Symplectic structure on colorings, 
Lagrangian tangles and Tits buildings 

by 
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Presented by the Editors

Summary. We define a symplectic form $\hat{\varphi}$ on a free $R$-module $R^{2n-2}$ associated to $2n$ points on a circle. Using this form, we establish a relation between submodules of $R^{2n-2}$ induced by Fox $R$-colorings of an $n$-tangle and Lagrangians or virtual Lagrangians in the symplectic structure $(R^{2n-2}, \hat{\varphi})$ depending on whether $R$ is a field or a PID. We prove that when $R = \mathbb{Z}_p$, $p > 2$, all Lagrangians are induced by Fox $R$-colorings of some $n$-tangles and note that for $p = 2$ and $n > 3$ this is no longer true. For any ring, every $2\pi/n$-rotation of an $n$-tangle yields an isometry of the symplectic space $R^{2n-2}$. We analyze invariant Lagrangian subspaces of this rotation and we partially answer the question whether an operation of rotation (generalized mutation) defined in [A-P-R] preserves the first homology group of the double branched cover of $S^3$ along a given link.

1. Alternating form on the space of boundary points colorings.

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(1) This relation might be viewed as a parallel of the well-known result that 3-manifolds yield Lagrangians in $H_1(\partial M, \mathbb{Q})$. DOI: 10.4064/ba201230-2-3 [1] © Instytut Matematyczny PAN, 2021
whether a rotation operation (generalized mutation) as defined in [A-P-R] preserves the first homology group of the double branched cover of $S^3$ along a given link; and we address the problem of “imprisoned colorings”.

Let $R$ be a commutative ring with identity and $p_1, \ldots, p_{2n}$ be points on a circle (or a square) as in Figure 1.

![Fig. 1. $R$-coloring of $2n$ points $p_1, \ldots, p_{2n}$ by $a_1, \ldots, a_{2n} \in R$. Different shapes of a tangle are useful for different applications.](image)

Regarding $R$ as a set of colors allows us to identify the set of all $R$-colorings of $2n$ points with the elements of the free $R$-module $R^{2n}$ with basis $e_1, \ldots, e_{2n}$, where $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ with $1 \in R$ in the $i$th position.

Let

$$W_{2n-1} = \left\{ \sum_{i=1}^{2n} a_i e_i \in R^{2n} : \sum_{i=1}^{2n} (-1)^i a_i = 0 \right\} \cong R^{2n-1},$$

$f_i = e_i + e_{i+1}$, $i = 1, \ldots, 2n-1$, and $f_{2n} = e_{2n} + e_1$. Clearly $f_1, \ldots, f_{2n-1}$ are in $W_{2n-1}$ and form a basis. Notice that for any $w \in R^{2n}$ we have

$$w = \sum_{i=1}^{2n} a_i e_i = a_1 f_1 + (a_2 - a_1) f_2 + \cdots + (a_{2n} - a_{2n-1} + \cdots - a_1) f_{2n-1} + \left( \sum_{i=1}^{2n} (-1)^i a_i \right) e_{2n}.$$

Let $W$ denote an $R$-module. Recall that an $R$-bilinear form $\varphi : W \times W \rightarrow R$ is called alternating (or skew-symmetric) if for all $u, v \in W$,

$$\varphi (u, v) = -\varphi (v, u).$$

Note that if 2 is invertible in $R$, then $\varphi$ is skew-symmetric if and only if $\varphi (u, u) = 0$ for all $u \in W$.

(2) We note that $f_{2n} = e_{2n} + e_1 = \sum_{i=1}^{n} f_{2i-1} - \sum_{i=1}^{n-1} f_{2i}$. 

If a skew-symmetric form \( \varphi \) is non-degenerate we call it a symplectic form \(^{(3)}\)

Consider an alternating form \( \varphi \) on \( W_{2n-1} \) of nullity 1 given by

\[
\varphi(f_i, f_j) = \begin{cases} 
  j - i & \text{if } |i-j| = 1, \\
  0 & \text{if } |i-j| \neq 1,
\end{cases}
\]

and let

\[
v_0 = \sum_{i=1}^{2n} e_i = \sum_{i=1}^{n} f_{2i-1} = \sum_{i=1}^{n} f_{2i}.
\]

Then \( \varphi(v_0, v) = 0 \) for all \( v \in W_{2n-1} \), i.e. \( v_0 \) is an isotropic vector. We will write \( v_0 \perp W_{2n-1} \) as \( v_0 \) is \( \varphi \)-orthogonal to \( W_{2n-1} \). It is useful to introduce the free \( R \)-module \( W_{2n} \) with the basis \( f_1, \ldots, f_{2n-1}, f_{2n} \). We can extend \( \varphi \) to a symplectic form \( \varphi' \) on \( W_{2n} \) by setting \( \varphi'(f_{2n}, f_j) = 0, j = 2, \ldots, 2n - 2, \) \( \varphi'(f_{2n}, f_{2n-1}) = 1 \), and \( \varphi'(f_{2n}, f_1) = -1 \).

Let \( w_0 = \sum_{i=1}^{n} (f_{2i-1} - f_{2i}) \) and let

\[
Rw_0 = \{ rw_0 : r \in R \}
\]

be the \( R \)-submodule of \( W_{2n} \) generated by \( w_0 \). We have

\[
W_{2n-1} \cong W_{2n}/Rw_0,
\]

and if \( Rv_0 \) denotes the \( R \)-submodule of \( W_{2n-1} \) consisting of all monochromatic \( R \)-colorings of \( 2n \) points on the circle, then

\[
W_{2n-2} = W_{2n-1}/Rv_0.
\]

As one verifies, \( \varphi \) descends to a symplectic form \( \hat{\varphi} \) on the free \( R \)-module \( W_{2n-2} \).

Let \( r : W_{2n} \to W_{2n} \) be defined by \( r(f_i) = f_{i+1} \) for \( i = 1, \ldots, 2n - 1 \) and \( r(f_{2n}) = f_1 \). It is easy to see that \( r \) is induced by a counterclockwise \( \pi/n \)-rotation of the circle with \( 2n \) points and clearly \( r \) is a \( \varphi' \)-isometry of \( W_{2n} \), i.e.

\[
\varphi'(r(f_i), r(f_j)) = \varphi'(f_i, f_j)
\]

for all \( i, j \in \{1, \ldots, 2n\} \).

**Lemma 1.1.** Let \( r : W_{2n} \to W_{2n} \) be the \( \varphi' \)-isometry of \( W_{2n} \) defined above.

(i) \( r \) descends to a \( \varphi \)-isometry of \( W_{2n-1} \) and a \( \hat{\varphi} \)-isometry of \( W_{2n-2} \).

(ii) The map \( m_{yz} : W_{2n} \to W_{2n} \), \( m_{yz}(f_i) = f_{2n-i} \), induced by the reflection of the circle with \( 2n \) points in the \( yz \) plane in \( \mathbb{R}^3 \) is an anti-isometry, i.e. \( \varphi'(m_{yz}(u), m_{yz}(v)) = -\varphi'(u, v) \) for all \( u, v \in W_{2n} \).
Proof. Since for \( w_0 = \sum_{i=1}^{n} (f_{2i-1} - f_{2i}) \),
\[
r(w_0) = -w_0,
\]
we see that \(-1\) and \(w_0\) is an eigenpair for \(\varphi'\). Also \(Rv_0 \subset W_{2n-1}\) is an invariant \(R\)-submodule of \(r\). Thus (i) follows (recall that \(Rv_0\) is the null \(R\)-submodule for \(\varphi\)).

Since \(\varphi'(m_{yz}(f_i), m_{yz}(f_j)) = -\varphi'(f_i, f_j)\) for all \(i, j = 1, \ldots, 2n\), the statement (ii) also follows. ■

Let \(V\) be a free \(R\)-module and \(\varphi\) be a symplectic form on \(V\). We say that \(W\) is an isotropic \(R\)-submodule of \(V\) if \(\varphi(w_1, w_2) = 0\) for all \(w_1, w_2 \in W\). A maximal isotropic \(R\)-submodule \(W\) of \(V\) is called a Lagrangian submodule. If \(R\) is a field, we will see it later, the isotropic subspaces \(W\) of \(W_{2n-2}\) form a flag structure in \((W_{2n-2}, \hat{\varphi})\), hence a Tits building. An \((n-1)\)-dimensional isotropic subspace of \((W_{2n-2}, \hat{\varphi})\) is called a Lagrangian subspace and when \(R = \mathbb{Z}_p\) (\(p\) prime) there are \(\prod_{i=1}^{n-1}(p^i + 1)\) of them (see [O]).

The above lemma will be used in Section 4 to analyze Fox \(R\)-colorings of rotors (see Section 2 and [Pr-1] for introduction to Fox 3-colorings).

2. Tangles as Lagrangians. For every \(n\)-tangle \(T = \begin{tikzpicture} \draw (-.5,.5) -- (0,0) -- (1,0); \end{tikzpicture}\) define the \(R\)-module \(\text{Col}_R(T)\) of its Fox \(R\)-colorings \(^4\) as follows. Let \(x_1, \ldots, x_m\) be arcs of \(T\) (parts of the tangle diagram from tunnel to tunnel) and \(c_1, \ldots, c_t\) be its crossings. Then \(\text{Col}_R(T)\) is a quotient of the free \(R\)-module \(R^m\) with basis \(\{x_1, \ldots, x_m\}\) modulo its submodule generated by \(2x_i - x_j - x_k\) for each crossing \(c_s, 1 \leq s \leq t\) (see Figure 2). An element \(c \in \text{Col}_R(T)\) is called a Fox \(R\)-coloring of \(T\) and it can be regarded as a mapping from the arcs of \(T\) to \(R\). Every \(R\)-coloring of \(T\) induces a coloring of its \(2n\) boundary points \(p_1, \ldots, p_{2n}\). Therefore, there is a homomorphism of \(R\)-modules \(\psi : \text{Col}_R(T) \to R^{2n}\) with \(\psi(\text{Col}_R(T)) \subseteq W_{2n-1}\) (see Proposition 2.2), i.e. \(\psi : \text{Col}_R(T) \to W_{2n-1}\). Moreover, since \(Rv_0 \subseteq \text{Col}_R(T)\), the homomorphism \(\psi\) descends to a homomorphism of the quotient \(R\)-modules
\[
\hat{\psi} : \text{Col}_R(T)/Rv_0 \to W_{2n-1}/Rv_0.
\]
It is natural to ask which subspaces of \(W_{2n-2} \simeq W_{2n-1}/Rv_0\) are induced by \(n\)-tangles. We answer this question partially when \(R\) is a field.

\(^4\) If \(R = \mathbb{Z}_p\) we write \(\text{Col}_p(T)\) instead \(\text{Col}_{\mathbb{Z}_p}(T)\).
Theorem 2.1. Let \( \hat{\psi} : \text{Col}_R(T)/Rv_0 \to W_{2n-2} \) be the homomorphism of \( R \)-modules defined above.

(i) If \( R \) is a field then \( \hat{\psi}(\text{Col}_R(T)/Rv_0) \) is a Lagrangian subspace of \( (W_{2n-2}, \hat{\varphi}) \).

(ii) If \( R \) is a commutative ring with identity and \( T \) is a rational \( n \)-tangle then \( \hat{\psi}(\text{Col}_R(T)/Rv_0) \) is a Lagrangian submodule of \( (W_{2n-2}, \hat{\varphi}) \).

Before we give the proof of Theorem 2.1, let us recall some standard terminology concerning tangles and sketch a rather general inductive procedure which will be used later.

Define an \((n,k)\)-tangle as a tangle with \( n \) inputs (on the left) and \( k \) outputs (on the right). Tangles can be composed if the number of outputs of the first one equals the number of inputs of the second \((\text{5})\). Let \( \sigma_i \) be a crossing between the \( i \)th and \((i+1)\)th strands, \( U^m_i \) be a “local minimum” and \( U^M_i \) be a “local maximum” (while moving in the direction of the \( x \)-axis) that involve \( i \) and \( i+1 \) strands \((\text{6})\) respectively.

We say that an \( n \)-tangle is rational if it is obtained from an \( n \)-tangle with no crossings and trivial components \((\text{7})\) by adding a finite number of crossings \( \sigma_i^{\pm 1} \) \((i = 1, \ldots, 2n)\). We see that the generators \( \sigma_i, U^m_i \) and \( U^M_i \) (for \( n < i < 2n \)) yield maps

\[
\sigma_i : T_{2n} \to T_{2n}, \\
U^m_i : T_{2n} \to T_{2n+2}, \\
U^M_i : T_{2n} \to T_{2n-2}
\]

where \( T_{2k} \) denotes the set of all tangles with \( k \) inputs and \( k \) outputs. We

\((\text{5})\) Note that there is a general category of tangles with boundary points as objects and tangles as morphisms.

\((\text{6})\) If inputs and outputs are labeled as in Figure 1 (right), then the standard \( i \)th generator of the braid group \( B_n \) will be denoted by \( \sigma_{2n-i} \). However, it is convenient to be able to add crossings, minima and maxima at any place in a given tangle \( T \) as we can then take advantage of rotational symmetry that is important for this paper.

\((\text{7})\) Any \( n \)-tangle with no crossings or trivial components can be chosen as a starting tangle.
observe that each $n$-tangle $T$ is a composition of crossings $\sigma_i$, maxima $U_i^m$, and minima $U_i^M$ (for $n < i < 2n$). Thus, to prove a property $P$ for all tangles, it is sufficient to argue that $P$ holds for “basic” tangles and then show by induction if a tangle $T$ has property $P$, then so do the tangles $T\sigma_i^{\pm 1}, TU_i^m$, and $TU_i^M$. This idea will be used several times (including the proof of our main theorem) and, in particular, we use it below to show that an $R$-Fox coloring of a tangle $T$ induces an $R$-coloring of its $2n$ boundary points that satisfies the “alternating” condition.

**Proposition 2.2.** Let $T$ be an $n$-tangle and $\psi: \text{Col}_R(T) \to R^{2n}$ be the homomorphism of $R$-modules which assigns to an $R$-Fox coloring of arcs of $T$ the corresponding $R$-coloring of their $2n$ boundary points. If $c = \sum_{i=1}^{2n} a_i e_i$ and $c \in \psi(\text{Col}_R(T))$ then $c$ satisfies the alternating condition, i.e. $\sum_{i=1}^{2n} (-1)^i a_i = 0$.

![Diagram](Fig. 3. Tangles Id and B0)

**Proof.** Consider an $n$-tangle $B_0$ shown in Figure 3. Since each Fox $R$-coloring $a_1, \ldots, a_n$ of the $n$ arcs of $B_0$ that join $x_1$ and $x_2$, $x_3$ and $x_4$, ..., $x_{2n-1}$ and $x_{2n}$ gives the corresponding $R$-coloring $c = a_1 e_1 + a_1 e_2 + \cdots + a_n e_{2n-1} + a_n e_{2n} \in \psi(\text{Col}_R(T))$ of the boundary points of $B_0$, clearly the alternating condition holds for $B_0$. Assume that the alternating condition holds for a tangle $T$, and consider tangles $T\sigma_i, T\sigma_i^{-1}, TU_i^m$ and $TU_i^M$. The local modification of $T$ shown in Figure 3 yields $a - b = d - c$ for the first two tangles and $a - b = 0$ for the last two. Hence the alternating condition holds for $T\sigma_i, T\sigma_i^{-1}, TU_i^m$ and $TU_i^M$. Therefore, it must also be true for an arbitrary tangle $T'$, which completes the proof.

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*(8) Since an $n$-tangle $T$ is a 1-dimensional manifold properly embedded in a cube, $T$ is a Morse function along the $x$-axis with $\sigma_i$’s between extremal points.

*(9) In [Pr-1], we gave a proof of this result in the case when 2 is not a zero divisor in $R$.

*(10) We could take the identity $n$-tangle Id shown in Figure 3 as an $n$-tangle with no crossings or trivial components as a “basic” state, but due to our later considerations it is better for us to choose $B_0$.**
Proof. To prove Theorem 2.1, we use an inductive argument similar to that above.

(0) We easily verify that the $R$-submodule induced by the Fox $R$-colorings of $B_0$ is a Lagrangian. In fact $\hat{\psi}(B_0) \subseteq W_{2n-2}$ is a free $R$-module with basis $f_1, f_3, \ldots, f_{2n-3}$.

(1) If a tangle $T$ yields a Lagrangian then $T'$ obtained from $T$ by adding one crossing is also a Lagrangian. In fact, we can view adding a crossing as a transvection in $(W_{2n-2}, \hat{\varphi})$. More precisely, a crossing between the $i$th and $(i+1)$th strands ($1 \leq i \leq 2n$ and indices are mod $2n$) yields a transvection on $W_{2n-2}$ defined by

$$t_i(v) = v \pm \hat{\varphi}(v, f_i)f_i,$$

where the sign $\pm$ depends on the type of the crossing. It is well-known (see [O] for example) that transvections are isometries of a symplectic space, thus in particular they map Lagrangians in $W_{2n-2}$ into Lagrangians in $W_{2n-2}$.

(2) Minima as “expansions”. The symplectic space $R^{2n}$ is obtained from the symplectic space $R^{2n-2}$ by adding a 2-dimensional symplectic space (with the symplectic form given by a standard $2 \times 2$ “hyperbolic” matrix). Lagrangians in $R^{2n}$ are built from Lagrangians in $R^{2n-2}$ by adding a vector from the 2-dimensional subspace that was added.

(3) Maxima as “contractions”. Adding a “maximum” to $T$ results in connecting the $i$th and $(i+1)$th strands. As a result an $n$-tangle $T$ reduces to an $(n-1)$-tangle $T^M_i$. Consider the projection $p_i$ from $R^{2n}$ to $R^{2n-2}$ defined by

$$p_i \left( \sum_{j=1}^{2n} x_i e_i \right) = \sum_{j=1, j \neq i, (i+1)}^{2n} x_i e_i.$$

One can verify that $p_i$ restricted to the subspace spanned by

$$\{ f_1, \ldots, f_{i-2}, f_{i-1} - f_i + f_{i+1}, f_i + f_{i+2}, \ldots, f_{2n-1} \}$$

is a symplectic isometry. If $R$ is a field then by a simple dimension counting argument ([11]) the $(n-1)$-tangle $T^M$ also induces a Lagrangian in $R^{2n-2}$.

([11]) This is a special case of Turaev’s “contraction lemma” (see [Tur] p. 180).
This completes the proof of Theorem 2.1(i). Theorem 2.1(ii) also follows, because we note that step (1) also holds for an arbitrary ring. ■

It is also important to note that Theorem 2.1(i) is no longer true if $R$ is a ring rather than a field (not even for $R = \mathbb{Z}$).

Example 2.3. Let $R = \mathbb{Z}$ and $T$ be the 2-tangle $U_2^m \sigma_1^2 \sigma_3^{-2} U_2^M$ shown in Figure 5. Then $\hat{\psi}(T) \subset W_2$ is generated by $2f_1$.

Using our previous considerations we obtain the following:

Theorem 2.4. If $R$ is a PID then $\hat{\psi}(\text{Col}_R(T)/Rv_0)$ is a virtual Lagrangian of $(W_{2n-2}, \hat{\phi})$, i.e. a finite index $R$-submodule of a maximal isotropic $R$-submodule of $W_{2n-2}$.

An important question that arises at this point is whether every Lagrangian subspace can be realized by a tangle. The answer is affirmative when $R = \mathbb{Z}_p$, $p > 2$.

Theorem 2.5. For $p > 2$, every Lagrangian subspace of $\mathbb{Z}_p^{2n-2}$ is induced by an $n$-tangle.

Proof. This follows from the work of J. Assion [A] (for $p = 3$) and B. Wajnryb [Wa-2, Wa-3] (for general $p > 2$). ■

When $R = \mathbb{Z}_2$ or $\mathbb{Z}$, Theorem 2.5 is true for 2- and 3-tangles only and it is related via the double branched cover to the fact that the mapping class group of a torus and a genus two surface is generated by “horizontal” Dehn twists. An algebraic proof is based on the observation that the symplectic group $\text{Sp}(2n - 2, R)$, $n < 4$, is generated by $2n - 1$ transvections $t_i(v) = v + \hat{\phi}(v, f_i) f_i$. However, Theorem 2.5 is not true in general when $p = 2$.

Proposition 2.6. For $p = 2$ the number of Lagrangians of $(\mathbb{Z}_2^{2n-2}, \hat{\phi})$ is $\prod_{i=1}^{n-1} (2i + 1)$ while the number of Lagrangians induced by $n$-tangles equals $\prod_{i=1}^{n-1} (2i + 1)$. In particular, for $n \geq 4$ not all Lagrangians of $(\mathbb{Z}_2^{2n-2}, \hat{\phi})$ are realizable by $n$-tangles.

Proof. When $p = 2$, the space $\text{Col}_2(T)$ is an invariant of homotopy of tangles (fixed on the boundary). Equivalently $\text{Col}_2(T)$ is preserved by a
crossing change. Thus, \( \psi(\text{Col}_2(T)) \) depends only on connections between \( 2n \) boundary points. Since there are \((2n - 1)(2n - 3) \ldots 1\) such connections and each of them yields a different \( n \)-dimensional subspace of \( \mathbb{Z}_2^{2n-1} \) (and \( \mathbb{Z}_2^{2n-2} \)), the statement follows.

**Corollary 2.7.** Let \( \psi : \text{Col}_R(T) \rightarrow R^{2n} \) be the homomorphism of \( R \)-modules defined as before.

(a) \( \dim \psi(\text{Col}_p(T)) = n \) for any \( n \)-tangle \( T \).

(b) For every \( p \)-coloring \( c \) of \( 2n \) boundary points (an element of \( \mathbb{Z}_p^{2n-1} \)) there is an \( n \)-tangle \( T \) and a Fox \( p \)-coloring of \( T \) that induces \( c \), i.e.

\[
\mathbb{Z}_p^{2n-1} = \bigcup_T \psi(\text{Col}_p(T)).
\]

**Proof.** (a) Follows from Theorem 2.1.

(b) Follows from Theorem 2.5 for \( p \geq 3 \) as every vector in a symplectic space is in some Lagrangian.

For \( p = 2 \), let \((\epsilon_1, \ldots, \epsilon_{2n}) \in \mathbb{Z}_2^{2n}\) and assume that \( \sum_{i=1}^{2n} \epsilon_i = 0 \). Then the number of \( i \in \{1, \ldots, 2n\} \) such that \( \epsilon_i = 1 \) is even. Let \( T \) be an \( n \)-tangle with no arc that connects boundary points labeled by 1 to those labeled by 0. If we assign 1 to each arc of \( T \) with endpoints labeled by 1, and 0 to the remaining ones, we see that such a Fox 2-coloring of \( T \) induces \((\epsilon_1, \ldots, \epsilon_{2n})\) (see Figure 6).

![Fig. 6. A tangle T inducing (\( \epsilon_1, \ldots, \epsilon_{2n} \)](image)

**Remark 2.8 (Surfaces analogy).** The best known example of a symplectic \( \mathbb{Z} \)-module in topology is \( H_1(F_g, \mathbb{Z}) \) (i.e. the first homology of a closed surface \( F_g \) of genus \( g \)), with the symplectic form defined as the algebraic crossing number of closed curves representing elements of \( H_1(F_g, \mathbb{Z}) \). In fact, we can make the analogy more precise by considering the double branched cover of \( (D^3, \partial D^3) \) along a given \( n \)-tangle. In particular, \( \partial D^3 \) branched over \( 2n \) points lifts to a surface \( F_{n-1} \). The symplectic structure on the space \( \mathbb{Z}_p^{2n-2} \) of boundary points colorings lifts to the symplectic structure on \( H_1(F_g, \mathbb{Z}_p) \). As is well-known, any 3-manifold with boundary \( F_g \) yields a Lagrangian subspace in \( H_1(F_g, \mathbb{Z}_p) \). In particular, the Lagrangian induced by an \( n \)-tangle corresponds to the Lagrangian subspace induced by a 3-manifold of the double branched cover. As noticed before, (for \( p = 2 \)) not every Lagrangian is induced by an \( n \)-tangle. This in turn has its consequences for branched coverings and mapping class groups (see Remark 2.9).
Remark 2.9. The mapping class group of a surface is generated by a finite number of Dehn twists. Furthermore, one shows that it is enough to take a “horizontal” Dehn twist (for genus $g \leq 2$) together with one additional non-horizontal Dehn twist (for $g > 2$) \cite{Lic, Hu, Wa-1}. In fact horizontal Dehn twists correspond to crossing changes, and the argument we used for $\mathbb{Z}_2$-colorings shows the known fact (see \cite{B-H}) that for $g > 2$ horizontal twists are insufficient to generate the mapping class group of $F_g$. On the other hand the above reasoning shows that for 3-tangles we produce all Lagrangians.

As is easy to see, the set $T_{2n}$ of all $n$-tangles with the operation of composition is an $R$-algebra. We will prove the following lemma.

Lemma 2.10. Let $m : T_{2n} \to T_{2n}$ be an isomorphism of $R$-algebras that sends an $n$-tangle $T$ to its mirror image $\overline{T}$ in the xy plane. Then $m$ induces an anti-isometry $m_* : W_{2n-2} \to W_{2n-2}$ of the symplectic module $(W_{2n-2}, \hat{\varphi})$ given by

$$m_*(f_i) = (-1)^i f_i$$

that maps Lagrangian submodules of $W_{2n-2}$ determined by $T$ to Lagrangian submodules of $W_{2n-2}$ determined by $m(T) = \overline{T}$.

Proof. We see that

$$m_*(f_{2n-1}) = m_*(-f_1 - f_3 - \cdots - f_{2n-3}) = f_1 + f_3 + \cdots + f_{2n-3} = -f_{2n-1}.$$ 

and analogously

$$m_*(f_{2n}) = m_*(-f_2 - f_4 - \cdots - f_{2n-2}) = f_2 + f_4 + \cdots + f_{2n-2} = f_{2n}.$$

Using the method from our proof of Theorem 2.1, we first show that Lemma \ref{lem:2.10} holds for a basic tangle $B_0$. Then we show that if the lemma holds for an $n$-tangle $T$, it also holds for any $n$-tangle $T'$ obtained from $T$ by adding a crossing, a minimum or a maximum.

(i) (Crossing) As noted earlier, adding a crossing $\sigma^\epsilon_i$ induces a transvection on $W_{2n-2}$:

$$t_i(v) = v - \epsilon \hat{\varphi}(v, f_i) f_i.$$ 

Since the operation of taking the mirror image changes only $\epsilon$ to $-\epsilon$ in the transvection $t_i$, one easily verifies that the conclusion of Lemma \ref{lem:2.10} also holds for $T \sigma^\epsilon_i$.

(ii) (Minimum or maximum) Clearly a minimum and a maximum are fixed when taking the mirror image. Since, as we saw in the proof of Theorem 2.1, adding a maximum or a minimum changes by 2 the dimension of the symplectic $R$-module associated to the boundary points $R$-coloring, the conclusion holds for $TU_i^m$ and $TU_i^M$. ■
We note that an extension of the anti-isomorphism $m_*: W_{2n-2} \to W_{2n-2}$, $m_*(f_i) = (-1)^i f_i$, to $W_{2n-1}$ is defined by

$$m_*(f_{2n}) = m_*(f_1 + f_3 + \cdots + f_{2n-1} - f_2 - f_4 - \cdots - f_{2n-2})$$

$$= -f_1 - f_3 - \cdots - f_{2n-1} - f_2 - f_4 - \cdots - f_{2n-2}$$

$$= f_{2n} - (f_1 + f_2 + \cdots + f_{2n}) = f_{2n} - 2v_0.$$

Using Lemmas 1.1(ii) and 2.10 we obtain as a corollary the following result, important for our later analysis of rotors.

**Corollary 2.11.** Let $m_y : \mathcal{T}_{2n} \to \mathcal{T}_{2n}$ be an anti-isomorphism of the $R$-algebra of tangles induced by the $y$-rotation about the $y$-axis in $\mathbb{R}^3$. Then the corresponding isometry of $R$-modules $m_{sy} : W_{2n-2} \to W_{2n-2}$ defined by $m_{sy}(f_i) = (-1)^i f_{2n-i}$ establishes a bijection between the sets of Lagrangians induced by $T$ and by $m_y(T)$.

3. **Buildings.** Let $C$ be the Coxeter group defined by the presentation

$$G = \langle s_1, \ldots, s_{d+1} \mid (s_is_j)^{m_{ij}} \rangle,$$

where $m_{ii} = 1$, $m_{ij} = m_{ji}$ and $m_{ij} \geq 2$ is an integer for $i \neq j$ (or $\infty$ in which case there is no $ij$th relation). We also let $l(w)$ be the length of the shortest word representing $w$ in the generators $s_i$. A *Tits building* is a set $X$ with a $C$-valued distance function $d : X \times X \to C$ that has the following properties:

1. $d(x, y) = 1$ iff $x = y$.
2. If $d(x, y) = w$ and $d(y, z) = s_i$ then $d(x, z) = w$ or $d(x, z) = ws_i$.
3. For each $x, y \in X$ and $s_i$ there is $z \in X$ such that

$$d(y, z) = s_i, \quad d(x, z) = d(x, y)s_i.$$

Moreover, $z$ is unique if $l(d(x, y)s_i) < l(d(x, y))$.

We note that $X$ can be realized geometrically as a simplicial complex $X_t$ as follows:

- Label the $(d - 1)$-dimensional faces of the simplex $\Delta^d$ with $s_1, \ldots, s_{d+1}$.
- Form $X \times \Delta^d$ and, for any $w, w' \in X$ with $d(w, w') = s_i$, glue $w \times \Delta^d$ and $w' \times \Delta^d$ along the face labeled $[12]$ by $s_i$.

The same construction for $C$ equipped with $d(w, w') = w^{-1}w'$ in place of $X$ yields the *Coxeter complex* $C_t$ of $C$. The geometric realization of $X_t$

\[\text{(12) Note that faces that are glued together have induced labeling of their faces by the remaining generators of } C, \text{ and the gluing has to preserve them.}\]
contains plenty of embedded copies of $C_t$ (called apartments). Any pair of chambers (d-dimensional simplices of $X_t$) is contained in an apartment [13].

Remark 3.1. Let $c \in X$ be a chamber. Then the folding map $f_c : X \rightarrow C$ defined by

$$f_c(c') = d(c, c')$$

induces a simplicial morphism $f_c : X_t \rightarrow C_t$ which is an isomorphism when restricted to an apartment that contains $c$.

Assume that $R$ is a field and consider the symplectic space $(W_{2n-2}, \hat{\varphi})$. We show how to construct a Tits building using $(W_{2n-2}, \hat{\varphi})$ by describing its geometrical realization as a simplicial complex. Indeed, as shown in [Brown, Ga], the simplicial complex $X_{n-1}$ whose vertices are isotropic subspaces of $W_{2n-2}$ and which has flags [14] of isotropic subspaces as simplices forms a building. More precisely, each vertex of $X_{n-1}$ is labeled by the dimension of the corresponding isotropic subspace of $W_{2n-2}$, and codimension one faces of chambers are labeled using the corresponding labels of the opposite vertices. Furthermore, the Lagrangian subspaces of $W_{2n-2}$ are in one-to-one correspondence with vertices of $X_{n-1}$ labeled by $n-1$. We observe that chambers of $X_{n-1} = X_t$ (elements of $X$) correspond to maximal isotropic flags in $W_{2n-2}$. In our case, the Coxeter group $C$ is a signed permutation group determined by

$$m_{i,j} = \begin{cases} 4 & \text{if } i = n-1, j = n-1, \\ 3 & \text{if } 1 \leq i < n-2, \\ 2 & \text{if } |i-j| > 1, \\ m_{n-2,n-1} = 4, & m_{i,i+1} = 3 \end{cases}$$

and the corresponding Coxeter complex can be identified with the first barycentric subdivision of the boundary of an $(n-1)$-dimensional cube $I^{n-1}$.

We now describe how to define apartments for our Coxeter complex. Choose a basis $B = \{v_1, w_1, \ldots, v_{n-1}, w_{n-1}\}$ of $W_{2n-2}$ with the property

$$\hat{\varphi}(v_i, w_i) = -\hat{\varphi}(w_i, v_i) = 1,$$

$$\hat{\varphi}(v_i, v_j) = -\hat{\varphi}(w_i, w_j) = \hat{\varphi}(v_i, w_j) = 0$$

for $i \neq j$ and consider all isotropic subspaces [15] spanned by subsets of $B$. Vertices corresponding to these subspaces span an apartment $A(B)$ in $X_t$. Moreover, the $2^{n-1}$ Lagrangian subspaces in $A(B)$ consisting of all subspaces

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(13) One may use the family of apartments to give another definition of buildings (see [Brown, Ron]).

(14) A flag in a vector space is a strictly increasing sequence of subspaces.

(15) Such a subspace can be obtained by choosing at most one vector from each pair $\{v_i, w_i\}$. 
of $W_{2n-2}$ with exactly one vector from each pair $\{v_i, w_i\}$ correspond to vertices of the cube $I^{n-1}$.

**Lemma 3.2.** Let $\mathcal{L}$ be the family of Lagrangian subspaces corresponding to $(n-1)$-labeled vertices of an apartment $A \subset X_{n-1}$. Then for each Lagrangian $M \subset W_{2n-2}$ there exists an $L \in \mathcal{L}$ with $M \cap L = \{0\}$.

**Proof.** Suppose this is not the case. Then for each $L \in \mathcal{L}$ there is a length $\leq 2$ path $P_L$ in the 1-skeleton of $X_{n-1}$ connecting $M$ and $L$. Choose a chamber $c \in A$ and consider the folding map $f_c$. The family $\{f_c(P_L) \mid L \in \mathcal{L}\}$ is a family of length $\leq 2$ paths in the 1-skeleton of $C_t$ connecting $f_c(M)$ to every vertex of the cube $I^{n-1}$. However, $f_c(M)$ cannot be connected by such a path to the vertex opposite to it. In other words, if $L$ corresponds to the vertex antipodal in $A$ to $f_c(M)$, then $L$ intersects trivially with $M$. ■

To finish this section, we give an example of a family $\mathcal{L}$ as in the lemma describing its elements in terms of Catalan tangles (i.e. $n$-tangles with no crossings). Let $B = \{v_i, w_i : i = 1, \ldots, n - 1\}$, where

- $v_1 = f_{n-1}$, $w_1 = f_n$,
- $v_2 = f_{n-2} + f_n$, $w_2 = f_{n-1} + f_{n+1}$,
- $v_3 = f_{n-3} + f_{n-1} + f_{n+1}$, $w_3 = f_{n-2} + f_n + f_{n+2}$,

etc.

**Proposition 3.3.** Each Lagrangian subspace of $W_{2n-2}$ corresponding to a vertex of $A(B)$ is represented by exactly one $n$-tangle from the following family $\mathcal{L}$. A tangle $T$ is in $\mathcal{L}$ iff

(a) $T$ is a Catalan $n$-tangle;
(b) each boundary point $p_i$ is connected to one of $p_{i-1}$, $p_{i+1}$, $p_{2n-i}$, $p_{2n-i-2}$.

**Proof.** We construct tangles from the family $\mathcal{L}$ inductively, starting from the left of the square (see Figure 1), adding the $i$th arc while choosing a vector from the pair $\{v_i, w_i\}$ at the same time. In the first step the choice of $v_1$ results in joining $p_n$ to $p_{n-1}$ by an arc and the choice of $w_1$ corresponds to connecting $p_n$ and $p_{n+1}$ by an arc. If in the $k$th step $v_k$ was chosen, then in the next step one can choose $v_{k+1}$ (and, at the same time, add an arc connecting the lowest available boundary points) or $w_{k+1}$ (and, at the same time, connecting the two lowest available boundary points on the right hand side of the square).

If in the $k$th step $w_k$ was chosen, then in the next step one can choose $w_{k+1}$ (and add a strand across the square connecting the lowest available boundary points) or $v_{k+1}$ (and connect the two lowest available boundary points on the left hand side of the square).

Finally, one connects the remaining two boundary points. ■
4. Rotors. In this section we given criteria when a dihedral flype (see Definition 4.2) preserves the space of Fox $p$-colorings $\text{Col}_p(L)$ ($p > 2$) modulo trivial colorings (the first homology with $\mathbb{Z}_p$ coefficients of the double branched cover of $S^3$ branched along a link $[\text{Pr-1}]$).

**Definition 4.1** ([A-P-R]). Consider an $n$-tangle which is a part of the link diagram placed in the regular $n$-gon with $2n$ boundary points ($n$ inputs and $n$ outputs). We say that such an $n$-tangle is an $n$-rotor if it has rotational symmetry, that is, the tangle is invariant with respect to rotation about the $z$-axis by $2\pi/n$ (see Figure 7).

**Definition 4.2** ([A-P-R]; rotation operation on a link diagram). Consider a link $L_1$ whose diagram is divided into two $n$-tangles: the rotor part, as described in Definition 4.1 and its complement called a stator. A rotant of a link $L_1$ is the link $L_2$ (Figure 7) obtained from $L_1$ by a dihedral flype of the rotor part. Note that $L_2$ does not depend on the choice of the dihedral flype. We say that $L_2$ is obtained from $L_1$ by a rotation.

As noted before, there is a $\mathbb{Z}_{2n}$-action on the set $\mathcal{T}_{2n}$ induced by the rotation $r$ of $n$-tangles counterclockwise about the $z$-axis by the angle $\pi/n$. Moreover, on the space $\mathbb{Z}_p^{2n}$ of $p$-colorings of $2n$ boundary points, $r$ yields an isomorphism defined by $r(e_i) = e_{i+1}$ (where indices are modulo $2n$) and, when restricted to $W_{2n-1}$, the map $r$ is a $\phi$-isometry which maps basis vectors as follows:

$$r(f_i) = f_{i+1},$$
$$r(f_{2n-1}) = f_{2n} = \sum_{i=1}^{n-1} (f_{2i-1} - f_{2i}) + f_{2n-1},$$
$$r(f_{2n}) = f_1.$$  
We analyze invariant subspaces of $r^2$ and, in particular, we look for invariant
Lagrangians of the $2\pi/n$-rotation about the $z$-axis. We use our analysis to partially answer the question whether an operation of rotation (in the sense of Definition 4.2) preserves the first homology of the double branched cover of $S^3$ along a link.

In particular, we prove the following result.

**Theorem 4.3.** Let $L$ be a link diagram with an $n$-rotor part $R$. Let the rotant $m_y(L)$ be obtained from $L$ by rotating $R$ around the $y$-axis by the angle $\pi$ and keeping the stator, $L - R$, unchanged. Assume that either $n = p$, where $p$ is a prime number, or $n$ is co-prime to $p$ and such that there exists $s$ with $p^s \equiv -1 \pmod{n}$. Then the space $\text{Col}_p(L)$ is preserved by any $n$-rotation.

In the proof of Theorem 4.3 we analyze eigenspaces of the symplectic space $\mathbb{Z}_p^{2n-2}$ with respect to rotation. This will allow us to obtain conditions under which a rotation invariant Lagrangian subspace of $\mathbb{Z}_p^{2n-2}$ is also invariant under the dihedral flype $s$. Figure 7 illustrates a pair of 4-rotants which have different spaces of Fox 5-colorings, $\mathbb{Z}_5^2$ and $\mathbb{Z}_5^3$, respectively. One can also check for the tangle $R$ in Figure 7 that $\ker(\psi) = \mathbb{Z}_5$. Therefore, there exists a nontrivial 5-coloring of $R$ which is 0 on the boundary of $R$ \footnote{This follows from the fact that any rotation preserves the determinant of $L$ which is equal to the order of the first homology, $H_1(M^{(2)}_{\partial L}, \mathbb{Z})$, where $M^{(2)}_{\partial L}$ is the 2-fold branched cover of $S^3$ along $L$. [Pr-2] [Tra] [DIPY]. Thus, if a rotation changes the space of 5-colorings then always $\ker(\psi) \neq \{0\}$.} Analogous examples can be constructed whenever $\gcd(p - 1, n) > 2$.

Most of this section is devoted to the proof of Theorem 4.3. Let $\mu_n$ be the group of $n$th roots of 1, and let $k$ be a finite algebraic extension of $\mathbb{Z}_p$ containing $\mu_n$. To understand $r^2$-invariant Lagrangians of $W = W_{2n-2}$ we first consider the action of $r^2$ on two invariant subspaces

$$W_+ = \langle f_0, f_2, \ldots, f_{2n-2} \rangle \quad \text{and} \quad W_- = \langle f_1, f_3, \ldots, f_{2n-1} \rangle.$$  

After extending scalars to $k$ we can diagonalize this action. For each $\alpha \in \mu_n - \{1\}$ the vector

$$D_\alpha = \sum_{j=0}^{n-1} \alpha^j f_{2j}$$

(respectively $C_\alpha = \sum_{j=0}^{n-1} \alpha^j f_{2j+1}$) is an eigenvector of $r^2$ with eigenvalue $\alpha$. The vectors $D_\alpha$ (resp. $C_\alpha$) form a basis of $W_+$ (resp. $W_-$. One shows

**Formula 4.4.**

$$\hat{\varphi}(C_\alpha, D_\beta) = \begin{cases} n(\alpha - 1) & \text{if } \alpha \beta = 1, \\ 0 & \text{otherwise}, \end{cases}$$

$$\hat{\varphi}(C_\alpha, C_\beta) = \hat{\varphi}(D_\alpha, D_\beta) = 0 \quad (W_+ \text{ and } W_- \text{ are Lagrangians})$$

where $\hat{\varphi}$ is also used for the extension $k$ of $\mathbb{Z}_p$. 

\footnote{This follows from the fact that any rotation preserves the determinant of $L$ which is equal to the order of the first homology, $H_1(M^{(2)}_{\partial L}, \mathbb{Z})$, where $M^{(2)}_{\partial L}$ is the 2-fold branched cover of $S^3$ along $L$. [Pr-2] [Tra] [DIPY]. Thus, if a rotation changes the space of 5-colorings then always $\ker(\psi) \neq \{0\}$.}
The action of \( m_y \) on our eigenvectors is given by
\[
m_y(C_\alpha) = -\alpha^{-1}C_{\alpha-1}, \quad m_y(D_\alpha) = D_{\alpha-1}.
\]
Let
\[W_{\alpha\pm1} = \langle C_\alpha, C_{\alpha-1}, D_\alpha, D_{\alpha-1} \rangle.\]
It follows from \([1]\) that the subspaces \( W_{\alpha\pm1} \) are symplectic and \( \hat{\varphi} \)-orthogonal to each other. Moreover, they are all 4-dimensional except the 2-dimensional \( W_{-1} = \langle C_{-1}, D_{-1} \rangle \) which appears when \( n \) is even.

Using the above description we see that \( r^2 \)-invariant Lagrangian subspaces of \( W \otimes k \) are direct sums \( L_{-1} \oplus \bigoplus L_{\alpha\pm1} \), where
\[
L_{-1} \subset W_{-1}, \quad L_{\alpha\pm1} \subset W_{\alpha\pm1} \quad \text{and} \quad \dim L_{-1} = 1, \quad \dim L_{\alpha\pm1} = 2.
\]
Moreover, while \( L_{-1} \) can be an arbitrary 1-dimensional subspace of \( W_{-1} \), \( L_{\alpha\pm1} \) has to be \( r^2 \)-invariant and isotropic in \( W_{\alpha\pm1} \). There are three possibilities for the latter to hold:

(a) \( L_{\alpha\pm1} = \langle C_\alpha, D_\alpha \rangle \),
(b) \( L_{\alpha\pm1} = \langle C_{\alpha-1}, D_{\alpha-1} \rangle \),
(c) \( L_{\alpha\pm1} = \langle v, w \rangle \), where \( v \) is a vector in \( \langle C_\alpha, D_\alpha \rangle \) and \( w \) is the unique (up to scaling) vector in \( \langle C_{\alpha-1}, D_{\alpha-1} \rangle \) such that \( \hat{\varphi}(v, w) = 0 \).

**Lemma 4.5.** The vectors \( w \) in (c) and \( m_y(v) \) are proportional.

*Proof.* One checks using \([1]\) and \([2]\) that \( \hat{\varphi}(v, m_y(v)) = 0 \). \( \blacksquare \)

**Proposition 4.6.** An \( r^2 \)-invariant subspace of \( W \otimes k \) is also \( m_y \)-invariant unless it contains a subspace of the form \( \langle C_\alpha, D_\alpha \rangle \).

The next step is to analyze which \( r^2 \)-invariant Lagrangian subspaces of \( W \otimes k \) arise as \( r^2 \)-invariant Lagrangian subspaces of \( W \) by extension of scalars.

As \( r^2 \) has no multiple eigenvalues in \( W_+ \) and \( W_- \), both of these split into direct sums of \( r^2 \)-irreducible subspaces:
\[
W_\pm = \bigoplus_{\phi} W^\phi_\pm,
\]
where \( \phi \) runs through the factors of \( \frac{x^n-1}{x-1} \) over \( \mathbb{Z}_p \). Put \( W^\phi = W^\phi_+ \oplus W^\phi_- \). Every \( r^2 \)-invariant Lagrangian subspace \( L \) of \( W \) decomposes as
\[
L = \bigoplus_{\phi} (L \cap W^\phi).
\]
Furthermore, each \( L \cap W^\phi \) is either \( \{0\} \), \( W^\phi \), or the graph of an \( r^2 \)-equivariant linear map \( W^\phi_- \rightarrow W^\phi_+ \) (including the degenerate case of \( W^\phi_+ \)). Clearly, only when for some \( \phi \), \( L \cap W^\phi = W^\phi \), does the space \( L \otimes k \) contain a \( \langle C_\alpha, D_\alpha \rangle \). Now \( W^\phi \) is isotropic if and only if for each pair \( \alpha, \alpha^{-1} \in \mu_n \) at most one of \( \alpha, \alpha^{-1} \) is a root of \( \phi \).
Lemma 4.7. Consider the following condition:

(3) There exists $s$ such that $p^s \equiv -1 \mod n$.

(i) If (3) holds, then for each $\alpha$ and some $\phi$,

$$\phi(\alpha) = 0 \text{ if and only if } \phi(\alpha^{-1}) = 0$$

(ii) If (3) does not hold, then there exist $\alpha$ and $\phi$ such that $\phi(\alpha) = 0$ while $\phi(\alpha^{-1}) \neq 0$.

Proof. The Galois group $\text{Gal}(k/\mathbb{Z}_p)$ is generated by the Frobenius automorphism $x \mapsto x^p$. If $\phi(\alpha) = 0$, then the roots of $\phi$ are exactly the conjugates of $\alpha$, i.e.

$$\alpha, \alpha^p, \alpha^{p^2}, \alpha^{p^3}, \ldots$$

If (3) holds, this sequence contains $\alpha^{-1}$. If $\alpha$ is a primitive $n$th root of 1, (3) is also necessary for this sequence to contain $\alpha^{-1}$. This proves the lemma.

Theorem 4.8.

(i) If (3) holds, then all $r^2$-invariant Lagrangians in $W$ are $m_y$-invariant.

(ii) If (3) does not hold, then there are $r^2$-invariant Lagrangians in $W$ which are not $m_y$-invariant.

Proof. For (i) we observe that (3) implies that no $W^\phi$ is isotropic. That is, $L \cap W^\phi$ has to be the graph of a map $W^-_\phi \to W^\phi_+$ (or $W^\phi_-$). Then $(L \cap W^\phi) \otimes k$ is the graph of $W^-_\phi \otimes k \to W^\phi_+ \otimes k$ (or $W^\phi_- \otimes k$), and does not contain the subspace $\langle C_\alpha, D_\alpha \rangle$.

For (ii), we pick $\phi$ as in Lemma 4.7(ii). Among all the other irreducible factors of $\mathbb{F}_p^\times \mathbb{F}_p^{n-1}$ exactly one, call it $\phi'$, has as its roots the inverses of the roots of $\phi$. Put

$$L = W^\phi \oplus \bigoplus_{\psi \neq \phi, \phi'} W^\psi_+.$$

We see that $L$ is isotropic and

$$m_y(L) = W^{\phi'} \oplus \bigoplus_{\psi \neq \phi, \phi'} W^\psi_+.$$

For $n \leq 17$ we list of all pairs $(n,p \mod n)$ that satisfy (3):

- $(3,2), (4,3), (5,2), (5,3), (5,4), (6,5), (7,3), (7,5), (7,6), (8,7), (9,2), (9,5), (9,8), (10,3), (10,7), (10,9), (11,2), (11,6), (11,7), (11,8), (11,10), (12,11), (13,2), (13,4), (13,5), (13,6), (13,7), (13,8), (13,10), (13,11), (13,12), (14,3), (14,5), (14,13), (15,14), (16,15), (17,2), (17,3), (17,4), (17,5), (17,6), (17,7), (17,8), (17,9), (17,10), (17,11), (17,12), (17,13), (17,14), (17,15), (17,16).
**Corollary 4.9.** If \((n, p)\) satisfies (3) (e.g. for the pairs above), then the number of \(p\)-colorings is the same for a link and its rotant.

For instance, when \(\gcd(p - 1, n) > 2\) condition (3) does not hold, i.e. \(x^n - 1\) has a root \(\alpha\) in \(\mathbb{Z}_p\) different from 1 and \(-1\). We will show that in such a case there exist rotors whose Lagrangians are not \(m_y\)-invariant. In particular, the number of \(p\)-colorings can be different for a link and its rotant. Consider a rotor \(R\) with fundamental domain shown in Figure 8.

\[\hat{\psi}(\text{Col}_p(R)) \otimes k \supseteq \langle C\alpha, D\alpha \rangle,\]

i.e. each \(k\)-coloring \(v\) of \(2n\) boundary points of \(R\) satisfying \(r^2(v) = \alpha v\) is induced by the Fox \(p\)-coloring of \(R\). The following condition on \(B\) ensures that \(R\) has the above property (as one can see using \(r^2\)-equivariant colorings): For all \(a, b \in \mathbb{Z}_p\), there are \(c, d \in k\) such that the \(k\)-coloring of the six boundary points shown in Figure 9 is induced by a Fox \(p\)-coloring of \(B\).

\[\begin{array}{cccc}
  d & & & \alpha d \\
  c & & & \alpha c \\
  a & & & b
\end{array}\]

Fig. 9. Condition for boundary coloring

We observe that it is sufficient to verify the above condition for two \(k\)-linearly independent pairs \((a, b)\). Thus, for instance, we take \((1, 1)\) and \((\alpha, 1)\). For each such pair, we find \((c, d)\) and then check that the resulting boundary \(k\)-colorings are \(\hat{\varphi}\)-orthogonal (in \(W_4\)). For \((1, 1)\) we take \(c = d = \frac{1}{\alpha + 1}\) and for \((\alpha, 1)\) let \(c = 0\) and \(d = 1\). The resulting colorings are represented in \(W_4\).
by the following vectors:
\[ v_1 = \frac{1}{\alpha + 1} f_1 + f_3 + \frac{\alpha}{\alpha + 1} f_5, \]
\[ v_2 = \alpha f_3 + (1 - \alpha) f_4 + (\alpha - 1) f_5 + f_6. \]

It is straightforward to check that these are \( \hat{\varphi} \)-orthogonal. It follows from Theorem 2.5 that the subspace of \( W_4 \) spanned by \( v_1, v_2 \) is induced by some 3-tangle \( B \).

**Remark 4.10.** All rotors constructed above have non-trivial \( p \)-colorings (corresponding to \( v_1 \)) equal to 1 on the boundary and non-constant in the interior. After subtracting from this coloring the constant coloring of \( B \) (with value 1), one gets a non-trivial Fox \( p \)-coloring of \( B \) which can be extended \( r^2 \)-equivariantly to a \( p \)-coloring of \( R \) with zero values on the boundary.

Two such cases are discussed in the next example.

**Example 4.11.** As one can verify, for \( n = 4, p = 5 \), we can take the braid \( B = \sigma_2^{-2} \sigma_1 \sigma_2^{-1} \) and \( \alpha = 2 \) \((2^4 - 1 = 15 \equiv 0 \mod 5)\). For \( n = 3, p = 7 \) we take the braid \( B = \sigma_2^{-1} \sigma_1^3 \sigma_2^{-1} \) and \( \alpha = 2 \) \((2^3 - 1 = 7 \equiv 0 \mod 7)\).

**Remark 4.12.** In Example 4.11, \( \hat{\psi}(\text{Col}_5(R)) \) and \( \hat{\psi}(\text{Col}_5(m_y(R))) \) intersect along their \( L_{-1} \)-part, so that if we use \( R \) as both the rotor and the stator then
\[
\log_5 |\text{Col}_5(R \cup R)| = 2 + \log_5 |\text{Col}_5(R \cup m_y(R))|.
\]

Let us now consider the case \( n = p > 2 \). As before, we split \( W = W_{2n-2} \) into a direct sum \( W = W_+ \oplus W_- \), where
\[
W_+ = \langle f_0, f_2, \ldots, f_{2n-2} \rangle \quad \text{and} \quad W_- = \langle f_1, f_3, \ldots, f_{2n-1} \rangle.
\]

For \( j = 1, \ldots, p - 1 \) put
\[
(4) 
\begin{align*}
\quad w_{p-j} &= \sum_{k=0}^{j-1} (-1)^{j+k-1} \binom{j-1}{k} f_{2k}, \quad v_{p-j} = \sum_{k=0}^{j-1} (-1)^{j+k-1} \binom{j-1}{k} f_{2k+1}. 
\end{align*}
\]

The vectors \( w_{p-j} \) [\( v_{p-j} \)] form a basis of \( W_+ \) [\( W_- \)]. In these bases, the action of \( r^2 \) is
\[
(5) 
\begin{align*}
\quad r^2 w_i &= w_i + w_{i-1}, \quad r^2 v_i = v_i + v_{i-1}, \\
\quad r^2 w_1 &= 0, \quad r^2 v_1 = 0.
\end{align*}
\]

One checks directly that
\[
(6) 
\hat{\varphi}(w_{p-i}, v_{p-j}) = (-1)^{i+j} \binom{i+j-1}{j},
\]
where the binomial coefficient should be read modulo $p$. In particular $\hat{\varphi}(w_i, v_j) = 0$ when $2p - i - j - 1 \geq p$, i.e. when $i + j \leq p - 1$.

**Corollary 4.13.** Let $L_s = \langle w_1, v_1, \ldots, w_s, v_s \rangle$. Then $L_s$ is an isotropic subspace of $W$ if and only if $s \leq (p - 1)/2$.

We want to describe the $r^2$-invariant Lagrangian subspaces $L$ of $W$. Let $s$ be the largest number such that $L_s \subset L$ (it may well be 0) and put $t = p - s - 1$.

**Lemma 4.14.** $L \subset L_t$.

*Proof.* Suppose $L \not\subset L_t$. Then we can find in $L$ a vector

$$u = a_kw_k + a_{k-1}w_{k-1} + \cdots + a_1w_1 + b_{p-1}v_{p-1} + b_{p-2}v_{p-2} + \cdots + b_1v_1$$

with $k > t$, $a_k \neq 0$ (or a vector of this form with $w$ and $v$ switched) that is not in $L_t$. Then $\varphi((r^2 - 1)^{k-t-1}u, v_s) \neq 0$. ■

**Lemma 4.15.** $L$ contains a unique vector of the form

$$u_t = w_t + \sum_{i=s+1}^t a_iv_i$$

(or one with $w$ and $v$ switched), and then it has the following basis: $w_1, \ldots, w_s, v_1, \ldots, v_s, a_{s+1}, \ldots, a_{t}$ where

$$u_k = w_k + \sum_{i=s+1}^k a_{i+t-k}v_i.$$

*Proof.* If $u \in L \cap L_k$ is non-zero in $L_k/L_{k-1}$, then so is $(r^2 - 1)^{t}u$ in $L_{k-l}/L_{k-l-1}$. Therefore, $(L \cap L_k)/L_{k-1}$ is 1-dimensional for $s + 1 \leq k \leq t$ (it is 0-dimensional for $k > t$, 2-dimensional for $k \leq s$, while $L$ has dimension $t + s$ and does not contain $L_{s+1}$).

Pick a vector $u \in L$ that spans $L/L_{t-1}$. We may assume that $w_t$ (or $v_t$) appears in $u$ with coefficient 1. If $c_{t-l}w_{t-l}$ appears in $u$, it can be canceled by subtracting $c_{t-l}(r^2 - 1)^{t}u$. We can also remove $w_i$ and $v_i$ with $i \leq s$, as these are in $L_s \subset L$. After these operations, we get the vector $u_t$ as in (8). Now $u_k$ equals $(r^2 - 1)^{t-k}u_t$ (modulo $L_s$), thus it belongs to $L$. Thus all the vectors $w_1, \ldots, w_s, v_1, \ldots, v_s, a_{s+1}, \ldots, u_t$ are in $L$, and are clearly linearly independent. Since $\dim L = t + s$, these vectors form a basis of $L$. ■

**Lemma 4.16.** $a_t = 0$. i.e.

$$\hat{\varphi}(w_i, v_j) = (-1)^{i+j} \binom{2p - i - j - 1}{p - j},$$

(7)

(8)
Proof. We see that
\[ 0 = \varphi(u_t, u_{s+1}) = \varphi(w_t, a_t v_{s+1}) + \varphi(a_t v_t, w_{s+1}) \]
\[ = -a_t \left( \left( \frac{p - 1}{t} \right) - \left( \frac{p - 1}{p - t} \right) \right) = 2(-1)^{t+1} a_t. \]

Every \(r^2\)-invariant Lagrangian subspace \(L\) of \(W\) is therefore determined by choosing \(s, a_s, a_{s+1}, \ldots, a_{t-1}\), where \(t = p - s - 1\). However, not every choice of these parameters gives rise to an isotropic subspace of \(W\)—there are extra conditions \(\varphi(u_u, u_k) = 0\), which can be written as
\[ (9)_{u,k} \sum_{i=p+t-u-k}^{t-1} a_i (-1)^{u+k+i-t} \times \left( \left( \frac{2p - u - k + t - i - 1}{p - k - i + t} \right) - \left( \frac{2p - u - k + t - i - 1}{p - k - 1} \right) \right) = 0. \]
The above has to hold for each pair \(u, k\) such that \(t \geq u > k \geq s + 2, u + k > p\).

Lemma 4.17. \((-1)^k \binom{n}{k} = \left( \frac{p - 1 + k - n}{k} \right) \mod p.\)

Proof. We have
\[ \binom{n}{k} = \frac{n(n - 1) \ldots (n - k + 1)}{k!}, \]
\[ \binom{p - 1 + k - n}{k} = \frac{(p - n + k - 1) \ldots (p - n + 1)(p - n)}{k!}. \]
Now \(p - n \equiv -n \mod p, p - n + 1 \equiv -(n - 1) \mod p, \) etc.

Let us now turn to the invariance of \(L\) under \(m_y\). First, using (4), (3) and Lemma 4.17 one gets
\[ m_y(w_j) = (-1)^j \sum_{i=0}^{j} \binom{j + 1}{i} w_i, \quad m_y(v_j) = (-1)^{j+1} \sum_{i=0}^{j} \binom{j}{i} v_i. \]
Therefore (calculating modulo \(m_y\)-invariant \(A_s\))
\[ m_y(u_u) = m_y(w_u + \sum_{i=s+1}^{u-1} a_{t+i-u} v_i) \]
\[ = (-1)^u \sum_{j=0}^{u} \binom{u + 1}{j + 1} w_j + \sum_{i=s+1}^{u-1} a_{t+i-u} (-1)^{i+1} \sum_{l=s+1}^{i} \binom{i}{l} v_l \]
\[ = (-1)^u \sum_{j=0}^{u} \binom{u + 1}{j + 1} w_j + \sum_{l=s+1}^{u-1} \left( \sum_{i=l}^{u-1} a_{t+i-u} (-1)^{i+1} \binom{i}{l} \right) v_l. \]
Note that $m_y(u_u) \in L$ if $m_y(u_u)$ is a linear combination of $u_j$ with coefficients matching those of $w_j$ in the above sum. Comparing the $v_l$ terms (in the above sum) gives

$$(10)_{u,l} \sum_{i=t+l-u}^{t-1} a_i (-1)^u \left( \binom{u + 1}{t + l + 1 - i} + (-1)^{i-t} \binom{i + u - t}{l} \right) = 0.$$ 

To ensure $m_y$-invariance of $L$, this formula should hold for all pairs $u, l$ such that $t \geq u > l \geq s + 1$.

**Theorem 4.18.** If $n = p > 2$ then each $r^2$-invariant Lagrangian subspace of $W$ is also $m_y$-invariant.

**Proof.** We will prove that equations (9) imply (10).

Applying Lemma 4.17 to (10) we find that (10) $u,l$ is equivalent to (11) $u,l$, where

$$(11)_{u,l} \sum_{i=t+l-u+1}^{t-1} a_i (-1)^{t+l-u+i} \left( \binom{p + t + l - u - i}{l + t + 1 - i} - \binom{p + t + l - u - i}{l} \right) = 0.$$ 

So, we want to prove (11) $u,l$ for all $u, l$ such that $t \geq u > l \geq s + 1$. In other words, we want (11) $u,l$ for all $u, l$ such that $t + 1 \geq v \geq s + 3$, $v - 2 \geq l \geq s + 1$.

On the other hand, (9) $u,k$ is equivalent to (11) $u,l$ with $l = p - k - 1$, so (11) $x,p-y-1$ for $x, y$ such that $t \geq x > y \geq s + 2, x + y > p$, in other words, (11) $x,z$ for $x, z$ such that $t \geq x \geq s + 3, x - 2 \geq z \geq p - x - 2$. This directly covers part of the equations that we want to have. There are two cases left:

(a) $p - v - 2 > l \geq s + 1$,
(b) $v = t + 1$.

If (a) holds, we notice that (11) $v,l \leftrightarrow (11)_{p-l-1,p-v-1}$, and the latter is on our list of assumptions.

If (b) is true, we first notice that (11) $t+1,s+1$ is a linear combination of (11) $t,s+1$ and (11) $t,s$. But (11) $t,s$ has zero coefficients since \(\binom{n}{k} = \binom{n}{n-k}\). If $l > s + 1$, then (11) $t+1,l$ is a linear combination of (11) $t,l-1$ and (11) $t,l$. 

5. Symplectic form on $t$-colorings of tangles. In this section, we consider a generalization of the symplectic structure on Fox $p$-colorings that we studied in previous sections to Alexander–Burau–Fox $t$-colorings (briefly $ABF$-colorings) and a related $t$-invariant symplectic structure. Here we only sketch an idea which we plan to develop in the future. Consider a 2-dimensional disk with $2n$ boundary points $p_1, \ldots, p_{2n}$ ($n$ inputs and $n$ outputs, see Figure 1). For a fixed commutative ring $R$ with identity we consider the free $R[t^{\pm 1}]$-module $(R[t^{\pm 1}])^{2n}$ with a basis $e_1, \ldots, e_{2n}$ and we view elements of
this module as colorings of boundary points by elements of $R[t^{\pm 1}]$. Consider the submodule $W_{2n-1}$ of $R[t^{\pm 1}]^{2n}$ generated by all elements of the form $e_i - u e_{i+1}$, where $u$ equals $-1, t$, or $t^{-1}$ depending on whether $p_i, p_{i+1}$ are inputs or outputs. On the $R[t^{\pm 1}]$-module $W_{2n-1}$ regarded as an $R$-module, we define a $t$-invariant alternating form. The quotient of $W_{2n-1}$ by trivial (constant) colorings is a symplectic $R$-module. We show that $R[t^{\pm 1}]$-colorings of an oriented $n$-tangle $T$ which satisfy the Alexander–Burau relation at each crossing yield a Lagrangian in our symplectic $R$-module. We discuss a relation between $t$-colorings and homologies of the universal cyclic coverings and even-fold cyclic branched coverings. We note that our considerations in the previous sections correspond to the case when $t = -1$ and the double branched covering.

Let $R$ be a commutative ring with identity and assume that the ring $\mathbb{Z}[t^{\pm 1}]$ acts on $R$. Thus $R$ is also a $\mathbb{Z}[t^{\pm 1}]$-module. An $ABF$-coloring of an oriented tangle $T$ is an assignment of elements of $R$ to arcs of $T$, such that at each crossing the $ABF$ relation:

$$ (1 - t)a + tb - c = 0 $$

(or equivalently, $(1-t^{-1})a + t^{-1}c - b = 0$); see Figure 10. We are mostly interested in the cases when $R$ is $\mathbb{Z}[t^{\pm 1}], \mathbb{Z}[t^{\pm 1}]/(t^{2k-1})$ or $\mathbb{Z}_p[t^{\pm 1}]/(q(t))$, where $p$ is a prime number and $q(t)$ an irreducible polynomial. For a boundary point we put $\varepsilon_i = 1$ if the point is an output and $\varepsilon_i = -1$ for an input. Define the $t$-alternating condition for an element $\sum_{i=1}^{2n} a_i e_i \in R^{2n}$ to be $\sum_{i=1}^{2n} c_i a_i = 0$, where $c_{i+1} = \varepsilon_i \varepsilon_{i+1} t^{(\varepsilon_i + \varepsilon_{i+1})/2} c_{i}$.

Let $\text{Col}_{R,t}(T)$ be the $R$-module of $ABF$-colorings of a tangle $T$. There is a map

$$ \psi : \text{Col}_{R,t}(T) \to R^{2n} $$

which assigns to an $ABF$-coloring of $T$ a coloring of its $2n$ boundary points.

**Lemma 5.1.** Elements of $\psi(\text{Col}_{R,t}(T))$ satisfy the $t$-alternating condition.

**Proof.** The proof proceeds in the same manner as in the case $t = -1$. The most important observation is that the $t$-alternating condition holds locally. In fact for a crossing we can write the $ABF$-relation in the form $a + tb - ta - c = 0$. ■

As in the case $t = -1$ we can find a basis for the subspace satisfying the $t$-alternating condition. Let $W_{2n-1} = W_{2n-1}(\varepsilon_1, \ldots, \varepsilon_{2n})$ denote this $R$-module.
Lemma 5.2. \( W_{2n-1} \) is a free \( R \)-module of rank \( 2n - 1 \) with a basis \( \{ f_1, \ldots, f_{2n-1} \} \), where \( f_i = e_i + u_i e_{i+1} \) and \( u_i \) satisfies \( c_{i+1} = -u_i^{-1} c_i \), i.e.
\[
u_i = -\varepsilon_i \varepsilon_{i+1} t^{-(\varepsilon_i + \varepsilon_{i+1})/2}.
\]

Our plan is to define a \( t \)-invariant alternating form of nullity one on \( W_{2n-1} \) that, as before, satisfies the condition
\[
\varphi(f_i, f_j) = \begin{cases} 
0 & \text{if } |j - i| \neq 1, \\
 j - i & \text{otherwise},
\end{cases}
\]
and additionally
\[
\varphi(-tf_i, f_j) = \varphi(f_i, f_j).
\]

Of course, we did this before for \( t = -1 \), but to achieve it in a more general setting \((17)\) we need to regard \( R \) as a ring with a subring \( R_0 \). Let \( j : R_0 \to R \) be a homomorphism of rings and \( \varphi \) be defined over \( R_0 \). We show how it works in one important case:

Example 5.3. Let \( I \) be an ideal of \( R_0[t^{\pm 1}] \) such that \( (t + 1) \in I \) and \( R = R_0[t^{\pm 1}]/I \). Then the \( t \)-invariant alternating form \( \varphi \) with nullity one described above is well defined.

Theorem 5.4. If \( T \) is a rational tangle then \( \psi(\text{Col}_{R,t}(T)) \) is a Lagrangian of nullity one in \( W_{2n-1} \).

Proof. We proceed as in the proof of Theorem 2.1. We observe that the homomorphism corresponding to a crossing is \( (t + 1) \) approximation to a transvection, so \( \varphi \) is an isometry. \( \blacksquare \)

6. History of the paper. In May of 2000 while J. H. Przytycki visited Banach Center in Warsaw he attended T. Januszkiewicz’s talk on Coxeter groups and buildings. After the talk Przytycki asked a question concerning his student M. Dąbkowski’s calculations of the number of Fox 3-colorings induced on the boundary by 2-, 3- and 4-tangles. As Przytycki told Januszkiewicz, his student got the numbers 4, 40 and 1120. Januszkiewicz immediately noticed that these numbers suggested the number of Lagrangians in symplectic spaces over \( \mathbb{Z}_3 \). Soon after, J. Dymara and Przytycki found the appropriate symplectic form on the space of \( p \)-colorings of \( 2n \) boundary points on a circle. It was the beginning of our collaboration. The first version of this paper was ready in 2001, but its publication was delayed \((18)\) Przytycki gave several talks related to this paper, starting from \([Pr-6]\). Meanwhile

\((17)\) The general setting is a future project.

\((18)\) There was also a mathematical reason for this. In early winter of 2002 Dąbkowski and Przytycki solved the Montesinos–Nakanishi conjecture using the nonabelian version of Fox \( n \)-colorings called Burnside groups of links \([DP-1],[DP-2]\).
some parts of this work were cited in [Pr-3, Pr-4, Pr-5] and the approach described in Section 3 was developed by Cimasoni and Turaev [C-T-1, C-T-2].

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