Symplectic Solutions in Stress Analysis of Thin Plates

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Abstract — Aiming at the common elastic thin plate materials in engineering, the dual control differential equation and the complete solution space composed of zero eigensolution and non-zero eigensolution are established in the state space. On this basis, the non-homogeneous condition of the boundary and the solution method of the non-homogeneous equation are studied systematically. The stress distribution of thin plate bending problem is discussed with an example, and the stress concentration of material under special constraints is analyzed.

1. INTRODUCTION
Symplectic system method is a new algorithm in the field of solid mechanics in recent years. Because the eigensolution expansion method is adopted, due to the eigensolution expansion method, the problem is reduced to solving Hamiltonian operator matrix [1]. In symplectic system, the study of elasticity takes the primal variable and its dual variable as the basic variables, which makes the method of separating variables be implemented smoothly, thus forming a unique direct method. Xu et al. obtained a perfect analytical solution to the problem of elastic revolution [2]. Steele and Kim [3] established the hybrid variational principle method, and discussed the elastic dynamic problem. The significance of the transition from Lagrangian system to Hamiltonian system lies in the transition from the traditional Euclidean geometry to the symplectic geometry, breaking through the traditional concept, so that the dual mixed variable method can be applied to the vast field of applied mechanics, breaking the traditional solution system, and replacing the solution system of elastic mechanics. This method is of great significance to engineering mechanics system, mathematical physics method and its radiation to other disciplines, and provides a powerful tool for discussing this kind of problems and improves the solution of elasticity to a new platform. The research results of these works are a revelation to viscoelastic problems. Symplectic system is introduced into viscoelasticity in symplectic geometry space. This method, marked by all state vector (mixed vector), is more suitable for the development trend of computer [4].

The influence of temperature on the properties of viscoelastic materials is reflected in the non-homogeneous dual equation and lateral condition [5, 6]. The nonhomogeneous lateral condition problem can be transformed into homogeneous one by means of variable substitution, so the thermoviscoelastic problem can be reduced to the special solution problem of solving nonhomogeneous equation. In this paper, the original variable and its dual variable (mixed variable) are taken as the basic variables of the problem. With the help of Laplace transformation and correspondence principle, the
Hamiltonian system method with the characteristics of conservative system is introduced to viscoelastic materials and structural mechanics of non conservative system.

2. SOLUTION METHOD

In the temperature field, the stress-strain relationship of viscoelastic materials can be expressed as follows:

\[
S_{ij} + \eta_m \frac{\partial S_{ij}}{\partial t} = 2G_k e_{ij} + 2\eta G \frac{\partial e_{ij}}{\partial t},
\]

\[
\sigma = 3K e - \theta T \quad (G_k \eta_m \leq \eta)
\]

To derive the governing equations in the Hamiltonian system, we express the strain energy density as

\[
\hat{L} = \frac{E(1-v)}{2(1+v)(1-2v)} \left[ (\partial_{x} \tilde{w})^2 + \tilde{u}_t^2 \right] + \frac{E_v}{(1+v)(1-2v)} \tilde{u}_t \tilde{w} \partial_{x} \tilde{w} + \frac{E}{4(1+v)} \left( \tilde{w} + \partial_{x} \tilde{v} \right)^2 - \theta T \left( \tilde{u} + \partial_{x} \tilde{w} \right)
\]

Dual vectors of the displacement vector can be obtained by

\[
\tilde{p} = \partial_{x} \tilde{L} = \begin{bmatrix} \sigma_x \\ \tau_{xy} \end{bmatrix}
\]

Using the principle of minimum total potential energy

\[
\delta \int_{-l}^{l} \int_{-b}^{b} \hat{L} dx dy = 0
\]

the Hamiltonian function described by dual variable is expressed by

\[
\hat{\varphi} = H \varphi + \tilde{f}
\]

in which

\[
\tilde{f} = \begin{bmatrix} 0 \\ 0 \\ v \frac{\sigma_0 (b+y) + \sigma_0 (b-y)}{2b} + \theta T \\ \frac{1}{2b} [ \tau_{0} (b+y) + \tau_{0} (b-y) ] \end{bmatrix}
\]
The fundamental solutions of Eq. (5) can be expressed in two groups

\[
\psi = \begin{bmatrix}
\bar{\alpha}_1 \cos(\kappa y) + \bar{\alpha}_2 \sin(\kappa y) \\
\bar{\alpha}_2 \sin(\kappa y) + \bar{\alpha}_1 \cos(\kappa y) \\
\bar{\alpha}_3 \cos(\kappa y) + \bar{\alpha}_4 \sin(\kappa y) \\
\bar{\alpha}_4 \sin(\kappa y) + \bar{\alpha}_3 \cos(\kappa y)
\end{bmatrix}
\] (7)

and the functions

\[
\tilde{\psi} = \begin{bmatrix}
\bar{\alpha}_3 \sin(\kappa y) + \bar{\alpha}_4 \cos(\kappa y) \\
\bar{\alpha}_4 \cos(\kappa y) + \bar{\alpha}_3 \sin(\kappa y) \\
\bar{\alpha}_5 \sin(\kappa y) + \bar{\alpha}_6 \cos(\kappa y) \\
\bar{\alpha}_6 \cos(\kappa y) + \bar{\alpha}_5 \sin(\kappa y)
\end{bmatrix}
\] (8)

Substituting the Eqs. (7) and (8) into Eq. (6), we can get the integral constants. Based on Eq. (4), the eigenvalue equations are obtained as

\[
2\kappa b + \sin(2\kappa b) = 0 \quad (9)
\]

\[
2\kappa b - \sin(2\kappa b) = 0 \quad (10)
\]

The above equations show that the key of thermoviscoelastic problem is to solve the special solution of nonhomogeneous equation. In order to facilitate the theoretical derivation, the nonhomogeneous term in the nonhomogeneous governing equations is expanded in the form of fundamental eigensolutions

\[
\bar{f} = \sum_n \left[ a_n(x) \bar{\eta}_n^{(a)} + b_n(x) \eta_n^{(b)} \right]
\] (11)

According to the symplectic orthogonal relation, the expansion coefficients in the equation can be obtained by the following formula

\[
a_n(x) = \left[ f, J, \eta_n^{(a)} \right], \quad b_n(x) = -\left[ f, J, \eta_n^{(a)} \right]
\] (12)

The brackets in the formula represent symplectic inner product, and the special solution of nonhomogeneous equation can also be expressed in the same way as

\[
\eta_p = \sum_n \left[ A_n(x) \eta_n^{(a)} + B_n(x) \eta_n^{(b)} \right]
\] (13)

Then we introduce the new variables as

\[
\tilde{\psi}' = \tilde{\psi} - \tilde{\psi}''
\] (14)

Let's consider the following boundary conditions:
Eqs. (48) and (49) can be expressed by

\[ q = q_0 \quad (x = -a) \]
\[ p = p_0 \quad (x = a) \]  

(16)

Thus, the boundary condition turn to be homogeneous

\[ v \sigma_x' + E \epsilon_y \bar{w} = 0, \quad \tau_{xy}' = 0 \quad (y = \pm b) \]  

(17)

Since Eq. (14) has non-zero solution, its coefficient determinant must be zero

\[ A_n(x) = \int_0^x a_n(\zeta) d\zeta, \quad B_n(x) = \int_0^x b_n(\zeta) d\zeta \]  

(18)

Therefore, we can establish algebraic equations about integral constants

3. Numerical Results
Consider the case of external forces at the free end

\[ \sigma_{-l} = 1, \quad \tau_{-l} = 0 \quad (x = -l), \]  

(19)

The heat conditions are

\[ T_{-l} = 1 - y^2 \quad (x = -l) \]
\[ T_l = (y + 1) / 3 \quad (x = l) \]  

(20)
Fig.1 is the calculated temperature field. According to the treatment of boundary conditions, the corresponding stress and deformation can be obtained. Fig.2 and Fig.3 show the distribution of normal stress and shear stress respectively. The numerical results show that due to the limited displacement of the fixed end, there is a significant stress concentration near the fixed end, especially near the two sharp corners. According to the Saint-Venant principle, the stress effects should be confined in the regain very near the boundary, and cannot be transferred far away. This point of view is well verified by Fig. 4 and Fig. 5.
4. CONCLUSION
The mechanical properties of viscoelastic materials are very sensitive to temperature, and the temperature influence on the properties of the materials is reflected in the inhomogeneous dual equation and the lateral condition. The nonhomogeneous boundary condition problem can be transformed into homogeneous boundary condition problem by means of variable substitution, so the thermoviscoelastic problem can be reduced to the special solution problem of solving nonhomogeneous equation.

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