1. INTRODUCTION

Let \((X, d)\) be a metric space; and \(x \mapsto \varphi(x)\), a function from \(X\) to \(R_+ := [0, \infty]\). The following Ekeland’s variational principle [7] (in short: EVP) is our starting point. Assume that

(a01) \(d\) is complete: each \(d\)-Cauchy sequence is \(d\)-convergent
(b02) \(\varphi\) is \(d\)-lsc: \(\liminf_n \varphi(x_n) \geq \varphi(x)\), whenever \(x_n \xrightarrow{d} x\).

Theorem 1. Let these conditions hold; and \(u \in X\) be arbitrary fixed. There exists then \(v = v(u) \in X\) in such a way that

\[
\begin{align*}
    d(u, v) &\leq \varphi(u) - \varphi(v) \quad \text{(hence } \varphi(u) \geq \varphi(v)\text{)} \quad (1.1) \\
    d(v, x) &> \varphi(v) - \varphi(x), \quad \text{for all } x \in X \setminus \{v\}. \quad (1.2)
\end{align*}
\]

As a matter of fact, the original result is with \(\varphi : X \to R \cup \{\infty\}\) being proper (\(\text{Dom}(\varphi) \neq \emptyset\)) and bounded from below \((\inf[\varphi(X)] > -\infty)\). But, the author’s argument also works in this relaxed setting; just pass to the triplet \((X(u, \leq), d; \psi)\), where \(u \in \text{Dom}(\varphi)\), \(X(u, \leq) := \{x \in X; u \leq x\}\), \(\psi := \varphi - \inf[\varphi(X)]\), and \(\leq\) is the quasi-order (i.e.: reflexive and transitive relation) described as

(a03) \((x, y \in X) x \leq y \text{ iff } d(x, y) + \varphi(y) \leq \varphi(x)\).

This principle found some basic applications to control and optimization, critical point theory and global analysis; see the quoted paper for details. So, it cannot be surprising that many extensions of EVP were proposed. Here, we shall be interested in the structural generalizations; which amount to the ambient metrical structure being substituted by a uniform one. The basic result of this type is Hamel’s variational principle [1] (in short: HVP). By a pseudometric over \(X\) we shall mean any map \(d : X \times X \to R_+:\) if, in addition, \(d\) is reflexive \([d(x, x) = 0, \forall x \in X]\) and symmetric \([d(x, y) = d(y, x), \forall x, y \in X]\) we shall say that it is a rs-pseudometric. Let \((\Lambda, \leq)\) be some directed quasi-ordered structure. Take a family \(D = \{d_\lambda; \lambda \in \Lambda\}\) of rs-pseudometrics over \(X\); supposed to be sufficient \([d_\lambda(x, y) = 0, \forall \lambda \in \Lambda \text{ imply } x = y]\), \(\Lambda\)-monotone \([\lambda \leq \mu \text{ implies } d_\lambda(y, x) \leq d_\mu(y, x)]\) and \(\Lambda\)-triangular \([\forall \lambda \in \Lambda,\]

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∃µ ∈ Λ(λ, ≤), with \(d_λ(x, z) ≤ d_µ(x, y) + d_µ(y, z), \forall x, y, z ∈ X\]; the couple \((X, D)\) will be then termed a Fang uniform space. Suppose that

(a04) \(D\) is sequentially complete;

and let \(φ : X → R_+\) be some function with

(a05) \(φ\) is sequentially descending \(D\)-lsc.

We refer to Section 3 for all unexplained notions]. Finally, take some increasing map \(λ ↦→ h(λ)\) from \(Λ\) to \(R_+:=[0, ∞]\).

Theorem 2. Let the Fang uniform space \((X, D)\) and the function \(φ\) be as above. Then, for each \(u ∈ X\), there exists \(v = v(u) ∈ X\) with

\[h(λ)d_λ(u, v) ≤ φ(u) - φ(v), \forall λ ∈ Λ \text{ (hence } φ(u) ≥ φ(v))\]

(1.3)

\[∀x ∈ X \setminus \{v\}, ∃µ = µ(x) ∈ Λ : h(µ)d_µ(v, x) > φ(v) - φ(x).\]

(1.4)

This result – based on a maximal principle comparable with Brøndsted’s \[4\] – extends the related statement in Fang \[8\], obtained via Zorn maximal techniques. It also includes the contribution due to Hadžić and Žikić \[10\] (in short: HZVP), founded on the maximal principle in Hicks \[13\]. Actually, the argument in HZVP may be easily adapted to (a04)+(a05), so as to get HVP; we do not give details.

Note that HVP \(⇒\) EVP, if \(Λ\) is a singleton; so, it is natural asking whether the reciprocal inclusion is also retainable. As we shall see (along Section 3), a positive answer is available – under a direct approach – for the ”descending” versions of these principles. The complete inclusion chain is established in Section 4: both HVP and EVP are equivalent with the Dependent Choices Principle in Tarski \[21\] (cf. Section 2); hence, mutually equivalent. Further aspects will be discussed elsewhere.

2. Brezis-Browder principles

Let \(M\) be a nonempty set; and \(R ⊆ M × M\) stand for a (nonempty) relation over \(M\). For each \(x ∈ M\), denote \(M(x, R) = \{y ∈ M : xRy\}\). The following ”Dependent Choices Principle” (in short: DC) is in effect for us:

Proposition 1. Suppose that

(b01) \(M(c, R)\) is nonempty, for all \(c ∈ M\).

Then, for each \(a ∈ M\) there exists \((x_n) ⊆ M\) with \(x_0 = a\) and \(x_nRx_{n+1}, \forall n\).

This principle, due to Tarski \[21\], is deductible from AC (= the Axiom of Choice), but not conversely; cf. Wolk \[25\]. Moreover, it seems to suffice for a large part of the ”usual” mathematics; see Moore \[19\], Appendix 2, Table 4).

(A) Let \(M\) be some nonempty set. Take a quasi-order \((≤)\) over \(M\); as well as a function \(φ : M → R_+\). Call the point \(z ∈ M\), \((≤, φ)\)-maximal when: \(φ\) is constant on \(M(z, ≤)\). A basic result about the existence of such points is the Brezis-Browder ordering principle \[3\] (in short: BB).

Proposition 2. Suppose that

(b02) \((M, ≤)\) is sequentially inductive:

\(\text{each ascending sequence has an upper bound (modulo } (≤))\)

(b03) \(φ\) is \((≤)-decreasing (x ≤ y ⇒ φ(x) ≥ φ(y)).\)

Then, for each \(u ∈ M\) there exists a \((≤, φ)\)-maximal \(v ∈ M\) with \(u ≤ v\).
In this case, we shall mean any part of \( H_x \) which will be denoted \( M \) on \( (M, S) \). This will necessitate some conventions and auxiliary facts. Let one, to limit as

\[
\lambda \geq \beta
\]

properties of \( \lambda \) is (strictly) descending and bounded below; hence

\[
\beta(v) \geq \beta(v), \forall u
\]

relations yields (via (2.2))

\[
(1/2)(\varphi(v) + \beta(v)) > \varphi(w),
\]

for at least one \( w \) (belonging to \( M_u \)). The relation \( R \) over \( M_u \) introduced via (2.1) fulfills \( M_u(v, R) \neq \emptyset \), for all \( v \in M_u \). So, by (DC), there must be a sequence \( (u_n) \) in \( M_u \) with \( u_0 = u \) and

\[
u_n \leq u_{n+1}, (1/2)(\varphi(u_n) + \beta(u_n)) > \varphi(u_{n+1}), \text{ for all } n.
\]

We have thus constructed an ascending sequence \( (u_n) \) in \( M_u \) for which \( (\varphi(u_n)) \) is (strictly) descending and bounded below; hence \( \lambda := \lim_{n} \varphi(u_n) \) exists in \( R_+ \). Moreover, from (b02), \( (u_n) \) is bounded above in \( M \); there exists \( v \in M \) such that

\[
u_n \leq v, \text{ for all } n.
\]

Combining with (b03), gives \( \varphi(u_n) \geq \varphi(v), \forall n \); and (by the properties of \( \beta \)) \( \varphi(v) \geq \beta(v), \forall n \). The former of these relations gives

\[
\lambda \geq \varphi(v) \text{ (passing to limit as } n \rightarrow \infty). \]

On the other hand, the latter of these relations yields (via (2.1))

\[
(1/2)(\varphi(u_n) + \beta(v)) \geq \varphi(u_{n+1}), \text{ for all } n \in N.
\]

Passing to limit as \( n \rightarrow \infty \) yields \( (\varphi(v)) \geq \beta(v) \geq \lambda \); so, combining with the preceding one, \( \varphi(v) = \beta(v)(= \lambda) \), contradiction. Hence, our working assumption cannot be accepted; and the conclusion follows.

\( \square \)

(B) This principle, including (EVP) (see below) found some useful applications to convex and nonconvex analysis (cf. the above references). For this reason, it was the subject of many extensions; see, e.g., Kang and Park [15]. The obtained results are interesting from a technical viewpoint. However, we must emphasize that, whenever a maximality principle of this type is to be applied, a substitution of it by the Brezis-Browder’s (BB) is always possible. This raises the question of to what extent are these enlargements of Proposition effective. As it will be shown in Section 4, the answer is negative. For the moment, we note that, a way of obtaining structural extensions from BB is by "splitting" the key condition (b02) as

\[
(b04) \quad (\forall (x_n) \subseteq M) \text{ ascending } \Rightarrow \text{ Cauchy } \Rightarrow \text{ convergent } \Rightarrow \text{ bounded above.}
\]

This will necessitate some conventions and auxiliary facts. Let \( S(M, \leq) \) stand for the class of all ascending sequences in \( M \). By a (sequential) convergence structure on \( (M, \leq) \) we mean, as in Kasahara [16], any part \( C \) of \( S(M, \leq) \times M \) with

\[
(b05) \quad x_n = x, \forall n \in N \rightarrow ((x_n); x) \in C
\]

\[
(b06) \quad ((x_n); x) \in C \rightarrow (y_n); x) \in C, \text{ for each subsequence } (y_n) \text{ of } (x_n).
\]

In this case, \( ((x_n); x) \in C \) writes \( x_n \xrightarrow{C} x; \) and reads: \( x \) is the \( C \)-limit of \( (x_n) \). When such elements exist, we say that \( (x_n) \) is \( C \)-convergent; the class of all these will be denoted \( S_c(M, \leq) \). Further, by a (sequential) Cauchy structure on \( (M, \leq) \) we shall mean any part \( \mathcal{H} \) of \( S(M, \leq) \) with

\[
(b07) \quad x_n = x, \forall n \in N \rightarrow (x_n) \in \mathcal{H}
\]

\[
(b08) \quad (x_n) \in \mathcal{H} \rightarrow (y_n) \in \mathcal{H}, \text{ for each subsequence } (y_n) \text{ of } (x_n).
\]
Each element of $\mathcal{H}$ will be referred to as a $\mathcal{H}$-Cauchy sequence. [For example, a good choice is $\mathcal{H} = \mathcal{S}_c(M, \leq)$; but this is not the only possible one]. Suppose that we introduced such a couple $(\mathcal{C}, \mathcal{H})$; referred to as a convex-Cauchy structure. Roughly speaking, the objective to be attained is a realization of (b04). To this end, the following conditions will be considered

(b09) $\mathcal{H}$ is $(\leq)$-regular: each ascending sequence in $M$ is $\mathcal{H}$-Cauchy

(b10) $(\mathcal{C}, \mathcal{H})$ is (sequentially) $(\leq)$-complete:

- each ascending $\mathcal{H}$-Cauchy sequence in $M$ is $\mathcal{C}$-convergent

(b11) $(\leq)$ is $\mathcal{C}$-selfclosed: the $\mathcal{C}$-limit of each ascending $\mathcal{C}$-convergent sequence in $M$ is an upper bound of it.

The following structural version of Proposition 2 is then available. Let again $(M, \leq)$ be a quasi-ordered structure; and $\varphi : M \rightarrow \mathbb{R}$ be some $(\leq)$-decreasing function.

**Proposition 3.** Suppose that the convex-Cauchy structure $(\mathcal{C}, \mathcal{H})$ is such that (b09)-(b11) hold. Then, conclusion in BB is retainable.

The proof is immediate (via Proposition 2); just note that (b09)-(b11) imply (b02). Hence, Proposition 3 is deductible from the quoted result. But, the recipro-
cal deduction is also possible. To verify this, it will suffice to taking the (sequential) convergence structure over $(M, \leq)$ as the bounded from above property $B$ [introduced as: $x_n \xrightarrow{B} x$ if $x_n \leq x$, for all $n$]; and the (sequential) Cauchy structure over $(M, \leq)$ be identical with $\mathcal{S}_B(M, \leq)$; we do not give details.

(C) A basic application of these facts is to be done in the pseudo-uniform setting. Let $(M, \leq)$ be a quasi-ordered structure. Denote $\mathcal{I}(M) = \{(x, x) : x \in M\}$ (the diagonal of $M$); and let $\mathcal{V}$ be a family of parts in $M \times M$. Under a convention similar to that in Nachbin [20, Ch 2, Sect 2], we say that $\mathcal{V}$ is a pseudo-uniformity over it when $\cap \mathcal{V} \supseteq \mathcal{I}(M)$. Suppose that we introduced such an object. The associated (sequential) convergence structure $(\mathcal{V})$ on $(M, \leq)$ may be described as

$$x_n \xrightarrow{\mathcal{V}} x \text{ if } \forall V \in \mathcal{V}, \exists n(V) : n \geq n(V) \implies (x_n, x) \in V.$$ 

It will be referred to as: $x$ is the $\mathcal{V}$-limit of $(x_n)$; if such elements exist, we shall say that $(x_n)$ is $\mathcal{V}$-convergent. In addition, we may introduce the $\mathcal{V}$-Cauchy property for an ascending sequence $(x_n)$ as

$$\forall V \in \mathcal{V}, \exists n(V) : n(V) \leq p \leq q \implies (x_p, x_q) \in V,$$

the class of all these will be denoted as $\text{Cauchy}(\mathcal{V})$. Now, in this context, further interpretations of the regularity conditions above are possible. Call the (ascending) sequence $(x_n)$, $\mathcal{V}$-asymptotic provided $\forall V \in \mathcal{V}, \exists n(V) : n \geq n(V) \implies (x_n, x_{n+1}) \in V$. Clearly, each $\mathcal{V}$-Cauchy (ascending) sequence is $\mathcal{V}$-asymptotic too. The converse is also true, if all such sequences are involved; so that, the global conditions below are equivalent

(b12) each ascending sequence in $M$ is $\mathcal{V}$-Cauchy

(b13) each ascending sequence in $M$ is $\mathcal{V}$-asymptotic.

By definition, either of these will be referred to as $\mathcal{V}$ is $(\leq)$-regular; this is just (b09), relative to $\mathcal{H}$=Cauchy($\mathcal{V}$). Further, call the ambient structure $\mathcal{V}$, (sequentially) $(\leq)$-complete when each ascending $\mathcal{V}$-Cauchy sequence in $M$ is $\mathcal{V}$-convergent. As before, this is nothing but the condition (b10), relative to $\mathcal{C} = (\mathcal{V})$ and $\mathcal{H}$=Cauchy($\mathcal{V}$). Finally, let us say that $(\leq)$ is $\mathcal{V}$-selfclosed when the $\mathcal{V}$-limit of each ascending $\mathcal{V}$-convergent sequence in $M$ is an upper bound of it; i.e., (b11) holds relative to
\( C = (V) \). The following "uniform" type version of Proposition 3 is then available. [The general conditions about \((M, \leq)\) and \(\varphi\) prevail].

**Proposition 4.** Assume that \( V \) is \((\leq)\)-regular, \((\text{sequentially}) (\leq)\)-complete and \((\leq)\) is \( V \)-selfclosed. Then, conclusions of BB are retainable.

As a consequence of this, Proposition 4 is deductible from the quoted result; hence, a fortiori, from BB. Moreover, the reciprocal deduction is also possible. In fact, let the premises of BB hold; and put \( V = \{\text{gr}(\leq)\} \), where \( \text{gr}(\leq) := \{(x, y) \in M \times M; x \leq y\} \). It is easy to see that all conditions in Proposition 4 are fulfilled; hence the claim. Summing up, these ordering principles are mutually equivalent; and as such, equivalent with BB.

The discussed particular case is an "extremal" one. To get "standard" examples in the area, we need further conventions. Let \( V \) be some family of parts in \( M \times M \); we call it, a fundamental system of entourages for a uniformity over \( M \), when (cf. Bourbaki [2, Ch 2, Sect 1])

\[
\begin{align*}
(b14) \quad (V, \supseteq) & \text{ is directed and } \cap V \supseteq \mathcal{I}(M) \\
(b15) \forall V \in V, \exists W \in V : W \subseteq V^{-1}, W \circ W \subseteq V.
\end{align*}
\]

The uniformity in question is just \( U = \{P \subseteq M \times M; P \supseteq Q, \text{ for some } Q \in V\} \). As a rule, the "uniform" terminology refers to it. However (as results directly from (b14)+(b15)), all \( U \)-notions are in fact \( V \)-notions. For example, the (sequential) convergence structure \( (\mathcal{U}) \) on \( (M, \leq) \) (introduced as before) is nothing else than \( (V) \). Likewise, the (attached to \( \mathcal{U} \) Cauchy and asymptotic properties are identical with the (corresponding) ones related to \( V \).

The following "standard" version of Proposition 4 holds. (As before, \((M, \leq)\) is a quasi-ordered structure; and \( \varphi : M \to \mathbb{R}_+ \) is \((\leq)\)-decreasing).

**Proposition 5.** Assume \( V \) is (sequentially) \((\leq)\)-complete, \((\leq)\) is \( V \)-selfclosed and \( (V, \supseteq) \) is \((\leq, \varphi)\)-compatible \( \forall V \in V, \exists \delta = \delta(V) > 0: \)

\[
x, y \in M, x \leq y, \varphi(x) - \varphi(y) < \delta \implies (x, y) \in V.
\]

Then, for each \( u \in M \) there exists a \((\leq, \varphi)\)-maximal \( v \in M \) with \( u \leq v \).

**Proof.** It will suffice proving that \( V \) is \((\leq)\)-regular (see above). Let \((x_n)\) be an ascending (modulo \( \leq \)) sequence in \( M \). The sequence \((\varphi(x_n))\) is descending and bounded from below; hence a Cauchy one (for each \( \delta > 0 \), there exists \( n(\delta) \), such that: \( n(\delta) \leq p \leq q \implies \varphi(x_p) - \varphi(x_q) < \delta \)). This, along with (b16), gives us the conclusion we want. \( \square \)

Note that, a direct consequence of (b16) is (via (b14)+(b15))

\[
\text{if } x <> y \text{ and } \varphi(x) = \varphi(y) \text{ then } (x, y) \in \cap V. \tag{2.3}
\]

(Here, \( x <> y \) means: either \( x \leq y \text{ or } y \leq x \). This gives (under (b16))

\[
(\forall z \in M) \text{ \((\leq, \varphi)\)-maximal } \implies \text{ \((\leq, V)\)-maximal}; \tag{2.4}
\]

where the last property means: \( w \in M \) and \( z \leq w \) imply \((z, w) \in \cap V \). So, Proposition 3 is, at the same time, an existence principle for \((\leq, V)\)-maximal elements; and, as such, it may be compared with a related statement of Turinici [23]. In particular, when \( V \) is separated \((\cap V = \mathcal{I}(M))\), we have (again via (b16))

\[
(\forall z \in M) : \text{ \((\leq, V)\)-maximal } \iff \text{ \((\leq)\)-maximal } (M(z, \leq) = \{z\}); \tag{2.5}
\]
and Proposition 3 yields the maximal principle in Hamel [11] Theorem 1. On the other hand, (2.3) (and the separated property of \( \mathcal{V} \)) also gives (via (b03))

\[
(\forall x, y \in M): x \leq y, y \leq x \implies x = y;
\]  

(2.6)

wherefrom, \( (\leq) \) is *antisymmetric* (hence an order) on \( X \). Summing up, the "separated" variant of Proposition 3 is identical with the main result in Brøndsted [4, Theorem 1]; for this reason, it will be referred to as the Brøndsted Maximal Principle (in short: BMP). In addition, the developments above tell us that BMP is deductible from BB. The reciprocal inclusion is also possible; cf. Section 4. Further aspects may be found in Mizoguchi [18].

### 3. Metrical reduction

Under these preliminaries, we may now return to the question of the introductory part. Let \( X \) be a nonempty set; and \((\Lambda, \leq)\), a directed quasi-ordered structure. Further, take a family \( E = (e_\lambda; \lambda \in \Lambda) \) of \( \rho \)-pseudometrics over \( X \); which union addition is sufficient, \( \Lambda \)-monotone and \( \Lambda \)-triangular (cf. Section 1). Its associated family of relations \( \mathcal{V} = \{U(\lambda, r); \lambda \in \Lambda, r > 0\} \), where

\[
(c01) \quad U(\lambda, r) = \{(x, y) \in X \times X; e_\lambda(x, y) < r\}, \quad \lambda \in \Lambda, r > 0,
\]

is a fundamental system of entourages for a uniform structure \( \mathcal{U} = \mathcal{U}(\Lambda, E) \) over \( X \) (cf. Bourbaki [2, Ch 2, Sect 1]). This structure, introduced in 1996 by Fang [8], became a very useful instrument in the probabilistic metric spaces theory. As a rule, the "uniform" terminology refers to it. However (as results directly by definition), all \( \mathcal{U} \)-notions are in fact \( \mathcal{V} \)-notions; so, we shall work with \( \mathcal{V} \) in place of \( \mathcal{U} \). For simplicity reasons, we shall denote \((X, \mathcal{V})\) as \((X, E)\); and call it, a Fang uniform space. Let \( \mathcal{C} = (\mathcal{V}) \) and \( \mathcal{H} = \text{Cauchy}(\mathcal{V}) \) stand for the, respectively, (sequential) convergence and Cauchy structure attached to the quasi-uniformity \( \mathcal{V} \) (cf. Section 2). All notions related to these may now be translated in terms of \( E \). For example, the convergence relation \( x_n \xrightarrow{\mathcal{V}} x \) means: \( x_n \xrightarrow{e_\lambda} x \), for each \( \lambda \in \Lambda \). It will be also written as: \( x_n \xrightarrow{E^\lambda} x \); and reads: \( x \) is an \( E \)-limit of \( (x_n) \); when such elements exist, we say that \( (x_n) \) is \( E \)-convergent. On the other hand, the \( \mathcal{V} \)-Cauchy property of \( (x_n) \) may be written as: \( (x_n) \) is \( e_\lambda \)-Cauchy, for each \( \lambda \in \Lambda \); and will be referred to as: \( (x_n) \) is \( E \)-Cauchy. Clearly, each \( E \)-convergent sequence is \( E \)-Cauchy; the reciprocal is not in general valid.

Let \((X, E)\) be a Fang uniform space; and \( \varphi : X \to R_+ \) be a function. For convenience, we list our basic hypotheses:

\[
(c02) \quad E \text{ is sequentially } \varphi \text{-complete:}
\]

- each \( E \)-Cauchy sequence with \((\varphi(x_n))\) descending is \( E \)-convergent

\[
(c03) \quad \varphi \text{ is sequentially descending } E \text{-lsc:}
\]

\[
\lim_n \varphi(x_n) \geq \varphi(x), \text{ whenever } x_n \xrightarrow{E} x \text{ and } (\varphi(x_n)) \text{ is descending.}
\]

The following variational principle is our starting point.

**Theorem 3.** Let these general conditions be admitted; as well as \((c02)+(c03)\). Then, for each \( u \in X \) there exists \( v = v(u) \in X \) so that

\[
e_\lambda(u, v) \leq \varphi(u) - \varphi(v), \forall \lambda \in \Lambda \text{ (hence } \varphi(u) \geq \varphi(v))
\]  

\[
(3.1)
\]

\[
\forall x \in X \setminus \{v\}, \exists \mu = \mu(x) \in \Lambda : e_\mu(v, x) > \varphi(v) - \varphi(x).
\]  

(3.2)
In particular, when \( \Lambda \) (hence \( E \) as well) is a singleton, Theorem \( \ref{thm:variational} \) yields the following metric variational statement. Let \((X,d)\) be a metric space; and \( \varphi : X \to \mathbb{R}^+ \) be some function.

**Theorem 4.** Suppose that our data fulfill

(c04) \( d \) is \( \varphi \)-complete:

each \( d \)-Cauchy sequence with \((\varphi(x_n))\) descending is \( d \)-convergent

(c05) \( \varphi \) is descending \( d \)-lsc:

\[
\lim_n \varphi(x_n) \geq \varphi(x), \text{ whenever } x_n \xrightarrow{d} x \text{ and } (\varphi(x_n)) \text{ is descending.}
\]

Then, conclusions of EVP are retainable.

Now, the remarkable fact to be underlined may be expressed as: Theorem \( \ref{thm:variational} \) is deductible from (hence, equivalent with) this last statement:

Theorem \( \ref{thm:variational} \) = \( \Rightarrow \) Theorem \( \ref{thm:variational} \) [hence Theorem \( \ref{thm:variational} \) \( \iff \) Theorem \( \ref{thm:variational} \)]. (3.3)

The verification will necessitate a number of steps.

1) Denote for simplicity

(c06) \( \Delta(x, y) = \sup\{e_{\lambda}(x, y) ; \lambda \in \Lambda\}, \ x, y \in X \).

Since all members of \( E \) are rs-pseudometrics, \( \Delta \) is also endowed with such properties. Moreover (as \( E \) is \( \Lambda \)-triangular), \( \Delta \) is triangular \[ \Delta(x, z) \leq \Delta(x, y) + \Delta(y, z), \forall x, y, z \in X \]. Finally, \( \Delta \) is sufficient \[ \Delta(x, y) = 0 \implies x = y \]; because so is \( E \) (in the precise sense). Summing up, \( \Delta \) is a generalized metric on \( X \) in the Luxemburg-Jung sense \[ \ref{ref:luxemburg-jung} \], \[ \ref{ref:jung} \]. It allows us introducing the associated uniform structure \( U = U(\Delta) \); i.e., the one for which \( V = \{U(\varepsilon) ; \varepsilon > 0\} \), where

(c07) \( U(\varepsilon) = \{(x, y) \in M \times M ; \Delta(x, y) < \varepsilon\}, \varepsilon > 0 \),

is a fundamental system of entourages. This, in turn gives us the associated convergence \( (\Delta \rightarrow) \) and Cauchy structure, \( \text{Cauchy}(\Delta) \). The natural question to be posed is that of clarifying the relationships between these and the ones attached to the family \( E = (e_{\lambda} ; \lambda \in \Lambda) \). First, by these conventions, we have

**Lemma 1.** The generic local inclusions hold:

\[
(\forall (x_n), \forall x) : [x_n \xrightarrow{\Delta} x] \implies [x_n \xrightarrow{E} x].
\]

(for each sequence): \( \Delta \)-Cauchy \( \implies \) \( E \)-Cauchy. (3.4)

Generally, the reciprocal inclusions are not true; because the uniform structure attached to \( E \) is strictly finer than the one induced by the generalized metric \( \Delta \). The following completion of these facts is available.

**Lemma 2.** Under these notations,

\[
(\forall (x_n), \forall x) [(x_n) \text{ is } \Delta \text{-Cauchy}] \text{ and } [x_n \xrightarrow{E} x] \text{ imply } [x_n \xrightarrow{\Delta} x]
\]

\( E \) is sequentially \( \varphi \)-complete \( \implies \Delta \) is \( \varphi \)-complete. (3.6)

**Proof.** (Lemma \( \ref{lem:uniform} \)) The second part in the statement follows (via Lemma \( \ref{lem:sequential} \)) from the first part of the same; so, it will suffice proving that (3.6) holds. Let \((x_n)\) be a \( \Delta \)-Cauchy sequence in \( X \), so as (for some \( x \in X \))

\[
x_n \xrightarrow{E} x \text{ (hence } e_{\lambda}(x_n, x) \to 0, \text{ for each } \lambda \in \Lambda)\).
\]

By definition, for each \( \beta > 0 \) there exists some rank \( n(\beta) \) in such a way that \( \Delta(x_i, x_j) \leq \beta \) (hence \( e_{\lambda}(x_i, x_j) \leq \beta, \forall \lambda \in \Lambda \), whenever \( n(\beta) \leq i \leq j \). Let the rank
$i \geq n(\beta)$ be arbitrary fixed; and, for each $\lambda \in \Lambda$, let $\mu \in \Lambda(\lambda, \leq)$ be the index given by the $\Lambda$-triangular property of $E$. We have, for all such $(\lambda, \mu)$,
\[ e_\lambda(x_i, x) \leq e_\mu(x_i, x) + e_\mu(x_j, x) \leq \beta + e_\mu(x_j, x), ~ \forall j \geq i. \]
Passing to limit upon $j$ gives (for all $i$ like before)
\[ e_\lambda(x_i, x) \leq \beta, ~ \forall \lambda \in \Lambda \text{ (hence } \Delta(x_i, x) \leq \beta). \]
This, by the arbitrariness of $\beta$, yields $x_n \xrightarrow{\Delta} x$; as claimed.\hfill\Box

**Proof.** (relation (3.3)) Let $\Delta$ stand for the generalized metric over $X$ introduced via (c06); and put $X_u = \{x \in X; \Delta(u, x) \leq \varphi(u) - \varphi(x)\}$ (where $u$ is the point in the statement). For the moment, $\Delta$ is a standard metric over $X_u$. On the other hand, the imposed conditions assure us (via Lemma 1) that (c04)+(c05) hold (over $X_u$) with $\Delta$ in place of $d$. Summing up, Theorem 3 applies to $(X_u, \Delta)$ and $\varphi$. It gives us, for the starting point $u \in X_u$ some other point $v = v(u) \in X_u$ with the properties (1.1)+(1.2) (relative to $X_u$ and $\Delta$). This yields the conclusions (3.1)+(3.2) we want; and completes the argument.\hfill\Box

It remains now to discuss the relationships between Theorem 3 and HVP. So, let $(\Lambda, \leq)$ be as before; and $D = (d_\lambda; \lambda \in \Lambda)$ be a family of rs-pseudometrics over $X$; supposed to be sufficient, $\Lambda$-monotone and $\Lambda$-triangular, (cf. Section 1). Further, take an increasing map $\lambda \mapsto h(\lambda)$ from $\Lambda$ to $R^0_+$. Define another family $E = (e_\lambda; \lambda \in \Lambda)$ of rs-pseudometrics over $X$ according to
\[ (c08) \quad e_\lambda(x, y) = h(\lambda)d_\lambda(x, y), \quad x, y \in X. \]

**Lemma 3.** The family $E$ is sufficient, $\Lambda$-monotone and $\Lambda$-triangular. In addition, we have, for each $\lambda \in \Lambda$,
\[ (\forall(x_n)) \ D_\lambda \text{-Cauchy} \iff e_\lambda \text{-Cauchy} \quad (3.8) \]
\[ (\forall(x_n), \forall x) [x_n \xrightarrow{d_\lambda} x] \iff [x_n \xrightarrow{e_\lambda} x]. \quad (3.9) \]
Hence, all (sequential) $D$-concepts are equivalent with their attached $E$-concepts.

**Proof.** The second part of the statement is clear, by the very definition (c08); so, it remains to establish its first half. The sufficiency of $E$ results at once from the one of $D$; and the $\Lambda$-monotonicity of the same is directly reducible to the one of $\lambda \mapsto h(\lambda)$. Finally, let $\lambda \in \Lambda$ be arbitrary fixed. By the $\Lambda$-triangular property of $D$, there exists $\mu \in \Lambda(\lambda, \leq)$ in such a way that the precise condition be retainable. This, again by the increasing property of $\lambda \mapsto h(\lambda)$, yields
\[ e_\lambda(x, y) = h(\lambda)d_\lambda(x, y) \leq h(\lambda)[d_\mu(x, y) + d_\mu(y, z)] \leq e_\mu(x, y) + e_\mu(y, z), \quad \forall x, y, z \in X; \]
i.e., the $\Lambda$-monotonicity (relative to $E$) holds too.\hfill\Box

As a direct consequence of the last remark, (a04) $\implies$ (c02), (a05) $\iff$ (c03); wherefrom, HVP is deductible from Theorem 3.

The following completion of these facts is to be noted. Let $I$ be some nonempty set. Take a family $F = (f_i; i \in I)$ of rs-pseudometrics over $X$; supposed to be sufficient $[f_i(x, y) = 0, \forall i \in I \text{ imply } x = y]$, and $I$-triangular $[\text{for each } i \in I, \text{ there exist } j = j(i) \text{ and } k = k(i) \text{ in } I \text{ such that } f_j(x, z) \leq f_j(x, y) + f_k(y, z), \forall x, y, z \in X]$. In this case, the couple $(X, F)$ will be termed a BMLO uniform space. This concept, due to Benbrik, Mbarki, Lahrech and Ouahab, includes the
Theorem 3. Remember that, the associated family of relations

Proof. Let the Fang uniform space \((X, E)\) be a BMLO uniform space as well. But, the reciprocal inclusion is also true. In fact, let \(\Lambda\) stand for the class of all (nonempty) finite parts of \(I\), endowed with the usual inclusion, \((\subseteq)\); clearly, \((\Lambda, \subseteq)\) is a directed ordered structure. For each \(\lambda \in \Lambda\) define the \(\lambda\)-pseudometric \(d_\lambda\) over \(X\) as: \(d_\lambda(x, y) = \sup\{f_i(x, y); i \in \lambda\}, x, y \in X\). The family \(D = (d_\lambda; \lambda \in \Lambda)\) of all these is easily shown to be sufficient, \(\Lambda\)-monotone and \(\Lambda\)-triangular; i.e., \((X; D)\) is a Fang uniform space. In addition, all usual \(F\)-concepts (like \(F\)-convergence and \(F\)-Cauchy) are equivalent with their corresponding \(D\)-concepts. Hence, all variational results over BMLO uniform spaces established by these authors are completely reducible to the ones involving Fang uniform spaces we just presented; see also Hamel and Loehne [12]. In particular, this is retainable for the variational principles in standard uniform spaces due to Mizoguchi [13]; because any such structure is a BMLO uniform space. Finally, remember that the Fang uniform spaces are the natural model for the Menger probabilistic metric spaces (see Fang’s paper [8] for details) as well as (cf. Hadžić and Žikić [10]) for the fuzzy metric spaces. As a consequence, the obtained results incorporate the variational principle on Menger probabilistic metric spaces due to Hadžić and Ovcin [9].

4. Complete inclusion chain

By the developments above, we have the inclusions: \((\text{DC}) \implies (\text{BB}) \implies (\text{BMP})\) and \([\text{Theorem 1} \iff \text{Theorem 2}]\) \(\implies (\text{HVP}) \implies (\text{EVP})\). So, we may ask of to what extent is Theorem 3 or, equivalently, Theorem 4 deductible from (BMP). Moreover – when a positive answer to this is available – the natural question to be addressed is that of the obtained inclusion chain being reversible. Clearly, the natural setting for solving these problems is \((\text{ZF})(=\text{the standard Zermelo-Fraenkel system})\) without \((\text{AC})(=\text{the Axiom of Choice})\); referred to in the following as the reduced Zermelo-Fraenkel system.

(A) Concerning the former of these, the following (positive) answer is available.

Proposition 6. We have (in the reduced Zermelo-Fraenkel system) \((\text{BMP}) \implies \text{Theorem 3} \implies (\text{HVP})\); hence (see above) \((\text{BMP}) \implies \text{Theorem 4} \implies (\text{EVP})\).

Proof. Let the Fang uniform space \((X, E)\) and the function \(\varphi : X \to R^+\) be as in Theorem 3. Remember that, the associated family of relations \(V = \{U(\lambda, r); \lambda \in \Lambda, r > 0\}\) given by \((c01)\), is a fundamental system of entourages for a uniform structure \(U = U(\Lambda, E)\) over \(X\). Further, let the ordering \((\leq)\) over \(X\) be defined as

\[
(x, y \in X) : x \leq y \iff e_\lambda(x, y) \leq \varphi(x) - \varphi(y), \forall \lambda \in \Lambda.
\]

We claim that Proposition 3 is applicable to these data. First, \(V\) is \((\leq, \varphi)\)-admissible, via

\[
(\forall \delta > 0) : x \leq y, \varphi(x) - \varphi(y) < \delta \implies (x, y) \in \cap\{U(\lambda, \delta); \lambda \in \Lambda\}. \tag{4.1}
\]

It remains to verify that \((\leq)\) is \(V\)-selfclosed. Let \((x_n)\) in \(X\) be ascending; i.e.,

\[
(d02) \quad e_\lambda(x_n, x_m) \leq \varphi(x_n) - \varphi(x_m), \forall \lambda \in \Lambda, \text{ whenever } n \leq m;
\]

with, in addition, \(x_n \xrightarrow{E} x\). Note that, in such a case, \((\varphi(x_n))\) is descending; so that \(\varphi(x_n) \geq \varphi(x), \forall n\), if one takes \((c03)\) into account. Given \(\lambda \in \Lambda\), take \(\mu \in \Lambda(\lambda, \leq)\) according to the \(\Lambda\)-triangular property of \(E\); and let \(n\) be arbitrary fixed. For each \(m \geq n\) we have, by \((d02)\) and the remark above

\[
e_\lambda(x_n, x) \leq e_\mu(x_n, x_m) + e_\mu(x_m, x) \leq \varphi(x_n) - \varphi(x) + e_\mu(x_m, x).
\]
Passing to limit as $m \to \infty$ we get $e_{\lambda}(x_n, x) \leq \varphi(x_n) - \varphi(x), \ \forall \lambda \in \Lambda$ [i.e.: $x_n \leq x$], for all $n$; hence the claim.

\begin{remark}
As a consequence, Theorem 3 is deductible from BB. A direct proof of this may be obtained as follows. Let the conditions of Theorem 3 hold; and $(\leq)$ be the ordering (d01) on $X$. We have to verify that BB is applicable to $(X, \leq; \varphi)$; precisely, that (b02) holds for these data. Let $(x_n)$ be an ascending sequence in $X$ (according to (d02)). The sequence $(\varphi(x_n))$ is descending and bounded from below; hence a Cauchy one $[\forall \delta > 0, \exists n(\delta) : n(\delta) \leq p \leq q \implies \varphi(x_p) - \varphi(x_q) \leq \delta]$. This, along with (d03), tells us that $(x_n)$ is $\nu$-Cauchy; or, equivalently, $E$-Cauchy. Taking (c02) into account, it follows that $x_n \overset{E}{\to} x$ as $n \to \infty$, for some $x \in X$. Combining with the reasoning above gives $x_n \leq x$, for all $n$; hence the claim. From BB it then follows that, for the starting $u \in X$, there exists a $(\leq, \varphi)$-maximal $v \in X$ with $u \leq v$. This element has the properties (5.1) + (5.2), and the conclusion follows.

(B) Let $X$ be a nonempty set; and $(\leq)$ be an order on it. We say that $(\leq)$ has the inf-lattice property, provided: $x \land y := \inf(x, y)$ exists, for all $x, y \in X$. Further, we say that $z \in X$ is a $(\leq)$-maximal element if $X(z, \leq) = \{z\}$; the class of all these points will be denoted as $\max(X, \leq)$. In this case, $(\leq)$ is called a Zorn order when $\max(X, \leq)$ is nonempty and cofinal in $X$ [for each $x \in X$ there exists a $(\leq)$-maximal $v \in X$ with $u \leq v$]. Further aspects are to be described in a metric setting. Let $d : X \times X \to R_+$ be a metric over $X$; and $\varphi : X \to R_+$ be some function. Then, the natural choice for $(\leq)$ above is

$$x \leq_{(d, \varphi)} y \iff d(x, y) \leq \varphi(x) - \varphi(y);$$

referred to as the Brøndsted order [5] attached to $(d, \varphi)$. Denote $X(x, \rho) = \{u \in X : d(x, u) < \rho\}$, $x \in X$, $\rho > 0$ [the open sphere with center $x$ and radius $\rho$]. Call the ambient metric space $(X, d)$, discrete when for each $x \in X$ there exists $\rho = \rho(x) > 0$ such that $X(x, \rho) = \{x\}$. Note that, under such an assumption, any function $\psi : X \to R$ is continuous over $X$. However, the Lipschitz property $|\psi(x) - \psi(y)| \leq L(d(x, y), x, y \in X, \text{for some } L > 0)$ cannot be assured, in general.

Now, the statement below is a particular case of EVP:

\begin{theorem}
Let the metric space $(X, d)$ and the function $\varphi : X \to R_+$ satisfy

(d03) $(X, d)$ is discrete bounded and complete
(d04) $(\leq_{(d, \varphi)})$ has the inf-lattice property
(d05) $\varphi$ is $d$-nonexpansive and $\varphi(X)$ is countable.

Then, $(\leq_{(d, \varphi)})$ is a Zorn order.

\end{theorem}

We shall refer to it as: the discrete Lipschitz countable version of EVP (in short: (EVPdLc)). Clearly, (EVP) $\implies$ (EVPdLc). The remarkable fact to be added is that this last principle yields (DC); so, it completes the circle between all these.

\begin{proposition}
We have (in the reduced Zermelo-Fraenkel system) $(\text{EVPdLc}) \implies (\text{DC})$. So (by the above), the maximal/variational principles (BB), (BMP), (HVP) and (EVP) are all equivalent with (DC); hence, mutually equivalent.

\end{proposition}

For a detailed proof, see Turinici [23]. In particular, when the specific assumptions (d04) and (d05) are ignored in Theorem 5, Proposition 7 above reduces to the result in Brunner [6]. Further aspects will be delineated elsewhere.
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