Non-planar double-box, massive and massless pentabox Feynman integrals in negative dimensional approach

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Negative dimensional integration method (NDIM) is a technique which can be applied, with success, in usual covariant gauge calculations. We consider three two-loop diagrams: the scalar massless non-planar double-box with six propagators and the scalar pentabox in two cases, where six virtual particles have the same mass and in the case where all of them are massless. Our results are given in terms hypergeometric functions of Mandelstam variables and for arbitrary exponents of propagators and dimension $D$ as well.

Keywords: Quantum field theory, negative dimensional integration, pentabox and double-box integrals.

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I. INTRODUCTION

Studies in particle phenomenology require more and more sophisticated calculations [1], and the measurement of the $(g - 2)$ factor has now an error of 1 ppb order of magnitude [2] thanks to the perturbative approach. Within this perspective there have been some interest in massless double-box — planar and non-planar — and pentabox integrals. Smirnov [4] studied the former, using the Mellin-Barnes (MB) technique, in three cases: scalar and tensorial with four legs on-shell and scalar with one leg off-shell; Tausk and Smirnov and Veretin [7], also using the MB method, presented explicit results for non-planar, or crossed, double-box with seven and six propagators, in the special case where the exponents of propagators are equal to one; Kümmer et al came across with the same integrals studying the potential between quarks in the Coulomb gauge [8]; Anastasiou et al calculated the latter, in the integration-by-parts approach, for both the scalar and tensorial cases with massless internal particles [9]. The results which are obtained with the MB approach are, like the ones obtained by NDIM, expressed in terms of infinite series of hypergeometric type. Of course, progress along this line is greater in covariant gauges, however perturbative calculations in non-covariant ones are also carried out and need sometimes more powerful techniques than the former [10].

In this paper we choose to tackle the scalar on-shell non-planar double-box integral for arbitrary exponents of propagators, a result that is missing in the literature and the pentabox integral in two cases: where all particles are massless, and where six virtual particles have the same mass, a diagram which, for instance, contributes to photon-photon scattering, and as far as we know was not yet calculated before. NDIM gives us several results in terms of Mandelstam variables and masses, each valid in a certain kinematical region. We give results in terms of hypergeometric series for arbitrary exponents of propagators and dimension. In our approach no reduction formulas or integration-by-parts methods are used or even necessary. It is also worth observing that NDIM is a technique which can be applied to other gauge choices like the Coulomb and the light-cone gauges [11].

An important feature of NDIM is that it is not a regularization technique. It is worth remembering Dunne and Halliday [12]: the negative-dimensional integrals (in $D$-dimensions) can be related to positive dimensional ones (in $2N$-dimensions) over Grassmannian variables; in fact, one has just to make $D \leftrightarrow -2N$. So, in NDIM context there are no singularities, no poles etc. However, when we perform the analytic continuation, in order to allow negative exponents of propagators and positive dimension, then poles appear for specific values of those exponents and physical $D = 4$ dimension and we have the same results which other techniques provide. This is therefore a consistent method to solve Feynman loop integrals pertaining to usual covariant or to non-covariant algebraic gauges, like Coulomb and light-cone ones (even at the two-loop level).

The aim of our paper is not to establish the axiomatic foundations for NDIM nor to demonstrate in a rigorous mathematical sense its principles and basis. Instead, given the simple steps that the methodology requires to work
out complicated Feynman integrals we are interested in testing it to the limits of our present calculational abilities. For this purpose we are presenting here in another [13] true two-loop calculation the exact results yielded by this method. Such results must be compared with others obtained using different techniques so that not only previous answers be double-checked and the confidence in the novel method be increased but also in order to demonstrate the feasibility of the latter in performing the calculations.

The outline for our paper is as follows: in the next section we study pentabox integrals, where in one case virtual particles are massless and in the other case, we consider six of the virtual particles massive with equal masses, a graph which contributes to photon-photon scattering (which were considered recently in [14]), although a full calculation of such an effect as well as evaluation of beta functions in physical processes are beyond the scope of the present work. We solve also the scalar massless non-planar double-box with six propagators. In section 3, we present our conclusions.

II. PENTABOX AND NON-PLANAR DOUBLE-BOX INTEGRALS

Let us define immediately the relevant three negative-dimensional integrals, namely,

\[ \mathcal{P} = \int \int d^D q \, d^D k_1 \, \mathcal{P}(q, k_1, p, p', p_1), \]  

\[ \mathcal{M}\mathcal{P} = \int \int d^D q \, d^D k_1 \, \mathcal{M}\mathcal{P}(q, k_1, p, p', p_1, \mu), \]  

and

\[ \mathcal{N}\mathcal{P}_6 = \int \int d^D q \, d^D r \, \mathcal{N}\mathcal{P}_6(q, r, p_2, p_3, p_4), \]

where the integrands are respectively

\[ \mathcal{P} \equiv (q^2)^i (q - p)^{2j} (q - k_1)^{2k} (q - k_1 - p_1)^{2l} (k_1^2)^m (q - p_1)^{2n} (q - p - p')^{2r}, \]

\[ \mathcal{M}\mathcal{P} \equiv (q^2 - \mu^2)^i [(q - p)^2 - \mu^2]^j [(q - k_1)^2 - \mu^2]^k [(q - k_1 - p_1)^2 - \mu^2]^l (k_1^2)^m \times [(q - p_1)^2 - \mu^2]^n [(q - p - p')^2 - \mu^2]^r, \]

\[ \mathcal{N}\mathcal{P}_6 \equiv (q^2)^i (q - p_3)^{2j} (q + r)^{2k} (q + r + p_2)^{2l} (r^2)^m (r - p_4)^{2n}, \]

which represent massless pentabox, massive pentabox and non-planar double-box, respectively. Once they are introduced, we evaluate them using NDIM.

When Halliday and co-workers advanced the idea of negative-dimensional integration they proved that it is equivalent [13] to Grassmannian integration in positive dimension, the correspondence being as simple as \( D \leftrightarrow -2N \). This fact is implied in the very structure of the above integrands.
A. Massless Pentabox integral

\[
G_P = \int \int d^Dq \, d^Dp' \exp \left[ -\alpha q^2 - \beta (q - p)^2 - \gamma (q - r')^2 - \theta (q - r' - p_1)^2 - \phi r'^2 - \eta (q - p_1)^2 - \omega (q - p - p')^2 \right] \quad (7)
\]

\[
= \left( \frac{\pi^2}{\Lambda} \right)^{D/2} \exp \left[ -\frac{1}{\Lambda} (\alpha \lambda_2 \omega s + \beta \eta \lambda_2 t + \beta \theta \phi t) \right], \quad (8)
\]

where \( \lambda_2 = \gamma + \theta + \phi \) and \( \Lambda = (\alpha + \beta + \eta + \omega) \lambda_2 + \gamma \phi + \theta \phi \).

Taylor expanding (7) we have our negative-dimensional integral \( P \) as a factor in a seven-fold summation series,

\[
G_P = \sum_{i,j,k,l,m,n,r} (-1)^{i+j+k+l+m+n+r} \frac{\alpha^i \beta^j \gamma^k \theta^l \phi^m \eta^n \omega^r}{i!j!k!l!m!n!r!} P. \quad (9)
\]

On the other hand taking (8) and making an expansion (including a multinomial one) in power series, we obtain,

\[
G_P = \sum_{X,Y,Z=0}^{\infty} \frac{\alpha X_{123} + Y_{123} \beta X_{4567} + Y_{4567} \gamma X_{14} + Y_{14} \chi X_{257} + Y_{257} \rho X_{369} + Y_{369} \phi X_{135} + Y_{135} \eta X_{457} + Y_{457} \omega X_{123} + Z_{345}}{X_1! \ldots X_7! Y_1! \ldots Y_9! Z_1! \ldots Z_5!} \times (-s)^X_{123} (-t)^X_{4567} (-X_{123} + D/2)!, \quad (10)
\]

where sum over \( X, Y, Z \) is a shorthand notation for a 7-, 9-, and 5-fold sums respectively. Moreover, we define \( X_{12} = X_1 + X_2, \ldots \), so on and so forth.

Therefore, the exponential above generates a series indexed by 21 indices, while the 7 propagators in the argument of the integrand give rise to 7 equations and the multinomial expansion to another one, see eq. (9). Now, solving both equations for \( P \) we conclude that there must be some relations among the two sets of indices \( \{ X, Y, Z \} \) and \( \{ i, j, k, l, m, n, r \} \). It is a system of algebraic equations.

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FIG. 1. Scalar massless pentabox with all external legs on-shell. Mandelstam variables are defined as \( s = (p + p')^2 \) and \( t = (p - p_1)^2 \).
and the parameters of 3

\[
\begin{align*}
X_{123} + Y_{123} &= i \\
X_{456} + Y_{456} &= j \\
X_{14} + Y_{147} + Z_{13} &= k \\
X_{257} + Y_{258} + Z_{24} &= l \\
X_{367} + Y_{369} + Z_{125} &= m \\
X_{456} + Y_{789} + Z_{13} &= n \\
X_{123} + Z_{345} &= r \\
\Sigma X + \Sigma Y + \Sigma Z &= -D/2
\end{align*}
\]

for which there is not a unique way to solving it, since we have 21 “unknowns” and just 8 equations. In fact, there are 203,490 possible 8 $\times$ 8 systems which can be solved in terms of exponents of propagators $i, j, k, l, m, n, r, \text{dimension } D$ and some of $X, Y, Z$. The computer can calculate such 8 $\times$ 8 determinants easily: 134,890 of them are zero, i.e., give empty sets of solutions.

So, our result will be written in terms of a 13-fold series of hypergeometric type. This can be worked out conveniently and be simplified, since we have three possible variables: $t/s, s/t$, and unity. Series which depend on both $t/s$ and $s/t$ can not be convergent, so we will not consider them.

Next, our strategy is to search for the simplest hypergeometric series among the remaining 68,600 solutions. The criterion for this search is dictated by the fact that the more sums with unity argument the simpler it is, since one can sum them (at least in principle, respected some relations amongst the parameters) using Gauss’ summation formula [13].

\[
2F_1(a, b; c|1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad \text{Re}(c-a-b) > 0.
\]

In other words, when some of those 13 sums can be rewritten as gamma functions. Of course, not all sums with unity argument can be summed this way, as we will shortly see. These cases occur when the resulting function is of the type $3F_2(...|1)$, or even more complex ones.

There are many different ways in which the 13-fold series appear, and we can classify them according to the following general form,

\[
\text{(Momenta)(Gammas)} \sum_{n_1!...n_9!m_1!...m_4!} \frac{z^A}{n_1...n_9!m_1...m_4!},
\]

where $z$ is any one of the three possible variables $t/s, s/t$, or 1, and $A$ represents either $n_{123} = n_1 + n_2 + n_3$, or $n_{1234} = \sum_{i=1}^4 n_i$, or $n_{12345} = \sum_{i=1}^5 n_i$.

The simplest hypergeometric series representations for the scalar massless pentabox is given by hypergeometric series with three such “variables” (here we use the word “variables”, to denote the one variable $s/t$ appearing thrice as summation variables within the series):

\[
P_3 = \pi^D s^{\sigma-j} t^j P_{3AC}^Y \Sigma(Y_4, Y_5, Y_6) 3F_2(a_3, b_3, c_3; e_3, f_3|1)
\]

where

\[
\Sigma(Y_4, Y_5, Y_6) \equiv \sum_{y_4, y_5, y_6=0}^{\infty} \frac{(-k|Y_4)(j-k-l-m-n-D/2-Y_{456})(D+k+l+m+n+r|Y_{456})}{(1-k-m-D/2-Y_5)(D+k+l+m|Y_{456})} \\
\times \frac{(D+i+k+l+m+n|Y_{456})(k+l+D/2|Y_6)(-j|Y_{456})(s/t)|Y_{456}}{(1+\sigma-j|Y_{456})Y_4!Y_5!Y_6!},
\]

and the parameters of $3F_2$ are

\[
\begin{align*}
a_3 &= -k + Y_4, \\
b_3 &= j - k - l - m - n - D/2 - Y_{456}, \\
c_3 &= -k - l - m - D/2, \\
e_3 &= 1 - k - m - D/2 - Y_5, \\
f_3 &= -k - l - m - n - D/2
\end{align*}
\]

with $\sigma = i + j + k + l + m + n + r + D$ and
$P_3^{AC} = (-i|\sigma - j|(-k - l - m - n - D/2|j)(-l|k + l + m + D/2)(-m| - k - l - D/2)
\times (-r|k + l + r + D/2)(\sigma + D/2| - 2\sigma - D/2 + j)(k + l + m + D|i + n)$

(16)

In the above equations, we have used the subscript “3” and superscript “AC” to mean that we have three “variables” and the result is analytically continued into positive dimension $D$. Pochhammer symbols and some of their properties are used throughout the expressions,

$(a|b) \equiv (a)_b = \frac{\Gamma(a + b)}{\Gamma(a)}$, $(a| - k) = \frac{(-1)^k}{(1 - a|k)}$, $(a|b + c) = (a + b|c)(a|b)$.

(17)

Observe that in (13) there is a fourth sum, namely in $Z_1$, which has unity argument and could not be summed because it is a $3_F_2$ hypergeometric series which can be expressed in terms of gamma functions only in some special cases. This means that we were able to sum up up to 9 series (with unity arguments) using (12).

The second type of result provided by our method is given by hypergeometric series with four “variables”,

$P_4 = \pi^D s^a P_4^{AC} \sum_{X_n = 0}^{\infty} \frac{(-j|X_{4567})(-n|X_{456})(-k|X_4)(-l|X_{57})(-\sigma|X_{4567})}{(i - \sigma|X_{4567})(r - \sigma|X_{4567})(-k - l|X_{57})}
\times \frac{(D/2 + m|X_{456})(D/2 + k + l|X_6)(-k - l - m - D/2|X_7)}{(D + k + l + m|X_{456}) D_4^X X_5^X X_6^X X_7^X}$

(18)

where the subscript “4” means a 4-fold hypergeometric series representations for $P$, and

$P_4^{AC} = (-i|\sigma)(-r|\sigma)(-k - l - m - D/2|\sigma + D/2| - 2\sigma - D/2)(k + l + m + D| - m - D/2)(-n|2m + D/2)$.

(19)

Note that this four-fold series can be used to study forward scattering. Taking $t = 0$ we are left with only the first term in the series, which is equal to unity, that is, the series collapses and the result is merely (the superscript FS for forward scattering case)

$P_4^{[FS]} = \pi^D s^a P_4^{AC}$.

The last one, given by hypergeometric series with five “variables”,

$P_5 = \pi^D s^{i + j + r + D/2 (k + l + m + n + D/2) P_5^{AC}}$
\times \sum_{Y_7, Z_8 = 0}^{\infty} \frac{(-n|Y_{789})(-k|Y_7 + Z_1)(-k - l - m - D/2|Z_{12})(k + l + D/2|Y_6)}{(1 - k - m - D/2|Y_{789})(1 + j + k - l - m - n - D/2|Y_{789} + Z_{12})(D + k + l + m|Y_{789})}
\times \frac{(j + r + D/2|Y_{789} + Z_{12})(i + j + D/2|Y_{789} + Z_{12})}{(1 + i + j + D/2|Y_{789} + Z_{12}) Y_7^Y Y_8^Y Y_9^Y Y_1^Z Y_2^Z}$

(20)

where

$P_5^{AC} = (-i| - j - r - D/2)(-j|k + l + m + n + D/2)(k + l + m + D| - m - D/2)(-l| - k - m - D/2)$
\times \frac{(-n|i + j + m + D/2)(-r|k + m + r + D/2)(\sigma + D/2|j + r - \sigma)}{(-i| - j - r - D/2)(-j|k + l + m + n + D/2)(k + l + m + D| - m - D/2)(-l| - k - m - D/2)}$

(21)

Observe that in the last line of (20) when we take either $i = -1$ or $r = -1$ those pertinent Pochhammer symbols within the series do cancel out, simplifying it.

Of course, there are results which depend on the inverse of such variables, i.e., $P_3(t/s)$, $P_4(s/t)$ and $P_5(t/s)$, which means an interchange of one or more pairs of exponents of propagators and $s \leftrightarrow t$.

How are these different expressions related to one another? How do they relate to previous calculations by other methods, for example for special cases of the propagator powers? How unique or ambiguous is the analytic continuation in dimension to get back to answers in positive dimensions? What form of renormalization is necessary to connect with conventional calculations?

To answer the first question one must recall that if two series ($P_3$ and $P_5$, for instance) represent the same function (the integral $P$) then they must be related by analytic continuation $[10]$. So, the results provided by NDIM are always related by analytic continuation either directly (overlapping regions of convergence) or indirectly (no overlapping of regions). In the present case there are no previous calculations in the literature considering arbitrary exponents of propagators.
B. Massive Pentabox integral

Introducing masses in the NDIM context is very simple [17]. The generating functional becomes,

$$G_{MP} = \int \int d^Dq \, d^Dk_1 \, \exp \left\{ -\alpha(q^2 - \mu^2) - \beta((q-p)^2 - \mu^2) - \gamma((q-k_1)^2 - \mu^2) - \theta((q-k_1-p_1)^2 - \mu^2) \right\}$$

$$-\phi k_1^2 - \eta((q-p_1)^2 - \mu^2) - \omega((q-p-p')^2 - \mu^2) \right\}$$

$$= \left( \frac{\pi^2}{\Lambda} \right)^{D/2} \exp \left[ -\frac{1}{\Lambda} (\alpha \lambda_2 \omega s + \beta \eta \lambda_2 t + \beta \theta \phi t) \right] \exp \left[ (\alpha + \beta + \gamma + \theta + \eta + \omega) \mu^2 \right],$$

so that we have six additional sums, which are generated by the second exponential in the second line above, which corresponds to the massive sector. Besides, the resulting hypergeometric series will have variables of the form,

$$\frac{t}{s}, \frac{t}{\mu^2}, \frac{s}{\mu^2},$$

their inverses or unity. Powers of them also occur, like $\sqrt{s/t}$ and $(s/t)^2$.

Once more we look for convergent series. Among the very BIG number of possible systems — altogether $27!/8!19! = 2,220,075$ — there are 1,093,289 which have no solution, and the remaining 1,126,786 among which we are able to find solutions. So we are left with 49.24% of the total, from which we hunt for the most convenient ones.

![Pentabox diagram with six massive propagators.](image)

At two-loop level it contributes to photon-photon scattering.

The simplest hypergeometric series representation is given by a seven-fold summation,

$$\mathcal{M}_7^{\sigma \mu^2} = \pi^D (\mu^2)^{\sigma} P_7^{AC} \sum_{\text{all}=0}^{\infty} \frac{(-i|X_{123})(-j|X_{4567})(-k|X_{14})(-l|X_{257})(-m|X_{367})(-n|X_{456})(-r|X_{123})(-\sigma|X_{1234567})}{(-m - \sigma|X_{1245} + 2X_{367})(-k - l|X_{12457})(D/2|X_{1234567})}$$

$$\times \frac{(-i - j - m - n - r - D/2|X_{12457} + 2X_{367})(-k - l - m - D/2|X_7)(D/2 + m|X_{1245})}{(-i - j - n - r|2X_{123456} + X_7)(D/2 + m|X_{1245})} \left( \frac{s}{\mu^2} \right)^{X_{123}} \left( \frac{t}{\mu^2} \right)^{X_{4567}},$$

FIG. 2. Pentabox diagram with six massive propagators. We consider the case of equal masses and external particles on-shell. At two-loop level it contributes to photon-photon scattering.
where

$$P_7^{AC} = (D/2|m)(-\sigma - m)(-i - j - n - r - m - D/2)(-k - l - m - D/2). \tag{25}$$

As a sample numerical calculation we give in the Table 1 below an expansion in the $\epsilon = 2 - D/2$ parameter.

| Terms | $\epsilon^0$ | $\epsilon^1$ | $\epsilon^2$ |
|-------|--------------|--------------|--------------|
| 0     | 0.167027110  | -0.012614420 | 0.3914896    |
| 1     | 0.167553265  | -0.012917188 | 0.3927286    |
| 2     | 0.167558252  | -0.012920469 | 0.3927776    |
| 3     | 0.167558333  | -0.012920522 | 0.3927779    |
| 4     | 0.167558335  | -0.012920523 | 0.3927778    |

TABLE 1. Column "terms" indicates where we truncated all the series. Observe that even for a few terms in the series that were summed we get a good precision. The result is of the form $A + B\epsilon + C\epsilon^2$ where $A, B, C$ are given in the table.
Hypergeometric series with ten summation indices, nine “variables” also occur,

\[
(\mathcal{M}\mathcal{P})_9 = \pi^D t^\sigma P_{9AC}^{\mathcal{AC}} \sum_{\text{all}=0}^\infty \frac{(-i|X_{123} + W_1)(-k|W_3 + X_1 + Z_1)(-n|W_5)\cdots}{(1 + j - \sigma|X_{123} + W_{13456})(1 + j - \sigma|X_{123} + W_{13456})(1 + j - \sigma|X_{123} + W_{13456})} \times \frac{(-i|X_{123} + W_1)(-k|W_3 + X_1 + Z_1)(-n|W_5)\cdots}{(1 + j - \sigma|X_{123} + W_{13456})(1 + j - \sigma|X_{123} + W_{13456})(1 + j - \sigma|X_{123} + W_{13456})}
\]

Observe that there is a tenth series with unity argument, which is not summable since it is a 3\(_3\)F\(_2\)(a, b, c; e, f|1).

Finally the last result we present for the massive pentabox is a 11-fold series,

\[
(\mathcal{M}\mathcal{P})_{11} = \pi^D (\mu^2)^{i+j+r+D/2} \sum_{\text{all}=0}^\infty \frac{(-i|X_{123})(-k|X_{123} + W_1)(-n|W_3 + X_1 + Z_1)(-n|W_5 + Y_789)\cdots}{(1 + j - k - l - m - n - D/2|W_{345} + Y_{789} + Z_{12})} \times \frac{(1 - k - l - m - n - D - X_{123} + Y_{789} + Z_{12} + W_{345})(1 - k - l - m - n - D - X_{123} + Z_{12})}{(1 + j - k - l - m - n - D/2|W_{345} + Y_{789} + Z_{12})}
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Observe that there is a tenth series with unity argument, which is not summable since it is a 3\(_3\)F\(_2\)(a, b, c; e, f|1).

Finally the last result we present for the massive pentabox is a 11-fold series,
into another one which did not allow us to consider \( s = 4m^2 \). The answer is in \([14]\); when we carry out an analytic continuation and in the process cross a branch cut, this analytic continuation is not unique and poles do appear. In that case, there were simple and double poles, and we had two possible cases: \( F_3 \to \Sigma F_2 \) and \( F_3 \to H_2 \). In our present case, \((MP)_{11}\) and \((MP)_{10}\) have third order poles, and \((MP)_{17}\) is finite. It depends on the kinematical region we wish to study the pertinent integral.

Back in the 50’s and 60’s \([20]\) a lot of research was done in order to study singularities of Feynman integrals. One of the results we borrow from them is that a graph like the one considered in the present section can not have singularities in the physical sheet. This is a well-known theorem due to Eden (1952). So, in a full calculation of photon-photon scattering the poles must cancel out.

In order to extract specific pole structures of these integrals, we can proceed just like in our previous works on the subject \([18]\). Expand gamma functions around \( \varepsilon = 0 \) and use Taylor expansion in the hypergeometric series, so that we are left with parametric derivatives of hypergeometric functions. The reader can see detailed calculations in the above referred paper; also in the appendix of \([21]\) and in the next section.

C. Non-Planar Double-Box integral

Now we turn to massless non-planar double-box with six propagators, namely,

\[
\mathcal{NP}_6 = \int d^Dq \, d^Dr \ (q^2)\, (q^2)\, (q+r)^2 \, (q+r+p_2)^2 \, (r^2)^m \, (r-p_4)^2n, \tag{32}
\]

which represents the diagram of Figure 4. This diagram was also studied by Smirnov and Veretin \([3]\) that presented an explicit result in the case where all exponents were equal to minus one. On the other hand, we will write down results for arbitrary values of them.

The generating functional

\[
G_{\mathcal{NP}_6} = \int d^Dq \, d^Dr \ \exp \left\{-\alpha q^2 - \beta (q-p_4)^2 - \gamma (q+r)^2 - \theta (q+r+p_2)^2 - \phi r^2 - \omega (r-p_4)^2 \right\} \tag{33}
\]

\[
\left(\frac{\pi^2}{\zeta}\right)^{D/2} \ exp \left[-\frac{1}{\zeta} (\beta \gamma \omega s + \alpha \theta \omega t + \beta \theta \phi u)\right],
\]

can be integrated out without difficulty. We define \( \lambda_3 = \alpha + \beta + \gamma + \theta \) and \( \zeta = \alpha \gamma + \alpha \theta + \beta \gamma + \beta \theta + \lambda_3 (\phi + \omega) \). It is easy to see that Taylor expanding the above exponential will give us three series, while multinomial expansion for \( \zeta \) other twelve series. The equations that form the system to be solved come from the propagators, six altogether, and an additional constraint originates from the multinomial expansion.

So, at the end of the day we are left with \((15 - 7) = 8\)-fold series. Their variables are either,

\[
\frac{t}{s}, \frac{s}{t}, \frac{u}{t}, \frac{t}{u}, \frac{u}{s}, \frac{s}{u}, \frac{r}{t}, \frac{t}{r}
\tag{34}
\]

and/or unity. The simplest hypergeometric series representations for \( \mathcal{NP}_6 \) are double series,

\[
\mathcal{NP}_6 = \pi^D s^{\sigma'} \Gamma_{\mathcal{NP}}^{AC} \sum_{X_2, X_3=0}^{\infty} \frac{(-i|X_2)(-m|X_3)(-\sigma'|X_{23})(-l|X_{23})}{(1-k-\sigma'|X_{23})(1+n-\sigma'|X_{23})(1+j-\sigma'|X_2)} \frac{(-t/s)^{X_2} (u/s)^{X_3}}{X_2! X_3!}, \tag{35}
\]

where \( \sigma' = i + j + k + l + m + n + D \), is the sum of exponents and dimension and

\[
\Gamma_{\mathcal{NP}}^{AC} = (-j|\sigma') (-k|\sigma') (-n|\sigma') (\sigma' + D/2) - 2\sigma' - D/2 (i + j + m + n + D) - i - j - D/2 \times (k + l + m + n + D) - m - n - D/2 (i + j + k + l + D) - k - l - D/2. \tag{36}
\]

The above hypergeometric series reduces to an Appel \( F_2 \) function in the special case where \( k = -1 \),

\[
\mathcal{NP}_6 = \pi^D s^{\sigma'} \Gamma_{\mathcal{NP}}^{AC} (k = -1) \ F_2 \left( \begin{array}{cc}
-l, & -i, -m \\
1 + j - \sigma', & 1 + n - \sigma' \\
\frac{r}{s}, & \frac{s}{u}
\end{array} \right). \tag{37}
\]

If we take all the exponents to be equal to minus one, we have
\[ N_P = \pi^D s^{D-6} \frac{\Gamma(6-D)\Gamma^3(D/2-2)\Gamma^3(D-5)}{\Gamma(3D/2-6)\Gamma^3(D-4)} \sum_{X_2,X_3=0}^{\infty} \frac{(1|X_2)(1|X_3)(1|X_3)}{(6-D|X_4)(6-D|X_2)} \frac{(-t/s)^{X_2} (u/s)^{X_3}}{X_2!X_3!}. \]  

(38)

This series is the Appel \[ F_2 \] hypergeometric function which converges when \(|t/s| < 1, |u/s| < 1\) and \(|t/s|+|u/s| < 1\). Note that the above result has a double pole, \(1/\epsilon^2\), just like in Tausk’s \[ \] and in Smirnov and Veretin’s \[ \] works.

One could now ask: Would the divergent pieces come from a few terms in the series or all of them? Clearly, in the present case, divergent factors which generate double and simple poles come from the gamma functions. To write down these poles explicitly we have to Taylor expand also the hypergeometric function \( F_2 \), then we have around \( \epsilon = 0 \) \((D = 4 - 2\epsilon)\),

\[ N_P = \pi^D s^{D-6} \frac{\Gamma(6-D)\Gamma^3(D/2-2)\Gamma^3(D-5)}{\Gamma(3D/2-6)\Gamma^3(D-4)} F_2 \left( \frac{1,1,1}{6-D,6-D} \Bigg| \frac{-t}{s},\frac{u}{s} \right) \]

(40)

\[ = \pi^D s^{D-6} \left[ -\frac{3}{\epsilon^2} + 12 + O(\epsilon) \right] \]

\[ \times \left\{ F_2 \left( \frac{1,1,1}{2,2} \Bigg| \frac{-t}{s},\frac{u}{s} \right) + 2\epsilon(\partial_\gamma + \partial_\gamma')F_2 + 2\epsilon^2 \left[ \partial_\gamma^2 + \partial_{\gamma'}^2 + 4\partial_\gamma \partial_{\gamma'} \right] F_2 + O(\epsilon^3) \right\}, \]

where \( \gamma_E \) is the Euler’s constant and

\[ \partial_\gamma F_2 = \frac{\partial}{\partial\gamma} F_2 \left( \alpha,\beta,\beta' \Bigg| \gamma,\gamma' \right. \left| \frac{-t}{s},\frac{u}{s} \right. \right), \]

are called parametric derivatives of hypergeometric functions and can be calculated using Davydychev’s \[ \] approach. (\( T \) is given below.)

In order to rewrite the above result as polilogarithm functions, such as \( \text{Li}_2 \), \( \text{Li}_3 \) and usual logarithms we have to use an integral representation for \( F_2 \),

\[ \frac{\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma-\beta)\Gamma(\gamma'-\beta')}{\Gamma(\gamma)\Gamma(\gamma')} F_2 \left( \alpha,\beta,\beta' \Bigg| \gamma,\gamma' \right. \left| P_1,P_2 \right. \right) = \int_0^1 dx_1 dx_2 \frac{x_1^{\beta-1}x_2^{\beta'-1}(1-x_1)^{\gamma-\beta-1}(1-x_2)^{\gamma'-\beta'-1}}{(1-x_1P_1-x_2P_2)^\alpha}, \]

(41)

where \( x_1 + x_2 = 1 \). This task in not an easy one at all, since the second derivatives give a very cumbersome result and direct comparison between our result and Tausk’s \[ \] is not possible (we were not able to show that both results are equivalent analytically).

However, the important special case of forward scattering can not be read off directly from Tausk’s result. On the contrary, making \( t = 0 \) or \( u = 0 \) is extremely simple in our eq.(38). Let \( u = 0 \), then one of the series (in \( X_3 \) index) does not contribute and

\[ F_2 \left( \frac{1,1,1}{6-D,6-D} \Bigg| \frac{-t}{s},0 \right) = 2F_1 \left( \frac{1,1}{6-D} \Bigg| \frac{-t}{s},0 \right) = \frac{\Gamma(6-D)\Gamma(4-D)}{\Gamma^2(5-D)}, \]

(42)

since \( s + t + u = 0 \) we were able to sum up the series \( 2F_1 \), so \( \partial/\partial\gamma \) can be rewritten in terms of \( \partial/\partial\epsilon \) and parametric derivatives are easily calculated from eq.(42). We quote the result up to order \( \epsilon^3 \),

\[ N_P \mid_{u=0} = \pi^D s^{2-2\epsilon} T \left\{ -\frac{48}{\epsilon^2} + \frac{201}{2\epsilon^3} - \frac{204}{\epsilon^4} + \frac{8}{\epsilon} \left[ 83 + 16\psi''(1) - 16\psi''(2) \right] + 4 \left[ -134 + 35\psi''(1) - 35\psi''(2) \right] + O(\epsilon) \right\}, \]

(43)

where \( \psi(z)\Gamma(z) = \Gamma(z), \psi'(z), \psi''(z) \) their derivatives and

\[ T = \frac{\Gamma(5-D)\Gamma^3(D/2-1)}{\Gamma(3D/2-5)(5-D)}, \]

which in fact can be calculated to arbitrary order, since we start from an exact result.
Observe that the result has poles of higher orders, namely $\epsilon^{-4}$ and $\epsilon^{-3}$, which come from the fact we have taken $u = 0$, which also mean that the result has a branch point in $u = 0$, a well-known fact from the theory of hypergeometric functions.

Of course, the $7 \times 15$ system of linear algebraic equations defined by this integral, the $7 \times 7$ solvable subsystems are divided into the following categories: among the grand total of 6,435 possible solutions, 3,519 have no solution at all. Among the remaining 2,916 relevant solutions, NDIM provides other kind of series, such as 7-fold series and 5-fold series. And all of them have symmetries among $s, t$ and $u$, namely,

$$(p_3 \leftrightarrow p_4, \ j \leftrightarrow n, \ i \leftrightarrow m, \ t \leftrightarrow u), \quad (p_2 \leftrightarrow p_3, \ l \leftrightarrow n, \ k \leftrightarrow m, \ t \leftrightarrow s), \quad (p_2 \leftrightarrow p_4, \ j \leftrightarrow l, \ i \leftrightarrow k, \ s \leftrightarrow u),$$ (44)

So, for each hypergeometric series representations provided by NDIM there are other two, also originated from the system of algebraic equations, which represent the same integral and can be transformed into the first using (44). This is the case of (35), and it is stated also by Tausk [5], i.e., the diagram is completely symmetric (through analytic continuation) under external legs permutations.

![Figure 3. Scalar massless non-planar double-box with six propagators. The labels in the internal lines represent the exponents of propagators, see (3).](image)

Despite the complicated form of non-planar double-box with six propagators, the result we obtained is very simple, a double hypergeometric series, which in the special case of $k = -1$ reduces to $F_2$ Appel function. Tausk presented a result for the same graph in terms of di- and trilogarithms, in the special case where all exponents are equal to minus one. If we recall [18], Davydychev calculated a scalar integral for photon-photon scattering and transformed the four $F_2$ Appel hypergeometric functions into dilogarithms. The same occurs here, in one hand di and trilogarithms and in the other double hypergeometric series, namely $F_2$.

### III. CONCLUSION

In this work we considered covariant gauge scalar pentabox and non-planar double-box integrals. In the former we considered two cases: all virtual particles being massless, and in the other, six of them having the same mass $\mu$ while the seventh is massless. The latter was calculated in the massless case and arbitrary exponents of propagators, a result which was missing in the literature. No reduction formulas, i.e., rules to connect a given diagram with simpler ones, were used. The results are given in terms of hypergeometric series and in different kinematical regions.

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