Bernstein’s inequalities for general Markov chains

Bai Jiang\textsuperscript{*} \hspace{1cm} Qiang Sun\textsuperscript{†} \hspace{1cm} Jianqing Fan\textsuperscript{‡}

Abstract

We establish Bernstein’s inequalities for functions of general (general-state-space and possibly non-reversible) Markov chains. These inequalities achieve sharp variance proxies and encompass the classical Bernstein inequality for independent random variables as special cases. The key analysis lies in bounding the operator norm of a perturbed Markov transition kernel by the exponential of sum of two convex functions. One coincides with what delivers the classical Bernstein inequality, and the other reflects the influence of the Markov dependence. A convex analysis on these two functions then derives our Bernstein inequalities. As applications, we apply our Bernstein inequalities to the Markov chain Monte Carlo integral estimation problem and the robust mean estimation problem with Markov-dependent samples, and achieve tight deviation bounds that previous inequalities can not.

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\textsuperscript{*}Department of Operations Research and Financial Engineering, Princeton University, Princeton, NJ 08544, USA; Email: baij@princeton.edu.

\textsuperscript{†}Department of Statistical Sciences, University of Toronto, Toronto, ON M5G 1X6, Canada; Email: qiang.sun@utoronto.ca.

\textsuperscript{‡}Department of Operations Research and Financial Engineering, Princeton University, Princeton, NJ 08544, USA; Email: jqfan@princeton.edu.
Concentration inequalities are tail probability bounds regarding how the average of random variables deviates from its expectation. They have found enormous applications in statistics, machine learning, information theory and other related fields. Many concentration inequalities are derived by the Cramér-Chernoff method (Boucheron et al., 2013, Section 2.2). Let $Z_1, \ldots, Z_n$ be $n$ random variables with mean zero. Suppose the moment generating function (mgf) of $\sum_{i=1}^n Z_i$ is bounded as
\[
E\left[e^{t\sum_{i=1}^n Z_i}\right] \leq e^{ng(t)}, \quad \forall \ t \in \mathbb{R},
\] (1)
with some convex function $g(t)$. Let $g^*(\epsilon) := \sup\{t\epsilon - g(t) : \ t \in \mathbb{R}\}$ for $\epsilon > 0$ be the Fenchel conjugate of $g(t)$, then the Cramér-Chernoff method derives that
\[
\mathbb{P}\left(\frac{1}{n} \sum_{i=1}^n Z_i \geq \epsilon\right) \leq e^{-ng^*(\epsilon)}, \quad \forall \ \epsilon > 0.
\] (2)

According to the conjugation and second-order properties of convex functions (Gorni, 1991), if there exists a constant $V > 0$ such that
\[
g(t) = \frac{Vt^2}{2} + o(t^2) \quad \text{as } t \to 0,
\]
then
\[
g^*(\epsilon) = \frac{\epsilon^2}{2V} + o(\epsilon^2) \quad \text{as } \epsilon \to 0.
\]
That means, a convex function $g(t)$ in the mgf bound (1) with a sharp constant $V = \lim_{t \to 0} \frac{2g(t)}{t^2}$ leads to a tight concentration in the inequality (2).

We refer to this constant $V$ as the variance proxy of the concentration inequality, as it not only characterizes the shape of the probability tail in (2) but also bounds the variance of
Indeed, bringing \( \sum_{i=1}^{n} Z_i \) \( \sqrt{n} \). Hence,

\[
\mathbb{E} \left[ e^{t \sum_{i=1}^{n} Z_i} \right] = 1 + \mathbb{E} \left[ \sum_{i=1}^{n} Z_i \right] \cdot t + \mathbb{E} \left[ \left( \sum_{i=1}^{n} Z_i \right)^2 \right] \cdot \frac{t^2}{2} + \ldots
\]

\[
= 1 + \text{Var} \left[ \sum_{i=1}^{n} Z_i \right] \cdot \frac{t^2}{2} + o(t^2), \text{ and}
\]

\[
e^{ng(t)} = 1 + ng(t) + \frac{(ng(t))^2}{2} + \ldots
\]

\[
= 1 + nV \cdot \frac{t^2}{2} + o(t^2)
\]

into both sides of (1) yields \( \text{Var} \left( \sum_{i=1}^{n} Z_i \right) \cdot t^2 \leq nV \cdot t^2 + o(t^2) \). Dividing both sides by \( t^2 \) and taking \( t \to 0 \) yield \( \text{Var} \left( \sum_{i=1}^{n} Z_i \right) \leq nV \).

Bernstein (1946), Bennett (1962) and Hoeffding (1963) pioneered the study of concentration inequalities for independent random variables. Suppose \( Z_1, Z_2, \ldots, Z_n \) are independent random variables with \( \mathbb{E} Z_i = 0 \) and \( |Z_i| \leq c \). Hoeffding’s inequality holds in the form of (1) and (2) with

\[
g(t) = \frac{c^2 t^2}{2}, \quad g^*(\epsilon) = \frac{c^2}{2\epsilon^2}, \quad V = c^2.
\]

Bennett’s inequality takes variances of random variables into account and holds with

\[
g(t) = \frac{\sigma^2}{c^2}(e^{tc} - 1 - tc), \quad g^*(\epsilon) = \frac{\sigma^2}{c^2} h \left( \frac{\epsilon c}{\sigma^2} \right), \quad V = \sigma^2, \quad (3)
\]

where \( \sigma^2 = \frac{\sum_{i=1}^{n} \text{Var}(Z_i)}{n} \) and \( h : u \in [0, \infty) \mapsto (1 + u) \log(1 + u) - u \). Lower bounding \( h(u) \geq \frac{u^2}{2(1+u/3)} \) in Bennett’s inequality further derives Bernstein’s inequality

\[
\mathbb{P} \left( \frac{1}{n} \sum_{i=1}^{n} Z_i \geq \epsilon \right) \leq \exp \left( -\frac{ne^2}{2(\sigma^2 + c\epsilon/3)} \right). \quad (4)
\]

Much efforts have been made to extend concentration inequalities from independent random variables to dependent random variables. Some of them focus on the Markov dependence, an important dependence structure. In the setting of the Markov dependence, \( Z_1, Z_2, \ldots, Z_n \) are assumed to be functions of a Markov chain \( X_1, X_2, \ldots, X_n \), i.e., \( Z_i = f_i(X_i) \).

A large body of this literature utilize the spectral analysis of the Markov operator (transition kernel) in the Hilbert space \( L_2 \) to develop concentration inequalities for Markov chains. León and Perron (2004) proved a Hoeffding-type inequality with a sharp variance proxy by using a convex majorization of the transition kernel for a finite-state-space, reversible Markov chain and a time-independent function \( f_i = f \). Miasojedow (2014) obtained a variant of Léon and Perron’s inequality for a general (general-state-space, possibly non-reversible) Markov chain. Fan et al. (2021) improved upon them and established a Hoeffding-type inequality with the optimal variance proxy for general Markov chains and time-dependent functions \( f_i \)’s.
While Hoeffding-type inequalities use the bound of random variables $|Z_i| \leq c$ (or $|f_i| \leq c$ in the setting of the Markov dependence) only, both Bennett-type and Bernstein-type inequalities incorporate variances of $Z_i$ and thus could be sharper. For a finite-state-space, reversible Markov chain $\{X_i\}_{i \geq 1}$ and a time-independent function $f_i = f$, Lezaud (1998a) proved a Chernoff-type inequality which incorporates variances of $Z_i = f(X_i)$. Lezaud’s analysis recently inspired several Bernstein-type inequalities in (Paulin, 2015).

This paper establishes two Bernstein-type inequalities for general Markov chains. One regards the time-dependent function case in which $f_i$, $i = 1, 2, \ldots, n$ are different. This inequality encompasses the classical Bernstein’s inequality for independent random variables. The other regards the time-independent function case in which $f_i = f$, $i = 1, 2, \ldots, n$ are identical. This inequality encompasses the classical Bernstein’s inequality for independently and identically distributed (i.i.d.) random variables. These two Bernstein-type inequalities achieve optimal variance proxies and tightest concentration, as their exponential rates are reached asymptotically by a class of Markov chains.

A few other concentration inequalities for Markov chains were developed by exploiting the minorization and drift conditions (Glynn and Ormoneit, 2002; Douc et al., 2011; Adamczak and Bednorz, 2015), or information-theoretical ideas (Kontoyiannis et al., 2005, 2006) rather than the spectral analysis. These inequalities therefore do not require the Markov chain under study to have a non-zero spectral gap, a quantity measuring the convergence speed of a Markov transition kernel in the spectral analysis. But they usually have less explicit and often sub-optimal constants or more complicated expressions. A less related line of research includes Marton’s work on the concentration of measure phenomenon for contracting Markov chains (Marton, 1996a, b, 1998, 2003, 2004) and further progresses under various dependence structures (Samson, 2000; Chazottes et al., 2007; Kontorovich and Ramanan, 2008; Redig and Chazottes, 2009).

The rest of this paper proceeds as follows. Section 2 presents our Bernstein-type inequalities, extensions to the nonstationary Markov chains, and an impossibility result. Section 3 summarizes preliminary results about the spectral analysis. Section 4 collects technical proofs of main theorems. Section 5 presents two applications of our newly-derived Bernstein-type inequalities. Section 6 concludes the paper with a discussion on the novelty of our results.

2 Main results

2.1 Bernstein’s inequalities for general Markov chains

We introduce some notation before presenting main theorems. Denote by $\pi$ the stationary distribution of a Markov chain $\{X_i\}_{i \geq 1}$. Let $\pi(h) = \int h(x) \pi(dx)$ be the integral of real-valued function $h$ with respect to $\pi$. Denote by $L_2 := \{h : \pi(h^2) < \infty\}$ the Hilbert space of $\pi$-square-integrable functions endowed with the inner product $\langle h_1, h_2 \rangle_\pi = \pi(h_1 h_2)$, and by $L_2^0 := \{h \in L_2 : \pi(h) = 0\}$ the subspace of mean zero functions. The transition kernel $P(x, y)$
of the Markov chain induces an integral operator, called Markov operator, acting in \( L^2 \), i.e., 
\[ Ph(x) = \int P(x, y) h(y) dy. \]
Here we abuse the notation \( P \) for both the transition kernel and the Markov operator. Denote by \( P^* \) the adjoint of \( P \), i.e., \( P^*(x, y) = \pi(y) P(y, x) / \pi(x) \). Refer to \((P + P^*)/2\) as the additive reversibilization of \( P \), as it is self-adjoint and corresponds to a reversible transition kernel (Fill, 1991).

Let \( \lambda(P) \in [0, 1] \) be the operator norm of \( P \) acting in \( L^2 \), and refer to \( 1 - \lambda(P) \) as the absolute spectral gap of \( P \) (see Definition 1). Denote by \( \sigma_0(P) \) the spectrum of \( P \) acting in complexified \( L^2 \). A Markov chain is reversible if \( P^*(x, y) = P(x, y) \). For reversible Markov chains, \( P \) is self-adjoint and \( \sigma_0(P) \subseteq [-1, +1] \) are real-valued. Let \( \lambda_r(P) : = \sup \{ \theta : \theta \in \sigma_0(P) \} \) and refer to \( 1 - \lambda_r(P) \) as the right spectral gap of \( P \). An equivalent definition of \( \lambda_r(P) \) is given by 
\[ \lambda_r(P) = \sup \{ \langle h, Ph \rangle_\pi : h \in L^2, \langle h, h \rangle_\pi = 1 \}. \]
For non-reversible Markov chains, \( P \) is not self-adjoint, and \( \sigma_0(P) \) are not real-valued. However, \( \langle h, Ph \rangle_\pi = \langle h, ((P + P^*)/2) h \rangle_\pi \) for any \( h \in L^2 \) and \((P + P^*)/2\) is self-adjoint. We thereafter extend the definition of the right spectral gap for non-reversible Markov chains by letting \( \lambda_r(P) : = \lambda_r((P + P^*)/2) \) (see Definition 2). Evidently, \( 0 \leq \text{max} \{ \lambda_r(P), 0 \} \leq \lambda(P) \leq 1 \).

The next two theorems present the main results of this paper.

**Theorem 1.** Suppose \( \{X_t\}_{t \geq 1} \) is a stationary Markov chain with invariant distribution \( \pi \) and absolute spectral gap \( 1 - \lambda > 0 \), and \( f_i \)'s are functions with \( |f_i| \leq c \) and \( \pi(f_i) = 0 \). Let \( \sigma^2 = \sum_{i=1}^n \pi(f_i^2)/n \). Then, for any \( 0 \leq t < (1 - \lambda)/5c \),
\[
\mathbb{E}_\pi \left[ e^{t \sum_{i=1}^n f(X_i)} \right] \leq \exp \left( \frac{n \sigma^2}{c^2} (e^c - 1 - tc) + \frac{n \sigma^2 \lambda^2 t^2}{1 - \lambda - 5ct} \right). \tag{5}
\]
Moreover, for any \( \epsilon > 0 \),
\[
\mathbb{P}_\pi \left( \frac{1}{n} \sum_{i=1}^n f(X_i) \geq \epsilon \right) \leq \exp \left( -\frac{ne^{2}/2}{\alpha_1(\lambda) \cdot \sigma^2 + \alpha_2(\lambda) \cdot c \epsilon} \right), \tag{6}
\]
where
\[ \alpha_1(\lambda) = \frac{1 + \lambda}{1 - \lambda}, \quad \alpha_2(\lambda) = \begin{cases} \frac{1}{3} & \text{if } \lambda = 0, \\ \frac{5}{(1 - \lambda)} & \text{if } \lambda \in (0, 1). \end{cases} \]

**Theorem 2.** Suppose \( \{X_t\}_{t \geq 1} \) is a stationary Markov chain with invariant distribution \( \pi \) and right spectral gap \( 1 - \lambda_r > 0 \), and \( f \) is a function with \( |f| \leq c \) and \( \pi(f) = 0 \). Let \( \sigma^2 = \pi(f^2) \). Then, for any \( 0 \leq t < (1 - \text{max}\{\lambda_r, 0\})/5c \),
\[
\mathbb{E}_\pi \left[ e^{t \sum_{i=1}^n f(X_i)} \right] \leq \exp \left( \frac{n \sigma^2}{c^2} (e^c - 1 - tc) + \frac{n \sigma^2 \max\{\lambda_r, 0\}^2 t^2}{1 - \max\{\lambda_r, 0\} - 5ct} \right). \tag{7}
\]
Moreover, for any \( \epsilon > 0 \),
\[
\mathbb{P}_\pi \left( \frac{1}{n} \sum_{i=1}^n f(X_i) \geq \epsilon \right) \leq \exp \left( -\frac{ne^{2}/2}{\alpha_1(\text{max}\{\lambda_r, 0\}) \cdot \sigma^2 + \alpha_2(\text{max}\{\lambda_r, 0\}) \cdot c \epsilon} \right), \tag{8}
\]
where \( \alpha_1, \alpha_2 \) are defined in Theorem 1.
These results compare well with the classical Bernstein’s inequalities for independent random variables. The exponent on the right-hand side of (5) consists of two terms. The first term $\frac{\sigma^2}{c^2}(e^{tc} - 1 - tc)$ coincides with the convex function $g(t)$ in the mgf bound (3) of the classical Bennett’s and Bernstein’s inequalities. The second term $\frac{\sigma^2}{c^2}(e^{tc} - 1 - tc)$ results from the Markov dependence and decreases to 0 as $\lambda$ decreases to zero. Consequently, Theorem 1 encompasses the classical Bennett’s and Bernstein’s inequalities under independence as special cases. Indeed, independent random variables $Z_1, Z_2, \ldots, Z_n$ can be seen as transformations of i.i.d. random variables $U_1, \ldots, U_n ~ \text{Uniform}[0,1]$ via the inverse cumulative distribution functions $F_{Z_i}^{-1}$, i.e., $Z_i = f_i(U_i)$ with $f_i = F_{Z_i}^{-1}$. The i.i.d. random variables $\{U_i\}_{i=1}^n$ form a stationary Markov chain with $\lambda = 0$. In this case, (5) is simplified to be
\[
\mathbb{E}\left[e^{t\sum_{i=1}^n Z_i}\right] \leq \exp\left(\frac{n\sigma^2}{c^2}(e^{tc} - 1 - tc)\right),
\]
which is exactly the mgf bound deriving the classical Bennett’s and Bernstein’s inequalities. In a similar vein, Theorem 2 encompasses the classical Bennett’s and Bernstein’s inequalities for i.i.d. random variables as special cases.

The variance proxy of the Bernstein-type inequality (6) in Theorem 1 is optimal in the sense that it is attainable by a class of Markov chains. Consider a stationary Markov chain $\{X_i\}_{i=1}^n$ with transition kernel $P(x,B) = \lambda\delta(x \in B) + (1 - \lambda)\pi(B)$ for state $x$ and any measurable subset $B$ of the state space. This chain has invariant distribution $\pi$ and absolute (and right) spectral gap $1 - \lambda$. For function $f$ with $|f| \leq c$, $\pi(f) = 0$ and $\pi(f^2) = \sigma^2$, it follows from the Central Limit Theorem for reversible Markov chains (Kipnis and Varadhan, 1986; Geyer, 1992; Rosenthal, 2003) that the asymptotic variance is given by
\[
\sigma_{as}^2(f) := \lim_{n \to \infty} \text{Var}\left(\frac{\sum_{i=1}^n f(X_i)}{\sqrt{n}}\right) = \frac{1 + \lambda}{1 - \lambda} \sigma^2.
\]
Recall that any variance proxy upper bounds $\frac{\text{Var}(\sum_{i=1}^n f(X_i))}{\sqrt{n}}$ for any $n \geq 1$. It follows that $V \geq \sigma_{as}^2(f) = \frac{1 + \lambda}{1 - \lambda} \sigma^2$. This lower bound for the variance proxy is attained by (6). Similarly, the variance proxy of the Bernstein-type inequality (8) in Theorem 2 is optimal if $\lambda_r \geq 0$, which is the case for Markov chains generated by popular MCMC algorithms such as the Metropolis-Hastings algorithm (Atchadé and Perron, 2007).

Tables 1 and 2 summarize some of the recent results on Hoeffding-type and Bernstein-type inequalities for Markov chains along with the classical Hoeffding’s, Bennett’s and Bernstein’s inequalities for the time-dependent function case and the time-independent function case, respectively. Notation in both tables are the same as in Theorems 1 and 2. Paulin (2015) generalized the absolute spectral gap for non-reversible Markov chains in a different way to ours. His Bernstein-type inequalities for non-reversible Markov chains involve a quantity called “pseudo spectral gap”. The relation between his “pseudo spectral gap” and the absolute spectral gap we use is unclear, and thus his Bernstein-type inequalities for non-reversible Markov chains are not included in the tables.
Table 1: Concentration inequalities for time-dependent functions $f_i$ of Markov chains.

| type     | reference                  | condition                        | variance proxy |
|----------|----------------------------|----------------------------------|----------------|
| Hoeffding| Hoeffding, 1963            | independent                      | $c^2$          |
| Hoeffding| Fan et al., 2021           | general-state-space              | $\frac{1+\lambda}{1-\lambda} \cdot c^2$ |
| Bernstein| Bernstein, 1946            | independent                      | $\sigma^2$     |
| Bennett  | Bennett, 1962              | independent                      | $\sigma^2$     |
| Bernstein| Paulin, 2015, (3.22)       | general-state-space, reversible  | $\frac{1}{1-\lambda} \cdot \sigma^2$ |
| Bernstein| Theorem 1                  | general-state-space              | $\frac{1+\lambda}{1-\lambda} \cdot \sigma^2$ |

Table 2: Concentration inequalities for time-independent function $f$ of Markov chains.

| type     | reference                  | condition                        | variance proxy |
|----------|----------------------------|----------------------------------|----------------|
| Hoeffding| Hoeffding, 1963            | independent                      | $c^2$          |
| Hoeffding| León and Perron, 2004      | finite-state-space, reversible   | $\frac{1+\max\{\lambda,0\}}{1-\max\{\lambda,0\}} \cdot c^2$ |
| Hoeffding| Miasojedow, 2014           | general-state-space              | $\frac{1+\lambda}{1-\lambda} \cdot c^2$ |
| Hoeffding| Fan et al., 2021           | general-state-space              | $\frac{1+\max\{\lambda,0\}}{1-\max\{\lambda,0\}} \cdot c^2$ |
| Bernstein| Bernstein, 1946            | independent                      | $\sigma^2$     |
| Bennett  | Bennett, 1962              | independent                      | $\sigma^2$     |
| Chernoff | Lezaud, 1998a, (1)         | finite-state-space, reversible   | $\frac{2}{1-\lambda} \cdot \sigma^2$ |
| Chernoff | Lezaud, 1998a, (13)        | general-state-space              | $\frac{2}{1-\lambda} \cdot \sigma^2$ |
| Bernstein| Paulin, 2015, (3.20)$^1$   | general-state-space, reversible  | $\left(\frac{4+\lambda}{1-\lambda} + 0.8\right) \cdot \sigma^2$ |
| Bernstein| Paulin, 2015, (3.21)       | general-state-space, reversible  | $\frac{2}{1-\lambda} \cdot \sigma^2$ |
| Bernstein| Theorem 2                  | general-state-space              | $\frac{1+\max\{\lambda,0\}}{1-\max\{\lambda,0\}} \cdot \sigma^2$ |

$^1$ (3.20) of Paulin (2015) gives a variance proxy $\sigma_{asy}^2(f) + 0.8\sigma^2$. Since $\sigma_{asy}^2(f)$, the asymptotic variance of $n^{-1/2} \sum_{i=1}^n f(X_i)$, is unknown in practice and is equal to $\frac{1+\lambda}{1-\lambda} \cdot \sigma^2$ in the worst case for reversible Markov chains (Rosenthal, 2003, Proposition 1), we replace $\sigma_{asy}^2(f)$ with $\frac{1+\lambda}{1-\lambda} \cdot \sigma^2$ for clearness.

For a reversible Markov chain and a sequence of time-dependent functions, the tail probability inequality (6) in Theorem 1 improves over the previous Bernstein-type inequality of Paulin (2015, Equation 3.22 in Theorem 3.9) in terms of the sharpness of variance...
proxy. Specifically,

\[ \frac{1 + \lambda}{1 - \lambda} \cdot \sigma^2 < \frac{4}{1 - \lambda^2} \cdot \sigma^2, \quad \forall \lambda \in [0, 1). \]

### 2.2 The Extendable and Non-extendable

In this subsection, we extend our Bernstein-type inequalities to the nonstationary case and present an impossibility result that our Bernstein-type inequalities are not extendable to unbounded functions.

We first extend Theorems 1 and 2 to non-stationary Markov chains, provided that the initial distribution of the Markov chain exhibits finite moments with respect to the invariant distribution.

**Theorem 3.** Suppose \( \{X_i\}_{i \geq 1} \) is a Markov chain with initial distribution \( \nu \), invariant distribution \( \pi \) and absolute spectral gap \( 1 - \lambda > 0 \), and \( f_i \) are functions with \( |f_i| \leq c \) and \( \pi(f_i) = 0 \). Let \( \sigma^2 = \sum_{i=1}^{n} \pi(f_i^2)/n \). Suppose \( \nu \) is absolutely continuous with respect to \( \pi \) and its derivative \( \frac{d\nu}{d\pi} \) has a finite \( p \)-moment for some \( p \in (1, \infty) \), i.e.,

\[
\left\| \frac{d\nu}{d\pi} \right\|_{\| \cdot \|_{\nu},p} := \left\{ \int \left| \frac{d\nu}{d\pi}(x) \right|^p \pi(dx) \right\}^{1/p} \quad \text{if} \quad 1 < p < \infty
\]

\[
\quad \quad < \infty \quad \text{if} \quad p = \infty
\]

Let \( q = p/(p - 1) \in [1, \infty) \) then, for any \( 0 \leq t < (1 - \lambda)/5cq \),

\[
\mathbb{E}_\nu \left[ e^{\sum_{i=1}^{n} f_i(X_i)} \right] \leq \left\| \frac{d\nu}{d\pi} \right\|_{\| \cdot \|_{\nu},p} \exp \left( \frac{n\sigma^2}{qc}(e^{qtc} - 1 - qtc) + \frac{n\sigma^2 \lambda t^2}{1 - \lambda - 5cqt} \right)
\]

It follows that

\[
\mathbb{P}_\nu \left( \frac{1}{n} \sum_{i=1}^{n} f_i(X_i) \geq \epsilon \right) \leq \left\| \frac{d\nu}{d\pi} \right\|_{\| \cdot \|_{\nu},p} \exp \left( -\frac{1}{q} \cdot \frac{ne^2 / 2}{\alpha_1(\lambda) \cdot \sigma^2 + \alpha_2(\lambda) \cdot \epsilon} \right)
\]

where \( \alpha_1 \) and \( \alpha_2 \) have been defined in Theorem 1. If \( f_i = f \) for \( i = 1, \ldots, n \) then the above two inequalities holds with \( \max\{\lambda, 0\} \) in place of \( \lambda \).

The classical Bernstein’s inequality can be extended for some unbounded (e.g., sub-Exponential) independent random variables. Unfortunately, the boundedness of random variables is necessary for Bernstein-type inequalities to hold in the Markov chain setup. Below is an impossibility result for the existence of Bernstein’s inequalities for unbounded functions. It is a strengthened version of Fan et al. (2021, Theorem 4).

**Theorem 4.** Consider a stationary Markov chain \( \{X_i\}_{i \geq 0} \) on the state space \( \mathcal{X} = \mathbb{R} \) with invariant distribution \( \pi \sim \mathcal{N}(0, 1) \) and transition kernel

\[
P(x, B) = \lambda \mathbb{1}(x \in B) + (1 - \lambda) \pi(B), \quad \forall x \in \mathcal{X}, \forall \text{ measurable } B \subseteq \mathcal{X},
\]

for some \( 0 < \lambda < 1 \). There exists no function \( g(t) = Vt^2/2 + o(t^2) \) with finite variance proxy \( V > 0 \) and \( t_0 > 0 \) such that

\[
\mathbb{E}_\pi e^{\sum_{i=1}^{n} X_i} \leq e^{\epsilon g(t)}, \quad \forall n \geq 1, \forall t \in [0, t_0).
\]
3 Preliminaries

This section collects preliminary results needed for the technical proofs in this paper. Assume the state space of the Markov chain, denoted by $X$, is equipped with a sigma-algebra $\mathcal{B}$, and $(X, \mathcal{B})$ a standard Borel space\(^2\). This assumption is commonly seen in the literature for measure-theoretical studies of Markov chains. It holds in most practical examples, in which $X$ is a subset of a multi-dimensional real space and $\mathcal{B}$ is the Borel sigma-algebra over $X$. Let $(X_i)_{i \geq 1}$ be a Markov chain on the state space $(X, \mathcal{B})$ with invariant measure $\pi$.

### 3.1 Hilbert space and spectral gaps

For a real-valued, $\mathcal{B}$-measurable function $h : X \rightarrow \mathbb{R}$, let

$$\pi(h) := \int_X h(x)\pi(dx).$$

The set of all $\pi$-square-integrable functions

$$L_2(X, \mathcal{B}, \pi) := \{ h : \pi(h^2) < \infty \}$$

is a Hilbert space endowed with the inner product

$$\langle h_1, h_2 \rangle_\pi = \int_X h_1(x)h_2(x)\pi(dx), \quad \forall h_1, h_2 \in L_2(X, \mathcal{B}, \pi).$$

Define the norm of a function $h \in L_2(X, \mathcal{B}, \pi)$ as

$$\|h\|_\pi = \sqrt{\langle h, h \rangle_\pi},$$

which induces the norm of a linear operator $T$ on $L_2(X, \mathcal{B}, \pi)$ as

$$\|T\|_\pi = \sup\{ \|T h\|_\pi : \|h\|_\pi = 1 \}.$$  

Write $L_2$ in place of $L_2(X, \mathcal{B}, \pi)$ for simplicity, whenever the probability space $(X, \mathcal{B}, \pi)$ is clear in the context.

Each transition kernel $P(x, B)$ for $x \in X$ and $B \in \mathcal{B}$, if invariant with respect to $\pi$, corresponds to an integral operator $h \mapsto \int_X h(y)P(\cdot, dy)$ on $L_2$. We abuse notation $P$ to denote this operator. Let $1 : x \in X \mapsto 1$ be identically one function, and let $\Pi$ be the projection operator onto $1$, i.e.,

$$\Pi : h \mapsto \langle h, 1 \rangle_\pi 1 = \pi(h)1.$$

**Definition 1** (Absolute spectral gap). A $\pi$-invariant Markov operator $P$ has non-zero absolute spectral gap $1 - \lambda(P)$ if

$$\lambda(P) := \|P - \Pi\|_\pi < 1.$$  

\(^2\)A measurable space $(X, \mathcal{B})$ is standard Borel if it is isomorphic to a subset of $\mathbb{R}$. See Definition 4.33 in Breiman (1992).
Let $L^0_2 = \{ h \in L_2 : \pi(h) = 0 \}$ denote the subspace of $L_2$ consisting of mean-zero functions. An equivalent definition of $\lambda(P)$ is given by

$$\lambda(P) = \sup \left\{ \|Ph\|_\pi : \|h\|_\pi = 1, h \in L^0_2 \right\}.$$

Let $P^*$ denote the adjoint of $P$. Viewed as transition kernels, $P$ and $P^*$ are time-reversals of each other

$$P^*(x,y) = \frac{\pi(y)P(y,x)}{\pi(x)}.$$

The additive reversiblization of $P$, given by $(P + P^*)/2$, is a self-adjoint Markov operator. Denote by $\sigma_0(K)$ of an operator $K$ acting in $L^0_2$. If $K$ is self-adjoint then $\sigma_0(K)$ are real-valued.

**Definition 2 (Right spectral gap).** A $\pi$-invariant Markov operator $P$ has non-zero right spectral gap $1 - \lambda_r(P)$ if

$$\lambda_r(P) := \sup\{\theta : \theta \in \sigma_0((P + P^*)/2)\} < 1.$$

An equivalent definition of $\lambda_r(P)$ is given by

$$\lambda_r(P) = \sup \left\{ \langle h, Ph \rangle_\pi : \|h\|_\pi = 1, h \in L^0_2 \right\}.$$

It is elementary that $\lambda_r(P) \leq \lambda(P)$.

### 3.2 León-Perron: Convex majorization of Markov operator

Note that every convex combination of Markov operators is still a Markov operator.

**Definition 3 (León-Perron operator).** A Markov operator $\hat{P}_\lambda$ on $L_2$ is said León-Perron if it is a convex combination of operators $I$ and $\Pi$ with some coefficient $\lambda \in [0,1)$, that is

$$\hat{P}_\lambda = \lambda I + (1 - \lambda)\Pi.$$

This León-Perron operator associates with a transition kernel

$$\hat{P}_\lambda(x,B) = \lambda \mathbb{1}(x \in B) + (1 - \lambda)\pi(B), \quad \forall x \in X, \forall B \in \mathcal{B},$$

which characterizes a simple transition dynamics: at each step, the Markov chain either stays at the current state with probability $\lambda$ or jumps to a new state drawn from $\pi$ with probability $1 - \lambda$. Evidently, any León-Perron operator $\hat{P}_\lambda$ is invariant with $\pi$.

The León-Perron operator has played a central role as the convex majorization of the Markov operator in the argument of León and Perron (2004). This paper uses a few properties of León-Perron operators for general Markov chains, which were developed along the line of research on Hoeffding-type inequalities for Markov chains (León and Perron, 2004; Miasojedow, 2014; Fan et al., 2021).

Let $E^h$ denote the multiplication operator of function $e^h : x \mapsto e^{h(x)}$. The following three lemmas are taken from Fan et al. (2021).
Lemma 1. Suppose \( \{X_t\}_{t \geq 1} \) is a stationary Markov chain with invariant distribution \( \pi \) and absolute spectral gap \( 1 - \lambda > 0 \). For any bounded functions \( f_i \) and any \( t \in \mathbb{R} \),

\[
\mathbb{E}_\pi \left[ e^{\sum_{i=1}^n f(X_i)} \right] \leq \prod_{i=1}^n \| E^{f_i/2} \hat{P}_A E^{f_i/2} \|_\pi.
\]

Lemma 2. Suppose \( \{X_t\}_{t \geq 1} \) is a stationary Markov chain with invariant distribution \( \pi \) and right spectral gap \( 1 - \lambda_r > 0 \). For any bounded function \( f \) and any \( t \in \mathbb{R} \),

\[
\mathbb{E}_\pi \left[ e^{\sum_{i=1}^n f(X_i)} \right] \leq \| E^{f/2} \hat{P}_A E^{f/2} \|_\pi^n.
\]

Lemma 3. Suppose \( \{X_t\}_{t \geq 1} \) is a stationary Markov chain driven by a León-Perron operator \( \hat{P}_A = \lambda I + (1 - \lambda)\Pi \) for some \( \lambda \in [0, 1) \). For any bounded function \( f \) and any \( t \in \mathbb{R} \),

\[
\lim_{n \to \infty} \frac{1}{n} \log \mathbb{E}_\pi \left[ e^{\sum_{i=1}^n f(X_i)} \right] = \log \| E^{f/2} \hat{P}_A E^{f/2} \|_\pi.
\]

Lemma 1 bounds the mgf of \( \sum_{i=1}^n f_i(X_i) \) by the product of the operator norms of symmetrically perturbed León-Perron operators \( E^{f_i/2} \hat{P}_A E^{f_i/2} \). Lemma 2 refines Lemma 1 with \( \max(\lambda_r, 0) \) in place of \( \lambda \) in case that \( f_i \)'s are identical. Lemma 3 asserts that a Markov chain driven by \( \hat{P}_A = \lambda I + (1 - \lambda)\Pi \) is the extreme case of all Markov chains with absolute spectral gap \( 1 - \lambda \) in the sense that the inequality in Lemma 1 is asymptotically tight.

3.3 Kato’s perturbation theory

Another cornerstone of our technique is Kato’s analysis on the largest eigenvalue of a perturbed operator (Kato, 2013). This was the main tool to prove Chernoff-type and Bernstein-type inequalities for Markov chains by Lezaud (1998a) and Paulin (2015). Specifically, Kato’s analysis expends the largest eigenvalue of the perturbed Markov operator \( PE^{tf} \) as a series in \( t \).

Consider a reversible Markov chain \( \{X_t\}_{t \geq 1} \) with invariant distribution \( \pi \), self-adjoint Markov operator \( P \) and right spectral gap \( 1 - \lambda_r \). Recall that, for any function \( f \in \mathcal{L}^0 \), its asymptotic variance is given by

\[
\sigma_{as}^2(f) = \lim_{n \to \infty} \text{Var} \left[ n^{-1/2} \sum_{i=1}^n f(X_i) \right].
\]

By Rosenthal (2003, Proposition 1), for any \( h \in \mathcal{L}^0 \),

\[
\sigma_{as}^2(f) \leq \frac{1 + \lambda_r}{1 - \lambda_r} \cdot \| f \|^2_\pi.
\]

Let \( Z = (I - P + \Pi)^{-1} - \Pi \) be the negative of the reduced resolvent of \( P \) with respect to its largest eigenvalue 1. A few useful properties of \( Z \) are summarized in Lemma 4. Proofs of these properties can be found in Lezaud (1998b, Proposition 1.5) and Paulin (2015, Lemma 5.3).
Lemma 4. Suppose a self-adjoint Markov operator $P$ has invariant distribution $\pi$ and right spectral gap $1 - \lambda_r$. The following properties of $Z = (I - P + \Pi)^{-1} - \Pi$ hold.

(a) $Z\Pi = \Pi Z = 0$.
(b) $ZP = Z - (I - \Pi)$.
(c) $\langle h, Zh \rangle_\pi = (\sigma^2_a(h) + \|h\|^2_\pi)/2$, $\forall h \in L^0_\pi$.
(d) $\|Z\|_\pi = (1 - \lambda_r)^{-1}$.

Next lemma, adopted from Lezaud (1998a), expends the largest eigenvalue of the perturbed Markov operator $PE^{tf}$ as a series in $t$.

Lemma 5. Consider a reversible, irreducible Markov chain on finite state space $X$ with invariant distribution $\pi$, self-adjoint Markov operator (transition probability matrix) $P$ and right spectral gap $1 - \lambda_r > 0$ (in this case, $\lambda_r$ is the second largest eigenvalue of $P$). Let $D$ be the diagonal matrix with elements $\{f(x) : x \in X\}$ and let $T^{(m)} = PD^m/m!$ for any $m \geq 0$ with $D^0 = I$ by convention. Then

$$PE^{tf} = P \left( \sum_{m=0}^{\infty} \frac{D^m}{m!} t^m \right) = \sum_{m=0}^{\infty} T^{(m)} \cdot t^m.$$

Let $t_0 = \left( 2\|T^{(1)}\|_\pi (1 - \lambda_r)^{-1} + c_0 \right)^{-1}$ for some $c_0$ such that

$$\|T^{(m)}\|_\pi \leq 4T^{(1)}\|_\pi c_0^{m-1}, \forall m \geq 1.$$

Then, for any $t$ such that $|t| < t_0$, the largest eigenvalue of $PE^{tf}$, denoted by $\beta(t)$, admits an expansion

$$\beta(t) = \sum_{m=0}^{\infty} \beta^{(m)} t^m.$$

In this expansion, $\beta^{(0)} = 1$ is the largest eigenvalue of $T^{(0)} = P$ and, for any $m \geq 1$,

$$\beta^{(m)} = \sum_{p=1}^{m} \frac{-1}{p} \sum_{v_1 + \cdots + v_p = m, \ v_i \geq 1} \sum_{k_1 + \cdots + k_p = p - 1, \ k_j \geq 0} \text{trace} \left( T^{(v_1)}Z^{(k_1)} \cdots T^{(v_p)}Z^{(k_p)} \right),$$

where $Z^{(0)} = -\Pi$ and $Z^{(j)} = Z^j$ for $j \geq 1$ are powers of $Z$.

4 Proof of Theorems

This section proves four lemmas (Lemmas 6 to 9) and then main theorems.
Lemma 6. For any function \( f \) such that \( |f| \leq c \), \( \pi(f) = 0 \) and \( \pi(f^2) = \sigma^2 \), let

\[
\tilde{f}_k(x) = \left[ \frac{f(x) + c}{c/3k} \right] \times \frac{c}{3k} - c,
\]

where \( \lceil \cdot \rceil \) is the ceiling function, and

\[
\tilde{f}_k = \frac{\tilde{f}_k - \pi(\tilde{f}_k)}{1 + 1/3k}.
\]

Then, for any bounded linear operator \( T \) acting on \( \mathcal{L}_2 \) and any \( t \in \mathbb{R} \),

\[
\| E^{t^2/2} T E^{t^2/2} \|_\pi = \lim_{k \to \infty} \| E^{\tilde{f}_k^2/2} T E^{\tilde{f}_k^2/2} \|_\pi.
\]

Proof of Lemma 6. It is elementary that

\[
\sup_x |\tilde{f}_k(x)| \leq c
\]

and

\[
\sup_x |\tilde{f}_k(x) - \pi(\tilde{f}_k)| \leq \frac{1}{1 + 1/3k} \sup_x |\tilde{f}_k(x) - f(x)| \leq c/3k
\]

By definition, \( \pi(\tilde{f}_k) = 0 \). And, from the above facts regarding \( \tilde{f}_k \), it follows that

\[
\sup_x |\tilde{f}_k(x)| \leq \sup_x |\tilde{f}_k(x) - \pi(\tilde{f}_k)| \leq c
\]

Now, from the above facts regarding \( \tilde{f}_k \), it follows that

\[
\sup_x |\tilde{f}_k(x) - f(x)| \leq \sup_x |\tilde{f}_k(x) - \tilde{f}_k(x) - \pi(\tilde{f}_k)| + \sup_x |\tilde{f}_k(x) - f(x)| + |\pi(\tilde{f}_k)|
\]

For any bounded linear operator \( T \) acting on \( \mathcal{L}_2 \) and any \( t \in \mathbb{R} \),

\[
\| E^{t^2/2} T E^{t^2/2} \|_\pi = \| E^{(f-\tilde{f}_k)^2/2} E^{\tilde{f}_k^2/2} T E^{\tilde{f}_k^2/2} E^{(f-\tilde{f}_k)^2/2} \|_\pi
\]

\[
\leq \| E^{(f-\tilde{f}_k)^2/2} \|_\pi \| E^{\tilde{f}_k^2/2} T E^{\tilde{f}_k^2/2} \|_\pi \| E^{(f-\tilde{f}_k)^2/2} \|_\pi
\]

\[
\leq \exp \left( 2 \sup_x |\tilde{f}_k(x) - f(x)| \cdot t/2 \right) \| E^{\tilde{f}_k^2/2} T E^{\tilde{f}_k^2/2} \|_\pi
\]

\[
\leq e^{t/2k} \| E^{\tilde{f}_k^2/2} T E^{\tilde{f}_k^2/2} \|_\pi.
\]

Similarly,

\[
\| E^{\tilde{f}_k^2/2} T E^{\tilde{f}_k^2/2} \|_\pi = \| E^{(f-\tilde{f}_k)^2/2} E^{(f-\tilde{f}_k)^2/2} E^{(f-\tilde{f}_k)^2/2} \|_\pi
\]

\[
\leq \| E^{(f-\tilde{f}_k)^2/2} \|_\pi \| E^{(f-\tilde{f}_k)^2/2} T E^{(f-\tilde{f}_k)^2/2} \|_\pi \| E^{(f-\tilde{f}_k)^2/2} \|_\pi
\]

\[
\leq \exp \left( 2 \sup_x |\tilde{f}_k(x) - f(x)| \cdot t/2 \right) \| E^{(f-\tilde{f}_k)^2/2} T E^{(f-\tilde{f}_k)^2/2} \|_\pi
\]

\[
\leq e^{t/2k} \| E^{(f-\tilde{f}_k)^2/2} T E^{(f-\tilde{f}_k)^2/2} \|_\pi.
\]
Putting the last two displays together yields
\[ e^{-ct/k} E[i/2] T E[i/2] \leq \| E[i/2] T E[i/2] \| \leq e^{ct/k} E[i/2] T E[i/2]. \]

Taking \( k \to \infty \) completes the proof. \( \square \)

**Lemma 7.** Let \( \lambda \) be a León-Perron operator with \( \lambda \in [0, 1) \) on a general state space \( X \). Let \( f \) be a function on \( X \) taking finitely many possible values. On the finite state space
\[ Y = \{ y \in f(X) : \pi(x : f(x) = y) > 0 \}, \]
define a transition probability matrix \( \lambda \) consisting of elements \( \pi(x : f(x) = y) \) for each \( y \in Y \). Let \( E^Y \) denote the diagonal matrix with elements \( \{ y \in Y \} \). Then
\[ \| E[i/2] \lambda \| = \| E[i/2] \lambda \| \| E^Y \| = \| E[i/2] \lambda \| \| E^Y \| \| E^Y \| \| E^Y \| \| E^Y \| \| E^Y \| . \]

**Proof of Lemma 7.** Let \( \{ B_i \}_{i \geq 1} \) be a sequence of i.i.d. Bernoulli random variables with success probability \( \lambda \), and let \( \{ W_i \}_{i \geq 1} \) be a sequence of i.i.d. random variables following \( \pi \), respectively. Evidently,
\[ \hat{X}_1 = W_1, \quad \hat{X}_i = B_i \hat{X}_{i-1} + (1 - B_i) W_i, \quad \forall i \geq 2; \]
\[ \hat{Y}_1 = f(W_1), \quad \hat{Y}_i = B_i \hat{Y}_{i-1} + (1 - B_i) f(W_i), \quad \forall i \geq 2. \]

are stationary Markov chains with invariant distributions \( \pi, \mu \) and transition probabilities \( \lambda, \mu \) on state spaces \( X, Y \), respectively; and \( \hat{Y}_i = f(\hat{X}_i) \) for \( i \geq 1 \). Putting them with \( \text{Lemma 3} \) together yields
\[ \log \| E[i/2] \lambda \| = \lim_{n \to \infty} \frac{1}{n} \log \mathbb{E}_\pi \left[ \exp \left( \sum_{i=1}^{n} f(\hat{X}_i) \right) \right] \]
\[ = \lim_{n \to \infty} \frac{1}{n} \log \mathbb{E}_\mu \left[ \exp \left( \sum_{i=1}^{n} \hat{Y}_i \right) \right] \]
\[ = \log \| E[i/2] \lambda \| \| E[i/2] \| \| E[i/2] \| . \]

\( \square \)

**Lemma 8.** Assume the same setup as in \( \text{Lemma 5} \) and let \( \lambda = \lambda(P) \). Then for any \( 0 \leq t < (1 - \lambda_0)/5c \),
\[ \| E[i/2] \lambda \| \| E[i/2] \| \leq \exp \left( \frac{\sigma^2}{c^2} (e^{tc} - 1 - tc) + \frac{\sigma^2 \lambda_0^2}{1 - \lambda_0^2 - 5ct} \right). \]

**Proof of Lemma 8.** Elements in the matrix \( E[i/2] \lambda \) are non-negative. It follows from the Perron-Frobenius theorem that the operator norm of \( E[i/2] \lambda \) coincides with its largest eigenvalue. \( E[i/2] \lambda \) is similar to \( P \lambda \), so they share the same eigenvalues. Thus
\[ \|E^{t/2}PE^{t/2}\|_\pi \] is equal to the largest eigenvalue of \( PPE \). Denote by \( \beta(t) \) this eigenvalue. Lemma 5 asserts that
\[
\beta(t) = \sum_{m=0}^{\infty} \beta^{(m)} t^m, \quad \forall 0 \leq t < t_0,
\]
where \( \beta^{(0)} = 1 \) and
\[
\beta^{(m)} = \sum_{p=1}^{m} \frac{1}{p} \sum_{v_1 + \cdots + v_p = m, \ v_i \geq 1} - \frac{\text{trace} \left( PD^{v_1}Z^{(k_1)} \cdots PD^{v_p}Z^{(k_p)} \right)}{v_1! \cdots v_p!}, \quad \forall \ m \geq 1.
\]

We make the choice of \( t_0 = (1 - \lambda) / (3 - \lambda) c \) precise by taking \( c_0 = c \geq \| D \|_\pi \). Indeed,
\[
\| T^{(1)} \|_\pi = \| PD \|_\pi \leq c,
\]
\[
\| T^{(m)} \|_\pi = \frac{1}{m!} \| PD^m \|_\pi \leq \| PD \|_\pi m^{-1} \leq \| T^{(1)} \|_\pi c^{m-1}.
\]

Proceed to compute coefficients \( \beta^{(m)} \) for \( m \geq 1 \). It is straightforward that
\[
\beta^{(1)} = -\text{trace} \left( PD^1Z^{(0)} \right) = \text{trace} \left( PDIT \right) = \text{trace} \left( DIP \right) = \text{trace} \left( DI \right) = \pi(f) = 0.
\]

And,
\[
\beta^{(2)} = -\frac{\text{trace} \left( PD^2Z^{(0)} \right)}{2} - \frac{\text{trace} \left( PD^1Z^{(0)}PD^1Z \right)}{2} + \text{trace} \left( PD^1Z^1PD^1Z^1 \right)
\]
\[
= \frac{\text{trace} \left( D^2IT \right)}{2} + \text{trace} \left( PD^1PDZ + \text{trace} \left( PDZPDIT \right) \right)
\]
\[
= \frac{\text{trace} \left( D^2IT \right)}{2} + \text{trace} \left( ZPDITD \right) = \| f \|_\pi^2 + \langle ZPf, f \rangle_\pi = \frac{\sigma^2_\pi(f)}{2},
\]
where the last step uses parts (b) and (c) of Lemma 4. For \( m \geq 3 \), \( \beta^{(m)} \) is the sum of \( m \) terms \( \{ \beta^{(m)}_p : \ p = 1, \ldots, m \} \).
\[
\beta^{(m)}_p = \frac{1}{p} \sum_{v_1 + \cdots + v_p = m, \ v_j \geq 1} \frac{\text{trace} \left( PD^{v_1}Z^{(k_1)} \cdots PD^{v_p}Z^{(k_p)} \right)}{v_1! \cdots v_p!}.
\]

For \( p = 1 \),
\[
\beta^{(m)}_1 = -\frac{\text{trace} \left( PD^1Z^{(0)} \right)}{m!} = \frac{\pi(f^m)}{m!}.
\]

For \( p = 2, \ldots, m \), consider each term in the summation. Since \( k_1 + \cdots + k_p = p - 1 \), there exists an index \( j_1 \in \{1, \ldots, p\} \) such that \( k_{j_1} = 0 \). Let
\[
(k'_1, \ldots, k'_p) = (k_{j_1+1}, \ldots, k_p, k_1, \ldots, k_{j_1}).
\]
be the cyclic rotation of \((k_1, \ldots, k_p)\), and correspondingly
\[
(v'_1, \ldots, v'_p) = (v_{j_1+1}, \ldots, v_p, v_1, \ldots, v_{j_1})
\]
be the cyclic rotation of \((v_1, \ldots, v_p)\).

Using facts that \(\text{trace}(AB) = \text{trace}(BA)\) whenever the dimension of matrices \(A, B\) are appropriate, that \(Z^{(k)} = Z^{(k_j)} = Z^{(0)} = -\Pi\), and that \(\Pi \Pi = \Pi\), write
\[
-\text{trace}\left(PD^{v_j}Z^{(k_1)} \ldots PD^{v_p}Z^{(k_p)}\right) = -\text{trace}\left(PD^{v_j}Z^{(k_1)}PD^{v_j}Z^{(k_2)} \ldots PD^{v_p-1}Z^{(k_p)}PD^{v_p}Z^{(k_1)}\right)
\]
\[
= \text{trace}\left(D^{v_j}Z^{(k_1)}PD^{v_j}Z^{(k_2)} \ldots PD^{v_p-1}Z^{(k_p)}PD^{v_p}\Pi\right)
\]
\[
= \text{trace}\left(D^{v_j}Z^{(k_1)}PD^{v_j}Z^{(k_2)} \ldots PD^{v_p-1}Z^{(k_p)}PD^{v_p}\Pi\right)
\]
\[
= \langle f, D^{v_j-1}Z^{(k_1)}PD^{v_j}Z^{(k_2)} \ldots PD^{v_p-1}Z^{(k_p)}PD^{v_p-1}f \rangle_\pi.
\]

Since \(k'_1 + \cdots + k'_{p-1} = p - 1 \geq 1\) given \(p \geq 2\), there exists an index \(j_2 \in \{1, \ldots, p - 1\}\) such that \(k'_j \geq 1\). From the fact that \(Z\Pi = 0\) (Lemma 4, part (a)), it follows that \(Z^{(k'_2)}P = Z^{(k'_2)}(P - \Pi)\). Thus,
\[
-\text{trace}\left(PD^{v_j}Z^{(k_1)} \ldots PD^{v_p}Z^{(k_p)}\right) = \langle f, D^{v_j-1}Z^{(k_1)}PD^{v_j}Z^{(k_2)} \ldots Z^{(k_p)}P \ldots PD^{v_p-1}Z^{(k_p)}PD^{v_p-1}f \rangle_\pi
\]
\[
\leq \|f\|_e^2 \|P - \Pi\|_e \|D\|_e^{(p-1)\sum_{j=1}^p v'_j} \|Z\|_e^{(p-1)\sum_{j=1}^p k'_j} \|P\|_e^{p-2}.
\]

Putting it together with facts that \(\|f\|_e^2 = \sigma^2\), \(\|P - \Pi\|_e = \lambda\), \(\|D\|_e \leq c\), \(\sum_{j=1}^p v'_j = m\), \(\|Z\|_e = (1 - \alpha)\) (Lemma 4, part (d)), \(\sum_{j=1}^{p-1} k'_j = p - 1\) and \(\|P\|_e = 1\) yields
\[
-\text{trace}\left(PD^{v_j}Z^{(k_1)} \ldots PD^{v_p}Z^{(k_p)}\right) \leq \sigma^2 \lambda e^{m/2} (1 - \alpha)^{(p-1)}.
\]

Note that \(v_1 v_2 \ldots v_p \geq 2^{m-p}\) and that \(\sum_{p=1}^m \frac{1}{p(m-1)!} \left(\frac{2p-2}{p-1}\right)^{p-1} \leq 5^{m-2}\) for \(m \geq 3\) on (Lezard, 1998a, page 856), and that \(\frac{2}{\lambda} > 1\). For any \(m \geq 3\),
\[
\sum_{p=2}^m \frac{\beta_p^m}{p} \leq \sum_{p=2}^m \frac{1}{p} \sum_{v_1 + \cdots + v_p = m, v_j \geq 1} \sum_{k_1 + \cdots + k_p = p - 1, k_j \geq 0} \frac{\sigma^2 \lambda e^{m/2} (1 - \alpha)^{(p-1)}}{2^{m-p}}.
\]
Further,
\[
\beta^m = \beta_1^m + \sum_{p=2}^m \beta_p^m \leq \frac{\pi (f(m))}{m!} + \frac{\sigma^2 \lambda}{5c} \left(\frac{5c}{1 - \alpha}\right)^{m-1}.
\]
The above inequality also holds for \( m = 2 \), as

\[
\beta^{(2)} = \frac{\sigma_0^2}{2} \leq \sigma^2 \left( \frac{1}{2} + \frac{\lambda_t}{1 - \lambda_t} \right) = \frac{\pi(f^2)}{2!} + \frac{\sigma^2\lambda t}{5c} \left( \frac{5c}{1 - \lambda_t} \right)^{m-1} \leq \pi(f^2) + \frac{\sigma^2\lambda t}{5c} \left( \frac{5c}{1 - \lambda_t} \right)^{m-1}.
\]

Thus,

\[
\beta(t) = \beta^{(0)} + \beta^{(1)} t + \sum_{m=2}^{\infty} \beta^{(m)} t^m
\]

\[
\leq 1 + 0 + \sum_{m=2}^{\infty} \frac{\pi(f^m) t^m}{m!} + \sum_{m=2}^{\infty} \frac{\sigma^2\lambda t}{5c} \left( \frac{5ct}{1 - \lambda_t} \right)^{m-1}
\]

\[
\leq \exp \left\{ \sum_{m=2}^{\infty} \frac{\pi(f^m) t^m}{m!} + \sum_{m=2}^{\infty} \frac{\sigma^2\lambda t}{5c} \left( \frac{5ct}{1 - \lambda_t} \right)^{m-1} \right\}.
\]

In the exponent, the first term

\[
\sum_{m=2}^{\infty} \frac{\pi(f^m) t^m}{m!} \leq \sum_{m=2}^{\infty} \frac{\pi(f^2)}{m!} \sum_{m=2}^{\infty} \frac{e^{mt}}{m!} = \frac{\sigma^2}{c^2} \sum_{m=2}^{\infty} \frac{e^{mt}}{m!} = \frac{\sigma^2}{c^2} (e^{tc} - 1 - tc),
\]

and the second term, for any \( 0 \leq t < (1 - \lambda_t)/5c < t_0 \),

\[
\sum_{m=2}^{\infty} \frac{\sigma^2\lambda t}{5c} \left( \frac{5ct}{1 - \lambda_t} \right)^{m-1} = \frac{\sigma^2\lambda t^2}{1 - \lambda_t - 5ct},
\]

completing the proof. \( \square \)

**Lemma 9.** For \( \lambda \in [0, 1) \), define for any \( 0 \leq t < (1 - \lambda)/5c \)

\[
g_1(t) = \frac{\sigma^2}{e^2} (e^{tc} - 1 - tc), \quad g_2(t) = \frac{\sigma^2\lambda t^2}{1 - \lambda - 5ct}.
\]

(9)

If \( \lambda \in (0, 1) \) then

\[
(g_1 + g_2)^* (\varepsilon) \triangleq \sup \{ t \varepsilon - g_1(t) - g_2(t) : 0 \leq t < (1 - \lambda)/5c \}
\]

\[
\geq \frac{\varepsilon^2}{2} \left( \frac{1 + \lambda}{1 - \lambda} \cdot \sigma^2 + \frac{5c \varepsilon}{1 - \lambda} \right)^{-1}.
\]

(10)

If \( \lambda = 0 \) then

\[
(g_1 + g_2)^* (\varepsilon) = g_1^* (\varepsilon) \geq \frac{\varepsilon^2}{2} \left( \sigma^2 + \frac{c \varepsilon}{3} \right)^{-1}.
\]

**Proof of Lemma 9.** Extend functions \( g_1(t) \) and \( g_2(t) \) to the domain \( (-\infty, +\infty) \) as

\[
g_1(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{\sigma^2}{e^2} (e^{tc} - 1 - tc) & \text{if } t \geq 0 \end{cases}
\]

\[
g_2(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{\sigma^2\lambda t^2}{1 - \lambda - 5ct} & \text{if } 0 \leq t < \frac{1 - \lambda}{5c} \\ +\infty & \text{if } t \geq \frac{1 - \lambda}{5c} \end{cases}
\]

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Both $g_1(t)$ and $g_2(t)$ are closed proper convex functions. Their convex conjugates are given by

$$g_1^*(e_1) = \begin{cases} \frac{\sigma^2}{2 \lambda} h_1 \left( \frac{\sigma e_1}{\lambda} \right) & \text{if } e_1 \geq 0, \\ +\infty & \text{if } e_1 < 0, \end{cases}$$

(11)

with $h_1(u) = (1 + u) \log(1 + u) - u$ for $u \geq 0$, and

$$g_2^*(e_2) = \begin{cases} \frac{(1-\lambda)\sigma^2}{2 \lambda \sigma^2} h_2 \left( \frac{5 \sigma e_2}{\lambda \sigma^2} \right) & \text{if } e_2 \geq 0, \\ +\infty & \text{if } e_2 < 0, \end{cases}$$

(12)

with $h_2(u) = (\sqrt{1 + u} + u/2 + 1)^{-1}$.

The convex conjugate $(g_1+g_2)^*$ is still well-defined under this extension. Indeed, $g_1(t) = O(t^2)$ and $g_2(t) = O(t^2)$ as $t \to 0^+$, $t \varepsilon - g_1(t) - g_2(t) > 0$ for small enough $t > 0$; and $t \varepsilon - g_1(t) - g_2(t) \leq 0$ for $t \leq 0$. Therefore,

$$(g_1 + g_2)^* (\varepsilon) := \sup_{0 \leq c (1-\lambda)/5 \varepsilon} \{t \varepsilon - g_1(t) - g_2(t)\} = \sup_{t \in \mathbb{R}} \{t \varepsilon - g_1(t) - g_2(t)\}.$$ 

By the Moreau-Rockafellar formula (Rockafellar, 2015, Theorem 16.4), the convex conjugate of $g_1 + g_2$ is the infimal convolution of their conjugates $g_1^*$ and $g_2^*$. That is,

$$(g_1 + g_2)^* (\varepsilon) = \inf \{g_1^*(e_1) + g_2^*(e_2) : e_1 + e_2 = \varepsilon, e_1, e_2 \in \mathbb{R} \}.$$ 

Putting (11) and (12) into the last display yields that

$$(g_1 + g_2)^* (\varepsilon) = \inf \left\{ \frac{\sigma^2}{2 \lambda} h_1 \left( \frac{\sigma e_1}{\lambda} \right) + \frac{(1-\lambda)\sigma^2}{2 \lambda \sigma^2} h_2 \left( \frac{5 \sigma e_2}{\lambda \sigma^2} \right) : e_1 + e_2 = \varepsilon, e_1 \geq 0, e_2 \geq 0 \right\}.$$ 

Bounding $h_1(u) \geq \frac{u^2}{2(1+u/3)}$ and $h_2(u) \geq \frac{1}{2+u}$ for $u \geq 0$ yields that

$$(g_1 + g_2)^* (\varepsilon) \geq \inf \left\{ \frac{e_1^2}{2(\sigma^2 + \frac{e_1}{3})} + \frac{e_2^2}{2 \left( \frac{21}{14} \sigma^2 + \frac{5 \sigma e_2}{14} \right)} : e_1 + e_2 = \varepsilon, e_1 \geq 0, e_2 \geq 0 \right\}.$$ 

Using the fact that $e_1^2/a + e_2^2/b \geq (e_1 + e_2)^2/(a + b)$ for any non-negative $e_1, e_2$ and positive $a, b$ yields

$$(g_1 + g_2)^* (\varepsilon) \geq \inf \left\{ \frac{(e_1 + e_2)^2}{2(\sigma^2 + \frac{e_1}{3})} + \frac{21}{14} \sigma^2 + \frac{5 \sigma e_2}{14} : e_1 + e_2 = \varepsilon, e_1 \geq 0, e_2 \geq 0 \right\}.$$ 

$$= \frac{\varepsilon^2}{2 \left( \frac{14}{13} \sigma^2 + \frac{5 \sigma e_2}{14} \right)}.$$ 

For the case of $\lambda = 0$, we merely need to lower bound $h_1(u)$ with $u^2/2(1+u/3)$ as $g_1^*(\varepsilon) = (g_1 + g_2)^* (\varepsilon)$.
Proof of Theorem 1. By Lemma 1, it suffices to show for $f \in \{f_1, \ldots, f_n\}$ that
\[
|E^{t/2} \tilde{P}_t E^{t/2}| \leq e^{R(t)+K_2(t)},
\] (13)
where
\[
\sigma^2 = \pi(f^2), \quad g_1(t) = \frac{\sigma^2}{c^2} (e^{tc} - 1 - tc), \quad g_2(t) = \frac{\sigma^2 \lambda t^2}{1 - \lambda - 5ct}.
\]
To this end, we discretize $f$ as $\tilde{f}_k$ in the way of Lemma 6. Then
\[
|E^{t/2} \tilde{P}_t E^{t/2}| \to \lim_{k \to \infty} |E^{t/2} \tilde{P}_t E^{t/2}|.
\]
Note that $\tilde{f}_k$ takes at most $(6k + 1)$ possible values. For each $E^{t/2} \tilde{P}_t E^{t/2}$, let $(\tilde{X}_i)_{i \geq 1}$ be a Markov chain of $\tilde{P}_t$ and $Y_i = f_{\tilde{X}_i}(X_i)$. Then $(Y_i)_{i \geq 1}$ is a Markov chain in a finite state space $\mathcal{Y}_k = f_{\tilde{X}_k}(X_k)$. Let $\bar{Q}_k$ and $\mu_k$ denote the transition kernel (transition probability matrix) and invariant distribution (probability vector) of the Markov chain $(Y_i)_{i \geq 1}$, and let $E^{t/2} \bar{Q}_k$ be the diagonal matrix of $e^{t/2}$ for $y \in \mathcal{Y}_k$. By Lemma 7,
\[
|E^{t/2} \tilde{P}_t E^{t/2}| = |E^{t/2} \bar{Q}_k E^{t/2}|_\mu.
\]
Note that
\[
\sum_{y \in \mathcal{Y}_k} \mu_k y = \sum_{y \in \mathcal{Y}_k} \pi(\{x : \tilde{f}_k(x) = y\})y = \pi(\tilde{f}_k) = 0
\]
and that $\bar{Q}_k$ is a reversible, irreducible transition probability matrix on finite state space $\mathcal{Y}_k$ with both absolute and right spectral gaps being $1 - \lambda$. It follows from Lemma 8 that
\[
|E^{t/2} \bar{Q}_k E^{t/2}|_\mu \leq \exp \left( \frac{\pi(\tilde{f}_k^2)}{c^2} (e^{tc} - 1 - tc) + \frac{\pi(\tilde{f}_k^2) \lambda t^2}{1 - \lambda - 5ct} \right).
\]
Collecting these pieces together and letting $k \to \infty$ yield (13) and complete the proof. □

Proof of Theorem 2. Just substitute Lemma 1 with Lemma 2 in the proof of Theorem 1. □

Proof of Theorem 3. Let $g(t) = \frac{\sigma^2}{c^2} (e^{tc} - 1 - tc) + \frac{\sigma^2 \lambda t^2}{1 - \lambda - 5ct}$. Write
\[
\mathbb{E}_\nu \left[ e^{\sum_{i=1}^n f(X_i)} \right] = \mathbb{E}_\nu \left[ \frac{d\nu}{d\pi} (X_1) \cdot e^{\sum_{i=1}^n f(X_i)} \right]
\]
\[
\leq \mathbb{E}_\nu \left[ \left\| \frac{d\nu}{d\pi} (X_1) \right\|^{\frac{1}{p}} \right] \cdot \mathbb{E}_\nu \left[ e^{q \sum_{i=1}^n f(X_i)} \right]^{\frac{1}{q}} \quad \text{[Hölder’s inequality]}
\]
\[
\leq \left\| \frac{d\nu}{d\pi} \right\|_{\nu,p} \exp (n \cdot g(qt)/q). \quad \text{[Theorem 1]}
\]
This arrives at the first claimed inequality. The second claimed inequality follows from the first claimed inequality and the facts that the conjugate of $g_{a,b}(t) = ag(t/a) + b$ is given by $g_{a,b}^*(\epsilon) = g^*(\epsilon)/a - b$ (Boyd and Vandenberghe, 2004, Page 95) and Lemma 9. □

Proof of Theorem 4. The proof is similar to that of Fan et al. (2021, Theorem 4), and thus is omitted. □
Algorithm 1 Independent Metropolis-Hastings (IMH) Algorithm

Pick an initial state \((x_0, y_0, z_0) \sim \nu(x, y, z)\).

for \(i = 0, 1, 2, \ldots\) do

Generate a random candidate state \((x', y', z') \sim \nu(x, y, z)\).

Calculate the acceptance probability

\[
A(x_i, y_i, z_i, x', y', z') = \min\left\{1, \frac{\pi(x', y', z')\nu(x_i, y_i, z_i)}{\pi(x_i, y_i, z_i)\nu(x', y', z')}\right\}
\]

Generate a uniform random number \(u \in [0, 1]\).

if \(u \leq A(x_i, y_i, z_i, x', y', z')\) then

accept the new state \((x', y', z')\) and set \((x_{i+1}, y_{i+1}, z_{i+1}) = (x', y', z')\).

else

reject the new state \((x', y', z')\) and set \((x_{i+1}, y_{i+1}, z_{i+1}) = (x_i, y_i, z_i)\).

end if

end for

5 Applications

5.1 MCMC integral estimation with non-asymptotic confidence intervals

MCMC methods are widely used for numerical approximation of multi-dimensional integrals in Bayesian statistics, computational physics and computational biology (Gilks et al., 1995). To showcase the advantages of our new Bernstein-type inequalities, we derive an MCMC estimator with a non-asymptotic confidence interval for a multi-dimensional integral and compare this confidence interval with those derived by previous Bernstein-type and Hoeffding-type inequalities.

Consider a typical example of the MCMC estimation for an integral in \(\mathbb{R}^3\)

\[
\pi(f) = \int_0^1 \int_0^1 \int_0^1 f(x, y, z)\pi(x, y, z)dx dy dz,
\]

where

\[
f(x, y, z) = \sqrt{x^2 + y^2 + z^2}
\]

and

\[
\pi(x, y, z) \propto \exp\left(\sin\left(\frac{\pi xyz}{2}\right)\right).
\]

A reversible Markov chain \(\{(X_i, Y_i, Z_i)\}_{i \geq 1}\) is generated by an independent Metropolis-Hastings (IMH) algorithm with the uniform distribution

\[
\nu(x, y, z) = 1, \quad \forall (x, y, z) \in [0, 1]^3
\]

as the initial distribution and the proposal kernel. The details are collected in Algorithm 1. Note that the normalizing constant of the probability density function \(\pi(x, y, z)\) is not needed for the implementation of Algorithm 1, as only the ratio of \(\pi(x', y', z')/\pi(x, y, z)\) is used in the calculation of the acceptance probability.
An MCMC estimator for the desired integral $\pi(f)$ is given by

$$\hat{f}_n = \frac{1}{n} \sum_{i=0}^{n-1} f(X_{ik+1}, Y_{i+1}, Z_{ik+1})$$

for some positive integer $k$. Using evenly-spaced subsamples of a Markov chain is a popular trick in the practice of MCMC (Geyer, 1992). These sub-samples $\{(X_{ik+1}, Y_{i+1}, Z_{ik+1})\}_{i \in \mathbb{Z}}$ form a Markov chain with a transition kernel $P^k$, which is the $k$-th iterate of the transition kernel $P$ of the original Markov chain. It is clear that $\Lambda_k(P^k) = \Lambda_k(P)^k$ by the spectral mapping theorem.

Applying the Bernstein-type inequality in Theorem 3 yields

$$\mathbb{P}_\pi\left(|\hat{f}_n - \pi(f)| \geq \epsilon\right) \leq 2 \sup_{(x, y, z) \in [0, 1]^3} \frac{\nu(x, y, z)}{\pi(x, y, z)} \times \exp\left(-\frac{ne^2/2}{1+\epsilon^2\sigma^2 + 5\epsilon^2}\right).$$

Let $C$ be the normalizing constant of $\pi$, namely,

$$C := \int_0^1 \int_0^1 \int_0^1 \exp\left(\frac{\pi(x,y,z)}{2}\right) dx dy dz$$

then

$$\sup_{(x, y, z) \in [0, 1]^3} \frac{\nu(x, y, z)}{\pi(x, y, z)} = \frac{\nu(0, 0, 0)}{\pi(0, 0, 0)} = C.$$ 

It follows that, with probability at least $1 - 2\delta$, $\pi(f)$ is within the interval

$$\hat{f}_n \pm \sqrt{\frac{1 + \lambda^k}{1 - \lambda^k}} \cdot \sqrt{\frac{2\log(C/\delta)}{n}} + \frac{5}{1 - \lambda^k} \cdot \frac{2\log(C/\delta)}{n}. $$

For the unknown constant $C$, we substitute it with its upper bound $e$. For the unknown constant $\sigma^2 := \text{Var}_\pi(f)$, we substitute it with its sample estimate. For the unknown constant $\lambda$, we substitute it with its upper bound $1 - e^{-1}$. This upper bound is deduced by using Atchadé and Perron (2007, Theorem 2.2). Write

$$1 - \lambda(P) = 1 - \lambda_k(P) = 1 - \lambda = \inf_{(x, y, z) \in [0, 1]^3} \tilde{A}(x, y, z),$$

where $\tilde{A}(x, y, z)$ is the overall acceptance probability at state $(x, y, z)$

$$\tilde{A}(x, y, z) = \int_0^1 \int_0^1 \int_0^1 A(x, y, z, x', y', z') \nu(x', y', z') dx' dy' dz' = \int_0^1 \int_0^1 \int_0^1 \min\left\{1, \frac{\pi'(x', y', z')}{\nu(x', y', z')}\right\} \nu(x', y', z') dx' dy' dz'$$

Apparently, $(x_*, y_*, z_*) = (1, 1, 1)$ minimizes the density ratio $\frac{\nu(x, y, z)}{\pi(x, y, z)}$. Thus

$$\inf_{(x, y, z) \in [0, 1]^3} \tilde{A}(x, y, z) = \int_0^1 \int_0^1 \int_0^1 \min\left\{1, \frac{\pi'(x', y', z')}{\nu(x', y', z')}\right\} \nu(x', y', z') dx' dy' dz' = \int_0^1 \int_0^1 \frac{\pi(x, y, z) \nu'(x', y', z')}{\pi(x_*, y_*, z_*)} dx' dy' dz' = \frac{\nu(x_*, y_*, z_*)}{\pi(x_*, y_*, z_*)} = \inf_{(x, y, z) \in [0, 1]^3} \frac{\nu(x, y, z)}{\pi(x, y, z)} = Ce^{-1} \geq e^{-1}. $$
With $k = 10$, $\delta = 0.005$ and $n = 100000$, experimental results show that $\pi(f)$ is within the interval $0.5829 \pm 0.0052$ with at least 99% probability. Table 3 compares this confidence interval with those derived by other Bernstein-type and Hoeffding-type inequalities for Markov chains. The confidence interval derived by our Bernstein-type inequalities is at least 30% tighter than those derived by previous Bernstein-type and Hoeffding-type inequalities.

| Inequality Type     | Half Length of CI |
|---------------------|-------------------|
| Theorem 3           | Bernstein         | 0.0052           |
| Paulin (2015, Equation 3.20) | Bernstein     | 0.0068           |
| Paulin (2015, Equation 3.21) | Bernstein     | 0.0071           |
| Paulin (2015, Equation 3.22) | Bernstein     | 0.0104           |
| Fan et al. (2021, Theorem2) | Hoeffding  | 0.0080           |

Table 3: Comparing confidence intervals (CIs) derived by selective concentration inequalities for Markov chains

As a final remark, we note that, although a change-of-variable trick $t = xyz$ and some numerical integration method can approximate

$$C = \int_0^1 \exp\left(\sin\left(\frac{\pi t}{2}\right)\right) \frac{\log^2(t)}{2} dt \approx 1.2335$$

in this typical example and further $\lambda = 1 - Ce^{-1} \approx 0.5462$, the normalizing constant $C$ is usually difficult to compute in general cases. We use the upper bounds $\lambda \leq 1 - e^{-1} \approx 0.6321$ and $C \leq e \approx 2.7183$ rather than their approximate values $\lambda \approx 0.5462$ and $C \approx 1.2335$ to construct the confidence intervals in Table 3. Nonetheless, the approach for the IMH algorithm showcased in this section is applicable whenever the density ratio $\nu/\pi$ has known upper and lower bounds.

5.2 Robust mean estimation under Markov dependence

In our second example, we consider to robustly estimate the mean of a sequence of Markov-dependent samples. Suppose the data $y_1, \ldots, y_n \in \mathbb{R}$ are generated according to

$$y_i = \mu^* + e_i, \quad (14)$$

where $\mu^* \in \mathbb{R}$ is the underlying mean and $\{e_i : 1 \leq i \leq n\}$ with $\mathbb{E} e_i = 0$ is a stationary and general Markov chain with invariant measure $\pi$ and absolute spectral gap $1 - \lambda > 0$. With a slight overload of notation, we assume $\pi$ has only bounded second moment such that

$$\sigma^2 := \int x^2 \pi(dx) < \infty.$$ 

To estimate the mean $\mu^*$, the sample mean estimator $\sum_{i=1}^n y_i/n$ is known to achieve at best a polynomial-type non-asymptotic confidence width even in the independent case.
That is, for i.i.d. $y_i$, there is some distribution $F$ for $\varepsilon_i$ with mean 0 and variance $\sigma^2$ such that
\[
P \left( \left| \sum_{i=1}^{n} \frac{y_i}{n} - \mu \right| \leq c \sigma \sqrt{\frac{1}{n} \cdot \frac{1}{\delta}} \right) > 1 - 2\delta,
\]
for some constant $c$. Intuitively, this means that the sample mean does not concentrate on the true mean fast enough when errors have only finite variances. Catoni (2012) made an important step towards estimating the mean with faster concentration. Specifically, Catoni constructs a robust mean estimator $\hat{\mu}_r$, depending on some tuning parameter $\tau$, that deviates from the true mean $\mu$ logarithmically in $1/\delta$, that is
\[
P \left( \left| \hat{\mu}_r - \mu^* \right| \leq c \sigma \sqrt{\frac{1}{n} \cdot \log \left( \frac{1}{\delta} \right)} \right) > 1 - 2\delta, \tag{15}
\]
for some constant $c$.

However, no such results exist for Markov-dependent data with only second moment. Our goal is to estimate $\mu^*$ in (14) with a similar non-asymptotic guarantee as in (15). Similar to Sun et al. (2020), we will make use of the Huber loss: $\ell_\tau(x) = x^2/2$ if $|x| \leq \tau$; and $\ell_\tau(x) = \tau |x| - \tau^2/2$, elsewhere. We consider the following estimator
\[
\hat{\mu}_r = \arg\min_\mu \left\{ \frac{1}{n} \sum_{i=1}^{n} \ell_\tau(y_i - \mu) =: L_n(\mu) \right\}.
\]

We need the following curvature lemma, whose proof is provided later in this subsection.

**Lemma 10.** Suppose that $\tau \geq c(r + \sigma)$ and $n \geq c(1 - \lambda)^{-1}(1 + \lambda) \log(1/\delta)$ for some absolute constant $c$. With probability at least $1 - \delta$, we have
\[
\kappa_-(\mu^*, r) = \inf_\mu \left\{ \frac{\langle \nabla L_n(\mu) - \nabla L_n(\mu^*), \mu - \mu^* \rangle}{\| \mu - \mu^* \|^2} : \mu \in B_r(\mu^*) \right\} \geq 1/2.
\]

The above lemma indicates that the loss function $L_n(\mu)$ is locally one-point strongly convex with respect to (w.r.t.) a single point $\mu^*$. Intuitively, this lemma should hold with high probability since the Huber loss is quadratic in the center and linear at tails. In the independent case, Sun et al. (2020) showed a similar result holds with high probability in the independent case. With this lemma, we are ready to derive the non-asymptotic properties of $\hat{\mu}_r$.

**Theorem 5.** Take $\tau = c_0 \sigma \sqrt{n / \log(1/\delta)}$. Suppose $n \geq c(\sqrt{\alpha_1(\lambda)} \vee \alpha_2(\lambda)) \log(1/\delta)$ for some constant $c$, where $\alpha_1(\lambda)$ and $\alpha_2(\lambda)$ are defined in Theorem 1, that is
\[
\alpha_1(\lambda) = \begin{cases} 1 + \lambda & \text{if } \lambda = 0, \\ 1 - \lambda & \text{if } \lambda \in (0, 1), \end{cases} \quad \alpha_2(\lambda) = \begin{cases} 1/3 & \text{if } \lambda = 0, \\ 5/(1 - \lambda) & \text{if } \lambda \in (0, 1). \end{cases}
\]
Then with probability at least $1 - 3\delta$, 

$$|\bar{\mu}_t - \mu^*| \leq C\sigma \left( \sqrt{\alpha_1(\lambda) \vee \alpha_2(\lambda)} \right) \sqrt{\frac{\log(1/\delta)}{n}},$$

for some constant $C$.

The theorem above is a direct consequence of our newly-derived Bernstein-type inequalities, while using the Hoeffding-type inequality for general Markov chains by Fan et al. (2021) would give a significantly weaker, in fact non-diminishing, upper bound. Applying the Hoeffding-type inequality in the proof of Theorem 5, we would get instead, with probability at least $1 - 3\delta$, 

$$|\bar{\mu}_t - \mu^*| \leq C\sigma \sqrt{\alpha_1(\lambda)}.$$

6 Discussions

The generality and sharpness of our inequalities result from a few new elements we bring to Lezaud’s method (Lezaud, 1998a). To highlight the technical novelty of this paper, let us relook at Lezaud’s method. For a finite-state-space, reversible Markov chain $\{X_i\}_{i \geq 1}$ and time-dependent function $f$ with $|f| \leq c$, $\pi(f) = 0$, and $\pi(f^2) = \sigma^2$, Lezaud’s method consists of the following four steps:

(a) For any $t > 0$, $E_\pi \left[ e^{t \sum_{i=1}^n f(X_i)} \right] \leq \| E^{t/2} PE^{t/2} \|^n_\pi$;

(b) $\| E^{t/2} PE^{t/2} \|^\pi_\pi$ is equal to the largest eigenvalue of $E^{t/2} PE^{t/2}$, which coincides with that of $PE^{t/2}$;

(c) By Kato’s perturbation theory (Kato, 2013, Chapter 2), the largest eigenvalue of $PE^{t/2}$ is expanded as a series in $t$ and then upper bounded;

(d) The Cramér-Chernoff method derives final inequalities.

Our paper extends Lezaud’s method for general-state-space Markov chains. It is straightforward that steps (a) and (d) still hold on general state spaces. Step (c) had been justified by Lezaud (2001, in Appendix). However, step (b) does not follow directly, because the perturbed Markov operator $E^{t/2} PE^{t/2}$ in the general-state-space setting may be infinite-dimensional and its operator norm is not necessarily equal to its largest eigenvalue.

To overcome this difficulty, we take a Léon-Perron operator $\hat{P}$ as the extremal version of the original Markov operator $P$ and replace step (a) of Lezaud’s method with 

$$E_\pi \left[ e^{t \sum_{i=1}^n f(X_i)} \right] \leq \| E^{t/2} \hat{P} E^{t/2} \|^n_\pi.$$

Further, by discretizing $f$ as $\hat{f}_k$, we show that 

$$\| E^{t/2} \hat{P} E^{t/2} \|^\pi_\pi = \lim_{k \to \infty} \| E^{t/2} \hat{P} E^{t/2} \|^\pi_\pi.$$
$E^{t/2}PE^{t/2}$ is self-adjoint and behaves like an operator on a finite state space. We henceforth consider its finite-state-space counterpart $E^{t/2}QE^{t/2}$. Steps (b) and (c) of Lezaud’s method can then be applied to analyze $E^{t/2}QE^{t/2}$.

Both the Léon-Perron operator and the discretization technique were originally proposed for establishing Hoeffding-type inequalities for Markov chains (León and Perron, 2004; Miasojedow, 2014; Fan et al., 2021). Interestingly, they can help reduce the general-state-space cases to finite-state-space cases in the derivation of our Bernstein-type inequalities without losing sharpness.

In the finite state space, we expand the norm (largest eigenvalue) of the perturbed Markov operator as a series in $t$ in the same way with Lezaud’s method, but derive tighter bounds for coefficients. These tighter bounds lead to a more precise characterization of the mgf of $\sum_i f_i(X_i)$

$$\mathbb{E}_\pi \left[ e^{\sum_i f_i(X_i)} \right] \leq \exp \left( n \cdot \frac{\sigma^2 \lambda t^2}{g_1(t)} + n \cdot \frac{\sigma^2 A t^2}{g_2(t)} \right).$$

In the exponent, both $g_1(t)$ and $g_2(t)$ are convex functions of $t$. $g_1(t)$ coincides with the convex function that derives the classical Bernstein’s inequality (3) for independent random variables, and $g_2(x)$ goes to 0 as the Markov dependence degenerates to the independence. This preciseness eventually contributes to the exact reduction of our Bernstein inequalities to their classical counterparts under independence.

Another challenge arises when applying the Cramér-Chernoff method to convert the mgf bound to the deviation probability bound, due to the unavailability of an explicit expression of the convex conjugate of $g_1(t) + g_2(t)$. To address this challenge, we use the Moreau-Rockafellar formula (Rockafellar, 2015, Theorem 16.4) in the convex analysis theory and lower bound the convex conjugate. This derivation loses no sharpness in terms of the variance proxy.

We provide applications of our newly derived Bernstein-type inequalities to an MCMC integral estimation problem and a robust mean estimation problem. In the MCMC integral estimation problem, the non-asymptotic confidence interval by our Bernstein-type inequalities is at least 30% tighter than previous state-of-the-art (Paulin, 2015). For the robust mean estimation problem, the Hoeffding-type inequalities by Fan et al. (2021) can only achieve a non-diminishing upper bound. This is because Bernstein-type inequalities incorporate variances of the random variables and thus are generally sharper, especially when the random variables occasionally take on large values but have relatively small variances. We expect to see more use cases in the future.
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A Proofs for Section 5

This section presents the proofs to Lemma 10 and Theorem 5. We start with the proof of Lemma 10.

Proof of Lemma 10. For notational simplicity, let \( f_i(\mu) = \ell_{\tau}(y_i - \mu) \) and
\[
\psi_{\tau}(x) := \nabla \ell_{\tau}(x) = \text{sign}(x)(|x| \land \tau).
\]
By convexity, \( \langle \nabla f_i(\mu) - \nabla f_i(\mu^*), \mu - \mu^* \rangle \geq 0 \) for any \( i \), and hence
\[
D(\mu) := \langle \nabla L_n(\mu) - \nabla L_n(\mu^*), \mu - \mu^* \rangle
\]
\[
= \frac{1}{n} \sum_{i=1}^{n} \langle \psi_{\tau}(\epsilon_i) - \psi_{\tau}(y_i - \mu), \mu - \mu^* \rangle
\]
\[
\geq \frac{1}{n} \sum_{i=1}^{n} \langle \psi_{\tau}(\epsilon_i) - \psi_{\tau}(y_i - \mu), \mu - \mu^* \rangle 1_{E_i},
\]
where \( 1_{E_i} \) is the indicator function of the event
\[
E_i = \{ |\epsilon_i| \leq \tau/2 \} \cap \{ |\mu - \mu^*| \leq r \leq \tau/2 \} = \{ |\epsilon_i| \leq \tau/2 \}
\]
where the second equality automatically verifies as long as \( \tau \geq 2r \). On event \( E_i \), observe that \( \psi_{\tau}(\epsilon_i) = \epsilon_i \) and \( \psi_{\tau}(y_i - \mu) = y_i - \mu \) for all \( \mu \in B_r(\mu^*) \). Consequently,
\[
D(\mu) \geq \frac{1}{n} \sum_{i=1}^{n} (\mu - \mu^*, \mu - \mu^*) 1_{E_i}
\]
\[
= |\mu - \mu^*|^2 \cdot \frac{1}{n} \sum_{i=1}^{n} 1_{E_i}
\]
for any \( \mu \in B_r(\mu^*) \). Thus to prove the lemma, it suffices to lower bound \( \frac{1}{n} \sum_{i=1}^{n} 1_{E_i} \). By the Hoeffding-type inequality for general Markov chains (Fan et al., 2021, Theorem 3), we have with probability at least \( 1 - \delta \)
\[
\frac{1}{n} \sum_{i=1}^{n} 1_{E_i} \geq - \sqrt{\frac{1 + \lambda}{1 - \lambda} \cdot \log(1/\delta)} \cdot \frac{1}{2n}
\]
By Markov’s inequality,
\[
1 \geq \mathbb{P}(|\epsilon_i| \leq \tau/2) \geq 1 - \frac{4\sigma^2}{\tau^2} \geq \frac{3}{4}.
\]
Now because
\[
n \geq 32 \frac{1 + \lambda}{1 - \lambda} \log(1/\delta),
\]
thus it holds that
\[
\frac{1}{n} \sum_{i=1}^{n} 1_{E_i} \geq \mathbb{E}1_{E_i} - \sqrt{\frac{1 + \lambda}{1 - \lambda} \cdot \log(1/\delta)} \cdot \frac{1}{2n} \geq \frac{1}{2}.
\]
This finishes the proof. \( \square \)
Proof of Theorem 5. For simplicity of notation, write $\mathbb{E}_x$ as $\mathbb{E}$ and $\mathbb{P}_x$ as $\mathbb{P}$. By Lemma 10, we have, if $\tau \geq c(r + \sigma)$ and $n \geq c(1 - \lambda)^{-1}(1 + \lambda) \log(1/\delta)$ for some absolute constant $c$, then with probability at least $1 - \delta$

$$\kappa_-(\mu^*, r) = \inf_{\mu} \left\{ \frac{\langle \nabla L_\mu(\mu) - \nabla L_\mu(\mu^*), \mu - \mu^* \rangle}{|\mu - \mu^*|^2} : \mu \in \mathcal{B}_r(\mu^*) \right\} \geq 1/2.$$ 

From now on, we implicitly condition on the above probability event $\mathcal{E}$. Take $r$ such that

$$r = 4\sigma \left( 4c_0\alpha_2(\lambda) + \sqrt{2\alpha_1(\lambda) + c_0} \right) \sqrt{\frac{\log(1/\delta)}{n}}$$

and assume that $|\hat{\mu}_r - \mu^*| \leq r$. Then

$$\frac{1}{2} |\hat{\mu}_r - \mu^*|^2 \leq \left| \nabla L_\mu(\hat{\mu}) - \nabla L_\mu(\mu^*) \right| \cdot |\hat{\mu}_r - \mu^*|,$$

or equivalently

$$\frac{1}{2} |\hat{\mu}_r - \mu^*| \leq \left| \nabla L_\mu(\hat{\mu}) - \nabla L_\mu(\mu^*) \right| \leq \left| \nabla L_\mu(\mu^*) - \mathbb{E} \nabla L_\mu(\mu^*) \right| + \left| \mathbb{E} \nabla L_\mu(\mu^*) \right|,$$

where we use the first order optimality condition that $\nabla L_\mu(\hat{\mu}) = 0$ because $\hat{\mu}_r$ is a minimum of $L_\mu(\mu)$. Both terms in the right hand side of the above inequality involve the gradient of the loss function, which can be written as

$$\nabla L_\mu(\mu^*) = \frac{1}{n} \sum_{i=1}^{n} \text{sign}(\epsilon_i) \cdot (|\epsilon_i| \wedge \tau), \quad (16)$$

where sign(·) is the sign function. We start with the second term. Because $\mathbb{E}(\epsilon_i) = 0$, we have

$$\left| \mathbb{E} \nabla L_\mu(\mu^*) \right| = \frac{1}{n} \left| \sum_{i=1}^{n} \mathbb{E} \left( \text{sign}(\epsilon_i) \cdot (|\epsilon_i| \wedge \tau) - \epsilon_i \right) \right|$$

$$= \frac{1}{n} \left| \sum_{i=1}^{n} \mathbb{E} (\epsilon_i \cdot 1(|\epsilon_i| > \tau)) \right|$$

$$\leq \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left( \frac{\epsilon_i^2 \cdot 1(|\epsilon_i| > \tau)}{\tau} \right) \leq \frac{\sigma^2}{\tau},$$

where the last second inequality is due to Markov inequality. We then bound the first term. Rewrite $\nabla L_\mu(\mu^*) - \mathbb{E} \nabla L_\mu(\mu^*)$ as

$$\nabla L_\mu(\mu^*) - \mathbb{E} \nabla L_\mu(\mu^*) = \frac{1}{n} \sum_{i=1}^{n} \text{sign}(\epsilon_i) \cdot (|\epsilon_i| \wedge \tau) - \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ \text{sign}(\epsilon_i) \cdot (|\epsilon_i| \wedge \tau) \right]$$

$$= \frac{1}{n} \sum_{i=1}^{n} f(\epsilon_i), \quad (17)$$

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where \( f(e_i) = \text{sign}(e_i) \cdot (|e_i| \land \tau) - \mathbb{E}\{\text{sign}(e_i) \cdot (|e_i| \land \tau)\} : \mathbb{R} \rightarrow [-\tau - \frac{\alpha^2}{\tau}, \tau + \frac{\alpha^2}{\tau}] \). We have \( \sum_{i=1}^{n} p(f_i^2)/n \leq \sigma^2 \). Thus applying Theorem 1 with \( c = \tau + \frac{\alpha^2}{\tau} \) and \( f_i = f \), we obtain for any \( t > 0 \)

\[
P\left( \frac{1}{n} \sum_{i=1}^{n} f(e_i) > t \right) \leq \exp\left(-\frac{nt^2/2}{\alpha_1(\lambda) \cdot \sigma^2 + \alpha_2(\lambda) \cdot ct}\right), \tag{18}
\]

where \( \alpha_1(\lambda) \) and \( \alpha_2(\lambda) \) are defined in Theorem 1. Taking \( \tilde{t} = \frac{nt^2/2}{\alpha_1(\lambda) \cdot \sigma^2 + \alpha_2(\lambda) \cdot ct} \) in (18) and using a symmetry argument, we obtain for any \( \tilde{t} > 0 \)

\[
P\left( \frac{1}{n} \sum_{i=1}^{n} f(e_i) \geq \frac{2\alpha_2(\lambda) ct}{n} + \sqrt{\frac{2\alpha_1(\lambda) \sigma^2 \tilde{t}}{n}} \right) \leq 2 \exp(-\tilde{t}).
\]

Taking \( \delta = \exp(-\tilde{t}) \) in the above inequality and \( \tau = c_0 \sigma \sqrt{n/\log(1/\delta)} \geq \sigma \), we obtain with probability at least \( 1 - 2 \delta \)

\[
\left| \frac{1}{n} \sum_{i=1}^{n} f(e_i) \right| \leq \frac{2\alpha_2(\lambda) c \log(1/\delta)}{n} + \sqrt{\frac{2\alpha_1(\lambda) \sigma^2 \log(1/\delta)}{n}}
\leq \frac{4\alpha_2(\lambda) \log(1/\delta)}{n} + \sqrt{2\sigma \sqrt{\frac{2\alpha_1(\lambda) \log(1/\delta)}{n}}}
\leq \left(4c_0 \alpha_2(\lambda) + \sqrt{2\alpha_1(\lambda)}\right) \sigma \sqrt{\frac{\log(1/\delta)}{n}} \leq \sigma \left(\alpha_2(\lambda) + \sqrt{\alpha_1(\lambda)}\right) \sqrt{\frac{\log(1/\delta)}{n}}.
\]

Combining upper bounds for both terms, we acquire with probability at least \( 1 - 2 \delta \)

\[
\hat{\mu}_r - \mu^* \leq 2\sigma \left(4c_0 \alpha_2(\lambda) + \sqrt{2\alpha_1(\lambda)} + c_0\right) \sqrt{\frac{\log(1/\delta)}{n}}.
\]

Without conditioning on the probability event \( \mathcal{E} \), we obtain with probability at least \( 1 - 3 \delta \) the above inequality holds.

We then show that \( \hat{\mu}_r - \mu^* \leq r \) must hold. If not, we shall construct an intermediate solution between \( \mu^* \) and \( \hat{\mu}_r \), denoted by \( \mu_{r,\eta} = \mu^* + \eta(\hat{\mu}_r - \mu^*) \), such that \( |\mu_{r,\eta} - \mu^*| = r \). Specifically, we can choose some \( \eta \in (0, 1) \) so that \( |\mu_{r,\eta} - \mu^*| = r \). We then repeat the above calculation and obtain

\[
|\mu_{r,\eta} - \mu^*| \leq 2\sigma \left(4c_0 \alpha_2(\lambda) + \sqrt{2\alpha_1(\lambda)} + c_0\right) \sqrt{\frac{\log(1/\delta)}{n}} < r.
\]

which is a contradiction. Thus it must hold that \( \hat{\mu}_r - \mu^* \leq r \). Finally the conditions in the theorem verify the conditions used in the proof and this completes the proof.

□