One–Dimensional Primordial Gravitational Waves In Quadratic Gravity

Taimur Mohammadi,*
Behrooz Malekolkalami,†

June 29, 2021

Abstract

To study of Primordial Gravitational Waves, one dimensional toy model in pure Quadratic gravity is considered. Solutions to the primordial perturbations include simple incoming and outcoming waves. Interestingly, the perturbations with scale–invariant spectrum (confirmed by CMB data) can be found among the solutions.

Keywords: Gravitational waves, Power Spectrum, Quadratic Gravity.

1 Introduction

Gravitational waves (GW) are disturbances propagated in spacetime which have different sources as dynamo mechanism in binary systems and black holes. The GW was first predicted in 1916 by Einstein [1] and the first direct observation from merging massive black holes reported in February 2016 by the Virgo collaboration and LIGO scientific collaboration [2,3,4]. This observation can open a new way to a gravitational–wave astronomy, especially in helping to understand and explore different and yet inaccessible phenomena in astrophysics and cosmology.

The standard cosmology is based on the idea that the universe has reasonably homogeneous and isotropic structure (in the large scale). Based on the observational evidences, Universe is in an acceleration phase today and then to describe a relatively complete evolving universe, the descriptive equations must include parameters such as the composition of the universe, its current expansion rate and the initial spectrum of density perturbations. The initial density perturbations have been generated during inflation and are assumed to be the origin of large scale structure observable today. In addition, simple inflation models predict tensorial fluctuations which are the source of gravitational radiation so–called primordial GW (PGW). They form a stochastic background of GW with a nearly scale–invariant spectrum. PGW have been propagating in spacetime since the inflationary period and have not been detected to date. It is hoped that they could be detected by the next generations of more sensitive instruments if their amplitude is large enough. A detection of PGW can be a great help in revealing the mysteries of the very early universe as fluctuations amplified by inflationary mechanism, and energy scales of the inflationary period, because, PGW power spectrum is a direct measure of the expansion rate of the Universe at the time that wavelength was stretched beyond the horizon. In other words, the spectrum of energy carried by PGW can represent a measure of the energy of that period and challenge the ability of a direct detection of these relic waves by the current gravitational detector. The LISA is a complementary project to CMB experiments which aims to detect GW by using laser interferometry.

Although current technical knowledge is not able to detect PGW, there are other ways to reveal the traces of them. The Cosmic Microwave Background (CMB) observations have a key role in this era of precision cosmology. The data collected from a large number of experiments measuring the intensity and the polarization

*Department of Physics, University of Kurdistan, P.O.Box 66177-15175, Sanandaj, Iran. Email: t.mohammadi@uok.ac.ir
†Department of Physics, University of Kurdistan, P.O.Box 66177-15175, Sanandaj, Iran. Email: b.malakolkalami@uok.ac.ir
of the CMB anisotropies are in very good agreement with the predictions of the inflationary paradigm. The major challenge in current CMB Astronomy is the detection of the primordial B–mode polarization, which can be an important proof of the existence of a PGW produced in inflation era. Indeed, it is believed that both electric and magnetic polarization of CMB are generated by GW, specially, the detection of the B–mode signal from stochastic background GW is one of the important goals of these studies.

Due to the random nature of PGW, it is common to use the spectral methods to study and wave analysis when dealing with a such wave. In this regard, one of the important quantity is the spectral energy density. It is convenient to discuss a possible direct detection of the PGW and measures the energy density stored into the waves per logarithmic interval of frequency $f$, that is

$$\Omega_h(k, t) = \frac{1}{\rho_c} \frac{d\rho}{d \ln f},$$

(1)

where $\rho$ and $\rho_c$ are energy density and critical energy density respectively. More precisely, it characterizes the spectrum of the relic gravitational radiation present today ($t = t_0$) inside our horizon, and thus accessible to direct observations. In this work, we will also use this tool to study the PGW in Modified Gravity framework. The study of the PGW and their spectrum has been done in the framework of the General Relativity in the literature. It is true that General Relativity is one of the most successful physical theories which have been tested and confirmed experimentally in the Solar System and by binary pulsars. However, this does not imply that it is valid on all scales in any environments and therefore, its validity may be challenged. In other words, the tests of GR on different cosmological scales and context can identify its regime of validity. Especially that, recent detections of gravitational waves have opened a new window to test gravity in strong gravity regime as well as to test the propagation of gravitational waves.

Based on this motivation, studying the GW in Modified Gravity Theories can be justified and attractive. As such considering the GW produced in inflation scenarios [5], cosmic expansion at very early universe and the current period [6], or Cosmic Acceleration [7] [8] and Cosmological Constant Problem, based on modified gravity theories [9]. The most common model of the modified gravity is $f(R)$ gravity which replace Ricci scalar in the Einstein-Hilbert action by an arbitrary function of Ricci scalar [10]. Accordingly, the study of theoretical models may facilitate or accelerate the achievement of experimental goals. In the present work, we focus on the power spectrum of PGW through a one–dimension (toy) model when the gravity frame is described by a pure quadratic Ricci scalar. The motivation for such a model is due to including exact (and familiar) solutions including arbitrary functions or parameters. This allows the solutions to be adapted to observations or possible constraints. Also, it may be a good help in analyzing the models in two or three dimensions. In addition, we note that as far as the authors’ ability is concerned, the pure quadratic gravity has been selected from a number of tested functions ($f(R) = e^R, \sqrt{R}, ...$), for the reasons mentioned above.

The work is organized as follows:

In section 2, the motion equations for tensorial perturbations in quadratic gravity are presented. In section 3, the one–dimensional model is introduced and discussed. Conclusions are given in section 4.

2 Evolution of Perturbations in $f(R)$ gravity

In order to obtain the motion equations for gravitational perturbations, we consider a perturbed Friedmann–Robertson–Walker (FRW) metric with the line element shown as [14]

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = (\hat{g}_{\mu\nu} + h_{\mu\nu}) dx^\mu dx^\nu = a^2 \left[ -d\tau^2 + (\delta_{ij} + h_{ij}(t, x)) dx^i dx^j \right]$$

(2)

where $\tau$ is conformal time, $\hat{g}_{\mu\nu} = \text{diag}\{-a^2, a^2, a^2, a^2\}$ is the unperturbed FRW background and $h_{\mu\nu}$ are the perturbations satisfying the conditions: symmetric ($h_{ij} = h_{ji}$), traceless ($h^i_i = 0$) and transverse ($h^j_{ij} = 0$) and also $|h_{\mu\nu}| << 1$, $h_{00} = h_{0i} = 0$.

To write down the perturbational equations, we use the lagrangian formalism. For an isotropic and perfect fluid, the anisotropic stress of the energy–momentum tensor [15] is zero. Hence, the motion equations take the following form [16]

$$\partial_\mu \left( \sqrt{-g} \frac{\partial f(R)}{\partial (\partial_\mu h_{ij})} \right) = 0,$$

(3)

which by using the chain rule becomes

$$\partial_\mu \left( \sqrt{-g} \frac{df}{dR} \frac{\partial R}{\partial (\partial_\mu h_{ij})} \right) = 0,$$

(4)

\(^1\)One of the well–known of the quadratic action is Starobinsky model for Cosmic Inflation. Other kinds of $f(R)$ models can be found in [11] [12] [13].
where \( \bar{g} \) is the determinant of \( g_{\mu \nu} \) and Ricci scalar is given by \[14, 17\]:

\[
R = 
\left( \frac{-\bar{g}^{\mu \nu} \partial_{\mu} h_{ij} \partial_{\nu} h_{ij}}{64\pi G} \right).
\]

(5)

By substituting \( f(R) = R^2 \) into (4) and using the isotropic condition \( h_{ij}(\tau, x) = h(\tau, x) \), it reduces to

\[
\partial_{\tau} \left( \sqrt{-\bar{g}} R \bar{g}^{00} \partial_{\tau} h \right) + \partial_x \left( \sqrt{-\bar{g}} R \bar{g}^{11} \partial_x h \right) = 0,
\]

(7)

where Ricci scalar (5) takes the form

\[
R = \frac{\bar{g}^{00} \partial_{\tau} h + \bar{g}^{11} \partial_x h}{\sqrt{-\bar{g}} g_{\mu \nu} \partial_{\mu} h_{ij} \partial_{\nu} h_{ij}}.
\]

(6)

By simplifying this scalar into (7) and calculation, we find

\[
-\partial_{\tau} \left( \left[ (\partial_{\tau} h)^2 - (\partial_x h)^2 \right] \right) \partial_{\tau} h + \partial_x \left( \left[ (\partial_{\tau} h)^2 - (\partial_x h)^2 \right] \right) \partial_x h = 0,
\]

(8)

in the next section, we will use the latter equation to propagate one–dimensional perturbations.

\section{One–Dimensional Solutions for \( f(R) = R^2 \)}

By \( h = h(\tau, x) = h(\tau, x) \), we mean One–dimensional perturbations (say \( x \) dimension), therefore, equation (6) reduces to

\[
\partial_{\tau} \left( \sqrt{-\bar{g}} R \bar{g}^{00} \partial_{\tau} h \right) + \partial_x \left( \sqrt{-\bar{g}} R \bar{g}^{11} \partial_x h \right) = 0,
\]

where Ricci scalar (5) takes the form

\[
R = \frac{\bar{g}^{00} \partial_{\tau} h + \bar{g}^{11} \partial_x h}{\sqrt{-\bar{g}} g_{\mu \nu} \partial_{\mu} h_{ij} \partial_{\nu} h_{ij}}.
\]

(6)

By substituting this scalar into (7) and simplification, we find

\[
-\partial_{\tau} \left( \left[ (\partial_{\tau} h)^2 - (\partial_x h)^2 \right] \right) \partial_{\tau} h + \partial_x \left( \left[ (\partial_{\tau} h)^2 - (\partial_x h)^2 \right] \right) \partial_x h = 0,
\]

(8)

this is a nonlinear equation and as we know nonlinearity can have a different nature to the equations and their solutions. Unlike linear equations, there are much less literature exists on the solution of nonlinear partial differential equations. Thus, depending on form of the equation, methods to obtain solutions can be different. For example, one of the useful technics is to use the symmetric methods, from which the solution can be predicted. On the other hand, solving equations by computer algebra methods has also attracted many people. Fortunately, for equation (8), both symmetrical and computer algebra methods lead to the following general solutions:

\[
h(\tau, x) = \begin{cases} f(x + \tau), \\ g(x - \tau), \end{cases}
\]

where \( f, g \) are arbitrary functions. Note that due to the nonlinearity, the linear combination can’t be a solution.

The above general solutions includes also the familiar plane wave

\[
\epsilon^{i(x \pm \tau)},
\]

(9)

therefore, in such universe, the plane waves can be only in the incoming or outgoing mode. Anyway, such waves do not have the required power to detect today and this can be verified by the spectral power formula for perturbations in (FRW) expanding universe, that is \[18\]

\[
\Omega_b(\tau, k) = \frac{\Delta^2_{h,prim}}{12\pi^2(\tau) H^2(\tau)} \left[ \partial_\tau T(\tau, k) \right]^2
\]

(10)

where \( T(\tau, k) \) is the transfer function and \( \Delta^2_{h,prim} = \frac{16}{\pi} \left( \frac{H_{inf}}{m_{Pl}} \right)^2 \). So, to detect the primordial GW in the present age, one have to look for solutions that produce a proper spectral power. If, we consider the following damped solution

\[
h(\tau, x) = b e^{-\kappa |x - \tau|} \quad b, \kappa > 0,
\]

(11)

and use the equation (10) to calculate its power spectrum, the graph for such spectrum is similar to the graph illustrated in Fig.1. In this figure, \( \Omega(k) = \Omega_b(\tau_0, k), a(\tau_0) = 1 \) and we assume \( b = \kappa = 3 \). To be more precise, Fig.1 shows the spectrum of the (tensorial) primordial perturbations produced in the geometry of the very early universe, at the present time \( \tau = \tau_0 \), in terms of the comoving wave number. In the presented frequency range, it is seen that by increasing the wave number, the spectrum takes a scale–invariant character. Such a character for primordial fluctuations is predicted by the cosmic inflationary scenarios based on the symmetry breaking phase transition of a self–ordering scalar field \[20\].

At the end of this section, we draw the reader’s attention to two points about the spectral graph (Fig.1):

1) By increasing numerical coefficients \( b, \kappa \), the graph moves to the right and up (and vise versa).

2) This graph qualitatively resembles the spectral graph for a driven harmonic oscillator where the main similarities return to the presence of a spectrum peak and decreasing character at high frequencies.

\footnote{\( H_{inf} \) is the Hubble constant during inflation and \( m_{Pl} \) is the Planck Mass.}
4 Conclusions

We look for analytical solutions (in pure Quadratic Gravity) describing the (primordial) gravitational waves that can generate proper spectrum in the present time. To this end, a one–dimension (toy) model for perturbations is considered. The non–linearity of motion equations allows the variety of solutions which allows to select the perturbation solutions having required spectral power. The corresponding spectral graphs indicates, there is no chance to detect for Harmonic perturbations. Instead, a damped solution can provide a spectrum with relatively enough power representing a scale–invariant spectrum in high frequency regime (confirmed by CMB data).

References

[1] A. Einstein, Sitzungsber. K. Preuss. Akad. Wiss. 1, 688 (1916).
[2] B. P. Abbott, et al, Phys. Rev. Lett. 116, 061102 (2016).
[3] B. P. Abbott, et al, Phys. Rev. Lett. 119, 141101 (2017).
[4] B. P. Abbott, et al, Phys. Rev. Lett. 119, 161101 (2017).
[5] M.C Guzzetti, N. Bartolo, M. Liguori and S. Matarrese, La Rivista del Nuovo Cimento 39, 399 (2016).
[6] S. Nojiri, S. D. Odintsov and V. K. Oikonomou, Phys. Rept. 692, 1-104 (2017).
[7] M. Trodden, International Journal of Modern Physics D 16, 2065-2074 (2007).
[8] A. Borowiec, M. Kamionka, A. Kurek, M. Szydlowski, JCAP 2, 207 (2012).
[9] T. Clifton, P. G. Ferreira, A. Padilla and C. Skordis, Phys. Rept. 513, 1-189 (2012).
[10] T. Inagaki, M. Taniguchi, International Journal of Modern Physics D 29, 2050072 (2020).
[11] B. Soham and S. Shankaranarayanan, Eur. Phys. J. C 78, 737 (2018).
[12] J. Naf and P. Jetzer, Phys. Rev. D 84, 024027 (2011).
[13] T. P. Sotiriou and V. Faraoni, Rev. Mod. Phys. 82, 451 (2010).
[14] Latham A. Boyle and Paul J. Steinhardt, Phys. Rev. D 77, 063504 (2008).
[15] S. Weinberg, Phys. Rev. D 69, 023503 (2004).
[16] T. Mohammadi, B. Malekolkalami and X. Ghamari, arXiv: gr-qc/1912.12474 (2020).
[17] M. Gasperini, *Theory Of Gravitational Interaction* (Springer International Publishing, AG 2013, 2017), pp.167-169.

[18] Y. Watanabe and E. Komatsu, *Phys. Rev. D* **73**, 123515 (2006).

[19] N. Bernala and F. Hajkarim, *Phys. Rev. D* **100**, 063502 (2019).

[20] K. Jones-Smith, L. M. Krauss, and H. Mathur, *Phys. Rev. Lett.* **100**, 131302 (2008).