GENUS STATISTICS FOR STRUCTURE FORMATION WITH TOPOLOGICAL DEFECTS

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ABSTRACT

I study the efficiency of genus statistics in differentiating between different models of structure formation. Simple models that reproduce the salient features of the structure that are seeded by topological defects are examined. I consider accretion onto static point masses, modeling slow-moving cosmic string loops or other primordial pointlike sources. Filamentary structures and wakes are also considered as models of the structures seeded by slow- and fast-moving strings, respectively. The predictions of genus statistics for Gaussian fluctuations are compared to genus curves obtained by the CfA Redshift Survey. A generic class of density models with wakes and filaments is found to provide results comparable to or better than Gaussian models for this suite of tests.

Subject headings: accretion, accretion disks — cosmic strings — large-scale structure of universe — methods: statistical

1. INTRODUCTION

The two main paradigms for the origin of the seed perturbations are, on the one hand, the inflationary scenario that produces Gaussian fluctuations and, on the other, topological defects produced at phase transitions in the early universe that produce a non-Gaussian spectrum of density perturbations, especially on small scales. Because both models produce a nearly scale-invariant spectrum of fluctuations, the power spectrum is not the best statistic to distinguish between them; a higher order statistic is required in addition. In this article, I investigate the efficiency of genus statistics in distinguishing between different models of structure formation (such as inflationary and cosmic string models), and I compare these with observational genus curves obtained from the CfA Redshift Survey. Some of the models are oversimplified but aim at reproducing the salient features of the structure that are seeded by topological defects. I consider accretion onto static point masses, modeling slow-moving cosmic string loops or other primordial pointlike sources. Filamentary structures and wakes are also considered as models of the structures seeded by slow- and fast-moving, long, wiggly strings.

Previously, some work on genus curves of isodensity contours for toy models of structure formation seeded by topological defects was done by Brandenberger, Kaplan, & Ramsey (1993). By requiring that the structures all have the same size and mass, they appear not to have properly taken the scaling solution into account. The size of the defect seeds increases with time (proportionally to the horizon), and the amplitude of the density perturbations induced by larger defects is usually smaller because the perturbations have less time to grow by gravitational instability. This effect is properly accounted for in the present article, and so I consider this work an improvement over that of Brandenberger et al. (1993). Robinson & Albrecht (1996) performed a similar study with cosmic string wakes. Their cosmic string toy model consisted of a realization of a string power spectrum, in which the phases of the Fourier modes were chosen at random, plus a single cosmic string wake. They concluded that the genus statistic is not a good discriminator between their model and a model without the wake included. However, their model underestimates the presence of sheetlike features in the density field and, therefore, it should not be considered a good cosmic string toy model in this regard. In this article I show that the genus statistic is a good discriminator between different cosmic string toy models and is sensitive not only to the power spectrum of the perturbations but also to correlations between the phases of different Fourier modes.

This article is organized as follows. In § 2, the Zeldovich approximation is solved for accretion onto static point masses, filaments, and wakes, assuming that the dark matter is cold. At the end of the section, a modification of the Zeldovich approximation that accounts for neutrino free-streaming is introduced for a hot dark matter model. In § 3, the genus statistic is described. Analytic results for Gaussian perturbations are given, and topological measures of departures from Gaussianity are introduced. In § 4, I describe the way in which the fluctuations are generated for the models considered and relate that to the results of § 2. In § 5, the results are presented. The dependence of the genus curves on the type and number of topological defects present is demonstrated and error bars indicating sample variance are introduced. The density probability distributions for several of the models are also given, and the parameterization of the genus curve is discussed. The final section is a discussion of the results.

In this article, I assume that $h = H_0/(100 \, \text{km} \, \text{s}^{-1} \, \text{Mpc}^{-1}) = 0.5$ and $\Omega = 1$.

2. THE GROWTH OF PERTURBATIONS

2.1. Cold Dark Matter

I employ the Zeldovich approximation (Zeldovich 1970) to examine each model I study. The universe is assumed to be flat, with no cosmological constant, and the dark matter is assumed to be cold. (In the next section, we modify the Zeldovich approximation in order to describe hot dark matter as well.) In the matter era, the dynamical equations that describe the evolution of the scale factor are

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G \rho_b}{3}$$

(1)
and
\[ \frac{\ddot{a}}{a} = - \frac{4\pi G \rho_b}{3}, \] (2)
where \( \rho_b \) is the background density. The position of the cold dark matter (CDM) particles is given by
\[ r = a(t)[q + \psi(q, t)], \] (3)
where \( q \) is the unperturbed comoving position of the particle and \( \psi \) is the comoving displacement vector of the particle. In the presence of a perturbing string seed, the CDM particle will obey
\[ \frac{d^2r}{dt^2} = F_{\text{seed}} + F_{\text{matter}}, \] (4)
where the acceleration \( F_{\text{matter}} \) due to the surrounding matter is given by
\[ F_{\text{matter}} = -\nabla \Phi, \] (5)
with the gravitational potential \( \Phi \) satisfying the Poisson equation
\[ \nabla^2 \Phi = 4\pi G \rho = 4\pi G \rho_b (1 + \delta). \] (6)
The conservation of mass is described by the continuity equation that can be written in comoving coordinates to first order in \( \psi \) as
\[ \frac{\partial (\rho a^3)}{\partial t} + \nabla_q \cdot (\rho a^3 \dot{\psi}) = 0, \] (7)
which can be integrated in the limit of small perturbations to give
\[ \delta = \frac{\dot{\rho}}{\rho} \sim - \nabla_q \cdot \psi, \] (8)
again to first order in \( \psi \). The Poisson equation can then be integrated to give
\[ \nabla_r \Phi = \frac{4\pi G \rho_b}{3} \left( r - 3a\psi_{||} \right), \] (9)
where \( \psi_{||} \) and \( \psi_{\perp} \) are defined by
\[ \psi = \psi_{||} + \psi_{\perp}, \] (10)
\[ \nabla_r \times \psi_{||} = 0, \] (11)
and
\[ \nabla_r \cdot \psi_{\perp} = 0. \] (12)
In the linear regime with \(|\psi| \ll |q|\), we can use equations (2), (4), (5), and (9) to write Zeldovich approximation as
\[ a^2 \frac{\ddot{\psi}_{||}}{a^2} + \ddot{a} \frac{\dot{\psi}_{||}}{a \dot{a}} + \ddot{a} \frac{\dot{\psi}_{||}}{a \dot{a}} \psi_{||} = F_{\text{seed}}_{||}, \] (13)
\[ a^2 \frac{\ddot{\psi}_{\perp}}{a^2} + 2 \dot{a} \frac{\dot{\psi}_{\perp}}{a \dot{a}} \psi_{\perp} = F_{\text{seed}}_{\perp}. \] (14)
If the perturbations are irrotational to begin with, and if the source term is irrotational, \( \psi_{\perp} = 0 \) so that \( \psi = \psi_{||} \). In this case, the Zeldovich approximation takes the usual form
\[ \left( \frac{\ddot{\psi}_{||}^2}{a^2} + 2 \dot{a} \frac{\dot{\psi}_{||}}{a \dot{a}} + \frac{3}{a} \ddot{a} \right) = \frac{1}{a} F_{\text{seed}} = S_{\text{seed}}. \] (15)
If the seed is a static point mass, then \( S_{\text{seed}} \) is given by
\[ S_{\text{pmass}}(q, t) = - \frac{GM}{a^3 |X|^{3}}, \] (16)
where \( X = q - q_p \) and \( q_p \) is the comoving position of the point mass. If the seed is a static line of mass, then \( S_{\text{seed}} \) is given by
\[ S_{\text{lmass}}(q, t) = - \frac{2GM_A}{a^2 |X|^2}, \] (17)
where now \( X = q - q_l \) and \( q_l \) is the comoving position of the point in the line of mass nearer to \( q \), and \( M_A \) is the mass per unit length of the line of mass. The Zeldovich approximation can be solved using the Green’s function method
\[ \psi(q, t) = \int_{t_i}^{\infty} G(t, t') S_{\text{seed}} dt', \] (18)
where \( t_i \geq t_{eq} \) is the time at which the perturbations start to grow. Assuming that \( \psi(t_i) = 0 \) and \( \psi(t_{eq}) = 0 \), the matter era Green’s function \( G(t, t') \) is given by (see, for example, Veeraragavan & Stebbins 1990)
\[ G(t, t') = \frac{3}{5} \left[ \left( \frac{t}{t'} \right)^{2/3} - \frac{t}{t'} \right], \quad t > t', \] (19)
and
\[ G(t, t') = 0, \quad t < t'. \]
It will be convenient to normalize the scale factor so that \( a(t_0) = 1 \), where \( t_0 \) is the age of the universe today. The solutions to equation (15) for the static point mass and static line of mass are
\[ S_{\text{pmass}}(q, t) = \frac{3GMXt_0^{2/3}}{2 |X|^{3}} \left[ 1 - 2 \frac{t}{5} - \frac{3}{5} \left( \frac{t}{t_i} \right)^{2/3} \right], \] (20)
and
\[ S_{\text{lmass}}(q, t) = - \frac{6GM_Xt_0^2}{5 |X|^2} \times \left[ \left( \frac{t}{t_i} \right)^{2/3} - \frac{3}{5} \left( \frac{t}{t_i} \right)^{2/3} + \frac{3}{5} \left( \frac{t}{t_i} \right) \right]. \] (21)
The formation of wakes can be modeled given an initial velocity of the form
\[ \psi(q, t_i) = - \frac{u_i X}{|X|}, \] (22)
where \( X = q - q_w \) and \( q_w \) is the comoving position of the point in the plane of the wake nearer to \( q \). This is given to the particles in cosmic string models because of the conical spacetime with \( u_i = 8\pi G \mu \gamma, \) where \( \gamma \) is the string velocity. The solution of the Zeldovich equation in this case is
\[ \psi(q, t) = - \frac{3}{5} \frac{u_i t_i}{t_i^{1/3} t_0^{2/3}} \frac{X}{|X|} \left[ \left( \frac{t}{t_i} \right)^{2/3} - \frac{t_i}{t} \right]. \] (23)

### 2.2. Hot Dark Matter

If the dark matter is hot, then small-scale perturbations with wavelengths smaller than the neutrino (or other hot dark matter candidate) comoving free-streaming length, \( \lambda_{FS} \),
will be erased. The comoving distance that hot particles can move since $t \geq t_{eq}$ is

$$\lambda_{FS}(t) = \int_{t_{eq}}^{t} v_s(t') \frac{dt'}{a(t')} \sim 3v_s(z)z(t),$$  \hspace{1cm} (24)

where $v_s(t) \propto t^{-2/3}$ is the thermal velocity of the hot particles, $z(t)$ is the redshift, and $t_{eq}$ is the age of the universe today. The comoving free-streaming length, $\lambda_{FS}$, decreases proportionally to $t^{-1/3}$ in the matter era. The maximal free-streaming length is then

$$\lambda_{FS} = 3v_s(t_{eq})z_{eq} = \lambda_{FS},$$  \hspace{1cm} (25)

Here, I consider a hot dark matter model with two neutrino species. In particular, I assumed that these two species have mass, $m_n = 46 \text{ } h^2 \text{ } eV$, which is sufficient to make $\Omega = 1$. In this case, the velocity of the neutrinos at $t_{eq}$ is given by

$$v_s(t_{eq}) = T_{eq}^2/m_n \sim 0.09,$$  \hspace{1cm} (26)

so that $\lambda_{FS} \sim 3 \text{ } h^{-2} \text{ Mpc}$. The reason for choosing two neutrino species instead of only one is that, in the first case, the free-streaming length is increased by a factor of 2 if the particles have the same mass. This allows us to observe the effect that neutrino damping has on the genus curves for larger values of the smoothing scale (refer to § 5.2).

In order to properly describe the formation of structure in the context of a hot dark matter model, one would need to solve the Boltzmann equation for the dark matter particles. However, by studying the clustering of neutrinos in cosmic string-induced wakes, Perivolaropoulos, Brandenberger, & Stebbins (1990) showed that most of the results can be described correctly using a naive modification of the Zeldovich approximation. This modification is based on the fact that, on average, the dark matter particles will start to collapse only when the comoving free-standing length has fallen below $|X|$. So, I modify the Zeldovich approximation in the context of a hot dark matter model by setting $|\psi| = 0$ on scales $|X| < \lambda_{FS}$ and evolving scales $|X| > \lambda_{FS}$ as for the cold dark matter case.

3. GENUS STATISTICS

3.1. Definition of Genus

To measure the topology of isodensity contours, we use the Gauss-Bonnet theorem that relates the integrated Gaussian curvature (a local property) of a surface with the genus (a global property) of that surface. The Gaussian curvature of a two-dimensional surface at a particular point is

$$K = \frac{1}{a_1 a_2},$$  \hspace{1cm} (27)

where $a_1$ and $a_2$ are the two principal radii of curvature at that point. A surface has a positive (or negative) Gaussian curvature if the two radii of curvature point in the same (or in opposite) directions. For example, a sphere has positive radii of curvature given by $K = 1/r^2$, where $r$ is the radius of the sphere. A cylinder has $K = 0$ because one of the principal radii of curvature is infinite. Saddle points have negative curvature because $a_1$ and $a_2$ have opposite signs. The Gauss-Bonnet theorem relates the integral of the Gaussian curvature over the surface with the genus in the following way:

$$\int K dA = 4\pi(1 - g),$$  \hspace{1cm} (28)

where $g$ is the genus of the surface and $dA$ is a surface element. The genus measures the number of closed curves that may be drawn on a surface without separating it. It can also be defined as

$$g = \text{number of holes} - \text{number of compact regions} + 1.$$  \hspace{1cm} (29)

A curved surface may be approximated by a network of polygonal faces. When I use such a network to compute the genus, I find that

$$\sum D_i = 4\pi(1 - g),$$  \hspace{1cm} (30)

where $D_i = 2\pi - \sum V_i$ is the angle deficit at a vertex (in radians) and $V_i$ are the angles around the vertex. In this case, the curvature is effectively compressed into delta functions at the vertex. For example, a cube that approximates a compact surface of genus zero has only three squares along each vertex, and consequently $\sum D_i = 8 \times (2\pi - 3\pi/2) = 4\pi$, as expected.

Fig. 1.—Comparison of the genus curve obtained for a $64^3$ simulation of a $P(k) \propto k$ power spectrum smoothed on a scale $\lambda_{eq}$ equal to $2^{1/2}$ times the grid spacing, obtained analytically (a) and numerically (b) using my program to calculate the genus.
I used equation (30) to construct a numerical algorithm to calculate the genus of an isodensity surface by applying the method suggested by Gott, Mellot, & Dickinson (1986). The isodensity surface is constructed by binning the density onto a cubic lattice and identifying pixels with density above (1) and below (0) a certain threshold \( \delta_c \). This procedure approximates a closed surface by a network of squares. My program to compute the genus was tested against analytic results for Gaussian perturbations and also against well-known topological configurations for which I knew the genus beforehand (e.g., a network of isolated cubes).

3.2. Window Function

The data is smoothed with a Gaussian window function

\[
w(r) = \frac{1}{\pi^{1/2} \lambda_s^2} e^{-r^2/\lambda_s^2},
\]

where \( \lambda_s \) is the smoothing scale. This definition implies that the smoothing length is greater than the usual width of a Gaussian by a factor of \( 2^{1/2} \). The smoothing scales are always greater than the average interparticle spacing but...
not too large as to erase all the relevant features in the density map. The smoothing scales considered were 6, 8, 10, 12, 16, and 20 \( h^{-1} \text{Mpc} \) in order to make a direct comparison with the results of Vogeley et al. (1994) for the CfA survey.

### 3.3. Genus for Gaussian Random Fields

A comparison with predictions for the genus curve from inflationary models is essential if one has to decide which model better describes the kind of large-scale structures we observe in the universe today. The genus curves of random fields are well studied and some analytic results have been derived (Bardeen et al. 1986; Hamilton, Gott, & Weinberg 1986). The genus per unit volume is given by

\[
g_{v}(\nu) = N(1 - \nu^2)e^{-\nu^2/2}, \tag{32} \]

where \( \nu \) is the number of standard deviations above or below the mean density contour. The amplitude \( N \) depends on the power spectrum of density fluctuations \( \mathcal{P}(k) = V|\delta_k|^2/(2\pi)^3 \) as

\[
N = \frac{1}{4\pi^2} \left( \frac{\langle k^2 \rangle}{3} \right)^{3/2}, \tag{33} \]

where

\[
\langle k^2 \rangle = \int \frac{k^2 \mathcal{P}(k) d^3k}{\mathcal{P}(k) d^3k}. \tag{34} \]

If we smooth the structure on a scale \( \lambda_s \), the power spectrum becomes

\[
\mathcal{P}(k) = \mathcal{P}(k)e^{-k^2\lambda_s^2/2}, \tag{35} \]

and if the power spectrum is of the form \( \mathcal{P}(k) \propto k^n \), then \( N \) is given by

\[
N = \frac{1}{(2\pi)^2 \lambda_s^3} \left( \frac{3 + n}{3} \right)^{3/2}. \tag{36} \]

To compute the genus curve we must multiply equation (32) by the volume of the grid.

Gaussian, random perturbations with power spectrum \( \mathcal{P}(k) \) can be generated numerically using the fact that, in Fourier space, the density perturbations \( \delta \) are given by

\[
\delta_k = V^{-1/2}(2\pi)^{3/2}[\mathcal{P}(k)]^{1/2}e^{i\theta_k}, \tag{37} \]

with \( \theta_k \) chosen at random in the interval \( 0 \leq \theta < 2\pi \), subject to the condition \( \delta_k = \delta^*_k \) (\( \theta_k = -\theta_{-k} \)), in order to ensure that the perturbations are real. The density fluctuations \( \delta \) can then be calculated in position space making use of a fast Fourier transform algorithm.

In Figure 1 I plotted a random-phase genus curve obtained for a \( \mathcal{P}(k) \propto ke^{-k^2\Delta_x^2} \) power spectrum obtained both analytically, using equations (32) and (36), and numerically, using my program to compute the genus (here, \( \Delta x \) is the grid spacing). The two curves are almost identical, as expected if the program to compute the genus is working properly. Small differences between the two curves can be mainly attributed to the choice of periodic boundary conditions. Smaller differences are also caused by the sample variance (the volume of the box is not infinite) and to small, numerical imprecisions. The figure is symmetric with respect to the vertical axis, which puts in evidence the topological equivalence of positive and negative linear density perturbations for random-phase models of structure formation. The shape of the random-phase genus curve is independent of the power spectrum. For \( |\nu| < 1 \), the genus is always positive, the surface has more holes than compact regions, and so the surface is “spongelike.” For \( |\nu| > 1 \), the genus is negative and the surface has a lot of independent, compact regions. For nonrandom phase distributions, the genus curve will be asymmetric, in general because topological symmetry between high- and low-density regions is not expected in most cases.

### 3.4. Genus Metastatistics

To quantify departures from the random-phase curve, Vogeley et al. (1994) used genus-related statistics such as the amplitude, the width, and the shift of the genus curve. The
amplitude was defined as the amplitude of the best-fit random-phase curve, which is the genus curve that minimizes $\chi^2$. They used this definition because they wanted to compare the observational data from the CfA Redshift Survey with predictions from Gaussian models, so it seemed appropriate. Although I make a comparison between present observations and non-Gaussian models, I am going to retain the same definition in order to directly compare my results with those of Vogeley et al. (1994).

The width of the genus peak $W_\ell$ was defined as the difference between the zero crossings of the genus curve. For random phases, the genus curve is positive over the range $-1 < \ell < 1$ and negative elsewhere, and consequently

$$W_\ell = v_+ - v_- = 2.$$ This change of the genus sign is believed to coincide with the percolation thresholds for random-phase perturbations. In the range $-1 < \ell < 1$, both high-density ($\delta > \delta_c$) and low-density ($\delta < \delta_c$) phases percolate, while for $\ell > 1$, only the low-density phase percolates and for $\ell < 1$, only the high-density phase percolates.

The last statistic they used was the shift $\Delta \nu$ of the genus curve, which was quantified in the following form

$$\Delta \nu = \frac{\int_{-1}^{1} v G(v)_{\text{obs}} \, dv}{\int_{-1}^{1} G(v)_{\text{fit}} \, dv},$$

(38)

where $G(v)_{\text{obs}}$ is the measured genus curve and $G(v)_{\text{fit}}$ is the best-fit random phase curve. A negative value of $\Delta \nu$ indicates a density distribution that is more "meatball-like"
isolated cluster models) than random phase, while a positive value of $\Delta v$ is characteristic of a "bubble-like" topology ("swiss cheese" topology).

3.5. **Parameterization of the Genus Curve**

For a Gaussian density field, the volume fraction in the high-density region is given by

$$f(v > v_0) = \frac{1}{\sqrt{2}} \int_{v_0}^{\infty} e^{-t^2/2} dt.$$  \hspace{1cm} (39)

Vogeley et al. (1994) did not compute the mean and standard deviation $v$ of the density distribution and express the genus curve as a function of $v$. Instead, they determined the genus curve as a function of the volume fraction in the high-density region and used equation (39) to parameterize $f$ as a function of $v$. For Gaussian perturbations, these two methods of calculating the genus curves produce similar results. However, for some of the models I investigate in this article, the genus curves obtained by the methods above are quite different owing to their non-Gaussianity.

3.6. **Smoothing of the Genus Curve**

The genus curves were smoothed using a very simple procedure known as three-point boxcar smoothing (see, for
example, Vogeley et al. 1994) that was shown to give better estimates of the true genus curve for Gaussian random-phase models. It consists in determining the genus as

\[ G(l_i) = \frac{1}{2} [G(l_{i-1}) + G(v_i) + G(v_i-1)] \]

(40)

where \( v_{i+1} = v_i + 0.1 \).

4. MODELS AND OBSERVATIONS

4.1. Toy Models

Here I investigate simplified cosmic string models for structure formation, not taking into consideration the detailed properties of cosmic string networks known from numerical simulations. These models are, respectively, a network of fast-moving strings, a network of slow-moving, wiggly strings, and a network of slow-moving, small loops.

The gravitational effect of a slow-moving small loop is well approximated by that of a static point mass, thus generating spherical accretion. Also, a slow-moving wiggly string can be approximated by a static line of mass, thus generating filamentary structures, while a fast moving string generates a sheetlike wake. Hence, both the filament and wake models should give a rough approximation to the structures seeded by cosmic strings. Although the network of slow-moving small loops cannot be considered a realistic cosmic string toy model (because loops of cosmic string that are produced by a cosmic string network are born with relativistic velocities and are in much higher number than those considered here), there are scenarios that this toy model does approximate, notably those with loop nucleation during inflation (see, for example, Vilenkin & Shellard 1994).

FIG. 8.—Statistical comparison of genus curves obtained from the CfA survey with genus curves obtained from 10 realizations of the five wake model. Error bars on the five wake model are 1 \( \sigma \).

FIG. 9.—Density probability distribution for several models studied with hot dark matter as a function of \( \nu \) calculated directly from the density distribution; (a) five filament model, (b) five wake model, and (c) 25 sphere model.
After the friction-dominated era, cosmic string networks rapidly attain a scaling solution where the average properties of the network (such as the average number of defects and the average correlation length) remain the same at all times when scaled to the horizon size (Bennett & Bouchet 1990; Allen & Shellard 1990; Albrecht & Turok 1989). This means that, although fluctuations laid down at later times are usually smaller in amplitude because they have less time to grow by gravitational instability, they will have larger wavelengths (in proportion to the horizon size). These facts are essential ingredients of the toy models I consider in this article.

The kind of shapes I investigate also appear in other defect-seeded structure formation models such as those seeded by global monopoles or global textures. Although I want primarily to test if the genus statistic is a good discriminator between different models of structure formation (especially between different non-Gaussian models), I also want to see if some of the features of these simplified models match current observations. To test cosmic string models of structure formation, properly, one needs to go beyond these simplified models and perform large-scale network simulations (Avelino & Shellard 1995; Avelino 1996).

In Figure 2 I plotted the isodensity contours for some of the CDM toy models considered. We can see mainly filamentary, wake-like, and spherical structures, respectively, in the five filament, five wake, and 25 sphere models. Other kinds of shapes can be obtained owing to the superposition...
of density perturbations generated by several defects. In the sphere model, there are more small spheres than big ones because denser objects are generated later in this model, when there are fewer defects inside the box. In the filament and wake models, we can see more smaller wakes and filaments than larger ones. This is because smaller objects are seeded earlier in this model, when there are more defects inside the box. These give rise to larger density perturbations and so to thicker objects in the density contour plot.

4.2. The CfA Survey

Vogeley et al. (1994) studied the topology of large-scale structure in the CfA Redshift Survey. This survey includes \( \sim 12,000 \) galaxies with limit magnitude \( m_b \leq 15.5 \), and it allowed for the computation of the topology on smoothing scales from 6–20 \( h^{-1} \) Mpc. To ensure that the topology is not dominated by shot noise, the smoothing length must be larger than the average intergalaxy (or interparticle) separation. To determine the maximum distance \( r \) appropriate for a given choice of smoothing length, Vogeley et al. found \( r_{\max} \) such that

\[
 n(r_{\max}) = \lambda e^{-3} , \tag{41}
\]

where \( n \) is the average number density of galaxies with \( r < r_{\max} \). This means that the number of galaxies included is an increasing function of the smoothing length. Table 1 shows the volume of the survey as a function of the smoothing length.
length. The number of resolution elements, defined by \( N_{res} = V_{\text{survey}}/(\pi^{3/2}l_{c}^3) \), and the number of galaxies included in the topological analysis of the CfA survey are also given as a function of the smoothing length. The volume of my simulation boxes and the number of resolution elements were chosen to be the same as for the CfA survey analysis so that direct comparison with my results was possible. In my simulations, the smoothing length was always more than 2 times larger than the average interparticle spacing.

Vogeley et al. (1994) used genus statistics to test several variants of the cold dark matter (CDM) cosmology. All of them failed to match the observations to a high confidence level (\( \geq 90\% \)), even when the evolution of the perturbations into the nonlinear regime through the use of \( N \)-body codes was taken into account. This provides a good motivation for this work, which is based on non-Gaussian models of structure formation.

### 4.3. Generation of Fluctuations

I apply the genus statistics to smoothing scales between 6 and \( 20 \ h^{-1} \) Mpc, which are in the linear or mildly nonlinear regime by the present time. In this article I consider only linear theory, in the form of the Zeldovich approximation, to evolve the perturbations in the matter era. Matter accretion during the radiation era is not considered. For linear perturbations, the genus curves do not change with time if no additional defects enter the box. The effect of nonlinear evolution on the genus curves is discussed in work currently in preparation & Canavezes However, I(Avelino 1997). I expect this to be small on scales greater than \( 8 \ h^{-1} \) Mpc.

I account for the scaling solution by fixing the length and number of defects of each type per horizon volume at any given time. These were placed in the box with random positions and orientations. The number of defects is chosen to have a Poisson distribution with average \( N_t = N V_{\text{box}}/V_H \),

\[
P(n) = \frac{N_t^n}{n!} e^{-N_t},
\]

where \( P(n) \) is the probability function, \( N \) is the average number of defects per horizon volume, \( V_{\text{box}} \) is the volume of the box where the fluctuations are produced, and \( V_H = 4\pi H^{-3}/3 \) is the Hubble volume. The comoving wavelength of the perturbations induced by the defects is taken to be initially (at \( t_{eq} \)) \( \lambda_{eq} \) and \( \frac{3}{2}\lambda_{eq} \times \frac{3}{2}\lambda_{eq} \) for the filaments and wakes, respectively (\( \lambda_{eq} = t_{eq}/a_{eq} \)). Structures seeded at later times have a larger wavelength because of the scaling solution. These structures were produced by displacing the particles according to the symmetry point, axis, and plane and with a dependence on the spatial coordinate given by each of the solutions to the Zeldovich equation (see Appendix). I assumed that the biasing parameter is not scale-dependent so that I can directly compare the genus curves obtained from the CfA survey with those obtained for the toy models considered in this article (at least for scales that are in the linear regime), which deal only with the distribution of the dark matter.

### 4.4. Compensation

Arbitrary energy-momentum perturbations are not possible in a Friedmann-Robertson-Walker spacetime (Traschen 1985; Traschen, Turok, & Brandenberger 1986; Veeraragavan & Stebbins 1990; Robinson & Wandelt 1996). When the strings are formed in the early universe, the

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**TABLE 1**

**Volume Statistics for the CfA Survey**

| \( \lambda_c \) (h^{-1} Mpc) | \( V_{\text{survey}} \) (\( h^{-1} \) Mpc)³ | \( N_{res} \) | \( N_{\text{galaxies}} \) |
|-----------------------------|---------------------------------|-------------|-------------------|
| 6                           | \( 3.31 \times 10^7 \)           | 260         | 5546              |
| 8                           | \( 5.87 \times 10^7 \)           | 202         | 7257              |
| 10                          | \( 8.38 \times 10^7 \)           | 150         | 8139              |
| 12                          | \( 1.07 \times 10^8 \)           | 111         | 7777              |
| 16                          | \( 1.51 \times 10^8 \)           | 66          | 8234              |
| 20                          | \( 1.87 \times 10^8 \)           | 42          | 8404              |

Note.—The total volume and, consequently, the number of resolution elements for the toy model simulations is the same as for the CfA survey.
excess energy and linear momentum carried by the string is
compensated by an equal deficit in the background radia-
tion (Maguguejo 1992). Although it is difficult to mimic the
detailed effects of compensation in the genus curve, I can, at
least, as a first approximation, correct the genus curve for
the average shift of the peak caused by not including com-
ensation with
\[ \delta_{\text{new}} = \delta_{\text{old}} - \delta_{\text{comp}}, \]

such that \(\left< \delta_{\text{new}} \right> = 0\) inside the box.

5. RESULTS

5.1. Toy Models and CDM

In Figure 3 I show a comparison of the density proba-
bility distribution for three of the models studied in the
context of cold dark matter, averaged over 10 simulations.
These were the five filament model, the five wake model,
and the 25 sphere model.

We can see that, for \(\lambda = 6 \, h^{-1} \, \text{Mpc}\), the density distribu-
tion is very non-Gaussian and that it will approach Gauss-
ianity as we go to larger smoothing scales. We can also see
that the largest departures from Gaussianity occur for the
five filament model, while for wakes the density probability
distribution is very nearly Gaussian, even at small scales.
If all the models had the same number of objects per unit
volume, we would expect the sphere model to exhibit the
largest departures from Gaussianity, while the wake model
should be the most Gaussian. The reason for this is that,
while a pointlike overdensity can only be inside or outside a
chosen volume, linelike perturbations and planar pertur-
bations can be partially inside several volumes (a planar
perturbation with size \(1 \times 1\) being able to "touch" more
volumes than a linelike perturbation of size 1). Consequently,
the number of pieces contained in a chosen smoothing
volume will be maximized for the wake model, which makes
it more Gaussian, but minimized for sphere model, which
makes it less Gaussian. The reason that my particular
sphere model is more Gaussian than the filament model is
that there are more objects per unit volume in the sphere
model than in the filament toy model.

The large departure from Gaussianity on small scales
makes the genus curve very sensitive to the param-
eterization, so that different genus curves are obtained if the
genus is expressed directly as a function of the number of
standard deviations from mean density or if we use the
volume fraction parameterization for \(v\). A comparison of
two genus curves for the five filament model is shown in
Figure 4 in order to illustrate the effect of the param-
eterization by volume fraction. The genus curves are con-
siderably different (the one parameterized by volume
fraction being more like random phase), but as we have
seen, this effect should be most exaggerated for the five
filament model at the smallest scale considered \((\lambda = 6 \, h^{-1}
\, \text{Mpc})\). For other toy models and smoothing scales, the
dependence of the genus curve on the parameterization will
not be as considerable. However, for small smoothing
lengths, some information about the density probability dis-
bution is lost. In this article, I shall use the volume frac-
tion parameterization in order to compare the genus curves
for these toy models of structure formation with the genus
curves obtained from the CfA survey.

In Figures 5 and 6 I show a comparison of the genus
curves for our six CDM models of structure formation, with
the predictions from one of the models tried by Vogeley et
al. (1994) (the standard inflationary CDM model with
\(\Omega = 1, h = 0.5\), and biasing parameter of \(b = 1.5\)), as well as
the observations taken from the CfA survey. Error bars
indicating sample variance were not included for the sake of
clearly. The sizes of the boxes for the toy models studied in
this article were chosen such that a direct comparison
between their genus curves and CfA genus curves is possible
for each of the smoothing scales considered. It is readily
apparent from the graphs that, of all the models studied, the
one that best fits the observational data (at least from a
topological point of view) is the five wake toy model. The
amplitude of the genus curves seems to be only marginally
larger than the CfA genus curves. Also, the width of the
genus curves seems to mimic observations much better than
the standard CDM model. The 10 wake model also does
marginally better than standard CDM. The amplitude of
the genus curves is always smaller than that predicted by
the standard CDM scenario and is in better agreement with
observations.

The standard CDM model gives too large an amplitude
for the genus curve, especially on small scales, and it fails to
match other features of the observed genus curves. For
example, the standard CDM model gives \(W \sim 2\), and
although sample variance allows some fluctuations around
this value, it is not enough to explain genus peak widths as
large as 2.5 or 2.6 with a very large confidence level
(\(>90\%\)). It is possible to have other random-phase models
with a smaller amplitude of the genus curve, as in open
models or models with a nonzero cosmological constant.
However, as found by Vogeley et al. (1994), the problem of
matching the other statistics, especially the width of the
genus peak, remains unsolved.

The five and 10 filament models perform better than
spheres and marginally worse than standard CDM. The
genus amplitude is higher than for standard CDM. How-
ever, these models provide a better fitting to the width
of the observed genus curves than the standard CDM inflationary
scenario. The sphere model is clearly ruled out. The
amplitude of the genus curves is too large, and it fails to
match the shape of the observational genus curves.

It is also apparent from the graphs that, for most of the
smoothing scales considered, the genus amplitude is an
increasing function of the number of defects. This should be
expected because I am increasing the number of structures
present inside the box.

In Figure 7 the genus curves obtained for the best model
(the five wake model) are plotted, with the error bars prop-
erly included. The line represents the average genus curve
among 10 realizations of the model. The error bars are \(1 \sigma\)
error bars over these realizations. On small scales, the
asymmetry of the genus curves is visible, as we would expect
from almost any non-Gaussian model of structure forma-
tion. However, the parameterization by volume fraction
makes the genus curves look more like random-phase
curves.

In Figure 8 a statistical comparison of the genus curves
for the wake model with the predictions from the CfA
survey is shown. The genus amplitude for this model is in
agreement with the observed genus amplitude for all but
one of the scales considered. Again, the error bars we seen in
the plots are \(1 \sigma\) error bars of 10 simulations. The shift and
width are also in agreement with observations. This toy
model seems to fit the CfA genus curves better than any
random phase model tried by Vogeley et al. (1994).
5.2. Toy Models and HDM

In Figure 9 a comparison of the density probability distribution for three of the models studied is shown as a function of the smoothing length and averaged over 10 simulations. These were the five filament model, the five wake model, and the 25 sphere model with hot dark matter. These models are more Gaussian than the corresponding models with CDM, and consequently the dependence of the genus curves on the parameterization is smaller. This is a consequence of the additional smoothing caused by the neutrino free-streaming length. For the wake model, the density probability distribution is nearly Gaussian even for \( \lambda_s = 6 \, h^{-1} \text{Mpc} \).

In Figures 10 and 11 I show a comparison of the genus curves for the six models of structure formation that I studied in the context of hot dark matter with the predictions from the standard CDM model and observations taken from the CfA Redshift Survey. The amplitude of the genus curves is smaller with HDM than with CDM, especially on small scales, because an additional smoothing was introduced because of the free-streaming of the neutrinos. The models that best mimic the observational results are again the wake models (particularly the 10 wake model).

In Figure 12 we see the genus curves obtained for the best model (the 10 wake model), with the error bars properly included. These seem to be more like random-phase curves than the genus curves obtained for the five wake model. The line represents the average genus curves among 10 realizations of the model, and the error bars are 1 \( \sigma \) error bars.

Figure 13 shows a statistical comparison of the genus curves for the 10 wake model with the predictions from the CfA survey. The genus amplitude and width of the genus curves are in agreement with observations for all the scales considered. However, the shift of the genus curves does not seem to match the observations very well. Again, the error bars we see in the plots are 1 \( \sigma \) error bars of 10 simulations. This toy model also fits the observations better than any random phase model tried by Vogeley et al. (1994).

6. DISCUSSION

I conclude from the results presented in this article that the genus statistic is a good discriminator between different toy models of structure formation that is sensitive to shape, the number of structures seeded, and dark matter type. I have also shown that, at least for some of the models considered (in particular the wake models), there is a better agreement with the observations than that for random-phase models of structure formation. It is necessary to use string network simulations in combination with numerical codes that generate and evolve the density perturbations seeded by such networks in order to properly test the cosmic string model for structure formation. However, I have shown that there are some features of the observed genus curves that cannot be easily reproduced by random-phase models of structure formation, but which are matched by some toy models considered in this article over a range of smoothing lengths. Although this cannot be considered to be a serious quantitative test of the cosmic string paradigm for structure formation, it provides a good motivation for arguing that some of the topological features generated in defect models for structure formation may be in better agreement with observations than the ones produced by random-phase models predicted by most inflationary scenarios.

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APPENDIX

A1. THE SPHERE MODEL

In the sphere model, I chose a comoving position \( q \) at random for each sphere. Particles were then displaced according to the Zeldovich approximation. Consider a sphere laid down at an instant \( t_0 \). A CDM particle at a comoving position \( q' \) would move from that time \( t_0 \) until the present time \( t_1 \) a comoving distance given (in linear theory) by

\[
\psi = \frac{3}{2} \frac{GMt_0^2}{|X|^3} b_s(t),
\]

where \( X = q' - q \) and

\[
b_s(t) = 1 - \frac{2}{5} \frac{t_1}{t_0} - \frac{3}{5} \left( \frac{t_0}{t_1} \right)^{2/3}.
\]

If we account for the growth of the mass of the loops chopped off by the network of cosmic strings (\( M \propto t \)), we have that the quotient of the perturbations laid down at two different times, \( t_1 \) and \( t_2 \), at the same position in comoving space is given by

\[
\frac{\psi(X, t_1)}{\psi(X, t_2)} = \frac{t_2 \times b_s(t_1)}{t_1 \times b_s(t_2)}.
\]

Thus, the sphere model perturbations seeded at later times can have larger amplitude, but only for this model.

A2. THE FILAMENT MODEL

In the filament model, a comoving position \( q \) and a unit vector \( v \) were chosen at random for each filament. Particles were then displaced according to the Zeldovich approximation. Let us consider a CDM particle at a comoving position \( q' \) and let us define the vector \( y = q' - q \). Consider the vector \( X = y = (y \cdot v)v \), and assume that the filament has a size given by \( S_f \). If
\[(y \cdot v) < 0.5S_f, \text{ then a CDM particle at a position } q' \text{ would move a comoving distance given by} \]

\[
\psi = -\frac{6}{5} GM_L t_0^2 \frac{X}{|X|} b_f(t), \quad (A4)
\]

where

\[
b_f(t) = \left(\frac{t_0}{t_i}\right)^{2/3} \ln \left(\frac{t_0}{t_i}\right) - \frac{3}{5} \left(\frac{t_0}{t_i}\right)^{2/3} + \frac{3}{5} \frac{t_i}{t_0}. \quad (A5)
\]

If \[(y \cdot v) > 0.5S_f, \text{ the particle would not move at all. The mass per unit length of the cosmic strings is approximately constant over time } (M_L = \text{const}) \text{ and so we have that the quotient of the perturbations laid down at two different times, } t_i \text{ and } t_2, \text{ at the same position in comoving space is given by} \]

\[
\frac{\psi(X, t_i)}{\psi(X, t_2)} = b_f(t_1) / b_f(t_2). \quad (A6)
\]

Consequently, the perturbations seeded at earlier times will have larger amplitude. The initial filament comoving size \((t_{eq_i})\) was taken to be \(S_f = \lambda_{eq} (\text{where } \lambda_{eq} = t_{eq}/a_{eq})\), and its size increases with time proportionally to the horizon so that \(S_f \propto t^{1/3}\). Consequently, at later times larger structures are formed, but they will be less dense.

A3. THE WAKE MODEL

In the wake model, a comoving position \(q\) and a unit vector \(v\) were also chosen at random for each wake. Particles were then displaced according to the Zeldovich approximation. Let us consider the vector

\[
A = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
\]

and a vector perpendicular to it,

\[
B = \begin{pmatrix} \sin \alpha \\ \cos \alpha \\ 0 \end{pmatrix}.
\]

Consider the rotation matrices

\[
M1 = \begin{pmatrix} \cos \beta & 0 & \sin \beta \\ 0 & 1 & 0 \\ -\sin \beta & 0 & \cos \beta \end{pmatrix}, \quad (A7)
\]

\[
M2 = \begin{pmatrix} 0 & \cos \theta & \sin \theta \\ \cos \theta & 0 & \sin \theta \\ -\sin \theta & 0 & \cos \theta \end{pmatrix}. \quad (A8)
\]

The vector \(A' = M2 \cdot M1 \cdot A\) obtained as the result of multiplying the matrices \(M1\) and \(M2\) by \(A\) is a new vector given by \(A' = (\sin \beta, \sin \theta \cos \beta, \cos \theta \cos \beta)\). The vector \(B' = M2 \cdot M1 \cdot B\) is perpendicular to \(A'\) and is given by

\[
B' = \begin{pmatrix} \cos \beta \sin \alpha \\ \cos \theta \cos \alpha - \sin \theta \sin \beta \sin \alpha \\ -\sin \theta \cos \alpha - \cos \theta \sin \beta \sin \alpha \end{pmatrix}. \quad (A9)
\]

The angles \(\beta\) and \(\theta\) were chosen subject to the constraint \(v = A'\), and \(\alpha\) was chosen at random in the interval \(0 \leq \alpha < 2\pi\). We have now two perpendicular vectors \(A'\) and \(B'\), and we find \(C'\) perpendicular to these two such as

\[
C' \cdot A' = 0 \quad \text{and} \quad C' \cdot B' = 0, \quad (A10)
\]

with \(|C'| = 1\). To see how a CDM particle at a comoving position \(q'\) will move, let us consider the vector \(y = q' - q\) and the vectors \(x_1 = (y \cdot B')B', x_2 = (y \cdot C')C', \) and \(X = y - x_1 - x_2\). Let us assume the size of the wake to be \(S_w\). If \(|x_1| < 0.5S_w\) and \(|x_2| < 0.5S_w\), then a CDM particle at a comoving position \(q'\) would move a comoving distance given by

\[
\psi = -\frac{2}{5} u t_i^{1/3} t_{01}^{2/3} \frac{X}{|X|} b_w(t), \quad (A11)
\]

where

\[
b_w(t) = \left(\frac{t_0}{t_i}\right)^{2/3} - \frac{t_i}{t_0}. \quad (A12)
\]
If \( |x_1| > 0.5S_w \) or \( |x_2| > 0.5S_w \), the particle would not move at all. The quotient of the perturbations laid down at two different times, \( t_1 \) and \( t_2 \), at the same position in comoving space is given by

\[
\frac{\psi(X, t_1)}{\psi(X, t_2)} = \frac{b_1(t_1) \times t_1^{1/3}}{b_2(t_2) \times t_2^{1/3}}. \tag{A13}
\]

Again the perturbations seeded at earlier times will have larger amplitude. The initial wake comoving size was taken to be \( \frac{\lambda}{2} x_0 \), and it grows proportionally to the horizon as in the filament case.

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