Role of Higher-Order Interactions on the Modulational Instability of Bose-Einstein Condensate Trapped in a Periodic Optical Lattice

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Abstract
In this paper, we investigate the impact of higher-order interactions on the modulational instability (MI) of Bose-Einstein Condensates (BECs) immersed in an optical lattice potential. We derive the new variational equations for the time evolution of amplitude, phase of modulational perturbation, and effective potential for the system. Through effective potential techniques, we find that high density attractive and repulsive BECs exhibit new character with direct impact over the MI phenomenon. Results of intensive numerical investigations are presented and their convergence with the above semi analytical approach is brought out.

1 Introduction
Solitons in a Bose-Einstein condensate (BEC) trapped in an optical lattice (OL) potential have attracted great attention recently. Generally, BEC in OL is an ideal test bed for condensed matter theory with non-linearity, which gives variety of phenomena such as gap solitons, localization, breathers, diffusion, vortices in lattice, etc... Such a rich dynamics that arises in BECs in OL can be attributed to the phase transition that takes place from superfluid to mott-insulator. In particular, the dispersion dynamics in OL can give rise to stable localized matter wave states in the form of gap solitons. This is represented by

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stationary solutions of the associated Gross-Pitaevskii (GP) equation with the eigenvalue located at optically induced finite bandgap, in the repulsive condensate also [1].

In the ultracold regime, most of the results of experiments in BECs are reproduced and described by the theoretical model based on the nonlinear mean-field GP equation with two-body interaction [2, 3] and the number density dependent three-body interaction [4, 5]. However, it should be pointed out that the impact of higher-order interactions have not yet been considered [2, 6, 7] in this context. The fact that the BEC density increases in various experiments with strong compression of the trap [8, 9], makes the introduction of higher-order interaction more realistic and inevitable in the description of the dynamics. In very recent findings, higher-order interactions challenged the three-body interactions over the MI of BECs [10]. Earlier studies bring about stabilization in BECs even with repulsive two-body interaction with/without nonlinear management in which the role of higher order interactions has been made more crucial [11, 12].

The inclusion of the shape dependent higher-order interaction at higher density through residual nonlinearity is justified as new phenomena and may bring about changes into the dynamics of BECs, especially when immersed in traps like OL potentials. It can be recalled that the recent first real time solitary matter wave pulse was observed experimentally in Rb atoms by the process of modulational instability, where the condensate was kept in periodically varying optical lattice structure (optical waveguide). Thus, in the present work, we examine the effects of higher-order interactions and how they impact the MI [13] of plane wave for a collective interactions including residual nonlinearity of the system consisting of BECs trapped in an OL potential driven by harmonic trap. To produce localized patterns such as solitons, breathers and fundamental vortices which form the basis for energy transport mechanism in nonlinear physical systems, the investigation of the onset of MI is inevitable. Some of the recent investigations towards the exploration of MI parametric domain by means of linear stability analysis from the perspective of higher-order interactions in recent years include BECs with zero-nonlinearity in single component [14], residual nonlinearity in two-component condensate [15], Spin-Orbit coupled (SOC) BECs mixture [16], inter-component asymmetry interaction in quantum droplets and SOC-BECs in optical lattice [17, 18].

In the present investigation, through the time-dependent variational approach (TDVA) [10, 19, 20] and numerical calculations, we study the MI in BECs through the solutions of GP equation with the effect of periodic potential. For this purpose, we perform the TDVA to propose not only the MI conditions but also the time-dependence of the perturbation parameters. The paper is structured as follows: In Section 2, we present the theoretical model that describes the condensates under our consideration. Section 3 is devoted to the mathematical framework in which we derive the MI conditions of the system through the TDVA. Then, in Section 4, we discuss the onset of MI under the higher-order interaction and OL potential. In Section 5, we perform direct numerical simulation to check the validity of the MI conditions found by analytical methods. Finally, in Section 6, we summarize our results and present our conclusions.

2 The Model

At ultra-low temperatures, BECs with two-, three- body and higher-order interactions can be described by the following GP equation [7, 21]
\[
\hat{i} \hbar \frac{\partial \Psi(\mathbf{r}, t)}{\partial t} = \left( -\frac{\hbar^2}{2m} \nabla^2 + V_{\text{trap}} + g |\Psi(\mathbf{r}, t)|^2 \right) \Psi(\mathbf{r}, t) + (g_3 |\Psi(\mathbf{r}, t)|^4 + \eta_0 \nabla^2 |\Psi(\mathbf{r}, t)|^2) \Psi(\mathbf{r}, t),
\]

(1)

where \( \hbar \) is the reduced Planck’s constant, \( m \) is the mass of the boson, \( V_{\text{trap}} = \frac{m}{2} \left( \omega_r^2 r^2 + \omega_\perp^2 \perp^2 \right) + V_{\text{opt}}(x) \) upon which we emphasize that the OL potential is applied only in the longitudinal direction, i.e., \( V_{\text{opt}}(x) = V_m \cos^2(kx + \theta) \), with \( V_m \) being the potential depth of OL. The parameter \( \theta \) is an arbitrary phase and \( k = 2\pi/\lambda \) is a wave number of the OL that can be controlled by varying the angle between two counter propagating laser beams whose interference creates the OL [22]. \( \omega_r \) and \( \omega_\perp \), respectively, are the longitudinal and radial frequencies of the external trap, and \( \rho \) denotes the radial distance. The parameters \( g \) and \( g_3 \) are the strengths of the two- and three-body interatomic interactions, respectively. The role of \( g \) depends on \( a_i \) by \( g = 4\pi \hbar^2 a_i / m \). The last term describes the shape-dependent confinement correction of the two-body collision potential. The parameter \( \eta \) is the higher-order scattering coefficient which depends on both the s-wave scattering length and the effective range for collisions [23, 24]. This parameter reads \( \eta_0 = g g_2 \), where \( g_2 \) is defined by \( g_2 = a_r^2 / 3 - a_r r_e / 2 \), with \( r_e \) being effective range.

The radial motion can be strongly confined by making the radial trapping frequency \( \omega_\rho \) much larger than the axial frequency \( \omega_x \). In this case, the condensate is cigar-shaped, and owing to that, one can take

\[
\Psi(\mathbf{r}, t) = \phi_0(\rho) \phi(x, t)
\]

(2)

with \( \phi_0 = \sqrt{1/(\pi a_\perp^2)} \exp(-\rho^2/(2a_\perp^2)) \), \( \rho = \sqrt{r^2 + z^2} \) and \( a_\perp = \sqrt{\hbar/m \omega_\perp} \), is the ground state of the radial equation,

\[
-\frac{\hbar^2}{2m} \nabla_\rho^2 \phi_0 + \frac{m}{2} \omega_\rho^2 \rho^2 \phi_0 = \hbar \omega_\rho \phi_0.
\]

(3)

Then, multiplying both sides of the GP (1) by \( \phi_0^* \) and integrating over the transverse variable \( \rho \), we can get the following quasi-1D GP equation [7, 10, 22]:

\[
\hat{i} \hbar \frac{\partial \phi(x, t)}{\partial t} = \left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{m}{2} \omega_\rho^2 \rho^2 + V_m \cos^2(\kappa x) \right) \phi(x, t) + \left( g_1 |\phi|^2 + g_3 |\phi|^4 + \eta_0 \frac{\partial^2 |\phi|^2}{\partial x^2} \right) \phi(x, t),
\]

(4)

where, \( g_1 = g/(2\pi a_\perp^2) \), \( g_3 = g_3 / (3\pi^2 a_\perp^4) \) and \( \eta_0 \) is \( \eta_0 = g_2 / (2\pi a_\perp^2) \). It is more convenient to use the above (4) into a dimensionless form. For this purpose, we make the transformation of variables as \( \hat{t} = \mu t \), \( \hat{x} = x \kappa \), \( \psi = \phi \sqrt{2a_\rho \omega_\perp / \mu} \), where \( \mu = E_R / \hbar \) with \( E_R = \hbar^2 \kappa^2 / 2m \). In the case of deep optical potential wells combined to a weak confinement, the harmonic trapping potential can be neglected compared to the OL potential [20]. In this case, we can get the following normalized 1D GP equation:

\[
\hat{i} \frac{\partial \psi(x, t)}{\partial t} = \left( -\frac{\partial^2}{\partial x^2} + V \cos^2(x) + g |\psi(x, t)|^2 \right) \psi(x, t) + \left( \chi |\psi(x, t)|^4 + \eta \frac{\partial^2 |\psi(x, t)|^2}{\partial x^2} \right) \psi(x, t),
\]

(5)
$V_s = V_m/E_R$, \( g = a_s/a_{so} \), $\chi = g_3 t a_{so}^2 \omega_\perp^2 \kappa^4/E_R $, and $\eta = gg_2 \kappa^2 $, where $a_{so}$ is the constant scattering length.

### 3 Modulational Instability

Making use of the variational approach in ref. [10, 20], we write the lagrangian density as

$$
\mathcal{L} = \frac{i}{2} \left( \frac{\partial \psi}{\partial t} \psi^* - \frac{\partial \psi^*}{\partial t} \psi \right) - \frac{1}{2} \frac{\partial^2 |\psi|^2}{\partial z^2} - V_s \cos^2(x) |\psi|^2 - \frac{1}{2} g |\psi|^4 - \frac{1}{3} \chi |\psi|^6 - \frac{1}{2} \eta \frac{\partial^2 |\psi|^2}{\partial z^2} |\psi|^2.
$$

with the MI-motivated wavefunction of the form

$$
\psi(x, t) = (A_0 + \delta) \exp \left[ i (kx - (k^2 + gA_0^2 + \eta A_0^4 + \chi A_0^4) t) \right],
$$

where $\delta = a_1(t) \exp [i(qx + b_1(t))] + a_2(t) \exp [i(qx + b_2(t))]$. Using this ansatz in the procedure used in Ref. [19], the following equations are obtained (derivations given in Appendix A)

$$
\frac{\partial a}{\partial t} = A_0^2 a \sin(b) \left[ q^2 \eta - g - \chi (2A_0^2 + 6a^2) \right],
$$

$$
\frac{\partial b}{\partial t} = -2q^2 - V_s + 2A_0^2 q^2 \eta - 4A_0^4 \chi + 8q^2 \eta a^2 - \chi a^2 (36A_0^2 + 20a^2) - 2g(A_0^2 + 3a^2) + 2A_0^2 \cos(b) \left[ q^2 \eta - g - \chi (2A_0^2 + 12a^2) \right],
$$

with the effective potential given by

$$
V_{\text{eff}} = 2A^2 V_s \left[ q^2 + A_0^2 (g - q^2 \eta + 2A_0^2 \chi) + \frac{V_s}{4} \right] + 2A^2 \left[ q^2 \left( q^2 + 2gA_0^2 + 4A_0^4 \chi - 2q^2 \eta A_0^2 \right) \right] + 3A^3 \left[ g \left( \frac{1}{2} V_s + q^2 + gA_0^2 \right) + \chi A_0^2 \left( 3V_s + 6q^2 \right) + 4gA_0^2 \right] + 4A_0^4 \right] - \eta A^3 \left[ 4q^4 - 4q^4 A_0^2 \eta \right] + 2q^2 V_s + 7A_0^2 q^2 (g + 2A_0^2) \right] + A^4 \left[ \frac{\chi}{3} \left( 5V_s + 10q^2 \right) + \frac{175}{2} A_0^2 \chi + \frac{101}{2} A_0^2 \right] + \frac{9}{8} \left( \frac{g}{A_0^2} \right) \right] + \frac{25}{18} \chi^2 A^6 - \eta A^4 \left[ -2q^4 \eta + \frac{64}{3} A_0^2 q^2 \chi + 3q^2 g \right] + A^5 \chi \left[ \frac{5}{2} g + 15A_0^2 \chi - \frac{10}{3} q^2 \eta \right].
$$

Common derivation of stability analysis sets the time-dependent condition for the modula tionally unstable waves with wavenumber $q$ evolving in BECs trapped in an OL potential as:
\[ V_s^2 + 4q^2(V_s + q^2 + 2gA_0^2 + 4\chi A_0^4 - 2\eta q^2 A_0^2) + 4A_0^2 V_s(g + 2A_0^2\chi - q^2\eta) < 0. \] (11)

4 Onset of MI under Higher-order Interactions and OL Potential

Next, through the effective potential, we can analyze the role of the three-body interaction on the stability of the BECs along with both two-body and higher-order interactions. One can determine the stability of the dynamics from the assessment of the curvature of effective potential at \( A = 0 \). The dynamics is stable if the potential is convex, i.e. the second derivative \( \frac{\partial^2 V_{eff}}{\partial A^2} |_{A=0} > 0 \). Otherwise, the dynamics is unstable if the potential turns out to be concave, i.e. the second derivative \( \frac{\partial^2 V_{eff}}{\partial A^2} |_{A=0} < 0 \).

In Fig. 1, the effective potential curves are shown for the attractive two-body condensate. Panel (a) corresponds to the combination of negative three-body interaction \( \chi = -1 \) and positive higher-order interaction \( \eta = 1 \) for four different strengths of OL range chosen as \( V_s \in [-2, 4] \). The wavenumber \( q \) is modulationally stable for \( V_s = -2 \) and modulationally unstable for \( V_s = 0, 2, 4 \). In panel (b), for repulsive three-body \( \chi = 1 \) and attractive higher-order interactions \( \eta = 1 \) with the same range of optical potential \( V_s \in [-2, 4] \), the wavenumber gives stable modes for optical strength \( V_s = 0, 2, 4 \) and critically stable mode for \( V_s = -2 \). One can easily observe that for any change of sign of interaction, the stability of the system shows some kind of anti-symmetry nature in MI scenario. The physical insight has been observed from effective potential curve when we exchange the sign of higher-order interaction for attractive two-body condensates. The stability condition is also

![Fig. 1](image-url)
controlled by the strength of optical lattice energy reinforced with short range interactions and accordingly, the instability gets either enhanced or suppressed.

In Fig. 2, the effective potential curves are shown for the repulsive two-body condensates. Panel (a) corresponds to the combination of negative three-body interaction ($\chi = -1$) and positive higher-order interaction ($\eta = 1$) for four different strengths of OL range chosen as $V_s \in [-4, 2]$. The wavenumber $q$ is modulationally stable for $V_s = -4$, critically stable for $V_s = -2$, and unstable for $V_s = 0, 2$. In panel(b), for repulsive three-body ($\chi = 1$) and attractive higher-order interactions ($\eta = -1$) with the same range of optical potential $V_s \in [-4, 2]$, the wavenumber gives stable modes for optical strength $V_s = 0, 2$, critically stable mode for $V_s = -2$, and unstable mode for $V_s = -4$.

Thus, the repulsive two-body condensate also shows the anti-symmetric MI behavior with respect to the reversal of the sign of interaction strength similar to attractive condensates which means that energy excitation over the initial perturbation has exchanged symmetry with respect to three-body and higher-order interactions controlled by the strength of optical potential. Moreover, if we turn off the higher-order interaction also, we find the stability condition as in [9]. The removal of three-body interaction in our model results in the instability condition found in [25, 26]. The results of MI in a harmonic trap can also be obtained as a special case of our model when we remove the optical potential [21]. Figure 3 shows the modulationally stable and unstable domains in the $(V_s, q)$ plane for attractive two-body interaction ($g < 0$). The top row in Fig. 3 shows absence of three-body interaction ($\chi = 0$) for $g_2 = 0, g_2 = 0.25$ and $g_2 = -0.25$ from left to right respectively. Here, the three different regions made by the parabolas which are obtained through (11) represent stability domains for finite range of modulational wave

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**Fig. 2** Effective potential $V_{\text{eff}}$ $V_s$ perturbed amplitude $A$ for repulsive two body interaction $g = 1$.

(a) Negative three-body interaction $\chi = -1$ and positive higher order interaction $\eta = 1$. (b) positive three-body interaction $\chi = 1$ and negative higher-order interaction $\eta = -1$. All quantities plotted are dimensionless.
number. The region (II) which is in between two parabolas indicates unstable domain while the region (I) surrounded by first parabola and also the region (III) behind the second parabola represent the stable domains. When the higher-order interaction is turned off \((g_2 = 0)\), the modulational unstable region is quite large. For positive higher-order interactions \((g_2 = 0.25)\), the wave number shrinks in the tail of two parabolas which implies it suppresses instability for appropriate negative values of optical potential strength and the location of vertex of two parabolas is similar as in the case of absence of higher-order interaction. But for negative higher-order interaction \((g_2 = -0.25)\), the tails of the parabolas expand so that the area of the unstable region increases, which causes the enhancement of MI on modulated wave number \((q)\). Hence, the negative higher-order interaction with attractive two-body interaction supports system instability. Now when we turn on the three-body interaction as positive \((\chi = 0.25)\), the \((V_s, q)\) domain is shown in bottom of Fig. 3. The remaining parameters are the same as what we used in upper panel. In this case, similar to the upper panel, the parabolas have three regions (I,II,III). In this case, the vertex of second parabola shifts towards zero of optical potential strength for all the higher-order interaction values \((g_2 = 0, 0.25, -0.25)\). The area of region (II) in the bottom row, which represents the unstable domain has reduced. The inclusion of positive three-body interaction with higher-order interaction of attractive condensate suppresses the system instability. In Fig. 4, which is also obtained from (11), the \((V_s, q)\) domain has been shown for repulsive two-body interaction \((g > 0)\) condensate in the absence of three-body interaction (top row, \(\chi = 0\)) and positive three-body interaction (bottom row, \(\chi = 0.25\)). In both top and bottom rows, from left to right, higher-order interaction strength is \(g_2 = 0\), \(g_2 = 0.25\) and \(g_2 = -0.25\) respectively. In the absence of three-body interaction \((\chi = 0)\) for all values of higher-order interaction, the domain has two modulationally stable regions (I, III) and one unstable region (II). For \(g_2 = 0\) and \(g_2 = -0.25\), they have similar kind of MI domain

![Fig. 3 The stability/instability domain in the \((V_s, q)\) plane for attractive two-body interaction i.e., \(g < 0\). From left to right ([a,b,c] and [d,e,f]), \(g_2 = 0\), \(g_2 = 0.25\) and \(g_2 = -0.25\), respectively. The upper and lower rows correspond to \(\chi = 0\) and \(\chi = 0.25\), respectively. All quantities plotted are dimensionless](image)
as shown in between the vertex of two parabolas. But, the domain for $g_2 = -0.25$ has expanded more at the tails of the parabolas so that the MI has enhanced at this value of potential strength. The positive higher-order interaction ($g_2 = 0.25$) suppresses the MI near the tails of the two parabolas where the shrink has been more pronounced, but between the vertex, it supports instability of the system. In the bottom row of Fig. 4, the three-body interaction has been made positive ($\chi = 0.25$) with the higher-order interaction strengths ($g_2 = 0$, $g_2 = 0.25$ and $g_2 = -0.25$) being the same from left to right as in the case of absence of three-body interaction ($\chi = 0$). The combined impact of three-body interaction and higher-order interaction along with repulsive two-body interaction supports modulationally unstable regimes for the negative optical potential strength. One can observe from the figure that area of region II has been enhanced which corresponds to the negative domain of optical potential. Here, we observe that the inclusion of the three-body interaction with higher order interaction in the repulsive condensate can enhance the system instability. In general, this does not support instability dynamics in simple non-linear frame without trap (i.e. NLS equation) Besides this, the instability conditions in $(V_s, q)$ domain from Figs. 3 and 4 can also be justified by checking the time evolution equations of variational parameters $a(t)$ and $b(t)$ described by the ODEs in (8)-(9). The variational parameter $a(t)$ preserves its initial oscillatory profile for stable dynamics because the interplay doesn’t happen between non-linearity and linear dispersion. If the interplay takes place in the system, the dynamics becomes unstable. In that case, the variational parameters exponentially grow with time. Then, we numerically solve the set of equations given by (8)-(9) by means of a fourth order Runge-Kutta scheme. The initial condition values used here are $a(0) = 0.01$ and $b(0) = 0$. In Fig. 5, we have shown the dynamics of the system in the presence of attractive two-body interaction. Panels (a-b) and (c-d) correspond to the absence ($\chi = 0$) and presence.
\( \chi = 0.25 \) of three-body interaction, respectively. Also, panels (a, c) and (b, d) pertain to the presence of attractive \((\eta < 0)\) and repulsive \((\eta > 0)\) higher-order interaction, respectively. In each panel, we have shown the dynamics of the BECs for four different values of the optical potential which is chosen from Fig. 3(b-c and e-f). In Fig. 5(a), the initial perturbed amplitude oscillates with time for optical potential strengths \( V_s = 1, 3, 5 \) for \( g_2 = 0.25 \). Hence, it is obvious that the system is able to sustain its stability for this set of parameters. It shows exponential growth mode for \( V_s = -1 \) which means that excitation takes place and the system becomes unstable. In Fig. 5(b) the dynamics shows unstable modes for optical potential strengths \( V_s = -1 \) and 1 while stable modes exist up to 5 time units for \( V_s = 3 \). The OL potential for \( V_s = 5 \) with \( g_2 = -0.25 \) exhibits stable mode. The impact of the inclusion of three-body along with higher-order interactions is

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![Graphs showing response of variational parameter (a) with respect to time (t) for attractive two-body interaction (a) and (b) for absence of three-body interaction (a = 0) with \( g_2 = 0.25 \) and \( g_2 = -0.25 \) respectively. (c) and (d) for presence of three-body interaction (\( \chi = 0.25 \)) with \( g_2 = 0.25 \) and \( g_2 = -0.25 \) respectively. All quantities plotted are dimensionless.](image-url)
shown in Fig. 5(c) and (d) for the value of $g_2 = 0.25$ and $g_2 = -0.25$ with OL potential strengths in the range $V_s \in [-1, 3]$. The inclusion of three-body interaction in both cases ($g_2 = 0.25, -0.25$) enables the initial amplitude to oscillate with time against perturbation. The effects of higher-order interactions altogether suppress the instability dynamics in attractive condensates.

The dynamics of the BECs with repulsive two-body interaction has been shown in Fig. 6(a) and (b) in the absence of three-body interaction along with $g_2 = 0.25$ and $g_2 = -0.25$ respectively, while (c) and (d) correspond to the presence of three-body interaction along with $g_2 = 0.25$ and $g_2 = -0.25$ respectively. In Fig. 6(a-d), we have shown the dynamics of the BECs for four different values of the optical potential ($V_s \in [-6, 0]$) which is chosen from Fig. 4(b-c and e-f). In Fig. 6(a) and (b), the results are shown for the absence of three-body interaction along with $g_2 = 0.25$ and $-0.25$, respectively. In panel(a), the dynamics shows stability for $V_s = -6, 2$ and instability for $V_s = -2, -4$. In panel(b), the dynamics shows stability for $V_s = 0$ and instability for $V_s = -2, -4, -6$. In Fig. 6(c) and (d), the results are shown for the inclusion of three-body interaction with $g_2 = 0.25$ and $-0.25$, respectively. In both panels, the dynamics shows stability for $V_s = 0$ and instability for increasing negative strength of optical potential $V_s = -2, -4, -6$. From all above results, one can observe that the instability dynamics can be controlled by sign and strength of optical potential reinforced with higher-order interaction in high density condensates. This is well matched with analytical results from ($V_s, q$) domain (Figs. 4 and 5).

### 5 Numerical Results

The above analytical results are derived from linearization of the unperturbed carrier wave. The validity of such analysis is limited to small amplitudes of perturbation compared with the carrier wave. So, analytical analysis of MI needs confirmation through direct numerical...
simulations of the modified GP E (4). In the present study, we consider a limiting case of the following equation, where the condensate is initially prepared in OL and embedded in harmonic magnetic field [9, 25, 26]: \[ \phi(x, t) = \phi_{TF}[\phi_0 + \epsilon \cos(qx)] \]. Here, we use the Thomas-Fermi approximation considering \[ \phi_{TF} = \sqrt{\max[0, \mu - V(x)]} \] as the background wave function. The numerical values are chosen as \( \mu = 1.0, \phi_0 \equiv A_0 = 1.0, \alpha = 0.004, q = 1.0, \epsilon = 0.001 \) for both attractive and repulsive condensates during numerical integrations. The harmonic potential is only active at \( t = 0 \), when the optical potential is turned off. However, the condensate is prepared at \( t < 0 \) in an optically embedded harmonic magnetic field, where the initial wave function affects the propagation without influencing its modulational instability or stability. In Fig. 7, we show the dynamics of 3D space-time evolution of the square amplitude of the wave, \(|\phi(x, t)|^2\), for attractive condensate in OL embedded in harmonic potential with fixed values of higher-order non-linearity \( g_2 = -0.25 \) and three-body interaction \( \chi = 0.25 \). In order to check the analytical result, we pick up the appropriate optical potential strength from domain in Fig. 3(a). The right panel of Fig. 7 shows the dynamics for the optical potential strength \( V_s = 2 \) which represents the stable dynamics. In this case, one can observe that only a fraction of the condensate can oscillate in space and time. Moreover, the essential condition for dynamical stability that shows oscillation deviation of the square amplitude, is not moved away from its initial value \( \phi_0 = 1.0 \) [9]. Figure 7 (left panel) shows the dynamics of 3D space-time evolution of the square amplitude \(|\phi(x, t)|^2\) of the wave for optical potential strength \( V_s = 0 \) with the rest of parameters being the same as in Fig. 3(a).The dynamics exhibits instability as one could expect for these parameters in Fig. 3(a) and 5(d). The square amplitude \(|\phi(x, t)|^2\) drastically moves away from \( \phi_0 = 1.0 \) to \( \phi_0 = 4.0 \). Comparing the results displayed in two panels of Fig. 7, one can realize from the growth rates that the system is in fact unstable in left panel compared to right panel and also the wave amplitude is higher for unstable mode. In Fig. 8, we show the dynamics of 3D space-time evolution of the square amplitude of the wave, \(|\phi(x, t)|^2\), where values of repulsive condensate, strength of the higher-order non-linearity \( (g_2) \), three-body interaction \( (\chi) \) and optical potential strength \( (V_s) \), are picked from the Fig. 4(e). The left panel shows unstable dynamics for \( V_s = -4 \), where the initial amplitude moves away

![Figure 7](image-url)
from \(\phi_0 = 1.0\) to \(\phi_0 = 2.0\). The right panel shows that, for stable dynamics with \(V_s = 0\), only fraction of condensate atoms do oscillate and, the deviation of initial amplitude is \(\phi_0 = 1.0\) to \(\phi_0 = 1.2\). All these results are well matched with analytical results.

6 Conclusion

In this paper, we have derived the modulational instability condition (see (11)) for BECs trapped in an optical lattice potential. There arises corrective terms, as compared to previous available data (see ref. [10]), due to the amount of OL potential energy. First analytically, through a variational approach, the MI is analysed and predictions are built over the shrinking of thickness for stability bandwidth. These results match with reality, especially in the context of BECs on the surface of atomic chips and in atomic waveguides involving a strong compression of the traps thereby appreciably enhancing the density of the BECs. The nature of the condensate (attractive and repulsive) is nonsensitive to the effects of the MI criteria with OL potential.

Secondly, after trying analytical resolution of the modified GP equation through approximation method (TDVA), we obtained the ordinary differential equations in various perturbation variables. We have then analyzed in detail the role of higher-order interactions on the modulational instability of BECs immersed in OL potential through effective potential plots. For instance, we have made prominent predictions over feasible insight into the dynamics of high density BECs: When higher-order interactions are not taken into account, the MI may appear or not, depending on the strength and sign of optical lattice applied to the condensate. But in the presence of higher-order interaction, while its strength is controllable by the Feshbach resonance techniques, we have been able to suppress unstable zones thereby annihilating the modulational instability phenomena in the system. These semi analytical results are then supported by direct numerical integrations.

![Fig. 8](image-url) The dynamics of the condensate wave for \(g > 0\) with \(g_2 = 0.25\) and \(\chi = 0.25\). The 3D space-time evolution of the square amplitude of the wave for 2 modes chosen in Fig. 4(e). (Left panel) \(V_s = -4\) for unstable region and (right panel) \(V_s = 0\) for stable region. All quantities plotted are dimensionless.
Appendix A: Derivation of Variational Parameters $a(t)$ and $b(t)$

The MI-motivated trial wavefunction in (7) is substituted into the Lagrangian density in (6) and the effective Lagrangian is calculated by integrating the Lagrangian density as

$$ L_{\text{eff}} = \int L \, dx $$

But here, we consider an annular (one-dimensional) geometry, which imposes periodic boundary conditions on the wave function $\psi(x,t)$ and integration limits $0 \leq x < 2\pi$. This causes the quantization of the wave numbers, i.e. $k, q = 0, \pm 1, \pm 2, \pm 3, \ldots$. In this new geometry, calculating the effective Lagrangian yields

$$ L_{\text{eff}} = \pi \left\{ -V_s (A_0^2 + a_1^2 + a_2^2) - g \left[ 2A_0^2 (a_1^2 + a_2^2) - A_0^4 \right. \right. $$

$$ + 4A_0^2 a_1^2 + 4A_0^2 a_1 a_2 \cos(b_1 + b_2) + a_1^4 + a_2^4 \right] + \eta \left[ 8q^2 a_1^2 a_2^2 \right. $$

$$ + 4A_0^2 q^2 a_1 a_2 \cos(b_1 + b_2) + 2A_0^2 q^2 (a_1^2 + a_2^2) \right] - \chi \left[ \frac{2}{3} (a_1^6 + a_2^6) \right. $$

$$ + 4A_0^4 (a_1^2 + a_2^2) + 6A_0^2 (a_1^4 + a_2^4) + 6a_1^3 a_2 (a_2^2 + a_1^2) + \frac{4}{3} A_0^6 $$

$$ + 4A_0^2 a_1 a_2 \cos(b_1 + b_2) (2A_0^2 + 3a_1^2 + 3a_2^2) + 24A_0^2 a_1^2 a_2^2 \right. $$

$$ + 2a_1^2 (-2kq - q^2 - b_1) + 2a_2^2 (2kq - q^2 - b_2) \left. \right\} $$

The expression of this effective Lagrangian is such that the pair $(b_1(t), b_2(t))$ may be interpreted as the set of generalized coordinates of the system, while the pair $(A_1(t), A_2(t))$, with $A_1(t) = 2a_1^2(t)$ and $A_2(t) = 2a_2^2(t)$, gives the corresponding momenta. The Hamiltonian of the system is expressed as

$$ H = -L + \int_{-\infty}^{\infty} \frac{i}{2} \left( \frac{\partial \psi^*}{\partial t} \psi - \frac{\partial \psi}{\partial t} \psi^* \right) dx. $$

Considering the integration limits imposed by the new geometry, we have

$$ H = \pi \left\{ V_s \left( A_0^2 + \frac{A_1}{2} + \frac{A_2}{2} \right) + g \left[ 2A_0^2 (A_1 + A_2) + A_0^4 \right. \right. $$

$$ + \frac{1}{4} (A_1^3 + A_2^3) + 2A_0^2 \sqrt{A_1} \sqrt{A_2} \cos(b_1 + b_2) + A_1 A_2 \right. $$

$$ - \eta \left[ 2A_0^2 q^2 \sqrt{A_1} \sqrt{A_2} \cos(b_1 + b_2) + A_0^2 q^2 (A_1 + A_2) \right. $$

$$ + 2q^2 A_1 A_2 \right] - \chi \left[ \frac{1}{12} (A_1^3 + A_2^3) + 2A_0^3 (A_1 + A_2) \right. $$

$$ + \frac{3}{2} A_0^2 (A_1^2 + A_2^2) + \frac{3}{4} A_1 A_2 (A_1 + A_2) - \frac{4}{3} A_0^6 + 6A_0^2 A_1 A_2 $$

$$ + A_0^2 \sqrt{A_1} \sqrt{A_2} \cos(b_1 + b_2)(4A_0^2 + 3A_1 + 3A_2) \right. $$

$$ + A_1 (2kq + q^2 + k^2) + A_2 (-2kq + q^2 + k^2) + 2A_0^2 k^2 \left. \right\} $$

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In order to derive the evolution equations for the time-dependent parameters introduced in (7), we use the corresponding Euler-Lagrange equations based on the variational effective Lagrangian $L_{\text{eff}}$. In the generalized form, these equations read

$$\frac{d}{dt} \left( \frac{\partial L_{\text{eff}}}{\partial \dot{\xi}_i} \right) - \frac{\partial L_{\text{eff}}}{\partial \xi_i} = 0,$$

where $\xi_i$ and $\dot{\xi}_i$ are, respectively, the generalized coordinate and corresponding generalized momentum. Hence, the evolution equation corresponding to the variational parameter $a_1$ is

$$\frac{d a_1}{d t} = \left[ q^2 \eta - g - \chi \left( 2A_0^2 - 3a_1^2 - 3a_2^2 \right) \right] A_0^2 a_2 \sin(b_1 + b_2).$$

For the parameter $b_1$, the evolution equation reads

$$\frac{d b_1}{d t} = -2kq - q^2 - \frac{V_s}{2} - g \left( A_0^2 + a_1^2 + 2a_2^2 \right) + q^2 \eta \left( A_0^2 + 4a_2^2 \right) - \chi \left[ 2A_0^4 + 6A_0^2 (a_1^2 + a_2^2) + a_1^4 \right] + 3a_2^2 (2a_1^2 + a_2^2) + (q^2 \eta - g) A_0^2 \frac{a_2}{a_1} \cos(b_1 + b_2).$$

For the parameter $a_2$, we get

$$\frac{d a_2}{d t} = \left[ q^2 \eta - g - \chi \left( 2A_0^2 - 3a_1^2 - 3a_2^2 \right) \right] A_0^2 a_1 \sin(b_1 + b_2).$$

and for the parameter $b_2$, the evolution equation is

$$\frac{d b_2}{d t} = -2kq - q^2 - \frac{V_s}{2} - g \left( A_0^2 + 2a_1^2 + a_2^2 \right) + q^2 \eta \left( A_0^2 + 4a_1^2 \right) - \chi \left[ 2A_0^4 + 6A_0^2 (2a_1^2 + a_2^2) + a_2^4 \right] + 3a_1^2 (a_1^2 + 2a_2^2) + (q^2 \eta - g) A_0^2 \frac{a_1}{a_2} \cos(b_1 + b_2).$$

For simplicity, we may use a variant of the MI-motivated wavefunction (7) for which

$$a_1 = a_2 = a, \quad \text{and} \quad b_1 + b_2 = b.$$

Then the coupled ordinary differential equations for $a(t)$ and $b(t)$ are shown (8) and (9).

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**Declarations**

**Competing interests** The authors declare no competing interests.
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