FINITE AND INFINITE SYMMETRIES IN
(2+1)-DIMENSIONAL FIELD THEORY*

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Dedicated to Franco Iachello on his 50th Birthday

Recent Problems in Mathematical Physics, Salamanca, Spain, June 1992; XIX International Colloquium on Group Theoretical Methods in Physics, Salamanca, Spain, July 1992, Condensed Matter and High Energy Physics, Cagliari, Italy, September 1992.

CTP#2110 Typeset in \TeX{} by Roger L. Gilson June 1992
BU HEP 92-21

* This work is supported in part by funds provided by the U. S. Department of Energy (D.O.E.) under contract #DE-AC02-76ER03069 (RJ), and #DE-AC02-89ER40509(S-YP).
ABSTRACT

These days, Franco Iachello is the eminent practitioner applying classical and finite groups to physics. In this he is following a tradition at Yale, established by the late Feza Gursey, and succeeding Gursey in the Gibbs chair; Gursey in turn, had Pauli as a mentor. Iachello’s striking achievement has been to find an actual realization of arcane supersymmetry within mundane adjacent even-odd nuclei. Thus far this is the only physical use of supersymmetry, and its fans surely must be surprised at the venue. Here we describe the role of $SO(2,1)$ conformal symmetry in non-relativistic Chern–Simons theory: how it acts, how it controls the nature of solutions, how it expands to an infinite group on the manifold of static solutions thereby rendering the static problem completely integrable. Since Iachello has also used the $SO(2,1)$ group in various contexts, this essay is presented to him on the occasion of his fiftieth birthday.
1. INTRODUCTION

We shall discuss finite- and infinite-dimensional conformal symmetries of field theories with non-relativistic kinematics. Such field theories also describe the second quantization of non-relativistic particle mechanics. Particle mechanics, with its second order in time dynamics, has the structure of a relativistic field theory in one time and zero space dimensions, and a relativistic field theory in any dimension can enjoy conformal symmetry. Thus there are family relationships between the conformal symmetries of non-relativistic field theory, non-relativistic particle mechanics and relativistic field theory, and our first task is to describe these interrelations.

A conformal transformation in \((D + 1)\)-dimensional relativistic field theory changes the independent variables, viz. the space-time coordinates \(x^\mu\) of the fields (fields are dependent variables), and infinitesimally reads

\[
\delta f x^\mu = -f^\mu (x) \tag{1.1}
\]

where \(f^\mu\) is a conformal Killing vector, i.e. \(f^\mu\) satisfies the conformal Killing equation.

\[
\partial_\mu f_\nu + \partial_\nu f_\mu = \frac{2}{D + 1} g_{\mu\nu} \partial_\alpha f^\alpha \tag{1.2}
\]

Here \(g_{\mu\nu}\) is the Minkowski metric tensor with signature \((1, -1, -1, \ldots)\) and \(D\) is the spatial dimensionality.

As is well-known, Eq. (1.2) has the finite number of \(\frac{1}{2}(D+2)(D+3)\) solutions for \(D > 1\), and conformal transformations form an \(SO(2, D + 1)\) group. The solutions to (1.2) comprise

\[
D + 1 \text{ space-time translations } f^\mu (x) = a^\mu , \quad a^\mu \text{ constant } \tag{1.3a}
\]

\[
\frac{1}{2}D(D + 1) \text{ space-time rotations } f^\mu (x) = \omega^\mu_{\nu} x^\nu , \quad \omega_{\mu\nu} = -\omega_{\nu\mu} \tag{1.3b}
\]
a single scale transformation \( f^\mu(x) = ax^\mu, \) a constant (1.3c)

\[ D + 1 \] special conformal transformations \( f^\mu(x) = 2c \cdot x x^\mu - c^\mu x^2, \) \( c^\mu \) constant

\[ (1.3d) \]

The finite versions of these are, respectively,

\[ x^\mu \to x^\mu + a^\mu \] (1.4a)

\[ x^\mu \to \Lambda^\mu_\nu x^\nu, \quad \Lambda^\mu_\alpha \Lambda^\nu_\beta g_{\mu\nu} = g_{\alpha\beta} \] (1.4b)

\[ x^\mu \to \epsilon^\alpha x^\mu \] (1.4c)

\[ x^\mu \to \frac{x^\mu - c^\mu x^2}{1 - 2c \cdot x + c^2 x^2} \] (1.4d)

The last, the finite special conformal transformation, can also be seen as an inversion, \( x^\mu \to x^\mu / x^2 \), followed by a translation and another inversion, \( i.e. \) a translation in the inverted coordinate.

At \( D = 1 \) there exists an infinite number of solutions to (1.2) corresponding to arbitrary redefinition of \( x^\pm = \frac{1}{\sqrt{2}} (x^0 \pm x^1) \) and forming an infinite parameter group. Infinitesimally we have

\[ \delta f x^\pm = -f^\pm (x^\pm), \quad f^\pm \text{ arbitrary} \] (1.5)

while the finite version reads

\[ x^\pm \to X^\pm (x^\pm), \quad X^\pm \text{ arbitrary} \] (1.6)

A linear conformal transformation on a space-time multiplet of Lorentz covariant relativistic fields \( \varphi \), \( i.e. \) on the dependent variables, can be taken as

\[ \delta f \varphi = f^\alpha \partial_\alpha \varphi + \partial_\alpha f^\beta \left( \frac{\Delta}{D+1} g^{\alpha\beta} + \frac{1}{2} \Sigma^{\alpha\beta} \right) \varphi \] (1.7)
Here $\Sigma^{\alpha\beta}$ is the spin-matrix, acting on the space-time components of $\varphi$ and $\Delta$ is a constant, called the scale-dimension of $\varphi$. When the Lagrange density for $\varphi$ possesses a conventional relativistic kinetic term — quadratic in derivatives for Bose fields, linear for Fermi fields — the kinetic action is invariant against conformal transformations (1.7) provided

$$\Delta = \frac{D-1}{2} \quad \text{bosons} \quad (1.8a)$$

$$\Delta = \frac{D}{2} \quad \text{fermions} \quad (1.8b)$$

[These values for $\Delta$ correspond to the dimensionality of a field in units of inverse length when $\hbar$ and $c$ are scaled to unity.] Also the Bose field monomial

$$L_I = \varphi^2 (\frac{\Delta}{D+1})$$

leads to an invariant action $\int d^{D+1}x L_I$. At $D = 1$, Bose fields become dimensionless, see (1.8a), and the conformally monomial (1.9) cannot be formed. Nevertheless, there exists a non-trivial conformally invariant theory — the completely integrable Liouville theory,

$$L_{\text{Liouville}} = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \frac{\mu^2}{\beta^2} e^{\beta \varphi}$$

whose action is invariant provided the single-component scalar field $\varphi$ is transformed according to an inhomogenous generalization of (1.7),

$$\delta_f \varphi = f^\alpha \partial_\alpha \varphi + \frac{1}{\beta} \partial_\alpha f^\alpha$$

(1.11a)

or equivalently

$$\delta_f e^{\beta \varphi} = \partial_\alpha (f^\alpha e^{\beta \varphi})$$

(1.11b)
The hallmark of a conformally invariant theory is that an energy-momentum tensor $T^{\mu\nu}$ can be constructed, which is conserved (as a consequence of translation invariance) symmetric (as a consequence of Lorentz invariance) and traceless (as a consequence of conformal invariance). Thus conformal invariance in a relativistic field may be summarized by a relation between the energy density $\mathcal{E} = T^{00}$ and the trace of the spatial stress tensor $T^{ij}$.

\[
\mathcal{E} = \sum_{i=1}^{D} T^{ii}
\]  

(1.12)

The currents $j^\mu_f$ that are conserved as a consequence of conformal invariance are then constructed in the Bessel–Hagen form from a projection of $T^{\mu\nu}$ on the conformal Killing vector,

\[
j^\mu_f = T^{\mu\nu} f_\nu
\]  

(1.13a)

and the constants of motion read

\[
C_f = \int d^D r \left( \mathcal{E} f^0 - \mathbf{P} \cdot \mathbf{f} \right)
\]  

(1.13b)

where $\mathbf{P}$ is the momentum density.

\[
\mathbf{P}^i = T^{0i}
\]  

(1.14)

[Frequently it is necessary to “improve” the energy-momentum tensor obtained by Noether’s theorem or by general relativistic considerations.]

The kinetic Lagrangian for non-relativistic motion of point-particles in $d$-dimensional space is quadratic in derivatives with respect to time, which is the single independent variable. Hence it has the structure of a $(0+1)$-dimensional relativistic “field theory,” where the “field” is particle position $\mathbf{r}(t)$, now a dependent variable, while the $d$-spatial dimensions form an “internal” space and $\mathbf{r}$ is a vector in this space. “Conformal” transformations degenerate into
reparametrization of the single independent variable, \textit{i.e.} time. The previous discussion can be taken at \( D = 0 \), but the conformal Killing equation (1.2) becomes vacuous. Nevertheless, one easily shows that (1.3) and (1.4) [with (1.3b) and (1.4b) absent] are invariances of the kinetic term provided the dependent variable \( r \) transforms according to

\[
\delta r = f \dot{r} - \frac{1}{2} \ddot{f} r
\]

when the independent variable \( t \) changes by

\[
\delta t = -f(t) \quad (1.16a)
\]

\[
f(t) = a, at, at^2 \quad (1.16b)
\]

These comprise the \( D = 0 \) restriction of (1.7) and (1.8a) with \( \Delta = -1/2 \), and form an \( SO(2,1) \) group of transformations. It is seen that \( r \) has scale dimension of \(-1/2\), \textit{i.e.} it scales as \( \sqrt{t} \) — this is a consequence of having scaled \( \hbar \) to unity and with non-relativistic kinematics one can take \( m \) to be dimensionless. Further from (1.9) at \( D = 0 \) one sees that the \( r^{-2} \) potential also gives an invariant action since \( \dot{r}^2 \) and \( 1/r^2 \), the two terms comprising a Lagrangian or Hamiltonian, scale in the same way. When

\[
H = \frac{1}{2} m \dot{r}^2 + \frac{\lambda}{r^2} \quad (1.17)
\]

the three constants of motion

\[
C_f = H f - \frac{m}{4} (r \cdot \dot{r} + \dot{r} \cdot r) f + \frac{m}{4} r^2 \ddot{f} \quad (1.18)
\]

also generate the transformation (1.14) when the canonical momentum \( p \), conjugate to \( r \), is taken to be \( m \dot{r} \), and their algebra realizes the \( SO(2,1) \) Lie algebra. One can view (1.18) as the one-time, zero-space analog of the Bessel–Hagen expression (1.13).
In specific spatial dimensions $d$, other interactions in addition to the $1/r^2$ potential preserve conformal invariance. Examples for $d = 3$ and 2 are, respectively, interaction with a Dirac magnetic monopole and a point vortex. For these (1.17) and (1.18) retain the same form, but the relation between canonical momentum and velocity is modified by the presence of a vector potential

$$p = m \dot{r} + A$$  \hspace{1cm} (1.19)

where at $d = 3$, $A$ is the Dirac vector potential that gives rise to a monopole magnetic field of strength $g_m$

$$A = A_D, \quad B = \nabla \times A_D = \frac{g_m r}{r^3}$$ \hspace{1cm} (1.20)

while in planar physics at $d = 2$, $A$ is the vortex potential,

$$A = \frac{\Phi}{2\pi} \nabla \theta, \quad B = \nabla \times A = \Phi \delta^2(r)$$ \hspace{1cm} (1.21)

with $\tan \theta = y/x$ and $\Phi$ being the flux of the vortex. [In the above we use the amusing distributional formula $\nabla \times \nabla \theta = 2\pi \delta^2(r)$.

Furthermore, at $d = 2$, the $\delta$-function potential also scales as $r^{-2}$, hence it also appears to be conformally invariant. In the following, we shall consider the two-dimensional particle model, with a vortex (1.21) and $\delta$-function interactions; \textit{i.e.} the Hamiltonian is

$$H = \frac{1}{2} m \dot{\delta^2}(r) - g \delta^2(r) = \frac{1}{2m} (p - A)^2 - g \delta^2(r)$$ \hspace{1cm} (1.22)

Non-relativistic particle quantum mechanics may be second quantized, and in this way one is led to a non-relativistic quantum field theory, with field-theoretic symmetries that encode the above $SO(2,1)$ particle symmetries, but now realized in an action on the dependent
field variable $\psi$, which is a function of the independent variables $t$, and $r$, where $r$ is a two-dimensional vector.

In these lectures, we shall explain these symmetries of non-relativistic field theory, at $d = 2$ — planar physics. Also we shall show how the $SO(2,1)$ group can expand to an infinite-dimensional group of conformal reparametrizations of the two-dimensional spatial plane.

Contexts, wherein recently there is encountered a non-relativistic, planar field theory, are the following two:

1) **Second quantized, non-relativistic particles with Abelian or non-Abelian charge, interacting with a gauge field whose kinetic dynamics is provided by the Chern–Simons action.** The field theoretic action in the Abelian case, with gauge potentials eliminated in terms of matter variables, is

$$I = \int dt \ d^2r \left\{ i\bar{\psi} \partial_t \psi - \frac{1}{2m} |D\psi|^2 + \frac{g}{2} \rho^2 \right\}$$  \hspace{1cm} (1.23)$$

where the covariant derivative $D$ involves a gauge potential $A$

$$D = \nabla - iA$$  \hspace{1cm} (1.24)$$

which is determined by the matter density $\rho = |\psi|^2$

$$A(t,r) = \nabla \times \frac{1}{2\pi\kappa} \int d^2r' \ln |r - r'| \rho(t,r')$$  \hspace{1cm} (1.25)$$

so that the Chern–Simons Gauss law is satisfied.

$$B = \nabla \times A = -\frac{1}{\kappa} \rho$$  \hspace{1cm} (1.26)$$
Here $1/\kappa$ measures the interaction strength and without loss of generality we may take it to be non-negative. [In the plane, the cross product of two vectors defines a scalar, and cross multiplication with a single vector results again in a vector; in components:

$$s = \varepsilon^{ij}v^i_{(1)}v^j_{(2)}; \quad v^i_{(1)} = \varepsilon^{ij}v^j_{(2)}.$$]

Also there is present in (1.23) a quartic self-interaction of strength $g$, which is the second-quantized description of a two-body $\delta$-function interaction.

Thus (1.23) provides the second quantization of (1.22); it may also be presented as

$$I = \int dt d^2r \{i\psi^* \partial_\tau \psi - \mathcal{E}\}$$

(1.27)

where the energy density is given by the formula

$$\mathcal{E} = \frac{1}{2m} |D\psi|^2 - \frac{g}{2} \rho^2$$

(1.28)

2. **Effective action for gravity or Abelian vector gauge theories in the eikonal (large-$s$, fixed-$t$) limit.** It is found that in the eikonal regime, the conventional action [Einstein–Hilbert for gravity, Maxwell for gauge theory] can be written as a total derivative on a two-dimensional space-time plane imbedded in four-dimensional space-time. By integrating the total derivative onto a curve (parametrized by $\tau$) forming the boundary of that two-dimensional plane, the action [without sources] becomes

$$I_{\text{eikonal}} = \frac{1}{2} \int d\tau d^2r \left( \partial_i \Omega^+ \partial_i \Omega^- - \partial_i \Omega^- \partial_i \Omega^+ \right)$$

(1.29)

where the overdot denotes differentiation with respect to $\tau$, while $r$ and $\partial_i, i = 1, 2$, refer to the remaining two spatial directions, and $\Omega^\pm$ are the surviving field (gravitational, vector) degrees of freedom.\(^2\) Upon defining

$$\psi = \frac{1}{\sqrt{2}} (\partial_x + i\partial_y) (\Omega^+ - i\Omega^-)$$

(1.30)

(1.29) may be rewritten, apart from total derivative contributions, as

$$I_{\text{eikonal}} = \int d\tau d^2r i\psi^* \partial_\tau \psi$$

(1.31)

Precisely the same form as (1.27) is revealed, except now the energy density vanishes.
II. SYMMETRIES

The field theoretic Lagrangian

\[ L = \int d^2 r i \psi^* \partial_t \psi - H \] (2.1a)

\[ H = \int d^2 r \mathcal{E} \] (2.1b)

represents both the non-relativistic Chern–Simons model (1.27), (1.28) and the eikonal limits of relativistic theory (1.29), (1.31), where in the latter case \( \mathcal{E} \) vanishes. So we use (2.1) as the basis for our discussion of symmetries in both cases, keeping in mind that the vanishing of \( \mathcal{E} \) for the latter, renders much of the analysis vacuous, but as we shall see, not without relevance to the former. Throughout we shall solely deal with the Abelian Chern–Simons theory, though as far as symmetry properties are concerned, the non-Abelian model behaves similarly. Also discussion is confined to classical symmetries of the field theory, viewed as a classical non-linear system. Anomalies in the symmetries due to quantum effects will only be mentioned in the Conclusion.

The energy density (1.28) of the Chern–Simons model

\[ \mathcal{E} = \frac{1}{2m} |D\psi|^2 - \frac{g}{2} \rho^2 \] (2.2a)

is identically equal to

\[ \mathcal{E} = \frac{1}{2m} |D\psi|^2 - \frac{1}{2} \left( \nabla \times \mathbf{j} + \frac{1}{m} B \rho \right) - \frac{g}{2} \rho^2 \] (2.2b)

where the current \( \mathbf{j} \) is

\[ \mathbf{j} = \frac{1}{m} \overline{\psi} \gamma^1 \gamma^3 \psi \] (2.3)
and $D$ is the holomorphic, gauge covariant derivative.

\[ D \equiv D_x - iD_y \quad (2.4) \]

The curl of $j$ will not contribute to a variational derivation of the equations of motion, nor will it contribute to the integrated total energy, provided the current is sufficiently well-behaved at the edge of space [which lies at infinity]. With the assumption of requisite regularity for $j$ and the use of the Chern–Simons Gauss law constraint (1.26), the energy/Hamiltonian may be presented as

\[ E = H = \int d^2 r \mathcal{H} \quad , \quad \mathcal{H} = \frac{1}{2m} |D\psi|^2 - \frac{1}{2} \left( g - \frac{1}{m\kappa} \right) \rho^2 \quad (2.5) \]

This is the form of the Hamiltonian density that we shall scrutinize as regards to the symmetries of the model.

The symmetries are of two kinds: a) symmetries of the action, *i.e.* transformations which leave the action invariant and lead to constants of motion by Noether’s theorem — these are well-known and include the obvious Galilean transformations, and the $SO(2,1)$ time-reparametrization conformal symmetries specific to the planar model with which we are here concerned; and b) symmetries of the critical points of the action, *i.e.* transformations which leave selected equations of motion invariant, map solutions into solutions, but do not give rise to constants of motion because they do not leave invariant the action away from its critical points. These symmetries of the Chern–Simons model have not been previously studied systematically, though their occurrence in static solutions at $g = 1/m\kappa$ had been noted: they comprise conformal reparametrization symmetries of the two-dimensional plane.\(^\text{3, 4}\)
II.A. Finite-Dimensional Symmetry Group of the Action

As befits any respectable field theory, our model is invariant against time translation, space translation and rotation, as well as against Galileo boosts because dynamics is non-relativistic. The last invariance is perhaps unexpected in the presence of gauge fields, which conventionally are invariant against the Lorentz boosts of special relativity (indeed this led to the invention of special relativity!). What distinguishes the present situation is that the gauge dynamics are of the Chern–Simons variety, and the Chern–Simons term, being topological, is invariant against all space-time transformation, while the non-relativistic matter system is only Galileo invariant. The transformation laws on the fields are familiar. For the first three,

\begin{align*}
\text{time translations} & \quad t' = t + a \\
& \quad r' = r
\end{align*} \quad (2.6)

\begin{align*}
\text{space translation} & \quad t' = t \\
& \quad r' = r + a
\end{align*} \quad (2.7)

\begin{align*}
\text{space rotation} & \quad t' = t \\
& \quad r'^i = R^{ij} (\omega) r^j
\end{align*} \quad (2.8)

\text{[}R^{ij} (\omega) \text{ is the rotation matrix through angle } \omega \text{]} \text{ the field transforms as a scalar.}

$$\psi'(t', r') = \psi(t, r)$$ \quad (2.9)

The Galileo transform

\begin{align*}
\text{Galileo boost} & \quad t' = t \\
& \quad r' = r + vt
\end{align*} \quad (2.10)
requires a 1-cocycle in the field transformation law.

\[ \psi'(t', r') = e^{imv \cdot (r' + \frac{1}{2}vt)} \psi(t, r) \]  
(2.11)

Additionally, our system is invariant against conformal reparametrizations of time. These include three \( SO(2, 1) \) transformations of time: translation (2.6) and (2.9),

\begin{align*}
t' &= at \\
r' &= \sqrt{a} r
\end{align*}
(2.12)

for which the field transformation law acquires a weight factor,

\[ \psi'(t', r') = \frac{1}{\sqrt{a}} \psi(t, r) \]  
(2.13)

and translation of inverse time,

\begin{align*}
\text{conformal time transformation} \\
\frac{1}{t'} &= \frac{1}{t} + a \\
r' &= \frac{1}{1 + at} r
\end{align*}
(2.14)

where the field transformation law has both a weight factor and 1-cocycle.

\[ \psi'(t', r') = (1 + at) e^{imn^2 \frac{1}{1+at}} \psi(t, r) \]  
(2.15)

One can check that owing to the weight factors, which are square roots of the Jacobian, the density \( \rho \) transforms with the Jacobian, \( J \)

\[ \rho'(t', r') = J \rho(t, r) \]  
(2.16)

\[ J \equiv \det \left\{ \frac{\partial r^i}{\partial r'^j} \right\} \]  
(2.17)

and the vector potential, defined by (1.25), transforms covariantly.

\[ A'^i(t', r') = A^j(t, r) \frac{\partial r'^j}{\partial r^i} \]  
(2.18)
The action is invariant, and the conserved generators can be obtained from Noether’s theorem.

Alternatively one records the formula for the energy momentum tensor components:

- **Energy density**
  \[ T^{00} = E = \frac{1}{2m} |D\psi|^2 - \frac{g}{2} \rho^2 \]  
  (2.19)

- **Momentum density**
  \[ \mathcal{P} = m\mathfrak{j} = \Im\psi^* D\psi \]  
  (2.20)

These satisfy continuity equations with energy flux \( T \),

- **Energy flux**
  \[ T = -\frac{1}{2} \left( (D_t\psi)^* D\psi + (D\psi)^* D_t\psi \right) \]  
  (2.21)

and momentum flux — the stress tensor \( T^{ij} \).

- **Momentum flux**
  \[ T^{ij} = \frac{1}{2} \left( (D_i\psi)^* (D_j\psi) + (D_j\psi)^* (D_i\psi) - \delta^{ij} (D_k\psi)^* (D_k\psi) \right) \]
  
  \[ + \frac{1}{4} \left( \delta^{ij} \nabla^2 - 2\partial_i\partial_j \right) \rho + \delta^{ij} \mathcal{E} \]  
  (2.22)

Here \( D_i = \partial_i + iA^0 \), where \( A^0 \) solves the Chern–Simons equation that supplements (1.26).

\[ A^0(t, r) = -\frac{1}{2\pi\kappa} \int d^2 r' \epsilon^{ij} \frac{(r^i - r'^i)}{|r - r'|^2} j^j(t, r') \]  
(2.23)

The continuity equations read

\[ \partial_t \mathcal{E} + \nabla \cdot \mathbf{T} = 0 \]  
(2.24)

\[ \partial_t P^i + \partial_j T^{ij} = 0 \]  
(2.25)

Note that energy flux \( \mathbf{T} \) does not equal momentum density \( \mathcal{P} \), since our theory is not Lorentz invariant. But it is rotationally invariant; that is why the stress-tensor is symmetric in its spatial indices. Also \( T^{ij} \) satisfies

\[ 2\mathcal{E} = \sum_{i=1}^2 T^{ii} \]  
(2.26)
and this reflects the $SO(2,1)$ invariance, being the non-relativistic analog of (1.12).

Of course, the theory is also phase invariant; this produces one more continuity equation

$$\partial_t \rho + \nabla \cdot \mathbf{j} = 0$$  \hfill (2.27)

where the proportionality of the matter flux current $\mathbf{j}$ to the momentum density (2.20) is a consequence of Galileo invariance.

The constants of motion are now constructed from moments of the energy momentum tensor and $\rho$. They are, respectively

\begin{align*}
\text{energy} & \quad E = \int d^2 r \mathcal{E} \quad \hfill (2.28) \\
\text{momentum} & \quad \mathbf{P} = \int d^2 r \mathbf{P} \quad \hfill (2.29) \\
\text{angular momentum} & \quad M = \int d^2 r \mathbf{r} \times \mathbf{P} \quad \hfill (2.30) \\
\text{Galileo boost} & \quad \mathbf{B} = t \mathbf{P} - m \int d^2 r \mathbf{r} \rho \quad \hfill (2.31) \\
\text{dilation} & \quad D = t E - \frac{1}{2} \int d^2 r \mathbf{r} \cdot \mathbf{P} \quad \hfill (2.32) \\
\text{special conformal} & \quad K = -t^2 E + 2t D + \frac{m}{2} \int dr^2 r^2 \rho \quad \hfill (2.33) \\
\text{matter number} & \quad N = \int d^2 r \rho \quad \hfill (2.34)
\end{align*}

It is straightforward to verify that all these are conserved, as a consequence of the continuity equations and the special properties (symmetry and trace) of the stress tensor. Note that collectively the constants may be written analogously to (1.18) and to the Bessel–Hagen expression (1.13), as

$$C_f = \int d^2 r \mathcal{E}_{f_1} - \int d^2 r \mathbf{P}_{f_2} + \int d^2 r \rho_{f_3}$$  \hfill (2.35)

for suitable $f_i$. 

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The above transformations may be generalized in the following interesting manner. One may consider an arbitrary reparametrization of time [rather than the specific forms (2.6), (2.12), (2.14)]. Also one may shift \( r \) by a vector with arbitrary time dependence [rather than constant (2.7) or linear (2.10) in time]. Finally, one may rotate \( r \) as in (2.8), but with a time-dependent angle. Of course, these transformations are no longer symmetry operations of our theory, but they map our model onto another, closely related one: it is found that the above transformations introduce interactions with external fields. Specifically, after these transformations are carried out there arise external electric and magnetic fields, determined by the parameters of the transformations, hence the fields are time-dependent but constant in space; additionally there is an external harmonic potential, with time-varying frequency.\(^5\)

[With specific time-dependence, the parameters can conspire to produce static electric and magnetic fields as well as time-independent harmonic forces; also one can suppress selectively any of the external effects.]

Higher symmetries are widely studied these days in field theory, but it seems that rarely do they provide specific dynamical information about a model — rather they give an elegant frame for describing solutions and other properties.

As an exception that proves the rule, we now show that the conformal symmetries allow deriving the following useful result about the highly non-linear dynamics of our Chern–Simons theory: all static solutions carry zero energy.\(^6\) This follows immediately from (2.32) and/or (2.33): the left sides are time independent, and so are the last terms in the right sides for time independent \( \rho \) and \( P = m\mathbf{j} \), \( i.e. \) for static solutions. Thus \( H = E \), and also but less importantly \( D \), must vanish. Similarly, from (2.31) one sees that \( \mathbf{P} \) must vanish, but this is not surprising — we expect static solutions to carry no momentum. Note however: angular
momentum need not vanish for static configurations; it can be constructed from the current $\mathbf{m} \mathbf{j} = \mathbf{P}$, which in the static case must be divergence-free, according to (2.27).

Since $E$ can be given by (2.5) (provided there is sufficient regularity so that the integral $\int d^2 r \nabla \times \mathbf{j}$ vanishes) we see that static solutions can exist only for $g \geq 1/m\kappa$. Especially interesting is the limiting case $g = 1/m\kappa$, where the integrand is non-negative and therefore must vanish on static solutions. In this way the $SO(2,1)$ conformal symmetry demands that all static solutions (at $g = 1/m\kappa$) satisfy

$$D\psi = 0 \quad (2.36)$$

Together with the Chern–Simons constraint (1.26) this implies that $\rho$ satisfies the Liouville equation,

$$\nabla^2 \ln \rho = -\frac{2}{\kappa} \rho \quad (2.37)$$

which can be integrated explicitly in terms of two arbitrary functions, which are further specified by the physical requirements that one may wish to impose on static solutions.$^{1,3}$

[In the non-Abelian case, the analogous equations, with $\psi$ in the same fundamental representation as the gauge fields, realize a two-dimensional reduction of four-dimensional self-dual gauge field equations in a space with signature $(++--)$ and lead to many integrable systems, principally the Toda system.$^4$]

As is well-known and was remarked in the Introduction, the Liouville equation is invariant against conformal redefinition of the two-dimensional plane. In Euclidean space this involves the complex variable $x + iy = z$ transforming into an arbitrary function of $z$, but not of $z^*$: Our next task will be to understand the properties of the action (1.23) (at $g = 1/m\kappa$) that are responsible for this infinite symmetry. In fact its stationary points are conformally invariant.
But before turning to this topic, we point out that the above described transformations can be used to generate interesting new solutions from the explicitly determined static solution. First by Galileo [(2.10), (2.11)] or conformal [(2.14), (2.15)] boosting of static solutions, one obtains time dependent solutions to the Chern–Simons model. Moreover, by performing transformations with time-dependent parameters, one finds time-dependent solutions to the Chern–Simons model with external, appropriately constructed electric and magnetic fields as well as an external harmonic oscillator.\(^7\)

**B. Infinite-Dimensional Symmetry Group of Stationary Points of the Action**

The dilation transformation (2.12) and (2.13) rescales the spatial coordinate \( r \). Here we inquire about the response of the action (2.1a), (2.5) to a conformal redefinition of spatial coordinates,

\[
\mathbf{r}' = \mathbf{r}'(\mathbf{r}) \tag{2.38a}
\]

where

\[
x' + iy' \equiv z' = z'(z) \tag{2.38b}
\]

and time is unchanged, \( t' = t \).

Generalizing (2.13), we posit a field transformation law with a weight,

\[
\psi'(\mathbf{r}') = \left. \frac{\partial z^*}{\partial z^*} \psi(\mathbf{r}) \right|_{\mathbf{r}' = \mathbf{r}} \tag{2.39}
\]

which apart from a phase is the square root of the Jacobian, as in (2.13) and (2.15), while the choice of phase is dictated by the Hamiltonian (2.5). [Since time is not transformed, we
suppress the time argument.} This has the consequence that the density transforms with the Jacobian as in (2.16).

\[ \rho'(r') = J \rho(r) \quad (2.40) \]

\[ J = \det \left( \frac{\partial r^i}{\partial r'^j} \right) = \left| \frac{\partial z}{\partial z'} \right|^2 \quad (2.41) \]

For infinitesimal \( \delta z = -f(z) \), this transformation law coincides with (1.11b), taken in Euclidean space and \( e^{i\beta \varphi} \) identified with \( \rho \). It further follows that the gauge potential transforms covariantly.

\[ A'^i(r') = A^j(r) \frac{\partial r^j}{\partial r'^i} \quad (2.42) \]

This is most easily proven by first noting that \( A \), when given by Eq. (1.25), is transverse and satisfies \( \nabla \times A(r) = -\frac{1}{\kappa} \rho(r) \), and then verifying that \( A'(r') \) in (2.42) also is transverse and satisfies \( \nabla \times A'(r') = -\frac{1}{\kappa} J \rho(r) = -\frac{1}{\kappa} \rho'(r') \). [In carrying out the differentiations it is useful to pass the complex variables.] It follows that \( D \psi \equiv (\partial_x - i\partial_y - iA_x + A^y) \psi \) transforms with the Jacobian.

\[ D_{r'} \psi'(r') = JD_r \psi(r) \quad (2.43) \]

So finally we can state the transformation law for the Lagrange density.

\[ \mathcal{L} = i\psi^* \partial_t \psi - \frac{1}{2m} |D\psi|^2 + \frac{1}{2} \left( g - \frac{1}{m\kappa} \right) \rho^2 \]

\[ = i\psi^* \partial_t \psi - \mathcal{H} \quad (2.44) \]

Evidently it is true that

\[ \mathcal{L}'(r') = Ji\psi^*(r)\partial_t \psi(r) - J^2 \mathcal{H}(r) \quad (2.45) \]

so that the Lagrangian transforms as

\[ L' = \int d^2r' \mathcal{L}'(r') = \int d^2r i\psi^*(r)\partial_t \psi(r) - \int d^2r J^2 \mathcal{H}(r) \quad (2.46) \]

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One factor of the Jacobian disappears when changing spatial variables in the integration, and
the symplectic form \( \int d^2 r \, i \psi^* \partial_t \psi \) is invariant. But the Hamiltonian density \( H \) remains with
one factor \( J \), hence the total Lagrangian is not in general invariant, and neither is the action,
the time integral of \( L \) — because \( t \) is not changed in the present transformation rules [in
contrast to (2.12) and (2.14)]. [It does not appear possible to find a transformation of time
that would restore invariance.]

However, for static solutions we know that \( E = \int d^2 r \, H \) vanishes. If this vanishing is
due to the local vanishing of \( H \), as is true at \( g = 1 / m \kappa \), then the static critical points of the
action are invariant. This then shows that static solutions with zero \( H \) will be mapped into
each other by spatial conformal transformations — the dilation (2.12) expands to an infinite
symmetry group on the solutions, but there are no new constants of motion.

[Since in the non-Abelian generalization, with matter in the adjoint representation, equations
are dimensional reductions of self-dual Yang–Mills equations in four dimensions, the
finite-dimensional conformal invariance of the latter is seen to survive the dimensional re-
duction, and in two dimensions expands to the infinite-dimensional conformal group.]

On the other hand, in the effective field theories for the eikonal regime (1.29), (1.31),
where there is no Hamiltonian to begin with, the transformations (2.38), (2.39) are symmetries
of the action, and also \( \tau \) may be arbitrarily reparametrized. Note that owing to the derivative
relation (1.30) between \( \Omega^\pm \) and \( \psi \); \( \psi = \sqrt{2} \frac{\partial}{\partial r} (\Omega^+ - i \Omega^-) \), the transformation law for \( \Omega^\pm \) is
without the weight factor,

\[
\Omega'(r') = \Omega^\pm(r)
\]  

which arises for \( \psi \), as in (2.39), when the derivative is taken.
III. CONCLUSION AND SUGGESTIONS FOR FURTHER RESEARCH

The rigid scale invariance of the action for non-relativistic (2+1)-dimensional field theory with quartic self-interaction and coupling to a Chern–Simons gauge field, expands at the static critical points of the action to the infinite conformal group on the plane. The scale symmetry allows establishing the important result that static solutions carry zero energy, and the infinite conformal symmetry “explains” why the static system is completely integrable. The kinetic action of effective eikonal field theories also possesses the infinite symmetry.

The Chern–Simons model at \( g = 1/m\kappa \) is the bosonic partner of an \( N = 2 \) supersymmetric theory with fermions and the invariance of the extended action against the supersymmetric generalization of the bosonic symmetries \((2.6) – (2.15)\) has been established.\(^9\) While the invariances of the static critical points in the supersymmetric action have not been explicitly checked, they too presumably enjoy an infinite conformal symmetry, because the supersymmetric static equations retain the form of the bosonic equations.

In our considerations, the possibility of quantum symmetry breaking anomalies has been ignored. It is known that the quartic self-interaction, which as we have seen is formally scale invariant, suffers from quantum scale anomalies.\(^10\) This is particularly clear in the first quantized framework, where the two-dimensional \( \delta \)-function potential, while scaling classically as \( r^{-2} \), does not give rise to energy-independent phase shifts, as is required by scale invariance and is explicitly realized by the scale invariant \( 1/r^2 \) potential. There is a quantum scale anomaly — the simplest example of the anomaly phenomenon.\(^11\) On the other hand, anomalies in the theory with both quartic self-coupling and Chern–Simons interaction have thus far not been assessed; in fact there is some indication of anomaly cancellation, even without supersymmetry.\(^12\) Further research on this question would be interesting.\(^13\)
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