On the associated primes of generalized local cohomology modules

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1. Introduction

Throughout this paper, $R$ is a commutative Noetherian ring with identity. For an ideal $\mathfrak{a}$ of $R$ and $i \geq 0$, the $i$-th local cohomology module of $M$ is defined as:

$$H^i_{\mathfrak{a}}(M) = \lim_{\rightarrow n} \text{Ext}^i_R(R/\mathfrak{a}^n, M).$$

In [8], Huneke conjectured that if $M$ is a finitely generated $R$-module, then the set of associated primes of $H^i_{\mathfrak{a}}(M)$ is finite. Singh [15] provides a counter example for this conjecture. However, it is known that the conjecture is true in many situations. For example, in [11] it is shown that if $R$ is local and $\dim R/\mathfrak{a} = 1$, then for a finitely generated $R$-module $M$, the set $\text{Ass}_R(H^i_{\mathfrak{a}}(M))$ is finite for all $i \geq 0$.

Also, Brodmann and Lashgari [2] showed that the first non-finitely generated local cohomology module of a finitely generated $R$-module has only finitely many associated primes. Also, see [10] and [4] for a far reaching generalizations of this result.

The following generalization of local cohomology theory is due to Herzog [7] (see also [17]). The generalized local cohomology functor $H^i_{\mathfrak{a}}(\,\cdot\,.,\,\cdot\,)$ is defined by

$$H^i_{\mathfrak{a}}(M, N) = \lim_{\rightarrow n} \text{Ext}^i_R(M/\mathfrak{a}^nM, N)$$

for all $R$-modules $M$ and $N$. Clearly, this is a generalization of the usual local cohomology functor. Recently, there are some new interest in generalized local cohomology (see e.g. [1], [5], [6] and [18]). Our main aim in this paper is to establish the following.

**Theorem 1.1.** Let $\mathfrak{a}$ be an ideal of $R$ and let $M$ and $N$ be two finitely generated $R$-modules. Then the following statements hold.
(i) For any positive integer $t$,

\[
\text{Ass}_R(H^t_a(M, N)) \subseteq \bigcup_{i=0}^{t} \text{Ass}_R(\text{Ext}^i_R(M, H^{t-i}_a(N))).
\]

(ii) If $d = pd(M)$ and $n = \dim N$ are finite, then $H^{n+d}_a(M, N)$ is Artinian. In particular $\text{Ass}_R(H^{n+d}_a(M, N))$ consists of finitely many maximal ideals.

(iii) Suppose that $(R, \mathfrak{m})$ is local with dimension $n$ and that $d = pd(M)$ is finite. Then $\text{Supp}_R(H^{n+d-1}_a(M, N))$ is finite.

Clearly (i) extends the main results of [10, Theorem B], [2, Theorem 2.2] and [4, Corollary 2.7], (ii) extends [12, Theorem 2.2], and (iii) is an improvement of [11, Corollary 2.4].

2. The results

First, we recall the definition of a weakly Laskerian module. An $R$-module $M$ is said to be Laskerian if any submodule of $M$ is an intersection of a finite number of primary submodules. Obviously, any Noetherian module is Laskerian. In [4], as a generalization of this notion, we introduced the following definition.

**Definition 2.1.** An $R$-module $M$ is said to be *weakly Laskerian* if the set of associated primes of any quotient module of $M$ is finite.

**Example 2.2.** (i) Every Laskerian module is weakly Laskerian.

(ii) Any module with finite support is weakly Laskerian. In particular, any Artinian $R$-module is weakly Laskerian.

**Theorem 2.3.** Let $\mathfrak{a}$ be an ideal of $R$ and $M$ be a finitely generated $R$-module. Let $N$ be an $R$-module and $t$ a positive integer. Then

\[
\text{Ass}_R(H^t_a(M, N)) \subseteq \bigcup_{i=0}^{t} \text{Ass}_R(\text{Ext}^i_R(M, H^{t-i}_a(N))).
\]

**Proof.** By [14, Theorem 11.38], there is a Grothendieck spectral sequence

\[
E^{p,q}_2 := \text{Ext}^p_R(M, H^q_a(N)) \Rightarrow H^{p+q}_a(M, N).
\]

For all $i \geq 2$, we consider the exact sequence

\[
0 \longrightarrow \ker d^{0,t}_i \longrightarrow E^{0,t}_i \xrightarrow{d^{0,t}_i} E^{i,t-i+1}_i.(1)
\]
Since $E^0_{i,t} = \ker d^0_{i-1,t}/\im d^1_{i-1,t+i-2}$ and $E^j_i = 0$ for all $j < 0$, we may use (1) to obtain $\ker d^i_{t+2} \cong E^i_{t+2} \cong \cdots = E^i_{i,t}$ for all $0 \leq i \leq t$. There exists a finite filtration

$$0 = \phi^{i+1}H^t \subseteq \phi^iH^t \subseteq \cdots \subseteq \phi^1H^t \subseteq \phi^0H^t = H^t_a(M,N)$$

such that

$$E^{i,t-i}_\infty = \phi^iH^t/\phi^{i+1}H^t$$

for all $0 \leq i \leq t$.

Now, the exact sequences $0 \rightarrow \phi^{i+1}H^t \rightarrow \phi^iH^t \rightarrow E^{i,t-i}_\infty \rightarrow 0$ ($0 \leq i \leq t$) in conjunction with

$$E^{i,t-i}_\infty \cong \ker d^i_{t+2} \subseteq \ker d^i_{i,t} \subseteq E^i_{i,t}$$

yields

$$\Ass_R(H^t_a(M,N)) \subseteq \bigcup_{i=0}^t \Ass_R(\Ext^i_R(M,H^t_{a}(N))).$$

Next, we obtain an extension of [2, Theorem 2.2], [10, Theorem B], and [4, Corollary 2.7].

**Corollary 2.4.** Let $a$ be an ideal of a local ring $(R, m)$ and $M$ be a finitely generated $R$-module, and $N$ a weakly Laskerian $R$-module. If $H^i_a(N)$ is weakly Laskerian module for all $i < t$, then $\Ass_R(H_a^t(M,N))$ is finite.

**Proof.** This is immediate by 2.3 and [4, Lemma 2.3 and Corollary 2.7]. □

**Corollary 2.5.** Let $(R, m)$ be a local ring and let $\dim R \leq 2$. Let $M$ be a finitely generated $R$-module and $N$ a weakly Laskerian $R$-module. Then $\Ass_R(H^t_a(M,N))$ is finite for all $t \geq 0$.

**Proof.** By [11, Corollaries 2.3, 2.4], [3, Theorem 6.1.2] and 2.2(ii), $H^i_a(N)$ is weakly Laskerian for all $t \geq 1$. Also, $\Gamma_a(N)$ is weakly Laskerian by 2.1. So, by 2.4, $\Ass_R(H^t_a(M,N))$ is finite for all $t \geq 0$. □

**Corollary 2.6.** Let $a$ be an ideal of a local ring $(R, m)$ and $\dim R = n$. Let $M$ be a finitely generated $R$-module and $N$ be an $R$-module such that $H^i_a(N) = 0$ for all $i \neq n-1, n$. Then $\Ass_R(H^t_a(M,N))$ is finite for all $t \geq 0$.

**Proof.** By [11, Corollaries 2.3, 2.4] and hypothesis, $\Supp_R(H^t_a(N))$ is finite for all $t \geq 0$; so that, by 2.4, $\Ass_R(H^t_a(M,N))$ is finite for all $t \geq 0$. □
Corollary 2.7. Let $a$ be an ideal of a local ring $(R, m)$ with $\dim R/a = 1$. Let $M$ and $N$ be two $R$-modules. Then $\text{Ass}_R(H^t_a(M, N))$ is finite for all $t \geq 0$.

Proof. It is clear that $\text{Supp}_R(H^t_a(N))$ is finite for all $t \geq 0$; hence by 2.4, $\text{Ass}_R(H^t_a(M, N))$ is finite for all $t \geq 0$. □

Following [9], a sequence $x_1, \ldots, x_n$ of elements of $a$ is said to be an $a$-filter regular sequence on $N$, if

$$\text{Supp}_R((x_1, \ldots, x_{i-1})N :_N x_i/(x_1, \ldots, x_{i-1})N) \subseteq V(a)$$

for all $i = 1, \ldots, n$, where $V(a)$ denotes the set of all prime ideals of $R$ containing $a$.

The concept of an $a$-filter regular sequence is a generalization of the one of a filter regular sequence which has been studied in [16, Appendix 2(ii)] and has led to some interesting results. It is easy to see that the analogue of [16, Appendix 2(ii)] holds true whenever $R$ is Noetherian, $N$ is a finitely generated $R$-module and $m$ replaced by $a$; so that, if $x_1, \ldots, x_n$ be an $a$-filter regular sequence on $N$, then there is an element $y \in a$ such that $x_1, \ldots, x_n, y$ is an $a$-filter regular sequence on $N$. Thus for a positive integer $n$, there exists an $a$-filter regular sequence on $N$ of length $n$.

The following Lemma, which needs the concept of a filter regular sequence, is a generalization of [13, Lemma 3.4].

Lemma 2.8. Let $a$ be an ideal of $R$ and $M$ be a finitely generated $R$-module such that $d = \text{pd}(M)$ is finite. Let $N$ be an $R$-module and assume that $n \in \mathbb{N}$ and $x_1, \ldots, x_n$ is an $a$-filter regular sequence on $N$. Then $H^{i+n}_a(M, N) \cong H^i_a(M, H^n_{(x_1, \ldots, x_n)}(N))$ for all $i \geq d$.

Proof. Consider the spectral sequence

$$E^{p,q}_2 := H^p_a(M, H^q_{(x_1, \ldots, x_n)}(N)) \Longrightarrow H^{p+q}_a(M, N).$$

We have $E^{p,q}_2 = 0$ for $q > n$ (by Theorem 3.3.1 of [3]) and for $q = n$, $p > d$ (by Proposition 2.5 of [13] and Lemma 1.1 of [18]). It therefore follows $E^{i,n}_2 \cong E^{i,n}_\infty$ and $E^{i,n}_\infty \cong H^{i+n}_a(M, N)$. This proves the result. □

The following result is a generalization of [12, Theorem 2.2] and [6, Theorem 1.2].
Let $a$ be an ideal of $R$ and let $M$ and $N$ be two finitely generated $R$-modules. Assume that $d = \text{pd}(M)$ and $n = \dim N$ are finite. Then $H_{a}^{n+d}(M, N)$ is an Artinian $R$-module. In particular, $\text{Ass}_R(H_{a}^{n+d}(M, N))$ is a finite set consisting of maximal ideals.

**Proof.** Let $x_1, \ldots, x_n$ be an $a$-filter regular sequence on $N$. Then, by 2.8,

$$H_{a}^{n+d}(M, N) \cong H_{a}^{d}(M, H_{(x_1, \ldots, x_n)}^{n}(N))$$

and, by [3, Exercise 7.1.7], $H_{(x_1, \ldots, x_n)}^{n}(N)$ is Artinian. Put $S = H_{(x_1, \ldots, x_n)}^{n}(N)$. Then $H_{a}^{d}(M, S) \cong H^{d}(\text{Hom}(M, \Gamma_a(\hat{E})))$ by [6, Lemma 2.1], where $\hat{E}$ is an injective resolution of $S$ such that its terms are all Artinian modules. Therefore $H_{a}^{n+d}(M, N)$ is Artinian and $\text{Ass}_R(H_{a}^{n+d}(M, N))$ is a finite set consisting of maximal ideals. $\square$

The following theorem is an improvement of [11, Corollary 2.4].

**Theorem 2.10.** Let $(R, m)$ be a local ring of dimension $n$, $N$ an $R$-module, $M$ a finitely generated $R$-module, and $d = \text{pd}(M)$ is finite. Then $\text{Supp}_R(H_{a}^{n+d-1}(M, N))$ is finite.

**Proof.** Consider the Grothendieck spectral sequence

$$E_2^{p,q} := \text{Ext}_R^p(M, H_{a}^{q}(N)) \Longrightarrow H_{a}^{p+q}(M, N).$$

So, we have a finite filtration

$$0 = \phi^{d+n}H^{d+n-1} \leq \phi^{d+n-1}H^{d+n-1} \leq \cdots \leq \phi^1H^{d+n-1} \leq \phi^0H^{d+n-1} = H_{a}^{d+n-1}(M, N)$$

and the equalities $E_2^{i,d+n-i-1} = \phi^iH^{n+d-1}/\phi^{i+1}H^{n+d-1}$ for all $0 \leq i \leq n + d - 1$. Since $E_2^{i,n+d-i-1} = 0$ for all $i \neq d - 1$, $d$ and $E_2^{i,n+d-i-1}$ is a subquotient $E_2^{i,n+d-i-1}$, it follows that

$$\phi^{d+1}H^{n+d-1} = \phi^{d+2}H^{n+d-1} = \cdots = \phi^{d+n}H^{n+d-1} = 0$$

and that

$$\phi^{d-1}H^{n+d-1} = \phi^{d-2}H^{n+d-1} = \cdots = \phi^0H^{n+d-1} = H_{a}^{n+d-1}(M, N).$$

Now, using the above consequences in conjunction with [11, Corollaries 2.3, 2.4], it is easy to see that $\text{Supp}_R(E_{\infty}^{d,n-1})$ and $\text{Supp}_R(E_{\infty}^{d-1,n})$ are finite sets.

Next, consider the exact sequence

$$0 \rightarrow E_{\infty}^{d,n-1} \rightarrow H_{a}^{n+d-1}(M, N) \rightarrow E_{\infty}^{d-1,n} \rightarrow 0,$$
to deduce that \( \text{Supp}_R(H^{n+d-1}_a(M, N)) \) is a finite. □

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