BRST symmetry and $W$-algebra in higher derivative models

Rabin Banerjee $^{a,b}$, Biswajit Paul$^{a,c}$, Sudhaker Upadhyay $^{a,d}$

$^a$S. N. Bose National Centre for Basic Sciences, JD Block, Sector III, Salt Lake City, Kolkata -700 098, India

$rabin@bose.res.in$

$^{c}$bisu_1729@bose.res.in

d$sudhakerupadhyay@gmail.com$

Abstract

In this paper we discuss the (anti-)BRST symmetries and $W$-algebra of higher derivative theories of relativistic particles satisfying general gauge conditions. Using this formalism, the connection between the (anti-)BRST symmetries and $W$-algebra for the massless particle with rigidity is established. Incidentally, the full $W$-algebra emerges only when the anti-BRST transformations are considered in tandem with the BRST ones. Further, the BRST symmetry is made finite and coordinate-dependent. We show that such finite coordinate-dependent BRST symmetry changes the BRST invariant gauge-fixing fermion within the functional integration. This is exploited to connect two different arbitrary gauge conditions.

1 Introduction

It is usual to consider theories where the Lagrangian has only single time derivative of the fields. But in some cases we need to consider terms where higher time derivative of the fields appear. Such theories are known as higher derivative (HD) theories. The concept of introduction of HD field is not new and has been considered by many authors and applied in diverse fields like electrodynamics [1,2], supersymmetry [3,4], noncommutativive theory [5,6], cosmology [7,8], extended Maxwell-Chern-Simon theory [9,10], theory of anyons [11,12], relativistic particle with torsion [13], membrane model of the electron [15,16] etc. In gravity theories HD terms were added to ensure renormalizability [17]. There are various models of gravity where HD corrections are added to the Einstein-Hilbert action [18,21]. HD terms also frequently appear in the context of string theory [22,23]. The importance of HD terms, therefore, cannot be overemphasised. A useful and interesting HD model to be considered is the relativistic particle model with curvature. In this case the curvature term, which is higher derivative in nature, is added to the action of the usual massive relativistic particle. This model was introduced long ago by Pisarski [24] and still continues to be under active consideration [25,31]. Interestingly, the model has only one gauge symmetry identified as diffeomorphism symmetry although there are two independent primary first class constraint present in the theory [31], which is unusual. The presence of the extra primary first class constraint was successfully explained as an effect of the higher derivative nature. The massless version, known as massless particle model with rigidity, is shown to describe bosons and fermions [28]. This model has two gauge symmetries viz. diffeomorphism and $W$-symmetry alongwith two primary first class constraints [29,31]. One can also add a torsion term to the relativistic particle model and the theory emerges with very interesting results. When quantised, the relativistic particle model with torsion leads to Majorana equations [14]. The massive sector
contains infinite number of states when quantised in the Minkowski space and finite number of states in euclidean space [11]. Higher derivative models also shows Majorana equations [13] with the 2+1 dimensional analogous models lead to anyons. Another interesting line where the higher derivative models appear corresponds to finite-gap (algebra-geometric) systems [32]. There, the higher derivative models are given by the so called Novikov equation, with a space variable playing a role of the evolution parameter.

On the other hand, BRST symmetry is a very powerful tool to quantize a theory with gauge invariance which also helps in the proof of the renormalizability and unitarity of gauge theories [33–36]. This transformation, which is characterized by an infinitesimal, global and anticommuting parameter leaves the effective action as well as path integral of the effective theory invariant. In gauge field theories the usual BRST symmetry has been generalized to make it finite and field-dependent [37]. This finite field dependent BRST (FFBRST) transformations have found many applications in various contexts [38–45].

The implementation of BRST symmetries for HD theories is quite nontrivial and poses problems. In this context, therefore, a natural question arises regarding the application of BRST formalism to relativistic particle models. Indeed it is not surprising that in spite of a considerable volume of research on relativistic particle models, this aspect remains unstudied. A basic motivation of this paper is to bridge this gap.

In this paper, we consider a relativistic particle model with curvature as a HD theory possessing a gauge symmetry. The constraint analysis of this model and its massless analogue is discussed. Further, the gauge symmetry transformations in the case of massive relativistic particle model with curvature are identified with diffeomorphism invariance. However, for the massless particle model with rigidity it corresponds to W-symmetry in addition to diffeomorphism invariance. We construct the BRST symmetry and anti-BRST symmetry for these particle models. The difficulties of applying BRST transformations to HD theories are bypassed by working in the first order formalism developed in [26,28,31] instead of the conventional Ostrogradski approach [46]. It is shown that the (anti-)BRST symmetry transformations for all variables reproduce the diffeomorphism symmetry of the massive relativistic particle model including curvature. Furthermore, we also show that the massless particle model with rigidity yields both the diffeomorphism and W-invariances. We explicitly demonstrate the $W_3$-algebra. For BRST transformations this algebra is shown for all variables, excluding the anti-ghost. Exactly the same features are revealed, but now excluding the ghost variable instead of the anti-ghost, for anti-BRST transformations. To get the complete picture, therefore, both BRST and anti-BRST transformations have to be considered. Further, we implement the concept of FFBRST transformation [37] in the quantum mechanical relativistic particle model. The quantum mechanical version of FFBRST transformation [37] is called as finite coordinate-dependent BRST (FCBRST) transformation. We see that FCBRST transformation for the relativistic particle model is a symmetry of the action only, but not of the generating functional. Analogous to FFBRST the FCBRST transformation changes the Jacobian of path integral measure non-trivially. For an appropriate choice of finite coordinate dependent parameter FCBRST connects two different gauge-fixed action within functional integration.

The plan of the paper is as follows. In sections 2 and 3 we introduce the various relativistic particle models and discuss their gauge symmetries. The nilpotent BRST and anti-BRST transformation with emergence of $W_3$-algebra is demonstrated in section 4. In section 5, we construct the FCBRST transformation. Two arbitrary gauges are connected with the help of this FCBRST within a path integral formalism in section 6. We draw concluding remarks in the last section.
2 Massive relativistic particle model with curvature

The massive relativistic point particle theory with curvature has the action

\[ S = -m \int \sqrt{\dot{x}^2} d\tau + \alpha \int \frac{\left( (\dot{x}\ddot{x})^2 - \dot{x}^2 \dot{y}^2 \right)^{\frac{1}{2}}}{\dot{x}^2} d\tau. \]  

(1)

The model is meaningful for \( \alpha < 0 \) and \( \dot{x}^2 > 0 \). The energy spectrum for this model turns out to be the energy spectrum of the Majorana equation [26, 27]. By direct substitution we may verify that (1) is invariant under reparametrisation,

\[ \tau \rightarrow \tau + \Lambda(\tau), \]  

(2)

where \( \Lambda \) is an infinitesimal reparametrisation parameter. Under this reparametrisation \( x^\mu \) transforms as,

\[ \delta x^\mu = x^\mu(\tau - \Lambda) - x^\mu(\tau) = -\Lambda \dot{x}^\mu. \]  

(3)

Since this is a higher derivative model, the usual Hamiltonian formalism does not apply. There is, however, the well established Ostrogradski method where the momenta are defined in some non-trivial way [46]. Other than this, a first order formalism exists in the literature where the time derivatives of the coordinates are considered as independent variables to convert the theory into a first order one. There are different variants [26, 28, 31] of this formalism and we adopt the one that was developed by two of us in a collaborative work [31]. To convert the theory into a first order one we introduce the new coordinates

\[ q_1^\mu = x^\mu \quad q_2^\mu = \dot{x}^\mu. \]  

(4)

The Lagrangian in these coordinates has a first order form given by

\[ L = -m \sqrt{q_2^\mu} + \alpha \frac{\left( (q_2^\mu q_2^\mu) - q_2^2 q_2^\mu \right)^{\frac{1}{2}}}{q_2^\mu} + q_0^\mu (q_1^\mu - q_2^\mu), \]  

(5)

where \( q_0^\mu \) are the Lagrange multipliers that enforce the constraints

\[ q_1^\mu - q_2^\mu = 0. \]  

(6)

Let \( p_{0\mu}, p_{1\mu} \) and \( p_{2\mu} \) be the canonical momenta conjugate to \( q_{0\mu}, q_{1\mu} \) and \( q_{2\mu} \) respectively,

\[ p_{0\mu} = \frac{\partial L}{\partial \dot{q}_{0\mu}} = 0, \]

\[ p_{1\mu} = q_{0\mu}, \]

\[ p_{2\mu} = \frac{\alpha ((q_2^\mu q_2^\mu)q_2^\mu - q_2^2 q_2^\mu)}{q_2^\mu \sqrt{(q_2^\mu q_2^\mu)^2 - q_2^2 q_2^\mu}}. \]  

(7)

The primary constraints thus obtained are listed below [27,31],

\[ \Phi_{0\mu} = p_{0\mu} \approx 0, \]

\[ \Phi_{1\mu} = p_{1\mu} - q_{0\mu} \approx 0, \]

\[ \Phi_1 = p_{2\mu} q_2 \approx 0, \]

\[ \Phi_2 = p_{2\mu} q_2 + \alpha^2 \approx 0. \]  

(8)

\(^1\)contractions are abbreviated as \( A^\mu B_\mu = AB, A^\mu A_\mu = A^2 \). We consider the model in 3 + 1 dimensions. So \( \mu \) assumes the values 0, 1, 2, 3 [26,27,31].
Conservation of the constraints (8) yield the following secondary constraints,

\[
\omega_1 = q_0 q_2 + m \sqrt{q_2^2} \approx 0, \\
\omega_2 = q_0 p_2 \approx 0.
\]

We define \( \Phi'_2 \) as a combination of constraints,

\[
\Phi'_2 = \left( q_0^2 - m^2 \right) \Phi_2 - 2 p_2^2 (q_0 q_2) \omega_1.
\]

\( \Phi_1 \) and \( \Phi'_2 \) form the first class constraint set which are primary in nature. The second class constraints \( \Phi_0 \), \( \Phi_1 \), \( \omega_1 \) and \( \omega_2 \) are eliminated by replacing all the Poisson brackets by Dirac brackets. The nonzero Dirac brackets between the phase space variables are listed below

\[
\begin{align*}
\{ q_\mu, q_\nu \}_D &= \frac{1}{p_1^2 - m^2} \left( p_{\mu 2} q_{\nu 2} - q_{\mu 2} p_{\nu 2} \right), \\
\{ q_\mu, p_\nu \}_D &= \frac{q_{\mu 2} p_{\nu 2}}{p_1^2 - m^2}, \\
\{ p_\mu, q_\nu \}_D &= \frac{1}{p_1^2 - m^2} \left( m p_{\mu 2} q_{\nu 2} + p_{\mu 2} p_{\nu 2} \right), \\
\{ q_\mu, q_\nu \}_D &= \eta_{\mu \nu}, \\
\{ q_{\mu 2}, p_{\nu 2} \}_D &= \eta_{\mu \nu} - \frac{1}{p_1^2 - m^2} \left( p_{\mu 2} p_{\nu 2} + \frac{m}{\sqrt{q_2^2}} p_{\mu 2} q_{\nu 2} \right).
\end{align*}
\]

Now the generator \( G \) of the gauge symmetry is given by a combination of the first class constraints,

\[
G = \epsilon^1 \Phi_1 + \epsilon^2 (p_1^2 - m^2) \Phi_2.
\]

However, we have shown in [31] that there is only one independent gauge parameter which is a consequence of the higher derivative theory. The generator is then given by,

\[
G = \frac{q_{\mu 2}}{q_2^2} \frac{d}{d\tau} \left( 2 m p_2^2 \sqrt{q_2^2} \epsilon^2 q_{2}^\mu \right) \Phi_1 + \epsilon^2 (p_1^2 - m^2) \Phi_2.
\]

It may be noted that in the expression of the gauge generator there appears time derivative of the fields as well gauge parameter \( \epsilon^2 \). The apparent problem of taking gauge variation of the derivative of the phase space variables does not arise since they are multiplied by constraints which are set to zero after computing the brackets. Now the gauge transformation of the variables are given by

\[
\delta q_1^\mu = \{ q_1^\mu, G \}_D = \left( 2 \epsilon^2 p_2^2 m \sqrt{q_2^2} \right) q_2^\mu,
\]

and

\[
\delta q_2^\mu = \{ q_2^\mu, G \}_D = \left[ \frac{q_{\nu 2}}{q_2^2} \frac{d}{d\tau} \left( 2 m p_2^2 \sqrt{q_2^2} \epsilon^2 q_{2}^\nu \right) \right] q_2^\mu + \left[ 2 \epsilon^2 q_2^2 (p_1^2 - m^2) \right] p_2^\mu.
\]

Dirac brackets are denoted by \( \{,\}_D \). We consider \( p_1^2 - m^2 \neq 0 \), else it is a singular case. Explicit constraint structure and Dirac brackets of the singular case can be found in [31].
In terms of the reparametrization parameter, \( \Lambda = -2\epsilon^2 p_2^2 m \sqrt{q_2^2} \), the transformation for \( q_1^\mu \) may be expressed as,

\[ \delta q_1^\mu = -\Lambda q_2^\mu. \] (16)

Using the identification (16) the relation (16) reproduces the reparametrisation symmetry (3). Thus the gauge symmetry gets identified with the reparametrisation symmetry.

3 The model of massless particle with rigidity

The massless version of the model (1) is known as the model of massless particle with rigidity. The massless version is not obtained simply by putting \( m = 0 \) in (1) due to reasons of internal consistency [28]. This requires a modification in the curvature term and the model is given by,

\[ S = \alpha \int \frac{\dot{x}^2 \dddot{x}^2 - (\ddot{x} \dddot{x})^2}{\dot{x}^2} d\tau. \] (17)

The action of this model thus is proportional to the curvature and also its classical equation of motion is compatible only for super-relativistic motion of the particle. The importance of the model lies in the fact that it corresponds to massless modes of either integer or half-integer helicity states. The model finds its relevance when \( \dot{x}^2 < 0 \) [28]. Once again we adopt the first order formalism developed in [31]. We introduce the new coordinates

\[ q_1^\mu = x^\mu, \quad q_2^\mu = \dot{x}^\mu. \] (18)

The Lagrangian in these coordinates has a first order form given by

\[ L = \alpha \frac{\left(q_2^2 \ddot{q}_2^2 - (q_2 \dot{q}_2)^2\right)^{1/2}}{q_2^2} + q_0^\mu (\dot{q}_1^\mu - q_2^\mu), \] (19)

where \( q_0^\mu \) are the Lagrange multipliers that enforce the same constraints mentioned in equation (6).

Let \( p_{0\mu}, p_{1\mu} \) and \( p_{2\mu} \) be the canonical momenta conjugate to \( q_{0\mu}, q_{1\mu} \) and \( q_{2\mu} \) respectively, having same expressions as that of (7). Consequently, we obtain the following primary constraints

\[ \Phi_{0\mu} = p_{0\mu} \approx 0, \]
\[ \Phi_{1\mu} = p_{1\mu} - q_{0\mu} \approx 0, \]
\[ \Phi_1 = p_{2\mu} q_2 \approx 0, \]
\[ \Phi_2 = p_{2\mu} q_2^2 - \alpha^2 \approx 0. \] (20)

The secondary set of constraints obtained by time conserving the primary ones are,

\[ \omega_1 = q_0 q_2 \approx 0, \]
\[ \omega_2 = q_0 p_2 \approx 0. \] (21)

Finally, by conserving \( \omega_2 \) the tertiary constraint is obtained as

\[ \omega_3 = q_0^2 \approx 0. \] (22)
This completes the chain of constraints. In the above constraint structure the first class set is \{Φ_1, Φ_2, ω_1, ω_2, ω_3\} and all others are second class in nature. Once again we remove all second class constraints as described in the previous section. Fortunately the Dirac bracket comes out to be same as the Poisson brackets between the phase space variables. After removing the nondynamical variables \(q_0μ\) and \(p_0μ\) by solving the second class constraints \(Φ_0μ\) and \(Φ_1μ\), the final set of first class constraints are

\[
\begin{align*}
Ω_1 &= Φ_1 = p_2 q_2 ≈ 0, \\
Ω_2 &= Φ_1 = p_2^2 q_2^2 − α^2 ≈ 0, \\
Ω_3 &= ω_1 = p_1 q_2 ≈ 0, \\
Ω_4 &= ω_2 = p_1 p_2 ≈ 0, \\
Ω_5 &= ω_3 = p_3^2 ≈ 0.
\end{align*}
\]

Note that the original condition \(δ^2 < 0\) translates into \(q_2^2 < 0\). This ensures the reality of \(α\), as may be easily seen from the constraint \(Ω_2 ≈ 0\). The reality of \(α\) is connected to the helicity states of the particles as discussed [28].

As done previously, the gauge generator is written as a combination of all the first class constraints,

\[
G = \sum_{α=1}^{5} ε^α Ω_α.
\]

However, due to the presence of secondary first-class constraints, the parameters of gauge transformation (\(ε^α\)) are not independent [35][17]. It is found that only two out of the five gauge parameters are independent. For our convenience we take \(ε^3\) and \(ε^5\) as independent.

So, the expression for the gauge generator becomes [31],

\[
G = \left(ε^3 + \frac{q_2 q_2^2 ε^3 - α \sqrt{g} \frac{q_2}{q_2^2} ε^5}{q_2^2}\right) Ω_1 + \frac{1}{2 q_2^2} \left(ε^5 - \frac{q_2 q_2^2 ε^5 + α \sqrt{g}}{p_2 q_2^2} ε^3\right) Ω_2 \\
+ ε^3 Ω_3 + ε^5 Ω_4 + ε^5 Ω_5,
\]

which contains only two independent parameters (here \(g = q_2^2 q_2^2 - (q_2 q_2^2)^2\)).

We now calculate the gauge variations of the dynamical variables, defined as \(δq = \{q, G\}_D\). These are given by,

\[
\begin{align*}
δq_1^μ &= ε^3 q_2^μ + ε^5 p_2^μ + 2 ε^5 p_1^μ, \\
δq_2^μ &= \left(ε^3 + \frac{q_2 q_2^2 ε^3 - α \sqrt{g}}{q_2^2} ε^5\right) q_2^μ + \left(ε^5 - \frac{q_2 q_2^2 ε^5 + α \sqrt{g}}{p_2 q_2^2} ε^3\right) p_2^μ + ε^5 p_1^μ, \\
δp_1^μ &= 0, \\
δp_2^μ &= - \left(ε^3 + \frac{q_2 q_2^2 ε^3 - α \sqrt{g}}{q_2^2} ε^5\right) p_2^μ \\
&- \frac{p_2^2}{q_2^2} \left(ε^5 - \frac{q_2 q_2^2 ε^5 + α \sqrt{g}}{p_2 q_2^2} ε^3\right) q_2^μ - ε^3 p_1^μ.
\end{align*}
\]

The above transformations can be identified as diffeomorphism (\(D\)) and \(W\)-symmetry by putting \(ε^5 = 0\) and \(ε^3 = 0\) respectively, as done in [31]. Detailed calculations on all the phase-space variables
show that \[29–31\]
\[
\begin{align*}
\left[ \delta^D_{\epsilon^1}, \delta^D_{\epsilon^2} \right] &= \delta^D_{\epsilon^3}; \quad \text{with} \quad \epsilon^3 = \epsilon^1 \epsilon^2 - \epsilon^2 \epsilon^1 \\
\left[ \delta^D_{\epsilon^4}, \delta^W_{\epsilon^5} \right] &= \delta^W_{\epsilon^5}; \quad \text{with} \quad \epsilon^5 = -\epsilon^3 \epsilon^5 \\
\left[ \delta^W_{\epsilon^1}, \delta^W_{\epsilon^2} \right] &= \delta^W_{\epsilon^5}; \quad \text{with} \quad \epsilon^5 = \frac{p^2}{q_2^2} (\epsilon^5 \epsilon^5_1 - \epsilon^5 \epsilon^5_1).
\end{align*}
\]

This reproduces the usual \(W_3\)-algebra.

4 \((\text{Anti-})\)BRST symmetries and \(W_3\)-algebra

In this section we construct the nilpotent BRST and anti-BRST symmetries for the theory. For this purpose we need to fix a gauge before the quantization of the theory as the theory is gauge invariant and therefore has some redundant degrees of freedom. The general gauge condition in this case is chosen as:
\[
F_1[f(q)] = 0, \quad (27)
\]
where \(f(q)\) is a general function of all the generic variables \(q\). Some explicit examples of gauge conditions corresponding to relativistic particle models are \[28\].
\[
q^0_1 - \tau = 0, \quad q^0_2 - 1 = 0, \quad p^0_2 = 0, \quad q^2_2 = 0. \quad (28)
\]
The general gauge condition \[27\] can be incorporated at a quantum level by adding the appropriate gauge-fixing term to classical action.

The linearised gauge-fixing term using Nakanishi-Lautrup auxiliary variable \(B(q)\) is given by
\[
S_{gf} = \int d\tau \left[ \frac{1}{2} B^2 + BF_1[f(q)] \right]. \quad (29)
\]
To complete the effective theory we need a further Faddeev-Popov ghost term in the action. The ghost term in this case is constructed as
\[
S_{gh} = \int d\tau \left[ \bar{c} s F_1[f(q)] \right],
\]
\[
= -\int d\tau \left[ c \bar{s} F_1[f(q)] \right], \quad (30)
\]
where \(c\) and \(\bar{c}\) are ghost and anti-ghost variables. Now the effective action can be written as
\[
S_{\text{eff}} = S + S_{gf} + S_{gh}. \quad (31)
\]
The source free generating functional for this theory is defined as
\[
Z[0] = \int Dq \ e^{iS_{\text{eff}}}, \quad (32)
\]
where \(Dq\) is the path integral measure. The nilpotent BRST symmetry of the effective action in the case of relativistic particle model with curvature is defined by replacing the infinitesimal reparametrisation parameter \(\Lambda\) to ghost variable \(c\) in the gauge transformation given in equation \[16\] as
\[
\begin{align*}
\delta^D q^\mu_1 &= -c q^\mu_2, \quad \delta^D q^\mu_2 = -\dot{c} q^\mu_2 - c \dot{q}^\mu_2, \\
\delta^D c &= 0, \quad \delta^D B = B, \quad \delta^D B = 0, \quad (33)
\end{align*}
\]
where \( c, \bar{c} \) and \( B \) are ghost, anti-ghost and auxiliary variables respectively for relativistic particle model with curvature. This BRST transformation, corresponding to gauge symmetry identified with the diffeomorphism invariance, leaves both the effective action as well as generating functional, invariant. Similarly, we construct the anti-BRST symmetry transformation, where the roles of ghost and anti-ghosts are interchanged with some coefficients, as

\[
\begin{align*}
\bar{s}^D q_1^\mu &= -c q_2^\mu, \\
\bar{s}^D q_2^\mu &= -\dot{c} q_2^\mu - \dot{\bar{c}} \bar{q}_2^\mu, \\
\bar{s} D \bar{c} &= 0, \\n\bar{s} D c &= -B, \\n\bar{s} D B &= 0.
\end{align*}
\]  

These transformations are nilpotent and absolutely anticommuting in nature i.e.

\[
(s^D)^2 = 0, \quad (\bar{s}^D)^2 = 0, \quad s^D s^D + \bar{s}^D s^D = 0.
\]  

The above (anti-)BRST transformations are valid for both the models. On the other hand, the nilpotent BRST and anti-BRST symmetry transformations, identified with \( W \)-symmetry (with \( \epsilon^3 = 0 \)) in (26), for relativistic massless particle model with rigidity only, are constructed as

\[
\begin{align*}
s^W q_1^\mu &= \eta p_2^\mu + 2\eta p_1^\mu, \\
s^W q_2^\mu &= -\frac{\alpha \sqrt{g}}{q_2} \bar{q} q_2^\mu + \left( \bar{\eta} - \frac{q_2}{q_2^2} \dot{\bar{\eta}} \right) p_2^\mu + \dot{\eta} p_1^\mu, \\
s^W p_1^\mu &= 0, \quad s^W p_2^\mu = \frac{\alpha \sqrt{g}}{q_2} \bar{q} q_2^\mu - \frac{p_2^\mu}{q_2^2} \left( \bar{\eta} - \frac{q_2}{q_2^2} \dot{\bar{\eta}} \right) q_2^\mu, \\
s^W \eta &= 0, \quad s^W \bar{\eta} = B, \quad s^W B = 0,
\end{align*}
\]  

and

\[
\begin{align*}
\bar{s}^W q_1^\mu &= \dot{\bar{\eta}} q_2^\mu + 2\dot{\bar{\eta}} p_1^\mu, \\
\bar{s}^W q_2^\mu &= -\frac{\alpha \sqrt{g}}{q_2} \bar{q} q_2^\mu + \left( \bar{\eta} - \frac{q_2}{q_2^2} \dot{\bar{\eta}} \right) p_2^\mu + \dot{\bar{\eta}} p_1^\mu, \\
\bar{s}^W p_1^\mu &= 0, \quad \bar{s}^W p_2^\mu = \frac{\alpha \sqrt{g}}{q_2} \bar{q} q_2^\mu - \frac{p_2^\mu}{q_2^2} \left( \bar{\eta} - \frac{q_2}{q_2^2} \dot{\bar{\eta}} \right) q_2^\mu, \\
\bar{s}^W \bar{\eta} &= 0, \quad \bar{s}^W \eta = -B, \quad \bar{s}^W B = 0,
\end{align*}
\]  

where \( \eta, \bar{\eta} \) and \( B \) are ghost, anti-ghost and auxiliary variables, respectively, for relativistic massless particle model with rigidity.

Here we observe interestingly that the BRST symmetry transformations of all variables (excluding the anti-ghost variable) given in equations (33) and (36) also satisfy the \( W_3 \)-algebra as

\[
\begin{align*}
\left[ s^D_{c_1}, s^D_{c_2} \right] &= s^D_{c_3}, & \text{with} \quad c_3 &= c_2 \dot{c}_1 - \dot{c}_2 c_1, \\
\left[ s^D_c, s^W_\eta \right] &= s^W_{\bar{\eta}'}; & \text{with} \quad \eta' &= \dot{\bar{\eta}} c, \\
\left[ s^W_{\bar{\eta}}, s^W_{\eta} \right] &= s^W_{\eta_3}; & \text{with} \quad \eta_3 &= \frac{p_2^2}{q_2} (\eta_2 \bar{\eta}_1 - \eta_1 \dot{\bar{\eta}}),
\end{align*}
\]

and the anti-BRST symmetry transformations of all variables (excluding ghost variable) given in equations (34) and (37) also satisfy the \( W_3 \)-algebra as

\[
\begin{align*}
\left[ s^D_{\bar{c}_1}, s^D_{\bar{c}_2} \right] &= s^D_{\bar{c}_3}; & \text{with} \quad \bar{c}_3 &= \bar{c}_2 \dot{\bar{c}}_1 - \dot{\bar{c}}_2 \bar{c}_1.
\end{align*}
\]
\[
\begin{align*}
[s_c, s_\eta] &= s_\eta^\prime; \quad \text{with} \quad \eta' = \hat{\eta}c \\
[s_\eta^W, s_\eta] &= s_\eta^W; \quad \text{with} \quad \eta_3 = \frac{p^2}{q^2} (\eta_2 \hat{\eta}_1 - \eta_1 \hat{\eta}_2).
\end{align*}
\]

This completes our analysis of the connection between the (anti-)BRST symmetries and \(W_3\)-algebra.

5 FCBRST formulation for higher derivative theory

In this section we investigate the finite coordinate-dependent BRST (FCBRST) formulation for general higher derivative theory. To do so, we first define the infinitesimal BRST symmetry transformation with Grassmannian constant parameter \(\delta \rho\) as

\[
\delta_b q = s q \delta \rho,
\]

where \(sq\) is the BRST variation of generic variables \(q\) in the HD theories. The properties of the usual BRST transformation in equation (38) do not depend on whether the parameter \(\delta \rho\) is (i) finite or infinitesimal, (ii) variable-dependent or not, as long as it is anticommuting and global in nature. These observations give us a freedom to generalize the BRST transformation by making the parameter \(\delta \rho\) finite and coordinate-dependent without affecting its properties. We call such generalized BRST transformation in quantum mechanical systems as FCBRST transformation. In the field theory such generalization is known as FFBRST transformation [37]. Here we adopt a similar technique to generalize the BRST transformation in quantum mechanical systems as FCBRST transformation. In particular, we start by making the infinitesimal parameter coordinate-dependent with introduction of an arbitrary parameter \(\kappa\) \((0 \leq \kappa \leq 1)\). We allow the generalized coordinates, \(q(\kappa)\), to depend on \(\kappa\) in such a way that \(q(\kappa = 0) = q\) and \(q(\kappa = 1) = q'\), the transformed coordinate.

The usual infinitesimal BRST transformation, thus can be written generically as

\[
dq(\kappa) = s[q] \Theta'[q(\kappa)] d\kappa,
\]

where the \(\Theta'[q(\kappa)] d\kappa\) is the infinitesimal but coordinate-dependent parameter. The FCBRST transformation with the finite coordinate-dependent parameter then can be constructed by integrating such infinitesimal transformation from \(\kappa = 0\) to \(\kappa = 1\), to obtain [37]

\[
q' \equiv q(\kappa = 1) = q(\kappa = 0) + s(q) \Theta[q],
\]

where

\[
\Theta[q] = \int_0^1 d\kappa' \Theta'[q(\kappa')],
\]

is the finite coordinate-dependent parameter.

Such a generalized BRST transformation with finite coordinate-dependent parameter is the symmetry of the effective action in equation (31). However, the path integral measure in equation (32) is not invariant under such transformation as the BRST parameter is finite in nature. The Jacobian of the path integral measure for such transformations is then evaluated for some particular choices of the finite coordinate-dependent parameter, \(\Theta[q(x)]\), as

\[
Dq' = J(\kappa) Dq(\kappa).
\]
The Jacobian, $J(\kappa)$ can be replaced (within the functional integral) as

$$J(\kappa) \to \exp[iS_1[q(\kappa)]]$$

iff the following condition is satisfied [37]

$$\int Dq \left[ \frac{1}{J} \frac{dJ}{d\kappa} - i \frac{dS_1[q(x,\kappa)]}{d\kappa} \right] \exp [i(S_{\text{eff}} + S_1)] = 0,$$

where $S_1[q]$ is local functional of variables such that at $\kappa = 0$ it must vanish.

The infinitesimal change in the $J(\kappa)$ is written as [37],

$$\frac{1}{J} \frac{dJ}{d\kappa} = -i \int d\tau \left[ \pm s\kappa \frac{\partial \Theta'[q(\kappa)]}{\partial q} \right],$$

where $\pm$ sign refers to whether $q$ is a bosonic or a fermionic variable.

Thus, the FCBRST transformation with appropriate $\Theta$, changes the effective action $S_{\text{eff}}$ to a new effective action $S_{\text{eff}} + S_1(\kappa = 1)$ within the functional integration.

6 Connecting different gauges in relativistic particle models

Here we will exploit the general FCBRST formulation developed in the previous section to connect the path integral of relativistic particle models with different gauge conditions. The FCBRST transformations ($f_b$) for the relativistic particle model with curvature are constructed as follows:

$$f_b q_1^\mu = - cq_1^\mu \Theta[q], \quad f_b q_2^\mu = (-\dot{c} q_2^\mu - c \ddot{q}_2^\mu) \Theta[q], \quad f_b c = 0, \quad f_b \bar{c} = B \Theta[q], \quad f_b B = 0,$$

where $\Theta[q]$ is an arbitrary finite coordinate-dependent parameter. Now, we show how two different gauges (say $F_1(q) = 0$ and $F_2(q) = 0$) in the relativistic particle model may be connected by such transformations. For this purpose, let us choose the following infinitesimal coordinate dependent parameter (through equation (41))

$$\Theta'[q] = -i \int d\tau \bar{c}(F_1 - F_2).$$

Let us first calculate the infinitesimal change in the Jacobian $J(\kappa)$ for above $\Theta'[q]$ using the relation (45) as

$$\frac{1}{J} \frac{dJ}{d\kappa} = i \int d\tau [-B(F_1 - F_2) + s(F_1 - F_2) \bar{c}],$$

$$= -i \int d\tau [B(F_1 - F_2) + \bar{c} s(F_1 - F_2)].$$

To express the Jacobian as $e^{iS_1}$ [37], we take the ansatz,

$$S_1[\kappa] = \int d\tau [\zeta_1(\kappa) BF_1 + \zeta_2(\kappa) BF_2 + \zeta_3(\kappa) \bar{c} sF_1 + \zeta_4(\kappa) \bar{c} sF_2],$$

where $\zeta_i(\kappa)(i = 1, \ldots, 4)$ are constant parameters satisfying the boundary conditions

$$\zeta_i(\kappa = 0) = 0.$$
To satisfy the crucial condition (44), we calculate the infinitesimal change in $S_1$ with respect to $\kappa$ using the relation (39) as

\[
\frac{dS_1[q, \kappa]}{d\kappa} = \int d\tau [\zeta_1' BF_1 + \zeta_2' BF_2 + \zeta_3' c sF_1 + \zeta_4' \bar{c} sF_2 + (\zeta_1 - \zeta_2) B(sF_1) \Theta' + (\zeta_2 - \zeta_4) B(sF_2) \Theta'],
\]

(51)

where prime denotes the differentiation with respect to $\kappa$. Exploiting equations (48) and (51), the condition (44) simplifies to,

\[
\int Dq \left[ (\zeta_1' + 1) BF_1 + (\zeta_2 - 1) BF_2 + (\zeta_3' + 1) c sF_1 + (\zeta_4' - 1) \bar{c} sF_2 + (\zeta_1 - \zeta_3) B(sF_1) \Theta' + (\zeta_2 - \zeta_4) B(sF_2) \Theta' \right] e^{i(S_{eff} + S_1)} = 0.
\]

(52)

The comparison of coefficients from the terms of the above equation gives the following constraints on the parameters $\zeta_i$

\[
\begin{align*}
\zeta_1' + 1 &= 0, \quad \zeta_2' - 1 = 0, \quad \zeta_3' + 1 = 0, \quad \zeta_4' - 1 = 0, \\
\zeta_1 - \zeta_3 &= 0, \quad \zeta_2 - \zeta_4 = 0.
\end{align*}
\]

(53)

The solutions of the above equations satisfying the boundary conditions (50) are

\[
\begin{align*}
\zeta_1 &= -\kappa, \quad \zeta_2 = \kappa, \quad \zeta_3 = -\kappa, \quad \zeta_4 = \kappa.
\end{align*}
\]

(54)

With these values of $\zeta_i$ the expression of $S_1[\kappa]$ given in equation (49) becomes

\[
S_1[\kappa] = \int d\tau [-\kappa BF_1 + \kappa BF_2 - \kappa c sF_1 + \kappa \bar{c} sF_2],
\]

(55)

which vanishes at $\kappa = 0$. Now, by adding $S_1(\kappa = 1)$ to the effective action ($S_{eff}$) given in equation (31) we get

\[
S_{eff} + S_1(\kappa = 1) = S + \int d\tau \left[ \frac{1}{2} B^2 + BF_2[f(q)] + \bar{c}sF_2[f(q)] \right],
\]

(56)

which is nothing but the effective action for relativistic particle models satisfying the different gauge condition $F_2[f(q)] = 0$. Thus, under FCBRST transformation, the generating functional of HD models changes from one gauge condition ($F_1[f(q)] = 0$) to another gauge ($F_2[f(q)] = 0$) as

\[
\int d\tau e^{iS_{eff}} \overset{FCBRST}{\longrightarrow} \left( \int d\tau e^{i[S_{eff} + S_1(\kappa = 1)]]} \right).
\]

(57)

We end this section by noting that the FCBRST transformation with appropriate finite coordinate-dependent parameter is able to connect two different (arbitrary) gauges of the relativistic particle model.

7 Conclusions

The relativistic particle models have always been an interesting area of research as it led to the Polyakov action of string theory [22]. When a curvature term is added to the action of the relativistic particle
model it becomes a higher derivative (HD) theory. Due to HD nature, it shows an inconsistency in counting the independent gauge degrees of freedom. The apparent mismatch is due to the interrelation between the variables with higher derivatives. Whereas, if we consider the mass term to be zero (with proper condition on the particle velocities as in [28]) the mismatch vanishes and the number of gauge degrees of freedom and number of independent primary first class constraints are same [31], as happens for all standard theories [47, 48]. So, it would be interesting to study the BRST symmetries of both these models. But here we are faced an obstacle. For HD theories there is no well defined prescription for analysing BRST symmetry. In the present case this is avoided by working in the first order formalism developed in [31].

In this paper, we have analysed the different constraint structures of the models of relativistic particle with curvature and of massless relativistic particle with rigidity. The relativistic particle model with curvature is shown to have the diffeomorphism symmetry whereas the gauge symmetries of the model of relativistic massless particle with rigidity contain both diffeomorphism and $W$-morphisms. The nilpotent BRST and anti-BRST symmetries for these model have also been investigated. A remarkable feature for such symmetries is the manifestation of $W_3$-algebra. The BRST symmetries for all variables (excluding for anti-ghost variable) corresponding to diffeomorphism and $W$-morphisms satisfy the $W_3$-algebra. Likewise, apart from the ghost variable, the anti-BRST symmetry transformations for all other variables also satisfy the same $W_3$-algebra. Thus the full $W_3$-algebra for all variables is obtained by taking into account both BRST and anti-BRST transformations.

The finite coordinate-dependent BRST (FCBRST) symmetry, which is quantum mechanical analog of finite field-dependent BRST (FFBRST), has also been analysed in full generality for higher derivative particle models. It has been shown that although such a transformation is a symmetry of the effective action, it breaks the invariance of the generating functional of the path integral. The Jacobian of path integral measure changes non-trivially for FCBRST symmetry transformation. We have shown that FCBRST transformation with a suitable coordinate-dependent parameter changes the effective action from one gauge to another within a functional integral. Thus, FCBRST formulation is very useful to connect two different Greens functions for models of relativistic particles. The results were explicitly presented for the massive case. For the massless version, all results go over trivially in the appropriate limit. Finally, we feel that although our analysis was done for relativistic particle models, it is general enough to include other higher derivative models.

Acknowledgement

One of the authors (BP) gratefully acknowledges the Council of Scientific and Industrial Research (CSIR), Government of India, for financial assistance.

References

[1] B. Podolsky, Phys. Rev. 62 (1942) 68.
[2] B. Podolsky, C. Kikuchi, Phys. Rev. 65 (1944) 228.
[3] J. Iliopoulos, B. Zumino, Nucl. Phys. B 76 (1974) 310.
[4] F. S. Gama, M. Gomes, J. R. Nascimento, A. Yu. Petrov, A. J. da Silva. Phys. Rev. D 84 (2011) 045001.

[5] C. S. Chu, J. Lukierski, W. J. Zakrzewski, Nucl. Phys. B 632 (2002) 219.

[6] P. D. Alvarez, J. Gomis, K. Kamimura, M. S. Plyushchay, Phys. Lett. B 659 (2008) 906 [arXiv:0711.2644].

[7] I. P. Neupane, JHEP 09 (2000) 040.

[8] S. Nojiri, S. D. Odintsov, S. Ogushi, Phys. Rev. D 65 (2001) 023521.

[9] C. M. Reyes, Phys. Rev D 80, 105008 (2009).

[10] P. Mukherjee, B. Paul, Phys. Rev. D 85 (2012) 045028.

[11] M. S. Plyushchay, Nucl. Phys. B 362 (1991) 54.

[12] Peter A. Horváthy, M. S. Plyushchay, JHEP 0206 (2002) 033 [hep-th/0201228]

[13] M. S. Plyushchay, Electron. J. Theor. Phys. 3N10 (2006) 17 [math-ph/0604022].

[14] M. S. Plyushchay, Phys. Lett. B 262 (1991) 71.

[15] R. Cordero, A. Molgado, E. Rojas, Class. Quantum Grav. 28 (2011) 065010.

[16] B. Paul, Phys. Rev. D 87 (2013) 045003.

[17] K. S. Stelle, Phys. Rev. D 16 (1977) 953.

[18] A. J. Accioly, Revista Brasileira de Fisica 18 (1988) 593.

[19] T. P. Sotiriou, V. Faraoni, Rev. Mod. Phys. 82 (2010) 451 [arXiv:0805.1726].

[20] I. Gullu, T. C. Sisman, B. Tekin, Phys. Rev. D 81 (2010) 104017.

[21] N. Ohta, Class. Quantum Grav. 29 (2012) 015002.

[22] A. M. Polyakov, Nucl. Phys. B 268 (1986) 406.

[23] D. A. Eliezer, R. P. Woodard, Nucl. Phys. B 325 (1989) 389.

[24] R. D. Pisarski, Phys. Rev. D 34 (1986) 670.

[25] V. V. Nesterenko, J. Phys. A 22 (1989) 1673.

[26] M. S. Plyushchay, Mod. Phys. Lett. A 3 (1988) 1299;

[27] M. S. Plyushchay, Int. J. Mod. Phys. A 4 (1989) 3851.

[28] M. S. Plyushchay, Mod. Phys. Lett. A 4 (1989) 837, Phys. Lett. B 243 (1990) 383.

[29] E. Ramos, J. Roca, Nucl. Phys. B 436 (1995) 529.

[30] E. Ramos, J. Roca, Nucl. Phys. B 452 (1995) 705.
[31] R. Banerjee, P. Mukherjee, B. Paul, JHEP **1108** (2011) 085.

[32] S. Novikov, S. V. Manakov, L. P. Pitaevskii, V. E. Zakharov, *Theory of solitons: The inverse Scattering Method*, Contemporary Soviet Mathematics, Consultants Bureau [Plenum], New York, 1984.

[33] C. Becchi, A. Rouet, R. Stora, Annals Phys. **98** (1974) 287.

[34] I. V. Tyutin, LEBEDEV-**75-39** (1975).

[35] M. Henneaux, C. Teitelboim, *Quantization of gauge systems*, Princeton, USA: Univ. Press (1992).

[36] S. Weinberg, *The quantum theory of fields, Vol-II: Modern applications*, Cambridge, UK Univ. Press (1996).

[37] S. D. Joglekar, B. P. Mandal, Phys. Rev. D **51** (1995) 1919.

[38] S. D. Joglekar, B. P. Mandal, Int. J. Mod. Phys. A **17**, (2002) 1279.

[39] S. D. Joglekar, A. Misra, J. Math. Phys **41**, (2000) 1755.

[40] B. P. Mandal, S. K. Rai, S. Upadhyay, Eur. Phys. Lett. **92** (2010) 21001.

[41] S. Upadhyay, S. K. Rai, B. P. Mandal, J. Math. Phys. **52** (2011) 022301.

[42] S. Upadhyay, B. P. Mandal, Mod. Phys. Lett. A **40** (2010) 3347; Eur. Phys. Lett. **93** (2011) 31001; Eur. Phys. J. C **72**, 2059 (2012); Eur. Phys. J. C **72** (2012) 2065; Annals of Physics **327** (2012) 28850.

[43] R. Banerjee, B. P. Mandal, Phys. Lett. B **27** (2000) 488.

[44] S. Upadhyay, M. K. Dwivedi, B. P. Mandal, Int. J. Mod. Phys. A **28** (2013) 1350033.

[45] M. Faizal, B. P. Mandal, S. Upadhyay, Phys. Lett. B **721** (2013) 159.

[46] M. Ostrogradsky, *Mem. Ac. St. Petersbourg* **V14** (1850) 385.

[47] R. Banerjee, H. J. Rothe, K. D. Rothe, Phys. Lett. B **463** (1999) 248, [hep-th/9906072]; Phys. Lett. B **479** (2000) 429, [hep-th/9907217]; J. Phys. A **33** (2000) 2059; [hep-th/9909039].

[48] A. Hanson, T. Regge, C. Tietelboim, *Constrained Hamiltonian System*, (Accademia Nazionale Dei Lincei, Roma, 1976).