THE NUMBER OF ORIENTABLE SMALL COVERS OVER CUBES

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Abstract. We count orientable small covers over cubes. We also get estimates for $O_n/R_n$, where $O_n$ is the number of orientable small covers and $R_n$ is the number of all small covers over an $n$-cube up to the Davis-Januszkiewicz equivalence.

1. Introduction

A small cover, defined by Davis and Januszkiewicz [2], is an $n$-dimensional closed smooth manifold $M$ with an effective real torus $(S^0)^n (=: T^n)$-action such that the action is locally isomorphic to a standard $T^n$-action on $\mathbb{R}^n$ and the orbit space $M/T^n$ can be identified with a simple combinatorial polytope. For instance, $\mathbb{R}P^n$ with a natural $T^n$-action is a small cover over an $n$-simplex. In general, a real toric manifold, the set of real points of a toric manifold, provides an example of small covers. Hence, small covers can be seen as a topological generalization of real toric manifolds in algebraic geometry.

A small cover over a cube is known as a real Bott manifold which is obtained as iterated $\mathbb{R}P^1$ bundles starting with a point, where each fibration is the projectivization of a Whitney sum of two real line bundles. These manifolds are well-studied in numerous papers such as [3] and [4]. The author also found a strong relation between small covers and acyclic digraphs, and he calculated the number of them up to several senses in [1].

In the present paper, we restrict our attention to the case of orientable small covers over a cube. Thankfully, Nakayama and Nishimura [5] found a simple criterion for a small cover to be orientable. Using this criterion, we establish the formula of the number of orientable small covers over a cube and show that the ratio $O_n/R_n$ is approximately $\frac{1.262}{2^n}$, where $O_n$ is the number of orientable small covers and $R_n$ is the number of small covers over an $n$-cube up to the Davis-Januszkiewicz equivalence.

2. Orientable small covers over cubes

Let $P$ be an $n$-dimensional simple polytope with $m$ facets. Two small covers $M_1$ and $M_2$ over $P$ are Davis-Januszkiewicz equivalent (or simply, $D$-$J$ equivalent) if there is a weak $T^n$-equivariant homeomorphism $f: M_1 \rightarrow M_2$ which makes the

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It is well-known by [2] that all small covers over \( P \) can be distinguished by the map \( \lambda \) from the set of facets of \( P \) to \( \mathbb{Z}_2^n = \{0,1\}^n \), called the characteristic function, which satisfies the non-singularity condition: \( \{\lambda(F_1), \ldots, \lambda(F_m)\} \) is a basis of \( \mathbb{Z}_2^n \) whenever the intersection \( F_1 \cap \cdots \cap F_m \) is non-empty, where \( \{F_1, \ldots, F_m\} \) is the set of facets of \( P \). Let \( M_1, M_2 \) be two small covers over \( P \) corresponding to characteristic functions \( \lambda_1, \lambda_2 \), respectively. By [2], \( M_1 \) is D-J equivalent to \( M_2 \) if and only if there is an automorphism \( \sigma \in \text{Aut}(\mathbb{Z}_2^n) \) such that \( \lambda_1 = \sigma \circ \lambda_2 \). Hence, the D-J equivalence classes are independent of the choice of basis for \( \mathbb{Z}_2^n \).

One may assign an \( n \times m \) matrix \( \Lambda \) to \( \lambda \) by ordering the facets and choosing a basis for \( \mathbb{Z}_2^n \) as the follow:

\[
\Lambda = (\lambda(F_1) \cdots \lambda(F_m)).
\]

If we additionally assume that the first \( n \) facets meet at a vertex, by the non-singularity condition, we can choose an appropriate basis of \( \mathbb{Z}_2^n \) such that \( \Lambda = (E_n | \Lambda_*) \), where \( E_n \) is the identity matrix of size \( n \) and \( \Lambda_* \) is an \( n \times (m-n) \) matrix. Hence, the D-J equivalence classes of small covers over \( P \) are classified by \( \Lambda_* \).

Now, we consider the case where \( P \) is an \( n \)-cube. Note that \( P \) has \( 2^n \) facets. We order the facets of \( P \) satisfying \( F_j \cap F_{n+j} = \emptyset \) for \( 1 \leq j \leq n \). Then the first \( n \) facets meet at a vertex. Hence, for each \( \lambda \), the corresponding matrix \( \Lambda \) can be expressed as \( \Lambda = (E_n | \Lambda_*) \), where \( \Lambda_* \) is an \( n \times n \) matrix. One can check that the non-singularity condition holds if and only if all of principal minors of \( \Lambda_* \) are 1. Therefore, there is a bijection between small covers over cubes up to the D-J equivalence and square \( \mathbb{Z}_2 \)-matrices all of whose principal minors are 1.

Let \( M(n) \) be the set of square \( \mathbb{Z}_2 \)-matrices of size \( n \) all of whose principal minors are 1 and let \( G_n \) be the set of acyclic digraphs with labelled \( n \) vertices. By [1], we have a bijection \( \phi : G_n \rightarrow M(n) \) by

\[
\phi : G \mapsto A(G)^t + E_n,
\]

where \( A(G)^t \) is the transpose matrix of the vertex adjacency matrix of \( G \) (see Figure 1).

**Remark 2.1.** In the classical theory of real Bott manifolds, the representative matrix of real Bott manifold is the transpose matrix of its characteristic function.
matrix $\Lambda_*$. This is a reason why we use $A(G)^t$ instead of $A(G)$ in the definition of $\phi$.

On the other hand, we have a nice orientability condition for small covers due to Nakayama and Nishimura in [5].

**Theorem 2.2** (Nakayama and Nishimura [5]). Let $P$ be an $n$-dimensional simple polytope with $m$ facets and let $M$ be a small cover over $P$ with $\Lambda$. Then $M$ is orientable if and only if the sum of entries of the $i$-th column of $\Lambda$ is odd for all $i = 1, \ldots, m$.

**Corollary 2.3.** The number of orientable small covers over an $n$-cube up to D-J equivalence is equal to the number of acyclic digraphs with labelled $n$ vertices all of whose vertices have even out-degrees.

**Proof.** Let $G$ be a digraph and $A(G)$ its vertex adjacency matrix. Then the sum of entries of the $i$-th row of $A(G)$ means the out-degree of the $i$-th vertex of $G$ (see Appendix). Let $M$ be a orientable small cover over an $n$-cube corresponding to $\Lambda_*$. Since $\Lambda_* \in M(n)$, the transpose $\Lambda_*^t$ of $\Lambda_*$ is also in $M(n)$. Note that the sum of entries of each row of $\Lambda_*^t - E_n$ is even by Theorem 2.2 and hence, every vertex of $\phi^{-1}(\Lambda_*)$ has an even out-degree. Since the D-J equivalence classes are classified by $\Lambda_*$ and $\phi$ is a bijection, we prove the corollary. □

3. **The number of orientable small covers**

Let $R_n$ be the number of acyclic digraphs with labelled $n$ vertices. The following is the recursive formula for $R_n$ due to R. W. Robinson in [6].

$$R_n = \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} 2^{k(n-k)} R_{n-k}.$$ 

Let $\mathcal{O}_n \subset \mathcal{G}_n$ be the set of acyclic digraphs all of whose vertices have even out-degrees and let $O_n$ be the cardinality of $\mathcal{O}_n$ (we use the alphabet ‘O’ instead of ‘E’ although they have only ‘even’ out-degree vertices, because the ‘O’ is the abbreviation of the word ‘Orientable’).

**Theorem 3.1.** Let $R_k$ be the number of acyclic digraphs with labelled $k$ vertices. Then,

$$O_n = \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} 2^{(k-1)(n-k)} R_{n-k}.$$ 

**Proof.** We count matrices in $M(n)$ all of whose the sum of entries of each column are odd. Let us denote the sum of entries of the $i$-th column of an $n \times n$ matrix $A$ by $c_i(A)$. Since an acyclic digraph always has a vertex of out-degree 0, there is at least one $i$ such that $c_i(A) = 1$ for each $A \in M(n)$. Assume $c_{i_1}(A) = \cdots = c_{i_k}(A) = 1$, where $k \geq 1$. Since all principal minors of $A$ are 1, the diagonal entries of $A$ are all 1. Thus, by a replacement of labels, we may assume that $A$ is of the following form:

$$\begin{pmatrix} E_k & S \\ 0 & T \end{pmatrix},$$

where $E_k$ is the identity matrix of size $k$, $T$ is an $(n-k) \times (n-k)$-matrix and $S$ is a $k \times (n-k)$-matrix. Note that $A \in M(n)$ if and only if $T \in M(n-k)$. Thus we may control only one row of $S$ for making all $c_i(A)$’s are odd. This implies the number of $A$’s of the form [8] whose $c_i(A)$’s are odd for all $i$ is $2^{(k-1)(n-k)}R_{n-k}$. To avoid counting repeatedly, we apply the Principle of Inclusion-Exclusion and we get the formula for $O_n$. □

Here are a few values of $R_n$ and $O_n$.

| $n$ | 1   | 2   | 3   | 4   | 5   | 6   | 7   |
|-----|-----|-----|-----|-----|-----|-----|-----|
| $R_n$ | 1   | 3   | 25  | 543 | 29,281 | 3,781,503 | 1,138,779,265 |
| $O_n$ | 1   | 1   | 4   | 43  | 1,156 | 74,581 | 11,226,874 |

Let us consider the chromatic generating functions of $R_n$ and $O_n$, namely, we set

$$R(x) = \sum_{n=0}^{\infty} R_n \frac{x^n}{n!2^{\binom{n}{2}}}$$
and
$$O(x) = \sum_{n=0}^{\infty} O_n \frac{x^n}{n!2^{\binom{n}{2}}}.$$

Corollary 3.2. Let $F(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!2^{\binom{n}{2}}}$. Then

$$O(x) = \frac{1 - F(-x)}{F(-\frac{x}{2})}.$$

Proof. Let us consider chromatic generating functions $A(x)$, $B(x)$ and $C(x)$ with respect to the sequences $A_n$, $B_n$ and $C_n$, respectively. Note that if $C(x) = A(x)B(x)$, then $C_n = \sum_{k=0}^{n} A_k B_{n-k} \binom{n}{k} 2^{k(n-k)}$. Thus, we have $F(-x)R(x) = 1$ (see [7]) and

$$R\left(\frac{x}{2}\right) F(-x) + O(x) = \sum_{n=0}^{\infty} \frac{R_n}{2^n} \frac{x^n}{n!2^{\binom{n}{2}}} = R\left(\frac{x}{2}\right).$$

Hence we have $O(x) = F\left(-\frac{x}{2}\right)^{-1} (1 - F(-x))$. □

Let $G(x) = \frac{F\left(\frac{x}{2}\right)}{1 - F(x)}$. We obtain estimates for $O_n$ by analyzing the behavior of the function $G(x)$. Since $F(x)$ has an isolated zero $\alpha \approx -1.488$ (see [8, Section 2]), $G(x)$ has an isolated zero $2\alpha$. Hence, standard techniques provide the asymptotic formula

$$G(x) \sim G'(2\alpha)(x - 2\alpha).$$

Hence we have

$$O(x) = \frac{1}{G(-x)} \sim \frac{1}{G'(2\alpha)(-x - 2\alpha)}.$$

Note that $F'(x) = F\left(\frac{x}{2}\right)$. Therefore, the following asymptotic formula

$$O(x) \sim -\frac{1 - F(2\alpha)}{\alpha F'\left(\frac{\alpha}{2}\right)} \sum_{n=0}^{\infty} \left(-\frac{x}{2\alpha}\right)^n$$

immediately follows two facts $\frac{1}{G'(2\alpha)} = \frac{2(1-F(2\alpha))}{F'(\alpha)}$ and $\frac{1}{-x - 2\alpha} = \frac{1}{-2\alpha} \sum_{n=0}^{\infty} \left(-\frac{x}{2\alpha}\right)^n$.

Therefore $O_n \sim K 2^{\binom{n}{2}} n! \left(-\frac{1}{2\alpha}\right)^n$, where $K = \frac{1 - F(2\alpha)}{\alpha F\left(\frac{\alpha}{2}\right)} \approx 2.197.$
Corollary 3.3. We have estimates for the orientable small covers ratio as
\[ \frac{O_n}{R_n} \sim \frac{K}{C^{2^n}}, \]
where \( K \approx 1.262 \).

**Proof.** Since \( R(x)F(-x) = 1 \) and \( F(x) \) has an isolated zero \( \alpha \), we have \( R(x) = \frac{1}{F(-x)} \sim \frac{1}{F(\alpha)} \sum_{n=0}^{\infty} \frac{(-x)^n}{\alpha^n} \). Hence, we have \( R_n \sim C 2^{\binom{n}{2}} n! \left( -\frac{1}{\alpha} \right)^n \), where \( C = -\frac{1}{\alpha F(\alpha)} \approx 1.739 \). Therefore \( \frac{O_n}{R_n} \sim \frac{K}{C^{2^n}} \approx \frac{1.262}{2^n} \).

**Appendix. Graph theory terminology**

We review the terminology in graph theory, following [8]. A directed graph or digraph \( G \) is a triple \( (V, E, \varphi) \), where \( V = \{v_1, \ldots, v_n\} \) is a set of vertices, \( E \) is a set of directed edges, and \( \varphi \) is a map from \( E \) to \( V \times V \). If \( \varphi(e) = (u, v) \), then \( e \) is called an edge from \( u \) to \( v \) with the initial vertex \( u \) and the final vertex \( v \). If \( u = v \) then \( e \) is called a loop. If \( \varphi \) is injective and has no loops, then \( G \) is said to be simple. In this case, we denote \( e \) by \( (u, v) \) for simplicity and represent \( G \) by \( (V, E) \).

Throughout this paper, every graph is simple. A walk of length \( k \) from vertex \( u \) to \( v \) is a sequence \( v_0, v_1, \ldots, v_k \) such that \( v_0 = u \) and \( v_k = v \), where \( (v_i, v_{i+1}) \in E \) for all \( i = 0, \ldots, k-1 \). If all the \( v_i \)'s are distinct except for \( v_0 = v_k \), then the walk is called a cycle. \( G \) is acyclic if there is no cycle of any length in \( G \). The out-degree of a vertex \( v \) is the number of edges of \( G \) with the initial vertex \( v \). Similarly the in-degree of \( v \) is the number of edges of \( G \) with the final vertex \( v \).

All digraphs can be represented by matrices. Define an \( n \times n \) matrix \( A(G) = (A_{ij}) \) by
\[ A_{ij} = \begin{cases} 1, & \text{if } (v_i, v_j) \in E; \\ 0, & \text{otherwise}. \end{cases} \]
The matrix \( A(G) \) is called the vertex adjacency matrix of \( G \). We remark that the sum of entries of the \( i \)-th column of \( A \) is equal to the in-degree of \( v_i \) and the sum of entries of the \( j \)-th row of \( A \) is equal to the out-degree of \( v_j \).

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