A perturbation algorithm for the pointers of Franke-Gorini-Kossakowski-Lindblad-Sudarshan equation

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Abstract

This paper is devoted to the study of behavior of open quantum systems consistently based on the Franke-Gorini-Kossakowski-Lindblad-Sudarshan (FGKLS) equation which covers evolution in situations when decoherence can be distinguished. We focus on the quantum measurement operation which is determined by final stationary states of an open system - so called pointers. We find pointers by applying the FGKLS equation to asymptotically constant density matrix. In seeking pointers, we have been able to propose a perturbative scheme of calculation, if we take the interaction components with an environment to be weak. Thus, the Lindblad operators can be used in some way as expansion parameters for perturbation theory. The scheme we propose is different for the cases of non-degenerate and degenerate Hamiltonian. We illustrate our scheme by particular examples of quantum harmonic oscillator with spin in external magnetic field. The efficiency of the perturbation algorithm is demonstrated by its comparison with the exact solution.

Keywords: density matrix, Franke-Gorini-Kossakowski-Lindblad-Sudarshan equation, pointers, perturbation theory, decoherence.

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1 Introduction

In the last decade the interest to evolution of open quantum systems was growing with the development of quantum information technology. The evolution is accompanied by dissipation of quantum states and by decoherence, the process, in which pure quantum states transform into mixed ones, that contain classical probabilities, beside quantum ones. The quantum information contained in an initial state is getting partially lost in the environment. In studying such systems one may progress further in understanding of how the quantum world becomes the classical one we live in. See [1, 2, 3] for more details.

The examination of behavior of open quantum systems is based on the equation which covers a dynamical semigroup evolution when decoherence can be distinguished,

\[
\dot{\rho} = -i[H, \rho] + \sum_a L^{(a)} \rho L^{(a),\dagger} - \frac{1}{2} \left\{ \sum_a L^{(a),\dagger} L^{(a),\dagger} \rho \right\}
\]

(1)

where \(\rho\) is a density matrix of the system under study, \(\{L^{(a)}\}\) is a set of operators carrying information about an interaction of the system with an environment. \(\{\}\) denotes an anticommutator.

This equation was obtained independently by G. Lindblad [6] and V.A. Franke [4] as well as by V. Gorini, A. Kossakowski and E.C.G.Sudarshan [5] and we name it further on as the FGKLS equation.

For this exact form of equation there are several premises. When focusing on non-dissipative processes we assume that the density matrix in evolution in time must remain Hermitian, positive, and its trace must be equal to 1. It implicates that density matrix always sticks to its definition: it contains in itself a notion of non-negative probabilities of the realisation of some pure quantum states and an assumption that the entire probability stays the same. The opposite of the latter would mean a decay of the system or some other event of the leak of the total probability. However if the dissipation holds one could factor it out to proceed to conditional probabilities for conditional density matrices [11]. The reduced equation becomes nonlinear. Such situations are not considered in this work. If all the properties mentioned above are preserved, and we require the master equation to be linear, providing the state superposition, then the only possible form of equation is the FGKLG form (for this statement a more strict property - complete positivity - is necessary [7], [8]).

One could also consider the composite system (system of interest + environment) to obey the Liouville law of evolution in time with a Hamiltonian \(H\) (usually of the form \(H_S + H_E + H_{int}\)) associated to the whole system. Then after tracing out the environmental degrees of freedom the density matrix of the space \(H_S \otimes H_E\) is reduced to the density matrix on the subspace \(H_S\) and in the weak coupling limit the equation approaches the FGKLS form.
An example of open quantum system is one or a pair of optically active non-interacting atoms (or molecules) weakly coupled to a heat bath. This model was studied in [7], [8].

A variety of application areas exists for the FGKLS equation: from solid state physics and quantum optics[1] to high energy physics [13, 14], and even in the search for quantum mechanics breaking in vacuum due to tiny tracks of quantum gravity (see review in [15]).

Recently, the discussion of open quantum systems has been initiated by S. Weinberg, in regard to the problem of quantum measurement [9]. He considered a measuring apparatus as an environment that interferes into a quantum state of the system (particle, molecule, other). As a result, a system evolves, constantly interacting with an environment, in such a way as to produce the result of a measurement. An appropriate description of decoherence of a state of an open system turns out to be the density matrix instead of a usual state vector, because mixed states cannot be discussed in any other way. But, still, the standard approach is understanding the mixed state as a composition of state vectors with various classical probabilities. On this subject S. Weinberg also made his point, that we should give up the state language and consider the density matrix as a primary mathematical object to describe quantum world [10].

The measurement operation is determined by final stationary states of an open system - so called pointers. We expect them to be steady, as they are the result of a smooth measurement and they appear at the end of evolution when $\|L^{(a)}\|^2 \ll \|H\|$. We find pointers by applying to the FGKLS equation (1) an extra condition:

$$\dot{\rho} = 0.$$ (2)

The term "pointer" was introduced by W.H. Zurek in [16], [17]. In this manner he called the environment-superselected preferred states. The idea was to relate them to a measuring device.

These pointer states can have significant applications to quantum cryptography, since in that science the robust (stationary) states, not being destroyed by external intervention, are in great need [18]. A mathematically more rigorous treatment of non-Hamiltonian quantum mechanics can be found in [19].

The organization of this paper is as follows. In Sec.2, the notations are given, and the main equations are prepared in a suitable form using the energy eigenstate basis. In Sec.3, in seeking pointers, we propose a scheme of calculation taking the interaction with an environment to be weak, $\|L^{(a)}\|^2 \ll \|H\|$. Thus, the jump operators $L^{(a)}$ can be used in some way as expansion parameters for perturbation theory. The scheme we propose is different for the cases of non-degenerate and degenerate Hamiltonian in (1). The degeneracy of the Hamiltonian spectra may arise either as a consequence of global symmetry of the system or due to a (quasi) level crossing in the spectrum of (non-Hermitian) Hamiltonians [25], [26]. In this paper we re-
strict ourselves to Hermitian Hamiltonians only. The perturbation algorithm for a degenerate spectrum is presented in Subsect. 3.2.

A few considerations and attempts to elaborate the perturbation expansion for the FGKLS equation have been undertaken in [21, 22, 23, 24] at a more symbolic form using a deviation from the bare Hamiltonian taken as a small expansion parameter. As compared with them our expansion treats the operators generating decoherence as perturbations and as well includes the option of (almost) degenerate energy levels when the jump coefficients break the Hamiltonian degeneracy and tune respective energy eigenstates in a particular direction in the Hilbert space: interaction with an environment directs the system towards a set of steady states, pointers.

In Sec. 4 we study particular examples in order to illustrate our scheme of calculation. They include a number of models of quantum harmonic oscillator with spin interacting with external magnetic field. The efficiency of the perturbation algorithm is demonstrated by its comparison with the exact solution. In Conclusions possible developments of perturbation algorithms for Hamiltonians with exceptional points are outlined and prospective research directions in the quantum models design with the help of supersymmetric methods [20] are mentioned.

2 The first steps

Planning to develop the perturbation approach to construction of asymptotically steady states - pointers of the FGKLS equation (1) for the density matrix $\rho$, we start from the solution $\rho_0$ of the Liouville–von Neumann equation:

$$\dot{\rho}_0 = -i [H, \rho_0] = 0,$$

with some Hermitian Hamiltonian $H$ which does not depend on time. Further on, we suppose that the Lindblad operators $L^{(a)}$ in (1) are small, in comparison with Hamiltonian $H$, $\|L^{(a)}\|^2 \ll \|H\|$, and they serve as expansion parameters of perturbation theory. For brevity (as long as it does not change the content) in all formulas below, we will assume that we are dealing with a single Lindblad operator $L$. The appearance of additional Lindblad operators $L^{(a)}$, $a = 1, 2, \ldots$, will lead to an extra summation over the index $a$.

Thus, we take

$$\rho = \rho_0 + \Delta \rho$$

where $\Delta \rho$ is presumed to be small, compared to $\rho_0$: $\|\Delta \rho\| \ll \|\rho_0\|$.

The next step is to decompose all the operators in the energy basis: the orthonormal eigenvectors $\{|E_k\} \rangle$ of the Hamiltonian $H$:

$$H = \sum_k \varepsilon_k |E_k\rangle \langle E_k|,$$
where both eigenvalues \( \{\varepsilon_k\} \) and state vectors \( \{|E_k\}\) do not depend on \( t \).

We shall use the following notations for expansions:

\[
L = \sum_{ij} l_{ij}(t) |E_i\rangle \langle E_j|; \\
\rho = \sum_{ij} f_{ij}(t) |E_i\rangle \langle E_j|; \\
\rho_0 = \sum_{ij} g_{ij}(t) |E_i\rangle \langle E_j|
\]

with a usual condition for a density matrix:

\[
\text{Tr} \rho = \text{Tr} \rho_0 = 1.
\]

The coefficients \( \{l_{ij}\} \) of (6) will be our perturbation theory expansion parameters. It seems to be evident that \( \Delta \rho \) must be small compared to \( \rho_0 \). Namely, the expansion for \( \rho_0 \) in terms of \( l_{ij} \) should start with the zeroth order terms \( \sim l^0 \) (by \( l \) we denote any matrix element \( l_{ij} \)) while the expansion for \( \Delta \rho \) should start with the higher order, i.e. \( \sim l^k, k > 0 \). Nonetheless, it is not the case. As we will see, it is not possible to fix \( \rho_0 \) so that we could choose such small \( \Delta \rho \) for \( \rho_0 + \Delta \rho \) to satisfy equation (1) with the 'pointer' condition (2). It does not matter what \( \rho_0 \) we had in Liouville case, after turning on the FGKLS interaction, the resulting pointer state \( \rho \) would become exactly one obeying certain equations. This is the reason why we seek \( \rho \), not \( \Delta \rho \), and after finding \( \rho \) we compare it to \( \rho_0 \) to see if they are close.

Substituting now (5), (6), (7) into (1) (and noting that \( \{|E_k\}\) is a stationary basis) we get for the pointer (2):

\[
-\sum \sum l_{mk} f_{kl} l_{nl} + \frac{1}{2} \sum \sum l_{km}^* l_{kl} f_{ln} - \frac{1}{2} \sum \sum l_{kl} f_{mk}^* l_{lm} = 0
\]

for all \( m \) and \( n \). Here, \( \varepsilon_n \) has the zeroth order, \( l_{ij} \) are small expansion parameters, and \( f_{mn} \) are amenable to expansion in a power series in \( l_{ij} \).

Below, two cases will be considered separately. Namely, the case \( m = n \) with the first term in (10) vanishing

\[
\sum \sum l_{mk} f_{kl} l_{nl} - \frac{1}{2} \sum \sum l_{km}^* l_{kl} f_{ln} - \frac{1}{2} \sum \sum f_{mk} l_{kl}^* l_{lm} = 0
\]

will be called as equation 1, and the case \( m \neq n \), for which the corresponding equation (10) will be called as equation 2. The perturbation expansion differs crucially for these two cases.

By the substitution of the expansion (8) into the equation (3), one obtains the Liouville pointers:

\[
g_{mn} = -i g_{mn}(\varepsilon_m - \varepsilon_n) = 0, \quad t \to \infty,
\]
for arbitrary indices $m$ and $n$. This means that asymptotically the diagonal elements $g_{mm}(t)$ are arbitrary constants while for off-diagonal $m \neq n$ we have to consider two options:

- $H$ is non-degenerate, i.e. for $m \neq n$, $\varepsilon_m \neq \varepsilon_n$ and asymptotically $g_{mn}(t) \to 0$. Thus, the off-diagonal elements of density matrices $\rho_0$ for the Liouville pointers vanish at large $t$.

- $H$ is degenerate, i.e. there are some different states $|E_m\rangle$ and $|E_n\rangle$ ($m \neq n$) for which $\varepsilon_m = \varepsilon_n$. From (12) it is clear that in this case, $g_{mn}(t)$ are asymptotically arbitrary constants both for such indices $m, n$, and for diagonal elements with $m = n$, while $g_{mn}(t) \to 0$ for all other pairs of $m, n$.

3 The perturbation scheme of calculation of FGKLS pointers

To start with solving (1) perturbatively we replace the Lindblad operator as follows:

$$L \rightarrow \lambda L; \quad l_{ij} \rightarrow \lambda l_{ij}; \quad \lambda \ll 1.$$  (13)

Now, we suppose that $f_{mn}$ have a decomposition

$$f_{mn} = f^{(0)}_{mn} + f^{(1)}_{mn} + \ldots; \quad f^{(k)}_{mn} = c_k \lambda^{2k}$$  (14)

where $c_k$ are c-numbers which include dependence on finite values of $l_{ij}$. We expect the expansion in even powers $\lambda^{2k}$, because equations 1 and 2 are quadratic in small coefficients $\lambda$. Thus, eventually the expansion has the form:

$$f_{mn} = f^{(0)}_{mn} + f^{(1)}_{mn} + f^{(2)}_{mn} + \ldots = c_0 \lambda^0 + c_1 \lambda^2 + c_2 \lambda^4 + \ldots.$$  (15)

The trace restriction (9) provides a set of conditions:

$$\sum_{m} f^{(0)}_{mm} = 1; \quad \sum_{m} f^{(k)}_{mm} = 0, \ k = 1, 2, \ldots$$  (16)

The abovementioned equations 1 and 2 include an arbitrary fixed order $s$ of perturbation expansion for coefficients $f^{(s)}_{ij}$ as follows:

**Equation 1:**

$$\sum_i |l_{mi}|^2 f^{(s)}_{ii} - \sum_i |l_{im}|^2 f^{(s)}_{mm} + \sum_{i,j; i \neq j} l_{mi} f^{(s)}_{ij} l_{mj}^* - \frac{1}{2} \sum_{i,j; i \neq m} l_{im}^* l_{ij} f^{(s)}_{jm} - \frac{1}{2} \sum_{i,j; i \neq m} f^{(s)}_{mj} l_{ij}^* l_{im} = 0$$  (17)
for an arbitrary fixed index \( m \).

**Equation 2:**

\[
-i f^{(s)}_{mm}(\varepsilon_m - \varepsilon_n) + \lambda^2 \sum_i l_{mi} f^{(s-1)}_{ii} l^*_{ni} - \frac{1}{2} \lambda^2 \sum_i l^*_{im} l_{m}(f^{(s-1)}_{mm} + f^{(s-1)}_{nn}) + \\
+ \lambda^2 \sum_{i,j:i \neq j} l_{mi} f^{(s-1)}_{ij} l^*_{nj} - \frac{1}{2} \lambda^2 \sum_{i,j:i \neq n} l^*_{im} l_{ij} f^{(s-1)}_{jn} - \\
- \frac{1}{2} \lambda^2 \sum_{i,j:j \neq m} f^{(s-1)}_{mj} l^*_{ij} l_{in} = 0, 
\]

(18)

for arbitrary fixed pair \( m \neq n \).

Now we have to consider a Hamiltonian of non-degenerate and degenerate kinds separately, since in the latter case the first term in **Equation 2** (18) disappears for \( \varepsilon_n = \varepsilon_m \), substantially changing an application of the perturbation theory.

### 3.1 Non-degenerate Hamiltonian

- As the first step, we shall rewrite equations 1 (17) by gathering the diagonal elements \( f^{(s)}_{mm} \) in the l.h.s., but the non-diagonal elements \( f^{(s)}_{mn}, m \neq n \) of the same order \( s \) in the r.h.s. In the matrix form, the result is:

\[
\begin{pmatrix}
- \sum_{t \neq 1} |l_{t1}|^2 \\
|l_{12}|^2 \\
|l_{31}|^2 \\
\vdots
\end{pmatrix}
\begin{pmatrix}
|l_{12}|^2 \\
- \sum_{t \neq 2} |l_{t2}|^2 \\
|l_{32}|^2 \\
\vdots
\end{pmatrix}
\begin{pmatrix}
|l_{13}|^2 \\
- \sum_{t \neq 3} |l_{t3}|^2 \\
\vdots
\end{pmatrix}
\ldots
\begin{pmatrix}
|f^{(s)}_{11}| \\
|f^{(s)}_{22}| \\
|f^{(s)}_{33}| \\
\vdots
\end{pmatrix}
\]

\[= \begin{pmatrix}
- \sum_{i,j:i \neq 1} l_{i} f^{(s)}_{ij} l^*_{j1} + \frac{1}{2} \sum_{i,j,i \neq j} l^*_{i1} l_{ij} f^{(s)}_{j1} + \frac{1}{2} \sum_{i,j:j \neq 1} f^{(s)}_{j1} l_{ij} l^*_{i1} \\
- \sum_{i,j:i \neq 2} l_{i} f^{(s)}_{ij} l^*_{j2} + \frac{1}{2} \sum_{i,j,i \neq j} l^*_{i2} l_{ij} f^{(s)}_{j2} + \frac{1}{2} \sum_{i,j:j \neq 2} f^{(s)}_{j2} l_{ij} l^*_{i2} \\
- \sum_{i,j:i \neq 3} l_{i} f^{(s)}_{ij} l^*_{j3} + \frac{1}{2} \sum_{i,j,i \neq j} l^*_{i3} l_{ij} f^{(s)}_{j3} + \frac{1}{2} \sum_{i,j:j \neq 3} f^{(s)}_{j3} l_{ij} l^*_{i3} \\
\vdots
\end{pmatrix}. 
\]

The existence of solutions \( f^{(s)}_{nm} \) to this system of equations depends on the properties of the matrix in the l.h.s. and on the column in the r.h.s. Summing up all the lines in the matrix, we get the null line, i.e. its determinant vanishes. In the case of homogeneous system (with zero column in the r.h.s.), the solution can be found, and it contains one or more free parameters depending on the rank of the matrix. One free parameter is just necessary to fulfill the trace condition (9). In the case of inhomogeneous system (with nonzero column in the r.h.s.), according to Kronecker–Capelli theorem, a solution of system of equations (19) exists if the rank of the matrix is equal to the rank of the augmented matrix (the matrix with the right part of the equation glued to it).
The number of parameters in the solution also depends on the rank of the matrix, but, anyway, it is more than 0, so the trace condition (9) can be imposed. If the mentioned ranks are not equal to each other, then there is no solution. In such a case, our scheme would not work at all. But we have a sign that the solution of (19) actually can exist. One can easily check, that the sum of all the equations leads to identity $0 = 0$. So, the system of equations is consistent, at least.

- The second step is finding non-diagonal elements $f_{mn}^{(s)}$ from equations 2 (18), if we know diagonal $f_{mn}^{(s-1)}$ and non-diagonal $f_{mn}^{(s-1)}$, $m \neq n$ elements of the previous order $s - 1$. Let us rewrite equations 2 (18) for $m \neq n$ as:

$$f_{mn}^{(s)} = \frac{i\lambda^2}{\varepsilon_m - \varepsilon_n} \left[ - \sum_i l_{mi} f_{ii}^{(s-1)} l_{ni}^* + \frac{1}{2} \sum_i l_{im}^* l_{in} (f_{mm}^{(s-1)} + f_{nn}^{(s-1)}) - \sum_{i,j;i\neq j} l_{mi} f_{ij}^{(s-1)} l_{nj}^* + \frac{1}{2} \sum_{i,j;i\neq n} l_{im}^* l_{ijn} f_{jn}^{(s-1)} + \frac{1}{2} \sum_{i,j;i\neq m} f_{mj}^{(s-1)} l_{ij}^* l_{in} \right]$$

(20)

Our scheme of calculations is as follows:

- To start with, due to the obvious fact that $f_{mn}^{(-1)} \equiv 0$ for $m \neq n$, one obtains from equation (20) that in the zeroth order $f_{mn}^{(0)} = 0$ for $m \neq n$.

- Then, we substitute $f_{mn}^{(0)} = 0, m \neq n$ into equation (19). The column in the r.h.s. vanishes, and we get a homogeneous equation with degenerate matrix. Therefore the solution $\{f_{mn}^{(0)}\}$ exists with one or more free parameters, and after that, one parameter is eliminated by the trace condition $\sum_m f_{mm}^{(0)} = 1$.

- We put the zeroth order $f_{mn}^{(0)} = 0, m \neq n$ and $f_{mn}^{(0)}$ obtained above into equation (20) to find the first order nondiagonal $f_{mn}^{(1)}$, $m \neq n$.

- We substitute $f_{mn}^{(1)}, m \neq n$ into equation (19). According to Kronecker–Capelli theorem, we find either the solution for diagonal $f_{mn}^{(1)}$ with one or more parameters, or no solutions. In the former case, one parameter of all is eliminated by the trace condition $\sum_m f_{mm}^{(1)} = 0$.

- If the previous step is successful, we find $f_{mn}^{(2)}, m \neq n$ from equation (20) by substitution of $f_{mn}^{(1)}, m \neq n$ and $f_{mn}^{(1)}$.

- And so on.

At the steps of this scheme, in which we must apply Kronecker–Capelli theorem, there is a probability that the scheme crashes down (if the theorem states that there is no solution).
We must also note here that the FGKLS pointers have an evident difference from Liouville pointers: their diagonal elements are already not arbitrary at the zeroth order. After "turning Lindblad on", the density matrix changes essentially - it chooses a direction in which it "goes". And conversely, after slowly "turning Lindblad off" (i.e. taking the solution for the Lindblad pointers \( \rho \), setting all the terms with \( L \) to vanish ) some "traces" of Lindblad equation remain.

3.2 Degenerate Hamiltonian

In this case, more delicate study has to be implemented. **Equation 2** (18) differs for \((m, n)\) such that \(\varepsilon_m = \varepsilon_n\) (let us call this pair \((m, n)\) "internal") and \((m, n)\) for which \(\varepsilon_m \neq \varepsilon_n\) (we will call this pair \((m, n)\) "external"). Separating \(\{f_{mn}^{(s)}\}\) with internal and external indices, we write **Equation 1** (17) and the two versions of **Equation 2** (18) in the following form:

**Equation 1:**

\[
\sum_i |l_{mi}|^2 f_{ii}^{(s)} - \sum_i |l_{im}|^2 f_{mm}^{(s)} + \sum_{i,j; i \neq j; (i,j) - \text{int}} l_{im}^* f_{ij}^{(s)} l_{mj}^* =
\]

\[
-\frac{1}{2} \sum_{i,j; j \neq m; (j,m) - \text{int}} l_{im}^* l_{ij} f_{jm}^{(s)} - \frac{1}{2} \sum_{i,j; j \neq m; (m,j) - \text{int}} f_{mj}^* l_{ij} l_{im} =
\]

\[
= - \sum_{i,j; i \neq j; (i,j) - \text{ext}} l_{mi}^* f_{ij}^{(s)} l_{mj}^* + \frac{1}{2} \sum_{i,j; j \neq m; (j,m) - \text{ext}} l_{im}^* l_{ij} f_{jm}^{(s)} + \frac{1}{2} \sum_{i,j; j \neq m; (m,j) - \text{ext}} f_{mj}^* l_{ij} l_{im}
\]  (21)

for some \(m\).

**Equation 2:**

\[
\sum_i l_{mi} f_{ii}^{(s)} l_{mi}^* - \frac{1}{2} \sum_i l_{im}^* l_{in} (f_{mm}^{(s)} + f_{nn}^{(s)}) +
\]

\[
+ \sum_{i,j; j \neq i; (i,j) - \text{int}} l_{mi} f_{ij}^{(s)} l_{nj}^* - \frac{1}{2} \sum_{i,j; j \neq m; (i,j) - \text{int}} l_{im}^* l_{ij} f_{jn}^{(s)} - \frac{1}{2} \sum_{i,j; j \neq m; (m,j) - \text{int}} f_{mj}^* l_{ij} l_{in} =
\]

\[
= - \sum_{i,j; i \neq j; (i,j) - \text{ext}} l_{mi} f_{ij}^{(s)} l_{nj}^* + \frac{1}{2} \sum_{i,j; j \neq n; (j,n) - \text{ext}} l_{im}^* l_{ij} f_{jn}^{(s)} + \frac{1}{2} \sum_{i,j; j \neq n; (m,j) - \text{ext}} f_{mj}^* l_{ij} l_{in}
\]  (22)

for internal \((m, n), m \neq n\).
The scheme of calculations is as follows:

Analogously to non-degenerate case, due to $f^{(s-1)}_{mn} \equiv 0$ for any $(m,n)$, one obtains from equation (23) that in the zeroth order $f^{(0)}_{mn} = 0$, $m \neq n$, $(m,n) - \text{external}$.

Then, we substitute $\{f^{(0)}_{mn} = 0, m \neq n, (m,n) - \text{external}\}$ into equations (21) and (22). A composition of these two equations is a system of equations for $\{f^{(0)}_{mn}, f^{(0)}_{mn}, m \neq n, (m,n) - \text{internal}\}$ with vanishing r.h.s.. If the determinant of the matrix corresponding to the system of equations is equal to 0, then the solution has one or more free parameters. In such a case, we will then impose, as usual, the trace condition $\sum_m f^{(0)}_{mn} = 1$, reducing the number of parameters by one. Otherwise, the solution has no parameters, and we are not able to impose the trace condition.

We insert $\{f^{(0)}_{mn}\}$ with $(m,n)$ any possible pair of indexes (equal, internal or external) into equation (23). Therefore, the r.h.s. is completely known, and we obtain $\{f^{(1)}_{mn}, m \neq n, (m,n) - \text{external}\}$.

We substitute $\{f^{(1)}_{mn}, m \neq n, (m,n) - \text{external}\}$ into equations (21) and (22). We have inhomogeneous system of equations for $\{f^{(1)}_{mn}, f^{(1)}_{mn}, m \neq n, (m,n) - \text{internal}\}$. Comparing the rank of the matrix corresponding to the system of equations and the rank of the augmented matrix with applying Kronecker–Capelli theorem, we again have a set of options. There are no solutions, or the solution doesn’t have any parameters, or the solution has one or more free parameters. In the latter case, we can continue: impose the trace condition $\sum_m f^{(1)}_{mn} = 0$, and to reduce the number of parameters by one.

We find $\{f^{(2)}_{mn}, m \neq n, (m,n) - \text{external}\}$ from equation (23) (substituting the whole set $f^{(1)}_{mn}$, with $(m,n)$ equal, internal or external), if we succeed in finding $\{f^{(1)}_{mn}, f^{(1)}_{mn}, m \neq n, (m,n) - \text{internal}\}$ in the previous step.

$$f^{(s)}_{mn} = \frac{i\lambda^2}{\varepsilon_m - \varepsilon_n} \left[ - \sum_i l_{mi} f^{(s-1)}_{ii} l^*_n + \frac{1}{2} \sum_i l^*_m l_{in} (f^{(s-1)}_{mn} + f^{(s-1)}_{nm}) - \sum_{i,j \neq j; (i,j) - \text{int}} l_{mi} f^{(s-1)}_{ij} l^*_n + \frac{1}{2} \sum_{i,j \neq n; (j,n) - \text{int}} l^*_m l_{in} f^{(s-1)}_{jn} + \frac{1}{2} \sum_{i,j \neq m; (m,j) - \text{int}} f^{(s-1)}_{mj} l^*_m l_{in} ight]$$

for external $(m,n)$, $m \neq n$.
• And so on.

4 Examples

Since for the case of a degenerate Hamiltonian matrix analysis of the system of linear equations is unclear to some extent, we will have a look at some examples. The first two will be the models of a one-dimensional particle that moves in the potential of harmonic oscillator and possesses a spin interacting with external magnetic field. Interaction with the environment is described by one or two rather simple Lindblad operators, correspondingly. In the third example, the system with two-dimensional Hilbert space and one off-diagonal Lindblad operator will be considered. As in the general discussion, our goal is to find pointers - asymptotically (at \( t \to \infty \)) stationary solutions of the Lindblad equation (1).

4.1 First oscillator example

To start with, we choose a model of one-dimensional harmonic oscillator in a constant magnetic field \( B \) along \( z \)-axis:

\[
H = H_0 \otimes I + \delta \cdot I \otimes \sigma_3; \quad H_0 \equiv \frac{p^2}{2m} + \frac{1}{2} m \omega x^2; \quad \delta \equiv \frac{1}{2} \mu B \quad (24)
\]

and the single Lindblad operator \( L \) in the form:

\[
L = \frac{1}{2} \lambda I \otimes \sigma_+ ; \quad \sigma_+ \equiv \sigma_1 + i \sigma_2, \quad (25)
\]

where \( \lambda \) is a small \( c \)-number.

An appropriate energy basis \( |\Psi_M\rangle \) will be a direct product of eigenvectors of \( H_0 \) and eigenvectors of \( \sigma_3 \):

\[
\{ |\psi_n\rangle \otimes |a\rangle \}, \quad n = 0, 1, \ldots; a = 0, 1; \quad H_0 |\psi_n\rangle = E_n |\psi_n\rangle , \quad E_n = \omega (n + \frac{1}{2}) , \quad (26)
\]

where \( \{ |\psi_n\rangle \} \) is a state vector of harmonic oscillator with well known coordinate wave function:

\[
|\psi_n\rangle(x) = \frac{1}{\sqrt{2^n n!}} \left( \frac{m \omega}{\pi} \right)^{\frac{1}{4}} e^{-m \omega x^2} H_n (\sqrt{m \omega} x) . \quad (27)
\]

\( H_n(x) \) are Hermite polynomials, and as usual, \( \hbar \) was taken to be equal to 1. In turn, spin eigenvectors and eigenvalues of \( \sigma_3 \) are as follows:

\[
|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} ; \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} ; \quad \sigma_3 |a\rangle = (-1)^a |a\rangle , \quad a = 0, 1 . \quad (28)
\]

For brevity, hereinafter we will use the unique index \( M \equiv (m, a) \), where \( m \) is from the oscillator space and \( a \) is from the spin space. Correspondingly,
eigenvectors of $H$ are now $\{ |\Psi_M\rangle \} = \{ |\psi_n\rangle \otimes |a\rangle \}$. Two indices coincide $M = M'$ only if $m = m', a = a'$.

Accordingly, an arbitrary operator $O$ can be represented by the coefficients $o_{MN}$ as

$$O = \sum_{M,N} o_{MN} |\Psi_M\rangle \langle \Psi_N| = \sum_{M,N} o_{MN} |\psi_m\rangle \otimes |a\rangle \langle b|; \quad (29)$$

$$M = (m,a), \quad N = (n,b).$$

From the preceding relations it follows that

$$H |\Psi_M\rangle = \left( \omega \left( m + \frac{1}{2} \right) + \delta(-1)^a \right) |\Psi_M\rangle,$$

and the corresponding matrix elements of $H$ are

$$\varepsilon_{MN} = \varepsilon_M \delta_{MN} = \delta_{mn} \delta_{ab} \left( \omega \left( m + \frac{1}{2} \right) + \delta(-1)^a \right). \quad (30)$$

The matrix elements of $L$ are as follows:

$$l_{MN} \equiv l_{mnab} = \lambda \delta_{mn} \delta_{ab} \delta_{b1}. \quad (31)$$

Let us write equations 1 (17) and 2 (18) for this system, keeping in mind the general scheme of calculation for degenerate Hamiltonian described above:

**Equation 1:**

$$\sum_P |l_{MP}|^2 f_P^{(s)} - \sum_P |l_{PM}|^2 f_M^{(s)} + \sum_{PR,R\neq M} l_{MP} l_{PR} f_R^{(s)} - \frac{1}{2} \sum_{PR,R\neq M} l_{PM} l_{PR} f_R^{(s)} - \frac{1}{2} \sum_{PR,R\neq M} f_M^{(s)} l_{PR}^* l_{PM} = 0 \quad (32)$$

for arbitrary fixed index $M = (m,a)$.

**Equation 2:**

$$-i f_M^{(s)} (\varepsilon_M - \varepsilon_N) + \lambda^2 \sum_P l_{MP} f_P^{(s-1)} l_{NP}^* - \frac{1}{2} \lambda^2 \sum_P l_{PM}^* l_{PN} (f_M^{(s-1)} + f_N^{(s-1)}) + \lambda^2 \sum_{PR,R\neq M} l_{MP} f_P^{(s-1)} l_{MR}^* - \frac{1}{2} \lambda^2 \sum_{PR,R\neq M} l_{PM}^* l_{PR} (f_R^{(s-1)} + f_N^{(s-1)}) + \frac{1}{2} \lambda^2 \sum_{PR,R\neq M} f_M^{(s-1)} l_{PR}^* l_{PN} = 0 \quad (33)$$

for arbitrary fixed pairs $M = (m,a)$ and $N = (n,b)$, $M \neq N$.

Substituting $\varepsilon_M$ (30) and $l_{MN}$ (31) and simplifying the equations, we get

**Equation 1:**

$$f_M^{(s)} = 0 \quad (34)$$
for $M = (m, 1)$.

and **Equation 2:**

$$-i f^{(s)}_{MN} (\varepsilon_M - \varepsilon_N) + |\lambda|^2 \left( f^{(s-1)}_{mn11} \delta_{a0} \delta_{b0} (1 - \delta_{mn}) - \frac{1}{2} f^{(s-1)}_{mn1b} \delta_{a1} (1 - \delta_{mn} \delta_{b1}) - \frac{1}{2} f^{(s-1)}_{mna1} \delta_{b1} (1 - \delta_{mn} \delta_{a1}) \right) = 0. \quad (35)$$

for all pairs $M = (m, a)$ and $N = (n, b)$, such that $M \neq N$.

To separate non-degenerate and degenerate cases in this model, let us find the pairs of indexes $M = (m, a)$ and $N = (n, b)$ which are "internal" according to the terminology introduced above, i.e. $\varepsilon_M = \varepsilon_N$. Recalling (30), one can conclude that the only options to get degeneracy are:

$$M = (m, 0) \quad \text{and} \quad N = (m + q, 1); \quad q \equiv \frac{2\delta}{\omega} \quad (36)$$

$$M = (m, 1) \quad \text{and} \quad N = (m - q, 0); \quad m = 0, 1, 2, \ldots . \quad (37)$$

which are possible only if $q = \frac{2\delta}{\omega}$ is an integer.

### 4.1.1 Non-degenerate Hamiltonian

Let the parameter $q$ is not an integer. The form of **Equation 1** looks just as (34), and, according to (35), **Equation 2** is,

$$f^{(s)}_{MN} = -\frac{i|\lambda|^2}{\varepsilon_M - \varepsilon_N} \left( f^{(s-1)}_{mn11} \delta_{a0} \delta_{b0} (1 - \delta_{mn}) - \frac{1}{2} f^{(s-1)}_{mn1b} \delta_{a1} (1 - \delta_{mn} \delta_{b1}) - \frac{1}{2} f^{(s-1)}_{mna1} \delta_{b1} (1 - \delta_{mn} \delta_{a1}) \right) \quad (38)$$

for the pairs $M = (m, a)$ and $N = (n, b)$, such that $M \neq N$.

Let us follow the scheme of calculation for non-degenerate Hamiltonian described earlier.

- We look at equation 2 (38) and see that $f^{(0)}_{MN} = 0, M \neq N$, since in the right part $f^{(s-1)}_{PR} = 0$ for any indices $P, R$.

- **Equation 1** (34) states that $f^{(0)}_{mn11} = 0$ and, consequently, $f^{(0)}_{mn00}$ are arbitrary real (because of hermiticity of the density matrix) numbers only restricted by the trace condition $\sum_m f^{(0)}_{mm00} = 1$.

- We must substitute all matrix elements of the zeroth order - diagonal elements $f^{(0)}_{mm00}, f^{(0)}_{mn11}$ and non-diagonal elements $f^{(0)}_{mnab}, M \neq N$ - into equation 2 (38) in order to find first-order non-diagonal elements $f^{(1)}_{MN}, M \neq N$. The only elements of the zeroth order that do not vanish are $f^{(0)}_{mm00}$. Thus, the right part of equation 2 (38) is equal to 0, i.e. $f^{(1)}_{MN} = 0, M \neq N$. 

13
• Keeping in mind (34), we know that \( f_{mn11}^{(1)} = 0 \), therefore \( f_{mn00}^{(1)} \) are arbitrary real numbers only restricted by the trace condition \( \sum_m f_{mn00}^{(1)} = 0 \).

• Again the right part of equation 2 (38) is equal 0. Therefore, \( f_{MN}^{(2)} = 0, M \neq N \).

• And so on.

Summing up, \( f_{mn00} \) are arbitrary real numbers only restricted by the trace condition \( \sum_m f_{mn00} = 1 \), while all other matrix elements \( f_{MN} \) vanish.

If we recall the Liouville pointers, they correspond to diagonal density matrices with arbitrary elements, that means, with arbitrary \( f_{mn00} \) and \( f_{mn11} \) only obeying the trace condition: \( \sum_m (f_{mn00} + f_{mn11}) = 1 \). We demonstrated above that the Lindblad pointers differ essentially from the Liouville ones: while the part \( f_{mn00} \) of diagonal elements of the density matrix after turning on an interaction with an environment may coincide with those without interaction, the part \( f_{mn11} \) definitely vanishes. We can say that after turning an interaction on, the resulting density matrix "receives the direction", namely, along \( f_{mn11} = 0 \). If vice versa, we decide to turn off the interaction with an environment, the resulting density matrix "remembers" it keeping \( f_{mn11} = 0 \).

4.1.2 Degenerate Hamiltonian

In this case the incoming parameters are such that \( \frac{2\delta}{\omega} \) is an integer. According to the scheme of calculation for degenerate Hamiltonian, we need two versions of equation 2 (35): one - for internal (36), (37)) pairs of indices, and second - for external pairs. The result is:

**Equation 1:**

\[
\begin{align*}
\text{Equation 1:} & \\
f_{mn11}^{(s)} &= 0 \\
\end{align*}
\]

**Equation 2:**

• For internal pairs:

\[
\begin{align*}
\text{Equation 2:} & \\
f_{m,m+q,0,1}^{(s)} &= 0; \quad f_{m,m,q,1,0}^{(s)} = 0. \\
\end{align*}
\]

• For external pairs \( M = (m, a), N = (n, b), M \neq N \):

\[
\begin{align*}
\text{Equation 2:} & \\
f_{MN}^{(s)} &= -\frac{i|\lambda|^2}{\varepsilon_M - \varepsilon_N} \left( f_{mn11}^{(s-1)} \delta_{a0} \delta_{b0} (1 - \delta_{mn}) - \frac{1}{2} f_{mn1b}^{(s-1)} \delta_{a1} (1 - \delta_{mn} \delta_{1b}) - \frac{1}{2} f_{mna1}^{(s-1)} \delta_{b1} (1 - \delta_{mn} \delta_{a1}) \right). \\
\end{align*}
\]

Let us find all the orders \( f_{mnab}^{(s)} \) now, following the procedure above.
• First we look at equation 2 for external pairs (41) and obtain that
\( f_{MN}^{(0)} = 0 \) for external \( M \neq N \), since \( f_{PR}^{(-1)} = 0 \) for all possible indices \( P, R \).

• Then, in equation 1 (39) and equation 2 for internal indices (40), we insert \( f_{MN}^{(0)} = 0 \) for external \( M \neq N \) in order to find diagonal \( f_{MM}^{(0)} \) and the rest of non-diagonal \( f_{MN}^{(0)} = 0 \) for internal \( M \neq N \). It is evident from (39) and (40) that:
  1. Diagonal elements: \( f_{mm}^{(0)}_{11} = 0; f_{mm}^{(0)}_{00} \) are arbitrary except for the limitation: \( \sum_m f_{mm}^{(0)}_{00} = 1 \) (trace condition).
  2. Non-diagonal internal elements: \( f_{m,m+q,0,1}^{(0)} = 0, f_{m,m-q,1,0}^{(0)} = 0 \) (there are no other internal pairs of indices).

• We look again at equation 2 for external pairs of indices (41). We need to insert there all matrix elements of the zeroth order \( f_{MN}^{(0)} \). But we know that the only nonvanishing ones are \( f_{mm}^{(0)}_{00} \). Therefore, the r.h.s. is equal to 0, and \( f_{MN}^{(1)} = 0 \) for external \( M \neq N \).

• We analyze equations 1 (39) and 2 for internal pairs of indices (40). We again easily find:
  1. Diagonal elements: \( f_{mm}^{(1)}_{11} = 0; f_{mm}^{(1)}_{00} \) are arbitrary except for the limitation: \( \sum_m f_{mm}^{(1)}_{00} = 0 \) (trace condition).
  2. Non-diagonal internal elements: \( f_{m,m+q,0,1}^{(1)} = 0, f_{m,m-q,1,0}^{(1)} = 0 \). There are no other internal pairs of indices.

• Having a look at equation 2 for external pairs of indices (41), again we conclude that \( f_{MN}^{(2)} = 0 \) for external \( M \neq N \).

• And so on.

As a result, we have that \( f_{mm00} \) are arbitrary except for the trace condition: \( \sum_m f_{mm00} = 1 \). All the other matrix elements vanish.

Let us again compare the Liouville and the Lindblad pointers. The first ones are the density matrices with arbitrary diagonal elements \( f_{mm00} \) and \( f_{mm11} \), restricted only by the trace condition \( \sum_m (f_{mm00} + f_{mm11}) = 1 \), and with arbitrary elements of the kind \( f_{m,m+q,0,1}^{(s)} \) and \( f_{m,m-q,1,0}^{(s)} \) (indices \( (m,0) \) and \( (m+q,1) \) correspond to the states with the same energy, the same concerns \( (m,1) \) and \( (m-q,0) \)). We see that interaction with an environment leads to the Liouville pointer with destroyed degeneracy, in addition to disposing of \( f_{mm11} \), as in the non-degenerate case (see Subsection 4.1.1).

4.2 Second oscillator example

As a second example, we shall take the same system of one-dimensional harmonic oscillator with spin 1/2 in a magnetic field presented by the Hamiltonian (24) but with another Lindblad operator. Instead of a single operator \( L \)
along $\sigma_+ = \sigma_1 + i\sigma_2$, we shall use two Lindblad operators $L^{(1)}, L^{(2)}$ directed along $\sigma_1, \sigma_2$, correspondingly:

$$L^{(1)} = \gamma_1 \cdot I \otimes \sigma_1; \quad L^{(2)} = \gamma_2 \cdot I \otimes \sigma_2,$$

(42)

where $\gamma_1$ and $\gamma_2$ are arbitrary (in general, complex) small numbers $|\gamma_{1,2}| \ll 1$. We shall keep the same definitions and notations as in Example 1, i.e. Eqs. (26) - (30) unchanged, but with the following expressions for the matrix elements of the Lindblad operators:

$$L^{(1)}_{MN} = l^{(1)}_{mnab} = \gamma_1 \delta_{mn} (\delta_{a0} \delta_{b1} + \delta_{a1} \delta_{b0});$$

$$L^{(2)}_{MN} = l^{(2)}_{mnab} = \gamma_2 \delta_{mn} (-i \delta_{a0} \delta_{b1} + i \delta_{a1} \delta_{b0}).$$

(43)

Again, the spectrum of the Hamiltonian (24) can be degenerate for integer values of $q = \frac{2\delta}{\omega}$. In such a case, due to (30) the "internal" pairs of degenerate levels are again as in (36), (37).

Let us now consider the situation with degenerate Hamiltonian postponing non-degenerate case up to the end of the Subsection. Substituting $\varepsilon_M$ (30) and $L^{(1),(2)}_{MN}$ (43) into equation 1 and equation 2 (equations (21) and (22)-(23), correspondingly), simplifying them, and including an additional summation over the set of the Lindblad operators $L^{(a)}, a = 1, 2$, we get (here and below, matrix elements with internal pairs of indices are in bold)

**Equation 1:**

$$f_{mm00}^{(s)} = f_{mm11}^{(s)}$$

(44)

**Equation 2:**

- For internal pairs of indices:

$$f_{m,m+q,0,1}^{(s)} = \frac{|\gamma_1|^2 - |\gamma_2|^2}{|\gamma_1|^2 + |\gamma_2|^2} f_{m,m+q,1,0}^{(s)}$$

(45)

$$f_{m,m−q,0,1}^{(s)} = \frac{|\gamma_1|^2 - |\gamma_2|^2}{|\gamma_1|^2 + |\gamma_2|^2} f_{m,m−q,0,1}^{(s)}$$

(46)

- And for external pair of indices:

$$f_{m,m−q,0,1}^{(s)} = -\frac{i}{4\delta} \left[ (|\gamma_1|^2 - |\gamma_2|^2) f_{m,m−q,1,0}^{(s−1)} - (|\gamma_1|^2 + |\gamma_2|^2) f_{m,m−q,0,1}^{(s−1)} \right];$$

(47)

$$f_{m,m+q,1,0}^{(s)} = \frac{i}{4\delta} \left[ (|\gamma_1|^2 - |\gamma_2|^2) f_{m,m+q,1,0}^{(s−1)} - (|\gamma_1|^2 + |\gamma_2|^2) f_{m,m+q,0,1}^{(s−1)} \right];$$

(48)

$$f_{mm01}^{(s)} = -\frac{i}{\omega (m−n) + 2\delta} \left[ (|\gamma_1|^2 - |\gamma_2|^2) f_{mm10}^{(s−1)} - (|\gamma_1|^2 + |\gamma_2|^2) f_{mm01}^{(s−1)} \right],$$

$$n \neq m \pm q;$$

(49)
Therefore, the only difference is that in this case matrix elements calculation for degenerate Hamiltonian works. All $f_{mn01}$ from equations (45), (46). Diagonal elements with internal pairs of indices, that is $(m,m)$, of indices $(0)$, $(1)$ must be skipped (together with equations (45)-(48)), since they do not vanish.

Finally, we conclude that $f_{MN} = 0$ for non-equal pairs $M \neq N$, since $f_{MN}^{(s)} = 0$ for each order $s$ of perturbation theory. At last, we need to deal with $f_{MM}$. Equation (44) simply gives that $f_{mm00} = f_{mm11}$, since $f_{mm00}^{(s)} = f_{mm11}^{(s)}$ in each order $s$ of perturbation theory. It means that all matrix elements $f_{mm00}$ are arbitrary, $\{f_{mm11}\}$ are defined from them. Next, the arbitrariness is reduced by the trace condition: $\sum f_{mm00} = \frac{1}{2}$ (it follows from $\sum f_{mmaa} = 1$).

If we consider now the non-degenerate case, we will get the same results. The only difference is that in this case matrix elements $f_{m,m+q,0,1}^{(s)}$, $f_{m,m+q,1,0}^{(s)}$ and so on must be skipped (together with equations (45)-(48)), since they do
not exist for non-integer values of $q = \frac{2\delta}{\omega}$. Nonetheless, the reasoning stays the same, so the result still is true.

As we have argued in the very end of Section 2 for the general case, the Liouville pointers are density matrices with arbitrary diagonal elements and elements with unequal internal indices. Thus, in the example considered here Liouville pointers are density matrices with arbitrary $f_{mm00}$, $f_{mm11}$ and $f_{m,m+q,0,1}$, $f_{m,m-q,1,0}$ (if Hamiltonian is degenerate). $\{f_{mm00}, f_{mm11}\}$ are restricted by the trace condition: $\sum_m f_{mm00} + \sum_m f_{mm11} = 1$.

It is interesting to compare the forms of Lindblad and Liouville pointers for the considered system. This will be done with the help of two tables: separately for degenerate and non-degenerate Hamiltonians.

- **Degenerate Hamiltonian** ($q = \frac{2\delta}{\omega}$ is an integer)

| Liouville pointers | Lindblad pointers |
|--------------------|-------------------|
| $f_{mm00}$ are arbitrary*, $f_{mm11}$ are arbitrary*, $f_{m,m+q,0,1}$ are arbitrary, $f_{m,m-q,1,0}$ are arbitrary | $f_{mm00}$ are arbitrary*, $f_{mm11} = f_{mm00}$ |
| *but restricted by the trace condition $\sum_m f_{mm00} + \sum_m f_{mm11} = 1$ | *but restricted by the trace condition $\sum_m f_{mm00} = \frac{1}{2}$ |

We know that when we turn on an interaction of a system with environment (described in this example by Lindblad operators (42)), Liouville pointers (expected final quantum states of a closed system) transform into Lindblad pointers (expected final quantum states of an open system). Thus, our result shows that this kind of interaction aligns the populations of the spin up and spin down quantum states. Moreover, this interaction completely destroys degeneracy.

- **Non-degenerate Hamiltonian** ($q = \frac{2\delta}{\omega}$ is **not** an integer)

| Liouville pointers | Lindblad pointers |
|--------------------|-------------------|
| $f_{mm00}$ are arbitrary*, $f_{mm11}$ are arbitrary*, *but restricted by the trace condition $\sum_m f_{mm00} + \sum_m f_{mm11} = 1$ | $f_{mm00}$ are arbitrary*, $f_{mm11} = f_{mm00}$ *but restricted by the trace condition $\sum_m f_{mm00} = \frac{1}{2}$ |

The conclusion is the same. Populations of spin up and spin down states become equal as a result of the interaction of the system with the environment.

What is interesting, the result does not depend on the values of coupling constants $\gamma_1$, $\gamma_2$ (see (42)).
4.3 Third example: two-dimensional space of states

Let us consider an example, where a Hamiltonian and a single Lindblad operator are given by the following $2 \times 2$ matrices in the energetic basis like in (5) - (8). $H$ is non-degenerate:

$$H = \begin{pmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{pmatrix}, \varepsilon_1 \neq \varepsilon_2,$$

and $L$ - off-diagonal matrix:

$$L = \begin{pmatrix} 0 & l_{12} \\ l_{21} & 0 \end{pmatrix}.$$ 

4.3.1 Pointers for FGKLS equation by means of perturbation theory

1. The general expression (20) of Section 3 for non-diagonal elements is essentially simplified now since first two sums in the r.h.s. vanish automatically for arbitrary off-diagonal two-dimensional matrix $l_{kl}$. Finally, this equation can be written as:

$$f^{(s)}_{12} = \frac{i}{\varepsilon_1 - \varepsilon_2} \left[ -l_{12} f^{(s-1)}_{21} f^{*}_{21} + \frac{1}{2} (|l_{21}|^2 + |l_{12}|^2) f^{(s-1)}_{12} \right].$$

The equation for $f^{(s)}_{21}$ is similar due to hermiticity of $\rho$. Since $f^{(-1)}_{12} = f^{(-1)}_{21} = 0$, we obtain from (55) that $f^{(0)}_{12} = f^{(0)}_{21} = 0$, and all the orders $f^{(s)}_{12} = f^{(s)}_{21} = 0$, as well. Thus, all non-diagonal elements of $\rho$ vanish.

2. The equation (19) for diagonal elements of $\rho$ becomes also very simple in this example due to vanishing of off-diagonal elements $f_{ij}$, $i \neq j$ : the system (19) becomes homogeneous:

$$\begin{pmatrix} -|l_{21}|^2 & |l_{12}|^2 \\ |l_{21}|^2 & -|l_{12}|^2 \end{pmatrix} \begin{pmatrix} f^{(s)}_{11} \\ f^{(s)}_{22} \end{pmatrix} = 0$$

Applying the trace conditions (16), we get expressions for diagonal elements of $\rho$ :

$$f_{11} = \frac{|l_{12}|^2}{|l_{12}|^2 + |l_{21}|^2};$$

$$f_{22} = \frac{|l_{21}|^2}{|l_{12}|^2 + |l_{21}|^2}.$$ 

Summing up, the only solution for the pointer of FGKLS equation in our scheme of calculation is:

$$\begin{pmatrix} \frac{|l_{12}|^2}{|l_{12}|^2 + |l_{21}|^2} & 0 \\ 0 & \frac{|l_{21}|^2}{|l_{12}|^2 + |l_{21}|^2} \end{pmatrix}.$$
4.3.2 Pointers by means of exact solution

An arbitrary diagonal non-degenerate Hamiltonian \((53)\) and off-diagonal \(L\) as in \((54)\) can be expressed as:

\[
H = \varepsilon_0 I + \varepsilon_3 \sigma_3, \quad \varepsilon_0 \neq \varepsilon_3 \in \mathbb{R};
\]

\[
L = (a_1 + i b_1) \sigma_1 + (a_2 + i b_2) \sigma_2, \quad a_1, a_2, b_1, b_2 \in \mathbb{R}.
\]

Correspondingly, the FGKLS equation \((1)\) takes the form:

\[
\dot{\rho} = 2 \left[ \hat{h} - a_0 \hat{b} + b_0 \hat{a}, \rho \right] - 2| \hat{a} |^2 \rho_{\perp \hat{a}} - 2| \hat{b} |^2 \rho_{\perp \hat{b}} + 2 \left[ \hat{a}, \hat{b} \right]
\]

where

\[
\rho_{\perp \hat{a}} = \rho - \frac{(\hat{a} \cdot \rho)}{| \hat{a} |^2} \hat{a}
\]

and in components \((62)\) reads:

\[
\dot{\rho}_1 = -2(a_1^2 + b_2^2) \rho_1 + 2(-\varepsilon_3 + a_1 a_2 + b_1 b_2) \rho_2;
\]

\[
\dot{\rho}_2 = 2(\varepsilon_3 + a_1 a_2 + b_1 b_2) \rho_1 - 2(a_1^2 + b_1^2) \rho_2;
\]

\[
\dot{\rho}_3 = -2(a_1^2 + a_2^2 + b_1^2 + b_2^2) \rho_3 + 2(a_1 b_2 - a_2 b_1).
\]

- The last equation \((66)\) can be solved straightforwardly:

\[
\rho_3(t) = \frac{a_1 b_2 - a_2 b_1}{a_1^2 + a_2^2 + b_1^2 + b_2^2} + \left( \rho_3(0) - \frac{a_1 b_2 - a_2 b_1}{a_1^2 + a_2^2 + b_1^2 + b_2^2} \right) e^{-2(a_1^2 + a_2^2 + b_1^2 + b_2^2)t}
\]

with the limit value:

\[
\rho_3(\infty) = \frac{a_1 b_2 - a_2 b_1}{a_1^2 + a_2^2 + b_1^2 + b_2^2}.
\]

which provides the diagonal part of the pointer $\frac{1}{2} I + \rho_3(\infty) \sigma_3$. Comparing explicit expressions \((68)\) with Eq.\((59)\), we can check that:

\[
\frac{|l_{12}|^2}{|l_{12}|^2 + |l_{21}|^2} = \frac{1}{2} + \frac{a_1 b_2 - a_2 b_1}{a_1^2 + a_2^2 + b_1^2 + b_2^2}.
\]

- At last, we need to solve the first two equations \((64), (65)\) and make sure that off-diagonal elements vanish in the late time limit. Indeed, \(\rho_2\) can be expressed from the first equation:

\[
\rho_2 = \frac{\dot{\rho}_1 + 2(a_1^2 + b_2^2) \rho_1}{2(-\varepsilon_3 + a_1 a_2 + b_1 b_2)}.
\]

After substituting this expression the second equation becomes the second-order differential equation:

\[
\ddot{\rho}_1 + 2(a_1^2 + a_2^2 + b_1^2 + b_2^2) \dot{\rho}_1 + 4 \rho_1 [(a_1 b_2 - a_2 b_1)^2 + \varepsilon_3^2] = 0,
\]
and the roots $\lambda$ are:

$$\lambda = -(a_1^2 + a_2^2 + b_1^2 + b_2^2) \pm \sqrt{(a_1^2 + a_2^2 + b_1^2 + b_2^2)^2 - 4[(a_1b_2 - a_2b_1)^2 + \varepsilon_3^2]}.$$  

(72)

The exponent $e^{\lambda t}$ vanishes in the limit $t \to +\infty$ for any sign of $\pm$ above, if $4[(a_1b_2 - a_2b_1)^2 + \varepsilon_3^2] > 0$, which is fulfilled due to $\varepsilon_3 \neq 0$. Even if the square root in the r.h.s. of (72) vanishes, i.e. $\lambda_1 = \lambda_2$, the solution $\rho_1(t)$ vanishes in the limit $t \to +\infty$, because the exponent dominates over the polynomial. The same is true for $\rho_2(t)$, because they are related by (70), and the same exponents diminish $\rho_2$ in the late time limit. Thus, the off-diagonal part of $\rho(t)$ vanishes asymptotically, and the complete result coincides with that obtained in (59) by means of perturbation algorithm.

5 Conclusion

Our first goal has been to construct FGKLS pointers, given an interaction with an environment is weak, and perturbation theory can be applied. We have succeeded in presenting the formulas for finding FGKLS pointers in each order of perturbation theory for non-degenerate and degenerate Hamiltonians. We have obtained that turning an interaction with an environment on completely changes the final states. When the system is closed, they are much more arbitrary. If the system becomes open, its final states obey some specific sets of equations. It means that an interaction directs the system towards a set of fixed states, pointers, that we have been looking for throughout this work.

We have also studied particular examples of quantum harmonic oscillator with spin interacting with external magnetic field. The first one is easy and has been solved completely. The second one is more complicated, but we have been able to write the formulas for finding all the orders of perturbation theory. In the third example $H$ is an arbitrary non-degenerate $2 \times 2$ matrix and $L$ is an arbitrary $2 \times 2$ non-zero off-diagonal matrix. The pointer is predicted by our perturbation theory scheme and is shown to coincide with the exact solution.

Supersymmetry properties of open quantum systems have been recently studied. See [20] that suggests a technique for quantum engineering of systems with controllable decoherence. Non-Hermitian Hamiltonians may have exceptional spectral points, in this case normalized states include not only the eigenstates but also associated states [25], [26]. The extension of the perturbation approach to the FGKLS equation for Hamiltonians with exceptional points is wanted for such systems and is under elaboration.
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