On maximal totally real embeddings

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Abstract

We consider complex structures with totally real zero section of the tangent bundle. We assume that the complex structure tensor is real-analytic along the fibers of the tangent bundle. This assumption is quite natural in view of a well known result by Bruhat and Whitney [Br-Wh]. We provide explicit integrability equations for such complex structures in terms of the fiberwise Taylor expansion. In a particular geometric case considered in the literature, we explicit much further the fiberwise Taylor expansion of the complex structure as well as the integrability equations.

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1 Introduction and statement of the main result

Let \((E, \pi_E, M)\) be a smooth vector bundle over a manifold \(M\). Let \(E_p\) be the fiber of \(E\) over a point \(p \in M\) and let \(\eta \in E_p\). We consider the transition map \(\tau_\eta(v) := \eta + v\) acting over \(E_p\) and we consider its differential

\[ d\tau_\eta : T_{E_p,0} \rightarrow T_{E_p,\eta}, \]

at the point 0. Composing \(d\tau_\eta\) with the canonical isomorphism \(E_p \cong T_{E_p,0}\) we obtain an isomorphism map

\[ T_\eta : E_p \rightarrow T_{E_p,\eta}. \quad (1.1) \]

We denote by \(0_M\) the zero section of \(E\). Differentiating the identity \(\text{id}_M = \pi_E \circ 0_M\) we obtain \(\mathbb{T}_{M,p} = d_0\pi_E \circ d_00_M\). This implies the decomposition

\[ T_{E,0_p} = d_00_M (T_{M,p}) \oplus \ker d_0\pi_E. \]

We notice also the obvious equalities \(\ker d_0\pi_E = d_0\tau_\eta (T_{E_p,0}) = T_\eta (E_p) \cong E_p\), for any \(\eta \in E_p\). Now applying this to \(\eta = 0_p\), using the previous decomposition and the canonical isomorphism \(d_00_M (T_{M,p}) \cong T_{M,0_p}\), we infer the existence of the canonical isomorphism \(T_{E,0_p} \cong T_{M,0_p} \oplus E_p\), that we rewrite as

\[ T_{E|M} \cong T_M \oplus E. \quad (1.2) \]

**Definition 1** A real sub-manifold \(M\) of an almost complex manifold \((X, J)\) is called totally real if \(T_{M,p} \cap J(T_{M,p}) = 0_p\) for all \(p \in M\). A totally real sub-manifold \(M\) of an almost complex manifold \((X, J)\) is called maximally totally real if \(\dim \mathbb{C} M = \dim \mathbb{C} X\).

1.1 \(M\)-totally real almost complex structures over \(T_M\)

We consider \(M\) included inside \(T_M\) via the zero section. We know by the isomorphism \(T_{\tau_\eta} \cong E_p\) with \(E = T_M\), that this embedding induces the canonical isomorphism \(T_{T_M|M} \cong T_M \oplus T_M\). The vector bundle \(T_{T_M|M}\) is a complex one with the canonical complex structure \(J_{\text{can}} : (u, v) \mapsto (-v, u)\) acting on the fibers.

Any almost complex structure which is a continuous extension of \(J_{\text{can}}\) in a neighborhood of \(M\) inside \(T_M\) makes \(M\) a maximally totally real sub-manifold of \(T_M\).

Over an arbitrary small neighborhood of \(M\) inside \(T_M\) the complex distribution \(T_{T_M}^{0,1}\) is horizontal with respect to the natural projection \(\pi : T_M \rightarrow M\).

We remind that the data of a smooth complex horizontal distribution over \(T_M\) coincides with the one of section

\[ A \in C^\infty (T_M, \pi^* \mathbb{C}T_M^* \otimes \mathbb{C}T_M), \]

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such that \( d\pi \cdot A = \mathbb{I}_{\pi^*T_M} \).

For any complex vector field \( \xi \in C^\infty (M, \mathbb{C}T_M) \) we will denote by abuse of notation \( A(\xi) \equiv A \cdot (\xi \circ \pi) \). The section \( A \) evaluated at the point \( \eta \in T_M \) will be denoted by \( A_\eta \).

We notice that we can write \( A = \alpha + i\beta \), with \( \alpha, \beta \in C^\infty (T_M, \pi^*T_M \otimes \pi^*T_M) \), such that \( d\pi \cdot \alpha = \mathbb{I}_{\pi^*T_M} \) and \( \beta_\eta = T_\eta B_\eta \), with \( B \in C^\infty (T_M, \pi^* \text{End} (T_M)) \).

The section \( A \) determines an almost complex structure \( J_A \) over \( T_M \) such that

\[
T_{T_M, J_A, \eta}^{0,1} = A_\eta \left( \mathbb{C}T_{M, \pi(\eta)} \right) \subset \mathbb{C}T_{T_M, \eta},
\]

if and only if

\[
A_\eta \left( \mathbb{C}T_{M, \pi(\eta)} \right) \cap \overline{A_\eta \left( \mathbb{C}T_{M, \pi(\eta)} \right)} = 0.
\]

This condition is equivalent to the property:

\[
A_\eta (\xi_1) = A_\eta (\xi_2),
\]

implies \( \xi_1 = \xi_2 = 0 \). Taking \( d_\eta \pi \) in the equality (1.4) we infer \( \xi_1 = \xi_2 \). Thus equality (1.4) is equivalent to \( (A - A) (\xi_1) = 0 \) and the previous property is equivalent to \( \text{Ker} (A - A) = 0 \), i.e.

\[
B \in C^\infty (T_M, \pi^* \text{GL} (T_M)).
\]

We notice that with respect to the canonical complex structure of \( T_{T_M|U} \) we have the equality \( (u, v)^{0,1} = (\xi, i\xi) \), with \( \xi := (u - iv)/2 \). Then \( J_A \) is an extension of this complex structure over an open neighborhood \( U \subseteq T_M \) of \( M \) if and only if for any point \( p \in M \) we have \( \alpha_{0p} = d_p 0_M \) and \( B_{0p} = \mathbb{I}_{T_M,p} \). We denote by

\[
T \in C^\infty (T_M, \pi^*T_M \otimes \pi^*T_M),
\]

the canonical section which at the point \( \eta \in T_M \) takes the value \( T_\eta \).

**Definition 2** Let \( M \) be a smooth manifold. An \( M \)-totally real almost complex structure over an open neighborhood \( U \subseteq T_M \) of the image of the zero section \( 0_M \) is a couple \((\alpha, B)\) with

\[
\alpha \in C^\infty (U, \pi^*T_M \otimes \pi^*T_M),
\]

and

\[
B \in C^\infty (U, \pi^* \text{GL} (T_M)),
\]

such that \( d\pi \cdot \alpha = \mathbb{I}_{\pi^*T_M} \) over \( U \) and such that \( \alpha_{0p} = d_p 0_M \), \( B_{0p} = \mathbb{I}_{T_M,p} \), for all \( p \in M \). With \( A := \alpha + iTB \), the almost complex structure \( J_A \) associated to \((\alpha, B)\) is the one which satisfies

\[
T_{T_M, J_A, \eta}^{0,1} = A_\eta \left( \mathbb{C}T_{M, \pi(\eta)} \right) \subset \mathbb{C}T_{T_M, \eta},
\]

for all \( \eta \in U \subseteq T_M \).
Every almost complex smooth extension of the canonical complex structure $J^{\text{can}}$ of $T_{T_M/M}$ over a neighborhood of $M$ inside $T_M$ can be expressed, over a sufficiently small neighborhood $U \subseteq T_M$ of $M$, as the almost complex structure associated to a unique $M$-totally real almost complex structure over $U$.

We provide below an explicit formula for the almost complex structure $J_A$. For this purpose we notice first that for any vector $\xi \in T_{T_M,J_M}$, we have

$$\xi_{J_A}^{0,1} = \frac{1}{2} A_{\eta} \left[ d_{\eta} \pi - i B^{-1}_{\eta} T^{-1}_{\eta} \left( \| T_{T_M} - \alpha_{\eta} d_{\eta} \pi \right) \right] \xi,$$

$$\xi_{J_A}^{1,0} = \frac{1}{2} \overline{A}_{\eta} \left[ d_{\eta} \pi + i B^{-1}_{\eta} T^{-1}_{\eta} \left( \| T_{T_M} - \alpha_{\eta} d_{\eta} \pi \right) \right] \xi.$$

Indeed $\xi_{J_A}^{0,1} \in T_{T_M,J_M,\eta}^{0,1}$, $\xi_{J_A}^{1,0} \in T_{T_M,J_M,\eta}^{1,0}$ and $\xi = \xi_{J_A}^{1,0} + \xi_{J_A}^{0,1}$. We deduce the expression

$$J_{A,\eta} = -\alpha_{\eta} B^{-1}_{\eta} T^{-1}_{\eta} \left( \| T_{T_M} - \alpha_{\eta} d_{\eta} \pi \right) + T_{\eta} B_{\eta} d_{\eta} \pi. \quad (1.5)$$

This shows that for any $\alpha$-horizontal vector $\xi \in T_{T_M,\eta}$, i.e. $\xi = \alpha_{\eta} d_{\eta} \pi \xi$, we have

$$J_{A,\eta} \xi = T_{\eta} B_{\eta} d_{\eta} \pi \xi.$$

In equivalent terms

$$J_{A,\eta} \alpha_{\eta} v = T_{\eta} B_{\eta} v, \quad (1.6)$$

for any $\eta \in U \subseteq T_M$ and any $v \in T_{M,\pi(\eta)}$. Moreover (1.5) implies

$$J_{A,\eta} \text{Ker} d_{\eta} \pi = -\alpha_{\eta} B^{-1}_{\eta} T^{-1}_{\eta}. \quad (1.7)$$

A well known theorem by Bruhat and Whitney [Br-Wh] states that for any real-analytic manifold $M$ there exist a complex manifold $(X,J)$ and a real-analytic embedding of $M$ in $X$ such that as a sub-manifold of $X$, $M$ is maximally totally real. In addition one can arrange that $X$ is an open neighborhood $U \subseteq T_M$ of the zero section and $J_M = J^{\text{can}}$.

Moreover Bruhat and Whitney show [Br-Wh] that if $X$ is a real-analytic manifold equipped with two different real-analytic complex structures $J_1$ and $J_2$ which contains a real analytic sub-manifold $M$ which is maximally totally real with respect to both $J_1$ and $J_2$, then there exist neighborhoods $U_1$ and $U_2$ of $M$ inside $X$ and a real-analytic diffeomorphism $\kappa: U_1 \rightarrow U_2$ which is the identity on $M$ and is a holomorphic mapping of $(U_1,J_1)$ onto $(U_2,J_2)$.

In other words the structure $J$ constructed by Bruhat and Whitney in [Br-Wh] is unique up to complex isomorphisms.

We state below our results on the integrability conditions for $J$.

### 1.2 The integrability equations for $M$-totally real almost complex structures

Let $(E, \pi_E, M)$ be a vector bundle over a manifold $M$. For an arbitrary section $B \in C^\infty(E, \pi_E^* (T^*_M \otimes E))$, we define the derivative along the fiber

$$DB \in C^\infty \left( E, \pi_E^* (E^* \otimes T^*_M \otimes E) \right),$$

for any section $B$. We will denote the derivative $DB$ of $B$ with respect to the fiber $E$ or the base $M$.

We consider the following conditions:

1. **Integrability condition**
   
   $$(DB)^{1,0} = 0,$$

2. **Compatibility condition**
   
   $$(DB)^{0,1} = 0.$$

We prove that these conditions imply the integrability condition $\xi_{J_A}^{1,0} = 0$ for any vector $\xi \in T_{T_M,J_M}$.

Furthermore, we prove that if the integrability condition holds, then the compatibility condition also holds.

Finally, we show that if both the integrability and compatibility conditions hold, then $J_M = J^{\text{can}}$.
by the formula
\[ D_\eta B(v) := \frac{d}{dt} B_{\eta + tv} \in T_{M,p}^* \otimes E_p, \]
for any \( \eta, v \in E_p \). We denote by \( \text{Alt}_2 \) the alternating operator (without normalizing coefficient!) which acts on the first two entries of a tensor. For any morphism \( A : T_M \to E \) and any bilinear form \( \beta : E \times T_M \to E \) we define the contraction operation
\[ A \l vert \beta := \text{Alt}_2 (\beta \circ A), \]
where the composition operator \( \circ \) act on the first entry of \( \beta \). For a given covariant derivative operator \( \nabla \) acting on the smooth sections of \( T_M \), we denote by \( \nabla^\perp \) the linear projection to the associated horizontal distribution. (See lemmas 14, 16 and definition 5 in subsection 9.1 of the appendix for precise definitions and properties of \( \nabla^\perp \).)

**Theorem 1** Let \( M \) be a smooth manifold and let \( J_A \) with \( A = \alpha + iTB \) be an \( M \)-totally real almost complex structure over an open neighborhood \( U \subseteq T_M \) of the image of the zero section. Let also \( \nabla \) be a covariant derivative operator acting on the smooth sections of \( T_M \). Then \( J_A \) is integrable over \( U \) if and only if the complex section \( S := T^{-1}(\nabla^\perp - A) \) satisfies the equation
\[ H_{\nabla^\perp} \left( \nabla^{\text{End}(T_M),\pi} S \right)_\eta - S_{\eta} \nabla S + S_{\eta} \tau^\nabla + R^\nabla \cdot \eta = 0, \tag{1.8} \]
for any point \( \eta \in U \), where \( \nabla^{\text{End}(T_M),\pi} \) is the covariant derivative operator acting on the smooth sections of \( \pi^* \text{End}(T_M) \) induced by \( \nabla \) and where \( \tau^\nabla \) and \( R^\nabla \) are respectively the torsion and curvature forms of \( \nabla \).

We notice that \( S_{|M} = iI_{T_M} \) by the conditions \( \alpha_0 = H_{\nabla^\perp} \beta = d_p 0_M \) and \( B_0 = I_{T_M,p} \).

**Notation for the statement of the main theorem.**

For any \( A \in T_M^* \otimes \text{End}_\mathbb{C}(CT_M) \) and for any \( \theta \in T_M^* \otimes \mathbb{C}T_M \), the product operations of tensors \( A \cdot \theta, A \l vert \theta \in T_M^* \otimes \mathbb{C}T_M \) are defined by
\[ (A \cdot \theta) (u_1, \ldots, u_p, v_1, \ldots, v_q) := A(u_1, \ldots, u_p) \cdot \theta(v_1, \ldots, v_q), \]
\[ (A \l vert \theta) (u_1, \ldots, u_p, v_1, \ldots, v_q) := \sum_{j=1}^q \theta(v_1, \ldots, A(u_1, \ldots, u_p) \cdot v_j, \ldots, v_q). \]
We will denote for notation simplicity \( R^\nabla \cdot \theta := R^\nabla \cdot \theta - R^\nabla \cdot \theta \). We will denote by \( \text{Circ} \) the circular operator
\[ (\text{Circ} \theta) (v_1, v_2, v_3, \bullet) := \theta(v_1, v_2, v_3, \bullet) + \theta(v_2, v_3, v_1, \bullet) + \theta(v_3, v_1, v_2, \bullet), \]
acting on the first three entries of any \( q \)-tensor \( \theta \), with \( q \geq 3 \). We define also the permutation operation \( \theta_2 (v_1, v_2, \bullet) := \theta(v_2, v_1, \bullet). \)
For any covariant derivative $\nabla$ acting on the smooth sections of $\mathbb{C}T_M$ we define the operator
\[ d^\nabla : C^\infty (M, T_M^* \otimes_m \mathbb{C}T_M) \rightarrow C^\infty (M, \Lambda^2 T_M^* \otimes_m \mathbb{C}T_M), \]
with $k \geq 1$ as follows
\[ d^\nabla A (\xi_1, \xi_2, \mu) := \nabla_{\xi_1} A (\xi_2, \mu) - \nabla_{\xi_2} A (\xi_1, \mu), \]
with $\xi_1, \xi_2 \in T_M$ and $\mu \in T_M^{\oplus (k-1)}$. Moreover for any
\[ A \in C^\infty (M, T_M^* \otimes_m \Lambda^2 T_M^* \otimes_m \mathbb{C}T_M), \]
\[ B \in C^\infty (M, T_M^* \otimes_m \Lambda^2 T_M^* \otimes_m \mathbb{C}T_M), \]
we define the exterior product
\[ A \wedge_1 B \in C^\infty (M, \Lambda^2 T_M^* \otimes_m \Lambda^2 T_M^* \otimes_m \mathbb{C}T_M), \]
as
\[ (A \wedge_1 B) (\xi_1, \xi_2, \eta, \mu) := A (\xi_1, B (\xi_2, \eta), \mu) - A (\xi_2, B (\xi_1, \eta), \mu), \]
with $\xi_1, \xi_2 \in T_M$, $\eta \in T_M^{\oplus l}$ and $\mu \in T_M^{\oplus (k-1)}$. We denote by $\text{Sym}_{r_1, \ldots, r_s}$ the symmetrizing operator (without normalizing coefficient!) acting on the entries $r_1, \ldots, r_s$ of a multi-linear form. We use in this paper the common convention that a sum and a product running over an empty set is equal respectively to 0 and 1.

With these notation we can state our main theorem.

**Theorem 2 (Integrability in the fiberwise real analytic case).**

Let $M$ be smooth manifold equipped with a torsion free covariant derivative operator $\nabla$ acting on the smooth sections of the tangent bundle $T_M$, let $U \subseteq T_M$ be an open neighborhood of the image of the zero section with connected fibers let $J_A$ be an $M$-totally real almost complex structure over $U$, real-analytic along the fibers of $U$ and consider the fiberwise Taylor expansion at the origin
\[ T_{\eta}^{-1} (H^\eta - A) \eta \cdot \xi = i\xi + \sum_{k \geq 1} S_k (\xi, \eta^k), \]
with $\eta \in T_M$ in a neighborhood of the image of the zero section, with $\xi \in T_M, \pi (\eta)$ arbitrary, with
\[ S_k \in C^\infty (M, T_M^* \otimes_m \Lambda^k T_M^* \otimes_m \mathbb{C}T_M), \]
with $\eta^k := \eta \otimes k \in T_M^{\oplus k}$ and let $\nabla^S_{\xi}$ be the complex covariant derivative operator acting on the smooth sections of $\mathbb{C}T_M$ defined by
\[ \nabla^S_{\xi} \eta := \nabla_\xi \eta + S_1 (\xi, \eta). \]
Then $J_A$ is integrable over $U$ if and only if $S_1 \in \mathcal{C}^\infty (M, S^2 T^*_M \otimes_n \mathbb{C} T_M)$, i.e. $\nabla^{S_1}$ is torsion free and for all $k \geq 2$,

$$S_k = \frac{i}{k} \nabla^{S_1} \sigma_{k-1} + \frac{1}{(k+1)!} \text{Sym}_{2,\ldots,k+1} \beta_{k-1} (\sigma_{k-2}) + \sigma_k,$$

$$\sigma_k \in \mathcal{C}^\infty (M, S^{k+1} T^*_M \otimes_n \mathbb{C} T_M),$$

$$\text{Circ} \beta_{k+1} (\sigma_k) = 0,$$

where $\sigma_1 := 0$, $\beta_1 (\sigma_0) := R^{\nabla^{S_1}}$, $\beta_2 (\sigma_1) := -\frac{i}{4} (\nabla^{S_1} R^{\nabla^{S_1}})_2$ and for all $k \geq 3$,

$$\beta_k (\sigma_{k-1}) := \frac{i}{k} R^{\nabla^{S_1}} \sigma_{k-1} + \frac{1}{(k+1)!k!} \text{Sym}_{3,\ldots,k+2} \theta_k (\sigma_{k-1}),$$

$$\theta_k (\sigma_{k-1}) := \frac{i}{k} \sum_{r=3}^{k-1} \frac{(r+1)!}{r!} (i d^{\nabla^{S_1}})^{r-1} (R^{\nabla^{S_1}} \sigma_{r-1})$$

$$- 2 i (i d^{\nabla^{S_1}})^{k-2} (\nabla^{S_1} R^{\nabla^{S_1}})_2$$

$$+ \sum_{r=4}^{k+1} \sum_{p=2}^{r-2} (i d^{\nabla^{S_1}})^{r+1-r} (pS_p \wedge_1 S_{r-p}).$$

In more explicit terms

$$S_2 = S_0^0 + \sigma_2,$$  \hspace{1cm} (1.9)

$$S_0^0 (\xi_1, \xi_2, \xi_3) := \frac{i}{6} \left[ R^{\nabla^{S_1}} (\xi_1, \xi_2) \xi_3 + R^{\nabla^{S_1}} (\xi_1, \xi_3) \xi_2 \right],$$  \hspace{1cm} (1.10)

$$\sigma_2 \in \mathcal{C}^\infty (M, S^3 T^*_M \otimes_n \mathbb{C} T_M),$$  \hspace{1cm} (1.11)

$$\text{Circ} \beta_3 (\sigma_2) = 0,$$  \hspace{1cm} (1.12)

$$\beta_3 (\sigma_2) := \frac{i}{3} R^{\nabla^{S_1}} \sigma_2 + \frac{1}{4!3!} \text{Sym}_{3,4,5} \theta_3 (\sigma_2),$$  \hspace{1cm} (1.13)

$$\theta_3 (\sigma_2) := 2 d^{\nabla^{S_1}} (\nabla^{S_1} R^{\nabla^{S_1}})_2 + 4! S_2 \wedge_1 S_2.$$  \hspace{1cm} (1.14)

The assumption that the complex structure tensor is real-analytic along the fibers of the tangent bundle is quite natural. Indeed in the case $M$ is real analytic then the $M$-totally real complex structure constructed by Bruhat and Whitney [Br-Wh] is also real analytic with respect to the real analytic structure of the tangent bundle induced by $M$.

In this paper we request from the readers very good knowledge of the geometric theory of linear connections. Basics of such theory can be found in the appendix.
1.3 Application of the main integrability result

Over a Riemannian manifold $(M, g)$, we denote by

$$V^g \ni (\eta, t) \mapsto -\Phi^g_t(\eta) \in T_M,$$

the geodesic flow, where $V^g \subset TM \times \mathbb{R}$ is an open neighborhood of $TM \times \{0\}$. Let $\nabla g$ be the Levi-Civita connection of the metric $g$. We denote by $H^g$ the linear projection to the associated horizontal distribution. We state the following corollary of the main theorem 2.

**Corollary 1** Let $(M, g)$ be a smooth Riemannian manifold, let $U \subseteq TM$ be an open neighborhood of the image of the zero section with connected fibers, let $J \equiv J_A$ be an $M$-totally real almost complex structure over $U$, real analytic along the fibers of $U$ and consider the fiberwise Taylor expansion at the origin

$$T_{\eta}^{-1}(H^g - \mathcal{A}) \eta \cdot \xi = i\xi + \sum_{k \geq 1} S_k(\xi, \eta^k),$$

with $\eta \in TM$ in a neighborhood of the image of the zero section, with $\xi \in TM, \pi(\eta)$ arbitrary, with

$$S_k \in C^\infty(M, T^*_{TM} \otimes_{\mathbb{R}} S^k TM \otimes_{\mathbb{R}} CT_M),$$

and with $\eta^k := \eta^{\otimes k} \in T^{\otimes k}_{M, \pi(\eta)}$. Then the statements (a) and (b) below are equivalent.

(a) The almost complex structure $J$ is integrable over $U$ and for any $\eta \in U$ the smooth map $\psi_\eta : t + is \mapsto s\Phi^g_t(\eta)$, defined in a neighborhood of $0 \in \mathbb{C}$ is $J$-holomorphic.

(b) The components $S_k$ satisfy $S_1 = 0$, $S_k = i(k + 1)! \sum_{r=1}^{k-k+1} \Theta_k(g)$, for all $k \geq 2$, with $\Theta_2(g) := 2R^g$ and with

$$\Theta_k(g) := -2i(id_1^{\otimes k})^{k-3} (\nabla^g R^g)_2 + \sum_{r=4}^{k} r! \sum_{p=2}^{r-2} r^{r-2} (id_1^{\otimes 2})^{k-r} (pS_p \wedge_1 S_{r-p}),$$

for all $k \geq 3$ and the equations $\text{Circ Sym}_{3, \ldots, k+1} \Theta_k(g) = 0$, are satisfied for all $k \geq 4$.

It has been 64 years since the existence of complex structures on Grauert Tubes was proven for the first time by Bruhat-Whitney [Br-Wh]. Still, up to now, the explicit form of the Taylor expansion has remained mysterious. This is finally clarified in the main theorem 1.5 in [Pal-Sal], which is based on the statement of corollary 1. Indeed in [Pal-Sal], theorem 1.5, we obtain a rather simple and explicit global expression for the complex structure on Grauert tubes.

The expression in theorem 1.5 in [Pal-Sal] (see also theorem 1.6 there for a more general statement) is important for applications to analytic microlocal analysis over manifolds. It allows indeed an explicit global construction of the complex extension of a given global Fourier integral operator defined on a real analytic manifold.
The expression in theorem 1.5 in [Pal-Sal] allows also to perform useful explicit global intrinsic operator computations in the sense of [Pal]. In more explicit terms, given a global intrinsic section over the Grauert tube, an explicit formula for the complex structure such as the one in theorem 1.5 in [Pal-Sal], allows to determine if the section is holomorphic or not.

The proof of corollary 4 will be given in the sub-section 6.2. In the case $(M,g)$ is a compact real analytic Riemannian manifold, the complex structure in the statement of corollary 4 exist thanks to the work of Guillemin-Stenzel [Gu-St], Lempert [Lem], Lempert-Szöke [Le-Sz1, Le-Sz2], Szöke [Szo1, Szo2] as well as Bielawski [Bie]. Thus in this case the integrability conditions

$$I_k := \text{Circ Sym}_{3,\ldots,k+1} \Theta_k (g) = 0,$$

in the statement of corollary 4 are satisfied for all $k \geq 4$. We notice in particular that in the case $k = 4$, the equation $I_4 := \text{Circ Sym}_{3,4,5} \Theta_4 (g) = 0$, expands out to

$$\text{Circ Sym}_{3,4,5} \left[ 3d^{\nabla^g} (\nabla^g R)^2_2 - 2 \tilde{R}^g \wedge_1 \tilde{R}^g \right] = 0,$$

with $\tilde{R}^g := \text{Sym}_{2,3} R^g$. We will show in a quite general set-up that the previous equation is an identity. We have indeed the following result which shows the vanishing of $I_4$.

**Proposition 1** Let $\nabla$ be a torsion free complex covariant derivative operator acting on the smooth sections of the bundle $\mathbb{C}T_M$ with curvature operator $R^\nabla (\cdot,\cdot,\cdot) \equiv R^\nabla (\cdot,\cdot)$. Let $\tilde{R}^\nabla := \text{Sym}_{2,3} R^\nabla$. Then

$$\text{Circ Sym}_{3,4,5} \left[ 3d^{\nabla^g} (\nabla^g R)^2_2 - 2 \tilde{R}^g \wedge_1 \tilde{R}^g \right] = 0. \quad (1.15)$$

The proof will be provided in section 7. In subsection 5.1 in [Pal-Sal] we provide a shorter proof of the vanishing of $I_4$ in proposition 4 by using some more advanced combinatorial techniques. In subsection 5.2 in [Pal-Sal] we show also the vanishing of $I_5$. Using computer algebra (see sections 2 and 5 in [Pa-Sa-Ge]) we can show the vanishing of $I_k$ for $k = 4,\ldots,7$. In section 5 in [Pa-Sa-Ge] we use the explicit expression in theorems 1.5, 1.6 in [Pal-Sal] and we observe that in the case $k = 7$, the computer perform the computation in approximately one second, but we expect that the case $k = 8$ would take a computation of approximately two weeks. We feel confident at this point to formulate the following conjecture.

**Conjecture 1** Let $M$ be a smooth manifold and let $\nabla$ be a torsion free complex covariant derivative operator acting on the smooth sections of the bundle $\mathbb{C}T_M$. Then the sequence of tensors $S_k \in C^\infty (M, T^*_M \otimes_s S^k T_M \otimes_s \mathbb{C}T_M)$, $k \geq 2$, defined by the inductive rule

$$S_k := \frac{i}{(k+1)!k!} \text{Sym}_{2,\ldots,k+1} \Theta_k^\nabla,$$

with $\Theta_2^\nabla := 2R^\nabla$ and with

$$\Theta_k^\nabla := -2i(id^\nabla)^{k-3}(\nabla R^\nabla)_2 + \sum_{r=4}^k \sum_{p=2}^{r-2} i(pS_p \wedge_1 S_{r-p}),$$
for all $k \geq 3$, satisfies the identities

$$I_k := \text{CircSym}_{1 \ldots k} \Theta_k^\nabla = 0,$$

for all $k \geq 4$.

A general mathematical proof for the vanishing of all the integrability conditions $I_k$ is part of a long and difficult work in progress. A corollary of the solution of the above conjecture and of the main theorem \[2\] will be the following striking result which allow canonical construction of maximal totally real embeddings.

**Corollary 2** (Canonical maximal totally real embeddings).

Let $M$ be a real analytic manifold and let $\nabla$ be a torsion free complex covariant derivative operator acting on the real analytic sections of the complexified tangent bundle $\mathbb{C}T_M$. Then there exists an open neighborhood $U \subseteq T_M$ of the image of the zero section with connected fibers and a fiberwise real-analytic section $S$ of $\pi^* \text{End}(T_M)$ over $U$ with fiberwise Taylor expansion at the origin

$$S_\eta \cdot \xi = \sum_{k \geq 2} S_k(\xi, \eta^k),$$

for any $\eta \in U$ and any $\xi \in T_{M, \pi(\eta)}$, with $S_k \in C^\infty(M, T^*_M \otimes_{\pi} S^{k}T^*_M \otimes_{\pi} \mathbb{C}T_M)$ for all $k \geq 2$, (we denote by $\eta^k := \eta \otimes k \in T_{M, \pi(\eta)}$) given by the recursive formula

$$S_k := \frac{i}{(k+1)!k!} \text{Sym}_{1 \ldots k+1} \Theta_k^\nabla,$$

with $\Theta_2^\nabla := 2R^\nabla$ and

$$\Theta_k^\nabla := -2i \left( \text{id}^\nabla \right)^{k-3} (\nabla R^\nabla)_2 + \sum_{r=4}^{k} \sum_{p=2}^{r-2} r! \left( \text{id}^\nabla \right)^{k-r} (pS_p \wedge 1 S_{r-p}),$$

for all $k \geq 3$, such that $J_A$ with

$$\mathcal{A} := -iT_{\mathbb{R}T_M} + H^\nabla - TS,$$

is an $M$-totally real complex structure over $U$ which is real-analytic over $U$.

Indeed in the statement of the main theorem \[2\] we set $\sigma_k = 0$ for all $k \geq 2$ and we identify the torsion free complex covariant derivative $\nabla S^1$ with the arbitrary torsion free complex covariant derivative $\nabla$ in the statements of corollary \[2\] and conjecture \[2\]. Then the integrability equations in the statement of the main theorem \[2\] reduce to the identities $I_k = 0$, for all $k \geq 4$ in the statement of the conjecture \[1\].

We notice that the notation $H^\nabla$ in the above definition of the section $A$ is slightly abusive. We mean there by $H^\nabla$ the restriction to $T_M$ of the horizontal map over $\mathbb{C}T_M$ associated to the complex covariant derivative operator $\nabla$. We must observe here the obvious inclusion $T_{\mathbb{R}T_M}|_{T_M} \subset \mathbb{C}T_M$.

The expression of $S_k$ above can an should be replaced with the explicit global expression in the theorems 1.5 and 1.6 in [Pal-Sal]. That expression shows that
in the case \((M, \nabla)\) with smooth regularity we can assume weaker conditions on the growth of the covariant derivatives of the curvature and still obtain convergence along the fibers.

We obtain in this more general setting a canonical \(M\)-totally real complex structure over \(U\) which is real-analytic along the fibers of \(U\). This is sufficient for the applications to micro local analytic analysis over manifolds.

We wish to point out that in the general setting of a torsion free complex covariant derivative operator \(\nabla\) acting on the sections of the complexified tangent bundle \(\mathbb{C}T_M\) there are no geodesics associated to \(\nabla\). (Cauchy’s existence theorem does not apply).

Therefore there exist no geodesic flow associated to \(\nabla\) and the Jacobi field techniques of the authors \([\text{Gu-St}1, \text{Lem}, \text{Le-Sz}1, \text{Le-Sz}2, \text{Szo}1, \text{Szo}2, \text{Bie}]\) do not apply.

We wish also to point out that in mathematics and in theoretical physics there are many important natural complex differential operators that are defined via complex connections as above.

The set up of corollary 1 is inspired by the articles \([\text{Gu-St}1, \text{Lem}, \text{Le-Sz}1, \text{Le-Sz}2, \text{Szo}1, \text{Szo}2, \text{Bie}]\). The genesis of their approach will be reminded in subsection 6.1 and is needed for the proof of corollary 1.

The long series of articles due to Guillemin-Stenzel \([\text{Gu-St}1, \text{Lem}, \text{Lempert}, \text{Szo}1, \text{Szo}2, \text{Burns}, \text{Burns-Halverscheid-Hind}, \text{Aslam-Burns-Irvine}, \text{Burns-Halverscheid-Hind}, \text{Aslam-Burns-Irvine}]\) as well as Aslam-Burns-Irvine \([\text{ABI}]\) and Szőke \([\text{Szo}1, \text{Szo}2]\) are inspired by the fundamental work of Grauert \([\text{Gra}]\).

Their existence results are needed in a crucial way in analytic micro-local analysis, in pluri-potential theory (see the work by Zelditch \([\text{Zel}]\)) as well as in Hamiltonian dynamics and in geometric quantization (see the work by Morao-Nunes \([\text{Mo-Nu}]\) and Hall-Kirwin \([\text{Ha-Ki}]\).

\section{General connections over vector bundles}

\subsection{Basic definitions}

\textbf{Definition 3} Let \((E, \pi_E, M)\) be a smooth vector bundle over a manifold \(M\). A connection form over \(E\) is a section \(\gamma \in C^\infty(E, T^*_E \otimes T_E)\) such that \(d\pi_E \cdot \gamma = 0\) and \(\gamma|_{\text{Ker} \, d\pi_E} = \Pi_{\text{Ker} \, d\pi_E}\).

We will denote by \(\gamma_\eta\) the connection form \(\gamma\) evaluated at the point \(\eta \in E\).

\textbf{Lemma 1} For any connection \(\gamma \in C^\infty(E, T^*_E \otimes T_E)\) the map

\[ d_\eta \pi_{E|\text{Ker} \, \gamma_\eta} : \text{Ker} \, \gamma_\eta \rightarrow T_{M, \pi_E(\eta)}, \]

is an isomorphism for all \(\eta \in E\).

\textbf{Proof} The assumption \(\gamma|_{\text{Ker} \, d\pi_E} = \Pi_{\text{Ker} \, d\pi_E}\) implies \(\gamma \cdot (\Pi_{T_E} - \gamma) = 0\). Thus \(\text{Im} \,(\Pi_{T_E} - \gamma) \subseteq \text{Ker} \, \gamma\). Then \(\text{Im} \,(\Pi_{T_E} - \gamma) = \text{Ker} \, \gamma\). Indeed if \(\gamma(u) = 0\) then \(u = (\Pi_{T_E} - \gamma) u\). On the other hand we notice that the condition \(d\pi_E \cdot \gamma = 0\) implies \(d\pi_E \cdot (\Pi_{T_E} - \gamma) = d\pi_E\) and thus

\[ d_\eta \pi_{E|\text{Ker} \, \gamma_\eta} \cdot (\Pi_{T_E} - \gamma) = d\pi_E. \]
This equality shows that the map (2.1) is surjective. The injectivity follows from the fact that if \( u, v \in \text{Ker} \gamma \) and \( d_\eta \pi_E (u - v) = 0 \) then \( u - v = \gamma (u - v) = 0 \) by the assumption \( \gamma |_{\text{Ker} \pi_E} \oplus \text{Ker} \pi_E. \)

We denote by \( H^\gamma_{\eta} := (d_\eta \pi_E |_{\text{Ker} \gamma})^{-1} \) the horizontal map. We deduce the existence of a section

\[
H^\gamma = C^\infty (E, \pi_E^* T_M^* \otimes T_E),
\]

such that \( d\pi_E \cdot H^\gamma = \pi_E^* T_M. \) (We notice that \( d\pi_E \in C^\infty (E, T_E^* \otimes \pi_E^* T_M) \)). Composing both sides of (2.2) with \( H^\gamma_{\eta} \) we infer

\[
\gamma = \Pi_{T_E} - H^\gamma \cdot d\pi_E,
\]

and the smooth vector bundle decomposition \( T_E = \text{Ker} d\pi_E \oplus \text{Ker} \gamma. \)

The data of a connection form \( \gamma \) is equivalent with the data of a horizontal form \( H^\gamma. \) The connection form is called linear if the horizontal form \( H^\gamma \) satisfies

\[
d((\eta_1, \eta_2)) (sm_{E}) \cdot (H^\gamma_{\eta_1} \oplus H^\gamma_{\eta_2}) = H^\gamma_{\eta_1 + \eta_2},
\]

\[
H^\gamma_{\lambda \eta} = d_\eta (\lambda \Pi_E) \cdot H^\gamma_{\eta},
\]

where \( sm_{E} : E \oplus E \rightarrow E \) is the sum bundle map where \( \eta_1, \eta_2, \eta \in E \) with \( \pi_E (\eta_1) = \pi_E (\eta_2), \) and \( \lambda \) is a scalar.

**Definition 4** The **curvature form** \( \theta^\gamma \in C^\infty (E, \Lambda^2 T_E^* \otimes T_E) \) of a connection form \( \gamma \) is defined as

\[
\theta^\gamma (\xi_1, \xi_2) := - \gamma ([\Pi_{T_E} - \gamma] \xi_1, (\Pi_{T_E} - \gamma) \xi_2],
\]

for all \( \xi_1, \xi_2 \in C^\infty (E, T_E). \)

The definition is tensorial. Indeed if \( f \in C^\infty (E, \mathbb{R}) \) then

\[
[(\Pi_{T_E} - \gamma) f \xi_1, (\Pi_{T_E} - \gamma) \xi_2] = f([(\Pi_{T_E} - \gamma) \xi_1, (\Pi_{T_E} - \gamma) \xi_2]
\]

\[
- [(\Pi_{T_E} - \gamma) \xi_2, f] (\Pi_{T_E} - \gamma) \xi_1.
\]

The conclusion follows from the fact that \( \gamma \cdot (\Pi_{T_E} - \gamma) = 0. \) We notice that

\[
\theta^\gamma \in C^\infty (E, \Lambda^2 (\text{Ker} \gamma)^* \otimes \text{Ker} d\pi),
\]

and such element is uniquely determined by the **curvature field** \( \Theta^\gamma \) defined as

\[
\Theta^\gamma (\xi_1, \xi_2) (\eta) := T^{-1}_\eta \Theta^\eta (H^\eta_{\eta} \xi_1, H^\eta_{\eta} \xi_2),
\]

for all \( \xi_1, \xi_2 \in T_{M, \pi_E(\eta)}. \) In the case \( \gamma \) is linear then

\[
\Theta^\gamma \in C^\infty (M, \Lambda^2 T_M^* \otimes \text{End} (E)),
\]

is called the **curvature operator**. The terminology is consistent with the fact that if we denote by \( \nabla^\gamma \) the covariant derivative associated to \( \gamma \) then the identity \( H^{\nabla^\gamma} = \Theta^\gamma \) holds, thanks to lemma [19] in the appendix.
Parallel transport. Given any horizontal form \( \alpha \in C^\infty (E, \pi^*T_M^* \otimes a^*T_E) \) over a vector bundle \( E \), the parallel transport with respect to \( \alpha \) is defined as follows. We consider a smooth curve \( c : (-\varepsilon, \varepsilon) \to M \) and the section \( \sigma \in C^1 ((-\varepsilon, \varepsilon), c^*E) \) which satisfies the equation
\[
\dot{\sigma} = (\alpha \circ \sigma) \cdot \dot{c},
\]
over \( (-\varepsilon, \varepsilon) \) with \( \sigma(0) = \eta \in E_{c(0)} \). We define the parallel transport map \( c_{\varepsilon,t} : E_{c(0)} \to E_{c(t)}, t \in (-\varepsilon, \varepsilon) \) along \( c \) with respect to \( \alpha \) as \( \tau_{c_{\varepsilon,t}}^\alpha (\eta) = \sigma (t) \).

We consider now a \( C^1 \)-vector field \( \xi \) over \( M \) and let \( \varphi_{\xi,t} \) be the associated 1-parameter sub-group of transformations of \( M \). Let \( \Phi_{\xi,t}^\alpha : E \to E \) be the parallel transport map along the flow lines of \( \varphi_{\xi,t} \). In equivalent terms the map \( \Phi_{\xi,t}^\alpha \) is determined by the ODE
\[
\dot{\Phi}_{\xi,t}^\alpha = (\alpha \circ \Phi_{\xi,t}^\alpha) \cdot (\xi \circ \varphi_{\xi,t} \circ \pi_E),
\]
with initial condition \( \Phi_{\xi,0}^\alpha \equiv 1_E \). We observe that by definition of parallel transport, the map \( \Phi_{\xi,t}^\alpha \) satisfies \( \pi_E \circ \Phi_{\xi,t}^\alpha = \varphi_{\xi,t} \circ \pi_E \). This follows also from the equalities
\[
(d\pi_E \circ \Phi_{\xi,t}^\alpha) \cdot \dot{\Phi}_{\xi,t}^\alpha = \xi \circ \varphi_{\xi,t} \circ \pi_E
\]
\[
= \dot{\varphi}_{\xi,t} \circ \pi_E.
\]
Moreover the vector field \( \Xi^\alpha := \alpha \cdot (\xi \circ \pi_E) \) over \( E \) satisfies \( \dot{\Phi}_{\xi,t}^\alpha = \Xi^\alpha \circ \Phi_{\xi,t}^\alpha \).

Indeed
\[
\Xi^\alpha \circ \Phi_{\xi,t}^\alpha = (\alpha \circ \Phi_{\xi,t}^\alpha) \cdot (\xi \circ \pi_E \circ \Phi_{\xi,t}^\alpha)
\]
\[
= (\alpha \circ \Phi_{\xi,t}^\alpha) \cdot (\xi \circ \varphi_{\xi,t} \circ \pi_E).
\]
We deduce that \( t \mapsto \Phi_{\xi,t}^\alpha \) is also a 1-parameter sub-group of transformations of \( E \).

2.2 The geometric meaning of the curvature field

The following result provides a clear geometric meaning of the curvature field.

**Lemma 2** Let \( (E, \pi_E, M) \) be a smooth vector bundle over a manifold \( M \) and consider a horizontal form \( \alpha \in C^\infty (E, \pi^*T_M^* \otimes a^*T_E) \) over bundle \( E \). Then the curvature field \( \Theta^\alpha \) associated to \( \alpha \) satisfies
\[
\Theta^\alpha (\xi_1, \xi_2) (\eta) = T^{-1}_\eta \frac{\partial^2}{\partial \xi_1 \partial \xi_2} \left( \Phi_{\xi_1,-s}^\alpha \circ \Phi_{\xi_2,-t}^\alpha \circ \Phi_{\xi_1,s}^\alpha \circ \Phi_{\xi_2,t}^\alpha (\eta) \right).
\]
for any \( \xi_1, \xi_2 \in C^\infty (M, T_M) \) such that \( [\xi_1, \xi_2] \equiv 0 \) and for any \( \eta \in E \).

**Proof** We observe first that if we have a family of transformations \( (\Psi_s)_s \) over a manifold with \( \Psi_0 = \text{id} \) and a curve \( c \) then
\[
\frac{d}{ds}_{|s=0} \Psi_s (c_s) = \dot{\Psi}_0 (c_0) + d\Psi_0 (\dot{c}_0)
\]
\[
= \dot{\Psi}_0 (c_0) + \dot{c}_0.
\]
Applying the last equality to $\Psi_s = \varphi_{\xi_2,-s}$ and $c_s := \varphi_{\xi_1,-t} \circ \varphi_{\xi_2,s} \circ \varphi_{\xi_1,t}$, we infer

$$\frac{d}{ds}_{|s=0} \left( \varphi_{\xi_2,-s} \circ \varphi_{\xi_1,-t} \circ \varphi_{\xi_2,s} \circ \varphi_{\xi_1,t} \right) = -\xi_2 + \frac{d}{ds}_{|s=0} \left( \varphi_{\xi_1,-t} \circ \varphi_{\xi_2,s} \circ \varphi_{\xi_1,t} \right),$$

and thus

$$[\xi_1, \xi_2] = \frac{d}{dt}_{|t=s} \left( \frac{d}{ds}_{|s=0} \left( \varphi_{\xi_1,-t} \circ \varphi_{\xi_2,s} \circ \varphi_{\xi_1,t} \right) \right) = \frac{d}{dt}_{|t=s} \left( \varphi_{\xi_2,-s} \circ \varphi_{\xi_1,-t} \circ \varphi_{\xi_2,s} \circ \varphi_{\xi_1,t} \right).$$

In a similar way

$$[\Xi_1^2, \Xi_2^1] = \frac{d}{dt}_{|t=s} \frac{d}{ds}_{|s=0} \left( \Phi_{\xi_1,-s}^\alpha \circ \Phi_{\xi_2,-t}^\alpha \circ \Phi_{\xi_1,s}^\alpha \circ \Phi_{\xi_2,t}^\alpha \right),$$

with $\Xi_j^2 := \alpha \cdot (\xi_j \circ \pi_E)$, $j = 1, 2$. Let $\eta \in E_p$ and observe that

$$\Phi_{\xi_1,-s}^\alpha \circ \Phi_{\xi_2,-t}^\alpha \circ \Phi_{\xi_1,s}^\alpha \circ \Phi_{\xi_2,t}^\alpha (\eta) \in E_p,$$

for all parameters $t, s$, since $\varphi_{\xi_1,-s} \circ \varphi_{\xi_2,-t} \circ \varphi_{\xi_1,s} \circ \varphi_{\xi_2,t} (p) = p$ thanks to the assumption $[\xi_1, \xi_2] \equiv 0$. We conclude the required geometric identity \hfill $\Box$

### 2.3 Comparison of the curvature fields of two connections

We consider now two connection forms $\gamma_j$, $j = 1, 2$ over $E$ and let $\alpha_j := H^\gamma_j$ be the corresponding horizontal forms. The fact that $d\pi_E (\alpha_1 - \alpha_2) = 0$ implies that there exist a section

$$B := T^{-1} (\alpha_1 - \alpha_2) \in C^\infty \left( E, \pi_E^* (T_M^* \otimes E) \right),$$

which satisfies

$$\gamma_1 = \gamma_2 - TB \cdot d\pi_E.$$

We want to compare the curvature fields $\Theta_j := \Theta^\gamma_j$. We will denote by abuse of notation $\alpha_j \xi \equiv \alpha_j \cdot (\xi \circ \pi_E)$ and $B \xi \equiv B \cdot (\xi \circ \pi_E)$ for any $\xi \in C^\infty (M, T_M)$.

**Lemma 3** In the above set up, the identity

$$\Theta_1 (\xi_1, \xi_2) = (\Theta_2 - B \circ DB) (\xi_1, \xi_2) - T^{-1} \left[ [\alpha_2 \xi_1, TB \xi_2] - [\alpha_2 \xi_2, TB \xi_1] \right] + B \left[ \xi_1, \xi_2 \right], \quad (2.3)$$

holds for any $\xi_1, \xi_2 \in C^\infty (M, T_M)$.
Proof. We notice first the equalities
\[ T\Theta_1 (\xi_1, \xi_2) = \theta_1 (\alpha_1 \xi_1, \alpha_1 \xi_2) \]
\[ = -\gamma_1 \left[ \alpha_1 \xi_1, \alpha_1 \xi_2 \right] \]
\[ = -\gamma_2 \left[ \alpha_1 \xi_1, \alpha_1 \xi_2 \right] + TB \cdot d\pi_E \left[ \alpha_1 \xi_1, \alpha_1 \xi_2 \right] \]
\[ = T\Theta_2 (\xi_1, \xi_2) \]
\[ - \gamma_2 \left[ \alpha_2 \xi_1, TB \xi_2 \right] + \left[ TB \xi_1, \alpha_2 \xi_2 \right] + \left[ TB \xi_1, TB \xi_2 \right] \]
\[ + TB \left[ \xi_1, \xi_2 \right]. \]

In the last line we use the well known identity
\[ d\pi_E \left[ \alpha_1 \xi_1, \alpha_1 \xi_2 \right] = \left[ \xi_1, \xi_2 \right] \circ \pi_E, \]
which follows from the fact that
\[ d\pi_E \alpha_j = \xi_j \circ \pi_E, \quad j = 1, 2. \]

Let now \( \Phi_{TB^{\xi_2},t} \) be the 1-parameter sub-group of transformations of \( E \) associated to the vertical vector field \( TB\xi_2 \). It satisfies \( \pi_E \circ \Phi_{TB^{\xi_2},t} = \pi_E \). Using the standard expression of the Lie bracket
\[ \left[ \alpha_2 \xi_1, TB \xi_2 \right] = \frac{d}{dt}_{|t=0} \frac{d}{ds}_{|s=0} \left( \Phi_{\xi_1, -t} \circ \Phi_{TB^{\xi_2}, s} \circ \Phi_{\xi_1, t} \right), \]
we deduce that this vector field is vertical. In the same way \( [TB\xi_1, \alpha_2 \xi_2] \) is vertical. It is obvious that the vector field \( \left[ TB\xi_1, TB\xi_2 \right] \) is also vertical. We infer the identity
\[ T\Theta_1 (\xi_1, \xi_2) = T\Theta_2 (\xi_1, \xi_2) - \left[ TB\xi_1, TB\xi_2 \right] \]
\[ - \left[ \alpha_2 \xi_1, TB\xi_2 \right] - \left[ TB\xi_1, \alpha_2 \xi_2 \right] + TB \left[ \xi_1, \xi_2 \right]. \]

The required formula (2.3) follows from the identity
\[ [TB\xi_1, TB\xi_2] = T(B\cdot DB) (\xi_1, \xi_2), \quad \text{(2.4)} \]
that we show now. We first remind the reader that for any vector space \( V \), the canonical translation operator \( T : C^\infty (V, V) \to C^\infty (V, T_V) \) defined as \( (T\xi) (v) := T_v \xi_v \) is a Lie algebra isomorphism, where the Lie algebra structure over \( C^\infty (V, V) \) is defined by \( [\xi, \eta]_v := D_v \eta \cdot \xi_v - D_v \xi \cdot \eta_v \). Indeed if we define the action of \( C^\infty (V, V) \) over \( C^\infty (V, \mathbb{R}) \) as
\[ (\xi, f) (v) := D_v f \cdot \xi_v \]
\[ = \frac{d}{dt}_{|t=0} f (v + t\xi_v) \]
\[ = [(T\xi).f] (v), \]
then
\[(\xi,\eta.f) (v) = \frac{d}{dt}\bigg|_{t=0} (\eta.f) (v + t\xi_v)
= \frac{d}{dt}\bigg|_{t=0} \left( D_{v+t\xi_v} f \cdot \eta_{v+t\xi_v} \right)
= D^2_v f (\xi_v, \eta_v) + D_v f \cdot D_v \eta \cdot \xi_v. \]

The fact that the bilinear form \(D^2_v f\) is symmetric implies
\[\xi,\eta.f - \eta,\xi.f = [\xi,\eta].f.\]

On the other hand by definition
\[T\xi.T\eta.f - T\eta.T\xi.f = \xi,\eta.f - \eta,\xi.f,\]
\[[\xi,\eta].f = T[\xi,\eta].f.\]

We conclude the required identity \([T\xi, T\eta] = T[\xi, \eta]\). We apply this remark to our set-up. For any point \(p \in M\), we denote by \(B\xi (p) \in C^\infty (E_p, E_p)\) the map \(\eta \in E_p \mapsto B\xi (p) \in E_p\), and we denote by \(TB\xi (p) \in C^\infty (E_p, T_{E_p})\) the section \(\eta \in E_p \mapsto T_{\eta}B\xi (p) \in T_{E_p}\). Then for any \(\eta \in E_p\)
\[[TB\xi_1, TB\xi_2]_{\eta} = [TB\xi_1 (p), TB\xi_2 (p)]_{\eta}
= T_{\eta} [B\xi_1 (p), B\xi_2 (p)]_{\eta}
= T_{\eta} [D_{\eta}B (B_{\eta}\xi_1 (p)) \xi_2 (p) - D_{\eta}B (B_{\eta}\xi_2 (p)) \xi_1 (p)],\]
which shows (2.4). \(\square\)

We notice now that for any covariant derivative \(\nabla\) over \(E\), the identity (9.8) can be expressed as
\[H^\nabla \xi, T\eta.s = T\pi^*_E (\nabla_\xi s), \quad (2.5)\]
for any vector field \(\xi \in C^\infty (M, T_M)\) and any section \(s \in C^\infty (E, E)\). We need to show the following more general formula.

**Lemma 4** Let \((E, \pi_E, M)\) be a smooth vector bundle over a manifold \(M\) and let \(\nabla\) be a covariant derivative operator acting on the smooth sections of \(E\). Then the equality holds
\[H^\nabla \xi, T\sigma = T\nabla^\pi_E H^\nabla_\xi \sigma, \quad (2.6)\]
for any vector field \(\xi \in C^\infty (M, T_M)\) and for any section \(\sigma \in C^\infty (E, \pi^*_E E)\).

We observe that (2.6) implies (2.5), since \(\nabla^\pi_E H^\nabla_\xi = \pi^*_E (\nabla_\xi s)\), thanks to the functorial property (9.6).

**Proof** In order to show the identity (2.6) we notice first that the assumption \(\sigma \in C^\infty (E, \pi^*_E E)\) means that \(\sigma\) is a map \(\sigma : E \to E\) such that \(\pi_E \circ \sigma = \pi_E\). Then the 1-parameter subgroup of transformations of \(E\) associated to the vector
The field $T\sigma$ satisfies $\Phi_{T\sigma,t}(\eta) = \eta + t\sigma(\eta)$. Moreover with the notation in the proof of identity (9.8) 

$$\left[H\nabla\xi, T\sigma\right] = \frac{d}{dt}{|_{t=0}} \frac{d}{ds}{|_{s=0}} (\Phi_{\xi,-t} \circ \Phi_{T\sigma,s} \circ \Phi_{\xi,t}).$$

The fact that $\Phi_{\xi,-t}$ is linear on the fibers of $E$ implies

$$\Phi_{\xi,-t} \circ \Phi_{\Sigma,s} \circ \Phi_{\xi,t} = \Phi_{\xi,-t}[\Phi_{\xi,t} + s\sigma \circ \Phi_{\xi,t}]$$

$$= \mathbb{I}_E + s\Phi_{\xi,-t} \cdot \sigma \circ \Phi_{\xi,t}.$$ 

We infer

$$\frac{d}{ds}{|_{s=0}} (\Phi_{\xi,-t} \circ \Phi_{T\sigma,s} \circ \Phi_{\xi,t})(\eta) = T\eta \Phi_{\xi,-t} \cdot \sigma \circ \Phi_{\xi,t}(\eta),$$

for any $\eta \in E_p$. We observe that $\sigma \circ \Phi_{\xi,t}(\eta) \in E_{\varphi_{\xi,t}(p)}$. Indeed using the property $\pi_E \circ \sigma = \pi_E$ we deduce

$$\pi_E \circ \sigma \circ \Phi_{\xi,t}(\eta) = \pi_E \circ \Phi_{\xi,t}(\eta)$$

$$= \varphi_{\xi,t}(p).$$

We remind now that if $t \mapsto \eta_t \in E$ is a smooth curve such that $c_t := \pi_E(\eta_t)$ then

$$T^{-1}_{\eta_t} \sigma \circ \Phi_{\xi,t}(\eta) = \frac{d}{dt}{|_{t=0}} \left[\sigma \circ \Phi_{\xi,t}(\eta)\right],$$

thanks to formula (9.7). We apply the previous identity to the curve $\eta_t := \sigma \circ \Phi_{\xi,t}(\eta) \in E_{\varphi_{\xi,t}(p)}$. We obtain

$$T^{-1}_{\eta_t} \nabla_{\eta_t} \Phi_{\xi,t}(\eta) = \frac{d}{dt}{|_{t=0}} \left[\Phi_{\xi,-t} \cdot \sigma \circ \Phi_{\xi,t}(\eta)\right]$$

$$= T^{-1}_{\eta} \left[H\nabla\xi, T\sigma\right](\eta).$$

Moreover

$$\frac{d}{dt}{|_{t=0}} \left[\sigma \circ \Phi_{\xi,t}(\eta)\right] = d_{\eta} \sigma \cdot \dot{\Phi}_{\xi,0}(\eta)$$

$$= d_{\eta} \sigma \cdot H_{\eta} \xi(p).$$

We conclude the equality

$$T^{-1}_{\eta} \nabla_{\eta}(H_{\eta} \xi(p)) = T^{-1}_{\eta} \left[H\nabla\xi, T\sigma\right](\eta),$$

which represents the required formula (2.6). \[\square\]

We can show now the following result.
Lemma 5 Let \((E, \pi_E, M)\) be a smooth vector bundle over a manifold \(M\) and let \(\nabla\) and \(\nabla^{TM}\) be covariant derivative operators acting respectively on the smooth sections of the bundles \(E\) and \(TM\).

Then for any section \(B \in C^\infty(E, \pi_E^*(T^*_M \otimes E))\) the curvature field \(\Theta^\alpha\) of the horizontal form \(\alpha := H^\nabla + TB\) satisfies

\[
\Theta^\alpha = -H^\nabla + \nabla^{TM} \otimes E, \pi_E B - B^\nabla DB - \tau B^\nabla + R^\nabla, \tag{2.7}
\]

where \(\nabla^{TM} \otimes E, \pi_E\) is the covariant derivative on the smooth sections of the bundle \(\pi_E^*(T^*_M \otimes E)\), induced by \(\nabla\) and \(\nabla^{TM}\) and where \(\tau\) is the torsion form of \(\nabla^{TM}\).

Proof In the case \(\alpha_2 = H^\nabla\) in the identity (2.3) we can apply the formula (2.6) to the sections \(B\xi_j \in C^\infty(E, \pi_E^*E)\). We obtain

\[
\Theta_1 (\xi_1, \xi_2) = \left(R^\nabla - B^\nabla DB\right) (\xi_1, \xi_2) - \nabla^E_{H^\nabla \xi_1} (B\xi_2) + \nabla^E_{H^\nabla \xi_2} (B\xi_1) + B[\xi_1, \xi_2].
\]

Using functorial properties of the pull-back we have (with no abuse of notation)

\[
\nabla^E_{H^\nabla \xi_1} (B \cdot \pi_E \xi_2) = \nabla^E_{H^\nabla \xi_1} B \cdot \pi_E \xi_2 + B \cdot \nabla^E_{H^\nabla \xi_1} (\pi_E \xi_2)
\]

\[
= \nabla^{TM} \otimes E, \pi_E B \cdot \pi_E \xi_2 + B \cdot \pi_E \nabla^E_{H^\nabla \xi_1} (\xi_2).
\]

We conclude by (2.3) that if \(\alpha_1 = \alpha = H^\nabla + TB\) then the curvature field \(\Theta^\alpha\) of \(\alpha\) satisfies the identity

\[
\Theta^\alpha (\xi_1, \xi_2) = \left(R^\nabla - B^\nabla DB\right) (\xi_1, \xi_2) - \nabla^{TM} \otimes E, \pi_E B\xi_2 + \nabla^{TM} \otimes E, \pi_E B\xi_1 - B^\nabla \tau^{TM} (\xi_1, \xi_2),
\]

We infer the required formula (2.7). \(\square\)

3 First reduction of the integrability equations

Proof of theorem 1

Proof Let \(\gamma^A\) be the connection form associated to the horizontal form \(A\). Then the integrability of \(J_A\) is equivalent to the condition

\[
\gamma^A[A\xi_1, A\xi_2] = 0, \tag{3.1}
\]

for all smooth complex vector fields \(\xi_1, \xi_2\) over \(M\). (We remind here the use of the abusive notation \(A\xi \equiv A(\xi \circ \pi]\). We denote respectively by \(\Theta^A\) and \(\Theta^\alpha\) the curvature fields of the horizontal distributions \(A\) and \(\alpha\). The integrability condition (3.1) is equivalent to the condition \(\Theta^A \equiv 0\). Then applying the identity (2.3) with \(\alpha_1 = A, \alpha_2 = \alpha\) and separating real and imaginary parts we deduce that the integrability of \(J_A\) is equivalent to the system

\[
\begin{cases}
\Theta^\alpha + B^\nabla DB = 0, \\
TB[\xi_1, \xi_2] = [\alpha_1, TB\xi_2] - [\alpha_2, TB\xi_1].
\end{cases} \tag{3.2}
\]
Let $\Gamma \in C^\infty(U, \pi^* \text{End}(T_M))$ such that $\alpha = H^\nabla - TT$. Using the formula (2.7) in the case $E = T_M$ and $\nabla = \nabla^{T_M}$ we can write the previous equation of the system (3.2) as

$$H^\nabla - \nabla^{\text{End}(T_M),\pi} \Gamma - \Gamma \nabla^D B + R^\nabla = 0.$$  

We express the second equation of the system (3.2) as

$$TB[\xi_1, \xi_2] = [H^\nabla \xi_1, TB\xi_2] - [TT\xi_1, TB\xi_2]$$

which can be expressed as

$$H^\nabla - \nabla^{\text{End}(T_M),\pi} B - \Gamma \nabla^D B + B^\nabla = 0.$$  

We conclude that the system (3.3) is equivalent to the complex equation (1.8).

**Remark 1** We notice that in the case $(\alpha, B) = (H^\nabla, \mathbb{I}_{\pi^* T_M})$, i.e. in the case $J_A = J_{H^\nabla}$, the system (3.3) reduces to

$$R^\nabla = 0,$$

$$\tau^\nabla = 0.$$  

In this way we re-obtain the statement of lemma 13.

**Lemma 6** Under the assumptions of the theorem 2 the $M$-totally real almost complex structure $J_A$ is integrable over $U$ if and only if

$$S_1 \in C^\infty(M, S^2 T^*_M \otimes \mathbb{C} T_M),$$

(i.e. $\nabla^{S_1}$ is torsion free),

$$R^{\nabla^{S_1}} = -2i \text{Alt}_2 \bar{S}_2,$$

$$d^{\nabla^{S_1}} S_k + \sum_{p=2}^{k-1} p S_{k-p+1} \Lambda_1 S_{k-p+1} + i (k + 1) \text{Alt}_2 S_{k+1} \left(\bar{\xi}_1, \bar{\xi}_2, \eta^k\right) = 0.$$
Proof Let $S := T^{-1}(H^V - \overline{A})$. In the case the connection $\nabla$ is torsion free the equation (1.8) reduces to

$$H^V - \nabla^{\text{End}(T_M),\pi} S - S \nabla D S + R^V = 0.$$  \hfill (3.6)

The identification $S_{k,\eta} \cdot \xi \equiv S_k (\xi, \eta^k)$ shows that $S_{k,\eta} \in T^*_M, \pi(\eta) \otimes \mathbb{C} T_M, \pi(\eta)$, i.e.

$$S_k \in C^\infty \left( T_M, \pi^* (T^*_M \otimes \mathbb{C} T_M) \right),$$

and

$$S = \sum_{k \geq 0} S_k.$$ \hfill (3.7)

We remind the reader of the formula

$$\nabla_H^{\text{End}(T_M),\pi} S_k \cdot \xi_2 |_{\eta} = \nabla^{\text{End}(T_M),\pi} S_k \cdot \xi_2 + S_k \cdot \nabla^{\xi_2} \xi_2,$$

for any vector field $\xi_1, \xi_2$ over $M$. On the other hand, by definition

$$\nabla_H^{\text{End}(T_M),\pi} S_k \cdot \xi_2 |_{\eta} = T^{-1}_{S_k(\xi_2,\eta^k)} \nabla_{S_k(\xi_2,\eta^k)} d_{\eta} (S_k \cdot \xi_2) (H^V \xi_1) = T^{-1}_{S_k(\xi_2,\eta^k)} \frac{d}{dt} \bigg|_{t=0} \left[ S_k \left( \xi_2 \circ \varphi_{\xi_1,t} \circ \pi (\eta), \Phi_{\xi_1,t} (\eta)^k \right) \right].$$

Let now $\eta$ be the vector field over $\text{Im} (\varphi_{\xi_1,\pi(\eta)})$ defined by

$$\eta (\varphi_{\xi_1,t} \circ \pi (\eta)) = \Phi_{\xi_1,t} (\eta).$$

Then

$$\nabla_H^{\text{End}(T_M),\pi} S_k \cdot \xi_2 |_{\eta} = \nabla^{\xi_1} \left[ S_k \left( \xi_2, \eta^k \right) \right]_{\pi(\eta)} = \nabla^{\xi_1} S_k \left( \xi_2 \circ \pi (\eta), \eta^k \right) + S_k \left( \nabla^{\xi_2} \xi_2 |_{\pi(\eta)}, \eta^k \right),$$

since $\nabla^{\xi_1} \eta = 0$. We conclude the identity

$$\left( \nabla^{\text{End}(T_M),\pi} S_k \right) |_{\eta} \cdot \xi_2 = \nabla^{\xi_1} S_k \left( \xi_2, \eta^k \right),$$

$\xi_1, \xi_2 \in T_M, \pi(\eta)$. We infer the formula

$$H^V - \nabla^{\text{End}(T_M),\pi} S_k = d^V_1 S_k,$$ \hfill (3.8)

We notice now the equalities

$$D_{\eta} S_k (v) \cdot \xi = \frac{d}{dt} \bigg|_{t=0} \left[ S_k \left( \xi, (\eta + tv)^k \right) \right] = \sum_{j=1}^k S_k \left( \xi, \eta^{j-1}, v, \eta^{k-j} \right) = k S_k \left( \xi, v, \eta^{k-1} \right),$$

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and
\[(S_l \neg DS_k)_{\eta} (\xi_1, \xi_2) = k S_k (\xi_2, S_{l, \eta} \cdot \xi_1, \eta^{k-1}) - k S_k (\xi_1, S_{l, \eta} \cdot \xi_2, \eta^{k-1}).\]

We infer the equality
\[\begin{align*}
(S_l \neg DS_k)_{\eta} (\xi_1, \xi_2) & = -k (S_k \wedge_1 S_l) (\xi_1, \xi_2, \eta^{k+1}) \\
& = -k (S_k \wedge_1 S_l) (\xi_1, \xi_2, \eta^{k+1}) - k S_k (\xi_1, S_{l, \eta} \cdot \xi_2, \eta^{k+1}).
\end{align*}\]

Let \(W \subset U\) be any set containing the zero section of \(T_M\) such that \(W \cap T_{M,p}\) is a neighborhood of \(0_p\) for any \(p \in M\) and such that the fiberwise expansion \((3.7)\) converges over \(W \cap T_{M,p}\). The fact that by assumption \(U \cap T_{M,p}\) is connected implies by the fiberwise real analyticity of \(S\) that \(S\) is a solution of \((3.6)\) over \(U\) if and only if it satisfies \((3.6)\) over \(W\).

Using \((3.8)\) we can write the equation \((3.6)\) under the form
\[\begin{align*}
\sum_{k \geq 1} d^\nabla S_k - \sum_{l,p \geq 0} (S_l \neg DS_p) + R^\nabla & = 0,
\end{align*}\]

over \(W\). We decompose the sum
\[\begin{align*}
\sum_{l,p \geq 0} (S_l \neg DS_p) & = \sum_{l \geq 0, p \geq 1} (S_l \neg DS_p) \\
& = \sum_{l \geq 0, p \geq 1} (S_l \neg DS_p) + i \sum_{k \geq 0} (\Pi_{T_M} \neg DS_{k+1}) \\
& = \sum_{k \geq 1} (S_{k-p+1} \neg DS_p) + i \sum_{k \geq 0} (\Pi_{T_M} \neg DS_{k+1}) \\
& = -\sum_{k \geq 1} p (S_p \wedge_1 S_{k-p+1}) - i \sum_{k \geq 1} (k+1) (S_{k+1} \wedge_1 \Pi_{T_M}),
\end{align*}\]

thanks to the equality \((3.9)\). If we denote by \(\deg_\eta\) the degree with respect to the fibre variable \(\eta \in \mathcal{E}_{\pi(\eta)}\) we have
\[\begin{align*}
\deg_\eta d^\nabla S_k &= \deg_\eta (S_p \wedge_1 S_{k-p+1}) = k, \\
\deg_\eta (S_{k+1} \wedge_1 \Pi_{T_M}) &= k, \\
\deg_\eta R^\nabla &= 1.
\end{align*}\]

Thus by homogeneity the equation \((3.10)\) is equivalent to the countable system
\[\begin{align*}
S_1 \wedge_1 \Pi_{T_M} &= 0, \\
d^\nabla S_1 + S_1 \wedge_1 S_1 + 2i S_2 \wedge_1 \Pi_{T_M} + R^\nabla &= 0, \\
\left[d^\nabla S_k + \sum_{p=1}^k p (S_p \wedge_1 S_{k-p+1}) + i (k+1) S_{k+1} \wedge_1 \Pi_{T_M}\right] (\xi_1, \xi_2, \eta^k) &= 0, \\
\forall k \geq 2, \forall \xi_1, \xi_2, \eta \in T_M.
\end{align*}\]

(3.11)
The first equation in the system means $S_1 \in C^\infty(M, S^2 T_M^* \otimes \mathbb{C}T_M)$, i.e. the complex connection $\nabla^S_1$ is torsion free. The second equation in the system (3.11) rewrites as (3.3). We show now that the equation for $k \geq 2$ in the system (3.11) rewrites as (3.4). Indeed using the formula

$$\nabla^\Gamma_\xi \theta (v_1, \ldots, v_p) = \nabla^\xi \theta (v_1, \ldots, v_p) + \Gamma (\xi, \theta (v_1, \ldots, v_p)) - \sum_{j=1}^p \theta (v_1, \ldots, v_{j-1}, \Gamma (\xi, v_j), v_{j+1}, \ldots, v_p),$$

where $\Gamma \in C^\infty(M, T_M^* \otimes \wedge^p \mathbb{C}T_M)$, $\theta \in C^\infty(M, T_M^* \otimes \wedge^p \mathbb{C}T_M)$ and $\xi, v_k \in T_M$, we infer

$$d^\Gamma_1 S_k (\xi_1, \xi_2, \eta^k) = \nabla^\xi_1 S_k (\xi_2, \eta^k) - \nabla^\xi_2 S_k (\xi_1, \eta^k) + S_1 (\xi_1, S_k (\xi_2, \eta^k)) - S_k (S_1 (\xi_1, \xi_2), \eta^k) - k S_k (\xi_2, S_1 (\xi_1, \eta), \eta^{k-1})$$

$$- S_1 (\xi_2, S_k (\xi_1, \eta^k)) + S_k (S_1 (\xi_2, \xi_1), \eta^k) + k S_k (S_1 (\xi_2, \xi_1), \eta^{k-1}) = \left[ d^\Gamma_1 S_k + S_1 \wedge_1 S_k + k S_k \wedge_1 S_1 \right] (\xi_1, \xi_2, \eta^k),$$

since $S_1$ is symmetric and $S_k$ is symmetric in the last $k$ variables. We conclude (3.5).

**Remark 2** In the case $S_k = 0$, for all $k \geq 2$, the previous system reduces to the equation

$$d^\Gamma_1 S_1 + S_1 \wedge_1 S_1 + R^\Gamma = 0. \quad (3.12)$$

The equation (3.12) means that the complex connection $\nabla^S_1$ acting on sections of $\mathbb{C}T_M$ is flat. In the case $B = \mathbb{R}^n$, the second equation in the system (3.11) implies

$$d^\Gamma_1 \Gamma_1 + \Gamma_1 \wedge_1 \Gamma_1 + R^\Gamma = 0,$$

with $\Gamma_1 := S_1$. This means that the real connection $\nabla^\Gamma_1$ is flat.

**4 Second reduction of the integrability equations**

In this section we will prove the following result.
Proposition 2 Under the assumptions of the theorem, the $M$-totally real almost complex structure $J_A$ is integrable over $U$ if and only if

\[ S_1 \in \mathcal{C}^\infty \left( M, S^2 T_M^* \otimes \mathbb{R} \mathcal{C} T_M \right), \] i.e. $\nabla S_1$ is torsion free,

\[ S_2 = S_2^0 + \sigma_2, \]

\[ S_2^0 (\xi_1, \xi_2, \xi_3) := \frac{i}{6} \left[ R^{\nabla S_1} (\xi_1, \xi_2) \xi_3 + R^{\nabla S_1} (\xi_1, \xi_3) \xi_2 \right], \]

\[ \sigma_2 \in \mathcal{C}^\infty \left( M, S^3 T_M^* \otimes \mathbb{R} \mathcal{C} T_M \right), \]

\[ S_3 = \frac{i}{3} \nabla S_1 \sigma_2 + \frac{1}{4!} \text{Sym}_{2,3,4} (\nabla S_1 R^{\nabla S_1})_2 + \sigma_3, \]

\[ \sigma_3 \in \mathcal{C}^\infty \left( M, S^4 T_M^* \otimes \mathbb{R} \mathcal{C} T_M \right), \]

\[ (\nabla S_1 R^{\nabla S_1})_2 (\xi_1, \xi_2, \xi_3, \xi_4) := \nabla S_1 R^{\nabla S_1} (\xi_1, \xi_1, \xi_3) \xi_1, \quad \text{for all} \ \xi_1, \xi_2, \xi_3, \xi_4 \in T_{M, \pi(\xi_3)} \]

and for all $k \geq 3$,

\[ \left[ d^{\nabla S_1} S_k + \sum_{p=2}^{k-1} p S_p \wedge S_{k-p+1} + i \left( k + 1 \right) \text{Alt}_2 S_{k+1} \right] (\xi_1, \xi_2, \eta^k) = 0, \]

for all $\xi_1, \xi_2, \eta \in T_{M, \pi(\eta)}$.

We first remind the reader that for any complex connection $\nabla$ acting over the sections of $\mathcal{C} T_M$ its torsion $\tau^\nabla$ satisfies the identity

\[ \tau^\nabla = d^\nabla \Pi_{T_M}, \]

where $d^\nabla$ is the covariant exterior differentiation and $\Pi_{T_M} \in \mathcal{C}^\infty \left( M, T^*_M \otimes T_M \right)$. Then

\[ d^\nabla \tau^\nabla = R^\nabla \wedge \Pi_{T_M}, \]

and

\[ (R^\nabla \wedge \Pi_{T_M}) (\xi_1, \xi_2, \xi_3) = R^\nabla (\xi_1, \xi_2) \xi_3 + R^\nabla (\xi_2, \xi_3) \xi_1 + R^\nabla (\xi_3, \xi_1) \xi_2. \]

We conclude that if a connection is torsion free then its curvature operator satisfies the algebraic Bianchi identity.

We denote by $\text{Alt}_p$ the alternating operator (without normalizing coefficients!) acting on the first $p \geq 2$ entries of a tensor, counted from the left to the right. We notice the following very elementary fact.

Lemma 7 Let $V$ be a vector space over a field $\mathbb{K}$ of characteristic zero. Then for any integer $p \geq 2$, the sequence

\[ 0 \longrightarrow S^{p+1} V^* \longrightarrow V^* \otimes S^p V^* \xrightarrow{\text{Alt}_2} \Lambda^2 V^* \otimes S^{p-1} V^* \xrightarrow{\text{Alt}_2} \Lambda^3 V^* \otimes S^{p-2} V^*, \]

is exact.
Proof The equality
\[ S_p + 1 V^* = \ker \left( V^* \otimes S_p V^* \xrightarrow{\text{Alt}_2} \Lambda^2 V^* \otimes S^{p-1} V^* \right), \]
is obvious. We show now the equality
\[
\text{Im} \left( V^* \otimes S_p V^* \xrightarrow{\text{Alt}_2} \Lambda^2 V^* \otimes S^{p-1} V^* \right) = \ker \left( \Lambda^2 V^* \otimes S^{p-1} V^* \xrightarrow{\text{Alt}_2} \Lambda^3 V^* \otimes S^{p-2} V^* \right). \tag{4.1}
\]
We show first the inclusion \( \subseteq \) in (4.1). We notice the equality
\[
\left( \Lambda^2 V^* \otimes S^{p-1} V^* \xrightarrow{\text{Alt}_2} \Lambda^3 V^* \otimes S^{p-2} V^* \right) = \left( \Lambda^2 V^* \otimes S^{p-1} V^* \xrightarrow{\text{Circ}} \Lambda^3 V^* \otimes S^{p-2} V^* \right).
\]
Let now \( \beta := \text{Alt}_2 \alpha \), with \( \alpha \in V^* \otimes S^p V^* \). Then summing up the two equalities
\[
\beta (v_1, v_2; v_3, v_4, \ldots, v_{p+1}) = \alpha (v_1; v_2, v_3, v_4, \ldots, v_{p+1}) - \alpha (v_2; v_1, v_3, v_4, \ldots, v_{p+1}),
\]
\[-\beta (v_1, v_3; v_2, v_4, \ldots, v_{p+1}) = -\alpha (v_1; v_3, v_2, v_4, \ldots, v_{p+1}) + \alpha (v_3; v_1, v_2, v_4, \ldots, v_{p+1}),
\]
we obtain
\[
\beta (v_1, v_2; v_3, v_4, \ldots, v_{p+1}) - \beta (v_1, v_3; v_2, v_4, \ldots, v_{p+1}) \]
\[= - \alpha (v_2; v_3, v_1, v_4, \ldots, v_{p+1}) + \alpha (v_3; v_2, v_1, v_4, \ldots, v_{p+1}) \]
\[= - \beta (v_2, v_3; v_1, v_4, \ldots, v_{p+1}), \]
which rewrites as
\[
\beta (v_1, v_2; v_3, v_4, \ldots, v_{p+1}) + \beta (v_2, v_3; v_1, v_4, \ldots, v_{p+1}) + \beta (v_3, v_1; v_2, v_4, \ldots, v_{p+1}) = 0,
\]
i.e. \( \text{Circ} \beta = 0 \), which shows the inclusion \( \subseteq \) in (4.1). In order to show the reverse inclusion in (4.1) we consider \( \beta \in \Lambda^2 V^* \otimes S^{p-1} V^* \) with \( \text{Circ} \beta = 0 \) and we will prove that \( \beta = C_p \text{Alt}_2 \alpha \), with
\[
\alpha := \text{Sym}_{2,\ldots,p+1} \beta \in V^* \otimes S^p V^*,
\]
and with \( C_p := p / (p+1)! \). Indeed
\[
\frac{1}{(p-1)!} \alpha (v_1; v_2, \ldots, v_{p+1}) = \sum_{j=2}^{p+1} \beta (v_1, v_j; v_2, \ldots, v_{p+1})
\]
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and
\[
\frac{1}{(p-1)!} \left( \text{Alt}_2 \alpha \right) (v_1, v_2; \ldots, v_{p+1})
= \frac{1}{(p-1)!} \alpha (v_1; v_2, \ldots, v_{p+1}) - \frac{1}{(p-1)!} \alpha (v_2; v_1, \hat{v}_2, \ldots, v_{p+1})
= \sum_{j=2}^{p+1} \beta (v_1, v_j; v_2, \ldots, \hat{v}_j, \ldots, v_{p+1})
- \sum_{\hat{j}=1}^{p+1} \beta (v_2, v_j; v_1, \hat{v}_2, \ldots, \hat{v}_j, \ldots, v_{p+1})
= \beta (v_1, v_2; v_3, \ldots, v_{p+1}) + \sum_{j=3}^{p+1} \beta (v_1, v_j; v_2, v_3, \ldots, \hat{v}_j, \ldots, v_{p+1})
+ \beta (v_1, v_2; v_3, \ldots, v_{p+1}) + \sum_{j=3}^{p+1} \beta (v_j, v_2; v_1, \hat{v}_2, v_3, \ldots, \hat{v}_j, \ldots, v_{p+1})
\]

Using the circular identity \( \text{Circ} \beta = 0 \), we obtain
\[
\frac{1}{(p-1)!} \left( \text{Alt}_2 \alpha \right) (v_1, v_2; \ldots, v_{p+1})
= 2 \beta (v_1, v_2; v_3, \ldots, v_{p+1}) - \sum_{j=3}^{p+1} \beta (v_2, v_1; v_j, v_3, \ldots, \hat{v}_j, \ldots, v_{p+1})
\]

This combined with the fact that \( \beta \in \Lambda^2 V^* \otimes S^{p-1} V^* \) implies
\[
\frac{1}{(p-1)!} \left( \text{Alt}_2 \alpha \right) (v_1, v_2; \ldots, v_{p+1})
= 2 \beta (v_1, v_2; v_3, \ldots, v_{p+1}) + (p-1) \beta (v_1, v_2; v_3, \ldots, v_{p+1})
= (p+1) \beta (v_1, v_2; \ldots, v_{p+1})
\]
which shows the required identity. \( \square \)

A direct consequence of the proof of lemma 7 is the following fact.

**Corollary 3** Let \( R \in C^\infty (M, \Lambda^2 T_M^* \otimes \Lambda^1 T_M^* \otimes \Lambda^0 T_M^* \otimes \mathbb{C} T_M) \) satisfying the algebraic Bianchi identity. Then a tensor \( S \in C^\infty (M, T_M^* \otimes S^2 T_M^* \otimes \mathbb{C} T_M) \) satisfies \( 3R = \text{Alt}_2 S \) if and only if \( S = \text{Sym}_2, \) with \( \sigma \in C^\infty (M, S^3 T_M^* \otimes \mathbb{C} T_M) \).

We infer by corollary 3 that the equation (3.4) is satisfied by \( S^2 = S^0 + \sigma_2 \), with
\[
S^0_2 (\xi_1, \xi_2, \xi_3) = \frac{1}{6} \left[ R^{\xi_1} (\xi_1, \xi_2) \xi_3 + R^{\xi_2} (\xi_1, \xi_3) \xi_2 \right], \quad (4.2)
\]
and with $\sigma_2 \in C^\infty(M, S^3 T^*_M \otimes_m CT_M)$. We consider now the equation (4.3) for $k = 2$, which writes as

$$\left[ d^{\nabla S_1}_1 S_2 + 3i \text{Alt}_2 S_3 \right] (\xi_1, \xi_2, \eta^2) = 0. \quad (4.3)$$

The fact that the tensor $d^{\nabla S_1}_1 S_2 + 3i \text{Alt}_2 S_3$, is symmetric in the last two variables implies that the equation (4.3) is equivalent to the equation

$$d^{\nabla S_1}_1 S_2 + 3i \text{Alt}_2 S_3 = 0,$$

that we can rewrite under the form

$$d^{\nabla S_1}_1 S_0^0 + 3i \text{Alt}_2 \hat{S}_3 = 0, \quad (4.4)$$

with

$$\hat{S}_3 := S_3 - \frac{i}{3} \nabla^{S_1} \sigma_2.$$

Then using the expression (4.2) we can rewrite equation (4.4) in the explicit form

$$\nabla^{S_1}_1 R^{\nabla S_1}_1 (\xi_2, \xi_3) \xi_4 + \nabla^{S_1}_1 R^{\nabla S_1}_1 (\xi_2, \xi_4) \xi_3$$

$$- \nabla^{S_1}_1 R^{\nabla S_1}_1 (\xi_1, \xi_3) \xi_4 - \nabla^{S_1}_1 R^{\nabla S_1}_1 (\xi_1, \xi_4) \xi_3$$

$$= -18 \left[ \hat{S}_3 (\xi_1, \xi_2, \xi_3, \xi_4) - \hat{S}_3 (\xi_2, \xi_1, \xi_3, \xi_4) \right]. \quad (4.5)$$

We notice that the fact that the complex connection $\nabla^{S_1}_1$ is torsion free implies that the tensor $\rho$ given by $\rho(\xi_1, \xi_2, \xi_3, \xi_4) := \nabla^{S_1}_1 R^{\nabla S_1}_1 (\xi_2, \xi_3) \xi_4$ satisfies the circular identity with respect to the first and last three entries. Moreover $\rho$ is obviously skew-symmetric with respect to the variables $\xi_2, \xi_3$.

**Lemma 8** Let $\rho$ be a 4-linear form which satisfies the circular identity with respect to the first and last three entries and which is skew-symmetric with respect to the second and third variables. Then a 4-linear form $S$ which is symmetric with respect to the last three entries satisfies the equation

$$\text{Alt}_2 [8 \text{Sym}_{3,4} \rho - S] = 0, \quad (4.6)$$

if and only if

$$S = -2 \text{Sym}_{2,3,4} \rho_2 + \sigma$$

$$= 2 \text{Sym}_{2,3,4} \rho_3 + \sigma,$$

with $\rho_2 (\xi_1, \xi_2, \xi_3, \xi_4) := \rho (\xi_2, \xi_3, \xi_1, \xi_4)$, with $\rho_3 (\xi_1, \xi_2, \xi_3, \xi_4) := \rho (\xi_2, \xi_3, \xi_1, \xi_4)$, for all $\xi_1, \xi_2, \xi_3, \xi_4 \in T_{M,\pi}(\xi_1)$ and with $\sigma$ a 4-linear form which is symmetric with respect to all its entries.
\textbf{Proof} We observe first that the assumptions on \( \rho \) imply \( \text{Circ Alt}_2 \text{Sym}_{3,4} \rho = 0 \). Indeed

\[
(\text{Alt}_2 \text{Sym}_{3,4} \rho) (\xi_1, \xi_2, \xi_3, \xi_4) = \rho (\xi_1, \xi_2, \xi_3, \xi_4) + \rho (\xi_1, \xi_2, \xi_4, \xi_3)
- \rho (\xi_2, \xi_1, \xi_3, \xi_4) + \rho (\xi_2, \xi_1, \xi_4, \xi_3),
\]

and

\[
(\text{Circ Alt}_2 \text{Sym}_{3,4} \rho) (\xi_1, \xi_2, \xi_3, \xi_4)
= \rho (\xi_1, \xi_2, \xi_3, \xi_4) + \rho (\xi_1, \xi_2, \xi_4, \xi_3) - \rho (\xi_2, \xi_1, \xi_3, \xi_4) - \rho (\xi_2, \xi_1, \xi_4, \xi_3)
+ \rho (\xi_2, \xi_3, \xi_1, \xi_4) - \rho (\xi_2, \xi_3, \xi_4, \xi_1) - \rho (\xi_3, \xi_2, \xi_1, \xi_4) - \rho (\xi_3, \xi_2, \xi_4, \xi_1),
\]

where we denote by \( \rho (\cdot, \cdot, \cdot, \cdot) \) the terms we group together. Using the assumption \( \rho \) is skew-symmetric with respect to the second and third variables we infer

\[
(\text{Circ Alt}_2 \text{Sym}_{3,4} \rho) (\xi_1, \xi_2, \xi_3, \xi_4)
= 2\rho (\xi_1, \xi_2, \xi_3, \xi_4) + \rho (\xi_1, \xi_2, \xi_4, \xi_3) + \rho (\xi_2, \xi_1, \xi_3, \xi_4)
+ 2\rho (\xi_2, \xi_3, \xi_1, \xi_4) + \rho (\xi_2, \xi_3, \xi_4, \xi_1)
+ 2\rho (\xi_3, \xi_1, \xi_2, \xi_4) + \rho (\xi_3, \xi_1, \xi_4, \xi_2).
\]

Using the circular assumptions on \( \rho \) we infer

\[
(\text{Circ Alt}_2 \text{Sym}_{3,4} \rho) (\xi_1, \xi_2, \xi_3, \xi_4)
= -\rho (\xi_1, \xi_3, \xi_2, \xi_4) - \rho (\xi_2, \xi_1, \xi_3, \xi_4) - \rho (\xi_3, \xi_2, \xi_1, \xi_4)
= 0.
\]

Then by the proof of lemma 4.6 in the case \( p = 3 \), we infer that a 4-linear form \( S \) which is symmetric with respect to the last three entries satisfies the equation \( (4.6) \) if and only if

\[
S = \text{Sym}_{2,3,4} \text{Alt}_2 \text{Sym}_{3,4} \rho + \sigma,
\]

with \( \sigma \) any 4-linear form which is symmetric with respect to all its entries,
We infer the equation 

$$(\text{Sym}_{2,3,4} \text{Alt}_2 \text{Sym}_{3,4} \rho)(\xi_1, \xi_2, \xi_3, \xi_4)$$

$$= \rho(\xi_1, \xi_2, \xi_3, \xi_4)_1 + \rho(\xi_1, \xi_2, \xi_4, \xi_3)_2 - \rho(\xi_2, \xi_1, \xi_3, \xi_4)_3 - \rho(\xi_2, \xi_1, \xi_4, \xi_3)_4$$

$$+ \rho(\xi_1, \xi_2, \xi_4, \xi_3)_3 + \rho(\xi_1, \xi_2, \xi_3, \xi_4)_4 - \rho(\xi_2, \xi_1, \xi_3, \xi_4)_5 - \rho(\xi_2, \xi_1, \xi_4, \xi_3)_6$$

$$+ \rho(\xi_1, \xi_3, \xi_2, \xi_4)_1 + \rho(\xi_1, \xi_3, \xi_4, \xi_2)_2 - \rho(\xi_3, \xi_1, \xi_2, \xi_4)_3 - \rho(\xi_3, \xi_1, \xi_2, \xi_4)_4$$

$$+ \rho(\xi_1, \xi_3, \xi_4, \xi_2)_3 + \rho(\xi_1, \xi_3, \xi_2, \xi_4)_4 - \rho(\xi_3, \xi_1, \xi_2, \xi_4)_5 - \rho(\xi_3, \xi_1, \xi_2, \xi_4)_6$$

$$+ \rho(\xi_1, \xi_4, \xi_2, \xi_3)_1 + \rho(\xi_1, \xi_4, \xi_3, \xi_2)_2 - \rho(\xi_4, \xi_1, \xi_2, \xi_3)_3 - \rho(\xi_4, \xi_1, \xi_2, \xi_3)_4$$

$$+ \rho(\xi_1, \xi_4, \xi_3, \xi_2)_5 + \rho(\xi_1, \xi_4, \xi_2, \xi_3)_6 - \rho(\xi_4, \xi_1, \xi_2, \xi_3)_7 - \rho(\xi_4, \xi_1, \xi_2, \xi_3)_8$$. 

The fact that $\rho$ is skew-symmetric with respect to the second and third variables implies that $\text{Sym}_{2,3,4} \rho = 0$. We infer 

$$(\text{Sym}_{2,3,4} \text{Alt}_2 \text{Sym}_{3,4} \rho)(\xi_1, \xi_2, \xi_3, \xi_4)$$

$$= -2\rho(\xi_2, \xi_1, \xi_3, \xi_4) - 2\rho(\xi_2, \xi_1, \xi_4, \xi_3)$$

$$- 2\rho(\xi_3, \xi_1, \xi_2, \xi_4) - 2\rho(\xi_3, \xi_1, \xi_4, \xi_2)$$

$$- 2\rho(\xi_4, \xi_1, \xi_2, \xi_3) - 2\rho(\xi_4, \xi_1, \xi_3, \xi_2)$$

$$= 2\rho(\xi_2, \xi_3, \xi_1, \xi_4) + 2\rho(\xi_2, \xi_4, \xi_1, \xi_3)$$

$$+ 2\rho(\xi_3, \xi_2, \xi_1, \xi_4) + 2\rho(\xi_3, \xi_4, \xi_1, \xi_2)$$

$$+ 2\rho(\xi_4, \xi_2, \xi_1, \xi_3) + 2\rho(\xi_4, \xi_3, \xi_1, \xi_2)$$,

which shows the required expressions for $S$. 

By the equation (4.4), we can apply lemma § to the tensor $\rho := \nabla S_i R^{\sigma_1}$. We infer the equation

$$S_3 = \frac{1}{4!3} \text{Sym}_{2,3,4}(\nabla S_i R^{\sigma_1})_2 + \frac{i}{3} \nabla S_i \sigma_2 + \sigma_3,$$  

(4.7)

We deduce that the equation (4.4) is equivalent to the equation (4.7). This concludes the proof of the proposition § thanks to lemma §.
5 Third reduction of the integrability equations and proof of the main theorem

In this section we will prove the following result.

Lemma 9 Under the assumptions of the theorem \( \mathbb{2} \) the \( M \)-totally real almost complex structure \( J_A \) is integrable over \( U \) if and only if

\[
S_1 \in \mathcal{C}^\infty (M, S^2 T^*_M \otimes_n CT_M), \quad \text{i.e. } \nabla^{S_1} \text{ is torsion free,}
\]

\[
S_2 = S^0_2 + \sigma_2,
\]

\[
S^0_2 (\xi_1, \xi_2, \xi_3) := \frac{i}{6} \left[ R^{S_1} (\xi_1, \xi_2) \xi_3 + R^{S_1} (\xi_1, \xi_3) \xi_2 \right],
\]

\[
\sigma_2 \in C^\infty \left( M, S^3 T^*_M \otimes_n CT_M \right),
\]

and for all \( k \geq 3 \),

\[
S_k = \frac{i}{k} \nabla^{S_1} \sigma_{k-1} + \frac{i}{(k+1)!} \text{Sym}_{2, \ldots, k+1} \beta_{k-1} + \sigma_k,
\]

\[
\sigma_k \in C^\infty \left( M, S^{k+1} T^*_M \otimes_n CT_M \right),
\]

\[
\beta_k := \frac{i}{k} d_1^{S_1} \nabla^{S_1} \sigma_{k-1} + \frac{i}{(k+1)!} d_1^{S_1} \text{Sym}_{2, \ldots, k+1} \beta_{k-1}
\]

\[
+ \frac{1}{k!} \text{Sym}_{3, \ldots, k+2} \left( \sum_{p=2}^{k-1} p S_p \wedge_1 S_{k-p+1} \right),
\]

\[
\beta_2 := -\frac{i}{3} (\nabla^{S_1} R^{S_1})_2,
\]

\[
\text{Circ} \beta_k = 0.
\]

Proof We show that the statement of proposition \( \mathbb{2} \) is equivalent to the statement of lemma \( \mathbb{9} \). We show indeed by induction on \( k \geq 3 \) the following statement.

Statement 1 The tensors \( S_h, h = 3, \ldots, k+1 \), satisfy the equations

\[
\left[ d_1^{S_1} S_h + \sum_{p=2}^{h-1} p S_p \wedge_1 S_{h-p+1} + i (h+1) \text{Alt}_2 S_{h+1} \right] (\xi_1, \xi_2, \eta^h) = 0, \quad (5.1)
\]

for all \( h = 3, \ldots, k \), for all \( \xi_1, \xi_2, \eta \in T_{M, \pi((\eta))} \) and

\[
S_3 = \frac{i}{3} \nabla^{S_1} \sigma_2 + \frac{1}{4!} \text{Sym}_{2,3,4} (\nabla^{S_1} R^{S_1})_2 + \sigma_3,
\]

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with \( \sigma_3 \in C^\infty \left( M, S^4 T^*_M \otimes C T_M \right) \), if and only if the tensors \( S_h \) satisfy for all \( h = 3, \ldots, k+1 \), the identities

\[
S_h = \frac{i}{h} \nabla^{S_1} \sigma_{h-1} + \frac{i}{(h+1)!} \text{Sym}_{2,\ldots,h+1} \sigma_{h-1} + \sigma_h ,
\]

(5.2)

with \( \sigma_h \in C^\infty \left( M, S^{h+1} T^*_M \otimes C T_M \right) \) and where for all \( r = 3, \ldots, k \),

\[
\beta_r := \frac{i}{r} d_1^{S_1} \nabla^{S_r} \sigma_{r-1} + \frac{i}{(r+1)!} d_1^{S_1} \text{Sym}_{2,\ldots,r+1} \beta_{r-1}
\]

\[+ \frac{1}{r!} \text{Sym}_{3,\ldots,r+2} \left( \sum_{p=2}^{r-1} p S_p \wedge_1 S_{r-p+1} \right) ,
\]

with \( \beta_2 := -\frac{i}{4} (\nabla^{S_1} R^{S_1})_2 \) satisfies the equation \( \text{Circ} \beta_r = 0 \).

The statement 1 follows directly from the following fact.

**Fact 1** Let \( S_h \), for some \( h = 3, \ldots, k \), be the tensor given by (5.2). Then the tensor \( S_{h+1} \) satisfies the equation (5.1) if and only if \( S_h \) satisfies the identity (5.2), with \( h \) replaced by \( h+1 \) and \( \beta_h \) satisfies the equation \( \text{Circ} \beta_h = 0 \).

In order to show the fact 1 we observe first that (5.1) rewrites as

\[
d_1^{S_1} S_h + \frac{1}{h!} \text{Sym}_{3,\ldots,h+2} \left( \sum_{p=2}^{h-1} p S_p \wedge_1 S_{h-p+1} \right) + i (h+1) \text{Alt}_2 S_{h+1} = 0 .
\]

Using the expression (5.2) for \( S_h \) and the definition of \( \beta_h \), we can rewrite the previous identity as

\[
\beta_h = -\text{Alt}_2 \left[ \nabla^{S_1} \sigma_h + i (h+1) S_{h+1} \right] .
\]

(5.3)

By the proof of lemma 9 we deduce \( \text{Circ} \beta_h = 0 \) and

\[-\nabla^{S_1} \sigma_h - i (h+1) S_h = C_{h+1} \text{Sym}_{2,\ldots,h+2} \beta_h - i (h+1) \sigma_{h+1} .
\]

Therefore the identity (5.3) is equivalent to; \( \text{Circ} \beta_h = 0 \) and \( S_{h+1} \) satisfies (5.2), with \( h \) replaced by \( h+1 \). This concludes the proof fact 1. We infer the required conclusion of lemma 9. \( \square \)

**Proof of the main theorem**

**Proof** We show that the recursive definition of \( \beta_k \) in the statement of lemma 9 yields the formula

\[
\beta_k = \frac{i}{k} d_1^{S_1} \nabla^{S_k} \sigma_{k-1} + \frac{1}{(k+1)!} \text{Sym}_{3,\ldots,k+2} \theta_k ,
\]

(5.4)

\[
\theta_k := \sum_{r=2}^{k-2} \frac{(r+2)!}{r+1} (i d_1^{S_1})^{k-r} \nabla^{S_r} \sigma_r + \frac{1}{k} \left( i d_1^{S_1} \right)^{k-2} \beta_2
\]

\[+ \sum_{r=3}^{k} \frac{(r+1)!}{r-1} \sum_{p=2}^{r-1} (i d_1^{S_1})^{k-r} (p S_p \wedge_1 S_{r-p+1}) ,
\]

(5.4)
for all \( k \geq 3 \). We show (5.4) by induction on \( k \). We notice first that the recursive definition of \( \beta_k \) rewrites as

\[
\beta_k = \frac{i}{k} d_1^{\nabla^1 \nabla^1} \sigma_{k-1} + \text{Sym}_{3,\ldots,k+2} \left[ \frac{i}{(k+1)!} d_1^{\nabla^1 \nabla^1} \beta_{k-1} + \frac{1}{k!} \sum_{p=2}^{k-1} p S_p \land 1 S_{k-p+1} \right],
\]

and we write

\[
\beta_{k+1} = \frac{i}{k+1} d_1^{\nabla^1 \nabla^1} \sigma_k + \text{Sym}_{3,\ldots,k+3} \left[ \frac{i}{(k+2)!} d_1^{\nabla^1 \nabla^1} \beta_k \right],
\]

Using the inductive assumption (5.4) we infer the expressions

\[
\frac{i}{(k+2)!} d_1^{\nabla^1 \nabla^1} \beta_k = \frac{1}{(k+1)!} \sum_{r=2}^{k-2} (r+2)! \left( \frac{i}{(k+1)!} d_1^{\nabla^1 \nabla^1} \right)^{k+1-r} \nabla^1 \sigma_r + 3! \left( \frac{i}{(k+1)!} d_1^{\nabla^1 \nabla^1} \right)^{k-1} \beta_2
\]

This combined with the identity \( \text{Sym}_{4,\ldots,k+1} \text{Sym}_{4,\ldots,k+3} = k! \text{Sym}_{3,\ldots,k+3} \), yields

\[
\beta_{k+1} = \frac{i}{k+1} d_1^{\nabla^1 \nabla^1} \nabla^1 \sigma_k + \frac{1}{(k+2)!} \text{Sym}_{3,\ldots,k+3} (i d_1^{\nabla^1 \nabla^1})^3 \nabla^1 \sigma_{k-1}
\]

Putting the terms together we obtain (5.4) for \( \beta_{k+1} \). Then the obvious identity \( d_1^{\nabla^1 \nabla} = \text{Alt}_2 \nabla^2 \), combined with the formula (5.5) below, allows to conclude the
required expression of $\beta_k \equiv \beta_k (\sigma_{k-1})$ in the statement of the main theorem. (We perform the change of indices $r' := r + 1$ in the above expression of $\theta_k$). This concludes the proof of the main theorem. □

We remind first the following elementary and well known fact.

**Lemma 10** For any covariant derivative operator $\nabla$ acting on the smooth sections of $\mathbb{C}T_M$ and for any tensor $\theta \in C^\infty (X, T^*_M \otimes \mathbb{C}T_M)$ holds the identity

$$\text{Alt}_2 \nabla^2 \theta = R^\nabla \theta.$$ \hspace{1cm} (5.5)

## 6 The symplectic approach

### 6.1 General facts

Let $M$ be a smooth manifold and let $\theta \in C^\infty (T^*_M, T^*_{TM})$ be the canonical 1-form on the total space of the cotangent bundle defined as $\theta_\lambda := \lambda \cdot d\pi_T \circ \pi_T$, for any $\lambda \in T^*_M$. The canonical symplectic form over the total space $T_M$ is defined as $\Omega := -d\theta$. Let now $g$ be a Riemann metric over $M$ viewed as a vector bundle map $\pi : T_M \rightarrow T^*_M$. We define also the forms $\theta^g := g^* \theta$ and $\Omega^g := g^* \Omega = -d\theta^g$ over the total space of the tangent bundle. In explicit terms $\theta^g_\eta = g(\eta) \cdot d\pi_{TM}$, for all $\eta \in T_M$, i.e.

$$\theta^g_\eta (\xi) = g_{\pi_{TM}(\eta)} (\eta, d\pi_{TM} \cdot \xi),$$

for all $\xi \in T_{TM,\eta}$. Let $\nabla^g$ be the Levi-Civita connection, defined as

$$2 \nabla^g_\xi \eta := g^{-1} \left[ \xi \cdot d (g\eta) + \eta \cdot d (g\xi) + d(\langle \xi, \eta \rangle)_g \right] + [\xi, \eta],$$

for any $\xi, \eta \in C^\infty (M, T_M)$. Let also $\gamma^g \in C^\infty (T_M, T_M^* \otimes T^*_M)$ be the Levi-Civita 1-form, which is determined along any section $\eta \in C^\infty (M, T_M)$, by the identity $\gamma^g_\eta \cdot d\eta = T^g_\eta \nabla^g \eta$.

For any curve $\eta : t \mapsto \eta_t \in T_M$, we define the covariant derivative

$$\frac{\nabla^g \eta_t}{dt} := T_{\eta_t}^{-1} \gamma^g_{\eta_t} \hat{\eta}_t \in T_{TM, \pi(\eta_t)}.$$

We consider now two curves $\eta_j : t \mapsto \eta_{j,t} \in T_M$, $j = 1, 2$, such that $\pi_{TM} (\eta_{1,t}) = \pi_{TM} (\eta_{2,t}) = x_t$. Then

$$\frac{d}{dt} g_{|x_t} (\eta_{1,t}, \eta_{2,t}) = g_{|x_t} \left( \frac{\nabla^g \eta_1}{dt}, \eta_{2,t} \right) + g_{|x_t} \left( \eta_{1,t}, \frac{\nabla^g \eta_2}{dt} \right).$$

With the previous notation hold the following well known lemma (see also Klingenberg’s book [KII] for a proof using local coordinates).

**Lemma 11** The formula

$$\Omega^g_\eta (\xi_1, \xi_2) = g_p (d\pi_{TM} \cdot \xi_1, T_{\eta}^{-1} \gamma^g_\eta \xi_2) - g_p (d\pi_{TM} \cdot \xi_2, T_{\eta}^{-1} \gamma^g_\eta \xi_1),$$

hold for any $\eta \in T_M$, $p = \pi_{TM} (\eta)$ and for any $\xi_1, \xi_2 \in T_{TM, \eta}$. 

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Proof With respect to a local coordinate trivialization of the tangent bundle we can extend in a linear way the vectors ξ₁, ξ₂ in to vector fields Ξ₁, Ξ₂ in a neighborhood of \( T_{M,p} \) inside \( T_M \). In this way \([Ξ₁, Ξ₂] = 0\) and thus \( Ω^g (Ξ₁, Ξ₂) = Ξ₂, θ^g (Ξ₁) − Ξ₁, θ^g (Ξ₂) \). We denote by \( η_{j,t}, j = 1, 2 \) the corresponding flow lines starting from \( η \). Then

\[
Ω^g_η (ξ₁, ξ₂) = \frac{d}{dt}|_{t=0} \left[ g_{πT_M(η_{2,t})} \left( η_{2,t}, d_{n_{ξ₂}} πT_M, Ξ₁ (η_{2,t}) \right) \right]
\]

\[
− \frac{d}{dt}|_{t=0} \left[ g_{πT_M(η_{1,t})} \left( η_{1,t}, d_{n_{ξ₁}} πT_M, Ξ₂ (η_{1,t}) \right) \right].
\]

We distinguish two cases.

• In the case when \( d_η πT_M ξ_j = 0 \) for some \( j \), say \( j = 1 \), then \( d_{n_{ξ₂}} πT_M, Ξ₁ (η_{2,t}) = 0 \) and

\[
\frac{d}{dt}|_{t=0} d_{n_{ξ₁}} πT_M, Ξ₂ (η_{1,t}) = 0,
\]

by the linear nature of the local extension. Then

\[
Ω^g_η (ξ₁, ξ₂) = − g_π (T_{η}^{−1} γ^g_η ξ₁, d_η πT_M, ξ₂).
\]

The case \( j = 2 \) is quite similar.

• In the case when \( d_η πT_M ξ_j \), do not vanish for \( j = 1, 2 \), then the vector fields \( ξ_j := d_π T_M, Ξ_j \) are well defined and \( [ξ₁, ξ₂] = 0 \). Then

\[
Ω^g_η (ξ₁, ξ₂) = g_π (T_{η}^{−1} γ^g_η ξ₂, d_η πT_M, ξ₁) + g_π (η, \nabla^g_{ξ₂ (p)} ξ₁ − \nabla^g_{ξ₁ (p)} ξ₂)
\]

\[
− g_π (T_{η}^{−1} γ^g_η ξ₁, d_η πT_M, ξ₂)
\]

\[
= g_π (T_{η}^{−1} γ^g_η ξ₂, d_η πT_M, ξ₁) + g_π (η, [ξ₁, ξ₂] (p))
\]

\[
− g_π (T_{η}^{−1} γ^g_η ξ₁, d_η πT_M, ξ₂),
\]

which implies the required conclusion.  

We need to remind in detail also the following very well known lemma (see also [Kli]).

Lemma 12 Let \( 2 ξ^g := Ω^{g−1} |d|_{η}^2 \) and let \( Φ^η_1 \) be the corresponding 1-parameter sub-group of transformations of \( T_M \). Then for any \( η \in T_M \) the curve \( c_t := πT_M \circ Φ^η_1 (η) \) is the geodesic with initial speed \( c_0 = η \) and \( c_t = Φ^η_1 (η) \).

Proof For any \( η \in T_M \) and for any \( ξ \in T_{T_M,η} \), let \( t \mapsto η_t \in T_M \) be the curve such that \( η_0 = ξ \). Then

\[
ξ : | |^2 = \frac{d}{dt}|_{t=0} \left[ g_{πT_M(η_t)} (η_t, η_t) \right]
\]

\[
= 2 g_π (η, T_{η}^{−1} γ^g_η ξ),
\]

and thus

\[
Ω^g_η (ξ^g, ξ) = g_π (η, T_{η}^{−1} γ^g_η ξ),
\]

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by the definition of the vector field $\zeta_\eta^g$. Using lemma [11] we infer

$$g_p(d_\eta \pi_{TM} \zeta_\eta^g, T_\eta^{-1} \gamma^g_\eta \xi) - g_p(d_\eta \pi_{TM} \xi, T_\eta^{-1} \gamma^g_\eta \zeta_\eta^g) = g_p \left( \eta, T_\eta^{-1} \gamma^g_\eta \xi \right).$$

In the case $d_\eta \pi_{TM} \xi = 0$, the identity (6.1) yields

$$g_p(d_\eta \pi_{TM} \zeta_\eta^g, T_\eta^{-1} \gamma^g_\eta \xi) = g_p \left( \eta, T_\eta^{-1} \gamma^g_\eta \xi \right),$$

and thus $d_\eta \pi_{TM} \zeta_\eta^g = \eta$. In the case $\gamma^g_\eta \xi = 0$, the identity (6.1) yields

$$g_p(d_\eta \pi_{TM} \xi, T_\eta^{-1} \gamma^g_\eta \zeta_\eta^g) = 0,$$

and thus $\gamma^g_\eta \zeta_\eta^g = 0$. We deduce the formula

$$\zeta_\eta^g = H^g_\eta \cdot \eta.$$ 

Thus the flow line $\eta_t := \Phi_t^g(\eta)$ satisfies the identity

$$\dot{\eta}_t = H^g_{\eta_t} \cdot \eta_t.$$ 

We deduce

$$\dot{c}_t = d_\eta \pi_{TM} \cdot \dot{\eta}_t$$

$$= d_\eta \pi_{TM} \cdot H^g_{\eta_t} \cdot \eta_t$$

$$= \eta_t,$$

and $\ddot{c}_t = H^g_{\dot{c}_t} \cdot \dot{c}_t$, which is the geodesic equation. □

We provide now a proof of the following well known result due to Lempert-Szöke [Le-Sz1]. See also Guillemin-Stenzel [Gu-St], Burns [Bu1, Bu2] and Burns-Halverscheid-Hind [BHH].

**Corollary 4** Let $(M, g)$ be a smooth Riemannian manifold. A complex structure $J$ over the total space of the tangent bundle $T_M$ satisfies the conditions

$$J_M = J^{can},$$

$$2\theta^g = d| \cdot |^2_g \cdot J,$$

if and only if for any $\eta \in T_M$, the smooth map $\psi_\eta: t + is \mapsto s\Phi_t^g(\eta)$, defined in a neighborhood of $0 \in \mathbb{C}$, is $J$-holomorphic.

**Proof** We define the Reeb vector field $\Xi := \Omega^{g^{-1}} \theta^g$. This vector field is independent of the metric $g$. Indeed by lemma [11] the identity

$$g_p(\eta, d_\eta \pi_{TM} \xi) = g_p(d_\eta \pi_{TM} \Xi, T_\eta^{-1} \gamma^g_\eta \xi) - g_p(d_\eta \pi_{TM} \xi, T_\eta^{-1} \gamma^g_\eta \Xi),$$

holds for any $\xi \in T_{TM, \eta}$. Thus if $d_\eta \pi_{TM} \xi = 0$ we deduce the equality

$$g_p(d_\eta \pi_{TM} \Xi, T_\eta^{-1} \xi) = 0,$$
and thus $d_\eta \pi T_M \Xi_\eta = 0$. Then the identity (6.6) reduces as
\[ g_p(\eta, d_\eta \pi T_M \xi) = - g_p(d_\eta \pi T_M \xi, T_\eta^{-1} \Xi_\eta), \]
for any $\xi \in T_{TM, \eta}$. We infer the formula
\[ \Xi_\eta = - T_\eta \cdot \eta, \tag{6.7} \]
for all $\eta \in T_M$. We notice now that the identity (6.5) is equivalent to the identity
\[ \Omega^g(2 \Xi, \xi) = d | \cdot |^2 g J \xi, \]
and is also equivalent to the identity $\theta^g = - d_\eta \cdot | \cdot |^2 g$. Thus
\[ \Omega^g = d_\eta \cdot | \cdot |^2 g = i \partial \overline{\partial} | \cdot |^2 g, \]
thanks to the fact that $J$ is integrable. We infer that the symplectic form $\Omega^g$ is $J$-invariant. Thus
\[ \Omega^g(2 J \Xi, J \xi) = d | \cdot |^2 g J \xi, \tag{6.8} \]
i.e.
\[ J \Xi = \Xi^g. \tag{6.9} \]
This combined with (6.7) and with (6.2) implies that (6.5) is equivalent to the identity
\[ J_\eta H^g_\eta \cdot \eta = T_\eta \cdot \eta. \tag{6.9} \]
We show now that the later combined with (6.4) is equivalent to the $J$-holomorphy of the maps $\psi_\eta$. For this purpose we observe that the differential of such maps is given by
\[ d_{t_0 + is_0} \psi_\eta \left( a \frac{\partial}{\partial t} + b \frac{\partial}{\partial s} \right) = a d(\eta_0 \psi_{T_{TM}}) \Phi^g_{t_0} (\eta) + b T_{\eta_0} \Phi^g_{t_0} (\eta). \]
But
\[ \Phi^g_{t_0} (\eta) = \zeta^g \circ \Phi^g_{t_0} (\eta) = H^g_{\Phi^g_{t_0} (\eta)} \cdot \Phi^g_{t_0} (\eta), \]
thanks to (6.2). Then using the property (6.3) of the linear connection $\nabla^g$ we infer
\[ d_{t_0 + is_0} \psi_\eta \left( a \frac{\partial}{\partial t} + b \frac{\partial}{\partial s} \right) = \left( a H^g_{\Phi^g_{t_0} (\eta)} + b T_{\eta_0} \Phi^g_{t_0} (\eta) \right) \cdot \Phi^g_{t_0} (\eta). \tag{6.10} \]
The smooth map $\psi_\eta$ is $J$-holomorphic if and only if
\[ d_{t_0 + is_0} \psi_\eta \left( -b \frac{\partial}{\partial t} + a \frac{\partial}{\partial s} \right) = J d_{t_0 + is_0} \psi_\eta \left( a \frac{\partial}{\partial t} + b \frac{\partial}{\partial s} \right), \]
thus, if and only if
\[ \left( -b H^g_{\Phi^g_{t_0} (\eta)} + a T_{\eta_0} \Phi^g_{t_0} (\eta) \right) \cdot \Phi^g_{t_0} (\eta) = J \left( a H^g_{\Phi^g_{t_0} (\eta)} + b T_{\eta_0} \Phi^g_{t_0} (\eta) \right) \cdot \Phi^g_{t_0} (\eta). \]
For $s_0 \neq 0$ this is equivalent to (6.3). For $s_0 = 0$ this is equivalent to (6.4). We deduce the required conclusion.

The condition (6.3) implies that $J$ is an $M$-totally real complex structure.

We can provide now the proof of corollary 1.
6.2 Proof of corollary 1

Proof If we write \( A = \alpha + iTB \) and \( \alpha = H^g - TT \), then
\[
S := T^{-1}(H^g - \mathcal{T}) = \Gamma + iB.
\]
We set \( S_k = \Gamma_k + iB_k \). From the proof of corollary 4 we know that in the case \( J \) is integrable over \( U \), the curve \( \psi_\eta \) is \( J \)-holomorphic if and only if hold (6.3).
The later rewrites as
\[
H^g_\eta \cdot \eta = -J_\eta T_\eta \cdot \eta.
\]
Using (1.7) we infer that the previous identity is equivalent to
\[
H^g_\eta \cdot \eta = \alpha_\eta B^{-1}_\eta \cdot \eta.
\]
(6.11)
Taking \( d_\eta \pi \) on both sides of (6.11) we deduce \( \eta = B^{-1}_\eta \cdot \eta \). Therefore (6.11) is equivalent to the system
\[
\begin{align*}
\sum_{k \geq 1} B_k (\eta^{k+1}) &= 0, \\
\sum_{k \geq 1} \Gamma_k (\eta^{k+1}) &= 0.
\end{align*}
\]
and thus as \( S_k (\eta^{k+1}) = 0 \) for all \( k \geq 1 \). We remind now that, according to theorem 2 the integrability of the structure \( J \) implies the condition \( S_1 \in C^\infty(M, S^2T^*_M \otimes \mathbb{C}T_M) \). We infer \( S_1 = 0 \). We notice that, with the notation of the statement of theorem 2 the equation \( \text{Circ} \beta_k = 0 \) hold for all \( k \geq 1 \). This combined with the identity
\[
[Circ, \text{Sym}_{2,\ldots,k+2}] = 0,
\]
implies
\[
\text{Circ} \beta_{2,\ldots,k+2} \beta_k = 0,
\]
(6.13)
for all \( k \geq 1 \). So if we apply the Circ operator to both sides of the definition of \( S_2 \) in the statement of theorem 2 we infer \( \text{Circ} S_2 = \text{Circ} \sigma_2 = 3\sigma_2 \). If we evaluate this equality to \( \eta^3 \) we infer \( S_2 (\eta^3) = \sigma_2 (\eta^3) \), which implies \( \sigma_2 = 0 \). We show now by induction that \( \sigma_k = 0 \) for all \( k \geq 2 \). Indeed by the inductive assumption
\[
S_{k+1} = \frac{i}{(k+2)!} \text{Sym}_{2,\ldots,k+2} \beta_k + \sigma_{k+1}.
\]
Applying the Circ operator to both sides of this identity and using the equation (6.13), we infer \( \text{Circ} S_{k+1} = \text{Circ} \sigma_{k+1} = 3\sigma_{k+1} \), which evaluated at \( \eta^{k+2} \) gives \( S_{k+1} (\eta^{k+2}) = \sigma_{k+1} (\eta^{k+2}) \). We deduce \( \sigma_{k+1} = 0 \). Using the identity
\[
\text{Sym}_{2,\ldots,k+1} \text{Sym}_{3,\ldots,k+1} = (k-1)! \text{Sym}_{2,\ldots,k+1},
\]
(6.14)
we infer from the statement of theorem 2 and with the notation there
\[ S_k = i \frac{k!}{(k+1)!} \theta_{k-1}, \]
for \( k \geq 2 \), with \( \theta_1 := 2Rg \) and
\[ \theta_k := -2i(id_1^\nabla g)^{k-2} (\nabla g R)^2 + \sum_{r=4}^{k+1} r! \sum_{p=2}^{r-2} (id_1^\nabla g)^{k+1-r} (pS_p \wedge S_{r-p}), \]
for all \( k \geq 2 \). Moreover we observe that the equation \( \text{Circ} \beta_k = 0, k \geq 3 \) rewrites as
\[ \text{Circ} \text{Sym}_{3, \ldots, k+2} \theta_k = 0. \]
If we set \( \Theta_k (g) := \theta_k - 1 \), for all \( k \geq 2 \) we obtain the required expansion.
On the other hand if the expansion in the statement of the lemma under consideration hold then \( J \) is integrable thanks to theorem 2 and \( \text{Circ} S_k = 0 \) for all \( k \geq 2 \), \( (S_1 = 0) \). Indeed for \( k = 2, 3 \) this equality follows from the identities \( \text{Circ} \Theta_k (g) = 0 \) and
\[ [\text{Circ}, \text{Sym}_{3, \ldots, k+1}] = 0. \] (6.15)
For \( k \geq 4 \), we use the identities (6.13) (6.14) and the integrability equations satisfied by the metric \( g \). We deduce \( S_k (g^{k+1}) = 0 \), for all \( k \geq 1 \), which is equivalent to (6.11) and so to the fact that the curves \( \psi_\eta \) are \( J \)-holomorphic. □

7 Proof of the proposition 1

Proof We expand first the term
\[ d_1^\nabla (\nabla R^\nabla)_2 (\xi_1, \xi_2, \xi_3, \xi_4, \xi_5) \]
\[ = \nabla_{\xi_1} (\nabla R^\nabla)_2 (\xi_2, \xi_3, \xi_4, \xi_5) - \nabla_{\xi_2} (\nabla R^\nabla)_2 (\xi_1, \xi_3, \xi_4, \xi_5) \]
\[ = \nabla^2 R^\nabla (\xi_1, \xi_3, \xi_2, \xi_4, \xi_5) - \nabla^2 R^\nabla (\xi_2, \xi_3, \xi_1, \xi_4, \xi_5) \]
\[ = \nabla^2 R^\nabla (\xi_1, \xi_3, \xi_2, \xi_4, \xi_5) + \nabla^2 R^\nabla (\xi_2, \xi_3, \xi_4, \xi_1, \xi_5) \]
\[ = \nabla^2 R^\nabla (\xi_3, \xi_1, \xi_2, \xi_4, \xi_5) + \nabla^2 R^\nabla (\xi_3, \xi_2, \xi_4, \xi_1, \xi_5) \]
\[ + (R^\nabla \cdot R^\nabla) (\xi_1, \xi_3, \xi_2, \xi_4, \xi_5) + (R^\nabla \cdot R^\nabla) (\xi_2, \xi_3, \xi_4, \xi_1, \xi_5), \]
thanks to formula (5.5). Using the differential Bianchi identity we infer
\[ d_1^\nabla (\nabla R^\nabla)_2 (\xi_1, \xi_2, \xi_3, \xi_4, \xi_5) \]
\[ = -\nabla^2 R^\nabla (\xi_3, \xi_4, \xi_1, \xi_2, \xi_5) \]
\[ + (R^\nabla \cdot R^\nabla) (\xi_1, \xi_3, \xi_2, \xi_4, \xi_5) + (R^\nabla \cdot R^\nabla) (\xi_2, \xi_3, \xi_4, \xi_1, \xi_5) . \]

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In order to simplify the notation in the computations that will follow we will use from now on the identification

$$\theta (\xi_1, \xi_2, \xi_3, \xi_4, \xi_5) \equiv \theta (12345) ,$$

for any tensor \(\theta\). We expand now the term

$$\text{Circ Sym}_{3,4,5} d_1^\nabla (\nabla R) .$$

We let

$$\theta (12345) := \nabla^2 R \nabla (34125) ,$$

and we observe the identities

\[
\begin{align*}
\text{(Sym}_{3,4,5} \theta) (12345) &= \nabla^2 R (34125) + \nabla^2 R (35124) + \nabla^2 R (43125) \\
&+ \nabla^2 R (45123) + \nabla^2 R (53124) + \nabla^2 R (54123) , \\
\text{(Sym}_{3,4,5} \theta) (23145) &= \nabla^2 R (14235) + \nabla^2 R (15234) + \nabla^2 R (41235) \\
&+ \nabla^2 R (45231) + \nabla^2 R (51234) + \nabla^2 R (54231) , \\
\text{(Sym}_{3,4,5} \theta) (31245) &= \nabla^2 R (24315) + \nabla^2 R (25314) + \nabla^2 R (42315) \\
&+ \nabla^2 R (45312) + \nabla^2 R (52314) + \nabla^2 R (54312) ,
\end{align*}
\]

Summing up we obtain

\[
\text{(Circ Sym}_{3,4,5} \theta) (12345)
\]

\[
= \nabla^2 R (34125) + \nabla^2 R (14235) + \nabla^2 R (24315) \\
+ \nabla^2 R (35124) + \nabla^2 R (15234) + \nabla^2 R (25314) \\
+ \nabla^2 R (43125) + \nabla^2 R (41235) + \nabla^2 R (42315) \\
+ \nabla^2 R (45123) + \nabla^2 R (45231) + \nabla^2 R (45312) \\
+ \nabla^2 R (53124) + \nabla^2 R (51234) + \nabla^2 R (52314) \\
+ \nabla^2 R (54123) + \nabla^2 R (54231) + \nabla^2 R (54312) ,
\]

where we denote by \(\nabla^2 R (\cdots \cdots)\) the terms that summed up together equal zero thanks to the differential Bianchi identity for \(j = 1, 3\) and thanks to the
algebraic Bianchi identity for $j = 2, 4$. Using formula (5.5) we infer

\[(\text{Circ Sym}_{3,4,5} \theta) (12345)\]

\[= \nabla^2 R^\nabla (43125)_i + \nabla^2 R^\nabla (41235)_i + \nabla^2 R^\nabla (42315)_i ,\]

\[+ \nabla^2 R^\nabla (53124)_i + \nabla^2 R^\nabla (51234)_i + \nabla^2 R^\nabla (52314)_i ,\]

\[+ (R^\nabla \cdot R^\nabla) (34125) + (R^\nabla \cdot R^\nabla) (14235) + (R^\nabla \cdot R^\nabla) (24315)\]

\[+ (R^\nabla \cdot R^\nabla) (35124) + (R^\nabla \cdot R^\nabla) (15234) + (R^\nabla \cdot R^\nabla) (25314),\]

where as before we denote by $\nabla^2 R^\nabla (\cdots)$ the terms that summed up together equal zero thanks to the differential Bianchi identity. We deduce the expression

\[(\text{Circ Sym}_{3,4,5} \theta) (12345)\]

\[= (R^\nabla \cdot R^\nabla) (34125) + (R^\nabla \cdot R^\nabla) (14235) + (R^\nabla \cdot R^\nabla) (24315)\]

\[+ (R^\nabla \cdot R^\nabla) (35124) + (R^\nabla \cdot R^\nabla) (15234) + (R^\nabla \cdot R^\nabla) (25314) .\] (7.1)

We set now for notation simplicity $\rho := R^\nabla \cdot R^\nabla$ and let

\[\Theta (12345) := \rho (13245) + \rho (23415) .\]

We observe that, by definition, the tensor

\[\rho \in C^\infty \left( M, \Lambda^2 T^*_M \otimes_\mathbb{R} \Lambda^2 T^*_M \otimes_\mathbb{R} T^*_M \otimes_\mathbb{R} \mathbb{C} T_M \right) ,\]

satisfies the circular identity with respect to its last three entries. We expand now the term

\[\text{Circ Sym}_{3,4,5} \Theta .\]

We observe the identities

\[(\text{Sym}_{3,4,5} \Theta) (12345) = \rho (13245) + \rho (23415)\]

\[+ \rho (13254) + \rho (23514)\]

\[+ \rho (14235) + \rho (24315)\]

\[+ \rho (14253) + \rho (24513)\]

\[+ \rho (15234) + \rho (25314)\]

\[+ \rho (15243) + \rho (25413) ,\]
\[(\text{Sym}_{3,4,5,\Theta} (23145) = \rho (21345) + \rho (31425)
+ \rho (21354) + \rho (31524)
+ \rho (24315) + \rho (34125)
+ \rho (24351) + \rho (34521)
+ \rho (25314) + \rho (35124)
+ \rho (25341) + \rho (35421),\]

\[(\text{Sym}_{3,4,5,\Theta} (31245) = \rho (32145) + \rho (12435)
+ \rho (32154) + \rho (12534)
+ \rho (34125) + \rho (14235)
+ \rho (34152) + \rho (14532)
+ \rho (35124) + \rho (15234)
+ \rho (35142) + \rho (15432).\]
Summing up we obtain

\[(\text{Circ Sym}_{3,4,5} \Theta) (12345)\]

\[= \rho(13245)_1 + \rho(23415)_2 + \rho(21345)_3 + \rho(31425)_i + \rho(32145)_2 + \rho(12435)_3 + \rho(13254)_4 + \rho(23514)_5 + \rho(21354)_6 + \rho(31524)_7 + \rho(34125)_8 + \rho(14235)_9 + \rho(24315)_10 + \rho(14253)_11 + \rho(15234)_12 + \rho(24513)_10 + \rho(24351)_11 + \rho(35124)_12 + \rho(35124)_11 + \rho(15432)_10 + \rho(15243)_10 + \rho(15243)_11 + \rho(25413)_11 + \rho(25341)_11 + \rho(35421)_12 + \rho(35421)_11 + \rho(35142)_12 + \rho(15432)_11 + \rho(15234)_10 ,\]

where we denote by \(\rho(\cdots)_j\) the terms that we sum up together using the symmetries of \(\rho\). We obtain

\[(\text{Circ Sym}_{3,4,5} \Theta) (12345)\]

\[= 2\rho(13245) + 2\rho(23415) + 2\rho(12435) + 2\rho(13254) + 2\rho(23514) + 2\rho(15234) + 3\rho(14235) + 3\rho(24315) + 3\rho(34125) + 3\rho(15234) + 3\rho(25314) + 3\rho(35124) .\]
We conclude the expression
\[
\left[ \text{Circ Sym}_{3,4,5} \partial_1^\nabla (\nabla R^\nabla) \right] (12345) \\
= 2\rho (13245) + 2\rho (23415) + 2\rho (12435) \\
+ 2\rho (13254) + 2\rho (23514) + 2\rho (12534) \\
+ 2\rho (14235) + 2\rho (24315) + 2\rho (34125) \\
+ 2\rho (15234) + 2\rho (25314) + 2\rho (35124)
\] (7.2)

We expand now the term
\[
\text{Circ Sym}_{3,4,5}(\tilde{R}^\nabla \wedge_1 \tilde{R}^\nabla).
\]

From now on we will denote for notation simplicity \((123) \equiv R^\nabla (123)\) and
\[
[123] := (123) + (132).
\]

We observe the identities
\[
\left[ \text{Sym}_{3,4,5}(\tilde{R}^\nabla \wedge_1 \tilde{R}^\nabla) \right] (12345) = [1 [234] 5] - [2 [134] 5] \\
+ [1 [235] 4] - [2 [135] 4] \\
+ [1 [243] 5] - [2 [143] 5] \\
+ [1 [245] 3] - [2 [145] 3] \\
+ [1 [253] 4] - [2 [153] 4] \\
+ [1 [254] 3] - [2 [154] 3],
\]
\[
\left[ \text{Sym}_{3,4,5}(\tilde{R}^\nabla \wedge_1 \tilde{R}^\nabla) \right] (23145) = [2 [314] 5] - [3 [214] 5] \\
+ [2 [315] 4] - [3 [215] 4] \\
+ [2 [341] 5] - [3 [241] 5] \\
+ [2 [345] 1] - [3 [245] 1] \\
+ [2 [351] 4] - [3 [251] 4] \\
+ [2 [354] 1] - [3 [254] 1].
\]
\[
\left[\text{Sym}_{3,4,5}(\tilde{R}^\nabla \wedge_1 \tilde{R}^\nabla)\right] (31245) = [3 [124] 5] - [1 [324] 5] \\
+ [3 [125] 4] - [1 [325] 4] \\
+ [3 [142] 5] - [1 [342] 5] \\
+ [3 [145] 2] - [1 [345] 2] \\
+ [3 [152] 4] - [1 [352] 4] \\
+ [3 [154] 2] - [1 [354] 2] .
\]

Summing up using the symmetries of \([\cdots]\) and \((\cdots)\) we obtain
\[
\left[\text{Circ Sym}_{3,4,5}(\tilde{R}^\nabla \wedge_1 \tilde{R}^\nabla)\right] (12345) \\
= 6 [1 (234) 5] + 6 [2 (314) 5] + 6 [3 (124) 5] \\
+ 6 [1 (235) 4] + 6 [2 (315) 4] + 6 [3 (125) 4] \\
+ 2 [1 [245] 3]_1 - 2 [2 [145] 3]_2 \\
+ 2 [2 [345] 1]_1 - 2 [3 [245] 1]_1 \\
+ 2 [3 [145] 2]_2 - 2 [1 [345] 2]_2 .
\]

We combine now the terms \([\cdots]_j\), for each \(j = 1, 2, 3\) and we explicit and simplify them by using the algebraic Bianchi identity. We obtain
\[
\left[\text{Circ Sym}_{3,4,5}(\tilde{R}^\nabla \wedge_1 \tilde{R}^\nabla)\right] (12345) \\
= 6 [1 (234) 5] + 6 [2 (314) 5] + 6 [3 (124) 5] \\
+ 6 [1 (235) 4] + 6 [2 (315) 4] + 6 [3 (125) 4] \\
+ 6 (13 [245]) + 6 (32 [145]) + 6 (21 [345]) .
\]
Expanding further we obtain the complete expansion

\[
\left[ \text{Circ Sym}_3,4,5 \left( \tilde{R}^\nabla \wedge_1 \tilde{R}^\nabla \right) \right] (12345) \\
= 6 \left( 1 \left( 234 \right) 5 \right) + 6 \left( 15 \left( 234 \right) \right) \\
+ 6 \left( 2 \left( 314 \right) 5 \right) + 6 \left( 25 \left( 314 \right) \right) \\
+ 6 \left( 3 \left( 124 \right) 5 \right) + 6 \left( 35 \left( 124 \right) \right) \\
+ 6 \left( 1 \left( 235 \right) 4 \right) + 6 \left( 14 \left( 235 \right) \right) \\
+ 6 \left( 2 \left( 315 \right) 4 \right) + 6 \left( 24 \left( 315 \right) \right) \\
+ 6 \left( 3 \left( 125 \right) 4 \right) + 6 \left( 34 \left( 125 \right) \right) \\
+ 6 \left( 13 \left( 245 \right) \right) + 6 \left( 13 \left( 254 \right) \right) \\
+ 6 \left( 32 \left( 145 \right) \right) + 6 \left( 32 \left( 154 \right) \right) \\
+ 6 \left( 21 \left( 345 \right) \right) + 6 \left( 21 \left( 354 \right) \right) .
\]

Expanding the terms \( \rho \) present in the expression (7.2) we obtain the complete expansion of the term

\[
\left\{ \text{Circ Sym}_3,4,5 \left[ 3d_i^\nabla \left( \nabla R^\nabla \right)_2 - 2 \tilde{R}^\nabla \wedge_1 \tilde{R}^\nabla \right] \right\} (12345) ,
\]
given by

\[
\left\{ \text{Circ Sym}_{3,4,5} \left[ 3d_1^F \left( \nabla R^F \right)_2 - 2 \tilde{R}^F \wedge_1 \tilde{R}^F \right] \right\}_{(12345)}
\]

\[
= 6 \left( 13 (245) \right)_1 - 6 \left( (132) 45 \right)_2 - 6 \left( 2 (134) 5 \right)_3 - 6 \left( 24 (135) \right)_4
\]

\[
+ 6 \left( 23 (15) \right)_5 - 6 \left( 2 (234) 15 \right)_6 - 6 \left( 4 (231) 5 \right)_7 - 6 \left( 41 (235) \right)_8
\]

\[
+ 6 \left( 12 (435) \right)_9 - 6 \left( (124) 35 \right)_10 - 6 \left( 4 (123) 5 \right)_11 - 6 \left( 43 (125) \right)_12
\]

\[
+ 6 \left( 13 (254) \right)_{13} - 6 \left( (132) 54 \right)_{14} - 6 \left( 2 (135) 4 \right)_{15} - 6 \left( 25 (134) \right)_{16}
\]

\[
+ 6 \left( 23 (514) \right)_{17} - 6 \left( (235) 14 \right)_{18} - 6 \left( 5 (231) 4 \right)_{19} - 6 \left( 51 (234) \right)_{20}
\]

\[
+ 6 \left( 12 (534) \right)_{21} - 6 \left( (125) 34 \right)_{22} - 6 \left( 5 (123) 4 \right)_{23} - 6 \left( 53 (124) \right)_{24}
\]

\[
+ 6 \left( 14 (235) \right)_{25} - 6 \left( (142) 35 \right)_{26} - 6 \left( 2 (143) 5 \right)_{27} - 6 \left( 23 (145) \right)_{28}
\]

\[
+ 6 \left( 24 (315) \right)_{29} - 6 \left( (243) 15 \right)_{30} - 6 \left( 3 (241) 5 \right)_{31} - 6 \left( 31 (245) \right)_{32}
\]

\[
+ 6 \left( 34 (125) \right)_{33} - 6 \left( (341) 25 \right)_{34} - 6 \left( 1 (342) 5 \right)_{35} - 6 \left( 12 (345) \right)_{36}
\]

\[
+ 6 \left( 15 (234) \right)_{37} - 6 \left( (152) 34 \right)_{38} - 6 \left( 2 (153) 4 \right)_{39} - 6 \left( 23 (154) \right)_{40}
\]

\[
+ 6 \left( 25 (314) \right)_{41} - 6 \left( (253) 14 \right)_{42} - 6 \left( 3 (251) 4 \right)_{43} - 6 \left( 31 (254) \right)_{44}
\]

\[
+ 6 \left( 35 (124) \right)_{45} - 6 \left( (351) 24 \right)_{46} - 6 \left( 1 (352) 4 \right)_{47} - 6 \left( 12 (354) \right)_{48}
\]

\[
- 12 \left( 1 (234) 5 \right)_{49} - 12 \left( 15 (234) \right)_{50}
\]

\[
- 12 \left( 2 (314) 5 \right)_{51} - 12 \left( 25 (314) \right)_{52}
\]

\[
- 12 \left( 3 (124) 5 \right)_{53} - 12 \left( 35 (124) \right)_{54}
\]

\[
- 12 \left( 1 (235) 4 \right)_{55} - 12 \left( 14 (235) \right)_{56}
\]

\[
- 12 \left( 2 (315) 4 \right)_{57} - 12 \left( 24 (315) \right)_{58}
\]

\[
- 12 \left( 3 (125) 4 \right)_{59} - 12 \left( 12 (345) \right)_{60} - 12 \left( 12 (345) \right)_{61} - 12 \left( 13 (245) \right)_{62}
\]

\[
- 12 \left( 32 (145) \right)_{63} - 12 \left( 32 (154) \right)_{64} - 12 \left( 21 (345) \right)_{65} - 12 \left( 21 (354) \right)_{66}
\]
where as before we denote by $(\cdots \cdots)$ the terms that we sum up together using the symmetries of the curvature tensor $R^\nabla$. All the terms summed up together cancel up. This is obvious for all the sub indexes $j$ with the exception of $j = 3, 6, 9, 13, 16, 19$ for which we must provide the detail of the computation. Indeed for $j = 3$ we have

$$-6 ((341) 25) - 6 (2 (134) 5) - 6 (2 (143) 5) - 12 (2 (314) 5)$$

$$= 6 (2 (341) 5) + 6 (2 (413) 5) - 6 (2 (314) 5)$$

$$= -6 (2 (134) 5) - 6 (2 (314) 5)$$

$$= 0.$$  

For $j = 6$ we have

$$-12 (1 (234) 5) - 6 ((234) 15) - 6 ((243) 15) - 6 (1 (342) 5)$$

$$= 6 (1 (243) 5) + 6 (1 (432) 5) - 6 (1 (234) 5)$$

$$= -6 (1 (324) 5) - 6 (1 (234) 5)$$

$$= 0.$$  

For $j = 9$ we have

$$-6 ((124) 35) - 6 ((142) 35) - 6 (3 (241) 5) - 12 (3 (124) 5)$$

$$= 6 (3 (142) 5) + 6 (3 (421) 5) - 6 (3 (124) 5)$$

$$= -6 (3 (214) 5) - 6 (3 (124) 5)$$

$$= 0.$$  

For $j = 13$ we have

$$-6 ((351) 24) - 6 (2 (135) 4) - 6 (2 (153) 4) - 12 (2 (315) 4)$$

$$= 6 (2 (351) 4) + 6 (2 (513) 4) - 6 (2 (315) 4)$$

$$= -6 (2 (135) 4) - 6 (2 (315) 4)$$

$$= 0.$$
For $j = 16$ we have

$$-12 (1 \, 235 \, 4) - 6 ((235) \, 14) - 6 ((253) \, 14) - 6 (1 \, 352 \, 4)$$

$$= 6 (1 \, 235 \, 4) + 6 (1 \, 352 \, 4) - 6 (1 \, 235 \, 4)$$

$$= -6 (1 \, 325 \, 4) - 6 (1 \, 235 \, 4)$$

$$= 0.$$  

For $j = 19$ we have

$$-6 ((125) \, 34) - 6 ((152) \, 34) - 6 (3 \, 251 \, 4) - 12 (3 \, 125 \, 4)$$

$$= 6 (3 \, 152 \, 4) + 6 (3 \, 521 \, 4) - 6 (3 \, 125 \, 4)$$

$$= -6 (3 \, 215 \, 4) - 6 (3 \, 125 \, 4)$$

$$= 0.$$  

We infer the required identity (1.16) in the statement of proposition 1. □

8 The almost complex structure associated to a connection over the tangent bundle

This section is not needed for the proof of the results in the paper. We include it in order to clarify the integrability of an $M$-totally real almost complex structure over $T_M$ associated to the horizontal distribution of a linear connection. We include it also to remind and to prove in modern terms a well known result due to Dombrowsky [Dom].

It is well known (see [Dom]) that we can construct an $M$-totally real almost complex structure over $T_M$ by using the horizontal distribution $\mathcal{H} \subset T_M$ associated to a linear connection $\nabla$ acting on the sections of $T_M$. Indeed in this case we set $\alpha_\eta := H_\eta$ and $B_\eta := \Pi_{T_M, \pi(\eta)}$, where $\eta \mapsto H_\eta$ is the horizontal map associated to $\mathcal{H}$. We will denote $J_\mathcal{H} := J_A$. If we define for any $\eta \in T_{M, p}$ the vertical projection $\text{Vert}_\eta : T_{T_M, \eta} \rightarrow T_{T_M, p, \eta}$ as

$$\text{Vert}_\eta := \Pi_{T_{T_M, \eta}} - H_\eta d_\eta \pi,$$

where $\pi : T_M \rightarrow M$ is the canonical projection, then

$$J_{\mathcal{H}, \eta} := -H_\eta T_\eta^{-1} \text{Vert}_\eta + T_\eta d_\eta \pi.$$  

If we decompose any vector $\xi \in \mathbb{C}T_{T_M, \eta}$ in its horizontal and vertical parts
\[ \xi = \xi^h + \xi^v \] with \[ \xi^v := \text{Vert}_\eta (\xi) \] then we have the expressions
\[ J_{H, \eta} \xi = -H_\eta T^{-1}_\eta \xi^v + T_\eta d_\eta \pi \xi^h, \]
\[ (J_{H, \eta} \xi)^h = -H_\eta T^{-1}_\eta \xi^v, \]
\[ (J_{H, \eta} \xi)^v = T_\eta d_\eta \pi \xi^h. \]

We infer
\[ \xi_{J_H}^{0,1} (\eta) = \frac{1}{2} \left[ \xi^h - i H_\eta T^{-1}_\eta \xi^v + \xi^v + i T_\eta d_\eta \pi \xi^h \right] \]
\[ = \frac{1}{2} \left[ \xi^h + H_\eta \mu + i T_\eta \left( d_\eta \pi \xi^h + \mu \right) \right], \]
with \( \mu := -i T^{-1}_\eta \xi^v \). We notice also the identity
\[ T_{T_{M}, J_H, \eta}^{0,1} = \frac{1}{2} \left( H_\eta + i T_\eta \right) \mathcal{C} T_{M, p}, \] (8.1)
for any \( \eta \in T_{X, p} \). The distribution \( T_{T_{M}, J_H, \eta}^{0,1} \) is horizontal, but the associated map does not satisfy the condition (9.5) of linear connections thanks to the identity (9.4). Therefore this distribution does not identify a linear connection. However its integrability implies that the vector bundle \( T_M \) is flat. Indeed we have the following well known lemma due to Dombrowsky [Dom].

**Lemma 13** The torsion form \( \tau^{J_H} \) of the almost complex structure \( J_H \) satisfies at the point \( \eta \in T_M \) in the directions \( V_1, V_2 \in T_{T_{M}, J_H, \eta}^{0,1} \) the identity
\[ 8 \tau^{J_H} (V_1, V_2) (\eta) = -H_\eta \left[ \tau^\nabla (v_1, v_2) + i R^\nabla (v_1, v_2) \eta \right] \]
\[ + T_\eta \left[ i \tau^\nabla (v_1, v_2) - R^\nabla (v_1, v_2) \eta \right], \]
where \( R^\nabla := \nabla^2 \) is the complex linear extension of the curvature tensor of \( \nabla \), where \( \tau^\nabla \) is the torsion of the complex connection \( \nabla \) and where \( v_j := d_\eta \pi V_j \), \( j = 1, 2 \). In particular \( J_H \) is a complex structure if and only if the linear connection \( \nabla \) is flat and torsion free.

**Proof** Let \( \xi_j \) be vector field local extensions of \( v_j \) such that \( [\xi_1, \xi_2] \pi (\eta) = 0 \). Then
\[ \Xi_j := \frac{1}{2} (H + i T) \xi_j, \]
are local vector field extensions of \( V_j \). We expand the bracket
\[ 4 \left[ \Xi_1, \Xi_2 \right] (\eta) = \left( [H \xi_1, H \xi_2] + i [H \xi_1, T \xi_2] + i [T \xi_1, H \xi_2] - [T \xi_1, T \xi_2] \right) (\eta) \]
\[ = H_\eta [\xi_1, \xi_2] - T_\eta [R^\nabla (v_1, v_2) \eta] + i T_\eta [\nabla \xi_1 \xi_2 - \nabla \xi_2 \xi_1]. \]
The last equality follows from the computation at the end of the proof of lemma 19 and thanks to the identity \((9.8)\) in the appendix. (We notice that 
\([T_{\xi_1}, T_{\xi_2}] \equiv 0\), since the vector fields \(T_{\xi_1}\) are tangent constant along the fibers).

Thanks to the assumption \([\xi_1, \xi_2] \pi (\eta) = 0\), we infer the equality

\[
4 [\Xi_1, \Xi_2] (\eta) = T_\eta \left[ i \tau^\nabla (v_1, v_2) - R^\nabla (v_1, v_2) \eta \right].
\]

The required formula follows from the identity

\[
\xi_{J_{\eta}}^{1,0} (\eta) = \frac{1}{2} \left[ \xi^h + i H_\eta T^{-1} \xi^v + \xi^v - i T_\eta d_\eta \pi \xi^h \right].
\]

The fact that the distribution \(T_{J_{\eta}}^{1,0}\) is horizontal implies that \(\tau^{J_{\eta}} (V_1, V_2) (\eta)\) vanishes for all \(V_j\) if and only if the quantity

\[
\tau^\nabla (v_1, v_2) + i R^\nabla (v_1, v_2) \eta,
\]

vanishes for all \(v_j\). In particular for real vectors \(v_j\) this implies that \(R^\nabla\) and \(\tau^\nabla\) vanish at the point \(\pi (\eta)\).

We observe that a connection over \(T_M\) is flat and torsion free if and only if there exist local parallel frames with vanishing Lie brackets.

## 9 Appendix

In this appendix we provide some well known basic facts about the geometric theory of linear connections needed for the reading of the paper. (See also [Gau]). We strongly recommend its reading even to experts.

### 9.1 The horizontal distribution associated to a linear connection

We start with the following fact.

**Lemma 14** Let \(\nabla\) be a linear connection acting on sections of a vector bundle \(E\) over a manifold \(M\). Then the linear map

\[
T_{M,p} \ni \xi \longmapsto H_\eta (\xi) := d_\eta \sigma (\xi) - T_\eta \nabla_\xi \sigma \in T_{E,\eta},
\]

is independent of the sections \(\sigma\) such that \(\sigma (p) = \eta\).

**Proof** Let \(e = (e_k)_{k=1}^r\) be a local frame of \(E\) over an open set \(U \subset M\). We consider the local expression \(\sigma = e \cdot f\) with \(f \in C^1 (U, \mathbb{R}^r)\). Let \(A \in C^\infty (U, T_M^* \otimes \mathbf{Mat}_{r \times r} (\mathbb{R}))\) be the connection form of \(\nabla\) with respect to the local frame \(e\), i.e \(\nabla e = e \cdot A\). Then \(\nabla \sigma = e \otimes (df + A \cdot f)\). If we denote by \(\theta_e : U \times \mathbb{R}^r \to E_{U(\eta)}\) then the differential of this map at the point \((p, f (p))\) provides an isomorphism

\[
d_{p,f(p)} \theta_e : T_{U,p} \oplus \mathbb{R}^r \to T_{E,\sigma(p)}.
\]

With respect to it, the equality hold

\[
d_{p,f(p)} \theta_e [\xi \oplus d_\eta f (\xi)] = d_\eta \sigma (\xi).
\]
We observe now the linear identity \( d_{r\sigma(p)} \cdot d_{p,0}\theta_e|_{0\oplus\mathbb{R}^r} = d_{p,f(p)}\theta_e|_{0\oplus\mathbb{R}^r} \). We infer
\[
T_{\sigma(p)} \cdot \theta_e|_{(p)\times\mathbb{R}^r} = d_{p,f(p)}\theta_e|_{0\oplus\mathbb{R}^r},
\] (9.1)
and
\[
T_{\sigma(p)} \left[ e(p) \cdot \left( d_{p,f}(\xi) + A(\xi) \cdot f(p) \right) \right] = d_{p,f(p)}\theta_e \left[ 0 \oplus \left( d_{p,f}(\xi) + A(\xi) \cdot f(p) \right) \right],
\]
\[
T_{\sigma(p)} \nabla \xi \sigma = d_{p,f(p)}\theta_e \left[ 0 \oplus \left( d_{p,f}(\xi) + A(\xi) \cdot f(p) \right) \right].
\]
Thus
\[
H_{\sigma(p)}(\xi) = d_{p,f(p)}\theta_e \left[ \xi \oplus \left( -A(\xi) \cdot f(p) \right) \right],
\]
i.e. if \( \eta = e \cdot h \), then
\[
H_{\eta}(\xi) = d_{p,h}\theta_e \left[ \xi \oplus \left( -A(\xi) \cdot h \right) \right],
\]
which shows the required conclusion. \( \square \)

Let \( \pi_E : E \to M \) be the projection map and notice the equality \( \text{Ker} d_\eta \pi_E = T_{E_\eta,\eta_1} \), for any \( \eta \in E_p \). The identity \( \pi_E \circ \sigma = \text{id}_M \) implies
\[
d_{\sigma(p)} \pi_E \circ d_p \sigma(\xi) = \xi.
\]

We deduce the identity \( d_\eta \pi_E \circ H_{\eta}(\xi) = \xi \). We define the horizontal distribution \( H \subset T_E \) associated to \( \nabla \) as
\[
H_\eta := H_{\eta}(T_{M,\pi_E(\eta)}) \subset T_{E,\eta}.
\]

We notice now that the tangent bundle of the vector bundle \( E \oplus E \) is given by the fibers
\[
T_{E \oplus E,(\eta_1,\eta_2)} = \left\{ (v_1,v_2) \in T_{E,\eta_1} \oplus T_{E,\eta_2} \mid d_{\eta_1} \pi_E(v_1) = d_{\eta_2} \pi_E(v_2) \right\},
\]
and that the differential of the sum bundle map \( sm_E : E \oplus E \to E \) satisfies
\[
d_{(\eta_1,\eta_2)}(sm_E)(v_1,v_2) = T_{\eta_1,\eta_2}(T_{\eta_1}^{-1}v_1 + T_{\eta_2}^{-1}v_2),
\]
for any \( (v_1,v_2) \in T_{E,\eta_1} \oplus T_{E,\eta_2} \) such that \( d_{\eta_1} \pi_E(v_1) = d_{\eta_2} \pi_E(v_2) = 0 \). We infer that for any sections \( \sigma_j \) of \( E \) such that \( \sigma_j(p) = \eta_j, \ j = 1,2 \), the equalities
\[
H_{\eta_1,\eta_2}(\xi) = d_\mu(\sigma_1 + \sigma_2)(\xi) - T_{\eta_1,\eta_2} \nabla_\xi(\sigma_1 + \sigma_2)
\]
\[
= d_{(\eta_1,\eta_2)}(sm_E)(d_\mu \sigma_1(\xi),d_\mu \sigma_2(\xi)) - T_{\eta_1,\eta_2} \nabla_\xi \sigma_1 - T_{\eta_1,\eta_2} \nabla_\xi \sigma_2
\]
\[
= d_{(\eta_1,\eta_2)}(sm_E)(d_\mu \sigma_1(\xi) - T_{\eta_1} \nabla_\xi \sigma_1,d_\mu \sigma_2(\xi) - T_{\eta_2} \nabla_\xi \sigma_2).
\]
hold. We conclude that
\[
H_{\eta_1,\eta_2}(\xi) = d_{(\eta_1,\eta_2)}(sm_E)(H_{\eta_1}(\xi),H_{\eta_2}(\xi)).
\] (9.2)
Lemma 15  For any section \( \sigma \in C^1(M,E) \) and for any function \( u \in C^1(M,\mathbb{R}) \) the identity holds

\[
 d_p (u\sigma) = d_p u \otimes T_{u\sigma(p)}\sigma (p) + d_{\sigma(p)}[u (p) \mathbb{I}_E] \cdot d_p \sigma ,
\]

for any point \( p \in M \).

Proof  With the notation in the proof of lemma \[14\]

\[
 d_p (u\sigma)(\xi) = d_{p,uf(p)} \theta_e \left[ \xi \oplus d_p (uf)(\xi) \right] = d_{p,uf(p)} \theta_e \left[ \xi \oplus d_p u (\xi) f (p) \right] + d_{p,uf(p)} [u (p) f (\xi)] ,
\]

thanks to (9.1). Using the identity

\[
 d_p (\lambda\sigma) = d_{\sigma(p)} (\lambda \mathbb{I}_E) \cdot d_p \sigma ,
\]

for any \( \lambda \in \mathbb{R} \), we conclude

\[
 d_p (u\sigma)(\xi) = d_p u (\xi) T_{u\sigma(p)}\sigma (p) + d_{\sigma(p)}[u (p) \mathbb{I}_E] \cdot d_p \sigma (\xi) .
\]

\[\square\]

We observe also the elementary identity

\[
 d_\eta (\lambda \mathbb{I}_E) \cdot T = \lambda T_{\lambda \eta} ,
\]

for all \( \eta \in E \). We show now the identity

\[
 H_{\lambda \eta} = d_\eta (\lambda \mathbb{I}_E) \cdot H_\eta ,
\]

for all \( \eta \in E \). Indeed let \( \sigma \) be a section such that \( \sigma (p) = \eta \). Using \[9.3\] and \[9.4\] we obtain the equalities

\[
 H_{\lambda \eta} = d_p (\lambda \sigma) - T_{\lambda \eta} \nabla (\lambda \sigma) = d_{\sigma(p)}(\lambda \mathbb{I}_E) \cdot d_p \sigma - \lambda T_{\lambda \eta} \nabla \sigma = d_{\sigma(p)}(\lambda \mathbb{I}_E) \cdot [d_p \sigma - T_{\eta} \sigma] = d_\eta (\lambda \mathbb{I}_E) \cdot H_\eta .
\]

The property \[9.5\] implies in particular \( H_{0_p} = d_p 0_M \), where \( 0_M \) is the zero section of \( T_M \).
Definition 5 A distribution $\mathcal{H} \subset T_E$, is called horizontal if the map

$$d_\eta \pi_{E|\mathcal{H}_\eta} : \mathcal{H}_\eta \rightarrow T_{M, \pi_E(\eta)},$$

is an isomorphism for all $\eta \in E$.

Lemma 16 Any horizontal distribution $\mathcal{H} \subset T_E$, which satisfies the conditions (9.2) and (9.5) with $H_\eta := (d_{\eta} \pi_E|_{\mathcal{H}_\eta})^{-1}$, determines a connection $\nabla$ over $E$ with associated horizontal distribution $\mathcal{H}$.

Proof The connection $\nabla$ is defined by the formula

$$\nabla_{\xi} \sigma = T_{\sigma(p)}^{-1} \left[ d_{\sigma(p)} \sigma - H_{\sigma(p)} \right] \left( \xi \right),$$

for any $\xi \in T_{M,p}$. The definition is well posed because

$$\left[ d_{\sigma(p)} \sigma - H_{\sigma(p)} \right] \left( \xi \right) \in T_{\sigma(p)},$$

which follows from the identity

$$d_{\sigma(p)} \pi_E \cdot \left[ d_{\sigma(p)} \sigma - H_{\sigma(p)} \right] \left( \xi \right) = 0.$$

It is obvious that the additive property of $\nabla$ is equivalent to the condition (9.2).

We observe now that with the previous definition, the covariant Leibniz property

$$\nabla_{\xi} \left( u \sigma \right) = d_{\sigma(p)} \left( \xi \right) \sigma(p) + u(p) \nabla_{\xi} \sigma,$$

is equivalent to the identity

$$d_{\sigma(p)} \left( u \sigma \right)(\xi) - H_{u \sigma(p)}(\xi) = d_{\sigma(p)} \left[ u(p) T_{\sigma(p)}^{-1} \cdot \left( d_{\sigma(p)}(\xi) - H_{\sigma(p)}(\xi) \right) \right].$$

We develop the right hand side using (9.2). We infer that the previous identity is equivalent to the following one

$$d_{\sigma(p)} \left( u \sigma \right)(\xi) - H_{u \sigma(p)}(\xi) = d_{\sigma(p)} \left( u(p) \mathbb{I}_{E} \cdot \left[ d_{\sigma(p)}(\xi) - H_{\sigma(p)}(\xi) \right] \right).$$

The later hold true thanks to lemma 15 and the assumption (9.5). □

The data of a smooth horizontal distribution over $E$ coincides with the one of section $H \in C^\infty (E, \pi_E^* T_M \otimes T_E)$ such that $d_{\pi_E} \cdot H = \mathbb{I}_{E \otimes T_E}$. (We notice that $d_{\pi_E} E \in C^\infty (E, T_E \otimes \pi_E^* T_M)$). Such type of section determines a connection if and only if it satisfies the identity (9.5).

For any vector $\Xi \in T_{E,\eta}$ we denote by

$$\gamma_{\mathcal{H}}^\eta (\Xi) := \Xi - H_\eta \circ d_{\eta} \pi_E(\Xi),$$

its vertical component with respect to the horizontal distribution $\mathcal{H}$. In particular

$$\gamma_{\sigma(p)}^\mathcal{H} \cdot d_{\sigma(p)}(\xi) = T_{\sigma(p)}[\nabla_{\xi} \sigma(p)].$$
9.2 The induced connection

Let \( \psi : N \rightarrow M \) be a smooth map. We define the vector bundle \( \psi^*E := N \times_{\psi} E \) over \( N \). In explicit terms

\[
\psi^*E = \left\{ (y, \eta) \in N \times E \mid \psi(y) = \pi_E(\eta) \right\},
\]

and the projection over \( N \) is given by the restriction of the projection to the first factor. We will denote by \( \Psi : \psi^*E \rightarrow E \) the restriction of the projection to the second factor. The sections of \( \psi^*E \) are identified with the maps \( \sigma : N \rightarrow E \) such that \( \pi_E \circ \sigma = \psi \). In this way, if \( s \) is a section of \( E \) then the section \( \psi^*s := s \circ \psi \) is a section of \( \psi^*E \). More in general if \( \alpha \) is a section of \( \Lambda^pT^*_M \otimes \psi^*E \), we define the section \( \psi^*\alpha \in \Lambda^pT^*_N \otimes \psi^*E \) as

\[
(\psi^*\alpha)(y) := (\alpha \circ \psi)(y) \cdot \Lambda^p(d_y\psi).
\]

We provide a generalization of lemma 15.

**Lemma 17** For any section \( \sigma \in C^1(N, \psi^*E) \) and for any function \( u \in C^1(N, \mathbb{R}) \) the identity holds

\[
d_p(u\sigma) = d_pu \otimes T_{u\sigma(p)}\sigma(p) + d_{\sigma(p)}[u(p) \mathbb{I}_E] \cdot d_p\sigma,
\]

for any point \( p \in N \).

**Proof** A local frame \( e \) of \( E \) induces a local frame \( \psi^*e \) of \( \psi^*E \) over the open set \( \psi^{-1}(U) \). Then \( \sigma = \psi^*e \cdot f \) with \( f \in C^1(\psi^{-1}(U), \mathbb{R}^r) \). We denote by \( \theta_e : U \times \mathbb{R}^r \rightarrow E_{|U} \) the trivialization map induced by the local frame \( e \) of \( E \). Then the differential of this map at the point \( (\psi(p), f(p)) \) provides an isomorphism

\[
d_{\psi(p),f(p)}\theta_e : T_{U,\psi(p)} \oplus \mathbb{R}^r \rightarrow T_{E,\sigma(p)},
\]

and

\[
d_p\sigma(\xi) = d_{\psi(p),uf(p)}\theta_e \left[ d_p\psi(\xi) \oplus d_p f(\xi) \right].
\]

for any \( \xi \in T_{N,p} \) we have

\[
d_p\left(u\sigma\right)(\xi) = d_{\psi(p),uf(p)}\theta_e \left[ d_p\psi(\xi) \oplus d_p\left(u(f)(\xi)\right) \right]
\]

\[
= d_{\psi(p),uf(p)}\theta_e \left\{ d_p\psi(\xi) \oplus \left[ d_pu(\xi) f(p) + u(p) d_p f(\xi) \right] \right\}
\]

\[
= d_{\psi(p),uf(p)}\theta_e \left[ 0 \oplus d_p\left(u(f)(\xi)\right) \right]
\]

\[
+ d_{\psi(p),uf(p)}\theta_e \left[ d_p\psi(\xi) \oplus u(p) d_p f(\xi) \right]
\]

\[
= T_{u\sigma(p)} \cdot \theta_e \left( \psi(\xi), d_p u(\xi) f(p) \right) + d_p\left(u(p)\sigma(\xi)\right),
\]

thanks to (15). Using the equality

\[
d_p\left(\lambda\sigma\right) = d_{\sigma(p)}\left(\lambda\mathbb{I}_E\right) \cdot d_p\sigma,
\]

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for any \( \lambda \in \mathbb{R} \), we conclude the required identity
\[
d_p (u \sigma) (\xi) = d_p u (\xi) T_{u \sigma(p)} \sigma (p) + d_{\sigma(p)} [u (p) \Pi_E] \cdot d_p \sigma (\xi)
\]
\( \blacksquare \)

The induced connection \( \nabla^\psi \) over \( \psi^* E \) is defined by the formula
\[
\nabla^\psi_{\xi} \sigma := T^{-1}_{\sigma(p)} H_{\sigma(p)} d_p \sigma (\xi)
\]
\[
= T^{-1}_{\sigma(p)} \left[ d_p \sigma (\xi) - H_{\sigma(p)} d_p \psi (\xi) \right],
\]
for any \( \xi \in T_{N,p} \). It is obvious that the additive property of \( \nabla^\psi \) follows from the condition \( \text{(9.2)} \). We show now that \( \nabla^\psi \) satisfies the Leibniz property
\[
\nabla^\psi_{\xi} (u \sigma) = d_p u (\xi) \sigma (p) + u (p) \nabla^\psi_{\xi} \sigma.
\]
Indeed using lemma \( \text{(17)} \) and the identity \( \text{(9.5)} \) we have
\[
\nabla^\psi_{\xi} (u \sigma) = T^{-1}_{u \sigma(p)} \gamma^H_{u \sigma(p)} d_p (u \sigma) (\xi)
\]
\[
= T^{-1}_{u \sigma(p)} \gamma^H_{u \sigma(p)} \left[ d_p u (\xi) T_{u \sigma(p)} \sigma (p) + d_{\sigma(p)} [u (p) \Pi_E] d_p \sigma (\xi) \right]
\]
\[
= d_p u (\xi) \sigma (p) + T^{-1}_{u \sigma(p)} \left[ d_{\sigma(p)} [u (p) \Pi_E] d_p \sigma (\xi) - H_{u \sigma(p)} d_p \psi (\xi) \right]
\]
\[
= d_p u (\xi) \sigma (p) + d_p u (\xi) \sigma (p) + u (p) \nabla^\psi_{\xi} \sigma.
\]
We observe also that for any \( s \in C^\infty (M, E) \) and \( \xi \in T_{N,p} \) we have the equalities
\[
\nabla^\psi_{\xi} (\psi^* s) = T^{-1}_{s \circ \psi(p)} \gamma^H_{s \circ \psi(p)} d_{\psi(p)} s \cdot d_p \psi (\xi)
\]
\[
= \nabla s (\psi (p)) \cdot d_p \psi (\xi),
\]
in other terms the functorial formula
\[
\nabla^\psi (\psi^* s) = \psi^* (\nabla s),
\]
holds.
9.2.1 The induced connection (second approach)

We observe that the tangent space of $\psi^*E$ at the point $(y, \eta)$ is given by the equality

$$T_{\psi^*E,(y,\eta)} = \{ (\xi, \theta) \in T_{N,y} \oplus T_{E,\eta} \mid d_y\psi (\xi) = d_{\eta}\pi_E (\theta) \}.$$ 

Given any horizontal distribution $H \in C^\infty (E, \pi^*_E T_N \otimes T_E)$ over $E$, we define the horizontal distribution

$$H^\psi := \Psi^*H \in C^\infty (\psi^*E, \pi^*_\psi E T_N \otimes T_{\psi^*E}).$$

In explicit terms

$$H^\psi (y, \eta) = \Pi_{T_{N,y}} \oplus H \eta \cdot d_y\psi.$$

If $H$ satisfies the identities (9.2) and (9.5) then so does $H^\psi$. This follows indeed from the identities

$$d_{(y,\eta_1,\eta_2)} (sm_{\psi^*E}) = \Pi_{T_{N,y}} \oplus d_{(\eta_1,\eta_2)} (sm_E),$$

$$d_{(y,\eta)} (\lambda \Pi_{\psi^*E}) = \Pi_{T_{N,y}} \oplus d_{\eta} (\lambda \Pi_E).$$

By definition of $H^\psi$ we infer that the induced connection $\nabla^\psi$ over $\psi^*E$ satisfies the formula

$$\nabla^\psi \sigma = T_{\psi^*E}^{-1} (\xi) \cdot \left[ d_y\sigma (\xi) - H_{\sigma(y)} \cdot d_y\psi (\xi) \right],$$

for any $\xi \in T_{N,y}$.

The local frame $e$ induces a local frame $\eta := e \circ \psi$ of $\psi^*E$ over $\psi^{-1} (U)$. We compute the local connection $A^\psi$ form of $\nabla^\psi$ with respect to such frame. We notice that $\nabla^\psi \eta = \psi^* (e \cdot A) = \eta \cdot \psi^* A$ by the previous remark. We infer the equality $A^\psi = \psi^* A$.

9.2.2 Parallel transport

We consider a smooth curve $\gamma : (-\varepsilon, \varepsilon) \longrightarrow M$ and a section $\sigma \in C^1 ((-\varepsilon, \varepsilon), \gamma^*E)$ which satisfies the equation

$$\nabla^\gamma \sigma = 0,$$

over $(-\varepsilon, \varepsilon)$ with $\sigma (0) = \eta \in E_{\gamma(0)}$. If we write $\sigma (t) = e (\gamma (t)) \cdot f (t)$ then

$$\nabla^\gamma \sigma = e (\gamma (t)) \cdot \left[ f (t) + A (\dot{\gamma} (t)) \cdot f (t) \right].$$

We infer that the parallel transport map $\tau_{\gamma,t} : E_{\gamma(0)} \longrightarrow E_{\gamma(t)}$, $t \in (-\varepsilon, \varepsilon)$ given by $\tau_{\gamma,t} (\eta) = \sigma (t)$, is linear. We show the following fact.

**Lemma 18** For any smooth curve $\gamma : (-\varepsilon, \varepsilon) \longrightarrow M$ and for any section $\sigma \in C^1 ((-\varepsilon, \varepsilon), \gamma^*E)$, holds the identity

$$\nabla^\gamma \sigma (0) = \frac{d}{dt}_{t=0} \left[ \tau_{\gamma,t}^{-1} \cdot \sigma (t) \right],$$

(9.7)
Proof  We notice first that the term $\tau_{\gamma,t}^{-1} \cdot \sigma (t)$ is given by the intrinsic identities

$$\frac{du_t}{ds} + A (\dot{\gamma} (s)) \cdot u_t (s) = 0,$$

$$u_t (t) = f (t),$$

$$e (\gamma (0)) \cdot u_t (0) = \tau_{\gamma,t}^{-1} \cdot \sigma (t).$$

Integrating the first equation we infer

$$u_t (t) - u_t (0) = - \int_0^t A (\dot{\gamma} (s)) \cdot u_t (s) \, ds.$$

Using the second equation we obtain

$$f (t) - u_t (0) = - \int_0^t A (\dot{\gamma} (s)) \cdot u_t (s) \, ds.$$

Deriving with respect to the variable $t$ we obtain

$$\frac{d}{dt} u_t (0) = \dot{f} (t) + A (\dot{\gamma} (t)) \cdot u_t (t)$$

$$= \dot{f} (t) + A (\dot{\gamma} (t)) \cdot f (t).$$

Evaluating at $t = 0$ and multiplying both sides with $e (\gamma (0))$ we infer the required conclusion. \hfill \Box

We consider now a $C^1$-vector field $\xi$ over $M$ and let $\varphi_{\xi,t}$ be the associated 1-parameter subgroup of transformations of $M$. Let $\Phi_{\xi,t} : E \rightarrow E$ be the parallel transport map along the flow lines of $\varphi_{\xi,t}$. It is obvious by definition, that the map $\Phi_{\xi,t}$ satisfies $\pi_E \circ \Phi_{\xi,t} = \varphi_{\xi,t} \circ \pi_E$.

The vector field $\Xi := \dot{\Phi}_{\xi,0}$ over $E$ satisfies the equality $\Xi (\eta) = H_{\eta} (\xi)$, for any $\eta \in E$. This is a direct consequence of the definition of the induced connection along the flow lines of $\xi$.

To any section $\sigma \in C^1 (M,E)$ we can associate a $C^1$-vector field $\Sigma$ over $E$ defined as $\Sigma (\eta) := T_{\eta} [\sigma \circ \pi_E (\eta)]$. Let $\Phi_{\Sigma,t}$ be the associated 1-parameter subgroup of transformations of $E$. In explicit terms it satisfies

$$\Phi_{\Sigma,t} (\eta) = \eta + t \sigma \circ \pi_E (\eta).$$

Then

$$[\Xi, \Sigma] = \frac{d}{dt} \bigg|_{t=0} \frac{d}{ds} \bigg|_{s=0} (\Phi_{\xi,-t} \circ \Phi_{\Sigma,s} \circ \Phi_{\xi,t}).$$

The fact that the map $\Phi_{\xi,-t}$ is linear on the fibers implies

$$\Phi_{\xi,-t} \circ \Phi_{\Sigma,s} \circ \Phi_{\xi,t} = \Phi_{\xi,-t} [\Phi_{\xi,t} + s \sigma \circ \pi_E \circ \Phi_{\xi,t}]$$

$$= \Pi_E + s \Phi_{\xi,-t} \cdot \sigma \circ \pi_E \circ \Phi_{\xi,t}$$

$$= \Pi_E + s \Phi_{\xi,-t} \cdot \sigma \circ \varphi_{\xi,t} \circ \pi_E.$$
Thus for any $\eta \in E_p$ holds
\[ \Phi_{\xi,-t} \circ \Phi_{\Sigma,s} \circ \Phi_{\xi,t} (\eta) = \eta + s\Phi_{\xi,-t} \cdot \sigma \circ \varphi_{\xi,t} (p) \in E_p. \]

We conclude
\[ \left[ \Xi, \Sigma \right] (\eta) = \frac{d}{dt} |_{t=0} T_{\eta} \left[ \Phi_{\xi,-t} \cdot \sigma \circ \varphi_{\xi,t} (p) \right] \]
\[ = T_{\eta} \left[ \nabla_{\xi} \sigma (p) \right], \]
i.e. for any $\eta \in E$ the equality holds
\[ \left[ \Xi, \Sigma \right] (\eta) = T_{\eta} \left[ \nabla_{\xi} \sigma \circ \pi_E (\eta) \right]. \tag{9.8} \]

Iterating twice we deduce the identity
\[ \left[ \Xi_1, \left[ \Xi_2, \Sigma \right] \right] (\eta) = T_{\eta} \left[ \nabla_{\xi_1} \nabla_{\xi_2} \sigma \circ \pi_E (\eta) \right]. \tag{9.9} \]
Moreover the fact that by (9.8) the vector fields $[\Xi_j, \Sigma]$, $j = 1, 2$ are tangent to the fibers of $E$ and constant along them implies
\[ \left[ \left[ \Xi_1, \Sigma \right], \left[ \Xi_2, \Sigma \right] \right] = 0. \tag{9.10} \]

### 9.3 The geometric meaning of the curvature tensor

**Lemma 19** Let $R := \nabla^2$ be the curvature tensor of the connection $\nabla$. Then for any vector fields $\xi_1, \xi_2$ over $M$ and for any $\eta \in E$ the identity holds
\[ \gamma_{\eta} \left[ \left[ \Xi_1, \Xi_2 \right] (\eta) \right] = T_{\eta} \left[ R (\xi_2, \xi_1) \eta \right]. \]

**Proof** Let $\sigma$ be a local section of $E$ such that $\sigma (p) = \eta$. By definition of horizontal lift $\Xi$ of a vector field $\xi$ we have
\[ \Xi (\eta) = \left[ d\sigma (\xi) \circ \pi_E (\eta) - T_{\eta} \left[ \nabla_{\xi} \sigma \circ \pi_E (\eta) \right] \right]. \]

We infer by (9.8) the identity
\[ [d\sigma (\xi)] \circ \pi_E = \Xi + [\Xi, \Sigma]. \]
We infer $\sigma \cdot \xi = \Xi + [\Xi, \Sigma]$ over $\text{Im} \sigma$. Thus
\[ \sigma \cdot [\xi_1, \xi_2] = \left[ \sigma \cdot \xi_1, \sigma \cdot \xi_2 \right] \]
\[ = \left[ \Xi_1, \Xi_2 \right] + \left[ \Xi_1, \left[ \Xi_2, \Sigma \right] \right] + \left[ \Xi_1, \Sigma \right], \Xi_2 \right], \]
thanks to (9.10). We rewrite the previous equality as
\[ \left[ \Xi_1, \Xi_2 \right] = \left[ \Xi_2, \left[ \Xi_1, \Sigma \right] \right] - \left[ \Xi_1, \left[ \Xi_2, \Sigma \right] \right] - \sigma \cdot [\xi_2, \xi_1]. \]
Using (9.9) we deduce

\[
[\Xi_1, \Xi_2](\eta) = T_\eta \left[ (\nabla_{\xi_2} \nabla_{\xi_1} \sigma - \nabla_{\xi_1} \nabla_{\xi_2} \sigma) (p) \right] - d_p \sigma([\xi_2, \xi_1])
\]

\[
= T_\eta \left[ (\nabla_{\xi_2} \nabla_{\xi_1} \sigma - \nabla_{\xi_1} \nabla_{\xi_2} \sigma - \nabla_{[\xi_2, \xi_1]} \sigma) (p) \right] - H_\eta([\xi_2, \xi_1])
\]

\[
= T_\eta [R(\xi_2, \xi_1) \sigma (p)] + H_\eta([\xi_1, \xi_2])
\]

We infer the required conclusion. \qed

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