ON FUNCTIONS WHOSE MEAN VALUE ABSICASSAS ARE STOLARSKY MEANS OF ENDPOINTS

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ABSTRACT. We provide a new approach to determine all differentiable functions whose mean value abscissas are Stolarsky means of endpoints.

1. INTRODUCTION AND THE MAIN RESULT

Let $I \subset \mathbb{R}$ be an open interval and let $f : I \to \mathbb{R}$ be a differentiable function. For any finite interval $(a, b) \subset I$, Lagrange’s Mean Value Theorem guarantees the existence of $c \in (a, b)$ such that the tangent to the graph of $f$ at the point $(c, f(c))$ is parallel to the secant through its endpoints $(a, f(a))$ and $(b, f(b))$.

It is one of the many beautiful properties of the parabolas with horizontal directrix that the mean value in the Lagrange Mean Value Theorem always corresponds to the arithmetic mean of the endpoints. That is, we have

$$f(b) - f(a) = (b - a)f'(\frac{a + b}{2}) \quad \text{for all } a, b \in \mathbb{R},$$

(1)

if $f$ is any quadratic polynomial. It is then a natural question to ask whether quadratic polynomials are the only members of the vector space of differentiable functions having the property (1). More generally, by asking for which $f$ the mean value $c$ in Lagrange’s Mean Value Theorem depends on the endpoints in a certain given manner, one arrives at so-called functional-differential equations.

It was Haruki [4] and Aczél [1] who showed independently that the quadratic polynomials are the only differentiable functions that solve (1). The functional-differential equation (1) was one of the starting points for the rich literature devoted to various and more general functional equations including the ones for higher order Taylor expansions and also in the abstract setting of groups. To name just a few, we mention [8], [5], [3], [7], [2] and also [9] for a textbook reference.

In this note we are interested in the case when the mean value in Lagrange’s Mean Value Theorem corresponds to the so-called Stolarsky mean of end-points. More precisely, we solve the functional-differential equation

$$f(b) - f(a) = (b - a)f'(S_\alpha(a, b)), \quad a, b > 0,$$

(2)

where, for $\alpha \in (-\infty, \infty)$, $S_\alpha : (0, \infty)^2 \to (0, \infty)$ stands for the Stolarsky mean which is a continuous function defined by

$$S_\alpha(a, b) = \begin{cases} \frac{(b^\alpha - a^\alpha)}{\alpha(b - a)} & \text{if } \alpha \notin \{0, 1\}, \\ S_0(a, b) & \text{if } \alpha = 0, \\ S_1(a, b) & \text{if } \alpha = 1, \end{cases}$$

(3)

for all \( a, b > 0 \) with \( a \neq b \), and \( S_\alpha(a, a) := a \) for all \( a > 0 \), where

\[
S_0(a, b) := \frac{b - a}{\log b - \log a}, \quad S_1(a, b) := \frac{1}{e} \exp \left( \frac{b \log b - a \log a}{b - a} \right),
\]

see e.g. [10]. For example, the case \( \alpha = 2 \) corresponds to the arithmetic mean.

Our main result reads as follows.

**Theorem 1.** Let \( \alpha \in \mathbb{R} \) be arbitrary and let \( f : (0, \infty) \to \mathbb{R} \) be a differentiable function satisfying (2).

(i) If \( \alpha \notin \{0, 1\} \), then there exist constants \( c_1, c_2, c_3 \in \mathbb{R} \) such that

\[
f(x) = c_1 x^\alpha + c_2 x + c_3, \quad x > 0.
\]  

(ii) If \( \alpha \in \{0, 1\} \), then there exist constants \( c_1, c_2, c_3 \in \mathbb{R} \) such that

\[
f(x) = c_1 x^\alpha \log(x) + c_2 x + c_3, \quad x > 0.
\]

In particular, we have the following result for the case when the mean value in Lagrange’s Mean Value Theorem corresponds to the geometric mean of endpoints.

**Corollary 2.** Let \( f : (0, \infty) \to \mathbb{R} \) be a differentiable function such that

\[
f(b) - f(a) = (b - a)f'(\sqrt{ab}) \quad \text{for all} \quad a, b > 0.
\]  

Then there exist constants \( c_1, c_2, c_3 \in \mathbb{R} \) such that

\[
f(x) = \frac{c_1}{x} + c_2 x + c_3, \quad x > 0.
\]

It is easy to check that all the functions of the form (3)-(4) indeed satisfy (2). To the best of our knowledge, the first and the only work in the current literature which considers (2) is the paper [6]. The purpose of this note is to provide a direct approach based on reducing the problem to elementary differential equations with a novel trick inspired by the methods of [2].

2. Proof of Theorem [1]

First we establish a preliminary result on the smoothness of the solutions of (2).

**Lemma 1.** Let \( \alpha \in \mathbb{R} \) be arbitrary. Every differentiable function \( f : (0, \infty) \to \mathbb{R} \) satisfying (2) is infinitely differentiable on \((0, \infty)\).

**Proof.** We provide a detailed proof for the case \( \alpha \notin \{0, 1\} \). The analyses of the cases \( \alpha = 0 \) and \( \alpha = 1 \) are completely similar and are left to the interested reader. Let us take arbitrary \( x_0 > 0 \). Below we show infinite differentiability of \( f \) in a neighborhood of \( x_0 \). It is easy to see that there are \( y_0 > 0 \) and \( h_0 > 0 \) such that \((y_0 + h_0)^\alpha - y_0^\alpha = \alpha h_0 x_0^{\alpha-1}\). For example, we can take

\[
y_0 = h_0 = \left( \frac{2^{\alpha - 1}}{\alpha} \right)^{1/\alpha} x_0.
\]

Next, we consider the function

\[
\psi(x, y) = (y + h_0)^\alpha - y^\alpha - \alpha h_0 x_0^{\alpha-1}.
\]

We have \( \psi(x_0, y_0) = 0 \). Clearly, \( \psi \) is continuously differentiable with

\[
\frac{\partial}{\partial y} \psi(x, y) = \alpha(y + h_0)^{\alpha-1} - \alpha y^{\alpha-1}
\]

which does not vanish since \( \alpha \neq 1 \) and \( \alpha \neq 0 \). By the implicit function theorem there is a neighborhood \( U \) of \( x_0 \) and a unique continuously differentiable function \( \phi : U \to (0, \infty) \) such that \( \phi(x_0) = y_0 \) and \( \psi(x, \phi(x)) = 0 \) for all \( x \in U \). Furthermore,
Proof of Theorem 1. First let \( f \) be \( k \) times differentiable, then (8) implies its \( k+1 \) times differentiability in \( U \). By induction, it follows that \( f \) is infinitely differentiable in the neighborhood \( U \) of \( x_0 \).

**Proof of Theorem** 1. First let \( \alpha \notin \{0, 1\} \). Let \( t > 0 \) be fixed arbitrarily. For every \( r > 0 \), there is \( x_0 > 0 \) such that \((t + r)^\alpha - t^\alpha = \alpha rx_0^{\alpha - 1}\). Next, we consider the function

\[
\Psi(h, y) = (y + h)^\alpha - y^\alpha - \alpha h x_0^{\alpha - 1}
\]

on \((0, \infty)^2\). We have \( \Psi(r, t) = 0 \). It is easy to check that the implicit function theorem applies and yields the existence of a neighborhood \( V \) of \( r \), and a unique continuously differentiable function \( g: V \to (0, \infty) \) such that \( g(r) = t \) and \( \Psi(h, g(h)) = 0 \) for all \( h \in V \). Moreover, we have

\[
g'(h) = \frac{1}{\alpha h} \frac{(g(h) + h)\alpha - g(h)^\alpha - \alpha h(g(h) + h)^{\alpha - 1}}{(g(h) + h)^{\alpha - 1} - g(h)^{\alpha - 1}}. \tag{10}
\]

By inserting \( g \) and \( h + g \) in (2) for \( a \) and \( b \), respectively, and taking into account the fact that \( \Psi(h, g(h)) = 0 \) for all \( h \in V \), we get

\[
f(h + g(h)) - f(g(h)) = h f'(x_0), \quad h \in V. \tag{11}
\]

Next, let us differentiate (11) with respect to \( h \) to get

\[
f'(h + g(h)) + f'(h + g(h))g'(h) - f'(g(h))g'(h) = f'(x_0). \tag{12}
\]

If we multiply (12) by \( \frac{1}{r^2} \) and differentiate the resulting identity with respect to \( h \), and insert (10), then for \( h = r \), we obtain,

\[
\frac{1}{\alpha r^3} \left( f'(r + t) - f'(t) \right) = \frac{R(r, t)}{S(r, t)} + (f'(x_0) - f'(t)) \frac{T(r, t)}{S(r, t)}. \tag{13}
\]

for all \( r > 0 \), where

\[
R(r, t) := \frac{f''(r + t)}{r^3} \phi(r, t)^2 - \frac{f''(t)}{r^3} \psi(r, t)^2
\]

\[
S(r, t) := \frac{\alpha}{r^3} \left( (r + t)^{\alpha - 1} - t^{\alpha - 1} \right) \phi(r, t) - \frac{\alpha - 1}{r^2} t^{\alpha - 2} \psi(r, t)
\]

\[
T(r, t) := \frac{\alpha - 1}{r^3} \left( (r + t)^{\alpha - 2} \phi(r, t) - t^{\alpha - 2} \psi(r, t) \right) - \frac{1}{r^4} \left( (r + t)^{\alpha - 1} - t^{\alpha - 1} \right)
\]

with

\[
\phi(r, t) := \frac{(r + t)^\alpha - t^\alpha}{\alpha r} - t^{\alpha - 1}, \quad \psi(r, t) := \frac{(r + t)^\alpha - t^\alpha}{\alpha r} - (r + t)^{\alpha - 1}.
\]
It is not difficult to check that the following asymptotic expansions hold

\[
R(r,t) = \frac{(\alpha - 1)^2 (\alpha - 2)}{4} t^{2\alpha - 4} f'''(t) - \frac{(\alpha - 1)^2 (\alpha - 2)}{6} t^{2\alpha - 5} f''(t) + o(1),
\]
\[
S(r,t) = \frac{\alpha (\alpha - 1)^2 (\alpha - 2)}{12} t^{2\alpha - 5} + o(1),
\]
\[
T(r,t) = -\frac{(\alpha - 1)^2 (\alpha - 2)}{12} t^{2\alpha - 6} + o(1),
\]
as \(r \to 0\). Furthermore, from the equation \(\Psi(r,t) = 0\), we get

\[
x_0 = \left(\frac{(t + r)^{\alpha} - t^{\alpha}}{\alpha r}\right)^{1/\alpha} = t + o(1), \quad \text{as} \quad r \to 0.
\]

In view of these asymptotic expansions, letting \(r \to 0\) in (13), we deduce the elementary differential equation

\[
f'''(t) - \frac{\alpha - 2}{t} f''(t) = 0.
\]

Since \(t > 0\) was fixed arbitrarily, (14) must hold for all \(t > 0\). Taking into account the hypothesis that \(\alpha (\alpha - 1) \neq 0\), it follows immediately that the solution space of the differential equation in (14) is given by \(\text{span}\{1, t, t^{\alpha}\}\), completing the proof of part (i).

For the cases \(\alpha = 0\) and \(\alpha = 1\), in the same way as above one obtains the differential equation (14) (for the corresponding values of \(\alpha\)), the solution spaces of which are \(\text{span}\{1, t, \log t\}\) and \(\text{span}\{1, t, t \log t\}\), respectively.

\[\Box\]

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