Torsional Alfvén waves in stratified and expanding magnetic flux tubes

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Abstract

The effects of both density stratification and magnetic field expansion on torsional Alfvén waves in magnetic flux tubes are studied. The frequencies, the period ratio $P_1/P_2$ of the fundamental and its first-overtone, and eigenfunctions of torsional Alfvén modes are obtained.

Our numerical results show that the density stratification and magnetic field expansion have opposite effects on the oscillating properties of torsional Alfvén waves.

Keywords: Sun: corona . Sun: magnetic fields . Sun: oscillations

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1 Introduction

Hannes Alfvén (1942) predicted the existence of Alfvén wave which is one of the magnetohydrodynamic (MHD) waves propagating in magnetized plasmas such as the solar atmosphere. Torsional Alfvén waves can be observed as temporal and spatial variations in spectral emission along the coronal loops (Zaqarashvili 2003). They are an ideal tool for coronal seismology as their phase speed depends on plasma quantities within the loop alone, while wave speeds of magnetosonic oscillations are influenced by plasma conditions in the ambient medium (Zaqarashvili & Murawski 2007). More recently, torsional Alfvén waves in the solar atmosphere were discovered by Jess et al. (2009) using the high-resolution Swedish Solar Telescope.

Torsional Alfvén modes can have a significant role in coronal heating and solar wind acceleration, based upon the ability of torsional waves to penetrate easily into the corona (see e.g. Ruderman 1999; Copil, Voitenko & Goossens 2008). Possible Alfvén wave dissipation processes that may be responsible for heating the solar plasma have been investigated by various authors, e.g., dissipation in resonant layers (Poedts, Goossens & Kerner 1989, 1990a,b,c; Erdélyi & Goossens 1995; Ruderman et al. 1997a,b; Erdélyi 1998; Ruderman 1999; Narain et al. 2001; Safari et al. 2006; Karami, Nasiri & Amiri 2009; Karami & Bahari 2010) and phase mixing (Heyvaerts & Priest 1983; Ruderman, Nakariakov & Roberts 1998; Smith, Tsiklauri & Ruderman 2007; Karami & Ebrahimi 2009).

People have also paid special attention to the nonlinear effects of torsional Alfvén modes. It was demonstrated numerically that the observed spiky intensity profiles due to impulsive energy releases could be obtained from nonlinear torsional waves (see e.g. Moriyasu et al. 2004; Antolin et al. 2008). Taroyan (2009) showed that small-amplitude Alfvén waves can be amplified into the nonlinear regime by the presence of siphon flows in coronal loops.

Zaqarashvili & Murawski (2007) investigated the evolution of torsional Alfvén waves in longitudinally inhomogeneous coronal loops. They concluded that the inhomogeneous mass density field leads to the reduction of a wave frequency of torsional oscillations, in comparison to that estimated from mass density at the loop apex. Also this frequency reduction results from the decrease of an average Alfvén speed as far as the inhomogeneous loop is denser at its footpoints.

Copil, Voitenko & Goossens (2010) studied torsional Alfvén waves in twisted small scale current threads of the solar corona. They showed that the trapped Alfvén eigenmodes do exist and are localized in thin current threads where the magnetic field is twisted. They pointed out that the wave spectrum is discrete in phase velocity, and the number of modes is finite and depends on the amount of the magnetic field twist. Also the phase speeds of the modes are between the minimum of the Alfvén speed in the interior and the exterior Alfvén speed.

Vasheghani Farahani, Nakariakov & Van Doorselaere (2010) investigated torsional axisymmetric long wavelength MHD modes of solar coronal plasma structures with the use of the second order thin flux tube approximation. They concluded that the phase speed of torsional waves depends upon the direction of the wave propagation, and also the waves are compressible.

Verth, Erdélyi & Goossens (2010) studied the observable properties of torsional Alfvén waves in both thin and finite-width stratified and expanding magnetic flux tubes. They demonstrated that for thin flux tubes, observation of the eigenmodes of torsional Alfvén waves can provide temperature diagnostics of both the internal and surrounding plasma. They also showed that in the finite-width flux tube regime, these waves are the ideal magneto-seismological tool for probing radial plasma inhomogeneity in solar waveguides.

All mentioned in above motivate us to have further investigate on torsional Alfvén waves by considering the effects of both density stratification and magnetic field expansion on the frequencies and eigenfunctions of torsional Alfvén modes in the magnetic flux tubes. This paper
is organized as follows. In Section 2 we introduce the model and derive the equations of motion. In Section 3 we give numerical results. Section 4 is devoted to conclusions.

2 Model and equations of motion

We consider an expanding magnetic flux tube of length $2L$ with longitudinal plasma density as typical coronal loop. The tube is assumed to be thin, $r_a/L \ll 1$, where $r_a$ is the tube radius at the apex. Following Ruderman, Verth & Erdélyi (2008) and Verth & Erdélyi (2008), the background magnetic field is assumed to have both radial and axial components with $r$- and $z$-dependence, i.e. $B_r = B_r(r, z)$ and $B_z = B_z(r, z)$. The coronal plasma is nearly zero-$\beta$ and this yields the magnetic field to be force free. For the selected magnetic field, the electrical current is in the $\phi$-direction. Hence the force free condition, i.e. $\mathbf{J} \times \mathbf{B} = 0$, is satisfied when the electrical current $\mathbf{J} = \nabla \times \mathbf{B} = 0$. The background magnetic field can be related to a vector potential field $\mathbf{A}$ as

$$\mathbf{B} = \nabla \times \mathbf{A},$$

where

$$\mathbf{A} = \frac{\psi(r, z)}{r} \mathbf{e}_\phi.$$

Therefore the radial and axial components of the magnetic field can be expressed in terms of the scalar potential $\psi$ as

$$B_r = -\frac{1}{r} \frac{\partial \psi}{\partial z}, \quad B_z = \frac{1}{r} \frac{\partial \psi}{\partial r},$$

where the magnetic field here is perpendicular to $\nabla \psi(r, z)$, i.e. the magnetic field lines lie in the surface $\psi(r, z) = \text{constant}$. Hence, the equation of the tube boundary is given by $\psi(r, z) = \psi_0$, where $\psi_0$ is a constant. Here, our aim is to study the torsional Alfvén waves in which the surface of the flux tube $\psi = \psi_0$ has an oscillating motion in the azimuthal direction. If we apply the force free condition $\nabla \times \mathbf{B} = 0$ to the equilibrium magnetic field (1), we obtain a partial differential equation for $\psi$ as

$$\frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2} = 0.$$ (4)

Verth & Erdélyi (2008) solved the above equation and found the $z$-component of the magnetic field as

$$B_z(z) = B_{z,f} \left\{ 1 + \frac{(1 - \Gamma^2)}{\Gamma^2} \left[ \frac{\cosh \left( \frac{zL}{r_f} \right) - \cosh(1)}{1 - \cosh(1)} \right] \right\},$$ (5)

and the radius of the magnetic flux tube boundary as

$$r(z) = r_f \left\{ 1 + \frac{(1 - \Gamma^2)}{\Gamma^2} \left[ \frac{\cosh \left( \frac{zL}{r_f} \right) - \cosh(1)}{1 - \cosh(1)} \right] \right\}^{-1/2},$$ (6)

where $B_{z,f} = B_z(\pm L)$ and $r_f = r(\pm L)$ are $z$-component of the magnetic field and the radius of flux tube at the loop footpoints, respectively. Also $\Gamma = \frac{r_a}{r_f} = \frac{r(a)}{r_f}$ is the tube expansion factor which is defined as ratio of the tube radius at the apex ($z = 0$) to the tube radius at the footpoints ($z = \pm L$). For a tube with constant cross section, the expansion factor is unity but for an expanding flux tube we have $\Gamma > 1$. As Ruderman, Verth & Erdélyi (2008) emphasized
the important property of this particular model is that it can describe only magnetic tubes with relatively small expansion factors, definitely smaller than 1.87. We also take into account the effect of density stratification and assume that the density varies exponentially with the height, $h$, in the atmosphere as $\rho = \rho_f e^{-h/H}$. Here $\rho_f$ is the density at the footpoints and $H$ is the density scale height. Following Ruderman, Verth & Erdélyi (2008) for a half-circle loop with length $2L$ the density can be written as

$$\rho(z) = \rho_f \exp\left[-2\mu \cos\left(\frac{\pi z}{2L}\right)\right], \quad \mu := \frac{L}{\pi H},$$

(7)

where $\mu$ is defined as stratification parameter.

Note that the Alfvén velocity in our selected model varies only along the background magnetic field. Since we used the thin tube approximation, hence the variation across the magnetic field lines is negligible. Ruderman et al. (1997a,b) to study the resonant absorption of torsional Alfvén waves considered the variation of Alfvén velocity only across the magnetic field.

The linearized MHD equations for a zero-$\beta$ plasma are

$$\frac{\partial^2 \xi}{\partial t^2} = \frac{1}{4\pi \rho} \left( \nabla \times \delta \mathbf{B} \right) \times \mathbf{B},$$

(8)

$$\delta \mathbf{B} = \nabla \times (\xi \times \mathbf{B}),$$

(9)

where $\xi = (0, 0, \xi_\phi)$ is the Lagrangian displacement of the plasma and $\delta \mathbf{B} = (0, 0, \delta B_\phi)$ is the Eulerian perturbation in the magnetic field. Note that in Eq. (8) due to the force free background magnetic field, the term $\frac{1}{4\pi \rho} \left( \nabla \times \mathbf{B} \right) \times \delta \mathbf{B}$ is absent.

We rewrite Eqs. (8) and (9) in components as

$$\frac{\partial^2 \xi_\phi}{\partial t^2} = \frac{1}{4\pi \rho} \left[ B_r \frac{\partial (r \delta B_\phi)}{\partial r} + B_z \frac{\partial \delta B_\phi}{\partial z} \right],$$

(10)

$$\delta B_\phi = \frac{\partial (B_r \xi_\phi)}{\partial r} + \frac{\partial (B_z \xi_\phi)}{\partial z}.$$  

(11)

Now like Ruderman, Verth & Erdélyi (2008) and Verth, Erdélyi & Goossens (2010) we use a non-orthogonal flux coordinate system in which $\psi$ becomes an independent variable instead of $r$, i.e. $r = r(\psi, z)$. In this coordinate system, an arbitrary function $f(r, z)$ will be transformed to another function $F(\psi, z)$ as $f(r, z) = F(\psi(r, z), z)$. Therefore, using Eq. (3) the $r$ and $z$ partial derivatives of $f$ transform to

$$\left( \frac{\partial f}{\partial r} \right)_z = r B_z \frac{\partial F}{\partial \psi}, \quad \left( \frac{\partial f}{\partial z} \right)_\psi = \frac{\partial F}{\partial z} - r B_r \frac{\partial F}{\partial \psi}.$$ 

(12)

Differentiating the identities $\psi = \psi(r(\psi, z), z)$ and $r = r(\psi(r, z), z)$ with respect to $z$ and using Eq. (3) one can get

$$\frac{\partial r}{\partial z} = \frac{B_z}{B_r}, \quad \frac{\partial r}{\partial \psi} = \frac{1}{r B_z}.$$ 

(13)

Using Eqs. (12) and (13) one can rewrite the field Eqs. (10) and (11) as

$$\frac{\partial^2 \xi_\phi}{\partial t^2} = \frac{B_z}{4\pi \rho r} \frac{\partial (r \delta B_\phi)}{\partial z},$$

(14)
\[ \delta B_\phi = r B_z \frac{\partial}{\partial z} \left( \frac{\xi_\phi}{r} \right). \] (15)

Substituting Eq. (15) into (14) and considering the time-dependence as \( e^{-i\omega t} \) the result yields

\[ \frac{B_z}{4\pi r} \frac{\partial}{\partial z} \left[ r^2 B_z \frac{\partial}{\partial z} \left( \frac{\xi_\phi}{r} \right) \right] + \rho \omega^2 \xi_\phi = 0, \] (16)

where the variables \( B_z(z), r(z) \) and \( \rho(z) \) are given by Eqs. (5), (6) and (7), respectively. Equation (16) is the same as Eq. (24) in Verth, Erdélyi & Goossens (2010) and also the same as the equation for continuum Alfvén waves on individual magnetic surfaces for a purely poloidal magnetic field for any azimuthal wave number \( m \) (see Eq. 71 in Poedts, Hermans & Goossens 1985).

Following Verth, Erdélyi & Goossens (2010) in the thin flux tube limit, all magnetic surfaces oscillate with the same frequency, i.e., the frequency is independent of magnetic surface of constant \( \psi \) as long as \( r_a/L \ll 1 \). Therefore, the torsional Alfvén waves described by Eq. (16) are global in which the whole loop oscillates with one frequency. For the finite width flux tube, Verth, Erdélyi & Goossens (2010) found the local torsional Alfvén wave frequencies that vary across the magnetic surfaces.

One notes that the MHD waves have mixed properties. For instance, torsional Alfvén waves can be coupled to kink waves (see e.g. De Groof & Goossens 2000, 2002; Goossens & De Groof 2001; De Groof, Paes & Goossens 2002; Goossens, De Groof & Andries 2002; Goossens, Andries & Arregui 2006; Goossens 2008). Indeed, the pure torsional Alfvén waves require an axisymmetric background with a purely poloidal magnetic field (no azimuthal component) and axisymmetric motions.

In the next section we solve the above differential equation using the suitable boundary conditions to obtain the eigenvalues \( \omega \) and the eigenfunctions \( \xi_\phi(z) \).

### 3 Numerical results

Here, we solve Eq. (16) using the shooting method to obtain both the eigenfrequencies and eigenfunctions of torsional Alfvén waves in stratified and expanding magnetic flux tube. We use the rigid boundary conditions and assume that \( \xi_\phi(-L) = \xi_\phi(L) = 0 \). As typical parameters for a coronal loop, we assume \( 2L = 10^5 \) km, \( \rho_f = 2 \times 10^{-14} \) gr cm\(^{-3} \) and \( B_{zf} = 100 \) G. For such a loop, one finds Alfvén speed \( v_A = \frac{B_{zf}}{\sqrt{4\pi\rho_f}} = 2000 \) km s\(^{-1} \) and Alfvén frequency \( \omega_A := \frac{v_A}{2\pi} = 0.02 \) rad s\(^{-1} \).

The effects of magnetic field expansion and density stratification on the frequencies of the fundamental and first-overtone \( n = 1, 2 \) torsional Alfvén modes are displayed in Figs. 1 and 2, respectively. Figure 1 shows that for a given stratification parameter \( \mu \), the frequencies of the fundamental and first-overtone modes decrease with increasing the expansion factor \( \Gamma \). This result is in agreement with that obtained by Ruderman, Verth & Erdélyi (2008) for the kink modes. Figure 2 presents that for a given expansion factor \( \Gamma \), the frequencies of the fundamental and first-overtone modes increase when the stratification parameter increases. This also is in good concord with the result derived by Karami, Nasiri & Amiri (2009) for the kink and fluting body waves. One notes that Verth, Erdélyi & Goossens (2010) showed that for a vertical stratified and expanding thin magnetic flux tube, for the isothermal case in which the Alfvén velocity is constant along the magnetic field, the frequency of torsional Alfvén waves remains unchanged. But for the non-isothermal case in which the Alfvén velocity varies along the flux tube, the frequency of oscillations for hot and cool tubes decreases and increases, respectively.
The results of Verth, Erdélyi & Goossens (2010) for hot and cool tubes are in good agreement with our results presented in Figs. 1 and 2, respectively. In the other words, one can say that the cool and hot loops behave like those tubes in which the effect of density stratification and magnetic field expansion is dominant, respectively.

The period ratio $P_1/P_2$ of the fundamental and first-overtone $n = 1, 2$ torsional Alfvén modes versus the expansion factor and stratification parameter is plotted in Figs. 3 and 4, respectively. Note that the period ratio is used as a seismological tool to investigate e.g., longitudinal structure (Andries et al. 2005; Andries, Arregui & Goossens 2005) and radial structure (Verth, Erdélyi & Goossens 2010) of magnetic loops. Figure 3 reveals that for a given $\mu$, the period ratio $P_1/P_2$ increases with increasing the expansion factor. This is in agreement with the result obtained by Verth & Erdélyi (2008) for the kink body modes. Figure 4 illustrates that for a given $\Gamma$, the period ratio $P_1/P_2$ decreases when the stratification parameter increases. This also is in good concord with the result reported by Karami, Nasiri & Amiri (2009) for the kink and fluting body waves. Note that Verth, Erdélyi & Goossens (2010) pointed out that for a vertical stratified and expanding thin magnetic flux tube, the period ratio of torsional Alfvén modes for the isothermal case is 2. Whereas for the non-isothermal case, the period ratio for both cool and hot tubes is bigger than 2. They also considered a semicircular thin magnetic flux tube and showed that the period ratio for the isothermal case is 2. But for the non-isothermal case, the period ratio for hot and cool tubes becomes bigger and smaller than 2, respectively. The results of Verth, Erdélyi & Goossens (2010) for hot and cool semicircular thin tubes are in good agreement with our results illustrated in Figs. 3 and 4, respectively.

From Figs. 1 to 4 one can conclude that the magnetic field expansion and density stratification have opposite effects on the frequencies and period ratio $P_1/P_2$ of torsional Alfvén modes. This is expected because as a result of magnetic field expansion, Alfvén speed becomes smaller at the loop apex and the travel time of wave between the two footpoints becomes greater. But density stratification causes Alfvén speed becomes greater at the loop apex. Hence, the travel time of wave between the two footpoints becomes smaller.

Here, we also investigate the effects of both the magnetic field expansion and density stratification on the eigenfunctions of torsional Alfvén modes. Figures 5 and 6 show the normalized anti-node shift $\Delta z/L$ of the first-overtone eigenfunction versus the magnetic expansion factor and stratification parameter, respectively. The anti-node shift of the first-overtone mode is one of the important quantities in coronal seismology. For instance, it was pointed out by Verth (2007) that if the magnetic field becomes weaker towards the loop apex, a useful magneto-seismological signature is the anti-node of the 1st harmonic since it shifts towards the loop apex. Also Andries, Arregui & Goossens (2009) used the anti-node shift in the presence of longitudinal density stratification as signature of the eigenmode modifications. They showed that the anti-node shift of the eigenfunctions of the higher kink-oscillation overtones are even more than the first overtone. Figure 5 clears that for a given stratification parameter, the normalized anti-node shift increases with increasing the expansion factor. This is in good agreement with the result obtained by Verth & Erdélyi (2008) for the kink body modes. Figure 6 presents that for a given expansion factor, the normalized anti-node shift decreases when the stratification parameter increases. In the other words, the magnetic field expansion causes positive anti-node shift and density stratification causes negative anti-node shift. To illustrate this in more detail, the eigenfunctions of the first-overtone torsional Alfvén modes for the different values of $\mu$ and $\Gamma$ are displayed in Fig. 7. It shows that as a result of density stratification, the anti-nodes of the first-overtone eigenfunction shift away from the loop apex ($z = 0$). This result was also obtained by Andries, Arregui & Goossens (2009) for kink modes. Figure 7 clears also that in the presence of magnetic field expansion, the anti-nodes shift towards the loop apex which is in
agreement with the result obtained by Verth & Erdélyi (2008) for the kink body modes. It is remarkable that Verth, Erdélyi & Goossens (2010) illustrated that for a vertical stratified and expanding thin magnetic flux tube, the antinodes of the eigenfunctions for the isothermal case remains unshifted. But for the non-isothermal case for a cool tube, the distance between the antinodes becomes more spaced out with height and for a hot tube the antinodes become closer together with height. The results of Verth, Erdélyi & Goossens (2010) for cool and hot tubes are in agreement with our results displayed in the up and down panels of Fig. 7, respectively.

Note that in some cases in above we compared our results for the frequencies, period ratio and eigenfunctions of torsional Alfvén modes with those obtained by others for kink modes. Although torsional Alfvén waves \((m = 0)\) and kink waves \((m = 1)\) have axisymmetric and nonaxisymmetric motions, respectively, in the presence of the magnetic field expansion and density stratification they show the similar behaviour. This may be caused by this fact that the restoring force in both of them is the magnetic tension force. This is why that Goossens et al. (2009) used the adjective Alfvénic for kink modes.

4 Conclusions

Here, the effects of density stratification and magnetic filed expansion on torsional Alfvén waves in coronal loops are studied. To do this, a typical coronal loop is considered as an expanding magnetic flux tube that undergoes a density varying along the tube. The linearized MHD equations are reduced to an eigenvalue problem for the azimuthal component of the Lagrangian displacement of the plasma. Using the shooting method and under the rigid boundary conditions for the loop footpoints, both the eigenfrequencies and eigenfunctions of the fundamental and first-overtone torsional Alfvén modes are obtained. Our numerical results show the following.

i) For a given density stratification parameter \(\mu\), the frequencies of the fundamental and first-overtone torsional Alfvén modes decrease and the period ratio \(P_1/P_2\) increases when the magnetic field expansion factor \(\Gamma\) increases.

ii) For a given \(\Gamma\), the frequencies of the fundamental and first-overtone torsional Alfvén modes increase and the period ratio \(P_1/P_2\) decreases when \(\mu\) increases.

iii) Both the density stratification and magnetic field expansion shift the location of the anti-nodes of the first-overtone torsional Alfvén modes but in the apposite directions.

All mentioned in above illustrate that the density stratification and magnetic field expansion have opposite effects on the oscillating properties of torsional Alfvén waves.

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Figure 1: Frequencies of the fundamental and first-overtone torsional Alfvén modes versus the expansion factor, $\Gamma = \frac{r_a}{r_f}$, for different stratification parameter $\mu = 0$ (solid line), $2/\pi$ (dashed line) and $1$ (dash-dotted line). The loop parameters are: $2L = 10^5$ km, $\rho_f = 2 \times 10^{-14}$ gr cm$^{-3}$, $B_{z,f} = 100$ G. Frequencies are in units of Alfvén frequency, $\omega_A = 0.02$ rad s$^{-1}$. 
Figure 2: Frequencies of the fundamental and first-overtone torsional Alfvén modes versus the stratification parameter, $\mu = \frac{L}{\pi H}$, for different expansion factor $\Gamma = 1$ (solid line), 1.35 (dashed line) and 1.7 (dash-dotted line). Auxiliary parameters as in Fig. 1.

Figure 3: The period ratio $P_1/P_2$ of the fundamental and first-overtone torsional Alfvén waves versus the expansion factor for different stratification parameter $\mu = 0$ (solid line), $2/\pi$ (dashed line) and 1 (dash-dotted line). Auxiliary parameters as in Fig. 1.
Figure 4: The period ratio $P_1/P_2$ of the fundamental and first-overtone torsional Alfvén waves versus the stratification parameter for different expansion factor $\Gamma = 1$ (solid line), 1.35 (dashed line) and 1.7 (dash-dotted line). Auxiliary parameters as in Fig. 1.

Figure 5: Normalized shift of the first-overtone anti-node versus the expansion factor for different stratification parameter $\mu = 0$ (solid line), $2/\pi$ (dashed line) and 1 (dash-dotted line). Auxiliary parameters as in Fig. 1.
Figure 6: Normalized shift of the first-overtone anti-node versus the stratification parameter for different expansion factor $\Gamma = 1$ (solid line), 1.35 (dashed line) and 1.7 (dash-dotted line). Auxiliary parameters as in Fig. 1.

Figure 7: Eigenfunctions of the first-overtone torsional Alfvén modes against fractional length $\zeta = z/L$ for different expansion factor and stratification parameter. In the up panel $\Gamma = 1.5$ and in the down panel $\mu = 0.5$. Auxiliary parameters as in Fig. 1.