Static Friction between Elastic Solids due to Random Asperities

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Several workers have established that the Larkin domains for two three dimensional nonmetallic elastic solids in contact with each other at a disordered interface are enormously large, implying that there should be negligible static friction per unit area in the macroscopic solid limit. The present work argues that the fluctuations in the heights of the random asperities at the interface that occur in the Greenwood-Williamson model can account for static friction.

It is well known that one must apply a minimum force (known as static friction) in order to get two solids, which are in contact, to slide relative to each other. Several workers have provided evidence, however, that there might be no static friction for nonmetallic crystalline surfaces, incommensurate with each other. Aubry showed this for the weak potential limit of one dimensional Frenkel-Kontorova model[1], and recently He, et. al.[2] and Muser and Robbins[3] have shown for weakly interacting two dimensional incommensurate interfaces that the force of static friction per unit area falls to zero as $A^{-1/2}$ in the thermodynamic limit, where A is the interface area. Even solids made of identical materials are incommensurate if their crystalline axes are rotated with respect to each other. Disorder, however, can pin contacting solids, just as it pins sliding charge density waves[4,5] and vortices in a superconductor[6]. Recently, it has been shown that Larkin domains (i.e., domains over which the solids are able to distort to accommodate the disorder at the interface) for contacting three dimensional elastic solids are enormously large compared to typical solid sizes[7-9], implying that the force of static friction per unit area due to interface disorder should also fall off as $A^{-1/2}$ in the thermodynamic limit. In Refs. 2 and 3 it was proposed that the presence of a submonolayer film of mobile molecules at the interface is a requirement for the occurrence of static friction between incommensurate surfaces, which appears to imply that clean interfaces are frictionless. In the present work, it is argued that the Greenwood-Williamson model (GW)[10,11] predicts the existence of sparsely spaced higher than average asperities at the interface which interact more strongly with the second surface than with each other. Since these are in the strong pinning limit, they will exhibit static friction.

Here, scaling methods, like those used by Fisher for charge density waves (CDW)[12], are used to study static friction for non-smooth interfaces. This can be accomplished by minimizing the potential energy of the solid in contact with a rigid disordered substrate at $z=0$ with respect to the size of a Larkin domain[4], which is expected to give qualitatively correct results for the problem of two elastic disordered solids in contact. Given that the energy density of the elastic solid is given approximately by

$$E = (1/2)E'\left|\nabla u_2^2 + V(r)\delta(z), \right.$$  \hspace{1cm} (1)

where $E'$ is an effective Young’s modulus and $V(r)$ is the potential per unit area of the disordered substrate and $u(r)$ is the local displacement of the solid, the energy of a single Larkin domain is given by

$$E = (1/2)L' E' \alpha \left|\nabla_u u_2^2 / L^2 + V_0 a L, \right.$$  \hspace{1cm} (2)

where $a$ is a local length scale (e.g., a lattice constant), $L$ is the width and $L'$ is the height of the domain, $|\nabla_u u_2|^2 = |\partial u'/\partial x'|^2 + |\partial u'/\partial y'|^2$, where we assume that the local displacement $u'$ varies on length scales $L$ and $L'$ in the x and y and the z directions, respectively. That is, we assume that $u'(x,y,z)$ has the form $u'(x',y',z')$, where the function $u'$ varies by an amount of the order of local length scales (e.g., a lattice constant) when $x'$, $y'$ and $z'$, defined by $(x',y',z')=(x/L,y/L,z/L')$, each vary by an amount of order unity. Here, $V_0$ is a typical value of the potential per unit area. When Eq. (2) is minimized with respect to $L'$ one finds that $L' \approx L$ and the energy per unit volume within a Larkin domain at the interface is given by

$$E/L^3 \approx [(1/2)E'\left|\nabla_u u_2^2 - V_0 a \right|/L^2,$$  \hspace{1cm} (3)

(where we use the average value of $|\nabla u_2|^2$ here) whose absolute minimum occurs for infinite $L$ (more correctly $L$ comparable to the interface length)$V_0 a$, implying that the static friction per atom decreases as the reciprocal of the square root of the surface area. These scaling arguments apply equally well to disorder on the sub-asperity level due to atomic level defects and to the multi-micron length scale level due to asperities. When applied at the sub-asperity level, they show that the contact force between asperities should be proportional to the square root of area of contact. On the multi-asperity length scale, the atoms are replaced by asperities and the defects by contacting asperities.

If the sliding solid has dimensions normal to the interface much smaller than those along it, $L'$ can only be as large as the thickness. Minimization of Eq. (3) with respect to $L$ with $L'$ fixed at the thickness shows that the
Larkin length is comparable to the film thickness. Thus, the interface consists of many Larkin domains. Since the pinning force scales with the number of Larkin domains at the interface, the force of static friction per unit area is non-zero value, in the large solid limit.

Fisher[12] has shown that above the critical dimensions of 4, charge density waves are not pinned for typical impurity strengths, but fluctuations in the impurity concentration and strength lead to pinning. (The critical dimension for two solids in contact is 3, as seen above.) Consider the effect of fluctuations in the defect concentration for thick solids, by dividing the solid into boxes of length L and examining the percentage of blocks at the interface of sufficiently large defect concentration to be in the “strong pinning” regime, where the substrate force on each block dominates over the inter-block elastic force. Consider the parameter \( h = V_0/\alpha a^2 \), where \( \alpha \) is the interatomic force constant, \( a \) is a lattice spacing and \( V_0 \) is the strength of the potential due to a defect acting on an atom. Let \( n_c = c'L^2 \) be the number of defects within a particular block and \( c' \) (where \( c' > c \), where \( c \) is the average defect concentration for the interface), the defect concentration strong enough for the block to be considered a strong block. Then the ratio of the interaction of a typical strong block with the substrate to \( \alpha a^2 \) is \( b(c'L^2)^{1/2} \). The interface area surrounding each strong block is the total interface area \( A \) divided by the number of strong blocks at the interface, \( PA/L^2 \), where \( P \) is the probability of a particular block being a strong one. Then \( L^2/P \) is the interface area surrounding each block and the typical length \( L' \) over which the elastic interaction between two strong blocks acts is its square root, \( L' = L/P^{1/2} \). Then the total elastic energy associated with each strong block is the product of the volume per strong block\( = (L')^2 \) times the elastic energy density, which is proportional to \( \nabla u^2 \) (which scales as \( (L')^{-2} \)) or \( L' \). The criterion for a block to be a strong one is \( h(c'L^2)^{1/2} >> L' \), or \( h >> (c'P)^{-1/2} \). Since \( c'P < 1 \), this violates our previous assumption that \( h << 1 \), implying that such fluctuations cannot result in strong pinning. There are also fluctuations in the locations of the points of contact within the defect potential wells within each Larkin domain, it too does not lead to static friction.[13]

Defect strength fluctuations, however, can lead to static friction, as we shall see. On the multi-asperity scale, the surface asperity height distribution is likely to produce such fluctuations for sufficiently large surfaces even those that are quite smooth. This is the situation for the GW model[10,11,13,14], in which there are elastic spherical asperities on a surface with an exponential or Gaussian height distribution in contact with a rigid substrate. The GW model is generally accepted to be a correct way to account for Anonon’s laws in most cases[11], especially for relatively light loads. Volmer and Nattermann’s discussion of static friction[14] is not qualitatively different from that of Ref. 10. In the GW model, the total contact area is of the order of

\[
A_c = 2\pi \sigma bN \int_h^\infty ds \phi(s)(s-h),
\]

where \( \phi(s) \) is the distribution of asperity heights \( z \), where \( s = z/\sigma \), \( \sigma \) is a length scale associated with the height distribution, and \( b \) is the the ratio of the distance of the lower part of the bulk part of the sliding solid, from the surface in which it is in contact, to \( \sigma \), \( b \) is the radius of curvature of an asperity and \( N \) is the number of asperities above a certain size, independent of whether they are in contact[10,11,13,14]. The interaction of a single asperity with the substrate is equal to the product of the contact area and a shear strength for the interface. Actually, to be consistent with our scaling arguments and Refs. 2 and 3 we should assume that the friction force on a single asperity is proportional to the square root of the asperity contact area, but when this was done, the results were not changed qualitatively.

The energy of the interface consists of two parts. One part is the single asperity energy, which consists of the interaction energy of an asperity with the substrate plus the elastic energy cost necessary for each asperity to seek its minimum energy, neglecting its elastic interaction with other asperities, which is independent of the asperity density. The second part is the elastic interaction between asperities within the same solid, which depends on the asperity density. In order to determine these energies, let us model the interaction of the \( \ell^\text{th} \) asperity with the substrate by a spherically symmetric harmonic potentials of force constant \( \alpha_{\ell} \). Assume that in the absence of distortion of the solid, the \( \ell^\text{th} \) asperity lies a distance \( \Delta \), from the center of its potential well. Let \( u_{\ell} \) be the displacement of the \( \ell^\text{th} \) asperity from its initial position. We use the usual elastic Green’s function tensor of the medium at a distance \( r \) from the point at which a force is applied at the interface, but for simplicity, we approximate it by the simplified form \( G(r) = (E' r)^{-1} \), where \( E' \) is Young’s modulus[15]. Then the equilibrium conditions on the \( u_{\ell} \)’s are

\[
u_{\ell} = (E'a)^{-1}\alpha_{\ell} \Delta_{\ell} - u_{\ell}) + \sum_j(E' R_{\ell,j})^{-1}\alpha_j (\Delta_j - u_j),
\]

where \( a \) is a parameter of the order of the size of the asperity. To lowest order in the interasperity interaction, approximate the solution for \( u_{\ell} \) is

\[
u_{\ell} = u_{\ell}^0 + [1 + (E'a)^{-1}\alpha_{\ell}]^{-1} \sum_j(E' R_{\ell,j})^{-1}\alpha_j (\Delta_j - u_j^0),
\]

where \( u_{\ell}^0 \) is the zeroth order approximation (i.e., the solution to Eq. (5) neglecting the second term on the right hand side of the equation). Since the contacting asperities are randomly distributed over the interface, we can estimate the second term (i.e., the summation over \( j \)) on the right hand side of Eq. (6) by its root mean square (RMS) average which is estimated by integrating the square of the summand over the position of the \( j^\text{th} \)
asperity, which is in contact with the substrate, over its position and multiplying by the density of asperities in contact with the substrate \( \rho \). Since the angular integrals only give a factor of order unity, we need only evaluate the integral over the magnitude of \( R_{0,j} \), giving an RMS value of the sum over \( R^{-1} \) of order \( [\rho h(L/a)]^{1/2} \) where here \( L \) is the length of the interface and \( a \) is the asperity size. For \( L \approx 1\,\text{cm} \) and \( a \approx 10^{-5}\,\text{cm} \), \([\ln(L/a)]^{1/2}\) is of order unity. The energy of the system can be written as

\begin{equation}
(1/2) \sum_j \alpha_j |\Delta_j - u_j|^2 +
(1/2)E' \sum_j \int d^3 r |\nabla G(r)| (\alpha_j |\Delta_j - u_j|)^2
\end{equation}

It follows from Eqs. (5-7) that the two lowest order non-vanishing terms in an expansion of the energy of the system in powers of \( \rho^{1/2} \) are the zeroth and first order ones. Since the shearing of the junction at the area of contact of two asperities involves the motion of two atomic planes relative to each other, the distance over which the contact potential varies must be of the order of atomic distances. Then, if we denote the width of the asperity contact potential well by \( b \), of the order of atomic spacings, we must choose a typical value for \( \alpha \) such that \( \alpha b \) is of the order of the shear rupture strength of the asperity contact junction. Thus, \( \alpha >> E'a \). Zeroth order in the asperity density in Eq. (7) is of the order of \( \alpha \Delta^2 \), where \( \alpha \) is a typical value of \( \alpha_j \), and \( \Delta \) is a typical value of \( \Delta_j \). The term linear in \( \rho^{1/2} \) is easily shown to be of the order of \( E'a^2 \Delta^2 \rho^{1/2} \) to zeroth order in \( E'a/\alpha \). Since it depends on \( \rho \) it represents an interaction energy between the asperities. Then, the mean interasperity interaction is proportional to the square root of the number of contacting asperities per unit surface area, given by

\begin{equation}
\rho(h) = (N/A) \int_h^\infty ds \phi(s).
\end{equation}

where \( A \) is the total surface area and \( N \) is the total number of asperities whether in contact with the substrate or not. The integral in Eq. (4) divided by the integral in Eq. (8), which is proportional to the contact area per asperity and the square root of the integral in Eq. (8) are plotted as a function of the load, which is given in this model by

\begin{equation}
F_L = (4/3)E'N(b/2)^{1/2}[\sigma^{3/2} \int_h^\infty ds \phi(s)(s-h)^{3/2},
\end{equation}

in Fig. 1.

A Gaussian distribution is assumed here for \( \phi(s) \) (i.e., \( \phi(s) = (2\pi)^{-1/2}e^{-s^2/2} \)). Since the square root of Eq. (8) drops to zero in the limit of vanishing load, whereas Eq. (4) divided by Eq. (8) approaches a nonzero value, this implies that the interface will approach the strong pinning regime (i.e., the regime in which the asperity-substrate interaction dominates over the inter-asperity interaction) in the limit of vanishing load.

Let us now give sample numerical values for some of the quantities which occur in the application of the GW model to this problem. For example, for a typical plastic, Young’s modulus is \( 4.0 \times 10^6 \, \text{N/m}^2 \) and the shear rupture strength is \( 3.5 \times 10^7 \, \text{N/m}^2 \). Following Ref. 11, we choose \( \sigma = 2.4 \times 10^{-4} \, \text{mm} \) and \( b = 6.6 \times 10^{-7} \, \text{mm} \), and assume that there is a density of \( 4.0 \times 10^3 \) asperities/\text{mm}^2. Then by performing the integrals in Eqs. (4), (8) and (9), we find that for \( F_L/A = 3.98 \times 10^{-4} \, \text{N/mm}^2 \), where \( A \) is the apparent area of the interface, the total contact area divided by \( A \) is \( 3.03 \times 10^{-5} \), and the contact area per asperity from the ratio of Eqs. (4) and (8) is \( 2.44 \times 10^{-5} \, \text{mm}^2 \). Also, \( \rho(h)^{1/2} \), which is equal to the square root of Eq. (8) is \( 1.11 \, \text{mm}^{-1} \). The mean interasperity interaction force is approximately equal to the derivative of the first order term in \( \rho^{1/2} \) given above Eq. (8) with respect to \( \Delta \) or \( E'a^2 \rho(h)^{1/2} \Delta \), where \( a \) is taken as the square root of the mean contact area per asperity divided by \( \pi \). The mean strength of the force acting on an asperity, due to the solid with which it is in contact, will be estimated by the product of its contact area and the shear rupture strength \( E_r \). Then, the condition for the latter quantity to dominate over the asperity-asperity interaction, \( E_r \pi a^2 > E'4\pi a^2 \rho^{1/2} \Delta \) or \( E_r/E' > 4\rho(h)^{1/2} \Delta \), is easily satisfied by the above calculated quantities since the right hand side is \( 4 \times 10^{-7} \) and the left hand side is \( 8.75 \times 10^{-3} \).

Although for higher loads the system appears to move towards the "weak pinning" limit, the latter conclusion is most likely incorrect because it does not take into account the fact that the distribution of asperity heights contains asperities which are much higher than average. These
asperities will be compressed much more than a typical asperity, making the friction force on them considerably larger than average. Since the probability of such asperities occurring is relatively small, however, they will be typically far apart, putting them in the strong pinning limit. For example, the probability of the ratio of an asperity height to $\sigma$ being greater than a value $h_L$ is

$$P(h_L) = \int_{h+h_L}^{\infty} ds \phi(s),$$

whose mean height and hence contact area is proportional to

$$P(h_L)^{-1} \int_{h+h_L}^{\infty} ds \phi(s)(s-h).$$

These two quantities are plotted in Fig. 2. It is seen that even for $h_L$ only equal to $1/2$ (corresponding to an asperity height equal to $(1/2)\sigma$), Eq. (11) remains larger than the square root of Eq. (10).

Although it has been argued here that the GW model predicts the occurrence of a sufficiently dilute concentration of asperities with stronger than average forces acting on them due to the second solid to consider the asperities to be essentially uncorrelated, this still does not necessarily guarantee that there will be static friction, since it has been argued that even for uncorrelated asperities static friction will only occur if the asperities exhibit multistability[9,16]. The condition for multistability to occur at an interface[9], namely that the force constant due to the asperity contact potential be larger than that due to the elasticity of the asperity ($\approx E'a$), however, will be satisfied, as discussed earlier.

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In conclusion, although at first sight it appeared that arguments based on Larkin domains indicate that the disorder at an interface between two nonmetallic elastic solids in contact would not result in static friction, when one takes into account the fluctuations in asperity height that occur in models such as the GW model, there will always exist asperities with greater than average height to put them in the "strong pinning regime," resulting in static friction. For light loads, even typical height asperities (as opposed to height fluctuations) can easily be in the "strong pinning limit."

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