Temporal Graph Classes: A View Through Temporal Separators

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Abstract

We investigate the computational complexity of separating two distinct vertices \( s \) and \( z \) by vertex deletion in a temporal graph. In a temporal graph, the vertex set is fixed but the edges have (discrete) time labels. Since the corresponding Temporal \((s, z)\)-Separation problem is \( \text{NP} \)-hard, it is natural to investigate whether relevant special cases exist that are computationally tractable. To this end, we study restrictions of the underlying (static) graph—there we observe polynomial-time solvability in the case of bounded treewidth—as well as restrictions concerning the “temporal evolution” along the time steps. Systematically studying partially novel concepts in this direction, we identify sharp borders between tractable and intractable cases.

1 Introduction

Reachability, connectivity, and robustness in networks depend often on time. For instance, in public transport or human contact networks, available connections or contacts are time-dependent. To model such time-dependent aspects, one turns from static graphs to temporal graphs. Formally, an undirected temporal graph \( G = (V, E, \tau) \) is an ordered triple consisting of a set \( V \) of vertices, a set \( E \subseteq \binom{V}{2} \times \{1, \ldots, \tau\} \) of time-edges, and a maximal time label \( \tau \in \mathbb{N} \). We study the problem of finding a small set of vertices in a temporal graph whose removal disconnects two designated terminals: a classic, polynomial-time solvable problem in (static) graph theory.

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**Temporal \((s,z)\)-Separation**

**Input:** A temporal graph \(G = (V, E, \tau)\), two distinct vertices \(s, z \in V\), and \(k \in \mathbb{N}\).

**Question:** Does \(G\) admit a temporal \((s,z)\)-separator of size at most \(k\)?

Herein, a vertex set \(S\) is a temporal \((s,z)\)-separator if there is no temporal \((s,z)\)-path in \(G - S := (V \setminus S, \{(v, w), t\} \in E \mid v, w \in V \setminus S, \tau)\). A temporal \((s,z)\)-path of length \(\ell\) in \(G\) is a sequence \(P = ((\{v_0, v_1\}, t_1), (\{v_1, v_2\}, t_2), \ldots, (\{v_{\ell-1}, v_\ell\}, t_\ell))\) of time-edges in \(E\), where \(s = v_0, z = v_\ell, v_i \neq v_j\) for all \(i, j \in \{0, \ldots, \ell\}\) with \(i \neq j\), and \(t_i \leq t_{i+1}\) for all \(i \in \{1, \ldots, \ell - 1\}\). **Temporal \((s,z)\)-Separation** is \(\text{NP}\)-hard [12]. In this work, we study **Temporal \((s,z)\)-Separation** on restricted classes of temporal graphs with the goal to identify computationally tractable cases.

So far, in the literature one basically finds two different directions concerning the definition of temporal graph classes. One direction is defining temporal graph classes through the underlying graph (that is, essentially, the graph obtained by forgetting about the time labels of the edges) [3, 10, 21].

Herein, one restricts the input temporal graph to have its underlying graph being contained in some specific graph class. The other direction consist of properties expressible through temporal aspects [7, 11, 14, 18]. Such properties are, for instance, each layer being a subgraph of its succeeding layer, or the temporal graph being periodic, that is, having a subsequence of layers which is repeated in the same order for some periods. In this work, we study **Temporal \((s,z)\)-Separation** on temporal graph classes from both directions.

**Our contributions.** We show that **Temporal \((s,z)\)-Separation** remains \(\text{NP}\)-complete on many restricted temporal graph classes.

- **Temporal \((s,z)\)-Separation** remains \(\text{NP}\)-hard on temporal graphs whose underlying graph falls into a class of graphs containing complete-but-one graphs (that is, complete graphs where exactly one edge is missing) or line graphs. However, if the underlying graph is of bounded treewidth, **Temporal \((s,z)\)-Separation** becomes polynomial-time solvable (see Figure 1 for an overview).

- **Temporal \((s,z)\)-Separation** remains \(\text{NP}\)-hard on temporal graphs where each layer contains only one edge (Corollary 3.1). In contrast, if we require each layer to be a unit interval graph with respect to the same global vertex ordering, then **Temporal \((s,z)\)-Separation** becomes solvable in polynomial time (Theorem 3.1).

- Regarding temporal graph classes defined through temporal aspects,
Figure 1: Computational complexity of Temporal \((s, z)\)-Separation for some graph classes of the underlying graph. An edge between two classes indicates containment of the lower in the upper class. For the classes of line, complete-but-one, bipartite, and planar graphs, we provide for which values of the maximum time label \(\tau\) \(\text{NP}\)-hardness is proven as well as the parameterized complexity of Temporal \((s, z)\)-Separation when parameterized by the solution size \(k\). We point out that in the case of planar graphs, neither a bound on \(\tau\) nor the parameterized complexity regarding \(k\) is known.

Temporal \((s, z)\)-Separation becomes solvable in polynomial time on single-peaked temporal graphs, on graphs where all layers are identical (1-periodic or 0-steady), or when the number of periods is at least the number of vertices. In all other considered cases Temporal \((s, z)\)-Separation remains \(\text{NP}\)-complete. (See Table 1 in Section 4 for an overview.)

Related work. Kempe et al. [12] proved Temporal \((s, z)\)-Separation to be \(\text{NP}\)-complete. Zschoche et al. [21] proved that Temporal \((s, z)\)-Separation remains \(\text{NP}\)-complete on temporal graphs with bipartite or planar underlying graphs. Moreover, Temporal \((s, z)\)-Separation is \(\text{W}[1]\)-hard when parameterized by the solution size \(k\) [21].

Casteigts et al. [7] defined twelve different classes of temporal graphs and showed a corresponding inclusion diagram. Among these classes, they define temporal graph classes with recurrence or periodicity of edges. On a slightly different notion of the latter class, Flocchini et al. [11] studied the problem of exploring a temporal graph. Kuhn et al. [14] studied the problem of token dissemination on temporal graphs where for each time-interval of length \(T\),
there is a static subgraph present in all layers in the interval connecting all vertices of the temporal graph.

The class of temporal graphs with underlying graphs of bounded treewidth are considered in the context of temporal graph exploration [10] and single-source temporal connectivity [3]. Erlebach et al. [10] studied the problem of temporal graph exploration on temporal graphs with underlying graphs being planar and of bounded vertex degree. They also introduced the class of temporal graphs with regularly present edges, where the absence of each edge in consecutive time steps is lower- and upper-bounded by two values. Michail and Spirakis [18] studied a temporal version of the traveling salesperson problem on temporal graphs with bounded dynamic diameter, where the dynamic diameter is the smallest number $d$ such that every vertex can reach any other vertex at any time in at most $d$ time steps.

2 Preliminaries

As a convention, $\mathbb{N}$ denotes the natural numbers without zero. For $n \in \mathbb{N}$, we use $[n] := \{1, n\} = \{1, \ldots, n\}$. For a sequence $x_1, \ldots, x_n$ and $a, b \in [n]$, $a < b$, we write $x_{[a:b]}$ for subsequence $x_a, \ldots, x_b$.

**Static graphs.** We use basic notations from (static) graph theory [9], Let $G = (V, E)$ be an undirected, simple graph. We use $V(G)$, $E(G)$, and $\Delta(G)$ to denote the set of vertices, set of edges, and the maximum vertex degree of $G$, respectively. We denote by $G - V' := (V \setminus V', \{v, w\} \in E \mid v, w \in V \setminus V')$ the graph $G$ without the vertices in $V' \subseteq V$. For $V' \subseteq V$, $G[V'] := G - (V \setminus V')$ denotes the induced subgraph of $G$ on the vertices $V'$. A path of length $\ell$ is sequence of edges $P = (\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_\ell, v_{\ell+1}\})$ where $v_i \neq v_j$ for all $i, j \in [\ell - 1]$ with $i \neq j$. We set $V(P) = \{v_1, v_2, \ldots, v_{\ell+1}\}$. Path $P$ is an $(s, z)$-path if $s = v_1$ and $z = v_{\ell+1}$. A set $S \subseteq V$ of vertices is an $(s, z)$-separator in $G$ if there is no $(s, z)$-path in $G - S$.

A tree decomposition of a graph $G$ is a pair $T := (T, (B_i)_{i \in V(T)})$ consisting of a tree $T$ and a family $(B_i)_{i \in V(T)}$ of bags $B_i \subseteq V(G)$, such that (i) for all vertices $v \in V(G)$ the set $B^{-1}(v) := \{i \in V(T) \mid v \in B_i\}$ is non-empty and induces a subtree of $T$, and (ii) for every edge $e \in E(G)$ there is an $i \in V(T)$ with $e \subseteq B_i$. The width of $T$ is $\max\{|B_i| - 1 \mid i \in V(T)\}$. The treewidth $tw(G)$ of $G$ is defined as the minimal width over all tree decompositions of $G$. Computing for graph $G$ a tree decomposition of width $tw(G)$ is computable in $tw(G)^{O(tw(G)^3)} \cdot |V(G)|$ time (cf. Cygan et al. [8]).
**Temporal graphs.** Let $G = (V, E, \tau)$ be a temporal graph. The graph $G_i(G) = (V, E_i(G))$ is called layer $i$ of the temporal graph $G = (V, E, \tau)$ if and only if $\{v, w\} \in E_i(G) \Leftrightarrow (\{v, w\}, i) \in E$. The underlying graph $G_i(G)$ of a temporal graph $G = (V, E, \tau)$ is defined as $G_i(G) := (V, E_i(G))$, where $E_i(G) = \{e \mid (e, t) \in E\}$. (We drop $G$ in the notations if it is clear from the context.) For $X \subseteq V$ we define the induced temporal subgraph of $G$ by $X$ by $G[X] := (X, \{(\{v, w\}, t) \in E \mid v, w \in X\}, \tau)$. We say that a temporal graph $G$ is connected if its underlying graph $G_i(G)$ is connected. We define the maximum degree of a temporal graph $\Delta(G)$ as the maximum degree of its underlying graph $\Delta(G_i)$. Let $s, z \in V$. The departure time (arrival time) of a temporal $(s, z)$-path $P = ((e_1, t_1), \ldots, (e_\ell, t_\ell))$ is $t_1$ ($t_\ell$), the traversal time of $P$ is $t_\ell - t_1$, and the length of $P$ is $\ell$. The vertices visited by $P$ are denoted by $V(P) = \bigcup_{i=1}^{\ell} e_i$. Throughout the whole paper we assume that the temporal input graph $G$ is connected and that there is no time-edge between $s$ and $z$. Furthermore, in accordance with Wu et al. [20] we assume that the time-edge set $E$ is ordered by ascending labels. The concatenation of two temporal graphs $G_1 = (V, E_1, \tau_1)$, $G_2 = (V, E_2, \tau_2)$ is denoted by $G_1 \circ G_2 = (V, E_1 \cup \{(e, t + \tau_1) \mid (e, t) \in E_2\}, \tau_1 + \tau_2)$. Furthermore, we define that $G^1_x = G_1$ and $G^1_x = G^x_1 \circ G_1$, for all integers $x \geq 2$.

**Lemma 2.1.** Given a temporal graph $G = (V, E, \tau)$ and two distinct vertices $s$ and $z$, a temporal $(s, z)$-path can be computed in $O(|E|)$ time.

**Proof.** Let $G = (V, E, \tau)$ be a temporal graph and let $V = \{v_1, \ldots, v_{n-2}\} \cup \{s, z\}$. For each $v \in \{v_1, \ldots, v_{n-2}\}$, we define the sets $\phi(v) := \{t \mid t \in [\tau], 3w : (\{v, w\}, t) \in E\}$ and $\bar{\phi}(v) := \{(t, t') \mid \phi(v)^2 \mid t < t' \wedge t' \in \phi(v) : t < t'' < t\}$.

The static expansion of $(G, s, z)$ is a directed graph $H := (V', A)$ where $V' = \{s, z\} \cup \{u_{i,j} \mid j \in [n-2], \forall t < \phi(v_{j})\}$ and $A = A' \cup A_s \cup A_z \cup A_{col}$, $A' := \{(u_{i,j}, u_{i,j}') \mid (u_{i,j}, v_{j}) \in E\}, A_s := \{(s, u_{i,j}) \mid \{s, v_{j}\}, i \in E\}, A_z := \{(u_{i,j}, z) \mid \{v_{j}, z\}, i \in E\}$, and $A_{col} := \{(u_{i,j}, u'_{i,j}) \mid (t, t') \in \bar{\phi}(v_{j}) \wedge j \in [n-2]\}$ (referred to as column-edges of $H$). Observe that each temporal $(s, z)$-path in $G$ has a one-to-one correspondence to some $(s, z)$-path in $H$ and that $H$ can be computed in $O(|E|)$ time [21]. Thus we can find a temporal $(s, z)$-path in $G$, using a breadth-first search on the static expansion of $(G, s, z)$. This gives us a overall running time of $O(|E|)$. \qed

**Parameterized complexity.** A parameterized problem is in XP if there is an algorithm that solves each instance $(I, r)$ in $|I|^{O(r)}$ time, as well as
it is fixed-parameter tractable (in FPT) if there is an algorithm that solves each instance \((I, r)\) in \(f(r) \cdot |I|^{O(1)}\) time, where \(f\) is a computable function depending only on the parameter \(r\) [8]. There is a hierarchy of hardness classes for parameterized problems, of which the most important one is \(\text{W}[1]\). If a parameterized problem is \(\text{W}[1]\)-hard, then it is (presumably) not in FPT.

3 Structural Restrictions

Two approaches to define temporal graph classes contain (i) restricting each layer to be contained in specific graph class or (ii) restricting the underlying graph to be contained in a graph class. We point out that both are independent of the order of the graphs and hence appear to not fully capture the temporal characteristics of a given temporal graph. Indeed, our results support this fact as we obtain intractability for many restricted graph classes.

3.1 Layer-wise Restrictions

Restricting the layers to fall into a specific graph class neither captures any temporal aspect of the temporal graph nor the full picture drawn by all layers together. In fact, we show that such restrictions are not helpful: the problem is already NP-hard when each layer consists of at most one edge.

Lemma 3.1. There exists a polynomial-time many-one reduction that maps any instance \((G = (V, E, \tau), s, t, k)\) of TEMPORAL \((s,z)\)-SEPARATION to an equivalent instance \((G' = (V, E', \tau'), s, t, k)\) such that each layer in \(G'\) has at most one edge and \(\tau' \leq \tau \cdot |V|^4\).

Proof. Let \(G = (V, E, \tau)\) be a temporal graph. We construct \(G' = (V, E', \tau')\) in the following way. For each layer \(i\) of \(G\) we construct a temporal graph \(G_i = (V, E_i, \tau_i)\) by fixing an arbitrary order on the edge set \(E_i = \{e_1, \ldots, e_m\}\) of layer \(i\) in \(G\) and set the time-edge set of layer \(j\) of \(G_i\) to be \(\{(e_j, j)\}\). Now, we build \(G' = G_1^{E_1} \circ G_2^{E_2} \circ \cdots \circ G_\tau^{E_\tau}\), where \(|E_i|\) is the number of edges in layer \(i\) of \(G\) for all \(i \in [\tau]\). This is obviously a polynomial-time construction. Since for all \(i \in [\tau] : |E_i| \leq |V|^2\) and each \(G_i\) has \(|E_i|\) many layers, we know that \(\tau' \leq \tau \cdot |V|^4\).

Observe, that the underlying graph of \(G_i\) and \(G_i^{E_i}\) is the layer \(i\) of \(G\). Since every temporal path is also a path in the underlying graph, it is easy to see that for each temporal \((v, w)\)-path in \(G_i^{E_i}\) there is a \((v, w)\)-path in layer \(i\) of \(G\) which visits the vertices in the same order, where \(i \in [\tau]\), and \(v, w \in V\). Let \(i \in [\tau]\). We claim for any \(v, w \in V\) we have that for each \((v, w)\)-path
P of length ℓ in layer i of G there is a temporal (v, w)-path in \( G_i^\ell \) which visits the vertices in the same order. Let \( V(P) = \{v = v_0, \ldots, v_{\ell+1} = w\} \) such that \( v_j \) is visited before \( v_{j+1} \), for all \( j \in \{0, \ldots, \ell\} \). We prove this by induction over \( \ell \). If \( \ell = 1 \), then we know there is a time-edge between \( v \) and \( w \) in \( G_1 \). For the induction step we observe that there is a time-edge between \( v_0 = v \) and \( v_1 \) in \( G_i \) and, by the induction hypothesis, there is a temporal \((v_1, w)\)-path of length \( \ell - 1 \) in \( G_i^{\ell-1} \) which visits the vertices in the same order as \( P \). Since \( \ell \leq |E_i| \), we have that for each \((v, w)\)-path in layer \( i \) of \( G \) there is a temporal \((v, w)\)-path in \( G_i^{\{|E_i|\}} \) which visits the vertices in the same order, where \( v, w \in V \) and \( i \in [\tau] \). From here, we can conclude that a vertex set \( S \subseteq V \setminus \{s, z\} \) is a temporal \((s, z)\)-separator in \( G \) if and only if \( S \) is a temporal \((s, z)\)-separator in \( G' \), because in the construction of \( G' \) we replaced the layer \( i \) of \( G \) with \( G_i^{\{|E_i|\}} \).

Lemma 3.1 (together with known hardness results [12, 21]) implies the following.

**Corollary 3.1.** Temporal \((s, z)\)-Separation is NP-complete and W[1]-hard when parameterized by the solution size \( k \) even if each layer has at most one edge.

Now we consider a scenario where temporal networks have a certain geometric interpretation. For example in data sets where vertices are individuals and edges model physical proximity (see e.g. [4]), it is a reasonable assumption that the individual layers are disc intersection graphs (assuming the individuals only move in the plane). We move to the one-dimensional case, where we get (unit) interval graphs, and investigate this restriction as a starting point for further research. We show in the following that if each layer of a given temporal graph \( G \) is restricted to be a unit interval graph and there is an ordering on the vertices that matches the relative positions of the intervals in all layers, then we can solve Temporal \((s, z)\)-Separation on \( G \) in polynomial time. We first give a formal definition of the restriction.

In the following we introduce temporal interval graphs. We call a temporal graph \( G = (V, E, \tau) \) a *temporal interval graph* if every layer \( G_i \) is an interval graph. We say that a temporal graph \( G = (V, E, \tau) \) is a *temporal unit interval graph* if every layer \( G_i \) is a unit interval graph. By Lemma 3.1, Temporal \((s, z)\)-Separation on temporal unit interval graph is NP-hard.

We call a total ordering \( <_V \) on a vertex set \( V \) compatible with a unit interval graph \( G = (V, E) \) if there are unit intervals \([a_v, a_v + 1]\) with \( a_v \in \mathbb{R} \) for all vertices \( v \in V \) that induce the graph \( G \) and for all \( u, v \in V \) with \( u <_V v \)
we have that $a_n \leq a_v$. Note that for every unit interval graph there is a total ordering on the vertices that is compatible with it.

**Definition 3.1.** A temporal graph $G = (V, E, \tau)$ is an *order-preserving temporal unit interval graph* if $G$ is a temporal unit interval graph and there is a total ordering $<_V$ on the vertex set $V$ that is compatible with every layer $G_i$.

Given an order-preserving temporal unit interval graph $G = (V, E, \tau)$, we denote by $<_V$ a compatible total ordering on $V$, we denote by $n := |V|$, and we enumerate the vertices in $V$ as $\{v_1, v_2, \ldots, v_n\}$ such that $v_i <_V v_j \Leftrightarrow i \leq j$. Furthermore, we use the following notation: $V_{<i} := \{v_j \mid 1 \leq j < i\}$ and $V_{>i} := \{v_j \mid n \geq j > i\}$ and $N_{G_i}^>(v_i) := N_{G_i}(v_i) \cap V_{>1}$. If the ordering $<_V$ is clear from the context, we refer to vertices as smaller or larger than other vertices to express that they appear before or after, respectively, in the ordering $<_V$.

We state some useful properties of temporal paths and separators in order-preserving temporal unit interval graphs.

**Lemma 3.2.** Let $G = (V, E, \tau)$ be an order-preserving temporal unit interval graph with ordering $<_V$.

(i) For all $1 \leq a \leq b \leq \tau$ and for all $S \subseteq V$ we have that $G_{[a:b]} - S$ is also an order-preserving temporal unit interval graph.

(ii) If for some $1 \leq i < j \leq n$ there is a temporal $(v_i, v_j)$-path $P$ in $G$, then there is temporal $(v_i, v_j)$-path $P'$ in $G$ that visits its vertices in the order given by $<_V$.

(iii) Let $S \subseteq V$ be a temporal $(v_i, v_j)$-separator in $G$ for some $1 \leq i < j \leq n$. Then $S' = S \setminus (V_{<i} \cup V_{>j})$ is also a temporal $(v_i, v_j)$-separator in $G$.

(iv) A temporal $(v_i, v_j)$-separator in $G$ is also a temporal $(v_i', v_j')$-separator in $G$ for all $1 \leq i', j' \leq n$.

(v) Let $S \subseteq V \setminus \{s, z\}$ such that $v_i$ is the largest vertex reachable from $s$ in $G - S$. Let $t$ denote the first time $v_i$ is reachable from $s$ in $G - S$, and let $t \leq t' \leq \tau$ such that $|N^>_{G_{t'}}(v_i)| = \max_{1 \leq t'' \leq \tau} |N^>_{G_{t''}}(v_i)|$. Then $N^>_{G_{t'}}(v_i) \subseteq S$.

(vi) Let $S_1 \subseteq V \setminus \{s, z\}$ such that $v_i$ is the largest vertex reachable from $s$ in $G_{[1:t]} - S$ for some $t \in [\tau - 1]$. Let $S_2 \subseteq V \setminus \{s, z\}$ such that $v_j$ is the largest vertex reachable from $s$ in $G_{[t+1:\tau]} - S$. If $i \leq j$, then $S = S_1 \cup S_2$ is a temporal $(s, z)$-separator in $G$ such that there is no vertex reachable from $s$ in $G - S$ that is larger than $v_j$.

(vii) Let $S \subseteq V$ be an inclusion-wise minimal temporal $(s, z)$-separator in $G$ with the property that a given $v_i$ is the largest vertex that is reachable
from $s$ in $G - S$ and let $v_j$ be the smallest vertex that is not in $S$ such that $S$ is also a temporal $(s, v_j)$-separator in $G$. Then for all $v_i <_V v <_V v_j$ with $v_i \neq v \neq v_j$ we have that $v \in S$, and we have that $S \cap V_{> j} = \emptyset$.

Proof. (i): Obvious.

(ii): Let $v_{i'}$ be the last vertex on $P$ such that $i' \leq i$, and let $t \in [\tau]$ be the index of the layer where $P$ contains the edge $\{v_{i'}, v_x\}$, where $v_x$ is the successor of $v_{i'}$ on $P$. As $G_t$ is an unit interval graph with order $<_V$, the edge $\{v_i, v_x\}$ is present in $G_t$. Our path $P'$ starts with this edge. Then we repeat the argument from $v_x$ until a vertex $v_y$ is reached such that the above argument leads to a vertex $v_z$ in time step $t'$ with $j < z$. In this case, with an analogue argument, we can add the edge $\{v_y, v_j\}$ being present in $G_{t'}$ to $P'$, which finishes the construction.

(iii): Follows directly from (ii).

(iv): Follows directly from (ii).

(v): Suppose not. Then there is a time step $t''$ with larger neighborhood and hence there is a vertex $v_j \in N_{G_{t''}}^<(v_i) \setminus N_{G_{t''}}^>(v_i)$. Hence, $v_j$ with $j > i$ is reachable from $s$ in $G_{[1, t''] - S}$, contradicting the definition of $v_i$.

(vi): Follows directly from (ii).

(vii): Assume towards a contradiction that there is a vertex $v \notin S$ with $v_i <_V v <_V v_j$ and $v_i \neq v \neq v_j$. Then either $v$ is reachable from $S$ in $G - S$, which would be a contradiction to $v_i$ being the largest vertex reachable from $s$ in $G - S$, or $v$ is not reachable from $s$ in $G - S$, a contradiction to the assumption that $v_j$ is the smallest vertex such that $S$ is also a temporal $(s, v_j)$-separator in $G$. Furthermore, $S \cap V_{> j} = \emptyset$ follows from the assumption that $S$ is inclusion-wise minimal and Lemma 3.2(iii). □

Due to Lemma 3.2(iii), we can assume without loss of generality that $v_1 = s$ and $v_n = z$.

Theorem 3.1. Temporal $(s,z)$-Separation on order-preserving temporal unit interval graphs with given ordering $<_V$ is solvable in polynomial time.

Let $G = (V, E, \tau)$ be a given order-preserving temporal unit interval graph with total ordering $<_V$ and $k$ be a given upper-bound on the temporal separator size. We assume that there is no layer with an edge between $s$ and $z$. In order to solve the problem, we define the following dynamic programming table $T$ of size $\tau \times (n - 1)$. In the table entry $T[t, i]$ we store a
minimum temporal \((s, z)\)-separator \(S\) for \(G_{[1:t]}\) with the property that there is no vertex reachable from \(s\) in \(G_{[1:t]} - S\) that is larger than \(v_i\). Let

\[
N(v, t, t') := \begin{cases} 
\{N_{G_{[1:t]}}^{\geq t''}(v) \mid t \leq t'' \leq t'\}, & \text{if } \forall t \leq t'' \leq t' : (\{v, z\}, t'') \notin E, \\
\{V \setminus \{s, z\}\}, & \text{otherwise}.
\end{cases}
\]

Let \(T\) be defined in the following way:

\[
T[1, 1] = N_{G_1}(s),
\]

\[
T[t, 1] = \arg \max_{S \in N(s, t)} |S|, \quad (1)
\]

\[
T[1, i] = \arg \min_{S \in Y_i} |S|, \text{ where } Y_i = \{T[1, i-1]\} \cup N(v_i, 1, 1), \quad (2)
\]

\[
T[t, i] = \arg \min_{S \in X_{t,i}} |S|, \text{ where } X_{t,i} = \{T[t', i'] \cup \arg \max_{S \in N(v_i, t'+1, t)} |S| \mid i' \in [i-1] \land t' \in [t-1]\} \\
\cup \{T[t, i-1]\} \cup \{\arg \max_{S \in N(v_i, 1, t)} |S|\}. \quad (3)
\]

We decide whether we face a yes-instance by checking if there is an \(i \in [n-1]\) such that \(|T[\tau, i]| \leq k\).

It is easy to see that each table entry can be computed in polynomial time and the table has polynomial size. Hence, the algorithm has polynomial running time.

**Proof of Correctness.** We prove via induction over both dimensions of \(T\) that \(T[t, i]\) is a minimum temporal \((s, z)\)-separator \(S\) for \(G_{[1:t]}\) with the property that there is no vertex reachable from \(s\) in \(G_{[1:t]} - S\) that is larger than \(v_i\). First, observe that Lemma 3.2(v) implies that \(T[1, 1]\) and \(T[t, 1]\) are correctly filled in Equations (1) and (2). Hence, we have that the base for our induction is correct.

We proceed with the proof of the cases specified by Equations (3) and (4) in two steps. First we show that for all \(T[t, i]\) with \(t \geq 1\) and \(i > 1\), we have that \(T[t, i]\) is a temporal \((s, z)\)-separator \(S\) for \(G_{[1:t]}\) with the property that there is no vertex reachable from \(s\) in \(G_{[1:t]} - S\) that is larger than \(v_i\). Then, in a second step, we show that said separator is *minimum*.

It is easy to check that if \(t = 1\), then for all \(i \in [n-1]\) we have that \(T[1, i]\) (as specified in Equation (3)) is a temporal \((s, z)\)-separator with the desired properties. Next, we consider the case that \(t, i > 1\). We show that every set in \(X_{t,i}\) is a temporal \((s, z)\)-separator with the desired
properties. By induction we know that this holds for $T[t, i - 1]$. It is also easy to check that it holds for $S' = \arg\max_{S \subseteq N(v_i, t, i)} |S|$. For arbitrary $i' \in [i - 1]$ and $t' \in [t - 1]$ (Equation (4)) it is also straightforward to see that $S' = T[t', i'] \cup \arg\max_{S \subseteq N(v_i, t', i + 1)} |S|$ has the desired properties.

By induction, $T[t', i']$ contains a temporal $(s, z)$-separator for $G_{[1:t']}$ with the property that there is no vertex reachable from $s$ in $G_{[1:t']}$ - $T[t', i']$ that is larger than $v_t'$. The set $S'' = \arg\max_{S \subseteq N(v_i, t', i + 1, t)} |S|$ either equals $V \setminus \{s, z\}$, in which case we clearly have a separator with the desired properties, or it forms a temporal $(s, z)$-separator for $G_{[t'+1:t]}$ with the property that there is no vertex reachable from $s$ in $G_{[t'+1:t]} - S'$ that is larger than $v_i$. Then by Lemma 3.2(vi) we get that we have a separator with the desired properties.

Now we show that for all $t \geq 1$ and $i > 1$, the separator contained in $T[t, i]$ is of minimum size. Let $S^* \subseteq V \setminus \{s, z\}$ be a minimum set of vertices such that in $G_{[1:t]}$ - $S^*$ the vertex $v_j$, $j \leq i$, is the largest reachable vertex from $s$. If $j < i$, then by induction hypothesis (both for $t = 1$ and $t > 1$) we have that $|S^*| \geq |T[t, i - 1]|$ and hence $|T[t, i]| \leq |S^*|$. We continue with the case that $j = i$. If $v_i$ is reachable in $G_{[1:t]}$ - $S^*$ from $s$, then by Lemma 3.2(v) we know that $N_{G_{[1:t]}}(v_i) \subseteq S^*$ for all $t' \in [t]$. As $S^*$ is minimum, it holds true that $|S^*| = \max_{S \subseteq N(v_i, 1, t)} |S|$, and we have that $\arg\max_{S \subseteq N(v_i, 1, t)} |S| \in X_{t,i}$ (if $t = 1$, then $\arg\max_{S \subseteq N(v_i, 1, t)} |S| \in Y_i$) which implies that $|T[t, i]| \leq |S^*|$. Now assume that $t > 1$ and $v_i$ is not reachable from $s$ in $G_{[1:t]}$ - $S^*$. Let $t'$ be the largest time-step in which $v_i$ is not reachable from $s$ in $G_{[1:t']}$. Let $i' < i$ be the largest index such that $v_{i'}$ is reachable from $s$ in $G_{[1:t']}$. By Lemma 3.2(v), we know that $S'' := N_{G_{[1:t']}}(v_{i'})$, where $t' + 1 \leq t'' \leq t$ achieves the maximum cardinality, is contained in $S^*$. Let $S'$ be the smallest subset of $S^*$ such that in $G_{[1:t']}$ - $S'$ the vertex $v_{i'}$ is the largest reachable vertex from $s$. By induction hypothesis, we have that $|S'| \geq |T[t', i']|$. From Lemma 3.2(vii) it follows that $S' \cap S'' = \emptyset$. Hence, because $S^*$ is minimum, we can write $S^* = S' \cup S''$. Hence, we have

$$|S| = |S'| + |S''| \geq |T[t', i']| + |N_{G_{[1:t']}}(v_{i'})| \geq \min_{S \subseteq X_{t,i}} |S| = |T[t, i]|,$$

where the second inequality follows from the fact that $T[t', i'] \cup N_{G_{[1:t']}}(v_{i'}) \subseteq X_{t,i}$. □

### 3.2 Underlying-wise Restrictions

We next study temporal graphs where the underlying graph is contained in some graph class. A graph is complete-but-one if all but one possible edges
Lemma 3.3. There exists a polynomial-time many-one reduction that maps any instance \((G = (V,E,\tau), s,t,k)\) of Temporal \((s,z)\)-Separation to an equivalent instance \((G' = (V,E',\tau'), s,t,k)\) such that \(E(G_i(G')) = (\binom{V}{2}) \setminus \{s,t\}\).

Proof. We construct \(G'\) as follows. Let \(G'\) be initially \((V,E' = \emptyset,\tau' = 1)\). For each \((e,t) \in E'\), add the edge \((e,t+1)\) to \(E'\). Next, for each edge \(\{v,w\} \notin G_i\) with \(v, w \in V \setminus \{s\}\) and \(v \neq w\), add \((\{v,w\},1)\) to \(E'\). Finally, for each edge \(\{s,v\} \notin G_i, v \in V \setminus \{z\}\), add \((\{s,v\},\tau+2)\) to \(E'\). The one-to-one correspondence of the temporal \((s,z)\)-separators in \(G\) and \(G'\) is immediate.

Lemma 3.3 implies that Temporal \((s,z)\)-Separation remains \(\text{NP}\)-hard on all temporal graphs where the underlying graph falls into a graph class containing all complete-but-one graphs, for instance the classes of unit interval or threshold graphs. We refer to Figure 1 in Section 1 for an overview.

Note that complete-but-one graphs are no line graphs, as each complete-but-one graph (with at least five vertices) contains the forbidden \(K_5 - e\) as induced subgraph \([5, G_3]\). Hence, we next study Temporal \((s,z)\)-Separation on temporal graphs where the underlying graph is a line graph.

Lemma 3.4. Temporal \((s,z)\)-Separation on temporal graphs where the underlying graph is a line graph is \(\text{NP}\)-complete.

Proof. A temporal \((s,z)\)-path \(P = (\{s = v_0, v_1\}, t_1), \ldots, (\{v_{\ell-1}, v_\ell = z\}, t_\ell)\) is called strict if \(t_i < t_{i+1}\) for all \(i \in \{1, \ldots, \ell - 1\}\). In the literature, strict temporal paths are also known as journeys \([1, 2, 16, 17]\). A vertex set \(S\) is a strict temporal \((s,z)\)-separator if there is no strict temporal \((s,z)\)-path in the temporal graph \(G - S\). The Strict Temporal \((s,z)\)-Separation problem is the “strict” variant of Temporal \((s,z)\)-Separation and asks for a strict temporal \((s,z)\)-separator instead of a temporal \((s,z)\)-separator.

We reduce from the \(\text{NP}\)-hard Strict Temporal \((s,z)\)-Separation where each layer is equal and there is no vertex in the underlying graph of degree at most one \([21]\). Our reduction is close to the reduction from Strict Temporal \((s,z)\)-Separation to Temporal \((s,z)\)-Separation due to Zschoche et al. \([21]\). Let \((G = (V,E,\tau), s,t,k)\) be an instance of Strict Temporal \((s,z)\)-Separation with \(G_i(G) = G_j(G)\) for all \(i, j \in [\tau]\). We construct an instance \((G' = (V',E',\tau'), s'^*, t'^*, k)\) of Temporal \((s,z)\)-Separation, where \(G_i(G')\) is a line graph, as follows.

are present.

Lemma 3.3. There exists a polynomial-time many-one reduction that maps any instance \((G = (V,E,\tau), s,t,k)\) of Temporal \((s,z)\)-Separation to an equivalent instance \((G' = (V,E',\tau'), s,t,k)\) such that \(E(G_i(G')) = (\binom{V}{2}) \setminus \{s,t\}\).

Proof. We construct \(G'\) as follows. Let \(G'\) be initially \((V,E' = \emptyset,\tau' = 1)\). For each \((e,t) \in E'\), add the edge \((e,t+1)\) to \(E'\). Next, for each edge \(\{v,w\} \notin G_i\) with \(v, w \in V \setminus \{s\}\) and \(v \neq w\), add \((\{v,w\},1)\) to \(E'\). Finally, for each edge \(\{s,v\} \notin G_i, v \in V \setminus \{z\}\), add \((\{s,v\},\tau+2)\) to \(E'\). The one-to-one correspondence of the temporal \((s,z)\)-separators in \(G\) and \(G'\) is immediate.

Lemma 3.3 implies that Temporal \((s,z)\)-Separation remains \(\text{NP}\)-hard on all temporal graphs where the underlying graph falls into a graph class containing all complete-but-one graphs, for instance the classes of unit interval or threshold graphs. We refer to Figure 1 in Section 1 for an overview.

Note that complete-but-one graphs are no line graphs, as each complete-but-one graph (with at least five vertices) contains the forbidden \(K_5 - e\) as induced subgraph \([5, G_3]\). Hence, we next study Temporal \((s,z)\)-Separation on temporal graphs where the underlying graph is a line graph.

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We reduce from the \(\text{NP}\)-hard Strict Temporal \((s,z)\)-Separation where each layer is equal and there is no vertex in the underlying graph of degree at most one \([21]\). Our reduction is close to the reduction from Strict Temporal \((s,z)\)-Separation to Temporal \((s,z)\)-Separation due to Zschoche et al. \([21]\). Let \((G = (V,E,\tau), s,t,k)\) be an instance of Strict Temporal \((s,z)\)-Separation with \(G_i(G) = G_j(G)\) for all \(i, j \in [\tau]\). We construct an instance \((G' = (V',E',\tau'), s'^*, t'^*, k)\) of Temporal \((s,z)\)-Separation, where \(G_i(G')\) is a line graph, as follows.
Let $G = (V, E) := G_\downarrow(G)$. We construct a graph $G' = (V', E')$ which will be the underlying graph of $G'$ (we refer to Figure 2 for an illustration). Let $G'$ be initially a copy of $G$. As a first step, iteratively replace each vertex $v$ by a set $W_v$ of $\deg(v) + 1$ vertices such that each edge incident with $v$ is incident with exactly one vertex from $W_v$ and every vertex in $W_v$ is of degree at most one. Note that there is exactly one vertex in $W_v$ not being incident with an edge, and we call this vertex $v^*$. Denote the edge set of $G'$ after the first step by $E''$. Next, replace each edge $\{x,y\} \in E'$ by two paths and of length three and for each identify one of its endpoints with $x$ and one with $y$. Denote by $e^x_{(x,y)}$, $e^y_{(x,y)}$ and by $e^x_{(y,x)}$, $e^y_{(y,x)}$ the inner vertices of each path respectively, where $e^x_{(x,y)}$, $e^y_{(y,x)}$ are the neighbors of $x$ and $e^y_{(x,y)}$, $e^x_{(y,x)}$ are neighbors of $y$ on the paths. Next, connect the neighbors of $x$ on both paths by an edge, and connect the neighbors of $y$ on both paths by an edge (we refer to these edges as path stilts). Finally, for each $v \in V$, make $W_v$ a clique (and refer to all edges in the clique not incident with $v^*$ as clique stilts). This finishes the construction of $G'$. It is not hard to see that $G_\downarrow(G') = G'$.

We construct $G'$ with vertex set $V'$ and underlying graph $G'$ as follows. Add the set $\{(e, 1) \mid e \in E' \text{ is a stilt}\}$. For each $2 \leq t \leq 2\tau + 1$, add the set $\{\{v^*, w\}, t \mid w^* \in W_v \setminus \{v^*\}\}$. For each $1 \leq t \leq \tau$, add the set of temporal edges $\{\{x, e^x_{(x,y)}\}, 2t\}, \{e^x_{(x,y)}, e^y_{(x,y)}\}, 2t), \{\{y, e^y_{(y,x)}\}, 2t\}, \{e^x_{(y,x)}, e^y_{(y,x)}\}, 2t\} \mid \{x, y\} \in E''\}$ and $\{\{x, e^x_{(y,x)}\}, 2t+1\}, \{\{y, e^y_{(y,x)}\}, 2t+1\} \mid \{x, y\} \in E''\}$. This finishes the construction of $G'$. It is not difficult to see that $G_\downarrow(G') = G'$.

For the correctness, it is enough to observe the following. There is no
temporal \((s^*, z^*)\)-path starting at time step one. It holds true that \(\{v, w\} \in E\) if and only if there is a temporal \((v^*, w^*)\)-path starting at \(t\) and ending at \(t + 1\) for every \(2 \leq t \leq 2\tau\) that does not contain a \(u^*\) different to \(v^*, w^*\).

We can assume a minimal temporal \((s^*, z^*)\)-separator in \(G'\) to only contain vertices in \(\{v^* \mid v \in V\}\). Hence, the following is immediate: if \(S \subseteq V\) is a strict temporal \((s, z)\)-separator in \(G\), then \(\{v^* \mid v \in S\}\) is a temporal \((s^*, z^*)\)-separator in \(G'\), and vice versa. 

\[\square\]

**Classification through parameterization.** An alternative way to classify an instance of a graph-theoretic problem is through its (graph) parameters. Through the lens of parameterized complexity theory, fixed-parameter tractability is considered as tractable. Accordingly, we study **Temporal \((s, z)\)-Separation** according to some parameterizations. Any upper-bound on the maximum length of a temporal \((s, z)\)-path leads to a straightforward search-tree algorithm.

**Lemma 3.5.** **Temporal \((s, z)\)-Separation** is solvable in \(O(\ell^k \cdot |E|)\) time, and thus is fixed-parameter tractable when parameterized by \(k + \ell\), where \(k\) is the solution size and \(\ell\) is the maximum length of a temporal \((s, z)\)-path.

**Proof.** We present a depth-first search algorithm (see Algorithm 1) to show fixed-parameter tractability. Let \(\mathcal{I} := (G = (V, E, \tau), s, z, k)\) be an instance of **Temporal \((s, z)\)-Separation**. The basic idea of our algorithm is simple: at least one vertex of each temporal \((s, z)\)-path must be in the temporal \((s, z)\)-separator. Thus, we compute an arbitrary temporal \((s, z)\)-path (Line 4) and branch over all visited vertices of that temporal \((s, z)\)-path (Line 9) until we cannot find a temporal \((s, z)\)-path in \(G - S\) or until we already picked \(k\) vertices to be in temporal \((s, z)\)-separator, in which case the algorithm outputs \(\text{no}\). Hence, if the algorithm outputs \(\text{yes}\), then \(S\) is a temporal \((s, z)\)-separator.

It remains to show that if there is a temporal \((s, z)\)-separator in \(G\), then the algorithm outputs \(\text{yes}\). We show this by induction. We call a tuple \((S', k')\) a **partial solution** if there is a temporal \((s, z)\)-separator \(S\) of size \(k\) such that \(S' \subseteq S\) and \(k' \geq k - |S'|\). Note that \((\emptyset, k)\) is a trivial partial solution. Now assume \(\text{getSeparator}\) is called with a partial solution \((S', k')\), then we have that either \(S'\) is already a temporal \((s, z)\)-separator in which case the algorithm outputs \(\text{yes}\), or there is a temporal \((s, z)\)-path \(P\) in \(G - S'\) and a temporal \((s, z)\)-separator \(S\) such that \(S' \subseteq S\). It is clear that \(S \cap V(P) \neq \emptyset\), let \(v \in S \cap V(P)\). At some point the algorithm chooses the vertex \(v\) in the for-loop in Line 9 and thus invokes a recursive call with \((S' \cup \{v\}, k' - 1)\).
Algorithm 1: The algorithm behind Lemma 3.5.

Data: A temporal graph $G = (V, E, \tau)$, two distinct vertices $s, z \in V$, and an integer $k \in \mathbb{N}$.

Result: Whether $G$ admits a temporal $(s, z)$-separator of size at most $k$.

1. getSeparator($\emptyset, k$);
2. output no;
3. function getSeparator($S, k$)
4. compute temporal $(s, z)$-path $P$ in $G - S$;
5. if there is no temporal $(s, z)$-path in $G - S$ then
6. output yes;
7. exit;
8. else if $k > 0$ then
9. for $v \in V(P) \setminus \{s, z\}$ do
10. getSeparator($S \cup \{v\}, k - 1$);
11. end
12. end

is clear that $(S' \cup \{v\}) \subseteq S$, we additionally have that $k' - 1 \geq k - |S' \cup \{v\}|$ since $v \notin S'$. Hence, we have that $(S' \cup \{v\}, k' - 1)$ is a partial solution. Furthermore, we have that $|S'| < |S' \cup \{v\}|$. It is easy to see that if there is partial solution $(S^*, k^*)$ with $|S^*| = k$, then $S^*$ is a temporal $(s, z)$-separator. This implies that the algorithm eventually finds a temporal $(s, z)$-separator if one exists and hence is correct.

From Lemma 2.1, we know that we can compute Line 4 in $O(|E|)$ time. Now, we upper-bound the size of the search tree in which each node is a call of the getSeparator() function. We can upper-bound the maximum depth of the search tree by $k$ as in each recursive call we decrease $k$ by one, until $k = 0$. Furthermore, a temporal $(s,z)$-path of length at most $\ell$ visits at most $\ell - 1$ vertices different from $s$ and $z$. Thus we can upper-bound the running time of Algorithm 1 by $O(\ell^k \cdot |E|)$.

From Lemma 3.5 we can derive that Temporal $(s,z)$-Separation is linear-time solvable on temporal graph classes where the underlying graph has a constant vertex cover number\(^2\).

\(^2\)The vertex cover number of a graph is the smallest number of vertices that cover all edges in the graph.
Corollary 3.2. Temporal \((s, z)\)-Separation can be solved in \(O(\text{vc} \cdot |E|)\) time, and thus is fixed-parameter tractable when parameterized by the vertex cover number \(\text{vc}\) of the underlying graph.

Proof. Let \(I := (G = (V, E, \tau), s, z, k)\) be an instance on Temporal \((s, z)\)-Separation and \(\text{vc}\) be the vertex cover number of the underlying graph. Since at least one endpoint of each edge of the underlying graph \(G = (V, E)\) must be in the vertex cover, the maximum length of a path in \(G\), and hence the maximum length of a temporal \((s, z)\)-path, is at most \(2 \cdot \text{vc}\).

Without loss of generality we assume that there is no temporal \((s, z)\)-path \(P\) of length two, because each vertex \(v \in V(P) \setminus \{s, z\}\) must be contained in any temporal \((s, z)\)-separator. We can find such a temporal \((s, z)\)-path by restricting the breath-first search of Lemma 2.1 such that it explores only vertices which are reachable by a path which contains at most two non-column edges in the static expansion. Let \(V' \subseteq V\) be a vertex cover of size at most \(\text{vc}\) for \(G\). The graph \(G' = (V' \setminus \{s, z\})\) contains at most \((s, z)\)-paths of length two, because all remaining edges are incident with \(s\) or \(z\). By our assumption, we know that neither of these \((s, z)\)-paths correspond to a temporal \((s, z)\)-path in \(G\). Hence, \(k < \text{vc}\) or \(I\) is a yes-instance. It is folklore that if \(G\) admits a vertex cover of size \(\text{vc}\), then we can compute one in \(O(2^{\text{vc}} \cdot |E|)\) time. The application of Lemma 3.5 completes the proof. \(\square\)

Another graph parameter which upper-bounds the maximum length of a \((s, z)\)-path in the underlying graph is the tree-depth of the underlying graph.

Corollary 3.3. Temporal \((s, z)\)-Separation is solvable in \(O(2^{\text{td}(G)} \cdot k \cdot |E|)\) time, and thus is fixed-parameter tractable when parameterized by \(k + \text{td}(G)\), where \(k\) is the solution size and \(\text{td}(G)\) is the tree-depth of the underlying graph.

First, we provide a formal definition of tree-depth.

Definition 3.2. The tree-depth for graph \(G\) with connected components \(G_1, \ldots, G_p\) is recursively defined by:

\[
\text{td}(G) := \begin{cases} 
1 & \text{if } G \text{ has only one vertex,} \\
\max_{i \in [p]} \text{td}(G_i) & \text{if } G \text{ is not connected, and} \\
1 + \min_{v \in V(G)} \text{td}(G - \{v\}) & \text{if } G \text{ is connected.}
\end{cases}
\]

For details, refer to Nešetřil and de Mendez [19].
A1: A2: A3: S: Z:
s z1
2
2
2
2
2
2
3 2, 3 3 3 3
3
3 3

Figure 3: The idea for the dynamic program from Theorem 3.2 for a temporal graph $G$. Vertices in $S$ are the temporal $(s,z)$-separator, vertices in $Z$ are not reachable from $s$ in $G - S$, and vertices in $A_t$ are not reachable from $s$ in $G - S$ before time $t$.

Proof of Corollary 3.3. The tree-depth of a graph $G$ can be (roughly) approximated by $\log_2(h) \leq \text{td}(G) \leq h$ [19], where $h$ denotes the height of a depth-first search tree of $G$. Hence, all path in $G$ are of length at most $2^{\text{td}(G)}$. The application of Lemma 3.5 completes the proof.

One of the tools from the repertoire for designing fixed-parameter algorithms for (static) graph problems are tree decompositions [8, 9]. A tree decomposition is a mapping of a graph into a related tree-like structure. For many graph problems this tree-like structure can be used to formulate a bottom-up dynamic program that starts at the leaves and ends at the root of the tree decomposition [8]. Indeed, if we parameterize by $\text{tw}_{\downarrow}(G)$, where $\text{tw}_{\downarrow}(G)$ is defined as the treewidth of the underlying graph $G_{\downarrow}(G)$, then we obtain an XP-algorithm by dynamic programming.

Theorem 3.2. Temporal $(s,z)$-Separation is solvable in time $O((\tau + 2)^{\text{tw}_{\downarrow}(G)} \cdot \text{tw}_{\downarrow}(G) \cdot |V| \cdot |E|)$, if a nice tree-decomposition of the underlying graph with treewidth $\text{tw}_{\downarrow}(G)$ is given, and where $\tau$ is the maximum label.

Note that a tree-decomposition of width $O(\text{tw}(G))$ is computable in $2^{O(\text{tw}(G))}$ n time [6] and can be turned into a nice tree-decomposition in polynomial-time [8, Lemma 7.4], where $G$ is a graph with $n$ vertices. The dynamic program is based on the fact that for each vertex $v \in V$ in a temporal graph $G = (V,E,\tau)$ there is a point of time $t \in \tau$ such that $v$ cannot be reached from $s \in V$ before time $t$. In particular, we guess a partition $V = A_1 \uplus \ldots \uplus A_\tau \uplus S \uplus Z$ such that $S$ is the temporal $(s,z)$-separator and in $G - S$, no vertex contained in $Z$ is reachable from $s$ and no vertex $v \in A_t$ can be reached from $s$ before time step $t$, where $t \in \tau$. See Figure 3 for an illustrative example.

We prove Theorem 3.2 by introducing a dynamic program which is executed on a nice tree decomposition.
Definition 3.3. A tree decomposition \( T := (T, (B_i)_{i \in V(T)}) \) of a graph \( G \) is a **nice tree decomposition** if \( T \) is rooted, every node of the tree \( T \) has at most two children nodes, and for each node \( i \in V(T) \) the following conditions are satisfied:

(i) If \( i \) has two children nodes \( k, j \in V(T) \) in \( T \), then \( B_i = B_k = B_j \). Node \( i \) is called a **join node**.

(ii) If \( i \) has one child node \( j \), then one of the following conditions must hold:
   
   (a) \( B_i = B_j \cup \{v\} \). Node \( i \) is called an **introduce node** of \( v \).
   
   (b) \( B_i = B_j \setminus \{v\} \). Node \( i \) is called a **forget node** of \( v \).

(iii) If \( i \) is a leaf in \( T \), then \( |B_i| = 1 \). Node \( i \) is called a **leaf node**.

For the node \( i \in V(T) \), the tree \( T_i \) denotes the subtree of \( T \) rooted at \( i \). The set \( B(T_i) := \bigcup_{j \in V(T_i)} B_j \) is the union of all bags of \( T_i \).

We are going to color \( V \) with \( \tau + 2 \) colors \( \langle A_{[1: \tau]}, S, Z \rangle \). A vertex \( v \in V \) of color \( Y \in \{A_{[1: \tau]}, S, Z\} \) is denoted by \( v \in Y \) and hence each color is a set of vertices. The meaning of colors is that if \( v \in S \), then \( v \) is in the temporal \((s,z)\)-separator, if \( v \in Z \), then \( v \) is not reachable from \( s \) in \( G - S \), and if \( v \in A_i \), then \( v \) cannot be reached before time point \( i \) from \( s \).

Definition 3.4. We say \( \langle A_{[1: \tau]}, S, Z \rangle \) is a **coloring** of \( X \subseteq V(G) \) if \( X = A_1 \uplus \cdots \uplus A_{\tau} \uplus S \uplus Z \). A coloring \( \langle A_{[1: \tau]}, S, Z \rangle \) of \( X \subseteq V(G) \) is **valid** if

(i) \( s \in A_1 \), (ii) \( z \in Z \), and (iii) for all \( a \in A_i, a' \in A_j, \text{ and } b \in Z \)
   
   - there is no temporal \((a,b)\)-path with departure time at least \( i \) in \( G[X] - S \), and
   
   - there is no temporal \((a,a')\)-path with departure time at least \( i \) and arrival time at most \( j - 1 \) in \( G[X] - S \).

We call a coloring \( \langle A_{[1: \tau]}, S, Z \rangle \) of \( X \subseteq Y \subseteq V(G) \) **extendable** to \( Y \) if there is a valid coloring \( \langle A'_{[1: \tau]}, S', Z' \rangle \) of \( Y \) such that \( S \subseteq S', Z \subseteq Z' \), and \( A_i \subseteq A'_i \), for all \( i \in [\tau] \).

Lemma 3.6. Let \( G = (V, E, \tau) \) be a temporal graph, and \( s, z \in V \). There is a valid coloring \( \langle A_{[1: \tau]}, S, Z \rangle \) of \( V \) such that \( |S| = k \) if and only if there is a temporal \((s,z)\)-separator \( S' \) of size \( k \) in \( G \).

Proof. \( \Rightarrow \): Let \( \langle A_{[1: \tau]}, S, Z \rangle \) be a valid coloring of \( V \) such that \( |S| = k \). The vertex \( s \) has the color \( A_1 \) and the vertex \( z \) has the color \( Z \). We know that there is no temporal \((s,z)\)-path in \( G[V] - S = G - S \), otherwise condition (iii) of the definition of a valid coloring is violated. Hence, the vertex set \( S \) is a temporal \((s,z)\)-separator of size \( k \) in \( G \).
Hence, at time point $s$ we compute a table $\text{temporal}(\mathcal{T})$. We can observe that there are no consequence, there is no the earliest time point in which $v$ can be reached from $s$. We set $v \in A_t$. As a consequence, there is no a temporal $(w,v)$-path with departure time at least $t'$ and arrival time at most $t-1$, as otherwise there is a temporal $(s,v)$-path with arrival time at most $t-1$ contradicting that $t$ is the earliest time point in which $v$ is reachable from $s$. Finally, we can observe that there are no $a \in A_i$ and $b \in Z$ such that there is a temporal $(a,b)$-path with departure time at least $i$, because $a$ can be reached at time point $i$ from $s$ and all vertices of color $Z$ are not reachable in $G - S$. Hence, $(A_{[1:\tau]}, S, Z)$ is a valid coloring for $V$.

Let $G = (V, E, \tau)$ be a temporal graph, $s, z \in V$, $G_i$ be the underlying graph of $G$, and $\mathcal{T} = (T, (B_t)_{t \in V(T)})$ be a nice tree decomposition of $G_i$ of width $tw(G_i)$. We add $s$ and $z$ to every bag of $\mathcal{T}$. Thus, $\mathcal{T}$ is of width at most $tw(G_i) + 2$.

In the following, we give a dynamic program on $\mathcal{T}$. For each node $x$ in $T$ we compute a table $D_x$ which stores for each coloring $(A_{[1:\tau]}, S, Z)$ of $B_x$ the minimum size of $S'$ over all valid colorings $(A'_{[1:\tau]}, S', Z')$ of $B(T_x)$ such that $S \subseteq S'$, $Z \subseteq Z'$, and $A_i \subseteq A'_i$ for all $i \in [\tau]$.

$$D_x[A_{[1:\tau]}, S, Z] := \begin{cases} \min |S'|, & \text{there is a valid coloring } (A'_{[1:\tau]}, S', Z') \\ \infty, & \text{otherwise} \end{cases}$$

Let $r \in V(T)$ be the root of $T$. If $D_r[A_{[1:\tau]}, S, Z] = k' < \infty$, then the coloring $(A_{[1:\tau]}, S, Z)$ of $B_r$ is extendable to $B(T_r) = V(G)$ and there is a temporal $(s, z)$-separator of size $k'$ in $G$. Hence, the input instance $(G, s, z, k)$ is a yes-instance if and only if $k' \leq k$.

The dynamic program first computes the tables for all leaf nodes of $T$ and then in a “bottom-up” manner, all tables of nodes of which all child nodes are already computed. The computation of $D_x$, $x \in V(T)$, depends on the type of $x$, that is, whether $x$ is a leaf, introduce, forget, or join node.

**Leaf node.** Let $x \in V(T)$ be a leaf node of $\mathcal{T}$. Thus, $B_x = \{s, v, z\}$. We test each coloring of $B_x$ and set $D_x[A_{[1:\tau]}, S, Z] = \infty$ if $s \notin A_1$ or $z \notin A_1$. 

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Z, because the coloring cannot be valid. Assume \( s \in A_1 \) or \( z \in Z \). We distinguish three cases.

**Case 1:** If \( v \in S \), then this is a valid coloring. We set \( D_x[A_{1:τ}], S, Z] = 1 \).

**Case 2:** If \( v \in Z \), then we set \( D_x[A_{1:τ}], S, Z] = \infty \), if there is a \( \{(s, v), t \} \in E(G[B_x]) \), and \( D_x[A_{1:τ}], S, Z] = 0 \) otherwise.

**Case 3:** If \( v \in A_i, i \in [τ], \) then we set \( D_x[A_{1:τ}], S, Z] = \infty \), if there is a \( \{(s, v), t \} \in E(G[B_x]) \) with \( t < i \) or if there is a \( \{(v, z), t \} \in E(G[B_x]) \) with \( i \leq t \), and \( D_x[A_{1:τ}], S, Z] = 0 \) otherwise.

**Lemma 3.7.** Let \( G \) be a temporal graph and \( T \) be a tree decomposition of \( G \) as described above, \( x \in V(T) \) be a leaf node, and \( \langle A_{1:τ}, S, Z \rangle \) be a coloring of \( B_x \). Then the following holds:

(i) \( D_x[A_{1:τ}], S, Z] < \infty \) if and only if \( \langle A_{1:τ}, S, Z \rangle \) is a valid coloring of \( B_x \).

(ii) If \( \langle A_{1:τ}, S, Z \rangle \) is a valid coloring of \( B_x \), then \( D_x[A_{1:τ}], S, Z] = |S| \).

(iii) The table entry \( D_x[A_{1:τ}], S, Z \rangle \) can be computed in \( O(|E|) \) time.

**Proof.** We first prove (i).

\( \Leftarrow \): Let \( D_x[A_{1:τ}], S, Z] = \infty \). There are five cases in which \( D_x[A_{1:τ}], S, Z] \) is set to \( \infty \). Either \( s \notin A_1, z \notin Z, v \in Z \) and there is a time-edge \( \{(s, v), t \} \in E(G[B(T_x)]) \), or \( v \in A_i \) and there is a time-edge \( \{(s, v), t \} \in E(G[B(T_x)]) \) with \( t < i \) or there is a time-edge \( \{(v, z), t \} \in E(G[B(T_x)]) \) with \( i \leq t \), where \( i \in [τ] \). It follows that \( \langle A_{1:τ}, S, Z \rangle \) is no valid coloring of \( B_x \).

\( \Rightarrow \): Let \( D_x[A_{1:τ}], S, Z] < \infty \). Note that \( s \) must be of color \( A_1 \) and \( z \) must be of color \( Z \). Observe that \( D_x[A_{1:τ}], S, Z] = 0 \) or \( D_x[A_{1:τ}], S, Z] = 1 \). Consider the case of \( D_x[A_{1:τ}], S, Z] = 1 \). Thus, \( v \in S \). This implies that \( G[B(T_x)] - S \) is time-edgeless and therefore \( \langle A_{1:τ}, S, Z \rangle \) is a valid coloring of \( B_x \). Next, consider the case of \( D_x[A_{1:τ}], S, Z] = 0 \). If \( v \in Z \), then there is no time-edge from \( s \) to \( v \) which means \( \langle A_{1:τ}, S, Z \rangle \) is a valid coloring of \( B_x \). If \( v \in A_i \), then there is no time-edge \( \{(s, v), t \} \) with \( t < i \) and there is no time-edge \( \{(v, z), t \} \) with \( i \leq t \). In both cases \( \langle A_{1:τ}, S, Z \rangle \) is a valid coloring of \( B_x \).

If \( \langle A_{1:τ}, S, Z \rangle \) is a valid coloring of \( B_x \), then \( D_x[A_{1:τ}], S, Z] = |S| \) as we set \( D_x[A_{1:τ}], S, Z] = 1 \) if and only if \( v \in S \). This proves (ii). Furthermore, we can check by iterating over all time-edges whether \( \langle A_{1:τ}, S, Z \rangle \) is a valid coloring of \( B_x \). This proves (iii), and hence (i)–(iii) hold true.

**Introduce node.** Let \( x \in V(T) \) be an introduce node of \( T \), \( y \in V(T) \) denote its child node, and \( B_x \setminus B_y = \{v\} \). We distinguish three cases.

**Case 1:** If \( v \in S \), then we set \( D_x[A_{1:τ}], S, Z] = D_y[A_{1:τ}], S \setminus \{v\}, Z] + 1 \).
Case 2: If \( v \in Z \), then we set \( D_x[A_{1:1}, S, Z] = D_y[A_{1:1}, S, Z \setminus \{v\}] \) if for all \( w \in V \) with \( \{(w, v), t\} \in E(G[B(T_x)]) \) holds \( w \in A_i \Rightarrow t < i \). Otherwise, we set \( D_x[A_{1:1}, S, Z] = \infty \).

Case 3: If \( v \in A_i, i \in [\tau] \), then we set \( D_x[A_{1:1}, S, Z] = D_y[A_{1:i-1}, A_i \setminus \{v\}, A_{i+1:1}, S, Z] \), if for all \( \{(v, w), t\} \in E(G[B(T_x)]) \) holds: \( t \geq i \Rightarrow w \in \bigcup_{j=1}^{t} A_j \cup S \) and \( t < i \Rightarrow w \in \bigcup_{j=t+1}^{\tau} A_j \cup S \cup Z \). Otherwise, we set \( D_x[A_{1:1}, S, Z] = \infty \).

We prove the correctness for each case separately. First, we prove the correctness of the first case.

Lemma 3.8. Let \( G \) and \( T \) be as described above, \( x \in V(T) \) be an introduce node of \( v, y \in V(T) \) be the child node of \( x, \langle A_{1:1}, S, Z \rangle \) be a coloring of \( B_x \) and \( v \in S \). Then the following holds:

1. Coloring \( \langle A_{1:1}, S \setminus \{v\}, Z \rangle \) of \( B_y \) is extendable to \( B(T_y) \) if and only if coloring \( \langle A_{1:1}, S, Z \rangle \) of \( B_x \) is extendable to \( B(T_x) \).
2. The value of \( D_x[A_{1:1}, S, Z] \) corresponds to Equation (5) and can be computed in \( O(1) \) time.

Proof. \( \Rightarrow \): Let \( \langle A_{1:1}, S \setminus \{v\}, Z \rangle \) be a coloring of \( B_y \) which is extendable to \( B(T_y) \). Then there is a valid coloring \( \langle A'_{1:1}, S', Z' \rangle \) of \( B(T_y) \) such that \( S \setminus \{v\} \subset S', Z \subset Z' \), and \( A_i \subset A'_i \), for all \( i \in [\tau] \), where \( S' \) is a temporal \((s, z)\)-separator in \( G[B(T_y)] \) of size \( D_y[A_{1:1}, S \setminus \{v\}, Z] \). Note that \( v \notin S' \), because \( v \notin B(T_y) \), because \( x \) is the introduce node for \( v \). Since \( B(T_y) \setminus B(T_y) = \{v\} \), we know that \( G[B(T_y)] - S' \) is the same temporal graph as \( G[B(T_x)] - (S' \cup \{v\}) \). Hence, the coloring \( \langle A_{1:1}, S, Z \rangle \) is extendable to \( B(T_x) \) and \( |S' \cup \{v\}| = |S'| + 1 \) implies that the table entry \( D_x[A_{1:1}, S, Z] = D_y[A_{1:1}, S \setminus \{v\}, Z] + 1 \).

\( \Leftarrow \): Let \( \langle A_{1:1}, S \setminus \{v\}, Z \rangle \) be not extendable to \( B(T_y) \) then \( \langle A_{1:1}, S, Z \rangle \) is not extendable to \( B(T_x) \) because \( G[B(T_y)] \) is a temporal subgraph of \( G[B(T_x)] \), where \( v \notin B(T_y) \). Hence, \( D_x[A_{1:1}, S, Z] = \infty + 1 = \infty \).

Note that \( D_x[A_{1:1}, S, Z] \) can be computed in \( O(1) \) time because we just have to look up the value of \( D_y[A_{1:1}, S, Z] \).

Second, we show the correctness of the second case.

Lemma 3.9. Let \( G \) and \( T \) be as described above, \( x \in V(T) \) be an introduce node of \( v, y \in V(T) \) be the child node of \( x, \langle A_{1:1}, S, Z \rangle \) be a coloring of \( B_x \) and \( v \in Z \). Then the following holds:
1. Coloring \( \langle A_{[1,\tau]}, S, Z \rangle \) of \( B_x \) is extendable to \( B(T_x) \) if and only if coloring \( \langle A_{[1,\tau]}, S, Z \setminus \{v\} \rangle \) of \( B_y \) is extendable to \( B(T_y) \) and for all \((\{w,v\}, t) \in E(G[B(T_x)])\) it holds that \( w \in A_i \implies t < i \).

2. The value of \( D_x[A_{[1,\tau]}, S, Z] \) corresponds to Equation (5) and can be computed in \( O(|E|) \) time.

**Proof.** \( \Rightarrow \): Let coloring \( \langle A_{[1,\tau]}, S, Z \setminus \{v\} \rangle \) of \( B_y \) be extendable to \( B(T_y) \). Then, there is a valid coloring \( \langle A'_{[1,\tau]}, S', Z' \rangle \) of \( B(T_x) \) such that \( S \subseteq S', Z \subseteq Z' \), and \( A_i \subseteq A'_i \), for all \( i \in [\tau] \). Since \( B(T_y) = B(T_x) \setminus \{v\} \) and \( (Z \setminus \{v\}) \subseteq Z \subseteq Z' \), the coloring \( \langle A_{[1,\tau]}, S, Z \setminus \{v\} \rangle \) of \( B_y \) is extendable to \( B(T_y) \). Furthermore, \( v \in Z \) implies that for all time-edges \((\{w,v\}, t) \in E(G[B(T_x)])\) it holds that \( w \in A_i \implies t < i \).

\( \Leftarrow \): First, if coloring \( \langle A_{[1,\tau]}, S, Z \setminus \{v\} \rangle \) of \( B_y \) is not extendable to \( B(T_y) \) then coloring \( \langle A_{[1,\tau]}, S, Z \rangle \) of \( B_x \) cannot be extendable to \( B(T_x) \) because \( G[B(T_y)] \) is a temporal subgraph of \( G[B(T_x)] \). Hence, \( D_x[A_{[1,\tau]}, S, Z] = \infty \).

Let \( \langle A_{[1,\tau]}, S, Z \setminus \{v\} \rangle \) be a coloring of \( B_y \) which is extendable to \( B(T_y) \) and for all \((\{w,v\}, t) \in E(G[B(T_x)])\) it holds that \( w \in A_i \implies t < i \). Since \( \langle A_{[1,\tau]}, S, Z \setminus \{v\} \rangle \) is extendable to \( B(T_y) \) we know that there is a valid coloring \( \langle A'_{[1,\tau]}, S', Z' \rangle \) of \( B(T_y) \) such that \( S \subseteq S', Z \setminus \{v\} \subseteq Z' \), and \( A_i \subseteq A'_i \), for all \( i \in [\tau] \). We claim that \( \langle A'_{[1,\tau]}, S', Z' \cup \{v\} \rangle \) is a valid coloring of \( B(T_x) \).

As \( \langle A'_{[1,\tau]}, S', Z' \rangle \) is a valid coloring of \( B(T_y) \), we have that \( s \in A'_{i_1}, z \in Z' \), and for all \( a \in A'_{i_1} \) and \( a' \in A'_{i_2} \) there is no temporal \((a,a')\)-path with departure time at least \( i \) and arrival time at most \( j - 1 \) in \( G[B(T_y)] \) \( - S \), for all \( i, j \in [\tau] \).

Suppose there exist \( a \in A'_{i_1} \) and \( b \in Z' \) such that there is a temporal \((a,b)\)-path \( P \) in \( G[B(T_x)] \) \( - S \) with departure time at least \( i \), for some \( i \in [\tau] \). Since \( B(T_x) \) \( - B(T_y) = \{v\} \) and \( \langle A'_{[1,\tau]}, S', Z' \rangle \) is a valid coloring of \( B(T_y) \), vertex \( v \) is the first vertex of color \( Z \) which is visited by \( P \). Hence, there is a time-edge \((\{w,v\}, t) \in E(G[B(T_x)])\) such that \( w \in A_i \) and \( i \leq t \), contradicting \( w \in A_i \implies t < i \). It follows that \( \langle A'_{[1,\tau]}, S', Z' \cup \{v\} \rangle \) is a valid coloring of \( B(T_x) \) and hence \( \langle A_{[1,\tau]}, S, Z \rangle \) is extendable to \( B(T_x) \).

Since \( v \in Z \), we have \( D_x[A_{[1,\tau]}, S, Z] = D_y[A_{[1,\tau]}, S, Z \setminus \{v\}] \).

Note that \( D_x[A_{[1,\tau]}, S, Z] \) can be computed in \( O(|E|) \) time, since we can decide whether for all \((\{w,v\}, t) \in E(G[B(T_x)])\) it holds that \( w \in A_i \implies t < i \) by iterating once over the time-edges in \( E \). \( \square \)

Last, we show the correctness of the third case.

**Lemma 3.10.** Let \( G \) and \( T \) be as described above, \( x \in V(T) \) be an introduce node of \( v \), \( y \in V(T) \) be the child node of \( x \), \( \langle A_{[1,\tau]}, S, Z \rangle \) be a coloring of \( B_x \) and \( v \in A_i \), where \( i \in [\tau] \). Then the following holds:
1. Coloring \( \langle A_{[1:\tau]}, S, Z \rangle \) of \( B_x \) is extendable to \( B(T_x) \) if and only if coloring \( \langle A_{[1:\tau]}, A_i \setminus \{v\}, A_{[i+1:\tau]}, S, Z \rangle \) of \( B_y \) is extendable to \( B(T_y) \) and for each \( \{v, w\}, t \in E(G[B(T_x)]) \) it holds that: \( t \geq i \implies w \in \bigcup_{j=1}^{t-1} A_j \cup S \) and \( t < i \implies w \in \bigcup_{j=t+1}^{\tau} A_j \cup S \cup Z \).

2. The value of \( D_x[A_{[1:\tau]}, S, Z] \) corresponds to Equation (5) and can be computed in \( O(|E|) \) time.

Proof. \( \Rightarrow \) Let coloring \( \langle A_{[1:\tau]}, S, Z \rangle \) of \( B_x \) be extendable to \( B(T_x) \). Then, there is a valid coloring \( \langle A'_{[1:\tau]}, S', Z' \rangle \) of \( B(T_x) \) such that \( S \subseteq S', Z \subseteq Z' \), and \( A_j \subseteq A'_j \) for all \( j \in [\tau] \). Since \( B(T_y) = B(T_x) \setminus \{v\} \) and \( \langle A_i \setminus \{v\} \rangle \subseteq A_i \subseteq A'_i \), the coloring \( \langle A_1, \ldots, A_i \setminus \{v\}, A_{[i+1:\tau]}, S, Z \rangle \) of \( B_y \) is extendable to \( B(T_y) \). Let \( \langle \{v, w\}, t \rangle \in E(G[B(T_x)]) \). We distinguish into two cases.

First, let \( t \geq i \). Note that \( w \in B_y \) since \( x \) is a introduce node for \( v \). Since \( \langle A'_{[1:\tau]}, S', Z' \rangle \) is a valid coloring of \( B(T_x) \), \( w \notin Z \) since there is no temporal \( (v, w) \)-path with departure time \( t \) in \( G[B(T_x)] - S' \). Assume towards a contradiction that \( w \in A_j \), where \( j \in \{t+1, \ldots, \tau\} \). Then the time-edge \( \{\{v, w\}, t\} \) is a temporal \( (v, w) \)-path with departure time at least \( i \) and arrival time at most \( j - 1 \), contradicting the fact that \( \langle A'_{[1:\tau]}, S', Z' \rangle \) is a valid coloring of \( B(T_x) \). Hence, \( w \in \bigcup_{j=1}^{t-1} A_j \cup S \).

Second, let \( t < i \). Again, \( \langle A'_{[1:\tau]}, S', Z' \rangle \) is a valid coloring of \( B(T_x) \) and therefore \( w \notin \bigcup_{j=1}^{t-1} A_j \) because otherwise there would be a temporal \( (w, v) \)-path in \( G[B(T_x)] - S' \) with departure time at least \( t \) and arrival time \( t < i \), contradicting the fact that \( \langle A'_{[1:\tau]}, S', Z' \rangle \) being a valid coloring. Hence \( w \in \bigcup_{j=t+1}^{\tau} A_j \cup S \cup Z \).

\( \Leftarrow \) First, if coloring \( \langle A_1, \ldots, A_i \setminus \{v\}, \ldots, A_{[\tau]}, S, Z \rangle \) of \( B_y \) is not extendable to \( B(T_y) \) then coloring \( \langle A_{[1:\tau]}, S, Z \rangle \) of \( B_x \) cannot be extendable to \( B(T_x) \) because \( G[B(T_y)] \) is a temporal subgraph of \( G[B(T_x)] \). Hence, \( D_x[A_{[1:\tau]}, S, Z] = \infty \).

Let coloring \( \langle A_1, \ldots, A_i \setminus \{v\}, \ldots, A_{[\tau]}, S, Z \rangle \) of \( B_y \) be extendable to \( B(T_y) \) and for each \( \{v, w\}, t \in E(G[B(T_x)]) \) it holds that: \( t \geq i \implies w \in \bigcup_{j=1}^{t-1} A_j \cup S \) and \( t < i \implies w \in \bigcup_{j=t+1}^{\tau} A_j \cup S \cup Z \). As the coloring \( \langle A_1, \ldots, A_i \setminus \{v\}, \ldots, A_{[\tau]}, S, Z \rangle \) is extendable to \( B(T_y) \), we have a valid coloring \( \langle A'_i, S', Z' \rangle \) of \( B(T_y) \) such that \( S \subseteq S', Z \subseteq Z' \), \( A_i \setminus \{v\} \subseteq A'_i \), and \( A_j \subseteq A'_j \), for all \( j \in [\tau] \setminus \{i\} \). We claim that \( \langle A'_i, \ldots, A'_i \setminus \{v\}, \ldots, S', Z' \rangle \) is a valid coloring for \( B(T_x) \). We know \( s \in A'_i \) and \( z \in Z' \).

Suppose towards a contradiction that there exist \( a \in A'_j \) and \( a' \in A'_{j'}, j, j' \in [\tau] \), such that there is a temporal \( (a, a') \)-path \( P \) with departure time at least \( j \) and arrival time at most \( \ell - 1 \). Since coloring \( \langle A'_{[1:\tau]}, S', Z' \rangle \) of \( B(T_y) \) is valid, we know that \( v \) appears in \( P \). Thus, there are time-
edges $\{\{w_1, v\}, t_1\}, \{\{v, w_2\}, t_2\} \in E(G[B(T_y)])$ in $P$ such that $t_1 \leq t_2$ and $w_1$ appears before $v$ and $v$ appears before $w_2$ in $P$, where $w_1 \in A'_{u_1}, w_2 \in A'_{u_2}$. Note that $w_1 \in A_{u_1}$ and $w_2 \in A_{u_2}$ as $x$ is an introduce node of $v$. Refer to Figure 4 for an illustration.

We know that
\begin{itemize}
    \item $u_1 \leq t_1$, otherwise there is a temporal $(a, w_1)$-path with departure time at least $j$ and arrival time at most $u_1 - 1$ in $G[B(T_y)]$, contradicting the fact that $(A'_{1:|\tau|}, S', Z')$ is valid.
    \item $i \leq t_1$, otherwise either $w_1 \notin \bigcup_{j=t_1+1}^{t} A_j \cup S \cup Z$ contradicting the fact that for each $\{(v, w), t\} \in E(G[B(T_y)])$ it holds that $t < i \implies w \in \bigcup_{j=t+1}^{t} A_j \cup S \cup Z$, or $w_1 \in \bigcup_{j=t_1+1}^{t} A_j \cup S \cup Z$ and $w \in A_{u_1}$, contradicting the fact that $(A'_{1:|\tau|}, S', Z')$ is a coloring of $B(T_y)$.
    \item $i \leq t_2$, otherwise $i > t_1$ since $t_1 \leq t_2$.
    \item $u_2 \leq t_2$, otherwise $i \leq t_2$ and $w_2 \notin \bigcup_{j=1}^{t} A'_j$, contradicting the fact that for each $\{(v, w), t\} \in E(G[B(T_y)])$ it holds that $t < i \implies w \in \bigcup_{j=1}^{t} A_j \cup S \cup Z$, or $w_2 \in \bigcup_{j=1}^{t} A'_j$ and $w \in A_{u_2}$, contradicting the fact that $(A'_{1:|\tau|}, S', Z')$ is a coloring of $B(T_y)$.
\end{itemize}

It follows that $P$ contains the temporal $(w_2, a')$-path as temporal subpath with departure time at least $u_2 \leq t_2$ and arrival time $\ell - 1$. As this temporal subpath also exists in $B(T_y)$, this contradicts the fact that coloring $(A'_{1:|\tau|}, S', Z')$ of $B(T_y)$ is valid. We conclude that $P$ does not exist.

Next, suppose towards a contradiction that there exist $a \in A'_j, j \in [\tau]$, and $b \in Z$ such that there is a temporal $(a, b)$-path $P'$ with departure time at least $j$. The vertex $v \in A_i$ is the last vertex visited by $P'$ which is not colored by $Z$, otherwise we would be able to find a subsequence of $P'$ similar to $P$. Thus, there are time-edges $\{\{w_1, v\}, t_1\}, \{\{v, b\}, t_2\} \in E(G[B(T_y)])$ which are in $P'$ such that $w_1$ is visited before $v$ and $v$ is visited before $b$, where $w_1 \in A'_{u_1}$. We conclude analogously to the case of $P$ that $u_1 \leq t_1, i \leq t_1, i \leq t_2$. Since $i \leq t_2$, we have that either $b \notin \bigcup_{j=1}^{t} A_j \cup S$, contradicting the fact that for each $\{(v, w), t\} \in E(G[B(T_y)])$ it holds that $t \geq i \implies w \in \bigcup_{j=1}^{t} A_j \cup S$, or $b \in \bigcup_{j=1}^{t} A_j \cup S$ and $b \in Z$, contradicting
the fact that \(\langle A_1, \ldots, A_i \setminus \{v\}, \ldots, A_r, S, Z \rangle\) is a coloring of \(B_y\). Hence, \(P'\) does not exist.

Clearly, \(D_x[A_{1:1}, S, Z] = D_y[A_{1:1}, A_i \setminus \{v\}, \ldots, A_r, S, Z]\) because \(v \notin S\).

Note that \(D_x[A_{1:1}, S, Z]\) can be computed in \(O(|E|)\) time, because we can iterate once over the time-edge set \(E\) and decide if for all \((\{w, v\}, t) \in E(G[B(T_y)])\) it holds that if \(t \geq i\) then \(w \in \bigcup_{j=i}^{t} A_j \cup S\) and if \(t < i\) then \(w \in \bigcup_{j=t+1}^{i} A_j \cup S \cup Z\). □

**Forget node.** Let \(x \in V(T)\) be a forget node of \(T\), \(y \in V(T)\) its child, and \(B_y \setminus B_x = \{v\}\). We set

\[
D_x[A_{1:1}, S, Z] = \min \left\{ \min_{i \in [1]} D_y[A_{i+1}, A_i \cup \{v\}, A_{i+1}, S, Z], \right.
\]

\[
\left. D_y[A_{1:1}, S \cup \{v\}, Z], \quad D_y[A_{1:1}, S, Z \cup \{v\}] \right\}
\]

**Lemma 3.11.** Let \(G\) and \(T\) be as described above, \(x \in V(T)\) be a forget node of \(v\), \(y \in V(T)\) be the child node of \(x\), and \(A_{1:1}, S, Z\) be a coloring of \(B_x\). Then the following holds:

1. Coloring \(\langle A_{1:1}, S, Z \rangle\) of \(B_x\) is extendable to \(B(T_x)\) if and only if there is a coloring \(\langle A'_{1:1}, S', Z' \rangle\) of \(B_y\) which is extendable to \(B(T_y)\) such that \(S \subseteq S', Z \subseteq Z', A_i \subseteq A_i',\) for all \(i \in [\tau]\).

2. The value of \(D_x[A_{1:1}, S, Z]\) corresponds to Equation (5) and can be computed in \(O(|E|)\) time.

**Proof.** ⇒: Let coloring \(\langle A_{1:1}, S, Z \rangle\) of \(B_x\) be extendable to \(B(T_x)\). Then there is a valid coloring \(\langle A''_{1:1}, S'', Z'' \rangle\) of \(B(T_x)\) such that \(S \subseteq S'', Z \subseteq Z'',\) and \(A_i \subseteq A_i''\), for all \(i \in [\tau]\). Since \(y\) is a child of \(x\) and \(B_x \subseteq B_y\), we know that \(B(T_x) = B(T_y)\) and therefore there is a coloring \(\langle A'_{1:1}, S', Z' \rangle\) of \(B_y\) which is extendable to \(B(T_y)\), where \(S' \subseteq S'', Z' \subseteq Z'',\) and \(A_i \subseteq A_i',\) for all \(i \in [\tau]\). It follows from \(B_x \subseteq B_y\), that \(S \subseteq S', Z \subseteq Z',\) and \(A_i \subseteq A_i',\) for all \(i \in [\tau]\).

⇐: It is easy to see that coloring \(\langle A_{1:1}, S, Z \rangle\) of \(B_x\) is extendable to \(B(T_x)\) if there is a coloring \(\langle A'_{1:1}, S', Z' \rangle\) of \(B_y\) which is extendable to \(B(T_y)\), where \(S \subseteq S', Z \subseteq Z'\) and \(A_i \subseteq A_i',\) for all \(i \in [\tau]\), because \(G[B(T_y)]\) is a temporal subgraph of \(G[B(T_y)]\). Since we want to extend the coloring of \(B_x\) such that we have a minimum size \(S\), we take the minimum of all possible colorings \(\langle A'_{1:1}, S', Z' \rangle\) of \(B_y\) such that \(S \subseteq S', Z \subseteq Z'\) and \(A_i \subseteq A_i',\) for all \(i \in [\tau]\).
Note that we can compute the table entry \( D_x[A_{1:τ}, S, Z] \) in \( O(|E|) \) time, because we have to look up \( τ + 2 \) entries in \( D_y \) and \( τ \leq |E| \), see [21].

**Join node.** Let \( x \in V(T) \) be a join node of \( T \), \( y_1, y_2 \in V(T) \) be children of \( x \), and hence \( B_x = B_{y_1} = B_{y_2} \). We set \( D_x[A_{1:τ}, S, Z] = D_{y_1}[A_{1:τ}, S, Z] + D_{y_2}[A_{1:τ}, S, Z] - |S| \).

**Lemma 3.12.** Let \( G \) be a temporal graph and \( T \) be a tree decomposition of \( G \) as described above, \( x \in V(T) \) be a join node of \( v \), \( y_1, y_2 \in V(T) \) be the child nodes of \( x \), and \( \langle A_{1:τ}, S, Z \rangle \) be a coloring of \( B_x \). Then the following holds:

1. Coloring \( \langle A_{1:τ}, S, Z \rangle \) of \( B_x = B_{y_1} = B_{y_2} \) is extendable to \( B(T_x) \) if and only if it is extendable to \( B(T_{y_1}) \) and \( B(T_{y_2}) \).
2. The value of \( D_x[A_{1:τ}, S, Z] \) corresponds to Equation (5) and can be computed in \( O(1) \) time.

**Proof.** \( \Rightarrow \): Let coloring \( \langle A_{1:τ}, S, Z \rangle \) of \( B_x = B_{y_1} = B_{y_2} \) be extendable to \( B(T_x) \). Then there is a valid coloring \( \langle A'_{1:τ}, S', Z' \rangle \) of \( B(T_x) \) such that \( S \subseteq S' \), \( Z \subseteq Z' \), and \( A_i \subseteq A_i' \), for all \( i \in [τ] \). Since \( B(T_{y_1}), B(T_{y_2}) \subseteq B(T_x) \) and \( B_x = B_{y_1} = B_{y_2} \), we know that \( \langle A_{1:τ}, S, Z \rangle \) is extendable to \( B(T_{y_1}) \) and \( B(T_{y_2}) \).

\( \Leftarrow \): Let coloring \( \langle A_{1:τ}, S, Z \rangle \) of \( B_x \) be extendable to \( B(T_{y_1}) \) and \( B(T_{y_2}) \). Thus, there is a valid coloring \( \langle A'_{1:τ}, S', Z' \rangle \) for \( B(T_{y_1}) \) such that \( S \subseteq S' \), \( Z \subseteq Z' \), \( A_i \subseteq A_i' \), and there is a coloring \( \langle A''_{1:τ}, S'', Z'' \rangle \) for \( B(T_{y_2}) \) such that \( S \subseteq S'' \), \( Z \subseteq Z'' \), \( A_i \subseteq A_i'' \), for all \( i \in [τ] \). We claim that \( \langle A'_i \cup A''_i, \ldots, A'_i \cup A''_i, S' \cup S'', Z' \cup Z'' \rangle \) is a valid coloring of \( B(T_x) \). Suppose not, that is, \( \langle A'_i \cup A''_i, \ldots, A'_i \cup A''_i, S' \cup S'', Z' \cup Z'' \rangle \) is a coloring but not valid, or it forms no coloring.

In the first case, each \( s \notin A'_i \cup A''_i \) or \( z \notin Z' \cup Z'' \) contradicts the fact that \( \langle A'_{1:τ}, S', Z' \rangle \) and \( \langle A''_{1:τ}, S'', Z'' \rangle \) are valid colorings. Next, suppose there are \( a \in A'_i \cup A''_i \), \( i \in τ \), and \( b \in Z' \cup Z'' \) such that there is a temporal \((a, b)\)-path \( P \) with departure time at least \( i \) in \( G[B(T_x)] - (S' \cup S'') \). Then, either \( P \) exists in \( G[B(T_{y_1})] \) or in \( G[B(T_{y_2})] \), contradicting the fact that \( \langle A'_{1:τ}, S', Z' \rangle \) and \( \langle A''_{1:τ}, S'', Z'' \rangle \) are valid colorings, or \( P \) contains an edge \( \{v, w\}, t \) that is neither in \( G[B(T_{y_1})] \) nor in \( G[B(T_{y_2})] \). It follows that \( \{v, w\} \notin B_{y_1} \cap B_{y_2} \) but \( \{v, w\} \subseteq B_x \), contradicting the fact that \( T \) is a nice tree decomposition. It is not difficult to see that the case of \( a \in A'_i \cup A''_i \) and \( a \in A'_j \cup A''_j \), \( i, j \in τ \), such that there is a temporal \((a, a')\)-path \( P \) with departure time at least \( i \) at arrival time at most \( j - 1 \) in \( G[B(T_x)] - (S' \cup S'') \), follows the same argumentation.
In the second case, that is \( A'_1 \cup A''_1, \ldots, A'_\tau \cup A''_\tau, S' \cup S'', Z' \cup Z'' \) forms no coloring, there is a vertex \( v \in B(T_{y_2}) \cap B(T_{y_1}) \) which has different colors in \( A'_{1:\tau}, S', Z' \) and \( A''_{1:\tau}, S'', Z'' \). If \( v \not\in B_x = B_{y_1} = B_{y_2} \), then \( B^{-1}(v) \) is not a connected subtree of \( T \), contradicting the fact that \( T \) is a nice tree decomposition. If \( v \in B_x \), then \( v \) has different colors in \( A'_{1:\tau}, S, Z \), contradicting the fact that \( A'_{1:\tau}, S, Z \) is a coloring of \( B_x \). Altogether, it follows that \( A'_1 \cup A''_1, \ldots, A'_\tau \cup A''_\tau, S' \cup S'', Z' \cup Z'' \) is a valid coloring of \( B(T_x) \).

Furthermore, this implies that for all vertices \( w \in B(T_x) \) it holds that \( w \in S' \cap S'' \) implies \( w \in S \). Hence, \( |S'| + |S''| - |S| = |S'| + |S''| - |S' \cap S''| = |S' \cup S''| \).

Note that by a look up one table entry of \( D_y \) and one in \( D_{y_2} \), we can compute the table entry \( D_x[A'_{1:\tau}, S, Z] \) in \( \mathcal{O}(1) \) time. \( \square \)

Having Lemmata 3.6 to 3.12, we now prove Theorem 3.2.

**Proof of Theorem 3.2.** The algorithm works as follows on input instance \( \mathcal{I} = (G = (V, E, \tau), s, z, k) \) of TEMPORAL \((s, z)\)-SEPARATION. Let \( T \) be a nice tree decomposition for the underlying graph \( G_1 \) of width at most \( \text{tw}(G_1) \).

1. Add \( s \) and \( z \) to every bag in \( \mathcal{O}(\text{tw}_1(G) \cdot |V|) \) time. Note that \( |V(T)| \in \mathcal{O}(\text{tw}_1(G) \cdot |V|) \) and that each bag is of size at most \( \text{tw}_1(G) + 2 \).
2. Compute the dynamic program of Equation (5) on \( T \). This can be done in \( \mathcal{O}((\tau + 2)^{\text{tw}_1(G) + 2} \cdot \text{tw}_1(G) \cdot |V| \cdot |E|) \), because there are at most \( (\tau + 2)^{\text{tw}_1(G) + 2} \) possible colorings for each bag, there are at most \( \mathcal{O}(\text{tw}_1(G) \cdot |V| \cdot |E|) \) many bags, and table entry for one coloring can be computed in \( \mathcal{O}(|E|) \) time, see Lemmata 3.7 to 3.12.
3. Iterate over the root table \( D_y \). If there is an entry of size at most \( k \), then output yes, otherwise output no. The correctness of this step follows from Lemma 3.6.

Hence, the input instance \( \mathcal{I} \) can be decided in \( \mathcal{O}((\tau + 2)^{\text{tw}_1(G) + 2} \cdot \text{tw}_1(G) \cdot |V| \cdot |E|) \) time. \( \square \)

It remains open whether TEMPORAL \((s, z)\)-SEPARATION is fixed-parameter tractable when parameterized by \( \text{tw}_1 \) or by \( k + \text{tw}_1 \).

## 4 Temporal Restrictions

In Section 3 we considered restrictions on the layers and the underlying graph. Observe that these restrictions do not cover the temporal aspects of a temporal graph, that is, any reordering of the layers yields a different temporal graph having the same restrictions. In this section, we study temporal
Table 1: Let $\tau$ denote the maximum time label and $r$ the number of periods in $G$.

| Graph Class                        | Temporal ($s, z$)-Separation | polynomial-time | NP-hard |
|------------------------------------|------------------------------|-----------------|---------|
| $p$-monotone temporal graphs       | single-peaked                | $p \geq 2$      |         |
| $p$-periodic temporal graphs       | $p = 1$, or $r \geq n$       | $p \geq 2$      |         |
| $T$-interval connected temporal graphs | -                            | $T \geq 1$      |         |
| $\lambda$-steady temporal graphs  | $\lambda = 0$ or $(\lambda, \tau \text{ const.})$ | $\lambda \geq 1$ |         |

graph classes whose definitions rely on the temporal aspect of any temporal graph, that is, the ordering of the layers. Herein, we study monotone, periodic, consecutively connected, and steady temporal graphs.

Note that monotone, periodic, and consecutively connected are quite specific temporal graph classes [7]. Unfortunately, even on these specific temporal graph classes, except for trivial cases we obtain hardness by straightforward arguments. We refer to Table 1 for an overview on our results.

**Monotone Temporal Graphs.** Intuitively, a temporal graph is monotone if it can be decomposed into time-intervals on which the layers are consecutively subgraphs or supergraphs.

**Definition 4.1.** A temporal graph $G = (V, E, \tau)$ is $p$-monotone if $p \in \mathbb{N}$ is the smallest number such that there are $1 = i_1 < i_2 < \ldots < i_{p+1} = \tau$ such that for all $\ell \in [p]$ holds $E_j \subseteq E_{j+1}$ or $E_j \supseteq E_{j+1}$ for all $i_\ell \leq j < i_{\ell+1}$.

Khodaverdian et al. [13] call a temporal graph monotone if whenever an edge is contained in a layer, this edge is contained in all succeeding layers. Their motivation is activation of proteins, or more general, temporal graphs that model activation by connected components. Casteigts et al. [7, Class 6] call this property of temporal graphs while additionally requiring the underlying graph to be connected as “recurrence of edges”. Since we only consider temporal graphs with connected underlying graphs, both definitions form special cases of our $1$-monotone temporal graphs where each layer is a subgraph of its successor.

A peak in a $p$-monotone temporal graph is an index $i_\ell \in \{i_1, i_2, \ldots, i_{p+1}\}$ such that there exists $i_{\ell-1} \leq j < i_\ell$ with $E_j \subseteq E_{j+1}$ or $i_\ell \leq j < i_{\ell+1}$ with $E_j \supseteq E_{j+1}$. As a convention, 1-monotone temporal graphs are single-peaked, that is, they have only one peak. Indeed, observe that for TEMPORAL
(s, z)-Separation only the peaks matter. Hence, we obtain the following reduction rule.

**Observation 4.1.** Given an instance \( I = (G = (V, E, \tau), s, t, k) \) with \( G \) being \( p \)-monotone with \( \ell \) peaks, we can compute in polynomial time an instance \( I' = (G' = (V, E', \tau'), s, t, k) \) such that \( I \) is equivalent to \( I' \) and \( \tau' \leq \ell \).

**Observation 4.1 at hand,** the following is straightforward:

**Observation 4.2.** Temporal \((s, z)\)-Separation is solvable in polynomial time on single-peaked temporal graphs.

Surprisingly, the situation changes when the temporal graph is already 2-monotone but not single-peaked. We can make every temporal graph \( \tau \)-monotone by simply adding edge-free layers between any two consecutive layers, formally:

**Observation 4.3.** There is a polynomial-time many-one reduction that maps any instance \((G = (V, E, \tau), s, t, k)\) of Temporal \((s, z)\)-Separation to an equivalent instance \((G' = (V, E', 2\tau - 1), s, t, k)\) such that for all \( i \in \{\tau\} \) it holds that \( E_{2i-1}(G') = E_i(G) \) and for all \( i \in \{\tau-1\} \) it holds that \( E_{2i}(G') = \emptyset \).

As Temporal \((s, z)\)-Separation is already \( \text{NP} \)-complete for \( \tau = 2 \) [21], we get the following.

**Observation 4.4.** For all \( p \geq 2 \), Temporal \((s, z)\)-Separation on \( p \)-monotone temporal graphs is \( \text{NP} \)-complete.

**Periodic Temporal Graphs.** In several real-world scenarios one observes periodicity; Indeed, whenever one observes mobile entities with periodic movements [7], as satellites or (scheduled) public transport, over longer time periods, periodic patterns appear. Such models motivate the following class of temporal graphs.

**Definition 4.2.** A temporal graph \( G = (V, E, \tau) \) is \( p \)-periodic if \( p \in \mathbb{N} \) is the smallest number such that \( G = G'^r \) where \( G' = (V, E', p) \) and \( r \) is called the number of periods.

Different notions of periodic temporal graphs exist in the literature. Flocchini et al. [11] consider periodic temporal graphs obtained from “carriers”, that is, a set of strict temporal paths define a network. Liu and Wu [15] consider delay tolerant networks where nodes have some cyclic movement
pattern and get connected when they are in reach: if the time steps are large enough, periodicity is observed. In both cases, the smallest common multiple of the time spans of the entities define the length of a period. Casteigts et al. [7, Class 8] define periodic temporal graphs by periodicity of edges, that is, for all edges e, time steps t, and c ∈ N, edge e is present at time step t if and only if e is present at time step t + c · p, where p is the periodicity. They require the underlying graph to be connected, but they do not require minimality on the periodicity.

We know that Temporal (s,z)-Separation is NP-complete on 2-periodic temporal graphs [21]. Contrarily, on 1-periodic temporal graphs, Temporal (s,z)-Separation collapses to (s,z)-Separation in the underlying graph. Surprisingly, if the number of periods is large enough, then the problem becomes polynomial-time solvable.

Let P be an (s,z)-path of length ℓ in the underlying graph G ↓ of the temporal graph G = (V,E,τ). A time-edge sequence (e1,t1),..., (eℓ,tℓ) ∈ E ℓ is a realization of P (P' ≃ P) if (e1,...,eℓ) is P. The distance to temporality of P in G is min_{P' ≃ P} |f_{P'}| − 1, where |f_{P'}| is the number of monotonically increasing intervals of the function f_{P'} : [ℓ] → [τ], f_{P'}(x) = t_x where t_x is the label of the x-th time-edge of P'. Furthermore, the distance to temporality from s to z in G is the maximum distance to temporality over all (s,z)-path in G ↓.

Lemma 4.1. Let G = G''' be a p-periodic temporal graph such that the number of periods r is at least the distance to temporality from s to z in G'. Then Temporal (s,z)-Separation is solvable in polynomial time.

Proof. Let G = G''' be a p-periodic temporal graph such that the number of periods r is at least the distance to temporality from s to z in G'. Then every (s,z)-path in G ↓(G) forms a temporal (s,z)-path in G. Hence, we can compute a minimum (s,z)-separator in G ↓(G) to solve Temporal (s,z)-Separation in polynomial time. □

Observe that the distance to temporality from s to z is two in the temporal graph from the reduction of Zschoche et al. [21] for maximum label τ = 2. Thus Temporal (s,z)-Separation is NP-hard, even if the input temporal graph G = G''' is p-periodic and the number of periods r is the distance to temporality from s to z in G' plus one.

However, the distance to temporality is clearly upper-bounded by the number of vertices. Hence, we obtain the following.
Corollary 4.1. Let \( G = (V, E, \tau) \) be a \( p \)-periodic temporal graph. If the number of periods \( r \geq |V| \), then Temporal \((s, z)\)-Separation is solvable in polynomial time.

Interval-Connected Temporal Graphs. Kuhn et al. [14, Definition 2.1] introduced the following class of temporal graphs.

Definition 4.3. A temporal graph \( G = (V, E, \tau) \) is \( T \)-interval connected for \( T \geq 1 \) if for every \( t \in [\tau - T + 1] \) the static graph \( G = (V, \bigcap_{i=t}^{t+T-1} E_i(G)) \) is connected.

Kuhn et al. [14] studied \( T \)-interval connected temporal graphs in the context of counting and token dissemination. Note that temporal graphs where each layer is connected are 1-interval connected temporal graphs, but are not necessarily \( T \)-interval connected for some \( T \geq 2 \). On the other hand, for every \( T \)-interval connected temporal graph it holds true that each layer is connected.

Observation 4.5. There is a polynomial-time many-one reduction that maps any instance \((G = (V, E, \tau), s, t, k)\) of Temporal \((s, z)\)-Separation to an equivalent instance \((G' = (V', E', \tau), s, t, k + 1)\) such that \( G' \) is \( T \)-interval connected for every \( T \geq 1 \).

Proof. Let instance \( \mathcal{I} = (G = (V, E, \tau), s, t, k) \) of Temporal \((s, z)\)-Separation be given. Obtain the temporal graph \( G' \) from \( G \) by adding a vertex \( v \) to \( G \) and connect \( v \) to all other vertices in \( V \) in each layer of \( G \). Clearly, every temporal \((s, z)\)-separator in \( G' \) contains vertex \( v \). As \( G = G' - v \), instance \((G', s, t, k + 1)\) is equivalent to \( \mathcal{I} \). Moreover, for any \( T \geq 1 \) it holds true that for every \( t \in [\tau - T + 1] \) the graph \( G = (V, \bigcap_{i=t}^{t+T-1} E_i(G)) \) is a supergraph of the star graph with center \( v \) and set \( V \) of leaves. \( \square \)

Steady Temporal Graphs. When observing a network over time with high resolution, we expect evolutionary instead of revolutionary changes in one time step. For instance, observing any contact network per second, we do not expect many contacts to appear in the same second. More generally, in several real-world scenarios we do not expect big changes from one time step to the other. This assumption motivates the following class of temporal graphs.

Definition 4.4. A temporal graph \( G = (V, E, \tau) \) is \( \lambda \)-steady if \( \lambda \in \mathbb{N} \) is the smallest number such that for each point in time \( t \in [\tau - 1] \) the size of the symmetric difference of two consecutive edge sets \( |E_t \triangle E_{t+1}| \) is at most \( \lambda \).
To the best of our knowledge, this class is not considered in the literature.

One can expect that hardness results for temporal graphs translate to steady temporal graphs, even if $\lambda \leq 1$.

**Lemma 4.2.** There is a polynomial-time many-one reduction that maps any instance $(G = (V,E,\tau), s,t,k)$ of **Temporal (s,z)-Separation** to an equivalent instance $(G' = (V',E',\tau'), s,t,k)$ such that $G'$ is 1-steady.

**Proof.** Let instance $I = (G = (V,E,\tau), s,t,k)$ of **Temporal (s,z)-Separation** be given. We construct $G' = (V',E',\tau')$ in the following way. Intuitively, for each layer $i$ of $G$ we slowly construct $E_i$ and deconstruct it afterwards. Formally, for each $i \in [\tau]$ we construct the sequences of edge-sets $E_i^1 := E_i^1, \ldots, E_i^{2\cdot |E_i|+1}$ such that (i) $E_i^1 = E_i^{2\cdot |E_i|+1} = \emptyset$, (ii) $E_i^{1|E_i|+1} = E_i$, (iii) for each $j \in [|E_i|]$ we have $E_i^j \subseteq E_i^{j+1}$, for each $j \in \{|E_i|+1, \ldots, 2|E_i|\}$ we have $E_i^j \supseteq E_i^{j+1}$, and

Now we construct the time-edge set $E' := \{(\{v,w\},i+t-1) | \{v,w\} \in E_i^1\}$. Hence, $\tau' = 2 \cdot \sum_{i=1}^{\tau} |E_i| + 1$. Observe, that for all $i \in [\tau-1]$ and all $j \in [2|E_i|]$ we have $|E_i^j \triangle E_i^{j+1}| = 1$. Moreover, for all $i \in [\tau-1]$ the last entry $E_i^{2|E_i|}$ of $E_i$ and the first entry $E_i^{1|E_i|+1}$ of $E_i^{1|E_i|+1}$ have $|E_i^2| \triangle E_i^{1|E_i|+1}| = 1$. It follows that $G'$ is 1-steady.

Observe that $G'$ is indeed $\tau$-monotone with $\tau$ peaks, where the $i$th peak correspond to layer $E_i$. Hence, the correctness follows then from Observation 4.1. □

The reduction of Lemma 4.2 increases the maximum label by a factor depending on the input size. Indeed, from previous results [21] it follows that **Temporal (s,z)-Separation** on $\lambda$-steady temporal graphs is fixed-parameter tractable when parameterized by $\tau$.

**Corollary 4.2.** **Temporal (s,z)-Separation** on $\lambda$-steady temporal graphs is fixed-parameter tractable when parameterized by the maximum label $\tau$.

**Proof.** Zschoche et al. [21] showed that **Temporal (s,z)-Separation** is fixed-parameter tractable when parameterized by the size of the temporal core.

For a temporal graph $G = (V,E,\tau)$, the vertex set $W = \{v \in V | \exists \{v,w\} \in (\bigcup_{i=1}^{\tau} E_i) \setminus (\bigcap_{i=1}^{\tau} E_i)\} \subseteq V$ is called the temporal core of $G$.

This corollary follows directly from the fact that the temporal core of a $\lambda$-steady temporal graph $G = (V,E,\tau)$ is upper-bounded by $2 \cdot \lambda \cdot \tau$. □
5 Conclusion

We studied the problem Temporal \((s, z)\)-Separation on different temporal graph classes—with structural and temporal restrictions on temporal graph models. We proved Temporal \((s, z)\)-Separation to remain NP-complete on the majority of the considered classes of restricted temporal graphs. Polynomial-time solvability is achieved for temporal graphs where the underlying graph has bounded treewidth, on single-peaked temporal graphs, temporal graphs with many periods, and temporal graphs where each layer is a unit interval graph with respect to the same vertex ordering.

Our results call into question to which extent currently in the literature considered notions of temporal graph classes address the features of temporal graphs and hence impose useful restrictions on temporal graphs. For instance, the introduced class of order-preserving temporal unit interval graphs is more restrictive than just requiring the layers to fall into a specific graphs class; however, also this notion does not capture temporal aspects. Exploring further, more sophisticated structural restrictions of temporal graphs, whose definitions may rely on global properties and on temporal aspects, is of particular interest when asking for computationally tractable cases of Temporal \((s, z)\)-Separation.

A specific direction for future work would be to use the derived polynomial-time algorithms as a basis for distance-to-triviality parameterizations. For instance, for a temporal unit interval graph one may introduce a parameter \(\kappa\) that bounds how much the vertex orderings of two consecutive layers of a temporal unit interval graph differ. More specifically, given a temporal unit interval graph \(G = (V, E, \tau)\), we define \(\kappa\) as the smallest integer such that there are vertex orderings \(<^1_V, \ldots, <^\tau_V\) such that \(<^t_V\) is compatible with layer \(G_t\) for all \(t \in [\tau]\), and the orderings of any two consecutive layers have Kendall tau distance\(^3\) at most \(\kappa\), that is, for all \(t \in [\tau - 1]\) we have that \(K(<^t_V, <^{t+1}_V) \leq \kappa\). Clearly for order-preserving temporal unit interval graphs we have that \(\kappa = 0\) and it is easy to observe (with the help of Lemma 3.1) that we get NP-hardness for \(\kappa = 1\). We conjecture that we can achieve fixed-parameter tractability for the parameter combination \((\kappa, \tau)\) for Temporal \((s, z)\)-Separation on temporal unit interval graphs.

\(^3\)The Kendall tau distance is a metric that counts the number of pairwise disagreements between two total orderings; it is also known as “bubble sort distance”.

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References

[1] E. C. Akrida, L. Gąsieniec, G. B. Mertzios, and P. G. Spirakis. On temporally connected graphs of small cost. In Proc. of 13th WAOA, volume 9499, pages 84–96. Springer, 2015. 12

[2] E. C. Akrida, J. Czyzowicz, L. Gąsieniec, Ł. Kuszner, and P. G. Spirakis. Temporal flows in temporal networks. In Proc. of 10th CIAC, volume 10236, pages 43–54. Springer, 2017. 12

[3] K. Axiotis and D. Fotakis. On the size and the approximability of minimum temporally connected subgraphs. In Proc. of 43rd ICALP, volume 55, pages 149:1–149:14. Dagstuhl Publishing, 2016. 2, 4

[4] A. Barrat and J. Fournet. Contact patterns among high school students. PLoS ONE, 9(9):e107878, 2014. 7

[5] L. W. Beineke. Characterizations of derived graphs. Journal of Combinatorial Theory, 9(2):129–135, 1970. 12

[6] H. L. Bodlaender, P. G. Drange, M. S. Dregi, F. V. Fomin, D. Lokshtanov, and M. Pilipczuk. A $c^kn$ 5-approximation algorithm for treewidth. SIAM Journal on Computing, 45(2):317–378, 2016. 17

[7] A. Casteigts, P. Flocchini, W. Quattrociocchi, and N. Santoro. Time-varying graphs and dynamic networks. International Journal of Parallel, Emergent and Distributed Systems, 27(5):387–408, 2012. 2, 3, 28, 29, 30

[8] M. Cygan, F. V. Fomin, Ł. Kowalik, D. Lokshtanov, D. Marx, M. Pilipczuk, M. Pilipczuk, and S. Saurabh. Parameterized Algorithms. Springer, 2015. 4, 6, 17

[9] R. Diestel. Graph Theory, 5th Edition, volume 173 of Graduate Texts in Mathematics. Springer, 2016. 4, 17

[10] T. Erlebach, M. Hoffmann, and F. Kammer. On temporal graph exploration. In Proc. of 42nd ICALP, volume 9134, pages 444–455. Springer, 2015. 2, 4

[11] P. Flocchini, B. Mans, and N. Santoro. On the exploration of time-varying networks. Theoretical Computer Science, 469:53–68, 2013. 2, 3, 29
[12] D. Kempe, J. Kleinberg, and A. Kumar. Connectivity and inference problems for temporal networks. *Journal of Computer and System Sciences*, 64(4):820–842, 2002. 2, 3, 7

[13] A. Khodaverdian, B. Weitz, J. Wu, and N. Yosef. Steiner network problems on temporal graphs. *CoRR*, abs/1609.04918v2, 2016. 28

[14] F. Kuhn, N. A. Lynch, and R. Oshman. Distributed computation in dynamic networks. In *Proc. of 42nd STOC*, pages 513–522. ACM, 2010. 2, 3, 31

[15] C. Liu and J. Wu. Scalable routing in cyclic mobile networks. *IEEE Transactions on Parallel and Distributed Systems*, 20(9):1325–1338, 2009. 29

[16] G. B. Mertzios, O. Michail, I. Chatzigiannakis, and P. G. Spirakis. Temporal network optimization subject to connectivity constraints. In *Proc. of 40th ICALP*, volume 7966, pages 657–668. Springer, 2013. 12

[17] O. Michail. An introduction to temporal graphs: An algorithmic perspective. *Internet Mathematics*, 12(4):239–280, 2016. 12

[18] O. Michail and P. G. Spirakis. Traveling salesman problems in temporal graphs. *Theoretical Computer Science*, 634:1–23, 2016. 2, 4

[19] J. Nešetřil and P. O. de Mendez. *Sparsity: graphs, structures, and algorithms*. Springer Science & Business Media, 2012. 16, 17

[20] H. Wu, J. Cheng, Y. Ke, S. Huang, Y. Huang, and H. Wu. Efficient algorithms for temporal path computation. *IEEE Transactions on Knowledge and Data Engineering*, 28(11):2927–2942, 2016. 5

[21] P. Zschoche, T. Fluschnik, H. Molter, and R. Niedermeier. On efficiently finding small separators in temporal graphs. *arXiv preprint arXiv:1711.00963*, 2017. URL https://arxiv.org/abs/1711.00963. 2, 3, 5, 7, 12, 26, 29, 30, 32