Removing the Curse of Superefficiency: an Effective Strategy For Distributed Computing in Isotonic Regression

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Abstract

We propose a strategy for computing the isotonic least-squares estimate of a monotone function in a general regression setting where the data are distributed across different servers and the observations across servers, though independent, can come from heterogeneous sub-populations, thereby violating the identically distributed assumption. Our strategy fixes the super-efficiency phenomenon observed in prior work on distributed computing in the isotonic regression framework, where averaging several isotonic estimates (each computed at a local server) on a central server produces super-efficient estimates that do not replicate the properties of the global isotonic estimator, i.e. the isotonic estimate that would be constructed by transferring all the data to a single server. The new estimator proposed in this paper works by smoothing the data on each local server, communicating the smoothed summaries to the central server, and then computing an isotonic estimate at the central server, and is shown to replicate the asymptotic properties of the global estimator, and also overcome the super-efficiency phenomenon exhibited by earlier estimators. For data on $N$ observations, the new estimator can be constructed by transferring data just over order $N^{1/3}$ across servers [as compared to transferring data of order $N$ to compute the global isotonic estimator], and requires the same order of computing time as the global estimator.

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1 Background

Distributed computing has now become significant in the practice of statistics as well as other branches of data science. Large volumes of data, often relating to the same or closely related studies or experiments, are no longer stored on one single computer; rather, they are distributed across a number of platforms in some structured manner, owing partly to natural memory constraints on individual machines, and partly for convenience. This, typically, poses problems for computing optimal estimates of parameters of interest from the data at hand. Conventional statistical estimates are generally obtained under the premise that the totality of the data is accessible to a single computing device and can be processed at one stroke, yielding estimates that
are optimal in some quantitatively defined sense. However, this is not automatically the case in a distributed environment. The calculation of global estimates that require simultaneous processing of all available data then entails transferring the entire bulk of data from different computers to a central machine, which in itself can be both time and resource consuming, followed by a potentially complex computation on the aggregated data (of massive volume), which may be infeasible under many circumstances.

Divide and conquer algorithms are a standard approach to addressing these issues in a distributed computing environment. The idea behind this is as follows: suppose the entire data set is stored across a number of machines. On each machine, calculate a natural estimate of the parameter of interest from the data on it and transfer this estimate to a central machine. Next, combine the estimators thus obtained, at the central machine in a judicious way to produce a final estimate, the so-called pooled estimate, which replicates the properties of the natural global estimate, i.e. the one we could have computed were it feasible to store and analyze all available data on one machine. The term ‘replicates the properties’ can be understood in various ways and is often specific to the problem at hand: one might be able to show that the pooled and the global estimates have the same rate of convergence, or are comparable, up to constants, in terms of a certain measure of risk, or it might even be possible to demonstrate that the pooled estimate and the global estimate have the same limit distributions under appropriate conditions. The other important factor is computational burden: one would expect that the divide and conquer algorithm is not substantially more computationally onerous than the global estimator. As the literature on distributed computing is enormous, here we provide a selection of instances of research on distributed computing problems in a variety of statistical/machine-learning contexts: see, e.g. (Hsieh et al., 2014), (Li et al., 2013), (Zhang et al., 2013), (Zhao et al., 2014), (Battey et al., 2015), (Shang and Cheng, 2017), (Volgushev et al., 2017). The above papers illustrate that the sample splitting approach buys computational dividends, yet statistical optimality [in the sense that the resulting estimator is as efficient (or minimax rate optimal) as the global estimate based on applying the estimation algorithm to the entire data set] is retained.

In the nonparametric function estimation context, most results of the divide and conquer type focus on estimation under smoothness constraints, where the essence of the strategy is to compute a smoothed estimator of the unknown function at each server and combine the estimators at the central server, by averaging; this strategy is employed, for example, in (Li et al., 2013), (Zhang et al., 2013), (Zhao et al., 2014). However, the averaging strategy leads to highly problematic pooled estimators in non-regular function estimation problems, e.g. function estimation under a monotonicity constraint, where the least squares estimates under the monotonicity constraint are non-standard/non-regular in the sense that they are highly non-linear in the data, and exhibit non-Gaussian limit distributions. This is the core content of the recent work by (Banerjee et al., 2018) [henceforth BDS] where it is demonstrated that in monotone function estimation problems, the ‘pooled-by-averaging’ estimator [henceforth, generally referred to as BDSE] becomes super-efficient: its asymptotic relative efficiency (in terms of MSE) with respect to the global monotone least squares estimator computed at any single model goes to
infinity, whereas, in the uniform sense, the ARE goes to 0, i.e. the maximal MSE of BDSE over a collection of models relative to that of the global least squares estimator goes to $\infty$. Furthermore, BDSE has a distribution different from that of the global estimator which converges to a Chernoff limit (discussed in details below). Indeed, the super-efficiency property just alluded to, has also been observed for the pooled-by-averaging estimator in the genre of non-standard problems exhibiting cube-root asymptotics in the sense of (Kim and Pollard, 1990); see, (Shi et al., 2017).

Our goal in this paper is to construct a new estimator in the monotone function estimation problem which does not suffer from the super-efficiency problem and which also exhibits the limiting properties of the global estimator. To this end, we provide some details of the problem considered in BDS and the results obtained, as they are crucial to understanding the goal and the approach of the current work.

Consider a sample of size $N$ (very large) from the model $Y_i = \mu(X_i) + \epsilon_i$ which is distributed across $m$ different servers, each server containing a subsample of size $n$, and $m = o(N)$. The function $\mu$ is known to be monotone and the $X_i$ come from a density on $[0, 1]$. The residual $\epsilon_i$ is assumed to satisfy $E(\epsilon_i | X_i) = 0$. Computing the global isotonic estimate at a point $t_0 \in (0, 1)$ involves moving all the data to a central server and performing the isotonization on all $N$ data-points on the central server. This can be time-consuming when $N$ is really large. The construction of BDSE involves computing the isotonic estimate of $\mu$, say $\hat{\mu}_j$, on the $j$'th server, and then obtaining the average of these isotonic estimates. Hence, the pooled estimate at the point $t_0$ is given by: $\overline{\mu}(t_0) := \frac{1}{m-1} \sum_{j=1}^{m} \hat{\mu}_j(t_0)$. Computing BDSE at a particular point only requires transferring $m$ numbers (from the $m$ machines) to the central server, where $m = o(N)$.

One can compare the computational burden involved in the calculation of the global estimator to that for BDSE. For the global estimator, once all the data-points have been transferred to the central machine, sorting of the $X_i$'s (resulting in an induced sorting of the $Y_i$'s) can be accomplished typically in $O(N \log N)$ time. Post-sorting, one can implement isotonic regression via the PAVA algorithm (Robertson et al., 1988) (Chapter 1) which takes $O(N)$ time. Thus, the total computational burden is $O(N \log N)$ computing time plus the transferring of $N$ bivariate pairs to the central machine. On the other hand, for the pooled estimator, on each machine, the isotonic estimate based on the subsample stored in that machine takes $O(n \log n)$ computing time, leading to a total computing time of $O(mn \log n)$. At the central server, averaging takes $O(m)$ time. If $n \sim N^\gamma$ for some $0 < \gamma < 1$, this gives a total computing time of order $O(N \log N)$, and in addition, one transfers $m \sim N^{1-\gamma}$ scalars (the values $\hat{\mu}_j(t_0)$ for $j = 1, 2, \ldots, m$) to the central machine. Thus, the pooled estimator is computationally less burdensome than the global estimator. Similar considerations apply to the computation of the global and pooled isotonic estimators of the inverse function $\mu^{-1}$.

BDS showed that their pooled-by-averaging estimator (BDSE) of the inverse function has dichotomous behavior. We briefly revisit this important result. For convenience and the sake of completeness, we state these results essentially in their entirety. Consider a nonincreasing and continuously differentiable function $\mu_0$ on $[0, 1]$ with
\[ 0 < c < |\mu_0'(t)| < d < \infty \text{ for all } t \in [0,1]. \] For an \( x_0 \in (0,1) \), define a neighborhood \( \mathcal{M}_0 \) of \( \mu_0 \) as the class of all continuous nonincreasing functions \( \mu \) on \([0,1]\) that are continuously differentiable on \([0,1]\), coincide with \( \mu_0 \) outside of \((x_0 - \epsilon_0, x_0 + \epsilon_0)\) for some (small) \( \epsilon_0 > 0 \), satisfy \( 0 < c < |\mu'(t)| < d < \infty \) for all \( t \in [0,1] \), and such that \( \mu^{-1}(a) \in (x_0 - \epsilon_0, x_0 + \epsilon_0) \) where \( a = \mu_0(x_0) \). Now, consider \( N \) i.i.d. observations \( \{(X_i, Y_i)\}_{i=1}^N \) from \((X,Y)\) where \( Y_i = \mu_0(X_i) + \epsilon_i \) and \( X_i \sim \text{Uniform}(0,1) \) is independent of \( \epsilon_i \sim N(0, \nu^2) \). Then, the isotonic estimate \( \hat{\theta}_N \) of \( \theta_0 := \mu_0^{-1}(a) \) (which is \( x_0 \)) satisfies
\[ N^{1/3} (\hat{\theta}_N - \theta_0) \overset{d}{\to} G, \] as \( N \to \infty \), where \( G =_d \kappa Z \), with \( Z \) following the Chernoff distribution, and \( \kappa > 0 \) being a constant. Writing \( N = m \times n \), where \( m \) and \( n \) are as defined above, as \( N \to \infty \), the BDSE of \( \mu^{-1}(a) \), say \( \bar{\theta}_m \) satisfies:
\[ N^{1/3} (\bar{\theta}_m - \theta_0) \overset{d}{\to} m^{-1/6} H, \] where \( H \) has the same variance as \( G \) but is distributed differently from \( G \). Furthermore,
\[ \mathbb{E}_{\mu_0} \left[ N^{2/3} (\hat{\theta}_N - \theta_0)^2 \right] \to \text{Var}(G) \quad \text{and} \quad \mathbb{E}_{\mu_0} \left[ N^{2/3} (\bar{\theta}_m - \theta_0)^2 \right] \to m^{-1/3} \text{Var}(G), \] as \( N \to \infty \). Hence, BDSE outperforms the global inverse isotonic regression estimator in terms of point wise MSE.

This phenomenon is reversed when one looks at the maximal MSEs of the two estimators over the class of models defined by \( \mathcal{M}_0 \), as described in Theorem 5.1 of BDS.

**Theorem 1.1 (Theorem 5.1 of BDS)** Let
\[ E := \limsup_{N \to \infty} \sup_{\mu \in \mathcal{M}_0} \mathbb{E}_{\mu} \left[ N^{2/3} (\hat{\theta}_N - \mu^{-1}(a))^2 \right] \quad \text{and} \quad E_m := \liminf_{N \to \infty} \sup_{\mu \in \mathcal{M}_0} \mathbb{E}_{\mu} \left[ N^{2/3} (\bar{\theta}_m - \mu^{-1}(a))^2 \right] \] where the subscript \( m \) indicates that the maximal risk of the \( m \)-fold pooled estimator (\( m \) fixed) is being considered. Then \( E < \infty \) while \( E_m \geq m^{2/3} c_0 \), for some \( c_0 > 0 \). When \( m = m_n \) diverges to infinity,
\[ \liminf_{N \to \infty} \sup_{\mu \in \mathcal{M}_0} \mathbb{E}_{\mu} \left[ N^{2/3} (\bar{\theta}_{m_n} - \mu^{-1}(a))^2 \right] = \infty. \]

Therefore, from Theorem 1.1 it follows that the asymptotic maximal risk of BDSE diverges to \( \infty \) at rate (at least) \( m^{2/3} \). Thus, the better off we are with BDSE for a fixed function, the worse off we are in the uniform sense over the class of functions \( \mathcal{M}_0 \). Hence, unfortunately, while maintaining a computational burden that is better than the global estimator, BDSE has undesirable statistical properties as seen above.

As mentioned above, we will construct a corrected estimator in the isotonic regression problem that does not suffer from the undesirable ‘super-efficiency phenomenon’. Our new estimator recovers all the desirable properties of the global isotonic estimator, and is computationally not
anymore onerous than the latter. Furthermore, for our analysis, we address a much broader scenario than is conventionally considered in isotonic regression problems. Since we are thinking of large $N$ problems, with the data being stored separately across different servers, it is natural to allow heterogeneity in data. Thus, while considering the pairs $\{X_i, Y_i\}_{i=1}^{N}$ to be independent, we will no longer consider them to be identically distributed; rather, they will be assumed to come from a number of different ($m$) sub-populations with the $(X_i, Y_i)$ pairs in each sub-population being i.i.d. What links the different sub-populations is the common mean function $\mu$ of interest: $E(Y_i|X_i) = \mu(X_i)$ for all $i$, for a common monotone function $\mu$. Furthermore, the $N$ pairs will be scrambled across a number of different servers (say $L$), with the same server hosting data from different sub-populations, as well as data from the same sub-population potentially stored on multiple servers. Our new estimator essentially reverses the steps involved in constructing BDSE. BDSE involves isotonization on local servers followed by smoothing (as in simple averaging) on the central server, while, in this paper, we do the opposite: first smooth (by local averaging) on each local server, and then isotonize the smoothed data on the central server.

2 The Set-Up and the Estimator

Assume that we have $m$ samples of respective sizes $n_1, \ldots, n_m$ and that for all $j = 1, \ldots, m$, the $j$-th sample is composed of i.i.d. pairs of real valued random variables $(X_{ji}, Y_{ji}), i = 1, \ldots, n_j$, such that $E(Y_{ji}|X_{ji}) = \mu(X_{ji})$ for all $i, j$ and an unknown regression function $\mu$ defined on $[0, 1]$. We denote by $F_{Xj}$ the common distribution function of the covariates $X_{ji}, i = 1, \ldots, n_j$ in the $j$-th sample. The data are stored on several servers numbered $1, \ldots, L$ for some integer $L \geq 1$. The allocation of data on the different servers is arbitrary in the sense that a sample can be spread on several servers, a server can host data from several different samples, and the number of stored observations can vary across the different servers. The number $L$ of different servers can even grow as $N \to \infty$. The total sample size is

$$N = \sum_{j=1}^{m} n_j.$$ 

For ease of exposition, when considering simultaneously all the samples, we relabel the observations from the $m$ samples to obtain independent pairs $(X_i, Y_i), i = 1, \ldots, N$ such that $E(Y_i|X_i) = \mu(X_i)$, where the distribution function of $X_i$ is one of $F_{X1}, \ldots, F_{Xm}$. Let $K$ be a positive integer that grows to infinity as $N \to \infty$, and for all $k \in \{1, \ldots, K\}$, let $I_k = ((k-1)/K, k/K]$. Let $S_{\ell}$ denote the set of indices $i$, such that $(X_i, Y_i)$ is stored in the $\ell$'th server. Now, for each server $\ell$ ($1 \leq \ell \leq L$) record

$$T_{\ell k} = \sum_{i=1}^{N} Y_i \mathbb{1}_{i \in S_{\ell}} \mathbb{1}_{X_i \in I_k}$$

1In BDS, the number of servers was designated by $m$, while in this paper we change notation and call it $L$. As we will see below, it is the number of different sub-populations that really enters into the properties of the pooled estimator in general and not the number of servers. When each sub-population has its own server, then obviously $L = m$. 
and
\[ C_{\ell k} = \sum_{i=1}^{N} \mathbb{1}_{i \in S_{\ell}} \mathbb{1}_{X_i \in I_k}, \]
for \( k \in \{1, \ldots, K\} \). Next, for each \( \ell \), transfer \( \{(T_{\ell k}, C_{\ell k})\}_{k=1}^{K} \) to a central server. Compute a regressogram estimate on the central server in the following manner: for each \( k \in \{1, \ldots, K\} \),
\[ \overline{y}_k = \frac{1}{\sum_{\ell=1}^{L} C_{\ell k}} \sum_{\ell=1}^{L} T_{\ell k} = \frac{1}{\sum_{i=1}^{N} \mathbb{1}_{X_i \in I_k}} \sum_{i=1}^{N} Y_i \mathbb{1}_{X_i \in I_k}. \]
Our final estimator of \( (\mu(\overline{x}_1), \ldots, \mu(\overline{x}_K))^T \), where \( \overline{x}_k = k/K \), is
\[ \hat{y} = \arg \min_{h \in \mathbb{R}^K : h_1 \geq \cdots \geq h_K} \sum_{k=1}^{K} w_k (\overline{y}_k - h_k)^2, \] (4)
where
\[ w_k = \frac{1}{N} \sum_{i=1}^{N} \mathbb{1}_{X_i \in I_k} = \frac{\sum_{\ell=1}^{L} C_{\ell k}}{N}. \]
Note that the estimator does not depend on the way the observations were stored across different servers.

**Computational considerations:** Consider the computational burden for the new estimator. Assume, for now, that \( K \sim N^{\zeta} \) for some \( 0 \leq \zeta < 1 \). First, focus on the computational time it takes for calculating \( (T_{\ell k}, C_{\ell k}) \) for all \( \ell \) and \( 1 \leq k \leq K \). For each \( X_i \), one has to determine in which interval \( I_k \) it falls, and then assign the pair \( (X_i, Y_i) \) to the interval \( I_k \). This can be accomplished in \( O(\log N^{\zeta}) = O(\log N) \) time. Since there are \( N \) such points (scrambled across the different servers), the total time taken is \( O(N \log N) \). Next, computing \( (C_{\ell k}, T_{\ell k}) \) for a fixed \( \ell \) involves less than \( 2 n_{\ell k} \) additions, where \( n_{\ell k} \) is the number of \( (X_i, Y_i) \) pairs assigned to \( I_k \) on server \( \ell \). Hence, computing the vector \( \{C_{\ell k}, T_{\ell k}\}_{1 \leq k \leq K} \) takes \( O(\sum_{k=1}^{K} n_{\ell k}) \) time. Summing up across the different \( \ell \)'s, we are looking at a total of \( O(N \log N) \) time, i.e. \( O(N \log N) \) time.
After the pairs \( \{T_{\ell k}, C_{\ell k}\}_{1 \leq k \leq K} \) have been transferred to the central server, computing the vector \( \{(w_k, \overline{y}_k)\}_{1 \leq k \leq K} \) takes \( O(LN^{\zeta}) \) time, and the final isotonization step takes \( O(N^{\zeta}) \) time. Thus, the total computing time is \( O(LN^{\zeta}) \vee O(N \log N) \) which is dominated by \( O(N \log N) \) provided \( L \) (which could grow with \( N \)) and \( \zeta \) are not too large. In addition to the total computing time, the burden also involves transferring about \( 2LK \sim LN^{\zeta} \) numbers between machines, which is larger than the amount of data transferred in the construction of BDSE. As shall be seen below, with \( K \) slightly larger than \( N^{1/3} \) — say \( K \sim N^{1/3 + \eta_1} \) \((\eta_1 \text{ small})\) — and \( m \) of a smaller order than \( N^{1/3} \), the new estimator is able to recover the properties of the global estimator: hence, so long as the number of machines is not too large — say \( L = N^{1/3 - \eta_2} \) — the total amount of data required to be
transferred is of order $N^{2/3+\eta_1-\eta_2} = o(N^{2/3})$ when $\eta_2 > \eta_1$.

Note that the computation of the global isotonic estimator in this situation would require transferring all data points to the central server which is exactly $O(N)$ and the isotonic algorithm at the central server would take $O(N \log N)$ time. Note also that the minimum amount of data transferring needed for the new estimator above is of order $K$ (this happens when the number of servers $L$ is held fixed) and therefore of larger order than $N^{1/3}$. On the other hand, in the scenario of BDS, where $L = m$, the BDSE is constructed using $m$ sub-samples where $m$ is of order at most $N^{1/4}$: this corresponds to a data-transfer of order at most $N^{1/4}$ numbers to construct the super-efficient estimator at any given point. The additional amount of data that needs to be transferred to construct the new estimator can be viewed as the cost of alleviating the super-efficiency phenomenon exhibited by BDSE.

**Characterization of the new estimator:** It is a standard result in isotonic regression that the minimum in (4) is achieved at a unique vector $(\hat{y}_1, \ldots, \hat{y}_K)^T$. We give below a characterization of the minimizer. In the sequel, we consider the piecewise-constant left-continuous estimator $\hat{\mu}_N$ that is constant on the intervals $[0, \pi_1]$, and $[\pi_{k-1}, \pi_k]$ for all $k = 2, \ldots, K$, and such that

$$\hat{\mu}_N(\pi_k) = \hat{y}_k$$

for all $k = 1, \ldots, K$. Let $F_N$ be the empirical distribution function corresponding to $X_1, \ldots, X_N$

$$F_N(x) = \frac{1}{N} \sum_{i=1}^{N} 1_{X_i \leq x}, \ x \in \mathbb{R},$$

(5)

and let $\Lambda_N$ be the piecewise-constant right-continuous process on $[0,1]$ that is constant on the intervals $[0, \pi_1)$, and $[\pi_{k-1}, \pi_k)$ or all $k = 2, \ldots, K$ such that

$$\Lambda_N(\pi_j) = \sum_{k=1}^{j} w_k \hat{y}_k = \frac{1}{N} \sum_{i=1}^{N} Y_i 1_{X_i \leq \pi_j}$$

for all $j = 1, \ldots, K$, and $\Lambda_N(0) = 0$. Then,

$$F_N(\pi_j) = \sum_{k=1}^{j} w_k$$

and $\hat{\mu}_N$ is the left-hand slope of the least concave majorant of the cumulative sum diagram defined by the set of points $\{(F_N(\pi_k), \Lambda_N(\pi_k)), k = 0, \ldots, K\}$ where $\pi_0 = 0$. We define the corresponding inverse estimator as follows:

$$U_N(a) = \argmax_{u \in [\pi_0, \ldots, \pi_K]} \{\Lambda_N(u) - a F_N(u)\}$$

(6)

where $\pi_0 = 0$, argmax denotes the greatest location of the maximum, and where we recall that for every nonincreasing left-continuous function $h : [0,1] \to \mathbb{R}$, the generalized inverse of $h$ is defined
as: for every $a \in \mathbb{R}$, $h^{-1}(a)$ is the greatest $t \in [0, 1]$ that satisfies $h(t) \geq a$, with the convention that the supremum of an empty set is zero. To see that $U_N = \hat{\mu}_N^{-1}$, note that from the characterization above of $\hat{\mu}_N$ as the slope of a least concave majorant, it follows that for all $a \in \mathbb{R}$ and $t \in (0, 1]$, we have the equivalences

$$\hat{\mu}_N(t) < a \iff (\exists t_i < t) (\forall x_j \geq t) : \frac{\Lambda_N(x_j) - \Lambda_N(x_i)}{F_N(x_j) - F_N(x_i)} < a$$

$$\iff (\exists t_i < t) (\forall x_j \geq t) : \Lambda_N(x_j) - aF_N(x_j) < \Lambda_N(x_i) - aF_N(x_i)$$

$$\iff \underset{u \in \{x_0, \ldots, x_K\}}{\text{argmax}} \{\Lambda_N(u) - aF_N(u)\} < t$$

whereas for $t = 0$, we have the equivalence

$$\hat{\mu}_N(0) < a \iff \underset{u \in \{x_0, \ldots, x_K\}}{\text{argmax}} \{\Lambda_N(u) - aF_N(u)\} = 0.$$  

We study below the asymptotic properties of $U_N(a)$ for arbitrary $a$ and use these to deduce the asymptotic properties of $\hat{\mu}_N(t)$ for a fixed $t \in (0, 1)$ using the switch relation

$$\hat{\mu}_N(t) \geq a \iff t \leq U_N(a),$$

(7)

that holds for all $t \in (0, 1]$ and $a \in \mathbb{R}$.

It will be useful to also record similar characterizations of the global estimator $\hat{\mu}_{N,G}$ of $\mu$, for the sake of completeness. Recall that the global estimator is the isotonic estimator that we would compute if all the data $\{X_i, Y_i\}_{i=1}^N$ could have been brought over (or were already there) on a central server. Letting $\Lambda_{N,G}(t) = N^{-1} \sum_{i=1}^N Y_i \mathbb{1}_{X_i \leq t}$, for $a \in \mathbb{R}$, define

$$U_{N,G}(a) = \underset{u \in [0,1]}{\text{argmax}} \{\Lambda_{N,G}(u) - aF_N(u)\}.  \hspace{1cm} (8)$$

Then $U_{N,G}(a) = \hat{\mu}_{N,G}^{-1}(a)$ and similar to the pooled estimator, we have the following characterization:

$$\hat{\mu}_{N,G}(t) \geq a \iff t \leq U_{N,G}(a),$$

(9)

that holds for all $t \in (0, 1]$ and $a \in \mathbb{R}$.

3 Asymptotic properties of the new estimators

In the sequel, we denote by $g$ the generalized inverse of $\mu$ and by $\mathbb{E}^X$ the conditional expectation given $X_1, \ldots, X_N$. Being the inverse of $\mu$, $g$ is only defined on the interval $[\mu(1), \mu(0)]$. In the sequel, we expand $g$ to the whole real line by setting $g(a) = 0$ for all $a > \mu(0)$ and $g(a) = 1$ for all $a < \mu(1)$.

Furthermore, for all $x \geq 0$, $[x]$ denotes the integer part of $x$. We denote by $F_X$ the mixing distribution function

$$F_X(x) = \sum_{j=1}^m \frac{n_j}{N} F_{X_j}(x).$$

(10)
Note that the function depends on $N$ but that for notational convenience, this is not made explicit in the notation.

To develop the asymptotic properties of the proposed estimator, we will impose some further conditions on the model. These are:

A1. Assume that $F_X$ has a density function $f_X$ on $[0, 1]$ that satisfies

$$C_1 < \inf_{t \in [0, 1]} f_X(t) \leq \sup_{t \in [0, 1]} f_X(t) \leq C_2$$

for some positive numbers $C_1$ and $C_2$ that do not depend on $N$.

A2. With $\varepsilon_i = Y_i - \mu(X_i)$ for all $i = 1, \ldots, N$, assume that there exists $\sigma > 0$ such that $E[\varepsilon_i^2 | X_i] \leq \sigma^2$ for all $i$, with probability one.

A3. The regression function $\mu$ satisfies:

$$C_3 < \frac{|\mu(t) - \mu(x)|}{t - x} < C_4$$

for all $t \neq x \in [0, 1]$, for positive numbers $C_3$ and $C_4$.

A4. The number of bins $K$ satisfies $K^{-1} = o(N^{-1/3})$ and there exists $\lambda \in (0, 1]$ that may depend on $N$ and satisfies

$$\min_{1 \leq j \leq m} \frac{n_j}{N} \geq \lambda > 0$$

and

$$\liminf_{N \to \infty} N^{1/3} \lambda (\log N)^{-3} = \infty.$$  

Remarks on the assumptions: Assumption (A1) is fulfilled for instance if each $F_{X_j}$, $j = 1, \ldots, m$ has a density function $f_{X_j}$ such that $C_1 < f_{X_j}(x) < C_2$ for all $x \in [0, 1]$. Note that Assumption (A3) is weaker than differentiability, it implies that $\mu$ is both Lipschitz and so to speak inverse Lipschitz. It also implies that the inverse function $g$ defined above is continuous. Assumption (A4) is critical to recovering the Chernoff-type asymptotics for the pooled estimator; that $K$ grows faster than $N^{1/3}$ ensures that the data are averaged over bins of length smaller than $N^{-1/3}$, so that the isotonic algorithm operating on these averages at the central machine can still recover the $N^{-1/3}$ convergence rate. If $K$ were to grow exactly at the rate $N^{1/3}$ or slower, the pooled estimator would no longer demonstrate Chernoff-type cube-root asymptotics. Furthermore, in (A4), we assume that the proportion $n_j/N$ of observations from the $j$-th sample is at least of order $N^{-1/3}(\log N)^3$. This also plays a critical role in the subsequent analysis. Since,

$$1 = \sum_{j=1}^{m} \frac{n_j}{N} \geq m \min_{1 \leq j \leq m} \frac{n_j}{N},$$

the conditions in (13) imply that the number $m$ of different sub-samples cannot grow to fast: we must have $m \ll N^{1/3}(\log N)^{-3}.$
3.1 Uniformly Bounded MSE Property of the New Estimators

The Inverse Problem: We first demonstrate that the new estimator in the inverse problem exhibits uniformly bounded maximal risk (MSE) over an appropriate class of models, as $N$ grows to $\infty$. This is an analogue of the first result in Theorem 4.1 of BDS for the global isotonic estimator of the inverse function, though it is established here under weaker conditions. For this task, we denote by $F_1$ the class of non-increasing functions $\mu$ on $[0, 1]$ that satisfy (12) and $\sup_t |\mu(t)| \leq C_5$, where $C_5 > 0$ is a positive number.

Theorem 3.1 Under assumptions (A1) through (A4), there exists $C > 0$ that depends only on $\sigma^2, C_1, C_2, C_3, C_4$ such that for all $a \in \mathbb{R},$

$$\limsup_{N \to \infty} \sup_{\mu \in F_1} N^{2/3} \mathbb{E}_\mu (U_N(a) - \mu^{-1}(a))^2 \leq C.$$ 

The proof of the above theorem relies on a number of preliminary results which are presented, next. In the remainder of this section, we assume that assumptions (A1) to (A4) are always satisfied (though some results may require only a subset of these assumptions). Additional assumptions will be imposed when required.

Lemma 3.2 Let $\theta > 0$ be arbitrary. Then, there exist (i) a number $c > 0$ that depends only on $C_1, C_3,$ (ii) an integer $N_0 > 0$ that depends only on $C_1, C_2, C_3, C_4, \theta$ and (iii) an event $\mathcal{E}_N$ that depends only on $C_2$, such that for all $N \geq N_0$, we have $\mathbb{P}(\mathcal{E}_N) \geq 1 - N^{-\theta}$ and on $\mathcal{E}_N$,

$$E^X \Lambda_N(u) - E^X \Lambda_N \left( \frac{[Kg(a)]}{K} \right) - a \left( F_N(u) - F_N \left( \frac{[Kg(a)]}{K} \right) \right) \leq -c(u - g(a))^2$$ (14)

for all $a \in \mathbb{R}$ and all $u \in \{x_0, \ldots, \bar{x}_K\}$ such that $|u - g(a)| \geq N^{-1/3}$.

The proof of this lemma is long and technical and is available in the appendix. The next result gives a polynomial tail bound on the estimation error $U_N(a) - g(a)$ over a high-probability set that is eventually used to bound the MSE.

Lemma 3.3 With $\mathcal{E}_N$ and $N_0$ taken from Lemma 3.2 there exists $C > 0$ that depends only on $\sigma^2, C_1, C_2, C_3, C_4$ such that for all $a \in \mathbb{R}$ and $x > 0$,

$$\mathbb{P}(|U_N(a) - g(a)| \geq x, \mathcal{E}_N) \leq \frac{C}{N x^3}$$ (15)

for all $N \geq N_0$.

Proof. The inequality in the lemma is obvious for $x \in (0, N^{-1/3})$ since for such $x$’s, it suffices to choose $C \geq 1$ so that the right hand side is larger than one. Hence, in the sequel we consider $x \geq N^{-1/3}$. For all $a \in \mathbb{R}$ and all $u \in \{x_0, \ldots, \bar{x}_K\}$ such that $|u - g(a)| \geq x$, define $e(a, u)$ as in (33) and $M_N(u) = \Lambda_N(u) - E^X (\Lambda_N(u))$. The characterization in (3) proves the following inclusion

10
of events:

\[
\{ U_N(a) - g(a) \geq x \} 
\subset \left\{ \max_{u \in \{x_0, \ldots, x_K\}, \ u - g(a) \geq x} \left\{ \Lambda_N(u) - a F_N(u) \right\} \geq \Lambda_N \left( \frac{\left[ K g(a) \right]}{K} \right) - a F_N \left( \frac{\left[ K g(a) \right]}{K} \right) \right\}
\]

\[
= \left\{ \max_{u \in \{x_0, \ldots, x_K\}, \ u - g(a) \geq x} \left\{ M_N(u) - M_N \left( \frac{\left[ K g(a) \right]}{K} \right) + c(a,u) \right\} \geq 0 \right\}.
\]

Since \( x \geq N^{-1/3} \), combining this with Lemma 3.2 shows that there exists \( c > 0 \) that depends only on \( C_1, C_3 \) such that

\[
\mathbb{P} \left( U_N(a) - g(a) \geq x, \mathcal{E}_N \right) 
\leq \mathbb{P} \left( \max_{u \in \{x_0, \ldots, x_K\}, \ u - g(a) \geq x} \left\{ M_N(u) - M_N \left( \frac{\left[ K g(a) \right]}{K} \right) - c(u - g(a))^2 \right\} \geq 0 \right)
\]

for \( N \geq N_0 \). Hence,

\[
\mathbb{P} \left( U_N(a) - g(a) \geq x, \mathcal{E}_N \right) 
\leq \sum_{k \geq 0} \mathbb{P} \left( \max_{u \in \{x_0, \ldots, x_K\}, \ u - g(a) \in [x2^k, x2^{k+1}] \} \left\{ M_N(u) - M_N \left( \frac{\left[ K g(a) \right]}{K} \right) - c(u - g(a))^2 \right\} \geq 0 \right)
\]

\leq \sum_{k \geq 0} \mathbb{P} \left( \max_{u \in \{x_0, \ldots, x_K\}, \ u - g(a) \in [0, x2^{k+1}] \} \left\{ M_N(u) - M_N \left( \frac{\left[ K g(a) \right]}{K} \right) \right\} \geq c(x2^k)^2 \right). \quad (16)

Let \( \mathbb{P}^X \) denote the conditional probability given \( X_1, \ldots, X_N \). By definition, for all \( u \in \{x_0, \ldots, x_K\} \) we have

\[
M_N(u) = \frac{1}{N} \sum_{i=1}^N \varepsilon_i \mathbb{1}_{X_i \leq u} \quad (17)
\]

where \( \varepsilon_i = Y_i - \mu(X_i) \). The process \( M_u \) can be extended to all \( u \in \mathbb{R} \) using the same definition as above. Then, \( M_N \) is a centered martingale under \( \mathbb{P}^X \) that satisfies

\[
\mathbb{E}^X \left( M_N(u) - M_N(v) \right)^2 = \frac{1}{N^2} \sum_{i=1}^N \mathbb{E}^X \left( \varepsilon_i^2 \mathbb{1}_{u < X_i \leq v} \right) \leq \frac{\sigma^2}{N} \left( F_N(u) - F_N(v) \right) \quad (18)
\]

for all \( u \leq v \), using that \( \mathbb{E}^X \left( \varepsilon_i^2 \right) \leq \sigma^2 \) for all \( i \) by assumption. Hence, it follows from the Doob inequality that for all \( k \geq 0 \),

\[
\mathbb{P}^X \left( \max_{u \in \{x_0, \ldots, x_K\}, \ u - g(a) \in [0, x2^{k+1}] \} \left\{ M_N(u) - M_N \left( \frac{\left[ K g(a) \right]}{K} \right) \right\} \geq c(x2^k)^2 \right)
\]

\leq \frac{\sigma^2}{c^2 N (x2^k)^4} \left( g(a) + x2^{k+1} \right) - F_N \left( \frac{\left[ K g(a) \right]}{K} \right).
Taking the expectation on both sides of the preceding inequality yields for large enough $N$ that

\[
\mathbb{P}\left(\max_{u \in \{x_0, \ldots, x_K\}} \left\{ M_N(u) - M_N\left(\frac{\lceil Kg(a)\rceil}{K}\right)\right\} \geq c(x^{2^k})^2 \right)
\leq \frac{\sigma^2 \left\{ F_X\left(\frac{g(a) + x^{2^{k+1}}}{c^2 N(x^{2^k})^2}\right) - F_X\left(\frac{\lceil Kg(a)\rceil}{c^2 N(x^{2^k})^2}\right)\right\}}{c^2 N(x^{2^k})^4}
\leq \frac{\sigma^2 C_2 (x^{2^{k+1}} + K^{-1})}{c^2 N(x^{2^k})^4}
\leq \frac{2\sigma^2 C_2 x^{2^{k+1}}}{c^2 N(x^{2^k})^4},
\]

where $C_2$ is taken from (11). For the penultimate inequality, we used that $x^{2^k+1} \geq N^{-1/3}$ for all $k$ whereas $K^{-1} = o(N^{-1/3})$, implying that $K^{-1} \leq x^{2^{k+1}}$ for all $k$ provided that $N$ is sufficiently large. Putting the previous inequality in (16) we obtain that for sufficiently large $N$,

\[
\mathbb{P}(U_N(a) - g(a) \geq x, \mathcal{E}_N) \leq \sum_{k \geq 0} \frac{4\sigma^2 C_2}{c^2 N(x^{2^k})^4}.
\]

Since $C := \sum_{k \geq 0} 2^{-3k}$ is finite, we conclude that

\[
\mathbb{P}(U_N(a) - g(a) \geq x, \mathcal{E}_N) \leq \frac{4\sigma^2 C_2 C}{c^2 N x^3}.
\]

Similar arguments show that

\[
\mathbb{P}(g(a) - U_N(a) \geq x, \mathcal{E}_N) \leq \frac{4\sigma^2 C_2 C}{c^2 N x^3},
\]

and therefore,

\[
\mathbb{P}(|g(a) - U_N(a)| \geq x, \mathcal{E}_N) \leq \frac{8\sigma^2 C_2 C}{c^2 N x^3}.
\]

The lemma follows. \qed

We are now ready to prove the theorem.

**Proof of Theorem 3.1** Fix $\mu \in \mathcal{F}_1$ arbitrarily. Since both $U_N$ and $\mu^{-1}$ take values in $[0, 1]$, we have $|U_N(a) - \mu^{-1}(a)| \leq 1$ for all $a$ and therefore, with $\overline{\mathcal{E}_N}$ the complementary event to $\mathcal{E}_N$ taken from Lemma 3.2 where we set $\theta = 2/3$, we have

\[
\mathbb{E}_\mu\left(|U_N(a) - \mu^{-1}(a)|^2 1_{\overline{\mathcal{E}_N}}\right) \leq \mathbb{P}_\mu(\overline{\mathcal{E}_N}) \leq N^{-2/3}
\]

(19)
for $N$ sufficiently large. On the other hand, it follows from the Fubini theorem that

$$
\mathbb{E}_\mu \left( |U_N(a) - \mu^{-1}(a)|^2 \mathbb{1}_{\mathcal{E}_N} \right) = \int_0^\infty \mathbb{P}_\mu \left( |U_N(a) - \mu^{-1}(a)| > \sqrt{t}, \mathcal{E}_N \right) dt
$$

$$
= \int_0^\infty 2y \mathbb{P}_\mu \left( |U_N(a) - \mu^{-1}(a)| > y, \mathcal{E}_N \right) dy
$$

$$
\leq \int_0^\infty 2y \left( \frac{C}{Ny^3} \wedge 1 \right) dy.
$$

For the last inequality, we used [15] together with the fact that a probability cannot be larger than one. Hence,

$$
\mathbb{E}_\mu \left( |U_N(a) - \mu^{-1}(a)|^2 \mathbb{1}_{\mathcal{E}_N} \right) \leq \int_0^{N^{-1/3}} 2y dy + \int_{N^{-1/3}}^\infty \frac{2C}{Ny^2} dy
$$

$$
\leq N^{-2/3} (1 + 2C).
$$

Combining with [19] yields

$$
\mathbb{E}_\mu \left( |U_N(a) - \mu^{-1}(a)|^2 \right) \leq N^{-2/3} (2 + 2C),
$$

which completes the proof of the Theorem (by taking $C$ to be $2 + 2C$) where $C$ is the constant from Lemma [3.3].

\[\square\]

**The Direct Problem:** We now establish an analogue of the second result in Theorem 4.1 of BDS to demonstrate that the new estimator fixes the super-efficiency phenomenon in the direct problem as well, i.e. it has bounded uniform MSE as $N \to \infty$ over the class $\mathcal{F}_1$.

Denote by $\tilde{F}_N$ the step function on $[0, 1]$ such that $\tilde{F}_N(\bar{x}_k) = F_N(\bar{x}_k)$ for all $k = 0, \ldots, K$, and $\tilde{F}_N$ is constant on all intervals $[\bar{x}_{k-1}, \bar{x}_k)$ for $k = 1, \ldots, K$. We denote by $\tilde{F}_N^{-1}$ the corresponding inverse function:

$$
\tilde{F}_N^{-1}(t) = \inf \{ x \in [0, 1] \mid \tilde{F}_N(x) \geq t \}.
$$

Since $\tilde{F}_N^{-1}(\bar{x}_k) = \bar{x}_k$ for all $k = 0, \ldots, K$, it follows from the characterization in [6] that

$$
U_N(a) = \tilde{F}_N^{-1}(V_N(a))
$$

(20)

for all $a \in \mathbb{R}$, where

$$
V_N(a) = \arg\max_{u \in \{ \tilde{F}_N(\bar{x}_0), \ldots, \tilde{F}_N(\bar{x}_K) \}} \{ \Lambda_N \circ \tilde{F}_N^{-1}(u) - au \}.
$$

The following lemma provides tail bound probabilities for $V_N$.

**Lemma 3.4** With $\varepsilon_i = Y_i - \mu(X_i)$ for all $i = 1, \ldots, N$, assume that there exists $\sigma > 0$ such that $\mathbb{E}[\varepsilon_i^p | X_i] \leq \sigma^p$ for all $i$ and some $p \geq 2$, with probability one. Assume that $F_X$ has a density function $f_X$ on $[0, 1]$ that satisfies [11] for some positive numbers $C_1, C_2$. Then, there exists $C > 0$ that depends only on $p, C_2$ and $\sigma$ such that

$$
\mathbb{P} \left( V_N(a) \geq x \right) \leq \frac{C}{N^{p/2}x^{p-1}(a - \mu(0))^p}
$$

13
for all \( a > \mu(0) \) and

\[
P(1 - V_N(a) \geq x) \leq \frac{C}{N^{\nu/2} x^{\alpha-1}(\mu(1) - a)^\beta}.
\]

for all \( a < \mu(1) \).

For a proof of this lemma, see the Appendix.

The following theorem establishes the desired property for the pooled direct estimator.

**Theorem 3.5** Fix \( \delta \in (0, 1/2) \). Then, there exists \( C > 0 \) that depends only on \( \sigma, p, C_1, C_2, C_3, C_4, C_5, \delta \) such that for all \( t \in [\delta, 1 - \delta] \)

\[
\limsup_{N \to \infty} \sup_{\mu \in \mathcal{F}_1} N^{2/3} \mathbb{E}_\mu(\hat{\mu}_N(t) - \mu(t))^2 \leq C.
\]

**Proof.** Similar to the proof of Theorem 3.1 for the inverse problem, we would like to restrict ourselves to the event \( E_N \) from Lemma 3.2, where \( \theta \) can be chosen arbitrarily large. However, we do not have an analogue of (19) for the direct problem since \( \hat{\mu}_N \) is not bounded as is \( U_N \). Hence, we first prove that \( \hat{\mu}_N \) remains bounded by a power of \( N \) apart possibly on a negligible set. For this task, consider an arbitrary \( A > 0 \) such that \( A + \mu(0) > 0 \), and note that for all \( t \in [0, 1] \), and all non-increasing functions \( \mu \) on \([0, 1]\), we have

\[
\mathbb{E}_\mu [\hat{\mu}_N^2(t) \mathbb{1}_{\hat{\mu}_N(t) > A + \mu(0)}] \leq \mathbb{E}_\mu [\hat{\mu}_N^2(0) \mathbb{1}_{\hat{\mu}_N(0) > A + \mu(0)}].
\]

Hence, it follows from the Fubini theorem that for all non-increasing \( \mu \in \mathcal{F}_1 \),

\[
\mathbb{E}_\mu [\hat{\mu}_N^2(t) \mathbb{1}_{\hat{\mu}_N(t) > A + \mu(0)}] \leq \int_0^{\infty} \mathbb{P}_\mu(\hat{\mu}_N(0) \mathbb{1}_{\hat{\mu}_N(0) > A + \mu(0)} > \sqrt{y})dy
\]

\[
= (A + \mu(0))^2 \mathbb{P}_\mu(\hat{\mu}_N(0) > A + \mu(0)) + \int_{A + \mu(0)}^{\infty} 2y \mathbb{P}_\mu(\hat{\mu}_N(0) > y)dy.
\]

Note that if \( \hat{\mu}_N(0) > y \) for some \( y \in \mathbb{R} \), then for the inverse we must have \( U_N(y) > 0 \). Since \( U_N \) can only assume values in the set of jump points of \( \hat{\mu}_N \) it is of the form \( \mathcal{F}_k = k/K \) for some \( k \geq 1 \). Next, \( \hat{\mu}_N \) can have jumps only at those \( \mathcal{F}_k \) where \( F_N \) has a jump, i.e. \( F_N(\mathcal{T}_k) > F_N(\mathcal{T}_{k-1}) \). Since the size of a jump of \( F_N \) is at least \( N^{-1} \), we must have \( F_N(\mathcal{T}_k) \geq N^{-1} \) and therefore, \( F_N(\mathcal{T}_k) = F_N(\mathcal{T}_k) \geq N^{-1} \). Thus,

\[
V_N(y) = \bar{F}_N(U_N(y)) = F_N(U_N(y)) = F_N(\mathcal{T}_k) \geq N^{-1},
\]

implying that for all \( \mu \in \mathcal{F}_1 \),

\[
\mathbb{E}_\mu [\hat{\mu}_N^2(t) \mathbb{1}_{\hat{\mu}_N(t) > A + \mu(0)}] \leq (A + \mu(0))^2 \mathbb{P}_\mu(V_N(A + \mu(0)) \geq N^{-1}) + \int_{A + \mu(0)}^{\infty} 2y \mathbb{P}_\mu(V_N(y) \geq N^{-1})dy.
\]

14
With $C$ taken from Lemma 3.4 where it is assumed that $p > 2$, we arrive at

$$
\mathbb{E}_\mu \left[ \hat{\mu}_N^2(t) 1_{\hat{\mu}_N(t) > A + \mu(0)} \right] \leq C N^{-1+p/2} (A + \mu(0))^2 A^{-p} + 2C N^{-1+p/2} \int_{A+\mu(0)}^\infty y(y - \mu(0))^{-p} dy
$$

$$
= C N^{-1+p/2} (A + \mu(0))^2 A^{-p} + 2 C N^{-1+p/2} \left\{ \frac{A^{2-p}}{p-2} + \mu(0) \frac{A^{1-p}}{p-1} \right\}
$$

$$
\leq C N^{-1+p/2} (A + C_5)^2 A^{-p} + 2 C N^{-1+p/2} \left\{ \frac{A^{2-p}}{p-2} + C_5 \frac{A^{1-p}}{p-1} \right\}.
$$

With $A = N^{(3p-2)/(6(p-2))}$, this proves that there exists $C' > 0$ that depends only on $\sigma$, $p$, $C_2$ and $C_5$ such that

$$
\mathbb{E}_\mu \left[ \hat{\mu}_N(t)^2 1_{\hat{\mu}_N(t) > A + \mu(0)} \right] \leq C' N^{-2/3}
$$

for all $t \in [0, 1]$ and $\mu \in \mathcal{F}_1$. Now, with $A = N^{(3p-2)/(6(p-2))}$,

$$
\mathbb{E}_\mu \left[ (\hat{\mu}_N(t) - \mu(t))^2 1_{\hat{\mu}_N(t) > A + \mu(0)} \right] \leq \mathbb{E}_\mu \left[ 2 \left( \hat{\mu}_N^2(t) + \mu^2(t) \right) 1_{\hat{\mu}_N(t) > A + \mu(0)} \right]
$$

$$
\leq 2C' N^{-2/3} + 2 \max\{|\mu(0)|, |\mu(1)|\} \mathbb{P}(\hat{\mu}_N(0) > A + \mu(0))
$$

$$
\leq 2C' N^{-2/3} + 2 \max\{|\mu(0)|, |\mu(1)|\} \mathbb{P}(V_N(A + \mu(0)) \geq N^{-1})
$$

similar as above, whence

$$
\mathbb{E}_\mu \left[ (\hat{\mu}_N(t) - \mu(t))^2 1_{\hat{\mu}_N(t) > A + \mu(0)} \right] \leq 2C' N^{-2/3} + 2C_5 C N^{-1+p/2} A^{-p}
$$

$$
\leq C'' N^{-2/3}
$$

(21)

where $C''$ depends only on $\sigma$, $p$, $C_2$, and $C_5$. This enables us to restrict to the event $\mathcal{E}_N$ of Lemma 3.2 provided that $\theta$ is chosen sufficiently large in the lemma. Indeed, with $\theta > (5p - 6)/(3(p - 2))$, the previous inequality implies that with $A = N^{(3p-2)/(6(p-2))}$ and $N$ sufficiently large,

$$
\mathbb{E}_\mu \left[ (\hat{\mu}_N(t) - \mu(t))^2 1_{\bar{\mathcal{E}}_N} \right] \leq (A + \mu(0) - \mu(t))^2 \mathbb{P}(\bar{\mathcal{E}}_N) + C'' N^{-2/3}
$$

$$
\leq (A + 2C_5)^2 \mathbb{P}(\bar{\mathcal{E}}_N) + C'' N^{-2/3}
$$

(22)

$$
\leq 2C'' N^{-2/3}
$$

(23)

for all $t \in [0, 1]$ and all $\mu \in \mathcal{F}_1$. It can be shown similarly that for $N$ sufficiently large,

$$
\mathbb{E}_\mu \left[ (\hat{\mu}_N(t) - \mu(t))^2 1_{\bar{\mathcal{E}}_N} \right] \leq 2C'' N^{-2/3}
$$

for all $t \in [0, 1]$ and all $\mu \in \mathcal{F}_1$, implying that

$$
\limsup_{N \to \infty} \sup_{\mu \in \mathcal{F}_1} N^{2/3} \mathbb{E}_\mu \left[ (\hat{\mu}_N(t) - \mu(t))^2 1_{\bar{\mathcal{E}}_N} \right] \leq 4C''
$$
for all $t \in [0, 1]$. Hence, it now suffices to prove that there exists $C > 0$ that depends only on $\sigma, p, C_1, C_2, C_3, C_4, C_5, \delta$ such that

$$
\limsup_{N \to \infty} \sup_{\mu \in \mathcal{F}_1} N^{2/3} \mathbb{E}_{\mu} \left[ (\hat{\mu}_N(t) - \mu(t))^2 \mathbbm{1}_{\mathcal{E}_N} \right] \leq C. 
$$

(24)

To prove this, fix $\mu \in \mathcal{F}_1$ arbitrarily, and invoke the Fubini Theorem to obtain that

$$
\mathbb{E}_{\mu} \left[ (\hat{\mu}_N(t) - \mu(t))^2 \mathbbm{1}_{\mathcal{E}_N} \right] = \int_0^\infty 2y \mathbb{P}_\mu (\hat{\mu}_N(t) - \mu(t) \geq y, \mathcal{E}_N) \, dy 
$$

using the switch relation (7) for the last equality. We split the above integral into the sum of two integrals and first consider

$$
I_1 = \int_0^{\mu(0) - \mu(t)} 2y \mathbb{P}_\mu (U_N(\mu(t) + y) \geq t, \mathcal{E}_N) \, dy.
$$

With $C_4$ taken from the definition of $\mathcal{F}_1$ we have

$$
t = \mu^{-1}(\mu(t)) \geq \mu^{-1}(\mu(t) + y) + yC_4^{-1}
$$

for all $t \in [0, 1]$ and $y \in [0, \mu(0) - \mu(t)]$. Combining Lemma 3.3 with the fact that a probability cannot be larger than one then yields

$$
I_1 \leq N^{-2/3} + \int_{N^{-1/3}}^{\mu(0) - \mu(t)} 2y \mathbb{P}_\mu (U_N(\mu(t) + y) - \mu^{-1}(\mu(t) + y) \geq yC_4^{-1}, \mathcal{E}_N) \, dy 
$$

$$
\leq N^{-2/3} + \int_{N^{-1/3}}^\infty \frac{2CC_4^3}{Ny^2} \, dy 
$$

$$
\leq N^{-2/3} \left( 1 + 2CC_4^3 \right). 
$$

(26)

Next, Lemma 3.3 yields

$$
I_2 := \int_{\mu(0) - \mu(t)}^{\infty} 2y \mathbb{P}_\mu (U_N(\mu(t) + y) \geq t, \mathcal{E}_N) \, dy 
$$

$$
\leq \int_{\mu(0) - \mu(t)}^{\mu(0) - \mu(t) + N^{1/6}} 2y \mathbb{P}_\mu (U_N(\mu(t) + y) \geq t, \mathcal{E}_N) \, dy + \int_{\mu(0) - \mu(t) + N^{1/6}}^{\infty} 2y \mathbb{P}_\mu (U_N(\mu(t) + y) \geq t, \mathcal{E}_N) \, dy 
$$

$$
\leq \frac{C}{Nt^3} \int_{\mu(0) - \mu(t)}^{\mu(0) - \mu(t) + N^{1/6}} 2y \, dy + \int_{\mu(0) - \mu(t) + N^{1/6}}^{\infty} 2y \mathbb{P}_\mu (U_N(\mu(t) + y) \geq t, \mathcal{E}_N) \, dy,
$$

where the first term on the right-hand side is equal to

$$
\frac{C}{Nt^3} \left( (\mu(0) - \mu(t) + N^{1/6})^2 - (\mu(0) - \mu(t))^2 \right) = \frac{C}{Nt^3} \left( 2(\mu(0) - \mu(t))N^{1/6} + N^{1/3} \right) 
$$

$$
\leq \frac{C}{Nt^3} \left( 4C_5N^{-1/6} + 1 \right) N^{1/3} 
$$

$$
\leq \frac{2C}{\delta^3} N^{-2/3}
$$
for sufficiently large $N$, for all $t \geq \delta$ and $\mu \in \mathcal{F}_1$. Using the connection (20) between $U_N$ and $V_N$ yields
\[
I_2 \leq \frac{2C}{\delta^3} N^{-2/3} + \int_{\mu(0)-\mu(t)+N^{1/6}}^{\infty} 2y^{\mathbb{P}}_{\mu} \left( V_N(\mu(t) + y) \geq \tilde{F}_N(t), \mathcal{E}_N \right) dy,
\]
where $\tilde{F}_N(t) = \tilde{F}_N([KtK^{-1}]) = F_N([KtK^{-1}])$ by definition of $\tilde{F}_N$ and $F_N$. Regarding the proof of Lemma 3.2, it can be seen that on $\mathcal{E}_N$ we have
\[
\sup_{t \in [0,1]} |F_N(t) - F_X(t)| \leq C_2 N^{-1/3}
\]
whence
\[
I_2 \leq \frac{2C}{\delta^3} N^{-2/3} + \int_{\mu(0)-\mu(t)+N^{1/6}}^{\infty} 2y^{\mathbb{P}}_{\mu} \left( V_N(\mu(t) + y) \geq F_X([KtK^{-1}]) - C_2 N^{-1/3} \right) dy
\]
\[
\leq \frac{2C}{\delta^3} N^{-2/3} + \int_{\mu(0)-\mu(t)+N^{1/6}}^{\infty} 2y^{\mathbb{P}}_{\mu} \left( V_N(\mu(t) + y) \geq C_1 (t - K^{-1}) - C_2 N^{-1/3} \right) dy
\]
\[
\leq \frac{2C}{\delta^3} N^{-2/3} + \int_{\mu(0)-\mu(t)+N^{1/6}}^{\infty} 2y^{\mathbb{P}}_{\mu} \left( V_N(\mu(t) + y) \geq C_1 \delta/2 \right) dy,
\]
for all $t \in [\delta, 1 - \delta]$, provided that $N$ is sufficiently large. Hence, it follows from Lemma 3.4 where it is assumed that $p > 2$, that
\[
I_2 \leq \frac{2C}{\delta^3} N^{-2/3} + \frac{2pC}{(C_1 \delta)^p} \int_{\mu(0)-\mu(t)+N^{1/6}}^{\infty} \frac{yN^{-p/2}}{(y + \mu(t) - \mu(0))^p} dy.
\]
For the integral on the right-hand side we have
\[
\int_{\mu(0)-\mu(t)+N^{1/6}}^{\infty} \frac{yN^{-p/2}}{(y + \mu(t) - \mu(0))^p} dy
\]
\[
= \int_{\mu(0)-\mu(t)+N^{1/6}}^{\infty} \frac{N^{-p/2}}{(y + \mu(t) - \mu(0))^{p-1}} dy + \int_{\mu(0)-\mu(t)+N^{1/6}}^{\infty} \frac{(\mu(0) - \mu(t))N^{-p/2}}{(y + \mu(t) - \mu(0))^{p-1}} dy
\]
\[
= \int_{N^{1/6}}^{\infty} \frac{N^{-p/2}}{u^{p-1}} du + \int_{N^{1/6}}^{\infty} \frac{(\mu(0) - \mu(t))N^{-p/2}}{u^{p}} du
\]
\[
\leq \frac{1}{p-2} N^{(1-2p)/3} + \frac{2C_5}{p-1} N^{(1-4p)/6}.
\]
Hence, we can find $\tilde{C}$ that depends only on $p, \sigma, C_1 - C_5$ such that
\[
I_2 \leq \frac{2pC}{(C_1 \delta)^p} \tilde{C} N^{-2/3}
\]
for sufficiently large $N$, for all $\mu \in \mathcal{F}_1$ and $t \in [\delta, 1 - \delta]$.
Combining this with (20) and (25) proves that there exists $C > 0$ that depends only on $\sigma, p, C_1, C_2, C_3, C_4, C_5, \delta$ such that
\[
\limsup_{N \to \infty} \sup_{\mu \in \mathcal{F}_1} N^{2/3} \mathbb{E}_\mu [ (\tilde{\mu}_N(t) - \mu(t))^2 ] \leq C.
\]
It can be proved similarly that

$$\limsup_{N \to \infty} \sup_{\mu \in \mathcal{F}_i} N^{2/3} \mathbb{E}_\mu \left[ (\hat{\mu}_N(t) - \mu(t))^2 1_{\mathcal{E}_N} \right] \leq C,$$

which completes the proof \[24\], and hence the proof of the theorem. \hfill \Box

In the next section, we show that under a fixed $\mu$, the new estimator recovers the asymptotic distribution of the global estimator with the same convergence rate.

### 3.2 Asymptotic distributions

To establish asymptotic distributions for our new estimators, we make additional assumptions in the case that the number $m$ of different samples goes to infinity, and we clarify the asymptotic setting further.

When considering the case where $m$ is allowed to grow to infinity as $N \to \infty$, we assume that there is a sequence of unknown distinct distributions $\{P_j\}_{j \geq 1}$ such that our set of observations is part of an infinite sequence of pairs $\{(X_i, Y_i)\}_{i \geq 1}$, where for all $i$ the distribution of $(X_i, Y_i)$ takes the form $P_j$ for some $j \geq 1$. Hence, $m = m_N$ is the number of different distributions that appear across the first $N$ observations $(X_1, Y_1), \ldots, (X_N, Y_N)$. To fix ideas, possibly rearranging the probabilities in the sequence $\{P_j\}_{j \geq 1}$, we assume without loss of generality in the sequel that for all $N$, the $m = m_N$ distributions that appear across the first $N$ observations are $P_1, \ldots, P_m$. Note that the setting does not exclude that $m_N = 1$ for all $N$, i.e. that all observations are drawn from the same distribution $P_1$. In the case where $m_N > 1$ for sufficiently large $N$, it is not excluded that $m_N$ remains bounded. In the sequel, for all $j \geq 1$, we denote by $\sigma_j$ the function such that

$$\sigma_j^2(u) = \mathbb{E}[(Y - \mu(X))^2 | X = u]$$

for all $u \in [0, 1]$ and by $f_j$ the density function of $X$, which is assumed to exist, where $(X, Y)$ has distribution $P_j$. Then, the distribution function $F_X$ in (10) has a density function $f_X$ on $[0, 1]$ given by

$$f_X(u) = \sum_{j=1}^{m} \frac{n_j}{N} f_j(u) \tag{27}$$

for all $u \in [0, 1]$.

We next make the following technical assumptions.

$\tilde{A}_0$. The functions $\{f_j\}$ are uniformly bounded in $j$ on the interval $[0, 1]$.

$\tilde{A}_1$. Let

$$\omega(\delta) = \sup_{j \geq 1} \left\{ \sup_{|u-v| \leq \delta} |\sigma_j^2(u) - \sigma_j^2(v)|, \sup_{|u-v| \leq \delta} |f_j(u) - f_j(v)| \right\}$$

for all $\delta \geq 0$. Then, $\omega(\delta) \to 0$ as $\delta \to 0$. \hfill 18
\( \tilde{A}_2 \). The density function \( f_X \) converges pointwise [and hence, uniformly] on \([0, 1] \) as \( N \to \infty \) to a continuous function \( f_\infty \) that is bounded away from zero. This implies that \( \tilde{A}_1 \) holds for some positive numbers \( C_1, C_2 \) that do not depend on \( N \), provided that \( N \) is sufficiently large.

\( \tilde{A}_3 \). The function \( \sigma^2_X \) defined by

\[
\sigma^2_X(u) := \sum_{j=1}^m \frac{n_j}{N} \sigma^2_j(u) f_j(u)
\]

for all \( u \in [0, 1] \) converges pointwise [and hence, uniformly] to a continuous function \( \sigma^2_\infty \), bounded away from 0, as \( N \to \infty \).

\( \tilde{A}_4 \). With \( \varepsilon_i := Y_i - \mu(X_i) \) for all \( i = 1, \ldots, N \), there exists \( \sigma > 0 \) such that \( \mathbb{E}[|\varepsilon_i|^p | X_i = t] \leq \sigma^p \) for all \( i, t \) and some \( p > 2 \).

\( \tilde{A}_5 \). The function \( \mu \) is decreasing and has a continuous first derivative on \([0, 1] \) such that \( \inf_{u \in [0, 1]} |\mu'(u)| > 0 \)

For notational convenience, we do not make it explicit in the notation that \( F_X, f_X, \sigma_X, m \) may depend on \( N \).

**Remark:** The pointwise convergence of \( f_X \) to \( f_\infty \) implies uniform convergence because by assumption \( \tilde{A}_1 \), the class of functions \( \{f_j\} \) is uniformly equicontinuous, which then implies that the class \( \{f_X\} \) is also uniformly equicontinuous. Also, the pointwise convergence of \( \sigma^2_X \) to \( \sigma^2_\infty \) guarantees uniform convergence, because the class of functions \( \{\sigma^2_X\} \) is uniformly equicontinuous: this follows from the uniform boundedness of the class \( \{f_j\} \) assumed in \( \tilde{A}_0 \), the uniform boundedness of \( \{\sigma^2_j\} \), which is a consequence of \( \tilde{A}_4 \), and the uniform equicontinuity of the classes \( \{f_j\} \) and \( \{\sigma^2_j\} \) assumed in \( \tilde{A}_1 \).

**Theorem 3.6** With \( t \in (0, 1) \) fixed, and \( a = \mu(t) + N^{-1/3}x \) for some fixed \( x \in \mathbb{R} \), under Assumptions \( \tilde{A}_1 \) through \( \tilde{A}_4 \) and \( A_4 \), we have

\[
N^{1/3}(U_N(a) - g(a)) \to_d \left( \frac{2\sigma_\infty(t)}{\mu'(t)|f_\infty(t)|} \right)^{2/3} Z \quad \text{as } N \to \infty,
\]

where \( Z := \arg\max_{u \in \mathbb{R}} \{W(u) - u^2\} \), \( W \) being a standard two-sided Brownian motion starting at 0, has the so-called Chernoff’s distribution.

An interesting feature of the estimator \( U_N \) is that its asymptotic behavior does not depend on the way the \( N \) data are allocated on the different servers. The direct estimator \( \hat{\mu}_N \) shares this feature, as is shown in the next result.

**Theorem 3.7** Under the same assumptions as in Theorem 3.6, with \( t \in (0, 1) \) fixed, we have

\[
N^{1/3}(\hat{\mu}_N(t) - \mu(t)) \to_d \left( \frac{4\sigma^2_\infty(t)|\mu'(t)|}{f^2_X(t)} \right)^{1/3} Z \quad \text{as } N \to \infty,
\]

where \( Z \) is as defined in Theorem 3.6.
Remark: The estimators \( \hat{\mu}_N(t) \) and \( U_N(a) \) have the same asymptotic distributions (when centered around their respective estimands and scaled by the factor \( N^{1/3} \)) as the corresponding global isotonic estimators, \( \hat{\mu}_{N,G} \) and \( U_{N,G} \) defined in (9) and (8) respectively. In other words, the asymptotic distributions of the estimators \( N^{1/3}(U_{N,G} - g(a)) \) and \( N^{1/3}(\hat{\mu}_{N,G}(t) - \mu(t)) \) are those arising in Theorems 3.6 and 3.7 respectively. The limit distributions of the global estimators can be established by the same set of techniques as used in the proofs of Theorems 3.6 and 3.7. Thus, the new estimators proposed in this paper not only circumvent the super-efficiency phenomenon but recover the asymptotic properties of their corresponding global versions. We note that the global isotonic estimators \( \hat{\mu}_{N,G}(t) \) and \( U_{N,G}(a) \) also possess the uniformly bounded maximal MSE property for their respective estimands, i.e. exact analogues of the results in Theorems 3.5 and 3.1 hold for \( N^{1/3}(U_{N,G} - g(a)) \) and \( N^{1/3}(\hat{\mu}_{N,G}(t) - \mu(t)) \) respectively, and can be established by similar techniques as used in the proofs of these two theorems.

Remark: The setting of the theorems in this section with a growing sequence of sub-populations such that conditions A1 through A5 hold is not difficult to satisfy. Consider, for example, \( m = \lceil N^{1/4} \rceil \) and \( P_j \) has density \( f_j(u) = (1 - \epsilon_j) f_0(u) + \epsilon_j f_1(u) \) where \( f_0 \) and \( f_1 \) are Lipschitz continuous densities bounded away from 0 and \( m \). Let the distribution of the \( X_i \)’s be \( P_1 \) for \( i = 1, 2, \ldots, \lceil N/m \rceil \), \( P_2 \) for \( \lceil N/m \rceil + 1 \leq i \leq 2\lceil N/m \rceil \), . . . , and \( P_m \) for \( (m - 1)\lceil N/m \rceil \leq i < N \). For each \( i \), the regression model is \( Y = \mu(X_i) + \epsilon_i \) where the \( \epsilon_i \)'s are i.i.d. \( N(0, \sigma^2) \) (say) and independent of the \( X_i \)'s, which are also mutually independent, and \( \mu \) satisfies all the desired conditions in this manuscript, in particular \( \tilde{A}_5 \). Then, it is easy to check that all the five conditions at the beginning of this section hold, with \( f_\infty = f_0 \) and \( \sigma_\infty^2(u) = \sigma^2 f_\infty(u) \).

The proof of Theorem 3.6 is in the appendix. The proof of Theorem 3.7 follows.

Proof of Theorem 3.7. It follows from the switch relation (7) that for all fixed \( t \in (0, 1) \), with \( a = \mu(t) + N^{-1/3} x \) we have

\[
\mathbb{P}\left( N^{1/3}(\hat{\mu}_N(t) - \mu(t)) < x \right) = \mathbb{P}\left( \tilde{\mu}_N(t) < \mu(t) + N^{-1/3} x \right) = \mathbb{P}(t > U_N(a)) = \mathbb{P}\left( N^{1/3}(U_N(a) - g(a)) < N^{1/3}(t - g(a)) \right).
\]

Now, \( N^{1/3}(t - g(a)) = x g'(\mu(t)) + o(1) = x|\mu'(t)|^{-1} + o(1) \), so it follows from Theorem 3.6 that

\[
\lim_{N \to \infty} \mathbb{P}\left( N^{1/3}(\hat{\mu}_N(t) - \mu(t)) < x \right) = \mathbb{P}\left( \left( \frac{2\sigma_\infty(t)}{|\mu'(t)| f_X(t)} \right)^{2/3} Z < \frac{x}{|\mu'(t)|} \right),
\]

using that the Chernoff distribution \( Z \) is continuous (see e.g. (Groeneboom and Wellner, 2001)).  
\( \square \)
4 Discussion

We have proposed new estimators for distributed computing in the isotonic regression problem whose computations are not anymore onerous than that of the respective global isotonic estimators, which replicate the properties of the global estimators, and do not suffer from the superefficiency phenomenon unlike the BDSE. The key change from the BDS procedure lies in smoothing the data on local servers followed by isotonization on the central server, an ‘SI’ (smoothing-isotonization) procedure. We note here that such ‘SI’ procedures and their converse (‘IS’) procedures have been studied in monotone function problems, though not in distributed computing environments and not under the heterogeneity setting of our paper. See, for example, (Mukerjee, 1988), (Mammen, 1991), (Van Der Vaart and Van Der Laan, 2003), (Anevski et al., 2006) and (Groeneboom et al., 2010).

The ideas in this paper also have certain connections to other work in the monotone function literature which are worth mentioning. (Zhang et al., 2001) study isotonic estimation of a decreasing density with histogram-type data based on i.i.d. data under a once differentiable assumption on the density. The domain of the density is split into bins, and the counts in each bin are available. When the number of bins grows at a rate faster than $n^{1/3}$, Theorem 4.6 of this paper shows that the isotonic estimate based on binned data recovers the Chernoff-type asymptotic distribution of the classical Grenander estimator. A similar phenomenon transpires in our problem. The $(C_{\ell k}, T_{\ell k})$ pair records the number of observations in the bin $I_k$ and the sum of the responses in that bin respectively, for the $\ell$th server. Once these are transferred to the central server, we sum across $\ell$ to find the total number of observations in $I_k$ and the sum of the responses corresponding to all those observations and construct our isotonic estimator using these statistics.

In our problem, $K$ grows faster than $N^{1/3}$ and we obtain a Chernoff limit for the pooled estimator. This naturally raises the question as to how the number of bins $K$ for the smoothing step on the local servers would influence the distribution of the estimators developed in this paper. When $N^{1/3} = o(K)$, the grid is sufficiently dense and the corresponding bins sufficiently small, so that our isotonized regressogram estimator recovers the asymptotics of the classical, i.e. global isotonic regression estimator, but this will no longer be the case when $K \sim N^{1/3}$ or $K = o(N^{1/3})$. When $K \sim N^{1/3}$, the results of (Zhang et al., 2001) (Theorem 3.3 and Corollary 4.4) and (Tang et al., 2012) (Theorem 3.7) who study monotone function estimation with covariates supported on a grid indicate that the limit distribution of the isotonized regressogram estimator at a point will neither be normal, nor will it be given by Chernoff’s distribution. When $K = o(N^{1/3})$, the grid is sparse enough so that the regressogram estimates are ordered with probability increasing to one, so that the isotonized regressogram estimator agrees with the original estimator with increasing probability, and the results in (Zhang et al., 2001) (Theorem 4.1) and (Tang et al., 2012) (Theorem 3.1) suggest an asymptotic normal distribution for our proposed estimator. We do not go into a full investigation of the details of these asymptotics in the distributed setting, however, since this is not relevant to the goal of the current work: produce a pooled estimator whose properties mimic the global estimator.

We believe that similar estimators can be proposed for distributed convex regression. For convex regression, a BDS type estimator is expected to fail completely, since the global convex
least squares estimator is itself asymptotically biased, as suggested by the extensive simulation experiments in [Azadbakhsh et al., 2014]. However, a convexified regressogram estimator in the spirit of the one considered in this paper, ought to be able to recover the properties of the global convex LS estimator provided \( K \) is selected appropriately: we conjecture that in the convex case \( K \) should be taken to be \( K^{-1} = o(N^{-1/5}) \). This will provide a possible avenue for future research.

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## 5 Appendix

**Lemma 5.1** Assume that the distribution function \( F_X \) taken from (10) has a density function \( f_X \) on \([0, 1]\) that satisfies (11) for some positive numbers \( C_1, C_2 \). Let \( F_N \) be the empirical distribution function taken from (5) and let \( F_{N}^{-1} \) be the corresponding empirical quantile function. We then have

\[
\mathbb{P} \left( \sup_{t \in [0,1]} |F_N(t) - F_X(t)| > x \right) \leq 2 \sum_{j=1}^{m} \exp(-2n_j x^2) \tag{28}
\]

and

\[
\mathbb{P} \left( \sup_{t \in [0,1]} |F_{N}^{-1}(t) - F_{X}^{-1}(t)| > x \right) \leq 4 \sum_{j=1}^{m} \exp(-2n_j C_1^2 x^2) \tag{29}
\]

for all \( N \) and \( x > 0 \).

**Proof** Let \( F_{Xj} \) denote the common distribution function of the \( X_i \)’s from sample \( j \) and denote by \((X_{ji}, Y_{ji})\), \( i = 1, \ldots, n_j \) the observations from sample \( j \). It follows from the triangle inequality that

\[
\sup_{t \in [0,1]} |F_N(t) - F_X(t)| \leq \sum_{j=1}^{m} \frac{n_j}{N} \sup_{t \in [0,1]} \left| \frac{1}{n_j} \sum_{i=1}^{n_j} \mathbb{1}_{X_{ji} \leq t} - F_{Xj}(t) \right|
\]

where we recall that \( \sum_{j=1}^{m} n_j = N \). Hence, for all \( x > 0 \) we have

\[
\mathbb{P} \left( \sup_{t \in [0,1]} |F_N(t) - F_X(t)| > x \right) \leq \mathbb{P} \left( \sup_{t \in [0,1]} \left| \frac{1}{n_j} \sum_{i=1}^{n_j} \mathbb{1}_{X_{ji} \leq t} - F_{Xj}(t) \right| > x \text{ for some } j \in \{1, \ldots, m\} \right)
\]

\[
\leq \sum_{j=1}^{m} \mathbb{P} \left( \sup_{t \in [0,1]} \left| \frac{1}{n_j} \sum_{i=1}^{n_j} \mathbb{1}_{X_{ji} \leq t} - F_{Xj}(t) \right| > x \right). \]
Since for all fixed \( j \), the random variables \( X_{ji} \), \( i = 1, \ldots, n_j \) are i.i.d. with distribution function \( F_{Xj} \), it follows from Corollary 1 in (Massart, 1990) that

\[
\mathbb{P} \left( \sup_{t \in [0,1]} \left| \frac{1}{n_j} \sum_{i=1}^{n_j} I_{X_{ji} \leq t} - F_{Xj}(t) \right| > x \right) \leq 2 \exp(-2n_jx^2).
\]

Combining the two preceding displays completes the proof of (28).

Now, consider (29). Since \( f_X \) is supported on \([0,1]\), both \( F_N^{-1} \) and \( F_X^{-1} \) take values in \([0,1]\) so the sup-distance between those functions is less than or equal to one. This means that the probability on the left hand side of (29) is equal to zero for all \( x \geq 1 \). Hence, it suffices to prove (29) for \( x \in (0,1) \). As is customary, we use the notation \( y_+ = \max(y,0) \) and \( y_- = -\min(y,0) \) for all real numbers \( y \). This means that \(|y| = \max(y_-, y_+). \) Recall the switching relation for the empirical distribution and empirical quantile functions: for arbitrary \( a \in [0,1] \) and \( t \in [0,1] \), we have

\[
F_N(a) \geq t \iff a \geq F_N^{-1}(t).
\]

For all \( x \in (0,1) \) we then have

\[
\mathbb{P} \left( \sup_{t \in [0,1]} (F_N^{-1}(t) - F_X^{-1}(t))_+ > x \right) = \mathbb{P} \left( \exists t \in [0,1] : F_N^{-1}(t) > x + F_X^{-1}(t) \right)
\]

\[
= \mathbb{P} \left( \exists t \in [0,1] : t > F_N(x + F_X^{-1}(t)) \right).
\]

Using \( t = F_X(F_X^{-1}(t)) \) together with the change of variable \( u = x + F_X^{-1}(t) \) we obtain

\[
\mathbb{P} \left( \sup_{t \in [0,1]} (F_N^{-1}(t) - F_X^{-1}(t))_+ > x \right) \leq \mathbb{P} (\exists u \geq x : F_X(u-x) > F_N(u))
\]

\[
= \mathbb{P} (\exists u \in (x,1) : F_X(u-x) > F_N(u)).
\]

For the last equality, we use the fact that \( F_X(u-x) \leq 1 = F_N(u) \) for all \( u \geq 1 \), and \( F_X(u-x) = 0 \leq F_N(u) \) for all \( u \leq x \). With \( C_1 \) taken from (11) we have \( F_X(u-x) < F_X(u) - C_1x \) for all \( x \in (0,1) \) and \( u \in (x,1) \). Combining this to the previous display yields

\[
\mathbb{P} \left( \sup_{t \in [0,1]} (F_N^{-1}(t) - F_X^{-1}(t))_+ > x \right) \leq \mathbb{P} (\exists u \in (x,1) : F_X(u) - F_N(u) > C_1x)
\]

\[
\leq \mathbb{P} \left( \sup_{u \in \mathbb{R}} |F_X(u) - F_N(u)| > C_1x \right)
\]

\[
\leq 2 \sum_{j=1}^{m} \exp(-2n_jC_1^2x^2).
\]

For the last inequality, we used (28). On the other hand, for all \( x \in (0,1) \) we have

\[
\mathbb{P} \left( \sup_{t \in [0,1]} (F_N^{-1}(t) - F_X^{-1}(t))_- > x \right) \leq \mathbb{P} (\exists t \in [0,1] : F_N^{-1}(t) < F_X^{-1}(t) - x)
\]

\[
\leq \mathbb{P} (\exists u \in (x,1) : F_N^{-1}(F_X(u)) \leq u - x).
\]
using the change of variable $u = F_X^{-1}(t)$. Hence, with the switching relation we obtain

$$
P \left( \sup_{t \in [0,1]} (F_N^{-1}(t) - F_X^{-1}(t))_+ > x \right) \leq P \left( \exists u \in (x,1) : F_X(u) \leq F_N(u - x) \right)
$$

$$\leq P \left( \exists u \in (x,1) : F_X(u - x) + C_1 x < F_N(u - x) \right),$$

using that $F_X(u - x) < F_X(u) - C_1 x$ for all $x \in (0,1)$ and $u \in (x,1)$. Using again (28) together with the change of variable $v = u - x$, we arrive at

$$P \left( \sup_{t \in [0,1]} (F_N^{-1}(t) - F_X^{-1}(t))_+ > x \right) \leq P \left( \sup_{v \in \mathbb{R}} |F_X(v) - F_N(v)| > C_1 x \right)
$$

$$\leq 2 \sum_{j=1}^{m} \exp(-2n_j C_1^2 x^2).$$

Combining the previous display with (31) completes the proof of (29) since $|y| \leq y_- + y_+$ for all $y \in \mathbb{R}$. \qed

**Lemma 5.2** Under the assumptions of Theorem 3.6, for all $p > 0$ there exists $K_p > 0$ such that for all $N$,

$$E \left( \sup_{t \in [0,1]} |F_N(t) - F_X(t)|^p \right) \leq K_p N^{-p/2}. \quad (32)$$

**Proof.** It follows from the Fubini theorem that

$$E \left( \sup_{t \in [0,1]} |F_N(t) - F_X(t)|^p \right) = \int_0^\infty P \left( \sup_{t \in [0,1]} |F_N(t) - F_X(t)|^p > x \right) dx
$$

$$\quad = \int_0^\infty pa^{p-1} \left( \sup_{t \in [0,1]} |F_N(t) - F_X(t)| > x \right) dx.$$

Combining this with (28) and the fact that a probability cannot be larger than one then yields

$$E \left( \sup_{t \in [0,1]} |F_N(t) - F_X(t)|^p \right) \leq N^{-p/3} + 2 \sum_{j=1}^{m} \int_{N^{-1/3}}^\infty pa^{p-1} \exp(-2n_j x^2) dx
$$

$$\quad \leq N^{-p/3} + 2N \int_{N^{-1/3}}^\infty pa^{p-1} \exp(-2N^{2/3}(\log N)^3 x^2) dx$$

for sufficiently large $N$, where we used (13) for the last inequality. The result follows by computing the integral on the right-hand side. \qed

**Proof of Lemma 3.2** For all $a \in \mathbb{R}$ and $u \in \{ \overline{a}, \ldots, \overline{K} \}$ such that $|u - g(a)| \geq N^{-1/3}$, define

$$e(a,u) = E^X \Lambda_N(u) - E^X \Lambda_N \left( \frac{[Kg(a)]}{K} \right) - a \left( F_N(u) - F_N \left( \frac{[Kg(a)]}{K} \right) \right). \quad (33)$$
By definition of $\Lambda_N$ we have

$$e(a, u) = \frac{1}{N} \sum_{i=1}^{N} \mu(X_i) \left( \mathbb{1}_{X_i \leq u} - \mathbb{1}_{X_i \leq [Kg(a)]K^{-1}} \right) - a \left( F_N(u) - F_N \left( \frac{[Kg(a)]}{K} \right) \right).$$

Now, $X_i \neq [Kg(a)]K^{-1}$ for all $i$, almost surely since $X_i$ has a continuous distribution function, so (12) implies that

$$\left| \mu(X_i) - \mu \left( \frac{[Kg(a)]}{K} \right) \right| \geq \left| X_i - \frac{[Kg(a)]}{K} \right| C_3,$$

implying that

$$e(a, u) \leq \left( \mu \left( \frac{[Kg(a)]}{K} \right) - a \right) \left( F_N(u) - F_N \left( \frac{[Kg(a)]}{K} \right) \right) - C_3f(a, u)$$

with a decreasing function $\mu$, where

$$f(a, u) = \frac{1}{N} \sum_{i=1}^{N} \left( X_i - \frac{[Kg(a)]}{K} \right) \left( \mathbb{1}_{X_i \leq u} - \mathbb{1}_{X_i \leq [Kg(a)]K^{-1}} \right).$$

Using again (12), we obtain that for all $a \in [\mu(1), \mu(0)]$,

$$\left| \mu \left( \frac{[Kg(a)]}{K} \right) - a \right| = \left| \mu \left( \frac{[Kg(a)]}{K} \right) - \mu \circ g(a) \right| \leq C_4K^{-1}.$$ 

(36)

On the other hand, since $F_X$ has a bounded derivative that satisfies (11) we have

$$\left| F_X(u) - F_X \left( \frac{[Kg(a)]}{K} \right) \right| \leq |F_X(u) - F_X(g(a))| + \left| F_X(g(a)) - F_X \left( \frac{[Kg(a)]}{K} \right) \right|$$

$$\leq C_2 \left( |u - g(a)| + K^{-1} \right) \leq 2C_2|u - g(a)|$$

(37)

for sufficiently large $N$, using that $K^{-1} = o(N^{-1/3})$ whereas $|u - g(a)| \geq N^{-1/3}$ for the last inequality. Next, since $m \leq N$, it follows from (25) in the appendix that

$$\mathbb{P} \left( \sup_{t \in [0,1]} |F_N(t) - F_X(t)| > x \right) \leq 2N \exp \left( -2x^2 \min_{1 \leq j \leq m} n_j \right)$$

for all $x > 0$. With (13), we obtain

$$\mathbb{P} \left( \sup_{t \in [0,1]} |F_N(t) - F_X(t)| > x \right) \leq 2N \exp \left( -2x^2 N \lambda \right)$$

25
for all $x > 0$. With $\tilde{E}_N$ the event that
\[
\sup_{t \in [0,1]} |F_N(t) - F_X(t)| \leq C_2 N^{-1/3} (\log N)^{-1}
\] (38)
we conclude from the previous display that
\[
1 - \mathbb{P}(\tilde{E}_N) \leq 2N \exp \left( -2C_2^2 N^{1/3} \lambda (\log N)^{-2} \right) \ll N^{-\theta},
\] (39)
where we used (13) for the last claim. Combining (34), (36) and (37) proves that on $\tilde{E}_N$, we have
\[
e(a, u) \leq C_4 K^{-1} \left( |F_X(u) - F_X \left( \frac{K\lambda(a)}{K} \right) | + 2 \sup_{t \in [0,1]} |F_N(t) - F_X(t)| \right) - C_3 f(a, u)
\] 
\[
\leq 2C_2 C_4 K^{-1} (|u - g(a)| + N^{-1/3}) - C_3 f(a, u)
\] (40)
for all $a \in [\mu(1), \mu(0)]$. The inequality in (34) holds also for $a \not\in [\mu(1), \mu(0)]$, and in that case,
\[
\frac{K\lambda(a)}{K} = g(a) = \begin{cases} 0 & \text{if } a > \mu(0) \
1 & \text{if } a < \mu(1), \end{cases}
\]
implying that
\[
\left( \mu \left( \frac{K\lambda(a)}{K} \right) - a \right) \left( F_X(u) - F_N \left( \frac{K\lambda(a)}{K} \right) \right) \leq 0.
\]
Hence, the inequality in (40) holds for all $a \in \mathbb{R}$ and $u \in \{x_0, \ldots, x_K\}$. Using that $K^{-1} = o(N^{-1/3})$ whereas $|u - g(a)| \geq N^{-1/3}$, we conclude that on $\tilde{E}_N$,
\[
e(a, u) \leq o(u - g(a))^2 - C_3 f(a, u)
\]
uniformly over all $a$ and $u$ such that $|u - g(a)| \geq N^{-1/3}$. Hence, it suffices to prove that with $f(a, u)$ taken from (35), there exists $\hat{c} > 0$ that only depends on $C_1$ such that on an event $E_N$ whose probability is larger than $1 - N^{-\theta}$, and such that $E_N \subset \tilde{E}_N$, we have
\[
f(a, u) \geq \hat{c} (u - g(a))^2 \text{ for all } a \in \mathbb{R}, \ u \in \{x_0, \ldots, x_K\} \text{ such that } |u - g(a)| \geq N^{-1/3}.
\] (41)
Similar to (39), if follows from (29) in the appendix that
\[
\mathbb{P} \left( \sup_{x \in [0,1]} |F_N^{-1}(x) - F_X^{-1}(x)| > \frac{N^{-1/3}}{\log N} \right) \leq 4N \exp \left( -2C_1^2 N^{1/3} (\log N)^{-2} \lambda \right) \ll N^{-\theta}.
\] (42)
In the sequel, we consider
\[
E_N = \tilde{E}_N \cap \left\{ \sup_{x \in [0,1]} |F_N^{-1}(x) - F_X^{-1}(x)| \leq N^{-1/3} (\log N)^{-1} \right\}.
\]
It follows from (39) and (42) that \(1 - \mathbb{P}(\mathcal{E}_N) \ll N^{-\delta}\) so in particular, \(\mathbb{P}(\mathcal{E}_N) \geq 1 - N^{-\delta}\) for sufficiently large \(N\). It remains to show that (41) holds on \(\mathcal{E}_N\). Since \(X_1, \ldots, X_N\) are independent with a continuous distribution function, they are all distinct from each other and for all \(i\), there exists a (unique) random \(j\) such that \(X_i = F_N^{-1}(j/N)\), where \(F_N^{-1}\) is the empirical quantile function corresponding to \(X_1, \ldots, X_N\). Hence, reordering the terms in the sum in (35), we obtain that

\[
\begin{align*}
f(a, u) &= \frac{1}{N} \sum_{i=1}^{N} \left( F_N^{-1}(i/N) - \frac{[Kg(a)]}{K} \right) \left( \mathbb{I}_{F_N^{-1}(iN-1) \leq u} - \mathbb{I}_{F_N^{-1}(iN-1) \leq [Kg(a)K^{-1}]} \right) \\
&= \frac{1}{N} \sum_{i=1}^{N} \left( F_N^{-1}(i/N) - \frac{[Kg(a)]}{K} \right) \left( \mathbb{I}_{iN-1 \leq F_N(u)} - \mathbb{I}_{iN-1 \leq F_N([Kg(a)K^{-1}])} \right).
\end{align*}
\]

Using that \(F_N^{-1}\) is constant on all intervals \(((i-1)N^{-1}, iN^{-1}]\) we arrive at

\[
f(a, u) = \int_{F_N([Kg(a)K^{-1}])}^{F_N(u)} \left( F_N^{-1}(x) - \frac{[Kg(a)]}{K} \right) \, dx.
\]

Hence, on \(\mathcal{E}_N\) we have

\[
\begin{align*}
|f(a, u) - \int_{F_N([Kg(a)K^{-1}])}^{F_N(u)} \left( F_N^{-1}(x) - \frac{[Kg(a)]}{K} \right) \, dx| &\leq |F_N(u) - F_N([Kg(a)K^{-1}]| \times \sup_{x \in [0, 1]} |F_N^{-1}(x) - F_N^{-1}(x)| \\
&\leq C_2 \left( |u - g(a)| + K^{-1} + 2N^{-1/3}(\log N)^{-1}\right) N^{-1/3}(\log N)^{-1}
\end{align*}
\]

for all \(a, u\). Hence,

\[
\begin{align*}
&\left| f(a, u) - \int_{F_N([Kg(a)K^{-1}])}^{F_N(u)} \left( F_N^{-1}(x) - g(a) \right) \, dx \right| \\
&\leq C_2 \left( |u - g(a)| + K^{-1} + 2N^{-1/3}(\log N)^{-1}\right) N^{-1/3}(\log N)^{-1} \\
&\quad + \int_{F_N([Kg(a)K^{-1}])}^{F_N(u)} \left( F_N^{-1}(x) - \frac{[Kg(a)]}{K} \right) \, dx - \int_{F_N([Kg(a)K^{-1}])}^{F_N(u)} \left( F_N^{-1}(x) - g(a) \right) \, dx
\end{align*}
\]

It follows that

\[
\begin{align*}
&\left| f(a, u) - \int_{F_N([Kg(a)K^{-1}])}^{F_N(u)} \left( F_N^{-1}(x) - g(a) \right) \, dx \right| \\
&\leq C_2 \left( |u - g(a)| + K^{-1} + 2N^{-1/3}(\log N)^{-1}\right) N^{-1/3}(\log N)^{-1} + K^{-1} |F_N(u) - F_N(g(a))| \\
&\quad + \int_{F_N([Kg(a)K^{-1}])}^{F_N(u)} \left( F_N^{-1}(x) - \frac{[Kg(a)]}{K} \right) \, dx - \int_{F_N([Kg(a)K^{-1}])}^{F_N(u)} \left( F_N^{-1}(x) - \frac{[Kg(a)]}{K} \right) \, dx.
\end{align*}
\]
Now, on $\mathcal{E}_N$ we also have

$$
\left| F_X^{-1}(x) - \frac{[Kg(a)]}{K} \right| = \left| F_X^{-1}(x) - F_X^{-1} \circ F_X \left( \frac{[Kg(a)]}{K} \right) \right|
\leq \frac{1}{C_1} \left| x - F_X \left( \frac{[Kg(a)]}{K} \right) \right|
\leq \frac{1}{C_1} \left( \left| F_N(u) - F_N \left( \frac{[Kg(a)]}{K} \right) \right| + C_2 N^{-1/3}(\log N)^{-1} \right),
$$

for all $x$ lying between $F_N(u)$ and $F_N([Kg(a)]^{-1})$. For such $x$’s, we obtain on $\mathcal{E}_N$ that

$$
\left| F_X^{-1}(x) - \frac{[Kg(a)]}{K} \right| \leq \frac{1}{C_1} \left( \left| F_X(u) - F_X \left( \frac{[Kg(a)]}{K} \right) \right| + 3C_2 N^{-1/3}(\log N)^{-1} \right)
\leq \frac{3C_2}{C_1} \left( |u - g(a)| + K^{-1} + N^{-1/3}(\log N)^{-1} \right)
$$

for all $a$ and $u$, for sufficiently large $N$. Therefore, with $K \geq 1$ we obtain on $\mathcal{E}_N$ that

$$
\left| f(a, u) - \int_{F_X(g(a))}^{F_X(u)} \left( F_X^{-1}(x) - g(a) \right) dx \right|
\leq 2C_2 \left( |u - g(a)| + K^{-1} + 2N^{-1/3}(\log N)^{-1} \right) N^{-1/3}(\log N)^{-1}
+ \frac{3C_2}{C_1} \left( |u - g(a)| + K^{-1} + N^{-1/3}(\log N)^{-1} \right) \left( 2 \sup_{u \in [0, 1]} |F_N(u) - F_X(u)| + C_2 K^{-1} \right)
= O \left( |u - g(a)| + K^{-1} + N^{-1/3}(\log N)^{-1} \right) \left( N^{-1/3}(\log N)^{-1} + K^{-1} \right)
$$

on $\mathcal{E}_N$, uniformly over $a \in \mathbb{R}$ and $u \in \{\bar{a}_0, \ldots, \bar{a}_K\}$. Now, we can do the change of variable $t = F_X^{-1}(x)$ to get on $\mathcal{E}_N$ that

$$
f(a, u) = \int_{g(a)}^{u} (t - g(a)) f_X(t) dt
+ O \left( |u - g(a)| + K^{-1} + N^{-1/3}(\log N)^{-1} \right) \left( N^{-1/3}(\log N)^{-1} + K^{-1} \right) \quad (43)
$$

uniformly over $a \in \mathbb{R}$ and $u \in \{\bar{a}_0, \ldots, \bar{a}_K\}$. Here,

$$
\int_{g(a)}^{u} (t - g(a)) f_X(t) dt \geq C_1 \int_{g(a)}^{u} (t - g(a)) dt
$$

where $C_1$ is taken from (111), for all $a, u$. Since it is assumed that $K^{-1} = o(N^{-1/3})$, we conclude that on $\mathcal{E}_N$,

$$
f(a, u) \geq \frac{C_1}{2} (u - g(a))^2 + o((g(a) - u)^2),
$$

28
where the small o-term is uniform over all \( u \) and \( a \) such that \(|u - g(a)| \geq N^{-1/3}\).
Hence, (11) holds on \( \mathcal{E}_N \) provided that \( \tilde{c} < C_1/2 \) and \( N \) is sufficiently large. It follows that on \( \mathcal{E}_N \), for all sufficiently large \( N \),
\[
e(a, u) \leq o((u - g(a))^2 - C_3 \tilde{c}(g(a) - u)^2)
\]
where in view of the above proof, the small-o term can be chosen of the form
\[
o((u - g(a))^2 = 2C_2C_4K^{-1}(|u - g(a)| + N^{-1/3}).
\]
Therefore, for any \( c < C_3 \tilde{c} \), for all sufficiently large \( N \), \( e(a, u) \leq -c(g(a) - u)^2 \) on \( \mathcal{E}_N \). This completes the proof of the lemma.

**Proof of Lemma 3.4.** For all \( a \not\in [\mu(1), \mu(0)] \) and \( u \in \{\tau_0, \ldots, \tau_K\} \), define \( e(a, u) \) as in (34). We then have (34) where
\[
\frac{[Kg(a)]}{K} = g(a) = \begin{cases} 0 & \text{if } a > \mu(0) \\ 1 & \text{if } a < \mu(1). \end{cases}
\]
and \( f \) is given by (35). Note that (36) is no longer true for \( a \not\in [\mu(1), \mu(0)] \) since in such a case, \( a \neq \mu \circ g(a) \). Instead, we will use
\[
\left( \mu \left( \frac{[Kg(a)]}{K} \right) - a \right) \left( F_N(u) - F_N \left( \frac{[Kg(a)]}{K} \right) \right) = \begin{cases} (\mu(0) - a)F_N(u) & \text{if } a > \mu(0) \\ (\mu(1) - a)(F_N(u) - 1) & \text{if } a < \mu(1), \end{cases}
\]
using that \( F_N(0) = 0 \) and \( F_N(1) = 1 \). Since \( f(a, u) \geq 0 \) for all \( a, u \) (34) yields
\[
e(a, u) \leq \begin{cases} (\mu(0) - a)F_N(u) & \text{if } a > \mu(0) \\ (\mu(1) - a)(F_N(u) - 1) & \text{if } a < \mu(1) \end{cases}
\]
(44)
for all \( a \not\in [\mu(1), \mu(0)] \) and \( u \in \{\tau_0, \ldots, \tau_K\} \).

Since \( \Lambda_N(\tau_0) - aF_N(\tau_0) = 0 \), it follows from the definition of \( V_N \) that the following inequalities hold for all \( x > 0 \) and \( a > \mu(0) \):
\[
\mathbb{P}(V_N(a) \geq x) \leq \mathbb{P}\left( \max_{u \in (\tilde{F}_N(\tau_0), \ldots, \tilde{F}_N(\tau_K))} \{\Lambda_N \circ \tilde{F}_N^{-1}(u) - au \geq 0\} \right) 
\]
\[
= \mathbb{P}\left( \max_{u \in (\tilde{F}_N(\tau_0), \ldots, \tilde{F}_N(\tau_K))} \{M_N \circ \tilde{F}_N^{-1}(u) + e(a, \tilde{F}_N^{-1}(u)) \geq 0\} \right)
\]
where \( M_N(u) = \Lambda_N(u) - \mathbb{E}^N(\Lambda_N(u)) \) takes the form (17). The first inequality in (44) then yields
\[
\mathbb{P}(V_N(a) \geq x) \leq \mathbb{P}\left( \max_{u \in (\tilde{F}_N(\tau_0), \ldots, \tilde{F}_N(\tau_K))} \{M_N \circ \tilde{F}_N^{-1}(u) + (\mu(0) - a)u \geq 0\} \right) 
\]
\[
\leq \sum_{k \geq 0} \mathbb{P}\left( \max_{u \in [x2^k, x2^{k+1}]} \{M_N \circ \tilde{F}_N^{-1}(u) \geq (a - \mu(0))x2^k\} \right).
\]
Let $p \geq 2$ and $\sigma > 0$ such that $\mathbb{E}[\varepsilon_i^p | X_i] \leq \sigma^p$ for all $i$, almost surely. The process $M_n$ is a centered martingale under $\mathbb{P}^X$ which, according to Theorem 3 in [Rosenthal, 1970], satisfies

$$\mathbb{E}^X |M_N(u)|^p \leq \frac{A_p}{N^p} \max \left\{ \sum_{i=1}^N \mathbb{E}^X |\varepsilon_i|^p \mathbb{1}_{X_i \leq u}; \left( \sum_{i=1}^N \mathbb{E}^X |\varepsilon_i|^2 \mathbb{1}_{X_i \leq u} \right)^{p/2} \right\}$$

$$\leq \frac{A_p \sigma^p}{N^p} \max \left\{ NF_N(u); (NF_N(u))^{p/2} \right\}$$

$$\leq \frac{A_p \sigma^p F_N(u)}{N^{p/2}}$$

for all $u \in [0, 1]$ and $A_p = (p/2)^{p/2}2^p + \sigma^p/4$. For the penultimate inequality, we used that $\mathbb{E}^X |\varepsilon_i|^2 \leq (\mathbb{E}^X |\varepsilon_i|^p)^{2/p}$ thanks to the Hölder inequality whereas for the last inequality, we used that $N \leq N^{p/2}$ and $F_N^{p/2}(u) \leq F_N(u)$. Combining the two preceding displays with the Doob inequality yields that for all $x > 0$,

$$\mathbb{P}(V_N(a) \geq x) \leq \sum_{k \geq 0} \mathbb{E} \left[ \mathbb{P}^X \left( \max_{u \in \{F_N(x), \ldots, F_N(x_k)\}, x \in [x^k, x^{k+1}]} \{M_N \circ \tilde{F}_N^{-1}(u) \geq (a - \mu(0))x^{2k} \} \right) \right]$$

$$\leq \sum_{k \geq 0} \mathbb{E} \left[ \frac{A_p \sigma^p F_N(x^{2k+1})}{N^{p/2}(a - \mu(0))^p(x^{2k})^p} \right].$$

With $C_2$ taken from (11) we conclude that for all $x > 0$,

$$\mathbb{P}(V_N(a) \geq x) \leq \sum_{k \geq 0} \frac{A_p \sigma^p F_N(x^{2k+1})}{N^{p/2}(a - \mu(0))^p(x^{2k})^p} \leq \sum_{k \geq 0} \frac{2A_p C_2 \sigma^p}{N^{p/2}(a - \mu(0))^p(x^{2k})^{p-1}}.$$

Since $C := 2A_p C_2 \sigma^p \sum_{k \geq 0} 2^{-k(p-1)}$ is finite, we conclude that

$$\mathbb{P}(V_N(a) \geq x) \leq \frac{C}{N^{p/2}(a - \mu(0))^px^{p-1}},$$

which proves the first assertion. For the second assertion, since $\pi_K = \tilde{F}_N(\pi_K) = 1$, we write for $a < \mu(1)$ and $x > 0$:

$$\mathbb{P}(1 - V_N(a) \geq x)$$

$$\leq \mathbb{P} \left( \max_{u \in \{\tilde{F}_N(x^{k_0}), \ldots, \tilde{F}_N(x^{k_1})\}, 1 - u \geq x} \{A_N \circ \tilde{F}_N^{-1}(u) - au \} \geq A_N(1) - a \right)$$

$$= \mathbb{P} \left( \max_{u \in \{\tilde{F}_N(x^{k_0}), \ldots, \tilde{F}_N(x^{k_1})\}, 1 - u \geq x} \{M_N \circ \tilde{F}_N^{-1}(u) - M_N(1) + e(a, \tilde{F}_N^{-1}(u)) \} \geq 0 \right).$$
The first inequality in (44) then yields

\[ P(1 - V_N(a) \geq x) \leq P \left( \max_{u \in \{\bar{F}_N(x_0), \ldots, F_N(x_K)\}} 1_{u \geq x} \left\{ M_N \circ \bar{F}_N^{-1}(u) - M_N(1) - (\mu(1) - a)(1 - u) \right\} \geq 0 \right) \leq \sum_{k \geq 0} P \left( \max_{u \in \{\bar{F}_N(x_0), \ldots, F_N(x_K)\}} 1_{u \geq x^2k+1} \left\{ M_N \circ \bar{F}_N^{-1}(u) - M_N(1) \right\} \geq (\mu(1) - a)x^2k \right), \]

and we use the Doob inequality, similar as above. Details are omitted. \[ \square \]

**Proof of Theorem 3.6** It follows from (44) together with Lemma 3.3 that with probability tending to one,

\[ N^{1/3}(U_N(a) - g(a)) = \max_{u \in H_N} \left\{ \Lambda_N(u) - aF_N(g(a) + N^{-1/3}u) \right\} \]

where \( H_N \) is the set of all \( u \in \mathbb{R} \) such that \( g(a) + N^{-1/3}u \in \{x_0, \ldots, x_K\} \) and \( |u| \leq v_N \), where \( v_N \) is an arbitrary sequence that diverges to infinity as \( N \to \infty \). In the sequel, we consider a sequence \( v_N \) such that \( v_N \leq \log N \) for all \( N \). Hence, with probability that tends to one we have

\[ N^{1/3}(U_N(a) - g(a)) = \max_{u \in H_N} \left\{ N^{2/3} \left( M_N(g(a) + N^{-1/3}u) - M_N \left( \frac{[Kg(a)]}{K} \right) \right) + N^{2/3}e(a, g(a) + N^{-1/3}u) \right\} \]  

where \( M_N(u) = \Lambda_N(u) - E_N(\Lambda_N(u)) \) for all \( u \in \{x_0, \ldots, x_K\} \) and \( e \) is taken from (33), that is

\[ e(a, g(a) + N^{-1/3}u) = \frac{1}{N} \sum_{i=1}^{N} \mu(X_i) \left( \mathbb{1}_{X_i \leq g(a) + N^{-1/3}u} - \mathbb{1}_{X_i \leq [Kg(a)]K^{-1}} \right) - a \left( F_N(g(a) + N^{-1/3}u) - F_N \left( \frac{[Kg(a)]}{K} \right) \right). \]  

(46)

We extend \( M_N \) and \( e(a, \cdot) \) as constant functions in between two consecutive points in \( H_N \) so that

\[ N^{1/3}(U_N(a) - g(a)) = \max_{|u| \leq v_N} \left\{ N^{2/3} \left( M_N(g(a) + N^{-1/3}u) - M_N \left( \frac{[Kg(a)]}{K} \right) \right) + N^{2/3}e(a, g(a) + N^{-1/3}u) \right\} + o_p(1). \]  

(47)

Now, since \( a = \mu(t) + N^{-1/3}x \) for some fixed \( x \in \mathbb{R} \) and \( t \in (0, 1) \), and \( g' = 1/\mu' \circ g \) on \( (\mu(1), \mu(0)) \) is bounded by assumption, we have

\[ g(a) = t + O(N^{-1/3}). \]  

(48)
Hence, for sufficiently large \( N \), every \( X_i \) that lies between \([Kg(a)] K^{-1}\) and \( g(a) + N^{-1/3} u \) for some \(|u| \leq v_N \) also lies in \([t - N^{-1/3} \log N, t + N^{-1/3} \log N]\). This implies that for all such \( X_i \)'s there exists \( \theta_i \in [t - N^{-1/3} \log N, t + N^{-1/3} \log N] \) such that

\[
\mu(X_i) = \mu\left( \frac{[Kg(a)]}{K} \right) + \left( X_i - \frac{[Kg(a)]}{K} \right) \mu'(\theta_i)
\]

\[
= \mu\left( \frac{[Kg(a)]}{K} \right) + \left( X_i - \frac{[Kg(a)]}{K} \right) \left( \mu'(t) + o(1) \right)
\]

where the small \( o \)-term is uniform, by continuity of \( \mu' \) over the compact interval \([t - N^{-1/3} \log N, t + N^{-1/3} \log N]\). Plugging this in (46), and using the notation \( f \) in (35), yields

\[
e(a, g(a) + N^{-1/3} u) = \left( \mu'(t) + o(1) \right) f(a, g(a) + N^{-1/3} u) + \left( \mu \left( \frac{[Kg(a)]}{K} \right) - a \right) \left( F_N(g(a) + N^{-1/3} u) - F_N \left( \frac{[Kg(a)]}{K} \right) \right)
\]

(50)

It can be seen from the proof of Lemma 3.2 that (35) holds on the event \( \mathcal{E}_N \), whose probability tends to one as \( N \to \infty \), implying that

\[
F_N(g(a) + N^{-1/3} u) - F_N \left( \frac{[Kg(a)]}{K} \right) = F_X(g(a) + N^{-1/3} u) - F_X \left( \frac{[Kg(a)]}{K} \right) + O_p(N^{-1/3}(\log N)^{-1})
\]

\[
= O_p(N^{-1/3} v_N + K^{-1} + N^{-1/3}(\log N)^{-1})
\]

\[
= O_p(N^{-1/3} v_N)
\]

uniformly over \( u \in H_N \). Since (36) holds for all \( a \in [\mu(1), \mu(0)] \), combining the two preceding displays yields

\[
e(a, g(a) + N^{-1/3} u) = \left( \mu'(t) + o(1) \right) f(a, g(a) + N^{-1/3} u) + O_p(K^{-1} N^{-1/3} v_N).
\]

Next, we invoke (43), that holds on the event \( \mathcal{E}_N \) uniformly over \( a \) and \( u \), to conclude that

\[
e(a, g(a) + N^{-1/3} u) = \left( \mu'(t) + o(1) \right) \int_{g(a)}^{g(a)+N^{-1/3} u} (z - g(a)) f_X(z)dz + o_p(N^{-2/3})
\]

uniformly over \( u \in H_N \), provided that \( v_N \ll \min\{\log N; N^{-1/3} K\} \). By assumption, \( N^{-1/3} K \) diverges to infinity as \( N \to \infty \), so we can find a sequence \( v_N \) that satisfies the above condition and that diverges to infinity as \( N \to \infty \), as required in the definition of \( H_N \). In the sequel, we consider a sequence \( v_N \) that satisfies the above conditions and in addition, the below condition:

\[
v_N \ll \left( \max \left\{ \sup_{|z-t| \leq N^{-1/3} \log N} |f_X(z) - f_\infty(z)|, \sup_{|z-t| \leq N^{-1/3} \log N} |f_\infty(t) - f_\infty(z)| \right\} \right)^{-1/2}.
\]

Note that by assumption, the right-hand side of the inequality in the above display diverges to infinity as \( N \to \infty \), which ensures existence of such a sequence \( v_N \). We then have

\[
e(a, g(a) + N^{-1/3} u) = \left( \mu'(t) + o(1) \right) \int_{g(a)}^{g(a)+N^{-1/3} u} (z - g(a)) f_\infty(z)dz + o_p(N^{-2/3}),
\]

32
using that for \( u \geq 0 \) (and similarly for \( u \leq 0 \),
\[
\left| \int_{g(a)}^{g(a)+N^{-1/3}u} (z - g(a)) (f_X(z) - f_\infty(z)) \, dz \right| \leq \int_{g(a)}^{g(a)+N^{-1/3}u} (z - g(a)) |f_X(z) - f_\infty(z)| \, dz
\]
uniformly for all \( |u| \leq v_N \), which implies
\[
\left| \int_{g(a)}^{g(a)+N^{-1/3}u} (z - g(a)) (f_X(z) - f_\infty(z)) \, dz \right| \leq \frac{N^{-2/3} v_N^2}{2} \sup_{|z-g(a)| \leq N^{-1/3} v_N} |f_X(z) - f_\infty(z)|
\]
thanks to (45), (51) and the assumption that \( v_n \ll \log N \). Similarly,
\[
\left| \int_{g(a)}^{g(a)+N^{-1/3}u} (z - g(a)) (f_\infty(z) - f_\infty(t)) \, dz \right| \leq \frac{N^{-2/3} v_N^2}{2} \sup_{|z-t| \leq N^{-1/3} \log N} |f_\infty(z) - f_\infty(t)|
\]
and therefore,
\[
e(a, g(a) + N^{-1/3} u) = \left( \mu'(t) + o(1) \right) \int_{g(a)}^{g(a)+N^{-1/3}u} (z - g(a)) f_\infty(z) \, dz + o_p(N^{-2/3})
\]
Hence we obtain
\[
N^{2/3} e(a, g(a) + N^{-1/3} u) = -\left( |\mu'(t)| + o(1) \right) f_\infty(t) \frac{u^2}{2} + o_p(1). \tag{51}
\]
On the other hand, with
\[
Z_N(u) = N^{2/3} \left( M_N(g(a) + N^{-1/3} u) - M_N \left( \frac{[K g(a)]}{K} \right) \right);
\]
where \( M_N \) is as defined in (17) for all \( u \in \{ \bar{x}_0, \ldots, \bar{x}_K \} \), we have
\[
Z_N(u) = N^{-1/3} \sum_{j=1}^{m} \sum_{i=1}^{n_j} \varepsilon_{ji} \left( 1_{X_{ji} \leq g(a) + N^{-1/3} u} - 1_{X_{ji} \leq [K g(a)] K^{-1}} \right) \tag{52}
\]
where we denote by \((X_{ji}, Y_{ji})\), \(i = 1, \ldots, n_j\) the observations from sample \(j\), for \(j = 1, \ldots, m\), and 
\(\varepsilon_{ji} = Y_{ji} - \mu(X_{ji})\). Note that the process \(Z_N\) is centered and has been extended to \(\mathbb{R}\) by being constant in between two consecutive points in \(H_N\). For all \(u \geq v \geq 0\) in \(H_N\) we have

\[
\mathbb{E}[Z_N(u)Z_N(v)] = N^{-2/3} \sum_{j=1}^{m} \sum_{i=1}^{n_j} \mathbb{E}\left[\varepsilon_{ji}^2 1_{[Kg(a)]K^{-1} < X_{ji} \leq g(a) + N^{-1/3}u} 1_{[Kg(a)]K^{-1} < X_{ji} \leq g(a) + N^{-1/3}v}\right]
\]

\[
= N^{-2/3} \sum_{j=1}^{m} \sum_{i=1}^{n_j} \mathbb{E}\left[\sigma_j^2(X_{ji}) 1_{[Kg(a)]K^{-1} < X_{ji} \leq g(a) + N^{-1/3}v}\right],
\]

where the last equality is obtained by conditioning with respect to \(X_{ji}\) and using that \(u \geq v \geq 0\). With \(u, v\) fixed, this implies that

\[
\mathbb{E}[Z_N(u)Z_N(v)] = N^{-2/3} \sum_{j=1}^{m} \sum_{i=1}^{n_j} \mathbb{E}\left[\sigma_j^2(t) 1_{[Kg(a)]K^{-1} < X_{ji} \leq g(a) + N^{-1/3}v}\right] + o(1)
\]

using that for \(u, v \in H_N\)

\[
\mathbb{E}[Z_N(u)Z_N(v)] - N^{-2/3} \sum_{j=1}^{m} \sum_{i=1}^{n_j} \mathbb{E}\left[\sigma_j^2(t) 1_{[Kg(a)]K^{-1} < X_{ji} \leq g(a) + N^{-1/3}v}\right]
\]

\[
\leq N^{-2/3} \sum_{j=1}^{m} \sum_{i=1}^{n_j} \mathbb{E}\left[\sigma_j^2(X_{ji}) - \sigma_j^2(t) 1_{[Kg(a)]K^{-1} < X_{ji} \leq g(a) + N^{-1/3}v}\right]
\]

\[
\leq N^{-2/3} \omega(N^{-1/3} \log N) \sum_{j=1}^{m} \sum_{i=1}^{n_j} \mathbb{E}\left[1_{[Kg(a)]K^{-1} < X_{ji} \leq g(a) + N^{-1/3}v}\right]
\]

where \(\omega(\delta) \to 0\) as \(\delta \to 0\) by assumption, and

\[
N^{-2/3} \sum_{j=1}^{m} \sum_{i=1}^{n_j} \mathbb{E}\left[1_{[Kg(a)]K^{-1} < X_{ji} \leq g(a) + N^{-1/3}v}\right] = N^{1/3} \left| F_X(g(a) + N^{-1/3}v) - F_X([Kg(a)]K^{-1}) \right| = O(1).
\]

Hence,

\[
\mathbb{E}[Z_N(u)Z_N(v)] = N^{-2/3} \sum_{j=1}^{m} \sum_{i=1}^{n_j} \sigma_j^2(t) \mathbb{P}\left([Kg(a)]K^{-1} < X_{ji} \leq g(a) + N^{-1/3}v\right) + o(1)
\]

\[
= N^{-2/3} \sum_{j=1}^{m} \int_{[Kg(a)]K^{-1}}^{g(a) + N^{-1/3}v} f_j(z)dz + o(1)
\]

34
for all fixed real numbers \( u \geq v \geq 0 \). It follows that

\[
\left| \mathbb{E} [Z_N(u)Z_N(v)] - N^{-2/3} \sum_{j=1}^m n_j \sigma_j^2(t) \int_{[Kg(a)]K^{-1}} \kappa(t) \, dz \right| \\
\leq N^{-2/3} \sum_{j=1}^m n_j \sigma_j^2(t) \omega(N^{-1/3} \log N) \left( N^{-1/3} v + O(K^{-1}) \right) + o(1)
\]

since \( \omega(\delta) \to 0 \) as \( \delta \to 0 \). The Jensen inequality for conditional expectation combined with Assumption \( \tilde{A}_4 \) shows that \( \sigma_j^2(t) \leq \sigma^2 \) for all \( i \) and \( t \) and therefore, \( N^{-1} \sum_{j=1}^m n_j \sigma_j^2(t) \leq \sigma^2 \). This implies that

\[
\mathbb{E} [Z_N(u)Z_N(v)] = N^{-2/3} \sum_{j=1}^m n_j \sigma_j^2(t) f_j(t) (N^{-1/3} v + o(N^{-1/3})) + o(1) = \sigma_X^2(t) v + o(1).
\]

We conclude that for all \( u \geq v \geq 0 \), \( \mathbb{E} [Z_N(u)Z_N(v)] = \text{cov}(Z_N(u), Z_N(v)) \) converges to \( \sigma^2(t) v \). The case of negative \( u \) and \( v \) can be treated likewise and therefore, \( \text{cov}(Z_N(u), Z_N(v)) \) converges to \( \sigma^2(t)(u| \wedge |v|) \) if \( uv \geq 0 \). It can be seen similarly that it converges to zero if \( uv < 0 \) (hence \( u \) and \( v \) have different signs). Hence, the covariance converges to \( \sigma_\infty(t) \text{cov}(W(u), W(v)) \), so we conclude from the Lindeberg-Feller theorem that jointly,

\[
(Z_N(u_1), \ldots, Z_N(u_k)) \xrightarrow{d} \sigma_\infty(t)(W(u_1), \ldots, W(u_k))
\]

for all fixed \( u_1, \ldots, u_k \in \mathbb{R} \), as \( N \to \infty \). Now, consider the restriction of \( Z_N \) to the compact interval \([-M, M]\), for a fixed \( M > 0 \) and \( \epsilon > 0 \) we have

\[
\mathbb{P} \left( \sup_{|s| \leq \delta} |Z_N(s) - Z_N(t)| \geq \epsilon \right) \leq \sum_{k=1}^{M[\delta^{-1}]} \mathbb{P} \left( \sup_{|t-k\delta| \leq 2\delta} |Z_N(k\delta) - Z_N(t)| \geq \epsilon \right). \tag{54}
\]

Let \( \pi \) be the permutation such that the \( X_{\pi(j)} \) are ordered in \( j \), that is \( X_{\pi(1)} < \cdots < X_{\pi(N)} \) a.s. Let \( \mathbb{P}_X \) denote the conditional probability given \( X_1, \ldots, X_N \). Since \( \varepsilon_{\pi(1)}, \ldots, \varepsilon_{\pi(N)} \) are centered and independent under \( \mathbb{P}_X \), the process \( \{Z_N(k\delta) - Z_N(t), \ t \geq k\delta\} \) is a forward centered martingale whereas \( \{Z_N(k\delta) - Z_N(t), \ t \leq k\delta\} \) is a reverse centered martingale conditionally on \( X_1, \ldots, X_N \), for all \( k \). Hence, it follows from the Doob inequality that for all \( k \),

\[
\mathbb{P} \left( \sup_{|t-k\delta| \leq 2\delta} |Z_N(k\delta) - Z_N(t)| \geq \epsilon \right) \leq \frac{2p}{\epsilon^p} (\mathbb{E} |Z_N(k\delta) - Z_N((k-2)\delta)|^p + \mathbb{E} |Z_N(k\delta) - Z_N((k+2)\delta)|^p). \tag{55}
\]
Note that the inequalities above are first obtained for the conditional probabilities and then integrated over the distribution of $X$ for the unconditional. Now, it follows from the Rosenthal inequality, see (Rosenthal, 1970), that for all $k$ and a constant $C$ that depends only on $p$, we have

$$\mathbb{E}|Z_N(k\delta) - Z_N((k+2)\delta)|^p \leq CN^{-p/3}\left(\sum_{i=1}^N \mathbb{E}(\varepsilon_i^p 1_{X_i \in I_k}) + \left(\sum_{i=1}^N \mathbb{E}(\varepsilon_i^2 1_{X_i \in I_k})\right)^{p/2}\right).$$

Here, $I_k = (g(a) + N^{-1/3}k\delta, g(a) + N^{-1/3}(k+2)\delta)$ (at least if $g(a) + N^{-1/3}k\delta, g(a)$ and $N^{-1/3}(k+2)\delta$ both belong to $H_N$) and $p$ is taken from Assumption $\tilde{A}_4$. Hence, with $f_X$ taken from (27) we have

$$\mathbb{E}|Z_N(k\delta) - Z_N((k+2)\delta)|^p \leq C\sigma^p N^{-p/3}\left(\sum_{i=1}^N \mathbb{E}(1_{X_i \in I_k}) + \left(\sum_{i=1}^N \mathbb{E}(1_{X_i \in I_k})\right)^{p/2}\right) = C\sigma^p N^{-p/3}\left(N \int_{I_k} f_X(u) du + N^{p/2}\left(\int_{I_k} f_X(u) du\right)^{p/2}\right).$$

It follows from the Assumption $\tilde{A}_1$ that $f_X$ is bounded by a constant $A$ that does not depend on $N$ and therefore,

$$\mathbb{E}|Z_N(k\delta) - Z_N((k+2)\delta)|^p \leq C\sigma^p N^{-p/3}\left(2AN^{2/3}\delta + \left[2AN^{2/3}\delta\right]^{p/2}\right) (1 + o(1)) \leq 2C\sigma^p N^{-p/3}\left[2AN^{2/3}\delta\right]^{p/2} = 2C\sigma^p [2A\delta]^{p/2}$$

for $N$ sufficiently large. Arguing similarly for $\mathbb{E}|Z_N(k\delta) - Z_N((k - 2)\delta)|^p$ we conclude from (55) that there exists $C > 0$ that depends only on $p$ and $A$ such that

$$\mathbb{P}\left(2 \sup_{|t-k\delta| \leq 2\delta} |Z_N(k\delta) - Z_N(t)| \geq \epsilon\right) \leq C\sigma^p \epsilon^{-p}\delta^{p/2}$$

for all $k$. Summing up this inequality over all $k$ on the right-hand side of (54), we obtain that there exists $C > 0$ that depends only on $p$ and $A$ such that for all $\delta > 0$ and $\epsilon > 0$,

$$\mathbb{P}\left(\sup_{|t-s| \leq \delta ; s, t \in [-M,M]} |Z_N(s) - Z_N(t)| \geq \epsilon\right) \leq CM\sigma^p \epsilon^{-p}\delta^{-1+p/2}.$$ 

Since $p > 2$, this converges to zero as $\delta \to 0$. Using (53), it follows from (Billingsley, 2013, Theorem 7.5) that $Z_N$ converges weakly to $\sigma_\infty W$ on all compact intervals $[-M, M]$. Combining this with (47) and (51) we conclude that $N^{1/3}(U_N(a) - g(a))$ is the location of the maximum of a process that weakly converges to the continuous Gaussian process

$$\sigma_\infty(t) W(u) - \frac{|\mu'(t)| f_\infty(t)}{2} u^2, \ u \in \mathbb{R}.$$
The above process achieves its maximum at a unique point $T$ by Lemma 2.6 of (Kim and Pollard, 1990), and it follows from Lemma 3.3 that $N^{1/3}(U_N(a) - g(a))$ is uniformly tight. Hence, Corollary 5.58 in van der Vaart shows that $N^{1/3}(U_N(a) - g(a))$ converges in distribution to $T$. Now, $T$ is also the unique location of the maximum of the process

$$W(u) - \frac{\mu'(t)f_\infty(t)}{2\sigma_\infty(t)}u^2, \; u \in \mathbb{R}.$$  

Changing scale in the Brownian motion finally shows that

$$\left(\frac{\mu'(t)f_\infty(t)}{2\sigma_\infty(t)}\right)^{2/3} T$$

has the same distribution as $Z$, which completes the proof. □

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