Non-relativistic strings in expanding spacetime

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Abstract
We obtain a non-relativistic diffeomorphism invariant string action as a special limit of the Nambu–Goto action in a FLRW background. We use this action to study non-relativistic string dynamics in an expanding universe and construct an analytic model describing the macroscopic properties of non-relativistic string networks. The non-relativistic constraint equations allow arbitrarily small string velocities and thus a ‘frustrated’ equation of state for non-interacting strings can be obtained without the need of a velocity damping mechanism. Assuming that colliding string segments reconnect by the exchange of partners, non-relativistic string networks exhibit scaling behaviour, but with enhanced energy densities due to the smaller average string velocity. Non-relativistic string networks can be relevant in several contexts in condensed matter physics and cosmology.

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1. Introduction

Progress in solving string theory in various backgrounds can be done by considering sectors of the theory that decouple from the rest of the degrees of freedom in a suitable limit. Such decoupled sectors are characterized by having an altogether different asymptotic symmetry compared to that of the parent string theory. A well-known example to such a truncation is the BMN [1] sector of string theory in AdS$_5 \times$ S$^5$. Once a consistent sector is found, a complete worldsheet theory with the appropriate symmetries can be written down without further reference of the parent theory.

Non-relativistic string theory [2] (see also [3]) in flat space is another consistent sector of string theory, whose worldsheet conformal field theory description has the appropriate Galilean symmetry [4]. Non-relativistic superstrings and non-relativistic superbranes [5, 6] are obtained as a certain decoupling limit of the full relativistic theory. The basic idea behind the decoupling limit is to take a particular non-relativistic limit in such a way that the light
states satisfy a Galilean-invariant dispersion relation, while the rest decouple. For the case of strings, this can be accomplished by considering wound strings in the presence of a background $B$-field and tuning the $B$-field so that the energy coming from the $B$-field cancels the tension of the string. In flat space, once kappa symmetry and diffeomorphism invariance are fixed, non-relativistic strings are described by a free field theory in flat space. In $\text{AdS}_5 \times S^5$ [7], the worldsheet theory reduces to a supersymmetric free field theory in $\text{AdS}_2$.

It is an interesting question whether similar non-relativistic string actions can be constructed in an expanding spacetime and if so, whether non-relativistic strings could play a cosmological role in the form of cosmic strings. The study of cosmic strings\footnote{The dynamics of cosmic strings can be described by considering perturbations around a static solitonic string solution. Keeping all orders in the perturbations results in a relativistic effective string action, while keeping only up to quadratic order gives rise to the non-relativistic string action we will consider in this paper [5].} has been catalyzed in the past few years mainly due to theoretical motivations, in particular the realization that they are generic in supersymmetric grand unified theory (SU(5) GUT) models [8] and brane inflation [9, 10]. The latter possibility is of particular significance as it provides a potential observational window to superstring physics [11, 12]. Further, the fact that the Planck satellite and laser interferometers such as LISA and LIGO may be able to probe a significant part of cosmic string tensions relevant to these models [13], opens the possibility of detecting cosmic strings in the foreseeable future.

One can think of situations in which ordinary cosmic strings could behave non-relativistically. Network simulations in a matter or radiation dominated universe [14] suggest that, at late times, string segments move relatively slowly and coherently on the largest scales, but also show evidence that small-scale-structure [15, 16] which is largely responsible for damping energy from the network through the formation of minuscule loops, remains relativistic as Hubble damping is inefficient at scales much smaller than the horizon [17]. However, the situation is different for strings in de Sitter spacetime, where Hubble damping can be very efficient rendering the strings essentially non-relativistic. This may be relevant for late time cosmology as observations [18, 19] suggest that the universe is already entering a de Sitter phase. Further, non-relativistic string networks have been considered as solid dark matter (SDM) [20, 21] and more recently [22] as an alternative explanation of galactic rotation curves.

It would thus be desirable to have an effective diffeomorphism invariant action\footnote{Note that [20] considers an action applicable to a ‘continuous medium’ with internal structure, which is invariant under limited reparametrizations preserving the worldlines of the constituent particles. Here we consider a \textit{diffeomorphism invariant} non-relativistic action.} describing the dynamics of non-relativistic strings in a cosmological context. On the other hand, the fact that one can construct a consistent worldsheet theory of non-relativistic strings at quantum level (in flat space) also motivates the study of \textit{fundamental} non-relativistic strings in an expanding spacetime. In this paper, we point out that a non-relativistic diffeomorphism invariant action can be obtained in the case of a Friedmann–Lemaître–Robertson–Walker (FLRW) spacetime as a limit of the relativistic Nambu–Goto action, and study the dynamics and cosmological evolution of non-relativistic strings.

The structure of the paper is as follows. In section 2, we review some basic results about the Nambu–Goto action and the dynamics of relativistic strings in an expanding universe, which will be useful for comparison to the non-relativistic case. In section 3, we obtain a non-relativistic diffeomorphism invariant worldsheet action by taking a particular limit of the Nambu–Goto action in expanding spacetime. We move on in section 4 to study non-relativistic string dynamics as described by this action. The physical interpretation of non-relativistic strings as well as their possible coupling to cosmology—in particular the effective equation...
of state of an ‘ideal gas’ of non-interacting, non-relativistic strings—is discussed in section 5. The effect of string interactions is left for section 6, where macroscopic models for the cosmological evolution of both relativistic and non-relativistic string networks are discussed. In section 7, we solve numerically the non-relativistic network model for a wide range of parameters and compare the results to those of the relativistic model. In section 8, we discuss possible applications of non-relativistic strings in condensed matter and cosmological contexts. Finally, we have three appendices which describe an alternative derivation of our non-relativistic string action as a semiclassical expansion [23] around the vacuum solution (appendix A), the Hamiltonian formulation of relativistic and non-relativistic string dynamics (appendix B) and the construction of a spacetime energy–momentum tensor for the non-relativistic string (appendix C).

2. Relativistic string in expanding spacetime

Let us first consider a string moving in a $D + 1$ dimensional spacetime with metric $G_{MN}(M, N = 0, 1, 2, \ldots, D)$. Its world history is described by a two-dimensional spacetime surface, the string worldsheet $x^M = x^M(\sigma^i), i = 0, 1$. The dynamics is governed by the Nambu–Goto action

$$S_{NG} = -T_R \int \sqrt{-\gamma} \, d^2 \sigma,$$

(2.1)

where $T_R$ is the string tension and $\gamma$ is the determinant of the pullback of the background metric on the worldsheet, $\gamma_{ij} = G_{MN}(x) \partial_i x^M \partial_j x^N$.

The equations of motion for the fields $x^M$ obtained from this action are given by

$$\nabla^2 x^M + \Gamma^M_{NA} \gamma^{ij} \partial_i x^N \partial_j x^A = 0,$$

(2.2)

where $\Gamma^M_{NA}$ is the $(D + 1)$-dimensional Christoffel symbol and $\nabla^2 x^M$ the covariant Laplacian of the worldsheet fields $x^M$.

By varying the action with respect to the background metric $G_{MN}$ we obtain a spacetime energy–momentum tensor

$$T^{MN}(y^A) = \frac{1}{\sqrt{-G}} T_R \int d^2 \sigma \sqrt{-\gamma} \gamma^{ij} \partial_i x^M \partial_j x^N \delta^{(D+1)}(y^A - x^A(\sigma^i)).$$

(2.3)

The rigid symmetries of (2.1) are given by the Killing vectors of $G_{MN}$. The Nambu–Goto action is also invariant under 2D diffeomorphisms of the worldsheet coordinates $\sigma^i$. One can use this freedom to fix the gauge by imposing two conditions on $x^M(\sigma^i)$.

Now consider string propagation in an expanding universe described by a flat FLRW metric

$$G_{MN} = a(x^0)^2 \eta_{MN}$$

(2.4)

in conformal time $x^0 \equiv \eta$.

A convenient gauge choice in this case is the transverse temporal gauge given by

$$\begin{align*}
\dot{x}x' &= 0 \\
\tau &= x^0.
\end{align*}$$

(2.5)

The equations of motion (2.2) become [24]

$$\begin{align*}
\dot{\epsilon} &= -2 \frac{\dot{a}}{a} \epsilon^2 \\
\dot{\epsilon} x \dot{x} + 2 \frac{\dot{a}}{a} (1 - \dot{x}^2) \dot{x} &= \left( \frac{\dot{x}^2}{\epsilon} \right)^{\epsilon^{-1}}.
\end{align*}$$

(2.6)
where
\[ \epsilon = -\frac{x'^2}{\sqrt{-\gamma}} = \left( \frac{x'^2}{1 - x'^2} \right)^{1/2}. \]  
(2.7)

The variable \( \epsilon \) is related to the canonical momentum associated with the field \( x^0(\tau) \). Indeed, in the transverse gauge \( \gamma_{01} = \gamma_{\tau\sigma} = 0 \), we have
\[ p_0 = -\frac{T_R}{2} \sqrt{-\gamma} \frac{\partial y_{ij}}{\partial \tau} x^0 = -T_R a(x^0)^2 \epsilon x^0, \]  
(2.8)
which, after imposing the temporal gauge condition \( \dot{x}^0 = 1 \), becomes
\[ -T_R a(x^0)^2 \epsilon. \]

In the transverse temporal gauge, the energy–momentum tensor (2.3) of a relativistic string in a FLRW background is [17]
\[ T_{MN}(\eta, y^I) = \frac{1}{a(\eta)^{D+1}} T_R \int d\sigma (\epsilon \dot{x}^M \dot{x}^N - \epsilon^{-1} \dot{x}^M \dot{x}^N) \delta^{(D)}(y^I - x^I(\sigma, \eta)), \]  
(2.9)
where, having integrated out \( \delta(\eta - x^0(\tau)) \), \( I \) runs from 1 to \( D \).

To construct the string energy one projects \( T_{MN} \) on a spatial hypersurface \( \eta = \text{const} \), with induced metric \( h \) and normal covectors \( n_M = (a(\eta), 0) \), integrating over the \( D \) spatial coordinates
\[ E(\eta) = -\int \sqrt{h} n_M n_N T_{MN} d^D y = \int \sqrt{h} T_0^0 d^D y. \]  
(2.10)
Thus, due to the foliation, the energy can be constructed from the 00 component of the energy–momentum tensor. Since, \( \sqrt{h} = a(\eta)^D \), equation (2.10) becomes
\[ E(\eta) = \int T_0^0 a(\eta)^D d^D y = a(\eta) T_R \int \epsilon d\sigma. \]  
(2.11)
Therefore, the energy of the relativistic string is simply the tension times the physical string length, taking into account relativistic length contraction.

3. Non-relativistic limit of Nambu–Goto action in FLRW

Now consider a string, charged under a background antisymmetric 2-tensor field \( B_{MN} \), propagating in \( D + 1 \) FLRW spacetime. The string couples to \( B \) through a topological Wess–Zumino term, so that the total action reads
\[ S = S_{NG} + S_{WZ} = -T_R \int \sqrt{-\gamma} d^3 \sigma + q \int B^*, \]  
(3.1)
where \( q \) is the string charge and \( B^* \) the pullback of \( B \) on the worldsheet. We consider a relativistic string aligned in the \( x^0, x^1 \) directions, its transverse coordinates being \( x^a \) with \( a = 2, 3, \ldots, D \).

The non-relativistic limit [2, 3, 5] of this string consists of rescaling the longitudinal coordinates
\[ x^\mu \rightarrow \omega x^\mu, \quad \mu = 0, 1 \]  
(3.2)
and taking the limit \( \omega \rightarrow \infty \). This yields a divergent term, coming from \( S_{NG} \), which (in some geometries) can be cancelled by an appropriate choice of a closed \( B_{MN} \). If we assume that the string is wrapped on a spatial circle
\[ x^1 \sim x^1 + 2\pi R \]  
(3.3)
then the chosen \( B_{01} \) cannot be set to zero by a gauge transformation.
The above procedure generally works in flat spacetime (in fact one only needs the longitudinal part of the metric be flat \[5\]), but for a curved background it is not guaranteed that there is a choice of closed \(B\) which cancels the diverging piece of the action. Non-relativistic superstring actions have been obtained in the case of \(\text{AdS}_5 \times S^5\) \[7\].

We will now see that the non-relativistic limit can also be taken in the case of a FLRW background. We write the Lagrangian density of the Nambu–Goto piece as

\[
L_{\text{NG}} = -T_R \sqrt{-\det \left[ g_{ij} + G_{ab}(\eta) \partial_i X^a \partial_j X^b \right]}
\]

where

\[
g_{ij} = G_{\mu\nu} \partial_i x^\mu \partial_j x^\nu\]

and

\[
G_{ab}(\eta) = a(\eta)^2 \delta_{ab}.
\]

Then, assuming a power law expansion

\[a(\eta) = \eta^{\alpha/2}\]

(for example \(\alpha = 2\) resp. 4 in radiation resp. matter dominated era) we obtain the non-relativistic limit of \(S_{\text{NG}}\) by the rescaling \(3.2\), which implies

\[
a(\eta) \to \omega^{\alpha/2} a(\eta).
\]

Expanding the Lagrangian density in powers of the parameter \(\omega\) we then obtain

\[
L_{\text{NG}} = -T_R \omega^\alpha \left\{ \omega^2 \sqrt{1 - \det g} + \frac{1}{2} \omega \sqrt{1 - \det g} g^{ij} G_{ab}(\eta) \partial_i X^a \partial_j X^b \right\} + \mathcal{O} \left( \frac{1}{\omega^2} \right).
\]

We can then rescale the string tension by

\[
T_R \omega^\alpha \to T_0
\]

and take the limit \(\omega \to \infty\), yielding a finite and a divergent piece

\[
L_{\text{reg}} = -\frac{T_0}{2} \sqrt{-\det g} g^{ij} G_{ab}(\eta) \partial_i X^a \partial_j X^b
\]

\[
L_{\text{div}} = -T_0 \omega^2 \sqrt{-\det g} = -T_0 \omega^2 a(\eta)^2 \sqrt{-\det(\eta_{\mu\nu} \partial_\mu X^a \partial_\nu X^b)}.
\]

The divergent piece can be cancelled by choosing an appropriate closed \(B_{\mu\nu}\). Indeed if we choose\(^3\) \(B_{\mu\nu} = a(\eta)^2 \epsilon_{\mu\nu}\) the Wess–Zumino part of the Lagrangian becomes

\[
\frac{1}{2} \omega^2 (q_o \omega^\alpha) a(\eta)^2 \epsilon_{\mu\nu} e^i \partial_\mu x^a \partial_\nu x^b.
\]

This term precisely cancels the divergent piece \(3.9\) if one tunes the rescaled charge \((q_o \omega^\alpha)\) with the string tension \(T_0\). We are thus left with the non-relativistic string action

\[
S_{\text{NR}} = -\frac{T_0}{2} \int \sqrt{-\det g} g^{ij} G_{ab}(\eta) \partial_i X^a \partial_j X^b \ d^2 \sigma.
\]

This action can also be derived by a ‘semiclassical approximation’ \[23\] from the classical solution

\[
x_0^M = \begin{cases} \tau, & M = 0 \\ \lambda \sigma, & M = 1 \\ 0, & M = a \in \{ 2, \ldots, D \} \end{cases}
\]

(see appendix A).

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3 The chosen \(B_{\mu\nu}\) is closed. Working in zweibeins \(e^\mu\) we have \(d\theta = \frac{1}{2} [d(a(x^0)^2 \epsilon_{\mu\nu} e^\mu \wedge e^\nu) = \frac{1}{2} [d(a(x^0)^2 \epsilon_{\mu\nu} e^\mu \wedge e^\nu) = 2 a \partial_\mu \partial_\nu \epsilon_{\mu\nu} e^\mu \wedge e^\nu + a^2 (\partial_\mu \epsilon^0 \wedge e^\nu + \partial_\nu \epsilon^0 \wedge e^\mu) = a^2 (\partial_0 \wedge e^\nu + \partial_\nu \wedge e^0) = a^2 (\partial_0 \wedge e^0 + \partial_0 \wedge e^0) = 0\), where we have used \(\partial_0 \wedge e^0 = 0\), Cartan’s structure equation with zero torsion.
4. Non-relativistic string dynamics

The action (3.11) is characterized by 2D diffeomorphism invariance with respect to the worldsheet coordinates $\sigma^i$ and global Galilei invariance (modulo time translations due to time dependence of the metric) with respect to the transverse spacetime coordinates $X^a$. The canonical variables satisfy two primary constraints

$$p_\mu \epsilon^{\mu \nu} \xi^\nu + \frac{1}{2} \left( \frac{P_a P_b}{T_0} G^{ab}(x^0) + T_0 X^a X^b G_{ab}(x^0) \right) = 0 \quad (4.1)$$

$$p_\mu x^\mu + P_a X^a = 0, \quad (4.2)$$

where $p_\mu, P_a$ are the canonical momenta corresponding to $x^\mu$ and $X^a$.

Varying the action with respect to the transverse and longitudinal fields $X^a$ and $x^\mu$ respectively, one obtains the equations of motion

$$\partial_i \left( \sqrt{-g} g^{ij} \partial_j X^a \right) + \sqrt{-g} g^{ij} \Gamma^a_{bc} \partial_i X^b \partial_j X^c + \sqrt{-g} g^{ij} \partial_i X^b \partial_j x^0 \left( \partial_0 \Gamma_{bc} \right) G^{ca} = 0 \quad (4.3)$$

$$\partial_i \left[ \sqrt{-g} \partial_k \eta_{\mu \nu} a(x^0)^2 \left( g^{ik} g^{mn} - 2 g^{im} g^{kn} \right) \partial_m X^a \partial_n X^b G_{ab}(x^0) \right] = \sqrt{-g} g^{mn} \partial_m X^a \partial_n X^b \frac{\partial G_{ab}(x^0)}{\partial x^\mu} \quad (4.4)$$

subject to the boundary condition (3.3). For the metric (2.4) the Christoffel symbols $\Gamma^a_{bc}$ vanish and the transverse equations of motion (4.3) relate the covariant divergence of the transverse fields $X^a$ to the time derivative of the transverse metric $G_{ab}$. We can use the 2D reparametrization invariance of the action to fix the gauge. For our discussion it will be convenient to work in the static gauge

$$x^0 - \tau = 0, \quad x^1 - \lambda \sigma = 0, \quad (4.5)$$

identifying worldsheet and background times, while allowing for multiple windings of the non-relativistic string. Indeed, defining $\sigma \in [0, 2\pi)$, the periodicity condition (4.5) requires that

$$\lambda = n R, \quad (4.6)$$

where $n$ is the string winding number.

After fixing the gauge, the physical degrees of freedom of the non-relativistic string are the transverse coordinates $X^a$ and the corresponding momenta $P_a$. The equation of motion (4.3) becomes

$$\dot{X}^a = -2 \frac{\ddot{a}}{a} X^a + \frac{\lambda}{a} X^a, \quad (4.7)$$

which is the wave equation with a cosmological damping term $-2 \frac{\ddot{a}}{a} X^a$. This equation (for $\lambda = 1$) has been used by Vilenkin [25] to describe small perturbations around a straight cosmic string, and was obtained by taking the limit $\dot{X}^2 \ll 1, X^2 \ll 1$ of the relativistic equations of motion in the static gauge. Here, there is also a winding number $\lambda$. One might be tempted to say that, for $\dot{a}/a = 0$, equation (4.7) implies a wave propagation velocity

$$v_0^2 = \frac{\lambda}{a}. \quad (4.8)$$

However, one should remember that the physical coordinates are not $\sigma, \tau$ but rather $x^1 = \lambda \sigma, x^0 = \tau$, so rewriting (4.7) in terms of the physical variables we get (in the case $\dot{a}/a = 0$)

$$\partial^2_{x^0} X^a = \partial_{x^1}^2 X^a, \quad (4.9)$$
which describes a wave propagating at the velocity of light. The non-relativistic string allows the propagation of waves along the longitudinal directions with the speed of light. However, the transverse velocities are not restricted, in contrast to the case of the relativistic string.

An ‘energy’

\[ P_0 = \frac{1}{2\lambda} \int d\sigma \left( \frac{P_a P_b}{T_0} G^{ab}(x^0) + T_0 X^a X^b G_{ab}(x^0) \right) \]  

(4.10)

can be obtained from the constraint (4.1), which in the gauge (4.5) becomes

\[ P_0 = \frac{1}{2} \int d\sigma (\lambda T_0 X^a X^b + \lambda^{-1} T_0 X^a X^b) G_{ab}(x^0). \]  

(4.11)

This can be interpreted as the sum of the kinetic and potential energies of transverse excitations along the string. The actual string energy, obtained by integrating the projection of the energy–momentum tensor on a constant \( x^0 \) hypersurface is (see appendix C)

\[ E(x^0) = a(x^0)^{-1} P_0. \]  

(4.12)

Since \( x^0 \) translation is not an isometry of (2.4) the Lagrangian is not time translationally invariant and \( p_0 \) is not conserved. In fact, its time evolution can be found from the longitudinal equations of motion (4.4). In the gauge (4.5) the \( \mu = 0 \) component of (4.4) becomes

\[ \frac{1}{2} (\dot{X}^a X_a + \lambda^{-2} X^a X_a) = \lambda^{-2} (\dot{X}^a X_a) + \frac{\lambda}{4} (\dot{X}^a X_a - \dot{X}^a X_a). \]  

(4.13)

Integrating we obtain

\[ P_0 = a\lambda T_0 \int d\sigma (\lambda^{-2} X^a X^b - \dot{X}^a X^b) \delta_{ab}, \]  

(4.14)

where the boundary term gives no contribution due to the periodicity condition (3.3).

Similarly, from the constraint (4.2) we define the momentum \( P_1 \) along the string

\[ P_1 = -\frac{1}{\lambda} \int P_a X^a d\sigma = -T_0 \int X_a X^a d\sigma \]  

(4.15)

in the gauge (4.5). Translational invariance then dictates that \( P_1 \) is conserved, as can be easily verified using the equations of motion.

5. Physical interpretation and cosmology

5.1. The NR particle versus NR string limit

The non-relativistic limit is generally understood as a low-velocity limit, which can be formally obtained by sending the speed of light \( c \) to infinity. This procedure works, at least in the case of the point particle, although there are some conceptual issues involved when taking limits of dimensionful constants like \( c \) [26, 27]. A safer route is to keep \( c \) constant and rescale the time coordinate by a dimensionless parameter, say \( \omega \), taking the limit \( \omega \to \infty \). One can thus obtain a reparametrization invariant, non-relativistic action for the point particle. The naive application of this to the case of the string fails\(^4\) but this problem was solved with the realization [2] (see also [3]) that in order to obtain a Galilei invariant string action one has to rescale both longitudinal coordinates, not the time coordinate only. In a sense, one can speak of a non-relativistic ‘particle’ limit, obtained by taking \( v \ll 1 \) and a non-relativistic limit for extended objects for which one has to rescale all worldvolume coordinates, as we did in section 3 for the case of the string.

\(^4\) The string obtained in this limit has a fixed length and no physical oscillations (see [28]).
The rescaling of the longitudinal string direction corresponds to the assumption 
\( (\partial y/\partial x)^2 \ll 1 \), which one makes when deriving the wave equation by applying Newton’s 2nd 
law on an infinitesimal string segment. In the rescaling prescription we followed, the waves 
move along the string at the speed of light as the string tension equals the mass per unit length. 
One usually thinks of non-relativistic strings as ‘violin-type’ having a small tension compared 
to their mass per unit length and thus a subluminal ‘sound speed’ along the string. In this sense, 
the strings we consider here are ‘hybrid’, having a relativistic speed of propagation along the 
string, but transverse Galilean invariance. However, it is precisely this hybrid action (in flat 
space) which arises in the simplest Lorentz invariant field theories when one studies the low-
energy dynamics of domain wall solutions. Strings with subluminal propagation speeds (which 
would correspond to a differentiation between the string mass per unit length and the tension) 
can arise in more complicated models, which allow for spontaneously broken longitudinal 
Lorentz invariance through a current generation mechanism on the string worldsheet\(^5\) 
[30, 31]. To obtain such string actions as a non-relativistic limit of the Nambu–Goto action, 
one would have to rescale each of the longitudinal coordinates by a different factor and take 
both factors to infinity while keeping their ratio constant.

Note that, in order to ensure that the antisymmetric field \( B \) used to cancel the divergent 
piece of the action cannot be gauged away, the non-relativistic string had to wind a compact 
dimension, say \( x^1 \sim x^1 + 2\pi R \). In fact, the divergent piece of the action is a total derivative 
with respect to the worldsheet coordinates [5], so if the action (3.11) was to be interpreted as 
an effective non-relativistic action, one could simply drop this term without requiring that the 
string is wound. However, if the non-relativistic string is to be interpreted as a fundamental 
object, consistency requires a non-trivial winding. In this case, there are two distinct scales, 
namely \( T_0 \) of dimension mass squared, which appears in the action (3.11), and the mass scale 
\( m = 2\pi n RT_0 \), related to the geometry (through the compactification radius \( R \)) and the string 
winding number \( n \). In fact, when quantizing the non-relativistic string [2], one encounters 
again the necessity of winding, as the mass \( m \) is needed to define the energy states of the 
non-relativistic string spectrum. In the flat case there are no physical states with zero winding 
number [2]. Also note that in deriving the non-relativistic string action (3.11) we have defined 
the tension \( T_0 \) by a rescaling of the relativistic string tension \( T_R \), appearing in the Nambu–Goto 
action (see equation (3.7))

\[
T_0 = \omega^\alpha T_R, 
\]

where the expansion exponent \( \alpha \) is positive and \( \omega \) is taken to infinity. Interpreting \( T_0 \) as the 
physically relevant quantity which is to be kept constant, the relativistic tension \( T_R \) goes to 
zero as \( \omega \) tends to infinity.

Finally, we comment on the stability of the non-relativistic string. A closed non-relativistic 
string is more stable to breakage than its relativistic counterpart\(^6\). This is a consequence of the 
winding, which only allows a discrete number of potential ‘splitting points’ along the string. 
From an astrophysical perspective, ordinary cosmic string loops decay through gravitational 
radiation, which mainly couples to the kinetic energy of the fluctuations. In particular, the 
power in gravitational radiation scales with the sixth power of the root-mean-square (rms) 
velocity (see, for example, [17, 32]). Thus, if such non-relativistic strings were to play an 
astrophysical role, their decay rate would be power-law suppressed. For long strings, the main 
energy-loss mechanism is through string intercommutation, which removes string length from 
the long string network. This is also expected to be suppressed for non-relativistic strings

\(^5\) See [29] for a discussion of the relation between strings with broken longitudinal Lorentz invariance and Kaluza–
Klein strings in one-dimension higher.

\(^6\) We thank F Passerini for discussions of this point.
as the interaction rates are proportional to the string velocities. We shall now consider the possibility of coupling non-relativistic strings to cosmology.

5.2. Coupling to gravity and cosmology

The non-relativistic action we have analysed describes the dynamics of the independent degrees of freedom of the non-relativistic string, namely the transverse excitations. In obtaining this action we have introduced a closed $B$ field, which cancels the divergent piece corresponding to the rest energy of the string. Alternatively, if we are not interested in quantization, we can simply drop the divergent part of the action without introducing the $B$ field because it is a total derivative (cf the case of the point particle). Here, we will follow the latter approach. The energy–momentum tensor of the non-relativistic string (appendix C) therefore describes the energy of the transverse excitations but does not include a contribution from the rest mass of the string. However, when one couples non-relativistic matter to general relativity it is necessary to include the rest mass $m_0 c^2$ in the energy–momentum tensor, which gives the main contribution to the $T^{00}$ part while kinetic contributions are subdominant. Following this logic we will add the rest mass of the string to the $T^{00}$ part of the energy–momentum tensor of (appendix C), which can then be coupled to Einstein’s equations. From now on we work in $D = 3$ spatial dimensions.

Consider a cosmological setup where the cosmic fluid has a component due to a gas of non-interacting, non-relativistic strings. To obtain the energy density of the string fluid, one has to sum the contributions of all string segments in the network and, as we discussed, it is the rest energy of the segments which will give the dominant contribution. This is in analogy to a gas of non-relativistic particles (dust), where the dominant contribution to the energy density is

$$\rho = \langle T_{00} \rangle = \frac{m_0 n + \mathcal{O}(v^3)}{\gamma^2},$$

where $m_0$ is the particle rest mass, $n$ the rest frame number density and $v$ the rms particle velocity. The off-diagonal terms of the energy–momentum tensor of the particle fluid average out to zero by summing over all particles with random velocities in all directions, whereas the $T_{ii}$ components are proportional to the kinetic energy density, which for non-relativistic particles is negligible so that $p \ll \rho$.

In the case of a ‘string gas’ one can obtain an effective energy–momentum tensor in an analogous manner, by approximating the string network as a collection of straight string segments moving with average velocity $v$, and averaging over string orientations and directions of motion. Let us first consider the relativistic case. The effective energy–momentum tensor can be constructed by considering a straight string oriented in the $\hat{z}$ direction say, and Lorentz boosting its energy–momentum tensor in the $\pm \hat{x}$ and $\pm \hat{y}$ directions [33]. The result is

$$\langle T_{\mu}^{\nu} \rangle = \frac{\mu}{3L^2} \begin{pmatrix} 3\gamma^2 & 0 & 0 & 0 \\ 0 & (1 - v^2\gamma^2) & 0 & 0 \\ 0 & 0 & (1 - v^2\gamma^2) & 0 \\ 0 & 0 & 0 & (1 - v^2\gamma^2) \end{pmatrix},$$

where $\mu$ is the string tension, $L$ the average separation between nearby strings in the network and $\gamma = (1 - v^2)^{-1/2}$ a Lorentz factor corresponding to $v$. From (5.2) the equation of state can be read

$$-p = \langle T_{i}^{i} \rangle = \frac{1}{\gamma^2}(\gamma^2 - v^2)\langle T_{00}^{00} \rangle = \frac{1}{\gamma^2}(1 - 2v^2)\langle T_{00}^{0} \rangle \Rightarrow p = -\frac{1}{\gamma^2}(1 - 2v^2)\rho.$$  (5.3)

A similar procedure can be followed for non-relativistic strings, which are generally expected to have much smaller string velocities. Indeed, for relativistic strings the constraint $\dot{x}^2 + \dot{x}^2 = 0$ in the conformal gauge imposes that critical points on the string move with the
speed of light, but for non-relativistic strings the physical string velocities can take any value. One can thus obtain the equation of state for such a non-relativistic string gas by using Lorentz boosts with \( y = 1 \) or, alternatively, by performing transverse Galilean boosts instead. The result is again \( p = -\frac{1}{3}(1 - 2v^2)\rho \), but with the difference that one can safely assume \( v \ll 1 \), unlike the relativistic network case, where the strings oscillate relativistically at small scales, while there is no known mechanism which is efficient enough to damp these excitations. Indeed, Hubble damping is inefficient at scales much smaller than the horizon, and for large scales, of order the string correlation length, numerical simulations (see, for example, [14]) demonstrate that string segments move more slowly and coherently, but at speeds large enough to produce significant deviations from the equation of state \( w \equiv p/\rho = -1/3 \).

Note that one can apply an analogous procedure for strings which have a tension \( T \) smaller than their mass per unit length \( \mu (T < \mu) \). In this case the resulting equation of state is

\[
p = -\frac{1}{3}\left[T/\mu(1-v^2) - v^2\rho\right] = -\frac{1}{3}\left[v_0^2 - (1 + v_0^2)v^2\right]\rho,
\]

(5.4)

where we have defined the ‘sound speed’ along the string \( v_0 = \sqrt{T/\mu} \). Equation (5.4) can in general lead to both positive or negative equation of state with \( p > -\rho/3 \). This is in contrast to vacuum (non-interacting) cosmic strings with \( \mu = T \), where the rms speed does not exceed \( 1/\sqrt{2} \) so the equation of state is nonpositive (5.3) with \( p \geq -\rho/3 \). However, this is to be expected because in the limit \( T \to 0 \) the ‘string’ describes a line-like structure of dust particles with \( 0 < p \ll \rho \). In fact, taking \( v_0 \to 0 \) in equation (5.4) gives \( p = \rho v^2/3 \), or, in terms of the kinetic energy density \( \rho_k \),

\[
p = \frac{2}{3}\rho_k,
\]

(5.5)

which is precisely the equation of state for a gas of non-relativistic particles, following from ordinary kinetic theory considerations. In connection to the discussion of the previous sections, obtaining this kind of non-relativistic string from the Nambu–Goto action involves a rescaling of the longitudinal directions by different factors, the ratio of which determines the propagation speed \( v_0 \).

Finally, note that this discussion only applies to a ‘perfect’ gas of non-interacting strings. String intercommutations typically result in the removal of energy from the network in the form of closed string loops, significantly altering the above picture. Thus, a frustrated string network, with \( w \simeq -1/3, \rho \propto a^{-2} \) eventually dominates over matter or radiation, but turning on string interactions will result in a different equation of state. For Abelian string networks, where interactions are efficient, the resulting scaling law is \( \rho \propto t^{-2} \), where \( t \) is cosmic time, which scales like radiation in the radiation era and like matter in the matter era. The cosmological evolution of non-relativistic string networks, including the possible effects of string intercommutation will be discussed in the following section.

6. Velocity dependent one-scale (VOS) models

In this section we discuss analytic models for the evolution of macroscopic variables describing the large-scale properties of a string network. We will first review results for relativistic strings and then construct a macroscopic evolution model for non-relativistic strings, based on the action (3.11). To set up the physical picture we briefly summarize Kibble’s one-scale model [34], which captures the basic qualitative features of network evolution.

Monte Carlo simulations of cosmic string formation suggest that to a good approximation the strings have the shapes of random walks at the time of formation [35]. Such ‘Brownian’ strings can be described by a characteristic length \( L \), which determines both the typical radius of curvature of strings and the typical distance between nearby string segments in the network.
On average there is a string segment of length $L$ in each volume $L^3$ and thus the density of the cosmic string network at formation is

$$\rho = \frac{\mu L}{L^3} = \frac{\mu}{L^2}, \quad (6.1)$$

where $\mu$ is the string mass per unit length, which for relativistic strings is equal the ‘tension’ $T_R$ appearing in the Lagrangian. Assuming that the strings are simply stretched by the cosmological expansion we have $\rho \propto a(t)^{-2}$. This decays slower than both the matter ($\propto a(t)^{-3}$) and radiation ($\propto a(t)^{-4}$) energy densities and so such non-interacting strings would soon dominate the universe.

Now consider the effect of string interactions. As the network evolves, the strings collide or curl back on themselves creating small loops, which oscillate and radiatively decay. Via these interactions enough energy is lost from the network to ensure that string domination does not actually take place. Each string segment travels on average a distance $L$ before encountering another nearby segment in a volume $L^3$. Assuming relativistic motion ($v \approx 1$) and that the produced loops have an average size $L$, the corresponding energy loss is given by $\dot{\rho}_{\text{loops}} \approx L^{-4} \mu L$. The energy-loss rate equation is therefore

$$\dot{\rho} \approx -2 \frac{d}{dt} \frac{\rho}{L}. \quad (6.2)$$

Equation (6.2) has an attractor ‘scaling’ solution in which the characteristic length $L$ stays constant relative to the horizon $d_H \sim t$ [34]. The approach of string networks to a scaling regime has been verified by high-resolution simulations [14, 36]. Equation (6.2) was derived on physical grounds and it only captures the basic processes involved in string evolution, namely the stretching and intercommuting of strings. It does not take into account other effects like the redshifting of string velocities due to Hubble expansion. In fact, it neglects completely the evolution of string velocities, making the crude approximation that they remain constant throughout cosmic history. However, we can construct a more accurate velocity-dependent one-scale (VOS) model, based on the Nambu–Goto action (2.1).

6.1. Relativistic strings

The relativistic VOS model [32, 37] extends Kibble’s one-scale model, abandoning the constant string velocity approximation and introducing an extra variable, the rms velocity of string segments, whose dynamics is governed as we will see by a macroscopic version of the relativistic equations of motion (2.6). Although the simple one-scale model captures most of the qualitative features of macroscopic string evolution, this correction is crucial for quantitative modelling. Indeed, the average string velocity enters linearly in the loop production term, which provides the main energy-loss mechanism of the string network, and so the evolution of string velocities significantly affects the string energy density. The resulting VOS model is still very simple depending on only one free parameter but, remarkably, it has been shown to accurately fit numerical simulation data throughout cosmic history [38]. We briefly sketch how the model is constructed from the microscopic equations of section 2. This will be useful for comparison to the non-relativistic case.

7 Strictly speaking there are two parameters in the VOS model, the loop production efficiency $\tilde{c}$ and the momentum parameter $k$. For the second parameter, however, there exists a physically motivated ansatz (6.15), which expresses it in terms of the rms velocity $v(t)$. Once this choice is made, one is only left with the freedom of tuning $\tilde{c}$ when trying to fit numerical simulations.
Consider the relativistic string energy defined in section 2 (equation (2.11))

\[ E(\eta) = a(\eta) T_R \int \epsilon \ d\sigma \]

and take the first derivative with respect to conformal time \( \eta \). Using the equation of motion (2.6) for \( \epsilon \), one finds

\[ \dot{E} = \frac{\dot{a}}{a} (1 - 2v^2) E, \quad (6.3) \]

where \( v^2 = \int \epsilon \hat{x}^2 \ d\sigma / \int \epsilon \ d\sigma \equiv \langle \hat{x}^2 \rangle \) is the worldsheet average of the square of transverse velocities. For a network of strings the energy density \( \rho \) is related to the total string energy by \( E \propto \rho a(\eta)^3 \). Therefore

\[ \frac{\dot{\rho}}{\rho} = \frac{\dot{E}}{E} - \frac{3}{a} \frac{\dot{a}}{a} = -2 \frac{\dot{a}}{a} (1 + v^2). \quad (6.4) \]

To this we add a phenomenological term [17, 34] describing the production of loops when strings collide and curl back on themselves. The resulting network density evolution equation is

\[ \frac{\dot{\rho}}{\rho} = -2 \frac{\dot{a}}{a} (1 + v^2) \rho - \tilde{c} \frac{\rho v}{L}, \quad (6.5) \]

where \( \tilde{c} \) is the loop production efficiency related to the integral of an appropriate loop production function over all relevant loop sizes [17]. This is treated as a free parameter which can be determined by comparison to numerical simulations.

In the VOS model, the rms velocity \( v \) appearing in equation (6.5) is promoted to a dynamical variable whose evolution is given by a macroscopic version of the Nambu–Goto equation of motion (6.2). This equation can be obtained by differentiating \( v^2 \) and eliminating \( \ddot{x} \) using the equation of motion. This introduces the second spatial derivative \( x'' \) which corresponds to string curvature and can be expressed in terms of the mean curvature radius of the network. Differentiating \( v^2 \) and using equation (2.6) we find

\[ 2v \ddot{v} = \langle \hat{x}^2 \rangle = 2 \langle \hat{x} \cdot \hat{x} \rangle - 2 \frac{\dot{a}}{a} (\langle \hat{x}^2 \rangle^2 - \langle \hat{x}^4 \rangle). \quad (6.6) \]

The second term is of purely statistical nature and has the effect of ‘renormalizing’ the coefficient of the \( \frac{\dot{a}}{a} v^2 \) term which will find later. It has been demonstrated numerically [37] to have small magnitude and thus can be neglected.

Keeping only the first term and using the equation of motion for \( x \) we find

\[ v \ddot{v} = \frac{\int \hat{x} \cdot x' e^{-1} \ d\sigma}{\int \epsilon \ d\sigma} + \frac{\int \hat{x} \cdot x' (e^{-1})' \ d\sigma}{\int \epsilon \ d\sigma} - 2 \frac{\dot{a}}{a} (\langle \hat{x}^2 \rangle - \langle \hat{x}^4 \rangle). \quad (6.7) \]

The second term vanishes due to the gauge condition \( \hat{x} \cdot x' = 0 \). Further, within our approximations \( \langle \hat{x}^4 \rangle \simeq \langle \hat{x}^2 \rangle^2 \) so the third term becomes \( 2 \frac{\dot{a}}{a} v^2 (1 - v^2) \). For the first term we need to express \( x'' \) in terms of the local curvature vector. We define

\[ ds = \sqrt{\epsilon x^2} \ d\sigma = \sqrt{1 - \epsilon x^2} \ d\sigma \quad (6.8) \]

and the physical (local) radius of curvature by

\[ \frac{d^2 x}{ds^2} = \frac{a(\eta)}{R} \hat{u}, \quad (6.9) \]

where \( \hat{u} \) is a unit vector. Then

\[ x'' = \frac{d^2 x}{d\sigma^2} = x^2 \frac{d^2 x}{ds^2} + x \frac{d}{ds} \frac{d\sqrt{x^2}}{ds}. \quad (6.10) \]
Due to the constraint $\mathbf{x} \cdot \dot{\mathbf{x}} = 0$ the second term vanishes on dotting with $\dot{\mathbf{x}}$ so we have

$$
\int \mathbf{x} \cdot \dot{\mathbf{x}}' \epsilon^{-1} d\sigma = \int \mathbf{x} \cdot \frac{d^2\mathbf{x}}{ds^2} (1 - \dot{\mathbf{x}}^2) \epsilon d\sigma = a(\eta) (\langle \mathbf{x} \cdot \dot{\mathbf{u}} \rangle (1 - \dot{\mathbf{x}}^2)/\mathcal{R}) \int \epsilon d\sigma.
$$

(6.11)

We define the momentum parameter $k$ [32] by the equation

$$
\langle (\mathbf{x} \cdot \dot{\mathbf{u}}) (1 - \dot{\mathbf{x}}^2)/\mathcal{R} \rangle = k \nu \mathcal{R} (1 - \nu^2),
$$

(6.12)

where $\mathcal{R}$ is now the average string radius of curvature, numerically close to the correlation length $L$ for Brownian networks [17, 37, 39]. With this definition, equation (6.7) becomes

$$
\dot{v} = \frac{a(\eta)}{\mathcal{R}} k (1 - v^2) - 2 \frac{\dot{a}}{a} v (1 - v^2).
$$

(6.13)

Changing to cosmic time $t$, with $dt = a d\eta$ and $\dot{v} = a \frac{d\nu}{dt}$ we finally obtain

$$
\frac{d\nu}{dt} = (1 - v^2) \left( \frac{k}{\mathcal{R}} - 2 H \nu \right),
$$

(6.14)

where $H = a^{-1} \frac{da}{dt}$ is the Hubble parameter. Note that, since

$$
v^2 = \langle \dot{\mathbf{x}}^2 \rangle = \left( \frac{dx}{d\eta} \right)^2 = \left( \frac{a}{dt} \right)^2,
$$

and the physical coordinates $x_{\text{phys}}$ are given in terms of the comoving ones $x$ by $x_{\text{phys}} = a x$, the rms velocity $\nu$ has the interpretation of physical peculiar velocity of string segments. Equation (6.14) has therefore a clear physical meaning: the rms peculiar velocities of string segments are produced by string curvature and damped by cosmological expansion.

The momentum parameter $k$ is a measure of the angle between the curvature vector and the velocity of string segments and thus it is related to the smoothness of the strings. As $v$ increases towards relativistic values the accumulation of small-scale structure renders the strings wiggly. Velocities become uncorrelated to curvature and $k$ decreases. In particular it can be shown analytically that for flat space, where $v^2 = 1/2$, the momentum parameter vanishes for a wide range of known solutions [37, 40].

An accurate ansatz for the momentum parameter $k$ for relativistic strings has been proposed in [32]

$$
k = k(v) = \frac{2 \sqrt{2} (1 - 8 \nu^6)}{\pi (1 + 8 \nu^6)},
$$

(6.15)

satisfying $k(1/\sqrt{2}) = 0$.

Note that the fact that $v = 1/\sqrt{2}$ in flat spacetime can be shown analytically for closed loops only, but for long strings it is observed in numerical simulations [17]. For expanding or contracting spacetimes, $v$ is less or greater than $1/\sqrt{2}$, respectively. Hence for an expanding universe, string velocities are subject to the constraint

$$
v^2 \leq \frac{1}{2}.
$$

(6.16)

In a matter or radiation dominated universe, Hubble expansion is too weak to significantly reduce string velocities, which remain close to $1/2$ at short scales [17]. This limitation does not apply to non-relativistic strings.
6.2. Non-relativistic strings

For the non-relativistic string the energy of the excitations is given by (see appendix C)

\[ E_{\text{exc}} = a(\eta) \left( \frac{1}{2} \int d\sigma (\mu X^2 + \mu \lambda^{-2} X'{}^2) \right) = a(\eta)^{-1} P_0, \]  

(6.17)

where \( X \) are the transverse string coordinates and we have defined the tension \( \mu = \lambda T_0 \). To that we must add the string mass

\[ E_0 = a(\eta) \mu \int d\sigma, \]  

(6.18)

so that the total energy is

\[ E = E_0 + E_{\text{exc}} = a(\eta) \mu \int d\sigma + a(\eta)^{-1} P_0. \]  

(6.19)

Then, differentiating with respect to conformal time \( (\dot{\sigma} = \frac{d}{d\eta}) \), we have

\[ \dot{E} = \frac{\dot{a}}{a} E_0 + (a^{-1} P_0) = \frac{\dot{a}}{a} E_0 - \frac{\dot{a}}{a} E_{\text{exc}} + a^{-1} \dot{P}_0 \]
\[ = \frac{\dot{a}}{a} \left( 1 + \frac{1}{2} W^2 - \frac{3}{2} V^2 \right) E_0, \]  

(6.20)

where we have used equations (4.11), (4.13) and defined the rms quantities

\[ V^2 = \frac{\int d\sigma X^2}{\int d\sigma} \equiv \langle X^2 \rangle, \]  

(6.21)

and

\[ W^2 = \frac{\int d\sigma \lambda^{-2} X'^2}{\int d\sigma} \equiv \langle \lambda^{-2} X'^2 \rangle = \langle (\partial_x X)^2 \rangle, \]  

(6.22)

corresponding to the average velocity of string segments and the average magnitude of string tangent vectors. The latter quantity parametrizes small-scale perturbations on the string, \( W = 0 \) corresponding to strings which are straight at scales smaller than the correlation length\(^8\). Thus, the term \( W^2/2 \) in equation (6.20) corresponds to the average elastic energy due to short-scale string deformations. In the non-relativistic limit one has \( W^2 \ll 1 \).

Defining the energy density \( \rho \propto E a^{-3} \), and using

\[ \frac{\dot{E}}{E_0} \sim \frac{\dot{E}}{E} = \frac{\dot{\rho}}{\rho} + 3 \frac{\dot{a}}{a}, \]  

(6.23)

we find

\[ \dot{\rho} = -\frac{\dot{a}}{a} \left( 2 - \frac{1}{2} W^2 + \frac{3}{2} V^2 \right) \rho - \dot{\rho} V^2 \rho, \]  

(6.24)

where we have included a phenomenological loop production term, as in the relativistic case. From (6.21) we have

\[ 2 V V = \langle X^2 \rangle = 2 \langle X \cdot X \rangle - 2 \frac{\dot{a}}{a} \langle (X^2)^2 \rangle - \langle (X^4) \rangle \]  

(6.25)

as before. We neglect the statistical terms and using the non-relativistic equation of motion (4.7) we find

\[ V \dot{V} = \frac{\int X \cdot \dot{X} d\sigma}{\int d\sigma} = \frac{\int \lambda^{-2} X \cdot X'' d\sigma}{\int d\sigma} = 2 \frac{\dot{a}}{a} V^2. \]  

(6.26)

\(^8\) With this interpretation, one expects that \( W \) should have the effect of reducing the effective radius of curvature of the network. As we will see later, this is indeed the case.
In order to express \( X'' \) in terms of the string curvature vector we define
\[
d s = \sqrt{1 + (\dot{a}_i X)^2} \, d x_i = \lambda \sqrt{1 + \lambda^{-2} X^2} \, d \sigma
\] (6.27)
and the physical radius of curvature
\[
\frac{d^2 Y}{d \sigma^2} = a(\eta) \frac{k V}{R},
\] (6.28)
where we have introduced the 3-vector \( Y = (x^1, X) \) and a unit 3-vector \( \hat{u} \). Now
\[
X'' = \frac{d^2 X}{d \sigma^2} = \lambda^2 (1 + \lambda^{-2} X^2) \frac{d^2 X}{d \sigma^2} + \lambda X \frac{d \sqrt{1 + \lambda^{-2} X^2}}{d \sigma}.
\] (6.29)
In this case, the second term will not cancel on dotting with \( X \), because \( X \cdot X' \neq 0 \) for the non-relativistic string. Instead we have two terms
\[
\lambda^{-2} \int \dot{X} \cdot X' \, d \sigma = \int \dot{X} \cdot \frac{d^2 X}{d \sigma^2} (1 + \lambda^{-2} X^2) \, d \sigma + \lambda^{-2} \int \dot{X} \cdot X' (\ln \sqrt{1 + \lambda^{-2} X^2})' \, d \sigma.
\] (6.30)
For the first term we note that, since \( \dot{X} \) is normal to \( (x^1, 0) \) in Cartesian coordinates,
\[
\dot{X} \cdot \frac{d^2 X}{d \sigma^2} = X \cdot \frac{d^3 Y}{d \sigma^2}
\] (6.31)
and so we can use equation (6.28) to write
\[
\int \dot{X} \cdot \frac{d^2 X}{d \sigma^2} (1 + \lambda^{-2} X^2) \, d \sigma = a(\eta) \frac{k V}{R} (1 + W^2) \int d \sigma.
\] (6.32)
Here, in analogy to the relativistic case, we have defined a momentum parameter \( k \) by
\[
\langle (1 + \lambda^{-2} X^2)(X \cdot \hat{u}) / R \rangle = \frac{k V}{R} (1 + W^2).
\] (6.33)
For the second term in (6.30) we have
\[
\lambda^{-2} \int \dot{X} \cdot X' (\ln \sqrt{1 + \lambda^{-2} X^2})' \, d \sigma = \lambda^{-2} \int \dot{X} \cdot X' \frac{X' \cdot X'' \lambda^{-2}}{1 + \lambda^{-2} X^2} \, d \sigma
\]
\[
= \lambda^{-2} \int (\dot{X} \cdot X') (X' \cdot \hat{u}) a(\eta) \frac{k V W^2}{R} \, d \sigma + \lambda^{-2} \int (\dot{X} \cdot X') X^2 \frac{X' \cdot X'' \lambda^{-2}}{(1 + \lambda^{-2} X^2)^2} \, d \sigma
\]
\[
= a(\eta) \frac{k' V W^2}{R} \int d \sigma + O(V W^4),
\] (6.34)
where we have used equation (6.29) and defined the parameter \( k' \) by
\[
\langle \lambda^{-2} (X \cdot X') (X \cdot \hat{u}) / R \rangle = \frac{k' V W^2}{R}.
\] (6.35)
Putting all the terms together, equation (6.26) can be rewritten (in terms of cosmic time \( t \)) as
\[
\frac{d V}{d t} = \frac{1}{R} (k + k'' W^2) - 2 HV,
\] (6.36)
with
\[
k'' \equiv k + k'.
\] (6.37)

Equations (6.24), (6.36) form the non-relativistic velocity dependent one-scale (NRVOS) model. In principle one should consider \( W \) as a third dynamical variable and try to derive an evolution equation, as in the case of \( V \). As a first approximation we will assume that time variations in \( W \) do not have a significant impact, \( W \) remaining always small, and we will treat
it as a constant parameter. This approximation will be tested in the following section, where we will solve the NRVOS equations numerically, for different choices of the $W$ parameter.

Finally, one comment is in order regarding the magnitude of the parameter $k' \equiv \frac{1}{2} \langle \mathbf{X} \cdot \mathbf{X}' \rangle \langle \mathbf{X}' \cdot \mathbf{u} \rangle$. The expression for $k'$ measures the average value of $(\mathbf{X} \cdot \mathbf{X}') \langle \mathbf{X}' \cdot \mathbf{u} \rangle$, the first factor of which contains uncorrelated vectors, while for the second factor string tangents will generally be normal to the local curvature vector. On the other hand $k$ corresponds to the average value of $\mathbf{X} \cdot \mathbf{u}$ and these two vectors are correlated, at least for smooth strings/small excitation velocities. Given that the $k''$ term in (6.34) is already suppressed by a factor $O(W^2)$ it is a good approximation to set $k'' \simeq k$. Then, $W$ has the effect of renormalizing the effective radius of curvature $R \rightarrow R/(1+W^2)$ (or equivalently the momentum parameter $k \rightarrow k(1+W^2)$), as may be expected from its interpretation as a short-scale structure parameter.

7. Relativistic versus non-relativistic network evolution

In this section we solve numerically the NRVOS equations for a non-relativistic string network and compare to the relativistic case. The naive expectation is that non-relativistic networks are denser than their relativistic counterparts because the small string velocities reduce the effect of the loop production term. Physically, the transverse excitations on strings are non-relativistic so fewer loops are produced per unit time due to string self-intersections. Long string segment interactions are also suppressed due to the low collision rate corresponding to small velocities.

To close the NRVOS equations we need to specify an ansatz for the non-relativistic momentum parameter $k$. For a velocity dependent model like the one we have developed, it is not consistent to treat $k$ as a constant parameter. Further, in the relativistic case, its dependence on the rms velocity $v$ (equation (6.15)) is important in determining the scaling values of the network variables. The functional dependence of the momentum parameter on $v$ can be obtained by considering ‘curvature’ and ‘bulk’ contributions to string velocities, as explained in [32]. Following the discussion in that reference we take

$$k(v) = k_0(1 - v^2),$$

where $k_0$ is a constant. This has the same functional dependence as the low-velocity limit of $k(v)$ in [32], but here we have left the overall normalization $k_0$ as a free parameter. This reflects the fact that the non-relativistic string limit is not merely a low-velocity one. There is a difference between slowly moving, straight, relativistic strings and wiggly, non-relativistic strings. The defining property of the non-relativistic string is that its transverse excitations be Galilei, as opposed to Lorentz, invariant. The difference between relativistic and non-relativistic strings is in the transverse perturbations. In an effective description, non-relativistic strings can be thought of as having a short wavelength cut-off on the string excitations. As a result, arbitrarily small-wavelength relativistic perturbations are not excited and this translates into a reduced curvature parameter normalization $k_0$. The string can be thought of as a massive rigid rod, but with tension $T$ equal to its mass per unit length $\mu$. In analogy to the relativistic case, where the overall normalization was determined by comparison to a known analytic solution [32], $k_0$ can be obtained in the non-relativistic case by comparison to a given model of non-relativistic string. In the general discussion below we will simply treat it as a free parameter and examine its effect on the network evolution.

9 Relativistic invariance in the longitudinal directions implies that the waves along the string travel at the speed of light $c$ [5]. Note the difference to the other notion of non-relativistic string with $T < \mu$ and longitudinal speed $v < c$.
Equations (6.24), (6.36) and (7.1) have been solved numerically for a range of parameters $k_0$ and $W$. This was done by rewriting equation (6.24) in terms of the correlation length $L = \sqrt{\mu/\rho}$ and then introducing a function $\gamma(t) = L/t$. Under the assumption $L \simeq R$, the resulting equation for $\gamma(t)$ together with (6.36) form a non-autonomous system of coupled ODE’s, which can be integrated numerically. During matter or radiation domination, this system has an attractor solution in which both $\gamma(t)$ and $v(t)$ tend to constant values (scaling). Here, we present numerical results for a radiation dominated universe.

In figure 1 we plot the evolution of the string energy density and rms velocity for both non-relativistic and relativistic string networks, that is, the solution of equations (6.24), (6.36), (7.1) in the former case and (6.5), (6.14), (6.15) in the latter. To highlight the effect of non-relativistic velocities, we have chosen a value of the parameter $k_0$ which gives a scaling value of $V \simeq 0.1$ and taken $W < V$. We have also assumed that both networks have the same loop production efficiency parameter $\tilde{c}$ and chosen the value $\tilde{c} = 0.23$, suggested by relativistic network simulations [37]. As expected, the non-relativistic network has a much higher scaling string density compared to the non-relativistic one. Of course, non-interacting strings ($\tilde{c} = 0$) do not converge to a scaling solution.

We now explore the dependence of non-relativistic string evolution on the parameters $k_0$ and $W$. In figure 2 we plot the normalized string density $\rho t^2/\mu = \gamma^{-2}$ and the rms string velocity $V$ as functions of cosmic time $t$ for different choices of $k_0$ producing string velocities $0 < V < 1$. We have assumed a constant value of $W < V$, but below we will consider the effect of varying $W$ also, allowing for $W > V$. It is apparent from figure 2 that the rms string velocities are controlled by the parameter $k_0$. Reducing $k_0$ leads to smaller $V$, which in turn
implies a higher string density, due the reduced energy-loss term. The fact that the scaling value of the rms velocity is not universal for non-relativistic strings, but instead depends on the parameter $k_0$, is not surprising. In the relativistic case, there is a distinct upper speed limit $c = 1$ and the relativistic constraint implies that the rms velocities are smaller than, but not far off, $1/\sqrt{2}$ (see, for example, [17]). On the other hand, in any non-relativistic theory velocities are unbounded.

We then consider the impact of varying the parameter $W$. Looking at the first term of equation (6.24), which describes dilution due to cosmic expansion, one observes that $W^2$ and $V^2$ appear with opposite signs, so a large $W$ could counterbalance (or even reverse) the effect of $V$ on this term. However, if both $V$, $W \ll 1$ they play no significant role in that term. Thus, one only needs to check the case $W > V$ when $V$, $W$ are not negligible. Figure 3 shows the time evolution of $\rho$ for a choice of $k_0$ leading to $V \simeq 0.1$, for the cases $W = 0, 0.1, 0.5$. The first two figures show identical evolutions, even though in the second one $W \simeq V$. In the third
Figure 3. Dependence of normalized string density on the parameter $W$ for a network with $V \approx 0.1$. The plots correspond to $W = 0, 0.1$ and 0.5 respectively. Increasing $W$ does not significantly alter the scaling density until $W$ becomes greater than $V$. For $W = 0.5 = 5V$, the scaling is reduced by 10%, so it remains two orders of magnitude greater than that of relativistic strings.

case, however, where $W^2 = 25V^2$ the effect of $W$ counterbalances that of $V$ in the dilution term of (6.24), resulting in an appreciable reduction of the string scaling density, at the 10% level. Since the most important impact of string velocities is through the loop production term of (6.24), the basic prediction of the model, which is a dramatic enhancement of the string scaling density (figure 1), remains robust.

8. Discussion

So far we have studied the dynamics and macroscopic evolution of non-relativistic strings in some generality, without discussing any specific setup in which they could be relevant. However, non-relativistic string-like objects arise in several contexts and have been considered before in the literature.
For example, [41] studied non-relativistic vortex strings with motivations from both cosmology [17] and condensed matter physics [42, 43]. In that reference, the non-relativistic limit was taken at the level of the equations of motion by requiring small string velocities $X^2 \ll 1$. Here, we have taken the non-relativistic limit at the level of the string action but this involved a rescaling procedure which corresponds to having both $X^2 \ll 1$ and $(\partial X/\partial \zeta)^2 \ll 1$, where $\zeta$ is the physical length along the string. The non-relativistic evolution model we have developed in section 6.2 can be applied to the condensed matter context considered in [41] by introducing a friction term relevant to that situation. Adding this term and setting $\dot{a}/a = 0$ equation (6.24), expressed in terms of the correlation length, reads

$$\frac{2dL}{dt} = \tilde{c} V + \frac{L}{\ell_d} V^2,$$

(8.1) as in [41], where $\ell_d$ is the relevant damping length scale. The velocity evolution equation is also modified by the addition of a friction term $-\ell_d/L$, again as in [41]. The system has a solution with $L \propto t^{1/2}$, which is actually observed experimentally for defects in condensed matter systems and liquid crystals [44–46].

In cosmology, slowly evolving string networks have been invoked in order to obtain a negative equation of state [47]. Bucher and Spergel [20] have proposed a solid dark matter (SDM) model, which could be realized in terms of a frustrated string or domain wall [48] network. Rigidity and stability in this scenario have been studied in [21]. More recently, a string network of the SDM kind was revived [22] in an attempt to explain the flat rotational curves and the Tully–Fisher relation observed in galaxies, which were the main motivation for the development of MOND theories. The fundamental difficulty [21] with the SDM scenario is to explain how an essentially non-relativistic network can naturally arise from an initial tangle of (relativistic) Nambu–Goto strings like those believed to be produced in cosmological phase transitions. Indeed, Hubble damping is inefficient at subhorizon scales [17] and there is no known mechanism efficient enough to damp the relativistic short-scale excitations on strings. These affect the equation of state through the velocity dependent term in equation (5.3), leading to $w > -1/3$.

Further, numerical evidence is now accumulating supporting that scaling behaviour in field theory strings and domain walls is rather generic [50], so that frustrated networks seem hard to obtain. On the other hand the analysis we did in section 7 points towards a SDM picture for non-relativistic strings, where the above problems are not present. Here, string velocities can be arbitrarily small and, as we saw in section 7, network densities are dramatically enhanced so that strings could even dominate the universe before scaling is reached.

Note that the procedure for obtaining the non-relativistic string action (3.11) required at least one of the spatial directions to be compact. If the action (3.11) is to be treated as a classical effective action this global property can be ignored, but if it is taken to describe a fundamental object, then the winding around a compact dimension is required at quantum level. The fact that a consistent non-relativistic string theory based on the action (3.11) can be constructed [2] allows one to take the view that there is a fundamental winding string obeying this action. Then, a cosmological setup like that of sections 6.2 and 7 can still be considered as long as the compactification radius is larger than the horizon. This possibility of having a universe with non-trivial topology is not observationally excluded. Cosmological

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10 For a recent review on the MOND scenario see [49].

11 One could argue that the velocity which enters the equation of state is the coherent string velocity at the scale of the string correlation length rather than the rms short-scale velocity. While it is true that the coherent velocities are typically smaller, numerical simulations [14] suggest $v_{coh} \approx 0.15$ so one still expects significant departure from $w = -1/3$. Furthermore, small-scale structure has the effect of ‘renormalizing’ the string mass per unit length [15, 17] and string tension so that equation (5.4) should be used instead of (5.3). This also increases the value of $w$. 

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observations constrain the local geometry as described by the metric to be nearly flat [19], but the global topology of spatial hypersurfaces need not be that of the covering space. Indeed, topological identifications under freely acting subgroups of the isometry group are allowed, and the WMAP sky maps appear to be compatible with finite flat topologies with fundamental domain significantly greater than the distance to the decoupling surface [51] (see also [52]).

One can therefore imagine a situation where fundamental non-relativistic strings are wound around 1-cycles in a non-simply-connected universe, in a setup analogous to that of the Brandenberger–Vafa scenario [53]. If the compactification radius is larger than the horizon, as required by cosmological observations, a network of such wound strings behaves like an open string network. An analogous situation occurs in ordinary cosmic string simulations, where the network evolves in a periodic box and there is a class of long strings (determined mainly by initial conditions) which wind around the box. As the universe expands these strings tend to straighten out and behave essentially non-relativistically [54]. These strings are usually discarded as artefacts of the periodicity of the box, but in a universe of compact topology, such configurations can play a physical role.

Finally, in theories with compact extra dimensions one has the possibility of non-relativistic strings winding 1-cycles in the internal space. Analogous (but relativistic) objects have been considered in the context of brane inflation [55–57], which are topologically trapped and behave like monopoles. Although the copious production of such objects in the early universe is inconsistent with the existence of an early radiation era, there are regions in parameter space where they are allowed and in some cases can provide candidates for dark matter. The situation of non-relativistic strings wrapping an internal dimension is qualitatively similar, but the corresponding energy spectrum is different than in the relativistic case.

The outstanding question arising from the above is to what extent such non-relativistic strings are ‘natural’ or ‘generic’ objects in cosmology. Even though non-relativistic strings exist in some part of the moduli space of string theory, there is at present no mechanism which produces them in a cosmological setup. Nevertheless, it is clear that the non-relativistic string action and the VOS model developed here are applicable at least as effective descriptions of cosmic string and vortex string in certain situations. Indeed, the action we have considered is the only sensible non-relativistic limit, having $T = \mu$, of the standard Nambu–Goto action, and is precisely the action one obtains when considering the low-energy dynamics of topological defects in field theory. The macroscopic NRVOS model based on this action, provides a semi-analytic tool for the study of the cosmological evolution of non-relativistic strings. Possible situations of cosmological interest involving non-relativistic strings include strings in de Sitter space, solid dark matter, wound strings, etc, as discussed above. Further, in a condensed matter application we have noted that our model reproduces the correct scaling law, as experimentally observed.

It would be interesting to go one step further and perform numerical simulations of string network evolution based on the non-relativistic string action presented here. The comparison of macroscopic string evolution and small-scale structure to the relativistic case could provide an independent means of probing the effect of small-scale structure on string networks, which is an area of current interest and active research.

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Appendix A. Non-relativistic string from semiclassical approximation

In this section we derive the action \((3.11)\) as a semiclassical expansion around the vacuum solution. Non-relativistic D-brane actions on \(\text{AdS}_5 \times S^5\) have been recently constructed \([23]\) with this method, by considering the Dirac–Born–Infeld (DBI) action and expanding around a classical solution. Here, we apply the method to the case of the Nambu–Goto action in an expanding FLRW background.

We start from the Nambu–Goto action

\[
S_{\text{NG}} = -T \int \sqrt{-\gamma} \, d^2 \sigma
\]

and write the induced metric \(\gamma_{ij} = G_{MN} \partial_i x^M \partial_j x^N\) as

\[
\gamma_{ij} = e^M_i e^N_j \eta_{MN},
\]

where \(e^M_i = a(x^0) \partial_i X^M\) is the (worldsheet induced) vielbein for the FLRW metric \((2.4)\). We now consider the vacuum field configuration

\[
x_0^M = \begin{cases} 
\tau, & M = 0 \\
\lambda \sigma, & M = 1 \\
0, & M = a \in (2, \ldots, D)
\end{cases}
\]

which is a solution of the equations of motion \((2.2)\). The only non-trivial vielbeins evaluated on the solution are the longitudinal ones

\[
e^\mu_0_i = a(x^0) \partial_i X^\mu, \quad \mu = 0, 1,
\]

the transverse ones \(e^a_0\) being zero. The induced metric on the static solution is then

\[
\gamma_{ij}^0 = e^M_0 e^N_0 \eta_{MN} = e^\mu_0_i e^\nu_0_j \eta_{\mu\nu}.
\]

Next we introduce transverse fluctuations around this solution, namely

\[
x^a = X^a(\sigma^i).
\]

The induced metric can then be written as the sum of the static solution plus a piece quadratic in the fluctuations

\[
\gamma_{ij} = \gamma_{ij}^0 + \gamma_{ij}^2,
\]

where

\[
\gamma_{ij}^0 = a(x^0)^2 \text{diag}(-(-\partial_x x^0)^2, (\partial_x x^1)^2) = a(x^0)^2 \text{diag}(-1, \lambda^2)
\]

\[
\gamma_{ij}^2 = a(x^0)^2 \partial_i X^a \partial_j X^b \delta_{ab}.
\]

Expanding \(\sqrt{-\gamma}\) we have

\[
\sqrt{-\text{det} \gamma} = \sqrt{-\text{det}(\gamma_0 + \gamma_2)} = \sqrt{-\text{det}\left[\gamma_0 \left(1 + \gamma_0^{-1} \gamma_2\right)\right]}
\]

\[
= \sqrt{-\text{det} \gamma_0 \left(1 + \frac{1}{2} \gamma_0^{-1/2} \gamma_{ij}^2 + \cdots\right)}
\]

\[
= \sqrt{-\text{det} \gamma_0 + \frac{1}{2} \sqrt{-\text{det} \gamma_0 \gamma_{ij}^2} + \cdots}.\]
For the zero-order (in the fluctuations) piece we can write
\[ \sqrt{-\det \gamma_0} d^2 \sigma = \det(e_0)^{\mu}_i d^2 \sigma = \frac{1}{2} \epsilon_{\mu \nu} e_0^\mu e_0^\nu, \] \hspace{1cm} (A.11)
where our conventions are such that $\epsilon_{01} = 1$. This can be therefore cancelled by choosing a closed $B_\mu^\nu$ field as in section 3. We are left with the second-order piece, which corresponds to the non-relativistic string action (3.11) we obtained in the last section. Indeed, from equations (A.1), (A.9) and (A.10) we get
\[ S_2 = -\frac{T}{2} \int a(x^0)^2 \sqrt{-\det \gamma_0} \gamma_{ij} \delta_{ab} \partial_i X^a \partial_j X^b d^2 \sigma \]
\[ = -\frac{T}{2} \int \sqrt{-\det \gamma_0} G_{ab}(x^0) \partial_i X^a \partial_j X^b d^2 \sigma. \] \hspace{1cm} (A.12)

Appendix B. Hamiltonian formulation

Here we discuss the Hamiltonian formulation of the Nambu–Goto and non-relativistic strings.

B.1. Relativistic string

We first consider the relativistic Nambu–Goto string. The Hessian of the Lagrangian (2.1) has two null eigenvalues and as a result the canonical variables $x^M, p_M$ satisfy two primary constraints, namely

\[ \frac{p_M p_N}{T} G^{MN}(x^0) + T x^M x^N G_{MN}(x^0) = 0 \] \hspace{1cm} (B.1)
\[ p_M x^M = 0, \] \hspace{1cm} (B.2)

which are first class. From these the Dirac Hamiltonian can be constructed

\[ H = \int \mathcal{H} d\sigma = \int \left[ \frac{f(\sigma, \tau)}{2} \left( \frac{p_M p_N}{T} G^{MN}(x^0) + T x^M x^N G_{MN}(x^0) \right) + h(\sigma, \tau) p_M x^M \right] d\sigma, \] \hspace{1cm} (B.3)

where the Lagrange multipliers $f, g$ are arbitrary functions on the worldsheet. The Poisson brackets for $x^M$ and $p_M$ are

\[ \{X^M(\sigma), P_N(\sigma')\} = \delta(\sigma - \sigma') \delta^M_N \] \hspace{1cm} (B.4)
\[ \{X^M(\sigma), X^N(\sigma')\} = 0 \] \hspace{1cm} (B.5)
\[ \{P_M(\sigma), P_N(\sigma')\} = 0 \] \hspace{1cm} (B.6)

and, choosing $h = 0$, the equations of motion read

\[ \begin{cases}
\dot{x}^M = \{x^M, H\} = f T^{-1} p_N G^{NM}(x^0) \\
p_M = \{p_M, H\} = T (f x^N) G_{NM}(x^0) + \frac{a}{2} f T^{-1} p_N G^{NA}(x^0) - T x^N x^A G_{NA}(x^0) \delta^B_M.
\end{cases} \] \hspace{1cm} (B.7)

Thus, for the spacelike fields $x^I, I = 1, \ldots, D$, we have

\[ \ddot{x}^I = f T^{-1} (p_N G^{NI}(x^0) + p_N G^{NI}(x^0)) + \dot{f} T^{-1} p_N G^{NI}(x^0) \]
\[ = f (f x^N) \delta^I_A - 2 \frac{a}{T} f T^{-1} p_J a(x^0) - 2 \delta^{IJ}. \] \hspace{1cm} (B.8)
In the temporal gauge $\dot{x}^0 = 1$ we have from equation (B.7) that
\[ f = -Ta(x^0)^2 p_0^{-1} \]  
and so
\[ p_I = Ta(x^0)^2 f^{-1}\dot{x}^I = -p_0 \dot{x}^I . \]  
Then equation (B.8) becomes
\[ \ddot{x}^I = T^2 a(x^0)^4 p_0^{-1} (p_0^{-1} \dot{x}^I)' - p_0 p_0^{-1} \dot{x}^I . \]  
Now $p_0$ can be found from the constraint (B.1)
\[ p_0 = Ta(x^0)^2 \left( \frac{x^I}{1 - \dot{x}^I} \right)^{1/2} = Ta(x^0)^2 \epsilon . \]  
In view of our previous results for $\dot{\epsilon}$ in the Lagrangian formulation (see equation (2.6)) we see that we are going to recover the equation of motion for $x^I$. Indeed, from (B.7) we have
\[ p_0^{-1} p_0 = -\dot{a} a T^2 p_0^{-2} \left[ T^{-1} a(x^0)^{-2} (p_0^2 \dot{x}^I - p_0^2) - Ta(x^0)^2 \dot{x}^I \right] = 2 \frac{\dot{a}}{a} (1 - \dot{x}^I) , \]  
where we have used (B.12). Thus, (B.11) becomes
\[ \dot{x}^I = \epsilon^{-1} (\epsilon^{-1} x^I)' - 2 \frac{\dot{a}}{a} (1 - \dot{x}^I) \dot{x}^I , \]  
as in (2.6). Finally, the constraint (B.2) in this gauge becomes $\dot{x}^I x^I = 0$, as before.

B.2. Non-relativistic string

We now turn to the Hamiltonian formulation of the non-relativistic string. The Hamiltonian density can be constructed from the constraints (4.1)–(4.2) introducing arbitrary functions $f(\sigma, \tau)$ and $h(\sigma, \tau)$ as Lagrange multipliers
\[ H = f \left( p_\mu \eta^\mu_{\nu} x^\nu(\sigma) + \frac{1}{2} \left( \frac{P_\mu}{T_0} G^{ab}(x^0) + T_0 X^a X^b G_{ab}(x^0) \right) \right) + h(p_\mu x^\mu + P_a X^a) . \]  
The canonical variables satisfy the following Poisson brackets:
\[ \{X^M(\sigma), P_N(\sigma')\} = \delta(\sigma - \sigma') \delta^M_N \]  
\[ \{X^M(\sigma), X^N(\sigma')\} = 0 \]  
\[ \{P_M(\sigma), P_N(\sigma')\} = 0 . \]  
Then, choosing $h = 0$ the transverse equations of motion are
\[ \dot{X}^a = [X^a, H] = f T_0^{-1} P_a(\sigma) G^{ab}(x^0) \]  
\[ \dot{P}_a = \{P_a, H\} = T_0 (f X^b(\sigma))' G_{ab}(x^0) \]  
while the longitudinal ones read
\[ \dot{x}^\mu = \{x^\mu, H\} = f e^{\mu \nu} \eta_{\nu \rho} x^\rho(\sigma) \]  
\[ \dot{p}_\mu = \{p_\mu, H\} = (f p_\nu(\sigma)') e^{\nu \rho} \eta_{\rho \mu} + \frac{1}{2} f \left( \frac{P_\mu}{T_0} G^{ab}(x^0) - T_0 X^a X^b G_{ab}(x^0) \right) \delta^0_{\mu} . \]  
Now choose the gauge (4.5) by setting $x^0 = 1$ and $x^I = \lambda$. Equations (B.20) then imply
\[ f = -\lambda^{-1} \]
and the transverse equations of motion (B.19) give

\[ X^a = -(\lambda T_0)^{-1} (P_a G^{ab}(x^0) + P_b G^{ab}(x^0)) = \lambda^{-2} X^{\alpha a} - 2\frac{\dot{a}}{a} X^a \] (B.22)

recovering equation (4.7).

From equations (B.20) we have

\[ \dot{p}_0 = \lambda^{-1} \left[ p'_0 - \frac{\dot{a}}{a} \left( \frac{P_a P_b}{T_0} G^{ab}(x^0) - T_0 X^\alpha X^b G^{ab}(x^0) \right) \right] \] (B.23)

\[ = -\lambda^{-2} (P_a X^\alpha)' + \lambda^{-1} \frac{\dot{a}}{a} \left( T_0 X^\alpha X^b G^{ab}(x^0) - \frac{P_a P_b}{T_0} G^{ab}(x^0) \right), \] (B.24)

using the constraint (4.2). Since \( p_0 \) is given by equation (4.1) and \( P_a = -\lambda T_0 \dot{X}_a \) (from (B.20)) we recover equation (4.13).

Appendix C. Energy–momentum tensor of non-relativistic action

In order to obtain the energy–momentum tensor of the non-relativistic string, we vary the action (3.11) with respect to the background metric \( G_{MN} \)

\[ T^{MN} = \frac{-2}{\sqrt{-\det G_{MN}}} \frac{\delta S}{\delta G_{MN}} = \frac{-2}{\sqrt{-\det G_{MN}}} \left( \begin{array}{cc} \frac{\delta S}{\delta G_{\mu\nu}} & 0 \\ 0 & \frac{\delta S}{\delta G_{\mu\nu}} \end{array} \right). \] (C.1)

The transverse part reads

\[ T^{ab} = \frac{T_0}{\sqrt{-\det G_{MN}}} \int d^2 \sigma \sqrt{-\det g} g^{ij} \partial_i X^a \partial_j X^b \delta^{(D+1)}(y^M - x^M(\sigma^i)), \] (C.2)

where as the longitudinal part is

\[ T^\mu\nu = \frac{T_0}{\sqrt{-\det G_{MN}}} \int d^2 \sigma \sqrt{-\det g} \left[ \frac{1}{2} g^{m\mu} \partial_m x^\alpha \partial_\alpha x^\nu \partial_i X^a \partial_j X^b G_{ab} \right. \\
\left. - g^{m\mu} \partial_m x^\alpha \partial_\alpha x^\nu \partial_i X^a \partial_j X^b G_{ab} \right] \delta^{(D+1)}(y^M - x^M(\sigma^i)). \] (C.3)

In particular, the 00 component in the gauge (4.5) becomes

\[ T^{00}(\eta, y^K) = -\frac{1}{a(\eta)^{D+1}} \int \lambda \frac{T_0}{2} (X^a X^b + \lambda^{-2} X^a X^b) \delta_{ab} \delta^{(D)}(y^K - x^K(\sigma, \eta)) d\sigma, \] (C.4)

where, having integrated out \( \delta(\eta - \tau) \), \( K \) runs from 1 to \( D \). Note that the explicit scale-factor dependence of the integrand cancels, because of the presence of \( \sqrt{-\det g^{-1} g^{-1} G} \) in equation (C.3). The first factor scales like \( a(\eta)^2 \), the next two factors as \( a(\eta)^{-2} \) each, and the last factor as \( a(\eta)^2 \), giving a scale-factor-independent result. Of course \( T^{00} \) still depends on time through the time dependence of the fields \( X^a \).

The string energy can be defined as in section 2, by considering a spatial hypersurface \( \eta = \text{const.} \), with normal (co)vector \( n_M = (a(\eta), 0) \), and integrating the energy density \( n_M n_N T^{MN} = a(\eta)^2 T^{00} = T_0 \) over the relevant D-volume

\[ E(\eta) = -\int \sqrt{h} n_M n_N T^{MN} d^D y = -\int a(\eta)^{D+2} T^{00} d^D y = a(\eta) \frac{\lambda T_0}{2} \int d\sigma \left( X^a X^b + \lambda^{-2} X^a X^b \right) \delta_{ab}. \] (C.5)
Similarly, the $ab$ components of the energy–momentum tensor in the gauge (4.5) are

$$T^{ab}(\eta, y^K) = \frac{1}{a(\eta)^{D+1}} \int \lambda T_0(-X^a X^b + \lambda^{-2} X^a X^b) \delta^{(D)}(y^K - x^K(\sigma, \eta)) \, d\sigma.$$  

(C.6)

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