A Geometric Application for the $\text{det}^{S^2}$ Map

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Abstract. We discuss properties of the $\text{det}^{S^2}$ map, present a few explicit computations, and give a geometrical interpretation for the condition $\text{det}^{S^2}((v_{i,j})_{1 \leq i < j \leq 4}) = 0$.

1. Introduction

The determinant of an $n \times n$ matrix has countless applications in mathematics, so it is not surprising that attempts to generalize it have been made over the years. A modern approach to this problem is presented in [2] where the notion of hyperdeterminant is discussed in detail, but similar ideas can be traced back to Cayley’s work.

The $\text{det}^{S^2}$ map (for a vector space of dimension $d = 2$) was introduced in [4] as the unique multilinear function that satisfies a certain universality property. This map does not seem to fit in the setting of hyperdeterminants described in [2], but just like the classical determinant, the $\text{det}^{S^2}$ map is associated to an exterior algebra "like" construction (more precisely an exterior GSC-operad). The construction of the GSC operad was inspired by results on Higher Hochschild homology ([1], [3]), and the Swiss-Cheese Operad ([5]). Even though the operad construction exists for vector spaces of any dimension, the existence of the $\text{det}^{S^2}$ map is not known for dimension $d > 2$ (but it was conjectured in [4]).

In this paper we outline a series of properties and results about $\text{det}^{S^2}$. In section 2 we recall a few notations and formulas from [4]. In section 3 we show that $\text{det}^{S^2}$ is invariant under the action of the group $SL_2(k)$, and under the action of the symmetric group $S_4$. We also give some explicit computations with geometrical flavor.

In section 4 we present an analog of the well known fact that the determinant of a square matrix vanishes if and only if the column vectors are linearly dependent. For $(v_{i,j})_{1 \leq i < j \leq 4}$ a collection of six vectors, we describe the relationship between $\text{det}^{S^2}((v_{i,j})_{1 \leq i < j \leq 4})$ vanishing and the existence of a quadrilateral $Q_1Q_2Q_3Q_4$ with edges $Q_iQ_j$ a multiple of $v_{i,j}$. We conclude the paper with a few remarks and examples.

2. Preliminaries

In this paper $k$ is a field with $\text{char}(k) = 0$, and $V$ is a $k$-vector space of dimension 2. For geometrical applications we will take $k = \mathbb{R}$. Some of the results presented here work also over a commutative ring.

The following convention was used in [4] to represent an element from $V^6$. Consider $v_{i,j} \in V$ for all $1 \leq i < j \leq 4$, we denote

$$\mathcal{V} = (v_{i,j})_{1 \leq i < j \leq 4} = \begin{pmatrix} 0 & v_{1,2} & v_{1,3} & v_{1,4} \\ 0 & v_{2,3} & v_{2,4} & v_{3,4} \\ 0 & 0 & v_{3,4} & 0 \end{pmatrix} \in V^6.$$
This notation is convenient when we want to keep track of the positions of the elements in $\mathfrak{M}$. The zeros on the diagonal of $\mathfrak{M}$ do not play any role, they are mostly for symmetry and help keep track of rows and columns. When there is no danger of confusion we will use the notation $\mathfrak{M}$ for a generic element in $V^6$.

Notice that we have an natural action of the symmetric group $S_4$ on $V^6$ given by

$$\sigma \cdot \begin{pmatrix} 0 & v_{1,2} & v_{1,3} & v_{1,4} \\ 0 & v_{2,3} & v_{2,4} \\ 0 & v_{3,4} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & v_{\sigma(1),\sigma(2)} & v_{\sigma(1),\sigma(3)} & v_{\sigma(1),\sigma(4)} \\ 0 & v_{\sigma(2),\sigma(3)} & v_{\sigma(2),\sigma(4)} \\ 0 & v_{\sigma(3),\sigma(4)} \\ 0 \end{pmatrix}$$

with the conventions that $v_{i,j} = v_{j,i}$ for $i > j$.

We recall from [4] the formula for the map $\text{det}^2 : V^6 \to k$. For $v_{i,j} = (\alpha, \beta, \gamma) \in k^2$ we have

$$\text{det}^2 \begin{pmatrix} 0 & (\alpha_{1,2}, \beta_{1,2}) & (\alpha_{1,3}, \beta_{1,3}) & (\alpha_{1,4}, \beta_{1,4}) \\ 0 & (\alpha_{2,3}, \beta_{2,3}) & (\alpha_{2,4}, \beta_{2,4}) \\ 0 & (\alpha_{3,4}, \beta_{3,4}) \end{pmatrix} = \alpha_{1,2} \alpha_{2,3} \alpha_{3,4} \beta_{1,3} \beta_{2,4} \beta_{1,4} + \alpha_{1,2} \beta_{2,3} \alpha_{3,4} \beta_{1,3} \beta_{2,4} \alpha_{1,4} + \alpha_{1,2} \beta_{2,3} \beta_{3,4} \alpha_{1,3} \alpha_{2,4} \beta_{1,4} + \beta_{1,2} \beta_{2,3} \beta_{3,4} \alpha_{1,3} \alpha_{2,4} \beta_{1,4} + \beta_{1,2} \alpha_{2,3} \beta_{3,4} \alpha_{1,3} \alpha_{2,4} \beta_{1,4} + \beta_{1,2} \alpha_{2,3} \alpha_{3,4} \beta_{1,3} \beta_{2,4} \alpha_{1,4} - \beta_{1,2} \beta_{2,3} \beta_{3,4} \alpha_{1,3} \alpha_{2,4} \beta_{1,4} - \beta_{1,2} \alpha_{2,3} \beta_{3,4} \alpha_{1,3} \alpha_{2,4} \beta_{1,4} - \beta_{1,2} \alpha_{2,3} \alpha_{3,4} \beta_{1,3} \beta_{2,4} \alpha_{1,4} - \alpha_{1,2} \beta_{2,3} \beta_{3,4} \alpha_{1,3} \alpha_{2,4} \beta_{1,4} - \alpha_{1,2} \beta_{2,3} \alpha_{3,4} \beta_{1,3} \beta_{2,4} \alpha_{1,4} - \alpha_{1,2} \beta_{2,3} \alpha_{3,4} \beta_{1,3} \beta_{2,4} \alpha_{1,4}.$$

This formula was obtained from an exterior algebra "like" construction. Essentially, $\text{det}^2$ is the unique nontrivial multilinear map defined on $V^6$ which has the property that $\text{det}^2 \begin{pmatrix} 0 & v_{1,2} & v_{1,3} & v_{1,4} \\ 0 & v_{2,3} & v_{2,4} \\ 0 & v_{3,4} \end{pmatrix} = 0$ if there exists $1 \leq i < j < k \leq 4$ such that $v_{i,j} = v_{i,k} = v_{j,k}$. We refer to [4] for more details.

Remark 2.1. One should notice that the $\text{det}^2$ map does not coincide with any of the hyperdeterminant maps discussed in [2]. Due to dimension restrictions, the only possible candidate is the hyperdeterminant of multidimensional the matrices of type $2 \times 2 \times 3$ (see [2] page 463). However if in the formula of the hyperdeterminant we take $a_{0,0,1} = a_{1,0,1} = a_{1,1,0} = 1$ and all of the other entries $a_{i,j,k} = 0$ then $Det(A) = -1$, while our $\text{det}^2$ map is automatically 0 if only four of the twelve scalar entries in $\mathfrak{M}$ are nonzero.

3. Properties and Computations

In this section we establish some basic properties of $\text{det}^2$, and give a few explicit computations. Most of these results can be checked by direct computation, but they also follow from Lemma 4.1.

With the notations from the previous section we have the following

Lemma 3.1. (a) If $T : V \to V$ is a linear map then

$$\text{det}^2 \begin{pmatrix} 0 & T(v_{1,2}) & T(v_{1,3}) & T(v_{1,4}) \\ 0 & T(v_{2,3}) & T(v_{2,4}) \\ 0 & T(v_{3,4}) \end{pmatrix} = \text{det}(T)^3 \text{det}^2 \begin{pmatrix} 0 & v_{1,2} & v_{1,3} & v_{1,4} \\ 0 & v_{2,3} & v_{2,4} \\ 0 & v_{3,4} \end{pmatrix}.$$

In particular $\text{det}^2$ is invariant under the action of the group $SL_2(k)$.

(b) For any $\sigma \in S_4$ we have

$$\text{det}^2 \sigma \begin{pmatrix} 0 & v_{1,2} & v_{1,3} & v_{1,4} \\ 0 & v_{2,3} & v_{2,4} \\ 0 & v_{3,4} \end{pmatrix} = \text{sgn}(\sigma) \text{det}^2 \begin{pmatrix} 0 & v_{1,2} & v_{1,3} & v_{1,4} \\ 0 & v_{2,3} & v_{2,4} \\ 0 & v_{3,4} \end{pmatrix}.$$

Proof. This follows by direct computations. $\square$
Example 3.2. Let $P_1, P_2, P_3, P_4$ be four points in the plane $\mathbb{R}^2$, and take $v_{i,j} = \overrightarrow{P_iP_j}$, then
\[
\text{det}^{S^2} \begin{pmatrix}
0 & v_{1,2} & v_{1,3} & v_{1,4} \\
0 & v_{2,3} & v_{2,4} & 0 \\
0 & v_{3,4} & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} = 0.
\]
The converse of this example will be discussed in detail in the next section.

Example 3.3. Let $L_1, L_2, L_3, L_4$ be lines in $\mathbb{R}^2$ passing through the origin, with $\theta_i$ the angle between the positive $x$-axis and $L_i$. If $v_{i,j} = \begin{pmatrix} \cos(\theta_j - \theta_i) \\ \sin(\theta_j - \theta_i) \end{pmatrix}$ then
\[
\text{det}^{S^2} \begin{pmatrix}
0 & v_{1,2} & v_{1,3} & v_{1,4} \\
0 & v_{2,3} & v_{2,4} & 0 \\
0 & v_{3,4} & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} = \sin(\theta_2 - \theta_1) \sin(2(\theta_3 - \theta_2)) \sin(\theta_4 - \theta_3).
\]
Notice that because $\text{det}^{S^2}$ is a geometric invariant the result depends only on the angle between the $L_i$ and $L_{i+1}$, so if we take $\phi_i = \theta_{i+1} - \theta_i$ we have
\[
\text{det}^{S^2} \begin{pmatrix}
0 & v_{1,2} & v_{1,3} & v_{1,4} \\
0 & v_{2,3} & v_{2,4} & 0 \\
0 & v_{3,4} & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} = \sin(\phi_1) \sin(2\phi_2) \sin(\phi_3).
\]

One can also check the following description for $\text{det}^{S^2}$.

Remark 3.4. Take $(v_{i,j})_{1 \leq i < j \leq 4} \in V^6$, and let $\langle \cdot, \cdot \rangle$ be the standard inner product on $\mathbb{R}^2$. Then
\[
\text{det}^{S^2} ((v_{i,j})_{1 \leq i < j \leq 4}) = \text{det}(v_{1,4}, v_{2,4}) \langle v_{1,2}, v_{3,4} \rangle \langle v_{1,3}, v_{2,3} \rangle + \text{det}(v_{3,4}, v_{1,4}) \langle v_{1,3}, v_{2,4} \rangle \langle v_{1,2}, v_{2,3} \rangle + \text{det}(v_{2,4}, v_{3,4}) \langle v_{1,4}, v_{2,3} \rangle \langle v_{1,2}, v_{1,3} \rangle.
\]
Where $\text{det}(v, w)$ is the determinant of the $2 \times 2$ matrix with $v$ and $w$ as columns. In particular, we have
\[
\text{det}^{S^2} \begin{pmatrix}
0 & v_1 & v_2 & v_3 \\
0 & v_3^\perp & v_2^\perp & v_1^\perp \\
0 & v_2^\perp & v_1^\perp & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} = 0
\]
for any $v_1, v_2, v_3 \in \mathbb{R}^2$ and $v_i^\perp$ any non-zero vector orthogonal to $v_i$.

4. Main Result

In this section we present the converse of the Example 3.2. First we need the following lemma.

Lemma 4.1. Let $(v_{i,j})_{1 \leq i < j \leq 4} \in V^6$, then
\[
\text{det}^{S^2} ((v_{i,j})_{1 \leq i < j \leq 4}) = \text{det}
\begin{pmatrix}
\alpha_{1,2} & \alpha_{2,3} & 0 & -\alpha_{1,3} & 0 & 0 \\
\beta_{1,2} & \beta_{2,3} & 0 & -\beta_{1,3} & 0 & 0 \\
\alpha_{1,2} & 0 & 0 & 0 & \alpha_{2,4} & -\alpha_{1,4} \\
\beta_{1,2} & 0 & 0 & 0 & \beta_{2,4} & -\beta_{1,4} \\
0 & 0 & \alpha_{3,4} & \alpha_{1,3} & 0 & -\alpha_{1,4} \\
0 & 0 & \beta_{3,4} & \beta_{1,3} & 0 & -\beta_{1,4}
\end{pmatrix}.
\]

Proof. It follows by direct computations. \qed

We are now ready to prove the main result of this note.
Theorem 4.2. Take $V = \mathbb{R}^2$, and let $(v_{i,j})_{1 \leq i < j \leq 4} \in V^6$, then the following are equivalent.

(a) \( \det S^2 \begin{pmatrix} 0 & v_{1,2} & v_{1,3} & v_{1,4} \\ 0 & v_{2,3} & v_{2,4} \\ 0 & v_{3,4} \end{pmatrix} = 0. \)

(b) There exist points $Q_1, Q_2, Q_3, Q_4$ in the plane $\mathbb{R}^2$, and $\lambda_{i,j} \in \mathbb{R}$ for $1 \leq i < j \leq 4$ not all zero such that $\lambda_{i,j}v_{i,j} = Q_iQ_j$.

Proof. First notice that assuming (b), from Example 3.2 we know that

\[
\det S^2 \begin{pmatrix} 0 & \lambda_{1,2} & \lambda_{1,3} & \lambda_{1,4} \\ 0 & \lambda_{2,3} & \lambda_{2,4} \\ 0 & \lambda_{3,4} \end{pmatrix} = 0.
\]

However we cannot conclude (a) since since we don’t know that all $\lambda_{i,j}$ are nonzero.

Regardless, condition (b) is satisfied if and only if there is a non-trivial solution $(\lambda_{i,j})_{1 \leq i < j \leq 4}$ to the system of vector equations

\[
\begin{align*}
\lambda_{1,2}v_{1,2} + \lambda_{2,3}v_{2,3} - \lambda_{1,3}v_{1,3} &= 0 \\
\lambda_{1,2}v_{1,2} + \lambda_{2,4}v_{2,4} - \lambda_{1,4}v_{1,4} &= 0 \\
\lambda_{1,3}v_{1,3} + \lambda_{3,4}v_{3,4} - \lambda_{1,4}v_{1,4} &= 0 \\
\lambda_{2,3}v_{2,3} + \lambda_{3,4}v_{3,4} - \lambda_{2,4}v_{2,4} &= 0.
\end{align*}
\]

This induces a system of linear equations described by

\[
\begin{pmatrix}
\alpha_{1,2} & \alpha_{2,3} & 0 & -\alpha_{1,3} & 0 & 0 \\
\beta_{1,2} & \beta_{2,3} & 0 & -\beta_{1,3} & 0 & 0 \\
\alpha_{1,2} & 0 & 0 & 0 & \alpha_{2,4} & -\alpha_{1,4} \\
\beta_{1,2} & 0 & 0 & 0 & \beta_{2,4} & -\beta_{1,4} \\
0 & 0 & \alpha_{3,4} & \alpha_{1,3} & 0 & -\alpha_{1,4} \\
0 & 0 & \beta_{3,4} & \beta_{1,3} & 0 & -\beta_{1,4} \\
0 & \alpha_{2,3} & \alpha_{3,4} & 0 & -\alpha_{2,4} & 0 \\
0 & \beta_{2,3} & \alpha_{3,4} & 0 & -\beta_{2,4} & 0
\end{pmatrix}
\begin{pmatrix}
\lambda_{1,2} \\
\lambda_{2,3} \\
\lambda_{3,4} \\
\lambda_{1,3} \\
\lambda_{2,4} \\
\lambda_{1,4}
\end{pmatrix} = 0.
\]

And so, condition (b) is equivalent to the above matrix being rank deficient (i.e. rank 5 or smaller).

Denote the row vectors in the matrix from equation 4.2 as $\{R_i\}_{1 \leq i \leq 8}$. The matrix is rank deficient if and only if the submatrices formed by each choice of 6 rows from $\{R_i\}_{1 \leq i \leq 8}$ results in a zero determinant. Notice the set $\{R_1, R_3, R_5, R_7\}$ is linearly dependent. This is true about the set $\{R_2, R_4, R_6, R_8\}$ as well. This shows that the only possibility of a non-zero 6 by 6 determinant arises if we choose 3 rows from each of the two sets $\{R_1, R_3, R_5, R_7\}$ and $\{R_2, R_4, R_6, R_8\}$.

We want to show that in order to compute its rank we can chose the first six rows of the above matrix. Suppose we have a submatrix with 3 rows from $\{R_1, R_3, R_5, R_7\}$ and 3 rows from $\{R_2, R_4, R_6, R_8\}$. If $R_1$ is not chosen as one of the rows then $R_3, R_5, R_7$ must be chosen, and the linear dependence relation on the set $\{R_1, R_3, R_5, R_7\}$ gives a sequence of elementary row operations that replaces the row $R_7$ with $R_1$. The same argument allows us to replace $R_7$ with whichever row of the set $\{R_1, R_3, R_5\}$ is missing. A similar argument on the set $\{R_2, R_4, R_6, R_8\}$ means we only need to consider the rows $R_2, R_4, R_6$ as being rows in our submatrix. Thus the determinant of any such matrix is a nonzero multiple the determinant of the matrix consisting of rows $\{R_1, R_2, R_3, R_4, R_5, R_6\}$. Lemma 4.1 now shows that the determinant of this matrix is given by $\det S^2 \begin{pmatrix} 0 & v_{1,2} & v_{1,3} & v_{1,4} \\ 0 & v_{2,3} & v_{2,4} \\ 0 & v_{3,4} \end{pmatrix}$. 


Thus the $6 \times 8$ matrix from equation 4.2 is rank deficient if and only if $det^{S^2} \begin{pmatrix} 0 & v_{1,2} & v_{1,3} & v_{1,4} \\ 0 & v_{2,3} & v_{2,4} \\ 0 & v_{3,4} & 0 \end{pmatrix} = 0$, which shows the equivalence of (a) and (b).

**Remark 4.3.** Geometrically, Theorem 4.2 says that the vectors $v_{i,j}$ give the directions of all diagonals in a quadrilateral if and only if $det^{S^2} ((v_{i,j})_{1 \leq i < j \leq 4}) = 0$.

**Remark 4.4.** It should be noted that different values of $det^{S^2}$ can come from different orderings of the same 6 vectors. The orbit of $(v_{i,j})_{1 \leq i < j \leq 4}$ under the action of the group $S_4$ is bounded above by $|S_4| = 24$, but the total number of arrangements of 6 vectors is 720. Thus we cannot relax condition (b) in Theorem 4.2 to be $\{\lambda_{i,j}v_{i,j} \mid 1 \leq i < j \leq 4\} = \{Q_iQ_j^* \mid 1 \leq i < j \leq 4\}$ as sets.

For example if $v_{1,2} = v_{2,3} = v_{1,3} = e_1$ and $v_{1,4} = v_{2,4} = v_{3,4} = e_2$ then $\lambda_{1,2} = \lambda_{2,3} = \lambda_{1,3} = 0$ and $\lambda_{1,4} = \lambda_{2,4} = \lambda_{3,4} = 1$ is a solution for the system 4.1. On the other hand if $w_{1,2} = w_{2,3} = w_{3,4} = e_1$ and $w_{1,3} = w_{2,4} = w_{1,4} = e_2$ we have that $det^{S^2} \begin{pmatrix} 0 & w_{1,2} & w_{1,3} & w_{1,4} \\ 0 & w_{2,3} & w_{2,4} \\ 0 & w_{3,4} & 0 \end{pmatrix} = 1$ and so the equation 4.1 has no non-trivial solution. Even though $\{v_{i,j} \mid 1 \leq i < j \leq 4\} = \{w_{i,j} \mid 1 \leq i < j \leq 4\}$ as sets, the $det^{S^2}$ map takes different values on the corresponding elements in $V^6$.

**Remark 4.5.** From a homological perspective, Theorem 4.2 can be seen as studying relations among relations. More precisely, if we denote $R_{i,j,k} = \lambda_{i,j}v_{i,j} + \lambda_{j,k}v_{j,k} - \lambda_{i,k}v_{i,k}$, then we have already imposed the condition $R_{1,2,3} - R_{1,2,4} + R_{1,3,4} - R_{2,3,4} = 0$. This is very similar with Cayley’s theory of elimination (see Appendix B in [2]). It is natural to ask if this idea can be used to define a $det^{S^2}$ map for any finite dimensional vector space $V$.

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