Invariant Polynomial Functions on $k$ qudits

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Abstract

We study the polynomial functions on tensor states in $(\mathbb{C}^{n})^{\otimes k}$ which are invariant under $SU(n)^{k}$. We describe the space of invariant polynomials in terms of symmetric group representations. For $k$ even, the smallest degree for invariant polynomials is $n$ and in degree $n$ we find a natural generalization of the determinant. For $n, d$ fixed, we describe the asymptotic behavior of the dimension of the space of invariants as $k \to \infty$. We study in detail the space of homogeneous degree 4 invariant polynomial functions on $(\mathbb{C}^{2})^{\otimes k}$.

1 Introduction

In quantum mechanics, a combination of states in Hilbert spaces $H_1, \ldots, H_k$ leads to a state in the tensor product Hilbert space $H_1 \otimes \cdots \otimes H_k$. Such a state will be called here a tensor state. In this paper we take $H_1 = \cdots = H_k = \mathbb{C}^n$ where $n > 1$. Then a tensor state is a joint state of $k$ qudits. It would be very interesting to classify tensor states in $(\mathbb{C}^n)^{\otimes k}$ up to the action of the product $U(n)^{k}$ of unitary groups of local symmetries. A natural approach to this is to study the algebra of invariant polynomials. This approach was developed by Rains [R], by Grassl, Rötteler and Beth [G-R-B1] [G-R-B2], by Linden and Popescu [L-P] and by Coffman, Kundu and Wootters [C-K-W]. These authors study the ring of invariant polynomials in the components of a tensor state in $(\mathbb{C}^n)^{\otimes k}$ and in their complex-conjugates. For $k$ qubits, explicit descriptions of invariants are given in [G-R-B1], [G-R-B2], [L-P] and in [C-K-W].

In this paper the symmetry group we consider is the product $G = SU(n)^{k}$ of special unitary groups; one thinks of $G$ as the special group of local symmetries. We study the $G$-invariant polynomial functions $Q$ on the tensor states in $(\mathbb{C}^n)^{\otimes k}$ (we discuss in §2 how this is relevant to the description of the $G$-orbits). We consider polynomials in the entries of a tensor state, in other words, holomorphic polynomials.

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Let $\mathcal{R}_{n,k,d}$ be the space of homogeneous degree $d$ polynomial functions on tensor states in $(\mathbb{C}^n)^{\otimes k}$. Let $\mathcal{R}_{n,k,d}^G$ be the space of $G$-invariants in $\mathcal{R}_{n,k,d}$. See §3 for more discussion. We reduce the problem of computing $\mathcal{R}_{n,k,d}^G$ to a problem in the invariant theory of the symmetric group $\mathfrak{S}_d$ (Proposition 2.1). In particular, $\mathcal{R}_{n,k,d}^G$ is non-zero only if $d$ is a multiple of $n$. So the “first” case is $d = n$; we examine this in §4. We find that if $k$ is odd then $\mathcal{R}_{n,k,n}^G = 0$ while if $k$ is even then $\mathcal{R}_{n,k,n}^G$ is 1-dimensional. In the latter case we write down (§5) explicitly the corresponding invariant polynomial $P_{n,k}$ in $\mathcal{R}_{n,k,n}$; we find $P_{n,k}$ is a natural generalization of the determinant of a square matrix.

For fixed $n, d$ the direct sum $\oplus_k \mathcal{R}_{n,k,d}$ is an associative algebra. We study the asymptotic behavior of $\dim \mathcal{R}_{n,k,d}$ as $k \to \infty$ in §4. In §5, we specialize to the case of $k$-qubits, i.e. $n = 2$. We compute the dimension of the space $\mathcal{R}_{2,k,4}^G$ of degree 4 invariants as well as the dimension of the space of invariants in $\mathcal{R}_{2,k,4}^G$ under the natural action of $\mathfrak{S}_k$. We show that $\oplus_k \mathcal{R}_{2,k,4}^G$ is a polynomial algebra on 2 generators. For $k \leq 5$ we describe the representation of $\mathfrak{S}_k$ on $\mathcal{R}_{2,k,4}^G$. For $k = 4$ we find some interesting relations with the results on classification of tensor states in $(\mathbb{C}^2)^{\otimes 4}$ given in [3].

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## 2 Polynomial invariants of tensor states

We will consider $(\mathbb{C}^n)^{\otimes k}$ as a space of contravariant tensor states $u$. Then (once we fix a basis of $\mathbb{C}^n$) $u$ is given by $n^k$ components $u^{p_1p_2\cdots p_k}$. We consider the algebra $\mathcal{R}_{n,k}$ of polynomial functions on $(\mathbb{C}^n)^{\otimes k}$. So $\mathcal{R}_{n,k}$ is the polynomial algebra $\mathbb{C}[x_{p_1p_2\cdots p_k}]$ in the $n^k$ coordinate functions $x_{p_1p_2\cdots p_k}$. We have a natural algebra grading $\mathcal{R}_{n,k} = \bigoplus_{d=0}^{\infty} \mathcal{R}_{n,k,d}$ where $\mathcal{R}_{n,k,d}$ is the space of homogeneous degree $d$ polynomial functions.

A function in $\mathcal{R}_{n,k,d}$ amounts to a symmetric degree $d$ covariant tensor $Q$ in $(\mathbb{C}^n)^{\otimes k}$. So $Q$ has $n^{dk}$ components $Q_{i_1\cdots i_{dk}}$ where we think of the indices $i_{ab}$ as being arranged in a rectangular array of $d$ rows and $k$ columns and $Q_{i_{11}\cdots i_{dk}}$ is invariant under permutations of the rows of the array. Then $Q$ defines the function

$$
    u \mapsto Q_{i_{11}\cdots i_{dk}} u^{i_{11}i_{12}\cdots i_{1k}} u^{i_{21}i_{22}\cdots i_{2k}} \cdots u^{i_{d1}i_{d2}\cdots i_{dk}}
$$

where we used the usual Einstein summation convention. In this way, $\mathcal{R}_{n,k}$ identifies with $S^d((\mathbb{C}^n)^{\otimes k})$.

Now the group $G = SU(n)^k$ acts on our tensor states $u$ and tensors $Q$ as follows. Let the matrix $g_{ij}$ live in the $m$-th copy of $SU(n)$ and let $g^{ij}$ be the inverse matrix. Then $g_{ij}$ transforms $u^{p_1p_2\cdots p_k}$ into $g_{pi_{p_{11}p_{12}\cdots p_{1k}}} u^{q_1q_2\cdots q_k}$ and $Q_{i_{11}\cdots i_{dk}}$ into $Q_{j_{11}\cdots j_{dk}} g^{j_{1m}i_{1m}} g^{j_{2m}i_{2m}} \cdots g^{j_{dm}i_{dm}}$. The identification of $\mathcal{R}_{n,k,d}$ with $S^d((\mathbb{C}^n)^{\otimes k})$ is $G$-equivariant.

We are interested in the algebra $\mathcal{R}_{n,k,d}^G = \bigoplus_{d=0}^{\infty} \mathcal{R}_{n,k,d}^G$ of $G$-invariants. We view this as a first step towards studying the orbits of $G$ on $(\mathbb{C}^n)^{\otimes k}$. One can first study the orbits of the complex group $G_C = SL(n, \mathbb{C})^k$ and then decompose the $G_C$-orbits under the $G$-action. Note that a polynomial is $G$-invariant if and only if its is $G_C$-invariant. The closed $G_C$ orbits play a special role—they are the most degenerate orbits. Given any orbit $Y$, its closure contains a unique closed orbit $Z$; then points in $Y$ degenerate to
points in $Z$. The $G_C$-invariant functions separate the closed orbits; they take the same values on $Y$ and on $Z$. The set of closed orbits of $G_C$ in $(\mathbb{C}^n)^{\otimes k}$ has the structure of an affine complex algebraic variety with $\mathcal{R}^G_{n,k}$ as its algebra of regular functions. Thus a complete description of $\mathcal{R}^G_{n,k}$ would lead to a precise knowledge of the closed $G_C$-orbits.

Our approach is thus somewhat different from that of [IKR-B1] [G-R-B2] [L-P] [C-K-W] who study the invariant functions on $(\mathbb{C}^n)^{\otimes k}$ which are polynomials in the $x_{pi\cdots pk}$ and in their complex conjugates; these can also be described as the invariant polynomial functions on $(\mathbb{C}^n)^{\otimes k} \oplus (\overline{\mathbb{C}^n})^{\otimes k}$.

At this point it is useful to examine the case $k = 2$. We can identify $(\mathbb{C}^n)^{\otimes 2}$ with the space $M_n(\mathbb{C})$ of square matrices and then $G = SU(n)^2$ acts on $M_n(\mathbb{C})$ by $(g, h) \cdot u = guh$. So $\mathcal{R}^G_{n,k,d}$ is the space of homogeneous degree $d$ polynomial functions $Q$ of an $n$ by $n$ matrix $u$ which are bi-$SL(n, \mathbb{C})$-invariant, i.e. $Q(guh^{-1}) = Q(u)$ for $g, h \in SL(n, \mathbb{C})$. Then $Q$ is, up to scaling, the $r$th power of the determinant $D$ for some $r$. Hence $d = rn$. It follows that $\mathcal{R}^G_{n,2}$ is the polynomial algebra $\mathbb{C}[D]$. Thus the space of closed orbits for $SL(n, \mathbb{C})$ identifies with $\mathbb{C}$, where $\lambda$ corresponds to the unique closed orbit $Z_\lambda$ inside the set $X_\lambda$ of matrices of determinant $\lambda$. For $\lambda \neq 0$, $Z_\lambda = X_\lambda$ while for $\lambda = 0$, $Z_0$ reduces to the zero matrix.

We view $S^d((\mathbb{C}^n)^{\otimes k})$ as the space of invariants for the symmetric group $\mathfrak{S}_d$ acting on $((\mathbb{C}^n)^{\otimes d})^{\otimes k}$. So

$$\mathcal{R}^G_{n,k,d} = (((\mathbb{C}^n)^{\otimes k})^{\otimes d})^{G \times \mathfrak{S}_d} = (((\mathbb{C}^n)^{\otimes d})^{\otimes k})^{G \times \mathfrak{S}_d}$$

(2.2)

Recall the Schur decomposition $(\mathbb{C}^n)^{\otimes d} = \bigoplus_\alpha S^\alpha(\mathbb{C}^n) \otimes E_\alpha$ where $\alpha$ ranges over partitions of $d$ with at most $n$ rows, $S^\alpha(\mathbb{C}^n)$ is the irreducible covariant representation of $SU(n)$ given by the Schur functor $S^\alpha$, and $E_\alpha$ is the corresponding irreducible representation of $\mathfrak{S}_d$. We use the convention that $E_\alpha$ is the trivial representation if $\alpha = [d]$, while $E_\alpha$ is the sign representation if $\alpha = [1^d]$. Thus we have

$$((\mathbb{C}^n)^{\otimes d})^{\otimes k} = \sum_{|\alpha_1| = \cdots = |\alpha_k| = d} S^{\alpha_1}(\mathbb{C}^n) \otimes \cdots \otimes S^{\alpha_k}(\mathbb{C}^n) \otimes E_{\alpha_1} \otimes \cdots \otimes E_{\alpha_k}$$

(2.3)

Now taking the invariants under $G \times \mathfrak{S}_d$ we get

$$\mathcal{R}^G_{n,k,d} = \sum_{|\alpha_1| = \cdots = |\alpha_k| = d} S^{\alpha_1}(\mathbb{C}^n)^{SU(n)} \otimes \cdots \otimes S^{\alpha_k}(\mathbb{C}^n)^{SU(n)} \otimes (E_{\alpha_1} \otimes \cdots \otimes E_{\alpha_k})^{\mathfrak{S}_d}$$

(2.4)

The representation $S^\alpha(\mathbb{C}^n)$, since it is irreducible, has no $SU(n)$-invariants except if $S^\alpha(\mathbb{C}^n) = \mathbb{C}$ is trivial. This happens if and only if $\alpha_j$ is a rectangular partition with all columns of length $n$. This proves:

**Proposition 2.1.** If $n$ does not divide $d$, then $\mathcal{R}^G_{n,k,d} = 0$. If $d = nr$, then $\mathcal{R}^G_{n,k,d}$ is isomorphic to $(E^{\otimes r})^{\mathfrak{S}_d}$ where $\pi = [r^n]$.

The permutation action of $\mathfrak{S}_k$ on $(\mathbb{C}^n)^{\otimes k}$ induces an action of $\mathfrak{S}_k$ on $\mathcal{R}^G_{n,k,d}$.

**Corollary 2.2.** The isomorphism of Proposition 2.1 intertwines the $\mathfrak{S}_k$-action on $\mathcal{R}^G_{n,k,d}$ with the action of $\mathfrak{S}_k$ on $(E^{\otimes r})^{\mathfrak{S}_d}$ given by permuting the $k$ factors $E_\pi$.  

3
3 The generalized determinant function

Given $n$ and $k$, we want to find the smallest positive value of $d$ such that $\mathcal{R}_{n,k,d}^G \neq 0$. By Proposition [2.4], the first candidate is $d = n$.

**Corollary 3.1.** $\mathcal{R}_{n,k,n}^G \neq 0$ iff $k$ is even. In that case, $\mathcal{R}_{n,k,n}^G$ is one-dimensional and consists of the multiples of the function $P_{n,k}$ given by

$$P_{n,k}(u) = \sum_{\sigma_2, \ldots, \sigma_k \in \mathfrak{S}_n} \epsilon(\sigma_2) \cdots \epsilon(\sigma_k) \prod_{h=1}^n u^{h_{\sigma_2} \cdots h_{\sigma_k}} \quad (3.1)$$

where $h_{\sigma_j} = \sigma_j(h)$.

**Proof.** By Proposition [2.4] we need to compute $(E^{\otimes k}_\pi)_{\otimes d}$. For $d = n$, $\pi = [1^n]$ and so $E_\pi$ is the sign representation of $\mathfrak{S}_n$. Then $(E^{\otimes k}_\pi)$ is one-dimensional and carries the trivial representation if $k$ is even, or the sign representation if $k$ is odd.

Now for $k$ even, we can easily compute a non-zero function $P = P_{n,k}$ in $\mathcal{R}_{n,k,n}$. For $S^*(\mathbb{C}^n)$ is the top exterior power $\Lambda^n \mathbb{C}^n$. Thus $P$ is a non-zero element of the one-dimensional subspace $(\Lambda^n \mathbb{C}^n)^{\otimes k}$ of $((\mathbb{C}^n)^{\otimes n})^{\otimes k}$. The tensor components of $P$ are then given by $P_{i_1 \cdots i_n k} = \frac{1}{n!} \epsilon(\sigma_1) \cdots \epsilon(\sigma_k)$ if for each $j$, the column $i_1, \ldots, i_n$ is a permutation $\sigma_j$ of $1, \ldots, n$ and 0 otherwise. Then we get

$$P_{n,k}(u) = \frac{1}{n!} \sum_{\sigma_1, \ldots, \sigma_k \in \mathfrak{S}_n} \epsilon(\sigma_1) \cdots \epsilon(\sigma_k) \prod_{h=1}^n u^{h_{\sigma_1} \cdots h_{\sigma_k}} \quad (3.2)$$

where $h_{\sigma_i} = \sigma_i(h)$. The expression is very redundant, as each term appears $n!$ times. We remedy this by restricting the first permutation $\sigma_1$ to be 1. This gives [3.1].

$P_{n,k}$ is a *generalized determinant*; $P_{n,k}$ is invariant under the $\mathfrak{S}_k$-action. For $k = 2$, [3.1] reduces to the usual formula for the matrix determinant.

Recall that the rank $s$ of a tensor state $u$ in $(\mathbb{C}^n)^{\otimes k}$ is the smallest integer $s$ such that $u$ can be written as $u = v_1 + v_2 + \cdots + v_s$, where the $v_i$ are decomposable tensor states $v_i = w_{i1} \otimes w_{i2} \otimes \cdots \otimes w_{ik}$. There is a relation between the rank and the vanishing of $P_{n,k}$ as follows:

**Corollary 3.2.** If the tensor state $u$ in $(\mathbb{C}^n)^{\otimes k}$ has rank less than $n$, then $P_{n,k}(u) = 0$.

It is easy to find a tensor state $u$ of rank $n$ such that $P_{n,k}(u)$ is non-zero. For instance, $P_{n,k}(u) = 1$ if $u$ has all components zero except $u^{1 \cdots 1} = \cdots = u^{n \cdots n} = 1$. For $k = 2$, $P_{n,k}(u) = 0$ implies $u$ has rank less than $n$. For bigger (even) $k$, this is false, if $n$ is large enough. This happens essentially because the rank of $u$ can be very large (at least $\frac{n^k}{kn-k+1}$). Thus $P_{n,k}$ gives only partial information about the rank.

4 Asymptotics as $k \to \infty$

Suppose we fix $n$ and $d$ where $d = rn$. Then there is a $G$-invariant associative graded algebra structure $P \circ Q$ on the direct sum $\oplus_k \mathcal{R}_{n,k,d}^G$. Indeed, the product of tensors
induces a \((G \times \mathfrak{S}_d)\)-invariant map \(V^\otimes k \otimes V^\otimes l \to V^\otimes (k+l)\) where \(V = (\mathbb{C}^n)^\otimes d\). The induced multiplication on the spaces of \((G \times \mathfrak{S}_d)\)-invariants gives the product on \(\oplus_k \mathcal{R}^G_{n,k,d}\), where we use the identification in \((2)\). This multiplication corresponds, under the isomorphism of Proposition \((2.1)\), to the product map \(E^\otimes_k \otimes E^\otimes_l \to E^\otimes (k+l)\). This structure is very useful. For instance, if \(d = n\), then \(P_{n,k} \circ P_{n,l} = \frac{1}{m!} P_{n,k+l}\). Thus the determinant \(P_{n,2}\) determines \(P_{n,2m}\) in that the \(m\)-fold product \(P_{n,2} \circ \cdots \circ P_{n,2}\) is equal to \((n!)^{-m+1} P_{n,2m}\).

We will study the size of the algebra \(\oplus_k \mathcal{R}^G_{n,k,d}\) by finding an asymptotic formula for the dimension of \(\mathcal{R}^G_{n,k,d}\). We do this for \(r \geq 2\). Indeed for \(r = 1\) we already know \(\dim \mathcal{R}^G_{n,k,n}\) is 1 if \(k\) is even or 0 if \(k\) is odd; we call this the static case. The asymptotics involve the number

\[ p = \dim E_\pi = d! \prod_{m=0}^{n-1} \frac{m!}{(m+r)!} \quad (4.1) \]

where \(\pi = [r^n]\) as in Proposition \((2.1)\). Our formula for \(p\) is immediate from the hook formula for the dimension of an irreducible symmetric group representation.

**Proposition 4.1.** Assume \(d = rn\) with \(r \geq 2\). Then \(\dim \mathcal{R}^G_{n,k,d} \sim c \frac{p^k}{d!} \) as \(k \to \infty\), where \(c = 1\) with one exception: \(c = 4\) if \(n = 2, d = 4\).

*Proof.* Let \(s = \dim \mathcal{R}^G_{n,k,d} = \dim (E_\pi^\otimes k)^{\mathfrak{S}_d}\). Then \(s = \frac{1}{d!} \sum_{\sigma \in \mathfrak{S}_d} \chi(\sigma)^k\) where \(\chi : \mathfrak{S}_d \to \mathbb{Z}\) is the character of \(E_\pi\). If \(\sigma\) acts trivially on \(E_\pi\), then \(\chi(\sigma) = p\). If \(\sigma\) acts non-trivially, we claim \(|\chi(\sigma)| < p\). To show this, it suffices to show that \(\sigma\) has at least two distinct eigenvalues on \(E_\pi\); this is because \(\chi(\sigma)\) is the sum of the \(p\) eigenvalues of \(\sigma\). Now the set \(\Sigma_d\) of \(\sigma \in \mathfrak{S}_d\) which act on \(E_\pi\) by a scalar is a normal subgroup of \(\mathfrak{S}_d\). So if \(d \geq 5\), then \(\Sigma_d = \{1\}\), the alternating group \(A_d\) or \(\mathfrak{S}_d\). We can easily rule out the latter two possibilities, so \(\Sigma_d = \{1\}\), which proves our claim. If \(d \leq 4\), then (since \(r > 1\) and \(n > 1\), we have \(d = 4, n = 2\) and \(\pi = [2, 2]\)). Our claim is clear here since \(\mathfrak{S}_4\) acts on \(E_\pi\) through the reflection representation of \(\mathfrak{S}_3\) on \(\mathbb{C}^2\).

Therefore we have \(s = c \frac{p^k}{d!} + O(p^r)\) as \(k \to \infty\) where \(c\) is cardinality of the kernel of \(\mathfrak{S}_d \to \text{Aut} E_\pi\). Our work in the previous paragraph computes \(c\). \(\square\)

Proposition \((4.1)\) implies that the algebra \(\oplus_k \mathcal{R}^G_{n,k,d}\) is far from commutative, as it has roughly \(1/N\) times the size of the tensor algebra \(\oplus_k (\mathbb{C}^n)^\otimes k\). We note however that the \(\mathfrak{S}_k\)-invariants in \(\oplus_k \mathcal{R}^G_{n,k,d}\) form a commutative subalgebra, isomorphic to \(S(E_\pi)^{\mathfrak{S}_d}\).

## 5 Quartic invariants of \(k\) qubits

The case \(n = 2\) is of particular interest, as here the qudits are qubits, and this is the case being most discussed in quantum computation. Here we can give some precise non-asymptotic results for the first non-static case, namely \(\mathcal{R}^G_{2,k,4}\). We put \(E = E_\pi = E_{[2,2]}\). The proof of Proposition \((4.1)\) easily gives

**Corollary 5.1.** We have \(\dim \mathcal{R}^G_{2,k,4} = \frac{1}{3}(2^{k-1} + (-1)^k)\).
The first few values of \( \dim R_{2,k,4}^G \), starting at \( k = 1 \), are 0, 1, 1, 3, 5, 11, 21, 43. For \( k = 2 \) and \( k = 3 \) the unique (up to scalar) invariants are, respectively, the squared determinant \( P_{2,2}^2 \) and the Cayley hyperdeterminant \( H_{2,3} \) (see [C-K-Z]). We note that the hyperdeterminant is very closely related to the relative tangle of 3 entangled qubits discussed in [C-K-W].

It would be useful to study \( R_{2,k,4}^G \) as a representation of \( \mathcal{G}_k \), where \( \mathcal{G}_k \) acts by permuting the \( k \) qubits. The \( \mathcal{G}_k \)-invariants in \( R_{2,k,4}^G \) are the \((\mathcal{G}_k \ltimes G)\)-invariants in \( R_{2,k,4} \). These \((\mathcal{G}_k \ltimes G)\)-invariant polynomials are very significant as they separate the closed orbits of the extended symmetry group \( \mathcal{G}_k \ltimes SL(2, \mathbb{C})^k \) acting on \((\mathbb{C}^2)^{\otimes k}\). We can compute the dimension of the \( \mathcal{G}_k \)-invariants as follows:

**Proposition 5.2.** The dimension of the space of \( \mathcal{G}_k \ltimes G \)-invariants in \( R_{2,k,4} \) is \( M_k = \left\lfloor \frac{k}{6} \right\rfloor + r_k \) where \( r_k = 0 \) if \( k \equiv 1 \mod 6 \), or \( r_k = 1 \) otherwise. Furthermore the algebra \( \oplus_k R_{2,k,4}^{\mathcal{G}_k \ltimes G} \) is the polynomial algebra \( \mathbb{C}[P_{2,2}^2, H_{2,3}] \).

**Proof.** We have isomorphisms \( R_{2,k,4}^{\mathcal{G}_k \ltimes G} \sim (E^{\otimes k})^{\mathcal{G}_k} \times E_3 \sim S^k(E)^{\mathcal{G}_3} \) since the representation of \( \mathcal{G}_4 \) on \( E \) factors through \( \mathcal{G}_3 \). Thus the algebra \( \oplus_k R_{2,k,4}^{\mathcal{G}_k \ltimes G} \) identifies with \( S(E)^{\mathcal{G}_3} \). Now \( S(E)^{\mathcal{G}_3} \) is the algebra of \( \mathcal{G}_3 \)-invariant polynomial functions on traceless \( 3 \times 3 \) diagonal matrices, and so is a polynomial algebra on the functions \( A \mapsto Tr(A^2) \) and \( A \mapsto Tr(A^3) \). These invariants correspond (up to scaling) to \( P_{2,2}^2 \) and \( H_{2,3} \). The formula for the dimension follows easily.

For instance, we have: \( M_1 = 0 \), \( M_k = 1 \) for \( 2 \leq k \leq 5 \), and \( M_6 = 2 \). We remark that by replacing \( S(E)^{\mathcal{G}_3} \) by \( \wedge(E)^{\mathcal{G}_3} \), it is easy to prove that the sign representation of \( \mathcal{G}_k \) does not occur in \((E^{\otimes k})^{\mathcal{G}_4}\) for any \( k \geq 2 \).

We can determine the \( \mathcal{G}_k \)-representation on \( R_{2,k,4}^G \) for small \( k \) by explicit trace computations. For \( k = 2 \) and \( k = 3 \) we have the trivial 1-dimensional representation. For \( k = 4 \), we find \( R_{2,4,4}^G \) is the direct sum \( E_{[4]} \oplus E_{[2,2]} \). The trivial representation \( E_{[4]} \) of \( \mathcal{G}_4 \) is spanned by \( P_{2,2}^2 \), while the 2-dimensional representation \( E = E_{[2,2]} \) is spanned by the determinants \( \Delta(ijkl) \) introduced in [B]. Here \( (ijkl) \) is a permutation of \((1234)\). Given a tensor state \( u \in (\mathbb{C}^2)^{\otimes 4} \), we can view it as an element \( v \) of \( \mathbb{C}^4 \otimes \mathbb{C}^4 \), where the first (resp. second) \( \mathbb{C}^4 \) is the tensor product of the \( i \)-th and \( j \)-th copies of \( \mathbb{C}^2 \) (resp. of the \( k \)-th and \( l \)-th copies). Then \( \Delta(ijkl)(u) \) is the determinant of \( v \). As shown in [B], the \( \Delta(ijkl) \) span the representation \( E \) of \( \mathcal{G}_4 \). The significance of the \( \Delta(ijkl) \) is that their vanishing describes the closure of the set of tensor states in \((\mathbb{C}^2)^{\otimes 4}\) of rank \( \leq 3 \). For \( k = 5 \) the representation \( R_{2,5,4}^G \) of \( \mathcal{G}_5 \) is \( E_{[5]} \oplus E_{[2,1,1,1]} \).

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