Gravitational Dressing of 3D Conformal Galileon

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Could a conformal Galileon be describing a gauge mode of a broader diffeomorphism invariant theory? We answer this question affirmatively in 3D by using a coset construction for a nonlinearly realized conformal symmetry. In particular, we show that the conformal Galileon emerges as a Stückelberg field of a local Weyl symmetry. The coset construction gives us a diffeomorphism and Weyl invariant 3D theory containing first, second, and third powers of the curvatures, their couplings to certain tensors made of the Galileon and its derivatives, and the conformal Galileon terms. In a theory with a boundary additional surface terms are required for the boundary Weyl anomaly cancellation. When gravity is switched off, the theory reduces to the conformal Galileon plus a boundary term. On the other hand, it reduces to a generalization of “New Massive Gravity” by A. Sinha, if the Weyl symmetry is gauge-fixed in a particular way. Last but not least, there exists a parameter space for which the theory has no propagating ghosts.
1 Introduction and summary

A Lagrangian for a free scalar field can be amended by the mass and nonlinear interaction terms that are polynomial in the field. Theories with up to a quartic polynomial are motivated by the renormalisability in 4D (notwithstanding the Landau pole), and are often used in particle physics and cosmology. For a more general polynomial such a theory is not renormalizable but can be quantized as an effective field theory; the quantum loop-induced interactions with derivatives would be suppressed by powers of a momentum as compared to the renormalized polynomial terms.

Alternatively, one could think of a single scalar field theory that is made nontrivial by interaction terms containing derivatives of the field, while the polynomials would not appear due to symmetry reasons (e.g., field space shift symmetry or Galilean invariance). Generically such theories cannot be truncated to a finite number of interaction terms as this would give rise to a ghost; instead, they are usually viewed as a low energy approximation to a certain microscopic theory. A well known example is the chiral Lagrangian for the Nambu-Goldstone mesons of massless QCD, that gets completed into the asymptotically free quark-gluon Lagrangian; embedding of General Relativity (GR) into string theory is another example.

However, there seems to be an exception from the above lore for the higher derivative theories within a class that yields only second order equations of motion [1]. A subset of this class contains a scalar theory that emerged within the brane induced gravity [2] in a certain limit [3], and then in a more general setting [4], where astrophysical and cosmological applications were discussed and the name Galileon introduced.

The Galileon action exhibits field space galilean invariance while its Lagrangian transforms as a total derivative. Owing to this property, there are only a finite number of Galileon terms in any finite number of dimensions, and they contain less derivatives per field than a generic galilean invariant Lagrangian term would. Because of this, the generic Lagrangian terms – that are exactly Galilean invariant – are suppressed by powers of derivatives as compared to the Galileons, and hence one can focus on the latter in a low energy approximation\(^1\). Importantly, the truncation of the theory to the Galileon terms does not lead to additional degrees of freedom. This is unusual but a perturbative quantum low energy effective field theory with a finite number of Galileons has a well defined range of validity, and is predictive [4–7].

In spite of this progress in the perturbative quantum effective field theory, there are issues in the classical theory related to the lack of global hyperbolicity of the respective nonlinear equations of motion: while they contain only second derivatives acting on a field, \(\partial\partial\pi\), they are also at least quadratic in \(\partial\partial\pi\), leading in general to issues of conceptual and computational nature [8].

Understanding whether or not such issues represent an impasse is of great interest for both a formal field theory of Galileons and their potential applications in cosmology. The present paper does not aim to study hyperbolicity for Galileons, but rather to provide a possibly useful framework for addressing it as follows: It might be easier to tackle the issue if the Galileon is embedded into a diff invariant theory of gravity

\(^1\)A trivial example: \((\partial\pi)^2\) is a Galileon, while the galilean invariant term, \((\partial\partial\pi)^2\), is not.
in which it would just be a gauge mode of a tensor field that enters with no more
derivatives than it does in GR. For instance, in both the brane induced gravity [2]
and the diff invariant massive gravity [9, 10], the Galileon appears as a gauge mode
describing the helicity-0 state of a massive graviton in a certain limit. Only in that
specific limit the Galileon acquires a gauge-invariant meaning. Away from the limit one
could remove it by turning to the unitary gauge, i.e., absorb it into the metric $g_{\mu\nu}$. Its
classical equations of motion are Einstein’s equations amended by a polynomial in the
metric and its inverse, but no extra derivatives. Moreover, the Bianchi identities lead
to a constraint that is similar to a harmonic gauge-fixing condition of GR. Hence, the
issue of hyperbolicity in massive gravity in unitary gauge seems to be at least somewhat
similar to that in GR\textsuperscript{2}.

Motivated by this, we set a goal to embed the conformal Galileon into a diffeo-
morphism invariant theory, where the conformal Galileon would be a gauge mode. In
the present paper this will be done in 3D. A sequel will address a 4D construction.

The paper is organized as follows: In section 2 we briefly collect the well known
facts about the coset construction for a conformal group that has a Poincaré subgroup
realized linearly, and the rest nonlinearly.

Section 3 constitutes the bulk of the paper. Using the coset fields we first construct
the effective Lagrangian in the absence of a dynamical gravitational field; we show that
the inverse Higgs constraint emerges as a solution to the classical equations of motion,
and after solving it, reproduce the Lagrangian of the conformal Galileon field theory.
\textit{En route} we find that the conformal Galileons can be obtained from a Lagrangian
that is identical to that of the ghost free massive gravity [10], but with a different
Stückelberg sector.

Next, we introduce a dynamical gravitational field by gauging the tangent-space
Poincaré symmetry. We then find the effective Lagrangian. It contains four inde-
pendent sets of terms that enter with arbitrary coefficients not specified by the coset
construction. The key point is that the conformal Galileon naturally emerges as a
Stückelberg field of a local Weyl symmetry. The obtained theory is diffeomorphism
and Weyl invariant. In addition to the conventional terms of a Weyl invariant 3D GR,
it contains second and third powers of the curvatures, their couplings to certain ten-
sors made of the Galileon and its derivatives, and the full set of the conformal Galileon
terms. If a boundary is introduced, then additional surface terms are required for the
boundary Weyl anomaly cancellation. When gravity is switched off, the theory reduces
to the conformal Galileon plus a boundary term. On the other hand, it reduces to a
generalization of “New Massive Gravity” [12] by A. Sinha [13], if the Weyl symmetry
is gauge-fixed in a particular way. In section 4 we argue that the parameter space for
which the theory has no propagating ghosts is that of “New Massive Gravity” [12]. Al-
ternatively, one can gauge-fix a conformal mode of the metric tensor by imposing, e.g.,
unimodularity, $\det g = 1$, in which case one would obtain the covariantized conformal
Galileons coupled to unimodular 3D gravity.

\textsuperscript{2}The results of [11] might be interpreted as supporting a thought that the embedding of the
Galileons into a gravitational theory helps with hyperbolicity.
2 The $SO(n,2)/ISO(n-1,1)$ coset and conformal Galileons

We start by a brief summary of the coset construction for spacetime symmetries [14–16] and in particular for a conformal group spontaneously broken down to the Poincaré group [15, 17]. The method has been nicely reviewed in [18] and was used to construct the Lagrangians of conformal Galileons in a flat space. In the next section we will generalize the method to arbitrary gravitational backgrounds by gauging the tangent-space Poincaré symmetry (see [19] and references therein).

The $n$ dimensional conformal algebra is realized through the $n(n+1)/2$ generators of the Poincaré algebra with the generators conventionally denoted by $J_{ab}, P_a$, plus $n$ generators of special conformal transformations, $K_a$, and a generator of dilatations, $D$, with the well-known commutation relations:

\[
\begin{align*}
[P_a, D] &= P_a, \\
[J_{ab}, K_c] &= \eta_{ac} K_b - \eta_{bc} K_a, \\
[K_a, P_b] &= 2 J_{ab} - 2 \eta_{ab} D.
\end{align*}
\]

Following the coset construction method we should look at a Maurer-Cartan form built out of the $SO(n,2)/ISO(n-1,1)$ coset element $\Sigma$. A convenient parametrization for $\Sigma$ is:

\[\Sigma = e^{\pi D} e^{\xi^a K_a}.\]

Since the Maurer-Cartan form is an element of the Lie algebra we may expand it in the basis of the generators:

\[\Sigma^{-1} (d + dx^a P_a) \Sigma = E^a P_a + \omega_K^a K_a + \omega_D D - \frac{1}{2} \omega_J^{ab} J_{ab},\]

where:

\[
\begin{align*}
E^a &= e^\pi \delta^a, \\
\omega_K^a &= d\xi^a - \xi^2 E^a + \xi^a \omega_D, \\
\omega_J^{ab} &= -2 E^a \xi^b + 2 E^b \xi^a.
\end{align*}
\]

Here $\pi$ and $\xi^a$ are the coset fields of nonlinearly realized dilatations and special conformal transformations, which may or may not end up giving rise to the respective dynamical Nambu-Goldstone fields. Indeed, since $[K_a, P_b] \sim \eta_{ab} D$, not all the Nambu-Goldstone fields parametrizing the coset should be physical and it should be possible to eliminate the ones associated with the generator $K_a$. This could be achieved by imposing the inverse Higgs constraint (IHC) $\omega_D = 0$ [16, 18].

In the present paper we will take a different path, instead of imposing the IHC by hand at the level of the algebra we will show that it emerges as a consequence of the classical equation of motion for a properly built action.

\footnote{Some clarifications: $\delta^a \equiv \delta^a_{\mu} dx^\mu$. We are distinguishing the Latin indices from the Greek indices for the later convenience; when we introduce gravity the former will refer to a Local Lorentz group, while the latter to a space-time group. $\xi^a$ is a zero form in space-time but a vector of the Local Lorentz group.}
3 The effective Lagrangian

3.1 Flat background metric

Simplest actions in 3D constructed via the coset elements of the last section are:

\[ S_0 = \int \varepsilon^{abc} E^a \wedge E^b \wedge E^c; \quad \delta_\xi S_0 = 0, \]  
\[ S_1 = \int \varepsilon^{abc} E^a \wedge E^b \wedge \omega_K^c; \quad \delta_\xi S_1 \simeq - \int \varepsilon_{abf} \omega_D \wedge E^a \wedge E^b \delta_\xi f, \]  
\[ S_2 = \int \varepsilon^{abc} \omega_K^a \wedge \omega_K^b \wedge E^c; \quad \delta_\xi S_2 \simeq 2 \int \varepsilon_{abf} \omega_D \wedge E^a \wedge \omega_K^b \delta_\xi f, \]  
\[ S_3 = \int \varepsilon^{abc} \omega_K^a \wedge \omega_K^b \wedge \omega_K^c; \quad \delta_\xi S_3 \simeq 9 \int \varepsilon_{abf} \omega_D \wedge \omega_K^a \wedge \omega_K^b \delta_\xi f. \]  

We did not include derivative terms as they'd give additional degrees of freedom. \( \omega_D \) has no Latin indices and does not enable to construct any nontrivial coset term (see, however, the section on the Wess-Zumino terms). We've also written out above the variations with respect to \( \xi \) (the sign \( \simeq \) denotes equality up to a total derivative); these expressions confirm that IHC, \( \omega_D = 0 \), is a solution of the equations of motion.

The total action is a sum of the above four actions each weighted with some coefficient which cannot be determined by the effective action formalism but instead should be constrained by various consistency requirements and experimental/observational results whenever available.

Here, we note that one can rewrite the above actions in a form equivalent to that of 3D massive gravity potentials [10] but with a different matrix \( K \) entering those potentials. Indeed, define the matrix \( K \) as follows:

\[ \omega_K^a = E^a_\nu K_\nu^\mu dx^\mu, \]  
then use IHC, \( \omega_D = 0 \), to deduce the expression for \( K \) in terms of \( \pi \):

\[ K_\mu^\nu = - \frac{1}{2} e^{-2\pi} \left( \partial_\mu \partial_\nu \pi - \partial_\mu \pi \partial_\nu \pi + \frac{1}{2} (\partial \pi)^2 \delta_\mu^\nu \right), \]  
and finally, substitute this expression into the above four actions, to get the four conformal Galileon terms:

\[ S_0 = -6 \int d^3 x \, e^{3\pi}, \]  
\[ S_1 = -2 \int d^3 x \, e^{3\pi} [K] \simeq -\frac{1}{2} \int d^3 x \, e^{\pi} (\partial \pi)^2, \]  
\[ S_2 = - \int d^3 x \, e^{3\pi} \left( [K]^2 - [K^2] \right) \simeq -\frac{1}{8} \int d^3 x \, e^{-\pi} (\partial \pi)^2 L_1^{TD}, \]  
\[ S_3 = - \int d^3 x \, e^{3\pi} \left( [K]^3 - 3 [K] [K^2] + 2 [K^3] \right) \simeq \]  
\[ \simeq - \frac{3}{16} \int d^3 x \, e^{-3\pi} \left( (\partial \pi)^2 L_2^{TD} - \frac{3}{2} (\partial \pi)^4 L_1^{TD} + (\partial \pi)^6 \right), \]  

where, \( L_1^{TD}, L_2^{TD} \) are the total derivative terms of the respective order, \( L_n^{TD} \sim (\partial \partial \pi)^n \).
3.2 Dynamical metric

The goal of this section is to dress the conformal Galileons with a dynamical gravitational field. In particular, we’d like to arrive at a theory in which a conformal Galileon field describes a gauge mode of a certain local symmetry. For this, we will follow a method used in [19]; it consists of gauging the Poincaré group by introducing new gauge fields associated with the translations, $e^a$ (vielbein), and rotations $\omega^{ab}$ (spin connection). The exterior derivative in (2.5) must be replaced with the exterior covariant derivative. As before, we use the expansion of the Maurer-Cartan form in the basis of generators:

$$\Sigma^{-1}\left(d + e^a P_a - \frac{1}{2} \omega^{ab} J_{ab}\right)\Sigma = E^a P_a + \omega^a K_a + \omega_D D - \frac{1}{2} \omega^{ab} J_{ab}. \quad (3.11)$$

The covariantized counterparts of (2.6)-(2.9) are:

$$E^a = e^\pi e^a, \quad (3.12)$$

$$\omega_D = d\pi + 2E^a \xi_a, \quad (3.13)$$

$$\omega^a K_a = D\xi^a - \xi^2 E^a + \xi^a \omega_D, \quad (3.14)$$

$$\omega^{ab}_J = \omega^{ab} - 2E^a \xi^b + 2E^b \xi^a. \quad (3.15)$$

It should be emphasized that $e^a = e^a_\mu dx^\mu$ is the vielbein of the pure gravitational field, while $D$ is the covariant derivative with respect to $e^a$, and $\omega^{ab}$ is the spin connection of $e^a$. Using $\omega^{ab}_J$ we can build another coset element - the Riemann curvature two form:

$$R^{ab} = d\omega^{ab}_J + \omega^{ac}_J \wedge \omega^{cb}_J = R^{ab} + 2E^a \wedge \omega^K_b + 2\omega^K_a \wedge E^b, \quad (3.16)$$

where

$$R^{ab} = d\omega^{ab} + \omega^{ac}_J \wedge \omega^{cb}_J. \quad (3.17)$$

The curvature two form can be used, alongside with the fields utilized in the last section, to construct the effective Weyl invariant actions.

Before proceeding to this task, we recall that a gravitationally covariantized conformal unitary field theory must be Weyl invariant [20]. To incorporate this property, we define the Weyl transformations as:

$$e^a \to e^\sigma e^a, \quad \pi \to \pi - \sigma, \quad \xi^a \to \xi^a + \frac{1}{2} e^{-\sigma} \partial^a \sigma. \quad (3.18)$$

It is easy to verify that $E^a$, $\omega_D$, and $R^{ab}$ are all invariant under these transformations. Moreover, since $\omega^K_a$ is part of $R^{ab}$ it would be logical to construct the effective Weyl invariant actions using the above three elements only.

In 3D the Weyl tensor is identically zero and $R^{ab}$ is entirely determined in terms of the Ricci tensor and scalar, so it is more convenient to work in terms of these quantities rather than the Riemann two form $R^{ab}$ itself:

$$R^{ab} = e^a \wedge \bar{S}^b + \bar{S}^a \wedge e^b, \quad \bar{S}^a = R^a - \frac{1}{4} Re^a, \quad (3.19)$$
\[ \mathcal{R}^{ab} = E^a \wedge \Omega^b + \Omega^a \wedge E^b, \quad \Omega^a = \frac{1}{2} \epsilon^{-\pi} S^a + \omega_K^a. \] (3.20)

Here \( R^a \equiv e^a_{\mu} R^\mu dx^\mu \) and \( R \) are the Ricci one form and scalar respectively. \( S^a \) is the first order analogue of the Schouten tensor, we will refer it as a Schouten one form. This quantity has interesting properties in 3D which are inherited from the curvature two form. In particular, from (3.19) we see that it satisfies both Bianchi identities:

\[ D S^a = 0, \quad e_a \wedge S^a = 0. \] (3.21)

Expressions (3.19) and (3.21) enable us to conclude that, in 3D, \( S^a \) fully replaces the curvature two form, \( R^{ab} \). Following the same logic, from (3.20) we also see that \( \Omega^a \) can replace \( \mathcal{R}^{ab} \) as a coset element. Thus there are three Weyl invariant coset elements that can be used to build effective actions: \( \omega_D, E^a \) and \( \Omega^a \). The rest is straightforward, building the actions follows the same receipt as in the case of the flat metric. Actions and their variations with respect to \( \xi \) are given by:

\[ S_0 = \int \epsilon_{abc} E^a \wedge E^b \wedge E^c; \quad \delta_\xi S_0 = 0, \] (3.22)

\[ S_1 = \int \epsilon_{abc} E^a \wedge E^b \wedge \Omega^c; \quad \delta_\xi S_1 \simeq - \int \epsilon_{abf} \omega_D \wedge E^a \wedge E^b \delta \xi^f, \] (3.23)

\[ S_2 = \int \epsilon_{abc} E^a \wedge \Omega^b \wedge \Omega^c; \quad \delta_\xi S_2 \simeq 2 \int \epsilon_{abf} \omega_D \wedge E^a \wedge \Omega^b \delta \xi^f, \] (3.24)

\[ S_3 = \int \epsilon_{abc} \Omega^a \wedge \Omega^b \wedge \Omega^c; \quad \delta_\xi S_3 \simeq 9 \int \epsilon_{abf} \omega_D \wedge \Omega^a \wedge \Omega^b \delta \xi^f. \] (3.25)

The actions, \( S_0, S_1, S_2, \) and \( S_3 \), are respectively: the Weyl invariant analogues of the cosmological term, the Einstein-Hilbert kinetic term, the Bergshoeff, Hohm & Townsend term [12] (usually referred as “New Massive Gravity”), and the curvature cubed term [13]. It is clear that IHC remains to be a solution of the equations of motion. Substituting that solution, the actions can be rewritten in the following form:

\[ S_0 = -6 \int d^3 x \sqrt{g} e^{3\pi}, \] (3.26)

\[ S_1 = -2 \int d^3 x \sqrt{g} e^{\pi} \left( \frac{1}{8} R + e^{2\pi} [\mathcal{K}] \right), \] (3.27)

\[ S_2 = - \int d^3 x \sqrt{g} e^{-\pi} \left[ - \frac{1}{4} \left( R_{\mu \nu} R^{\mu \nu} - \frac{3}{8} R^2 \right) - e^{2\pi} \left( R_{\mu}^{\mu} - \frac{1}{2} \delta_{\mu}^{\mu} R \right) \mathcal{K}^{\mu}_\mu + e^{4\pi} \left( \mathcal{K}^{2} - 2 \mathcal{K}^{2} \right) \right], \] (3.28)

\[ S_3 = - \int d^3 x \sqrt{g} e^{-3\pi} \left[ \frac{1}{4} R_{\mu}^{\mu} R_{\nu}^{\nu} R_{\rho}^{\rho} - \frac{9}{32} R R_{\mu \nu} R^{\mu \nu} + \frac{17}{256} R^3 + \frac{3}{4} e^{2\pi} \left( 5 R_{\nu}^{\nu} \delta_\nu^{\nu} - \frac{3}{2} R R_{\nu}^{\nu} - R_{\alpha \beta} R^{\alpha \beta} \delta_\nu^{\nu} + 2 R_{\nu}^{\mu} R_{\nu}^{\nu} \right) \mathcal{K}^{\mu}_\mu + 3 e^{4\pi} \left( R_{\nu}^{\nu} - \frac{3}{8} R \delta_\nu^{\nu} \right) \left( \mathcal{K}^{\mu}_\mu \mathcal{K}^{\rho}_\rho - [\mathcal{K}] \mathcal{K}^{\mu}_\mu \right) + e^{6\pi} \left( \mathcal{K}^{3} - 3 [\mathcal{K}] [\mathcal{K}^{2}] + 2 [\mathcal{K}^{3}] \right) \right], \] (3.29)
with
\[ K^\nu_\mu = -\frac{1}{2} e^{-2\pi} \left( \nabla_\mu \nabla^\nu \pi - \partial_\mu \pi \partial^\nu \pi + \frac{1}{2} (\partial \pi)^2 \delta^\nu_\mu \right). \] (3.30)

Thus we’ve arrived at the action in which the conformal galileon is a Stuckelberg field for the local Weyl invariance. It is also clear that in the limit when gravity is switched off \( h_{\mu\nu} \to 0 \) (where \( h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu} \); \( g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab} \) being the dynamical metric) the actions \( S_i, i = 0, 1, 2, 3 \) reduce to their flat space counterparts \( S_n, n = 0, 1, 2, 3 \).

### 3.3 Wess-Zumino terms and anomalies

It is known that there are no conformal anomalies in odd dimensions \([21]\). We can verify this in the case at hand by explicitly building the Wess-Zumino terms. This is done by invoking \( \omega_D \), which does not have a Lorentz index and can be used to create a four form in 3D. From the expressions below we see that the WZ terms coincide with \((3.22)-(3.25)\) and therefore do not contain any new information:

\[ 3 \int \epsilon_{abc} \omega_D \wedge E^a \wedge E^b \wedge E^c = S_0, \] (3.31)
\[ \int \epsilon_{abc} \omega_D \wedge E^a \wedge E^b \wedge \Omega^c = S_1, \] (3.32)
\[ -\int \epsilon_{abc} \omega_D \wedge E^a \wedge \Omega^b \wedge \Omega^c = S_2, \] (3.33)
\[ -3 \int \epsilon_{abc} \omega_D \wedge \Omega^a \wedge \Omega^b \wedge \Omega^c = S_3. \] (3.34)

The absence of anomalies allows us to eliminate the gauge mode \( \pi \) from the actions, therefore the obtained theory in the unitary gauge coincides with “New Massive Gravity” \([12]\) and its extensions \([13]\).

What happens if we include a 2D boundary? In that case and for classically scale invariant degrees of freedom there is a quantum trace anomaly on the boundary \([22, 23]\):

\[ \int d^3 x \sqrt{g} T^\mu_\mu = \int d^3 x \sqrt{g} \delta(x) \left( c_1 r + c_2 k^\mu_\mu k^\mu_\mu \right), \] (3.35)

where the coefficients \( c_1 \) and \( c_2 \) depend on a field content and boundary conditions considered; \( \delta(x) \) has support only on the boundary, \( r \) and \( k^\mu_\mu \) are its intrinsic and traceless part of extrinsic curvature respectively. Under the Weyl transformation \( r \) transforms, while the \( \sqrt{g} k^\mu_\mu k^\mu_\mu \) is invariant. The effective action that reproduces the trace anomaly can be expressed as \([24, 25]\):

\[ S_{\text{anomaly}} = \int d^3 x \sqrt{g} \delta(x) \left[ c_1 (\pi r + (\partial \pi)^2) + c_2 \pi k^\mu_\mu k^\mu_\mu \right]. \] (3.36)

This action should be included, with the opposite sign, for the Weyl symmetry to remain exact at the quantum level. If so then the \( \pi \) mode can be eliminated by a gauge fixing. Alternatively, one could choose to gauge fix a conformal mode of the metric tensor, by imposing, e.g., a unimodularity condition, \( \det g = 1 \), and thus obtaining unimodular gravity coupled to covariantised conformal Galileons.
4 Degrees of freedom

Last but not least, important comments concerning the degrees of freedom are in order here. The total bulk action

$$S_{\text{tot}} = \sum_{i=0}^{3} \alpha_i S_i,$$  \hspace{1cm} (4.1)

is a linear combination of the four actions with arbitrary coefficients denoted by $\alpha_i$. These coefficients cannot be calculated within the low energy effective field theory approach. However, various consistency conditions can be imposed on them.

Since (4.1) in general propagates ghosts, we impose the condition that any type of ghost to be absent from the theory. Note that these ghosts are neither due to the conformal Galileons, nor the nonlinear terms of a conformal mode of the 3D tensor field [26], which are both ghost free. Instead, the ghosts arise because of the tensor in the terms quadratic and cubic in the curvatures.

To reveal the ghosts let us begin with small perturbations above the Minkowski space in the unitary gauge $\pi = 0$. In that case the graviton propagator received contributions from $\alpha_1 S_1$ and $\alpha_2 S_2$, while the cubic curvature terms give rise to the interactions but not to a modification of the graviton propagator. Schematically, the graviton propagator is proportional to

$$\frac{1}{\alpha_1 \partial^2 + \alpha_2 \partial^4},$$ \hspace{1cm} (4.2)

where we have ignored the tensorial structure for simplicity. The second derivative arises from $S_1$, while the quartic derivative comes form $S_2$.

The above expression has two poles, describing two states, one massless and one massive, as it can also be seen from the equivalent rewriting of (4.2):

$$\frac{1}{\alpha_1 \partial^2} - \frac{1}{\alpha_1 (\partial^2 + \alpha_1 / \alpha_2)}.$$ \hspace{1cm} (4.3)

The latter shows that either the first or the second pole has to describe a ghost. The situation seems hopeless, except that in 3D a massless tensor does not propagate any dynamical degrees of freedom, hence one could afford it to have a ghost-like sign of the kinetic term. Thus, we choose the coefficients $\alpha_1$ and $\alpha_2$ as in New Massive Gravity, $\alpha_1 = -1$, $\alpha_2 > 0$ [12]. This guarantees that the massive tensor mode with two dynamical degrees of freedom has a good kinetic term and non-tachyonic mass; it also guarantees that the Galileon has a “right-sign” kinetic term.

To reiterate, the above choice corresponds to a “wrong” sign of the Einstien-Hilbert term in the action (4.1); as a result, the massive tensor mode has good kinetic and mass terms, while the massless one has a ghost-like kinetic term. Since a massless tensor field in 3D does not propagate any dynamical degrees of freedom, such a ghost-like term would not lead to the usual uncontrollable instabilities; it would instead manifest itself as “antigravity” for certain classical solutions in 3D [27].
Regarding the $\alpha_3 S_3$ term, as mentioned before it gives new interactions on Minkowski space, however would yield further four-derivative modifications of the propagator on a generic curved background; hence we will also put $\alpha_3 = 0$ to avoid additional ghosts on generic curved backgrounds.

With the above choice of the coefficients the theory is just a Weyl invariant generalization of New Massive Gravity, and could have been obtained from the latter by a Weyl transformation of the metric field. It is interesting that this very theory is derived via the coset construction for a nonlinearly realized conformal symmetry. Moreover, it can be rewritten as a theory with two tensor fields, one being dynamical and another algebraically determined [12] so that only linear terms in curvature appear in the action; the issue of hyperbolicity that we alluded to in the first section might be easier to study in this formulation than it is for conformal Galileons.

Alternatively, the theory provides a covariant formulation of the conformal Galileon in which the conformal mode of the tensor field can be gauged away instead of the Galileon. In that case, the dynamics of the gravitational conformal mode is entirely encoded in the conformal Galileon.

If a boundary is present, then the boundary terms should be introduced as in (3.36) to maintain the Weyl invariance of the full theory at the quantum level.

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