Metric perturbations in a cosmological model with gravitational particle production

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Abstract

The present paper addresses the issue of cosmological metric perturbations in models of cosmology accompanied by gravitational particle production. The general structure of such theories is constructed and then the paper focuses on metric perturbations of the de Sitter space in the early Universe full of radiation, such spaces can occur in presence of gravitational particle production. The main focus of the calculations is on scalar metric perturbations. The paper briefly opines on vector and tensor perturbations of the de Sitter space in question. The results show that the metric perturbations can have finite instability in the short scale and these instabilities may produce the inhomogeneities in the early radiation dominated universe. The instabilities are in general finite and do not tend to blow up to infinity. The tensor perturbations do not show any instability in the present model.

1 Introduction

In warm inflation models [1, 2, 3, 4] the universe enters a de Sitter phase in presence of radiation. The de Sitter phase (strictly speaking a quasi de Sitter phase) is full of radiation arising from the decay of the scalar inflaton field. The inflaton has thermal fluctuations and these thermal fluctuations produce the density perturbations necessary for the matter dominated phase of the universe. In the present case we present a de Sitter space which is again full of radiation, but this radiation does not arise from the decay of any inflaton field. This radiation arises from the cosmological creation of massless fields in the de sitter phase. Gravitational particle creation in open systems was studied in detail in Ref. [5]. Parker’s original work [6] showed that it is impossible to cosmologically produce massless particles in a radiation dominated universe where the scale-factor of the universe $a(t) \propto t^{1/2}$. Later it has been shown that one can think of cosmological production of massless particles in a universe filled with radiation if $a(t)$ is not proportional to $t^{1/2}$. If the radiation filled universe is in a de Sitter phase then one can actually think of

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cosmological production of massless particles [7]. Authors have previously worked on de Sitter phase of the universe in presence of gravitational particle production [8, 9]. In this paper we present the cosmological metric perturbation of the universe passing through a de Sitter phase but which is primarily full of radiation. We show that such a universe is perturbatively stable for long wavelength modes but for short wavelengths the modes may turn out to be perturbatively unstable although the perturbations do not diverge and radiation fluid clumping may result in such theories. By instability we mean that the perturbations do not oscillate and keeps on increasing, the increment can take the system out of linear perturbation regime for some modes.

As particle creation can happen in a dynamical spacetime as the Friedmann-Lemaitre-Robertson-Walker (FLRW) spacetime, the produced particles can slowdown the rate at which energy density gets diluted in an expanding universe. For a long time people have been working phenomenologically on big bang models of the Universe with particle creation [10, 11]. The particle creation rate is predicted from quantum field theory in curved spacetime but quantum theory cannot produce a phenomenological number using which calculations can be done. Consequently people have assumed various forms of the particle creation rate as in Ref. [12, 13]. The slowdown of the rate of the energy density of particles can be taken care of by a negative pressure component arising purely due to the creation of particles. Thermodynamics of such cosmological models have been studied in detail in Ref. [14]. Various models of cosmological particle creation have also been used by many authors to model the late development of the universe [15, 16, 17]. The negative pressure arising in such cases can also produce a de Sitter space in the early universe if the rate of particle production in the early universe was high. If the universe is full of radiation then one can have a de Sitter phase full of radiation. The cosmological models for such interesting spaces were discussed in detail by previous authors in Ref. [7]. Although this kind of a de Sitter space may appear in the early phase of the universe where the universe was predominantly full of radiation, more rigorous calculations are needed to convince ourselves that this phase can indeed be a good competitor for the traditional super cool inflationary models where the de Sitter space originates due to the potential energy of one scalar field [18, 19, 20]. In the traditional models of inflation the inflaton field produces a de Sitter expansion and the perturbation of the inflaton field in conjunction with other metric perturbations produce the inhomogeneities present in the the early universe. In the present case although we have a de Sitter phase in the early universe, the scenario lacks the inflaton perturbations. Only metric perturbations and the accompanying perturbations in the fluid parameters are present in our present model of accelerated expansion of the early Universe. The metric perturbation of the model suggests that there can arise short scale inhomogeneities in the early phase of the de Sitter universe which may produce the signatures of inhomogeneities of the present day Universe.

Out of the various models of particle creation cosmology, in this paper we have chosen to follow the work presented in Ref. [7]. In this reference the authors briefly mentioned about an initial de Sitter phase in a Universe full of radiation. The previous analysis did not discuss about the cosmological perturbations during such a de Sitter phase in the very early Universe. For the first time we present the results of cosmological perturbations in the analysis of de Sitter spaces in particle creation cosmology. The material in the paper is organized in the following way. In the next section we present the relevant calculations and results required to understand the background (unperturbed) cosmology
accompanied by gravitational particle production. In section 3 we present the details about the de Sitter phase in presence of radiation. Section 4 presents the results related to scalar cosmological perturbations in the de Sitter model in presence of radiation. In section 5 we briefly present the results related to vector and tensor perturbations in such models. The next section concludes the paper by summarizing the results obtained in it.

2 The cosmological model with gravitational particle production

In the present case we work with spatially flat FLRW metric

$$ds^2 = dt^2 - a^2(t) \left[ dr^2 + r^2 d\theta^2 + \sin^2 \theta d\phi^2 \right], \quad (1)$$

where $a(t)$ is the dimensionless scale-factor. The universe is pervaded by a fluid whose elements move with 4-velocity $u^\mu$ and the 4-velocity is normalized as $u^\mu u_\mu = 1$. For the background evolution $\bar{u}^\mu = (1, 0)$. In our notation all the barred quantities will stand for background values. The fluid 4-velocity is $u^\mu = \bar{u}^\mu + \delta u^\mu$, where $\delta u^\mu$ is the fluid 4-velocity perturbation. Unbarred quantities include the perturbative fluctuations. Here we are talking about metric perturbations. The energy-momentum tensor of the fluid is

$$T^\mu_\nu = (\rho + P)u^\mu u^\nu - Pg^\mu_\nu, \quad (2)$$

where $\rho$ and $P$ are the energy density and pressure of the fluid. Here we assume the fluid to be a barotropic one with an equation of state

$$P = \omega \rho, \quad (3)$$

where $\omega$ is a constant. The fluid is made up of particles whose number may increase with time. The particle flow vector is defined as:

$$N^\mu = nu^\mu, \quad (4)$$

where $n$ is the particle number density and is given by $n = N/V$ where the number of particles present in a physical volume $V = a^3$ is $N$. Due to matter creation there arises a pressure which gives rise to a new energy-momentum tensor as $[21]

$$T^\mu_\nu_c = P_c(u^\mu u^\nu - g^\mu_\nu), \quad (5)$$

where $P_c$ is called the particle creation pressure. The Einstein equation in presence of particle creation is then given by

$$G^\mu_\nu = \kappa(T^\mu_\nu + T^\mu_\nu_c), \quad (6)$$

where $\kappa = 8\pi G$. Since the two fluids exchange energy-momentum we have

$$D_\mu(T^\mu_\nu + T^\mu_\nu_c) = 0, \quad (7)$$

where the covariant derivative acting on a second rank contravariant tensor is $D_\mu T^\mu_\nu = \partial_\mu T^\mu_\nu + \Gamma^\mu_\mu T^\alpha_\nu + \Gamma^\nu_\mu T^\mu_\alpha$, and in our convention

$$\Gamma^\mu_\alpha_\beta = \frac{1}{2} g^\mu_\sigma (\partial_\beta g_{\alpha \sigma} + \partial_\alpha g_{\beta \sigma} - \partial_\sigma g_{\alpha \beta}).$$
In the present case the energy-momentum conservation equation for the unperturbed fluid is

$$\dot{\bar{\rho}} + 3H(\bar{\rho} + \bar{P} + \bar{P}_c) = 0.$$  \((8)\)

Here \(H\) is the Hubble parameter defined as \(H \equiv \dot{a}/a\). As particles are being produced due to cosmological expansion, we must have a conservation equation as \([21]\)

$$D_\mu N^\mu = n\Gamma$$  \((9)\)

where \(\Gamma\) stands for the rate of change of particle number in a physical volume \(V\) and \(n\) is the number of particles per unit physical volume. The above relation is assumed to be true for cosmological models with particle production on dimensional grounds although we do not know the value of \(\Gamma\) precisely. We assume throughout our work that \(\Gamma \geq 0\). From the above relation one can work for the unperturbed case

$$D_\mu \bar{N}^\mu = \dot{\bar{n}} + \bar{n}\Gamma_{\mu}^{\mu} = \dot{\bar{n}} + 3H\bar{n} = \bar{n}\Gamma.$$  \((10)\)

From the above equation we can write

$$\frac{\dot{\bar{n}}}{\bar{n}} = \bar{\Gamma} - 3H.$$  

These equations hold true for the unperturbed variables. Next we write down the general thermodynamic relation in the fluid.

The energy conservation equation, using the assumption of local equilibrium, can be written as

$$TdS = d\left(\frac{\bar{\rho}}{\bar{n}}\right) + Pd\left(\frac{1}{\bar{n}}\right),$$  \((11)\)

where \(T\) is the temperature of the fluid and \(s\) is the entropy per particle, or the specific entropy. Assuming adiabatic particle creation where specific entropy is conserved, \(\dot{s} = 0\), the unperturbed variables satisfy

$$0 = \dot{\bar{\rho}} - \frac{\dot{\bar{n}}}{\bar{n}}(\bar{\rho} + \bar{P}) = \dot{\bar{\rho}} + 3H\left(1 - \frac{\bar{\Gamma}}{3H}\right)(\bar{\rho} + \bar{P}).$$  \((12)\)

Using the above result and the relation in Eq. \([8]\) we get the expression for the unperturbed creation pressure as

$$\bar{P}_c = -\frac{\bar{\Gamma}}{3H}(\bar{\rho} + \bar{P}) = -\frac{\bar{\Gamma}}{3H}(1 + \omega)\bar{\rho}.$$  \((13)\)

As long as \(\omega > -1, \bar{\rho} > 0\) and \(\bar{\Gamma} > 0\) we will have \(\bar{P}_c < 0\). The above equation is like an equation of state (EOS) connecting \(\bar{P}_c\) and \(\bar{\rho}\) and \(\bar{\Gamma}\). As in cosmology the EOS of barotropic fluid does not depend upon orders of perturbation we assume the above EOS to be valid in general to all orders of metric perturbation and write

$$P_c = -\frac{\Gamma}{3H}(\rho + P) = -\frac{\Gamma}{3H}(1 + \omega)\rho,$$  \((14)\)

where up to first order \(P_c = \bar{P}_c + \delta P_c, \Gamma = \bar{\Gamma} + \delta \Gamma\) and \(\rho = \bar{\rho} + \delta \rho\). We will work with \(\delta P_c\) later when we introduce metric perturbation. This calculation shows that the pressure of
the fluid due to particle creation is negative to the zeroth order of perturbation but \( \delta P_c \) may not be smaller than zero.

From Eq. (12) it is seen that \( \dot{\rho} + 3H\bar{\rho}(1 + \omega) = \bar{\Gamma}\dot{\bar{\rho}}(1 + \omega) \) producing

\[
\frac{\dot{\bar{\rho}}}{\bar{\rho}} = (1 + \omega)(\bar{\Gamma} - 3H).
\]

Comparing this result with the expression of \( \dot{n}/\bar{n} \) we can write

\[
(1 + \omega)\frac{\dot{n}}{n} = \frac{\dot{\bar{\rho}}}{\bar{\rho}},
\]

which has a solution as [22]

\[
\bar{n} = n_0\left(\frac{\bar{\rho}}{\rho_0}\right)^{1/(1+\omega)},
\]

where \( n_0 \) and \( \rho_0 \) are the values of the variables at some given instant of cosmic time.

Before we proceed we can try to figure out how temperature scales in such theories of cosmology where massless particles can be created due to cosmological expansion [23, 24]. Assuming \( \bar{T}, \bar{n} \) to be the basic thermodynamic variables and taking \( \bar{\rho} = \bar{\rho}(\bar{n}, \bar{T}) \) one can show

\[
\frac{\dot{\bar{T}}}{\bar{T}} = (\frac{\partial P}{\partial \bar{\rho}})_{n}\frac{\dot{\bar{n}}}{\bar{n}},
\]

giving us the time dependence of temperature in cosmological evolution accompanied by gravitational particle production when specific entropy is conserved. Using the equation of state in Eq. (3) we can write the above equation as

\[
\frac{\dot{\bar{T}}}{\bar{T}} = \omega\frac{\dot{\bar{n}}}{\bar{n}},
\]

whose solution is [22]

\[
\bar{n}(\bar{T}) = n_0\left(\frac{\bar{T}}{T_0}\right)^{1/\omega},
\]

More over using the result from Eq. (15) in Eq. (18) we immediately get

\[
\bar{\rho}(\bar{T}) = \rho_0\left(\frac{\bar{T}}{T_0}\right)^{(1+\omega)/\omega},
\]

where in the above equations \( n_0, T_0 \) and \( \rho_0 \) are the values of the respective variables at some specific time. The above relations give us that in presence of radiation fluid when \( \omega = 1/3 \) we have \( \bar{\rho}_r \propto \bar{T}^4 \) as in standard cosmological models without particle creation [22].

One can now write down the Einstein equations from Eq. (6) as:

\[
\frac{\kappa}{3}\bar{\rho}, \quad 2\dot{H} + 3H^2 = -\kappa(\bar{P} + \bar{P}_c).
\]

The above two equations can be combined and written as

\[
\dot{H} = -\frac{\kappa}{2}\bar{\rho}(1 + \omega) \left[ 1 - \frac{\bar{\Gamma}}{3H} \right],
\]
where one can use the Einstein equation and finally write [7]

\[ \dot{H} = -\frac{3}{2} (1 + \omega) H^2 \left( 1 - \frac{\bar{\Gamma}}{3H} \right). \]  

(22)

The above equation shows that as long as \( \bar{\Gamma} \ll 3H \) one recovers the standard FLRW solutions in GR. If \( \bar{\Gamma} = 3H \), irrespective of the value of \( \omega \) the model produces a de Sitter phase where \( \dot{H} = 0 \) and \( H \) remains a constant. In this paper we will focus on the de Sitter phase in a universe full of radiation. In general \( \bar{\Gamma} \) will be a function of time and the de Sitter phase can be obtained for a short time interval and after that interval \( \bar{\Gamma} < 3H \) and the system will start to evolve towards standard radiation dominated cosmological evolution if \( \omega = 1/3 \).

### 3 de Sitter phase in presence of radiation

In this present case one can indeed have a de Sitter phase in presence of radiation as discussed in Ref. [7]. In the above reference the authors briefly mention about such a phase which can exist in the very early universe. In this paper we elaborate on this idea and try to see what happens to the model Universe if there are metric perturbations. Let us write down the specific equations relevant for studying this phase when \( \omega = 1/3 \) and the energy density is \( \bar{\rho}_r \) due to radiation. The equations are

\[ H^2 = \frac{\kappa}{3} \bar{\rho}_r, \]

(23)

and

\[ \dot{H} = -2H^2 \left( 1 - \frac{\bar{\Gamma}}{3H} \right), \]

(24)

where \( \Gamma \) is the particle creation rate during this phase when the dominant matter present is radiation but the universe is passing through a de Sitter phase. In this phase we have

\[ \bar{\rho}_r = A\bar{T}^4, \]

(25)

where \( A = \frac{\pi^2 g_*}{30}\) \( g_* \) being the effective relativistic internal degrees of freedom. As discussed in the introduction, there are various ways in which people have tried to write a phenomenological form of \( \bar{\Gamma} \). In this paper we will follow the form of \( \bar{\Gamma} \) as discussed in Ref. [7]. If we model the system with particle creation rate as

\[ \bar{\Gamma} = \frac{3H^2}{H_I}, \]

(26)

where \( H_I \) is the value of the Hubble parameter when \( \dot{H} \) is exactly zero during the initial phase of expansion. In reality \( H \) will be near \( H_I \) during the de Sitter phase and as long as \( H \sim H_I \) the quasi de Sitter phase will exist. As \( \dot{H} \sim 0 \) in the de Sitter phase \( H \) is a slowly varying function and consequently from Eq. (23) we see that \( \bar{\rho}_r \) is also a slowly varying function during this phase. From Eq. (25) it becomes clear that during the quasi de Sitter expansion the temperature of the universe will approximately remain constant. As \( \dot{H} \sim 0 \) during the de Sitter phase we must have
Figure 1: Evolution of \( n(t) \) with \( t \) in the radiation dominated de Sitter phase, more details given in text.

\[ a(t) = a_i e^{H_i t}, \]  

(27)
during the quasi de Sitter phase. Here \( a_i \) is the scale-factor at the initial moment when inflation starts. To define the quasi de Sitter phase one must have some variables which mimic the “slow roll parameter” of standard inflationary scenario. In our case we can define one parameter as

\[ \epsilon \equiv -\frac{\dot{H}}{H^2}, \]  

(28)
and assume that quasi de Sitter phase exists as long as

\[ \epsilon \ll 1. \]  

(29)

From Eq. (24) we can immediately write

\[ \epsilon = 2 \left[ 1 - \frac{\bar{\Gamma}}{3H} \right], \]  

(30)
which yields

\[ H = \left( 1 - \frac{\epsilon}{2} \right) H_I. \]  

(31)

In the absence of any potential function for the inflaton field only one slow parameter \( \epsilon \) is meaningful. Fig. 1 and Fig. 2 show the variation of particle number density and temperature during radiation dominated de Sitter phase in the cosmological model with particle creation where the particle creation rate is given in Eq. (26). In producing the plots we have used the initial values of the energy density and number density of particles as

\[ \rho_0 = \frac{\pi^2}{30} g_* T_0^4, \quad n_0 = \frac{\zeta(3)}{\pi^2} g_* T_0^4 \]  

(32)
where \( T_0 = .008 \) and \( g_* \sim 100 \). Here \( \zeta(3) \) is the Riemann-Zeta function of 3. All the quantities are in Planck units. To get the value of the variables in natural units where all the quantities are expressed in GeV or its inverse powers one has to multiply each
quantity with a power of Planck mass where the power is specified by the mass dimension of each quantity. The figures above show that during radiation dominated de Sitter phase the number density and temperature essentially remains constant. This fact is in stern contrast with standard cold inflation scenarios where temperature and number density of particles rapidly vanish to zero. On the other hand these features may be a bit (superficially) similar to warm inflation predictions [1, 2].

4 Scalar cosmological perturbations

Following the conventions set in Ref. [25] we can write the perturbed line element as

$$ds^2 = a^2(\eta) \left\{ (1 + 2\phi) - 2\partial_i B dx^i d\eta - [(1 - 2\psi)\delta_{ij} + 2\partial_i \partial_j E] dx^i dx^j \right\} ,$$

where \(\phi, \psi, B\) and \(E\) are scalar functions of space and time. Here \(d\eta = dt/a\) where \(\eta\) is the conformal time. In conformal time

$$\mathcal{H} = \frac{a'}{a} ,$$

where \(a' = da/d\eta\). In this section we will use conformal time exclusively. From now on we work in the longitudinal gauge where

$$B = E = 0 .$$

As in the present case the spatial part of the energy-momentum tensor is diagonal we have \(\phi = \psi\). The particle production mechanism particularly affects the scalar perturbations. To see how it affects various sectors of the theory let us first see how the perturbed continuity equation gets affected in cosmological models with particle production.

To find out how particle production affects the perturbed continuity equation we calculate

$$\delta(D_\mu T^\mu_{\nu}) = \partial_\mu \delta T^\mu_{\nu} + (\delta \Gamma^\mu_{\mu\nu}) \tilde{T}^\nu_{\nu} + \tilde{\Gamma}^\mu_{\mu\nu} \delta T^\nu_{\nu} - (\delta \Gamma^\alpha_{\mu\nu}) \tilde{T}^\mu_{\alpha} - \tilde{\Gamma}^\alpha_{\mu\nu} \delta T^\nu_{\alpha} .$$

In conformal time

$$\tilde{\Gamma}^0_{00} = \mathcal{H}; \quad \tilde{\Gamma}^0_{0i} = \mathcal{H}\delta^i_j; \quad \tilde{\Gamma}^i_{ij} = \mathcal{H}\delta_{ij}; \quad \tilde{\Gamma}^0_{0i} = \tilde{\Gamma}^0_{00} = \tilde{\Gamma}^i_{jk} = 0 ,$$

and the perturbed connection coefficients are:

$$\delta \Gamma^0_{00} = \phi' , \quad \delta \Gamma^0_{0i} = \partial_i \phi , \quad \delta \Gamma^0_{0j} = \partial^i \phi , \quad \delta \Gamma^0_{ij} = -4\mathcal{H}\phi\delta_{ij} - \phi'\delta_{ij} , \quad \delta \Gamma^i_{0j} = -\phi'\delta^i_j ,$$

and

$$\delta \Gamma^i_{jk} = -\delta^i_k \partial_j \phi - \delta^i_j \partial_k \phi + \partial^i \phi \delta_{jk} .$$

We can now write,

$$\delta(D_\mu T^\mu_{00}) = \delta \rho' + 3\mathcal{H}(\delta \rho + \delta P) + (\bar{\rho} + \bar{P}) a \nabla^2 V - 3\phi'(\bar{\rho} + \bar{P}) .$$

In our notation

$$\bar{T}^0_{00} = \bar{\rho} , \quad \bar{T}^i_{ij} = -P\delta^i_j , \quad \bar{T}^i_{\nu j} = -P_e\delta^i_j ,$$

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and
\[ \delta T^0_{i0} = \delta \rho, \quad \delta T^i_{0} = -\bar{\rho} + \bar{P}a\partial^i V, \quad \delta T^0_{i} = -(\bar{\rho} + \bar{P})a^{-1}\partial_i V, \quad \delta T^i_{ij} = -\delta^j_{\bar{\partial}}\delta P, \]

whereas
\[ \delta T^0_{0} = 0, \quad \delta T^i_{c0} = -a\bar{P}_c\partial^i V, \quad \delta T^0_{ci} = -\bar{P}_ca^{-1}\partial_i V, \quad \delta T^i_{cj} = -\delta^j_{\bar{\partial}}\delta P_c. \]

In the above calculations we will use the scalar part of velocity perturbation given by \( \delta u^i = -\partial^i V \) where \( V \) is the fluid velocity potential. Now we can write taking \( \nu = 0, \)
\[ \delta(D_{\mu}T^\mu) = \bar{\partial}_i\delta T_{i0} - (\delta \Gamma^i_{j0})\bar{T}^j_i - \bar{\Gamma}^i_{j0}\delta T^j_i = a\bar{P}_c\nabla^2 V - 3\phi'\bar{P}_c + 3\bar{H}\delta P_c. \quad (37) \]

As \( \delta[(D_{\mu}T^\mu) + (D_{\mu}T^\mu)] = 0 \) we have from Eq. (37) and Eq. (36)
\[ \delta\rho' + 3\bar{H}(\delta \rho + \delta P) + (\bar{\rho} + \bar{P})a\nabla^2 V - 3\phi'(\bar{\rho} + \bar{P}) + \bar{P}_c(a\nabla^2 V - 3\phi') + 3\bar{H}\delta P_c = 0, \quad (38) \]
giving us the continuity equation in a cosmological model with particle creation. One can also find out the Euler equation by working out \( \delta[(D_{\mu}T^\mu) + (D_{\mu}T^\mu)] = 0. \) Here \( \delta P_c \) is obtained from Eq. (14) as
\[ \delta P_c = -\frac{a}{3\bar{H}}(1 + \omega)(\bar{\rho} \delta \Gamma + \bar{\Gamma} \delta \rho). \quad (39) \]

From the above expression we see that \( \delta P_c \) vanishes when both \( \bar{\Gamma} \) and \( \delta \Gamma \) vanishes. When \( \Gamma = 0 \) the above continuity equation reduces to the standard continuity equation in cosmology.

### 4.1 The other relevant scalar perturbations

Let us particularly focus on perturbation of \( N^\mu \) which gives \( \delta N^\mu = \bar{u}^\mu \delta n + \bar{\bar{u}} \delta u^\mu \) where a bar over the variables specify the unperturbed values. For the scalar part \( \delta u^i = -\partial^i V \) where \( V \) is the fluid velocity potential. This gives us
\[ \delta N^0 = \bar{u}^0 \delta n, \quad \delta N^i = -\bar{\bar{u}}^0 \partial^i V. \]

First we perturb the number conservation equation in Eq. (9) and get
\[ \delta(D_{\mu}N^\mu) = \delta[(\partial_{\mu}n)u^\mu + n(\partial_{\mu}u^\mu)] + \delta(n\Gamma^\mu_{\mu\alpha}u^\alpha) = (\partial_{\mu}n)\bar{u}^0 + (\partial_{\bar{\mu}}\bar{u}) \delta u^\mu + \delta n(\partial_{\bar{\mu}}\bar{u}^0) + \bar{\bar{u}}(\partial_{\bar{\mu}}\delta u^\mu) + (\delta n)\bar{\bar{u}}^0 + \bar{\bar{u}}(\partial_{\bar{\mu}}\delta u^\mu) = \bar{\bar{\Sigma}} \delta n + \bar{\bar{\Sigma}} \delta \Gamma. \]

Using conformal time we have \( \bar{u}^0 = a^{-1}(1, 0) \) and \( \bar{\bar{u}}_{\mu} = a(1, 0), \partial_{\bar{\mu}}\bar{u} = 0, \) we have
\[ (\delta n)' + 3\bar{H}\delta n + \bar{\bar{u}} \delta \bar{\bar{\Gamma}} + a\bar{n}\nabla^2 V = a\bar{\bar{\Sigma}} \delta n + a\bar{n} \delta \Gamma, \]
where \( \nabla^2 \equiv -\partial_{\bar{\mu}}\partial^{\bar{\mu}}. \) The above equation can also be written as
\[ (\delta n)' + (3\bar{H} - a\bar{\bar{\Gamma}}) \delta n + a\bar{n}\nabla^2 V = a\bar{n} \delta \Gamma. \]

\(^1\text{In conformal time we have } \bar{u}^0 = a^{-1} \text{ instead of } 1. \text{ In coordinate time we should have } \bar{u}^0 = 1 \text{ as used in the initial part of this paper.}\)
Now, Eq. (9) in conformal time is \( \ddot{\bar{n}} + (3 \mathcal{H} - a \dot{\Gamma}) \bar{n} = 0 \) which implies

\[
(\delta n)' - \frac{\bar{n}'}{\bar{n}} \delta n + \ddot{\bar{n}} \Gamma'_{\mu0} + a \bar{n} \nabla^2 V = a \bar{n} \delta \Gamma.
\]

As \( \delta \Gamma'_{\mu0} = -2\phi' \), in the longitudinal gauge, we can write the final expression of the perturbed particle number conservation becomes

\[
(\delta n)' - \frac{\bar{n}'}{\bar{n}} \delta n - 2\phi' \bar{n} + a \bar{n} \nabla^2 V = a \bar{n} \delta \Gamma.
\]

where

\[
\frac{\bar{n}'}{\bar{n}} = a \dot{\Gamma} - 3 \mathcal{H},
\]

giving us \( \bar{n}(\eta) \) as function of conformal time.

In the present case the energy momentum tensor is \( T^{mn} = T_{\mu}^{mn} + T_{\nu}^{\mu \nu} \) as given in Eq. (6) and consequently we have

\[
T_0^0 = \rho, \quad T_j^i = -(P + P_c) \delta_j^i.
\]

Following the method of gauge invariant scalar cosmological perturbations in Ref. [25] one can write in the longitudinal gauge,

\[
\nabla^2 \phi - 3 \mathcal{H} \phi' - 3 \mathcal{H}^2 \phi = \frac{k}{2} a^2 \delta \rho,
\]

\[
D_i(a\phi)' = \frac{k}{2} (\bar{\rho} + \bar{P} + \bar{P}_c) a^2 \delta u_i,
\]

\[
\phi'' + 3 \mathcal{H} \phi' + (2 \mathcal{H}' + \mathcal{H}^2) \phi = \frac{k}{2} a^2 (\delta P + \delta P_c).
\]

and Eq. (39) giving the expression of \( \delta P_c \). Using the fact that \( \delta P = \omega \delta \rho \) one can simplify the above equations as

\[
\phi'' - \omega \nabla^2 \phi + 3 \mathcal{H}(1 + \omega) \phi' + [2 \mathcal{H}' + \mathcal{H}^2(1 + 3 \omega)] \phi = \frac{k}{2} a^2 \delta P_c.
\]

For the sake of generality we will try to keep the general equation of state explicit in the equations written below but for all operational purpose we mean \( \omega = 1/3 \) in this paper. One can put the form of \( (k/2)a^2 \delta \rho \) from Eq. (43) in the first term on the right hand side of the above equation and obtain:

\[
\phi'' - \omega \nabla^2 \phi + 3 \mathcal{H}(1 + \omega) \phi' + [2 \mathcal{H}' + \mathcal{H}^2(1 + 3 \omega)] \phi = \frac{k}{2} a^2 \delta P_c.
\]

Using the expression of \( \delta P_c \) one can write the above expression as

\[
\phi'' - \left[ \omega - \frac{a(1 + \omega) \dot{\Gamma}}{3 \mathcal{H}} \right] \nabla^2 \phi + \left( 3 \mathcal{H} - a \dot{\Gamma} \right) (1 + \omega) \phi' + [2 \mathcal{H}' + \mathcal{H}^2(1 + 3 \omega)] \phi = \frac{a}{2} (1 + \omega) \mathcal{H} \delta \Gamma.
\]

This equation specifies the growth of \( \phi \) once \( \delta \Gamma \) is known. From Eq. (44) we can write

\[
\nabla^2 V = \frac{2}{\kappa a^2} \frac{a^2 \nabla^2 \phi + a \nabla^2 \phi'}{\kappa (\bar{\rho} + \bar{P} + \bar{P}_c)}.
\]

Using this equation we can write Eq. (40) as

\[
\delta n' - \frac{\bar{n}'}{\bar{n}} \delta n - \left[ 2\phi' + \frac{2(\mathcal{H} \nabla^2 \phi + \nabla^2 \phi')}{\kappa (\bar{\rho} + \bar{P} + \bar{P}_c)} \right] \bar{n} = a \bar{n} \delta \Gamma.
\]

These above equations specify the metric perturbation equations of the cosmological model with particle creation. The equations become solvable once \( \delta \Gamma \) is specified.
4.2 The basic ansatz to solve the perturbation equations when $\delta \Gamma = 0$

We assume that the particle production rate $\Gamma$ depends only on the property of the background evolution as given in Eq. (26). One can find out the value of $\Gamma$ only if one knows the microphysics of the universe as $\Gamma$ depends upon quantum field theory in curved background. In our phenomenological model we have assumed a form of $\tilde{\Gamma}$ in the FLRW background. Under a metric perturbation the the Lagrangian density of the quantum field can be affected, but the gravitational particle production rate is assumed to be unchanged. As a consequence $\delta \Gamma$ can be ignored and the gravitational particle production rate remains a property solely dictated by the background evolution.

In the simplest gravitational particle production model of cosmology we can assume

$$\delta \Gamma = 0,$$  \hspace{1cm} (50)

where $\delta \Gamma$ arises due to metric perturbations, and as a consequence the relevant perturbation equations which requires to be solved are,

$$\phi'' - \left[ \omega - \frac{a(1 + \omega)\tilde{\Gamma}}{3\mathcal{H}} \right] \nabla^2 \phi + (3\mathcal{H} - a\tilde{\Gamma})(1 + \omega)\phi' + [2\mathcal{H}' + \mathcal{H}^2(1 + 3\omega)] - a(1 + \omega)\tilde{\Gamma}\mathcal{H}\phi = 0,$$  \hspace{1cm} (51)

$$\delta n' - \frac{n'}{n} \delta n - 2 \left[ \phi' + \frac{(\mathcal{H} \nabla^2 \phi + \nabla^2 \phi')}{\kappa(\bar{\rho} + \bar{P} + \bar{P}_c)} \right] \bar{n} = 0,$$  \hspace{1cm} (52)

$$V + \frac{2}{\kappa a^2 (\bar{\rho} + \bar{P} + \bar{P}_c)} = 0.$$  \hspace{1cm} (53)

Using Eq. (18) and the expression of $\dot{n}/n$ we can write

$$\frac{\dot{T}'}{T} = \omega \frac{\dot{n}'}{n} = \omega(a\tilde{\Gamma} - 3\mathcal{H}),$$  \hspace{1cm} (54)

where primes naturally denote differentiation with respect to conformal time $\eta$. The
temperature fluctuations can be approximately figured out by assuming that the metric perturbations do not take the system out of local thermal equilibrium. Assuming the perturbed system to be very near local thermodynamic equilibrium one can still assume

\[ \rho = \bar{\rho} + \delta \rho \sim \frac{\pi^2}{30} g_*, T^4 \]  

(55)

where \( T = \bar{T} + \delta T \). Up to first order the relation gives

\[ \delta T \sim \frac{15}{2\pi^2 g_* T^3} \]  

(56)

We will see later that metric perturbations may induce fluid flow in the initial stages of the de Sitter phase and consequently local thermodynamic equilibrium can only hold approximately. The above simple estimation can give us the temperature fluctuations induced by metric perturbations assuming the system is near about local thermal equilibrium. In this paper we have assumed that the universe to be full of radiation where \( \omega = 1/3 \) so that Eq. (54) can easily be integrated to see how the background temperature evolves when one has

\[ \bar{\Gamma} = \frac{3\mathcal{H}^2}{a^2 H_I} , \]  

(57)

which can be obtained from Eq. (26). For plane wave like perturbations where the perturbation modes are proportional to \( e^{ik \cdot x} \) we will have the basic perturbation equations as:

\[ \phi''_k + k^2 \left[ \omega - \frac{a(1 + \omega)\bar{\Gamma}}{3\mathcal{H}} \right] \phi_k + (3\mathcal{H} - a\bar{\Gamma})(1 + \omega)\phi'_k + [2\mathcal{H}' + \mathcal{H}^2(1 + 3\omega) - a(1 + \omega)\bar{\Gamma}H]\phi_k = 0 , \]  

(58)

\[ \delta n'_k - \frac{n'}{\bar{n}} \delta n_k - 2 \left[ \phi'_k - \frac{k(H\phi_k + \phi'_k)}{\kappa(\bar{\rho} + \bar{P} + \bar{P}_c)} \right] \bar{n} = 0 , \]  

(59)

\[ V_k + 2 \frac{a'd\phi_k + a\phi'_k}{\kappa a^2 (\bar{\rho} + \bar{P} + \bar{P}_c)} = 0 , \]  

(60)

where \( \phi_k(\eta) \), \( \delta n_k(\eta) \) and \( V_k(\eta) \) are the \( k \)th Fourier mode of the respective perturbations. Now

\[ \bar{\rho} + \bar{P} + \bar{P}_c = (1 + \omega)\bar{\rho} \left[ 1 - \frac{a\bar{\Gamma}}{3\mathcal{H}} \right] = \frac{3\mathcal{H}^2}{\kappa a^2} (1 + \omega) \left[ 1 - \frac{\mathcal{H}}{aH_I} \right] , \]  

and so we can write Eq. (59) and Eq. (60) as

\[ \delta n'_k - \frac{n'}{\bar{n}} \delta n_k - 2 \left[ \phi'_k - \frac{a^2k^2(H\phi_k + \phi'_k)}{3\mathcal{H}^2(1 + \omega)(1 - \frac{\mathcal{H}}{aH_I})} \right] \bar{n} = 0 , \]  

(61)

\[ V_k + 2 \frac{a'd\phi_k + a\phi'_k}{3\mathcal{H}^2 (1 + \omega)(1 - \frac{\mathcal{H}}{aH_I})} = 0 . \]  

(62)

In the present case

\[ \epsilon = \frac{\mathcal{H}^2 - \mathcal{H}'}{\mathcal{H}^2} , \quad \frac{\mathcal{H}}{aH_I} = 1 - \frac{\epsilon}{2} , \]  

(63)
Figure 5: Evolution of $V_k$ with $\eta$ in the radiation dominated de Sitter phase for $\delta \Gamma = 0$ and $k = 10^{-5}$.

and

$$\mathcal{H}' = (1 - \epsilon)\mathcal{H}^2.$$  

The above equations have to be solved for $k < \mathcal{H}$ which corresponds to the superhorizon limit and $k > \mathcal{H}$ corresponding to the subhorizon limit. The perturbation equations can also be written as:

$$\phi_k'' + \frac{3}{2} (1 + \omega) \mathcal{H} \epsilon \phi_k' + \left[ \frac{\epsilon}{2} k^2 (1 + \omega) - k^2 + 2 \mathcal{H}' + \mathcal{H}^2 (1 + 3 \omega) - 3 (1 + \omega) \mathcal{H}^2 (1 - \frac{\epsilon}{2}) \right] \phi_k = 0,$$

and

$$\delta n''_k + \frac{3}{2} \mathcal{H} \delta n_k - 2 \left[ \phi_k' - \frac{2 a^2 k^2 (\mathcal{H} \phi_k + \phi'_k)}{3 \mathcal{H}^2 (1 + \omega) \epsilon} \right] \bar{n} = 0,$$

$$V_k + \frac{4 a' \phi_k + a \phi'_k}{3 \mathcal{H}^2 (1 + \omega) \epsilon} = 0.$$  

It must be noted that $\delta n_k \ll \bar{n}$ but in an absolute sense $\delta n_k$ need not be smaller than one as the fluctuation in the number of particles in unit volume cannot be less than one.

The plots of the perturbations are shown in Fig. 3, Fig. 4 and Fig. 5. In all of these plots we have taken $k = 10^{-5}$ and $\omega = 1/3$. We have used Planck units as described earlier. While plotting the above figures we used $T_0 = .008$ and $g_\ast \sim 100$ and $\mathcal{H}_0 = .001$. The zero subscript specify initial value of any variable. The scale-factor initially is taken to be unity. For the perturbations we have used $\phi_k(\eta_0) = .01$, $\phi_k'(\eta_0) = .0001$, and $\delta n_k(\eta_0) = .001 \times n_0$ where $n_0$ is specified in Eq. (32). In all of the plots $\eta_0 = -10^3$. The plots show well behaved perturbations of the relevant quantities in the radiation dominated de Sitter phase.

The main point about the above perturbation equations is that they may produce unstable behavior when $k > \mathcal{H}$. The perturbations do not turn out to be singular in this regime but tend to be unstable and may slowly attain value near to or greater than one. From Eq. (65) it is seen that when $k^2 > \mathcal{H}^2$, $\phi_k$ can become unstable\footnote{In this limit we must also have $k^2 > \mathcal{H}'$ as $\mathcal{H}' = (1 - \epsilon)\mathcal{H}^2$ and $\epsilon \ll 1$.}. To see this explicitly we write Eq. (65) when $\epsilon \ll 1$ as

$$\phi_k'' \sim k^2 \phi_k', \quad \text{when } k > \mathcal{H}.$$  

\chapter{The}
showing that \( \phi_k \) can increase without bound in this limit. The condition \( k^2 > \mathcal{H}^2 \) will not stay forever and consequently \( \phi_k \) will not become singular. On the other hand \( \phi_k \) can become non-perturbative, particularly if the initial value of it is not too small, when \( k^2 > \mathcal{H}^2 \). On the other hand when \( k^2 < \mathcal{H}^2 \) and \( \epsilon \ll 1 \), Eq. (65) becomes \( \phi_k'' \sim 2(\mathcal{H}^2 - \mathcal{H}')(\phi_k = \epsilon \mathcal{H}^2 \phi_k) \), and in our approximation scheme we can write the last condition as

\[
\phi_k'' \sim 0, \quad \text{when} \quad k < \mathcal{H}
\]

showing that \( \phi_k \) will be approximately a linear function of conformal time in this limit. In particular if \( \phi_k' \sim 0 \) then the scalar perturbation will be a constant in this limit.

As \( \epsilon \to 0 \) when \( \bar{\Gamma} \to 3\mathcal{H}/a \) we see Eq. (66) can also be written as

\[
\left( \frac{\delta n_k}{\bar{n}} \right)' \sim 2 \left[ \phi_k' - \frac{2a^2k^2(\mathcal{H}\phi_k + \phi_k')}{3\mathcal{H}^2(1 + \omega)\epsilon} \right]. \tag{70}
\]

From the above expression we see the fractional number density can build up if \( \phi_k \) increases in the limit \( k > \mathcal{H} \). Depending upon the sign of \( \phi_k' \) and \( \phi_k \) we see that \( \delta n_k/\bar{n} \) can be positive or negative implying that at some length scales there will be more fractional particle and at some length scales the particle fraction is less. These results show that one can have inhomogeneities in the radiation filled universe during the de Sitter phase. These inhomogeneities build up in the radiation filled universe and as slowly the de Sitter phase transforms to a normal radiation phase the inhomogeneities will show up as patches of over grown gravitational potential and energy density. It requires further investigation to answer whether these inhomogeneities can act as seed inhomogeneities which appear in cosmic microwave spectrum. These calculations show that in the de Sitter phase with radiation we expect inhomogeneities and out of local equilibrium behavior for some time. Because of the movement of the radiation fluid there can be other kinds of transport phenomena which we do not discuss in this paper. Transport phenomenon induced by metric perturbations is an interesting topic by itself and will be addressed in a separate paper.

5 Brief comment on the other kind of metric perturbations

The de Sitter phase in a radiation filled universe can produce interesting features in pure vector perturbation sector. The tensor perturbations on the other hand do not produce any novel features. The pure vector perturbation can get amplified due to the particle production process inducing radiation fluid movement in the shorter length scales whereas and tensor perturbations in the present case do not get amplified and its properties remain similar to standard inflationary result. In this section we will discuss briefly about the vector and tensor perturbations in cosmology with particle creation. The basic results related to the metric perturbations quoted in the present section follows the conventions set in Ref. [25].
5.1 Vector perturbations

When only vector perturbations are excited the metric is given as

\[ g_{\mu\nu} = a^2(\eta) \begin{pmatrix} 1 & S_i \\ -\delta_{ij} + \partial_j F_i + \partial_i F_j \\ S_i \end{pmatrix} \],

where \( S^i \) and \( F^i \) are 3-vectors satisfying the constraint \( \partial^i S^i = \partial_i F^i = 0 \). In the Newtonian gauge where \( F^i = 0 \), the standard theory of vector perturbation gives

\[ v^i_k = \frac{k^2}{2a^2(\bar{\rho} + P + P_c)} S^i_k, \]  

and

\[ S^i_k(\eta) = \frac{C^i_k}{a(\eta)^2}, \]  

where \( C^i_k \) is the Fourier component of a constant 3-vector. Here \( v^i_k \) is the kth Fourier component of the pure vector part of the fluid velocity 3-vector \( v^i \). Here \( S^i_k \) and \( C^i_k \) stands for the kth Fourier mode of the specific variables. For all these variables \( k \cdot v^i_k = k \cdot S^i_k = k \cdot C^i_k = 0 \). In this case we see that the metric perturbation \( S^i_k \) becomes negligible as the universe expands but from Eq. (71) it is seen that the initial values of \( v^i_k \) may not be negligible when \( \epsilon \) is small and \( k > H \). The pure vector part of fluid velocity perturbations cannot be neglected in this case and this may have important cosmological consequences. In pure radiation domination the pure vector part of fluid velocity perturbations remains constant, and if they were negligible initially they remain so throughout the radiation dominated phase. In the present case the pure vector part of fluid velocity perturbations cannot be neglected in the de Sitter phase.

5.2 Tensor perturbations

In order to tackle tensor perturbations

\[ ds^2 = a^2(\eta) \left[ d\eta^2 - (\delta_{ij} - h_{ij})dx^i dx^j \right], \]  

where \( h_{ij} \) is a symmetric, traceless tensor with \( \partial^i h_{ij} = 0 \) one can rescale \( h_{ij} \) and write

\[ h_{ij} = \frac{v}{a} e_{ij}, \]  

where \( v \) is a function of \( \eta, x \) and \( e_{ij} \) is the time-independent polarization tensor. Expanding \( v \) in plane waves we get,

\[ v''_k + \left( k^2 - \frac{a''}{a} \right) v_k = 0. \]  

In our case during the initial stages of quasi-de Sitter expansion \( v_k \) has a solution which is similar to standard inflationary pictures. The tensor perturbations do not carry any special signature from the particle production model. Once the particle production rate decreases considerably normal radiation domination returns, when \( a(\eta) \propto \eta \), and then \( v_k \propto \exp(\pm ik\eta) \).
6 Conclusion

In this article we have tried to formulate metric perturbations in a cosmological model where gravitational particle production is taking place. The formulation requires various new assumptions about the particle production process in cosmology and to our understanding we present the first results of such a calculation. Previous authors have presented the cosmological background (unperturbed) calculations of radiation filled universe in a de Sitter phase where the de Sitter negative pressure is supplied by the particle creation pressure. If the particle creation rate attains a specific value then a radiation filled universe may enter a de Sitter phase where one may try to see whether such a phase can emulate the properties of standard slow roll inflation. The expression of the creation pressure turns out to be proportional to the particle production rate and energy density and always remains negative. We have used the expression connecting the creation pressure and energy density like a new equation of state from which one can get the perturbed creation pressure. In the initial part of the present paper we have presented the basic equations governing the background cosmological evolution in a cosmological model with particle production. It has been pointed out how such a cosmological model can enter a de Sitter phase in presence of radiation. In the model of particle production, as studied in the paper, specific entropy does not increase and the created radiation can locally equilibrate with the existing radiation. As long as the radiation filled universe remains quasi de Sitter massless particle production can go on and as slowly the system comes out of the de Sitter phase massless particle production goes down as predicted by Leonard Parker [6]. The background evolution of such cosmological models are stable. The rate of particle production actually comes from microphysics involving quantum field theoretical methods in curved spacetimes. Those methods give a formal expression of the rate but do not give a phenomenological value. In our model we have assumed the rate to be proportional to the square of the Hubble parameter.

The main results of the paper are related to the perturbation of the cosmological model with gravitational particle creation. Initially the scalar cosmological perturbation calculations are presented. We have consistently used the longitudinal gauge in our calculations for the scalar perturbations. While working out the scalar perturbations we have put the perturbation of the particle production rate to be zero. This assumption is based on the fact that in the gravitational particle production models metric perturbations introduce new interactions between the quantized scalar field and the classical metric perturbations but do not alter the particle production process which comes purely from the background model of cosmology. With this assumption and the equation of state connecting creation pressure and energy density one can write down the scalar perturbation equations in a cosmological model accompanied by gravitational particle production. When the equations are particularly solved for the de Sitter universe full of radiation then various interesting results come out. The perturbation equations show that all the relevant Fourier modes of the perturbations in longitudinal gauge show finite instability in the subhorizon limit where as on the superhorizon case the the Fourier mode of the metric perturbation in the longitudinal gauge linearly increases with time or becomes constant. In general for a radiation dominated universe the short wavelength perturbations oscillate in a stable fashion. The effect of the de Sitter phase is to make the small scale perturbations finitely unstable which may cause inhomogeneities in the early universe. These inhomogeneities may act as sources for the inhomogeneities in the cosmic microwave spectrum.
and help in structure formation in the later phase of evolution. In this work we have not dealt in detail how these inhomogeneities are sustained, further investigation in this line is required. Moreover the initial inhomogeneities may produce out of (local) equilibrium behavior although in this article we have assumed near local equilibrium is always maintained.

In the previous section we have briefly discussed the fate of pure vector and tensor perturbations in cosmology where gravitational particle production takes place. The vector metric perturbation dies down quickly in an exponentially expanding spacetime where as the pure vector velocity perturbations may not die down so easily. It has been shown that in the de Sitter phase the fluid pure vector perturbation may be amplified in the subhorizon limit indicating directed flow of radiation from one region to other. These strong local flows can produce interesting effects in the de Sitter model studied. Further investigation is required to unravel the effects of such flow in the universe full of radiation. The tensor part of the perturbation sector does not carry any special signature of gravitational particle production process, there are no amplification or instability in this sector. The tensor perturbations have a solution which are similar to the solutions in standard tensor perturbation calculation in de Sitter space in inflation.

The work presented in this paper is a first attempt to look at the perturbation spectrum in the cosmological models where gravitational particle production takes place. The results show that such particle production mechanism can induce instabilities and inhomogeneities in the small scale, something unlike standard inflationary theories. Initial investigation shows that one can indeed have inhomogeneities in this gravitational particle production models but it is not clear whether such inhomogeneities will sustain as they are not “frozen” in time. Further analysis in this field will unravel whether the de Sitter model presented here can in reality compete with the standard inflationary models of cosmology.

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