On Infravacua and the Localisation of Sectors

Walter Kunhardt
Institut für Theoretische Physik der Universität Göttingen
Bunsenstraße 9, 37073 Göttingen, Germany
e-mail: kunhardt@theorie.physik.uni-goettingen.de
June 5, 1998

Abstract
A certain class of superselection sectors of the free massless scalar field in 3 space dimensions is considered. It is shown that these sectors, which cannot be localised with respect to the vacuum, acquire a much better localisation, namely in spacelike cones, when viewed in front of suitable “infravacuum” backgrounds. These background states coincide, essentially, with a class of states introduced by Kraus, Polley and Reents as models for clouds of infrared radiation.

1 Introduction
In the analysis of superselection sectors, the localisability properties of charges are crucial for defining notions as charge composition and conjugation, statistics or a (global) gauge group. In a classical work [1], Doplicher, Haag and Roberts carried through such a programme for charges which fulfill what is now called the DHR criterion, i.e., which are compactly localised. Now this criterion is very restrictive, and Buchholz and Fredenhagen [2] established that sectors of theories in 3 space dimensions without massless particles in general only comply with the so-called BF criterion, i.e., they are localised in spacelike cones. Still, these authors could extend the analysis of [1] to charges with such a weaker localisation behaviour.

The situation is more difficult for theories with massless particles. Typically, these theories possess sectors whose localisation is too poor for the DHR framework to be applicable. Motivated by what is expected to happen in QED, it has been proposed by Buchholz in [3] to improve the localisation by viewing the charges in front of some suitable background field instead of the vacuum. In QED, such background fields should correspond to clouds...
of infrared radiation. An appropriate mathematical description of such infrared clouds has been introduced by Kraus, Polley and Reents in [4].

Here, we want to verify this mechanism in a simpler model, namely in the theory of the free massless scalar field in 3 space dimensions [5]. More precisely, we will consider a certain class of (non-Lorentz invariant) sectors described by automorphisms of the observable algebra and analyze their localisation properties in terms of the BF criterion. In particular, we will show that the sectors under consideration do satisfy this criterion with respect to a KPR-like background but do not satisfy it with respect to the vacuum. (Calling the background fields ”KPR-like” should indicate that they are very similar, yet not identical, to those of [4].)

As to the consistency of such an approach, it should be kept in mind that, in a theory whose charges are compactly localised, the superselection structure can be described without any difference with respect to the vacuum as well as with respect to so-called infravacua, the latter being generalisations of the KPR-like background states considered here. As has been shown in [5], when viewed in front of such an infravacuum, the charges remain compactly localised and have the same fusion structure and statistics as in front of the vacuum. Moreover, positivity of the energy in a sector does not depend on the background chosen, nor do the masses of massive particles possibly contained in such a theory.

The above-mentioned class of sectors of the free massless field has been studied recently by Buchholz et al. [5] with the purpose of modeling charges of electromagnetic type. In the following, we will stick very closely to the notations introduced there, but we should emphasize that our point of view is slightly different from that adopted in [5]: Buchholz et al. achieved a better localisation of the sectors by restricting them to a (non-Lorentz invariant) subnet $A_0 \subset A$ of the observable net. The sectors then even became localised in the DHR sense, which permitted them to carry through a DHR-like analysis, even though the net $A_0$ does not fulfill Haag duality. Here, in contrast, the subnet $A_0$ will play no rôle, and the localisation obtained will be in a weaker sense.

We end this Introduction by recalling the definition of the model under consideration. The observable algebra of the free massless scalar field is defined in its vacuum representation. More precisely, let $\mathcal{K} := L^2(\mathbb{R}^3, d^3k)$ be the Hilbert space of momentum space wave functions, $\omega(k) := |k|$ the one-particle energy and $U(t, \vec{x}) = e^{i\omega(\vec{k})t - \vec{r} \cdot \vec{k}}$ the usual representation of the spacetime translations. The vacuum Hilbert space of our model will be the bosonic Fock space $\mathcal{H}$ over $\mathcal{K}$; the induced unitary representation of
the spacetime translations will still be denoted by \( U(t, \vec{x}) \) without any risk of confusion. For any \( v \in \mathcal{K} \), \( W(v) \in \mathcal{B}(\mathcal{H}) \) will denote the corresponding Weyl operator. The normalisation is chosen such that the Weyl relations read \( W(u)W(v) = e^{-\frac{4}{i} \text{Im}(u,v)}W(u + v) \). For any real linear subspace \( \mathcal{L} \subset \mathcal{K} \), \( \mathcal{W}(\mathcal{L}) \) denotes the \( \mathbb{C}^* \)-subalgebra of \( \mathcal{B}(\mathcal{H}) \) generated by the operators \( W(f) \), \( f \in \mathcal{L} \). The net of observables now is given as

\[
\mathcal{O} \mapsto \mathfrak{A}(\mathcal{O}) := \mathcal{W}(\mathcal{L}(\mathcal{O}))'',
\]

where \( \mathcal{O} \mapsto \mathcal{L}(\mathcal{O}) \) is the isotonous, local and covariant net of symplectic subspaces in \( \mathcal{K} \) (indexed by the set of open double cones in Minkowski space) defined as follows: If \( \mathcal{O} := \{0\} \times O \)" is the causal completion of an open ball \( O \subset \mathbb{R}^3 \) at time \( t = 0 \), one has

\[
\mathcal{L}(\mathcal{O}) := \omega^{-\frac{1}{2}}D_R(O) + i\omega^{\frac{1}{2}}\hat{D}_R(O),
\]

where \( D_R(O) \) is the set of all real-valued smooth functions with support in \( O \) and \( \hat{\cdot} \) denotes the Fourier transform. For other double cones \( \mathcal{O} \), the space \( \mathcal{L}(\mathcal{O}) \) is defined by translation covariance and additivity. The symplectic form \( \sigma \) on \( \mathcal{L} := \bigcup_{\mathcal{O}} \mathcal{L}(\mathcal{O}) \subset \mathcal{K} \) reads

\[
\sigma(f_1, f_2) := -\text{Im} \langle f_1, f_2 \rangle,
\]

and locality for the net \( \mathcal{L}(\cdot) \) just means \( \sigma(\mathcal{L}(\mathcal{O}_1), \mathcal{L}(\mathcal{O}_2)) = 0 \) whenever \( \mathcal{O}_1 \) and \( \mathcal{O}_2 \) are spacelike to each other. As usual, we also associate symplectic subspaces of \( \mathcal{L} \) (resp. \( \mathbb{C}^* \)-subalgebras of \( \mathcal{B}(\mathcal{H}) \)) to unbounded regions in \( \mathbb{R}^{1+3} \) by additivity (resp. additivity and norm closure) and simply denote by \( \mathfrak{A} \) the quasi-local algebra \( \mathfrak{A}(\mathbb{R}^{1+3}) \).

The charges under consideration are given in terms of net automorphisms \( \gamma \in \text{Aut} \mathfrak{A} \) which are labeled uniquely by elements of the (additive) abelian group

\[
\mathcal{L}_\Gamma := \omega^{-\frac{1}{2}}D_R(\mathbb{R}^3) + i\omega^{\frac{1}{2}}\hat{D}_R(\mathbb{R}^3).
\]

Any element \( \gamma \in \mathcal{L}_\Gamma \) gives rise to a linear form \( l_\gamma : \mathcal{L} \rightarrow \mathbb{C} \),

\[
l_\gamma(f) := -\text{Im} \int d^3k \overline{\gamma(\vec{k})} f(\vec{k})
\]

and hence to an automorphism, again denoted by \( \gamma \), of \( \mathfrak{A} \) by

\[
\gamma(W(f)) := e^{il_\gamma(f)}W(f).
\]

As explained in \[5\], \( \gamma \) is indeed a well-defined automorphism of the quasi-local algebra \( \mathfrak{A} \) since, by Huygens' principle, it turns out to be locally normal;
as a consequence, it can be extended by weak continuity from the local Weyl algebras $\mathcal{W}(\mathcal{L}(\mathcal{O}))$ to the local von Neumann algebras $\mathfrak{A}(\mathcal{O})$.

There will be no risk of confusion in viewing the real vector space $L_\Gamma$ as an abelian subgroup of $\text{Aut}\mathfrak{A}$. In particular, a sum $\gamma_1 + \gamma_2$ in $L_\Gamma$ corresponds to the composition $\gamma_1 \circ \gamma_2$ in $\text{Aut}\mathfrak{A}$. Moreover, $\gamma_1$ and $\gamma_2$ define the same sector of $\mathfrak{A}$, i.e., they are unitarily equivalent in $\mathcal{B}(\mathcal{H})$, iff $\gamma_1 - \gamma_2 \in L_\Gamma \cap \mathcal{K}$. In this case, the Weyl operator $W(\gamma_1 - \gamma_2)$ is well defined and implements the unitary equivalence $\gamma_1 \sim \gamma_2$ on $\mathfrak{A}$.

Any $\gamma \in L_\Gamma$ can be written uniquely in the form $\gamma = \omega^{-\frac{i}{2}} \hat{\sigma} + i \omega^{-\frac{i}{2}} \hat{\rho}$ with functions $\sigma, \rho \in D_\mathbb{R}(\mathbb{R}^3)$. Since $\hat{\sigma}$ and $\hat{\rho}$ are analytic, it is obvious that $\gamma$ is square integrable, i.e., $\gamma \in \mathcal{K}$, iff $\hat{\rho}(0) = 0$. As a consequence, the sectors considered are labeled by a single real parameter

$$q_\gamma := \hat{\rho}(0) = \int d^3 x \, \rho(\vec{x})$$

which is interpreted as the charge of the sector $[\gamma]$. In particular, this shows that the sectors are transportable; as a matter of fact, they even have positive energy $\mathbb{R}$.

2 Bad localisation of the sectors in front of the vacuum

It has been shown in [5] that the automorphisms $\gamma \in L_\Gamma$ do not satisfy the DHR localisation criterion. Here, we want to strengthen this result and show with closely related methods that they do not even satisfy the BF criterion, that is, that they are not localisable in spacelike cones. To this end, it is sufficient to prove the following

**Proposition 2.1** Let $C \subset \mathbb{R}^3$ be an open convex cone having 0 as its apex and denote with $\mathcal{C} := (\{0\} \times C)^\prime\prime$ its causal completion. Then, for any $\gamma \in L_\Gamma$,

$$\gamma|_{\mathfrak{A}(C)} \cong \text{id}|_{\mathfrak{A}(C)} \quad \text{iff} \quad q_\gamma = 0.$$

The “if” part of this proposition is trivial, and before proving the “only if” part, we recall some facts about the dilation covariance of the model. The dilation group $\mathbb{R}_{>0}$ acts unitarily on $\mathcal{K}$ and leaves the space $\mathcal{L}$ invariant. More precisely, $f \in \mathcal{L}(\mathcal{O})$ is mapped onto $f_\lambda \in \mathcal{L}(\lambda \mathcal{O})$, where $f_\lambda(\vec{k}) := \lambda^{\frac{3}{2}} f(\lambda \vec{k})$. Writing $f = \omega^{-\frac{i}{2}} \hat{h} + i \omega^{\frac{i}{2}} \hat{g}$, it is verified by a straightforward computation that this entails for the linear form $l_\gamma$, $\gamma \in L_\Gamma$:

$$l_\gamma(f_\lambda) = \int \frac{d^3 k}{\omega^{\frac{i}{2}}} \overline{\hat{\sigma}(\vec{k} / \lambda)} \hat{h}(\vec{k}) - \frac{1}{\lambda} \int d^3 k \, \overline{\hat{\sigma}(\vec{k} / \lambda)} \hat{g}(\vec{k}).$$
In the limit $\lambda \to \infty$, $\vec{k} \mapsto \frac{1}{\lambda} \hat{\rho}(\vec{k}/\lambda) \hat{h}(\vec{k})$ converges to $\hat{\rho}(0) \hat{h}$ in the space of test functions, and since $\frac{2\pi^2}{r}$ is the Fourier transform of $\frac{1}{\omega^2}$ in the sense of distributions, one obtains

$$\lim_{\lambda \to \infty} l_\gamma(f_\lambda) = q_\gamma \kappa_f \quad \text{with} \quad \kappa_f := 2\pi^2 \int \frac{d^3x}{|x|} h(x).$$

This allows us to prove the following

**Lemma 2.2** Let $f \in \mathcal{L}(\mathcal{O}')$, where $\mathcal{O} \subset \mathbb{R}^{1+3}$ is a neighbourhood of 0. For any $\gamma \in \mathcal{L}_\Gamma$, one then has

$$\text{w-lim}_{\lambda \to \infty} \gamma(W(f_\lambda)) = e^{iq_\gamma \kappa_f} e^{-\frac{1}{4} \|f\|^2} 1.$$ 

Proof: Since the dilations act geometrically, it follows by locality from the special form of the localisation region of $f$ that $\lim_{\lambda \to \infty} \sigma(f_\lambda, f') = 0$ for any $f' \in \mathcal{L}$. Hence, $(W(f_\lambda))_{\lambda > 0}$ is a central sequence of unitaries in $W(\mathcal{L})$ whose set of weak limits is, by the irreducibility of the vacuum representation, a (nonempty) subset of $\mathbb{C} 1$. On the other hand, unitarity of the dilations permits us to evaluate this limit in the vacuum state: $\omega_0(W(f_\lambda))_{\lambda \to \infty} \to e^{-\frac{1}{4} \|f\|^2}$. But this means that $W(f_\lambda)$ has $e^{-\frac{1}{4} \|f\|^2} 1$ as its unique weak limit for $\lambda \to \infty$, establishing thus the assertion for $\gamma = 0$. For arbitrary $\gamma \in \mathcal{L}_\Gamma$, it now follows easily in view of the discussion in the preceding paragraph.

Physically, the sequence $(W(f_\lambda))_{\lambda \to \infty}$ is interpreted as a measurement of the asymptotic behaviour (in the spatial directions determined by the smearing function $h$) of the “Coulomb potential” of the “charge density” $\rho$. In QED, one expects that operators measuring the asymptotic electric flux distribution play a similar rôle, cf. [3]. In the present case, the leading $1/r$ behaviour of the Coulomb potential is isotropic in all sectors $[\gamma]$. This fact, reflected by the factorizing of $\lim_{\lambda \to \infty} l_\gamma(f_\lambda)$ as seen above, is relevant in the Proof of Prop. 2.1: Let $\gamma \in \mathcal{L}_\Gamma$ with $q_\gamma \neq 0$. Choose a nonvanishing, nonnegative test function $h \in D_{\mathbb{R}}(C)$. Letting $f := \omega^{-\frac{1}{2}} \hat{h}$, this implies $\kappa_f \neq 0$ and $f \in \mathcal{L}(\mathcal{C} \cap \mathcal{O}')$ for some neighbourhood $\mathcal{O} \subset \mathbb{R}^{1+3}$ of 0. Since $e^{iq_\gamma \kappa_f} \neq 1$ can always be achieved by a mere rescaling of $h$, Lemma 2.2 shows that the weak limits (as $\lambda \to \infty$) of $W(f_\lambda)$ and $\gamma(W(f_\lambda))$ are different scalar multiples of the unit operator. But since $W(f_\lambda) \in \mathfrak{A}(\mathcal{C})$ for all $\lambda > 0$, this implies $\gamma|_{\mathfrak{A}(\mathcal{C})} \neq \text{id}|_{\mathfrak{A}(\mathcal{C})}$. ■
3 Infravacuum background states

In this section we introduce a class of background states in front of which the automorphisms $\gamma$ will be shown (in Section 4) to have better localisation properties. Apart from two modifications necessitated by the present model, these background states are of the same type as those introduced by Kraus, Polley and Reents \[4\] as a model for infrared clouds in QED or, more generally, in any theory containing massless particles.

3.1 Preliminaries on quasifree states

First, we recall that a quasifree state on $\mathfrak{A}$ is a locally normal state $\omega_T$ which is, on the Weyl operators $W(f) \in \mathfrak{A}$, $f \in \mathcal{L}$ of the form

$$\omega_T(W(f)) = e^{-\frac{1}{4}||Tf||^2}.$$ 

Here, $T : D_T \rightarrow \mathcal{K}$ is a real linear, symplectic (i.e., fulfilling $\text{Im}(T v, T w) = \text{Im}(v, w), v, w \in D_T$) operator defined on a dense, real linear subspace $D_T$ which contains $\mathcal{L}$. In the case at hand, we will have in addition $TT' = \mathcal{K}$, which entails that $\omega_T$ is a pure state. Its GNS representation $\pi_T$ acts irreducibly on the vacuum Hilbert space $\mathcal{H}$ as $\pi_T(W(f)) = W(T f), f \in \mathcal{L}$.

Next, we describe the real linear operator $T$ in terms of a pair of complex linear operators $T_1, T_2$ defined on complex linear subspaces $D_T$, $j = 1, 2$.

**Lemma 3.1** Let $\Gamma : \mathcal{K} \rightarrow \mathcal{K}$ be an antiunitary involution. Then, the formulae

$$T := T_2 \frac{1+\Gamma}{2} + T_1 \frac{1-\Gamma}{2}$$

$$D_T := \{v \in \mathcal{K} | \frac{1+\Gamma}{2} v \in D_T, \frac{1-\Gamma}{2} v \in D_T\}$$

establish a bijection between

- densely defined, $\Gamma$-invariant $\mathbb{R}$-linear operators $T : D_T \rightarrow \mathcal{K}$ and
- densely defined, $\Gamma$-invariant $\mathbb{C}$-linear operators $T_j : D_{T_j} \rightarrow \mathcal{K}$, $j = 1, 2$.

Moreover, $T$ is symplectic iff $\langle T_1 u_1, T_2 u_2 \rangle = \langle u_1, u_2 \rangle$ for all $u_j \in D_{T_j}$.

\[1\] Here, $T : D_T \rightarrow \mathcal{K}$ being $\Gamma$-invariant means $\Gamma D_T = D_T$ and $[\Gamma, T] = 0$ on $D_T$. 


Since all assertions can be checked by simple calculations, we omit the formal proof of this Lemma and merely point out that the converse formulae expressing $T_1$ and $T_2$ in terms of $T$ read

\[
D_{T_2} = \{ v \in K \mid \frac{1+i}{2} \Gamma v \subset DT \}, \quad T_2 = T \frac{1+i}{2} - iT \frac{1+i}{2} i \\
D_{T_1} = \{ v \in K \mid \frac{1-i}{2} \Gamma v \subset DT \}, \quad T_1 = T \frac{1-i}{2} + iT \frac{1-i}{2} i.
\]

**Remark:** The involution $\Gamma$ induces the notion of real and imaginary parts of vectors $v \in K$: $\text{Re} v = \frac{1+i}{2} v$, $\text{Im} v = \frac{1-i}{2} v$. Then, $T_2$ acts on the real and $T_1$ on the imaginary parts:

\[
\text{Re}Tv = T_2 \text{Re}v, \quad \text{Im}Tv = T_1 \text{Im}v, \quad v \in DT.
\]

From now on, we will fix $\Gamma$ to be pointwise complex conjugation in position space. In terms of momentum space wave functions $v \in \mathcal{K}$, this means

\[
(\Gamma v)(\vec{k}) := \overline{v(-\vec{k})}.
\]

[For the sake of completeness, we point out that Kraus et al. used pointwise conjugation in momentum space for defining their background states in [4]. In their case as well as in ours, the choice of the involution $\Gamma$ is dictated by the set of sectors under consideration.]

### 3.2 Quasifree states with positive energy

Before describing in detail the operators $T_1, T_2$, we introduce some notation: for any $\epsilon > 0$, let $P_{\epsilon} : \mathcal{K} \to \mathcal{K}$ be the projector onto the subspace $P_{\epsilon} \mathcal{K} = \{ v \in \mathcal{K} \mid v(\vec{k}) = 0 \text{ if } |\vec{k}| < \epsilon \}$ and denote by

\[
D_0 := \bigcup_{\epsilon > 0} P_{\epsilon} \mathcal{K}
\]

the dense subspace of functions vanishing in some neighbourhood of $\vec{k} = 0$. Note that $[P_{\epsilon}, \Gamma] = 0$ and $\Gamma D_0 = D_0$. The subspace $D_0$ will serve as a provisional domain for $T_1$ and $T_2$.

Now we follow [4] and choose

- a sequence $(\epsilon_i)_{i \in \mathbb{N}}$ in $\mathbb{R}_{>0}$ satisfying $\epsilon_{i+1} < \epsilon_i$ and $\epsilon_i \xrightarrow{i \to \infty} 0$.

This sequence induces a decomposition of momentum space into concentric spherical shells. The projections onto the associated spectral subspaces of $\mathcal{K}$ will be denoted by $P_i := P_{\epsilon_{i+1}} - P_{\epsilon_i}$. For notational convenience, we also put $P_0 := P_{\epsilon_1}$. 

7
• a sequence \((Q_i)_{i \in \mathbb{N}}\) of orthogonal projections in \(\mathcal{K}\) with finite rank \(\text{rk}Q_i\) satisfying \(Q_i\Gamma = \Gamma Q_i, \ Q_iP_i = Q_i\).

• a sequence \((b_i)_{i \in \mathbb{N}}\) in \([0, 1[\) satisfying \(b_i \xrightarrow{i \to \infty} 0\) and \(\sum_i \frac{1}{b_i}\text{rk}Q_i < \infty\).

If, e.g., the \(\epsilon_i\) decrease exponentially and \(\text{rk}Q_i\) is polynomially bounded, this can be satisfied by \(b_i \propto \frac{1}{i^{\alpha}}, \alpha > 0\).

With these data, define \(\mathbb{C}\)-linear operators \(T_1, T_2\) on the subspace \(D_0\) by

\[
T_1 := 1 + s\text{-lim}_{n \to \infty} \sum_{i=1}^{n} (b_i - 1)Q_i, \quad T_2 := 1 + s\text{-lim}_{n \to \infty} \sum_{i=1}^{n} \frac{1}{b_i} - 1)Q_i.
\]

Since, on every \(v \in D_0\), the number of terms which contribute on the right-hand side is finite, these operators are well defined and map \(D_0\) into itself. Moreover, the relations

\[
T_1P_i = ((1 - Q_i) + b_iQ_i)P_i, \quad T_2P_i = ((1 - Q_i) + \frac{1}{b_i}Q_i)P_i
\]

show that the subspace \(P_i\mathcal{K}\) decomposes into a subspace \((1 - Q_i)P_i\mathcal{K}\) where both \(T_1\) and \(T_2\) act trivially and an orthogonal subspace \(Q_iP_i\mathcal{K} = Q_i\mathcal{K}\) where they act as multiplications with the scalars \(b_i\) and \(\frac{1}{b_i}\), respectively. As a consequence, \(T_1\) and \(T_2\) are inverses of each other. Because of \(\lim_{i \to \infty} b_i = 0\), \(T_1\) is bounded (\(\|T_1\| = 1\)), whereas \(T_2\) is not. Also, it is clear that \(T_1\) and \(T_2\) are \(\Gamma\)-invariant and symmetric. In particular, it follows that \(\langle T_1u_1, T_2u_2\rangle = \langle u_1, T_1T_2u_2\rangle = \langle u_1, u_2\rangle\) for any \(u_1, u_2 \in D_0\). We are thus in the situation of Lemma 3.1 and obtain an unbounded symplectic operator

\[
T : D_0 \longrightarrow \mathcal{K}, \quad T = T_2 \frac{1+\Gamma}{2} + T_1 \frac{1-\Gamma}{2}.
\]

In the next step, \(T\) has to be extended to a larger domain \(D_T \supset \mathcal{L}\). To this end, we analyze its singular behaviour for \(|\vec{k}| \to 0\) by comparing it with powers of (a regularized version \(\omega_r\) of ) the one-particle Hamiltonian \(\omega\). Setting

\[
\omega_r := \omega (1 - P_0) + \epsilon_1 P_0 = \begin{cases} \omega & \text{on} \ (1 - P_0)\mathcal{K}, \\ \epsilon_1 1 & \text{on} \ P_0\mathcal{K} \end{cases}
\]

and noting that \(\omega_r^{1/2}D_0 \subset D_0\), we obtain:

**Lemma 3.2** \(T_2\omega_r^{1/2}\) is bounded.
Proof: Making use of \( \|\omega_i P_i\| = \epsilon_i \) for \( i \in \mathbb{N} \), one obtains for \( v \in D_0 \)

\[
\left\| (T_2 - 1) \omega^\frac{1}{2} v \right\|^2 = \left\| \sum_i (\frac{1}{b_i} - 1) Q_i \omega^\frac{1}{2} v \right\|^2 = \sum_i (\frac{1}{b_i} - 1)^2 \left\langle \omega^\frac{1}{2} v, Q_i \omega^\frac{1}{2} v \right\rangle 
\leq \sum_i (\frac{1}{b_i} - 1)^2 \text{rk} Q_i \left\langle \omega^\frac{1}{2} v, P_i \omega^\frac{1}{2} v \right\rangle \leq \sum_i (\frac{1}{b_i} - 1)^2 \text{rk} Q_i \epsilon_i \|v\|^2.
\]

From the conditions imposed on the \( b_i \), it follows that \( \sum_i (\frac{1}{b_i} - 1)^2 \text{rk} Q_i \epsilon_i \) is finite. Thus \( (T_2 - 1) \omega^{1/2} \) is bounded, hence also \( T_2 \omega^{1/2} \).

We now can extend \( T_1 \) by continuity to all of \( \mathcal{K} =: D_{T_1} \) and \( T_2 \) by the formula

\[
T_2 v := T_2 \omega^\frac{1}{2} \omega^{-\frac{1}{2}} v, \quad v \in \omega^\frac{1}{2} \mathcal{K}
\]

to the dense subspace \( \omega^\frac{1}{2} \mathcal{K} =: D_{T_2} \). (Strictly speaking, the symbol \( T_2 \omega^\frac{1}{2} \) on the right-hand side stands for the continuous extension to \( \mathcal{K} \) of the operator considered in the previous Lemma.) Note that \( T_1 \) and \( T_2 \) still are \( \Gamma \)-invariant.

We collect the relevant properties in the following Lemma:

**Lemma 3.3**

1. \( D_T := \{ v \in \mathcal{K} | \frac{1+\Gamma}{2} v \in \omega^\frac{1}{2} \mathcal{K} \} \) is a real linear dense subspace of \( \mathcal{K} \).
2. \( T = T_2 \frac{1+\Gamma}{2} + T_1 \frac{1-\Gamma}{2} \) is well defined on \( D_T \).
3. \( T : D_T \rightarrow \mathcal{K} \) is a symplectic operator.
4. \( \mathcal{L} \subset D_T \) and \( T \mathcal{L} \) is dense in \( \mathcal{K} \).

Proof: Part 1 is obvious, since \( D_0 \subset D_T \); part 2 has been shown in the previous paragraph. For 3, we have to show that \( \langle T_1 u_1, T_2 u_2 \rangle = \langle u_1, u_2 \rangle \) remains true for all \( u_1 \in D_{T_1} \) and \( u_2 \in D_{T_2} \). First, assume \( u_1 \in D_0 \). Since \( D_0 \) is dense in \( \mathcal{K} \) and invariant under \( \omega^{1/2} \), there exists a sequence \( u_2^{(n)} \in D_0 \), \( n \in \mathbb{N} \) such that \( \omega^{-\frac{1}{2}} u_2 = \lim_{n \to \infty} \omega^{-\frac{1}{2}} u_2^{(n)} \), implying \( u_2 = \lim_{n \to \infty} u_2^{(n)} \). Using the boundedness of \( T_2 \omega^{1/2} \), we can compute

\[
\langle T_1 u_1, T_2 u_2 \rangle = \langle T_1 u_1, T_2 \omega^\frac{1}{2} \omega^{-\frac{1}{2}} u_2 \rangle = \langle T_1 u_1, T_2 \omega^\frac{1}{2} \lim_{n \to \infty} \omega^{-\frac{1}{2}} u_2^{(n)} \rangle = \lim_{n \to \infty} \langle T_1 u_1, T_2 \omega^\frac{1}{2} \omega^{-\frac{1}{2}} u_2^{(n)} \rangle = \lim_{n \to \infty} \langle u_1, u_2^{(n)} \rangle = \langle u_1, u_2 \rangle.
\]
Since $T_1$ is bounded, the restriction on $u_1$ can now be dropped by continuity, thus yielding the assertion. Finally, $\mathcal{L} \subset D_T$ is obvious, and the remaining part of 4 is equivalent, in terms of $T_1$ and $T_2$, to

$$
\frac{1+i}{2} T \mathcal{L} = T_2 \frac{1+i}{2} \mathcal{L} = T_2 \omega^{-1/2} \hat{D}_R \quad \text{is dense in} \quad \frac{1+i}{2} \mathcal{K}.
$$

$$
\frac{1-i}{2} T \mathcal{L} = T_1 \frac{1-i}{2} \mathcal{L} = T_1 \omega^{1/2} \hat{D}_R \quad \text{is dense in} \quad \frac{1-i}{2} \mathcal{K}.
$$

By $\mathbb{C}$-linearity, this in turn is equivalent to $T_2 \omega^{-1/2} \hat{D}_C = T_2 \omega^{1/2} \omega^{-1/2} \hat{D}_C$ and $T_1 \omega^{1/2} \hat{D}_C$ both being dense in $\mathcal{K}$. But this is implied by the fact that, on the one hand, both operators $T_2 \omega^{1/2}$ and $T_1$ are bounded and have dense images (since they are invertible on the dense, invariant subspace $D_0$) and that, on the other hand, the subspaces $\omega^{-1/2} \hat{D}_C$ and $\omega^{1/2} \hat{D}_C$ are dense in $\mathcal{K}$ (by the spectral calculus of $\omega$).

With the above preparations, we can define a state $\omega_T : \mathfrak{A} \to \mathbb{C}$ and analyze its main properties.

**Proposition 3.4** The quasifree state $\omega_T$, defined on $\mathcal{W}(\mathcal{L})$ by

$$
\omega_T(W(f)) = e^{-\frac{i}{2} \|Tf\|^2}, \quad f \in \mathcal{L}
$$

extends to a unique locally normal state $\omega_T$ over the quasilocal algebra $\mathfrak{A}$. This state is pure and has positive energy.

Proof: The difficult part of this proof is to obtain local normality of $\omega_T$ on the net $\mathcal{O} \hookrightarrow \mathcal{W}(\mathcal{L}(\mathcal{O}))$ of Weyl algebras. To this end, recall that $T$ is (on $\mathcal{L}$) the strong limit of symplectic operators $T_n$ such that $T_n - 1$ have finite rank. As a consequence, the associated quasifree states $\omega_{T_n}$ are vector states in the vacuum representation and converge weakly to $\omega_T$ on $\mathcal{W}(\mathcal{L})$. Now since the Fredenhagen-Hertel compactness condition $C_\sharp$ is known to be fulfilled in the present model, we can conclude that $\omega_T$ is locally normal if the sequence $(\omega_{T_n})_{n \in \mathbb{N}}$ is bounded with respect to some exponential energy norm $\|\cdot\|_\beta$, $\beta > 0$ defined by $\|\omega\|^2_\beta := \omega(e^{2\beta H})$. But this follows from $\sum_i \frac{n_i}{\beta} \text{rk} Q_i < \infty$, as F. Hars has shown in [9], adapting ideas from [4]. (Although our involution $\Gamma$ differs from that of [4], the arguments leading to this conclusion are still valid.) Hence, $\omega_T$ is locally normal on $\mathcal{W}(\mathcal{L})$ and thus extends uniquely to a locally normal state on $\mathfrak{A}$. Since it is a weak limit of states in the vacuum representation with positive energy, the arguments of Buchholz and Doplicher [10] can be applied to show that $\omega_T$ has positive energy, too. Finally, the relation $T \mathcal{L} = \mathcal{K}$, established in Lemma 3.3, implies that $\omega_T$ is pure, as has been noted at the very beginning of this section. ■
Remark: The inequality $\sum_i \frac{\varepsilon_i}{b_i^2} \text{rk} Q_i < \infty$, which played a crucial rôle in the previous proof, has a direct physical interpretation. Indeed, performing the limit $\omega_{T_n} \to \omega_T$ corresponds to the excitation of more and more low-energy “photon” modes in comparison to the vacuum, namely those singled out by the projections $Q_i$, $i = 1, \ldots, n$ which appear in $T$. Since $\frac{1}{b_i^2}$ measures the amplitude of these modes, each of them carries an energy of about $\varepsilon_i b_i^2$.

Hence the modes in the energy interval $[\varepsilon_{i+1}, \varepsilon_i]$ contribute with (at most) $\frac{\varepsilon_i}{b_i^2} \text{rk} Q_i$ to the mean energy of the state $\omega_T$, and the above inequality thus means that $\omega_T$ describes an infrared cloud with finite total energy. We conjecture that these arguments can be sharpened in order to prove that the transition energy [11] between the sectors $\pi_0$ and $\pi_T$ vanishes. In terms of [6], the properties of $\pi_T$ could then be summarised by saying that it is an “infravacuum representation”, and we will indeed use this terminology in the sequel.

3.3 KPR-like quasifree states

We reach our goal of improving the localisation of the automorphisms $\gamma$ by considering a special class of infravacuum representations. The main idea, due to [4], is to control the angular momentum carried by the low-energy modes. It may be formalised as follows.

Definition: A state $\omega_T$ over $A$ based on the sequences $\varepsilon_i, Q_i, b_i$ as described above is called a KPR-like state (and $\pi_T$ (resp. $T$) a KPR-like representation (resp. symplectic operator)) if the following additional conditions are fulfilled:

1. $(\ln \frac{\varepsilon_i}{\varepsilon_{i+1}})_{i \in \mathbb{N}}$ is polynomially bounded, and $\sum_i b_i^2 \ln \frac{\varepsilon_i}{\varepsilon_{i+1}} < \infty$.

2. With respect to the tensor product structure of the subspace $P_i K$, $P_i K \cong L^2([\varepsilon_{i+1}, \varepsilon_i], \omega^2 d\omega) \otimes L^2(S^2)$, the projections $Q_i$ read

$$Q_i = \frac{\langle \xi_i | \xi_i \rangle}{\langle \xi_i | \xi_i \rangle} \otimes \tilde{Q}_i$$

with $\tilde{Q}_i := \sum_{0 < l \leq i} \sum_{m = -l}^{l} |Y_{lm}(\omega)| \langle Y_{lm} | Y_{lm} \rangle$;

here the vector $\xi_i \in L^2([\varepsilon_{i+1}, \varepsilon_i], \omega^2 d\omega)$ is given by $\xi_i(\omega) = \omega^{-\frac{3}{2}}$ and $Y_{lm} \in L^2(S^2)$ are the spherical harmonics.

This definition has been formulated so as to imply the regularity property of the bounded operator $T_1$ formulated in the next Lemma. It is only through this result that the two additional properties of KPR-like infravacua enter
the analysis of Section 4. It is apparent from the ensuing proof that the above definition may be generalised in several respects. However, we refrain from discussing these possibilities here.

In contrast, we draw the reader’s attention to the following crucial difference between our KPR-like states and the “true” KPR states as defined in [4]: In our case, the projection $\tilde{Q}_i$ contains no summand $|Y_{00}\rangle\langle Y_{00}|$. In physical terms, this means that the infrared cloud does not contain any spherically symmetric low-energy modes. Such a restriction is necessary, since it is precisely by such modes or, equivalently, by the isotropic long-range behaviour of the “Coulomb potential”, that the sectors $[\gamma]$ differ from each other. Too strong an $l = 0$ contribution to the infrared cloud would therefore render the sectors indistinguishable in front of that background. (Indeed, if one had $0 \leq l \leq i$ in the definition of $\tilde{Q}_i$, one would obtain, instead of Lemma 3.6 below, that $\pi_T \circ \gamma \cong \pi_T$ for all $\gamma \in L_T$.) This seemingly artificial restriction on the background states mimics the situation in QED, where the Coulomb field $\vec{E}(\vec{k}) \sim ik/\omega^2$ cannot be compensated by transverse photons.

**Lemma 3.5** Let the sequences $\epsilon_i, Q_i, b_i$ be such that $\omega_T$ is a KPR-like state. Let $u \in K$ have, in a neighbourhood of $\vec{k} = 0$, the form $u(\vec{k}) = \eta(\vec{k}/|\vec{k}|)$ with some $\eta \in C^\infty(S^2) \subset L^2(S^2)$. Then the sequence $(T_1 \omega^{-\frac{3}{2}} P_n u)_{n \in \mathbb{N}}$ converges iff $\eta \perp Y_{00}$.

Proof: Without any restriction, one may assume $u = c \otimes \eta$ with $c(\omega) = 1$ if $\omega < \epsilon_1$. Let $\eta \perp Y_{00}$. For $0 < m < n$, one computes

$$T_1 \omega^{-\frac{3}{2}} P_n u - T_1 \omega^{-\frac{3}{2}} P_m u = T_1 \omega^{-\frac{3}{2}} \sum_{i=m}^{n-1} P_i (c \otimes \eta) = \sum_{i=m}^{n-1} T_1 P_i (\xi_i \otimes \eta) = \sum_{i=m}^{n-1} \xi_i \otimes ((1 - \tilde{Q}_i) \eta + b_i \tilde{Q}_i \eta).$$

Now $\eta \in C^\infty(S^2)$ entails that $\|(1 - \tilde{Q}_i) \eta\|^2 = \sum_{l > i} \sum_m |\langle Y_{lm}, \eta \rangle|^2$, $i \in \mathbb{N}$ is a sequence of rapid decrease (since $\eta \in D(\tilde{L}_2^N)$ for any $N$, $\tilde{L}$ denoting the angular momentum operator). Thus, using $\|\xi_i\|^2 = \int_{\epsilon_i+1}^{\epsilon_i+1} \omega^2 d\omega \frac{1}{\omega^3} = \ln \frac{\epsilon_i}{\epsilon_{i+1}}$
and \( \| b_i \hat{Q}_i \eta \|^2 \leq b_i^2 \| \eta \|^2 \), one obtains
\[
\left\| T_1 \omega^{-\frac{3}{2}} P_{e_n} u - T_1 \omega^{-\frac{3}{2}} P_{e_n} u \right\|^2 \leq \sum_{i=m}^{n-1} \| \xi_i \|^2 \left( (1 - \hat{Q}_i) \eta \| \| + b_i^2 \| \hat{Q}_i \eta \| ^2 \right)
\]
\[
\leq \sum_{i=m}^{n-1} \ln \frac{e_i}{e_{i+1}} \left( \frac{c_N}{e_i} + b_i^2 \right).
\]
With suitably chosen \( N \), the right-hand side vanishes as \( m, n \rightarrow \infty \) due to the conditions imposed on \( e_i \) and \( b_i \). Hence \( (T_1 \omega^{-\frac{3}{2}} P_{e_n} u)_{n \in \mathbb{N}} \) is a Cauchy sequence. Conversely, assume \( \langle Y_{00}, \eta \rangle \neq 0 \). With \( \eta = (Y_{00}, \eta) Y_{00} + \eta_1 \), \( (T_1 \omega^{-\frac{3}{2}} P_{e_n} (c \otimes \eta_1))_{n \in \mathbb{N}} \) is convergent, hence \( (T_1 \omega^{-\frac{3}{2}} P_{e_n} u)_{n \in \mathbb{N}} \) is divergent because \( (T_1 \omega^{-\frac{3}{2}} P_{e_n} (c \otimes Y_{00}))_{n \in \mathbb{N}} = (\omega^{-\frac{3}{2}} P_{e_n} (c \otimes Y_{00}))_{n \in \mathbb{N}} \) is.

We end this section with a result which shows that the KPR-like infravacua do not affect the superselection structure of the present model. As the previous Lemma, it makes essential use of the fact that \( T f = f \) for all rotation invariant elements \( f \in D_T \).

**Lemma 3.6** Let \( \pi_T \) be a KPR-like infravacuum representation. Then, for any \( \gamma_1, \gamma_2 \in \mathcal{L}_T \), one has

\[
\pi_0 \circ \gamma_1 \cong \pi_0 \circ \gamma_2 \quad \text{iff} \quad \pi_T \circ \gamma_1 \cong \pi_T \circ \gamma_2.
\]

Proof: Let \( \pi_0 \circ \gamma_1 \cong \pi_0 \circ \gamma_2 \). Then \( \gamma := \gamma_1 - \gamma_2 \in \mathcal{L}_T \) has charge \( q_\gamma = 0 \), as noted in the Introduction, which does not only yield \( \gamma \in \mathcal{K} \), but even \( \gamma \in D_T \). Hence, the unitary \( W(T\gamma) \) is well defined and intertwines the representations \( \pi_T \circ \gamma_1 \) and \( \pi_T \circ \gamma_2 \). Conversely, assume \( \pi_0 \circ \gamma_1 \ncong \pi_0 \circ \gamma_2 \), i.e., \( q_{\gamma_1} \neq q_{\gamma_2} \). For any rotation invariant test function \( h \in \mathcal{D}_R(\mathbb{R}^3 \setminus \{0\}) \), one has \( \omega^{-\frac{1}{2}} h := f \in \mathcal{L}(\mathcal{O}) \) for some open neighbourhood \( \mathcal{O} \subset \mathbb{R}^{1+3} \) of \( 0 \). Since \( T f_\lambda = f_\lambda \), Lemma 2.2 implies

\[
\pi_T \circ \gamma_j(W(f_\lambda)) = \gamma_j(W(f_\lambda)) \xrightarrow{\lambda \rightarrow \infty} e^{i q_j \kappa_j} e^{-\frac{1}{2} \| f \|^2} \mathbf{1}.
\]

As it is always possible to obtain \( e^{i q_{\gamma_1} \kappa_j} \neq e^{i q_{\gamma_2} \kappa_j} \) by a rescaling of \( h \), it follows that \( \pi_T \circ \gamma_1 \ncong \pi_T \circ \gamma_2 \). 

## 4 Better localisation of the sectors in front of KPR-like infravacua

The main aim of this section is to prove the following result which establishes some (non-Lorentz invariant) version of BF localisation. In the sequel, we
will denote by $C = \{t\} \times C''$ an “upright” spacelike cone whose basis is the open convex cone $C \subset \mathbb{R}^3$ at time $t$. Note that the set of upright spacelike cones is translation invariant and that an arbitrary spacelike cone can be obtained from an upright one by a Lorentz transformation.

**Proposition 4.1** Let $\pi_T$ be a KPR-like infravacuum representation, and let $\gamma \in \mathcal{L}_\Gamma$. Then one has for any upright spacelike cone $\mathcal{C}$:

$$\pi_T \circ \gamma|_{\mathfrak{a}(C')} \cong \pi_T|_{\mathfrak{a}(C')}.$$

To prove this assertion, we will first deal with a special case in which the relevant computations can be carried out quite explicitly. Eventually, the formal proof will consist in reducing the general case to the special one.

The case discussed first amounts to the following two assumptions:

- $\mathcal{C} = \{0\} \times C''$ and the apex of $C$ is the origin $0 \in \mathbb{R}^3$;
- $\gamma \in \mathcal{L}_\Gamma$ has the special form $\gamma = i\omega - \frac{3}{2}\hat{\rho}$, where $\rho \in \mathcal{D}_\mathbb{R}(\mathbb{R}^3)$ satisfies $\rho = -\Delta \Phi$ with a rotation invariant function $\Phi \in C_\mathbb{R}^\infty(\mathbb{R}^3)$ obeying, for some $0 < r_1 < r_2 < \infty$,

$$\Phi(\vec{x}) = \begin{cases} 
0 & \text{if } |\vec{x}| < r_1, \\
\frac{q_0}{\vec{x} |\vec{x}|} & \text{if } |\vec{x}| > r_2.
\end{cases}$$

To proceed, we note that the cone $C \subset \mathbb{R}^3$ determines, by projection onto the unit sphere $S^2$, a subset of $S^2$ which we denote by $\mathcal{C}$, too. Now we choose a function $\chi^C \in C_\mathbb{R}^\infty(S^2)$ with the properties

(i) $\chi^C|_{S^2\setminus C} = 1$ and (ii) $\langle Y_{00}, \chi^C \rangle = 0$

and denote by $\Phi^C \in C_\mathbb{R}^\infty(\mathbb{R}^3)$ the product

$$\Phi^C(\vec{x}) := (\Phi \cdot \chi^C)(\vec{x}) := \Phi(\vec{x}) \chi^C(\frac{\vec{x}}{|\vec{x}|}).$$

This function will now be used to construct a unitary intertwiner from $\pi_T$ to $\pi_T \circ \gamma$ on the $C^*$-algebra $\mathcal{W}(\mathcal{L}(C'))$.

For this purpose, we calculate (using spherical coordinates)

$$-\Delta \Phi^C = \rho \cdot \chi^C + \frac{\Phi}{r^2} \cdot \vec{L}^2 \chi^C.$$
This function is square-integrable, hence its Fourier transform \( u^C := -\widehat{\Delta \Phi^C} \) lies in \( K \), and
\[
v_n^C := i\omega^{-\frac{3}{2}}P_{\epsilon_n}u^C, \quad n \in \mathbb{N}
\]
is a well-defined sequence in \( D_0 \) which approximates the linear form \( l_\gamma \) on \( C' \) in the following sense:

**Lemma 4.2** For any \( f \in \mathcal{L}(C') \), one has \( l_\gamma(f) = -\lim_{n \to \infty} \text{Im} \langle v_n^C, f \rangle \).

Proof: Write \( f = \omega^{-\frac{1}{2}}h + i\omega^{\frac{3}{2}}g \) with \( h, g \in D_{\mathbb{R}}(\mathbb{R}^3 \setminus C) \) and consider
\[
-\text{Im} \langle v_n^C, f \rangle = -\text{Im} \langle i\omega^{-\frac{3}{2}}P_{\epsilon_n}u^C, \omega^{-\frac{1}{2}}h + i\omega^{\frac{3}{2}}g \rangle = \int_{|k| > \epsilon_n} d^3k \omega^{-2}u^C(k) \hat{h}(\vec{k}).
\]
Since \( u^C \in L_\text{loc}^\infty(\mathbb{R}^3) \) (cf. Lemma 4.3), whence \( \hat{\Phi^C} = \omega^{-2}u^C \in L_\text{loc}^1(\mathbb{R}^3) \), it follows that this sequence converges for \( n \to \infty \) to
\[
\int_{\mathbb{R}^3} d^3k \Phi^C(k) \hat{h}(\vec{k}) = \Phi^C(h) = \Phi(h) = \int_{\mathbb{R}^3} d^3k \Phi(h) \hat{h}(\vec{k}).
\]
In the previous line, we have viewed \( \Phi^C \) and \( \Phi \) as distributions and made use of the fact that they coincide on \( \text{supp} h \). The proof is now completed by a straightforward computation showing that the last expression equals \( l_\gamma(f) \).

Whereas property (i) of \( \chi^C \) was essential for the previous Lemma, the following one will show how property (ii) determines the behaviour of \( u^C \) in a neighbourhood of \( \vec{k} = 0 \).

**Lemma 4.3** There exists a smooth function \( \eta \in C^\infty(S^2) \) with \( \langle Y_{00}, \eta \rangle = 0 \) and an analytic function \( R : \mathbb{R}^3 \to \mathbb{C} \) with \( R(0) = 0 \) such that
\[
u^C(\vec{k}) = \eta(\frac{\vec{k}}{|\vec{k}|}) + R(\vec{k}) \quad \text{for} \ \vec{k} \neq 0.
\]

Proof: Let \( S_{00} \) denote the set of all rotation invariant test functions. Since \( \langle Y_{00}, \widehat{L^2 \chi^C} \rangle = 0 \), there exists a unique distribution \( F_1 \) on \( \mathbb{R}^3 \) which is homogeneous of degree \(-3\) and which coincides on \( \mathbb{R}^3 \setminus \{0\} \) with \( \frac{\Phi^C}{4\pi \omega} \). By Thms. 7.1.16 and 18 of [13] it follows that its Fourier transform \( \widehat{F_1} \) is homogeneous of degree 0 and restricts on \( \mathbb{R}^3 \setminus \{0\} \) to a smooth function, i.e., \( \widehat{F_1}(\vec{k}) = \eta(\frac{\vec{k}}{|\vec{k}|}) \) for \( \vec{k} \neq 0 \) with some \( \eta \in C^\infty(S^2) \). Moreover, since \( F_1|_{S_{00}} = 0 \) and \( S_{00} \) is stable under Fourier transformations, it follows that \( \langle Y_{00}, \eta \rangle = 0 \). Now consider the distribution \( F_2 := -\Delta \Phi^C - F_1 \). For \( \vec{r} \neq 0 \), it is given by
\( F_2 = \rho \cdot \chi^C + (\Phi - \frac{\omega_n}{\Delta_\pi}) \frac{1}{r^2} \cdot \hat{L}^2 \chi^C \) and thus has compact support. Hence, its Fourier transform is an analytic function \( R: \hat{F}_2(\vec{k}) = R(\vec{k}), \vec{k} \in \mathbb{R}^3 \). As \( \chi^C \) was assumed to fulfill \( \langle Y_{00}, \chi^C \rangle = 0 \), it follows in particular that \( R(0) = \int d^3x (\rho \cdot \chi^C)(x) = 0 \). To sum up, we have \(-\Delta \Phi^C = \hat{F}_1 + \hat{F}_2\) (in the sense of distributions) and thus \(-\Delta \Phi^C = \hat{F}_1 + \hat{F}_2\). Since all three terms of this last equation are smooth on \( \mathbb{R}^3 \setminus \{0\} \), this implies the identity
\[
\hat{u}(\vec{k}) = -\hat{\Delta} \Phi^C(\vec{k}) = \eta(\vec{k}/|\vec{k}|) + R(\vec{k}) \text{ for all } \vec{k} \neq 0. \]

The knowledge of \( u^C \) at \( k = 0 \) now permits us to establish the connection with the KPR-like infravacuum representations described in Section 3.

**Lemma 4.4** Let \( T \) be a KPR-like symplectic operator. Then:

1. The limit \( v^C_T := \lim_{n \to \infty} T v^C_n \) exists in \( \mathcal{K} \).

2. The unitary \( W(v^C_T) \) satisfies
   \[
   \text{Ad} W(v^C_T) \circ \pi_T = \pi_T \circ \gamma \quad \text{on } \mathfrak{A}(\mathcal{C}').
   \]

Proof: Since \( \Delta \Phi^C \) is real-valued, one has \( \Gamma u^C = u^C \), hence \( \Gamma v^C_n = -v^C_n \).

Thus, \( T v^C_n = T_1 v^C_n = i T_1 \omega^{-\frac{9}{4}} P_n u^C = i T_1 \omega^{-\frac{9}{4}} P_n (u^C_2 + u^C_3) \) with \( u^C_1, u^C_2, u^C_3 \in \mathcal{K} \)

defined by \( u^C_1 := (1 - P_0)(1 \cdot \eta) \) and \( u^C_2 := u^C - u^C_1 = P_0(1 \cdot \eta) + R \), where \( \eta \) and \( R \) are as in the previous Lemma. In particular, \( R(0) = 0 \) yields \( u^C_2 \in D_{\omega^{-3/2}} \) which implies \( T_1 \omega^{-\frac{9}{4}} P_n u^C_2 = T_1 P_n \omega^{-\frac{9}{4}} u^C_2 \xrightarrow{n \to \infty} T_1 \omega^{-\frac{9}{4}} u^C_2 \) by the boundedness of \( T_1 \). On the other hand, it follows from \( \langle Y_{00}, \eta \rangle = 0 \)

by Lemma 3.5 that the sequence \( (T_1 \omega^{-\frac{9}{4}} P_n u^C_1)_{n \in \mathbb{N}} \) is convergent, which completes the proof of Part 1. Part 2 is a straightforward computation:

Let \( f \in \mathcal{L}(\mathcal{C}') \); then, by Lemma 4.2, \( \text{Im}(v^C_T, T f) = \lim_{n \to \infty} \text{Im}(T v^C_n, T f) = \lim_{n \to \infty} \text{Im}(v^C_n, f) = -l(f) \), which implies

\[
\text{Ad} W(v^C_T)(\pi_T(W(f))) = W(v^C_T)W(T f)W(v^C_T)^* = e^{-i\text{Im}(v^C_T, T f)}W(T f) = e^{il(f)}W(T f) = \pi_T \circ \gamma(W(f)).
\]

This establishes the stated equivalence on \( W(\mathcal{L}(\mathcal{C}')) \) and, hence, by local normality (of both \( \pi_T \) and \( \gamma \)) also on \( \mathfrak{A}(\mathcal{C}'). \]

With these preparations, we are ready for the Proof of Prop. 4.1: By standard arguments using transportability of the charges, \( \gamma_x \cong \gamma \), and translation covariance of the representation \( \pi_T \), it can always be assumed that the apex of \( \mathcal{C} \) is \( 0 \in \mathbb{R}^{1+3} \). To remove the assumption on the special form of \( \gamma \) as well, we note that, for any \( \gamma \in \mathcal{L}_\Gamma \), there exists some equivalent \( \gamma_0 \in \mathcal{L}_\Gamma \) with the special form considered.
Such a $\gamma_0$ automatically satisfies $\gamma_0 - \gamma \in D_T$ and thus provides a unitary $W(T(\gamma_0 - \gamma))$ performing the equivalence $\pi_T \circ \gamma \cong \pi_T \circ \gamma_0$ on all of $\mathcal{A}$. Taking into account all the above, this proves Prop. 4.1.

5 Conclusions

The present work has shown in a concrete example that choosing a background different from the vacuum can improve the localisability properties of superselection sectors in theories with massless particles. Typically, such backgrounds correspond to clouds of infrared radiation which exist in great variety in any such theory, but it appears that suitable background states have to be chosen carefully in order to match with the sectors under consideration. To illustrate this point, we recall that this led us, in particular, to choose complex conjugation in position space as the involution $\Gamma$. If we had, on the other hand, chosen complex conjugation $\hat{\Gamma}$ in momentum space instead of $\Gamma$, our sectors would only have been localisable in upright spacelike double cones, i.e., in regions of the form $\mathcal{C} \cup (a - \mathcal{C})$, where $\mathcal{C}$ is an upright spacelike cone and $a \in \mathbb{R}^{1+3}$, but not in spacelike cones any more. (The former statement can be verified with the method of Section 4, the latter can be reduced to an application of Lemma 2.2.)

Finally, we remark that we were unable, with our methods, to establish the full BF localisation criterion for the sectors $[\gamma]$, i.e., $\pi_T \circ \gamma|_{\mathcal{A}(\mathcal{C}')} \cong \pi_T|_{\mathcal{A}(\mathcal{C}')}^\prime$ also for spacelike cones $\mathcal{C}$ which do not contain an upright one. Apart from the obvious fact that the KPR-like states break Lorentz covariance explicitly, there seem to be other indications that such a result might indeed not be true. However, we do not pursue this point further, since the localisation properties obtained here should be sufficient for carrying through a DHR-like analysis (along the lines of [2]) in front of the infravacuum background. If this is indeed possible, it has to be studied in a subsequent step under which conditions (to be imposed on the infravacuum) the superselection structure thus obtained will be independent from the particular background.

Acknowledgements: I would like to thank Prof. D. Buchholz for numerous helpful discussions and stimulating interest in this work. Partial financial support from the Deutsche Forschungsgemeinschaft ("Graduiertenkolleg Theoretische Elementarteilchenphysik" at Hamburg University) is also gratefully acknowledged.
References

[1] S. Doplicher, R. Haag, J. E. Roberts: *Local Observables and Particle Statistics I*, Commun. Math. Phys. **23** (1971) 199–230 and *II*, Commun. Math. Phys. **35** (1974) 49–85

[2] D. Buchholz, K. Fredenhagen: *Locality and the Structure of Particle States*, Commun. Math. Phys. **84** (1982) 1–54

[3] D. Buchholz: *The Physical State Space of Quantum Electrodynamics*, Commun. Math. Phys. **85** (1982) 49–71

[4] K. Kraus, L. Polley, G. Reents: *Models for Infrared Dynamics: I. Classical Currents*, Ann. Inst. H. Poincaré **A 26** (1977) 109–162

[5] D. Buchholz, S. Doplicher, G. Morchio, J. E. Roberts, F. Strocchi: *A Model for Charges of Electromagnetic Type*, Operator Algebras and Quantum Field Theory (Roma 1996 Conference proceedings), eds.: S. Doplicher et.al. (International Press, Cambridge, 1997), hep-th/9705089

[6] W. Kunhardt: *On infravacua and superselection theory*, J. Math. Phys. **39** (1998) 6353–6363, hep-th/9704212

[7] K. Fredenhagen, J. Hertel: unpublished manuscript (1979)

[8] D. Buchholz, M. Porrmann: *How Small is the Phase Space in Quantum Field Theory*, Ann. Inst. H. Poincaré **52** (1990) 237–257

[9] F. Hars: *Infravakua in der Quantenelektrodynamik*, Diploma thesis, Hamburg University (1993)

[10] D. Buchholz, S. Doplicher: *Exotic Infrared Representations of Interacting Systems*, Ann. Inst. H. Poincaré **40** (1984) 175–184

[11] R. Wanzenberg: *Energie und Präparierbarkeit von Zuständen in der lokalen Quantenfeldtheorie*, Diploma thesis, Hamburg University (1987)

[12] L. Hörmander: *The Analysis of Linear Partial Differential Operators I*, (Spriger Verlag, Berlin, Heidelberg, 1990)