The Finite-Difference Analysis and Time Flow

Ashot Yu. Shahverdian
(Yerevan Physics Institute)

1. Introduction

The present paper introduces a method of analysis of one dimensional systems. Its application to study of Poincare recurrence time flow is considered. The approach suggested consists of reducing the research of a given system’s orbits $X = (x_i)_{i=1}^{\infty}$ to analysis of alternations of the monotone increase and decrease of higher order absolute finite differences, taken from $X$. Through some special representation of finite orbits, we associate with $X$ some numerical sequence, is called the conjugate orbit. We formulate two conditions, when the study of $X$ is reduced to analysis of the special orbits’ asymptotical intersections with some base set from numerical interval $(0, 1)$ of zero Lebesgue measure. Hence, the approach can be characterized as an “asymptotical” analogy of Poincare’s classical ”section” method.

The method distinguishes two cases: the continuous, when some numerical quantities $\rho$ can take arbitrary values from interval $(0, 1)$, and the discrete case, when all of them belong to a finite set. The continuous was applied in Ref. 1 to study of neural activity. In the present work, we are more interested in the discrete case, which has an additional specific. Thus, as soon as we have stated the method’s applicability, the mechanism generating time series $X$, is replaced by the return map $R$, generating the conjugate series. While the actual exact description of this mechanism may remain unknown (e.g., for earthquake time series or neuron spike trains), the function $R$ does not depend on it and has a simple analytic shape.

This approach is applicable to the study of various time series, arising in modern applied science (e.g., medicine, astronomy). The innovation consists of a new way to measure the fast oscillating time series – this measure is the Hausdorff dimension of some thin spaces $\mathcal{A}$, to which the conjugate series are attracted. As an example of such an application, the computational analysis of the sequences of fractional parts is presented. We show that the results obtained testify on the possibility of Cantorian structure of our usual time flow. In this context, we recall some well known (qualitative) conclusions on irregular and intricate time flow, as indicated in Bergson’s philosophical
treatments \[2\] and in some works on quantum physics \[3, 4\].

2. Description of the Method

2.1 Finite differences and conjugate orbits

We deal with one dimensional, deterministic or stochastic systems, generating numerical sequences $X = (x_i)_{i=1}^{\infty}$ $(0 \leq x_i \leq 1)$. We impose no restrictions on the system, and it may also possess different inner states changing with time. Below, we set two conditions on a given orbit $X$, under which it should be regarded as chaotic. We introduce the notion of conjugate (with $X$) orbit $\bar{\nu}$, in terms of which the irregular nature of $X$ can be established and studied.

Let us first give some special representation of finite orbit $X_k = (x_i)_{i=1}^{k}$, which reflects its higher order differential structure. For the sequence $X_k$ and number $1 \leq s \leq k - 1$ we let

$$
\Delta_i^{(0)} = x_i, \quad \Delta_i^{(s)} = |\Delta_{i+1}^{(s-1)} - \Delta_i^{(s-1)}| \quad (i = 1, 2, \ldots, k - s).
$$

It is not difficult to obtain

$$
\Delta_i^{(s-1)} = \mu_{k,s-1} + \sum_{p=1}^{i-1} (-1)^{\delta_p^{(s)}} \Delta_p^{(s)} - \min_{0 \leq i \leq k-s} \left( \sum_{p=1}^{i} (-1)^{\delta_p^{(s)}} \Delta_p^{(s)} \right) \quad (1)
$$

where

$$
\delta_p^{(s)} = \begin{cases} 
0 & \Delta_{p+1}^{(s)} \geq \Delta_p^{(s)} \\
1 & \Delta_{p+1}^{(s)} < \Delta_p^{(s)} 
\end{cases} \quad \text{(it is supposed that } \sum_{1}^{0} = 0) ,
$$

and

$$
\mu_{k,s} = \min\{\Delta_i^{(s)} : 1 \leq i \leq k - s\} .
$$

From Eq. (1) it is easy to see, that if for a given $m$, $0 \leq m \leq k$ all of the finite binary sequences

$$
\delta_1^{(s)}, \delta_2^{(s)}, \ldots, \delta_{k-s}^{(s)} \quad (s = 0, 1, \ldots, m)
$$

as well as all of the numbers

$$
\mu_{k,1}, \mu_{k,2}, \ldots, \mu_{k,m} \quad \text{and} \quad \Delta_1^{(m)}, \Delta_2^{(m)}, \ldots, \Delta_{k-m}^{(m)}
$$





are known, one can completely restore the initial finite orbit $X_k = (x_i)_{i=1}^k$. Hence, it follows from (1), we can consider that finite orbits with length $k$ are given namely in the next special representation

$$\bar{\zeta}_k = (r_1^{(k)}, r_2^{(k)}, \ldots, r_m^{(k)}; \mu_{k,1}, \mu_{k,2}, \ldots, \mu_{k,m}; \rho_1, \rho_2, \ldots, \rho_{k-m})$$  \hspace{1cm} (2)

where

$$r_s^{(k)} = 0.\delta_1^{(s)}\delta_2^{(s)}\ldots\delta_{k-s}^{(s)} \quad (1 \leq s \leq m)$$  \hspace{1cm} (3)

are the rational numbers, and

$$\mu_{k,1}, \mu_{k,2}, \ldots, \mu_{k,m} \quad \text{and} \quad \rho_1, \rho_2, \ldots, \rho_{k-m} \quad (\text{where} \quad \rho_i = \Delta_i^{(m)})$$

are some numbers from interval $[0, 1]$. Here $m = m_k$ and $m_k$ tend to $\infty$ as $k \to \infty$. It is easy to see, that after applying the recurrent procedure (1) the sequence $X_k$ is completely recovered by $\bar{\zeta}_k$. Now, we can introduce the basic tool of the method – the notion of conjugate orbit: we say, that numerical sequence $\bar{\nu} = (\nu_s)_{s=1}^\infty$ is the conjugate orbit, associated with the given orbit $\bar{X} = (x_i)_{i=1}^\infty$, if for each $s \geq 1$ the terms $\nu_s$ are defined as follows

$$\nu_s = \lim_{k \to \infty} r_s^{(k)} \quad (= 0.\delta_1^{(s)}\delta_2^{(s)}\delta_3^{(s)}\ldots)$$

were $r_s^{(k)}$ are the $r$-coordinates from (2).

Let us introduce a thin set $\mathcal{B}$ and its subsets $\mathcal{B}_k$, are the base sets, mentioned in Sec. 1. We consider the numbers $0 < x < 1$ represented in the form of binary expansion,

$$x = 0.\delta_1\delta_2\delta_3\ldots \quad (= \sum_{k=1}^{\infty} 2^{-k}\delta_k, \ \delta_k = 0, 1) \ ;$$  \hspace{1cm} (4)

further, we also operate with the corresponding binary sequences $\bar{x}$,

$$\bar{x} = (\delta_1, \delta_2, \delta_3, \ldots, \delta_n, \ldots) .$$

For a given natural $k \geq 2$ we define the subsets $\mathcal{B}_k$ of numerical interval $(0, 1): \mathcal{B}_k$ is the set all of those real numbers $x \in (0, 1)$ for each of which

$$n_{i+1} - n_i \leq k \quad (i = 1, 2, 3, \ldots)$$

where $n_i$ denote all the consecutive positions where the changes of binary symbol from (4) occur, $\delta_{n_{i+1}} = 1 - \delta_{n_i}$. Let also $\mathcal{B}$ be the union of all $\mathcal{B}_k$
\( (k \geq 2) \). All of the sets \( B_k, \ 2 \leq k \leq \infty \), have zero Lebesgue measure (see Ref. 1) (hereafter, in order to reduce some formulations, we use also the notation \( B_\infty \ (\equiv B) \)).

Let us now suppose, that the orbit \( \bar{X} \) is such, that each of the sequences \( \bar{X}_s = (\delta^{(s)}_1, \delta^{(s)}_2, \ldots, \delta^{(s)}_n, \ldots) \ (s = 0, 1, \ldots) \) (5)

where \( s \) is fixed, has bounded lengths of series with the same binary symbol. In other words, we assume that for increasing sequence of indeces \( n_{i+1}^{(s)} \), designating all of those positions in natural series, where the changes of binary symbol occur, \( \delta^{(s)}_{n_{i+1}} = 1 - \delta^{(s)}_{n_{i}} \), we have

\[
\begin{align*}
n_{i+1}^{(s)} - n_{i}^{(s)} & \leq K_s < \infty \quad (i = 1, 2, \ldots) .
\end{align*}
\] (6)

Hence, taking both numbers \( k \) and \( k - m \) large enough (for determity it can be chosen \( m \) is equal to entire part of \( k/2, m = [k/2] \)), we can consider each of the sequences (5) as the binary expansion of some number \( \nu_s \in B_K \),

\[
\nu_s = 0.\delta^{(s)}_1 \delta^{(s)}_2 \ldots \delta^{(s)}_n \ldots
\]

or, in other words, each of the sequences \( \bar{r}^{(k)}_s \) of the form (3) converges (as \( k \to \infty \)) to a number \( \nu_s \) from \( B_K \). Here, \( 2 \leq K \leq \infty \), and according to (6) it should be taken \( K = \sup \{K_s : s \geq 1\} \).

**Continuous Case.** Now, we assume also, that the quantities

\[
\mu_k = \sum_{i=1}^{m_k} \mu_{k,i}^2 + \sum_{i=1}^{k-m_k} \rho_i^2
\]

converge to zero,

\[
\lim_{k \to \infty} \mu_k = 0 .
\] (7)

In such a way, the limitations we impose on the time series \( \bar{X} \) are the following:

\( (C_1) \): for every \( k \geq 1 \) the sequence \( \bar{X}^k \in B \)

\( (C_2) \): the quantities \( \rho_{i,k} \) and \( \mu_{i,k} \) are such, that: \( \mu_k = o(1) \) as \( k \to \infty \).
Then, one can see [1], these two restrictions imply that the transformed sequence $\tilde{\zeta}_k$ from (2) tends to space $B^\infty (= B \times B \times \ldots)$,
\[ ||\tilde{\zeta}_k - B^\infty|| = o(1) \quad (k \to \infty) \quad (8) \]
where $||.||$ is the usual metric in Euclidean spaces $R^k$, $||x|| = (\sum_{i=1}^{k} x_i^2)^{1/2}$. Particularly, (8) implies that for every $s$-th ($1 \leq s \leq m$) coordinate of $\tilde{\zeta}_k$ we have
\[ \lim_{k \to \infty} \tilde{\zeta}_k^{(s)} = \lim_{k \to \infty} r_s^{(k)} = \nu_s \in B \]
- namely this relation was meant when we mentioned the "asymptotical" intersections (of transformed into a special form orbits) with a thin space.

**Discrete Case.** In this case we deal with one dimensional time series with upper bounded terms and with restricted measurement accuracy, known in advance. If it is some number of the form $10^{-m}$, $m \geq 1$, then multiplying all of the terms of our series by this quantity, we will obtain some time series $\tilde{X}$, consisting of natural numbers bounded by a pregiven $N$,
\[ \tilde{X} = (n_1, n_2, n_3, \ldots, n_k, \ldots), \quad n_k \in [0, N] \quad (k = 1, 2, 3, \ldots) \quad (9) \]
In other words, we deal with the random sequences [3]; particularly, such kind of sequences arise as a result of the realizations of hazard games. However, we consider only a special class of sequences (9) – as above, we impose two limitations (below $d \geq 1$ is some natural number; we may assume $d = 1$) on the differential structure of the series $\tilde{X}$:

\[ (D_1): \quad \text{for every } k \geq 0 \text{ the sequence } \tilde{X}^k \in B_K \]

\[ (D_2): \quad \text{for all large enough } k \geq 1 \text{ and all } i \geq 1 \text{ the quantities } \rho_i^k \in \{0, d\}. \]

It should be noticed, that condition $(D_1)$ (and $(C_1)$) implies the dimensionality limitations (see next section), while the $(D_2)$, an analogy of $(C_2)$, is a requirement on certain regularity of the process, generating $\tilde{X}$.

The restriction $(D_2)$ means, that we consider such processes (9), which are reduced to binary processes $\tilde{X}$ (with components $x_i = 0, 1$). Let us call the binary sequence $\tilde{X}$, satisfying the conditions $(D_1)$ and $(D_2)$ as $\beta_K$-sequence and denote $B_K$ the collection all of the numbers $x \in (0, 1)$ for
which its binary expansion $\bar{x}$ is $\beta_K$-sequence. From the definition, we have the next characteristic of these sets; below $x_n = \Delta^{(n)}(x)$ and $\Delta$ is the shift transformation (or the "tent map", see Ref. 6):

\[
\Delta(x) = \begin{cases} 
2x & 0 < x < 1/2 \\
2x - 1 & 1/2 \leq x < 1 
\end{cases}
\]

**Proposition 1** The number $x \in \hat{B}_\infty$ iff for some numbers $0 < K_p < \infty$ the relation

\[
|x_n - \Delta^{(p)}(x_n)| \geq 2^{-K_p}
\]  \hspace{1cm} (10)

holds for every $n \geq 0$ and $p \geq 1$. The number $x \in \hat{B}_K$, $2 \leq K < \infty$, iff the inequality (10) holds with $K_p \equiv K$ and for every $n \geq 0$ and $p \geq 1$.

According to Poincare recurrence theorem (see, e.g., Ref. 6), the relation

\[
\lim \inf_{p \to \infty} |x - \Delta^{(p)}(x)| = 0
\]

holds for each $0 < x < 1$ except some set of zero Lebesgue measure. In the case $K_p \equiv K$ the condition (10) means recurrence break (by analogy with "ergodicity break" from Ref. 7) for all points $x_n$. Hence, all the sets $\hat{B}_k$ are found within exceptional set, mentioned in just quoted Poincare theorem. Note also certain resemblance of the relation (10) with the basic inequalities in Li - Yorke known theorems [8].

### 2.2 Conjugate attractor and return map

Let us now denote $\mathcal{A}$ the closure of conjugate orbit $(\nu_s)_{s=1}^{\infty}$, i.e. $\mathcal{A} \subset [0,1]$ is the union of the set $\{\nu_s : s \geq 1\}$ with the collection of all its cluster points. Then this orbit should be considered as a chaotic one, whenever $\mathcal{A}$ fills in either an interval or a Cantor set. We note, that for the most actual systems the constants from Eq. (6) for all $s \geq 1$ are upper bounded by the same number $K \geq 2$ and, consequently,

\[
\mathcal{A} \subset \mathcal{B}_K \text{ and } \dim(\mathcal{A}) \leq \dim(\mathcal{B}_K).
\]  \hspace{1cm} (11)

Hence, just mentioned criterion of chaosity simply means that $\mathcal{A}$ is a Cantor set. If $\mathcal{A}$ appears to be the same set for almost each orbit $\bar{X}$, then we can conclude that $\mathcal{A}$ is the attractor of the system. The numerical values of
Hausdorff dimension of the sets $B_k$ ($2 \leq k \leq \infty$) can be obtained from the formula \[ z_k(2^{\dim(B_k)}) = 1 \quad \text{where} \quad z_k(s) = \sum_{n=1}^{k} s^{-n} \quad (s > 0) \]

- hence, (11) implies an upper estimate of Hausdorff dimension of attractor $A$.

If we consider the numbers $x$ from unit interval are given in the form (4), then it is not difficult to obtain the next statement (below, $a \oplus b = |a - b|$ is the logical sum of the binary symbols $a, b \in \{0, 1\}$):

**Proposition 2** If $\bar{X}$ is a binary sequence, then the return map $R : \nu_k \to \nu_{k+1}$ of the conjugate orbit $\bar{\nu}$ has the form

$$ R(x) = \sum_{n=1}^{\infty} 2^{-n}(\epsilon_n \oplus \epsilon_{n+1}) $$

where $\epsilon_n$ are the coefficients of binary expansion of number $x$.

One can find (in the operator form) the function $R$ in Sinai’s early constructions on weak isomorphism of dynamical systems [10, 11]. For more details on this function see Ref. 9, where a fractal dynamical system is introduced.

3. Sequences of Fractional Parts and Time Flow

Some works in quantum mechanics [3] introduce the concept "Cantorian Space-Time". The computational study of the flow of our usual (but in some sense discretized) time, implemented in this section, confirms again the possibility of such a concept. We are based on the method introduced – an approach completely different from those considered in other works on time flow [4, 12, 13].

Let us start from the statement and computational analysis of the following number-theoretical problem. Let us have two numbers $0 < \alpha, \lambda < 1$, $\alpha$ is irrational and let

$$ \{\alpha\}, \{2\alpha\}, \{3\alpha\}, \ldots, \{n\alpha\}, \ldots \quad (12) $$

be the sequence of the fractional parts, generated by $\alpha$ (hereafter $\{x\}$ denotes the fractional part of number $x$). We are interested in the behavior of the
time series

\[ \bar{X} = (n_k)_{k=1}^{\infty} \quad \text{where} \quad n_k = m_{k+1} - m_k \quad (k \geq 1) ; \]

here \( m_k \) are all of the natural numbers, arranged in increasing order, \( m_{k+1} > m_k \), such that

\[ \{\alpha m_k\} \in (0, \lambda). \quad (13) \]

According to H. Weyl theorem \([10, 14]\) the sequence (12) is uniformly distributed in the interval \((0, 1)\), so that we have infinite number of \( m_k \), satisfying (13). If we take into attention the ergodicity of the transformation \( \pi \),

\[ \pi(x) = \{x + \alpha\}, \quad \pi : (0, 1) \to (0, 1), \quad (14) \]

this statement can also be deduced (for a.e. irrational \( \alpha \)) from Poincare’s recurrence theorem. From numerous theoretical results on the sequences of fractional parts (see details in Ref. 14), we note the following Slater-Florecc theorem: for given irrational \( \alpha \) there exist two natural numbers \( p \) and \( q \), such that for all \( k \geq 1 \) we have \( n_k \in \{p, q, p + q\} \). We may also note a formula, obtained in Ref. 15 – it expresses the exact value of the total number of indeces \( m_k \), satisfying (13) and not exceeding a given \( N \), through the coefficients of some expansions of the numbers \( N \) and \( \lambda \).

Our basic conclusion in the problem stated, deduced from computational experiments, can be formulated in the next form: for each irrational \( 0 < \alpha < 1 \) there exist the numbers \( 0 < \lambda < 1 \) such that the conjugate to \( \bar{X} \) orbit \( \bar{\nu} \) is asymptotically close to a Cantor set; and conversely, for each \( 0 < \lambda < 1 \) there exist irrational numbers \( 0 < \alpha < 1 \), such that the conjugate to \( \bar{X} \) orbit \( \bar{\nu} \) is asymptotically close to a Cantor set.

Some of these computational results, on the Fig. 2 are presented. Here, the numerical value \( \lambda = 0.13 \) and several values of parameter \( \alpha \) are considered. In dependence of control parameter \( \alpha \) the conjugate orbits \( \bar{\nu} \) demonstrate all possible types of motion: the periodic, when they are attracted to some finite set, Cantorian, when the attractor \( A \) has Cantorian self-similar structure, and the completely chaotic motion, when the orbits fill in an interval from \((0, 1)\). For instance, for \( \alpha = 0.1250002 \) the Cantorian attractor in Fig. 2.b is shown. The graph Fig. 2.a has been obtained for the value \( \alpha = 0.124999 \); in this case the orbit \( \bar{\nu} \) consists of 13 straight lines (some of them are not shown). In some respect, the behavior (in dependence of control parameter \( \alpha \)) of conjugate orbits in this problem reminds Feigenbaum’s
transitions (see e.g., Ref. 6), and this needs further detailed computational study.

To end, let us consider the following interpretation of the above introduced number-theoretical problem; in particular, this will explain the title of present section. Let us have in a two dimensional plane a circle \( E \) and a point \( S \), situated outside of disc, bounded by \( E \). Let \( \omega \) be the angle, under which the circle \( E \) from the point \( S \) is seen, and let also \( L \) be an arc of \( E \) with the length \(|L| = \lambda < \omega \) with the center at the point of intersection of circle \( E \) with the rectilinear segment, connecting its center with \( S \). Let us assume that the point \( M \in E \) is rotated in discrete time \( T = \{1, 2, 3, \ldots \} \) with the frequency \( \alpha \). Then, the consecutive moments of falling the point \( M \) in the arc \( L \) constitute a sequence of natural numbers \( m_k \in T \), coinciding with the sequence \( \bar{X} \) from Eq. (13). Also, it is not difficult to see, that the sequence of fractional parts (12) coincides with the sequence of iterates of the function \( \pi(x) \) from Eq. (14), i.e.

\[
\{\alpha n\} = \pi^{(n)}(\alpha) \quad (n \geq 1)
\]

and, consequently, the numbers \( m_k \) represent the Poincare recurrence (of point \( M \) into the arc \( L \)) time. Now, let us interpret \( E \) as "Earth" and \( S \) as "Sun". Then, replacing the continuous rotation of Earth by discrete motion, we obtain that \( m_k \) appears to be an approximate of the flow of our usual time. In such a way, the above described computations demonstrate, that in principle, it is possible that our time possesses the Cantorian structure. The numerical values of the quantities \( \alpha, \lambda, \) and \( \omega \) can be specified for this "astronomical" case, but here we do not dwell on this point.

4. Some Remarks and Discussions

Let us discuss two problems, relating to the brain activity. In the paper Ref. 1, through the method above, applied to experimental data of actual neurons, the Cantorian structure of brain activity has been deduced. The method permits further development [9] by means of introducing some dynamical system \( \mathcal{F} = (\mathcal{A}, T, \mu) \) on the Cantor spaces \( \mathcal{A} \). The evolution operator \( T \) is easily expressed through the return map \( \mathcal{R} \), while the attractor \( \mathcal{A} \) and invariant (singular) measure \( \mu \) remain the main quantities, distinguishing different processes. In practical applications, the measure \( \mu \) can be calculated through frequency analysis, accepted in mathematical-linguistic works (see,
e.g. Ref. 5). Hence, we can operate with the theoretical-information characteristics of system \( \mathcal{F} \). Whether there exists an analogy of known Billingsley-Eggleston formula \[10\]

\[
Dimension = Entropy,
\]

relating to Eggleston thin spaces, for the fractal dynamical systems \( \mathcal{F} \)? Note also Mandelbrot’s work \[16\], who considered the applications of this formula to problems of turbulence.

We know \[17\], that there exists some mechanism of time perception in the brain. Its activity, it follows from our previous results \[1\], must be found in some thin (multidimensional) Cantor space. How to relate the measurable spaces \( \mathcal{F}_t \) and \( \mathcal{F}_p \), which correspond to time flow and time perception, with each other? Particularly, can we compare the values of \( \dim(\mathcal{A}_t) \) and \( \dim(\mathcal{A}_p) \)?

It seems, these problems can be resolved after detailed computational frequency analysis, in order to compute the invariant measures \( \mu \) (and hence another characteristics, say, entropy) of both time flow and brain activity.

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Figure Captions

Figure 1. The graph of inverse function $\mathcal{M}(x) = \mathcal{R}^{-1}(x)$. The graph is constructed through 15,000 points, computed by means of random numbers generator. From definition of the function $\mathcal{R}$ it is easy to see that $\mathcal{M}$ is a multivalent function, defined on the interval $(0, 1)$. The graph of the return map $\mathcal{R}$ can be obtained as a result of symmetric reflection of this ”fractal letter $M$” in respect of diagonal $(0, 0), (1, 1)$.

Figure 2. Two different types of conjugate orbit’s behavior in dependence of control parameter $\alpha$. The graph a) demonstrates the case when the attractor $\mathcal{A}$ consists of the finite set, while in the graph b) these orbits are attracted to some Cantor set. The graph c) shows the self-similar structure of the Cantor set $\mathcal{A}$ from the previous graph.
