ON THE HAMILTONIAN FOR WATER WAVES

WALTER CRAIG

Abstract. Many equations that arise in a physical context can be posed in the form of a Hamiltonian system, meaning that there is a symplectic structure on an appropriate phase space, and a Hamiltonian functional with respect to which time evolution of their solutions can be expressed in terms of a Hamiltonian vector field. It is known from the work of VE Zakharov that the equations for water waves can be posed as a Hamiltonian dynamical system, and that the equilibrium solution is an elliptic stationary point. In this article we generalize the Hamiltonian formulation of water waves by Zakharov to a general coordinatization of the dynamical free surface, which allows it to apply to situations that include overturning wave profiles. This answers a question posed to the author by T. Nishida during the RIMS Symposium on Mathematical Analysis in Fluid and Gas Dynamics that took place during July 6 - 8 2016.

1. Introduction

The equations for water waves describe the flow of an incompressible and irrotational fluid with a free surface, under the additional restoring forces of gravity and with the possibility to include the effects of surface tension. The fluid velocity $u(t, x, y)$, expressed in Eulerian coordinates, satisfies the conditions

\begin{equation}
\nabla \cdot u = 0, \quad \nabla \wedge u = 0, 
\end{equation}

in a fluid domain $\Omega(t) \subseteq \mathbb{R}^{d-1} \times \mathbb{R}_y^1$ whose boundary consists of two components, a bottom described by a hypersurface $s \in \mathbb{R}^{d-1} \mapsto b(s) \in \mathbb{R}^d$ and a free surface given by a time dependent hypersurface $s \in \mathbb{R}^{d-1} \mapsto \gamma(t, s) \in \mathbb{R}^d$. It is possible that the bottom is unbounded below, and indeed it is common to consider the case that the bottom boundary lies at $\{y = -\infty\}$. The free surface of the fluid domain itself, defined by the hypersurface $\gamma(t, s)$, is one of the unknowns. Because of the constraints (1.1) the fluid motion is given by a potential flow

\begin{equation}
\begin{aligned}
\nu = \nabla \varphi, \quad \Delta \varphi &= 0, \quad \text{in the fluid domain } \Omega(t); \\
\partial_N \varphi &= 0, \quad \text{bottom boundary conditions on } (x, y) \in \{b(s) : s \in \mathbb{R}^{d-1}\}.
\end{aligned}
\end{equation}

Denote the horizontal and vertical components of the hypersurface defining the free surface of $\Omega(t)$ by $\gamma(t, s) = (\gamma_1(t, s), \gamma_2(t, s))$ (that is, $x = \gamma_1 \in \mathbb{R}^{d-1}$ and $y = \gamma_2 \in \mathbb{R}^1$),

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and the space-time unit normal vector to the free surface to be \( \mathbf{N}_{t,x,y} \). Furthermore define the space-time vector describing the fluid velocity field as \( \mathbf{T}_{t,x,y} = (1, u(t, x, y))^T = (1, \nabla_{x,y} \varphi)^T \). Then the kinematic free boundary condition on the free surface \( \{(x, y) \in \gamma\} \) is the geometrical condition that

\[
(1.3) \quad \mathbf{N}_{t,x,y} \cdot \mathbf{T}_{t,x,y} = 0 .
\]

The physics of the flow of Euler’s equations is described by the second nonlinear boundary condition;

\[
(1.4) \quad \partial_t \varphi = -g \gamma^2 - \frac{1}{2} |\nabla \varphi|^2 + \sigma H(\eta) \quad \text{Bernoulli condition} .
\]

The force of surface tension is given by the term \( \sigma H \), where \( H(\eta) \) is the mean curvature of the free surface. We study both of the cases \( \sigma > 0 \) and \( \sigma = 0 \).

In the case that the free surface is given as a graph, \( y = \eta(t, x), x \in \mathbb{R}^{d-1} \) the conditions (1.3) (1.4) can be rewritten as

\[
(1.5) \quad \begin{align*}
\partial_t \eta &= \partial_y \eta - \partial_x \eta \cdot \partial_x \varphi \quad \text{kinematic boundary conditions} \\
\partial_t \varphi &= -g \eta - \frac{1}{2} |\nabla \varphi|^2 + \sigma H(\eta) \quad \text{Bernoulli condition} .
\end{align*}
\]

This dynamic free boundary problem was recognized by VE Zakharov ([10] 1968) to be a Hamiltonian PDE, which is to say that equations (1.2) (1.5) can be given the form of a Hamiltonian system

\[
\dot{z} = J^H(z) , \quad \text{where} \quad X^H(z) = J \operatorname{grad}_z H(z) ,
\]

with the Hamiltonian function \( H \) the total energy of the system (1.1) (1.2) (1.5), namely

\[
(1.6) \quad H = \frac{1}{2} \iint_{\Omega(t)} |\nabla \varphi|^2 \, dy \, dx + \frac{g}{2} \int \eta^2 \, dx + \sigma \int_{\mathbb{R}^{d-1}} \sqrt{1 + |\partial_x \eta|^2} - 1 \, dx .
\]

A more subtle aspect is the choice of canonical variables for the phase space, which as in [10] is normally given by \( z(x) = (\eta(x), \xi(x) := \varphi(x, \eta(x))) \), namely

\[
\partial_t \begin{pmatrix} \eta \\ \xi \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} \delta_\eta H \\ \delta_\xi H \end{pmatrix} , \quad \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} := J ,
\]

which is in Darboux coordinates in that the symplectic form is defined by \( J \) as given above.

The goals of this article are to explain this fact, and to extend the formulation of the equations of water waves as a Hamiltonian system to free surfaces given in general coordinates, satisfying equations (1.1) (1.2) and boundary conditions (1.3) (1.4). In particular this allows the equations to describe the evolution of smooth overturning wave profiles.

2. **Hamiltonian for overturning waves**

During the RIMS Symposium on Mathematical analysis in fluid and gas dynamics in Kyoto, T. Nishida asked the author whether Zakharov’s formulation of the water waves problem (1.5) as a Hamiltonian PDE could be extended to take into consideration the case of geometries of free surfaces that are not graphs, and in particular waves that are overturning. A subtle issue in Zakharov’s formulation is the specific choice of canonical conjugate variables, which appears to require remarkable insight, but in retrospect can
be deduced from a principle of least action à la Lagrange and a subsequent Legendre transform \[4\]. It turns out that similar considerations are useful when seeking to describe the water wave problem in general coordinates.

After a first version of this article was circulated, Tom Bridges brought to my attention the article of T.B. Benjamin and P. Olver \[1\], which gives a ‘quasi-Hamiltonian’ structure for the problem of water waves with arbitrary parametrization of the free surface. It also is a formulation that can describe overturning wave profiles. The result was elaborated in T. Bridges & N. Donaldson \[2\]. The principal difference between the formulation in these two articles and the present one, in addition to being derived for both two- and three-dimensional cases, is that in the former the symplectic form is degenerate due to the extra degrees of freedom of the parametrization of the free surface, and it depends upon the phase space variables, in particular on gradient of the velocity potential. In the present article Hamilton’s canonical equations are nondegenerate, given in Darboux coordinates, and by using the Dirichlet – Neumann operator the evolution equations can be restricted entirely to the free surface.

We will address Nishida’s question essentially on a formal level, and in the case \(d = 2\), for which \(\Omega(t) \subseteq \mathbb{R}^2\). Configuration space is taken to be the space of curves \(\Gamma := \{s \mapsto \gamma(s) : s \in \mathbb{R}\}\). To actually perform analysis for data in this configuration space, including solving Laplace’s equation on the fluid domain \(\Omega(t)\), we should give some topology to this space, such as \(\gamma \in C^1(\mathbb{R})\), and we should consider free surfaces \(\gamma(s)\) that have a limit \(\lim_{s \to \pm \infty} \gamma(s) = 0\). Furthermore we should ask that there be a uniform lower bound on the distance between \(\gamma\) and the bottom boundary \(\{b(s)\}\), and also that \(\gamma\) satisfy a global chord - arc condition. However in the present context we will ignore these details.

2.1. Free surface boundary conditions. Given a one parameter family of curves \(\gamma(t, s)\), the velocity, which is the time derivative \(\partial_t \gamma(t, s) = \dot{\gamma}(t, s)\) defines a vector field in the tangent space over the curve \(\gamma\). A natural orthonormal frame for the tangent space over \(\gamma(s)\) is given by \((T(s), N(s))\), where

\[
T(s) = \frac{\partial_s \gamma(s)}{|\partial_s \gamma(s)|}, \quad N(s) = -JT(s), \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]

In the frame \((T(s), N(s))\) the velocity \(\dot{\gamma}\) vector field can be represented by its coordinates

\[
n(t, s) = N \cdot \dot{\gamma}, \quad \tau(t, s) = T \cdot \dot{\gamma}.
\]

The equations for water waves determine the evolution of the fluid domain \(\Omega(t)\) and the velocity field \(u(t, x, y)\) defined in \(\Omega(t)\). Because of the condition of irrotationality \[1.1\] they can be reduced to two nonlinear boundary conditions posed on the free surface \(\gamma(t, s)\). The first of these is the kinematic condition \[1.3\], which in this frame is written

\[
0 = N_{t,x,y} \cdot T_{t,x,y} = c(t, s)[N \cdot \dot{\gamma} - N \cdot u],
\]

where \(c(t, s)\) is a normalization irrelevant to the present discussion. When interpreted in terms of the velocity potential \(\nabla_{x,y} \varphi(t, x, y) = u(t, x, y)\) this is the statement that

\[
N \cdot \nabla \varphi|_{\gamma} = N \cdot \dot{\gamma} = n(t, s).
\]

(2.1)
With the foresight of Zakharov’s formulation, define \( \xi(s) = \varphi(\gamma(s)) \) to be the boundary values of the velocity potential on the free surface \( \gamma(s) \). Then the function \( n(s) \) can be expressed in terms of the Dirichlet – Neumann operator for the fluid domain 

\[
(2.2) \quad n(s) = G(\gamma)\xi(s)
\]

(and these quantities will depend parametrically on time \( t \)). Namely, \( \varphi \) is the solution of Laplace’s equation on the fluid domain \( \Omega \) satisfying Neumann boundary conditions on the bottom \( (x, y) = b(s) \) and with boundary data on the free surface \( \gamma(s) \) given by \( \varphi(\gamma(s)) = \xi(s) \), where the operator \( G(\gamma) \) is then defined by

\[
\xi(s) \mapsto \varphi(x, y) \mapsto \nabla \varphi \cdot N := G(\gamma)\xi(s) .
\]

The normalization for the Dirichlet – Neumann operator is that \( |N| = 1 \) (which differs slightly from what is commonly used for the problem posed in graph coordinates). This is an elliptic boundary value problem which can be solved for \( \varphi(x, y) \), hence the map \( \xi \mapsto G(\gamma)\xi \) is well defined.

The Bernoulli condition (1.4) expresses the physics described by the Euler equations on the free surface. Written in terms of \( \xi(t, s) = \varphi(\gamma(t, s)) \), for which \( \partial_t \xi(t, s) = \partial_t \varphi(t, \gamma(t, s)) = \varphi_t + \nabla \varphi \cdot \dot{\gamma} \), this is

\[
(2.3) \quad \partial_t \xi - \nabla \varphi \cdot \gamma = -g\gamma_2 - \frac{1}{2} |\nabla \varphi|^2 .
\]

Recalling the definition that \( \dot{\gamma} = nN(t, s) + \tau T(t, s) \) (and using that \( |T| = |N| = 1 \) and \( T \cdot N = 0 \)),

\[
\nabla \varphi \cdot \dot{\gamma} = (\nabla \varphi \cdot N)n + (\nabla \varphi \cdot T)\tau = (G(\gamma)\xi)n + \frac{\partial_s \xi}{|\partial_s \gamma|} \tau .
\]

Therefore (2.3) is rewritten as

\[
(2.4) \quad \partial_t \xi = -g\gamma_2 + \frac{1}{2} \left[ (G(\gamma)\xi)^2 - \frac{1}{|\partial_s \gamma|^2} (\partial_s \xi)^2 + 2 \frac{\partial_s \xi}{|\partial_s \gamma|} \tau \right] ,
\]

where we have used the definition (2.2) for \( n \) in terms of \( \xi \).

In general the geometry of the curve \( \gamma \) can be recovered from \( T(s) \) (or equivalently from \( N(s) \)) by integration, but not its parametrization. However so far in this discussion we have not addressed the issues of ambiguity that have been introduced by allowing arbitrary (nonsingular) coordinatization of curves \( \gamma(s) \). There exist numerous useful possibilities to specify this parametrization, one standard one being as a graph, but another is to parametrize by arc length. In this latter case

\[
\partial_s \gamma = T , \quad |T(s)| = 1 , \quad \partial_s T = \kappa(s)N , \quad \partial_s N = -\kappa(s)T ,
\]

which describes the evolution in \( s \) of the Frenet frame, \( \kappa(s) \) being the curvature. In these coordinates one recovers \( \tau(s) \) from \( n(s) \); indeed because

\[
0 = \partial_t |T(t, s)|^2 = 2\partial_t T \cdot T = 2\partial_s \dot{\gamma} \cdot T
\]

then one has

\[
(2.5) \quad \partial_s \tau = \partial_s \dot{\gamma} \cdot T + \dot{\gamma} \cdot \partial_s T = \kappa \dot{\gamma} \cdot N = \kappa n .
\]
In arc length coordinates, equation (2.4) is somewhat simpler, namely
\begin{equation}
\partial_t \xi = -g\gamma^2 + \frac{1}{2} \left[ (G(\gamma)\xi)^2 - (\partial_s \xi)^2 + 2\partial_s \xi \tau \right].
\end{equation}
In this case, the tangential component of the velocity is recovered from (2.2) (2.5), namely \( \partial_s \tau = \kappa n = \kappa G(\gamma)\xi \).

2.2. Legendre transform. The Lagrangian for free surface water waves corresponds to the total energy of the system, which consists of two terms, the kinetic energy \( K \) and the potential energy \( U \);
\begin{equation}
L = K - U.
\end{equation}
The Legendre transform is the classical approach to transfer a Lagrangian system into the canonical conjugate coordinates of a Hamiltonian system. When the Lagrangian functional \( L \) is expressed in terms of the variables \( \gamma \) and \( \dot{\gamma} \), by analogy with classical mechanics one defines conjugate momentum variables via the Legendre transform as 
\[ \xi = \delta_\dot{\gamma} L. \]
The kinetic energy is given by the Dirichlet integral
\[ K = \int_\Omega \frac{1}{2} |\nabla \varphi(x, y)|^2 \, dy \, dx \]
and the potential energy is respectively
\[ U = \int_\Omega g \gamma \, dy \, dx + C, \]
which is, as usual, only defined up to an additive constant. If the effects of surface tension were to be included in the equations of motion, then the potential energy has an additional term, namely
\[ U = \int_\Omega g \gamma \, dy \, dx + \sigma \int_\gamma dS_\gamma + C', \]
where \( dS_\gamma = |\partial_s \gamma(s)| \, ds \). Our derivation below is in the case that \( \sigma = 0 \), but by modifications of the argument the case \( \sigma \neq 0 \) is also able to be included.

Integrating by parts in \( K \) and using the boundary conditions, we can express the kinetic energy in terms of integrated quantities on the free surface
\begin{equation}
K = \int_\gamma \frac{1}{2} \xi G(\gamma)\xi \, dS_\gamma.
\end{equation}
We note that the normalization for the Dirichlet – Neumann operator \( G(\gamma) \) is different from that used in [10] and [5], so that it is Hermetian with respect to the line element \( dS_\gamma \). Using (2.2) the kinetic energy can be written in terms of \( \gamma \) and \( \dot{\gamma} \);
\begin{equation}
K(\gamma, \dot{\gamma}) := \int_\gamma \frac{1}{2} n G^{-1}(\gamma)n \, dS_\gamma.
\end{equation}
The potential energy $U$ can be expressed with respect to the divergence theorem, using a vector field $V(x, y) := (0, \frac{g}{2}y^2)^T$;

\begin{equation}
U(\gamma) = \int \int_{\Omega} \nabla \cdot V(x, y) \, dvol = \int_{\gamma} V \cdot N \, dS_\gamma + C = \int_{\gamma} \frac{g}{2} \frac{\partial_1 \gamma_1}{|\partial_s \gamma|} \, dS_\gamma + C .
\end{equation}

In arc length parametrization this would read

\begin{equation}
U(\gamma) = \int_{\gamma} \frac{g}{2} \gamma_1^2 \partial_s \gamma_1 \, ds + C .
\end{equation}

In the case of general coordinates for the free surface $\gamma(s) = (\gamma_1(s), \gamma_2(s))$, gradients are expressed with respect to the metric given by $\int_{\gamma} N \cdot dS_\gamma$. The gradient of the kinetic energy $K$ above, and because of the decomposition $\dot{\gamma}(s) = \tau(s)T(s) + n(s)N(s)$, this is

\begin{equation}
\delta_\gamma K = \delta_n K + \delta_\tau K = G^{-1}(\gamma)n + 0 .
\end{equation}

Thus $\xi = G^{-1}(\gamma)n$ is the canonical conjugate variable to normal perturbations of a given free surface $\gamma$, while $\tau$ remains undefined without further specification of the parametrization of the curve $\gamma$. This degeneracy will be resolved when a particular form of parametrization is imposed.

Following the prescription of the Legendre transform (2.11) the Hamiltonian is given by

\begin{equation}
H = K + U = \frac{1}{2} \int_{\gamma} \xi G(\gamma) \xi \, dS_\gamma + \frac{g}{2} \int_{\gamma} \frac{\partial_1 \gamma_1}{|\partial_s \gamma|} \, dS_\gamma .
\end{equation}

The remaining questions are to how to best express the variables that are canonically conjugate to $\xi(s)$, and to show that the resulting equations of motion (2.2) (2.4) coincide with the Hamiltonian vector field, namely

\begin{equation}
\partial_t z = J \text{grad} H(z) .
\end{equation}

The gradient of the kinetic energy $K$ with respect to $\xi$ is

\begin{equation}
\text{grad}_\xi K = G(\gamma)\xi ,
\end{equation}

which corresponds to the conjugate of the normal variations of $K$ with respect to $\gamma$. 
Using the expression (2.10), the gradient of the potential energy is given by

\[
(\delta U, \delta \gamma) = \int g_2(s) \left( \frac{-2\gamma_1^2 \partial_s \gamma_2}{\gamma_2^2 \partial_s \gamma_1} \right) \cdot \left( \frac{\delta \gamma_1}{\delta \gamma_2} \right) ds
\]

\[
= \int g_2(s) N \cdot \left( \frac{\delta \gamma_1}{\delta \gamma_2} \right) dS_{\gamma},
\]

corresponding to the gradient of \(U\) with respect to normal variation of \(\gamma\) itself, namely \(\text{grad}_{N, \delta \gamma} U\).

The gradient of the kinetic energy \(K\) with respect to \(\gamma\) is the more subtle quantity in this formulation. Consider a fluid domain \(\Omega\) with free surface \(\gamma(s)\) and a family of nearby domains \(\Omega_1\) with nearby free surfaces \(\gamma_1(s) = \gamma(s) + \delta \gamma(s)\). Denote the outward unit normal by \(N(s)\) and \(N_1(s)\) respectively. We consider the Dirichlet integrals

\[
K(\gamma, \xi) = \frac{1}{2} \int_\gamma \xi(s) G(\gamma) \xi(s) dS_{\gamma}, \quad K_1 = K(\gamma_1, \xi) = \frac{1}{2} \int_{\gamma_1} \xi(s) G(\gamma_1) \xi(s) dS_{\gamma_1},
\]

for which we impose that the boundary values of the velocity potentials \(\Phi_1(x, y)\) on \(\gamma_1\) and \(\Phi(x, y)\) on \(\gamma\) coincide

\[
\Phi(\gamma(s)) = \xi(s) = \Phi_1(\gamma_1(s)),
\]

while we vary the boundary curve \(\gamma(s)\) to \(\gamma_1(s) = \gamma(s) + \delta \gamma(s)\). This is to say that one takes the partial derivative of the kinetic energy with respect to variations of the domain, while fixing the boundary conditions for the velocity potential on the free surface. To this effect, the boundary values of \(\Phi(x, y)\) on the curve \(\gamma_1(s)\) are given by

\[
\Phi(\gamma_1(s)) = \Phi(\gamma(s)) + \nabla \Phi(\gamma(s)) \cdot \delta \gamma(s) + \mathcal{O}(\delta^2)
\]

\[
= \Phi(\gamma(s)) + (\nabla \Phi \cdot N) N \cdot \delta \gamma(s) + (\nabla \Phi \cdot T) T \cdot \delta \gamma(s) + \mathcal{O}(\delta^2).
\]

Therefore

\[
(2.14) \quad \Phi_1(\gamma_1(s)) - \Phi(\gamma_1(s)) = -(\nabla \Phi \cdot N) N \cdot \delta \gamma(s) - (\nabla \Phi \cdot T) T \cdot \delta \gamma(s) + \mathcal{O}(\delta^2)
\]

Furthermore, given a harmonic function \(\Phi(x, y)\) defined on a neighborhood that includes \(\Omega \cup \Omega_1\), by Green’s theorem the difference of the boundary integral expressions for their Dirichlet integrals is given by

\[
(2.15) \quad \frac{1}{2} \int_{\gamma_1} \Phi(\gamma_1(s)) N_1 \cdot \nabla \Phi(\gamma_1(s)) dS_{\gamma_1} - \frac{1}{2} \int_{\gamma} \Phi(\gamma(s)) N \cdot \nabla \Phi(\gamma(s)) dS_{\gamma}
\]

\[
= \frac{1}{2} \int_{\Omega \setminus \Omega_1} |\nabla \Phi|^2 dvol \simeq \frac{1}{2} \int_{\gamma} |\nabla \Phi|^2 N \cdot \delta \gamma(s) dS_{\gamma}.
\]
Therefore the variation of the kinetic energy $K$ with fixed boundary data $\xi(s)$ is calculated as the limit in small $\delta$ of

$$K_1 - K = \frac{1}{2} \int_{\gamma_1} \xi(s)G(\gamma_1)\xi(s)\,dS_{\gamma_1} - \frac{1}{2} \int_{\gamma} \xi(s)G(\gamma)\xi(s)\,dS_{\gamma}$$

$$= \frac{1}{2} \int_{\gamma_1} \Phi_1(\gamma_1)N_1 \cdot \nabla\Phi_1(\gamma_1)\,dS_{\gamma_1} - \frac{1}{2} \int_{\gamma} \Phi(\gamma)N \cdot \nabla\Phi(\gamma)\,dS_{\gamma}$$

$$= \int_{\gamma_1} (\Phi_1 - \Phi)(\gamma_1)N_1 \cdot \nabla\Phi_1(\gamma_1)\,dS_{\gamma_1}$$

$$+ \frac{1}{2} \int_{\gamma_1} \Phi(\gamma_1)N_1 \cdot \nabla\Phi(\gamma_1)\,dS_{\gamma_1} - \frac{1}{2} \int_{\gamma} \Phi(\gamma)N \cdot \nabla\Phi(\gamma)\,dS_{\gamma} + \mathcal{O}(\delta^2).$$

Using (2.14) in the first term and (2.15) in the second and third,

$$K_1 - K = \int_{\gamma} -(\nabla\Phi \cdot N)^2 N \cdot \delta\gamma(s) - (\nabla\Phi \cdot N)(\nabla\Phi \cdot T)\,T \cdot \delta\gamma(s)dS_{\gamma}$$

$$+ \frac{1}{2} \int_{\gamma} |\nabla\Phi|^2 N \cdot \delta\gamma\,dS_{\gamma} + \mathcal{O}(\delta^2).$$

Furthermore, both of the velocity potentials $\Phi$ and $\Phi_1$ satisfy Neumann boundary conditions on the bottom $(x, y) = b(s)$. Thus $N \cdot \nabla\Phi(\gamma(s)) = G(\gamma)\xi(s)$ and $T \cdot \nabla\Phi(\gamma(s)) = \frac{1}{|\partial_s\gamma|}\partial_s\xi(s)$, giving an expression in the limit as $\delta \to 0$ for $\text{grad}_s K$, namely

$$\langle \delta K \cdot \delta\gamma \rangle_\gamma = \int_{\gamma} \text{grad}_s K \cdot \delta\gamma\,dS_{\gamma}$$

$$= \frac{1}{2} \int_{\gamma} -(G(\gamma)\xi)^2 N \cdot \delta\gamma + \left(\frac{1}{|\partial_s\gamma|}\partial_s\xi\right)^2 N \cdot \delta\gamma - 2\left(\frac{1}{|\partial_s\gamma|}\partial_s\xi G(\gamma)\xi\right)T \cdot \delta\gamma\,dS_{\gamma}.$$

With these expressions in hand, we conclude that the equations of motion for the problem of water waves takes the canonical form of a Hamiltonian system;

$$N \cdot \partial_t \gamma = \text{grad}_\xi H$$

$$\partial_t \xi = -\text{grad}_N \partial_t H.$$

In general the choice of coordinatization of the free surface is made separately from the decomposition of the tangent space $T_\gamma$ into its normal and tangential components. Variations $Y(s) = \delta\gamma(s)$ of $\gamma$ are necessarily constrained by the coordinate choice to the class of admissible variations. The choice of coordinatization determines the tangential component of the velocity $T = T \cdot \partial_t \gamma$ as a function of the normal component, through the constraints imposed by the coordinatization of the free surface. This applies in particular to the time derivative of the curve, $\dot{\gamma}(s) \in T_\gamma$. That is, coordinatization dictates a relation between $T \cdot \delta\gamma$ and $N \cdot \delta\gamma$, say $T \cdot \delta\gamma = T(\gamma)(N \cdot \delta\gamma)$ in somewhat abstract terms. Thus, in terms of such a coordinate choice,

$$\text{grad}_{N,\delta\gamma} K = \frac{1}{2} \left[ \left(\frac{1}{|\partial_s\gamma|}\partial_s\xi\right)^2 - (G(\gamma)\xi)^2 \right] - 2\left(\frac{1}{|\partial_s\gamma|}\partial_s\xi G(\gamma)\xi T(\gamma)\right).$$
This gradient is worked out in detail for several standard choices of parametrization in the subsection below.

2.3. **Particular coordinates.** Common choices for the parametrization of the free surface are: (1) the classical case of free surfaces given as a graph in \( x \in \mathbb{R}^1 \), which does not allow for overturning free surfaces. (2) arc length parametrization of \( \gamma(s) \) which are able to describe overturning wave profiles. In these coordinates we have seen that \( \partial_s \tau = \kappa n \).

(3) Lagrangian coordinates, for which fluid particle positions are advected by the flow, \( \partial_t (X(t), Y(t)) = u(X(t), Y(t)) = \nabla \varphi(\gamma(t, \cdot)) \), or (4) conformal mapping coordinates as used in [7]. Specifying the coordinatization of free surface curves \( \gamma \) in cases (1)(2) and (4) gives rise to systems of constraints which may be considered to be holonomic as they are imposed independently of the velocity \( \dot{\gamma} \). The parametric specification by Lagrangian coordinates in contrast is a nonholonomic constraint.

The traditional choice of parametrization is (1) to write the surface as a graph; in such graph coordinates, \( \gamma = (x, \eta(x)) \), and the pair of variables \( (\eta(x), \xi(x)) \) are canonically conjugate as given by Zakharov [10]. With the expression for the kinetic energy \( K \) in terms of the Dirichlet – Neumann operator \( G(\eta) \) as in [5], then

\[
H(\eta, \xi) = \int_{\mathbb{R}^1} \frac{1}{2} \xi G(\eta) \sqrt{1 + (\partial_x \eta)^2} \, dx + \frac{g}{2} \int_{\mathbb{R}^1} \eta^2 \, dx .
\]

In these graph coordinates, \( \gamma = (x, \eta(x)) \) so that admissible variations are \( \delta \gamma = (0, \delta \eta) \), and the relationship between \( N \cdot \delta \gamma \) and \( T \cdot \delta \gamma \) is given by

\[
T \cdot \delta \gamma = \partial_x \eta N \cdot \delta \gamma .
\]

The gradient of the kinetic energy is thus

\[
\frac{1}{2} \left[ \left( \frac{\partial_x^2 \xi}{1 + (\partial_x \eta)^2} \right) - \left( G(\eta) \xi \right)^2 - 2 \left( \frac{1}{\sqrt{1 + (\partial_x \eta)^2}} \partial_x \xi G(\eta) \xi \partial_x \eta \right) \right] .
\]

Because of this,

\[
N \cdot \dot{\gamma} = \frac{1}{\sqrt{1 + (\partial_x \eta)^2}} \left( \begin{array}{c} -\partial_x \eta \\ 1 \end{array} \right) \cdot \left( \begin{array}{c} 0 \\ \partial_t \eta \end{array} \right)
\]

and the resulting equations (1.3) for water waves are given by

\[
\frac{1}{\sqrt{1 + (\partial_x \eta)^2}} \partial_t \eta = \delta_x H = G(\eta) \xi
\]

\[
\partial_t \xi = -\delta_x H = -g \eta + \frac{1}{2} \left[ (G(\eta) \xi)^2 - \frac{(\partial_x \xi)^2}{1 + (\partial_x \eta)^2} + 2 \partial_x \xi G(\eta) \xi \partial_x \eta \right] .
\]

Calculating for an independent verification of (2.4), one finds that in graph coordinates

\[
\tau = \frac{\partial_t \eta \partial_x \eta}{\sqrt{1 + (\partial_x \eta)^2}} = G(\eta) \xi \partial_x \eta .
\]

This system of equations, modulo the difference in normalization of the Dirichlet – Neumann operator \( G(\eta) \), appears in [5], and is used in the existence theory for water waves and many of its distinguished scaling limits in [8][9].
Coordinates given in terms of arc length along the free surface $\gamma(s)$ allow the system \((1.3)\) to describe overturning wave profiles. This choice of coordinates implies in particular that $\partial_s \gamma(s) = T(s)$ and $\partial_s T \perp T$, since

$$|\partial_s \gamma(s)|^2 = 1, \quad 0 = \partial_s |\partial_s \gamma(s)|^2 = 2\partial_s \gamma \cdot \partial_s^2 \gamma.$$  

Indeed any vector field $Y(s)$ along the curve $\gamma(s)$ that arises from an infinitesimal motion which preserves the arc length parametrization must satisfy

$$0 = \frac{d}{d\delta} \bigg|_{\delta=0} |\partial_s \gamma + \delta Y|^2 = 2\partial_s \gamma \cdot Y = 2T \cdot Y.$$  

Admissible variations $\delta \gamma(s)$ are arc length preserving in the present case, implying that $Y = \partial_s \delta \gamma$ is as above, and hence

$$0 = \partial_s (T \cdot \delta \gamma) = \partial_s T \cdot \delta \gamma + T \cdot \partial_s \delta \gamma = \kappa N \cdot \delta \gamma .$$  

This is the relationship between tangential and normal variations that applies to the gradient of the kinetic energy, an interesting geometrical aspect of this choice of coordinates. The resulting Bernoulli equations of motion are

$$\partial_t \xi = -g\gamma^2 - \frac{1}{2} \left[ \left( \frac{1}{|\partial_s \gamma|} \partial_s \xi \right)^2 - (G(\gamma)\xi)^2 - 2 \left( \frac{1}{|\partial_s \gamma|} \partial_s \xi G(\gamma)\xi T(\gamma) \right) \right]$$

where $T(\gamma)$ satisfies

$$\partial_s (G(\gamma)\xi T) = \kappa G(\gamma)\xi .$$

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