A suspended rope wrapped around a cylinder

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Abstract
The problem of a suspended rope wrapped around a fixed cylinder is studied. If the suspension force is larger than a certain threshold (which is larger than the weight of the rope), the rope would remain tightly wrapped around the cylinder. For suspension forces smaller than that threshold but larger than another threshold, the rope becomes loose (loses contact with the cylinder) at some points, but still remains at rest. These thresholds are obtained when there is no friction. The system is then analysed also with friction, and the threshold of tight-wrapping is obtained for that case as well.

Keywords: suspended rope, loose rope, equilibrium

(Some figures may appear in colour only in the online journal)

1. Introduction

A standard problem in elementary mechanics is the study of the equilibrium of a particle on a surface. If the particle moves on the surface, there could be a point where the normal force changes direction. As a nonadhesive surface cannot produce a normal force which pulls the particle toward the surface, at this point the normal force becomes zero and the particle loses contact with the surface. Finding the point where a particle sliding on a frictionless sphere loses contact with the sphere has been studied in many books on elementary mechanics ([1–4] for example). In [5], this problem was extended to arbitrary surfaces, and conditions were found for the point where the particle leaves the surface.

The problem could be more complex if the moving object is not a point particle, for example when it is one dimensional (such as a rope). A simple example of this, is the problem of a rope wrapped around a pole, where due to friction, a small force on one side can keep the rope from sliding. Some other examples regarding flexible chains and strings have been considered in [6]. In [7], the case of a rope wrapped around a circular post has been studied, where the rope crosses over itself several times. The equilibrium of a rope wrapped around a
solid body, and the problem of two ropes crossing each other lying on a surface, have been investigated in [8].

The problem addressed here is that a massive uniform rope is wrapped around a fixed cylinder, so that the rope has two vertical ends supported by external forces. The system is in a uniform gravitational field. It could be that there is friction between the rope and the cylinder, or not. Both cases are studied. It could be that the rope is tightly wrapped around the cylinder, as in figure 1, or that the rope is not touching the bottom of the cylinder, as in figure 2. The questions are the following.

• What is the minimum value of the supporting force, in order that the rope is tightly wrapped around the cylinder?
If the supporting force is less that the above value, how much does the rope drop for a given supporting force?

What is the minimum value of the supporting force, in order that the rope does not completely fall?

In section 2, conditions are investigated for the rope to remain tightly wrapped around the cylinder. It is shown that there is a minimum value for the external force holding the rope, in order that the rope remains tightly wrapped around the cylinder, and that the minimum is determined. In section 3, a situation is studied where the rope has lost contact with some of the bottom part of the cylinder. When this happens, the ends of the rope, where the supporting force is applied, fall by a certain amount $S$. It is shown that two constraints are to be satisfied, one is that the normal force density applied to the rope by the cylinder should be outward, and the other is that the loose part of the rope should be supported by the tension of the rope. It is further shown that this second constraint is stronger. The angle at which the rope loses contact with the cylinder is found, in terms of the supporting force. It is shown that there is a minimum value for the supporting force in order to prevent the rope from falling indefinitely. Also, it is seen that the relation of the angle of contact-loss and the supporting force consists of a stable part and an unstable part. In section 4, the effect of friction is taken into account and conditions are found for the rope to be at the threshold of losing contact with the cylinder. Section 5 is devoted to concluding remarks.

2. The tight rope

Consider a uniform rope of linear mass density $\lambda$ and length $L$. The rope is wrapped around a stationary cylinder of radius $R$, is in a vertical plane, and each of its ends are pulled upward with the force $F$, as in figure 1. An element of the rope which is in contact with the cylinder, spanning an arc angle of $\delta\theta$, has the mass $\delta m$:

$$\delta m = R \lambda \delta\theta. \tag{1}$$

The linear weight density is denoted by $w$:

$$w = \lambda g, \tag{2}$$

where $g$ is the acceleration due to gravity. Denoting the tension of the rope by $T$ and the normal force acting on this element by $\delta N$, Newton’s equations for this element (at rest), projected on the tangent and the normal, are

$$\left(T + \delta T\right) \cos \frac{\delta\theta}{2} - T \cos \frac{\delta\theta}{2} - (wR\delta\theta) \sin \theta = 0, \tag{3}$$

$$\delta N - \left(T + \delta T\right) \sin \frac{\delta\theta}{2} - T \sin \frac{\delta\theta}{2} + (wR\delta\theta) \cos \theta = 0. \tag{4}$$

Dividing the above equations by $\delta\theta$, and sending $\delta\theta$ to 0, one arrives at a differential equation for $T$ and an algebraic equation for $\delta N/\delta\theta$. It would be convenient to introduce the new parameters $\tau$ and $n$ through

$$\tau := \frac{T}{wR}, \quad n := \frac{1}{wR} \frac{\delta N}{\delta\theta}. \tag{5}$$

Basically, $\tau$ is dimensionless tension and $n$ is dimensionless angular density of the normal force. In terms of these, equations (3) and (4) become

$$\frac{d\tau}{d\theta} = \sin \theta. \tag{6}$$
\( n = \tau - \cos \theta. \) \hspace{1cm} (8)

These result in

\[ \tau(\theta) = \tau \left( \frac{\pi}{2} \right) - \cos \theta. \] \hspace{1cm} (9)

\[ n(\theta) = \tau \left( \frac{\pi}{2} \right) - 2 \cos \theta. \] \hspace{1cm} (10)

\( \tau(\pi/2) \) is the dimensionless tension at the point A. The normal force density should be non-negative, resulting in

\[ \tau \left( \frac{\pi}{2} \right) \geq 2. \] \hspace{1cm} (11)

This gives a criterion for \( F \):

\[ F \geq F_0, \quad F_0 := \frac{wL}{2} + wR \left(2 - \frac{\pi}{2}\right). \] \hspace{1cm} (12)

If the external force \( F \) is less than \( F_0 \), then the rope would lose contact with the cylinder at some bottom part of the cylinder. One notes that this force is larger than half the weight of the rope: \((2F_0)\) should cancel the weight of the rope as well as the downward component of the normal force.

### 3. The loose rope

Suppose the external force \( F \) is less than \( F_0 \). Then the rope would lose contact with the cylinder till some angle \( \theta_0 \) (figure 2). Equations (5) through (10) still hold for larger angles.

The condition that \( n \) vanishes at \( \theta_0 \), gives the tension at the angle \( \theta_0 \) through

\[ \tau_1(\theta_0) = \cos \theta_0. \] \hspace{1cm} (13)

But at least for large \( \theta_0 \), this is obviously wrong, as it says that \( \tau_1(\pi/2) \) tends to 0 for \( \theta_0 \) tending to \((\pi/2)\). What would then cancel the weight of the rope?

The problem is that there is another constraint as well, apart from the fact that the normal force should be non-negative, and that constraint is stronger.

The other constraint that should be taken into account, comes from that part of the rope which is not in contact with the cylinder. In that part, for an element of the length \( \delta s \) of the rope, (see figure 3) one has

\[ \delta(T \cos \beta) = 0. \] \hspace{1cm} (14)
\[ \delta(T \sin \beta) = w \delta s, \]  

(15)

which gives

\[ \tau \cos \beta = c, \]  

(16)

\[ \frac{d(\tau \sin \beta)}{dx} = \frac{1}{R \cos \beta}, \]  

(17)

where \( c \) is a constant, \( \tan \beta \) is the slope of the hanging part of the rope, \( x \) is the horizontal coordinate and \( y \) is the vertical coordinate (upward), so

\[ \tan \beta = \frac{dy}{dx}. \]  

(18)

One then has

\[ cR \frac{d^2y}{dx^2} = \left[ 1 + \left( \frac{dy}{dx} \right)^2 \right]^{1/2}, \]  

(19)

leading to

\[ \frac{dy}{dx} = \sinh \left( \frac{x}{cR} \right), \]  

(20)

where \( x = 0 \) has been taken the point at which the rope is horizontal. So,

\[ y = cR \cosh \left( \frac{x}{cR} \right), \]  

(21)

with an additive constant appropriately chosen. This is the well-known hanging-rope equation. Denoting by \( \ell \) half the length of the hanging part,

\[ \ell = \int_0^{R \sin \theta_0} dx \left[ 1 + \left( \frac{dy}{dx} \right)^2 \right]^{1/2}, \]  

\[ = cR \sinh \left( \frac{\sin \theta_0}{c} \right), \]  

(22)

the vertical component of the equilibrium equation for the hanging part of the rope is

\[ \frac{\ell}{R} = \tau(\theta_0) \sin \theta_0. \]  

(23)

Combining this with (16), results in

\[ \frac{\ell}{cR} = \tan \theta_0, \]  

(24)

so that

\[ \tan \theta_0 = \sinh \left( \frac{\sin \theta_0}{c} \right). \]  

(25)

Hence

\[ c = \frac{\sin \theta_0}{\sinh^{-1}(\tan \theta_0)}, \]  

(26)

\[ \frac{\ell}{R} = \frac{\sin \theta_0 \tan \theta_0}{\sinh^{-1}(\tan \theta_0)}, \]  

(27)

\[ \tau_2(\theta_0) = \frac{\tan \theta_0}{\sinh^{-1}(\tan \theta_0)}. \]  

(28)
Comparing equations (13) and (26), one notes that $\tau_1(\theta_0)$ is always smaller than 1, and $\tau_2(\theta_0)$ is always larger than 1, showing that $\tau_2(\theta_0)$ is larger than $\tau_1(\theta_0)$, hence the constraint on tension due to the hanging part of the rope is stronger than the constraint due to the normal force being non-negative.

Regarding the force $F$, one has

$$F = (wR)\tau\left(\frac{\pi}{2}\right) + w\left[\frac{L}{2} - \ell - \left(\frac{\pi}{2} - \theta_0\right)R\right],$$

$$= (wR)[\tau(\theta_0) + \cos \theta_0] + w\left[\frac{L}{2} - \ell - \left(\frac{\pi}{2} - \theta_0\right)R\right],$$

$$= F_0 + wR \left[\tau(\theta_0) + \cos \theta_0 + \theta_0 - \frac{\ell}{R}\right],$$

where

$$\tau(\theta_0) = \frac{(1 - \sin \theta_0) \tan \theta_0}{\sinh^{-1}(\tan \theta_0)} + \cos \theta_0 + \theta_0 - 2.$$  \hfill (29)

Also the tension at point A would satisfy

$$\tau_A(\theta_0) = \frac{\tan \theta_0}{\sinh^{-1}(\tan \theta_0)} + \cos \theta_0.$$  \hfill (31)

For $\theta_0 = 0$, the right-hand side is equal to 2, as expected.

If the rope is loosened, so that

$$\tau_A(\theta_0) = 2 - \epsilon,$$  \hfill (32)

the ends of the rope (where the supporting force is applied) fall by a certain amount $S$. For a small value of $\epsilon$, one would have

$$2 - \epsilon = \frac{\tan \theta_0}{\sinh^{-1}(\tan \theta_0)} + \cos \theta_0,$$  \hfill (33)

$$= 2 \left(1 - \frac{\theta_0^2}{6}\right) + O(\theta_0^4),$$  \hfill (34)

so that

$$\theta_0 = (3 \epsilon)^{1/2} + O(\epsilon^{3/2}).$$  \hfill (35)

For $S$, one has regarding $\ell$,

$$S = \ell - R\theta_0,$$

$$= R\left[\sin \theta_0 \tan \theta_0 - \theta_0\right],$$

$$= R\left[\frac{2\theta_0^3}{45} + O(\theta_0^7)\right].$$  \hfill (36)

so that

$$S = R\left[\frac{2}{5} (3 \epsilon^5)^{1/2} + O(\epsilon^{7/2})\right].$$  \hfill (37)

Inspection of (31) shows that (see figure 4)

$$\tau_A(\theta_0)$$ is decreasing with $\theta_0$, \hspace{1em} $\theta_0 < \theta_m$,

$$\tau_A(\theta_0)$$ is increasing with $\theta_0$, \hspace{1em} $\theta_0 > \theta_m$,

$$\tau_A(\theta_0) \to \infty$, \hspace{1em} $\theta_0 \to \frac{\pi}{2}$.  \hfill (38)
where $\theta_m$ is the point at which the derivative of $\theta_A$ vanishes:

$$\sinh^{-1}(\tan \theta_m) - (\sin \theta_m)[1 + (\cos^2 \theta_m)[\sinh^{-1}(\tan \theta_m)]^2} = 0,$$

$$\theta_m = 0.9855 \text{ rad or } 56.46^\circ. \quad (39)$$

Denoting $\tau_A(\theta_m)$ by $\tau_m$,

$$\tau_m = \tau_A(\theta_m),$$

$$= 1.810, \quad (41)$$

the following are observed.

- For $\tau_A < \tau_m$, there are no solutions for $\theta_0$. This means that if $\tau_A < \tau_m$, the tension cannot support the rope and the rope falls indefinitely.
- For $\tau_m < \tau_A < 2$, there are two solutions for $\theta_0$. One of these is less than $\theta_m$, the other is larger than $\theta_m$. The first solution corresponds to a stable equilibrium, which means that changing $\tau_A$ slightly results in a slight change in $\theta_0$. The second solution corresponds to an unstable equilibrium: increasing $\tau_A$ slightly results in $\theta_0$ moving towards the first solution and a stable equilibrium being established near the first solution, while decreasing $\tau_A$ slightly results in the rope falling indefinitely.
- For $\tau_A > 2$, there is only one solution for $\theta_0$, and that solution is larger than $\theta_m$. The equilibrium is unstable. Increasing $\tau_A$ slightly results in the rope becoming tight ($\theta_0$ tending to zero), while decreasing $\tau_A$ slightly results in the rope falling indefinitely.

### 4. Rope with friction

If the rope is tight, and there is friction, Newton’s equations change to

$$\frac{d\tau}{d\theta} + f = \sin \theta,$$

$$n + \cos \theta = \tau,$$
where \( f \) is the dimensionless angular density of the normalized friction:

\[
f := \frac{1}{w R} \frac{\delta F_{fr}}{\delta \theta},
\]

\( \delta F_{fr} \) being the friction felt by a part of the rope of angular size \( \delta \theta \). Differentiating the second equation, to eliminate the tension, one arrives at

\[
\frac{dn}{d\theta} + f = 2 \sin \theta.
\]

If there is a point \( \theta' \) where \( n(\theta') \) vanishes, then \( f(\theta') \) vanishes as well, so that \( dn/d\theta \) is positive at \( \theta' \), unless \( \theta' = 0 \). But if \( dn/d\theta \) is positive at \( \theta' \), then \( n(\theta') \) is negative at some \( \theta \) with \( \theta < \theta' \), which is not acceptable. So \( n \) can vanish only at \( \theta = 0 \). So relaxing the rope, the first point which loses contact with the cylinder is the lowest point of the rope. Knowing that

\[
f(\theta) \geq \mu n(\theta),
\]

it is seen that at the beginning of sliding,

\[
n_3(0) = 0,
\]

\[
f_3(\theta) = \mu n_3(\theta),
\]

\[
\frac{dn_3}{d\theta} + \mu n_3 = 2 \sin \theta.
\]

So,

\[
n_3(\theta) = \frac{2[\mu \sin \theta - \cos \theta + \exp(-\mu \theta)]}{1 + \mu^2},
\]

resulting in

\[
f_3(\theta) = \frac{2 \mu [\mu \sin \theta - \cos \theta + \exp(-\mu \theta)]}{1 + \mu^2},
\]

\[
\tau_3(\theta) = \frac{2 \mu \sin \theta + (\mu^2 - 1) \cos \theta + 2 \exp(-\mu \theta)}{1 + \mu^2},
\]

from which

\[
\tau_3 \left( \frac{\pi}{2} \right) = \frac{2 [\mu + \exp(-\mu \pi/2)]}{1 + \mu^2},
\]

as expected, for \( \mu \to 0 \) these tend to the previous results. For large \( \mu \),

\[
n_3(\theta) = \frac{2 \sin \theta}{\mu} + O(\mu^{-2}),
\]

\[
f_3(\theta) = 2 \sin \theta + O(\mu^{-1}),
\]

\[
\tau_3(\theta) = \cos \theta + O(\mu^{-1}),
\]

\[
\tau_3 \left( \frac{\pi}{2} \right) = \frac{2}{\mu} + O(\mu^{-2}).
\]

5. Concluding remarks

A system was studied which consists of a massive uniform rope wrapped around a fixed cylinder, with the rope having two vertical ends supported by external forces. It was found that in order that the rope remains tightly wrapped around the cylinder, the supporting force should be larger than a minimum value. Then a situation was studied where the rope loses contact at some bottom part of the cylinder. Conditions for such an equilibrium were investigated, and a relation was found between the supporting force and the amount by which the ends of the rope fall. Finally, the effect of the friction was taken into account and the system was particularly studied at the threshold where the rope loses contact with some part of the cylinder, for which the value of the supporting force was determined in terms of the friction coefficient.
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