XXZ spin chain in transverse field as a regularization of the sine-Gordon model

Silvio Pallua and Predrag Prester

Department of Theoretical Physics, University of Zagreb
Bijenička c.32, POB 162, 10001 Zagreb, Croatia

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We consider here XXZ spin chain perturbed by the operator $\sigma^x$ ("in transverse field") which is a lattice regularization of the sine-Gordon model. This can be shown using conformal perturbation theory. We calculated mass ratios of particles which lie in a discrete part of the spectrum and obtained results in accord with the DHN formula and in disagreement with recent calculations in literature based on numerical Bethe Ansatz and infinite momentum frame methods. We also analysed a short distance behavior of this states (UV or conformal limit). Our result for conformal dimension of the second breather state is different from the conjecture in [Klassen and Melzer, Int. J. Mod. Phys. A 8, 4131 (1993)] and is consistent with this paper for other states.

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I. INTRODUCTION

The sine-Gordon (SGM) and massive Thirring (MTM) models in two dimensions belong to a group of the most studied QFT’s and are certainly the best understood nontrivial massive field theories. A large number of different techniques have been successfully tested on these models and they led us to a number of interesting results, including the famous duality relation between them [1–3].

Regarding a mass spectrum, we can classify all methods in basically three groups: (a) the semi-classical DHN method [4], (b) factorized scattering theory [5] and (c) methods based on Bethe Ansatz, which can be further subdivided in continuum ones [6] and in discrete ones [7] (some lattice regularizations were used). Results of all methods were the same; beside soliton and antisoliton (fermion and antifermion in MTM language) there are bound states (breathers) and their masses are given by:

$$m_n = 2m \sin \frac{n\pi \beta^2}{2(8\pi - \beta^2)} , \quad n = 1, 2, \ldots < \frac{8\pi}{\beta^2} - 1 \quad (1.1)$$

where $m$ is the soliton mass and $\beta$ is the coupling constant in SGM (see [2]). Because of the Coleman’s theorem of equivalence between the SGM and the MTM in soliton number (charge) zero sector (proved using perturbative expansion in mass), the same spectrum should be valid for the MTM. Using standard conventions (as in [4]), a connection ("duality relation") between $\beta$ and the MTM coupling constant $g_0$ (in the Schwinger normalization) is given by:

$$1 + \frac{g_0}{\pi} = \frac{4\pi}{\beta^2}$$

However, recently appeared claims [10–12] that mass spectrum of the MTM is different than (1.1) and that there is only one breather in the whole interval $g_0 > 0$ (for negative values of $g_0$ fermion and antifermion repel each other and there are no bound states, like in (1.1)). In [6], using the infinite momentum frame technique and working only in $q\bar{q}$ sector of the Fock space (neglecting $qqqq$ and higher fermion components), authors obtained the mass of the (only) breather:

$$M = 2m \cos \alpha \quad (1.2)$$

where the parameter $0 \leq \alpha < \pi/2$ is obtained by solving equation:

$$\frac{\tan \alpha}{\tan \alpha - \alpha} = \frac{g}{\pi} \left[ 1 + \frac{1}{\cos^2 \alpha} \left( 1 - \frac{g}{4\pi} \right) \right]$$

and $g$ is the MTM coupling constant in Johnson’s normalization which is connected to that in Schwinger’s normalization by:

$$g_0 = \frac{2g}{2 - \frac{g^2}{\pi}}$$

Afterwards, in [11] authors reexamined an analysis of [10], but contrary to [6] they numerically solved Bethe Ansatz equations for a finite space extension and a finite number of quasi-particles, and after that made an extrapolation to infinity. Their analysis confirmed results of [10]; they found only one breather, with the mass in good agreement with (1.2).

In this paper we propose ourselves to calculate certain properties of the SGM like mass ratios and scaling dimensions of operators creating particle states. Using the conformal perturbation theory [13,14] it can be shown that XXZ spin chain with an even number of sites and periodic boundary conditions in a transverse magnetic field ($\sigma^x$ perturbation) is spin chain regularization of the SGM (see Appendix B in [14]). We numerically diagonalize the spin chain Hamiltonian up to 16 sites and extrapolate results to infinite length continuum limit using the BST extrapolation algorithm [15,16]. The same method was previously applied to conformal unitary models perturbed by some relevant (usually thermal) operator [17–19]. In this way we can obtain estimates of mass ratios without further assumptions, particularly those criticized in [10–12].
Results of our analysis are as follows. For a whole range of the coupling constants we can cover (0 < β ≤ \sqrt{2π}) our results agree with the DHN formula (1.1) and disagree with (1.2), i.e. results of [10,11]. Of course, we couldn’t say anything about breathers higher than third because they lie in a continuum part of the spectrum (m_n > 2m_1 for n ≥ 3). We should also say that precision in this method is far from that achieved by, e.g. Bethe Ansatz methods, so we can’t claim that DHN formula is exact.

Finally, as a by product, we studied UV limit of particle states. It agrees with that conjectured in [4] for (anti)soliton and first breather. However, for the second breather we obtain the same scaling dimension as for the first, contrary to [4].

II. THE SGM AS A MASSIVE PERTURBATION OF THE GAUSSIAN MODEL

The SGM is a 1 + 1 dimensional field theory of a pseudoscalar field \( \varphi \), defined classically by the Lagrangian:

\[
\mathcal{L}_{SG} = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi + \lambda \cos(\beta \varphi) \quad (2.1)
\]

Here \( \lambda \) is a mass scale (with mass dimension depending on a regularization scheme), \( \beta \) is a dimensionless coupling (which does not renormalize) and one identifies field configurations that differ by a period \( 2\pi/\beta \) of the potential (because we want to have “ordinary” QFT with a unique vacuum).

In [14] it was shown that SGM can be viewed as a perturbed CFT when second term in (2.1) is treated as a (massive) perturbation. We’ll now repeat here relevant results of their analyses.

An unperturbed theory \( \lambda = 0 \) (approached in UV limit) is the free massless compactified pseudoscalar CFT (known as Gaussian model). It is conventional to use \( \Phi \equiv \sqrt{\pi} \varphi \), so that the radius of compactification \( r \), defined by equivalence \( \Phi \sim \Phi + 2\pi r \), is connected to \( \beta \) with

\[
r = \frac{\sqrt{\pi}}{\beta} \quad (2.2)
\]

Solution of the equation of motion in Euclidean space, \( \partial^2 \Phi(z, \bar{z}) = 0 \), is

\[
\Phi(z, \bar{z}) = \frac{1}{2} (\phi + \bar{\phi})
\]

The Gaussian model is a CFT with central charge \( c = 1 \) and an operator algebra generated by the primary fields \( V_{m,n} \)

\[
V_{m,n} = e^{iz\Phi(z, \bar{z}) + i2\pi(\varphi - \bar{\varphi})/(2\pi)}
\]

where \( \Phi \equiv (\phi - \bar{\phi})/2 \). Conformal dimensions of \( V_{m,n} \) are

\[
(\Delta_{m,n}, \bar{\Delta}_{m,n}) = \left( \frac{1}{2} \left( \frac{m}{2r} + nr \right)^2, \frac{1}{2} \left( \frac{m}{2r} - nr \right)^2 \right) \quad (2.4)
\]

so that it’s scaling dimension and (Lorentz) spin are:

\[
d_{m,n} = \Delta_{m,n} + \bar{\Delta}_{m,n} = \left( \frac{m}{2r} \right)^2 + (nr)^2 = \frac{m^2\beta^2}{4\pi} + \frac{n^2\pi}{\beta^2}
\]

\[
s_{m,n} = \Delta_{m,n} - \bar{\Delta}_{m,n} = mn
\]

It is understood that \( V_{m,n} \) are generated so that

\[
\langle V_{m,n}(z, \bar{z})V_{m,n}(0, 0) \rangle = \delta_{m,-m}\delta_{m,-m'}z^{-2\Delta_{m,n}}\bar{z}^{-2\bar{\Delta}_{m,n}}
\]

Because of \( V_{m,n}^+=V_{-m,-n} \), we can define hermitian combinations

\[
V^{(+)}_{m,n} = \frac{1}{2} (V_{m,n} + V_{-m,-n})
\]

\[
V^{(-)}_{m,n} = \frac{i}{2} (V_{m,-n} - V_{m,n})
\]

which will be useful later.

In [14] it is argued that an UV limit of the SGM is generated by

\[
L_B = \{ V_{m,n} | m, n \in \mathbb{Z} \} \quad (2.5)
\]

We suppose that Hilbert space of the full (perturbed) theory is isomorphic to that of the unperturbed theory. From (2.2) and (2.3) follows that a (properly normalized) perturbing operator in the SGM (2.1) is

\[
\cos(\beta \varphi) = V^{(+)}_{1,0} \quad (2.6)
\]

which means that \( \lambda \) has mass dimension \( y = 2 - d_{1,0} = 2 - \frac{\beta^2}{4\pi} \). From the condition of relevancy of the perturbation, i.e. \( y > 0 \), we obtain Coleman’s bound \( \beta^2 < 8\pi \). Also, from (2.3) and (2.4) we can see that SGM has \( \hat{U}(1) \times Z_2 \times \hat{Z}_2 \) internal symmetry group. The \( \hat{U}(1) \) acts as shift on \( \Phi \), i.e. \( V_{m,n} \rightarrow e^{i\alpha n}V_{m,n} \), while \( Z_2 \) and \( \hat{Z}_2 \) are generated by \( R: (\Phi, \bar{\Phi}) \rightarrow (-\Phi, \bar{\Phi}) \) (i.e. \( V_{m,n} \rightarrow V_{-m,n} \)) and \( \hat{R}: (\Phi, \bar{\Phi}) \rightarrow (\Phi, -\bar{\Phi}) \) (i.e. \( V_{m,n} \rightarrow V_{m,-n} \)), respectively.

To conclude this section, consider the SGM defined on a cylinder with infinite time dimension and space extension equal to \( L \). There are three independent constants with which we can express all quantities in the theory, \( \beta, \lambda \) and \( L \) with mass dimensions \( d_\beta = 0 \), \( d_\lambda = 2 - d_{1,0} = 2 - \beta^2/(4\pi) \) and \( d_L = -1 \). It is useful to define dimensionless scaling parameter \( \mu \)

\[
\mu \equiv LM^{d_X} = \lambda L^{2-\frac{d_\lambda}{2\pi}} \quad (2.7)
\]

and use \( \beta, \mu \) and \( \lambda \) as a set of independent parameters. Now, from ordinary dimensional analysis follows that any quantity \( X \) in the theory, with mass dimension \( d_X \), can be written as

\[
X = \lambda^{\frac{d_X}{2\pi}}g_X(\beta, \mu) = \lambda^{\frac{d_X}{2\pi}}g_X(\beta, \mu) \quad (2.8)
\]

where \( g_X \) is the scaling function connected to \( X \). We see that all dimensionless quantities depend only on \( \beta \) and \( \mu \). Especially, we have for masses of particles:
Now, there are two interesting limits. The first one is the infinite length limit, \( L \to \infty \), which is equal to \( \mu \to \infty \) (see (2.7)). We are interested here in mass ratios:

\[
    r_i(\beta) = \lim_{\mu \to \infty} \frac{m_{i+1}(\beta, \mu, \lambda)}{m_i(\beta, \mu, \lambda)} = \frac{G_{i+1}(\beta, \mu)}{G_i(\beta, \mu)}
\]

The second interesting limit is the UV limit given by \( L \to 0 \) (\( \mu \to 0 \)). Basic assumption of conformal perturbation theory is that the perturbed QFT should approach smoothly to CFT in UV limit. It means that if we write (2.8) in the form:

\[
    X = X_{CFT}(\beta, L) + \lambda^{\frac{2}{d}} h_X(\beta, \mu) \tag{2.10}
\]

where \( X_{CFT} \) is the value for \( X \) in conformal point (\( \lambda = 0 \)), than a Taylor expansion for \( \mu^d x/d_0 h_X(\beta, \mu) \) around \( \mu = 0 \) will have finite radius of convergence and \( h_X(\beta, 0) = 0 \). Specially, for the mass gaps we have well-known formula:

\[
(m_i)_{CFT} = \frac{2\pi}{L} d_i
\]

where \( d_i \) is the scaling dimension of the operator which creates that state from the vacuum. Now from (2.9), (2.10) and (2.7) follows:

\[
    m_i(\beta, \mu, \lambda) = \frac{2\pi}{L} d_i + \lambda^{\frac{2}{d}} H_i(\beta, \mu)
\]

\[
= \frac{\lambda^{\frac{2}{d}}}{d} \left[ 2\pi d_i \mu^{-\frac{2}{d}} + H_i(\beta, \mu) \right] \tag{2.11}
\]

\[
= \lambda^{2-\frac{2}{d^2}} H_i(\beta, \mu)
\]

Now, what are scaling dimensions of zero-momentum one-particle states in SGM, i.e. of soliton, antisoliton and breathers? In Table I we show values conjectured in [14]. In Sec. V we’ll show that we obtain a different result for the second breather.

### III. SPIN CHAIN REGULARIZATION OF THE SGM

It was proposed (Appendix B in [14]) that XXZ spin chain with periodic boundary conditions in a transverse magnetic field defined by the Hamiltonian:

\[
    H = -\sum_{n=1}^{N} (\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \Delta \sigma_n^z \sigma_{n+1}^z + h \sigma_n^z)
\]

\[\bar{\sigma}_{N+1} \equiv \bar{\sigma}_1 \tag{3.1}\]

The argument has two steps; first one must show that unperturbed theories are equivalent, i.e. that (3.1) with \( h = 0 \) is a spin chain regularization of \( L_b \) CFT (2.5), and second that in the unperturbed theory (\( h = 0 \)) perturbation operator \( \sigma_n^x \) is a lattice regularization of \( V_{1,0}^{(+)}(x) \).

For a first step one must take \( h = 0 \) in (3.1), i.e. to consider periodic XXZ spin chain

\[
    H_{XXZ} = -\sum_{n=1}^{N} (\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \Delta \sigma_n^z \sigma_{n+1}^z)
\]

\[\bar{\sigma}_{N+1} \equiv \bar{\sigma}_1 \tag{3.2}\]

\[H_{XXZ} \text{ commutes with } S^z = 1/2 \sum_{n=1}^{N} \sigma_n^z
\]

We denote eigenvalues of \( S^z \) by \( Q \). \( Q \) is integer (half-odd integer) when \( N \) is even (odd) and \(-N/2 \leq Q \leq N/2 \). \( H_{XXZ} \) is also translation-invariant where translations by one site are generated by:

\[
    T = \prod_{n=1}^{\tilde{N}-1} \left( \frac{1}{2} (\bar{\sigma}_n \cdot \bar{\sigma}_{n+1} + 1) \right) \tag{3.3}\]

and we define (lattice) momentum operator by \( T = \exp(-i P) \). From (3.1) follows that \( T^N = 1 \), so eigenvalues \( P_k \) of the lattice momentum \( P \) are given by:

\[
P_k = \frac{2\pi}{N} k, \quad k = 0, 1, \ldots, N - 1 \tag{3.4}\]

Obviously, \( P_k \) are defined mod \( 2\pi \).

Now, in [14] it has been shown that energy-momentum spectrum of the periodic XXZ chain in charge sector \( Q \) has following asymptotic form for large \( N \):

\[
    E_{Q,\nu} = N \epsilon_{\infty} + \frac{2\pi \zeta}{N} \left( \Delta_{Q,\nu} - \Delta_{Q,\nu}^{\tilde{N}} - \frac{e}{12} \right) \tag{3.5a}
\]

\[
P_{Q,\nu} = \frac{2\pi}{N} \left( \Delta_{Q,\nu} - \Delta_{Q,\nu}^{\tilde{N}} + \pi \kappa_{Q,\nu} \right) \tag{3.5b}
\]

where \( \nu \in \mathbb{Z} \), \( n, \tilde{N} \in \mathbb{N}^0 \), central charge \( c = 1 \), \( \kappa_{Q,\nu} \in [0, 1] \), and conformal dimensions \( \Delta_{Q,\nu} \) and \( \Delta_{Q,\nu}^{\tilde{N}} \) are given by

\[
(\Delta_{Q,\nu}, \Delta_{Q,\nu}^{\tilde{N}}) = \left( \frac{1}{2} \left[ \frac{Q}{2} + \nu \right]^2 + n, \frac{1}{2} \left[ \frac{Q}{2} - \nu \right]^2 + \tilde{N} \right) \tag{3.6}
\]

where compactification radius is \( r = [2(1 - \gamma/\pi)]^{-1/2} \). From (3.3) and (3.6) we can infer that continuum limit of \( H_{XXZ} \) defined by:

\[
H_{XXZ}^{cont} \equiv \lim_{N \to a} \frac{1}{\zeta_{N \to a}} (H_{XXZ} - N \epsilon_{\infty}) \tag{3.7a}
\]

\[
P_{Q,\nu}^{cont} \equiv \lim_{N \to a} \frac{1}{\tilde{a}} (P - \pi \kappa) \tag{3.7b}
\]

\((a \text{ is lattice constant and } L = Na \text{ is kept fixed}) \text{ defines } c = 1 \text{ CFT, and in fact contains } L_b \text{ of the Gaussian}\)
model as we shall see. In (3.7b) $\kappa$ is an operator which project states having “nonuniversal macroscopic momentum” equal to $\pi$ (see \[23\]). We shall comment more on this at the end of this section. $\zeta$ is a normalization factor and $c_\infty$ is (c-number) nonuniversal bulk energy density. Nonuniversal quantities are subtracted in QFT limit.

Let us see how one can obtain $L_b$ and $L_f$ from $H_{XXZ}^{cont}$. First, from (3.6) and it is obvious that

$$H_{SGM} = (\Delta_{Q,\nu}^0, \Delta_{Q,\nu}^0) = (\Delta_{Q,\nu}^0, \Delta_{Q,\nu}^0)$$

where $\Delta_{m,n}$ and $\bar{\Delta}_{m,n}$ are conformal dimensions (2.4) of the vertex operator $V_{m,n}$ in Gaussian model. Comparing \[3\] with \[22\], it is obvious that $Q$ must be integer, so $N$ must be even, and

$$L_b (r = [2 \left(1 - \frac{\gamma}{\pi}\right)]^{-\frac{1}{2}}) \iff H_{XXZ}^{cont}(\gamma) \quad (3.8)$$

So, in (3.8) is given the first half of equivalence between (3.1) and the SGM, that unperturbed CFT’s are equivalent. Now one must show the second part, that operator $\sigma_n^x$ is lattice counterpart of $V_{1,0}^c(x)$ ($x = na$) in the Gaussian model. In \[22\] it was shown (in the leading order in the lattice constant $a$) that:

$$\sigma_n^x \propto a^{d_1} V_{1,0}^c(a, x = a \vec{x})$$

where $x = na$. The constant of proportionality in (3.9) is in fact known \[24\] \[25\] but we’ll not need it here. So, from (3.9) we see that:

$$\sigma_n^x \propto V_{1,0}^c(x) \quad x = na \quad (3.10)$$

in the leading order. That finally completes the argument that Hamiltonian (3.1) is a spin chain regularization of the SGM where connection between coupling constants is

$$\beta = \sqrt{\frac{\pi}{\zeta}} = \sqrt{2(\pi - \gamma)} \quad (3.11)$$

Let us made a comment on internal symmetries of continuum and lattice models. As we emphasized in the last section SGM posses $Z_2 \times Z_2 \times U(1)$ symmetry and is integrable. But spin chain (3.1) is only symmetric on $Z_2$ generated by “charge conjugation operator” $C$:

$$C = \prod_{n=1}^{N} \sigma_n^x$$

and in fact is believed to be non-integrable. That spin chain representation of a QFT has less symmetries is not something new \[17\].

Now, what are the relations between dimensionfull parameters ($L, \lambda, \mu$) in the (continuum) SGM and parameters ($N, h$) in (lattice) (3.1). From (3.7a) and (3.8) follows

$$H_{SGM} = \frac{1}{\zeta} \lim_{\lambda \rightarrow \infty, a \rightarrow L/N} \frac{H}{a}$$

So, if we denote by $\tilde{m}_i$ mass gaps in the spin chain, we have:

$$m_i(L) = \frac{1}{\zeta} \lim_{\lambda \rightarrow \infty, a \rightarrow L/N} \frac{\tilde{m}_i}{a} \quad (3.12)$$

Also, from (3.11) we have:

$$h \propto \lim_{a \rightarrow 0} \lambda a^{d_\lambda} = \lim_{a \rightarrow 0} \lambda a^{2} \frac{a^2}{\pi^2} \quad (3.13)$$

where factor of proportionality is finite. Of course, we have $L = Na$ and $\lambda$ fixed. We can see from (3.13) that $h \rightarrow 0$ because $d_\lambda > 0$. We can now express scaling parameter $\mu$ using lattice constants:

$$\mu = \lambda L^{d_\lambda} \propto \lim_{a \rightarrow 0} h N^{d_\lambda} \quad (3.14)$$

Constant of proportionality is not important for us because we are interested here only in $L \rightarrow \infty$ ($\mu \rightarrow \infty$) and $L \rightarrow 0$ ($\mu \rightarrow 0$) limits. If we define now

$$\tilde{\mu} \equiv h N^{d_\lambda} = h N^{2 - 2\beta} = h N^{\frac{3}{2} + \frac{\pi}{2\beta}} \quad (3.15)$$

from (2.9), (3.12), (3.14) and (3.15) we can see that:

$$\tilde{m}_i = h (2 - \frac{a^2}{\pi^2})^{-1} G_i(\gamma, \tilde{\mu}) \quad (3.16)$$

where $\gamma$ is connected to $\beta$ by (3.11). Strictly speaking, scaling law (3.15) should be exactly valid only in the continuum limit $N \rightarrow \infty$, $a \rightarrow 0$ and $h \rightarrow 0$ where $L = Na$ and $\lambda \propto h a^{\frac{\pi^2}{2}}$ are kept fixed. For finite $N$ (3.15) is only approximate and we expect that scaling is worser for $N$ smaller.

* * *

To keep our promise, we shall now comment subtraction of “nonuniversal momentum” $\pi$ mentioned in the part of the text following eq. (3.7b), which doesn’t sound very natural (maybe “too statistical”). A more natural explanation is based on the fact that SGM is equivalent to (2.1) when number of lattice sites $N$ is even. Let’s suppose that lattice is staggered, i.e. that (in continuum limit terms) real space translations are given by translations by even number of sites, and translation by one site is some internal state transformation \[26\]. A consequence is that $T^2$ is the “real” lattice translation operator, so $2P$ is the “real” momentum which is also defined $\mod 2\pi$. But, now we must multiply (3.5a) by 2, so how can we obtain the same conformal dimensions $\Delta$ and $\bar{\Delta}$. An explanation is that the continuum spatial extension of the system is now $L = a N/2$, so we must put $N/2$ in place of $N$ in (3.5a). In (3.5a) it just compensates factor 2, and in (3.5a) we already needed scaling factor $\zeta$ which should be now halved.
IV. MASS SPECTRUM

Now we are ready to calculate particle mass-ratios in SGM $L \to \infty$ limit using connection with spin chain (3.2). First we must numerically calculate mass gaps of spin chain for finite $N$ and $h$. Then we must make continuum limit, i.e. $N \to \infty$ keeping $L = Na$ and $\mu$ fixed (obviously $a \to 0$ and from (3.13) $h \to 0$). Finally we should make $L \to \infty$, i.e. $\mu \to \infty$ (see (3.14)) limit. In practice, it is preferable to do following [17–19]; first take $N \to \infty$ with $h$ fixed and afterwards extrapolate to $h \to 0$. A difference is that in latter case one does $\mu \to \infty$ before $h \to 0$. This limits are performed using BST extrapolation metod [15].

We numerically diagonalized Hamiltonian (3.1) for up to 16 sites using Lanczos algorithm. But before doing numerics, one should maximally exploit symmetries. The Hamiltonian (3.1) commutes with translation operator $T$ (given by (3.5)) and with charge conjugation operator $C$. So, we can break Hamiltonian (3.1) in blocks, each marked with eigenvalues of the operators $P = i \ln T$ and $C$ which can be $P_k = \frac{2\pi k}{h} \mod 2\pi$, (see (3.1)) and $C = \pm 1$ (because $C^2 = 1$). We are interested in mass ratios, so we only need zero-momentum sector. But, because “true” space translations are generated by $T^2$ (or because we must subtract “nonuniversal macroscopic momentum” $\pi$, if you like it more) zero-momentum sector is a union of $P = \pi$ and $P = \pi$ sectors. So we must diagonalize four blocks which we will denote by $0$, and it is expected from the DHN formula (1.1) and Fujita et al. formula (1.2).

In Figs. 1-3 we present numerical results for the spin chain (3.1). We saw in Sec. 1 that it is obtained when $\mu(\mu) \to 0$. Using (3.12) and (1.13) in the continuum result (2.11) we obtain that the scaling relation for mass gaps of spin chain should have form

$$\tilde{m}_a(\gamma, \tilde{\mu}, h) = \zeta \sqrt{\frac{\pi}{\gamma}} \left[ 2\pi d_a \tilde{\mu} - \frac{\pi}{\gamma} + \tilde{H}_a(\gamma, \tilde{\mu}) \right]$$

where we must now include proper normalization factor $\zeta$ for the spin chain Hamiltonian. Because it doesn’t depend on $h$ we can take it from unperturbed XXZ spin chain (3.2), where it is well known

$$\zeta = \frac{\pi \sin \gamma}{\gamma}$$

Before we plot reduced scaling functions $\tilde{H}_a(\gamma, \tilde{\mu})$ we must know scaling dimension $d_a$ of the corresponding state. On the other hand, we can choose $d_a$ and see does it gives the right behaviour of $\tilde{H}_a(\gamma, \tilde{\mu})$ when $\tilde{\mu} \to 0$ (which is the same as for $H_a$ mentioned below (2.10)).

In Table 3 we have presented scaling dimensions of zero-momentum particle states of SGM as conjectured in [14]. But our numerical results clearly indicate that first and second breather ($B1$ and $B2$) have exactly the same scaling dimensions. In Figs. 1[15] we show numeric results for reduced scaling functions, where we used for scaling dimensions values from Table 3.

We can see in Figs. 1[16] that finite size effects are stronger for $A$ closer to $-1$ (where they are in fact logarithmic because of the appearance of marginal operators), which is expected from [28].

V. UV (CONFORMAL) LIMIT OF PARTICLE STATES

Let’s now turn our attention to the opposite UV limit of our results for the spin chain (3.1). We saw in Sec. 1 that it is obtained when $\mu(\mu) \to 0$. Using (3.12) and (3.13) in the continuum result (2.11) we obtain that the scaling relation for mass gaps of spin chain should have form

To make an extrapolation $h \to 0$ one should obtain results for smaller $h$, at least $h \approx 0.1$. From Figs. 1[18] one can see that for that one should diagonalize Hamiltonian with $N \geq 26$, which is too demanding even for most powerful machines today.

Finally, (partialy) extrapolated mass ratios

$$\tilde{r}_a(\Delta, h) = \lim_{N \to \infty} \frac{\tilde{m}_a}{m_{B1}} = \lim_{N \to \infty} \frac{\tilde{G}_a}{G_{B1}}, \quad a \in \{S, A, B2\}$$

are given in Tables 7[19] and Fujita et al. formula (1.2). One can see that our results confirm DHN and reject Fujita et al.

VI. CONCLUSION

In this paper we use the XXZ spin chain in transverse field as a lattice regularization of the sine-Gordon model.
This equivalence can be understood e.g. from conformal perturbation theory. One of our goals was to calculate by numerical analysis masses in the sine-Gordon theory. This is now of interest because recent calculations based on numerical treatment of Bethe Ansatz [14] and infinite momentum frame technique [10] are in disagreement with previous approaches used in literature [4-7]. Our results are in agreement with DHN formula [16]. We stress that previous approaches used in literature [11] and infinite momentum frame technique [10] are in disagreement with previous approaches used in literature [11].

We also analyse conformal limit and find conformal dimensions of various states. We find that the conformal dimension of second breather state disagrees with the conjecture by [14]. Our calculations for dimensions of other states agree with those in [4].

TABLE I. Scaling dimensions of particle states in SGM as conjectured in [4].

| State          | Operator | Scaling dimension |
|----------------|----------|-------------------|
| soliton        | $V_{0,1}$ | $\frac{\pi}{\beta}$ |
| antisoliton    | $V_{0,-1}$ | $\frac{\pi}{\beta}$ |
| p-th breather  | $V_{p,0}^{(-p)}$ | $\frac{\beta^2}{4\pi}$ |

TABLE II. Estimates for the scaled gaps $\tilde{G}_{\alpha}(\beta, \infty)$ as a function of $h$ at $\Delta = -0.9$ ($\beta^2 = 5.38$). The numbers in brackets give the estimated uncertainty in the last given digit.

| $h$  | $\tilde{G}_{B1}$  | $\tilde{G}_{B2}$  | $\tilde{G}_{A}$  | $\tilde{G}_{B1}$ |
|------|------------------|------------------|------------------|------------------|
| 0.8  | 4.85922 (5)      | 5.2274 (1)       | 7.358 (2)        | 8.706 (6)        |
| 0.5  | 4.9421 (7)       | 5.368 (1)        | 7.25 (1)         | 8.93 (3)         |
| 0.3  | 5.012 (6)        | 5.49 (1)         | 7.10 (5)         | 9.2 (1)          |
| 0.2  | 5.04 (2)         | 5.55 (3)         | 6.9 (1)          | 8.7 (2)          |

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TABLE III. The same as Table II but now for $\Delta = -0.6$ ($\beta^2 = 4.43$).

| $h$ | $\tilde{G}_{B1}$ | $\tilde{G}_S$ | $\tilde{G}_A$ | $\tilde{G}_{B2}$ |
|-----|------------------|---------------|---------------|------------------|
| 0.8 | 4.48354 (1)      | 5.9727 (1)    | 7.477 (1)     | 8.305 (4) |
| 0.5 | 4.51002 (3)      | 6.199 (1)     | 7.386 (6)     | 8.41 (1) |
| 0.3 | 4.537 (1)        | 6.38 (1)      | 7.28 (3)      | 8.49 (5) |
| 0.2 | 4.548 (5)        | 6.47 (3)      | 7.16 (7)      | 8.56 (13) |

TABLE IV. The same as Table III but now for $\Delta = -0.1$ ($\beta^2 = 3.34$).

| $h$ | $\tilde{G}_{B1}$ | $\tilde{G}_S$ | $\tilde{G}_A$ | $\tilde{G}_{B2}$ |
|-----|------------------|---------------|---------------|------------------|
| 0.8 | 3.795834 (2)     | 7.21140 (8)   | 7.7036 (2)    | 7.261 (5) |
| 0.5 | 3.75549 (3)      | 7.483 (1)     | 7.715 (2)     | 7.21 (1) |
| 0.3 | 3.7372 (3)       | 7.63 (1)      | 7.73 (1)      | 7.16 (1) |
| 0.2 | 3.728 (1)        | 7.65 (3)      | 7.71 (4)      | 7.11 (2) |

TABLE V. Estimates for the mass gap ratios $\tilde{r}_a(\Delta, h)$ as a function of $h$ at $\Delta = -1$ ($\beta^2 = 2\pi$). We also added predictions obtained from (1.1) (DHN) and (1.2) (Fujita at al).

| $\tilde{r}_a$ | $h$ | S | A | B2 | DHN | Fujita |
|---------------|-----|---|---|----|-----|--------|
| 0.8           | 1   | 1.4703 (7) | 1.419 (4) | 1.36 (1) | 1.32 (2) | 1.07577 (3) |
|               | 0.5 | 1.766 (7)  | 1.74 (2)  | 1.62 (5) | 1.732 -  | 1.5142 (5) |

TABLE VI. The same as Table V but now for $\Delta = -0.9$ ($\beta^2 = 5.38$).

| $\tilde{r}_a$ | $h$ | S | A | B2 | DHN | Fujita |
|---------------|-----|---|---|----|-----|--------|
| 0.8           | 1   | 1.07577 (3) | 1.0862 (2) | 1.095 (3) | 1.101 (7) | 1.724  |
|               | 0.5 | 1.5142 (5)  | 1.467 (3)  | 1.42 (1)  | 1.37 (3)  | 1.205  |
|               | 0.3 | 1.792 (1)   | 1.807 (7)  | 1.84 (3)  | 1.73 (5)  | 1.88 |

TABLE VII. The same as Table V but now for $\Delta = -0.6$ ($\beta^2 = 4.43$).

| $\tilde{r}_a$ | $h$ | S | A | B2 | DHN | Fujita |
|---------------|-----|---|---|----|-----|--------|
| 0.8           | 1   | 1.33214 (3) | 1.3745 (2) | 1.37 (5)  | 1.38 (7)  | 1.517  |
|               | 0.5 | 1.6677 (2)  | 1.638 (1)  | 1.605 (8) | 1.57 (2)  | 1.517  |
|               | 0.3 | 1.8523 (9)  | 1.865 (3)  | 1.87 (1)  | 1.88 (3)  | 1.888  |

TABLE VIII. The same as Table VII but now for $\Delta = -0.4$ ($\beta^2 = 3.96$).

| $\tilde{r}_a$ | $h$ | S | A | B2 | DHN | Fujita |
|---------------|-----|---|---|----|-----|--------|
| 0.8           | 1   | 1.53365 (3) | 1.5970 (2) | 1.639 (3) | 1.654 (8) | 1.724  |
|               | 0.5 | 1.7927 (1)  | 1.779 (1)  | 1.762 (6) | 1.74 (1)  | 1.724  |
|               | 0.3 | 1.880 (1)   | 1.886 (3)  | 1.885 (5) | 1.90 (2)  | 1.914  |

TABLE IX. The same as Table V but now for $\Delta = -0.1$ ($\beta^2 = 3.34$).

| $\tilde{r}_a$ | $h$ | S | A | B2 | DHN | Fujita |
|---------------|-----|---|---|----|-----|--------|
| 0.8           | 1   | 1.89982 (2) | 1.9926 (3) | 2.042 (4) | 2.052 (9) | 2.096  |
|               | 0.5 | 2.02949 (8) | 2.0543 (7) | 2.068 (3) | 2.07 (1)  | 2.096  |
|               | 0.3 | 1.913 (1)   | 1.920 (3)  | 1.916 (4) | 1.907 (7) | 1.942  |

TABLE X. Scaling dimensions of particle states in SGM as conjectured from our numerical results.

| State | Operator | Scaling dimension |
|-------|----------|-------------------|
| soliton | $V_{0,1}$ | $\tilde{\beta} = \frac{1}{\pi} \left(1 - \frac{\pi}{2\pi} \right)^{-1}$ |
| antisoliton | $V_{0,-1}$ | $\tilde{\beta} = \frac{1}{\pi} \left(1 - \frac{\pi}{2\pi} \right)$ |
| 1st breather | $V_{1,0}^{(+)1}$ | $\tilde{\beta} = \frac{1}{\pi} \left(1 - \frac{\pi}{2\pi} \right)$ |
| 2nd breather | $V_{1,0}^{(-)1}$ | $\tilde{\beta} = \frac{1}{\pi} \left(1 - \frac{\pi}{2\pi} \right)$ |
Figure 1
Figure 2
Figure 3
Figure 4
Figure 5
Figure 6