GERBES, HOLONOMY FORMS AND REAL STRUCTURES

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Abstract. We study geometry on real gerbes in the spirit of Cheeger-Simons theory. The concepts of adaptations and holonomy forms are introduced for flat connections on real gerbes. Their relations to complex gerbes with connections are presented, as well as results in loop and map spaces.

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1. Introduction

The concept of (complex) gerbes, especially non-abelian ones, was first introduced by J. Giraud in 1971. Then the Chern-Weil theory of gerbes was carried out in the book of Brylinski [B]. Abelian gerbes were further refined and clarified by Murray [M] and Chatterjee [C] from two different viewpoints. The two approaches have their own advantages, and to some degree, it is analogous to the bundle theory where one could opt for either the principal or vector bundle approaches. Gerbes have found applications in Physics for the so-called \(B\)-fields and in generalized index theory for twisted vector bundles, see for example [A, BM, MMS] among many others.

In this paper, we will be undertaking the real version of gerbes by adopting an approach more in line with Chatterjee and Hitchin [H]. Just like real vector bundles, one should substitute the Cheeger-Simons theory for the Chern-Weil theory. Thus we focus on flat connections on real gerbes and introduce the concept of adaptations in order to characterize the holonomy form of flat connections. It turns out that one can relate real gerbes with complex gerbes, provided the latter admit real structures in a certain sense. This will be the second topic in the paper. Our third topic is to extend the main results in Brylinski [B] and Chatterjee [C] to the real gerbe case. These involve the loop space of a manifold and the map space between a surface and a manifold.

Here is an outline of the paper: in Section 2 we review some basic facts on real line bundles with flat connections. We show how to get a flat connection from a holomorphic structure on a complex line bundle (Theorem 2.4). This

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will be used in later sections. In Section 3, we present the basic theory on real gerbes and flat connections. The concepts of adaptations and holonomy forms are introduced in Definition 3.2 as well as their important properties (Theorems 3.3, 3.5). Similar to the case of line bundles, we relate real and complex gerbes. The last section deals with the loop and map spaces. By using the holonomy of a flat gerbe connection, one constructs a line bundle on the loop space (Theorem 4.1). Through a certain boundary map, one pulls back this bundle to the map space and the interesting result here is that the pull-back line bundle is canonically trivial (Theorem 4.3). In the complex gerbe case of [B], such a result has the origin in Topological Quantum Field Theory.

It is noted that throughout the paper, Riemannian metrics on a real line bundle \( l \) occupy a significant role, one reason being that they lead to canonical trivializations on \( l \otimes l \).

2. Holonomy forms, real and complex line bundles

In this section, we start with a few basic but perhaps not exactly familiar facts on real line bundles. We then relate real to complex line bundles, which serves as a prototype for gerbes. Some of the results are new and have not appeared elsewhere to our knowledge.

Let \( l \rightarrow X \) be a real line bundle with structure group \( \mathbb{R}^* \) over a smooth manifold. We emphasize that throughout the paper, \( X \) is allowed to be non-orientable or orientable but not oriented. The space of connections on \( l \) is affine modeled on \( \Omega^1(X) \), since \( \text{End}l = l \otimes l^* \) is canonically trivial, and acted by the gauge group \( \text{Map}(X, \mathbb{R}^*) \). Whereas the space of flat connections on \( l \) is affine modeled on the space \( Z^1(X) \) of closed 1-forms and is preserved by the gauge group action.

Proposition 2.1. For any fiber metric \( h = (\quad) \) on \( l \), there is a unique compatible connection \( \nabla_h \), namely such that

\[
d(s, t) = (\nabla_h s, t) + (s, \nabla_h t)
\]

where \( s, t \) are sections. Moreover \( \nabla_h \) is flat.

Proof. Locally under a frame \( s_i \) of \( l \) with \( h_i = (s_i, s_i) > 0 \), the connection is given as \( \nabla_h = d + \frac{1}{2}d \ln h_i \), which is flat since \( \frac{1}{2}d \ln h_i \) is closed. \( \square \)

In the case of a Hermitian complex line bundle, one usually concentrates on unitary connections. Proposition 2.1 indicates that it would be too restrictive to consider only Riemannian connections in the real case. Instead one should expand to consider all flat connections. Then one is led to measure how far a flat connection is from being a metric one.

Definition 2.2. Given a flat connection \( \nabla \) on \( l \), the holonomy form (or compatibility form), with respect to a metric \( h \), is defined to be the closed 1-form \( \theta(\nabla, h) = \nabla - \nabla_h \in Z^1(X) \). Locally under a frame \( s_i \),

\[
\theta(\nabla, h) = a_i - \frac{1}{2}d \ln h_i
\]
where one writes $\nabla = d + a_i$ under the same frame.

Note that even if $\nabla$ is not flat, $\theta(\nabla, h)$ is still a well-defined 1-form, although it is not closed anymore.

More generally for a higher rank Riemannian vector bundle with a flat connection, by taking determinant, one can define a similar closed 1-form, which was used in a fundamental way by Bismut-Zhang [BZ] in connection with analytic torsions.

Of course $\theta(\nabla, h)$ will be trivial if $\nabla$ is compatible with $h$. In general one sees readily that the class $[\theta(\nabla, h)] \in H^1(X, \mathbb{R})$ is independent of $h$, so it makes sense to denote it by $\tilde{c}(\nabla)$. One can think of $\tilde{c}(\nabla)$ as an obstruction to the existence of a metric compatible with $\nabla$. Note that $\tilde{c}(\nabla)$ is invariant under the gauge transformation hence it descends to a map on the quotient of flat connections under the gauge action,

$$\tilde{c} : \mathcal{F}/\mathcal{G} \to H^1(X, \mathbb{R}).$$

This map is in fact bijective. Moreover the image of $\mathcal{H} \to \mathcal{F}/\mathcal{G}, h \mapsto [\nabla_h]$ is precisely $\tilde{c}^{-1}(0)$, where $\mathcal{H}$ is the space of metrics on $\mathcal{L}$. In particular there exists a flat connection not compatible with any metric, provided $H^1(X, \mathbb{R}) \neq \{0\}$.

Recall that to each flat connection $\nabla$, one can associate its holonomy class $c(\nabla) \in H^1(X, \mathbb{R}^*)$ by using parallel transport. The class $c(\nabla)$ relates to $\theta(\nabla, h)$ as follows, explaining the terminology used in Definition 2.2.

**Proposition 2.3.** (a) The homomorphism $\ln |\cdot| : \mathbb{R}^* \to \mathbb{R}$ induces one from $H^1(X, \mathbb{R}^*)$ to $H^1(X, \mathbb{R})$, under which the holonomy class $c(\nabla)$ is mapped to $\tilde{c}(\nabla)$.

(b) The sign map $\mathbb{R}^* \to \mathbb{Z}_2 = \{\pm 1\}$ gives rise to the homomorphism $H^1(X, \mathbb{R}^*) \to H^1(X, \mathbb{Z}_2)$, under which $c(\nabla)$ is mapped to $w_1(\mathcal{L})$ hence independent of the flat connection $\nabla$.

(c) Conversely the natural homomorphisms $\mathbb{Z}_2 \hookrightarrow \mathbb{R}^*, \exp : \mathbb{R} \to \mathbb{R}^*$ lead to the decomposition

$$H^1(X, \mathbb{R}^*) = H^1(X, \mathbb{Z}_2) \times H^1(X, \mathbb{R})$$

as Abelian groups, under which $c(\nabla) = (w_1(\mathcal{L}), \tilde{c}(\nabla))$.

**Proof.** All three parts amount essentially to the following basic fact: along a loop $\rho : S^1 \to X$, the holonomy of $\nabla$ (not necessarily flat) is given by

$$\text{hol}_\rho(\nabla) = \epsilon \cdot \exp \int_{S^1} \rho^* \theta(\nabla, h)$$

where $\epsilon = w_1(\mathcal{L})[\rho], [\rho] \in H_1(X)$ and $w_1(\mathcal{L})$ is viewed as a map $H_1(X) \to \{\pm 1\}$. Compare with the complex line bundle case in [MP]. Indeed since $\rho^* \nabla$ is flat, there is a cover $\{I_i, 1 \leq i \leq n\}$ of $S^1$ by open intervals such that on each $I_i$, the pull-back bundle $\rho^* \mathcal{L}$ is trivialized by a flat frame $s_i$ of $\rho^* \nabla$.

(Actually one can choose $n = 2$.) Then under $s_i$, the connection 1-form of
\[ \rho^* \nabla \] is \( a_i = 0 \) and
\[ \rho^* \theta(\nabla, h) = \theta(\rho^* \nabla, \rho^* h) = -\frac{1}{2} d \ln h_i \]
where we set \( h_i = \rho^* h(s_i, s_i) \) which is subject to \( h_{i+1}/h_i = g_{i,i+1}^2 \) on \( I_i \cap I_{i+1} \), with \( g_{i,i+1} \) denoting the transition function of \( \rho^* l \). Pick any point \( t_i \in I_i \cap I_{i+1} \) for \( i = 1, \cdots , n - 1 \) and \( t_n \in I_n \cap I_1 \). For convenience, set \( t_0 = t_n, h_{n+1} = h_1 \). One calculates the right side of (2) as
\[ \varepsilon \cdot \exp \int_{g_1} \rho^* \theta(\nabla, h) = \varepsilon \cdot \exp \sum_{i=1}^n \int_{t_{i-1}}^{t_i} -\frac{1}{2} d \ln h_i \]
\[ = \varepsilon \cdot \exp \sum_{i=1}^n \frac{1}{2} \ln \left[ \frac{h_i(t_{i-1})}{h_i(t_i)} \right] \]
\[ = \varepsilon \cdot \prod_{i=1}^n \sqrt{h_i(t_{i-1})/h_i(t_i)} \]
\[ = \varepsilon \cdot \prod_{i=1}^n g_{i,i+1}(t_i) \cdot \sqrt{h_1(t_0)/h_{n+1}(t_n)} \]
\[ = \varepsilon \cdot \prod_{i=1}^n g_{i,i+1}(t_i) \]
where \( \varepsilon_i = \text{sgn} g_{i,i+1}(t_i) \). The last term is the holonomy of \( \nabla \) along \( \rho \) by definition, thus verifying equation (2). The signs cancel out because \( \varepsilon = w_1(l)[\rho] \) is the total sign change \( \prod_{i=1}^n [\text{sgn} g_{i,i+1}(t_i)] \) along \( \rho \).

Of course \( c(\nabla)[\rho] = \text{hol}_\rho(\nabla) \); the rest of the proof is now pretty clear. \( \square \)

Part (c) says that \( w_1(l), \tilde{c}(\nabla) \) are complementary each other in the sense that they capture respectively the topological and geometrical aspects of the bundle \( l \). The picture here contrasts with the Chern-Weil theory, where the real Chern classes alone capture both the topological and the geometrical aspects of a complex bundle. Of course the picture here should be viewed as an instance of the Cheeger-Simons theory [CS].

It is also possible to describe the above results in terms of Cech cohomology. As usual let \( R, R^* \) denote the sheaves of \( R^* \) or \( R \)-valued functions on \( X \), while \( R, R^* \) denote the constant sheaves. On any open cover of \( X \), the transition functions of \( l \) is a co-closed 1-Cech cycle in \( R^* \) hence determines a class \( [l] \in \tilde{H}^1(X, R^*) \). The short exact sequence \( \mathbb{Z}_2 \to R^* \to R \) yields a long exact sequence of cohomology groups, which in turn proves that \( \tilde{H}^1(X, R^*) \) is naturally isomorphic to \( H^1(X, \mathbb{Z}_2) \), because the fineness of \( R_\mathbb{R} \) implies \( H^k(X, R) \) vanishes for \( k > 0 \). Then under this isomorphism, \( [l] \) is mapped to \( w_1(l) \). Likewise, by using local \( \nabla \)-flat trivializations of \( l \), the transitions are local constants and form a co-closed 1-Cech cycle in \( R^* \), representing a class \( \tilde{c}(\nabla) \in \tilde{H}^1(X, R^*) \). Under the natural isomorphism \( \tilde{H}^1(X, R^*) \to H^1(X, R^*) \), \( \tilde{c}(\nabla) \) becomes the holonomy class \( c(\nabla) \).

We now compare this real picture with the complex and holomorphic pictures. First let \( M \) be any smooth manifold with an involution \( \sigma \) and \( L \to M \) a complex line bundle. Since \( \sigma \) is to be considered as a real structure, we assume \( L \) admits a fiberwise conjugate linear lifting \( \tau \) of \( \sigma \). We adopt the usual convention to call \( (L, \tau) \) a real complex line bundle or simply a Real line bundle. It will turn out to be useful also to take the conjugate linear
lifting $\sigma^*: T^*M \otimes \mathbb{C} \to T^*M \otimes \mathbb{C}$ on the $\mathbb{C}$-factor, hence a conjugate linear
extension $\sigma^*: \Omega^p_C(M) \to \Omega^p_C(M)$ on complex valued forms. Then we call a
$\sigma^*$-invariant complex form on $M$ a Real form (as oppose to a real form in
$\Omega^p(X)$). In the same spirit, a $\tau$-invariant section of $L$ is refereed as a Real
section and in general, a Real object is simply a complex object invariant
under some real structure.

Set $X = M_{\mathbb{R}} := \text{Fix} \sigma$, which we assume to be a smooth manifold. Set
also the real line bundle $l = L_{\mathbb{R}} := \text{Fix} \tau \to X$. Given a Real connection $d_A$
on $L$, namely $d_{\mathbb{R}} = d_A$ where by definition $d_{\mathbb{R}} s = \tau d_A \sigma^{-1} s$ for a section $s$, it
obviously restricts to a connection $\nabla_A$ on $l$. The curvature 2-form $F$ of $d_A$
satisfies $\nabla F = F$, where $\nabla = \sigma^* F$. Thus $F$ restricts to a Real 2-form $F'$ on $X$
($F'$ of course is the curvature of $\nabla_A$). If $d_A$ is also unitary then $F'$ is purely
imaginary, hence $F'$ must be trivial and $\nabla_A$ is flat. Alternatively if $d_A$
is compatible with a Real Hermitian metric $H$ on $L$ (i.e. $H(u, v) = H(u, v)$),
then $\nabla_A$ is compatible with $h$ hence flat, where the Riemannian metric $h$
on $l$ is the restriction of $H$.

Now move on to the holomorphic picture and assume $M$ is a complex
manifold such that $\sigma$ is anti-holomorphic. A holomorphic structure $\alpha$
on $L$ is characterized by its Dolbeault operator $\overline{\partial}_\alpha : \Omega^p(L) \to \Omega^{p,1}(L)$, which satisfies

$$\overline{\partial}_\alpha(f s) = \overline{\partial} f \cdot s + f \cdot \overline{\partial}_\alpha s$$

and $\overline{\partial}_\alpha \circ \overline{\partial}_\alpha = 0$. The set of such operators is affine modeled on the space
$Z^{0,1}(M)$ of $\overline{\partial}$-closed $(0, 1)$ forms. This is quite analogous to flat connections
on $l$. Given a Hermitian metric $H$ on $L$, one is perhaps also tempted to
consider a compatibility form $\Theta(\overline{\partial}_\alpha, H) \in Z^{0,1}(M)$. Borrowing from formula
(11), one might try to set locally

$$\Theta(\overline{\partial}_\alpha, H) = \alpha_s - \frac{1}{2} \overline{\partial} \ln H_s$$

where $\overline{\partial}_\alpha = \overline{\partial} + \alpha_s$ and $H_s = H(s, s) > 0$ under a local frame $s$ of $L$. (In
particular, if $s$ is holomorphic, $\Theta(\overline{\partial}_\alpha, H) = -\overline{\partial} \ln H_s$. Compare with the
$\alpha, H$-compatible connection $d_{\alpha, H}$, which is given by the local 1-form $\partial \ln H_s$.)
Of course $\Theta(\overline{\partial}_\alpha, H)$ is not going to be well-defined, as the above expression
depends on the choice of $s$. Nonetheless if $s$ is $\tau$-invariant, $\Theta(\overline{\partial}_\alpha, H)$ can
be restricted to a well-defined real 1-form on $X$, using which we prove the
following result which is somewhat surprising at first glance.

**Theorem 2.4.** Any holomorphic structure $\alpha$ on $L$ determines a unique flat
connection $\nabla_\alpha$ on $l$. In particular when $L$ is the trivial bundle with
the trivial $\alpha, \nabla_\alpha$ recovers the differential $d$ on $X$.

**Proof.** We can assume $\alpha$ is Real, namely $\overline{\partial}_\alpha$ commutes with $\tau$; otherwise
we replace $\overline{\partial}_\alpha$ by its average $\overline{\partial}_\alpha + \frac{1}{2} (\tau^*(\overline{\partial}_\alpha) - \overline{\partial}_\alpha)$. Take any Real Hermitian
metric $H$ on $L$. Let $h$ be its restriction to $l$ and $\nabla_h$ the compatible flat
connection. The idea of proof is to first show that under a local $\tau$-invariant
frame $s$ of $L$ near $X$, the $(0, 1)$ form
\[
\Theta_s(\partial_s, H) := \alpha_s - \frac{1}{2} \overline{\partial} \ln H_s
\]
restricts to a well-defined real closed 1-form $\theta(\partial_s, H)$ on $X$, namely independent of $s$. Then show that the flat connection defined as
\[
\nabla_\alpha := \nabla_h + 2\theta(\overline{\partial}_\alpha, H)
\]
is independent of $H$.

Since $\alpha, H$ are both Real, $\alpha_s$ and $\partial \ln H_s$ are Real; hence $\Theta_s(\partial_s, H)$ restricts to a real local 1-form $\theta_s(\partial_s, H)$ on $X$. Let $s' = fs$ be another $\tau$-invariant frame, so $H_{s'} = |f|^2 H$. Then
\[
\Theta_{s'}(\partial_{s'}, H) = \alpha_{s'} - \frac{1}{2} \overline{\partial} \ln H_{s'}
= \alpha_s + f^{-1} \overline{\partial} f - \frac{1}{2} \overline{\partial} \ln H_s - \overline{\partial} \ln |f|
= \Theta_s(\partial_s, H) + f^{-1} \overline{\partial} f - |f|^{-1} \overline{\partial} |f|.
\]
When restricted to $X$, $f$ is real and positive. Hence $f^{-1} \overline{\partial} f - |f|^{-1} \overline{\partial} |f| = 0$, from which it follows that $\theta_{s'}(\partial_{s'}, H) = \theta_s(\partial_s, H)$, giving a well-defined form $\theta(\overline{\partial}_s, H)$ on $X$. The form $\theta(\overline{\partial}_s, H)$ is closed since $\alpha_s$ and $\partial \ln H_s$ are $\partial$-closed as well as Real.

Next we show that $\nabla_\alpha$ as defined in (3) is independent of $H$. Let $H' = e^f H$ be a second Real Hermitian metric where $f$ is some real valued $\sigma$-invariant function on $M$. For their restrictions to $l$, $h' = e^f h$ holds as well. We need to show
\[
\nabla_h + 2\theta(\overline{\partial}_s, H) = \nabla_{h'} + 2\theta(\overline{\partial}_s, H').
\]
As before, $H_s', h_s', h_s'$ denote the norm squares of a local frame $s$ under the various metrics so $H_s' = e^f H_s$ and $h_s' = e^f h_s$ are still valid. Then locally we need
\[
\frac{1}{2} \ln h_s + 2[\alpha_s - \frac{1}{2} \overline{\partial} \ln H_s]|_X = \frac{1}{2} \ln h_s' + 2[\alpha_s - \frac{1}{2} \overline{\partial} \ln H_s']|_X
\]
which simplifies to $2\overline{\partial} f = df$ on $X$. The last holds true since $\overline{\partial} f$ restricts to a real 1-form: $\overline{\partial} f = \overline{\partial} f = \partial f = \partial f$ on $X$.

When $\alpha$ is trivial, we set $H_s = 1$ under the global trivial holomorphic frame so that $\overline{\partial} s = 1, \theta(\overline{\partial}_s, H) = 0$ and $\nabla_\alpha = d$. \hfill $\Box$

**Corollary 2.5.** Let $\mathcal{F}^c$ denote the space of all holomorphic structures on $L$ and $\mathcal{F}$ be the space of all flat connections on $\ell$. The following diagram commutes:
\[
\begin{array}{ccc}
\mathcal{F}^c \times Z^{0,1}(M) & \rightarrow & \mathcal{F}^c \\
\downarrow & & \downarrow \\
\mathcal{F} \times Z^1(X) & \rightarrow & \mathcal{F},
\end{array}
\]
where the map $Z^{0,1}(M) \rightarrow Z^1(X)$ sends $\omega$ to the restriction of $(\omega + \overline{\omega})/2$ on $X$. 

It is possible to construct $\nabla_\alpha$ in Theorem 2.3 directly without using a Hermitian metric. Let $s$ be $\tau$-invariant local holomorphic frame of $\alpha$. Then $s$ restricts to a local frame of $l$. One simply declares $t$ to be the flat frame of a connection $\nabla_\alpha$ on $l$; one can check that $\nabla_\alpha$ is well-defined, i.e. independent of the choice of $s$. The flatness of $\nabla_\alpha$ follows from $\nabla_\alpha \circ \nabla_\alpha = 0$. Since $\alpha$ is Real, $\nabla_\alpha$ induces a map between the spaces of Real forms: $\Omega^0(l)_R \rightarrow \Omega^{0,1}(l)_R$. It is possible to show that the restriction maps $\Omega^0(l)_R \rightarrow \Omega^0(l), \Omega^{0,1}(l)_R \rightarrow \Omega^1(l)$ are both surjective. Then $\nabla_\alpha$ is the unique map making the following diagram commute:

$$
\begin{array}{ccc}
\Omega^0(l)_R & \longrightarrow & \Omega^{0,1}(l)_R \\
\downarrow & & \downarrow \\
\Omega^0(l) & \longrightarrow & \Omega^1(l).
\end{array}
$$

Clearly the theorem does not hold for a higher rank holomorphic bundle $E$ with a real lifting, since the real bundle $E_R$ may not be flat in the first place.

The theorem suggests that the flat connection $\nabla_\alpha$ can be considered appropriately as the counter-part of a holomorphic structure $\alpha$. In the holomorphic set up, $\alpha$ and $H$ determine a unique connection $d_{\alpha,H}$, while in the real set up, $\nabla_\alpha$ and $h$ determine a unique 1-form $\theta(\nabla_\alpha, h)$. (By equation (3), $\theta(\nabla_\alpha, h) = 2\theta(\nabla_\alpha, H)$. The flat connection $\nabla_h + \theta(\nabla_\alpha, H)$ will depend on $H$. Note that $d_{\alpha,H}$ restricts to the metric connection $\nabla_h$, hence dependent on $H$ but having nothing to do with $\alpha$.)

Thus a holomorphic structure $\alpha$ on $L$ determines the topological property of the line bundle $l$ via $\nabla_\alpha$. As an application, one could study the link between the complex and real analytic torsions developed by Bismut et al [BCS, BL], which are defined on holomorphic and real determinant line bundles of elliptic operators. We hope to follow up this in a future work.

3. **Real gerbes, connections, and adaptations**

Real gerbes are next in the hierarchy after real line bundles. On a smooth manifold $X$, a real gerbe $G = \{U_i, l_{ij}, s_{ijk}\}$ consists of the following data:

- $\{U_i\}$ is an open cover of $X$,
- $l_{ij}$ is a real line bundle on $U_i \cap U_j$ with a given isomorphism $l_{ij} \otimes l_{ji} = \mathbb{R}$,
- $s_{ijk}$ is a trivialization of the bundle $l_{ijk} = l_{ij} \otimes l_{jk} \otimes l_{ki}$ over $U_i \cap U_j \cap U_k$.

Take a refinement cover if necessary so that each $l_{ij}$ is a trivial bundle and $l_{ijk}$ has a second trivialization. Then $s = \{s_{ijk}\}$ can be viewed as a Čech 2-cycle of the sheaf $\mathbb{R}^*$. One imposes that $s$ be co-closed: $\delta s = 1$.

Thus the *gerbe class* $[G] := [s] \in \check{H}^2(X, \mathbb{R}^*)$ is defined. We say that two gerbes are isomorphic if they become the same on a common refinement. Then the isomorphic class of $G$ is determined by $[G]$. Note that one can also view $[G] \in H^2(X, \mathbb{Z}_2)$ using the natural isomorphism $H^2(X, \mathbb{Z}_2) \rightarrow \check{H}^2(X, \mathbb{Z}_2) \rightarrow \mathbb{R}^*$, which comes from the short exact sequence $\mathbb{Z}_2 \rightarrow \mathbb{R}^* \rightarrow \mathbb{R}$ as
Examples. (1) Any manifold \( X \) is locally spin, hence admits a collection of spinor bundles \( S_i \to U_i \) which satisfy \( S_i = S_j \otimes l_{ij} \) for some line bundle \( l_{ij} \) on \( U_{ij} \). In the language of [M] for example, \( \{ S_i \} \) form a twisted vector bundle, the failure of which to be a global bundle is the real gerbe \( \{ l_{ij} \} \) with the gerbe class \( w_2(X) \).

(2) Take any codimension 2 submanifold \( \Sigma \subset X \). Trivializing the tubular neighborhood, one can construct easily a real gerbe that has the gerbe class \( \text{PD}[\Sigma] \in H^2(X, \mathbb{Z}_2) \).

So far this is in a complete analogy with a complex gerbe. On the other hand, to do geometry we need a connection etc., which will be different from the complex case. A gerbe connection \( \nabla = \{ \nabla_{ij} \} \) on \( \mathcal{G} \) consists of connections \( \nabla_{ij} \) on \( l_{ij} \) such that \( s_{ijk} \) is a covariant constant trivialization under the induced product connection \( \nabla_{ijk} = \nabla_{ij} \otimes \nabla_{jk} \otimes \nabla_{ki} \). We call \( \nabla \) flat if all \( \nabla_{ij} \) are flat. Likewise, a gerbe metric \( h = \{ h_{ij} \} \) on \( \mathcal{G} \) consists of a family of fiber metrics \( h_{ij} \) on \( l_{ij} \) such that each trivialization \( s_{ijk} \) has norm 1 under the induced product metric \( h_{ijk} \). The usual argument of partitions of unity shows that connections and metrics always exit on any gerbe \( \mathcal{G} \). Furthermore, a metric \( h \) leads to a flat connection \( \nabla_h = \{ \nabla_{ij}^h \} \), where \( \nabla_{ij}^h \) is the flat connection compatible with \( h_{ij} \). Hence we have established the existence of flat gerbe connections on any gerbe as well.

Remark 3.1. In the complex case [C, H], a gerbe connection \( \nabla^c = \{ \nabla_{ij}^c, F_i \} \) consists of two parts: the part of local connections \( \nabla_{ij}^c \) on \( U_i \cap U_j \) (called the 0-connection), and the part of local 2-forms \( F_i \) on \( U_i \) (called the 1-connection) which are subject to a certain condition. Likewise it is possible to incorporate a collection of local 2-forms into our real gerbe connection \( \nabla = \{ \nabla_{ij} \} \). However our main interest lies in flat gerbe connections, for which we can simply take trivial local 2-forms. Thus 1-connections do not play any significant role here.

Given a flat gerbe connection \( \nabla = \{ \nabla_{ij} \} \), \( \nabla_{ij} \)-flat trivializations on \( l_{ij} \) induce a \( \nabla_{ijk} \)-flat trivialization on each \( l_{ijk} \), which together with the original \( \nabla_{ijk} \)-flat trivialization \( s_{ijk} \) define a closed Čech 2-cycle of the constant sheaf \( \mathbb{R}^* \). We call the resulted class \( c(\nabla) \in \tilde{H}^2(X, \mathbb{R}^*) = H^2(X, \mathbb{R}^*) \) the holonomy of \( \nabla \). Given a smooth map defined on a closed oriented surface, \( f : \Sigma \to X \), the holonomy around \( \Sigma \) is by definition the value \( c(\nabla)(f_*[\Sigma]) \in \mathbb{R}^* \). As in the case of line bundles, under the natural homomorphism \( H^2(X, \mathbb{R}^*) \to H^2(X, \mathbb{R}) \), the image of \( c(\nabla) \) recovers the gerbe class \( [\mathcal{G}] \). (Indeed both classes are represented by the same 2-cocycle.) Let \( \bar{c}(\nabla) \in H^2(X, \mathbb{R}) \) be the image class of \( c(\nabla) \) under the other natural homomorphism \( H^2(X, \mathbb{R}^*) \to H^2(X, \mathbb{R}) \). To do a Cheeger-Simons type theory here means to find a differential form representing \( \bar{c}(\nabla) \). Such a form will be
the gerbe version of the holonomy form introduced in the previous section. 
The following is a main definition of the paper.

**Definition 3.2.** Consider a gerbe metric $h = \{h_{ij}\}$ and a flat connection 
$\nabla = \{\nabla_{ij}\}$ on $\mathcal{G}$. An adaptation to $(\nabla, h)$ is a collection of 1-forms $\beta = \{\beta_i\}$ on $\{U_i\}$ such that

$$\beta_i - \beta_j = \theta(\nabla_{ij}, h_{ij}) \text{ on } U_i \cap U_j,$$

where $\theta(\nabla_{ij}, h_{ij})$ is the holonomy form of $\nabla_{ij}$ with respect to $h_{ij}$. Since each 
$\theta(\nabla_{ij}, h_{ij})$ is closed, $\{d\beta_i\}$ fit together to yield a global 2-form $B$ on $X$. We 
call $B$ the holonomy form of $(\nabla, h, \beta)$ (or of $\nabla$ with respect to $h, \beta$).

Note that even if $\nabla$ is not flat, it still makes sense to define adaptations. 
Of course the 2-form $B$ is no longer well-defined globally but can be viewed 
instead as a “twisted” 2-form on $\theta$ where $\theta = \nabla_{ij}$.

The standard sheaf theory shows the existence of adaptations: Since $\nabla, h$ are both compatible with the trivialization $s_{ijk}$ locally, $\{\theta(\nabla_{ij}, h_{ij})\}$ is a 
closed Cech 1-cocycle in the sheaf $\mathcal{A}^1$ of 1-forms on $X$. The fineness of $\mathcal{A}^1$ 
guarantees the existence of a 0-cycle $\beta = \{\beta_i\}$ of 1-forms.

Adaptations are however not unique for a given metric and a flat connection: any two differ by a global 1-form on $X$. To some degree, $\beta$ resembles a 1-connection in the complex gerbe case, while $B$ is analogous to the curvature 
3-form of the 1-connection. The following is parallel to Proposition 2.3.

**Theorem 3.3.** Suppose $B$ is the holonomy form of a flat gerbe connection 
$\nabla$ with respect to a gerbe metric $h$ and an adaptation $\beta$. Then the class 
$[B] \in H^2(X, \mathbb{R})$ depends on $\nabla$ only. In fact $B$ represents the class $c(\nabla)$. 
Under the natural product $H^2(X, \mathbb{R}^*) = H^2(X, \mathbb{Z}_2) \times H^2(X, \mathbb{R})$, we have 
$c(\nabla) = ([\mathcal{G}], [B])$.

**Proof.** First we show that $[B]$ is independent of the choice of adaptations: 
If $\beta' = \{\beta'_i\}$ is another adaptation to $(\nabla, h)$, then there is a global 1-form 
$\alpha$ on $X$ such that $\beta'_i = \beta_i + \alpha$. Then the corresponding holonomy form $B'$ 
satisfies $[B'] = [B + d\alpha] = [B]$.

To see $[B]$ is independent of $h$, let $h' = \{h'_{ij}\}$ be another gerbe metric. 
Then there is a positive function $f_{ij}$ on each $U_i \cap U_j$ such that $h'_{ij} = f_{ij} h_{ij}$. 
Since the metrics $h_{ijk}, h'_{ijk}$ both normalize the trivialization $s_{ijk}$, $\{f_{ij}\}$ form 
a co-closed Cech 1-cycle of the sheaf $\mathbb{R}^+$. But $\check{H}^1(X, \mathbb{R}^+)$ is trivial (due to 
the fineness of $\mathbb{R}^+$), consequently, there is a 0-cycle $\{f_i\}$ with co-boundary 
$\{f_{ij}\}$. Now one checks easily that $\beta'_i = \beta_i - \frac{1}{2} d \ln f_i$ is an adaptation to 
$(\nabla, h')$ and the associated holonomy form $B' = d\beta'_i = d\beta_i = B$, from which 
$[B'] = [B]$ for sure.

Let $\Sigma$ be a closed surface and $f : \Sigma \to X$ a smooth map representing a 
class $\alpha \in H_2(X, \mathbb{Z})$. The remaining statement in the theorem is essentially
due to that the holonomy of $\nabla$ around $\Sigma$ is
\begin{equation}
(4) \quad c(\nabla)(\alpha) = \varepsilon \cdot \exp \int_{\Sigma} f^* B,
\end{equation}
where $\varepsilon = [G](\alpha) \in \{\pm 1\}$. By functorality, it is enough to show for the case that $X = \Sigma$ and $f = id$, where (4) becomes $c(\nabla) = \varepsilon \cdot \exp \int_{\Sigma} B$. To compute the integral on the right side, the idea is to partition $\Sigma$ suitably and apply Stokes’ Theorem repeatedly. For example when $\Sigma$ is the 2-sphere $S^2$, view $S^2$ as a rectangle $I^2$ with all four sides collapsed to the base point of $I^2$. Take a small enough rectangular subdivision of $I^2$ so that $G$ is trivialized around all grid lines. On each sub-rectangle, apply Stokes’ Theorem twice to reduce the integral first to the four sides and then to the four corners. The final outcome is exactly the value $c(\nabla) \in H^2(S^2, R^*) = R^*$. The complete details are left to the interested reader (compare with the complex gerbe case in [MP]).

\textbf{Corollary 3.4.} Fix any metric $h$ on $G$. A flat gerbe connection $\nabla$ on $G$ has trivial holonomy iff the gerbe class $[G]$ is trivial and there exists an adaptation to $\nabla, h$ with the holonomy form vanishing identically.

\textbf{Proof.} In general, start with adaptation $\beta$ to $(\nabla, h)$ with holonomy form $B$. Then every form $B'$ representing the class $[B]$ can be realized by the holonomy form of another adaptation $\beta'$ to $(\nabla, h)$. In fact, $B' = B + d\gamma$ for some 1-form $\gamma$ on $X$ and one simply takes $\beta' = \{\beta'_i + \gamma\}$ where $\beta = \{\beta_i\}$ with respect to some open cover $\{U_i\}$. In the situation of the corollary, if $c(\nabla) = 0$ then $[B] = 0$ and one uses $B' = 0$. The rest is clear.

The following sums up the main properties that will be quite useful for Section 4.

\textbf{Theorem 3.5.} Suppose the gerbe class $[G] \in \check{H}^2(X, R^*)$ is trivial.

(a) Then $G$ admits a trivialization, namely a collection of line bundles $\{l_i\}$ on $\{U_i\}$ together with isomorphisms $l_i \otimes l_j^* = l_{ij}$ on $U_i \cap U_j$. Given a second trivialization $\{l'_i\}$, a global line bundle $\xi$ on $X$ is resulted by patching all $l_i \otimes (l'_i)^*$ together.

(b) Given any gerbe connection $\{\nabla_{ij}\}$ on $G$, there is a collection of connections $\nabla_i$ on $l_i$ such that $\nabla_i \otimes \nabla_j^* = \nabla_{ij}$ under the isomorphism $l_i \otimes l_j^* = l_{ij}$. From a second trivialization $\{l'_i\}$ with connections $\{\nabla'_i\}$, $\nabla_i \otimes (\nabla'_i)^*$ together form a well-defined global connection $D$ on the bundle $\xi$.

(c) Suppose further that $\nabla = \{\nabla_{ij}\}$ is a flat gerbe connection with trivial holonomy, $\varepsilon(\nabla) = 0$. Then one can choose all connections $\{\nabla_i\}$ in part (b) to be flat. Given a second trivialization with flat local connections, the induced connection $D$ from the last part is flat as well.

(d) Let $h = \{h_{ij}\}$ be a gerbe metric on $G$. Then there is a collection of metrics $h_i$ on $l_i$ such that $h_{ij}/h_{ij} = h_{ij}$ on $U_i \cap U_j$. For the flat connections $\{\nabla_i\}$ constructed in part (c), the 1-form collection $\beta_0 = \{\theta(\nabla_i, h_i)\}$ is an adaptation to $\nabla$ and $h$. Moreover the holonomy form $B^0$ of $(\nabla, h, \beta^0)$ is trivial, which in particular proves Corollary 3.4 for a second time.
(e) The second part of (d) has a partial converse in the following sense. Fix a gerbe metric \( h = \{ h_{ij} \} \) and a collection of local metrics \( \{ h_i \} \) as in (d). Suppose \( \nabla = \{ \nabla_{ij} \} \) is a flat gerbe connection with trivial holonomy and \( \beta = \{ \beta_i \} \) is any adaptation to \( (\nabla, h) \). If holonomy form \( B \) is trivial, then there exist flat gerbe connections \( \nabla' = \{ \nabla'_{ij} \} \) with trivial holonomy and a collection of flat local connections \( \{ \nabla'_i \} \) subordinate to \( \nabla' \) as in (c) such that \( \beta_i = \theta(\nabla'_i, h_i) \). In other words, (d) and (e) imply essentially that for a given \( \beta \), equations \( \beta_i = \theta(\nabla'_i, h_i) \) admit solutions for \( \nabla'_i \) iff \( B = 0 \).

Proof. (a) Take a refinement cover of \( \{ U_i \} \) if necessary, so that each \( l_{ij} \) is trivialized on \( U_{ij} \) and \( G \) is represented by a 2-cocycle \( s = \{ s_{ijk} \} \). Since \( [G] = [s] = 0 \in \check{H}^2(X, \mathbb{R}^*) \), there is a Cech 1-cocycle \( f = \{ f_{ij} \} \in \check{C}^1(X, \mathbb{R}^*) \) with coboundary \( \delta f = s \). Then the trivial bundle \( l_i \to U_i \) is glued with the trivial bundle \( l_j \otimes l_{ij} \) on \( U_i \cap U_j \) via \( f_{ij} \). By utilizing \( f_{ij} \) as local transition functions, this gives a desired trivialization on the original cover when \( l_{ij} \) is not necessarily trivial. For another trivialization \( \{ l'_i \} \) of \( G \), we have \( l_i \otimes l'_j = l'_i \otimes (l'_j)^* \) on \( U_i \cap U_j \). Hence \( l_i \otimes (l'_i)^* = l_j \otimes (l'_j)^* \), namely the local bundles \( \{ l_i \otimes (l'_i)^* \} \) glue together to form a global real line bundle \( \xi \) on \( X \).

(b) Still assume \( G = \{ l_{ij} \} \) and \( \{ l_i \} \) are both locally trivialized as in (a), so that \( \nabla_{ij} = d + a_{ij} \) and \( \nabla_i = d + a_i \) for some 1-forms \( a_{ij}, a_i \). To find the required \( \nabla_i \), one needs to have some \( a_i \) such that

\[
(5) \quad a_i - a_j = a_{ij} + f^{-1}_{ij} df_{ij},
\]

namely \( T := \{ a_{ij} + f^{-1}_{ij} df_{ij} \} \) is a coboundary Cech cycle in the sheaf \( \mathcal{A}^1 \) of 1-forms on \( X \). This is the case as \( T \) is closed and the Cech cohomology \( \check{H}^1(X, \mathcal{A}^1) \) is trivial from the fineness of \( \mathcal{A}^1 \). For a second trivialization \( \{ l'_i \} \) with local connections \( \{ \nabla'_i \} \), clearly \( \nabla_i \otimes \nabla'_i = \nabla_j \otimes \nabla'_j \) under the same isomorphism \( l_i \otimes (l'_i)^* = l_j \otimes (l'_j)^* \), forming a connection \( D \) on \( \xi \) by gluing.

(c) When \( \nabla = \{ \nabla_{ij} \} \) is flat, we choose flat trivializations for each bundle \( l_{ij} \) in parts (a), (b) above, so that \( a_{ij} = 0 \) and \( G \) is represented by the 2-cocycle \( s = \{ s_{ijk} \} \) which now lives in the constant subsheaf \( \mathbb{R}^* \subset \check{R}^* \). Since the holonomy \( c(\nabla) = [s] = 0 \in \check{H}^2(X, \mathbb{R}^*) \), one can choose a local constant 1-cycle \( \{ f_{ij} \} \) with coboundary equal to \( s \). Thus in equation (5) above, the right side is identically zero, which means we can choose all \( a_i = 0 \) to get the desired flat connections \( \{ \nabla_i \} \). For a second trivialization of \( G \) together with local flat connections, the induced connection \( D \) is flat.

(d) As above, refine the open cover so that \( G \) is trivialized locally. Then each \( h_{ij} \) is a positive function on \( U_i \cap U_j \). Since \( h_{ij} h_{jk} h_{ki} = 1 \) on \( U_i \cap U_j \cap U_k \), \( \{ h_{ij} \} \) form a closed 1-cocycle in the sheaf \( \check{R}^+ \) of positive functions. As \( \check{H}^1(X, \check{R}^+) \) vanishes by fineness of \( \check{R}^+ \), there is a 0-cycle \( \{ h_i \} \) with coboundary \( h_i h_j^{-1} = h_{ij} \).

For flat connections \( \nabla_i \) constructed in (c), by definition \( \theta(\nabla_i, h_i) = a_i - \frac{1}{2} d \ln h_i \) and \( \theta(\nabla_{ij}, h_{ij}) = a_{ij} - \frac{1}{2} d \ln h_{ij} \). It is then easy to see that \( \{ \theta(\nabla_i, h_i) \} \)
is an adaptation to $\nabla$ and $h$, namely

$$\theta(\nabla_i, h_i) - \theta(\nabla_j, h_j) = \theta(\nabla_{ij}, h_{ij}).$$

The holonomy form $B^0$ is trivial, since $\theta(\nabla_i, h_i)$ are all closed.

(e) Start with some local flat connections $\{\nabla_i\}$ subordinate to $\nabla$ as in (c). Set $\gamma_i = \beta_i - \theta(\nabla_i, h_i)$. Then on $U_i \cap U_j$,

$$\gamma_i - \gamma_j = [\beta_i - \beta_j] - \left[\theta(\nabla_i, h_i) - \theta(\nabla_j, h_j)\right]$$

$$= \theta(\nabla_{ij}, h_{ij}) - \theta(\nabla_{ij}, h_{ij}) = 0.$$

Hence we have a global 1-form $\gamma$ on $X$ and $d\gamma = B$.

If $B = 0$, then on a possibly refined open cover, $\beta_i - \theta(\nabla_i, h_i) = dg_i$ for some function $g_i$. Put $g_{ij} = g_i - g_j$ on $U_i \cap U_j$, and define $\nabla'_{ij} = \nabla_{ij} + dg_{ij}$.

Since $\{g_{ij}\}$ is a closed Cech 1-cycle in the sheaf $R$, $\{\nabla'_{ij}\}$ form a flat gerbe connection with trivial holonomy. Moreover the flat local connections $\nabla'_i = \nabla_i + dg_i$ are subordinate to $\{\nabla'_{ij}\}$ in the sense of part (c). One checks readily that $\beta_i = \theta(\nabla'_i, h_i)$.

In (c), without assuming $c(\nabla) = 0$, there may not exist flat local connections $\nabla_i$ satisfying (4), because the sheaf $\mathcal{Z}^1$ of closed 1-forms on $X$ has the non-trivial cohomology $\tilde{H}^1(X, \mathcal{Z}^1)$ in general. (In fact $\tilde{H}^1(X, \mathcal{Z}^1) = H^1(X, R).$)

**Definition 3.6.** Adapting the terms from Chatterjee [C], we call $\{l_i\}, \{\nabla_i\}, \{h_i\}$ an object bundle, object connection and object metric, which will all be referred to as objects conveniently. They are respectively subordinate to the gerbe $\mathcal{G}$, the connection $\nabla$ and the metric $h$. Furthermore, two objects of $\mathcal{G}$ determine the difference bundle $\xi$, two objects of $\nabla$ determine the difference connection $D$, etc.

From another point of view, as in [M, Y] for example, $\{l_i\}$ can also be called a twisted line bundle on $X$ over the gerbe $\mathcal{G}$, while $\{\nabla_i\}$ a twisted connection over the gerbe connection $\nabla = \{\nabla_{ij}\}$.

In the final part of the section, we consider Real complex gerbes and relate them to real ones. As in the previous section, $\sigma : M \to M$ is a smooth involution with fixed point set $X$. Take a complex gerbe $\mathcal{G}^c = \{L_{ijk}, s_{ijk}, U_i^c; i, j, k \in I\}$ on $M$. Assume the cover $\{U_i^c; i \in I\}$ is Real, namely there is an involution $I \to I, i \mapsto \overline{i}$, such that $\sigma : U_i^c \to U_i^c$ is a diffeomorphism for any $i \in I$, and $i = \overline{i}$ whenever $X \cap U_i^c \neq \emptyset$. Then we say $\mathcal{G}^c$ is Real if $\sigma : U_i^c \cap U_j^c \to U_i^c \cap U_j^c$ has a Real lifting $\tau_{ij} : L_{ij} \to L_{ij}^\tau$. Obviously the real part of $\mathcal{G}^c$ yields a real gerbe $\mathcal{G} = \{l_{ijk}, s_{ijk}, U_i; i, j, k \in I\}$ on $X$ where $I_R = \{i \in I; X \cap U_i^c \neq \emptyset\}$.

A gerbe connection $\nabla^c = \{\nabla^c_{ij}, F_i\}$ on $\mathcal{G}^c$ is called Real if each $\nabla^c_{ij}$ is Real with respect to $\tau_{ij}$ and $\overline{F_i} = F_i$ (i.e. $F_i$ is Real also), where $\overline{F_i} = \sigma^*F_i$ and $\sigma^*$ is conjugate linear on complex forms as before. A Hermitian metric $H = \{H_{ij}\}$ on $\mathcal{G}^c$ is called Real in the obvious sense.

According to Chatterjee [C], a gerbe connection $\nabla^c = \{\nabla^c_{ij}, F_i\}$ is compatible with a gerbe Hermitian metric $H$ if each $\nabla^c_{ij}$ is compatible with $H_{ij}$.
in the usual sense and $F_i$ is purely imaginary, the latter being so required
in view that the curvature of each $\nabla^c_{ij}$ is purely imaginary. Clearly a Real
Hermitian connection $\nabla^c$ restricts to a flat connection $\nabla = \{\nabla_{ij}; i, j \in \mathcal{I}_R\}$
on $\mathcal{G}$. Note that $F_i$ restricts trivially on $X$, suggesting once more that there
is no need to include any 2-forms as a part of any real gerbe connection.

Suppose further that $M$ is a complex manifold. By definition a holomorphic structure $\alpha = \{\alpha_{ij}\}$ on $\mathcal{G}^c$ consists of a collection of holomorphic structures $\alpha_{ij}$ on $L_{ij}$ such that $s^c_{ijk}$ is a holomorphic section. A connection $\nabla^c$ is compatible with $\alpha$ if each $\nabla^c_{ij}$ is so with $\alpha_{ij}$ and $F_i$ has no $(0, 2)$-component (see [C]). Thus a Hermitian holomorphic gerbe $\mathcal{G}^c$ admits a unique compatible set $\{\nabla^c_{ij}\}$ of local connections (the 0-connection), while the 1-connection $\{F_i\}$ is now a collection of local imaginary $(1, 1)$-forms which however are not unique.

Assume $\sigma : M \to M$ is anti-holomorphic and $\alpha$ is Real in the sense that each $\alpha_{ij}$ is Real with respect to $\tau_{ij}$. Then Theorem 2.4 says that any Real holomorphic structure $\alpha$ on $\mathcal{G}^c$ determines a unique flat connection $\nabla^\alpha$ on $\mathcal{G}$. Note that $\alpha, H$ do not determine an adaptation subordinated to $(\nabla^\alpha, h)$, nor does a 1-connection $\{F_i\}$. However we can obtain a unique adaptation from objects:

**Example.** Suppose the holomorphic gerbe class $[\alpha] = 0 \in H^2(M, \mathcal{O}^*)$ so that $(\mathcal{G}^c, \alpha)$ admits a holomorphic object $\{L_i, \alpha_i\}$. Endow $\{L_i\}$ with a Hermitian object metric $H^{ob} = \{H_i\}$ of $H = \{H_{ij}\}$. If the objects are both Real, then one has a well-defined adaptation to $(\nabla^\alpha, h)$, given by $\{\theta^\alpha_{ij}(h_i)\}$, where $\nabla_i^{ob}$ is the flat connection induced by $\alpha_i$ (using Theorem 2.4) and $h_i$ is the restriction of $H_i$.

Going in the opposite direction, one can complexify a real gerbe to get a complex gerbe on the same open cover. Through complexification, a flat real gerbe connection becomes a flat complex gerbe connection with 1-connection trivial.

### 4. Holonomy Bundles on Loop Spaces and Map Spaces

Consider the free loop space $LX = \{\rho \mid \rho : S^1 \to X \text{ smooth}\}$. The evaluation map $LX \times S^1 \to X$ induces a homomorphism

$$H^2(X, G) \to H^2(LX \times S^1, G).$$

Composing this with the slant product $H^2(LX \times S^1, G) \to H^1(LX, G)$ over the generator of $H_1(S^1, G)$, we have the homomorphism

$$\mu : H^2(X, G) \to H^1(LX, G).$$

The following results can be viewed as geometric interpretations of $\mu$ in the cases that $G$ is $\mathbb{Z}_2, \mathbb{R}^*$, or $\mathbb{R}$. 

Theorem 4.1. (a) There is a well-defined line bundle $\tilde{l} \to LX$ associated to each gerbe $\mathcal{G}$ with connection $\nabla$. The isomorphism class of $\tilde{l}$ is independent of the choice of $\nabla$ and the association $[\mathcal{G}] \mapsto [\tilde{l}]$ recovers the homomorphism $\mu : H^2(X, \mathbb{Z}_2) \to H^1(LX, \mathbb{Z}_2)$.

(b) If $\nabla$ is flat then $\tilde{l}$ carries a natural flat connection $\tilde{\nabla}$ as well. The holonomies of $\nabla$ and $\tilde{\nabla}$ re-establish the homomorphism $\mu : H^2(X, \mathbb{R}^*) \to H^1(LX, \mathbb{R}^*)$.

(c) Given a gerbe metric $h$ and a flat connection $\nabla$ on $\mathcal{G}$, each adaptation $\beta$ to $(\nabla, h)$ corresponds to a unique metric $\tilde{h}$ on $\tilde{l}$. Furthermore, the map $[B] \mapsto [\theta]$ realizes the homomorphism $\mu : H^2(X, \mathbb{R}) \to H^1(LX, \mathbb{R})$, where $B, \theta$ are respectively the holonomy forms of $(\nabla, \beta, h)$ and $(\nabla, \tilde{\beta}, \tilde{h})$.

Proof. We focus on the constructions, leaving most of the verifications to the interested reader, since they can be checked as in the complex gerbe case.

(a) Start with an open cover $\{V_a\}$ of $X$ such that $\{LV_a\}$ covers $LX$, $H^2(V_a, \mathbb{R}^*)$ is trivial and $V_a \cap V_b$ consists of contractible components for any $a \neq b$. (For example take a small tubular neighborhood of each loop in $X$. The cover $\{V_a\}$ does not have to with the cover $\{U_i\}$ where $\mathcal{G}$ is locally trivialized.) On each $V_a$, $\mathcal{G}$ restricts to a trivial gerbe $\mathcal{G}|_{V_a}$ because of the triviality of $H^2(V_a, \mathbb{Z}_2)$. Applying parts (a), (b) of Theorem 3.5 to $\mathcal{G}|_{V_a}$ with the gerbe connection $\nabla|_{V_a}$, we have an object bundle with object connection. Repeat this with $V_b$. On $V_a \cap V_b$, we have now two restricted objects with connections as well as their difference line bundle $\xi_{ab} : V_a \cap V_b$ with a connection $D_{ab}$. Introduce a map $g_{ab} : LV_a \cap LV_b \to \mathbb{R}^*$, where at each loop $\rho \in LV_a \cap LV_b = L(V_a \cap V_b)$, $g_{ab}(\rho)$ is the holonomy of $D_{ab}$ along $\rho$. Then our bundle $\tilde{l}$ is defined by the transition functions $\{g_{ab}\}$ with respect to the open cover $\{LV_a\}$.

(b) In this case, the restricted gerbe connection $\nabla|_{V_a}$ is flat and has a trivial holonomy from the triviality of $H^2(V_a, \mathbb{R}^*)$. By part (c) of Theorem 3.5, the difference connection $D_{ab}$ is flat, hence $g_{ab}$ are all locally constant, because $V_a \cap V_b$ is component-wise contractible. Thus $\{g_{ab}\}$ gives the expected flat connection $\tilde{\nabla}$ on $\tilde{l}$.

(c) On each $V_a$, apply the proof of part (e) of Theorem 3.5 to the restrictions of $\mathcal{G}, \nabla, \beta$, so that we have a global 1-form $\gamma_a$ (depending on objects). Then the metric $\tilde{h}$ is given by the family of functions $\{\tilde{h}_a\}$ on the open cover $\{LV_a\}$, where $\tilde{h}_a : LV_a \to \mathbb{R}^+, \rho \mapsto \exp(2\int_\rho \gamma_a)$. To see $\{\tilde{h}_a\}$ can be glued via the transitions $\{g_{ab}\}$, we just need to check that at any loop $\rho \in LV_a \cap LV_b$, $\exp[2\int_\rho (\gamma_a - \gamma_b)] = g_{ab}^2(\rho)$, or equivalently

$$
\pm \exp\left[\int_\rho (\gamma_a - \gamma_b)\right] = g_{ab}(\rho)
$$

where the right side is by definition the holonomy of $D_{ab}$ along $\rho$. To compute the left side, let $\mathcal{G}$ be locally trivialized on some open cover $\{U_i\}$ of $X$ and...
let \( \{U_i^a\} \) be the induced cover of \( V_a \) so that we have an object, an object connection, an object metric \( \{\nu_i^a\} \), \( \{\gamma_i^a\} \), \( \{h_i^a\} \) of \( G|_{V_a} \), \( \nabla|_{V_a} \), \( h|_{V_a} \) respectively, as in Theorem 3.5. Then \( \gamma_a = \beta_i^a - \theta(\gamma_i^a, h_i^a) \) on \( U_i^a \subset V_a \) from (e) of Theorem 3.5. Likewise, \( \gamma_b = \beta_i^b - \theta(\gamma_i^b, h_i^b) \) on \( U_i^b \subset V_b \) by working with restrictions to \( V_b \). Now cover the loop \( \rho \subset V_a \cap V_b \) with some common open sets \( W_k = U_i^a = U_i^b \). On each \( W_k \), \( \gamma_a - \gamma_b = \theta(\gamma_i^a, h_i^a) - \theta(\gamma_i^b, h_i^b) \), since \( \beta_i^a = \beta_i^b \), both being restrictions of the same adaptation \( \beta \) from \( X \). On the other hand, \( \theta(\gamma_i^a, h_i^a) - \theta(\gamma_i^b, h_i^b) = a_k - \frac{1}{2} \ln h_k \), where \( a_k \) is the connection 1-form of \( D_{ab} \) and \( h_k = h_i^a / h_i^b \). Here we use the fact that \( D_{ab} \) is the difference connection of \( \{\gamma_i^a\}, \{\gamma_i^b\} \). Equation (7) now holds true because with a suitable sign,

\[
\pm \exp\left[\int_\rho (\gamma_a - \gamma_b)\right] = \pm \exp\left[\sum_k \int_{\rho \cap W_k} \left(a_k - \frac{1}{2} \ln h_k\right)\right]
\]

calculates the holonomy of \( D_{ab} \) along \( \rho \), given the (difference) metric \( h_{ab} = \{h_i\} \) on the difference bundle of \( \{\nu_i^a\}, \{\nu_i^b\} \). Indeed one can check that the integrand is the global holonomy form of \( (D_{ab}, h_{ab}) \) along \( \rho \). Compare with formula (2) in the the proof of Proposition 2.3.

A priori \( \gamma_a \) depends on the choice of various objects \( \{\nu_i^a\}, \{\gamma_i^a\}, \{h_i^a\} \) on \( V_a \). But by taking \( a = b \), the above argument shows easily that the function \( h_a \) actually does not depend on such choices, since the difference connection \( D_{aa} \) has trivial holonomy.

It is possible to describe the principal bundle of \( \mathcal{L} \to LX \) explicitly, which will be useful later.

**Corollary 4.2.** (a) The principal \( \mathbb{R}^* \)-bundle \( P_l \) of \( \mathcal{L} \) can be constructed as follows. At a point \( \rho \in LX \), the fiber \( P_{l,\rho} \) consists of equivalence classes of flat object connections of \( (\rho^*G, \rho^*\nabla) \) on \( S^1 \), where two flat object connections are equivalent if their difference connection has trivial holonomy.

(b) If the gerbe connection \( \nabla \) is flat, then the associated flat connection \( \bar{\nabla} \) on \( P_l \) can also be described explicitly.

**Proof.** (a) For a dimension reason, the pull-back gerbe connection \( \rho^*\nabla \) on \( S^1 \) is flat and with trivial holonomy. By part (c) of Theorem 3.5, the fiber \( P_{l,\rho} \) is well-defined and is acted transitively by the group of isomorphic flat difference connections on \( S^1 \). The last group is \( H^1(S^1, \mathbb{R}^*) = \mathbb{R}^* \), hence \( P_l \) is a principal \( \mathbb{R}^* \)-bundle on \( LX \). Clearly \( P_l \) associates with \( l \), as the transition functions of \( P_l \) on the open cover \( \{LV_a\} \) constructed in the proof of Theorem 4.1 are also given by the holonomy of difference connections on \( S^1 \).

(b) Take a path \( f : S^1 \times [0, 1] \to X \) in \( LX \) between two loops \( \rho_0, \rho_1 \in LX \). Then the pull-back flat gerbe connection \( f^*\nabla \) has trivial holonomy, since \( H^2(S^1 \times [0, 1], \mathbb{R}^*) = \{1\} \). Part (c) of Theorem 3.5 tells us that \( (f^*G, f^*\nabla) \) admits a flat object on \( S^1 \times [0, 1] \). The restricted flat objects to \( \rho_0, \rho_1 \) characterize the parallel transport of \( \bar{\nabla} \) along the path \( f \). (Note that here one
needs $\nabla$ to be flat in order to get a flat $f^*\nabla$. This is slightly different from the complex gerbe case, where the pull-back gerbe connection is automatically flat as the curvature 3-form has to vanish on the 2-dimensional manifold $S^1 \times [0, 1]$.)

Next let $\Sigma$ be an oriented surface with boundary consisting of $m$ components. On each component $\partial_k \Sigma$, fix an orientation-preserving parameterization $S^1 \to \partial_k \Sigma$. Let $\text{Map}(\Sigma, X)$ denote the space of smooth maps from $\Sigma$ to $X$. Then we have $m$ natural maps $b_k : \text{Map}(\Sigma, X) \to LX$, where for $f : \Sigma \to X$, the loop $b_k(f)$ is the composition of $S^1 \to \partial_k \Sigma$ with $f$. The following extends the main result in Brylinski [B] to the real case.

**Theorem 4.3.** Suppose $\nabla = \{\nabla_{ij}\}$ is a flat connection defined on a gerbe $\mathcal{G} = \{l_{ij}\}$.

(a) The line bundle $\tilde{l} = \bigotimes_{k=1}^{m} b_k^* \tilde{l} \to \text{Map}(\Sigma, X)$ carries a canonical trivialization $s$, where $\tilde{l}$ is the bundle constructed in Theorem 4.1.

(b) The trivialization $s$ is flat with respect to the pull-back connection $\nabla = \bigotimes b_k^* \tilde{\nabla}$, where $\tilde{\nabla}$ is the flat connection on $\tilde{l}$ induced by $\nabla$.

(c) Let $h = \{h_{ij}\}$ be a metric on $\mathcal{G}$ and $\beta = \{\beta_i\}$ an adaptation to $\nabla$.

\[
\text{Suppose } \tilde{h} \text{ is the metric on } \tilde{l} \text{ induced by } (\nabla, h, \beta). \text{ Then } s \text{ is of norm } 1 \text{ with respect to the pull-back metric } \tilde{h} = \bigotimes b_k^* \tilde{h}. \text{ Consequently, } \nabla \text{ and } \tilde{h} \text{ are compatible each other.}
\]

**Proof.** For a better presentation, we will work with principal bundles. Let $P = P_l$ be constructed as in Corollary 4.2 and let

\[
P_w = \prod_{k=1}^{m} b_k^* P = b_1^* P \times \cdots \times b_m^* P
\]

be the fiber product bundle on $\text{Map}(\Sigma, X)$. $P_w$ has the structure group $\mathbb{R}^* \times \cdots \times \mathbb{R}^*$ and the associated bundle $\overline{\mathcal{G}} = P_w \times_{\eta} \mathbb{R}^*$ is exactly the principal frame bundle of $\tilde{l}$, where $\eta : \mathbb{R}^* \times \cdots \times \mathbb{R}^* \to \mathbb{R}^*$ is the multiplication homomorphism. (Since both bundles use the same transition functions.) Thus an element in any fiber of $\overline{\mathcal{G}}$ is represented by some $(d_1, \ldots, d_m) \in P_w$ inside the equivalence class

\[
[d_1, \ldots, d_m] = \{(a_1d_1, \ldots, a_md_m) \in P_w \mid a_1, \ldots, a_m \in \mathbb{R}^* \text{ and } \prod_{k} a_k = 1\}.
\]

(a) We show $\overline{\mathcal{G}}$ is canonically trivial by constructing a canonical section. Take any point $f \in \text{Map}(\Sigma, X)$, and consider the pull-back flat gerbe connection $f^*\nabla$ on $\Sigma$. Since $H^2(\Sigma, \mathbb{R}^*) = \{1\}$, $f^*\nabla$ must have trivial holonomy hence admits a flat object connection $\nabla_{ob}$ on some object bundle $\mathcal{G}_{ob}$ of $f^*\mathcal{G}$ by Theorem 3.5. Let $\nabla_{ob}^1, \ldots, \nabla_{ob}^m$ be the restrictions to the boundary components of $\partial \Sigma$. So by Corollary 4.2 we have an element $\nabla_{ob}^k \in P_{f_k}$ in the fiber over $f_k = b_k(f) \in LX$, and $\nabla_{ob}^k \in (b_k^* P)_f$ as well. Now construct a
section $s$ of $\mathcal{P}$ by using the equivalence class

$$s(f) = [\nabla_{ob}^1, \ldots, \nabla_{ob}^m] \in \mathcal{P}_f.$$ 

It remains to show that $s(f)$ is independent of the choice of objects. Let $\nabla'_{ob}$ be a second flat object connection of $f^*\nabla$ on another object bundle $\mathcal{G}'_{ob}$ of $f^*\mathcal{G}$. These result in the difference flat connection $D$ on the difference bundle $\xi$ according to Theorem 3.5. The restricted objects also yield difference flat connections $D_k$ on the difference bundles $\xi_k$ over all components of $\partial \Sigma$. Of course $D_k, \xi_k$ are just restrictions of $D, \xi$ to $\partial_k \Sigma$. Let $\nabla_{ob}^k$ denote the restricted flat object connection to the boundary. By construction of $P_{f_k}$,

$$\nabla_{ob}^k = \alpha(D_k) \nabla_{ob}^k,$$

where $\alpha(D_k) \in \mathbb{R}^*$ is the holonomy of $D_k$ along the loop $f_k$. To show $s(f)$ is well defined, from (8), we need to check $\prod_{k=1}^m \alpha(D_k) = 1$. We will apply our holonomy formula (2) to calculate $s(f)$.

Let $h^t = \{h^t_{ij}\}$ be any gerbe metric on $f^*\mathcal{G}$. Choose any object metrics $h_{ob}, h'_{ob}$ on $\mathcal{G}_{ob}, \mathcal{G}'_{ob}$, both subordinate to $h^t$. So we have the difference metric $H$ on $\xi$, with a restriction $H_k$ on $\xi_k$. By (2), $\alpha(D_k) = \pm \exp \int_{\partial_k \Sigma} \theta(D_k, H_k)$. Consequently

$$\prod_{k=1}^m \alpha(D_k) = \left[ \prod_{k=1}^m (\pm 1) \right] \left[ \exp \int_{\partial \Sigma} \theta(D, H) \right]$$

where we note $\theta(D, H)|_{\partial_k \Sigma} = \theta(D_k, H_k)$. Now the total sign $\prod_{k=1}^m (\pm 1) = 1$ since the gerbe class $f^*\mathcal{G}$ is trivial, and by Stokes’ Theorem, $\int_{\partial \Sigma} \theta(D, H) = \int_{\Sigma} d\theta(D, H) = 0$ as the holonomy form $\theta(D, H)$ is closed. Putting together we arrive at the desired formula: $\prod_{k=1}^m \alpha(D_k) = 1$.

(b) Recall from Theorem 4.1 the flat connection $\nabla$ on $P$ is constructed via locally constant transition functions that are holonomy of difference connections of some object connections. In particular, the constant $t = 1$ is a local flat section of $P$. By using the pull-backs of the afore-mentioned object connections, it is not hard to see that $t$ pulls back to $s$ locally. Thus $s$ is locally hence globally flat with respect to $\nabla$. Alternatively one can use the construction of $\nabla$ given in the proof of Corollary 1.2 to show the flatness of $s$.

(c) Similar to (a) and (b), the main idea is to use objects on the whole $\Sigma$ to calculate the pull-back metrics on boundary components via restrictions. Choose $\nabla_{ob}, \mathcal{G}_{ob}$ as in (a). Together with the pull-back adaptation $f^*\beta$ and gerbe metric $f^*h$, we have a global 1-form $\gamma$, as constructed in the proof of Theorem 3.5. On each $\partial_k \Sigma$, using the restrictions of $\nabla_{ob}, \mathcal{G}_{ob}, f^*\beta, f^*h$, we have also the global 1-form $\gamma_k$, which of course is the restriction of $\gamma$. Then according to Theorem 4.1 under the pull-back metric $h$, the norm of $s$ at $f \in \text{Map}(\Sigma, X)$ is

$$\prod_k \exp(2 \int_{\partial_k \Sigma} \gamma_k) = \exp(2 \int_{\partial \Sigma} \gamma).$$
By Stokes’ Theorem, \( \int_{\partial \Sigma} \gamma = \int_{\Sigma} d\gamma = \int_{\Sigma} B \), where \( B \) is the holonomy form of \((f^*G, f^*\nabla, f^*\beta)\). Since the holonomy class of \( f^*\nabla \) is trivial, \( \int_{\Sigma} B = 0 \) by Theorem 3.3, and the norm \( |s(f)| = 1 \) consequently. \( \square \)

**Remark 4.4.** To be logically correct, we check that the argument in (a) is independent of the gerbe metric \( h^t \) on \( f^*G \). Let \( h^{tt} = \{ h^{tt}_{ij} \} \) be a second choice. Then there exist positive functions \( f_{ij} \) such that \( h^{tt}_{ij} = f_{ij} h^t_{ij} \). Since the sheaf \( \mathbb{R}^+ \) of positive functions is fine, there is a Cech 0-cocycle \( \{ f_i \} \) with coboundary \( \{ f_{ij} \} \). Multiplying \( h^{ob}, h'^{ob} \) by \( \{ f_i \} \) we get two object metrics subordinate to \( h^t \). However their difference metric on \( \xi \) remains the same, and so does the rest argument. We can also check that the argument does not depend upon the choice of object metrics \( h^{ob}, h'^{ob} \). On \( G^{ob}, G'^{ob} \) take another pair of object metrics, still subordinate to \( h^t \). For the resulting metric \( H^{o} \) on \( \xi \), the 1-form \( \theta(D, H^{o}) \) differs from \( \theta(D, H) \) by an exact 1-form on \( \Sigma \). Hence the integral on \( \partial \Sigma \) remains the same.

Let us now incorporate the real picture of the section. Suppose \( \sigma : M \to M \) is a smooth involution with fixed point set \( M_{\mathbb{R}} = X \). For a loop \( \rho \in LM \), define \( \overline{\rho} \in LM \) by \( \overline{\rho}(t) = \sigma(\rho(t)) \). Then the fixed loop space \( (LM)^{\mathbb{R}} = LX \).

Given a complex gerbe \( G^c \) on \( M \) with connection \( \nabla^c = \{ \nabla^c_{ij}, F_i \} \), a well-defined complex line bundle \( \tilde{L} \to LM \) is constructed in [B, C] together with a connection \( \tilde{\nabla}^c \). The following sums up the relation between the two gerbe pictures on \( M \) and \( X \). Its proof is essentially book-keeping.

**Proposition 4.5.** (a) Suppose the gerbe \( G^c \) and connection \( \nabla^c \) are both Real. Then the associated complex line bundle \( \tilde{L} \) and connection \( \tilde{\nabla}^c \) are also Real. Taking real parts, we have a real line bundle \( \tilde{L}^{\mathbb{R}} \to LX \) and a connection \( \tilde{\nabla}^{\mathbb{R}} \). They are naturally identified with the bundle \( \overline{L} \) and connection \( \overline{\nabla} \), which are constructed by using the real gerbe \( G \) and connection \( \nabla \). If \( \nabla^c \) is flat or unitary, then \( \tilde{\nabla}^{\mathbb{R}} = \overline{\nabla} \) is flat as well.

(b) Each Hermitian gerbe metric \( H \) on \( G^c \) induces a unique metric \( \tilde{h} \) on \( \tilde{L} \).

(c) Suppose further that \( \sigma \) is anti-holomorphic on a complex manifold \( M \) and \( G^c \) is a holomorphic gerbe with a holomorphic structure \( \alpha \). Then we have a second line bundle \( \tilde{l}^{\alpha} \to LX \) with a flat connection \( \tilde{\nabla}^{\alpha} \). Moreover \( \tilde{l}^{\alpha} \) is isomorphic with \( \tilde{l} \).

**Proof.** (a) By construction in [B, C], \( \tilde{L}, \overline{\nabla} \) are obviously Real. To see \( \tilde{L}^{\mathbb{R}} = \overline{L} \) amounts to comparing the holonomies of a Real connection and its real part.

Let \( G = dF_i \) denote the curvature 3-form of \( \nabla^c \). According to [B, C] the curvature \( K \) of \( \overline{\nabla} \) is evaluated at \( \rho \in LM \) as

\[
K(u, v) = \exp \int_{\rho} i_{u,v} G,
\]
where \( u, v \in T_\rho(LM) \) are two vector fields along \( \rho \) and \( i_{u,v} \) is the contraction.

If \( \nabla^c \) is flat, \( G = 0 \), then \( K = 0 \) and \( \tilde{\nabla}^c \) is flat, and so is its real part \( \tilde{\nabla}_R^c \).

(But note that the gerbe connection \( \nabla \) on \( \mathcal{G} \) may not be flat.)

If \( \nabla^c \) is unitary, then \( \nabla \) is flat and so is \( \tilde{\nabla} = \tilde{\nabla}_R^c \).

(b) \( H \) restricts to a Riemann metric \( h = \{ h_{ij} \} \) on \( \mathcal{G} \). Take the unique compatible connection \( \nabla_{ij}^h \) of \( h_{ij} \) on \( l_{ij} \) with \( \theta(\nabla_{ij}^h, h_{ij}) = 0 \). Together with the trivial adaptation \( \beta_i = 0 \), we get a metric \( \tilde{h} \) by Theorem 4.1. Note that there is no Hermitian metric on \( L \) directly induced by \( H \).

(c) Let \( \nabla^\alpha \) be the flat gerbe connection on \( \mathcal{G} \) that is associated to \( \alpha \). Then using \( (\mathcal{G}, \nabla^\alpha) \), we have a real line bundle \( \tilde{l}^\alpha \to LX \) with flat connection \( \tilde{\nabla}^\alpha \) by Theorem 4.1. One sees that \( \tilde{l}^\alpha, \tilde{l} \) have the same isomorphism type, since both are determined by that of \( \mathcal{G} \).

Note that for the sake of our next discussion, we have extended \( \sigma : M \to M \) to \( \tilde{\sigma} : LM \to LM \) by using the identity map on \( S^1 \). One could also use the antipodal or complex conjugation maps on \( S^1 \), in which cases we no longer have the fixed loop spaces \( (LM)_R = LX \). Such cases also appear interesting to study.

Our next purpose is to study the Real analogue of Theorem 4.6 in connection with map spaces. Let \( \Sigma \) be a closed complex curve with a real structure (namely an anti-holomorphic involution). The real part \( \Sigma_R \) consists of \( m \) circles \( \Sigma_1, \ldots, \Sigma_m \). Continue assuming \( \sigma : M \to M \) to be a smooth involution such that \( X = M_R \) is a smooth manifold, as before. Fix any orientation on each \( \Sigma_k \) and orientation-preserving diffeomorphism \( S^1 \to \Sigma_k \). Obviously \( \text{Map}(\Sigma, M) \) inherits a natural smooth involution and the real part \( \text{Map}_R(\Sigma, M) \) contains all the Real maps from \( \Sigma \) to \( M \). Using the fixed diffeomorphism \( S^1 \to \Sigma_k \) we have a map \( b_k : \text{Map}_R(\Sigma, M) \to LX \). Take a Real complex gerbe \( \mathcal{G}^c \) with connection \( \nabla^c \) on \( M \) and let \( \tilde{l} \to LX \) be the real line bundle from Proposition 4.6. Set the real line bundle \( \tilde{l} \to \text{Map}_R(\Sigma, M) \) to be \( \otimes_k b_k^* \tilde{l} \). The main question here is about the triviality of \( \tilde{l} \). (Incidentally, reversing orientation on any \( \Sigma_k \) will not change the isomorphism type of \( \tilde{l} \).) Certainly the answer does not come directly from Theorem 4.6 for one thing \( \text{Map}_R(\Sigma, M) \) explicitly involves \( M \) while \( LX \) does not. In fact we are able to answer the question only in a special case.

Recall that the complement \( \Sigma \setminus \Sigma_R \) contains at most two connected components; cf. Wilson [W] for example. The real structure on \( \Sigma \) is called dividing or non-dividing, depending on whether there are two or one components. In the dividing case, there is a well-defined pair of opposite orientations on all \( \Sigma_R \), which are induced by orientations on the two components.

**Theorem 4.6.** Let \( \Sigma \) be a closed complex curve carrying a dividing real structure and \( M \) carrying a smooth involution with real part \( X \). Use the real part \( \Sigma_R \) to define maps \( b_k : \text{Map}_R(\Sigma, M) \to LX \) as above.

(a) Suppose \( \tilde{l} \to LX \) with connection \( \tilde{\nabla} \) is constructed as in part (a) of Proposition 4.6. Then \( \tilde{l} := \otimes_k b_k^* \tilde{l} \) is canonically trivialized. If the gerbe
connection $\nabla^c$ is flat or unitary, then the trivialization is flat with respect to the pull-back connection $\nabla = \otimes_k b_k^* \nabla$.

(b) Suppose we are in the complex set up of part (c) of Proposition 4.5, so we have the real line bundle $L^a \to LX$ with flat connection $\nabla^a$. Then the pull-back bundle $\tilde{L}^a = \otimes_k b_k^* L^a$ has a flat trivialization with respect to the connection $\tilde{\nabla}^a = \otimes_k b_k^* \nabla^a$.

Proof. (a) Pick one of the two components in $\Sigma \setminus \Sigma_R$ and orient $\Sigma_R$ accordingly. Fix any orientation-preserving diffeomorphisms $S^1 \to \Sigma_k$ where $\Sigma_R = \coprod_{k=1}^m \Sigma_k$. Label the chosen component as $\Sigma^+$, with the boundary included. Extending the usual complex gerbe case for surface with boundary, we consider the map $b_k : \text{Map}(\Sigma, M) \to LM$ and the complex line bundle $\mathcal{T} = \otimes b_k^* L$, where $L$ is as in Proposition 4.3. (This is a generalization of the usual complex case, as $\Sigma$ does not have a boundary – the real part $\Sigma_R$ plays the same role here.) Since $\Sigma_k$ contains real points only, the map $b_k$ is clearly Real with respect to the natural real structures. The restriction to real parts $b_k : \text{Map}_R(\Sigma, M) \to LX$ is exactly the one defined early. Note $\mathcal{L}$ has a real lifting as $L$ is Real, and $\tilde{l} \to \text{Map}_R(\Sigma, M)$ is exactly the real part of $\mathcal{T}$ under such real lifting.

To show $\tilde{l}$ is canonically trivial, it is enough to show $\mathcal{T}$ is so with a Real trivialization. This in turn follows essentially the same kind of proof as Theorem 4.3. Indeed, at $f \in \text{Map}(\Sigma, M)$, let $f_+$ be its restriction to $\Sigma^+$. For dimension reason, the pull-back gerbe connection $f_+^* \nabla^c$ is flat. And its holonomy is trivial as well as $H^3(\Sigma^+, \mathbb{C}^*)$ is certainly trivial. Thus $f_+^* \mathcal{G}^c$ admits object bundle $\mathcal{G}^{c}_{ob}$ with connection $\nabla^{c}_{ob}$ by the complex version of Theorem 3.5 (see [B, C]). The restrictions $\nabla^{c}_{ob, 1}, \cdots, \nabla^{c}_{ob, m}$ of $\nabla^c$ to the circles $\Sigma_1, \cdots, \Sigma_m$ yield an element $s(f)$ in the fiber of the principal bundle $\mathcal{T}^c$ associated to $\mathcal{T}$. That $s(f)$ is independent of the choice of objects follows again from the holonomy formula along circles and the Stokes Theorem applied to $\Sigma^+$ with boundary $\Sigma_R$. Consequently we have $\mathcal{T}^c$ hence $\mathcal{L}$ canonically trivialized by the section $s$. Since $\mathcal{G}^c, \nabla^c$ are both Real, it is not hard to check that $s$ is invariant under the real lifting on $\mathcal{T}$: essentially this is due to the fact that the other component $\Sigma^- \setminus \Sigma_R$ induces the opposite orientation on $\Sigma_R$ as $\Sigma^+$ does. Restricting to the real part $\text{Map}_R(\Sigma, M)$, $s$ becomes the canonical trivialization of $\mathcal{T}$.

When $\nabla^c$ is flat or unitary, the connection $\nabla$ is flat hence $\nabla$ is flat as well. The trivialization $s$ is $\nabla$-flat, following a similar proof to Theorem 4.3.

(b) By Proposition 4.5, $\mathcal{T}$ is isomorphic with $\mathcal{L}$ hence $\mathcal{T}$ is trivial. Moreover the induced trivialization from $\tilde{l}$ is $\nabla^a$-flat.

When the real structure on $\Sigma$ is non-dividing, we are unable to obtain any definitive result but conjecture that $\tilde{l}$ is not trivial in general. A basic result in real algebraic geometry says that $\Sigma$ is dividing iff the class $[\Sigma_R] \in H_1(\Sigma, \mathbb{Z}_2)$ is trivial, see [W1] for example. Thus in the non-dividing case, the Poincare dual $PD[\Sigma_R] \in H^1(\Sigma, \mathbb{Z}_2)$ associates with a non-trivial
real line bundle on \( \Sigma \). We conjecture that \( \mathcal{L} \) should be related to this line bundle.

We end the paper with a speculation concerning real Gromov-Witten invariants. Continue using the real complex set-up from part (b) of Theorem 4.6. Let \( \mathcal{M}_R \subset \text{Map}_R(\Sigma, M) \) be the moduli space of real holomorphic curves. Take a multi-degree cohomology class \( \beta = (\beta_1, \cdots, \beta_m) \), where \( \beta_k \in H^{n_k}(LX, \mathcal{L}) \) and \( \mathcal{L} = \mathcal{L}^\alpha \) carries the flat connection \( \nabla^\alpha \) so that the cohomology with twisted coefficients is defined. Assume the total degree of \( \beta \) is \( \sum_k n_k = \dim \mathcal{M}_R \). Set \( b^*\beta = b_1^*\beta_1 \cup \cdots \cup b_m^*\beta_m \), which is an ordinary cohomology class of degree \( \dim \mathcal{M}_R \) on \( \text{Map}_R(\Sigma, M) \), since the pull-back bundle \( \mathcal{L} \) is trivial. Then one might attempt to define a real type of Gromov-Witten invariant with respect to a real holomorphic gerbe \((\mathcal{G}^\alpha, \alpha)\) as the map

\[
GW_{R,\alpha} : H^{n_1}(LX, \mathcal{L}) \times \cdots \times H^{n_m}(LX, \mathcal{L}) \longrightarrow \mathbb{R}
\]

by sending \( \beta \) to \((b^*\beta, [\mathcal{M}_R])\), where \( [\mathcal{M}_R] \) is a compactified real moduli space. Here implicitly we have asserted that the real moduli space \( \mathcal{M}_R \) is orientable, which seems plausible under our assumption that \( \Sigma \) has a dividing real structure (as in Theorem 4.6). Compare with Katz-Liu [KL].

In the non-dividing case, \( b^*\beta \) belongs to the cohomology of twisted coefficients \( \mathcal{L} \) and \( \mathcal{M}_R \) may not be orientable either. However, if the orientation bundle of \( \mathcal{M}_R \) matches \( \mathcal{L} \), then the above definition for \( GW_{R,\alpha} \) can make sense again.

A Gromov-Witten invariant with respect to a complex gerbe has been introduced by Pan, Ruan, and Yin [PRY]. We hope to return to the study of the real case in a future work.

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