EXPANSION OF REAL VALUED MEROMORPHIC FUNCTIONS INTO FOURIER TRIGONOMETRIC SERIES

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Abstract. In the main part of the paper, on the basis of contour integration of complex meromorphic functions whose singularities lie onto an integration contour, in the first step, a concept of improper integrals absolute existence of meromorphic functions, as more general one with respect to the concept of improper integrals convergence (existence), is introduced into analysis. In the second step, in the case when a modulus of complex parameter tends to infinity, an interval of improper integrals convergence of parametric meromorphic functions is defined. In accordance with this, it is shown that the class of real valued meromorphic functions, whose finitely many isolated singularities lie onto a real axis segment \([t_0, t_1]\), may be expanded into Fourier trigonometric series, separately. At all points of the segment, at which the meromorphic functions are continuous ones, the Fourier trigonometric series is summable and its sum is equal to the function values at those points. Finally, that all is illustrated by two representative examples.

1. Introduction

From the author’s viewpoint, proper attention should be paid to a class of mathematical expressions reducing, in the boundary case, to the difference of infinities \(\infty - \infty\). Namely, in general case, the difference of infinities \(\infty - \infty\) is an indefinite expression taking any value from the extended numeric straight-line (real axis) \(\mathbb{R}^*\), that is, from the segment \([-\infty, +\infty]\). Causality related to the value itself of an indefinite expression is behavior of the mathematical expression reducing to it in the boundary case. Therefore, a case of so-called alternative numerical series is indicative one. \[\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \]

In fact, since each alternative series can be expressed, according to its definition - \textit{Definition 1, Section 2.6, Chapter 2, p. 28, [2] - by difference of two positive numerical series, then in the case when each of them definitely diverges the alternative series reduces to an indefinite expression \(\infty - \infty\). The typical representative of such a class of series, is the alternative numerical series \(\sum_{k=1}^{\infty} (-1)^k\). By means of Caushy’s definition of both the sequence convergence and the numerical series convergence \[\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \]

it can be immediately shown that the alternative numerical series \(\sum_{k=1}^{\infty} (-1)^k\), indefinitely diverges. In other words, since in this emphasized

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case, the limiting value of partial sums \( \sum_{k=1}^{n} (-1)^k \) of the numerical series \( \sum_{k=1}^{+\infty} (-1)^k \) does not exist when \( n \to +\infty \), then the sum of the observed numerical series, in the Caushy’s sense, does not exist too. Accordingly, it is natural to ask the following questions: How much is the sum value of this numerical series, more exactly, is this numerical series summable? Closely related to these questions is another: How much is, in this emphasized instance, the numerical value of an indefinite expression of difference of infinities \( \infty - \infty \)? Clearly, conceptually distinction should be made between summation of series in the Caushy’s sense and its summability. In the modern mathematical analysis, more exactly, in the series theory, and from the point of view of the general convergence (summability) of numerical series, an answer to the former questions was given by Frobenius, Holder and Cesaro \[2\]. In this paper it is presented slightly different and more indirect answer more related to the problem of exact determining rather than to the problem of redefining the sum itself of indefinitely divergent series. It is more indirect because the fundamental conclusions will be based on the results of the complex analysis theory, more exactly, on contour integration of the functions of a complex variable. In this case also, a proper attention will be paid to the mathematical expressions reducing, in the boundary case, to an indefinite expression of difference of infinities \( \infty - \infty \).

2. The main results

2.1. Contour integration and improper integral. The concept of an improper integral absolute existence of a meromorphic function \( f(z): C^1 \to C^1 \); \( C^- \) is the set of complex numbers, clearly in the case when the singularities of the function lie onto the integration contour, is based on concept of total value (v.t.) of an improper integral of a meromorphic function which is defined to be the sum of Caushy’s principal value (v.p.) and Jordan’s singular value (v.s.) of an improper integral of a meromorphic function \( f(z) \). Jordan’s singular value (v.s.) of an improper integral of a meromorphic function is defined to be a limiting value, as \( \varepsilon \to 0^+ \), of an integral of the function \( f(z) \) over a certain part \( \hat{PQ} \) of a circular path of integration \( G_\varepsilon \): \( G_\varepsilon = \{ z \mid z (\theta) = \varepsilon e^{i\theta}; \theta \in [0, 2\pi] \} \), bypassing a singularity of the function, where the points of the complex plane: \( P \) and \( Q \), are intersection points of the circular contour \( G_\varepsilon \) and an integration contour \( G \), \( i \) - denotes an imaginary unit and \( e \) - is a base of natural logarithm. In fact, in other words a concept of improper integrals absolute existence of meromorphic functions generalizes the fundamental concept of improper integrals convergence (existence).

By results of both so-called Jordan’s lemma - Theorem 1, Subsection 3.1.4, Section 3.1, Chapter 3, p. 52, \[3\] - and the fundamental Caushy’s theorem on residues - Theorem 1, Subsection 3.6.2, Section 3.6, Chapter 3, p. 226, \[4\] - the sum of Caushy’s principal value (v.p.) and Jordan’s singular value (v.s.) of an improper integral of a meromorphic function \( f(z) \) whose only singularity (simple pole) lies
onto closed integration contour $G$, can be proved to be

$$v.t. \int_{G} f(z) \, dz = v.p. \int_{G} f(z) \, dz + v.s. \int_{G_{\varepsilon}} f(z) \, dz =$$

$$= v.p. \int_{G} f(z) \, dz + \begin{cases} -i\alpha A \\ i(2\pi - \alpha) A \end{cases} = \begin{cases} 0 \\ i2\pi A \end{cases}$$

where $\alpha$ is an absolute value of an angular difference of arguments of intersection points: $Q$ and $P$, with respect to the point $z_0$, respectively, in the limit as $\varepsilon \to 0^+$, and $A$ is a residue of the function $f(z)$ at the point $z_0$, that is: $A = \lim_{z \to z_0} (z - z_0)^k f(z)$, on condition that such limiting value exists, $\mathbb{3}$ and $\mathbb{3}$. 

In this emphasized case, as distinguished from Caushy’s principal value ($v.p.$), Jordan's singular value ($v.s.$), just as well as the total value ($v.t.$), of an improper integral of a meromorphic function $f(z)$, are not uniquely defined ones, already they depend upon the choice of the part of a circular path bypassing the singularity of the function.

In the case when the singularity of the meromorphic function $f(z)$ at the point $z_0$ is a pole of a higher order, the conclusion essentially differs from the preceding one. Namely, in that case, both the Caushy’s principal value ($v.p.$) and Jordan’s singular value ($v.s.$) of an improper integral do not exist in the limit as $\varepsilon \to 0^+$. The improper integral reduces to an indefinite expression of the difference of infinities $\infty - \infty$. Thus, as for the meromorphic function $z \mapsto \frac{1}{(z-z_0)^k}$, where $k \geq 2$ ($k \in \mathbb{N}$) and $\mathbb{N}$ is a set of natural numbers, the improper integral along an any closed integration path passing through the point $z_0$ absolutely exists and its unique total value ($v.t.$) is identically zero. This is in agreement with both the general Caushy-Goursat’s integral theorem - Theorem 1, Subsection 3.5.2, Section 3.5, Chapter 3, p. 203, $\mathbb{3}$ - and the results of contour integration of rational functions - Subsubsection 3.1.2.3, Subsection 3.1.2. Section 3.1, Chapter 3, p. 46, $\mathbb{3}$. Now then, in this emphasized case, a sum of values of integrals of meromorphic function $f(z)$ over a part of any integration path $G$ between intersection points as well as over the part $\hat{PQ}$ of circular path $G_{\varepsilon}$, is identically zero for each $\varepsilon$. Since choice of an arc path: $\hat{PQ}$ or $\hat{PQ}$, bypassing a singularity of the function is arbitrary, the above-mentioned unique sum remains zero in the limit as $\varepsilon \to 0^+$.

2.1.1. Example. Let the part of an integration path between points: $P$ and $Q$, be a part of circumference of a circle centred at the orgin and of radius $a$: $a \in \mathbb{R}_{+}^{*}$ ($\mathbb{R}_{+}^{*}$ is a set of positive real numbers) and $a$ be also a singularity of the function $f(z)$: $f(z) = \frac{1}{(z-a)^{k^{*}}}$, $k^{*} \geq 2$ and $k^{*} \in \mathbb{N}$. Then, an integral of the function $f(z)$, over the part of a circular integration path $G$ from the point $P$ to the point $Q$ which

$\text{Symbol } \int_{G} \text{ denotes an integration over the closed contour of integration } G, \text{ in this case in the positive mathematical direction}$
dose not contain the singularity $a$, reduces to the integral

$$(2.2) \quad a \int_{\alpha^*}^{2\pi-\alpha^*} \frac{e^{i\theta^*}}{(ae^{i\theta^*} - a)^{k+2}} d\theta^* =$$

$$= -\left(\frac{-i}{2a}\right)^k \left[ \int_{\alpha}^{\pi-\alpha} \frac{\cos(k\theta)}{(\sin \theta)^{k+2}} d\theta - i \int_{\alpha}^{\pi-\alpha} \frac{\sin(k\theta)}{(\sin \theta)^{k+2}} d\theta \right],$$

where $\alpha^*$ is an argument of the point $P$ with respect to the origin ($\alpha = \frac{\alpha^*}{2}$), $\theta = \frac{\theta^*}{2}$ and $k = k^* - 2$ ($k \in N_0; N_0 = \{0, 1, 2, \ldots\}$).

As results of partial integration the following integral dependencies are obtained:

$$(2.3) \quad i (1 + k) \int_{\alpha}^{\pi-\alpha} \frac{2ae^{2i\theta}}{(ae^{2i\theta} - a)^{k+2}} d\theta = \left(\frac{-i}{2a}\right)^{k+1} \left[ \frac{\cos(k\theta)}{(\sin \theta)^k} \cot \theta |_{\alpha}^{\pi-\alpha} - \right.$$}

$$\left. -k \int_{\alpha}^{\pi-\alpha} \frac{\sin[(k-1)\theta]}{(\sin \theta)^{k+1}} d\theta - i (k + 1) \int_{\alpha}^{\pi-\alpha} \frac{\sin(k\theta)}{(\sin \theta)^{k+2}} d\theta \right]$$

and

$$(2.4) \quad (k + 2) \int_{\alpha}^{\pi-\alpha} \frac{\sin[(k+1)\theta]}{(\sin \theta)^{k+3}} d\theta = -2 \frac{\cos(k\theta)}{(\sin \theta)^k} \cot \theta |_{\alpha}^{\pi-\alpha} -$$

$$-2k \int_{\alpha}^{\pi-\alpha} \frac{\sin[(k-1)\theta]}{(\sin \theta)^{k+1}} d\theta - \frac{\sin(k\theta)}{(\sin \theta)^{k+1}} \frac{1}{\sin \theta} |_{\alpha}^{\pi-\alpha}.$$

As, in this emphasized case, $2a \sin \alpha = \varepsilon$ a by-pass integral value is:

$$(2.5) \quad \varepsilon^{-(k+1)} \int_{\frac{\pi}{2} + \alpha}^{\pi - \alpha} e^{-i(k+1)\theta} d\theta = \left\{ \begin{array}{ll}
\frac{(n)}{n} \frac{\sin(2n\alpha)}{(2a \sin \alpha)^{2n}}; & k = 2n - 1 \\
\frac{2(-1)^n \cos(2n+1)\alpha}{2n+1} & k = 2n \end{array} \right. , \quad n \in N.$$

By the comparative analysis of preceding equalities it can be easily shown that if the sum of the integral value of the function $f(z)$ over the circular integration contour $G$ from the point $P$ to the point $Q$ and the by-pass integral value, for arbitrarily chosen $\alpha$ and $k = 2n - 1$ respectively, is identically zero, just as well as the value of the integral $\int_{\alpha}^{\pi-\alpha} \frac{\sin[(k-1)\theta]}{(\sin \theta)^{k+1}} d\theta$ for $k = 2n - 1$, see the equation (2.4), then for $k = 2n - 1$ and $n \in N$ it holds

$$(2.6) \quad \int_{\alpha}^{\pi-\alpha} \frac{\sin(k\theta)}{(\sin \theta)^{k+2}} d\theta = \frac{1}{n} \frac{\sin(2n\alpha)}{(\sin \alpha)^{2n}}.$$
Since the second integral on the right-hand side of the equation (2.3) is identically zero for \( k = 2n \), then in view of the preceding result (2.4) it follows that

\[
(2.7) \quad \int_{\alpha}^{\pi-\alpha} \frac{2a e^{2i\theta}}{(ae^{2i\theta} - a)^{2(n+1)}} d\theta = \frac{2(-1)^n}{2n+1} \left\{ \frac{-\cos[(2n+1)\alpha]}{(2a \sin \alpha)^{2n+1}} \right\}, \ n \in \mathbb{N}.
\]

If one considers the fact that, for a corresponding natural number \( n (n \in \mathbb{N}) \), the sum of functional expressions on right-hand sides of relations: (2.5) (for \( k = 2n \) and 2.7), is identically zero for arbitrarily chosen \( \alpha \), then the sum of integrals on the left-hand sides of these equations is identically zero too. Thus, the improper integral of the function \( f \rightarrow \alpha \) exists and its total value, as limiting value of a sum of integrals (2.2) and (2.5) as a case, there exists a sequence of circular contours of integration \( G_r \) and (2.8), is identically zero for each \( k^* \geq 2 \) and \( k^* \in \mathbb{N} \), and what has been just proved by method of a mathematical induction.

**Comment:** As it has been just illustrated by the previous example, the concept of an improper integral absolute existence is more general concept with respect to the concept of an improper integral convergence. This is in connection with indefinite expression of the difference of infinities \( \infty - \infty \) to which the improper integral value is reduced in a boundary case. Namely, independently from the fact that the improper integral absolutely exists, in some of the concrete cases its principal value does not exist. Hence, by introducing a by-pass integral into the analysis the concept itself of improper integral convergence (existence) is generalized to the concept of improper integral absolute existence.

### 2.2. Fourier trigonometric series of real valued meromorphic functions.

#### 2.2.1. An analysis of an idea

Without loss of the generality, one may assume that a complex meromorphic function \( g(z, t) \), where the variable \( t \) is independent one with respect to the complex variable \( z \), has infinitely but a count of many simple poles: \( a_1, a_2, ... \) onto the imaginary axis of the complex plane \( C^1 \). In that emphasized case, there exists a sequence of circular contours of integration \( G_r \), centred at the origin and of radius \( r \), such that onto theirs boundaries there are no singularities of the function \( g(z, t) \). Hence, by the fundamental Cauchy’s theorem on residues the sequence of the partial sums can be formed

\[
(2.8) \quad \sum_{k=1}^{n} A_k(t) = \frac{1}{2 \pi i} \oint_{G_r} g(z, t) \, dz,
\]

where \( A_k(t) : A_k(t) = \text{Res}_z g(z, t), z = a_k \), are residues of the function \( g(z, t) \) at the points: 

\[
z = a_k, \ k = 1, 2, ..., n.
\]

For \( k = 0 \) and \( k = 1 \) i.e. \( k^* = 2 \) and \( k^* = 3 \), on the basis of the relation (2.3), it holds

\[
\int_{\alpha}^{\pi-\alpha} \frac{2a e^{2i\theta}}{(ae^{2i\theta} - a)^{k^*}} d\theta = -\cot \frac{\alpha}{a} \quad \text{and} \quad \int_{\alpha}^{\pi-\alpha} \frac{2a e^{2i\theta}}{(ae^{2i\theta} - a)^{k^*+1}} d\theta = \frac{\cot \alpha}{2a^{k^*+1}}, \ \text{respectively. From the relation (2.5), the values of by-pass integrals, in these emphasized cases, are equal to:} \quad \frac{1}{e^{\pm i\alpha}} \int_{\frac{\pi}{2} + \alpha}^{-\frac{\pi}{2} + \alpha} e^{-2i\theta} d\theta = \frac{\cot \alpha}{2a^{k^*+1}}, \ \text{respectively. Accordingly, the total value of improper integral} i \int_{\alpha}^{\frac{\pi}{2} + \alpha} \frac{2a e^{i\theta}}{(ae^{i\theta} - a)^{k^*}} d\theta, \ \text{is identically zero, for both} \ k^* = 2 \ \text{and} \ k^* = 3.
\]
On the one hand, on the basis of the second Jordan’s lemma - Theorem 2, Subsection 3.1.4, Section 3.1, Chapter 3, p. 52, 3 - if there exists an unique limiting value: \[ \lim_{|z| \to +\infty} [z g(z,t)]; \] for each \( z \in C^1 \), then the sequence of the partial sums \[ \sum_{k=1}^{n} A_k(t) \] converges, in other words there exists a sum of the infinite functional series \[ \sum_{k=1}^{+\infty} A_k(t) \] in the Cauchy’s sense:

\[
(2.9) \quad \sum_{k=1}^{+\infty} A_k(t) = - \text{Res} \ g(z,t),
\]

where \( 2\pi i \text{ Res} \ g(z,t) = \lim_{r \to +\infty} \oint_{G_{t}} g(z,t) \, dz = -2\pi i \lim_{|z| \to +\infty} [z g(z,t)]. \)

However, on the other hand, by the same Jordan’s lemma, if there exists no an above mentioned limiting value: \[ \lim_{|z| \to +\infty} [z g(z,t)]; \] for each \( z \in C^1 \), already there exist only partial limiting values: \[ \lim_{|z| \to +\infty} [z g(z,t)]; \] \( \text{Re} \ z > 0 \) and \( \lim_{|z| \to +\infty} [z g(z,t)]; \) \( \text{Re} \ z < 0 \), then there exists a infinite sum of the residues of the function \( g(z,t) \) that is equal to the limiting sum of integral values:

\[
(2.10) \quad \sum_{k=1}^{+\infty} A_k(t) = \frac{1}{2\pi i} \lim_{r \to +\infty} \left[ \int_{G_{k}^{R}} g(z,t) \, dz + \int_{G_{k}^{L}} g(z,t) \, dz \right],
\]

where the integral paths: \( G_{k}^{R} = \{ z \mid z(\theta) = r e^{i\theta}; \theta \in [-\frac{\pi}{2} + \delta(r), \frac{\pi}{2} - \delta(r)] \} \) and \( G_{k}^{L} = \{ z \mid z(\theta) = r e^{i\theta}; \theta \in [\frac{\pi}{2} + \delta(r), \frac{3\pi}{2} - \delta(r)] \} \), are arc parts of the circular path of integration \( G_{t} \) in the right-hand and left-hand half-plane of the complex plane \( C^1 \), respectively, and an arbitrary angular function \( \delta(r) \), which is of sufficiently small real positive values for any positive values of \( r \), satisfies the condition:

\[ \lim_{r \to +\infty} \delta(r) = 0. \]

In other words, although in this emphasized case there exists no a sum of the infinite functional series \( \sum_{k=1}^{+\infty} A_k(t) \) in the Cauchy’s sense, this infinite functional series is summable. Note that in this case too:

\[
\sum_{k=1}^{+\infty} A_k(t) = - \text{Res} \ g(z,t), \quad \text{where} \quad 2\pi i \text{ Res} \ g(z,t) = \lim_{r \to +\infty} \left[ \int_{G_{k}^{R}} g(z,t) \, dz + \int_{G_{k}^{L}} g(z,t) \, dz \right].
\]

2.2.2. Cauchy’s formula. As it is well-known, during the deriving Cauchy’s formula for expansion of real valued functions into an infinite functional series, 1 and 2 (taken over from 3) - Formula (9), Subsection 4.6.2, Section 4.6, Chapter 4, p. 94, 3 - in a real axis interval \( (t_0, t_1) \)

\[
(2.11) \quad f(t) = \sum_{k=1}^{+\infty} \frac{w(a_k)}{dz} \left[ t_k \int_{t_0}^{t_k} e^{a_k(t-\tau)} f(\tau) \, d\tau; \, t \neq t_{ai} \right],
\]
where $t_{si}$ are break points of the function $t \mapsto f(t)$ in $(t_0, t_1)$, the conditions for existence of finite limiting values of the functional expressions:

\[
(2.12) \quad \lim_{|z| \to +\infty} \left[ z \frac{p(z)}{q(z)} \int_{t_0}^{t} e^{z(t-\tau)} f(\tau) \, d\tau - z \frac{p(-z)}{q(-z)} \int_{t_0}^{t} e^{-z(t-\tau)} f(\tau) \, d\tau \right];
\]

\[
(2.13) \quad \lim_{|z| \to +\infty} \left[ z \frac{w(z)}{q(z)} \int_{t}^{t_1} e^{z(t-\tau)} f(\tau) \, d\tau - z \frac{w(-z)}{q(-z)} \int_{t}^{t_1} e^{-z(t-\tau)} f(\tau) \, d\tau \right],
\]

which have to be satisfied by the function $t \mapsto f(t)$ in $(t_0, t_1)$, are of the most importance.

Namely, let $g_1(z, t) = \frac{p(z)}{q(z)} \int_{t_0}^{t} e^{z(t-\tau)} f(\tau) \, d\tau$ and $g_2(z, t) = \frac{w(z)}{q(z)} \int_{t}^{t_1} e^{z(t_\tau)} f(\tau) \, d\tau$, where an analytic function $q(z)$: $q(z) = p(z) + w(z)$, has infinitely but a count of many simple poles: $a_1, a_2, \ldots$ onto the imaginary axis as singularities. If under an assumption that: \[ \lim_{|z| \to +\infty} \frac{w(z)}{q(z)} = 0 \] and \[ \lim_{|z| \to +\infty} \frac{w(-z)}{q(-z)} = 1 \] as well as

\[ \lim_{|z| \to +\infty} \frac{w(z)}{q(z)} = 0 \] and \[ \lim_{|z| \to +\infty} \frac{w(-z)}{q(-z)} = 1 \]

the following functional expressions:

\[ \lim_{|z| \to +\infty} \int_{t_0}^{t} e^{-z(t_\tau)} f(\tau) \, d\tau \] and \[ \lim_{|z| \to +\infty} \int_{t}^{t_1} e^{-z(t_\tau)} f(\tau) \, d\tau, \]

in such a way that: \[ \lim_{|z| \to +\infty} \int_{t}^{t_1} e^{-z(t_\tau)} f(\tau) \, d\tau = f(t) \] and \[ \lim_{|z| \to +\infty} \int_{t_0}^{t} e^{-z(t_\tau)} f(\tau) \, d\tau = f(t); \ t \neq t_{si}, \] then on the basis of previous analyzed idea it can be proved that at all points of $(t_0, t_1)$, at which a function $t \mapsto f(t)$ is continuous, there exists a sum of an infinite functional series on the right-hand side of the equation (2.11) which is just equal to the function values at those points, more exactly, in the general case, Cauchy’s infinite functional series of the function $t \mapsto f(t)$ is summable.

If a function $t \mapsto f(t)$ satisfies the general well-known Dirichlet’s conditions in $(t_0, t_1)$, then partial sums of Cauchy’s infinite functional series of the function $t \mapsto f(t)$ at all points of $(t_0, t_1)$, at which the function $t \mapsto f(t)$ is continuous, converge to the function values at those points \[ \lim_{|z| \to +\infty} \int_{t}^{t_1} e^{-z(t_\tau)} f(\tau) \, d\tau = f(t) \] \[ \lim_{|z| \to +\infty} \int_{t_0}^{t} e^{-z(t_\tau)} f(\tau) \, d\tau = f(t); \ t \neq t_{si}, \] At the break points $t_{si}$ of the function $t \mapsto f(t)$ in $(t_0, t_1)$ the partial sums of Cauchy’s infinite functional series of the function $t \mapsto f(t)$ converge to the following functional values

\[
(2.14) \quad \frac{1}{2} \left[ \lim_{\varepsilon \to 0^+} f(t_{si} + \varepsilon) + \lim_{\eta \to 0^+} f(t_{si} - \eta) \right].
\]

At the extreme points of the segment $[t_0, t_1]$: $t_0$ and $t_1$, at which a function $t \mapsto f(t)$ is continuous on one’s right and left respectively, the sum of Cauchy’s infinite functional series of the function $t \mapsto f(t)$ is equal to the following functional value

\[
(2.15) \quad \frac{1}{2} \left[ \lim_{\varepsilon \to 0^+} f(t_0 + \varepsilon) + \lim_{\eta \to 0^+} f(t_1 - \eta) \right].
\]
2.2.3. Interval of improper integrals convergence of real valued functions. Let $\nu \mapsto f(\nu)$, be an analytic function of complex variable $\nu$ on some neighborhood $V_0$ of the point $\nu = 0$ at which the function $\nu \mapsto f(\nu)$ has a pole of arbitrary order as a singularity. A function $\nu \mapsto f(\nu) e^{-z(t-\nu)}$, where $z \in C^1$ is a complex parameter and $t \in R^1_+$ ($Re \nu = t$) is fixed point belonging to the neighborhood $V_0$, is parametric analytic function on $V_0$. Further, a smooth one-one mapping $\nu(\theta): R^1 \rightarrow C^1$ ($\nu(\theta) = \frac{t-t_0}{2} e^{i\theta} + \frac{t_0+1}{2}$) of the real axis segment $[-\pi, 0]$ ($\theta \in [-\pi, 0]$) onto the set of complex points $\nu$ of the complex plane $C^1$ is defined. A fixed point $Re \nu = t_0$ ($t_0 < 0$) also belongs to the neighborhood $V_0$ of the zero point $\nu = 0$. An arbitrary $n$-division $P_n$: $P_n = \{\theta_0 = -\pi, \theta_1, ..., \theta_i, ..., \theta_n = 0\}$, where $n \in N$, is one of all possible $n$-divisions of the segment $[-\pi, 0]$.

Accordingly, since a complex function $f[\nu(\theta)] e^{-z(t-\nu(\theta))}$ of a real variable $\theta$ is a regular that in the segment $[-\pi, 0]$, and in those circumstances its both a real and an imaginary part satisfies all conditions of Langrange’s mean value theorem of the differential calculus in the segment $[-\pi, 0]$, then, for each partial segment $[\theta_{i-1}, \theta_i]$ of the segment $[-\pi, 0]$, it holds

\[
\text{Re} \left\{ \left\{ \frac{d}{d\theta} \left\{ f[\nu(\theta)] e^{-z(t-\nu(\theta))} \right\} \right\}_{\theta=\theta_i^*} \right\} = \frac{\text{Re} \left\{ f[\nu(\theta_i)] e^{-z(t-\nu(\theta_i))} \right\} - \text{Re} \left\{ f[\nu(\theta_{i-1})] e^{-z(t-\nu(\theta_{i-1}))} \right\}}{\theta_i - \theta_{i-1}};
\]

\[
\text{Im} \left\{ \left\{ \frac{d}{d\theta} \left\{ f[\nu(\theta)] e^{-z(t-\nu(\theta))} \right\} \right\}_{\theta=\theta_i^*} \right\} = \frac{\text{Im} \left\{ f[\nu(\theta_i)] e^{-z(t-\nu(\theta_i))} \right\} - \text{Im} \left\{ f[\nu(\theta_{i-1})] e^{-z(t-\nu(\theta_{i-1}))} \right\}}{\theta_i - \theta_{i-1}},
\]

where $\{\theta_i^*, \theta_i^{**}\} \in [\theta_{i-1}, \theta_i]$.

By virtue of (2.16) and (2.17), it is possible to form the integral sums

\[
\sum_{i=1}^{n} \text{Re} \left\{ \left\{ \frac{d}{d\theta} \left\{ f[\nu(\theta)] e^{-z(t-\nu(\theta))} \right\} \right\}_{\theta=\theta_i^*} \right\} (\theta_i - \theta_{i-1}) = \text{Re} \left[ f(t) \right] - \text{Re} \left[ f(t_0) e^{-z(t-t_0)} \right];
\]

\[
\sum_{i=1}^{n} \text{Im} \left\{ \left\{ \frac{d}{d\theta} \left\{ f[\nu(\theta)] e^{-z(t-\nu(\theta))} \right\} \right\}_{\theta=\theta_i^*} \right\} (\theta_i - \theta_{i-1}) = \text{Im} \left[ f(t) \right] - \text{Im} \left[ f(t_0) e^{-z(t-t_0)} \right],
\]

that is, after the performed differentiation

\[
\sum_{i=1}^{n} \text{Re} \left\{ \left\{ \frac{d}{d\theta} f[\nu(\theta)] e^{-z(t-\nu(\theta))} \right\}_{\theta=\theta_i^*} \right\} (\theta_i - \theta_{i-1}) +
\sum_{i=1}^{n} \text{Re} \left\{ z f[\nu(\theta)] e^{-z(t-\nu(\theta))} \right\}_{\theta=\theta_i^*} (\theta_i - \theta_{i-1}) =
\]
Re \{ f (t) \} - Re \left[ f (t_0) e^{-z(t-t_0)} \right];

(2.21) \sum_{i=1}^{n} \text{Im} \left\{ \left( \frac{d}{d\theta} f [\nu (\theta)] e^{-z[t-\nu(\theta)\]} \right) \right\} \bigg|_{\theta=\theta_i*} (\theta_i - \theta_{i-1}) +

+ \sum_{i=1}^{n} \text{Im} \left\{ zf [\nu (\theta)] e^{-z[t-\nu(\theta)\]} \right\} \bigg|_{\theta=\theta_i*} (\theta_i - \theta_{i-1}) =

= \text{Im} \{ f (t) \} - \text{Im} \left[ f (t_0) e^{-z(t-t_0)} \right].

Having in view the fact that \( \nu (\theta) = \frac{t-t_0}{2} e^{i\theta} + \frac{t_0+t}{2}, \) it is possible to define an interval (a semi-interval) of a change of the argument \( \varphi: \varphi \in R^1, \) of the complex parameter \( z, \) for which it holds: \( \lim_{|z| \to +\infty} e^{-z[t-\nu(\theta)\]} = 0; \theta \in (-\pi, 0). \) Namely, since for \( \text{Re} \, z \neq 0 \)

(2.22)

\[ z \left[ t - \nu (\theta) \right] = z \left( \frac{t - t_0}{2} \right) (1 - e^{i\theta}) = \]

\[ = \left( \frac{t - t_0}{2} \right) \left[ \text{Re} \, z (1 - \cos \theta) \right] \left( 1 + \frac{\text{Im} \, z}{\text{Re} \, z 1 - \cos \theta} \sin \theta \right) + \]

\[ + i \left[ \text{Im} \, z (1 - \cos \theta) - \text{Re} \, z \sin \theta \right], \]

then, if the condition

(2.23) \[ \text{Re} \, z (1 - \cos \theta) \left( 1 + \frac{\text{Im} \, z}{\text{Re} \, z 1 - \cos \theta} \sin \theta \right) > 0 \]

is satisfied, it follows that \( \lim_{|z| \to +\infty} e^{-z[t-\nu(\theta)\]} = 0; \theta \in (-\pi, 0). \)

In view of the fact that \( \frac{\sin \theta}{1 - \cos \theta} = \cot \frac{\theta}{2} = \left( \tan \frac{\theta}{2} \right)^{-1} \) and \( \frac{\text{Im} \, z}{\text{Re} \, z} = \tan \varphi, \) the condition (2.23) is satisfied if and only if \( \varphi \in (-\frac{\pi}{2}, 0], \) in other words for each \( \varphi \in (-\frac{\pi}{2}, 0] \) it holds \( \lim_{|z| \to +\infty} e^{-z[t-\nu(\theta)\]} = 0; \theta \in (-\pi, 0). \)

Hence, and on the basis of derived relations: (2.20) and (2.21), in the limit as \( n \to +\infty, \) more exactly, when a maximum partial segment \([\theta_{i-1}, \theta_i]\) of the segment \([-\pi, 0]\) vanishes, the condition \( \varphi \in (-\frac{\pi}{2}, 0] \) becomes a condition of convergence of a parametric contour integral

(2.24) \[ \lim_{|z| \to +\infty} z \int_G f (\nu) e^{-z(t-\nu)} d\nu = f (t), \]

where \( G = \{ \nu \mid \nu = \frac{t-t_0}{2} e^{i\theta} + \frac{t_0+t}{2} \} \) and \( \theta \in [-\pi, 0]. \)

In other words, the semi-interval \((-\frac{\pi}{2}, 0]\) is a semi-interval of a complex parametric contour integral convergence of the function \( f (\nu) e^{-z(t-\nu)} \) along the given contour of integration \( G, \) in the limit as \( |z| \to +\infty. \)

In the next step, instead of the above integration path \( G, \) a complex plane curve \( G^* \) consisting of parts of real axis defined by segments: \([t_0, -\varepsilon]\) and \([\varepsilon, t]\) \((\varepsilon \in R^1_+)\) as well as of a part of a circular path defined by a smooth one-one mapping \( \nu (\theta): R^1 \to C^1 \left( \nu (\theta) = \varepsilon e^{i\theta} \right) \) of the segment \([-\pi, 0]\) of a real axis onto a set of points of the complex plane, is taken for a contour of integration. The integral
\[ \int_{G^*} f(\nu) e^{-z(t-\nu)} d\nu, \]

defined over the integration contour \( G^*_\epsilon = \{ \nu \mid \nu(\theta) = \epsilon e^{i\theta} \} \), bypassing a singularity of a function \( f(\nu) e^{-z(t-\nu)} \) at the point \( \nu = 0 \), is a by-pass integral. In this case too, similarly to the previous analysis, it is possible to define an interval (a semi-interval) of a parametric contour integral convergence of the function \( f(\nu) e^{-z(t-\nu)} \). Namely, since in this case

\[
(2.25) \quad z[t - \nu(\theta)] = z[t - \epsilon (\cos \theta + i \sin \theta)] = \\
= \text{Re} z(t - \epsilon \cos \theta) + \epsilon \text{Im} z \sin \theta + \\
i[\text{Im} z(t - \epsilon \cos \theta) - \epsilon \text{Re} z \sin \theta],
\]

then the condition (2.23), for the contour of integration \( G \), reduces to the condition

\[ (2.26) \quad \text{Re} z \left[ (t - \epsilon \cos \theta) + \epsilon \frac{\text{Im} z}{\text{Re} z} \sin \theta \right] > 0; \quad \text{Re} z > 0,
\]

for the contour of integration \( G^*_\epsilon \).

In view of the fact that \( t > \epsilon > 0 \), there exists a positive real number \( k: k \in \mathbb{R}_+^1 \), such that \( t = (1 + k) \epsilon \). Hence, the condition (2.26) reduces to the condition

\[ (2.27) \quad \epsilon \text{Re} z[k + (1 - \cos \theta) + \tan \varphi \sin \theta] > 0; \quad \text{Re} z > 0.
\]

As \( |\sin \theta| \leq 1 \), for \( \theta \in (-\pi, 0) \), the condition (2.27) is satisfied if and only if \( \varphi \in (-\frac{\pi}{2}, \arctan k] \), in other words for each \( \varphi \in (-\frac{\pi}{2}, \arctan k] \) it holds

\[ (2.28) \quad \lim_{|z| \to +\infty} \int_{G^*_\epsilon} f(\nu) e^{-z(t-\nu)} d\nu = 0.
\]

On the other hand, since the real meromorphic function \( f[(\text{Re} \nu)] \) satisfies in the semi-segment \([t_0, 0)\), as well as in the semi-interval \((0, t]\), general well-known Dirichlet’s conditions, 2 and 3, then, for each \( \text{Re} z > 0 \), it holds:

\[ \lim_{|z| \to +\infty} \int_{t_0}^t f(\tau) e^{-z(t-\tau)} d\tau = 0 \quad \text{and} \quad \lim_{|z| \to +\infty} \int_{t}^\infty f(\tau) e^{-z(t-\tau)} d\tau = f(t),
\]

respectively, where \( \tau = \text{Re} \nu \). Therefore, the semi-interval \((-\frac{\pi}{2}, \arctan k] \) is a semi-interval of convergence of the parametric contour integral

\[ (2.29) \quad \lim_{|z| \to +\infty} \int_{G^*_\epsilon} f(\nu) e^{-z(t-\nu)} d\nu = f(t).
\]

Taking into consideration the fact that in the limit as \( \epsilon \to 0^+ \) and for each \( t > 0 \):

\( k \to +\infty \) (\( t = (1 + k) \epsilon \), the interval \((-\frac{\pi}{2}, \frac{\pi}{2}) \) \( (\varphi \in (-\frac{\pi}{2}, \frac{\pi}{2})) \) becomes an interval of convergence of an improper integral

\[ (2.30) \quad \lim_{|z| \to +\infty} \left[ v.t. \int_{t_0}^t f(\tau) e^{-z(t-\tau)} d\tau \right] = f(t)
\]

absolutely existing in the segment \([t_0, t] \); \( t > 0 \). In other words, for each \( \text{Re} z > 0 \), it holds (2.30).
In the similar manner it can be proved to be

\[
\lim_{|z| \to +\infty} \frac{z}{|z|} \left[ \text{v.t.} \int_{t_0}^{t} f(\tau) e^{-z(\tau-t_0)} d\tau \right] = f(t_0),
\]

for each \( \text{Re } z > 0 \) and \( t > 0 \) (\( t \in (t_0, t_1) \)).

**Comment:** It should be emphasized that as distinguished from limiting values of contour integrals:

\[
\lim_{|z| \to +\infty} z \int_{G} f(\nu) e^{-z(\nu-t)} d\nu \quad \text{and} \quad \lim_{|z| \to +\infty} z \int_{G^*} f(\nu) e^{-z(\nu-t)} d\nu,
\]

which are equal, their intervals of convergence are different. ▼

2.2.4. **Fourier formula.** Based on the results obtained by previous analysis one may say that in addition to a class of real valued functions satisfying so-called general Dirichlet’s conditions in the real axis segment \([t_0, t_1]\), for which the functional expressions: (2.12) and (2.13), converge for each \( \text{Re } z \geq 0 \), there exists an one other class of real valued functions for which Cauchy’s formula is still in effect, a class of the real valued meromorphic functions whose finitely many isolated singularities lie onto the segment \([t_0, t_1]\).

From Cauchy’s formula, and for \( p(z) = -1 \) and \( w(z) = e^{az} \) \((a \in R_0^+)\) i.e. \( q(z) = e^{az} - 1 \), it is immediately obtained that for \( t_1 - a < t < t_0 + a \) and \( t \neq t_{\text{si}} \)

\[
f(t) = \frac{1}{a} \left[ \text{v.t.} \int_{t_0}^{t_1} f(\tau) d\tau \right] + \frac{2}{a} \sum_{k=1}^{+\infty} \left[ \text{v.t.} \int_{t_0}^{t_1} f(\tau) \cos \frac{2k\pi (t-\tau)}{a} d\tau \right].
\]

If \( a = 2\pi \), \( t_0 = -\pi \) and \( t_1 = \pi \), then the equation (2.32) represents an expansion of a real valued meromorphic function \( t \mapsto f(t) \) into a Fourier trigonometric series in the interval \((-\pi, \pi)\), more exactly, for each \(-\pi < t < \pi \) and \( t \neq t_{\text{si}} \), where \( t_{\text{si}} \) are break points of the function \( t \mapsto f(t) \) in the interval \((-\pi, \pi)\), it holds

\[
f(t) = \frac{1}{2} A_0 + \sum_{k=1}^{+\infty} [A_k \cos (kt) + B_k \sin (kt)],
\]

where

\[
A_k = \frac{1}{\pi} \left[ \text{v.t.} \int_{-\pi}^{\pi} f(\tau) \cos (k\tau) d\tau \right] ; \ k \in N_0
\]

and

\[
B_k = \frac{1}{\pi} \left[ \text{v.t.} \int_{-\pi}^{\pi} f(\tau) \sin (k\tau) d\tau \right] ; \ k \in N.
\]

**Comment:** According to the well-known result of Dirichlet’s theorem, see - *Theorem 1, Section 5.4, Chapter 5, p. 65, [3]* - in the general case of a class of real valued functions \( t \mapsto f_d(t) \) satisfying the general Dirichlet’s conditions in the segment \([-\pi, \pi]\), the Fourier trigonometric series on the right-hand side of the equation (2.33) can be said to converge to a function \( F_d(t) \). Clearly, at all points of the interval \((-\pi, \pi)\) at which the function \( t \mapsto f_d(t) \) is continuous, a convergent value of Fourier series is equal to
the function value: \( F_d(t) = f_d(t) \). Considering the consequence of whether Bessel’s inequality \( 3 \) or Riemann-Lebesque’s theorem - Theorem 2, Section 6.2, Chapter 5, p. 96, \( 3 \) - Fourier’s coefficients of the function \( t \mapsto f_d(t) \): \( A_k \) and \( B_k \), tend to zero as \( k \to +\infty \). This is important from the viewpoint of the convergence of infinite numerical series obtained by expansion of functions \( t \mapsto f_d(t) \) into Fourier trigonometric series. Note that the conditions of Dirichlet’s theorem are only sufficient conditions for convergence of Fourier trigonometric series of functions \( t \mapsto f_d(t) \).

A nature of Fourier trigonometric series convergence of a class of real valued meromorphic functions \( t \mapsto f_m(t) \), whose finitely many isolated singularities lie onto the segment \([−π, π]\), can be said to be different from a case to a case. The same holds also for Fourier’s coefficients of a function \( t \mapsto f_m(t) \) in the limit as \( k \to +\infty \). Namely, in the general case of real meromorphic functions \( t \mapsto f_m(t) \), at all points of the interval \((−π, π)\) at which a function \( t \mapsto f_m(t) \) is continuous, Fourier trigonometric series of the function \( t \mapsto f_m(t) \) is summable, more exactly it has defined sum \( F_m(t) = f_m(t) \). The concept of the sum of Fourier trigonometric series, in this case, is generalization of the concept of the sum in the Cauchy’s sense. At the break points \( t_{si} \) of the function \( t \mapsto f_m(t) \) in the segment \([−π, π]\), the sum of Fourier trigonometric series of the real valued meromorphic function \( t \mapsto f_m(t) \), in the general case, is not defined. ▼

3. Examples

3.1. Example 1. An expansion of the function \( t \mapsto \frac{1}{2} \frac{\sin t}{1 - \cos t} \) into a Fourier trigonometric series in the segment \([−π, π]\). The function \( f(t) = \frac{1}{2} \frac{\sin t}{1 - \cos t} \) having at the point a simple pole as a singularity is a real valued meromorphic function in the segment \([−π, π]\). Cauchy’s principal value (v.p.) of an improper integral of the function \( f(t) \) is equal to:

\[
\text{v.p.} \int_{-\pi}^{\pi} f(t) \, dt = \frac{1}{2} \lim_{\varepsilon \to 0^+} \left[ \int_{-\pi}^{-\varepsilon} \frac{\sin t}{1 - \cos t} \, dt + \int_{\varepsilon}^{\pi} \frac{\sin t}{1 - \cos t} \, dt \right] = 0.
\]

As \( \lim_{\varepsilon \to 0^+} \frac{\sin z}{\varepsilon (1 - \cos z)} = 1; z \in C^1 \), the by-pass integral value in the limit as \( \varepsilon \to 0^+ \) (Jordan’s singular value (v.s.) of the improper integral) is equal to:

\[
\lim_{\varepsilon \to 0^+} \frac{1}{2} \int_{G_{\varepsilon}} \frac{\sin z}{1 - \cos z} \, dz = \begin{cases} -i\pi; & \kappa = 1 \\ i\pi; & \kappa = 2 \end{cases},
\]

dependently on the choice of a circular arc \( G_{\varepsilon} \) bypassing the singularity of the function \( f(t) \) in the complex plane: \( G_{\varepsilon} = \left\{ z \mid z = \varepsilon e^{i\theta}; \theta, \kappa \in \left[ -\pi, \frac{\pi}{\kappa} \right]; \kappa = 1 \text{ or } 2 \right\} \).

The total value (v.t.) of the improper integral, as a sum of Cauchy’s principal value (v.p.) and Jordan’s singular value (v.s.):

\[
\frac{1}{\pi} \text{v.t.} \int_{-\pi}^{\pi} f(t) \, dt = \frac{1}{2\pi} \text{v.t.} \int_{-\pi}^{\pi} \frac{\sin t}{1 - \cos t} \, dt = \begin{cases} -i \frac{1}{\pi} \pi \\ i \frac{1}{\pi} \pi \end{cases}.
\]

absolutely exists in this case and as one can see is not unique.
On the other hand, since \[ \frac{1}{\pi} \int_0^\pi \frac{\sin^2 \left( (k + 1/2)t \right)}{\sin \left( \frac{1}{2}t \right)} dt = 1 \; ; \; k \in \mathbb{N} \; \text{ - Formula (6) Section 6.2, Chapter 6, p. 95, [2]} \], and \[ \frac{1}{\pi} \int_{-\pi}^\pi \cos (kt) dt = 0 \; ; \; k \in \mathbb{N} \], as well as \[ \lim_{z \to 0} \frac{\sin z \cos (kz)}{2(1 - \cos z)} = 1, \]
then it follows that

\[ B_k = \frac{1}{\pi} \sqrt{\int_{-\pi}^\pi \sin t \sin (kt) \frac{dt}{2(1 - \cos t)} = \frac{1}{\pi} \int_{-\pi}^\pi \cot \frac{t}{2} \sin (kt) dt = 1} \; \; , \; k \in \mathbb{N}, \] as well as

\[ A_k = \frac{1}{\pi} \sqrt{\int_{-\pi}^\pi \sin t \cos (kt) \frac{dt}{2(1 - \cos t)} = \left\{ -i \; ; \; k \in \mathbb{N} \right\}, \] (3.4)

Accordingly, and by the Fourier formula (2.33), a Fourier trigonometric series of the function \( t \mapsto \frac{1}{2} \sin t \frac{1}{1 - \cos t} \) can be expressed by the following functional form

\[ \frac{1}{2} \sin t \frac{1}{1 - \cos t} = \sum_{k=1}^{+\infty} \sin (kt) \mp i \frac{1}{2} \left[ 1 + 2 \sum_{k=1}^{+\infty} \cos (kt) \right], \] (3.6)

that is, the equalities

\[ \frac{1}{2} \sin t \frac{1}{1 - \cos t} = \sum_{k=1}^{+\infty} \sin (kt) \; \text{ and } \; 1 + 2 \sum_{k=1}^{+\infty} \cos (kt) = 0 \; \text{i.e.} \]

\[ \frac{1}{2} \frac{1 - e^{\pm it}}{1 - \cos t} = -\sum_{k=1}^{+\infty} e^{\pm k t} \]
hold for each \( t \in (-\pi, \pi) \) and \( t \neq 0 \), respectively.

By relation (2.15), for \( t = \pm \pi \), it follows that

\[ \sum_{k=1}^{+\infty} \sin (k\pi) = 0 \; \text{ and } \; 1 + 2 \sum_{k=1}^{+\infty} \cos (k\pi) = 1 + 2 \sum_{k=1}^{+\infty} (-1)^k = 0. \] (3.8)

3.2. Example 2. An expansion of the function \( t \mapsto \frac{1}{2} \sin t \frac{1}{1 - \cos t} \) into a Fourier trigonometric series in the segment \([-\pi, \pi]\). The real valued meromorphic function \( f (t) = \frac{1}{2} \frac{1}{1 - \cos t} \) in the segment \([-\pi, \pi]\) has the second order pole at the point \( t = 0 \) as a singularity. The improper integral \[ \frac{1}{\pi} \int_{-\pi}^\pi \frac{dt}{1 - \cos t} \] absolutely exists and reduces to the indefinite expression of difference of infinities \( \infty - \infty \). Namely, independently on the choice of the circular arc bypassing the singularity \( z = 0 \) of the function \( z \mapsto \frac{1}{2} \frac{1}{1 - \cos z} \) in the complex plane (for example \( G_\varepsilon = \{ z \mid z = \varepsilon e^{i\theta}; \; \theta \in [-\pi, 0) \} \)), on the one hand it holds

\[ \frac{1}{2} \sqrt{\int_{-\pi}^\pi \frac{dt}{1 - \cos t} = \frac{1}{2} \lim_{\varepsilon \to 0^+} \left[ \int_{-\pi}^{-\varepsilon} \frac{dt}{1 - \cos t} + \int_{G_\varepsilon} \frac{dz}{1 - \cos z} + \int_{\varepsilon}^{\pi} \frac{dt}{1 - \cos t} \right] = \]

\[ = \lim_{\varepsilon \to 0^+} \left[ \frac{\sin \varepsilon}{1 - \cos \varepsilon} + \frac{1}{2} \int_{G_\varepsilon} \frac{dz}{1 - \cos z} \right]. \] (3.9)
On the other hand, since - see Definition 4, Section 2.2, Chapter 2, p. 82, [7]

\[
\int_{G_{\varepsilon}} \frac{dz}{1 - \cos z} = \int_{-\pi}^{0} \frac{d\theta}{1 - \cos \left(z(\theta)\right)} = \int_{-\pi}^{0} \frac{i\varepsilon e^{i\theta}}{1 - \cos (\varepsilon e^{i\theta})} d\theta,
\]

that is

\[
\int_{G_{\varepsilon}} \frac{dz}{1 - \cos z} = -\sin \left[\frac{z(\theta)}{1 - \cos \left(z(\theta)\right)}\right]_{-\pi}^{0} = -\sin \left(\frac{\varepsilon e^{i\theta}}{1 - \cos (\varepsilon e^{i\theta})}\right)_{-\pi}^{0} = \frac{2\sin \varepsilon}{1 - \cos \varepsilon},
\]

then finally it follows that the total value (v.t.) of the improper integral \( \int_{-\pi}^{\pi} \frac{dt}{2(1 - \cos t)} \)
is equal to the value zero:

\[
v.t. \int_{-\pi}^{\pi} \frac{dt}{2(1 - \cos t)} = 0.
\]

The complex function \( z \mapsto \frac{z^k}{z - 1}; k \in \mathbb{N} \) of complex variable \( z \) is a meromorphic function having at the point \( z = 1 \) a simple pole as singularity. Since \( \lim_{z \to 1} \left[(z - 1) \frac{z^k}{z - 1}\right] = 1 \) then, according to the result (3.1) in the Section 2.1 of the paper, Cauchy’s principle value (v.p.) of the improper integral \( \int_{G_{\varepsilon}} \frac{dz}{z - 1} \) over the circular contour of integration \( G: G = \{z \mid z = \varepsilon e^{i\theta}; \theta \in [-\pi, \pi]\} \) is equal to:

\[
v.p. \int_{G_{\varepsilon}} \frac{dz}{z - 1} = i\pi.
\]

With regard to the fact that \( z = \varepsilon e^{i\theta} \) onto the integration contour \( G \), it follows that

\[
v.p. \int_{G} \frac{z^k}{z - 1} dz = \int_{-\pi}^{\pi} \frac{ie^{ik\theta}}{1 - e^{-i\theta}} d\theta = \int_{-\pi}^{\pi} \frac{i \cos (k\theta)}{2(1 - \cos \theta)} d\theta - \int_{-\pi}^{\pi} \frac{i \cos [(k + 1)\theta]}{2(1 - \cos \theta)} d\theta = i\pi,
\]

since \( v.p. \int_{-\pi}^{\pi} \frac{\cos(k\theta) \sin \theta}{1 - \cos \theta} d\theta = 0 \) and \( \int_{-\pi}^{\pi} \sin (k\theta) d\theta = 0 \); for each \( k \in \mathbb{N} \).

On the other hand, for \( k \in \mathbb{N} \): \( \lim_{z \to 0} \left[z \cos (kz)\right] = 0 \) and \( \lim_{z \to 0} \frac{z \sin(kz) \sin z}{1 - \cos z} = 0 \).

According to the result of Jordan’s lemma - Theorem 1, Subsection 3.1.4, Section 3.1, Chapter 3, p. 52, [3] - it holds

\[
\left(3.14 \right) \quad \frac{1}{2\pi i} \lim_{\varepsilon \to 0^+} \int_{G_{\varepsilon}} \left\{ \frac{\cos (kz) - \cos [(k + 1)z]}{2(1 - \cos z)} \right\} dz = \frac{1}{4\pi i} \lim_{\varepsilon \to 0^+} \int_{G_{\varepsilon}} \left[ \cos (kz) - \frac{\sin (kz) \sin z}{1 - \cos z} \right] dz = 0,
\]
where the singularity $z = 0$ of the meromorphic function $z \mapsto \frac{1}{2\, (1 - \cos z)}$ is bypassed by the parts $G_{\varepsilon \kappa} : G_{\varepsilon \kappa} = \{ z \mid z = e^{i\theta}, \theta \in [ -\pi, 0]; \kappa = 2 \}$, of a circular path of integration: $G_{\varepsilon} = \{ z \mid z = e^{i\theta}, \theta \in [ -\pi, \pi] \}$.

Finally, from results: (3.13) and (3.14), the integral equation is obtained

$$
\frac{1}{2\pi} \int_{-\pi}^{\pi} \left[ \frac{\cos (k\theta)}{2 (1 - \cos \theta)} d\theta - \frac{\cos [(k+1)\theta]}{2 (1 - \cos \theta)} d\theta \right] = \frac{1}{2}; \, k \in N,
$$

and that is in agreement with result (3.4).

Further, since $\cos (2\theta) = 1 - 2 \sin^2 \theta$, it follows that

$$
\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\cos (2\theta)}{2 (1 - \cos \theta)} d\theta = -2,
$$

in view of the results: (3.4) and (3.12).

Consequently, considering (3.15) it has been just proved by a method of mathematical induction that for each $k \in N$ it holds

$$
A_k = \frac{1}{\pi} \frac{\pi}{2\pi} \int_{-\pi}^{\pi} \frac{\cos (kt)}{2 (1 - \cos t)} dt = -k.
$$

As for an improper integral $\frac{\pi}{2\pi} \frac{\sin (kt)}{2 (1 - \cos t)} dt$, its total value (v.t.) reduces to the value of by-pass integrals $\int_{G_{\varepsilon \kappa}} \frac{\sin (kz)}{2 (1 - \cos z)} dz$ in the limit as $\varepsilon \to 0^+$. Taking into account that for each $k \in N$: $\lim_{z \to 0} \frac{\sin (kz)}{z (1 - \cos z)} = k$, it follows that

$$
B_k = \frac{1}{\pi} \frac{\pi}{2\pi} \int_{-\pi}^{\pi} \frac{\sin (kt)}{2 (1 - \cos t)} dt =
$$

$$
= \frac{1}{\pi} \lim_{\varepsilon \to 0^+} \int_{G_{\varepsilon \kappa}} \frac{\sin (kz)}{2 (1 - \cos z)} dz = \left\{ \begin{array}{ll}
-ik; & \kappa = 1 \\
\quad ik; & \kappa = 2
\end{array} \right.; \, k \in N.
$$

Therefore, Fourier trigonometric series of the function $t \mapsto \frac{1}{2\, (1 - \cos t)}$ in the segment $[ -\pi, \pi ]$, according to results: (3.12) and (3.17) as well as (3.18), can be expressed by the following functional form

$$
\frac{1}{2\, (1 - \cos t)} = \sum_{k=1}^{+\infty} k \cos (kt) \mp i \sum_{k=1}^{+\infty} k \sin (kt) = \sum_{k=1}^{+\infty} k e^{\pm ikt},
$$

that is, the equalities

$$
\frac{1}{2\, (1 - \cos t)} = \sum_{k=1}^{+\infty} k \cos (kt) \quad \text{and} \quad \sum_{k=1}^{+\infty} k \sin (kt) = 0,
$$

hold for each $t \in ( -\pi, \pi )$ and $t \neq 0$, respectively.
In the extreme points of the segment \([-\pi, \pi]\), from (2.15), it follows that
\[
(3.21) \quad \sum_{k=1}^{+\infty} k \cos (k\pi) = \sum_{k=1}^{+\infty} k (-1)^k = -\frac{1}{4} \quad \text{and} \quad \sum_{k=1}^{+\infty} k \sin (k\pi) = 0. \blacksquare
\]

Comment: By an expansion of the real valued functions of real variable \(t\):
\[
f(t) = \begin{cases} \frac{\sin t}{2(1 - \cos t)} ; & \tau_0 \leq |t| \leq \pi \\ 0 , & |t| < \tau_0 \end{cases} \quad \text{and} \quad g(t) = \begin{cases} b; & \tau_0 \leq t \leq \pi \\ 0; & |t| < \tau_0 \\ a; & -\pi \leq t \leq -\tau_0 \end{cases}
\]

satisfying Dirichlet’s conditions in the segment \([-\pi, \pi]\), into Fourier trigonometric series, it is obtained that
\[
f(t) = \sum_{k=1}^{+\infty} \frac{1}{\pi} \left[ \int_{\tau_0}^{\pi} \frac{\sin \tau}{1 - \cos \tau} \sin (k\tau) \, d\tau \right] \sin (kt) ; \quad |t| \in (\tau_0, \pi),
\]
\[
g(t) = \frac{1}{2\pi} \left( \int_{-\pi}^{\tau_0} a \, d\tau + \int_{\tau_0}^{\pi} b \, d\tau \right) + \sum_{k=1}^{+\infty} \frac{1}{\pi} \left\{ \int_{-\pi}^{\tau_0} a \sin (k\tau) \, d\tau + \int_{\tau_0}^{\pi} b \sin (k\tau) \, d\tau \right\} \sin (kt) +
\]

\[
\quad + \left[ \int_{-\pi}^{-\tau_0} a \cos (k\tau) \, d\tau + \int_{\tau_0}^{\pi} b \cos (k\tau) \, d\tau \right] \cos (kt) \}; \quad |t| \in (\tau_0, \pi),
\]

that is
\[
(3.22) \quad f(t) = \sum_{k=1}^{+\infty} \left[ 1 + \frac{\tau_0}{\pi} - 2 \sum_{\kappa=0}^{k-1} \frac{\sin (k\tau_0)}{k\pi} - \frac{\sin (k\tau_0)}{k\pi} \right] \sin (kt) ; \quad |t| \in (\tau_0, \pi),
\]
\[
(3.23) \quad g(t) = \frac{a + b}{2} - \frac{a + b}{2\pi} \left[ \frac{1}{2} + \sum_{k=1}^{+\infty} \frac{\sin (k\tau_0)}{k\tau_0} \cos (kt) \right] \tau_0 +
\]

\[
\]

By the well-known trigonometrical equalities: \(\sin [(k + 1) t] = \sin (kt) \cos (t) + \cos (kt) \sin t\) and \(\frac{\sin t \sin [(k + 1) t]}{1 - \cos t} = \frac{\sin t \sin (kt)}{1 - \cos t} \sin (kt) + (1 + \cos t) \cos (kt)\), as well as \(\sin (kt) \cos t = \frac{1}{2} \{ \sin [(k - 1) t] + \sin [(k + 1) t]\}\) and \(\cos t \cos (kt) = \frac{1}{2} \{ \cos [(k - 1) t] + \cos [(k + 1) t]\}\), the following recurrent formula for Fourier’s coefficients of the function \(f(t)\) in the segment \([-\pi, \pi]\) is obtained
\[
\frac{1}{\pi} \int_{\tau_0}^{\pi} \frac{\sin t \sin [(k + 1) t]}{1 - \cos t} \, dt = \frac{1}{\pi} \int_{\tau_0}^{\pi} \frac{\sin t \sin [(k - 1) t]}{1 - \cos t} \, dt - \frac{2 \sin (k\tau_0)}{k\pi} \frac{\sin [(k + 1) \tau_0]}{(k + 1) \pi} - \frac{\sin [(k - 1) \tau_0]}{(k - 1) \pi} ; \quad k \in N
\]

On the other hand, for \(k = 1\), that is, for \(k = 2\), it holds: \(\frac{1}{\pi} \int_{\tau_0}^{\pi} \frac{(\sin t)^2}{1 - \cos t} \, dt = \frac{1}{\pi} \int_{\tau_0}^{\pi} (1 + \cos t) \, dt = 1 - \frac{\tau_0}{\pi} - \sin \frac{\tau_0}{\pi}\), that is, \(\frac{1}{\pi} \int_{\tau_0}^{\pi} \frac{\sin (2t) \sin t}{1 - \cos t} \, dt = \frac{1}{\pi} \int_{\tau_0}^{\pi} (1 + \cos t) \cos t \, dt = 1 - \frac{\tau_0}{\pi} - \frac{2 \sin \frac{\tau_0}{\pi}}{\pi} - \frac{\sin (2\tau_0)}{2\pi}\), respectively.
\[ + \frac{b}{\pi} - a \sum_{k=1}^{+\infty} \frac{\cos(k\tau) - (-1)^k}{k} \frac{\sin(kt)}{k}; \quad |t| \in (\tau_0, \pi). \]

From the functional relation (3.23) it follows for \( a = b, \ |t| \in (\tau_0, \pi) \) and \( \tau_0 > 0 \) that
\[
\frac{1}{2} + \sum_{k=1}^{+\infty} \frac{\sin(k\tau)}{k\tau} \cos(kt) = 0. \tag{3.24}
\]

Thus, for \( t \in (\tau_0, \pi) \) and \( \tau_0 > 0 \) it holds
\[
\frac{\pi}{2} = \sum_{k=1}^{+\infty} \left( \cos(\tau) - (-1)^k \right) \frac{\sin(kt)}{k}, \tag{3.25}
\]

that is
\[
\sum_{k=1}^{+\infty} \cos(\tau) \frac{\sin(kt)}{k} = \frac{\pi}{2} - \frac{t}{2}; \quad t \in (\tau_0, \pi), \tag{3.26}
\]

since for \( t \in (-\pi, \pi), \ [3] \),
\[
\sum_{k=1}^{+\infty} (-1)^k \frac{\sin(kt)}{k} = -\frac{t}{2}. \tag{3.27}
\]

On the other hand, taking into account the fact that
\[
\lim_{k \to +\infty} \left[ 1 + \frac{\tau_0}{\pi} - \frac{1}{2} \sum_{k=0}^{k-1} \frac{\sin(k\tau)}{k\pi} - \frac{\sin(k\tau_0)}{k\pi} \right] = 0,
\]

see the last Comment in the preceding Section of this paper, finally it follows for \( \tau_0 \in (0, \pi) \) that
\[
\sum_{k=0}^{+\infty} \frac{\sin(k\tau)}{k} = \frac{\pi}{2} + \frac{\tau_0}{2}. \tag{3.28}
\]

Since a real parameter \( \tau_0 \) takes any value from the interval \((0, \pi)\), even if that is finitely small value, it would be reasonable to ask: Whether the functional expressions: (3.22) and (3.24) as well as (3.25) and (3.26), hold in the limit as \( \tau_0 \to 0^+ \)? In other words: Are the limiting values of sums, in these emphasized cases, equal to sums of limiting values of the functional expressions, as \( \tau_0 \to 0^+ \), respectively? On the basis of previously derived results in the Example 1 and of the above obtained result (3.28) as well as of the well-known result of the series theory: \( \frac{\tau}{4} = \sum_{k=1}^{+\infty} \sin(2k-1)t \); for \( t \in (0, \pi), \ [3] \), an answer to the former questions is yes. However, the problem of generalization of a preceding conclusion stays open and can be a subject of a separate analysis.

Similarly, since \( \frac{d}{dt} \left[ \frac{\sin t}{2(1-\cos t)} \right] = \frac{1}{2(1-\cos t)} \) and \( \frac{d}{dt} \left\{ \frac{1}{2} \ln [2(1 - \cos t)] \right\} = \frac{\sin t}{2(1-\cos t)} \) for \( |t| \in (0, \pi) \), then closely related to results: (3.7) and (3.20), of the paper, as well as to the well-known result of the series theory: \( -\frac{1}{2} \ln [2(1 - \cos t)] = \sum_{k=1}^{+\infty} \frac{\cos(kt)}{k}, \) for \( |t| \in (0, \pi), \ [3] \), is the following question: In which general cases
the derivative of a sum of infinite functional series is equal to the sum of the derivative of any series member, separately?
This question also stays open for a separate analysis.

4. Conclusion

Taking into consideration the fact that obtained results are theoretical news, one can say that the certain possibilities for expansion of some mathematical analysis knowledge connecting to the problems to which a proper attention has been paid in this paper are opening up. Thus, from viewpoint of results, derived in the Subsubsection 2.2.4 of the paper for instance, and having in mind the fact that causality related to the area of Fourier trigonometric series of real valued functions is the theory of partial differential equations, it is obvious which possibilities are opening up in this area of mathematics.

On the other hand, disregarding the fact that the results of the paper are, in a certain sense, the theoretical news, some of them have been predictable. So, the alternative numerical series: \( \sum_{k=0}^{\infty} (-1)^k \), has the defined sum, more exactly it is summable and its sum is equal to \( \frac{1}{2} \), just as it has been assumed yet by Euler and Dalamber. Making use of this assumption they obtained absolutely exact results. It is nothing other to be left than to prove validity of this assumption. As for the results: (3.20) and (3.21), from the Example 2, one can say that they are theoretical news and causality related to the result (3.8). Namely, since \( \sum_{k=1}^{\infty} k \sin (kt) = 0 \) for \( t = \frac{\pi}{2} \), that is \( \sum_{k=0}^{\infty} (2k+1) (-1)^k = 0 \), it follows that \( \sum_{k=0}^{\infty} 2k (-1)^k = -\sum_{k=0}^{\infty} (-1)^k = -\frac{1}{2} \).

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