Extension of Renormalizability

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Abstract

Arguments are provided which show that extension of renormalizability in quantum field theory is possible. A dressed scheme for the perturbation expansion is proposed. It is proven that in this scheme a nonrenormalizable interaction becomes renormalizable in the restrictive sense, i.e. its ultraviolet divergences can be cancelled by a finite number of counterterms included in the Lagrangian. As an illustration, the renormalization of the $\pi$-nucleon pseudovector interaction is discussed in some detail.

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It is now generally believed that renormalizability is not a fundamental requirement of quantum field theory. In fact, the widely acknowledged effective field theory [1, 2, 3] contains nonrenormalizable interactions. It has been especially emphasized by Weinberg [1] that renormalizability is unnecessary for the following main reasons: (1) it places a too stringent restriction on the possible types of renormalizable interactions and (2) as regards the cancellation of ultraviolet (UV) divergences, nonrenormalizable theories are actually also renormalizable, if all of the possible interactions allowed by symmetries are included in the Lagrangian, because then there will be enough counterterms to cancel every UV divergence. However, it is still desirable to find means to broaden the extent of renormalizability, since for a renormalizable interaction only a finite number of counterterms in the Lagrangian is needed for the elimination of infinities, while an infinite number is necessary, if it is a nonrenormalizable (NR or nr) interaction. Hereafter we shall always understand renormalizability in the above restrictive sense specified by finite number. We would like to show that such an extension is indeed possible. In this letter we shall only consider ordinary quantum field theory based on special relativity. Consider, for instance, a fermion propagator

\[ G_{\alpha\beta}(x) = \langle T[\psi_\alpha(x_1)\bar{\psi}_\beta(x_2)] \rangle = \int \frac{d^4 k}{(2\pi)^4} e^{ikx} G_{\alpha\beta}(k), \]  

where \( kx = k_\mu x^\mu = \vec{k} \cdot \vec{x} - k_0 t \) and \( k_\mu \equiv (\vec{k}, ik_0) \). Let the superscript "0" indicate a zeroth order approximation. The Dyson-Schwinger equation for \( G_{\alpha\beta}(k) \) reads

\[ G(k) = G^0(k) + G^0(k)\Sigma(k)G(k). \]  

If an appropriate approximation \( \Sigma_d(k) \) to the self-energy \( \Sigma(k) = \Sigma_d(k) + \Sigma_r(k) \) has been found, we may introduce a dressed propagator \( G_d(k) \) and rewrite Eq.(2-1) as follows

\[ G_d(k) = G^0(k) + G^0(k)\Sigma_d(k)G_d(k), \]

\[ G(k) = G_d(k) + G_d(k)\Sigma_r(k)G(k). \]

According to perturbation theory it is not difficult to see that a perturbation series can also be expanded in terms of \( G_d(k) \) (dressed scheme, DS) instead of \( G^0(k) \) (ordinary scheme, OS), if proper care has been taken to avoid redundancy of diagrams. Clearly, the same remark also applies to boson propagators. Consider an arbitrary connected one-particle irreducible Feynman diagram \( F \). Let us assume that each interaction \( i \) in the Lagrangian is characterized by \( n_\kappa \) fields of type \( \kappa \) and \( d_i \) derivatives acting on these fields. Following the argument given in [1, 4], one finds easily
that the superficial degree of divergence $d_F$ of diagram F can be written in the form:

$$d_F = 4 - \Sigma_\kappa E_\kappa(2 - p_\kappa) - \Sigma_i N_i r_i - \Sigma_i d_i,$$

(3-1)

$$\tilde{d}_F = d_F + \Sigma_i' d_i,$$

(3-2)

$$r_i = 4 - d_i - \Sigma_\kappa n_i \kappa (2 - p_\kappa),$$

(3-3)

where we have expressed the asymptotic behavior of the propagator $\Delta_\kappa(k)$ of field $\kappa$ (except the fermion propagator which is denoted by $G$) $\Delta_\kappa(k) \sim k^{-2p_\kappa}$, $E_\kappa$ is the number of external lines of field $\kappa$, $N_i$ the number of vertices of interaction $i$ in F and $r_i$ is defined by Eq. (3-1), where the prime over $\Sigma_i$ means that $i$ only runs over those vertices which are connected with external lines and whose momentum factor becomes an external momentum (see Eq. (8)). Since external momenta are not involved in the momentum integration, their $d_i$-contribution to $d_F$ should be subtracted. However, if we consider the asymptotic behavior of diagram $F$, it is given by $\tilde{d}_F$ (see Eq. (3-2)) [1], because all the external momentum factors should be included. Consider, for instance, the pseudovector $\pi$-N interaction (PVI) $\mathcal{L}_{pv} = ij \bar{\psi} \gamma_\mu \gamma_5 \vec{\tau} \cdot (\partial_\mu \vec{\phi}) \psi$, we have $d_{pv} = 1$, and $G^0(k) \sim k^{-1}$, $\Delta^0_{\pi}(k) \sim k^{-2}$, or $p_\pi = 1/2$ and $p_\pi = 1$ if we assume the ordinary scheme (OS). According to Eq. (3-3) $r_{pv} = -1$. Eq. (3-1) says that $d_F$ grows with $N_i(i = pv)$, thus as is wellknown, PVI is nonrenormalizable. Now let us study DS. The one loop approximation to the nucleon self-energy reads

$$\Sigma_{pv}(k) = 3 f^2 \int \frac{d^4 q}{(2\pi)^4} \gamma_\mu q_\mu \gamma_5 G(k - q) \gamma_\nu q_\nu \gamma_5 \Delta_\kappa(q).$$

(4)

It is known that we have $\Sigma(k) = \gamma_\mu k_\mu a(k^2) - iMb(k^2)$. Substituting

$$G^0(k) = -[\gamma_\mu k_\mu - i(M - i\varepsilon)]^{-1}, \Delta^0_{\pi}(k) = -i[k^2 + m_\pi^2 - i\varepsilon]^{-1}$$

(5)

in Eq. (4), one finds that $\Sigma_{pv}(k)$ is divergent. However, as usual, one may calculate it by means of regularization, namely by Feynman’s parametrization and dimensional regularization or by the
counterterm method, i.e. by introducing counterterms in the Lagrangian and eliminating the divergences by cancellation. From both of these methods one obtains that the regularized $a_{pv}(k^2)$ and $b_{pv}(k^2)$ behave asymptotically as

$$a_{pv}(k^2) \sim k^2, \quad b_{pv}(k^2) \sim k^2$$  \hspace{1cm} (6)

(see Eqs.(10) and (16-17)). A formal solution to Eq.(2) can be written in the form

$$G(k) = -[\gamma_\mu k_\mu - iM + \Sigma(k)]^{-1}. \hspace{1cm} (7)$$

Let us choose $\Sigma_d(k) = \Sigma_{pv}(k)$. From Eq.(2-2) we obtain $G_d(k)$. According to Eqs.(6) and (7) we assert $G_d(k) \sim k^{-3}$ which implies $p_\pi = 3/2$. Thus, if the expansion is in terms of $G_d(k)$, from Eq.(3-2) we get $r_{pv} = 1$, even though $\Delta_\pi(k)$ is still taken as $\Delta^0_\pi(k)$. Eq.(3-1) shows that now PVI becomes super renormalizable. Since $G_d(k)$ is derived from $G^0(k)$ through consideration of higher order terms (in fact, through the Dyson-Schwinger equation, Eq.(2-2), it has summed an infinite series produced by $G^0$ and $\Sigma_d$), one may feel strange why there is such a drastic change of convergence behavior between OS and DS. This is clearly due to the fact that $\Sigma_d = \Sigma_{pv}$ is divergent, PVI is nonrenormalizable (see below) and we have made $\Sigma_{pv}$ finite by regularization. Hence DS may differ from OS essentially. However, for example, for the renormalizable pseudoscalar $\pi$-N interaction the one loop contribution to the nucleon self-energy yields $a_{ps}(k^2) \sim \ln k^2$ and $b_{ps}(k^2) \sim \ln k^2$, therefore $G_\pi(k) \sim (\ln k^2)^{-1}$ as given by Weinberg’s theorem. This shows that $p_\pi$ is essentially equal to 1/2 and no unexpected change of renormalizability occurs. Let us now consider the $\pi$–meson self-energy. The one loop contribution is

$$\Pi(k) = -3 f^2 \int \frac{d^4q}{(2\pi)^4} Tr[\gamma_\mu k_\mu \gamma_5 G(k + q) \gamma_\nu k_\nu \gamma_5 G(q)]. \hspace{1cm} (8)$$

It is seen that the momentum factors $k$ coming from derivatives acting on the $\pi$–fields now belong to external momenta, thus they do not contribute to the superficial degree of divergence $d_\pi$ of $\Pi(k)$ as mentioned in Eq.(3-1). In OS $d_\pi = 2$ and $\Pi_\pi(k)$ is divergent. Hence, for its calculation regularization is necessary. However, in DS since in Eq.(8) $G$ should be replaced by $G_d$, whose asymptotic behavior is $G_d(k) \sim k^{-3}$, we have $d_{ln} = -2$, thus $\Pi_d(k)$ is convergent.

From Eq.(2-2) it is seen that Eq.(4) is an integral equation for $\Sigma_{pv}(k)$. Whether its solution exists concerns the self-consistency of our method. In the case of renormalizable interactions such integral equation has been considered previously [5, 6, 7, 8]. Here, we shall only consider the case where the asymptotic behavior of $G(k) \sim k^{-2p_\pi}$ with $1 \leq 2p_\pi \leq 4$. The right-hand
side of Eq. (4) is then divergent and has to be regularized. Our previous result indicated in Eq. (6) corresponds to a solution to Eq. (4) obtained by a first iteration with the initial input given by Eq. (5). Since it has been regularized and is finite, we may take it as a new input and continue the iteration. However, it does not converge. The iteration results oscillate between $\Sigma_{\nu} \sim k^3$ and $\Sigma_{\nu} \sim k^1$. This is owing to the fact that different regularization will render the integral equation different. Following the above naive iteration procedure, one is actually solving two integral equations: one with $G(k) \sim k^{-3}$ and the other with $G(k) \sim k^{-1}$ in Eq. (4). Therefore, one has to look for a formulation which requires only one regularization for the entire iteration process. This can be achieved by the Källen-Lehmann spectral representation, which for $G(k)$ can be written as

$$G(k) = -Z_i \int_0^\infty \frac{dm^2 \gamma_\mu k_\mu f_\alpha(-m^2) + iM_i f_\beta(-m^2)}{k^2 + m^2 - i\epsilon},$$

(9-1)

$$f_\gamma(-m^2) = \delta(m^2 - M_i^2) + \theta(m^2 - M_\tau^2)\gamma(-m^2),$$

(9-2)

where $(-Z_i)$ is the residue of $G(k)$ at the physical pole $\gamma_\mu k_\mu = iM_i$, $\gamma = \alpha$ or $\beta$, $m_i = M_i + m_\tau$ and $\theta$ is the step function. Substituting Eq. (9) into Eq. (4), choosing $\Delta_\alpha(q) = \Delta_\alpha(q)$ and again using Feynman’s parametrization and dimensional regularization, we find

$$a_R(k^2) = \frac{3f^2Z_i}{16\pi^2} \int_0^\infty dm^2 f_\alpha(-m^2) \int_0^1 dx \left\{ (1 + 3x) \ln K^2 - 2x \right\} K^2 + x^2(1 - x)k^2 \ln K^2),$$

(10-1)

$$b_R(k^2) = \frac{3f^2Z_i M_i}{16\pi^2 M} \int_0^\infty dm^2 f_\beta(-m^2) \int_0^1 dx \left\{ x^2k^2 \ln K^2 + K^2(1 - 2 \ln K^2) \right\},$$

(10-2)

$$K^2 = x(1 - x)k^2 + xm^2 + (1 - x)m_\tau^2,$$

(10-3)

where to regularize $a$ and $b$ we have used $\overline{MS}$ (modified minimal subtraction) and deleted terms proportional to $(1/\epsilon - \gamma_E + \ln 4\pi)$. From Eqs. (7) and (9)) one can further derive the following relations

$$Z_i \alpha(k^2) = \frac{1}{\pi} \text{Im} \frac{1 + a_R(k^2)}{D(k^2)}, \quad Z_i \beta(k^2) = \frac{1}{\pi M_i} \text{Im} \frac{1 + b_R(k^2)}{D(k^2)},$$

(11)

$$D(k^2) = k^2 \left[ 1 + a_R(k^2) \right]^2 + M^2 \left[ 1 + b_R(k^2) \right]^2.$$

We still need to know how to determine $M$ and $Z_i$. By definition and Eq. (7) one easily finds

$$M_i \left[ 1 + a(-M_i^2) \right] - M \left[ 1 + b(-M^2) \right] = 0,$$

$$Z_i^{-1} = 1 + a(-M_i^2) + 2M_i \left[ Mb'(-M_i^2) - M_i a'(-M_i^2) \right],$$

(12)

where $C'(k^2) \equiv dC(k^2)/dk^2$. Eqs. (10)-(12) build a closed set of equations for the determination of $a_R, b_R, \alpha$ and $\beta$. They can be solved by the method of iteration. However, we have found that the
iteration procedure does not converge, though $\bar{MS}$ has been proved successful in other aspects. In
the following we shall describe our solution to Eq.(4) obtained by the counterterm method. The
counterterm CTL to be included in the Lagrangian can be written as

$$CTL = -\bar{\psi} \left[ M_\mu + \sum_{\ell=1}^{3} \frac{1}{\ell!} \eta_\ell (\gamma_\mu \partial_\mu)^\ell \right] \psi.$$  (13)

Its contribution to the fermion self-energy CTS is given by

$$CTS = i \left[ M_\mu + \sum_{\ell=1}^{3} \frac{1}{\ell!} \eta_\ell (i\gamma_\mu k_\mu)^\ell \right], \quad (14-1)$$

$$\Sigma_R(k) = \Sigma(k) + CTS = \gamma_\mu k_\mu a(k^2) - iMb(k^2) + CTS,$$  (14-2)

where $\Sigma_R(k)$ means the renormalized self-energy. We note that for our purpose there is no need
to redefine $(\gamma_\mu \partial_\mu)^\ell$ in Eq.(14-1), because the contribution of CTL can always be made to form a
pair with the self-energy $\Sigma(k)$ and so its net effect is included in $\Sigma_R(k)$ (see FIG.1). Hereafter the
counterterm method will be referred to as the method of renormalization. In order to determine
the parameters in Eq.(13), we shall use the on-shell renormalization conditions which read

$$\Sigma_R(k)|_{\bar{\gamma}=iM_t} = \frac{\partial \Sigma_R}{\partial \bar{\gamma}}|_{\bar{\gamma}=iM_t} = 0,$$  \hspace{1cm} (15-1)

$$\frac{\partial^2 \Sigma_R}{\partial \bar{\gamma}^2}|_{\bar{\gamma}=iM_t} = i\kappa; \quad \frac{\partial^3 \Sigma_R}{\partial \bar{\gamma}^3}|_{\bar{\gamma}=iM_t} = \lambda,$$  \hspace{1cm} (15-2)

where $\bar{\gamma} \equiv \gamma_\mu k_\mu$. The two conditions in Eq.(15-1) imply $M = M_t$ and $Z_t = 1$. Usually one also put
$\kappa = \lambda = 0$. Here we leave them to be two constant free parameters which may be determined by
other requirements or by fitting experimental data. we have the following relations

$$a_\alpha(k^2) = \hat{a}_\alpha(k^2) + M_t \kappa - \frac{1}{2} \lambda M_t^2 + \frac{1}{6} \lambda k^2$$
$$b_\alpha(k^2) = \hat{b}_\alpha(k^2) + \frac{1}{2} M_t \kappa - \frac{1}{6} \lambda M_t^2 - \frac{1}{2} \kappa - \lambda M_t k^2,$$  \hspace{1cm} (16)

where $\hat{a}_R$ and $\hat{b}_R$ are the results for $\kappa = \lambda = 0$. Their explicit expressions are

$$\hat{a}_R(k^2) = -\frac{3f^2}{16\pi^2} \int_{0}^{\infty} dm^2 f_\alpha(-m^2) \int_{0}^{1} dx \left\{ x^2(1-x)k^2 + (1+3x)k^2(k^2) \right\} \ln \frac{K^2(-M_t^2)}{K^2(k^2)}$$

$$\left( k^2 + M_t^2 \right) C_\alpha + 4M_t^2(2M_t^2 - k^2) C_\alpha(1) - 2M_t^2(2M_t^2 - k^2) C_\alpha(2) + 4M_t^2(M_t^2 - 1/3k^2) C_\alpha(3),$$  \hspace{1cm} (17-1)

$$\hat{b}_R(k^2) = -\frac{3f^2}{16\pi^2} \int_{0}^{\infty} dm^2 f_\beta(-m^2) \int_{0}^{1} dx \left\{ 2K^2(k^2) - x^2 k^2 \right\} \ln \frac{K^2(k^2)}{K^2(-M_t^2)} - (k^2 + M_t^2) C_\beta$$

$$- M_t^2(10k^2 - 2M_t^2) C_\beta(1) + 4M_t^2 k^2 C_\beta(2) - 4M_t^2(k^2 - 1/3M_t^2) C_\beta(3),$$  \hspace{1cm} (17-2)
where \( C_\gamma \) and \( C_\gamma(l) \) (\( \gamma = \alpha \) or \( \beta \), \( l=1 \) to \( 3 \)) are \( k^2 \)-independent constants. Their explicit expressions can be found easily from Eqs.(14) and (15). As they are long and space consuming, we shall not write them down here. Eqs.(11), (16) and (17) now build a closed set of equations. We have solved this set by iteration. The initial input is taken as \( \alpha = \beta = 0 \). It is found that the iteration series converges quite quickly. In FIG.2 and FIG.3 we have plotted our numerical results for the self-consistent sets \((\alpha, \beta)\) and \((a_R, b_R)\). Besides \( \kappa = \lambda = 0 \), we have further calculated two cases of \( \kappa \neq 0 \) and \( \lambda \neq 0 \) as an illustration. From FIG.2 one observes that through appropriate choice of their values the peak can be made sharper and more pronounced. Thus, the introduction of these two additional parameters is physically meaningful and worth considering. Finally we would like to emphasize that the solution to the regularized integral equation, Eq.(4), exists and can be obtained by the above iteration procedure also offers a noteworthy support of our proposal for the extension of renormalizability.

We would like to point out that the above results of renormalizability for PVI are general, namely if an interaction is nonrenormalizable in OS, it is always possible to find a DS such that it becomes renormalizable. It is interesting to note that the possibility comes from the condition \( r_{nr} < 0 \). Let \( i = \lambda \) be a NRI and use a superscript * to label OS. Consider a lowest order fermion self-energy diagram \( \Sigma_f(k) \) with each of its external vertices A and B (see FIG.1) being a \( \lambda \). According to Eq.(3) the asymptotic behavior of \( \Sigma_f(k) \) can be written as \( \tilde{d}_f^* = d_f^* + \Sigma d_i = 2 \left[ p_f^* + |r_f^*| \right] \), because \( r_f^* < 0 \) and \( E_f = N_\lambda = 2 \). Since regularization will not affect \( \tilde{d}_f^* \) (see, for instance, Eqs(10) and Eqs.(16, 17)), if we substitute the regularized \( \Sigma_f(k) \) for \( \Sigma_d(k) \) in Eq.(2-2), its solution \( G_{df}(k) \), as

![FIG. 2: Baryon spectral functions for \( \pi \) model with pseudovector \( \pi - N \) coupling: (a) \( \alpha(k^2) \) and (b) \( \beta(k^2) \).](attachment:fig2.png)
shown by Eq.(7), behaves asymptotically as $k^{-d_f} = k^{-2}\pi_f$ or $\pi_f = 2/\pi_f$. From Eq.(3-3) one easily finds that in DS $r_\lambda = r_\lambda^* + n_{sf} (\tilde{d}_f^2/2 - p_f^*) = (n_{sf} - 1)|r_\lambda^*|$, which shows $r_\lambda \geq 0$, because $n_{sf} \geq 1$. Thus, NRI $\lambda$ becomes renormalizable or super-renormalizable in DS. If $\lambda$ contains no fermion fields, clearly we may instead consider a lowest order boson self-energy diagram $\Pi_B(k)$ with each of its external vertices A and B being $\lambda$. Following the same argument, we again find that the asymptotic behavior of $\Pi_B(k)$ is given by $\tilde{d}_B^2 = 2(p_B^* + |r_A^*|)$, and the dressed propagator $\Delta_{db}(k)$ behaves asymptotically as $k^{-d_b} = k^{-2}\pi_{bf}$ or $\pi_{bf} = \tilde{d}_f^2/2$, i.e. in DS determined by $\Delta_{db}(k)$ $r_f = r_f^* + n_{fb} (\tilde{d}_b^2/2 - p_b^*) = (n_{fb} - 1)|r_f^*|$. Thus $r_f \geq 0$, which again confirms our above conclusion. Clearly the above argument also applies to the case of more than one NRI. Say we have $\zeta$ different interactions which can contribute to the fermion self-energy and among which there are two NRIs $\lambda$ and $\eta$ with $r_\lambda^* < r_\eta^*$. Altogether we can build $\zeta \times \zeta$ fermion self-energy insertions $\Sigma_{ij}(k)$ with the two external vertices being interaction $i$ and $j$, respectively. $\Sigma_{ij}(k)$ should be regularized, if it is divergent. Now set $\Gamma_d(k) = \Sigma_{ij}(k)$ in Eq.(2), where $\zeta$ may be smaller than $\xi$. The choice of $\zeta$ will affect the efficiency of calculation, but is irrelevant to our present discussion. We shall only require that $i=j=\lambda$ is included in $\Gamma_d(k)$. From Eqs.(3) and (7) one finds that the asymptotic behavior of $G_{df}(k)$ is given by $k^{-d_f} = k^{-2}\pi_{df}$ with $\pi_{df} = p_f^* + |r_\lambda^*|$. Thus, in DS $r_\lambda = (n_{sf} - 1)|r_\lambda^*| \geq 0$, while $r_{\eta} = n_{sf} |r_\lambda^*| - |r_\eta^*| > 0$. Clearly our conclusion holds generally. Moreover one observes that all the renormalizable interactions in OS become super renormalizable in the above DS, if the latter is constructed by means of NRI as shown above. Note that DS is derived from OS in
a simple and natural way. As we have emphasized, the reason that they may differ significantly in their property of renormalizability is because of regularization and nonrenormalizability. We have demonstrated that DS can be constructed by dressed propagators determined by the Dyson-Schwinger equation with the regularized fermion or boson self-energy as its kernel. It is seen that besides being renormalizable, DS further offers a non-perturbative method for the calculation. To show that DS exists, we have only made use of Eq.(3) and the existence of regularized expression for the self-energy. Thus, we may conclude that the present quantum field theory based on special relativity with interactions not too exotic is actually a renormalizable theory, if a proper framework of representation is established. A more stringent condition is that the solution to the self-energy integral equation should exist. In principle this would not be a problem, if it were not for the fact that regularization is necessary. Though a general mathematical existence proof is beyond the scope of this letter, in text we have suggested a method which can be used to study and check each special case individually.

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