands to the right (as described above when showing our desired elements span), we obtain a relation in $R$. On the other hand, there must be a $\psi_w y^a$ with nontrivial coefficient maximal in Bruhat order compared to all other diagrams with non-trivial coefficients. Since pulling right only adds correction terms strictly smaller in Bruhat order, we have a relation in $R$ where the corresponding diagram to $\psi_w y^a$ has non-trivial coefficient. Since these elements are a basis, this coefficient must be trivial, giving a contradiction. Thus, this relation is trivial and we have a basis of $\tilde{T}^\lambda$.

**Proposition 2.5** For any fixed choice of reduced word for each permutation, the elements $\psi_w$ generate $P^\kappa_i$ as a module over the subalgebra generated by the $y^\iota_i$’s.

**Proof.** Clear from the fact that $\tilde{T}^\Lambda$ surjects onto $T^\Lambda$. 

In order to organize our computations, we must keep track of leading terms in this basis under multiplication; the term “straightening” suggests that these will roughly correspond to the multiplication of permutations. The reality is a bit more subtle. In order to do this, we consider the category $O_n$ whose objects are ordered $n$ elements sets labeled with simple roots of our algebra, and whose morphisms are label preserving maps. Obviously, every diagram in $\tilde{T}^\Lambda$ gives such a map by simply tracing out the black strands (we ignore red strands for the time being). We now wish to put a slightly strange composition on these maps which will give us a different category from the naive one with these morphisms.

In order to compose morphisms $a$ and $b$, we factor each in a minimal length way into the naive product of a number of simple involutions, i.e. those that switch
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adjacent elements in the order. Now, we consider the concatenation of these words, which we endeavor to simplify. We impose the usual braid relations on involutions, but we change how they square. If $s_i = (i, i + 1)$ in cycle notation, we impose that $s_i^2 = 1$ if the $i$th and $i + 1$th have different labels and $s_i^2 = s_i$ is the labels are the same.

Note that if the concatenation is not a reduced word, we can apply braid relations until there are two adjacent $s_i$’s in the word, which we can simplify to obtain a shorter word. This process terminates at a reduced word for a unique permutation. We note that morphisms in this category can be given the usual Bruhat order.

Proposition 2.6 Given any diagram $x \in \tilde{\mathcal{T}}_\lambda$ with associated morphism $\omega_x$ in $\mathcal{U}_n$, when $x$ is written in terms of basis elements, all diagrams which appear have associated morphisms shorter than or equal to $\omega_x$ in $\mathcal{U}_n$.

Proof. This is clear from the quiver Hecke relations of Figure 4 and the algorithm for writing a morphism in terms of the basis, since all relations for reducing the “length” of a diagram, or to adjust it to fit a particular reduced word of a permutation only introduce extra terms shorter in Bruhat order. We must use $\mathcal{U}_n$ because these relations will sometimes remove a $s_i$ which permutes two like colored strands from a word where $s_i^2$ appears. This could increase the length in the usual multiplication of the symmetric group, but will not in $\mathcal{U}_n$. □

This proposition has another important consequence. Let $\kappa_1, \kappa_2$ be two weakly increasing functions $[1, \ell] \rightarrow [0, n]$ and assume that for some $j$ we have $\kappa_i(j) = \kappa_i(j+1)$ for $i = 1, 2$. Then, we let $\Lambda'$ denote $\Lambda$ with the block $\lambda_k, \lambda_{k+1}$ replaced by $\lambda_k + \lambda_{k+1}$ and let

$$\kappa'_i(k) = \begin{cases} \kappa_i(k) & k \leq j \\ \kappa_i(k + 1) & k > j. \end{cases}$$

There is an obvious map

$$\tilde{c} : e(i, \kappa'_1)\tilde{T}_\lambda e(i, \kappa'_2) \rightarrow e(i, \kappa_1)\tilde{T}_\lambda e(i, \kappa_2)$$

given by separating the $k$th red strand into 2 strands, labeled with $\lambda_k$ and $\lambda_{k+1}$, and also an induced map on quotients

$$c : e(i, \kappa'_1)\tilde{T}_\lambda e(i, \kappa'_2) \rightarrow e(i, \kappa_1)\tilde{T}_\lambda e(i, \kappa_2).$$

Corollary 2.7 The maps $\tilde{c}$ and $c$ are isomorphisms.

Proof. The fact for $\tilde{c}$ simply follows from the fact that the bases of Proposition 2.4 correspond under this map.

Note further that under $\tilde{c}$ that any element of $e(i, \kappa_1)\tilde{T}_\lambda e(i, \kappa_2)$ which has a violating strand can be rewritten by sliding all crossings and dots out of the space between the $k$ and $k + 1$st strands to be the image of an element with a violating strand under $\tilde{c}$.
Since the kernels to the projections to the domain and target of \( c \) correspond under \( \tilde{c} \), we must have that \( c \) is an isomorphism.

2.4. **Relationship to quiver Hecke algebras.** If \( \underline{\lambda} = (\lambda) \), then we will simplify notation by writing \( T^\lambda \) for \( T_{\underline{\lambda}} \), and \( P_i \) for \( P_i^0 \).

**Theorem 2.8** \( R^\lambda \cong T^\lambda \).

*Proof.* By composing the inclusion \( R \hookrightarrow \tilde{T}^\lambda \) given by adding a red line at the left and the projection \( \tilde{T}^\lambda \to T^\lambda \), we obtain a map. This map is a surjection since any element of the basis of Proposition 2.4 not in the image contains a strand to the left of the single red strand and thus is sent to 0.

The image of \( R \) in \( \tilde{T}^\lambda \) is readily identifiable: it is the span of all diagrams where both at \( y = 0 \) and \( y = 1 \), the single red strand is left of all blacks. The image is clearly contained in this space, since the image of a diagram in \( R \) satisfies this condition for all values of \( y \), and any diagram with this condition can be rewritten using the Theorem 2.4 as a sum of elements where no two strands cross twice. Since the red strand is at the far left both at \( y = 0 \) and \( y = 1 \) it cannot cross a black strand exactly once, and thus must not cross with any of them; that is, we have written our element in terms of basis vectors in the image of \( R \). Let \( e_0 \) be the idempotent given by the image of the identity in \( R \). We note that left multiplication by \( e_0 \) kills exactly the diagrams which do not have the red strand at the far left at the bottom and similarly for right multiplication and the top, so \( R = e_0 \tilde{T}^\lambda e_0 \).

The kernel of the map \( R \to T^\lambda \) is thus the intersection \( K^\lambda \cap R \); we must show that this coincides with the cyclotomic ideal. First note that \( K^\lambda \cap R = e_0 K^\lambda e_0 \). By definition, \( K^\lambda \) is spanned by elements with a violating strand, so \( K^\lambda \cap R \) is spanned by all elements with a violating strand where the red strand is at the left at the top and bottom.

In such a diagram, we can slide all violating black strands back over the red. We thus obtain \( \lambda^i \) dots on all \( \alpha_i \)-colored strands that were violating in the earlier diagram. In particular, any one of these strands which has no other strand to its left at the point where it was violating carries \( \lambda^i \) dots, and thus lies in the cyclotomic ideal. On the other hand, for any element in the cyclotomic ideal, when can simply slide the leftmost strand left at the point where it carries \( \lambda^i \) dots to obtain a violating strand. This gives the equality of ideals and thus the desired isomorphism.

This cyclotomic quotient plays several important roles in “controlling” the representation theory of \( T_{\underline{\lambda}} \). Consider the projectives where \( \kappa(i) = 0 \) for all \( i \), in which case we will simply denote the projective for \( \kappa \) by \( P_i^0 \). We note that \( P_i^0 \) carries an obvious action of \( R \) by composition on the bottom. We let \( P^0 = \oplus_i P_i^0 \) be the sum of all such projectives with \( \kappa(i) = 0 \), and \( P = \oplus_i P_i \) be the corresponding module over \( T^\lambda \).
Proposition 2.9  \( \text{End}_{\mathcal{T}_\lambda}(P^0) \cong T^\lambda \cong R^\lambda \).

Proof. The first isomorphism follows from repeated application of Corollary \([2.7]\). The second is just a restatement of Proposition \([2.8]\). \(\square\)

2.5. The module category structure. Based on the graphical calculus developed in the Section I, we can define an action of \( \mathcal{U} \) on the categories \( \mathcal{V}_\lambda \). First, we define a candidate functors by a simple extension of our graphical calculus. Each of these is defined sending a module \( M \) to a module spanned by diagrams containing a coupon that carries elements of \( M \).

The induction \( \tilde{\delta}_i M \) of an \( \tilde{T}_\lambda \)-module \( M \) is the vector space generated by diagrams as in Figure \([10]\) for \( m \in M \), modulo the relation that the sum of diagrams which are identical outside the coupon is given by adding the labels on the coupon.

The algebra \( \tilde{T}_\lambda \) acts by multiplication on the top, simplifying using Proposition \([2.5]\) so that all crossings of strands connecting the coupon occur below the new strand, and absorbing these into the coupon.

More algebraically, this is an extension of scalars; We have a map \( \nu_i : \tilde{T}_\lambda \to \tilde{T}_\lambda \) given by adding a \( i \)-colored strand at the far right, and \( \tilde{\delta}_i M \cong \tilde{T}_\lambda \otimes \tilde{T}_\lambda M \) where the tensor product is taken over the ring map \( \nu_i \).

Definition 2.10 Induction for \( T_\lambda \)-modules is defined by \( \mathcal{F}_i M = \tilde{\delta}_i M \otimes \tilde{T}_\lambda T_\lambda \) for \( M \in \mathcal{V}_\mu \).

Analogous restriction functors \( \tilde{\mathcal{E}}_i, \mathcal{E}_i \) right adjoint to these are defined by the second set of pictures in Figure \([10]\).

These functors give an action of \( \mathcal{U} \), as we will show momentarily; we should note that in order for this action to make sense, we must assign a category to each weight, refining the category that corresponds to the entire representation. To calculate the weight in which \( P^\kappa \) belongs, one should add the weights on the red lines minus the roots on the black strands.

More generally, we can imagine labeling the regions of the diagram starting with 0 at the left, and using the rule given in [KLR, §3.1.1], which the additional rule that the label on the region right of a red strand minus that to its left is the label of the strand itself. The weight we identify above would be the label at the far right of the diagram.

Proposition 2.11 The assignment \( \mathcal{E}_i \mapsto \tilde{\mathcal{E}}_i, \mathcal{F}_i \mapsto \tilde{\mathcal{F}}_i \) gives an action of \( \mathcal{U} \) on \( \mathcal{V}_\lambda \), where the action of 2-morphisms is simply by composition on the bottom of the diagram, perhaps followed by simplification.

In particular, the functors \( \tilde{\mathcal{F}}_i \) and \( \mathcal{E}_i \) are exact.
We have added the orientations in Figure 10 in order to make the action of 2-morphism easier to visualize.

**Proof.** First note that it is enough to show that the correct relations hold if the functors are applied to $M = P_i^\kappa$ for any $(i, \kappa)$.

This can be proven by constructing an auxiliary category which clearly has a $\mathcal{U}$ action and which has $\tilde{T}_\lambda$ as a quotient. This category is quite close in spirit to $\tilde{T}_\lambda$, but we must use an enlargement of it. Thus, we define a 2-category $\tilde{\mathcal{U}}$ whose

- objects are weights,
- 1-morphisms are sequences of $E_i$'s, $F_i$'s and $I_\lambda$'s such that sum of the corresponding weights is the difference between target and image. We translate these into sequences of colored dots as usual by sending $I_\lambda$ to red dots marked with $\lambda$.
- 2-morphisms between two of these objects are $\mathbb{K}$-linear combinations of immersed oriented diagrams where no red strands cross or self-intersect that match, subject to the relations of Figures 1, 2, 3 and 4, and the relations for $T_\lambda$ (remember, all these relations are local and imposed up to isotopy, but they do take into account orientations of red and black strands.). Furthermore, we must impose similar relations between red strands and oppositely oriented red strands.

![Figure 10. The functors $E_i$ and $F_i$](image-url)
This category acts on $\bigoplus_{\mu, \nu} V_\mu \otimes V_\nu$ by the usual action of $E_i$ and $F_i$, and letting $I_\lambda$ act by sending $M$ to the same module considered as a module over $T^{\mu+\lambda}$. On the level of 2-morphisms, this action sends the crossing $\times$ to the obvious projection map $I_\lambda E_i \to I_\lambda$ and $\times$ to the map multiplying at the bottom by $\lambda^i$ dots on the new strand formed by applying $F_i$.

In particular, the inclusion of $U$ by horizontally composing with any set of red lines to the left is injective. It follows by the same arguments as Theorem 2.4 that $\tilde{U}$ has a basis analogous to that of Khovanov and Lauda for $U$.

Now consider the $U$-module subcategory of $\tilde{U}$ where the red lines are fixed to have labels $\Delta$ in order, and consider its quotient by all 1-morphisms of the form $AE_i$ and all 2-morphisms given by positive degree bubbles at the far left of the diagram. The argument that the idempotent completion of this category is the category of projective $T^\Delta$ modules is precisely the same as the proof of Proposition 1.6.

This shows, in particular, that $K_0(T^\Delta)$ is a module over $U^Z_\Delta(g)$, which we will show in the next section is isomorphic to the tensor product $V^Z_\Delta$.

3. Standard modules

3.1. Standard modules defined. When analyzing the structure of representation-theoretic categories, such as the categories $O$ appearing in Stroppel’s construction of Khovanov homology [Str], a crucial role is played by the Verma modules and their analogues. The property of “having objects like Verma modules” was formalized by Cline-Parshall-Scott as the property of being quasi-hereditary [CPS88]. Unfortunately, this is too strong of an assumption for us; as we noted earlier, the cyclotomic
QHA is Frobenius, and thus very far from being quasi-hereditary (any ring which is both Frobenius and quasi-hereditary is semi-simple).

Luckily, our categories satisfy a weaker condition: they are **standardly stratified**, as defined by the same authors [CPS96]. To show this, we must construct a collection of modules which are called **standard**, and show that projectives have a filtration by these modules compatible with a preorder.

We define a preorder on \((i, \kappa)\)'s by calling \((i, \kappa) \leq (i', \kappa')\) if

\[
\sum_{k \leq \kappa(j)} \alpha_{i_k} \leq \sum_{k \leq \kappa'(j)} \alpha_{i'_k} \quad \text{for all } j \in [1, \ell].
\]

This preorder can be packaged as the dominance order for a function \(\alpha_{i, \kappa} : [1, \ell] \to X(g)\) which we call a **root function** given by

\[
\alpha_{i, \kappa}(k) = \sum_{\kappa(k-1) < j \leq \kappa(k)} \alpha_{i_j}.
\]

Note that this preorder is entirely insensitive to permutations of the black strands which do not cross any red strands.

**Definition 3.1** By convention, we call a red/black crossing where black strands go from NW to SE **left** and the mirror image of such a crossing **right**.

Note that this terminology does not apply to black/black crossings; if we call a crossing left or right we are implicitly assuming it is black/red.

\[
\begin{array}{cc}
\times & \times \\
\text{a “left” crossing} & \text{a “right” crossing}
\end{array}
\]

The significance of these definitions is that a map induced between projectives by adding a left crossing on the bottom always sends a projective to one smaller in the preorder \(\leq\), and **vice versa** for right crossings. We will call a black strand that makes a left crossing below all right crossings **standardly violating**.

Let \(L_i^\kappa \subset P_i^\kappa\) be the submodule generated by diagrams with no right crossings, and at least one left crossing; that is, the module spanned by all diagrams with standardly violating strands.

**Proposition 3.2** The image of any map from a projective higher than \((i, \kappa)\) in the preorder \(\leq\) is contained in \(L_i^\kappa \subset P_i^\kappa\), and these images generate \(L_i^\kappa\). That is, the submodule \(L_i^\kappa\) is the “trace” of these projectives.

**Proof.** Generation is clear: any diagram with only left crossings defines a map from a higher projective to \(P_i^\kappa\) with the image of the idempotent being the original diagram.
To show that any such image lands in $L_\kappa^i$, consider an arbitrary map from a higher projective. This is given by a sum of diagrams in $P_\kappa^i$ whose upper end points are given by the idempotent for that projective. Now, apply Proposition 2.5 with a set of representatives that do all crossings within blocks consisting of a red strand and the black strands to its immediate left before doing any others. By the definition of the preorder, all these diagrams must have at least one left crossing which occurs before we make any crossings between these blocks, and all right crossings involve strands from different blocks; thus the image lies in $L_\kappa^i$. □

**Definition 3.3** We define $S_\kappa^i = P_\kappa^i / L_\kappa^i$ to be the standard module for $\kappa$ and $i$.

Proposition 3.2 shows that this matches the definition of a standard module for an algebra with preorder on its projectives given in (for instance) [MSb], so our terminology matches theirs. Below, when we speak of a group of black strands, we will always mean the set of black strands which originate between two consecutive red strands at the bottom of the diagram.

Let $e_\alpha$ be the idempotent which is 1 on projectives $P_\kappa^i$ with $\alpha_{1,k} = \alpha$. We let $S_\alpha$ be the standard quotient of the projective $e_\alpha T^A$. Let $C^\alpha$ be the subcategory of modules with a presentation in $\add(S_\alpha)$ for fixed $\alpha$.

![Figure 11. The action of $T_{a(1)}^{\lambda_1} \otimes \cdots \otimes T_{a(\ell)}^{\lambda_\ell}$ on $e_\alpha T^A$.](image)

Acting by black-black crossings on just each group of strands as in Figure 11 gives a map $T_{a(1)}^{\lambda_1} \otimes \cdots \otimes T_{a(\ell)}^{\lambda_\ell} \to \End_{\mathcal{T}_2}(S_\alpha)$, so we can think of $S_\alpha$ as a $T_{a(1)}^{\lambda_1} \otimes \cdots \otimes T_{a(\ell)}^{\lambda_\ell} - T_{a(\lambda)}^A$ bimodule, and $S = \bigoplus \alpha S_\alpha$ as a $T^{\lambda_1} \otimes \cdots \otimes T^{\lambda_\ell} - T^A$-bimodule.

**Definition 3.4** The standardization functor is the tensor product with this bimodule:

$$S^{A}(-) = - \otimes_{T^{\lambda_1} \otimes \cdots \otimes T^{\lambda_\ell}} S : \mathcal{B}^{\lambda_1, \ldots, \lambda_\ell} \to \mathcal{B}^{A}$$

More generally, we can construct partial standard modules, where we only kill the left crossings for some of the red strands. This will give us a standardization
functor

$$\mathcal{S}_{\lambda_1 \cdots \lambda_m} : \mathcal{U}_{\lambda_1 \cdots \lambda_m} \rightarrow \mathcal{U}_\Delta$$

for any sequence of sequences $\lambda_1, \ldots, \lambda_m$ such that the concatenation $\lambda_1 \cdots \lambda_m$ is equal to $\lambda$.

Of particular interest is the standardization functor which corresponds to adding a new red strand labeled $\mu$ and no black ones, since this categorifies the inclusion of $V_\lambda \otimes \{ v_{\text{high}} \} \hookrightarrow V_\lambda \otimes V_\mu$. We denote this functor $\mathcal{S}_{\lambda \mu}(- \otimes P_\emptyset) = \mathfrak{I}_\mu$.

We can think of the standardization functor as a (very far from full) inclusion of the naive tensor product category into $\mathfrak{U}_\Delta$. This functor is full when only considered on objects corresponding to a single root function, but there are, of course, many “new” maps between the different values.

### 3.2. Decategorification.

In order to understand the Grothendieck group $K_0(T^\Delta)$, we need to better understand its Euler form. In particular, we need a candidate bilinear form on $V_\lambda$. There is a system of non-degenerate $U_q(g)$-invariant sesquilinear forms $\langle , \rangle$ on all tensor products $V_\lambda^\mathbb{Z}$ defined by $\langle v, w \rangle = \langle \Theta_0^{(\ell)} v, w \rangle_p$, where $\Theta_0^{(\ell)}$ is the $\ell$-fold quasi-$R$-matrix and $\langle -, -, \rangle_p$ is the term-wise $q$-Shapovalov form. The usual $q$-R-matrix on two tensor factors is defined in [Lus93, §4]; the $\ell$-fold one is defined inductively by $\Theta_0^{(\ell)} = (\Theta_0^{(2)} \otimes 1 \otimes \cdots \otimes 1) : \Lambda \otimes 1 \otimes \cdots (1 \otimes \cdots (1 \otimes 1))$.

**Proposition 3.5** This form is non-degenerate, and $\tau$-Hermitian in the sense that we have $\langle u \cdot v, v' \rangle = \langle v, \tau(u \cdot v') \rangle$ for any $v, v' \in V_\Delta$ and $u \in U_q(g)$, where $\tau$ is the antiautomorphism defined in Section 1.3.

Furthermore, for any $j < \ell$, the natural map $V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_1} \otimes \{ \cdot v_{1}^{j+1} \} \otimes \cdots \otimes \{ v_{1}^{\ell} \} \hookrightarrow V_\Delta$ is an isometric embedding.

**Proof.** The first statement follows from

$$\langle u \cdot v, v' \rangle = \langle \Theta_0^{(\ell)} \Delta(u)v, v' \rangle_p = \langle \Delta(u)\Theta_0^{(\ell)} v, v' \rangle_p = \langle \Theta_0^{(\ell)} v, (\tau \otimes \cdots \otimes \tau) \Delta(u)\tau v' \rangle_p = \langle \Theta_0^{(\ell)} v, \Delta(\tau(u))\tau v' \rangle_p = \langle v, \tau(u \cdot v') \rangle.$$

The second reduces to case of two factors, since $\langle -, -, \rangle$ is a multiple of the $q$-Shapovalov form on any simple submodule of a tensor product. In this case it follows form the fact that $\Theta_0^{(2)} \in U^- \otimes U^+$ and $\Theta_0^{(2)} = 1 \times 1$, so $\Theta_0^{(2)}$ fixes $v \otimes v_{\text{high}}$. \qed

Let $v_i^\kappa \in V_\lambda$ be defined inductively by

- if $\kappa(\ell) = n$, then $v_i^\kappa = v_i^\kappa \otimes v_{\ell}$ where $v_{\ell}$ is the highest weight vector of $V_{\lambda_\ell}$, and $\kappa^-$ is the restriction to $[1, \ell - 1]$.
- If $\kappa(\ell) \neq n$, so $v_i^\kappa = F_i v_i^\kappa$, where $i^\kappa = (i_1, \ldots, i_{n-1})$.

**Theorem 3.6** There is a canonical isomorphism $\eta : K_0(T^\Delta) \rightarrow V_\lambda^\mathbb{Z}$ given by $[P_i^\kappa] \mapsto v_i^\kappa$ intertwining the inner product defined above with the Euler form.
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Proof. First, note that
\[ \dim_q \text{Hom}(P_i^\kappa, P_i^{\kappa'}) = \langle v_i^{\kappa'}, v_i^{\kappa'} \rangle. \]
We prove this by induction on \( n \) and \( \ell \). Unless \( n = \kappa(\ell) = \kappa'(\ell) \), we can move a \( \mathcal{E}_i \) from one side to become a \( \mathcal{F}_i \) on the other (up to shift). The decompositions of \( \mathcal{E}_i P_i^\kappa \) into \( P_i^{\kappa''} \)'s matches that of the vector since both are done using the commutation relations between \( \mathcal{E}_i \) and \( \mathcal{F}_i \) or \( \mathcal{E}_i \) and \( \mathcal{F}_i \), which we already know match.

If \( n = \kappa(\ell) = \kappa'(\ell) \), then the dimension of the \( \text{Hom} \)-space and the inner product are both unchanged by simply removing the red line (obviously, this holds if we use \( \tilde{P}_i^\kappa \) and \( \tilde{P}_i^{\kappa'} \) by the basis of Theorem 2.4, and the isomorphism only sends elements with violating strands to elements with violating strands). This shows the equality.

Thus, if we are given any linear relation satisfied by \( P_i^\kappa \)'s, the corresponding linear combination of \( v_i^{\kappa'} \)'s is in the kernel of this form, and thus 0 in \( V_\lambda \). Thus, the map \( \eta \) is well-defined and surjective.

A surjective map of finitely generated free \( \mathbb{Z}[q, q^{-1}] \) modules is an isomorphism if and only if they have the same rank (the kernel must be a summand, which is zero if and only if its complement has the rank of the whole module). Thus, we need only prove that \( \mathcal{B}_\lambda \) has at most \( \dim(V_\lambda) \) simple modules.

Consider a simple module \( L \), and let \( \alpha \) be maximal among root functions for which \( L e_\alpha \neq 0 \). Let \( K \) be any simple constituent of \( L e_\alpha \) as a \( T^{\lambda_1} \otimes \cdots \otimes T^{\lambda_{\ell}} \)-module. Then, by adjunction, we have a surjective map \( S^\lambda(K) \to L \). As modules over \( T^{\lambda_1} \otimes \cdots \otimes T^{\lambda_{\ell}} \), we have a surjective map \( K \to S^\lambda(K)e_\alpha \) which is an isomorphism, so by assumption composing with the map to \( L \) gives an inclusion of \( K \). This implies that every proper submodule of \( S^\lambda(K) \) is killed by \( e_\alpha \); thus, the sum of all proper submodules is itself killed by \( e_\alpha \) and is thus proper itself. This implies that the cosocle of \( S^\lambda(K) \) is simple and thus \( L \). Thus \( L \) is uniquely determined by \( K \), and there no more simple objects in \( \mathcal{B}_\lambda \) than there are in \( \mathcal{B}^{\lambda_1, \ldots, \lambda_{\ell}} \), which has exactly \( \dim(V_\lambda) \) simple modules. Thus, the ranks coincide, and we are done. □

Now, we let \( s_i^\kappa = F_{i(2)} \cdots F_{i_1} v_1 \otimes \cdots \otimes F_{i_n} \cdots F_{i(\ell+1)} v_{\ell+1} \).

Proposition 3.7 \( \eta([S_i^\kappa]) = s_i^\kappa \).

Proof. This proof depends on two inequalities, which we will use to “squeeze” the inner products of the two sides of the equality with projectives. First, we prove by induction that
\[ \dim_q \text{Hom}(P_i^\kappa, S_i^{\kappa''}) \leq \langle v_i^{\kappa'}, v_i^{\kappa''} \rangle. \]
Consider the restriction of a standard module \( \mathcal{E}_i S_i^\kappa \). This carries a filtration by submodules \( q_i \), where \( q_i \) is the submodule generated by the collection of diagrams where the rightmost strand at the top lands to the right of the \( i \)th strand and left of the \( i + 1 \)st at the bottom.
We let $\kappa_m$ and $i_m$ be associated to the sequence pictured at the bottom of Figure 12. Then we have a natural map

\[(3.4) \quad S_i = S^A(\cdots \boxtimes P_{i_{m-1}} \boxtimes \mathcal{E}_i P_{i_m} \boxtimes P_{i_{m+1}} \boxtimes \cdots) \left( \sum_{j=1}^{m-1} \langle \alpha_i, \lambda_j - \alpha(j) \rangle \right) \rightarrow q_i/q_{i+1}.
\]

sending a $\boxtimes$ of diagrams to the horizontal composition of those diagrams with the strand attaching to $\mathcal{E}_i$ pulled through the bottom of all the diagrams to its right (see Figure 13). This map is clearly surjective, so applying the induction hypothesis, we see that

\[
\dim \text{Hom}(\gamma_i, P_i^\kappa, S_i^\alpha) = \dim \text{Hom}(P_i^\kappa, \mathcal{E}_i S_i^\alpha) \leq \sum_{j=1}^{\ell} \dim \text{Hom}(P_i^\kappa, S_j)
\]

\[
\leq \sum_{j=1}^{\ell} \langle v_i^\kappa, E_j^{(i)} s_i^\alpha \rangle_1 = \langle v_i^\kappa, E_i s_i^\alpha \rangle_1 = \langle F_i v_i^\kappa, s_i^\alpha \rangle_1,
\]

where $E_i^{(j)}$ is $E_i$ just acting in the $j$th tensor factor.
If \( \kappa(\ell) \neq n \), then we can write \( P_\kappa^\ell \) as the image of a \( \overline{\mathcal{F}}_i \), and this shows the induction step. If \( \kappa(\ell) = n \), then either \( \text{Hom} \) to a standard is 0, or the red strand can be removed from both. This shows the inequality (3.3).

Now, we move on to showing the equality

\[
\dim \text{Hom}(P_\kappa^1, S_\kappa^2) = \langle v_1^\kappa, s_2^\kappa \rangle.
\]

This will immediately imply the desired result by non-degeneracy of the Euler form.

Consider the module \( \overline{\mathcal{F}}_i S_\kappa^i \), equipped with the filtration consisting of submodules \( p_m \) generated by diagrams where the black strand starting at the far right never passes left of the \( m \)th red strand.

Acting on the element \( x_m \) depicted in the Figure 14 induces a map

\[
S_\kappa^m \left( - \sum_{j=m+1}^{\ell} \langle \alpha_i, \lambda_j - \alpha(j) \rangle \right) \rightarrow p_m / p_{m-1}
\]

which is clearly surjective.

Thus, we obtain a second inequality

\[
\dim \text{Hom}(\mathcal{E}_i P_\kappa^\ell, S_i^\kappa') = \dim \text{Hom}(P_\kappa^\ell, \overline{\mathcal{F}}_i S_i^\kappa') \leq \sum_{j=1}^{\ell} \dim \text{Hom}(P_\kappa^j, S_i^\kappa') \leq \sum_{j=1}^{\ell} \langle v_i^\kappa, s_i^\kappa' \rangle_1 = \langle v_i^\kappa, E_i s_i^\kappa' \rangle_1 = \langle E_i v_i^\kappa, s_i^\kappa' \rangle_1.
\]

Since the initial and final quantities are equal by induction, the above can only hold if the inequality (3.3) is always an equality. □

We note that we have now shown that the morphisms between standard modules and successive quotients of \( \mathcal{F}_i S_i^\kappa \) and \( \mathcal{E}_i S_i^\kappa \) must be isomorphisms for dimension reasons. That is, these standard filtrations directly categorify the identities

\[
\Delta^{(\ell)}(E_i) = E_i \otimes 1 \otimes \cdots \otimes 1 + \tilde{K}_i \otimes E_i \otimes 1 \otimes \cdots \otimes 1 + \cdots + \tilde{K}_i \otimes \cdots \otimes \tilde{K}_i \otimes E_i, \quad \tilde{K}_i \otimes \cdots \otimes \tilde{K}_i \otimes E_i.
\]
\[ \Delta^\ell(F_i) = F_i \otimes \tilde{K}_{-i} \otimes \cdots \otimes \tilde{K}_{-i} + 1 \otimes F_i \otimes \tilde{K}_{-i} \otimes \cdots \otimes \tilde{K}_{-i} + \cdots + 1 \otimes \cdots \otimes 1 \otimes F_i. \]

3.3. **Simple modules and crystals.** Lauda and Vazirani show that there is a natural crystal structure on simple representations of \( R^\lambda = T^\lambda \), which is isomorphic to the usual highest weight crystal \( B(\lambda) \). A similar crystal structure exists for simples of \( T^\lambda \); we denote the set of isomorphism classes of simple modules by \( \mathcal{B}^\lambda \).

Note that we have a candidate for a map \( h : \mathcal{B}^{\lambda_1} \times \cdots \times \mathcal{B}^{\lambda_\ell} \to \mathcal{B}^\lambda \), given by \( \text{cosoc} S^\lambda(L_1 \boxtimes \cdots \boxtimes L_\ell) \); it’s not immediately obvious that this module is simple, but in fact, we have already shown in the proof of Theorem \ref{thm:3.6} that this map is well-defined, and surjective. Since \( \mathcal{B}^\lambda \) and \( \mathcal{B}^{\lambda_1, \ldots, \lambda_\ell} \) has the same number of simples, we have that

**Theorem 3.8** The map \( h \) defines a bijection.

For a simple module \( L \), the modules \( \text{cosoc}(\tilde{\gamma}_i L) \), and \( \text{soc}(E_i L) \) are both several copies of a single simple module. We define \( \tilde{f}_i(L) \) and \( \tilde{e}_i(L) \) to be these simples.

**Theorem 3.9** These operators make the classes of the simple modules a perfect basis of \( K_0(T^\lambda) \) in the sense of Berenstein and Kazhdan \cite[Definition 5.30]{BK07}. In particular, they define a crystal structure on simple modules.

**Proof.** This proof uses entirely standard techniques. If \( a \) is the largest integer such that \( \tilde{e}_i^{(a)}(L) \neq 0 \), then \( \tilde{e}_i^{(a)}(L) \) is semi-simple; in fact, it is a sum of copies of \( \tilde{e}_i^{(a)}(L) \) (since \( \tilde{S}_i^{(a)}(\tilde{e}_i^{(a)}(L)) \) surjects onto \( L \)). In particular, any other simple constituent of \( E_i(L) \) is killed by \( \tilde{e}_i^{(a-1)} \). This is the definition of a perfect basis. \( \square \)

**Proposition 3.10** Any simple module \( L \in \mathcal{B}^\lambda \) is isomorphic to its dual: \( L \cong L^* \).

**Proof.** First, we note that if \( L \) is self-dual, then so is \( \tilde{\gamma}_i L \), since \( \tilde{\gamma}_i \) commutes with \( \star \). Thus, the socle of \( \tilde{\gamma}_i L \) is isomorphic to the dual of the cosocle. On the other hand, since \( E_i \) and \( \tilde{\gamma}_i \) are biadjoint, we have an induced map \( \tilde{\gamma}_i L \to \tilde{\gamma}_i L \) sending the cosocle to the socle, which induces an isomorphism from the obvious quotient copy of \( L \) to the obvious sub copy of \( L \) inside \( E_i \tilde{\gamma}_i L \) (after applying \( E_i \) to the map). This map thus must induce an isomorphism between a summand of the socle to a summand of the sococle. Thus, \( \tilde{f}_i(L) \) is self-dual. Of course, an analogous argument shows that \( \tilde{e}_i \) also preserves self-dual modules. Thus, the result is clear for \( \mathcal{B}^\lambda \).

Now, consider the general case. On the subcategory of modules killed by \( e_{a'} \) for \( a' \not\leq a \), the functor \( M \mapsto M e_a \) is a functor to \( \mathcal{B}^{\lambda_1, \ldots, \lambda_\ell} \), which has a left adjoint \( S^\lambda \) and
right adjoint $\star \circ S^\lambda \circ \star$. Obviously, the socle of $S^\lambda(L_1 \boxtimes \cdots \boxtimes L_\ell)^\star$ and the cosocle of $S^\lambda(L_1 \boxtimes \cdots \boxtimes L_\ell)$ are dual simple modules.

On the other hand, we have a map

$$S^\lambda(L_1 \boxtimes \cdots \boxtimes L_\ell) \to S^\lambda(L_1^\star \boxtimes \cdots \boxtimes L_\ell^\star)^\star$$

This map is non-zero, since the induced map

$$S^\lambda(L_1 \boxtimes \cdots \boxtimes L_\ell)e_\alpha \to S^\lambda(L_1^\star \boxtimes \cdots \boxtimes L_\ell^\star)^\star e_\alpha$$

is the identity map from $L_1 \boxtimes \cdots \boxtimes L_\ell$ to $L_1 \boxtimes \cdots \boxtimes L_\ell$. On the other hand, its image lies in the socle, and thus is an isomorphism from $h(L_1, \ldots, L_\ell)$ to its dual, since we already know $L_i$ is self-dual. □

**Conjecture 3.11** The crystal structure induced on $B^\lambda$ by $h$ has Kashiwara operators given by $\tilde{f}_i$ and $\tilde{e}_i$, where $B^{\lambda_1} \times \cdots \times B^{\lambda_\ell}$ is endowed with the tensor product crystal structure.

Since $K_0(T^\lambda) \cong V_\lambda$, this implies that an isomorphism of crystals exists between $B^\lambda$ and $B^{\lambda_1} \times \cdots \times B^{\lambda_\ell}$ without actually determining what it is.

Choose any infinite sequence $\{i_1, \ldots, \}$ of simple roots such that each root appears infinitely often. For any element $v$ of a highest weight crystal, there are unique integers $\{a_1, \ldots, \}$ such that $\cdots \tilde{e}_{i_j}^{a_j} \tilde{e}_{i_1}^{a_1} v = v_{\text{high}}$ and $\tilde{e}_{i_j}^{a_j+1} \cdots \tilde{e}_{i_1}^{a_1} v = 0$; the parametrization of the elements of the crystal by this tuple is called the “string parametrization.”

Our system of projectives $P^\kappa_i$ is quite redundant; there are many more of them than there are simple modules, as Proposition 3.8 shows. We can produce a smaller projective generators by using string parametrizations.

**Definition 3.12** We call a sequence $(i, \kappa)$ stringy if the sequence of $i$’s between the $j$th and $j+1$st red lines is the string parametrization of a crystal basis vector in $V_\lambda_j$.

We will implicitly use the canonical identification between stringy sequences and $B^\lambda$ via $h$.

As in Khovanov and Lauda [KLa, §3.2], we order the elements of the crystal by first decreasing weight (so that the smallest element is the highest weight vector) and then lexicographically by the string parametrization.

For the tensor product crystal, we use the dominance order on $\alpha$’s, with the order on nodes in the factors used lexicographically to break ties.

**Proposition 3.13** The projective cover of any simple appears as a summand of $P^\kappa_i$ where $(i, \kappa)$ is the corresponding stringy sequence. This cover is, in fact, the unique indecomposable summand which doesn’t appear in $P^\kappa_{i'}$ for $(i', \kappa') > (i, \kappa)$.
As a matter of convention, we call the root function of the stringy sequence where an indecomposable projective first appears the root function of that projective.

**Proof.** Obviously, \( P_\iota^\kappa \rightarrow S_\iota^\kappa = \bigotimes (\delta_{i_\iota}^{a_\iota}) \cdots (\delta_{i_1}^{a_1}) \bigotimes (\delta_{i_0}^{a_0}) \cdots (\delta_{i_{k_\iota}}^{a_{k_\iota}+1}) P_0 \) which in turn surjects to the corresponding simple, by the definition of Kashiwara operators on simple modules, and the map \( h \). Thus, the indecomposable projective cover is a summand of \( P_\iota^\kappa \).

For a simple \( L \), there is only a map of \( P_\iota^\kappa \) to \( L \) if \( L \) is the image under \( h \) of simples that appear in \( \delta_{i_\iota}^{a_\iota} \cdots \delta_{i_{k_\iota}+1} P_0 \) or \( L \) is higher in the dominance order. Since only larger simples in Khovanov and Lauda’s order appear in \( \delta_{i_\iota}^{a_\iota} \cdots \delta_{i_{k_\iota}+1} P_0 \) by [KLa, Lemma 3.7], the projective cover of any simple which appears other than that for our chosen stringy sequence is a summand in a projective for a higher stringy sequence. \( \square \)

For an indecomposable projective \( P \), its **standard quotient** is its quotient under the sum of all images of maps from projectives with higher root sequences. This coincides with its image in \( S_\iota^\kappa \), the standard quotient for its associated stringy sequence. This standard quotient is indecomposable, since it is a quotient of an indecomposable projective.

**Proposition 3.14** Consider \( (\iota, \kappa) \) with the associated root function \( \alpha \). Then the sum of indecomposable summands of \( P_\iota^\kappa \) that have the same root function surject to \( S_\iota^\kappa \), which is a direct sum of the standard quotients of those projectives.

**Proof.** If an indecomposable summand of \( P_\iota^\kappa \) has a different root function, it must be higher, so this summand is in the image of a higher stringy projective and thus in \( L_\iota^\kappa \). Thus, the other summands must surject.

Similarly, it is clear that the intersection of any indecomposable with the same root function with \( L_\iota^\kappa \) is exactly the trace of the projectives with higher root functions. \( \square \)

Finally, we prove a result which, while somewhat technical in nature, is very important for understanding how to decategorify our construction. As in [BGS96, §2.12], we let \( C^i(T_\Lambda) \) denote the category of complexes of graded modules satisfying \( C_j = 0 \) for \( i \gg 0 \) or \( i + j \ll 0 \).

**Theorem 3.15** Every simple module over \( T_\Lambda \) has a projective resolution in \( C^i(T_\Lambda) \). In particular, each simple module \( L \) has a well-defined class in \( K_0(T_\Lambda) \otimes_{\mathbb{Z}[q, q^{-1}]} \mathbb{Z}((q)) \cong V_\Lambda \).

**Proof.** The proof is by induction on our order above. First, we do the base case of \( \Lambda = (\lambda) \) and \( \lambda - \alpha = k\alpha \). This case, \( T_{(\lambda)} \) is Morita equivalent to its center, which is
the cohomology ring on a Grassmannian of \( k \)-planes in \( \lambda^i \)-dimensional space. In particular, it is positively graded, so such a resolution exists.

Now, we bootstrap to the case where \( \lambda = (\lambda) \) but \( \alpha \) is arbitrary. In this case, we may assume that \( L' = \tilde{e}_i^1 L \) has this type of resolution. Now, we consider

\[
M = \text{Ind}_{\alpha + a_1, \alpha, \lambda} L' \boxtimes L(\tilde{e}_i^1),
\]

where here we use the notation of [KLa §3.2]. The module \( M \) has a projective resolution of the prescribed type, by inducing the outer tensors of the resolutions on the two factors. Furthermore, there is a surjection \( M \to L \) whose kernel has composition factors smaller in the order given above on simples, by [KLa Theorem 3.7]. Since each of these has an appropriate resolution by induction, we may lift the inclusion of each composition factor to a map of projective resolutions, and take the cone to obtain a resolution of \( L \) in \( C^\uparrow(T^\lambda) \).

Finally, we deal with the general case using standardization; let \( L = h([L_i]) \). By standardizing the resolutions of \( L_i \), we obtain a standard resolution of \( \mathcal{S}(L_1 \boxtimes \cdots \boxtimes L_\ell) \). Replacing each standard with its finite projective resolution, we obtain a projective resolution of the same module. As before, the kernel of the surjection of this module to \( L \) has composition factors all smaller in the partial order, so we may attach projective resolutions of each composition factor to obtain a projective resolution of \( L \) in \( C^\uparrow(T^\lambda) \). □

3.4. Standard stratification. Now, we proceed to showing that the algebra \( T^\lambda \) is standardly stratified. Consider the set \( \Phi \) of permutations of the bottom ends of the strands which only move black strands into blocks to their left and are minimal coset representatives for the permutations of the strands at the top of the diagram. We first give these a partial order which only depends only on the resulting idempotent at the top of the diagram.

So, we first preorder \( \Phi \) according to this preorder on the idempotent \((i_\Phi, \kappa_\Phi)\) which appears at the top of the diagram. Then within the permutations giving a single idempotent, we use the Bruhat order. Unlike the preorder above, this is a partial order.

![Figure 15. The element \( x_\phi \)](image-url)
Let \( x_\phi \) be an element where we permute the strands exactly according to a chosen reduced word of \( \phi \in \Phi \). Let

\[
P_{\leq \phi} = \langle x_\phi' | \phi' \leq \phi \rangle \subset P^\kappa_i \quad P_{< \phi} = \langle x_\phi' | \phi' < \phi \rangle \subset P^\kappa_i
\]

The element \( x_\phi \) is not unique, since it depends on a choice of reduced word; however, any two choices differ by an element of \( L_{< \phi} \), so the filtration described above is unique.

**Proposition 3.16** \( P_{\leq \phi}/P_{< \phi} \cong S^\kappa_{\theta_\phi} \).

We note that some of these subquotients are trivial, but in this case the corresponding standard module is trivial as well.

**Proof.** Since this map is surjective, we have \( \dim P_{\leq \phi}/P_{< \phi} \leq \dim S^\kappa_{\theta_\phi} \). On the other hand, we have \( v^\kappa_i = \sum_{\phi \in \Phi} q^{-\deg x_\phi} S^\kappa_{\theta_\phi} \), so taking inner product with \([T\Delta]\), we obtain \( \dim P^\kappa_i = \sum_{\phi \in \Phi} \dim S^\kappa_{\theta_\phi} \).

Thus we must have equality above, and the map is an isomorphism for degree reasons. \(\square\)

**Corollary 3.17** The algebra \( T^\Delta \) is standardly stratified with standard modules given by the standard quotients of indecomposable projectives, and the preorder on simples/standards/projectives given by the dominance order on root functions \( \alpha \).

**Corollary 3.18** Every standard module has a finite length projective resolution.

This is a standard fact about finite dimensional standardly stratified algebras; in particular, any module with a standard filtration has a well-defined class in \( K_0(T^\Delta) \).

**Proof.** First note that if a module \( M \) is filtered by modules which have finite length projective resolutions, these resolutions can be glued to give a finite length resolution of the entire module.

Now, we induct on the partial order \( \leq \). If a standard is maximal in this order, it is projective. For an arbitrary standard, there is a map \( P^\kappa_i \to S^\kappa_i \) with kernel filtered by standards higher in the partial order. Since each of these has a finite length projective resolution, \( S^\kappa_i \) does as well. \(\square\)

We note that \( e(i, \kappa)T^\Delta e(i, 0) \) has a unique element consisting of a diagram with no dots and no crossings between black strands which simply pulls red strands to the left and black to the right. As before, we call this element \( \theta_\kappa \) (leaving \( i \) implicit).

**Lemma 3.19** The map from \( P^\kappa_i \to P^0_i \) given by the action of \( \theta_\kappa \) is injective.
Proof. Obviously, this map is filtered, where we include \( \Phi_{i,k} \subset \Phi_{i,k} \) by precomposing with the permutation that pushes all black strands to the right. Furthermore, it induces an isomorphism on each successive quotient in this image. Thus, it is injective.

We let \( \Psi_a^{\lambda_1;\ldots;\lambda_n} = T_{\alpha(1)}^{\lambda_1} \otimes \cdots \otimes T_{\alpha(\ell)}^{\lambda_\ell} \mod \), and let \( C^a \) be the subcategory of modules which have a presentation by standard modules with root function \( \alpha \).

**Proposition 3.20** We have a natural isomorphism

\[
\text{End}_{T}^\lambda(S_\alpha) \cong T_{\alpha(1)}^{\lambda_1} \otimes \cdots \otimes T_{\alpha(\ell)}^{\lambda_\ell}.
\]

In particular, \( C^a \) is equivalent to \( \Psi_a^{\lambda_1;\ldots;\lambda_n} \). The triangulated subcategories generated by \( C^a \) form a semi-orthogonal decomposition of the derived category \( D^+(V^\lambda_\alpha) \) with respect to dominance order.

Proof. Since every standard with root function \( \alpha \) is a summand of \( S_\alpha \) and \( S_\alpha \) has trivial higher Exts

\[
C^a \cong \text{End}^{\text{op}}(S_\alpha) \mod.
\]

Let us calculate this endomorphism algebra. By the projective property, every endomorphism of \( S_\alpha \) is induced by an endomorphism of \( e_\alpha T_\lambda \). Thus \( \text{End}^{\text{op}}(S_\alpha) \) is the quotient of the subalgebra of \( e_\alpha T_\lambda e_\alpha \) which preserves the kernel of the standard quotient modulo those that send everything to the kernel.

Apply Proposition 2.5 in the case where each reduced word puts each group of black strands and red immediately to its left in the correct order first, followed by a shortest coset representative for this Young subgroups. This implies that the diagram from any permutation which has a left crossing has at least one before any right crossings. By the definition of the standard quotient such a diagram is 0. On the other hand, an element of \( e_\alpha T_\lambda e_\alpha \) must have equal numbers of the two types of crossings, so our element can be “straightened” so that no red and black strands ever cross. Thus, we have a surjective map from \( \tilde{T}_{\alpha(1)}^{\lambda_1} \otimes \cdots \otimes \tilde{T}_{\alpha(\ell)}^{\lambda_\ell} \).

By definition of a standard quotient, the cyclotomic ideal of this tensor product is killed by the map to \( \text{End}^{\text{op}}(S_\alpha) \), so we have a surjective map

\[
T_{\alpha(1)}^{\lambda_1} \otimes \cdots \otimes T_{\alpha(\ell)}^{\lambda_\ell} \rightarrow \text{End}^{\text{op}}(S_\alpha),
\]

which we need only show is also injective. Since \( \text{Ext}^{>0}(S_\alpha, S_\alpha) = 0 \), this is equivalent to showing that

\[
\dim \text{End}(S_\alpha, S_\alpha) = \langle [S_\alpha], [S_\alpha] \rangle_1 = \dim T_{\alpha(1)}^{\lambda_1} \otimes \cdots \otimes T_{\alpha(\ell)}^{\lambda_\ell}.
\]

The second equality follows from the equality \( \langle a \otimes b, a' \otimes b' \rangle = \langle a, b \rangle \langle a', b' \rangle \) if \( a,a' \) and \( b,b' \) are weight vectors with each pair having the same weight, which follows, in turn, from the upper-triangularity of \( \Theta(2) \).

Finally, we establish the semi-orthogonal decomposition: by Proposition 3.16 the subcategory generated by \( C^a' \) for \( a' > a \) in the dominance order is the same as that
generated by $P_i^\kappa$ such that $\alpha_{i,\kappa} > \alpha$. Since all the simple modules in $S_i^\kappa$ are given by idempotents $e_{i,\kappa}$ such that $\alpha_{i,\kappa} \leq \alpha$, we have
\[
\text{Ext}^*(S_i^{\kappa'}, S_i^\kappa) = 0
\]
whenever $\alpha_{i,\kappa} < \alpha_{i,\kappa'}$, and higher Ext's vanish when equality holds. \hfill \Box

3.5. Self-dual projectives. One interesting consequence of the module structure over $\mathcal{U}$ and standard stratification is the understanding it gives us of the self-dual projectives of our category. Self-dual projectives have played a very important role in understanding the structure of representation theoretic categories like $\mathfrak{sl}_2$. For example, the unique self-dual projective in BGG category $O$ for $\mathfrak{g}$ was key in Soergel’s analysis of that category [Soe90, Soe92], and the self-dual projectives in category $O$ for a rational Cherednik algebra provide an important perspective on the Knizhnik-Zamolodchikov functor defined by Ginzburg, Guay, Opdam and Rouquier [GGOR03]. In particular, as Mazorchuk and Stroppel show [MSb], these modules also play an important role in the identification of the Serre functor; we will apply their results to describe the Serre functor of the perfect derived category of $T^\perp$-modules in the sequel [Web] to this paper.

**Theorem 3.21** If $P$ is an indecomposable projective $T^\perp$-module, then the following are equivalent:

1. $P$ is injective.

2. $P$ is a summand of the injective hull of an indecomposable standard module.

3. $P$ is isomorphic (up to grading shift) to a summand of $P^0$.

**Proof.** (3) $\rightarrow$ (1): To establish this, we show that $P^0$ is self-dual; that is, there is a non-degenerate pairing $P_i^0 \otimes P^0_i \rightarrow \mathbb{k}$. This is given by $(a, b) = \tau_\lambda(ab)$, where $\tau_\lambda$ is the Frobenius trace on $\text{End}(P^0) \cong T^\lambda$ given in Section 1.3. Thus $P^0$ is both projective and injective, so any summand of it is as well.

(1) $\rightarrow$ (2): Since $P$ is indecomposable and injective, it is the injective hull of any submodule of $P$. Since $P$ has a standard stratification, it has a submodule which is standard.

(2) $\rightarrow$ (3): We have already established that $P^0$ is injective, so we need only establish that any simple in the socle of $S_i^\kappa$ is a summand of the cosocle of $P^0$ (since the injective hull of $S_i^\kappa$ coincides with that of its socle). It suffices to show that there is no non-trivial submodule of $S_i^\kappa$ killed by $e_{0,\kappa}$. If such a submodule $M$ existed, then we would have $M\bar{\theta}_\kappa = 0$. Thus, its preimage $M'$ in $P_i^\kappa$ satisfies $M'\bar{\theta}_\kappa \subset L_i^0$. But the injectivity of Lemma 3.19 and the fact that $L_i^\kappa\bar{\theta}_\kappa = L_i^0 \cap P_i^\kappa\bar{\theta}_\kappa$, this implies that $M = 0$. \hfill \Box

For two rings $A$ and $B$, we say an $A$-$B$ bimodule $M$ has the **double centralizer property** if $\text{End}_B(M) = A$ and $\text{End}_A(M) = B$. In particular, this implies that the
functor

$$- \otimes_A M : A \mod A \to B \mod B$$

is fully faithful on projectives (it could be quite far from being a Morita equivalence, as the theorem below shows).

**Corollary 3.22** The projective-injective $P^0$ has the double centralizer property for the actions of $T^\lambda$ and $T^\lambda$ on the left and right.

**Proof.** By [MSb, Corollary 2.6], this follows immediately from the fact that the injective hull of an indecomposable standard is also a summand of $P^0$. □

Thus, in this case, our algebra can be realized as the endomorphisms of a collection of modules over $R^\lambda$, in a way analogous to the realization of a regular block of category $O$ as the modules over endomorphisms of a particular module over the coinvariant algebra, or of the cyclotomic $q$-Schur algebra as the endomorphisms of a module over the Hecke algebra.

In fact, these modules are easy to identify. Given $(i, \kappa)$, we consider the element $y_{i, \kappa}$ of $P_i^0$ given by

$$y_{i, \kappa} = e_i \prod_{j=1}^{\ell} \prod_{k=\kappa(j)+1}^{n} y_{i, \kappa}^{j,k}.$$ 

Pictorially this is given by multiplying the element with no black/black crossings going from $(i, 0)$ to $(i, \kappa)$ (which we denote $\theta_{\kappa}$) by its horizontal reflection $\hat{\theta}_{\kappa}$, and then straightening the strands.

![Figure 16. The element $y_{(1,5,4,2),(0,1,1,3)}$.](image)

**Proposition 3.23** The algebra $T^\lambda$ is isomorphic to the algebra $\text{End}_{T^\lambda}(\bigoplus_{\kappa} y_{i, \kappa} T^\lambda)$.

**Proof.** Based on Corollary 3.22 all we need to show is that $\text{Hom}_{T^\lambda}(P_i^0, P_i^\lambda) \cong y_{i, \kappa} P_i^0$ as a $T^\lambda$ representation. A map $m$ from $P_i^0$ to $P_i^\lambda$ is simply a linear combination of diagrams starting at $i$ with the correct placement of red strands and ending at $i'$ with all red strands to the right. By Proposition 2.5 we can assure that all red/black crossings occur above all black/black ones, so $m = \hat{\theta}_{\kappa} m'$, where $m \in T^\lambda$. 

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Thus, we have maps
\[
\text{Hom}_{\mathfrak{T}_λ}(P^0, P^0_i) \xrightarrow{δ_κ} \text{Hom}_{\mathfrak{T}_λ}(P^0, P^κ_i) \xrightarrow{δ_κ} \text{Hom}_{\mathfrak{T}_λ}(P^0, P^0_i)
\]
given by composition. The first of these is surjective, as we argued above. Furthermore, the latter is injective, by Proposition 3.19. Thus, Hom_{\mathfrak{T}_λ}(P^0, P^κ_i) is isomorphic to the image of the composition of these maps, which is \(y_{i,κ}T^λ\). □

4. Quiver varieties

4.1. Background. We now turn to the geometric construction discussed in the introduction. Throughout this section, we will work in parallel between

- the case where \(\mathbb{k} = \bar{Q}_ℓ\) and \(k = \mathbb{F}_p\) is the field with \(p\) elements, for primes \(p \neq ℓ\), and
- the case where \(\mathbb{k} = \mathbb{Q}\) and \(k = \mathbb{C}\).

Also, throughout, we let \(D(X)\) for a Artin stack or variety \(X\) over \(k\) denote

- its bounded below derived category of fppf sheaves with constructible cohomology with coefficients in \(\mathbb{k}\) if \(k\) is a finite field or
- its bounded below derived category of analytic sheaves with constructible cohomology if \(k = \mathbb{C}\).

In the finite field case, we have an extra piece of structure, whose characteristic 0 counterpart we will not discuss: the Frobenius action arising from the absolute Galois group of \(k\).

We note that if we pick an isomorphism \(\bar{Q}_ℓ \cong \mathbb{C}\), and an integral form of our variety/stack, then these categories are closely akin, though each contains “pathological objects” the other cannot see; however, we will restrict our attention to subcategories where they match up.

**Definition 4.1** For each orientation \(Ω\) of \(Γ\) (thought of as a subset of the edges of the oriented double), a \(k\)-representation of \((Γ, Ω)\) with shadows is

- a pair of finite dimensional \(k\)-vector spaces \(V\) and \(W\), graded by the vertices of \(Γ\), and
- a map \(x_σ : V_{σ(ω)} \to V_{σ(α)}\) for each oriented edge (as usual, \(α\) and \(ω\) denote the head and tail of an oriented edge), and
- a map \(z : V \to W\) with that preserves grading.

We let \(\textbf{w}\) and \(\textbf{v}\) denote \(Γ\)-tuples of integers.
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For now, we fix an orientation $\Omega$, though we will sometimes wish to consider the collection of all orientations. With this choice, we have the **universal** $(w, v)$-dimensional representation

$$E_{v,w} = \bigoplus_{i \rightarrow j} \text{Hom}(k^v_i, k^{2v}_j) \oplus \bigoplus_i \text{Hom}(k^v_i, k^w_i).$$

In moduli terms, this is the moduli space of actions of the quiver (in the sense above) on the vector spaces $k^v, k^w$, with their chosen bases considered as additional structure.

If we wish to consider the moduli space of representations where $V$ has fixed graded dimension (rather than of actions on a fixed vector space), we should quotient by the group of isomorphisms of quiver representations: $G_v = \prod_i \text{GL}(k^v_i)$ acting by pre- and post-composition. The result is the **moduli stack of $v$-dimensional representations shadowed by $k^w$**, which we can define as the stack quotient

$$X^w_v = E_{v,w}/G_v.$$

This is not a scheme in the usual sense, but rather a smooth Artin stack. Those readers made skittish by the mention of stacks can consider this as a purely formal symbol whose derived category is the equivariant derived category of $E_{v,w}$ in whatever sense they like, for example, as in the book of Bernstein and Lunts [BL94] or as described by the author and Williamson [WW].

By convention, if $w_i = \alpha_i^\vee(\lambda)$ and $\mu = \lambda - \sum v_i \alpha_i$, then $X^\lambda_w = X^w_v$ (if the difference is not in the positive cone of the root lattice, then this is by definition empty), and $X^\lambda = \sqcup_{\xi} X^\lambda_{\xi,\nu}$.

We let $X^\lambda_{\xi,\nu}$ denote the moduli stack of short exact sequences (“Hecke correspondences”) where the subobject belongs in $X^\lambda_{\xi}$, the total object in $X^\lambda_{\xi,\nu}$ and the quotient in $X^0_{\xi,\nu}$. This is a quotient of the variety of pairs consisting of a point in $E_{w,v}$, and an invariant collection of subspaces of $k^v_i$ of codimension $\alpha_i^\vee(v)$ by the natural action of the group $G_v$. Thus, this moduli stack is equipped with projections

$$\begin{array}{c}
X^\lambda_{\xi,\nu} \\
| \downarrow p_1 \quad \downarrow p_2 \quad \downarrow p_3 \\
X^\lambda_{\xi} \quad X^\lambda_{\xi,\nu} \quad X^0_{\nu}
\end{array}$$

From this geometric setup, we can construct a monoidal actions by $D(X^0)$ on $D(X^1)$, where the former category is endowed with the Hall monoidal structure defined by Lusztig in [Lus91, §3] by convolution

$$\mathcal{F} \star \mathcal{G} = (p_2)_! p_3^! \mathcal{F} \otimes p_1^! \mathcal{G} [\dim X^1_{\xi} + \dim X^0_{\xi,\nu} - \dim X^1_{\xi,\nu}].$$

The Hall monoidal structure, by definition, is the special case of this where $\lambda = 0$. 

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Zheng’s functors $F_i^{(n)}$ are given by convolution with Hecke correspondences when $i$ is a source (we may switch between different graph orientations by Fourier transform, as discussed extensively in the references [Zheb, Lus91]).

We can rephrase this as an action of the 2-category $\mathcal{L}$ whose objects are weights $\lambda$, and the Hom category from $\lambda$ to $\lambda'$ is $D(X^0_{\lambda'} - \lambda)\times D(X^0_{\lambda' - \lambda}) \to D(X^0_{\lambda'' - \lambda})$ given by Lusztig’s convolution product $\star$.

This category is actually quite closely related to the categories $U^-$ and $U^+$ we discussed before. This is essentially proven in [VV], but since it is not stated there in the form most convenient for us, let us state it here:

**Proposition 4.2** There is a fully faithful 2-functor $\vartheta: U^- \to \mathcal{L}$ which on objects sends $F_i$ to the sheaf $\mathbb{k}_{X^0_{-\alpha_i}}$. The essential image of this functor is the category of sheaves $\mathcal{D}$ defined by Lusztig in [Lus91, §2.2], which in finite type is all sums of shifts of semi-simple perverse sheaves in $\mathcal{L}$.

This proposition is a natural categorification of the observation of Ringel that $U(n_-)$ maps to the Hall algebra of the corresponding quiver [Rin90].

**Proof.** Vasserot and Varagnolo [VV] 3.6 show that we have an isomorphism

$$\text{Hom}_{D(X^0_{-\nu})}(\mathbb{k}_{X^0_{-\alpha_1}} \star \cdots \star \mathbb{k}_{X^0_{-\alpha_j}} \star \cdots \star \mathbb{k}_{X^0_{-\alpha_n}}) \cong e_j \mathcal{R}e_i,$$

sending composition to multiplication. We define the functor on objects to send $E_i \mapsto \mathbb{k}_{X^0_{-\alpha_i}} \star \cdots \star \mathbb{k}_{X^0_{-\alpha_n}}$ and on morphisms using the isomorphism above. This is monoidal on objects essentially by definition, so we need only consider whether it is on morphisms. This is clear from the definition of Vasserot and Varagnolo’s map, since it constructs the crossings of pairs of strands precisely from maps relating the convolutions of pairs.

□

In [Zheb], Zheng constructs a geometric categorification of tensor products using the geometry of these varieties. He identifies a subcategory $\mathcal{N}$ in the derived category $D(X^\lambda)$ which geometrically corresponds to the unstable locus (in the sense of Nakajima’s papers [Nak94]) in $T^* X^\lambda$. Here, we use the stability condition on $T^* X^\lambda$ such that a representation with shadows is a stable point if it has no subrepresentation killed by all the shadow maps.

**Definition 4.3** When $k = \mathbb{C}$, we define the subcategory $\mathcal{N}$ to be the full subcategory of $D(X^\lambda)$ whose objects have singular supports lying in the unstable locus of $T^* X^\lambda$.

In the case $k = \mathbb{F}_p$, there is an analogous category $\mathcal{N}$ which has a similar definition. Since we are working over a field of positive characteristic, the conventional definition of singular support is not sensible and we cannot use Definition 4.3. Lusztig
shows that for any two orientations $\Omega'$ and $\Omega$, the corresponding categories of sheaves are related by a Fourier transform functor $\mathcal{F}_{\Omega', \Omega} : D(X^A(\Omega')) \to D(X^A(\Omega))$.

**Definition 4.4** When $k = \mathbb{F}_p$, we define $N$ to be the thick subcategory of $D(X^A)$ generated by the image under $\mathcal{F}_{\Omega', \Omega}(A)$ for every orientation $\Omega'$ of objects $A \in D(X^A(\Omega'))$ whose support is contained in the set where there for some source $v$ there is an element $x \in V_v$ killed by all outgoing maps from that source.

Thus, the category $D(X^A)/N$ where we localize $N$ (that is, declare any map whose cone lies in $N$ to be an isomorphism) should be thought of as the category of sheaves on a hypothetical space whose cotangent bundle is a Nakajima quiver variety. Since the underlying objects of this category are the same as those of $D(X^A)$, we will not distinguish between them notationally, and just indicate which category Hom-spaces are computed in. Zheng shows in the positive characteristic case [Zheb, Proposition 2.3.4] that $N$ is preserved by the monoidal action of $\mathcal{U}^-$ (since the image of this category in Lusztig’s category is tensor generated by Hecke correspondences), so $D(X^A)/N$ inherits a action of this category. Furthermore, on $D(X^A)/N$ each of the functors $\mathcal{F}_{\mu_i}^{(n)}$ has a natural biadjoint. This defines a second action of Lusztig’s category generated by functors $\mathcal{E}_{\mu_i}^{(n)}$.

We now establish the analogue of this fact for the analytic case:

**Lemma 4.5** There is an action of $\mathcal{U}^-$ on $D(X^A)/N$ induced by the 2-functor $\delta$, and an action of $\mathcal{U}^+$ by functors adjoint to these.

**Proof.** If $k$ is finite, this follows from the work of Zheng as described above. If $k = \mathbb{C}$, then morally, the proof is precisely the same as Zheng’s but slightly different phrasing is necessary. First, note that composing with $\delta$, we get an action of $\mathcal{U}_i^-$ on $D(X^A)$. We wish to show that this action preserves $N$. This is because in a short exact sequence, if the sub is unstable, the total space is as well. The exact same definition of $\mathcal{E}_i$ given in [Zheb] carries through as well; it is, of course, tempting to try to define it by simply reversing the correspondence that defines $\mathcal{F}_{\mu}$, but this doesn’t preserve the subcategory $N$. Instead, we Fourier transform until $i$ is a source, and then pullback to the subvarieties $\tilde{X}_{\mu_i}^A, \tilde{X}_{\mu_i}^{A'}$ defined as the loci where the map $x_{out} : V_i \to W_i \oplus \bigoplus_{i \to j} V_j$ is injective.

Given the projection maps

$\tilde{X}_{\mu_i}^A \xrightarrow{\tilde{p}_1} \tilde{X}_{\mu_i}^{A, \mu_{i}} \xrightarrow{\tilde{p}_2} \tilde{X}_{\mu_i}^{A - \mu_{i}, \mu_{i}}$,

we have that $\mathcal{F}_{i}^{(n)}$ is induced by $(\tilde{p}_2)_{\dagger} \tilde{p}_1^\ast$, so its biadjoint (up to shift) is $\mathcal{E}_i^{(n)} = (\tilde{p}_1)_\ast \tilde{p}_2^\ast$. Here we use freely the fact that both $\tilde{p}_i$ are projective space bundles, and thus both proper and flat. Now that we have restricted to the subspace where the outward map is injective, the total space of a exact sequence has an unstable subrepresentation.
As before, there is an obvious projection map $p$ preserves the image of $N$, and induces an endofunctor of $D(X^A)/N$. This issue is dealt with in a more canonical way by Li in [Li] by working with the left and right adjoints to the localization functor.

**Theorem 2.5.2**

We let $\mathfrak{Q}_\lambda$ be the smallest full, idempotent complete, triangulated subcategory of $D(X^A)/N$ closed under these monoidal actions containing the constant sheaf $\mathcal{V}$ on $X^A_\lambda = pt$, which Zheng denotes $\mathfrak{Q}_{c,pt}$.

This category has a more geometric definition. Let $X^A_\lambda$ be the moduli space of representations with flags $\cdots \subset V(j) \subset \cdots \subset V$ such that $V(j)/V(j-1)$ is 1-dimensional and concentrated on the node $i$. We can include multiplicities $\delta_i$ by requiring the dimension of $V(j)/V(j-1)$ to be this dimension instead. There is an obvious forgetful map $p_i : X^A_\lambda \to X^A_i$.

**Proposition 4.7**

The category $\mathfrak{Q}_\lambda$ is the smallest full, idempotent complete, triangulated subcategory of $D(X^A)/N$ which contains the pushforward $(p_i)_!\mathbb{K}_{X^A_\lambda}$ of the constant sheaf on $X^A_\lambda$ for all $i$.

**Proof.** This is essentially just a restatement of the definition; it’s immediate for the definition that

$$\mathcal{F}_{i_{n}} \cdots \mathcal{F}_{i_{1}} \mathcal{V} \cong (p_i)_!\mathbb{K}_{X^A_\lambda}[\dim X^A_\lambda],$$

so the desired category is contained in $\mathfrak{Q}_\lambda$. On the other hand, Zheng [Zheb] Theorem 2.5.2] has proven that $\mathcal{E}^{(m)}_i \mathcal{F}_{i_{n}} \cdots \mathcal{F}_{i_{1}} \mathcal{V}$ has a decomposition as we would expect for a 2-quantum group action, and thus our desired category is closed under both monoidal actions.

This can be generalized to sequences $\underline{\lambda}$; let $\underline{\lambda}_{(j)} = (\lambda_1, \ldots, \lambda_j)$ and $\lambda_{(j)} = \lambda_1 + \cdots + \lambda_j$ and $w_{(j)}$ the corresponding dimension vector. Picking a filtration of $k^w$ by the smaller $\Gamma$-graded spaces $k^{w_{(\ell)}}$, we obtain a filtration of $X^A$ by nested subvarieties $X^{\lambda_{(\ell)}}$. Using the pushforward functor, which we denote $I_j$, we can thus view all the categories $\mathfrak{Q}_{\lambda_{(\ell)}}$ as subcategories of $D(X^A)/N$, though this pushforward does not commute with the monoidal action, so the image is no longer closed under the monoidal action.

Given $(i, \kappa)$, we can define a more refined moduli space $X^A_{i,\kappa}$ which is the moduli space of representations with flags as with $X^A_i$ with the additional condition that if $m \leq \kappa(j)$, then the image of $V(m)$ under the shadow maps is in $k^{w_{(\ell)}}$. As before, there is an obvious projection map $p_{i,\kappa} : X^A_{i,\kappa} \to X^A_i$. This allows us to construct sheaves

$$\mathcal{F}_{(i,\kappa)} \mathcal{V} := \mathcal{F}_{i_{n}} \cdots \mathcal{F}_{i_{(j)+1}} I_f \mathcal{F}_{i_{(j)}} \cdots \mathcal{F}_{i_{(2)+1}} I_f \mathcal{F}_{i_{(2)}} \cdots \mathcal{F}_{i_{1}} I_f \mathcal{V} \cong (p_{i,\kappa})_!\mathbb{K}_{X^A_{i,\kappa}}[\dim X^A_i].$$
Definition 4.8 We let $Q_\Lambda$ be the smallest full, idempotent complete, triangulated subcategory of $D(X^\Lambda)/\mathcal{N}$ which contains $\mathcal{F}_{(i,\kappa)} \mathcal{V}$ for all $(i, \kappa)$.

Geometrically, this is roughly the subcategory of sheaves whose singular supports are contained in Nakajima’s tensor product quiver variety [Nak01].

Throughout this section, we fix a particular choice of $Q_{\ast,\ast}$, which coincides with the choice used in [VV, §3.3] and [Roub, §3.2.4]. This choice is forced on us by geometry and is of the following nature: we let $e_{ij}$ denote the number of edges oriented from $i$ to $j$ in our chosen orientation $\Omega$, and fix

$$Q_{ij}(u, v) = (-1)^{e_{ij}}(u - v)^{c_{ij}}.$$ 

We can construct a candidate action of $\mathcal{U}$ using these actions and the biadjunction of $E_i$ and $F_i$ (up to shift) defined by Zheng in [Zheb, Theorem 2.5.1].

Theorem 4.9 The actions of Lemma 4.5 combine to give an action of $\mathcal{U}$.

Proof. This follows immediately from [CL, Thm. 1.1], since we know that

- there are actions of $\mathcal{U}^+$ and $\mathcal{U}^-$ by functors which are left and right adjoint,
- the weight decomposition is given by the dimension vector of the representations,
- $K^0(Q_\Lambda)$ is integrable by [Zheb, 3.3.4].
- The negative degree maps from any simple perverse object to itself is trivial and the degree 0 maps are all scalar multiplication; thus the identity functor on $Q_\Lambda$ has trivial negative degree maps and degree 0 maps spanned by the identity.
- the decompositions (3) and (5) from [CL, Def. 1.2] are confirmed by Zheng [Zheb, 2.5.2].

4.2. Comparison with algebraic constructions. Having established the action of a 2-quantum group, we now wish to compare this construction to ours.

Proposition 4.10 $\text{Ext}^\bullet_{D(X^\Lambda)}\left(\bigoplus_{i,\kappa} \mathcal{F}_{i,\kappa} \mathcal{V}\right) \cong T^\Lambda_\Lambda$.

We note that the Ext-algebra, considered as an $A_\infty$-algebra, is formal (all higher differentials vanish) in the finite field case because it is pure (the Frobenius acts on the $i$th degree portions by $p^{i/2}$ for some fixed square root $p^{1/2} \in \mathbb{Q}_\ell$), so this is even an isomorphism of $A_\infty$-algebras. This fact is one of the points where is useful to consider the finite field perspective.

Formality in the $k = \mathbb{C}$ case follows from comparison theorems in étale cohomology (see, for example [BBD82, §6]), but can also be shown more directly by appealing to the formality of Borel-Moore homology on the fiber products whose BM homology is identified with the Ext-algebra above by [CG97, 8.6.7].
Proof. The proof is essentially the same as that of Vasserot-Varagnolo. The Ext-algebra can be identified with a convolution algebra in equivariant homology, which is free of the correct rank over $H^*(BT_v)$, the classifying space of the torus in $G_v$, which we identify with the polynomial algebra generated by $y_i$'s. Furthermore, from the usual formalism of convolution algebras, we obtain a faithful action on a sum of polynomial rings, which we need only check is the polynomial representation defined in the proof of Theorem 2.4.

The arguments of Vasserot-Varagnolo carry over directly to show the action of black strands. Thus we need only calculate the action of the red/black crossings.

A left crossing corresponds to pullback of equivariant cohomology to a subvariety which is an equivariant deformation retract, and thus is the obvious isomorphism between these polynomial rings.

A right crossing of the $m$th red strand and $i$th black strand, on the other hand, corresponds to pushforward from, and thus multiplication by the Euler class of the normal bundle of, the moduli stack of quiver representations with flags where the $i$th space of the flag lands in $k^{w(m-1)}$ inside the space where it lands in $k^{w(m)}$. This normal bundle is the bundle of maps from the $i$th tautological bundle to $k^{w(m)}/k^{w(m-1)}$, and so its Euler class is $y_k^{i,m}$. □

Applying the localization functor, we thus have a natural map

$$\phi : T_{\lambda} \rightarrow \text{Ext}_{\mathbb{C}_A}^\bullet \left( \bigoplus_{i,k} F_{i,k} \mathbb{V} \right).$$

**Theorem 4.11** The map $\phi$ is a surjection with kernel $K_{\lambda}$. Furthermore, thus

- If $k$ is finite, then $\mathbb{S}_{\lambda}$ is canonically equivalent to the derived category $D^b(\mathbb{S}_{\lambda})$.
- If $k = \mathbb{C}$, then $\mathbb{S}_{\lambda}$ is canonically equivalent to the category of dg-modules over $T_{\lambda}$ (considered as a dg-algebra with trivial differential) which are bounded below and finite-dimensional in each degree.

We should note that in the finite field case, this is an equivalence of graded categories in a slightly unusual way. It would be more compatible with the usual conventions of homological algebra to identify $\mathbb{S}_{\lambda}$ with bounded below perfect dg-modules over $T_{\lambda}$ equipped with a second grading as $T_{\lambda}$-modules that the differential preserves. This is, of course, the same as the category of complexes we describe by a simple skew of the gradings. In particular, if we forget the mixed structure on $\mathbb{S}_{\lambda}$, we arrive at the category of $T_{\lambda}$-dg-modules, as in the $k = \mathbb{C}$ case.

Proof. Consider a diagram with a violating strand (as illustrated in Figure [7]); the corresponding morphism factors through $F_{(i,k)}$ with $k(1) \neq 0$; this is supported on the unstable locus and is in $N$. Thus, the kernel contains $K_{\lambda}$. 55
Also note that we know the dimensions coincide, since the graded Euler form is uniquely specified by the condition that it is Hermitian, $U_q(g)$ invariant, and preserved by tensoring with highest weight vector.

Thus, we need only show surjectivity. First note that the subcategory of $\mathfrak{Q}_\lambda$ given by

- the objects $F_{i,\kappa} V$, together with any subobjects given by a projection which is in the image of $\tilde{T}^\lambda$, and
- the morphisms are given by the image of $\tilde{T}^\lambda$

itself carries an action of $\mathcal{U}$ along with the correct compatibilities with inclusion functors $I$, and so its graded Euler form must coincide with with that of the full category. This establishes the surjectivity, and thus the isomorphism.

The Ext-algebra is formal as an $A_\infty$-algebra, since it is a quotient of a formal algebra. Thus, the equivalence of categories follows from the fact that the $F_{i,\kappa} V$ generate $\mathfrak{Q}_\lambda$.

Of course, it is quite unsurprising that our final result (up to the issue of gradings) is independent of whether we work in the finite field or analytic situation. Since all our perverse sheaves arose from pushing forward constant sheaves, all local systems we see are Gauss-Manin connections and thus defined over the integers. Thus, we can apply the results of [BBD82, §6] to see that the fppf version of Lusztig’s category in the finite field situation and the analytic version in the complex situation are equivalent, and of course this equivalence sends Zheng’s thick subcategory to the subcategory supported on the unstable locus. Thus, we obtain a functor from the fppf to the analytic version of $\mathfrak{Q}_\lambda$, which Theorem 4.11 shows is an equivalence.

4.3. **Koszulity and geometry.** Recall that we call a graded artinian abelian category **Koszul** if there exists a complete (up to shift) collection of simples $S_i$ such that $\text{Ext}^*(S, S)$ for the sum $S = \oplus S_i$ has internal and homological gradings coincide. We call a graded algebra **Koszul** if its representation category is Koszul. The **Koszul dual** of a Koszul category is the category of finitely generated graded representations of $\text{Ext}^*(S, S)$.

It is tempting to conclude from Theorem 4.11 that $T^\lambda$ is Koszul, since we know that it can be realized as a graded Ext algebra with internal and homological gradings matching. However, this guess is easily disproven by examples. For example, $\mathbb{F}[x]/(x^n)$ appears for all $n$ in the case of $\mathfrak{sl}_2$, and this algebra is only Koszul if $n = 1$ (when $n = 2$, this algebra has a Koszul grading, but it is not the one we have chosen).

\[ \text{Many authors require Koszul algebras to be positively graded, but this is inconvenient for us. However, an algebra is Koszul in our sense if and only it is graded Morita equivalent to a positively graded algebra (given by the endomorphisms of the graded projective cover of $S$) that is Koszul in the usual sense.} \]
Let $\mathcal{Q}_\lambda$ be the heart of the perverse $t$-structure in $\mathcal{Q}_\lambda$. Then $\mathcal{F}_i \mathcal{V}$ is a collection of semi-simple objects in $\mathcal{Q}_\lambda$ in which every simple appear as a summand.

Let $S$ be the sum of one simple object from each isomorphism class of $\mathcal{Q}_\lambda$ and let $\mathcal{T}_\lambda = \text{Ext}^\bullet_{\mathcal{Q}_\lambda}(S, S)$. For a simple $L \in \mathcal{B}_\lambda$, we let $P_L$ be the projective cover of $L$ with the grading in which its appears in $P^\kappa_1$ for a stringy sequence $(i, \kappa)$.

**Proposition 4.12** $\mathcal{T}_\lambda \cong \text{End}(\bigoplus P_L)$.

**Proof.** This follows from the fact that if we apply the geometric functors for the stringy sequence to $\mathcal{V}$ we obtain a Verdier self-dual object which contains exactly one simple summand which is a shift of the simple corresponding to $P_L$. Thus, this simple must be itself Verdier self-dual and thus perverse. □

We note that this algebra is positively graded, and graded Morita equivalent to $\mathcal{T}_\lambda$.

**Theorem 4.13** The category $\mathcal{Q}_\lambda$ is naturally equivalent to the category of representations of the quadratic dual of $\mathcal{T}_\lambda$. If the inclusion $D^+(\mathcal{Q}_\lambda) \rightarrow \mathcal{Q}_\lambda$ is an equivalence, then $\mathfrak{B}_\lambda$ is a Koszul category and $\mathcal{Q}_\lambda$ is its Koszul dual.

**Proof.** The equivalence of $\mathcal{Q}_\lambda$ to $\mathcal{V}_\lambda$ sends elements of $\mathcal{Q}_\lambda$ to linear complexes of projectives over $\mathcal{T}_\lambda$ in the sense of [MOS09, §3]. Thus, [MOS09, Theorem 12] shows that $\mathcal{Q}_\lambda$ is equivalent to the representations of the quadratic dual of $\mathcal{T}_\lambda$.

The Koszul duality statement follows from [MOS09, Theorem 30]. □

**Conjecture 4.14** The inclusion $D^+(\mathcal{Q}_\lambda) \rightarrow \mathcal{Q}_\lambda$ is an equivalence if and only if the weights $\lambda_i$ are all miniscule.

In particular, we expect that if $\mathfrak{g} = sl_h$, and all the weights $\lambda_i$ are fundamental, then we have that $\mathcal{T}_\lambda$ is Koszul. Indeed, in the case of $\omega_1$, this follows from directly from geometry:

**Proposition 4.15** If $\mathfrak{g} = sl_h$ and $\lambda_i = \omega_1$, then the hypotheses of Theorem 4.13 hold, and so $\mathcal{T}_\lambda$ is graded Morita equivalent to a Koszul algebra; in fact, it is equivalent to a sum of blocks of category $\mathcal{O}$ for $\mathfrak{gl}_\ell$.

We will show a more general version of this equivalence with category $\mathcal{O}$ in the next section using algebraic methods; this geometric method has the advantage of showing in this case that $\mathcal{T}_\lambda$ is Koszul in the grading we have given it.

**Proof.** In this case, we need only work in the orientation where all arrows point towards $\omega_1$. An object is in $\mathcal{N}$ if and only if it supported on the bad locus in this orientation; this is essentially equivalent to Nakajima’s proof in [Nak94, §7] that in
this case, the quiver variety is isomorphic to the cotangent bundle of the partial flag variety. Thus, $\mathcal{Q}_A$ in this case is a category of perverse sheaves on the cotangent bundle to the flag variety. The IC complex of an any particular Schubert variety can be realized adding one red line, and then performing the $F_i$'s corresponding to the sizes of subspaces containing the line, adding another red line, doing the $F_i$'s for the sizes of subspaces which have larger intersection with the plane than the line, etc. This is the right number of simples, so this shows $\mathcal{Q}_A$ is precisely the Schubert smooth perverse sheaves.

The inclusion $D^+(\mathcal{Q}_A) \rightarrow \mathcal{Q}_A$ is an equivalence for Schubert smooth perverse sheaves by [BGS96, Corollaries 3.3.1-2]. □

5. Comparison to category $O$

5.1. Cyclotomic degenerate Hecke algebras. Now, we specialize to the case where $\mathfrak{g} \cong sl_n$. In this case, we can reinterpret our results in terms of the work of Brundan and Kleshchev [BK08, BK09] who have shown that in this case, the cyclotomic Khovanov-Lauda algebra is a cyclotomic degenerate affine Hecke algebra (cdAHA).

Recall that the degenerate affine Hecke algebra (dAHA) is the algebra with generators $x_1, \ldots, x_d$ and $w \in S_d$ such that

$$s_i x_j = x_{s_i j} s_i - \delta_{ji} + \delta_{ji+1} \quad x_i x_j = x_j x_i$$

for the simple reflections in $s_i \in S_d$ and the usual relations between permutations.

We have a natural action of $H_d$ on the $\mathfrak{gl}_N$ module $P \otimes V^\otimes d$ for any $\mathfrak{gl}_n$ representation $P$, where $V = \mathbb{C}^N$ is the defining representation of $\mathfrak{gl}_N$:

- $S_d$ acts on the $d$ copies of $V$, and
- $x_1$ acts by $C \otimes 1^{\otimes d-1}$ where $C$ is the Casimir element of $\mathfrak{gl}_N$.

We will be most interested in the case where $P$ is a certain parabolic Verma module for a parabolic $\mathfrak{p}$; in this case, by the definition of induction,

$$P \otimes V^\otimes d \cong U(\mathfrak{gl}_n) \otimes_{U(\mathfrak{p})} (W \otimes V^\otimes d)$$

for a finite dimensional representation $W$ of $\mathfrak{p}$.

**Definition 5.1** Parabolic category $O$, which we denote $O^\mathfrak{p}$, is the full subcategory of $\mathfrak{gl}_N$-modules with a weight decomposition where $\mathfrak{p}$ acts locally finitely.

Since induction sends finite-dimensional modules to $\mathfrak{p}$-locally finite modules, $P \otimes V^\otimes d$ lies in this category.

Attached to each parabolic $\mathfrak{p} \subset \mathfrak{gl}_N$, we have a unique composition $\pi = (\pi_1, \ldots, \pi_\ell)$ such that $\mathfrak{p}$ is conjugate to block-diagonal matrices for this composition (the composition $\pi$ can be recovered as the gaps in the finest flag $\mathfrak{p}$ preserves). These can be used to define a weight $\lambda = \sum_i \omega_{\pi_i} \in Y(\mathfrak{g})$; that is, $\lambda^i = \#\{i| \pi_i = j\}$.
Definition 5.2  The cyclotomic degenerate affine Hecke algebra is the quotient of the dAHA given by

\[ H^\lambda = \bigoplus_{d \geq 0} H_d / \left( \prod_{i=1}^{n} (x_1 - i)^{\lambda_i} \right). \]

This has a natural system of orthogonal idempotents \( e_d \) for all \( d \geq 0 \) which project to the image of \( H_d \). Brundan and Kleshchev show that when \( P \) is the parabolic Verma module associated to the “ground state” tableau on \( \pi \), then the action of dAHA on \( P \otimes V^{\otimes d} \) factors through its cyclotomic quotient.

Thus, we have a functor \( \text{Hom}_{\text{dAHA}}(P \otimes V^\otimes, -) : O^\pi \rightarrow H^\lambda -\text{mod} \). This functor is very far from being an equivalence, but on each block of \( O^\pi \) it is either 0, or fully faithful on projectives. Thus, certain blocks of \( O^\pi \) can be described in terms of endomorphism rings of modules over \( H^\lambda \).

In [BK09], Brundan and Kleshchev show that each category \( \mathcal{B}^\lambda_{\mu} \) is equivalent to a block of the representations of \( H^\lambda \). Thus, using this isomorphism, we can also express \( \mathcal{B}^\lambda_{\mu} \) in terms of endomorphisms of modules over \( H^\lambda \).

There is an idempotent of \( H_d \) associated to any length \( d \) sequence of integers. We let \( e_g \) be the sum of these idempotents corresponding to sequences of integers in \([1, n]\). In this section, we use the polynomials \( Q_{ij} \) as defined in the previous section for a fixed orientation of the type A (or later, affine type A) quiver. The most obvious choice is

\[ Q_{ij}(u, v) = \begin{cases} 1 & i \neq j \pm 1 \\ u - v & i = j + 1 \\ v - u & i = j - 1 \end{cases} \]

Proposition 5.3 ([BK09]) There is an isomorphism \( \Upsilon : T^\lambda \rightarrow e_g H^\lambda e_g \) such that \( \Upsilon(y_{j, i}) = e(i)(x_j - i_j) \).

5.2. Comparison of categories. First, let us endeavor to understand how we can translate the \( T^\lambda \)-modules \( y_{i, k} T^\lambda \) defined in Section 2.4 into the language of the cdAHA using \( \Upsilon \). It’s immediate from Proposition 5.3 that

\[ \Upsilon(y_{i, \kappa}) = e(i) \prod_{j=1}^{\ell} \prod_{k=\kappa(j)+1}^{n} (x_k - i_k)^{\lambda_j^k}. \]

However, the strong dependence of this element on \( e(i) \) makes it problematic for use in the Hecke algebra.

We first specialize to the case where \( \lambda_j = \omega_{\pi_j} \) for some \( \pi_j \). As suggested by the notation, we will later want to think of \( \pi_j \) as a composition. This bit of notation allows us to associate to each \( \kappa \) an element of \( H^\lambda_{\mu} \) (note that there is no dependence
(5.5) \[ z_\kappa = \prod_{j=1}^\ell \prod_{k=1}^{\kappa(j)} (x_k - \pi_j) \]

We let \( M^\kappa_i = e(i)z_\kappa H^{\lambda,n} \) and \( M^\kappa = z_\kappa H^{\lambda,n} \).

**Proposition 5.4** For all \( i \), we have \( y_{i,\kappa} H^{\lambda,n} = M^\kappa_i \). In particular, we have an isomorphism \( T^\lambda \cong \text{End}(\oplus_{\kappa} M^\kappa) \).

**Proof.** If \( a \neq i,j \), then we can rewrite \( e(i) \) as

\[ e(i) = (x_j - a)e(i) \left( \frac{-1}{a - i_j} \right) - \frac{x_j - i_j}{(a - i_j)^2} - \frac{(x_j - i_j)^2}{(a - i_j)^3} - \cdots \]

since \( (x_j - i_j)e(i) \) is nilpotent. It follows that

(5.6) \[ e(i)(x_k - \pi_j)H^{\lambda,n} = e(i)(x_k - i_k)H^{\lambda,n} \]

since \( \lambda_k = \delta_{\pi,j,k} \). Thus, applying (5.6) to each term in \( z_\kappa \), the result follows. \( \square \)

We note that the modules \( M^\kappa \) are closely related to the permutation modules discussed by Brundan and Kleshchev in [BK08, §6]. Each way of filling \( \pi \) as a tableau such that the column sums are \( \kappa(i) - \kappa(i-1) \) results in a permutation module which is a summand of \( M^\kappa \).

Now we wish to understand how the modules \( M^\kappa \) are related to parabolic category \( O \). Let \( N = \sum_j \pi_j \) be the number of boxes in \( \pi \). As before, the \( \pi_i \) give a composition of \( N \), and thus a parabolic subgroup \( p \subset gl_N \), which is precisely the operators preserving a flag \( V_1 \subset V_2 \subset \cdots \subset V \). If, as usual, \( \kappa \) is a weakly increasing function on \([1,\ell]\) with non-negative integer values and further \( \kappa(\ell) \leq d \), then we let

\[ V^d_\kappa = V^{\otimes(1)}_1 \otimes V^{\otimes(2)-\kappa(1)}_2 \otimes \cdots \otimes V^{d-\kappa(\ell)} \]

as a \( p \)-representation. We can induce this representation to an object in \( O^p \) which we denote

\[ P^\kappa_d \cong U(gl_n) \otimes_{U(p)} (C_{-\rho} \otimes V^d_\kappa), \]

where \( C_{-\rho} \) is the 1-dimensional \( p \)-module defined in [BK08, pg. 4].

All the objects \( P^\kappa_d \) live in the subcategory we denote \( O^p_{>0} \) which is generated by all parabolic Verma modules whose corresponding tableau has positive integer entries. We also consider a much smaller subcategory which has only finitely many simple objects: let \( O^p_n \) be the subcategory of \( O^p \) generated by all parabolic Vermas whose corresponding tableau only uses the integers \([1,n]\). Let \( pr_n : O^p \to O^p_n \) be the projection to this subcategory \( (O^p_n \) is a sum of blocks, so there is a unique projection).

**Proposition 5.5** If one ranges over all \( \kappa \) and all integers \( d \), then \( \oplus_{\kappa,d} V^d_\kappa \) is a projective generator for \( O^p_{>0} \).
Proof. This follows from a simple modification of the proof of [BK08, Theorem 4.14]. In the notation of that proof, we have that $P^\kappa_d \cong R(\mathcal{P}_\kappa^{\ell}(t) \otimes \mathcal{C}_{-\rho}) \otimes V^{\otimes \kappa - \kappa(t)}$, where $\kappa^{\ell}$ is the restriction of $\kappa$ to $[1, \ell - 1]$. As noted in that proof, by induction, this is two functors which preserve projective modules applied to a projective module; thus $P^\kappa_d$ is projective.

Each of Brundan and Kleshchev’s divided power modules is a summand in one of the $P^\kappa_d$’s, as we noted earlier. Since any indecomposable projective of $\mathcal{O}_p$ is a summand of a divided power module, the same is true of the $P^\kappa_d$’s. □

**Proposition 5.6** For all $d, \kappa$, we have

\[
z_\kappa H^\lambda e_d \cong \text{Hom}(P \otimes V^{\otimes d}, P^\lambda_d) \\
M^\kappa e_d \cong \text{Hom}(P \otimes V^{\otimes d}, \text{pr}_\kappa(P^\lambda_d)).
\]

Proof. This rests on a single computation, which is that the image in $P \otimes V$ of the action of $\prod_{i=1}^{\ell} (x_1 - \pi_i)$ is $U(\mathfrak{gl}_n) \otimes_{U(\mathfrak{p})} (\mathcal{C}_{-\rho} \otimes V) \subset U(\mathfrak{g}_n) \otimes_{U(\mathfrak{p})} (\mathcal{C}_{-\rho} \otimes V) \cong P \otimes V$; this follows from [BK08, Lemma 3.3]. This shows that the image of $z_\kappa$ acting on $P \otimes V^{\otimes d}$ is $P^\lambda_d$, so by the projectivity of $P \otimes V^{\otimes d}$, every homomorphism to $P^\lambda_d$ factors through this one.

We can identify those homomorphisms whose image is in $\text{pr}_\kappa(P^\lambda_d) \subset P^\lambda_d$ as those killed by some power of $\chi^n_j = \prod_{i=1}^{n} (x_j - i)$ for each $j$ (if a number $m$ appears in a tableau, then $x_j - m$ is nilpotent for some $j$, and so if $m \notin [1, n]$, then $\chi^n_j$ is invertible for that $j$). Thus, this homomorphism space is the subspace of $z_\kappa H^\lambda e_d$ on which all $\chi^n_j$ act nilpotently, which is precisely $M^\kappa e_d$. □

**Corollary 5.7** We have an equivalence $\Xi : \mathcal{B}_\lambda \cong \mathcal{O}_p^n$.

We can generalize this statement a bit further: let us now consider the case where the weights $\lambda_i$ are not fundamental. In this case, to each weight $\lambda$, we have a unique Young diagram given by writing it as a sum of fundamental weights, and we obtain a pyramid $\pi$ by concatenating these horizontally (this is the pyramid associated earlier to the refinement of $\mathcal{A}$ into fundamental weights). We associate a parabolic $\mathcal{P}$ with the pyramid as before.

For each collection of semi-standard\footnote{In [BK08], these are called “standard.”} tableaux $T_i$ on each of these diagrams which only use the integers $[1, n]$, this gives a tableau on $\pi$ (now just column-strict). Such a tableau can be converted into a module in $\mathcal{O}_p$ for $\mathfrak{g}_N$ (where $N = \sum \pi_i$) by taking the projective cover of the $\mathcal{P}$-parabolic Verma module corresponding to this tableau. Let $\mathcal{O}_p^n$ be the subcategory of modules presented by these projectives.
Proposition 5.8 The functor \( \Xi \) induces an equivalence of \( O^p_\lambda \) and \( \mathfrak{g}_\lambda^\Delta \).

Proof. What is clear from Corollary 5.7 is that \( \mathfrak{g}_\lambda^\Delta \) is equivalent to the subcategory of \( O^p_\lambda \) consisting of objects presented by projectives \( \text{pr}_n(P_d^\lambda) \) for the sequence of weights obtained by breaking \( \lambda \) into fundamental weights, where we require \( \kappa \) to be constant on the blocks of fundamental weights obtained by breaking up \( \lambda_i \). In terms of category \( O \), we only induce finite-dimensional \( p \) vector spaces obtained by tensoring the vector spaces which appear in a particular flag preserved by \( \mathfrak{g}_\lambda \), the gaps of which encode the sequence \( \lambda_i \).

That is, the indecomposable projectives of \( \mathfrak{g}_\lambda^\Delta \) are sent to the indecomposable projectives which appear as summands of these \( \text{pr}_n(P_d^\lambda) \). Thus these are in bijection, and there can only be \( \dim V_\lambda \) of the latter. Since there is exactly that number of tableaux which are semi-standard in blocks as described above, we need only show that these occur as summands.

This follows from the relationship between the crystal structure on tableaux and projectives in category \( O \). Specifically, since any tableau which is semi-standard in blocks can be obtained from the empty tableau by the operations of attaching a fresh Young diagram filled with the ground state tableau and of applying crystal operators, the argument from [BK08, Corollary 4.6] shows that the projective corresponding to such a tableau is a summand of an appropriate \( P^\kappa_d \).

We note that this shows that our categorification corresponds to that for twice fundamental weights of \( \mathfrak{sl}_n \) recently given by Hill and Sussan [HS].

The category \( O^p \) has a natural endofunctor given by tensoring with \( V \). Restricting to \( O^p_n \) we can take the functor \( f_* = \text{pr}_n(- \otimes V) \). This functor has a natural decomposition \( f_* = \bigoplus_{i=1}^n f_i \) in terms of the generalized eigenspaces of \( x_1 \) acting on \( - \otimes V \); we need only take \( i \in [0, n] \) since these are the only eigenvalues of \( x_1 \) on the projection to \( O^p_n \).

Proposition 5.9 We have a commutative diagram

\[
\begin{array}{ccc}
O^p_n & \xrightarrow{f_i} & O^p_n \\
\downarrow{\Xi} & & \downarrow{\Xi} \\
\mathfrak{g}_\lambda^\Delta & \xrightarrow{\tilde{\gamma}_i} & \mathfrak{g}_\lambda^\Delta
\end{array}
\]

Proof. The functor \( f_* \) corresponds to tensoring a \( H^{\lambda,n}_d \)-module with \( H^{\lambda,n}_{d+1} \). This in turn corresponds to all ways of going from \( d \) black strands to \( d + 1 \), that is the functor \( \bigoplus_{i=1}^n \tilde{\gamma}_i \). Via Brundan and Kleshchev’s isomorphism, \( x_n \) acts on \( \tilde{\gamma}_i M \) for any \( M \) by
For any parabolic subalgebra $q \supset p$ with Levi $l = q/\text{rad } q$, we have an induction functor $\text{ind}_{l}^{gl} \overset{\text{def}}{=} U(gl_{N}) \otimes U(q) - : O^{p}(l) \to O^{p}$ where $O^{p}(l)$ denotes the parabolic category $O$ for $l$ and the parabolic $p/\text{rad } q$ (here $l$-representations are considered as $q$ representations by pullback).

Choices of $q$ are in bijection with partitions of $\lambda$ into consecutive blocks $\lambda_{1}, \ldots, \lambda_{k}$. Let $\Xi_{l} : \mathfrak{X}_{\lambda_{1}, \ldots, \lambda_{k}} \to O^{p}(l)$ be the comparison functor analogous to $\Xi$ for $l$.

**Proposition 5.10** We have a commutative diagram

\[
\begin{array}{ccc}
O^{p}_{l}(l) & \xrightarrow{\text{ind}_{l}^{gl}} & O^{p}_{n} \\
\Xi_{l} \downarrow & & \downarrow \Xi \\
\mathfrak{X}_{\lambda_{1}, \ldots, \lambda_{k}} & \xrightarrow{\mathfrak{S}_{\lambda_{1}, \ldots, \lambda_{k}}} & \mathfrak{X}_{\lambda}
\end{array}
\]

*Proof.* We need only check this on projectives: consider a representation of $l$ given by an exterior product of projectives in category $O$ for each of its $gl_{j}$-factors

\[P = P_{1} \boxtimes \cdots \boxtimes P_{k}.
\]

Then the induction $\text{ind}_{l}^{gl} P$ is a quotient of the projective $P'$ corresponding to the concatenation $T$ of the tableaux $T_{i}$ for the $P_{i}$. The kernel is the image of all maps from projectives higher than $T$ in Bruhat order through a series of transpositions which change the content of at least one of the $T_{i}$.

Similarly, the standardization $\mathfrak{S}_{\lambda_{1}, \ldots, \lambda_{k}}(\Xi_{l}^{-1}(P))$ is a quotient of $\Xi_{l}^{-1}(P')$; the kernel is the image of all maps from projectives that correspond to idempotents for sequences where at least one black strand has been moved left from one block to the other. Thus, these functors agree on the level of projective objects.

They agree on morphisms between projectives because of the compatibility of the isomorphism $\Upsilon$ with the natural inclusions $H_{d'} \otimes H_{d''} \hookrightarrow H_{d'+d''}$ and $R(v') \otimes R(v'') \hookrightarrow R(v' + v'')$.

Some care is required here on the subject of gradings. Brundan and Kleshchev’s results relating category $O$ to Khovanov-Lauda algebras are ungraded; they imply no connection between the usual graded lift of $\tilde{O}^{p}$ of category $O$ and the graded category of modules over $T_{\lambda}$. However, such an equivalence does exist.

**Proposition 5.11** The equivalence $\Xi$ has a graded lift.
Figure 17. The idempotent corresponding to $S_k$.

**Proof.** Obviously, it’s enough to check this for the case where $\Lambda$ only contains fundamental weights.

In the case when $\lambda_i = \omega_1$ for all $i$, we know by Proposition 4.13 that $T_\Lambda$ is Koszul with its usual grading, and since Koszul gradings on basic algebras are unique up to isomorphism [BGS96, Corollary 2.5.2], this agrees with the standard grading. In fact, this shows that furthermore, the grading on $U$ gives a graded lift of projective functors.

Now, recall that $O^n_n$ is the subcategory of $O_n$ which only uses a certain set of simple modules in its composition series: exactly those in the subcrystal corresponding to the inclusion

$$\wedge^{\pi_1} C^n \otimes \cdots \otimes \wedge^{\pi_i} C^n \subset (C^n)^{\otimes n}.$$  

Now, we claim that $\mathfrak{B}_\Lambda$ is graded equivalent to the subcategory of $\mathfrak{B}^N$ corresponding to the same subcrystal, closed under the action of $U$. To spare ourselves a great deal of unpleasantness with signs, we apply the isomorphism, discussed in the final section of [KLb], to the same category with $Q_{i,i+1}(u,v) = u + v$. We should note that this isomorphism only works in finite type $A$; in affine type $A$ or any other generalized Dynkin diagram with cycles, there is no such obvious isomorphism.

The desired equivalence $G$ sends $P^\kappa_i$ to

$$\mathfrak{F}(\lambda) \otimes \cdots \otimes \mathfrak{F}(\lambda) \otimes \mathfrak{F}(\lambda) \otimes L_{\pi_1} \otimes L_{\pi_1}$$

where $L_{\kappa}$ is the unique simple of $\mathfrak{B}^N$ which is of weight $\omega_k$ and highest weight (i.e., killed by all $E_i$). This simple is the unique simple quotient of the standard module $S_k$ for the sequence $(1, 2, \ldots, k - 1, 1, 2, \ldots, k - 2, 1, \ldots, k - 3, \ldots, 1, 2, 1)$, with a red strand labeled with $\omega_1$ coming before each 1, and at the end, that is, with

$$k(m) = k(m - 1) - m(m - 1)/2.$$  

Under the functor $\Xi$, these are sent to the objects $\Xi_\nu(P^\kappa_i)$, where $\Xi_\nu$ is the equivalence $\mathfrak{B}^N \cong O^n_n$ we defined earlier (to distinguish it from that for the larger category $\mathfrak{B}^N$); so assuming that we can define such a functor $G$ which is full and faithful, it will define an equivalence with the correct subcategory.

Now, we must consider morphisms. The functor $G$ sends a morphism $a : P^\kappa_i \rightarrow P^\kappa'_i$ to the morphism obtained by simply exploding every red strand labeled with $\omega_k$ to a copy of the simple $L_k$, times a sign. To each left crossing of strands labeled with $\omega_k$ and $\alpha_g$, we attach $\max(k - g, 0)$, and the necessary sign is $-1$ raised to the sum of
all these weights over all left crossings. Let us say a little more precisely what this means: \( G(P_i^K) \) is a quotient of

\[
\tilde{\delta}_{i_0} \cdots \tilde{\delta}_{i_{d(t)-1}} S(\tilde{\delta}_{i_{d(t)}} \cdots \tilde{\delta}_{i_{d(t)-1}}) (S(\cdots) \otimes S_{\pi_{d(t)-1}}) \otimes S_{\pi_{d(t)}}
\]

in a natural way, and the morphism between two such modules defined by an “exploded” diagram (such a morphism exists, because an exploded diagram never has a left crossing involving strands in the \( L_k \)) descends to a morphism between the corresponding modules of the form \( G(P_i^K) \).

We need only check that this map behaves well under composition. Of course, relations not involving red lines give no trouble. Now, consider the bigon relation (2.2), which says that if we straighten a strand labeled with \( \alpha_g \) which makes a bigon over a strand labeled with \( \omega_k \), we obtain a dot on that strand if \( g = k \), and otherwise, it pulls across with no dots. We must consider 4 cases separately:

- If the black line is labeled with \( \alpha_g \) for \( g > k \), the relation is trivial.
- If \( g = k \), then one can straighten the additional black strand left until it strikes the only strand in the diagram labeled with \( \alpha_{k-1} \). Pulling this straight creates a term where the \( \alpha_{k-1} \) strand carries a dot, which is 0, since this kills \( L_k \) and one where the \( \alpha_k \) strand carries a dot, which proceeds unimpeded out of the diagram.
- If \( 1 < g < k \), then in each group between red lines, the only action occurs in pulling across a sequence of the form \((\ldots, g-1, g, g+1, \ldots)\). That this pulls through, only picking up a sign, is dealt with in Figure 18. Finally, we reach the group that ends \((\ldots, g-1, g)\), which we thus leave with the additional black attached to the \( \alpha_g \) labeled strand in that block, like the bottom left step of Figure 18 but with the \( \alpha_{g+1} \) absent.
  
  We can pull the crossing through until it reaches the end of the next block, which is \((\ldots, g-1)\), at which point we apply the calculation is the bottom line of Figure 18 to remove the crossing. Since the number of signs picked up is \( k - g \), this accounts for the sign we introduced earlier.
- If \( g = 1 \), then the argument is precisely the same with the sequence \((g - 1, g, g + 1)\) replaced with the red line and \((1, 2)\).

The only other relation whose proof is non-trivial is the first line of (2.1), where we assume the crossing strands are labeled with \( \alpha_g \),

- If \( g > k \), the the crossing commutes past all strands that appear.
- If \( g = k \), then this follows from the analogous QHA relation when this crossing strikes the unique line labeled with \( \alpha_{k-1} \).
- If \( g < k \), then we must just show that all correction terms that appear when sliding through are trivial. In all cases, they will contain a bigon in a single color with no dots, so they are all 0.
Thus, the graded equivalence between $\mathfrak{g}^{\lambda N}$ and $O_n$ induces a graded Morita equivalence between $\mathfrak{g}^\Delta$ and $O_n^\Delta$ (note that we do not claim that this is the equivalence $\Xi_n$), so $\mathfrak{g}^\Delta$ is Koszul. We use the uniqueness of Koszul gradings again to show that $\Xi_n$ sends the grading on $\mathfrak{g}^\Delta$ to a standard grading on $O_n^\Delta$.

We note that both the action of projective functors and of the induction functors have graded lifts which are unique up to grading shift, and thus are determined by their action on the Grothendieck group. Thus the graded lifts given by the action of $\mathcal{U}$ and $\mathcal{S}$ agree, up to an easily understood shift, with those used in other papers on graded category $O$ (most importantly for us, the work of Mazorchuk-Stroppel [MSa] and Sussan [Sus07]) on link homologies, which we address in this paper's sequel [Web]).

5.3. The affine case. We note that the constructions of the previous subsection generalize in an absolutely straightforward way to the affine case by simply replacing the results of Section 3 of [BK09] with Section 4.

We let $\hat{H}_d$ denote the affine Hecke algebra (not the degenerate one we considered earlier). Fix an element $\zeta \in \overline{k}$, the separable algebraic closure of $k$ such that

$$1 + \zeta + \zeta^2 + \cdots + \zeta^{n-1} = 0,$$

and $n$ is smallest integer for which this holds (for example, if $k$ is characteristic 0, these means that $\zeta$ is a primitive $n$th root of unity). The cyclotomic affine Hecke algebra or Ariki-Koike algebra (introduced in [AK94]) for $\lambda$ is the quotient

$$\hat{H}^\lambda = \oplus_d \hat{H}_d / \langle (x_i - \zeta)^{\epsilon(\lambda)} \rangle,$$

where we adopt the slightly strange convention that if $\zeta \in \mathbb{Z}$, then $\zeta^i = \zeta + i$, and otherwise it is the usual power operation.
Theorem 5.12 ([BK09, Main Theorem]) When $\mathfrak{g} \cong \hat{\mathfrak{sl}}_n$, there is an isomorphism $T^\lambda \cong \hat{\mathcal{H}}^\lambda$.

This symmetric Frobenius algebra has a natural quasi-hereditary cover, called the **cyclotomic $q$-Schur algebra**, defined by Dipper, James and Mathas [DJM98]. Indecomposable projectives over this algebra are indexed by ordered $k = \sum_{i=0}^n \alpha_i^\vee(\lambda)$-tuples of partitions.

**Proposition 5.13** When $\mathfrak{g} = \hat{\mathfrak{sl}}_n$, then $\mathfrak{g}^\lambda$ is equivalent to the subcategory of representations of the cyclotomic $q$-Schur algebra consisting of objects presented by certain projective modules.

If all $\lambda_i$ are fundamental, then these are exactly the projectives for the multipartitions where each constituent partitions are $n$-regular.

In general, we break the multipartition into smaller ones consisting of the first $k_1 = \sum_{i=0}^n \alpha_i^\vee(\lambda_1)$ partitions, the next $k_2$, etc, and take the projectives for multipartitions where each of these smaller multi-partitions is $n$-Kleshchev.

**Proof.** By Corollary 3.22, $T^\lambda$ is the endomorphism algebra of certain modules over $T^\lambda$, which one can see by the same arguments as Proposition 5.6 are of the form $\hat{\zeta}_i T^\lambda$ where

$$\hat{\zeta}_k = \prod_{j=1}^\ell \prod_{k=1}^{\nu(j)} (x_k - \zeta^{\nu(j)}).$$

These are permutation modules for the Ariki-Koike algebra, exactly those corresponding to multi-partitions where all constituent partitions have all parts size 1. Thus, in the case where all $\lambda$’s are fundamental, the category of modules over $T^\lambda$ is the subcategory of representations of the cyclotomic $q$-Schur algebra generated by summands of these, and in the case where not all representations are fundamental, we must restrict these projectives further.

The descriptions above follow from the fact that for the permutation module of the multipartition where all parts are 0 except for the last, which has all parts 1, the indecomposable projectives which appear are exactly those for $n$-Kleshchev multipartitions. \qed

Thus, our categorification can be seen a generalization of the Ariki categorification theorem [Ari96]. As mentioned in the introduction, the author and Stroppel will address the question of how to describe the entirety of the cyclotomic $q$-Schur algebra diagrammatically in a future paper [SW].
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