Mock theta functions and indefinite modular forms II

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Abstract

In this paper, we compute the Zwegers’s modification of the mock theta functions $\Phi^{[m,0]}_*$ and study the modular transformation properties of the indefinite modular forms which appear in the explicit formula for the modified functions $\tilde{\Phi}^{[m,0]}_*$.

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1 Introduction

In the previous paper [17] we studied indefinite modular forms obtained from the Zwegers’ modification of the mock theta function $\Phi^{(-)[m,p]}$ for $m \in \frac{1}{2}\mathbb{N}_{\text{odd}}$. In the current paper, we study the case $m \in \mathbb{N}$ where, in order that the Zwegers’ modification works, we consider the function

$$\Phi^{(\pm)[m,s]} := \Phi_1^{(\pm)[m,s]} + \Phi_2^{(\pm)[m,s]}$$

(1.1)

with $\Phi_i^{(\pm)[m,s]}$ ($i = 1, 2$) defined in [17].

Comparing the results in the current paper with those in the previous paper [17], there appear strange difference between the case $m \in \mathbb{N}$ and the case $m \in \frac{1}{2}\mathbb{N}_{\text{odd}}$. In the case $m \in \frac{1}{2}\mathbb{N}_{\text{odd}}$ in [17], we need 3 series of functions $g_k^{(i)[m,p]}(\tau)$ ($i = 1, 2, 3$) to get $SL_2(\mathbb{Z})$-invariant family for each $m \in \frac{1}{2}\mathbb{N}_{\text{odd}}$ whereas, in the case $m \in \mathbb{N}$ in the current paper, only one series of functions $g_k^{(1)[m,p]}(\tau)$ span $SL_2(\mathbb{Z})$-invariant spaces for each $m \in \mathbb{N}$.

This paper is organized as follows.

In section 2 we make preparation on basic properties of $\Phi^{(\pm)[m,s]}$. In section 3 we derive the explicit formula for $\Phi^{[m,0]}(\tau, z_1, z_2, 0)$ by using the Kac-Peterson’s identity just in the similar way with [19]. In section 4 we deduce the formula for $\Phi^{[m,0]}(\tau, z_1 + p\tau, z_2 - p\tau, 0)$ in the case when

$$(z_1, z_2) = \left(\frac{z}{2} + \frac{\tau}{2} - \frac{1}{2}, \frac{z}{2} - \frac{\tau}{2} + \frac{1}{2}\right) \quad \text{and} \quad (z_1, z_2) = \left(\frac{z}{2} + \frac{\tau}{2}, \frac{z}{2} - \frac{\tau}{2}\right).$$

(1.2)

In section 5 we compute the Zwegers’s correction function $\Phi_{\text{add}}^{[m,0]}(\tau, z_1, z_2, 0)$ for $(z_1, z_2)$ given by (1.2). In section 6 using the results obtained in 5 we make detail investigation on the relation between $\Phi^{[m,0]}(\tau, z_1 + p\tau, z_2 - p\tau, 0)$ and the modified function $\Phi_{\text{add}}^{[m,0]}(\tau, z_1 + p\tau, z_2 - p\tau, 0)$ when $(z_1, z_2)$ is given by (1.2). In section 7 we obtain the explicit formula for $\Phi_{\text{add}}^{[m,0]}(\tau, z_1, z_2, 0)$ in the case when $(z_1, z_2)$ is given by (1.2). In section 8 we introduce functions $\Xi^{(i)[m,p]}(\tau, z)$ and $\Upsilon^{(i)[m,p]}(\tau, z)$ for $p \in \mathbb{Z}$ such that $0 \leq p \leq 2m$ and $i \in \{1, 2\}$, and compute the modular transformation properties of these functions. In section 9 we define the functions $G^{(i)[m,p]}(\tau, z)$ and $g_k^{(i)[m,p]}(\tau)$ by

$$G^{(i)[m,p]}(\tau, z) := \Xi^{(i)[m,p]}(\tau, z) - \Upsilon^{(i)[m,p]}(\tau, z)$$

$$= \sum_{k \in \mathbb{Z}} g_k^{(i)[m,p]}(\tau)[\theta_{k,m} + \theta_{-k,m}](\tau, z)$$

and compute modular transformation of these functions. In section 10 using the relation between $g_k^{(1)[m,p]}(\tau)$ and $g_k^{(2)[m,p]}(\tau)$, we define the indefinite modular forms $g_k^{[m,p]}(\tau)$ and obtain their modular transformation properties.
2 Preliminaries

Lemma 2.1. Let \( m \in \frac{1}{2} \mathbb{N} \), \( s \in \frac{1}{2} \mathbb{Z} \) and \( a \in \mathbb{Z} \). Then
\[
\Phi^{(\pm)[m,s]}(\tau, z_1 + a\tau, z_2 - a\tau, t) = (\pm 1)^a e^{2\pi ima(z_1 - z_2)} q^{ma^2} \Phi^{(\pm)[m,s-2am]}(\tau, z_1, z_2, t) \tag{2.1}
\]

Proof. This follows immediately from Lemma 2.3 in [17] and definition of \( \Phi^{(\pm)[m,s]} \).

Lemma 2.2. Let \( m \in \frac{1}{2} \mathbb{N} \), \( s \in \frac{1}{2} \mathbb{Z} \) and \( a \in \mathbb{Z}_{\geq 0} \). Then
\[
\Phi^{(\pm)[m,s]}(\tau, z_1 + a\tau, z_2 - a\tau, 0) = (\pm 1)^a e^{2\pi ima(z_1 - z_2)} q^{ma^2} \left\{ \Phi^{(\pm)[m,s]}(\tau, z_1, z_2, 0) + \sum_{k \in \mathbb{Z}} e^{-\pi i(k-s)(z_1 - z_2)} q^{-\frac{1}{4m}(k-s)^2} \left[ \theta^{(\pm)}_{s-k,m} + \theta^{(\pm)}_{-(s+k),m} \right](\tau, z_1 + z_2) \right\} \tag{2.2}
\]

Proof. This lemma can be shown in the similar way with the proof of Lemma 2.4 in [17] as follows. By Lemma 2.1 in [16] with Remark 2.1 in [17], we have
\[
\Phi^{(\pm)[m,s-a]}(\tau, z_1, z_2, 0) - \Phi^{(\pm)[m,s]}(\tau, z_1, z_2, 0) = \sum_{k \in \mathbb{Z}} e^{\pi i(s+k)(z_1 - z_2)} q^{-\frac{(s+k)^2}{4m}} \left[ \theta^{(\pm)}_{s+k,m} + \theta^{(\pm)}_{-(s+k),m} \right](\tau, z_1 + z_2)
\]

Letting \( a \to 2am \), we have
\[
\Phi^{(\pm)[m,s-2am]}(\tau, z_1, z_2, 0) - \Phi^{(\pm)[m,s]}(\tau, z_1, z_2, 0) = \sum_{k \in \mathbb{Z}} e^{\pi i(s+k)(z_1 - z_2)} q^{-\frac{(s+k)^2}{4m}} \left[ \theta^{(\pm)}_{s+k,m} + \theta^{(\pm)}_{-(s+k),m} \right](\tau, z_1 + z_2)
\]

Substituting this equation (2.3) into (2.1), we obtain (2.2), proving Lemma 2.2.

Lemma 2.3. Let \( m \in \frac{1}{2} \mathbb{N} \), \( s \in \frac{1}{2} \mathbb{Z} \) and \( p \in \mathbb{Z} \) such that \( mp \in \mathbb{Z} \). Then

1) if \( p \geq 0 \),
\[
\Phi^{(\pm)[m,s]}(\tau, z_1, z_2 + p\tau, 0) = e^{-2\pi impz_1} \Phi^{(\pm)[m,s]}(\tau, z_1, z_2, 0)
\]
\[
- e^{-2\pi impz_1} \sum_{k=0}^{mp-1} e^{\pi i(s+k)(z_1 - z_2)} q^{-\frac{(s+k)^2}{4m}} \left[ \theta^{(\pm)}_{s+k,m} + \theta^{(\pm)}_{-(s+k),m} \right](\tau, z_1 + z_2)
\]
Proof. These formulas follow immediately from Lemma 2.5 in [16] and definition of \( \Phi(\pm)[m,s] \).

Following formulas for theta functions can be shown easily by using Lemma 1.1 in [17] and Note 1.1 in [17], and will be used in the proof of Lemmas 4.1 and 4.2.

Note 2.1. For \( m \in \mathbb{N} \) and \( p \in \mathbb{Z} \), the following formulas hold:

1) \[
\frac{\theta^{(-)}_{\frac{1}{2},m+\frac{1}{2}}(\tau, \frac{m(2p+1)\tau - m}{m+\frac{1}{2}})}{2m+1} = e^{-\frac{\pi}{2}(m+\frac{1}{2})^2(2p+1)^2} \frac{\theta^{(-)}_{p,m+\frac{1}{2}}(\tau, 0)}{2m+1, m+\frac{1}{2}}(\tau, 0)
\]

2) \[
\frac{\theta^{(-)}_{\frac{1}{2},m+\frac{1}{2}}(\tau, \frac{m(2p+1)\tau}{m+\frac{1}{2}})}{2m+1} = q^{\frac{m^2}{2}(2m+1)^2(2p+1)^2} \theta^{(-)}_{2m+1, p}(\tau, 0)
\]

Note 2.2. For \( m \in \frac{1}{2} \mathbb{N} \) and \( p \in \mathbb{Z} \), the following formulas hold:

1) (i) \[
\frac{\theta^{(-)}_{-\frac{1}{2},m+\frac{1}{2}}(\tau, \frac{z + (2p+1)\tau - 1}{2m+1})}{2m+1} = e^{\frac{\pi}{(2m+1)^2}} q^{-\frac{1}{16(m+\frac{1}{2})}(2p+1)^2} \frac{\theta^{(-)}_{p,m+\frac{1}{2}}(\tau, 0)}{2m+1, m+\frac{1}{2}}(\tau, z)
\]

(ii) \[
\frac{\theta^{(-)}_{\frac{1}{2},m+\frac{1}{2}}(\tau, \frac{z - (2p+1)\tau - 1}{2m+1})}{2m+1} = e^{\frac{\pi}{2(2m+1)^2}} q^{-\frac{1}{16(m+\frac{1}{2})}(2p+1)^2} \frac{\theta^{(-)}_{-p,m+\frac{1}{2}}(\tau, 0)}{2m+1, m+\frac{1}{2}}(\tau, z)
\]

2) (i) \[
\frac{\theta^{(-)}_{-\frac{1}{2},m+\frac{1}{2}}(\tau, \frac{z + (2p+1)\tau}{2m+1})}{2m+1} = q^{-\frac{1}{16(m+\frac{1}{2})}(2p+1)^2} e^{-\frac{\pi}{2}(2p+1)^2} \frac{\theta^{(-)}_{p,m+\frac{1}{2}}(\tau, 0)}{2m+1, m+\frac{1}{2}}(\tau, z)
\]

(ii) \[
\frac{\theta^{(-)}_{\frac{1}{2},m+\frac{1}{2}}(\tau, \frac{z - (2p+1)\tau}{2m+1})}{2m+1} = q^{-\frac{1}{16(m+\frac{1}{2})}(2p+1)^2} e^{-\frac{\pi}{2}(2p+1)^2} \frac{\theta^{(-)}_{-p,m+\frac{1}{2}}(\tau, 0)}{2m+1, m+\frac{1}{2}}(\tau, z)
\]

Note 2.3. For \( p \in \mathbb{Z} \), the following formulas hold:

1) (i) \[
\frac{\vartheta_{11}(\tau, \frac{z}{2} + \frac{(2p+1)\tau - 1}{2})}{2} = q^{-\frac{1}{8}(2p+1)^2} e^{-\frac{\pi}{2}(2p+1)^2} \vartheta_{0,\frac{1}{2}}(\tau, z)
\]

(ii) \[
\frac{\vartheta_{11}(\tau, \frac{z}{2} - \frac{(2p+1)\tau - 1}{2})}{2} = -q^{-\frac{1}{8}(2p+1)^2} e^{\frac{\pi}{2}(2p+1)^2} \vartheta_{0,\frac{1}{2}}(\tau, z)
\]

2) (i) \[
\vartheta_{11}(\tau, \frac{z}{2} + \frac{(p + \frac{1}{2})\tau}{2}) = -i (-1)^p q^{-\frac{1}{8}(2p+1)^2} e^{-\frac{\pi}{2}(2p+1)^2} \vartheta_{0,\frac{1}{2}}(\tau, z)
\]

(ii) \[
\vartheta_{11}(\tau, \frac{z}{2} - \frac{(p + \frac{1}{2})\tau}{2}) = i (-1)^p q^{-\frac{1}{8}(2p+1)^2} e^{\frac{\pi}{2}(2p+1)^2} \vartheta_{0,\frac{1}{2}}(\tau, z)
\]
3 Explicit formula for \( \Phi[m,0]^{\tau}(\tau, z_1, z_2, 0) \)

Proposition 3.1. For \( m \in \frac{1}{2} \mathbb{N} \), \( \Phi[m,0]^{\tau}(\tau, z_1, z_2, 0) \) is given by the following formula:

\[
\theta_{\frac{1}{2}, m + \frac{1}{2}}^{(-)}(\tau, z_1 - z_2) \Phi[m,0]^{\tau}(\tau, z_1, z_2, 0) = \left[ \sum_{j \in \mathbb{Z}} - \sum_{j, k \in \mathbb{Z}} \right] (-1)^j q^{(m + \frac{1}{2})^{j + \frac{1}{4} \left( m + \frac{1}{2} \right)^2} - \frac{k^2}{4m}} \\
\hspace{5cm} \times \sum_{j \in \mathbb{Z}} \sum_{j, k \in \mathbb{Z}} \left[ \frac{1}{\pi} \right] (-1)^j q^{(m + \frac{1}{2})^{j + \frac{1}{4} \left( m + \frac{1}{2} \right)^2}} \cdot \sum_{j \in \mathbb{Z}} \sum_{j, k \in \mathbb{Z}} \left( \theta_{k,m}^{(-)}(\lambda, \mu) + \theta_{-k,m}^{(-)}(\lambda, \mu) \right) \right.
\]

\[
+ \frac{i \eta(\tau)^3}{2 \pi m (j + \frac{1}{4} \left( m + \frac{1}{2} \right)^2)(z_1 - z_2)} e^{-\pi i k (z_1 - z_2)} \left[ \theta_{k,m}^{(-)}(\lambda, \mu) + \theta_{-k,m}^{(-)}(\lambda, \mu) \right] \right]
\]

\[
(3.1)
\]

Proof. Letting \( z_1 = z_2 \) in the formulas (3.1a) and (3.1b) in [18] and making their sum, we have

\[
\sum_{j \in \mathbb{Z}} (-1)^j q^{(m + \frac{1}{2})^{j + \frac{1}{4} \left( m + \frac{1}{2} \right)^2}} e^{-2\pi i j m (z_1 - z_3)} \left[ \Phi_1[m,0]^{\tau}(\tau, z_1, z_3, -z_3 - 2j\tau, 0) \right] = e^{-\pi i z_1} \sum_{k \in \mathbb{Z}} (-1)^k q^{(m + \frac{1}{2})k^2 - \frac{k^2}{4k}} e^{2\pi i k m (z_1 - z_3)} \Phi_1^{(-)}(\frac{1}{2}, m) (\tau, z_1, z_3 - 2k\tau, 0)
\]

\[
+ e^{-\pi i z_3} \sum_{k \in \mathbb{Z}} (-1)^k q^{(m + \frac{1}{2})k^2 - \frac{k^2}{4k}} e^{2\pi i k m (z_1 - z_3)} \Phi_1^{(-)}(\frac{1}{2}, m) (\tau, z_3, z_3 - 2k\tau, 0)
\]

Letting \((j, k) \rightarrow (-j, -k)\) and changing the notation \(-z_3 \rightarrow z_2\), this formula becomes:

\[
\sum_{j \in \mathbb{Z}} (-1)^j q^{(m + \frac{1}{2})^{j + \frac{1}{4} \left( m + \frac{1}{2} \right)^2}} e^{2\pi i j m (z_1 + z_2)} \Phi[m,0]^{\tau}(\tau, z_1, z_2 + 2j\tau, 0) = e^{-\pi i z_1} \sum_{k \in \mathbb{Z}} (-1)^k q^{(m + \frac{1}{2})k^2 + \frac{k^2}{4k}} e^{-2\pi i k m (z_1 + z_2)} \Phi_1^{(-)}(\frac{1}{2}, m) (\tau, z_1, -z_1 + 2k\tau, 0)
\]

\[
+ e^{\pi i z_2} \sum_{k \in \mathbb{Z}} (-1)^k q^{(m + \frac{1}{2})k^2 + \frac{k^2}{4k}} e^{-2\pi i k m (z_1 + z_2)} \Phi_1^{(-)}(\frac{1}{2}, m) (\tau, -z_2, z_2 + 2k\tau, 0)
\]

(3.2)

First we compute the LHS of this equation (3.2):

LHS of (3.2):

\[
= q^{-\frac{1}{16(m + \frac{1}{2})}} \sum_{j \in \mathbb{Z}_{\geq 0}} (-1)^j q^{(m + \frac{1}{2})^{j - \frac{1}{4(m + \frac{1}{2})^2}}} e^{2\pi i j m (z_1 + z_2)} \Phi[m,0]^{\tau}(\tau, z_1, z_2 + 2j\tau, 0)
\]
\[
+ q \frac{1}{16(m+\frac{1}{2})} \sum_{j \in \mathbb{Z}_{<0}} (-1)^j q^{(m+\frac{1}{2})(j-\frac{1}{4(m+\frac{1}{2})})^2} e^{2\pi ij m(z_1+z_2)} \Phi_{m,0}^*[\tau, z_1, z_2 + 2j\tau, 0]
\]

where (I) and (II) are computed by using Lemma 2.3 as follows:

\[
(I) = q \frac{1}{16(m+\frac{1}{2})} \sum_{j \in \mathbb{Z}_{\geq 0}} (-1)^j q^{(m+\frac{1}{2})(j-\frac{1}{4(m+\frac{1}{2})})^2} e^{-2\pi ij m(z_1-z_2)} \Phi_{m,0}^*[\tau, z_1, z_2, 0]
\]
\[
- q \frac{1}{16(m+\frac{1}{2})} \sum_{j \in \mathbb{Z}_{\geq 0}} \sum_{k=0}^{2mj-1} (-1)^j q^{(m+\frac{1}{2})(j-\frac{1}{4(m+\frac{1}{2})})^2} q^{\frac{k^2}{4m}} e^{-2\pi ij m(z_1-z_2)} e^{\pi ik(z_1-z_2)} \theta_{k,m} + \theta_{-k,m} (\tau, z_1 + z_2)
\]

\[
(II) = q \frac{1}{16(m+\frac{1}{2})} \sum_{j \in \mathbb{Z}_{\geq 0}} (-1)^j q^{(m+\frac{1}{2})(j-\frac{1}{4(m+\frac{1}{2})})^2} e^{-2\pi ij m(z_1-z_2)} \Phi_{m,0}^*[\tau, z_1, z_2, 0]
\]
\[
+ q \frac{1}{16(m+\frac{1}{2})} \sum_{j, k \in \mathbb{Z}} (-1)^j q^{(m+\frac{1}{2})(j-\frac{1}{4(m+\frac{1}{2})})^2} q^{\frac{k^2}{4m}} e^{-2\pi ij m(z_1-z_2)} e^{\pi ik(z_1-z_2)} \times \theta_{k,m} + \theta_{-k,m} (\tau, z_1 + z_2)
\]

Then we have

LHS of (3.2) = (I) + (II)

\[
= q \frac{1}{16(m+\frac{1}{2})} \sum_{j \in \mathbb{Z}} (-1)^j q^{(m+\frac{1}{2})(j-\frac{1}{4(m+\frac{1}{2})})^2} e^{-2\pi ij m(z_1-z_2)} \Phi_{m,0}^*[\tau, z_1, z_2, 0]
\]
\[
\underbrace{e^{\frac{\pi im}{2m+\frac{1}{2}}(z_1-z_2)} \theta_{\frac{m}{2}, m+\frac{1}{2}}(\tau, \frac{m}{m+\frac{1}{2}}(z_1-z_2))}_{\text{restriction}}
\]
\[
- q \frac{1}{16(m+\frac{1}{2})} \left[ \sum_{j, k \in \mathbb{Z}} - \sum_{j, k \in \mathbb{Z}} \right] \times (-1)^j q^{(m+\frac{1}{2})(j-\frac{1}{4(m+\frac{1}{2})})^2} q^{\frac{k^2}{4m}} e^{-2\pi ij m(z_1-z_2)} e^{\pi ik(z_1-z_2)} \theta_{k,m} + \theta_{-k,m} (\tau, z_1 + z_2)
\]

(3.3a)

Next we compute the RHS of (3.2) by using Lemma 2.1 in [19]:

RHS of (3.2)

\[
= e^{-\pi iz_1} \sum_{k \in \mathbb{Z}} (-1)^k q^{\frac{k^2 + \frac{1}{2}k}{4}} e^{-2\pi i km(z_1+z_2)} \Phi_{1,0}^*\left(\frac{1}{2}, \frac{1}{2}\right)_{\tau, z_1, -z_1 + 2k\tau, 0}
\]
\[
- i e^{-2\pi ikz_1} \frac{\eta(\tau)^3}{\eta_1(\tau, z_1)}
\]
\[ + e^{i2z} \sum_{k \in \mathbb{Z}} (-1)^k q^{(m+\frac{1}{2})k^2 + \frac{1}{2} k} e^{2\pi i km(z_1 + z_2)} \Phi_{\frac{1}{2}}^{-\frac{1}{2}}(\tau, -z_2, z_2 + 2k\tau, 0) ] \\
- i e^{2\pi i z_2} \eta(\tau)^3 \quad \frac{d}{d\tau} \frac{d}{d\tau} \frac{\eta(\tau)}{\theta_{11}(\tau, z_1)} \]

\[ \begin{align*}
&= - i q - \frac{1}{16(m + \frac{1}{2})} e^{-\frac{2\pi i m}{2m+1} (z_1 - z_2)} \\
&\times \sum_{k \in \mathbb{Z}} (-1)^k q^{(m+\frac{1}{2})k^2 + \frac{1}{2} k} e^{2\pi i (m+\frac{1}{2})(k+\frac{1}{4(m+\frac{1}{2})})} \eta(\tau)^3 \quad \frac{d}{d\tau} \frac{d}{d\tau} \frac{\eta(\tau)}{\theta_{11}(\tau, z_1)} \\
&+ i q - \frac{1}{16(m + \frac{1}{2})} e^{-\frac{2\pi i m}{2m+1} (z_1 - z_2)} \\
&\times \sum_{k \in \mathbb{Z}} (-1)^k q^{(m+\frac{1}{2})k^2 + \frac{1}{2} k} e^{2\pi i (m+\frac{1}{2})(k+\frac{1}{4(m+\frac{1}{2})})} \eta(\tau)^3 \quad \frac{d}{d\tau} \frac{d}{d\tau} \frac{\eta(\tau)}{\theta_{11}(\tau, z_2)} \\
&\end{align*} \]

Then by (3.3a) and (3.3b), we have

\[ q - \frac{1}{16(m + \frac{1}{2})} \left[ \sum_{j, k \in \mathbb{Z}} - \sum_{j, k \in \mathbb{Z}} \right] \\
0 \leq k < 2mj \quad 2mj \leq k < 0 \\
\times (-1)^j q^{(m+\frac{1}{2})(j+\frac{1}{4(m+\frac{1}{2})})^2} q^{-\frac{2\pi i (k-2jm)(z_1 - z_2)}{4m}} [ \theta_{k,m + \theta - k,m} ] (\tau, z_1 + z_2) \]

\[ \begin{align*}
&= - i q - \frac{1}{16(m + \frac{1}{2})} e^{-\frac{2\pi i m}{2m+1} (z_1 - z_2)} \theta_{\frac{1}{4}, m + \frac{1}{2}}(\tau, z_1 + z_2 + \frac{z_1 - z_2}{2m+1}) \frac{\eta(\tau)^3}{\theta_{11}(\tau, z_1)} \\
&+ i q - \frac{1}{16(m + \frac{1}{2})} e^{-\frac{2\pi i m}{2m+1} (z_1 - z_2)} \theta_{\frac{1}{4}, m + \frac{1}{2}}(\tau, z_1 + z_2 - \frac{z_1 - z_2}{2m+1}) \frac{\eta(\tau)^3}{\theta_{11}(\tau, z_2)} \\
&\end{align*} \]

Multiplying \( q - \frac{1}{16(m + \frac{1}{2})} e^{\frac{2\pi i m}{2m+1} (z_1 - z_2)} \) to both sides, we have

\[ \theta_{\frac{1}{4}, m + \frac{1}{2}}(\tau, \frac{m(z_1 - z_2)}{m + \frac{1}{2}}) \frac{\eta(\tau)^3}{\theta_{11}(\tau, z_1, z_2, 0)} \]
For then by (3.4a) and (3.4b) we obtain (3.1), proving Proposition 3.1. The 1st term in the RHS of this equation (3.4a) is rewritten by putting \( j = -j' \) and \( k = -k' \) as follows:

the 1st term in the RHS of (3.4a)

\[
\left[ \sum_{j, k \in \mathcal{Z}} - \sum_{j, k \in \mathcal{Z}} \right] (-1)^j q^{(m+\frac{1}{2})(j-\frac{1}{4(m+\frac{1}{2})})^2} q^{-\frac{\nu^2}{4m}} e^{-2\pi i m (j-\frac{1}{4(m+\frac{1}{2})})(z_1-z_2)}
\]

\[
\times e^{\pi i k(z_1-z_2)} \eta(\tau, z_1 + z_2)
\]

\[\times e^{-i \theta(-) \frac{1}{2} \frac{m+\frac{1}{2}}{m+1} \left( \tau, z_1 + z_2 + \frac{z_1 - z_2}{2m+1} \right) \eta(\tau, z_1)}\]

\[\times e^{-i \theta(-) \frac{1}{2} \frac{m+\frac{1}{2}}{m+1} \left( \tau, z_1 + z_2 + \frac{z_1 - z_2}{2m+1} \right) \eta(\tau, z_2)}\]

(3.4a)

Then by (3.4a) and (3.4b) we obtain (3.1), proving Proposition 3.1

In order to rewrite the formula (3.1), we use the following:

Note 3.1. For \( m \in \frac{1}{2} \mathbb{N} \), the following formula holds:

\[
\left[ \sum_{j, k \in \mathcal{Z}} - \sum_{j, k \in \mathcal{Z}} \right] (-1)^j q^{(m+\frac{1}{2})(j-\frac{1}{4(m+\frac{1}{2})})^2} q^{-\frac{\nu^2}{4m}} e^{-2\pi i m (j-\frac{1}{4(m+\frac{1}{2})})(z_1-z_2)}
\]

\[
\times e^{-\pi i k(z_1-z_2)} \eta(\tau, z_1 + z_2)
\]

\[= \left[ \sum_{j, r \in \mathcal{Z}} - \sum_{j, r \in \mathcal{Z}} \right] \sum_{s \in \mathcal{Z}} (-1)^j q^{(m+\frac{1}{2})(j-\frac{1}{4(m+\frac{1}{2})})^2} q^{-\frac{\nu^2}{4m}} e^{-2\pi i m (j-\frac{1}{4(m+\frac{1}{2})})(z_1-z_2)}
\]

\[
\times e^{-\pi i (2mr-s)(z_1-z_2)} \eta(\tau, z_1 + z_2)
\]

\[+ \left[ \sum_{j, r \in \mathcal{Z}} - \sum_{j, r \in \mathcal{Z}} \right] \sum_{s \in \mathcal{Z}} (-1)^j q^{(m+\frac{1}{2})(j-\frac{1}{4(m+\frac{1}{2})})^2} q^{-\frac{\nu^2}{4m}} e^{-\pi i (2mr+s)(z_1-z_2)} \eta(\tau, z_1 + z_2)
\]

\[= \left[ \sum_{j, r \in \mathcal{Z}} - \sum_{j, r \in \mathcal{Z}} \right] \sum_{s \in \mathcal{Z}} (-1)^j q^{(m+\frac{1}{2})(j-\frac{1}{4(m+\frac{1}{2})})^2} q^{-\frac{\nu^2}{4m}} e^{-2\pi i m (j-\frac{1}{4(m+\frac{1}{2})})(z_1-z_2)}
\]

\[
\times e^{-\pi i (2mr-s)(z_1-z_2)} \eta(\tau, z_1 + z_2)
\]

\[+ \left[ \sum_{j, r \in \mathcal{Z}} - \sum_{j, r \in \mathcal{Z}} \right] \sum_{s \in \mathcal{Z}} (-1)^j q^{(m+\frac{1}{2})(j-\frac{1}{4(m+\frac{1}{2})})^2} q^{-\frac{\nu^2}{4m}} e^{-\pi i (2mr+s)(z_1-z_2)} \eta(\tau, z_1 + z_2)
\]

(3.5)
Proof. In order to prove (3.5), we need only to note the following for \( j \in \mathbb{Z} \):

\[
\begin{align*}
\{ k \in \mathbb{Z} : 0 < k \leq 2mj \} &= \{ k = 2mr - s ; \ r, s \in \mathbb{Z} \ \text{and} \ 0 < r \leq j, 0 \leq s < 2m \} \\
\{ k \in \mathbb{Z} : 2mj < k \leq 0 \} &= \{ k = 2mr - s ; \ r, s \in \mathbb{Z} \ \text{and} \ j < r \leq 0, 0 \leq s < 2m \}
\end{align*}
\]

Then the LHS of (3.5) is rewritten as follows:

\[
\text{LHS of (3.5)} = (I) + (II)
\]

where

\[
(I) := \left[ \sum_{j, r \in \mathbb{Z}} \sum_{s \in \mathbb{Z}} (-1)^j q^{(m+\frac{1}{2})(j+\frac{1}{4(m+\frac{1}{2})})^2} q^{-(m-r-s)^2/4m} \right] \\
\times e^{2 \pi im(j+\frac{1}{4(m+\frac{1}{2})})(z_1 - z_2)} e^{-\pi i(2mr-s)(z_1 - z_2)} \left[ \theta(s,r,m) + \theta(r,m,s) \right](\tau, z_1 + z_2) \tag{3.6a}
\]

\[
(II) := \left[ \sum_{j, r \in \mathbb{Z}} \sum_{s \in \mathbb{Z}} (-1)^j q^{(m+\frac{1}{2})(j+\frac{1}{4(m+\frac{1}{2})})^2} q^{-(m-r-s)^2/4m} \right] \\
\times e^{2 \pi im(j+\frac{1}{4(m+\frac{1}{2})})(z_1 - z_2)} e^{-\pi i(2mr-s)(z_1 - z_2)} \left[ \theta(s,r,m) + \theta(r,m,s) \right](\tau, z_1 + z_2)
\]

Putting \( s' := 2m - s \), we have

\[
m < s < 2m \quad \iff \quad 0 < s' < m
\]

so (II) is rewritten as follows:

\[
(II) = \left[ \sum_{j, r \in \mathbb{Z}} \sum_{s \in \mathbb{Z}} (-1)^j q^{(m+\frac{1}{2})(j+\frac{1}{4(m+\frac{1}{2})})^2} q^{-(m-r-s)^2/4m} \right] \\
\times e^{2 \pi im(j+\frac{1}{4(m+\frac{1}{2})})(z_1 - z_2)} e^{-\pi i(2mr-s)(z_1 - z_2)} \left[ \theta(s',r,m) + \theta(r,s',m) \right](\tau, z_1 + z_2) \\
= \left[ \sum_{j, r' \in \mathbb{Z}} \sum_{s' \in \mathbb{Z}} (-1)^j q^{(m+\frac{1}{2})(j+\frac{1}{4(m+\frac{1}{2})})^2} q^{-(m-r'+s')^2/4m} \right] \\
\times e^{2 \pi im(j+\frac{1}{4(m+\frac{1}{2})})(z_1 - z_2)} e^{-\pi i(2mr'+s')(z_1 - z_2)} \left[ \theta(s',r,m) + \theta(r,s',m) \right](\tau, z_1 + z_2) \tag{3.6b}
\]

Then by (3.6a) and (3.6b), we have

\[
(I) + (II) = \text{RHS of (3.5)}, \quad \text{proving Note 3.1}
\]
By Note 3.1, the formula (3.1) in Proposition 3.1 is rewritten as follows:

**Proposition 3.2.** For $m \in \frac{1}{2} \mathbb{N}$, $\Phi^{[m,0]}(\tau, z_1, z_2, 0)$ is given by the following formula:

$$
\theta_{\frac{1}{2}, m+\frac{1}{2}}(\tau, m(z_1 - z_2) \pm \frac{1}{2}) \Phi^{[m,0]}(\tau, z_1, z_2, 0)
$$

$$
= \eta(\tau)^3 \left\{ - \frac{\theta_{\frac{1}{2}, m+\frac{1}{2}}(\tau, z_1 + z_2 + \frac{z_1 - z_2}{2m + 1})}{\vartheta_{11}(\tau, z_1)} + \frac{\theta_{\frac{1}{2}, m+\frac{1}{2}}(\tau, z_1 + z_2 - \frac{z_1 - z_2}{2m + 1})}{\vartheta_{11}(\tau, z_2)} \right\} + \eta(\tau)^3 \left( \sum \sum_{j, r \in \mathbb{Z}} (-1)^j q^{(m+\frac{1}{2})(j+\frac{1}{4(m+\frac{1}{2})})^2} q^{-\frac{(2mr-s)^2}{4m}} \right)
$$

4 $\Phi^{[m,0]}(\tau, z_1, z_2, t) \sim$ the case $z_1 - z_2 = 2a\tau + 2b$

4.1 $\Phi^{[m,0]}(\tau, z_1, z_2, t) \sim$ the case $z_1 - z_2 = (1 + 2p)\tau - 1$

**Lemma 4.1.** For $m \in \mathbb{N}$ and $p \in \mathbb{Z}$, the following formula holds:

$$
\theta_{\frac{1}{2}, m+\frac{1}{2}}(\tau, 0) \Phi^{[m,0]}\left(\tau, \frac{z}{2} + \frac{\tau}{2} - \frac{1}{2} + pr, \frac{z}{2} - \frac{\tau}{2} + \frac{1}{2} - pr, 0\right)
$$

$$
= \frac{\eta(\tau)^3}{\theta_{\frac{1}{2}, m+\frac{1}{2}}(\tau, z)} \cdot \left[ \theta_{p, m+\frac{1}{2}} + \theta_{-p, m+\frac{1}{2}} \right](\tau, 0)
$$

$$
= \frac{q^{(2p+1)^2}}{\theta_{\frac{1}{2}, m+\frac{1}{2}}(\tau, z)} \sum \sum_{j, r \in \mathbb{Z}_{\text{odd}}} \left( \sum_{k \in \mathbb{Z}} (-1)^j - \frac{1}{2} + k q^{(m+\frac{1}{2})(j+\frac{2km}{2m+1})^2} \right)
$$

$$
= \frac{q^{-\frac{1}{4m}(2mr+k+2mp)^2 + \frac{mp}{2}(2p+1)^2}}{\theta_{\frac{1}{2}, m+\frac{1}{2}}(\tau, z)} \left[ \theta_{k,m} + \theta_{-k,m} \right](\tau, z)
$$
\[
\theta^{(-)}_{2mp+m+\frac{1}{2}, m+\frac{1}{2}}(\tau, 0) \sum_{k \in \mathbb{Z}} (-1)^k q^{-\frac{1}{4m}k^2+\frac{1}{2}(2p+1)} [\theta_{k,m} + \theta_{-k,m}](\tau, z) 
\]

(4.1)

**Proof.** Letting \( z_1 = \frac{t}{2} + \frac{\tau}{2} - \frac{1}{2} + p\tau \) and \( z_2 = \frac{t}{2} - \frac{\tau}{2} + \frac{1}{2} - p\tau \) namely \( z_1 + z_2 = z \) in the formula (3.7) in Proposition 3.2, we have

\[
\theta^{(-)}_{\frac{1}{2}, m+\frac{1}{2}}(\tau, \frac{m((2p+1)\tau - 1)}{m+1}) \Phi_{[m,0]}(\tau, z_1, z_2, 0) 
\]

\[
= i\eta(\tau)^3 \left\{ -\frac{\theta^{(-)}_{\frac{1}{2}, m+\frac{1}{2}}(\tau, z + \frac{(2p+1)\tau - 1}{2m+1})}{\vartheta_{11}(\tau, z_1)} + \frac{\theta^{(-)}_{\frac{1}{2}, m+\frac{1}{2}}(\tau, z - \frac{(2p+1)\tau - 1}{2m+1})}{\vartheta_{11}(\tau, z_2)} \right\} 
\]

\[
= \left[ \sum_{j, r \in \mathbb{Z}} - \sum_{j, r \in \mathbb{Z}} \right] \sum_{s \in \mathbb{Z}} (-1)^j q^{\left(m+\frac{1}{2}\right)(j+\frac{1}{4(m+\frac{1}{2})})^2} q^{-\frac{(2mr-s)^2}{4m}} e^{-\pi i (2mr-s)(2p+1)\tau-1} [\theta_{s,m} + \theta_{-s,m}](\tau, z) 
\]

\[
= i\eta(\tau)^3 \times (1) 
\]

\[
e - e^{-\pi i m} \left[ \sum_{j, r \in \mathbb{Z}} - \sum_{j, r \in \mathbb{Z}} \right] \sum_{s \in \mathbb{Z}} (-1)^j q^{\left(m+\frac{1}{2}\right)(j+\frac{1}{4(m+\frac{1}{2})})^2 + m(j+\frac{1}{4(m+\frac{1}{2})})+(2p+1)} 
\]

\[
\times q^{-\frac{(2mr-s)^2}{4m}} e^{-\pi i (2mr-s)(2p+1)\tau-1} [\theta_{s,m} + \theta_{-s,m}](\tau, z) 
\]

\[
e - e^{-\pi i m} \left[ \sum_{j, r \in \mathbb{Z}} - \sum_{j, r \in \mathbb{Z}} \right] \sum_{s \in \mathbb{Z}} (-1)^j q^{\left(m+\frac{1}{2}\right)(j+\frac{1}{4(m+\frac{1}{2})})^2 + m(j+\frac{1}{4(m+\frac{1}{2})})+(2p+1)} 
\]

\[
\times q^{-\frac{(2mr+s)^2}{4m}} e^{-\pi i (2mr+s)(2p+1)\tau-1} [\theta_{s,m} + \theta_{-s,m}](\tau, z) 
\]

(4.2)

The LHS of this equation (4.2) becomes by Note 2.1 as follows:

LHS of 4.2 = \[ e^{-\pi i m} q^{-\frac{m^2}{2(2m+1)}(2p+1)} \theta^{(-)}_{2mp+m+\frac{1}{2}, m+\frac{1}{2}}(\tau, 0) \cdot \Phi_{[m,0]}(\tau, z_1, z_2, 0) \]
Also (I) is computed by using Notes 2.2 and 2.3 as follows:

\[
(I) = - \frac{\theta^{(-)}_{0, \frac{m}{2} + 1}(\tau, z + \frac{(2p + 1)\tau - 1}{2m + 1})}{\psi_{11}(\tau, \frac{z}{2} + \frac{(2p + 1)\tau - 1}{2})} + \frac{\theta^{(-)}_{\frac{m}{2} + 1}(\tau, z - \frac{(2p + 1)\tau - 1}{2m + 1})}{\psi_{11}(\tau, \frac{z}{2} - \frac{(2p + 1)\tau - 1}{2})}
\]

\[
= - e^{\frac{\pi i}{2m + 1}} q \frac{-1}{4m + 1} (2p + 1)^2 e^{-\frac{\pi i}{2}(2p + 1)z} \theta_{p, m + \frac{1}{2}}(\tau, 0) - \frac{1}{4m + 1} (2p + 1)^2 \cdot \frac{1}{\theta_{0, \frac{1}{2}}(\tau, z)} \cdot \left[ \theta_{p, m + \frac{1}{2}} + \theta_{-p, m + \frac{1}{2}} \right](\tau, 0)
\]

Then substituting these into (4.2) and multiplying \( e^{\frac{\pi im}{2m+1}} \) and rewriting the 2nd terms in the RHS of (4.2) by using

\[
\sum_{j, r \in \mathbb{Z}} - \sum_{0 < r \leq j, j \leq r < 0} = \left[ \sum_{j, r \in \mathbb{Z}} - \sum_{j, r \in \mathbb{Z}} \right] - \sum_{r = 0} - \sum_{j \in \mathbb{Z}}
\]

the above formula (4.2) becomes as follows:

\[
q^{-\frac{m^2}{2(2m + 1)}(2p + 1)^2} \theta^{(-)}_{2mp + m + \frac{1}{2}, m + \frac{1}{2}}(\tau, 0) \Phi_{[m, 0]}(\tau, z1, z2, 0)
\]

\[
= q^{m(\frac{1}{2}m + 1)} (2p + 1)^2 \frac{\eta(\tau)^3}{\theta_{0, \frac{1}{2}}(\tau, z)} \cdot \left[ \theta_{p, m + \frac{1}{2}} + \theta_{-p, m + \frac{1}{2}} \right](\tau, 0)
\]

\[
- \left[ \sum_{j, r \in \mathbb{Z}} - \sum_{j, r \in \mathbb{Z}} \right] \sum_{s \in \mathbb{Z}} (-1)^{j + s} q^{(m + \frac{1}{2})(j + \frac{1}{2} + \frac{2mp}{2m + 1})^2} \left[ \theta_{s, m} + \theta_{-s, m} \right](\tau, z)
\]

\[
\times q^{-\frac{1}{4m} \left[ 2m(r + \frac{1}{2}) - s + 2mp \right]^2 + \frac{\pi}{4} (2p + 1)^2} \left[ \theta_{s, m} + \theta_{-s, m} \right](\tau, z)
\]

\[
+ \sum_{j \in \mathbb{Z}} (-1)^j q^{(m + \frac{1}{2})(j + \frac{1}{2} + \frac{2mp}{2m + 1})^2} q^{-\frac{m^2}{2(2m + 1)}(2p + 1)^2} \theta^{(-)}_{2mp + m + \frac{1}{2}, m + \frac{1}{2}}(\tau, 0)
\]

\[
\times \sum_{s \in \mathbb{Z}} (\tau)^s q^{-\frac{1}{4m} s^2 + \frac{\pi}{2}(2p + 1)} \left[ \theta_{s, m} + \theta_{-s, m} \right](\tau, z)
\]

\[
0 \leq s \leq m
\]
Lemma 4.2. For \( q = \frac{m^2}{4m} [2(m+1)+s+2mp] + \frac{m^2}{4} (2p+1)^2 \) and replacing \( \sum_{j,r} \) with \( \sum_{j,r}^{\text{odd}} \) by putting \( j + \frac{1}{2} = j' \) and \( r + \frac{1}{2} = r' \) in the above formula (4.4), we obtain

\[
\theta^{(-)}_{2mp+m+\frac{1}{2},m+\frac{1}{2}}(\tau, 0) \cdot \Phi^{m,0}[\tau, z_1, z_2, 0)]
\]

proving Lemma [4.1]

4.2 \( \Phi^{m,0}[\tau, z_1, z_2, t] \sim \text{the case } z_1 - z_2 = (1 + 2p) \tau \)

Lemma 4.2. For \( m \in \mathbb{N} \) and \( p \in \mathbb{Z} \), the following formula holds:

\[
\theta^{(-)}_{2mp+m+\frac{1}{2},m+\frac{1}{2}}(\tau, 0) \cdot \Phi^{m,0}[\tau, \tau, \frac{z}{2} + \frac{\tau}{2} + pt, \frac{z}{2} - \frac{\tau}{2} + pt, 0)
\]

\[
= (-1)^p \cdot q^m (2p+1)^2 \cdot \eta(\tau)^3 \cdot [\theta^{(-)}_{p,m+\frac{1}{2}} + \theta^{(-)}_{-p,m+\frac{1}{2}}](\tau, 0)
\]

\[
- \left[ \sum_{j,r \in \mathbb{Z}^{\text{odd}}} - \sum_{j,r \in \mathbb{Z}^{\text{odd}}} \right] \sum_{k \in \mathbb{Z}} \sum_{0 \leq r < j', j' < r < 0} (-1)^{j'-\frac{1}{2}+k} q^{(m+\frac{1}{2})(j'+\frac{2pm}{2m+1})^2}
\]

\[
\times q^{-\frac{1}{4m} [2(m+1)+s+2mp] + \frac{m^2}{4} (2p+1)^2} \sum_{0 \leq k < m} \sum_{0 \leq r' < j', j' < r < 0} (-1)^{k-\frac{1}{2}+k} q^{k+\frac{2pm}{2m+1}} \cdot \Phi^{m,0}[\tau, \tau, \frac{z}{2} + \frac{\tau}{2} + pt, \frac{z}{2} - \frac{\tau}{2} + pt, 0)
\]
Letting Proof. in Proposition 3.2, we have

\[ \sum_{k \in \mathbb{Z}} (-1)^j \frac{1}{2} \left( \frac{m}{2} + \frac{1}{2} \right)^2 q^{(m+1)(j+\frac{1}{2})^2} \]

\[ \times q^{-\frac{1}{4m} (2mr-k+2mp)^2 + \frac{1}{2p+1} (2p+1)^2} \left[ \theta_{k,m} + \theta_{-k,m} \right] (\tau, z) \]

\[ \theta_{2mp+m+\frac{1}{2},m+\frac{1}{2}}^{-} (\tau, 0) \sum_{k \in \mathbb{Z}} q^{-\frac{1}{4m} k^2 + \frac{k}{2} (2p+1)} \left[ \theta_{k,m} + \theta_{-k,m} \right] (\tau, z) \quad (4.5) \]

Proof. Letting \( \begin{cases} z_1 = \frac{x}{2} + \frac{y}{2} + p \tau \\ z_2 = \frac{x}{2} - \frac{y}{2} - p \tau \end{cases} \) namely \( \begin{cases} z_1 + z_2 = z \\ z_1 - z_2 = (2p+1) \tau \end{cases} \) in the formula (3.7) in Proposition 3.2, we have

\[ \theta_{\frac{1}{2},m+\frac{1}{2}}^{-} \left( \tau, \frac{m(2p+1)\tau}{m+\frac{1}{2}} \right) \Phi_{[m,0]} (\tau, z_1, z_2, 0) \]

\[ = i \eta(\tau)^3 \left\{ \theta_{\frac{1}{2},m+\frac{1}{2}}^{-} \left( \tau, z + \frac{(2p+1)\tau}{2m+1} \right) \right. \]

\[ \left. \theta_{\frac{1}{2},m+\frac{1}{2}}^{-} \left( \tau, z - \frac{(2p+1)\tau}{2m+1} \right) \right\} \quad (1) \]

\[ - \left[ \sum_{j, r \in \mathbb{Z}} - \sum_{j, r \in \mathbb{Z}} \right] \sum_{s \in \mathbb{Z}} (-1)^j \frac{1}{2} \left( \frac{m}{2} + \frac{1}{2} \right)^2 q^{(m+1)(j+\frac{1}{2})^2} \]

\[ \times e^{2\pi i m(j+\frac{1}{4(m+\frac{1}{2})}) (2p+1) \tau} e^{-\pi i (2ms-s)(2p+1) \tau} \left[ \theta_{s,m} + \theta_{-s,m} \right] (\tau, z) \]

\[ - \left[ \sum_{j, r \in \mathbb{Z}} - \sum_{j, r \in \mathbb{Z}} \right] \sum_{s \in \mathbb{Z}} (-1)^j \frac{1}{2} \left( \frac{m}{2} + \frac{1}{2} \right)^2 q^{(m+1)(j+\frac{1}{2})^2} \]

\[ \times e^{2\pi i m(j+\frac{1}{4(m+\frac{1}{2})}) (2p+1) \tau} e^{-\pi i (2ms-s)(2p+1) \tau} \left[ \theta_{s,m} + \theta_{-s,m} \right] (\tau, z) \]

\[ = i \eta(\tau)^3 \times (1) \]

\[ - \left[ \sum_{j, r \in \mathbb{Z}} - \sum_{j, r \in \mathbb{Z}} \right] \sum_{s \in \mathbb{Z}} (-1)^j \frac{1}{2} \left( \frac{m}{2} + \frac{1}{2} \right)^2 q^{(m+1)(j+\frac{1}{2})^2} + m(j+\frac{1}{4(m+\frac{1}{2})}) (2p+1) \]

\[ \times q^{-\frac{1}{4m} (2mr-s)^2 - \frac{1}{2} (2mr-s)(2p+1)} \left[ \theta_{s,m} + \theta_{-s,m} \right] (\tau, z) \]

\[ - \left[ \sum_{j, r \in \mathbb{Z}} - \sum_{j, r \in \mathbb{Z}} \right] \sum_{s \in \mathbb{Z}} (-1)^j \frac{1}{2} \left( \frac{m}{2} + \frac{1}{2} \right)^2 q^{(m+1)(j+\frac{1}{2})^2} + m(j+\frac{1}{4(m+\frac{1}{2})}) (2p+1) \]

\[ \times q^{-\frac{1}{4m} (2mr-s)^2 - \frac{1}{2} (2mr-s)(2p+1)} \left[ \theta_{s,m} + \theta_{-s,m} \right] (\tau, z) \]
\[
\times q^{\frac{(2mr+s)^2}{4m}} - \frac{1}{2}(2mr+s)(2p+1) \left[ \theta_{s,m} + \theta_{-s,m} \right](\tau, z)
\]

The LHS of this equation (4.6) becomes by Note 2.1 as follows:

\[
\text{LHS of (4.6)} = q^{-\frac{m^2}{2(2m+1)}(2p+1)^2} \theta_{2mp+m+\frac{1}{2}, m+\frac{1}{2}}(\tau, 0) \Phi[[m,0]^\ast(\tau, z_1, z_2, 0)
\]

Also (I) is computed by using Notes 2.2 and 2.3 as follows:

\[
(I) = - \theta_{\frac{\tau}{2}, m+\frac{1}{2}}(\tau, z + \frac{2p+1}{2m+1} \tau) + \theta_{\frac{\tau}{2}, m+\frac{1}{2}}(\tau, z - \frac{2p+1}{2m+1} \tau)
\]

\[
\times q^{-\frac{1}{16(m+\frac{1}{2})^2}(2p+1)^2} e^{\frac{-im(2p+1)z}{2(2m+1)^2}} \vartheta_{0, \frac{1}{2}}(\tau, 0) \Phi[(\tau, z)
\]

Then substituting these into (4.6) and rewriting the 2nd term in the RHS of (4.6) by using (4.3), the above formula (4.6) becomes as follows:

\[
q^{-\frac{m^2}{2(2m+1)}(2p+1)^2} \theta_{2mp+m+\frac{1}{2}, m+\frac{1}{2}}(\tau, 0) \Phi[[m,0]^\ast(\tau, z_1, z_2, 0)
\]

\[
= (-1)^p q^{\frac{m}{8(m+\frac{1}{2})^2}(2p+1)^2} \eta(\tau)^3 \frac{\vartheta_{0, \frac{1}{2}}(\tau, z)}{\vartheta_{0, \frac{1}{2}}(\tau, z)} \left[ \theta_{p, m+\frac{1}{2}} + \theta_{-p, m+\frac{1}{2}} \right](\tau, 0)
\]

\[
- \left[ \sum_{j, r \in \mathbb{Z}} - \sum_{j, r \in \mathbb{Z}} \right] \sum_{s \in \mathbb{Z}} (-1)^j \left[ q^{(m+\frac{1}{2})(j+\frac{1}{2} + \frac{2m+1}{2m+1})^2} - q^{-\frac{m^2}{4(m+\frac{1}{2})^2}(2p+1)^2}ight)
\]

\[
\times q^{-\frac{1}{4m}(2m(r+\frac{1}{2})-s+2m)^2 + \frac{m}{4}(2p+1)^2} \left[ \theta_{s, m} + \theta_{-s, m} \right](\tau, z)
\]

\[
+ \sum_{j \in \mathbb{Z}} (-1)^j q^{(m+\frac{1}{2})(j+\frac{1}{2} + \frac{2m}{2m+1})^2} - q^{-\frac{m^2}{4(m+\frac{1}{2})^2}(2p+1)^2}
\]

\[
\times \theta_{2mp+m+\frac{1}{2}, m+\frac{1}{2}}(\tau, 0)
\]

\[
\times \sum_{s \in \mathbb{Z}} q^{-\frac{1}{4m} z^2 + \frac{1}{2}(2p+1) \left[ \theta_{s, m} + \theta_{-s, m} \right](\tau, z)
\]

\[
- \left[ \sum_{j, r \in \mathbb{Z}} - \sum_{j, r \in \mathbb{Z}} \right] \sum_{s \in \mathbb{Z}} (-1)^j q^{(m+\frac{1}{2})(j+\frac{1}{2} + \frac{2m}{2m+1})^2} - q^{-\frac{m^2}{2(2m+1)}(2p+1)^2}
\]
Multiplying \( q^{m^2/(2m+1)(2p+1)^2} \) and replacing \( \sum_{j,r \in \mathbb{Z}} \) with \( \sum_{j',r' \in \mathbb{Z}_{\text{odd}}} \) by putting \( j + \frac{1}{2} = j' \) and \( r + \frac{1}{2} = r' \) in the above formula (4.7), we obtain

\[
\theta_{-2mp+m+\frac{1}{2},m+\frac{1}{2}}^{-}(\tau, 0) \Phi_{m,0}^{[m,0]}(\tau, z_1, z_2, 0)
\]

\[
= (-1)^p q^{m/(2p+1)^2} \frac{\eta(\tau)^3}{\eta_{0,\frac{1}{2}}^{-}(\tau, z)} \cdot \left[ \theta_{-p,m+\frac{1}{2}}^{-} + \theta_{-p,m+\frac{1}{2}}^{-}\right](\tau, 0)
\]

\[
- \left[ \sum_{j',r' \in \mathbb{Z}_{\text{odd}}} - \sum_{j',r' \in \mathbb{Z}_{\text{odd}}} \right] \sum_{0 \leq k < m} (-1)^{j'-\frac{1}{2}} q^{(m+\frac{1}{2})(j'+\frac{2pm}{2m+1})^2} q^{-\frac{1}{4m}[2mr'+k+2mp]^2} \frac{(2p+1)^2}{2} \left[ \theta_{k,m} + \theta_{-k,m}\right](\tau, z)
\]

\[
+ \sum_{j \in \mathbb{Z}} (-1)^j q^{(m+\frac{1}{2})(j'+\frac{2pm}{2m+1})^2} q^{-\frac{1}{4m}[2mr'+k+2mp]^2} \frac{(2p+1)^2}{2} \left[ \theta_{k,m} + \theta_{-k,m}\right](\tau, z)
\]

proving Lemma [4.2].

---

5 \( \Phi_{\text{add}}^{[m,0]}(\tau, z_1, z_2, t) \sim \) the case \( z_1 - z_2 = 2a \tau + 2b \)

**Lemma 5.1.** Let \( m \in \frac{1}{2} \mathbb{N}, \ j, a \in \frac{1}{2} \mathbb{Z} \) and \( b \in \mathbb{Q} \) such that \( 4mb \in \mathbb{Z} \). Then the functions \( P_{j,m}(\tau, z) := P_{j,m}^{(+)}(\tau, z) \) and \( Q_{j,m}(\tau, z) := Q_{j,m}^{(+)}(\tau, z) \) defined by the formulas (4.1a) and (4.1b) in [77] satisfy the following:

1) \( P_{j,m}(\tau, a \tau + b) + e^{4\pi ij} e^{8\pi imab} P_{-j,m}(\tau, a \tau + b) = 0 \)

2) \( Q_{j,m}(\tau, a \tau + b) + e^{4\pi ij} e^{8\pi imab} Q_{-j,m}(\tau, a \tau + b) \)

\[
= e^{2\pi ij} e^{8\pi imab} \sum_{k \in \mathbb{Z}} e^{4\pi imbk} \left[ q^{-\frac{1}{4m}(j+2m(2a-k))(j-2mk)} + q^{\frac{1}{4m}(j-2m(2a-k))(j+2mk)} \right]
\]
Proof. The claim 1) follows from (4.4) in [17] and the claim 2) follows from (4.6b) in [17], since 
\[ e^{-8\pi \text{im}ab} = e^{8\pi \text{im}ab}. \]

Lemma 5.2. Let \( m \in \mathbb{N}, a, b \in \frac{1}{2}\mathbb{Z} \) and \( j \in \mathbb{Z} \). Then

1) \( R_{j,m}(\tau, a\tau + b) + R_{2m-j,m}(\tau, a\tau + b) = 2e^{2\pi ij b} \sum_{k \in \mathbb{Z}} q^{-\frac{1}{m}(j+2m(2a-k))(j-2mk)} \)

2) \( R_{0,m}(\tau, a\tau + b) = \sum_{k \in \mathbb{Z}} q^{mk(2a-k)} \)

Proof. The claim 1) is obtained immediately from Lemma 4.1 in [17]. To prove the claim 2) we note, by letting \( j = 0 \) in Lemma 5.1, that

\[ P_{0,m}(\tau, a\tau + b) = 0 \]

and that

\[ Q_{0,m}(\tau, a\tau + b) = \frac{1}{2} \sum_{k \in \mathbb{Z}} \left\{ q^{-\frac{1}{m}2m(2a-k)(-2mk)} + q^{-\frac{1}{m}2m(2a-k)2mk} \right\} \]

\[ = \sum_{k \in \mathbb{Z}} q^{mk(2a-k)} \]

Thus we have

\[ R_{0,m}(\tau, a\tau + b) = P_{0,m}(\tau, a\tau + b) + Q_{0,m}(\tau, a\tau + b) = \sum_{k \in \mathbb{Z}} q^{mk(2a-k)} \]

proving Lemma 5.2.

Using the above Lemmas 5.1 and 5.2, the Zwegers’ additional function

\[ \Phi_{\text{add}}^{[m,0]}(\tau, z_1, z_2, 0) := \Phi_{1,\text{add}}^{[m,0]}(\tau, z_1, z_2, 0) + \Phi_{2,\text{add}}^{[m,0]}(\tau, z_1, z_2, 0) \]

is obtained as follows:

**Proposition 5.1.** Let \( m \in \mathbb{N} \) and \( a, b \in \frac{1}{2}\mathbb{Z} \). Then, for \( z_1 \) and \( z_2 \) satisfying \( z_1 - z_2 = 2a\tau + 2b \), the correction function \( \Phi_{\text{add}}^{[m,0]}(\tau, z_1, z_2, 0) \) is given by the following formula:

\[ \Phi_{\text{add}}^{[m,0]}(\tau, z_1, z_2, 0) = - \sum_{k \in \mathbb{Z}} q^{mk(2a-k)} \theta_{0,m}(\tau, z_1 + z_2) \]

\[ - \frac{1}{2} \sum_{j \in \mathbb{Z}} e^{2\pi ij b} \sum_{k \in \mathbb{Z}} q^{-\frac{1}{m}(j+2m(2a-k))(j-2mk)} \left[ \theta_{j,m} + \theta_{-j,m} \right](\tau, z_1 + z_2) \] (5.1)
Proof. By the formula for $\Phi^{[m,0]}_{\text{add}}$ in [17], we have

$$\Phi^{[m,0]}_{\text{add}}(\tau, z_1, z_2, 0) = -\frac{1}{2} \sum_{j \in \mathbb{Z}} R_{j,m}(\tau, a\tau + b) \left[ \theta_{j,m} + \theta_{-j,m} \right](\tau, z_1 + z_2)$$

$$\sum_{j=0}^{0 \leq j \leq 2m} + \sum_{j \in \mathbb{Z}}^{0 < j < 2m}$$

$$= -\frac{1}{2} \sum_{j \in \mathbb{Z}}^{0 < j < 2m} R_{0,m}(\tau, a\tau + b) \times 2 \theta_{0,m}(\tau, z_1 + z_2)$$

$$\sum_{k \in \mathbb{Z}}^{0 \leq k \leq 2a} q^{mk(2a-k)} \quad \text{by Lemma 5.2}$$

$$-\frac{1}{4} \sum_{j \in \mathbb{Z}}^{0 < j < 2m} R_{j,m}(\tau, a\tau + b) \left[ \theta_{j,m} + \theta_{-j,m} \right](\tau, z_1 + z_2)$$

$$-\frac{1}{4} \sum_{j \in \mathbb{Z}}^{0 < j < 2m} R_{2m-j,m}(\tau, a\tau + b) \left[ \theta_{2m-j,m} + \theta_{-(2m-j),m} \right](\tau, z_1 + z_2)$$

$$= -\sum_{k \in \mathbb{Z}}^{0 \leq k \leq 2a} q^{mk(2a-k)} \theta_{0,m}(\tau, z_1 + z_2)$$

$$-\frac{1}{4} \sum_{j \in \mathbb{Z}}^{0 < j < 2m} \left\{ R_{2m-j,m}(\tau, a\tau + b) + R_{j,m}(\tau, a\tau + b) \right\} \left[ \theta_{j,m} + \theta_{-j,m} \right](\tau, z_1 + z_2)$$

Then, using Lemma 5.2, this is rewritten as follows:

$$= -\sum_{k \in \mathbb{Z}}^{0 \leq k \leq 2a} q^{mk(2a-k)} \theta_{0,m}(\tau, z_1 + z_2)$$

$$-\frac{1}{4} \sum_{j \in \mathbb{Z}}^{0 < j < 2m} 2e^{2\pi ijb} \sum_{k \in \mathbb{Z}}^{1 \leq k \leq 2a} q^{-\frac{1}{4m}(j+2m(2a-k))(j-2mk)} \left[ \theta_{j,m} + \theta_{-j,m} \right](\tau, z_1 + z_2)$$

$$= -\sum_{k \in \mathbb{Z}}^{0 \leq k \leq 2a} q^{mk(2a-k)} \theta_{0,m}(\tau, z_1 + z_2)$$
For Proposition 5.1, proving Proposition 5.1.

Using the above Proposition 5.1, the Zwegers’s additional functions $\Phi_{\text{add}}^{[m,0]}(\tau, z_1, z_2, 0)$ for $(z_1, z_2) = \left(\frac{\tau}{2} + \frac{\tau}{2} - \frac{1}{2}, \frac{\tau}{2} - \frac{1}{2} + \frac{1}{2}\right)$ and $(z_1, z_2) = \left(\frac{\tau}{2} + \frac{\tau}{2}, \frac{\tau}{2} - \frac{\tau}{2}\right)$ are obtained as follows:

Lemma 5.3. For $m \in \mathbb{N}$, the following formulas hold:

1) $\Phi_{\text{add}}^{[m,0]}(\tau, z_1, z_2, 0) = -\sum_{j \in \mathbb{Z}} (-1)^j q^{-\frac{1}{12}j(j-2m)} \theta_{j,m}(\tau, z_1 + z_2)$

2) $\Phi_{\text{add}}^{[m,0]}(\tau, z, z) = -\sum_{j \in \mathbb{Z}} (-1)^j q^{-\frac{1}{12}j(j-2m)} \theta_{j,m}(\tau, z)$

Proof. 1) In the case $\left\{ z_1 = \frac{\tau}{2} + \frac{\tau}{2} - \frac{1}{2}, z_2 = \frac{\tau}{2} - \frac{1}{2} + \frac{1}{2}\right\}$, letting $\left\{ \frac{2a}{2b} = -1 \right\}$ in (5.1), we have

$$\Phi_{\text{add}}^{[m,0]}(\tau, z_1, z_2, 0) = -\sum_{k \in \mathbb{Z}} q^{mk(1-k)} \theta_{0,m}(\tau, z_1 + z_2)$$

$$-\sum_{j \in \mathbb{Z}} e^{2\pi ij} \sum_{k \in \mathbb{Z}} q^{-\frac{1}{12}(j+2m(1-k))(j-2m)} \theta_{j,m}(\tau, z_1 + z_2)$$

$$= -2 \theta_{0,m}(\tau, z_1 + z_2) - \sum_{j \in \mathbb{Z}} (-1)^j q^{-\frac{1}{12}j(j-2m)} \theta_{j,m}(\tau, z_1 + z_2)$$

$$= -2 \theta_{0,m}(\tau, z) - \sum_{j \in \mathbb{Z}} (-1)^j q^{-\frac{1}{12}j(j-2m)} \theta_{j,m}(\tau, z) - \sum_{j \in \mathbb{Z}} (-1)^j q^{-\frac{1}{12}j(j-2m)} \theta_{j,m}(\tau, z)$$

proving 1).
2) In the case \( z_1 = \frac{\tau}{2} + \frac{\tau}{2} \), letting \( \begin{cases} 2a = 1 \\ 2b = 0 \end{cases} \) in (5.1), we have

\[
\Phi^{[m,0]}_{\text{add}}(\tau, z_1, z_2, 0) = - \sum_{k \in \mathbb{Z}} q^{mk(1-k)} \theta_{0,m}(\tau, z_1 + z_2)
\]

\[
- \frac{1}{2} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} q^{-\frac{1}{4m}(j+2m(1-k))(j-2mk)} \left[ \theta_{j,m} + \theta_{-j,m} \right](\tau, z)
\]

\[
= -2 \theta_{0,m}(\tau, z_1 + z_2) - \frac{1}{2} \sum_{j \in \mathbb{Z}} q^{-\frac{1}{4m}(j-2m)} \left[ \theta_{j,m} + \theta_{-j,m} \right](\tau, z)
\]

\[
= -2 \theta_{0,m}(\tau, z) - \frac{1}{2} \sum_{j \in \mathbb{Z}} q^{-\frac{1}{4m}(j-2m)} \theta_{j,m}(\tau, z) - \frac{1}{2} \sum_{j \in \mathbb{Z}} q^{-\frac{1}{4m}(j-2m)} \theta_{-j,m}(\tau, z)
\]

proving 2).

\[
\Phi^{[m,0]}(\tau, z_1 + p\tau, z_2 - p\tau, 0)
\]

6.1 \( \Phi^{[m,0]}(\tau, z_1 + p\tau, z_2 - p\tau, t) \sim \text{the case } z_1 - z_2 = \tau - 1 \)

Lemma 6.1. For \( m \in \mathbb{N} \) and \( p \in \mathbb{Z} \), the following formula holds:

\[
\Phi^{[m,0]}(\tau, z_2 - \frac{\tau}{2} + \frac{\tau}{2} - \frac{1}{2} + p\tau, z_2 - \frac{\tau}{2} + \frac{1}{2} - p\tau, 0)
\]

\[
= q^{mp(p+1)} \left\{ \Phi^{[m,0]}(\tau, z_2 - \frac{\tau}{2} + \frac{1}{2} - p\tau, z_2 - \frac{\tau}{2} + \frac{1}{2} - \frac{1}{2} - p\tau, 0) \right\}
\]

\[
+ \sum_{r \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} (-1)^k q^{-\frac{1}{4m}(2mr+k)(2m(r-1)+k)} \left[ \theta_{k,m} + \theta_{-k,m} \right](\tau, z)
\]
\[ -2 \sum_{r \in \mathbb{Z}} q^{-mr(r+1)} \theta_{0,m}(\tau, z) + (-1)^m \sum_{r \in \mathbb{Z}} q^{-\frac{4}{2m}(2r+1)(2r-1)} \theta_{m,m}(\tau, z) \]  

\text{(6.1)}

\text{Proof.} \text{ Letting } \begin{cases} z_1 = \frac{z}{2} + \frac{\tau}{2} - \frac{1}{2} \\ z_2 = \frac{z}{2} - \frac{\tau}{2} + \frac{1}{2} \end{cases} \text{ namely } \begin{cases} z_1 - z_2 = \tau - 1 \\ z_1 + z_2 = z \end{cases} \text{ in the formula (2.2) in Lemma 2.2, we have } \n
\begin{align*}
\Phi^{[m,0]}(\tau, z_1 + p\tau, z_2 - p\tau, 0) &= e^{2\pi mp(\tau - 1)} q^{mp^2} \left\{ \Phi^{[m,0]}(\tau, z_1, z_2, 0) + \sum_{k \in \mathbb{Z}} e^{-\pi ik(\tau - 1)} q^{-\frac{1}{4m}k^2} [\theta_{k,m} + \theta_{-k,m}](\tau, z) \right\} \\
&= q^{mp(p+1)} \left\{ \Phi^{[m,0]}(\tau, z_1, z_2, 0) + \sum_{k \in \mathbb{Z}} (-1)^k q^{-\frac{1}{4m}(k^2 + 2mk)} [\theta_{k,m} + \theta_{-k,m}](\tau, z) \right\} 
\end{align*}

\text{(6.2)}

\text{We compute (I) by putting } k = 2mr + k' (0 \leq r < p, \ 1 \leq k' \leq 2m) \text{ as follows:}

\begin{align*}
(I) &= \sum_{r \in \mathbb{Z}} \sum_{k' \in \mathbb{Z}} (-1)^{2mr+k'} q^{-\frac{1}{4m}(2mr+k')(2mr+k'+2m)} [\theta_{2mr+k',m} + \theta_{-(2mr+k'),m}](\tau, z) \\
&= \sum_{r \in \mathbb{Z}} \sum_{k' \in \mathbb{Z}} (-1)^k q^{-\frac{1}{4m}(2mr+k')(2m(r+1)+k')} \theta_{k',m}(\tau, z) \hfill \text{(A)} \\
&\quad + \sum_{r \in \mathbb{Z}} \sum_{k' \in \mathbb{Z}} (-1)^{k'} q^{-\frac{1}{4m}(2mr+k')(2m(r+1)+k')} \theta_{-k',m}(\tau, z) \hfill \text{(B)}
\end{align*}

\text{where (A) is computed by using}

\[ \sum_{k \in \mathbb{Z}} = \sum_{1 \leq k \leq 2m} - \sum_{0 \leq k < 2m} + \sum_{k = 2m} \]  

\text{as follows:}

\begin{align*}
(A) &= \sum_{r \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} (-1)^k q^{-\frac{1}{4m}(2mr+k)(2m(r+1)+k)} \theta_{k,m}(\tau, z) \\
&\quad + \sum_{r \in \mathbb{Z}} \sum_{k' \in \mathbb{Z}} (-1)^{-k'} q^{-\frac{1}{4m}(2mr+k')(2m(r+1)+k')} \theta_{-k',m}(\tau, z)
\end{align*}

\text{(6.3)}
And (B) becomes by putting \( k' = 2m - k \) and \( r + 2 = -r' \) as follows:

\[
(B) = \sum_{r' \in \mathbb{Z}} \sum_{k' \in \mathbb{Z}} (-1)^{k'} q^{- \frac{1}{4m}(2mr + k')(2m(r+1) + k')} \theta_{k',m}(\tau, z)
\]

\[
- \sum_{k' \in \mathbb{Z}} (-1)^{k'} q^{- \frac{1}{4m}(-2m+k')k'} \theta_{k',m}(\tau, z)
\]

Then (I) becomes as follows:

\[
(I) = (A) + (B)
\]

\[
= \sum_{k \in \mathbb{Z}} (-1)^k \sum_{r \in \mathbb{Z}} q^{- \frac{1}{4m}(2mr + k)(2m(r+1) + k)} \theta_{k,m}(\tau, z)
\]

\[
- \sum_{k \in \mathbb{Z}} q^{- \frac{1}{4m}(-2m+k)k} \theta_{k,m}(\tau, z) - \theta_{0,m}(\tau, z) + q^{-mp(p+1)} \theta_{0,m}(\tau, z)
\]

\[
\text{(6.4)}
\]

\[
- \sum_{k \in \mathbb{Z}} (-1)^k q^{- \frac{1}{4m}(-2m+k)k} \theta_{k,m}(\tau, z)
\]

\[
0 \leq k < 2m
\]
Then substituting (6.4) into (6.2), we have

\[
\Phi[m,0]^\ast(\tau, z_1 + p\tau, z_2 - p\tau, 0) = q^{mp(p+1)} \left\{ \Phi[m,0]^\ast(\tau, z_1, z_2, 0) + (I) \right\}
\]

\[
= q^{mp(p+1)} \left\{ \Phi[m,0]^\ast(\tau, z_1, z_2, 0)
\right.
\]
\[
+ \sum_{r \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} (-1)^k q^{-\frac{1}{4m}(2mr+k)(2m(r-1)+k)} \theta_{k,m}(\tau, z)
\]
\[
\left. - \sum_{k \in \mathbb{Z}} (-1)^k q^{-\frac{1}{4m}(k-k+2m)} \theta_{k,m}(\tau, z) + q^{-mp(p+1)} \theta_{0,m}(\tau, z) \right\}
\]

\[
\Phi[m,0]^\ast_{\text{add}}(\tau, z_1, z_2, 0) \quad \text{by Lemma 5.3}
\]

\[
= q^{mp(p+1)} \left\{ \tilde{\Phi}[m,0]^\ast(\tau, z_1, z_2, 0) + \sum_{r \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} (-1)^k q^{-\frac{1}{4m}(2mr+k)(2m(r-1)+k)} \theta_{k,m}(\tau, z)
\right.
\]
\[
\left. + q^{-mp(p+1)} \theta_{0,m}(\tau, z) \right\}
\]

(6.5)

We go further to compute

\[
(II) = (II)_A + (II)_B
\]

where

\[
(II)_A := \sum_{r \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} (-1)^k q^{-\frac{1}{4m}(2mr+k)(2m(r-1)+k)} \theta_{k,m}(\tau, z)
\]

\[
- p \leq r \leq p \quad 0 \leq k \leq m
\]

\[
(II)_B := \sum_{r \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} (-1)^k q^{-\frac{1}{4m}(2mr+k)(2m(r-1)+k)} \theta_{k,m}(\tau, z)
\]

\[
- p \leq r \leq p \quad m < k < 2m
\]

First we compute (II)_A :

\[
(II)_A = \sum_{r \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} (-1)^k q^{-\frac{1}{4m}(2mr+k)(2m(r-1)+k)} \theta_{k,m}(\tau, z)
\]

\[
- p \leq r \leq p \quad 0 \leq k \leq m
\]

\[
\sum_{k \in \mathbb{Z}} + \sum_{k=m}^{m+1}
\]

\[
0 \leq k < m
\]

\[
= \sum_{r \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} (-1)^k q^{-\frac{1}{4m}(2mr+k)(2m(r-1)+k)} \theta_{k,m}(\tau, z)
\]

\[
- p \leq r \leq p \quad 0 \leq k < m
\]
\[ + (-1)^m \sum_{r \in \mathbb{Z}} q^{-\frac{3}{4m}(2r+1)(2r-1)} \theta_{m,m}(\tau, z) \tag{6.6a} \]

Next, \((II)_B\) is rewritten as follows by putting \(k = 2m - k'\):

\[
(II)_B = \sum_{r \in \mathbb{Z}} \sum_{k' \in \mathbb{Z}} (-1)^{k'} q^{-\frac{1}{4m}(2m(r+1)-k')(2mr-k')} \theta_{-k',m}(\tau, z) \]

\[
\stackrel{r=-r'}{=} \sum_{r' \in \mathbb{Z}} \sum_{k' \in \mathbb{Z}} (-1)^{k'} q^{-\frac{1}{4m}(2m(r'-1)+k')(2mr'+k')} \theta_{-k',m}(\tau, z) \]

\[
- \sum_{r' \in \mathbb{Z}} \sum_{k' \in \mathbb{Z}} q^{-m(r'-1)} \theta_{0,m}(\tau, z) \]

\[
-2 \sum_{r \in \mathbb{Z}} q^{-mr(r+1)} \theta_{0,m}(\tau, z) - q^{-mp(p+1)} \theta_{0,m}(\tau, z) \tag{6.6b} \]

Then by (6.6a) and (6.6b), we have

\[
(II) + q^{-mp(p+1)} \theta_{0,m}(\tau, z) \]

\[
= \sum_{r \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} (-1)^k q^{-\frac{1}{4m}(2mr+k)(2m(r-1)+k)} \left[ \theta_{k,m} + \theta_{-k,m} \right](\tau, z) \]

\[- 2 \sum_{r \in \mathbb{Z}} q^{-mr(r+1)} \theta_{0,m}(\tau, z) + (-1)^m \sum_{r \in \mathbb{Z}} q^{-\frac{3}{4m}(2r+1)(2r-1)} \theta_{m,m}(\tau, z) \tag{6.7} \]

Substituting this equation (6.7) into (6.2), we obtain

\[
\Phi^{[m,0]}(\tau, z_1 + p\tau, z_2 - p\tau, 0) = q^{mp(p+1)} \Phi^{[m,0]}(\tau, z_1, z_2, 0) \]

\[+ \sum_{r \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} (-1)^k q^{-\frac{1}{4m}(2mr+k)(2m(r-1)+k)} \left[ \theta_{k,m} + \theta_{-k,m} \right](\tau, z) \]
Lemma 6.2. For \( m \leq p \), we have

\[
-2 \sum_{r \in \mathbb{Z}, 0 \leq r \leq p-1} q^{-mr(r+1)} \theta_{0,m}(\tau, z) + (-1)^m \sum_{r \in \mathbb{Z}} q^{-\frac{\pi}{4m}(2r+1)(2r-1)} \theta_{m,m}(\tau, z) \]

proving Lemma 6.1. \( \square \)

6.2 \( \Phi^{[m,0]} e^{(\tau, z_1 + p \tau, z_2 - p \tau, t)} \sim \) the case \( z_1 - z_2 = \tau \)

Lemma 6.2. For \( m \in \mathbb{N} \) and \( p \in \mathbb{Z} \), the following formula holds:

\[
\Phi^{[m,0]} e^{(\tau, z_1 + \frac{z}{2} + p \tau, z_2 - \frac{z}{2} - p \tau, 0)} = q^{mp(p+1)} \left[ \Phi^{[m,0]} e^{(\tau, z_1 + \frac{z}{2} + \tau, z_2 - \frac{z}{2} - \tau, 0)} \right]
\]

\[
+ \sum_{r \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} q^{-\frac{1}{4m}(2mr+k)(2m(r-1)+k)} \frac{1}{\tau, z} \left[ \theta_{k,m} + \theta_{-k,m} \right] (\tau, z)
\]

\[
-2 \sum_{r \in \mathbb{Z}, 0 \leq r \leq p-1} q^{-mr(r+1)} \theta_{0,m}(\tau, z) + \sum_{r \in \mathbb{Z}} q^{-\frac{\pi}{4m}(2r+1)(2r-1)} \theta_{m,m}(\tau, z) \]

(6.8)

Proof. Letting \( \begin{cases} z_1 = \frac{z}{2} + \frac{\tau}{2} \\ z_2 = \frac{z}{2} - \frac{\tau}{2} \end{cases} \)

namely \( \begin{cases} z_1 - z_2 = \tau \\ z_1 + z_2 = z \end{cases} \)

in the formula (2.2) in Lemma 2.2, we have

\[
\Phi^{[m,0]} e^{(\tau, z_1 + \frac{z}{2} + p \tau, z_2 - \frac{z}{2} - p \tau, 0)} = q^{mp(p+1)} \left[ \Phi^{[m,0]} e^{(\tau, z_1 + \frac{z}{2} + \frac{\tau}{2}, z_2 - \frac{z}{2} - \frac{\tau}{2}, 0)} \right]
\]

\[
+ \sum_{r \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} q^{-\frac{1}{4m}(2mr+k)(2m(r-1)+k)} \frac{1}{\tau, z} \left[ \theta_{k,m} + \theta_{-k,m} \right] (\tau, z)
\]

(6.9)

We compute (I) by putting \( k = 2mr + k', (0 \leq r < p, 1 \leq k' \leq 2m) \) as follows:

(I) := \[ \sum_{r \in \mathbb{Z}} \sum_{k' \in \mathbb{Z}, 0 \leq r < p, 1 \leq k' \leq 2m} q^{-\frac{1}{4m}(2mr+k')(2mr+k'+2m)} \frac{1}{\tau, z} \left[ \theta_{2mr+k', m} + \theta_{-(2mr+k'), m} \right] (\tau, z) \]

(6.10)
\[
\begin{align*}
+ \sum_{r \in \mathbb{Z}} \sum_{k' \in \mathbb{Z}} q^{-\frac{1}{4m}(2mr+k')(2m(r+1)+k')} \theta_{-k',m}(\tau, z) \\
&= \left( A \right) + \left( B \right)
\end{align*}
\]

where \( (A) \) is computed by using (6.3) as follows:

\[
(A) = \sum_{r \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} q^{-\frac{1}{4m}(2mr+k)(2m(r+1)+k)} \theta_{k,m}(\tau, z)
\]

\[
- \sum_{r \in \mathbb{Z}} q^{-\frac{1}{4m}(2mr+2m(r+1)} \theta_{0,m}(\tau, z) + \sum_{r \in \mathbb{Z}} q^{-\frac{1}{4m}(2mr+2m)(2m(r+1)+2m)} \theta_{2m,m}(\tau, z)
\]

\[
\sum_{r \in \mathbb{Z}} q^{-mr(r+1)} \theta_{0,m}(\tau, z)
\]

And \( (B) \) becomes by putting \( k' = 2m - k \) and \( r + 2 = -r' \) as follows:

\[
(B) = \sum_{r' \in \mathbb{Z}} \sum_{k' \in \mathbb{Z}} q^{-\frac{1}{4m}(2mr'+k')(2m(r'+1)+k')} \theta_{k',m}(\tau, z)
\]

\[
- \sum_{k' \in \mathbb{Z}} q^{-\frac{1}{4m}(-2m+k') \theta_{k',m}(\tau, z)}
\]

Then (I) becomes as follows:

\[
(I) = (A) + (B)
\]
\[
\sum_{k \in \mathbb{Z}} \sum_{r \in \mathbb{Z}} q^{-\frac{1}{4m}(2mr+k)(2m(r+1)+k)} \theta_{k,m}(\tau, z) \]

Then substituting (6.10) into (6.9), we have

\[
\Phi[m,0]^* (\tau, z_1 + pr, z_2 - pr, 0) = q^{mp(p+1)} \left\{ \Phi[m,0]^* (\tau, z_1, z_2, 0) + (I) \right\}
\]

\[
= q^{mp(p+1)} \left\{ \Phi[m,0]^* (\tau, z_1, z_2, 0) \right. \\
\left. + \sum_{r \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} q^{-\frac{1}{4m}(2mr+k)(2m(r+1)+k)} \theta_{k,m}(\tau, z) \right. \\
\left. - \sum_{k \in \mathbb{Z}} q^{-\frac{1}{4m}(k-2m)} \theta_{k,m}(\tau, z) \right. \\
\left. + q^{-mp(p+1)} \theta_{0,m}(\tau, z) \right\} \\
\Phi[m,0]^* (\tau, z_1, z_2, 0) \text{ by Lemma 5.3}
\]

\[
= q^{mp(p+1)} \left\{ \Phi[m,0]^* (\tau, z_1, z_2, 0) + \sum_{r \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} q^{-\frac{1}{4m}(2mr+k)(2m(r-1)+k)} \theta_{k,m}(\tau, z) \right. \\
\left. + q^{-mp(p+1)} \theta_{0,m}(\tau, z) \right\}
\]

(II) \quad (II) = (II)_A + (II)_B

We go further to compute

(II) = (II)_A + (II)_B

where

(II)_A := \sum_{r \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} q^{-\frac{1}{4m}(2mr+k)(2m(r-1)+k)} \theta_{k,m}(\tau, z)
\[ (\text{II})_B := \sum_{r \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} q^{-\frac{1}{4m}(2mr+k)(2m(r-1)+k)} \theta_{k,m}(\tau, z) \]

First we compute \((\text{II})_A\) :

\[ (\text{II})_A = \sum_{r \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} q^{-\frac{1}{4m}(2mr+k)(2m(r-1)+k)} \theta_{k,m}(\tau, z) \]

\[ = \sum_{r \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} q^{-\frac{1}{4m}(2mr+k)(2m(r-1)+k)} \theta_{k,m}(\tau, z) \]

\[ + \sum_{r \in \mathbb{Z}} q^{-\frac{m}{2}(2r+1)(2r-1)} \theta_{m,m}(\tau, z) \]  

Next, \((\text{II})_B\) is rewritten as follows by putting \(k = 2m - k'\) :

\[ (\text{II})_B = \sum_{r' \in \mathbb{Z}} \sum_{k' \in \mathbb{Z}} q^{-\frac{1}{4m}(2m(r'+1)-k')(2mr'+k')} \theta_{-k',m}(\tau, z) \]

\[ = \sum_{r' \in \mathbb{Z}} \sum_{k' \in \mathbb{Z}} q^{-\frac{1}{4m}(2m(r'+1)-k')(2mr'+k')} \theta_{-k',m}(\tau, z) \]

\[ - \sum_{r' \in \mathbb{Z}} q^{-mr'(r'-1)} \theta_{0,m}(\tau, z) \]  

Then by (6.13a) and (6.13b), we have

\[ (\text{II}) + q^{-mp(p+1)} \theta_{0,m}(\tau, z) \]
Proposition 7.1. \[ \eta = -\Phi \left[ \sum_{j, r} q^{-\frac{1}{4m}(2mr+k)(2m(r-1)+k)} \theta_{k,m} + \theta_{-k,m} \right] (\tau, z) \]

Substituting this equation (6.14) into (6.9), we obtain

\[ \Phi^{[m,0]}(\tau, z_1 + pr, z_2 - pr, 0) = q^{mp(p+1)} \left\{ \Phi^{[m,0]}(\tau, z_1, z_2, 0) \right\} \]

proving Lemma 6.2

7 Modified function \( \tilde{\Phi}^{[m,0]} \) with specialization

Proposition 7.1. For \( m \in \mathbb{N} \) and \( p \in \mathbb{Z}_{\geq 0} \), the following formulas hold:

1) \(( -1)^p q^{-\frac{m}{p-m}} \theta^{(-)}_{p-m-\frac{1}{2}, m+\frac{1}{2}}(\tau, 0) \) \( \tilde{\Phi}^{[m,0]}(\tau, z, \frac{z}{2} + \frac{\tau}{2} - \frac{1}{2}, \frac{z}{2} - \frac{\tau}{2} + \frac{1}{2}, 0) \]

\[ = \frac{\eta(\tau)^3}{\theta_{0,\frac{1}{2}}(\tau, z)} \left[ \theta_{p, m + \frac{1}{2}} + \theta_{-p, m + \frac{1}{2}} \right](\tau, 0) \]

\[ - \left[ \sum_{j, r \in \frac{1}{2} \mathbb{Z}_{\text{odd}}} - \sum_{j, r \in \frac{1}{2} \mathbb{Z}_{\text{odd}}} \right] \sum_{k \in \mathbb{Z}} (-1)^{j+\frac{1}{2}+k} q^{-\frac{1}{4m}(2mr+k+2mp)^2} \theta_{k,m} + \theta_{-k,m} \]

\[ \times \left[ \sum_{j, r \in \frac{1}{2} \mathbb{Z}_{\text{odd}}} - \sum_{j, r \in \frac{1}{2} \mathbb{Z}_{\text{odd}}} \right] \sum_{k \in \mathbb{Z}} (-1)^{j+\frac{1}{2}+k} q^{-\frac{1}{4m}(2mr-k+2mp)^2} \theta_{k,m} + \theta_{-k,m} \]

\[ - (-1)^p q^{-\frac{1}{p-m}} \theta^{(-)}_{p-m-\frac{1}{2}, m+\frac{1}{2}}(\tau, 0) \sum_{r \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} (-1)^{k} q^{-\frac{1}{4m}(m(2r-1)+k)^2} \theta_{k,m} + \theta_{-k,m} \]

\[ + 2(-1)^p \theta^{(-)}_{p-m-\frac{1}{2}, m+\frac{1}{2}}(\tau, 0) \sum_{r \in \mathbb{Z}} q^{-m(r+\frac{1}{2})^2} \theta_{0,m}(\tau, z) \]
2) $(-1)^p \eta^{(−)}_{−p+m+\frac{1}{2},m+\frac{1}{2}}(\tau,0) \Phi_{[m,0]}(\tau, \frac{z}{2} + \frac{\tau}{2}, \frac{z}{2} - \frac{\tau}{2}, 0) = (-1)^p \frac{\eta^{(−)}(\tau)}{\theta^{(−)}_{0,\frac{1}{2}}(\tau, z)} \left[ \theta^{(−)}_{p,m+\frac{1}{2}}(\tau, z) + \theta^{(−)}_{−p,m+\frac{1}{2}}(\tau, z) \right]$

$- \sum_{j, r \in \frac{1}{2} Z_{\text{odd}}} \sum_{0 \leq r < j} \sum_{j \leq r < 0} \sum_{0 < k < m} (-1)^{j-k} q^{(m+\frac{1}{2})(j+\frac{2pm}{2m+1})^2} q^{-\frac{1}{4m}(2mr+k+2mp)^2} \left[ \theta_{k,m} + \theta_{-k,m} \right] (\tau, z)$

$- \sum_{j, r \in \frac{1}{2} Z_{\text{odd}}} \sum_{0 \leq r \leq j} \sum_{j \leq r < 0} \sum_{0 \leq k < m} (-1)^{j-k} q^{(m+\frac{1}{2})(j+\frac{2pm}{2m+1})^2} q^{-\frac{1}{4m}(2mr+k+2mp)^2} \left[ \theta_{k,m} + \theta_{-k,m} \right] (\tau, z)$

$- \theta^{(−)}_{2mp+m+\frac{1}{2},m+\frac{1}{2}}(\tau,0) \sum_{r \in Z} \sum_{p < r \leq p} q^{-\frac{1}{4m}(m(2r-1)+k)^2} \left[ \theta_{k,m} + \theta_{-k,m} \right] (\tau, z)$

$+ 2 \theta^{(−)}_{2mp+m+\frac{1}{2},m+\frac{1}{2}}(\tau,0) \sum_{r \in Z} q^{-m(r+\frac{1}{2})^2} \theta_{0,m}(\tau, z)$

$- \theta^{(−)}_{2mp+m+\frac{1}{2},m+\frac{1}{2}}(\tau,0) \sum_{r \in Z} q^{-mr^2} \theta_{m,m}(\tau, z)$

### Proof
1) In the case $\left\{ \begin{array}{l} z_1 = \frac{z}{2} + \frac{\tau}{2} - \frac{1}{2} \\ z_2 = \frac{z}{2} - \frac{\tau}{2} + \frac{1}{2} \end{array} \right.$, substituting (6.1) into (4.1), we have

$\theta^{(−)}_{2mp+m+\frac{1}{2},m+\frac{1}{2}}(\tau,0) \times \left\{ q^{mp(p+1)} \Phi_{[m,0]}(\tau, \frac{z}{2} + \frac{\tau}{2} - \frac{1}{2}, \frac{z}{2} - \frac{\tau}{2} + \frac{1}{2}, 0) \right.$

$+ q^{mp(p+1)} \sum_{r \in Z} \sum_{k \in Z} (-1)^k q^{-\frac{1}{4m}(2mr+k)(2m(r-1)+k)} \left[ \theta_{k,m} + \theta_{-k,m} \right] (\tau, z)$

$- 2 q^{mp(p+1)} \sum_{r \in Z} q^{-mr^2} \theta_{0,m}(\tau, z)$

$+ (-1)^m q^{mp(p+1)} \sum_{r \in Z} q^{-\frac{1}{4m}(2r+1)(2r-1)} \theta_{m,m}(\tau, z) \left\} \right.$

$- \sum_{r \in Z} q^{-mr^2} \theta_{m,m}(\tau, z)$

$\right.$
\[= \eta^{(p+1)^2} \frac{\eta(\tau)^3}{\theta_{0,\frac{1}{2}}(\tau, z)} \cdot \left[ \theta_{p,m+\frac{1}{2}} + \theta_{-p,m+\frac{1}{2}} \right](\tau, 0) \]

\[- q^{m(2p+1)^2} \left[ \sum_{j, r \in \mathbb{Z}_{\text{odd}}} - \sum_{j, r \in \mathbb{Z}_{\text{odd}}} \right] \sum_{k \in \mathbb{Z}} \sum_{0 < k < m} (-1)^j - \frac{k}{2} q^{(m+\frac{1}{2})(j+\frac{2m}{2m+1})^2} \]

\[- q^{m(2p+1)^2} \left[ \sum_{j, r \in \mathbb{Z}_{\text{odd}}} - \sum_{j, r \in \mathbb{Z}_{\text{odd}}} \right] \sum_{k \in \mathbb{Z}} \sum_{0 < k < m} (-1)^j - \frac{k}{2} q^{(m+\frac{1}{2})(j+\frac{2m}{2m+1})^2} \]

\[+ \theta_{2mp+m+\frac{1}{2}, m+\frac{1}{2}}^{(-)}(\tau, 0) \sum_{k \in \mathbb{Z}} (-1)^k q^{-\frac{2k^2}{4m}(2p+1)} \left[ \theta_{k,m} + \theta_{-k,m} \right](\tau, z) \]

Multiplying \(q^{-\frac{mp(p+1)}{2}}\) to both sides, we have

\[q^{-\frac{mp}{2}(2p+1)^2} \theta_{r,m}^{(-)}\left[ (\tau, 0) \right] \frac{d}{d(\tau, z)} \left( \tau, \frac{\tau}{2} + \frac{1}{2}, \frac{\tau}{2} - \frac{1}{2} \right) \]

\[+ \frac{q^{-\frac{mp}{2}(2p+1)^2}}{2}\theta_{r,m}^{(-)}\left[ (\tau, 0) \right] \sum_{p < r < m} (-1)^k q^{-\frac{k^2}{4m}} \left[ \theta_{k,m} + \theta_{-k,m} \right](\tau, z) \]

\[- 2 q^{-\frac{mp}{2}(2p+1)^2} \theta_{r,m}^{(-)}\left[ (\tau, 0) \right] \sum_{r \in \mathbb{Z}} q^{-mr(r+1)} \theta_{0,m}(\tau, z) \]

\[+ (-1)^m q^{-\frac{mp}{2}(2p+1)^2} \theta_{r,m}^{(-)}\left[ (\tau, 0) \right] \sum_{r \in \mathbb{Z}} q^{-mr(r+1)} \theta_{0,m}(\tau, z) \]

\[= \frac{\eta(\tau)^3}{\theta_{0,\frac{1}{2}}(\tau, z)} \cdot \left[ \theta_{p,m+\frac{1}{2}} + \theta_{-p,m+\frac{1}{2}} \right](\tau, 0) \]

\[- \left[ \sum_{j, r \in \mathbb{Z}_{\text{odd}}} - \sum_{j, r \in \mathbb{Z}_{\text{odd}}} \right] \sum_{k \in \mathbb{Z}} \sum_{0 < k < m} (-1)^j - \frac{k}{2} q^{(m+\frac{1}{2})(j+\frac{2m}{2m+1})^2} \]

\[- \left[ \sum_{j, r \in \mathbb{Z}_{\text{odd}}} - \sum_{j, r \in \mathbb{Z}_{\text{odd}}} \right] \sum_{k \in \mathbb{Z}} \sum_{0 < k < m} (-1)^j - \frac{k}{2} q^{(m+\frac{1}{2})(j+\frac{2m}{2m+1})^2} \]

\[- \left[ \sum_{j, r \in \mathbb{Z}_{\text{odd}}} - \sum_{j, r \in \mathbb{Z}_{\text{odd}}} \right] \sum_{k \in \mathbb{Z}} \sum_{0 < k < m} (-1)^j - \frac{k}{2} q^{(m+\frac{1}{2})(j+\frac{2m}{2m+1})^2} \]

\[- \left[ \sum_{j, r \in \mathbb{Z}_{\text{odd}}} - \sum_{j, r \in \mathbb{Z}_{\text{odd}}} \right] \sum_{k \in \mathbb{Z}} \sum_{0 < k < m} (-1)^j - \frac{k}{2} q^{(m+\frac{1}{2})(j+\frac{2m}{2m+1})^2} \]

\[- \left[ \sum_{j, r \in \mathbb{Z}_{\text{odd}}} - \sum_{j, r \in \mathbb{Z}_{\text{odd}}} \right] \sum_{k \in \mathbb{Z}} \sum_{0 < k < m} (-1)^j - \frac{k}{2} q^{(m+\frac{1}{2})(j+\frac{2m}{2m+1})^2} \]

\[- \left[ \sum_{j, r \in \mathbb{Z}_{\text{odd}}} - \sum_{j, r \in \mathbb{Z}_{\text{odd}}} \right] \sum_{k \in \mathbb{Z}} \sum_{0 < k < m} (-1)^j - \frac{k}{2} q^{(m+\frac{1}{2})(j+\frac{2m}{2m+1})^2} \]

\[- \left[ \sum_{j, r \in \mathbb{Z}_{\text{odd}}} - \sum_{j, r \in \mathbb{Z}_{\text{odd}}} \right] \sum_{k \in \mathbb{Z}} \sum_{0 < k < m} (-1)^j - \frac{k}{2} q^{(m+\frac{1}{2})(j+\frac{2m}{2m+1})^2} \]
\[\begin{align*}
\times q^{-\frac{1}{4m}[2mr-k+2mp]^2} [\theta_{k,m} + \theta_{-k,m}] (\tau, z) \\
+ \theta^{(-)}_{2mp+m+\frac{1}{2}, m+\frac{1}{2}} (\tau, 0) \sum_{k \in \mathbb{Z}} \sum_{0 \leq k \leq m} (-1)^k q^{-\frac{1}{4m}[k-m(2p+1)]^2} [\theta_{k,m} + \theta_{-k,m}] (\tau, z)
\end{align*}\]

namely

\[q^{-\frac{m}{4}} \theta^{(-)}_{2mp+m+\frac{1}{2}, m+\frac{1}{2}} (\tau, 0) \Phi_{m,0} \Phi_{0,0} [\theta_{0,0} + \theta_{-0,0}] (\tau, 0)
\]

\[= \frac{\eta(\tau)^3}{\theta_{0,0}^3 (\tau, z)} \sum_{0 \leq r < j} \sum_{j \leq r < 0} \sum_{k \in \mathbb{Z}} \sum_{0 \leq k \leq m} (-1)^j q^{(m+\frac{1}{2})(j+2mp)\tau, z} \theta_{k,m} + \theta_{-k,m} (\tau, z)
\]

\[= \frac{\eta(\tau)^3}{\theta_{0,0}^3 (\tau, z)} \sum_{0 \leq r \leq j} \sum_{j \leq r < 0} \sum_{k \in \mathbb{Z}} \sum_{0 \leq k \leq m} (-1)^j q^{(m+\frac{1}{2})(j+2mp)\tau, z} \theta_{k,m} + \theta_{-k,m} (\tau, z)
\]

\[+ \theta^{(-)}_{2mp+m+\frac{1}{2}, m+\frac{1}{2}} (\tau, 0) \sum_{k \in \mathbb{Z}} \sum_{0 \leq k \leq m} (-1)^k q^{-\frac{1}{4m}[k-m(2p+1)]^2} [\theta_{k,m} + \theta_{-k,m}] (\tau, z)
\]

\[= \frac{1}{4m} (2mr+k)(2m(r-1)+k) - \frac{m}{4} = -\frac{1}{4m} (m(2r-1)+k)^2
\]
(I) \(_B\) becomes as follows:

\[
(I) \_B = - \theta^{(-)}_{2mp+m+1, m+1}(\tau, 0) \, \sum_{r \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} (-1)^k q^{- \frac{1}{4m}(m(2r-1)+k)^2} \left[ \theta_{k,m} + \theta_{-k,m} \right] (\tau, z)
\]

\[
- \sum_{r \in \mathbb{Z}} + \sum_{r = -p}^{r=p} \left| -p < r \leq p \right|
\]

\[
= - \theta^{(-)}_{2mp+m+1, m+1}(\tau, 0) \, \sum_{r \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} (-1)^k q^{- \frac{1}{4m}(m(2r-1)+k)^2} \left[ \theta_{k,m} + \theta_{-k,m} \right] (\tau, z)
\]

\[
- \theta^{(-)}_{2mp+m+1, m+1}(\tau, 0) \, \sum_{k \in \mathbb{Z}} (-1)^k q^{- \frac{1}{4m}(-m(2p+1)+k)^2} \left[ \theta_{k,m} + \theta_{-k,m} \right] (\tau, z)
\]

\[
\uparrow
\]

\[
- (I) \_A
\]

So

\[
(I) \_A + (I) \_B = - \theta^{(-)}_{2mp+m+1, m+1}(\tau, 0) \sum_{r \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} (-1)^k q^{- \frac{1}{4m}(m(2r-1)+k)^2} \left[ \theta_{k,m} + \theta_{-k,m} \right] (\tau, z)
\]

Substituting (7.2b) into (7.2a) and noticing that

\[
\theta^{(-)}_{2mp+m+1, m+1}(\tau, 0) = (-1)^p \theta^{(-)}_{p-m+1, m+1}(\tau, 0) = (-1)^p \theta^{(-)}_{p-m-\frac{1}{2}, m+\frac{1}{2}}(\tau, 0)
\]

we obtain (7.1a), proving the claim 1).

2) In the case \( \left\{ \begin{array}{l} z_1 = \frac{\tau}{2} + \frac{\tau}{2} \\ z_2 = \frac{\tau}{2} - \frac{\tau}{2} \end{array} \right. \), substituting (6.8) into (4.5), we have

\[
\theta^{(-)}_{2mp+m+1, m+1}(\tau, 0) \times \left\{ q^{mp(p+1)} \left( \frac{\tau}{2} + \frac{\tau}{2} \right) \Phi[m, 0] (\tau, 0) \right. \\
+ q^{mp(p+1)} \sum_{r \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} q^{- \frac{1}{4m}(2mr+k)(2m(r-1)+k)} \left[ \theta_{k,m} + \theta_{-k,m} \right] (\tau, z) \\
- 2 q^{mp(p+1)} \sum_{r \in \mathbb{Z}} q^{- mr(r+1)} \theta_{0,m} (\tau, z) \\
+ q^{mp(p+1)} \sum_{r \in \mathbb{Z}} q^{- \frac{1}{2}(2r+1)(2r-1)} \theta_{m,m} (\tau, z) \left\}
\]

\[
\sum_{r \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} (-1)^k q^{- \frac{1}{4m}(m(2r-1)+k)^2} \left[ \theta_{k,m} + \theta_{-k,m} \right] (\tau, z)
\]

\[
- \sum_{r \in \mathbb{Z}} + \sum_{r = -p}^{r=p} \left| -p < r \leq p \right|
\]
\[
= (-1)^p q^\frac{m}{2}(2p+1)^2 \frac{\eta (\tau)^3}{\theta(- \tau, z)} \cdot \left[ \theta(- \tau, z) + \theta(- \tau, z) \right] (\tau, 0) \\
- q^\frac{m}{2}(2p+1)^2 \left[ \sum_{j, r < j, r \in \mathbb{Z}_{odd}} - \sum_{j, r < j, r \in \mathbb{Z}_{odd}} \right] \sum_{k \in \mathbb{Z}} \sum_{0 < k < m} (-1)^{j-\frac{1}{2}} q^{(m+\frac{1}{2})(j+\frac{2}{2m+1})^2} \times q^{-\frac{1}{4m} [2mr+k+2mp]^2} [\theta_{k,m} + \theta_{-k,m}] (\tau, z) \\
- q^\frac{m}{2}(2p+1)^2 \left[ \sum_{j, r < j, r \in \mathbb{Z}_{odd}} - \sum_{j, r < j, r \in \mathbb{Z}_{odd}} \right] \sum_{k \in \mathbb{Z}} \sum_{0 \leq k \leq m} (-1)^{j-\frac{1}{2}} q^{(m+\frac{1}{2})(j+\frac{2}{2m+1})^2} \times q^{-\frac{1}{4m} [2mr-k+2mp]^2} [\theta_{k,m} + \theta_{-k,m}] (\tau, z) \\
+ \theta(- \tau, z) \sum_{k \in \mathbb{Z}} q^{-\frac{1}{4m} k^2+\frac{1}{2}(2p+1)} [\theta_{k,m} + \theta_{-k,m}] (\tau, z) \\
\]

Multiplying \( q^{-\frac{m}{2}(2p+1)^2} = q^{-mp(p+1)-\frac{m}{2}} \) to both sides, we have
\[
q^{-\frac{m}{2}} \theta(- \tau, z) \Phi^{[m,0]} \left( \tau, \frac{r}{2} + \tau, \frac{z}{2} - \frac{1}{2}, \frac{z}{2} - \frac{1}{2} \right) (\tau, 0) \\
+ q^{-\frac{m}{2}} \theta(- \tau, z) \\
\times \sum_{r \in \mathbb{Z}} \sum_{0 \leq k < m} q^{-\frac{1}{4m} [(2mr+k)(2m(r-1)+k)] [\theta_{k,m} + \theta_{-k,m}] (\tau, z) \\
- 2 q^{-\frac{m}{2}} \theta(- \tau, z) \sum_{r \in \mathbb{Z}} q^{-mr(r+1)} [\theta_{0,m} (\tau, z) \\
+ q^{-\frac{m}{2}} \theta(- \tau, z) \sum_{r \in \mathbb{Z}} q^{-\frac{1}{2}(2r+1)(2r-1)} [\theta_{m,m} (\tau, z) \\
= (-1)^p \frac{\eta (\tau)^3}{\theta(- \tau, z)} [\theta(- \tau, z) + \theta(- \tau, z) (\tau, 0) \\
- \left[ \sum_{j, r < j, r \in \mathbb{Z}_{odd}} - \sum_{j, r < j, r \in \mathbb{Z}_{odd}} \right] \sum_{k \in \mathbb{Z}} \sum_{0 < k < m} (-1)^{j-\frac{1}{2}} q^{(m+\frac{1}{2})(j+\frac{2}{2m+1})^2} \times q^{-\frac{1}{4m} [2mr+k+2mp]^2} [\theta_{k,m} + \theta_{-k,m}] (\tau, z) \\
\]
\[- \left[ \sum_{j, r \in \frac{1}{2} \mathbb{Z}_{\text{odd}}} - \sum_{j, r \in \frac{1}{2} \mathbb{Z}_{\text{odd}}} \right] \sum_{k \in \mathbb{Z}} \sum_{0 \leq r \leq j} (-1)^{j-\frac{1}{2}} q^{(m+\frac{1}{2})(j+\frac{2m}{2m+1})^2} \times q^{-\frac{1}{4m} [2mr-k+2mp]^2} [\theta_{k,m} + \theta_{-k,m}] (\tau, z) \]

\[+ \theta^{(-)}_{2mp+m+\frac{1}{2},m+\frac{1}{2}} (\tau, 0) \sum_{k \in \mathbb{Z}} q^{-\frac{1}{4m} [k-m(2p+1)]^2} [\theta_{k,m} + \theta_{-k,m}] (\tau, z) \]

namely

\[q^{-\frac{m}{4}} \theta^{(-)}_{2mp+m+\frac{1}{2},m+\frac{1}{2}} (\tau, 0) \tilde{\Phi}^{[m,0]} (\tau, \frac{z}{2} + \frac{1}{2}, \frac{z}{2} - \frac{1}{2}) \]

\[= (-1)^p \frac{\eta(\tau)^3}{\eta_{0,\frac{1}{2}}(\tau, z)} [\theta^{(-)}_{p,m+\frac{1}{2}} + \theta^{(-)}_{-p,m+\frac{1}{2}}] (\tau, 0) \]

\[- \left[ \sum_{j, r \in \frac{1}{2} \mathbb{Z}_{\text{odd}}} - \sum_{j, r \in \frac{1}{2} \mathbb{Z}_{\text{odd}}} \right] \sum_{k \in \mathbb{Z}} \sum_{0 \leq r \leq j} (-1)^{j-\frac{1}{2}} q^{(m+\frac{1}{2})(j+\frac{2m}{2m+1})^2} \times q^{-\frac{1}{4m} [2mr-k+2mp]^2} [\theta_{k,m} + \theta_{-k,m}] (\tau, z) \]

\[+ \theta^{(-)}_{2mp+m+\frac{1}{2},m+\frac{1}{2}} (\tau, 0) \sum_{k \in \mathbb{Z}} q^{-\frac{1}{4m} [k-m(2p+1)]^2} [\theta_{k,m} + \theta_{-k,m}] (\tau, z) \]

\[\overset{(1)A}{=} \]

\[- q^{-\frac{m}{4}} \theta^{(-)}_{2mp+m+\frac{1}{2},m+\frac{1}{2}} (\tau, 0) \sum_{r \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} q^{-\frac{1}{4m} [2mr-k(2m(r-1)+k)]} [\theta_{k,m} + \theta_{-k,m}] (\tau, z) \]

\[\overset{(1)B}{=} \]

\[+ 2 q^{-\frac{m}{4}} \theta^{(-)}_{2mp+m+\frac{1}{2},m+\frac{1}{2}} (\tau, 0) \sum_{r \in \mathbb{Z}} q^{-mr(r+1)} \theta_{0,m}(\tau, z) \]

\[+ q^{-\frac{m}{4}} \theta^{(-)}_{2mp+m+\frac{1}{2},m+\frac{1}{2}} (\tau, 0) \sum_{r \in \mathbb{Z}} q^{-mr^2+\frac{mp}{2}} \theta_{m,m}(\tau, z) \]

(7.4a)
where \((I)_B\) becomes as follows:

\[
(I)_B = -\theta_{2mp+m+\frac{1}{2},m+\frac{1}{2}}(\tau, 0) \sum_{r \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} q^{-\frac{1}{4m} \left( (m(2r-1)+k)^2 \left[ \theta_{k,m} + \theta_{-k,m} \right] (\tau, z) \right)}
\]

\[
= -\theta_{2mp+m+\frac{1}{2},m+\frac{1}{2}}(\tau, 0) \sum_{r \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} q^{-\frac{1}{4m} \left( (m(2r-1)+k)^2 \left[ \theta_{k,m} + \theta_{-k,m} \right] (\tau, z) \right)} - (I)_A
\]

so

\[
(I)_A + (I)_B = -\theta_{2mp+m+\frac{1}{2},m+\frac{1}{2}}(\tau, 0) \sum_{r \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} q^{-\frac{1}{4m} \left( (m(2r-1)+k)^2 \left[ \theta_{k,m} + \theta_{-k,m} \right] (\tau, z) \right)}
\]

\[
\times \left[ \theta_{k,m} + \theta_{-k,m} \right] (\tau, z)
\]

Substituting (7.4b) into (7.4a) and using (7.3), we obtain (7.1b), proving the claim 2). □

8 Modular transformation of \(\tilde{\psi}^{(i)[m]}(\tau, z)\)

8.1 \(\tilde{\psi}^{(i)[m]}(\tau, z)\)

For \(m \in \mathbb{N}\) and \(i \in \{1, 2\}\), we consider functions \(\tilde{\psi}^{(i)[m]}(\tau, z)\) defined by

\[
\tilde{\psi}^{(1)[m]}(\tau, z) := \Phi^{[m,0]}(\tau, \frac{z}{2} + \frac{\tau}{2} - \frac{1}{2}, \frac{z}{2} - \frac{\tau}{2} + \frac{1}{2}, 0)
\]

\[
= \Phi^{[m,0]}(\tau, \frac{z}{2} + \frac{\tau}{2} + \frac{1}{2}, \frac{z}{2} - \frac{\tau}{2} - \frac{1}{2}, 0)
\]

\[
\tilde{\psi}^{(2)[m]}(\tau, z) := \Phi^{[m,0]}(\tau, \frac{z}{2} + \frac{\tau}{2} - \frac{1}{2}, \frac{z}{2} - \frac{\tau}{2} + \frac{1}{2}, 0)
\]

The function \(\tilde{\psi}^{(1)[m]}(\tau, z)\) satisfies the following modular transformation properties.

Lemma 8.1. Let \(m \in \mathbb{N}\), then

1) \(\tilde{\psi}^{(1)[m]}(\tau, \frac{z}{\tau}) = (-1)^m \tau e \frac{\pi i m}{24} e^{-\frac{\pi i m}{24}} q^{-\frac{m}{8}} \tilde{\psi}^{(1)[m]}(\tau, z)\)
2) \( \tilde{\psi}^{(1)[m]}(\tau + 1, z) = \tilde{\psi}^{(2)[m]}(\tau, z) \)

Proof. These are obtained easily from Lemma 2.1 in [17] as follows.

1) \( \tilde{\psi}^{(1)[m]}\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) = \Phi[m,0] * \left(-\frac{1}{\tau}, \frac{z}{2\tau} - \frac{1}{2\tau} + \frac{z}{2\tau} + \frac{1}{2\tau}, 0\right) \)

\[ = \Phi[m,0] * \left(-\frac{1}{\tau}, \frac{z}{\tau} - \frac{1}{\tau} + \frac{z}{\tau} + \frac{1}{\tau}, 0\right) = \tau e^{\frac{2\pi i m}{\tau} \left(\frac{z}{\tau} - \frac{1}{\tau} + \frac{z}{\tau} + \frac{1}{\tau}\right)} \Phi[m,0] * \left(\tau, \frac{z}{2} + \frac{1}{2\tau}, 0\right) = (-1)^m \tau e^{\frac{2\pi i m}{\tau} z^2} e^{-\frac{2\pi i m}{\tau} q - \frac{m}{\tau} \tilde{\psi}^{(1)[m]}(\tau, z)}, \text{ proving 1).} \]

2) \( \tilde{\psi}^{(1)[m]}(\tau + 1, z) = \Phi[m,0] * \left(\tau + 1, \frac{z}{2\tau} + \frac{1}{2\tau} + \frac{z}{2\tau} - \frac{1}{2\tau}, 0\right) \)

\[ = \Phi[m,0] * \left(\tau + 1, \frac{z}{2\tau} + \frac{1}{2\tau}, 0\right) = \Phi[m,0] * \left(\tau, \frac{z}{2\tau} + \frac{1}{2\tau} - \frac{\tau}{2\tau}, 0\right) = \tilde{\psi}^{(2)[m]}(\tau, z), \text{ proving 2).} \]

\[ \square \]

8.2 \( \Xi^{(i)[m,p]} * (\tau, z) \) and \( \Upsilon^{(i)[m,p]} * (\tau, z) \)

For \( m \in \mathbb{N} \) and \( p \in \mathbb{Z} \) such that \( 0 \leq p \leq 2m \) and \( i \in \{1, 2, 3\} \), we define functions \( \Xi^{(i)[m,p]} * (\tau, z) \) and \( \Upsilon^{(i)[m,p]} * (\tau, z) \) as follows:

\[ \Xi^{(1)[m,p]} * (\tau, z) := (-1)^p q^{-\frac{m}{2}} \theta^{(-)}_{p,m-rac{1}{2}, m+rac{1}{2}}(\tau, 0) \cdot \tilde{\psi}^{(1)[m]}(\tau, z) \quad (8.1a) \]

\[ \Xi^{(2)[m,p]} * (\tau, z) := q^{-\frac{m}{2}} \theta^{(-)}_{p,m-rac{1}{2}, m+rac{1}{2}}(\tau, 0) \cdot \tilde{\psi}^{(2)[m]}(\tau, z) \quad (8.1b) \]

and

\[ \Upsilon^{(1)[m,p]} * (\tau, z) := \eta(\tau)^3 \cdot \frac{\theta^{(-)}_{p,m+rac{1}{2}}(\tau, z) + \theta^{(-)}_{-p,m+rac{1}{2}}(\tau, z)}{\theta^{(-)}_{0,1}(\tau, z)} \quad (8.2a) \]

\[ \Upsilon^{(2)[m,p]} * (\tau, z) := \eta(\tau)^3 \cdot \frac{\theta^{(-)}_{p,m+rac{1}{2}}(\tau, z) + \theta^{(-)}_{-p,m+rac{1}{2}}(\tau, z)}{\theta^{(-)}_{0,1}(\tau, z)} \quad (8.2b) \]

To compute modular transformation of these functions, we use the following formulas which are obtained easily from Lemmas 1.3 and 1.4 in [17].

Note 8.1. Let \( m \in \mathbb{Z}_{\geq 0} \) and \( p \in \mathbb{Z} \). Then

1) (i) \( \theta^{(-)}_{p,m-rac{1}{2}, m+rac{1}{2}} \left(-\frac{1}{\tau}, 0\right) = -i (-1)^{m+p} \left(-i\tau\right)^{\frac{3}{2}} \frac{2m}{\sqrt{2m+1}} \sum_{p'=0}^{2m} (-1)^p e^{-\frac{2\pi i}{2m+1} p' p'} \theta^{(-)}_{p', m-rac{1}{2}, m+rac{1}{2}}(\tau, 0) \)
\[(ii) \, \theta^{(-)}_{p-m-\frac{1}{2},m+\frac{1}{2}}(\tau, z) = (-1)^p e^{\frac{\pi i}{2}(m+\frac{1}{2})} e^{\frac{2\pi i}{2m+1} r^2} \theta^{(-)}_{p-m-\frac{1}{2},m+\frac{1}{2}}(\tau, z)\]

2) \(\begin{align*}
(i) & \, [\theta_{p,m+\frac{1}{2}} + \theta_{-p,m+\frac{1}{2}}] \left( -\frac{1}{\tau}, \frac{z}{\tau} \right) \\
\quad & = \left( (\frac{-ir}{2}) \right) e_{\frac{2\pi i}{2m+1} r^2} \sum_{p' \in \mathbb{Z}/(2m+1)} e^{\frac{2\pi i}{2m+1} (p' - p) (r^2)} \left[ \theta_{p',m+\frac{1}{2}} + \theta_{-p',m+\frac{1}{2}} \right] (\tau, z) \\
\quad & = \left( (\frac{-ir}{2}) \right) e_{\frac{2\pi i}{2m+1} r^2} \sum_{p' \in \mathbb{Z}/(2m+1)} e^{-\frac{2\pi i}{2m+1} (p' - p) (r^2)} \left[ \theta_{p',m+\frac{1}{2}} + \theta_{-p',m+\frac{1}{2}} \right] (\tau, z).
\end{align*}\]

(ii) \(\left[ \theta_{p,m+\frac{1}{2}} + \theta_{-p,m+\frac{1}{2}} \right] (\tau, z) = e^{\frac{2\pi i}{2m+1} r^2} \left[ \theta^{(-)}_{p,m+\frac{1}{2}} + \theta^{(-)}_{-p,m+\frac{1}{2}} \right] (\tau, z)\)

Then modular transformation properties of \(\Xi^{(1)}[m,p]^*\) and \(Y^{(1)}[m,p]^*\) are given by the following formulas:

**Lemma 8.2.** Let \(m \in \mathbb{N}\) and \(p \in \mathbb{Z}\). Then

1) \(\Xi^{(1)}[m,p]^* \left( -\frac{1}{\tau}, \frac{z}{\tau} \right) = \left( (\frac{-ir}{2}) \right) e_{\frac{2\pi i}{2m+1} r^2} \sum_{p' = 0}^{2m} e^{-\frac{2\pi i}{2m+1} (p' - p) (r^2)} \Xi^{(1)}[m,p']^* (\tau, z)\)

2) \(\Xi^{(1)}[m,p]^* (\tau + 1, z) = e^{\frac{\pi i}{2m+1} r^2} \Xi^{(2)}[m,p]^* (\tau, z)\)

**Proof.** 1) \(\Xi^{(1)}[m,p]^* \left( -\frac{1}{\tau}, \frac{z}{\tau} \right)\)

\(= \left( (\frac{-ir}{2}) \right) e^{-\frac{\pi i}{2m+1} r^2} \sum_{p' = 0}^{2m} e^{-\frac{2\pi i}{2m+1} (p' - p) (r^2)} \Xi^{(1)}[m,p']^* (\tau, z)\)

\(= \left( (\frac{-ir}{2}) \right) e^{\frac{\pi i}{2m+1} r^2} \sum_{p' = 0}^{2m} e^{-\frac{2\pi i}{2m+1} (p' - p) (r^2)} \Xi^{(1)}[m,p']^* (\tau, z)\)

proving 1).

2) \(\Xi^{(1)}[m,p]^* (\tau + 1, z) = \left( (\frac{-ir}{2}) \right) e^{-\frac{\pi i}{2m+1} (r + 1) (r^2)} \sum_{p' = 0}^{2m} e^{-\frac{2\pi i}{2m+1} (p' - p) (r^2)} \Xi^{(1)}[m,p']^* (\tau + 1, z)\)

\(= \left( (\frac{-ir}{2}) \right) e^{-\frac{\pi i}{2m+1} r^2} \sum_{p' = 0}^{2m} e^{-\frac{2\pi i}{2m+1} (p' - p) (r^2)} \Xi^{(1)}[m,p']^* (\tau, z)\)

proving 2).
Lemma 8.3. Let $m \in \mathbb{N}$ and $p \in \mathbb{Z}$. Then

1) $\Upsilon^{(1)[m,p]} \left( -\frac{1}{\tau}, \frac{z}{\tau} \right) = \frac{(-i\tau)^{\frac{3}{2}}}{\sqrt{2\tau m + 1}} e^{\frac{x_{im}z^2}{2m+1}} \sum_{p' = 0}^{2m} e^{-\frac{2\pi i}{2m+1} pp'} \Upsilon^{(1)[m,p']} \left( \tau, z \right)$

2) $\Upsilon^{(1)[m,p]} (\tau + 1, z) = e^{x_{im}p^2 + \pi i \frac{x_{im}}{4}} \Upsilon^{(2)[m,p]} (\tau, z)$

Proof. 1) $\Upsilon^{(1)[m,p]} \left( -\frac{1}{\tau}, \frac{z}{\tau} \right) = \eta \left( -\frac{1}{\tau} \right)^3 \cdot \frac{\theta_{p,m+\frac{1}{2}} \left( -\frac{1}{\tau}, \frac{z}{\tau} \right) + \theta_{-p,m+\frac{1}{2}} \left( -\frac{1}{\tau}, \frac{z}{\tau} \right)}{\theta_{0,\frac{1}{2}} \left( -\frac{1}{\tau}, \frac{z}{\tau} \right)}$

$$\begin{align*}
&= \frac{(-i\tau)^{\frac{3}{2}} \eta(\tau)^3}{\sqrt{2m+1}} e^{\frac{x_{im}z^2}{2m+1}} \sum_{p' \in \mathbb{Z}(2m+1)} e^{-\frac{2\pi i}{2m+1} pp'} \left[ \frac{\theta_{p',m+\frac{1}{2}} + \theta_{-p',m+\frac{1}{2}}}{\theta_{0,\frac{1}{2}} \left( \frac{p'}{\tau}, \frac{z}{\tau} \right)} \right] \\
&= \frac{(-i\tau)^{\frac{3}{2}}}{\sqrt{2m+1}} e^{\frac{x_{im}z^2}{2m+1}} \sum_{p' \in \mathbb{Z}(2m+1)} \eta(\tau)^3 \left[ \frac{\theta_{p',m+\frac{1}{2}} + \theta_{-p',m+\frac{1}{2}}}{\theta_{0,\frac{1}{2}} \left( \frac{p'}{\tau}, \frac{z}{\tau} \right)} \right],
\end{align*}$$

proving 1).

2) $\Upsilon^{(1)[m,p]} (\tau + 1, z) = \eta(\tau + 1)^3 \frac{\theta_{p,m+\frac{1}{2}} + \theta_{-p,m+\frac{1}{2}}}{\theta_{0,\frac{1}{2}} \left( \frac{p}{\tau}, \frac{z}{\tau} \right)} (\tau + 1, z)$

$$\begin{align*}
&= \left[ e^{\frac{\pi i}{2}} \eta(\tau) \right]^3 \frac{e^{\frac{x_{im}p^2}{2m+1}} \left[ \theta_{p,m+\frac{1}{2}} + \theta_{-p,m+\frac{1}{2}} \right]}{\theta_{0,\frac{1}{2}} \left( \frac{p}{\tau}, \frac{z}{\tau} \right)} \\
&= e^{\frac{x_{im}p^2}{2m+1} + \frac{\pi i}{4}} \eta(\tau)^3 \left[ \frac{\theta_{p,m+\frac{1}{2}} - \theta_{-p,m+\frac{1}{2}}}{\theta_{0,\frac{1}{2}} \left( \frac{p}{\tau}, \frac{z}{\tau} \right)} \right],
\end{align*}$$

proving 2).

9 $G^{(i)[m,p]} (\tau, z)$ and $g_k^{(i)[m,p]} (\tau)$

9.1 $G^{(i)[m,p]} (\tau, z)$

For $m \in \mathbb{N}$ and $p \in \mathbb{Z}$ such that $0 \leq p \leq 2m$ and $i \in \{1, 2\}$, we put

$$G^{(i)[m,p]} (\tau, z) := \Xi^{(i)[m,p]} (\tau, z) - \Upsilon^{(i)[m,p]} (\tau, z) \quad (9.1a)$$

Then, by Proposition $7.1$, $G^{(i)[m,p]} (\tau, z)$ can be written in the following form:

$$G^{(i)[m,p]} (\tau, z) = \sum_{k \in \mathbb{Z}} g_k^{(i)[m,p]} (\tau) \left[ \theta_{k,m} + \theta_{-k,m} \right] (\tau, z)$$
\[ = \sum_{k \in \mathbb{Z}} g_k^{(i)[m,p]}(\tau) \left[ \theta_{k,m} + \theta_{-k,m} \right](\tau, z) + 2g_0^{(i)[m,p]}(\tau)\theta_{0,m}(\tau, z) \]
\[ + 2g_m^{(i)[m,p]}(\tau)\theta_{m,m}(\tau, z) \] (9.1b)

The modular transformation properties of \( G^{(1)[m,p]}(\tau, z) \) are obtained immediately from (9.1a) and Lemma 8.2 and Lemma 8.3 as follows:

**Proposition 9.1.** Let \( m \in \mathbb{N} \) and \( p \in \mathbb{Z} \geq 0 \) such that \( 0 \leq p \leq 2m \). Then

1) \[ G^{(1)[m,p]}(-1, \frac{z}{\tau}, \frac{z}{\tau}) = (-i\tau)^2 \sqrt{2m + 1} e^{\frac{\pi m}{2m+1} z^2} \sum_{p' = 0}^{2m} e^{-\frac{2\pi i}{2m+1} pp'} G^{(1)[m,p']}(\tau, z) \] (9.2a)

2) \[ G^{(1)[m,p]}(\tau + 1, z) = e^{\frac{\pi i}{2m+1} p^2 + \frac{\pi i}{4}} G^{(2)[m,p]}(\tau, z) \] (9.2b)

The formula (9.2a) is rewritten as follows:

**Proposition 9.2.** Let \( m \in \mathbb{N} \) and \( \ell \in \mathbb{Z} \geq 0 \) such that \( 0 \leq \ell \leq 2m \). Then

\[ \sum_{p = 0}^{2m} e^{2\pi i\ell p} G^{(1)[m,p]} (-1, \frac{z}{\tau}) = (-i\tau)^2 \sqrt{2m + 1} e^{\frac{\pi m}{2m+1} z^2} \sum_{p' = 0}^{2m} e^{-\frac{2\pi i}{2m+1} pp'} G^{(1)[m,p']}(\tau, z) \] (9.3)

**Proof.** Applying \( \sum_{p = 0}^{2m} e^{2\pi i\ell p} \) to (9.2a), we have

\[ \sum_{p = 0}^{2m} e^{2\pi i\ell p} G^{(1)[m,p]} (-1, \frac{z}{\tau}) \]
\[ = \sum_{p = 0}^{2m} e^{2\pi i\ell p} (-i\tau)^2\sqrt{2m + 1} e^{\frac{\pi m}{2m+1} z^2} \sum_{p' = 0}^{2m} e^{-\frac{2\pi i}{2m+1} pp'} G^{(1)[m,p']}(\tau, z) \]
\[ = (-i\tau)^2\sqrt{2m + 1} e^{\frac{\pi m}{2m+1} z^2} \sum_{p' = 0}^{2m} e^{2\pi i\ell p'} e^{-\frac{2\pi i}{2m+1} pp'} G^{(1)[m,p']}(\tau, z) \]
\[ = (-i\tau)^2\sqrt{2m + 1} e^{\frac{\pi m}{2m+1} z^2} G^{(1)[m,\ell]}(\tau, z), \quad \text{proving Proposition 9.2} \]

9.2 Modular transformation of \( g_k^{(i)[m,p]}(\tau) \)

To compute modular transformation of \( g_k^{(i)[m,p]}(\tau) \), we use the following formulas which are obtained easily from Lemmas 1.3 and 1.4 in [117].

**Note 9.1.** Let \( m \in \mathbb{N} \) and \( k \in \mathbb{Z} \). Then
1) \[ \left[ \theta_{k,m} + \theta_{-k,m} \right] \left( -\frac{1}{\tau}, \frac{z}{\tau} \right) = (-i\tau)^{\frac{1}{2}} e^{\frac{\pi im}{2\tau} z^2} \sum_{j \in \mathbb{Z}/2m\mathbb{Z}} \cos \frac{\pi j k}{m} \left[ \theta_{j,m} + \theta_{-j,m} \right] (\tau, z) \]

\[ = (-i\tau)^{\frac{1}{2}} \sqrt{\frac{2}{m}} e^{\frac{\pi im}{2\tau} z^2} \times \left\{ \sum_{0<j<m} \cos \frac{\pi j k}{m} \left[ \theta_{j,m} + \theta_{-j,m} \right] (\tau, z) + \theta_{0,m}(\tau, z) + (-1)^k \theta_{m,m}(\tau, z) \right\} \]

(ii) \[ \theta_{0,m} \left( -\frac{1}{\tau}, \frac{z}{\tau} \right) = (-i\tau)^{\frac{1}{2}} e^{\frac{\pi im}{2\tau} z^2} \sum_{0<j<m} \left\{ \theta_{j,m} + \theta_{-j,m} \right\} (\tau, z) + \theta_{0,m}(\tau, z) + \theta_{m,m}(\tau, z) \]

(iii) \[ \theta_{m,m} \left( -\frac{1}{\tau}, \frac{z}{\tau} \right) = (-i\tau)^{\frac{1}{2}} e^{\frac{\pi im}{2\tau} z^2} \sum_{0<j<m} \left\{ \theta_{j,m} + \theta_{-j,m} \right\} (\tau, z) + \theta_{0,m}(\tau, z) + \theta_{m,m}(\tau, z) \]

2) \[ \left[ \theta_{k,m} + \theta_{-k,m} \right] (\tau + 1, z) = e^{\frac{\pi i k^2}{2m}} \left[ \theta_{k,m} + \theta_{-k,m} \right] (\tau, z) \]

(ii) \[ \theta_{0,m}(\tau + 1, z) = \theta_{0,m}(\tau, z) \]

(iii) \[ \theta_{m,m}(\tau + 1, z) = e^{\frac{\pi i m}{2}} \theta_{m,m}(\tau, z) \]

The relation between modular transformation properties of \( G^{(i)[m,p]}(\tau, z) \) and those of \( g_{k}^{(i)[m,p]}(\tau) \), for \( i \in \{1, 2\} \), is obtained by using the above formulas as follows:

**Lemma 9.1.** Let \( m \in \mathbb{N} \) and \( p \in \mathbb{Z} \) and \( i \in \{1, 2\} \) Then

1) \[ G^{(i)[m,p]} \left( -\frac{1}{\tau}, \frac{z}{\tau} \right) = (-i\tau)^{\frac{1}{2}} \sqrt{\frac{2}{m}} e^{\frac{\pi im}{2\tau} z^2} \left( \sum_{k \in \mathbb{Z}} g_{k}^{(i)[m,p]} \left( -\frac{1}{\tau} \right) \left\{ \sum_{j \in \mathbb{Z}} \cos \frac{\pi j k}{m} \left[ \theta_{j,m} + \theta_{-j,m} \right] (\tau, z) + \theta_{0,m}(\tau, z) + (-1)^k \theta_{m,m}(\tau, z) \right\} \right. \]

\[ + \left. g_{0}^{(i)[m,p]} \left( -\frac{1}{\tau} \right) \left\{ \sum_{j \in \mathbb{Z}} \left[ \theta_{j,m} + \theta_{-j,m} \right] (\tau, z) + \theta_{0,m}(\tau, z) + \theta_{m,m}(\tau, z) \right\} \right) \]

\[ + \left. g_{m}^{(i)[m,p]} \left( -\frac{1}{\tau} \right) \left\{ \sum_{j \in \mathbb{Z}} \left( -\frac{1}{\tau} \right) \left[ \theta_{j,m} + \theta_{-j,m} \right] (\tau, z) + \theta_{0,m}(\tau, z) + (-1)^m \theta_{m,m}(\tau, z) \right\} \right) \]

(9.4a)
2) \( G^{(i)}[m,p]^* (\tau + 1, z) = \sum_{k \in \mathbb{Z}} e^{\frac{\pi i k^2}{2m}} g_k^{(i)}[m,p]^* (\tau + 1) [\theta_{k,m} + \theta_{-k,m}] (\tau, z) + 2 g_0^{(i)}[m,p]^* (\tau + 1) \theta_{0,m}(\tau, z) + 2 e^{\frac{\pi i m}{2}} g_m^{(i)}[m,p]^* (\tau + 1) \theta_{m,m}(\tau, z) \) (9.4b)

Proof. By (9.1b) and Note 9.1, we have

\[
G^{(i)}[m,p]^* \left( -\frac{1}{\tau}, \frac{z}{\tau} \right) = \sum_{k \in \mathbb{Z}} g_k^{(i)}[m,p]^* \left( -\frac{1}{\tau} \right) [\theta_{k,m} + \theta_{-k,m}] \left( -\frac{1}{\tau}, \frac{z}{\tau} \right)
+ 2 g_0^{(i)}[m,p]^* \left( -\frac{1}{\tau} \right) \theta_{0,m} \left( -\frac{1}{\tau}, \frac{z}{\tau} \right) + 2 g_m^{(i)}[m,p]^* \left( -\frac{1}{\tau} \right) \theta_{m,m} \left( -\frac{1}{\tau}, \frac{z}{\tau} \right)
= (-i\tau)^2 \sqrt{\frac{2}{m}} e^{\frac{\pi i m}{2} z^2} \left( \sum_{k \in \mathbb{Z}} g_k^{(i)}[m,p]^* \left( -\frac{1}{\tau} \right) \sum_{j \in \mathbb{Z}, 0<j<k} \cos \frac{\pi j \tau k}{m} [\theta_{j,m} + \theta_{-j,m}] (\tau, z) + \theta_{0,m} (\tau, z) + (-1)^k \theta_{m,m}(\tau, z) \right)
+ g_0^{(i)}[m,p]^* \left( -\frac{1}{\tau} \right) \sum_{j \in \mathbb{Z}, 0<j<m} [\theta_{j,m} + \theta_{-j,m}] (\tau, z) + \theta_{0,m}(\tau, z) + \theta_{m,m}(\tau, z)
+ g_m^{(i)}[m,p]^* \left( -\frac{1}{\tau} \right) \sum_{j \in \mathbb{Z}, 0<j<m} (-1)^j [\theta_{j,m} + \theta_{-j,m}] (\tau, z) + \theta_{0,m}(\tau, z) + (-1)^m \theta_{m,m}(\tau, z) \right)
\]
and

\[
G^{(i)}[m,p]^* (\tau + 1, z) = \sum_{k \in \mathbb{Z}} g_k^{(i)}[m,p]^* (\tau + 1) [\theta_{k,m} + \theta_{-k,m}] (\tau + 1, z) e^{\frac{\pi i k^2}{2m}} [\theta_{k,m} + \theta_{-k,m}] (\tau, z)
+ 2 g_0^{(i)}[m,p]^* (\tau + 1) \theta_{0,m}(\tau + 1, z) + 2 g_m^{(i)}[m,p]^* (\tau + 1) \theta_{m,m}(\tau + 1, z)
+ 2 g_0^{(i)}[m,p]^* (\tau + 1) \theta_{0,m}(\tau, z) + 2 e^{\frac{\pi i m}{2}} g_m^{(i)}[m,p]^* (\tau + 1) \theta_{m,m}(\tau, z)
\]

proving Lemma 9.1.
Proposition 9.3. Let \( m \in \mathbb{N} \) and \( p, k \in \mathbb{Z} \) such that \( 0 \leq p \leq 2m \) and \( 0 \leq k \leq m \). Then

1) \( S \)-transformation of \( g_k^{(1)[m,p]}(\tau) \) are as follows:

\[
\begin{align*}
(i) \quad & g_j^{(1)[m,p]} \left( -\frac{1}{\tau} \right) = \left\{ \begin{array}{ll}
-\frac{i\tau}{\sqrt{2m(m+1/2)}} \sum_{p'=0}^{2m} \sum_{k \in \mathbb{Z}} e^{i\tau p'} \cos \frac{\pi j k}{m} g_k^{(1)[m,p']}(\tau) & (0 < j < m) \\
+ \frac{i\tau}{\sqrt{2m(m+1/2)}} \sum_{p'=0}^{2m} e^{i\tau p'} \left\{ g_0^{(1)[m,p']}(\tau) + (-1)^j g_m^{(1)[m,p']}(\tau) \right\} & (0 < j < m) \\
\end{array} \right. \\
(ii) \quad & g_0^{(1)[m,p]} \left( -\frac{1}{\tau} \right) = \left\{ \begin{array}{ll}
-\frac{i\tau}{2\sqrt{2m(m+1/2)}} \sum_{k \in \mathbb{Z}} e^{i\tau p'} g_k^{(1)[m,p']}(\tau) & (0 < k < m) \\
+ \frac{i\tau}{2\sqrt{2m(m+1/2)}} \sum_{p'=0}^{2m} e^{i\tau p'} \left\{ g_0^{(1)[m,p']}(\tau) + g_m^{(1)[m,p']}(\tau) \right\} & (0 < k < m) \\
\end{array} \right. \\
(iii) \quad & g_m^{(1)[m,p]} \left( -\frac{1}{\tau} \right) = \left\{ \begin{array}{ll}
-\frac{i\tau}{2\sqrt{2m(m+1/2)}} \sum_{k \in \mathbb{Z}} (-1)^k e^{i\tau p'} g_k^{(1)[m,p']}(\tau) & (0 < k < m) \\
+ \frac{i\tau}{2\sqrt{2m(m+1/2)}} \sum_{p'=0}^{2m} e^{i\tau p'} \left\{ g_0^{(1)[m,p']}(\tau) + (-1)^m g_m^{(1)[m,p']}(\tau) \right\} & (0 < k < m) \\
\end{array} \right. \\
\end{align*}
\]

2) \( T \)-transformation of \( g_k^{(1)[m,p]}(\tau) \) are as follows:

\[
\begin{align*}
(i) \quad & g_j^{(1)[m,p]}(\tau+1) = e^{i\tau p^2 + i\tau - \frac{\pi j^2}{2m}} g_j^{(2)[m,p]}(\tau) & (0 < j < m) \\
(ii) \quad & g_0^{(1)[m,p]}(\tau+1) = e^{i\tau p^2 + i\tau} g_0^{(2)[m,p]}(\tau) \\
(iii) \quad & g_m^{(1)[m,p]}(\tau+1) = e^{i\tau p^2 + i\tau - \frac{\pi m^2}{2m}} g_m^{(2)[m,p]}(\tau) \\
\end{align*}
\]

Proof. Substituting (9.1b) and (9.4a) into (9.3), we have

\[
\begin{align*}
& \sum_{p=0}^{2m} e^{2\pi i \tau p^2} \left(-\frac{i\tau}{\sqrt{2m}}\right)^{-\tau} \left( \begin{array}{c}
\sum_{k \in \mathbb{Z}} g_k^{(1)[m,p]} \left( -\frac{1}{\tau} \right) \left\{ \begin{array}{ll}
\sum_{j \in \mathbb{Z}} \cos \frac{\pi j k}{m} \left[ \theta_{j,m} + \theta_{-j,m} \right](\tau, z) + \theta_{0,m}(\tau, z) + (-1)^k \theta_{m,m}(\tau, z) \\
+ g_0^{(1)[m,p]} \left( -\frac{1}{\tau} \right) \left\{ \begin{array}{ll}
\sum_{j \in \mathbb{Z}} \left[ \theta_{j,m} + \theta_{-j,m} \right](\tau, z) + \theta_{0,m}(\tau, z) + \theta_{m,m}(\tau, z) \\
\end{array} \right. \\
\end{array} \right) \\
\end{align*}
\]
\[ + g_m^{(1)[m,p]} \left( -\frac{1}{\tau} \right) \left\{ \sum_{j \in \mathbb{Z},\; 0<j<m} (-1)^j [\theta_{j,m} + \theta_{-j,m}] (\tau, z) + \theta_{0,m}(\tau, z) + (-1)^m \theta_{m,m}(\tau, z) \right\} \]

\[ = (-i\tau)^{\frac{3}{2}} \sqrt{2m+1} \ e^{i\pi m z^2} \left\{ \sum_{j \in \mathbb{Z},\; 0<j<m} g_j^{(1)[m,\ell]} (\tau) [\theta_{j,m} + \theta_{-j,m}] (\tau, z) \right. \]

\[ + 2 g_0^{(1)[m,\ell]} (\tau) \theta_{0,m}(\tau, z) + 2 g_m^{(1)[m,\ell]} (\tau) \theta_{m,m}(\tau, z) \right\} \]

namely

\[ (-i\tau)^{-1} \sqrt{\frac{2}{m(2m+1)}} \sum_{p=0}^{2m} e^{\frac{2\pi i}{m+1} \ell p} \sum_{k \in \mathbb{Z},\; 0<k<m} g_k^{(1)[m,p]} \left( -\frac{1}{\tau} \right) \]

\[ \times \left\{ \sum_{j \in \mathbb{Z},\; 0<j<m} \cos \frac{\pi j k}{m} [\theta_{j,m} + \theta_{-j,m}] (\tau, z) + \theta_{0,m}(\tau, z) + (-1)^k \theta_{m,m}(\tau, z) \right\} \]

\[ + (-i\tau)^{-1} \sqrt{\frac{2}{m(2m+1)}} \sum_{p=0}^{2m} e^{\frac{2\pi i}{m+1} \ell p} g_0^{(1)[m,p]} \left( -\frac{1}{\tau} \right) \]

\[ \times \left\{ \sum_{j \in \mathbb{Z},\; 0<j<m} \left[ \theta_{j,m} + \theta_{-j,m} \right] (\tau, z) + \theta_{0,m}(\tau, z) + \theta_{m,m}(\tau, z) \right\} \]

\[ + (-i\tau)^{-1} \sqrt{\frac{2}{m(2m+1)}} \sum_{p=0}^{2m} e^{\frac{2\pi i}{m+1} \ell p} g_m^{(1)[m,p]} \left( -\frac{1}{\tau} \right) \]

\[ \times \left\{ \sum_{j \in \mathbb{Z},\; 0<j<m} (-1)^j \left[ \theta_{j,m} + \theta_{-j,m} \right] (\tau, z) + \theta_{0,m}(\tau, z) + (-1)^m \theta_{m,m}(\tau, z) \right\} \]

\[ = \sum_{j \in \mathbb{Z},\; 0<j<m} g_j^{(1)[m,\ell]} (\tau) [\theta_{j,m} + \theta_{-j,m}] (\tau, z) + 2 g_0^{(1)[m,\ell]} (\tau) \theta_{0,m}(\tau, z) + 2 g_m^{(1)[m,\ell]} (\tau) \theta_{m,m}(\tau, z) \]

Comparing the coefficients of \([\theta_{j,m} + \theta_{-j,m}] (\tau, z),\; \theta_{0,m}(\tau, z)\) and \(\theta_{m,m}(\tau, z)\) in this equation, we have

\[ g_j^{(1)[m,\ell]} (\tau) = \frac{(-i\tau)^{-\frac{3}{2}}}{\sqrt{m(m+\frac{1}{2})}} \sum_{p=0}^{2m} \sum_{k \in \mathbb{Z},\; 0<k<m} e^{\frac{2\pi i}{m+1} \ell p} \cos \frac{\pi j k}{m} g_k^{(1)[m,p]} \left( -\frac{1}{\tau} \right) \]
2) Substituting \((9.1b)\) and \((9.4b)\) into \((9.2b)\), we have

\[
2 \, g_0^{(1)}[m,l] \ast (\tau) = \frac{(-i\tau)^{-1}}{\sqrt{m(m+\frac{1}{2})}} \sum_{p=0}^{2m} \sum_{k \in \mathbb{Z}} e^{\frac{2\pi i}{2m+1}p\ell} \left( g_0^{(1)}[m,l] \ast \left( -\frac{1}{\tau} \right) + (-1)^j \, g_0^{(1)}[m,l] \ast \left( -\frac{1}{\tau} \right) \right)
\]

Then, replacing \(\tau\) with \(-\frac{1}{\tau}\) in these equations, we obtain the formulas in the claim 1).

2) Substituting \((9.1b)\) and \((9.4b)\) into \((9.2b)\), we have

\[
\sum_{j \in \mathbb{Z}} e^{\frac{\pi i}{2m} j^2} \, g_j^{(1)}[m,p] \ast (\tau + 1) [\theta_{j,m} + \theta_{-j,m}] (\tau, z)
\]

\[
+ \, 2 \, g_0^{(1)}[m,p] \ast (\tau + 1) \theta_{0,m}(\tau, z)
\]

\[
+ \, 2 e^{\frac{\pi i}{2m+1}p\ell} \, g_0^{(1)}[m,p] \ast (\tau + 1) \, \theta_{m,m}(\tau, z)
\]

\[
= \, e^{\frac{\pi i \ell}{2m+1}} \left\{ \sum_{j \in \mathbb{Z}} g_j^{(2)}[m,p] \ast (\tau) [\theta_{j,m} + \theta_{-j,m}] (\tau, z)
\right. 
\]

\[
+ \, 2 \, g_0^{(2)}[m,p] \ast (\tau) \, \theta_{0,m}(\tau, z)
\]

\[
+ \, 2 g_0^{(2)}[m,p] \ast (\tau) \, \theta_{m,m}(\tau, z)
\}
\]

Comparing the coefficients of \([\theta_{j,m} + \theta_{-j,m}] (\tau, z)\), \(\theta_{0,m}(\tau, z)\) and \(\theta_{m,m}(\tau, z)\) in this equation, we have

\[
\left\{ \begin{array}{l}
\left( \frac{\pi i \ell}{2m+1} \right)^2 \, g_j^{(1)}[m,p] \ast (\tau + 1) = e^{\frac{\pi i \ell}{2m+1}p\ell + \frac{\pi i}{4}} \, g_j^{(2)}[m,p] \ast (\tau) \\
\left( \frac{\pi i \ell}{2m+1} \right)^2 \, g_0^{(1)}[m,p] \ast (\tau + 1) = e^{\frac{\pi i \ell}{2m+1}p\ell + \frac{\pi i}{4}} \, g_0^{(2)}[m,p] \ast (\tau) \\
\left( \frac{\pi i \ell}{2m+1} \right)^2 \, g_0^{(1)}[m,p] \ast (\tau + 1) = e^{\frac{\pi i \ell}{2m+1}p\ell + \frac{\pi i}{4}} \, g_0^{(2)}[m,p] \ast (\tau) \\
\end{array} \right.
\]
namely
\[
\begin{align*}
\left\{ \begin{array}{l}
g_j^{(1)[m,p]}(\tau + 1) &= e^{\pi i \frac{m}{2(m+1)}} p^2 + \pi i \frac{1}{4} - \pi i \frac{1}{2(m+1)}^2 g_j^{(2)[m,p]}(\tau) \\
g_0^{(1)[m,p]}(\tau + 1) &= e^{\pi i \frac{m}{2(m+1)}} p^2 + \pi i \frac{1}{4} g_0^{(2)[m,p]}(\tau) \\
g_m^{(1)[m,p]}(\tau + 1) &= e^{\pi i \frac{m}{2(m+1)}} p^2 + \pi i \frac{1}{4} - \pi i \frac{m}{2(m+1)} g_m^{(2)[m,p]}(\tau)
\end{array} \right. \\
\end{align*}
\]
proving the claim 2).

9.3 Explicit formula for \( g_k^{(i)[m,p]}(\tau) \)

The explicit formulas for \( G^{(i)[m,p]}(\tau, z) \) \((i \in \{1, 2\})\) are obtained from Proposition 7.1 and the formulas (8.1a), (8.1b), (8.2a), (8.2b) and (9.1a) as follows:

\[
G^{(1)[m,p]}(\tau, z) = \\
- \left[ \sum_{j, r \in \frac{1}{2}Z_{\text{odd}}} \left( \sum_{k \in Z} \frac{(-1)^j - \frac{1}{2} + k}{m} \left( \sum_{j \leq r < 0} q^{m \left( j + \frac{2(m+1)}{2m+1} \right)^2} \right) \right) \right] \\
- \left[ \sum_{j, r \in \frac{1}{2}Z_{\text{odd}}} \left( \sum_{k \in Z} \frac{(-1)^j - \frac{1}{2} + k}{m} \left( \sum_{j \leq r < 0} q^{m \left( j + \frac{2(m+1)}{2m+1} \right)^2} \right) \right) \right] \\
- \frac{\theta_{-2mp+m+\frac{1}{2},m+\frac{1}{2}}^{(-)}(\tau, 0)}{2} \sum_{r \in Z} q^{m(r-\frac{1}{2})^2} \theta_0(\tau, z) \\
+ 2 \frac{\theta_{-2mp+m+\frac{1}{2},m+\frac{1}{2}}^{(-)}(\tau, 0)}{2} \sum_{r \in Z} q^{-mr^2} \omega_m(\tau, z) \\
- (-1)^m \frac{\theta_{-2mp+m+\frac{1}{2},m+\frac{1}{2}}^{(-)}(\tau, 0)}{2} \sum_{r \in Z} q^{-mr^2} \omega_m(\tau, z)
\]

\[
G^{(2)[m,p]}(\tau, z) = \\
- \left( \sum_{j, r \in \frac{1}{2}Z_{\text{odd}}} \left( \sum_{k \in Z} \frac{(-1)^j - \frac{1}{2} + k}{m} \left( \sum_{j \leq r < 0} q^{m \left( j + \frac{2(m+1)}{2m+1} \right)^2} \right) \right) \right) \\
- \left[ \sum_{j, r \in \frac{1}{2}Z_{\text{odd}}} \left( \sum_{k \in Z} \frac{(-1)^j - \frac{1}{2} + k}{m} \left( \sum_{j \leq r < 0} q^{m \left( j + \frac{2(m+1)}{2m+1} \right)^2} \right) \right) \right] \\
- \frac{\theta_{-2mp+m+\frac{1}{2},m+\frac{1}{2}}^{(-)}(\tau, 0)}{2} \sum_{r \in Z} q^{m(r-\frac{1}{2})^2} \theta_0(\tau, z) \\
+ 2 \frac{\theta_{-2mp+m+\frac{1}{2},m+\frac{1}{2}}^{(-)}(\tau, 0)}{2} \sum_{r \in Z} q^{-mr^2} \omega_m(\tau, z) \\
- (-1)^m \frac{\theta_{-2mp+m+\frac{1}{2},m+\frac{1}{2}}^{(-)}(\tau, 0)}{2} \sum_{r \in Z} q^{-mr^2} \omega_m(\tau, z)
\]
\[ - (-1)^p \left[ \sum_{j, r \in \frac{1}{2} \mathbb{Z}_{\text{odd}}}^{0 \leq r \leq j} - \sum_{j, r \in \frac{1}{2} \mathbb{Z}_{\text{odd}}}^{j \leq r < 0} \right] \sum_{k \in \mathbb{Z}}^{0 \leq k \leq m} (-1)^{j-\frac{1}{4}} q^{m+\frac{1}{2}}(j+\frac{2m}{2m+k})^2 \times q^{-\frac{1}{4m}(2mr-k+2mp)^2} \left[ \theta_{k,m} + \theta_{-k,m} \right](\tau, z) \]

\[ - (-1)^p \theta^{(-)}_{2mp+m+\frac{1}{2}, m+\frac{1}{2}}(\tau, 0) \sum_{r \in \mathbb{Z}}^{0 \leq r \leq p} q^{-\frac{1}{4m}(2r-1+k)^2} \left[ \theta_{k,m} + \theta_{-k,m} \right](\tau, z) \]

\[ + 2 (-1)^p \theta^{(-)}_{2mp+m+\frac{1}{2}, m+\frac{1}{2}}(\tau, 0) \sum_{r \in \mathbb{Z}}^{0 \leq r \leq p-1} q^{-m(r+\frac{1}{2})^2} \theta_{0,m}(\tau, z) \]

Then the explicit formulas for \( g_{k}^{(i)[m,p] \ast}(\tau) \) follow immediately from (9.1b) and the above formulas as follows:

**Proposition 9.4.** Let \( m \in \mathbb{N} \) and \( p, k \in \mathbb{Z} \) such that \( 0 \leq p \leq 2m \) and \( 0 \leq k \leq m \). Then \( g_{k}^{(i)[m,p] \ast}(\tau) \) \((i \in \{1, 2\})\) are as follows:

1) (i) \( g_{k}^{(1)[m,p] \ast}(\tau) \)

\[ = - \left[ \sum_{j, r \in \frac{1}{2} \mathbb{Z}_{\text{odd}}}^{0 \leq r \leq j} - \sum_{j, r \in \frac{1}{2} \mathbb{Z}_{\text{odd}}}^{j \leq r < 0} \right] \sum_{k \in \mathbb{Z}}^{0 \leq k \leq m} (-1)^{j-\frac{1}{4}+k} q^{m+\frac{1}{2}}(j+\frac{2m}{2m+k})^2 q^{-\frac{1}{4m}(2mr-k+2mp)^2} \]

\[ - \theta^{(-)}_{2mp+m+\frac{1}{2}, m+\frac{1}{2}}(\tau, 0) \sum_{r \in \mathbb{Z}}^{0 \leq r \leq p} q^{-\frac{1}{4m}(2r-1+k)^2} \left( \sum_{r \in \mathbb{Z}}^{0 \leq r \leq p-1} (-1)^k q^{-\frac{1}{4m}(2r-1+k)^2} \right) \]

(ii) \( 2 g_{0}^{(1)[m,p] \ast}(\tau) \)

\[ = - 2 \left[ \sum_{j, r \in \frac{1}{2} \mathbb{Z}_{\text{odd}}}^{0 \leq r \leq j} - \sum_{j, r \in \frac{1}{2} \mathbb{Z}_{\text{odd}}}^{j \leq r < 0} \right] \sum_{k \in \mathbb{Z}}^{0 \leq k \leq m} (-1)^{j-\frac{1}{4}+k} q^{m+\frac{1}{2}}(j+\frac{2m}{2m+k})^2 q^{-\frac{1}{4m}(2mr-k+2mp)^2} \]

\[ - 2 \theta^{(-)}_{2mp+m+\frac{1}{2}, m+\frac{1}{2}}(\tau, 0) \sum_{r \in \mathbb{Z}}^{0 \leq r \leq p-1} q^{-m(r+\frac{1}{2})^2} \]

\[ - 2 \theta^{(-)}_{2mp+m+\frac{1}{2}, m+\frac{1}{2}}(\tau, 0) \sum_{r \in \mathbb{Z}}^{0 \leq r \leq p-1} q^{-m(r+\frac{1}{2})^2} \]
(iii) \(2g_{m}^{(1)[m,p]}(\tau)\)

\[
= -2(-1)^{m} \left[ \sum_{j, r \in \mathbb{Z}_{\text{odd}}} - \sum_{j, r \in \mathbb{Z}_{\text{odd}}} \right] (\tau)_{m, \frac{1}{2}} q^{(m+\frac{1}{2})(j+\frac{2pm}{2m+1})^2} - m(r+p-\frac{1}{2})^2
\]

\[-(-1)^{m} \theta^{(-)}_{m} (\tau, 0) \sum_{r \in \mathbb{Z}} q^{-mr^2} \]

2. (i) \(g_{k}^{(2)[m,p]}(\tau)\) (0 < k < m)

\[
= -(-1)^{p} \left[ \sum_{j, r \in \mathbb{Z}_{\text{odd}}} - \sum_{j, r \in \mathbb{Z}_{\text{odd}}} \right] (\tau)_{m, \frac{1}{2}} q^{(m+\frac{1}{2})(j+\frac{2pm}{2m+1})^2} - \frac{1}{4m} [2mr+k+2mp]^2
\]

\[-(-1)^{p} \theta^{(-)}_{2mp+m+\frac{1}{2}, \frac{1}{2}} (\tau, 0) \sum_{r \in \mathbb{Z}} q^{-\frac{1}{4m} (m(2r-1)+k)^2} \]

(ii) \(2g_{0}^{(2)[m,p]}(\tau)\)

\[
= -2(-1)^{p} \left[ \sum_{j, r \in \mathbb{Z}_{\text{odd}}} - \sum_{j, r \in \mathbb{Z}_{\text{odd}}} \right] (\tau)_{m, \frac{1}{2}} q^{(m+\frac{1}{2})(j+\frac{2pm}{2m+1})^2} - m(r+p)^2
\]

\[-2(-1)^{p} \theta^{(-)}_{2mp+m+\frac{1}{2}, \frac{1}{2}} (\tau, 0) \sum_{r \in \mathbb{Z}} q^{-m(r+\frac{1}{2})^2} \]

(iii) \(2g_{m}^{(2)[m,p]}(\tau)\)

\[
= -2(-1)^{p} \left[ \sum_{j, r \in \mathbb{Z}_{\text{odd}}} - \sum_{j, r \in \mathbb{Z}_{\text{odd}}} \right] (\tau)_{m, \frac{1}{2}} q^{(m+\frac{1}{2})(j+\frac{2pm}{2m+1})^2} - m(r+p-\frac{1}{2})^2
\]

\[-(-1)^{p} \theta^{(-)}_{2mp+m+\frac{1}{2}, \frac{1}{2}} (\tau, 0) \sum_{r \in \mathbb{Z}} q^{-mr^2} \]

\[-(-1)^{p} \theta^{(-)}_{2mp+m+\frac{1}{2}, \frac{1}{2}} (\tau, 0) \sum_{r \in \mathbb{Z}} q^{-mr^2} \]

\[-(-1)^{p} \theta^{(-)}_{2mp+m+\frac{1}{2}, \frac{1}{2}} (\tau, 0) \sum_{r \in \mathbb{Z}} q^{-mr^2} \]
10 Indefinite modular forms $g_j^{[m,p]}(\tau)$

By Proposition 9.4 we observe that the following formula

$$g_j^{(2)[m,p]}(\tau) = (-1)^{j+p} g_j^{(1)[m,p]}(\tau)$$  \hspace{1cm} (10.1)

holds for all $m \in \mathbb{N}$ and $p, j \in \mathbb{Z}$ such that $0 \leq p \leq 2m$ and $0 \leq j \leq m$. Then, simplifying the notation, we define the functions $g_j^{[m,p]}(\tau)$ by

$$g_j^{[m,p]}(\tau) = g_j^{(1)[m,p]}(\tau)$$  \hspace{1cm} (10.2)

Then the modular transformation formulas for these functions $g_j^{[m,p]}(\tau)$ are obtained from Proposition 9.4 as follows:

**Proposition 10.1.** Let $m \in \mathbb{N}$ and $p, j \in \mathbb{Z}$ such that $0 \leq p \leq 2m$ and $0 \leq j \leq m$. Then

1. $g_j^{[m,p]}(\tau) = g_j^{[m,p]}(\tau)$

2. $g_j^{[m,p]}(\tau + 1) = e^{\frac{\pi i}{2m+1}(p+\frac{2m+1}{2})^2 - \frac{\pi i}{2m}(j+m)^2} g_j^{[m,p]}(\tau)$
11 An example \sim the case \( m = 1 \)

The functions which take place in the case \( m = 1 \) are \( g_k^{[1,p]^*}(\tau) \) \((p = 0, 1, 2; \ k = 0, 1)\) and they are, by Proposition 9.4, as follows:

\[
2 g_0^{[1,p]^*}(\tau) = -2 \left[ \sum_{j, r \in \frac{1}{2}\mathbb{Z}_{\text{odd}}} - \sum_{j, r \in \frac{1}{2}\mathbb{Z}_{\text{odd}}} \right] (-1)^j - \frac{1}{2} q^{\frac{3}{2}(j + \frac{2p}{3})^2 - (r+p)^2} \\
- 2 \theta_{2p+\frac{3}{2}, \frac{3}{2}}^{(-)}(\tau, 0) \sum_{r \in \mathbb{Z}} q^{-(r+\frac{1}{2})^2}
\]

(11.1a)

\[
2 g_1^{[1,p]^*}(\tau) = 2 \left[ \sum_{j, r \in \frac{1}{2}\mathbb{Z}_{\text{odd}}} - \sum_{j, r \in \frac{1}{2}\mathbb{Z}_{\text{odd}}} \right] (-1)^j - \frac{1}{2} q^{\frac{3}{2}(j + \frac{2p}{3})^2 - (r+p-\frac{1}{2})^2} \\
+ \theta_{2p+\frac{3}{2}, \frac{3}{2}}^{(-)}(\tau, 0) \sum_{r \in \mathbb{Z}} q^{-r^2}
\]

(11.1b)

Putting \( \frac{j - \frac{1}{2}}{2} = j' \) \( \frac{r - \frac{1}{2}}{2} = r' \), the above formulas are rewritten as follows:

\[
g_0^{[1,p]^*}(\tau) = - \left[ \sum_{j', r' \in \mathbb{Z}} - \sum_{j', r' \in \mathbb{Z}} \right] (-1)^j' q^{\frac{3}{2}(j' + \frac{2p}{3})^2 - (r'+\frac{1}{2}+p)^2} \\
- \theta_{2p+\frac{3}{2}, \frac{3}{2}}^{(-)}(\tau, 0) \sum_{r \in \mathbb{Z}} q^{-(r+\frac{1}{2})^2}
\]

(11.2a)

\[
g_1^{[1,p]^*}(\tau) = \left[ \sum_{j', r' \in \mathbb{Z}} - \sum_{j', r' \in \mathbb{Z}} \right] (-1)^j' q^{\frac{3}{2}(j' + \frac{2}{3}(r'+\frac{1}{2}+p)^2 - (r'+p)^2} \\
+ \frac{1}{2} \theta_{2p+\frac{3}{2}, \frac{3}{2}}^{(-)}(\tau, 0) \sum_{r \in \mathbb{Z}} q^{-r^2}
\]

(11.2b)

These functions are written explicitly as follows:

**Note 11.1.**

1) \( g_0^{[1,0]^*}(\tau) = - \left[ \sum_{j, r \in \mathbb{Z}} - \sum_{j, r \in \mathbb{Z}} \right] (-1)^j q^{\frac{3}{2}(j + \frac{1}{2})^2 - (r+\frac{1}{2})^2} = - q^{\frac{3}{2}} + \ldots \)
Lemma 11.1. Define the functions $f_i(\tau)$ ($i = 0, 1, 2, 3$) by

\[
\begin{align*}
  f_0(\tau) & := -\frac{2g_1^{[1,0]}(\tau)}{\eta(\tau)^2}, \\
  f_1(\tau) & := -\frac{g_0^{[1,0]}(\tau)}{\eta(\tau)^2}, \\
  f_2(\tau) & := -\frac{g_1^{[1,0]}(\tau)}{\eta(\tau)^2}, \\
  f_3(\tau) & := \frac{2g_0^{[1,1]}(\tau)}{\eta(\tau)^2}.
\end{align*}
\] (11.3)

Then,
1) these functions \( f_i(\tau) \) satisfy the following \( S \)-transformation properties:

\[
\begin{align*}
(0) \quad f_0\left(-\frac{1}{\tau}\right) &= \frac{1}{\sqrt{6}} \left\{ f_0(\tau) + 2 f_1(\tau) + 2 f_2(\tau) + f_3(\tau) \right\} \\
(i) \quad f_1\left(-\frac{1}{\tau}\right) &= \frac{1}{\sqrt{6}} \left\{ f_0(\tau) + f_1(\tau) - f_2(\tau) - f_3(\tau) \right\} \\
(ii) \quad f_2\left(-\frac{1}{\tau}\right) &= \frac{1}{\sqrt{6}} \left\{ f_0(\tau) - f_1(\tau) - f_2(\tau) + f_3(\tau) \right\} \\
(iii) \quad f_3\left(-\frac{1}{\tau}\right) &= \frac{1}{\sqrt{6}} \left\{ f_0(\tau) - 2 f_1(\tau) + 2 f_2(\tau) - f_3(\tau) \right\}
\end{align*}
\]

2) the leading terms of \( f_i(\tau) \) are as follows:

\[
\begin{align*}
\begin{cases}
    f_0(\tau) &= q^{-\frac{1}{24}} + \ldots \\
    f_1(\tau) &= q^{\frac{1}{24}} + \ldots \\
    f_2(\tau) &= q^{\frac{3}{24}} + \ldots \\
    f_3(\tau) &= 2 q^{\frac{17}{24}} + \ldots
\end{cases}
\end{align*}
\]

Next we consider the Jacobi’s theta function

\[
\theta_{j,3}(\tau, z) = \sum_{n \in \mathbb{Z}} q^{3(n+\frac{j}{3})^2} e^{6\pi i (n+\frac{j}{3}) z}
\]

The \( S \)-transformation of \( \theta_{j,3}(\tau, 0) \) is computed by using Lemmas 1.2 and 1.3 in [17] as follows:

\[
\begin{align*}
(i) \quad \theta_{0,3}\left(-\frac{1}{\tau}, 0\right) &= \frac{(-i\tau)^{\frac{j}{6}}}{\sqrt{6}} \left\{ \theta_{0,3}(\tau, 0) + 2 \theta_{1,3}(\tau, 0) + 2 \theta_{2,3}(\tau, 0) + \theta_{3,3}(\tau, 0) \right\} \\
(ii) \quad \theta_{1,3}\left(-\frac{1}{\tau}, 0\right) &= \frac{(-i\tau)^{\frac{j}{6}}}{\sqrt{6}} \left\{ \theta_{0,3}(\tau, 0) + \theta_{1,3}(\tau, 0) - \theta_{2,3}(\tau, 0) - \theta_{3,3}(\tau, 0) \right\} \\
(iii) \quad \theta_{2,3}\left(-\frac{1}{\tau}, 0\right) &= \frac{(-i\tau)^{\frac{j}{6}}}{\sqrt{6}} \left\{ \theta_{0,3}(\tau, 0) - \theta_{1,3}(\tau, 0) - \theta_{2,3}(\tau, 0) + \theta_{3,3}(\tau, 0) \right\} \\
(iv) \quad \theta_{3,3}\left(-\frac{1}{\tau}, 0\right) &= \frac{(-i\tau)^{\frac{j}{6}}}{\sqrt{6}} \left\{ \theta_{0,3}(\tau, 0) - 2 \theta_{1,3}(\tau, 0) + 2 \theta_{2,3}(\tau, 0) - \theta_{3,3}(\tau, 0) \right\}
\end{align*}
\]

And the leading terms of \( \theta_{j,3}(\tau, 0) \) are

\[
\begin{align*}
\begin{cases}
    \theta_{0,3}(\tau, 0) &= q^0 + \ldots \\
    \theta_{1,3}(\tau, 0) &= q^{\frac{1}{12}} + \ldots \\
    \theta_{2,3}(\tau, 0) &= q^{\frac{1}{3}} + \ldots \\
    \theta_{3,3}(\tau, 0) &= 2 q^{\frac{3}{12}} + \ldots
\end{cases}
\end{align*}
\]

Then putting

\[
h_j(\tau) := \frac{\theta_{j,3}(\tau, 0)}{\eta(\tau)} \quad (j = 0, 1, 2, 3) \quad (11.4)
\]

we have

Lemma 11.2. 1) Functions \( h_j(\tau) \) satisfy the following \( S \)-transformation properties:
Proof. By Lemmas 11.1 and 11.2, we see that the functions
satisfy the same
Proposition 11.1.

\begin{align*}
(0) \quad h_0\left(-\frac{1}{\tau}\right) &= \frac{1}{\sqrt{6}} \left\{ h_0(\tau) + 2 h_1(\tau) + 2 h_2(\tau) + h_3(\tau) \right\} \\
(i) \quad h_1\left(-\frac{1}{\tau}\right) &= \frac{1}{\sqrt{6}} \left\{ h_0(\tau) + h_1(\tau) - h_2(\tau) - h_3(\tau) \right\} \\
(ii) \quad h_2\left(-\frac{1}{\tau}\right) &= \frac{1}{\sqrt{6}} \left\{ h_0(\tau) - h_1(\tau) - h_2(\tau) + h_3(\tau) \right\} \\
(iii) \quad h_3\left(-\frac{1}{\tau}\right) &= \frac{1}{\sqrt{6}} \left\{ h_0(\tau) - 2 h_1(\tau) + 2 h_2(\tau) - h_3(\tau) \right\} \\

\end{align*}

2) the leading terms of \( h_j(\tau) \) are as follows:

\[
\begin{align*}
 h_0(\tau) &= q^{-\frac{1}{24}} + \cdots \\
 h_1(\tau) &= q^{\frac{1}{24}} + \cdots \\
 h_2(\tau) &= q^{\frac{3}{24}} + \cdots \\
 h_3(\tau) &= 2 q^{\frac{17}{24}} + \cdots 
\end{align*}
\]

From these formulas, we obtain the following:

**Proposition 11.1.**

\[
\begin{align*}
(i) \quad \left[ \sum_{j,r \in \mathbb{Z}} \sum_{0 \leq r \leq j, j < r < 0} \right] (-1)^j q^{\frac{3}{2}(j+\frac{1}{2})^2 - (r+\frac{1}{2})^2} &= \eta(\tau) \theta_{1,3}(\tau, 0) \\
(ii) \quad \left[ \sum_{j,r \in \mathbb{Z}} \sum_{0 \leq r \leq j, j < r < 0} \right] (-1)^j q^{\frac{3}{2}(j+\frac{1}{2})^2 - r^2} &= \eta(\tau) \theta_{2,3}(\tau, 0) \\
(iii) \quad \left[ \sum_{j,r \in \mathbb{Z}} \sum_{0 \leq r < j, j \leq r < 0} \right] (-1)^j q^{\frac{3}{2}(j+\frac{1}{2})^2 - (r+\frac{1}{2})^2} &= \frac{1}{2} \eta(\tau) \theta_{3,3}(\tau, 0) \\
(iv) \quad \left[ \sum_{j,r \in \mathbb{Z}} \sum_{0 \leq r < j, j \leq r < 0} \right] (-1)^j q^{\frac{3}{2}(j+\frac{1}{2})^2 - r^2} &= \frac{1}{2} \left\{ \eta(\tau) \theta_{0,3}(\tau, 0) + \theta_{\frac{1}{2},\frac{1}{2}}^{(-)}(\tau, 0) \right\} 
\end{align*}
\]

**Proof.** By Lemmas 11.1 and 11.2 we see that the functions \( \{f_i(\tau)\}_{i=0,1,2,3} \) and \( \{h_i(\tau)\}_{i=0,1,2,3} \) satisfy the same S-transformation properties and have the same polar parts. Then, by Lemma 4.8 in [14], we have

\[ f_i(\tau) = h_i(\tau) \quad \text{for all } i, \]

namely

\[
\begin{align*}
 2 g_0^{[1,1]}(\tau) &= \eta(\tau) \theta_{3,3}(\tau, 0) \\
 -2 g_1^{[1,1]}(\tau) &= \eta(\tau) \theta_{0,3}(\tau, 0) \\
 -g_0^{[1,0]}(\tau) &= \eta(\tau) \theta_{1,3}(\tau, 0) \\
 g_1^{[1,0]}(\tau) &= \eta(\tau) \theta_{2,3}(\tau, 0) 
\end{align*}
\]

by (11.3) and (11.4). Rewriting \( g_k^{[1,j]}(\tau) \) by using Note 11.1 we obtain the formulas in Proposition 11.1. \( \square \)
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