Conformal Geodesics Cannot Spiral

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Abstract

We show that conformal geodesics on a Riemannian manifold cannot spiral: there does not exist a conformal geodesic which becomes trapped in every neighbourhood of a point.

In memory of Bernd Schmidt (1941–2023)

1 Introduction

A pseudo-Riemannian manifold, \((M, g)\) defines a distinguished set of curves which are metric geodesics of \(g\). These curves are solutions to a set of 2nd order ODEs and are uniquely specified locally by an initial point and an initial tangent direction. However, with the exception of null geodesics in the Lorentzian case, there is in general no relation between the metric geodesics of two conformally related metrics. By analogy with metric geodesics, conformal geodesics can be thought of as a distinguished set of curves defined on a conformal manifold \((M, [g])\). These curves are solutions to a system of conformally invariant 3rd order ODEs and are uniquely specified locally by an initial position, unit tangent direction and perpendicular acceleration [12, 11, 11]. Although metric geodesics have been studied extensively, relatively little is known about conformal geodesics.

Conformal geodesics were systematically introduced into general relativity by Friedrich and Schmidt [6]. Their motivation was to construct good co-ordinate systems for the local study of conformal boundaries, as introduced in Penrose’s process of conformal compactification. Normal or Gaussian co-ordinates based on timelike geodesics cannot be expected to behave well near such conformal boundaries - recall for example how, in compactified Minkowski space, timelike geodesics do not pass through null infinity but instead all focus at timelike infinity. It was shown in [6] (following an earlier suggestion in [10]) that one could instead define conformal normal or conformal Gaussian co-ordinates constructed from conformal geodesics. These would provide good co-ordinates near the conformal boundaries of asymptotically Minkowskian, asymptotically de Sitter or anti-de Sitter spacetimes.
An unresolved issue stemming from this earlier work, which was also known as an unsolved problem in Riemannian-signature conformal geometry, was the question of spiralling for conformal geodesics. A curve can be said to spiral at a point \( p_* \) if it enters and remains in every neighbourhood of \( p_* \) but does not pass through \( p_* \) itself. It is a classical result, based on the existence of geodesically-convex neighbourhoods [5], that metric geodesics cannot spiral, although until now this has not been shown for conformal geodesics. It is a question of interest both abstractly and because spiralling raises the possibility of a new kind of co-ordinate singularity to guard against.

In [11], spiralling was ruled out in some special cases using 1st integrals, and integrability (see also [3] for integrable, non-spiralling examples on four-manifolds). For example, it was shown that conformal geodesics defined on a Riemannian Einstein manifold cannot spiral. The conformal geodesic equations were also integrated for some example geometries and it was observed that the solutions did not spiral.

In this paper we will prove a general no-spiralling theorem for conformal geodesics (Theorem 2.3). We begin by considering the analogous result for metric geodesics. The standard proof that metric geodesics cannot spiral relies on the existence of a geodesically convex neighbourhood at each point [5, Proposition 4.2]. Such a proof will not work for conformal geodesics since, despite their variational origins [4], these are not local length minimisers. With this in mind, we begin by constructing a proof of the no-spiralling theorem for metric geodesics which does not rely on length minimisation arguments. This will then form the basis of our proof in the conformal geodesic case.

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### 2 The Conformal Geodesic Equations

Conformal geodesics are curves defined on a conformal manifold \((M, [g])\), which we assume to be Riemannian. Given any choice of metric \(g \in [g]\), these curves can be defined by the following set of conformally invariant third order differential equations [11]:

\[
\nabla_u a = -(|a|^2 + L(u, u))u + L^# u
\]

where \(u\) is the unit tangent vector to the curve (i.e. \(g(u, u) = 1\)), \(a := \nabla_u u\) is the perpendicular acceleration (i.e. \(g(u, a) = 0\)) and \(|a| := \sqrt{g(a, a)}\). The Schouten tensor \(L : T_p M \times T_p M \mapsto \mathbb{R}\) is defined by

\[
L = \frac{1}{d-2} \left( \text{Ric} - \frac{R}{2(d-1)} g \right)
\]
in terms of the Ricci tensor $Ric$, the Ricci scalar $R$ and the dimension of the manifold $d$. We denote by $L^\#: T_pM \mapsto T_pM$ the corresponding endomorphism defined by $g(L^\#u, v) = L(u, v)$ for all vector fields $u, v$.

Changing the metric to $\hat{g} = \Omega^2g$, where $\Omega : M \to \mathbb{R}$ results in changes to the Schouten tensor, the Levi–Civita connection, unit tangent vector and the acceleration

$$\hat{L} = L - \nabla \Upsilon + \Upsilon \otimes \Upsilon - \frac{1}{2}\Upsilon^2 g,$$

$$\hat{\nabla}_v w = \nabla_v w + \Upsilon(v)w + \Upsilon(w)v - g(v, w)\Upsilon^2,$$

$$\hat{u} = \Omega^{-1}u,$$

$$\hat{a} = \Omega^{-2}(a - \Upsilon^2 + \Upsilon(u)u),$$

where $\Upsilon \equiv \Omega^{-1}d\Omega$, and $\Upsilon^2$ is a vector field defined by $\Upsilon(w) = g(\Upsilon^2, w)$. It is now a matter of explicit calculation to verify that the conformal geodesic equations (2.1) are conformally invariant.

In the rest of the paper we shall fix a metric $g$ in the conformal class, and consider (2.1) for this choice of metric. We will denote the set of unit vectors in $T_pM$ by $S(T_pM)$ and $S(TM)$ will denote the corresponding unit tangent bundle over $M$.

Picard’s theorem applied to equation (2.1) shows that, locally, a conformal geodesic is uniquely specified by an initial point $p \in M$, an initial unit tangent, $u_0$, at $p$ and an initial acceleration, $a_0$, at $p$ which is perpendicular to $u_0$ [11, Theorem 1.1]. We will denote the conformal geodesic with these initial conditions and arc length parameter $t$ by $\gamma_{p, u_0, a_0}(t)$.

**Definition 2.1.** A curve, $\gamma$, with arc length parameter $t$, spirals towards a point $p^* \in M$ if for any neighbourhood $N$ containing $p^*$, there exists some $T$ such that $\gamma(t) \in N$ for all $t > T$.

We demand that the conformal geodesic has a well defined unit tangent vector and thus exclude the trivial case where it consists only of a single point.

Throughout this paper we will refer to balls in $(M, g)$ and in various tangent spaces. We make the following definition

$$B(q, r) := \{q' \in M : d(q, q') < r\}, \quad \bar{B}(q, r) := \{q' \in M : d(q, q') \leq r\} \quad (2.3)$$

where $d : M \times M \to \mathbb{R}_{\geq 0}$ denotes the Riemannian distance function induced by the metric $g$.

We will also refer to balls in the tangent space $T_{u_0}(T_pM)$, where $u_0 \in S(T_pM)$. We identify $T_{u_0}(T_pM)$ with $T_pM$ and these balls are defined using the metric $g$, which in turn defines a notion of length on the vector space $T_pM$.

### 2.1 Proof for Metric Geodesics

**Theorem 2.2.** Let $(M, g)$ be a Riemannian manifold. Then geodesics on $(M, g)$ cannot spiral.
Proof (without using length minimisation arguments)\textsuperscript{1}

- **Step 1:** Define the exponential map and geodesic ball at $p \in M$ and establish that all metric geodesics through $p$ reach the boundary of this geodesic ball.

The exponential map, $\exp_p$ at a point $p \in M$ \textsuperscript{9} Definition 1.10\textsuperscript{10} is a diffeomorphism from some neighbourhood of $U \subset T_pM$ containing 0 to a neighbourhood $V \subset M$ containing $p$. A geodesic ball is any open ball centred on $p$ and contained in $V$. The exponential map at $p$ allows us to construct a co-ordinate system on any geodesic ball. Geodesic segments foliate the geodesic ball with the point $p$ removed. For the purposes of this proof, the important property of the geodesic ball is that all metric geodesics through $p$ must reach its boundary.

- **Step 2:** Show that for any compact $K \subset M$ there exists $\epsilon > 0$ such that for all $p \in K$, $B(p, \epsilon)$ is a geodesic ball.

We define a function on $M$ which maps each point $p \in M$ to the injectivity radius at $p$ (defined to be the supremum of the set $\{r \in \mathbb{R}_{\geq 0} : B(p, r) \text{ is a geodesic ball}\}$), capped at 1 to avoid any complications involving infinite sizes. Since the exponential map defined at any $p \in M$ is a local diffeomorphism, this function is strictly positive. By the continuity of the injectivity radius \textsuperscript{8} Proposition 2.1.10\textsuperscript{10}, it is also continuous. The result then follows from the fact that a continuous function on a compact set is bounded and attains its bounds.

- **Step 3:** Use the fact that metric geodesics intersect the boundary of the geodesic ball to deduce that they cannot spiral.

Suppose the metric geodesic $\gamma$ spirals towards some $p_\ast \in M$. Then for any $R < d(p_\ast, \gamma(0))$ there is a unique $T > 0$ such that $d(p_\ast, \gamma(T)) = R$ and $d(p_\ast, \gamma(t)) < R$ for all $t > T$. Denote the point $\gamma(T)$ by $p$. This is the point at which $\gamma$ enters $B(p_\ast, R)$ for the final time.

Let $K \subset M$ containing $p_\ast$ be compact. Choose $R = \epsilon/2$ (where $\epsilon$ is as in Step 2) and find the corresponding $p$ (reducing $\epsilon$ if necessary so that $R < d(p_\ast, \gamma(0))$). By reducing $\epsilon$ further if necessary, we can assume that $p \in K$. Then, using the results of Step 2, $B(p, \epsilon)$ is a geodesic ball. From Step 1 we know that $\gamma$ must leave this geodesic ball at some later parameter value. In doing so, it must once again reach a distance of at least $R = \epsilon/2$ from $p_\ast$. This is a contradiction since we assumed that $d(p_\ast, \gamma(t)) < R$ for all $t > T$. This argument is illustrated in Figure 1\textsuperscript{1}.

\textsuperscript{1}It was pointed out to us by the referee that since this proof does not rely on length minimisation arguments, Theorem 2.2 also applies to geodesics of any affine connection.
Figure 1: We find $T \in \mathbb{R}$ and $\epsilon > 0$ such that there is a geodesic ball of radius $\epsilon$ centred at $p := \gamma(T)$, where $d(p_*, \gamma(T)) = \epsilon/2$ and $d(p_*, \gamma(t)) < \epsilon/2$ for all $t > T$. However, we have also shown that $\gamma$ must leave this geodesic ball and hence reach a distance of $\epsilon/2$ from $p_*$ at some $t > T$. This is our contradiction. We show geodesic balls as circles for ease of illustration.

2.2 Outline of Proof for Conformal Geodesics

The exponential map at $p \in M$ was useful because it allowed us to identify a subset of $M$ which was foliated by segments of metric geodesics through $p$. Inspired by this, we will define an exponential map adapted to conformal manifolds which allows us to see how conformal geodesics foliate some subset of $M$.

Despite some additional complexities in the conformal geodesic case, we are able to prove a no-spiralling theorem by following three steps analogous to those used in the proof of Theorem 2.2. These three steps correspond to the three main sections of this paper.

Theorem 2.3. Let $(M, [g])$ be a conformal manifold such that metrics in $[g]$ are smooth. Then conformal geodesics on $(M, [g])$ cannot spiral.

Outline of Proof (see Section 5 for the complete proof using results derived in Sections 3 and 4):

- Step 1: Define an exponential map adapted to conformal geodesics and identify a domain and range on which it is a homeomorphism.

In Section 3, we define a conformal geodesic analogue of the exponential map at some $(p, u_0) \in S(TM)$. Unlike for metric geodesics, there is no subset of the domain of this exponential map on which it is a homeomorphism onto some ball centred at $p$. Instead, in Section 3.3, we find that it is a homeomorphism onto some set which we call a heart at $p$ (see Definition 3.5). We say that this heart has direction $u_0$. Crucially, in Section 3.4, we are able to show that conformal geodesics
re-intersect the boundary of this heart. This is analogous to the fact that metric geodesics through some point re-intersect the boundary of a geodesic ball centred at that point.

**Result of Step 1 (Theorem 3.7):** For any \((p, u_0) \in S(TM)\), any conformal geodesic with unit tangent \(u_0\) at \(p\) re-intersects the boundary of a heart at \(p\) with direction \(u_0\).

- **Step 2:** Define a notion of size for a heart and show that given any compact \(K \subset M\), there exists \(\epsilon > 0\) such that there is a heart at \(p\) with direction \(u_0\) which has size at least \(\epsilon\), for any \(p \in K\) and any \(u_0 \in S(T_p M)\).

In the metric geodesic case we considered geodesic balls in \(M\) since these had an obvious size associated to them. In the conformal geodesic setting it is simpler to consider certain half balls in \(T_{u_0}(T_p M)\) which are mapped to hearts in \(M\). The size of a heart is defined by considering the radii of balls in \(M\) which have \(u_0\) tangent to their boundary and are contained inside this heart. Note that this size has units of length, regardless of the dimension of the conformal manifold. We show that as we vary \(p\) and \(u_0\) (as well as the radius of the domain ball\(^2\)) in \(T_{u_0}(T_p M)\) we are able to find a lower bound for this size which is strictly positive and varies continuously with \(p\) and \(u_0\). The result follows from the fact that a continuous function on a compact set is bounded and attains its bound.

**Result of Step 2 (Theorem 4.2):** Let \((M, [g])\) be a conformal manifold such that metrics in \([g]\) are smooth and let \(W\) be a compact subset of \(S(TM)\). Then there exists some \(\epsilon > 0\) such that at any \((p, u_0) \in W\) there is a heart \(H_{p,u_0}\) at \(p\) with direction \(u_0\) which has size at least \(\epsilon\).

- **Step 3:** Using the fact that a conformal geodesic with unit tangent \(u_0\) at \(p\) must re-intersect the boundary of \(H_{p,u_0}\), deduce that conformal geodesics cannot spiral.

Suppose \(\gamma\) is a conformal geodesic which spirals towards some \(p_* \in M\). We consider a metric geodesic ball centred on \(p_*\) (i.e. a set \(B(p_*, r)\) for some \(r\)) which is entered by \(\gamma\) for the final time at some point \(p\) where it has unit tangent \(u_0\) (i.e. \(\gamma\) enters the ball at \(p\) and never re-intersects its boundary). Using steps 1 and 2, we

\(^1\)This is important since we need to rule out making the heart have zero size by simply choosing the domain to be trivial. To avoid this, we choose the domain to be the ball in \(T_{u_0}(T_p M)\) with \(u_0\) perpendicular to its boundary and radius equal to

\[\frac{1}{2} \times \sup\{r : \exp_{p, u_0} \textit{is a homeomorphism on the ball of radius } r \textit{ with } u_0 \textit{ perpendicular to its boundary}\}.\]
show that it is possible to choose this ball (and the corresponding \( p \) and \( u_0 \)) such that it is contained inside some heart \( H_{p,u_0} \) with position \( p \) and direction \( u_0 \) (see (3.21)). By Theorems 3.7 and 4.2 \( \gamma \) must re-intersect the boundary of this heart, so we deduce that it must also re-intersect the boundary of the ball. This is our contradiction. □

**Result of Step 3 (Theorem 2.3):** Let \((M, [g])\) be a conformal manifold such that metrics in \([g]\) are smooth. Then conformal geodesics on \((M, [g])\) cannot spiral.

### 3 The Exponential Map for Conformal Geodesics

In this section we will complete Step 1 of the proof of Theorem 2.3 as outlined in Section 2.2.

**Step 1:** Define an exponential map adapted to conformal geodesics and identify a domain and range on which it is a homeomorphism.

For any \((p, u_0) \in S(TM)\), we define the following analogue of the exponential map appropriate to conformal geodesics (see [9, Definition 1.10] for a definition of the exponential map for metric geodesics):

**Definition 3.1 (The Exponential Map).**

\[
\exp_{p,u_0} : T_{u_0} (S (T_p M)) \ni N_0 \to M
\]

\[
A \mapsto \begin{cases} 
\gamma_{p,u_0,A_{\perp}/|A|^2} (2\pi|A|) & \text{if } A \neq 0 \\
p & \text{if } A = 0
\end{cases}
\]

(3.4)

where \( A_{\perp} := A - g(A, u_0)u_0 \) denotes the component of \( A \) which is perpendicular to \( u_0 \in S(T_p M) \) and \( N_0 \) is a neighbourhood of 0 in \( T_{u_0} (S (T_p M)) \) on which this map is well defined (see [11, Theorem 1.1]). This map is a continuous function of \( A \), as we will see in the next section.

#### 3.1 The Directional Derivative of The Exponential Map

The exponential map for metric geodesics is usually analysed by considering its directional derivatives. We will do the same thing for the conformal geodesic exponential map in Definition 3.1. Calculating these directional derivatives at \( A = 0 \) will allow us to understand this map in neighbourhoods of \( p \).

The directional derivative at 0 is

\[
D \left( \exp_{p,u_0} \right)_0 : T_0(N_0) \to T_p M
\]

\[
A \mapsto \frac{d}{d\lambda} \exp_{p,u_0} (\lambda A) \big|_{\lambda=0}
\]

(3.5)

\[
= \frac{d}{d\lambda} \gamma_{p,u_0,A_{\perp}/|A|^2} (2\pi\lambda|A|) \big|_{\lambda=0}.
\]
To calculate this, we use a power series expansion:

$$\gamma_{p,u_0,a_0}(t) = p + \sum_{n=0}^{\infty} \frac{1}{n!} t^{n+1} u^{(n)}(0).$$

(3.6)

This expression (and subsequent similar expressions) should be understood in terms of co-ordinates.

Using repeated applications of the conformal geodesic equation (2.1) and working in geodesic normal co-ordinates centred at $p$, we can express the power series (3.6) in terms of $u_0$ and $a_0$ (as well as the Schouten tensor and its derivatives evaluated at $p$ and derivatives of the Christoffel symbols evaluated at $p$). We see that, for a conformal geodesic with initial data $(p, u_0, A_\perp/(\lambda|A|^2))$, we have

$$u'(0) = a_0 = A_\perp/(\lambda|A|^2) = O(\lambda^{-1})$$

$$u''(0) = \nabla_u a|_{t=0} + O(1) = -\frac{1}{\lambda^2} \frac{|A_\perp|^2}{|A|^4} u_0 + O(1)$$

(3.7)

where $u_0$ denotes the initial unit tangent velocity and by $O(1)$ we mean terms which are bounded in the limit $\lambda \to 0$. Continuing recursively we have

$$u^{(n)}(0) = \nabla_u^{(n)} u|_{t=0} + O(\lambda^{-(n-2)}) = O(\lambda^{-n})$$

(3.8)

as $\lambda \to 0$. Here $\nabla_u^{(n)}$ denotes the directional derivative $\nabla_u$ applied $n$ times. In particular all terms involving the Schouten tensor $L$ and its derivatives, as well as terms involving derivatives of the Christoffel symbols, are $O(\lambda^{-(n-2)})$ (i.e. they are sub-leading in the limit $\lambda \to 0$). To calculate the directional derivative (3.5) we must evaluate the power series (3.6) at $t = 2\pi \lambda|A| = O(\lambda)$. We see that the terms in this power series which are first order in $\lambda$ are exactly those that arise in Euclidean space, where $L \equiv 0$. For the Euclidean metric on $\mathbb{R}^n$, conformal geodesics with initial data $(p, u_0, A_\perp/(\lambda|A|^2))$ are circles with equations of the form

$$x(t) = p + u_0|a_0|^{-1} \sin(|a_0|t) + a_0|a_0|^{-2} (1 - \cos(|a_0|t)) .$$

(3.9)

The directional derivative of the exponential map at 0 is therefore

$$D(\exp_{p,u_0})_0 : A \mapsto u_0 \frac{|A|^2}{|A|} \sin \left(2\pi \frac{|A_\perp|}{|A|} \right) + A_\perp \frac{|A|^2}{|A|} \left(1 - \cos \left(2\pi \frac{|A_\perp|}{|A|} \right) \right).$$

(3.10)

Note that this expression is finite in the limit $A_\perp \to 0$.

It is clear from the definition of the exponential map that if $A$ and $A'$ are related by a reflection in a plane perpendicular to $u_0$ (i.e. $A_\perp = A'_\perp$ and $|A| = |A'|$), then $\exp_{p,u_0}(A) = \exp_{p,u_0}(A')$. Since our aim is to find a region on which the exponential map
is a homeomorphism, we restrict our attention to the half space \( g(A, u_0) \geq 0 \). Based on equation (3.10), we define the following quantities related to \( A \perp, A \in T_pM \)

\[
\sin \theta := \frac{|A\perp|}{|A|}, \quad \theta \in [0, \pi/2] \\
\hat{a}_0 := \frac{A\perp}{|A\perp|}.
\]

(3.11)

We have assumed that \( A\perp \neq 0 \) and \( A \neq 0 \) since this will simplify expressions later on, although these cases can be obtained from the results below by taking the limits \( A\perp \to 0 \) and \( A \to 0 \) respectively.

We see that a vector \( A \) in the half space \( g(u_0, A) \geq 0 \) is uniquely defined by \( \theta, \hat{a}_0 \) and \( |A| \). The vector \( \hat{a}_0 \) defines a “quarter space” in \( T_pM \) by \( g(A, \hat{a}_0) \geq 0 \) and \( g(A, u_0) \geq 0 \). The variable \( \theta \) is the angle between \( A \) and \( 3u_0 \).

In terms of \( |A|, \theta \) and \( \hat{a}_0 \), we can re-write the directional derivative (3.10) as

\[
D(\exp_{p, u_0})_0 : A \mapsto u_0 \frac{|A|}{\sin \theta} \sin (2\pi \sin \theta) + \hat{a}_0 \frac{|A|}{\sin \theta} (1 - \cos (2\pi \sin \theta)).
\]

(3.12)

We therefore have the following leading order expression for the exponential map

\[
\exp_{p, u_0}(|A|, \theta, \hat{a}_0) = p + u_0 \frac{|A|}{\sin \theta} \sin (2\pi \sin \theta) + \hat{a}_0 \frac{|A|}{\sin \theta} (1 - \cos (2\pi \sin \theta)) + O(|A|^3).
\]

(3.13)

If we hold \( \hat{a}_0 \) and \( \sin \theta/|A| \) fixed while varying \( |A| \), we see from Definition 3.1 that the image of this curve under the exponential map describes a conformal geodesic parameterised by \( |A| \). Suppose we extend \( u_0, \hat{a}_0 \) to obtain an orthonormal basis of \( T_pM \). Using the metric geodesic exponential map, this basis corresponds to a set of geodesic normal co-ordinates for a neighbourhood of \( p \in M \). This co-ordinate expression for the conformal geodesic is the same as the co-ordinate expression for a circle in \( \mathbb{R}^n \) to first order in \( |A| \), with the \( O(|A|^2) \) terms vanishing. Recall that conformal geodesics in Euclidean space are exactly circles (all higher order terms in \( |A| \) vanish). By analogy, if we do a power series expansion of metric geodesics in these same co-ordinates we also find that the 2nd order term in the expansion parameter vanishes. Recall that metric geodesics in Euclidean space are exactly straight lines (i.e. there exist co-ordinates in which all non-linear terms in the expansion parameter vanish).

### 3.2 The Image of a Ray Under the Exponential Map

In this section we will study the image of a ray in the half space \( g(u_0, A) \geq 0 \) originating from \( A = 0 \). Such a ray is defined by fixing \( \theta \) and \( \hat{a}_0 \). As in Section 3.1, \( \hat{a}_0 \) defines a quarter space in \( T_pM \) via the conditions \( g(u_0, A) \geq 0 \), \( g(\hat{a}_0, A) \geq 0 \) and \( \theta \) denotes the angle between \( A \) and \( u_0 \).

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\(^3\)For fixed \((u_0, \hat{a}_0)\) we will draw sketches in the \( u_0 - \hat{a}_0 \) half space defined by \( g(A, u_0) \geq 0 \). This actually corresponds to considering \( \theta \in [0, \pi/2] \) for both \( \hat{a}_0 \) and \( -\hat{a}_0 \).
Figure 2: The directional derivative in some direction in the $u_0 - \hat{a}_0$ plane which makes an angle of $\theta$ with the positive $u_0$ axis is a vector in the same plane making an angle of $\pi \sin \theta$ with the positive $u_0$ axis.

Lemma 3.2. Let $(p, u_0) \in S(TM)$. Under the exponential map at $(p, u_0)$, a ray from $A = 0$ defined by $\hat{a}_0$ and $\theta = \theta_0$ is mapped to a curve in $M$ whose tangent at $p$ lies in the half space of span{$u_0, \hat{a}_0$} defined by $g(u_0, \hat{a}_0) \geq 0$ and makes an angle of $\theta = \pi \sin \theta_0$ with $u_0$ (see Figure 2).

Proof: The directional derivative at 0 in the direction tangent to a particular ray is given by equation (3.12) and is non-zero provided $A \neq 0$ and $\theta \neq \pi/2$ (i.e. $A \perp \neq A$). This directional derivative gives the tangent direction at $p$ of the curve which is the image of this ray under the exponential map. We see that this image curve lies in the same quarter space as $A$ and that the angle, $\theta$, between the directional derivative and the vector $u_0$ satisfies

$$\tan \theta = \frac{1 - \cos(2\pi \sin \theta_0)}{\sin(2\pi \sin \theta_0)}$$

$$= \tan(\pi \sin \theta_0)$$

$$\Rightarrow \quad \theta = \pi \sin \theta_0.$$ (3.14)

3.3 The Exponential Map is a Homeomorphism onto a Heart

Lemma 3.2 tells us that the exponential map takes distinct rays through the origin in $T_pM$ to curves with distinct tangent directions at $p$. We will show that for rays confined to a wedge in $T_pM$, we can find a lower bound on the arc length parameter (equivalently a lower bound on $|A|$ in Definition 3.4) at which their image curves can re-intersect. This tells us that the exponential map is a homeomorphism on this wedge. We will then consider a union of such wedges. We find that as we increase the opening angle of the wedge towards $\pi$, the lower bound on the arc length parameter we obtain decreases to 0. We conclude that
the exponential map is a homeomorphism on some region in the half space \( g(A, u_0) \geq 0 \) with cross-sections like the one shown on the right of Figure 6.

**Definition 3.3.** For any \( u_0 \in S(T_pM) \), let \( C(u_0, r) \) denote the open ball in \( T_{u_0}(T_pM) \) of radius \( r \) (defined using the metric \( g \), where we identify \( T_{u_0}(T_pM) \) with \( T_pM \)) which has \( u_0 \) inward pointing and perpendicular to its boundary at \( p \).

**Theorem 3.4.** Let \( (p, u_0) \in S(TM) \). Then the exponential map at \( (p, u_0) \) is a homeomorphism on an open ball \( C(u_0, r) \) for some \( r > 0 \).

In Section 3.1 we calculated an expression for the directional derivative of the exponential map at \( A = 0 \). We now calculate the derivative at \( A \neq 0 \). To do this we calculate the terms in \( \exp_{p,u_0}(A + \delta A) - \exp_{p,u_0}(A) \) which are first order in the variation \( \delta A \).

Recall that for any \( u_0 \in S(T_pM) \) and \( A \in T_{u_0}(T_pM) \) we can identify \( T_{u_0}(T_pM) \) with \( T_pM \) and write \( A \) as a linear combination of the perpendicular unit vectors \( u_0 \) and \( \hat{a}_0 \) (but note that if \( A \) is parallel to \( u_0 \) then no such \( \hat{a}_0 \) is defined or required). It follows that any \( \delta A \in T_{u_0}T_pM \) can be written as \( \delta A = \alpha u_0 + \beta \hat{a}_0 + \gamma \hat{a}_0' \), where \( \hat{a}_0' \) is a unit vector perpendicular to both \( u_0 \) and \( \hat{a}_0 \) and we take \( \gamma \geq 0 \) without loss of generality. Once again note that if \( A \) is parallel to \( u_0 \) then \( \hat{a}_0 \) is not required. Equation (3.13), suggests it will be more convenient to express \( \delta A \) in terms of variations in \( |A| \) and the quantities \( \sin \theta \) and \( \hat{a}_0 \) introduced in (3.11). Similar to before we assume that \( A_\perp \neq 0 \) (i.e. \( \sin \theta \neq 0 \)) since this will simplify the expressions, however this case can once again be obtained by taking the
limit \( A_\perp \to 0 \) in the final result and setting \( \beta \) to zero. We have

\[
\delta |A| = \alpha \cos \theta + \beta \sin \theta + O(2),
\]

\[
\delta \sin \theta = \frac{\beta \cos^2 \theta}{|A|} - \frac{\alpha \sin \theta \cos \theta}{|A|} + O(2),
\]

\[
\delta \hat{a}_0 = \frac{\gamma}{|A| \sin \theta} \hat{a}_0' + O(2).
\]

(3.15)

where by \( O(2) \) we mean terms which are second order in \( \alpha, \beta \) and \( \gamma \) (i.e. second order in \( \delta A \)).

By the chain rule, we have

\[
D(\exp_{p,u_0} A)(\delta A) = \frac{\partial \exp_{p,u_0}(A)}{\partial |A|} \delta |A| + \frac{\partial \exp_{p,u_0}(A)}{\partial \sin \theta} \delta \sin \theta + \frac{\partial \exp_{p,u_0}(A)}{\partial \hat{a}_0^\alpha} \delta \hat{a}_0^\alpha. \quad (3.16)
\]

Since we are interested in calculating \( D(\exp_{p,u_0} A)(\delta A) \) we will now drop higher order terms in \( \delta A \) (i.e. the \( O(2) \) terms in (3.15)). Evaluating each of the three terms in (3.16) separately, we have

\[
\frac{\partial \exp_{p,u_0}(A)}{\partial |A|} \delta |A| = [\alpha \cos \theta + \beta \sin \theta] \\
\times \left[ u_0 \frac{\sin (2\pi \sin \theta)}{\sin \theta} + \hat{a}_0 \frac{(1 - \cos (2\pi \sin \theta))}{\sin \theta} + O(|A|^2) \right].
\]

(3.17)

\[
\frac{\partial \exp_{p,u_0}(A)}{\partial \sin \theta} \delta \sin \theta = [\beta \cos^2 \theta - \alpha \sin \theta \cos \theta] \\
\times \left[ u_0 \frac{2\pi \cos (2\pi \sin \theta)}{\sin \theta} - \frac{\sin(2\pi \sin \theta)}{\sin^2 \theta} \right] + \hat{a}_0 \left( \frac{2\pi \sin(2\pi \sin \theta)}{\sin \theta} - \frac{1 - \cos(2\pi \sin \theta)}{\sin^2 \theta} \right) + O(|A|^2)
\]

\[
\frac{\partial \exp_{p,u_0}(A)}{\partial \hat{a}_0^\alpha} \delta \hat{a}_0^\alpha = \gamma \hat{a}_0' \left[ 1 - \cos(2\pi \sin \theta) \frac{1}{\sin^2 \theta} + O(|A|^2) \right].
\]

We are interested in when \( D(\exp_{p,u_0} A)(\delta A) \) can be zero. It is straightforward to check that if \( \theta \in [0, \theta_0) \) for some \( \theta_0 \in [0, \frac{\pi}{2}) \), each of the three terms above are linearly independent at \( |A| = 0 \) (assuming they are non-zero), and hence also at \( |A| < c_1 \), for some constant \( c_1 \) which depends on \( \theta_0 \). It follows that \( D(\exp_{p,u_0} A)(\delta A) \) can only vanish if each of the three terms above vanish individually. We calculate the norm of each quantity and define
functions $f_i(\theta, |A|) \ (i = 1, 2, 3)$ as follows.

\[
\begin{align*}
&\left| \frac{\partial \exp_{p,u_0}(A)\delta |A|}{\partial |A|} \right| = 2 |\alpha \cos \theta + \beta \sin \theta| \times \left[ \frac{\sin(\pi \sin \theta)}{\sin \theta} + O(|A|^2) \right] \\
&\left| \frac{\partial \exp_{p,u_0}(A)\delta \sin \theta}{\partial \sin \theta} \right| = 2 \left| \beta \cos^2 \theta - \alpha \sin \theta \cos \theta \right| \\
&\left| \frac{\partial \exp_{p,u_0}(A)}{\partial \tilde{a}_0^\alpha \delta \tilde{a}_0^\alpha} \right| = 2 \gamma \times \left[ \frac{\sin^2(\pi \sin \theta)}{\sin^2 \theta} + O(|A|^2) \right]
\end{align*}
\]

Plots of $f_1(\theta, 0)$ and $f_2(\theta, 0)$ are shown in Figures 4 and 5 respectively (note that $f_3(\theta, 0) = f_1(\theta, 0)^2$). We see that $f_i(\theta, 0) > 0 \ (i = 1, 2, 3)$ for $\theta \in [0, \pi/2)$, however $f_1(\pi/2, 0) = f_3(\pi/2, 0) = 0$. Now suppose we restrict to $\theta \in [0, \theta_0]$ for some $\theta_0 \in [0, \pi/2)$. Then there exists some constant $c_2 > 0$ (depending on $\theta_0$) such that $f_i(\theta, |A|) > 0 \ (i = 1, 2, 3)$ for $\theta \in [0, \theta_0]$ and $|A| < c_2$. This means that, for $\theta \in [0, \theta_0]$ and $|A| < c_2$, the three quantities in (3.17) all vanish if and only if $\alpha = \beta = \gamma = 0$, or equivalently if and only if $\delta A = 0$. It follows that $D(\exp_{p,u_0})_A$ has non-zero determinant at any $A$ with $\theta \in [0, \theta_0]$ and $|A| < c := \min(c_1, c_2)$. We refer to such a set as a wedge of radius $c$ and opening angle $\theta_0$.

As we let $\theta_0 \to \pi/2$, we see that $c \to 0$, so the radius of the wedge on which we can guarantee that $D(\exp_{p,u_0})_A$ has non-zero determinant tends to zero. We conclude that $D(\exp_{p,u_0})_A$ has non-zero determinant on a series of wedges whose radii tend to zero as their opening angle tends to $\pi$. Taking the union over all such wedges, we find that $D(\exp_{p,u_0})_A$ has non-zero determinant on some set with boundary perpendicular to $u_0$ at $p$ (see Figure 6). Such a set contains an open ball in $T_{u_0}(\mathcal{T}_p M)$ which has $u_0$ inward pointing and perpendicular to its boundary - i.e. it contains $C(u_0, r)$ for some $r > 0$.

Next we show that the exponential map is an injection on this set $C(u_0, r)$. Suppose $A, A' \in C(u_0, r)$. Since $C(u_0, r)$ is an open ball in $T_{u_0}(\mathcal{T}_p M)$, it is convex and hence $f(t) := \exp_{p,u_0}(A + t(A' - A)) \in C(u_0, r)$ for any $t \in [0, 1]$. This function is differentiable on $(0, 1)$ so, by the mean value theorem, there exists some $t_* \in (0, 1)$ with

\[
f'(t_*) = f(1) - f(0) \implies D(\exp_{p,u_0})_{A+t_*(A' - A)}(A' - A) = \exp_{p,u_0}(A') - \exp_{p,u_0}(A).
\]

Since $D(\exp_{p,u_0})_{A+t_*(A' - A)}$ has non-zero determinant, it follows that if $\exp_{p,u_0}(A') = \exp_{p,u_0}(A)$ then we must have $A = A'$. 

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Figure 4: Plot of $f_1(\theta, 0)$. We see that $f_1(\theta, 0) > 0$ for $\theta \in [0, \pi/2)$.

Figure 5: Plot of $f_2(\theta, 0)$. We see that $f_2(\theta, 0) > 0$ for $\theta \in [0, \pi/2]$.

Figure 6: The derivative of the exponential map has non-zero determinant on a series of wedges whose radii decrease to 0 as their opening angle increases towards $\pi$. Taking the union of such sets gives a set with cross sections like the one shown on the right of this figure. In particular we conclude that the derivative of the exponential map has non-zero determinant on $C(u_0, r)$ for some $r > 0$. 
We conclude that the exponential map is injective on $C(u_0, r)$ and hence is a homeomorphism from this set to its image. □

This theorem allows us to formally define the notion of a heart at $(p, u_0) \in S(TM)$ which was referred to in Section 2.2.

**Definition 3.5.** A heart at $(p, u_0) \in S(TM)$ is defined to be a set $\exp_{p,u_0}(C(u_0, r))$ where $r > 0$ is such that $\exp_{p,u_0}$ is a homeomorphism on $C(u_0, r)$.

The reason for this name is due to the shape of cross-sections of this set. As the radius of a wedge shrinks to 0, so too must the size of the smallest ball which contains its image under the exponential map. Furthermore, by Lemma 3.2 we see that as the opening angle of this wedge increases to $\pi$, the opening angle of the image set at $p$ must increase to $2\pi$ (consider two rays defined by $\pm \hat{a}_0$ and let $\theta_0 \to \frac{\pi}{2}$). In particular, the images of these two rays are both tangent to $-u_0$ at $p$. Consequently, a heart is a set with “heart shaped” cross-sections which feature a cusp at $p$ (shown on the right of Figure 3)

Note that if we choose $|A_\perp| = \epsilon^3 < 1$ and $|A| = \epsilon^2$ then the vector $A$ will be mapped to a point on a conformal geodesic which has perpendicular acceleration of size $1/\epsilon$ at $p$. By choosing $\epsilon$ arbitrarily small, this can be made arbitrarily large while simultaneously ensuring that $A \in C(u_0, r)$. As a result, given any $r > 0$, the image set $\exp_{p,u_0}(C(u_0, r))$ will always contain at least some portion of every conformal geodesic with unit tangent vector $u_0$ at $p$, even those with arbitrarily large perpendicular acceleration at this point. This is crucial for the lemma below and for our proof of the no-spiralling theorem (Theorem 2.3) in Section 5.

**Example (the Euclidean metric on $\mathbb{R}^n$):** In the case where $g$ is the Euclidean metric on $\mathbb{R}^n$, the $O(|A|^3)$ terms in equation (3.13) are identically 0 and conformal geodesics are circles. The images of distinct rays from 0 in the half space $g(u_0, A) > 0$ are also rays. These do not re-intersect, so the exponential map is a homeomorphism (in fact a diffeomorphism) from this half space onto $\mathbb{R}^n$ with the ray from 0 in the direction of $-u_0$ removed. This is essentially the familiar result that we can foliate $\mathbb{R}^n$ by circles tangent to $u_0$ at $p$ (shown for $n = 2$ in Figure 7). Since we have insisted that the conformal geodesic has initial direction $u_0$ (rather than $-u_0$) we cannot reach points on the $-u_0$ axis. These points correspond to the circle through infinity with initial direction $u_0$ at $p$.

If the Schouten tensor is not identically zero then the higher order terms in equation (3.13) become important as $|A|$ increases. These terms may cause the images of the rays to intersect. This would reduce the size of the domain on which the exponential map is a homeomorphism.

---

4As pointed out by the referee, in flat Euclidean space the shape of these cross-sections resembles that of a cardioid but is in fact slightly different. The image of a round sphere in $T_pM$ of radius $R$ with $u_0$ inward pointing and perpendicular at its boundary has cross-sections with boundary given in plane polar co-ordinates $(r, \phi)$ by $r = \frac{4R\sqrt{\pi^2 - \phi^2}}{\phi} \sin \phi$ for $\phi \in (-\pi, \pi]$. 

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Figure 7: \( \mathbb{R}^2 \) can be foliated by circles which are tangent to \( u_0 \) at \( p \).

3.4 Conformal geodesics re-intersect the heart boundary

Given any \((p, u_0) \in S(TM)\), we would like to consider the largest open ball \( C(u_0, r) \) (see Definition 3.3) on which \( \exp_{p,u_0} \) is a homeomorphism. It is of course possible this map is a homeomorphism on \( C(u_0, r) \) for all \( r > 0 \), as is the case in Euclidean space (see Section 3.3). Our proof of the no-spiralling theorem in Section 5 will be simpler if we work with a finite sized domain with boundary. With this in mind, we make the following definition

**Definition 3.6.** Given any \((p, u_0) \in S(TM)\), define

\[
    r_{(p,u_0)} := \frac{1}{2} \times \min\{1, \sup\{r : \exp_{p,u_0} \text{ is a homeomorphism on } C(u_0, r)\}\}. \tag{3.20}
\]

The factor of \( 1/2 \) is included since it will be convenient in the proof of Theorem 4.2 that the exponential map is a homeomorphism onto the heart and its boundary. We will be particularly interested in the heart which is the image of the set \( C(u_0, r_{(p,u_0)}) \) under the exponential map at \((p, u_0)\). We denote this heart by

\[
    H_{p,u_0} = \exp_{p,u_0}(C(u_0, r_{(p,u_0)})). \tag{3.21}
\]

For the proof of the no-spiralling theorem (Theorem 2.3), we will require the following property of the heart \( H_{p,u_0} \).

**Theorem 3.7.** For any \((p, u_0) \in S(TM)\), any conformal geodesic with unit tangent \( u_0 \) at \( p \) re-intersects the boundary of the heart \( H_{p,u_0} \).

**Proof:** As discussed in Section 3.1 a conformal geodesic is the image under the exponential map of a curve in \( T_pM \) defined by holding \( \hat{a}_0 \) and \( \sin \theta / |A| \) constant while...
varying $|A|$. Therefore, a conformal geodesic intersects the boundary of $H_{p,u_0}$ when this curve reaches the boundary of $C(u_0,r_{(p,u_0)}) \subset T_pM$. \qed

In fact, the curve in $T_pM$ defined by holding $\hat{a}_0$ and $\sin \theta/|A|$ constant reaches the boundary of $C(u_0, r_{(p,u_0)})$ at some $|A| \leq r_{(p,u_0)}$. Hence from the definition of the exponential map (Definition 3.1) we see that the arc length parameter distance travelled along the conformal geodesic before it reaches the boundary of $H_{p,u_0}$ is at most $2\pi r_{(p,u_0)}$.

4 Size of the Heart

The aim of this section is to follow Step 2 of the proof of Theorem 2.3 as outlined in Section 2.2.

Step 2: Define a notion of size for a heart and show that given any compact $K \subset M$, there exists $\epsilon > 0$ such that there is a heart at $p$ with direction $u_0$ which has size at least $\epsilon$, for any $p \in K$ and any $u_0 \in S(T_pM)$.

Consider the heart $H_{p,u_0}$ and let $k(p,u_0,r)$ denote the set of all closed balls in $M$ of radius $r$ with $p$ on their boundary and $u_0$ pointing into or tangent to the ball. If $r$ is sufficiently small, then all balls in $k(p,u_0,r)$ will be contained inside the closure of $H_{p,u_0}$, which we denote $\bar{H}_{p,u_0}$ (see Figure 8).

**Definition 4.1.** At any point $(p, u_0) \in S(TM)$, we define the size of the heart $H_{p,u_0}$ by

$$R : S(TM) \to \mathbb{R}_{>0} \quad (p, u_0) \mapsto \sup \{r : \bar{B} \subset \bar{H}_{p,u_0} \forall \bar{B} \in k(p,u_0,r)\}. \quad (4.22)$$

**Theorem 4.2.** Let $(M,[g])$ be a conformal manifold such that metrics in $[g]$ are smooth and let $W$ be a compact subset of $S(TM)$. Then there exists some $\epsilon > 0$ such that $R(p,u_0) \geq \epsilon$ for any $(p,u_0) \in W$.

To prove this theorem, we begin by considering the following ODE with an initial condition

$$y'(t) = f(t,y) \text{ and } y(0) = y_0. \quad (4.23)$$

**Theorem 4.3** ([7, Section V.3 Corollary 3.3 and Section V.4 Corollary 4.1]). Suppose $f(t,y)$ is of class $C^m$, $m \geq 1$, on an open $(t,y)$-set. Then (4.23) has unique solution $y = \eta(t,y_0)$ which is of class $C^m$ on its domain of existence.

**Corollary 4.4.** Suppose the metric $g$ is $C^4$. Then the heart boundary is a $C^2$ surface away from the cusp.
Figure 8: This figure shows a sample of balls in the set \( k(p,u_0,r) \). We have chosen \( r < R(p,u_0) \) which means that all of these balls are contained in the heart \( H_{p,u_0} \).

Proof: We will parameterise conformal geodesics \( x^\alpha(t) \) by their arc length, \( t \). The conformal geodesic equation can then be written as three separate 1st order ODEs on some co-ordinate patch

\[
\frac{dx^\beta}{dt} = u^\beta \\
\frac{du^\beta}{dt} = a^\beta - \Gamma^\beta_\gamma u^\gamma u^\delta \\
\frac{da^\beta}{dt} = -\Gamma^\beta_\gamma u^\gamma a^\delta - \left(g_\gamma^\delta a^\gamma a^\delta + L_\gamma u^\gamma a^\delta\right) u^\beta + L^\beta_\gamma u^\gamma. (4.24)
\]

Since the metric is \( C^4 \), the Christoffel symbols \( \Gamma^\beta_\gamma \delta \), the Schouten tensor \( L \) and the corresponding endomorphism \( L^\# \) are all \( C^2 \). Theorem 4.3 implies that the solution \( x^\alpha = \eta^\alpha(t,x_0,u_0,a_0) \) is also \( C^2 \). This tells us that, away from \( A = 0 \), the exponential map considered as a function of \( (p,u_0,A) \) is \( C^2 \). We conclude that the heart boundary is \( C^2 \) away from the cusp. \( \square \)

Next we define the following function.

Definition 4.5.

\[
R_1(p,u_0) := \sup\{r : \text{any open ball in } M \text{ of radius } r \text{ with } u_0 \text{ tangent to its boundary is contained in } H_{p,u_0}\} (4.25)
\]

Lemma 4.6. For any compact \( W \subset S(TM) \), there exists \( \epsilon > 0 \) such that \( R_1(p,u_0) \geq \epsilon \) for any \( (p,u_0) \in W \).
Proof: Let \( W \subset S(TM) \) be compact and let \( (p, u_0) \in W \). This specifies a value \( r_{(p,u_0)} \) such that the exponential map at \( (p, u_0) \) is a homeomorphism from the open ball \( C(u_0, r_{(p,u_0)}) \) to the heart \( H_{p,u_0} \). Next we choose a unit vector \( \hat{a}_0 \) which is perpendicular to \( u_0 \). By restricting to elements of \( T_pM \) in span\{\( u_0, \hat{a}_0 \}\}, we obtain a curve through 0 on the boundary of the ball \( C(u_0, r_{(p,u_0)}) \subset T_{0}(T_pM) \) which we call a circle of radius \( r_{(p,u_0)} \). This circle is mapped by the exponential map to a curve \( \lambda_{p,u_0,\hat{a}_0} \) on the heart boundary. Finally, we choose some value \( x \in (-\pi/2, \pi/2) \). This corresponds uniquely to a point on this circle (by considering the angle between \( A \) and \( u_0 \)). The exponential map takes this point to a point on the curve \( \lambda_{p,u_0,\hat{a}_0} \). Note that by choosing \( x \in (-\pi/2, \pi/2) \) we do not include the problem point at 0 which is mapped to the cusp.

By Corollary 4.4, \( \lambda_{p,u_0,\hat{a}_0} \) is a \( C^2 \) curve away from the cusp. Hence for any \( x \in (-\pi/2, \pi/2) \) we can define the following continuous function

\[
F(p, u_0, \hat{a}_0, x) = \left| \frac{d^2}{dx^2} \lambda_{p,u_0,\hat{a}_0}(x) \right|.
\]

(4.26)

There is a subtlety at the point \( p \), where the boundary of the heart contains a cusp and directional derivatives have discontinuities. However, if we approach \( p \) along the curve \( \lambda_{p,u_0,\hat{a}_0} \) in two different directions then the function \( F \) tends to a finite limit (note that these two limits may not be equal). If we join the two ends of \( \lambda_{p,u_0,\hat{a}_0} \) at \( p \), then these two limits correspond to the two values we get for the magnitude of the curvature of \( \lambda_{p,u_0,\hat{a}_0} \) at \( p \) if we use one-sided derivatives in either direction.

This means that, for fixed \((p, u_0, \hat{a}_0)\), the function \( F(p, u_0, \hat{a}_0, x) \) is bounded. Moreover, this bound is itself a continuous function of \((p, u_0, \hat{a}_0)\). We conclude that if we consider \((p, u_0) \in W \subset S(TM), x \in (-\pi/2, \pi/2) \) and any choice of \( \hat{a}_0 \), then \( F(p, u_0, \hat{a}_0, x) \) is bounded.

In short, given any \((p, u_0) \in W \) and any circle of radius \( r_{(p,u_0)} \) in \( T_{u_0}(T_pM) \) with \( u_0 \) perpendicular to its boundary, the magnitude of the sectional curvature at a point on the image of this circle, taken in the direction of the circle’s image, is bounded by \( 1/\epsilon > 0 \), for some \( \epsilon > 0 \), where we understand that at \( p \) itself we consider two different sectional curvatures corresponding to the different directions along the image of the circle.

As a result, for any \((p, u_0) \in W \), the heart \( H_{p,u_0} \) must contain every open ball in \( M \) of radius \( \epsilon \) which has \( u_0 \) tangent to its boundary at \( p \), i.e. \( R_1(p, u_0) \geq \epsilon \) (see Figure 9). \( \square \)

Next we consider balls in \( M \) with \( u_0 \) pointing inwards on the boundary. We make the following definitions.

Definition 4.7.

\[
B_{1/2}(r, p, u_0) := B(r, p) \cap \{ q : g(\exp_p^{-1}(q), u_0) \geq 0 \} \quad (4.27)
\]

Here \( \exp_p \) denotes the metric geodesic map at \( p \) (we can restrict ourselves to sufficiently small \( r \) so that this map is invertible on \( B(p,r) \) for all \((p,u_0) \in W \)). The set we have
Figure 9: If every sectional curvature at every point on the heart boundary is bounded above by $\epsilon > 0$, then any ball of radius $1/\epsilon$ with $u_0$ tangent to its boundary at $p$ is contained inside the heart.

Figure 10: $B_{1/2}(r, p, u_0)$ is the half ball in $M$ of radius $r$ centred on $p$ consisting of points $q$ such that $g(\exp^-_p(q), u_0) \geq 0$. 
If $R_1(p, u_0) > r$ and $R_2(p, u_0) > 2r$ then $R(p, u_0) > r$.

The result follows by observing that $R(p, u_0) > R_1(p, u_0)$ and $R(p, u_0) > R_2(p, u_0)$ (Figure 11).
Figure 12: If $\gamma$ enters $B(p_*, R_W) \subset H_{p,u_0}$, then in order to reach the boundary of $H_{p,u_0}$ it must first leave $B(p_*, R_W)$.

5 No-Spiralling Theorem

The aim of this section is to follow Step 3 of the proof of Theorem 2.3 as outlined in Section 2.2.

Step 3: Using the fact that a conformal geodesic with unit tangent $u_0$ at $p$ must re-intersect the boundary of $H_{p,u_0}$, deduce that conformal geodesics cannot spiral.

Theorem 2.3 Let $(M, [g])$ be a conformal manifold such that metrics in $[g]$ are smooth. Then conformal geodesics on $(M, [g])$ cannot spiral.

Proof: Suppose there is a conformal geodesic, $\gamma$, which spirals towards some point $p_* \in M$. We will show that there exists $r > 0$ such that $B(p_*, r) \subset H_{p,u_0}$, where $p$ and $u_0$ are the point and unit tangent at which $\gamma$ enters $\bar{B}(p_*, r)$ for the final time. This would then contradict Theorem 3.7 since if $\gamma$ does not leave $\bar{B}(p_*, r)$ then it cannot re-intersect the boundary of $H_{p,u_0}$. This is illustrated in Figure 12.

Define a compact subset of $S(TM)$ by $W := \{(p, u_0) \subset S(TM) : p \in \bar{B}(p_*, 1)\}$. Invoking Theorem 4.2 we can then define:

$$R_W := \inf_{(p, u_0) \in W} \{R(p, u_0)\} > 0. \quad (5.29)$$

So for any $(p, u_0) \in S(TM)$ with $d(p, p_*) \leq 1$, any closed ball in $M$ of radius at most $R_W$ and $u_0$ inward pointing or tangent to its boundary at $p$ is contained inside $\bar{H}_{p,u_0}$.

Let $p$ denote the point at which $\gamma$ enters $\bar{B}(p_*, R_W)$ for the final time and let $u_0$ denote its unit tangent at this point (so $u_0$ is either inward pointing or tangent to the boundary of $\bar{B}(p_*, R_W)$ at $p$). By construction we then have $\bar{B}(p_*, R_W) \subset H_{p,u_0}$. But since $\gamma$ remains trapped in $\bar{B}(p_*, R_W)$, it cannot re-intersect the boundary of $H_{p,u_0}$. This contradicts Theorem 3.7. \(\square\)
6 Summary

We have shown that conformal geodesics defined on a Riemannian conformal manifold cannot spiral. This proof was inspired by a proof of the no-spiralling theorem for metric geodesics which did not rely on length minimisation arguments, instead focusing on properties of the exponential map. However, in the conformal geodesic version there were some additional complications. The most striking of these was that the exponential map we defined for conformal geodesics was not a homeomorphism onto any ball centred at $p \in M$. Instead we found that given any $(p, u_0) \in S(TM)$, conformal geodesics with unit velocity $u_0$ at $p$ foliate some set with cross-sections which resemble a heart.

In this paper we have only considered Riemannian conformal manifolds. In the Lorentzian case, our proof fails from the very beginning. In particular, the exponential map (Definition (3.1)) is not well defined since it is now possible to have $|A| = 0$ for $A \neq 0$.

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