A new approach to combine generalized interval valued fuzzy numbers based on average width of fuzzy set concept

Palash Dutta

Received 14 July 2015; Revised 01 September 2015; Accepted 08 August 2016

ABSTRACT. The objective of this paper is to devise a new technique to combine generalized interval valued fuzzy numbers (GIVFNs) based on average width of fuzzy set concept. The effectiveness and applicability of the proposed technique are illustrated with an example.

2010 AMS Classification: 03E72

Keywords: Fuzzy set, Generalized interval valued fuzzy numbers, Average width of fuzzy number.

Corresponding Author: Palash Dutta (palash.dtt@gmail.com)

1. Introduction

In general, real world problems are ill defined, i.e., objectives and parameters are not precisely known. Initially, the obstacles of lack of precision have been dealt with using the well established and age old classical probability approach. But due to the fact that the requirement on the data and on the environment are very high and that many real world problems are fuzzy nature and not random, the probability applications are inappropriate in lots of cases. Also, the applications of fuzzy set theory in real world problems give better results. More often, type-I fuzzy set theory[40] is used to deal with imprecision, vagueness, lack of data etc. However, in some situations it is not always possible for a membership function of the type to precisely assign one point from [0,1] so it is more realistic to assign interval value. According to Gehrke et al.[15] many people believe that assigning an exact number to expert’s opinion is too restrictive and the assignment of an interval valued is more realistic. In such situations interval valued fuzzy set (IVFS) comes into picture. In May, 1975 Sambuc[30] presented in his doctoral research (thesis) the concept of IVFS named as Φ-fuzzy set. After development of IVFVs, different researchers have been studied this issue and applied in different areas.
Sambuc [30] and Grattan [18] noted that the presentation of a linguistic expression in the form of fuzzy sets is not enough. Interval valued fuzzy sets were suggested by Gorzczany [16, 17] and Turksen [32]. Wnag and Li [36] defined interval valued fuzzy numbers (IVFN) and gave their extended operations. Turksen [33] studied interval valued logic and applied in preference modeling. Wang and Li [35] presented the correlation coefficient of interval-valued fuzzy numbers and some of their properties. Lin [24] used interval-valued fuzzy numbers to represent vague processing time in job-shop scheduling problems. Yao and Lin [38] used interval-valued fuzzy numbers to represent unknown job processing time for constructing a fuzzy flow-shop sequencing model.

Chen et al. [7] proposed a fuzzy risk analysis method based on similarity measure of interval valued fuzzy number. Chen [8], Chen et al. [6], Wei et al. [37] have developed a fuzzy risk analysis based on the similarity measure of interval-valued fuzzy number. Carlsson et al. [5] introduced a possibilistic mean value for interval-valued fuzzy numbers as the arithmetic mean of the possibilistic mean values of its upper and lower fuzzy numbers. Carlsson et al. [3] introduced the concept that means value and variance can be utilized as a ranking method for interval-valued fuzzy numbers. Tomas et al. [31] discussed Group Multi-Criteria decision making based upon Interval-Valued Fuzzy numbers and extended the MULTIMOORA Method, Jozsef [20] combined interval-valued fuzzy sets and OWA operators to create new aggregation methods and proved that the new operators satisfy some important properties.

Some researchers [3, 19, 23, 27, 39, 41] investigated and suggested some methods for measuring distance between interval valued fuzzy sets (IVFS). Application of IVFSs in medical diagnosis can be found in [1, 2, 10, 14, 22, 25, 26, 28].

An IVFS is a set in which every element has degree of membership in the form of an interval. One can say, IVFS consist of two membership function, one is upper membership function (UMF) and other is lower membership function (LMF). In this study we consider these UMF and LMF as generalized fuzzy set. In 1985, Chen [9] developed the concept of generalized fuzzy number. A generalized fuzzy number is a fuzzy number in which height of the fuzzy number is less than unit. Dutta [12] proposed new method to study all basic arithmetic operations between Generalized triangular Fuzzy Numbers using $\alpha$-cut technique apart from Chen’s approach. Also proposed an alternative approach to perform arithmetic operations between GFNs and dubbed as Normalized approach. An uncertainty measure technique called average width is proposed too for GFNs.

Rashid et al. [29] proposed a new method to aggregate opinion of several decision makers using generalized interval valued fuzzy numbers. Das et al. [11] discussed permanent of interval valued fuzzy matrices.

In this paper, an attempt has been made to combine generalized IVFNs based on average width of fuzzy set concept in which LFM of IVFNs are treated as generalized fuzzy numbers. In literature, in the operation of triangular generalized IVFNs, it is observed that resulting fuzzy number is obtained in the form of generalized triangular IVFN, but in the present study it is found that the resulting fuzzy number is obtained in the form of generalized IVFN in which UMF is type-I normal fuzzy number while LMF is trapezoidal type fuzzy number. Furthermore, the resultant IVFN will be
generalized trapezoidal type IVFN if LMF and UMF are considered to be generalized triangular IVFNs.

2. Preliminaries

Fuzzy set theory provides a way to characterize the imprecisely defined variables, define relationships between variables based on expert human knowledge and use them to compute results. In this section, some necessary backgrounds and notions of type-I fuzzy set theory [9, 13, 40] and IVFS theory [16, 17, 32, 33, 34] that will be required in the sequel are reviewed.

Definition 2.1. Let $X$ be a universal set. Then the fuzzy subset $A$ of $X$ is defined by its membership function

$$
\mu_A : X \rightarrow [0, 1]
$$

which assign a real number $\mu_A(x)$ in the interval $[0, 1]$, to each element $x \in A$, where the value of $\mu_A(x)$ at $x$ shows the grade of membership of $x$ in $A$.

Definition 2.2. Given a fuzzy set $A$ in $X$ and any real number $\alpha \in [0, 1]$.

(i) The $\alpha$-cut of $A$, denoted by $\alpha A$ is the crisp set

$$
\alpha A = \{x \in X : \mu_A(x) \geq \alpha\}.
$$

(ii) The strong $\alpha$-cut, denoted by $\alpha^+ A$ is the crisp set

$$
\alpha^+ A = \{x \in X : \mu_A(x) > \alpha\}.
$$

Definition 2.3. The support of a fuzzy set $A$ on $X$ is a crisp set defined as

$$
Supp(A) = \{x \in X : \mu_A(x) > 0\}.
$$

Definition 2.4. The height of a fuzzy set $A$, denoted by $h(A)$ is the largest membership grade obtain by any element in the set and it is denoted as

$$
h(A) = \sup_{x \in X} \mu_A(x).
$$

Definition 2.5. Generalized Fuzzy Numbers (GFN): The membership function of GFN $A = [a, b, c, d; w]$, where $a \leq b \leq c \leq d, 0 < w \leq 1$ is defined as

$$
\mu_A(x) = \begin{cases} 
0, & x < a \\
w, & \frac{x-a}{b-a}, a \leq x \leq b \\
w, & \frac{x-c}{d-c}, c \leq x \leq d \\
0, & x > d.
\end{cases}
$$

If $w = 1$, then GFN $A$ is a normal trapezoidal fuzzy number $A = [a, b, c, d]$. If $a = b$ and $c = d$, then $A$ is a crisp interval. If $b = c$ then $A$ is a generalized triangular fuzzy number. If $a = b = c = d$ and $w = 1$ then $A$ is a real number. Compared to normal fuzzy number the GFN can deal with uncertain information in a more flexible manner because of the parameter $w$ that represent the degree of confidence of opinions of decision maker’s.
Definition 2.6. An interval valued fuzzy set $A$ defined in the universe of discourse $X$ is represented by
\[ A = \{(x, [\mu_A^L(x), \mu_A^U(x)] : x \in X)\}, \]
where $0 \leq \mu_A^L(x) \leq \mu_A^U(x) \leq 1$ and the membership grade $\mu_A(x)$ of elements of $x$ to the interval valued fuzzy set $A$ is represented by an interval $[\mu_A^L(x), \mu_A^U(x)]$ (i.e., $\mu_A(x) = [\mu_A^L(x), \mu_A^U(x)]$).

Definition 2.7. If an interval valued fuzzy set $A$ satisfies the following properties:
(i) $A$ is normal,
(ii) $A$ is defined in a closed bounded interval,
(iii) $A$ is convex set, then $A$ is called an interval valued fuzzy number.

Definition 2.8 ([21], $\alpha$-cut of IVFN). Let $A$ be a continuous and convex IVFN with UMF $\mu_A^L(\alpha A)$ and LMF $\mu_A^U(\alpha A)$. Let $\alpha$-cut of UMF ($\bar{A}$) be $\alpha A = [L_{\bar{A}}, R_{\bar{A}}]$ and of LMF ($\bar{A}$) be $\alpha A = [L_{\bar{A}}, R_{\bar{A}}]$. Then, $\alpha$-cut of IVFS $A$ can be calculated by the following formula:
\[ \alpha A = \begin{cases} ([L_{\bar{A}}, R_{\bar{A}}], [L_{\bar{A}}, R_{\bar{A}}]), & \alpha \leq h(\bar{A}) \land \alpha \leq h(\bar{A}) \\ ([L_{\bar{A}}, R_{\bar{A}}], \Phi), & \alpha \leq h(\bar{A}) \land \alpha > h(\bar{A}) \\ (\Phi, \Phi), & \alpha > h(\bar{A}), \end{cases} \]
where $\forall \alpha : L_{\bar{A}} \leq L_{\bar{A}} \leq R_{\bar{A}} \leq R_{\bar{A}}$, $h(\bar{A})$ is the height of LMF, $h(\bar{A})$ is the height of UMF and $\Phi$ is an empty set.

Definition 2.9. $\alpha$-cut of Generalized IVFN: Let $A$ be a generalized IVFN and its membership function is given as
\[ \bar{\mu}_A(x) = \begin{cases} \frac{x-a_1}{c_1-a_1}, & x \in [a_1, c_1] \land UMF \\ \frac{x-a_1}{c_1-a_1}, & x \in [c_1, e_1] \land UMF \\ \frac{x-b_1}{c_1-b_1}, & x \in [b_1, c_1] \land LMF \\ \frac{d_1-x}{d_1-c_1}, & x \in [c_1, d_1] \land LMF \end{cases} \]

Then, $\alpha$-cut of the generalized IVFN $A$ is
\[ \alpha A = \begin{cases} \left[\left(\frac{\alpha}{w_A^L}(c_1 - a_1) + a_1, c_1 - \frac{\alpha}{w_A^L}(e_1 - c_1)\right), \left(\frac{\alpha}{w_A^L}(c_1 - b_1) + b_1, d_1 - \frac{\alpha}{w_A^L}(d_1 - c_1)\right)\right], & \alpha \leq w_A^L; \alpha \leq w_A^L \\ \left(\left(\frac{\alpha}{w_A^L}(c_1 - a_1) + a_1, c_1 - \frac{\alpha}{w_A^L}(e_1 - c_1)\right), \Phi\right), & \alpha \leq w_A^L; \alpha > w_A^L \end{cases} \]
where $w_A^U \geq w_A^L$. 

362
3. Average Width of GFN

Let $A$ be a GFN whose membership function is

$$
\mu_A(x) = \begin{cases} 
0, & x < a \\
w, & \frac{x-a}{b-a}, a \leq x \leq b \\
w, & \frac{c-x}{c-b}, b \leq x \leq c \\
0, & x > c.
\end{cases}
$$

Then the $\alpha$-cut of $A$ is

$$
\alpha \in [0, w].
$$

We calculate average width of GFN $A$ following the steps below:

Step 1: Consider consecutive $N$ numbers of values from $[0, w]$.

Step 2: Then find $\alpha$-cuts for each $\alpha$-values.

Step 3: Calculate width of each $\alpha$-cuts.

That is, $c - \frac{\alpha}{w}(c - b) - \left\{ \frac{\alpha}{w}(b - a) + a \right\} = (c - a)(1 - \frac{\alpha}{w})$.

Step 4: Sum up all the widths and divide by $N$.

That is, $\frac{\sum (c-a)(1-\frac{\alpha}{w})}{N}$, which will give the average width of the generalized fuzzy number $A$.

4. Arithmetic operations on Generalized IVFNs

In this section, we perform arithmetic operations between generalized interval valued fuzzy numbers (GIVFNs) based on width of interval concept where LMFs are considered as generalized triangular fuzzy sets.

Let $A$ and $B$ be two generalized IVFNs and their membership functions are given as

$$
\bar{\mu}_A(x) = \begin{cases} 
w^U_A \frac{x-a_1}{c_1-a_1}, & x \in [a_1, c_1] \\
w^L_A \frac{x+b_1}{c_1-b_1}, & x \in [b_1, c_1] \\
\end{cases}
$$

$UMF$

$$
\bar{\mu}_B(x) = \begin{cases} 
w^U_B \frac{c_1-x}{c_1-b_1}, & x \in [c_1, d_1] \\
w^L_B \frac{d_1-x}{d_1-c_1}, & x \in [c_1, d_1] \\
\end{cases}
$$

$LMF$
where $w_A^U = 1 = w_B^U$ and $w_A^L, w_B^L < 1$. Then, $\alpha-$ cut of GIVFNs $A$ and $B$ are
\[
\alpha A = \begin{cases} 
\left\{ \left( \left\lceil \frac{\alpha}{w_A^U} (c_1 - a_1) + a_1, e_1 - \frac{\alpha}{w_A^U} (e_1 - c_1) \right\rceil, \left\lfloor \frac{\alpha}{w_A^U} (e_1 - c_1) \right\rfloor \right) \right\}, & \text{when } \alpha \leq w_A^U, \\left\lfloor \frac{\alpha}{w_A^U} (c_1 - a_1) + a_1, e_1 - \frac{\alpha}{w_A^U} (e_1 - c_1) \right\rfloor, & \text{when } \alpha \leq w_A^L, \end{cases}
\]
\[
\alpha B = \begin{cases} 
\left\{ \left\lceil \frac{\alpha}{w_B^U} (c_2 - a_2) + a_2, e_2 - \frac{\alpha}{w_B^U} (e_2 - c_2) \right\rceil, \left\lfloor \frac{\alpha}{w_B^U} (e_2 - c_2) \right\rfloor \right\}, & \text{when } \alpha \leq w_B^U, \\left\lfloor \frac{\alpha}{w_B^U} (c_2 - a_2) + a_2, e_2 - \frac{\alpha}{w_B^U} (e_2 - c_2) \right\rfloor, & \text{when } \alpha \leq w_B^L, \end{cases}
\]
respectively.

It is well known that $\alpha-$ cut of GIVFN gives closed intervals, to perform arithmetic operation $\otimes$ between $A$ and $B$ it is necessary to consider all the four combinations (4.1) (i.e., $\alpha A \otimes \alpha B$). In the first combination, it is taken as $\alpha \leq \min(u_A^U, u_B^U)$ (i.e., $\alpha \leq h(A)$ & $\alpha \leq h(B)$). Similarly, for the other combinations $\alpha \leq \min(u_A^L, u_B^L)$, $\alpha \leq \min(u_A^U, u_B^L)$ and $\alpha \leq \min(u_A^L, u_B^U)$. Then
\[
\alpha A \otimes \alpha B = \begin{cases} 
\left\{ \left\lceil \frac{\alpha}{w_A^U} (c_1 - a_1) + a_1, e_1 - \frac{\alpha}{w_A^U} (e_1 - c_1) \right\rceil \otimes \left\lfloor \frac{\alpha}{w_B^U} (c_2 - a_2) + a_2, e_2 - \frac{\alpha}{w_B^U} (e_2 - c_2) \right\rfloor, & \text{when } \alpha \in [0, \min(u_A^U, u_B^U)] \right\}, & \text{when } \alpha \leq \min(w_A^L, w_B^L)] \end{cases}
\]
Where $\otimes$ indicates the basic four arithmetic operations $(+, -, \times, \div)$. Using Decomposition theorem four fuzzy sets will be obtained and among of these fuzzy sets UMF and LMF of the resulting IVFS can be evaluated using width of generalized fuzzy
number. Maximum and minimum average width will represent UMF and LMF of the resultant IVFS respectively.

**Theorem 4.1.** Addition of two generalized interval valued triangular fuzzy numbers produces a generalized interval valued fuzzy number in which UMF is triangular fuzzy set and LMF is trapezoidal fuzzy set.

**Proof.** Consider the first combination of $^\alpha A + ^\alpha B$ of (4.1). That is,

$$\left[ \frac{\alpha}{w_A}(c_1 - a_1) + a_1, e_1 - \frac{\alpha}{w_A}(e_1 - c_1) \right] + \left[ \frac{\alpha}{w_B}(c_2 - a_2) + a_2, e_2 - \frac{\alpha}{w_B}(e_2 - c_2) \right],$$

where $\alpha \in [0, \min(w_A, w_B)].$

(4.2)

$$\left[ \frac{\alpha}{w_A}(c_1 - a_1) + a_1 + \frac{\alpha}{w_B}(c_2 - a_2) + a_2, e_1 + \alpha(e_1 - c_1) + e_2 - \alpha(e_2 - c_2) \right].$$

To find the membership function $\mu_{^1A+B}(x)$, we equate both the first and second component of (4.2) to $x$, which gives

$$x = (a_1 + a_2) + \alpha \left( \frac{c_1 - a_1}{w_A} + \frac{c_2 - a_2}{w_B} \right)$$

and

$$x = (e_1 + e_2) + \alpha \left( \frac{e_1 - c_1}{w_A} + \frac{e_2 - c_2}{w_B} \right).$$

Now, expressing $\alpha$ in terms of $x$, and setting $\alpha = 0$ and $\alpha = 1$ in (4.2), we get $\alpha$ together with the domain of $x$, that is

(4.3) $\alpha = \begin{cases} \frac{x - (a_1 + a_2)}{\frac{c_1 - a_1}{w_A} + \frac{c_2 - a_2}{w_B}}, & x \in \left( a_1 + a_2, (a_1 + a_2) + w \left( \frac{c_1 - a_1}{w_A} + \frac{c_2 - a_2}{w_B} \right) \right) \\ (a_1 + a_2), & x \in \left( (a_1 + a_2), (a_1 + a_2) + w \left( \frac{c_1 - a_1}{w_A} + \frac{c_2 - a_2}{w_B} \right) \right) \end{cases}$

and

(4.4) $\alpha = \begin{cases} \frac{(e_1 + e_2) - x}{\frac{(e_1 - c_1)}{w_A} + \frac{(e_2 - c_2)}{w_B}}, & x \in \left( (e_1 + e_2) - w \left( \frac{e_1 - c_1}{w_A} + \frac{e_2 - c_2}{w_B} \right), (e_1 + e_2) \right) \\ (e_1 + e_2), & x \in \left( (e_1 + e_2), (e_1 + e_2) - w \left( \frac{e_1 - c_1}{w_A} + \frac{e_2 - c_2}{w_B} \right) \right) \end{cases}$

which gives the membership function of the fuzzy number $A + B$.
In a similar fashion, another three membership functions for the remaining combinations $\alpha A + \alpha B$ of (4.1) can be evaluated and respectively given below.

\[
\mu_{A+B}^1(x) = \begin{cases}
  x-(a_1+a_2), & x \in (a_1 + a_2), (a_1 + a_2) + w \left( \frac{c_1-a_1}{w_A} + \frac{c_2-a_2}{w_B} \right) \\
  \frac{a_1-a_2}{w_A} + \frac{c_2-a_2}{w_B}, & w, x \in (a_1 + a_2) + w \left( \frac{c_1-a_1}{w_A} + \frac{c_2-a_2}{w_B} \right), \\
  (e_1 + e_2) - w \left( \frac{e_1-c_1}{w_A} + \frac{e_2-c_2}{w_B} \right), & (e_1 + e_2) \\
  \frac{e_1-c_1}{w_A} + \frac{e_2-c_2}{w_B}, & (e_1 + e_2) = \frac{e_1-c_1}{w_A} + \frac{e_2-c_2}{w_B}.
\end{cases}
\]

\[
\mu_{A+B}^2(x) = \begin{cases}
  x-(a_1+b_2), & x \in (a_1 + b_2), (a_1 + b_2) + w \left( \frac{c_1-a_1}{w_A} + \frac{c_2-b_2}{w_B} \right) \\
  \frac{a_1-b_1}{w_A} + \frac{c_2-b_2}{w_B}, & w, x \in (a_1 + b_2) + w \left( \frac{c_1-a_1}{w_A} + \frac{c_2-b_2}{w_B} \right), \\
  (e_1 + d_2) - w \left( \frac{e_1-c_1}{w_A} + \frac{d_2-c_2}{w_B} \right), & (e_1 + d_2) \\
  \frac{e_1-c_1}{w_A} + \frac{d_2-c_2}{w_B}, & (e_1 + d_2) = \frac{e_1-c_1}{w_A} + \frac{d_2-c_2}{w_B}.
\end{cases}
\]

\[
\mu_{A+B}^3(x) = \begin{cases}
  x-(b_1+a_2), & x \in (b_1 + a_2), (b_1 + a_2) + w \left( \frac{c_1-b_1}{w_A} + \frac{c_2-a_2}{w_B} \right) \\
  \frac{c_1-b_1}{w_A} + \frac{c_2-a_2}{w_B}, & w, x \in (b_1 + a_2) + w \left( \frac{c_1-b_1}{w_A} + \frac{c_2-a_2}{w_B} \right), \\
  (d_1 + e_2) - w \left( \frac{d_1-c_1}{w_A} + \frac{e_2-c_2}{w_B} \right), & (d_1 + e_2) \\
  \frac{d_1-c_1}{w_A} + \frac{e_2-c_2}{w_B}, & (d_1 + e_2) = \frac{d_1-c_1}{w_A} + \frac{e_2-c_2}{w_B}.
\end{cases}
\]
The average width of these four membership function can be evaluated at the common height \( \min(w_i^U, w_i^L, w_i^U, w_i^L) \). The membership function whose average width is maximum represents UMF while the membership function whose average width is minimum represents LMF of the resultant IVFN. Shape of the UMF is triangular fuzzy set (as \( w_j^U = w_j^L = 1 \)), on the other hand LMF is generalized trapezoidal fuzzy set with height \( \min(w_i^U, w_i^L) \).

**Remark 4.2.** If \( w_i^U \) and \( w_i^L \) \(< 1\) then shape of the UMF will also be generalized trapezoidal fuzzy set with height \( \min(w_i^U, w_i^L) \).

**Theorem 4.3.** Subtraction of two generalized interval valued triangular fuzzy numbers produces a generalized interval valued fuzzy number in which UMF is triangular fuzzy set and LMF is trapezoidal fuzzy set.

**Proof.** Consider the first combination of \( ^\alpha A - ^\alpha B \) of (4.1). That is,

\[
\frac{\alpha}{w_A^U} (c_1 - a_1) + a_1, e_1 - \frac{\alpha}{w_A^U} (e_1 - c_1) - \left[ \frac{\alpha}{w_B^U} (c_2 - a_2) + a_2, e_2 - \frac{\alpha}{w_B^U} (e_2 - c_2) \right],
\]

where \( \alpha \in [0, \min(w_i^U, w_i^L)] \)

\[
= \left\{ \frac{\alpha}{w_A^U} (c_1 - a_1) + a_1 \right\} - \left\{ e_2 - \frac{\alpha}{w_B^U} (e_2 - c_2) \right\}, \left\{ e_1 - \frac{\alpha}{w_A^U} (e_1 - c_1) \right\} - \left\{ \frac{\alpha}{w_B^U} (c_2 - a_2) + a_2 \right\}
\]

which gives

\[
(a_1 - e_1) + \alpha \left\{ \frac{c_1 - a_1}{w_A^U} + \frac{c_2 - a_2}{w_B^U} \right\}, (a_1 - e_1) + \alpha \left\{ \frac{c_1 - a_1}{w_A^U} + \frac{c_2 - a_2}{w_B^U} \right\}.
\]

To find the membership function \( \mu_{A-B}^\alpha(x) \), we equate both the first and second component of (4.5) to \( x \), which gives

\[
x = (a_1 - e_2) + \alpha \left( \frac{c_1 - a_1}{w_A^U} + \frac{c_2 - a_2}{w_B^U} \right) \quad \text{and} \quad x = (e_1 - a_2) + \alpha \left( \frac{c_1 - a_1}{w_A^U} + \frac{c_2 - a_2}{w_B^U} \right).
\]
Now expressing $\alpha$ in terms of $x$, and setting $\alpha = 0$ and $\alpha = 1$ in (4.5), we obtain $\alpha$ together with the domain of $x$, that is

$$
\alpha = \frac{x - (a_1 - e_2)}{\frac{a_1 - a_2}{w_A^I} + \frac{c_2 - c_2}{w_B^I}}, \; x \in \left[ (a_1 - e_2), (a_1 - e_2) + w \left\{ \frac{c_1 - a_1}{w_A^I} + \frac{c_2 - c_2}{w_B^I} \right\} \right]
$$

$$
\alpha = \frac{(e_1 - a_2) - x}{\frac{e_1 - c_1}{w_A^I} + \frac{c_2 - c_2}{w_B^I}}, \; x \in \left[ (e_1 - a_2) - w \left\{ \frac{e_1 - c_1}{w_A^I} + \frac{c_2 - a_2}{w_B^I} \right\}, (e_1 - a_2) \right]
$$

where $w = \min(w_A^I, w_B^I)$ and $\alpha \in [0, w]$. Which gives the following membership function

$$
\mu_{A-B}^1(x) = \begin{cases}
    \frac{x - (a_1 - e_2)}{\frac{a_1 - a_2}{w_A^I} + \frac{c_2 - c_2}{w_B^I}}, & x \in \left[ (a_1 - e_2), (a_1 - e_2) + w \left\{ \frac{c_1 - a_1}{w_A^I} + \frac{c_2 - c_2}{w_B^I} \right\} \right] \\
    \frac{e_1 - a_2 - x}{\frac{e_1 - c_1}{w_A^I} + \frac{c_2 - a_2}{w_B^I}}, & x \in \left[ (e_1 - a_2) - w \left\{ \frac{e_1 - c_1}{w_A^I} + \frac{c_2 - a_2}{w_B^I} \right\}, (e_1 - a_2) \right]
\end{cases}
$$

Similarly, membership functions of other remaining combinations can be evaluated and respectively depicted below.

$$
\mu_{A-B}^2(x) = \begin{cases}
    \frac{x - (a_1 - d_2)}{\frac{a_1 - a_1}{w_A^I} + \frac{d_2 - c_2}{w_B^I}}, & x \in \left[ (a_1 - d_2), (a_1 - d_2) + w \left\{ \frac{c_1 - a_1}{w_A^I} + \frac{d_2 - c_2}{w_B^I} \right\} \right] \\
    \frac{e_1 - b_2 - x}{\frac{e_1 - c_1}{w_A^I} + \frac{c_2 - b_2}{w_B^I}}, & x \in \left[ (e_1 - b_2) - w \left\{ \frac{e_1 - c_1}{w_A^I} + \frac{c_2 - b_2}{w_B^I} \right\}, (e_1 - b_2) \right]
\end{cases}
$$
\[
\begin{align*}
\mu_{A-B}^3(x) &= \begin{cases}
\frac{x-(b_1-e_2)}{c_1-b_1 + \frac{e_2-c_2}{w_B}}, & x \in [(b_1-e_2), (b_1-e_2) + w \left(\frac{c_1-b_1}{w_A} + \frac{e_2-c_2}{w_B}\right)] \\
\frac{c_1-b_1 + \frac{e_2-c_2}{w_B} - x}{d_1-c_1 + \frac{e_2-b_2}{w_B}}, & x \in [(b_1-e_2) - w \left(\frac{c_1-b_1}{w_A} + \frac{e_2-c_2}{w_B}\right), (d_1-a_2)]
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\mu_{A-B}^4(x) &= \begin{cases}
\frac{x-(b_1-d_2)}{c_1-b_1 + \frac{d_2-c_2}{w_B}}, & x \in [(b_1-d_2), (b_1-d_2) + w \left(\frac{c_1-b_1}{w_A} + \frac{d_2-c_2}{w_B}\right)] \\
\frac{c_1-b_1 + \frac{d_2-c_2}{w_B} - x}{d_1-c_1 + \frac{d_2-b_2}{w_B}}, & x \in [(b_1-d_2) - w \left(\frac{c_1-b_1}{w_A} + \frac{d_2-c_2}{w_B}\right), (d_1-d_2)]
\end{cases}
\end{align*}
\]

Similarly, the UMF and LMF of the resultant IVFN can be evaluated using average width concept at the common height \(\min(w_A^L, w_B^L, w_A^U, w_B^U)\). Here also, maximum and minimum average width represent UMF and LMF of the resultant IVFN respectively. The shape of the UMF is triangular fuzzy set (as \(w_A^U = w_B^U = 1\)), on the other hand LMF is generalized trapezoidal fuzzy set with height. \(\square\)

**Remark 4.4.** Here also, if \(w_A^L \& w_B^L < 1\), then shape of the UMF will also be generalized trapezoidal fuzzy set with height \(\min(w_A^L, w_B^L)\).

**Theorem 4.5.** Multiplication of two generalized interval valued triangular fuzzy numbers produces a generalized interval valued fuzzy number in which UMF is triangular type fuzzy set and LMF is trapezoidal type fuzzy set.

**Proof.** Consider the first combination of \(^\alpha A^\alpha B\) of (4.1). That is,

\[
\left[\frac{\alpha}{w_A^c_1} (c_1 - a_1) + a_1, c_1 - \frac{\alpha}{w_A^c_1} (e_1 - c_1)\right] \left[\frac{\alpha}{w_B} (c_2 - a_2) + a_2, c_2 - \frac{\alpha}{w_B} (e_2 - c_2)\right],
\]

where \(\alpha \in [0, \min(w_A^L, w_B^L)]\).

\[
(4.6) \left\{\begin{array}{ll}
\frac{\alpha}{w_A^c_1} (c_1 - a_1) + a_1 & \frac{\alpha}{w_B^c_1} (c_2 - a_2) + a_2 \\
\frac{\alpha}{w_A^c_1} (e_1 - c_1) & \frac{\alpha}{w_B^c_1} (e_2 - c_2)
\end{array}\right\}
\]

369
To find the membership function $\mu_{AB}(x)$, we equate both the first and second component of (4.6) to $x$, which gives

$$x = \left\{ \frac{\alpha}{w_A}(c_1 - a_1) + a_1 \right\} \left\{ \frac{\alpha}{w_B}(c_2 - a_2) + a_2 \right\}.$$

Then

$$x = \frac{\alpha^2}{w_A w_B} (c_1 - a_1)(c_2 - a_2) + \frac{\alpha}{w_A} a_2 (c_1 - a_1) + \frac{\alpha}{w_B} a_1 (c_2 - a_2) + a_1 a_2.$$

Thus

$$\alpha^2 \frac{(c_1-a_1)(c_2-a_2)}{w_A w_B} + \alpha \left\{ \frac{a_2 (c_1-a_1)}{w_A} + \frac{a_1 (c_2-a_2)}{w_B} \right\} + (a_1 a_2 - x) = 0.$$

This is a quadratic equation and solving. So we get

$$\alpha = \frac{\left\{ a_2 (c_1-a_1) w_B^U + a_1 (c_2-a_2) w_A^U \right\} \sqrt{\{a_2(c_1-a_1)w_B^U + a_1(c_2-a_2)w_A^U\}^2 - 4w_A^U w_B^U (c_1-a_1)(c_2-a_2) (a_1 a_2 - x)}}{2(c_1-a_1)(c_2-a_2)}.$$

Similarly, $x = \left\{ e_1 - \frac{\alpha}{w_A} (e_1 - c_1) \right\} \left\{ e_2 - \frac{\alpha}{w_B} (e_2 - c_2) \right\}$ gives

$$\alpha = \left\{ e_2 (e_1-c_1) w_B^U + e_1 (e_2-c_2) w_A^U \right\} - \sqrt{\{e_2(e_1-c_1)w_B^U + e_1(e_2-c_2)w_A^U\}^2 - 4w_A^U w_B^U (e_1-c_1)(e_2-c_2)(e_1 e_2 - x)}}{2(e_1-c_1)(e_2-c_2)}.$$

Now, setting $\alpha = 0$ and $\alpha = 1$ in (4.6), we have the membership function $\mu_{AB}(x)$. 

370
\[
\mu_{1AB}(x) = \begin{cases} 
-\{a_2(c_1-a_1)w_B^{2}+a_1(c_2-a_2)w_A^{2}\} + \sqrt{\{a_2(c_1-a_1)w_B^{2}+a_1(c_2-a_2)w_A^{2}\}^2 - 4w_A^{4}w_B^{4}(c_1-a_1)(c_2-a_2)(a_1a_2-x)} & \\
\frac{1}{2(c_1-a_1)(c_2-a_2)} 
\end{cases} 
\]

if \( x \in \left[ a_1a_2, a_1a_2 + \frac{w_A^{2}}{w_A^{4}w_B^{4}}(c_1-a_1)(c_2-a_2) + \frac{w_B}{w_A}a_2(c_1-a_1) + \frac{w_B}{w_A}a_1(c_2-a_2) \right] \)

\[
\mu_{2AB}(x) = \begin{cases} 
-\{b_2(c_1-a_1)w_B^{2}+a_1(c_2-a_2)w_A^{2}\} + \sqrt{\{b_2(c_1-a_1)w_B^{2}+a_1(c_2-a_2)w_A^{2}\}^2 - 4w_A^{4}w_B^{4}(c_1-a_1)(c_2-a_2)(a_1b_2-x)} & \\
\frac{1}{2(c_1-a_1)(c_2-b_2)} 
\end{cases} 
\]

if \( x \in \left[ a_1b_2, a_1b_2 + \frac{w_A^{2}}{w_A^{4}w_B^{4}}(c_1-a_1)(c_2-b_2) + \frac{w_B}{w_A}b_2(c_1-a_1) + \frac{w_B}{w_A}a_1(c_2-b_2) \right] \)

Similarly, membership functions for remaining combinations can be calculated and respectively listed below.
\[ \begin{align*}
\mu_{AB}(x) &= \begin{cases} 
\frac{-\{a_2(c_1-b_1)w_A^L + b_1(c_2-a_2)w_A^L\} + \sqrt{\{a_2(c_1-b_1)w_A^L + b_1(c_2-a_2)w_A^L\}^2 - 4w_A^L w_A^L (c_1-b_1)(c_2-a_2)(b_1a_2-x)}}{2(c_1-b_1)(c_2-a_2)} 
&\text{if } x \in \left[b_1a_2, b_1a_2 + \frac{w_A^L}{w_A^L} (c_1 - b_1)(c_2 - a_2) + \frac{w_B^L}{w_B^L} b_1(c_2 - a_2)\right], \\
&\frac{w - x}{w_A^L} \left(1 - \frac{x}{w_B^L}\right) + b_1(c_2-a_2), \\
\end{cases} \\
\mu_{AB}(x) &= \begin{cases} 
\frac{-\{b_2(c_1-b_1)w_B^L + b_2(c_2-b_2)w_B^L\} + \sqrt{\{b_2(c_1-b_1)w_B^L + b_2(c_2-b_2)w_B^L\}^2 - 4w_A^L w_B^L (c_1-b_1)(c_2-b_2)(b_1b_2-x)}}{2(c_1-b_1)(c_2-b_2)} 
&\text{if } x \in \left[b_1b_2, b_1b_2 + \frac{w_A^L}{w_B^L} (c_1 - b_1)(c_2 - b_2) + \frac{w_B^L}{w_B^L} b_2(c_1 - b_1) + \frac{w_B^L}{w_B^L} b_2(c_1 - b_2)\right], \\
&\frac{w - x}{w_A^L} \left(1 - \frac{x}{w_B^L}\right) + b_2(c_2-b_2), \\
&\frac{w - x}{w_B^L} \left(1 - \frac{x}{w_B^L}\right) + b_2(c_2-b_2), \\
\end{cases} \\
\{d_2(d_1-c_1)w_A^L + d_1(d_2-c_2)w_A^L\} - \sqrt{\{d_2(d_1-c_1)w_A^L + d_1(d_2-c_2)w_A^L\}^2 - 4w_A^L w_A^L (d_1-c_1)(d_2-c_2)(d_1d_2-x)}}{2(d_1-c_1)(d_2-c_2)} 
&\text{if } x \in \left[d_1d_2, d_1d_2 + \frac{w_A^L}{w_A^L} (d_1 - c_1)(d_2 - c_2) + \frac{w_A^L}{w_A^L} d_2(d_1 - c_1) + \frac{w_A^L}{w_A^L} d_2(d_2 - c_2)\right]. \\
\end{align*} \]
Here too, the UMF and LMF of the resultant IVFN can be evaluated using average width concept at the common height $\min(w_A^U, w_A^L, w_B^U, w_B^L)$. The maximum and minimum average width represent UMF and LMF of the resultant IVFN respectively. The Shape of the UMF is triangular type fuzzy set (as $w_A^U = w_B^L = 1$), on the other hand LMF is generalized trapezoidal type fuzzy set with height $\min(w_A^L, w_B^L)$.

Remark 4.6. If $w_A^U \& w_B^U < 1$ then shape of UMF will also be generalized trapezoidal type fuzzy set with height $\min(w_A^U, w_B^U)$.

Theorem 4.7. Division of two generalized interval valued triangular fuzzy numbers produces a generalized interval valued fuzzy number in which UMF is triangular type fuzzy set and LMF is trapezoidal type fuzzy set.

Proof. Consider the first combination of $\alpha A \div \alpha B$ of (4.1). That is,

$$\left[ \frac{\alpha}{w_A}(c_1 - a_1) + a_1, e_1 - \frac{\alpha}{w_A}(e_1 - c_1) \right] \div \left[ \frac{\alpha}{w_B}(c_2 - a_2) + a_2, e_2 - \frac{\alpha}{w_B}(e_2 - c_2) \right],$$

where $\alpha \in [0, \min(W_A^U, W_B^U)]$. That is,

$$\left\{ \frac{\alpha}{w_A}(c_1 - a_1) + a_1, e_1 - \frac{\alpha}{w_A}(e_1 - c_1) \right\} \left\{ e_2 - \frac{\alpha}{w_B}(e_2 - c_2), \frac{\alpha}{w_B}(c_2 - a_2) + a_2 \right\}.$$  

(4.7)

To find the membership function $\mu_{A \div B}^1$ we equate both the first and second component of (4.7) to $x$, we have

$$x = \left\{ \frac{\alpha}{w_A}(c_1 - a_1) + a_1 \right\} \text{ and } x = \left\{ e_1 - \frac{\alpha}{w_A}(e_1 - c_1) \right\}.$$  

Now expressing $\alpha$ in terms of $x$, and setting $\alpha = 0$ and $\alpha = 1$ in (4.7), we get $\alpha$ together with the domain of $x$, Now expressing in terms of $x$, and setting and in (4.7), we get together with the domain of $x$, that is,

$$\alpha = \left\{ \frac{e_2x - a_1}{(c_1 - a_1 + e_2 - c_2)x} \right\} , x \in \left\{ \frac{\alpha}{w_A}(c_1 - a_1) + a_1, a_1 \right\}.$$  

and

$$\alpha = \left\{ \frac{e_1 - b_2x}{(c_1 - a_1 + e_2 - c_2)x} \right\} , x \in \left\{ \frac{\alpha}{w_A}(c_1 - a_1) + a_1, a_1 \right\}.$$

Then the membership function of $\mu_{A \div B}^1(x)$ the resultant INFN is obtained as

$$\mu_{A \div B}^1(x).$$
which are given below. Similarly, membership functions for other combinations can be evaluated and which are given below.

\[
\begin{align*}
\mu_{A \oplus B}^1(x) &= W, \quad x \in \left[ \frac{a_1}{e_1 - a_1 + a_1}, \frac{e_2 - a_1 + a_1}{w_A} \right], \\
\mu_{A \oplus B}^2(x) &= W, \quad x \in \left[ \frac{a_1}{e_1 - a_1 + a_1}, \frac{e_2 - a_1 + a_1}{w_A} \right], \\
\mu_{A \oplus B}^3(x) &= W, \quad x \in \left[ \frac{a_1}{e_1 - a_1 + a_1}, \frac{e_2 - a_1 + a_1}{w_A} \right], \\
\mu_{A \oplus B}^4(x) &= W, \quad x \in \left[ \frac{a_1}{e_1 - a_1 + a_1}, \frac{e_2 - a_1 + a_1}{w_A} \right].
\end{align*}
\]
In this case also, the UMF and LMF of the resultant IVFN can be evaluated using average width concept at the common height $\min(w_A^U, w_B^U, w_A^L, w_B^L)$ where maximum and minimum average width represent UMF and LMF of the resultant IVFN respectively. The Shape of the UMF is triangular type fuzzy set (as $w_A^U = w_B^U = 1$), the LMF is generalized trapezoidal type fuzzy set with height $\min(w_A^L, w_B^L)$.

**Remark 4.8.** In this case also it is observed that if $w_A^U \& w_B^U < 1$ then shape of UMF will also be generalized trapezoidal type fuzzy set with height $\min(w_A^L, w_B^L)$.

5. **Numerical Examples**

Let $A$ and $B$ be two interval valued generalized fuzzy sets whose membership functions are given as

$$
\mu_A(x) = \begin{cases} 
\frac{x-10}{15}, & x \in [10, 25] \\
\frac{40-x}{15}, & x \in [25, 40] \\
0.8\frac{x-20}{5}, & x \in [20, 25] \\
0.8\frac{40-x}{5}, & x \in [25, 30]
\end{cases}
\mu_B(x) = \begin{cases} 
\frac{x-2}{7}, & x \in [2, 6] \\
\frac{10-x}{4}, & x \in [6, 10] \\
0.7\frac{x-4}{2}, & x \in [4, 6] \\
0.7\frac{10-x}{2}, & x \in [6, 8]
\end{cases}
$$

Then $\alpha$-Cut of the fuzzy members $A$ and $B$ are

$$
A^\alpha = \{[15\alpha + 10, 40 - 15\alpha], \alpha \in [0, 1]; \frac{5}{0.8}\alpha + 20, 30 - \frac{5}{0.8}\alpha], \alpha \in [0.0, 0.8]\}
$$

$$
B^\alpha = \{[4\alpha + 2, 10 - 4\alpha], \alpha \in [0, 1]; \frac{2}{0.7}\alpha + 4.8 - \frac{2}{0.7}\alpha], \alpha \in [0.0, 0.7]\}
$$

respectively.

Now, to add fuzzy numbers $A$ and $B$,

$$
A^\alpha + B^\alpha = \begin{cases} 
[19\alpha + 12, 50 - 19\alpha], & \alpha \in [0, 0.1]
\end{cases}
\begin{cases} 
\left[\left(15 + \frac{2}{0.7}\right)\alpha + 14.48 - (15 + \frac{2}{0.7}\alpha)ight], & \alpha \in [0, 0.1]
\end{cases}
\begin{cases} 
\left[\left(\frac{2}{0.7}\right)\alpha + 20 - (\frac{2}{0.7}\alpha)ight], & \alpha \in [0, 0.4]
\end{cases}
\begin{cases} 
\left[\left(\frac{5}{0.8} + \frac{2}{0.7}\right)\alpha + 24.38 - \left(\frac{5}{0.8} + \frac{2}{0.7}\alpha\right), & \alpha \in [0, 0.7, 0.8]\right]
\end{cases}
$$


Then the four membership functions are $\mu_{A+B}^1(x) = \begin{cases} 
\frac{x-12}{19}, & x \in [12, 31]
\end{cases}$ $\begin{cases} 
\frac{50-x}{19}, & x \in [31, 50]
\end{cases}$, $\mu_{A+B}^2(x) = \begin{cases} 
\frac{0.7(x-14)}{12.5}, & x \in [14, 26.5]
\end{cases}$ $\begin{cases} 
\frac{0.7(38-x)}{12.5}, & x \in [35.5, 48]
\end{cases}$, $\mu_{A+B}^3(x) = \begin{cases} 
\frac{0.8(x-22)}{8.2}, & x \in [22, 30.2]
\end{cases}$ $\begin{cases} 
\frac{0.8(40-x)}{8.2}, & x \in [31.8, 40]
\end{cases}$, $\mu_{A+B}^4(x) = \begin{cases} 
\frac{0.7(x-24)}{6.375}, & x \in [24, 30.375]
\end{cases}$ $\begin{cases} 
\frac{0.7(38-x)}{6.375}, & x \in [31.625, 38]
\end{cases}$.

Thus the average width of the four membership functions are 24.7, 21.5, 10.825 and 7.625, respectively. Here, maximum average width is 24.7 and minimum average width is 7.625 and hence membership function of the resulting IVFS is $\mu_{A+B}(x) = \begin{cases} 
\frac{x-12}{19}, & x \in [12, 31]
\end{cases}$ $\begin{cases} 
\frac{50-x}{19}, & x \in [31, 50]UMF
\end{cases}$ $\begin{cases} 
\frac{0.7(x-24)}{6.375}, & x \in [24, 30.375]
\end{cases}$ $\begin{cases} 
\frac{0.7(38-x)}{6.375}, & x \in [31.625, 38]LMF.
\end{cases}$
Similarly, membership functions for subtraction, multiplication and division can be obtained and which are respectively.

\[ \mu_{A-B}(x) = \begin{cases} 
\frac{x}{19}, & x \in [0, 19] \\
\frac{38-x}{19}, & x \in [19, 38] \text{UMF} \\
0.7\frac{x-12}{6.375}, & x \in [12, 18.375] \\
0.7\frac{38-x}{5.375}, & x \in [19.625, 26] \text{LMF},
\end{cases} \]

\[ \mu_{A\times B}(x) = \begin{cases} 
\frac{-35+\sqrt{25+60x}}{60}, & x \in [20, 150] \text{UMF} \\
\frac{60}{60}, & x \in [150, 400] \\
0.7-\frac{46+\sqrt{324+22.25x}}{14}, & x \in [80, 146.25] \text{LMF} \\
0.7-\frac{76-\sqrt{400+22.25x}}{14}, & x \in [153.75, 240],
\end{cases} \]

\[ \mu_{A/B}(x) = \begin{cases} 
\frac{10x-10}{4x+15}, & x \in [1, 4.166] \\
\frac{40-2x}{2x+15}, & x \in [4.166, 20] \text{UMF} \\
0.7\frac{8x-20}{2x+4.375}, & x \in [2.5, 4.0625] \\
0.7\frac{30-x}{2x+4.375}, & x \in [4.2708, 7.5] \text{LMF}.
\end{cases} \]

6. Conclusions

In general, real world problems are ill defined due to lack of data, imprecision, vagueness, partial information etc. More often, type-I fuzzy set is explored to deal with such real word problems. However, in some situations, generalized IVFNs may come into picture. In this paper, a new approach has been devised to combine generalized IVFNs based on average width of fuzzy set concept. In this study, it is found that the shape of resultant fuzzy number is generalized IVFN where shape of UMF is triangular type fuzzy set while LMF is trapezoidal type fuzzy number. This approach can also combine completely generalized IVFNs (i.e., IVFNs whose height of UMF and LMF are strictly less than 1). In this case, the resultant IVFN is completely generalized IVFNs in which shape of both the UMF and LMF are trapezoidal type fuzzy sets. Numerical illustrations have been exhibited to check the validity of the proposed approach.

Acknowledgements. Author would like to thank anonymous referees for their helpful comments and suggestions. Author also would like to thank Ms Rashmi Rekha Bhuyan for her continuous inspiration and motivation to complete this work.

References

[1] J. Y. Ahn, K. H. Choi and J. H. Park, A Headache Diagnosis Method Using an Aggregate Operator, Communications of the Korean Statistical Society 19 (2012) 359–365.
[2] J. Y. Ahn, K. S. Han, S. Y. Oh and C. D. Lee, An application of interval-valued intuitionistic fuzzy sets for medical diagnosis of headache, International Journal of Innovative Computing, Information and Control 7 (2011) 2755–2762.
[3] H. Bustince and P. Burillo, Vague sets are intuitionistic fuzzy sets, Fuzzy Sets and Systems 79 (1996) 403–405.
[4] C. Carlson, R. Fuller and J. Mezei, On Mean Value and Variance of Interval-Valued Fuzzy Numbers, Communications in Computer and Information Science 299 (2012) 19–28.
[5] C. Carlsson and R. Fuller and Mezei, Project Selection With Interval-Valued Fuzzy Numbers, Int. Symp. On IEEE 12th Computational Intelligence and Informatics (Cinti) (2011) 23–26.
[6] S. M. Chen and J. H. Chen, Fuzzy risk based on similarity measures between interval-valued fuzzy numbers and interval-valued fuzzy number arithmetic operators, Expert Systems with Applications 36 (2009) 6309–6317.
[7] S. J. Chen and S. M. Chen, Fuzzy risk analysis based on measures of similarity between intervals valued fuzzy numbers, Computers and Mathematics with Applications 55 (2008) 1670–1685.
[8] S. J. Chen, A new method for handling the similarity measure problems of interval valued fuzzy numbers, Proc. of the Second International Conference on Natural Computation and the Third International Conference on Fuzzy Systems and Knowledge Discovery, China (2006) 25–334.
[9] S. H. Chen, Operations on fuzzy numbers with function principal, Tamkang Journal of Management Sciences 6 (1985) 13–25.
[10] H. Choi, G. Mun, J. Ahn and C. Korea, A medical diagnosis based on interval valued fuzzy set, Biomedical Engineering: APPLICATIONS, Basis and Communications 24 (2012) 349–354.
[11] A. Das , M. Pal and M. Bhowmik , Permanent of interval-valued and triangular number fuzzy matrices, Ann. Fuzzy Math. Inform. 10 (3) (2015) 381–395.
[12] P. Dutta, Comparison of Arithmetic Operations of Generalized Fuzzy Numbers: Case Study in Risk Assessment, cybernetics and systems 47 (2016) 290–320.
[13] P. Dutta, H. Boruah, and T. Ali, Fuzzy Arithmetic with and without using α-cut method: A Comparative Study, International Journal of Latest Trends in Computing 2 (2011) 99–107.
[14] S. Elizabeth and L. Sujatha, Medical Diagnosis Based on Interval Valued Fuzzy Number Matrices, Annals of Pure and Applied Mathematics 7 (2014) 91–96.
[15] M. Gehrke, C. Walker and E. Walker, Some Comments on Interval-Valued Fuzzy Sets, Int. Journal of Intelligent Systems 11 (1996) 751–759.
[16] M. B. Gorzalczany, A method of inference in approximate reasoning based on interval-valued fuzzy sets, Fuzzy Sets and Systems 21 (1987) 1–17.
[17] M. B. Gorzalczany, Approximate inference with interval-valued fuzzy sets—an outline, In: Proceedings of the Polish Symposium on Interval and Fuzzy Mathematics, Poland, Oznan (1983) 89-95.
[18] G. I. Grattan, Fuzzy membership mapped onto interval and many-valued quantities, MLQ Math. Log. Q. 22 (1975) 149–160.
[19] P. Grzegorzewski, Distances between intuitionistic fuzzy sets and/or interval-valued fuzzy sets based on the hausdorff metric, Fuzzy Sets and Systems 48 (2004) 319–328.
[20] R. W. Jozsef, Aggregation Operators and Interval-Valued Fuzzy Numbers in Decision Making, Adv. Intell. Syst. Comput. 206 (2013) 535–544.
[21] H. Hamrawi, Type-2 Fuzzy Alpha-cuts' Ph. D. Dissertation, De Montfort University 2011.
[22] M. M. Khalaf, Medical diagnosis via interval valued intuitionistic fuzzy sets, Ann. Fuzzy Math. Inform. 6 (2) (2013) 245–249.
[23] C. Li, Distances between interval-valued fuzzy sets, Proceedings of the 28th North American Fuzzy Information Processing Society Annual Conference (NAFIPS2009), Cincinnati, USA, DOI:10.1109/NAFIPS.2009.5156438, June, 14-17, 2009.
[24] F. T. Lin, Fuzzy job-shop scheduling based on ranking level (λ, 1) interval-valued fuzzy numbers, IEEE Transactions on Fuzzy Systems 10 (2002) 510–522.
[25] A. R. Meenakshi and M. Kaliraja, An application of interval valued fuzzy matrices in medical diagnosis, International Journal of Mathematical Analysis 5 (2011) 1791–1802.
[26] A. Mukherjee and S. Sarkar, Similarity measures of interval-valued fuzzy soft sets and their application in decision making problems, Ann. Fuzzy Math. Inform. 8 (9) (2014) 447–460.
[27] J. H. Park, K. M. Lim, J. S. Park and Y. C. Kwun, Distances between interval-valued intuitionistic fuzzy sets, Journal of Physics: Conference Series 96 (2008) 1–8.
[28] P. Rajarajewari, P. Dhanalakshmi, An application of interval valued intuitionistic fuzzy soft matrix theory in medical diagnosis, Ann. Fuzzy Math. Inform. 9 (3) (2015) 463–472.
[29] T. Rashid, I. Beg and S. M. Husnine, Robot selection by using generalized interval-valued fuzzy numbers with TOPSIS, Applied Soft Computing 21 (2014) 462-468.
[30] R. Sambuc, Fonctions f-floues application ‘l aide an diagnostic en pathologie thyroidienne, Ph.D. Thesis, University of Marseille 1975.

[31] B. Tomas and S. Zeng, Group Multi-Criteria Decision Making Based Upon Interval-Valued Fuzzy Numbers: An Extension of the MULTIMOORA Method, Expert Systems with Applications 40 (2013) 543–550.

[32] I. B. Turksen, Interval-valued strict preference with Zadeh triples, Fuzzy Sets and Systems 78 (1996) 183–195.

[33] I. B. Turksen, Interval-valued fuzzy sets and compensatory, Fuzzy Sets and Systems 51 (1992) 295–307.

[34] I. B. Turksen, Interval valued fuzzy sets based on normal forms, Fuzzy Sets and Systems 20 (1986) 191–210.

[35] G. Wang and X. Li, Correlation and information energy of interval-valued fuzzy number, Fuzzy Sets and Systems 103 (2001) 169–175.

[36] G. Wang and X. Li, The applications of interval-valued fuzzy numbers and interval-distribution numbers, Fuzzy Sets and Systems 98 (1998) 331–335.

[37] S. H. Wei and S. M. Chen, Fuzzy risk analysis based on interval-valued fuzzy numbers, Expert Systems with Applications 36 (2009) 228–2299.

[38] J. S. Yao and F. T. Lin, Constructing a fuzzy flow-shop sequencing model based on statistical data, International Journal of Approximate Reasoning 29 (2002) 215–234.

[39] Z. Yue, Deriving decision makers weights based on distance measure for interval-valued intuitionistic fuzzy group decision making, Expert Systems with Applications 38 (2011) 11665–11670.

[40] L. A. Zadeh, Fuzzy set, Information and Control 8 (1965) 338–353.

[41] W. Y. Zeng and P. Guo, Normalized distance, similarity measure, inclusion measure and entropy of interval-valued fuzzy sets and their relationship, Inform. Sci. 178 (2008) 1334–1342.

Palash Dutta (palash.dtt@gmail.com)
Dept. of Mathematics, Dibrugarh University, Dibrugarh-786004, India