ISR corrections to associated $HZ$ production at future Higgs factories

Mario Greco$^{a,b}$, Guido Montagna$^{c,d}$, Oreste Nicrosini$^d$, Fulvio Piccinini$^{d,*}$, Gabriele Volpi$^c$

$^a$Dipartimento di Matematica e Fisica, Università di Roma 3, via della Vasca Navale 84, 00146 Roma – Italy
$^b$INFN, Sezione di Roma 3, via della Vasca Navale 84, 00146 Roma – Italy
$^c$Dipartimento di Fisica, Università di Pavia, via A. Bassi 6, 27100 Pavia – Italy
$^d$INFN, Sezione di Pavia, via A. Bassi 6, 27100 Pavia – Italy

Abstract

We evaluate the QED corrections due to initial state radiation (ISR) to associated Higgs boson production in electron–positron ($e^+e^-$) annihilation at typical energies of interest for the measurement of the Higgs properties at future $e^+e^-$ colliders, such as CEPC and FCC–ee. We apply the QED Structure Function approach to the four–fermion production process $e^+e^- \rightarrow \mu^+\mu^- b\bar{b}$, including both signal and background contributions. We emphasize the relevance of the ISR corrections particularly near threshold and show that finite third order collinear contributions are mandatory to meet the expected experimental accuracy. We analyze in turn the rôle played by a full four–fermion calculation and beam energy spread in precision calculations for Higgs physics at future $e^+e^-$ colliders.

Keywords: Higgs boson, electron–positron colliders, QED corrections

1. Introduction

The discovery of a new scalar particle at the LHC in 2012 by the ATLAS [1] and CMS collaborations [2] has opened a new chapter in particle physics, and immediately has triggered the question of the real nature of this boson. The determination of the spin–parity quantum numbers and the couplings to other Standard Model (SM) particles strongly suggest it to be the Higgs boson, i.e. the elementary scalar particle responsible for the mechanism of electroweak symmetry breaking. However, from the available data it can not be concluded yet that we have found the SM Higgs boson and not, for instance, one of the scalars postulated within possible extensions of the SM.

*Corresponding author

Email address: fulvio.piccinini@pv.infn.it (Fulvio Piccinini)
Therefore, various lepton collider Higgs factories \[3, 4, 5, 6, 7, 8, 9, 10\] are under consideration to study in detail the properties of the new particle to great accuracies, because of the much more favorable experimental environment than that at hadron colliders. Amongst the many candidates of Higgs factories, one can distinguish two main categories.

The first one exploits the possibility of \(s\)-channel Higgs resonant production, which is especially important, due to the narrow width of the Higgs boson, of about 4 MeV, as predicted by the SM. In particular, a muon collider Higgs factory \[3, 6\] could produce the Higgs boson in the \(s\)-channel and perform an energy scan to map out the Higgs resonance line shape at tens of MeV level \[11\]. This approach would provide the most direct measurement of the Higgs boson total width, the Yukawa coupling to muons and other fermions, and would also enable to simply investigate the existence of other scalar bosons predicted by natural extensions of the SM. This case has been studied in detail in Refs. \[12, 13, 14\], with a particular emphasis on the rôle of the Initial State Radiation (ISR) effects – which are quite important \[15, 16\] due to the \(s\)-channel resonant production of the very narrow Higgs boson – and the evaluation of the background processes.

The second category includes the possibility of ultra–high luminosity electron–positron (\(e^+e^-\)) colliders, such as the CEPC, FCC–ee, the International Linear Collider and CLIC, which have been proposed \[4, 5, 7, 8, 9, 10\] with the precise aim of observing the Higgs signal mainly through the reaction \(e^+e^- \rightarrow HZ\) at different center of mass (c.m.) energies, but also measuring possibly the Yukawa coupling to electrons. In these cases, the ISR effects could be quite sizable because of the smallness of the electron mass, in particular in the vicinity of the threshold of \(HZ\) production. A study at LEP time of those effects, as well as of the background processes, was given in Ref. \[17\].

In this paper, we focus on the process of associated \(HZ\) production in \(e^+e^-\) annihilation, with the aim of providing a comprehensive and accurate theoretical approach to precision measurements of the Higgs boson parameters at future Higgs factories. To this end, we consider for definitiveness the cleanest production channel given by the four–fermion process \(e^+e^- \rightarrow \mu^+\mu^- b\bar{b}\), including both signal and background contributions, and induced in the signal process by the dominant Higgs decay \(H \rightarrow b\bar{b}\) and the leptonic \(Z\) boson decay \(Z \rightarrow \mu^+\mu^-\). To model photon emission in the ISR process, we use a set of representative and state–of–the–art choices for the QED Structure Functions, as largely employed in the context of LEP precision phenomenology.

Next–to–leading electroweak corrections to \(e^+e^- \rightarrow HZ\) were calculated time ago in Refs. \[18, 19, 20, 21\], while mixed QCD–electroweak contributions have been computed recently in Refs. \[22, 23\]. A very preliminary investigation of the ISR contribution to the \(HZ\) signal process at \(\sqrt{s} = 250\) GeV has been performed in Ref. \[24\] using MadGraph. Hence, our study improves the existing analyses of associated Higgs boson production at the proposed \(e^+e^-\) Higgs factories. The case of resonant Higgs production in electron–positron collisions and the modifications induced by ISR corrections on the Higgs boson line shape have been recently studied in Refs. \[13, 14\].
The paper is organized as follows. In Sect. 2, we describe the formulation and parameterization of the ISR corrections in the general framework of the QED Structure Function approach. In Sect. 3, we quantify their effects on the Higgs boson associated production cross section at various c.m. energies of interest for the different projects of Higgs factories. We analyze in turn the role played by a full four–fermion calculation and beam energy spread in precision calculations for Higgs physics at future lepton colliders. Our conclusions and perspectives are drawn in Sect. 4.

2. Structure Function formulation of ISR effects

According to factorization theorems of soft and collinear singularities, the contribution of ISR to Higgs boson associated production can be evaluated using the following master formula:

\[ \frac{d\sigma}{d\sigma_0}(x_1 x_2, s) = \int dx_1 dx_2 D(x_1, s) D(x_2, s) d\sigma_0(x_1 x_2 s) \Theta(\text{cuts}). \]  

In Eq. (1), \( d\sigma_0 \) is any tree–level differential cross section including the signal \( e^+ e^- \rightarrow ZH \rightarrow \mu^+ \mu^- b\bar{b} \) and all the background matrix elements, as computed in Ref. [17], taken at the reduced squared c.m. energy \( x_1 x_2 s \). The \( \Theta \) function represents the imposed kinematical cuts. The function \( D(x, s) \) is the non–singlet collinear Structure Function modeling initial state photon radiation and giving the probability of finding inside a parent electron an electron with momentum fraction \( x \) at the energy scale \( s \). They were first introduced in Ref. [25] and later improved for precision physics at LEP in Ref. [26] to add second order finite contributions to the resummation of leading logarithmic corrections. More recently, also third order finite terms have been computed analytically in Refs. [27, 28, 29, 30]. In particular, in Ref. [27] the explicit analytical expression of finite additive third order terms was given, together with an iterative formula to compute higher and higher order contributions. The explicit analytical expression of finite factorized contributions can be found in Refs. [28, 29]. In Ref. [30] the analytical expression for the radiator function, defined as a convolution of two Structure Functions, is given up to third order by using the Structure Functions of Ref. [27]. In Ref. [31] finite forth and fifth order additive finite terms are provided in distributional form. A review of the QED Structure Function method can be found in Refs. [32, 33].

The explicit expressions of the Structure Functions used in the present study are listed in the following. The all–order Structure Function, valid in the soft

\[^1\]We do not consider in our study the contribution of beamstrahlung, its effect being largely dependent from the considered Higgs factory project.

\[^2\]The complete four–fermion calculation of the background contributions includes 24 Feynman diagrams corresponding to the neutral–current processes of \( \gamma\gamma, \gamma Z \) and \( ZZ \) production and decay [17]. We compute them and the signal matrix element in the fermion massless approximation.
photon limit and dubbed as Gribov–Lipatov solution, is given by

\[
D_{GL}(x, s) = \exp \left[ \frac{1}{2} \beta \left( \frac{3}{4} - \gamma_E \right) \right] \frac{1}{\Gamma(1 + \frac{1}{2} \beta)} \frac{1}{2} \Gamma(1 + \frac{1}{2} \beta - 1),
\]  

(2)

where

\[
\beta = \frac{2\alpha}{\pi} (L - 1), \quad L = \ln(s/m_e^2).
\]  

(3)

In the above equations, \(\alpha\) is the fine structure constant, \(m_e\) is the electron mass, \(\Gamma\) and \(\gamma_E\) are the Gamma function and the Euler–Mascheroni constant, respectively. According to Eq. (2) and following equations, photon radiation is treated in strictly collinear approximation and the collinear logarithmic enhancements are encoded in the large \(\beta\) factor, with \(\beta \approx 0.11\) at the HZ threshold. The additive Structure Function including up to third order finite terms reads as follows [27]:

\[
D_A(x, s) = \sum_{i=0}^{3} d_A^{(i)}(x, s),
\]

\[
d_A^{(0)}(x, s) = D_{GL}(x, s),
\]

\[
d_A^{(1)}(x, s) = -\frac{1}{4} \beta (1 + x),
\]

\[
d_A^{(2)}(x, s) = \frac{1}{32} \beta^2 \left[ (1 + x) \left( -4 \ln(1 - x) + 3 \ln(x) \right) - 4 \frac{\ln x}{1 - x} - 5 - x \right],
\]

\[
d_A^{(3)}(x, s) = \frac{1}{384} \beta^3 \left\{ (1 + x) \left[ 18 \zeta(2) - 6 \text{Li}_2(x) - 12 \ln^2(1 - x) \right] + \frac{1}{1 - x} \left[ -\frac{3}{2} (1 + 8x + 3x^2) \ln x - 6(x + 5)(1 - x) \ln(1 - x) \right] - 12(1 + x^2) \ln x \ln(1 - x) + \frac{1}{2} (1 + 7x^2) \ln^2 x \right.
\]

\[
\left. - \frac{1}{4} (39 - 24x - 15x^2) \right\}.
\]  

(4)

where \(\zeta\) is the Riemann \(\zeta\) function and \(\text{Li}_2\) is the dilogarithm. The factorized Structure Function with up to third order finite terms can be written as [28 29]:

\[
D_F(x, s) = D_{GL}(x, s) \sum_{i=1}^{3} d_F^{(i)}
\]

\[
d_F^{(1)}(x, s) = \frac{1}{2} (1 + x^2),
\]

\[
d_F^{(2)}(x, s) = \frac{1}{4} \beta \left[ -\frac{1}{2} (1 + 3x^2) \ln x - (1 - x)^2 \right],
\]

\[
d_F^{(3)}(x, s) = \frac{1}{8} \left( \frac{\beta}{2} \right)^2 \left[ (1 - x)^2 + \frac{1}{2} (3x^2 - 4x + 1) \ln x \right].
\]
\[ + \frac{1}{12} (1 + 7x^2) \ln^2 x + (1 - x^2) \ln(1 - x) \]  

In the next Section, a detailed comparison between the results obtained by using all the above Structure Functions for the modeling of ISR corrections to associated Higgs production is discussed.

3. Numerical results

For the presentation of the numerical results, we consider two distinct event selection conditions:

1. No Cuts. In this situation, we simply require that the invariant masses of the $\mu^+\mu^-$ and $b\bar{b}$ pairs satisfy the loose constraint $M_{\mu^+\mu^-} \geq 12$ GeV and $M_{b\bar{b}} \geq 12$ GeV. These cuts are imposed in order to suppress the contribution from the background processes of $\gamma\gamma$ and $Z\gamma$ production and decay.

2. Cuts. According to this event selection, in order to mimic the finite $b\bar{b}$ mass resolution foreseen at future $e^+e^-$ colliders, we apply a cut on the $b\bar{b}$ pair invariant mass given by $M_H - 3 \text{ GeV} \leq M_{b\bar{b}} \leq M_H + 3 \text{ GeV}$ \[34\], with $M_H = 125$ GeV, in association to the cut $M_{\mu^+\mu^-} \geq 12$ GeV.

We use as input parameters $M_Z = 91.1876$ GeV, $\Gamma_Z = 2.4952$ GeV, $M_W = 80.385$ GeV, $M_H = 125$ GeV and $m_b = 4.7$ GeV, the latter being necessary for the evaluation of the $Hb\bar{b}$ Yukawa coupling. All the other derived quantities are computed at the three level, using the so-called $G_\mu$ scheme.

From the numerical simulation, we observe that at $\sqrt{s} = 240$ GeV, in the absence of a cut on the $b\bar{b}$ invariant mass around the Higgs mass, the signal+background ($S+B$) $\mu^+\mu^-$ invariant mass distribution is about a factor of four larger than the signal alone ($S$), whereas the cut reduces the background such that the signal alone and the signal plus background $\mu^+\mu^-$ invariant mass distributions differ at the per cent level. The corresponding cross sections at $\sqrt{s} = 240$ GeV are $\sigma_S = 5.899(3) \text{ fb}$ and $\sigma_{S+B} = 24.39(2) \text{ fb}$ when no cuts are imposed, while we obtain $\sigma_S = 5.899(3) \text{ fb}$ and $\sigma_{S+B} = 5.960(3) \text{ fb}$ in the presence of a $b\bar{b}$ invariant mass cut around the Higgs mass. The digits in parenthesis correspond to the $1\sigma$ Monte Carlo error estimate. These cross section values are obtained by including ISR QED corrections in the Gribov–Lipatov approximation as in Eq. \[3\]. This simple analysis points out that the theoretical predictions must rely upon a full four–fermion calculations in order to meet the foreseen experimental precision, which is presently estimated at the level of $0.4 \times 10^{-3}$ \[35\], at least for the CEPC and FCC–ee facilities.

The relevance of ISR QED contributions to associated Higgs boson production, neglected in previous studies, is shown in Fig. \[4\] where the line shape of the signal process $e^+e^- \rightarrow ZH \rightarrow \mu^+\mu^- b\bar{b}$ in the lowest order approximation (Born) is compared to the corresponding QED corrected cross section (QED) computed according to Eq. \[1\] (upper panel); the relative impact of ISR corrections is shown in the lower panel of Fig. \[4\]. The contribution of ISR has been
Figure 1: The total cross section of the process $e^+e^- \rightarrow ZH \rightarrow \mu^+\mu^-b\bar{b}$ for the $ZH$ signal contribution in the Born approximation (Born), for the full four–fermion calculation in the Born approximation ($e^+e^- \rightarrow \mu^+\mu^-b\bar{b}$ Born) and with ISR QED corrections to the signal process (QED), as a function of the c.m. energy (upper panel). The relative effect of ISR corrections is shown in lower panel. Numerical results correspond to the event selection 2. described in the text.

evaluated by using the Structure Function of Eq. (4) including finite third order contributions. We also show in Fig. 1 the line shape of the full four–fermion calculation in the tree–level approximation ($e^+e^- \rightarrow \mu^+\mu^-b\bar{b}$ Born). These results have been obtained according to the event selection condition 2. above. It can be noticed that the overall effect of ISR QED corrections is strongly varying with $\sqrt{s}$ and quite large, at the level of -35% in the threshold region and between -10% and -20% at several c.m. energy values. The large impact of the ISR corrections at threshold can be simply understood in terms of the cut–off on the maximum photon energy induced by the finite width of the $Z$ and $H$ bosons and therefore mainly derives from the contribution of multiple soft photon radiation. Incidentally, one can also see, as already remarked, that the full four–fermion prediction differs from the signal calculation at the percent level at and above threshold, thus emphasizing the importance of a full $4f$ calculation also in the presence of a tight cut on the invariant mass of the $b\bar{b}$ fermion pair.

Figure 2 shows the relative effect of the different finite order approximations included in the electron Structure Function. The main conclusions that can be drawn from Fig. 2 are the following. The first order additive finite effects, as compared to the Gribov–Lipatov approximation describing multiple soft photon emission, are of the order of some percent, increasing with the c.m. energy above threshold. This can be understood as hard photon radiation becomes
more and more important well above threshold. The second order additive finite contributions, as compared to the up to first order ones, amount to some per mille and are therefore strictly necessary for a precision measurement of the Higgs production cross section. The third order additive finite effects are at the level of $10^{-4}$ in comparison with the Structure Function containing up to second order contributions. In view of the foreseen precision at future Higgs factories, these $O(\beta^3)$ collinear effects have to be carefully taken into account.

A final remark concerns the relationship between the additive corrections of Eq. (4) and the factorized contributions of Eq. (5). The comparison between the two prescriptions, as illustrated in Fig. 2, shows that the relative difference between the third order additive and factorized corrections is at the level of $10^{-5}$ or below it, and thus negligible as compared to the foreseen experimental precision. In other words, the two approximations can be considered as equivalent in comparison with the expected experimental accuracy. Since the additive and factorized Structure Functions including finite effects at a given order $\beta^n$ differ for finite terms at the next order $\beta^{n+1}$, one can take the difference between third order additive and factorized solutions as an estimate of fourth order finite contributions. Should in a future a higher precision in the calculation of ISR QED corrections be needed, higher order finite effects could be included using the available analytical expressions for the fourth and/or fifth order collinear contributions given in Ref. [31].

In Fig. 3 we show, for completeness, the impact due to the beam energy spread and related uncertainty on the cross section of the $HZ$ signal process as a function of $\sqrt{s}$. We simulated the profile of the machine energy spread in terms of a Gaussian distribution of the c.m. energy, with a standard deviation $\sigma_{BES}$ associated to the energy spread given by $\sigma_{BES}(\sqrt{s})/\sqrt{s} = (0.086\pm0.009)\%$, where the uncertainty corresponds to a knowledge of about 10% to its value [36]. As it can be seen from Fig. 3 the contribution of the machine energy spread is only relevant at threshold, being of the order of 0.5% with an uncertainty of

Figure 2: The relative effect of different approximations for the electron Structure Functions modeling ISR corrections to the $e^+e^- \rightarrow ZH \rightarrow \mu^+\mu^- b\bar{b}$ signal process, as a function of the c.m. energy. Results corresponding to the event selection 2. described in the text.
4. Conclusion

We have computed the ISR QED corrections to the process $e^+ e^- \rightarrow \mu^+ \mu^- b \bar{b}$ from $ZH$ associated production at typical energies of interest for the measurement of the properties of the Higgs particle at future $e^+ e^-$ facilities. Using different prescriptions for the electron Structure Function at various levels of sophistication, we have shown that the QED radiative corrections are quite substantial, especially in the vicinity of the $ZH$ threshold. We have also provided clear evidence that third order collinear contributions must be taken into account in order to meet the expected experimental accuracy of future Higgs factories. We have also evaluated the impact due to the beam energy spread and related uncertainty, to conclude that this effect is only relevant at threshold but is not in general a limiting factor in precise predictions for associated Higgs boson production at future $e^+ e^-$ accelerators.

Our study improves the existing analyses of the proposed Higgs factories and can serve as a guideline for the target accelerator designs with respect to the physics goals.

Possible perspectives of the present work include the evaluation of ISR corrections to other relevant signatures for physics at future $e^+ e^-$ accelerators, such as $WW$, $t \bar{t}$, $ZHH$ and $t \bar{t}H$ production processes.

Acknowledgements

We are grateful to A. Blondel, P. Janot and R. Tenchini for useful discussions.
References

[1] G. Aad, et al., Observation of a new particle in the search for the Standard Model Higgs boson with the ATLAS detector at the LHC, Physics Letters B716 (2012) 1–29.

[2] S. Chatrchyan, et al., Observation of a new boson at a mass of 125 GeV with the CMS experiment at the LHC, Physics Letters B716 (2012) 30–61.

[3] Y. Alexahin, et al., Muon Collider Higgs Factory for Snowmass 2013, in Community Summer Study 2013: Snowmass on the Mississippi (CSS2013) Minneapolis, MN, USA, July 29-August 6, 2013 [arXiv:1308.2143].

[4] M. Aicheler, et al., A Multi-TeV linear collider based on CLIC technology: CLIC Conceptual Design Report, Tech. Rep. CERN-2012-007 (2012). URL http://project-clic-cdr.web.cern.ch/project-CLIC-CDR/CDR_Volume1.pdf

[5] H. Baer, et al., The International Linear Collider Technical Design Report - Volume 2: Physics [arXiv:1306.6352].

[6] C. Rubbia, A complete demonstrator of a muon cooled Higgs factory [arXiv:1308.0612].

[7] M. Bicer, et al., First Look at the Physics Case of TLEP, JHEP 01 (2014) 164. [arXiv:1308.6176].

[8] M. Ahmad, et al., CEPC-SPPC Preliminary Conceptual Design Report, Volume I: Physics and Detector. (2015). URL http://cepc.ihep.ac.cn/preCDR/volume.html

[9] K. Fujii, et al., Physics Case for the 250 GeV Stage of the International Linear Collider [arXiv:1710.07621].

[10] S. Asai, J. Tanaka, Y. Ushiroda, M. Nakao, J. Tian, S. Kanemura, S. Matsumoto, S. Shirai, M. Endo, M. Kakizaki, Report by the Committee on the Scientific Case of the ILC Operating at 250 GeV as a Higgs Factory [arXiv:1710.08639].

[11] V. Barger, M. Berger, J. Gunion, T. Han, Higgs Boson physics in the $s$ channel at $\mu^+\mu^-$ colliders, Physics Reports 286 (1997) 1–51. [arXiv:hep-ph/9602415].

[12] M. Greco, On the study of the Higgs properties at a muon collider, Mod. Phys. Lett. A30 (2015) 1530031. [arXiv:1503.05046].

[13] M. Greco, T. Han, Z. Liu, ISR effects for resonant Higgs production at future lepton colliders, Phys. Lett. B763 (2016) 409–415. [arXiv:1607.03210 doi:10.1016/j.physletb.2016.10.078].
[14] S. Jadach, R. A. Kycia, Lineshape of the Higgs boson in future lepton colliders, Phys. Lett. B755 (2016) 58–63. arXiv:1509.02406 doi:10.1016/j.physletb.2016.01.065

[15] M. Greco, G. Pancheri-Srivastava, Y. Srivastava, Radiative Corrections for Colliding Beam Resonances, Nucl. Phys. B101 (1975) 234–262. doi:10.1016/0550-3213(75)90304-1

[16] M. Greco, G. Pancheri-Srivastava, Y. Srivastava, Radiative Effects for Resonances with Applications to Colliding Beam Processes, Phys. Lett. 56B (1975) 367–372. doi:10.1016/0370-2693(75)90321-4

[17] G. Montagna, O. Nicrosini, F. Piccinini, Higgs boson production in $e^+e^- \rightarrow \mu^+\mu^-bb$, Phys. Lett. B348 (1995) 496–502.

[18] J. Fleischer, F. Jegerlehner, Radiative Corrections to Higgs Production by $e^+e^- \rightarrow ZH$ in the Weinberg-Salam Model, Nucl. Phys. B216 (1983) 469–492. doi:10.1016/0550-3213(83)90296-1

[19] B. A. Kniehl, Radiative corrections for associated $ZH$ production at future $e^+e^-$ colliders, Z. Phys. C55 (1992) 605–618. doi:10.1007/BF01561297

[20] A. Denner, J. Kublbeck, R. Mertig, M. Bohm, Electroweak radiative corrections to $e^+e^- \rightarrow HZ$, Z. Phys. C56 (1992) 261–272. doi:10.1007/BF01555523

[21] G. Belanger, F. Boudjema, J. Fujimoto, T. Ishikawa, T. Kaneko, K. Kato, Y. Shimizu, Full one loop electroweak radiative corrections to single Higgs production in $e^+e^-$, Phys. Lett. B559 (2003) 252–262. arXiv:hep-ph/0212261 doi:10.1016/S0370-2693(03)00339-3

[22] Y. Gong, Z. Li, X. Xu, L. L. Yang, X. Zhao, Mixed QCD-EW corrections for Higgs boson production at $e^+e^-$ colliders, Phys. Rev. D95 (9) (2017) 093003. arXiv:1609.03955 doi:10.1103/PhysRevD.95.093003

[23] Q.-F. Sun, F. Feng, Y. Jia, W.-L. Sang, Mixed electroweak-QCD corrections to $e^+e^- \rightarrow HZ$ at Higgs factories, Phys. Rev. D96 (5) (2017) 051301. arXiv:1609.03995 doi:10.1103/PhysRevD.96.051301

[24] C. Chen, Z. Cui, G. Li, Q. Li, M. Ruan, L. Wang, Q.-s. Yan, $H \rightarrow e^+e^-$ at CEPC: ISR effect with MadGraph arXiv:1705.04486

[25] E. Kuraev, V. Fadin, On Radiative Corrections to $e^+e^-$ Single Photon Annihilation at High-Energy, Sov. J. Nucl. Phys. 41 (1985) 466–472. arXiv:1503.04830

[26] O. Nicrosini, L. Trentadue, Soft Photons and Second Order Radiative Corrections to $e^+e^- \rightarrow Z^0$, Phys. Lett. B196 (1987) 551.

[27] M. Cacciari, A. Deandrea, G. Montagna, O. Nicrosini, QED structure functions: A Systematic approach, Europhys. Lett. 17 (1992) 123–128.
[28] M. Skrzypek, S. Jadach, Exact and approximate solutions for the electron nonsinglet structure function in QED, Z. Phys. C49 (1991) 577–584.

[29] M. Skrzypek, Leading logarithmic calculations of QED corrections at LEP, Acta Phys. Pol. B23 (1992) 135–172.

[30] G. Montagna, O. Nicrosini, F. Piccinini, The QED radiator at order $\alpha^3$, Phys. Lett. B406 (1997) 243–248.

[31] A. Arbuzov, Nonsinglet splitting functions in QED, Phys. Lett. B470 (1999) 252–258.

[32] G. Montagna, O. Nicrosini, F. Piccinini, Precision physics at LEP, La Rivista del Nuovo Cimento 21 n. 9 (1998) 1–162.

[33] A. B. Arbuzov, V. V. Bytev, E. A. Kuraev, E. Tomasi-Gustafsson, Yu. M. Bystritskiy, Structure function approach in QED for high energy processes, Phys. Part. Nucl. 41 (2010) 394–424. [doi:10.1134/S1063779610030020]

[34] R. Tenchini, private communication.

[35] F. Piccinini, Talk given at FCC–ee meeting, Berlin May 2017.

[36] A. Blondel, private communication.