Chaotic spin dynamics of a long nanomagnet driven by a current

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Abstract

We study the spin dynamics of a long nanomagnet driven by an electrical current. In the case of only dc current, the spin dynamics has a sophisticated bifurcation diagram of attractors. One type of attractors is a weak chaos. On the other hand, in the case of only ac current, the spin dynamics has a rather simple bifurcation diagram of attractors. That is, for small Gilbert damping, when the ac current is below a critical value, the attractor is a limit cycle; above the critical value, the attractor is chaotic (turbulent). For normal Gilbert damping, the attractor is always a limit cycle in the physically interesting range of the ac current. We also developed a Melnikov integral theory for a theoretical prediction on the occurrence of chaos. Our Melnikov prediction seems to be performing quite well in the dc case. In the ac case, our Melnikov prediction seems to be predicting transient chaos. The sustained chaotic attractor seems to have extra support from parametric resonance leading to a turbulent state.

Mathematics Subject Classification: 35, 65, 37, 78

1. Introduction

The greatest potential of the theory of chaos in partial differential equations lies in its abundant applications in science and engineering. The variety of specific problems demands continuing innovation of the theory [16–19, 21, 22, 24, 25]. In these representative publications, two theories were developed. The theory developed in [17–19] involves transversal homoclinic orbits, and the shadowing technique is used to prove the existence of chaos. This theory is very complete. The theory in [16, 21, 22, 24, 25] deals with Silnikov homoclinic orbits, and geometric construction of Smale horseshoes is employed. This theory is not very complete. The main machineries for locating homoclinic orbits are (1) Darboux
transformations, (2) isospectral theory, (3) persistence of invariant manifolds and Fenichel fibres and (4) Melnikov analysis and shooting technique. Overall, the two theories on chaos in partial differential equations are results of combining integrable theory, dynamical system theory and partial differential equations [20].

In this paper, we are interested in the chaotic spin dynamics in a long nanomagnet driven by an electrical current. We hope that the abundant spin dynamics revealed by this study can generate experimental studies on long nanomagnets. To illustrate the general significance of the spin dynamics, in particular, the magnetization reversal issue, we use a daily example: the memory of the hard drive of a computer. The magnetization is polarized along the direction of the external magnetic field. By reversing the external magnetic field, magnetization reversal can be accomplished, thereby, generating 0 and 1 binary sequence and accomplishing memory purpose. Memory capacity and speed via such a technique have reached their limits. The ‘bit’ writing scheme based on such an Oersted–Maxwell magnetic field (generated by an electrical current) encounters a fundamental problem from classical electromagnetism: the long range magnetic field leads to unwanted writing or erasing of closely packed neighbouring magnetic elements in the extremely high density memory device and the induction laws place an upper limit on the memory speed due to slow rise-and-decay time imposed by the law of induction. Discovered by Slonczewski [37] and Berger [1], electrical current can directly apply a large torque to a ferromagnet. If electrical current can be directly applied to achieve magnetization reversal, such a technique will dramatically increase the memory capacity and speed of a hard drive. The magnetization can then be switched on the scale of nanoseconds and nanometres [42]. The industrial value will be tremendous. Nanomagnets driven by currents have been intensively studied recently [3,4,7,8,10–14,27–29,31–33,36,40–42,46]. The investigations have gone beyond the original spin valve system [1,37]. For instance, current driven torques have been applied to magnetic tunnel junctions [6,38], dilute magnetic semiconductors [43] and multi-magnet couplings [10,14]. Ac currents were also applied to generate spin torque [7,36]. Such ac current can be used to generate the external magnetic field [36] or applied directly to generate spin torque [7].

Mathematically, the electrical current introduces a spin torque forcing term in the conventional Landau–Lifshitz–Gilbert (LLG) equation. The ac current can induce novel dynamics of the LLG equation, such as synchronization [7,36] and chaos [26,44]. Both synchronization and chaos are important phenomena to understand before implementing the memory technology. In [26,44], we studied the dynamics of synchronization and chaos for the LLG equation by ignoring the exchange field (i.e. LLG ordinary differential equations). When the nanomagnetic device has the same order of length along every direction, exchange field is not important, and we have a so-called single domain situation where the spin dynamics is governed by the LLG ordinary differential equations. In this paper, we will study what we call ‘long nanomagnet’ which is much longer along one direction than other directions. In such a situation, the exchange field will be important. This leads to a LLG partial differential equation. In fact, we will study the case where the exchange field plays a dominant role.

The paper is organized as follows: section 2 presents the mathematical formulation of the problem. Section 3 is an integrable study of the Heisenberg equation. Based upon section 3, section 4 builds the Melnikov integral theory for predicting chaos. Section 5 presents the numerical simulations.

2. Mathematical formulation

To simplify the study, we will investigate the case where the magnetization depends on only one spatial variable and has periodic boundary condition in this spatial variable. The application
of this situation will be a large ring-shaped nanomagnet. Thus, we shall study the following forced LLG equation in the dimensionless form,

$$\frac{\partial m}{\partial t} = -m \times H - \epsilon m \times (m \times H) + \epsilon (\beta_1 + \beta_2 \cos \omega_0 t) m \times (m \times e_z),$$  \hspace{1cm} (2.1)

subject to the periodic boundary condition

$$m(t, x + 2\pi) = m(t, x),$$  \hspace{1cm} (2.2)

where \(m\) is a unit magnetization vector \(m = (m_1, m_2, m_3)\) in which the three components are along \((x, y, z)\) directions with unit vectors \((e_x, e_y, e_z)\), \(|m(t, x)| = 1\); the effective magnetic field \(H\) has several terms

$$H = H_{\text{exch}} + H_{\text{ext}} + H_{\text{dem}} + H_{\text{ani}}$$

$$= \partial^2_x m + \epsilon a e_x - \epsilon m_3 e_z + \epsilon b m_1 e_x,$$  \hspace{1cm} (2.3)

where \(H_{\text{exch}} = \partial^2_x m\) is the exchange field, \(H_{\text{ext}} = \epsilon a e_x\) is the external field, \(H_{\text{dem}} = -\epsilon m_3 e_z\) is the demagnetization field and \(H_{\text{ani}} = \epsilon b m_1 e_x\) is the anisotropy field. For the materials of the experimental interest, the dimensionless parameters are in the ranges

$$a \approx 0.05, \quad b \approx 0.025, \quad \alpha \approx 0.02, \quad \beta_1 \in [0.01, 0.3], \quad \beta_2 \in [0.01, 0.3]$$  \hspace{1cm} (2.4)

and \(\epsilon\) is a small parameter measuring the length scale of the exchange field. One can also add an ac current effect in the external field \(H_{\text{ext}}\), but the results on the dynamics are similar.

Our goal is to build a Melnikov function for the LLG equation around domain walls. The roots of such a Melnikov function provide a good indication of chaos.

We will use the following notation for the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  \hspace{1cm} (2.5)

Let

$$m_+ = m_1 + i m_2, \quad m_- = m_1 - i m_2,$$  \hspace{1cm} (2.6)

i.e. \(m_+ = m_-\). Let

$$\Gamma = m_+ \sigma_1 = \begin{pmatrix} m_3 & m_- \\ m_+ & -m_3 \end{pmatrix}.$$  \hspace{1cm} (2.7)

Thus, \(\Gamma^2 = 1\) (the identity matrix). Let

$$\hat{H} = -H - \alpha m \times H + \beta m \times e_z, \quad \Pi = \begin{pmatrix} \hat{H}_3 & \hat{H}_1 - i \hat{H}_2 \\ \hat{H}_1 + i \hat{H}_2 & -\hat{H}_3 \end{pmatrix}.$$  \hspace{1cm} (2.8)

Then the LLG can be written in the form

$$i \hbar \partial_t \Gamma = \frac{1}{2} [\Gamma, \Pi],$$  \hspace{1cm} (2.9)

where

$$[\Gamma, \Pi] = \Gamma \Pi - \Pi \Gamma.$$
3. Isospectral integrable theory for the Heisenberg equation

Setting $\epsilon$ to zero, the LLG (2.1) reduces to the Heisenberg ferromagnet equation,
\[ \dot{m} = -m \times m_{xx}. \tag{3.1} \]
Using the matrix $\Gamma$ introduced in (2.7), the Heisenberg equation (3.1) has the form
\[ i\partial_t \Gamma = -\frac{i}{2} [\Gamma, \Gamma_{xx}], \tag{3.2} \]
where the bracket $[\ ,\ ]$ is defined in (2.9). Obvious constants of motion of the Heisenberg equation (3.1) are the Hamiltonian,
\[ \frac{1}{2} \int_0^{2\pi} |m_x|^2 \, dx, \]
the momentum,
\[ \int_0^{2\pi} \frac{m_1 m_{2x} - m_2 m_{1x}}{1 + m_3} \, dx, \]
and the total spin,
\[ \int_0^{2\pi} m \, dx. \]
The Heisenberg equation (3.1) is an integrable system with the following Lax pair [39],
\[ \partial_x \psi = i\lambda \Gamma \psi, \tag{3.3} \]
\[ \partial_t \psi = -\frac{\lambda}{2} (4i\lambda \Gamma + [\Gamma, \Gamma_x]) \psi, \tag{3.4} \]
where $\psi = (\psi_1, \psi_2)^T$ is complex-valued, $\lambda$ is a complex parameter, $\Gamma$ is the matrix defined in (2.7) and $[\Gamma, \Gamma_x] = \Gamma \Gamma_x - \Gamma_x \Gamma$. The inverse scattering method was applied to the Heisenberg equation (3.1) by Takhtajan [39]. In fact, there is a connection between the Heisenberg equation (3.1) and the 1D integrable focusing cubic nonlinear Schrödinger (NLS) equation via a nontrivial gauge transformation established by Zakharov and Takhtajan [45] and Lakshmanan [15], see also [5] (part II, chapter 1, section 4).

3.1. A simple linear stability calculation

Consider the domain wall
\[ \Gamma_0 = \begin{pmatrix} 0 & e^{-i\xi} \\ e^{i\xi} & 0 \end{pmatrix}, \quad \xi \in \mathbb{Z}; \quad \text{i.e. } m_1 = \cos \xi, \quad m_2 = \sin \xi, \quad m_3 = 0, \]
which is a fixed point of the Heisenberg equation. Linearizing the Heisenberg equation at this fixed point, one gets
\[ i\partial \Gamma = -\frac{i}{2} [\Gamma_0, \Gamma_{xx}] - \frac{1}{2} [\Gamma, \Gamma_{0xx}]. \]
Let
\[ \Gamma = \begin{pmatrix} m_3 & e^{-i\xi} (m_1 - im_2) \\ e^{i\xi} (m_1 + im_2) & -m_3 \end{pmatrix}, \]
we get
\[ \partial_x m_1 = 0, \quad \partial_x m_2 = m_{3xx} + \xi^2 m_3, \quad \partial_x m_3 = -m_{2xx} - 2\xi m_{1x}. \]
Let

\[ m_j = \sum_{k=0}^{\infty} (m^+_j(t) \cos kx + m^-_j(t) \sin kx), \]

where \( m^\pm_j(t) = c^\pm_{jk} e^{\pm \Omega t}, c^\pm_{jk} \) and \( \Omega \) are constants. We obtain that

\[ \Omega = \sqrt{k^2(\xi^2 - k^2)}, \]

which shows that only the modes \( 0 < |k| < |\xi| \) are unstable. Such an instability is called a modulational instability, also called a side-band instability. Comparing the Heisenberg ferromagnet equation (3.1) and the LLG equation (2.1), we see that if we drop the exchange field \( H_{\text{exch}} = \partial^2_x m \) in the effective magnetic field \( H (2.3) \), such a modulational instability will disappear, and the LLG equation (2.1) reduces to a system of three ordinary differential equations, which has no chaos as verified numerically. Thus the modulational instability is the source of the chaotic magnetization dynamics.

In terms of \( m^\pm_{jk} \), we have

\[ \frac{d}{dt} m^\pm_{1k} = 0, \quad \frac{d}{dt} m^\pm_{2k} = (\xi^2 - k^2)m^\pm_{3k}, \quad \frac{d}{dt} m^\pm_{3k} = k^2 m^\pm_{2k} \mp 2\xi km^\pm_{1k}. \]

Choosing \( \xi = 2 \), we have for \( k = 0 \),

\[
\begin{pmatrix}
  m^+_{10} \\
  m^\pm_{20} \\
  m^\pm_{30}
\end{pmatrix} = \begin{pmatrix}
  1 \\
  0 \\
  0
\end{pmatrix}
\]

for \( k = 1 \),

\[
\begin{pmatrix}
  m^+_{11} \\
  m^\pm_{21} \\
  m^\pm_{31}
\end{pmatrix} = \begin{pmatrix}
  1 \\
  \pm 4 \\
  0
\end{pmatrix}
\]

for \( k = 2 \),

\[
\begin{pmatrix}
  m^+_{12} \\
  m^\pm_{22} \\
  m^\pm_{32}
\end{pmatrix} = \begin{pmatrix}
  1 \\
  0 \\
  1
\end{pmatrix}
\]

for \( k > 2 \),

\[
\begin{pmatrix}
  m^+_{1k} \\
  m^\pm_{2k} \\
  m^\pm_{3k}
\end{pmatrix} = \begin{pmatrix}
  1 \\
  \pm 4/k \\
  0
\end{pmatrix} + \begin{pmatrix}
  0 \\
  \sqrt{k^2 - 4} \cos(k\sqrt{k^2 - 4}t) \\
  k \sin(k\sqrt{k^2 - 4}t)
\end{pmatrix}
\]

\[
\begin{pmatrix}
  0 \\
  k \cos(k\sqrt{k^2 - 4}t)
\end{pmatrix},
\]

where \( c_1, c_2 \) and \( c_3 \) are arbitrary constants.

The nonlinear foliation of the above linear modulational instability can be established via a Darboux transformation.
3.2. A Darboux transformation

A Darboux transformation for (3.3)–(3.4) can be obtained.

**Theorem 3.1.** Let \( \phi = (\phi_1, \phi_2)^T \) be a solution to the Lax pair (3.3)–(3.4) at (\( \Gamma_1, \nu \)). Define the matrix

\[
G = N \begin{pmatrix}
(\nu - \lambda)/\nu & 0 \\
0 & (\bar{\nu} - \lambda)/\bar{\nu}
\end{pmatrix} N^{-1},
\]

where

\[
N = \begin{pmatrix}
\phi_1 & -\bar{\phi}_2 \\
\phi_2 & \bar{\phi}_1
\end{pmatrix}.
\]

Then if \( \psi \) solves the Lax pair (3.3) and (3.4) at (\( \Gamma_1, \lambda \)),

\[
\hat{\psi} = G\psi
\]

solves the Lax pair (3.3)–(3.4) at (\( \hat{\Gamma}, \lambda \)), where \( \hat{\Gamma} \) is given by

\[
\hat{\Gamma} = N \begin{pmatrix}
e^{-i\theta} & 0 \\
0 & e^{i\theta}
\end{pmatrix} N^{-1} \Gamma N \begin{pmatrix}
e^{i\theta} & 0 \\
0 & e^{-i\theta}
\end{pmatrix} N^{-1},
\]

and \( e^{i\theta} = \nu/|\nu| \).

The transformation (3.6)–(3.7) is called a Darboux transformation. This theorem can be proved either through the connection between the Heisenberg equation and the NLS equation (with a well-known Darboux transformation) [2] or through a direct calculation.

Note also that

\[
\hat{\Gamma}^2 = I. \quad \text{Let}
\]

\[
\begin{pmatrix}
\Phi_1 & -\bar{\Phi}_2 \\
\Phi_2 & \bar{\Phi}_1
\end{pmatrix} = N \begin{pmatrix}
e^{-i\theta} & 0 \\
0 & e^{i\theta}
\end{pmatrix} N^{-1}
\]

\[
= \frac{1}{|\phi_1|^2 + |\phi_2|^2} \begin{pmatrix}
e^{-i\theta} |\phi_1|^2 + e^{i\theta} |\phi_2|^2 & -2i \sin \theta \phi_1 \bar{\phi}_2 \\
-2i \sin \theta \phi_1 \bar{\phi}_2 & e^{i\theta} |\phi_1|^2 + e^{-i\theta} |\phi_2|^2
\end{pmatrix}.
\]

Then

\[
\hat{\Gamma} = \begin{pmatrix}
\hat{m}_3 & \hat{m}_1 - i\hat{m}_2 \\
\hat{m}_1 + i\hat{m}_2 & -\hat{m}_3
\end{pmatrix},
\]

where

\[
\begin{align*}
\hat{m}_1 &= \hat{m}_1 + i\hat{m}_2 = \Phi_1 (m_1 + im_2) - \Phi_2 (m_1 - im_2) + 2\Phi_1 \Phi_2 m_3, \\
\hat{m}_3 &= (|\Phi_1|^2 - |\Phi_2|^2) m_3 - \Phi_1 \Phi_2 (m_1 + im_2) - \Phi_1 \Phi_2 (m_1 - im_2).
\end{align*}
\]

One can generate the figure eight connecting to the domain wall, as the nonlinear foliation of the modulational instability, via the above Darboux transformation.

3.3. Figure eight connecting to the domain wall

Let \( \Gamma \) be the domain wall

\[
\Gamma = \begin{pmatrix}
0 & e^{-i2\chi} \\
e^{i2\chi} & 0
\end{pmatrix}.
\]
i.e. $m_1 = \cos 2x$, $m_2 = \sin 2x$ and $m_3 = 0$. Solving the Lax pair (3.3)–(3.4), one gets two Bloch eigenfunctions

$$\psi = e^{\Omega t} \begin{pmatrix} 2\lambda \exp \left\{ \frac{i}{2} (k - 2)x \right\} \\ (k - 2) \exp \left\{ \frac{i}{2} (k + 2)x \right\} \end{pmatrix}, \quad \Omega = -i\lambda k, \quad k = \pm 2\sqrt{1 + \lambda^2}. \quad (3.10)$$

To apply the Darboux transformation (3.7), we start with the two Bloch functions with $k = \pm 1$,

$$\phi^+ = \begin{pmatrix} \sqrt{3}e^{-i\chi} \\ ie^{it} \end{pmatrix} \exp \left\{ \frac{\sqrt{3}}{2} t + i\frac{1}{2}x \right\},$$

$$\phi^- = \begin{pmatrix} -ie^{-i\chi} \\ \sqrt{3}e^{it} \end{pmatrix} \exp \left\{ -\frac{\sqrt{3}}{2} t - i\frac{1}{2}x \right\}. \quad (3.11)$$

The wise choice for $\phi$ used in (3.7) is

$$\phi = \sqrt{c^+ c^-}\phi^+ + \sqrt{c^+ c^-}\phi^- = \left( (\sqrt{3}e^{i\chi} - ie^{-i\chi})e^{-it} \right),$$

where $c^+/c^- = \exp[\sigma + iy]$, $\tau = \frac{1}{2}(\sqrt{3}t + \sigma)$ and $\chi = \frac{1}{2}(x + y)$. Then from the Darboux transformation (3.7), one gets

$$\hat{m}_1 + i\hat{m}_2 = -e^{i2x} \left\{ 1 - \frac{2 \text{sech} 2\tau \cos 2\chi}{(2 - \sqrt{3} \text{sech} 2\tau \sin 2\chi)^2} \text{sech} 2\tau \cos 2\chi \right\},$$

$$\hat{m}_3 = \frac{2 \text{sech} 2\tau \tanh 2\tau \cos 2\chi}{(2 - \sqrt{3} \text{sech} 2\tau \sin 2\chi)^2}. \quad (3.13)$$

As $t \to \pm \infty$,

$$\hat{m}_1 \to -\cos 2x, \quad \hat{m}_2 \to -\sin 2x, \quad \hat{m}_3 \to 0. \quad (3.14)$$

The expressions (3.13)–(3.14) represent the two-dimensional figure eight separatrix connecting to the domain wall $(m_+ = -e^{i2x}, m_3 = 0)$, parametrized by $\sigma$ and $y$. See figure 1 for an illustration. Choosing $y = 0, \pi$, one gets the figure eight curve section of figure 1. The spatial–temporal profiles corresponding to the two lobes of the figure eight curve are shown in figure 2. In fact, the two profiles corresponding to the two lobes are spatial translates of each other by $\pi$. Inside one of the lobes, the spatial–temporal profile is shown in figure 3(a). Outside the figure eight curve, the spatial–temporal profile is shown in figure 3(b). Here the inside and outside spatial–temporal profiles are calculated by using the integrable finite difference discretization [9, 34, 35] of the Heisenberg equation (3.1),

$$\frac{d}{dt} m(j) = -\frac{2}{h^2} m(j) \times \left( \frac{m(j - 1) + m(j + 1)}{1 + m(j) \cdot m(j + 1)} + \frac{m(j - 1)}{1 + m(j - 1) \cdot m(j)} \right), \quad (3.15)$$

where $m(j) = m(t, jh), j = 1, \ldots, N, Nh = 2\pi$, and $h$ is the spatial mesh size. The lattice (3.15) is a special case of the Sklyanin lattice [35]. The Darboux transformation of (3.15) was developed in [9]. For the computation of figure 3, we choose $N = 128$.

By a translation $x \to x + \theta$, one can generate a circle of domain walls:

$$m_+ = -e^{2i(x+\theta)}, \quad m_3 = 0,$$
where $\theta$ is the phase parameter. The three-dimensional figure eight separatrix connecting to the circle of domain walls, parametrized by $\sigma$, $\gamma$ and $\theta$, is illustrated in figure 4.

In general, the unimodal equilibrium manifold can be sought as follows: Let

$$m_j = c_j \cos 2x + s_j \sin 2x, \quad j = 1, 2, 3,$$

then the uni-length condition $|m|(x) = 1$ leads to

$$|c| = 1, \quad |s| = 1, \quad c \cdot s = 0,$$

where $c$ and $s$ are the two vectors with components $c_j$ and $s_j$. Thus the unimodal equilibrium manifold is three-dimensional and can be represented as in figure 5.

Using the formulae (3.13)–(3.14), we want to build a Melnikov integral. The zeros of the Melnikov integral will give a prediction on the existence of chaos. To build such a Melnikov integral, we need to first develop a Melnikov vector. This requires Floquet theory of (3.3).
3.4. Floquet theory

Focusing on the spatial part (3.3) of the Lax pair (3.3)–(3.4), let $Y(x)$ be the fundamental matrix solution of (3.3), $Y(0) = I$ (2 × 2 identity matrix), then the Floquet discriminant is defined by

$$\Delta = \text{trace } Y(2\pi).$$

The Floquet spectrum is given by

$$\sigma = \{\lambda \in \mathbb{C} \mid -2 \leq \Delta(\lambda) \leq 2\}.$$

Periodic and anti-periodic points $\lambda^{\pm}$ (which correspond to periodic and anti-periodic eigenfunctions, respectively) are defined by

$$\Delta(\lambda^{\pm}) = \pm 2.$$

The critical point $\lambda^{(c)}$ is defined by

$$\frac{d\Delta}{d\lambda}(\lambda^{(c)}) = 0.$$

The multiple point $\lambda^{(m)}$ is a periodic or an anti-periodic point which is also a critical point. The algebraic multiplicity of $\lambda^{(m)}$ is defined as the order of the zero of $\Delta(\lambda) \pm 2$ at $\lambda^{(m)}$. When the order is 2, we call the multiple point a double point and denote it by $\lambda^{(d)}$. The order can exceed 2. The geometric multiplicity of $\lambda^{(m)}$ is defined as the dimension of the periodic or anti-periodic eigenspace at $\lambda^{(m)}$ and is either 1 or 2.

Counting lemmas for $\lambda^{\pm}$ and $\lambda^{(c)}$ can be established as in [24, 30], which leads to the existence of the sequences $\{\lambda^{\pm}_{j}\}$ and $\{\lambda^{(c)}_{j}\}$ and their approximate locations. Nevertheless, counting lemmas are not necessary here. For any $\lambda \in \mathbb{C}$, $\Delta(\lambda)$ is a constant of motion of the Heisenberg equation (3.1). This is the so-called isospectral theory.
Figure 6. The periodic and anti-periodic points corresponding to the potential of domain wall $m_1 = e^{i2x}, m_3 = 0$. The open circles are double points, the solid circle at the origin is a multiple point of order 4 and the two bars intersect the imaginary axis at two periodic points which are not critical points.

**Example 3.2.** For the domain wall $m_1 = \cos 2x, m_2 = \sin 2x$ and $m_3 = 0$; the two Bloch eigenfunctions are given in (3.10). The Floquet discriminant is given by

$$\Delta = 2 \cos[2\pi \sqrt{1 + \lambda^2}].$$

The periodic points are given by

$$\lambda = \pm \sqrt{\frac{n^2}{4} - 1}, \quad n \in \mathbb{Z}, \quad n \text{ is even.}$$

The anti-periodic points are given by

$$\lambda = \pm \sqrt{\frac{n^2}{4} - 1}, \quad n \in \mathbb{Z}, \quad n \text{ is odd.}$$

The choice of $\phi^+$ and $\phi^-$ corresponds to $n = \pm 1$ and $\lambda = \nu = i\sqrt{3}/2$ with $k = \pm 1$.

$$\Delta' = -4\pi \frac{\lambda}{\sqrt{1 + \lambda^2}} \sin[2\pi \sqrt{1 + \lambda^2}],$$

$$\Delta'' = -4\pi (1 + \lambda^2)^{-3/2} \sin[2\pi \sqrt{1 + \lambda^2}] - 8\pi^2 \frac{\lambda^2}{1 + \lambda^2} \cos[2\pi \sqrt{1 + \lambda^2}].$$

When $n = 0$, i.e. $\sqrt{1 + \lambda^2} = 0$, by L’Hospital’s rule

$$\Delta' \to -8\pi^2 \lambda, \quad \lambda = \pm i.$$

That is, $\lambda = \pm i$ are periodic points, not critical points. When $n = \pm 1$, we have two imaginary double points:

$$\lambda = \pm i\sqrt{3}/2.$$

When $n = \pm 2, \lambda = 0$ is a multiple point of order 4. The rest periodic and anti-periodic points are all real double points. Figure 6 is an illustration of these spectral points.

### 3.5. Melnikov vectors

Starting from the Floquet theory, one can build Melnikov vectors.

**Definition 3.3.** An important sequence of invariants $F_j$ of the Heisenberg equation is defined by

$$F_j(m) = \Delta(\lambda_j^e)(m), m).$$
Lemma 3.4. If \( \lambda_j^{(c)} \) is a simple critical point of \( \Delta \), then
\[
\frac{\partial F_j}{\partial m} = \frac{\partial \Delta}{\partial m} |_{\lambda = \lambda_j^{(c)}}.
\]

Proof. We know that
\[
\frac{\partial F_j}{\partial m} = \frac{\partial \Delta}{\partial m} |_{\lambda = \lambda_j^{(c)}} + \frac{\partial \Delta}{\partial \lambda_j^{(c)}} \frac{\partial \lambda_j^{(c)}}{\partial m}.
\]
Since
\[
\frac{\partial \Delta}{\partial \lambda} |_{\lambda = \lambda_j^{(c)}} = 0,
\]
we have
\[
\frac{\partial^2 \Delta}{\partial \lambda^2} |_{\lambda = \lambda_j^{(c)}} \frac{\partial \lambda_j^{(c)}}{\partial m} \bigg|_{\lambda = \lambda_j^{(c)}} = 0.
\]
Since \( \lambda_j^{(c)} \) is a simple critical point of \( \Delta \),
\[
\frac{\partial^2 \Delta}{\partial \lambda^2} |_{\lambda = \lambda_j^{(c)}} \neq 0.
\]
Thus
\[
\frac{\partial \lambda_j^{(c)}}{\partial m} = - \left[ \frac{\partial^2 \Delta}{\partial \lambda^2} |_{\lambda = \lambda_j^{(c)}} \right]^{-1} \frac{\partial^2 \Delta}{\partial \lambda \partial m} |_{\lambda = \lambda_j^{(c)}}.
\]
Note that \( \Delta \) is an entire function of \( \lambda \) and \( m \) [24], then we know that \( \frac{\partial \lambda_j^{(c)}}{\partial m} \) is bounded, and
\[
\frac{\partial F_j}{\partial m} = \frac{\partial \Delta}{\partial m} |_{\lambda = \lambda_j^{(c)}}.
\]

Theorem 3.5. As a function of two variables, \( \Delta = \Delta(\lambda, m) \) has the partial derivatives given by Bloch functions \( \psi^\pm \) (i.e. \( \psi^\pm(x) = e^{\pm i \lambda x} \tilde{\psi}^\pm(x) \)), where \( \tilde{\psi}^\pm \) are periodic in \( x \) of period \( 2\pi \) and \( \Lambda \) is a complex constant:

\[
\begin{align*}
\frac{\partial \Delta}{\partial m_+} &= -i \lambda \frac{\sqrt{\Delta^2 - 4} W(\psi^+, \psi^-)}{W(\psi^+, \psi^-)} \psi_1^+ \psi_1^-,
\frac{\partial \Delta}{\partial m_-} &= i \lambda \frac{\sqrt{\Delta^2 - 4} W(\psi^+, \psi^-)}{W(\psi^+, \psi^-)} \psi_2^+ \psi_2^-,
\frac{\partial \Delta}{\partial m_3} &= i \lambda \frac{\sqrt{\Delta^2 - 4} W(\psi^+, \psi^-)}{W(\psi^+, \psi^-)} (\psi_1^+ \psi_2^- + \psi_2^+ \psi_1^-),
\frac{\partial \Delta}{\partial \lambda} &= \frac{i \sqrt{\Delta^2 - 4} W(\psi^+, \psi^-)}{W(\psi^+, \psi^-)} \left[ m_3 (\psi_1^+ \psi_2^- - \psi_2^+ \psi_1^-) - m_+ \psi_1^+ \psi_1^- + m_- \psi_2^+ \psi_2^- \right] dx,
\end{align*}
\]
where \( W(\psi^+, \psi^-) = \psi_1^+ \psi_2^- - \psi_2^+ \psi_1^- \) is the Wronskian.
Proof. Recalling that $Y$ is the fundamental matrix solution of (3.3), we have the equation for the differential of $Y$

$$\frac{\partial}{\partial x} dY = i\lambda \Gamma dY + i(d\lambda \Gamma + \lambda d\Gamma)Y, \quad dY(0) = 0.$$ 

Using the method of variation of parameters, we let

$$dY = YQ, \quad Q(0) = 0.$$ 

Thus

$$Q(x) = i \int_0^x Y^{-1}(d\lambda \Gamma + \lambda d\Gamma)Y \, dx$$

and

$$dY(x) = iY \int_0^x Y^{-1}(d\lambda \Gamma + \lambda d\Gamma)Y \, dx.$$ 

Finally

$$d\Delta = \text{trace } dY(2\pi) = i \text{ trace } \left\{ Y(2\pi) \int_0^{2\pi} Y^{-1}(d\lambda \Gamma + \lambda d\Gamma)Y \, dx \right\}.$$ \hspace{1cm} (3.16)

Let

$$Z = (\psi^+ \psi^-),$$

where $\psi^\pm$ are two linearly independent Bloch functions (for the case that there is only one linearly independent Bloch function, L'Hôpital's rule has to be used; for details, see [24]), such that

$$\psi^\pm = e^{\pm \Lambda x} \tilde{\psi}^\pm,$$

where $\tilde{\psi}^\pm$ are periodic in $x$ of period $2\pi$ and $\Lambda$ is a complex constant. (The existence of such functions is the result of the well-known Floquet theorem.) Then

$$Z(x) = Y(x)Z(0), \quad Y(x) = Z(x)[Z(0)]^{-1}.$$ 

Note that

$$Z(2\pi) = Z(0)E,$$ 

where $E = \begin{pmatrix} e^{\Lambda 2\pi} & 0 \\ 0 & e^{-\Lambda 2\pi} \end{pmatrix}$.

Then

$$Y(2\pi) = Z(0)E[Z(0)]^{-1}.$$ 

Thus

$$\Delta = \text{trace } Y(2\pi) = \text{trace } E = e^{\Lambda 2\pi} + e^{-\Lambda 2\pi}$$

and

$$e^{\pm \Lambda 2\pi} = \frac{1}{2} [\Lambda \pm \sqrt{\Lambda^2 - 4}].$$

In terms of $Z$, $d\Delta$ as given in (3.16) takes the form

$$d\Delta = i \text{ trace } \left\{ Z(0)E[Z(0)]^{-1} \int_0^{2\pi} Z(0)[Z(x)]^{-1}(d\lambda \Gamma + \lambda d\Gamma)Z(x)[Z(0)]^{-1} \, dx \right\}$$

$$= i \text{ trace } \left\{ E \int_0^{2\pi} [Z(x)]^{-1}(d\lambda \Gamma + \lambda d\Gamma)Z(x) \, dx \right\},$$

from which one obtains the partial derivatives of $\Delta$ as stated in the theorem. \hspace{1cm} $\square$

It turns out that the partial derivatives of $F_j$ provide the perfect Melnikov vectors rather than those of the Hamiltonian or other invariants [24], in the sense that $F_j$ is the invariant whose level sets are the separatrices.
3.6. An explicit expression of the Melnikov vector along the figure eight connecting to the domain wall

We continue the calculation in section 3.3 to obtain an explicit expression of the Melnikov vector along the figure eight connecting to the domain wall. Applying the Darboux transformation (3.6) to $\phi^\pm$ (3.11) at $\lambda = \nu$, we obtain

$$\hat{\phi}^\pm = \pm \frac{\tilde{v} - \nu}{\tilde{v}} \exp\left[\mp \frac{1}{2} \sigma \mp i \frac{1}{2} \gamma\right] W(\phi^+, \phi^-) \left( \frac{\bar{\phi}_2}{\phi_1} \right).$$

In formula (3.6), for general $\lambda$,

$$\det G = (\nu - \lambda)(\bar{\nu} - \lambda) \left| \frac{\phi_1}{|\phi_1|^2} \right|^2 W(\hat{\psi}^+, \hat{\psi}^-) = \det GW(\psi^+, \psi^-).$$

In the neighbourhood of $\lambda = \nu$,

$$\frac{\Delta^2 - 4}{W(\hat{\psi}^+, \hat{\psi}^-)} \to \frac{\sqrt{\Delta(v)\Delta''(v)}(\nu - \bar{\nu})}{|\nu|^2}.$$

As $\lambda \to \nu$, by L'Hospital's rule

$$\frac{\sqrt{\Delta^2 - 4}}{W(\hat{\psi}^+, \hat{\psi}^-)} \to \frac{\nu - \tilde{v}}{|\nu|^2} W(\phi^+, \phi^-).$$

Note, by the calculation in example 3.2, that

$$\nu = i \sqrt{3}, \quad \Delta(v) = -2, \quad \Delta''(v) = -24\pi^2,$$

then by theorem 3.5,

$$\frac{\partial \Delta}{\partial \hat{m}_1} \bigg|_{\hat{m}=\hat{m}} = 12\sqrt{3}\pi \frac{i}{(|\phi_1|^2 + |\phi_2|^2)^2} \phi_2^2,$$

$$\frac{\partial \Delta}{\partial \hat{m}_2} \bigg|_{\hat{m}=\hat{m}} = 12\sqrt{3}\pi \frac{-i}{(|\phi_1|^2 + |\phi_2|^2)^2} \phi_1^2,$$

$$\frac{\partial \Delta}{\partial \hat{m}_3} \bigg|_{\hat{m}=\hat{m}} = 12\sqrt{3}\pi \frac{2i}{(|\phi_1|^2 + |\phi_2|^2)^2} \phi_1 \phi_2,$$

where $\hat{m}$ is given in (3.13)–(3.14). With the explicit expression (3.12) of $\phi$, we obtain the explicit expressions of the Melnikov vector,

$$\frac{\partial \Delta}{\partial \hat{m}_1} \bigg|_{\hat{m}=\hat{m}} = \frac{3\sqrt{3}\pi}{2} \frac{i \operatorname{sech} 2\tau}{(2 - \sqrt{3} \operatorname{sech} 2\tau \sin 2\chi)^2} \left[ (1 - 2 \tanh 2\tau) \cos 2\chi 
+ i(2 - \tanh 2\tau) \sin 2\chi - i\sqrt{3} \operatorname{sech} 2\tau \right] e^{-i2\chi},$$

$$\frac{\partial \Delta}{\partial \hat{m}_2} \bigg|_{\hat{m}=\hat{m}} = \frac{3\sqrt{3}\pi}{2} \frac{-i \operatorname{sech} 2\tau}{(2 - \sqrt{3} \operatorname{sech} 2\tau \sin 2\chi)^2} \left[ (1 + 2 \tanh 2\tau) \cos 2\chi 
- i(2 + \tanh 2\tau) \sin 2\chi + i\sqrt{3} \operatorname{sech} 2\tau \right] e^{i2\chi},$$

$$\frac{\partial \Delta}{\partial \hat{m}_3} \bigg|_{\hat{m}=\hat{m}} = \frac{3\sqrt{3}\pi}{2} \frac{2i \operatorname{sech} 2\tau}{(2 - \sqrt{3} \operatorname{sech} 2\tau \sin 2\chi)^2} \left[ 2 \operatorname{sech} 2\tau - \sqrt{3} \sin 2\chi 
- i\sqrt{3} \operatorname{tanh} 2\tau \cos 2\chi \right].$$
where again
\[
m_\pm = m_1 \pm im_2, \quad \tau = \frac{\sqrt{3}}{2} t + \frac{\sigma}{2}, \quad \chi = \frac{1}{2} (x + \gamma),
\]
and \(\sigma\) and \(\gamma\) are two real parameters.

4. A Melnikov function

The forced LLG equation (2.1) can be rewritten in the form,
\[
\partial_t m = -m \times m_{xx} + \epsilon f + \epsilon^2 g, \quad (4.1)
\]
where \(f\) is the perturbation
\[
f = -am \times e_\perp + m_3(m \times e_\parallel) - bm_1(m \times e_\perp)
- am \times (m \times m_{xx}) + (\beta_1 + \beta_2 \cos \omega_0 t) m \times (m \times e_\perp),
\]
\[
g = -am \times [m \times (ae_\perp - m_3 e_\parallel + bm_1 e_\perp)].
\]
The Melnikov function for the forced LLG (2.1) is given as
\[
M = \int_0^\infty \int_{-\infty}^{\infty} \left[ \frac{\partial \Delta}{\partial m_+} (f_1 + if_2) + \frac{\partial \Delta}{\partial m_-} (f_1 - if_2) + \frac{\partial \Delta}{\partial m_3} f_3 \right] \bigg|_{m=\hat{m}} \, dx \, dt,
\]
where \(\hat{m}\) is given in (3.13)–(3.14), and \((\partial \Delta/\partial w) (w = m_+, m_- m_3)\) are given in (3.17)–(3.19). The Melnikov function depends on several external and internal parameters
\[
M = M(a, b, \alpha, \beta_1, \beta_2, \omega_0, \sigma, \gamma),
\]
where \(\sigma\) and \(\gamma\) are internal parameters. We can split \(f\) as follows:
\[
f = af^{(a)} + f^{(0)} + bf^{(b)} + \alpha f^{(\alpha)} + \beta_1 f^{(\beta_1)}
+ \beta_2 \left[ \cos \left( \frac{\sigma \omega_0}{\sqrt{3}} \right) f^{(c)} + \sin \left( \frac{\sigma \omega_0}{\sqrt{3}} \right) f^{(s)} \right],
\]
where
\[
f^{(a)} = -m \times e_\perp,
\]
\[
f^{(0)} = m_3(m \times e_\parallel),
\]
\[
f^{(b)} = -m_1(m \times e_\perp),
\]
\[
f^{(\alpha)} = -m \times (m \times m_{xx}),
\]
\[
f^{(\beta_1)} = m \times (m \times e_\perp),
\]
\[
f^{(c)} = \cos \left( \frac{2 \omega_0}{\sqrt{3}} \right) m \times (m \times e_\perp),
\]
\[
f^{(s)} = \sin \left( \frac{2 \omega_0}{\sqrt{3}} \right) m \times (m \times e_\perp).
\]

Thus \(M\) can be split as
\[
M = aM^{(a)} + M^{(0)} + bM^{(b)} + \alpha M^{(\alpha)} + \beta_1 M^{(\beta_1)}
+ \beta_2 \left[ \cos \left( \frac{\sigma \omega_0}{\sqrt{3}} \right) M^{(c)} + \sin \left( \frac{\sigma \omega_0}{\sqrt{3}} \right) M^{(s)} \right], \quad (4.2)
\]
where \(M^{(\xi)} = M^{(\xi)}(\gamma), \xi = a, 0, b, \alpha, \beta_1\) and \(M^{(\xi)} = M^{(\xi)}(\omega_0), \xi = c, s.\)
In general [20], the zeros of the Melnikov function indicate the intersection of certain centre-unstable and centre-stable manifolds. In fact, the Melnikov function is the leading order term of the distance between the centre-unstable and centre-stable manifolds. In some cases, such an intersection can lead to homoclinic orbits and homoclinic chaos. Here in the current problem, we do not have an invariant manifold result. Therefore, our calculation on the Melnikov function is purely from a physics, rather than rigorous mathematics, point of view.

In terms of the variables $m_+$ and $m_3$, the forced LLG equation (2.1) can be rewritten in the form that will be more convenient for the calculation of the Melnikov function,

\begin{align}
\partial_t m_+ &= i(m_+ m_{3xx} - m_3 m_{+xx}) + \epsilon f_+ + \epsilon^2 g_+ , \\
\partial_t m_3 &= \frac{1}{24}(m_+ m_{++xx} - m_+ m_{+++xx}) + \epsilon f_3 + \epsilon^2 g_3 ,
\end{align}

where

\begin{align*}
f_+ &= f_1 + i f_2 = a f_+^{(a)} + f_+^{(b)} + b f_+^{(a)} + \alpha f_+^{(a)} + \beta_1 f_+^{(b)} \\
&+ \beta_2 \left[ \cos \left( \frac{\sigma \omega_0}{\sqrt{3}} \right) f_+^{(c)} + \sin \left( \frac{\sigma \omega_0}{\sqrt{3}} \right) f_+^{(c)} \right] , \\
f_3 &= a f_3^{(a)} + f_3^{(b)} + b f_3^{(a)} + \alpha f_3^{(a)} + \beta_1 f_3^{(b)} \\
&+ \beta_2 \left[ \cos \left( \frac{\sigma \omega_0}{\sqrt{3}} \right) f_3^{(c)} + \sin \left( \frac{\sigma \omega_0}{\sqrt{3}} \right) f_3^{(c)} \right] , \\
g_+ &= g_1 + i g_2 = a a g_+^{(a)} + a g_+^{(b)} + a b g_+^{(b)} , \\
g_3 &= a a g_3^{(a)} + a g_3^{(b)} + a b g_3^{(b)} , \\
f_+^{(a)} &= -i m_3 , \\
f_+^{(b)} &= -i m_3 m_+ , \\
f_+^{(a)} &= -i \frac{1}{2} m_3 (m_+ + m_+^\tau ) , \\
f_+^{(b)} &= \frac{1}{2} m_3 (m_+ - m_+^\tau ) - m_3^2 , \\
f_+^{(c)} &= \cos \left( \frac{2}{\sqrt{3}} \omega_0 \tau \right) \left[ \frac{1}{2} m_3 (m_+ - m_+^\tau ) - m_3^2 \right] , \\
f_+^{(c)} &= \sin \left( \frac{2}{\sqrt{3}} \omega_0 \tau \right) \left[ \frac{1}{2} m_3 (m_+ - m_+^\tau ) - m_3^2 \right] , \\
f_3^{(a)} &= \frac{1}{24} (m_+ - m_+^\tau ) , \\
f_3^{(b)} &= 0 , \\
f_3^{(b)} &= \frac{1}{4} (m_+^2 - m_+^\tau ) , \\
f_3^{(a)} &= m_{3xx} m_+^2 - \frac{1}{2} m_3 (m_+ m_{+++xx} + m_+ m_{+=+xx} ) ,
\end{align*}
Figure 7. (a) The graph of $M^{(b_1)}$ as a function of $\gamma$, and $M^{(b_1)}$ is independent of $\omega_0$. (b) The graph of $M^{(c)}$ as a function of $\gamma$ and $\omega_0$. (c) The graph of the imaginary part of $M^{(s)}$ as a function of $\gamma$ and $\omega_0$.

\[ f^{(b_1)}_3 = \frac{1}{2} m_3 (m_+ + \bar{m}_+) , \]
\[ f^{(c)}_3 = \frac{1}{2} \cos \left( \frac{2}{\sqrt{3}} \omega_0 \tau \right) m_3 (m_+ + \bar{m}_+) , \]
\[ f^{(a)}_3 = \frac{1}{2} \sin \left( \frac{2}{\sqrt{3}} \omega_0 \tau \right) m_3 (m_+ + \bar{m}_+) , \]
\[ g^{(a)}_3 = m_3^2 - \frac{1}{2} m_+ (m_+ - \bar{m}_+) , \]
\[ g^{(b)}_3 = m_3^2 m_+ , \]
\[ g^{(b)}_3 = \frac{1}{4} m_3^2 (m_+ + \bar{m}_+) - \frac{1}{4} m_+ (m_+^2 - \bar{m}_+^2) , \]
\[ g^{(a)}_3 = - \frac{1}{2} m_3 (m_+ + \bar{m}_+) , \]
\[ g^{(b)}_3 = - m_3 |m_+|^2 , \]
\[ g^{(b)}_3 = - \frac{1}{4} m_3 (m_+ + \bar{m}_+)^2 . \]
A direct calculation gives that
\[ M^{(a)}(\gamma) = M^{(b)}(\gamma) = M^{(e)}(\gamma) = 0, \quad M^{(d)}(\gamma) = 91.3343, \]
and \( M^{(d)} \) and \( M^{(e)} \) are real, while \( M^{(c)} \) is imaginary. The graph of \( M^{(d)} \) is shown in figure 7(a) (note that \( M^{(d)} \) is independent of \( \omega_0 \)). The graph of \( M^{(e)} \) is shown in figure 7(b). The imaginary part of \( M^{(c)} \) is shown in figure 7(c). In the case of only dc current \((\beta_2 = 0), M = 0 \) (4.2) leads to
\[ \alpha = -\beta_1 M^{(d)}(\gamma)/91.3343, \quad (4.5) \]
where \( M^{(d)}(\gamma) \) is a function of the internal parameter \( \gamma \) as shown in figure 7(a). In the general case \((\beta_2 \neq 0), M^{(e)}(\gamma, \omega_0) = 0 \) determines the curves
\[ \gamma = \gamma(\omega_0) = 0, \pi/2, \pi, 3\pi/2, \quad (4.6) \]
and \( M = 0 \) (4.2) leads to
\[ |\beta_2| > |(91.3343\alpha + \beta_1 M^{(d)})/M^{(c)}|, \quad (4.7) \]
where \( M^{(d)} \) and \( M^{(e)} \) are evaluated along curve (4.6), \( M^{(d)} = \pm 43.858 \) (+ for \( \gamma = \pi/2, 3\pi/2; - \) for \( \gamma = 0, \pi \)), \( M^{(c)} \) is plotted in figure 8 (the upper curve corresponds to \( \gamma = \pi/2, 3\pi/2 \), the lower curve corresponds to \( \gamma = 0, \pi \)) and
\[ \cos\left(\frac{\sigma \omega_0}{\sqrt{3}}\right) = -\frac{91.3343\alpha + \beta_1 M^{(d)}}{\beta_2 M^{(c)}}. \]

5. Numerical simulation

In the entire paper, we use the finite difference method to numerically simulate the LLG (2.1). Due to an integrable discretization [9] of the Heisenberg equation (3.1), the finite difference performs much better than Galerkin Fourier mode truncations. As in (3.15), let \( m(j) = m(t, jh), \quad j = 1, \ldots, N, \quad Nh = 2\pi \) and \( h \) is the spatial mesh size. Without further notice, we always choose \( N = 128 \) (which provides enough precision). The only tricky part in the finite difference discretization of (2.1) is the second derivative term in \( H \); for the rest of the terms, just evaluate \( m \) at \( m(j) \):
\[ \frac{\partial^2 m(j)}{h^2} = \frac{m(j+1)}{1 + m(j+1) \cdot m(j)} + \frac{m(j-1)}{1 + m(j-1) \cdot m(j)}. \]
Figure 9. The bifurcation diagram for the attractors and typical spatial profiles on the attractors in the case of only dc current where $\beta_1$ is the bifurcation parameter, $c_1 \cdots c_6$ are the bifurcation thresholds and $c_1 \in [0, 0.01], c_2 \in [0.0205, 0.021], c_3 \in [0.0231, 0.0232], c_4 \in [0.025, 0.026], c_5 \in [0.08, 0.1], c_6 \in [0.13, 0.15]$.

Figure 10. The spatio–temporal profiles of solutions in the attractors in the case of only dc current. (a) $\beta_1 = 0.0205$ spatially non-uniform fixed point. (b) $\beta_1 = 0.023$ spatially non-uniform and temporally periodic or quasiperiodic attractor. (c) $\beta_1 = 0.0235$ weak chaotic attractor.

5.1. Only dc current case

In this case, $\beta_2 = 0$ in (2.1), and we choose $\beta_1$ as the bifurcation parameter, and the rest of the parameters as

$$a = 0.05, \quad b = 0.025, \quad \alpha = 0.02, \quad \epsilon = 0.01.$$  (5.1)

The computation is first run for the time interval $[0, 8120\pi]$, then the figures are plotted starting from $t = 8120\pi$. The bifurcation diagram for the attractors and the typical spatial profiles
on the attractors are shown in figure 9. This figure indicates that interesting bifurcations happen over the interval $\beta_1 \in [0, 0.15]$ which is the physically important regime where $\beta_1$ is comparable to values of other parameters. There are six bifurcation thresholds $c_1 \ldots c_6$ (figure 9). When $\beta_1 < c_1$, the attractor is the spatially uniform fixed point $m_1 = 1$ ($m_2 = m_3 = 0$). When $c_1 \leq \beta_1 \leq c_2$, the attractor is a spatially non-uniform fixed point as shown in figure 10(a). When $c_2 < \beta_1 < c_3$, the attractor is spatially non-uniform and temporally periodic (a limit cycle) or quasiperiodic (a limit torus) as shown in figure 10(b). Here as the value of $\beta_1$ is increased, first there is one basic temporal frequency, then more frequencies enter and the temporal oscillation amplitude becomes bigger and bigger. When $c_3 \leq \beta_1 < c_4$, the attractor is chaotic, i.e. a strange attractor as shown in figure 10(c). Even though the chaotic nature is not very apparent in figure 10(c), due to the smallness of the perturbation parameter together with smallness of all other parameters, the temporal evolution is chaotic, and we have used the Liapunov exponent and power spectrum devices to verify this. When $c_4 \leq \beta_1 < c_5$, the attractor is spatially non-uniform and temporally periodic (a limit cycle) as shown in figure 11(a). The spatial modulation is small, and it is not even apparent in figure 11(a). But it is apparent on the individual typical spatial profiles as seen in region V in figure 9. With the increase in $\beta_1$, the spatial modulation becomes smaller and smaller. When $c_5 \leq \beta_1 < c_6$, the attractor is spatially uniform and temporally periodic (a limit cycle, the so-called procession) as shown in figure 11(b). When $\beta_1 \geq c_6$, the attractor is the spatially uniform fixed point $m_1 = -1$ ($m_2 = m_3 = 0$).

When $\beta_2 = 0$, the Melnikov function predicts that around $\beta_1 = 0.041$ (4.5), there is probably chaos; while the numerical calculation shows that there is an interval $[0.0231, 0.026]$ for $\beta_1$ where chaos is the attractor. Since the perturbation parameter $\epsilon = 0.01$ in the numerical calculation, the Melnikov function prediction seems in agreement with the numerical calculation.

Some of the attractors in figure 9 are attractors of the corresponding ordinary differential equations by setting $\partial_1 = 0$ in (2.1), i.e. the single domain case. These attractors are the ones in regions I, VI and VII in figure 9 [26, 44]. Because we are studying the only dc current case, the ordinary differential equations do not have any chaotic attractor [26, 44].
On the other hand, the partial differential equation (2.1) does have a chaotic attractor (region IV in figure 9).

In general, when $\beta_1 < 0$, the Gilbert damping dominates the spin torque driven by dc current and $m_1 = 1$ is the attractor. When $\beta_1 > 0.15$, the spin torque driven by dc current dominates the Gilbert damping, $m_1 = -1$ is the attractor, and we have a magnetization reversal. In some technological applications, $\beta_1 > 0.15$ may correspond to too a high dc current that can burn the device. On the other hand, in the technologically advantageous interval $\beta_1 \in [0, 0.15]$, magnetization reversal may be hard to achieve due to the sophisticated bifurcations in figure 9.

5.2. Only ac current case

In this case, $\beta_1 = 0$ in (2.1), and we choose $\beta_2$ as the bifurcation parameter, and the rest of the parameters as

$$a = 0.05, \quad b = 0.025, \quad \alpha = 0.0015, \quad \epsilon = 0.01, \quad \omega_0 = 0.2. \quad (5.2)$$

Unlike the dc case, here the figures are plotted starting from $t = 0$. It turns out that the types of attractors in the ac case are simpler than those of the dc case. When $\beta_2 = 0$, the attractor is a spatially non-uniform fixed point as shown in figure 12. In this case, the only
Figure 13. Homotopy deformation of the attractors under different initial conditions. (a) $\beta_2 = 0.21$ spatially uniform initial condition. (b) $\beta_2 = 0.21$ spatial non-uniform initial condition. (c) $\beta_2 = 0.21$ spatial more non-uniform initial condition.

perturbation is the Gilbert damping which damps the evolution to such a fixed point. When $0 < \beta_2 < \beta_2^* \in [0.18, 0.19]$, the attractor is a spatially non-uniform and temporally periodic solution. When $\beta_2 \geq \beta_2^*$, the attractor is chaotic as shown in figure 12. Our Melnikov prediction (4.7) predicts that when $|\beta_2| > 0.003$, certain centre-unstable and centre-stable manifolds intersect. Our numerics shows that such an intersection seems to lead transient chaos. Only when $|\beta_2| > \beta_2^*$, can be the chaos sustained as an attractor. It seems that such a sustained chaotic attractor gains extra support from parametric resonance due to the ac current driving [23], as can be seen from the turbulent spatial structure of the chaotic attractor (figure 12), which diverges quite far away from the initial condition. Another factor that may be relevant is the fact that higher-frequency spatially oscillating domain walls have more and stronger linearly unstable modes (3.5). By properly choosing initial conditions, one can find the homotopy deformation from the ODE limit cycle (procession) [26, 44] to the current PDE chaos as shown in figure 13 at the same parameter values.

We also simulated the case of normal Gilbert damping $\alpha = 0.02$. For all values of $\beta_2 \in [0.01, 0.3]$, the attractor is always non-chaotic. That is, the only attractor we can find is a spatially uniform limit cycle with small temporal oscillation as shown in figure 14.

Of course, when neither $\beta_1$ nor $\beta_2$ is zero, the bifurcation diagram is a combination of the dc only and ac only diagrams.
Figure 14. The attractor when $\alpha = 0.02, \beta = 0.21$ and all other parameters' values are the same with figure 12.

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