ON DIFFUSION PROCESSES WITH $B(\mathbb{R}^2, VMO)$ COEFFICIENTS AND “GOOD” GREEN’S FUNCTIONS OF THE CORRESPONDING OPERATORS

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Abstract. The solvability in Sobolev spaces with special mixed norms is proved for nondivergence form second order parabolic equations. The leading coefficients are assumed to be measurable in the time variable and two coordinates of space variables, and be almost in VMO (vanishing mean oscillation) with respect to the other coordinates. This solvability result implies the weak uniqueness of solutions of the corresponding stochastic Itô equations in the class of “good” solutions (which is nonempty). This also implies uniqueness of a Green’s function in the class of “good” ones (which is always nonempty).

1. Introduction

Let $\mathbb{R}^d$ be a Euclidean space of points $x = (x^1, ..., x^d)$ and let $d \geq 3$. We write $x = (x', x'')$, where $x' = (x^1, x^2)$ and $x'' = (x^3, ..., x^d)$. Set

$$D_i u = u_{x^i}, \quad D_{ij} u = u_{x^i x^j}, \quad \partial_t u = \partial u/\partial t.$$ 

By $Du$ and $D^2 u$ we mean the gradient and the Hessian matrix of $u$.

In this paper we are dealing with diffusion processes corresponding to parabolic equations in nondivergence form:

$$Lu - \lambda u = f,$$  

where $\lambda \geq 0$ is a constant, $f \in L_{p,q}$ (space defined later), and

$$Lu = \partial_t u + a^{ij} D_{ij} u + b^i D_i u - cu.$$  

We assume that all the coefficients are measurable and

$$c \geq 0, \quad |b^i| + c \leq K, \quad a^{ij} = a^{ji}, \quad \delta |\xi|^2 \leq a^{rs} \xi^r \xi^s \leq \delta^{-1} |\xi|^2$$

for all $i, j = 1, ..., d$, $\xi \in \mathbb{R}^d$, where $K$ and $\delta > 0$ are fixed constants.

For $p, q \in (1, \infty)$ we introduce $L_{p,q}$ as the space of (measurable) functions on $\mathbb{R}^{d+1} = \{(t, x) : t \in \mathbb{R}, x \in \mathbb{R}^d\}$ with finite norm given by

$$\|u\|_{L_{p,q}}^q = \int_{\mathbb{R}^{d-2}} \left( \int_{\mathbb{R}^3} |u|^p \, dx' \, dt \right)^{q/p} \, dx''.$$
Then we introduce the function space in which we are going to consider $L$ by setting

$$W^{1,2}_{p,q} = \{ u : u, \partial_t u, Du, D^2 u \in L_{p,q} \}.$$  

One of the main motivations to consider these particular Sobolev spaces with mixed norm comes from the theory of stochastic diffusion processes. Namely, we know that for any $(t, x) \in \mathbb{R}^{d+1}$ there exists a probability space $(\Omega, \mathcal{F}, P)$, a $d$-dimensional random process $x_s, s \geq 0$, and a $d$-dimensional Wiener process $w_s, s \geq 0$, such that $w_{s+h} - w_s$ is independent of $\mathcal{F}_s := \sigma\{x_r; r \leq s\}$ for $s, h \geq 0$, $w_s$ is $\mathcal{F}_s$-measurable, and, with probability one, for all $s \geq 0$

$$x_s = x + \int_0^s a^{1/2}(t+r, x_r) \, dw_r + \int_0^s b(t+r, x_r) \, dr, \quad (1.3)$$

where $a = (a^{ij}), b = (b^i)$.

We consider the case that $a$ is only measurable in $(t, x')$. It is more regular with respect to $x''$, it is almost in VMO. In such situations, as Uraltseva’s examples show, it is not possible to prove the solvability of (1.1) in usual Sobolev spaces $W^{1,2}_p(\mathbb{R}^{d+1})$ with $p > 2 + \gamma$, where $\gamma > 0$ is independent of $\delta$. Therefore, the usual method of proving the weak uniqueness for (1.3) based on applying Itô’s formula to $u(t+s, x_s)$, where $u$ is a solution of (1.1), cannot be justified if $d \geq 3$. The common knowledge until now is that $u$ should be in $W^{1,2}_p$ with $p < d+1$ but sufficiently close to $d+1$ in order to apply Itô’s formula to $u(t+s, x_s)$.

At the same time it is proved in [2] that (1.1) is uniquely solvable in $W^{1,2}_p(\mathbb{R}^{d+1})$ with $2 < p < 2 + \gamma$, where $\gamma = \gamma(d, \delta) > 0$, for any $f \in L_p(\mathbb{R}^{d+1})$. Then a very natural and puzzling question arose: can equation (1.3) on perhaps different probability spaces have solutions with different distributions when the corresponding parabolic equation is uniquely solvable in some space containing $C_0^\infty(\mathbb{R}^{d+1})$?

Under our assumptions we still do not know the answer to this question. We only show that weak uniqueness for (1.3) holds in the set of solutions whose Green’s functions belongs to $L_{p', q'}$, where $p' = p/(p-1), q' = q/(q-1)$ and $p$ and $q$ will be specified later. Of course, we also prove that under our assumptions such solutions do exist and, as a consequence, if for any reason weak uniqueness holds for (1.3), then its Green’s function is in $L_{p',q'}$. This consequence is new even for the equations with the coefficients measurable in time and VMO in space variables and, for that matter, new in the case that the coefficient are measurable in time and continuous in $x$ uniformly with respect to $t$, the classical case treated in [7]. Analogously, we introduce the notion of “good” Green’s functions for the operator $L$ and prove that such “good” functions exist and are unique.

Our methods are based on [2] and a simple consequence of the Rubio de Francia extrapolation theorem presented in [1] or [3].
2. Main results

First we introduce some notation. On many occasions we need to take derivatives with respect to only part of variables. The reader understands the meaning of the following notation:

\[ D_{x'} u = u_{x'}, \quad D_{x''} u = u_{x''}, \quad D_{x',x''} u = u_{x',x''}, \]
\[ D_{x',x''} u = u_{x',x''}, \quad D_{x''} u = D_{x''} u = u_{x''}. \]

\[ W^{1,2}_{p}(\mathbb{R}^{d+1}) = \{ u : u, Du, D^2 u, \partial_t u \in L_p(\mathbb{R}^{d+1}) \}. \]

We also use the abbreviations

\[ C_0^\infty = C_0^\infty(\mathbb{R}^{d+1}), \quad L_p = L_p(\mathbb{R}^{d+1}), \quad W^{1,2}_{p} = W^{1,2}_{p}(\mathbb{R}^{d+1}), ... \]

For matrix-valued functions \( a(t, x) \) on \( \mathbb{R}^{d+1} \) we understand \( \|a\|_{L_p} \) as

\[ \int_{\mathbb{R}^{d+1}} |\text{trace } a|^{|p|/2} \, dx \, dt. \]

Accordingly are introduced the norms in \( W \) spaces.

If \( B \) is a Borel subset of a plane \( \Gamma \) in a Euclidean space, we denote by \( |B| \) its volume relative to \( \Gamma \). This notation is somewhat ambiguous because \( B \) also belongs to the ambient space, where its volume can be zero. However, we hope that from the context it will be clear relative to which plane we take the volume in each instance. If there is a measurable function \( f \) on \( B \) which is integrable with respect to the Lebesgue measure \( \ell \) on \( \Gamma \) we set

\[ f_B = \int_B f(x) \, \ell(dx) := \frac{1}{|B|} \int_B f(x) \, \ell(dx). \]

Let

\[ B'_r(x') = \{ y' \in \mathbb{R}^2 : |x' - y'| < r \}, \]
\[ B''_r(x'') = \{ y'' \in \mathbb{R}^{d-2} : |x'' - y''| < r \}, \quad B_r(x) = B'_r(x') \times B''_r(x''), \quad Q_r(t, x) = (t + r^2, t) \times B_r(x), \quad Q_r = Q_r(0, 0), \]

and let \( Q \) be the collection of all \( Q_r(t, x) \). We call \( r \) the radius of \( Q = Q_r(t, x) \).

We require a quite mild regularity assumption on \( a^{ij} \). They are assumed to be measurable in \( t \) and \( x' \), and almost VMO with respect to \( x'' \). More precisely, we impose the following assumption in which \( \theta > 0 \) will be specified later and \( R_0 > 0 \) is a fixed number. Set

\[ \text{tr}_2 a = a^{11} + a^{22}. \]

Assumption 2.1 (\( \theta \)). For any \( Q = (s, t) \times B' \times B'' \in Q \) with radius \( \rho \leq R_0 \)

\[ \int_Q |a(r, x) - a_{B''}(r, x')| \, dx \, dr \leq \theta, \quad (2.1) \]
\[ \int_Q |\text{tr}_2 a_{B''}(r, x') - \text{tr}_2 a_{B' \times B''}(r)| \, dx \, dr \leq \theta, \quad (2.2) \]
where
\[
a_{B''}(r, x') = \int_{B''} a(r, x) \, dx'',
\]
\[
a_{B' \times B''}(r) = \int_{B'} \int_{B''} a(r, x) \, dx = \int_{B'} a_{B''}(r, x') \, dx'.
\]

Observe that if \(a(t, x)\) is independent of \(x''\), then the left-hand side of (2.1) is zero. It can be made as close to zero as we wish on the account of \(R_0\) if \(a(t, x)\) is continuous with respect to \(x''\) uniformly with respect to \((t, x')\). Also if \(tr_2 a(t, x)\) depends only on \(t, x''\) (for instance, constant), then the left-hand side of (2.2) is zero. It can be made as close to zero as we wish on the account of \(R_0\) if \(tr_2 a(t, x)\) is continuous with respect to \(x''\) uniformly with respect to \((t, x')\).

In the following theorems we use the constants (small) \(\gamma_0 = \gamma_0(\delta) \in (0, 1/2)\) and \(\theta = \theta(p, q, d, \delta) > 0\) and (large) \(N = N(p, q, d, \delta, K, R_0)\) which will be determined later.

**Theorem 2.2.** Let \(p \in (2, 2+\gamma_0), q > (pd-2p)/(2p-4)\) and let Assumption 2.1 \((\theta)\) be satisfied. Take \((t_0, x_0) \in \mathbb{R}^{d+1}\). Then there exists a probability space \((\Omega, \mathcal{F}, P)\) carrying a \(d\)-dimensional Wiener process \((w_t, F_t)\), \(t \geq 0\), and there exists a solution \(x_t\) of the Itô equation
\[
x_t = x_0 + \int_0^t a_{1/2}(t_0 + s, x_s) \, dw_s + \int_0^t b(t_0 + s, x_s) \, ds, \quad t \geq 0,
\]
(2.3)
such that it possesses the following property (a): for any nonnegative Borel \(f(t, x)\) we have
\[
E \int_0^\infty e^{-t} f(t, x_t) \, dt \leq N \|f\|_{L_{p,q}};
\]
(2.4)
(b) for any \(u \in W^{1,2}_{p,q}, \lambda \in \mathbb{R}\), and bounded \((\mathcal{F}_t)\)-stopping time \(\tau\) we have
\[
u(t_0, x_0) = E \int_0^\tau (\lambda u - Lu)(t_0 + t, x_t)e^{-\lambda \tau - \phi_t} \, dt + E u(t_0 + \tau, x_\tau)e^{-\lambda \tau - \phi_\tau},
\]
where
\[
\phi_t = \int_0^t c(t_0 + s, x_s) \, ds.
\]

**Definition 2.3.** We call any solution of (2.3) (on any probability space) “good” if for some \(p, q\) as in Theorem 2.5 and any \(T \in (0, \infty)\) there exists a constant \(N\) such that
\[
E \int_0^T f(s, x_s) \, ds \leq N \|f\|_{L_{p,q}}
\]
(2.6)
for any nonnegative Borel \(f\).
Observe that the solution existence of which is asserted in Theorem 2.5 is “good”. Also note that condition (2.6) means that there exists a function $G(s,y) \geq 0$ such that for any nonnegative Borel $f$

$$E \int_0^\infty f(s,x_s) \, ds = \int_0^\infty \int_{\mathbb{R}^d} G(s,y) f(s,y) \, dy \, ds$$

and for any $T \in (0, \infty)$ we have

$$\int_{\mathbb{R}^{d-2}} \left( \int_0^T \int_{\mathbb{R}^2} \frac{|G|'}{|p'| \, dx'dt} \right)^{q'/p'} \, dx'' \leq N^q,$$

where $N$ is taken from (2.6), $p' = \frac{p}{p-1}$, $q' = \frac{q}{q-1}$.

**Theorem 2.4.** Under the assumptions of Theorem 2.5 all “good” solutions of (2.3) have the same finite-dimensional distribution (weak uniqueness of solutions of (2.3)).

The restrictions on $q$ come from the following.

**Theorem 2.5.** Let one of the following conditions be satisfied:

(a) $d \geq 4$, $p > 2$, $q > (pd - 2p)/(2p - 4)$,

(b) $d = 3$, $2 < p < 4$, $q > p/(2p - 4)$,

(c) $d = 3$, $p \geq 4$, $q \in (1, \infty)$.

Then for any $u \in W^{1,2}_{p,q}$ and $(t, x) \in \mathbb{R}^{d+1}$ we have

$$|u(t,x)| \leq N \|I_t(\Delta u + \partial_t u - u)\|_{L_{p,q}} \leq N\|u\|_{W^{1,2}_{p,q}},$$

where $N = N(p, q, d)$ and $I_t(s,y) = I_0(s) = I(s, \infty)$.

First we need a lemma.

**Lemma 2.6.** For $\lambda > 0$ and fixed $\alpha \geq 1$ introduce

$$I(\lambda) = \int_0^\infty \frac{s}{(1 + s^2)^{\alpha/2}} e^{-\lambda s} \, ds.$$

Then $I(\lambda)$ is bounded for $\alpha > 2$, and $\lambda^\beta I(\lambda) \to 0$ as $\lambda \downarrow 0$ for any $\beta > 2 - \alpha$ if $2 \geq \alpha > 1$.

Proof. The assertion in case $\alpha > 2$ is obvious. If $\alpha = 1$, as is easy to see after the substitution $s = t/\lambda$, $\lambda I(\lambda) \to 1$ as $\lambda \downarrow 0$. In the remaining case $2 \geq \alpha > 1$ and we integrate by parts to get

$$I(\lambda) = -\frac{1}{\lambda} \int_0^\infty (1 - e^{-\lambda s}) g(s) \, ds, \quad g(s) = \frac{d}{ds} \frac{s}{(1 + s^2)^{\alpha/2}}.$$

Clearly, $|g(s)| \leq N(1 + s)^{-\alpha}$, where and below by $N$ we denote constants depending only on $\alpha$. Also, for any $\kappa \in (0,1]$, $1 - e^{-\lambda s} \leq N \lambda^\kappa s^\kappa$, which with $\kappa < \alpha - 1$ allows us to write

$$|I(\lambda)| \leq N \lambda^{\kappa-1} \int_0^\infty (1 + s)^{\kappa-\alpha} \, ds.$$
The last integral is finite since $\kappa - \alpha < -1$, 
\[ \lim_{\lambda \to 0} \lambda^{1-\kappa} |I(\lambda)| < \infty, \]
and since for any $\beta$ such that $\beta > 2 - \alpha$ there is a $\kappa \in (0, \alpha - 1)$ such that $\beta > 1 - \kappa$, the lemma is proved.

**Proof of Theorem 2.5.** We may assume that $u \in C_0^\infty$ and $(t, x) = (0, 0)$. In that case set $f = u - \Delta u - \partial_t u$. Then for a constant $c_d$
\[
 u(0, 0) = c_d \int_{\mathbb{R}^{d-2}} \left( \int_0^\infty \int_{\mathbb{R}^2} t^{-d/2} e^{-t-|x|^2/(4t)} f(t, x) \, dx \, dt \right) \, dx',
\]
and since for any $\beta$ such that $\beta > 2 - \alpha$ there is a $\kappa \in (0, \alpha - 1)$ such that $\beta > 1 - \kappa$, the lemma is proved.

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\]
and since for any $\beta$ such that $\beta > 2 - \alpha$ there is a $\kappa \in (0, \alpha - 1)$ such that $\beta > 1 - \kappa$, the lemma is proved.
(c) If $d = 3$ and $p \geq 4$, then $1 < p' < 4/3$ and $\alpha = 3p' - 2 \in (1, 2]$. In that case $|x''|^{4-3p'}I(\mu|x''|) = |x''|^{2-\alpha}I(\mu|x''|)$ blows up at the origin slower than $|x|^{-\varepsilon}$ for any $\varepsilon > 0$ and

$$u(0, 0) \leq N\|I_0 f\|_{L_p,q} \left( \int_{\mathbb{R}^{d-2}} e^{-q'\mu|x''|/\mu|x''|^{1-\varepsilon}} dx'' \right)^{1/q'}$$

(with a different arbitrary small $\varepsilon > 0$). This proves the theorem.

Recall that given $(t_0, x_0)$ a nonnegative function $G(s, y)$ is called a Green’s function of the operator $L$ with pole at $(t_0, x_0)$ if the equality

$$u(t_0, x_0) = -\int_0^\infty \int_{\mathbb{R}^d} G(s, y)Lu(s, y) dyds$$

holds for all $u \in C_0^\infty$. We call $G$ a “good” Green’s function if, for each $T \in (0, \infty)$, the left-hand side of (2.7) is finite ($p$ and $q$ are always taken from Theorem 2.5).

**Theorem 2.7.** Let assumption of Theorem 2.5 be satisfied and $(t_0, x_0) \in \mathbb{R}^{d+1}$. Then there exists a unique “good” Green’s function of $L$ with pole at $(t_0, x_0)$.

The existence part in this theorem is just a simple consequence of Theorem 2.5. Indeed, fix $T \in (0, \infty)$. Then

$$E \int_0^T e^{-\phi t} f(t, x_t) dt$$

is a nonnegative linear bounded functional on $L_{p,q}$. Hence there exists $G \geq 0$ for which the left-hand side of (2.7) is finite and

$$E \int_0^T e^{-\phi t} f(t, x_t) dt = \int_0^T \int_{\mathbb{R}^d} G(s, y)f(s, y) dyds.$$

Obviously, $G$ is independent of $T$ and

$$E \int_0^\infty e^{-\phi t} f(t, x_t) dt = \int_0^\infty \int_{\mathbb{R}^d} G(s, y)f(s, y) dyds$$

for any nonnegative Borel $f$ or for any $f$ for which at least one side is finite. By taking $f = -Lu$, where $u \in C_0^\infty$ and using (2.5) with $\tau$ so large that $u(t_0 + \tau + s, x) = 0$ for $s \geq 0$, we immediately get (2.10).

The most important ingredient in the proof of Theorems 2.5, 2.4, and the uniqueness part in Theorem 2.7 is the following result which we prove in Section 4. It generalizes one of the main results of [2] in the respect that the continuity of $tr_2 a$ with respect to $x''$ uniform with respect to $(t, x'')$ is replaced by a kind of VMO condition. Recall that $\gamma_0 = \gamma_0(\delta) \in (0, 1/2)$, $\theta = \theta(p, q, d, \delta) > 0$ and $N = N(p, q, d, \delta, K, R_0)$ are mentioned before Theorem 2.5.

**Theorem 2.8.** There exists $\lambda_0 > 0$, depending only on $p, q, d, \delta, K, R_0$, such that if $p \in (2, 2 + \gamma_0)$, $q \in (2, \infty)$, and Assumption 2.1 ($\theta$) is satisfied, then
(i) For any $u \in W^{1,2}_{p,q}$ and $\lambda \geq \lambda_0$

$$\lambda \|u\|_{L_{p,q}} + \sqrt{\lambda} \|Du\|_{L_{p,q}} + \|D^2 u, \partial_t u\|_{L_{p,q}} \leq N \|\lambda u - Lu\|_{L_{p,q}};$$

(2.12)

(ii) For any $\lambda \geq \lambda_0$ and $f \in L_{p,q}$, there exists a unique solution $u \in W^{1,2}_{p,q}$ of equation (1.1) in $\mathbb{R}^{d+1}$.

**Remark 2.9.** Theorem 2.8 and formula (2.5) imply the solvability of the Cauchy problem as in [5]. Indeed, if we take $f(t, x) = 0$ for $t \geq T$, then for the solution $u$ of (1.1) we get from (2.5) that $|u(t_0, x_0)| \leq e^{-\lambda(S-t_0)} \sup |u|$ for any $t_0 \geq T$ and $S > T$, where the last sup is finite due to Theorem 2.5. By sending $S \to \infty$ we conclude that $u(t, x) = 0$ for $t \geq T$ and hence $u$ is a solution of the Cauchy problem for (1.1) for $t < T$ with terminal data $u = 0$.

**Proof of Theorem 2.5.** First note that there is a sequence of operators $L_n$ with infinitely differentiable coefficients $a_n, b_n, c_n$ which satisfy the same assumptions as the original $a, b, c$ and which converge to $a, b, c$ almost everywhere in $\mathbb{R}^{d+1}$. To see that it suffices to use mollifiers with nonnegative kernels. Next, take a probability space $(\Omega', \mathcal{F}', P')$ carrying a $d$-dimensional Wiener process $w'_t$, $t \geq 0$, and define $x'^n_t, t \geq 0$, as unique solutions of

$$x'^n_t = x_0 + \int_0^t a^{1/2}_n(t_0 + s, x'^n_s) \, dw'_s + \int_0^t b_n(t_0 + s, x'^n_s) \, ds, \quad t \geq 0.$$

We know (consequence of the parabolic Aleksandrov estimates and Skorokhod’s embedding method, see, for instance, Section 2.6 of [4]) that the collection of the distributions of $x'^n$ on $C([0, \infty), \mathbb{R}^d)$ is tight and for any weakly convergent subsequence there exist probability space $(\Omega, \mathcal{F}, P)$ carrying a $d$-dimensional Wiener process $(w_t, \mathcal{F}_t)$, $t \geq 0$, and a solution $x_t$ of equation (2.3) such that a subsequence of distributions of $x'^n_t$ converges to the distribution of $x$. Furthermore, along the same subsequence, for any Borel $\mathbb{R}^m$-valued bounded function $g(t, x)$, the distributions on $C([0, \infty), \mathbb{R}^d) \times C([0, \infty), \mathbb{R}^m)$ of

$$(x'^n, \int_0^t g(s, x'^n_s) \, ds)$$

(2.13)

weakly converge to the distribution of

$$(x, \int_0^t g(s, x_s) \, ds).$$

Our first goal is to show that $x$ possesses property (a) with $p, q$ as in the statement of the theorem for which it holds automatically that $q > p$ because $\gamma_0 \in (0, 1/2)$.

Note that for any $f \in C_0^\infty$ there exists a smooth solution $u^n$ of $\lambda_0 u^n - L_n u^n = f$ for which Theorem 2.8 is valid. In case $c_n \equiv 0$, by Itô’s formula

$$u^n(t_0, x_0) = E \int_0^\infty e^{-t} [f - (\lambda_0 - 1)u^n](t_0 + t, x^n_t) \, dt,$$
which implies by Theorem 2.8 and Theorem 2.5 that
\[ E \int_0^\infty e^{-t} f(t_0 + t, x^a_t)\,dt \leq (1 + \lambda_0) \sup_{\mathbb{R}^{d+1}} |u^n| \leq N\|f\|_{L_{p,q}}. \]

By passing to the limit we see that (2.4) holds for \( f \in C^\infty_0 \) and then a standard argument shows that it also holds for all Borel nonnegative \( f \).

Next, take \( u \in W^{1,2}_{p,q} \) and let \( u^{(\varepsilon)} \) be mollified \( u \) with smooth kernels supported in \((-\varepsilon, \varepsilon) \times B_r\) such that \( u^{(\varepsilon)} \rightarrow u \) in \( W^{1,2}_{p,q} \) as \( \varepsilon \downarrow 0 \). In particular, \( u^{(\varepsilon)} \rightarrow u \) uniformly and \( \lambda u^{(\varepsilon)} - Lu^{(\varepsilon)} \rightarrow \lambda u - Lu \) in \( L_{p,q} \). Then after writing Itô’s formula for \( u^{(\varepsilon)}(t_0 + t, x_t) \exp[-\lambda t - \phi_t] \), passing to the limit, and using (2.4) (recall that \( \tau \) is bounded) we immediately arrive at (2.5). The theorem is proved.

**Proof of Theorem 2.4.** The last paragraph in the above proof convinces us that (2.5) holds for any “good” solution of (2.3). Moreover, for \( f \in C^\infty_0 \) by Theorem 2.8 there is a unique solution \( u \in W^{1,2}_{p,q} \) of \( \lambda_0 u - Lu^n = f \). Since
\[ \lambda_0(u^n - u) - L^n(u^n - u) = (L - L_n)u, \]
estimate (2.12) and the dominated convergence theorem imply that \( u^n \rightarrow u \) in \( W^{1,2}_{p,q} \), and then \( u^n \rightarrow u \) uniformly on \( \mathbb{R}^{d+1} \). Furthermore, it is a classical fact that if \( T \in (0, \infty) \) and \( f(t_0 + t, x) = 0 \) for \( t \geq T \), then \( u^n(t_0 + t, x) = 0 \) and hence \( u(t_0 + t, x) = 0 \) for \( t \geq T \). By using (2.5) with \( \tau = T \) we conclude that
\[ E \int_0^T f(t_0 + t, x_t)e^{-\lambda_0 t - \phi_t}\,dt = u(t_0, x_0) \]
and hence the left-hand side is independent of the choice of solution of (2.3) provided \( f \in C^\infty_0 \) and \( f(t_0 + t, x) = 0 \) for \( t \geq T \). By usual measure theoretic argument one shows that the independence holds for any Borel \( f \) which is bounded on \([0, T] \times \mathbb{R}^d \). In particular,
\[ E \int_0^T c(t_0 + t, x_t)e^{-\phi_t}\,dt = 1 - E \exp\left( - \int_0^T c(t_0 + t, x_t)\,dt \right) \]
is independent of the choice of solution of (2.3) for any bounded \( c \). This easily implies weak uniqueness. The theorem is proved.

**Corollary 2.10.** The property of weak uniqueness and the tightness of distributions of \( x^n \) obviously imply that the whole sequence of distributions of \( x^n \) weakly converges to the distribution of \( x \).

We fix \( \theta \) from Theorem 2.5 and suppose that Assumption 2.1 (\( \theta \)) is satisfied. Next, denote by \( \Omega \) the set of \( \mathbb{R}^{d+1} \)-valued functions \( \omega = \omega_s = (t + s, x_s), s \in [0, \infty) \), such that \( x_s \in C([0, \infty), \mathbb{R}^d) \) and, if \( \omega = \{(t + \cdot, x,.)\} \) and \( s \in [0, \infty) \), we set \( t_s(\omega) = t + s \) and \( x_s(\omega) = x_s \). As usual the argument \( \omega \) is almost always dropped. Introduce \( N_r = \sigma\{((t_s, x_s), 0 \leq s \leq \tau)\}, N^r = \sigma\{((t_s, x_s), 0 \leq s < \infty)\} \).

If \( x_t, t \geq 0 \), is a solution of (2.3) on a probability space, then the function \( (t_0 + t, x_t), t \geq 0 \), is an \( \Omega \)-valued, \( N^r \)-measurable random variable. If the
solution is “good”, its distribution on $\Omega$ we denote by $P_{t_0,x_0}$. Obviously $P_{t_0,x_0}$ are defined for any $(t_0, x_0)$.

**Theorem 2.11.** The triplet consisting of $\Omega$, the family $N_t$, $t \geq 0$, and the family $P_{t,x}$, $(t, x) \in \mathbb{R}^{d+1}$, is a strong Markov process which is strong Feller in the sense that for any $T \in (0, \infty)$ and Borel bounded $f(x)$ on $\mathbb{R}^d$ the function

$$v(t, x) = \int_{\Omega} f(x_{T-t}) P_{t,x}(d\omega)$$

(2.14)

is $\alpha$ Hölder continuous in $(t, x)$ such that $t < T$, where $\alpha \in (0, 1)$ depends only on $d$ and $\delta$.

**Proof.** Take $f \in C_0^\infty$ and set

$$u(t, x) = \int_{\Omega} \int_0^\infty f(t, x_t)e^{-\lambda_0 t} dt P_{t,x}(d\omega).$$

Then let $v$ be any $W^{1,2}_{p,q}$-solution of $\lambda_0 v - L v = f$ with $c \equiv 0$ (and $p, q$ as in Theorem 2.5). By (2.5), in which we take $\lambda = \lambda_0$, $\tau = T$ and let $T \to \infty$, we see that

$$v(t_0, x_0) = E \int_0^\infty f(t_0 + t, x_t)e^{-\lambda_0 t} dt.$$

It follows that $u = v$ at $(t_0, x_0)$ and at every other point as well. Now the strong Markov property follows directly from (2.5).

The strong Feller property for the Markov process generated by smooth coefficients $a_n, b_n$, taken from the proof of Theorem 2.5, is a classical result. The fact that the Hölder exponent and constants are under control independent of the smoothness of $a_n, b_n$ is the Krylov-Safonov result. Furthermore, Corollary 2.10 says that the functions (2.14) corresponding to $a_n, b_n$ converge to $v$ if $f$ is continuous. However, the above mentioned estimates of the Hölder continuity do not involve anything from $f$ apart from $\sup |f|$ and this and a usual measure theoretic argument proves the Hölder continuity for any Borel bounded $f$. The theorem is proved.

**Proof of Theorem 2.7.** We only need to prove uniqueness assuming without loss of generality that $(t_0, x_0) = 0$. Take $T \in (0, \infty)$ and observe that, owing to obvious approximations and the assumption concerning (2.7), (2.10) holds with $(t_0, x_0) = 0$ for “good” Green’s functions not only for $u \in C_0^\infty$ but also for $u \in W^{1,2}_{p,q}$ if $u(t, x) = 0$ for $t \geq T$.

Then take $f \in L_{p,q}$ such that $f(t, x) = 0$ if $t \geq T$, let $u \in W^{1,2}_{p,q}$ be a unique solution of (1.1) with $\lambda = \lambda_0$, and set $v(t, x) = e^{\lambda_0 t} u(t, x)$. Observe that $v \in W^{1,2}_{p,q}$ since $u(t, x) = 0$ for $t \geq T$ (see Remark 2.9) and $Lv = -e^{\lambda_0 t} f$. Then

$$\int_0^T \int_{\mathbb{R}^d} G(s, y)e^{\lambda_0 s} f(s, y) dy ds$$

is the same for all “good” Green’s functions, because it is equal to $v(0)$ by the above. The arbitrariness of $f \in L_{p,q}$ and the assumption that the left-hand side of (2.7) is finite bring the proof to an end.
3. Preliminary results

We first consider equations in \( \mathbb{R} \times \mathbb{R}^2 \) with measurable coefficients.

**Lemma 3.1.** Let \( d = 2 \) and

\[
Lu = \partial_t u + \sum_{i,j=1}^{2} a^{ij}(t, x) D_{ij} u.
\]

Assume that \( tr_2 a \) depends only on \( t \). Then there exists a \( \gamma_0 = \gamma_0(\delta) > 0 \) such that for any \( p \in (2 - \gamma_0, 2 + \gamma_0) \), \( u \in W^{1,2}_p(\mathbb{R}^3) \), and \( \lambda \geq 0 \), we have

\[
\| D^2 u \|_{L^p(\mathbb{R}^3)} + \| \partial_t u \|_{L^p(\mathbb{R}^3)} + \sqrt{\lambda} \| Du \|_{L^p(\mathbb{R}^3)} + \lambda \| u \|_{L^p(\mathbb{R}^3)} \leq N \| Lu - \lambda u \|_{L^p(\mathbb{R}^3)},
\]

where \( N = N(\delta, p) \). Moreover for any \( \lambda > 0 \) and \( f \in L^p(\mathbb{R}^3) \) there exists a unique \( u \in W^{1,2}_p(\mathbb{R}^3) \) solving \( Lu - \lambda u = f \) in \( \mathbb{R}^3 \).

This lemma is proved in [2] as Lemma 3.1.

**Lemma 3.2.** For any \( p, q \in (1, \infty) \) and \( \varepsilon > 0 \) there exist \( N_0 = N_0(d, p, q) \) and \( N = N(d, p, q, \varepsilon) \) such that for any \( u \) such that \( u, Du, D^2 u \in L_{p,q} \) and \( \lambda \geq 0 \) we have

\[
\lambda \| D^2 u \|_{L_{p,q}} + \lambda^{1/2} \| Du \|_{L_{p,q}} + \| D^2 u \|_{L_{p,q}} \leq N \| Lu - \lambda u \|_{L_{p,q}}, \quad (3.2)
\]

and

\[
\| D_{x',x''} u \|_{L_{p,q}} \leq \varepsilon \| D_{x',x''} u \|_{L_{p,q}} + N \| D_{x''} u \|_{L_{p,q}}. \quad (3.3)
\]

Proof. First observe that if (3.2) is true, then

\[
\| D_{x',x''} u \|_{L_{p,q}} \leq N \| D_{x',x''} u \|_{L_{p,q}} + N \| D_{x''} u \|_{L_{p,q}}
\]

and (3.2) follows owing to the different homogeneity of the above terms with respect to scalings in \( x' \).

Owing to the possibility of mollification, while proving (3.2) we may assume that \( \partial_t u, \partial^2_{t} u \in L_{p,q} \). Then the possibility to use scalings in \( t \) shows that to prove (3.2) it suffices to show that

\[
\lambda \| D^2 u \|_{L_{p,q}} + \lambda^{1/2} \| Du \|_{L_{p,q}} + \| D^2 u \|_{L_{p,q}} \leq N \| Lu - \lambda u - \partial^2_{t} u \|_{L_{p,q}}. \quad (3.4)
\]

That (3.4) holds for sufficiently large \( \lambda \) (with \( N \) independent of \( \lambda \)) follows from Theorem 5.5 of [1] as a very particular case. Then the fact that it holds for any \( \lambda \geq 0 \) follows by scaling in \( (t, x) \). The lemma is proved.

An immediate corollary of Lemmas 3.1 and 3.2 is the following estimate.

**Corollary 3.3.** Assume that \( tr_2 a \) depends only on \( (t, x'') \). Let

\[
Lu = u_t + \sum_{i,j=1}^{d} a^{ij}(t, x) D_{ij} u.
\]

Then for any \( p \in (2 - \gamma_0, 2 + \gamma_0) \), where \( \gamma_0 \) is taken from Lemma 3.1, any \( q \in (1, \infty) \), and any \( u \in W^{1,2}_{p,q} \) and \( \lambda \geq 0 \), we have

\[
\lambda \| u \|_{L_{p,q}} + \sqrt{\lambda} \| Du \|_{L_{p,q}} + \| D^2 u \|_{L_{p,q}} + \| \partial_t u \|_{L_{p,q}}
\]
where $N = N(\delta, d, p, q)$.

Proof. We first fix $x''$ and apply Lemma 3.1 to get
\[
\lambda^q \|u(\cdot, \cdot, x'')\|_{L^p(R^3)}^q + \|D^2_{x'} u(\cdot, \cdot, x'')\|_{L^p(R^3)}^q + \|\partial_t u(\cdot, \cdot, x'')\|_{L^p(R^3)}^q 
\leq N \sum_{i,j=1}^2 a_{ij} D_{ij} u(\cdot, \cdot, x'') + \partial_t u(\cdot, \cdot, x'') - \lambda u(\cdot, \cdot, x'')\|_{L^p(R^3)}^q. 
\] (3.6)

Upon integrating (3.6) with respect to $x''$ we arrive at
\[
\lambda \|u\|_{L^p,q} + \|D^2_{x'} u\|_{L^p,q} + \|\partial_t u\|_{L^p,q} 
\leq N \|Lu - \lambda u\|_{L^p,q} + \|D_{xx''} u\|_{L^p,q}. 
\] (3.7)

By using Lemma 3.2, we deduce from (3.7) that
\[
\lambda \|u\|_{L^p,q} + \|D^2_{x'} u\|_{L^p,q} + \|\partial_t u\|_{L^p,q} 
\leq N \|Lu - \lambda u\|_{L^p,q} + \|D_{x''} u\|_{L^p,q}. 
\]

To estimate $\|Du\|_{L^p,q}$, we again use Lemma 3.2 showing that
\[
\sqrt{\lambda} \|Du\|_{L^p,q} \leq N \lambda \|u\|_{L^p,q} + N \|D^2_{x'} u\|_{L^p,q}. 
\]

The corollary is proved.

4. PROOF OF THEOREM 2.8

We consider the operator
\[
Lu = \partial_t u + a^{ij} D_{ij} u + b^i D_i u - cu,
\]
where $(a^{ij})$ satisfy Assumption 2.1 ($\theta$) with some $\theta > 0$ to be specified later. First we deal with assertion (i) of Theorem 2.8.

Introduce
\[
\text{Var}_Q f = \int_Q \int_{R^d+1} |f - f_Q|^2 dx dt.
\]
Take a nonnegative $\zeta \in C_0^\infty(Q_{R_0})$ whose square integrates to one.

**Lemma 4.1.** For any $Q \in \mathbb{Q}$ and $u$ such that $D_{x''}^2 u \in L^2(Q)$, we have
\[
\int_{R^{d+1}} \int_Q \text{Var}_Q [D^2_{x''}(u \zeta(\cdot - (s, y)))] dy ds \geq \text{Var}_Q v, 
\] (4.1)

where
\[
v = (\|D^2_{x''} u\|^2 + \chi |u|^2)^{1/2}, 
\] (4.2)
and $\chi = \chi(d, R_0)$.  


Proof. We know that
\[
2 \text{Var}_Q f = \int_Q \int_Q |f(t_1, x_1) - f(t_2, x_2)|^2 \, dx_1 dx_2 dt_1 dt_2.
\]
We also know that
\[
\int_{\mathbb{R}^{d+1}} |f(s, y) - g(s, y)|^2 \, dy ds 
\geq \left( \left( \int_{\mathbb{R}^{d+1}} |f(s, y)|^2 \, dy ds \right)^{1/2} - \left( \int_{\mathbb{R}^{d+1}} |g(s, y)|^2 \, dy ds \right)^{1/2} \right)^2.
\]
Furthermore,
\[
\int_{\mathbb{R}^{d+1}} |D_{x''}^2 (u(t_1, x_1)\zeta(t_1-s, x_1-y))|^2 \, dy ds 
= |D_{x''}^2 u(t_1, x_1)|^2 + \chi|u(t_1, x_1)|^2,
\]
where
\[
\chi = \int_{\mathbb{R}^{d+1}} |D_{x''}^2 \zeta|^2 \, dy ds.
\]
Upon combining the above we come to (4.1). The lemma is proved.

Then, we extend Theorem 5.1 of [2] in which \(u\) was required to have support in a translate of \(Q_{R_0}\). This assumption in [2] was harmless because we could use partitions of unity for the particular operators under consideration. Here we do not know how to use partitions of unity in the mixed norm setting.

**Theorem 4.2.** Let \(\alpha = \alpha(d, \eta) \in (0, 1)\) be the constant in Theorem 4.4 of [2], \(\tau, \sigma \in (1, \infty)\), \(1/\tau + 1/\sigma = 1\). Take a \(u \in W^{1,2}_{2,\text{loc}}\), introduce \(v\) by (4.2), and set \(f = Lu\). Then under Assumption 2.1 (\(\theta\)) there exists a positive constant \(N\) depending only on \(d, \eta, \tau, \sigma\) such that, for any \((t_0, x_0) \in \mathbb{R}^{d+1}\), \(r \in (0, \infty)\), and \(\kappa \geq 4\),
\[
\text{Var}_{Q_{(t_0, x_0)}} v \leq N\kappa^{d+2} (|f|^2 + |Du|^2 + |u|^2)_{Q_{\kappa r}(t_0, x_0)} 
+ N\kappa^{d+2} \theta^{1/\sigma} (|D^2 u|^{2\tau})_{Q_{\kappa r}(t_0, x_0)}^{1/\tau} 
+ N\kappa^{-2\alpha} (|D_{x''}^2 u|^2)_{Q_{\kappa r}(t_0, x_0)}.
\]

Proof. Without losing generality we assume that \((t_0, x_0) = (0, 0)\), then we fix \(\kappa \geq 4\) and \(r \in (0, \infty)\). Owing to the fact that \(|Du|\) and \(|u|\) enter the right-hand side of (4.9), we may also assume that \(b = 0\) and \(c = 0\).

Next, for \((s, y) \in \mathbb{R}^{d+1}\) set \(Q = Q[s, y] = (s_1, s_2) \times B' \times B''\) to be \(Q_{\kappa r}\) if \(\kappa r < R_0\) and \(Q_{R_0}(s, y)\) if \(\kappa r \geq R_0\). For such \(Q[s, y]\) we denote \(B' = B'[s, y], B'' = B''[s, y]\). Recall the definitions given in Assumption 2.1 and set
\[
a_{[s, y]}(t) = a_{B'[s, y] \times B''[s, y]}(t)
\]
\[
a_{[s, y]} = \frac{a_{B''[s, y]}}{tr_2 a_{B''[s, y]}} tr_2 a_{[s, y]}, \quad \hat{f}_{[s, y]} = a_{[s, y]}^{ij} D_{ij} u + \partial_t u.
\]
Let \(a\) depend only on \((t, x')\), \(\text{tr}_2 a = \text{tr}_2 a_{[s, y]}\) depends only on \(t\) and takes values between \(2\delta\) and \(2\delta^{-1}\) and for any \((s, y)\),

\[
\int_{Q[s, y]} |a - a_{[s, y]}| \, dx dt \leq N \int_{Q[s, y]} |a \text{tr}_2 a_{B'v[s, y]} - a_{[s, y]} \text{tr}_2 a_{B'v[s, y]}| \, dx dt
\]

\[
\leq N \int_{Q[s, y]} |a - a_{B'v[s, y]}| \, dx dt + N \int_{Q[s, y]} |\text{tr}_2 a_{B'v[s, y]} - \text{tr}_2 a_{[s, y]}| \, dx dt \leq N\theta.
\]

Note that

\[
\hat{f}_{[s, y]} = \left(a_{[s, y]}^{ij} - a^{ij}\right) D_{ij} u + f,
\]

and for any values of the parameters \((s, y)\) and

\[
w_{[s, y]}(t, x) := u(t, x)\zeta(t - s, x - y)
\]

we have

\[
a_{[s, y]}^{ij} D_{x'x} w_{[s, y]} + \partial_t w_{[s, y]} = \hat{f}_{[s, y]}(\cdot - (s, y)) + 2a_{[s, y]}^{ij} D_{x'} D_x \zeta(\cdot - (s, y)) + u a_{[s, y]}^{ij} D_{x'x} \zeta(\cdot - (s, y)) + u \partial_t \zeta(\cdot - (s, y)) =: \tilde{f}_{[s, y]}.
\]

Since \(\text{tr}_2 a\) depends only on \(t\), by Theorem 4.5 of [2] with an appropriate translation

\[
\text{Var}_{Q_{[s, y]}} D^2_{x'x} w_{[s, y]} \leq N \kappa^{d+2} \left(\|\tilde{f}_{[s, y]}\|^2\right)_{Q_{[s, y]}} + N\kappa^{-2\alpha} \left(\|D^2_{x'x} w_{[s, y]}\|^2\right)_{Q_{[s, y]}},
\]

where \(N\) and \(\alpha\) depend only on \(d\) and \(\delta\). By the definition of \(\tilde{f}_{[s, y]}\),

\[
\int_{Q_{[s, y]}} |\tilde{f}_{[s, y]}|^2 \, dx dt \leq N \int_{Q_{[s, y]}} |f|^2 \zeta^2(\cdot - (s, y)) \, dx dt + NI_{[s, y]} + NJ_{[s, y]},
\]

where

\[
I_{[s, y]} = \int_{Q_{[s, y]}} |a_{[s, y]} - a|^2 |D^2 u|^2 \zeta^2(\cdot - (s, y)) \, dx dt,
\]

\[
J_{[s, y]} = \int_{Q_{[s, y]}} (|D\zeta|^2 + |D^2 \zeta|^2 + |\partial_t \zeta|^2)(\cdot - (s, y))(|Du|^2 + |u|^2) \, dx dt.
\]

Observe that owing to the facts that \(a\) and \(\zeta\) are bounded functions and \(\zeta\) is square integrable, and taking into account H"older’s inequality, we have

\[
\int_{\mathbb{R}^{d+1}} I_{[s, y]} \, dy ds = \int_{Q_{[s, y]}} \left(\int_{\mathbb{R}^{d+1}} |a_{[s, y]} - a|^2 \zeta^2(\cdot - (s, y)) \, dy ds\right) |D^2 u|^2 \, dx dt
\]

\[
\leq NI_1^{1/\sigma} I_2^{1/\tau},
\]

where

\[
I_1 = \int_{Q_{[s, y]}} \int_{\mathbb{R}^{d+1}} |a_{[s, y]} - a| \zeta^2(\cdot - (s, y)) \, dy ds \, dx dt,
\]

\[
I_2 = \int_{Q_{[s, y]}} |D^2 u|^{2\tau} \, dx dt.
\]
Note that, if $\kappa r < R_0$, we have $Q[s, y] = Q_{\kappa r}$ and $a_{[s, y]}$ is independent of $(s, y)$ and is constructed on the basis of $Q_{\kappa r}$. In light of this and (2.1) in that case $I_1 \leq N\theta$.

However, if $\kappa r \geq R_0$, we know that $\zeta$ is supported in $Q_{R_0}$ and therefore,

$$I_1 = N(\kappa r)^{-d-2} \int_{\mathbb{R}^{d+1}} \left( \int_{Q_{\kappa r} \cap Q_{R_0}(s, y)} |a_{[s, y]} - a| \zeta^2(\cdot - (s, y)) \, dx \right) \, dy \, ds.$$

Here the intersection is nonempty only if $|y| \leq \kappa r + 2R_0$ and $-R_0 \leq s \leq (\kappa r)^2$. Such couples are occupying the volume less than $N(\kappa r)^{d+2}$ (since $\kappa r \geq R_0$). For any of those couples the interior integral in (4.8) is less than the integral over $Q_{R_0}(s, y)$ and is dominated by the max $\zeta^2$ times the volume of $Q_{R_0}$ times the first expression in (4.4) with $Q[s, y] = Q_{R_0}(s, y)$. It follows that in this case $I_1 \leq N\theta$ again.

Estimating the integrals with respect to $(s, y)$ of $J[s, y]$ and of the last term in (4.5) is straightforward. This together with (4.5)-(4.7) yields (4.9). The theorem is proved.

Next we extract some consequences from Theorem 4.2 in terms of maximal and sharp functions in parabolic setting. Recall that the maximal function of $u$ is defined by

$$M_u(t, x) = \sup_{Q \in \Omega} \int_Q |u| \, dy \, ds.$$

Obviously, at each point of $Q_r(t_0, x_0)$ and even of $Q_{\kappa r}(t_0, x_0)$ the right-hand side of (4.9) is less than the expression which you get from it by replacing all averages with the corresponding maximal functions. On the other hand, the sharp function of $u$ is defined by

$$u^\#(t, x) = \sup_{Q \in \Omega} \int_Q |u - u_Q| \, dy \, ds.$$

In addition

$$\left( \frac{\text{Var}}{Q_r(t_0, x_0)} \right)^{1/2} \geq \int_{Q_r(t_0, x_0)} |v - v_{Q_r(t_0, x_0)}| \, dx \, dt.$$

It follows from the above that owing to (4.9) for any $Q \in \Omega$ and $(t, x) \in Q$ we have

$$\left( \int_Q |u - u_Q| \, dy \, ds \right)^2 \leq N\kappa^{d+2} M \left( \frac{|f|^2 + |Du|^2 + |u|^2}{2} \right)(t, x),$$

$$+ N\kappa^{d+2} \theta^{1/\sigma} \left( M\left( \frac{|D^2 u|^{2\tau}}{2} \right) \right)^{1/\tau}(t, x) + N\kappa^{-2\alpha} M(D_{x, u}^2)^2(t, x).$$

Obviously we can replace the left-hand side here with $(v^\#(t, x))^2$ and get

$$v^\#/2 \leq N\kappa^{d+2} M \left( \frac{|f|^2 + |Du|^2 + |u|^2}{2} \right),$$

$$+ N\kappa^{d+2} \theta^{1/\sigma} \left( M\left( \frac{|D^2 u|^{2\tau}}{2} \right) \right)^{1/\tau} + N\kappa^{-2\alpha} M(D_{x, u}^2)^2.$$  (4.9)
Now we remind the reader some well-known properties of the $A_p$-weights ($p \in (1, \infty)$). First, the Hardy-Littlewood theorem is true for $A_p$-weights: if $w \in A_p$ then
\[ \int_{\mathbb{R}^{d+1}} M^p f \, w(t, x) \, dx \, dt \leq N \int_{\mathbb{R}^{d+1}} f \, w \, dx \, dt, \]
where $N$ depends only on $d, p$, and the $A_p$-constant $[w]_p$ of $w$. In particular, if $r > 1$ and $w \in A_r$, then
\[ \int_{\mathbb{R}^{d+1}} M^p (|f|^2) \, w \, dx \, dt \leq N \int_{\mathbb{R}^{d+1}} |f|^{2r} w \, dx \, dt, \]
Also, if $w \in A_r$, then there exists $q \in (1, r)$ (close to $r$) and a constant $N$, depending only on $d, r, [w]_r$, such that $w \in A_q$. In particular, if $r/\tau \geq q$, then $w \in A_{r/\tau}$ and
\[ \int_{\mathbb{R}^{d+1}} (M(|D^2 u|^{2r}))^{r/\tau} \, w \, dx \, dt \leq N \int_{\mathbb{R}^{d+1}} |D^2 u|^{2r} w \, dx \, dt. \]
This shows how to choose $\tau > 1$ depending only on $d, r, [w]_r$.

The final piece of information we need to transform (4.9) is the Fefferman-Stein Theorem which says that for $w \in A_p$ we have
\[ \int_{\mathbb{R}^{d+1}} |v|^p w \, dx \, dt \leq N \int_{\mathbb{R}^{d+1}} |v|^p w \, dx \, dt \]
if the left-hand side is finite, where $N$ depends only on $d, p$, and $[w]_p$.

By combining all these facts with (4.9) we come to the following.

**Corollary 4.3.** Let $p > 2$, $K_0 \in (1, \infty)$, $u \in W^{1,2}_p(\mathbb{R}^{d+1})$ and let $w$ be an $A_{p/2}$-weight with $[w]_{p/2} \leq K_0$. Let $\alpha$ be the constant in Theorem 4.4 of [2] and $\theta \in (0, 1]$. Then under Assumption 2.1 ($\theta$) there exists constants $N$ and $\sigma > 1$, depending only on $d, R_0, \delta, p, \sigma$, and $K_0$, such that, for any $\kappa \geq 4$,
\[ \int_{\mathbb{R}^{d+1}} |D^2 u|^p w \, dx \, dt \leq N \kappa^{d+2} \int_{\mathbb{R}^{d+1}} (|Lu| + |Du| + |u|^p) w \, dx \, dt \]
\[ + N \kappa^{d+2} \sigma^{1/\sigma} \int_{\mathbb{R}^{d+1}} |D^2 u|^p w \, dx \, dt + N \kappa^{-2\alpha} \int_{\mathbb{R}^{d+1}} |D^2 u|^p w \, dx \, dt. \quad (4.10) \]

To choose $K_0$ we use Theorem 8.1 of [3] or Theorem 2.5 of [1] according to which, for $p, q \in (2, \infty)$ and two functions $f$ and $g$, the inequality
\[ \int_{\mathbb{R}^{d+1}} |f|^{p/2} w \, dx \, dt \leq \int_{\mathbb{R}^{d+1}} |g|^{p/2} w \, dx \, dt, \]
valid for any $A_{p/2}$-weight $w$ with $A_{p/2}$-constant majorated by $K_0(p, q, d)$, for certain constant $K_0(p, d)$, implies that
\[ \int_{\mathbb{R}^{d-2}} \left( \int_{\mathbb{R}^3} |f|^{p/2} \, dx' \, dt \right)^{q/p} \, dx'' \leq N(p, d) \int_{\mathbb{R}^{d-2}} \left( \int_{\mathbb{R}^3} |g|^{p/2} \, dx' \, dt \right)^{q/p} \, dx''. \]

We fix $p, q \in (2, \infty)$ take $K_0 = K_0(p, q, d)$ in Corollary 4.3 and after that take $\kappa$ in (4.10) so large that the last term is absorbed by the left-hand side.
Then we arrive at

**Corollary 4.4.** Let $p, q > 2$ and $u \in W_p^{1,2}(\mathbb{R}^{d+1})$. Then under Assumption 2.1 (θ) there exists constants $N = N(d, \delta, p, q)$ and $\sigma = \sigma(p, q, d) > 1$, such that

$$\|D_x^2 u\|_{L^{p,q}} \leq N \theta^{1/(pq)} \|D_x^2 u\|_{L^{p,q}} + N \|\partial_t u + |D u| + |u|\|_{L^{p,q}}.$$  \hspace{1cm} (4.11)

**Proof of Theorem 2.8.** Let $L_0 u = \partial_t u + a^{ij} D_{ij} u$. By Corollary 3.3

$$\|D^2 u\|_{L^{p,q}} \leq N \|L_0 u\|_{L^{p,q}} + N \|D_x^2 u\|_{L^{p,q}}.$$  

It follows that

$$\|D^2 u\|_{L^{p,q}} \leq N \|\partial_t u + |D u| + |u|\|_{L^{p,q}} + N \|D_x^2 u\|_{L^{p,q}}.$$  

This and Corollary 4.4 shows how to choose small $\theta = \theta(p, q, d, \delta) > 0$ in order to get

$$\|D^2 u\|_{L^{p,q}} \leq N \left( \|\partial_t u\|_{L^{p,q}} + \|D u\|_{L^{p,q}} + \|u\|_{L^{p,q}} \right).$$  \hspace{1cm} (4.12)

After that to prove assertion (i) of the theorem it suffices to use Agmon’s idea whose implementation in the mixed norms case the reader can find in the proof of Lemma 7.2.3 of [6].

In light of the method of continuity to prove assertion (ii) for general $L$ it suffices to prove it for $L = \partial_t + \Delta$. Furthermore, assertion (i) implies that it suffices to show that, if $f \in C_0^\infty$, then for $L = \partial_t + \Delta$ there exists a solution of (1.1) in $\mathbb{R}^{d+1}$ of class $W^{1,2}_{p,q}$.

Take such an $f$. Then by the classical theory there exists a solution $u$ of (1.1) all derivatives of whose are bounded. Next, take a $\zeta \in C_0^\infty$ such that $\zeta(0, 0) = 1$ and for $n = 1, 2, \ldots$ introduce $\zeta_n(t, x) = \zeta(t/n, x/n)$, $u_n = u\zeta_n$. Obviously, $u_n \in W^{1,2}_{p,q}$ and, owing to assertion (i) applied to $u_n$,

$$\lambda \|u\zeta\|_{L^{p,q}} + \sqrt{\lambda} \|\zeta_n D u\|_{L^{p,q}} + \|\zeta_n D^2 u, \zeta_n \partial_t u\|_{L^{p,q}} \leq N \|f \zeta_n\|_{L^{p,q}} + N (\sup |u| + \sup |D u|) \left( \|\zeta_n\|_{L^{p,q}} + \|D^2 \zeta_n\|_{L^{p,q}} + \|\partial_t \zeta_n\|_{L^{p,q}} \right),$$

where the constants $N$ are independent of $n$. By letting $n \to \infty$ we obtain the desired result. The theorem is proved.

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