Research on Bifurcation Characters of Rotor-SMA Bearing System

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Abstract. Based on Landau-Devonshire model, the bifurcation characteristic of rotor-shape memory alloy bearings(SMAB) system was investigated in this paper. Heteronomous system was transformed into autonomous system in averaging method and Van der Pol transformation, and the existence of Hopf bifurcation was proved in theory. The concept of broadened set of equilibrium point was introduced to improve centre manifold method to be adapted to heteronomous system. The equation of the flow on the centre manifold of rotor-SMAB system was obtained, and the existence of transcritical bifurcation and supercritical pitchfork bifurcation was proved in theory. Finally the results in centre manifold method and averaging method were compared with each other. The comparison shows that the results of the two methods were both the parts of global dynamic characteristic of rotor-SMAB system, while centre manifold method can be applied to research bifurcation behavior in the case of more dimensions. It means that the two methods both have limitation, and global dynamic characteristic must be obtained in kinds of method.

1. Introduction

Active control to rotor system with shape memory alloy bearings (SMAB) is a new direction in vibration control field. SMA has characteristics of shape memory, variable stiffness and super-elasticity, based on which the variable stiffness SMA bearing used in rotor system can be designed to reduce vibration amplitude of rotor system. Meng et al. researched the nonlinear response of a balanced flexible rotor-bearing system [1]. Zhang et al. investigated the multi-pulse chaotic motions of a rotor-active magnetic bearing (AMB) system with time-varying stiffness [2]. Wang and Jiang analyzed the multiple stabilities in a magnetic bearing system with time delays [3]. Straub and Friedrich obtained control rule of rotor vibration of helicopter engine using SMA [4]. Liu et al. studied active vibration control of rotor system with SMA spring bearing [5]. Yan et al. designed SMA support system for passing through critical speed of high-speed rotor [6]. We analyzed the stability of rotor-SMAB system [7], and the effect of parameters on system characteristic [8], and obtained optimal control discipline [9].

This research aims at dynamic characteristic of rotor-SMAB system. Hopf bifurcation, transcritical bifurcation and supercritical pitchfork bifurcation in rotor-SMAB system were proved by experiment and numerical simulation, but not explained well in theory because of heteronomous characteristic of

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that system. In this paper, the three kinds of bifurcation in rotor-SMAB system were proved in averaging method and centre manifold method respectively, and the results of the two methods were compared with each other.

2. Dynamic Model of Rotor-SMAB System

The structure of SMAB is shown in the left part of Fig.1, and the structure of rotor system is shown in the right part of Fig.1. The rotor system is made up of rotor, SMA variable stiffness bearing and flexible support. The strain-stress curve of SMA in the super-elasticity state is shown in Fig.2.

![Fig.1 the structure of rotor system](image1)

![Fig.2 the strain-stress curve of SMA](image2)

There are many mechanical models of SMA [10-15]. Most of them are based on thermodynamics theory and micromechanics theory, and make the percentage content of martensite as main variable of stress-strain-temperature equation. As results, those mechanical models of SMA are mostly shown as equations with subsection function and hard to be analyzed. Usually, research results to those models can only be obtained by numerical method. In this paper, Landau-Devonshire model was adopted to set up the dynamic model of rotor-SMAB system.

Landau-Devonshire model is described with continuous function. It can be shown as

\[
\sigma(T, \varepsilon) = \rho[2a_2(T - T_M)\varepsilon - 4a_4\varepsilon^3 + 6a_6\varepsilon^5]
\]

(1)

where \(\sigma\) is stress, \(\varepsilon\) is strain, \(T\) is temperature, \(\rho\) is elasticity modulus, \(T_M\) is initiative temperature of phase conversion of SMA, \(a_2, a_4, a_6\) are coefficients of Landau-Devonshire SMA model.

According to the structure of SMA bearing, we obtain

\[
\begin{aligned}
F &= \sigma\ast S \\
\varepsilon &= \varepsilon\ast L
\end{aligned}
\]

(2)

where \(F\) is resilience of SMA, \(x\) is displacement, \(S\) is the area of cross section of SMA wire, \(S = \pi R^2\), \(L\) is the length of SMA wire.

Based on Eqs.(1) and (2), the dynamic model of rotor-SMAB system can be shown as

\[
M\ddot{x} + C\dot{x} + Kx - E\dot{x}^3 + Hx^5 = Me\Omega^2 \cos(\Omega t) + Me\xi \sin(\Omega t)
\]

(3)

where \(M\) is mass, \(x\) is displacement, \(C\) is damping, \(C = C_0 + C_1\), \(C_0\) is inherent damping, \(C_1\) is equivalent damping of SMA, \(K\) is rigidity, \(K = K_0 + \frac{2a_2\rho(T - T_M)\mu S}{L}\), \(K_0\) is inherent rigidity, \(\mu\) is
structure coefficient of SMA bearing, \( E = \frac{4a_s \rho S}{L^3} \), \( H = \frac{6a_s \rho S}{L^3} \), \( e \) is eccentric distance, \( \Omega \) is angular velocity, \( \zeta \) is angular acceleration, \( t \) is time.

Eq.(3) can also be shown as
\[
\ddot{x} + \zeta \dot{x} + \omega^2 x + ax^3 + bx^5 = e\Omega^2 \cos(\Omega t) + e\zeta \sin(\Omega t)
\]
where \( \zeta \) is damping coefficient, \( \zeta = \frac{C}{M} \), \( \omega \) is inherent frequency, \( \omega = \sqrt[2]{\frac{K}{M}} \), \( a \) and \( b \) are nonlinear coefficient, \( a = -\frac{E}{M} < 0 \), \( b = \frac{H}{M} > 0 \).

Let \( A = e\sqrt{\Omega^2 + \zeta^2} \) and ignore the original phase, we obtain
\[
\ddot{x} + \zeta \dot{x} + \omega^2 x + ax^3 + bx^5 = A \cos \Omega t
\]
(4)

3. Bifurcation Analysis in Averaging Method

Eq.(4) can be transformed into autonomous system to be studied in averaging method. Van der Pol transformation was introduced as follow:

\[
\begin{align*}
    x &= \zeta \cos \omega t + \eta \sin \omega t \\
    \dot{y} &= \dot{x} = \omega(-\zeta \sin \omega t + \eta \cos \omega t)
\end{align*}
\]

The additive condition is
\( \zeta \cos \omega t + \eta \sin \omega t = 0 \)

Averaging Eq.(4) in 2 \( \pi \) period, we obtain
\[
\begin{align*}
    \frac{d\zeta}{dt} &= \frac{\omega^2 - \Omega^2}{2\Omega} \eta - \frac{3}{2}\zeta + \frac{3a}{8\Omega} \eta(\zeta^2 + \eta^2) + \frac{5b}{16\Omega} \eta(\zeta^2 + \eta^2)^2 \\
    \frac{d\eta}{dt} &= -\frac{\omega^2 - \Omega^2}{2\Omega} \zeta + \frac{A}{2\Omega} \zeta - \frac{3a}{8\Omega} \eta(\zeta^2 + \eta^2) - \frac{5b}{16\Omega} \zeta(\zeta^2 + \eta^2)^2
\end{align*}
\]
(5)

Let \( f = \frac{\omega^2 - \Omega^2}{2\Omega} \), the linear part of Eq.(5) can be shown as
\[
\begin{pmatrix}
    \zeta' \\
    \eta'
\end{pmatrix} =
\begin{pmatrix}
    -\frac{3}{2} & f \\
    -f & -\frac{3}{2}
\end{pmatrix}
\begin{pmatrix}
    \zeta \\
    \eta
\end{pmatrix}
\]

The eigenvalues were decided by the follow equation:
\[
(\lambda + \frac{3}{2})^2 + f^2 = 0
\]

Thus
\[
\lambda = -\frac{3}{2} \pm if
\]
Obviously, there is a pair of conjugate complex eigenvalues in the system. According to Hopf bifurcation theorem, there is Hopf bifurcation in the system when $\xi = 0$.

4. Bifurcation Analysis in Averaging Method

According to the above result, the existence of Hopf bifurcation in rotor-SMAB system was proved in theory. However, the existence of transcritical bifurcation and supercritical pitchfork bifurcation, which have been found in experiment and numerical simulation, can not be explained by the above result. In fact, Eq.(5) even can not explain the jump and hysteretic phenomenon in Eq.(4), which are familiar phenomenon in nonlinear system. It means that many nonlinear character of original system maybe be lost in averaging course. In this section, centre manifold method were used to research the bifurcation characteristic of Eq.(4).

Original centre manifold method can only be applied in autonomous system, while Eq.(4) is a nonlinear heteronomous differential equation. We introduced the concept of broadened set of equilibrium point into centre manifold method, and regard $t$ as variable constant, then the period orbit can be regard as the set of equilibrium points corresponding to the variable constant $t$. Because $t \in \mathbb{R}$ and real set being numerable set, the set of equilibrium points is the subset of numerable set, so is also numerable.

Let $\dot{x} = y$, Eq.(4) can be written as

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -\xi \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{bmatrix} 0 \\ -ax^3 - bx^5 + A \cos \Omega t \end{bmatrix}$$

The broadened set of balance points of system is $\{(d,0)\}$, where $d$ is defined by

$$-\omega^2 d - ad^3 - bd^5 + A \cos \Omega t = 0$$

(6)

Let $u = x - d$ and $v = y$. According to Eq.(6), we obtain

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega^2 - 3ad^2 - 5bd^4 & -\xi \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{bmatrix} 0 \\ -3adu^2 - au^4 - bu^5 - 5bdu^4 - 10bd^2u^3 - 10bd^3u^5 \end{bmatrix}$$

(7)

Eq.(7) is an autonomic system, and $(0,0)$ is the balance points of new system now. Let $f(u,v,t) = -3abu^2 - au^4 - bu^5 - 5bdu^4 - 10bd^2u^3 - 10bd^3u^5$

The linear system can be written as

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega^2 - 3ad^2 - 5bd^4 & -\xi \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

whose eigenvalues are defined by

$$\lambda \begin{pmatrix} \lambda & -1 \\ \omega^2 + 3ad^2 + 5bd^4 & \lambda + \xi \end{pmatrix} = \lambda^2 + \xi \lambda + \omega^2 + 3ad^2 + 5bd^4 = 0$$

so

$$\lambda = -\frac{\xi \pm \sqrt{\xi^2 - 4(\omega^2 + 3ad^2 + 5bd^4)}}{2}$$

Now we discuss the eigenvalues of the system:
• There are double zero eigenvalues in the system when $\xi = 0$ and $\omega^2 + 3ad^2 + 5bd^4 = 0$, so there is Hopf bifurcation in the system;
• There are a pair of pure imaginary eigenvalues in the system when $\xi = 0$ and $\omega^2 + 3ad^2 + 5bd^4 > 0$, so there is Hopf bifurcation in the system;
• There are single zero eigenvalue in the system when $\xi \neq 0$ and $\omega^2 + 3ad^2 + 5bd^4 = 0$;
• There are not zero eigenvalue in the system when $\xi \neq 0$ and $\omega^2 + 3ad^2 + 5bd^4 \neq 0$;
• There are not zero eigenvalue in the system when $\xi = 0$ and $\omega^2 + 3ad^2 + 5bd^4 < 0$.

There is also the possibility of the existence of bifurcation in the system in the third case above, so we investigate the bifurcation motion in center manifold method.

Obviously, there are $\lambda_1 = 0$ and $\lambda_2 = -\xi$ when $\xi \neq 0$ and $\omega^2 + 3ad^2 + 5bd^4 = 0$. The eigenvector corresponding to $\lambda_1 = 0$ is $(1,0)$, and the eigenvector corresponding to $\lambda_2 = -\xi$ is $(1,-\xi)$.

Let

$$
\begin{align*}
(u & w) = \begin{bmatrix} 1 & 1 \\ 0 & -\xi \end{bmatrix} \begin{bmatrix} w \\ z \end{bmatrix}, \quad T = \begin{bmatrix} 1 & 1 \\ 0 & -\xi \end{bmatrix}, \quad T^{-1} = \begin{bmatrix} 1 & \frac{1}{\xi} \\ 0 & -\frac{1}{\xi} \end{bmatrix}
\end{align*}
$$

The equation of the flow on the centre manifold can be obtained in center manifold method as follow

$$
\dot{w} = \lambda_1 w - \frac{3ad + 10bd^3}{\xi} w^2 - \frac{18a^2d^2 + 120abd^4 + 200b^2d^6 + a\xi^2 + 10bd^2\xi^2}{\xi^3} w^3 + \cdots
$$

The truncation equation is

$$
\dot{w} = \lambda_1 w - \frac{3ad + 10bd^3}{\xi} w^2 - \frac{18a^2d^2 + 120abd^4 + 200b^2d^6 + a\xi^2 + 10bd^2\xi^2}{\xi^3} w^3
$$

Now we discuss the bifurcation characteristic of the system:

1. if $18a^2d^2 + 120abd^4 + 200b^2d^6 + a\xi^2 + 10bd^2\xi^2 = 0$, Eq.(8) can be written as

$$
\dot{w} = \lambda_1 w - \frac{3ad + 10bd^3}{\xi} w^2
$$

According to bifurcation theory, there is the transcritical bifurcation in the system.

2. if $18a^2d^2 + 120abd^4 + 200b^2d^6 + a\xi^2 + 10bd^2\xi^2 = 0$, Eq.(8) can be written as

$$
\dot{w} = \lambda_1 w - \frac{3ad + 10bd^3}{\xi} w^2 - \frac{18a^2d^2 + 120abd^4 + 200b^2d^6 + a\xi^2 + 10bd^2\xi^2}{\xi^3} w^3
$$

According to bifurcation theory, there is the supercritical pitchfork bifurcation in the system.

5. **Comparison between Results in the two Method**

From the above results, we can see that:

Averaging method can only explain Hopf bifurcation phenomenon, and centre manifold method can explain Hopf bifurcation, transcritical bifurcation and supercritical pitchfork bifurcation phenomena. The result in averaging method shows that there is Hopf bifurcation in the system when $\xi = 0$, while the result in centre manifold method shows that there are not zero eigenvalue or pure
imaginary eigenvalues in the system when $\xi = 0$ and $\omega^2 + 3ad^2 + 5bd^4 < 0$. Considering that Hopf bifurcation, transcritical bifurcation and supercritical pitchfork bifurcation in rotor system with SMA bearing were proved by experiment and numerical simulation, it means that the results of the two methods were both the parts of global dynamic characteristic of rotor system with SMA bearing;

There are many nonlinear characters of original system being lost in averaging course;

Centre manifold method can be applied to research bifurcation characteristic in the case of more dimensions through broadening the truncation condition, while averaging method can not.

The results imply that the two methods both have limitation and global dynamic characteristic must be obtained in kinds of method.

6. Conclusion

Hopf bifurcation, transcritical bifurcation and supercritical pitchfork bifurcation in rotor-SMAB system were proved by experiment and numerical simulation, but not explained well in theory because of heteronomous characteristic of the system. Based on Landau-Devonshire SMA model, the bifurcation characteristic of rotor-SMAB system was investigated in this paper. Heteronomous system was transformed into autonomous system in averaging method and Van der Pol transformation, and the existence of Hopf bifurcation was proved in theory. The concept of broadened set of equilibrium point was introduced to improve centre manifold method to be adapted to heteronomous system. The equation of the flow on the centre manifold of rotor system with SMA bearing was obtained, and the existence of transcritical bifurcation and supercritical pitchfork bifurcation was proved in theory.

The comparison between results in the two methods shows that the results in averaging method and centre manifold method were both the parts of global dynamic characteristic of rotor system with SMA bearing. It means that the two methods both have limitation, and global dynamic characteristic must be obtained in kinds of method.

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