GRAPHS OF PARTITIONS AND RAMANUJAN’S $\tau$-FUNCTION

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Abstract. The invariant $z_\lambda$ attached to a partition $\lambda$ sits in the denominator of the Girard-Waring solution to Newton’s symmetric function relations. We interpret Ramanujan’s $\tau$-function in terms of $z_\lambda$, and interpret $z_\lambda$ in terms of the automorphisms of a graph.

1. Introduction

The partition function $p$ and Ramanujan’s $\tau$-function are related by similar generating functions and consequently by similar recursions involving the sum-of-divisors function $\sigma(n)$. These recursions are formally similar to Newton’s symmetric function relations, and one of our goals in this paper is to point out that the Girard-Waring solution of Newton’s relations yields a formula for $\tau(n)$ as a sum over the partitions of $n$. (We will also reproduce for comparison a similar result of Bruniere, Kohm and Ono in this direction.) The terms of the sum are fractions; the numerators are multiples of various $\sigma(k)$ and the denominators are $z_\lambda$, a standard partition invariant. It will become apparent that analyzing the terms of the sum algebraically is an unattractive task; they are too intricate. On the other hand, $z_\lambda$ is a combinatorial object, so it is natural to hope that $z_\lambda$ counts something. Our second aim in this paper is to interpret $z_\lambda$ as a counting function. The objects it counts are automorphisms of graphs representing partitions. The physicist C. M. Bender and his co-authors interpreted Bell numbers in terms of automorphisms of Feynman diagrams that represent partitions [B-B-M1,2]. These Feynman diagrams suggested the structure of the graph representation we are going to describe.

2. Background

A. The cusp form $\Delta$. This is the modular form on $\mathfrak{H}$, the upper half of the complex $z$-plane; the only facts we require are, first, that it has a product expansion in terms of the variable $q = q(z) = \exp(2\pi iz)$, namely,

$$\Delta(z) = q \prod_{n=1}^{\infty} (1 - q^n)^{24},$$

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and, second, that it is the generating function of $\tau$:

\[(2-1)\quad q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \sum_{n=1}^{\infty} \tau(n)q^n.\]

**B. Ramanujan’s recursion.** Ramanujan obtained a recursion for $\tau(n)$ by doing logarithmic differentiation on the product expansion of $\Delta$ [R]. We state the result of this technique in a more general setting. It has been codified as follows [A]:

If

\[(2-2)\quad \prod_{n=1}^{\infty} (1 - x^n)^{f(n)/n} = \sum_{n=0}^{\infty} r(n)x^n\]

then

\[(2-3)\quad nr(n) + \sum_{k=1}^{n} r(n - k) \sum_{d|k} f(d) = 0.\]

Let $x = q, f(n) = 24n, r(n) = \tau(n + 1)$. Then (2-1) is the hypothesis in this result and so, by (2-3),

\[(2-4)\quad n\tau(n + 1) + 24 \sum_{k=1}^{n} \tau(n + 1 - k)\sigma(k) = 0\]

for $n = 1, 2, \ldots$.

**C. Solving recursions.** Equation (2-3) is solved by means of the Girard-Waring formula. Our exposition follows [M] and all the facts we cite from the theory of symmetric functions appear there. A partition $\lambda$ of a positive integer $n$ is represented by a non-increasing list $(\lambda_1, \lambda_2, \ldots, \lambda_i, \ldots)$ with $\sum \lambda_i = n$; we call this a *list representation* to distinguish it from the graph representations introduced below. A variable number of zero terms are permitted in list representations, so they aren’t unique. We write $l(\lambda)$ (“length” of $\lambda$) for the number of positive parts and $|\lambda|$ for $n$ (“content” of $\lambda$). The multiplicity of a positive number $k$ in $\lambda$ is the number of occurrences of $k$ in a list representation of $\lambda$. This is well-defined and is denoted as $m_k(\lambda)$. The invariant $z_\lambda$ is defined as follows:

$$z_\lambda = \prod_k k^{m_k(\lambda)} m_k(\lambda)!.$$

Partitions operate on sequences as follows. If $\{s\}$ is a sequence with $s_0 = 1$, then

$$s_\lambda = \prod_i s_{\lambda_i}.$$
The condition \( s_0 = 1 \) ensures that the definition is independent of the choice of list-representation of \( \lambda \).

MacDonald proves the following version of the Girard-Waring formula for certain families of symmetric functions \( h_n, p_n \) indexed by \( n \). (Their arguments \( x_i \) are suppressed.) These families satisfy the recursion

\[
(2-5) \quad nh_n = \sum_{r=1}^{n} p_r h_{n-r},
\]

from which he shows it follows that

\[
h_n = \sum_{|\lambda|=n} p_{\lambda}/z_\lambda.
\]

MacDonald comments that the symmetric functions \( h_n \) are algebraically independent over \( \mathbb{Z} \), so we may specialize them in any way, and forget about the original variables \( x_i \). This remark also applies to the \( p_n \) because they are algebraically independent over \( \mathbb{Q} \). Replacing the symmetric functions \( h_n, p_n \) with arithmetic functions \( r(n), F(n) \), respectively, we get

**Lemma 2.1.** If we make partitions operate on arithmetic functions by writing \( F_\lambda = F(\lambda_1) \cdot F(\lambda_2) \ldots = \prod_k F(k)^{m_k(\lambda)} \) and we have an identity

\[
(2-6) \quad nr(n) = \sum_{k=1}^{n} F(k)r(n-k),
\]

then we also have an identity

\[
r(n) = \sum_{|\lambda|=n} F_{\lambda}/z_\lambda.
\]

In our applications of this fact, typically \( F \) is a divisor sum \( F(k) = \sum_{d|k} f(d) \).

**D. Graph automorphisms.** An automorphism of a graph \( g \) is a graph isomorphism of \( g \) to itself. We can identify graph automorphisms with certain permutations of the vertex set as follows. We define an action of the symmetric group \( S_n \) on size \( n \) square matrices \( M = (M_{i,j}) \) by setting \((\rho(M))_{i,j} = M_{\rho(i),\rho(j)}\) for \( \rho \in S_n \). Then we identify a graph \( g \) on \( n \) vertices with its adjacency matrix and say that \( \rho \) is an automorphism of \( g \) if \( g = \rho(g) \).

3. The sum-decomposition of \( \tau(n) \)

The substitutions \( h(n) = \tau(n+1) \) and \( f(k) = -24\sigma(k) \) transform equation (2-4) into equation (2-6) and so, by Lemma 2.1,
Theorem 3.1.

\[(3-1) \quad \tau(n + 1) = \sum_{|\lambda| = n} \frac{(-24)^{l(\lambda)}}{z_\lambda} \prod_k \sigma(k)^{m_k(\lambda)}.\]

Other modular forms have convenient product decompositions and as a result their Fourier expansion coefficients can be decomposed into sums like the right side of (3-1). Some of these modular forms are eigenfunctions of Hecke operators with Fourier coefficients constituting the values of multiplicative functions like \(\tau\).

We also want to mention that because of equation (2-5),

\[(3-2) \quad h_n = \frac{1}{n!} \begin{vmatrix} p_1 & -1 & 0 & \ldots & 0 \\ p_2 & p_1 & -2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_{n-1} & p_{n-2} & \ldots & \ldots & -n + 1 \\ p_n & p_{n-1} & \ldots & \ldots & p_1 \end{vmatrix}.

In view of Lemma 2.1 this fact translates to a determinant expression for Ramanujan’s function:

Theorem 3.2.

\[(3-2) \quad \tau(n + 1) = \frac{(-1)^n}{n!} \begin{vmatrix} 24\sigma(1) & 1 & 0 & \ldots & 0 \\ 24\sigma(2) & 24\sigma(1) & 2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 24\sigma(n-1) & 24\sigma(n-2) & \ldots & \ldots & n - 1 \\ 24\sigma(n) & 24\sigma(n-1) & \ldots & \ldots & 24\sigma(1) \end{vmatrix}.

As far as we know, the original appearance of equation (3-2) as a solution of equation (2-5) occurs in the theory of partitions, not the theory of symmetric functions, and is due to Brioschi [Br,D].

It is interesting to compare Theorem 3.1 with Theorem 3 of [B-K-O]. Let \(\mathcal{F}\) denote the usual fundamental domain of the action of \(SL_2(\mathbb{Z})\) on \(\mathcal{F}\). Let \(j(z) = q^{-1} + 744 + 196884q + \ldots\) denote the usual elliptic modular function on \(SL_2(\mathbb{Z})\). Let \(j_0(z) := 1\), and for every positive integer \(m\) let \(j_m(z)\) be the unique modular function which is holomorphic on \(\mathcal{F}\) whose Fourier expansion is of the form

\[j_m(z) = q^{-m} + \sum_{n=1}^{\infty} c_m(n)q^n.

The [B-K-O] authors remark that \(j_m(z) = j_1(z)T_0(m)\), where \(T_0(m)\) is the usual \(m\)th weight zero Hecke operator, and they give the first few \(j_m\) as \(j_0(z) = 1, j_1(z) = j(z) - 744, j_2(z) = j(z)^2 - 1488j(z) + 159768\), etc. Their Theorem 3 then states:
For every \( n \geq 2 \) define \( F_n(x_1, ..., x_{n-1}) \in \mathbb{Q}[x_1, ..., x_{n-1}] \) by

\[
F_n(x_1, ..., x_{n-1}) :=
- \frac{2x_1\sigma_1(n - 1)}{n - 1} + \sum_{m_1, ..., m_{n-2} \geq 0 \at op.\} \sum_{m_1 + 2m_2 + ... + (n-2)m_{n-2} = n-1} (-1)^{m_1 + ... + m_{n-2}, m_1 + ... + m_{n-2} - 1} \cdot \frac{m_1!...m_{n-2}!}{m_1!...m_{n-2}!} \cdot x_2^{m_1} ... x_{n-1}^{m_{n-2}}.
\]

If \( f = q + \sum_{n=2}^{\infty} a_f(n)q^n \) is a weight \( k \) meromorphic modular form on \( SL_2(\mathbb{Z}) \), then for every integer \( n \geq 2 \) we have

\[
a_f(n) = F_n(k, a_f(2), ..., a_f(n-1)) - \frac{1}{n-1} \sum_{\tau \in \mathbb{H}} e_{\text{ord}_\tau} \cdot j_{n-1}(\tau).
\]

In particular, the authors of [B-K-O] remark that (since \( \Delta \) has no zeros in \( \mathbb{H} \)) for every \( n \geq 2 \):

\[
\tau(n) = F_n(\tau(2), ..., \tau(n-1)).
\]

4. Graph Automorphisms Counted by \( z_\lambda \)

In this section we give the details of a combinatorial interpretation of \( z_\lambda \). (We don’t know if it possible to handle the numerator on the right side of (3.1) with the same tools.) In [B-B-M1] a partition \( \lambda \) is represented by a graph with \( l(\lambda) \) connected components, one for each positive part. The \( i \)th component of this graph is a complete graph on \( \lambda_i \) vertices. Thus the representation of \( \lambda \) comprises \( m_j(\lambda) \) complete graphs \( K_j \) for each positive integer \( j \). Graphs composed of completely connected components are called transitive graphs in [B-B-M1]. Obviously, there is a bijection between the transitive graphs on \( n \) vertices and the partitions of \( n \). Physicists define the symmetry number of a graph as the reciprocal of the number of its automorphisms. Bender and his co-authors showed that the sum of the symmetry numbers of the graphs representing unrestricted partitions of \( n \) is \( B(n)/n! \) where \( B(n) \) is the \( n \)th Bell number. (Bell numbers count partitions of labeled sets.) They proposed the project of finding similar interpretations of other combinatorial functions in terms of Feynman diagrams. In this section, we represent the partitions \( \lambda \) by another family of graphs and show that the functions \( 1/z_\lambda \) are the symmetry numbers of these graphs. But we make no claim that the graphs are Feynman diagrams.

To each positive part \( \lambda_i \) of a partition \( \lambda \) we associate a directed graph \( g(\lambda_i) \) as follows. If \( \lambda_i \geq 2 \) then \( g(\lambda_i) \) is a directed cycle on \( \lambda_i \) vertices. (Thus \( g(2) \) has two edges.) If \( \lambda_i = 1 \) then \( g(\lambda_i) \) is a directed loop on one vertex. Finally, we associate to the partition \( \lambda \) the graph \( G(\lambda) = \cup_i g(\lambda_i) \). We choose directions for the edges by viewing \( G(\lambda) \) as an adjacency matrix and insisting that at most \( l(\lambda) \) nonzero entries, representing at most one edge from each component, lie below the diagonal. (If \( \lambda_i \geq 2 \), there should be exactly one nonzero entry representing an edge of the corresponding component below the diagonal; of course if \( \lambda_i = 1 \) then the associated loop is represented by an entry \( = 1 \) on the diagonal.) The number
of vertices of $G(\lambda)$ is $|\lambda|$. The number of connected components of $G(\lambda)$ is $l(\lambda)$, one for each positive part $\lambda_i$. There are $m_i(\lambda)$ components of size $\lambda_i$ (measured in either vertices or edges), one for each part of size $\lambda_i$. Graph automorphisms of $G(\lambda)$ belong to the symmetric group $S_{|\lambda|}$. (Robin Chapman has pointed out to the number-theorist author that “the graph $G(\lambda)$ is just the directed graph associated to a permutation, and its automorphisms are the permutations commuting with that given permutation.”)

The main result of this section is

**Theorem 4.1.**

The symmetry number of $G(\lambda)$ is $1/z_\lambda$.

*Proof.* Let $|\lambda| = n$ and $\rho \in S_n$ be a graph automorphism of $G(\lambda)$. Then, for each $i$, $\rho$ must map $g(\lambda_i)$ to an isomorphic component $g(\lambda_j)$ with $\lambda_i = \lambda_j$. If it happens that $i = j$, then the restriction $\rho|_{g(\lambda_i)}$ is a member of the subgroup of size $\lambda_i$ in the dihedral group $D_{\lambda_i}$ that acts by rotation. (Permutations that act by reflection change the orientation of the graph and the definition of automorphism rules this out.) To summarize: $\rho$ acts on $G(\lambda)$ by rotating components and permuting components of equal size. There are $m_k(\lambda)!$ permutations of the components of size $k$, and there are $k$ rotations for each component of size $k$, for a total of $k^{m_k(\lambda)}$ combinations of rotations of the components of size $k$. Hence there are $m_k(\lambda)!k^{m_k(\lambda)}$ possible restrictions of $\rho$ to the components of size $k$. To count the graph automorphisms of $G(\lambda)$, we multiply this expression over all positive integers $k$ and get $z_\lambda$.

\[ \Box \]

We have the following analogy to the result of Bender et. al. concerning Bell numbers.

**Corollary 4.2.** The sum of the symmetry numbers of the $G(\lambda)$ as $\lambda$ runs over the partitions of $n$ is one.

*Proof.* Sylvester showed that $\sum_\lambda 1/z_\lambda = 1$ [S]. This follows from Lemma 2.1 with $r = F$ = the constant function with value 1.

\[ \Box \]

We close by quoting Robin Chapman again: “Theorem 4.1 [has] the natural combinatorial interpretation that $n!/z_\lambda$ is the number of permutations with cycle structure $\lambda$ (and then corollary 4.2 says that the total number of permutations is $n!$.)”

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