TEICHMÜLLER SPACE HAS NON-BUSEMANN POINTS

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Abstract. In this paper, we shall show that the metric boundary of the Teichmüller space with respect to the Teichmüller distance contains non-Busemann points when the complex dimension of the Teichmüller space is at least two.

1. Introduction

1.1. Metric boundary and horofunction boundary. The metric boundary of a metric space was defined by M. Rieffel in [20] as the boundary of a metric compactification. The metric compactification of a metric space \((M, \rho)\) with the base point \(x_0 \in M\) is the compactification given via Gelfand’s theorem as the maximal ideal space of the \(C^*\)-algebra generated by constant functions, continuous functions vanishing at infinity, and continuous functions which form

\[ \varphi_y(x) = \rho(x, x_0) - \rho(x, y) \]

for all \(y \in M\). He observed that the metric compactification is naturally identified with the compactification given by M. Gromov in [6], which recently called the horofunction compactification (cf. §4 in [20]. See also §8.12 of Chapter II in [2]).

In [20], he also defined geodesic-like sequences in a metric space with the base point, which called almost geodesics (cf. §3.2). He observed that any almost geodesic admits the limit in the metric boundary. He defined Busemann points in the metric boundary as the limits of almost geodesics, and posed a question which asks to determine whether every point in the metric boundary of a given metric space is a Busemann point (see the paragraph after Definition 4.8 in [20]). For this problem, C. Webster and A. Winchester [23] gave geometric conditions which determine whether or not every point on the metric boundary of a graph with the standard path metric is a Busemann point, and an example of a graph which admits non-Busemann points in its metric boundary.

1.2. Results. Let \(X\) be a Riemann surface of type \((g, n)\) with \(2g - 2 + n > 0\). The Teichmüller space \(T(X)\) of \(X\) is a quasiconformal deformation space of marked Riemann surfaces with same type as \(X\). Teichmüller space \(T(X)\) admits a canonical distance, called the Teichmüller distance \(d_T\) (cf. §2.3).

The aim of this paper is to show the following.

Theorem 1.1 (Non-Busemann points). When \(3g - 3 + n \geq 2\), the metric boundary of the Teichmüller space with respect to the Teichmüller distance contains non-Busemann points.

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When $3g - 3 + n = 1$, the Teichmüller space equipped with the Teichmüller distance is isometric to the Poincaré hyperbolic disk. Hence, every point in the metric boundary is a Busemann point. Furthermore, in this case, the metric boundary of the Teichmüller space equipped with the Teichmüller space coincides with the Thurston boundary (cf. e.g. [16]).

Recently, in [22], C. Walsh defined the horofunction boundaries for asymmetric metric spaces, and observed that the horofunction boundary of the Teichmüller space with respect to the Thurston’s (non-symmetrized) Lipschitz metric is canonically identified with the Thurston boundary. He also showed that every point in the Thurston boundary is a Busemann point with respect to the Thurston’s Lipschitz metric (cf. Theorem 4.1 of [22]).

The Thurston’s Lipschitz metric is the length spectrum asymmetric metric with respect to the hyperbolic lengths of simple closed curves, meanwhile the Teichmüller distance is recognized as the length spectrum metric with respect to the extremal lengths of simple closed curves via Kerckhoff’s formula (cf. (2.5). See also [11]). Since hyperbolic lengths and extremal lengths are fundamental geometric quantities in the Teichmüller theory, it is natural to compare properties of these two distances. Theorem [11] and Walsh’s results above imply that the asymptotic geometry with respect to the Teichmüller distance is more complicated than that with respect to the Thurston’s Lipschitz metric.

It is known that the metric boundary of a complete CAT(0)-space consists of Busemann points (cf. Corollary II.8.20 of [2]). Therefore, we conclude the following which is already well-known (cf. [14]).

**Corollary 1.1.** When $3g - 3 + n \geq 2$, the Teichmüller space equipped with the Teichmüller distance is not a CAT(0)-space.

1.3. **The Gardiner-Masur boundary.** Let $S$ be the set of homotopy classes of non-trivial and non-peripheral simple closed curves on $X$. We denote by $\text{Ext}_y(\alpha)$ the extremal length of $\alpha$ for $y \in T(X)$ (cf. §2.3.1). In a beautiful paper [5], F. Gardiner and H. Masur proved that the mapping

$$T(X) \ni y \mapsto (S \ni \alpha \mapsto \text{Ext}_y(\alpha)^{1/2}) \in \mathbb{P}\mathbb{R}^S_+$$

is an embedding and the image is relatively compact, where $\mathbb{R}_+ = \{x \geq 0\}$ and $\mathbb{P}\mathbb{R}^S_+ = (\mathbb{R}^S_+ - \{0\})/\mathbb{R}_{>0}$. The closure of the image is called the Gardiner-Masur compactification and the Gardiner-Masur boundary $\partial_{\text{GM}}T(X)$ is the complement of the image from the Gardiner-Masur compactification. They showed that the Gardiner-Masur boundary contains the space $\mathcal{PMF}$ of projective measured foliations (cf. Theorem 7.1 in [5]).

In [12], L. Liu and W. Su have shown that the horofunction boundary with respect to the Teichmüller distance is canonically identified with the Gardiner-Masur boundary of Teichmüller space. Hence, to conclude Theorem [11] we will show the following.

**Theorem 1.2** (Non-accessibility via almost geodesics). When $3g - 3 + n \geq 2$, the projective class of a maximal rational measured foliation cannot be the limit of any almost geodesic in the Gardiner-Masur compactification.

In contrast, from Theorem 7.1 in [5] and Theorem 3 in [18], when a measured foliation $G$ is either a weighted simple closed curve or a uniquely ergodic measured
foliation, the projective class \([G]\) is the limit of the Teichmüller ray associated to \([G]\), and hence it is a Busemann point with respect to the Teichmüller distance.

In \[17\], the author have already observed that any Teichmüller geodesic ray does not converge to the projective class \([G]\) when \(G\) is a rational foliation whose support consists of at least two curves. However, the author does not know whether this induces Theorem 1.2.

This paper is organized as follows. In §2, we recall the definitions and properties of ingredients in the Teichmüller theory, including the extremal length and the Teichmüller distance. In §3, we discuss the metric boundaries of metric spaces, and check that any almost geodesic converges in the Gardiner-Masur compactification. Though this convergence follows from properties of the metric boundary and Liu and Su’s work in \[12\], we shall give a simple proof of the convergence from the Teichmüller theory for the completeness of readers.

We treat measured foliations whose projective classes are the limits of almost geodesics in §4 and §5. Indeed, in §5, we will observe that when a measured foliation whose projective class is the limit of an almost geodesic has a foliated annulus as its component, any simple closed curve is not so twisted in the characteristic annulus corresponding to the foliated annulus through the almost geodesic (cf. Lemma 5.2). This is a key for getting our result. In §6, we give the proof of Theorem 1.2 by contradiction. Indeed, under the assumption that the projective class of maximal measured foliation \(G\) is the limit of an almost geodesic, we calculate the limit of the intersection number function with respect to \(G\). For getting the limit, we will apply the Kerckhoff’s calculation in \[9\] of the extremal length along the Teichmüller ray. One of the reason why the Kerckhoff’s calculation works is such non-twisted property of simple closed curves along the core curve of the characteristic annuli discussed in §5 (see \[8\]).

2. Extremal length and Teichmüller theory

2.1. Extremal length. Let \(\Gamma\) be a family of rectifiable curves on a Riemann surface \(R\). The extremal length of \(\Gamma\) (on \(R\)) is defined by

\[
\text{Ext}(\Gamma) = \sup_\rho \frac{L_\rho(\Gamma)^2}{A(\rho)}
\]

where supremum runs over all measurable conformal metric \(\rho = \rho(z)|dz|^2\) and

\[
L_\rho(\Gamma) = \inf_{\gamma \in \Gamma} \int_\gamma \rho(z)^{1/2}|dz| \quad \text{and} \quad A(\rho) = \int_R \rho(z)\,dxdy.
\]

The extremal length is a conformal invariant in the sense that

\[
\text{Ext}(h(\Gamma)) \leq K \text{Ext}(\Gamma)
\]

for a \(K\)-quasiconformal mapping \(h : R \to h(R)\), a Riemann surface \(R\), and a family \(\Gamma\) of rectifiable curves on \(R\).

**Proposition 2.1** (See \[1\]). Let \(\Gamma_1\) and \(\Gamma_2\) be two families of rectifiable curves on a Riemann surface \(R\).

1. If any curve in \(\Gamma_1\) is contained in a subdomain \(D_1\) of \(R\), the extremal length of \(\Gamma_1\) on \(R\) is equal to the extremal length of \(\Gamma_1\) on \(D_1\).
2. If any curve in \(\Gamma_2\) contains a curve in \(\Gamma_1\), \(\text{Ext}(\Gamma_1) \leq \text{Ext}(\Gamma_2)\).
(3) If curves of $\Gamma_1$ and $\Gamma_2$ are mutually disjoint, $\text{Ext}(\Gamma_1 + \Gamma_2) \geq \text{Ext}(\Gamma_1) + \text{Ext}(\Gamma_2)$.

where $\Gamma_1 + \Gamma_2 = \{\gamma_1 + \gamma_2 \mid \gamma_i \in \Gamma_i\}$.

2.1.1. Extremal length and modulus of annulus. For an annulus $A$, we denote by $\text{Ext}(A)$ the extremal length of the family of simple closed curves which homotopic to the core curve of $A$. The *modulus* of $A$ is the reciprocal of the extremal length of $A$. If $A$ is conformally equivalent to the flat annulus $\{r_1 < |z| < r_2\}$, it holds that $\text{Mod}(A) = (\log(r_2/r_1))/2\pi$.

**Proposition 2.2** (cf. Proposition 9.1 of [17]). Let $A$ be an annulus. Let $\{\beta_k\}_{k=1}^N$ be mutually disjoint Jordan arcs joining components of $\partial A$ such that $\beta_{k-1}$ and $\beta_{k+1}$ divides $\beta_k$ from the other arcs (set $\beta_{N+1} = \beta_1$). Let $\Gamma_k$ be the set of paths in $A - \bigcup_{j=1}^N \beta_j$ connecting $\beta_k$ and $\beta_{k+1}$. Let $\rho$ be the extremal metric for $\text{Ext}(A)$ on $A$ such that $A(\rho) = 1$. Suppose that the $\rho$-length of $\beta_k$ is bounded for all $k = 1, \cdots, N$. Then,

$$\text{Ext}(A)^{1/2} \leq \left( \sum_{k=1}^N \text{Ext}(\Gamma_k) \right)^{1/2} + B$$

where $B$ is the totality of $\rho$-lengths of $\beta_k$’s.

2.1.2. Extremal lengths of simple closed curves. For a Riemann surface $Y$ and a simple closed curve $\beta$ on $Y$, we define the extremal length $\text{Ext}_Y(\beta)$ of $\beta$ on $Y$ is the extremal length of the family of rectifiable closed curves on $Y$ homotopic to $\beta$. The extremal length is characterized geometrically as

$$(2.3) \quad \text{Ext}_Y(\beta) = 1/\sup\{\text{Mod}(A) \mid A \text{ runs all annuli on } Y \text{ whose core is homotopic to } \beta\}$$

where $A$ runs all annuli on $Y$ whose core is homotopic to $\beta$ (cf. e.g. [9] and [21]).

2.2. Measured foliations. The formal product $\mathbb{R}_+ \otimes S = \{t\alpha \mid t \geq 0, \alpha \in S\}$ is embedded into $\mathbb{R}_+^2$ via the intersection number function:

$$\mathbb{R}_+ \otimes S \ni t\alpha \mapsto [\beta \mapsto t i(\alpha, \beta)] \in \mathbb{R}_+^S.$$ 

The closure $\mathcal{MF} = \mathcal{MF}(X)$ of the image in $\mathbb{R}_+^S$ is called the space of measured foliations on $X$. The space $\mathcal{PMF} = \mathcal{PMF}(X)$ of projective measured foliations is the quotient space $(\mathcal{MF} - \{0\})/\mathbb{R}_+$. It is known that $\mathcal{MF}$ and $\mathcal{PMF}$ are homeomorphic to $\mathbb{R}^{6g-6+2n}$ and $S^{6g-7+2n}$, respectively (cf. [3]). It is also known that when we put $i(t\alpha, s\beta) = ts i(\alpha, \beta)$ for $t, s \in \mathbb{R}_+ \otimes S$, the intersection number function extends continuously on $\mathcal{MF} \times \mathcal{MF}$. To a measured foliation $G$, we associate a singular foliation and a transverse measure to the underlying foliation (cf. [3]). In this paper, we denote by $\int_\beta G$ the integration of the corresponding transverse measure over a path $\beta$.

A measured foliation $G$ is called *rational* if $G$ satisfies

$$i(\beta, G) = \sum_{i=1}^k w_i i(\beta, \alpha_i)$$

for some $w_i > 0$ and $\alpha_i \in S$ such that $i(\alpha_i, \alpha_j) = 0$ and $\alpha_i \neq \alpha_j$ for $i \neq j$. We write $G = \sum_{i=1}^k w_\alpha \alpha_k$ for such measured foliation. A rational measured foliation $G = \sum_{i=1}^k w_\alpha \alpha_k$ is *maximal* if any component of $X - \bigcup_{i=1}^k \alpha_i$ is a pair of pants. In this case, $k = 3g - 3 + n$. 
In [9], S. Kerckhoff showed that when we put $\text{Ext}_X(t\beta) = t^2\text{Ext}_X(\beta)$ for $t \beta \in \mathbb{R}_+ \otimes \mathcal{S}$, the extremal length extends continuously on $\mathcal{MF}$. We define
\[ \mathcal{MF}_1 = \{ F \in \mathcal{MF} \mid \text{Ext}_X(F) = 1 \}, \]
which is homeomorphic to $\mathcal{PMF}$ via the projection $\mathcal{MF} \to \mathcal{PMF}$.

In [19], Y. Minsky showed the following inequality, which recently called the Minsky’s inequality:

(2.4) \[ i(F, G)^2 \leq \text{Ext}_X(F) \text{Ext}_X(G) \]

for all $F, G \in \mathcal{MF}$ (cf. Lemma 5.1 of [19]). From Theorem 5.1 in [5], Minsky’s inequality is sharp in the sense that for any $G \in \mathcal{MF} - \{0\}$, there is an $F \in \mathcal{MF}$ which satisfies the equality in (2.4).

2.3. Teichmüller space. The Teichmüller space $T(X)$ of $X$ is the set of equivalence classes of marked Riemann surfaces $(Y, f)$ where $Y$ is a Riemann surface and $f : X \to Y$ a quasiconformal mapping. Two marked Riemann surfaces $(Y_1, f_1)$ and $(Y_2, f_2)$ are Teichmüller equivalent if there is a conformal mapping $h : Y_1 \to Y_2$ which homotopic to $f_2 \circ f_1^{-1}$. Throughout this paper, we consider the Teichmüller space as a pointed space with the base point $x_0 = (X, id)$.

2.3.1. Teichmüller distance and Kerckhoff’s formula. The Teichmüller distance between $y_1 = (Y_1, f_1)$ and $y_2 = (Y_2, f_2) \in T(X)$ is, by definition, the half of the logarithm of the extremal quasiconformal mapping between $Y_1$ and $Y_2$ preserving markings.

In [9], S. Kerckhoff gave the geometric interpretation of the Teichmüller distance by using the extremal lengths of measured foliations as follows. For $F \in \mathcal{MF}$ and $y = (Y, f) \in T(X)$, we define the extremal length of $F$ on $y$ by

$\text{Ext}_y(F) = \text{Ext}_Y(f(F))$.

Then, the following equality holds:

(2.5) \[ d_T(y_1, y_2) = \frac{1}{2} \log \sup_{H \in \mathcal{MF} - \{0\}} \frac{\text{Ext}_{y_1}(H)}{\text{Ext}_{y_2}(H)} = \frac{1}{2} \log \max_{H \in \mathcal{MF}, \text{Ext}_y(H) > 0} \frac{\text{Ext}_{y_1}(H)}{\text{Ext}_{y_2}(H)} \]

The Teichmüller space is topologized with the Teichmüller distance. Under this topology, the extremal length of a measured foliation varies continuously on $T(X)$ from the conformal invariance (2.2).

2.3.2. Quadratic differentials and Hubbard-Masur’s theorem. For a holomorphic quadratic differential $q = q(z)dz^2$ on a Riemann surface $Y$, we define a singular flat metric $|q| = |q(z)||dz|^2$. We call here this metric the $q$-metric.

In [7], Hubbard and Masur observed that for $y = (Y, f) \in T(X)$ and $G \in \mathcal{MF} - \{0\}$, there is a unique holomorphic quadratic differential $J_{G, y}$ on $Y$ whose vertical foliation is equal to $f(G)$. Namely,

$\inf_{\beta \in \mathcal{S}} \int_{\beta} |\text{Re}\sqrt{|J_{G, y}|}|$

holds for all $\beta \in \mathcal{S}$. In this case, we can see that

$\text{Ext}_y(G) = ||J_{G, y}|| = \int_Y |J_{G, y}|.$

Namely, the extremal length is the area of the $J_{G, y}$-metric. When $G = \beta \in \mathcal{S}$, we call the differential $J_{\beta, y}$ the Jenkins-Strebel differential for $\beta$. 
2.4. Teichmüller rays. Let \( x = (X, f) \in T(X) \) and \([G] \in \mathcal{PMF}\). By Ahlfors-Bers theorem, we can define an isometric embedding

\[ [0, \infty) \ni t \mapsto R_{G, x_0}(t) \in T(X) \]

with respect to the Teichmüller distance by assigning the solution of the Beltrami equation defined by the Teichmüller Beltrami differential

\[
\tanh(t) \frac{|J_{G, x_0}|}{J_{G, x_0}}
\]

for \( t \geq 0 \). We call \( R_{G, x_0} \) the Teichmüller (geodesic) ray associated to \([G] \in \mathcal{PMF}\).

Notice that the differential (2.6) depends only on the projective class of \( G \). It is known that

\[ \mathcal{PMF} \times [0, \infty) \ni ([G], t) \mapsto R_{G, x_0}(t) \in T(X) \]

is a homeomorphism (cf. [8]). One can see that

\[
\text{Ext}_{R_{G, x_0}(t)}(G) = e^{-2t} \text{Ext}_{x_0}(G)
\]

for \( G \in \mathcal{MF} \).

2.5. Gardiner-Masur boundary revisited. For \( y \in T(X) \), we let \( K_y = e^{2d_T(x_0, y)} \). Consider a continuous function on \( \mathcal{MF} \)

\[ \mathcal{E}_y(F) = \left( \frac{\text{Ext}_y(F)}{K_y} \right)^{1/2} \]

for \( y \in T(X) \). Then, in [17], the author observed that for any \( p \in \partial_G T(X) \), there is a function \( \mathcal{E}_p \) on \( \mathcal{MF} \) such that the function \( S \ni \beta \mapsto \mathcal{E}_p(\beta) \) represents \( p \) and when a sequence \( \{y_n\}_n \subset T(X) \) converges to \( p \) in the Gardiner-Masur compactification, there are \( t_0 > 0 \) and a subsequence \( \{y_{n_j}\}_j \) such that \( \mathcal{E}_{y_{n_j}} \) converges to \( \mathcal{E}_p \) uniformly on any compact set of \( \mathcal{MF} \).

3. Metric boundary and horofunction boundary

3.1. Metric boundary and horofunction boundary. Let \((M, \rho)\) be a locally compact metric space. Let \( C(M) \) be the space of continuous functions on \( M \), equipped with the topology of uniform convergence on compact subsets of \( M \). Let \( C^*_u(M) \) be the quotient space of \( C(M) \) via constant functions. For \( y \in M \) we set \( \psi_y(x) = \rho(x, y) \). Then, \( M \ni y \mapsto \psi_y \) is a continuous embedding into \( C(M) \). This embedding descends a continuous embedding into \( C^*_u(M) \). The closure \( \mathcal{Cl}(M) \subset C^*_u(M) \) of the image of this embedding is called the horofunction compactification and the complement \( \mathcal{Cl}(M) \setminus M \) is said to be the horofunction boundary of \( M \) (cf. [1], [2], and [20]). M. Rieffel pointed out that the metric boundary of \( M \) is canonically identified with the horofunction boundary of \( M \) as discussed in the introduction (cf. §4 in [20]).

In [12], L. Liu and W. Su showed that the horofunction compactification of the Teichmüller space with the Teichmüller distance is identified with the Gardiner-Masur compactification.
3.2. Almost geodesics. Let \((M, \rho)\) be a metric space. Let \(T \subset [0, \infty)\) be an unbounded set with \(0 \in T\). A mapping \(\gamma : \mathbb{R} \to M\) is said to be an \textit{almost geodesic} if for any \(\epsilon > 0\) there is a \(N > 0\) such that for all \(t, s \in T\) with \(t \geq s \geq N\),
\[
|\rho(\gamma(t), \gamma(s)) + \rho(\gamma(s), \gamma(0)) - t| < \epsilon
\]
(cf. Definition 4.3 of [20]). By definition, any geodesic ray is an almost geodesic. When \((M, \rho)\) is a pointed metric space, we assume in addition that \(\gamma(0)\) is the base point (cf. the assumption of Lemma 4.5 in [20]). By definition, for an unbounded subset \(T_0 \subset T\) with \(0 \in T_0\), the restriction \(\gamma|_{T_0} : T_0 \to M\) is also an almost geodesic. We call the restriction a \textit{subsequence} of an almost geodesic \(\gamma : T \to M\). A point of the metric boundary or the horofunction boundary of \(M\) is said to be a \textit{Busemann point} if it is the limit of an almost geodesic (cf. Definition 4.8 of [20]).

3.3. Convergence of almost geodesics. In this section, we shall check that any almost geodesic in \(T(X)\) converges in the Gardiner-Masur compactification. Though this follows from a fundamental property of the metric boundary (cf. [20]) and Liu and Su’s work [12], we now try to give a simple proof from the Teichmüller theory and it seems to be intriguing in itself. Notice that the author observed in [18] that any Teichmüller ray \(R_{G,x}(t)\) admits the limit for all \([G] \in \mathcal{PMF}\) by the different idea.

Let \(\gamma : T \to T(X)\) be an almost geodesic with the base point \(x_0 \in T(X)\). By definition, \(\gamma\) satisfies that \(\gamma(0) = x_0\) and for any \(\epsilon > 0\), there is an \(N > 0\) such that
\[
|d_T(\gamma(t), \gamma(s)) + d_T(\gamma(s), \gamma(0)) - t| < \epsilon
\]
for all \(t \geq s \geq N\). From Kerckhoff’s formula (2.5), (3.1) is equivalent to
\[
e^{t-\epsilon} \leq \max_{H \in \mathcal{MF}} \frac{\text{Ext}_{\gamma(t)}(H)^{1/2}}{\text{Ext}_{\gamma(s)}(H)^{1/2}} \cdot K^{1/2}_{\gamma(t)} \leq e^{t+\epsilon}.
\]

In particular, we have
\[
e^{t-\epsilon} \leq K^{1/2}_{\gamma(t)} \leq e^{t+\epsilon}
\]
when we set \(s = t\) in (3.2). Therefore, we deduce
\[
\frac{\text{Ext}_{\gamma(t)}(H)^{1/2}}{\text{Ext}_{\gamma(s)}(H)^{1/2}} \cdot K^{1/2}_{\gamma(s)} \leq e^{t} \cdot e^{\epsilon} \leq e^{\epsilon} \cdot e^{t} \cdot K^{1/2}_{\gamma(t)},
\]
and hence
\[
E_{\gamma(t)}(H) \leq e^{2\epsilon} E_{\gamma(s)}(H)
\]
for all \(H \in \mathcal{MF}\) and \(t \geq s \geq N\).

We set
\[
E'(F) = \inf_{t \in T} E_{\gamma(t)}(F)
\]
for \(F \in \mathcal{MF}\). From (3.4), for all \(\beta \in S\), the limit of any converging subsequence in \(\{E_{\gamma(t)}(\beta)\}_{t \in T}\) coincides with \(E'(\beta)\), which implies that \(\gamma : T \to T(X)\) converges in the Gardiner-Masur compactification as \(t \to \infty\).
4. Measured foliations as Busemann points

Let \( G \in \mathcal{MF}_1 \). Suppose that the projective class \([G]\) is a Busemann point in the horofunction compactification of Teichmüller space with respect to the Teichmüller metric. By definition and Liu and Su’s work \[12\], there is an almost-geodesic \( \gamma : T \to T(X) \) such that \( \gamma(t) \to [G] \) in the Gardiner-Masur closure. This means that there is a \( t_0 > 0 \) such that \( E_{\gamma(t)} \) converges to \( t_0 i(\cdot, G) \) uniformly on any compact sets of \( \mathcal{MF} \). We take \( G_t \in \mathcal{MF}_1 \) with \( R_{G_t, x_0}(d_T(x_0, \gamma(t))) = \gamma(t) \).

Lemma 4.1. Under the notation above, it holds \( t_0 = 1 \).

Proof. Let \( G_\infty \) be an accumulation point of \( \{G_t\}_{t \in T} \). By taking a subsequence if necessary, we may assume that \( G_t \) converges to \( G_\infty \). Let \( \alpha \in S \). From (2.7) and (3.3), we have

\[
i(\alpha, G_\infty) = \lim_{T \ni t \to \infty} i(\alpha, G_t) \leq \lim_{T \ni t \to \infty} \text{Ext}_{\gamma(t)}(\alpha)^{1/2} \text{Ext}_{\gamma(t)}(G_t)^{1/2}
\]

\[
\leq e^{\epsilon} \lim_{T \ni t \to \infty} e^{-\epsilon \text{Ext}_{\gamma(t)}(\alpha)^{1/2}} \leq e^{2\epsilon} \lim_{T \ni t \to \infty} E_{\gamma(t)}(\alpha)
\]

(4.1)

Since \( \epsilon > 0 \) is taken arbitrary, we get

(4.2)

\[i(\alpha, G_\infty) \leq i(\alpha, t_0 G)\]

for all \( \alpha \in S \). Thus, it follows from the Marden-Strebel’s minimal norm property that

(4.3) \[1 = \text{Ext}_{x_0}(G_\infty) = \|J_{G_\infty, x_0}\| \leq \|J_{t_0 G, x_0}\| = t_0^2 \|J_{G, x_0}\| = t_0^2,\]

and hence \( t_0 \geq 1 \) (see Theorem 3.2 of [13]. See also [4]).

From (3.3), by dividing every term in (5.2) by \( K^{1/2}_t = e^{d_T(x_0, \gamma(t))} \) and letting \( t \to \infty \), we get

(4.4)

\[e^{-2\epsilon} \leq \max_{H \in \mathcal{MF}_1} \frac{t_0 i(H, G)}{\text{Ext}_{\gamma(t)}(H)^{1/2}} \max_{H \in \mathcal{MF}_1} \text{Ext}_{\gamma(s)}(H)^{1/2} \leq e^{2\epsilon}\]

for \( s \geq N \). From Minsky’s inequality (2.3) and Kerckhoff’s formula, we have

\[\max_{H \in \mathcal{MF}_1} \frac{i(H, G)}{\text{Ext}_{\gamma(s)}(H)^{1/2}} = \sup_{H \in \mathcal{MF}_1-\{0\}} \frac{i(H, G)}{\text{Ext}_{\gamma(s)}(H)^{1/2}} = \text{Ext}_{\gamma(s)}(G)^{1/2}\]

and

\[e^{s-\epsilon} \leq \max_{H \in \mathcal{MF}} \text{Ext}_{\gamma(s)}(H)^{1/2} = K(x_0, \gamma(s)) \leq e^{s+\epsilon}.\]

Hence, we get

\[t_0 \text{Ext}_{\gamma(s)}(G)^{1/2} \leq e^{3\epsilon} e^{-s}.\]

On the other hand, from the distortion property, \( \text{Ext}_{\gamma(s)}(G)^{1/2} \geq e^{-\epsilon} e^{-s} \) holds in general. Therefore, we have

(4.5)

\[t_0 e^{-\epsilon} e^{-s} \leq t_0 \text{Ext}_{\gamma(s)}(G)^{1/2} \leq e^{3\epsilon} e^{-s}\]

and \( t_0 \leq 1 \), which is what we wanted.

From the proof of the lemma above, we also observe the following.

Corollary 4.1. \( G_t \) converges to \( G \) as \( t \to \infty \).
Proof. Let $G_\infty$ be an accumulation point of $\{G_t\}_{t \in T}$ as above. Recall from (1.2) and Lemma 4.1 above that $i(\alpha, G_\infty) \leq i(\alpha, G)$ for all $\alpha \in S$. Since $\|J_{G_\infty, x_0}\| = 1 = \|J_{G, x_0}\|$, by the calculation in (1.3) and the conclusion from the equality of the minimal norm property, we get $J_{G_\infty, x_0} = J_{G, x_0}$ and $G_\infty = G$.

Notice from (4.5) and Lemma 4.1 that
\begin{equation}
\lim_{t \to \infty} \|J_{G, \gamma(t)}\| \cdot K_{\gamma(t)} = \lim_{t \to \infty} \text{Ext}_{\gamma(t)}(G) \cdot K_{\gamma(t)} = 1.
\end{equation}

5. Measured foliations with foliated annuli

In this section, we devote to give asymptotic behaviors of moduli of characteristic annuli corresponding to foliated annuli and the twisting number of closed geodesics on the characteristic annuli. These observations will be used for proving Theorem 1.2 in the next section.

As the previous section, we continue to suppose that the projective class $[G]$ of $G \in \mathcal{MF}_1$ is the limit of an almost geodesic $\gamma: T \to T(X)$. Throughout this section, we suppose in addition that $G$ has a component of a foliated annulus with core $\alpha \in S$. Namely, $G = w_0 \alpha + F$ for some $w_0 > 0$ and $F \in \mathcal{MF}$. For the simplicity, we set $J_t = J_{G, \gamma(t)}$. Let $\gamma(t) = (Y_t, f_t)$ for $t \in T$ and $A_t \subset Y_t$ be the characteristic annulus of $J_t$ for $\alpha$.

We now fix a notation. For two functions $f(t)$ and $g(t)$ with variable $t$, $f(t) \asymp g(t)$ means that $f(t)$ and $g(t)$ are comparable in the sense that there are positive numbers $B_1$ and $B_2$ independent of the parameter $t$ such that $B_1 g(t) \leq f(t) \leq B_2 g(t)$.

5.1. Moduli of characteristic annuli. The asymptotic behavior of the modulus of $A_t$ is given as follows.

**Lemma 5.1.** Mod($A_t$) $\asymp K_{\gamma(t)}$ as $t \to \infty$.

**Proof.** By (2.3),
\[
\text{Mod}(A_t) \leq 1/\text{Ext}_{\gamma(t)}(\alpha) \leq K_{\gamma(t)}/\text{Ext}_{x_0}(\alpha)
\]
for all $t \in T$. In addition, by (1.6),
\[
1/\text{Mod}(A_t) = \ell_{J_t}(\alpha)/w_0 = w_0^{-2}\text{area of } A_t.
\]
\[
\leq w_0^{-2}\|J_t\| = w_0^{-2}\text{Ext}_{\gamma(t)}(G) \asymp K_{\gamma(t)}^{-1}
\]
as $t \to \infty$. □

5.2. Twisting numbers of paths in flat annuli. We here define the twisting numbers of proper paths in flat annuli. Let $S^1_L$ be the Euclidean circle of length $L$. Let $A = [0, m] \times S^1_L$ be a flat annulus. Let $\beta \subset A$ be an (unoriented) path connecting components of $\partial A$. Take a universal cover $[0, m] \times \mathbb{R} \to A$. Let $\tilde{\beta}$ be a lift of $\beta$. Let $(0, y_1), (m, y_2) \in [0, m] \times \mathbb{R}$ be the endpoints of $\tilde{\beta}$. Then, we define a twisting number $\text{tw}_A(\beta)$ of $\beta$ in $A$ by
\[
\text{tw}_A(\beta) = |y_1 - y_2|/L.
\]
One can easily check that the twisting number is defined independently of the choice of lifts.
5.3. **Twisting numbers of geodesics.** Let $\beta \in S$ with $\beta \neq \alpha$. For $t \in T$, we set $\beta^*_t$ be the geodesic representative of $\beta$ in $Y_t$ with respect to the $J_t$-metric. If $J_t$ admits a flat annulus whose core is homotopic to $\beta$, we choose one of closed trajectories in the flat annulus to define $\beta^*_t$.

Let $\beta^*_t \cap A_t = \{\sigma^1_s\}_{s=1}^{n_0}$ be the set of straight segment in $\beta^*_t$ in the part of $A_t$ counting multiplicity, where $n_0 = i(\beta, \alpha)$. Let $\{\sigma^2_j\}_j$ be a collection of maximal straight segments in $\beta^*_t \setminus \cup_{s=1}^{n_0} \sigma^1_s$, counting multiplicity. In this section, for a measured foliation $F$ and a path $\sigma$ transverse to the underlying foliation of $F$, we define $i(\sigma, F)$ as the infimum of the integrals of the transversal measure of $F$ over all paths homotopic to $\sigma$ rel endpoints.

**Lemma 5.2 (Twisting number).** For $s = 1, \ldots, n_0$, the twisting number of $\sigma^1_s$ in $A_t$ satisfies

$$tw_{A_t}(\sigma^1_s) = o(K_{\gamma(t)})$$

as $t \to \infty$.

**Proof.** When $n_0 = i(\beta, \alpha) = 0$, the geodesic representative $\beta^*_t$ does not intersect the interior of $A_t$. Hence, the conclusion automatically holds. Therefore, we may assume that $n_0 \neq 0$.

Let $q_t = J_t / \|J_t\|$. Then, the vertical foliation $V_{q_t}$ of $q_t$ is equal to $\|J_t\|^{-1/2} G$ for all $t \in T$. Especially, the $q_t$-height $w_t$ of the characteristic annulus $A_t$ is equal to $\|J_t\|^{-1/2} w_0$. Let $H_{q_t}$ be the horizontal foliation of $q_t$. Since each $\sigma^1_s$ is a $q_t$-straight segment,

$$i(\sigma^1_s, V_{q_t}) = \int_{\sigma^1_s} V_{q_t} \quad \text{and} \quad i(\sigma^1_s, H_{q_t}) = \int_{\sigma^1_s} H_{q_t}$$

for $j = 1, 2$. Hence,

$$\ell_{q_t}(\beta^*_t) = \sum_{s=1}^{n_0} \sqrt{i(\sigma^1_s, H_{q_t})^2 + i(\sigma^1_s, V_{q_t})^2} + \sum_j \sqrt{i(\sigma^2_j, H_{q_t})^2 + i(\sigma^2_j, V_{q_t})^2}. \tag{5.1}$$

Since $\|q_t\| = 1$, $\ell_{q_t}(\beta^*_t) \leq \text{Ext}_{\gamma(t)}(\beta)^{1/2}$ from (2.1). Therefore,

$$\|J_t\|^{-1/2} i(\beta, G) = i(\beta, V_{q_t}) \leq \ell_{q_t}(\beta^*_t)$$

$$= \sum_{s=1}^{n_t} \sqrt{i(\sigma^1_s, H_{q_t})^2 + i(\sigma^1_s, V_{q_t})^2} + \sum_j \sqrt{i(\sigma^2_j, H_{q_t})^2 + i(\sigma^2_j, V_{q_t})^2}$$

$$= \sum_{s=1}^{n_t} \sqrt{i(\sigma^1_s, H_{q_t})^2 + \|J_t\|^{-1} w_0^2} + \sum_j \sqrt{i(\sigma^2_j, H_{q_t})^2 + i(\sigma^2_j, V_{q_t})^2}$$

$$\leq \text{Ext}_{\gamma(t)}(\beta)^{1/2}.$$
Thus, we obtain
\[
\begin{align*}
i(\beta, G) &\leq \sum_{s=1}^{n_0} \sqrt{\|J_t\| i(\sigma_s^1, H_{q_t})^2 + w_0^2} \\
&\quad + \sum_j \sqrt{\|J_t\| i(\sigma_j^2, H_{q_t})^2 + \|J_t\| i(\sigma_j^2, V_{q_t})^2} \\
&= \sum_{s=1}^{n_0} \sqrt{i(\sigma_s^1, H_{J_t})^2 + w_0^2} + \sum_j \sqrt{i(\sigma_j^2, H_{J_t})^2 + i(\sigma_j^2, V_{J_t})^2}
\end{align*}
\]
(5.2) \quad \leq \|J_t\|^{1/2} \text{Ext}_{\gamma(t)}(\beta)^{1/2}.

From the assumption, Lemma 4.1 and (4.6),
\[
\|J_t\|^{1/2} \text{Ext}_{\gamma(t)}(\beta)^{1/2} = (1 + o(1)) \text{Ext}_{\gamma(t)}(\beta)^{1/2} K_{\gamma(t)}^{1/2} = (1 + o(1)) \mathcal{E}_{\gamma(t)}(\beta)
\]
tends to \(i(\beta, G)\) as \(t \to \infty\). Since
\[
i(\beta, G) = \int_{\beta_t^1} V_{J_t} = n_0 w_0 + \sum_j i(\sigma_j^2, V_{J_t}),
\]
we deduce from (5.2) that the summation
\[
\sum_{s=1}^{n_0} \left( \sqrt{i(\sigma_s^1, H_{J_t})^2 + w_0^2} - w_0 \right) + \sum_j \left( \sqrt{i(\sigma_j^2, H_{J_t})^2 + i(\sigma_j^2, V_{J_t})^2} - i(\sigma_j^2, V_{J_t}) \right)
\]
tends to zero as \(t \to \infty\). Since every term in (5.3) is non-negative, we get
\[
\lim_{t \to \infty} i(\sigma_s^1, H_{J_t}) = 0
\]
for \(s = 1, \ldots, n_0\).

We now fix \(s = 1, \ldots, n_0\). Let \([0, w_t] \times \mathbb{R} \to [0, w_t] \times S_{\ell_t}^1 \cong A_t\) be the universal cover, where \(\ell_t\) is the \(q_t\)-circumference of \(A_t\). Let \((0, y_1)\) and \((w_t, y_2)\) be the endpoints of a lift of \(\sigma_s^1\). From the definition,
\[
|y_1 - y_2| = i(\sigma_s^1, H_{q_t}).
\]
Since
\[
\text{Mod}(A_t) = w_t/\ell_t = \|J_t\|^{-1/2} w_0/\ell_t = (1 + o(1)) K_{\gamma(t)}^{1/2} w_0/\ell_t,
\]
from Lemma 5.1 we obtain
\[
\ell_t \asymp K_{\gamma(t)}^{-1/2}
\]
(5.5) for \(s = 1, \ldots, n_t\). Thus, it follows from (5.4) that
\[
\text{tw}_{A_t}(\sigma_s^1) = |y_1 - y_2|/\ell_t \asymp i(\sigma_s^1, H_{q_t}) K_{\gamma(t)}^{1/2}
\]
\[
= \|J_t\|^{-1/2} i(\sigma_s^1, H_{J_t}) K_{\gamma(t)}^{1/2} = (1 + o(1)) i(\sigma_s^1, H_{J_t}) K_{\gamma(t)}
\]
\[
= o(K_{\gamma(t)}),
\]
which implies what we wanted.
5.4. **Twisting deformations on flat annuli.** In this section, we shall recall a canonical quasiconformal mapping of the twisting deformations along the core curve on a flat annulus (cf. [15]).

Let \( A = \{ e^{-2\pi m} < |z| < 1 \} \) be a flat annulus of modulus \( m \). For \( t > 0 \), we consider a quasiconformal self-mapping \( W_t \) of \( A \) by

\[
W_t(z) = z |z|^{-\frac{\tau}{2\pi m}}.
\]

Then, the Beltrami differential of \( W_t \) is equal to

\[
\frac{\partial W_t}{\partial \overline{W_t}} = \frac{-i(\tau/m)}{4\pi - i(\tau/m)} \frac{z}{z} \frac{dz}{\overline{dz}}.
\]

We can check that

\[
tw_A(W_t(\sigma)) = tw_A(\sigma) - \tau.
\]

Especially, when a proper path \( \sigma \) in \( A \) has the twist parameter \( \tau \), \( tw_A(W_t(\sigma)) = 0 \).

### 6. Proof of Theorem 1.2

In this section, we shall show Theorem 1.2. Throughout this section, we assume that \( G = \sum_{i=1}^{k} w_i \alpha_i \) is a maximal rational foliation and \( k = 3g - 3 + n \geq 2 \). As before, we also assume that the projective class \([G]\) is the limit of an almost geodesic \( \gamma : T \to T(X) \). We continue to use symbols given in the previous sections.

6.1. **Notation.** Let \( A_{i,t} \subset Y_t \) be the characteristic annulus of \( q_t = J_t/||J_t|| \) for \( \alpha_i \). Let \( \Sigma_i \) be the critical graph of \( q_t \) and consider the \( K_{\gamma(t)}^{-1/2} \)-neighborhood \( N_t \) of \( \Sigma_i \) in \( Y_t \) with respect to the \( q_t \)-metric. Let \( A_{i,t}^0 = A_{i,t} \setminus N_t \) (cf. Figure 1). Since \( q_t \)-height of \( A_{i,t} \) is \((1 + o(1))K_{\gamma(t)}^{1/2}w_t \), when \( t \in T \) is sufficiently large, \( A_{i,t}^0 \) is a well-defined foliated subannulus of \( A_{i,t} \) with height

\[
w_{i,t}^\eta := (1 + o(1))K_{\gamma(t)}^{1/2}w_t - 2K_{\gamma(t)}^{-1/2} \asymp K_{\gamma(t)}^{-1/2}.
\]
Let \( \partial A^0_{i,t} = \partial_1 A^0_{i,t} \cup \partial_2 A^0_{i,t} \). Since \( \partial A^0_{i,t} \) consists of closed leaves in \( A_{i,t} \) and the heights of the remaining annuli in \( A_{i,t} - A^0_{i,t} \) are at most \( K^{-1/2} \) from (5.5), the moduli of remaining annuli in \( A_{i,t} - A^0_{i,t} \) are uniformly bounded, and hence

\[
\text{Mod}(A_{i,t}) = \text{Mod}(A^0_{i,t}) + O(1)
\]
as \( t \to \infty \).

6.2. Calculation of extremal length: Lower estimate. We take \( \beta \in \mathcal{S} \). We here devote to estimate the extremal length of \( \beta \) from below. From now on, we suppose that \( i(\beta,G) \neq 0 \).

Let \( A_t \) be the characteristic annulus of the Jenkins-Strebel differential \( J_{\beta,\gamma}(t) \) for \( \beta \). Fix \( i = 1, \ldots, k \). The intersection \( A_t \cap A^0_{i,t} \) contains at least \( n_i = i(\beta,\alpha_i) \)-components \( \{D_i\}_{i=1}^{n_i} \) such that \( D_i \) contains a path connecting \( A_t \cap \partial_1 A^0_{i,t} \) and \( A_t \cap \partial_2 A^0_{i,t} \).

Let \( \Gamma(D_i) \) be the family of rectifiable curves in \( D_i \) connecting \( A_t \cap \partial_1 A^0_{i,t} \) and \( A_t \cap \partial_2 A^0_{i,t} \). Let \( \rho_i \) be the restriction of the \( q_t \)-metric to \( D' = \bigcup_{i=1}^{n_i} D_i \). From (6.1), any curve in \( \sum_{i=1}^{n_i} \Gamma(D_i) \) has \( \rho_i \)-length at most \( n_i w_{i,t}'' \). Since the critical graph of the Jenkins-Strebel differential of \( \beta \) on \( Y_t \) has measure zero,

\[
A(\rho_i) = \int_{D'} \rho_i \leq (\text{|}q_t\text{-area of } A^0_{i,t}) = \ell_{i,t} w_{i,t}''.
\]

By the definition of the extremal length, we have

\[
\text{Ext} \left( \sum_{i=1}^{n_i} \Gamma(D_i) \right) \geq L_{\rho_i} \left( \sum_{i=1}^{n_i} \Gamma(D_i) \right)^2 / A(\rho_i) \geq (n_i w_{i,t}'')^2 / \ell_{i,t} w_{i,t}'' = n_i^2 \text{Mod}(A^0_{i,t}) = n_i^2 \text{Mod}(A_{i,t}) + O(1).
\]

Since any non-trivial simple closed curve in \( A_t \) traverses each \( D_i \) between \( A_t \cap \partial_1 A^0_{i,t} \) and \( A_t \cap \partial_2 A^0_{i,t} \), such simple closed curve contains a curve in \( \sum_{i=1}^{n_i} \Gamma(D_i) \). Therefore, from (3) of Proposition 2.1, we conclude

\[
\text{Ext}_{\gamma(t)}(\beta) \geq \sum_{i=1}^{k} \text{Ext} \left( \sum_{i=1}^{n_i} \Gamma(D_i) \right) \geq \sum_{i=1}^{k} n_i^2 \text{Mod}(A_{i,t}) + O(1)
\]
as \( t \to \infty \).

6.3. Calculation of extremal length: Upper estimate. Before discussing the upper bound, we deform \( Y_t \) slightly as follows. For \( i = 1, \ldots, k \), we fix a component \( \sigma_{s_i}^1 \) of \( \beta_t^* \cap A_{i,t} \). We put the Beltrami differential [5.0] on each flat annulus \( A_{i,t} \) with \( \tau = tw_{A_{i,t}}(\sigma_{s_i}^1) \). We extend the Beltrami differential to \( Y_t \) by putting 0 on the remaining part. Then, we obtain a quasiconformal deformation of \( Y_t \) with respect to the Beltrami differential to get \( \gamma'(t) \in T(X) \). By Lemmas 5.1 and 5.2

\[
tw_{A_{i,t}}(\sigma_{s_i}^1) / \text{Mod}(A_{i,t}) \to 0
\]
as \( t \to \infty \) for all \( i \), and hence,

\[
d_T(\gamma(t),\gamma'(t)) \to 0
\]
when \( t \to \infty \). This means that \( \gamma'(t) \) has the same limit as that of \( \gamma(t) \) in the Gardiner-Masur compactification. Thus, for simplifying of the notation, we may suppose that \( \gamma'(t) = \gamma(t) \).
Notice from \([5.7]\) that after this deformation, the twist parameter of each \(\sigma^j_t\) is zero. Hence, any segment in \(\beta^j_t \cap A_{i,t}\) has the twisting number at most one in \(A_{i,t}\) for all \(i\), because \(\beta\) is a simple closed curve and any two segments in \(\beta^j_t \cap A_{i,t}\) do not intersect transversely in \(A_{i,t}\). By taking a subsequence, we may assume that there is a (non-connected) graph \(\Sigma_0\) on \(X\) such that the making \(f_t : X \to Y_t\) induces an isomorphism \(\Sigma_0\) and \(\Sigma_t\).

### 6.3.1. The Idea for Getting an Appropriate Upper Bound

To give an upper estimate, from \([2.3]\), it suffices to construct a suitable annulus \(A_t\) on \(Y_t\) whose core is homotopic to \(f_t(\beta)\). The procedure given here is originally due to S. Kerckhoff in \([9]\), when a given almost geodesic \(\gamma\) is actually a geodesic (See also §8 of \([17]\)). We briefly recall the case when \(\gamma\) is a geodesic. We first cut each characteristic annuli \(A_{i,t}\) of \(J_t\) into \(n_i = i(\beta, \alpha_t)\)-congruent horizontal rectangles. The annulus \(A_t\) is made by composing appropriately such (slightly modified) \(n_t\)-congruent horizontal rectangles and ties (quadrilaterals) in \(N_t\) (cf. \([6.5]\)). We can take such ties with uniform extremal length (cf. Claim 1). Then, by applying Proposition 2.2, we obtain an upper bound of the extremal length of \(A_t\).

One of the essential reasons why we can get an appropriate upper bound in the case above is that, through the Teichmüller ray associated to the projective class of \(G = \sum_{i=1}^k w_i \alpha_i\), there are “no” twisting deformation along \(\alpha_t\) on the characteristic annuli, because the Teichmüller deformation is done by stretching in the horizontal and vertical directions. Indeed, the major part of the upper bound comes from the extremal length of congruent rectangles (cf. \([6.5]\)). The ‘no-twisting’ property implies that the totality of the extremal lengths of such rectangles is equal to the major part of the lower estimate \([6.2]\) (cf. \([6.4]\)).

In the case when \(\gamma\) is an almost geodesic, we have already observed in \([6.3]\) that \(\beta\) is not so twisted on the characteristic annuli too much. Hence, we can apply the similar argument for getting an appropriate upper bound of \(\text{Ext}_{g_t}(\beta)\).

### 6.3.2. Ties \(\{B^j_t\}_s\)

Since \(G\) is maximal, any component \(N^j_t\) of \(N_t\) is one of the three types: a pair of pants, an annulus with one distinguished point (a singularity of angle \(\pi\) or a flat point), or a half-pillow with two cone singularities of angle \(\pi\) (cf. Figure 2). In the the case when \(N^j_t\) is either an annulus or a half-pillow, we can deal with the same manner, and hence we now assume that \(N^j_t\) is a pair of pants.

Notice from \([5.5]\) that the length of any component of \(\partial N^j_t\) is of order \(K^{-1/2}_{\gamma_t}\) with respect to the metric \(\rho_t^j := |q_l|_{N^j_t}\). For simplifying of the notation, we assume that components of \(\partial N^j_t\) are \(\alpha_{i_1}, \alpha_{i_2},\) and \(\alpha_{i_3}\). Then, the critical graph \(\Sigma_t \cap N^j_t\) forms one of the graphs in Figure 2 (cf. \([5]\)).

We make equally spaced \(n_{i_l}\)-cuts in \(\alpha_{i_l}\) where \(n_{i_l} = i(\beta, \alpha_{i_l})\) \((l = 1, 2, 3)\). Let \(C^j_{i_l}\) be a component of \(N^j_t \setminus \Sigma_t\) which contains \(\alpha_{i_l}\) in the boundary. Let \(C^0_{i_l,j}\) be a subannulus of \(C^j_{i_l}\) with height \((2K_{\gamma_t})^{-1/2}\) and \(\alpha_{i_l} \subset C^0_{i_l,j}\). We cut \(C^0_{i_l,j}\) along the vertical slits with endpoints in the \(n_{i_l}\)-cuts in \(\alpha_{i_l}\) and get a family of Euclidean rectangles. Since the circumference and the height of \(C^0_{i_l,j}\) are of order \((K_{\gamma_t})^{-1/2}\), the moduli of such Euclidean rectangles are uniformly bounded above and below.

**Claim 1** (See Figure 3). *There is a family \(\{B^j_t\}_s\) of (singular) quadrilaterals such that*
Figure 2. Graphs in $N_j^t$.

Figure 3. Typical examples of ties $B_j^1$ and $B_j^2$ in $N_j^t$. $B_j^1$ is a ‘regular’ quadrilateral and $B_j^2$ is a ‘singular’ quadrilateral.

(1) $B_j^s \cap C_{0,j}^i$ is a rectangle above for all $s$ and $l$,
(2) the arc system given by correcting cores of $B_j^1$’s is homotopic to $\beta \cap N_j^t$, where the core of $B_j^s$ is a path in $B_s$ connecting between facing arcs in $B_j^s \cap \partial N_j^t$ and
(3) the extremal length of family of paths in $B_j^s$ homotopic to the core is uniformly bounded above.

Proof of Claim 1. Notice from (5.5) and the uniformity the moduli of $\{C_{j,l}^i\}_{l=1}^3$ that the conformal structure of $N_j^t$ is precompact in the reduced Teichmüller space. Since the intersection numbers $\{n_{i,l}\}_{l=1}^3$ are independent of $t$, we can take $B_j^s$ such that the width of each $B_j^s$ with respect to the $q_t$-metric are comparable with $K_{\gamma(t)}^{-1/2}$. By definition, the $|q_t|$-area of each $B_j^s$ is $O(K_{\gamma(t)}^{-1})$.

From the reciprocal relation between the module and the extremal length for quadrilateral or Rengel’s type inequality, the extremal length $\text{Ext}(B_j^s)$ of the family of paths in $B_j^s$ homotopic to the core satisfies

\[
\text{Ext}(B_j^s) \leq \frac{|q_t|\text{-area}}{(|q_t|\text{-width})^2} = O(1)
\]

for all $s$ (see §4 in Chapter I of [10]).

6.3.3. Construction of a model $\mathcal{A}_t$ of the extremal annulus. We divide each $A_{i,t}$ into congruence $n_i = i(\beta, \alpha_i)$-rectangles $\{R_{i,l}\}_{l=1}^{n_i}$ via proper horizontal segments. We may assume that for any $l$ and $j$, there is an $s$ such that $R_{i,l} \cap C_{i,j}^{0,s}$ is congruent
to $B_j \cap C^{0,j}_i$. We set
\[ R_{i,t}^0 = R_{i,t} \setminus (N_i \cup_j C^{0,j}_i) \]
(cf. Figure 4). Since twisting numbers of segments in $\beta^* \cap A_{i,t}$ on each $A_{i,t}$ are at most one for all $i$, from (5.6) and the Dehn-Thurston’s parametrization of simple closed curves (cf. [3]), we can glue all $A_{i,t}$ and $N_i$ appropriately at the part $C^{0,j}_i$ to get a Riemann surface $Y_t'$ and an annulus
\[ A'_t = (\bigcup_i R_{i,t}^0) \cup (\bigcup_j B_j^t) \]
(6.5) such that after deforming $Y_t'$ by a quasiconformal mapping with maximal dilatation $1 + o(1)$, we obtain $Y_t$ and the core of the image $A_t$ of the annulus $A'_t$ is homotopic to $f_t(\beta)$. Thus, we conclude
\[ \text{Ext}_{Y_t}(\beta) = \text{Ext}_{Y_t}(f_t(\beta)) \leq \text{Ext}(A_t) = (1 + o(1))\text{Ext}(A'_t) \]
as $t \to \infty$. Therefore, to get the upper estimate of the extremal length of $\beta$ on $y_t$, it suffices to give an upper estimate of the extremal length of $A'_t$.

6.3.4. Estimate of extremal length of $A'_t$. Let $\rho_t^A$ be the extremal metric on $A'_t$ for the extremal length $\text{Ext}(A'_t)$ with $A(\rho_t^A) = 1$. Let $\{S_u\}_u$ be a collection of all rectangles of form $C^{0,j}_i \cap B_j^t$ for all $i, j, s$. By the same argument as Claim 1 in §9.6, we can see the following.

**Claim 2.** For any $u$, there is a vertical line $\eta'_u$ in $S_u$ such that
\[ \sum_u \ell_{\rho_t^A}(\eta'_u) = O(1) \]
as $t \to \infty$.

Let us continue the calculation. Let $\{D_u\}_u$ be a collection of components of $A'_t \setminus \cup_u \eta'_u$. By labeling correctly, $\partial D_u$ contains $\eta'_u$ and $\eta'_{u+1}$, where $\eta'_u$ is labeled cyclically in $u$. By definition, each $D_u$ is contained in either $R_{i,t}$ or $B_j^t$ for some $i, t, j, s$. Let $\Gamma(D_u)$ be the family of paths connecting vertical segments $\eta'_u$ and $\eta'_{u+1}$. Let $\Gamma(R_{i,t})$ be the family of paths in $R_{i,t}$ connecting vertical boundary segments. Since
\[ \text{Ext}(\Gamma(R_{i,t})) = (\text{height of } A_{i,t})/(\ell_{i,t}/n_t) = n_t \text{Mod}(A_{i,t}), \]
(6.7)
by (1) and (2) of Proposition 2.2 and Claim 2, we have

\[ \text{Ext}(A')^{1/2} = \left( \sum_u \text{Ext}(\Gamma(D_u)) \right)^{1/2} + O(1) \]

\[ \leq \left( \sum_{D_u \subseteq R_{i,t}} \text{Ext}(\Gamma(R_{i,t})) + \sum_{D_u \subseteq B'_k} \text{Ext}(B'_k) \right)^{1/2} + O(1) \]

\[ \leq \left( \sum_{D_u \subseteq R_{i,t}} n_i \text{Mod}(A_{i,t}) + O(1) \right)^{1/2} + O(1) \]

\[ = \left( \sum_{i=1}^k n_i^2 \text{Mod}(A_{i,t}) + O(1) \right)^{1/2} + O(1). \]

(6.8)

Thus we get an upper bound of the extremal length of $A'$ as we desired.

6.4. **Conclusion.** From Lemma 5.1, by taking a subsequence if necessary, we may assume that $\text{Mod}(A_{i,t})/K_i(t)$ tends to a positive number $M_i$ for any $i = 1, \ldots, k$. From (6.2), (6.6), and (6.8), we deduce that

\[ i(\beta, G) = \lim_{t \to \infty} \left( \frac{\text{Ext}_{\gamma(t)}(\beta)}{K_{\gamma(t)}} \right)^{1/2} = \left( \sum_{i=1}^k M_i \right)^{1/2} = \left( \sum_{i=1}^k M_i i(\beta, \alpha_i)^2 \right)^{1/2} \]

for all $\beta \in S$. From the density of $\mathbb{R}_+ \otimes S$ in $MF$, the above also holds for all measured foliations. Thus, for $x, y > 0$ and $\beta_1, \beta_2 \in S$ with $i(\beta_1, \beta_2) = 0$, by substituting $\beta = x\beta_1 + y\beta_2$ to (6.9), we get

\[ \left( \sum_{i=1}^k M_i n_{1,i}^2 \right)x^2 + 2 \left( \sum_{i=1}^k M_i n_{1,i} n_{2,i} \right)xy + \left( \sum_{i=1}^k M_i n_{2,i}^2 \right)y^2 = (xi(\beta_1, G) + yi(\beta_2, G))^2, \]

where $n_{j,i} = i(\beta_j, \alpha_i)$. Therefore, the discriminant of the quadratic form above is zero. Namely, we have

\[ \left( \sum_{i=1}^k M_i n_{1,i} n_{2,i} \right)^2 - \left( \sum_{i=1}^k M_i n_{1,i}^2 \right) \left( \sum_{i=1}^k M_i n_{2,i}^2 \right) = 0, \]

for all such $\beta_1, \beta_2 \in S$. Hence, two vectors

\[ (\sqrt{M_1 n_{1,1}}, \ldots, \sqrt{M_k n_{1,k}}), \quad (\sqrt{M_1 n_{2,1}}, \ldots, \sqrt{M_k n_{2,k}}), \]

are parallel for all $\beta_1, \beta_2 \in S$ with $i(\beta_1, \beta_2) = 0$. However, this is impossible as we already observed in §6 of [17].

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