Modelling of Scalar Wave Propagation Problems in Heterogeneous Media by the Explicit Green’s Approach Method

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Abstract. The Explicit Green’s approach is an effective method for solving initial-boundary value problems in any kind of medium and geometry. This is due to the fact that numerical Green’s functions are adopted instead of analytical ones, rendering a more general approach. When the Explicit Green’s approach is applied to the scalar wave equation and Green’s functions are computed by the finite element method (FEM) in conjunction with the central difference time integration scheme, unstable results are observed. To circumvent this drawback, the present paper discusses the application of a time substep procedure to the central difference time integration for the Green’s functions computation. The substep procedure has the advantage of stabilizing the solution as well as increasing the accuracy order in the time domain from two to four. Numerical examples that demonstrate the accuracy and effectiveness of the proposed methodology are provided.

1. Introduction
Green’s functions are a very powerful tool and widely adopted to solve differential equations in many branches of science and engineering [1]. It is well known that the solution of a differential equation can be easily obtained once a Green’s function for the problem is known. For instance, if Green’s functions that satisfy all homogeneous boundary conditions of the problem under consideration are known, the solution is readily calculated. However, this technique, even though very important to construct benchmark solutions, is not feasible in practice due to the difficulty of finding Green’s functions for arbitrary geometries and/or nonhomogeneous media. Besides, in certain applications when Green’s functions are represented by infinite series, one may face with convergence problems. With the goal of obtaining a more general approach, Mansur et al. [3] proposed a formulation called Explicit Green’s Approach (ExGA) in which Green’s functions of the same problem under consideration are computed numerically and, as a result, the restriction about the medium and the geometry have been removed. Since the ExGA method adopts a numerical method as part of its formulation to compute Green’s functions, care should be taken in the selection of the numerical technique, specially the discretization in the time domain. In the present paper, an alternative way of computing the Green’s functions in the time is investigated when standard time integration schemes

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are employed. More precisely, an unequal time substeps are study giving rise to an improvement into the ExGA method as discussed hereafter.

2. Integral Expression for the Explicit Green’s Approach

The wave equation for the acoustic pressure \( p(\mathbf{x}, t) \) in a bounded, nonhomogeneous medium considering a constant density is given by [1]:

\[
\nabla^2 p(\mathbf{x}, t) + b(\mathbf{x}, t) = \frac{1}{c^2(\mathbf{x})} \ddot{p}(\mathbf{x}, t) \quad \text{in } \Omega \times (0, T)
\]

(1)

where \( c(\mathbf{x}) \) stands for the wave velocity and \( b(\mathbf{x}, t) \) is the source function per unit volume.

The solution of Eq. (1) can be accomplished by many numerical techniques such as the FEM, BEM and FDM. In the present paper, the ExGA method is adopted and its derivation can be initialized from the following integral identity over the space-time domain \( \Omega \times [0, T] \) [4]:

\[
\int_0^T \int_{\Omega} G(\mathbf{x}, \mathbf{y}, \tau - \tau) \left( \nabla^2 p(\mathbf{x}, \tau) + b(\mathbf{x}, \tau) - \frac{1}{c^2(\mathbf{x})} \frac{\partial^2 p(\mathbf{x}, \tau)}{\partial \tau^2} \right) d\Omega d\tau = 0
\]

(2)

The ExGA method resembles the standard time domain boundary element method (TD-BEM) for which the test function is assumed to be a Green’s function. Unlike the BEM, the Green’s function, in the ExGA method, satisfies the homogeneous boundary conditions of the problem under consideration. In this way, the Green’s function must be computed numerically since its expression is seldom known for any kind of geometry and medium. As a result, after applying the divergent theorem and integration by parts in the time to Eq. (2) and carrying out a discretization of the closed domain \( \bar{\Omega} = \Omega \cup \Gamma \) by suitable elements such as triangles or quadrilaterals over which all functions are interpolated; we obtain the following integral expression written in a recursive manner [3,5]:

\[
P_{n+1} = \tilde{G}(\Delta t)M P_n + G(\Delta t)\dot{M}P_n + \int_0^{\Delta t} G(\Delta t - \tau) F(t_n + \tau) d\tau
\]

(3)

\[
\dot{P}_{n+1} = \tilde{G}(\Delta t)MP_n + \tilde{G}(\Delta t)\dot{M}P_n + \int_0^{\Delta t} \dot{G}(\Delta t - \tau) F(t_n + \tau) d\tau
\]

(3)

where \( P_n \in \mathbb{R}^N \) represents the pressure nodal vector at time instant \( t_n = n\Delta t \), \( F \in \mathbb{R}^N \) the nodal load vector and \( G(\Delta t) \in \mathbb{R}^{N \times N} \) the Green’s function matrix at time instant \( t = \Delta t \) which is the discrete counterpart of the Green’s functions at nodal points with \( N \) being the number of unknowns nodal values. Furthermore, the matrix \( M \in \mathbb{R}^{N \times N} \) has the same expression of the standard FEM mass matrix [2].

3. Numerical Green’s functions

In order to initialize the time-marching process of Eq. (3) one must obtain only the Green’s matrix and its time derivatives at the first time step. To do so, the FEM [2] is employed here for the spatial discretization, yielding the following semidiscrete Galerkin approximation for the Green’s function for a given source point \( \mathbf{y} \) :

\[
\left( w^h(\mathbf{x}), \frac{1}{c^2(\mathbf{x})} G^h(\mathbf{x}, \mathbf{y}, t) \right) + \left( \nabla w^h(\mathbf{x}), \nabla G^h(\mathbf{x}, \mathbf{y}, t) \right) = 0
\]

(4)

\[
\left( w^h(\mathbf{x}), G^h(\mathbf{x}, \mathbf{y}, 0) \right) = 0
\]

(5)
\[ \left( w^h(x), \frac{1}{c^2(x)} \hat{G}^h(x,y,0) \right) = (w^h, \delta(x-y)) \]  

(6)

where \( \langle \cdot, \cdot \rangle \) represents the \( L_2(\Omega) \) inner product and \( w^h(x) \) the usual test function of polynomial type.

When time integration schemes are adopted in Eqs.(4)-(6), only the time interval \([0, \Delta t]\) must be considered in the analysis. Among many time integration schemes available in the literature [2], the well-known central difference scheme is adopted here since it is an explicit scheme and very easy to be implemented from a computational point of view. The expression for the central difference scheme is evaluated until the time instant \( t = \Delta t \) is reached.

It was shown by Loureiro [4] that when the central difference scheme is evaluated considering either one step or many equal substeps an unstable algorithm is derived. The main idea of the present paper is to still use the central difference scheme but consider an alternative strategy to yield a stable algorithm. It has been found that the time substeps proposed by Tarnow and Simo [6] can be used successfully for this purpose. Actually, they proposed a three substeps procedure that when applied to compute the Green’s matrix, the solution considering the following time substeps must be accomplished, namely: i) \( h_1 = \alpha \Delta t \); ii) \( h_2 = (1-2\alpha)\Delta t \), iii) \( h_3 = \alpha \Delta t \) where the final time instant is given by \( t_f = h_1 + h_2 + h_3 = \Delta t \). The parameter \( \alpha \) is determined so that a fourth order scheme \( O(\Delta t^4) \) is derived, yielding \( \alpha = \frac{2 + 2^{1/3} + 2^{2/3}}{3} \). A detailed study into the ExGA method in conjunction with the Tarnow and Simo’s substeps performed by Loureiro [4] revealed the following properties, namely: i) the fourth order accuracy is still retained; ii) the substeps are carried out only once for the first time step; afterwards steps of size \( \Delta t \) is employed to compute the pressure vector according to Eq. (3); and iii) the stability constraint is about 21.4% less than that of the standard central difference scheme. Hence, a novel accurate and efficient time-stepping procedure is derived.

4. Numerical Results and Discussion

The wave propagation in a region consisting of two materials due to a source point located at \((x = 0, y = a/4)\) with time variation expressed by \( f(T) = -400(T-0.1)T, T < 0.1 \) (\( T = ct/a \)) is analysed (see Fig. 1). Only half of the domain is considered in the modelling due to the symmetry and 40000 uniform quadrilateral elements are employed to discretize the domain. The proposed formulation gives stable results only if the time step satisfies the relation \( \beta \leq 0.786 \) where \( \beta = \frac{c_{\text{max}} \Delta t}{l} \) with \( l \) being the element length (here \( \beta = 0.72 \) is employed). On the boundary, a null pressure is imposed in the upper surface while a rigid boundary condition is prescribed in all the other boundaries.

![Figure 1: Problem geometry with the velocity model, source and receivers.](image-url)
The dimensionless pressure computed at receivers $1 (x = 0, y = a/5)$ and $2 (x = a/5, y = a/5)$ also shown in Fig. 1 considering the standard ExGA, the proposed one as well as the standard central difference scheme (given as a reference solution) are plotted in Fig. 2(a). It is clearly observed that unstable results are furnished by the standard ExGA as time advances (just after the wave has passed through the receivers), whereas stable results are achieved by the proposed ExGA. A snapshot of the pressure at dimensionless time instant $\tau = 0.504$ is displayed in Fig. 2(b) where we can see the primary reflection with a phase change at the upper boundary and the reflection and diffraction occur at the interface between the two media. When compared to the standard central difference scheme, the proposed ExGA method is more accurate since its accuracy order in time is two order higher than that of the standard central difference scheme and, as a consequence, a high-order time stepping procedure is established [4].

5. Conclusions
An improved Explicit Green’s approach formulation has been presented in this paper for solving the scalar wave equation in heterogeneous media. The use of the time substeps proposed by Tarnow and Simo [6] to compute Green’s functions at the first time step was capable of not only stabilizing the solution but also increasing the accuracy order from two to four in the time for the ExGA method. It is worth pointing out that the time substeps are applied only once and only for the Green’s functions computation with the pressure solution being computed by the ExGA expression with time step size $\Delta t$. The numerical example analysed here has confirmed the effectiveness and robustness of the proposed ExGA formulation.

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