Harmonic Complex Networks

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We report the possibility of obtaining complex networks with diverse topology, henceforth called harmonic networks, by taking into account the consonances and dissonances between sound notes as defined by scale temperaments. Temperaments define the intervals between musical notes of scales. In real-world sounds, several additional frequencies (partials) accompany the respective fundamental, influencing the consonance between simultaneous notes. We use a method based on Helmholtz’s consonance approach to quantify the consonances and dissonances, between each of the pairs of notes in a given scale temperament. We adopt two distinct partials structures: (i) harmonic; and (ii) shifted, obtained by taking the harmonic components to a given power \( \beta \), which is henceforth called the anharmonicity index. When these estimated consonances/dissonances are taken along several octaves, respective harmonic complex networks can be obtained, in which nodes and weighted edge represent notes, and consonance/dissonance, respectively. We consider five scale temperaments (i.e., equal, meantone, Werckmeister, just, and Pythagorean). The obtained results can be organized into two major groups, those related to complex networks and musical implications. Regarding the former group, we have that the harmonic networks can provide, for varying values of \( \beta \), a wide range of topologies spanning the space comprised between traditional models. The musical interpretations of the results include the confirmation of the more regular consonance pattern of the equal temperament, obtained at the expense of a wider range of consonances such as that obtained in the meantone temperament. We also have that scales derived for shifted partials tend to exhibit a wide range of consonance/dissonance behavior depending on the adopted temperament and anharmonicity strength.

“Non sai tu che la nostra anima è composta di armonia...”

Leonardo da Vinci

I. INTRODUCTION

Complexity corresponds to one of the most important subjects in science, which is often used to situate not only real-world data, but also several approaches to respective analysis and modeling. Not surprisingly, the particularly influential recent research areas of complex systems and network science are directly related to the above mentioned simple/complex dichotomy.

A great deal of the developments in network science is related to the systematic study of diverse connectivity patterns, as manifested in the topology of the respective networks. One of the simplest possible connectivity pattern consists of rings and lattices, characterized by full degree regularity, and uniformly random networks such as Erdős-Rényi (ER) can be understood as stochastic instances of those perfectly regular structures. At the other extreme of complexity, we have networks characterized by heterogeneous connectivity patterns, such as scale free and modular networks.

Complex network topologies are particularly important because they can substantially influence dynamic processes related to changes in states associated to each network node (e.g., neuronal activation) or on the network topology itself (e.g., resilience to attacks). Network complexity is directly associated to non-homogeneity of one or more of its topological properties, such as node degree or clustering coefficient, among many others. For example, scale free networks are characterized by a wide distribution of node degree facilitating the appearance of hubs.

Several real-world structures — such as protein interactions, airports, the Internet, etc. — are intrinsically heterogeneous, so that many complex network models were developed in order to explain and reproduce such specific real-world structures. Alternatively, complex network models can also be motivated by more abstract approaches, such as the Apollonian circle packings, convex polytopes, and non-Hamiltonian maximal planar graphs, among other models derived from number theory.

Both real-world and theoretically inspired networks typically have parameters that influence their specific properties. For instance, in the case of the ER model, the parameter corresponding to the connecting probability will respectively control the density of connections in the result networks. Frequently, small changes of parameter values tend to induce relatively small modifications in the obtained structures.

The present work focuses on a potentially new type of networks, namely those derived from the concept of consonance and dissonance between two distinct sounds or notes along a given scale and temperament. This type of graphs is henceforth called harmonic complex networks, or harmonic networks for short. More specifically, we consider Helmholtz’s interesting approach to sound consonance, involving the comparison of the frequencies of the partials involved in each of the two sounds, in order to estimate the consonance and disso-
nance between pairs of notes.

Every real-world sound signal includes, in addition to its fundamental frequency, a sequence of partial frequencies which are ultimately responsible why the same note can sound different when played at two distinct types of musical instruments. Helmholtz postulated \cite{14, 15} that the consonance/dissonance between two sounds, as perceived by human beings, would be related to the relative position of the involved partials, with dissonance arising when two simultaneous partials from different nodes are not identical but still are too close for proper discrimination by our aural perception. So, by comparing the several involved partials, it is possible to derive overall indices quantifying the resulting consonance and dissonance between any pair of notes.

Interestingly, while theoretically the sound produced by an ideal string would exhibit partials following the harmonic sequence, this is hardly the case in real-world instruments, where nonlinearities imply shifts from the harmonic relative positions. For instance, the notes produced by a piano can exhibit relatively intense partial shifts, especially at both the frequency extremities (bass and treble) \cite{16}.

In the Western musical tradition, sequences of notes are typically arrange in terms of respective scales, in particular those corresponding to two of the greek modes, namely the major and minor diatonic scales. Even if we focus on the major scale (which we will adopt henceforth, for simplicity’s sake), a large number of variations will still exist, corresponding to the several temperaments that have been developed along centuries as the means to allow modulations and transpositions employed to provide diversity and variation during the musical experience.

The organization of notes according to a particular scale (mode and temperament) gives rise to an immediate important question: which of these notes will sound consonant or dissonant when played together? This issue is of particular relevance because it provides much of the basis for musical composition, in the sense that the consonant/dissonant properties of pairs of notes can be considered for producing specific effects during music composition and performance.

Helmholtz’s approach to consonance provides an interesting means for quantifying, in an objective and automated manner, the pairwise consonance/dissonance of each of the pairs of nodes in a given scale \cite{14, 15}. Now, if we represent each note as a node, and interconnect them according to the consonance/dissonance, harmonic complex networks can be obtained. Figure 1(a) illustrates one such network, corresponding to 9 octaves of the C-major scale under equal temperament, obtained by using the Helmholtz-inspired methodology considered in the current work, while assuming the partials as being perfectly matched to the harmonic sequence. Figure 1(b) also illustrates the same scale in (a), but now in presence of partials shift, which is controlled by the anharmonicity parameter $\beta$.

This simple preliminary example illustrates interesting features exhibited by harmonic networks. First, a remarkably regular pattern of consonance has been obtained in Figure 1(a), with the notes having the same name defining more strongly interconnected groups or clusters. Moreover, as it will be further discussed in this work, many of the obtained consonances/dissonances tend to agree with the traditional harmonic view of interval consonances. Striking results were also obtained regarding the effect of the partials shift, illustrated in Figure 1(b).

More specifically, we have that such shifts can strongly influence the consonance/dissonance connecting patterns, in this specific case implying in making the network much less regular.

Figure 1(c) contains a third harmonic network, obtained in the same way as the network in (b) but considering an almost identical anharmonicity index. More specifically, the network in (b) was obtained with $\beta = 1.0056$, while the structure in (c) used $\beta = 1.0058$. So, interestingly the harmonic networks not only can be complex by themselves, but their properties can change drastically even for minute changes in the controlling parameters.

In the light of the surprising properties of harmonic networks, as illustrated by only a few examples so far, it becomes interesting to investigate further the properties of this potentially new type of networks, stemming from musical motivation. This constitutes the main objective of the present work. More specifically, two main aspects will be approached: (a) to define and characterize harmonic networks from the perspective of network science research; and (b) to discuss musical interpretations and implications of the obtained results.

As it will be shown, harmonic networks are remarkably interesting because of the their intrinsic potential to generate a wide diversity of topologies, especially in the presence of partials shifting controlled by the anharmonicity parameter $\beta$. More specifically, the harmonic networks produced for varying intensities of this parameter cover almost completely the region comprised between more traditional models such as Barabási-Albert, Erdos-Rényi, Watts-Strogatz, stochastic block model and spatial networks. This interesting feature of harmonic networks is related to number theoretical aspects related to the distribution of partials along the frequency domain. We postulate that harmonic networks, especially considering a wider range of temperaments, could be understood as an overall generative model capable of producing a substantially large range of topologies possibly encompassing those produced by more traditional models.

From the musical point of views, we will verify that the obtained results tend to agree substantially with the expected properties of the adopted temperaments. In addition to that, we will also observe that anharmonicity can greatly change the consonance/dissonance patterns induced by the considered temperaments.

This work starts by presenting the adopted methodology, presenting the considered temperaments, explaining
the concept of partial structure, describing the methodology used to estimate consonance and dissonance, explain how harmonic networks can be obtained, briefly describing the main measurements employed in order to characterize the topology of the obtained networks, and outlining the important statistical method known as principal component analysis (PCA) [17], which was used to obtain visualizations of the properties of the considered networks. In the following section, results are presented and discussed regarding two main types of harmonic networks, namely respective to non-shifted and shifted harmonic networks. Conclusions and suggestions for future work complete this article.

II. METHODOLOGY

In this section, we describe the employed concepts and methods, from the types of scales temperaments, to PCA, also explaining the model that converts musical notes into networks and the adopted measurements that have been employed in the analysis of the respective topologies.

A. Types of temperaments

A scales can be defined as a sequence of notes, at subsequent frequency intervals, starting at a specific given node that gives the name to the scale. In this study, we focus on major scales starting at the C1, \( f_1 = 32.7032 \text{Hz} \).

The actual frequencies of a given scale, let’s say C1 major, are specified by the respective temperament. Therefore, the choice of a temperament defines the actual extent of intervals between any pair of notes belonging to a given scale. In this study, we employ five different temperaments including Equal, just, Meantone, Pythagorean, and Werckmeister. In the case of the Equal temperament, the frequency ratios are computed as follows

\[
f_i = 2^{\frac{i-1}{12}},
\]

Table I illustrates the employed frequency ratios for each of the adopted temperaments.

The Pythagorean temperament is one of the earliest approaches to defining the pitch of scales. Its main characteristic is to have all fifth intervals perfectly aligned (and therefore consonant) at the expense of reducing the consonance between other intervals such as the third. The just temperament, which is also very old, tends to have an opposite effect as observed for the Pythagorean counterpart. The meantone temperament applies mostly for keyboard instruments, dating back to the early 1500. Again, major thirds are defined so as to be more consonant, as well as several other intervals, so that a more balanced consonance is achieved along the initial modes (e.g., C major, A minor, G major, etc.). However, transpositions or modulations to further away tones typically lead to considerable dissonances. The Werckmeister temperament was introduced by Andreas Werckmeister in 1691, presenting three alternative systems, being oriented to organ tuning [18, 19]. This temperament preserves the perfect fifths while favoring transposition and modulations. The equal temperament has been considered for a long time in western music, being more systematically formulated around 1600, and consolidated by composers such as J. S. Bach. This system is specially important as it implies that the extent of intervals are preserved at any possible tonality. In particular, the semitone ratio corresponds to \( \sqrt[12]{2} \). This is achieved by increasing the dissonances of some intervals, such as the major third, but now the effects of transposing or modulating to any possible tonality are all perfectly equivalent, favoring these transitions.

Figura 1. Example network of equal temperament, where items (a), (b), and (c) represent variations of \( \beta \) parameter. Interestingly, a small difference of \( \beta \) can lead to a substantial change in the network topology.
| Temperaments | Frequency Ratios |
|--------------|-----------------|
| Equal        | 1.0000, 1.0595, 1.1225, 1.1892, 1.2599, 1.3333, 1.4142, 1.5000, 1.5874, 1.6818, 1.7889, 1.8877 |
| Just         | 1.0000, 1.0449, 1.1180, 1.1963, 1.2500, 1.3375, 1.3975, 1.4953, 1.5625, 1.6719, 1.7889, 1.8692 |
| Meantone     | 1.0000, 1.0535, 1.1250, 1.1852, 1.2528, 1.3333, 1.4047, 1.4949, 1.5802, 1.6704, 1.7778, 1.8792 |
| Pythagorean  | 1.0000, 1.0535, 1.1250, 1.1852, 1.2530, 1.3333, 1.4062, 1.4949, 1.5802, 1.6667, 1.8000, 1.8750 |
| Werckmeister | 1.0000, 1.0535, 1.1174, 1.1852, 1.2528, 1.3333, 1.4047, 1.4949, 1.5802, 1.6667, 1.7778, 1.8792 |

Tabela I. Frequency ratios characteristics of the five considered temperaments. Given a specific initial node defining the scale, the subsequent notes are obtained by multiplying the frequency of that note by the indicated values. Observe that the values in this table are at lower resolution (less digits) that the values used in the reported experiments.

B. Harmonic frequencies

Taking the harmonic sequences in a string, defined as \( l = 1, 1/2, 1/3, 1/4, \ldots \), the respective frequencies are given by \( h_i = 1, 2, 3, 4, \ldots \), where \( f_1 = 1 \) if the fundamental frequency, and the remaining terms are known as harmonic partials. The complete sequence that defines a sound is given by

\[
f_1 h_1, f_1 h_2, f_1 h_3, \ldots, f_1 h_N, \tag{2}
\]

where \( f_i h_i \) represents the harmonic partials. More specifically, the frequencies of the harmonics can be calculated as

\[
f_n = f_1 h_n. \tag{3}
\]

We also consider non-harmonic sounds, more specifically those defined as above, but subjected to an anharmonic transformation implemented by the following power operation: \( h_n \) as

\[
h_n = n^\beta, \tag{4}
\]

where \( \beta \) induces a partial shift, implying a respective deviation from the respective harmonic partial. In the specific case of \( \beta = 1 \), we recover the perfectly harmonic partials. Strictly speaking, an harmonic component can be understood as a partial, but not vice-versa.

In order to find the highest frequency to be taken into account, we considered the maximum limit of audible frequency, which is 20,000Hz. So, the maximum \( n \) was respectively found.

It is also necessary to specify the amplitudes of these partials, and here we adopt an exponential decay, as follows

\[
M_n = \exp (-\alpha h_n), \tag{5}
\]

where \( \alpha \) is a constant that controls the decay.

C. Approaches to consonance and dissonance

Helmholtz believed that if two tones had frequencies with minute differences, these tones could be considered as being consonant \[14, 15\]. Similarly, Helmholtz considered dissonant frequencies being similar, but less than in the consonant cases. Because there are divergent theories regarding consonance and dissonance, some researchers studied approaches that take into account human perception \[15, 20\]. Guthrie and Morrill \[20\] analyzed and compared the human perception of pleasant and consonant sounds. With basis on previous studies, Plomp and Levelt \[15\] also examined people’s perception by using a consonance score. Interestingly, they found a strong relationship between their results and Helmholtz’s theory. More specifically, they showed that specific bandwidths define the relationship between consonant and dissonant sounds.

Other related aspects have also studied, which include the local consonance and the relationship between consonant scales and consonant timbres \[21\]. Furthermore, this author described that the choice of timbres, for a given set of tones, can be understood as a problem of optimization. More recently, Berezovsky \[22\] employed statistical mechanics-based techniques to explore patterns in music. In particular, he proposed a framework that included theoretical formalism, mean-field approximation, and numerical simulations.

D. Harmonic Connectivity

We compute the consonance and dissonance from a given pair of notes, \( X \) and \( Y \). In the case of consonance, we compare each of the \( X \) frequencies with all \( Y \) frequencies, and we consider as consonants the frequencies within the interval \( \Delta_{\text{min}} \). More specifically, the comparisons between a given frequency \( X_i \) and all \( y \) frequencies is given by

\[
c_i = \sum_{j \text{ so that } |X_i - Y_j| < \Delta_{\text{min}}} A(X_i)A(Y_j), \tag{6}
\]

where \( A(\cdot) \) accounts for the amplitude of a given partial. Finally, the overall level of consonance is computed as

\[
C = \sum_{i=1}^{N} c_i, \tag{7}
\]

where \( N \) is the number of harmonics of \( X \). We consider a pair of notes as dissonant when their pairs of partials
are close, but not as close as to be consonant, as follows

\[ d_i = \sum_{j \text{ so that } |X_i - Y_j| \geq \Delta_{\text{Min}} \text{ and } |X_i - Y_j| < \Delta_{\text{Max}}} A(X_i)A(Y_j), \]  

where \( \Delta_{\text{Max}} \) defines the dissonant maximum interval. The overall level of dissonance is computed as

\[ D = \sum_{i=1}^{N} d_i. \]  

Figure 2 illustrates an example of consonant and dissonant calculation. More specifically, a single frequency is within the interval defined by \( \Delta_{\text{Min}} \). Furthermore, another frequency comparison is within \( \Delta_{\text{Max}} \), but is not within \( \Delta_{\text{Min}} \), so it is considered dissonant.

By considering these estimated levels of consonance and dissonance, we created weighted networks. Each node represents a note, while the edges are weighted according to the consonance or dissonance levels between the respective pair of notes. In order to allow the calculation of the many of the existing complex network measurements, we considered unweighted versions of the networks. More specifically, we removed the edges with the lowest weights aiming at having all the networks with the same number of edges. After this edge removal, we considered only the largest connected component. So, the number of nodes and edges can present some small deviations from the adopted reference values.

### E. Complex network measurements

Many distinct measurements have been proposed to grasp the characteristics of complex networks. Here we considered the following measurements: Degree, clustering coefficient, neighborhood (undirected and directed, in and out, versions), accessibility, backbone and merged symmetries (for both versions we considered two, three, and four concentric levels), eigenvector centrality (undirected version), betweenness centrality (directed version), edge betweenness centrality-directed (undirected version), and eccentricity (undirected version). For each of the computed measurement, we estimate the respective average and standard deviations.

In order to characterize the networks obtained in this work, we compare them with some reference, traditional models. Each of these models has specific topological features, as described in the following. The uniformly random model proposed by Erdös-Rényi (ER), which is created from random connections, is defined by a single probability. As a consequence, most degrees result similar, approaching a regular network. The Watts-Strogatz model (WS), proposed as an approach to small-world networks, has small average shortest path distance and high clustering (here we consider the initial network as a 2D toroidal lattice). Barabási–Albert approach (BA) consists of a network with scale-free degree distribution, favoring the existence of hubs. As a geographical model, we employ the Random geometric graph (GEO) (we set the initial positions of the nodes as a 2D lattice). The Stochastic Block Model (SBM) is a model incorporating communities. We configured this model to present two communities with the same size. In order to obtain more reliable statistics, we considered 100 network samples for each of the above models.

### F. Principal component analysis

Principal component analysis (PCA) is an orthogonal transformation of the matrix of data \( X \), in which each line consists of a vector of features of a given sample. More specifically, the covariance matrix \( K \) of the data \( X \) is initially computed. From this matrix, the respective eigenvectors and eigenvalues are calculated. The projected axes are described by the eigenvectors that are ordered in decreasing order according to their respective eigenvalues, which describe the percentage of variance explanation provided by each subsequent axis. In other words, this transformation represents a rotation, in which the transformed axes are ordered in decreasing order according to the data dispersion (see Figure 3). One of the advantages of using PCA is that the projection completely decorrelates the involved random variables. By allowing the dimensionality of the data to be reduced, it can provide valuable visualizations of the data dispersion.
in lower dimensions.

In order to better understand the characteristics of the proposed network representations, we compare their topologies with some well-known models. First, we measure many different network features from the proposed networks and the network models. By considering only the models, we compute the PCA projections, which allow both types of networks to be visualized as 2D scatterplots.

III. RESULTS AND DISCUSSION

In this section, we present the obtained results and the respective discussion. The first part addresses the non-shifted partials ($\beta = 1$), followed by the analysis of shifted partials. In both cases, the obtained harmonic networks are compared with some well-known traditional models.

A. Non-Shifted Partials

First, we discuss the results obtained for the harmonic networks without shifting the respective partials, so that all partials can be understood as being harmonics. These networks are better appreciated when compared to more traditional complex network models, such as BA, ER, GEO, SBM, and WS.

In order to perform such a comparison, we create 100 network samples for each of the models described in Section IV-E. The parameters of these models were chosen so that the average degrees resulted similarly to those of the considered harmonic networks. More specifically, $\langle k \rangle = 10.37$, which was chosen for the Equal temperament network through visual inspection. We calculated the network measurements (described in Section IV-E) for several instances of the considered traditional models, and then obtained the respective PCA projection. For all considered temperaments, both respective consonance and dissonance harmonic networks were projected using the same PCA matrix, yielding the overall map shown in Figure 4.

As far as the consonance harmonic networks are concerned, we have that they resulted near the GEO, WS1 and WS2, with the Werckmeister network falling within the GEO cluster.

Figure 5 illustrates the harmonic networks obtained for each of the considered temperaments. The networks defined by the equal temperament (Figure 5(a)) exhibits a well-organized, symmetric topology that is a direct reflex of the constant frequency intervals adopted in this type of scale. However, the consonance is mostly concentrated between notes of the same name and intervals between $C$ and $G$ (specific fifth), with relatively minor consonances between most of the other intervals. The meantone network (Figure 5(b)) presents a more widespread and ample pattern of consonances, with respect to several intervals, which is in agreement with its original musical motivation (aimed at achieving more consonances at the expense of transposition and modulation). Both the Werckmeister and Pythagorean networks (Figure 5(c) and (e)) present a circular pattern of consonances, but respective to different intervals. The harmonic network obtained for the just temperament exhibits a remarkably symmetric topological organization, suggesting a ‘crystalline’ structure (Figure 5(d)). However, the consonance is again found between notes of the same type.

As shown in Figure 5, the patterns of predominant dissonances varies substantially among the considered temperaments, but a more dissonant core can be observed in all cases. As expected, the more symmetric, and also widespread, structure can be observed in the equal temperament (Figure 5(f)). In the other 4 cases, the dissonances are less widespread among the notes outside the central core.

B. Shifted Partials

Now we turn our attention to harmonic networks obtained in presence of shifted partials, i.e. presenting an anharmonicity level determined by the parameter $\beta$. Figure 6 illustrates the PCA projections, respective to each temperament, obtained for the consonance shifted-partial together with the more traditional complex networks considered in this work. The color bar indicates the values of $\beta$, which controls the shifting effect.

Remarkably, in all cases (i.e. temperaments), the harmonic networks tended to span the space comprised within the considered traditional complex networks models.

In the case of the equal temperament (Figure 6(a)), the obtained harmonic networks approach the BA model in the PCA space as $\beta$ increases from 0.80 to 1.20.
The effect of $\beta$ in defining progression of these harmonic networks is evident in Figure 4(a). This effect is substantially less prominent in the other temperaments, with the networks exhibiting substantial topological changes as $\beta$ is progressively increased. The just, Pythagorean and Werckmeister temperaments yielded the harmonic networks closest to the middle of the space defined by the more traditional complex network models. The meantone temperament generated harmonic networks overlapping the ER model, therefore suggesting more uniform topological properties.

The dissonance harmonic networks obtained for each of the considered temperaments, together with the more traditional complex network models, are shown in Figure 7.

As verified for the consonant harmonic networks, their dissonance counterparts also tended to populated the space defined between the more traditional models in the PCA space. The equal temperament networks exhibited the widest dispersion in this space (Figure 7(a)), with the other 4 temperaments yielding more compact harmonic network distributions. Interestingly, the progression of the harmonic networks in the PCA space exhibited an opposite behavior as observed for the consonance cases. More specifically, we have that the harmonic networks underwent a relatively smooth shifting in the PCA space in terms of the parameter $\beta$, which was not observed for the equal temperament harmonic networks.

One of the most interesting results discussed so far regards the impressive ability of the partials-shifted harmonic networks to produce a wide range of topological organizations, all comprised between the considered more traditional complex network models. As such, harmonic networks provide an intrinsic flexibility for generating several topologies, therefore allowing a flexible method for network construction that is completely independent of the musical relationships and implications of the reported harmonic networks. We also postulate that the suggested methodology to obtain harmonic networks, especially when considering other types of temperaments not necessarily limited by musical concerns as well unharmonicity transformations other than taking the power of the partials, has the potential to act as a kind of universal generator of networks. In other words, it is possible that harmonic networks underlie a general principle of complex network topology.

In order to better illustrate the topological diversity obtainable from partials-shifted harmonic networks, we provided the visualization of several networks, which is shown in Figure 8. Recall that each network maps into a point in the PCA space. It is easy to verify from this figure that the harmonic networks near to the GEO models tended to present a more regular and symmetric topology that, nevertheless, vary substantially even with small increases of $\beta$. Interestingly, some obtained networks (such as Figure 8(q)) even presented modular organization (two communities). Spiral organization can also be noticed in some cases (e.g., Figure 8(f) and (p)). As the values of $\beta$ increases, the respective networks tend to present hubs and compact cores such as that in Figure 8(i).

Figure 9 shows the visualizations of several consonance partials-shifted harmonic networks obtained for the just temperament, superimposed on the respective PCA diagram. Though the obtained networks have substantial topological diversity, reflected in the respective dispersion in the PCA space, the intrinsic topological differences are more difficult to be appreciated visually. Similar visualizations were obtained for the Werckmeister, Pythagorean and Meantone temperaments.
(a) Equal (consonance)  (b) Meantone (consonance)  (c) Werckmeister (consonance)  
(d) Just (consonance)  (e) Pythagorean (consonance)  
(f) Equal (dissonance)  (g) Meantone (dissonance)  (h) Werckmeister (dissonance)  
(i) Just (dissonance)  (j) Pythagorean (dissonance)

Figura 5. Visualization of consonance and dissonance networks we created from non-shifted partials, for all of the considered temperaments.
IV. CONCLUSION

Complex networks have achieved great importance and popularity mainly thanks to their ability to represent a wide range of heterogeneous connectivity patterns found in the real world. In this work, we suggest a procedure to obtain complex networks as induced by consonances and dissonances determined by specific scale modes and temperaments. The consonances and dissonances were estimated by employing a methodology moti-
Figura 7. PCA projection of the dissonance networks (with shifted partials) for all of the considered temperaments (squares). The circles represent the network models.

Voted on Helmholtz’s approach, reflecting the alignment between neighboring partials in two given sounds. Complex networks were obtained by taking into account the obtained levels of consonance and dissonance with respect to five important traditional temperaments, while assuming all scales as beginning at C1 and extending through five octaves. Focus was kept on the major mode. Partial contents considering or not frequency shifting were considered, leading to quite diverse features.

Several results were obtained. From the musical pers-
Figura 8. Example of consonance networks for some selected parameters of shifted partials and the respective positions in PCA. Here, we considered the equal temperament. In this projection, PC1 and PC2 represent 48.61% and 23.83% of the data variance, respectively.

perspective, we have that the obtained harmonic networks reflected many of the expected characteristics, with the equal temperament yielding particularly regular topological organization. An interesting result regards the fact that shifted partials can exhibit a remarkable diversity of consonance/dissonance relationships as a consequence of even small variations of $\beta$. This implies, among other things, that the consonance/dissonance of intervals can vary substantially in musical instruments presenting diverse levels and types of anharmonicity.

Particularly remarkable results have been obtained regarding network science aspects of the obtained harmonic networks. It has been shown that shifted partial scales can present an impressive diversity of topological properties that tend to span the region comprised between traditional models such as scale free, uniform random, geographical, and modular. Harmonic networks derived from equal temperament resulted particularly interesting as they undergo gradual changes of topology as the anharmonicity parameter $\beta$ is changed. These results suggest that the proposed harmonic network could correspond to a universal way to generate networks with diverse topology, especially when considering other temperaments and anharmonicity transformations. This would be related to the almost infinite possibilities of partial alignments implied by the number theoretical relationships between the obtained partial sequences. It would be interesting to investigate this possibility further. Also
Figura 9. Example of consonance networks for some selected parameters of shifted partials and the respective positions in PCA. Here, we considered the just temperament. In this projection, PC1 and PC2 represent 48.61% and 23.83% of the data variance, respectively.

It would be interesting to extend the characterization and comparison of topological properties of the considered networks by taking into account all the obtained topological measurements instead of PCA projections.

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