$n$-point functions of 2d Yang-Mills theories on Riemann surfaces

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Abstract

Using the simple path integral method we calculate the $n$-point functions of field strength of Yang-Mills theories on arbitrary two-dimensional Riemann surfaces. In $U(1)$ case we show that the correlators consist of two parts, a free and an $x$-independent part. In the case of non-abelian semisimple compact gauge groups we find the non-gauge invariant correlators in Schwinger-Fock gauge and show that it is also divided to a free and an almost $x$-independent part. We also find the gauge-invariant Green functions and show that they correspond to a free field theory.
Introduction

In recent years there have been many efforts to understand the two dimensional Yang-Mills theories. The partition function of these theories on $\Sigma_g$, a two-dimensional Riemann surface of genus $g$, has been calculated in the context of lattice gauge theory [1, 2]. On the other hand the string interpretation of 2d Yang-Mills theory was discussed in [3] and [4] by studying the $1/N$ expansion of the partition function for $SU(N)$ gauge group. It was shown that the coefficients of this expansion are determined by a sum over maps from a two-dimensional surface onto the two-dimensional target space.

Two-dimensional Yang-Mills theories have also been studied by means of path integral method [5, 6]. In [7] and [8] the partition function and the expectation values of Wilson loops have been calculated and in [9 – 11] these quantities were calculated by using the abelianization technique.

In this paper we study the correlation functions of field strengths of 2d Yang-Mills theories by path integral method in a simple way. We drive all $n$-point functions on arbitrary surface. In the first part we consider the $U(1)$ gauge group and by calculating $Z[J]$, the partition function in the presence of an external source, we compute the gauge-invariant correlators on $\Sigma_g$. We show that the results consist of two parts, a part which comes from a free field theory and a second part which is independent of coordinates of fields. We also rederive the results by means of the expectation value of Wilson loops.

In the second part, we investigate the non-abelian gauge theories. Using the fermionic path integral representation of the trace of Wilson loops, we find the $Z[J]$ in Schwinger-Fock gauge and calculate the angle ordered $n$-point functions. We see that the non-gauge invariant correlators consist of a gauge-invariant part, and another part which is almost independent of the coordinates of the fields. We also extract the gauge-invariant part of the correlators and show that they correspond to a free theory. At the end we justify our results by using the Wilson loop correlators.

When this paper was nearly finished, we became aware of the preprint [13] in which the two and four-point functions of $U(N)$ gauge group have been derived by lengthy abelianization method.

1 - Maxwell theory

Consider the partition function $Z[J]$ on $\Sigma_g$:

$$Z[J] = \int D\xi e^{-\frac{1}{\pi} \int \xi^2 d\mu + \int \xi J d\mu \delta^2(\frac{1}{2\pi} \int \xi d\mu)}$$ (1)
where the scalar field $\xi(x)$ is defined by $F_{\mu\nu} = \xi(x)\epsilon_{\mu\nu}$, $d\mu = \sqrt{g(x)}d^2x$ and

$$\delta^n\left(\frac{1}{2\pi}\int \xi d\mu\right) = \sum_n \delta\left(\frac{1}{2\pi}\int \xi d\mu - n\right) = \sum_n e^{in\int \xi d\mu}. \quad (2)$$

The insertion of $\delta^n\left(\frac{1}{2\pi}\int \xi d\mu\right)$ ensures that we are not integrating over arbitrary two-forms but over curvatures of connections. Performing the Gaussian integral (1), we find

$$Z[J] = Z_1[J]Z_2[J], \quad (3)$$

where:

$$Z_1[J] = e^{\frac{\epsilon}{2}\int J^2d\mu} \quad (4)$$

which is the partition function of a free field theory, and

$$Z_2[J] = \sum_n \exp\left[ine\int Jd\mu - \frac{\epsilon}{2}n^2A(\Sigma_g)\right] \quad (5)$$

in which $A(\Sigma_g)$ is the area of $\Sigma_g$. The gauge-invariant correlators of $\xi(x)$ is defined via:

$$<\xi(x_1)...\xi(x_n)> = \frac{1}{Z[0]} \delta\frac{\partial}{\partial J(x_1)}...\delta\frac{\partial}{\partial J(x_n)}Z[J]|_{J=0} \quad (6)$$

Now $Z_1$ is the partition function of a free theory, so that its $n$-point functions are zero unless $n$ is even. The $2n$-point functions factorize to the two point function

$$G(x, y) = e^{\epsilon\delta(x - y)}. \quad (7)$$

So

$$<\xi(x_1)...\xi(x_{2n})>_1 = \sum_p G(x_{i_1}, x_{i_2})...G(x_{i_{2n-1}}, x_{i_{2n}}), \quad (8)$$

where the summation is over all distinct pairings of the $2n$ indeces with $\frac{(2n)!}{(2!)^nn!}$ terms.

The correlators corresponding to $Z_2$ are also calculated to be

$$<\xi(x_1)...\xi(x_{2n})>_2 = \frac{1}{Z[0]} \sum_m (im\epsilon)^{2n}e^{-\frac{\epsilon^2m^2A(\Sigma_g)}{2}}, \quad (9)$$

and the odd-point functions are zero. The complete $n$-point function is simply obtained, using these two correlators. The odd-point functions are zero and the even-point functions are:

$$<\xi(x_1)...\xi(x_{2n})> = \sum_{m=0}^{2n} \sum_c <\xi(x_1)...\xi(x_m)>_1 <\xi(x_{m+1})...\xi(x_{2n})>_2; \quad (10)$$

where the inner summation is over all different ways of choosing $m$ indeces from $2n$ indeces. As mentioned in the introduction the correlators consist of a free and an $x$-independent parts.
Now it is useful to reproduce the above results from another method, that is using the expectation value \( \langle e^{i\alpha \int_{\gamma} A} \rangle \) where \( \gamma \) is a homologically trivial loop on \( \Sigma_g \):

\[
\langle e^{i\alpha \int_{\gamma} A} \rangle = \frac{1}{Z[0]} \int \mathcal{D}\xi e^{-\frac{1}{\epsilon} \int \xi^2 d\mu + i\alpha \int_{D} \xi d\mu \delta(p(\frac{1}{2\pi} \int \xi d\mu))}
\]

\[
= \frac{1}{Z[0]} \sum_n \exp\{-\frac{\epsilon}{2} [n^2 A(\Sigma) + \alpha^2 A(D) + 2\alpha n A(D)]\}
\]

\[
= 1 + \alpha^2 \frac{\epsilon/2}{Z[0]} \sum_n (\epsilon n^2 A^2(D) - A(D)) e^{-\frac{\epsilon}{2} n^2 A(\Sigma)} + o(\alpha^2),
\] (11)

where \( A(D) \) is the area of disk \( D \), the boundary of which is \( \gamma \).\] Expanding the left hand side of eq.(11) in terms of \( \alpha \), gives

\[
\langle e^{i\alpha \int_{\gamma} A} \rangle = 1 + i\alpha \int_{D} <\xi(x)> d\mu - \frac{\alpha^2}{2} \int_{D} <\xi(x)\xi(y)> d\mu(x)d\mu(y) + o(\alpha^2). \] (12)

By comparing the two side of eq.(11) , it is seen that \( \langle \xi(x) \rangle = 0 \). In order to find the two point function we use the following ansatz

\[
\langle \xi(x)\xi(y) \rangle = M\delta(x - y) + N. \] (13)

In fact, as the theory is topological, it is reasonable that the two point function consists of an \( x \)-independent term and a term which just sees if the two point are equal or not. This term should be a delta term, because of the Gaussian nature of the integrand in the partition function. Using this ansatz, it is readily seen that

\[
\langle \xi(x)\xi(y) \rangle = \epsilon\delta(x - y) - \frac{\epsilon^2}{Z[0]} \sum_m m^2 e^{-\frac{\epsilon}{2} m^2 A(\Sigma)} , \] (14)

which is the result obtained previously. Other \( n \)-point functions can also be obtained in this way.

2 - Yang-Mills theory

In this section we are going to calculate \( Z[J] \) for a non-abelian semisimple compact gauge group \( G \). To begin, we consider the wavefunction \( \psi_D[J] \) on the disk \( D \). If \( \gamma \) is the boundary of \( D \), \( \gamma = \partial D \), we choose the boundary condition to be \( \text{Pexp} \int_{\gamma} A = g_1 \in G \) (modulo conjugation), and therefore \( \psi_D[J] \) is defined as:

\[
\psi_D[J] = \int \mathcal{D}\xi e^{-\frac{1}{\epsilon} \int \xi^a \epsilon_a d\mu + \int \xi^a J_a d\mu} \delta(\text{Pexp} \int_{\gamma} A, g_1). \] (15)

We have

\[
\delta(h, g_1) = \sum_{\lambda \in \hat{G}} \chi_\lambda(h) \chi_\lambda(g^{-1}_1) \] (16)
where the summation is over all irreducible unitary representation of the group, and $\chi_{\lambda}$ is the character of the representation. We then use the fermionic path integral representation of the Wilson loop [7, 12]

$$\chi_{\lambda}(P \exp \int_{\gamma} A) = \int \mathcal{D}\eta \mathcal{D}\bar{\eta} e^{\int_{0}^{1} dt \eta(t) \dot{\eta}(t) + \int_{0}^{1} \bar{\eta}(t) A(t) \eta(0)},$$

(17)

where $\eta$ is a Grassmann valued vector in the representation $\lambda$. We also use the Schwinger-Fock gauge:

$$A^a_{\mu}(x) = \int_{0}^{1} dss' F^a_{\nu\mu}(sx).$$

(18)

In this gauge, one can write

$$\oint_{\gamma} \bar{\eta}(t) A(t) \eta(t) = \int_{D} \bar{\eta} F \eta.$$  

(19)

Using this, it is easily seen that

$$\psi_D[J] = \sum_{\lambda} \chi_{\lambda}(g^{-1}_{1}) \int \mathcal{D}\eta \mathcal{D}\bar{\eta} e^{\int_{0}^{1} dt \eta(t) \dot{\eta}(t)} e^{\int (J_{a} + \bar{\eta} T_{a,\lambda} \eta)(J_{a} + \bar{\eta} T_{a,\lambda} \eta) \sqrt{g} ds dt} \bar{\eta}(1) \eta(0),$$

(20)

where $T_{a,\lambda}$’s are the generators of the group in the representation $\lambda$. This integral is also calculated to be (see the appendix of [8])

$$\psi_D[J] = Z_1[J] \psi_{2,D}[J],$$

(21)

where

$$Z_1[J] = e^{\int J_a J_a dp},$$

(22)

and

$$\psi_{2,D}[J] = \sum_{\lambda} \chi_{\lambda}(g^{-1}_{1}) e^{-\frac{1}{2} c_2(\lambda) A(D)} \chi_{\lambda}(\mathcal{P} \exp \int dt \int ds \sqrt{g} J(s,t)).$$

(23)

Here the ordering is according to $t$ (the angle coordinate). The disk is parametrized by the coordinates $s$ (the radial coordinate) and $t$ (the angle coordinate), and $c_2(\lambda)$ is the quadratic Casimir of the representation $\lambda$.

It is now easy to see that

$$<\xi^{a_1}(x_1) ... \xi^{a_n}(x_n)>_{2,D} = \frac{1}{Z_D[0]} \sum_{\lambda} \chi_{\lambda}(g^{-1}_{1}) e^{-\frac{1}{2} c_2(\lambda) A(D)} e^\nu \chi_{\lambda}(T^{a_1} ... T^{a_n})$$

(24)

for $t(x_1) < ... < t(x_n)$.

To find the correlators on an arbitrary closed surface, it suffices to glue the disk to $\Sigma_{g,1}$, a genus-$g$ surface with a boundary, with boundary condition $P \exp \int_{\gamma} A = g^{-1}_{1}$:

$$<\xi^{a_1}(x_1) ... \xi^{a_n}(x_n)>_{2,\Sigma_g} = \frac{1}{Z_{\Sigma_g}[0]} \int dg_1 <\xi^{a_1}(x_1) ... \xi^{a_n}(x_n)>_{2,D} \psi_{\Sigma_{g,1}}(g^{-1}_{1}),$$

(25)
where (from [7]) we have

$$Z_{\Sigma_g}\{0\} = \sum_\lambda d(\lambda) e^{ -2g c_2(\lambda) A(\Sigma_g)},$$  \hspace{1cm} (26)

in which $d(\lambda)$ is the dimension of the representation $\lambda$, and

$$\psi_{\Sigma_{g,1}}(g_1^{-1}) = \sum_\lambda d(\lambda) e^{ -2g c_2(\lambda) A(\Sigma_{g,1}).}$$ \hspace{1cm} (27)

Using the orthogonality relation

$$\int \chi_\lambda(g) \chi_\mu(g^{-1}) dg = \delta_{\lambda\mu},$$ \hspace{1cm} (28)

one finds

$$\langle \xi^{a_1}(x_1) ... \xi^{a_n}(x_n) >_{2,\Sigma_g} = \frac{1}{Z_{\Sigma_g}\{0\}} \sum_\lambda d(\lambda) e^{ -2g c_2(\lambda) A(\Sigma_g)} \chi_\lambda(T^{a_1} ... T^{a_n}).$$ \hspace{1cm} (29)

where it is understood that $t_1 < ... < t_n$. This result obviously depends on the choice of the coordinates and hence is not gauge invariant.

$Z_1$ is the partition function of a free theory. So again we have (like eq.(8))

$$\langle \xi^{a_1}(x_1) ... \xi^{a_n}(x_2n) >_{1,\Sigma_g} = \sum_\mu G^{a_1 a_2}(x_1, x_2) ... G^{a_{2n-1} a_{2n}}(x_{2n-1}, x_{2n}).$$ \hspace{1cm} (30)

where

$$G^{ab}(x, y) = \epsilon^{ab} \delta(x - y).$$ \hspace{1cm} (31)

This completes the expression of the correlators of the Yang-Mills theory on $\Sigma_g$. Again they consist of a free part and a part which is almost $x$-independent, that is, it depends only on the angular ordering of the coordinates.

Now the important question that arises is that which part of the above results are gauge-invariant. First notice that the wavefunction (23) is not gauge-invariant. This can be checked by noting that it is not invariant under the transformation $J(x) \rightarrow U(x) J(x) U^{-1}(x)$ (which induces the gauge transformation), because of the character term $\chi_\lambda(P \exp \int J)$. Now try to divide the disk $D$ to $N$ parts and consider the wavefunction (23) for the disk $D_N = D/N$, (with area $A/N$). Then try to find the wavefunction of the disk $D$ by gluing the wavefunction of the small disks. This is justified only if the wavefunction is gauge invariant, because the radial and angle variables in each disk is not the same as the other disks. However, as $N$ tends to infinity, it is enough to calculate the wavefunction of the small disks only up to first order of $(A/N)$. But, up to first order, we have

$$\psi_{2, D/N}[J] = \sum_\lambda \chi_\lambda(g_1^{-1}) e^{ -2g c_2(\lambda) A(\Sigma_g)} d(\lambda),$$ \hspace{1cm} (32)
which is gauge invariant. Gluing these we find
\[ Z_{\Sigma_g}^{G.I.}[J] = e^{\frac{\nu}{2} \int J^\mu J_\mu} Z_{\Sigma_g}[0]. \] (33)

Therefore the gauge invariant part of the correlators are those quoted in eq.(30) which are free.

Another method of calculating the gauge-invariant correlators is using the expectation value of Wilson loops (which are gauge-invariant) [7]:
\[ <\chi_\mu(\text{Pexp} \int_\gamma A)> = \frac{1}{Z_{\Sigma_g}[0]} \sum_\lambda \sum_{\rho \in \lambda \otimes \mu} d(\lambda)d(\rho)^{1-2g} \exp\{-\epsilon[c_2(\lambda)A(D)+c_2(\rho)(A(\Sigma_g)-A(D))]\} \] (34)

where \( \gamma = \partial D \). If, by symmetry consideration, we use the following ansatz for the gauge-invariant two point function:
\[ <\xi^a(x)\xi^b(y)>^{G.I.} = M\delta^{ab}\delta(x-y), \] (35)

and, for small \( A(D) \), equate the linear term of both sides of eq.(34), we will find:
\[ M = -\frac{\epsilon}{d(\mu)c_2(\mu)Z_{\Sigma_g}[0]} \sum_\lambda \sum_{\rho \in \lambda \otimes \mu} d(\lambda)d(\rho)^{1-2g} \exp\{-\frac{\nu}{2}c_2(\rho)(A(\Sigma_g)-A(D))\}(c_2(\rho) - c_2(\lambda)). \] (36)

Then, using the identities:
\[ \sum_{\rho \in \lambda \otimes \mu} d(\rho)c_2(\rho) = d(\lambda)d(\mu)(c_2(\lambda) + c_2(\mu)) \]
\[ \sum_{\rho \in \lambda \otimes \mu} d(\rho) = d(\lambda)d(\mu). \] (37)

it is seen that \( M = \epsilon \), which is consistent with (31).

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