The \( r \)-largest four parameter kappa distribution

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Abstract

The generalized extreme value distribution (GEVD) has been widely used to model the extreme events in many areas. It is however limited to using only block maxima, which motivated to model the GEVD dealing with \( r \)-largest order statistics (rGEVD). The rGEVD which uses more than one extreme per block can significantly improves the performance of the GEVD. The four parameter kappa distribution (K4D) is a generalization of some three-parameter distributions including the GEVD. It can be useful in fitting data when three parameters in the GEVD are not sufficient to capture the variability of the extreme observations. The K4D still uses only block maxima. In this study, we thus extend the K4D to deal with \( r \)-largest order statistics as analogy as the GEVD is extended to the rGEVD. The new distribution is called the \( r \)-largest four parameter kappa distribution (rK4D). We derive a joint probability density function (PDF) of the rK4D, and the marginal and conditional cumulative distribution functions and PDFs. The maximum likelihood method is considered to estimate parameters. The usefulness and some practical concerns of the rK4D are illustrated by applying it to Venice sea-level data. This example study shows that the rK4D gives better fit but larger variances of the parameter estimates than the rGEVD. Some new \( r \)-largest distributions are derived as special cases of the rK4D, such as the \( r \)-largest logistic (rLD), generalized logistic (rGLD), and generalized Gumbel distributions (rGGD).

Keywords: Annual maximum sea level; Bias-variance trade-off; Delta method; Hydrology;
1 Introduction

The generalized extreme value distribution (GEVD) has been widely used to analyse univariate extreme values (Coles 2001). The GEVD encompasses all three possible asymptotic extreme value distributions predicted by large sample theory. The cumulative distribution function (cdf) of the GEVD is as follows (Hosking and Wallis 1997):

\[
F_3(x) = \exp \left\{- \left(1 - \frac{k(x - \mu)}{\sigma} \right)^{1/k} \right\},
\]

where \(1 - k(x - \mu)/\sigma > 0\) and \(\sigma > 0\), where \(\mu, \sigma,\) and \(k\) are the location, scale, and shape parameters, respectively. The particular case for \(k = 0\) in (1) is the Gumbel distribution. Note that the sign of \(k\) is changed from the book of Coles (2001).

One difficulty of applying the GEVD is using the limited amount of data for model estimation. Since extreme values are scarce, making effective use of the available information is important in extremes. This issue has motivated the search for a model to use more data other than just block maxima. The above univariate result was extended to the \(r\)-largest order statistics model, which gives the joint density function of the limit distribution (Coles 2001);

\[
f_3(x^{(1)}, x^{(2)}, \ldots, x^{(r)}) = \exp \left\{-w(x^{(r)})^{1/k} \right\} \times \prod_{s=1}^{r} \sigma^{-1} w(x^{(s)})^{\frac{1}{k}-1},
\]

where \(x^{(1)} \geq x^{(2)} \geq \cdots \geq x^{(r)}\), and \(w(x^{(s)}) = 1 - k \frac{x^{(s)} - \mu}{\sigma} > 0\) for \(s = 1, 2, \ldots, r\). This model is referred to the rGEVD. As a modeling tool, the technique was first developed in the Gumbel case by Smith (1986), building on theoretical developments in Weissman (1978). The general case in the GEVD was developed by Tawn (1988). The inclusion of more data up to \(r\)-th order statistics in each block other than just maxima will improve precision of model estimation, but the interpretation of parameters is unaltered from the univariate GEVD for block maxima. The rGEVD was encouraged to use by Zhang (2004), and has been employed in some real applications (Dupuis 1997; Soares and Scotto 2004; An and Pandey 2007; Wang and Zhang 2008; Feng and Jiang 2015; Naseef and Kumar 2017). The number \(r\) comprises a bias-variance trade-off: small values of \(r\) generate few data leading to high variance; large values of \(r\) are likely to violate the asymptotic support for the model, leading to bias (Coles 2001). Bader et al. (2017) developed automated methods of selecting \(r\) from the rGEVD.
The distributions derived by extreme value theory are not always applied to real data analysis. For example, the log-Pearson type III (PE3) or the generalized logistic (GLO) distributions have been widely used in hydrology and its use is sometimes mandated by government agencies (Vogel and Wilson 1996). For small to moderate sample sizes, the GEVD sometimes yields inadequate results. It may be because the GEVD is derived by a large sample theory for the extremes of independent sequences. For example, Salinas et al. (2014) showed that the GEVD fails to represent the kurtosis dispersion of flood annual maximum series of 13 nations in Europe. As a generalization of some common three-parameter distributions including the GEVD, the four parameter kappa distribution (K4D) was introduced by Hosking (1994). It can be useful in fitting data when three parameter distributions including the GEVD are not sufficient to capture the variability of observations. Some researchers studied on the K4D (Dupuis and Winchester 2001; Singh and Deng 2003; Park and Kim 2007; Murshed et al. 2014).

The probability density function (pdf) of K4D is, for $k \neq 0$, $h \neq 0$, $\sigma > 0$,

$$f_4(x) = \sigma^{-1} w(x)^{(1/k)-1} F_4(x)^{1-h}, \quad (3)$$

where

$$w(x) = 1 - k \frac{x - \mu}{\sigma}, \quad (4)$$

and

$$F_4(x) = \left\{ 1 - h w(x)^{1/k} \right\}^{1/h}, \quad (5)$$

is the cdf of the K4D. Note that a new shape parameter $h$ is added from the GEVD. The subscripts 3 and 4 in $f$ or $F$ are hereafter used to indicate the corresponding functions of the (3-parameter) GEVD and of the K4D, respectively.

The K4D includes many distributions as special cases, as shown in Figure 1: the generalized Pareto distribution (GPD) for $h = 1$, the GEVD for $h = 0$, the GLO distribution for $h = -1$, the generalized Gumbel distribution for $k = 0$, the Gumbel distribution for $h = 0$, $k = 0$. The K4D is flexible and widely applicable to the data including not only extreme values but also skewed data. It has been used in many fields, particularly in hydrology and atmospheric sciences, for fitting extreme values or skewed data (e.g., Parida 1999; Park and Jung 2002; Wallis et al. 2007; Seo et al. 2015; Brunner et al. 2019; Jung and Schindler 2019). Hosking and Wallis (1997) employed the K4D in regional frequency analysis as a parent distribution from which the samples are drawn. Fruh et al. (2010) found that the K4D works better with smaller bias than the GPD, but at the expense of higher uncertainty, in estimating the return
values of extreme daily precipitation in southwest Germany. Blum et al. (2017) found that the K4D provides a very good representation of daily streamflow across most physiographic regions in the conterminous United States. Kjeldsen et al. (2017) showed using the observed flood flow records in UK that the three parameter distributions, such as the GLO, GEV, and PE3 traditionally used in regional flood frequency analysis, can be replaced by a more flexible K4D.

In analyzing extreme values, the K4D has the same limitation of using only the block maxima as the GEVD has. Like as the GEVD was extended to rGEVD, an extension of the K4D to r-largest order statistic model may be very useful to address this limitation. The inclusion of more observations up to r-th order statistics other than just maxima will improve precision of model estimation. The extension in the K4D is not published yet. In this study, we thus developed an r-largest order statistics model as an extension of the K4D as well as of the rGEVD. It is referred to the rK4D. Figure 2 illustrates our motivic schema.

The remainder of this paper is organized as follows. Section 2 includes the definition of the rK4D, and its marginal and conditional distributions. The maximum likelihood estimation of parameters and quantile are considered in Section 3. Section 4 illustrates the usefulness and some practical concerns of the rK4D by applying it to Venice sea-level data. In Section 5, some new r-largest distributions as special cases of the rK4D are derived. Section 6 concludes with discussion.
2 r-largest four parameter kappa distribution

2.1 Definition of the rK4D

Let us denote \( \mathbf{x}^r = (x^{(1)}, x^{(2)}, \ldots, x^{(r)}) \) are the \( r \)-largest order statistics. The \( r \)-largest four parameter kappa distribution (rK4D) is not the result from any theoretical derivation but just an analogous extension from the K4D and the rGEVD. To define the joint probability density function (pdf) of the rK4D, we considered and followed the generalization processes from the GEVD to the K4D and to the rGEVD.

We define the joint pdf of the rK4D; under \( k \neq 0, \ h \neq 0 \),

\[
f_4(\mathbf{x}^r) = \sigma^{-r} C_r \times g(\mathbf{x}^r) \times F_4(x^{(r)})^{1-rh},
\]

where

\[
C_r = \begin{cases} 
\prod_{i=1}^{r-1} [1 - (r - i)h], & \text{if } r \geq 2 \\
1, & \text{if } r = 1,
\end{cases}
\]

(7)

\( F_4 \) is the cumulative distribution function (cdf) of K4D as in (5), and

\[
g(\mathbf{x}^r) = \prod_{s=1}^{r} w(x^{(s)})^{\frac{1}{h} - 1},
\]

(8)

where \( w(x) \) is defined as in (4).
The supports of this pdf are \(x^{(1)} \geq x^{(2)} \geq \cdots \geq x^{(r)}\), \(\sigma > 0\), \(w(x^{(s)}) > 0\) for \(s = 1, 2, \cdots, r\), \(C_r > 0\), and \(1 - h w(x^{(r)})^{1/k} > 0\). When \(r = 1\), this pdf is same as the pdf of the K4D in (3). When \(h \to 0\), this pdf goes to the pdf of the rGEVD in (2).

To check the consistency of the definition (6), we first derive the joint pdf of the \((r-1)\)K4D from that of the rK4D by integration. Then we check whether the pdf of the \((r-1)\)K4D still has the same pattern with the pdf (6) of the rK4D or not. If the answer is yes, the definition (6) is consistent.

\[
f_4(x^{r-1}) = \int_{-\infty}^{x^{(r-1)}} f_4(x^r) dx^r
\]

\[
= \sigma^{-r} C_r \times g(x^{r-1}) \times \int_{-\infty}^{x^{(r-1)}} w(x^r)^{\frac{1}{k} - 1} \times \left\{1 - h w(x^r)^{\frac{1}{k}}\right\} \frac{1-rh}{k} dx^r
\]

substitute \(v = 1 - h w(x^r)^{\frac{1}{k}}\), \(dx^r = \frac{\sigma}{h} \times w(x^r)^{-\frac{1}{k}+1} dv\)

\[
= \sigma^{-r} \times C_{r-1} \times g(x^{r-1}) \times \int_0^{1-h w(x^{r-1})^{\frac{1}{k}}} \frac{1-rh}{v^{\frac{1-k}{k}}} \, dv
\]

\[
= \sigma^{-(r-1)} \times C_{r-1} \times g(x^{r-1}) \times \left\{1 - h w(x^{r-1})^{\frac{1}{k}}\right\} \frac{1-(r-1)h}{k}
\]

Thus our definition (3) is consistent.

### 2.2 Marginal distributions of the rK4D

The marginal pdf of \(s\)-th order statistic from the rK4D is derived to the following by consecutive integrals of \(f_4(x^s)\) with respect to \((x^{(1)}, \cdots, x^{(s-1)})\):

\[
f_4(x^s) = \int_{x^{(s)}}^{\infty} \int_{x^{(s-1)}}^{\infty} \cdots \int_{x^{(2)}}^{\infty} f_4(x^s) \, dx^{(1)} \, dx^{(2)} \cdots \, dx^{(s-1)}
\]

\[
= \sigma^{-s} C_s F_4(x^{(s)})^{1-sh} \int_{x^{(s)}}^{\infty} \cdots \int_{x^{(2)}}^{\infty} g(x^{(1)}, \cdots, x^{(s)}) \, dx^{(1)} \cdots \, dx^{(s-1)}
\]

substitute \(v = 1 - \frac{k(x^{(1)} - \mu)}{\sigma}\), \(dx^{(1)} = -\frac{k}{\sigma} \, dv\)

\[
= \sigma^{-s} C_s F_4(x^{(s)})^{1-sh} g(x^{(2)}, \cdots, x^{(s)}) \int_{x^{(s)}}^{\infty} \cdots \int_{x^{(2)}}^{\infty} w(x^{(1)})^{\frac{1}{k} - 1} \, dx^{(1)} \cdots \, dx^{(s-1)}
\]
\[ \begin{align*}
&= \sigma^{-(s-1)} C_s F_4(x^{(s)})^{1-h} g(x^{(2)}, \ldots, x^{(s)}) \int_{x^{(s)}}^{\infty} \cdots \int_{x^{(3)}}^{\infty} w(x^{(2)})^{1/2} dx^{(2)}, \ldots, dx^{(s-1)} \\
&= \sigma^{-(s-1)} C_s F_4(x^{(s)})^{1-h} g(x^{(3)}, \ldots, x^{(s)}) \int_{x^{(s)}}^{\infty} \cdots \int_{x^{(3)}}^{\infty} w(x^{(2)})^{3/2-1} dx^{(2)}, \ldots, dx^{(s-1)} \\
&= \sigma^{-(s-2)} C_s F_4(x^{(s)})^{1-h} g(x^{(4)}, \ldots, x^{(s)}) \int_{x^{(s)}}^{\infty} \cdots \int_{x^{(4)}}^{\infty} \frac{1}{2} w(x^{(3)})^{3/2-1} dx^{(3)}, \ldots, dx^{(s-1)} \\
&\vdots \\
&= \sigma^{-1} \frac{C_s}{(s-1)!} w(x^{(s)})^{s-1} \times F_4(x^{(s)})^{1-h},
\end{align*} \]

for 2 \( \leq s \leq r \), where \( g(x) \) is defined as in (8), \( w(x) \) is defined as in (4), and \( F_4 \) is the cdf of K4D as in (5). We can see, as \( h \to 0 \), that the above marginal pdf (10) goes to the corresponding marginal pdf of the rGEVD,

\[ f_3(x^{(s)}) = \sigma^{-1} \frac{1}{(s-1)!} w(x^{(s)})^{s-1} \times \exp[-\tau(x^{(s)})], \]

where

\[ \tau(x) = w(x)^{1/2}. \]

This marginal pdf of rGEVD is also obtained by differentiating the marginal cdf of the rGEVD which is:

\[ H_3(x^{(s)}) = \exp[-\tau(x^{(s)})] \sum_{i=0}^{s-1} \frac{\tau(x^{(s)})^i}{i!}, \]

as provided in Coles (2001, p.67).

The marginal cdf of \( s \)-th order statistic from the rK4D is obtained by integrating \( f_4(x^{(s)}) \)
as follows:

\[ H_4(x^{(s)}) = \int_{-\infty}^{t} f_4(x^{(s)}) \, dx^{(s)} \]

\[ = \int_{-\infty}^{t} \frac{C_s}{\sigma (r-1)!} \times w(x^{(s)})^{\frac{1}{2}} \times \left\{ 1 - h \, w(x^{(s)})^{\frac{1}{2}} \right\} \frac{1}{h} \, dx^{(s)} \]

substitute \( v = 1 - h \, w(x^{(s)})^{\frac{1}{2}} \), \( dx^{(s)} = \frac{\sigma}{h} w(x^{(s)})^{-\frac{1}{2}} + 1 \, dv \)

\[ = \frac{1}{\sigma} \frac{C_s}{(s-1)!} \times w(x^{(s)})^{\frac{s-1}{2}} \times \int_{0}^{1-h \, w(t)^{\frac{1}{2}}} v^{\frac{1}{2}} \, dv \]

\[ = \frac{C_{s-1}}{(s-1)!} \times w(x^{(s)})^{\frac{s-1}{2}} \times \int_{0}^{1-h \, w(t)^{\frac{1}{2}}} v^{\frac{1}{2}} \, dv \]

\[ = C_{s-1} \times w(x^{(s)})^{\frac{s-1}{2}} \times F_4(t)^{1-(s-1)h} \]

When \( h \to 0 \), this marginal cdf goes to the marginal cdf of the rGEVD in \([13]\). The quantiles from this marginal cdf goes to the marginal cdf of the rGEVD in \([13] \). The property is used in generating random numbers from the rK4D.

### 2.3 Conditional distributions of the rK4D

The conditional pdf of \( X^{(s)} \) given \( X^{s-1} \) is, for \( 2 \leq s \leq r \):

\[ f_4(x^{(s)}|x^{s-1}) = \frac{\sigma^{-s} \times C_s \times g(x^{(s)}) \times F_4(x^{(s)})^{1-sh}}{\sigma^{-(s-1)} \times C_{s-1} \times g(x^{s-1}) \times F_4(x^{(s-1)})^{1-(s-1)h}} \]

\[ = \frac{\sigma^{-1} \times (1 - (s-1)h) \times g(x^{(s)}) \times F_4(x^{(s)})^{1-sh}}{F_4(x^{(s-1)})^{1-(s-1)h}} \]

which is actually same to \( f_4(x^{(s)}|x^{(s-1)}) = \frac{f_4(x^{(s)})}{F_4(x^{(s-1)})} \) under \( x^{(s)} \leq x^{(s-1)} \). The Markov property is thus satisfied.

The conditional cdf of \( X^{(s)} \) given \( X^{s-1} \) is,

\[ H_4(x^{(s)}|x^{s-1}) = \frac{F_4(x^{(s)})^{1-(s-1)h}}{F_4(x^{(s-1)})^{1-(s-1)h}} = \left( \frac{F_4(x^{(s)})}{F_4(x^{(s-1)})} \right)^{1-(s-1)h} \]

under \( x^{(s)} \leq x^{(s-1)} \). This is same as the cdf of the K4D with right truncated at \( x^{(s-1)} \) (Johnson et al. 1994). This property is used in generating random numbers from the rK4D.
2.4 Random number generation from the rK4D

The above property, which is the cdf of the K4D with right truncated at \( x^{(s-1)} \), can be exploited to generate the \( r \) components in a realized rK4D observation (Bader et al. 2017). The pseudo algorithm to generate a single observation is the following:

- Generate the first value \( x^{(1)} \) from the unconditional K4D.
- For \( i = 2, \ldots, r \): Generate \( x^{(i)} \) from the K4D right truncated by \( x^{(i-1)} \).

The resulting vector \( (x^{(1)}, \ldots, x^{(r)}) \) is a single observation from the rK4D. A simple way of generating random numbers from the K4D right truncated is; generate \( x^{(i)} \) from the unconditional K4D, then accept \( x^{(i)} \) if \( x^{(i)} < x^{(i-1)} \), or reject otherwise. Repeat until acceptance.

3 Maximum likelihood estimation

Let us denote \( x_i^r = ((x_i^{(1)}, x_i^{(2)}, \ldots, x_i^{(r)}), \ldots, (x_m^{(1)}, x_m^{(2)}, \ldots, x_m^{(r)})) \), which be the \( i \)-th observation of the \( r \)-largest order statistics, for \( i = 1, 2, \ldots, m \). By assuming \( \{x_1^r, x_2^r, \ldots, x_m^r\} \) follow the rK4D, the likelihood function of \( \mu, \sigma, h, k \) is as follows, for \( k \neq 0, h \neq 0 \):

\[
L(\mu, \sigma, h, k | \mathbf{x}^r) = \prod_{i=1}^{m} \left[ \sigma^{-r} C_r F_4(x_i^{(r)})^{1-rh} \times \prod_{j=1}^{r} \left( 1 - k \frac{x_i^{(j)} - \mu}{\sigma} \right)^{\frac{1}{1-k}} \right],
\]

(17)

under constraints. The details of the constraints (Hosking and Wallis 1997) which should be specified in minimizing numerically the negative log-likelihood function with respect to the parameters are; \( \sigma > 0, h < 1/(r - 1) \) for \( r \geq 2 \),

\[
h < \min_{1 \leq i \leq m} w(x_i^{(r)})^{-1/k},
\]

(18)

and

\[
\begin{cases}
\max_{1 \leq i \leq m} w(x_i^{(1)}) < \frac{\sigma}{|k|} + \mu, & \text{if } k > 0 \\
\min_{1 \leq i \leq m} w(x_i^{(r)}) > \frac{\sigma}{|k|} + \mu, & \text{if } k < 0.
\end{cases}
\]

(19)

We implemented a numerical algorithm using the ‘optim’ package in R program by consulting the ‘ismev’ package (Coles 2001).

The standard errors of the maximum likelihood estimates (MLE) are obtained approximately by the squared root of the diagonal terms of the inverse of the observed Fisher information matrix.
### 3.1 Quantiles of the block maxima

The quantiles of the GEVD are obtained by inverting (1):

\[
z_p = \mu + \sigma k \left[ 1 - \left( -\log(1 - p) \right)^k \right],
\]

(20)

where \( F_3(z_p) = 1 - p \). Here, \( z_p \) is known as the return level associated with the return period \( 1/p \), since the level \( z_p \) is expected to be exceeded on average once every \( 1/p \) years (Coles 2001). For example, a 20-year (50-year) return level is computed as the 95th (98th) quantile of the fitted GEVD.

The quantiles of the K4D are

\[
z_p = \mu + \sigma k \left\{ 1 - \left( \frac{1 - (1 - p)^{h/k}}{h} \right)^k \right\},
\]

(21)

where \( F_4(z_p) = 1 - p \).

### 3.2 Delta method for variance estimation

The variance estimation of the \( 1/p \)-year return level \((z_p)\) can be calculated by the delta method (Coles 2001). We present the details of the procedure including the derivatives of \( z_p \) with respect to each parameter as follows;

\[
\text{Var}(\hat{z}_p) \approx \nabla \hat{z}_p^t \text{ } V \nabla \hat{z}_p,
\]

(22)

where \( V \) is the covariance matrix of parameter estimates which is approximated by the inverse of the observed Fisher information matrix, and

\[
\nabla \hat{z}_p = \begin{bmatrix} \frac{\partial z_p}{\partial \mu}, \frac{\partial z_p}{\partial \sigma}, \frac{\partial z_p}{\partial k}, \frac{\partial z_p}{\partial h} \end{bmatrix}
\]

(23)

\[
\frac{\partial z_p}{\partial \mu} = 1, \quad \frac{\partial z_p}{\partial \sigma} = \frac{1 - y_p^k}{k}
\]

(24)

\[
\frac{\partial z_p}{\partial k} = -\sigma \frac{\ln(y_p) \times y_p^h}{k} - \frac{\sigma (1 - y_p^h)}{k^2}
\]

(25)

\[
\frac{\partial z_p}{\partial h} = \frac{\sigma \left\{ (1 - p)^h \ln(1 - p) + 1 - (1 - p)^h \right\} \times y_p^{k-1}}{h^2}
\]

(26)

where \( y_p = \frac{1 - (1 - p)^h}{h} \), evaluated at \((\hat{\mu}, \hat{\sigma}, \hat{k}, \hat{h})\).

For the confidence interval of the return level, the profile likelihood approach (Coles 2001) can be useful even though it is not obtained in this study.
4 Real application: Venice sea-level data

These data consist of the 10 largest sea-levels in Venice over the period 1931-1981, except for the year 1935 (Coles 2001). The rK4D model is fitted to the values for \( r = 1, 2, \cdots, 10 \). The MLE of parameters and the 20-year return levels with standard errors in the parenthesis for several values of \( r \) are given in Table 1. For comparison, similar results from the fitted rGEVD are also presented. The upper table is for the rGEVD and the lower one is for the rK4D. In Table 1, the standard errors of parameter estimates decrease with increasing values of \( r \) for the rGEVD. That is not obvious in the rK4D but generally shows a decreasing trend. These non-monotonic decreasing cases may be because of the trouble in numerical optimization with 4 parameters in the rK4D or the intrinsic property of the rK4D. The SEs of \( \hat{\mu}, \hat{\sigma}, \) and \( \hat{k} \) in the rK4D are generally bigger than those in the rGEVD. The SEs of \( h \) estimates in the rK4D are much larger compared to those of the other parameter estimates.

The 20-year return levels and its standard errors (SE) decrease with \( r \) in rGEVD, whereas those values for rK4D do not show a monotonic decrease. This phenomenon for the return levels of the rK4D is probably explained by that the return level and its SE are obtained for the annual maximum while the rK4D is fitted to the \( r \)-largest order statistics. Because the parameter estimates of the rK4D are obtained to take account into all data up to the \( r \)-largest observations, it may not work good for the annual maximum only. This phenomenon may be more serious for the rK4D than the rGEVD because the standard errors of the return levels of the rK4D are greater than those of the rGEVD. This is a re-confirmation of the general rule that the model with more parameters usually results in bigger variance (and less bias) than the model with fewer parameters (James et al. 2013).

Table 2 provides the Akaike information criteria (AIC), the Bayesian information criteria (BIC), and the trace and the determinant of the covariance matrix \((V)\) of the parameter estimates. The AIC and the BIC are defined as

\[
AIC(p) = -2 l(\hat{\theta}) + 2p, \quad BIC(p) = -2 l(\hat{\theta}) + p \ln(m),
\]

where \( l(\hat{\theta}) \) is the log-likelihood function evaluated at the parameter estimates \( \hat{\theta} \), \( m \) is the sample size, and \( p \) is the number of parameters. In Venice sea-level data, \( m = 50 \). These criteria are employed to select a preferred model by the rule that smaller is better. The trace \( tr(V) \) is the sum of variances, and the determinant \( |V| \) is interpreted as the volume of \( V \) occupied by the probability dispersion it describes. The \( |V| \) is thus sometimes called the generalized variance. It increases as the variances of parameter estimates increase; but also decreases as
Table 1: The estimates of parameters and 20-year return level ($r_{20}$) with standard errors (se) of the estimates in parenthesis which are obtained from the $r$-largest order statistic models fitted to Venice sea-level data with different values of $r$. Upper table is for the rGEVD and lower one is for the rK4D. ‘nllh’ stands for the negative log-likelihood function value.

| $r$ | nllh | $\hat{\mu}$ (se) | $\hat{\sigma}$ (se) | $\hat{k}$ (se) | rGEV $r_{20}$ (se) |
|-----|------|-------------------|-------------------|----------------|--------------------|
| 1   | 222.7| 111.1 (2.6)       | 17.2 (1.8)        | -0.077 (0.074) | 156.7 (6.2)        |
| 2   | 379.5| 114.5 (1.9)       | 15.0 (1.2)        | -0.056 (0.057) | 155.6 (5.6)        |
| 3   | 515.4| 117.3 (1.8)       | 14.8 (0.9)        | -0.097 (0.040) | 155.6 (4.4)        |
| 4   | 632.2| 118.3 (1.7)       | 14.3 (0.8)        | -0.099 (0.035) | 155.0 (4.1)        |
| 5   | 732.0| 118.6 (1.6)       | 13.7 (0.8)        | -0.088 (0.033) | 154.3 (4.0)        |
| 6   | 829.6| 118.8 (1.5)       | 13.4 (0.7)        | -0.086 (0.031) | 154.0 (3.9)        |
| 7   | 916.5| 119.1 (1.5)       | 13.2 (0.7)        | -0.090 (0.029) | 153.6 (3.7)        |
| 8   | 995.7| 119.6 (1.4)       | 13.1 (0.7)        | -0.097 (0.025) | 153.3 (3.4)        |
| 9   | 1064.3| 119.8 (1.4)      | 12.9 (0.6)        | -0.098 (0.024) | 153.0 (3.3)        |
| 10  | 1139.1| 120.5 (1.4)      | 12.8 (0.5)        | -0.113 (0.020) | 152.8 (2.9)        |

| $r$ | nllh | $\hat{\mu}$ (se) | $\hat{\sigma}$ (se) | $\hat{h}$ (se) | rK4D $r_{20}$ (se) |
|-----|------|-------------------|-------------------|-------------|--------------------|
| 1   | 221.8| 120.0 (5.2)       | 9.0 (2.4)         | -1.67 (1.34) | 153.6 (7.6)        |
| 2   | 372.6| 116.9 (2.4)       | 10.2 (1.3)        | -1.31 (0.58) | 159.5 (9.3)        |
| 3   | 499.8| 118.0 (2.1)       | 10.4 (1.1)        | -1.03 (0.32) | 153.8 (6.3)        |
| 4   | 610.6| 117.2 (1.9)       | 10.9 (1.0)        | -0.83 (0.24) | 154.8 (6.5)        |
| 5   | 705.4| 116.9 (2.0)       | 11.5 (1.1)        | -0.77 (0.21) | 157.9 (7.5)        |
| 6   | 803.8| 117.0 (1.9)       | 12.0 (1.1)        | -0.61 (0.17) | 158.4 (7.6)        |
| 7   | 889.4| 116.9 (1.8)       | 12.2 (1.0)        | -0.49 (0.14) | 157.5 (7.0)        |
| 8   | 961.9| 117.1 (1.8)       | 11.9 (0.9)        | -0.49 (0.13) | 154.5 (6.2)        |
| 9   | 1023.0| 117.2 (1.8)    | 11.8 (0.9)        | -0.52 (0.13) | 155.2 (6.0)        |
| 10  | 1089.1| 117.2 (1.7)    | 11.4 (0.8)        | -0.49 (0.12) | 151.9 (4.9)        |

the correlations among the parameter estimates increase (Wilks 2011).

In Table 2, the AIC and the BIC are smaller in the rK4D for each $r$ than the corresponding values in the rGEVD, except for the case $r = 1$. The rK4D is preferrable to the rGEVD for every $r$ except for $r = 1$. The $tr(V)$s in the rK4D for each $r$ are greater than those in the rGEVD, whereas the $log|V|$s in the rK4D are smaller than those in the rGEVD. This
Table 2: The Akaike information criteria (AIC), the Bayesian information criteria (BIC), and the trace and the log determinant of the covariance matrix (V) of the parameter estimates which are obtained from the $r$-largest order statistic models (the rGEVD and the rK4D) fitted to Venice sea-level data with different values of $r$.

| $r$ | rGEVD | rK4D |
|-----|-------|------|
|     | AIC   | BIC  | $tr(V)$ | $log|V|$ | AIC   | BIC  | $tr(V)$ | $log|V|$ |
| 1   | 451.4 | 457.2 | 10.16   | -2.31   | 451.1 | 459.4 | 34.72   | -3.56   |
| 2   | 764.9 | 770.7 | 5.12    | -4.55   | 753.2 | 761.0 | 7.92    | -6.14   |
| 3   | 1036.8| 1042.6| 4.16    | -6.01   | 1007.5| 1015.2| 5.49    | -7.73   |
| 4   | 1270.5| 1276.3| 3.49    | -6.98   | 1229.1| 1236.9| 4.84    | -8.83   |
| 5   | 1469.9| 1475.7| 3.06    | -7.63   | 1418.7| 1426.4| 5.01    | -9.31   |
| 6   | 1665.3| 1671.1| 2.86    | -8.09   | 1615.5| 1623.2| 4.85    | -9.84   |
| 7   | 1839.0| 1844.8| 2.67    | -8.60   | 1786.7| 1794.5| 4.41    | -10.68  |
| 8   | 1997.4| 2003.2| 2.48    | -9.10   | 1931.7| 1939.4| 4.05    | -11.22  |
| 9   | 2134.6| 2140.4| 2.34    | -9.48   | 2054.0| 2061.7| 4.00    | -11.48  |
| 10  | 2284.2| 2292.0| 2.16    | -10.06  | 2186.2| 2194.0| 3.41    | -12.25  |

means that there are more correlations among parameter estimates in the rK4D than in the rGEVD. The $tr(V)$ and the $log|V|$ in the rK4D (and in the rGEVD) decrease monotonically as $r$ increases. The biggest decreases in these values occur at the change from $r = 1$ to $r = 2$. That is, the variance decreases relatively a lot while the bias is not much increase, as $r$ changes from 1 to 2. This observation leads to the interpretation that the biggest benefit of employing the rK4D over the K4D is obtained at $r = 2$, for this data.

Figure 3 shows quantile-per-quantile plots obtained from the largest ($s = 1$) order statistics for the rK4D fit (red points) and for the rGEV fit (blue points) to Venice sea-level data with several values of $r$. In this figure, one can see that the rK4D fits the data better than the rGEVD. We thus infer the rK4D provides less biased predictions than the rGEVD, because the rK4D with 4 parameters is more flexible than the rGEVD with 3 parameters.

Figure 4 shows graphs of the first marginal pdfs from the rK4D fit to Venice sea-level data with $r = 1, 2, 3, 4$. We can see from this figure that the first marginal pdf changes a little as $r$ increases. Figure 5 shows graphs of the $s$-th marginal pdfs for $s = 1, 2, 3, 4$ from the rK4D fit with $r = 4$. The dot plot of the $s$-th order statistics are provided at bottom. We just see the
Figure 3: Quantile-per-quantile plots obtained from the largest order statistics for the rK4D fit and for the rGEV fit to Venice sea-level data with several values of r.

shapes of the s-th marginal pdfs and their differences as s changes.

5 More r-largest distributions

We can define some new r-largest distributions as special cases of the rK4D. When \( h \to -1 \), the pdf of r-largest generalized logistic distribution (rGLD) is obtained as follows:

\[
\lim_{h \to -1} f_4(x^r) = \sigma^{-r} C_r(-1) \times g(x^r) \times \left\{ 1 + h \, w(x^r)^{\frac{1}{r}} \right\}^{-(1+r)},
\]

(28)

where \( C_r(-1) \) is same as \( C_r \) defined in [7] with \( h = -1 \), \( g(x^r) \) is defined as in [8], and \( w(x) \) is defined as in [4]. This is a r-largest extension from a generalized logistic distribution (Ahmad et al. 1988; Hosking and Wallis 1997).

When \( h \to -1 \) and \( k \to 0 \), the pdf of r-largest logistic distribution (rLD) is obtained as a
Figure 4: Graphs of the first marginal probability density functions (pdf) from the rK4D fit to Venice sea-level data with $r = 1, 2, 3, 4$. The 20-year return levels obtained from each pdf are marked at the right bottom with notation $r_{20}^{(r)}$.

A special case of the rGLD as follows:

$$
\lim_{(h, k) \to (-1, 0)} f_4(x^r) = \sigma^{-r} C_r(-1) \times \left\{1 + \exp[-\alpha(x^r)]\right\}^{-(1+r)} \times \prod_{j=1}^{r} \exp[-\alpha(x^{(j)})],
$$

where $\alpha(x) = (x - \mu)/\sigma$.

When $k \to 0$, the pdf of $r$-largest generalized Gumbel distribution (rGGD) is obtained as follows:

$$
\lim_{k \to 0} f_4(x^r) = \sigma^{-r} C_r \times \left\{1 - h \exp[-\alpha(x^r)]\right\}^{\frac{1-rh}{k}} \times \prod_{j=1}^{r} \exp[-\alpha(x^{(j)})].
$$

This is a $r$-largest extension from a generalized Gumbel distribution (Jeong et al. 2014).

Since the support condition ($C_r > 0$) of the rK4D is same to $h < 1/(r - 1)$ for $r \geq 2$, the rK4D does not include the $r$-largest extensions of the generalized Pareto and the exponential distributions (the K4D with $h = 1$), except for the case $r = 1$. Figure 6 shows relationship of the rK4D to other $r$-largest distributions. These three parameter $r$-largest distributions derived from the rK4D may serve to model the extreme values with comparable performance to the rGEVD.
Figure 5: Graphs of the $s$-th marginal probability density functions (pdf) from the rK4D fit to Venice sea-level data with $r = 4$, where $s = 1, 2, 3, 4$.

Figure 6: Relationship of the $r$-largest four parameter kappa distribution (rK4D) to other $r$-largest distributions.

6 Conclusion and discussion

In this study, we introduced the $r$-largest four parameter kappa distribution (rK4D). The joint pdf, marginal and conditional distributions of the rK4D are derived. Application to Venice
sea-level data is presented with comparison to the $r$-largest GEVD. This study illustrates that the rK4D gives better fitting or less biases but larger variances of the parameter estimates than the rGEVD. Some new $r$-largest distributions are also derived as special cases of the rK4D.

The pdf definition of the rk4d may not be unique, because it is not a result from any theoretical derivation but just an analogous extension from the K4D and the rGEVD. A point process approach for extremes (Smith 1989; Coles 2001) may provide a theoretical insight.

The standard error (SE) of the MLE of the rK4D parameters decrease in general as $r$ increase, but the SE of the return level does not show a monotonic reduction trend. When sample size is large such as more than 100, it seems that the variance reduction effect by the addition of the $r$-largest to the first maximum is small, based on our experience. This is maybe because the GEVD or the K4D already estimate the return levels well with large sample, so may not really need to add the $r$-largest observations. We need more study on this matter. Moreover, the variances of the return levels in the rK4D are greater than those in the rGEVD. Some techniques to reduce the variance of the return level of the rK4D are anticipated in the future work. Since the rK4D generally will result in less bias than the rGEVD, one can consider the mean squared error (MSE) criterion for selecting better model between the rGEVD and the rK4D. Moreover, one may choose the best model among the $r$-largest Gumbel distribution, the rGEVD, and the rK4D based on some selection criteria such as Akaike information criterion, Bayesian information criterion, and cross-validation based MSE.

The use of the $r$-largest values as extremes enhances the power of estimation for moderate values of $r$, but the use of larger values of $r$ may lead to bias in the estimation (Zhang et al. 2004). The selection of $r$ is thus important in the rGEVD or in the rK4D. Bader et al.(2017) developed automated methods of selecting $r$ from the rGEVD. Their approach with modifications may be applied to the rK4D even though we did not try it in this study. Nonstationary modeling in the rK4D is another future work.

Making effective use of the available information is important in extremes, because extreme values are scarce. Thus, the use of an $r$-largest method is encouraged (Zhang et al. 2004). The rK4D, as an extention of the rGEVD, can serve to model the $r$-largest observations flexibly with less bias than the rGEVD, specially when three parameters in the rGEVD are not enough to capture the variability of observations well. Even though there are defects such as larger estimation variance in the rK4D compared to the rGEVD, the introduction of the rK4D will enrich and improve the modelling methodology for extreme events.
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