Some elementary rigorous remark about the replica formalism in the Statistical Physics’ approach to threshold phenomena in Computational Complexity Theory

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I. INTRODUCTION

The adoption of methods and ideas from Statistical Physics in the analysis of threshold phenomena in Computational Complexity Theory \cite{1} is a very interesting research field \cite{2}, \cite{3}.

Unfortunately some of these methods and ideas has not reached yet the level of mathematical rigor of Mathematical Physics and Theoretical Computer Science.

This applies in particular as to the replica formalism and as to the concept of replica symmetry breaking.

As prototype of this situation let us consider the satisfability threshold conjecture for the problem random $k$-SAT: given a uniformly-distributed random boolean formula in conjunctive normal form involving $n$ boolean variables $x_1, \cdots, x_n$ and $m$ clauses each of length $k$ (i.e. containing $k$ literals) such a conjecture states the existence of a critical value $\alpha_c$ for the clause density $\alpha := m/n = \lim_{n,m \to \infty} m/n$ such that for every $\epsilon > 0$ in the limit $n \to \infty$ the probability that a formula is satisfiable tends to 1 if $\alpha < (1 - \epsilon)\alpha_c$ while tends to 0 if $\alpha > (1 + \epsilon)\alpha_c$.

Introduced the spin variables $s_i := 2x_i - 1$ and introduced the clause matrix $J$ such that:

$$J_{ji} := \begin{cases} +1, & \text{if clause } j \text{ includes the literal } x_i; \\ -1, & \text{if clause } j \text{ includes the literal } \overline{x}_i; \\ 0, & \text{otherwise}. \end{cases}$$ (1.1)

it follows that that the number of violated clauses may be expressed as:

$$H(s_1, \cdots, s_n) := \frac{1}{2^k} \sum_{j=1}^m \prod_{i=1}^n (1 - J_{ji}s_i)$$ (1.2)

The Statistical Physics’ approach to the random $k$-SAT problem consists in considering $H$ as the hamiltonian of a spins’ system $Sys$ at thermodynamical equilibrium at temperature $T$ whose canonical partition function (we adopt units in with $k_B = 1$) is hence:

$$Z := \sum_{s_1, \cdots, s_n} \exp\left(-\frac{H(s_1, \cdots, s_n)}{T}\right)$$ (1.3)

Denoting the thermodynamic average over the spins with brackets $\langle \cdot \rangle$ and the average over the random instances with a bar $\bar{}$ one has then that the averaged number of violated clauses can be expressed as:

$$\bar{E} := \langle H \rangle = \lim_{T \to 0} \lim_{n \to \infty} T \log Z$$ (1.4)

Using the formula:

$$\log Z = \lim_{r \to 0} \frac{Z^r - 1}{r}$$ (1.5)

one can then express $\bar{E}$ as:

$$\bar{E} = \lim_{T \to 0} T \lim_{n \to \infty} \lim_{r \to 0} \frac{Z^r - 1}{r}$$ (1.6)

The first step of the replica formalism consists in expressing $Z^r$, for $r \in \mathbb{N}_+$, as the partition function of $r$ non-interacting replicas $Sys_1, \cdots, Sys_r$ of the system $Sys$:

$$Z^r = \sum_{s_1^1, \cdots, s_n^1} \cdots \sum_{s_1^r, \cdots, s_n^r} \exp\left(-\sum_{a=1}^r \frac{H(s_1^a, \cdots, s_n^a)}{T}\right)$$ (1.7)

The next step in the replica formalism consists in "prolonging analytically" such an expression of $Z^r$ to $r \in \mathbb{R}$ and to substitute it into equation (1.6) obtaining:

$$\bar{E} = \lim_{T \to 0} T \lim_{n \to \infty} \lim_{r \to 0} \frac{\sum_{s_1^1, \cdots, s_n^1} \cdots \sum_{s_1^r, \cdots, s_n^r} \exp\left(-\sum_{a=1}^r \frac{H(s_1^a, \cdots, s_n^a)}{T}\right) - 1}{r}$$ (1.8)

As correctly stated in \cite{3} as well as in the section 5.4.3 "The replica approach" of \cite{2}, anyway, from a mathematical point of view such a formalism is absolutely nonsense:
1. The existence of the thermodynamical limit $n \to \infty$ is not obvious and has to be proved.

2. Assuming the existence of the thermodynamical limit, the fact that for finite $n$ the probability distribution of $Z$ is determined by its moments $\{Z_r, r \in \mathbb{N}\}$ ceases to hold when $n \to \infty$.

3. An analytic function is not determined by the values assumed on a countable set $\mathbb{N}$.

Hence equation 1.8 has no mathematical meaning.

Actually what one obtains in such a formalism is an expression of the form:

$$Z_r = \int \prod_{\sigma} du_{\sigma} \delta(\sum_{\sigma} u_{\sigma} - 1) \exp(nF(\bar{u}))$$  \hspace{1cm} (1.9)

where $\bar{\sigma} \in \{-1, 1\}^r$, $u_{\sigma} \in \mathbb{R}^2$ while $F: \mathbb{R}^{2^r} \to \mathbb{R}$ is a suitable function.

One then uses a saddle-point approximation of such an integral:

$$Z_r = \exp(n \exp F_{\text{max}} + o_{n \to \infty}(n))$$  \hspace{1cm} (1.10)

where with $o_{n \to \infty}(n)$ we denote a quantity tending to infinity (for $n \to \infty$) more slowly than $n$.

The function $F$ is symmetric under permutations of the replicas; hence as long as a certain vector $\bar{u}^*$ maximizes $F$, so too does any vector $\bar{u}$ such that $u_{\sigma_1, \ldots, \sigma_r} = u_{\sigma_{\pi(1)}, \ldots, \sigma_{\pi(r)}} \sigma \in S_r$.

The assumption that $F$ has a unique maximum that is itself invariant under replicas’ permutation:

$$u_{\sigma_1, \ldots, \sigma_r}^* = u_{\sigma_{\pi(1)}, \ldots, \sigma_{\pi(r)}}^* \forall \pi \in S_r$$  \hspace{1cm} (1.11)

is known as the assumption of replica symmetry while its negation is called replica symmetry breaking since from a group theoretical viewpoint, it consists in a breakage of the permutational symmetry $S_r$ (under which the hamiltonian $\sum_{a=1}^n H(s_a^1, \ldots, s_a^n)$ is of course invariant) reducing the symmetry of the system to a suitable subgroup $G_r \subset S_r$.

Under the assumption of replica symmetry the computation of $Z_r$ would seem to support the satisfiability threshold conjecture since one finds a threshold value $\alpha_c$ such that:

$$\bar{E} = 0 \text{ for } \alpha < \alpha_c$$  \hspace{1cm} (1.12)

$$\bar{E} > 0 \text{ for } \alpha > \alpha_c$$  \hspace{1cm} (1.13)

For the exposed reasons, anyway, such an argument has no mathematical consistence.

The situation is even worse as to the investigations of the phase structure of random $k$-SAT involving replica symmetry breaking, a concept of which no consistent mathematical formalization exists.

The original explanation of such a concept given by Parisi and coworkers in the section 3.5 “Replica Symmetry Breaking: the Final Formulation” of [5] reminds Ionesco’s Absurd’s Theater\(^1\): indeed the breakage of the permutational symmetry $S_r \to G_r \subset S_r$ is therein augmented with a nonsense “analytic continuation” to $r \in \mathbb{R}$ that in the limit $r \to 0$ is claimed to imply that the group of permutations of zero objects $S_0$ would contain itself as a subgroup.

Of course nothing of such an explanation is mathematical meaningful:

if Tom has zero apples the number of ways in which he can order them is of course zero.

Indeed, despite the many efforts to recast Parisi’s theory concerning the mean field approximation of the Sherrington-Kirkpatrick’s model into a mathematically meaningful form [7] (not to speak about the more critical viewpoints concerning such a theory such as those exposed in [8], [9]) the whole replica formalism still lacks of any mathematical rigor.

In this brief notes we will present some elementary but rigorous argument that could be useful to recast some feature of such a formalism in a mathematically consistent framework.

\(^1\) so reaching a ”dramatic tension” strongly higher than the Shakespeare’s one cited in the introduction of [5].
II. PERMUTATION GROUP OF A SET

Given a set $X$ let us introduce the following:

**Definition II.1**

*permutation on $X$:*

a bijective map $p : X \mapsto X$

**Definition II.2**

*permutation group of $X$:*

$$(\text{Perm}(X), \cdot)$$

where:

- $\text{Perm}(X) := \{ p : \text{permutation on } X \}$
- $\cdot$ is the map composition

Let us recall the following basic [9]:

**Theorem II.1**

CAYLEY THEOREM:

HP:

$G_1$ group

TH:

$$\exists X \text{ set, } \exists G_2 \text{ subgroup of } \text{Perm}(X) : G_1 \sim_{is} G_2$$

where $\sim_{is}$ denotes the isomorphism equivalence relation.

**Example II.1**

Let us suppose that $|X| = n \in \mathbb{N}_+$. Then $\text{Perm}(X) = S_n$ is the $n^{th}$ symmetric group. One has clearly that $|\text{Perm}(X)| = n!$. Theorem [II.1] implies that any finite group of order $n$ is isomorphic to a subgroup of $S_n$.

Let us now consider the set $S_0 := \text{Perm}(\emptyset)$.

One has clearly that:

$$|S_0| = 0 \quad (2.1)$$

and hence:

$$S_0 = \emptyset \quad (2.2)$$

Let us now suppose to have a set $X$ such that $\text{Perm}(X)$ possesses the property that Parisi and coworkers erroneously ascribe to $S_0$: the property of being isomorphic to a subgroup $G$ of its$^2$:

$$\text{Perm}(X) \sim_{is} G \subset \text{Perm}(X) \quad (2.3)$$

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$^2$ Actually we don’t know if such a set $X$ exists; here we assume the existence of such a set to derive some property that, if it exists, $X$ must possess.
Since $G$ is in particular a subset of $\text{Perm}(X)$, $\text{Perm}(X)$ is bijective to a proper subset of its and hence it is an infinite set:

$$|\text{Perm}(X)| \geq \aleph_0$$  \hspace{1cm} (2.4)

from which it follows that:

$$|X| \geq \aleph_0$$  \hspace{1cm} (2.5)

In particular $X \neq S_0$. 
III. ONE PARAMETER FAMILIES OF PERMUTATION GROUPS

Let us consider a one-parameter family of sets \( \{X_\alpha, \alpha \in \mathbb{R}\} \) such that:

\[
|X_n| = n \quad \forall n \in \mathbb{N} \tag{3.1}
\]

So in particular:

\[
|X_0| = 0 \tag{3.2}
\]

and hence:

\[
X_0 = \emptyset \tag{3.3}
\]

Let us now observe that:

\[
|\text{Perm}(X_n)| = n! = \Gamma(n + 1) \quad \forall n \in \mathbb{N}_+ \tag{3.4}
\]

where:

\[
\Gamma(z) := \int_0^\infty t^{z-1} \exp(-t) \, dt \text{ for } \text{Re}(z) > 0 \tag{3.5}
\]

is the Euler Gamma function \[4\].

Let us remark that:

\[
|\text{Perm}(X_0)| \neq 0! = \Gamma(1) = 1 \tag{3.6}
\]

Clearly the right-hand side of equation (3.4) is well defined on the whole interval \((-1, +\infty)\) and in particular:

\[
\lim_{n \to 0} \Gamma(n + 1) = \Gamma(1) = 0! = 1 \tag{3.7}
\]

Let us observe anyway that:

\[
|\text{Perm}(X_\alpha)| \neq \Gamma(\alpha + 1) \quad \forall \alpha \in \mathbb{R} - \mathbb{N} \tag{3.8}
\]

since for every set \(S\):

\[
|S| \in \mathbb{N} \cup \{\aleph_n, n \in \mathbb{N}\} \tag{3.9}
\]
IV. CONSECUTIVE REPLICA SYMMETRY BREAKINGS

Let us consider a system \( S := \{s_1, \cdots, s_n\} \) consisting of \( n \in \mathbb{N} \) sub-systems. Let us suppose that initially the \( n \) sub-systems are identical. This means that for every property \( P \) one has that:

\[
P(\{s_1, \cdots, s_n\}) = P(\{s_{\pi(1)}, \cdots, s_{\pi(n)}\}) \quad \forall \pi \in S_n
\]  

(4.1)

In physical terms this means that \( S_n \) is a symmetry of the system.

Given a number \( m_1 \in \mathbb{N} : \frac{m_1}{m_1} \in \mathbb{N} \) let us divide the system \( S \) in \( \frac{m_1}{m_1} \) groups \( g_1^1 := \{s_1, \cdots, s_{m_1}\}, \cdots, g_{\frac{m_n}{m_1}}^1 := \{s_{n-m_1}, \cdots, s_n\} \) each consisting of \( m_1 \) elements.

Let us now suppose to differentiate the systems belonging to a group \( g_i^1 \) from those belonging to a different group \( g_j^1 \), \( i \neq j \); this means that for every property \( P \):

\[
P(x) = P(y) \quad \forall x, y \in g_i^1, \forall i = 1, \cdots, \frac{n}{m_1}
\]  

(4.2)

but that exists a property \( P \) such that:

\[
P(x) \neq P(y) \quad \forall x \in g_i^1, y \in g_j^1 : i \neq j
\]  

(4.3)

In physical terms this means to perform the symmetry breaking \( S_n \rightarrow G_{n;m_1} \) where:

\[
G_{n;m_1} := (S_{m_1})^{\frac{m_1}{m_1}}
\]  

(4.4)

In fact the system is now symmetric only under the \( m_1! \) permutations of the elements inside each group.

Clearly:

\[
|G_{n;m_1}| = (m_1!)^{\frac{m_1}{m_1}}
\]  

(4.5)

Let us observe that, contrary to what is claimed in the section 3.5 "Replica Symmetry Breaking: the Final Formulation" of [2], the system now is not invariant under the \( \left(\frac{m_1}{m_1}\right)! \) permutations of the groups \( g_1^1, \cdots, g_{\frac{m_n}{m_1}}^1 \) since now there exists a property distinguishing these groups.

Given a number \( m_2 \in \mathbb{N} \) such that \( \frac{m_1}{m_2} \in \mathbb{N} \) let us divide each group \( g_i^1 \) in \( \frac{m_1}{m_2} \) sub-groups each consisting of \( m_2 \) elements; so the group \( g_i^1 \) is divided in the subgroups \( g_i^2, \cdots, g_{\frac{m_1}{m_2}}^2 \) and so on.

Let us now suppose to differentiate the systems belonging to a group \( g_i^2 \) from those belonging to a different group \( g_j^2 \), \( i \neq j \); this means that for every property \( P \):

\[
P(x) = P(y) \quad \forall x, y \in g_i^2, \forall i = 1, \cdots, \frac{n}{m_2}
\]  

(4.6)

but that exists a property \( P \) such that:

\[
P(x) \neq P(y) \quad \forall s_1 \in g_i^2, s_2 \in g_j^2 : i \neq j
\]  

(4.7)

In physical terms this means to perform the symmetry breaking \( G_{n;m_1} \rightarrow G_{n;m_2} \) where of course:

\[
G_{n;m_2} := (S_{m_2})^{\frac{m_2}{m_2}}
\]  

(4.8)

Such a procedure can be iterated a certain number of times; supposed that there exist \( k+1 \in \mathbb{N} \) natural numbers \( m_1, \cdots, m_k \) such that:

\[
\frac{n}{m_1} \in \mathbb{N} \text{ and } \frac{m_i}{m_{i+1}} \in \mathbb{N} \quad i = 1, \cdots, k
\]  

(4.9)

at the \( k^{th} \) step one performs the symmetry breaking:

\[
G_{n;m_k} \rightarrow G_{n;m_{k+1}}
\]  

(4.10)

where of course:

\[
G_{n;m_k} := (S_{m_k})^{\frac{m_k}{m_k}}
\]  

(4.11)
Let us observe, anyway, that exists a maximum number $k_{\text{max}} \in \mathbb{N}$ of possible consecutive replica symmetry breakings.

If one was interested in performing the limit $n \to \infty$ one could argue that the maximum number of possible consecutive replica symmetry breaking "tends to infinity" in the following sense: for every $k \in \mathbb{N}$ there exists an $n \in \mathbb{N}$ and $k+1$ numbers $m_1, \cdots, m_{k+1} \in \mathbb{N}$ such that:

$$\frac{n}{m_1} \in \mathbb{N} \land \left( \frac{m_i}{m_{i+1}} \in \mathbb{N} \land i = 1, \cdots, k \right)$$

Unfortunately Parisi's theory involves instead a mathematically inconsistent limit of the iteration's procedure for $n \to 0$. 

$$G_{n;m_{k+1}} := (S_{m_{k+1}})_{m_{k+1}}^n$$ (4.12)
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