Toward an intuitive understanding of the structure of near-vector spaces

K.-T. Howell\textsuperscript{a,b} and S. Marques\textsuperscript{a,b}

\textsuperscript{a}Department of Mathematical Sciences, Stellenbosch University, Stellenbosch, South Africa; \textsuperscript{b}NITheCS (National Institute for Theoretical and Computational Sciences), Johannesburg, South Africa

ABSTRACT

In this article we analyze the definition André proposed for near-vector spaces to make it more transparent. We also study the class of near-vector spaces over division rings and give a characterization of regularity that gives a new insight into the decomposition of near-vector spaces into regular subspaces. We explicitly describe span and deduce a new characterization of subspaces.

ARTICLE HISTORY

Received 17 December 2020
Revised 3 February 2022
Communicated by Alberto Facchini

KEYWORDS

Near-vector spaces; nearrings; nearfields; division rings; span; subspaces; linear algebra

2020 MATHEMATICS SUBJECT CLASSIFICATION

16Y30; 12K05

1. Introduction

The notion of a structure with less linearity than a traditional vector space has been studied by a number of authors. First Beidleman [2] used near-ring modules to construct a near-vector space; whereas André [1] used an additive group together with a set of endomorphisms of the group, satisfying certain conditions. Next Karzel [3] defined a near-vector space structure mimicking a vector space structure but without the scalars acting as endomorphisms on the underlying group.

In the first part of the article we analyze André’s definition and in doing so we attempt to highlight how André’s definition seems to be the most suitable and natural definition to work with since it allows a lot of flexibility in the structure. We state a structural Lemma 3.4 for general vector spaces that hopefully reveals the connection between vector spaces and near-vector spaces and how the generalization affects the structure. We also study an important property of near-vector spaces, namely the non-unique additive structures on the underlying multiplicative group. In particular, we state the Key Lemma 3.8 for this article that gives a necessary condition for two additive structures to be the same. Several examples are exhibited to highlight the special features of near-vector spaces.

In the second part of the article we focus on constructions of near-vector spaces over division rings. We first focus on regularity. The main result (Theorem 4.2) of this section fully characterizes the uniqueness of the induced additive structure in terms of regularity, division ring and quasi-kernel structure. This permitted us to give an alternative proof of the decomposition into regular subspaces when the underlying set can be endowed with a division ring structure. We
then focus on the structure of Span. Our main result in that section, \textit{Theorem 4.5}, describes precisely the Span of any one element set in $V$. Surprisingly, it reveals that Span and linear combinations agree in that setting as they do in classical linear algebra. Generalizing this result of Span to a general set, we also deduce a very useful characterization of subspaces which actually corresponds to the usual characterization in classical linear algebra while the initial definition required a difficult check for the generating condition.

\section{Preliminary material}

In this section we define concepts that are analogous to those that are central to traditional linear algebra.

In [1] the notion of a near-vector space was defined as:

\textbf{Definition 2.1.} ([1], Definition 4.1). A near-vector space is a pair $(V, A)$ which satisfies the following conditions:

1. $(V, +)$ is a group and $A$ is a set of endomorphisms of $V$;
2. $A$ contains the endomorphisms $0$, $id$ and $-id$;
3. $A'$ is a subgroup of the group $\text{Aut}(V)$;
4. If $ax = bx$ with $x \in V$ and $a, b \in A$, then $a = b$ or $x = 0$, i.e. $A$ acts fixed point free on $V$;
5. The quasi-kernel $Q(V)$ of $V$, generates $V$ as a group. Here, $Q(V) = \{x \in V| \forall a, b \in A, \exists \gamma \in A \text{ such that } ax + bx = \gamma x\}$.

We will write $Q(V)'$ for $Q(V) \backslash \{0\}$ throughout this article and just $Q$ for $Q(V)$ if it does not cause confusion. Also, we note that we write scalars on the left and and as a result, make use of left near-fields. We will use $F$ to denote a near-field and $Fd$ its distributive elements.

From André’s definition, it is natural to define the concept of a homomorphism as follows:

\textbf{Definition 2.2.} ([5], Definition 3.2) Two near-vector spaces $(V_1, A_1)$ and $(V_2, A_2)$ are homomorphic when there is a group homomorphism $\theta : (V_1, +) \rightarrow (V_2, +)$ and a group isomorphism $\eta : (A_1', \cdot) \rightarrow (A_2', \cdot)$ such that $\theta(ax) = \eta(a)\theta(x)$ for all $x \in V_1$ and $a \in A_1'$. A homomorphism is written as a pair $(\theta, \eta)$.

This permits us to compare near-vector spaces. The reason we require $\eta$ to be an isomorphism is so that we have a fixed multiplicative structure over which to work. Relaxing this condition, would require a relaxation in terms of other definitions to be coherent with the usual properties we would expect.

In a near-vector space linear independence is defined in terms of the elements of the quasi-kernel $Q(V)$.

\textbf{Definition 2.3.} ([1]) Let $(V, A)$ be a near-vector space. We say that a set $S \subseteq Q(V)$ is a linearly independent subset if for any $v_1, \ldots, v_n$ in $S$ and $a_1, \ldots, a_n \in A$ such that if

$$a_1v_1 + \cdots + a_nv_n = 0$$

then

$$a_1 = \cdots = a_n = 0.$$

Otherwise we say $S$ is linearly dependent.

We will also need the notion of a subspace and Span as they will have an important role to play as in traditional linear algebra.
Definition 2.4. ([4], Definition 2.3) If \( (V, A) \) is a near-vector space and \( \emptyset \neq V' \subseteq V \) is such that \( V' \) is the subgroup of \( (V, +) \) generated additively by \( AX = \{ ax \mid x \in X, a \in A \} \), where \( X \) is a linearly independent subset of \( Q(V) \), then \( (V', A) \) is a subspace of \( (V, A) \), or simply \( V' \) is a subspace of \( V \) if \( A \) is clear from the context.

Definition 2.5. ([6], Definition 3.2) Let \((V, A)\) be a near-vector space. The Span of a set \( S \) of vectors is defined to be the intersection \( W \) of all subspaces of \( V \) that contain \( S \), denoted \( \text{Span}(S) \).

We will write \( \text{Span}(v) \) instead \( \text{Span}(\{v\}) \). It is straightforward to verify that \( W \) is a subspace, called the subspace spanned by \( S \), or conversely, \( S \) is called a spanning set of \( W \) and we say that \( S \) spans \( W \). It is not difficult to check that \( \text{Span}(S) \) is the set of all possible linear combinations of \( S \) if \( S \subseteq Q(V) \). However, if \( S \) contains elements outside the quasi-kernel, then it is not clear that these two coincide (see Theorem 4.5). For this reason, we define:

Definition 2.6. Let \( V \) be a near-vector space. For every \( v \in V \), the linear combinations of \( v \) is defined as the set

\[
L(v) = \{ a_1v + \cdots + a_tv \mid t \in \mathbb{N} \text{ and } a_i \in A, \text{for } i \in \{1, \ldots, t\} \}.
\]

Definition 2.7. ([1]) Let \((V, A)\) be a near-vector space. A linearly independent generating set for \( Q(V) \) is called a basis of \( V \) and its cardinality is called the dimension of \( V \).

As for a vector space, one can prove that a near-vector space has a basis by showing that from the set of elements of \( Q(V) \) generating \( V \), one can always extract a basis. Thus any near-vector space admits a basis in \( Q(V) \). It is routine as in linear algebra to prove that there is a well-defined notion of dimension, i.e. if a near-vector space has a finite basis all the bases have the same number of elements. See [1] for more details on this and a proof that any near-vector space admits a basis by enlarging an existing linear independent set.

The dimension of an element is defined as follows:

Definition 2.8. ([6], Definition 3.5) For \( v \in V \setminus \{0\} \), the dimension of \( v \) is defined to be

\[
n = \min\left\{ m \in \mathbb{N} \mid v = \sum_{i=1}^{m} a_iu_i, \text{ with } u_i \in Q(V) \setminus \{0\}, a_i \in A \setminus \{0\}, i = 1, \ldots, m \right\}.
\]

It is denoted by \( \text{dim}(v) = n \) and by definition, if \( v \) is the zero vector, \( \text{dim}(v) = 0 \).

The concept of regularity is a central notion in the study of near-vector spaces. André called the regular spaces the building blocks of near-vector space theory. They happen to be well-behaved, as we will see. This led to the Decomposition Theorem, where any near-vector space is decomposed into regular parts. See Theorem 4.13 in [1].

Definition 2.9. ([1] Definition 4.7) A near-vector space is regular if any two vectors of \( Q(V) \setminus \{0\} \) are compatible, i.e. if for any two vectors \( u \) and \( v \) of \( Q(V) \) there exists a \( \lambda \in A \setminus \{0\} \) such that \( u + \lambda v \in Q(V) \).

Note that every near-vector space \((V, A)\) with \( \text{dim}(V) \leq 1 \) is regular and has \( Q(V) = V \).

The addition in \( V \) naturally gives rise to an addition in \( A \) as follows:

Definition 2.10. ([1]) Let \((V, A)\) be a near-vector space and let \( v \in Q(V)^+ \). Define the operation \( +_v \) on \( A \) by

\[
(x +_v \beta)v := \alpha v + \beta v (\alpha, \beta \in A).
\]

With this addition, \((A, +_v, \cdot)\) is a near-field (see [1]). The essentiality of this definition will become clear to the reader in Sections 3 and 4.
This addition gives rise to

Definition 2.11. ([1], Definition 2.6) Let \((V, A)\) be a near-vector space and let \(u \in Q(V) \setminus \{0\}\). Define the kernel \(R_u(V) = R_u\) of \((V, A)\) by the set
\[
R_u := \{v \in V \mid (α + uβ)v = αv + βv \text{ for every } α, β \in A\}.
\]

In Theorem 4.2 in Section 4, we will see how the notion of the kernel relates to regularity.

3. Analysis of the definition of near-vector spaces

Definition 2.1 might be a bit disconcerting to some. As a result, we decided to revisit the definition and try to understand how essential each assumption is to a good notion of what a near-vector space should be as the natural way to widely generalize a vector space after the introduction of near-fields. As part of this we will later see how the concepts of subspace, Span and linear combinations interplay. Traditionally in linear algebra, a vector space is a field together with an abelian group endowed with an endomorphism action by the field on the abelian group, called the scalar multiplication. A near-vector space structure does not just result in a weakening with an abelian group but instead fix a multiplicative group. From André’s definition (Definition 2.1), one can construct a near-field such that the near-vector space can be viewed in the expected way mentioned above. Nevertheless, the choice of the near-field is not unique, as we will see later on. As result of a lack of uniqueness of the underlying near-field, the direct generalization of the traditional construction is weaker than what is proposed by André.

The essence of the geometry behind linear algebra stems from the existence of a coordinate system. The geometry resides in the notion of a basis which gives rise to unit vectors and coordinate axes which are one-dimensional subspaces. In order to reproduce this idea in a more general setting, we start with an additive group \(V\) and a set \(A\) such that there exist elements \(v \in V\) for which \(Av\) represents a set of coordinate axes and \(A\) becomes a near-field induced by \((V, +)\) that is in bijection with \(Av\).

Cleverly, André noted that one does not need this property to hold for every \(v \in V\) to guarantee the existence of a coordinate system, one only needs to ensure that such \(v\)’s generate \(V\). As we will see this is guaranteed with the property that
\[
Q(V) = \{v \in V | L(v) = Av\} = \{v \in V | Av = \text{Span}(v)\}
\]
generates \(V\) (see Definition 2.1). Note that, for \(v \in Q(V)\), we have
\[
L(v) = \text{Span}(v) = Av.
\]

Thus picking \(v \in Q(V)\) guarantees that \(Av = \text{Span}(v) = L(v)\). This shows that the coordinate axes \(Av\) are now subspaces of \(V\), which we would expect. The fact that \(Q(V)\) generates \(V\) allows us to see that
\[
V = \text{Span}(\{v \in V | L(v) = Av\}).
\]

The notion of a basis will give us access to a coordinate system from some \(Av\) with \(v \in Q(V)\). We are still expecting an underlying near-field structure. We will now explain how this structure can be revealed without being fixed in advance. To define a well-defined operation on \(A\) induced from that of \(V\), we need for all \(α, β \in A\) that \(αv + βv = γv \in Av\), for a unique \(γ \in A\). The existence of \(γ\) is equivalent to \(v\) belonging to \(Q(V)\) and its uniqueness is guaranteed by requiring fixed point freeness (see Definition 2.1). As a consequence of the existence and uniqueness of \(γ\), if we fix a nonzero \(v\) in \(Q(V)\), then for any \(α, β \in A\), we can define an operation \(+_v\) on \(A\) that sets \(α+_v β\) to be this \(γ\).
What is left to show now is that the addition of \( V \) naturally induces a structure of a near-field on \( A \). This mimics what we have for traditional vector spaces, where the underlying structure would be a field. The difference is that the underlying near-field structure is not fixed beforehand and is not unique.

We will use \( 0_A \) and \( 0_V \) to denote the identities of \( A \) and \( V \), respectively. In order for the group structure of \( V \) to induce a group structure on \( A \), we need that for all \( v \in Q(V) \), \( 0_A v = 0_V \). Therefore \( 0_A \) acts as the zero endomorphism on \( V \) which is generated by elements of \( Q(V) \). In order to have a meaningful endomorphism action of \( A \) on \( V \) that induces a near-field structure on \( A \), we will also need a multiplicative structure on \( A^* = A \setminus \{0\} \).

We will use \( 1_A \) to denote the multiplicative identity of \( A \). In order to ensure \( A \) acts as an endomorphism on \( V \) we will need \( 1_A \) to act as the identity endomorphism \( id_A \).

If \( v \in Q(V) \), then

\[
1_A v - 1_A v = 0_V,
\]

by the group structure of \((V, +)\), thus \((1_A + v(-1_A))v = 0_V \). Now by the fixed point free property \((1_A + v(-1_A))v = 0_A \). Thus \(-1_A \) is the inverse of \( 1_A \), so that for all \( v \in V \), \((-1_A)(v) = -v \) and \(-1_A \) act as \(-id_A \) in \( V \).

Moreover, since \( A \) induces an endomorphism action on \( V \), if we use \(-a\) to denote the additive inverse of \( a \in A \) we then have

\[
(-1_A a)v = (-1_A)(av) = -(av).
\]

As before, we can prove that \(-a = -1_A(a)\) is the additive inverse of \( A \). So that \( a \in A \), implies \(-a \in A \).

Note that as a consequence of \(-id_A \) acting as an endomorphism of \((V, +)\) we have that \(- (v + w) = -w - v = -v - w \), for all \( v, w \in V \). Therefore \((V, +)\) must be an abelian group.

Let us summarize what has been identified. As in traditional linear algebra we start with a group \((V, +)\) and a set of endomorphisms of \( V \), \((A, \cdot)\).

To have access to a coordinate system (basis) formed from an underlying near-field structure so that we are able to study the geometric properties of the system, we need:

1. A set of coordinate axes \( Av \) that generate \((V, +)\) as an additive group. This is guaranteed by the property” The quasi-kernel \( Q(V) \) of \( V \), generates \( V \) as a group.” in Definition 2.1.
2. That \((V, +)\) induces a near-field structure on \((A, \cdot)\), more precisely there exists a group operation \( +' \) on \( A \) induced by the operation \( + \) on \( V \) such that \((A, +', \cdot)\) is a near-field. The properties:” \( A \) contains the endomorphisms 0, \( id_A \) and \(-id_A \)”, \( A^* = A \setminus \{0\} \) is a subgroup of the group \( Aut(V) \)” and the fixed point free property precisely ensures that \((A, +', \cdot)\) is a near-field for any \( v \in Q(V) \setminus \{0\} \).

**Remark 3.1.** We note that André’s definition (Definition 2.1) requires the existence of elements in \( A \) that act as \( id_A \) and \(-id_A \) on \( V \). By Cauchy and Lagrange’s Theorem, \((A^+, \cdot)\) is a group with even order if and only if there is a \( x \neq 1_A \in A^* \) such that \( x^2 = 1_A \). For any \( v \in V \), we have that \( x(v + xv) = v + xv \) since \( V \) is abelian and by the fixed point free property, since \( x \neq 1_A \), we have that \( v + xv = 0 \) and \( xv = -v \). When the characteristic of \( V \) is not 2, this element \( x \) acts as \(-1_A \) in \( A \). However, when the characteristic of \( V \) is 2, \( x \) and \( 1_A \) have the same action, contradicting the fixed point freeness. To conclude, when the characteristic of \( V \) is not 2, \( A^* \) will have exactly one element of order 2 while in characteristic 2, \( A^* \) cannot have any element of order 2. Note that if \( A^* \) is finite this will imply that if the characteristic of \( V \) is 2, \( A^* \) will have even order, while if the characteristic is not 2, the order of \( A^* \) has to be odd.

From linear algebra the most basic example of a vector space is a field over itself, hence it would be essential to have that the additive group of a near-field be a near-vector space over its multiplicative group.
Example 3.2. ([1]) Let $F$ be a near-field that is not $M_C(\mathbb{Z}_2)$ (see [7], Proposition 8.1). Then we have that:

1. $(F, +)$ is a group and $(F, \cdot)$ is a set of endomorphisms of $F$;
2. $0_F$ acts as an endomorphism by assumption since $F$ is zero-symmetric. $1_F$ acts an endomorphism. Also, $-1_F \in F$ and acts as an endomorphism by Proposition 8.10 in [7];
3. It is clear that $F^* = F \setminus \{0\}$ is a subgroup of the group Aut($F$);
4. $F$ acts fixed point free on itself since $F$ has multiplicative inverses;
5. The quasi-kernel is the set $Q(F) = \{\lambda(k_i) \mid \lambda \in F, k_i \in F_d\} = F$ (Theorem 4.4 in [1]). For any nonzero $x \in F$, $\{x\}$ is a basis of $(F, F)$ of dimension 1.

Thus $(F, F)$ is a near-vector space.

In order to state the next lemma, we need some definitions.

**Definition 3.3.** For an (possibly infinite) index set $I$, we define

$$A(I) = \{a : I \rightarrow A | a \text{ is zero for all but finitely many } i \in I\},$$

and the standard basis elements $e_i \in A(I)$ for any $i \in I$ defined on $I$ as

$$e_i(j) = \delta_{i,j} = \begin{cases} 
0 & \text{if } i \neq j \\
1 & \text{otherwise}.
\end{cases}$$

The following lemma gives a way to visually compare near-field theory to classical linear algebra, emphasizing the structure that completes the analysis above.

**Remark 3.4.** Let $(V, A)$ be a near-vector space. $V$ induces a near-vector space structure on $A(I)$ as follows. Since $V$ admits a basis $\{v_i \mid i \in I\}$, the additive structure of the near-vector space $(A(I), A)$ is defined point-wise for any $a, b \in A(I)$ by

$$(a \oplus b)(i) = a(i) +_v b(i)$$

and the scalar multiplication is given by the multiplication in $A$.

These operations and the basis naturally define a near-vector space isomorphism as follows:

$$V \simeq \bigoplus_{i \in I} A v_i.$$

Then, the near-vector space isomorphism $V \simeq A(I)$ can be obtained by composing the previous isomorphism with the isomorphism

$$\bigoplus_{i \in I} A v_i \simeq A(I)$$

that sends $v_i$ to $e_i$ extended by linearity.

To emphasize the importance of the existence of an element acting as $-id_A \in A$ and the quasi-kernel generating $V$ beyond the induced structure of the near-field in $A$, we give the following two examples:

**Example 3.5.** Take

$$V = \mathbb{R}$$

and

$$A = \mathbb{R}^+ \cup \{0\}.$$
Then

$$Q(V) = \mathbb{R}.$$  

Note that all the axioms of Definition 2.1 are satisfied, except that $A$ contains no element acting as $-id_V$ in $V$. As a result, we cannot naturally obtain the structure of a near-field in $\mathbb{R}^+$ from the field structure of $\mathbb{R}$ as explained above. The set $\{-1, 1\}$ is a generating set of $\mathbb{R}$, but it is not linearly independent since for $x, y \in \mathbb{R}$, both nonzero,

$$x + y(-1) = 0$$

implies that $x = y$. Thus we also do not have a basis.

$A$-groups have been studied in [1]. They meet all the requirements of André’s definition, except for Assumption 5. of Definition 2.1. As an illustration, we give the following example where only this assumption is not satisfied. It shows that the notion of an $A$-group would not lead to a good notion of what intuitively we could expect a near-vector space to be.

**Example 3.6.** Take

$$V_1 = \mathbb{Z}$$

and

$$A = \{-1, 0, 1\}.$$  

Then

$$Q(V_1) = \{0\}.$$  

As another option, we could take,

$$V_2 = \mathbb{Z} \oplus \mathbb{Z}$$

and

$$A = \{-1, 0, 1\}.$$  

Then

$$Q(V_2) = \frac{\mathbb{Z}}{3\mathbb{Z}} \oplus \{0\}.$$  

In both of these examples the only missing assumption in the definition of a near-vector space is that $Q(V)$ does not generate $V$.

The following example illustrates the flexibility that the definition of a homomorphism allows. In the next section we will identify the important properties of near-vector spaces which will explain the unnatural phenomena in the example below.

**Example 3.7.** We use $+_3$ to denote the addition on $\mathbb{R}$ defined for all $x, y \in \mathbb{R}$ by

$$x+_3y = \sqrt[3]{x^3 + y^3}.$$  

Then we can show that $\mathbb{R}_{+_3} = (\mathbb{R}, +_3, \cdot)$ is a field. It is clear that $\mathbb{R}$ is closed under $+_3$, 0 is the identity element, $+_3$ is commutative and $-x$ is an inverse for $x$. As for distributivity, we have

$$x(x+_3y) = x\sqrt[3]{x^3 + y^3} = \sqrt[3]{(zx)^3 + (zy)^3} = (zx)+_3(zy).$$

Note that any odd power could be used to define the addition, giving an infinite number of field structures on $\mathbb{R}$.  

---

K.-T. HOWELL AND S. MARQUES

---
Next we prove that the mapping 
\[ \phi : (\mathbb{R}, +, \cdot) \rightarrow (\mathbb{R}, +, \cdot) \]
\[ x \mapsto x^3 \]
is a field isomorphism. It is well-known that \( \phi \) is a bijection and for all \( x, y \in \mathbb{R} \),

\[ \phi(x + y) = x^3 + y^3 = \phi(x) + \phi(y). \]

We can construct a near-vector space isomorphism from this. Since \( \phi \) induces both a multiplicative and additive isomorphism, we define \( \cdot_3 \) for all \( x, a \in \mathbb{R} \) by

\[ a \cdot_3 x = x^3 = \phi(x) \cdot x. \]

This homomorphism induces a commutative diagram:

\[ (\mathbb{R}, \cdot) \times (\mathbb{R}, +) \overset{\phi \times \phi}{\longrightarrow} (\mathbb{R}, \cdot) \times (\mathbb{R}, +) \]
\[ 
\begin{array}{c}
\downarrow m \\
(\mathbb{R}, +) \end{array} \\
\begin{array}{c}
\downarrow m \\
(\mathbb{R}, +) \\
\end{array} \\
\]

where \( m \) sends \((x, y)\) to \( xy\), the usual multiplication of \( x \) and \( y \) in \( \mathbb{R} \).

Note that \( \mathbb{R}_{+, 3} \) is a vector space over \( \mathbb{R}_{+, +} \), but only a near-vector space over \( \mathbb{R} \). Clearly, \( \mathbb{R} \) is a vector space over \( \mathbb{R} \). Even though they are not isomorphic in the traditional vector space sense, \((\phi, \phi)\) is a near-vector space isomorphism that gives more flexibility than traditional linear algebra would have allowed by only fixing the multiplicative structure of the underlying set.

Given \( v \in Q(V) \setminus \{0\} \), the near-fields \((A, +_w, \cdot)\) associated with the near-vector space \((V, A)\) are known to be isomorphic when \( w \in Av (= \text{Span}(V)) \), (Theorem 2.5 in [1]).

The main results of the section rely upon the following result that holds for any near-vector space and reveals an interesting necessary condition for \( +_v = +_w \), where \( v, w \in Q(V) \setminus \{0\} \).

**Lemma 3.8** (Key Lemma). Let \((V, A)\) be a near-vector space with \( \dim(V) > 1 \), \( Q(V) = V \) and \( S = \{ v_i \in Q(V) | i \in I \} \) a linearly independent set (possibly infinite) of \( V \). Suppose \( v = \sum_{i \in I} \theta_i v_i \) and \( v' = \sum_{i \in I} \theta'_i v_i \) are both in \( Q(V) \setminus \{0\} \) where all but finitely many of the \( \theta_i \) and \( \theta'_i \) are zero and we have that there exist \( i_0, j_0 \in I \) with \( i_0 \neq j_0 \), such that \( +_{\theta_i} v = +_{\theta'_i} v \). Then \( +_v = +_v' = +_{\theta_i} v = +_{\theta'_i} v \), for all \( i, j \in I \) where \( \theta_i, \theta'_j \) are nonzero.

**Proof.** By assumption \( v, v', v_i \in Q(V) \), for all \( i \in I \) therefore for all \( x, \beta \in A \) we get that

\[ (x +_v \beta) v = x(v) + \beta(v) \]
\[ = \sum_{i \in I} x \theta_i v_i + \sum_{i \in I} \beta \theta_i v_i = \sum_{i \in I} (x \theta_i v_i + \beta \theta_i v_i) \]
\[ = \sum_{i \in I} (x +_\theta \beta) \theta_i v_i. \]

Moreover,

\[ (x +_v \beta) v = \sum_{i \in I} (x +_\beta \beta) \theta_i v_i. \]

Therefore,

\[ 0 = \sum_{i \in I} (x +_v \beta) \theta_i v_i - \sum_{i \in I} (x +_\beta \beta) \theta_i v_i \]
\[ = \sum_{i \in I} ((x +_v \beta) +_{\theta_i} (x +_\beta \beta)) \theta_i v_i. \]
Since $S$ is linearly independent and for any $i \in I$ with $\theta_i$ nonzero,

$$(x + \alpha\beta) - \theta_i v_i (x + \theta_i v_i) = 0,$$

we have that

$$x + \alpha\beta = x + \theta_i v_i,$$

for all $\theta_i v_i$ (this is deduced from the fixed point freeness of the scalar multiplication). Thus

$$+ v = + \theta_i v_i$$

for all $i \in I$ with $\theta_i$ nonzero. The same can be done to show that $+ v = + \theta_j v_j$ for all $j \in I$ with $\theta_j$ nonzero. Thus proving that

$$+ v = + \theta_i v_i = + \theta_j v_j = + v.$$

**Remark 3.9.** Let $(V, A)$ be a near-vector space with $S = \{v_i \in Q(V) | i \in I\}$ a linearly independent set (possibly infinite) of $V$. If $v = \sum_{i \in I} \theta_i v_i \in Q(V)$, then $+ v = + \theta_i v_i$, for all $i \in I$ for which $\theta_i$ is nonzero.

### 4. Structural results of near-vector spaces constructed from division rings

#### 4.1. Regular spaces and their decomposition

In the next lemma, it is shown how the addition $+ v$ that turns $A$ into a near-field for $v \in Q(V) \setminus \{0\}$, results in a special structure on $(A, + v, \cdot)$.

**Lemma 4.1.** Let $V$ be a near-vector space and $v \in Q(V) \setminus \{0\}$. Then the following two conditions are equivalent:

1. $+ v = + \theta v$ for all $\theta \in A \setminus \{0\}$;
2. $(A, + v, \cdot)$ is a division ring.

**Proof.** If

$$+ v = + \theta v$$

for all nonzero $\theta \in A$, then for $x, \beta \in A$ we have that

$$(x + \alpha\beta) \theta v = (x + \theta v) \theta v = x \theta v + \beta \theta v = (x \theta + \alpha \theta \beta) v.$$  

By the fixed point free property,

$$(x + \alpha\beta) \theta = x \theta + \alpha \theta \beta.$$  

This is the only axiom we still require for $(A, + v, \cdot)$ to be a division ring. For the converse, if $(A, + v, \cdot)$ is a division ring, then for all $x, \beta \in A$ and $\theta \in A \setminus \{0\}$, we have that

$$(x + \alpha\beta) \theta v = (x \theta + \alpha \theta \beta) v = x \theta v + \beta \theta v = (x \alpha \theta + \beta \theta \beta) v.$$  

Since $v$ is nonzero, $\theta v$ is nonzero, so again by the fixed point free property, $+ v = + \theta v$.□

The following theorem attempts to describe the relationship between properties of $Q(V)$, $+ v$ and regularity.

**Theorem 4.2.** Let $(V, A)$ be a near-vector space.

The following statements are equivalent:

1. For any $v \in Q(V) \setminus \{0\}$, $V$ is a vector space over the near-field $(A, + v, \cdot)$;
2. There is a \( v \in Q(V) \setminus \{0\} \), such that \( V \) is a vector space over the near-field \((A, +_v, \cdot)\);

1' For any \( v \in Q(V) \setminus \{0\} \), \( V \) is a vector space over the near-field \((A, +_v, \cdot)\) and \((A, +_v, \cdot)\) is a division ring;

2' There is a \( v \in Q(V) \setminus \{0\} \), such that \( V \) is a vector space over the near-field \((A, +_v, \cdot)\) and \((A, +_v, \cdot)\) is a division ring;

3. \( Q(V) = V \) and \((A, +_v, \cdot)\) is a division ring, for all \( v \in Q(V) \setminus \{0\} \);

4. \( +_v = +_w \) for all \( v, w \in Q(V) \setminus \{0\} \);

5. \( R_w(V) = V \) for all \( w \in Q(V) \setminus \{0\} \);

6. \( V \) is regular and \( +_v = +_{\theta_v} \) for all \( v \in Q(V) \setminus \{0\} \) and \( \theta \in A \);

7. \( V \) is regular and \((A, +_v, \cdot)\) is a division ring for any \( v \in Q(V) \setminus \{0\} \).

Proof. (1) \( \iff \) (2) \( \iff \) (1') \( \iff \) (2') is not hard to prove using Lemma 4.1.

(2) \( \Rightarrow \) (3) If there is \( v \in V \) such that \( V \) is a vector space over the near-field \((A, +_v, \cdot)\), then for any \( w \in V \), we have \(zw + bw = (\alpha + \beta)v\). Therefore, \( w \in Q(V) \).

(3) \( \Rightarrow \) (4) Let \((V, A)\) be a near-vector space such that \( Q(V) = V \). Let \( B = \{v_i \in Q(V) | i \in I\} \) be a basis for \( V \). For any two nonzero \( v, w \in V \), either \( v = \theta w \) and \( +_v = +_w \), according to Lemma 4.1, since \((A, +_v, \cdot)\) is a division ring. Otherwise, \( dim(V) > 1 \), so we have that \( v = \sum_{i \in I} \theta_i v_i \in V \) and \( w = \sum_{i \in I} \theta_i v_i \in V \). There exist \( i \neq j \in I \) such that \( \theta_i \) and \( \theta_j \) are nonzero, therefore we can take \( w = \theta_i v_i + \theta_j v_j \). By assumption \( w \in Q(V) \), we can apply Lemma 3.8 to obtain that \( +_v = +_w = +_v \).

(4) \( \iff \) (5) \( \iff \) (6) \( \iff \) (7) is trivial. The last equivalence being a consequence again of Lemma 4.1.

(6) \( \Rightarrow \) (2) Suppose that \( V \) is regular and for any \( v \in V \), \( +_v = +_{\theta_v} \) for all \( v \in V \) and \( \theta \in A \). Let \( \{v_i \in Q(V) | i \in I\} \) be a basis for \( V \). For any \( i \neq j \in I \), from the regularity of \( V \), there is a \( \lambda_{ij} \in A \) such that

\[
\omega_{ij} = v_i + \lambda_{ij} v_j \in Q(V).
\]

Again applying Proposition 3.8 and the assumption, we get that

\[
+_\omega_{ij} = +v_i = +v_j = +v.
\]

For \( v \in V \) nonzero, we have \( v = \sum_{i \in I} \theta_i v_i \). Indeed, for any \( x, \beta \in A \),

\[
xv + b v = \sum_{i \in I} (x\theta_i v_i + b \theta_i v_i) = \sum_{i \in I} (x\theta_i + b \theta_i) \theta_i v_i = \sum_{i \in I} (x + b) \theta_i v_i = (x + b) v,
\]

since \( +_v = +_{\theta v} \), by assumption and since \( A \) is an endomorphism set of \( V \) by the definition of near-vector space.

This proves that \( V \) is a vector space over the near-field \((A, +_v, \cdot)\).

\( \square \)

Classical linear algebra is a particular case of a near-vector space corresponding precisely to the regular case, when \((A, +_v, \cdot)\) is a field for every nonzero \( v \in Q(V) \).

As an application of Theorem 4.2, we reprove the Decomposition Theorem (see [1]) for the division ring case.

Corollary 4.3. Let \((V, A)\) be a near-vector space for which \((A, +_v, \cdot)\) is a division ring, for all \( v \in Q(V)^* \). Then \( V \) is the direct sum of regular near-vector spaces \( V_j \) for \( j \in K \) for some index set \( K \). Every \( v \in V \) can be expressed uniquely as a sum of \( u_j \in V_j \) such that each \( u_j \in Q(V)^* \) lies in precisely one summand \( V_j \). The subspaces \( V_j \) are maximal regular near-vector spaces.

Proof. Let \( B = \{v_i \mid i \in I\} \) be a canonical basis of \( V \) for some index set \( I \). We put \( V_j = \{v \in V \mid +_v = +_{v_j}\} \). We use \( K_i \) to denote the following subset of \( I \), \( K_i = \{k \in I \mid +_{v_k} = +_{v_j}\} \). We first prove that
\[ \mathcal{V}_i = \text{Span}(\{v_k | k \in K_i\})(*) \]

Note that \( \text{Span}(\{v_k | k \in K_i\}) = \bigoplus_{k \in K} A v_k \). Therefore, clearly \( \text{Span}(\{v_k | k \in K_i\}) \subseteq \mathcal{V}_i \). Suppose by contradiction that there is \( v \in \mathcal{V}_i \) such that \( v \notin \text{Span}(\{v_k | k \in K_i\}) \). Then \( v = \sum x_i v_i \) with some \( i_0 \notin K_i \) and \( x_{i_0} \neq 0 \). But then \( +_v = +_{x_0} v_0 = +_{v_0} = -v \), by Lemmas 3.8 and 4.1, leading to a contradiction and proving (*)}. This also proves \( V = \bigoplus_{i \in K} \mathcal{V}_i \) and that the \( \mathcal{V}_i \) are maximal regular subspaces of \( V \) where \( K \) is a index set such that for each \( i \in I \) there is a unique \( k \in K \) such that \( v_i \in \mathcal{V}_k \). Indeed, the \( \mathcal{V}_i \) are regular by Theorem 4.2, since \( (A,+_v, \cdot) \) is a division ring and we have that \( +_v = +_w \) for all \( v, w \in \mathcal{V}_i \). Moreover, \( Q(\mathcal{V}_i) = \mathcal{V}_i \) for \( i \in K \). The rest is clear from (*) and the fact that \( B \) is a basis. \( \square \)

### 4.2. The structure of \( \text{Span} \)

It is immediate to prove the following lemma:

**Lemma 4.4.** Let \( (V, A) \) be a near-vector space, \( v_0, \ldots, v_n \in Q(V) \) are linearly dependent if there is \( i_0 \in \{1, \ldots, n\} \) such that \( v_{i_0} \in \text{Span}(\{v_1, \ldots, v_n\} \backslash\{v_i\}) \).

We have explicitly seen in Lemma 3.4 how \( V \simeq A^{(I)} \) allows us to retranslate the following lemma to any near-vector space.

**Theorem 4.5.** Let \( V = A^{(I)} \) as in Remark 3.4 and \( (A,+_w) \) be a division ring for any nonzero \( w \in Q(V) \backslash \{0\} \). Let \( v \in V \). According to Corollary 4.3, we have the regular decomposition \( V = \bigoplus_{j \in K} \mathcal{V}_j \) where \( K \subseteq I \), so that we can write \( v \) uniquely as \( v = \sum_{j \in K} v_j \) where \( v_j \in \mathcal{V}_j \). If \( N \) is the subset of \( K \), \( N = \{i \in K | v_i \neq 0\} \), we have that

\[ \text{Span}(v) = L(v) = \bigoplus_{i \in N} A v_i \]

and

\[ \dim(v) = |N|. \]

**Proof.** Let \( v \) be as in the statement, i.e.

\[ v = \sum_{i \in N} v_i. \]

For convenience we order the elements in \( N \) as \( \{i_1, \ldots, i_n\} \) where \( n = |N| \).

Since the \( v_i \) are in different regular components and we are assuming the division ring condition, using Theorem 4.2, we know that \( +_{v_i} \neq +_{v_j} \) for any \( i \neq j \in N \).

Therefore there exist \( \alpha, \beta \in A \) such that

\[ \alpha +_{v_i} \beta \neq \alpha +_{v_j} \beta. \]

Then, taking

\[ w = (\alpha v + \beta v) - (\alpha +_{v_i} \beta) v \in L(v) \subseteq \text{Span}(v), \]

we have that \( w = \sum_{i \in N_1} w_i \) where \( N_1 \subseteq N \backslash \{i_2\}, i \in N_1 \) and \( w_i \in \mathcal{V}_i \) nonzero.

Repeating this process successively starting with \( w \) instead of \( v \) in the first step, we can construct an element \( \omega_{i_1} = \theta_{v_i} v \in L(v) \subseteq \text{Span}(v) \) and more generally restarting from \( v \) we can construct \( \omega_{i_j} = \theta_{v_i} v_j \in L(v) \subseteq \text{Span}(v) \).

Since \( A \theta_{v_i} v = A v_i \), this proves that

\[ \bigoplus_{i \in N} A v_i \subseteq \text{Span}(v). \]
But since $\bigoplus_{i \in N} A v_i$ is a near-vector space contained in $L(v)$, we therefore have

$$\bigoplus_{i \in N} A v_i = L(v) = \text{Span}(v).$$

The fact that

$$\text{dim}(v) = |N|$$

follows from the definition of a direct sum and the previous lemma. \qed

We now give an example to illustrate the previous result.

**Example 4.6.** Consider the near-vector space $((\mathbb{Z}_{11})^3, \mathbb{Z}_{11})$, where scalar multiplication is defined for all $(x, y, z) \in (\mathbb{Z}_{11})^3$ and $\alpha \in \mathbb{Z}_{11}$ by

$$\alpha(x, y, z) = (\alpha^3x, \alpha^5y, \alpha^3z).$$

Note that

$$Q((\mathbb{Z}_{11})^3) = \{(a, 0, c) | a, c \in \mathbb{Z}_{11}\} \cup \{(0, b, 0) | b \in \mathbb{Z}_{11}\}.$$

Take for example, $v = (2, 5, 6) \in (\mathbb{Z}_{11})^3$, then $v \notin Q((\mathbb{Z}_{11})^3)$, and

$$\text{Span}((2, 5, 6)) = \mathbb{Z}_{11}(2, 0, 6) + \mathbb{Z}_{11}(0, 5, 0).$$

If we take, for example $w = (3, 0, 4)$, then $w \in Q((\mathbb{Z}_{11})^3)$, and

$$\text{Span}((3, 0, 4)) = \mathbb{Z}_{11}(3, 0, 4).$$

An easy but useful consequence of the previous theorem is the following.

**Corollary 4.7.** Let $(V, A)$ be a near-vector space. For every $S \subseteq Q(V)$,

$$\text{Span}(S) = L(S) = \bigoplus_{v \in T} A v,$$

where $T$ is the minimal spanning subset of $S$.

More generally we have,

**Corollary 4.8.** Let $(V, A)$ be a near-vector space and suppose $(A, +, \cdot)$ is a division ring for all $v \in Q(V) \setminus \{0\}$. Let $S \subseteq V$. Then there is $T \subseteq Q(V)$ such that

$$\text{Span}(S) = L(S) = \bigoplus_{v \in T} A v.$$

**Proof.** Let $S = \{v_i \in V | i \in I\}$ where $I$ is an index set, possibly infinite. For any $j \in I$,

$$v_j = \sum_{i \in N_j} v_{ji}$$

according to the regular decomposition where $v_{ji} \in V_i$ nonzero, where

$$N_j = \{i \in I | v_{ji} \neq 0\}.$$

It is not hard to see that

$$\text{Span}(v_j) = \bigoplus_{i \in N_j} A v_{ji} \subseteq \text{Span}(S).$$

From the previous corollary we know that

$$\text{Span}(\{v_j | j \in I, i \in N_j\}) = \bigoplus_{v \in T} A v$$

where $T$ is the minimal spanning subset of $\{v_j | j \in I, i \in N_i\}$. 
Moreover,

\[ \sum_{i \in I} \text{Span}(v_i) \subseteq \text{Span}(S). \]

We leave the details to the reader to conclude that

\[ \text{Span}(S) = L(S) = \bigoplus_{v \in T} Av, \]

proving the result.

From the previous corollary, we can rectify an error made in Lemma 2.4 in [4], where it was shown that a subset of a near-vector space is a subspace if and only if it is closed under addition and scalar multiplication. Of course, the one direction is obvious. The problem with the converse was that the proposed generating set was not necessarily contained in the subspace. We rectify it here for the case where we assume \((A, +_v, \cdot)\) is a division ring for all non-zero \(v \in Q(V)\).

**Corollary 4.9.** Let \((V, A)\) be a near-vector space and suppose \((A, +_v, \cdot)\) is a division ring for all nonzero \(v \in Q(V)\). \(W\) is a subspace of \(V\) if and only if it is non-empty and closed under addition and scalar multiplication.

The following two corollaries shed more light on why taking elements outside of \(Q(V)\) as basis elements would be counter-intuitive to our general intuition with respect to a basis.

**Corollary 4.10.** Let \((V, A)\) be a near-vector space and suppose \((A, +_v, \cdot)\) is a division ring for all \(v \in Q(V)\) such that \(\theta \neq 0\). Then there exist \(v\) and \(w \in V \setminus Q(V)\) such that \(Av \neq Aw\) but \(\text{Span}(v) = \text{Span}(w)\).

**Proof.** Take \(v_1, v_2 \in Q(V)\) linearly independent and not in the same regular component. We define \(v := v_1 + v_2\) and \(w := \theta v_1 + v_2 \in V \setminus Q(V), \) where \(1 \neq \theta \in A\). Then, \(Av \neq Aw\), but \(\text{Span}(v) = \text{Span}(w)\), by Corollary 4.5.

**Corollary 4.11.** Let \((V, A)\) be a near-vector space with \(\text{dim}(V) > 2\) and suppose \((A, +_v, \cdot)\) is a division ring for all \(v \in Q(V)\) such that \(\theta \neq 0\). Then there exists \(v\) and \(w \in V \setminus Q(V)\) such that \(\theta \neq 0\) and \(w \notin \text{Span}(v) \) and \(\text{Span}(v) \cap \text{Span}(w) \neq \{0\}\).

**Proof.** Take \(v_1, v_2, v_3 \in Q(V)\) linearly independent not in the same regular subspace in the decomposition of \(V\) and \(v = v_1 + v_2\) and \(w = v_2 + v_3 \in V \setminus Q(V), \) then we have \(v \notin \text{Span}(w)\) and \(w \notin \text{Span}(v), \) but

\[ \text{Span}(v) \cap \text{Span}(w) = Av_2 \neq \{0\}. \]

**Acknowledgments**

The authors would like to thank Charlotte Kestner for Example 3.6 and Jacques Rabie for Example 3.5 and his helpful comments.

**Funding**

The authors would like to express their gratitude for funding to the National Research Foundation (South Africa), Grant number 93050.
References

[1] André, J. (1974). Lineare Algebra über Fastkörpern. Math. Z. 136(4):295–313. DOI: 10.1007/BF01213874.
[2] Beidleman, J. C. (1966). On near-rings and near-ring modules. PhD dissertation. Pennsylvanian State University, PA.
[3] Karzel, H. (1984). Fastvektorräume, unvollständige Fastkörper und ihre abgeleiteten Strukturen. Erscheint in Mitt. Sem. Univ. Giessen.
[4] Howell, K.-T. (2015). On subspaces and mappings of near-vector spaces. Commun. Algebra. 43(6): 2524–2540. DOI: 10.1080/00927872.2014.900689.
[5] Howell, K.-T., Meyer, J. H. (2014). Near-vector spaces determined by finite fields. J. Algebra. 398:55–62. DOI: 10.1016/j.jalgebra.2013.09.019.
[6] Howell, K.-T., Sanon, S. P. (2018). On spanning sets and generators of near-vector spaces. Turk. J. Math. 42(6):3232–3241. DOI: 10.3906/mat-1807-155.
[7] Pilz, G. (1983). Near-Rings: The Theory and its Applications. Revised Edition. New York, NY: North Holland.