The pion at finite temperature
Nobuaki Kodama and Makoto Oka
Department of Physics, Tokyo Institute of Technology
Meguro, Tokyo 152 Japan

Abstract
Properties of the pion are studied at finite temperature with the help of the PCAC and the QCD sum rule. The pionic mode is treated consistently with the thermal pions which consist of the ground state at finite temperature. The temperature dependence of the pion decay constant is estimated for \( m_q = 0 \). No consistent solution is found above the critical temperature \( T_c = \sqrt{2} f_\pi^{T=0} \). The QCD sum rule shows that the finite quark mass makes the critical temperature move up to a temperature where the dilute pion gas approximation is no more valid. It also suggests that the pion decay constant and the continuum threshold decrease by \( \sim 15\% \) and \( \sim 10\% \), respectively, and that the pion mass slightly (\( \sim 3\% \)) increases at \( T = 160\text{MeV} \), where the pion gas approximation seems to break down.

\footnote{e-mail address: kodama@th.phys.titech.ac.jp}
1 Introduction

Behaviors of the hadron properties at finite temperature have been studied extensively in the context of the phase transition of QCD. In particular, the temperature dependencies of hadron masses, coupling constants and so on have been calculated in various approaches [1]. One of them is the QCD sum rule generalized for finite temperature by the authors of ref. [2].

The QCD sum rule is based on the operator product expansion (OPE) of a current correlator and the hadron duality [3]. At first [2, 4], the Matsubara Green function [5] was applied for the massless quark propagator in the OPE of current correlators at finite temperature. It was pointed out, however, that the use of the finite temperature Green function is not fully consistent for the calculation at temperature below the critical temperature, \( T < T_c \) [6, 7, 8]. In the hadronic phase quarks have discrete spectrum due to confinement, so that the use of the finite temperature Green function requires the full range of their interactions.

On the other hand, OPE is supposed to achieve a factorization of the soft scale and the hard scale. The soft scale dynamics is contained in the matrix elements of local operators, while the hard scale dynamics is taken into account in the Wilson coefficients. Therefore it is natural to include the temperature dependence only in the matrix elements of operators as far as \( T \sim \Lambda_{QCD} \ll Q \), where \( Q \) is the Euclidean momentum of currents. We conclude that construction of a finite temperature ground state is essential for the QCD sum rules at finite, but not high, temperature.

In this paper we investigate the pion at finite temperature, since the pion is the dominant particle consisting of the finite temperature ground state. At temperature, \( T < T_c \), the particles constructing the heat bath (finite temperature ground state) are hadrons, especially the pion, since the heavier states are suppressed by the Boltzmann factor \( e^{-m/T} \) at low temperature.

We employ the thermo field dynamics of pions in describing the ground state [9] and show that the PCAC is valid even at finite temperature. The finite temperature ground state \( |T\rangle \) gives clear understanding of the temperature dependence of the state and naturally avoids a problem of the analyticity at finite temperature [8] which was considered to unjustify the QCD sum rule approach.

The outline of this paper is as follows. In Sect.2, we define notations and construct the QCD ground state at finite temperature as a pion gas in the framework of thermo field dynamics. In Sect.3, we consider the PCAC in the ground state at finite temperature constructed in Sect.2. In Sect.4, the temperature dependence of the pion decay constant is investigated in the chiral limit. There we use the dilute pion gas approximation to estimate the finite temperature matrix elements. In Sect.5, the effects of the finite quark mass to the critical temperature in Sect.4 and other physical parameters of pion are studied by the QCD sum rule technique. In Sect.6, the conclusion and discussion are given.
2 QCD ground state at finite temperature

We define the creation and annihilation operators of the pion, \( a^\dagger(p) \) and \( a(p) \), which create or annihilate a pion with isospin \( a \) and the three dimensional momentum \( p \). These operators satisfy the following relations:

\[
\begin{align*}
[a^a(p), a^b(q)] &= \delta^3(q - p)\delta^{ab} \\
a(p)|n\rangle &= \sqrt{n}|n - 1\rangle \\
a^\dagger(p)|n - 1\rangle &= \sqrt{n}|n\rangle
\end{align*}
\]

where \( |n\rangle \) is a state that contains \( n \) pions with momentum \( p \). The free pion field is defined by

\[
\Phi^a(x) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2k_0}} [a^a(k)e^{-ikx} + a^\dagger a(k)e^{ikx}]
\]

where \( k = (k_0, k) \) and \( k_0 = \sqrt{k^2 + m^2_\pi} \).

We express the ground state (vacuum) at finite temperature as \( |T\rangle \), which is regarded as a gas of free pions. Then the free pion annihilation operator cannot annihilate \( |T\rangle \). Here we follow the thermo field dynamics formulated by Umezawa et.al. in ref.\[9\]. They introduce the Bogoliubov transformation and define a new operator \( \tilde{\alpha}^\dagger(k) \) by

\[
a(k) = \cosh \theta \ a(k) + \sinh \theta \ \tilde{\alpha}^\dagger(k)
\]

Eq.(4) is canonical transformation. The mixing angle \( \theta = \theta(k, T) \) depends on the momentum \( k \) and the temperature \( T \).

The tilde operator annihilates a hole, or creates a particle in a state occupied in the finite temperature vacuum. In eq.\[4\] the annihilation operator in the true vacuum is divided into an annihilation operator of a particle and a creation operator of a hole in the finite temperature vacuum.

Now we define the finite temperature vacuum \( |T\rangle \) such that

\[
\alpha(k)|T\rangle = 0, \ \tilde{\alpha}(k)|T\rangle = 0
\]

for all \( k \).

We determine \( \theta \) by calculating the ground state average of the particle number density which must coincide with the Bose-Einstein distribution, i.e.

\[
n_B(k, T) = \langle T|a^\dagger(k)a(k)|T\rangle/\delta^3(0) = (\sinh \theta)^2
\]

with \( n_B(\omega) = 1/(e^{\omega\beta} - 1) \) and \( \beta = 1/T \). Then we get

\[
cosh \theta = (1 + n_B)^{1/2}, \ \sinh \theta = n_B^{1/2}.
\]

As we have argued in Introduction, it is regarded that the quark degree of freedom does not explicitly depend on temperature in the confined hadron phase. This assumption means that the temperature effect is only in the non-perturbative contributions of QCD such as the quark condensate. This is of course valid only when the temperature is below the deconfinement phase transition. We assume furthermore that \( |T\rangle \) consists only of pions, since heavier degrees of freedom are suppressed by \( e^{-m_\beta} \) at low temperature.
3 PCAC at finite temperature

In order to study properties of the pion at finite temperature, we consider the following currents,

\[ J^A_\mu = \overline{q} \gamma_\mu \gamma_5 t^a q \]
\[ J^a_5 = \overline{q} i \gamma_5 t^a q \]

where \( t^a = \frac{i}{2} \) is the isospin operator. In the chiral limit the axial-vector current is conserved, \( \partial^\mu J^A_\mu = 2m_q \overline{q} i \gamma_5 t^a q \to 0 \). In following we omit the isospin indices \( a \) for simplicity.

It is important to note that the time-ordered correlator is not analytic in the \( q^2 \) plane at finite temperature [10]. Instead, the pionic mode can be studied in the retarded correlation function,

\[
\Pi(q) = \int d^4xe^{i\omega x} i\partial_\mu \langle T[R[J^A_\mu(x), J_5(0)]T] \rangle \\
= q_\mu \int d^4xe^{i\omega x} \langle T[R[J^A_\mu(x), J_5(0)]T] \rangle \\
= q_\mu \int d^4xe^{i\omega x} \sum_{n,m} \theta(t) \left[ e^{-i(p_n-p_m)x} \langle T[J^A_\mu n\bar{m}] \langle n\bar{m}|J_5|T\rangle - e^{i(p_n-p_m)x} \langle T[J_5 n\bar{m}] \langle n\bar{m}|J^A_\mu|T\rangle \right]
\]

Here we insert the complete set \( 1 = \sum_{n,m} |n\bar{m}\rangle \langle n\bar{m}| \) as the intermediate state. \( |n\bar{m}\rangle \) denotes a \( n \)-particle \( \bar{m} \)-hole state and \( p_n^\mu \) is the 4-momentum of the \( n \)-particle state, while \(-p_n^\mu\) is the 4-momentum of the \( \bar{m} \)-hole state.

The PCAC equation, \( \partial^\mu J^A_\mu = 2m_q J_5 \), and \( \partial^\mu \langle T[J^A_\mu|n\bar{m}\rangle = -i(p_n-p_m)^\mu \langle T[J^A_\mu|n\bar{m}\rangle \) leads to

\[ \langle T[J_5|n\bar{m}\rangle \langle n\bar{m}|J^A_\mu|T\rangle = -\langle T[J^A_\mu|n\bar{m}\rangle \langle n\bar{m}|J_5|T\rangle \]

When we take the rest-frame, i.e. \( q = 0 \) and \( q_0 = \omega \), the above correlation function is given by

\[
\Pi(\omega) = \omega \int_0^\infty dt \int d\sigma (e^{i(\omega-\sigma)t} + e^{i(\omega+\sigma)t}) \\
\times (2\pi)^3 \sum_{n,m} \delta(\sigma - (E_n - E_m)) \delta^3(p_n - p_m) \langle T[J^A_\mu|n\bar{m}\rangle \langle n\bar{m}|J_5|T\rangle \\
= i \int d\sigma \rho(\sigma) \frac{2\omega^2}{(\omega + i\epsilon)^2 - \sigma^2}
\]

On the other hand, \( \Pi(\omega) \) can be evaluated directly, giving

\[
\Pi(\omega) = \int d^4xe^{i\omega t} i\partial_\mu \langle T[R[J^A_\mu(x), J_5(0)]T] \rangle \\
= \int d^4xe^{i\omega t} i\delta(t) \langle T[J^A_\mu(x), J_5(0)]T \rangle \\
= \langle T[2\overline{q}(0)|T \rangle
\]
Thus we obtain
\[ i \int d\sigma \rho(\sigma) \frac{2\omega^2}{(\omega + i\epsilon)^2 - \sigma^2} = 2\langle T|\bar{q}q|T\rangle \] (12)

Eq. (12) is valid for any \( \omega \), i.e. even at \( \omega \to 0 \) and the RHS does not depend on \( \omega \). Then LHS also should not depend on \( \omega \). Therefore, we conclude that if \( \langle T|\bar{q}q(0)|T\rangle \neq 0 \), this correlator is saturated by a massless pole, \( \rho(\sigma) = A\delta(\sigma^2) \). Other states do not contribute to this correlator at all in the chiral limit.

The spectral function \( \rho(\sigma) \) is defined in eq. (11),
\[ \rho(\sigma) = (2\pi)^3 \sum_{m,n} \delta(\sigma - (E_n - E_m))\delta^3(p_n - p_m)\langle T|J^A_0|n\bar{m}\rangle\langle n\bar{m}|J_5|T\rangle \] (13)

The finite temperature vacuum satisfies \( \langle T|J|T\rangle = 0 \) for \( J = J_5 \) or \( J^A_0 \) because of parity conservation. Then the lowest energy contribution to the intermediate states comes from \( |n\bar{m}\rangle = |1\bar{0}\rangle \) and \( |0\bar{1}\rangle \). Here we remember that the tilde state is a hole and has the opposite quantum numbers of a corresponding particle state. Therefore the condition \( \rho(\sigma) = A\delta(\sigma^2) \) indicates that the two states \( |1\bar{0}\rangle \) and \( |0\bar{1}\rangle \) represent a massless particle and a massless hole, respectively, and are the only states that contribute in the chiral limit. These are the pion and its hole at finite temperature.

One might wonder why there are two Goldstone modes at finite temperature. This is caused by the existence of the tilde current which also satisfies the current conservation law.

At zero temperature a massless pole in the correlator requires existence of an asymptotic massless field, \( J^A_0 \to_{t \to \infty} -\sqrt{2}f_\pi \partial_\mu \Phi \). Although the finite temperature vacuum is not Lorentz invariant, we expect that the same relation for \( J^A_0 \) is valid in the rest frame at finite temperature except that the decay constant may depend on temperature, i.e. \( J^A_0 \to_{t \to \infty} -\sqrt{2}f^T_\pi \partial_0 \Phi \). Now we find
\[ \Pi(\omega) = 2\langle T|\bar{q}q|T\rangle = 2i(2\pi)^3 \left( \langle T|J^A_0|1\bar{0}\rangle\langle 1\bar{0}|J_5|T\rangle + \langle T|J^A_0|0\bar{1}\rangle\langle 0\bar{1}|J_5|T\rangle \right) \]
\[ = 2i(2\pi)^3 \left[ \langle T| - \sqrt{2}f^T_\pi \partial_0 \Phi \parallel 1\bar{0}\rangle\langle 1\bar{0}|\sqrt{2}f^T_\pi (m_T^2) \Phi|T\rangle + \langle T| - \sqrt{2}f^T_\pi \partial_0 \Phi \parallel 0\bar{1}\rangle\langle 0\bar{1}|\sqrt{2}f^T_\pi (m_T^2) \Phi|T\rangle \right] \]
\[ = - \frac{2(f^T_\pi)^2(m_T^2)^2}{2m_q} \{n_B + 1 - n_B\} \]
\[ = - \frac{2(f^T_\pi)^2(m_T^2)^2}{2m_q} \]
(14)

On the other hand, the retarded correlator of the free pion at finite temperature is given by
\[ \langle T|R[\partial_0 \Phi(x), \Phi(0)]|T\rangle \]

\(^2\) The finite temperature ground state is connected to the true vacuum by \( |T\rangle = G^{-1}(\theta)|0\rangle \) with \( G(\theta) = \exp \left[ \int d^3k \theta[a(k)\bar{a}(-k) - a^\dagger(k)\bar{a}^\dagger(-k)] \right] \).
the heat bath, the second term in eq (17), i.e., the matrix element, the soft-pion theorem. Since pions which consist of the pion gas are also affected by the 4-pion decay constant, of a hole and a particle contributions. This relation is same as that at zero temperature except that LHS means the sum of the strengths of a particle and a hole state is normalized to the zero temperature pion propagator. The difference between eqs. (14) and (15) comes from the temperature dependence of a particle and a hole state is normalized to the zero temperature pion propagator. It should be noted that the sum of the strengths as that at zero temperature. Finally we obtain the finite temperature version of the Gell-Mann-Oakes-Renner relation.

\[
\frac{(f_\pi^T)^2 (m_\pi^T)^2}{2m_q} = -\langle T|\bar{q}q|T\rangle
\]

This relation is same as that at zero temperature except that LHS means the sum of a hole and a particle contributions.

4 Pion decay constant, \( f_\pi^T \), at finite temperature.

We estimate the quark condensate at finite temperature in the dilute pion gas approximation. As far as \( T \) is not high, the expectation value of an operator, \( O \), at finite temperature, i.e. \( \langle T|O|T\rangle \), is given by

\[
\langle T|O|T\rangle \sim \langle 0|O|0 \rangle + \sum_{a=1}^{3} \int d^3p (\pi^a(p)|O|\pi^a(p))n_B(\epsilon)
\]

where \( \epsilon = \sqrt{p^2 + m^2} \), and \( a \) denotes the isospin index. To estimate \( \langle \pi|O|\pi \rangle \) we use the soft-pion theorem. Since pions which consist of the pion gas are also affected by the heat bath, the second term in eq (17), i.e., the matrix element, \( \langle \pi^a(p)|\bar{q}q|\pi^a(p)\rangle \), must also be temperature dependent. As we have seen in eq. (13) the free one-pion state \( |\pi\rangle = a^\dagger|0\rangle \) does not depend on the temperature. We conjecture that this effect can be taken into account by changing the reduction formula from \( a^\dagger \rightarrow \frac{1}{\sqrt{2f_\pi}} \partial^\mu J^A_\mu \) to \( a^\dagger \rightarrow \frac{1}{\sqrt{2f_\pi}} \partial^\mu J^A_\mu \). Then we get

\[
\frac{(f_\pi^T)^2 (m_\pi^T)^2}{2m_q} = -\langle \bar{q}q \rangle_0 \left[ 1 - \frac{T^2}{8(f_\pi^T)^2} \right].
\]
This is easily solved self-consistently under the assumption $m_T^T/m_T^{T=0} = 1$ and we find the temperature dependence of the pion decay constant, 

$$\left( f_T^T \right)^2 = \frac{1}{2} \left( f_T^{T=0} \right)^2 + f_T^{T=0} \sqrt{\left( f_T^{T=0} \right)^2 - \frac{T^2}{2}}$$

The $T$ dependence given by (19) is illustrated in Fig.1. It is clear that there exists no solution of eq.(18) above $T > T_c = \sqrt{2} f_T^{T=0}$. $f_T^T$ decreases towards $f_T^{T=0}$ when the temperature increases from 0 to $\sqrt{2} f_T^{T=0}$. This singular behavior comes from the consistency condition that the thermal pions are modified themselves in the finite temperature ground state. At $T = T_c$, the pion gas picture seems to break down and there is no more consistent solution above $T_c$.

5 QCD sum rules for the pion at finite temperature

So far, we have studied the properties of the pion in the chiral limit, i.e., at $m_q = m_\pi^2 = 0$. We study effects of the finite quark mass to the critical temperature and the other physical parameters of the pion by the QCD sum rule technique.
5.1 spectral function

We reconsider the spectral function of the retarded pion correlation function defined by

\[ \Pi_{\mu}(q) = i \int d^4x e^{iqx} \langle T[R[J_5^{A}(x), J_5^{A}(0)]T] \rangle \]  

(20)

Similarly to eq.(9), we find at the rest frame

\[ \Pi_{\mu=0}(\omega) = i \int d^4xe^{i\omega t} \sum_{n,m} \theta(t) \times \left[ e^{-i(p_n-p_m)x} \langle T|J_0^{A}|n\tilde{m}\rangle \langle n\tilde{m}|J_5^{A}|T \rangle - e^{i(p_n-p_m)x} \langle T|J_5^{A}|n\tilde{m}\rangle \langle n\tilde{m}|J_0^{A}|T \rangle \right] \]  

(21)

Using a relation of the thermo field dynamics \[9\],

\[ \langle T|J_5^{A}|n\tilde{m}\rangle \langle n\tilde{m}|J_0^{A}|T \rangle = e^{\beta(E_n-E_m)} \langle T|J_0^{A}|m\tilde{n}\rangle \langle m\tilde{n}|J_5^{A}|T \rangle \]  

(22)

we obtain the following expression,

\[ \omega \Pi(\omega) = \Pi_{\mu=0}(\omega) = \frac{1}{\omega - \sigma + i\epsilon} \]  

(23)

where the spectral function is defined by

\[ \rho_1(\sigma) = (2\pi)^3(1 - e^{-\beta\sigma}) \times \sum_{n,m} \delta(\sigma - (E_n - E_m))\delta^3(p_n - p_m) \langle T|J_0^{A}|n\tilde{m}\rangle \langle n\tilde{m}|J_5^{A}|T \rangle \]  

(24)

From eq.(22) and the relation

\[ \langle T|J_5^{A}|n\tilde{m}\rangle \langle n\tilde{m}|J_0^{A}|T \rangle = -\langle T|J_0^{A}|n\tilde{m}\rangle \langle n\tilde{m}|J_5^{A}|T \rangle \]

we obtain the following symmetry relation for the spectral function \( \rho_1(\sigma) \),

\[ \rho_1(\sigma) = \rho_1(-\sigma) \]  

(25)

Then we can rewrite the correlation function,

\[ \omega \Pi(\omega^2) = -\int_0^\infty d\sigma 2\omega \rho_1(\sigma) \frac{1}{(\omega + i\epsilon)^2 - \sigma^2} \]

\[ = \omega \int_0^\infty d\sigma \frac{\rho_1(\sigma)}{\sigma^2 - \omega^2 - i\epsilon} \]

\[ = \omega \int_0^\infty d\sigma \frac{2\rho_1(\sigma)}{\sigma^2 - \omega^2 - i\epsilon} \]  

(26)
Here the last equality is given for the delta function part in the spectral function. Eq.(26) corresponds to eq.(10) with \( \rho_1(\sigma) = \rho(\sigma) + \rho(-\sigma) \). From eq.(14), we expect that the strength of the pion pole in this spectral function, \( \rho_1(\sigma) \), is given by \( \frac{(f_T^2 m_T^2)}{2m_q} \). Then we may assume the spectral function, \( \rho_1(\sigma^2) \), in the following form for a finite quark mass,

\[
2\rho_1(\sigma^2) = \frac{2(f_T^2 m_T^2)^2}{2m_q} \delta(\sigma^2 - (m_T^2)^2) + Perturbation \cdot \theta(\sigma^2 - s_0)
\] (27)

The \textit{Perturbation} term is estimated by the bare-loop calculation. This form is same as that at the zero temperature except that the pion pole consists of a particle state and a hole state.

From our derivation of the spectral function, it is clear that the correlation function, \( \Pi(\omega^2) \), has an analytic property in the \( \omega^2 \) plane. Therefore we use the QCD sum rule techniques for this correlation function in the usual manner.

Here we write down the imaginary part of correlation function, \( \Pi(\omega^2) \).

\[
\frac{1}{\pi} \text{Im}\Pi(s) = \frac{2(f_T^2 m_T^2)^2}{2m_q} \delta(s - (m_T^2)^2) + Perturbation \cdot \theta(s - s_0)
\] (28)

5.2 The construction of the sum rule.

Next we construct the theoretical side of the QCD sum rule. The scalar part of this correlator can be easily calculated using the Ward identity and the pseudo-scalar correlator [11].

\[
\omega \Pi(\omega^2) =
-\omega \left[ \frac{3}{8\pi^2} \log(-\omega^2/\mu^2) \{1 + \frac{\alpha_s(17/3)}{\pi} \log(-\omega^2/\mu^2)\} + \frac{\alpha_s}{8\pi\omega^4} G_{\mu\nu} G_{\mu\nu} \\
- \frac{4\pi\alpha_s}{\omega^6} \left\{ \bar{d}\gamma_5 \sigma_{\mu\nu} t^a u \bar{u}\gamma_5 \sigma_{\mu\nu} t^a d - \frac{1}{3} (\bar{u}\gamma_\mu t^a u + \bar{d}\gamma_\mu t^a d) \sum Q \gamma_\mu t^a q \right\} \right]
\] (29)

At finite temperature the OPE side contains operators carrying spin [11]. We do not consider operators which have Lorentz indices except for the operators of dimension 3, since those operators are proportional to higher powers of temperature and also are multiplied by the small quark mass.

The only possible dimension 3 non-scalar operator is \( S.T.(\bar{d}\gamma_\mu \gamma_\nu d + \bar{u} \gamma_\nu \gamma_\mu u) \), where \( S.T. \) means the symmetric and traceless tensor. This happens to be zero in the isospin \( SU(2) \) symmetry limit, and therefore no spin operator contributes in this case.

Same as the phenomenological side, we only consider the same theoretical side with the zero temperature sum rule except for the temperature dependent condensate.
As is mentioned previously, as far as $T$ is not so high and the thermal pion gas is dilute, $\langle T|O(\mu^2)|T\rangle$ is approximated as

$$\langle T|O(\mu^2)|T\rangle \simeq \langle 0|O(\mu^2)|0\rangle + \sum_{a=1}^{3} \int d^3p (\pi^a(p)|O(\mu)|\pi^a(p)) n_B(\epsilon/T),$$

We estimate the temperature dependence of the condensate in the soft pion limit. We write down only the results.

$$\langle \bar{q}q\rangle_T = \langle \bar{q}q\rangle_0 (1 - \frac{T^2}{8(f_\pi^2)^2} B_1 (\frac{m_T}{T}))$$

$$\langle G \cdot G\rangle_T = \langle G \cdot G\rangle_0$$

$$\langle d\gamma_5 \sigma_{\mu \nu} t^a u \bar{u} \gamma_5 \sigma_{\mu \nu} t^a d \rangle_T - \frac{1}{3} \langle (\pi \gamma_5 t^a u + \bar{d} \gamma_5 t^a d) \sum q \gamma_5 t^a q \rangle_T = \frac{28}{27} \langle \bar{q}q\rangle_0 (1 - \frac{3T^2}{14(f_\pi^2)^2} B_1 (\frac{m_T}{T})).$$

Here the function $B_1(z)$ is defined by

$$B_1(z) = \frac{6}{\pi^2} \int_{z}^{\infty} dy \sqrt{y^2 - z^2} (e^y - 1)^{-1}.$$

Equating the theoretical side and the phenomenological side of the correlation function with the help of the dispersion relation, and making the Borel transformation, we obtain the sum rule for the pion at finite temperature:

$$\frac{2(f_\pi^T)^2 (m_\pi^T)^2 \exp(-m_\pi^T/M^2)}{2m_q} = 2m_q \left[ \frac{3}{8\pi^2} (1 + \frac{17}{3} + 2\gamma_E) \frac{\alpha_s}{\pi} (1 - \exp(-s_0^T/M^2)) \right]$$

$$- \frac{\alpha_s}{8\pi M^4} \langle GG \rangle_0 - \frac{56\pi \alpha_s}{27 M^6} \langle \bar{q}q\rangle_0^2 (1 - \frac{3T^2}{14(f_\pi^T)^2} B_1 (\frac{m_T}{T})))$$

$$- \frac{2}{M^2} \langle \bar{q}q\rangle_0 (1 - \frac{T^2}{8(f_\pi^T)^2} B_1 (\frac{m_T}{T})).$$

### 5.3 The estimation of temperature dependence.

In its early stages, the QCD sum rule for the pion was known to have a difficulty mainly because the coupling of the pion pole is not dominant. For the pseudo-scalar correlator, the excited pion states and the instanton effects become important, while for the axial-vector correlator, the axial-vector meson $a_1$ will disturb the sum rule.

These difficulties, however, can be circumvented by adapting the pseudoscalar-axialvector off-diagonal correlator since it is almost saturated by the pion. We, however, should not take a logarithmic derivative of the lowest moment sum rule to eliminate the coupling strength since the logarithmic derivative strongly enhances the contribution from the higher resonances. Therefore we evaluate the values of physical parameters by fitting the LHS and RHS of the sum rule.

The fitting analysis of the sum rule is performed in an interval of $M$, $\Omega(M_{\min} < M < M_{\max})$, the lower and upper limits of which are determined in the following argument.
According to the Ward identity, the deviation from the chiral limit is given by the pseudo-scalar correlator. The convergence of the OPE of the off-diagonal correlator depends on one of the pseudo-scalar correlator. Then we set the lower limit to ensure that the power correction of the pseudo-scalar correlator is smaller than 5% of the perturbative contribution. This condition is also expected to suppress the instanton contribution in the pseudo-scalar correlator. The upper limit is set to suppress the continuum contribution to less than 20% of the total in the pseudo-scalar correlator.

Because \( \Omega \) generally depends on temperature, we determine \( \Omega \) at each temperature. At zero temperature the above conditions give at \( s_0 \sim 2\text{GeV}^2 \),

\[
\Omega(0.99\text{GeV} < M < 1.18\text{GeV})
\]

We determine \( m_\pi^T \) and \( \lambda(T) = 2(f_\pi^T)^2(m_\pi^T)^2 \) by matching the RHS, \( R(s_0^T, T, M) \), and the LHS, \( L(m_\pi^T, \lambda(T), M) \) of the sum rule in the interval \( \Omega \). Let us rewrite the sum rule in the form

\[
\lambda(T) \equiv F(m_\pi^T, s_0^T, T, M) = (m_u + m_d)R(s_0^T, T, M)M^2\exp((m_\pi^T)^2/M^2)
\]

The coupling parameter \( \lambda(T) \) should not depend on \( M \). Therefore we choose \( m_\pi^T \) and \( s_0^T \) so that \( F \) is least dependent on \( M \) in \( \Omega \) and \( \lambda(T) \) is determined as the mean value \( \overline{F}(m_\pi^T, s_0^T, T) \) of the function \( F(m_\pi^T, s_0^T, T, M) \) in \( \Omega \). Then we define

\[
\delta = \text{Max}(|F(m_\pi^T, s_0^T, T, M) - \overline{F}(m_\pi^T, s_0^T, T)|/\overline{F}(m_\pi^T, s_0^T, T))
\]

which represents the variation of \( F(m_\pi^T, s_0^T, T, M) \) in \( \Omega \). At a given temperature we search the best fit parameters which minimize \( \delta \).

Because the condensate depends on the \( m_\pi^T \) and \( f_\pi^T \) at finite temperature, we have to determine \( m_\pi^T \) and \( f_\pi^T \) self-consistently, i.e., until the difference between the input \( m_\pi^T \), \( f_\pi^T \) and the output \( m_\pi^T \), \( f_\pi^T \) becomes less than 1%.

The results are shown in Figures 2-5. We find the similar behavior of \( f_\pi^T \) as that in the chiral limit. The consistent solution of \( f_\pi^T \) disappears above the critical temperature \( T_c \approx 180\text{MeV} \). At \( T_c \), \( m_\pi = 146\text{MeV} \), \( s_0 = 1.93 \) and \( f_\pi = 70.9\text{MeV} \). The critical value of the pion decay constant is almost the same as that in the chiral limit. It is, however, generally suggested that the pions can no more dominate the finite temperature ground state at \( T > 160\text{MeV} \) \[12\]. Therefore we trust our results only at \( T < 160\text{MeV} \). In this region the pion mass is almost stable, the decay constant decreases by \( \sim 15\% \) and the continuum threshold decreases by \( \sim 10\% \).

In this analysis the quark mass and quark condensate are determined by the pion sum rules at zero temperature and the 4-quark condensate is evaluated in the rho meson sum rule. These values are \( m_u + m_d = 15\text{MeV} \), \( \langle \overline{q}q \rangle_0 = -(220\text{MeV})^3 \), \( \langle \overline{q}q \rangle^2_0 = 7.3 \cdot 10^{-4}\text{GeV}^6 \), and the gluon condensate \( \langle \overline{q}G \cdot G \rangle_0 = (360\text{MeV})^4 \).

6 Conclusion and Discussion

The QCD sum rule is applied to the analysis of the pion properties at finite temperature. We find that the pion decay constant decreases, while the pion mass
Figure 2: The temperature dependence of the pion mass.

Figure 3: The temperature dependence of the continuum threshold $s_0$. 
Figure 4: The temperature dependence of pion decay constant $f_\pi^T$.

Figure 5: The ratios $m_\pi^T/m_\pi^{T=0}$ (solid line), $s_0^T/s_0^{T=0}$ (dashed) and the $f_\pi^T/f_\pi^{T=0}$ (dotted).
increases as the temperature increases. These results qualitatively coincide with those in other approaches, such as the NJL model [13]. The increase of the pion mass is very small, i.e., ∼3% at $T = 160\text{MeV}$, which is again consistent with the result in the chiral perturbation theory [14].

It is found that the consistency of the pion properties with those of the thermal pions suggests a critical behavior of the pion decay constant. In the chiral limit the critical temperature, $T_c$, is

$$T_c \approx \sqrt{2} \frac{f_{\pi}}{f_{\pi} = 94\text{MeV}} \approx 133\text{MeV}.$$  

Above $T_c$, there exists no consistent solution of $f_{\pi}$. The higher power corrections of $T$ for $\langle \bar{q}q \rangle$ can be expressed in the chiral limit as follows [12].

$$\langle \bar{q}q \rangle = \langle \bar{q}q \rangle_0 (1 - \frac{T^2}{8f_{\pi}^2} - \frac{T^4}{384f_{\pi}^4} - \frac{T^6}{288f_{\pi}^6} \frac{\Lambda}{T} + ... )$$

(31)

One sees that they tend to accelerate the breakdown. Furthermore the contribution of the higher resonances is expected to suppress the quark condensate. Therefore these corrections to our analysis are expected to enhance the critical behavior of the pion decay constant.

Our results of the "phase transition" seems to be the first order phase transition. The QCD phase transition with two massless quarks is expected to be the second order [15]. We conjecture that the discrepancy comes from the breaking of hadronic description of the finite temperature vacuum. At the temperature near $T_c$ (=$\sqrt{2} f_{\pi}^{T=0}$), the quark degrees of freedom are no more frozen, and the self-energy of quarks will break the PCAC relation. Such an effect might make the "phase transition" smooth.

The QCD sum rule with a finite quark mass indicates the same critical behavior at $T = 180\text{MeV}$. The difference of the critical temperature from the chiral limit can be understood from the consideration that the small difference of the quark mass leads to a large (compared with the quark mass) scale difference in the hadronic world, such as the pion mass. This, however, may not justify the hadronic description of the finite temperature vacuum at $T = 180\text{MeV}$, since the higher resonance contribution is expected to dominate the number density at $T > 160\text{MeV}$.

It is worthwhile to mention about the relation between $s_0$ and $\langle \bar{q}q \rangle_T$. The pion mass is approximately calculated by taking a logarithmic derivative of eq.(30). If we omit a few negligible terms and take $M \sim 1\text{GeV}$ then we get

$$(m_{\pi}^T)^2 = \frac{2m_q 3^{2\pi^2} [1 - (1 + s_0^T) e^{-s_0^T}]}{-2\langle \bar{q}q \rangle_T}$$

When the pion mass increases at finite temperature this equation gives the following constraint:

$$-2\langle \bar{q}q \rangle_T (m_{\pi}^{T=0})^2 < 2m_q 3^{2\pi^2} [1 - (1 + s_0^T) e^{-s_0^T}]$$

If we regard the continuum threshold $s_0$ as an order parameter for the deconfinement phase transition [16], then the above constraint indicates that the critical temperature $T_d$, defined by $s_0^{T=T_d} = 0$, cannot be lower than the chiral transition temperature $T_{ch}$ with $\langle \bar{q}q \rangle_{T=T_{ch}} = 0$. If the pion mass does not depend on the temperature, these two temperatures are equal in our estimation.
References

[1] R. Pisarski, Phys. Lett. B110 (1982) 222; Y. Koike, M. Fukugita and A. Ukawa, Phys. Lett. B213 (1988) 687; G. E. Brown and M. Rho, Phys. Rev. Lett. 66 (1991) 2720; T. Schäfer and E. V. Shuryak, SUNY-NTG-95-17.

[2] A. I. Bochkarev and M. E. Shaposnikov, Nucl. Phys. B268 (1986) 220.

[3] M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, Nucl. Phys. B147 (1979) 385, 448.

[4] H. G. Dosch, and S. Narison, Phys. Lett. B203 (1988) 155; C. A. Dominguez and M. Lowe, Phys. Lett. B233 (1989) 201; R. J. Furnstahl, T. Hatsuda and S. H. Lee, Phys. Rev. D42 (1990) 1744;

[5] I. Matzubara, Progr. Theor. Phys. 14 (1955) 351.

[6] T. Hatsuda, Y. Koike and S. H. Lee, Nucl. Phys. B394 (1993) 221.

[7] M. Dey, V. L. Eletsky and B. L. Ioffe, Phys. Lett. B252 (1990) 620.

[8] V. L. Eletsky and B. L. Ioffe, Phys. Rev. D47 (1993) 3083.

[9] H. Umezawa, H. Matsumoto and M. Tachiki, Thermo Field Dynamics and Condensed States, (North-Holland, Amsterdam, 1982)

[10] L. D. Landau, ZhETF 37 (1958) 805

[11] S. Narison, QCD Spectral Sum Rules, (World Scientific, Singapore, 1989)

[12] P. Gerber and H. Leutwyler, Nucl. Phys. B321 (1989) 387.

[13] T. Hatsuda and T. Kunihiro, Phys. Lett. B185 (1987) 304; T. Kunihiro, Nucl. Phys. B351 (1991) 593.

[14] J. Gasser and H. Leutwyler, Phys. Lett. B184 (1987) 83; ibid. B188 (1987) 477.

[15] R. D. Pisarski and F. Wilczek, Phys. Rev. D29 (1984) 338.

[16] C. A. Dominguez, UCT-TP-218/94.