This paper proposes an algorithm that finds the optimal sets of phasor measurement units (PMUs) to achieve a fault observable system, while it addresses the multi-estimation issue. The optimal PMU placement (OPP) problem here is to find a set of PMU locations with minimum number of members that enables fault observability in a system and satisfies a defined minimum sensitivity requirement in measurements. The proposed algorithm generalizes the impedance method in fault analysis and optimizes PMU utilization to maintain a required minimum sensitivity in each set of measurements, given the required fault detection accuracy. Also, the set of measurements is unique and distinctive for each fault scenario, preventing multi-estimation. A fault is referred to as a set of affected faulty line, fault location, and fault impedance. A sensitivity analysis is performed and sensitivity indices are derived to evaluate measurements quality to detect changes in fault line, location, or resistance. The algorithm is executed on IEEE 7-bus, 14-bus, and 30-bus test systems. Subsequently, artificial neural networks (ANN) are employed to build fault locators through offline training. ANN use an optimal PMU set obtained by the proposed algorithm to uniquely map between the corresponding measurements set and the faults.
determinable. It should be mentioned that normal observability does not guarantee fault observability [16]. Thus, a normal observable power system may not be fully observable during fault condition since fault alters the system structure. While many approaches are proposed to solve OPP problem for power system normal observability (under normal operating condition), there are a very limited number of studies that target OPP for fault observability. Optimal PMU placement for fault observability is investigated in [17] and [18]. Authors in [17] employ the popular one-bus-spaced strategy to find the OPP by genetic algorithm and using only PMU voltage measurements. The topic is expanded in [18] by considering zero injection buses (that reduce the system size) using both PMU voltage and current measurements followed by integer linear programming (ILP) methodology. Authors in [19] study deterministic and stochastic algorithms for placing minimum number of PMUs in a power system in order to locate any fault in the power system. Though the available approaches take advantage of various algorithms to impose observability constraints, the important issue of measurement sensitivity (quality) and its impact on OPP set and fault location is considered in very few literatures. In [20] the branch sensitivity indicator is developed and introduced into the PMU placement by giving higher priority to terminals pertaining to the higher sensitive lines. A pattern search algorithm (PSA) is proposed in [21] to minimize and optimally allocate PMUs considering nonlinear sensitive constraints of buses. However, in these publications the sensitivity analysis is considered for normal observability (sensitivity of either line current or voltage to the load change) rather than fault observability (sensitivity of both voltage and current to fault impedance and location). Authors of [22] utilize a minimization algorithm to reduce the number of sensors. This effort was followed by considering the measurement precision in the fault location problem [23] given the sensor locations; however, the precision has not been used in the measurement optimal placement. The effect of the measurement precision in PMUs placement is of paramount importance and adds additional constraints to the available methods while this has not been given enough attention in OPP solution methods.

This paper considers PMU direct measurements with adequate channel availability for voltage and current measurements. A more comprehensive definition of fault observable system than [16–18, 24] is adopted in this paper. If location and impedance of all faults of interest in a power system can be determined with predefined accuracy through a set of voltage and current measurements, the system is considered fault observable. A unique function mapping between measurements and faults is discussed and obtained in a systematic manner for the first time to the authors’ best knowledge. The contributions of this research include:

1. Introducing sensitivity analysis in OPP problem for networks fault observability. The quality of measurements is assessed at PMU locations using the proposed sensitivity indices. Thus, through sets of faults, one can judge whether a specific network bus is a proper measurement location for PMU placement. Using the proposed sensitivity analysis, measurement precision or inaccuracy instigated by the current transformers (CTs), potential transformers (PTs), and PMUs can be incorporated into the OPP problem. Considering measurement quality is also vital for other system analyses such as voltage stability, contingency studies, etc., which are mostly fault related.
2. Formulating minimal PMU placement and finding pertinent optimal PMU sets for fault observability and fault location. The proposed algorithm based on the defined accuracy of PMUs finds the optimal PMU sets such that using minimum number of PMUs the faults are located uniquely, avoiding multi-estimation. Multi-estimation is a condition where different faults result in similar measurements in a selected PMU set.
3. Developing a fault locator by utilizing the obtained optimal PMU set along with artificial neural networks (ANNs). Since there is assumed to be a function that relates the fault location and impedance to the voltage and current measurements, the function approximation property of the ANNs is employed to map between the faults and the measurements of the optimal PMU set.

Machine learning algorithms have already revolutionized many fields. Machine learning techniques such as reinforcement learning and extreme learning have been employed in power systems due to their generalization performance [25–27]. The reinforcement learning in the power system literature has been used to tune the learning mechanism based on a critic response or a target objective while enables the design of model-free approaches. On the other hand, the applications of learning methods such as ANNs is growing in power systems due to the ability of ANNs to learn complex non-linear relationships [28]. In this application, we found ANNs relatively easier than other learning mechanisms given the availability of the data and certainty of the system topology and parameters. Although deep learning mechanisms are great solutions to systems with large number of variables and uncertainties, they are generally time consuming and require large data, whereas the ANNs used in this paper are simpler and with the multistep training method proposed here, it does not impose much computation burden.

The remainder of this paper is organized in the following order: Section 2 presents the proposed sensitivity analysis and introduces the sensitivity indices. In Section 3, the sensitivity and multi-estimation criteria are presented, followed by the proposed algorithm for solving OPP problem in Section 4. Section 5 includes simulation results obtained from applying the proposed method on the IEEE 7-bus, IEEE 14-bus, and IEEE 30-bus test systems followed by artificial neural network fault locator results. Finally, concluding remarks are provided in Section 6.

2 | SENSITIVITY ANALYSIS

The approach presented in this paper is built upon the classical fault analysis and is considered for symmetrical three-phase systems. However, the approach can be generalized to
single-phase and unsymmetrical networks as well [29, 30]. A fault in power system changes the structure of the network, while its location and impedance are unknown. Therefore, previously known system states, impedance matrix $Z_0$, and admittance matrix $Y_0$ should be altered to accommodate the fault analysis (see Figure 1) [30].

This study considers faults on power system lines (note that faults on grid buses is a special case). That is, an extra bus $p \in N+1$ is designated at the point of fault, where $N$ is the network total number of buses. Figure 1 shows the procedure of adding a fault to the system. The unfaulty power system with known impedance matrix $Z_0$, voltages, and currents is depicted in Figure 1a. Figure 1d depicts the faulty system with a fault on one of the network lines, where the impedance matrix is $Z_1$ (fault not included). The line exposed to the fault is located between system buses $l$ and $k$ that are unknown due to the random nature of fault with unknown fault distance $D$ and fault resistance $R_f$.

**Definitions.** The following terms are used frequently in this paper.

**Normal value:** The value of a bus voltage or a line current in an unfaulty power system is called normal value.

**Fault:** A fault is referred to by $F = (l_f, D, R_f)$, where $l_f \in L_f = \{1, 2, ..., L\}$ is the line number where fault occurs with $L$ being the total number of lines in the studied power network, $D \in D = [0, 1]$ is the normalized distance of the fault with respect to one of the line-end buses ($D = \frac{\text{bargh}(lp)}{\text{bargh}(lk)}$ from Figure 1), and $R_f \in R_f = [0, R_{\text{max}}]$ is the fault line-to-ground resistance in the single-phase equivalent circuit with $R_{\text{max}}$ being the maximum fault resistance of interest. If $R_{\text{max}}$ is selected very small (short circuit), the loads can be ignored in the proposed method. Otherwise, the load information may be needed to locate the fault accurately.

**Observant bus:** Bus $b \in \{1, 2, ..., N\}$, where a measurement device capable of measuring the bus voltage and currents (of connected lines to that bus) is installed, is an observant bus.

**Observant set:** A set $H \subseteq \{1, 2, ..., N\}$ of observant buses is called an observant set.

**Adjacent bus:** Bus $u$ is called an adjacent bus to observant bus $b$ if $u \in U_b$ with $U_b$ is the set of all connected buses to observant bus $b$. Also, $U_b$ is called adjacent set to observant bus $b$ and has $b$, many members; i.e., there are $b$, many connected buses (lines) to observant bus $b$.

**Multi-estimation:** Multi-estimation is a condition where different faults cause similar measured values in an observant set.

Four steps are required to modify $Z_0$ and obtain $Z_4$ (dashed elements in Figure 1b imply faulty line removal from $Z_{\text{bus}}$). Each of these steps results in a new system with impedance matrix subscripted by the step number as shown in Figure 1 [24, 30]. By using the standard fault analysis, the voltage changes at observant bus $b$ (when fault $F$ occurs at bus $p$) can be described in Equation (1):

$$\Delta V_{h,F} = \frac{Z_3 (b, p)}{Z_3 (p, p) + R_f} \times V_{pre,f},$$  

where $Z_3 (b, p)$ is the $(b, p)$ entree of $Z_3$, $Z_3 (p, p)$ is the system Thevenin impedance seen from imaginary bus $p$, and $V_{pre,f}$ is the prefault voltage at the point of fault in the system. With the assumption of linear voltage drop along the transmission lines between buses and by ignoring line capacitances to avoid complexity, $V_{pre,f}$ can be calculated in Equation (2):

$$V_{pre,f} = V_l + (1 - D) \times (V_l - V_k).$$

For more accurate calculation in long transmission lines, hyperbolic voltage drop can be considered [24]. From the previous discussion, voltage and current changes in all buses of the system can be calculated by using original impedance matrix $Z_0$ along with $D$ and $R_f$, as will be explained next.

### 2.1 Voltage sensitivity indices

If deviation from the initial value of voltage at the observant bus $b$ after fault $F = (l_f, D, R_f)$ is represented by $\Delta V_{h,F}$, then the voltage sensitivity indices are defined as derivatives of fault distance $D$ and impedance $R_f$ with respect to $\Delta V_{h,F}$ as shown in Equation (3):

$$S^D_{h,F} = \frac{\partial D}{\partial \Delta V_{h,F}} \quad \text{and} \quad S^R_{h,F} = \frac{\partial R_f}{\partial \Delta V_{h,F}}.$$  

respectively. One can use derivatives of $\Delta V_{h,F}$ with respect to $D$ and $R_f$ instead and use the inverse function to achieve voltage sensitivity indices (3). That is, $S^D_{h,F} = \left(\frac{\partial \Delta V_{h,F}}{\partial D}\right)^{-1}$. Sensitivity index $S^R_{h,F}$ can be found in a similar manner. The derivation of indices (3) are given in the appendix.

### 2.2 Current sensitivity indices

An installed PMU on any grid bus measures the phasor of the bus voltage as well as that of the currents of all the connected lines. The number of line currents that a PMU is capable of measuring depends on the PMU’s available channels that are assumed to be large enough in this study. This assumption is widely acceptable due to recent developments in PMU.
technology and the need for acquiring the line current data for various purposes [17, 18].

Similar to voltage sensitivity indices, current sensitivity indices can be defined for each line connected between observant bus \( b \) and an adjacent bus \( u \) where \( u \in U_b \) and \( U_b \) is the set of all connected buses to observant bus \( b \). These indices are defined in Equation (4):

\[
S_{I_{hu,F}}^{\Delta I} = \frac{\partial \Delta I_{hu,F}}{\partial \Delta V_{h,F}} \quad \text{and} \quad S_{I_{hu,F}}^{R_f} = \frac{\partial R_f}{\partial \Delta V_{h,F}}.
\]  

(4)

The number of current sensitivity indices derived for each observant bus \( b \) in (4) is equal to the number of lines connected to bus \( b \). Figure 2 illustrates an example of a line current in the state of fault.

At any observant bus \( b \) within the network, line current changes can be expressed as in Equation (5):

\[
\Delta I_{hu,F} = \frac{\Delta V_{h,F} - \Delta V_{u,F}}{Z_{hu}} = -Y_{hu} \left( \begin{array}{c} b_n \end{array} \right) \times \left( \Delta V_{u,F} - \Delta V_{h,F} \right),
\]  

(5)

where \( Z_{hu} \) is the impedance of line \( hu \), \( Z_{hu} = (-Y_{hu} \left( \begin{array}{c} b_n \end{array} \right))^{-1} \), and \( Y_{hu} \left( \begin{array}{c} b_n \end{array} \right) \) is the \( (b, u) \) entree of the admittance matrix that corresponds to \( Z_{hu} \) according to Figures 1 and 2. Note that manipulating the admittance matrix is much easier than that of the impedance matrix and it is not detailed here. Admittance matrix modification procedure results in obtaining \( Y_{hu} \left( \begin{array}{c} b_n \end{array} \right) \) elements, many of them are not functions of \( D \) or \( R_f \). Derivation of the current sensitivity indices are also given in the appendix.

**Definition.** Consider an observant set \( H \subseteq \{1, 2, \ldots, N\} \). Measurement set \( M_{HF} \) corresponding to fault \( F \) is defined as \( M_{HF} = \{\Delta V_{h,F}, \Delta I_{hu,F} \mid b \in H, u \in U_b \} \), where \( U_b \) is an adjacent set to observant bus \( b \).

In order to demonstrate the concept through some numerical results, IEEE 7-bus test system is utilized for exemplar sensitivity analyses. Figure 3 depicts the IEEE 7-bus test system where the line numbers are presented in parenthesis. Figure 4 shows an example for sensitivity indices calculation where bus 2 is the observant bus and faults with maximum 0.1 p.u. resistance occur on line 8 in Figure 3, between buses 5 and 2. Figure 4a shows the fault bus and faults with maximum 0.1 p.u. resistance occur on line 8 in Figure 3, between buses 5 and 2. Figure 4a shows the changes in the measured voltage magnitude on bus 2, and Figure 4b depicts the voltage phase angle changes. Figures 4c and 4d illustrate the numerical values for \( \frac{\partial \Delta V_{h,F}}{\partial D} \) and \( \frac{\partial \Delta V_{h,F}}{\partial R_f} \), respectively, which are easier to formulate.
where ε terms indicate desired sensitivity thresholds. That is, for example, set $\Theta_{DV}^{D\mathcal{F}(l_f)}$ contains all faults on line $l_f$ for which voltage at observant bus $b$ is sensitive to the fault distance $(D)$. Similarly, set $\Theta_{Rf}^{R\mathcal{F}(l_f)}$ contains all faults on line $l_f$ for which current in line $b m$ (that is measured at observant bus $b$) is sensitive to the fault impedance $(R_f)$. Now, defining $\Theta_{Dh,F}^{D}(l_f) = \Theta_{DV}^{D\mathcal{F}(l_f)} \cup (\cup_{u \in U_h} \Theta_{DV}^{D\mathcal{F}(l_f)}), \Theta_{Rf}^{R\mathcal{F}(l_f)} = \Theta_{Rf}^{R\mathcal{F}(l_f)} \cup (\cup_{u \in U_h} \Theta_{Rf}^{R\mathcal{F}(l_f)}),$

and $\Theta_{h,F}^{R}(l_f) = \Theta_{h,F}^{R\mathcal{F}(l_f)}. \cap \Theta_{h,F}^{R\mathcal{F}(l_f)}$.

Set $\Theta_{h,F}^{D\mathcal{F}(l_f)}$ includes all faults on the line $l_f$ with fault distances for which voltage or some current measurements at observant bus $b$ are sensitive to. Similarly, set $\Theta_{h,F}^{R\mathcal{F}(l_f)}$ includes all faults on the line $l_f$ with fault impedances for which voltage or some current measurements at observant bus $b$ are sensitive to. Set $\Theta_{h,F}^{R\mathcal{F}(l_f)}$ includes all faults on the line $l_f$ with distances and impedances for which voltage or some current measurements at observant bus $b$ are sensitive to. Set $\Theta_{h,F}^{R\mathcal{F}(l_f)}$ may include all or some faults of interest on the line $l_f$ for $\exists l_f \in L_f$. Thus, in general, several observant buses might be needed to include all faults of interest on all system lines; i.e. for $\forall l_f \in L_f$.

Fault location (for all faults $F$) is possible if an observant set can find all faults in regions $D \times R_f$ for all power system lines. That is, for any faulty line $l_f \in L_f$, there must exist an observant set $H$ such that $\cup_{l_f \in H} \Theta_{h,F}^{R\mathcal{F}(l_f)} = D \times R_f$.

Figure 5 provides a numerical example for equation (6) using sensitivity indices depicted in Figures 4e and 4f. Areas where the sensitivity indices pass the desired sensitivity threshold (ε) are depicted in blue. Figure 5a illustrates that not all possible faults on line 8 in terms of fault location $(D)$ will cause distinguishable measurements on observant bus 2. That is, the red region, which correlates to a set of possible faults, needs line current measurements or other buses coverage so that we achieve fault detection coverage on line 8 for full observability. Figure 5b illustrates that all sensitivity indices pass the threshold criteria, which shows the capability of bus 2 voltage measurement in detecting line 8 fault resistances up to 0.1 p.u. In a similar manner
presented in Figures 4 and 5, current sensitivity indices can be calculated and demonstrated. Note that a bus can have several connecting lines.

In practice, realization of such condition may be difficult, especially for high values of fault impedance. Thus, a slightly simpler (and probably more conservative) approach is proposed here. One of the objectives in this paper is to benefit the provided fault location results and select observant buses that are able to locate at least 90% of all possible faults in the region \( \mathbf{D} \times \mathbf{R}_f \) for each faulty line \( l_f \in \mathcal{L}_f \). This practical criterion adds some flexibility in observant bus selection. By adding the 90% coverage criterion one has the option to lower the number of required PMUs. This also affects the implementation cost due to the lower number of required PMUs. The case with 100% coverage is also executed in this paper and the OPP results are compared with the ones of 90% coverage. The fault observability (neural network accuracy) of 90% and 100% coverage are almost the same; however, the number of PMUs in the latter case is higher. Due to the piecewise continuity of the defined sets in (6), an observant bus \( h \) is chosen if for \( \exists l_f \in \mathcal{L}_f = \{1, 2, \ldots, L\} \) condition (7a) or (7b) is satisfied:

\[
SVID = \left\{ \int \int_{\Theta_{\mathbf{D} \times \mathbf{R}_f}^{DV}(l_f)} dDdR_f \geq S_{DR} \right\} \lor \left\{ \int \int_{\Theta_{\mathbf{D} \times \mathbf{R}_f}^{RI}(l_f)} dDdR_f \geq S_{DR} \right\}
\]

(7a)

\[
SVIR_f = \left\{ \int \int_{\Theta_{\mathbf{D} \times \mathbf{R}_f}^{R_f V}(l_f)} dDdR_f \geq S_{DR} \right\} \lor \left\{ \int \int_{\Theta_{\mathbf{D} \times \mathbf{R}_f}^{R_f I}(l_f)} dDdR_f \geq S_{DR} \right\}
\]

(7b)

where \( S_{DR} = 0.9 \int \int_{\mathbf{D} \times \mathbf{R}_f} dDdR_f = 0.9R_{\text{max}} \). Condition SVID implies that observant bus \( h \) is sensitive to the distance of 90% of the faults, indicated by the region \( \mathbf{D} \times \mathbf{R}_f \) on the line \( l_f \). Similarly, SVIR implies that observant bus \( h \) is sensitive to the impedance of 90% of the faults indicated by region \( \mathbf{D} \times \mathbf{R}_f \) on the line \( l_f \). Subsequently as in Equation (8),

\[
SVIDR_f = SVID \land SVIR_f.
\]

Binary value \( SVIDR_f \) indicates whether the observant bus \( b \) is capable of illustrating (using its measurements) the changes in distance and/or impedance of a vast majority of the faults of interest that occur on line \( l_f \) with the desired precisions indicated by (6). Condition (8) will be checked for all the power system lines to find observant bus \( b \)'s domain of fault coverage. This step will reduce the number of required observant buses in obtaining fault observability in the entire system. In practice, one observant bus may not cover the faults on all power system lines and thus other observant buses must be exploited so that faulty lines that are not observed by one observant bus are observed by others. Thus, the above process is repeated for all the power system’s buses to lay out an initial mapping between the faults of interest and the power system buses as potential observant buses. A group of observant buses (i.e. an observant set, if one exists), that satisfies condition (8) for all \( l_f \in \mathcal{L}_f \) provides a solution to the fault location problem and thus renders the power system fault observable. This is equivalent to an observant set whose measurements (measurement set) are sensitive to 90% of distances or impedances of the faults on all power system lines.

### 3.2 Uniqueness and multi-estimation

After finding sensitive bus locations for measurement placement, multi-estimation is a necessary criterion to check to ensure that a measurement set is capable of locating all possible faults in the power system uniquely. The ability of precisely
locating a fault in the system depends on distinguishable measurements for any two different faults in the system.

Multi-estimation exists if for an observant set \[ H \subseteq \{1, 2, \ldots, N\} \] and two faults \([ l_f = (l_1, D_1, R_1) \] and \([ l_f = (l_2, D_2, R_2) \), where \( l_f \neq l_g \) all corresponding measurements from the observant set \( H \) are the same; i.e., \( M_{l_f} = M_{l_g} \) (see Section 2). Analytically, for any pair of faulty lines \( l_1, l_2 \in L_f \) and observant bus \( h \in H \), this results in the nonlinear equalities, shown in Equation (9), in terms of \( D_1, R_1, D_2, \) and \( R_2 \) for \( \forall u \in U_f^T \):

\[
\begin{aligned}
\Delta V_{h,l_1} - \Delta V_{h,l_2} &= 0 \\
\Delta I_{h,l_1} - \Delta I_{h,l_2} &= 0 \quad .
\end{aligned}
\]

Total number of faulty line pairs \( (l_1, l_2) \in L_f \) is \( \frac{L(L+1)}{2} \) where \( L \) is the number of power lines in a power system. This number includes combinations of any two different lines plus the number of power system lines \( (L) \) to account for multi-estimations on the individual lines. Thus, for each observant bus \( h \) in set \( H \), (9) represents \( \frac{L(L+1)}{2} (h+1) \) many equations, where \( h \) is the number of connected buses (lines) to observant bus \( h \). For unique fault location and fault observability, multi-estimation must not occur. That is, for \( l_1 \neq l_2 \), (9) must result in no solutions whereas for \( l_1 = l_2 \) it must yield \( D_1 = D_2 \) and \( R_1 = R_2 \). Equations (9) can be formed by employing (1) and (5) that lead to nonlinear equations that can be solved numerically.

This approach in the simplest form can be represented as an optimization problem in the form of \( \min \, w^T H \) under constraints (7) and (9), where \( H \) is an \( N \times 1 \) vector with its elements \( 0 \) or \( 1 \) represents selection of an observant bus, and \( w = [w_1, w_2, \ldots, w_N]^T \) is a weight matrix that reflects practical or operational priorities in selecting observant buses with \( 0 \leq w_i \leq 1 \). The optimization problem can be developed further to include other constraints such as contingencies, etc., but this is not the objective of this paper and neither discussed further here. Thus, an exhaustive search is used in this paper to solve the OPP problem.

## 4 | PROPOSED ALGORITHM FOR OPP AND ARTIFICIAL NEURAL NETWORK FAULT LOCATOR

Previous works consider optimal PMU placement with much emphasis on the PMU cost as a weight vector in the optimization problem [10–15]. However, measurement precision and bus suitability for fault observability are mostly neglected in assigning PMU locations. PMU fault location capability is a function of its location in the system. Measurement from a PMU installed in an improper location may cause significant inaccuracy in fault location. The proposed formulation and algorithm in this paper aims to thoroughly consider this issue. Power system buses should be checked for conditions (7) and (9) to obtain the most proper observant set \( H \). These conditions can be translated as sensitivity and uniqueness conditions required for fault observability and location, and are evaluated for all grid buses so that a set of appropriate observant buses are selected. Numerical solutions can be sought to evaluate observant buses which are explained next.

Before we proceed, the following discussions need to be conducted.

### Remark 1. (Measurement precision): IEEE C57.13 standard for instrumentation transformers suggests 0.3% error for current and voltage transformers [31,32]. Since PMU measurement precision is usually higher than that of the instrumentation, precisions of 0.1% for current and 0.1% for voltage measurements total vector error are considered in this study, denoted by \( TVE \) and \( TVE' \), respectively, where \( TVE = \frac{|X_{\text{measured}} - X_{\text{theoretical}}|}{X_{\text{theoretical}}} \times 100\% \).

It is worth mentioning that accurate phasor estimation can be made during fault transients [33–35]. Nevertheless, in this study fault duration is considered to be 0.1 second, which is 6 cycles at 60 Hz and is equal to the operating time of the circuit breakers. Since the transients caused by the faults are generally damped within two cycles [36], an installed PMU has enough time to measure the steady-state fault phasors. In case, a severe fault occurs at a PMU location, the amplitude of the measured voltage or current phasors can be very inaccurate; however, the proposed method exploits multiple measurements across the grid to assure that enough accurate measurements are taken.

### Fault location precision: Define \( TP_D \) as “target precision for fault distance \( D' \).” Also, define \( TP_{R_f} \) as “target precision for fault resistance \( R_f \).” Note that fault location range is \( 0 \leq D \leq 1 \) on a power line and thus for a given \( TP_D \leq 1 \), fault can be located on one of \( \frac{1}{TP_D} + 1 \) equally spaced points on any power lines. Also, if fault resistance range of interest is \( 0 \leq R_f \leq R_{\text{max}} \) for the given \( TP_{R_f} \) the fault resistance can be any of \( \frac{R_{\text{max}}}{TP_{R_f}} + 1 \) equally spaced resistances between 0 and \( R_{\text{max}} \).

### Remark 2. From the above discussion, the desired upper limits for sensitivity indices (Equations (3) and (4)) can be calculated in Equation (10):

\[
\begin{aligned}
S_{h,F}^{DV} &\leq TP_D \quad \frac{\varepsilon_{DV} \cdot TVE}{TVE} \quad S_{h,F}^{R_f/V} &\leq TP_{R_f} \quad \frac{\varepsilon_{R_f/V} \cdot TVE}{TVE} \\
S_{h,F}^{LV} &\leq TP_D \quad \frac{\varepsilon_{DLV} \cdot TVE}{TVE} \quad S_{h,F}^{R_f/L} &\leq TP_{R_f} \quad \frac{\varepsilon_{R_f/L} \cdot TVE}{TVE}
\end{aligned}
\]

for all \( h \in \{1, 2, \ldots, N\} \) and \( u \in U_f^T \). For example, for \( TP_D = 0.01, TP_{R_f} = 0.05, TVE = 0.1\% \), and \( TVE = 0.1\% \), one has \( \varepsilon_{DLV} = 10, \varepsilon_{R_f/L} = 50, \varepsilon_{DLV} = 10, \) and \( \varepsilon_{R_f/L} = 50 \).

So far, the relationship between sensitivity indices (3) and (4) and the fault location and impedance accuracy are explained. Thresholds (10) can be utilized to evaluate the quality of observant bus \( h \). Once the sensitivity measures (3) and (4) are obtained as functions of fault \( F = (l_f, D, R_f) \), they can be compared with thresholds (10) across all variations of faulty line...
\( l_f \), location \( D \), and impedance \( R_f \) to determine if observant bus \( b \) is a suitable choice. In addition, multi-estimation conditions should be verified.

### 4.1 Proposed algorithm

The algorithm to find optimal PMU sets is explained as follows:

1. Enter the algorithm inputs: \( TP^D \), \( TP^R \), \( TVE^V \), \( TVE^I \) and \( S_{D,R} \). Calculate the sensitivity thresholds \( \varepsilon_{D,V}, \varepsilon_{R,V}, \varepsilon_{D,I} \) and \( \varepsilon_{R,I} \) using (10).
2. Select an observant bus \( b \) and a faulty line \( l_f \in L_f \) and obtain the sensitivity indices (3) and (4) for fault \( F = (l_f, D, R_f) \) for all \( D \in D \) and \( R_f \in R_f \) with \( TP^D \) and \( TP^R \) steps (target precisions), respectively. That is, sensitivity indices are evaluated on all \( 1/T_P^D + 1 \) equally spaced points on line \( l_f \) and all the \( R_{max}^f \) equally spaced resistances between 0 and \( R_{max}^f \).
3. Obtain sensitivity ranges (6) by comparing the sensitivity indices of Step 2 with thresholds \( \varepsilon_{D,V}, \varepsilon_{R,V}, \varepsilon_{D,I} \) and \( \varepsilon_{R,I} \) of Step 1.
4. Check sensitivity criteria, i.e., the ability to find the fault distance and impedance with desired accuracies \( TP^D \) and \( TP^R \) (of Step 1) of observant bus \( b \) for fault location on line \( l_f \) through evaluating (7a), (7b), and (8).
5. Repeat Steps 2 to 4 for all lines \( l_f \in \{1, 2, ..., L_f\} \) and store the lines for which fault distance and impedance can be determined with desired accuracy using observant bus \( b \). This step also determines how many faulty lines are observable by observant bus \( b \) (rank of bus \( b \)).
6. Repeat Step 5 for all observant buses \( b \in \{1, 2, ..., N\} \).
7. Form all possible observant sets passing sensitivity criteria. An individual observant bus may not satisfy criterion (8) for all faults of interest in the power system. Thus, a set of observant buses (that may not be unique) may be capable of observing all faulty lines; that is, there may exist an observant set that satisfies criterion (8) for all grid lines. Such an observant set is capable of finding faulty lines, distances, and impedances of all faults through voltage and current measurements. In many power systems, a trivial set of such observant buses is the entirety of the power system buses. However, in the majority of power systems, a smaller number of observant buses forming an observant set can serve and observe all faulty lines (determining fault locations and impedances). In this step, combinations of observant buses forming such observant sets, with minimum number of observant buses are obtained and saved. This starts by examining one-observant-bus sets, two-observant-bus sets, three-observant-bus sets, etc., using an improved enumeration. That is, at least one measurement from the measurement set must pass the introduced sensitivity criteria for a given fault from the set of faults of interest; otherwise, the set is dismissed. Once an observant set that satisfies (8) is found for all faults of interest, it is retained and must be checked against multi-estimation.
8. Check against multi-estimation criterion using exhaustive search. The observant set obtained in Step 7 must be checked against multi-estimation criterion. Select a pair of faulty lines \( l_{f_1}, l_{f_2} \in L_f \). Equations (9) must yield no solutions but the trivial solution \( F_1 = F_2 \), for selected lines \( l_{f_1}, l_{f_2} \) and measurements of selected observant set. Discard the measurement set and go back to Step 7 for another measurement set selection if Equations (9) yield non-trivial solutions; i.e., two faults yield similar measurement sets in the selected observant set. Otherwise, go to Step 9.
9. Repeat Step 8 for all \( \frac{L_f(L_f+1)}{2} \) power line pairs. If the measurement set passes Step 8 for all possible line pairs, measurement set with minimal observant set is obtained since Steps 7 and 8 start with smaller observant sets and go up. Then, go to Step 10. Otherwise, go back to Step 7 for another measurement set selection.
10. Collect all the observant sets that pass Step 9. Typically, the observant set obtained in Step 9 is not unique. Thus, search can be continued to obtain more qualified observant sets that pass Step 9 with the number of observant buses equal to that of the last obtained observant set. This in turn determines multiple optimal PMU locations. Among the optimal sets, the set with the minimum number of measurements (voltages and currents) outperforms and is chosen.

The flowchart of the proposed algorithm is depicted in Figure 6. The proposed algorithm may include other constraints for practical applications such as excluding inaccessible measurement points (out of reach) and considering topological changes due to operational requirements (especially in micro grids). Thus, certain measurements can be removed from observant sets. For topological changes, the algorithm should be applied to the new topologies.

### 4.2 ANN fault locator

Once the optimal observant set is obtained, it is ensured that the set can locate all faults of interest uniquely without multi-estimation. Thus, a one-to-one map exists between the corresponding measurement set and the faults of interest including the faulty line, the fault distance, and impedance. Consequently, artificial neural networks (ANNs) are capable of and used to map the measurement set from the optimal observant set to corresponding faults comprising faulty line \( l_f \), distance \( D \), and resistance \( R_f \).

It is worth mentioning that, in this paper, the ANN is used as a fault locator rather than fault detector or identifier. Since the fault location is a matter of fault place and intensity, there is no associated time component. That is, the fault location strategy is the same at all times. In addition, fault location in the proposed method is performed in a very short time after the observant set is obtained. Therefore, the complications of spatiotemporal
Flowchart representation of the proposed algorithm

FIGURE 6

ANNs are intelligent mechanisms that can approximate complex nonlinear functions through employing a set of input and output data [39]. The function approximation property of ANNs is used here to estimate the function that maps the measurement set as the input and corresponding fault as the output. ANNs are trained offline and weights and bias values are obtained in MATLAB using the Levenberg-Marquardt optimization method [39] as an efficient method in training of feedforward ANNs. The ANNs here have one hidden layer and one output layer with sigmoid and linear activation functions, respectively.

In this study, instead of using one large neural network, a structure of networks is employed to have a more precise fault locator. That is, faulty line \( l_f \) is found in the first neural network using input data from the measurement set. Then, based on the detected faulty line, a pertinent neural network is activated to determine fault distance \( D \) and resistance \( R_f \), as shown in Figure 7. Input vector \( X \) of the first ANN is the measurement set's (corresponding to the obtained OPP) voltage and current magnitudes and angles. Output vector \( Y_1 \) is the faulty line \( l_f \). That is, \( Y_1 = W_1^T \Phi(V_1^T X) \), where \( W_1 \) is the output layer weight matrix, \( \Phi \) is the Sigmoid activation function, and \( V_1 \) is the hidden layer weight matrix. Next, a second ANN is selected based on the resultant faulty line from the first ANN. In the second ANN, the input vector is \( X \) as explained and output vector \( Y_2 = [D \ R_f] \) is the location and resistance of the fault located on faulty line \( l_f \). That is, \( Y_2 = W_{lf}^T \Phi(V_{lf}^T X) \), where \( W_{lf} \) is the output layer weight matrix, \( \Phi \) is the Sigmoid activation function, and \( V_{lf} \) is the hidden layer weight matrix corresponding to faulty line \( l_f \).

The individual ANNs are trained separately using relevant generated fault data. All ANNs utilize one hidden layer whose number of neurons vary with the size of the grid (e.g., 20–40 neurons for 7-bus and 35–65 neurons for 30-bus grid) where higher number of neurons are used for higher precision scenarios (lower TP\(D\) and TP\(R_f\)). Approximately 20% of the generated fault data is separated and used to test the trained neural networks. Neural network fault locator results presented in the next section are the percentage of the correct estimations for this portion of data. ANN design data is also provided in the tables in the next section.

FIGURE 7 Neural networks structure

5 | RESULTS AND DISCUSSION

The proposed algorithm is applied to the IEEE 7-bus (Figure 3), IEEE 14-bus, and IEEE 30-bus [40] test systems to assess the performance of the algorithm and the obtained optimal PMU set in fault location. The test systems consist of 3, 2, and 6 generators as well as 10, 21, and 42 transmission lines, respectively [40]. The IEEE 14-bus and 30-bus test cases one-line diagram are provided in the appendix. Once the proposed algorithm finds the optimal observant set(s) for each power system, artificial neural networks are utilized to obtain a fault locator using the observant set. It should be mentioned that despite ignoring the network capacitance in developed methodology (Equation 2), the shunt capacitance of transmission lines is available during the algorithm execution to include their effect.
TABLE 1 IEEE 7-bus OPP and ANN results for various target precisions and different measurements accuracies

| IEEE 7-bus OPP (R_f max 0.1 p.u.) | Optimal observant sets (PMU locations) | ANN |
|----------------------------------|----------------------------------------|-----|
| TVE^V | TVE^I | # PMUs | | Percentage estimation accuracy |
|       |       |       | | l | D(ave) (min) | R_f(ave) (min) |
| TP^V = 0.01, TP^I = 0.01, Total generated faults: 11,000 |
| 10^{-2} | 10^{-2} | 2 | (1,2)-(2,3) | | 99.6 | 99.9 | 99.9 |
| 10^{-3} | 10^{-2} | 2 | (1,2)-(2,3) | | 99.8 | 100 | 100 |
| 10^{-3} | 10^{-3} | 1 | (1)-(2)-(3)-(5) | | 99.9 | 100 | 100 |

| TP^V = 0.05, TP^I = 0.05, Total generated faults: 600 |
| 10^{-2} | 10^{-2} | 1 | (3)-(5) | | 99.1 | 99.1 | 100 |
| 10^{-3} | 10^{-2} | 1 | (3)-(5) | | 99.1 | 99.1 | 100 |
| 10^{-3} | 10^{-3} | 1 | (1)-(2)-(3)-(4)-(5)-(6) | | 100 | 99.1 | 100 |

|   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|

on OPP solutions and fault location accuracy. The algorithm is also executed with zero shunt capacitance and results compared with those considering the lines’ capacitance showing negligible differences. The ANNs are trained by known fault data that are measured by the optimal PMU set (observant set) and create a one-to-one map between the measurement set and the corresponding fault; i.e. fault line, distance, and impedance. After the training is completed, the ANN fault locator is tested by new fault data and accuracy of fault location is examined.

Fault impedance is considered to be purely resistive in this study [24]. The maximum fault resistance of the interest is considered to be $R_{\text{max}} = 17.4$ for all test cases, i.e. 0.1 p.u in 132 kV base voltage. By increasing the maximum fault resistance of interest the number of PMUs in the set may increase since the measurement set is demanded to cover higher resistance faults. Various voltage and current measurement precisions are used to solve the OPP problem to include various PT and CT precisions.

5.1 Proposed algorithm results

Table 1 presents OPP results for the IEEE 7-bus system. Two cases are performed in the simulation: with target precisions of 1% for fault distance $D$ and resistance $R_f$, and with target precisions of 5% for $D$ and $R_f$. These precisions are the desired fault location accuracies and thus they are used to generate faults for training and testing the ANN fault locator.

The first two columns of Table 1 show voltage and current measurement precisions. These precisions are used in solving the OPP in the proposed algorithm where sensitivity indices are utilized. Columns 3 and 4 represent the minimum number of required PMUs and the optimal observant set(s) suggested by the proposed algorithm. The ANN fault locator is trained by employing the optimal observant set shown in bold. For the case with 1% target precision, 11,000 fault scenarios are generated throughout the system for ANN training, and 2,200 fault data are used for test and validation. Similarly, for the case with 5% target precision, 600 fault scenarios are used for training and 120 fault scenarios are used for validation. The remaining columns show the accuracy of fault location using the trained ANN fault locator. The fault locator dedicates an artificial neural network for each line in the second stage when the faulty line is found, as shown in Figure 7. The top and bottom percentage values in the last two columns show the average and minimum estimation accuracies, respectively, across all network lines. One can observe that by using current and voltage precision of $10^{-2}$ (1%) only two optimal observant sets with 2 PMUs in each set are found by the proposed algorithm. By only improving the voltage precision to $10^{-3}$, the minimum number of PMUs and the optimal observant sets remain the same. However, if the current measurement precision is also improved to $10^{-3}$ (0.1%), one PMU would be enough for the system to be fault observable. On the other hand, by reducing the preferred precision for the fault location to 5%, only one PMU is adequate for observing all system faults.

Tables 2 and 3 present the results of the proposed OPP and ANN locators for IEEE 14-bus and 30-bus systems, respectively. The bus numbers given in [40] are adopted in this paper. It is observed that higher current measurement precision is more effective than that of voltage in reducing the number of required PMUs. Overall, these results show the impact of measurement precision on OPP solutions. In addition, provided results illustrate a significant improvement over the conventional one-bus-spaced method where approximately 50% of buses are required for the system fault observability [16, 18]. For example the number of suggested PMUs for one-bus-spaced method is 17 [18] for the IEEE 30-bus system as opposed to 13 PMUs obtained here based on $10^{-2}$ measurement precision. Moreover, [18] proposes 14 PMUs for the IEEE 30-bus system when considering
### TABLE 2  
IEEE 14-bus OPP and ANN results for various target precisions and different measurements accuracies

| IEEE 14-bus OPP ($R_f$ max 0.1 p.u.) | Optimal observant sets (PMU locations) | ANN | Percentage estimation accuracy |
|--------------------------------------|----------------------------------------|--|------------------------------|
| $T_{PE}^V = 0.01$, $T_{PE}^H = 0.01$, Total generated faults: 22,000 | $10^{-2}$ | $10^{-2}$ | 7 | (2,5,7,9,12,13,14)- (2,5,8,9,12,13,14)- (2,6,7,9,12,13,14)- (2,6,8,9,12,13,14)- (2,7,9,11,12,13,14)- (2,8,9,11,12,13,14) | $99.6$ | $99.9$ | $100$ |
| | $10^{-3}$ | $10^{-2}$ | 3 | (2,6,9)- (2,9,12)- (2,9,13) | $99.5$ | $99.9$ | $99.9$ |
| | | $10^{-3}$ | 1 | $\frac{3}{2}$ | $98.2$ | $99.9$ | $100$ |

| $T_{PE}^V = 0.05$, $T_{PE}^H = 0.05$, Total generated faults: 1200 | $10^{-2}$ | $10^{-2}$ | 2 | (2,6)- (2,12)- (2,13) | $96.7$ | $100$ | $100$ |
| | | $10^{-3}$ | 2 | (1,6)- (2,6)- (2,10)- (2,11)- (2,12)- (2,13)- (2,14)- (5,6)- (5,10)- (5,11)- (5,12)- (5,13)- (5,14) | $96.3$ | $100$ | $100$ |
| | | $10^{-3}$ | 1 | $1-2-3-4-5-5$ | $97.1$ | $100$ | $100$ |

### TABLE 3  
IEEE 30-bus OPP and ANN results for various target precisions and different measurements accuracies

| IEEE 30-bus OPP ($R_f$ max 0.1 p.u.) | Optimal observant sets (PMU locations) | ANN | Percentage estimation accuracy |
|--------------------------------------|----------------------------------------|--|------------------------------|
| $T_{PE}^V = 0.01$, $T_{PE}^H = 0.01$, Total generated faults: 45,000 | $10^{-2}$ | $10^{-2}$ | 13 | (2,4,6,10,12,15,19,22,25,26,27,29,30)- (1,5,6,9,12,15,19,21,25,26,27,29,30)- and 52 more. | $99.1$ | $99.7$ | $99.9$ | $96.8$ | $99.5$ |
| | | $10^{-3}$ | 8 | (1,5,6,12,15,18,22,29)- (2,4,6,12,15,18,22,29)- and 302 more. | $98.9$ | $99.9$ | $100$ |
| | | $10^{-3}$ | 2 | (6,12)- (6,15) | $96.1$ | $97.7$ | $100$ | $83.3$ | $100$ |
| $T_{PE}^V = 0.05$, $T_{PE}^H = 0.05$, Total generated faults: 2460 | $10^{-2}$ | $10^{-2}$ | 4 | (4,9,15,22)- (4,10,12,27)- (4,10,13,27)- (4,10,14,27)- (4,10,15,27)- (4,15,22,27)- (4,15,22,29)- (4,15,22,30) | $97.0$ | $97.7$ | $99.8$ | $83.3$ | $99.8$ |
| | | $10^{-3}$ | 3 | (2,14,24)- (2,14,25)- (2,15,27)- and 30 more. | $92.7$ | $99.6$ | $100$ | $91.7$ | $100$ |
| | | $10^{-3}$ | 1 | $4-6-12-13$ | $88.2$ | $99.4$ | $100$ | $83.3$ | $100$ |
six zero-injection buses (that reduce grid size), and [16] proposes eight PMUs using 15 additional flow measurements to achieve fault observability. By contrast, the proposed algorithm suggests 13 PMUs based on $10^{-2}$ measurement precision and two PMUs based on $10^{-3}$ measurement precision with the desired fault location accuracy of 1% for both fault distance and impedance. Table 4 summarizes the results of references [16] and [18] that employ ILP in the context of one-bus-spaced strategy for full system fault observability. Note that the measurement precision is not considered and elaborated in these works, whereas the precision plays an important role in the number of required measurement units. That is, higher fault location and/or impedance precision need larger number of PMUs.

6 | CONCLUSION

A new algorithm has been introduced for power system optimal PMU location using sensitivity analysis, where the fault location accuracy is specifically taken into account. With the proposed sensitivity analysis, appropriate indices are defined that can be used to qualify the measurements’ locations in the network in detecting fault location and impedance. Also, multi-estimation is introduced and checked in the proposed algorithm to guarantee a unique mapping between a PMU measurement set and all faults of interest in the system. The proposed algorithm finds the minimum number of PMUs required for system fault observability. By using obtained optimal PMU sets, an ANN-based fault locator is developed that map the measurements of the optimal PMU set on the system faults.

ORCID
Hamidreza Nazaripouya https://orcid.org/0000-0001-6555-9997

References
1. Phadke, A.: Synchronized phasor measurements in power systems. IEEE Comput. Appl. Power 6(2), 10–15 (1993)
2. Xu, B., Abur, A.: Observability analysis and measurement placement for systems with PMUs. In: IEEE PES Power Systems Conference and Exposition, 2004, New York, NY, vol. 2, pp. 943–946 (2004). https://doi.org/10.1109/PSCE.2004.1397683
3. Usman, M., Faruque, M.: Applications of synchrophasor technologies in power systems. J. Mod. Power Syst. Clean Energy 7(2), 211–226 (2019)
4. Ali, I., et al.: Performance comparison of IEC 61850-90-5 and IEEE C37.118.2 based wide area PMU communication networks. J. Mod. Power Syst. Clean Energy 4(3), 487–495, 2016
5. Cui, M., et al.: A novel event detection method using PMU data with high precision. IEEE Trans. Power Syst. 34(1), 454–466, 2019
6. Carvalho, M., et al.: Observability of power systems with optimal PMU placement. Comput. Oper. Res. 96), 330–349, 2018
7. Golshan, M.E.H., et al.: Determining minimum number and optimal placement of PMUs for fault observability in one-terminal algorithms. IET Gener. Transm. Distrib. 12(21), 5789–5797 (2018)
8. Chen, H., et al.: Synchrophasor-based real-time state estimation and situational awareness system for power system operation. J. Mod. Power Syst. Clean Energy 4(3), 370–382 (2016)
9. Aminifar, F., et al.: Contingency-constrained PMU placement in power networks. IEEE Trans. Power Syst. 25(1), 516–523 (2010)
10. Wu, F.F., Monticelli, A.: Network observability: Theory. IEEE Trans. Power Appar. Syst. PAS-104(5), 1042–1048 (1985)
11. Ghosh, P.K., et al.: Optimal PMU placement solution: graph theory and MCDM-based approach. IET Gener. Transm. Distrib. 11(13), 3371–3380, 2017
12. Dalal, M., Karegar, H.K.: Optimal PMU placement for full observability of the power network with maximum redundancy using modified binary cuckoo optimisation algorithm. IET Gener. Transm. Distrib. 10(11), 2817–2824 (2016)
13. Babul, R., Bhattacharyya, B.: Optimal allocation of phasor measurement unit for full observability of the connected power network. Int. J. Electr. Power Energy Syst. 79, 89–97, 2016
14. Hassanin, K.M., et al.: Optimal PMU’s placement for full observability of electrical power systems using flower pollination algorithm. In: 2017 IEEE International Conference on Smart Energy Grid Engineering (SEGE), Oshawa, ON, Canada, pp. 20–25 (2017). https://doi.org/10.1109/SEGE.2017.8052770
15. Korres, G.N., et al.: Optimal phasor measurement unit placement for numerical observability in the presence of conventional measurements using semi-definite programming. IET Gener. Transm. Distrib. 9(15), 2427–2436 (2015)
16. Kavasseri, R., Srinivasan, S.: Joint placement of phasor and conventional power flow measurements for fault observability of power systems. IET Gener. Transm. Distrib. 5(10), 1019–1024 (2011)
17. Geramian, S., et al.: Determination of optimal PMU placement for fault location using genetic algorithm. In: 2008 13th International Conference on Harmonics and Quality of Power, Wollongong, NSW, Australia, pp. 1–5 (2008). https://doi.org/10.1109/ICHQP.2008.4668810
18. Pokharel, S., Brahma, S.: Optimal PMU placement for fault location in a power system. In: 41st North American Power Symposium (NAPS), Starkville, MS, USA (2009)
19. Theodorakatos, N.P.: Fault location observability using phasor measurement units in a power network through deterministic and stochastic algorithms. Electr. Power Compon. Syst. 47(5), 212–229 (2019)
20. Makram, E., et al.: An improved model in optimal PMU placement considering sensitivity analysis. In: 2011 IEEE/PES Power Systems Conference and Exposition, Phoenix, AZ, USA, pp. 1–6 (2011). https://doi.org/10.1109/PSCE.2011.5772490

| Test system | Reference [7] | # PMUs | PMU bus locations | Reference [15] | # PMUs | PMU bus locations |
|-------------|---------------|--------|-------------------|---------------|--------|-------------------|
| IEEE 7-bus  | 5             | (1,2,4,5,6) | n/a              |              | n/a    | n/a               |
| IEEE 14-bus | 8             | (2,4,5,8,9,11,12,13) | 8          | (1,2,4,6,8,9,10,13) |
| IEEE 30-bus | 17            | (2,3,6,7,10,11,12,13,15,17,19,22,24,26,27,28,29) | 17          | (2,3,6,7,8,10,11,12,13,15,17,19,22,24,26,27,29) |
APPENDIX

Voltage sensitivity indices: Differentiation of $V_{pref}$ with respect to $D$ and $R_f$ can be performed using (2). Next, derivative of term $\frac{Z_3(h,p)}{Z_0(p,h)+R_f}$ in (1) is discussed. From the transition from impedance matrices $Z_1$ to $Z_2$ [24], one concludes that for any fault, term $Z_2(p,h)$ is the only $D$-dependent variable in $Z_2$ shown as $Z_2(p,h) = Z_1(k,k) + (1-D) × Z_{lk}$. Subsequently, the transition from impedance matrices $Z_2$ to $Z_3$, resulting from the addition of impedance $DZ_{lk}$ between buses $p$ and $l$, leads to

$$Z_3(p,h) = Z_2(p,h) + \left( Z_2(p,h) - Z_2(l,l) \right) \times \left( Z_2(p,p) - Z_2(l,l) \right)$$

and

$$Z_3(p,p) = Z_2(p,p) - \left( Z_2(p,h) - Z_2(l,l) \right) \times \left( Z_2(p,p) - Z_2(l,l) \right).$$

Thus, the derivative of $Z_3(h,p)$ with respect to $D$ is

$$\frac{\partial Z_3(h,p)}{\partial D} = \frac{\partial Z_1(k,k)}{\partial D} × Z_{lk}.$$

Similarly, the derivative of $Z_3(p,p)$ with respect to $D$ is

$$\frac{\partial Z_3(p,p)}{\partial D} = \frac{\partial Z_1(k,k)}{\partial D} × Z_{lk}.$$

Note that derivatives of $Z_3(h,p)$ and $Z_3(p,p)$ with respect to $R_f$ are zero.

Current sensitivity indices: Five elements that are $D$ dependent and one element that is $R_f$ dependent are obtained, for which $\frac{\partial Y_{lk}}{\partial D}$ and $\frac{\partial Y_{lk}}{\partial R_f}$ are calculated as

$$\frac{\partial Y_1(l,l)}{\partial D} = -\frac{\partial Y_1(h,p)}{\partial D} = \frac{Y_{lk}}{D^2},$$

$$\frac{\partial Y_1(k,k)}{\partial D} = -\frac{\partial Y_1(k,p)}{\partial D} = \frac{Y_{lk}}{(1-D)^2},$$

$$\frac{\partial Y_2(p,p)}{\partial D} = \left( \frac{1}{(1-D)^2} - \frac{1}{D^2} \right) Y_{lk},$$

$$\frac{\partial Y_2(p,h)}{\partial R_f} = -\frac{1}{R_f^2}.$$
Using chain rule on (5), current sensitivity indices (4) are obtained as

\[
S_{D_{h,F}} = \left( \frac{\partial Y_{2}(h,u)}{\partial D} \right) \left( \Delta V_{h,F} - \Delta V_{a,F} \right) + \left( \frac{\partial \Delta V_{a,F}}{\partial D} - \frac{\partial \Delta V_{h,F}}{\partial D} \right) Y_{hu}\right)^{-1}
\]

and

\[
S_{R_{F}} = \left( \frac{\partial Y_{2}(h,u)}{\partial R_{F}} \right) \left( \Delta V_{h,F} - \Delta V_{a,F} \right) + \left( \frac{\partial \Delta V_{a,F}}{\partial R_{F}} - \frac{\partial \Delta V_{h,F}}{\partial R_{F}} \right) Y_{hu}\right)^{-1}
\]

It should be mentioned that for cases where fault is on the line whose current is measured, \(S_{D_{h,F}}\) and \(S_{R_{F}}\) are calculated with \(p = N + 1\) due to an additional bus at fault location.

IEEE 14-bus and IEEE 30-bus test cases are shown in Figures 8 and 9, respectively. Line numbers are provided in parenthesis.