Finite-element analysis of plate stability under conditions of nonlinear creep

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Abstract. Resolving equations are obtained for the finite element analysis of the stability of plates and shells with allowance for nonlinear creep. The issue of plate stability under creep process is investigated by the example of a round plate rigidly clamped along the contour with an initial deflection under the action of radial compressive forces. It has been established that for plates made of a material that obeys the nonlinear Maxwell-Gurevich law, there is a long critical load $p_\infty$. When the load is less than the long critical ($p < p_\infty$), the deflection growth rate decays, i.e. buckling does not occur, at $p = p_\infty$ the deflection increases at a constant speed, and at $p > p_\infty$, the rate of growth of the deflection increases.

Introduction
The stability analysis of thin-walled structures in the form of plates and shells deserves much attention, since such structures are widely used in construction and other fields of technology. One of the urgent problems of the theory of plates and shells is their calculation under creep conditions, which confirms the large number of recent publications on this topic. So, in [1-7], questions of buckling during creep of composite thin-walled structures are investigated. Articles [8-10] are devoted to the problem of stability of viscoelastic plates and shells under dynamic and tracking loads, and medium-thickness plates are considered in [11]. The mathematical difficulties arising in solving these problems lead to the fact that many researchers restrict themselves to the linear laws of viscoelastic deformation or consider the case of steady-state creep. There is a need for universal calculation methods suitable for arbitrary laws of the relationship between stresses and creep deformations, including nonlinear ones.

The aim of this work is to obtain resolving equations for the stability problem of nonlinearly viscoelastic plates under the action of forces in the median plane, taking into account geometric nonlinearity. It should be noted that the problem of buckling during creep is not a problem of pure stability. Initial imperfections are required, which, as a rule, are accepted in the form of initial curvature or eccentricity of the load application.

Methods
We obtain the resolving equations for a plane finite element experiencing bending and the action of forces in the median plane, taking into account anisotropy, forced deformations, thickness inhomogeneity, and geometric nonlinearity. Forced deformations are understood as the sum of temperature deformations, creep deformations, shrinkage, etc. The relationship between stresses and strains, taking into account the above factors, can be written as
\{\sigma\} = \begin{bmatrix} \bar{D} \end{bmatrix}\{\varepsilon^{el}\},
\end{equation}

where \(\begin{bmatrix} \bar{D} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\) - matrix of elastic constants, which can be functions of \(z\) (coordinate \(z\) varies from \(-h/2\) to \(h/2\) within the thickness of the element), \(\{\sigma\} = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}^T\) - stress vector, 
\(\{\varepsilon^{el}\}\) - vector of elastic strains representing the difference between total and forced strains:
\begin{equation}
\{\varepsilon^{el}\} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \varepsilon_x^* \\ \varepsilon_y^* \\ \gamma_{xy}^* \end{bmatrix} = \{\varepsilon_{tot}\} - \{\varepsilon^*\}.
\end{equation}

We apply the variational Lagrange principle to derive equations. The potential strain energy is written as:
\begin{equation}
\Pi = \frac{1}{2} \int_{V} \{\sigma\}^T \{\varepsilon^{el}\} dV.
\end{equation}

Total deformations include deformations of the middle surface, as well as deformations caused by a change in curvature:
\begin{equation}
\{\varepsilon_{tot}\} = \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix} + z \begin{bmatrix} \chi_x \\ \chi_y \\ 2\chi_{xy} \end{bmatrix} = \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix} + z \{\chi\}.
\end{equation}

Deformations of the middle surface with large deflections include a linear and nonlinear component:
\begin{equation}
\{\varepsilon^0\} = \begin{bmatrix} \varepsilon_i^0 \\ \varepsilon_n^0 \end{bmatrix} + \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{bmatrix} + \frac{1}{2} \begin{bmatrix} \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial y} \end{bmatrix}.
\end{equation}

Taking into account equalities (2), (4), (5), the potential strain energy takes the form:
\begin{align}
\Pi &= \frac{1}{2} \int_{V} \{\sigma\}^T \left( \begin{bmatrix} \varepsilon_i^0 \\ \varepsilon_n^0 \end{bmatrix} + z \{\chi\} - \{\varepsilon^*\} \right) dV = \frac{1}{2} \int_{V} \left( \begin{bmatrix} \varepsilon_i^0 \\ \varepsilon_n^0 \end{bmatrix}^T + z \{\varepsilon^0\}^T - \{\varepsilon^*\}^T \right) \begin{bmatrix} \bar{D} \end{bmatrix} \left( \begin{bmatrix} \varepsilon_i^0 \\ \varepsilon_n^0 \end{bmatrix} \right) + \\
+ \left( \begin{bmatrix} \varepsilon_i^0 \\ \varepsilon_n^0 \end{bmatrix} + z \{\chi\} - \{\varepsilon^*\} \right) d\tilde{V} = \frac{1}{2} \int_{A} \left( \begin{bmatrix} \varepsilon_i^0 \\ \varepsilon_n^0 \end{bmatrix}^T \right) \begin{bmatrix} \bar{D} \end{bmatrix} d\tilde{V} + \int_{A} \left( \begin{bmatrix} \varepsilon_i^0 \\ \varepsilon_n^0 \end{bmatrix}^T \right) d\tilde{V} d\chi - \int_{A} \left( \begin{bmatrix} \bar{D} \end{bmatrix} \{\varepsilon^*\} \right) dz - \\
- \left( \begin{bmatrix} \varepsilon_i^0 \\ \varepsilon_n^0 \end{bmatrix} \right) \begin{bmatrix} \bar{D} \end{bmatrix} d\tilde{V} - 2 \left( \begin{bmatrix} \varepsilon_i^0 \\ \varepsilon_n^0 \end{bmatrix} \right) \begin{bmatrix} \bar{D} \end{bmatrix} d\tilde{V} d\chi - \int_{A} \left( \begin{bmatrix} \bar{D} \end{bmatrix} \{\varepsilon^*\} \right) dz + \end{align}
where \( \{ \epsilon \} = [B]\{U\} \),

\[
\{ \epsilon \} = \begin{bmatrix}
\epsilon_x^0 & \epsilon_y^0 & \gamma_{xy}^0 & \chi_x & \chi_y & 2\chi_{xy}
\end{bmatrix}^T
\]

and \( \{ N^* \} = \{ N_x^* \ N_y^* \ S^* \ M_x^* \ M_y^* \ H^* \} \), \( N_x, N_y \) - longitudinal forces, \( S \) - shear force.

With any approximation of displacements, the relationship between nodal displacements and deformations can be represented as:

\[
\{ \epsilon \} = [B]\{U\},
\]

where \( \{ U \} \) is the vector of nodal displacements.

The first term in (8), taking into account (9), will take the form:

\[
\frac{1}{2} \int_A \{ \epsilon \}^T (D) \{ \epsilon \} - 2\{ N^* \} dA = \frac{1}{2} \int_A \{U\}^T [B]^T [D][B]\{U\} dA - \frac{1}{2} \int_A \{U\}^T [B]^T dA \{ N^* \}. 
\]

If the approximation of the deflection is

\[
w(x, y) = \{ \Psi \} \{ U \},
\]

where \( \{ \Psi \} \) are the functions of the form, then the second integral in (8) can be written as

\[
\int_A \{ \epsilon \}_n^T [N_x \ S] dA = \frac{1}{2} \int_A \left( N_x \left( \frac{\partial w}{\partial x} \right)^2 + N_y \left( \frac{\partial w}{\partial y} \right)^2 + 2S \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right) dA =
\]

\[
= \frac{1}{2} \int_A \{U\}^T \left( N_x \frac{\partial \{ \Psi \}}{\partial x} \frac{\partial \{ \Psi \}}{\partial x} + N_y \frac{\partial \{ \Psi \}}{\partial y} \frac{\partial \{ \Psi \}}{\partial y} + S \left( \frac{\partial \{ \Psi \}}{\partial x} \frac{\partial \{ \Psi \}}{\partial x} + \frac{\partial \{ \Psi \}}{\partial y} \frac{\partial \{ \Psi \}}{\partial y} \right) \right) dA \{ U \}.
\]
By minimizing the total energy representing the difference between the potential energy of deformation and the work of external forces we obtain:

\[
([K] + [K_g])\{U\} = \{F\} + \{F^*\},
\]

where \([K] = \int [B]^T [D] [B] dA\) - stiffness matrix of geometrically linear FE, \([K_g] = \int A N_x \frac{\partial \{\Psi\}^T}{\partial x} \frac{\partial \{\Psi\}}{\partial x} + N_y \frac{\partial \{\Psi\}^T}{\partial y} \frac{\partial \{\Psi\}}{\partial y} + S \left( \frac{\partial \{\Psi\}^T}{\partial x} \frac{\partial \{\Psi\}}{\partial y} + \frac{\partial \{\Psi\}^T}{\partial y} \frac{\partial \{\Psi\}}{\partial x} \right) dA\) - geometric stiffness matrix, \(\{F^*\} = \int [B]^T dA \{N^*\}\) - contribution of forced deformations to the load vector.

The geometrically nonlinear problem is solved using the Newton – Raphson method, the essence of which consists in sequentially calculating additional displacements of elements caused by the residual forces. At the first stage, the calculation on the static load is carried out, then the creep calculation is performed. Creep strains are determined using linear approximation by the Euler method.

**Results and Discussion**

We consider the issue of plate stability in terms of creep on example of a round plate rigidly clamped along the contour with an initial deflection under the action of radial compressive forces (figure 1).

![Figure 1. Round plate, rigidly clamped along the contour, compressed by radial forces](image-url)

For this problem, a system of differential equations was obtained with respect to the stress function \(\Phi\) and deflection \(w\):
\[ D \left( \frac{d^3w}{dr^3} + \frac{1}{r} \frac{d^2w}{dr^2} - \frac{1}{r^2} \frac{dw}{dr} \right) = \frac{h}{r} \frac{d\Phi}{dr} \left( \frac{dw}{dr} + \frac{dw_0}{dr} \right) - \frac{dM^*_r}{dr} - \frac{M^*_\theta}{r} \] 
\[ \frac{d^3\Phi}{dr^3} + \frac{1}{r} \frac{d^2\Phi}{dr^2} - \frac{1}{r^2} \frac{d\Phi}{dr} = -E \left( \frac{1}{2} \frac{d^2w}{dr^2} \right)^2 + \frac{dw}{dr} \frac{dw_0}{dr} \frac{1}{r} \frac{d\varepsilon^*_r}{dr} + \frac{d\varepsilon^*_\theta}{dr} + \frac{\varepsilon^*_r - \varepsilon^*_\theta}{r} \right), \] 

(14)

where \( \varepsilon^*_r, \varepsilon^*_\theta \) - mid-surface creep deformations, \( D = Eh^2 / (12\(1 - \nu^2\)) \) - cylindrical stiffness.

The boundary conditions for the system (14) have the form:
\[ w \bigg|_{r=c} = 0, \quad \frac{dw}{dr} \bigg|_{r=c} = 0, \quad \sigma_{r,m} \bigg|_{r=c} = \frac{1}{r} \frac{d\Phi}{dr} \bigg|_{r=c} = -p. \] 
\[ (15) \]

For an ideal plate, an instantaneous loss of stability occurs at the following critical pressure [12]:
\[ p_{cr} = \frac{14.68D}{c^2h}. \] 
\[ (16) \]

As the creep law, we will use the nonlinear Maxwell-Gurevich equation, which can be found for example in [13]. If the Maxwell-Gurevich law is valid for the plate material, then replacing the instantaneous cylindrical stiffness with a long-term stiffness \( D_\infty \), we obtain the value of the long critical load:
\[ p_\infty = \frac{14.68D_\infty}{c^2h}. \] 
\[ (17) \]

Long-term cylindrical stiffness is determined by the formula [14]:
\[ D_\infty = \frac{\alpha h^3}{12(\alpha^2 - \beta^2)}, \] 
\[ (18) \]

where \( \alpha = 1 / E + 1 / E_\infty, \beta = \nu / E + 1 / (2E_\infty), \ E_\infty \) - high elasticity modulus.

The calculation was performed at \( \nu = 0.3, \ E = 3035 \) MPa, \( E_\infty = 2310 \) MPa, \( c = 1 \) m, \( h = 5 \) mm. For such a plate, \( p_\infty = 0.1 \) MPa, \( p_\infty = 0.47 \) \( p_{cr} \). The expression for the initial deflection was taken in the form:
\[ w_0 = f_0 \left( 1 - \frac{r^2}{c^2} \right)^2, \] 
\[ (19) \]

Figure 2 shows graphs of the deflection growth at various load values \( p < p_\infty, p = p_\infty \) and \( p > p_\infty \). The solid lines correspond to the solution of the system (14) using finite differences method, and the dashed lines correspond to the solution based on the FEM.

The results obtained by the two methods are in fairly good agreement with each other. Figure 2 shows that for \( p < p_\infty \), the deflection growth rate decays, for \( p = p_\infty \), the deflection increases at a constant rate, and for \( p > p_\infty \), the deflection growth rate increases. A similar character of the curves of the deflection growth was obtained for compressed rods based on the linear Maxwell – Thompson equation in [15].

**Summary**

The considered problem is very unlikely to be encountered in practical calculations, but it was chosen as a test one, since it has an analytical solution in the elastic formulation. In addition, the one-dimensional nature of this problem makes it relatively easy to obtain an independent solution using other methods for debugging the FEM algorithm. The introduced concept of a long critical load can be generalized to the case of more complex stability problems for creep of plates and shells.
Figure 2. Graphs of the deflection growth

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