Fluctuation Theorem for a Small Engine and Magnetization Switching by Spin Torque

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We consider a reversal of the magnetic moment of a nano-magnet by the fluctuating spin-torque induced by a non-equilibrium current of electron spins. This is an example of the problem of the escape of a particle from a metastable state subjected to a fluctuating non-conservative force. The spin-torque is the non-conservative force and its fluctuations are beyond the description of the fluctuation-dissipation theorem. We estimate the joint probability distribution of work done by the spin torque and the Joule heat generated by the current, which satisfies the fluctuation theorem for a small engine. We predict a threshold voltage above which the spin-torque shot noise induces probabilistic switching events and below which such events are blocked. We adopt the theory of the full-counting statistics under the adiabatic pumping of spin angular momentum. This enables us to account for the backaction effect, which is crucial to maintain consistency with the fluctuation theorem.

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The thermodynamics of small systems, the stochastic thermodynamics [1], is of growing importance in nanoscience. The key ingredient is the fluctuation theorem (FT) [1–3], which has been applied to the solid state physics recently and extends the fluctuation-dissipation theorem as well as the Onsager relations far from equilibrium (see e.g. Refs. 2–4). Recent studies suggest that the FT is also useful to analyze small engines [10–12]. In a small engine, during a short time step $\Delta t$ at finite temperature $T$, the input heat $q$ and the output work $w$ fluctuate and can take positive and negative values [Fig. 1 (a)]. The FT ensures that the joint probability distribution satisfies

$$P_{R,\Delta t}(-q,-w) = P_{\Delta t}(q,w)e^{-\beta(q+w)}, \quad \beta = (k_BT)^{-1},$$

where the subscript $R$ indicates that the external driving is reversed. From Jensen’s inequality, this equation reproduces the Carnot theorem, $(w)/(q) \leq 1$. The FT is applicable even when a cycle is not defined. The work can be attributed to a non-conservative force originating from a heat flow between two baths [Fig. 1 (a)]. Let us couple the small engine to a small system. The energy variation of the small system is equal to the fluctuating work:

$$\Delta E = w.$$  \hspace{1cm} (2)

We expect that Eqs. (1) and (2) are applicable to a wide spectrum of mesoscopic systems driven by non-conservative forces.

In the present paper, we apply this idea to the problem of the escape of a particle from a metastable state [13] subjected to a fluctuating non-conservative force. We consider the following specific setup: a nano-magnet connected to a left ferromagnetic lead (source) and a right normal metal lead (drain) [Fig. 1 (b)]. The magnetization vector of the bulk left ferromagnetic lead $M_L$ is fixed. Let us assume that the magnetization of the nano-magnet $M$ is anti-parallel to $M_L$. By applying a source-drain bias voltage $V$, spin polarized electrons are injected from the ferromagnetic lead, which exert a torque on the nano-magnet [14]. When the magnetic moment $MV$ ($V$ is the volume of the nano-magnet) is small, above a critical voltage $V^*$, $M$ is reversed and aligns parallel to $M_L$. The spin-torque is generated by the non-equilibrium current and thus the non-conservative force. It performs the work $w$ on the small system (the nano-magnet) and is accompanied by the Joule heat $q$. Since the spin angular momentum exchanged between electrons and the nano-magnet is discretized by $\hbar$, the spin-torque fluctuates and even under the critical voltage $V^*$, it can switch the magnetic moment probabilistically. The exponent $\Delta$ of the switching probability

$$P_\tau \sim e^{-\Delta},$$ \hspace{1cm} (3)

is well studied for equilibrium thermal fluctuations, which are Gaussian-distributed (see e.g. Refs. 15–18 and references therein). However, this is not the case for the non-equilibrium fluctuations. In current experiments [19], an MgO-insulating tunnel barrier is sandwiched between the nano-magnet and the ferromagnetic lead, which generates a Poisson-distributed shot-noise out of equilibrium [20]. Previous studies analyzing the non-equilibrium spin-torque shot noise [21–23] limited themselves to the Gaussian fluctuations. The non-Gaussian fluctuations are beyond the description of the fluctuation-dissipation theorem and, to our knowledge, have not been reliably described.

In the present paper, we determine the distribution
of non-Gaussian fluctuations by using the full-counting statistics [24] under the adiabatic pumping [25, 26], which gives the joint probability distribution consistent with the FT for a small engine. We evaluate the switching exponent $\Delta$ and predict another threshold voltage $V_{th}$, under which the probabilistic switching is completely blocked. This is a result of the backaction, i.e., the adiabatic pumping of the spin angular momentum [24], as a consequence of the FT.

Langevin equation in the energy coordinate – We take the $z$-axis parallel to the direction of the left magnetization, $e_z = (0, 0, 1) = M_L/|M_L|$, which is fixed [Fig. 1(b)]. We assume the uniaxial anisotropy of the nano-magnet in the $z$-direction. The anisotropic energy is,

$$E = -\frac{MH_K V(e_z \cdot m)^2}{2} = -\frac{MH_K V \cos^2 \theta}{2}, \quad (4)$$

where $M = |M|$ is the saturation magnetization and $m = M/M$. In the spherical coordinates, it is expressed as $m = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$. The anisotropic magnetic field is typically $H_K > 0$, and thus the magnetic moment tends to align with $m = e_z$ or $m = -e_z$. These 2 states are separated by the energy barrier $MH_K/2$. Because of this bistability, the setup is applicable to a memory device [19].

The dynamics of the nano-magnet is described by the stochastic Landau-Lifshitz-Gilbert equation,

$$\mathbf{m} = -\gamma (H_{\text{eff}} + \mathbf{h}) + \alpha \mathbf{m} \times \dot{\mathbf{m}} = -\gamma I_S/(M\mathbf{V}), \quad (5)$$

where $\gamma = 2\mu_B/h$ is the gyromagnetic ratio and $\mu_B$ is the Bohr magneton. The effective magnetic field is $H_{\text{eff}} = -V^{-1}\partial E/\partial M = H_K \cos \theta e_z$, and $\mathbf{h}$ is its fluctuation induced by thermally excited magnons. It is a Gaussian white noise, i.e., $\langle h_j(t) \rangle = 0$ ($j = x, y, z$), and the correlation is instantaneous and isotropic:

$$\langle h_j(t) h_k(t') \rangle = 2\alpha k_B T \delta_{jk} \delta(t - t')/(\gamma MV).$$

The Gilbert damping constant $\alpha$ also appears in the second term of the lhs of Eq. (5), indicating the relaxation to $m = e_z$ or $m = -e_z$. The spin-torque $I_S = \mathbf{m} \times (e_z \times \mathbf{m})$ aligns $\mathbf{M}$ parallel to $M_L$ [24].

Since typically the damping and the spin torque are weak, the variation of the energy after a single precession is small [15, 17, 18]. Therefore, the magnetic moment precesses along the $z$ axis with the frequency $\omega = \gamma H_K \cos \theta \equiv \Omega$ along a constant energy trajectory given by Eq. (4). In the following, we will concentrate on the negative branch, $\Omega = -\sqrt{-2\gamma^2 H_K E/(MV)}$, i.e., $-1 \leq m_z \leq 0$. It is convenient to consider the time derivative of the energy [14] averaged over a single precession:

$$\overline{E}(t) = \Omega \int_{t}^{t+\Delta t} dt' \langle E(t') \rangle/(2\pi).$$

In the first order in $\alpha$ and $\mathcal{Z}$, we obtain Eq. (2) for our system:

$$\overline{E} = \mathbf{M} \cdot \partial E/\partial \mathbf{M} = p_{\alpha} - p_{\alpha}, \quad (6)$$

where $p_{\alpha} = \gamma MV(\mathbf{m} \times H_{\text{eff}}) \cdot (\alpha \mathbf{m} \times H_{\text{eff}} + \mathbf{h})$ is the sum of the power dissipated by the Gilbert damping and that generated by the thermally fluctuating magnetic field. The average is $\langle p_{\alpha} \rangle = G_{\alpha}(h\Omega)^2 \sin^2 \theta$, where $G_{\alpha} = \sigma_\alpha MV/(h\mu_B)$. The variance is proportional to the temperature times the average $\langle \delta p_{\alpha}(t) \delta p_{\alpha}(t') \rangle = 2k_B T \langle \overline{E} \rangle \delta(t - t')/(\gamma MV) = 2k_B T \langle \overline{E} \rangle$, which is a consequence of the fluctuation-dissipation theorem [28].

The power gain by the spin-torque is

$$p_{\alpha} = 2\mu_B I_S \cdot H_{\text{eff}}/h = \Omega T_{\alpha}.$$  \quad (7)

For the uniaxial anisotropy case, only the $z$ component of the spin torque is necessary $T_{\alpha} = I \sin^2 \theta$. Our main task is to determine the probability distribution of the fluctuating $T_{\alpha}/2$ to be consistent with the FT for a small engine [1].

**Fluctuation theorem for non-conservative force** – During a time interval $\Delta t$, which is short but sufficiently longer than the period of the precession $2\pi/\Omega$, $n$ electrons are transmitted through the nano-magnet from left to right leads and the $s$ electron spins flip from $\uparrow$ to $\downarrow$. They are given by $n = \int_{t}^{t+\Delta t} dt' \langle T_{\alpha}(t') \rangle/e$, where $I$ is the charge current, and $s = \int_{t}^{t+\Delta t} dt' \langle T_{\alpha}(t') \rangle/h$. When the energy change is slow enough, we can calculate the joint probability distribution $P_{\Delta t}(n, s)$ using the full-counting statistics under the adiabatic pumping with the pumping frequency $\Omega$ [25, 26]. The scaled cumulant generating function (SCGF) $F_G$ is introduced as

$$\sum_{n, s} P_{\Delta t}(n, s; \Omega) e^{\lambda n + i s} \approx e^{\Delta F_G(\lambda, \chi; \Omega)}, \quad (8)$$

where $\lambda$ and $\chi$ are counting fields for the numbers of transmitted electrons and flipped spins. Electrons in the left ferromagnetic lead and those in the right metal lead obey the Fermi distribution: $f_r(E) = 1/[e^{\beta(E - \mu_r)} + 1]$ ($r = L, R$). In equilibrium, the chemical potentials are at the Fermi level $\mu_L = \mu_R = E_F$. The source drain bias voltage $V$ shifts the chemical potential of the lead last as $\mu_L = E_F + eV$.

For now, to keep the discussion simple and specific, we keep the general form of the SCGF under the adia-
batic pumping later, Eq. (20), and assume that the nano-magnet is ferromagnetic-insulating, although the current experiments use an insulator/metalllic ferromagnet nano-structure \[19\]. The SCGF acquires the bi-directional Poisson form \[20\]:

\[
\mathcal{F}_G(\lambda, \chi; \Omega) = \sum_{\nu, \nu'} \Gamma_{\nu \nu'}(\Omega) \left( e^{t \nu \lambda + i \nu' \chi - 1} + \sum_{\pm} \Gamma_{\pm}(e^{i \nu \lambda} - 1) \right).
\]  

(9)

The first line corresponds to the spin-flip tunneling process. The tunneling rate is

\[
\Gamma_{\nu \nu'}(\Omega) = \sin^2 \theta G_{\nu \nu'} \frac{\nu e V - \nu' e \hbar \Omega}{1 - e^{-\beta(\nu e V - \nu' e \hbar \Omega)}},
\]  

(10)

where \(G_{++} = G_{--} = G_p\) and \(G_{+-} = G_{-+} = G_a\) are spin-flip tunnel conductances. Their dimension is \(\hbar^{-1}\) and \(G_{+-}\) connects \(L \uparrow / L \downarrow\) and \(R \downarrow / R \uparrow\) states. The second line of Eq. (9) corresponds to the spin-preserving tunneling process.

\[
\Gamma_{\nu} = [G_p \cos^2(\theta/2) + G_{AP} \sin^2(\theta/2) - \sin^2 \theta (G_{+} + G_{-})](\nu e V)/(1 - e^{-\beta e V}).
\]  

(11)

Similar to the free energy \[30\], from the derivative of the SCGF, we can calculate the charge/spin current. For example, we obtain the spin-valve expression \[31\],

\[
\frac{\tau}{e} = \frac{\partial}{\partial \lambda} \left|_{\lambda=0} \right. \mathcal{F}_G(\lambda, 0; \Omega) = \left[ G_p \cos^2 \frac{\theta}{2} + G_{AP} \sin^2 \frac{\theta}{2} \right] e V,
\]

where \(G_p\) and \(G_{AP}\) are conductances in parallel and antiparallel alignments.

The SCGF is symmetric under the time reversal in the backward driving \(\Omega \rightarrow -\Omega\). It leads the spintronic FT \[8\]:

\[
\mathcal{F}_G(\lambda, \chi; \Omega) = \mathcal{F}_{G,R}(\lambda - \chi, \Omega) = \mathcal{F}_{G,R}(\lambda + \chi, -\Omega),
\]  

(12)

where the subscript \(R\) means that the magnetizations are also reversed, \(M \rightarrow -M\) and \(M_L \rightarrow -M_L\) (which results in \(G_{+} \leftrightarrow G_{-}\)). After the inverse Fourier transform and identifying the work as \(w = sh\Omega\) [see Eq. (7)] and the Joule heat as \(q = neV\), we obtain the FT for a small engine \[11\]. Our SCGF \[10\] together with the Langevin equation in the energy coordinate \[6\] enables us to calculate the switching exponent consistent with the FT.

**Magnetization switching** – The average value of the power \[7\] is given by

\[
\langle \rho_S(\Omega) \rangle = \hbar \Omega \frac{\partial}{\partial \chi} \mathcal{F}_G(0, \chi; \Omega) \bigg|_{\chi=0} = \Omega \mathcal{I}_{Sz}^{\Omega=0} - p_{\text{pump}}.
\]  

(13)

The first term is the power gain by the spin torque: \(p_{Sz}^{\Omega=0} = \hbar \sin^2 \theta (G_{+} - G_{-}) e V\). The second term is the power dissipation by the adiabatic pumping of spin angular momentum \[27\]:

\[
p_{\text{pump}} = \sin^2 \theta (h \Omega)^2 (G_{+} + G_{-}),
\]

which accounts for the backaction effect. We assume that initially the magnetizations are in antiparallel alignment, \(m_z = \cos \theta = -1\). Then for \(G_+ < G_-\), which means that the spin-flip process \(L \downarrow \rightarrow \uparrow \) is the majority process, at positive \(e V\), there exists a frequency \(\Omega^*\) at which the power gain and the power dissipation balance: \(\langle \rho_S(\Omega^*) \rangle = \langle \rho_S(\Omega) \rangle\). The condition leads, \(h \Omega^* = (G_{+} - G_{-}) e V/(G_{+} + G_{-} + G_{a})\). When the magnitude of the precession frequency at \(m_z = -1\), \(-\Omega = \gamma H_K\) becomes smaller than \(-\Omega^*\), \(m_z\) starts to increase to \(m_z = 0\) and eventually reaches \(m_z = 1\). The critical voltage \(e V^*\) above which the magnetization is reversed even in the absence of thermal fluctuations and spin-torque shot noise is

\[
e V^* = \frac{2 \mu_B H_K}{G_+ + G_{+} + G_{a}}.
\]  

(14)

Since the spin-torque shot noise is intrinsic and remains even at zero temperature, the nano-magnet switches probabilistically under \(e V^*\). A convenient way to calculate such switching probability is the path-integral approach of the Langevin equation \[6\] \[22\]. The switching probability \(P_\tau\) is the conditional probability to find \(m_z = -1\) at \(t = 0\) and \(m_z = 0\) at \(t = \tau\). It is given by

\[
P_\tau = \int E(\xi) = 0 \frac{DE}{e \xi^S} e^{i S},
\]

\[
i S = -\int_0^\tau dt \left[ i \langle \xi(t) \rangle E(t) - \mathcal{F}_G(0, \xi(t); \Omega(t)) - \mathcal{F}_\alpha(-\xi(t)) \right],
\]

where we added the SCGF of Gaussian thermal noise,

\[
\mathcal{F}_\alpha(\xi) = G_{a} \sin^2 \theta (h \Omega)^2 i \xi (1 + i \xi / \beta).
\]

Since the number of magnetic moments in the nanomagnet \(M V / \mu_B\) is typically large, we utilize the optimal-path approximation. The resulting switching probability acquires the form of Eq. (3) with the switching exponent:

\[
\Delta = -i S^* = -\frac{MV}{2 \mu_B} \int_{-\gamma H_K}^{\Omega^*} d\Omega \frac{i \xi^*}{\gamma H_K}.
\]  

(16)

When the Gilbert damping is absent \(\alpha = 0\), \(i \xi^* = \ln|\langle \Gamma_{+} + \Gamma_{-}\rangle/\langle \Gamma_{+} + \Gamma_{-}\rangle|\). The solid lines in Fig. 2 are the switching exponents as a function of the bias voltage at a finite temperature and at zero temperature. We find that, at zero temperature below \(e V_{b} = 2 \mu_B H_K\), the exponent diverges, which means that the switching is completely blocked. This is because the spin flip process \(\downarrow \rightarrow \uparrow\) is blocked: \(\Gamma_{+} \leftrightarrow \Gamma_{-} = 0\). At finite temperature, this divergence disappears and at \(e V = 0\), we obtain the Arrhenius law: \(\Delta = M H_K V/(2 k_B T)\). The inset shows results at a finite \(\alpha\). We see that the divergence remains.
Close to the critical voltage, we approximate $i\chi^{*} \approx (h\Omega - h\Omega^{*})/(k_{B}T_{\text{eff}})$ and obtain the Arrhenius-like form

$$
\Delta = \frac{M H_{K} V}{2k_{B}T_{\text{eff}}} \left( \frac{V^{*} - V}{V^{*}} \right)^{2},
$$

(17)

which quadratically depends on the distance from the critical voltage. The effective temperature,

$$
T_{\text{eff}} = \sum_{k} G_{\alpha} (eV \pm h\Omega^{*}) \coth \frac{eV^{*} + h\Omega^{*}}{2k_{B}T_{\text{eff}}} + 2k_{B}T G_{\alpha}
$$

is reduced to the real temperature $T_{\text{eff}} \approx T$ for high temperatures, $eV, eV_{\text{th}} \ll k_{B}T$. Then Eq. (17) reproduces the previous result [16, 18]. At zero temperature and $G_{\alpha} = 0$, $T_{\text{eff}} \approx 2G_{+}G_{-} eV/(G_{+} + G_{-})^{2}$, which is proportional to the bias voltage [22], indicating that the spin-torque shot noise is the dominant source of fluctuations around the critical voltage. The dashed lines in Fig. 2 show Eq. (17). They fit well for finite temperature or around the critical voltage.

When the volume becomes very small, i.e., $\nu \sim \mu_{B}/M$, we have to go beyond the optimal path approximation [34, 35]. In such cases, the time scales of the source of Gaussian noise and that of Poisson noise should be treated carefully [35].

**FIG. 2:** The bias-voltage dependence of the switching exponent $\Delta$ for $\alpha = 0$ and $G_{-} = 2G_{+}$. The two solid lines are for different temperatures $k_{B}T/(2\mu_{B}H_{K}) = 0$ and 0.1. The inset shows a plot for finite $G_{\alpha} = 5G_{+}$. The dashed lines indicate an Arrhenius-like law, Eq. (17). The critical voltages are $eV^{*}/(2\mu_{B}H_{K}) = 3$ and 8 for $G_{\alpha} = 0$ and $5G_{+}$.

**Full-counting statistics under the adiabatic pumping** — An electron transferred through the nano-magnet is affected by its precession motion. This scattering process is described by a time-dependent $S$-matrix:

$$
S(\theta, \phi(t)) = e^{-i\phi(t)\sigma_{z}/2}S(\theta)e^{i\phi(t)\sigma_{z}/2}
$$

(18)

$$
S(\theta) = \begin{pmatrix}
\mathbf{r}(\theta) & \mathbf{t}(\theta) \\
\mathbf{t}^{\prime}(\theta) & \mathbf{r}^{\prime}(\theta)
\end{pmatrix},
$$

(19)

where the Pauli matrix $\sigma_{z}$ acts in the spin space and $\phi(t) \approx \Omega t + \phi(0)$. $\mathbf{r}$ and $\mathbf{t}$ ($\mathbf{r}^{\prime}$ and $\mathbf{t}^{\prime}$) are $2 \times 2$ matrix of the spin-dependent reflection amplitudes and that of the spin-dependent transmission amplitudes for an incoming wave from the left (right) lead. For example, the element $t_{\sigma^{+}\sigma}$ describes an electron transmission from the spin $\sigma$ state in the left lead to the spin $\sigma^{+}$ state in the right lead.

The SCGF [3] is expressed by using the $S$-matrix as [23, 26],

$$
F_{G}(\lambda, \chi; \Omega) = \sum_{\ell} \int \frac{dE}{\hbar} \ln \det \left[ 1 - f(E) \left( 1 - e^{i\lambda + \chi \sigma_{z}/2} S(\theta; E) e^{-i\lambda - \chi \sigma_{z}/2} S^{\ell}(\theta; E) \right) \right],
$$

(20)

where $S^{\ell}(E)$ is the $S$-matrix for the $\ell$-th transverse channel. The counting field matrix $\lambda = \text{diag}(\lambda, \lambda, 0, 0)$ counts the number of electrons flowing out of the left lead. The precession motion effectively splits the $\uparrow$-spin and $\downarrow$-spin chemical potentials of the 2 leads after the gauge transform, $f(E) = \text{diag}(f_{L}(E + \hbar \Omega/2), f_{L}(E - \hbar \Omega/2), f_{R}(E + \hbar \Omega/2), f_{R}(E - \hbar \Omega/2))$. The spin-splitting of the chemical potentials is a result of the backaction, which is crucial to be consistent with the FT [35]. It also blocks the spin-flip tunneling process $\downarrow \rightarrow \uparrow$ under the threshold voltage.

Although we considered a simple model, it is also possible to calculate the $S$-matrix using a realistic model. Then, from Eq. (20), we obtain Eq. (13) expressed with general $I^{\Omega}_{S_{\uparrow}=0}$ and $p_{\text{pump}}$:

$$
I^{\Omega=0}_{S_{\uparrow}=0} = \frac{eV}{4\pi} \sum_{\ell, \sigma = \uparrow, \downarrow} \left( |t_{\downarrow}^{\ell}(\theta, \phi; E_{F})|^{2} + |t_{\uparrow}^{\ell}(\theta, \phi; E_{F})|^{2} - |t_{\uparrow, \sigma}^{\ell}(\theta, \phi; E_{F})|^{2} - |t_{\downarrow, \sigma}^{\ell}(\theta, \phi; E_{F})|^{2} \right),
$$

$$
p_{\text{pump}} = \frac{\hbar \Omega}{4\pi} \sum_{\ell} \text{tr}(\partial_{\theta}S_{\ell}(\theta, \phi; E_{F})\partial_{\theta}S_{\ell}(\theta, \phi; E_{F}^{\dagger})),
$$

in the leading order of $eV$ and $\Omega$. It is straightforward to take the channel mixing scattering into account. Our $p_{\text{pump}}$ reproduces Ref. [27].

**Summary** — We demonstrate the switching probability driven by fluctuating non-conservative spin-torque. The theory of the full-counting statistics under the adiabatic pumping enables us to account for the backaction effect and to obtain a distribution of the fluctuating spin-torque consistent with the fluctuation theorem for a small engine. We find the threshold voltage $eV_{\text{th}} = 2\mu_{B}H_{K}$, above which the spin-torque shot noise causes the probabilistic switching. Under the threshold the spin-flip tunneling process is blocked because of the backaction and thus the probabilistic switching is suppressed.

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SUPPLEMENTAL MATERIAL

Technical details of derivations of a scattering matrix, a scaled cumulant generating function and a switching exponent.

Scattering matrix and the scaled cumulant generating function

We derive the S-matrix of the ferromagnet/ferromagnetic insulator/normal metal structure. We take the z-axis perpendicular to the interface and assume translational invariance in the x and y directions. The Schrödinger equation is

\[
\left( -\frac{\hbar^2}{2m} \nabla^2 + U(z) \right) \psi(x, y, z) = E \psi(x, y, z), \quad U(z) = \begin{cases} \mu_B H_m \sigma_z / 2 & (z < 0) \\ U_0 + \mu_B H_m m \cdot \vec{\sigma} & (0 \leq z < d) \\ 0 & (d \leq z) \end{cases},
\]

(21)

where \( \vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z) \) is the Pauli matrix vector. The thickness of the ferromagnetic insulator is \( d \) and \( U_0 > 0 \) is the potential barrier height. The molecular (exchange) fields in the ferromagnetic lead and in the ferromagnetic insulator are \( H_{mL} \) and \( H_m \), respectively. The wave function is written as \( \psi(x, y, z) \) in the ferromagnetic lead \((z < 0)\), \( \psi(x, y, z) \) in the ferromagnetic insulator \((0 \leq z < d)\), and \( \psi(x, y, z) \) in the normal metal \((d \leq z)\), respectively. The wave function reads

\[
\left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} + U(z) \right) \psi(z) = E_\ell \psi(z), \quad E_\ell = E - \frac{\hbar^2 \pi^2}{2m L^2} (\ell_x^2 + \ell_y^2),
\]

(22)

where we introduced the channel index \( \ell = (\ell_x, \ell_y) \). The wave number of an electron with the energy \( E \) is \( k_\sigma = \sqrt{2m(E_\ell - \sigma \mu_B H_{mL})} / \hbar \) in the ferromagnetic lead \((z < 0)\), \( i k_\sigma = \sqrt{2m(E_\ell - U_0 - \sigma \mu_B H_m/2)} / \hbar \) in the ferromagnetic
insulator \(0 < z < d\) and \(k = \sqrt{2mE_i}/\hbar\) in the normal metal lead \(d < z\). The \(S\)-matrix in the leading order of \(e^{-\kappa_x d}\) is calculated as

\[
S(\theta) = \begin{pmatrix} P & 0 \\ 0 & P' \end{pmatrix} \left( -1 - (iA + \tau^\dagger\tau/2) \right) \tau \left( 1 + (iA + \tau^\dagger\tau/2) \right) \begin{pmatrix} P & 0 \\ 0 & P' \end{pmatrix}.
\]

(23)

where an Hermite matrix \(A^\dagger = A\) is not relevant for our model. Further, we neglect \(H_m\) except when it appears in the exponent of \(e^{-\kappa_x d}\). Then we obtain the following \(2 \times 2\) matrix of the spin-dependent transmission amplitude:

\[
\tau = \frac{1}{2} \begin{pmatrix} (\tau_{\uparrow\uparrow} + \tau_{\downarrow\downarrow} + (\tau_{\uparrow\downarrow} - \tau_{\downarrow\uparrow}) \cos \theta) & (\tau_{\uparrow\downarrow} - \tau_{\downarrow\uparrow}) \sin \theta \\ (\tau_{\uparrow\downarrow} - \tau_{\downarrow\uparrow}) \sin \theta & (\tau_{\uparrow\uparrow} + \tau_{\downarrow\downarrow} + (\tau_{\uparrow\downarrow} - \tau_{\downarrow\uparrow}) \cos \theta) \end{pmatrix}, \quad \tau_{\sigma\sigma'} = 4e^{-\kappa_x d} \sqrt{\kappa_0 k_{\sigma\sigma'} \kappa_0 k_{\sigma\sigma'}}. 
\]

(24)

2 \times 2 sub-matrices \(P, P'\) become diagonal and \((\sigma,\sigma')\) component of \(P^2\) and \(P'^2\) are \((\kappa_0 + ik_\sigma)/(\kappa_0 - ik_\sigma)\) and \(-i(\kappa_0 + ik)/(\kappa_0 - ik)\), where \(k_0 = \sqrt{2m(U_0 - E_i)/\hbar}\).

We insert the \(S\)-matrix (23) into Eq. (20) in the main text:

\[
\mathcal{F}_G(\lambda, \chi; \hbar\Omega) = \rho_\parallel \int dE \int \frac{dE}{\hbar} \ln \det \left[ 1 + f(E) \left( e^{i\lambda + i\chi\sigma}/S(E - E_\parallel, \theta) \right) \right],
\]

(25)

where \(\rho_\parallel = 2\pi mL^2/\hbar^2\) is the DOS of the transverse channel. Since the energy dependence of \(\tau_{\sigma\sigma'}\) is small around the Fermi energy \(E_F\), it is possible to approximate \(\tau_{\sigma\sigma'}(E - E_\parallel) \approx \tau_{\sigma\sigma'}(E_F) \exp(-E_\parallel/(2\delta))\), where \(\delta^{-1} = 2d\partial\kappa_0(E_\ell = E_F)/\partial E_\ell\). After performing the integral and expanding up to the leading order in \(e^{-\kappa_x d}\), we obtain Eq. (9) in the main text. The conductances are

\[
G_+ = \frac{1}{\hbar} \rho_\parallel \delta |\tau_{\uparrow\downarrow}(E_F) - \tau_{\downarrow\uparrow}(E_F)|^2 ,
\]

(26)

\[
G_- = \frac{1}{\hbar} \rho_\parallel \delta |\tau_{\uparrow\uparrow}(E_F) - \tau_{\downarrow\downarrow}(E_F)|^2 ,
\]

(27)

\[
G_P = \frac{1}{\hbar} \rho_\parallel \delta (|\tau_{\uparrow\uparrow}(E_F)|^2 + |\tau_{\downarrow\downarrow}(E_F)|^2) ,
\]

(28)

\[
G_{AP} = \frac{1}{\hbar} \rho_\parallel \delta (|\tau_{\uparrow\downarrow}(E_F)|^2 + |\tau_{\downarrow\uparrow}(E_F)|^2) .
\]

(29)

The reversal of the magnetic moments \(M \rightarrow -M\) and \(M_L \rightarrow -M_L\), which corresponds to \(H_m \rightarrow -H_m\) and \(H_{mL} \rightarrow -H_{mL}\), changes the tunneling amplitude to \(\tau_{\sigma\sigma'} \rightarrow -\tau_{\sigma\sigma'}\) and thus the conductances to \(G_+ \leftrightarrow G_-\).

### Switching exponent

We analyze the Langevin equation (6) in the main text by exploiting the Martin-Siggia-Rose approach (see Section 4 in Ref. [1]). We first discretize time \(\tau\) into \(N = \tau/\Delta t\) steps. For now, we neglect the equilibrium power dissipation \(p_\alpha\). The variation of the energy during a short time step from \(t_j = \Delta t j\) to \(t_{j+1}\) is

\[
E_{j+1} - E_j \approx \hbar \Omega_j s_j, \quad s_j = \int_{t_j}^{t_{j+1}} dt T_{sz}(t),
\]

(30)

where \(E_j = E(t_j)\) and \(\Omega_j = \Omega((E_{j+1} + E_j)/2)\). The stochastic variable \(s_j\) is distributed according to the joint probability distribution Eq. (8) described in the main text. The conditional joint probability to find \(E_j\) at time \(t_j\) and \(E_{j+1}\) at \(t_{j+1}\) accompanied by \(n\) electron transmission is given by

\[
P_{\Delta t}(n_j, E_{j+1}|E_j) = \int d\epsilon_{Sj} \delta(E_{j+1} - E_j - \epsilon_{Sj}) \sum_{s,n} P_{\Delta t}(n, s; \Omega_j) \delta(\epsilon_{Sj} - \hbar \Omega_j s) \delta_{n_j,n}
\]

(31)

\[
= \int_{-\pi}^{\pi} \frac{d\lambda_j}{2\pi} \int \frac{d\epsilon_{Sj}}{2\pi} e^{-i\lambda_j n_j - i\epsilon_{Sj}(E_{j+1} - E_j)} + \mathcal{F}_G(\lambda_j, \hbar \Omega_j; \Omega_j) \Delta t.
\]

(32)

Then the conditional joint probability to find \(E(0)\) at \(t = 0\) and \(E(\tau)\) at \(\tau\) accompanied by \(n\) electron transmission is calculated by accumulating joint probabilities for short time steps as follows:

\[
P_T(n, E(\tau)|E(0)) = \sum_{n_0, \ldots, n_{N-1}} \int dE_1 \cdots dE_{N-1} P_{\Delta t}(n_{N-1}, E_{N}|E_{N-1}) \cdots P_{\Delta t}(n_0, E_1|E_0) \delta_{n, \sum_{j=0}^{N-1} n_j}
\]

(33)

\[
= \int_{-\pi}^{\pi} \frac{d\lambda}{2\pi} \cdots \int \frac{d\lambda_{N-1}}{2\pi} \int dE_1 \cdots dE_{N-1} e^{\sum_{j=0}^{N-1} (-i\epsilon_{Sj}(E_{j+1} - E_j) + \mathcal{F}_G(\lambda_j, \hbar \Omega_j; \Omega_j) \Delta t - i\lambda_n).}
\]
We can prove the detailed FT by Jarzynski [2] based on the FT (12) in the main text along the same line of the proof in Ref. [3].

\[ P_n(n, E(\tau)|E(0))/P_{R,\tau}(-n, E(0)|E(\tau)) = e^{[n\alpha(E(\tau)-E(0))]}, \]  

(34)

In order to account for the equilibrium power dissipation, we can just replace \( F_G \) with \( F_G(\lambda, \xi h\Omega; \Omega) + F_\alpha(-\xi) \). Further, for calculating the switching rate, we can sum over \( n \), \( P_\tau(E(\tau)|E(0)) \equiv \sum_n P_n(n, E(\tau)|E(0)) \). Then in the continuous limit, \( \Delta t \to 0 \), we obtain the path-integral form Eq. (15) in the main text.

Since \( E, G_\alpha \propto V = L^2d \) and \( F_G \propto L^2/d \), for a modestly large nano-magnet, it is possible to perform the optimal path approximation [1, 4–6]. Namely, from the variational principle, we derive the “canonical equation of motion”:

\[ \dot{E} = -\frac{\partial F}{\partial (i\xi)}, \quad i\xi = -\frac{\partial F}{\partial E}. \]  

(35)

The “momenta” \( i\xi \) measures the strength of the fluctuations. \( i\xi = 0 \) corresponds to the noiseless case, which is always an optimal path. The equation of motion possesses the integral of motion, which is the “energy,” \( F \). Since the normalization condition ensures \( F(\xi = 0; \Omega) = 0 \), the optimal paths always satisfy \( F(\xi; \Omega) = 0 \).

We are interested in an optimal path that starts from \( (E, i\xi) = (-MH_KV/2, 0) \) and reaches \( (E, i\xi) = (0, 0) \). For \( \alpha = 0 \), we find 4 simple solutions satisfying \( F(\xi; \Omega) = 0 \):

\[ \hbar\Omega(E) = \pm 2\mu_B H_K, \quad i\hbar\Omega(E)\xi^* = \ln \frac{\Gamma_{++} + \Gamma_{--}}{\Gamma_{+-} + \Gamma_{-+}}, \quad i\xi^* = 0. \]  

(36)

Figure 3 (a) shows the optimal paths. The horizontal axis is \( \Omega = -\sqrt{-2\gamma^2H_K E/(MV)} \) and thus \( E = -MH_KV/2 \) and \( E = 0 \) correspond to \( h\Omega = -2\mu_B H_K \) and \( h\Omega = 0 \), respectively. Arrows indicate the directions of motion determined from Eq. (35). The initial state is at M, i.e., \( (h\Omega, i\xi h\Omega) = (-2\mu_B H_K, 0) \), and the final state is at T, i.e., \( (h\Omega, i\xi h\Omega) = (0, 0) \). The optimal path is \( M \to M' \to U \to T \), where the intermediate state \( U \) is \( (h\Omega, i\xi h\Omega) = (h\Omega^*, 0) \). The action along this path is calculated as

\[ iS^* = -\int_{-MH_KV/2}^{E(\Omega^*)} dE (i\xi^*) = \frac{MV}{2\mu_B} \ln \frac{G_-(eV - 2\mu_B H_K)}{G_+(eV + 2\mu_B H_K)} + \frac{eV}{2\mu_B H_K} \ln \frac{4G_+G_- (eV)^2}{(G_+ + G_-)^2 [(eV)^2 - (2\mu_B H_K)^2]}. \]  

(37)

The integral \( \int d\Omega i\xi^*\hbar\Omega \) gives the area of the shaded region in Fig. 3 (a). This equation leads to Eq. (16) in the main text and the switching probability up to the single instanton contribution, \( P_s(\tau) = 0|E(0) = -MH_KV/2) \approx e^{-\Delta} \).

At zero temperature, the integral \( (37) \) can be performed easily. For \( eV_{th} = 2\mu_B H_K < eV < eV^* \), we obtain

\[ iS^* = \frac{MV}{2\mu_B} \left\{ \ln \frac{G_-(eV - 2\mu_B H_K)}{G_+(eV + 2\mu_B H_K)} + \frac{eV}{2\mu_B H_K} \ln \frac{4G_+G_- (eV)^2}{(G_+ + G_-)^2 [(eV)^2 - (2\mu_B H_K)^2]} \right\}. \]  

(38)

For \( eV < eV_{th} \), it diverges to \( iS^* = -\infty \), which means that the switching is completely blocked. Figure 3 (b) shows the optimal path at \( eV = eV_{th} \). \( M' \) approaches \( (h\Omega, i\xi h\Omega) = (0, -\infty) \) in the limit of zero temperature, and the area of the shaded region diverges. For \( G_\alpha \neq 0 \), the optimal path is modified and we determine it numerically.

With increasing bias voltage, the shaded area decreases and eventually \( M' \) and \( U \) meet at \( M \) [Fig. 3 (c)]. The exponent and the switching probability become \( i\xi^* = 0 \) and \( P_s \approx 1 \). This critical condition is achieved at \( h\Omega^* = -2\mu_B H_K \), which is equivalent to the balance condition \( (p_s(\Omega^*)) = (p_\alpha(\Omega^*)) \). Around the critical point (for \( G_\alpha \neq 0 \)), we can expand \( \xi^* \) around \( \Omega = \Omega^* \) and \( \xi = 0 \), up to the lowest order as

\[ i\xi^*\hbar\Omega \approx \frac{h\Omega - h\Omega^*}{k_B T_{eff}}. \]

By plugging this expression into Eq. (37), we obtain Eq. (17) in the main text.

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FIG. 3: The optimal paths (a) for $eV = 3\mu_B H_K$, (b) for the threshold voltage $eV = eV_{th} = 2\mu_B H_K$ and (c) for the critical voltage $eV = eV^* = 6\mu_B H_K$. The parameters are as follows: $G_+ = 2G_+$, $G_\alpha = 0$ and $k_B T = 0$. 