Adiabatic evolution of decoherence-free subspaces and its shortcuts

S. L. Wu(武松林),1 X. L. Huang (黄晓理),2 H. Li(李宏),3 and X. X. Yi(衣学喜)3

1School of Physics and Materials Engineering, Dalian Nationalities University, Dalian 116600 China
2School of Physics and Electronic Technology, Liaoning Normal University, Dalian 116029, China
3Center for Quantum Sciences and School of Physics, Northeast Normal University, Changchun 130024, China
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The adiabatic theorem and shortcuts-to-adiabaticity for time-dependent open quantum systems are explored in this paper. Starting from the definition of dynamical stable decoherence-free subspace, we show that, under a compact adiabatic condition, the quantum state remains in the time-dependent decoherence-free subspace with an extremely high purity, even though the dynamics of the open quantum system may be not adiabatic. The adiabatic condition mentioned here in the adiabatic theorem for open systems is very similar with that for closed quantum systems, except that the operators required to change slowly are the Lindblad operators. We also show that the adiabatic evolution of decoherence-free subspaces (will be referred to as adiabatic decoherence-free subspaces(ADFSs) later) depends on the existence of instantaneous decoherence-free subspaces, which requires that the Hamiltonian of open quantum systems has to be engineered according to the incoherent control protocol. Besides, the shortcuts-to-adiabaticity for the adiabatic decoherence-free subspaces is also presented based on the transitionless quantum driving method. Finally, we provide an example that consists of a two-level system coupled to a broad band squeezed vacuum field to show our theory. Our approach employs Markovian master equations and the theory can apply to finite-dimensional quantum open systems.

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I. INTRODUCTION

The adiabatic theorem is among the most useful results that has been known since the early days of quantum mechanics [1, 2]. The theorem states that if a quantum system is in an instantaneous eigenstate of its time-dependent Hamiltonian $H(t)$ at one time, it will remain to that eigenstate up to a phase factor at later times, provided that its Hamiltonian changes very slowly. Recent developments in quantum information processing makes the quantum adiabatic dynamics active again, since the adiabaticity possesses intrinsic robustness against the control errors in the Hamiltonian [3, 4]. However, for a long time the adiabatic theorem is almost exclusively concerned with closed systems. To take the couplings of qubits to environments into account, many efforts have been recently put forward to extending the adiabatic theorem from close systems to open systems. For example, to formulate the adiabatic theorem for open quantum systems by the Jordan block decomposition of the dissipative generator [5, 6], in the weak coupling limit [7, 8], at zero temperature [9], by the method of effective Hamiltonian [10], by the noiseless subsystem decomposition [11], or in terms of the instantaneous steady state of the Liouvillian [12].

Among those enlightening approaches, there is an approach that combines the adiabatic dynamics with decoherence-free subspaces, known as adiabatic evolution of decoherence-free subspaces (ADFSs) [13]. The existence of an ADFS is by no means trivial, and its presence often reflects a symmetry preserving evolution. When the symmetry preserving evolution is slow enough, the state of open quantum systems lying initially in a time-dependent decoherence-free subspace(t-DFS) remains inside the subspace in latter times, and it is rigidly transported in the Hilbert space together with the t-DFS. However, there is no answer to the question: How slow the evolution can guarantee the quantum state inside the t-DFS with extremely high purity. In other words, the adiabatic condition for t-DFS is still missing, although the t-DFS was generalized into nonadiabatic case [14, 15].

As shown, the coherent control is significant to stabilize quantum states in the t-DFS. Therefore, we may ask the other question: whether the ADFSs also needs the assistance of coherent control?

In this paper, we will explore the adiabatic theorem for t-DFSs and present a suitable adiabatic condition to answer those questions. We have the following observations. Firstly, the adiabatic theorem shows that the coherent control on open quantum systems is crucial for the ADFSs. Both the incoherent and coherent evolutions are necessary for the quantum state to stay steadily in the t-DFSs. Secondly, the adiabatic condition, which has similar form with the adiabatic condition for closed quantum systems [2], reveals that the quantum state would keep in the t-DFSs with extremely high purity, if the change rate of the t-DFSs is very small with respect to the effective frequency difference of the non-hermitian Hamiltonian. Even if the effective Hamiltonian of the open quantum system is degenerate, the ADFSs can still be realized...
due to the decoherence, and the condition is available in this case as well.

Following the steps of the adiabatic theorem of closed quantum systems, we will derive adiabatic conditions on operators of the open quantum system. Since the t-DFSs are spanned by a set of common degenerate eigenstates of Lindblad operators, the motion of t-DFSs attributes to the symmetry preserving evolution of the environment. This can be realized by the incoherent control protocol leading to the adiabatic condition on the Lindblad operators. The adiabatic condition on the Lindblad operators shows that the Lindblad operators have to change so slowly that the quantum state follows the t-DFS without purity loss.

Recall that an instantaneous eigenstate of a time-dependent Hamiltonian can evolve to the other instantaneous eigenstate at later time, which is faster than the adiabatic evolution by shortcuts-to-adiabaticity (STA) in closed systems\[16, 17\]. Besides, it can also be realized by the other particular techniques within the broad concept of shortcuts to adiabaticity such as the transitionless quantum driving method\[18–22\], the inverse engineering program\[16\], and the fast quench dynamics\[23\]. It is then interesting to ask whether the condition for ADFSs is sufficient or necessary in Sec. IV. The effect of the coherent evolution and the adiabatic dynamics of t-DFSs is defined as a collection of quantum states such that the pure state \(\rho(t) \in \mathcal{H}_{\text{DFS}}\) fulfills

\[
\frac{\partial}{\partial t} \rho(t) = \frac{\partial}{\partial t} \text{Tr}[\rho^2(t)] = 0, \tag{2}
\]

which leads to the following conditions for DFSs: The space

\[
\mathcal{H}_{\text{DFS}} := \text{span}\{\ket{\Phi_1}, \ket{\Phi_2}, \ldots, \ket{\Phi_M}\}
\]

is a M-dimensional DFS if and only if the bases fulfill the following conditions: (1) The orthogonal bases of the DFS are the degenerate eigenstates of the Lindblad operators, i.e.,

\[
\hat{F}_\alpha \ket{\Phi_j} = c_\alpha \ket{\Phi_j}, \ \forall \alpha, j; \tag{3}
\]

(2) \(\mathcal{H}_{\text{DFS}}\) is invariant under the effective Hamiltonian

\[
\hat{H}^0_{\text{eff}} = \hat{H}_0 + \frac{i}{2} \sum_\alpha \left( c_\alpha^* \hat{F}_\alpha - c_\alpha \hat{F}_\alpha^\dagger \right), \tag{4}
\]

which means that the quantum state in the DFS is still a state in such DFS after the acting of \(\hat{H}^0_{\text{eff}}\), i.e.,

\[
\langle \Phi_n^\perp | \hat{H}^0_{\text{eff}} | \Phi_j \rangle = 0, \ \forall n, j, \tag{5}
\]

where \(\ket{\Phi_n^\perp}\) is the n-th basis of complementary subspace \(\mathcal{H}_{\text{CS}}\).

Here, we notice that the bases of the DFS are common degenerate eigenstates of Lindblad operators. Notice that the Lindblad operators are not always hermitian, the eigenstates of the Lindblad operators may be nonorthogonal. Thus, the biorthogonal bases \(\{\ket{\Phi_i}, \ket{\Psi_i}\}\) have to be used\[26\]. Assume that \(\ket{\Phi_i}, \ket{\Psi_i}\) is the i-th common degenerate right (left) eigenstate of Lindblad operators with same eigenvalue \(c_\alpha\), the right eigenstates are nonorthogonal with each other, but are orthogonal to the left eigenstates, i.e., \(\langle \Psi_i | \Phi_j \rangle = \delta_{ij}\). By the Schmidt orthogonalization, we could orthogonalize the right eigenstates with the common eigenvalue \(c_\alpha\), which is the states \(\{\Phi_i\}\) used as the bases of DFSs in Eqs. (3) and (5). It is obvious that the biorthogonal bases set \(\{\Phi_i\}, \{\Psi_i\}\) is equivalent to the orthogonal set \(\{\Phi_i\}\), because either \(\ket{\Phi_i}\) or \(\ket{\Psi_i}\) is the common degenerate eigenstates of \(\hat{F}_\alpha\). Therefore, we use the orthogonal degenerate eigenvalues set \(\{\Phi_i\}\) as the bases of DFSs in the following discussion. On the other hand, even though the right eigenstates of Lindblad operators are not orthogonal with each

II. ADIABATIC EVOLUTION OF T-DFS

A. A theorem for ADFSs

In this section, we will present a sufficient and necessary condition for dynamical stable decoherence free subspaces\[27\]. Let us consider an open quantum system of N-dimension described by the following master equation,

\[
\partial_t \rho(t) = \hat{\mathcal{L}}(\rho),
\]

\[
\hat{\mathcal{L}}(\rho) = -i [\hat{H}_0, \rho] + \sum_\alpha \left( \hat{F}_\alpha \rho \hat{F}_\alpha^\dagger - \frac{1}{2} \{ \hat{F}_\alpha^\dagger \hat{F}_\alpha, \rho \} \right) \tag{1}
\]

where \(\hat{H}_0(t)\) and \(\{\hat{F}_\alpha(t)\}\) are the Hamiltonian and Lindblad operators in an interaction picture. Based on this master equation, a dynamical stable DFS \(\mathcal{H}_{\text{DFS}}\) is defined as a collection of quantum states such that the pure state \(\rho(t) \in \mathcal{H}_{\text{DFS}}\) fulfills

\[
\frac{\partial}{\partial t} \rho(t) = \frac{\partial}{\partial t} \text{Tr}[\rho^2(t)] = 0, \tag{2}
\]

which leads to the following conditions for DFSs: The space

\[
\mathcal{H}_{\text{DFS}} := \text{span}\{\ket{\Phi_1}, \ket{\Phi_2}, \ldots, \ket{\Phi_M}\}
\]

is a M-dimensional DFS if and only if the bases fulfill the following conditions: (1) The orthogonal bases of the DFS are the degenerate eigenstates of the Lindblad operators, i.e.,

\[
\hat{F}_\alpha \ket{\Phi_j} = c_\alpha \ket{\Phi_j}, \ \forall \alpha, j; \tag{3}
\]

(2) \(\mathcal{H}_{\text{DFS}}\) is invariant under the effective Hamiltonian

\[
\hat{H}^0_{\text{eff}} = \hat{H}_0 + \frac{i}{2} \sum_\alpha \left( c_\alpha^* \hat{F}_\alpha - c_\alpha \hat{F}_\alpha^\dagger \right), \tag{4}
\]

which means that the quantum state in the DFS is still a state in such DFS after the acting of \(\hat{H}^0_{\text{eff}}\), i.e.,

\[
\langle \Phi_n^\perp | \hat{H}^0_{\text{eff}} | \Phi_j \rangle = 0, \ \forall n, j, \tag{5}
\]

where \(\ket{\Phi_n^\perp}\) is the n-th basis of complementary subspace \(\mathcal{H}_{\text{CS}}\).

Here, we notice that the bases of the DFS are common degenerate eigenstates of Lindblad operators. Notice that the Lindblad operators are not always hermitian, the eigenstates of the Lindblad operators may be nonorthogonal. Thus, the biorthogonal bases \(\{\ket{\Phi_i}, \ket{\Psi_i}\}\) have to be used\[26\]. Assume that \(\ket{\Phi_i}, \ket{\Psi_i}\) is the i-th common degenerate right (left) eigenstate of Lindblad operators with same eigenvalue \(c_\alpha\), the right eigenstates are nonorthogonal with each other, but are orthogonal to the left eigenstates, i.e., \(\langle \Psi_i | \Phi_j \rangle = \delta_{ij}\). By the Schmidt orthogonalization, we could orthogonalize the right eigenstates with the common eigenvalue \(c_\alpha\), which is the states \(\{\Phi_i\}\) used as the bases of DFSs in Eqs. (3) and (5). It is obvious that the biorthogonal bases set \(\{\Phi_i\}, \{\Psi_i\}\) is equivalent to the orthogonal set \(\{\Phi_i\}\), because either \(\ket{\Phi_i}\) or \(\ket{\Psi_i}\) is the common degenerate eigenstates of \(\hat{F}_\alpha\). Therefore, we use the orthogonal degenerate eigenvalues set \(\{\Phi_i\}\) as the bases of DFSs in the following discussion. On the other hand, even though the right eigenstates of Lindblad operators are not orthogonal with each
other, the set of all right eigenstates is still a complete set of the total Hilbert space. Thus, the rest of right eigenstates of Lindblad operators can be used, by the Schmidt orthogonalization again, to determine bases of the complementary subspaces.

In the following, we consider that Lindblad operators presented in Eq.\([11]\) are time-dependent, which might come from the engineering parameters in the environment (e.g., incoherent control scheme)\[^{24}\] or in the environment-system couplings\[^{23}\]. If there is an instantaneous subspace \(H_{\text{DFS}}(t) := \text{span}\{|\Phi_1(t)\rangle, |\Phi_2(t)\rangle, \ldots, |\Phi_M(t)\rangle\}\) which fulfills

\[
\hat{F}_\alpha(t)|\Phi_j(t)\rangle = c_\alpha(t)|\Phi_j(t)\rangle,
\]

and

\[
\langle \Phi_n^\perp(t) | \hat{H}_{\text{eff}}(t) | \Phi_j(t) \rangle = 0,
\]

the subspace \(H_{\text{DFS}}(t)\) is known as a time-dependent DFS (t-DFS) of the open quantum system. It has been shown that, when the eigenvalues \(c_\alpha(t)\) equal to zero, a t-DFS can be used to formulate ADFSs without further condition except the adiabatic one\[^{13,14}\]. In the following, we generalize the ADFSs into the case where the eigenvalues \(c_\alpha(t)\) are not zero and time-dependent.

We are now ready to informally state the theorem for ADFSs: *If a pure quantum state is initialized at \(t = 0\) in \(H_{\text{DFS}}(0)\), the final state at \(t = T\) will be in \(H_{\text{DFS}}(T)\), provided the bases of the t-DFS and its complementary subspace fulfill*

\[
\Xi(t) := \text{Max}_{\{n,i\}} \left| \frac{4\langle \Phi_i^\perp(t) | \hat{H}_{\text{eff}}(t) | \Phi_n(t) \rangle}{\omega_{ni} + i\Gamma_n} \right| \ll 1, \forall n, i,
\]

where \(\omega_{ni} = \langle \Phi_i^\perp(t) | \hat{H}_{\text{eff}}(t) | \Phi_n(t) \rangle - \langle \Phi_i(t) | \hat{H}_{\text{eff}}(t) | \Phi_i(t) \rangle\) and \(\Gamma_n = \sum_\alpha \langle \Phi_i^\perp(t) | \hat{F}_\alpha^\perp(t) | \Phi_n(t) \rangle \langle \Phi_i^\perp(t) | \hat{F}_\alpha(t) | \Phi_n(t) \rangle / 2\) with \(\hat{F}_\alpha(t) = \hat{F}_\alpha^\perp(t) - c_\alpha(t)\). The details to derive the adiabatic condition in Eq.\((8)\) can be found in Appendix \(A\).

The adiabatic condition obtained here is similar to the condition for the adiabatic dynamics with the non-Hermitian Hamiltonian\[^{28}\].

We should emphasize that the theorem for ADFSs obtained here is valid if there is a t-DFS \(H_{\text{DFS}}(t)\) at arbitrary moment and the bases of the t-DFSs are continuous with time. Different from the configuration of Ref.\[^{13}\], the instantaneous subspace does not only require that the quantum state in \(H_{\text{DFS}}(t)\) is an eigenstate of Lindblad operators (see Eq.\((11)\)), but also has to satisfy the condition that the quantum state in the t-DFS is still a state in t-DFS after the acting of \(\hat{H}_{\text{eff}}^0\) (see Eq.\((7)\)). Plugging Eq.\((5)\) into Eq.\((7)\), we obtain

\[
\langle \Phi_j(t) | \hat{H}_0(t) | \Phi_n^\perp(t) \rangle = -\frac{i}{2} \sum_\alpha c_\alpha(t) \langle \Phi_j(t) | \hat{F}_\alpha(t) | \Phi_n^\perp(t) \rangle,
\]

which is the very relationship between the coherent evolution and the incoherent evolution. This restriction is very important in realizing ADFSs.

As shown in Ref.\[^{27}\], the quantum state \(|\varphi(t)\rangle\) in \(H_{\text{DFS}}(t)\) fulfills

\[
\frac{\partial}{\partial t} |\varphi(t)\rangle = -i\hat{H}_{\text{eff}}^0(t)|\varphi(t)\rangle.
\]

The operator \(\hat{H}_{\text{eff}}^0(t)\) plays a role of effective Hamiltonian for the open quantum system, when quantum states are in the t-DFSs. To this extent and from the viewpoint of quantum control, a coherent control satisfying Eq.\((9)\) can enable the quantum state to stay in the t-DFSs.

The adiabatic condition presented in Eq.\((8)\) is very similar with the adiabatic condition for closed quantum systems\[^{1,2}\], except that the decoherence has been considered in the adiabatic condition. It has been verified that the quantitative condition is insufficient in guaranteeing the validity of the adiabatic approximation for closed systems. In fact, the adiabatic condition for ADFSs does also face this difficulty. Therefore, in Sec. \(\text{II C}\) we will present the lower bound of purity to describe the performance of the adiabatic approximation, and show that the adiabatic condition is available, if the total evolution time is very large.

B. Adiabatic condition on Lindblad operators

Stimulating by the adiabatic theorem for closed quantum systems, we may wonder whether the adiabatic condition for open system can be expressed in terms of operators of the open quantum system.

Remember that \(\{|\Phi_i(t)\rangle\rangle_{i=1}^M\) are common degenerate eigenstates of \(\hat{F}_\alpha(t)\) as defined in Eq.\((4)\). A useful expression can be found by taking the time derivative of Eq.\((4)\) and multiplying the resulting expression by \(\langle \Phi_n^\perp(t) |\)

\[
\sum_{j=1}^M |\Phi_j(t)\rangle \langle \Phi_j(t) | + \sum_{m=1}^N |\Phi_m^\perp(t)\rangle \langle \Phi_m^\perp(t) | = \hat{I}
\]

into the last term of Eq.\((11)\), we obtain

\[
\langle \Phi_n^\perp(t) | \hat{F}_\alpha(t) | \partial_t |\Phi_i(t)\rangle = -\langle \Phi_n^\perp(t) | \hat{F}_\alpha(t) | \hat{K}(t) |\Phi_i(t)\rangle + \langle \hat{F}_\alpha(t) |\rangle_n \langle \Phi_i^\perp(t) | \partial_t |\Phi_i(t)\rangle,
\]

with \(\hat{K} = \sum_{m\neq n} |\Phi_m^\perp(t)\rangle \langle \Phi_m^\perp(t) |\) and \(\langle \hat{F}_\alpha(t) |\rangle_n = \langle \Phi_n^\perp(t) | \hat{F}_\alpha(t) | \Phi_n^\perp(t) \rangle\). Substituting the above expression back into Eq.\((11)\) and following the process in Appendix \(B\) we obtain

\[
\text{Max}_{\{n,i\}} \left| \frac{\langle \Phi_n^\perp(t) | \partial_t |\Phi_i(t)\rangle}{\omega_{ni} + i\Gamma_n} \right| \leq \left( \sum_{n=0}^{N-M-1} \Gamma_{n+1}^\perp (F_{\text{Max}}^\perp)^n \right) \frac{N-M}{N-M}
\]
between the bases of the complementary subspace systems mentioned above can be expressed as two independent terms. On the other hand, it is difficult to hold the adiabatic condition. The larger the complementary subspace is, the more difficult it is for the adiabatic condition to be held. If the total evolution time \( T \) of the open quantum system should be engineered in order to keep the system in the t-DFS, which is a function of the total evolution time, then we can reach the adiabatic condition when the total evolution time is long enough.

Firstly, we show that the t-DFSs can not be expanded into the complementary subspaces bands. The first condition (the t-DFSs condition) states that, the t-DFSs are spanned by common degenerate eigenstates of the Lindblad operators (see Eq. (9)); and the Hamiltonian of open quantum systems should be engineered in order to keep the quantum states in the t-DFSs (see Eq. (10)). This condition ensures that the evolution of quantum states in the t-DFSs is unitary. Besides, this condition also claims that the requirement of adiabaticity on the Lindblad operators partly comes from the requirement on the Hamiltonian. This can be found from Eq. (10), it shows that when the change of Lindblad operators is adiabatic, parts of \( \hat{H}_0(t) \) have to change slowly to ensure that the t-DFSs condition holds. From the other viewpoint, the adiabatic requirement mentioned here is not only on the incoherent evolution (governed by Lindbladian), but also on the coherent evolution (governed by Hamiltonian).

The second condition (the adiabatic condition) requires that the Lindblad operators have to change so slowly that the quantum state follows the t-DFS with extremely high purity (see Eq. (11)). In other words, the transition between the t-DFS and the complementary subspaces can be negligible, when the adiabatic condition Eq. (11) is satisfied. Moreover, there is the other term in the adiabatic condition, i.e. \( \sum_{a=0}^{N-1} P_{N-M}^a (F_{\text{Max}})^a \). If this term is finite, it does not affect the adiabatic condition evidently. On the one hand, we may notice that, this term is related to the size of complementary subspaces. The larger the complementary subspace is, the more difficult the adiabatic condition to be held. On the other hand, \( F_{\text{Max}} \) describes the maximal transition induced by Lindblad operators in the complementary subspace. If the Lindblad operators \( \{ \hat{F}_a \} \) do not induce any transition between the bases of the complementary subspace \( F_{\text{Max}} = 0 \), the adiabatic condition can be rewritten into a compact form (also see Eq. (12)),

\[
\Xi(t) = \text{Max}_{\{n,i\}} \left| \frac{4\langle \Phi_n^+ | \hat{F}_{\alpha} | \Phi_i \rangle}{(c_{\alpha} - \langle \hat{F}_\alpha \rangle_n)} \right| \lesssim 1 \quad (15)
\]

In this case, the adiabatic condition does not depend on the size of complementary subspaces.

C. Lower Bound of the Purity

For closed quantum systems, the practical applications of the adiabatic theorem rely on the slowness of the Hamiltonian, which is also required for the theorem for ADFSs. Here, we present a lower bound for the purity to keep the system in the t-DFS, which is a function of the total evolution time \( T \). As expected, the ADFSs can be reached, if the total evolution time is long enough.

Firstly, we show that the t-DFSs can not be expanded by eigenstates of the Hamiltonian \( \hat{H} \) of the open quantum system. Suppose that part of eigenstates of \( \hat{H} \) can be used as bases of the t-DFSs, i.e.,

\[
\hat{H}_0(t)|\Phi_j(t)\rangle = \varepsilon_j(t)|\Phi_j(t)\rangle
\]

Now we consider the condition for the ADFSs Eq. (7), it follows that,

\[
\langle \Phi_n^+ | \hat{H}_0^{\text{eff}}(t) | \Phi_j(t) \rangle = \sum_{\alpha} c_{\alpha}(t) \langle \Phi_n^+ | \hat{F}_\alpha(t) | \Phi_j(t) \rangle \neq 0
\]

As mentioned in Sec. 11A, the t-DFSs condition requires that the t-DFS is invariant under the operation of \( \hat{H}_0^{\text{eff}}(t) \) (see Eq. (7)). Therefore, the t-DFS can not be expanded by the eigenstate of \( \hat{H} \).

In fact, the eigenstates of \( \hat{H}_0^{\text{eff}}(t) \) can be chosen as the bases of the t-DFSs and its complementary subspaces, i.e.,

\[
\hat{H}_0^{\text{eff}}(t)|\Phi_1(t)\rangle = E_1(t)|\Phi_1(t)\rangle, \\
\hat{H}_0^{\text{eff}}(t)|\Phi_n^+(t)\rangle = E_n^+(t)|\Phi_n^+(t)\rangle,
\]

where \( E_1(t) (E_n^+(t)) \) is the eigenvalue corresponding to the eigenstate \( |\Phi_1(t)\rangle (|\Phi_n^+(t)\rangle) \). It is easy to find that \( \langle \Phi_n^+ | \hat{H}_0^{\text{eff}}(t) | \Phi_j(t) \rangle = 0 \). Hence the eigenstates of \( \hat{H}_0^{\text{eff}}(t) \) can be divided into two bands: the ADFSs bands and the complementary subspaces bands.

We start with the time derivative of the purity Eq. (10). It is convenient to choose eigenstates of the effective Hamiltonian \( \hat{H}_0^{\text{eff}} \) as the bases of t-DFSs and its complementary subspaces. Under adiabatic approximation, we rewrite Eq. (10) into scalar form

\[
\partial_t p(t) = 2\text{Tr} \left\{ \hat{p}_D \left( -i\hat{G} + \sum_{\alpha} \hat{F}_\alpha \hat{p}_C \hat{F}_\alpha^* \right) \right\},
\]
where \( \omega_{mj} = E_m - E_j \) and \( \Gamma_m = \langle \tilde{\Gamma} \rangle \). Following the same discussion in Appendix A, we discard the second term in Eq.(16), and turn it into an inequality,

\[
\partial_t p(t) \geq 4 \text{Re} \left\{ \sum_{ijm} \tilde{\rho}_{ji} \tilde{\rho}_{im} (\Phi_m^+ | \partial_t | \Phi_j) \times \exp \left( i \int_0^t (\omega_{mj} - i\Gamma_m) d\tau \right) \right\}.
\]

One may wonder why the second term in Eq.(18) can be discarded. As discussed in Appendix A, this term describes the transition from the complementary subspaces into the t-DFSs. Since the population in complementary subspaces is very small under the adiabatic approximation, the contribution from this term is negligible. On the other hand, when the population leaks into the complementary subspace, this term will push the quantum state back to the t-DFS. Therefore, this term will help to keep (even to increase) the system in the t-DFS, which can be understood as continuous measurements causing zero effect projecting the system onto the subspace \( \tilde{\Phi} \).

Therefore, if we discard this term, the purity must be larger than the lower bound we present here.

Recall that \( p(t) \) is given by

\[
p(t) = \frac{1}{M} \sum \sum_{j} \sum_{l} \tilde{\rho}_{ji} \tilde{\rho}_{il} (\Phi_m^+ | \Phi_j) \times \exp \left( i \int_0^t (\omega_{mj} - i\Gamma_m) d\tau \right).
\]

Integrating by part, we have

\[
p(t) - 1 \geq 4 \text{Re} \left\{ \sum_{ijm} \int_0^t \tilde{\rho}_{ji} \tilde{\rho}_{im} (\Phi_m^+ | \partial_t | \Phi_j) \times \exp \left( i \int_0^t (\omega_{mj} - i\Gamma_m) d\tau \right) dt' \right\}.
\]

Plugging the master equation Eq.(11) into above inequality, we can obtain the lower bound of the purity, see Appendix C.

In general, although it is difficult to calculate exactly the integrals in above equation, it is still possible to obtain a bound for the integrals, which would lead to a lower bound. Noting that \( |\rho_{ij}| \leq 1 \), \( |\rho_{im}| \leq 1 \) and \( \exp(i \int_0^t \omega_{mj} d\tau) = 1 \), we have

\[
1 - p(t) \leq 4M \sum_j \left| \frac{\Phi_m^+ | \partial_t | \Phi_j}{\omega_{mj} + i\Gamma_m} \right| + \int_0^t (A_j + B_m + C) \left| \frac{\Phi_m^+ | \partial_t | \Phi_j}{\omega_{mj} + i\Gamma_m} \right| d\tau + \int_0^t \frac{\partial}{\partial t'} \left( \frac{\Phi_m^+ | \partial_t | \Phi_j}{\omega_{mj} + i\Gamma_m} \right) d\tau.
\]

where

\[
A_j = \sum_k \left| \langle \partial_t \Phi_j(t) | \Phi_k(t) \rangle \right| + \sum_k \left| \langle \partial_t \Phi_j(t) | \Phi_k^+(t) \rangle \right|,
\]

\[
B_m = \sum_k \left| \langle \partial_t \Phi_k(t) | \Phi_m^+(t) \rangle \right| + \sum_k \left| \langle \partial_t \Phi_k^+(t) | \Phi_m^+(t) \rangle \right|
\]

\[
+ \sum_{n \neq m} \left| \langle \Phi_n^+(t) | \Gamma | \Phi_m^+(t) \rangle \right|,
\]

\[
C = \frac{1}{M} \sum_{in} \left( \left| \langle \partial_t \Phi_n^+(t) | \Phi_i(t) \rangle \right| + \left| \langle \partial_t \Phi_i(t) | \Phi_n^+(t) \rangle \right| \right)
\]

Since the sums on the right hand side of Eq.(18) are finite terms, the purity is very close to 1 if each of the terms can be omitted. Therefore, we can present stronger conditions for ADFSs:

\[
\left| \frac{\Phi_m^+ | \partial_t | \Phi_i}{\omega_{mj} + i\Gamma_m} \right| \ll 1, \quad t \in [0, T],
\]

\[
\int_0^T (A_{jm} + B_{jm} + C) \left| \frac{\Phi_m^+ | \partial_t | \Phi_j}{\omega_{mj} + i\Gamma_m} \right| d\tau' \ll 1,
\]

\[
\int_0^T \left| \frac{\partial}{\partial t'} \left( \frac{\Phi_m^+ | \partial_t | \Phi_j}{\omega_{mj} + i\Gamma_m} \right) \right| d\tau' \ll 1.
\]

where \( T \) is the total evolution time for which the adiabatic approximation is valid.

To relate the lower bound to the evolution time \( T \), we may simplify Eq.(18) by the following estimations. By introduce the maximal modulus of integrand in Eq.(18) for \( t \in [0, T] \), we have

\[
1 - p(T)
\]
\[ 1 - p(T) \leq 4M \sum_{jm} \left( \frac{\langle \Phi_m^+ | \partial_s | \Phi_j \rangle}{\omega_{mj} + i \Gamma_m} \right) + \sup_{s \in [0,1]} \left\{ \sum_{ij} u_{ij}(t) \langle \Phi_i(t) | \Phi_j \rangle \right\} \] 

Now we consider a quantum system defined by the parameterized Hamiltonian \( \hat{H}(s) \) and Lindblad operators \( \hat{F}_\alpha(s) \), where \( s = t/T, \ t \in [0, T] \). Substituting \( t = sT \) into Eq. (19), we obtain the lower bound of the purity as a function of the total evolution time \( T \).

Since all terms on the right-hand side of above equation can be arbitrarily small as \( T \) increases, the purity can reach 1 if \( T \) is large enough. We then arrive at the conclusion that the ADFS is valid for quantum systems which fulfills both the adiabatic condition Eq. (14) and the t-DFS conditions Eqs. (9) and (7) as long as \( T \) is large enough.

### III. SHORTCUTS TO ADIABATIC EVOLUTION OF DECOHERENCE-FREE SUBSPACE

Equipped with the theorem of ADFSs, we are now ready to present STA dynamics for the ADFSs with the language of the transitionless quantum driving method [18]. Consider an open quantum system described by the master equation Eq. (11) and assume that there are t-DFSs \( \mathcal{H}_{t-DFS}(t) \) which is spanned by the common eigenstates of the Lindblad operators \( \{|\Phi_i(t)\rangle\} \). Our purpose is to drive the quantum state from the initial t-DFS into the target t-DFS without any purity loss. In the adiabatic approximation, the states driven by \( \hat{H}_0(t) \) and \( \{\hat{F}_\alpha(t)\} \) would be

\[ |\psi(t)\rangle = \sum_{i=1}^M c_i(t)|\Phi_i(t)\rangle, \]

with normalized parameters \( c_i(t) \). In the transitionless quantum driving method adopted here, in order to realize the ADFS, we have to find a Hamiltonian \( \hat{H}(t) \) for the open quantum systems satisfying Eq. (14).

For the Hamiltonian and the Lindblad operators, the states must follow \( \mathcal{H}_{t-DFS}(t) \) exactly without any purity loss: there are no transition between the t-DFS and the complementary subspace. To obtain \( \hat{H}(t) \), we notice that the unitary operator \( \hat{U}_{t-DFS}(t) \) is the solution of

\[ i\partial_t \hat{U}_{t-DFS}(t) = \hat{H}_{t-DFS}(t) \hat{U}_{t-DFS}(t), \]

where

\[ \hat{H}_{t-DFS}(t) = i\partial_t \hat{U}_{t-DFS}(t) \hat{U}_{t-DFS}^\dagger(t). \]

Following the fact that \( \hat{U}_{t-DFS}(t) \) can be parameterized by the common degenerate eigenstates of Lindblad operators,

\[ \hat{U}_{t-DFS}(t) = \sum_{ij} u_{ij}(t)|\Phi_i(t)\rangle\langle \Phi_j(0)|. \]

where \( \sum_{ij} u_{ij}(t)u_{ij}^*(t) = \delta_{ik} \). Substituting Eq. (21) into Eq. (20), we immediately obtain,

\[ \hat{H}_{t-DFS}(t) = i \sum_{i,j,l} \langle \partial_t u_{ij}(t) u_{jl}^*(t) | \Phi_i(t) \rangle \langle \Phi_j(t) | \]

\[ + u_{ij}(t)u_{jl}^*(t) \langle \partial_t | \Phi_i(t) \rangle \langle \Phi_j(t) | = \hat{H}_{t-DFS}(t) + \hat{H}_1(t). \]

Here \( \hat{H}_1(t) \) is the operator mentioned in Eq. (14), satisfying Eq. (7). At this time, the effective Hamiltonian can be written as

\[ \hat{H}_{t-DFS}(t) = \hat{H}(t) + \frac{i}{2} \sum_{\alpha} \left( c_\alpha(t) \hat{F}_\alpha(t) - c_\alpha(t) \hat{F}_\alpha^\dagger(t) \right). \]

with \( \hat{H}(t) = \hat{H}_0(t) + \hat{H}_1(t) \). The elements of the off-diagonal block of \( \hat{H}_1(t) \) can be determined by considering the t-DFSs condition Eq. (7), which reads

\[ \langle \Phi_n^\perp(t) | \hat{H}_1(t) | \Phi_k \rangle = i \langle \Phi_n^\perp(t) | \partial_t | \Phi_k \rangle. \]

Notice that any operator \( \hat{K}(t) \) can be decomposed as two parts, \( \hat{K}(t) = \hat{K}_N(t) + \hat{K}_Y(t) \) with \( \hat{K}_N(t) \) denoting the off-diagonal parts and \( \hat{K}_Y(t) \) diagonal parts. The off-diagonal parts means it can be written as,

\[ \hat{K}_N(t) = \hat{P}(t) \hat{K}(t) \hat{Q}(t) + \hat{Q}(t) \hat{K}(t) \hat{P}(t), \]

with the projectors \( \hat{P}(t) = \sum_i | \Phi_i(t) \rangle \langle \Phi_i(t) | ) \) on \( \mathcal{H}_{t-DFS}(t) \) and \( \hat{Q}(t) = \sum_n | \Phi_n^\perp(t) \rangle \langle \Phi_n^\perp(t) | \) on \( \mathcal{H}_{\perp DFS}(t) \). With this notation, the off-diagonal block of the Hamiltonian of open quantum systems can be written as

\[ \langle \Phi_j(t) | \hat{H}(t) | \Phi_n^\perp(t) \rangle = i \langle \Phi_n^\perp(t) | \partial_t | \Phi_k \rangle \]

which is the very condition for t-DFSs discussed in Ref. [13]. In other words, when the open quantum system is engineered by an coherent control according to Eq. (20), the dynamics of the quantum state lying in \( \mathcal{H}_{t-DFS}(t) \) is unitary. Hence, the quantum state will follow the t-DFS
from the initial DFS into a target DFS without any fidelity loss.

The other parts of the Hamiltonian, i.e., the upper block and the lower block of the Hamiltonian have no contribution to shortcuts to ADFS. This can be shown in the definition of ADFS. I.e., by the ADFS, the quantum state can be transported unitarily from an initial DFS into an target DFS, but it is not necessary be transported from an initial state to a target state. The upper block of Hamiltonian, which can be rewritten as

$$\hat{H}_D(t) = \hat{P}(t)\hat{H}(t)\hat{P}(t),$$

(27)

is to induce the transition between the bases of the t-DFS. Thus, we do not need to take any special requirement on the other parts of Hamiltonian. On the other hand, if we need to engineer the quantum state into a special target state, the upper block of Hamiltonian have to be engineered. Since the dynamics of quantum states in t-DFS is unitary, we can follow the standard procedure of the transitionless quantum driving method to drive the quantum state into an target state [18, 31].

IV. EXAMPLE

A. The ADFSs for a Two-level System

As an example, we consider a two-level system with ground state $|0\rangle$ and excited state $|1\rangle$ coupled to both a broadband squeezed vacuum field and a coherent control field $\Omega(t)$. Under the Markov approximation, the dynamics of the two-level system is described by the following master equation [32],

$$\partial_t \rho(t) = -i[H_0(t), \rho(t)] + \mathcal{L}_D \rho(t).$$

(28)

The Hamiltonian of two-level atom can be written as

$$H_0(t) = \Omega(t)|0\rangle\langle 1| + h.c..$$

(29)

The dissipator caused by the coupling to the squeezed vacuum is

$$\mathcal{L}_D \rho(t) = \gamma \cosh^2(r) \left( \sigma_+ \rho(t) \sigma_- - \frac{1}{2} \{ \sigma_+ \sigma_- \rho(t) \} \right) + \gamma \sinh^2(r) \left( \sigma_- \rho(t) \sigma_+ - \frac{1}{2} \{ \sigma_- \sigma_+ \rho(t) \} \right) + \gamma \sinh(r) \cosh(r) \exp(-i\theta) \sigma_- \rho(t) \sigma_- + \gamma \sinh(r) \cosh(r) \exp(i\theta) \sigma_+ \rho(t) \sigma_+, \quad (30)$$

where $r$ is the squeezing strength and $\theta$ is the squeezing phase, $\sigma_- (\sigma_+)$ is the lowering (raising) operator, $\gamma$ is the spontaneous decay rate. In Eq. (30), we have assumed that the vacuum squeezing field is perfect. If we redefine the decoherence operator as follows,

$$L = \cosh(r) \exp(-i\theta/2) \sigma_+ + \sinh(r) \exp(i\theta/2) \sigma_+, \quad (31)$$

the dissipator could be transformed into the Lindblad form,

$$\mathcal{L}_D \rho(t) = \frac{\gamma}{2} \left( 2L \rho(t) L^\dagger - \{ L^\dagger L, \rho(t) \} \right).$$

(32)

According to the theorem for ADFSs mentioned in Sec. II A, two independent conditions are required by the ADFSs. The t-DFS condition and the adiabatic condition. On one hand, a subspace spanned by $\mathcal{H}_{DFS} = \{|\phi\rangle\}$

![FIG. 1: (Color online) (a) The purity $p(t)$ versus dimensionless parameter $\mu$ (in units of $\pi$) with parameters $\nu = 0$, $\mu = 0.01\gamma$ (the red solid line), $\mu = 0.1\gamma$ (the blue dash line) and $\mu = \gamma$ (the black dot line); (b) The purity versus dimensionless parameter $\nu t$ (in units of $\pi$) with parameters $\mu = 0$, $\nu = 0.1\gamma$ (the red solid line), $\nu = 0.1\gamma$ (the blue dash line) and $\nu = \gamma$ (the black dot line). The results are obtained by calculating the master equation without the coherent control field $\Omega(t)$ (the green dot dash line) and with the coherent control field (the other lines).]

![FIG. 2: (Color online) The purity $p(t)$ and the adiabatic condition $\Xi(t)$ versus dimensionless parameters $\mu t$ (in units of $\pi$) with $\nu = 0$ and $\mu = 0.1\gamma$. The initial state is same as in in FIG 1(a).]
is a t-DFS of the two-level system, if |φ⟩ is the eigenvector of the Lindblad operator L. The Lindblad operator L (Eq. (31)) gives two nonorthogonal eigenstates,

\[ |φ_1⟩ = \frac{1}{n} \left( \sqrt{\sinh(r)} \exp(i\theta/2)|0⟩ + \sqrt{\cosh(r)}|1⟩ \right) \tag{33} \\
|φ_2⟩ = \frac{1}{n} \left( -\sqrt{\sinh(r)} \exp(i\theta/2)|0⟩ + \sqrt{\cosh(r)}|1⟩ \right) \tag{34} \]

with eigenvalues \( λ_1 = \sqrt{\sinh(r)} \cosh(r) \) and \( λ_2 = -\sqrt{\sinh(r)} \cosh(r) \), in which \( n = \sinh(r) + \cosh(r) \) is normalizing factor. Anyone of the eigenstates can be the basis of subspace \( \mathcal{H}_{DFS} \). Without losing the generality, we choose \( |φ_1⟩ \) as the basis of \( \mathcal{H}_{DFS} \). Then the basis of the orthogonal complement space can be determined, which reads

\[ |φ_\perp⟩ = \frac{1}{n} \left( \sqrt{\cosh(r)} \exp(i\theta/2)|0⟩ - \sqrt{\sinh(r)}|1⟩ \right) \tag{35} \]

At the same time, the two-level system have to be engineered according to Eq. (4). Taking the Hamiltonian Eq. (29) and the Lindblad operator Eq. (31) into the effective Hamiltonian \( H_{eff}(t) \) and considering the condition Eq. (1), we are able to write the accurate function of the coherent control field as

\[ Ω(r, θ) = \frac{iγ \exp(-r - iθ) \sqrt{\sinh(r) \cosh(r)}}{2} \tag{36} \]

We engineer the surroundings of the two-level atom from the vacuum field to the squeezed vacuum field by means of engineering reservoir technology (the incoherent control program)\(^\text{[33, 34]}\), which results in the time dependence of the squeezed parameters. For simplicity, both the squeezed strength and the squeezed phase are set to depend on time linearly, i.e., \( r = r_0 + \mu t \) and \( θ = θ_0 + νt \) with initial squeezed parameters \( r_0 \) and \( θ_0 \). By combining the incoherent control Eq. (31) with the coherent control Eq. (10), the instantaneous DFS \( \mathcal{H}_{DFS} \) is dynamically stable.

On the other hand, when the Lindblad operator Eq. (31) is engineered by incoherent control continuously, the t-DFS \( \mathcal{H}_{DFS} \) is time-dependent. To compel the quantum state of the two-level system to follow the track of the t-DFS, the adiabatic condition Eq. (8) have to be satisfied. Taking Eqs. (31), (33) and (34) into Eq. (8), the adiabatic condition reads

\[ Ξ = \frac{4i(\mu + iν \sinh(r) \cosh(r))}{\gamma \sqrt{\sinh(r) \cosh(r) (\sinh(3r) + \cosh(3r))}} \tag{37} \]

which characterizes the broken adiabaticity. In fact, it can be verified that the spectrum of the effective Hamiltonian is degenerate, i.e., \( H_{eff}(t) = 0 \). But the adiabatic condition is still available as shown in Eq. (37), which results from the contribution of the decoherence process.

The numerical results of purity versus dimensionless parameters \( \mu t \) and \( νt \) were illustrated in FIG. (1). The initial states are prepared in the initial t-DFS \( \mathcal{H}_{DFS}(0) \), i.e., \( ρ(0) = |φ_1(0)⟩⟨φ_1(0)| \) with \( r_0 = 0 \) and \( θ_0 = 0 \) in FIG. (a) and with \( r_0 = 2\pi \) and \( θ_0 = 0 \) in FIG. (b). According to Eq. (37), the adiabatic condition can be held, if the parameters \( \mu \) and \( ν \) is far less than the spontaneous decay rate \( γ \). In other words, the smaller \( \mu \) and \( ν \) are, the larger the purity is at the end of evolution, which is verified by FIG. (1). Besides, we also consider the effect of the coherent control field \( Ω(t) \) on the purity. When the coherent control field is absent (green dot dash line), the quantum state becomes a maximally mixed state in minimal time scale, even the parameters \( \mu \) and \( ν \) are very small. Therefore, the cooperation between the coherent control and the incoherent control is significant for the ADFSs.

![FIG. 3: (Color online) The fidelity between the quantum states \( ρ(t) \) and \( ρ_{DFS}(t) \) versus \( \mu t \) (in units of \( π \)) and \( φ \) (in units of \( π \)) with \( μ = ν = 0.1γ \).](image)

It can be observed in FIG. (a) that the dependence of the purity on the parameter \( \mu t \) is not monotonic. The purity loses evidently at the beginning of the engineering stage. After that, the purity increases and tends to a stable value. This is due to the competition between the broken adiabaticity and the decay effect from the complementary subspace to the t-DFS. To illustrate the result in detail, the purity \( p(t) \) and the adiabatic condition \( Ξ(t) \) as a function of the dimensionless parameter \( \mu t \) are plotted in FIG. (2) in which the change rate of the squeezed parameters are chosen as \( μ = 0.1γ \) and \( ν = 0 \). The initial squeezed parameters are set to be \( r_0 = 0 \) and \( θ_0 = 0 \). It is convenient to break the abscissa axis into two intervals. The boundary is the black dot line in FIG. (2) where the purity is minimal and \( Ξ = 0.129 \). When the adiabatic condition is broken, the quantum state can not follow the t-DFS (the interval at the left side of the black dot line), so that the purity decreases. This can also be found in Eq. (37). With the evolution, \( Ξ \) decays into the region where the adiabatic condition is satisfied (at the right side of the block dot line). At this time,
the decay effect is stronger than the broken adiabaticity. Since the decoherence process induces the transition from the complementary subpace into the t-DFS, the fidelity of the quantum state increases gradually until it reaches a stable value, which can be explained as continuous Zeno measurements projecting onto the t-DFSs[30]. As shown in FIG. 4(a), the purity increase with the parameter $\mu t$ (in units of $\pi$) with $\mu = \nu = 0.01\gamma$ (the green dot dash line), $\mu = \nu = 0.1\gamma$ (the red dash line), and $\mu = \nu = \gamma$ (the blue solid line). (b) The Bloch vectors for $\rho(t)$ (the red dash line) and $\rho_{\text{DFS}}(t)$ in FIG. 4(b) in case of $\mu = 0.1\gamma$. It can be observed that the Bloch vector of $\rho(t)$ starts from the center of the Bloch sphere (the maximally mixed state), which is illustrated by the red dash line in FIG. 4(b). With the evolution, the quantum state $\rho(t)$ is approaching to $\rho_{\text{DFS}}(t)$, until they have a stable distance in the Bloch sphere. According to the above observation, we conclude that the requirement on the preparation of the initial state can be relaxed. When the adiabatic condition is satisfied, we still can engineer the quantum state of quantum systems into the target subspace for arbitrary initial states.

B. protocol for shortcuts to ADFSs

Let us now apply the transitionless quantum driving method by taking the interaction picture Hamiltonian Eq.(24) as the reference Hamiltonian $\hat{H}_0(t)$. Choosing $\{|\phi_1(t)\rangle, \phi^\perp(t)\rangle\}$ given by Eqs. (33) and (35) as the bases of the t-DFS and the complementary subspace, the driving Hamiltonian Eq. (24) becomes in this case,

$$H_1(t) = \Omega'(t)|0\rangle\langle 1| + \text{h.c.},$$

with the coherent control field

$$\Omega'(r, \theta) = \frac{i \exp(-r - i\theta)(\mu - i\nu \sinh(r) \cosh(r))}{2\sqrt{\sinh(r) \cosh(r)}}.$$  

As shown in FIG. 4(a), the purity increase with the parameter $\mu t$ which can also be explained as a result of the decay effect. What is more, we can observe that the final purities are sensitive with the adiabatic condition. When the adiabatic condition is satisfied, the quantum state is drawn back into t-DFSs perfectly (the green dot dash line). On the contrary, when the adiabatic condition is broken, the system can not be derived into the t-DFS (the blue solid line). In other words, the quantum state will be drawn back into t-DFSs if the adiabatic condition is satisfied. For illustrating the conclusion more intuitively, we plot the Bloch vectors of the quantum states $\rho$ and $\rho_{\text{DFS}}(t)$ in FIG. 4(b) in case of $\mu = 0.1\gamma$. The numerical result illustrates that, due to the decay, the fidelity increases with the evolution. The final fidelity approaches to 1, even though the decay effect is stronger than the broken adiabaticity.

We implement $\hat{H}(t) = \hat{H}_0(t) + \hat{H}_1(t)$ as the Hamiltonian of the two-level system to illustrate practicability of

![FIG. 4: (Color online) (a) The purity versus dimensionless parameters $\mu t$ (in units of $\pi$) with $\mu = \nu = 0.01\gamma$ (the green dot dash line), $\mu = \nu = 0.1\gamma$ (the red dash line), and $\mu = \nu = \gamma$ (the blue solid line). (b) The Bloch vectors for $\rho(t)$ (the red dash line) and $\rho_{\text{DFS}}(t)$ in FIG. 4(b) in case of $\mu = 0.1\gamma$. It can be observed that the Bloch vector of $\rho(t)$ starts from the center of the Bloch sphere (the maximally mixed state), which is illustrated by the red dash line in FIG. 4(b). With the evolution, the quantum state $\rho(t)$ is approaching to $\rho_{\text{DFS}}(t)$, until they have a stable distance in the Bloch sphere. According to the above observation, we conclude that the requirement on the preparation of the initial state can be relaxed. When the adiabatic condition is satisfied, we still can engineer the quantum state of quantum systems into the target subspace for arbitrary initial states.](Image 93x568 to 523x740)
the shortcuts to ADFSs. The numerical results are illustrated in Fig. 5. Both the squeezed parameter and the squeezed phase are set to depend on time linearly,

\[ r(t) = \mu t + o, \quad \theta = \nu t, \quad \text{(41)} \]

where \( o \) is an extremely small constant. In Fig. 5(a), we consider the purity as a function of \( \mu t \) with \( \mu = \nu = \gamma \). The initial state of the two-level system is prepared in \( \mathcal{H}_{DFS}(0) \). As shown in the adiabatic condition Eq. (37), the purity cannot hold in this case, when we use \( \hat{H}_0(t) \) as the Hamiltonian of the two-level system. The purity of the quantum state decreases rapidly, since the adiabatic condition is not fulfilled (the green dash line in Fig. 5(a)). On the other hand, if the transitionless quantum driving method is used, the system remains in the t-DFS without any purity loss (the blue solid line in Fig. 5(a)). We also plot the Bloch vectors for both the ADFSs and the transitionless quantum driving in Fig. 5(b), which confirms our conclusion obtained in Fig. 5(a). As a result, the protocol proposed in Sec. III is the shortcuts to adiabaticity for the ADFSs, which can be used to accelerate the ADFSs definitely.

V. CONCLUSION

Starting from the definition of dynamical stable DFSs, we have presented an adiabatic theorem for an open system in a time-dependent decoherence-free subspaces. An adiabatic condition to guarantee the quantum system in the time-dependent decoherence-free subspace is also presented.

The adiabatic theorem contains two independent conditions, i.e., the condition for t-DFSs and the condition for adiabatic evolution. The first condition states that both the coherent evolution (governed by the Hamiltonian) and the incoherent evolution (governed by the Lindbladian) are important to enable the DFSs dynamically stable. The second condition is the adiabatic condition Eq. (13) (also see Eqs. (15)), which suggests that if the variation of the effective frequency of the Lindblad operators is smaller than the transition frequencies \( |\omega_{ni} + i\Gamma_n| \), the quantum state of open quantum system would keep in the t-DFS.

To quantify the probability of the system in the ADFSs, we derive a lower bound for the probability. Further, we have also proposed a STA protocol (i.e., the shortcuts-to-adiabaticity for open systems) for ADFSs. Following the transitionless quantum driving method for closed quantum systems, we show that the quantum state can keep in the t-DFS without any purity loss by adjusting the coherent evolution.

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Appendix A: The Adiabatic Condition Eq. (8)

In this part of appendix, we show that Eq. (8) is the condition approved for ADFSs. Firstly, we introduce a transformation operator

\[ T(t) = \sum_j \exp \left( i \int_0^t \langle \hat{H}_0^0(\tau) \rangle_\tau d\tau \right) |\Phi_j(0)\rangle \langle \Phi_j(t)| \]
\[
\sum_n \exp \left( i \int_0^t \left( i\hat{H}_{\text{eff}}(\tau)\right)_n + i\langle \hat{\Gamma}(\tau)\rangle_n \right) |\Phi_n^+(0)\rangle |\Phi_n^+(t)\rangle,
\]
where \(\langle \hat{H}_{\text{eff}}(\tau)\rangle_i = \langle \Phi_i(\tau) | \hat{H}_{\text{eff}}(\tau) | \Phi_i(\tau) \rangle\), \(\langle \hat{H}_{\text{eff}}(\tau)\rangle_n = \langle \Phi_n^+(\tau) | \hat{H}_{\text{eff}}(\tau) | \Phi_n^+(\tau) \rangle\), and \(\langle \hat{\Gamma}(\tau)\rangle_n = \sum_{\alpha} \langle \Phi_n^+(\tau) | (\hat{F}_\alpha(\tau) - c_\alpha^*(\tau) (\hat{F}_\alpha(\tau) - c_\alpha(\tau)) |\Phi_n^+(\tau)\rangle / 2\). Besides, the inverse transformation operator can be written as
\[
\hat{T}^{-1}(t) = \sum_j \exp \left( -i \int_0^t (\langle \hat{H}_{\text{eff}}(\tau)\rangle_i \right) |\Phi_i(t)\rangle |\Phi_i(0)\rangle \right.
\]
\[
+ \sum_n \exp \left( -i \int_0^t (\langle \hat{H}_{\text{eff}}(\tau)\rangle_n + i\langle \hat{\Gamma}(\tau)\rangle_n \right) |\Phi_n^+(t)\rangle |\Phi_n^+(0)\rangle \right)
\]
We transform the density matrix \(\hat{\rho}(t)\) into a rotating frame by this transformation operators, i.e. \(\hat{\rho}(t) = \hat{T}(t)\hat{\rho}(t)\hat{T}^{-1}(t)\). At this time, the master equation Eq.11 reads
\[
\partial_t \hat{\rho}(t) = \hat{\mathcal{L}}(\hat{\rho}),
\]
where
\[
\hat{\mathcal{L}}(\hat{\rho}) = -i[\hat{H} + \hat{G}, \hat{\rho}] + \sum_\alpha \left( \hat{F}_\alpha \hat{\rho} \hat{F}_\alpha^\dagger - \frac{1}{2} \{ \hat{F}_\alpha \hat{F}_\alpha^\dagger, \hat{\rho} \} \right)
\]
with \(\hat{H} = \hat{T} \hat{\mathcal{H}}_0 \hat{T}^{-1}, \hat{F}_\alpha = \hat{T} \hat{F}_\alpha \hat{T}^{-1}\) and \(\hat{G} = i\partial_t \hat{T} \cdot \hat{T}^{-1}\). By defining the new Lindblad operator as \(\hat{F}_\alpha = \hat{F}_\alpha - c_\alpha\), the decoherence terms in Eq.13 can be rewritten as
\[
\hat{\mathcal{L}}(\hat{\rho}) = -i[\hat{H}_{\text{eff}} + \hat{G}, \hat{\rho}] + \sum_\alpha \left( \hat{F}_\alpha \hat{\rho} \hat{F}_\alpha^\dagger - \frac{1}{2} \{ \hat{F}_\alpha \hat{F}_\alpha^\dagger, \hat{\rho} \} \right),
\]
in which \(\hat{H}_{\text{eff}} = \hat{T} \hat{\mathcal{H}}_{\text{eff}}^{\text{eff}} \hat{T}^\dagger\) with the effective Hamiltonian mentioned in Eq.7. Hence, according to the instantaneous DFSs condition (see Eqs.10 and 7), the operators used in Eq.13 imply the following properties: (1) \(\hat{F}_\alpha |\Phi_i(0)\rangle = 0\), \forall i; (2) \(\langle \Phi_n^+(0) | \hat{H}_{\text{eff}} | \Phi_j(0)\rangle = 0\), \forall n, j.

With the definition of the dynamical stable DFSs, we consider the time derivative of the purity \(p(t)\) of the quantum state \(\hat{\rho}\),
\[
\partial_t p(t) = 2 \text{Tr}\{ \hat{\rho} \partial_t \hat{\rho} \} = 2 \text{Tr}\{ \hat{\rho} \partial_t \hat{\rho} \}.
\]
The quantum state \(\hat{\rho}\) can be decomposed into three parts,
\[
\hat{\rho} = \hat{\rho}_D + \hat{\rho}_N + \hat{\rho}_C,
\]
where \(\hat{\rho}_D (\hat{\rho}_C)\) is diagonal block corresponding to DFSs (complementary subspaces) and \(\hat{\rho}_N\) is off-diagonal part. It is straight forward to obtain following commutation relations,
\[
[\hat{\rho}_D, \hat{\rho}_N] \neq 0, \quad [\hat{\rho}_C, \hat{\rho}_N] \neq 0, \quad [\hat{\rho}_D, \hat{\rho}_C] = 0.
\]
By introducing the projectors \(\hat{P} = \sum_j |\Phi_i(0)\rangle \langle \Phi_i(0)|\) on \(\mathcal{H}_{\text{DFS}}(0)\) and \(\hat{Q} = \sum_n |\Phi_n^+(0)\rangle \langle \Phi_n^+(0)|\) on \(\mathcal{H}_{\text{CS}}(0)\), we can rewrite the operators as \(\hat{\rho}_D = \hat{P} \hat{\rho} \hat{P}\), \(\hat{\rho}_C = Q \hat{\rho} Q\), and \(\hat{\rho}_N = \hat{P} \hat{\rho} \hat{Q} + Q \hat{\rho} \hat{P}\). When the total operation time \(T\) goes to infinity, the adiabatic evolution is achieved reliably 12. In other words, in the adiabatic approximation, \(\text{lim}_{T \to \infty} \hat{\rho}_D = \hat{\rho}\) and \(\text{lim}_{T \to \infty} \hat{\rho}_C = 0\), \(\text{lim}_{T \to \infty} \hat{\rho}_N = 0\). Therefore, the purity can be expressed approximately as
\[
\partial_t p(t) = 2 \lim_{T \to \infty} \text{Tr}\{ \hat{\rho}_D \partial_t \hat{\rho} \}.
\]
Taking Eq.14 and Eq.13 into above equation, we immediately obtain
\[
\partial_t p(t) = 2 \lim_{T \to \infty} \text{Tr}\left\{ \hat{\rho}_D (-i[\hat{H}_{\text{eff}} + \hat{G}, \hat{\rho}_D + \hat{\rho}_N + \hat{\rho}_C] + \sum_\alpha \left( \hat{F}_\alpha (\hat{\rho}_D + \hat{\rho}_N + \hat{\rho}_C) \hat{F}_\alpha^\dagger - \frac{1}{2} \{ \hat{F}_\alpha \hat{F}_\alpha^\dagger, \hat{\rho}_D + \hat{\rho}_N + \hat{\rho}_C \} \right) \right\}
\]
\[
= 2 \lim_{T \to \infty} \text{Tr}\{ \hat{\rho}_D (-i[\hat{G}, \hat{\rho}_N] + \sum_\alpha \hat{F}_\alpha \hat{\rho}_C \hat{F}_\alpha^\dagger) \},
\]
(A6)
where the following relations have been used: \( \hat{P} \hat{H}_{\text{eff}} \hat{Q} = 0, \hat{F}_a \hat{P} = 0, \hat{P} \hat{F}_a^\dagger = 0, \) and \( [\hat{\rho}_D, \hat{\rho}_C] = 0. \) It’s worth noting that the second term of right-hand side of above equation describes the population transition from the complementary subspace to the t-DFS. Further, the population in the complementary subspace is very small under the adiabatic approximation. Therefore, the contribution of this term on the evolution of the purity is so negligible that this term can be ignored. Thus we have

\[
\partial_t p(t) = \lim_{T \to \infty} 2 \text{Tr} \{ \hat{\rho}_D (-i [\hat{G}, \hat{\rho}_N]) \} = -2i \lim_{T \to \infty} \text{Tr} \{ \hat{\rho} (\hat{\rho}_D \hat{G}_R - \hat{G}_L \hat{\rho}_D) \} = -2i \lim_{T \to \infty} \langle \hat{\rho}_D \hat{G}_R - \hat{G}_L \hat{\rho}_D \rangle,
\]

where \( \hat{G}_R = \hat{P} \hat{G} \hat{Q} \) and \( \hat{G}_L = \hat{Q} \hat{G} \hat{P}. \) The time derivative of the purity can be rewritten in terms of the bases of \( \mathcal{H}_{\text{DFS}}(t) \) and \( \mathcal{H}_{\text{CS}}(t) \) as

\[
\partial_t p(t) = -2i \sum_{i,j,n} \hat{p}_{ij}^D \hat{n}_{ij}^N (t) |\partial_{t} |\Phi_i(t)\rangle \exp \left( i \int_0^t (\omega_{ni}(\tau) + i(\Gamma(\tau))_n) d\tau \right)
+ \langle \partial_{t} |\Phi_j(t)\rangle \hat{p}_{ij}^N \exp \left( -i \int_0^t (\omega_{nj}(\tau) + i(\Gamma(\tau))_n) d\tau \right) \tag{A7}
\]

To obtain above expression, we have used the fact that

\[
\hat{p}_{ij}^N = \langle \Phi_j(0)|\hat{T} \hat{\rho}^{-1} \hat{T}^{-1} |\Phi_i(0)\rangle
= \langle \Phi_j(t)|\hat{\rho}(t) |\Phi_i(t)\rangle \exp \left( -i \int_0^t (\omega_{nj}(\tau) + i(\Gamma(\tau))_n) d\tau \right).
\]

Since the first term in Eq. (A8) is conjugate to the second term, we can rewrite above expression as

\[
\partial_t p(t) = 4 \text{Re} \left\{ \sum_{i,j,n} \hat{p}_{ij}^D \hat{n}_{ij}^N (t) |\partial_{t} |\Phi_i(t)\rangle \exp \left( i \int_0^t (\omega_{ni}(\tau) + i(\Gamma(\tau))_n) d\tau \right) \right\}
\]

It is convenient to consider a dimensionless parameter \( s = t/T, \) where \( T \) is total evolution time. We rewrite above expression as

\[
\partial_s p(s) = 4 \text{Re} \left\{ \sum_{i,j,n} \hat{p}_{ij}^D \hat{n}_{ij}^N (s) |\partial_{s} |\Phi_i(s)\rangle \exp \left( -iT \int_0^s (\omega_{ni} (s') + i(\Gamma(s'))_n) ds' \right) \right\}. \tag{A9}
\]

Integrating by parts, we have

\[
p(s) - 1 \approx 4 \text{Re} \left\{ \sum_{i,j,n} \hat{p}_{ij}^D \hat{n}_{ij}^N (s) \frac{|\partial_{s} |\Phi_i(s)\rangle}{\omega_{ni}(s) + i(\Gamma(s))_n} \exp \left( -iT \int_0^s (\omega_{ni} (s') + i(\Gamma(s'))_n) ds' \right) \right\},
\]

where we have omitted the terms higher than the second order of \( T^{-1}. \) Therefore, in case of

\[
\frac{|4 \langle \Phi_i(t) |\partial_{t} |\Phi_i(t)\rangle \rangle}{\omega_{ni}(t) + i(\Gamma(t))_n} \ll 1 \ \forall n, i,
\]

the purity very approaches to 1. In other words, the adiabatic DFS can be achieved, if the condition

\[
\text{Max}_{n,i} \left| \frac{4 \langle \Phi_i(t) |\partial_{t} |\Phi_i(t)\rangle \rangle}{\omega_{ni}(t) + i(\Gamma(t))_n} \right| \ll 1
\]

is satisfied, which is the adiabatic condition Eq. (2) presented in Sec. II A.
Appendix B: The Derivation of Eq. (13)

We start with the eigen-equation of Lindblad operators Eq. (6). Taking the derivative of time on the eigen-equation, we have

\[ \partial_t \hat{\Phi}_\alpha(t) |\Phi_\alpha(t)\rangle + \hat{\Phi}_\alpha(t) \partial_t |\Phi_\alpha(t)\rangle = \partial_t c_\alpha(t) |\Phi_\alpha(t)\rangle + c_\alpha(t) |\Phi_\alpha(t)\rangle. \]

Then we act \( |\Phi_\alpha(t)\rangle \) from the left to obtain

\[ \langle \Phi_\alpha^+ | \partial_t \hat{F}_\alpha(t) |\Phi_\alpha(t)\rangle + \langle \Phi_\alpha^+ | \hat{F}_\alpha(t) \partial_t |\Phi_\alpha(t)\rangle = c_\alpha(t) \langle \Phi_\alpha^+ | \partial_t |\Phi_\alpha(t)\rangle. \]

We plug a complete set into the second term of above equation and consider the eigen-equation of Lindblad operators, to obtain

\[ \langle \Phi_\alpha^+ | \hat{F}_\alpha(t) \partial_t |\Phi_\alpha(t)\rangle = \sum_{m \neq n} \langle \Phi_\alpha^+ | \hat{F}_\alpha(t) |\Phi_m^\perp(t)\rangle \langle \Phi_m^\perp(t) | \partial_t |\Phi_\alpha(t)\rangle + \langle \hat{F}_\alpha(t) \rangle_n \langle \Phi_\alpha^+ | \partial_t |\Phi_\alpha(t)\rangle. \]

Here, we assume that \( c_\alpha \neq \langle \hat{F}_\alpha \rangle_n (\forall n) \). Taking this term back into Eq. (B1), we have

\[ \langle \Phi_\alpha^+ | \partial_t |\Phi_\alpha(t)\rangle = \frac{\langle \Phi_\alpha^+ | \partial_t \hat{F}_\alpha |\Phi_\alpha(t)\rangle}{c_\alpha - \langle \hat{F}_\alpha \rangle_n} + \sum_{m \neq n} \frac{\langle \Phi_\alpha^+ | \hat{F}_\alpha |\Phi_m^\perp \rangle}{c_\alpha - \langle \hat{F}_\alpha \rangle_n} \langle \Phi_m^\perp | \partial_t |\Phi_\alpha(t)\rangle. \]

For brevity, we have omitted the time arguments on the above equation. By means of same procedure, we obtain

\[ \langle \Phi_m^\perp | \partial_t |\Phi_\alpha(t)\rangle = \frac{\langle \Phi_m^\perp | \partial_t \hat{F}_\alpha |\Phi_\alpha(t)\rangle}{c_\alpha - \langle \hat{F}_\alpha \rangle_m} + \sum_{l \neq m} \frac{\langle \Phi_m^\perp | \hat{F}_\alpha |\Phi_l^\perp \rangle}{c_\alpha - \langle \hat{F}_\alpha \rangle_m} \langle \Phi_l^\perp | \partial_t |\Phi_\alpha(t)\rangle. \]

Let us substitute Eq. (B3) into Eq. (B2),

\[ \langle \Phi_\alpha^+ (t) | \partial_t |\Phi_\alpha(t)\rangle = \frac{\langle \Phi_\alpha^+ | \partial_t \hat{F}_\alpha |\Phi_\alpha \rangle}{c_\alpha - \langle \hat{F}_\alpha \rangle_n} + \sum_{m \neq n} \frac{\langle \Phi_\alpha^+ | \hat{F}_\alpha |\Phi_m^\perp \rangle}{c_\alpha - \langle \hat{F}_\alpha \rangle_n} \langle \Phi_m^\perp | \partial_t |\Phi_\alpha \rangle + \sum_{l \neq m} \frac{\langle \Phi_m^\perp | \hat{F}_\alpha |\Phi_l^\perp \rangle}{c_\alpha - \langle \hat{F}_\alpha \rangle_m} \langle \Phi_l^\perp | \partial_t |\Phi_\alpha \rangle. \]

We can find that \( \langle \Phi_\alpha^+ | \partial_t |\Phi_\alpha \rangle \) emerges again. Therefore, we repeat this procedure again and again until all of bases of the complementary subspace are considered,

\[ \langle \Phi_\alpha^+ (t) | \partial_t |\Phi_\alpha(t)\rangle = \frac{\langle \Phi_\alpha^+ | \partial_t \hat{F}_\alpha |\Phi_\alpha \rangle}{c_\alpha - \langle \hat{F}_\alpha \rangle_n} + \sum_{m \neq n} F_{nm} \frac{\langle \Phi_\alpha^+ | \partial_t \hat{F}_\alpha |\Phi_m \rangle}{c_\alpha - \langle \hat{F}_\alpha \rangle_m} + \sum_{l \neq m} \sum_{m \neq n} F_{nm} F_{ml} \frac{\langle \Phi_{ml}^\perp | \partial_t \hat{F}_\alpha |\Phi_m \rangle}{c_\alpha - \langle \hat{F}_\alpha \rangle_l} + \ldots \]

where

\[ F_{nm} = \frac{\langle \Phi_\alpha^+ | \hat{F}_\alpha |\Phi_m^\perp \rangle}{c_\alpha - \langle \hat{F}_\alpha \rangle_n}. \]

At this time, the adiabatic condition can be rewrite as
Thus
\[
\left| \Phi_n^\dagger(t) \partial_t \Phi_n(t) \right| \leq \left| \frac{\Phi_n^\dagger \partial_t \tilde{F}_n}{\omega_n + i\Gamma_n}(c_a - \langle F_a \rangle_n) \right| + \sum_{m \neq n} \left| F_{nm} \right| \left| \frac{\Phi_m^\dagger \partial_t \tilde{F}_m}{\omega_m + i\Gamma_m}(c_a - \langle F_a \rangle_n) \right| 
+ \sum_{m \neq n} \sum_{i \neq m} \left| F_{nm} \right| \left| \frac{\Phi_i^\dagger \partial_t \tilde{F}_i}{\omega_i + i\Gamma_i}(c_a - \langle F_a \rangle_i) \right| + \ldots
\]

Here, we have used the facts that any complex number \(a\) and \(b\) fulfill \(|ab| \leq |a||b|\) and \(|a + b| \leq |a| + |b|\). Here, we assume that all of \(\{|F_{nm}|, \forall m, n\}\) are bounded by a finite real number \(F_{\text{Max}}\). Replacing all \(\left| \frac{\Phi_n^\dagger \partial_t \tilde{F}_n}{\omega_n(c_a - \langle F_a \rangle_n)} \right|\) terms in above equation by the maximum among them, we can rewrite the adiabatic condition Eq.(5) into
\[
\text{Max}_{(n,i)} \left| \frac{\Phi_n^\dagger \partial_t \tilde{F}_n}{\omega_n + i\Gamma_n}(c_a - \langle F_a \rangle_n) \right| \leq \frac{1}{N-M} \text{Max}_{(n,i)} \left| \frac{\Phi_n^\dagger \partial_t \tilde{F}_n}{\omega_n + i\Gamma_n}(c_a - \langle F_a \rangle_n) \right| 
\times \left[ (N-M + (N-M)(N-M-1)F_{\text{Max}} + (N-M)(N-M-1)(N-M-2)F_{\text{Max}}^2 + \ldots + (N-M)!F_{\text{Max}}^{N-M}] \right]^{(n)}
\]

\[
= \left( \sum_{a=0}^{N-M-1} \frac{P_{a+1}^a}{N-M} (F_{\text{Max}})^a \right) \text{Max}_{(n,i)} \left| \frac{\Phi_n^\dagger \partial_t \tilde{F}_n}{\omega_n + i\Gamma_n}(c_a - \langle F_a \rangle_n) \right| \],
\]

where \(P_{N-M}^{n+1}\) is the number of permutation.

**Appendix C: The lower bound of the Purity**

We start with the inequality mentioned in Eq.(17), i.e.,
\[
p(t) - 1 \geq 4\text{Re} \left\{ \sum_{ijm} -i\bar{\rho}_j \rho_{im} \frac{\Phi_m^\dagger \partial_t \Phi_j}{\omega_m + i\Gamma_m} \exp \left( i \int_0^t (\omega_{mj} + i\Gamma_m)d\tau \right) + i \int_0^t \partial_t \bar{\rho}_j \rho_{im} \frac{\Phi_m^\dagger \partial_t \Phi_j}{\omega_m + i\Gamma_m} \exp \left( i \int_0^{t'} (\omega_{mj} + i\Gamma_m)d\tau \right) dt' + \ldots \right\}.
\]

Since the lower bound considered here is under adiabatic condition, we check the dynamical equation if the adiabatic approximation are reached. After transforming by the operator \(\hat{\rho} = \tilde{F}_n \hat{\rho} \tilde{F}_n\), the master equation can be rewritten as Eq.(A1), i.e.,
\[
\partial_t \hat{\rho} = -i[H_{\text{eff}} + \hat{G}, \hat{\rho}] + \sum_{\alpha} \left( \hat{F}_\alpha \hat{\rho} \hat{F}_\alpha^\dagger - \frac{1}{2} \hat{F}_\alpha^\dagger \hat{F}_\alpha \hat{\rho} \right). \]

Hence, we can obtain
\[
\partial_t \bar{\rho}_{ji} = \langle \Phi_j(0) | \partial_t \tilde{F}_n | \Phi_i(0) \rangle
= -i\langle \Phi_j(0) | [H_{\text{eff}} + \hat{G}, \hat{\rho}] | \Phi_i(0) \rangle + \sum_{\alpha} \langle \Phi_j(0) | \left( \hat{F}_\alpha \hat{\rho} \hat{F}_\alpha^\dagger - \frac{1}{2} \hat{F}_\alpha^\dagger \hat{F}_\alpha \hat{\rho} \right) | \Phi_i(0) \rangle
= -i\langle \Phi_j(0) | ((H_{\text{eff}} + \hat{G})\hat{\rho} - \hat{\rho}(H_{\text{eff}} + \hat{G})| \Phi_i(0) \rangle
\]
\[
\begin{aligned}
&= \sum_k \left( \exp \left( i \int_0^t \omega_{jk} d\tau \right) \langle \partial_t \Phi_j(t) | \Phi_k(t) \rangle \bar{\rho}_{ki} - \exp \left( i \int_0^t \omega_{ki} d\tau \right) \bar{\rho}_{jk} \langle \partial_t \Phi_k(t) | \Phi_i(t) \rangle \right) \\
&\quad + \sum_n \left( \exp \left( i \int_0^t \omega_{jn} - i \langle \tilde{\Gamma}(\tau) \rangle_n d\tau \right) \langle \partial_t \Phi_j(t) | \Phi_n^\perp(t) \rangle \bar{\rho}_{ni} \\
&\quad - \exp \left( i \int_0^t (\omega_{ni} + i \langle \tilde{\Gamma}(\tau) \rangle_n) d\tau \right) \bar{\rho}_{jn} \langle \partial_t \Phi_n^\perp(t) | \Phi_i(t) \rangle \right) .
\end{aligned}
\]

and

\[
\partial_t \bar{\rho}_{im} = \langle \Phi_i(0) | \partial_t \bar{\rho}(t) | \Phi_m^\perp(0) \rangle \\
= \langle \Phi_i(0) | \left( -i [\hat{H}_{\text{eff}}, \bar{C}, \bar{\rho}] + \sum_\alpha \left( \bar{F}_\alpha \bar{\rho} \bar{F}_\alpha^\perp - \frac{1}{2} \{ \bar{F}_\alpha^\perp, \bar{F}_\alpha \} \right) \right) | \Phi_m^\perp(0) \rangle \\
= -i \left( \sum_j \exp \left( i \int_0^t \langle \hat{H}_{\text{eff}}(\tau) \rangle_j - \langle \hat{H}_{\text{eff}}(\tau) \rangle_j d\tau \right) \langle \partial_t \Phi_i(t) | \Phi_j(t) \rangle \bar{\rho}_{jm} \\
+ \sum_n \exp \left( i \int_0^t \omega_{jn} - i \langle \tilde{\Gamma}(\tau) \rangle_n d\tau \right) \langle \partial_t \Phi_i(t) | \Phi_n^\perp(t) \rangle \bar{\rho}_{nm} \\
- \sum_j \exp \left( i \int_0^t \omega_{jm} - i \langle \tilde{\Gamma}(\tau) \rangle_m d\tau \right) \bar{\rho}_{ij} \langle \partial_t \Phi_j(t) | \Phi_m(t) \rangle \\
- \sum_n \exp \left( i \int_0^t (\omega_{nm} + i \langle \tilde{\Gamma}(\tau) \rangle_n - i \langle \tilde{\Gamma}(\tau) \rangle_m) d\tau \right) \bar{\rho}_{in} \langle \partial_t \Phi_n^\perp(t) | \Phi_m(t) \rangle \\
- \sum_{n \neq m} \sum_n \bar{\rho}_{jm} \langle \Phi_n^\perp(t) | \tilde{\Gamma}(\tau) | \Phi_m(t) \rangle \exp \left( i \int_0^t \omega_{nm} + i \langle \tilde{\Gamma}(\tau) \rangle_n - i \langle \tilde{\Gamma}(\tau) \rangle_m d\tau \right) \right) .
\]

In this expression, we have also omitted the "jump" terms \( \bar{F}_\alpha \bar{\rho} \bar{F}_\alpha^\perp \) with the same reason in Appendix A. Taking them into Eq. (17), the lower bound can be rewritten as

\[
p(t) - 1 \geq 4\text{Re} \left\{ \sum_{ijm} -i \rho_{ji} \rho_{im} \frac{\langle \Phi_m^\perp(t) | \partial_t | \Phi_j(t) \rangle}{\omega_{mj} + i \Gamma_m} \\
+ \int_0^t \sum_k \langle \partial_t \Phi_j(t) | \Phi_k(t) \rangle \rho_{kj} \rho_{im} \frac{\langle \Phi_m^\perp(t) | \partial_t | \Phi_j(t) \rangle}{\omega_{mj} + i \Gamma_m} dt' \\
+ \int_0^t \sum_n \langle \partial_t \Phi_j(t) | \Phi_n^\perp(t) \rangle \rho_{jn} \rho_{im} \frac{\langle \Phi_m^\perp(t) | \partial_t | \Phi_j(t) \rangle}{\omega_{mj} + i \Gamma_m} dt' \\
- \int_0^t \sum_n \rho_{jn} \langle \partial_t \Phi_n^\perp(t) | \Phi_j(t) \rangle \rho_{im} \frac{\langle \Phi_m^\perp(t) | \partial_t | \Phi_j(t) \rangle}{\omega_{mj} + i \Gamma_m} dt' \\
+ \int_0^t \sum_n \rho_{jn} \langle \partial_t \Phi_n^\perp(t) | \Phi_j(t) \rangle \rho_{jm} \frac{\langle \Phi_m^\perp(t) | \partial_t | \Phi_j(t) \rangle}{\omega_{mj} + i \Gamma_m} dt' \\
- \int_0^t \sum_k \rho_{kj} \rho_{im} \langle \partial_t \Phi_k(t) | \Phi_j(t) \rangle \frac{\langle \Phi_m^\perp(t) | \partial_t | \Phi_j(t) \rangle}{\omega_{mj} + i \Gamma_m} dt' \\
- \int_0^t \sum_n \rho_{jn} \rho_{im} \langle \Phi_n^\perp(t) | \bar{\Gamma} | \Phi_m(t) \rangle \frac{\langle \Phi_m^\perp(t) | \partial_t | \Phi_j(t) \rangle}{\omega_{mj} + i \Gamma_m} dt' \\
+ i \int_0^t \rho_{ji} \rho_{im} \frac{\partial}{\partial t'} \left( \frac{\langle \Phi_m^\perp(t) | \partial_t | \Phi_j(t) \rangle}{\omega_{mj} + i \Gamma_m} \right) dt' \right\} ,
\]

\]
with \( \rho_{ij} = \langle \Phi_i(t)|\hat{\rho}(t)|\Phi_j(t) \rangle \) and \( \rho_{in} = \langle \Phi_i(t)|\hat{\rho}(t)|\Phi_n^\perp(t) \rangle \). In the general case, although it is difficult to estimate exactly the values of the integrals in above equation, it is still possible to obtain bounds on the integrals, which will lead to the lower bound. Noting that \(|\rho_{ij}| \leq 1\) and \(|\exp(i \int_0^t \omega_n(t) dt)| = 1\), we have

\[
1 - p(t) \leq 4 \sum_{jm} \left| \frac{\langle \Phi_m^\perp|\partial_t|\Phi_j \rangle}{\omega_m + i\Gamma_m} \right| + M \left[ \sum_{k} \left| \langle \partial_t \Phi_j(t)|\Phi_k(t) \rangle \right| + \sum_{n} \left| \langle \partial_t \Phi_j(t)|\Phi_n^\perp(t) \rangle \right| \right] \frac{\langle \Phi_m^\perp|\partial_t|\Phi_j \rangle}{\omega_m + i\Gamma_m} dt' + \sum_{i} \left[ \sum_{n} \left| \langle \partial_t \Phi_k(t)|\Phi_i(t) \rangle \right| + \left| \langle \partial_t \Phi_k(t)|\Phi_n^\perp(t) \rangle \right| \right] \frac{\langle \Phi_m^\perp|\partial_t|\Phi_j \rangle}{\omega_m + i\Gamma_m} dt' + M \left[ \int_0^t \left| \frac{\partial_t}{\omega_m + i\Gamma_m} \left( \frac{\Phi_m^\perp}{\omega_m + i\Gamma_m} \right) \right| dt' \right] = 4M \sum_{jm} \left| \frac{\langle \Phi_m^\perp|\partial_t|\Phi_j \rangle}{\omega_m + i\Gamma_m} \right| + \int_0^t (A_j + B_m + C) \frac{\langle \Phi_m^\perp|\partial_t|\Phi_j \rangle}{\omega_m + i\Gamma_m} dt' + \int_0^t \left| \frac{\partial_t}{\omega_m + i\Gamma_m} \left( \frac{\Phi_m^\perp}{\omega_m + i\Gamma_m} \right) \right| dt',
\]

where

\[
A_j = \sum_{k} \left| \langle \partial_t \Phi_j(t)|\Phi_k(t) \rangle \right| + \sum_{n} \left| \langle \partial_t \Phi_j(t)|\Phi_n^\perp(t) \rangle \right| \\
B_m = \sum_{k} \left| \langle \partial_t \Phi_k(t)|\Phi_m^\perp(t) \rangle \right| + \sum_{n} \left| \langle \partial_t \Phi_k(t)|\Phi_m^\perp(t) \rangle \right| + \sum_{n \neq m} \left| \langle \Phi_n^\perp|\Phi_m^\perp \rangle \right| \\
C = \sum_{i} \sum_{n} \left( \left| \langle \partial_t \Phi_n^\perp(t)|\Phi_i(t) \rangle \right| + \left| \langle \partial_t \Phi_n^\perp(t)|\Phi_n^\perp(t) \rangle \right| \right) / M
\]

In the derivation, we have used the fact that \(|ab| \leq |a||b|\) and \(|a+b| \leq |a|+|b|\) with arbitrary complex numbers \(a\) and \(b\). By uniting like terms, we reach Eq. (14) in Sec III C.
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