Source lifetime dependence of neutrino oscillations - a simple wave-function derivation

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Abstract

When neutrinos are produced from long lived particles like pions, kaons and muons, the life-time of the source particle affects the flavour conversion probability formula. Experiments like LSND which use muon decay neutrinos are two orders of magnitude more sensitive to lower values of mass square difference compared to other experiments where the sources are pions or kaons.

The standard formula for neutrino oscillations is derived with the assumption that the uncertainty in the initial position of the neutrino is small compared to the distance between the production and the detection sites of the neutrinos. We shall show here that in a covariant treatment, the effective spread of the neutrino wave function in space is given by \((\sigma_x + c\tau)\) where \(\sigma_x\) is the initial spatial uncertainty of the production point of the neutrinos and \(\tau\) is the lifetime of the sources (eg. pions, kaons, muons etc) whose decay produce the neutrinos. We find that the effect of long lifetime sources is to exponentially suppress the oscillation term in the conversion probability formula. The flavor conversion probability as a function of distance, for relativistic neutrinos produced from long lived resonances, turns out to be

\[
P(\nu_e \rightarrow \nu_\mu; X) = \frac{1}{2} \sin^2 2\theta \left( 1 - \cos \left( \frac{2.53\Delta m^2 X}{E} \right) \right) \exp \left( -\frac{1.79\Delta m^2 \tau}{E} \right) \]

(1)
where $\Delta m^2$ is the mass square difference in $eV^2$, $L$ where $\Delta m^2$ is the mass square difference in $eV^2$, $X$ is the detector distance in $m (km)$, $\tau$ is the lifetime of the source particle in the lab frame in $m (km)$ and $E$ is the energy in $MeV (GeV)$. When $c\tau$ is comparable to the detector distance spatial oscillations of the conversions probability is not seen. However the neutrino mass squared difference that can be probed is lower for longer lived sources. We find that for the LSND experiment where the neutrinos are produced from muons ($c\tau_\mu = 658.65 m$), when fitted with the covariant oscillation formula (1) gives a bound on $\Delta m^2$ which is two orders of magnitude lower than the corresponding bounds obtained from other similar experiments but where the neutrino sources are pions or kaons ($c\tau_\pi = 7.8 m$).

We start with an initial wave function of a general Gaussian form

$$\Psi_{in}^{a}(x,t) = \left[ \frac{1}{(2\pi\sigma_x\sigma_t)^{1/2}} \exp \left\{ i(P_a(x - x_i) - E_a(t - t_i)) \right\} - \frac{(x - x_i)^2}{4\sigma_x^2} - \frac{(t - t_i)^2}{4\sigma_t^2} \right] \quad (2)$$

The initial spread of the wavefunction in space around the mean initial position $x_i$ is denoted by $\sigma_x$ and the spread in the time direction around the mean initial time of production $t_i$ is denoted by $\sigma_t$. The magnitudes of $\sigma_x$ and $\sigma_t$ depends on how the state is prepared. The time evolved wavefunction can be obtained from the initial wavefunction (2) by the linear superposition principle, which can be written formally as

$$\Psi_{a}(x_f,t_f) = \int dx \, dt \, K(x_f - x, t_f - t) \, \Psi_{in}^{a}(x,t) \quad (3)$$

where $K(x_f - x, t_f - t)$ is the probability amplitude of a particle initially located at $(x, t)$ to be detected at another spacetime point $(x_f, t_f)$. The transition amplitude $K(x_f - x, t_f - t)$ for a free particle is given by the expression (3) (4)

$$K(x_f - x, t_f - t; m_a) = \left( \frac{i}{4\pi^2} \right) \left( \frac{m_a}{s} \right) K_1(ims) \quad (4)$$

where $s = ((t_f - t)^2 - (x_f - x)^2)^{1/2}$ is the invariant spacetime interval propagated by the $\nu_i$ mass eigenstate. If this interval is large ($s >> m_a^{-1}$) and time-like ($t_f - t \geq (x_f - x)$ ) then we can use the asymptotic expansion of the Bessel function

$$K_1(ims) \simeq \left( \frac{2}{\pi i ms} \right)^{1/2} \exp \{-ims\} \quad (5)$$
to obtain from (4) the expression for the propagation amplitude at large time-like separation

\[ K(x_f - x, t_f - t; m_a) = \left( \frac{m_a}{2\pi i \sqrt{(t_f - t)^2 - (x_f - x)^2}} \right)^{1/2} \exp\{-im_a \sqrt{(t_f - t)^2 - (x_f - x)^2}\} \]

(6)

This expression for the free particle propagator is valid for bosons. For fermions there is an extra factor \((i \partial + m_a)\) operating on the l.h.s of the expressions (4) and (6). This factor only changes the normalisation and the expressions for the conversion probability is identical for bosons and fermions. The time evolved wave-function can be evaluated by substituting the expression (6) for the propagator \(K(x_f - x, t_f - t)\) in the expression (3) for \(\Psi\),

\[ \Psi_a(x_f - x_i, t_f - t_i; m_a) = N \int dx \, dt \left( \frac{1}{(t_f - t)^2 - (x_f - x)^2} \right)^{1/4} \times \exp\{i\Phi(x, t) - \frac{(x - x_i)^2}{4\sigma_x^2} - \frac{(t - t_i)^2}{4\sigma_t^2} \} \]

(7)

where the phase factor in the exponential is of the form

\[ \Phi(x, t) = -m_a \sqrt{(t_f - t)^2 - (x_f - x)^2} + P_a(x - x_i) - E_a(t - t_i) \]

(8)

and the constant coefficients have been clubbed together as the factor \(N\) which normalises the wave-function. We perform the integrations in (7) by the method of stationary phases [5]. The integral is approximated by the integrand along the trajectory where the phase is an extremum. The extremum of the phase (8) is given by \((\partial\Phi/\partial t) = 0 \Rightarrow (m_a (t_f - t)/(t_f - t)^2 - (x_f - x)^2)^{1/2} = E_a\) and \((\partial\Phi/\partial x) = 0 \Rightarrow (m_a (x_f - x)/(t_f - t)^2 - (x_f - x)^2)^{1/2} = P_a\). One can solve for \(t\) and \(x\) using these two equations and substitute in the integrand of (7) to get the expression for the integral in the stationary phase approximation, which turns out to be

\[ \Psi_a(x_f - x_i, t_f - t_i) = N' \exp \left\{ -iE_a(t_f - t_i) + iP_a(x_f - x_i) - \frac{(x_f - x_i) - (t_f - t_i) \frac{P_a}{E_a})^2}{4(\sigma_x^2 + \sigma_t^2(\frac{P_a}{E_a})^2)} \right\} \]

(9)

where again we have clubbed together all the constants as the normalisation coefficient \(N'\). Denoting \(X = x_f - x_i, T = t_f - t_i\) and \(v_a = (P_a/E_a)\) which can be identified by distance of
propagation, time of propagation and the group velocity of the particle respectively we can write the expression for the time evolved wave-function compactly as

$$\Psi_a(X, T) = N_a \exp \left\{ -iE_a T + iP_a X - \frac{(X - v_a T)^2}{4(\sigma_x^2 + v_a^2 \sigma_t^2)} \right\}$$ (10)

The state vector of a mass eigenstate can be in the mass basis $|\nu_a >$ in the form

$$|m_a; X, T > = \Psi_a(X, T)|\nu_a >$$ (11)

On the other hand the state of a weak interaction state say $\nu_e$ is expressed as a linear combination

$$|\nu_e; X, T > = \sum_a \Psi_a(X, T)|\nu_a >$$ (12)

The probability amplitude of a neutrino to be produced at $(x_i, t_i)$ as a $\nu_e$ and to be detected at $(x_f, t_f)$ as another weak eigenstate $\nu_\mu$ is given by

$$A(\nu_e \rightarrow \nu_\mu; X, T) = \langle \nu_\mu | \nu_e; X, T > \sum_a \Psi_a(X, T) < \nu_\mu | \nu_a > < \nu_\mu | \nu_e >$$ (13)

and the corresponding probability is given by

$$P(X, T; \nu_e \rightarrow \nu_\mu) = | \langle \nu_\mu | \nu_e; X, T > |^2$$

$$= \sum_{a, b} \Psi_a(X, T) \Psi_b^*(X, T) < \nu_\mu | \nu_a > < \nu_\mu | \nu_e > < \nu_e | \nu_b > < \nu_b | \nu_\mu >$$ (14)

Restricting ourselves to mixing between two flavour generations, and denoting the mixing matrix elements as $< \nu_a | \nu_e > = - < \nu_b | \nu_\mu > = \sin \theta$ and $< \nu_a | \nu_\mu > = < \nu_b | \nu_e > = \cos \theta$, the expression (14) for the probability reduces to the form

$$P(\nu_e \rightarrow \nu_\mu; X, T) = \sin^2 \theta \cos^2 \theta \left\{ \Psi_a(X, T) \Psi_a^*(X, T) + \Psi_b(X, T) \Psi_b^*(X, T) \right\}$$

$$- \sin^2 \theta \cos^2 \theta \left\{ \Psi_a(X, T) \Psi_b^*(X, T) + \Psi_b(X, T) \Psi_a^*(X, T) \right\}$$ (15)

This expression expresses the probability of a $\nu_e$ produced at $(x_i, t_i)$ to be converted to a $\nu_\mu$ at some other spactime point $(x_f, t_f)$. In actual experiments only the distance $X$ between
the source and the detector is known but the arrival time of the neutrinos is not measured. The expression for the conversion probability as a function of only $X$ is obtained by taking the time average of (15). The normalisation is chosen so that $\Psi_a(X,T)$ represents a single particle wave-function and therefore,

$$\int_{-\infty}^{+\infty} dT \Psi_a(X,T)\Psi_a^*(X,T) = 1$$  \hspace{1cm} (16)$$

The non-trivial contribution to the time averaging comes from the interference term of (15)

$$\text{Re} \int_{-\infty}^{+\infty} dT \Psi_a(X,T)\Psi_b^*(X,T) = \text{Re} \int_{-\infty}^{+\infty} dT \exp \left\{ -i(E_a - E_b)T + i(P_a - P_b)X \right\}$$

$$\times \exp \left\{ \frac{(X - v_a T)^2}{4(\sigma_x^2 + v_a^2 \sigma_t^2)} - \frac{(X - v_b T)^2}{4(\sigma_x^2 + v_b^2 \sigma_t^2)} \right\}$$  \hspace{1cm} (17)$$

This integral can be evaluated by the completing the squares, to give

$$\text{Re} \int_{-\infty}^{+\infty} dT \Psi_a(X,T)\Psi_b^*(X,T) = \cos \left\{ \left( (E_a - E_b) \frac{(v_a + v_b)}{(v_a^2 + v_b^2)} - (P_a - P_b) \right)X \right\}$$

$$\times \exp \left\{ -\frac{1}{(v_a^2 + v_b^2)} (E_a - E_b)^2 \bar{\sigma}^2 - \frac{(v_a - v_b)^2 X^2}{4\bar{\sigma}^2} \right\}$$  \hspace{1cm} (18)$$

where

$$\bar{\sigma} \equiv (\sigma_x + \frac{(v_a^2 + v_b^2)}{(v_a + v_b)} \sigma_t).$$  \hspace{1cm} (19)$$

We express the energy, momentum and the masses in terms of their averages $E = (E_a + E_b)/2, P = (P_a + P_b)/2, m = (m_a + m_b)/2$ and differences $\Delta E = (E_a - E_b), \Delta P = (P_a - P_b), \Delta m = (m_a - m_b)$. In terms of these variables (18) reduces to the form

$$\frac{\sqrt{2}}{(v_a^2 + v_b^2)^{1/2}} \cos \left( (E\Delta E - P\Delta P) \frac{X}{P} \right) e^{-A},$$

$$A = (\Delta E)^2 \frac{\bar{\sigma}^2}{2} \left( \frac{E^2}{P^2} \right) + (\frac{\Delta P^2}{P^2})^2 \frac{X^2}{4\bar{\sigma}^2}$$  \hspace{1cm} (20)$$

where we have retained the terms to the first order in $\Delta E/E, \Delta P/P$ and $\Delta m/m$. The energy and momenta of the neutrinos are determined by the energy-momentum conservation laws at the production vertex. Since neither the energy nor the momenta of the remaining
outgoing particles are measured, one cannot fix either the energy or the momenta of the
different neutrino mass eigenstates in the linear combination state. So one cannot assume
either \( E_a = E_b \) or \( P_a = P_b \). Only the mass shell relation \( E_a^2 = P_a^2 + m_a^2 \) for each mass
eigenstate \( \nu_a \) (\[3\]). In terms of the average and difference, the mass shell conditions imply,

\[
E \Delta E - P \Delta P = \frac{\Delta m^2}{2}
\]  
(22)

Using the relation (22) in (21) we see that the interference term reduces to the form

\[
\frac{\sqrt{2}}{(v_a^2 + v_b^2)^{1/2}} \cos \left( \frac{\Delta m^2}{2P} X \right) e^{-A}
\]  
(23)

we see that the oscillation length is \( L_{osc} = (4\pi P/\Delta m^2) \) for both relativistic as well as non-
relativistic particles. The significant difference in the formula comes from the suppression
factor \( A \) (21). For interference term to be non-zero, both the terms of the suppression
factor must be small. The second term in (21) is \( << 1 \) as long as \( X << L_{coh} \) with the
coherence length for relativistic neutrinos given by ,

\[
L_{coh} = 4\sqrt{2}(\sigma_x + v \sigma_t) \left( \frac{E^2}{\Delta m^2} \right)
\]  
(24)

where \( v = (v_a^2 + v_b^2)/(v_a + v_b) \simeq 1 \). This expression for the coherence length differs from
the expression derived in the standard wave-packet treatments \[1\] by the presence of the \( \sigma_t \)
term. In all accelerator neutrino oscillation experiments the criterion \( X << L_{coh} \) is satisfied
, so the effect of this term is negligible. The first term of (21) plays a more significant role.
For relativistic neutrinos, the suppression factor (21) reduces to the form,

\[
A = \left( \frac{\Delta m^2 (\sigma_x + v \sigma_t)}{2\sqrt{2}E} \right)^2
\]  
(25)

In the accelerator experiments \( \sigma_x \) - the spread in the beam of primary particles is of the
order of a few cm. The neutrinos are produced from the secondary decays of pions and kaons
produced in the primary collision. The uncertainty in time of production of such neutrinos
\( \sigma_t \) is then given by the lifetime \( \tau \) of the pions, kaons or muons which are the source of
neutrinos for that particular experiment. In such situations \( \sigma_x << \sigma_t \) and \( (\sigma_x + v\sigma_t) \approx v\tau \) and the expression (25) reduces to

\[
A \approx \left( \frac{\Delta m^2 \tau}{2\sqrt{2}E} \right)^2. \tag{26}
\]

Using these results we see that the expression for the time average of the conversion probability, for relativistic neutrinos produced from long lived resonances is given by

\[
P(\nu_\mu \rightarrow \nu_e; X) = \frac{1}{2} \sin^2 2\theta \left( 1 - \cos\left( \frac{\Delta m^2 X}{2E} \right) \exp - \left( \frac{\Delta m^2 \tau}{2\sqrt{2}E} \right)^2 \right) \tag{27}
\]

where \( \Delta m^2 \) is the mass square difference, \( X \) is the detector distance, \( \tau \) is the lifetime of the source particle in the lab frame and \( E \) is the neutrino energy. In practice one averages over the energy flux \( n(E) \) of the neutrinos and the average probability which is fitted with the experimental number to obtain the allowed regions of \( \Delta m^2 \) and \( \sin^2 2\theta \) is given formally by the expression,

\[
\langle P(\nu_\mu \rightarrow \nu_e; X) \rangle = \frac{\int dE n(E) P(\nu_\mu \rightarrow \nu_e; X)}{\int dE n(E)} \tag{28}
\]

The limits on the values of \( \Delta m^2 \) and \( \sin^2 2\theta \) obtained by fitting the results of different experiments with the covariant oscillation formula (27) are listed in Table I and plotted in Fig 1, Fig 2 and Fig 3. The regions of the parameter space allowed by the covariant wavepacket formula and by the standard formula of the LSND (muon source) experiment are shown in Fig 1, and the Karmen (pion source) experiment are shown in Fig 2. Since the pion decay length is of the same order as the experimental baseline, the improvement in sensitivity to \( \Delta m^2 \) is marginal compared to LSND. The combined result of all experiments listed in Table I plotted using the covariant oscillation formula is shown in Fig 3.

The dependence of the oscillation term on the source lifetime can be understood as follows. When the time uncertainty is large the uncertainty in energy becomes small and the wave-functions are then eigenstates of energy. In the time smearing the overlap of two different energy eigenstates dissapears and therefore the oscillation term vanishes.

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TABLES

TABLE I. The asymptotic limits on $\Delta m^2$ and $\sin^2 2\theta$ from different experiments according to the oscillation formula (27). $\tau$ is the lifetime of the neutrino source in the lab frame, $< E_\nu >$ is the average $\nu$ energy source in the lab frame, $< E_\nu >$ is the average $\nu$ energy, $X$ is the detector distance and $P$ is the experimental value of the conversion probability.

| Experiment(Source) | $\tau$ (m) | $< E_\nu >$ (MeV) | $X$ (m) | $P$ | $\Delta m^2$ (eV$^2$) | $\sin^2 2\theta$ |
|--------------------|-----------|-----------------|--------|-----|----------------|-----------------|
| LSND ($\mu$)      | 658.6     | 30              | 30     | (0.16 − 0.47) × 10$^{-2}$ | (1.0 − 1.5) × 10$^{-3}$ | 0.003 − 0.009 |
| LSND ($\pi$)      | 17        | 130             | 30     | (0.26 ± 0.15) × 10$^{-2}$ | 0.4 − 0.8 | 0.002 − 0.0082 |
| Karmen ($\pi$)    | 7.8       | 29.8            | 17.5   | < 0.3 × 10$^{-2}$ | < 0.08 | < 0.6 × 10$^{-2}$ |
| E776 ($\pi$)      | 578       | 5 × 10$^3$      | 10$^3$ | < 0.15 × 10$^{-2}$ | < 0.1 | < 0.3 × 10$^{-2}$ |
| CCFR ($K$)        | 5.41 × 10$^3$ | 140 × 10$^3$ | 1.4 × 10$^3$ | < 0.9 × 10$^{-3}$ | < 0.6 | < 0.18 × 10$^{-3}$ |
| Bugey(U,Pu)       | 3 × 10$^{10}$ | 5              | 95     | < 0.75 × 10$^{-1}$ | < 10$^{-9}$ | < 0.15 |
FIG. 1. Lsnd $\mu$ decay at rest experiment allowed regions with the standard formula (between the dotted curves) and the covariant oscillation formula (continuous curves).
FIG. 2. Karmen $\pi$ decay at rest experiment allowed regions with the standard formula (dotted line) and the covariant oscillation formulas (dashed curve).
FIG. 3. The region between the continuous lines is allowed by the LSND $\mu$ experiment [6]. The region between the dotted lines is allowed by the LSND $\pi$ experiment [11]. Region ruled out by E776 [7] is above the dashed-dotted curve and by Karmen [8] is above dashed curve. The region above the top-most dashed curved is ruled out by CCFR [9].
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