ON THE SMALE CONJECTURE FOR DIFF(S^4)

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Abstract. Recently Watanabe disproved the Smale Conjecture for S^4, by showing Diff(S^4) \neq SO(5). He showed this by proving that their higher homotopy groups are different. Here we prove this more directly by showing \pi_0 Diff(S^4) \neq 0, otherwise a certain loose-cork could not possibly be a loose-cork.

0. AN EXOTIC Diffeomorphism

Here we prove \pi_0 Diff(S^4) \neq 0, by showing that if this is not true, then the loose-cork defined in [A2] could not possibly be a loose-cork. The group \pi_0 Diff(S^4) = \pi_0 Diff(B^4, S^3) can be calculated from the homotopy exact sequence of the following fibration (where I = [0, 1], \dot{I} = \partial I)

(1) Diff(S^3 \times I, S^3 \times \dot{I}) \to Diff(B^4, S^3) \to Emb(B^4, IntB^4)

as the free part part of \pi_0 Diff(S^3 \times I, S^3 \times \dot{I}) = \pi_0 Diff(B^4, S^3) \oplus \mathbb{Z}_2. Emb(B^4, IntB^4) is path connected, and the \mathbb{Z}_2 summand comes from Dehn twisting S^3 \times I = B^4 - Int(B^4) along S^3 by using \pi_1 SO(4).

A nontrivial element of \pi_0 Diff(B^4, S^3) can be described as follows: Let T \subset S^3 be a tubular neighborhood of the figure-8 knot K lying inside of the 3-ball B^3. Call C = T \times J \approx S^3 \times B^2 \times J \subset B^3 \times J \subset S^3 \times J, where J = [-1, 1]. Define a diffeomorphism f_n : C \to C (where n \in \mathbb{N})

f_n(x, y, t) = \left( xe^{-2\pi in^2 t}, y, t \right)

As t runs from -1 to 1, f_n rotates T forward (and f_n^{-1} backwards) Let \bar{C} = C \cup S^3 \times N(J) \subset S^3 \times J, where N(J) is a small closed neighborhood of the boundary \dot{J} \subset J. After resizing J and extending domain of definition we extend f_n : \bar{C} \to \bar{C}.

\begin{figure}[h]
\centering
\includegraphics[width=0.2\textwidth]{dehn_twist.png}
\caption{Dehn twist}
\end{figure}

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is a fibered knot with fibration $S^3 - T \to S^1$, and the fiber punctured torus $S$. Let $\delta_s : S \to S$ be Dehn twisting diffeomorphism along the boundary parallel curve. That is, if we identify a collar $N$ of $\partial S$ with $N \approx S^1 \times (0, 1]$, then $\delta_s : N \to N$, is given by $\delta_s(x, r) = (xe^{2\pi is}, r)$. Since $\delta_s$ commutes with monodromy of the fibration $S^3 - T \to S^1$, it induces a diffeomorphism $\bar{\delta}_s : S^3 - T \to S^3 - T$, which is $e^{2\pi is}$ rotation along $\partial T$ in $K$ direction. Clearly $\bar{\delta}_s$ is supported near $T$.

Now define $g_n : (S^3 - T) \times J \to (S^3 - T) \times J$ by $g_n(x, t) = (\bar{\delta}_s(x), t)$ where $s = t^2 n^2$. By considering orientations, we can decompose $S^4 = (S^3 - T) \times J \sim -(T \times J)$ and extend $g_n$ to a diffeomorphism $\phi_n : S^3 \times J \to S^3 \times J$ by letting it to be $f_n^{-1}$ on $-(T \times J)$ (accounting the change of orientation in gluing). So, as $t$ runs from $-1$ to $1$, at each $t$ level $\phi_n$ simultaneously Dehn twists the pages of the fibration $S^3 - T \to S^1$ by $2\pi n^2 t^2$ amount to left, while rotating $T$ in the longitudinal direction $-2\pi n^2 t^2$ to the right.

Since $\phi_n$ is supported near $C$, it fixes some small neighborhood of a vertical arc $p \times J$, where $p \in S^3 - T$. Let $N(p) \approx B^3$ be a small ball neighborhood of $p$, then by the identifications $B^4 \approx B^3 \times J \approx (S^3 - N(p)) \times J$. Then by restriction, we get a diffeomorphism $\phi_n^0 : (B^4, S^3) \to (B^4, S^3)$

![Figure 2. $\phi_n^0 : B^3 \times J \to B^3 \times J$](image)

The vertical outer boundary of the wall of Figure 2 corresponds to the vertical outer boundary of $N(p) \times J$, where $\phi_n^0$ is identity. We will call $\phi_n^0$ Dehn twisting diffeomorphism of $B^4$ along $C$. More generally, when $K \subset S^3$ is any fibered knot and $T^2$ is the boundary of its tubular neighborhood, we will also refer the resulting $\phi_n^0 : B^3 \times J \to B^3 \times J$ Dehn twisting of $B^4$ along $T^2$. We will show that $\phi_n^0$ is not isotopic to identity fixing the boundary, but if we allow it to move the boundary it is isotopic to identity by the so called “swallow-follow isotopy”.
Theorem 1. The diffeomorphisms $\phi^0_n : (B^4, S^3) \to (B^4, S^3)$ are not isotopic to each other fixing the boundary, for distinct integers $n > 0$.

Proof. We will prove this theorem by constructing a properly imbedded disk $D \subset B^4$ which can not be isotopic to $\phi^0_n(D)$ rel boundary, otherwise certain associated cork (which we will define in the next section) would not be a cork: Let $K_0$ be the punctured $K$, and let $D = K_0 \times J$ be the obvious disk in $B^4 = B^3 \times J$, which $K\# - K$ bounds.

Figure 3 is the same as Figure 2, drawn by bending the figure. Then take $D_n$ be the image of this disk $D$ under $\phi_n$. $D_n$ is the union of a concordance $H_n$ from $K\# - K$ to $\bar{K}\# - \bar{K}$ induced by $\phi_n$, and the disk $\bar{D}_n = \bar{K}_0 \times I$, where $\bar{K}$ is the other end of the concordance $H_n$ as shown in the second picture of Figure 3. So we have $\phi_n^o : (B^4, D) \to (B^4, \bar{D}_n)$.

We shall see, although $D$ and $\bar{D}_n$ are not isotopic fixing their boundaries on $\partial B^4$, they are isotopic moving the boundaries, through an isotopy which is called ‘swallow-follow’. This isotopy is a composition of two isotopies of $K\# - K$: By first moving $-K$ along $K$ in its tubular neighborhood, then moving $K$ along $-K$ in its tubular neighborhood, than capping the resulting $K\# - K$ by the obvious disk it bounds in $B^4$. Various pictures of these moves are described in Figures 4, 5).

We can also explain swallow-follow isotopy from Figure 3 by first constructing two disjoint 2-toruses $T_1$ and $T_2$ in $\partial(B^3 \times J)$, enclosing $K$ and $-K$ respectively, not intersecting $D$. This can be done by piping boundaries of the tubular neighborhoods $T_1$, $T_2$, of $K$ and $-K$ to large disjoint spheres $S_1$ and $S_2$, enclosing them. That is $T_i = T_i' \# S_i$, with $i = 1, 2$ as shown in Figure 3. There is an isotopy $S_1 \simeq S_2$, indicated by the arrows of Figure 3. This has an affect of replacing the core $K$ in its tubular neighborhood by $K\# - K$, and then replacing the core $-K$ in its tubular neighborhood by $-K\# K$. 

![Figure 3. $\phi_n^o : B^3 \times J \to B^3 \times J$](image-url)
$T_1$ and $T_2$ are the two ends of an imbedded copy of $T \times J \subset B^3 \times J$, which is disjoint from $D$. Here we can take this $T$ to be solid torus. Dehn twisting diffeomorphism of $B^3 \times J$ along $T \times J$ takes $D$ to $D_n$. This induces diffeomorphism between $B^3 \times J$ with $D$ removed (carved) and $B^3 \times J$ with $D_n$ removed. As can be seen from Figure 4 this diffeomorphism amount to gluing two isotopies, each taking place in tubular neighborhoods of $K$ and $-K$ respectively. The effect of this is to tie a small copy of $-K$ to $K$, and move along $K$, then tie a small copy of $K$ to $-K$ and move along $-K$.

![Figure 4. Forming $D_n$ by swallow-follow isotopies](image)

![Figure 5. Swallow-follow isotopies of $K\# - K$](image)
1. **Forming the infinite order cork**

Next, from \( \phi_n \) we form an infinite order cork \((W, \tau_n)\): We do this by removing the slice disk \( D \) from \( B^4 \) then attaching \(-1\) framed 2-handle to the meridian linking circle \( \gamma \) (Figure 6), i.e. \( W^* := B^4 - N(D) \) and

\[
W = W^* + h^2_{\gamma^{-1}}
\]

**Figure 6. W**

The cork twisting map \( \tau_n : \partial W \to \partial W \) is given by \( f_n \) (Figures 8, 9) which is induced from the diffeomorphism \( \phi_n : W^* \to W^*_n \), which keeps \( \gamma \) fixed and twists \( \partial W^* \) by \( f_n \). When viewed \( \phi_n \) as a self map of \( W^* \), \( f_n \) becomes visible as a combination of two Dehn twists in \( \partial W^* \) opposite direction. This appears as a rotation along the connecting torus in the middle of Figure 3 as indicated in Figure 7 (recall that \( \partial W^* \) consists of two copies of \( S^3 - N(K) \) glued along their boundary toruses \([A1]\)). This rotation \( \tau_n \) on the boundary can not extend inside \( W \).

**Figure 7. Rotating \( K# - K \)**

Next we will describe the images of the handles of \( S^3 \times J \) by \( \phi_n \). Figure 8 describes \( \bar{C} \) and \( f_n(\bar{C}) \). Here \( \bar{C} \) is drawn as a round 2-handle attachment to two disjoint copies of \( S^3 \times I \) (0- and 4- handles are not drawn since hey are attached uniquely). Handlebody of \( f_n(\bar{C}) \) is constructed similarly as a round 2-handle attachment, except in this case while attaching the round 2-handle, we rotate \( T \times J \) in \( J \) direction. That is, rotate \( n \) and \(-n\)-times near each of its boundary components. Recall that, \( W_n \) is carved from \( S^3 \times J \) by using \( f_n(C) \).
Figure 8 describes the complement $S^3 \times J - C \approx (S^3 - T) \times J$, and its image $g_n(S^3 \times J - C)$. The left picture of Figure 8 is the complement of the left picture of Figure 8 in $S^3 \times J$. The right picture of Figure 8 is the complement of the right picture of Figure 8 (recall decomposing $S^4$ using a slice knot, by attaching and carving 2-handles from the two hemispheres $B^4_{\pm}$ as in 14.3 of [A1]). Here $\phi_n$ is the important part, which is used in the construction of the cork.

The two right pieces are the nontrivial diffeomorphism induced by the “swallow-folllow” isotopy. $\phi_n|_{C}$ is longitudinal rotation of $T$ along $J$, and $g_n(S^3 \times J - C)$ describes a carved out disk $D_n \subset B^4$ bounded by $K\# - K$, which is the disk obtained by concatenating a swallow-follow concordance of $K\# - K$, followed by the standard disk $D$ which $K\# - K$ bounds in $B^4$ and $D_n = \phi_n(D)$.

2. Constructions and proofs

Let $M^4$ be a smooth 4-manifold, and $S^2 \times D^2 \subset M^4$ be the tubular neighborhood of an imbedded 2-sphere $S$, denoted as 0-framed circle in the left picture of Figure 10. Gluck twisting of $M$ along $S$ is the operation of cutting out $S^2 \times D^2$ from $M$, and regluing by the nontrivial diffeomorphism of $S^2 \times S^1$. The affect of this operation on the
handlebody is indicated in Figure 10 (\[A1\]). This is the operation of arbitrarily separating 2-handle strands going through \(S\) into two groups, and applying 1 twist across one group and \(-1\) twist across the other. If \(H \subset M\) is an imbedded cylinder with boundary components \(\delta_1, \delta_1'\) away from the 2-handles, this operation corresponds to twisting \(M\) along \(H\).

![Figure 10. Gluck twisting to \(M\) along \(S\)](image)

Now recall the infinite order loose-cork \((W, h)\), defined in \([A2]\) and \([G]\). As discussed in \([A2]\), \(W\) is the contractible manifold of Figure 11 and the cork automorphism \(h : \partial W \to \partial W\) is given by the “\(\delta\)-move”, which is indicated in Figure 11. This operation is similar to Gluck twisting, where \(S\) is replaced by the unknotted circle \(\delta\). Main difference is, here we allow 1-handles (circle-with-dots) go through the circle \(\delta\).

![Figure 11. \(\delta\)-move diffeomorphism \(W \approx W\)](image)

**Remark 1.** At first glance reader might think the right and left twists of \(\delta\)-move in Figure 11 would cancel each other and nothing happens. But this is not so, this induces a nontrivial diffeomorphism of the boundary of \(W\) (which is the cork automorphism). For example, the delta move in Figure 26 of \([A3]\), alters the position of \(\gamma\) in a nontrivial way. \(\gamma\) could be the attaching circle of a 2-handle on top of the cork.

Figure 6 is an equivalent definition of \(W\), which is the contractible manifold obtained by blowing down \(B^4\) (6.2 of \([A1]\)) along the obvious ribbon disk \(D \subset B^4\) bounding \(K\# - K\), where \(K\) is the figure 8 knot.
Figure 6 version of $W$ can be obtained from Figure 11 by ignoring the middle dotted circle (1-handle) of the left picture of Figure 11, then canceling other two circles-with-dots with the $-1$ framed 2-handles. This process takes the middle 1-handle circle to the dotted $K\# - K$.

Figure 12 indicates to where this process takes the curve $\delta_1$ from left to the right picture of this figure (this is a very crucial observation). In the left picture $\delta$-move corresponds twisting along two parallel copies of $\delta_1$ in the opposite direction. To see this in the right picture of Figure 12 we have to perform this operation along two parallel copies of the curve corresponding $\delta_1$, which is a copy of figure-8 knot. One of the copies of the knot can be slid over the ribbon 1-handle (shown in the figure) to put in the position of the left picture of Figure 13.

So oppositely twisting $W$ along $\delta_-$ and $\delta_+$ curves of Figure 13 will result the desired $\delta$-move. This corresponds to altering the position of the standard ribbon disk $D \subset B^4$ which $K\# - K$ bounds to another ribbon disk $D_n$, which is obtained by concatenating the concordance induced by the isotopy with $D$. From our construction $D_n = \phi_n(D)$. If $\phi_n$ was isotopic to identity relative to boundary, the infinite order cork automorphism $h : \partial W \to \partial W$ (as $n \to \infty$) would extend as a smooth diffeomorphism inside $W$, which is a contradiction.\qed
Remark 2. Diffeomorphism type of the carved-out $B^4$, carved along a properly imbedded disk $D \subset B^4$, does not change if we isotope $\partial D \subset S^3$ before carving, however blowing down isotopic disks $D, D' \subset B^4$, fixing $\partial D$ and $\partial D'$, can give corks as in ([A4]), and as in this example.

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