Invariant Cyclic Homology

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Abstract

We define a noncommutative analogue of invariant de Rham cohomology. More precisely, for a triple \((A, H, M)\) consisting of a Hopf algebra \(H\), an \(H\)-comodule algebra \(A\), an \(H\)-module \(M\), and a compatible grouplike element \(\sigma\) in \(H\), we define the cyclic module of invariant chains on \(A\) with coefficients in \(M\) and call its cyclic homology the invariant cyclic homology of \(A\) with coefficients in \(M\). We also develop a dual theory for coalgebras. Examples include cyclic cohomology of Hopf algebras defined by Connes-Moscovici and its dual theory. We establish various results and computations including one for the quantum group \(SL(q, 2)\).

1 Introduction

It is well known that cyclic homology replaces de Rham cohomology in noncommutative settings. For example, by a result of Connes \([6]\) the periodic cyclic homology of the algebra of smooth functions on a smooth manifold is isomorphic to the de Rham cohomology of the manifold. Invariant de Rham cohomology was introduced by Chevalley and Eilenberg in \([2]\) in part to relate the de Rham cohomology of a Lie group to the cohomology of its Lie algebra.

In this paper we define a noncommutative analogue of invariant de Rham cohomology. More precisely, for a triple \((A, H, M)\) consisting of a Hopf algebra \(H\), an \(H\)-comodule algebra \(A\) and an \(H\)-module \(M\), and for a suitably
chosen grouplike element $\sigma \in \mathcal{H}$, we define the \textit{cyclic module of invariant chains} on $A$, denoted $\{C^n_h(A, M)\}_n$, as the space of coinvariants of the paracyclic module $\{C_n(A, M)\}_n$ under the action of $\mathcal{H}$. Such triples are called \textit{$\sigma$-compatible Hopf triple} in this paper. We call the resulting cyclic homology groups, the \textit{invariant cyclic homology of the algebra} $A$ with coefficients in $M$. For example in complete analogy with the classical case, the cyclic module $\{\tilde{H}^n_{(\delta, \sigma)}\}_n$, defined in [12] for a Hopf algebra $\mathcal{H}$ endowed with a modular pair $(\delta, \sigma)$ in involution, is isomorphic with the invariant cyclic module of the algebra $\mathcal{H}$ with respect to the coaction of $\mathcal{H}$ on itself via the comultiplication map $\mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$. (Classically [2], the cohomology of the Lie algebra of a Lie group $G$ appeared as the invariant de Rham cohomology of $G$ with respect to the action of $G$ on itself via multiplication).

We also develop a theory for coalgebras. This is needed, for instance, in order to treat the Connes-Moscovici cocyclic module $\{\mathcal{H}^n_{(\delta, \sigma)}\}_n$ of a Hopf algebra $\mathcal{H}$ as the invariant cocyclic module of $\mathcal{H}$; but we go beyond this case and introduce \textit{$\delta$-compatible Hopf cotriples} $(C, \mathcal{H}, V)$. Thus for an $\mathcal{H}$-module coalgebra $C$, an $\mathcal{H}$-comodule $V$, and a compatible character $\delta$ we define the \textit{cocyclic module of invariant cochains} on $C$, $\{C^n_h(C, V)\}_n$, as the space of coinvariants of the paracocyclic module $\{C^n(C, V)\}_n$.

One feature of our approach is that even for $A = \mathcal{H}$ or $C = \mathcal{H}$ we allow coefficients to enter the picture and thus can interpret group homology and Lie algebra homology with coefficients as invariant cyclic homology.

There are two other situations, corresponding to Hopf module algebras and Hopf comodule coalgebras, where one can also define an invariant cyclic homology theory. In particular, the twisted cyclic cohomology of [14] is a special case of invariant cyclic cohomology theory corresponding to $G = \mathbb{Z}$. There are also interesting connections with covariant differential calculi on Hopf algebras and quantum groups. These matters will be pursued elsewhere [10].

This work grew out of our attempt to interpret the Connes-Moscovici cocyclic module for Hopf algebras as well as the cyclic module in [12] [19] as special case of a general invariant cyclic homology theory. The existence of invariant cyclic homology and the methods we use was inspired by their work. We also mention that M. Crainic in [7] has interpreted the Connes-Moscovici cocyclic module $\{\mathcal{H}^n_{(\delta, 1)}\}_n$ as the space of invariant cochains on $\mathcal{H}$.
2 Preliminaries

Let $k$ be a commutative unital ring. In this paper, by an algebra we mean a unital associative algebra over $k$. The same convention applies to coalgebras and Hopf algebras. We denote the coproduct of a coalgebra by $\Delta$, its counit by $\epsilon$, and the antipode of a Hopf algebra by $S$. The unadorned tensor product $\otimes$ means tensor product over $k$. We use Sweedler’s notation and through this paper write $\Delta(h) = h^{(1)} \otimes h^{(2)}$, $\Delta^2(h) := (\Delta \otimes \text{id}) \circ \Delta(h) = h^{(1)} \otimes h^{(2)} \otimes h^{(3)}$, where summation is understood. A character of a Hopf algebra $H$ is a (unital) algebra homomorphism $\delta : H \rightarrow k$. A grouplike element is an element $\sigma \in H$ such that $\Delta(\sigma) = \sigma \otimes \sigma$ and $\epsilon(\sigma) = 1$.

Let $H$ be a Hopf algebra. By an $H$-module we mean a module over the underlying algebra of $H$ and by an $H$-comodule we mean a comodule over the underlying coalgebra of $H$. We use Sweedler’s notation for comodules. Thus if $\rho : M \rightarrow C \otimes M$ is the structure map of a left $C$-comodule $M$, we write $\rho(m) = m^{(-1)} \otimes m^{(0)}$, where summation is understood.

Let $H$ be a Hopf algebra. An algebra $A$ is called a left $H$-comodule algebra if $A$ is a left $H$-comodule via $\rho : A \rightarrow H \otimes A$ and $\rho$ is an algebra map. Similarly, a coalgebra $C$ is called a left $H$-module coalgebra if $C$ is a left $H$-module via $\mu : H \otimes C \rightarrow C$ and $\mu$ is a coalgebra map.

Let $M$ be a left $H$-module and $\delta$ a character of $H$. The space of coinvariants of $M$ (with respect to $\delta$) is the $k$-module

$$M_\delta = M / \text{span}\{hm - \delta(h)m \mid h \in H, \ m \in M\} = k_\delta \otimes M,$$

where the right $H$-module $k_\delta$ is defined by $k_\delta = k$ and $H$ acts via the character $\delta$. Similarly, if $M$ is a left $H$-comodule, and $\sigma \in H$ is a grouplike element, the space of coinvariants of $M$ is the $k$-module

$$M^{\sigma H} = \{m \in M \mid \rho(m) = \sigma \otimes m\} = \text{Hom}_H(k_\sigma, M),$$

where $k_\sigma = k$ is the left $H$-comodule defined by $\sigma$.

These concepts are usually considered for $\delta = \epsilon$, the counit of $H$ and $\sigma = 1$, the unit of $H$. The work of Connes-Moscovici \[5, 4, 3\], however, shows that for cyclic cohomology of Non(co)commutative Hopf algebras, it is absolutely necessary to consider this more general case.

By a paracyclic module we mean a simplicial module $M = \{M_n\}_n$ endowed with $k$-linear maps $\tau_n : M_n \rightarrow M_n$, $n \geq 0$, such that the following
identities hold

\[
\begin{align*}
\delta_i \tau_n &= \tau_{n-1} \delta_{i-1} \\
\delta_0 \tau_n &= \delta_n \\
\sigma_i \tau_n &= \tau_{n+1} \sigma_{i-1} \\
\sigma_0 \tau_n &= \tau_n^2 \sigma_n.
\end{align*}
\]

Here \(\delta_i\) and \(\sigma_i\) denote the face and the degeneracy operators of \(M\). If \(\tau_n^{n+1} = id\), for all \(n \geq 0\), we have a cyclic module in the sense of Connes [4].

For example, if \(A\) is an algebra and \(g : A \rightarrow A\) is an automorphism of \(A\), one can check that the following operators define a paracyclic module denoted by \(A_g^2\), where \(A_g^2 = A^{\otimes(n+1)}\) and

\[
\begin{align*}
\delta_0(a_0 \otimes a_1 \otimes \ldots \otimes a_n) &= a_0 a_1 \otimes a_2 \otimes \ldots \otimes a_n, \\
\delta_i(a_0 \otimes a_1 \otimes \ldots \otimes a_n) &= a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n, \quad 1 \leq i \leq n-1, \\
\delta_n(a_0 \otimes a_1 \otimes \ldots \otimes a_n) &= g(a_n) a_0 \otimes a_1 \ldots \otimes a_{n-1}, \\
\sigma_i(a_0 \otimes a_1 \otimes \ldots \otimes a_n) &= a_0 \otimes \ldots \otimes a_i \otimes 1 \otimes \ldots \otimes a_n, \quad 0 \leq i \leq n-1, \\
\tau(a_0 \otimes a_1 \otimes \ldots \otimes a_n) &= g(a_n) \otimes a_0 \otimes a_1 \ldots \otimes a_{n-1}.
\end{align*}
\]

Since \(\tau^{n+1}(a_0 \otimes \ldots \otimes a_n) = ga_0 \otimes \ldots \otimes ga_n\), it is clear that \(A_g^2\) is a cyclic module if and only if \(g = id\).

Dually, we have the notion of a paracyclic module. For example if \(C\) is a coalgebra and \(\theta : C \rightarrow C\) an automorphism of \(C\), the paracyclic module \(C^\theta_s\) is defined by \(C^\theta_s = C^{\otimes(n+1)}, \ n \geq 0\), and

\[
\begin{align*}
\delta_i(c_0 \otimes c_1 \otimes \ldots \otimes c_n) &= c_0 \otimes \ldots \otimes c_i^{(1)} \otimes c_i^{(2)} \otimes c_n \quad 0 \leq i \leq n, \\
\delta_{n+1}(c_0 \otimes c_1 \otimes \ldots \otimes c_n) &= c_{n+1} \otimes c_1 \otimes \ldots \otimes c_n \otimes \theta(c_0^{(1)}), \\
\sigma_i(c_0 \otimes c_1 \otimes \ldots \otimes c_n) &= c_0 \otimes \ldots \otimes c_i \otimes \varepsilon(c_{i+1}) \otimes \ldots \otimes c_n \quad 0 \leq i \leq n-1, \\
\tau(c_0 \otimes c_1 \otimes \ldots \otimes c_n) &= c_1 \otimes c_2 \otimes \ldots \otimes c_n \otimes \theta(c_0).
\end{align*}
\]

We denote the cyclic (co)homology group of a (co)cyclic module \(M\) by \(HC_s(M)\) (resp. \(HC^\bullet(M)\)) and if \(M = A^2\) or \(M = C_2\), we denote them by \(HC_s(A)\) and \(HC^\bullet(C)\) respectively.

We recall the statement of the Eilenberg-Zilber theorem for cylindrical modules from [8] (cf. [11] for a purely algebraic proof). This result is needed in Subsection 3.3. Recall that a cylindrical module is a doubly graded \(k\)-module \(\{X_{p,q}\}_{p,q}\) such that each row and each column is a
paracyclic module, all horizontal operators commute with all vertical operators and for all \( p, q \), \( \tau^{p+1}t^{q+1} = \text{id} : X_{p,q} \rightarrow X_{p,q} \). Here \( t, \tau : X_{p,q} \rightarrow X_{p,q} \) denote the horizontal and vertical cyclic operators respectively. The horizontal simplicial operators are denoted by \( d_i, s_i \), and the vertical operators by \( \delta_i, \sigma_i \).

Given a cylindrical module \( X \), its diagonal, denoted by \( d(X) \) is defined by \( d(X) = X_{n,n} \) with simplicial and cyclic operators given by \( d_i\delta_i, s_i\sigma_i \) and \( t\tau \). It is a cyclic module. Associated to this cyclic module, we have a mixed complex denoted by \((d(X), b_d, B_d)\). The Total complex of \( X \) is also a mixed complex \((\text{Tot}(X), b_t, B_t)\) with \( \text{Tot}(X)_n = \bigoplus_{p+q=n} X_{p,q} \). The generalized cyclic Eilenberg-Zilber theorem \([8]\) states that these two mixed complexes are chain homotopy equivalent. In particular they have isomorphic cyclic homology groups.

Finally we make an important remark about the antipode of Hopf algebras in this paper. Throughout this paper we assume that the antipode is bijective. We need this hypothesis throughout this paper and in particular in the proof of Theorems 3.12 and 4.10. It is known that the class of Hopf algebras with bijective antipode is a very large class that contains quantum groups.

### 3 Invariant Cyclic Homology of Hopf Triples

In this section we define the concept of \( \sigma \)-compatible Hopf triple and its invariant cyclic homology. In Subsection 3.1 we give various examples and calculations including one involving the quantum group \( A(\text{SL}_q(2)) \) and a 2-dimensional module. We also prove a Morita invariance theorem.

In Subsection 3.2 we compare the invariant cyclic homology with the invariant de Rham cohomology for smooth affine algebras. The comparison maps should be isomorphisms but we can verify this only for algebraic groups.

In Subsection 3.4 we consider the invariant cyclic homology of smash product algebras \( A = \mathcal{H}\#B \), where \( B \) is an \( \mathcal{H} \)-module algebra. We derive a spectral sequence and show that it collapses if \( \mathcal{H} \) is semisimple. The ultimate goal here would be to establish a similar spectral sequence for all Hopf-Galois extensions, but it is not clear how to do this at the moment.
3.1 Definition and basic properties

Definition 3.1. By a left Hopf triple we mean a triple $(A, \mathcal{H}, M)$, where $\mathcal{H}$ is a Hopf algebra, $A$ is a left $\mathcal{H}$-comodule algebra and $M$ is a left $\mathcal{H}$-module. Right Hopf triples are defined in a similar way.

Example 3.2.

(i) (Trivial triples). Let $\mathcal{H} = k$, $M$ any $k$-module, and $A$ any $k$-algebra. Then $(A, k, M)$ is a left Hopf triple.

(ii) Let $\mathcal{H}$ be a Hopf algebra and $M$ a left $\mathcal{H}$-module. Then $(A, \mathcal{H}, M)$ is a left Hopf triple, where $A = \mathcal{H}$ is the underlying algebra of $\mathcal{H}$ and $\mathcal{H}$ coacts on $\mathcal{H}$ via its comultiplication. In particular, for $M = k$ and $\mathcal{H}$ acting on $k$ via a character $\delta$, we obtain a Hopf triple $(\mathcal{H}, \mathcal{H}, k_\delta)$.

(iii) Let $G$ be an affine algebraic group acting from left on an affine algebraic variety $X$. Let $\mathcal{H} = \mathbb{C}[G]$ and $A = \mathbb{C}[X]$ be the coordinate rings of $G$ and $X$, respectively. Then $A$ is a left $\mathcal{H}$-comodule algebra. The coaction $\rho : A \rightarrow \mathcal{H} \otimes A$ is induced by the action $G \times X \rightarrow X$. Thus we obtain a Hopf triple $(\mathcal{H}[X], \mathbb{C}[G], \mathbb{C})$.

(iv) Let $G$ be a group. It is easy to see that a $kG$-comodule algebra is nothing but a $G$-graded algebra $A = \bigoplus_{g \in G} A_g$. The coaction $\rho : A \rightarrow kG \otimes A$ is given by $\rho(a) = \sum g \otimes a_g$, where $a = \sum a_g$. Thus for any $G$-graded algebra $A$ and a $G$-module $M$, we obtain a Hopf triple $(A, kG, M)$.

Given a left Hopf triple $(A, \mathcal{H}, M)$, let $C_n(A, M) = M \otimes A^{\otimes (n+1)}$. We define the simplicial and cyclic operators on $\{C_n(A, M)\}_n$ by

\[
\begin{align*}
\delta_0(m \otimes a_0 \otimes a_1 \otimes \ldots \otimes a_n) &= m \otimes a_0 a_1 \otimes a_2 \otimes \ldots \otimes a_n, \\
\delta_i(m \otimes a_0 \otimes a_1 \otimes \ldots \otimes a_n) &= m \otimes a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n, \quad 1 \leq i \leq n - 1, \\
\delta_n(m \otimes a_0 \otimes a_1 \otimes \ldots \otimes a_n) &= a_n^{-1} m \otimes a_n^0 a_0 \otimes a_1 \otimes \ldots \otimes a_{n-1}, \\
\sigma_i(m \otimes a_0 \otimes a_1 \otimes \ldots \otimes a_n) &= m \otimes a_0 \otimes \ldots \otimes a_i \otimes 1 \otimes \ldots \otimes a_n, \quad 0 \leq i \leq n, \\
\tau(m \otimes a_0 \otimes a_1 \otimes \ldots \otimes a_n) &= a_n^{-1} m \otimes a_n^0 \otimes a_0 \otimes \ldots \otimes a_{n-1}.
\end{align*}
\]

Proposition 3.3. Endowed with the above operators, $\{C_n(A, M)\}_n$ is a para-cyclic module.
Next, we define a left $\mathcal{H}$-coaction $\rho : C_n(A, M) \rightarrow \mathcal{H} \otimes C_n(A, M)$ by

$$\rho(m \otimes a_0 \otimes \ldots \otimes a_n) = (a_0^{(-1)} \ldots a_n^{(-1)}) \otimes m \otimes a_0^{(0)} \otimes \ldots \otimes a_n^{(0)}.$$  

**Lemma 3.4.** Endowed with the above coaction, $C_n(A, M)$ is an $\mathcal{H}$-comodule.

To define the space of coinvariants of $C_n(A, M)$, we fix a grouplike element $\sigma \in \mathcal{H}$. Let

$$C_n^H(A, M) = C_n(A, M)^{\co \mathcal{H}} = \{ x \in C_n(A, M) \mid \rho(x) = \sigma \otimes x \},$$

be the space of coinvariants of $C_n(A, M)$ with respect to $\sigma$. We would like to find conditions that guarantee $\{C_n^H(A, M)\}_n$ is a cyclic module. This leads us to the following definitions and results.

**Definition 3.5.** Let $M$ be a left $\mathcal{H}$-module and $\sigma \in \mathcal{H}$ a grouplike element. We define the $(M, \sigma)$-twisted antipode $\hat{S} : M \otimes \mathcal{H} \rightarrow M \otimes \mathcal{H}$ by

$$\hat{S}(m \otimes h) = h^{(2)} m \otimes \sigma S(h^{(1)}).$$

**Definition 3.6.** Let $M$ be a left $\mathcal{H}$-module and $\sigma \in \mathcal{H}$ a grouplike element. We call $(M, \sigma)$ a matched pair if $\sigma m = m$ for all $m \in M$. We call the matched pair $(M, \sigma)$ a matched pair in involution if

$$(\hat{S})^2 = id : M \otimes \mathcal{H} \rightarrow M \otimes \mathcal{H},$$

where $\hat{S}$ is defined in Definition 3.5.

**Example 3.7.** Let $M = k_\delta$ be the one dimensional module defined by a character $\delta \in \mathcal{H}$. It is clear that $(M, \sigma)$ is a matched pair in involution if and only if $(\delta, \sigma)$ is a modular pair in involution in the sense of [12], i.e., $\delta(\sigma) = 1$ and $(\sigma \hat{S}_\delta)^2 = id$.

Let $k$ be a field of characteristic zero and $q \in k$, $q \neq 0$ and $q$ not a root of unity. The Hopf algebra $\mathcal{H} = A(SL_q(2, k))$ is defined as follows. As an algebra it is generated by symbols $x, u, v, y$, with the following relations:

\begin{align*}
ux &= qxu, & vx &= qvx, & yu &= quy, & yv &= qvy, \\
vu &= vu, & xy - q^{-1}uv &= yx - quv &= 1.
\end{align*}
The coproduct, counit and antipode of $H$ are defined by
\[
\Delta(x) = x \otimes x + u \otimes v, \quad \Delta(u) = x \otimes u + u \otimes y, \\
\Delta(v) = v \otimes x + y \otimes v, \quad \Delta(y) = v \otimes u + y \otimes y, \\
\epsilon(x) = \epsilon(y) = 1, \quad \epsilon(u) = \epsilon(v) = 0, \\
S(x) = y, \quad S(y) = x, \quad S(u) = -qu, \quad S(v) = -q^{-1}v.
\]
For more details about $H$ we refer to [13]. We give an example of a Hopf triple where $M$ is not one dimensional.

**Example 3.8.** Let $M$ be a free $k$-module generated by $m_1$ and $m_2$ and let $H = A(SL_q(2,k))$ act on $M$ as follows:
\[
\begin{align*}
xm_1 &= q m_2, & x m_2 &= q m_1, \\
um_1 &= um_2 = vm_1 = vm_2 = 0, & y m_1 &= q^{-1} m_2, & y m_2 &= q^{-1} m_1,
\end{align*}
\]
One can check that $(M,1)$ is a matched pair in involution.

The following lemma will play an important role in the proof of Theorem 3.12. Its proof is elementary and hence omitted.

**Lemma 3.9.** Let $(M,\sigma)$ be a matched pair. Then $\hat{S} : M \otimes H \rightarrow M \otimes H$ is invertible and we have $\hat{S}^{-1}(m \otimes h) = h^{(1)} m \otimes S^{-1}(h^{(2)}) \sigma$.

**Definition 3.10.** Let $(A,H,M)$ be a Hopf triple and $\sigma \in H$ a grouplike element. We say $(A,H,M)$ is $\sigma$-compatible if $(M,\sigma)$ is a matched pair in involution.

**Lemma 3.11.** Let $(A,H,M)$ be a $\sigma$-compatible left Hopf triple. Then for any $a \in A$ and $m \in M$
\[
a^{(-1)} \sigma S(a^{(-3)}) \otimes a^{(-2)} m \otimes a^{(0)} = \sigma \otimes a^{(-1)} m \otimes a^{(0)}.
\]

**Proof.** Since $\hat{S}^{-2} = id$, we have
\[
a^{(-1)} \sigma S(a^{(-3)}) \otimes a^{(-2)} m \otimes a^{(0)} = \hat{S}^{-2}(a^{(-1)} \sigma S(a^{(-3)}) \otimes a^{(-2)} m) \otimes a^{(0)}
= \sigma^{-1} S^{-2}(a^{(-1)} \sigma S(a^{(-3)})) \otimes S^{-1}(a^{(-1)} \sigma S(a^{(7)})) S \sigma a^{(-3)} S(a^{-5}) a^{(-4)} m \otimes a^{(0)}
= \sigma^{-1} S^{-2}(a^{(-2)}) \sigma S^{-1}(a^{(-4)}) \sigma \otimes a^{(-5)} \sigma^{-1} S^{-1}(a^{(-1)}) \sigma a^{(-3)} m \otimes a^{(0)}
= a^{(-1)} S^{-1}(a^{(-2)}) \sigma \otimes a^{(-3)} m \otimes a^{(0)}
= \sigma \otimes a^{(-1)} m \otimes a^{(0)}.
\]

The following theorem is the main result of this section.

**Theorem 3.12.** Let \((A, \mathcal{H}, M)\) be a \(\sigma\)-compatible Hopf triple. Then \(\{C^\mathcal{H}_n(A, M)\}_n\) endowed with simplicial and cyclic operators induced by (1), is a cyclic module.

**Proof.** As a first step we show that the induced simplicial and cyclic operators are well defined on \(\{C^\mathcal{H}_n(A, M)\}_n\). We just prove this for \(\tau\), and \(\delta_n\) and leave the rest to the reader. Let \((m \otimes a_0 \otimes \ldots \otimes a_n) \in C^\mathcal{H}_n(A, M)\). We have
\[
a_0^{(-1)} \ldots a_n^{(-1)} \otimes m \otimes a_0^{(0)} \otimes \ldots \otimes a_n^{(0)} = \sigma \otimes m \otimes a_0 \otimes \ldots \otimes a_n
\]
which implies
\[
a_0^{(-1)} \ldots a_{n-1}^{(-1)} \otimes m \otimes a_0^{(0)} \otimes \ldots \otimes a_{n-1}^{(0)} \otimes a_n = \sigma S(a_n^{(-1)}) \otimes m \otimes a_0 \otimes \ldots \otimes a_n^{(0)}
\]
and
\[
a_n^{(-1)} a_0^{(-1)} \ldots a_{n-1}^{(-1)} \otimes a_n^{(-2)} m \otimes a_n^{(0)} \otimes a_0^{(0)} \otimes \ldots \otimes a_{n-1}^{(0)} = \\
= a_n^{(-1)} \sigma S(a_n^{(-3)}) \otimes a_n^{(-2)} m \otimes a_n^{(0)} \otimes a_0 \otimes \ldots \otimes a_{n-1}.
\]

Applying Lemma 3.11 for \(a = a_n\) we have
\[
a_n^{(-1)} a_0^{(-1)} \ldots a_{n-1}^{(-1)} \otimes a_n^{(-2)} m \otimes a_n^{(0)} \otimes a_0^{(0)} \otimes \ldots \otimes a_{n-1}^{(0)} = \\
\sigma \otimes a_n^{(-1)} m \otimes a_n^{(0)} \otimes a_0 \otimes \ldots \otimes a_{n-1}
\]
which means \(\tau(m \otimes a_0 \otimes \ldots \otimes a_n) \in C^\mathcal{H}_n(A, M)\).

From (3) we obtain
\[
a_n^{(-1)} a_0^{(-1)} \ldots a_{n-1}^{(-1)} \otimes a_n^{(-2)} m \otimes a_n^{(0)} a_0^{(0)} \otimes \ldots \otimes a_{n-1}^{(0)} = \\
\sigma \otimes a_n^{(-1)} m \otimes a_n^{(0)} a_0 \otimes \ldots \otimes a_{n-1}
\]
which implies \(d_n(m \otimes a_0 \otimes \ldots \otimes a_n) \in C^\mathcal{H}_{n-1}(A, M)\).

Checking that the other simplicial operators are well defined on \(\{C^\mathcal{H}_n(A, M)\}_n\) is straightforward.
The only thing left is to show that \( \tau^{n+1} = id \). We have
\[
\tau^{n+1}(m \otimes a_0 \otimes \ldots \otimes a_n) = a_0^{(-1)} \ldots a_n^{(-1)} m \otimes a_0^{(0)} \otimes \ldots \otimes a_n^{(0)}.
\]
Now since we are in \( CH_n(A, M) \), and by (2) we have
\[
\tau^{n+1}(m \otimes a_0 \otimes \ldots \otimes a_n) = \sigma m \otimes a_0 \otimes \ldots \otimes a_n = m \otimes a_0 \otimes \ldots \otimes a_n,
\]
because \((\sigma, M)\) is a matched pair.

We denote the resulting Hochschild, cyclic and periodic cyclic homology groups of the cyclic module \( \{CH_n(A, M)\} \) by \( HH_\bullet^H(A, M) \), \( HC_\bullet^H(A, M) \) and \( HP_\bullet^H(A, M) \), respectively, and refer to them as invariant Hochschild, cyclic and periodic cyclic homology groups of the \( \sigma \)-compatible Hopf triple \((A, H, M)\).

We give a few examples of invariant cyclic homology. More examples can be found in Subsections 3.2, 3.3, and 3.4. It is clear that if \((A, k, k)\) is a trivial Hopf triple (Example 3.2(i)), then \( HC^k_\bullet(A, k) \cong HC_\bullet(A) \), i.e., in this case, invariant cyclic homology is the same as cyclic homology of algebras.

Next we consider the Hopf triple \((A, kG, k)\) defined in Example 3.2(iv). Computing the invariant cyclic homology group \( HC^kG_\bullet(A, k) \), except in special cases, is not easy in general. Let \( G = \mathbb{Z} \) and \( A = \bigoplus_{n \geq 0} A_n \) be a positively graded algebra. One can see that
\[
C^{kG}_n(A, k) \cong C_n(A_0) \quad n \geq 0.
\]
Therefore, we obtain
\[
HC^{k\mathbb{Z}}_n(A, k) \cong HC_n(A_0).
\]
This statement is false if \( A \) has nonzero negative components. For example, if \( A = k[z, z^{-1}] = k\mathbb{Z} \) is the (Hopf) algebra of Laurent polynomials, by Proposition 3.16 we have \( HC^{k\mathbb{Z}}_n(k\mathbb{Z}, k) \cong HC^{(\varepsilon, 1)}_n(k\mathbb{Z}) = k \) for \( n \geq 0 \).

The following lemma enables us to identify \( C^H_n(H, M) \) with \( M \otimes H^{\otimes n} \).

**Lemma 3.13.** Let \( V \) be a left \( H \)-comodule. Then \( (H \otimes V)^{coH} \cong V \).

**Proof.** Define \( \eta : V \longrightarrow (H \otimes V)^{coH} \) by
\[
\eta(v) = \sigma S(v^{(-1)}) \otimes v^{(0)}.
\]
We show that $\eta$ is well defined and is an isomorphism of $k$-modules. Let $\rho$ be the structure map of $V$ and $\bar{\rho}$ be the induced comodule structure on $H \otimes V$. Since

$$\bar{\rho}(\eta(v)) = \bar{\rho}((\sigma S(v(-1)) \otimes v(0)) = \sigma S((v(-2))v(-1) \otimes \sigma S(v(-3)) \otimes v(0)) = \sigma \otimes \sigma S(v(-1)) \otimes v(0),$$

we see that $\eta$ is well defined. Now consider the following map:

$$\theta : (H \otimes V)^{coH} \rightarrow V$$
$$\theta(h \otimes v) = \epsilon(h)v.$$

It is easy to see that $\theta \eta = id_V$. We complete the proof of this lemma by showing that $\eta \theta = id_{H \otimes V}$. Let $h \otimes v \in (H \otimes V)^{coH}$. Then

$$\eta(\theta(h \otimes v)) = \eta(\epsilon(h)v) = \sigma S(v(-1))\epsilon(h) \otimes v(0) = \sigma S(h^{(1)}v(-1))h^{(2)} \otimes v(0) = \sigma S(h \otimes v) = h \otimes v.$$

\[\square\]

Applying the above isomorphism to $V = M \otimes H^{\otimes(n+1)}$, one can identify $C^n_H(\mathcal{H},M)$ with $M \otimes H^{\otimes n}$ and its simplicial and cyclic operators as follows:

1. \(\delta_0(m \otimes h_1 \otimes \ldots \otimes h_n) = \epsilon(h_1)m \otimes h_2 \otimes \ldots \otimes h_n,\)
2. \(\delta_i(m \otimes h_1 \otimes \ldots \otimes h_n) = m \otimes h_1 \otimes \ldots \otimes h_i h_{i+1} \otimes \ldots \otimes h_n, \quad 1 \leq i \leq n - 1\)
3. \(\delta_n(m \otimes h_1 \otimes \ldots \otimes h_n) = h_n m \otimes h_1 \otimes \ldots \otimes h_{n-1},\)
4. \(\sigma_0(m \otimes h_1 \otimes \ldots \otimes h_n) = m \otimes 1 \otimes h_1 \otimes \ldots \otimes h_n,\)
5. \(\sigma_i(m \otimes h_1 \otimes \ldots \otimes h_n) = m \otimes h_1 \otimes \ldots \otimes h_i \otimes 1 \otimes \ldots \otimes h_n, \quad 1 \leq i \leq n\)
6. \(\tau(m \otimes h_1 \otimes \ldots \otimes h_n) = h_n^{(2)} m \otimes \sigma S(h_1^{(1)} \ldots h_n^{(1)}) \otimes h_1^{(2)} \otimes \ldots \otimes h_{n-1}^{(2)}\).

We see that the Hochschild complex of \(\{C^n_H(\mathcal{H},M)\}_n\) is isomorphic to the Hochschild complex of the algebra $\mathcal{H}$ with coefficients in the bimodule $M$ where the right action is via $\epsilon$.

Let us recall that a Hopf algebra is called semisimple if the underlying algebra of $\mathcal{H}$ is semisimple \[13\]. It is known that $\mathcal{H}$ is semisimple if and only if there is a normalized integral in $\mathcal{H}$, i.e., an element $t \in \mathcal{H}$ such that $th = \epsilon(h)t$ for all $h \in \mathcal{H}$ and $\epsilon(t) = 1$.

**Proposition 3.14.** Let $(\mathcal{H}, \mathcal{H}, M)$ be a $\sigma$-compatible Hopf triple. If $\mathcal{H}$ is semisimple then $HC^n_H(\mathcal{H}, M) = M_{\mathcal{H}}$ and $HC_{2n+1}^H(\mathcal{H}, M) = 0$ for all $n \geq 0$. 

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Proof. Using a normalized integral, we define the following homotopy operator \( h : M \otimes \mathcal{H}^\otimes n \to M \otimes \mathcal{H}^\otimes {n+1} \) by

\[
h(m \otimes h_1 \otimes \ldots \otimes h_n) = m \otimes t \otimes h_1 \otimes \ldots \otimes h_n.
\]

It can be checked that \( bh + h = id \), where \( b = \sum_{i=0}^{n} (-1)^i \delta_i \) is the Hochschild boundary map. It follows that \( H^q_\mathcal{H}(\mathcal{H}, M) = 0 \) for \( i \geq 1 \), and \( H^0_\mathcal{H}(\mathcal{H}, M) = M_\mathcal{H} \). The rest is obvious.

As another example we like to compute the invariant cyclic homology of the Hopf triple introduced in Example 3.8. We first compute the Hochschild homology \( H_\bullet(\mathcal{H}, M) \). One knows that \( H_\bullet(\mathcal{H}, M) = \text{Tor}^{\mathcal{H}}_\bullet(\mathcal{H}, M) \), where \( \mathcal{H}^e = \mathcal{H} \otimes \mathcal{H}^{op} \). We take advantage of the free resolution for \( \mathcal{H} \) given in [15]:

\[
\cdots \to M_2 \xrightarrow{d_2} M_1 \xrightarrow{d_1} M_0 \xrightarrow{\mu} \mathcal{H},
\]

where \( \mu \) is the augmentation map and \( M_\bullet \), is a family of free left \( \mathcal{H}^e \)-modules with their rank given by

\[
\begin{align*}
\text{rank}(M_0) &= 1 \\
\text{rank}(M_1) &= 4 \\
\text{rank}(M_2) &= 7 \\
\text{rank}(M_* &= 8, \quad * \geq 3.
\end{align*}
\]

We refer the interested reader to [15] for this resolution. After a lengthy computation we obtain the following theorem.

**Theorem 3.15.** For any \( q \in k \) which is not a root of unity one has \( HC^1_\mathcal{H}(A(SL_q(2)), M) = k \oplus k \) and \( HC^n_\mathcal{H}(A(SL_q(2)), M) = 0 \) for all \( n \neq 1 \).

In [12], and independently [19], for a given Hopf algebra \( \mathcal{H} \) endowed with a modular pair in involution \((\delta, \sigma)\) in the sense of [12], a cyclic module \( \{\tilde{H}_n^{(\delta, \sigma)}\}_n \) is defined. The resulting cyclic homology theory is, in a sense, dual to the cyclic cohomology theory of Hopf algebras defined by Connes and Moscovici. We show that this theory is an example of invariant cyclic homology theory defined in this section. Consider the \( \sigma \)-compatible Hopf triple \((\mathcal{H}, \mathcal{H}, k_\delta)\) defined in Example 3.2(iv). One can check that for \( M = k_\delta \) the operators in (1) are exactly the operators defined in [12]. This proves the following proposition.
Proposition 3.16. The cyclic modules \( \{\widetilde{H}_n^{(\delta, \sigma)}\}_n \) and \( \{C_n^H(H, k_\delta)\}_n \) are isomorphic.

We show that under suitable conditions, the invariant cyclic homology is a direct summand in cyclic homology of algebras. Consider a \( \sigma \)-compatible Hopf triple \((A, H, k)\) where the action of \(H\) on \(k\) is via the counit \(\epsilon\). It is easy to see that in this case the paracyclic module \(\{C_n(A, k)\}_n\) is isomorphic to \(\{C_n(A)\}_n\), the cyclic module of the algebra \(A\). We therefore obtain an inclusion

\[ i : C_n^H(A, k) \hookrightarrow C_n(A). \]

To define a left inverse for \(i\), we need a suitable linear functional on \(H\). Recall from [12] that a trace \(Tr : H \rightarrow k\) is called \(\sigma\)-invariant if for any \(h \in H\)

\[ Tr(h^{(1)})h^{(2)} = Tr(h)\sigma. \]

Given such a trace on \(H\), we define the averaging operator

\[ \gamma : C_n(A) \rightarrow C_n^H(A) \]

\[ \gamma(a_0 \otimes \ldots \otimes a_n) = Tr(a_0^{(-1)} \ldots a_n^{(-1)}) a_0^{(0)} \otimes a_1^{(0)} \otimes \ldots \otimes a_n^{(0)}. \]

Let \(\rho : A \rightarrow H \otimes A\) denote the coaction and \(\bar{\rho}\) the induced coaction. We have

\[ \bar{\rho}\gamma(a_0 \otimes \ldots \otimes a_n) = Tr(a_0^{(-2)} \ldots a_n^{(-2)}) a_0^{(-1)} a_1^{(-1)} \ldots a_n^{(-1)} \otimes a_0^{(0)} \otimes \ldots \otimes a_n^{(0)} \]
\[ = \sigma \otimes Tr(a_0^{(-1)} \ldots a_n^{(-1)}) a_0^{(0)} \otimes a_1^{(0)} \otimes \ldots \otimes a_n^{(0)}, \]

which shows that the image of \(\gamma\) is in fact in the subspace of invariant chains. The following proposition has an elementary proof.

Proposition 3.17. Let \((A, H, k)\) be a \(\sigma\)-compatible Hopf triple and \(H\) admits a \(\sigma\)-invariant trace \(Tr\). Then \(\gamma\) is a cyclic map and \(\gamma i = Tr(\sigma)id\).

Corollary 3.18. If \(Tr(\sigma)\) is invertible in \(k\), then \(HC_n^H(A, k)\) is a direct summand in \(HC_n(A)\).

Let \((\mathcal{H}, H, M)\) be the Hopf triple defined in Example 3.2(ii). One can easily see that \((M_n(\mathcal{H}), H, M)\) is also a Hopf triple, where the coaction of \(\mathcal{H}\) on \(M_n(\mathcal{H})\) is induced by the comultiplication of \(\mathcal{H}\), i.e., for all \(h \otimes u \in \mathcal{H}\).
\( \mathcal{H} \otimes M_n(k) = M_n(\mathcal{H}) \), \( \rho(h \otimes u) = h^{(1)} \otimes h^{(2)} \otimes u \). We have the following identifications

\[
C^n(M_k(\mathcal{H}), M) = M \otimes M_k(\mathcal{H})^\otimes (n+1) = \mathcal{H} \otimes M_k(M) \otimes M_k(\mathcal{H})^\otimes n.
\]

Now we can apply Lemma 3.13 to \( V = M_k(M) \otimes M_k(\mathcal{H})^\otimes n \), to get

\[
C^n_{\mathcal{H}}(M_k(M), M) \cong M_k(M) \otimes M_k(\mathcal{H})^\otimes n.
\]

**Theorem 3.19.** (Morita invariance) For any matched pair in involution \((M, \sigma)\), and any \( k \geq 1 \) one has

\[
HC^n_{\mathcal{H}}(M_k(M), M) \cong HC^n_{\mathcal{H}}(M_k(\mathcal{H}), M), \quad n \geq 0.
\]

**Proof.** Consider the inclusion map

\[
C^n_{\mathcal{H}}(\mathcal{H}, M) \hookrightarrow C^n_{\mathcal{H}}(M_k(\mathcal{H}), M).
\]

One can show that this map is well-defined and is a cyclic module map. On the other hand by the above explanation one can see that

\[
H^n_{\mathcal{H}}(M_k(\mathcal{H}), M) \cong H_n(M_k(\mathcal{H}), M) \cong H_n(\mathcal{H}, M) \cong H^n_{\mathcal{H}}(\mathcal{H}, M),
\]

where the second isomorphism is Morita invariant of ordinary Hochschild homology [16]. The theorem can be proved now by invoking the long exact sequence relating the (invariant) Hochschild and cyclic homology groups. \( \square \)

### 3.2 Relation with invariant de Rham cohomology

In [2], Chevalley and Eilenberg defined the invariant de Rham cohomology of a \( G \)-manifold where \( G \) is a Lie group. One is naturally interested to know to what extent the relation between Hochschild homology and differential forms and between cyclic homology and de Rham cohomology (Hochschild-Kostant-Rosenberg [9] and Connes [6]) extend to our invariant setting. While we do not have a proof, we believe these results can be extended to the invariant case and in this subsection present some evidence in this direction.

Let \( \mu : G \times V \longrightarrow V \) denote the action of an affine algebraic group on a smooth affine algebraic variety \( V \). Let \( \mathcal{H} = \mathbb{C}[G] \) and \( A = \mathbb{C}[V] \) be the coordinate rings of \( G \) and \( V \), respectively. Then \( A \) is an \( \mathcal{H} \)-comodule algebra via a map \( \rho : A \longrightarrow \mathcal{H} \otimes A \) which is obtained by dualizing \( \mu \). Let \( \Omega^\bullet A \)
denote the algebraic de Rham complex of $V$ and $\Omega^\bullet_{\text{inv}} A = \{ \omega \in \Omega^\bullet A \mid g^* \omega = \omega, \forall g \in G \}$ its invariant part. The invariant de Rham cohomology of $V$ is, by definition, the cohomology of the complex $\Omega^\bullet_{\text{inv}} A$. Consider the map

$$\pi : A^{\otimes (n+1)} \to \Omega^n A,$$

and the antisymmetrization map $\gamma : \Omega^n A \to HH_n(A)$,

$$\gamma(a_0 da_1 \ldots da_n) = \frac{1}{n!} \sum_{\alpha \in S_n} sgn(\alpha)(a_0 \otimes a_{\alpha(1)} \otimes \ldots \otimes a_{\alpha(n)}).$$

It is easy to check that $\pi$ and $\gamma$ induce maps

$$\pi^G : (A^{\otimes (n+1)})^{co H} \to \Omega^n_{\text{inv}} A$$

$$\Omega^n_{\text{inv}} A \to HH_n^H(A, k).$$

We therefore obtain maps

$$HH_n^H(A, k) \to \Omega^n_{\text{inv}} (A)$$

and

$$HP_n^H(A, k) \to \bigoplus_{i=0}^{n} H^i_{\text{dR, inv}} (V).$$

We believe that both maps are isomorphisms, but do not have a proof at this stage except for $V = G$ with translation action. In this case the Hochschild homology of the cyclic module $\{C_n^H(H, \mathbb{C})\}$ is isomorphic to $H_n(H, \mathbb{C})$. By Hochschild-Kostant-Rosenberg’s theorem, we obtain

$$H_n^H(H, \mathbb{C}) \cong \wedge^n (\mathfrak{g}^*),$$

where $\mathfrak{g} = \text{Lie}(G)$ is the Lie algebra of $G$. One can then check that

$$HP_n^H(H, \mathbb{C}) \cong \bigoplus_{i=0}^{n} H^i_{\text{Lie}} (\mathfrak{g}, \mathbb{C}) \cong \bigoplus_{i=0}^{n} H^i_{\text{dR, inv}} (G).$$
3.3 The cocommutative case

In this subsection we show that Lie algebra homology (with coefficients) and group homology (with coefficients) can be interpreted in our framework. In fact we prove a much more general result for all cocommutative Hopf algebras. Consider the Hopf triple $(\mathcal{H}, \mathcal{H}, M)$ defined in Example 3.2(ii). We will show that if $\mathcal{H}$ is cocommutative, then

$$HC_n^\mathcal{H}(\mathcal{H}, M) = \bigoplus_{i \geq 0} H_{n-2i}(\mathcal{H}, M)$$

$$HP_n^\mathcal{H}(\mathcal{H}, M) = \bigoplus_{i = n \mod 2} H_i(\mathcal{H}, M)$$

where in the right hand side we have Hopf homology groups. For $\mathcal{H} = U(\mathfrak{g})$ or $\mathcal{H} = kG$ we obtain the relation between invariant cyclic homology on one side and Lie algebra or group homology on the other side.

We recall the definition of Hopf homology groups. This notion extends group homology and Lie algebra homology with coefficients in a module to all Hopf algebras. Let $\mathcal{H}$ be a Hopf algebra. The ground ring $k$ is a right $\mathcal{H}$-module via the map $(r, h) \mapsto r \epsilon(h)$, for all $r \in k$, and $h \in \mathcal{H}$. It is clear that the functor of coinvariants $M \mapsto M^\mathcal{H} = k \otimes M$ from the category of left $\mathcal{H}$-modules to the category of $k$-modules is right exact. We denote the corresponding left derived functors by $H_n^\mathcal{H}(\mathcal{H}, M)$). For $\mathcal{H} = U(\mathfrak{g})$ or $\mathcal{H} = kG$ we obtain Lie algebra or group homology, respectively.

Recall that the path space $EM$ of a simplicial module $M = \{M_n\}_n$ is defined by $(EM)_n = M_{n+1}$. It is a simplicial module with the simplicial operators of $M$ shifted by one. If $M$ is a cyclic module, there is no natural cyclic structure on $EM$. It is therefore remarkable that if $\mathcal{H}$ is cocommutative the path space $\{EC_n^\mathcal{H}(\mathcal{H}, M)\}_n$ is a cyclic module in a natural way. Define the operators

$$t_n : EC_n^\mathcal{H}(\mathcal{H}, M) \rightarrow EC_n^\mathcal{H}(\mathcal{H}, M),$$

by

$$t_n(m \otimes h_0 \otimes \ldots \otimes h_n) = h_n(3) m \otimes h_0(1) \ldots h_n(1) \otimes S(h_1(2) \ldots h_n(2)) \otimes h_1(3) \otimes \ldots \otimes h_{n-1}(3).$$

The proof of the following proposition is similar to Lemma 4.1 in [12] hence we discard it.

**Proposition 3.20.** Let $\mathcal{H}$ be a cocommutative Hopf algebra. Then the path space $\{EC_n^\mathcal{H}(\mathcal{H}, M)\}_n$ is a cyclic module.
We define a map
\[ \theta : EC_n^H(\mathcal{H}, M) \to C_n^H(\mathcal{H}, M) \]
\[ \theta(m \otimes h_0 \otimes \ldots \otimes h_n) = \epsilon(h_0)m \otimes h_1 \otimes \ldots \otimes h_n. \]

In a natural way, \( \mathcal{H} \) has an action on \( EC_n^H(\mathcal{H}, M) \) defined by \( g(m \otimes h_0 \otimes \ldots \otimes h_n) = m \otimes gh_0 \otimes \ldots \otimes h_n \).

**Lemma 3.21.** If \( \mathcal{H} \) is a cocommutative Hopf algebra then \( \{EC_n^H(\mathcal{H}, M)\}_n \) is a cyclic module and \( \theta \) is a cyclic module map. Moreover
\[ k \otimes EC_n^H(\mathcal{H}, M) \cong C_n^H(\mathcal{H}, M). \]

Using the above lemma and the method used in the proof of Theorem 4.1 in [12], we can prove the following theorem.

**Theorem 3.22.** Let \( \mathcal{H} \) be a cocommutative Hopf algebra. Then
\[ HC_n^\mathcal{H}(\mathcal{H}, M) \cong \bigoplus_{i \geq 0} H_{n-2i}(\mathcal{H}, M), \]
where in the right hand side appears the Hopf homology groups.

**Corollary 3.23.** Let \( \mathfrak{g} \) be a Lie algebra and \( M \) a \( \mathfrak{g} \)-module. Then
\[ HC_n^U(\mathfrak{g}, M) \cong \bigoplus_{i \geq 0} H_{n-2i}(\mathfrak{g}, M), \]
where in the right hand we have the Lie algebra homology.

**Corollary 3.24.** Let \( G \) be a (discrete) group and \( M \) a \( G \)-module. Then
\[ HC_n^{kG}(kG, M) \cong \bigoplus_{i \geq 0} H_{n-2i}(G, M), \]
where in the right hand side we have group homology.

### 3.4 Invariant cyclic homology of smash products

Let \( B \) be a right \( \mathcal{H} \)-module algebra and let \( A = \mathcal{H} \# B \) be the smash product of \( \mathcal{H} \) and \( B \). Recall that, as a \( k \)-module, \( A = \mathcal{H} \otimes B \), and the product of \( A \) is defined by
\[(h \otimes a)(g \otimes b) = hg^{(1)} \otimes (a)g^{(2)}b.\]

Now \( A \) is a left \( \mathcal{H} \)-comodule algebra with coaction \( \rho : A \to \mathcal{H} \otimes A \) defined by
\[ \rho(h \otimes a) = h^{(1)} \otimes (h^{(2)} \otimes a). \]
Let $M$ be a left $\mathcal{H}$-module and $\sigma \in \mathcal{H}$ a grouplike element such that $(A, \mathcal{H}, M)$ is a $\sigma$-compatible Hopf triple. In this subsection we give a spectral sequence to compute the invariant cyclic homology groups $\{HC_n^\mathcal{H}(A, M)\}_n$.

The main idea is to define a cylindrical module $\{X_{p,q}\}_{p,q}$, in such a way that its diagonal be isomorphic to the cyclic module $\{C_n^\mathcal{H}(A, M)\}_n$ (cf. II for a similar spectral sequence for Hopf algebra equivariant cyclic homology which motivated this subsection). We can then use the Eilenberg-Zilber theorem for cylindrical modules to obtain the desired spectral sequence. Let

$$X_{p,q} = M \otimes \mathcal{H}^{\otimes p} \otimes B^{\otimes (q+1)}.$$  

We define the vertical and horizontal simplicial and cyclic operators by

$$\delta_0(m \otimes g_1 \otimes \ldots \otimes g_p \otimes a_0 \otimes \ldots \otimes a_q) = m \otimes g_1 \otimes \ldots \otimes g_p \otimes a_0 a_1 \otimes a_2 \otimes \ldots \otimes a_q$$

$$\delta_i(m \otimes g_1 \otimes \ldots \otimes g_p \otimes a_0 \otimes \ldots \otimes a_q) = m \otimes g_1 \otimes \ldots \otimes g_p \otimes a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_q$$

$$\delta_q(m \otimes g_1 \otimes \ldots \otimes g_p \otimes a_0 \otimes \ldots \otimes a_q) = m \otimes g_1^{(2)} \otimes \ldots \otimes g_p^{(2)} \otimes (a_q) \sigma S(g_1^{(1)} \ldots g_p^{(1)}) \ldots g_p^{(3)} a_0 \otimes a_1 \otimes \ldots \otimes a_{q-1}$$

$$\tau(m \otimes g_1 \otimes \ldots \otimes g_p \otimes a_0 \otimes \ldots \otimes a_q) = m \otimes g_1^{(2)} \otimes \ldots \otimes g_p^{(2)} \otimes (a_q) \sigma S(g_1^{(1)} \ldots g_p^{(1)}) g_1^{(3)} \ldots g_p^{(3)} a_0 \otimes a_1 \otimes \ldots \otimes a_{q-1}$$

$$d_0(m \otimes g_1 \otimes \ldots \otimes g_p \otimes a_0 \otimes \ldots \otimes a_q) = \epsilon(g_1) m \otimes g_2 \otimes \ldots \otimes g_p \otimes a_0 \otimes a_1 \otimes \ldots \otimes a_q$$

$$d_i(m \otimes g_1 \otimes \ldots \otimes g_p \otimes a_0 \otimes \ldots \otimes a_q) = m \otimes g_1 \otimes \ldots \otimes g_i g_{i+1} \otimes \ldots \otimes g_p \otimes a_0 \otimes a_1 \otimes \ldots \otimes a_q$$

$$d_p(m \otimes g_1 \otimes \ldots \otimes g_p \otimes a_0 \otimes \ldots \otimes a_q) = g_p^{(1)} m \otimes g_1 \otimes \ldots \otimes g_{p-1} \otimes (a_0) S^{-1}(g_p^{(q+2)}) \otimes \ldots \otimes (a_q) S^{-1}(g_p^{(2)})$$

$$t(m \otimes g_1 \otimes \ldots \otimes g_p \otimes a_0 \otimes \ldots \otimes a_q) = g_p^{(2)} m \otimes \sigma S(g_1^{(1)} \ldots g_p^{(1)}) \ldots g_p^{(2)} \otimes (a_0) S^{-1}(g_p^{(q+3)}) \otimes \ldots \otimes (a_q) S^{-1}(g_p^{(3)})$$

The proof of the following theorem involves several pages of verifications and we simply omit it.

**Theorem 3.25.** Endowed with the operators defined above, $\{X_{p,q}\}_{p,q}$ is a cylindrical module.

The diagonal of this cylindrical module can be identified with the invariant cyclic module $\{C_n^\mathcal{H}(A, M)\}_n$: 
Proposition 3.26. There exists a natural isomorphism of cyclic modules \( d(X) \cong \{C_n^H(A, M)\}_n \).

Proof. We have
\[
C_n^H(A, M) = (M \otimes A^{\otimes (n+1)})^{co \mathcal{H}} = [M \otimes (\mathcal{H} \otimes B)^{\otimes (n+1)}]^{co \mathcal{H}} \cong M \otimes B \otimes (\mathcal{H} \otimes B)^{\otimes n},
\]
where in the last isomorphism we have used Lemma 3.13. One can check that the following two maps are cyclic maps and inverse to one another.

\[
\phi : M \otimes B \otimes (\mathcal{H} \otimes B)^{\otimes (n+1)} \to M \otimes \mathcal{H}^{\otimes n} \otimes B^{\otimes (n+1)}
\]
\[
\phi(m \otimes a_0 \otimes g_1 \otimes a_1 \otimes \ldots \otimes g_n \otimes a_n) = m \otimes g_1^{(1)} \otimes \ldots \otimes g_n^{(1)} \otimes (a_0)g_1^{(2)} \ldots g_n^{(2)} \otimes (a_1)g_2^{(3)} \ldots g_n^{(3)} \otimes \ldots \otimes (a_{n-1})g_n^{(n+2)} \otimes a_n
\]
\[
\psi : M \otimes \mathcal{H}^{\otimes (n)} \otimes B^{\otimes (n+1)} \to M \otimes B \otimes (\mathcal{H} \otimes B)^{\otimes (n+1)}
\]
\[
\psi(m \otimes g_1 \otimes \ldots \otimes g_n \otimes a_0 \otimes \ldots \otimes a_n) = m \otimes (a_0)S^{-1}(g_1^{(2)} \ldots g_n^{(n+1)}) \otimes g_1^{(1)} \otimes (a_1)S^{-1}(g_2^{(2)} \ldots g_n^{(n)}) \otimes \ldots \otimes g_n^{(1)} \otimes (a_{n-1})S^{-1}(g_2^{(2)} \ldots g_n^{(n)}) \otimes a_n.
\]

Now we are in a position to apply the Eilenberg-Zilber theorem for cylindrical modules combined with Proposition 3.26 to conclude that
\[
\text{Tot}(X) \cong d(X) \cong \{C_n^H(A, M)\}_n.
\]

We use the following filtration to derive a spectral sequence for \( HC_n^H(A, M) \)
\[
F_i\text{Tot}X = \sum_{p+q=n, q \leq i} X_{p,q} = \sum_{p+q=n, q \leq i} M \otimes \mathcal{H}^{\otimes p} \otimes B^{\otimes (q+1)}.
\]

Lemma 3.27. The \( E^1 \)-term of the spectral sequence associated to the above filtration is given by the Hopf homology groups
\[
E^1_{p,q} = H_p(\mathcal{H}, M \otimes B^{\otimes (q+1)}),
\]
where the action of \( \mathcal{H} \) on \( M \otimes B^{\otimes (q+1)} \) is given by
\[
h(m \otimes b_0 \otimes \ldots \otimes b_n) = h^{(1)}m \otimes (b_0)S^{-1}(h^{(q+2)}) \otimes \ldots \otimes (a_q)S^{-1}(h^{(2)}).
\]
Proposition 3.28. For any \( p \geq 0 \), the Hopf homology groups \( \{ H_p(\mathcal{H}, M \otimes B^{\otimes(q+1)}) \} \) is a cyclic module.

Proof. This proposition is true for any cylindrical module with the same proof given in \( \S \).

Theorem 3.29. There is a spectral sequence \( E_{p,q} \) that converges to \( HC^n(A, M) \). Its \( E^2 \)-term is given by cyclic homology groups,

\[
E^2_{p,q} = HC_q(H_p(\mathcal{H}, M \otimes B^{\otimes(q+1)})).
\]

It is easy to see that if \( \mathcal{H} \) is semisimple, then the spectral sequence collapses at \( E^1 \)-term. Let \( B^2_{H} \otimes M = H_0(\mathcal{H}, M \otimes b^{\otimes(n+1)}) \) denote the first column of \( E^1 \).

Proposition 3.30. Let \( \mathcal{H} \) be a semisimple Hopf algebra, \( B \) a right \( \mathcal{H} \)-module algebra, and \( A = \mathcal{H}\#B \). Then we have \( HC^n_{\mathcal{H}}(A, M) \cong HC_n(B^2_{H} \otimes M) \).

Proof. Since \( \mathcal{H} \) is semisimple, we have \( E^1_{p,q} = H_p(\mathcal{H}, M \otimes B^{\otimes(q+1)}) = 0 \) for \( p \geq 1 \), and the spectral sequence collapses. Then \( E^2_{p,q} = 0 \) for \( p \geq 1 \) and \( E^2_{0,q} = HC_q(B^2_{H} \otimes M) \).

4 Invariant Cyclic Cohomology of Hopf Cotriples

In this section we define the invariant cyclic cohomology of Hopf module coalgebras. One example is the Connes-Moscovici cyclic homology of a Hopf algebra with a modular pair in involution in the sense of \( \S \) which turns out to be the invariant cyclic cohomology of the coalgebra \( \mathcal{H} \). This is implicit in \( \S \) and explicitly done in \( \S \) for \( \sigma = 1 \). We go, however, beyond this (fundamental) example and define a cocyclic module for any Hopf cotriple \((C, \mathcal{H}, V)\) consisting of an \( \mathcal{H} \)-module coalgebra \( C \), an \( \mathcal{H} \)-comodule \( V \) and a compatible character \( \delta \) on \( \mathcal{H} \). Some of the results that appear in Section \( \S \) e.g. Morita invariance and the spectral sequence in Subsection \( \S \) can be formulated and proved in the context of Hopf cotriples. For the sake of brevity we decide not to include these results here.

4.1 Definition and basic properties

Definition 4.1. By a left Hopf cotriple we mean a triple \((C, \mathcal{H}, V)\) where \( \mathcal{H} \) is a Hopf algebra, \( C \) is a left \( \mathcal{H} \)-module coalgebra and \( V \) is a left \( \mathcal{H} \)-comodule.
Example 4.2. Let $\mathcal{H}$ be a Hopf algebra and $V$ a left $\mathcal{H}$-comodule. Then $(C, \mathcal{H}, V)$ is a left Hopf cotriple, where $C = \mathcal{H}$ is the underlying coalgebra of $\mathcal{H}$, with $\mathcal{H}$ acting on $C$ via multiplication. In particular for $V = k_\sigma$ and $\mathcal{H}$ coacting on $k = k_\sigma$ via a grouplike element $\sigma \in \mathcal{H}$, we have a Hopf cotriple $(\mathcal{H}, \mathcal{H}, k_\sigma)$. This is the Hopf cotriple that is relevant to Connes-Moscovici theory [3, 4, 5].

Example 4.3. (Trivial cotriples). Let $C$ be a coalgebra, $\mathcal{H} = k$ and $V = k$. Then $(C, k, k)$ is a Hopf cotriple.

Given a Hopf cotriple $(C, \mathcal{H}, V)$, let $C^n(C, V) = V \otimes C^{\otimes (n+1)}$. We define cosimplicial and cyclic operators on $(C, C, V)$ by

$$
\delta_i(v \otimes c_0 \otimes c_1 \otimes \cdots \otimes c_n) = v \otimes c_0 \otimes \cdots \otimes c_i^{(1)} \otimes c_i^{(2)} \otimes \cdots \otimes c_n \quad 0 \leq i \leq n
$$

$$
\delta_{n+1}(v \otimes c_0 \otimes c_1 \otimes \cdots \otimes c_n) = v^{(0)} \otimes c_0^{(2)} \otimes c_1 \otimes \cdots \otimes c_n \otimes v^{(-1)}c_0^{(1)}
$$

$$
\sigma_i(v \otimes c_0 \otimes c_1 \otimes \cdots \otimes c_n) = v \otimes c_0 \otimes \cdots c_i \otimes \varepsilon(c_{i+1}) \otimes \cdots \otimes c_n \quad 0 \leq i \leq n - 1
$$

$$
\tau(v \otimes c_0 \otimes c_1 \otimes \cdots \otimes c_n) = v^{(0)} \otimes c_1 \otimes c_2 \otimes \cdots \otimes c_n \otimes v^{(-1)}c_0.
$$

Proposition 4.4. Endowed with the above operators, $(C, C, V)$ is a para-cyclic module.

We have a diagonal $\mathcal{H}$-action on $C^n(C, V)$, defined by

$$
h(v \otimes c_0 \otimes \cdots \otimes c_n) = v \otimes h^{(1)}c_0 \otimes h^{(2)}c_1 \otimes \cdots \otimes h^{(n+1)}c_n.
$$

It is easy to see that $C^n(C, V)$ is an $\mathcal{H}$-module.

To define the space of coinvariants, we fix a character of $\mathcal{H}$, say $\delta$. Let

$$
C^n_\mathcal{H}(C, V) = \frac{C^n(C, V)}{\text{span}\{hm - \delta(h)m \mid m \in C^n(C, V), h \in \mathcal{H}\}}
$$

be the space of coinvariants of $C^n(C, V)$ under the action of $\mathcal{H}$ and with respect to $\delta$. Our first task is to find sufficient conditions under which $(C^n_\mathcal{H}(C, V))_n$ is a cyclic module.

Let us recall the twisted antipode $\widetilde{S} : \mathcal{H} \to \mathcal{H}$, where $\widetilde{S}(h) = \delta(h^{(1)})S(h^{(2)})$, from [5]. We define the $V$-twisted antipode

$$
\widetilde{S}_V : V \otimes \mathcal{H} \rightarrow V \otimes \mathcal{H}
$$

by

$$
\widetilde{S}_V(v \otimes h) = v^{(0)} \otimes S^{-1}(v^{(-1)})\widetilde{S}(h).
$$
The following lemmas are very useful in our theory.

**Lemma 4.5.** The twisted antipode and the $V$-twisted antipode are invertible.

**Proof.** Define

\[ \tilde{S}^{-1} : \mathcal{H} \to \mathcal{H}, \quad \tilde{S}_V^{-1} : V \otimes \mathcal{H} \to V \otimes \mathcal{H} \]

by

\[ \tilde{S}^{-1}(h) = \delta(h^{(2)})S^{-1}(h^{(1)}), \quad \tilde{S}_V^{-1}(v \otimes h) = v^{(0)} \otimes \tilde{S}^{-1}(h)S^{-1}(v^{(-1)}). \]

One can check that $\tilde{S} \circ \tilde{S}^{-1} = \tilde{S}^{-1} \circ \tilde{S} = id_{\mathcal{H}}$, and $\tilde{S}_V \circ \tilde{S}_V^{-1} = \tilde{S}_V^{-1} \circ \tilde{S}_V = id_{V \otimes \mathcal{H}}$. \(\square\)

**Lemma 4.6.** (7) Let $M$ and $N$ be two left $\mathcal{H}$-modules. Then for all $h \in \mathcal{H}$, $m \in M$, and $n \in N$ we have

\[ hm \otimes n = m \otimes \tilde{S}(h)n \quad \text{in} \quad (M \otimes N)_\mathcal{H}, \]

where $-\mathcal{H}$ is the functor of coinvariants with respect to $\delta$.

**Definition 4.7.** We call the pair $(\delta, V)$ a comatched pair if

\[ v^{(0)}\delta(v^{(-1)}) = v \quad \text{for all} \quad v \in V. \]

We call the comatched pair $(\delta, V)$ a comatched pair in involution if

\[ (\tilde{S}_V)^2 = id_{V \otimes \mathcal{H}}. \]

**Definition 4.8.** Let $\delta$ be a character of $\mathcal{H}$. A Hopf cotriple $(C, \mathcal{H}, V)$ is called $\delta$-compatible if $(\delta, V)$ is a comatched pair in involution.

**Lemma 4.9.** If $(\delta, V)$ is a comatched pair, then for any $v \in V$ we have

\[ v = v^{(0)}\delta(v^{(-1)}) = v^{(0)}\delta(S(v^{(-1)})) = v^{(0)}\delta(S^{-1}(v^{(-1)})). \]

**Proof.** The first equality is the definition of comatched pair. We prove the second one. Indeed

\[ v^{(0)}\delta(S(v^{(-1)})) = v^{(0)}\delta(v^{(-1)})\delta(S(v^{(-2)})) = v^{(0)}\epsilon(v^{(-1)}) = v. \]

The last equality can be proved in a similar way. \(\square\)

Now we can prove the main result of this section.
Theorem 4.10. Let \((C, \mathcal{H}, V)\) be a \(\delta\)-compatible Hopf cotriple. Then \(\{C^n_\mathcal{H}(C, V)\}_n\) is a cocyclic module.

Proof. We should first show that the cosimplicial and cyclic operators on \(\{C^n_\mathcal{H}(C, V)\}_n\) are well defined. In the following we check this just for the cyclic operator and the last coface. The rest is easy to check.

Let \(h \in \mathcal{H}\) and \(v \otimes c_0 \otimes \cdots \otimes c_n \in C^n_\mathcal{H}(C, V)\). We prove that 
\[
\tau(h(v \otimes c_0 \otimes \cdots \otimes c_n)) = \delta(h)\tau(v \otimes c_0 \otimes \cdots \otimes c_n)
\]
in the coinvariant space. Indeed,
\[
\tau(h(v \otimes c_0 \otimes \cdots \otimes c_n)) = \tau(v \otimes h^{(1)}c_0 \otimes \cdots \otimes h^{(n+1)}c_n)
\]

Now let us prove that the operator \(\delta_{n+1}\) is also well defined on \(C^n_\mathcal{H}(C, V)\).

We have
\[
\delta_{n+1}(h(v \otimes c_0 \otimes \cdots \otimes c_n)) = \delta_{n+1}(v \otimes h^{(1)}c_0 \otimes \cdots \otimes h^{(n+1)}c_n)
\]
and by the same steps as above this is equal to \(\delta_{n+1}(\delta(h)(v \otimes c_0 \otimes \cdots \otimes c_n))\).

So \(\{C^n_\mathcal{H}(C, V)\}_n\) is at least a paracocyclic module. To prove that it is
cocyclic module we check that $\tau^{n+1} = id$.

$$
\tau^{n+1}(v \otimes c_0 \otimes \cdots \otimes c_n) = v^{(0)} \otimes (v^{(-n-1)})c_0 \otimes \cdots \otimes (v^{(-1)})c_n \\
= v^{(-1)}(v^{(0)} \otimes c_0 \otimes \cdots \otimes c_n) \\
= \delta(v^{(-1)})v^{(0)} \otimes c_0 \otimes \cdots \otimes c_n \\
= v \otimes c_0 \otimes \cdots \otimes c_n.
$$

Example 4.11. Let $\mathcal{H} = V = k$ Then for any coalgebra $C$ one has $\{C^n_k(C, k)\}_n$ is the natural cyclic module, $C_\natural$, of the coalgebra $C$.

Example 4.12. Let $\mathcal{H}$ be a Hopf algebra, $C = \mathcal{H}$, and $V = k_\sigma$. The Hopf cotriple $(\mathcal{H}, \mathcal{H}, k_\sigma)$ is $\delta$-compatible if and only if $(\delta, \sigma)$ is a modular pair in involution in the sense of [4]. In this case $\{C^n_\mathcal{H}(\mathcal{H}, V)\}_n$ is isomorphic to the Connes-Moscovici cocyclic module $\mathcal{H}_z^{(\delta, \sigma)}$.

Lemma 4.13. Let $V$ be a left $\mathcal{H}$-module and $\delta$ a character of $\mathcal{H}$ then

$$(\mathcal{H} \otimes V)_\mathcal{H} \cong V,$$

where $\mathcal{H}$ acts diagonally on $\mathcal{H} \otimes V$.

Example 4.14. We generalize the previous example. Let $\mathcal{H}$ be a Hopf algebra, $V$ be a $\mathcal{H}$-comodule, and assume that $C = \mathcal{H}$. Then if $(\mathcal{H}, \mathcal{H}, V)$ is $\delta$-compatible we can use Lemma 4.12 to simplify the cocyclic module $\{C^n_\mathcal{H}(\mathcal{H}, V)\}_n$ to $\{V \otimes \mathcal{H}^{\otimes n}\}_n$ with the following operators

$$
\begin{align*}
\delta_0(v \otimes h_1 \otimes \cdots \otimes h_n) &= v \otimes 1 \otimes h_1 \otimes \cdots \otimes h_n \\
\delta_i(v \otimes h_1 \otimes \cdots \otimes h_n) &= v \otimes h_1 \otimes \cdots \Delta(h_i) \otimes h_\cdots \otimes h_n, \quad 1 \leq i \leq n \\
\delta_{n+1}(v \otimes h_1 \otimes \cdots \otimes h_n) &= v^{(0)} \otimes h_1 \otimes \cdots \otimes h_n \otimes v^{(-1)} \\
\sigma_i(v \otimes h_1 \otimes \cdots \otimes h_n) &= v \otimes h_1 \otimes \cdots \otimes h_{i-1} \otimes \epsilon(h_i) \otimes h_n, \quad 0 \leq i \leq n-1 \\
\tau(v \otimes h_1 \otimes \cdots \otimes h_n) &= v^{(0)} \otimes S(h_1^{(1)})h_2 \otimes \cdots \otimes S(h_1^{(2)})h_n \otimes \widetilde{S}(h_1^{(1)})v^{(-1)}. 
\end{align*}
$$

By a cotrace in $\mathcal{H}$ we mean an element $t \in \mathcal{H}$ with $t^{(1)} \otimes t^{(2)} = t^{(2)} \otimes t^{(1)}$. Let $\delta$ be a character of $\mathcal{H}$.

Definition 4.15. An element $t \in \mathcal{H}$ is called a left $\delta$-integral if for all $g \in \mathcal{H}$,

$$tg = \delta(g)t.$$
Obviously, for $\delta = \epsilon$, an $\epsilon$-integral is simply a (left) integral in $\mathcal{H}$.

**Example 4.16.** Let $G$ be a finite group and $\delta$ a character of $G$. Then one can check that
\[ t = \sum_{g \in G} \frac{g}{\delta(g)} \]

is a cotracial $\delta$-integral in $\mathcal{H} = kG$.

**Proposition 4.17.** If $\mathcal{H}$ admits a cotracial $\delta$-integral $t$ with $\delta(t)$ invertible in $k$, then $HC^*_\mathcal{H}(C, k)$ is a direct summand in $HC^*(C)$.

**Proof.** Given such an element $t$, we define a right inverse for the projection map
\[ \pi : C^*(C) \longrightarrow C^*_\mathcal{H}(C, k) \]
by
\[ \gamma : C^*_\mathcal{H}(C, k) \longrightarrow C^*(C) \]
\[ \gamma(c_0 \otimes c_1 \otimes \cdots \otimes c_n) = t^{(1)}c_0 \otimes t^{(2)}c_1 \otimes \cdots \otimes t^{(n+1)}c_0. \]
One can check that $\gamma$ is a well-defined cyclic module map and $\pi \gamma = \delta(t) id$. □

### 4.2 The commutative case

In this subsection we show that if $\mathcal{H}$ is a commutative Hopf algebra, then the invariant cyclic cohomology groups of $(\mathcal{H}, \mathcal{H}, V)$ decompose as
\[ HC^m(\mathcal{H}, V) = \bigoplus_{i \geq 0} H^{n-2i}(\mathcal{H}, V), \]
where on the right hand side we have the Hopf cohomology groups of $\mathcal{H}$ with coefficients in $V$ as defined below.

Consider the functor of coinvariants
\[ V \mapsto V^{co\mathcal{H}} = \{ v \in V \mid \rho(v) = 1 \otimes v \} \]
from the category of left $\mathcal{H}$-comodules to the category of $k$-modules. Since $V^{co\mathcal{H}} = Hom_{comod}(k, V)$, this functor is left exact. The corresponding right derived functors are denoted by $H^i(\mathcal{H}, V)$, and they appear on (5). Alternatively, these groups can be calculated from the complex
\[ V \xrightarrow{d_0} V \otimes \mathcal{H} \xrightarrow{d_1} V \otimes \mathcal{H}^\otimes 2 \xrightarrow{d_3} \cdots \]
with the differentials $d_n$, $n \geq 1$, given by

$$d_n(v \otimes h_1 \otimes h_2 \otimes \cdots \otimes h_n) = v \otimes 1 \otimes h_1 \otimes h_2 \otimes \cdots \otimes h_n + \sum_{i=1}^{n} (-1)^i v \otimes h_1 \otimes h_2 \otimes \cdots \otimes h_i^{(1)} \otimes h_{i+1}^{(2)} \otimes \cdots \otimes h_n + (-1)^{n+1} v \otimes h_1 \otimes h_2 \otimes \cdots \otimes h_n \otimes v(-1)$$

and $d_0(v) = v \otimes 1 - v(0) \otimes v(-1)$.

For any cosimplicial module $M = \{M^n\}_n$, its “path space” $EM$ is defined by $(EM)^n = M^{n+1}$ with all operators shifted by 1. It is a cosimplicial module. In particular, for $\{C^n_H(H, V)\}_n$ we obtain the cosimplicial module $\{EC^n_H(H, V)\}_n$.

**Proposition 4.18.** If $H$ is commutative, then $\{EC^n_H(H, V)\}_n$ has a cocyclic module structure.

**Proof.** Define a cyclic action

$$t : EC^n_H(H, V) \rightarrow EC^n_H(H, V)$$

by

$$t(v \otimes h_0 \otimes h_1 \otimes \cdots \otimes h_n) = v \otimes h_0 \otimes h_1 \otimes \cdots \otimes h_n.$$

One can check that all the axioms for cocyclic modules are satisfied. \qed

Now $H$ has a coaction on $EC^n_H(H, V)$ by

$$\rho : EC^n_H(H, V) \rightarrow H \otimes EC^n_H(H, V)$$

$$\rho(v \otimes h_0 \otimes h_1 \otimes \cdots \otimes h_n) = h_0^{(1)} \otimes v \otimes h_0^{(2)} \otimes h_1 \otimes \cdots \otimes h_n.$$ 

One can see that $(EC^n_H(H, V))^\text{co}H \cong C^n_H(H, V)$. By the same method that we used in the proof of Proposition 3.22 we can prove the following proposition.

**Proposition 4.19.** Let $H$ be a commutative Hopf algebra. Then the invariant cyclic cohomology of the $\epsilon$-compatible Hopf cotriple $(H, H, V)$ are given by

$$HC^n_H(H, V) = \bigoplus_{i \geq 0} H^{n-2i}(H, V).$$
Let $G$ be a complex affine algebraic group and $G \times M \rightarrow M$ a linear action of $G$ on a finite dimensional complex vector space $M$. Then $V = M^*$ is a comodule over $\mathcal{H} = \mathbb{C}[G]$. The cohomology groups $H^i(\mathcal{H}, V)$ are easily seen to be isomorphic to (algebraic) group cohomology $H^i_{alg}(G, M)$, where cochains $f : G \times G \times \cdots \times G \rightarrow M$ are assumed to be algebraic functions. It therefore follows from Proposition 4.19, that

$$HC_n^\mathcal{H}(\mathcal{H}, M) = \bigoplus_{i \geq 0} H^{n-2i}(G, M).$$

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