ON CONJUGACY CLASSES AND DERIVED LENGTH

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Abstract. Let $G$ be a finite group and $A$, $B$ and $D$ be conjugacy classes of $G$ with $D \subseteq AB = \{xy \mid x \in A, y \in B\}$. Denote by $\eta(AB)$ the number of distinct conjugacy classes such that $AB$ is the union of those. Set $C_G(A) = \{g \in G \mid x^g = x \text{ for all } x \in A\}$. If $AB = D$ then $C_G(D)/(C_G(A) \cap C_G(B))$ is an abelian group. If, in addition, $G$ is supersolvable, then the derived length of $C_G(D)/(C_G(A) \cap C_G(B))$ is bounded above by $2\eta(AB)$.

1. Introduction

Let $G$ be a finite group, $A \subseteq G$ be a conjugacy class of $G$, i.e. $A = a^G = \{g^{-1}ag \mid g \in G\}$ for some $a$ in $G$. Let $X$ be a normal subset of $G$, i.e. $X^g = \{g^{-1}xg \mid x \in X\} = X$ for all $g \in G$. We can check that $X$ is a union of $n$ distinct conjugacy classes of $G$, for some integer $n > 0$. Set $\eta(X) = n$.

We can check that given any two conjugacy classes $A$ and $B$ of $G$, the product $AB = \{xy \mid x \in A, y \in B\}$ of $A$ and $B$ is a normal subset of $G$. Then $\eta(AB)$ is the number of distinct conjugacy classes of $G$ such that $AB$ is the union of those classes.

Denote by $C_G(X) = \{g \in G \mid x^g = x \text{ for all } x \in X\}$ the centralizer of $X$ in $G$. If $G$ is a solvable group, denote by $dl(G)$ the derived length of $G$. Let $Z(G)$ be the center of the group $G$.

In this note, we are exploring the relations between the structure of the group $G$ and the product $AB$ of some conjugacy classes $A$ and $B$ of $G$. More specifically, we are exploring the relation between the derived length of some section of $G$ and properties of $AB$.

Given a finite solvable group $G$ and conjugacy classes $A$ and $B$ of $G$, is there any relationship between the derived length of $G$ and $\eta(AB)$? In general, the answer seems to be no. For instance $A\{e\} = A$ for any finite group $G$ and any conjugacy class $A$ of $G$. Thus $\eta(AB)$ may not give us information about $dl(G)$, but it does give us a linear bound on the derived length of a section of $G$, namely on the section $C_G(D)/(C_G(A) \cap C_G(B))$.

Theorem A. Let $G$ be a finite group and $A$, $B$ be normal subsets of $G$. Then $C_G(AB)/(C_G(A) \cap C_G(B))$ is abelian.

Given any integer $m > 0$, we show in Example 2.1 that there exists a nilpotent group $G$ with conjugacy classes $A$, $B$ and $D$ such that $|C_G(D)/(C_G(A) \cap C_G(B))| = m$ and $AB = D$. It follows that although $C_G(D)/(C_G(A) \cap C_G(B))$ is abelian, its order is unbounded and even the number of distinct prime divisors is unbounded.

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Theorem B. Let $G$ be a supersolvable group, $A$, $B$ and $D$ conjugacy classes of $G$ such that $D \subseteq AB$. Then $\text{dl}(C_G(D)/(C_G(A) \cap C_G(B))) \leq 2\eta(AB)$.

We now mention the “dual” situation for characters, where the “dual” of a conjugacy class $A$ is an irreducible character $\chi$ and the “dual” of the kernel $\text{Ker}(\chi)$ of $\chi$ is the centralizer $C_G(A)$ of $A$. Let $\Xi$ and $\Psi$ be complex characters of $G$. Thus it can be written as an integral linear combination of irreducible characters of $G$. Let $\eta(\Xi\Psi)$ be the number of distinct irreducible constituents of the character $\Xi\Psi$. In Theorem A of [4], it is proved that there exist universal constants $c$ and $d$ such that for any solvable group $G$, any irreducible characters $\chi, \psi$ and $\theta$ such that $\theta$ is a constituent of $\chi\psi$, we have that $\text{dl}(\text{Ker}(\theta)/(\text{Ker}(\chi) \cap \text{Ker}(\psi))) \leq c\eta(\chi\psi) + d$.

In Theorem A of [7] it is proved that given any supersolvable group $G$ and any conjugacy class $A$, we have that $\text{dl}(G/C_G(A)) \leq 2\eta(\text{AA}^{-1}) - 1$. We conjecture in [7] that there exist universal constants $q$ and $r$ such that for any solvable group $G$ and any conjugacy class $A$ of $G$, we had that $\text{dl}(G/C_G(A)) \leq q\eta(\text{AA}^{-1}) + r$. In light of Theorem B and because of the “dual” situation with characters, namely Theorem A of [4], we wonder the following.

Conjecture C. There exist universal constants $r$ and $s$ such that for any solvable group $G$, any conjugacy classes $A$, $B$ and $D$ of $G$ such that $D \subseteq AB$, we have that $\text{dl}(C_G(D)/(C_G(A) \cap C_G(B))) \leq r\eta(AB) + s$.

We will show in Theorem 2.3 that the previous conjecture has an affirmative answer if and only if Conjecture of [7] has an affirmative answer. We would like to point out that there are several examples of “dual results” between products of conjugacy classes and products of character. For example, see [9], [2] and [8], [3] and [6]. However not every result in products of characters has a “dual” result in conjugacy classes, see for example Section 3.

In Section 4 we provide an example of a property in a conjugacy class $A$ of $G$ that bounds the nilpotent class of $G/C_G(A)$, and therefore it bounds the derived length of that section.

2. Proofs

Notation. Let $G$ be a group, $X$ be a subset of $G$ and $a \in G$. Set $[a, X] = \{a[x] \mid x \in X\}$. Observe that $a^X = \{a[a, x] \mid x \in X\} = a[a, X]$.

Proof of Theorem A. Write $N = C_G(AB)$ and $C = C_G(N)$ so $AB \subseteq C$. Let $a \in A$, $b \in B$ and $g \in C$. Then $ab$ and $ab^g$ lie in $AB$, and so lie in $C$, and since $C$ is a subgroup, it follows that $b^{-1}b^g$ also lies in $C$. It follows then that working in $G/C$, $b$ is central, so $[b, G] \subseteq C$ and $[b, G, N] = 1$. In particular, $[\{b\}, N, N] = 1$ and so by the three-subgroups lemma, $[N', \{b\}] = 1$. Since this holds for all $b \in B$, we have that $N' \subseteq C_G(B)$. Similarly, $N' \subseteq C_G(A)$ and the result follows. \hfill $\square$

Example 2.1. Let $m > 0$ be an integer. Write $m = \prod_{i=1}^l p_i$, where $p_i$ are primes not necessarily distinct. Let $P_i$ be nonabelian of order $p_i^3$, and let $G$ be the direct product of the groups $P_i$. Choose noncommuting elements $a_i$ and $b_i$ in $P_i$ and write $a = \prod a_i$ and $b = \prod b_i$. It is easy to check that $A = aG = aZ$, $B = bG = bZ$, where $Z = Z(G)$. Then $AB = abZ = D$, where $D = (ab)^G$. Also $C_G(A) \cap C_G(B) = Z$, which has order $\prod p_i$, while $C_G(D)$ has order $\prod (p_i)^2$. Then $|C_G(D)/(C_G(A) \cap C_G(B))| = \prod_{i=1}^l p_i = m$, as wanted.
Definition 2.2. Let \( F(n) \) be a nondecreasing function defined on the natural numbers. A group \( G \) is good for \( F \) if for every conjugacy class \( A \) of \( G \), the group \( G/C_G(A) \) is solvable with derived length at most \( F(\eta(AA^{-1})) \).

In Theorem A of [7], it is proved that \( F(n) = 2n - 1 \) is good for all supersolvable groups.

Theorem 2.3. Suppose that a function \( F \) is good for all homomorphic images of \( G \). Let \( A, B \) and \( D \) be conjugacy classes of \( G \) with \( D \subseteq AB \). Then \( C_G(D)/(C_G(A) \cap C_G(B)) \) is solvable with derived length at most \( 1 + F(\eta(AB)) \).

Proof. Let \( N = C_G(D) \), and observe that \( C_G(A) \cap C_G(B) \subseteq N \). It suffices to show that both \( N/(N \cap C_G(A)) \) and \( N/(N \cap C_G(B)) \) have derived length at most \( 1 + F(\eta(AB)) \). We will prove the required length inequality for \( N/(N \cap C_G(A)) \); the other inequality follows similarly.

Let \( C = C_G(N) \) and, using the standard bar convention, write \( \bar{G} = G/C \). Since \( D \subseteq C \), we see that the identity is an element of \( \bar{A}\bar{B} \). Also \( \bar{A} \) and \( \bar{B} \) are conjugacy classes of \( \bar{G} \), so they must be inverse classes. By Lemma 2.5 of [5] we have that \( \eta(\bar{A}\bar{B}) \leq \eta(AB) \), and so by hypothesis we have then

\[
(2.4) \quad \text{dl}(\bar{G}/C_G(\bar{A})) \leq F(\eta(\bar{A}\bar{B})) \leq F(\eta(AB)).
\]

Thus \( \text{dl}(G/K) = F(\eta(AA^{-1})) \), where \( K \) is the preimage in \( G \) of \( C_G(\bar{A}) \). It follows then that \( \text{dl}(N/(N \cap K)) \leq F(\eta(AA^{-1})) \). Now \( C_G(A) \subseteq K \) and so it will be enough to show that \( N \cap K/(N \cap C_G(A)) \) is abelian, yielding the desired derived length bound for \( N/(N \cap C_G(A)) \).

Now \( K \) centralizes \( A \) modulo \( C \), so \( [(a), K] \subseteq C \) and \( [(a), K, N] = 1 \) for \( a \in A \). Then \( [(a), (N \cap K), (N \cap K)'] = 1 \) and by the three-subgroups lemma, \( (N \cap K)' \) centralizes \( a \), and we have \( (N \cap K)' \subseteq N \cap C_G(A) \) as wanted. \( \square \)

Since \( F(n) = 2n - 1 \) is good for supersolvable groups (Theorem A of [7]), Theorem A follows from the previous result.

3. Conjugacy class sizes

Fix a prime \( p \). Let \( G \) be a finite \( p \)-group and \( A \) be conjugacy classes of \( G \). In Theorem A of [5], we proved that if \( |A| = p^n \) for some integer \( n \), then \( \eta(AA^{-1}) \geq n(p-1) + 1 \). Thus, in the particular case that \( G \) is \( p \)-group, there is a relation between the size of \( A \) and \( \eta(AA^{-1}) \). We want to point out a “dual” result in character theory, where the “dual” of a conjugacy class is an irreducible complex character \( \chi \) and the “dual” of the inverse of a conjugacy class is the complex conjugate character \( \overline{\chi} \) of \( \chi \), where \( \overline{\chi}(g) = \overline{\chi(g)} \) for all \( g \in G \). More specifically, in Theorem A of [3] is proved that if \( G \) is a \( p \)-group, \( \chi \) is an irreducible character of \( G \) and \( \chi(1) = p^n \), then the product \( \chi \overline{\chi} \) of \( \chi \) and \( \overline{\chi} \) has at least \( 2n(p-1) + 1 \) distinct irreducible constituents, i.e. \( \eta(\chi \overline{\chi}) \geq 2n(p-1) + 1 \).

In Theorem A of [11] is proved that if \( \chi \) is an irreducible character of a solvable group \( G \) with \( \chi(1) > 1 \), then \( \chi(1) \) has at most \( \eta(\chi \overline{\chi}) - 1 \) different prime factors. If, in addition, \( G \) is supersolvable, then \( \chi(1) \) has at most \( \eta(\chi \overline{\chi}) - 2 \) prime factors. Is there any “dual” result in conjugacy classes as Theorem A of [11] in characters? In other words,

Question 3.1. Does it exist a function \( f : Z \to Z \) such that for any solvable group \( G \) and any conjugacy class \( A \) of \( G \), we have that \( |A| \) has at most \( f(\eta(AA^{-1})) \) different prime factors?
The answer is no, such function can not exist. More specifically,

**Example 3.2.** Let \( p \) be a prime and \( P \) be a group of order \( p \). Let \( G \) be the group of order \( p(p - 1) \), where \( P \triangleleft G \) and \( G/P \) induces on \( P \) the full group of automorphisms of \( P \). Then \( P \) contains just one nontrivial conjugacy class \( A \) of \( G \), namely \( A = P \setminus \{ \} \). Observe that \( AA^{-1} = P \) and so \( \eta(AA^{-1}) = 2 \). Also \( P = C_G(A) \), and thus \( |G/C_G(A)| = p - 1 \). This is obviously unboundedly large, and by a result of Erdos [10], it has unboundedly many prime factors.

**Remark.** Let \( G \) be the group as in the previous Example. Let \( \lambda \) be an irreducible character of \( N \). Then \( \lambda \) is a linear character and the induced character \( \lambda^G \) is an irreducible character of degree \( p - 1 \). Set \( \chi = \lambda^G \). We can check then \( \eta(\chi) = p \), namely the irreducible constituents of \( \chi \) are \( \chi \) and all irreducible character with kernel containing \( N \). Since \( n - 1 \leq 2^{n-2} \) for any integer \( n > 0 \), then \( p - 1 \) has at most \( p - 2 \) distinct prime factors.

4. **Conjugacy classes and nilpotent class**

Let \( \mathbb{Z}_1(G) = \mathbb{Z}(G) \) be the center of the group \( G \) and by induction define the \( i \)-center of \( G \) as \( \mathbb{Z}_i(G)/\mathbb{Z}_{i-1}(G) = \mathbb{Z}(G/\mathbb{Z}_{i-1}(G)) \). The following is a well known result.

**Lemma 4.1.** Let \( N \) be a group. Write \( Z_m = Z_m(N) \) for the \( m \)-th center of \( N \) and write \( N^m \) for the \( m \)-th term of the lower central series of \( N \). Then \( [N^m, Z_m] = 1 \).

**Proof.** Induct on \( m \). For \( m = 1 \), we have \( [N^1, Z_1] = [N, Z(N)] = 1 \), as needed. For \( m > 1 \), we want \( [N^{m-1}, N, Z_m] = 1 \). We have \( [N, Z_m, N^{m-1}] \subseteq [Z_m, N^{m-1}] = 1 \) by the inductive hypothesis. Now work in \( \bar{N} = N/Z \) where \( Z = Z_1 = Z(N) \). Note that \( \bar{N}^{m-1} = (N)^{m-1} \) and \( \bar{Z}_m = Z_{m-1}(\bar{N}) \). Then
\[
1 = [Z_{m-1}(\bar{N}), (\bar{N})^{m-1}] = [\bar{Z}_m, \bar{N}^{m-1}] = [\bar{Z}_m, N^{m-1}],
\]
and we have \( [Z_m, N^{m-1}] \subseteq Z \). Then \( [Z_m, N^{m-1}, N] = 1 \), and the result follows by the three-subgroups lemma.

**Theorem 4.2.** Let \( N \trianglelefteq G \) and \( a \in G \). Assume that \( [N, a] \subseteq Z_m \), where \( Z_m = Z_m(G) \) is the \( m \)-th center of \( N \). Let \( C = C_N(a^N) \). Then \( N/C \) is nilpotent of class at most \( m \).

**Proof.** Induct on \( m \). if \( m = 0 \) we are assuming that \( [N, a] = 1 \) and so \( N \) centralizes all of \( a^N \) and \( C = N \). In this case \( N/C \) is nilpotent of class zero. We can assume therefore that \( m > 0 \). Our goal is to show that \( N^{m+1} \subseteq C \).

Now let \( x \in a^N \). We want \( [N^{m+1}, \langle x \rangle] = 1 \), or equivalently, \( [N^m, N, \langle x \rangle] = 1 \). Now \( [N, \langle x \rangle, N^m] \subseteq [Z_m, N^m] = 1 \) by the previous lemma.

Let \( Z = Z_1 = Z(N) \) and write \( \bar{G} = G/Z \). Then
\[
Z_{m-1}(\bar{N}) = \bar{Z}_m \supseteq [\bar{N}, a] = [\bar{N}, a],
\]
and thus by the inductive hypothesis, \( N/B \) is nilpotent of class at most \( m - 1 \), where \( B \) is the preimage in \( N \) of \( C_N((a)^N) = C_N(a^N) \). In particular, \( N^m \subseteq B \) so \( x \) centralizes \( N^m \) modulo \( Z \), and we have \( \langle x \rangle, N^m \subseteq Z \), and hence \( \langle x \rangle, N^m, N \) = 1. The three-subgroups lemma now yields \( [N^m, \langle x \rangle, N] = 1 \), as wanted. □
Remark. Let $G_1 = C_2 \wr C_2$ be the wreath product of $C_2$ by $C_2$, where $C_2$ is the cyclic group of order 2. Thus $|G_1| = 8$ and $G_1$ is non abelian. Let $a_1 \in G_1 \setminus Z(G_1)$. We can check that $a_1^{G_1} = aZ(G_1)$ and $G_1/C_{G_1}(a_1^{G_1})$ is abelian and so it is nilpotent of class 1.

Let $N = G_1 \times G_1$ and $a_2 = (a_1, a_1)$ in $N$. Observe that $C_2$ acts on $N$ by permuting the entries. Set $G_2 = C_2N$. We can check that $a_2^{G_2} \subset a_2Z_2(G_2)$ but $a_2^{G_2} \not\subset a_2Z_2(G_2)$, and $G_2/(C_{G_2}(a_2^{G_2}))$ is nilpotent of class 2.

The author wonders if given any integer $m > 2$, we can find a group $G$ with an element $a \in G$ such that $a^G \subseteq aZ_m(G)$ and $G/C_{G}(a^G)$ is nilpotent of class $m$.

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