A note on spherically symmetric isentropic compressible flows with density-dependent viscosity coefficients

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Abstract

In this note, by constructing suitable approximate solutions, we prove the existence of global weak solutions to the compressible Navier-Stokes equations with density-dependent viscosity coefficients in the whole space $\mathbb{R}^N$, $N \geq 2$ (or exterior domain), when the initial data are spherically symmetric. In particular, we prove the existence of spherically symmetric solutions to the Saint-Venant model for shallow water in the whole space (or exterior domain).

Keywords: Compressible Navier-Stokes equations; density-dependent viscosity coefficients

1 Introduction

In this note, we consider the following compressible Navier-Stokes equations with density-dependent viscosity coefficients

$$\rho_t + \text{div}(\rho U) = 0, \quad (1.1)$$

$$\rho U_t + \text{div}(\rho U \otimes U) - \text{div}(2h(\rho)D(U)) - \nabla (g(\rho)\text{div}U) + \nabla P(\rho) = 0, \quad (1.2)$$

where $t \in (0, +\infty)$ and $x \in \mathbb{R}^N$, $N \geq 2$, $\rho(x, t)$, $U(x, t)$ and $P(\rho) = \rho^\gamma$ ($\gamma \geq 1$) stand for the fluid density, velocity and pressure respectively,

$$D(U) = \frac{1}{2}(\nabla U + (\nabla U)^T)$$

is the strain tensor, $h(\rho)$ and $g(\rho)$ are the Lamé viscosity coefficients satisfying

$$h(\rho) \geq 0, \quad 2h(\rho) + Ng(\rho) \geq 0. \quad (1.3)$$

In the last several decades, significant progress on the system (1.1)-(1.2) with positive constant viscosity coefficients has been achieved by many authors. In the case that the initial data are sufficiently regular and the initial density is bounded away from zero, there exists a unique local strong solution, and the solution exists globally in time provided that the initial data are small perturbations of an uniform non-vacuum state. For details, we refer the readers to papers [7, 20] and the references therein. The situation becomes more complex in the general case of nonnegative initial density, and a number of important questions are still open. For example, the uniqueness of global weak solutions. The first general result on the existence of global weak solutions was obtained by Lions in [18]. There have been many generalizations of this result,
see [9, 13, 15, 16, 23]. Using the compatibility condition, Salvi-Stra˘ skraba [22] and Choe-Kim [6] obtained the existence and uniqueness of the local strong solution.

The results in [11, 19, 24] show that the compressible Navier-Stokes system with constant viscosity coefficients have the singularity in the presence of vacuum. By some physical considerations, Liu, Xin and Yang in [19] introduced the modified Navier-Stokes system with density-dependent viscosity coefficients. As remarked in [19], in the derivation of the Navier-Stokes equations from the Boltzmann equation through the Chapman-Enskog expansion to the second order, the viscosity is a function of the temperature, and correspondingly depends on the density for isentropic fluids. Meanwhile, in geophysical flows, many mathematical models correspond to (1.1)-(1.2). In particular, the viscous Saint-Venant system for shallow water is expressed exactly as (1.1)-(1.2) with \( N = 2, h(\rho) = \rho, g(\rho) = 0 \) and \( \gamma = 2 \) ([1, 2, 18]). As remarked in [10], new mathematical challenges are encountered for the shallow water equations and the multi-dimensional compressible Navier-Stokes equations (1.1)-(1.2). The main difficulty is that the velocity can not be defined in the vacuum state.

For one-dimensional compressible Navier-Stokes equations (1.1)-(1.2) with \( h(\rho) = \rho^\theta \) and \( g(\rho) = 0, \theta \in (0, 1) \), there are many literatures on the well-posedness theory of the solutions, see [8, 14, 17, 19, 25, 26]. Considering the free boundary problem of the spherically symmetric system, the local existence and uniqueness of the weak solution were obtain in [5], the large-time behavior of the global solution for data close to equilibrium was obtained in [27, 28]. However, few results are available for multi-dimensional problems. In [11], Bresch, Desjardins and Lin showed the existence of global weak solutions in dimension 2 or 3 for the Korteweg’s system with the Korteweg stress tensor \( k\rho \nabla \Delta \rho \). An interesting new entropy estimate is established in [1] in a priori way, which provided some high regularity for the density. Later, a similar result was obtained in [2] with an additional quadratic friction term \( r\rho |U|U \). Recently, Mellet and Vasseur [21] proved the \( L^1 \) stability of weak solutions of the system (1.1)-(1.2) with \( N = 2, 3 \) and \( \gamma > 1 \), based on the new entropy estimate, extending the results in [12] to the case \( r = k = 0 \). Bresch and Desjardins constructed approximate solutions for the viscous shallow water system with drag terms or capillarity term and for the compressible Navier-Stokes equations with the cold pressure in [3], and proved the global existence of weak solutions to these systems in [3, 4]. In [10], Guo, Jiu and Xin constructed a class of approximate solutions and proved the existence of global weak solutions for the spherically symmetric compressible Navier-Stokes equations with density-dependent viscosity in a bounded domain \( (N = 2, 3, \gamma > 1) \).

In this note, we will construct a class of approximate solutions and prove the global existence of weak solutions for the spherically symmetric compressible Navier-Stokes equations with density-dependent viscosity in the whole space or exterior domain \( (N \geq 2, \gamma \geq 1) \). Using the method in [10], we can construct the approximate solutions on the annular domain \( \{ \varepsilon < |x| < R \} \) by solving the approximate systems of (1.1)-(1.2) with \( h^\varepsilon(\rho) = h(\rho) + \varepsilon \rho^\theta \) and \( g^\varepsilon(\rho) = g(\rho) + (\theta - 1)\varepsilon \rho^\theta \) instead of \( h(\rho) \) and \( g(\rho) \). Then, using the usual zero extensions as in [12, 13], we can construct the approximate solutions on the entire domain \( \mathbb{R}^N \). But, the entropy estimates of approximate solutions do not hold on the entire domain \( \mathbb{R}^N \), only hold on the annular domain. Using some techniques in Proposition 3.3, we can prove that \( \nabla \sqrt{\rho} \) belongs to \( L^\infty(0, T ; L^2(\mathbb{R}^N)) \), so that the nonlinear diffusion terms in the definition of weak solutions will make sense. The extension method in [10], can preserves the uniform \( L^\infty(0, T ; H^1(\Omega)) \) estimate of \( \sqrt{\rho} \), but seems not applicable to build approximate solutions in the whole space or exterior domain.
2 Statement of the results.

The Cauchy problem of the compressible Navier-Stokes equations can be written as

\[ \rho_t + \text{div}(\rho U) = 0, \]  
\[ (\rho U)_t + \text{div}(\rho U \otimes U) - \text{div}(2h(\rho)D(U)) - \nabla(g(\rho)\text{div}U) + \nabla P(\rho) = 0, \]  

with initial conditions

\[ \rho|_{t=0} = \rho_0 \geq 0, \quad \rho U|_{t=0} = m_0. \]  

Before introducing the notion of weak solution, let us state the assumptions on the viscosity coefficients, as in [21].

**Conditions on \( h(\rho) \) and \( g(\rho) \):**

We assume that \( h(\rho) \) and \( g(\rho) \) are two \( C^2(0, \infty) \) functions satisfying

\[ g(\rho) = 2\rho h'(\rho) - 2h(\rho), \]  
\[ h'(\rho) \geq \nu, \quad h(0) \geq 0, \]  
\[ |g'(\rho)| \leq \frac{1}{\nu} h'(\rho), \]  
\[ \nu_1 h(\rho) \leq 2h(\rho) + Ng(\rho) \leq \nu_2 h(\rho), \]

where \( \nu \in (0, 1) \) and \( \nu_2 \geq \nu_1 > 0 \) are three constants satisfying

\[ \frac{4N - 4\sqrt{2N^2 - 4N + 4}}{N^2 - 4N + 4} < \frac{\nu_1 - 2}{N}, \quad \frac{4N + 4\sqrt{2N^2 - 4N + 4}}{N^2 - 4N + 4} > \frac{\nu_2 - 2}{N}, \quad N \geq 3. \]

When \( N \geq 3 \) and \( \gamma \geq \frac{N}{N-2} \), we also require that

\[ \liminf_{\rho \to \infty} \frac{h(\rho)}{\rho^\gamma \epsilon^{\gamma+\epsilon}} > 0, \]

for some small \( \epsilon > 0 \).

**Remark 2.1.** From the above conditions, one has

\[ \begin{cases} 
C\rho^{\frac{N-1}{N} + \frac{\nu}{2N}} \leq h(\rho) \leq C\rho^{\frac{N-1}{N} + \frac{\nu}{2N}}, & \rho \geq 1, \\
C\rho^{\frac{N-1}{N} + \frac{\nu}{2N}} \leq h(\rho) \leq C\rho^{\frac{N-1}{N} + \frac{\nu}{2N}}, & \rho \leq 1.
\end{cases} \]

**Definition 2.1.** We say that \((\rho, U)\) is a weak solution of (2.1)-(2.3) on \( \mathbb{R}^N \times [0, T] \), provided that

1. \( \rho \in L^\infty(0, T; L^1 \cap L^\gamma(\mathbb{R}^N)), \quad \sqrt{\rho} \in L^\infty(0, T; H^1(\mathbb{R}^N)), \)
\[ \sqrt{\rho} U \in L^\infty(0, T; (L^2(\mathbb{R}^N))^{N}), \]
\[ h(\rho)D(U) \in L^2(0, T; (W_{\text{loc}}^{-1,1}(\mathbb{R}^N))^{N \times N}), \quad g(\rho)\text{div}U \in L^2(0, T; W_{\text{loc}}^{-1,1}(\mathbb{R}^N)), \]

with \( \rho \geq 0 \);

2. For any \( t_2 > t_1 \geq 0 \) and \( \phi_1 \in C^1_c(\mathbb{R}^N \times [0, \infty)) \), the mass equation (2.1) holds in the following sense:

\[ \int_{\mathbb{R}^N} \rho \phi_1 dx |_{t_1}^{t_2} = \int_{t_1}^{t_2} \int_{\mathbb{R}^N} (\rho \phi_1 \partial_t + \rho U \cdot \nabla \phi_1) dx dt; \]
(3) The following equality holds for all smooth test function $\phi_2(t, x) \in (C_c^2(\mathbb{R}^N \times [0, \infty)))^N$ with $\phi_2(T, \cdot) = 0$:

$$
\int_{\mathbb{R}^N} m_0 \cdot \phi_2(0, x) dx + \int_0^T \int_{\mathbb{R}^N} (\sqrt{\rho} \sqrt{\rho} U) \cdot \partial_t \phi_2 + \sqrt{\rho} U \otimes \sqrt{\rho} U : \nabla \phi_2 \ dx \ dt
$$

$$
+ \int_0^T \int_{\mathbb{R}^N} \rho^2 \phi_2 \ dx \ dt - < 2h(\rho) D(U), \nabla \phi_2 > - < g(\rho) \text{div} U, \text{div} \phi_2 > = 0, \quad (2.12)
$$

where the diffusion terms make sense when written as

$$
< 2h(\rho) D(U), \nabla \phi > = - \int_{\mathbb{R}^N} \frac{h(\rho)}{\sqrt{\rho}} (\sqrt{\rho} U)_{ij} \partial_i \phi_j \ dx \ dt - \int_{\mathbb{R}^N} (\sqrt{\rho} U)_{ij} 2h'(\rho) \partial_i \sqrt{\rho} \partial_j \phi_j \ dx \ dt
$$

$$
- \int_{\mathbb{R}^N} \frac{h(\rho)}{\sqrt{\rho}} (\sqrt{\rho} U)_{ij} \partial_i \phi_j \ dx \ dt - \int_{\mathbb{R}^N} (\sqrt{\rho} U)_{ij} 2h'(\rho) \partial_j \sqrt{\rho} \partial_i \phi_i \ dx \ dt,
$$

and

$$
< g(\rho) \text{div} U, \text{div} \phi > = - \int_{\mathbb{R}^N} \frac{g(\rho)}{\sqrt{\rho}} (\sqrt{\rho} U)_{ij} \partial_j \phi_i \ dx \ dt - \int_{\mathbb{R}^N} (\sqrt{\rho} U)_{ij} 2g'(\rho) \partial_j \sqrt{\rho} \partial_i \phi_i \ dx \ dt.
$$

In this paper, we will construct global spherically symmetric weak solutions to (2.11)-(2.3). The initial data are assumed to satisfy

$$
\rho_0 \geq 0 \text{ a.e. in } \mathbb{R}^N, \quad m_0 = 0 \text{ a.e. on } \{ x \in \mathbb{R}^N | \rho_0(x) = 0 \}, \quad (2.13)
$$

$$
\rho_0 \in L^2(\mathbb{R}^N), \quad \nabla h(\rho_0) \sqrt{\rho_0} \in L^2(\mathbb{R}^N), \quad \frac{\rho_0^2}{\rho_0} (1 + \ln(1 + \frac{m_0^2}{\rho_0})) \in L^1(\mathbb{R}^N). \quad (2.14)
$$

The main result of this paper is the following:

**Theorem 2.1.** Assume that $\gamma \geq 1$, $h(\rho)$ and $g(\rho)$ satisfy conditions (2.4)-(2.9). If the initial data have the form

$$
\rho_0 = \rho_0(|x|), \quad m_0 = m_0(|x|) \frac{x}{r}
$$

and satisfy (2.13)-(2.14), then the initial-value problem (2.11)-(2.3) has a global spherically symmetric weak solution

$$
\rho = \rho(|x|, t), \quad U = U(|x|, t) \frac{x}{r}
$$

satisfying for all $T > 0$,

$$
\rho(x, t) \in C([0, T]; L^1(\mathbb{R}^N)), \quad (2.15)
$$

$$
\int_{\mathbb{R}^N} \rho(x, t) dx = \int_{\mathbb{R}^N} \rho_0(x) dx. \quad (2.16)
$$

Moreover, it holds that

$$
\sup_{t \in [0, T]} \int_{\mathbb{R}^N} \left( \frac{1}{\rho} |\nabla h(\rho)|^2 + \rho(1 + |U|^2)(1 + \ln(1 + |U|^2)) \right) dx \leq C, \quad (2.17)
$$

where $C$ is a constant.

**Remark 2.2.** Using the similar argument as that in [10], one can obtain that

$$
\sup_{t \in [0, T]} \int_{\mathbb{R}^N} \rho |U|^{2+\eta} dx \leq C,
$$

when $\int_{\mathbb{R}^N} \rho_0 |U_0|^{2+\eta} dx \leq C$ for some small $\eta \in (0, 1)$. 

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Remark 2.3. Similarly, using the usual zero extension method, one can obtain the similar result for the existence of global weak solutions for the spherically symmetric compressible Navier-Stokes equations with density-dependent viscosity in a bounded domain \((N \geq 2, \gamma \geq 1)\).

Remark 2.4. Under conditions \((2.4)-(2.9)\), using the similar argument as that in [21], one can easily obtain the similar result as that in [21] with \(N \geq 2\) and \(\gamma \geq 1\).

• Exterior problem

Using the similar proof of Theorem 2.1, we can study the following exterior problem:

\[
\rho_t + \text{div}(\rho U) = 0, \quad t > 0, x \in \Omega, \tag{2.18}
\]

\[
(\rho U)_t + \text{div}(\rho U \otimes U) - \text{div}(2h(\rho)D(U)) - \nabla (g(\rho)\text{div}U) + \nabla P(\rho) = 0, \tag{2.19}
\]

with boundary and initial conditions

\[
(\rho U)|_{x \in \partial \Omega} = 0, \quad \rho|_{t=0} = \rho_0 \geq 0, \quad \rho U|_{t=0} = m_0, \tag{2.20}
\]

where \(\Omega = \{x \in \mathbb{R}^N ||x| > 1\}, \ N \geq 2\).

Definition 2.2. We say that \((\rho, U)\) is a weak solution of \((2.18)-(2.20)\) on \(\Omega \times [0, T]\), provided

(a) The condition (1) in Definition 2.1 where \(\mathbb{R}^N\) is replaced by \(\Omega\);

(b) For any \(t_2 > t_1 \geq 0\) and \(\phi_1 \in C^1_c(\mathbb{R}^N \times [0, \infty))\), the mass equation (2.1) holds in the following sense:

\[
\int_{\Omega} \rho(\phi_1 - \Phi_1)dx = \int_{t_1}^{t_2} \int_{\Omega} (\rho \partial_t \phi_1 + \rho U \cdot \nabla \phi_1)dxdt; \tag{2.21}
\]

(c) The condition (3) in Definition 2.1 where \(\mathbb{R}^N\) is replaced by \(\Omega\).

Using the similar proof of Theorem 2.1 and \(\|\sqrt{\rho(r)}\|_{L^\infty([1, \infty))} \lesssim \|\sqrt{\rho(r)}\|_{H^1([1, \infty))}\), we can obtain the similar result without the condition (2.9). Here, we give the following theorem and omit the proof.

Theorem 2.2. Assume that \(\gamma \geq 1\), \(h(\rho)\) and \(g(\rho)\) satisfy conditions \((2.4)-(2.7)\). If the initial data have the form

\[
\rho_0 = \rho_0(|x|), \quad m_0 = m_0(|x|) \frac{x}{r}
\]

and satisfy \((2.13)-(2.14)\) where \(\mathbb{R}^N\) is replaced by \(\Omega\), then the initial-value problem \((2.1)-(2.3)\) has a global spherically symmetric weak solution

\[
\rho = \rho(|x|, t), \quad U = u(|x|, t) \frac{x}{r}
\]

satisfying \((2.15)-(2.17)\) where \(\mathbb{R}^N\) is replaced by \(\Omega\), for all \(T > 0\).

Remark 2.5. In particular, we get the existence of spherically symmetric solutions to the Saint-Venant model for shallow water system in the whole space or exterior domain.
3 Proof of Theorem 2.1

The key point of the proof of Theorem 2.1 is to construct smooth approximate solutions satisfying the a priori estimates required in the $L^1$ stability analysis. The crucial issue is to obtain lower and upper bounds of the density. To this end, we study the following system as an approximate system of (2.1)-(2.2).

\[ \rho_t + \text{div}(\rho U) = 0, \]  
\[ (\rho U)_t + \text{div}(\rho U \otimes U) - \text{div}((2h(\rho) + \epsilon \rho^\theta)D(U)) - \nabla((g(\rho) + (\theta - 1)\epsilon \rho^\theta)\text{div}U) + \nabla P(\rho) = 0, \]  
where $\epsilon > 0$ is a constant and $\theta = \frac{N-1+\alpha}{N}$ with $\alpha \in (0, 1)$ satisfying

\[ V_1\left(\frac{N}{1-\alpha}\right) < \min\{\frac{\nu-2}{N}, \frac{\alpha-1}{N}\}, \quad V_2\left(\frac{N}{1-\alpha}\right) > \frac{\nu-2}{N}, \]  
where

\[ V_1(m) = \frac{4N(m-1) - 4\sqrt{N^2(m-1)^2 + (N-1)(m-1)(m-2)^2}}{(N-1)(m-2)^2} \]  
and

\[ V_2(m) = \frac{4N(m-1) + 4\sqrt{N^2(m-1)^2 + (N-1)(m-1)(m-2)^2}}{(N-1)(m-2)^2}. \]  

Remark 3.1. From (2.3), we can choose a small constant $\alpha$ satisfying (3.3).

When $\rho(x, t) = \rho(r, t)$, $U(x, t) = u(r, t)\frac{\zeta}{r}$, (3.1)-(3.2) becomes

\[ \rho_t + (\rho u)_r + \frac{(N-1)\rho u}{r} = 0, \]  
\[ \rho u_t + \rho uu_r + (\rho^\gamma)_r + (2h + \epsilon \rho^\theta)_r \frac{(N-1)u}{r} = ((2h + g + \theta \epsilon \rho^\theta)(u_r + \frac{(N-1)u}{r}))_r, \]  
for $r > 0$. We will first construct the smooth solution of (3.4)-(3.5) in the truncated region $0 < \epsilon < r < R < \infty$ with the following boundary conditions and initial condition

\[ u(r, t)|_{r=\epsilon} = u(r, t)|_{r=R} = 0, \]  
\[ (\rho, u)(r, 0) = (\rho_{0, \epsilon, R, \delta}, u_{0, \epsilon, R, \delta}) := (\rho_{0, \epsilon, R}*J_\delta, u_{0, \epsilon, R}*J_\delta), \epsilon < r < R, \]  
where $J_\delta$ is a standard mollifier,

\[ \rho_{0, \epsilon, R}(r) = \begin{cases} \rho_0(\epsilon) + \epsilon, & r \in [0, \epsilon], \\ \rho_0(r) + \epsilon, & r \in [\epsilon, R], \\ \rho_0(R) + \epsilon, & r \in [R, \infty], \end{cases} \]  
and

\[ u_{0, \epsilon, R}(r) = \begin{cases} 0, & r \in [0, \epsilon + 2\delta], \\ \frac{m_0(r)}{\rho_0(r) + \epsilon}, & r \in [\epsilon + 2\delta, R - 2\delta], \\ 0, & r \in [R - 2\delta, \infty). \end{cases} \]  

We assume that $\epsilon$ and $R$ satisfy $\epsilon R^N \leq \sqrt{\epsilon}$. Letting $\epsilon \to 0$ and $R \to \infty$, we can easily obtain that $(\rho_{0, \epsilon, R}, u_{0, \epsilon, R})$ convergence to $(\rho_0, u_0)$ in spaces given in (2.14). From (3.3) and similar arguments as that in [10], one can obtain the smooth solutions $(\rho^{\epsilon, R, \delta}(r, t), u^{\epsilon, R, \delta}(r, t))$ to the approximate system (3.4)-(3.7).
Remark 3.2. To obtain the existence of \((\rho^\varepsilon,R^\varepsilon,\delta(r,t), u^\varepsilon,R^\varepsilon,\delta(r,t))\), we need to consider the following system in the Lagrangian coordinates:

\[
\rho_t + \rho^2(r^{N-1}u)_x = 0,
\]

\[
r^{1-N}u_t + (\rho^\gamma)_x = [(ph + pg + \varepsilon \theta \rho^{\theta+1})/r^{N-1}] - (h + \varepsilon \theta \rho^\theta) (N-1)u/r,
\]

\[
u(0,\tau) = u(1,\tau) = 0,
\]

\[\rho(\cdot,\cdot)(\cdot,0) = (\rho_0,\varepsilon,R,\delta, u_0,\varepsilon,R,\delta).\]

From (3.3) and similar arguments as that in [10], one can obtain that \(\rho \in L^\infty(0,T;L_x^{\frac{N}{1-\alpha}})\), \((\rho^\theta)_x \in L^\infty(0,T;L_x^{\frac{N}{1-\alpha}})\) and \(\rho^{-1} \in L^{\infty}(T)\) (for simplicity, we omit the superscript). To estimate \(\|u\|_{L^\infty(0,T;L_x^{\frac{N}{1-\alpha}})}\), we need to estimate the following terms

\[
-(m-1)(2h\rho + g\rho + \varepsilon\theta\rho^{\theta+1})r^{2N-2}u^{m-2}u_x^2
-2(N-1)^2\rho h + (N-1)^2pg + \varepsilon(N-1)\theta(N-1) - N + 2)\rho^{1+\theta}r^{N-2}\rho^{-2}u^m
-\rho gm(N-1) + \varepsilon m(N-1)(\theta - 1)\rho^{1+\theta}r^{N-2}\rho^{-1}u^{m-1}u_x,
\]

(3.8)

where \(m = \frac{N}{1-\alpha}\). From (3.3), we have

\[
\text{(3.8)} \leq -C \rho h + \varepsilon \rho^{\theta+1}(r^{2N-2}u^{m-2}u_x^2 + r^{-2}\rho^{-2}u^m).
\]

Then, using similar arguments as that in [10], one can obtain \(u \in L^\infty(0,T;L_x^{\frac{N}{1-\alpha}})\).

So far, \((\rho^\varepsilon,R^\varepsilon,\delta, u^\varepsilon,R^\varepsilon,\delta)\) are defined on \(\varepsilon \leq r \leq R\). To take the limit \((\varepsilon_j, R_j, \delta_j) \to (0,\infty, 0)\), we extend \((\rho^\varepsilon,R^\varepsilon,\delta, u^\varepsilon,R^\varepsilon,\delta)\) to the whole space \(\mathbb{R}^N\) in the following way

\[
\rho^\varepsilon,R^\varepsilon,\delta(r,t) = \begin{cases} 
\rho^\varepsilon,R^\varepsilon,\delta(r,t), & r \in [\varepsilon, R], \\
0, & \text{else},
\end{cases}
\]

(3.9)

\[
u^\varepsilon,R^\varepsilon,\delta(r,t) = \begin{cases} 
\rho^\varepsilon,R^\varepsilon,\delta(r,t), & r \in [\varepsilon, R], \\
0, & \text{else}.
\end{cases}
\]

(3.10)

For simplicity, we denote the obtained approximate solutions \(\{\rho^\varepsilon_j,R^\varepsilon_j,\delta_j, \nu^\varepsilon_j,R^\varepsilon_j,\delta_j\}\) by \(\{\rho^j, \nu^j\}\). Let \(\rho^j(x,t) = \rho^j(r,t), \quad U^j(x,t) = u^j(r,t)\), \(B_{\varepsilon,R} = \{x \in \mathbb{R}^N \mid \varepsilon < |x| < R\}\) and \(B = \{x \in \mathbb{R}^N \mid |x| < R\}\).

Using similar arguments as that in proofs of Lemmas 3.2 and 4.1 in [10], and the similar argument as that in [15] (§5.5), we have the following lemma.

Lemma 3.1. There exists a constant \(C\) independent of \(\varepsilon, R\) and \(\delta\) such that

\[
\sup_{t \in [0,T]} \int_{\mathbb{R}^N} p^j(x,t)dx \leq C,
\]

(3.11)

\[
\sup_{t \in [0,T]} \int_{\mathbb{R}^N} \left( \frac{1}{2} \rho^j |U^j|^2 + \frac{1}{\gamma - 1} (p^j)^\gamma \right)(x,t)dx + \int_{0}^{T} \int_{\mathbb{R}^N} (\nu h(p^j) |\nabla U^j|^2)(x,t)dxdt
\]
\begin{align}
N &= 2. \\
\text{Proof.} \quad \text{From (2.5), (3.11)-(3.14), we have (3.12) bounded in Lemma 3.2. The pressure } \rho \text{ implies that (3.12) holds. Moreover, the following uniform estimate hold (3.14).}
\end{align}

\begin{align}
\sup_{t \in [0,T]} \int_{B_{\varepsilon_j,R_j}} \left( \frac{1}{2} \rho^j |U^j|^2 + \rho^j \log \rho^j - \bar{\rho} \log \bar{\rho} - (\log \bar{\rho} + 1)(\rho^j - \bar{\rho}) \right) (x,t) dx \\
+ \int_0^T \int_{B_{\varepsilon_j,R_j}} (\nu h(\rho^j)|\nabla U^j|^2)(x,t) dx dt + \alpha \varepsilon \int_0^T \int_{\mathbb{R}^N} ((\rho^j)^\theta |\nabla U^j|^2)(x,t) dx dt
\leq C, \text{ if } \gamma = 1, \bar{\rho} = e^{-|x|}, \quad (3.13)
\end{align}

\begin{align}
\sup_{t \in [0,T]} \int_{B_{\varepsilon_j,R_j}} \left( \frac{1}{2} \rho^j |U^j|^2 + \frac{2h'(\rho^j) + \theta \varepsilon (\rho^j)^{\theta-1} \nabla \rho^j}{\rho^j} \right)^2 (x,t) dx \\
+ \int_0^T \int_{B_{\varepsilon_j,R_j}} \left( \frac{2h'(\rho^j) + \theta \varepsilon (\rho^j)^{\theta-1} \nabla \rho^j \nabla (\rho^j)^{\gamma}}{\rho^j} \right) (x,t) dx dt \leq C. \quad (3.14)
\end{align}

Moreover, the following uniform estimate hold (3.15). \sup_{t \in [0,T]} \|\sqrt{\rho^j}\|_{H^1(B_{\varepsilon_j,R_j})} \leq C.

From this lemma, we can obtain the following lemma.

**Lemma 3.2.** The pressure \((\rho^j)^\gamma\) is bounded in \(L^{\frac{N+\gamma}{N}}(\mathbb{R}^N \times [0,T])\) when \(N \geq 3\), in \(L^\beta(\mathbb{R}^N \times [0,T])\) for all \(\beta \in [1,2)\) when \(N = 2\).

**Proof.** From (2.5), (3.11)- (3.14), we have \((\rho^j)^{\gamma/2}\) is bounded in \(L^2(0,T;H^1(B_{\varepsilon_j,R_j}))\).

When \(N \geq 3\), we get \((\rho^j)^{\gamma/2}\) is bounded in \(L^2(0,T;L^{\frac{2N}{N-2}}(B_{\varepsilon_j,R_j}))\) or \((\rho^j)^\gamma\) is bounded in \(L^1(0,T;L^{\frac{N+\gamma}{N}}(B_{\varepsilon_j,R_j}))\). Since \((\rho^j)^\gamma\) is bounded in \(L^\infty(0,T;L^1(B_{\varepsilon_j,R_j}))\), Hölder’s inequality implies that \((\rho^j)^\gamma\) is bounded in \(L^{\frac{N+\gamma}{N}}(B_{\varepsilon_j,R_j} \times [0,T])\). From (3.9), we obtain that \((\rho^j)^\gamma\) is bounded in \(L^{\frac{N+\gamma}{N}}(\mathbb{R}^N \times [0,T])\).

Similarly, we can get that \((\rho^j)^\gamma\) is bounded in \(L^\beta(\mathbb{R}^N \times [0,T])\) for all \(\beta \in [1,2)\) when \(N = 2\). \(\square\)

In the following proposition, we will estimate \(\|\rho|U|^2 \ln(1 + |U|^2)\|_{L^1(\mathbb{R}^N)}\).

**Proposition 3.1.** If \(\nu_1 h \leq 2h + Ng \leq \nu_2 h\) and \(\int_0^\infty \rho_0(1 + |u_0|^2) \ln(1 + |u_0|^2)r^{N-1} dr \leq C\), then the following estimate is true

\begin{align}
\sup_{t \in [0,T]} \int_{\varepsilon_j}^{R_j} \rho^j \frac{|u^j|^2}{2} \ln(1 + |u^j|^2)r^{N-1} dr \leq C. \quad (3.16)
\end{align}

where \(C\) is a constant independent of \(\varepsilon_j, R_j\) and \(\delta_j\).

**Proof.** Multiplying (3.5) by \(r^{N-1}u^j(1 + \ln(1 + |u|^2))\), integrating the resulting equation and using (3.4) yield

\[
\frac{d}{dt} \int_{\varepsilon_j}^{R_j} \rho^j \frac{1 + |u^j|^2}{2} \ln(1 + |u^j|^2)r^{N-1} dr
\]
\begin{align*}
+ \int_{\xi_j}^{R_j} (2h + \varepsilon(\rho^j)^\theta)(1 + \ln(1 + |u^j|^2))(u^j)_r^2 + (N - 1) \frac{(u^j)^2}{r^2})r^{N-1}dr \\
+ \int_{\xi_j}^{R_j} (2h + \varepsilon(\rho^j)^\theta) \frac{2(u^j)^2}{1 + |u|^2} (u^j)_r^2 r^{N-1}dr \\
+ \int_{\xi_j}^{R_j} (g + (\theta - 1)\varepsilon(\rho^j)^\theta)(1 + \ln(1 + |u^j|^2))(u^j)_r + (N - 1) \frac{u^j}{r})^2 r^{N-1}dr \\
+ \int_{\xi_j}^{R_j} (g + (\theta - 1)\varepsilon(\rho^j)^\theta) \frac{2(u^j)^2}{1 + |u|^2} (u^j)_r r^{N-1}dr \\
+ \int_{\xi_j}^{R_j} ((\rho^j)^\gamma)_r (1 + \ln(1 + |u|^2)) u^j r^{N-1}dr = 0.
\end{align*}

Since \( \nu_1 h \leq 2h + Ng \leq \nu_2 h \) and \( (1 + N(\theta - 1))\varepsilon(\rho^j)^\theta = \alpha \varepsilon(\rho^j)^\theta \), we have
\[
\frac{d}{dt} \int_{\xi_j}^{R_j} \rho^j \frac{1 + |u^j|^2}{2} \ln(1 + |u^j|^2) r^{N-1}dr \\
+ \int_{\xi_j}^{R_j} (\nu_1 h + \alpha \varepsilon(\rho^j)^\theta)(1 + \ln(1 + |u^j|^2))(u^j)_r^2 + (N - 1) \frac{(u^j)^2}{r^2} r^{N-1}dr \\
+ \int_{\xi_j}^{R_j} ((\rho^j)^\gamma)_r (1 + \ln(1 + |u^j|^2)) u^j r^{N-1}dr \\
\leq C \int_{\xi_j}^{R_j} (h + \varepsilon(\rho^j)^\theta)(u^j)^2 + (N - 1) \frac{(u^j)^2}{r^2})r^{N-1}dr.
\tag{3.17}
\]

Using integration by parts and Young’s inequality, we have
\[
\left| \int_{\xi_j}^{R_j} ((\rho^j)^\gamma)_r (1 + \ln(1 + |u^j|^2)) u^j r^{N-1}dr \right| \\
\leq C \int_{\xi_j}^{R_j} |u^j|(1 + \ln(1 + |u^j|^2))(\rho^j)^\gamma r^{N-1}dr + C \int_{\xi_j}^{R_j} |u^j|(1 + \ln(1 + |u^j|^2))(\rho^j)^\gamma r^{N-2}dr \\
\leq \frac{\nu_1}{2} \int_{\xi_j}^{R_j} h(1 + \ln(1 + |u^j|^2))(u^j)_r^2 + (N - 1) \frac{(u^j)^2}{r^2} r^{N-1}dr \\
+ C \int_{\xi_j}^{R_j} h^{-1}(\rho^j)^2(1 + \ln(1 + |u^j|^2)) r^{N-1}dr \\
\leq \frac{\nu_1}{2} \int_{\xi_j}^{R_j} h(1 + \ln(1 + |u^j|^2))(u^j)_r^2 + (N - 1) \frac{(u^j)^2}{r^2} r^{N-1}dr \\
+ C \left( \int_{\xi_j}^{R_j} \left( \frac{(\rho^j)^{2\gamma - \frac{\delta}{2}}}{h} \frac{2}{2 - \delta} r^{N-1}dr \right)^{\frac{2}{2 - \delta}} \right)^{\frac{\delta}{2} - \frac{\delta}{2}} \\
+ C \left( \int_{\xi_j}^{R_j} \rho^j(1 + \ln(1 + |u^j|^2)) \frac{2}{2 - \delta} r^{N-1}dr \right)^{\frac{\delta}{2} - \frac{\delta}{2}}.
\]

Combining it with (3.11) - (3.13) and (3.17), we get
\[
\sup_{t \in [0, T]} \int_{\xi_j}^{R_j} \rho^j \frac{1 + |u|^2}{2} \ln(1 + |u|^2) r^{N-1}dr \\
\leq C + C_\delta \left( \int_{\xi_j}^{R_j} \left( \frac{(\rho^j)^{2\gamma - \frac{\delta}{2}}}{h} \frac{2}{2 - \delta} r^{N-1}dr \right)^{\frac{2}{2 - \delta}} \right)^{\frac{\delta}{2} - \frac{\delta}{2}}.
\tag{3.18}
\]
From (2.5), we have $h \geq \nu \rho$ and
\[
\left( \int_{\epsilon_j}^{R_j} \left( \frac{(\rho^j)^{2\gamma - \frac{3}{2}}}{h} \right)^{\frac{2}{2-\gamma}} r^{N-1} dr \right)^{\frac{2-\gamma}{2}} \leq C \left( \int_{\epsilon_j}^{R_j} \left( (\rho^j)^{2\gamma - 1 - \frac{3}{2}} \right)^{\frac{2}{2-\gamma}} r^{N-1} dr \right)^{\frac{2-\gamma}{2}}.
\]

Then, using Lemma 3.2, we check that the right hand side is bounded $L^1$ in time for some small $\delta$, without any condition when $N = 2$, and when $N \geq 3$ under the condition that
\[
2\gamma - 1 < \frac{N + 2}{N} \gamma,
\]
which gives rise to the restriction $\gamma < \frac{N}{N-2}$. In either cases, we have
\[
\sup_{t \in [0,T]} \int_{\epsilon_j}^{R_j} \rho^j \frac{1}{2} \frac{|u^j|^2}{\ln(1 + |u^j|^2)} r^{N-1} dr \leq C.
\]

When $N \geq 3$ and $\gamma \geq \frac{N}{N-2}$, we need the extra hypothesis (2.9) to show that the right hand side of (3.18) is bounded and to obtain the same result.

From (3.9)-(3.10), we deduce that
\[
\sup_{t \in [0,T]} \int_{\mathbb{R}^N} \rho^j \frac{|U^j|^2}{2} \ln(1 + |U^j|^2) dx \leq C. \tag{3.19}
\]

**Proposition 3.2.** The sequence $\{\rho^j\}$ is bounded in $L^\infty(0,T;L^{\frac{N}{N-2}}(\mathbb{R}^N))$ when $N \geq 3$, or $L^\infty(0,T;L^2(\mathbb{R}^2))$ for all $q \geq 1$. There exists a subsequence of $\{\rho^j\}$, still denoted by itself, such that
\[
\rho^j(x,t) \to \rho(x,t), \tag{3.20}
\]
strongly in $C([0,T];L^\beta_{\text{loc}}(\mathbb{R}^N))$, $\beta \in [1,\frac{N}{N-2}]$, as $j \to \infty$. Here, $\rho \in L^\infty(0,T;L^1 \cap L^{\frac{N}{N-2}}(\mathbb{R}^N))$ when $N \geq 3$, or $\rho \in L^\infty(0,T;L^2(\mathbb{R}^2))$ for all $q \geq 1$. Moreover, $\rho(x,t) = \rho(r,t)$ is a spherically symmetric function.

**Proof.** We only consider the case $N \geq 3$, since the proof of the case that $N = 2$ is similar.

It follows from (3.9) and (3.15) that $\{\sqrt{\rho^j}\}$ is bounded in $L^\infty(0,T;L^q(\mathbb{R}^N))$ for $q \in [2,\frac{2N}{N-2}]$. Thus, $\{\rho^j\}$ is bounded in $L^\infty(0,T;L^{\frac{N}{N-2}}(\mathbb{R}^N))$ and $\{\rho^j U^j\}$ is bounded in $L^\infty(0,T;L^2(\mathbb{R}^N))$ due to (3.12) (3.13). Then, the continuity equation yields that $\{\partial_t \rho^j\}_{\epsilon_j \leq \frac{R_j}{\nu}, R_j \geq \rho}$ is bounded in $L^\infty(0,T;W^{-1,\frac{N}{N-2}}(B_{\frac{R_j}{\nu},n}))$, for any $k \geq n^{2N}$. Moreover, since $\nabla \rho^j = 2\sqrt{\rho^j} \nabla \sqrt{\rho^j}$, we have $\{\nabla \rho^j\}_{\epsilon_j \leq \frac{R_j}{\nu}, R_j \geq \rho}$ is bounded in $L^\infty(0,T;L^{\frac{N}{N-2}}(B_{\frac{R_j}{\nu},n}))$. From the Aubin-Lions lemma, we get
\[
\rho^j(x,t) \to \rho(x,t), \text{ strongly in } C([0,T];L^{\frac{N}{N-2}}(B_{\frac{R_j}{\nu},n})), \text{ as } j \to \infty. \tag{3.21}
\]

Since
\[
\|\rho^j - \rho\|_{L^\infty([0,T];L^{\frac{N}{N-2}}(B_{n}))} \leq \frac{C}{k} \|\rho^j - \rho\|_{L^\infty([0,T];L^{\frac{N}{N-2}}(B_{\frac{R_j}{\nu},n}))} + \|\rho^j - \rho\|_{L^\infty([0,T];L^{\frac{N}{N-2}}(B_{\frac{R_j}{\nu},n}))},
\]
we get
\[
\rho^j(x,t) \to \rho(x,t), \text{ strongly in } C([0,T];L^{\frac{N}{N-2}}(B_{n})), \text{ as } j \to \infty. \tag{3.22}
\]
Clearly, (3.20) holds and $\rho(x,t)$ is spherically symmetric. 

\[
\boxed{10}
\]
From (3.9), Lemma 3.2 and Proposition 3.2, we immediately obtain the following lemma.

**Lemma 3.3.** There exists a subsequence of \( \{ \rho^{j} \} \), still denoted by itself, such that

\[
(\rho^{j})^{\gamma}(x, t) \to \rho^{\gamma}(x, t),
\]

strongly in \( L^{1}_{loc}(\mathbb{R}^{N} \times [0, T]) \), as \( j \to \infty \).

**Proposition 3.3.** For any \( k \geq n^{2N} \), there exists a subsequence of \( \{ \rho^{j} \}_{j \leq \frac{1}{k}, R_{j} \geq n} \), still denoted by itself, such that

\[
\nabla \sqrt{\rho^{j}(x, t)} \overset{*}{\rightharpoonup} \nabla \sqrt{\rho(x, t)}, \text{ weak-* in } L^{\infty}([0, T], L^{2}(B_{\frac{1}{k}, n})),
\]

\[
\nabla \bar{h}(\rho^{j}(x, t)) \overset{*}{\rightharpoonup} \nabla \bar{h}(\rho(x, t)), \text{ weak-* in } L^{\infty}([0, T], L^{2}(B_{\frac{1}{k}, n})),
\]

as \( j \to \infty \), where \( \bar{h} \) satisfies \( \bar{h}(0) = 0 \) and \( \bar{h}'(s) = \frac{h'(s)}{\sqrt{s}} \). Moreover, \( \nabla \sqrt{\rho} \in L^{\infty}([0, T], L^{2}(\mathbb{R}^{N})) \) and \( \nabla \bar{h}(\rho) \in L^{\infty}([0, T], L^{2}(\mathbb{R}^{N})) \).

**Proof.** It follows from (3.15) that \( \{ \nabla \sqrt{\rho^{j}} \}_{j \leq \frac{1}{k}, R_{j} \geq n} \) is bounded in \( L^{\infty}(0, T; L^{2}(B_{\frac{1}{k}, n})) \). Thus, there exists a function \( f \in L^{\infty}(0, T; L^{2}(B_{\frac{1}{k}, n})) \) such that, up to a subsequence,

\[
\nabla \sqrt{\rho^{j}(x, t)} \overset{*}{\rightharpoonup} f, \text{ weak-* in } L^{\infty}([0, T], L^{2}(B_{\frac{1}{k}, n})).
\]

Combining it with (3.20), one can easily obtain \( f = \nabla \sqrt{\bar{h}} \) and

\[
\| \nabla \sqrt{\rho^{j}} \|_{L^{\infty}([0, T], L^{2}(B_{\frac{1}{k}, n}))} \leq \lim \inf_{j \to \infty} \| \nabla \sqrt{\rho^{j}} \|_{L^{\infty}([0, T], L^{2}(B_{\frac{1}{k}, n}))} \leq C
\]

with a constant \( C \) independent of \( k \) and \( n \). Clearly, we have \( \nabla \sqrt{\rho} \in L^{\infty}([0, T], L^{2}(\mathbb{R}^{N})) \). Similarly, we can easily obtain the result for \( \bar{h} \).

From Propositions 3.1, 3.2 and Corollary 3.1 using similar arguments as that in the proof of Lemmas 4.4 and 4.6 in [21], we can obtain the following proposition.

**Proposition 3.4.** 1) Up to a subsequence, \( m^{j} = \rho^{j}U^{j} \) converges strongly in \( L^{1}_{loc}(\mathbb{R}^{N} \times [0, T]) \) and \( L^{2}(0, T; L^{\beta}_{loc}(\mathbb{R}^{N})) \) to some \( m(x, t) \), for all \( \beta \in [1, \frac{N}{N-1}] \).

2) \( \sqrt{\rho^{j}U^{j}} \) converges strongly in \( L^{2}_{loc}(\mathbb{R}^{N} \times [0, T]) \) to \( \frac{m}{\sqrt{\rho}} \) (defined to be zero when \( m = 0 \)). In particular, \( m(x, t) = 0 \) a.e. on \( \{ \rho(x, t) = 0 \} \) and there exists a function \( U(x, t) \) such that

\[
m(x, t) = \rho(x, t)U(x, t).
\]

**Proof.** We only consider the case \( N \geq 3 \), since the proof of the case that \( N = 2 \) is similar.

1) Since \( \{ \sqrt{\rho^{j}} \} \) is bounded in \( L^{\infty}(0, T; L^{2} \cap L^{\frac{2N}{N-2}}) \) and \( \{ \sqrt{\rho^{j}U^{j}} \} \) is bounded in \( L^{\infty}(0, T; L^{2}) \), we have that

\[
\{ \rho^{j}U^{j} \} \text{ is bounded in } L^{\infty}(0, T; L^{1} \cap L^{\frac{2N}{N-1}}(\mathbb{R}^{N})).
\]

Since \( \nabla(\rho^{j}U^{j}) = 2\sqrt{\rho^{j}U^{j}}\nabla \sqrt{\rho} + \sqrt{\rho^{j}}\sqrt{\rho^{j}U^{j}} \nabla U^{j} \), from (2.5) and (3.11)-(3.14), we obtain that \( \{ \nabla(\rho^{j}U^{j}) \}_{j \leq \frac{1}{k}, R_{j} \geq n} \) is bounded in \( L^{2}(0, T; L^{1}(B_{\frac{1}{k}, n})) \). In particular, we get

\[
\{ (\rho^{j}U^{j}) \}_{j \leq \frac{1}{k}, R_{j} \geq n} \text{ is bounded in } L^{2}(0, T; W^{1,1}(B_{\frac{1}{k}, n})).
\]
Since \( \{\rho^j\}_{\epsilon_j \leq \frac{1}{\epsilon}, R_j \geq n} \) is bounded in \( L^\infty(B_{\frac{1}{\epsilon}, n} \times [0, T]) \), from (3.2), we can obtain that
\[
\{\partial_t (\rho^j U^j)\}_{\epsilon_j \leq \frac{1}{\epsilon}, R_j \geq n} \text{ is bounded in } L^2(0, T; W^{-2, \frac{N}{N-1}}(B_{\frac{1}{\epsilon}, n})).
\]
From the Aubin-Lions lemma, we have
\[
\rho^j U^j \rightharpoonup m,
\]
strongly in \( L^2([0, T]; L^\beta(B_{\frac{1}{\epsilon}, n})) \) for all \( \beta \in [1, \frac{N}{N-1}) \). From (3.25), we can easily obtain that
\[
\rho^j U^j \rightharpoonup m,
\]
strongly in \( L^2([0, T]; L^\beta(B_n)) \) for all \( \beta \in [1, \frac{N}{N-1}) \).

2) Using the similar argument as that in the proof of Lemma 4.6 in [21], we can obtain the part 2) of Proposition 3.4 where
\[
U = \begin{cases} \frac{m}{\rho}, & \text{if } \rho \neq 0, \\ 0, & \text{if } \rho = 0, \end{cases}
\]
and omit the detail.

Then, using similar arguments as that in the proof of Corollary 4.2 in [10], we can obtain the following corollary and omit the details.

**Corollary 3.2.** Let \( m^j(r, t) = (\rho^j U^j)(r, t) \), then

1) there exists a function \( m(r, t) \) such that \( m(x, t) = m(r, t) \frac{x}{r} \) and \( m^j(r, t) \) converges to \( m(r, t) \)
strongly in \( L^2(0, T; L^\beta_{\text{loc}}([0, \infty); r^{-N+1} dr)) \) for all \( \beta \in [1, \frac{N}{N-1}) \);

2) there exits a function \( u(r, t) \) such that \( U(x, t) = u(r, t) \frac{x}{r} \) and \( \sqrt{\rho^j U^j} \) converges to \( \frac{m}{\sqrt{\rho}} \)
(defined to be zero when \( m = 0 \)) strongly in \( L^2(0, T; L^2_{\text{loc}}([0, \infty); r^{-N+1} dr)) \).

Now, we show that \((\rho, U)\) obtained in Propositions 3.1-3.4 satisfies the weak form of (2.1), that is (2.11) holds.

**Proposition 3.5.** Let \((\rho, U)\) be the limit described as in Propositions 3.1-3.4. Then (2.11) holds. Moreover, \( \rho \in C([0, \infty); L^1(\mathbb{R}^N)) \).

**Proof.** We only consider the case \( t_1 > 0 \), since the proof of the case that \( t_1 = 0 \) is similar.

At first, we derive the weak form of (3.4). For any \( \varphi \in C^1_\text{c}([0, \infty) \times [0, \infty)) \), there exists \( n > 0 \) such that \( \text{supp} \varphi(\cdot, t) \subset [0, n] \). It follows from (3.4), (3.6) and (3.9)-(3.10) that
\[
\int_0^\infty \rho^j \varphi r^{-N+1} dr |_{t_1}^{t_2} - \int_{t_1}^{t_2} \int_0^\infty (\rho^j \varphi_t + \rho^j U^j \varphi_r) r^{-N+1} dr dt = 0,
\]
for any \( j \) satisfying \( R_j \geq n \). From Proposition 3.2 we have
\[
\int_0^\infty \rho^j \varphi r^{-N+1} dr \rightarrow \int_0^\infty \rho \varphi r^{-N+1} dr,
\]
and
\[
\int_{t_1}^{t_2} \int_0^\infty \rho^j \varphi r^{-N+1} dr dt \rightarrow \int_{t_1}^{t_2} \int_0^\infty \rho \varphi r^{-N+1} dr dt
\]
for any \( \varphi \in C^1_\text{c}([0, \infty) \times [0, \infty)) \).
as $j \to 0$. From Proposition 3.2 and Corollary 3.2, we have

\[
\int_{t_1}^{t_2} \int_0^\infty \rho^j \varphi_r r^{N-1} \, dr \, dt = \int_{t_1}^{t_2} \int_0^\infty \sqrt{\rho^j} (\sqrt{\rho^j} \varphi_r) r^{N-1} \, dr \, dt \to \\
\int_{t_1}^{t_2} \int_0^\infty \sqrt{\rho} (\sqrt{\rho} \varphi_r) r^{N-1} \, dr \, dt,
\]

as $j \to 0$.

Therefore, taking limit $j \to \infty$ in (2.26), we get

\[
\int_0^\infty \rho \varphi r^{N-1} \, dr \bigg|_{t_1}^{t_2} - \int_{t_1}^{t_2} \int_0^\infty (\rho \varphi_t + \rho w \varphi_r) r^{N-1} \, dr \, dt = 0. \tag{3.28}
\]

For any $\phi \in C^1_b(\mathbb{R}^N \times [t_1, t_2])$, define

\[
\varphi(r, t) = \int_S \phi_1 (r y, t) \, dS_y,
\]

where the integral is over the unit sphere $S = S^{N-1}$ in $\mathbb{R}^N$. Then it follows from (3.28) that (2.11) holds.

Similarly, we can easily obtain

\[
\partial_t \sqrt{\rho} + \text{div}(\sqrt{\rho} u) - \frac{1}{2} Q = 0, \quad \text{in} \; \mathcal{D}',
\]

where $Q \in L^2(0, T; L^2(\mathbb{R}^N))$ is the weak limit of $\{\sqrt{\rho} \text{div} u^j\}$ in $L^2(0, T; L^2(\mathbb{R}^N))$. Thus, we have $\sqrt{\rho} \in L^2([0, \infty); H^1(\mathbb{R}^N))$. Since $\sqrt{\rho} \in L^\infty([0, \infty); L^2(\mathbb{R}^N))$, we can easily get that $\sqrt{\rho} \in C([0, \infty); L^2(\mathbb{R}^N))$.

In the following, we prove that $(\rho, u)$ satisfies (2.12).

**Proposition 3.6.** Let $(\rho, U)$ be the limit described as in Propositions 3.1-3.4. Then (2.12) holds.

**Proof.** For any $\phi \in C^2_b([0, \infty) \times [0, T])$ with $\phi(0, t) = \phi(r, T) = 0$, there exists $n > 0$ such that $\text{supp} \phi(\cdot, t) \subset [0, n]$. It follows from (3.5)-(3.6) and (3.9)-(3.10) that

\[
\int_{T_{\varepsilon, j}}^\infty \rho_0^j u_0^j \phi(r, 0) r^{N-1} \, dr \\
+ \int_0^T \int_0^\infty (\rho^j u^j \phi_t + \rho^j (u^j)^2 \phi_r + (\rho^j)^\gamma (\phi_r + \frac{(N-1) \phi}{r})) r^{N-1} \, dr \, dt \\
- \int_0^T \int_{T_{\varepsilon, j}}^\infty 2h(\rho^j) (u^j \phi_r + \frac{(N-1) u^j \phi}{r^2}) r^{N-1} \, dr \, dt \\
= - \int_0^T \int_{T_{\varepsilon, j}}^\infty \varepsilon_j (\rho^j)^\theta (\frac{(N-1) u^j \phi}{r}) + \frac{(N-1) u^j \phi_r}{r} + \frac{(N-1)(N-2) u^j \phi}{r^2}) r^{N-1} \, dr \, dt + \varepsilon_j^j \\
+ \int_0^T \int_{T_{\varepsilon, j}}^\infty (g(\rho^j) + \theta \varepsilon_j (\rho^j)^\theta (u^j) + \frac{(N-1) u^j}{r^2}) (\phi_r + \frac{(N-1) \phi}{r}) r^{N-1} \, dr \, dt, \tag{3.29}
\]

for any $j$ satisfying $\varepsilon_j \leq \frac{1}{k}$ and $R_j \geq n$, where

\[
\varepsilon_b^j = \int_0^T \left\{ [2h(\rho^j) + g(\rho^j) + \theta \varepsilon_j (\rho^j)^\theta (u^j)] (\varepsilon_j, t) \varepsilon_j^{N-1} \phi(\varepsilon_j, t) - \varepsilon_j^{N-1} \phi(\varepsilon_j, t) \phi(\varepsilon_j, t) \right\} \, dt. \tag{3.30}
\]
Claim:

\[
\lim_{\varepsilon_j \to 0^+} \varepsilon_j^j = 0.
\] (3.31)

Since

\[
\varepsilon_j^{N-1} \int_0^T [(\rho^j)^\gamma \phi](\varepsilon_j, t) dt \leq \max_{t \in [0,T]} |\phi(\varepsilon_j, t)| \varepsilon_j^{N-1} \int_0^T (\rho^j)^\gamma(\varepsilon_j, t) dt 
\]

\[
\leq C \max_{t \in [0,T]} |\phi(\varepsilon_j, t)| \int_0^T \int_{\varepsilon_j}^{\varepsilon_j + R_j} [(\rho^j)^\gamma + |\partial_r (\rho^j)^\gamma|] r^{N-1} dr dt,
\]

\[
\int_0^T \int_{\varepsilon_j}^{\varepsilon_j + R_j} (\rho^j)^\gamma r^{N-1} dr dt \leq C,
\]

\[
\int_0^T \int_{\varepsilon_j}^{\varepsilon_j + R_j} |\partial_r (\rho^j)^\gamma| r^{N-1} dr dt \leq C \int_0^T \int_{\varepsilon_j}^{\varepsilon_j + R_j} [(\rho^j)^\gamma + |\partial_r (\rho^j)^\gamma|] r^{N-1} dr dt \leq C,
\]

and \(\lim_{\varepsilon_j \to 0^+} \max_{t \in [0,T]} |\phi(\varepsilon_j, t)| = 0\), we have

\[
\lim_{\varepsilon_j \to 0^+} \varepsilon_j^{N-1} \int_0^T [(\rho^j)^\gamma \phi](\varepsilon_j, t) dt = 0. \tag{3.32}
\]

From (3.31) and \(u(\varepsilon_j, t) = 0\), we get

\[
\rho^j_r(\varepsilon_j, t) + \rho^j(\varepsilon_j, t) \partial_r u^j(\varepsilon_j, t) = 0.
\]

Thus, using (2.4), we have

\[
\varepsilon_j^{N-1} \int_0^T ((2h(\rho^j) + g(\rho^j))u^j_r \phi)(\varepsilon_j, t) dt = -\varepsilon_j^{N-1} \int_0^T (\frac{2h(\rho^j) + g(\rho^j)}{\rho} \partial_r \rho^j \phi)(\varepsilon_j, t) dt 
\]

\[
= -\varepsilon_j^{N-1} \int_0^T (2\partial_r h(\rho^j) \phi)(\varepsilon_j, t) dt 
\]

\[
= 2\varepsilon_j^{N-1} h(\rho^j_0(\varepsilon_j)) \phi(\varepsilon_j, 0) + 2\varepsilon_j^{N-1} \int_0^T (h(\rho^j) \partial_t \phi)(\varepsilon_j, t) dt.
\]

It is easy to obtain

\[
|\sqrt{\rho^j(\varepsilon_j, t)}| \leq C |\sqrt{\rho^j}|_{H^1([\varepsilon_j, R_j])} \leq C \varepsilon_j^{\frac{N-1}{2}},
\]

\[
|\sqrt{\rho^j(1,t)}| \leq C |\sqrt{\rho^j}|_{H^1([1, R_j])} \leq C,
\]

\[
|\tilde{h}(\rho^j(\varepsilon_j, t))| \leq C |\tilde{h}(\rho^j(1,t))| + \|\nabla \tilde{h}\|_{L^2([\varepsilon_j, 1])} \leq C + C \varepsilon_j^{\frac{N-1}{2}}, \tag{3.33}
\]

where \(\tilde{h}\) satisfies \(\tilde{h}(0) = 0\) and \(\tilde{h}'(s) = \frac{h'(s)}{\sqrt{s}}\). Since \(h(s) \leq \sqrt{\tilde{h}(s)}\), we have

\[
h(\rho^j(\varepsilon_j, t)) \leq C + C \varepsilon_j^{-(N-1)}.
\]

Thus, we can easily obtain

\[
\lim_{\varepsilon_j \to 0^+} \max_{t \in [0, T]} |\partial_t \phi(\varepsilon_j, t)| = 0
\]

and

\[
\varepsilon_j^{N-1} \int_0^T (h(\rho^j) \partial_t \phi)(\varepsilon_j, t) dt \leq C \max_{t \in [0, T]} |\partial_t \phi(\varepsilon_j, t)| \to 0,
\]

\[14\]
as $\varepsilon_j \to 0+$. Hence, we have

$$
\lim_{\varepsilon_j \to 0+} \varepsilon_j^{N-1} \int_0^T (2h(\rho^j) + g(\rho^j)) u_t^j \phi(\varepsilon_j, t) dt = 0.
$$

Similarly, one can obtain that

$$
\lim_{\varepsilon_j \to 0+} \varepsilon_j^N \int_0^T [(\rho^j)^\theta u_t^j \phi](\varepsilon_j, t) dt = 0.
$$

Thus, (3.31) holds.

Now, for any $\phi_2 \in (C_\infty^2(\mathbb{R}^N \times [0, T]))^N$ with supp $\phi_2(\cdot, t) \subset B_n$ and $\phi_2(x, T) = 0$, we set

$$
\phi(r, t) = \int_S \phi_2(ry, t) \cdot y dS_y,
$$

(3.34)

Since

$$
(r^{N-1} \phi)_r = \partial_r \int_{|x| \leq r} \text{div} \phi_2(x, t) dx = r^{N-1} \int_S \phi_2^i(x, ry, t) dS_y,
$$

we have by direct calculation that

$$
- \int_0^T \int_{\varepsilon_j}^{\infty} 2h(\rho^j)(u_r^j \phi_r + \frac{(N-1)u_r^j}{r^2}) r^{N-1} dr dt = - \int_0^T \int_{|x| > \varepsilon_j} 2h(\rho^j) D(U^j) : \nabla \phi_2 dxdt.
$$

Similarly, one has

$$
\int_0^t \int_{\varepsilon_j}^{\infty} (g(\rho^j) + \theta \varepsilon_j (\rho^j)^\theta) \frac{(N-1)u_r^j}{r^2} r^{N-1} dr dt
$$

$$
= \int_0^t \int_{|x| > \varepsilon_j} (g(\rho^j) + \theta \varepsilon_j (\rho^j)^\theta) \text{div} U^j \text{div} \phi_2 dxdt
$$

and

$$
\int_0^T \int_{\varepsilon_j}^{\infty} (\rho^j)^\theta \frac{(N-1)u_r^j}{r^2} r^{N-1} dr dt
$$

$$
= \int_0^t \int_{|x| > \varepsilon_j} (\rho^j)^\theta \text{div} U^j \text{div} \phi_2 - D(U^j) : \nabla \phi_2) dxdt.
$$

Thus, from (3.29), we have

$$
\int_{|x| > \varepsilon_j} \rho_0^j U^j_0 \cdot \phi_2(x, 0) dx + \int_0^T \int_{\mathbb{R}^N} \sqrt{\rho^j} \sqrt{\rho^j} U^j \partial_t \phi_2 + \sqrt{\rho^j} U^j \otimes \sqrt{\rho^j} U^j : \nabla \phi_2 dxdt
$$

+ $\int_0^T \int_{\mathbb{R}^N} (\rho^j)^\gamma \text{div} \phi_2 dxdt - \int_0^T \int_{|x| > \varepsilon_j} [2h(\rho^j) D(U^j): \nabla \phi_2 + g(\rho^j) \text{div} U^j \text{div} \phi_2] dxdt$

$$
= \varepsilon_j \int_0^T \int_{|x| > \varepsilon_j} [(\theta - 1)(\rho^j)^\theta \text{div} U^j \text{div} \phi_2 + (\rho^j)^\theta D(U^j) : \nabla \phi_2] dxdt + \varepsilon_j^2.
$$

(3.35)

We proceed to show that each term on the left hand side of (3.35) converges to corresponding term in (2.12), and each term on the right hand side of (3.35) vanishes as $j \to \infty$.

First, the proof of the convergence of $\rho^j U^j \partial_t \phi_2$ is similar to that of (3.27).
Next, from Proposition 3.3, we obtain
\[
\int_0^T \int_{\mathbb{R}^N} \sqrt{\rho^j} U^j \otimes \sqrt{\rho^j} U^j : \nabla \phi_2 \, dx \, dt \to \int_0^T \int_{\mathbb{R}^N} \sqrt{\rho} U \otimes \sqrt{\rho} U : \nabla \phi_2 \, dx \, dt, \quad \text{as } j \to \infty.
\]

From Lemma 3.3, we have
\[
\int_0^T \int_{\mathbb{R}^N} (\rho^j)^\gamma \text{div}\phi_2 \, dx \, dt \to \int_0^T \int_{\mathbb{R}^N} \rho^\gamma \text{div}\phi_2 \, dx \, dt, \quad \text{as } j \to \infty.
\]

Concerning the diffusion terms on the left hand side of (3.35), using (3.9) and integration by parts, we have
\[
\int_0^T \int_{|x| > \varepsilon_j} 2h(\rho^j) D(U^j) : \nabla \phi_2 \, dx \, dt = - \int_0^T \int_{\mathbb{R}^N} \left[ \frac{h(\rho^j)}{\sqrt{\rho^j}} (\sqrt{\rho^j} U^j) \cdot \Delta \phi_2 + \frac{h(\rho^j)}{\sqrt{\rho^j}} (\sqrt{\rho^j} U^j) \cdot \nabla \text{div}\phi_2 \right] \, dx \, dt
\]
\[- \int_0^T \int_{B_{\varepsilon_j,n}} \left[ (\sqrt{\rho^j} U^j) \cdot (\nabla \tilde{h}(\rho^j) \cdot \nabla) \phi_2 + (\sqrt{\rho^j} U^j) \cdot (\nabla \phi_2 \cdot \nabla) \tilde{h}(\rho^j) \right] \, dx \, dt. \quad (3.36)
\]

Using the similar argument as that in the proof of (3.33), we have
\[
\left\| \frac{h(\rho^j)}{\sqrt{\rho^j}} \right\|_{L^\infty([0,T];L^2(\mathbb{R}^n))} \leq C_n, \quad \text{and } \|\rho^j\|_{L^\infty([\frac{1}{2},1]\times[0,T])} \leq C_{k,n}.
\]

Then, using the similar argument as that in the proof of (3.27), we have
\[
\int_0^T \int_{\mathbb{R}^N} \frac{h(\rho^j)}{\sqrt{\rho^j}} (\sqrt{\rho^j} U^j) \cdot \Delta \phi_2 \, dx \, dt \to \int_0^T \int_{\mathbb{R}^N} \frac{h(\rho)}{\sqrt{\rho}} (\sqrt{\rho} U) \cdot \Delta \phi_2 \, dx \, dt, \quad (3.37)
\]
and
\[
\int_0^T \int_{\mathbb{R}^N} \frac{h(\rho^j)}{\sqrt{\rho^j}} (\sqrt{\rho^j} U^j) \cdot \nabla \text{div}\phi_2 \, dx \, dt \to \int_0^T \int_{\mathbb{R}^N} \frac{h(\rho)}{\sqrt{\rho}} (\sqrt{\rho} U) \cdot \nabla \text{div}\phi_2 \, dx \, dt, \quad (3.38)
\]
as \(j \to \infty\). From Corollary 3.1, Lemma 3.2, and Propositions 3.3, 3.4, we have
\[
\int_0^T \int_{B_{\varepsilon_j,\frac{1}{2}}} (\sqrt{\rho^j} U^j) \cdot (\nabla \tilde{h}(\rho^j) \cdot \nabla) \phi_2 \, dx \, dt
\]
\[\leq \int_0^T \int_{B_{\varepsilon_j,\frac{1}{2}} \cap \{|U^j| \leq M\}} + \int_{B_{\varepsilon_j,\frac{1}{2}} \cap \{|U^j| > M\}} (\sqrt{\rho^j} U^j) \cdot (\nabla \tilde{h}(\rho^j) \cdot \nabla) \phi_2 \, dx \, dt
\]
\[\leq C \|
abla \phi_2\|_{L^\infty} \|
abla \tilde{h}(\rho^j)\|_{L^\infty([0,T];L^2)} \left( M \|
abla \rho^j\|_{L^2 \cap L^2(\mathbb{R}^N \times [0,T])} \right)
\]
\[+ \frac{1}{1 + \ln(1 + M^2)} M \|\rho^j U^j\|_{L^2([0,T];L^1)} \quad (3.39)
\]
\[\to 0, \quad \text{as } M, k \to \infty,
\]
\[\int_0^T \int_{B_{\varepsilon_j,\frac{1}{2}}} (\sqrt{\rho} U) \cdot (\nabla \tilde{h}(\rho) \cdot \nabla) \phi_2 \, dx \, dt \to 0, \quad \text{as } k \to \infty,
\]
and
\[
\int_0^T \int_{B^1_{\frac{r}{n}}} (\sqrt{\rho}^j \cdot \nabla h(\rho^j)) \cdot \nabla \phi_2 dx dt \to \int_0^T \int_{B^1_{\frac{r}{n}}} (\sqrt{\rho} \cdot \nabla h(\rho)) \cdot \nabla \phi_2 dx dt
\]
as \(j \to \infty\). Thus, we have
\[
\int_0^T \int_{B^1_{\frac{r}{n}}} (\sqrt{\rho}^j \cdot \nabla h(\rho^j)) \cdot \nabla \phi_2 dx dt \to \int_0^T \int_{\mathbb{R}^N} (\sqrt{\rho} \cdot \nabla h(\rho)) \cdot \nabla \phi_2 dx dt,
\]
as \(j \to \infty\). Similarly, we can obtain
\[
\int_0^T \int_{B^1_{\frac{r}{n}}} (\sqrt{\rho}^j \cdot \nabla \phi_2 \cdot \nabla h(\rho^j)) \cdot \nabla \phi_2 dx dt \to \int_0^T \int_{\mathbb{R}^N} (\sqrt{\rho} \cdot \nabla \phi_2 \cdot \nabla h(\rho)) \cdot \nabla \phi_2 dx dt,
\]
as \(j \to \infty\). From (3.36)-(3.40), we obtain
\[
\int_0^T \int_{|x| > \varepsilon_j} g(\rho^j) \div U^j \div \phi_2 dx dt \to < g(\rho) \div U, \div \phi_2 >, \text{ as } j \to \infty.
\]
Similarly, we obtain
\[
\int_0^T \int_{|x| > \varepsilon_j} g(\rho^j) \div U^j \div \phi_2 dx dt \to < g(\rho) \div U, \div \phi_2 >, \text{ as } j \to \infty.
\]
Up to now, we have proved that each term on the left hand side of (3.35) converges to corresponding term in (2.12) as \(j \to \infty\). In the following, we prove that each term on the right hand side of (3.35) vanishes as \(j \to \infty\).

From Lemma 3.1, we get
\[
| \varepsilon_j \int_0^T \int_{\mathbb{R}^N} (\rho^j)^{\theta} \div U^j \div \phi_2 dx dt | \\ \\
\leq C \varepsilon_j \| \nabla \phi_2 \|_{L^\infty(\mathbb{R}^N)} \left( \varepsilon_j \int_0^T \int_{\mathbb{R}^N} (\rho^j)^{\theta} | \nabla U^j |^2 dx dt \right)^{\frac{1}{2}} \left( \int_0^T \int_{\mathbb{R}^N} \rho^j dx dt \right)^{\frac{\theta}{2}} |B_n|^{\frac{1}{2}} \\ \\
\leq C(T) \varepsilon_j^{\frac{N(1-\theta)}{2}},
\]
and
\[
| \varepsilon_j \int_0^T \int_{\mathbb{R}^N} (\rho^j)^{\theta} D(U^j) : \nabla \phi_2 dx dt | \leq C(T) \varepsilon_j \frac{N(1-\theta)}{2}.
\]
It follows from (3.31) and (3.41)-(3.42) that each term on the right hand side of (3.35) vanishes as \(j \to \infty\).

Taking the limit \(j \to \infty\) in (3.35), we finish the proof of this proposition.

From the above arguments, we can immediately finish the proof of Theorem 2.1.
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