A Tale of Three Equations: Breit, Eddington-Gaunt, and Two-Body Dirac
Peter Van Alstine
12474 Sunny Glenn Drive, Moorpark, Ca. 90125
Horace W. Crater
The University of Tennessee Space Institute
Tullahoma, Tennessee 37388

G. Breit’s original paper of 1929 postulates the Breit equation as a correction to an earlier defective equation due to Eddington and Gaunt, containing a form of interaction suggested by Heisenberg and Pauli. We observe that manifestly covariant electromagnetic Two-Body Dirac equations previously obtained by us in the framework of Relativistic Constraint Mechanics reproduce the spectral results of the Breit equation but through an interaction structure that contains that of Eddington and Gaunt. By repeating for our equation the analysis that Breit used to demonstrate the superiority of his equation to that of Eddington and Gaunt, we show that the historically unfamiliar interaction structures of Two-Body Dirac equations (in Breit-like form) are just what is needed to correct the covariant Eddington Gaunt equation without resorting to Breit’s version of retardation.
I. INTRODUCTION

Three score and seven years ago, Gregory Breit extended Dirac’s spin-1/2 wave equation to a system of two charged particles [1]. He formed his equation by summing two free-particle Dirac Hamiltonians with an interaction obtained by substituting Dirac $\alpha$’s for velocities in the semi-relativistic electrodynamic interaction of Darwin:

$$E\Psi = \left\{ \vec{\alpha}_1 \cdot \vec{p}_1 + \beta_1 m_1 + \vec{\alpha}_2 \cdot \vec{p}_2 + \beta_2 m_2 - \frac{\alpha}{r}\left[ 1 - \frac{1}{2}(\vec{\alpha}_1 \cdot \vec{\alpha}_2 + \vec{\alpha}_1 \cdot \vec{r} \vec{\alpha}_2 \cdot \vec{r}) \right] \right\}\Psi. \quad (1.1)$$

Although successful for Breit’s purpose - perturbative calculation of the electromagnetic bound-state spectrum of multi-electron atoms, Breit’s equation turned out to be neither relativistic nor a well-defined wave equation. Nonetheless, ever since, people have applied Breit’s equation in nuclear and particle physics in situations far from its electrodynamic origin.

In the last 20 years, some of us [2-8] have tried to remedy this by constructing new fully covariant multiparticle relativistic quantum descriptions. L. Horwitz has had a hand in this effort as participant and critic. As he and F. Rohrlich pointed out in 1981 [9], for such descriptions, the two-body problem is essentially simpler than the generic n-body problem - reducing (just as in non-relativistic physics) to an effective one-body problem (most simply described in terms of effective variables introduced earlier by I.T. Todorov [10]). For most of these 20 years the authors of this paper have concentrated their efforts on using the special two-body equations to find sensible versions of them for interactions actually found in nature as opposed to “toys” like relativistic rotors and harmonic oscillators. We have applied them (with surprising success) to the phenomenological calculation of the $q\bar{q}$ meson bound state spectrum [11,12].

But, of more fundamental but related importance, we have demonstrated that our equations reproduce the spectral consequences of the original work of Breit but in a surprising way: whereas Breit’s equation yielded these results only through first-order perturbation theory, our “Two-Body Dirac equations” yield the same results through non-perturbative solution of a fully-relativistic quantum wave-equation [13,14]. For two relativistic spin-one-half particles interacting electromagnetically, our equations are given as a pair of compatible Dirac equations on a single 16-component wave-function:

$$D_1 \psi = (\pi_1 \cdot \gamma_1 + m_1)\psi = 0 \quad (1.2)$$
$$D_2 \psi = (\pi_2 \cdot \gamma_2 + m_2)\psi = 0, \quad (1.3)$$

in which

$$\pi_i \equiv p_i - A_i, \quad i = 1, 2$$

and

$$A_1 = [1 - \frac{1}{2}(G + G^{-1})]p_1 + \frac{1}{2}(G - G^{-1})p_2 - \frac{i}{2}(\partial G \cdot \gamma_2)\gamma_2$$
$$A_2 = [1 - \frac{1}{2}(G + G^{-1})]p_2 + \frac{1}{2}(G - G^{-1})p_1 + \frac{i}{2}(\partial G \cdot \gamma_1)\gamma_1$$

where $G^2 = (1 - 2A/w)^{-1}$. $w$ is the total c.m. energy, while the world scalar potential $A$ is a function of the covariant spacelike particle separation

$$x^\mu_1 = x_\mu + \vec{P} \cdot (\hat{P} \cdot x) \quad (1.4)$$

which is perpendicular to the total four-momentum, $P$. For $O(\alpha^4)$ electrodynamics, $A = A(x_\perp) = -\alpha/r$ in which $r = \sqrt{x^2_1}$. In these coupled equations, the subscript $i = 1, 2$ stands for the $ith$ particle so that $m_1$, and $m_2$ are the masses of the interacting fermions. The potentials $A_i^\mu$ in this form of the Two-Body Dirac equations appear through minimal interaction substitutions on the particle four-momenta. They introduce mutual interactions as though each particle were in an external potential produced by the other. (Hence, we refer to these forms of the Two-Body Dirac equations as the “external potential forms” or the “minimal interaction forms”.) The specific forms of the covariant spin-dependent terms in the interactions as well as the dependence of $A$ on $x_\perp$ are consequences of the necessary compatibility of the two Dirac equations

$$[D_1, D_2]\psi = 0. \quad (1.5)$$

(In most of our previous works, we have set out these equations and their compatibility in terms of $S$ operators that result from the $D$‘s through multiplication by $\gamma_5$‘s.)
Because we were originally able to obtain their versions for world scalar interaction by rigorously “taking the operator square roots” of two Klein-Gordon equations, to us they seem the natural two-body extensions of Dirac’s one-body equation hence deserving the name “Two-Body Dirac equations”. For the electromagnetic interaction of Eqs.(1.2-1.3), these equations possess a family of exact solutions for para-positronium with correct spectrum through $O(\alpha^4)$[13]. Moreover, they have not only been derived from quantum field theory (through a relativistic Lippmann-Schwinger equation) as the ”quantum-mechanical transform” of the Bethe-Salpeter equation [15,16], but their spin-independent structures have been obtained as well from classical relativistic field theory through Wheeler-Feynman electrodynamics. Most recently, Jallouli and Sadzjijn have completed work begun for spinless charged particles by Todorov [10] more than 20 years ago, by obtaining our spin-one-half electrodynamic equations from the quantum-field-theoretic eikonal approximation [17].

For all their successes, people have been slow to adopt the Two-Body Dirac equations, preferring to stick with methods that on the one hand artificially truncate the fully-relativistic Bethe-Salpeter equation (and thereby produce well-known pathologies) or on the other abuse the Breit equation by modifying its interaction structure in hopes that its nonperturbative problems will heal themselves. The perturbative success of Breit’s equation in electrodynamics becomes a psychological barrier to use of the fully-relativistic nonperturbative equation.

II. HISTORY

The historical origins of this psychological effect are intriguing. To investigate them, we begin by noting that because the Two-Body Dirac equations are a compatible pair, they can be rearranged in a large number of equivalent forms. Among these is a “Breit-form” which in the c.m. rest-frame reduces to the sum of two free Dirac Hamiltonian operators plus a covariant interaction. We set out this Breit Form in a previous volume of Foundations of Physics (in honor of F. Rohrlich)[12]. (The special properties that allow this equation to function as a non-singular version of the Breit equation were recently investigated by one of us in [18].) However, we have come to realize that the particular form of this interaction is part of an interesting controversy in the history of two-body equations.

We decided to find out just what prompted Breit to formulate his famous equation. Reading his paper of 1929, we saw that his work was actually proposed as the correction of an older “defective” equation called by him “the Eddington-Gaunt equation”[19]

$$E\Psi = [\hat{\alpha}_1 \cdot \hat{p}_1 + \beta_1 m_1 + \hat{\alpha}_2 \cdot \hat{p}_2 + \beta_2 m_2 - \frac{\alpha}{\mu} (1 - \hat{\alpha}_1 \cdot \hat{\alpha}_2)]\Psi,$$

(2.1)

with an interaction structure first suggested by Heisenberg and Pauli (according to Breit). We were startled to see that this equation was one already familiar to us as the lowest-order approximation to ours (referred to by us in [12] as “the familiar form for four-vector interactions without the Darwin piece”). In fact, in "Breit-like" form, our equation reads [12]

$$w\Psi = [\hat{\alpha}_1 \cdot \hat{p}_1 + \beta_1 m_1 + \hat{\alpha}_2 \cdot \hat{p}_2 + \beta_2 m_2 + w(1 - \exp[-G(x_{\perp})(1 - \hat{\alpha}_1 \cdot \hat{\alpha}_2)])]\Psi,$$

(2.2)

in which $G = \ln|G|$.

Since our covariant equation automatically has correct retardation, apparently we have discovered in Eq.(2.2) a (higher order) corrected form of the (Heisenberg-Pauli) "Eddington-Gaunt" equation that has been "fixed" by a mechanism different from Breit’s!

What’s going on here? In fact closer reading of Breit and Eddington reveals a fundamental reason for all this. Breit judged the Eddington-Gaunt equation defective because of its failure after rearrangement as an equation for the “large components” of the wave function to reproduce the spinless Darwin interaction:

$$-\frac{\alpha}{\mu} [1 - \frac{1}{2}(\hat{v}_1 \cdot \hat{v}_2 + \hat{v}_1 \cdot \hat{r} \hat{v}_2 \cdot \hat{r})]$$

(2.3)

and its consequent failure to yield a spectrum for He in agreement with experiment. But we recall that the Darwin interaction [20] is straightforwardly obtained by breaking the manifest covariance of $\int \int J_1 G J_2 = \int dr_1 \int dr_2 \hat{j}_1 \hat{\chi}_{26}^2 G[(x_1 - x_2)^2]$ in Wheeler-Feynman electrodynamics [21] by expanding the (1/2 advanced + 1/2 retarded) Green’s function about the instantaneous limit. An integration by parts in the time leads to

$$-\frac{\alpha}{\mu} [(1 - \hat{v}_1 \cdot \hat{v}_2) + \frac{1}{2} \hat{v}_1 \cdot (1 - \hat{r} \hat{r} \cdot \hat{v}_2)]$$

(2.4)

with the 1/2 in the second ”retardative term” coming from a Taylor expansion of the Green’s function [22]. By redoing the Darwin interaction in terms of the $\alpha$’s, Breit was able to restore the missing terms. On the other hand, if we read
Eddington’s notorious paper [19](and ignore his musings about the value of the fine-structure constant $\alpha$), we find that he arrived at the $-\frac{2}{r^2}(1 - \vec{a}_1 \cdot \vec{a}_2)$ interaction by performing the covariant four-velocity goes to $(1, \vec{a})$ substitution on the manifestly covariant JGJ coupling (that later became the heart of Wheeler-Feynman electrodynamics) in the form of $\dot{x}_1^\mu / \dot{x}_2^\mu$. In fact, this covariant form appears explicitly in our Eq.(2.2) where it is inherited (but correctly) directly from classical field theory (if we start from the Wheeler-Feynman approach [23]) or from quantum field theory in the form of the Bethe-Salpeter kernel in covariant Feynman Gauge [14]. This structure is a fundamental feature of the manifestly covariant approach! By comparing Eddington and Gaunt’s Eq.(2.1) with our Eq.(2.2), we see immediately that where they went wrong was to stop at lowest order, thereby missing an all-important retardative recoil term!

In detail, in Eq.(2.2), if we use the $\gamma$ matrix algebra to compute the exponential of matrix form to all orders, we find that

$$w(1 - \exp[-G(\vec{x}_\perp)(1 - \vec{a}_1 \cdot \vec{a}_2)])$$

$$= \frac{w}{4}[1(3ch[\vec{G}] + ch[3\vec{G}]) + \gamma_{51}\gamma_{52}(3sh[\vec{G}] - sh[3\vec{G}])]$$

$$+ \vec{a}_1 \cdot \vec{a}_2 (sh[\vec{G}] + sh[3\vec{G}]) + \vec{a}_1 \cdot \vec{a}_2 (ch[\vec{G}] - ch[3\vec{G}])],$$

a corrected form of Eq.(69) in [12]. If we eliminate $G$ in terms of the potential $A = -\frac{\vec{F}}{r}$, we find the striking result that to all orders in the potential:

$$w(1 - \exp[-G(\vec{x}_\perp)(1 - \vec{a}_1 \cdot \vec{a}_2)])$$

$$= A(1 - \vec{a}_1 \cdot \vec{a}_2) - \frac{A^2}{w}(1 - \vec{a}_1 \cdot \vec{a}_2) - (1 - \gamma_{51}\gamma_{52} + \vec{a}_2 \cdot \vec{a}_2) \frac{A^3}{w^2 (1 - 2\frac{A}{w})}.$$ 

From this, we immediately see that the perturbative dynamics (through $O(\alpha^4)$) will be given by

$$A(1 - \vec{a}_1 \cdot \vec{a}_2) - \frac{A^2}{w}(1 - \vec{a}_1 \cdot \vec{a}_2).$$

(2.7)

Apparently, as a lowest-order perturbation, the second term of this must be dynamically equivalent to the term

$$-\frac{\alpha}{r^2}(1 - \vec{r} \cdot \vec{r}) \cdot \vec{a}_2$$

which when added to

$$-\frac{\alpha}{r}(1 - \vec{a}_1 \cdot \vec{a}_2)$$

makes Breit’s famous

$$-\frac{\alpha}{r}[1 - \frac{1}{2}(\vec{a}_1 \cdot \vec{a}_2 + \vec{a}_1 \cdot \vec{r} \vec{a}_2 \cdot \vec{r})].$$

III. EQUIVALENCE?

Not only must these terms be dynamically equivalent, but somehow our equation must have restored the content of the Darwin interaction that Breit discovered was spoiled by the Eddington-Gaunt equation. Breit uncovered this defect by performing a perturbative (in orders of $1/c^2$) reduction to an equation on the upper-upper component of the 16-component wave function for both his equation and the Eddington-Gaunt equation. This rearranges the Breit equation as a set of corrections to the non-relativistic Schrödinger equation with Coulomb potential. Expressed in the c.m. frame this rearrangement becomes

$$H\psi = w\psi$$

(3.1)

in which $w$ is the total c.m. energy and

$$H = (m_1 + \frac{\vec{p}_1^2}{2m_1} - \frac{(\vec{p}_1^2)^2}{8m_1^3}) + (m_2 + \frac{\vec{p}_2^2}{2m_2} - \frac{(\vec{p}_2^2)^2}{8m_2^3}) +$$
contributing only to the ground state). Together they are equivalent to the first term of the third line of Eq. (3.2).

We have a situation in which some of the original investigators wrote down an interaction that was covariant (including Eddington’s realization that the interparticle separation \( r \) should be something like \( \sqrt{x^2 + y^2} \) [19]) but incomplete, while another (Breit) replaced it with a semi-relativistic approximation to an interaction that was correct.

\[
\begin{align*}
-\alpha((1 - \frac{\vec{p}^2}{m_1 m_2}) \frac{1}{r} - \frac{1}{2m_2 m_2 r} \vec{p} \cdot (1 - \hat{r} \vec{r}) \cdot \vec{p}_{\text{ordered}}
- \frac{1}{2} \left( \frac{1}{m_1^2} + \frac{1}{m_2^2} \right) \delta(\vec{r}) - \frac{1}{4} \vec{L} \cdot \left[ \left( \frac{1}{m_1^2} + \frac{2}{m_1 m_2} \right) \vec{\sigma}_1 + \left( \frac{1}{m_2^2} + \frac{2}{m_1 m_2} \right) \vec{\sigma}_2 \right] \\
+ \frac{1}{4m_1 m_2} \left( -\frac{8\pi}{3} \vec{\sigma}_1 \cdot \vec{\sigma}_2 \delta(\vec{r}) + \frac{\vec{\sigma}_1 \cdot \vec{\sigma}_2}{r^3} - \frac{3\vec{\sigma}_1 \cdot \hat{r} \vec{\sigma}_2 \cdot \vec{r}}{r^5} \right). 
\end{align*}
\] (3.2)

To see what new features are contained in our truncated interaction Eq. (2.7), we perform the same type of semirelativistic reduction on our equation (see Appendix). We obtain a seemingly different Hamiltonian:

\[
H = (m_1 + \frac{\vec{p}^2}{2m_1} + (\vec{p}^2)^2) + (m_2 + \frac{\vec{p}^2}{2m_2} + (\vec{p}^2)^2) +
-\alpha\left( -\frac{1}{2m_1 m_2} \left\{ \frac{\vec{p}, 1}{r} \right\} - \frac{1}{2(m_1 + m_2)} \alpha/r^2 + \frac{\pi}{m_1 m_2} \delta(\vec{r}) \\
- \left\{ \frac{1}{4} \left( \frac{1}{m_1^2} + \frac{1}{m_2^2} \right) \left( \frac{i}{r^3} \frac{1}{r^3} + \delta(\vec{r}) \right) \\
- \frac{1}{4} \vec{L} \cdot \left[ \left( \frac{1}{m_1^2} + \frac{2}{m_1 m_2} \right) \vec{\sigma}_1 + \left( \frac{1}{m_2^2} + \frac{2}{m_1 m_2} \right) \vec{\sigma}_2 \right] \\
+ \frac{1}{4m_1 m_2} \left( -\frac{8\pi}{3} \vec{\sigma}_1 \cdot \vec{\sigma}_2 \delta(\vec{r}) + \frac{\vec{\sigma}_1 \cdot \vec{\sigma}_2}{r^3} - \frac{3\vec{\sigma}_1 \cdot \hat{r} \vec{\sigma}_2 \cdot \vec{r}}{r^5} \right). 
\right) 
\] (3.3)

But, note that the \( \vec{r} \cdot \vec{p} \) term and \( \delta(\vec{r}) \) term in the third line of Eq. (3.3) give equivalent expectation values (both contributing only to the ground state). Together they are equivalent to the first term of the third line of Eq. (3.2).

Now, we come to the feature that led Breit to discard the defective Eddington Gaunt equation in favor of his new equation: reproduction of the correct Darwin interaction. Here this issue boils down to the equivalence (or lack of equivalence) of the second lines of Eq. (3.3) and Eq. (3.2). As luck would have it, Schwinger [24] has shown us just how to do this. In order to use the virial theorem to evaluate expectation values in his treatment of the positronium spectrum, Schwinger introduced a canonical transformation that turns the expectation value of the second line of the reduced Breit equation Eq. (3.2) into precisely the expectation value of the set of terms appearing as the second line in the reduced version Eq. (3.3) of our equation. In fact, we found some years ago that this transformation may be used to derive the relativistic Todorov equation of electrodynamics from classical electromagnetic field theory [25]. This takes care of the important spin-independent Darwin interaction that served as Breit’s criterion for rejecting the Eddington Gaunt equation. But, what happened to the intriguing spin-structure of the new term? If we look into the details of our approximate reduction of our 16-component equation to a single 4-component equation on the upper-upper wave function (outlined in Appendix A), we find that the spin-spin term in the equation for the upper-upper component is exactly cancelled by a spin-spin term resulting from its coupling to the lower-lower component. Cancellations and simplifications of other spin dependences then yield the Fermi-Breit form of our equation - Eq. (3.3). This establishes that the unfamiliar interaction of our Eq. (2.7) is perturbatively quantum-mechanically equivalent to that of Breit thereby providing a corrected form of the historical Eddington Gaunt equation. The lesson here is that manifestly covariant techniques yield manifestly covariant equations that produce correct spectral results if they contain the correct interactions. We have a situation in which some of the original investigators wrote down an interaction that was covariant (including Eddington’s realization that the interparticle separation \( r \) should be something like \( \sqrt{x^2 + y^2} \) [19]) but incomplete, while another (Breit) replaced it with a semi-relativistic approximation to an interaction that was correct.
Appendix A - Semirelativistic reduction of Eq.(2.2) with approximate interaction Eq.(2.7)

We write the sixteen component Dirac spinor as

\[ \psi = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{bmatrix} \] (A.1)

in which the \( \psi_i \) are four component spinors. All the matrices that operate on this spinor are sixteen by sixteen. In the standard Dirac representation (the subscripts on the identity 1 give the dimensionality of the unit matrix)

\[ \beta_1 = \begin{pmatrix} 1_8 & 0 \\ 0 & -1_8 \end{pmatrix}, \quad \gamma_{51} = \begin{pmatrix} 0 & 1_8 \\ 1_8 & 0 \end{pmatrix}, \quad \beta_1 \gamma_{51} \equiv \rho_1 = \begin{pmatrix} 0 & 1_8 \\ -1_8 & 0 \end{pmatrix} \] (A.2)

\[ \beta_2 = \begin{pmatrix} \beta & 0 \\ 0 & \beta \end{pmatrix}, \quad \beta = \begin{pmatrix} 1_4 & 0 \\ 0 & -1_4 \end{pmatrix} \] (A.3)

\[ \gamma_{52} = \begin{pmatrix} \gamma_5 & 0 \\ 0 & \gamma_5 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 0 & 1_4 \\ 1_4 & 0 \end{pmatrix} \] (A.4)

\[ \beta_2 \gamma_{52} \equiv \rho_2 = \begin{pmatrix} \rho & 0 \\ 0 & \rho \end{pmatrix}, \quad \rho = \begin{pmatrix} 0 & 1_4 \\ -1_4 & 0 \end{pmatrix} \] (A.5)

\[ \beta_1 \gamma_{51} \gamma_{52} = \begin{pmatrix} 0 & \gamma_5 \\ -\gamma_5 & 0 \end{pmatrix} \] (A.6)

\[ \beta_2 \gamma_{52} \gamma_{51} = \begin{pmatrix} 0 & \rho \\ \rho & 0 \end{pmatrix} \] (A.7)

Using \( \vec{\alpha}_i = \gamma_5 \vec{\sigma}_i \), our truncated Breit equation becomes

\[ [w - m_1 - m_2 + \frac{\alpha}{r} + \frac{\alpha^2}{wr^2} (1 - \vec{\sigma}_1 \cdot \vec{\sigma}_2)] \psi_1 
+ \vec{p} \cdot \vec{\sigma}_2 \psi_2 - \vec{p} \cdot \vec{\sigma}_1 \psi_3 - \frac{\alpha}{r} \vec{\sigma}_1 \cdot \vec{\sigma}_2 \psi_4 = 0 \] (A.8)

\[ [w - m_1 + m_2 + \frac{\alpha}{r} + \frac{\alpha^2}{wr^2} (1 - \vec{\sigma}_1 \cdot \vec{\sigma}_2)] \psi_2 
+ \vec{p} \cdot \vec{\sigma}_2 \psi_1 - \vec{p} \cdot \vec{\sigma}_1 \psi_3 - \frac{\alpha}{r} \vec{\sigma}_1 \cdot \vec{\sigma}_2 \psi_4 = 0 \] (A.9)

\[ [w + m_1 - m_2 + \frac{\alpha}{r} + \frac{\alpha^2}{wr^2} (1 - \vec{\sigma}_1 \cdot \vec{\sigma}_2)] \psi_3 
+ \vec{p} \cdot \vec{\sigma}_2 \psi_4 - \vec{p} \cdot \vec{\sigma}_1 \psi_1 - \frac{\alpha}{r} \vec{\sigma}_1 \cdot \vec{\sigma}_2 \psi_2 = 0 \] (A.10)

\[ [w + m_1 + m_2 + \frac{\alpha}{r} + \frac{\alpha^2}{wr^2} (1 - \vec{\sigma}_1 \cdot \vec{\sigma}_2)] \psi_4 
+ \vec{p} \cdot \vec{\sigma}_2 \psi_3 - \vec{p} \cdot \vec{\sigma}_1 \psi_2 - \frac{\alpha}{r} \vec{\sigma}_1 \cdot \vec{\sigma}_2 \psi_1 = 0 \] (A.11)
By using Coulomb variables in which $\vec{p} \sim \alpha$ and $1/r \sim \alpha$ we can obtain an expansion involving just the upper-upper component wave function $\psi_1$. The expansion we desire is one through order $\alpha^4$. To obtain this requires successive substitutions into Eq.(A.8) of expressions for the lower component wave functions given in Eqs.(A.9-11) through appropriate orders. From Eq.(A.11) we obtain

$$\psi_4 = \frac{1}{w + m_1 + m_2 + \alpha/r}[(\alpha/r)\vec{\sigma}_1 \cdot \vec{\sigma}_2 - \vec{p} \cdot \vec{\sigma}_2 \psi_3 + \vec{p} \cdot \vec{\sigma}_1 \psi_2]$$ \hspace{1cm} (A.12)

Note that we have ignored the $\alpha^2$ spin-dependent term in the denominator since in the substitution into Eq.(A.8) it would have produced terms that are higher order than $\alpha^4$. We solve Eqs.(A.9-10) for $\psi_2$ and $\psi_3$ to obtain

$$\psi_2 = \frac{1}{(w + m_1 - m_2 + \alpha/r)(w - m_1 + m_2 + \alpha/r)} \times [(w + m_1 - m_2 + \alpha/r)(\vec{p} \cdot \vec{\sigma}_1 + \vec{p} \cdot \vec{\sigma}_2 \psi_1) + (\alpha/r)\vec{\sigma}_1 \cdot \vec{\sigma}_2 \psi_2]$$ \hspace{1cm} (A.13)

$$\psi_3 = \frac{1}{(w + m_1 - m_2 + \alpha/r)(w - m_1 + m_2 + \alpha/r)} \times [(w - m_1 + m_2 + \alpha/r)(\vec{p} \cdot \vec{\sigma}_1 - \vec{p} \cdot \vec{\sigma}_2 \psi_1) + (\alpha/r)\vec{\sigma}_1 \cdot \vec{\sigma}_2 \psi_1]$$ \hspace{1cm} (A.14)

In Eq.(A.8) we need $\psi_4$ through order $\alpha^2$ and thus in Eq.(A.12) we need $\psi_2$ and $\psi_3$ only through order $\alpha$. This in turn implies that we can drop the $\psi_4$ and interaction terms in Eqs.(A.13-14). Performing this substitution yields the following expression for $\psi_4$ of appropriate order for use in Eq.(A.8):

$$\psi_4 = \frac{1}{2(m_1 + m_2)}[(\alpha/r)\vec{\sigma}_1 \cdot \vec{\sigma}_2 - \frac{m_1 + m_2}{2m_1 m_2} \vec{p} \cdot \vec{\sigma}_1 \psi_1]$$ \hspace{1cm} (A.15)

For the direct $\psi_2$ and $\psi_3$ terms in Eq.(A.8) we need Eq.(A.13) and Eq.(A.14) through order $\alpha^3$, which in turn requires Eq.(A.15). This leads to

$$\psi_2 = -\frac{1}{w - m_1 + m_2 + \alpha/r} [\vec{p} \cdot \vec{\sigma}_2 - \frac{\vec{p} \cdot \vec{\sigma}_1}{2(m_1 + m_2)}(\frac{\alpha}{r} \vec{\sigma}_1 \cdot \vec{\sigma}_2 - \vec{p} \cdot \vec{\sigma}_1 \vec{p} \cdot \vec{\sigma}_2 \frac{m_1 + m_2}{2m_1 m_2}] \psi_1$$

$$+ (\frac{\alpha}{r}) \frac{\vec{\sigma}_1 \cdot \vec{\sigma}_2 \vec{p} \cdot \vec{\sigma}_1 \psi_1}{4m_1 m_2}$$ \hspace{1cm} (A.16)

$$\psi_3 = \frac{1}{w + m_1 - m_2 + \alpha/r} [\vec{p} \cdot \vec{\sigma}_1 - \frac{\vec{p} \cdot \vec{\sigma}_2}{2(m_1 + m_2)}(\frac{\alpha}{r} \vec{\sigma}_1 \cdot \vec{\sigma}_2 - \vec{p} \cdot \vec{\sigma}_1 \vec{p} \cdot \vec{\sigma}_2 \frac{m_1 + m_2}{2m_1 m_2}] \psi_1$$

$$- (\frac{\alpha}{r}) \frac{\vec{\sigma}_1 \cdot \vec{\sigma}_2 \vec{p} \cdot \vec{\sigma}_1 \psi_1}{4m_1 m_2}$$ \hspace{1cm} (A.17)

Combining all terms of Eq.(A.8) and replacing $w$ by $m_1 + m_2$ in the $\alpha^4$ terms we obtain

$$w \psi_1 = [m_1 + m_2 - \frac{\alpha^2}{r^2(m_1 + m_2)}(1 - \vec{\sigma}_1 \cdot \vec{\sigma}_2)] \psi_1$$

$$+ \frac{\vec{p} \cdot \vec{\sigma}_2}{w - m_1 + m_2 + \alpha/r} [\vec{p} \cdot \vec{\sigma}_2 - \frac{\vec{p} \cdot \vec{\sigma}_1}{2(m_1 + m_2)}(\frac{\alpha}{r} \vec{\sigma}_1 \cdot \vec{\sigma}_2 - \vec{p} \cdot \vec{\sigma}_1 \vec{p} \cdot \vec{\sigma}_2 \frac{m_1 + m_2}{2m_1 m_2}] \psi_1$$

$$+ \frac{\vec{p} \cdot \vec{\sigma}_1}{w + m_1 - m_2 + \alpha/r} [\vec{p} \cdot \vec{\sigma}_1 - \frac{\vec{p} \cdot \vec{\sigma}_2}{2(m_1 + m_2)}(\frac{\alpha}{r} \vec{\sigma}_1 \cdot \vec{\sigma}_2 - \vec{p} \cdot \vec{\sigma}_1 \vec{p} \cdot \vec{\sigma}_2 \frac{m_1 + m_2}{2m_1 m_2}] \psi_1$$

$$- \vec{p} \cdot \vec{\sigma}_2 \frac{\alpha}{4m_1 m_2} \vec{\sigma}_1 \cdot \vec{\sigma}_2 \vec{p} \cdot \vec{\sigma}_1 \psi_1 - \vec{p} \cdot \vec{\sigma}_1 \frac{\alpha}{4m_1 m_2} \vec{\sigma}_2 \vec{p} \cdot \vec{\sigma}_2 \psi_1$$
\[ + \frac{\alpha}{r} \vec{d}_1 \cdot \vec{d}_2 \left[ \frac{\alpha}{2(m_1 + m_2)r} \vec{d}_1 \cdot \vec{d}_2 - \frac{1}{4m_1 m_2} \vec{p} \cdot \vec{d}_1 \vec{p} \cdot \vec{d}_2 \right] \psi_1 \quad (A.18) \]

We omit most of the steps of the remaining reduction, involving powers of Pauli matrices, commenting on just two portions of the details. The first is that the spin-spin term in the first line cancels with the corresponding spin-spin term in the last line that results from the identity \((\vec{d}_1 \cdot \vec{d}_2)^2 = 3 - 2\vec{d}_1 \cdot \vec{d}_2\). The second is that the semirelativistic \(\vec{p}\) kinetic corrections not only result from terms at the end of the second and third lines that involve four \(\vec{p} \cdot \vec{d}_i\) factors but also from the terms at the beginning of each of those lines, in particular

\[ \left( \frac{1}{w - m_1 + m_2 + \alpha/r} \vec{p} \cdot \vec{d}_2 + \frac{1}{w - m_1 - m_2 + \alpha/r} \vec{p} \cdot \vec{d}_1 \right) \psi. \quad (A.19) \]

One brings the denominator through and operates on \(\psi_1\) using the lower order portions of wave equation Eq.(A.18). When we perform this operation and various other simplifications we obtain the semirelativistic reduction given in Eq.(3.3) in the text of the paper.

[1] G. Breit, Phys. Rev. 34, 553 (1929).
[2] M. Kalb and P. Van Alstine, Yale Reports, C00-3075-146 (1976), C00-3075-156 (1976); P. Van Alstine, Ph.D. Dissertation Yale University, (1976).
[3] I. T. Todorov, Dubna Joint Institute for Nuclear Research No. E2-10175, 1976; Ann. Inst. H. Poincare A28 207 (1978).
[4] A. Komar, Phys. Rev. D 18, 1881,1887 (1978).
[5] P. Droz-Vincent Rep. Math. Phys.,8,79 (1975).
[6] P. Van Alstine and H.W.Crater, J. Math. Phys. 23,1697 (1982), H. W. Crater and P. Van Alstine, Ann. Phys. (N.Y.) 148, 57 (1983).
[7] H. Sazdjian, Phys. Rev. D1 33, 3401(1986).
[8] H. Sazdjian, Phys. Rev. D33, 3425, (1986).
[9] L.P. Horwitz and F. Rohrlich, Phys. Rev. D24, 1928 (1981).
[10] I. T. Todorov, in “Properties of the Fundamental Interactions,” ed. by A. Zichichi (Editrice Compositori, Bologna, 1973), vol. 9, part C, pp. 953-979; Phys. Rev. D3, 2351 (1971).
[11] H. W. Crater and P. Van Alstine, Phys. Rev. Lett. 53, 1577 (1984), Phys. Rev. D1 37, 1982 (1988).
[12] H. W. Crater and P. Van Alstine, Found. Of Phys. 24, 297 (1994). A preliminary form of this “Breit Form” of the Two-Body Dirac equations (equivalent to our present form in the c.m. rest-frame) was presented by us in a contributed talk at the Spring Meeting of the American Physical Society (1984).
[13] Peter Van Alstine and H. W. Crater, Phys. Rev. D34, 1932 (1986).
[14] H. W. Crater, R.L. Becker, C.Y. Wong, and P. Van Alstine, Phys. Rev. D46 5117, (1992).
[15] H. Sazdjian, Phys. Lett. 156B, 381 (1985); Extended Objects and Bound Systems, Proceedings of the Karuizawa International Symposium, 1992, eds. O. Hara, S. Ishida and S.Nake (World Scientific, Singapore, 1992), p 117.
[16] H.W. Crater and P. Van Alstine, Phys. Rev. D15 36, 3007 (1987).
[17] H.Jallouli and H. Sazdjian, Phys. Lett. B366,409 (1996);preprint IPNO/TH 96-01. hep-ph/9602241.
[18] H. W. Crater, C. W. Wong, and C. Y. Wong, Singularity-Free Breit Equation from Constraint Two-Body Dirac Equations International Journal of Modern Physics -E, in press.
[19] A. Eddington, Proc. Roy. Soc. A122, 358 (1929); J.A. Gaunt, Phil. Trans. Roy. Soc, Vol 228, 151, (1929);Proc. Roy. Soc. A122, 153 (1929).
[20] C.G. Darwin, Philos. Mag. 39,537 (1929).
[21] J.A. Wheeler, R.P. Feynman, Rev. of Modern Phys. 17,157 (1945), 21,425 (1949); H. Tetrode, Zeits. f.Physik, 10,317, (1922), A.D. Fokker, Zeits. f. Physik, 58,386, (1929), E.C.G. Sudarshan, N. Mukunda, Classical Dynamics: A Modern Perspective,John Wiley & Sons,(1974).
[22] P. Van Alstine and H. W. Crater, Phys. Rev. D 33, 1037 (1986).
[23] H. W. Crater and P. Van Alstine, Phys. Rev. D30, 2585 (1984).
[24] J. Schwinger, Particles, Sources, and Fields ,Addison-Wesley, Reading (1973), Vol. 2,pp.348-349. The canonical transformation is (see Ref. 25 below also) \(\vec{r} \rightarrow \vec{p} = (1 + \frac{2m_1 + m_2}{m_1 + m_2})\vec{r}\) by \(\vec{p} \rightarrow \vec{p} = \vec{p} + \frac{2m_1 + m_2}{m_1 + m_2} (\vec{r} \times \vec{L})\).
[25] H. W. Crater and P. Van Alstine, Phys. Rev. D46 476, (1992); H.W. Crater and D. Yang, J. Math. Phys. 32 2374, (1991).