Quantum mechanics had been started with the theory of the hydrogen atom, so considering the Quantum mechanics in Riemannian spaces it is first natural step to turn to just this system. A common quantum-mechanical hydrogen atom model is based materially on the assumption of the Euclidean character of the physical 3-space geometry. In this context, natural questions arise: what in such a model description is determined by this assumption, and which changes will be entailed by allowing for other spatial geometries: for instance, Lobachevsky’s $H_3$, Riemann’s $S_3$, or de Sitter geometry. The question is of fundamental significance, even beyond its possible experimental testing.

Firstly, the hydrogen atom in 3-dimensional space of constant positive curvature $S_3$ was considered by Schrödinger [1]. He had been studied the so-called factorization method in quantum mechanics; in particular, application of this techniques to discrete part of the energy spectrum for hydrogen atom had been elaborated. An idea was to modify the basic atom system so as to cover all the energy spectrum including the region $E > 0$ as well. However, mere placing of the atom system inside a finite box in order to make the whole energy spectrum discrete did not seem attractive, so Schrödinger had placed the atom into the curved background of the Riemann space model $S_3$. Due its compactness, the spherical Riemann model may simulate the effect of the finite box – see Schrodinger [1] and Stevenson [2].

In spherical coordinates of $S_3$

$$dl^2 = d\chi^2 + \sin^2 \chi (d\theta^2 + \sin^2 d\phi^2)$$

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the Shrödinger Hamiltonian in dimensionless units has the form
\[ H = -\frac{1}{2} \frac{\sqrt{g}}{\partial x^\alpha} \sqrt{g} g^{\alpha\beta} \frac{\partial}{\partial x^\alpha} - \frac{e}{\tan \chi}; \]

\( \rho \) is a curvature radius, a unit for length; \( M \) is a mass of the electron; \( \hbar^2/M\rho^2 \) is taken as a unit for energy; \( e = \frac{\alpha}{\rho}/\frac{\hbar^2}{M\rho^2} \) stands for a Coulomb interaction constant; the sign at \( e/\tan \chi \) corresponds to the attracting Coulomb force. The energy spectrum is entirely discrete and given by
\[ \epsilon_n = -\frac{e^2}{2n^2} + \frac{1}{2} (n^2 - 1), \quad n = 1, 2, 3, \ldots \]

Hydrogen atom in the Lobachevsky space \( H_3 \) was considered firstly by Infeld and Shild \[3\]
\[ dl^2 = d\chi^2 + \cosh^2 \chi (d\theta^2 + \sin^2 \theta d\phi^2), \]
\[ H = -\frac{1}{2} \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\alpha} \sqrt{g} g^{\alpha\beta} \frac{\partial}{\partial x^\alpha} - \frac{e}{\tanh \chi}. \]

Energy spectrum contains a discrete and continuous parts. The number of discrete levels is finite, they are specified by
\[ -\frac{e^2}{2} \leq \epsilon \leq \left( \frac{1}{2} - e \right), \]
\[ \epsilon_n = -\frac{e^2}{2n^2} - \frac{1}{2} (n^2 - 1), \quad n = 1, 2, 3, \ldots, N. \]

In the region \( \epsilon \geq \left( \frac{1}{2} - e \right) \) the energy spectrum is continuous.

Thus, the models of the hydrogen atom in Euclid, Riemann, and Lobachevsky spaces significantly differ from each other, which is the result of differences in three spatial geometries: \( E_3, H_3, S_3 \). To present time, we see a plenty of investigations on this matter:

Higgs \[4\], Leemon \[5\], Kurochkin – Otchik \[6\], Bogush – Kurochkin – Otchik \[7\], Parker \[8\], \[9\], Ringwood – Devreese \[10\], Kobayshi \[11\], Bessis – Bessis – Shamseddine \[12\], Grinberg – Maranon – Vucetich \[13\], Bogush – Otchik – Red’kov \[14\], Bessis – Bessis – Shamseddine \[15\], \[16\], \[17\], Chondming – Dianyan \[18\], Xu – Xu \[19\], Melnikov – Shikin \[20\], Shamseddine \[21\], Otchik – Red’kov \[22\], Barut – Inomata – Junker \[23\], Bessis – Bessis – Roux \[24\], Bogush – Otchik – Red’kov \[25\], Gorbatsievich – Priobe \[26\], Groshe \[27\], Barut – Inomata – Junker \[28\], Katayama \[29\], Chernikov \[30\], Mardoyan – Sisakyan \[31\], Granovskii – Zhedanov – Lutsenko \[32\], Kozlov – Harin \[33\], Vinitskii – Marfoyan – Pogosyan – Sisakyan – Strizh \[34\], Shamseddine \[35\], Bogush – Kurochkin – Otchik \[36\], Otchik \[37\], Nersessian – Pogosyan \[38\], Red’kov \[39\], Bogush – Kurochkin – Otchik \[40\], Kurochkin – Otchik – Shoukavy \[41\], Kurochkin – Shoukavy \[42\], Bogush – Otchik – Red’kov \[43\], Bessis – Bessis \[44\], Iwai \[45\], Cohen – Powers \[46\].
1. Separation of the variables for Dirac equation in curved models

Let us consider procedure of separation of the variables in the Dirac equation on the background of curved model, for definiteness using a spherical Riemann model $S^3$, a diagonal tetrad is taken in the form

$$dS^2 = dt^2 - d\chi^2 - \sin^2 \chi (d\theta^2 + \sin^2 \phi d\phi^2),$$

$$e^0_\alpha = (1, 0, 0, 0), \quad e^\alpha_\beta = (0, 1, 0, 0),$$

$$e^{(1)}_\alpha = (0, 0, \sin^{-1} \chi, 0), \quad e^{(2)}_\alpha = (0, 0, 0, \sin^{-1} \chi \sin^{-1} \theta). \quad (1)$$

Generally covariant Dirac equation

$$\left[i\gamma^\alpha (e^{(c)}_\alpha \partial_\alpha + \frac{1}{2} j^{ab} \gamma_{abc}) - m \right] \Psi = 0$$

takes the form

$$\left[i\gamma^0 \frac{\partial}{\partial t} + i(\gamma^3 \frac{\partial}{\partial \chi} + \frac{\gamma^1 j^{31} + \gamma^2 j^{32}}{\tan \chi}) + \frac{1}{\sin \chi} \Sigma_{\theta\phi} - m \right] \Psi = 0, \quad (2)$$

where an angular operator $\Sigma_{\theta\phi}$ is defined by

$$\Sigma_{\theta\phi} = i \gamma^1 \partial_\theta + \gamma^2 \frac{i \partial + j^{12} \cos \theta}{\sin \theta}. \quad (3)$$

Allowing for

$$\frac{\gamma^1 j^{31} + \gamma^2 j^{32}}{\tan \chi} = \frac{\gamma^3}{\tan \chi}, \quad \Psi = \frac{1}{\sin \chi} \tilde{\Psi}$$

eq. (2) is simplified

$$\left[i\gamma^0 \frac{\partial}{\partial t} + i\gamma^3 \frac{\partial}{\partial \chi} + \frac{1}{\sin \chi} \Sigma_{\theta\phi} - m \right] \tilde{\Psi} = 0. \quad (4)$$

To diagonalize operators $i\partial_t, J^2, J_3$, one takes the wave function in the form [47], [48]

$$\tilde{\Psi} = e^{-i\epsilon t} \begin{vmatrix} f_1(\chi) D_{-1/2} \\ f_2(\chi) D_{1/2} \\ f_3(\chi) D_{-1/2} \\ f_4(\chi) D_{1/2} \end{vmatrix}; \quad (5)$$

where Wigner functions are noted as $D_\sigma = D^{j}_{-m,\sigma}(\phi, \theta, 0)$. After separation of the variables we get four radial equations (let $\nu = j + 1/2$):

$$\epsilon f_3 - i \frac{d}{d\chi} f_3 - i \frac{\nu}{\sin \chi} f_4 - mf_1 = 0, \quad \epsilon f_4 + i \frac{d}{d\chi} f_4 + i \frac{\nu}{\sin \chi} f_3 - mf_2 = 0,$$

$$\epsilon f_1 + i \frac{d}{d\chi} f_1 + i \frac{\nu}{\sin \chi} f_2 - mf_3 = 0, \quad \epsilon f_2 - i \frac{d}{d\chi} f_2 - i \frac{\nu}{\sin \chi} f_1 - mf_4 = 0. \quad (6)$$
In spherical tetrad, the space reflection operator is given by

\[ \Pi_{\text{sph.}} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \otimes \hat{P}. \]

From eigenvalues equations \( \Pi_{\text{sph.}} \Psi_{jm} = \Pi \Psi_{jm} \) we obtain

\[ \Pi = \delta (-1)^{j+1}, \quad \delta = \pm 1, \quad f_4 = \delta f_1, \quad f_3 = \delta f_2, \quad (7) \]

which simplifies (6):

\[ \begin{align*} 
\epsilon f_1 + i \frac{d}{d\chi} f_1 + i \frac{\nu}{\sin \chi} f_2 - \delta mf_2 &= 0, \\
\epsilon f_2 - i \frac{d}{d\chi} f_2 - i \frac{\nu}{\sin \chi} f_1 - \delta mf_1 &= 0. 
\end{align*} \quad (8) \]

In terms of new functions

\[ f = \frac{f_1 + f_2}{\sqrt{2}}, \quad g = \frac{f_1 - f_2}{i \sqrt{2}}, \]

the above system reads

\[ \begin{align*}
\left( \frac{d}{d\chi} + \frac{\nu}{\sin \chi} \right) f + (\epsilon + \delta m) g &= 0, \\
\left( \frac{d}{d\chi} - \frac{\nu}{\sin \chi} \right) g - (\epsilon - \delta m) f &= 0.
\end{align*} \quad (9) \]

2. Pauli equation for Kepler problem, flat Minkowski space

Let us consider the problem of spinor spherical waves in Pauli approximation. It is convenient to start with the radial system for a free particle case. As a first step, one should separate the rest energy – for this it is enough to make a formal replacement \( \epsilon \Rightarrow \epsilon + m \):

\[ \begin{align*}
\left( \frac{d}{dr} + \frac{\nu}{r} \right) f + (\epsilon + m + \delta m) g &= 0, \\
\left( \frac{d}{dr} - \frac{\nu}{r} \right) g - (\epsilon + m - \delta m) f &= 0.
\end{align*} \quad (10) \]

States with opposite parity are specified by

\[ \delta = +1, \quad \left( \frac{d}{dr} + \frac{\nu}{r} \right) f + (\epsilon + 2m) g = 0, \quad \left( \frac{d}{dr} - \frac{\nu}{r} \right) g - \epsilon f = 0; \quad (11) \]

\[ \delta = -1, \quad \left( \frac{d}{dr} + \frac{\nu}{r} \right) f + \epsilon g = 0, \quad \left( \frac{d}{dr} - \frac{\nu}{r} \right) g - (\epsilon + 2m) f = 0. \quad (12) \]
With additional assumption $\epsilon + 2m \approx 2m$, in each case one gets radial Pauli equation for a big component:

$\delta = +1, \quad f \gg g, \quad \left(\frac{d^2}{dr^2} + \frac{\nu^2 + \nu}{r^2}\right) f + 2m\epsilon f = 0;$

$\delta = -1, \quad g \gg f, \quad \left(\frac{d^2}{dr^2} + \frac{\nu^2 - \nu}{r^2}\right) g + 2m\epsilon g = 0.$ \hfill (13)

Corresponding 2-component nonrelativistic spherical functions with different parity are

$$
\psi_{jm,\delta=+1} = \frac{e^{i\epsilon t}}{r} \begin{vmatrix} f(r) \ D_{-1/2} \\ f(r) \ D_{+1/2} \end{vmatrix}, \quad \psi_{jm,\delta=-1} = \frac{e^{i\epsilon t}}{r} \begin{vmatrix} ig(r) \ D_{-1/2} \\ -ig(r) \ D_{+1/2} \end{vmatrix} . \hfill (14)
$$

To obtain equation in presence of the Coulomb field, it is enough to make a formal replacement $\epsilon \Rightarrow \epsilon + \alpha/r$ in (10):

$$
\left(\frac{d}{dr} + \frac{\nu}{r}\right)f + (\epsilon + \frac{\alpha}{r} + \delta m) g = 0 ,
\left(\frac{d}{dr} - \frac{\nu}{r}\right)g - (\epsilon + \frac{\alpha}{r} - \delta m) f = 0 . \hfill (15)
$$

After separating the rest energy, for states with different parities we get

$\delta = +1, \quad \left(\frac{d}{dr} + \frac{\nu}{r}\right)f + \left(\frac{\alpha}{r} + 2m\right) g = 0 ,
\left(\frac{d}{dr} - \frac{\nu}{r}\right)g - (\epsilon + \frac{\alpha}{r}) f = 0 ; \hfill (16)

\delta = -1, \quad \left(\frac{d}{dr} + \frac{\nu}{r}\right)f + (\epsilon + \frac{\alpha}{r}) g = 0 ,
\left(\frac{d}{dr} - \frac{\nu}{r}\right)g - \left(\frac{\alpha}{r} + 2m\right) f = 0 . \hfill (17)

To derive the Pauli equation with known structure, one must impose additional restriction

$$
\epsilon + \frac{\alpha}{r} + 2m \approx 2m
$$

which means in fact that Pauli description cannot be goof enough in the region close to the origin $r = 0$ where a source of the Coulomb field is located. Thus, we obtain

$\delta = +1, \quad \left(\frac{d}{dr} + \frac{\nu}{r}\right) f + 2m g = 0 ,
\left(\frac{d}{dr} - \frac{\nu}{r}\right) g - (\epsilon + \frac{\alpha}{r}) f = 0 ; \hfill (18)

\delta = -1, \quad \left(\frac{d}{dr} + \frac{\nu}{r}\right)f + (\epsilon + \frac{\alpha}{r}) g = 0 ,
\left(\frac{d}{dr} - \frac{\nu}{r}\right)g - 2m f = 0 . \hfill (19)

Respective radial equations for big components are

$$
\delta = +1, \quad \frac{d^2 f}{dr^2} - \left(\frac{\nu(\nu + 1)}{r^2} - \frac{2m\alpha}{r} - 2m\epsilon\right) f = 0 ; \hfill (20)

\delta = -1, \quad \frac{d^2 g}{dr^2} - \left(\frac{\nu(\nu - 1)}{r^2} - \frac{2m\alpha}{r} - 2m\epsilon\right) g = 0 ; \hfill (21)
$$
two last equations can be related by the formal change \( \nu \Rightarrow \nu - 1 \).

From (20), introducing a new variable \( x = 2\sqrt{-2m\varepsilon r} \), one gets

\[
x \frac{d^2 f}{dx^2} - \left( \frac{\nu(\nu + 1)}{x} + \frac{x}{4} - \alpha \sqrt{\frac{m}{2\varepsilon}} \right) f = 0.
\]  

(22)

Searching solutions in the form \( f = x^A e^{-Cx} F \), one derives

\[
x \frac{d^2 F}{dx^2} + (2A - 2Cx) \frac{dF}{dx} + \left( \frac{A(A - 1)}{x} - 2AC + C^2 x - \frac{\nu(\nu + 1)}{x} - \frac{x}{4} + \alpha \sqrt{\frac{m}{2\varepsilon}} \right) F = 0.
\]

(22)

At special choice of \( A \) and \( C \) (underlined values correspond to bound states)

\[ A = -\nu, \ 1 + \nu, \quad C = -1/2, +1/2 \]

we simplify the problem

\[
x \frac{d^2 F}{dx^2} + (2A - x) \frac{dF}{dx} - \left( A - \alpha \sqrt{\frac{m}{2\varepsilon}} \right) F = 0.
\]

(23)

It is a confluent hypergeometric equation with parameters

\[ a = A - \alpha \sqrt{\frac{m}{2\varepsilon}}, \quad c = 2A. \]

Making series a polynomial in usual way: \( a = -n, \ n = 0, 1, 2, \ldots \), one obtains the energy quantization rule (remembering \( \nu = j + 1/2 \))

\[ 1 + \nu - \alpha \sqrt{\frac{m}{2\varepsilon}} = -n \quad \Rightarrow \quad \epsilon = -\frac{m\alpha^2}{2(n + \nu + 1/2)}. \]  

(24)

Turning to eq. (21), by means of the formal replacement \( \nu \Rightarrow \nu - 1 \), we get

\[ \nu - \alpha \sqrt{\frac{m}{2\varepsilon}} = -n \quad \Rightarrow \quad \epsilon = -\frac{m\alpha^2}{2(n + \nu)^2}. \]  

(25)

3. Pauli equation for Kepler problem in spherical space

In the spherical model, again let us start with free radial equations (compare with (11), (12))

\[
\delta = +1, \quad \left( \frac{d}{d\chi} + \frac{\nu}{\sin \chi} \right) f + 2m g = 0, \quad \left( \frac{d}{d\chi} - \frac{\nu}{\sin \chi} \right) g - \epsilon f = 0; \quad (26) \\
\delta = -1, \quad \left( \frac{d}{d\chi} + \frac{\nu}{\sin \chi} \right) f + \epsilon g = 0, \quad \left( \frac{d}{d\chi} - \frac{\nu}{\sin \chi} \right) g - 2m f = 0. \quad (27)
\]

Correspondingly, in each case the Pauli radial equation for a big component is

\[
\delta = +1, \quad f >> g, \quad \frac{d^2 f}{d\chi^2} - \left( \frac{\nu(\nu + \cos \chi)}{\sin^2 \chi} - 2\epsilon m \right) f = 0; \\
\delta = -1, \quad g >> f, \quad \frac{d^2 g}{d\chi^2} - \left( \frac{\nu(\nu - \cos \chi)}{\sin^2 \chi} - 2\epsilon m \right) g = 0. 
\]

(28, 29)
After changing the variable in (28), \( y = (1 + \cos x)/2 \), eq. (28) reads
\[
y(1 - y) \frac{d^2 f}{dy^2} + \left( \frac{1}{2} - y \right) \frac{df}{dy} + \left[ -\frac{1}{4} \nu (\nu + 1) - \frac{1}{4} \nu (\nu - 1) \right] + 2 \epsilon m \right] f = 0.
\]
Searching solution in the form \( f = y^A (1 - y)^B F \), for \( F \) we obtain
\[
y(1 - y) \frac{d^2 F}{dy^2} + \left[ 2A + \frac{1}{2} - (2A + 2B + 1)y \right] \frac{dF}{dy} + \frac{A(A - 1/2) - 1/4\nu(\nu - 1)}{y} + \frac{B(B - 1/2) - 1/4\nu(\nu + 1)}{1 - y} - (A + B)^2 + 2 \epsilon m \right] F = 0.
\]
At \( A, B \) taken according to
\[
A = \nu/2, \quad (-\nu + 1)/2, \quad B = -\nu/2, \quad (\nu + 1)/2,
\]
we get to a hypergeometric equation
\[
y(1 - y) \frac{d^2 F}{dy^2} + \left[ 2A + \frac{1}{2} - (2A + 2B + 1)y \right] \frac{dF}{dy} - [(A + B)^2 - 2 \epsilon m] F = 0
\]
with parameters given by
\[
\alpha = A + B + \sqrt{2 \epsilon m}, \quad \beta = A + B - \sqrt{2 \epsilon m}, \quad \gamma = 2A + \frac{1}{2}.
\]
Bound states are separated by
\[
A = \nu/2, \quad B = (\nu + 1)/2, \quad \beta = \nu + \frac{1}{2} - \sqrt{2 \epsilon m} = -n, \quad \epsilon = + \frac{(n + \nu + 1/2)^2}{2m}.
\]
To treat eq. (29), it suffices to make a formal change \( \nu \mapsto -\nu \), so we arrive at
\[
A = -\nu/2, \quad (\nu + 1)/2, \quad B = \nu/2, \quad (-\nu + 1)/2,
\]
and further
\[
y(1 - y) \frac{d^2 F}{dy^2} + \left[ 2A + \frac{1}{2} - (2A + 2B + 1)y \right] \frac{dF}{dy} - [(A + B)^2 - 2 \epsilon m] F = 0
\]
with parameters
\[
\alpha = A + B + \sqrt{2 \epsilon m}, \quad \beta = A + B - \sqrt{2 \epsilon m}, \quad \gamma = 2A + \frac{1}{2}.
\]
Bound states are specified by
\[
A = (\nu + 1)/2, \quad B = \nu/2, \quad \beta = \nu + \frac{1}{2} - \sqrt{2 \epsilon m} = -n, \quad \epsilon = + \frac{(n + \nu + 1/2)^2}{2m}.
\]
Corresponding 2-component wave functions with opposite parities are given by
\[
\psi_{jm,\delta=+1} = \frac{e^{i\eta}}{\sin \chi} \begin{vmatrix} f(\chi) D_{-1/2} \\ f(\chi) D_{+1/2} \end{vmatrix}, \quad \psi_{jm,\delta=-1} = \frac{e^{i\eta}}{\sin \chi} \begin{vmatrix} i g(\chi) D_{-1/2} \\ -i g(\chi) D_{+1/2} \end{vmatrix}.
\]

Now let us add the Coulomb potential
\[
\delta = +1, \quad (\frac{d}{d\chi} + \frac{\nu}{\sin \chi}) f + 2mg = 0, \quad (\frac{d}{d\chi} - \frac{\nu}{\sin \chi}) g - (\epsilon + \frac{\alpha}{\tan \chi}) f = 0;
\]
\[
\delta = -1, \quad (\frac{d}{d\chi} + \frac{\nu}{\sin \chi}) f + (\epsilon + \frac{\alpha}{\tan \chi}) g = 0, \quad (\frac{d}{d\chi} - \frac{\nu}{\sin \chi}) g - 2mf = 0.
\]

Respective Pauli radial equations are
\[
\delta = +1, \quad \frac{d^2 f}{d\chi^2} - \left(\frac{\nu(\nu + \cos \chi)}{\sin^2 \chi} - 2\epsilon m - \frac{2m\alpha}{\tan \chi}\right) f = 0,
\]
\[
\delta = -1, \quad \frac{d^2 g}{d\chi^2} - \left(\frac{\nu(\nu - \cos \chi)}{\sin^2 \chi} - 2\epsilon m - \frac{2m\alpha}{\tan \chi}\right) g = 0.
\]

Behavior of the function from \([37]\) near the points \(\chi = 0, \pi\) is characterized by
\[
\chi \sim 0, \quad \frac{d^2 f}{d\chi^2} - \frac{\nu(\nu + 1)}{\sin^2 \chi} f = 0,
\]
\[
f = \sin^A \chi, \quad A = 1 + \nu, -\nu;
\]
\[
\chi \sim \pi = (\pi - \beta), \quad \frac{d^2 f}{d\chi^2} - \frac{\nu(\nu - 1)}{\sin^2 \chi} f = 0,
\]
\[
f = \sin^B (\pi - \beta), \quad B = +\nu, 1 - \nu.
\]

and in the case \([38]\)
\[
\chi \sim 0, \quad \frac{d^2 g}{d\chi^2} - \frac{\nu(\nu - 1)}{\sin^2 \chi} g = 0,
\]
\[
g = \sin^A \chi, \quad A = +\nu, 1 - \nu;
\]
\[
\chi \sim \pi = (\pi - \beta), \quad \frac{d^2 g}{d\chi^2} - \frac{\nu(\nu + 1)}{\sin^2 \chi} g = 0,
\]
\[
g = \sin^B (\pi - \beta), \quad B = 1 + \nu, -\nu.
\]

To simplify the problem \([37]\), it is convenient to transform it to a new variable \(e^{i\chi} = z\):
\[
\left[ \frac{d^2}{dz^2} + \frac{1}{z} \frac{d}{dz} - \frac{4\nu^2}{(z^2 - 1)^2} - \frac{2\nu(1 + z^2)}{z(z^2 - 1)^2} - \frac{2m\epsilon - 2i\alpha m}{z^2} \frac{z^2 + 1}{z^2(z^2 - 1)} \right] f = 0.
\]

Analogously, eq. \([38]\) gives
\[
\left[ \frac{d^2}{dz^2} + \frac{1}{z} \frac{d}{dz} - \frac{4\nu^2}{(z^2 - 1)^2} + \frac{2\nu(1 + z^2)}{z(z^2 - 1)^2} - \frac{2m\epsilon}{z^2} - 2i\alpha m \frac{z^2 + 1}{z^2(z^2 - 1)} \right] f = 0.
\]

They can be transformed into each other via the formal replacement \(\nu \mapsto -\nu\). It suffices to examine one of them. Let us consider eq. \([41]\) near the singular points \(z = \pm 1, 0\):
\[ z = +1 , \]
\[ \left[ \frac{d^2}{dz^2} + \frac{d}{dz} - \frac{\nu^2}{(z-1)^2} - \frac{\nu}{(z-1)^2} \right] f = 0 \quad f = (z-1)^A, \quad A = \nu + 1, \quad -\nu ; \]
\[ z = -1 , \]
\[ \left[ \frac{d^2}{dz^2} - \frac{d}{dz} - \frac{\nu^2}{(z+1)^2} + \frac{\nu}{(z+1)^2} \right] f = 0 \quad f = (z-1)^B, \quad B = \nu, \quad -\nu + 1 ; \]
\[ z = 0 , \]
\[ \left[ \frac{d^2}{dz^2} + \frac{1}{z} \frac{d}{dz} - \frac{2me}{z^2} + \frac{2i\alpha m}{z^2} \right] f = 0 \quad f \sim z^{\pm \sqrt{2me-2i\alpha m}}. \]
With the substitution $f = z^A(z - 1)^B(z + 1)^C F(z)$, eq. (49) gives

$$
\frac{d^2 F}{dz^2} + \left[ \frac{2A + 1}{z} + \frac{2B}{z - 1} + \frac{2C}{z + 1} \right] \frac{dF}{dz} + \\
\left[ \frac{A^2 + e - E}{z^2} + \frac{B^2 - B - 1/4\nu(\nu + 2)}{(z - 1)^2} + \frac{C^2 - C - 1/4\nu(\nu - 2)}{(z + 1)^2} + \right. \\
\left. + \frac{BC + B + 2AB + 1/4[\nu(\nu + 2) - 4e]}{z - 1} + \frac{C + 2AC - B - 2AB - \nu}{z} + \right. \\
\left. - \frac{-C - 2AC - BC + 1/4[4e - \nu(\nu - 2)]}{z + 1} \right] F = 0. \quad (50)
$$

At $A$, $B$, $C$ taken according to

$$
A^2 + e - E = 0 \quad \Rightarrow \quad A = \pm \sqrt{E - e}, \\
B^2 - B - 1/4\nu(\nu + 2) = 0 \quad \Rightarrow \quad B = -\frac{1}{2} \nu, 1 + \frac{1}{2} \nu, \\
C^2 - C - 1/4\nu(\nu - 2) = 0 \quad \Rightarrow \quad C = \frac{1}{2} \nu, 1 - \frac{1}{2} \nu, \quad (51)
$$

eq. (50) becomes simpler

$$
\frac{d^2 F}{dz^2} + \left[ \frac{2A + 1}{z} + \frac{2B}{z - 1} + \frac{2C}{z + 1} \right] \frac{dF}{dz} + \\
\left[ \frac{C + 2AC - B - 2AB - \nu}{z} + \frac{BC + B + 2AB + 1/4[\nu(\nu + 2) - 4e]}{z - 1} + \right. \\
\left. + \frac{-C - 2AC - BC + 1/4[4e - \nu(\nu - 2)]}{z + 1} \right] F = 0, \quad (52)
$$

what is a Hein equation for $G(p, q; \alpha, \beta, \gamma, \delta; z)$

$$
\frac{d^2 F}{dz^2} + \left[ \frac{\gamma}{z} + \frac{\delta}{z - 1} + \frac{\alpha + \beta - \delta - \gamma}{z - p} \right] \frac{dF}{dz} + \\
\left[ -\frac{q}{pz} + \frac{p\alpha\beta - q}{p(p - 1)(z - p)} + \frac{-\alpha\beta + q}{(p - 1)(z - 1)} \right] F = 0, \quad (53)
$$

when $p = -1$:

$$
\frac{d^2 F}{dz^2} + \left( \frac{\gamma}{z} + \frac{\delta}{z - 1} + \frac{\alpha + \beta - \delta - \gamma}{z - p} \right) \frac{dF}{dz} + \left( \frac{q}{z} + \frac{\alpha\beta - q}{2(z - 1)} - \frac{\alpha\beta + q}{2(z + 1)} \right) F = 0. \quad (54)
$$

Comparing (52) with (54), one finds expressions for parameters

$$
p = -1, \quad q = (C - B)(1 + 2A) - \nu, \\
\gamma = 2A + 1, \quad \delta = 2B; \quad (\epsilon = 2C); \quad (55)
$$

and

$$
\alpha + \beta = 2A + 2B + 2C, \\
\alpha\beta = B + C + 2(AB + AC + BC) - 2\epsilon + \nu^2/2, 
$$
that is
\[ \alpha = A + B + C - \sqrt{A^2 + B^2 + C^2 - B - C + 2e - \nu^2 / 2}, \]
\[ \beta = A + B + C + \sqrt{A^2 + B^2 + C^2 - B - C + 2e - \nu^2 / 2}. \] (56)

Let
\[ A = +\sqrt{E - e}; \quad B = 1 + \nu / 2; \quad C = \nu / 2, \] (57)
(positive values for \( B \) and \( C \) make solutions to be vanishing at the points \( z = \pm 1 \) \((\chi = 0, \pi)\), then
\[ \alpha = 1 + \nu + \sqrt{E - e - \sqrt{E + e}}, \quad \beta = 1 + \nu + \sqrt{E - e + \sqrt{E + e}}, \]
or (see (46))
\[ \alpha = 2(j + 1) + \sqrt{2E - 2ie - \sqrt{2E} + 2ie}, \]
\[ \beta = 2(j + 1) + \sqrt{2E - 2ie + \sqrt{2E} + 2ie}. \] (58)

Let us impose additional constraint (condition of polynomials)
\[ \beta = -2n \] (59)
then a quantization condition arises
\[ -\sqrt{2E - 2ie} - \sqrt{2E + 2ie} = 2(n + j + 1), \]
which after simple manipulation we have arrived at a formula for energy levels
\[ E = -\frac{e^2}{2(n + j + 1)^2} + \frac{(n + j + 1)^2}{2}. \] (60)

It must be noted that the spectrum produced is very similar to that for Schrödinger’s particle in Coulomb field; besides, when \( e = , \) it reduces to the exact formula for energy levels for a free particle in the space \( S_3 \). With the use of (60), one can readily obtain rather simple representation for all involved parameters. Indeed, (let \( N = n + j + 1; \) below we take the roots with negative real parts)
\[ \sqrt{2E - 2ie} = \sqrt{-\frac{e^2}{N^2} + N^2 - 2ie} = \sqrt{(N - \frac{ie}{\sqrt{N}})^2} = -(N - \frac{ie}{\sqrt{N}}), \]
\[ \sqrt{2E + 2ie} = \sqrt{-\frac{e^2}{N^2} + N^2 + 2ie} = \sqrt{(N + \frac{ie}{\sqrt{N}})^2} = -(N + \frac{ie}{\sqrt{N}}). \] (61)

Therefore, \( \alpha, \beta \) take the form
\[ \alpha = 2(j + 1) - (N - \frac{ie}{\sqrt{N}}) + (N + \frac{ie}{\sqrt{N}}) = 2(j + 1) + \frac{2ie}{n + j + 1}, \]
\[ \beta = 2(j + 1) - (N - \frac{ie}{\sqrt{N}}) - (N + \frac{ie}{\sqrt{N}}) = -2n. \] (62)
4. Pauli equation for Kepler problem, hyperbolic space

Let us start with free radial equations (in which the rest energy is separated with the help of the formal replacement $\epsilon \mapsto \epsilon + m, \text{ and the approximation } \epsilon = 2m \approx 2m$ is used):

\[
\delta = +1, \quad \left( \frac{d}{d\beta} + \frac{\nu}{\sinh \beta} \right) f + 2m \ g = 0, \quad \left( \frac{d}{d\beta} - \frac{\nu}{\sinh \beta} \right) \ g - \epsilon \ f = 0; \quad (63)
\]

\[
\delta = -1, \quad \left( \frac{d}{d\beta} + \frac{\nu}{\sinh \beta} \right) f + \epsilon \ g = 0, \quad \left( \frac{d}{d\beta} - \frac{\nu}{\sinh \beta} \right) \ g - 2m \ f = 0. \quad (64)
\]

In each case one gets a radial Pauli equation for a big 2-component:

\[
\delta = +1, \quad f >> g, \quad \frac{d^2 f}{d\beta^2} - \left( \frac{\nu(\nu + ch \beta)}{\sinh^2 \beta} - 2\epsilon m \right) f = 0; \quad (65)
\]

\[
\delta = -1, \quad g >> f, \quad \frac{d^2 g}{d\beta^2} - \left( \frac{\nu(\nu - ch \beta)}{\sinh^2 \beta} - 2\epsilon m \right) g = 0. \quad (66)
\]

Corresponding wave functions for states with different parity are of the form

\[
\psi_{jm,\delta=+1} = \frac{e^{i\epsilon t}}{\sinh \beta} \left| \begin{array}{c}
 f(\beta) D_{-1/2} \\
 f(\beta) D_{+1/2}
\end{array} \right|, \quad \psi_{jm,\delta=-1} = \frac{e^{i\epsilon t}}{\sinh \beta} \left| \begin{array}{c}
 ig(\beta) D_{-1/2} \\
 -ig(\beta) D_{+1/2}
\end{array} \right|. \quad (67)
\]

Now let us consider the Coulomb field. It is enough to make a formal replacement in $\delta = +1$:

\[
\delta = +1, \quad \left( \frac{d}{d\beta} + \frac{\nu}{\sinh \beta} \right) f + 2m \ g = 0, \quad \left( \frac{d}{d\beta} - \frac{\nu}{\sinh \beta} \right) \ g - (\epsilon + \frac{\alpha}{\tanh \beta}) \ f = 0; \quad (68)
\]

\[
\delta = -1, \quad g >> f, \quad \frac{d^2 g}{d\beta^2} - \left( \frac{\nu(\nu - ch \beta)}{\sinh^2 \beta} - 2\epsilon m \right) g = 0. \quad (66)
\]

For each value of parity one obtains its differential equation

\[
\frac{d^2 f}{d\beta^2} - \left( \frac{\nu(\nu + ch \beta)}{\sinh^2 \beta} - 2\epsilon m - \frac{2m\alpha}{\tanh \beta} \right) f = 0, \quad (70)
\]

\[
\frac{d^2 g}{d\beta^2} - \left( \frac{\nu(\nu - ch \beta)}{\sinh^2 \beta} - 2\epsilon m - \frac{2m\alpha}{\tanh \beta} \right) g = 0. \quad (71)
\]

Let us study eq. $\delta = +1$. To simplify the problem it is convenient to transform it to a new variable $e^\beta = z$. As in the spherical space we will use to dimensionless variables and use the notation:

\[
2\nu \mapsto \nu = 2j + 1, \quad 2E \mapsto E, \quad 2\epsilon \mapsto e, \quad (72)
\]

then

\[
\left[ \frac{d^2}{dz^2} + \frac{1}{z} \frac{d}{dz} - \frac{1}{4} \frac{\nu(\nu - 2)}{(z + 1)^2} + \frac{E - e}{z^2} + \frac{1}{4} \frac{\nu(\nu + 2) + 4e}{z - 1} - \frac{\nu}{z} - \frac{1}{4} \frac{\nu(\nu + 2)}{(z - 1)^2} - \frac{1}{4} \frac{4e + \nu(\nu - 2)}{z + 1} \right] f = 0. \quad (73)
\]
With the substitution \( f = z^A (z - 1)^B (z + 1)^C F(z) \), (73) gives

\[
\frac{d^2 F}{dz^2} + \left[ \frac{2A + 1}{z} + \frac{2B}{z - 1} + \frac{2C}{z + 1} \right] \frac{dF}{dz} + \\
+ \left[ \frac{A^2 + E - e}{z^2} + \frac{B^2 - B - 1/4\nu (\nu + 2)}{(z - 1)^2} + \frac{C^2 - C - 1/4\nu (\nu - 2)}{(z + 1)^2} \right] \\
+ \frac{BC + B + 2AB + 1/4\nu (\nu + 2) + e}{z - 1} + \frac{C + 2AC - B - 2AB - \nu}{z} \\
+ \left[ -\frac{C - 2AC - BC - e - 1/4\nu (\nu - 2)}{z + 1} \right] F = 0.
\]

(74)

At \( A, B, C \) taken according to

\[
A^2 + E - e = 0 \quad \Rightarrow \quad A = \pm \sqrt{e - E};
\]

\[
B^2 - B - 1/4\nu (\nu + 2) = 0 \quad \Rightarrow \quad B = -\frac{1}{2} \nu, 1 + \frac{1}{2} \nu;
\]

\[
C^2 - C - 1/4\nu (\nu - 2) = 0 \quad \Rightarrow \quad C = \frac{1}{2} \nu, 1 - \frac{1}{2} \nu,
\]

(75)

eq. (74) becomes simpler

\[
\frac{d^2 F}{dz^2} + \left[ \frac{2A + 1}{z} + \frac{2B}{z - 1} + \frac{2C}{z + 1} \right] \frac{dF}{dz} + \\
+ \left[ \frac{BC + B + 2AB + 1/4\nu (\nu + 2) + e}{z - 1} + \frac{C + 2AC - B - 2AB - \nu}{z} \\
+ \left[ -\frac{C - 2AC - BC - e - 1/4\nu (\nu - 2)}{z + 1} \right] \right] F = 0,
\]

(76)

what is a Heun equation for \( G(p, q; \alpha, \beta, \gamma, \delta; z) \)

\[
p = -1, \quad q = C + 2AC - B - 2AB - \nu;
\]

\[
\gamma = 2A + 1, \quad \delta = 2B,
\]

(77)

and

\[
\alpha + \beta = 2A + 2B + 2C;
\]

\[
\alpha\beta = B + C + 2(AB + AC + BC) + \frac{1}{2}\nu^2 + 2e;
\]

that is

\[
\alpha = A + B + C - \sqrt{A^2 + B^2 + C^2 - B - C - 1/2\nu^2 - 2e},
\]

\[
\beta = A + B + C + \sqrt{A^2 + B^2 + C^2 - B - C - 1/2\nu^2 - 2e}.
\]

(78)

Let

\[
A = -\sqrt{e - E}; \quad B = 1 + \frac{1}{2} \nu; \quad C = \frac{1}{2} \nu;
\]

(79)
the negative value of $A$ ensures vanishing the function at the infinity $\chi \to +\infty$. The positive value of $B$ ensures vanishing of the function in the origin. Then

$$\alpha = 1 + \nu - \sqrt{e - E} - \sqrt{-e - E}, \quad \beta = 1 + \nu - \sqrt{e - E} + \sqrt{-e - E},$$

or remembering about (72)

$$\alpha = 2(j + 1) - \sqrt{2e - 2E} - \sqrt{-2E - 2e},
\beta = 2(j + 1) - \sqrt{2e - 2E} + \sqrt{-2E - 2e}. \quad (80)$$

Imposing additional constraint (condition for polynomial solutions)

$$\alpha = -2n; \quad (81)$$

we obtain

$$\sqrt{2e - 2E} + \sqrt{-2E - 2e} = 2(n + j + 1),$$

which after simple manipulation gives a formula for energy levels

$$E = -\frac{e^2}{2(n + j + 1)^2} - \frac{(n + j + 1)^2}{2}. \quad (82)$$

With the use of (82), one can readily obtain rather simple representation for involved parameters

$$\alpha = 2(j + 1) - N - \frac{e}{N} - N + \frac{e}{N} = -2n,
\beta = 2(j + 1) - N - \frac{e}{N} + N - \frac{e}{N} = 2(j + 1) - \frac{2e}{n + j + 1}. \quad (83)$$

Author plans to consider relativistic Coulomb problem on the base of the Dirac equation in space of constant curvature. Such a problem turns to be much more complicated – it reduces to a second order differential equation with 6 singular points. With special mathematical manipulations we can reduce the problem to a differential equation with 5 singular points, however it still remains very difficult mathematical task.

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[1] E. Schrödinger. A method of determining quantum-mechanical eigenvalues and eigenfunctions. Proc. Roy. Irish. Soc. A. 46. 9–16 (1940).
[2] A.F. Stevenson. A note on the "Kepler problem" in a spherical space, and the factorization method of solving eigenvalue problems. Phys. Rev. 59, 842–843 (1941).

[3] L. Infeld, A. Schild. A note on the Kepler problem in a space of constant negative curvature. Phys. Rev. 67, No 3/4, 121–122 (1945).

[4] P.W. Higgs. Dynamical symmetries in a spherical geometry. I. J. Phys. A. 12, No 3, 309–323 (1979).

[5] H.I. Leemon. Dynamical symmetries in a spherical geometry. II. J. Phys. A. 12, No 14, 489–501. (1979).

[6] Yu.A. Kurochkin, V.S. Otchik. Analogue of the Runge-Lenz vector and energy spectrum for Kepler problem in 3-dimensional sphere. Doklady Akad. Nauk BSSR. 23, No 11. 987–990 (1979).

[7] A.A. Bogush, Kurochkin Yu.A., Otchik V.S. On quantum-mechanical Kepler problem in Lobachevsky space. Doklady Akad. Nauk BSSR. 24, No 1. 19–22 (1979).

[8] L. Parker. One-electron atom in curved space-time. Phys. Rev. lett. 44, No 23. 1559–1562 (1980).

[9] L. Parker. The atom as a probe of curved space-time. Gen. Relat. and Grav. 13, No 4. 307–311 (1981).

[10] G.A. Ringwood, J.T. Devreese. The hydrogen atom: Quantum mechanics on the quotient of a conformally flat manifold. J. Math. Phys. 21. 1390–1392 (1980).

[11] K. Kobayashi. A derivation of the Pauli-Lenz vector and its variants. J. Phys. A. 13, No 2. 425–430 (1980).

[12] N. Bessis, G. Bessis. R. Shamseddine. Atomic fine-structure in a space of constant curvature. J. Phys. A. 15, No 10. 3131–3144 (1982).

[13] H. Grinberg, J. Marañon, H. Vucetich. The hydrogen atom as a projection of an homogeneous space. Z. Phys. C. 20. 147–149 (1983).

[14] A.A. Bogush, V.S. Otchik, V.M. Red’kov. Separation of variables in Schrödinger equation and normed wave functions for the Kepler problem in tree-dimensional spaces of constant curvature. Proceedings of the National Academy of Sciences of Belarus. Ser. fiz.-mat. 3. 56–62 (1983).

[15] N. Bessis, G. Bessis, R. Shamseddine. Space-curvature effects in atomic fine- and hyperfine-structure calculations. Phys. Rev. A. 29, No 5. 2375–2388 (1984).

[16] N. Bessis, G. Bessis, D. Roux. Atomic fine-structure calculations in a space of constant negative curvature. Phys. Rev. A. 30, No 2. 1094–1097 (1984).

[17] N. Bessis, G. Bessis. Atomic fine and hyper-fine structure caculations in a space of constant curvature. Lectures Notes in Physics. 212. 143–153 (1984).

[18] C.M. Xu, D.Y. Xu. Dirac equation and energy levels of hydrogen-like atoms in Robertson – Walker metrics. Nuovo Cim. B. 83, No 2. 162–172 (1984).

[19] C.M. Xu, D.Y. Xu. Dirac equation and energy-levels of hydrogen like atoms in Robertson – Walker metrics. Nuovo Cim. B. 3, No 2. 162–172 (1984).

[20] V.N. Melnikov, G.N. Shikin. Hydrogen-like atom in gravitational field of the universe. Izvestiz Vuzov. Fizika. 1. 55–59 (1985).

[21] R. Shamseddine. Structure fine et hyperfine atomique dans un espace à courbure constante. J. Phys. A. 19, No 5. 717–724 (1986).
[22] V.S. Otchik, V.M. Red'kov. Quantum mechanical Kepler problem in spaces of constant curvature. Preprint 298, Institute of Physics, NANB. Minsk (1986).

[23] A.O. Barut, A. Inomata and G. Junker. Path integral treatment of the hydrogen atom in a curved space of constant curvature. J. Phys. A: Math. Gen. 20, No 18. 6271–6280 (1987).

[24] N. Bessis, G. Bessis, D. Roux Space-curvature effects in the interaction between atom and external fields: Zeeman and Stark effects in a space of constant positive curvature. Phys. Rev. A. 33, No 1. 324–336 (1988).

[25] A.A. Bogush, V.S. Otchik, V.M. Red’kov. Complex parabolic coordinates and hydrogen atom on the sphere. Minsk (1988) 40 pages. Deposited in VINITI 12.04.88, 2722 - B88.

[26] A.K. Gorbatsieievich, A. Priebe. On the hydrogen atom in Kerr space time. Acta Phys. Polon. B. 20, No 11. 901–909 (1989).

[27] C. Groshe. The path integral for the Kepler problem on the pseudosphere. Ann. Phys. (N.Y.). 204. 208–222 (1990).

[28] A.O. Barut, A. Inomata and G. Junker. Path integral treatment of the hydrogen atom in a curved space of constant curvature. II. Hyperbolic space curvature. J. Phys. A: Math. Gen. 23, No 7. 1179–1190 (1990).

[29] N. Katayama. A note on the Kepler problem in a space of constant curvature. Nuovo Cim. B. 105, No 1. 113–119 (1990).

[30] N.A. Chernikov. The Kepler problem in the Lobachevsky space and its solution. Acta Phys. Polonica. B. 23. 115–122 (1992).

[31] L.G. Mardoyan, A.N. Sisakyan. The hydrogen-atom in curved space – orthogonality of the radial wave-functions with respect to the orbital angular momentum. Soviet J. Nuclear Physics-USSR. 55, No 9. 1366–1367 (1992).

[32] Ya.I. Granovskii, A.S. Zhedanov, I.M. Lutsenko. Quadric algebras and dynamics in curved space. I. An oscillator. Theor. Math. Phys. 91. 474–480 (1992); Quadric algebras and dynamics in curved space. II. The Kepler problem. Theor. Math. Phys. 91. 604–612 (1992).

[33] V.V. Kozlov, A.O. Harin. Kepler’s problem in constant curvature spaces. Celest. Mech. and Dynam. Astron. 54. 393–399 (1992).

[34] S.I. Vinitskii, L.G. Mardoyan, G.S. Pogosyan, A.N. Sisakyan, T.A. Strizh. Hydrogen-atom in curved space – expansion in free solutions on a 3-dimensional sphere. Physics of Atomic Nuclei. 56, No 3. 321–327 (1993).

[35] R. Shamseddine. On the resolution of the wave equations of electron in a space of constant curvature. Can. J. Phys. 75. 805–811 (1997).

[36] A.A. Bogush, Yu.A. Kurochkin, V.S.Otchik. Algebra of conserved operators for the Kepler-Coulomb problem in the spaces of constant curvature. Yad. Fiz. 61, No 10. 1889–1892 (1998).

[37] V.S. Otchik. On the connection between spherical and parabolic bases in the quantum mechanical Kepler problem in Lobachevsky space. Proc. of the National Acad. of Science of Belarus. Phys. Math.
ser. 4. 67–72 (1999).

[38] A. Nersessian, G. Pogosyan. Relation of the oscillator and Coulomb systems on spheres and pseudospheres. Phys. Rev. A. 63, No 2. 020103(R) (2001).

[39] V.M. Red’kov. On WKB-quantization in Lobachevski and Riemann 3-spaces. Nonlinear phenomena in complex systems. 6, No 2. 654–668 (2003).

[40] A.A. Bogush, Yu.A. Kurochkin, V.S. Otchik. Coulomb scattering in the Lobachevsky space. Nonlinear Phenomena in Complex Systems. 6. 894–897 (2003).

[41] Yu.A. Kurochkin, V.S. Otchik, Dz.V. Shoukavy. MIC-Kepler scattering problem in the three-dimensional Lobachevsky space. Non-Euclidean Geometry in Modern Physics: Proc. of the International Conference BGL-5 (Bolyai - Gauss - Lobachevsky). 10-13 Oct 2006, Minsk, Belarus. 116–121 (2006).

[42] Yu. Kurochkin, Dz. Shoukavy. Regge trajectories of the Coulomb potential in the space of constant negative curvature J. Math. Phys. 47, No 2. 022103 (2006).

[43] A.A. Bogush, V.C. Otchik, V.M. Red’kov. The Runge-Lenz vector for quantum Kepler problem in the space of positive constant curvature and complex parabolic coordinates. Proc. of 5th International Conference Bolyai-Gauss-Lobachevsky: Non-Euclidean Geometry In Modern Physics (BGL-5). 10-13 Oct 2006, Minsk, Belarus. 135–144 (2006); arxiv:hep-th/0612178

[44] N. Bessis, G. Bessis. Electronic wave functions in a space of constant curvature. J. Phys. A. 12, No 11. 1991–1997 (1979).

[45] T. Iwai. Quantization of the conformal Kepler problem and its application to the hydrogen-atom. J. Math. Phys. 23, No 6. 1093–1099 (1982).

[46] J.M. Cohen, R.T. Powers. The general relativistic hydrogen-atom. Comm. Mat. Phys. 86, No 1. 69–86 (1982).

[47] V.M. Red’kov. General covariant Dirac equation: spherical symmetry and Wigner D-functions, spinor monopole harmonics. Minsk (1988) 39 pages. Deposited in VINITI 9.03.88, 4577 - B88.

[48] V.M. Red’kov. Generally relativistical Tetrode-Weyl-Fock-Ivanenko formalism and behaviour of quantum-mechanical particles of spin 1/2 in the Abelian monopole field. 25 pages, arXiv:quant-ph/9812002