Cubical setting for discrete homotopy theory, revisited

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Abstract

We construct a functor associating a cubical set to a (simple) graph. We show that cubical sets arising in this way are Kan complexes, and that the A-groups of a graph coincide with the homotopy groups of the associated Kan complex.

We use this to prove a conjecture of Babson, Barcelo, de Longueville, and Laubenbacher from 2006, and a strong version of the Hurewicz theorem in discrete homotopy theory.

Introduction

Discrete homotopy theory, introduced in [BKLW01, BBdLL06], is a homotopy theory in the category of simple graphs. It builds on the earlier work of Atkin [Atk74, Atk76], made precise in [KL98], on the homotopy theory of simplicial complexes, and can also be generalized to the homotopy theory of finite metric spaces [BCW14]. It has found numerous applications [BL05, §5–6], including in matroid theory, hyperplane arrangements, combinatorial time series analysis, and, more recently, in topological data analysis [MZ19].

The key invariants associated to graphs in discrete homotopy theory are the A-groups $A_n(G, v)$, named after Atkin, which are the discrete analogue of homotopy groups $\pi_n(X, x)$, studied in the homotopy theory of topological spaces. In [BBdLL06], Babson, Barcelo, de Longueville, and Laubenbacher construct an assignment $G \mapsto X_G$, taking a graph $G$ to a topological space, constructed as a certain cubical complex, and conjecture that the A-groups of $G$ coincide with the homotopy groups of $X_G$. They further prove [BBdLL06, Thm. 5.2] their conjecture under an assumption of a cubical approximation property [BBdLL06, Prop. 5.1], a cubical analogue of the simplicial approximation theorem, which remains an open problem as of July 2022.

The assignment $G \mapsto X_G$ arises as a composite

$$\text{Graph} \xrightarrow{\mathbb{N}_1} \text{cSet} \xrightarrow{|-|} \text{Top}.$$ 

Here $\text{cSet}$ denotes the category of cubical sets, a well studied combinatorial model for the homotopy theory of spaces [Cis06]. Informally, a cubical set consists of a family of sets $\{X_n\}_{n \in \mathbb{N}}$ of $n$-cubes together with a family of structure maps indicating how different cubes ‘fit together,’ e.g., that a certain $(n-1)$-cube is a face of another $n$-cube. More formally, it is a presheaf on the category $\square$ of combinatorial cubes. The functor $\mathbb{N}_1: \text{Graph} \to \text{cSet}$ is obtained by taking the $n$-cubes of $\mathbb{N}_1 G$ to be maps $I_i^{\square n} \to G$, where $I_i^{\square n}$ denotes the $n$-dimensional hypercube graph. The functor $|\cdot|: \text{cSet} \to \text{Top}$, called the geometric realization, assigns the topological $n$-cube $[0, 1]^n$ to each (formal) $n$-cube of $X$ and then glues these cubes together according to structure maps. (We will of course give precise definitions of all these notions later in the paper.)

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A reader familiar with [BBdLL06] will recognize a change in notation: the functor $N_1$ above was in [BBdLL06, §4] denoted by $M_*$. This change is intentional, as we wish to emphasize that this is only the first in a sequence of functors $\text{Graph} \to \text{cSet}$ and natural transformations:

$$
N_1 \rightarrow N_2 \rightarrow N_3 \rightarrow \cdots.
$$

It is this sequence and its colimit

$$
N := \text{colim}(N_1 \rightarrow N_2 \rightarrow N_3 \rightarrow \cdots)
$$

that we investigate. Our main technical results can be summarized by the following theorem.

**Theorem** (cf. Theorems 4.1 and 4.6).

1. For a graph $G$, the cubical set $N_G$ is a Kan complex.
2. The natural transformations in (1) are natural weak equivalences.
3. For a based graph $(G, v)$, there is a natural group isomorphism $A_n(G, v) \cong \pi_n(N_G, v)$.

A few words of explanation are in order. A Kan complex is a cubical set satisfying a certain lifting property making it particularly convenient for the development of homotopy theory. In particular, in a companion paper [CK22], we develop the theory of homotopy groups of Kan complexes and show that these agree with their topological analogues under the geometric realization functor. This establishes the key ingredient required for the proof of the conjecture of Babson, Barcelo, de Longueville, and Laubenbacher, which asks for the commutativity of the outer square in the diagram:

$$
\begin{array}{ccc}
\text{Graph}_* & \overset{N}{\longrightarrow} & \text{Kan}_* \\
\downarrow \pi_n & & \downarrow \pi_n \\
A_n & \overset{}{\longrightarrow} & \text{Grp}
\end{array}
$$

By the theorem above, the two triangles on the left commute, with the upper one commuting up to a natural weak equivalence. The upper right triangle commutes on the nose and the lower right triangle commutes since it expresses the compatibility between homotopy groups of cubical sets and those of topological spaces, established in [CK22]. Thus the $A$-groups of $(G, v)$ agree with those of $(N_G, v)$, which in turn agree with those of $([N_1 G], v)$, since the map $N_1 G \to N G$ is a weak equivalence.

**Theorem** (Conjectured in [BBdLL06]; cf. Theorem 5.1). There is a natural group isomorphism $A_n(G, v) \cong \pi_n([N_1 G], v)$.

The insight of [BBdLL06] cannot be overstated. It is a priori not clear, or at the very least it was not clear to us, that the $A$-groups of a graph should correspond to the homotopy groups of any space. The fact that the space can be obtained from a graph in such a canonical and simple way is what drove us to the problem and to the field of discrete homotopy theory. The title of our paper, “Cubical setting for discrete homotopy theory, revisited” pays tribute to this insight by alluding to the title “A cubical set setting for the $A$-theory of graphs” of [BBdLL06, §3].

It is also worth noting that the case of $n = 1$ of Theorem 5.1 was previously proven in [BKLW01, Prop. 5.12] and perhaps helped inspire the statement of the conjecture in the general case.

Our main theorem allows us to derive a few more consequences of interest in discrete homotopy theory. The first of those is a strong form of the Hurewicz theorem for graphs. The Hurewicz theorem relates the
first non-trivial homotopy group of a sufficiently connected space to its homology. In discrete homotopy theory, it relates the first non-trivial A-group of a graph to its reduced discrete homology, introduced in [BCW14].

**Theorem** (Discrete Hurewicz Theorem; cf. Theorem 5.8). Let \( n \geq 2 \) and \((G, v)\) be a connected pointed graph. Suppose \( A_i(G, v) = 0 \) for all \( i \in \{1, \ldots, n-1\} \). Then the Hurewicz map \( A_n(G, v) \to \tilde{DH}_n(G, v) \) from the \( n \)-th A-group to the \( n \)-th reduced discrete homology group is an isomorphism.

This generalizes the results of Lutz [Lut21, Thm. 5.10], who proves surjectivity of the Hurewicz map for a more restrictive class of graphs; and complements the result of Barcelo, Capraro, and White [BCW14, Thm. 4.1], who prove the 1-dimensional analogue of the Hurewicz theorem, namely that the Hurewicz map \( A_1(G, v) \to \tilde{DH}_1(G, v) \) is surjective with kernel given by the commutator \([A_1(G, v), A_1(G, v)]\) subgroup.

Lastly, our main theorem allows us to equip the category of graphs with additional structure making it amenable to techniques of abstract homotopy theory and higher category theory.

By our theorem, the functor \( N : \text{Graph} \to \text{cSet} \) takes values in the full subcategory \( \text{Kan} \) of \( \text{cSet} \) spanned by Kan complexes. This subcategory is known to carry the structure of a fibration category in the sense of Brown [Bro73]. In brief, a fibration category is a category equipped with two classes of maps: fibrations and weak equivalences, subject to some axioms. Fibration categories are one of the main frameworks used in abstract homotopy theory (see, for instance, [Szu16]). We declare a map \( f \) of graphs to be a fibration/weak equivalence if \( Nf \) is one in the fibration category of Kan complexes. Our theorem guarantees that this gives a well-defined fibration category structure on \( \text{Graph} \) (Theorem 5.9), hence allowing for the use of techniques from abstract homotopy theory in discrete homotopy theory. Furthermore, it follows that the weak equivalences of this fibration category are precisely graph maps inducing isomorphisms on all A-groups for all choices of the basepoint.

This also allows us to put the results of [BBdLL06, §6] in the context of abstract homotopy theory by proving that the loop graph functor constructed there is an exact functor in the sense of fibration category theory (Theorem 5.15).

In a different direction, we observe that the functor \( N : \text{Graph} \to \text{cSet} \) is lax monoidal (Lemma 5.16). The category of graphs is enriched over itself, meaning that the collection of graph maps between two graphs forms not merely a set, but a graph, and that the composition of graph maps defines a graph homomorphism. Enrichment can be transferred along lax monoidal functors, which means that the category of graphs is, via \( N \), canonically enriched over cubical sets, and, more precisely, over Kan complexes (Theorem 5.24). This in particular allows for the use of techniques of higher category theory, developed extensively by Joyal and Lurie [Lur09], in discrete homotopy theory, via the cubical homotopy coherent nerve construction of [KV20, §2].

**Organization of the paper.** This paper is organized as follows. In Sections 1 and 2, we review the background on discrete homotopy theory and cubical sets, respectively. In Section 3, we explain the link between graphs and cubical sets by defining the functors \( N_m \) and \( N \), and proving their basic properties. The technical heart of the paper is contained in Section 4, where we prove our main results. In Section 5, we proceed to deduce the consequences of our main theorem, as described above.

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1 Discrete homotopy theory

The category of graphs

We define the category of simple undirected graphs without loops as a reflexive subcategory of a presheaf category.

Let \( \mathcal{G} \) be the category generated by the diagram

\[
\begin{array}{c}
V \\
\downarrow s \\
E \\
\uparrow t \\
\sigma
\end{array}
\]

subject to the identities

\[
rs = rt = \text{id}_V \quad \sigma^2 = \text{id}_E \\
\sigma s = t \quad \sigma t = s \\
\sigma r = r.
\]

We write \( \hat{\mathcal{G}} \) for the functor category \( \text{Set}^{\mathcal{G}^{\text{op}}} \).

For \( G \in \hat{\mathcal{G}} \), we write \( G_V \) and \( G_E \) for the sets \( G(V) \) and \( G(E) \), respectively. Explicitly, such a functor consists of sets \( G_V \) and \( G_E \) together with the following functions between them

\[
\begin{array}{c}
G_V \\
\downarrow {Gs} \\
G_E \\
\uparrow {Gt} \\
G\sigma
\end{array}
\]

subject to the dual versions of identities in \( \mathcal{G} \).

Definition 1.1. A graph is a functor \( G \in \hat{\mathcal{G}} \) such that the map \((Gs,Gt): G_E \to G_V \times G_V\) is a monomorphism.

Let Graph denote the full subcategory of \( \hat{\mathcal{G}} \) spanned by graphs.

In more concrete terms, a graph \( G \) consists of a set \( G_V \) of vertices and a set \( G_E \) of ‘half-edges.’ A half-edge \( e \in G_E \) has source and target, and these are given by the maps \( Gs \) and \( Gt \), respectively. Each half-edge is paired with its other half via the map \( G\sigma: G_E \to G_E \). Note that the edges paired by \( G\sigma \) have swapped source and target, making the pair (i.e. whole edge) undirected. The map \( Gr: G_V \to G_E \) takes a vertex to an edge whose source and target is that vertex (i.e. a loop). Finally, the condition that \((Gs,Gt)\) is a monomorphism ensures that there is at most one (whole) edge between any two vertices.

A map of graphs \( f: G \to H \) is a natural transformation between such functors. However, since \((Hs, Ht)\) is a monomorphism, such a map is completely determined by a function \( f_V: G_V \to H_V \) that preserves incidence relation.

We may therefore assume that our graphs have no loops, but the maps between them, rather than merely preserving edges, are allowed to contract them to a single vertex. That is, a graph map \( f: G \to H \) is a function \( f: G_V \to H_V \) such that if \( v, w \in G_V \) are connected by an edge, then either \( f(v) \) and \( f(w) \) are connected by an edge or \( f(v) = f(w) \).

![Figure 1: An example depiction of a graph.](image)

Theorem 1.2.
1. The inclusion \( \text{Graph} \hookrightarrow \hat{G} \) admits a left adjoint.

2. The category \( \text{Graph} \) is (co)complete.

3. The functor \( \text{Graph} \to \text{Set} \) mapping a graph \( G \) to its set of vertices \( G_V \) admits both adjoints.

**Proof.** 1. The left adjoint is \( \text{Im} : \hat{G} \to \text{Graph} \) given by

\[
(\text{Im} G)_V = G_V \quad (\text{Im} G)_E = (Gs, Gt)(G_E)
\]

where \((Gs, Gt)(G_E)\) is the image of \( G_E \) under the map \((Gs, Gt) : G_E \to G_V \times G_V\).

2. The category \( \hat{G} \) is (co)complete as a presheaf category. The conclusion follows from (1) as \( \text{Graph} \) is a reflexive subcategory of a (co)complete category.

3. The left adjoint \( \text{Set} \to \text{Graph} \) takes a set \( A \) to the discrete graph with vertex set \( A \). The right adjoint takes a set \( A \) to the complete graph with vertex set \( A \).

**Remark 1.3.** Theorem 1.2 gives a procedure for constructing limits and colimits in \( \text{Graph} \). Given a diagram \( F : J \to \text{Graph} \), the set of vertices of \( \lim F \) is the limit \( \lim UF : J \to \text{Set} \) of the diagram \( UF : J \to \text{Set} \). The set of edges is the largest set such that the limit projections \( \lim F \to F_j \) are graph maps. The set of vertices of \( \text{colim} F \) is the colimit \( \text{colim} UF : J \to \text{Set} \) and the set of edges is the smallest set such that the colimit inclusions \( F_j \to \text{colim} F \) are graph maps.

**Examples of graphs**

**Definition 1.4.** For \( m \geq 0 \),

1. the \( m \)-interval \( I_m \) is the graph which has
   - as vertices, integers \( 0 \leq i \leq m \);
   - an edge between \( i \) and \( i + 1 \).

2. the \( m \)-cycle \( C_m \) is the graph which has
   - as vertices, integers \( 0 \leq i \leq m - 1 \);
   - an edge between \( i \) and \( i + 1 \) and an edge between \( m - 1 \) and 0.

3. the infinite interval \( I_\infty \) is the graph which has
   - as vertices, integers \( i \in \mathbb{Z} \);
   - an edge between \( i \) and \( i + 1 \) for all \( i \in \mathbb{Z} \).

**Figure 2:** The graphs \( I_3 \) and \( C_3 \), respectively.

**Remark 1.5.** The graphs \( I_0 \) and \( I_1 \) are representable when regarded as functors \( G^{\text{op}} \to \text{Set} \), represented by \( V \) and \( E \), respectively.
For $m \geq 0$, we have a map $l: I_{m+1} \to I_m$ defined by

$$l(v) = \begin{cases} 0 & \text{if } v = 0 \\ v - 1 & \text{otherwise.} \end{cases}$$

As well, we have a map $r: I_{m+1} \to I_m$ defined by

$$r(v) = \begin{cases} m & \text{if } v = m + 1 \\ v & \text{otherwise.} \end{cases}$$

We write $c: I_{m+2} \to I_m$ for the composite $lr = rl$. Explicitly, this map is defined by

$$c(v) = \begin{cases} 0 & \text{if } v = 0 \\ m & \text{if } v = m + 2 \\ v - 1 & \text{otherwise.} \end{cases}$$

We show the inclusion $\text{Graph} \hookrightarrow \hat{G}$ preserves filtered colimits and use this to show all finite graphs are compact, i.e. if $G$ is a finite graph then the functor $\text{Graph}(G, -): \text{Graph} \to \text{Set}$ preserves filtered colimits.

**Proposition 1.6.** The inclusion $\text{Graph} \hookrightarrow \hat{G}$ preserves filtered colimits.

**Proof.** Fix a filtered category $J$ and a diagram $D: J \to \text{Graph}$. Let $i$ denote the inclusion $\text{Graph} \hookrightarrow \hat{G}$. Recall that $\text{colim} D$ is computed by $\text{Im}(\text{colim}(iD))$. It suffices to show $\text{colim}(iD) \in \hat{G}$ is a graph, since then the unit map $\text{colim}(iD) \to \text{Im}(\text{colim}(iD))$ is an isomorphism $\text{colim}(iD) \cong i(\text{colim} D)$ natural in $D$.

Let $\lambda: iD \to \text{colim}(iD)$ denote the colimit cone. Suppose two edges $e, e' \in \text{colim}(iD)_E$ have the same source and target. Regarding $e, e'$ as maps $I_1 \to \text{colim}(iD)$, these maps factor as

$$
\begin{array}{ccc}
I_1 & \xrightarrow{e,e'} & \text{colim}(iD) \\
\downarrow \pi & & \downarrow \lambda \\
(iDx) & \xrightarrow{\lambda_x} & \text{colim}(iD) \\
\end{array}
$$

for some $x \in J$ since $I_1 \subset \hat{G}$ is representable. Let $s, s' \in (iDx)^V$ denote the sources of $\pi, \pi'$ and $t, t' \in (iDx)^V$ denote the targets, respectively. Using an explicit description of the colimit (Remark 1.3), since $\lambda_x(s) = \lambda_x(s')$ and $\lambda_x(t) = \lambda_x(t')$, there exist arrows $f, g: y \to x$ and $h, k: z \to x$ with vertices $v \in i Dy$ and $w \in i Dz$ such that

$$iDf(v) = s \quad iDg(v) = s' \quad iDh(w) = t \quad iDk(w) = t'.$$

As $J$ is filtered, there exists an arrow $l: x \to w$ in $J$ such that $lf = lg$ and $lh = lk$. This implies the edges $iDl(\pi), iDl(\pi') \in (iDw)_E$ have the same source and target. As $iDw$ is a graph, we have that $\pi = \pi'$, thus $e = e'$.

**Corollary 1.7.** For a finite graph $G$, the functor $\text{Graph}(G, -): \text{Graph} \to \text{Set}$ preserves filtered colimits.

**Proof.** Given a filtered category $J$ and a diagram $D: J \to \text{Graph}$, we have a natural isomorphism

$$\text{Graph}(G, \text{colim} D) \cong \hat{G}(G, \text{colim} D)$$

by Proposition 1.6 since $\text{Graph}$ is a full subcategory. This then follows since $G \in \hat{G}$ is a finite colimit of representable presheaves.
Monoidal structure on the category of graphs

Define a functor \( \otimes : G \times G \rightarrow \hat{G} \) by
\[
(V, V) \mapsto I_0 \quad (V, E) \mapsto I_1 \\
(E, V) \mapsto I_1 \quad (E, E) \mapsto C_4
\]

Left Kan extension along the Yoneda embedding yields a monoidal product \( \otimes : \hat{G} \times \hat{G} \rightarrow \hat{G} \).

\[
\begin{array}{c}
G \times G \xrightarrow{\otimes} \hat{G} \\
\downarrow \\
\hat{G} \times \hat{G}
\end{array}
\]

Note that Graph is closed with respect to this product i.e. if \( G, H \in \text{Graph} \) then \( G \otimes H \in \hat{G} \) is a graph. Thus, this product descends to a monoidal product \( \otimes : \text{Graph} \times \text{Graph} \rightarrow \text{Graph} \), called the cartesian product.

Definition 1.8. Let \( G \) and \( H \) be graphs. The graph hom \( \otimes (G, H) \) has

- as vertices, morphisms \( G \rightarrow H \) in Graph
- an edge from \( f \) to \( g \) if there exists \( H : G \otimes I_1 \rightarrow H \) such that \( H|_{G \otimes \{0\}} = f \) and \( H|_{G \otimes \{1\}} = g \).

This structure makes the category of graphs into a closed symmetric monoidal category.

Proposition 1.9. \( (\text{Graph}, \otimes, I_0, \text{hom}^{\otimes}(-,-)) \) is a closed symmetric monoidal category.

Homotopy theory of graphs

Definition 1.10. Let \( f, g : G \rightarrow H \) be graph maps. An A-homotopy from \( f \) to \( g \) is a map \( \eta : G \otimes I_m \rightarrow H \) for some \( m \geq 0 \) such that \( \eta|_{G \otimes \{0\}} = f \) and \( \eta|_{G \otimes \{1\}} = g \).

We refer to A-homotopies as simply homotopies.

Proposition 1.11. For graphs \( G \) and \( H \), homotopy is an equivalence relation on the set of graphs maps \( G \rightarrow H \).

As well, we define the A-homotopy groups of a graph. As in topological spaces, this requires the definition of based graph maps and based homotopies between them.

Definition 1.12. Let \( A \rightarrow G \) and \( B \leftarrow H \) be graph monomorphisms. A relative graph map, denoted \( (G, A) \rightarrow (H, B) \), is a morphism from \( A \rightarrow G \) to \( B \leftarrow H \) in the arrow category \( \text{Graph}^{[1]} \), where \( [1] \) denotes the poset \( \{0 \leq 1\} \) viewed as a category.

Explicitly, this data consists of maps \( G \rightarrow H \) and \( A \rightarrow B \) such that the following square commutes.
\[
\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow & & \downarrow \\
G & \rightarrow & H
\end{array}
\]
That is, \( A \) is a subgraph of \( G \) and the map \( A \to B \) is the restriction of the map \( G \to H \) to \( A \), whose image is contained in the subgraph \( B \). Given a relative graph map \( f: (G, A) \to (H, B) \), we write \( f|_A \) for the map from \( A \) to \( B \). We define relative homotopies between relative graph maps as follows.

**Definition 1.13.** Let \( f, g: (G, A) \to (H, B) \) be relative graph maps. A **relative homotopy** from \( f \) to \( g \) is a relative map \( (G \otimes I_m, A \otimes I_m) \to (H, B) \) for some \( m \geq 0 \) such that

- the map \( A \otimes I_m \to B \) is a homotopy from \( f|_A \) to \( g|_A \);
- the map \( G \otimes I_m \to H \) is a homotopy from \( f \) to \( g \).

Explicitly, a relative homotopy from \( f \) to \( g \) consists of

- a homotopy \( \eta: G \otimes I_m \to H \) from \( f \) to \( g \);
- a homotopy \( \eta|_A: A \otimes I_m \to B \) from \( f|_A \) to \( g|_A \)

such that the following square commutes.

\[
\begin{array}{ccc}
A \otimes I_m & \xrightarrow{\eta|_A} & B \\
\downarrow & & \downarrow \\
G \otimes I_m & \xrightarrow{\eta} & H
\end{array}
\]

As before, this is an equivalence relation on relative graph maps.

**Proposition 1.14.** Relative homotopy is an equivalence relation on the set of relative graph maps \((G, A) \to (H, B)\).

Given a relative graph map \( f: (G, A) \to (H, B) \), if the subgraph \( B \) consists of a single vertex \( v \) then we refer to \( f \) as a graph map **based at** \( v \) or a **based** graph map. We refer to a homotopy between two such maps as a **based** homotopy.

**Definition 1.15.** Let \( G \) be a graph and \( n \geq 1 \). For \( i = 0, \ldots, n \) and \( \varepsilon = 0, 1 \), a map \( f: I^n \to G \) is **stable in direction** \((i, \varepsilon)\) if there exists \( M \geq 0 \) so that for \( v_i > M \), we have

\[
f(v_1, \ldots, (2\varepsilon - 1)v_i, \ldots, v_n) = f(v_1, \ldots, (2\varepsilon - 1)M, \ldots, v_n).
\]

For \( n, M \geq 0 \), let \( I^n_M \) denote the subgraph of \( I^n \) consisting of vertices \((v_1, \ldots, v_n)\) such that \(|v_i| \geq M\) for some \( i = 1, \ldots, n \). Given a based graph map \( f: (I^n_M, I^n_M) \to (G, v) \) we may also regard \( f \) as a based graph map \((I^n_M, I^n_M) \to (G, v)\) for any \( K \geq M \). This gives a notion of based homotopy between maps \((I^n_M, I^n_M) \to (G, v)\) which, for some \( M \geq 0 \), are based at \( v \).

**Proposition 1.16.** Based homotopy is an equivalence relation on the set of based maps

\[
\{(I^n_M, I^n_M) \to (G, v) \mid M \geq 0\}.
\]

**Definition 1.17.** Let \( n \geq 0 \) and \( v \in G \) be a vertex of a graph \( G \). The **\( n \)-th \( A \)-homotopy group** of \( G \) at \( v \) is the set of based homotopy classes of maps \((I^n_M, I^n_M) \to (G, v)\) based at \( v \) for some \( M \geq 0 \).
Let $n \geq 1$ and $i = 1, \ldots, n$. Given $f : (I_\infty^\otimes n, I_\infty^\otimes n) \to (G, v)$ and $g : (I_\infty^\otimes n, I_\infty^\otimes n) \to (G, v)$, we define a binary operation $f \cdot_i g : (I_\infty^\otimes n, I_\infty^\otimes n) \to (G, v)$ by

$$(f \cdot_i g)(v_1, \ldots, v_n) := \begin{cases} f(v_1, \ldots, v_n) & v_i \leq M \\ g(v_1, \ldots, v_i - M - M', \ldots, v_n) & v_i > M. \end{cases}$$

This induces a group operation on homotopy groups $\cdot_i : A_n(G, v) \times A_n(G, v) \to A_n(G, v)$. For $n \geq 2$ and $1 \leq i < j \leq n$, it is straightforward to construct a homotopy witnessing that

$$[[f] \cdot_i [g]] \cdot_j [[f] \cdot_j [g]] = [[f] \cdot_j [g]] \cdot_i [[f] \cdot_i [g]]$$

for any $[f], [g] \in A_n(G, v)$. The Eckmann-Hilton argument gives $[f] \cdot_i [g] = [f] \cdot_j [g]$ and that this operation is abelian.

### Path and loop graphs

**Definition 1.18.** For a graph $G$, we define the path graph $P G$ to be the induced subgraph of $\text{hom}^\otimes(I_\infty, G)$ consisting of maps which stabilize in the $(1, 0)$ and $(1, 1)$ directions.

As vertices of $PG$ are paths that stabilize, we have graph maps $\partial_{1,0}, \partial_{1,1} : PG \to G$ which send a vertex $v : I_\infty \to G$ to its left and right endpoints, respectively. Note that the path graph is given by the following colimit.

**Proposition 1.19.** For a graph $G$, we have an isomorphism

$$PG \cong \text{colim} \left( \text{hom}^\otimes(I_1, G) \xrightarrow{\gamma} \text{hom}^\otimes(I_2, G) \xrightarrow{\gamma} \text{hom}^\otimes(I_3, G) \xrightarrow{\gamma} \ldots \right)$$

natural in $G$.

**Definition 1.20.** For a pointed graph $(G, v)$, the loop graph $\Omega(G, v)$ is the subgraph of $PG$ of paths $I_\infty \to G$ whose left and right endpoints are $v$.

From the definition, it is immediate that

**Proposition 1.21.** For a pointed graph $(G, v)$, the square

$$\begin{array}{ccc} \Omega(G, v) & \xrightarrow{\gamma} & PG \\
\downarrow & & \downarrow \\
I_0 & \xrightarrow{(v, v)} & G \times G \end{array}$$

is a pullback.

The loop graph of a pointed graph $(G, v)$ has a distinguished vertex which is the constant path at $v$. This gives an endofunctor $\Omega : \text{Graph}_* \to \text{Graph}_*$. From this, we define the notion of $n$-th loop graphs.

**Definition 1.22.** For $n \geq 0$, we define the $n$-th loop graph to be

$$\Omega^n(X, x) := \begin{cases} (X, x) & n = 0 \\ \Omega(\Omega^{n-1}(X, x)) & \text{otherwise}. \end{cases}$$

In [BBdLL06], it is shown that the $n$-th homotopy groups of a graph correspond to the connected components of $\Omega^n(G, v)$.

**Proposition 1.23** ([BBdLL06, Prop. 7.4]). For $n \geq 0$, we have an isomorphism

$$A_n(G, v) \cong A_0(\Omega^n(G, v)).$$
2 Cubical sets and their homotopy theory

Cubical sets

We begin by defining the box category $\Box$. The objects of $\Box$ are posets of the form $[1]^n = \{0 \leq 1\}^n$ and the maps are generated (inside the category of posets) under composition by the following four special classes:

- **faces** $\partial^n_{i,\varepsilon} : [1]^{n-1} \to [1]^n$ for $i = 1, \ldots, n$ and $\varepsilon = 0, 1$ given by:
  $$\partial^n_{i,\varepsilon}(x_1, x_2, \ldots, x_{n-1}) = (x_1, x_2, \ldots, x_{i-1}, \varepsilon, x_i, \ldots, x_{n-1})$$

- **degeneracies** $\sigma^n_i : [1]^n \to [1]^{n-1}$ for $i = 1, 2, \ldots, n$ given by:
  $$\sigma^n_i(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$$

- **negative connections** $\gamma^n_{i,0} : [1]^n \to [1]^{n-1}$ for $i = 1, 2, \ldots, n - 1$ given by:
  $$\gamma^n_{i,0}(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_{i-1}, \max(x_i, x_{i+1}), x_{i+2}, \ldots, x_n)$$

- **positive connections** $\gamma^n_{i,1} : [1]^n \to [1]^{n-1}$ for $i = 1, 2, \ldots, n - 1$ given by:
  $$\gamma^n_{i,1}(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_{i-1}, \min(x_i, x_{i+1}), x_{i+2}, \ldots, x_n)$$

These maps obey the following **cubical identities**:

$$\partial_j,\varepsilon\partial_i,\varepsilon = \partial_{i+1},\varepsilon\partial_j,\varepsilon' \quad \text{for } j \leq i;$$  
$$\sigma_j\partial_i,\varepsilon = \begin{cases} 
\partial_{i-1,\varepsilon}\sigma_j & \text{for } j < i; \\
\text{id} & \text{for } j = i; \\
\partial_{i,\varepsilon}\sigma_{j-1} & \text{for } j > i;
\end{cases}$$  
$$\sigma_i\sigma_j = \sigma_j\sigma_{i+1} \quad \text{for } j \leq i;$$  
$$\gamma_{j,\varepsilon'}\gamma_{i,\varepsilon} = \begin{cases} 
\partial_{j-1,\varepsilon}\gamma_{j,\varepsilon'} & \text{for } j < i; \\
\text{id} & \text{for } j = i, \varepsilon = \varepsilon'; \\
\partial_{i,\varepsilon}\sigma_i & \text{for } j = i - 1, \varepsilon = 1 - \varepsilon'; \\
\partial_{i,\varepsilon}\gamma_{j-1,\varepsilon'} & \text{for } j > i;
\end{cases}$$  
$$\gamma_{j,\varepsilon'}\partial_i,\varepsilon = \begin{cases} 
\partial_{i,\varepsilon}\gamma_{j,\varepsilon'} & \text{for } j < i - 1; \\
\text{id} & \text{for } j = i - 1, \varepsilon = \varepsilon'; \\
\partial_{i,\varepsilon}\sigma_i & \text{for } j = i - 1, \varepsilon = 1 - \varepsilon'; \\
\partial_{i,\varepsilon}\gamma_{j-1,\varepsilon'} & \text{for } j > i;
\end{cases}$$  
$$\gamma_{j,\varepsilon'}\sigma_i = \begin{cases} 
\gamma_{i-1,\varepsilon}\sigma_j & \text{for } j < i; \\
\sigma_i\sigma_i & \text{for } j = i; \\
\gamma_{i,\varepsilon}\sigma_{j+1} & \text{for } j > i.
\end{cases}$$

A **cubical set** is a presheaf $X : \Box^{\text{op}} \to \Set$. A **cubical map** is a natural transformation of such presheaves. We write $cSet$ for the category of cubical sets and cubical maps.

Given a cubical set $X$, we write $X_n$ for the value of $X$ at $[1]^n$ and write cubical operators on the right e.g. given an $n$-cube $x \in X_n$ of $X$, we write $x\partial_{1,0}$ for the $\partial_{1,0}$-face of $x$.

**Definition 2.1.** Let $n \geq 0$.

- The **combinatorial $n$-cube** $\Box^n$ is the representable functor $\Box(-, [1]^n) : \Box^{\text{op}} \to \Set$;
- The **boundary of the $n$-cube** $\partial\Box^n$ is the subobject of $\Box^n$ defined by
  $$\partial\Box^n := \bigcup_{j=0, \ldots, n, \eta=0,1} \text{im} \partial_{j,\eta}.$$
• Given $i = 0, \ldots, n$ and $\varepsilon = 0, 1$, the $(i, \varepsilon)$-open box $\cap_{i, \varepsilon}^n$ is the subobject of $\partial \square^n$ defined by

$$\cap_{i, \varepsilon}^n := \bigcup_{(j, \eta) \neq (i, \varepsilon)} \text{im} \partial_{j, \eta}.$$ 

Observe $\square^0 \in \text{cSet}$ is the terminal object in $\text{cSet}$.

**Example 2.2.** Define a functor $\square \to \text{Top}$ from the box category to the category of topological spaces which sends $[1]^n$ to $[0, 1]^n$ where $[0, 1]$ is the unit interval. Left Kan extension along the Yoneda embedding gives the geometric realization functor $|−|: \text{cSet} \to \text{Top}$.

$$\begin{array}{ccc}
\square & \xrightarrow{[1]^n \mapsto [0, 1]^n} & \text{Top} \\
\downarrow & & \\
\text{cSet} & \xrightarrow{|−|} & \\
\end{array}$$

This functor is left adjoint to the singular cubical complex functor $\text{Sing}: \text{Top} \to \text{cSet}$ defined by

$$(\text{Sing} S)_n := \text{Top}([0, 1]^n, S).$$

Define a functor $\otimes: \square \times \square \to \square$ on the cube category which sends $([1]^m, [1]^n)$ to $[1]^{m+n}$. Postcomposing with the Yoneda embedding and left Kan extending gives a monoidal product on cubical sets.

$$\begin{array}{ccc}
\square \times \square & \longrightarrow & \square \\
\downarrow & & \\
\text{cSet} & \xrightarrow{\otimes} & \\
\end{array}$$

This is the geometric product of cubical sets. This product is biclosed, i.e. for a cubical set $X$, we write $\text{hom}_L(X, −): \text{cSet} \to \text{cSet}$ and $\text{hom}_R(X, −): \text{cSet} \to \text{cSet}$ for the right adjoints to the functors $− \otimes X$ and $X \otimes −$, respectively. We will only make use of the functor $\text{hom}_R(X, −)$, thus we write $\text{hom}(X, −)$ for $\text{hom}_R(X, −)$.

**Kan complexes**

**Definition 2.3.**

1. A cubical map $X \to Y$ is a Kan fibration if it has the right lifting property with respect to open box inclusions. That is, if for any commutative square,

$$\begin{array}{ccc}
\cap_{i, \varepsilon}^n & \longrightarrow & X \\
\downarrow & & \downarrow f \\
\square^n & \longrightarrow & Y \\
\end{array}$$

there exists a map $\square^n \to X$ so that the triangles

$$\begin{array}{ccc}
\cap_{i, \varepsilon}^n & \longrightarrow & X \\
\downarrow & & \downarrow f \\
\square^n & \longrightarrow & Y \\
\end{array}$$

commutes.
2. A cubical set $X$ is a Kan complex if the unique map $X \to \square^0$ is a Kan fibration.

We write Kan for the full subcategory of cSet consisting of Kan complexes.

**Example 2.4.** For any $S \in \text{Top}$, we have that Sing $S$ is a Kan complex. A map $\prod^n \epsilon \to \text{Sing} S$ is, by adjointness, a map $|\prod^n \epsilon| \to |\square^n|$ has a retract in Top. Pre-composing with this retract gives a map $|\square^n| \to S$ which restricts to the open box map $|\prod^n \epsilon| \to S$.

By adjointness, this gives a suitable map $\square^n \to \text{Sing} S$.

**Definition 2.5.** A map $f : X \to Y$ is a weak equivalence if the map $|f| : |X| \to |Y|$ is a weak homotopy equivalence, i.e. for any $n \geq 0$ and $x \in |X|$, the map $\pi_n(|f|) : \pi_n(|X|, x) \to \pi_n(|Y|, |f|(x))$ is an isomorphism.

We move towards describing the fibration category of Kan complexes.

**Definition 2.6.** A fibration category is a category $\mathcal{C}$ with two subcategories of fibrations and weak equivalences such that (in what follows, an acyclic fibration is a map that is both a fibration and a weak equivalence):

1. weak equivalences satisfy two-out-of-three property; that is, given two composable morphisms:
   
   $X \xrightarrow{f} Y \xrightarrow{g} Z$

   if two of $f, g, gf$ are weak equivalences then all three are;

2. all isomorphisms are acyclic fibrations;

3. pullbacks along fibrations exist; fibrations and acyclic fibrations are stable under pullback;

4. $\mathcal{C}$ has a terminal object $1$; the canonical map $X \to 1$ is a fibration for any object $X \in \mathcal{C}$ (that is, all objects are fibrant);

5. every map can be factored as a weak equivalence followed by a fibration.

**Example 2.7** ([Hov99, Thm. 2.4.19]). The category Top of topological spaces is a fibration where

- fibrations are Serre fibrations;

- weak equivalences are weak homotopy equivalences; i.e. maps $f : S \to S'$ such that, for all $s \in S$ and $n \geq 0$, the map $\pi_n(f) : \pi_n(S, s) \to \pi_n(S', f(s))$ is an isomorphism.

**Definition 2.8.** A functor $F : \mathcal{C} \to \mathcal{D}$ between fibration categories is exact if it preserves fibrations, acyclic fibrations, pullbacks along fibrations, and the terminal object.

Given a fibration category $\mathcal{C}$ with finite coproducts and a terminal object, the category of pointed objects $1 \downarrow \mathcal{C}$ is a fibration category as well.

**Proposition 2.9.** Let $\mathcal{C}$ be a fibration category with finite coproducts and a terminal object $1 \in \mathcal{C}$.

1. The slice category $1 \downarrow \mathcal{C}$ under $1$ is a fibration category where a map is a fibration/weak equivalence if the underlying map in $\mathcal{C}$ is;
2. the projection functor $1 \downarrow \mathcal{C} \to \mathcal{C}$ is exact.

Proof. The first statement follows from [Hov99, Prop. 1.1.8].

For the second statement, the projection functor preserves fibrations/weak equivalences by definition. It is a right adjoint to the functor $- \sqcup 1 : \mathcal{C} \to 1 \downarrow \mathcal{C}$ which adds a disjoint basepoint, hence preserves finite limits. \qed

Theorem 2.10.

1. The category Kan of Kan complexes is a fibration category where fibrations are Kan fibrations and weak equivalences are as defined above.

2. $\text{Sing} : \text{Top} \to \text{Kan}$ is an exact functor.

3. The category $\text{Kan}_*$ of pointed Kan complexes is a fibration category where a map is a fibration/weak equivalence if the underlying map in $\text{Kan}$ is.

Proof. 1. This is shown in [CK22, Thm. 2.17].

2. As a right adjoint, Sing preserves limits. It preserves fibrations and acyclic fibrations by an argument analogous to the one used in Example 2.4.

3. Follows from Proposition 2.9. \qed

Anodyne maps

We move towards defining anodyne maps of cubical sets; that is, maps which are both monomorphisms and weak equivalences.

Definition 2.11. A class of morphisms $S$ in a cocomplete category $\mathcal{C}$ is saturated if it is closed under

- pushouts: if $s : A \to B$ is in $S$ and $f : A \to C$ is any map then the pushout $C \to B \cup C$ of $s$ along $f$ is in $S$;

- retracts: if $s : A \to B$ is in $S$ and $r : C \to D$ is a retract of $s$ in $\mathcal{C}[1]$ then $r$ is in $S$;

- transfinite composition: given a limit ordinal $\lambda$ and a diagram $\lambda \to \mathcal{C}$ whose morphisms lie in $S$,

writing $A$ for the colimit of this diagram, the components of the colimit cone $\lambda_i : A_i \to A$ are in $S$. 

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**Definition 2.12.** For a set of morphisms $S$ in a cocomplete category $\mathcal{C}$, the *saturation* $\text{Sat} S$ of $S$ is the smallest saturated class containing $S$.

For a saturated class of maps, closure under pushouts and transfinite composition gives that

**Proposition 2.13 ([Hov99, Lem. 2.1.13]).** Saturated classes of maps are closed under coproduct. That is, given a collection $\{ s_i : A_i \to B_i \mid i \in I \}$ of morphisms in a saturated class $S$, the coproduct

$$\coprod_{i \in I} s_i : \coprod_{i \in I} A_i \to \coprod_{i \in I} B_i$$

is in $S$.

**Definition 2.14.** A map of cubical sets is *anodyne* if it is in the saturation of open box inclusions

$$\text{Sat}\{ \cap_{i, \varepsilon}^{\Delta^n} \hookrightarrow \Box^n \mid n \geq 1, i = 0, \ldots, n, \varepsilon = 0, 1 \}.$$ 

We use the following property of anodyne maps.

**Theorem 2.15 ([DKLS20, Thm. 1.34]).** Let $f : A \to B$ be an anodyne map and $g : X \to Y$ be a Kan fibration. Given a commutative square,

$$\begin{array}{ccc}
A & \longrightarrow & X \\
\downarrow g & & \downarrow f \\
B & \longrightarrow & Y
\end{array}$$

there exists a map $B \to X$ so that the triangles

$$\begin{array}{ccc}
A & \longrightarrow & X \\
\downarrow g & & \downarrow f \\
B & \longrightarrow & Y
\end{array}$$

commute.

**Theorem 2.16.** A cubical map is anodyne if and only if it is a monomorphism and a weak equivalence.

**Proof.** This follows from [DKLS20, Thm. 1.34], as the saturation of open box inclusions is exactly the class of maps which have the left lifting property with respect to fibrations.

In particular, we use that anodyne maps are sent to weak homotopy equivalences under geometric realization.

**Corollary 2.17.** Let $f : X \to Y$ be anodyne. We have that, for all $n \geq 0$ and $x \in |X|$, the map $\pi_n|f| : \pi_n(\partial X, x) \to \pi_n(\partial Y, y)|f|(x))$ is an isomorphism.

**Homotopies and homotopy groups**

Using the geometric product, we may define a notion of homotopy between cubical maps.
Definition 2.18. Given cubical maps \( f, g : X \to Y \), a homotopy from \( f \) to \( g \) is a map \( H : X \otimes \square^1 \to G \) such that the diagram

\[
\begin{array}{c}
X \otimes \square^0 \\
\downarrow \partial_{1,0} \quad \downarrow f \\
X \otimes \square^1 \\
\downarrow \partial_{1,1} \quad \downarrow g \\
X \otimes \square^0 \\
\end{array}
\]

commutes.

Let \( \mathbf{cSet}_2 \) denote the full subcategory of \( \mathbf{cSet}[1] \) spanned by monomorphisms. Explicitly, its objects are monic cubical maps \( A \hookrightarrow X \). A morphism from \( A \hookrightarrow X \) to \( B \hookrightarrow Y \) is a pair of maps \((f, g)\) which form a commutative square of the following form.

\[
\begin{array}{ccc}
A & \xrightarrow{g} & B \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & Y
\end{array}
\]

We refer to the objects and morphisms of \( \mathbf{cSet}_2 \) as relative cubical sets and relative cubical maps, respectively. We denote a relative cubical set \( A \hookrightarrow X \) by \( (X, A) \), supressing the data of the map itself. An exception to this is when the domain of a relative cubical set \( A \hookrightarrow X \) is the 0-cube \( A = \square^0 \). This is exactly the data of a pointed cubical set, i.e. an object in the slice category \( \square^0 \downarrow \mathbf{cSet} \) over \( \square^0 \). We denote this data by \( (X, x) \), where \( x \) is the monomorphism \( x : \square^0 \hookrightarrow X \).

For a relative cubical map \((f, g) : (X, A) \to (Y, B)\), the map \( g \) is uniquely determined by \( f \): if a map \( h : A \to B \) also forms a commutative square with \( f \) then \( g = h \) as the map \( B \hookrightarrow Y \) is monic. Thus, we denote a relative cubical map by \( f : (X, A) \to (Y, B) \). We additionally write \( f : X \to Y \) for the bottom map in the square and \( f|_A : A \to B \) for the top map.

We have a corresponding notion of homotopy between relative cubical maps.

Definition 2.19. Let \( f, g : (X, A) \to (Y, B) \) be relative cubical maps. A relative homotopy from \( f \) to \( g \) is a morphism \((X \otimes \square^1, A \otimes \square^1) \to (Y, B)\) in \( \mathbf{cSet}_2 \) such that

\[
\begin{array}{c}
\text{• the map } A \otimes \square^1 \to B \text{ is a homotopy from } f|_A \text{ to } g|_A; \\
\text{• the map } X \otimes \square^1 \to Y \text{ is a homotopy from } f \text{ to } g.
\end{array}
\]

Proposition 2.20 ([CK22, Prop. 2.30]). If \( B, Y \) are Kan complexes then relative homotopy is an equivalence relation on relative cubical maps \((X, A) \to (B, Y)\).

We write \([(X, A), (Y, B)]\) for the set of relative homotopy classes of relative maps \((X, A) \to (Y, B)\). With this, we define the homotopy groups of a Kan complex.

Definition 2.21 ([CK22, Cor. 3.16]). Let \((X, x)\) be a pointed Kan complex. We define the \( n \)-th homotopy group \( \pi_n(X, x) \) of \((X, x)\) as the relative homotopy classes of relative maps \((\square^n, \partial\square^n) \to (X, x)\).

We give an explicit description of multiplication in the first homotopy group \( \pi_1(X, x) \).

Definition 2.22. Given two 1-cubes \( u, v : \square^1 \to X \) in a cubical set \( X \),

1. a concatenation square for \( u \) and \( v \) is a map \( \eta : \square^2 \to X \) such that

\[
\begin{array}{c}
\eta \partial_{1,0} = u;
\end{array}
\]
• $\eta\partial_{1,1} = v\partial_{1,1}\sigma_1$;
• $\eta\partial_{2,1} = v$.

2. a concatenation of $u$ and $v$ is a 1-cube $w: \square^1 \to X$ which is the $\partial_{2,0}$-face of some concatenation square for $u$ and $v$.

**Proposition 2.23** ([CK22, Thm. 3.11]). If $(X, x)$ is a pointed Kan complex then composition induces a well-defined binary operation

$$[(\square^1, \partial\square^1), (X, x)] \times [(\square^1, \partial\square^1), (X, x)] \to [(\square^1, \partial\square^1), (X, x)]$$

on relative homotopy classes of relative maps which gives a group structure on $[(\square^1, \partial\square^1), (X, x)]$.

As with spaces, the homotopy groups of a Kan complex are the connected components of its loop space, which we define.

**Definition 2.24.** For a pointed Kan complex $(X, x)$,

• the loop space $\Omega(X, x)$ of $X$ is the pullback

$$\begin{array}{ccc}
\Omega(X, x) & \longrightarrow & \text{hom}(\square^1, X) \\
\downarrow & & \downarrow (\partial_{0,0}, \partial_{1,1}) \\
\square^0 & \longrightarrow & X \times X \\
\end{array}$$

with a distinguished 0-cube $x\sigma_1: \square^0 \to \Omega(X, x)$.

• for $n \geq 0$, the $n$-th loop space $\Omega^n(X, x)$ of $X$ is defined to be

$$\Omega^n(X, x) := \begin{cases} (X, x) & n = 0 \\
\Omega(\Omega^{n-1}(X, x)) & n > 0. \end{cases}$$

**Proposition 2.25** ([CK22, Cor. 3.16]). For a pointed Kan complex $(X, x)$ and $0 \leq k \leq n$, we have an isomorphism

$$\pi_n(X, x) \cong \pi_{n-k}(\Omega^k(X, x))$$

natural in $X$.

Using Proposition 2.25, the group structure on higher homotopy groups is induced by the bijection $\pi_n(X, x) \cong \pi_1(\Omega^{n-1}(X, x))$.

This definition of homotopy groups agrees with the homotopy groups of its geometric realization.

**Theorem 2.26** ([CK22, Thm. 3.25]). There is an isomorphism

$$\pi_n(X, x) \cong \pi_n(|X|, x)$$

natural in $X$.

We have that the loop space functor is exact.

**Theorem 2.27** ([CK22, Thm. 3.6]). The loop space functor $\Omega : \text{Kan}_* \to \text{Kan}_*$ is exact.
3 Cubical nerve of a graph

Let $m \geq 1$. Geometrically, we view the graph $I^\otimes_m$ as an $n$-dimensional cube. Making this intuition formal, we have face, degeneracy, and connection maps defined as follows.

- the face map $\partial^n_{i,\varepsilon} : I^\otimes_{m,n-1} \to I^\otimes_m$ for $1 \leq i \leq n$ and $\varepsilon = 0$ or 1 is given by
  $$\partial^n_{i,\varepsilon}(v_1, \ldots, v_n) = (v_1, \ldots, v_{i-1}, \varepsilon m, v_i, \ldots, v_n);$$

- the degeneracy map $\sigma^n_i : I^\otimes_{m,n+1} \to I^\otimes_m$ for $1 \leq i \leq n$ is given by
  $$\sigma^n_i(v_1, \ldots, v_n) = (v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n);$$

- the negative connection map $\gamma^n_{i,0} : I^\otimes_m \to I^\otimes_{m,n-1}$ for $1 \leq i \leq n-1$ is given by
  $$\gamma^n_{i,0}(v_1, \ldots, v_n) = (v_1, \ldots, v_{i-1}, \max(v_i, v_{i+1}), v_{i+2}, \ldots, v_n);$$

- the positive connection map $\gamma^n_{i,1} : I^\otimes_m \to I^\otimes_{m,n-1}$ for $1 \leq i \leq n-1$ is given by
  $$\gamma^n_{i,1}(v_1, \ldots, v_n) = (v_1, \ldots, v_{i-1}, \min(v_i, v_{i+1}), v_{i+2}, \ldots, v_n).$$

It is straightforward to verify that these maps satisfy cubical identities. This defines a functor $\square \to \text{Graph}$ which sends $[1]^n$ to $I^\otimes_m$. Left Kan extension along the Yoneda embedding gives an adjunction $\text{cSet} \rightleftarrows \text{Graph}$.

\[ \square \longrightarrow \text{Graph} \]
\[ \text{cSet} \]

Definition 3.1. For $m \geq 1$,

1. the $m$-realization functor $|-|_m : \text{cSet} \to \text{Graph}$ is the left Kan extension of the functor $\square \to \text{Graph}$ which sends $[1]^n$ to $I^\otimes_m$.

2. the $m$-nerve functor $N_m : \text{Graph} \to \text{cSet}$ is the right adjoint of the $m$-realization functor defined by
   $$\text{(N}_mG)_n := \text{Graph}(I^\otimes_m, G).$$

Remark 3.2. The 1-nerve of a graph $G$ is constructed in [BBdLL06] as the cubical set associated to $G$, denoted $M(G)$. The geometric realization of this cubical set is referred to as the cell complex associated to $G$, denoted $X_G$.

For a cubical set $X \in \text{cSet}$, the graph $|X|_m$ may be explicitly described as the colimit
$$|X|_m := \text{colim}(G_0 \hookrightarrow G_1 \hookrightarrow G_2 \hookrightarrow \ldots)$$
where

- $G_0$ is the discrete graph whose vertices are $X_0$;
• \( G_{n+1} \) is obtained from \( G_n \) via the following pushout (where \((X_n)_{nd}\) is the subset of \( X_n \) consisting of nondegenerate cubes).

\[
\begin{array}{ccc}
\coprod_{x \in (X_n)_{nd}} \partial I_m^{\otimes n} & \rightarrow & G_n \\
\downarrow & & \downarrow \gamma \\
\coprod_{x \in (X_n)_{nd}} I_m^{\otimes n} & \rightarrow & G_{n+1}
\end{array}
\]

Here, \( \partial I_m^{\otimes n} \) is the subgraph of \( I_m^{\otimes n} \) defined by

\[
\partial I_m^{\otimes n} := \{(v_1, \ldots, v_n) \in I_m^{\otimes n} | \text{for some } i = 0, \ldots, n, v_i = 0 \text{ or } m \}
\]

where the edge set is discrete if \( m = n = 1 \) and full otherwise.

**Example 3.3.**

1. For \( X = \Box_{2,1}^2 \), we have that \((\Box_{2,1}^2)_0\) has four 0-cubes. Thus, the graph \( G_0 \) is simply the discrete graph with four vertices.

   Figure 3: The embedding of \( G_0 \hookrightarrow G_1 \) for \(|\Box_{2,1}^2|_1\).

   The open box \( \Box_{2,1}^2 \) has three non-degenerate 1-cubes. The pushout constructed to obtain \( G_1 \) glues three copies of \( I_1 \) to \( G_0 \). As \( \Box_{2,1}^2 \) contains only degenerate cubes above dimension 1, constructing \( G_1 \) completes the construction of the graph \(|\Box_{2,1}^2|_1\).

2. To construct \(|\Box_{2,1}^2|_3\), we instead glue three copies of \( I_3 \). Observe that this process adds new vertices to the graph.

   Figure 4: The graph \(|\Box_{2,1}^2|_3\). The image of the embedding \( G_0 \hookrightarrow |\Box_{2,1}^2|_3 \) is highlighted.

For a graph \( G \), let \( l^*, r^*: N_m G \to N_{m+1} G \) denote the cubical maps obtained by precomposition with the surjections \( l_m^{\otimes n}, r_m^{\otimes n}: I_m^{\otimes n} \to I_m^{\otimes n} \). We think of these maps as inclusions of \( n \)-cubes of size \( m \) into \( n \)-cubes of size \( m+1 \). We write \( c^*: N_m G \to N_{m+2} G \) for the composite \( l^* r^* = r^* l^* \).

**Remark 3.4.** While the maps \( l_m^{\otimes n}, r_m^{\otimes n} \) have sections \( I_m^{\otimes n} \to I_{m+1}^{\otimes n} \), these maps do not commute with face maps, hence do not give retractions \( N_{m+1} G \to N_m G \). To demonstrate this, we show the map \( l^*: N_1 I_2 \to N_2 I_2 \) does not have a retraction. The identity map \( \text{id}\_I_2: I_2 \to I_2 \) gives a 1-cube of \( N_2 I_2 \) whose faces are the 0-cubes 0 and 2. Observe a retraction of \( l^* \) must send the 0-cubes 0 and 2 to 0 and 2, respectively. There is no map \( f: I_1 \to I_2 \) such that \( f \partial_{1,0} = 0 \) and \( f \partial_{1,1} = 2 \). That is, there is no 1-cube of \( N_1 I_2 \) which \( \text{id}\_I_2 \in (N_2 I_2)_1 \) can be mapped to. Thus, the map \( l^* \) does not have a retraction.
For the composite $G$, one verifies these maps satisfy cubical identities, thus $N$ is given as follows.

Figure 5: For a 1-cube $f: I_1 \to G$ of $N_1 G$, the map $l^*: N_1 G \to N_2 G$ sends $f$ to the 1-cube $fl: I_2 \to G$ of $N_2 G$ whereas $r^*: N_1 G \to N_2 G$ sends $f$ to the 1-cube $fr: I_2 \to G$ of $N_2 G$.

Figure 6: For a 2-cube $g: I^2 \to G$ of $N_1 G$, the map $l^*: N_1 G \to N_2 G$ sends $g$ to the 1-cube $gl^2: I^2_2 \to G$ of $N_2 G$ whereas $r^*: N_1 G \to N_2 G$ sends $g$ to the 1-cube $gr^2: I^2_2 \to G$ of $N_2 G$.

For a cubical set $X$, we analogously have maps $l_*, r_*: |X|_{m+1} \to |X|_m$. We write $c_*: |X|_{m+2} \to |X|_m$ for the composite $l_* r_* = r_* l_*$. We define the nerve functor $N: \text{Graph} \to \text{cSet}$ by

$$\text{(NG)}_n := \{ f: I^n_1 \to G \mid f \text{ is stable in all directions } (i, \varepsilon) \}.$$  

Cubical operators of $NG$ are given as follows.

- The map $\partial^n_{i, \varepsilon}: (NG)_n \to (NG)_{n-1}$ for $i = 1, \ldots, n$ and $\varepsilon = 0, 1$ is given by $f\partial^n_{i, \varepsilon}: I^{n-1}_\infty \to G$ defined by

$$f\partial^n_{i, \varepsilon}(v_1, \ldots, v_{n-1}) = f(v_1, \ldots, v_{i-1}, (2\varepsilon - 1)M, v_i, \ldots, v_{n-1})$$

where $M$ is such that $f$ is stable in direction $(i, \varepsilon)$;

- The map $\sigma^n_i: (NG)_n \to (NG)_{n+1}$ for $i = 1, \ldots, n$ is given by $f\sigma^n_i: I^{n+1}_\infty \to G$ defined by

$$f\sigma^n_i(v_1, \ldots, v_{n+1}) = f(v_1, \ldots, v_{i-1}, v_i, v_{i+1}, \ldots, v_{n+1});$$

- The map $\gamma^n_{i, 0}: (NG)_n \to (NG)_{n+1}$ for $i = 1, \ldots, n-1$ is given by $f\gamma^n_{i, 0}: I^{n+1}_\infty \to G$ defined by

$$f\gamma^n_{i, 0}(v_1, \ldots, v_{n+1}) = f(v_1, \ldots, v_{i-1}, \max(v_i, v_{i+1}), v_{i+1}, \ldots, v_{n+1});$$

- The map $\gamma^n_{i, 1}: (NG)_n \to (NG)_{n+1}$ for $i = 1, \ldots, n-1$ is given by $f\gamma^n_{i, 1}: I^{n+1}_\infty \to G$ defined by

$$f\gamma^n_{i, 1}(v_1, \ldots, v_{n+1}) = f(v_1, \ldots, v_{i-1}, \min(v_i, v_{i+1}), v_{i+1}, \ldots, v_{n+1}).$$

One verifies these maps satisfy cubical identities, thus $NG$ is a cubical set. A straightforward computation gives the following statement.

**Proposition 3.5.** We have an isomorphism

$$NG \cong \text{colim}(N_1 G \xrightarrow{l^*} N_2 G \xrightarrow{r^*} N_3 G \xrightarrow{l^*} N_4 G \xrightarrow{r^*} \ldots)$$

natural in $G$. 

The nerve and realization functors satisfy the following categorical properties.

**Proposition 3.6.** For cubical sets $X, Y$ and $m \geq 1$, we have an isomorphism

$$|X \otimes Y|_m \cong |X|_m \otimes |Y|_m$$

natural in $X$ and $Y$.

**Proof.** The composite functors

$$|-|_m \otimes |-|_m : \mathsf{cSet} \times \mathsf{cSet} \to \mathsf{Graph}$$

preserve all colimits. As $\mathsf{cSet}$ is a presheaf category, every cubical set is a colimit of representable presheaves. Thus, it suffices to show these composites are naturally isomorphic on pairs $(\Box^a, \Box^b)$ for $a, b \geq 0$. We have that

$$|\Box^a \otimes \Box^b|_m \cong |\Box^{a+b}|_m \cong |\Box^a|^m \otimes |\Box^b|^m.$$ 



**Corollary 3.7.** Let $X$ be a cubical set and $G$ be a graph. For $m \geq 1$, we have isomorphisms

$$\hom_L(X, N_m G) \cong N_m(\hom^\otimes(|X|_m, G)) \cong \hom_R(X, N_m G)$$

natural in $X$ and $G$.

**Proof.** The square

$$\begin{array}{ccc}
\mathsf{cSet} & \xrightarrow{\otimes X} & \mathsf{cSet} \\
| \downarrow | \downarrow & \longleftarrow \otimes |X|_m & \downarrow | \downarrow \\
\mathsf{Graph} & \xrightarrow{\hom(\cdot)} & \mathsf{Graph}
\end{array}$$

commutes up to natural isomorphism by Proposition 3.6, thus the corresponding square of right adjoints

$$\begin{array}{ccc}
\mathsf{Graph} & \xrightarrow{\hom^\otimes(|X|_m, \cdot)} & \mathsf{Graph} \\
N_m \downarrow & \longleftarrow N_m & \downarrow N_m \\
\mathsf{cSet} & \xrightarrow{\hom_L(\cdot, \cdot)} & \mathsf{cSet}
\end{array}$$

commutes up to natural isomorphism. For naturality in $X$, the required square commutes by faithfulness of the Yoneda embedding $\mathsf{cSet} \to \mathsf{Set}^{\mathsf{cSet}^{\mathsf{op}}}$.

**Proposition 3.8.** The nerve functor $N : \mathsf{Graph} \to \mathsf{cSet}$ preserves finite limits.

**Proof.** By Proposition 3.5, the nerve of $G$ is a filtered colimit. This then follows as filtered colimits commute with finite limits and $N_m : \mathsf{Graph} \to \mathsf{cSet}$ is a right adjoint for all $m \geq 1$. 

We have that the nerve functors preserve filtered colimits, which we use to give an analogue of Corollary 3.7 for the nerve functor $N : \mathsf{Graph} \to \mathsf{cSet}$. 

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Proposition 3.9. For $m \geq 0$, the functors $N_m, N : \text{Graph} \to \text{cSet}$ preserve filtered colimits.

Proof. For $N_m$, it suffices to show filtered colimits are preserved component-wise, i.e. that

$$\text{Graph}(f_m\otimes n, -) : \text{Graph} \to \text{Set}$$

preserves filtered colimits. This follows from Corollary 1.7.

For $N$, this follows since $N_m$ preserves filtered colimits and colimits commute with colimits.

For a cubical set $X$, define a functor $P_X : \text{Graph} \to \text{Graph}$ by

$$P_X G := \text{colim} \left( \text{hom}^\otimes(|X|_1, G) \xrightarrow{(l_\ast)^*} \text{hom}^\otimes(|X|_2, G) \xrightarrow{(r_\ast)^*} \text{hom}^\otimes(|X|_3, G) \xrightarrow{(l_\ast)^*} \ldots \right).$$

As an example, the path graph $P_G$ of a graph is exactly $P_{□G}$.

Proposition 3.10. Let $X$ be a cubical set with finitely many non-degenerate cubes. We have isomorphisms

$$N(P_X G) \cong \text{hom}_L(X, NG) \cong \text{hom}_R(X, NG)$$

natural in $X$ and $G$.

Proof. By Proposition 3.5, the left term $N(P_X G)$ is the colimit

$$N(P_X G) \cong \text{colim} \left( \begin{array}{c}
N_1 \text{hom}^\otimes(|X|_1, G) \xrightarrow{(l_\ast)^*} N_2 \text{hom}^\otimes(|X|_1, G) \xrightarrow{(r_\ast)^*} \ldots \\
N_1 \text{hom}^\otimes(|X|_2, G) \xrightarrow{(l_\ast)^*} N_2 \text{hom}^\otimes(|X|_2, G) \xrightarrow{(r_\ast)^*} \ldots \\
\ldots \\
\end{array} \right),$$

where the vertical maps are monomorphisms since $l_\ast, r_\ast : |X|_{m+1} \to |X|_m$ are epimorphisms for all $m \geq 1$. Computing this colimit component-wise in $\text{Set}$, this colimit is naturally isomorphic to the colimit

$$N(P_X G) \cong \text{colim} \left( \begin{array}{c}
N_1 \text{hom}^\otimes(|X|_1, G) \longrightarrow N_2 \text{hom}^\otimes(|X|_2, G) \longrightarrow \ldots \\
\end{array} \right)$$

along the diagonal. Applying Corollary 3.7, we may write

$$N(P_X G) \cong \text{colim} \left( \begin{array}{c}
\text{hom}_L(X, N_1 G) \longrightarrow \text{hom}_L(X, N_2 G) \longrightarrow \ldots \\
\end{array} \right).$$

As $X$ has finitely many non-degenerate cubes, the functor $\text{hom}_L(X, -)$ preserves filtered colimits. Thus,

$$N(P_X G) \cong \text{hom}_L(X, NG).$$

An analogous proof applies in the case of $\text{hom}_R(X, NG)$.

We have that the nerve functors “detect” concatenation of paths. That is, they contain all possible composition squares.

Proposition 3.11. Let $f : (I_\infty, I_{\geq M}) \to (G, v)$ and $g : (I_\infty, I_{\geq N}) \to (G, v)$ be based graph maps for some $M, N \geq 0$ such that $f \partial_1 = g \partial_0$. The concatenation $f \cdot g$ of $f$ followed by $g$ is a concatenation of $f$ and $g$ in the $(M + N)$-nerve $N_{(M+N)}G$. 

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Proof. The horizontal concatenation of $f_{\gamma_{1,0}}: I_{M+1}^2 \to G$ on the left and $g_{\sigma_{2}}: I_{N}^2 \to G$ on the right gives a square $\eta: I_{M+N} \to G$ such that

$$
\begin{align*}
\eta\partial_{1,0} &= f \\
\eta\partial_{1,1} &= g\partial_{1,1}\sigma_{1} \\
\eta\partial_{2,0} &= f \cdot g \\
\eta\partial_{2,1} &= g.
\end{align*}
$$

This is exactly a composition square witnessing $f \cdot g$ as a composition of $f$ and $g$ in $N_{(M+N)}G$.

As well, the nerve functors reflect isomorphisms.

**Proposition 3.12.** The nerve functors reflect isomorphisms. That is, given a graph map $f: G \to H$, if either

1. $N_{m}f: N_{m}G \to N_{m}H$ is an isomorphism for some $m \geq 1$; or
2. $N_{f}: NG \to NH$ is an isomorphism

then $f$ is.

**Proof.** We prove that $N: \text{Graph} \to \text{cSet}$ reflects isomorphisms as the case for $N_{m}: \text{Graph} \to \text{cSet}$ is analogous.

The inverse of $N_{f}$ is, in particular, an inverse on 0-cubes $(NH)_0 \to (NG)_0$, i.e. an inverse of $f$ on vertices $H_{V} \to G_{V}$. It suffices to show this map $g: H_{V} \to G_{V}$ is a graph map.

An edge in $H$ gives a map $e: I_1 \to H$. This gives a 1-cube $\tau: \square_{1} \to NG$ by the inclusion $N_{1}H \hookrightarrow NH$. As $N_{f}: NG \to NH$ is an isomorphism, there is a unique 1-cube $\overline{\tau}: \square_{1} \to NG$ such that $N_{f}(\overline{\tau}) = \tau$. This corresponds to a map $p: I_{\infty} \to G$ such that $p$ stabilizes in both directions and $fp = e$.

As $f$ is injective on vertices and $e$ is a path of length 1 (i.e. it consists of 2 vertices and 1 edge), we have that $p$ is a path of length 1. That is, $p$ is an edge, hence $g$ is a graph map.

Recall that a functor which reflects isomorphisms also reflects any (co)limits which it preserves. Thus,

**Corollary 3.13.**

1. For $m \geq 1$, the $m$-nerve $N_{m}: \text{Graph} \to \text{cSet}$ reflects all limits.
2. The nerve functor $N: \text{Graph} \to \text{cSet}$ reflects finite limits.

### 4 Main result

**Statement**

Our main theorem is the following.

**Theorem 4.1.** For any graph $G$,

1. the nerve $NG$ of $G$ is a Kan complex;
2. the natural inclusion $N_{1}G \hookrightarrow NG$ is anodyne.
Before proving Theorem 4.1, we explain the proof strategy and establish some auxiliary lemmas.

To show that the nerve of any graph is a Kan complex, we construct a map \( \Phi: I_{3m}^{\otimes n} \to |\nabla_{i,\varepsilon}|_{m} \) so that the triangle

\[
\begin{array}{ccc}
|\nabla_{i,\varepsilon}|_{3m} & \xrightarrow{c_m} & |\nabla_{i,\varepsilon}|_{m} \\
\downarrow \Phi & & \downarrow \\
I_{3m}^{\otimes n} & \rightarrow & I_{m}^{\otimes n}
\end{array}
\]

commutes. We show that every map \(|\nabla_{i,\varepsilon}|_{m} \to NG\) must factor through some \(m\)-nerve \(|\nabla_{i,\varepsilon}|_{m} \to N_{m}G \to NG\).

The map \( \Phi \) gives a filler for the composite \(|\nabla_{i,\varepsilon}|_{m} \to N_{m}G \to N_{3m}G \), thus a filler in \(NG\).

To show that the map \(N_{1}G \hookrightarrow NG\) is anodyne, we show the maps \(l^{*}, r^{*}: N_{m}G \hookrightarrow N_{m+1}G\) are anodyne.

This is done by an explicit construction establishing \(l^{*}\) and \(r^{*}\) as a transfinite composition of pushouts along coproducts of open box inclusions. With this, we have that \(N_{1}G \hookrightarrow NG\) is anodyne as the nerve of \(G\) is a transfinite composition of \(l^{*}\) and \(r^{*}\).

**Proof of part (1)**

**Construction 4.2.** Fix \(m, n \geq 1, i \in \{1, \ldots, n\}\), and \(\varepsilon \in \{0, 1\}\). We construct the map \( \Phi: I_{3m}^{\otimes n} \to |\nabla_{i,\varepsilon}|_{m}\).

Observe that, for \(n = 1\), we have that \(|\nabla_{i,\varepsilon}|_{m} \cong I_{0}\). In this case, the map \( \Phi \) is immediate. Thus, we assume \(n \geq 2\).

Define a map \( \varphi: I_{3m} \to I_{3m} \) by

\[
\varphi v := \begin{cases} 
  m & v \leq m \\
  i & m \leq v \leq 2m \\
  2m & v \geq 2m.
\end{cases}
\]

It is straightforward to verify this definition gives a graph map. From this, we have a map \( \varphi^{\otimes n-1}: I_{3m}^{\otimes n-1} \to I_{3m}^{\otimes n-1} \) which sends \((v_{1}, \ldots, v_{n-1})\) to \((\varphi v_{1}, \ldots, \varphi v_{n-1})\).

For \(v = (v_{1}, \ldots, v_{n-1}) \in I_{3m}^{\otimes n-1}\), let \(d(v)\) denote the node distance between \(v\) and \(\varphi^{\otimes n-1}v\). That is,

\[
d(v) := \sum_{i=1}^{n-1} |v_{i} - \varphi v_{i}|.
\]

Observe this gives a graph map \(d: I_{3m}^{\otimes n-1} \to I_{\infty}\).

For \(t \geq 0\), we have a graph map \(\beta[t]: I_{\infty} \to I_{t}\) defined by

\[
\beta[t](v) := \begin{cases} 
  0 & v \leq 0 \\
  v & 0 \leq v \leq t \\
  t & v \geq t.
\end{cases}
\]

One thinks of \(\beta[t]\) as bounding the graph \(I_{\infty}\) between 0 and \(t\). Recall the map \(c_{m}: I_{3m} \to I_{m}\) is given by

\[
c_{m}v := \begin{cases} 
  0 & v \leq m \\
  v - m & m \leq v \leq 2m \\
  \frac{v}{2} & v \geq 2m.
\end{cases}
\]

For a vertex \((v_{1}, \ldots, v_{n}) \in I_{3m}^{\otimes n}\), we write \(\sigma_{i}v\) for the vertex \((v_{1}, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{n}) \in I_{3m}^{\otimes n-1}\). As well, let \(\alpha_{\varepsilon}^{v_{1}}\) denote the value

\[
\alpha_{\varepsilon}^{v_{1}} := \varepsilon m + (1 - 2\varepsilon)c_{m}v_{1}.
\]
We may also write this as
\[
\alpha_{\varepsilon}^{v_i} = \begin{cases} 
  c^m v_i & \varepsilon = 0 \\
  m - c^m v_i & \varepsilon = 1.
\end{cases}
\]

We define \( \Phi : I_3^{2n} \to |\Gamma_{h,\varepsilon}|_m \) by
\[
\Phi(v_1, \ldots, v_n) := (c^m v_1, \ldots, c^m v_{i-1}, (1 - \varepsilon)m + (2\varepsilon - 1)(\beta[\alpha_{v_i}^\varepsilon]\varepsilon(d(\sigma_i v) - \alpha_{v_i}^\varepsilon)), c^m v_{i+1}, \ldots, c^m v_n).
\]
That is,
\[
\Phi(v_1, \ldots, v_n) = \begin{cases} 
  (c^m v_1, \ldots, c^m v_{i-1}, m - \beta[m - c^m v_i]\varepsilon(d(\sigma_i v) - c^m v_i), c^m v_{i+1}, \ldots, c^m v_n) & \text{if } \varepsilon = 0 \\
  (c^m v_1, \ldots, c^m v_{i-1}, \beta[c^m v_i]\varepsilon(d(\sigma_i v) + c^m v_i - m), c^m v_{i+1}, \ldots, c^m v_n) & \text{if } \varepsilon = 1.
\end{cases}
\]

We first show this formula is well-defined, i.e., that this tuple lies in the subgraph \( |\Gamma_{h,\varepsilon}|_m \) of \( I_3^{2n} \). Observe that if \( c^m v_k = 0 \) or \( m \) for some \( k \neq i \) in \( \{1, \ldots, n\} \) then this tuple indeed lies in \( |\Gamma_{h,\varepsilon}|_m \). For \( k < i \), this tuple lies on the \((k, 0)\) or \((k, 1)\)-face. For \( k > i \), this tuple lies on the \((k+1, 0)\) or \((k+1, 1)\)-face. Otherwise, if \( 0 < c^m v_k < m \) for all \( k \neq i \) then \( d(\sigma_i v) = 0 \) since \( v_k = \varphi v_k \) for all \( k \neq i \). From this, we have that
\[
\beta[\alpha_{v_i}^{1-\varepsilon}](d(\sigma_i v) - \alpha_{v_i}^{\varepsilon}) = \beta[\alpha_{v_i}^{\varepsilon}](\alpha_{v_i}^{1-\varepsilon}) = 0,
\]
thus \( \Phi(v_1, \ldots, v_n) \) lies on the \((i, 1 - \varepsilon)\)-face; that is, the face opposite the missing face.

To see this formula gives a graph map, suppose \((v_1, \ldots, v_n)\) and \((w_1, \ldots, w_n)\) are connected vertices in \( I_3^{2n} \). By definition, there exists \( k = 1, \ldots, n \) so that \( v_k \) and \( w_k \) are connected in \( I_{3m} \) and \( v_j = w_j \) for all \( j \neq k \). We first consider the case where \( k \neq i \). Observe that if \( c^m v_k = c^m w_k \) then this is immediate. If \( c^m v_k \neq c^m w_k \) then \( \sigma_i v = \varphi(\sigma_i w) \). This gives that \( d(\sigma_i v) = d(\sigma_i w) \), hence \( \Phi(v_1, \ldots, v_n) \) and \( \Phi(w_1, \ldots, w_n) \) are equal on all components except the \( k \)-th component. That is, they are connected. In the case where \( k = i \), we have that \( d_v \) and \( d_w \) differ by at most 1. This implies \( (1 - \varepsilon)m + (2\varepsilon - 1)(\beta[\alpha_{v_i}^{1-\varepsilon}](d_v - \alpha_{v_i}^{\varepsilon})) \) and \( (1 - \varepsilon)m + (2\varepsilon - 1)(\beta[\alpha_{v_i}^{\varepsilon}](d_w - \alpha_{v_i}^{\varepsilon})) \) differ by at most 1, thus \( \Phi(v_1, \ldots, v_n) \) and \( \Phi(w_1, \ldots, w_n) \) are connected.

Figure 7: Vertices of \( I_6^{\otimes 2} \) labelled by their image under \( d : I_6^{\otimes 2} \to I_\infty \).
Example 4.3. We look at the map $\Phi: I_{3m}^n \to |\mathcal{P}_{1,\varepsilon}^n|_m$ in the case of $n = 3$, $m = 2$, and $(i, \varepsilon) = (3, 1)$.

Figure 8: The graph $|\mathcal{P}_{1,1}^3|_2$ as a net. Vertices connected by a dotted line are identical.

We may write the map $\Phi: I_6^3 \to |\mathcal{P}_{3,1}^3|_2$ as

$$\Phi(v_1, v_2, v_3) = \begin{cases} (c^2 v_1, c^2 v_2, 0) & \text{if } v_3 \leq 2 \\ (c^2 v_1, c^2 v_2, \beta[1](d(v_1, v_2) - 1)) & \text{if } v_3 = 3 \\ (c^2 v_1, c^2 v_2, \beta[2](d(v_1, v_2))) & \text{if } v_3 \geq 4. \end{cases}$$

If $v_3 \leq 2$ then $(v_1, v_2, v_3)$ is contained in the $\partial_{3,0}$-face of $|\mathcal{P}_{3,1}^3|_2$, which is the face opposite the missing face. For the cross-section where $v_3 = 3$, if $d(v_1, v_2) \geq 2$ then $(v_1, v_2, v_3)$ is sent to a vertex in the $v_3 = 1$ cross-section of $|\mathcal{P}_{3,1}^3|_2$. For $v_3 \geq 3$, if $d(v_1, v_2) = 1$ then $(v_1, v_2, v_3)$ is sent to a vertex in the $v_3 = 1$ cross-section. If $d(v_1, v_2) \geq 2$ then $(v_1, v_2, v_3)$ is sent to a vertex in the $v_3 = 2$ cross-section. In all three cross-section, if $d(v_1, v_2) = 0$ then $(v_1, v_2, v_3)$ is contained in the face opposite the missing face of $|\mathcal{P}_{3,1}^3|_2$.

Figure 9: The map $\Phi: I_6^3 \to |\mathcal{P}_{3,1}^3|_2$ split into cross-sections. Vertices are colored by their image as in Fig. 8.
Lemma 4.4. The diagram

\[ \begin{array}{ccc} |\Gamma_{i,\varepsilon}^n|_{3m} & \xrightarrow{(e_\varepsilon)^m} & |\Gamma_{i,\varepsilon}^n|_m \\ \Phi \downarrow & & \downarrow \\ |\Box^n|_{3m} & \xrightarrow{\Phi} & |\Sigma^n|_{3m} \end{array} \]

commutes.

Proof. For \( n = 1 \), we have \( |\Gamma_{1,\varepsilon}^1|_m \cong I_0 \). The diagram then commutes as \( I_0 \) is terminal in \( \text{Graph} \).

For \( n \geq 2 \), fix \( (v_1, \ldots, v_n) \in |\Gamma_{i,\varepsilon}^n|_{3m} \). It suffices to show

\[(1 - \varepsilon)m + (2\varepsilon - 1)(\beta[\alpha_{1-\varepsilon}^v](d(\sigma_i v) - \alpha_{1-\varepsilon}^v))) = e^m v_i.\]

If \( v_i = (1 - \varepsilon)(3m) \) then \( d(\sigma_i v) = 0 \). Thus,

\[(1 - \varepsilon)m + (2\varepsilon - 1)(\beta[\alpha_{1-\varepsilon}^v](d(\sigma_i v) - \alpha_{1-\varepsilon}^v))) = (1 - \varepsilon)m + (2\varepsilon - 1)(0) \]
\[= (1 - \varepsilon)m \]
\[= e^m((1 - \varepsilon)(3m)) \]
\[= e^m v_i.\]

Otherwise, if \( v_i \neq (1 - \varepsilon)(3m) \) then \( d(\sigma_i v) \geq m \) (since \( v_k = 0, 3m \) for some \( k \neq i \)). For \( \varepsilon = 0 \), we have that

\[(1 - \varepsilon)m + (2\varepsilon - 1)(\beta[\alpha_{1}^v](d(\sigma_i v) - \alpha_0^v))) = m - \beta[\alpha_1^v](d(\sigma_i v) - \alpha_0^v) \]
\[= m - \beta[m - e^m v_i](d(\sigma_i v) - e^m v_i) \]
\[= m - (m - e^m v_i) \]
\[= e^m v_i.\]

For \( \varepsilon = 1 \), we have that

\[(1 - \varepsilon)m + (2\varepsilon - 1)(\beta[\alpha_{1-\varepsilon}^v](d(\sigma_i v) - \alpha_{1-\varepsilon}^v))) = \beta[\alpha_0^v](d(\sigma_i v) - \alpha_1^v) \]
\[= \beta[e^m v_i](d(\sigma_i v) + e^m v_i - m) \]
\[= e^m v_i.\]

\[\square\]

Theorem 4.5. For any graph \( G \), the nerve \( NG \) of \( G \) is a Kan complex.

Proof. Fix a map \( f: |\Gamma_{i,\varepsilon}^n| \to NG \). We have that \( NG \cong \text{colim}(N_1 G \hookrightarrow N_2 G \hookrightarrow N_3 G \hookrightarrow \ldots) \). Recall that in a presheaf category, any map from a representable presheaf to a colimit must factor through some component of the colimit cone. Thus, for any \( k \geq 0 \) and \( x: \Box^k \to |\Gamma_{i,\varepsilon}^n| \), the map \( f x: \Box^k \to NG \) factors through an inclusion \( N_m G \hookrightarrow NG \) for some \( m \geq 1 \). As \( |\Gamma_{i,\varepsilon}^n| \) has only finitely many non-degenerate cubes, \( f \) factors through the natural inclusion \( N_m G \hookrightarrow NG \) as a map \( g: |\Gamma_{i,\varepsilon}^n| \to N_m G \) for some \( m \geq 0 \).

By adjointness, \( g \) corresponds to a map \( \Phi: |\Gamma_{i,\varepsilon}^n|_m \to G \). Lemma 4.4 shows that \( \Phi\Phi: |\Box^n|_{3m} \to G \) is a lift of the composite map \( \Phi(e)^m: |\Gamma_{i,\varepsilon}^n|_{3m} \to G \).
By adjointness, this gives a filler $\square^n \to N_{3m}G$ for the map $(c^*)^m g: \square^n \to N_{3m}G$. Post-composing with the natural inclusion $N_{3m}G \hookrightarrow NG$, this gives a filler $\square^n \to NG$ of $f$. \hfill $\square$

Via Theorem 4.5, we may speak of the homotopy groups of $NG$. We have that the $A$-homotopy groups of $G$ are exactly the cubical homotopy groups of $NG$.

**Theorem 4.6.** We have an isomorphism $A_n(G, v) \cong \pi_n(NG, v)$ natural in $(G, v)$.

**Proof.** It is straightforward to verify that a relative graph map $(I^n \otimes I^n, I^n \otimes I^n) \to (G, v)$ is exactly a relative cubical map $(\square^n, \partial \square^n) \to (NG, v)$. This gives a set bijection $A_n(G, v) \to \pi_n(NG, v)$. Proposition 3.11 shows this map is a group homomorphism. \hfill $\square$

**Remark 4.7.** Theorem 4.6 shows why the functor $N: \text{Graph} \to \text{cSet}$ fails to preserve arbitrary limits, even though, by Proposition 3.8, it preserves finite ones. To see that, we first note that

$$A_1 \left( \prod_{k \geq 5} (C_k, 0) \right) \cong \bigoplus_{k \geq 5} \mathbb{Z},$$

so a map $I_\infty \to \prod_{k \geq 5} C_k$ that stabilizes outside of a finite interval must necessarily be homotopic to a constant in all but finitely many $C_k$’s. On the other hand, since $\pi_1: \text{cSet} \to \text{Grp}$ preserves arbitrary products by [CK22, Prop. 4.1], we have

$$N \left( \prod_{k \geq 5} (C_k, 0) \right) \not\cong \prod_{k \geq 5} (NC_k, 0).$$

**Proof of part (2)**

Now we show that the inclusions $l^*, r^*: N_mG \hookrightarrow N_{m+1}G$ are anodyne. We first explain the intuition for why this statement holds.

The 1-nerve $N_1G$ of a graph $G$ contains, as 1-cubes, all paths of length 2 in $G$ (i.e. paths with 2 vertices and 1 edge). Consider the image of the embedding $l^*: N_1G \to N_2G$ as a cubical subset $X \subseteq N_2G$ of the 2-nerve of $G$. A 1-cube of $N_2G$ is exactly a path of length 3 in $G$; a 1-cube of $X$ is a path of length 2 regarded as a path of length 3 whose first two vertices are the same. Given a path $f: I_2 \to G$ of length 3 in $G$, we may define a $3 \times 3$ square $g: I_2^\otimes 3 \to G$ in $G$ by

$$g(v_1, v_2) := \begin{cases} f(l(v_1)) & v_2 < 2 \\ f(v_1) & v_2 = 2. \end{cases}$$

$\square$

Figure 10: The square $g: I_2^\otimes 3 \to G$ constructed from the path $f: I_2 \to G$.

Observe the $\partial_{1,0}$- and $\partial_{1,1}$- and $\partial_{2,0}$-faces of $g$ are 1-cubes of $X$, i.e. they are paths of length 3 whose first two vertices are the same, whereas the $\partial_{2,1}$-face of $g$ is $f$. From this, we have that the restriction
$g|_{\Delta^2,1} : \Delta^2,1 \to G$ of $g$ to the open box corresponds to a map $\Delta^2,1 \to N_2G$ whose image is contained in $X$. Let $Y \subseteq N_2G$ denote the cubical subset generated by $X$ and $g$; that is, the cubical subset containing $X$, the 2-cube $g \in (N_2G)_2$, and all faces, degeneracies and connections of $g$. We have that the square

$$
\begin{array}{ccc}
\Delta^2 & \xrightarrow{\gamma} & X \\
\downarrow & & \downarrow \\
\square^2 & \xrightarrow{g} & Y
\end{array}
$$

is a pushout by definition of $Y$. This gives exactly that the inclusion $X \hookrightarrow Y$ is anodyne. Following this approach, one may construct an anodyne inclusion $X \hookrightarrow X_n$ such that $X_n$ contains all $n$-cubes of $N_2G$. With this, the colimit of the sequence $X \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow \ldots$ is exactly $N_2G$ and the inclusion $X \hookrightarrow N_2G$ is anodyne by closure under transfinite composition.

For $n > 0$ and $j \in \{0, \ldots, n\}$, the maps $l, r : I_{m+1} \to I_m$ yield maps

$$(\text{id} \otimes I^{\otimes n-j}_m, (\text{id} \otimes I^{\otimes j}_{m+1} \otimes r^{\otimes n-j}) : I^{\otimes n}_{m+1} \to I^{\otimes j}_{m+1} \otimes I^{\otimes n-j}_m.$$

For $j \in \{0, \ldots, n-1\}$, we define maps $\lambda^{n}_{m,j}, \rho^{n}_{m,j} : I^{\otimes n}_{m+1} \otimes I_1 \to I^{\otimes j+1}_{m+1} \otimes I^{\otimes n-j-1}_m$ by

$$\lambda^{n}_{m,j}(v_1, \ldots, v_n, v_{n+1}) = \begin{cases} (v_1, \ldots, v_j, lv_{j+1}, lv_{j+2}, \ldots, lv_n) & v_{n+1} = 0 \\ (v_1, \ldots, v_{j+1}, lv_{j+2}, \ldots, lv_n) & v_{n+1} = 1. \end{cases}$$

$$\rho^{n}_{m,j}(v_1, \ldots, v_n, v_{n+1}) = \begin{cases} (v_1, \ldots, v_j, rv_{j+1}, rv_{j+2}, \ldots, rv_n) & v_{n+1} = 0 \\ (v_1, \ldots, v_{j+1}, rv_{j+2}, \ldots, rv_n) & v_{n+1} = 1. \end{cases}$$

That is, the restriction of $\lambda^{n}_{m,j}$ to $I^{\otimes n}_{m+1} \otimes \{0\}$ is the map $\text{id}^{\otimes j}_{I_{m+1}} \otimes I^{\otimes n-j}$ and its restriction to $I^{\otimes n}_{m+1} \otimes \{1\}$ is $\text{id}^{\otimes j+1}_{I_{m+1}} \otimes I^{\otimes n-j-1}$ (likewise for $\rho^{n}_{m,j}$).

Figure 11: The graph $I_2 \otimes I_1$ with vertices labelled by their image under $\lambda^1_{1,0} : I_2 \otimes I_1 \to I_2$. 
The subgraph $I_2^{\otimes 2} \otimes \{0\}$ under $\lambda_1^{2,0}$

The subgraph $I_2^{\otimes 2} \otimes \{1\}$ under $\lambda_1^{2,1}$

The subgraph $I_2^{\otimes 2} \otimes \{0\}$ under $\lambda_1^{2,1}$

The subgraph $I_2^{\otimes 2} \otimes \{1\}$ under $\lambda_1^{2,1}$

Figure 12: Cross-sections of the graph $I_2^{\otimes 2} \otimes I_1$ with vertices labelled by their image under the maps $\lambda_1^{2,0}: I_2^{\otimes 2} \otimes I_1 \to I_2 \otimes I_1$ and $\lambda_1^{2,1}: I_2^{\otimes 2} \otimes I_1 \to I_2^{\otimes 2}$.

The maps $l^m, r^m: I_{m+1} \to I_1$ denote application of the maps $l, r$ a total of $m$ times. That is,

$$l^m(v) = \begin{cases} 0 & v < m + 1 \\ 1 & v = m + 1, \end{cases}$$

$$r^m(v) = \begin{cases} 0 & v = 0 \\ 1 & v > 0. \end{cases}$$

We write $\lambda_{m,j}^n: I_{m+1}^{\otimes n} \to I_{m+1}^{\otimes n} \otimes I_{m+1}^{\otimes n-j-1}$ for the composition of $id_{I_{m+1}^{\otimes n}} \otimes l^m: I_{m+1}^{\otimes n} \to I_{m+1}^{\otimes n} \otimes I_1$ followed by $\lambda_{m,j}^n: I_{m+1}^{\otimes n} \otimes I_1 \to I_{m+1}^{\otimes n} \otimes I_{m+1}^{\otimes n-j-1}$ and we write $\overline{\lambda}_{m,j}^n: I_{m+1}^{\otimes n} \to I_{m+1}^{\otimes n} \otimes I_{m+1}^{\otimes n-j-1}$ for the composition of $id_{I_{m+1}^{\otimes n}} \otimes r^n: I_{m+1}^{\otimes n} \to I_{m+1}^{\otimes n} \otimes I_1$ followed by $\rho_{m,j}^n: I_{m+1}^{\otimes n} \otimes I_1 \to I_{m+1}^{\otimes n} \otimes I_{m+1}^{\otimes n-j-1}$, respectively.

**Proposition 4.8.** Let $m, n > 0$ and $j \in \{0, \ldots, n\}$. For $i = 1, \ldots, n$ such that $i \neq j + 1$ and $\varepsilon = 0, 1$, we have that

1. $\overline{\lambda}_{m,j}^n \partial_{i,\varepsilon}: I_{m+1}^{\otimes n} \to I_{m+1}^{\otimes n}$ factors through $\lambda_{m,j}^{n-1}: I_{m+1}^{\otimes n-1} \to I_{m+1}^{\otimes n-1}$;

2. $\overline{\rho}_{m,j}^n \partial_{i,\varepsilon}: I_{m+1}^{\otimes n} \to I_{m+1}^{\otimes n}$ factors through $\rho_{m,j}^{n-1}: I_{m+1}^{\otimes n-1} \to I_{m+1}^{\otimes n-1}$

**Proof.** We consider the result for $\overline{\lambda}_{m,j}^n$, as the result for $\overline{\rho}_{m,j}^n$ is analogous.
Fix \((v_1, \ldots, v_n) \in I_{m+1}^{\otimes n+1}\). If \(i < j + 1\) then we have
\[
\bar{\lambda}_{m,j}^n \partial_{i,\varepsilon}(v_1, \ldots, v_n) = \lambda_{m,j}^n (v_1, \ldots, v_{i-1}, \varepsilon(m + 1), v_i, \ldots, v_n)
\]
\[
= \lambda_{m,j}^n (v_1, \ldots, v_{i-1}, \varepsilon(m + 1), v_i, \ldots, v_{n-1}, l^mv_n)
\]
\[
= \begin{cases} 
(v_1, \ldots, v_{i-1}, \varepsilon(m + 1), v_i, \ldots, v_j, lv_{j+1}, \ldots, lv_{n-1}) & \text{if } l^mv_n = 0 \\
(v_1, \ldots, v_{i-1}, \varepsilon(m + 1), v_i, \ldots, v_{j+1}, lv_{j+2}, \ldots, lv_{n-1}) & \text{if } l^mv_n = 1
\end{cases}
\]
\[
= \partial_{i,\varepsilon} \lambda_{m,j}^n (v_1, \ldots, v_{n-1}, l^mv_n)
\]
\[
= \partial_{i,\varepsilon} \bar{\lambda}_{m,j}^n (v_1, \ldots, v_n).
\]

Thus, \(\bar{\lambda}_{m,j}^n \partial_{i,\varepsilon} = \partial_{i,\varepsilon} \bar{\lambda}_{m,j}^{n-1}\).

Otherwise, we have \(i > j + 1\). Consider the embedding \(\iota: I_{m+1}^{\otimes n-1} \to I_{m+1}^{\otimes n}\) given by
\[
\iota(v_1, \ldots, v_{n-1}) = (v_1, \ldots, v_{i-1}, \varepsilon m, v_i, \ldots, v_{n-1}).
\]

With this, we may write
\[
\bar{\lambda}_{m,j}^n \partial_{i,0}^{n+1}(v_1, \ldots, v_n) = \bar{\lambda}_{m,j}^n (v_1, \ldots, v_{i-1}, \varepsilon(m + 1), v_i, \ldots, v_n)
\]
\[
= \lambda_{m,j}^n (v_1, \ldots, v_{i-1}, \varepsilon(m + 1), v_i, \ldots, v_{n-1}, l^mv_n)
\]
\[
= \begin{cases} 
(v_1, \ldots, v_{i-1}, lv_{i+1}, \ldots, lv_{i-1}, l(\varepsilon(m + 1)), lv_i, \ldots, lv_{n-1}) & \text{if } l^mv_n = 0 \\
(v_1, \ldots, v_{i-1}, lv_{i+1}, \ldots, lv_{i-1}, l(\varepsilon(m + 1)), lv_i, \ldots, lv_{n-1}) & \text{if } l^mv_n = 1
\end{cases}
\]
\[
= \begin{cases} 
(v_1, \ldots, v_{i-1}, lv_{i+1}, \ldots, lv_{i-1}, \varepsilon m, lv_i, \ldots, lv_{n-1}) & \text{if } l^mv_n = 0 \\
(v_1, \ldots, v_{i-1}, lv_{i+1}, \ldots, lv_{i-1}, \varepsilon m, lv_i, \ldots, lv_{n-1}) & \text{if } l^mv_n = 1
\end{cases}
\]
\[
= \iota \lambda_{m,j}^n (v_1, \ldots, v_{n-1}, l^mv_n)
\]
\[
= \iota \bar{\lambda}_{m,j}^n (v_1, \ldots, v_n).
\]

Thus, \(\bar{\lambda}_{m,j}^n \partial_{i,0} = \iota \bar{\lambda}_{m,j}^{n-1}\).

\begin{proposition}
Let \(m, n > 0\) and \(j \in \{0, \ldots, n\}\). We have that
\begin{enumerate}
\item \(\bar{\lambda}_{m,j}^n \partial_{j+1,0}\) and \(\bar{\lambda}_{m,j}^n \partial_{j+1,1}\) factor through \(id_{I_{m+1}^{\otimes j}} \otimes I_{m+1}^{\otimes n-j}: I_{m+1}^{\otimes n} \to I_{m+1}^{\otimes j} \otimes I_{m+1}^{\otimes n-j}\);
\item \(\bar{\rho}_{m,j}^n \partial_{j+1,0}\) and \(\bar{\rho}_{m,j}^n \partial_{j+1,1}\) factor through \(id_{I_{m+1}^{\otimes j}} \otimes I_{m+1}^{\otimes n-j}: I_{m+1}^{\otimes n} \to I_{m+1}^{\otimes j} \otimes I_{m+1}^{\otimes n-j}\).
\end{enumerate}
\end{proposition}

\textbf{Proof}. We show the result for \(\bar{\lambda}_{m,j}^n\) as the result for \(\bar{\rho}_{m,j}^n\) is analogous.

Fix \((v_1, \ldots, v_n) \in I_{m+1}^{\otimes n+1}\). We have that
\[
\bar{\lambda}_{m,j}^n \partial_{j+1,0}(v_1, \ldots, v_n) = \bar{\lambda}_{m,j}^n (v_1, \ldots, v_j, 0, v_{j+1}, \ldots, v_n)
\]
\[
= \lambda_{m,j}^n (v_1, \ldots, v_j, 0, v_{j+1}, \ldots, l^mv_n)
\]
\[
= \begin{cases} 
(v_1, \ldots, v_j, 0, lv_{j+1}, \ldots, lv_{n-1}) & \text{if } l^mv_n = 0 \\
(v_1, \ldots, v_j, 0, lv_{j+1}, \ldots, lv_{n-1}) & \text{if } l^mv_n = 1
\end{cases}
\]
\[
= \bar{\lambda}_{m,j}^n (v_1, \ldots, v_n).
\]

Thus, \(\bar{\lambda}_{m,j}^n \partial_{j+1,0} = \partial_{j+1,0} \sigma_{\iota}(id_{I_{m+1}^{\otimes j}} \otimes I_{m+1}^{\otimes n-j})(v_1, \ldots, v_n)\).
Consider the map \( f: I_{m+1}^\otimes I_m^\otimes n \to I_{m+1}^\otimes I_m^\otimes n-j \) defined by

\[
f(a_1, \ldots, a_j, b_1, \ldots, b_{n-j}) = \begin{cases} 
(a_1, \ldots, a_j, m, b_1, \ldots, b_{n-j}) & \text{if } l^{m-1}b_{n-j} = 0 \\
(a_1, \ldots, a_j, m+1, b_1, \ldots, b_{n-j-1}) & \text{if } l^{m-1}b_{n-j} = 1.
\end{cases}
\]

It is straightforward to verify this is a graph map. With this, we may write

\[
\tilde{\lambda}_{m,j}^n \partial_{j+1} = f(id_{I_{m+1}}^\otimes I_m^\otimes n-j).
\]

\[\Box\]

**Lemma 4.10.** Let \( m, n > 0, j \in \{0, \ldots, n-1\} \), and \( G \) be a graph. Consider a subobject \( X \) of \( N_{m+1}G \) which contains

- all \( n \)-cubes of \( N_{m+1}G \) which factor through \( I_{m+1}^\otimes I_m^\otimes n \);
- \( (if n > 1) \) for any \( x: I_{m+1}^\otimes I_m^\otimes k-h \to G \) where \( k < n \) and \( h \leq k \), the \((k+1)\)-cube \( x\tilde{\lambda}_{m,h-1}^{k+1}: I_{m+1}^\otimes I_m^\otimes n \to G \);
- \( (if j > 0) \) for any \( x: I_{m+1}^\otimes I_m^\otimes n-j \to G \), the \((n+1)\)-cube \( x\tilde{\lambda}_{m,j}^n: I_{m+1}^\otimes I_m^\otimes n \to G \) of \( N_{m+1}G \).

We have that, for any \( n \)-cube of \( N_{m+1}G \) which factors through \( \text{id}_{I_{m+1}}^\otimes I_m^\otimes n-j \), \( I_{m+1}^\otimes I_m^\otimes n-j \to G \), the restriction of the \((n+1)\)-cube \( x\tilde{\lambda}_{m,j-1}^n: I_{m+1}^\otimes I_m^\otimes n \to G \) to the open box \( x\tilde{\lambda}_{m,j}^n|_{\gamma_{m+1,1}}: \gamma_{m+1,1}^n \to N_{m+1}G \) factors through the inclusion \( X \hookrightarrow N_{m+1}G \).

\[\text{Proof.}\] We first show that \( X \) contains all \( n \)-cubes which factor through the map \( \text{id}_{I_{m+1}}^\otimes I_m^\otimes n-j : I_{m+1}^\otimes I_m^\otimes n \to I_{m+1}^\otimes I_m^\otimes n-j \). If \( j = 0 \) then this follows by assumption. Otherwise, consider such an \( n \)-cube, which we write as \( y(\text{id}_{I_{m+1}}^\otimes I_m^\otimes n-j) : I_{m+1}^\otimes I_m^\otimes n-j \to G \) for some \( y: I_{m+1}^\otimes I_m^\otimes n-j \to G \). Recall \( \tilde{\lambda}_{m,j-1}^n \partial_{n+1,1} = \text{id}_{I_{m+1}}^\otimes I_m^\otimes n-j \). This gives that \( y(\text{id}_{I_{m+1}}^\otimes I_m^\otimes n-j) = y\tilde{\lambda}_{m,j-1}^n \partial_{n+1,1} \). By assumption, \( X \) contains \( y\tilde{\lambda}_{m,j-1}^n \). Thus, it contains all faces of \( y\tilde{\lambda}_{m,j-1}^n \), including \( y \).

To see that each face of \( x\tilde{\lambda}_{m,j}^n|_{\gamma_{m+1,1}} \) is contained in \( X \), fix \((i, \varepsilon) \neq (n+1, 1)\). For \( i = n+1 \) and \( \varepsilon = 0 \), this follows as \( \tilde{\lambda}_{m,j}^n \partial_{n+1,0} = \text{id}_{I_{m+1}}^\otimes I_m^\otimes n-j \). Otherwise, if \( i \neq j+1 \), this follows from Proposition 4.8. If \( i = j+1 \) then Proposition 4.9 gives that \( \tilde{\lambda}_{m,j}^n \) factors through \( \text{id}_{I_{m+1}}^\otimes I_m^\otimes n-j \).

We have an analogous result for \( \tilde{\rho}_{m,j}^n \) as well.

**Lemma 4.11.** Let \( m, n > 0, j \in \{0, \ldots, n-1\} \), and \( G \) be a graph. Consider a subobject \( X \) of \( N_{m+1}G \) which contains

- all \( n \)-cubes of \( N_{m+1}G \) which factor through \( r^\otimes n : I_{m+1}^\otimes I_m^\otimes n \).
• (if \(n > 1\)) for any \(x: \mathbb{I}_{m+1}^{\otimes h} \otimes \mathbb{I}_m^{\otimes k-h} \to G\) where \(k < n\) and \(h \leq k\), the \((k+1)\)-cube \(x\tilde{p}_{m,h-1}^n: \mathbb{I}_{m+1}^{\otimes n+1} \to G\),

• (if \(j > 0\)) for any \(x: \mathbb{I}_{m+1}^{\otimes j} \otimes \mathbb{I}_m^{\otimes n-j} \to G\), the \((n+1)\)-cube \(x\tilde{p}_{m,j-1}^n: \mathbb{I}_{m+1}^{\otimes n+1} \to G\) of \(N_{m+1}G\).

We have that for any \(n\)-cube of \(N_{m+1}G\) which factors through \(\text{id}_{\mathbb{I}_{m+1}^{\otimes j+1}} \otimes \mathbb{I}_m^{\otimes n-j-1}: \mathbb{I}_{m+1}^{\otimes n} \to \mathbb{I}_{m+1}^{\otimes j+1} \otimes \mathbb{I}_m^{\otimes n-j-1}\) as some \(x: \mathbb{I}_{m+1}^{\otimes j+1} \otimes \mathbb{I}_m^{\otimes n-j-1} \to G\), the restriction of the \((n+1)\)-cube \(x\tilde{p}_{m,j}^n: \mathbb{I}_{m+1}^{\otimes n+1} \to G\) to the open box \(x\tilde{p}_{m,j}^n|_{\cap_{n+1}^{n+1}} : \cap_{n+1}^{n+1} \to N_{m+1}G\) factors through the inclusion \(X \hookrightarrow N_{m+1}G\).

**Theorem 4.12.** Let \(m > 0\) and \(G\) be a graph. The maps \(l^*, r^* : N_mG \to N_{m+1}G\) are anodyne.

**Proof.** We show that \(l^*\) is anodyne as the result for \(r^*\) is analogous. Let \(X_0\) denote the image of \(N_mG\) in \(N_{m+1}G\) under the embedding \(l^* : N_mG \hookrightarrow N_{m+1}G\). We show \(N_{m+1}G\) can be obtained from \(X_0\) by a transfinite composition of pushouts along coproducts of open box inclusions. For \(n > 0\) and \(j \in \{1, \ldots, n\}\), let \(X_{n,j}\) be the subobject of \(N_{m+1}G\) generated by:

• \(X_0\),

• (if \(n > 1\)) for any \(x: \mathbb{I}_{m+1}^{\otimes l} \otimes \mathbb{I}_m^{\otimes k-l} \to G\) where \(k < n\) and \(l \leq k\), the \((k+1)\)-cube \(x\tilde{p}_{m,l-1}^k: \mathbb{I}_{m+1}^{\otimes n+1} \to G\) of \(N_{m+1}G\),

• for any \(x: \mathbb{I}_{m+1}^{\otimes i} \otimes \mathbb{I}_m^{\otimes n-i} \to G\) where \(i \leq j\), the \((n+1)\)-cube \(x\tilde{p}_{m,i-1}^n: \mathbb{I}_{m+1}^{\otimes n+1} \to G\) of \(N_{m+1}G\).

By construction, there is a sequence of embeddings

\[
X_0 \twoheadrightarrow X_{1,1} \twoheadrightarrow X_{2,1} \twoheadrightarrow X_{2,2} \twoheadrightarrow X_{3,1} \twoheadrightarrow \ldots
\]

Note that the subobject \(X_{n,n}\) contains all \(n\)-cubes of \(N_{m+1}G\). This is because any \(n\)-cube \(x: \mathbb{I}_{m+1}^{\otimes n} \to G\) is the \(\partial_{n+1,1}\)-face of \(x\tilde{p}_{m,n-1}^n: \mathbb{I}_{m+1}^{\otimes n+1} \to G\). By construction, \(X_{n,n}\) contains \(x\tilde{p}_{m,n-1}^n\). Thus, it contains all faces of \(x\tilde{p}_{m,n-1}^n\), including \(x\). With this, we have

\[
N_{m+1}G \cong \text{colim}(X_0 \twoheadrightarrow X_{1,1} \twoheadrightarrow X_{2,1} \twoheadrightarrow X_{2,2} \twoheadrightarrow \ldots).
\]

It remains to show \(X_{n,j+1}\) is a pushout of \(X_{n,j}\) along a coproduct of open box inclusions and \(X_{n+1,1}\) is a pushout of \(X_{n,n}\) along a coproduct of open box inclusions.

Fix \(n > 0\) and \(j \in \{1, \ldots, n-1\}\). Let \(S_{n,j+1}\) be the set of \(n\)-cubes \(\mathbb{I}_{m+1}^{\otimes n} \to G\) which factor through the map \(\text{id}_{\mathbb{I}_{m+1}^{\otimes j+1}} \otimes \mathbb{I}_m^{\otimes n-j-1}: \mathbb{I}_{m+1}^{\otimes n} \to \mathbb{I}_{m+1}^{\otimes j+1} \otimes \mathbb{I}_m^{\otimes n-j-1}\) and are not contained in \(X_{n,j}\). We write an element of \(S_{n,j+1}\) as \(x(\text{id}_{\mathbb{I}_{m+1}^{\otimes j+1}} \otimes \mathbb{I}_m^{\otimes n-j-1})\) for some \(x: \mathbb{I}_{m+1}^{\otimes j+1} \otimes \mathbb{I}_m^{\otimes n-j-1} \to G\). By construction, \(X_{n,j+1}\) contains all \(n\)-cubes of \(X_{n,j+1}\) such as this \(n\)-cube \(x(\text{id}_{\mathbb{I}_{m+1}^{\otimes j+1}} \otimes \mathbb{I}_m^{\otimes n-j-1})\) is the \(\partial_{n+1,1}\)-face of \(x\tilde{p}_{m,j}^n\) (which is contained in \(X_{n,j+1}\) by construction). This gives a map

\[
\prod_{x(\text{id}_{\mathbb{I}_{m+1}^{\otimes j+1}} \otimes \mathbb{I}_m^{\otimes n-j-1}) \in S_{n,j+1}} \mathbb{I}_{m+1}^{\otimes n+1} \xrightarrow{x\tilde{p}_{m,j}^n} X_{n,j+1}.
\]

For each \(x(\text{id}_{\mathbb{I}_{m+1}^{\otimes j+1}} \otimes \mathbb{I}_m^{\otimes n-j-1}) \in S_{n,j+1}\), the restriction of the map \(x\tilde{p}_{m,j}^n\) to the open box \(\cap_{n+1}^{n+1}\) factors through \(X_{n,j}\) by Lemma 4.10.

\[
\prod_{x(\text{id}_{\mathbb{I}_{m+1}^{\otimes j+1}} \otimes \mathbb{I}_m^{\otimes n-j-1}) \in S_{n,j+1}} \mathbb{I}_{m+1}^{\otimes n+1} \xrightarrow{x\tilde{p}_{m,j}^n|_{\cap_{n+1}^{n+1}}} X_{n,j}.
\]
As \( x(id^\otimes_{m+1} \otimes I^{n-j-1}) \) is not in \( X_{n,j} \), the \( n+1 \)-cube \( \bar{x}_{m,j}^n \) is also not in \( X_{n,j} \) (as one of its faces is \( x(id^\otimes_{m+1} \otimes I^{n-j-1}) \)). The generating cubes of \( X_{n,j+1} \) which are not contained in \( X_{n,j} \) are exactly those of the form \( x\bar{x}_{m,j}^n \) for some \( x \in S_{n,j+1} \). Thus, we may write \( X_{n,j} \) as the following pushout.

\[
\begin{array}{ccc}
\prod_{x(id^\otimes_{m+1} \otimes I^{n-j-1}) \in S_{n,j+1}} \bar{x}_{m,j}^n \rightarrow X_{n,j} \\
\downarrow \quad \downarrow \quad \downarrow \\
\prod_{x(id^\otimes_{m+1} \otimes I^{n-j-1}) \in S_{n,j+1}} x\bar{x}_{m,j}^n \rightarrow X_{n,j+1}
\end{array}
\]

Thus, the map \( X_{n,j} \rightarrow X_{n,j+1} \) is a pushout along a coproduct of open box inclusions.

We now show the map \( X_{n,n} \rightarrow X_{n+1,1} \) is a pushout along a coproduct of open box inclusions. Let \( S_{n+1,1} \) be the set of \((n+1)\)-cubes which factor through the map \( id_{I^{m+1}} \otimes I^{n+1} \rightarrow I^{m+1} \otimes I^{n} \) and are not contained in \( X_{n,n} \). Similar to before, the generating cubes of \( X_{n+1,1} \) which are not contained in \( X_{n,n} \) are exactly those of the form \( x\bar{x}_{m,0}^{n+1} \) for some \( x \in S_{n+1,1} \). Thus Lemma 4.10 similarly shows that \( X_{n+1,1} \) may be written as the following pushout.

\[
\begin{array}{ccc}
\prod_{x(id^\otimes_{m+1} \otimes I^{n-1}) \in S_{n+1,1}} \bar{x}_{m,0}^{n+1} \rightarrow X_{n,n} \\
\downarrow \quad \downarrow \quad \downarrow \\
\prod_{x(id^\otimes_{m+1} \otimes I^{n-1}) \in S_{n+1,1}} x\bar{x}_{m,0}^{n+1} \rightarrow X_{n+1,1} 
\end{array}
\]

Thus, the map \( X_{n,n} \rightarrow X_{n+1,1} \) is a pushout along a coproduct of open box inclusions.

From this, we have that the inclusion of the 1-nerve of a graph into its nerve is anodyne.

**Corollary 4.13.** The natural map \( N_1G \rightarrow NG \) is anodyne.

**Proof.** By Proposition 3.5, \( NG \) is a transfinite composition of the maps \( c^*: N_mG \rightarrow N_{m+2}G \) for \( m > 0 \). By Theorem 4.12, each \( c^* \) is anodyne. Thus, each component of the colimit cone \( N_m \rightarrow NG \) is anodyne. \( \square \)

This gives us our main theorem.

**Proof of Theorem 4.1.** The first result is proven in Theorem 4.5. The second result is proven in Corollary 4.13. \( \square \)

5 Consequences

**Proof of the conjecture of Babson, Barcelo, de Longueville, Laubenbacher**

Using our main result, we obtain a proof of [BBdLL06, Thm. 5.2] which does not rely on the cubical approximation property [BBdLL06, Prop. 5.1].

**Theorem 5.1.** There is a natural group isomorphism \( A_n(G, v) \cong \pi_n(|N_1G|, v) \).
Proof. We have

\[ A_n(G, v) \cong \pi_n(NG, v) \quad \text{by Theorem 4.6} \]
\[ \cong \pi_n(|NG|, v) \quad \text{by Theorem 2.26} \]
\[ \cong \pi_n(|N_1G|, v) \quad \text{by Theorem 4.1 and Corollary 2.17}. \]

\[ \square \]

Discrete homology of graphs

We begin by recalling the standard definition of homology (with integral coefficients) of a chain complex. Recall that a (bounded) chain complex (over \( \mathbb{Z} \)) consists of a collection \( \{C_n \mid n \geq 0\} \) of abelian groups and, for \( n \geq 1 \), a group homomorphism \( \partial_n : C_n \to C_{n-1} \) such that \( \partial_{n-1} \partial_n = 0 \). A map \( f : C \to D \) of chain complexes consists of maps \( f_n : C_n \to D_n \) for \( n \geq 0 \) which make the respective squares commute. We write \( \text{Ch} \) for the category of chain complexes. Define a functor \( \tilde{H}_* : \text{Ch} \to \text{Ab} \) by taking a chain complex \( C \) to a sequence of homology groups given by:

\[ H_n C := \begin{cases} C_0 / \text{im } \partial_0 & n = 0 \\ \text{ker } \partial_n / \text{im } \partial_{n+1} & n > 0 \end{cases} \]

We refer to \( H_n C \) as the \( n \)-th homology of \( C \) with integer coefficients. One verifies that a map of chain complexes \( C \to D \) induces maps \( H_n C \to H_n D \) between their \( n \)-th homologies.

Moreover, we have a functor \( N : c\text{Set}_* \to \text{Ch} \) defined by

\[ (NX)_n := F_* X_n / DX_n. \]

The functor \( F_* : \text{Set}_* \to \text{Ab} \) is the left adjoint to the forgetful functor from abelian groups to pointed sets. Explicitly, \( F_*(X, x_0) = \bigoplus_{x \in X \setminus \{x_0\}} \mathbb{Z} \). Finally, \( DX_n \) is the subgroup of \( F_* X_n \) generated by elements that lie in the image of a degeneracy or a connection \( X_{n-1} \to X_n \).

Definition 5.2. The reduced cubical homology with integer coefficients is the functor \( \tilde{H}_* : c\text{Set}_* \to \text{Ab}^N \) given by the composite

\[ c\text{Set}_* \xrightarrow{N} \text{Ch} \xrightarrow{H_*} \text{Ab}^N. \]

Definition 5.3 (cf. [BCW14, §2]). The reduced discrete homology functor \( \overline{D}H_* : \text{Graph}_* \to \text{Ab}^N \) is the composite of functors

\[ \text{Graph}_* \xrightarrow{N_1} c\text{Set}_* \xrightarrow{\tilde{H}_*} \text{Ab}^N. \]

By the homotopy invariance of cubical homology, we have the following fact.

Proposition 5.4. Let \( f : (X, x) \to (Y, y) \) be a pointed cubical map. If \( f \) is a weak equivalence then \( H_* f : H_*(X, x) \to H_*(Y, y) \) is an isomorphism of graded abelian groups. \( \square \)

From this, we deduce that the discrete homology of a graph is the same as the cubical homology of its nerve.

Corollary 5.5. For a pointed graph \( (G, v) \), the natural map \( N_1G \to NG \) induces an isomorphism

\[ \tilde{H}_*(N_1G, v) \cong \tilde{H}_*(NG, v). \]

of graded abelian groups.

Proof. Follows from Proposition 5.4 and Theorem 4.1. \( \square \)
For any pointed Kan complex \((X, x)\), using the unit of the adjunction
\[
F_* : \text{cSet}_* \rightleftarrows \text{Ab}^{\Delta^{op}} : U_*
\]
between pointed cubical sets and cubical abelian groups, we may construct a natural map \(\pi_n(X, x) \to \tilde{H}_n(X, x)\), since \(\pi_n(U_*(F_*(X, x))) \cong \tilde{H}_n(X, x)\). This is the Hurewicz homomorphism, cf. [CKT22, Def. 4.14].

We then have the classical theorem of Hurewicz, phrased in the language of cubical sets:

**Theorem 5.6** ([CKT22, Thm. 4.16]). Let \(n \geq 2\) and \((X, x)\) be a pointed connected Kan complex. Suppose \(\pi_i(X, x) = 0\) for all \(i \in \{1, \ldots, n-1\}\), i.e., \(X\) is \(n\)-connected. Then the Hurewicz homomorphism \(\pi_n(X, x) \to \tilde{H}_n(X, x)\) is an isomorphism.

**Definition 5.7.** For any pointed connected graph \((G, v)\) and \(n \geq 2\), we therefore obtain the discrete Hurewicz homomorphism \(A_n(G, v) \to \tilde{DH}_n(G, v)\) as the composite
\[
A_n(G, v) \cong \pi_n(NG, v) \quad \text{by Theorem 4.6}
\to \tilde{H}_n(NG) \quad \text{the Hurewicz homomorphism}
\cong \tilde{H}_n(N_i G) \quad \text{by Corollary 5.5}
= \tilde{DH}_n(G, v) \quad \text{by Definition 5.3.}
\]

One may verify that this map recovers the homomorphism defined in [Lut21, §5.2]. We then have the expected discrete analogue of the Hurewicz theorem.

**Theorem 5.8** (Discrete Hurewicz Theorem). Let \(n \geq 2\) and \((G, v)\) be a pointed connected graph. Suppose \(A_i(G, v) = 0\) for all \(i \in \{1, \ldots, n-1\}\). Then the induced Hurewicz map \(A_n(G, v) \to \tilde{DH}_n(G, v)\) is an isomorphism.

**Proof.** This is an immediate consequence of the definition of the discrete Hurewicz homomorphism and Theorem 5.6. 

**Fibration Category of Graphs**

Via Theorem 4.1, we may view the nerve functor as a functor \(N : \text{Graph} \to \text{Kan}\) taking values in Kan complexes. From this, we induce a fibration category structure on the category of graphs.

**Theorem 5.9.** The category \(\text{Graph}\) of graphs and graph maps carries a fibration category structure where:

- the weak equivalences are the weak homotopy equivalences, i.e., maps \(f : G \to H\) such that, for all \(v \in G\) and \(n \geq 0\), the map \(A_n f : A_n(G, v) \to A_n(H, f(v))\) is an isomorphism;

- the fibrations are maps \(f : G \to H\) which are sent to fibrations \(N f : NG \to NH\) under the nerve functor \(N : \text{Graph} \to \text{cSet}\).

Before proving this, we consider factorization of the diagonal map separately. Recall that, for any graph \(G\), we have a commutative triangle
\[
\begin{array}{ccc}
PG & \to & G \\
\downarrow & & \downarrow \text{(id}_G, \text{id}_G) \\
G & \to & G \times G \\
\end{array}
\]
where \(G \hookrightarrow PG\) sends a vertex \(v\) to the constant path on \(v\).
Lemma 5.10. For any graph $G$, applying $N : \text{Graph} \to \text{cSet}$ to the diagram

\[
\begin{array}{ccc}
PG & \stackrel{(\partial^*_1, \partial^*_1)}{\longrightarrow} & G \\
\downarrow & & \downarrow \\
G & \stackrel{(\text{id}_G, \text{id}_G)}{\longrightarrow} & G \times G
\end{array}
\]

gives a factorization of the diagonal map $NG \to NG \times NG$ as a weak equivalence followed by a fibration.

Proof. Applying naturality of the isomorphism in Proposition 3.10 to the maps $\square \to \square$ and $\partial \square \to \square$ gives that the diagram

\[
\begin{array}{ccc}
NG & \longrightarrow & N(PG) \longrightarrow N(G \times G) \\
\downarrow & \simeq & \downarrow \simeq \\
NG & \longrightarrow & \text{hom}_R(\square, NG) \longrightarrow NG \times NG
\end{array}
\]

commutes. [CK22, Prop. 2.22] shows the bottom composite is exactly the factorization of the diagonal map in the fibration category structure on $\text{Kan}$. Thus, the top left map is a weak equivalence and the top right map is a fibration.

Proof of Theorem 5.9. Both the two-out-of-six property for weak equivalences and closure of acyclic fibrations under isomorphisms follow from Theorem 2.10. Pullbacks exist as $\text{Graph}$ is complete; they preserve (acyclic) fibrations by Proposition 3.8. Theorem 4.1 shows that the nerve of every graph is a $\text{Kan}$ complex. Factorization of the diagonal map is shown in Lemma 5.10, and this gives the factorization of an arbitrary map via [Bro73, Factorization lemma].

By definition, the nerve functor $N : \text{Graph} \to \text{cSet}$ reflects (and preserves) fibrations. We have that it reflects weak equivalences as well.

Proposition 5.11. The nerve functor $N : \text{Graph} \to \text{cSet}$ reflects weak equivalences. That is, given a map $f : G \to H$, if $Nf : NG \to NH$ is a weak equivalence then $f$ is a weak equivalence.

Proof. Follows from Theorem 4.6 and [CK22, Thm. 4.7].

As the nerve functor preserves finite limits, it is straightforward to show it is exact.

Proposition 5.12. The nerve functor $N : \text{Graph} \to \text{cSet}$ is exact.

Proof. The nerve functor preserves fibrations and acyclic fibrations by definition and by Proposition 5.11, respectively. Proposition 3.8 shows that it preserves all finite limits.

A consequence of Proposition 5.12 is that fibration sequences induce a long exact sequence of homotopy groups.

Theorem 5.13. Let $f : (G, v) \to (H, w)$ be a fibration between pointed graphs and $(F, v)$ be the fiber of $f$ over $w$, i.e. the pullback

\[
\begin{array}{ccc}
(F, v) & \stackrel{i}{\longrightarrow} & (G, v) \\
\downarrow & \gamma & \downarrow f \\
(I_0, 0) & \stackrel{w}{\longrightarrow} & (H, w)
\end{array}
\]
in $\text{Graph}_*$. Then, there is a long exact sequence

\[
\cdots \rightarrow A_n(F, v) \xrightarrow{A_{n,i}} A_n(G, v) \xrightarrow{A_{n,f}} A_n(H, w) \\
\xrightarrow{A_{n-1}(F, v)} A_{n-1}(G, v) \xrightarrow{A_{n-1,f}} A_{n-1}(H, w) \\
\cdots \\
\xrightarrow{A_1(F, v)} A_1(G, v) \xrightarrow{A_1,f} A_1(H, w) \\
\xrightarrow{A_0(F, v)} A_0(G, v) \xrightarrow{A_0,f} A_0(H, w).
\]

**Proof.** By Proposition 5.12, the map $N_f: (NG, v) \rightarrow (NH, w)$ is a fibration and its fiber is naturally isomorphic to $(NF, v)$. The result then follows by applying [CK22, Cor. 4.6] and Theorem 4.6.

Proposition 5.14 also shows that the nerve functor preserves loop spaces.

**Proposition 5.14.** The square

\[
\begin{array}{ccc}
\text{Graph}_* & \xrightarrow{N} & \text{Kan}_* \\
\Omega \downarrow & & \downarrow \Omega \\
\text{Graph}_* & \xrightarrow{N} & \text{Kan}_*
\end{array}
\]

commutes up to natural isomorphism.

**Proof.** Fix a pointed graph $(G, v)$. By Proposition 1.21, the square

\[
\begin{array}{ccc}
\Omega(G, v) & \xrightarrow{\gamma} & PG \\
\downarrow & & \downarrow (\partial^*_{0,0}, \partial^*_{1,1}) \\
I^*_{0} & \xrightarrow{(v,v)} & G \times G
\end{array}
\]

is a pullback. Proposition 3.8 shows the nerve functor preserves this pullback, thus the square

\[
\begin{array}{ccc}
N(\Omega(G, v)) & \xrightarrow{\gamma} & N(PG) \\
\downarrow & & \downarrow (\partial^*_{0,0}, \partial^*_{1,1}) \\
N\Omega_{0} & \xrightarrow{(v,v)} & N(G \times G)
\end{array}
\]

is a pullback. Proposition 3.8 also shows the nerve preserves products and the terminal object. Lemma 5.10 implies that $N(PG) \cong \text{hom}_R(\square^1, NG)$, hence the square

\[
\begin{array}{ccc}
N(\Omega(G, v)) & \xrightarrow{\gamma} & \text{hom}_R(\square^1, NG) \\
\downarrow & & \downarrow (\partial^*_{0,0}, \partial^*_{1,1}) \\
\square^0 & \xrightarrow{(v,v)} & NG \times NG
\end{array}
\]

is a pullback. That is, $N(\Omega(G, v)) \cong \Omega(NG, v)$.

By Proposition 2.9, the category $\text{Graph}_*$ of pointed graphs has a fibration category structure as well. Thus, the loop graph functor $\Omega: \text{Graph}_* \rightarrow \text{Graph}_*$ is a functor between fibration categories.

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Theorem 5.15. The loop graph functor \( \Omega : \text{Graph} \to \text{Graph} \) is exact.

Proof. By Proposition 5.12 and Theorem 2.27, the composite \( \Omega(N(\cdot, -)) : \text{Graph} \to \text{cSet} \) is exact. Applying Proposition 5.14, this gives that the composite \( N(\Omega(N(\cdot, -))) : \text{Graph} \to \text{cSet} \) is exact. The nerve functor reflects fibrations and acyclic fibrations as, by definition, it creates them. It reflects finite limits by Corollary 3.13. From this, it follows that \( \Omega : \text{Graph} \to \text{Graph} \) preserves fibrations, acyclic fibrations, and finite limits. \(\square\)

Cubical enrichment of the category of graphs

Recall (for instance from [Rie14, Def. 3.4.1]) that a functor \( F : C \to D \) between monoidal categories \( (C, \otimes, I_C) \) and \( (D, \otimes, I_D) \) is \textit{lax monoidal} if there exist natural transformations \( F c \otimes D F c' \to F(c \otimes c') \) and \( I_D \to F I_C \) subject to the associativity and unitality conditions, which we omit here.

Lemma 5.16.

1. For \( m \geq 0 \), the functor \( N_m : \text{Graph} \to \text{cSet} \) is lax monoidal.
2. The functor \( N : \text{Graph} \to \text{cSet} \) is lax monoidal.

Proof. By Proposition 3.6, the functors \( |-|_m : \text{cSet} \to \text{Graph} \) are strong monoidal, and hence in particular, oplax monoidal. Thus, their right adjoints \( N_m : \text{Graph} \to \text{cSet} \) are lax monoidal, proving (1).

Clearly, \( NI_0 \cong \square^0 \). As the geometric product preserves colimits in each variable and colimits commute with colimits, the cubical set \( NG \otimes NH \) (for graphs \( G \) and \( H \)) is the colimit of the following diagram.

\[
\begin{align*}
NG \otimes NH &= \text{colim} \left( \begin{array}{c}
N_1 G \otimes N_1 H & \longrightarrow & N_2 G \otimes N_1 H & \longrightarrow & N_3 G \otimes N_1 H & \longrightarrow & \ldots \\
N_1 G \otimes N_2 H & \longrightarrow & N_2 G \otimes N_2 H & \longrightarrow & N_3 G \otimes N_2 H & \longrightarrow & \ldots \\
N_1 G \otimes N_3 H & \longrightarrow & N_2 G \otimes N_3 H & \longrightarrow & N_3 G \otimes N_3 H & \longrightarrow & \ldots \\
& \vdots & & \vdots & & \vdots & & \ddots 
\end{array} \right)
\end{align*}
\]

Computing this colimit component-wise in \( \text{Set} \), one verifies that the colimit of the diagonal

\[ \text{colim}(N_1 G \otimes N_1 H \longrightarrow N_2 G \otimes N_2 H \longrightarrow N_3 G \otimes N_3 H \longrightarrow \ldots) \]

computes the same cubical set. As \( N(G \otimes H) \) is the colimit

\[ N(G \otimes H) = \text{colim}(N_1(G \otimes H) \longrightarrow N_2(G \otimes H) \longrightarrow N_3(G \otimes H) \longrightarrow \ldots) \]

the lax monoidal maps \( N_m G \otimes N_m H \to N_m(G \otimes H) \) induce a map on colimits \( NG \otimes NH \to N(G \otimes H) \) which satisfies the required associativity and unitality conditions. \(\square\)

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Remark 5.17. An alternative proof of (2) can be given using [DKLS20, Prop. 1.24], which gives an explicit description of the geometric product of cubical sets. Explicitly, the \(n\)-cubes of \(NG \otimes NH\) are in bijective correspondence with pairs of cubes

\[
(\square^k \to NG, \square^l \to NH),
\]

where \(k + l = n\). By definition of \(N\), each such a pair corresponds in turn to pair of maps

\[
(I_\infty^k \to G, I_\infty^l \to H)
\]

that stabilize in all directions. Taking products of these maps, we obtain a map \(I_\infty^n \to G \otimes H\) that stabilizes in all directions.

Corollary 5.18. The nerve functor \(N : \text{Graph} \to \text{cSet}\) preserves homotopy equivalences.

We describe the notion of enriched categories informally, with a reference to [Rie14, Def. 3.3.1] in lieu of a fully formal statement.

Definition 5.19. For a monoidal category \((V, \otimes, 1)\), a \((V, \otimes, I)\)-enriched category \(C\) consists of

- a class of objects \(\text{ob } C\);
- for \(X, Y \in \text{ob } C\), a morphism object \(C(X, Y) \in V\);
- for \(X, Y, Z \in \text{ob } C\), a composition morphism \(\circ : C(Y, Z) \otimes C(X, Y) \to C(X, Z)\) in \(V\);
- for \(X \in \text{ob } C\), an identity morphism \(\text{id}_X : C(X, X)\) in \(V\),

subject to appropriate associativity and unitality axioms (cf. [Rie14, Def. 3.3.1]).

Example 5.20. Any locally-small category is a \((\text{Set}, \times, \{\ast\})\)-enriched category, where the objects, morphism sets, composition function, and identity morphisms are as usually defined.

Example 5.21. Any closed monoidal category is enriched over itself. In particular, \(\text{Graph}_{G}\) is a \((\text{Graph}, \otimes, I_0)\)-enriched category where

- \(\text{ob } \text{Graph}_{G}\) is the collection of all graphs;
- for graphs \(G, H\), the morphism graph is \(\text{hom}^\otimes(G, H)\);
- for graphs \(G, H, J\), the composition morphism is the graph map given by composition of graph maps regarded as vertices:

\[
\text{hom}^\otimes(H, J) \otimes \text{hom}^\otimes(G, H) \to \text{hom}^\otimes(G, J);
\]
- the identity morphism is the identity map on \(G\) as a vertex \(\text{id}_G : I_0 \to \text{hom}^\otimes(G, G)\).

Example 5.22. Since enrichment can be transferred along lax monoidal functors [Rie14, Lem. 3.4.3], Lemma 5.16 implies there is a \((\text{cSet}, \otimes, \square^0)\)-enriched category \(\text{Graph}_{G}\) of graphs where \(N(\text{hom}^\otimes(G, H))\) is the morphism cubical set. Composition and identity morphisms are defined analogously.

Definition 5.23. A \((\text{cSet}, \otimes, \square^0)\)-enriched category \(C\) is locally Kan if, for all \(X, Y \in \text{ob } C\), the cubical set \(C(X, Y)\) is a Kan complex.

Theorem 5.24. The \((\text{cSet}, \otimes, \square^0)\)-enriched category \(\text{Graph}_{G}\) of graphs is locally Kan.

Proof. Follows from Theorem 4.1.
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