Semi-Quantum Key Distribution

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Secure key distribution among two remote parties is impossible when both are classical, unless some unproven (and arguably unrealistic) computation-complexity assumptions are made, such as the difficulty of factoring large numbers. On the other hand, a secure key distribution is possible when both parties are quantum. What is possible when only one party (Alice) is quantum, yet the other (Bob) has only classical capabilities? Recently, a semi-quantum key distribution protocol was presented (Boyer, Kenigsberg and Mor, Physical Review Letters, 2007), in which one of the parties (Bob) is classical, and yet, the protocol is proven to be completely robust against an eavesdropping attempt. Here we extend that result much further. We present two protocols with this constraint, and prove their robustness against attacks: we prove that any attempt of an adversary to obtain information (and even a tiny amount of information) necessarily induces some errors that the legitimate parties could notice. One protocol presented here is identical to the one referred to above, however, its robustness is proven here in a much more general scenario. The other protocol is very different as it is based on randomization.

1. INTRODUCTION

Processing information using quantum two-level systems (qubits), instead of classical two-state systems (bits), has lead to many striking results such as the teleportation of unknown quantum states and quantum algorithms that are exponentially faster than their known classical counterpart. Given a quantum computer, Shor’s factoring algorithm would render many of the currently used encryption protocols completely insecure, but as a countermeasure, quantum information processing has also given quantum cryptography. Quantum key distribution was invented by Bennett and Brassard (BB84), to provide a new type of solution to one of the most important cryptographic problems: the transmission of secret messages. A key distributed via quantum cryptography techniques can be secure even against an eavesdropper with unlimited computing power, and the security is guaranteed forever.

The conventional setting is as follows: Alice and Bob have labs that are perfectly secure, they use qubits for their quantum communication, and they have access to a classical communication channel which can be heard, but cannot be jammed (i.e. cannot be tampered with) by the eavesdropper. The last assumption can easily be justified if Alice and Bob can broadcast messages, or if they already share some small number of secret bits in advance, to authenticate the classical channel.

In the well-known BB84 protocol as well as in all other QKD protocols prior to [1], both Alice and Bob perform quantum operations on their qubits (or on their quantum systems). The question of how much “quantum” a protocol needs to be in order to achieve a significant advantage over all classical protocols is of great interest. For example, [2, 3, 4, 5] discuss whether entanglement is necessary for quantum computation, [6] shows nonlocality without entanglement, and [7, 8] discuss how much of the information carried by various quantum states is actually classical. This discussion was extended into the quantum cryptography domain in [1] where we presented and analyzed a protocol in which one party (Bob) is classical. For our purposes, any two orthogonal states of the quantum two-level system can be chosen to be the computational basis $|0\rangle$ and $|1\rangle$. For reasons that will soon become clear, we shall now call the computational basis “classical” and we shall use the classical notations $\{0,1\}$ to describe the two quantum states $\{|0\rangle, |1\rangle\}$ defining this basis. In the protocols we discuss, a quantum channel travels from Alice’s lab to the outside world and back to her lab. Bob can access a segment of the channel, and whenever a qubit passes through that segment Bob can either let it go undisturbed or (1) measure the qubit in the classical $\{0,1\}$ basis; (2) prepare a (fresh) qubit in the classical basis, and send it; (3) reorder the qubits (by using different delay lines, for instance). If all parties were limited to performing only operations (1)–(3), or doing nothing, they would always be working with qubits in the classical basis, and could never obtain any quantum superposition of the computational basis states; the qubits can then be considered “classical bits”; the resulting protocol would then be equivalent to an old-fashion classical protocol, and therefore, the operations themselves shall here be considered classical. We term this kind of protocol “QKD protocol with classical Bob” or Semi-Quantum Key Distribution (SQKD). We discuss and analyze two different variants of such a protocol. In one Bob performs operations (1) and (2) or transfers the qubit back to Alice; this variant is therefore named measure-resend SQKD. The other variant is based on randomization and named randomization-based SQKD. In this variant Bob is restricted to perform operations (1) and (3), or do nothing. This work extends the results of [1], by first generalizing the conditions under which the results of [1] hold for the measure-resend SQKD, specifically, proving that robustness still holds when the qubits are sent one by
one and are attacked collectively. In addition we define and analyze a randomization-based SQKD which leaks no information at all and results with a secret string with entropy exponentially close to its length. We provide a full proof of robustness for this variant as well.

To define our protocols we follow the definition (see for instance [9]) of the most standard QKD protocol, BB84. The BB84 protocol consists of two major parts: a first part that is aimed at creating a sifted key, and a second (fully classical) part aimed at extracting an error-free, secure, final key from the sifted key. In the first part of BB84, Alice randomly selects a binary value and randomly selects in which basis to send it to Bob, either the computational (“Z”) basis \([|0\rangle, |1\rangle]\), or the Hadamard (“X”) basis \([|+\rangle, |--\rangle]\). Bob measures each qubit in either basis at random. An equivalent description is obtained if Alice and Bob use only the classical operations (1) and (2) above and the Hadamard\([20]\) quantum gate \(H\). After all qubits have been sent and measured, Alice and Bob publish which bases they used. For approximately half of the qubits Alice and Bob used mismatching bases and these qubits are discarded. The values of the rest of the bits make the sifted key. The sifted key is identical in Alice’s and Bob’s hands and is attacked collectively. In addition we define and results with a secret string with entropy exponentially close to its length. We provide a full proof of robustness for this variant as well.

II. ROBUSTNESS

An important step in studying security is a proof of robustness; see for instance [13] for robustness proof of the entanglement-based protocol, and [14, 15] for a proof of robustness against the photon-number-splitting attack. Robustness of a protocol means that any adversarial attempt to learn some information necessarily induces some disturbance. It is a special case (in zero noise) of the more general “information versus disturbance” measure which provides explicit bound on the information available to Eve as a function of the induced error. Robustness also generalizes the no-cloning theorem: while the no-cloning theorem states that a state cannot be cloned, robustness means that any attempt to make an imprint of a state (even an extremely weak imprint) necessarily disturbs the quantum state.

A protocol is said to be completely robust if nonzero information acquired by Eve on the INFO string implies nonzero probability that the legitimate participants find errors on the bits tested by the protocol. A protocol is said to be completely nonrobust if Eve can acquire the INFO string without inducing any error on the bits tested by the protocol. A protocol is said to be partly robust if Eve can acquire some limited information on the INFO string without inducing any error on the bits tested by the protocol.

FIG. 1: (a) Eve’s maximum (over all attacks) information on the INFO string vs. the allowed disturbance on the bits tested by Alice and Bob, in a completely robust (solid line), partly robust (dashed), and completely nonrobust (densely dotted) protocol. (b) Robustness should not be confused with security; Eve’s maximum information on the final key vs. allowed disturbance in a secure protocol; such a protocol could be completely or partly robust.

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Partially robust protocols could still be secure, yet completely nonrobust protocols are automatically proven insecure. See also Fig. 1. As one example, BB84 is fully robust when qubits are used by Alice and Bob but it is only partly robust if photon pulses are used and sometimes two-photon pulses are sent. The well known two-state protocol (also called Bennett92 protocol) is not fully robust even if perfect qubits are used, if realistic channel losses are taken into account. Such partly robust protocols can still lead to a secure final key if enough bits are sacrificed for privacy amplification. On the other hand, such partly robust protocols can become completely nonrobust (and therefore totally insecure) if the loss rate is sufficiently high.
III. MOCK PROTOCOL AND ITS COMPLETE NONROBUSTNESS

Consider the following mock protocol: Alice flips a coin to decide whether to send a random bit in the computational basis \(\{|0\rangle, |1\rangle\}\) (“\(Z\)”), or in the Hadamard basis \(\{|+\rangle, |-\rangle\}\) (“\(X\)”). Bob flips a coin to decide whether to measure Alice’s qubit in the computational basis (to “SIFT” it) or to reflect it back (“CTRL”), without causing any modification to the information carrier. In case Alice chose \(Z\) and Bob decided to SIFT, i.e. to measure in the \(Z\) basis, they share a random bit that we call SIFT or sifted bit (that may, or may not, be confidential). In case Bob chose CTRL, Alice can check if the qubit returned unchanged, by measuring it in the basis she sent it. In case Bob chose to SIFT and Alice chose the \(X\) basis, they discard that bit. The idea that just one basis, the \(Z\)-basis, is sufficient for the key generation (while the other basis is used for finding the actions of an adversary) appeared already in [10]. The above iteration is repeated for a predefined number of times. At the end of the quantum part of the protocol Alice and Bob share, with high probability, a considerable amount of SIFT bits (also known as the “sifted key”). In order to make sure that Eve cannot gain much information by measuring (and resending) all qubits in the \(Z\) basis, Alice can check whether they have a low-enough level of discrepancy on the \(X\)-basis CTRL bits. In order to make sure that their sifted key is reliable, Alice and Bob must sacrifice a random subset of the SIFT bits, which we denote as TEST bits, and remain with a string of bits which we call INFO bits (INFO and TEST are common in QKD, e.g., in BB84 as previously described).

By comparing the value of the TEST bits, Alice and Bob can estimate the error rate on the INFO bits. If the error rate on the INFO bits is sufficiently small, they can then use an appropriate Error Correction Code (ECC) in order to correct the errors. If the error rate on the \(X\)-basis CTRL bits is sufficiently small, Alice and Bob can bound Eve’s information, and can then use an appropriate Privacy Amplification (PA) in order to obtain any desired level of privacy.

At first glance, this protocol may look like a nice way to transfer a secret bit from quantum Alice to classical Bob: It is probably resistant to opaque (intercept-resend) attacks.

However, it is completely non-robust; Eve could learn all bits of the INFO string using a trivial attack that induces no error on the bits tested by Alice and Bob (the TEST and CTRL bits). She would not measure the incoming qubit, but rather perform a cNOT from it into a \(|0^E\rangle\) ancilla[21]. If Alice chose \(Z\) and Bob decides to SIFT (i.e. measures in the \(Z\)-basis), she measures her ancilla and obtains an exact copy of their common bit, thus inducing no error on TEST bits and learning the INFO string. If, however, Bob decides on CTRL, i.e. reflects the qubit, Eve would do another cNOT from the returning qubit into her ancilla. This would reset her ancilla, erase the interaction she performed, and induce no error on CTRL bits, thus removing any chance of her being caught. In the following Section we present two protocols which overcome this problem via two different methods.

IV. TWO SEMI-QUANTUM KEY DISTRIBUTION PROTOCOLS

The following two protocols remedy the above weakness by not letting Eve know which is a SIFT qubit (that can be safely measured in the computational basis) and which is a CTRL qubit (that should be returned to Alice unchanged). Both protocols are aimed at creating an \(n\)-bit INFO string to be used as the seed for an \(l\)-bit shared secret key.

A. Protocol 1: Randomization-based SQKD.

Two versions are presented, both based on randomizing the returned qubits: Protocol 1 depends on a single parameter \(\delta > 0\) and is not completely-robust; Protocol 1’, with an additional parameter \(\epsilon \leq 1\) such that \(0 \leq \epsilon < \delta\) and with Step 7’ replacing Step 7, is completely robust.

Let \(n\), the desired length of the INFO string, be an even integer and let \(\delta > 0\) be some fixed parameter.

1. Alice sends \(N = \lceil 8n(1 + \delta) \rceil\) qubits. For each of the qubits she randomly selects whether to send it in the computational basis (\(Z\)) or the Hadamard basis (\(X\)). In each basis she sends random bits.

2. For each qubit arriving, Bob chooses randomly whether to measure it (to SIFT it) or to reflect it (CTRL). Bob reorders randomly the reflected qubits so that no one, neither Alice nor Eve, could tell which of them were reflected.

3. Alice collects the reflected qubits in a quantum memory[22].

4. Alice publishes which were her \(Z\) bits. Bob publishes which were his CTRL qubits, and in which order they were reflected; Alice then measures all the returned CTRL qubits in the basis she prepared them.

It is expected that for approximately \(N/4\) bits, Alice used the \(Z\) basis and Bob chose to SIFT (these are the SIFT bits, which form the sifted key); for approximately \(N/4\) bits, Alice used the \(Z\) basis and Bob chose CTRL (we refer to these bits as Z-CTRL), and for approximately \(N/4\) bits, Alice used the \(X\) basis and Bob chose CTRL (we refer to these bits as X-CTRL). In the rest of the bits, Bob expects a uniform distribution. Cf. Fig 2.

5. Alice checks the error-rate on the CTRL bits and if either the \(X\) error-rate or the \(Z\) error-rate is higher than some predefined threshold \(P_{\text{CTRL}}\), the protocol aborts.
6. Alice chooses at random $n$ SIFT bits to be TEST bits. She publishes which are the chosen bits. Bob publishes the value of these TEST bits. Alice checks the error-rate on the TEST bits and if it is higher than some predefined threshold $P_{\text{TEST}}$ the protocol aborts. Else, let $v$ be the string of the remaining SIFT bits.

7. Alice and Bob select the first $n$ bits in $v$ to be used as INFO bits. If there is no errors or eavesdropping, Alice and Bob share the same string. Otherwise, Bob’s string is likely to differ from the the INFO string until corrected in Step 8 below.

Unfortunately, Protocol 1 is not robust: we will show how Eve can count the number of “0”s and “1”s measured by Bob (i.e. the Hamming weight of the measured string) without being detectable and get about 0.3 bits of information on the INFO string, whatever its length (and prove she can not do better).

To make sure Eve cannot use statistics of occurrence of “0”s and “1”s in the INFO string, Protocol 1’ will fix in advance a subset of $\{0, 1\}^n$ to be used for the $n$-bit INFO strings. A new parameter $\epsilon \leq 1$ such that $0 \leq \epsilon < \delta$ is introduced and the set of INFO strings is

$$I_{n,\epsilon} = \left\{ y \in \{0, 1\}^n \mid \left| \frac{|y|}{n} - \frac{1}{2} \right| \leq \frac{\epsilon}{2} \right\} \quad (1)$$

where $|y|$ denotes the Hamming weight of $y$. When $\epsilon = 0$, $I_{n,0}$ is the set of $n$-bit strings with Hamming weight $n/2$; for $\epsilon = 1$ (which can happen if $\delta > 1$), $I_{n,\epsilon} = \{0, 1\}^n$. We will prove that when $\epsilon > 0$, the information carried by a random $y \in I_{n,\epsilon}$ is exponentially close to $n$ bits (in the parameter $n$). In that case, the set $I_{n,\epsilon}$ is a “good set” of INFO strings. When $\epsilon = 0$, $I_{n,0}$ has entropy of the order $n - 0.5 \log_2(n)$ bits.

As for robustness, it is obtained by replacing Step 7 by Step 7’:

7’. (a) Alice chooses a substring $x$ of $v$ of length $2h$ with $h$ zeros and $h$ ones, where $h = \lceil (1 + \epsilon)n/2 \rceil$; if she can not choose such a string, the protocol aborts.

(b) Alice chooses randomly $y \in I_{n,\epsilon}$.

(c) Alice chooses randomly a list of distinct indices $q_1 \ldots q_n$ such that $x_{q_1} \ldots x_{q_n} = y$.

(d) Alice announces publicly $q_1 \ldots q_n$; Bob thus learns that $v_{q_1} \ldots v_{q_n}$ is the INFO string.

We will show that the protocol aborts with exponentially small probability and leaks no information to Eve as long as she is undetectable.

B. Protocol 2: Measure-Resend SQKD

Our second protocol does not require Bob to randomize the qubits as in Step 2. Instead, Bob either measures and resends the qubit (SIFTS it) or reflects it (CTRL). Furthermore, Alice does not need to delay the measurement of the returning qubits until Step 4, because immediately in Step 3 she knows in which basis to measure.

The protocol is essentially the same as the previous one, with steps 1 to 7, but with steps 2, 3 and 4 modified to correspond to the new simplified siiting procedure; the modified steps are:

2. For each qubit arriving, Bob chooses randomly whether to measure and resend it in the same state he found (to SIFT it) or to reflect it (CTRL). Again, no one, neither Alice nor Eve, can tell which of the qubits were reflected.

3. Alice measures each qubit in the basis she sent it.

4. Alice publishes which were her $Z$ bits and Bob publishes which ones he chose to SIFT.

C. Classical Post-Processing.

The full protocol for the generation of the final key comprises any one of the above “semi-quantum” protocols, plus the “classical” step:

8. Alice publishes ECC & PA data, from which she and Bob extract the $l$-bit final key from the INFO string.

If the ECC is of rank $R$, publishing the ECC data entails publishing the parities of $n - R$ substrings of the INFO string, i.e. up to $n - R$ bits of information on the INFO string. This step must thus be excluded from the definition of robustness or else no protocol would ever be robust unless the ECC is degenerate (of rank $n$) and unable to correct any error (the minimal distance being 1). The $l$-bit key is chosen such that $l \leq R$ and it is the information on this final $l$-bit key that needs to be proven negligible to prove the security of the above protocols.

V. PROOFS OF ROBUSTNESS

We first show that Eve cannot obtain information on INFO bits in Protocol 2 without being detectable for the
case in which the qubits are sent by Alice one by one as well as the case they are sent together. This is performed by considering the general case in which Alice sends the qubits one by one but does not wait for a returning qubit before sending the next one (so that Eve can collect the qubits and attack them collectively). The scenario analyzed in [1] is a specific case of the setup we analyze here. We then bound the information Eve can get with Protocol 1 without inducing errors on TEST and CTRL bits and finally prove the complete robustness of Protocol $1'$.

A. Complete Robustness of Protocol 2.

1. Modeling the protocol.

Each time the protocol is executed, Alice sends to Bob a state $|\phi\rangle$ which is a tensor product of $N$ qubits, each of which is either $|+\rangle$, $|-\rangle$, $|0\rangle$ or $|1\rangle$; those qubits are indexed from 1 to $N$. Each of those qubits is either measured by Bob in the standard basis and resent as it was measured or simply reflected. We denote $m$ the set of bit positions measured by Bob; this is a subset of $[1 \ldots N]$ that we represent by an increasing list of $r$ integer positions $m_1 \ldots m_r$ corresponding to Bob measuring the $r$ qubits with index $m_1, \ldots, m_r$. For $i \in \{0, 1\}^N$, we denote

$$i_m = i_{m_1} i_{m_2} \ldots i_{m_r}$$

the substring of $i$ of length $r$ selected by the positions in $m$; of course $|i_m\rangle = |i_{m_1} i_{m_2} \ldots i_{m_r}\rangle$.

In the protocol, it is assumed that Bob has no quantum register; he measures the qubits as they come in. The physics would however be exactly the same if Bob used a quantum register of $r$ qubits initialized in state $|0^B\rangle = |0^r\rangle$ ($r$ qubits equal to 0), applied the unitary transform defined by [23]

$$M_m |i\rangle |0^B\rangle = |i\rangle |i_m\rangle$$

for $i \in \{0, 1\}^N$, sent back $|i\rangle$ to Alice and postponed his measurement to be performed on that quantum register $|i_m\rangle$; the qubits indexed by $m$ in $|i\rangle$ are thus automatically both measured and resent, and those not in $m$ simply reflected; the $k$th qubit sent by Alice is a SIFT bit if $k \in m$ and is either $|0\rangle$ or $|1\rangle$; it is a CTRL bit if $k \notin m$. This physically equivalent modified protocol simplifies the analysis and we shall thus model Bob’s measurement and resending, or reflection, with $M_m$. In most cases, Bob’s measurement will be performed bitwise; for each $k$ in $m$ we will denote $M_k$ the unitary that performs an exclusive or between $k$-th qubit in $i$ and on the corresponding qubit $j_k$ in Bob’s probe i.e. $M_k |i_k\rangle |j_k\rangle = |i_k\rangle |j_k \oplus i_k\rangle$. It follows that

$$M_m = M_{m_r} \ldots M_{m_2} M_{m_1}.$$
to $|0^E\rangle$ but she is forced to attack qubits individually. For each qubit $k$ from 1 to $N$, Eve applies a unitary $U_E^{(k)}$ acting on $\mathcal{H}_E$ and $\mathcal{H}_E^{[24]}$ before sending it to Bob and applies a unitary $U_F^{(k)}$ acting on the same spaces on the way back. The robustness of the individual-qubit protocol follows immediately from the robustness of Protocol 2 under the limited class of attacks where $U_1 = U_E^{(1)}, U_k = U_E^{(k)} U_F^{(k-1)}$ for $1 \leq k < N$ and $U_{N+1} = U_F^{(N)}$ (and qubits are returned all together to Alice).

### 3. The final global state.

Delaying all measurements allows considering the global state of the Eve+Alice+Bob system before all actual measurements; Eve’s and Bob’s actions are described by unitary transforms. The initial state is $|0^E\rangle |\phi^B\rangle$; Eve’s unitary transforms $U_1, \ldots, U_{N+1}$ act on the first two Hilbert spaces whilst Bob’s measurements $M_k$ performed when he receives qubit $k$ with $k \in m$ act on the last two spaces. For instance, if $N = 4$ and $m = (1,3)$ then the final global state of the system is $U_5 U_4 M_3 U_2 M_1 U_1 |0^E\rangle |\phi^B\rangle$ where measurement $M_1$ on qubit 1 occurs immediately after Eve applies $U_1$ and measurement $M_3$ on qubit 3 occurs immediately after Bob applies $U_3$.

The attacks $\{U_k\}_{1 \leq k \leq N+1}$ we are interested in are only those for which Eve is completely undetectable. Such attacks put strong restrictions on the global evolution of the system. In what follows, when we say that an attack induces no error on CTRL and TEST, we mean that for any choice of CTRL and TEST bits whose probability of occurrence according to protocol 2 is not 0, the probability that Eve’s attack induces an error on them is 0.

**Proposition 1.** If the attack $\{U_k\}_{1 \leq k \leq N+1}$ induces no error on TEST and CTRL bits, and if Alice sent state $|i\rangle$ with $i \in \{0,1\}^N$, then there is a state $|F_i\rangle \in \mathcal{H}_E$ such that, for all $m$, the final global state of the system after applying $U_{N+1}$ is

$$|F_i\rangle |i\rangle |i_m\rangle.$$  \hspace{1cm} (3)

**Proof.** The final global state of the system can always be written as $\sum_{j,j'} |E_{ijj'}\rangle |j\rangle |j'\rangle$ where $|j\rangle$ is the standard basis of $\mathcal{H}_F$ and $|j'\rangle$ of Bob’s probe space; If the protocol induces no errors on TEST bits, it must be so that for all $m$, $|E_{ijj'}\rangle = 0$ for $j' \neq i_m$ and thus the final global state must be $\sum_j |E_{ijj}\rangle |j\rangle |i_m\rangle$. Moreover, if there is no error on CTRL bits, then the probability for Alice to measure any $|j\rangle$ that is not $|i\rangle$ must be zero. She can indeed choose any qubit not in $m$ as a $Z$-CTRL bit; she also checks all the qubits measured by Bob, which must also coincide with those she sent since $i \in \{0,1\}^N$. Consequently $|E_{ijj}\rangle = 0$ if $j \neq i$ and the final state must be $|E_{ijj}\rangle |i\rangle |i_m\rangle$.

We now prove that $|E_{ijj}\rangle$ does not depend on $i_m$. Let $Z$ be the linear map defined by $Z|e^{(j)}\rangle |j'\rangle = |e^{(j)}\rangle |e^{(j)}\rangle$ i.e. $Z$ is the linear map on Bob’s probe space that maps its standard basis states on the state $|0^B\rangle$. It is clear that $Z U_k = U_k Z$ and $Z M_k = Z$ for all $k$. If we look at the particular case where $N = 4$ and $m = (1,3)$, i.e. Bob measures qubits 1 and 3, this implies that $Z U_5 U_4 M_3 U_2 M_1 U_1 |0^E\rangle |\phi^B\rangle = U_5 U_4 U_3 U_2 Z |0^E\rangle |\phi^B\rangle = U_5 U_4 U_3 U_2 U_1 |0^E\rangle |\phi^B\rangle$. Applying $Z$ to the final state just gives the final state obtained if $m$ is empty. If we apply $Z$ to $|E_{ijj}\rangle |i\rangle |i_m\rangle$ we get $|E_{ijj}\rangle |i\rangle |0^B\rangle$ and this state must be equal to the final global state when $m$ is empty. This implies that for all values of $m$, the states $|E_{ijj}\rangle |i\rangle |i_m\rangle$ must be the same; we call them $|F_i\rangle$ and this gives $|F_i\rangle |i\rangle |i_m\rangle$ as the final global state. Note that the Eve’s state $|F_i\rangle$ is not entangled with the system $|i\rangle$ sent back to Alice, nor with Bob’s register $|i_m\rangle$.

We now show that if Eve’s attack is undetectable by Alice and Bob, then Eve’s final state $|F_i\rangle$ is independent of the string $i \in \{0,1\}^N$. More precisely,

**Proposition 2.** If $\{U_k\}_{1 \leq k \leq N+1}$ is an attack on Protocol 2 that induces no error on TEST and CTRL bits, then for all $i,i' \in \{0,1\}^N$,

$$i,i' \in \{0,1\}^N \implies |F_i\rangle = |F_{i'}\rangle.$$  \hspace{1cm} (4)

**Proof.** For any index $k$, let Alice’s $k$-th qubit be in state $|+\rangle$, and all the other qubits be prepared in the $Z$-basis. Alice’s state can be written $\left(\frac{1}{\sqrt{2}} |i\rangle + |i'\rangle\right)$ where $i,i' \in \{0,1\}^N$; $i_k = 0$, $i'_k = 1$, and $i_t = i'_t$ for $t \neq k$. Let Bob choose $m$ such that $k \notin m$; such an $m$ exists because $N \geq 2$ and then $i_m = i'_m$. By the previous proposition and linearity, the final global state is $\frac{1}{\sqrt{2}} [|F_i\rangle |i\rangle + |F_{i'}\rangle |i'\rangle] |i_m\rangle$; since we are interested only in Alice’s $k$-th qubit, we trace-out all the other qubits in Alice and Bob’s hands and get the state

$$\frac{1}{\sqrt{2}} [|F_i\rangle |0\rangle + |F_{i'}\rangle |1\rangle];$$

if $|0\rangle$ and $|1\rangle$ are replaced by their values in term of $|+\rangle$ and $|−\rangle$, this rewrites $\frac{1}{2} [|F_i\rangle + |F_{i'}\rangle] |+\rangle + \frac{1}{2} [|F_i\rangle - |F_{i'}\rangle] |−\rangle$ and since the probability that Alice measures $|−\rangle$ must be 0, $\frac{1}{2} [|F_i\rangle - |F_{i'}\rangle] = 0$ i.e. $|F_i\rangle = |F_{i'}\rangle$. The above holds for any $l$; any bit in $i$ can be flipped without affecting $|F_i\rangle$ and thus $|F_i\rangle$ is the same for all $i \in \{0,1\}^N$.

**Theorem 3.** For any attack $\{U_k\}_{1 \leq k \leq N+1}$ on Protocol 2 that induces no error on TEST and CTRL bits, Eve’s final state is independent of the state $|\phi\rangle$ sent by Alice, and Eve has thus no information on the INFO string.
Proof. By Proposition 2, there is a state \( F_{\text{final}} \) of Eve’s probe space such that for all \( i \in \{0,1\}^N \), Eve’s final state \( |F_i\rangle = |F_{\text{final}}\rangle \). By Proposition 1, for all \( i \in \{0,1\}^N \) and all \( m \), the final state after applying \( U_{N+1} \) if Alice sends \(|i\rangle\) is thus \(|F_{\text{final}}\rangle|i\rangle|\bar{m}\rangle\). For all superpositions \(|\phi\rangle = \sum_i c_i|i\rangle\) that Alice may send, and all \( m \), the final state of the Eve+Alice+Bob system after applying \( U_{N+1} \) is consequently

\[
|F_{\text{final}}\rangle \sum_i c_i|i\rangle|\bar{m}\rangle;
\]

Eve’s probe state \(|F_{\text{final}}\rangle\) is independent of \( \bar{m} \) and therefore of the SIFT bits and INFO bits — if Eve is to be undetectable.

The above theorem means that Protocol 2 is completely robust.

B. Partial robustness of Protocol 1.

1. Modeling the protocol.

The states \(|\phi\rangle\) sent by Alice are still products of \( N \) qubits each of which is either \(|+\rangle\), \(|-\rangle\), \(|0\rangle\) or \(|1\rangle\). In Step 2 of the protocol, Bob either measures a qubit, or reflects it; moreover, he reorders randomly the reflected qubits; let \( r \) be the number of reflected qubits and let \( s = s_1s_2\ldots s_r \) be the list of those \( r \) randomly ordered bit positions. For instance, if \( r = 4 \), and Bob reflects qubits 8, 1, 5 and 4 in that order then \( s = 8154 \) (examples will use positions from 1 to 9 to avoid comma separated lists). The list of non-reflected bits is indexed by the complement \( \bar{s} \) and will always be listed in ascending order; if \( N = 9 \) and \( s = 8154 \) then \( \bar{s} = 23679 \).

Bob’s measurement can still be postponed, but this time, since Bob keeps the qubits selected by \( \bar{s} \) without sending a copy, there is no need to copy. For all string \( s \) we will still denote \( i_s = i_{s_1}\ldots i_{s_r} \), the list of bits selected by \( s \) in the order specified by \( s \); Bob’s operation is then captured by

\[
U'_s|i\rangle = |i_s\rangle|i\rangle
\]

where \(|i_s\rangle\) is the state reflected to Alice, and \(|i_s\rangle\) the state (to be) measured by Bob. With \( N = 9 \) and \( s = 8154 \), and if Alice sent \(|i\rangle\ldots|i_9\rangle\) with \( i_1,\ldots,i_9 \in \{0,1\} \), the state reflected is \(|i_9i_1i_2i_3i_4\rangle\) and the state to be measured \(|i_2i_3i_6i_7i_9\rangle\). Of course, Alice can compare \( i_s \) with what she actually sent only when \( s \) is known and consequently keeps \(|i_s\rangle\) in quantum memory. With these notations, qubit \( k \) is CTRL if \( k \in s \) and it is SIFT if it is either \(|0\rangle\) or \(|1\rangle\) and \( k \notin s \).

2. Eve’s attack.

Eve’s most general attack is still comprised of two unitaries: \( U_E \) and \( U_F \) sharing a common probe space; \( U_E \) is applied on \(|0^F\rangle\) and \(|\phi\rangle\) and attacks qubits as they go from Alice to Bob; \( U_F \) is applied on Eve’s probe and \(|i_s\rangle\) as those bits go back from Bob to Alice; one slightly annoying problem is that the dimension of the space on which \( U_F \) acts is not fixed; it depends on the size of \( s \), i.e. the number of bits reflected by Bob; there is thus one unitary \( U_F \) for each \( r > 0 \).

3. The global final state.

Since Bob uses no probe space, the global state after Eve applies \( U_F \) is simply \( U_E|0^F\rangle|\phi\rangle \); then Bob applies \( U'_s \) to his part of the system, which corresponds to the global unitary \( I_E \otimes U'_s \) where \( I_E \) is the identity on Eve’s probe space. Then \( U_F \) is applied only on Eve’s probe and \(|i_s\rangle\); if we denote \( I_s \) the identity on the system left in Bob’s hands, given by the qubits selected by \( \bar{s} \), the final global state is then

\[
[U_F \otimes I_s]|I_E \otimes U'_s|U_E|0^F\rangle|\phi\rangle.
\]

Proposition 4. If \((U_E,U_F)\) is an attack on Protocol 1 such that \( U_E \) induces no error on test bits then there are states \(|E_i\rangle\) in Eve’s probe space such that for all \( i \in \{0,1\}^N \),

\[
U_E|0^F\rangle|i\rangle = |E_i\rangle|i\rangle.
\]

If moreover \( U_F \) induces no error on ctrl bits, then there are states \(|F_{s,i}\rangle\) of Eve’s probe space such that for all \( i \in \{0,1\}^N \), and all sequence \( s \) of distinct elements of \([1..N]\),

\[
U_F|E_i\rangle|i_s\rangle = |F_{s,i}\rangle|i_s\rangle.
\]

Proof. \( U_E|0^F\rangle|i\rangle \) can be expanded as \( \sum_j |E_{ij}\rangle\langle j| \) and since for any \( k \) there must be a 0 probability of getting \( j_k \) different from \( i_k \) (there is a non zero probability that Bob chooses bit \( k \) as a test bit), \(|E_{ij}\rangle = 0 \) for \( j \neq i \) and thus (7) holds with \(|E_i\rangle = |E_{ii}\rangle\). In Step 4, Bob publishes the bit positions \( s \) and, for Eve’s attack to be unnoticeable by Alice, the state held by Alice after \( U_F \) is applied to \(|E_i\rangle|i_s\rangle \) needs to be equal to \(|i_s\rangle\). By Hilbert-Schmidt, this implies that the bipartite state \( U_F|E_i\rangle|i_s\rangle \) must be of the form \(|F_{i}\rangle|i_s\rangle\). The pure state \(|F_{i}\rangle\) depends here on \( i \), both through \(|E_i\rangle\) and \( i_s \), and also on the string \( s \) chosen to select the reflected qubits, i.e. \(|F_{i}\rangle\) is a function \( i \) and \( s \) and will be written \(|F_{s,i}\rangle\), giving Eq. (8). □

When the attack \((U_E,U_F)\) induces no error on test and ctrl bits then, using (6), (7) and (8),

\[
[U_F \otimes I_B]|I_E \otimes U'_s|U_E|0^F\rangle|i\rangle = |F_{s,i}\rangle|i_s\rangle\langle i_s|.
\]

One can no longer expect Eve’s final state \(|F_{s,i}\rangle\) after Alice sent state \(|i\rangle\) and Bob reflected the qubits specified by \( s \) to be constant, as is shown in the following example:
Example 1. Let Eve’s probe space be of dimension $N + 1$ with basis states $|0\rangle \ldots |N\rangle$. Eve’s initial state is $|0\rangle$. Let $U_E|0\rangle = |i\rangle$ and $U_F|h\rangle = |h - |j\rangle|j\rangle$. This means that $U_E$ puts in the probe the Hamming weight $h = |i\rangle$ of the string $i \in \{0, 1\}^N$ if Alice sends state $|i\rangle$, and $U_F$ subtracts from the probe the Hamming weight of the string $|j\rangle$ returned by Bob. In particular $[F_{s,i} = |i\rangle - |i_s\rangle = |i_s\rangle]$. For $U_F$ to be defined on all basis states assume the difference is modulo $N + 1$. Bob can clearly detect no error on TEST bits. Moreover, if Alice sends $|\phi\rangle = \sum c_i |i\rangle$, the final state is $\sum c_i |i_s\rangle |i_s\rangle |i\rangle$ and, once Bob has measured $|i_s\rangle$, Eve’s probe $|i_s\rangle$ factors out and the resulting state in Alice’s hands is the same as if Eve had applied neither $U_E$ nor $U_F$, i.e. the final state had been $\sum_i c_i |0\rangle |i_s\rangle |i\rangle$; no error can thus be detected on CTRL bits.

Example 1 shows that Eve can learn the Hamming weight $|i_s\rangle$ of the string measured by Bob and stay completely invisible to Alice and Bob, i.e. induce no error on TEST and CTRL bits. Therefore, in order to make protocol 1 robust, the choice of the INFO bits must be done in a more careful way.

But first, we need to show that Eve can learn at most the Hamming weight of $|i_s\rangle$; this is a consequence of Eq. (13) below, which is derived from a sequence of lemmas. The first lemma states that all the bits in $i$ whose index are in $s$ can be flipped without changing $|F_{s,i}\rangle$; in Protocol 2, this was true for all qubits in $i$, but then, all the qubits were returned. In Protocol 1, only the qubits in $s$ are returned to Alice; the following lemma shows that for a fixed $s$, Eve’s state depends only on the bits kept by Bob.

Lemma 5. For any attack $(U_E, U_F)$ on Protocol 1 that induces no error on TEST and CTRL bits, if $|E_i\rangle$ and $|F_{s,i}\rangle$ are given by (7) and (8) then

$$i_s = i_s' \implies |F_{s,i}\rangle = |F_{s',i}\rangle. \quad (10)$$

Proof. The result is trivial if $s$ is empty. If not, we follow the steps of the proof of Proposition 2 and prove this bit-wise; let $k$ be an index in $s$ and $i$ and $i'$ be such that $i_k = 0$ and $i'_k = 1$, all other bits being the same. Assume wlg that $k$ is the first element of $s$ i.e. $s = ks'$ and thus $i_s = i_k i_s'$. If Alice sends the state $\frac{1}{\sqrt{2}}(|i\rangle + |i'\rangle)$ i.e. the $k$th qubit sent by Alice is $|+\rangle$ and all the other qubits are prepared in the $Z$-basis, with bit values according to $i$, then by linearity and Eq. (9) the final state of the Eve+Alice+Bob system is $\frac{1}{\sqrt{2}}(|F_{s,i}\rangle|0\rangle + |F_{s,i'}\rangle|1\rangle)|i_s\rangle |i_s'\rangle$; if we trace out all the qubits in $s'$ and $\bar{s}$ to keep only Eve’s probe and qubit $k$ in Alice’s hands, we get the state

$$\frac{1}{\sqrt{2}}(|F_{s,i}\rangle|0\rangle + |F_{s,i'}\rangle|1\rangle);$$

writing $|0\rangle$ and $|1\rangle$ in terms of $|+\rangle$ and $|−\rangle$ and considering only those terms in the resulting state that contain $|−\rangle$ gives $\frac{1}{2} [F_{s,i}|−\rangle|−\rangle]$; and since the probability that Alice measures $|−\rangle$ as the $k$th qubit must be 0 (because $k \in s$), $|F_{s,i}\rangle = |F_{s,i'}\rangle = 0$, i.e., $|F_{s,i}\rangle = |F_{s,i'}\rangle$. \hfill \square

The following lemma simply expresses the fact that, when Alice sends $|i\rangle$ and Bob reflects the qubits with indices in $s$ then Eve’s final state depends only on $|i\rangle$ and the state reflected by Bob.

Lemma 6. For any attack $(U_E, U_F)$ on Protocol 1 that induces no error on TEST and CTRL bits, if $|E_i\rangle$ and $|F_{s,i}\rangle$ are given by (7) and (8) then for all $i$, $s$ and $s'$,

$$i_s = i_{s'} \implies |F_{s,i}\rangle = |F_{s',i}\rangle. \quad (11)$$

Proof. If $i_s = i_{s'}$ then $U_F|E_i\rangle|i_s\rangle = U_F|E_i\rangle|i_{s'}\rangle$ and thus $|F_{s,i}\rangle|i_s\rangle = |F_{s',i}\rangle|i_{s'}\rangle$. \hfill \square

When Eq. (11) is used, we are using the fact that when Eve sees a qubit $|0\rangle$ (resp. a qubit $|1\rangle$) coming back from Bob, then she cannot tell to what qubit $|0\rangle$ (resp. $|1\rangle$) sent by Alice this qubit corresponds provided of course more than one $|0\rangle$ (resp. $|1\rangle$) had been sent by Alice. The preceding lemmas can be used to show that, if Eve induces no error on TEST and CTRL bits, then Eve’s intermediate state $|E_i\rangle$ just after $U_E$ is applied stays invariant when the bits in $i$ are permuted; let us first look at an example.

Example 2. Let $N = 4$ and $r = 2$ and let us see that $|E_{0111}\rangle = |E_{0111}\rangle$ i.e. Eve’s state after the attack $U_E$ on the qubits from Alice to Bob is the same whether Alice sends state $|1011\rangle$ or $|0111\rangle$. By Eq. (10), $|F_{14,0111}\rangle = |F_{14,0011}\rangle$ which is Eve’s final state when Bob reflects bits 1 and 4 and Alice sends either $|1011\rangle$ or $|0111\rangle$. Similarly $|F_{24,0111}\rangle = |F_{24,0011}\rangle$. We now use Eq. (11) to get $|F_{24,1111}\rangle = |F_{24,0111}\rangle$ (Eve cannot tell if the returning $|0\rangle$ is bit 1 or bit 2); those identities imply $|F_{24,1111}\rangle = |F_{24,1111}\rangle$. We now go back to the definition of $F$: $|F_{14,0111}\rangle$ is Eve’s final state if Alice sent $|1011\rangle$ and Bob reflected the bits 1 and 4 and from Eq. (8) we get $U_F|E_{0111}\rangle|11\rangle = |F_{14,0111}\rangle|11\rangle$ (bits 14 being 11). Similarly $U_F|E_{0111}\rangle|11\rangle = |F_{24,0111}\rangle|11\rangle$ and since the r.h.s. members are equal and $U_F$ is unitary, $|E_{0111}\rangle = |E_{1111}\rangle$.

Following the lines of Example 2, we prove the following lemma.

Lemma 7. For any attack $(U_E, U_F)$ on Protocol 1 that induces no error on CTRL and TEST bits, if $|E_i\rangle$ and $|F_{s,i}\rangle$ are given by (7) and (8) then for all $i$, $s'$,$i' \in \{0, 1\}^N$

$$|i\rangle = |i'\rangle \implies |E_i\rangle = |E_{i'}\rangle. \quad (12)$$

Proof. Eq. (12) means that $|E_i\rangle$ depends only on the number of “0”s and “1”s in $i$, not on their positions. We need only show that any two (distinct) bits in $i$ can be swapped without affecting $|E_i\rangle$ and, wlg, $|E_{011\rangle} = |E_{1001}\rangle$ for any $i'' \in \{0, 1\}^{N-2}$[25]. Let $s'$ be any sequence of distinct elements of $\{3 \ldots N\}$; $|F_{i'\rangle}00\rangle = |F_{i'\rangle00\rangle}$ and
two strings have the same Hamming weight. Let us see
Eve’s final state can be written
Let
\[ |s,i⟩′′ \]
\[ |i⟩|s′⟩ |j⟩ |s,i⟩′ ′′ \]
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\[ F_{s,i}′ ′′ \]
\[ |s′⟩ \]
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For any particular $n$-bit INFO string $x$, the entropy of $W$ given $x$ and $k$ is the entropy of the binomial distribution with $kn - 2n$ trials (for the $kn - 2n$ remaining bits) and is thus

$$H(W \mid x, k) = \frac{1}{2} \log_2 \left( \frac{1}{2} e(k - 2)n \right) + O \left( \frac{1}{n} \right).$$

(16)

The bits of the INFO string are random bits chosen by Alice and the strings $x$ are thus equally likely; this implies $H(W \mid X, k) = H(W \mid x, k)$. The information Eve gains on $X$ when $W$ is known is, for any fixed $k$, $H(X \mid k) - H(X \mid W, k)$. It is a basic fact from information theory that $H(X \mid k) - H(X \mid W, k) = H(W \mid k) - H(W \mid X, k)$ and Eve’s information is thus

$$I(W; X, k) = H(W \mid k) - H(W \mid X, k)$$

$$= \frac{1}{2} \log_2 \frac{k - 1}{k - 2} + O \left( \frac{1}{n} \right)$$

$$= \frac{1}{2} \log_2 \left( 1 + \frac{1}{k - 2} \right) + O \left( \frac{1}{n} \right).$$

(17)

For $k \geq 4$, $I(W; X, k) \leq 0.293 + O(n^{-1})$; the probability that $k < 4$ is exponentially small in $n$ and thus $I(W; X) \leq \sum_k I(W; X, k)p(k) < 0.293 + O(n^{-1})$.

\[\Box\]

C. Properties of Protocol 1'.

1. The information contained in the INFO string.

Alice chooses randomly $y \in I_{n, \epsilon}$ to send as the INFO string. The information contained in $y$ is thus the entropy of a uniform distribution on $I_{n,\epsilon}$.

**Proposition 11.** If $\epsilon > 0$, the entropy of the uniform distribution on $I_{n,\epsilon}$ is exponentially close to $n$ (its distance to $n$ is of order $e^{-\Omega(n)}$).

**Proof.** For any integer $N > 0$, the entropy of the uniform distribution on a set of $N$ elements is $\log_2(N)$. We are thus looking for a lower bound on $\log_2(|I_{n,\epsilon}|)$.

Let $Y = (Y_1, \ldots, Y_n)$ be a uniformly distributed random variable on $\{0, 1\}^n$; the $Y_i$ are independent Bernoullis with probability $p = 1/2$. Let $Y = \sum_{i=1}^n Y_i/n$; $Y$ is nothing but $|Y|/n$; the expectancy $E[|Y|]$ is $1/2$.

$$n - \log_2(|I_{n,\epsilon}|) = -\log_2 \left( \frac{1}{2^n} \right)$$

$$= -\log_2 \left( P \left( |Y - \frac{1}{2}| \leq \frac{\epsilon}{2} \right) \right).$$

By Hoeffding’s inequality (29)

$$P \left( |\bar{Y} - \frac{1}{2}| > \frac{\epsilon}{2} \right) \leq 2 \exp \left( -\frac{\epsilon^2}{2} \right),$$

and thus

$$n - \log_2(|I_{n,\epsilon}|) \leq -\frac{1}{\ln(2)} \ln \left( 1 - 2 \exp \left( -\frac{\epsilon^2}{2n} \right) \right).$$

For $0 < x < 0.5$, it is easy to verify that $-\ln(1 - x) \leq 3x/2$; thus, for $n$ large enough (e.g. $n > \ln(16)/\epsilon^2$),

$$n - \log_2(|I_{n,\epsilon}|) \leq \frac{3}{\ln 2} \exp \left( -\frac{\epsilon^2}{2n} \right).$$

\[\Box\]

While the entropy is $\approx n$ when $\epsilon > 0$, we now show that it has a gap of $0.5 \log_2(n)$ bits when $\epsilon = 0$.

**Proposition 12.** For $\epsilon = 0$, the entropy of the uniform distribution on $I_{n,0}$ is asymptotically equal to $n - 0.5 \log_2(n) - 0.5(\log_2(e) - 1)$.

**Proof.** Stirling’s formula gives

$$\lim_{n \to \infty} \left( \frac{n}{n/2} \right) = 1.$$ We get the result by taking the log.

Thus, by choosing $\epsilon > 0$, we avoid asymptotically losing more than $0.5 \log_2(n)$ bits of information.

2. Probability of aborting Protocol 1'.

The protocol aborts if there are less than $h$ zeros or $h$ ones left in the SIFT string after $n$ TEST bits have been chosen, where $h = \lceil (1 + \epsilon)n/2 \rceil$. We prove that this occurs with a probability that decreases exponentially with $n$.

**Proposition 13.** For any $0 \leq \epsilon < \delta$ and $\epsilon \leq 1$ fixed by the protocol, the probability that it aborts is exponentially small.

**Proof.** We begin with showing that, besides an exponentially small probability, the number of SIFT bits is larger than $N/4$. We follow by showing that this is enough for having at least $h$ zeros and ones, except for exponential probability. Let $\delta'$ be a real number such that $\epsilon < \delta' < \delta$. Let $N = \lceil 8n(1 + \delta) \rceil$. For $i$ such that $1 \leq i \leq N$, let $X_i = 1$ if the qubit $i$ is SIFT and $X_i = 0$ otherwise. The variables $X_i$ are clearly independent; their distribution is a Bernoulli with $p = 0.25$, as shown in Fig 2. The random variable $S$ giving the number of SIFT bits is $S = \sum_{i=1}^N X_i$. Denote $X = S/N$; it is clear that $E[X] = 1/4$, and we can bound $P[S \leq N/4]$ using Hoeffding (Theorem 19),

$$P \left[ S \leq 2n(1 + \delta') \right] \leq P \left[ X \leq \frac{11 + \delta'}{4(1 + \delta)} \right]$$

$$\leq P \left[ X - \frac{1}{4} \leq -\frac{\delta - \delta'}{4(1 + \delta)} \right]$$

$$\leq \exp \left( -\frac{1}{8} \left( \frac{\delta - \delta'}{1 + \delta} \right)^2 n \right).$$
and thus
\[
P \left[ S > 2n(1 + \delta') \right] \geq 1 - e^{-k_1 n}
\]
for \( k_1 = 1/8 \left( (\delta' - \delta)/(1 + \delta) \right)^2 \).

For each \( S > 2n(1 + \delta') \), the \( S \) bits are uniformly distributed. After \( n \) test bits are chosen, the remaining \( S-n > 2n(1+\delta')-n = n(1+2\delta') \) bits are still uniformly distributed. Every time there are at least \( h \) zeros and \( h \) ones after the \( n \) test bits are chosen, and in addition there are more than \( 2n(1+\delta') \) SIFT bits, the protocol succeeds. As a consequence, the probability of success is larger than or equal to the probability that \( S > 2n(1 + \delta') \) times the probability that the \( S-n \) remaining bits contain at least \( h \) zeros and \( h \) ones, given that \( S > 2n(1 + \delta') \). Let \( V \) be the length of the string \( v \), i.e. \( V = S - n > n(1 + 2\delta') \).

Let us index the bits in \( v \) from 1 to \( V \), let \( Z_i = 1 \) if bit \( i \) is 0 and \( Z_i = 0 \) otherwise, let \( Z = \sum_{i=1}^{V} Z_i \) and let \( Z = Z/V ; Z \) is thus the number of bits equal to 0 in \( v \); the \( Z_i \) are Bernoulli with \( p = 1/2 \) and are independent. Let us denote \( P_v \) the probability conditional to that particular value of \( V \). The probability that there are strictly less than \( h \) zeros in \( v \) is bounded by
\[
P_v[Z < h] \leq P_v[Z \leq (1 + \epsilon)n/2] \\
= P_v[Z \leq (1 + \epsilon)n/(2V)] \\
\leq P[Z \leq \frac{1 + \epsilon}{2(1 + 2\delta')}] \\
= P[Z - \frac{1}{2} \leq -\frac{2\delta' - \epsilon}{2(1 + 2\delta')}] 
\]
where \( \delta' > \epsilon \) by hypothesis and again, by Hoeffding (Theorem 19), the probability that there are not enough zeros when \( S > 2n(1 + \delta') \) is bounded by
\[
\exp \left( -\frac{1}{2} \left( \frac{2\delta' - \epsilon}{1 + 2\delta'} \right)^2 n \right)
\]
and the probability that there are at least \( h \) zeros and \( h \) ones when \( S > 2n(1 + \delta') \) is larger than or equal to \( 1 - 2e^{-k_2 n} \) with \( k_2 = \frac{1}{2} \left( \frac{2\delta' - \epsilon}{1 + 2\delta'} \right)^2 \). As a consequence, the probability that the protocol succeeds is at least
\[
(1-e^{-k_1 n})(1-2e^{-k_2 n}) = 1-e^{-k_1 n}-2e^{-k_2 n}+2e^{-(k_1+k_2)n}
\]
which is more that \( 1-3e^{-kn} \) with \( k = \min\{k_1, k_2\} \). It is exponentially close to 1 with \( n \).

**D. Complete robustness of Protocol \( \Gamma' \).**

The assumption is that Eve’s attack is undetectable, and we want to show that she gets no information on the INFO string. During the execution of the protocol, Eve learns which are the TEST bits, she learns their values, she learns the number of bits measured by Bob and, more importantly, her attack allows her to know their Hamming weight. We group all those data in the multivariate random variable \( R \) of which the details will be irrelevant; \( r \) will be a particular set of data. The execution of the protocol also gives Eve the set of indices \( q \) such that \( v_q = y \). What we want to show is that
\[
I(Y; Q, R) = 0,
\]
i.e. the mutual information between the INFO string \( y \) and what Eve knows, namely \( (q, r) \), is zero.

1. **Probabilistic setup.**

Let \( F \) be the set of indices measured by Bob. By Theorem 9, if Eve is unnoticeable, her final state may depend only on \( |i_F| \). Eve’s final state does not depend on \( y \) either. That implies that, whatever \( r \) Eve learns and for any value \( y \in \{0, 1\}^n \)
\[
|i_F| = |i_F'| \implies p(i \mid y, r) = p(i' \mid y, r).
\]

For \( h = \lfloor (1 + \epsilon)n/2 \rfloor \), Alice chooses \( 2h \) indices in \( F \) that are SIFT bits and not TEST bits, say \( E \). Let \( E_h \) be the set of all balanced strings \( x \) indexed by \( E \), i.e.
\[
E_h = \{ x \in \{0, 1\}^E \mid |x| = h \}.
\]

**Lemma 14.** For any \( x, x' \in E_h \),
\[
p(x \mid y, r) = p(x' \mid y, r) = \frac{1}{|E_h|} = \frac{h!^2}{(2h)!}. \tag{21}
\]

**Proof.** To simplify notations, and without loss of generality, assume that \( E = \{1, \ldots, 2h\} \) so that \( \{0, 1\}^E \) is the set of bitstrings with indices from 1 to \( 2h \), and \( F = \{1, \ldots, |F|\} \); \( p(x \mid y, r) = \sum_v p(xvv' \mid y, r) \) where \( v \) are all bitstrings with indices in \( \{2h+1, \ldots, |F|\} \) and \( v' \) are those with indices in \( \{|F| + 1, \ldots, N\} \); similarly \( p(x' \mid y, r) = \sum_v p(x'vv' \mid y, r) \); if we let \( i = xvv' \) and \( i' = x'vv' \) then \( xv = i_F, x'v = i'_F \) and \( |i_F| = |xv| = |x| + |v| = |x'| + |v| = |x'v| = |i'_F| \) and thus, by (19), \( p(i \mid y, r) = p(i' \mid y, r) \) and the two sums are equal. \( \square \)

2. **Combinatorial lemmas.**

Given a set \( E \) and \( k \leq |E| \), we denote \( \mathcal{P}(E, k) \) the set of permutations of \( k \) elements in \( E \), i.e. the set of strings \( q_1 \ldots q_k \) of \( k \) distinct elements in \( E \): \[ \mathcal{P}(E, k) = \frac{|E|!}{(|E| - k)!}. \tag{22} \]
From now on, $\epsilon$ such that $0 \leq \epsilon \leq 1$, $\epsilon < \delta$ will be fixed, as well as $h = \lfloor (1 + \epsilon)n/2 \rfloor$ and $E$, a set of $2h$ indices of SIFt bits that are not TEST bits. For $y \in I_{n,\epsilon}$ and $x \in E_h$ we let

$$Q(x, y) = \{q \in \mathcal{P}(E, n) \mid x_q = y\}.$$  

**Lemma 15.** For all $y \in I_{n,\epsilon}$ and $x \in E_h$ the number of elements $|Q(x, y)|$ of $Q(x, y)$ is

$$|Q(x, y)| = \frac{h!^2}{(h - n + |y|)! \times (h - |y|)!}$$ (23)

**Proof.** A string $y \in \{0, 1\}^n$ is in $I_{n,\epsilon}$ if and only if it contains at most $h$ zeros and $h$ ones. Let $E_0 = \{ j \in E \mid x_j = 0 \}$ and $E_1 = \{ j \in E \mid x_j = 1 \}$; $|E_0| = |E_1| = h$ and the permutations $q$ such that $x_q = y$ are in 1–1 correspondence with the elements of

$$\mathcal{P}(E_0, n - |y|) \times \mathcal{P}(E_1, |y|)$$

corresponding to the $n - |y|$ indices giving a 0 in $y$ and the $|y|$ indices giving a 1 in $y$. The result follows from (22). \qed

**Lemma 16.** For all $q \in \mathcal{P}(E, n)$ and $y \in I_{n,\epsilon}$

$$\{|x \in E_h \mid q \in Q(x, y)\} = \binom{2h - n}{h - |y|}$$ (24)

**Proof.** A string $x \in E_h$ is such that $q \in Q(x, y)$ if and only if it satisfies $x_q = y$; this means that $x_{q_1} = y_1, \ldots, x_{q_n} = y_n$ (bits indexed by $q$ are fixed), the other bits are arbitrary provided there is a total of $h$ bits equal to 0 and $h$ bits equal to 1; the desired strings are thus obtained by filling the $2h - n$ bit positions whose indices are not in the list $q$ with $h - |y|$ bits equal to 1 (and the others equal to 0); there are $\binom{2h - n}{h - |y|}$ such strings. \qed

Eq. (24) can be rewritten

$$\{|x \in E_h \mid x_q = y\} = \frac{(2h - n)!}{(h - n + |y|)! (h - |y|)!},$$ (25)

3. **Proof of robustness.**

We want to show that $q$ leaks no information on $y \in I_{n,\epsilon}$. For any fixed $x \in E_h$ and $y \in I_{n,\epsilon}$, the probability that Alice sends $q$ is $1/|Q(x, y)|$ if $q \in Q(x, y)$, 0 otherwise, independently of any value of $r$:

$$p(q \mid x, y, r) = \begin{cases} \frac{1}{|Q(x, y)|} & \text{if } x_q = y \\ 0 & \text{otherwise} \end{cases}$$ (26)

**Lemma 17.** For all values of $r$, all $y \in I_{n,\epsilon}$ and all $q \in \mathcal{P}(E, n)$

$$p(q \mid y, r) = \frac{(2h - n)!}{(2h)!}.$$ (27)

**Proof.**

$$p(q \mid y, r) = \sum_{x \in E_h} p(q \mid x, y, r)p(x \mid y, r)$$

$$= \sum_{x \in E_h} \frac{1}{|Q(x, y)|} \frac{h!^2}{(h - n + |y|)! (h - |y|)!}$$

$$= \sum_{x \in E_h} \frac{(h - n + |y|)! (h - |y|)!}{(2h)!}$$

$$= \frac{(2h - n)! (h - n + |y|)! (h - |y|)!}{(2h)!}$$

$$= \frac{(2h - n)!}{(2h)!}$$

where the second equality is due to (21) and (26), and the third and forth equalities are given by (23) and (25). \qed

**Theorem 18.** For all $\epsilon$ and $\delta$ such that $0 \leq \epsilon \leq 1$ and $\epsilon < \delta$, the protocol is completely robust, i.e. if Eve is undetectable by the legitimate parties, then $I(Y; Q, R) = 0$.

**Proof.** The parameters $n$ and $\epsilon$ are constants of the protocol; they are fixed before all random choices of Alice or Bob, and all measurements. So is the value $h = \lfloor (1 + \epsilon)n/2 \rfloor$. The right-hand side of Eq. (27) is thus a constant[28] and Lemma 17 implies that the random variables $Q$ and $(Y, R)$ are independent: $p(q, y, r) = p(q)p(y)p(r)$; the variables $Y$ and $R$ must also be independent, because Alice chooses $y$ randomly, independently of everything else: $p(y) = p(y)p(r)$. This implies that $p(q, y, r) = p(q)p(y)p(r)$, $Y$ is independent of $(Q, R)$, therefore $I(Y; Q, R) = 0$. \qed

VI. CONCLUSION

We presented two protocols for QKD with one party performing only classical operations: measure a qubit in the classical $\{0, 1\}$ basis, let the qubit pass undisturbed back to its sender, randomize the order of several qubits, or resend a qubit after its measurement. We proved the robustness of these protocols; this provides intuition why we believe they are secure. We hope that this work sheds light on “how much quantumness” is required in order to perform the classically-impossible task of secret key distribution. This work extends the previous work [17] and the conference version [18] by two aspects: it proves robustness of the measure-resend SQKD Protocol for a more general scenario and proves the full robustness of a randomization-based SQKD Protocol, eliminating any information leak to the adversary.
Note that in this work we assumed perfect qubits. We leave the examination of our protocol against PNS and other implementation-dependent attacks to future research. This work was partially supported by the Israeli MOD. We thank Moshe Nazarathy for providing the motivation for this research.

APPENDIX

Theorem 19 (Hoeffding). (Hoeffding [19]) If $X_1, \ldots, X_n$ are independent random variables with finite first and second moments, $P[a_i \leq X_i \leq b_i] = 1$ for $1 \leq i \leq n$, and

\[
\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i,
\]

then

\[
P\left[ \bar{X} - E[\bar{X}] \geq \kappa \right] \leq \exp \left( -\frac{2\kappa^2 n^2}{\sum_{i=1}^{n}(b_i - a_i)^2} \right)
\]

where $\exp(x) = e^x$. When $0 \leq X_i \leq 1$, this gives (by symmetry for (28) and summation for (29))

\[
P\left[ \bar{X} - E[\bar{X}] \geq \kappa \right] \leq \exp \left( -2\kappa^2 n \right)
\]

\[
P\left[ \bar{X} - E[\bar{X}] \leq -\kappa \right] \leq \exp \left( -2\kappa^2 n \right)
\]

(28)

\[
P\left[ |\bar{X} - E[\bar{X}]| \geq \kappa \right] \leq 2 \exp \left( -2\kappa^2 n \right).
\]

(29)

[1] M. Boyer, D. Kenigsberg, and T. Mor, Phys. Rev. Lett. 99, 140501 (2007).
[2] S. Braunstein, C. Caves, R. Jozsa, N. Linden, S. Popescu, and R. Schack, Phys. Rev. Lett. 83, 1054 (1999).
[3] R. Jozsa and N. Linden, Proc. of the Roy. Soc. of London series A 459, 2011 (2003).
[4] E. Biham, G. Brassard, D. Kenigsberg, and T. Mor, Theoretical computer science 320, 15 (2004).
[5] D. Kenigsberg, T. Mor, and G. Ratsaby, Quantum Information and Computation 6, 606 (2006).
[6] C. H. Bennett, D. P. DiVincenzo, C. A. Fuchs, T. Mor, E. Rains, P. W. Shor, J. A. Smolin, and W. K. Wootters, Phys. Rev. A 59, 1070 (1999).
[7] B. Groisman, S. Popescu, and A. Winter, Physical Review A 72, 032317 (2005).
[8] C. A. Fuchs and M. Sasaki, Quantum Information and Computation 3, 337 (2003).
[9] E. Biham, M. Boyer, P. O. Boykin, T. Mor, and V. Roychowdhury, Journal of Cryptology 19, 381 (2006).
[10] T. Mor, Phys. Rev. Lett. 80, 3137 (1998).
[11] D. Mayers, J. ACM 48, 351 (2001), also in quant-ph/9802025.
[12] P. W. Shor and J. Preskill, Phys. Rev. Lett. 85, 441 (2000).
[13] C. H. Bennett, G. Brassard, and N. D. Mermin, Phys. Rev. Lett. 68, 557 (1992).
[14] V. Scarani, A. Acín, G. Ribordy, and N. Gisin, Phys. Rev. Lett. 92, 057901 (2004).
[15] A. Acín, N. Gisin, and V. Scarani, Phys. Rev. A 69, 012309 (2004).
[16] P. Jacquet and W. Szpankowski, IEEE Trans. Inform. Theory 45, 1072 (1999).
[17] D. Kenigsberg, PhD in computer science, Technion - Israel Institute of Technology (2007).
[18] M. Boyer, D. Kenigsberg, and T. Mor, in ICQNM ’07: Proceedings of the First International Conference on Quantum, Nano, and Micro Technologies (IEEE Computer Society, Washington, DC, USA, 2007), p. 10, ISBN 0-7695-2759-0.
[19] W. Hoeffding, Journal of the American Statistical Association 58, 13 (1963).
[20] H[0] = |+>, H[1] = |−>.
[21] By the term “cNot from A into B” we mean that A is the control qubit and B is the target, as is commonly called; we prefer to use the term “control qubit” in a different meaning in our paper.
[22] Quantum memory is not strictly required, since instead of it (with a certain penalty to the protocol rate) Alice can measure each reflected qubit in a random basis.
[23] If |j⟩B is Bob’s register with j ∈ {0, 1}r, then Mm[|j⟩|y⟩B = |j⟩|m ⊕ j⟩B where ⊕ denotes a bitwise exclusive or.
[24] The transforms U(k)E and U(k)F act on $H_E \otimes \ldots \otimes H_k \otimes \ldots$ and leave $H_l$ fixed for $l \neq k$.
[25] If $n \geq 1$ then $N = [8n(1 + \delta)] \geq 8$ and $N - 2 \geq 1$.
[26] When $n$ is large, the binomial $B(n, p)$ is well approximated by a normal with variance $\sigma^2 = np(1 - p)$, whose entropy is $\log_2(\sigma) = \log_2(2\pi e)$, with a factor of $\frac{1}{\log_2(e)}$ this rewrites $\frac{1}{2} \log n + \frac{1}{2} + \log \sqrt{2\pi (1 - p)n}$ as in [16] where a proven result implying the error is of order $\frac{1}{n}$.
[27] For simplifying the notations, bits keep their indices even when they appear in substrings.
[28] From (22), we see that $P(q | y, r) = 1/|P(E, n)|$ which is the probability of a random n-permutation of |E| = 2h elements.