Zero-Cost Coercions for Program and Proof Reuse

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Abstract. We introduce the notion of identity coercions between non-indexed and indexed variants of inductive datatypes, such as lists and vectors. An identity coercion translates one type to another such that the coercion function definitionally reduces to the identity function. This allows us to reuse vector programs to derive list programs (and vice versa), without any runtime cost. This also allows us to reuse vector proofs to derive list proofs (and vice versa), without the cost of equational reasoning proof obligations. Our work is formalized in Cedille, a dependently typed programming language based on a type-annotated Curry-style type theory with implicit (or, erased) products (or, dependent functions), and relies crucially on erasure to introduce definitional equalities between underlying untyped terms.

Keywords: Dependent types; coercion; reuse; implicit products; cedille.

1 Introduction

In dependently typed languages (such as Agda [13], Coq [15], Idris [3], or Lean [12]) it is common to define traditional algebraic datatypes, as well as more refined indexed variants of algebraic datatypes, where the values of the indexed type are a restriction of the values of the original algebraic type to particular indices. An example of two such datatypes are lists and vectors, vectors being lists indexed by their length.

To prevent code duplication, a programmer may want to define a function over lists by reusing a function over vectors (or vice versa), which we refer to as program reuse. For example, we can derive list append (appendL) by reusing vector append (appendV) as follows:

appendL : ∀ A : * . List A → List A → List A
appendL = λ xs ys . v2l (appendV (l2v xs) (l2v ys))

This is achieved by coercing the list arguments to vectors (via l2v), passing them to the reused function appendV, and coercing the resulting vector to a list (via v2l). Unfortunately, this has the drawback of linear-time coercions back and forth between lists and vectors (via l2v and v2l) when we run our code.
A programmer may also want to prevent code duplication by defining a proof of a property about list functions (defined by reuse) in terms of a proof of a property about vector functions (or vice versa), which we refer to as proof reuse. For example, we may want to derive associativity of list append (\texttt{appendAssocL}) in terms of associativity of vector append (\texttt{appendAssocV}) as follows:

\begin{verbatim}
appendAssocL : \forall A : *. \Pi xs ys zs : List A .
appendL (appendL xs ys) zs ≃ appendL xs (appendL ys zs)
appendAssocL = \lambda xs ys zs . cong v2l
(appendAssocV (l2v xs) (l2v ys) (l2v zs))
\end{verbatim}

Unfortunately, reusing the proof of \texttt{appendAssocV} by casting our arguments to lists (via \texttt{v2l}), and by congruence (\texttt{cong}) applied to the cast back to vectors (via \texttt{l2v}), is not enough to get the proof above to type check. We must additionally perform equational reasoning, by appealing to the identity laws established by an isomorphism between lists and vectors. In other words, the proof would need to rewrite occurrences of ((\texttt{v2l} \circ \texttt{l2v}) xs) and ((\texttt{l2v} \circ \texttt{v2l}) xs) to \texttt{xs} in the appropriate places, which may only appear after previous rewrites and \(\beta\)-reductions.

We show that in a type-annotated implementation of a Curry-style type theory, coercions that definitionally reduce to the identity function (\(\lambda x.x\)) are derivable, and we call them “identity coercions”. Identity coercions enable zero-cost program reuse, avoiding runtime overhead, and zero-cost proof reuse, avoiding equational reasoning overhead (making \texttt{appendAssocL} above well-typed).

1.1 The Setting

In a Curry-style type theory with implicit products (such as ICC \[^1\]), an untyped Church-encoded vector can be assigned the vector type (\texttt{Vec}), but also the list type (\texttt{List}). This is possible because the types share the same class of untyped terms, and because vectors are a subtype of lists in ICC (\(\texttt{Vec A n} \leq \texttt{List A}\)).

A type-annotated version of a Curry-style calculus with implicit products (such as ICC* \[^1\] and i\(\lambda P2\) \[^1\]) adds typing information to terms, but compares erased terms (removing type annotations, implicit type applications, etc.) during conversion (\(\llbracket t \rrbracket =_{\alpha\beta\eta} \llbracket t' \rrbracket\)). The extra type annotations on terms allows them to be algorithmically type checked, making type-annotated versions of Curry-style calculi suitable as the basis of programming languages.

This paper is formalized in Cedille, a dependently typed programming language based on i\(\lambda P2\) \[^1\]. In a type-annotated setting, a vector cannot be used in the place of a list, as they have distinct types, despite the fact that their erased untyped values are equal. Nonetheless, Barras and Bernardo \[^1\] demonstrate

\[^1\] A pre-release of Cedille for evaluating our formalization is available here:
http://cs.uiowa.edu/~astump/cedille-prerelease.zip
The Cedille code accompanying this paper is here:
https://github.com/larrytheliquid/zero-cost-coercions
(in ICC*) that it is possible to write an identity coercion from Church-encoded vectors to Church-encoded lists, which can be thought of as a checkable term witness of the subtyping relationship: $\text{Vec } A n \leq \text{List } A$.

### 1.2 Contributions

In $\lambda P^2$, Stump [14] adds a dependent intersection type [8] and a heterogeneous equality type [9] to a type-annotated Curry-style calculus with implicit products, allowing inductive types (i.e. those supporting an induction principle) to be derived, but whose erased terms are untyped Church-encodings. Working in Cedille (based on $\lambda P^2$), our contributions are:

1. **Extending** the non-dependent identity coercion from Church-encoded vectors to lists, to an identity coercion from inductive vectors to lists ($v2l$ in Section 3.1). By working with inductive types, we can write proofs by induction but still support identity coercion.

2. **Introducing** the dependent identity coercion from inductive lists to vectors ($l2v$ in Section 3.2). This is a witness of the dependent subtyping relationship: $(xs : \text{List } A) \leq \text{Vec } A (\text{length } xs)$. Because the length of the output vector depends on the input list, the dependent identity coercion $l2v$ cannot be written using Church-encoded datatypes, which do not have induction principles [6].

3. **Introducing** the identity coercion from inductive vectors to length-constrained lists ($v2u$ in Section 4.2). This allows vectors to be coerced to lists, while “remembering” the constraint that the length of the output list should be the length of the input vector index.

4. **Introducing** a functorial map for inductive lists ($\text{mapL}$ in Section 5.1), whose partial application to an identity coercion results in an identity coercion.

After reviewing how to derive inductive datatypes in Cedille (Section 2), we show how to reuse a vector program ($\text{appendV}$) and proof ($\text{appendAssocV}$) to define a list program ($\text{appendL}$) and proof ($\text{appendAssocL}$) in Section 3 show how to reuse a list program ($\text{appendL}$) and proof ($\text{appendAssocL}$) to define a vector program ($\text{appendV}$) and proof ($\text{appendAssocV}$) in Section 4 show how to reuse a nested list program ($\text{concatL}$) and proof ($\text{concatDistAppendL}$) to define a nested vector program ($\text{concatV}$) and proof ($\text{concatDistAppendV}$) in Section 5 discuss related work in Section 6 and conclude in Section 7. We reiterate that all of our instances of program and proof reuse are zero-cost, as they are implemented in terms of identity coercions.

**Remark 1.1.** All lemmas and theorems in this paper are trivial consequences of definitional equality ($\|t\| =_{\alpha\beta\eta} \|t'\|$). Nonetheless, we prove them by hand to aid the reader in understanding why terms erase the way they do, and in particular how our carefully crafted identity coercions indeed erase to the identity function (up to $\alpha\beta\eta$-equality).

\[\text{Remark 1.1.}\] Barras and Bernardo did not name their technique, which we are calling “identity coercion”. 

\[\text{Remark 1.1.}\]
2 Background: Deriving Inductive Types

In this section we review how to derive inductive types in Cedille [14], whose erased terms are untyped Church-encodings. An inductive datatype is defined as the dependent intersection of 3 components:

1. The Church-encoding of the datatype.
2. The unary parametricity theorem of the Church-encoding.
3. The reflection theorem of the Church-encoding.

2.1 Church-Encoding

The first component (VecC) is the Church-encoded vector type, where the predicatively quantified \( X \) is a family of types indexed by the natural numbers.

**Convention 1** We include "C" in the suffix of an identifier to indicate that it relates to a Church-encoded datatype.

\[
\begin{align*}
\text{VecC} & : \, \star \rightarrow \text{Nat} \rightarrow \star = \lambda A : \star . \lambda n : \text{Nat} . \\
& \quad \forall X : \text{Nat} \rightarrow \star . \\
& \quad X \text{ zero } \rightarrow \\
& \quad (\forall n : \text{Nat} . A \rightarrow X n \rightarrow X (\text{suc} n)) \rightarrow \\
& \quad X n . \\
\end{align*}
\]

Implicit products are introduced by the \( \forall \) quantifier, and represent erased dependent function arguments. Implicit products can be used for type arguments (e.g. the type family \( X \)), but also for value arguments (e.g. \( n \) in the cons case, used for indexing).

**Constructors** Next, we define the Church-encoded vector constructors nilCV and consCV.

**Convention 2** We suffix an identifier with "V" to indicate that it relates to vectors.

\[
\begin{align*}
nilCV & : \forall A : \star . \text{VecC} \cdot A \text{ zero } = \\
& \quad \Lambda A . \Lambda X . \lambda cN . \lambda cC . cN . \\
\end{align*}
\]

\[
\begin{align*}
\text{consCV} & : \forall A : \star . \forall n : \text{Nat} . A \rightarrow \text{VecC} \cdot A n \rightarrow \text{VecC} \cdot A (\text{suc} n) = \\
& \quad \Lambda A . \Lambda n . \lambda x . \lambda xs . \Lambda X . \lambda cN . \lambda cC . \\
& \quad cC -n x \ (xs \cdot X \ cN \ cC) .
\end{align*}
\]

**Remark 2.1.** A full definition of term erasure for our base theory \( \iota\lambda P2 \) can be found in Figure 6 of Stump'17 [14].
All of the implicit abstractions ($\Lambda$) are erased, as are implicit applications (prefixed by minus, e.g. $-n$), and type applications (prefixed by a centered dot, e.g. $\cdot X$)\(^3\). We can verify that erasing the type-annotated $\text{nilCV}$ and $\text{consCV}$ results in the untyped Church-encodings of nil and cons, respectively:

**Lemma 2.2.** $|\text{nilCV}|$ is the Church-encoding of nil.

*Proof.*

\[
\begin{align*}
= & \quad |\Lambda A. X. \lambda c_n. c_e. c_n| \\
= & \quad \lambda c_n. c_e. c_n \\
\end{align*}
\]

$\square$

**Lemma 2.3.** $|\text{consCV}|$ is the Church-encoding of cons.

*Proof.*

\[
\begin{align*}
= & \quad |\Lambda A, n. \lambda x, xs. AX. \lambda c_n. c_e. c_e -n x (xs \cdot X c_n c_e)| \\
= & \quad \lambda x, xs, c_n, c_e. c_e -n x (xs \cdot X c_n c_e) \\
= & \quad \lambda x, xs, c_n, c_e. c_e x (xs c_n c_e) \\
\end{align*}
\]

$\square$

### 2.2 Unary Parametricity Theorem

The second component ($\text{VecP}$) is the unary parametricity predicate on Church-encoded vectors ($\text{VecC}$). It takes 4 types of abstract arguments, described below, and can be understood as an abstract version of an eliminator (i.e. an induction principle in type theory):

1. An abstract *return type* ($X$).
2. An abstract *motive* ($P$, an abstract predicate over $X$).
3. Abstract Church-encoded *constructors* ($c_N$ for nil and $c_C$ for cons).
4. Abstract parametricity *branches* ($p_N$ for nil branch and $p_C$ for cons branch).

The reader may wish to compare the vector parametricity theorem type below ($\text{VecP}$), which has abstract versions of arguments, with the type of the eliminator $\text{elimVec}$ in Section 2.4, which has concrete versions of arguments.

\[
\begin{align*}
\text{VecP} & \triangleq \Pi A : * . \Pi n : \text{Nat} . \text{VecC} \cdot A n \rightarrow * = \\
& \lambda A : * . \lambda n : \text{Nat} . \lambda xsC : \text{VecC} \cdot A n . \\
& \forall X : \text{Nat} \rightarrow * . \forall P : \Pi n : \text{Nat} . X n \rightarrow * . \\
& \forall cN : X \text{zero} . \forall cC : \forall n : \text{Nat} . A \rightarrow X n \rightarrow X (\text{suc n}) . \\
& \Pi pN : P \text{ zero } cN . \\
& \Pi pC : \forall n : \text{Nat} . \forall xs : X n . \\
& \quad \Pi x : A . P n xs \rightarrow P (\text{suc n}) (cC -n x xs) . \\
& P n (xsC \cdot X cN cC) .
\end{align*}
\]

\(^3\) While $\forall$ and $A$ quantify over and introduce (respectively) both type and value arguments, minus (-) is special syntax for implicit (value) applications, while center dot ($\cdot$) is special syntax for type applications.
Notice that the abstract constructors (cN and cC) are implicit arguments, but the abstract parametricity branches (pN and pC) are explicit arguments (introduced as non-erased dependent functions via \( \Pi \)). Furthermore, the number of explicit (non-erased) arguments in the types of cN and cC is equal the number of explicit arguments in the types of pN and pC, respectively. This coincidence has been arranged so that Church-encoded vectors (VecC) and their parametricity witnesses (VecP) share the same class of (erased) untyped term inhabitants.

**Convention 3** We include “P” in the suffix of an identifier to indicate that it relates to the parametricity theorem of a Church-encoded datatype.

**Constructors** Now we define “constructors” for witnessing parametricity in the nil case (nilPV) and the cons case (consPV):

\[
\begin{align*}
nilPV &: \forall A : \star . \text{VecP} \cdot A \text{ zero} (\text{nilCV} \cdot A) = A \ A . \\
& \quad \ A \ X . \ A \ P . \ A \ cN . \ A \ cC . \ \lambda pN . \ \lambda pC . pN .
\end{align*}
\]

\[
\begin{align*}
\text{consPV} &: \forall A : \star . \forall n : \text{Nat} . \forall xsC : \text{VecC} \cdot A \ n . \\
& \quad \text{Π x : A . VecP} \cdot A \ n \ xsC \to \\
& \quad \text{VecP} \cdot A \ (\text{suc} \ n) \ (\text{consCV} \cdot A \ -n \ x \ xsC) = \\
& \quad A \ A . \ A \ n . \ A \ xsC . \ \lambda x . \ \lambda xsP . \\
& \quad A \ X . \ A \ P . \ A \ cN . \ A \ cC . \ \lambda pN . \ \lambda pC . \\
& \quad pC \ -n \ -(xsC \cdot X \ cN \ cC) \ x \ (xsP \cdot X \cdot P \ -cN \ -cC \ pN \ pC) .
\end{align*}
\]

Any additional arguments that would get in the way of the parametricity witnesses erasing to their corresponding Church-encodings appear as implicit (erased) arguments, such as -(xsC \cdot X \ cN \ cC) in the definition of consPV. The parametricity witness of the nil (resp. cons) case erases to the Church-encoding of nil (resp. cons), just like the erasure of nilCV (resp. consCV).

**Lemma 2.4.** \( |\text{nilPV}| \) is the Church-encoding of nil.

**Proof.** Erase implicit abstractions. \qed

**Lemma 2.5.** \( |\text{consPV}| \) is Church-encoded cons.

**Proof.** Erase implicit abstractions and applications. \qed

### 2.3 Reflection Theorem

The third (and final) component (VecR) is the reflection theorem for Church-encoded vectors. It states that eliminating a vector as a vector, and using its constructors (nilCV and consCV) for the branches, results in the vector being eliminated:

\[
\begin{align*}
\text{VecR} &: \Pi A : \star . \Pi n : \text{Nat} . \text{VecC} \cdot A \ n \to \star = \\
& \quad \lambda A : \star . \lambda n : \text{Nat} . \lambda xsC : \text{VecC} \cdot A \ n . \\
& \quad xsC \cdot (\text{VecC} \cdot A) \text{ nilCV consCV} \simeq xsC .
\end{align*}
\]
We cannot derive this using the Church-encoded vector type (VecC), because it lacks an induction principle \[[6]\]. Hence, we include VecR as a component of the inductive vector definition, which will have an induction principle.

**Convention 4** We include “R” in the suffix of an identifier to indicate that it relates to the reflection theorem of a Church-encoded datatype.

**Constructors** Now we also define “constructors” for witnessing reflection in the nil and cons cases:

\[
\text{nilRV} \triangleq \forall A : * \cdot \text{VecR} \cdot A \text{ zero} (\text{nilCV} \cdot A) = \Lambda A . \beta .
\]

\[
\text{consRV} \triangleq \forall A : * \cdot \forall n : \text{Nat} . \forall x : A \cdot \forall \text{xsC} : \text{VecC} \cdot A n \cdot \forall q : \text{VecR} \cdot A n \text{ xsC} . \text{VecR} \cdot A (\text{suc n}) (\text{consCV} \cdot A -n x \text{ xsC}) = \Lambda A . \Lambda n . \Lambda x . \Lambda \text{ xsC} . \Lambda q . \rho q - \beta .
\]

Reflection for the nil case is proven trivially by $\beta$, the reflexive constructor of equality types. Reflection for the cons case is proven by first rewriting (using the equality elimination rule $\rho$) by the reflection proof for the tail of the vector ($q$), after which the proof becomes trivial ($\beta$).

**Remark 2.6.** A full definition of all introduction and elimination rules for our base theory $\text{\iota}\lambda\Pi2$ can be found in Figure 8 of Stump’17 \[[14]\].

### 2.4 Inductive Type

Finally, we define the inductive type of vectors (Vec) as the dependent intersection (using type former $\iota$) of the Church-encoded vector type (VecC from Section 2.1) and its parametricity theorem (VecP from Section 2.2), which is again intersected with the reflection theorem for Church-encoded vectors (VecR from Section 2.3):

\[
\text{Vec} \triangleq * \rightarrow \text{Nat} \rightarrow * = \lambda A : * . \lambda n : \text{Nat} . \iota \text{ xs} : (\iota \text{ xsC} : \text{VecC} \cdot A n , \text{VecP} \cdot A n \text{ xsC}) . \text{VecR} \cdot A n \text{ xs}.1 .
\]

A dependent intersection ($\iota$-type) is like a dependent pair ($\Sigma$-type) whose erased components must be equal, and whose pair constructor erases to its erased left component ($\iota[t , t’] = [t]$). VecC and VecP share the same class of erased inhabitants, so it makes sense to intersect them. But, why does it make sense to intersect these with proofs of equality (VecR)? The answer involves a modified reflexive equality introduction rule, accepting any term as an additional argument, where the erasure of the reflexive equality proof becomes the erasure of the term argument ($\iota[\beta{t}] = [t]$).

\[\text{(4) A reflexive equality proof without a term argument (}$\beta\text{)$ erases to the identity function by default, as in $\iota\lambda\Pi2$ \[[14]\].} \]
Constructor Helper Function Below, we define a helper function to construct a vector from the intersection of VecC and VecP, and the reflection theorem (VecR) as an implicit argument (⇒ is syntax for non-dependent ∀).

\[
\text{mkVec} \triangleright \forall A : \ast . \forall n : \text{Nat} . \\
\text{let} \; \text{xs} := (\updownarrow \text{xsC} : \text{VecC} \cdot A n . \text{VecP} \cdot A n \text{xsC}) . \\
\text{VecR} \cdot A n \text{xs.1} \Rightarrow \text{Vec} \cdot A n = \\
A A n . \lambda \text{xs} . A q \cdot [\text{xs} , \rho q - \beta \{\text{xs}\} ]
\]

The left component of the intersection pair is our (VecC/VecP) intersection xs. Although the right component expects the reflection theorem (q), we cannot return q immediately, because the intersection pair introduction rule requires the erasure of both components to be equal. Instead, we rewrite by our reflection proof, changing the goal from (xsC \cdot (VecC \cdot A) \text{nilCV consCV} \simeq \text{xsC}) to (xsC \simeq \text{xsC}). Then, we use \text{β}\{\text{xs}\} to construct a trivial equality that erases to the same term as the left component of the pair (xs).

**Assumption 1** To conserve space, henceforth all proofs assume that implicit abstractions and applications have already been erased.

Below, we can see that \text{mkVec} is our first example of an identity coercion (a function erasing to the identity). Additionally, the proof demonstrates how the intersection pair components erase to the same term (xs), making it a well-typed introduction of an intersection pair:

**Lemma 2.7.** \text{|mkVec| is the identity function:}

**Proof.**
\[
= \lambda \text{xs}. [\text{xs} , \rho q - \beta \{\text{xs}\}] \quad \text{Erase rewrite.} \\
= \lambda \text{xs}. [\text{xs} , \beta \{\text{xs}\}] \quad \text{Erase reflexive equality.} \\
= \lambda \text{xs}. [\text{xs} , \text{xs}] \quad \text{Erase pair.} \\
= \lambda \text{xs}. \text{xs} \quad \square
\]

**Notation 1** We use large pipes (within small pipes) to focus on the erasure of subterms, rather than erasing according to a depth-first strategy. For example, \(| f x \; y |\) denotes erasing the subterm y first.

**Constructors** Finally, it is straightforward to define constructors of our inductive vector type (Vec) from the helper \text{mkVec} and the 3 constructor components we defined previously.

\[
\text{nilV} \triangleright \forall A : \ast . \text{Vec} \cdot A \text{zero} = A A . \\
\text{mkVec} \cdot A -\text{zero} [\text{nilCV} \cdot A , \text{nilPV} \cdot A ] - (\text{nilRV} \cdot A) .
\]

\[
\text{consV} \triangleright \forall A : \ast . \forall n : \text{Nat} . A \rightarrow \text{Vec} \cdot A n \rightarrow \text{Vec} \cdot A (\text{suc} n) = \\
A A n . \lambda x . \lambda \text{xs} . \text{mkVec} \cdot A - (\text{suc} n) [\text{consCV} \cdot A - n x \text{xs.1.1} , \text{consPV} \cdot A - n - \text{xs.1.1} x \text{xs.1.2} ] \\
- (\text{consRV} \cdot A - n - x - \text{xs.1.1} - \text{xs.2}) .
\]
Below, we verify that the inductive constructors erase to their untyped Church-encoded equivalents:

**Theorem 2.8.** $\text{nilV}$ is the Church-encoding of nil:

**Proof.**

\[ \begin{align*}
&= \beta \left[ \text{mkVec} \left[ \text{nilCV} \cdot \text{nilPV} \right] \right] & \text{By Lemma 2.2} \\
&= \left[ \text{mkVec} \left[ \lambda c_n, c_c. c_n \cdot \text{nilPV} \right] \right] & \text{By Lemma 2.3} \\
&= \alpha \left[ \text{mkVec} \left[ \lambda \lambda c_n, c_c, c_n, \lambda p_n, p_c, p_n \cdot \text{nilPV} \right] \right] & \text{Erase pair.} \\
&= \left[ \text{mkVec} \left( \lambda c_n, c_c. c_n \right) \right] & \text{By Lemma 2.4} \\
&= \beta \lambda c_n, c_c. c_n \qed
\end{align*} \]

In the proof of Theorem 2.8 above, we can see that the intersection pair passed to $\text{mkVec}$ is type correct, as both of its components erase to the same ($\alpha$-equivalent) term.

**Theorem 2.9.** $\text{consV}$ is the Church-encoding of cons:

**Proof.** Same as the proof of Theorem 2.8, but erasing $\text{consCV}$ (instead of nilCV) by Lemma 2.3 in the first step, and $\text{consPV}$ (instead of nilPV) by Lemma 2.5 in the second step. \qed

**Eliminator** The whole point of defining the inductive vector type ($\text{Vec}$), as opposed to the Church-encoded vector type ($\text{VecC}$), is so we can define its eliminator (i.e. its induction principle in type theory):

\[
\text{elimVec} \triangleright \forall A : \star. \forall n : \text{Nat}. \Pi xs : \text{Vec} \cdot A \cdot n. \\
\forall P : \Pi n : \text{Nat}. \text{Vec} \cdot A \cdot n \rightarrow \star. \\
\Pi pN : P \text{ zero } (\text{nilV} \cdot A). \\
\Pi pC : \forall n : \text{Nat}. \forall xs : \text{Vec} \cdot A \cdot n. \Pi x : A. \\
P n xs \rightarrow P (\text{suc } n) (\text{consV} \cdot A - n \cdot x xs). \\
P n (xs.1.2 \cdot (\text{Vec} \cdot A) \cdot P -(\text{nilV} \cdot A) -(\text{consV} \cdot A)).
\]

We apply the parametricity theorem (via projection $\text{xs.1.2}$) to the motive $P$ and the concrete vector constructors $\text{nilV}$ and $\text{consV}$, instantiating the abstract constructor arguments of $\text{VecP}$. The result has the following type:

\[
P n (\text{xs.1.1} \cdot (\text{VecC} \cdot A) \cdot (\text{nilV} \cdot A) \cdot (\text{consV} \cdot A))
\]

Note that the second argument to $P$ is exactly one of the sides of our reflection theorem ($\text{VecR}$), so we rewrite (using $\rho$) by the reflection proof (via its projection $\text{xs.2}$) to arrive at our goal ($P n x s$).\footnote{Because $\rho$ rewrites the left side of an equality to the right side, we get the symmetric version of our reflection proof $\text{xs.2}$ by applying our symmetry operator $\varsigma$. The operator $\varsigma$ changes any equation $t_1 \simeq t_2$ to $t_2 \simeq t_1$.}
Because the projections and rewrites are erased, the eliminator $\text{elimVec}$ is actually a dependent identity coercion (from $\text{Vec}$ to the rest of the eliminator type, starting with $\forall \ P$ and ending with $P \ n \ xs$):

**Lemma 2.10.** $|\text{elimVec}|$ is the identity function:

**Proof.**

\[
\begin{align*}
\delta &\quad \lambda xs. |\rho \ cys| |xs.2| - |xs.1.2| \\
= &\quad \lambda xs. |\rho cys - xs| \\
= &\quad \lambda xs. xs
\end{align*}
\]

3 Reusing Vector Definitions

In this section we demonstrate reusing vector programs and proofs to define list-versions of the programs and proofs. Through the use of identity coercions, our program reuse does not introduce runtime overhead, and our proof reuse does not introduce equational reasoning overhead.

3.1 Identity Coercion from Vec to List

We extend Barras and Bernardo’s identity coercion from Church-encoded vectors to lists ($v2lC$), to an identity coercion between inductive versions of the types ($v2l$), i.e. those supporting induction principles. Identity coercions for inductive types are defined using the same 3 components as inductive constructors (a Church-encoding component, like in Section 2.1, a parametricity theorem component, like in Section 2.2, and a reflection theorem component, like in Section 2.3).

**Church-Encoding** A standard way of translating a Church-encoded vector to a Church-encoded list is to eliminate the vector at the concrete list type, as shown in Figure 1. Alternatively, we can “go underneath” the list codomain, and eliminate the vector using the abstract list return type ($X$) and abstract list constructors ($cN$ for nil and $cC$ for cons). This alternative way, which is possible when the domain is a subtype of the codomain, appears below.

\[
v2lC \quad \forall A : \star . \forall n : \text{Nat} . \ \text{Vec} \cdot A \ n \rightarrow \text{List} \cdot A
\]

\[
= \Lambda A . \ A n . \lambda xs . \ A \ X . \lambda cN . \lambda cC .
\]

\[
xs.1.1 . (\lambda _ : \text{Nat} . \ X) \ cN (\ A _ . \ cC).
\]
One minor difference, compared to $v2lC'$ in Figure 1, is that $v2lC$ takes an inductive (rather than Church) vector as its argument. Hence, we access the Church-encoded vector via the projection $x_s.1.1$. This difference only becomes necessary in Section 3.2, where it allows us to define a dependent identity coercion.

Barras and Bernardo point out that after erasure, the alternative abstract elimination $\eta$-contracts to the identity function, and for this reason we call it an “identity coercion”:

**Lemma 3.1.** $|v2lC|$ is the identity function:

**Proof.**

1. $\delta \lambda x_s. \lambda c_n. |x_s.1.1 \ c_n \ c_c|$
   Erase projection.

2. $\lambda x_s. (\lambda c_n. \lambda c_c. x_s \ c_n \ c_c)$
   Contract.

3. $\eta \lambda x_s. x_s$
   □

**Parametricity Theorem** Second, we translate the vector parametricity theorem to the list parametricity theorem, this time projecting out the vector parametricity theorem (via $x_s.1.2$). In addition to the abstract arguments that $v2lC$ receives from its codomain, $v2lP$ also receives an abstract motive ($P$) and abstract parametricity theorem branches ($pN$ and $pC$).

$v2lP \ ▷ \ \forall \ A : \star . \ \forall \ n : \text{Nat} .$

1. $\Pi x_s : \text{Vec} \cdot A \ n. \ \text{ListP} \cdot A (v2lC \cdot A -n x_s)$

2. $\Lambda A . \Lambda n . \lambda x_s. \lambda X. A P . A cN . A cC . \lambda pN . \lambda pC .$

3. $x_s.1.2 \cdot (\lambda _ : \text{Nat} . X) \cdot (\lambda _ : \text{Nat} . P)$

4. $-cN -cC pN (A _ . pC)$.

Because the abstract motive ($P$) is an implicit argument, and the abstract Church constructors ($cN$ and $cC$) are also implicit arguments, they get erased, thus $v2lP$ is also an identity coercion (albeit between parametricity theorems):

**Lemma 3.2.** $|v2lP|$ is the identity function:

**Proof.**

1. $\delta \lambda x_s. \lambda p_n. \lambda p_c. |x_s.1.2 \ p_n \ p_c|$
   Erase projection.

2. $\lambda x_s. (\lambda p_n. \lambda p_c. x_s \ p_n \ p_c)$
   Contract.

3. $\eta \lambda x_s. x_s$

**Reflection Theorem** Third, we reuse the Church vector reflection theorem (projection $x_s.2$) to prove the vector reflection theorem.

$v2lR \ ▷ \ \forall \ A : \star . \ \forall \ n : \text{Nat} .$

1. $\Pi x_s : \text{Vec} \cdot A \ n. \ \text{ListR} \cdot A (v2lC \cdot A -n x_s)$

2. $\Lambda A . \Lambda n . \lambda x_s. \lambda x_s. x_s.2 .$

**Convention 5** We suffix an identifier with “L” to indicate that it relates to lists.
A proof of vector reflection \((\text{xsC} \cdot (\text{VecC} \cdot A) \ni\text{nilCV} \ consCV \simeq \text{xsC})\) can be used as a proof of list reflection \((\text{xsC} \cdot (\text{ListC} \cdot A) \ni\text{nilCL} \ consCL \simeq \text{xsC})\), because their types are definitionally equal (where definitional equality is defined on erased terms). This works because the type applications \((\text{VecC} \cdot A)\) and \((\text{ListC} \cdot A)\) are erased, \(\text{nilCV}\) and \(\text{nilCL}\) both erase to the untyped Church-encoding of nil (by Lemma 2.2 for vectors, and similarly for lists), and \(\text{consCV}\) and \(\text{consCL}\) both erase to the untyped Church-encoding of cons (by Lemma 2.3 for vectors, and similarly for lists).

**Identity Coercion** Finally, we put together our 3 components \((v21C, v21P,\) and \(v21R)\) to translate inductive vectors to inductive lists, using the \text{mkList} helper constructor. This is analogous to defining the vector constructors in terms of their 3 components and \text{mkVec} in Section 2.4.

\[
v21 \vdash \forall A : \ast. \forall n : \text{Nat} . \text{Vec} \cdot A \ n \to \text{List} \cdot A
\]
\[
= \Lambda A . \Lambda n . \lambda xs . \text{mkList} \cdot A
[\ v21C \cdot A -n xs , v21P \cdot A -n xs \ ] -(v21R \cdot A -n xs) .
\]

We have successfully generalized Barras and Bernardo’s non-dependent identity coercion between Church-encoded vectors and lists, to a non-dependent identity coercion between inductive vectors and lists.

**Theorem 3.3.** \([v2l]\) is the identity function:

**Proof.**

\[
=_{\delta} \lambda xs. [\text{mkList} \ [v21C \ x] , v21P \ x] \]
By Lemma 5.1

\[
=_{\beta} \lambda xs. [\text{mkList} \ [xs , v21P \ x]] \]
By Lemma 5.2

\[
=_{\beta} \lambda xs. [\text{mkList} \ [xs , x]] \]
Erase pair.

\[
= \lambda xs. [\text{mkList} \ x] \]
By Lemma 5.7 (for lists).

\[
=_{\beta} \lambda xs. x
\]

Type checking requires that both components of the pair (introducing an intersection type), in the definition of \(v21\), are definitionally equal. The third step in the proof of Theorem 3.3 demonstrates that this requirement is satisfied, after erasing the left and right components in the first and second steps, respectively.

**Remark 3.4.** Although the coercion of the reflection theorem \((v21R)\) happens to erase to the identity function, we never emphasize the erasure of reflection theorem proofs. This is because they appear in erased argument positions in the definitions of identity coercions (e.g. the erased argument \((v21R \cdot A -n xs)\) of \text{mkList}, in the definition of \(v21\)).
3.2 Identity Coercion from List to Vec

Barras and Bernardo’s non-dependent identity coercion takes Church-encoded vectors to lists, which we have extended (in Section 3.1) to take inductive vectors to lists. Because we are using inductive types, we can now define the dependent identity coercion from inductive lists to vectors (\(l2v\)). Church-encoded types cannot be used to define \(l2v\), as the resulting vector length depends on the input vector in the type of \(l2v\) (i.e. \(\Pi \xs : \List \cdot A . \Vec \cdot A (\length \cdot A \xs)\)), as in Figure 2. Although we could express the type using Church-encoded data, we could not inhabit it, as reducing \(\length\) in the codomain (when defining the nil and cons branches of the coercion) requires an induction principle (i.e. \(\text{elimList}\)).

Figure 2 contains the definition of \(l2v\), which follows the same 3-component structure used to define \(v2l\) in Section 3.2. The primary difference is that the Church (\(l2vC\)) and parametricity (\(l2vP\)) components are defined by induction, using \(\text{elimList}\). It is crucial that the domains of \(l2vC\) and \(l2vP\) are inductive lists, because \(l2vC\) and \(l2vP\) need to be defined by induction.

**Lemma 3.5.** \(|l2vC|\) is the identity function:

**Proof.**

\[
=_{\delta} \lambda \xs. \lambda c_n. \lambda c_c. [\text{elimList} \ \xs \ c_n \ c_c] \quad \text{By Lemma 2.10 (for lists)}.
\]

\[
=_{\beta} \lambda \xs. (\lambda c_n. \lambda c_c. \xs \ c_n \ c_c) \quad \text{Contract}.
\]

\[
=_{\eta} \lambda \xs. \xs \quad \square
\]

It is not enough that we can define a coercion from lists to vectors, we also want \(l2v\) to be an identity coercion. This is established by Theorem 3.7, which relies on \(l2vC\) being an identity coercion (Lemma 3.5), and \(l2vP\) being an identity coercion (Lemma 3.6). The proof that \(l2vC\) is an identity coercion is similar to
Barras and Bernardo’s argument about $v2lC$ (Lemma 3.1), relying on erasure and $\eta$-contraction. The main difference is that we also rely on the fact that our induction principle ($\text{elimList}$) is also an identity coercion (Lemma 2.10 but for the list datatype).

**Lemma 3.6.** $|l2vP|$ is the identity function:

*Proof.* Same as the proof of Lemma 3.5, $\alpha$-renaming $c_n$ and $c_c$ to $p_n$ and $p_c$. $\Box$

**Theorem 3.7.** $|l2v|$ is the identity function:

*Proof.* Same as the proof of Theorem 3.3 but erasing $l2vC$ (instead of $v2lC$) by Lemma 5.5 in the first step, $l2vP$ (instead of $v2lP$) by Lemma 3.6 in the second step, and $\text{mkVec}$ (instead of $\text{mkList}$) by Lemma 2.7 in the final step. $\Box$

### 3.3 Program Reuse

We achieve program reuse by defining list append ($\text{appendL}$) in terms of vector append ($\text{appendV}$), in the standard way by applying $\text{appendV}$ to the result of coercing both arguments to vectors (using $l2v$), and coercing the result of vector append to a list (using $v2l$).

$\text{appendL} \triangleq \forall A : \star . \text{List} \cdot A \rightarrow \text{List} \cdot A \rightarrow \text{List} \cdot A$

$= \Lambda A . \lambda xs . \lambda ys . v2l \cdot A$

$= (\text{add} (\text{length} \cdot A xs) (\text{length} \cdot A ys))$

$(\text{appendV} \cdot A - (\text{length} \cdot A xs) (12v \cdot A xs) - (\text{length} \cdot A ys) (12v \cdot A ys)) .$

The important property is that program reuse (i.e. the definition of $\text{appendL}$ in terms of $\text{appendV}$) incurs no runtime penalty. We prove this below, showing that the erasure of our derived list append is equal to the erasure of vector append, which relies on $v2l$ and $12v$ being identity coercions:

**Theorem 3.8.** $|\text{appendL}| = |\text{appendV}|$

*Proof.* $\delta \lambda xs. \lambda ys. |v2l (\text{appendV} | l2v xs | l2v ys)|$ By Theorem 3.7

$\beta \lambda xs. \lambda ys. |v2l (\text{appendV} xs ys)|$ By Theorem 3.3

$\beta \lambda xs. \lambda ys. |\text{appendV}| xs ys$ Contraction.

$\eta |\text{appendV}|$ $\Box$

### 3.4 Proof Reuse

Proof reuse, proving that list append is associative ($\text{appendAssocL}$) in terms of a proof that vector append is associative ($\text{appendAssocV}$), is even easier than program reuse. We derive $\text{appendAssocL}$ by applying $\text{appendAssocV}$ to the result of coercing each argument from a list to a vector (using $12v$). We do not need
to coerce in the other direction (using \( v2l \)), because our result is already an equality type that erases to our goal:\(^6\)

\[
\text{appendAssocL} \triangleq \forall A : \star . \\
\Pi xs : \text{List} \cdot A . \Pi ys : \text{List} \cdot A . \Pi zs : \text{List} \cdot A . \\
\text{appendL}(\text{appendL} \, xs \, ys) \, zs \simeq \text{appendL} \, xs \,(\text{appendL} \, ys \, zs) \\
= \, A \cdot A \cdot \lambda xs . \lambda ys . \lambda zs . \, \text{appendAssocV} \cdot A \\
-(\text{length} \cdot A \, xs) \,(12v \cdot A \, xs) \\
-(\text{length} \cdot A \, ys) \,(12v \cdot A \, ys) \\
-(\text{length} \cdot A \, zs) \,(12v \cdot A \, zs) .
\]

The result of reusing \( \text{appendAssocV} \) has the following type:

\[
\text{appendV} \,(\text{appendV} \,(12v \, xs) \,(12v \, ys)) \,(12v \, zs) \simeq \\
\text{appendV} \,(12v \, xs) \,(\text{appendV} \,(12v \, ys) \,(12v \, zs))
\]

After erasure, this \( \beta \)-reduces to our goal because \( \text{appendV} \) erases to \( \text{appendL} \) by Theorem 3.8, and \( 12v \) erases to \( (\lambda x . x) \) by Theorem 3.7. Without identity coercions in the derived program \( \text{appendL} \) and derived proof \( \text{appendAssocL} \), proof reuse would require equational reasoning by appealing to the identity laws established by an isomorphism between lists and vectors (as mentioned in the introduction Section II).

4 Reusing List Definitions

In this section we demonstrate reuse in the other direction (compared to Section 3), reusing list programs and proofs to define vector-versions of the programs and proofs. This direction of reuse takes more effort, because we may want to write functions over vectors with \textit{index constraints} in terms of functions over lists without the constraints. Hence, we are required to prove that the list-based reused definition implies the constraints we explicitly state in the vector-based derived definition.

4.1 Vectors as Length-Constrained Lists

The \( v2l \) function is “lossy” in the sense that the input vector length does not appear in the list codomain. In Section 4.2, we create a version of \( v2l \) (named \( v2u \)) that “remembers” the index information, by taking a vector to a list and a constraint on its length. Below, we derive a new type (which will be the codomain of \( v2u \)), named \( \text{VecL} \), as the intersection of a list and its length constraint.

\[
\text{VecL} \triangleq \star \rightarrow \text{Nat} \rightarrow \star = \lambda A : \star . \lambda n : \text{Nat} . \\
\iota xs : \text{List} \cdot A . n \simeq \text{length} \cdot A \, xs .
\]

\(^6\) Our proof reuse does not require congruence (\( \text{cong} \)), as used in the introduction Section II, because converting two of our equality types (\( \simeq \)) only requires their erased terms to be equal, not their types.
lengthPres ◦ ∀ A : ⋆ . ∀ n : Nat . II xs : Vec · A n .
\[ n \simeq \text{length} · A (v2l · A \cdot n \cdot xs) \]

Fig. 3. Length is preserved by coercion.

Because we use an intersection type, the erasure of VecL will always be its left (List) component, so the additional constraint does not get in the way of definitional equality checking.

Below, we define the constructor helper function \( \text{mkVecL} \), taking a list and an erased constraint to a VecL. We rewrite by the proof of the constraint (q) in the right component, so that we may return \( \beta\{xs\} \) as the right component, allowing the erasure of the left and right sides to both be \( xs \).

\[
\text{mkVecL} ◦ ∀ A : ⋆ . ∀ n : Nat . Vec · A n \rightarrow VecL · A n = \Lambda A . \Lambda n . \lambda xs . \text{mkVecL} · A \cdot n = \Lambda A . \Lambda n . \lambda xs . \Lambda q . [ \text{mkVecL} · A \cdot n \cdot xs ] .
\]

Just like \( \text{mkVec} \) in Section 2.4, \( \text{mkVecL} \) is also an identity coercion:

Lemma 4.1. \(|\text{mkVecL}| \) is the identity function:
Proof. Same as the proof of Lemma 2.7. \( \square \)

4.2 Identity Coercion from Vec to VecL

Now we define \( \text{v2u} \), taking a vector to a list and the constraint that the length of the list is equal to the index of the vector (by using VecL as the codomain of \( \text{v2u} \)). The function \( \text{v2u} \) uses \( \text{mkVecL} \) to construct a VecL from a vector by coercing to a list (via \( \text{v2l} \)), and proving the constraint that \( \text{v2l} \) preserves the vector index length w.r.t. the output list length (via \( \text{lengthPres} \) in Figure 3):

\[
\text{v2u} ◦ ∀ A : ⋆ . ∀ n : Nat . Vec · A n \rightarrow VecL · A n = \Lambda A . \Lambda n . \lambda xs . \text{mkVecL} · A \cdot n = \Lambda A . \Lambda n . \lambda xs . \Lambda q . \Lambda \text{v2l} · A \cdot n \cdot xs . \text{mkVecL} · A \cdot n . \text{v2l} · A \cdot n \cdot xs . \text{lengthPres} · A \cdot n \cdot xs) .
\]

Convention 6 We include “u” in an identifier to indicate that it relates to length-constrained lists.

The function \( \text{v2u} \) is also an identity coercion, as it is defined in terms of other identity coercions (\( \text{mkListL} \) and \( \text{v2l} \)):

Theorem 4.2. \(|\text{v2u}| \) is the identity function:
Proof. \[ =_\delta \lambda xs. \text{mkListL} \cdot v2l \cdot z \cdot xs \] By Theorem 3.3.
\[ =_\beta \lambda xs. \text{mkListL} \cdot xs \] By Lemma 4.1.
\[ =_\beta \lambda xs. xs \] \( \square \)
4.3 Program Reuse

We achieve program reuse by defining vector append \((\text{appendV})\) in terms of list append \((\text{appendL})\). We must coerce the output of \(\text{appendL}\) to a vector by \(\text{l2v}\), and the arguments of \(\text{appendL}\) to lists by the first projection of \(\text{v2u}\). However, we must also perform rewrites to ensure that \(\text{appendV}\) produces a vector whose length is the sum of both input vectors \((\text{add} \ n \ m)\):

\[
\text{appendV} : \forall A : \star . \forall n : \text{Nat} . \text{Vec} \cdot A n \rightarrow \forall m : \text{Nat} . \text{Vec} \cdot A m \rightarrow \text{Vec} \cdot A (\text{add} n m)
\]

The result of reusing \(\text{appendL}\) has the following type:

\[
\text{Vec} \cdot A (\text{length} (\text{appendL} (\text{v2u} \ xs).1 (\text{v2u} \ ys).1))
\]

We rewrite by the property \((\text{lengthDistAppend} \text{in Figure 4})\) that length distributes through list append via addition:

\[
\text{Vec} \cdot A (\text{add} (\text{length} (\text{v2u} \ xs).1 (\text{v2u} \ ys).1))
\]

Rewriting by the length-constraints of both coerced lists, via the second projection of \(\text{v2u}\) for \(\text{xs}\) and \(\text{ys}\), results in our goal type.

Reusing the program \(\text{appendL}\) to define \(\text{appendV}\) incurs no runtime penalty:

**Theorem 4.3.** \(|\text{appendV}| \text{ is } |\text{appendL}|\):

**Proof.** Erase rewrites and projections, then same as the proof of Theorem 3.8, exchanging \(\text{appendL}\) for \(\text{appendV}\), and \(\text{v2u}\) (erased by Theorem 4.2) for \(\text{v2l}\). \(\Box\)

4.4 Proof Reuse

We achieve proof reuse by proving that vector append is associative \((\text{appendAssocV})\) in terms of a proof that list append is associative \((\text{appendAssocL})\). Once again, this is easier than program reuse, as we must only coerce the arguments to lists (using \(\text{v2l}\)), but must not coerce the result (using \(\text{l2v}\)) because it is already an equality type:
appendAssocV ▼ ∀ A : * .
∀ n : Nat . Π xs : Vec · A n .
∀ m : Nat . Π ys : Vec · A m .
∀ o : Nat . Π zs : Vec · A o .
appendV (appendV xs ys) zs ≃ appendV xs (appendV ys zs)
= Λ A . Λ n . λ xs . Λ m . λ ys . Λ o . λ zs .
appendAssocL · A (v2l · A -n xs) (v2l · A -m ys) (v2l · A -o zs) .

The result of reusing appendAssocL has the following type:

appendL (appendL (l2v xs) (l2v ys)) (l2v zs) ≃
appendL (l2v xs) (appendL (l2v ys) (l2v zs))

After erasure, this β-reduces to our goal because appendL erases to appendV by Theorem 4.3 and l2v erases to (λ x . x) by Theorem 3.7. Again, proof reuse is zero-cost as no equational reasoning needs to be performed.

Remark 4.4. Note that in the definition of both append and appendAssoc, we could exchange (v2l xs) for (v2u xs)1 and lemma (lengthPres xs) for (v2u xs)2, and vice versa, because the latter term in both pairs erases to the former term.

5 Reusing Nested List Definitions

In this section we demonstrate reuse for nested datatypes, reusing programs and proofs over lists of lists (List · (List · A)) to define programs and proofs over vectors of vectors (Vec · (Vec · A n) m). Like in Section 4, such reuse requires proving properties about the lists to satisfy the vector length requirements of the derived definitions.

5.1 List Map

To reuse a list of lists as a vector of vectors, we must be able to coerce the inner lists in addition to the outer list. This can be achieved by mapping v2l over the inner lists. However, to ensure that reused definitions are identity coercions, it is crucial that we define list map (mapL) in Barras and Bernardo's style of eliminating the input list at the abstract type, and using the abstract constructors, of the output list (as in Section 2.1). We define mapL in abstract-elimination style for inductive lists in terms of our familiar 3 components (mapCL, mapPL, and mapRL).

Church-Encoding The Church-component of map eliminates the Church-encoded input list (via projection xs.1.1) at abstract type X, using abstract constructors cN and cC. We define the head of our mapped list to be (f x) in the abstract cons case (cC):
mapCL ▮ ∀ A : ⋆ . ∀ B : ⋆ . Π f : A → B . List · A → ListC · B
    = Λ A . Λ B . Λ f . Λ xs . Λ X . Λ cN . Λ cC .
      xs.1.1 · X cN (Λ x . cC (f x)) .

Even though we have defined mapCL in abstract elimination style, it is not an identity coercion (we know nothing about the input function f, which may change the elements of the list). However, if we partially apply mapCL to the identity function, then the result is an identity coercion:

**Lemma 5.1.** [mapCL] (λx. x) is the identity function:

**Proof.**

\[= \delta (\lambda f. \lambda xs. \lambda cN. \lambda cC. |xs.1.1 \cdot cN (\lambda x. cC (f x))) (\lambda x. x)\] Erase projection.

\[= (\lambda f. \lambda xs. \lambda cN. \lambda cC. cN (\lambda x. cC (f x))) (\lambda x. x)\] Reduce.

\[= \eta \lambda xs. (\lambda cN. \lambda cC. xs cN cC)\] Contract.

\[= \eta \lambda xs. xs\] Contract.

**Parametricity Theorem** The parametricity-component of map eliminates the parametricity theorem input (via projection xs.1.2) at abstract type X and abstract motive P, and using abstract constructors cN and cC, and abstract parametricity branches pN and pC. This time we use (f x) for the head position of the cons branch of the parametricity theorem (pC):

mapPL ▮ ∀ A : ⋆ . ∀ B : ⋆ .
        Π f : A → B . Π xs : List · A .
        ListP · B (mapCL · A · B f xs)
    = Λ A . Λ B . Λ f . Λ xs .
      Λ X . Λ pN . Λ pC .
      xs.1.2 · X · pN -cN -((Λ x . cC (f x))
      pC -xsC (f x)) .

The partial application of the parametricity-component of map to the identity function is likewise an identity coercion:

**Lemma 5.2.** [mapPL] (λx. x) is the identity function:

**Proof.** Same as the proof of Lemma 5.1 exchanging xs.1.2 for xs.1.1, and α-renaming pN and pC to cN and cC. □

**Reflection Theorem** The reflection theorem component of the map is defined by a simple induction (using elimList) on the input list, and reusing the reflection theorem proof of the input:

mapRL ▮ ∀ A : ⋆ . ∀ B : ⋆ .
        Π f : A → B . Π xs : List · A .
        ListR · B (mapCL · A · B f xs)
    = Λ A . Λ B . Λ f . Λ xs . elimList · A xs ·
        (λ xs : List · A . ListR · B (mapCL · A · B f xs))
    β
    (Λ xs . λ x . λ ih . p+ ih - β) .
Remark 5.3. Neither mapRL, nor its partial application to the identity function, results in an identity coercion. This is not important, as explained in Remark 3.4, because we will only use it in an erased position in the definition of mapL (as an implicit argument to mkList).

List Map Finally, we define mapL for inductive lists in the usual way, by applying the constructor helper function mkList to the intersection pair of the Church and parametricity components, and the reflection component:

\[
\text{mapL} : \forall A : \star . \forall B : \star . \Pi f : A \rightarrow B . \text{List} \cdot A \rightarrow \text{List} \cdot B \\
= A \cdot A . A \cdot B . \lambda f . \lambda xs . \text{mkList} \cdot B \\
[ \text{mapCL} \cdot A \cdot B f xs , \text{mapPL} \cdot A \cdot B f xs ] \\
\neg (\text{mapRL} \cdot A \cdot B f xs).
\]

Because mkList is an identity coercion, as are the components mapCL and mapPL, the partial application of mapL to the identity function is also an identity coercion:

Theorem 5.4. \(|\text{mapL}| (\lambda x. x) is the identity function:

Proof.

\[
\begin{align*}
=_{\delta} (\lambda f . \lambda xs . |\text{mkList} [\text{mapCL} f xs , \text{mapPL} f xs]|) (\lambda x. x) & \quad \text{Reduce.} \\
=_{\beta} \lambda xs . |\text{mkList} [|\text{mapCL} | (\lambda x. x) xs , \text{mapPL} (\lambda x. x) xs]|) & \quad \text{By Lemma 5.1} \\
=_{\beta} \lambda xs . |\text{mkList} [xs , |\text{mapPL} | (\lambda x. x) xs]| & \quad \text{By Lemma 5.2} \\
=_{\beta} \lambda xs . |\text{mkList} [xs , xs]| & \quad \text{Erase pair.} \\
= \lambda xs . |\text{mkList} xs| & \quad \text{By Lemma 2.7} \\
=_{\beta} \lambda xs . xs & \quad \square
\end{align*}
\]

5.2 Nested Identity Coercions

In order to reuse a program over nested lists to derive a program over nested vectors, we must coerce the nested vectors input of the derived program to nested lists. Below, we define v2l-v2l to perform such a coercion between nested datatypes.

\[
v2l-v2l : \forall A : \star . \forall n : \text{Nat} . \forall m : \text{Nat} . \\
\text{Vec} \cdot (\text{Vec} \cdot A n) m \rightarrow \text{List} \cdot (\text{List} \cdot A) \\
= A \cdot A . A \cdot n . A \cdot m . \lambda xss . \text{mapL} \cdot (\text{Vec} \cdot A n) \cdot (\text{List} \cdot A) \cdot (v2l \cdot A -n) \cdot (v2l \cdot (\text{Vec} \cdot A n) -m xss).
\]

Unsurprisingly, we define v2l-v2l by mapping v2l (using mapL) over the result of coercing the outer vector to a list (again via v2l). However, we now have an instance of a mapL applied to an identity coercion (v2l), which allows v2l-v2l to be an identity coercion between nested types:
Theorem 5.5. \([v2l-v2l] \text{ is the identity function:}\)

Proof. 
\[
\begin{align*}
\delta & \lambda xss. [\text{mapL} | v2l | (v2l \ xss)] \\
& = \lambda xss. [\text{mapL} | \lambda x. x | (v2l \ xss)] \\
& = \beta \lambda xss. [v2l \ xss] \\
& = \beta \lambda xss. xss
\end{align*}
\]

Because we plan on reusing an unindexed function over nested lists to define an indexed function over nested vectors, we will need to remember the length constraints on coerced nested lists (like in Section 4). Thus, we also define the nested mapping function \([v2u-v2l]\), which maps the outer vector to a list, but remembers the inner list length constraints by mapping the inner vectors to length-constrained lists (\([VecL]\)):

\[
[v2u-v2l] \blacktriangleleft \forall A : \star. \forall n : \text{Nat} . \forall m : \text{Nat} . \\
\text{Vec} \cdot (\text{Vec} \cdot A \ n) \ m \rightarrow \text{List} \cdot (\text{VecL} \cdot A \ n) \\
= \Lambda A . \Lambda n . \Lambda m . \lambda xss . \\
(\text{mapL} \cdot (\text{Vec} \cdot A \ n) \cdot (\text{VecL} \cdot A \ n) \ (v2u \cdot A -n) \\
\quad (v21 \cdot (\text{Vec} \cdot A \ n) \ -m xss)) .
\]

Just like \([v2l-v2l]\), \([v2u-v2l]\) is also an identity coercion:

Theorem 5.6. \([v2u-v2l] \text{ is the identity function:}\)

Proof. Same as the proof of Theorem 5.5, but erasing \([v2u]\) (instead of \([v2l]\)) by Theorem 4.2 in the first step.  

5.3 Identity Coercion from VecL to List

If we have a length-constrained list (\([VecL]\)), we can retrieve the inner list as the first projection of intersection:

\[
u2l \blacktriangleleft \forall A : \star. \forall n : \text{Nat} . \text{VecL} \cdot A \ n \rightarrow \text{List} \cdot A \\
= \Lambda A . \Lambda n . \lambda xs . xs.1 .
\]

The nice thing about length-constrained lists is that they erase to their list component, preventing the constraint from interfering with definitional equalities. Similarly, the projection of the list from the length-constrained list is an identity coercion, preventing the constraint from incurring runtime overhead:

Lemma 5.7. \([u2l] \text{ is the identity function:}\)

Proof. Erase projection.  

We can use \([v2u-v2l]\) to coerce a vector of vectors to a list of length-constrained lists, allowing us to rewrite by the constraints to prove that derived vector programs have the appropriate indices. However, ultimately we want to reuse a nested list program, so we also define \([u2l-12l]\) to project away the constraints of the inner length-constrained lists:
Fig. 5. Length distributes through concat.

Once again, this results in a nested identity coercion:

**Lemma 5.8.** \([\ u2l\cdot l2l\ ]\) is the identity function:

**Proof.**

\[=_{\beta} \lambda xss. \ |mapL\mid \ u2l\mid xss|\]  
\[= \lambda xss. \ |mapL (\lambda x. x)\mid xss|\]  
\[=_{\beta} \lambda xss. xss\]

\[\square\]

### 5.4 Program Reuse

Now we reuse a list concatenation program (**concatL**, flattening a list of lists to a list) to derive a vector concatenation program (**concatV**). Once again, we coerce the result of reusing **concatL** to a vector (using **l2v**). However, this time we reuse **concatL** by applying it to the result of mapping the input vector of vectors to a list of lists (via **v2l-v2l**). Vector concatenation (**concatV**) requires the index of the resulting vector to equal the product of the outer and inner input vector lengths (**\(\text{mult} \ m \ n\)**), thus we must also perform rewrites to ensure that our reused list program (**concatL**) respects this indexing requirement.

\[\text{concatV} \leftrightarrow \forall A : \ast. \ \forall n : \text{Nat} . \ \forall m : \text{Nat} . \ \forall xss : \text{List} \cdot (\text{Vec} \cdot A \ m) \rightarrow \text{Vec} \cdot A \ (\text{mult} \ m \ n)\]

\[\rho (v2u \cdot (\text{Vec} \cdot A \ n) \ m \rightarrow \text{Vec} \cdot A \ (\text{mult} \ m \ n)) \cdot 2 \]

\[\rho (\text{lengthDistConcat} \cdot A \ n \ (\text{vec} \cdot (v2v \cdot A \ n \ m \ xss)) ) \]

\[\cdot (12v \cdot A \ (\text{concatL} \cdot A \ (v2v \cdot A \ n \ m \ xss))) \]

The result of reusing **concatL** has the following type:

\[\text{Vec} \cdot A \ (\text{length} \ (12v \ (\text{concatL} \ (v2v \cdot A \ n \ m \ xss))))\]

We rewrite by the property (**lengthDistConcat** in Figure 5) that length distributes through the list concatenation of the nested coercion (performed by **u2l-l2l**, which takes a list of \(n\)-length-constrained lists to a list of lists). The result of this distribution is the product of the length of the nested list and \(n\):

\[\text{length} (\text{concatL} \cdot A \ (u2l \cdot A \ n \ xss))\]
Vec · A (mult (length xss) n)

Note that the property lengthDistConcat relies on all nested lists having the same length (n), hence it is defined for a list of length-constrained lists. Yet, our type resulting from reusing concatL applies length to a list of (non-constrained) lists (12v (concatL (v2l-v2l -n -m xss))), so why does the rewrite using lengthDistConcat succeed? The reason is that both (concatL · A (u2l-u2l · A -n xss)) and (12v (concatL (v2l-v2l -n -m xss))) erase to (concatL xss)! The former is a consequence of identity coercion u2l-u2l (Lemma 5.8), and the latter is a consequence of identity coercions 12v (Theorem 5.5), and v2l-v2l (Theorem 5.5).

Finally, we rewrite by the length constraint on the length of the outer input vector (using projection (v2u xss).2), changing (length xss) to m, resulting in our goal type.

Remark 5.9. The rich definitional equalities introduced by erasure and identity coercions make program reuse of concatV in terms of concatL straightforward, allowing us to easily rewrite our goal type by lengthDistConcat. Program reuse in a non-erased setting requires more complex lemmas and rewrites, due to VecL being defined as a non-erased dependent pair (Σ-type), rather than an erased dependent intersection (ι-type).

Theorem 5.10. |concatV| is |concatL|:

Proof. Erase rewrites, then:

\[=_{\lambda} \lambda xss. |12v (concatL \mid v2l-v2l xss)|\] By Theorem 5.5
\[=_{\beta} \lambda xss. |12v (concatL xss)|\] By Theorem 3.7
\[=_{\beta} \lambda xss. |concatL| xss\] Contract.
\[=_{\eta} |concatL|\]

5.5 Proof Reuse

We achieve proof reuse by proving that vector concat distributes through vector append (concatDistAppendV) in terms of the corresponding proof for lists (concatDistAppendL). This only requires applying our reused proof of concatDistAppendL to the result of coercing our input nested vector arguments to nested lists (via v2l-v2l):

concatDistAppendV ◁ ∀ A : * .
∀ n1 : Nat . ∀ m1 : Nat . II xss : Vec · (Vec · A n1) m1 .
∀ n2 : Nat . ∀ m2 : Nat . II yss : Vec · (Vec · A n2) m2 .
appendV (concatV xss) (concatV yss) ≃ concatV (appendV xss yss)
= A A . A n1 . A m1 . λ xss . A n2 . A m2 . λ yss .
concatDistAppendL · A
(v2l-v2l A -n1 -m1 xss)
(v2l-v2l A -n2 -m2 yss) .
The result of reusing \texttt{concatDistAppendL} has the following type:

\[
\text{appendL} \left( \text{concatL} \left( \text{v2l-v2l} \; xss \right) \right) \left( \text{concatL} \left( \text{v2l-v2l} \; yss \right) \right) \\
\cong \\
\text{concatL} \left( \text{appendL} \left( \text{v2l-v2l} \; xss \right) \right) \left( \text{v2l-v2l} \; yss \right)
\]

After erasure, this \(\beta\)-reduces to our goal because \texttt{appendL} erases to \texttt{appendV} by Theorem 4.3, \texttt{concatL} erases to \texttt{concatV} by Theorem 5.10, and \texttt{v2l-v2l} erases to \((\lambda \; x \; . \; x)\) by Theorem 5.5.

6 Related Work

6.1 Coercible in Haskell

Breitner et al. describe a GHC extension to Haskell (available starting with GHC 7.8) for a type class \texttt{Coercible a b}, which allows casting from \texttt{a} to \texttt{b} when such a cast is indeed the identity function \cite{4}. The motivation is to support retyping of data defined using Haskell's \texttt{newtype} statement, which is designed to give programmers the power to erect abstraction barriers that cannot be crossed outside of the module defining the \texttt{newtype}. Within such a module, however, \texttt{Coercible a b} and associated cast function \texttt{coerce : a \rightarrow b} allow programmers to apply zero-cost casts to change between a \texttt{newtype} and its definition.

\texttt{Coercible} had to be added as primitive to GHC, along with a rather complex system of \texttt{roles} specifying how coercibility of application of type constructors follows from coercibility of arguments to those constructors. In contrast, in the present work, we have shown how to derive zero-cost coercions within the existing type theory of Cedille, with no extensions. On the other hand, much of the complexity of \texttt{Coercible} in GHC arises from (1) how it interoperates with programmer-specified abstraction (via \texttt{newtype}) and (2) the need to resolve \texttt{Coercible a b} class constraints automatically, similarly to other class constraints in Haskell. The present work does not address either issue. However, the present work does allow for dependent casts between indexed variants of datatypes, which \texttt{Coercible} does not cover.

6.2 Ornaments

Ornaments \cite{10} are used to define refined version of types (e.g. \texttt{Vec}) from unreified types (e.g. \texttt{List}) by “ornamenting” the unreified type with extra index information. In contrast, our work establishes a relationship between \texttt{Vec} and \texttt{List} after-the-fact, by defining identity coercions in both directions for existing types. By \texttt{defining} vectors as natural-number-ornamented lists, ornaments can be used to calculate the “patch” type necessary to adapt a function from one type to another type \cite{5}. For example, ornaments could calculate that \texttt{lengthDistAppend} is necessary to adapt \texttt{appendL} from lists to vectors (\texttt{appendV}).

Although ornaments can be used to derive coercions between types in an ornamental relationship \cite{10,7}, they will not be identity coercions. Besides refining the indices of existing datatypes, ornaments also allow data to be added to
existing datatypes. For example, vectors can be index-refined lists, but lists can also be natural numbers with elements added. Our work only covers the index refinement aspect of ornaments.

6.3 Type Theory in Color

Type Theory in Color (TTC) \[2\] generalizes the concept of erased arguments of types to various colors, which may be erased optionally and independently according to modalities in the type theory. In the vector datatype declaration, the index data can be colored. If a vector is passed to a function expecting a list (whose modality enforces the lack of the index data color), then a free non-dependent identity coercion (using our parlance) is performed.

Lists can also be used as vectors, via a free dependent identity coercion in the other direction. This works due to a mechanism to interpret lists as a predicate on natural numbers. The list predicate is generated as the erasure of its colored elements (like ornaments, colors can add data in addition to refining indices), which results in refining lists by the length function.

Our work can be used to define a dependent identity coercion from natural numbers to the datatype of finite sets (\(\text{Fin}\)). This is not possible with colors, because \(\text{Fin}\) is indexed by successor (\(\text{suc}\)) in both of its constructors, which would require generating a predicate on the natural numbers from a non-deterministic function (or relation). Colors allow identity coercions to be generated and implicitly applied because colors erase types, as well as values, whereas implicit products only erase values (e.g. \(\Lambda\) is erased, but not \(\forall\)). Thus, while identity coercions need to be explicitly crafted and applied in our setting, we are able to define identity coercions (like taking natural numbers to finite sets) for which there is no unique solution.

7 Conclusion

We have demonstrated how to achieve zero-cost program and proof reuse between lists and vectors, which scales to the nested datatype setting, through the use of identity coercions, which erase to the identity function. Our technique works for datatypes like lists and vectors, where vectors are the length-indexed version of lists. Vectors have a subtype relationship with lists, and vice versa, supporting identity coercion in both directions.

For future work, we would like to explore what sort of program and proof reuse is possible (via identity coercions) between types that only have a subtyping relationship in one direction, such as untyped and intrinsically typed versions of \(\lambda\)-calculus expressions. We would also like to explore integrating a notion of ornaments into our setting, to automate the generation of the “patch” types necessary for program reuse. Finally, we would like to generalize our results to a class of datatypes related by refinement, via a generic encoding of indexed datatypes.
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