POISSON-CHARLIER AND POLY-CAUCHY MIXED TYPE POLYNOMIALS

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ABSTRACT. In this paper, we consider Poisson-Charlier and poly-Cauchy mixed type polynomials and give various identities of those polynomials which are derived from umbral calculus.

1. Introduction and Preliminaries

For \( r \in \mathbb{Z}_{\geq 0} \), the Cauchy numbers of the first kind with order \( r \) are defined by the generating function to be

\[
\left( \frac{t}{\log (1 + t)} \right)^r = \sum_{n=0}^{\infty} C_n^{(r)} \frac{t^n}{n!}, \quad (\text{see } [3, 10, 11, 12]).
\]

In particular, when \( r = 1 \), \( C_n^{(1)} = C_n \) are called Cauchy numbers of the first kind.

The Cauchy numbers of the second kind with order \( r \) are defined by

\[
\left( \frac{t}{(1 + t) \log (1 + t)} \right)^r = \sum_{n=0}^{\infty} \hat{C}_n^{(r)} \frac{t^n}{n!}, \quad (\text{see } [3, 10, 11, 12]).
\]

When \( r = 1 \), \( \hat{C}_n^{(1)} = \hat{C}_n \) are called the Cauchy numbers of the second kind.

As is well known, the generating function for the Poisson-Charlier polynomials is given by

\[
e^{-t} \left( 1 + \frac{t}{a} \right)^x = \sum_{n=0}^{\infty} C_n (x : a) \frac{t^n}{n!}, \quad (a \neq 0), \quad (\text{see } [14, 15]).
\]

Recently, Komatsu has considered the poly-Cauchy polynomials of the first kind as follows:

\[
\frac{1}{(1 + t)^r} \text{Lif}_k (\log (1 + t)) = \sum_{n=0}^{\infty} C_n^{(k)} (x) \frac{t^n}{n!},
\]

where

\[
\text{Lif}_k (t) = \sum_{n=0}^{\infty} \frac{t^n}{n! (n + 1)^k}, \quad (\text{see } [10, 11]).
\]

He also introduced the poly-Cauchy polynomials of the second kind by

\[
(1 + t)^x \text{Lif}_k (-\log (1 + t)) = \sum_{n=0}^{\infty} \hat{C}_n^{(k)} (x) \frac{t^n}{n!}, \quad (\text{see } [10, 11]).
\]
In this paper, we consider Poisson-Charlier and poly-Cauchy of the first kind mixed type polynomials as follows:

\[ e^{-t} \text{Li}_k \left( \log \left( 1 + \frac{t}{a} \right) \right) \left( 1 + \frac{t}{a} \right)^{-x} = \sum_{n=0}^{\infty} PC_n^{(k)} (x : a) \frac{t^n}{n!}, \quad (a \neq 0). \tag{1.7} \]

The Poisson-Charlier and poly-Cauchy of the second kind mixed type polynomials are defined by the generating function to be

\[ e^{-t} \text{Li}_k \left( - \log \left( 1 + \frac{t}{a} \right) \right) \left( 1 + \frac{t}{a} \right)^{x} = \sum_{n=0}^{\infty} PC_n^{(k)} (x : a) \frac{t^n}{n!}, \quad (a \neq 0). \tag{1.8} \]

It is known that the Frobenius-Euler polynomials of order \( r \) are given by

\[ \left( \frac{1 - \lambda}{e^t - \lambda} \right)^{r} e^{xt} = \sum_{n=0}^{\infty} H_n^{(r)} (x|\lambda) \frac{t^n}{n!}, \quad (\text{see} \ [1, 4, 7, 9]). \tag{1.9} \]

where \( r \in \mathbb{Z}_{\geq 0} \), and \( \lambda \in \mathbb{C} \) with \( \lambda \neq 1 \).

The Bernoulli polynomials of order \( r \) are also defined by the generating function to be

\[ \left( \frac{t}{e^t - 1} \right)^{r} e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)} (x) \frac{t^n}{n!}, \quad (\text{see} \ [2, 5, 9, 10, 13]). \tag{1.10} \]

The Stirling number of the first kind is given by

\[ (x)_n = (x)(x-1) \cdots (x-n+1) = \sum_{l=0}^{n} S_1 (n, l) x^l, \quad (\text{see} \ [14, 15]), \tag{1.11} \]

and by \( (1.10) \), we get

\[ (\log (1+t))^m = m! \sum_{l=m}^{\infty} S_1 (l, m) \frac{t^l}{l!}, \quad (\text{see} \ [8, 9, 14, 15]). \tag{1.12} \]

From \( (1.11) \), we note that

\[ x^{(n)} = (-1)^n (-x)_n = \sum_{l=0}^{n} (-1)^{n-l} S_1 (n, l) x^l, \tag{1.13} \]

where \( x^{(n)} = x(x+1) \cdots (x+n-1) \), (see \([1-15]\)).

Let \( \mathbb{C} \) be the complex number field and let \( \mathcal{F} \) be the set of all formal power series in the variable \( t \):

\[ \mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \bigg| a_k \in \mathbb{C} \right\}. \tag{1.14} \]

Let \( \mathbb{P} = \mathbb{C}[x] \) and let \( \mathbb{P}^* \) be the vector space of all linear functionals on \( \mathbb{P} \).

\( \langle L | p(x) \rangle \) is the action of the linear functional \( L \) on the polynomial \( p(x) \), and we recall that the vector space operations on \( \mathbb{P}^* \) are defined by \( \langle L + M | p(x) \rangle = \langle L | p(x) \rangle + \langle M | p(x) \rangle \), \( c \langle L | p(x) \rangle = \langle cL | p(x) \rangle \), where \( c \) is complex constant in \( \mathbb{C} \).

For \( f(t) \in \mathcal{F} \), let us define the linear functional on \( \mathbb{P} \) by setting

\[ \langle f(t) | x^n \rangle = a_n, \quad (n \geq 0). \tag{1.15} \]

Thus, by \( (1.14) \) and \( (1.15) \), we get

\[ \langle t^k | x^n \rangle = n! \delta_{n,k}, \quad (n, k \geq 0), \quad (\text{see} \ [4, 8, 14]), \tag{1.16} \]

where \( \delta_{n,k} \) is the Kronecker’s symbol.

Let \( f_L(t) = \sum_{k=0}^{\infty} \frac{\langle L | x^k \rangle}{k!} t^k \). Then, by \( (1.15) \), we see that \( \langle f_L(t) | x^n \rangle = \langle L | x^n \rangle \).

The map \( L \mapsto f_L(t) \) is a vector space isomorphism from \( \mathbb{P}^* \) onto \( \mathcal{F} \). Henceforth,
\( \mathcal{F} \) denotes both the algebra of formal power series in \( t \) and the vector space of all linear functionals on \( \mathbb{P} \), and so an element \( f(t) \) of \( \mathcal{F} \) will be thought of as both a formal power series and a linear functional. We call \( \mathcal{F} \) the umbral algebra and the umbral calculus is the study of umbral algebra. The order \( O(f(t)) \) of a power series \( f(t) \neq 0 \) is the smallest integer \( k \) for which the coefficient of \( t^k \) does not vanish. If \( O(f(t)) = 1 \), then \( f(t) \) is called a delta series; if \( O(f(t)) = 0 \), then \( f(t) \) is called an invertible series. For \( f(t), g(t) \in \mathcal{F} \) with \( O(f(t)) = 1 \) and \( O(g(t)) = 0 \), there exists a unique sequence \( s_n(x) \) such that \( \langle g(t)f(t)^k \rangle s_n(x) = n! \delta_{n,k} \), for \( n, k \geq 0 \). The sequence \( s_n(x) \) is called an invertible series. For \( f(t) \), \( g(t) \in \mathcal{F} \) with \( O(f(t)) = 1 \) and \( O(g(t)) = 0 \), there exists a unique sequence \( s_n(x) \) such that \( \langle g(t)f(t)^k \rangle s_n(x) = n! \delta_{n,k} \), for \( n, k \geq 0 \). The sequence \( s_n(x) \) is called an invertible series. For \( f(t) \), \( g(t) \in \mathcal{F} \) with \( O(f(t)) = 1 \) and \( O(g(t)) = 0 \), there exists a unique sequence \( s_n(x) \) such that \( \langle g(t)f(t)^k \rangle s_n(x) = n! \delta_{n,k} \), for \( n, k \geq 0 \). The sequence \( s_n(x) \) is called an invertible series.

Let \( f(t), g(t) \in \mathcal{F} \) and \( p(x) \in \mathbb{P} \). Then we see that
\[
\langle f(t)g(t)p(x) \rangle = \langle f(t)g(t)p(x) \rangle = \langle g(t)f(t)p(x) \rangle ,
\]
and
\[
f(t) = \sum_{k=0}^{\infty} \langle f(t)|x^k \rangle \frac{t^k}{k!}, \quad p(x) = \sum_{k=0}^{\infty} \langle t^k|p(x) \rangle x^k \frac{t^k}{k!} .
\]

By (1.18), we get
\[
t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k}, \quad \text{and} \quad e^x p(x) = p(x + y), \quad \text{(see [14])}.
\]

For \( s_n(x) \sim (g(t), f(t)) \), we have the generating function of \( s_n(x) \) as follows:
\[
\frac{1}{g(f(t))} e^{f(t)} = \sum_{n=0}^{\infty} s_n(x) \frac{x^n}{n!} \quad \text{for all } x \in \mathbb{C},
\]
where \( f(t) \) is the compositional inverse of \( f(t) \) with \( f(f(t)) = t \).

Let \( s_n(x) \sim (g(t), f(t)) \). Then we have the following equations (see [8, 10, 14, 15]):
\[
\begin{align*}
(1.21) \quad f(t)s_n(x) &= ns_{n-1}(x), \quad (n \geq 0), \\
\frac{d}{dx}s_n(x) &= \sum_{l=0}^{n-1} \binom{n}{l} \langle f(t)|x^{n-l} \rangle s_l(x),
\end{align*}
\]
\[
\begin{align*}
(1.22) \quad s_n(x) &= \sum_{j=0}^{n} \frac{1}{j!} \langle g(f(t))^{-1}f(t)^j \rangle x^n x^j, \quad \langle f(t)|xp(x) \rangle = \langle \partial_x f(t)|p(x) \rangle,
\end{align*}
\]
\[
\begin{align*}
(1.23) \quad s_n(x+y) &= \sum_{j=0}^{n} \binom{n}{j} s_j(x) p_{n-j}(y), \quad \text{where } p_n(x) = g(t)s_n(x) .
\end{align*}
\]

For \( p_n(x) \sim (1, f(t)), q_n(x) \sim (1, g(t)) \), it is well known that
\[
(1.24) \quad q_n(x) = x \left( \frac{f(t)}{g(t)} \right)^{n} x^{-1} p_n(x), \quad (n \geq 1), \quad \text{(see [14, 15])} .
\]

For \( s_n(x) \sim (g(t), f(t)), r_n(x) \sim (h(t), l(t)) \), let us assume that
\[
s_n(x) = \sum_{m=0}^{\infty} C_{n,m} r_n(x), \quad (n \geq 0).
\]

Then we have
\[
(1.25) \quad C_{n,m} = \frac{1}{m!} \left\langle h(\mathcal{F}(t)) \left( \frac{f(t)}{g(f(t))} \right)^{m} \mathcal{F}(t)^{n} \right| x^n , \quad \text{(see [8, 10, 14])} .
\]

In this paper, we investigate some identities of Poisson-Charlier and poly-Cauchy mixed type polynomials arising from umbral calculus. That is, we give various
identities of the Poisson-Charlier and poly-Cauchy polynomials of the first and second kind mixed type polynomials which are derived from umbral calculus.

2. Poisson-Charlier and poly-Cauchy mixed type polynomials

From (1.6), (1.7) and (1.20), we note that

\[ PC_n^{(k)} (x : a) \sim e^{a(e^{-t}t - 1)} \frac{1}{\text{Lif}_k (-t)}, \ a (e^t - 1) \]

and

\[ \hat{P}C_n^{(k)} (x : a) \sim e^{a(e^t - 1)} \frac{1}{\text{Lif}_k (-t)}, \ a (e^t - 1) \]

Now, we observe that

\[ PC_n^{(k)} (y : a) \]

\[ = \left\langle \sum_{l=0}^{\infty} PC_l^{(k)} (y : a) \frac{1}{l!} x^n \right\rangle \]

\[ = \left\langle e^{-t} \text{Lif}_k \left( \log \left(1 + \frac{t}{a}\right) \right) \left(1 + \frac{t}{a}\right)^{-y} x^n \right\rangle \]

\[ = \left\langle e^{-t} \text{Lif}_k \left( \log \left(1 + \frac{t}{a}\right) \right) \left(1 + \frac{t}{a}\right)^{-y} x^n \right\rangle \]

\[ = \sum_{l=0}^{n} C_l^{(k)} (y) \frac{1}{a^l!} (n)_l \left\langle e^{-t} | x^{n-l} \right\rangle = \sum_{l=0}^{n} C_l^{(k)} (y) \left(\frac{n}{l}\right) \frac{(-1)^{n-l}}{a^l} \]

Therefore, by (2.4), we obtain the following proposition.

**Theorem 1.** For \( n \geq 0 \), we have

\[ PC_n^{(k)} (x : a) = \sum_{l=0}^{n} C_l^{(k)} (x) \left(\frac{n}{l}\right) (-1)^{n-l} \frac{1}{a^l}, \]

where \( a \neq 0 \).

Alternatively,

\[ PC_n^{(k)} (y : a) \]

\[ = \left\langle \text{Lif}_k \left( \log \left(1 + \frac{t}{a}\right) \right) e^{-t} \left(1 + \frac{t}{a}\right)^{-y} x^n \right\rangle \]

\[ = \sum_{l=0}^{n} C_l (-y : a) \frac{1}{l!} (n)_l \left\langle \text{Lif}_k \left( \log \left(1 + \frac{t}{a}\right) \right) | x^{n-l} \right\rangle \]

\[ = \sum_{l=0}^{n} C_l (-y : a) \left(\frac{n}{l}\right) C_{n-l}^{(k)} \frac{1}{a^{n-l}} \]

\[ = \sum_{l=0}^{n} \frac{\left(\frac{n}{l}\right) C_{n-l}^{(k)}}{a^{n-l}} C_l (-y : a). \]

Therefore, by (2.4), we obtain the following proposition.
Proposition 2. For \( n \geq 0, a \neq 0 \), we have

\[
PC_n^{(k)}(x : a) = \sum_{l=0}^{n} \binom{n}{l} \frac{C_{l}^{(k)}}{a^{n-l}} C_l(-x : a).
\]

Remark. By the same method as (2.3) and (2.4), we get

\[
(2.5) \quad PC_n^{(k)}(x : a) = \sum_{l=0}^{n} \left(\frac{n}{l}\right) \frac{(-1)^{n-l} C_{l}^{(k)}}{a^{l}} C_{l}^{(k)}(x),
\]

and

\[
(2.6) \quad PC_n^{(k)}(x : a) = \sum_{l=0}^{n} \left(\frac{n}{l}\right) \frac{C_{l}^{(k)}}{a^{l}} C_{n-l}^{(k)}(x : a).
\]

It is not difficult to show that

\[
(2.7) \quad \left(-\frac{1}{a}\right)^n x^{(n)} = a^{-n} \sum_{l=0}^{n} (-1)^{k} S_{1}(n, k) x^k \sim (1, a (e^{-t} - 1)),
\]

and

\[
(2.8) \quad a^{-n} (x)_n = a^{-n} \sum_{k=0}^{n} S_{1}(n, k) x^k \sim (1, a (e^t - 1)).
\]

By (2.1), we get

\[
(2.9) \quad e^{a(e^{-t} - 1)} \frac{1}{\text{Li}_k(-t)} PC_n^{(k)}(x : a) \sim (1, a (e^{-t} - 1)).
\]

From (2.7), (2.9), we have

\[
(2.10) \quad e^{a(e^{-t} - 1)} \frac{1}{\text{Li}_k(-t)} PC_n^{(k)}(x : a) = \left(-\frac{1}{a}\right)^n x^{(n)}.
\]

Thus, by (2.10) we get

\[
(2.11) \quad PC_n^{(k)}(x : a) = \text{Li}_k(-t) e^{-a(e^{-t} - 1)} \left(-\frac{1}{a}\right)^n x^{(n)}
\]

\[
= \left(-\frac{1}{a}\right)^n \text{Li}_k(-t) \sum_{l=0}^{n} \frac{a^l}{l!} (1 - e^{-t})^l x^{(n)}.
\]

By (1.13), we see that \( x^{(n)} \sim (1, 1 - e^{-t}) \). From (2.11) and (2.11), we have

\[
(1 - e^{-t})^l x^{(n)} = (n)_l x^{(n-l)}.
\]
and

\[(2.12) \quad P\tilde{C}_n^{(k)}(x : a) = -\frac{1}{a} n \sum_{l=0}^{n} \frac{a^l}{l!} \left(1 - e^{-t}\right)^l x^n \]

\[= -\frac{1}{a} n \text{Lif}_k (-t) \sum_{l=0}^{n} \frac{n!}{l!} a^n x^{n-l} \]

\[= -\frac{1}{a} n \sum_{l=0}^{n} \frac{n!}{l!} \sum_{m=0}^{n-l} (-1)^{n-l-m} \text{S}_1(n - l, m) \sum_{r=0}^{m} \frac{(-1)^r}{r!(r+1)^k} t^r x^m \]

\[= a^{-n} \sum_{l=0}^{n} \sum_{m=0}^{n-l} (-1)^{l+m+r} \binom{n}{l} \binom{m}{j} \frac{a^l}{(r+1)^k} \text{S}_1(n - l, m) x^{m-r} \]

\[= a^{-n} \sum_{j=0}^{n} \left\{ \sum_{m=j}^{n} \sum_{l=0}^{n-m} (-1)^{l+j} \binom{n}{l} \binom{m}{j} \frac{a^l}{(m-j+1)^k} \text{S}_1(n - l, m) \right\} x^j. \]

Therefore, by (2.12), we obtain the following theorem.

**Theorem 3.** For \( n \geq 0, k \in \mathbb{Z} \) and \( a \neq 0 \), we have

\[ P\tilde{C}_n^{(k)}(x : a) = a^{-n} \sum_{j=0}^{n} \left\{ \sum_{m=j}^{n} \sum_{l=0}^{n-m} (-1)^{l+j} \binom{n}{l} \binom{m}{j} \frac{a^l}{(m-j+1)^k} \text{S}_1(n - l, m) \right\} x^j. \]

**Remark.** By (2.2) and (2.8), we get

\[(2.13) \quad P\tilde{C}_n^{(k)}(x : a) = \text{Lif}_k (-t) e^{-a(t^i-1)a^{-n}} (x)_n \]

\[= a^{-n} \text{Lif}_k (-t) \sum_{l=0}^{n} \frac{(-a)^l}{l!} (t^i-1)^l (x)_n. \]

By the same method as (2.12), we get

\[ P\tilde{C}_n^{(k)}(x : a) = a^{-n} \sum_{j=0}^{n} \left\{ \sum_{m=j}^{n} \sum_{l=0}^{n-m} (-1)^{l+m+j} \binom{n}{l} \binom{m}{j} \frac{a^l}{(m-j+1)^k} \text{S}_1(n - l, m) \right\} x^j. \]
From (1.22) and (2.1), we note that

\[(2.14)\]

\[
P C_n^{(k)} (x : a)
= \sum_{l=0}^{n} \frac{1}{l!} \left( e^{-t} \text{Lif}_k \left( \log \left( 1 + \frac{t}{a} \right) \right) - \log \left( 1 + \frac{t}{a} \right) \right) \mid x^n \mid x^l
\]

\[
= \sum_{l=0}^{n} \frac{(-1)^l}{l!} \sum_{r=0}^{n-l} \frac{1}{r+1} S_1 (r + l, l) \left( e^{-t} \text{Lif}_k \left( \log \left( 1 + \frac{t}{a} \right) \right) \mid t^{r+n} x^n \mid \right) x^l
\]

\[
= \sum_{l=0}^{n} \left\{ \sum_{r=0}^{n-l} \frac{(-1)^l}{a^{r+l}} \right\} S_1 (n - r, l) PC_n^{(k)} (0 : a) x^l
\]

Therefore, by (2.14), we obtain the following theorem.

**Theorem 4.** For \( n \geq 0, k \in \mathbb{Z} \) and \( a \neq 0 \), we have

\[
PC_n^{(k)} (x : a) = \sum_{l=0}^{n} \frac{(-1)^l}{a^{n-l}} \binom{n}{r} S_1 (n - r, l) PC_r^{(k)} (0 : a) x^l.
\]

**Remark.** From (1.22) and (2.2), we can also derive the following equation.

\[(2.15)\]

\[
PC_n^{(k)} (x : a) = \sum_{l=0}^{n} \left\{ \sum_{r=0}^{n-l} \frac{(-1)^l}{a^{n-r}} S_1 (n - r, l) PC_r^{(k)} (0 : a) \right\} x^l.
\]

By (2.1), we easily see that

\[(2.16)\]

\[
e^{a(e^{-t} - 1)} \frac{1}{\text{Li}_k (-t)} PC_n^{(k)} (x : a) \sim \left( 1, a \left( e^{-t} - 1 \right) \right), \quad x^n \sim (1, t).
\]

Thus, by (1.24) and (2.10), for \( n \geq 1 \) we get

\[(2.17)\]

\[
e^{a(e^{-t} - 1)} \frac{1}{\text{Li}_k (-t)} PC_n^{(k)} (x : a)
= \left( \frac{t}{a \left( e^{-t} - 1 \right)} \right)^n x^{n-1} x^n = (-a)^n x \left( \frac{t}{e^{-t} - 1} \right)^n x^{n-1}
\]

\[
= (-a)^n x \sum_{r=0}^{n} \frac{(-1)^r}{r!} B_r (n) x^{n-r-1}
\]

\[
= (-a)^n x \sum_{r=0}^{n-1} \frac{(-1)^r}{r!} B_r (n - 1) x^{n-1}
\]

\[
= (-a)^n x \sum_{r=0}^{n-1} B_r (n) \left( \frac{-1}{r} \right) x^{n-r},
\]

where \( B_r^{(n)} = B_r^{(n)} (0) \) are called the Bernoulli numbers of order \( n \).
From (2.17), we have

\[ PC_n^{(k)}(x : a) \]

\[ = (-a^{-1})^n \sum_{r=0}^{n-1} (-1)^r \binom{n-1}{r} B_r^{(n)} \text{Lif}_k(-t) e^{-a(e^t - 1)} x^{n-r} \]

\[ = (-a^{-1})^n \sum_{r=0}^{n-1} (-1)^r \binom{n-1}{r} B_r^{(n)} \text{Lif}_k(-t) \sum_{t=0}^{n-r} \frac{(-a)^t}{t!} (e^t - 1)^t x^{n-r} \]

\[ = (-a^{-1})^n \sum_{r=0}^{n-1} (-1)^r \binom{n-1}{r} B_r^{(n)} \sum_{l=0}^{n-r} \frac{(-a)^t}{t!} \sum_{j=0}^{l} \frac{l!}{(j+l)!} S_2(j + l, l) \times (-1)^{j+l} \text{Lif}_k(-t) t^{j+l} x^{n-r} \],

where \( S_2(n, k) \) is the stirling number of the second kind.

Now, we observe that

\[ \text{Lif}_k(-t) t^{j+l} x^{n-r} \]

\[ = (n-r)_{j+l} \text{Lif}_k(-t) x^{n-r-j-l} \]

\[ = (n-r)_{j+l} \sum_{m=0}^{\infty} \frac{(-1)^m t^m}{m!(m+1)^k} x^{n-r-j-l} \]

\[ = (n-r)_{j+l} \sum_{m=0}^{n-r-j-l} \frac{(-1)^m}{m!(m+1)^k} (n-r-j-l)_m x^{n-r-j-l-m} \].

Therefore, by (2.18) and (2.19), we obtain the following theorem.

**Theorem 5.** For \( n \geq 1, k \in \mathbb{Z} \) and \( a \neq 0 \), we have

\[ PC_n^{(k)}(x : a) \]

\[ = a^{-n} \sum_{m=0}^{n} \left\{ \sum_{r=0}^{n-m} \sum_{l=0}^{n-m-r} \sum_{j=0}^{n-r} (-1)^{l+m} \binom{n-1}{r} \binom{n-r}{j+l} \right. \]

\[ \times \left( \binom{n-r-j-l}{m} a^l S_2(j + l, l) \frac{B_l^{(n)}}{(n-r-j-l-m+1)^k} \right) x^m. \]

**Remark.** We note that

\[ e^{a(e^t - 1)} \frac{1}{\text{Lif}_k(-t)} P\hat{C}_n^{(k)}(x : a) \sim (1, a(e^t - 1)), \quad x^n \sim (1, t). \]

Thus, for \( n \geq 1 \) we have

\[ e^{a(e^t - 1)} \frac{1}{\text{Lif}_k(-t)} P\hat{C}_n^{(k)}(x : a) = a^{-n} \sum_{l=0}^{n-1} \binom{n-1}{l} B_l^{(n)} x^{n-l}. \]
From \( (2.21) \), for \( n \geq 1 \) we can derive

\[
PC_n^{(k)} (x : a) = (-a^{-1})^n \sum_{m=0}^n \left[ \sum_{r=0}^{n-m} \sum_{l=0}^{n-m-r} \sum_{j=0}^{n-r-l} \left\{ (-1)^{r+j+m} \frac{(n-1) \binom{n-r}{j+l}}{(n-r-j-l-m+1)^x} \right. \right. \]
\[
\times \left. \left. \left( \frac{n-r-j-l}{m} \right) a^j S_2 (j+l, l) B_j^{(a)} \right] \right\} x^m.
\]

By \( (1.23), (2.1) \) and \( (2.2) \), we get

\[
PC_n^{(k)} (x + y : a) = \sum_{j=0}^n \binom{n}{j} PC_j^{(k)} (x : a) (-a^{-1})^{n-j} y^{(n-j)}
\]
\[
= \sum_{j=0}^n \binom{n}{j} PC_{n-j}^{(k)} (x : a) (-a^{-1})^j y^j
\]

and

\[
PC_n^{(k)} (x + y : a) = \sum_{j=0}^n \binom{n}{j} PC_j^{(k)} (x : a) a^{-j} (y)^{n-j}
\]
\[
= \sum_{j=0}^n \binom{n}{j} PC_{n-j}^{(k)} (x : a) a^{-j} (y)^j.
\]

From \( (1.21), (2.1) \) and \( (2.2) \), we have

\[
PC_n^{(k)} (x - 1 : a) - PC_n^{(k)} (x : a) = a^{-1} n PC_{n-1}^{(k)} (x : a),
\]

and

\[
PC_n^{(k)} (x + 1 : a) - PC_n^{(k)} (x : a) = a^{-1} n PC_{n-1}^{(k)} (x : a).
\]

For \( s_n (x) \sim (g(t), f(t)) \), we note that recurrence formula for \( s_n (x) \) is given by

\[
s_{n+1} (x) = \left( x - \frac{g'}{g(t)} \right) \frac{1}{f'(t)} s_n (x).
\]

Thus, by \( (2.1), (2.2) \) and \( (2.27) \), we get

\[
PC_{n+1}^{(k)} (x : a)
\]
\[
= -\frac{1}{a} x PC_n^{(k)} (x + 1 : a) - PC_n^{(k)} (x : a)
\]
\[
+ a^{-(n+1)} \sum_{j=0}^n \left\{ \sum_{m=j}^{n-m} \sum_{l=0}^{m-n} (-1)^{j+l} \binom{n}{j} \binom{m}{l} \frac{a^j}{(m-j+2)^x} S_1 (n-l, m) \right\} (x+1)^j,
\]
and

\[
\hat{P}^{(k)}_{n+1}(x : a) = \frac{1}{a} x \hat{P}^{(k)}_n(x - 1 : a) - \hat{P}^{(k)}_n(x : a) - a^{-(n+1)} \sum_{j=0}^{n} \left\{ \sum_{m=j}^{n-m} (-1)^{l+m+j} \binom{n}{l} \binom{m}{j} \right\} (x - 1)^j.
\]

Note that

\[
P^{(k)}_n(y : a) = \left\langle \sum_{l=0}^{\infty} P^{(k)}_l(y : a) \frac{t^l}{l!} x^n \right\rangle = \left\langle e^{-t \text{Lif}_k} \left( \log \left( 1 + \frac{t}{a} \right) \right) \left( 1 + \frac{t}{a} \right)^{-y} \left| x^n \right. \right\rangle
\]

\[
= \left\langle \partial_t \left( e^{-t \text{Lif}_k} \left( \log \left( 1 + \frac{t}{a} \right) \right) \right) \left( 1 + \frac{t}{a} \right)^{-y} \left| x^n \right. \right\rangle
\]

\[
= \left\langle \left( \partial_t e^{-t} \right) \text{Lif}_k \left( \log \left( 1 + \frac{t}{a} \right) \right) \left( 1 + \frac{t}{a} \right)^{-y} \left| x^n \right. \right\rangle
\]

Now, we observe that

\[
\partial_t \text{Lif}_k \left( \log \left( 1 + \frac{t}{a} \right) \right) = \frac{1}{a (1 + \frac{t}{a})} \left\{ \text{Lif}_{k-1} \left( \log \left( 1 + \frac{t}{a} \right) \right) - \text{Lif}_k \left( \log \left( 1 + \frac{t}{a} \right) \right) \right\}.
\]
From (2.31), we have

\[ (2.32) \quad \left\langle e^{-t} \left( \partial_t \text{Lif}_k \left( \log \left( 1 + \frac{t}{a} \right) \right) \right) \left( 1 + \frac{t}{a} \right)^{-y} \bigg\vert x^{n-1} \right\rangle \]

\[ = \frac{1}{a} \left\langle e^{-t} \text{Lif}_{k-1} \left( \log \left( 1 + \frac{t}{a} \right) \right) - \text{Lif}_k \left( \log \left( 1 + \frac{t}{a} \right) \right) \right( 1 + \frac{t}{a} \right)^{-y} \bigg\vert x^{n-1} \right\rangle \]

\[ = \frac{1}{n} \sum_{l=0}^{n-1} \left\langle \binom{n}{l} \frac{C_l}{a^l} \{ PC_{n-l}^{(k)} (y : a) - PC_{n-l}^{(k)} (y : a) \} \right. \bigg\rangle \].

Therefore, by (2.30) and (2.32), we obtain the following theorem.

**Theorem 6.** For \( n \geq 0, k \in \mathbb{Z} \) and \( a \neq 0 \), we have

\[ PC_n^{(k)} (x : a) = -PC_{n-1}^{(k)} (x : a) - \frac{1}{a} x PC_{n-1}^{(k)} (x + 1 : a) \]

\[ + \frac{1}{n} \sum_{l=0}^{n-1} \binom{n}{l} \frac{C_l}{a^l} \left\{ PC_{n-l}^{(k)} (x : a) - PC_{n-l}^{(k)} (x : a) \right\} \].

**Remark.** Note that

\[ (2.33) \quad \frac{1}{a} \left\langle e^{-t} \left( \text{Lif}_{k-1} \left( \log \left( 1 + \frac{t}{a} \right) \right) - \text{Lif}_k \left( \log \left( 1 + \frac{t}{a} \right) \right) \right) \right( 1 + \frac{t}{a} \right)^{-y} \bigg\vert x^{n-1} \right\rangle \]

\[ = \sum_{l=0}^{n-1} \frac{C_l}{a^l} \binom{n-1}{l} \cdot \left\langle e^{-t} \left( \text{Lif}_{k-1} \left( \log \left( 1 + \frac{t}{a} \right) \right) - \text{Lif}_k \left( \log \left( 1 + \frac{t}{a} \right) \right) \right) \right( 1 + \frac{t}{a} \right)^{-y} \bigg\vert x^{n-l} \right\rangle \]

\[ = \sum_{l=0}^{n-1} \left\langle \binom{n-1}{l} \frac{1}{n-l} \frac{C_l}{a^l} \left\langle e^{-t} \left( \text{Lif}_{k-1} \left( \log \left( 1 + \frac{t}{a} \right) \right) - \text{Lif}_k \left( \log \left( 1 + \frac{t}{a} \right) \right) \right) \right( 1 + \frac{t}{a} \right)^{-y} \bigg\vert x^{n-l} \right\rangle \]

\[ = \frac{1}{n} \sum_{l=0}^{n-1} \binom{n}{l} \frac{C_l}{a^l} \cdot \left\langle e^{-t} \left( \text{Lif}_{k-1} \left( \log \left( 1 + \frac{t}{a} \right) \right) - \text{Lif}_k \left( \log \left( 1 + \frac{t}{a} \right) \right) \right) \right( 1 + \frac{t}{a} \right)^{-y} \bigg\vert x^{n-l} \right\rangle \].
By (2.30) and (2.33), we also get the following equation:

\[ PC_n^{(k)}(x : a) = - PC_{n-1}^{(k)}(x : a) - \frac{1}{a} x PC_{n-1}^{(k)}(x + 1 : a) \]

\[ + \frac{1}{n} \sum_{l=0}^{n-1} \left( \frac{n}{l} \right) a^l \left\{ PC_{n-l}^{(k-1)}(x + 1 : a) - PC_{n-l}^{(k)}(x + 1 : a) \right\} . \]

By the same method as Theorem 5, we see that

\[ \hat{PC}_n^{(k)}(x : a) = - \hat{PC}_{n-1}^{(k)}(x : a) - \frac{1}{a} x \hat{PC}_{n-1}^{(k)}(x - 1 : a) \]

\[ + \frac{1}{n} \sum_{l=0}^{n-1} \left( \frac{n}{l} \right) a^l \left\{ \hat{PC}_{n-l}^{(k-1)}(x : a) - \hat{PC}_{n-l}^{(k)}(x : a) \right\} , \]

and

\[ \hat{PC}_n^{(k)}(x : a) = - \hat{PC}_{n-1}^{(k)}(x : a) - \frac{1}{a} x \hat{PC}_{n-1}^{(k)}(x - 1 : a) \]

\[ + \frac{1}{n} \sum_{l=0}^{n-1} \left( \frac{n}{l} \right) a^l \left\{ \hat{PC}_{n-l}^{(k-1)}(x - 1 : a) - \hat{PC}_{n-l}^{(k)}(x - 1 : a) \right\} . \]

Here, we compute

\[ \left\langle e^{-t \text{Lif}_k} \left( - \log \left( 1 + \frac{t}{a} \right) \right) \left( \log \left( 1 + \frac{t}{a} \right) \right)^m \bigg| x^n \right\rangle \]

in two different ways.

On the one hand,

\[ \left\langle e^{-t \text{Lif}_k} \left( - \log \left( 1 + \frac{t}{a} \right) \right) \left( \log \left( 1 + \frac{t}{a} \right) \right)^m \bigg| x^n \right\rangle \]

\[ = \sum_{l=0}^{n-m} \frac{m!}{a^{l+m}} \left( \frac{n}{l+m} \right) S_1(l + m, m) \left\langle e^{-t \text{Lif}_k} \left( - \log \left( 1 + \frac{t}{a} \right) \right) \bigg| x^{n-l-m} \right\rangle \]

\[ = \sum_{l=0}^{n-m} \frac{m!}{a^{l+m}} \left( \frac{n}{l+m} \right) S_1(l + m, m) \hat{PC}_{n-l-m}^{(k)}(0 : a) \]

\[ = \sum_{l=0}^{n-m} \frac{m!}{a^{n-l}} \left( \frac{n}{l} \right) S_1(n - l, m) \hat{PC}_{l}^{(k)}(0 : a) . \]

On the other hand,
Thus, by (2.38) and (2.39), we get

\[
\left\langle e^{-t} \operatorname{Li}_k \left( -\log \left( 1 + \frac{t}{a} \right) \right) \left( \log \left( 1 + \frac{t}{a} \right) \right)^m x^n \right\rangle
\]

\[
= \left\langle e^{-t} \operatorname{Li}_k \left( -\log \left( 1 + \frac{t}{a} \right) \right) \left( \log \left( 1 + \frac{t}{a} \right) \right)^m x \cdot x^{n-1} \right\rangle
\]

\[
= \left( \partial_t e^{-t} \right) \left( \operatorname{Li}_k \left( -\log \left( 1 + \frac{t}{a} \right) \right) \left( \log \left( 1 + \frac{t}{a} \right) \right)^m \right|_{x^{n-1}}
\]

\[
= \left( \partial_t e^{-t} \right) \left( \operatorname{Li}_k \left( -\log \left( 1 + \frac{t}{a} \right) \right) \left( \log \left( 1 + \frac{t}{a} \right) \right)^m \right|_{x^{n-1}}
\]

\[
= \left( \partial_t e^{-t} \right) \left( \operatorname{Li}_k \left( -\log \left( 1 + \frac{t}{a} \right) \right) \left( \log \left( 1 + \frac{t}{a} \right) \right)^m \right|_{x^{n-1}}
\]

\[
+ \frac{m-1}{a} \left( e^{-t} \left( \partial_t \operatorname{Li}_k \left( -\log \left( 1 + \frac{t}{a} \right) \right) \right) \left( \log \left( 1 + \frac{t}{a} \right) \right)^m \right|_{x^{n-1}}
\]

\[
+ \frac{1}{a} \left( e^{-t} \operatorname{Li}_{k-1} \left( -\log \left( 1 + \frac{t}{a} \right) \right) \left( 1 + \frac{t}{a} \right)^{n-1} \right) \left( \log \left( 1 + \frac{t}{a} \right) \right)^m x^{n-1}
\]

It is easy to show that

(2.39)

\[
\left\langle e^{-t} \operatorname{Li}_k \left( -\log \left( 1 + \frac{t}{a} \right) \right) \left( 1 + \frac{t}{a} \right)^{-1} \left( \log \left( 1 + \frac{t}{a} \right) \right)^m x^{n-1} \right\rangle
\]

\[
= \sum_{l=0}^{n-m} \frac{(m-1)!}{a^{n-l-1}} \left( n - 1 \right) S_1 \left( l + m - 1, m - 1 \right) PC_{n-l}^{(k)} \left( l + 1, m - 1 \right) \cdot \right.
\]

\[
= \sum_{l=0}^{n-m} \frac{(m-1)!}{a^{n-l-1}} \left( n - 1 \right) S_1 \left( l + m - 1, m - 1 \right) PC_{n-l}^{(k)} \left( l + 1, m - 1 \right).
\]

Thus, by (2.38) and (2.39), we get

(2.40)

\[
\left\langle e^{-t} \operatorname{Li}_k \left( -\log \left( 1 + \frac{t}{a} \right) \right) \left( \log \left( 1 + \frac{t}{a} \right) \right)^m \right|_{x^{n}}
\]

\[
= - \sum_{l=0}^{n-m} \frac{m!}{a^{n-l-1}} \left( n - 1 \right) S_1 \left( l + m - 1, m - 1 \right) PC_{n-l}^{(k)} \left( 0 + 1, m - 1 \right)
\]

\[
+ \frac{m-1}{m} \sum_{l=0}^{n-m} \frac{m!}{a^{n-l-1}} \left( n - 1 \right) S_1 \left( l + m - 1, m - 1 \right) PC_{n-l}^{(k)} \left( 0 + 1, m - 1 \right)
\]

\[
+ \frac{1}{m} \sum_{l=0}^{n-m} \frac{m!}{a^{n-l-1}} \left( n - 1 \right) S_1 \left( l + m - 1, m - 1 \right) PC_{n-l}^{(k-1)} \left( 0 + 1, m - 1 \right).
\]

Therefore, by (2.39) and (2.40), we obtain the following theorem.
Theorem 7. For \( n, m \geq 0 \) with \( n - m \geq 0, k \in \mathbb{Z} \) and \( a \neq 0 \), we have

\[
\sum_{l=0}^{n-m} \frac{n}{a^{n-l}} S_1(n-l, m) P C_i^{(k)}(0 : a)
+ \sum_{l=0}^{n-1-m} \frac{n-1}{a^{n-l-1}} S_1(n-1-l, m) P C_i^{(k)}(0 : a)
= \left(1 - \frac{1}{m}\right) \sum_{l=0}^{n-m} \frac{n}{a^{n-l}} S_1(n-1-l, m-1) P C_i^{(k)}(-1 : a)
+ \frac{1}{m} \sum_{l=0}^{n-1-m} \frac{n-1}{a^{n-l-1}} S_1(n-1-l, m-1) P C_i^{(k)}(-1 : a).
\]

Remark. From the computation of

\[
\left\langle e^{-t \text{Li}_k} \left( \log \left( 1 + \frac{t}{a} \right) \right) \left( \log \left( 1 + \frac{t}{a} \right) \right)^m \bigg| x^n \right\rangle
\]

in two different ways, we can also derive the following equation:

\[
(2.41) \quad \sum_{l=0}^{n-m} \frac{n}{a^{n-l}} S_1(n-l, m) P C_i^{(k)}(0 : a)
+ \sum_{l=0}^{n-1-m} \frac{n-1}{a^{n-l-1}} S_1(n-1-l, m) P C_i^{(k)}(0 : a)
= \left(1 - \frac{1}{m}\right) \sum_{l=0}^{n-m} \frac{n}{a^{n-l}} S_1(n-1-l, m-1) P C_i^{(k)}(1 : a)
+ \frac{1}{m} \sum_{l=0}^{n-1-m} \frac{n-1}{a^{n-l-1}} S_1(n-1-l, m-1) P C_i^{(k-1)}(1 : a).
\]

By (1.21), (2.1) and (2.2), we easily see that

\[
(2.42) \quad \frac{d}{dx} PC_n^{(k)}(x : a) = (-1)^n n! \sum_{l=0}^{n-1} \frac{(-1)^l}{(n-l)!} a^{n-l} P C_i^{(k)}(x : a),
\]

and

\[
(2.43) \quad \frac{d}{dx} PC_n^{(k)}(x : a) = (-1)^n n! \sum_{l=0}^{n-1} \frac{(-1)^{l-1}}{(n-l)!} a^{n-l} P C_i^{(k)}(x : a).
\]

For

\[
PC_n^{(k)}(x : a) \sim \left( e^{\alpha (e^t - 1)} \frac{1}{\text{Li}_k(-l)} a^{(e^t - 1)} \right), \quad (a \neq 0)
\]

and

\[
B_n^{(s)}(x) \sim \left( \left( \frac{t^s}{l} \right) a^s, \quad (s \in \mathbb{Z}_{\geq 0}) \right).
\]
let us assume that

\begin{equation}
(2.44) \quad PC_n^{(k)} (x : a) = \sum_{m=0}^{n} C_{n,m} B_{m}^{(s)} (x).
\end{equation}

From (1.25), we note that

\begin{equation}
(2.45) \quad C_{n,m} = (-1)^m \frac{m!}{n!} \left\langle e^{-t \text{Lif}_k \left( \log \left( 1 + \frac{t}{a} \right) \right) \left( 1 + \frac{t}{a} \right)^{-s} \left( \frac{t}{\log \left( 1 + \frac{t}{a} \right)} \right)^s \left( \log \left( 1 + \frac{t}{a} \right) \right)^m x^n} \right\rangle.
\end{equation}

Now, we observe that

\begin{equation}
(2.46) \quad \left( \log \left( 1 + \frac{t}{a} \right) \right)^m x^n = \sum_{l=0}^{n-m} \frac{m!}{a^{n-l}} \left( \frac{n}{l} \right) S_1 (n-l,m) x^l.
\end{equation}

By (2.45) and (2.46), we get

\begin{equation}
(2.47) \quad C_{n,m} = (-1)^m \sum_{l=0}^{n-m} \frac{m!}{a^{n-l}} \left( \frac{n}{l} \right) S_1 (n-l,m) \left\langle \left( \log \left( 1 + \frac{t}{a} \right) \right)^s \left( \frac{t}{\log \left( 1 + \frac{t}{a} \right)} \right)^s \left( \log \left( 1 + \frac{t}{a} \right) \right)^l x^l \right\rangle.
\end{equation}

Therefore, by (2.44) and (2.47), we obtain the following theorem.

**Theorem 8.** For \( n \geq 0 \), \( k \in \mathbb{Z} \) and \( a \neq 0 \), we have

\begin{equation}
PC_n^{(k)} (x : a) = \sum_{m=0}^{n} \left\{ (-1)^m \sum_{l=0}^{n-m} \frac{m!}{a^{n-l+i}} \left( \frac{n}{l} \right) \left( \frac{l}{i} \right) S_1 (n-l,m) C_i^{(s)} PC_{l-i}^{(k)} (s : a) \right\} B_{m}^{(s)} (x).
\end{equation}
Remark. By the same method as Theorem 8, we get

\begin{equation}
\hat{PC}_n^k (x : a)
= \sum_{m=0}^{n} \left\{ \sum_{l=0}^{n-m} \sum_{i=0}^{l} \frac{n(l)}{a^n-l+i} S_1 (n-l, m) \hat{C}_i^s PC_{l-i}^k (s : a) \right\} B_m^{(s)} (x).
\end{equation}

For

\[ PC_n^k (x : a) \sim \left( e^a (e^{-t}-1) \frac{1}{\text{Lif}_k (-t)} a (e^{-t}-1) \right), (a \neq 0), \]

and

\[ H_m^s (x | \lambda) \sim \left( \left( \frac{x^t - \lambda}{1 - \lambda} \right)^s, (s \in \mathbb{Z}_{\geq 0}) \right), \]

let us assume that

\begin{equation}
PC_n^k (x : a) = \sum_{m=0}^{n} C_{n,m} H_m^s (x | \lambda).
\end{equation}

From (1.25), we have

\begin{equation}
C_{n,m} = \frac{(-1)^m}{m! (1 - \lambda)^s}
\times \left( e^{-t} \text{Lif}_k \left( \log \left( 1 + \frac{t}{a} \right) \right) \left( 1 + \frac{t}{a} \right)^{-s} \left( 1 - \lambda - \frac{\lambda t}{a} \right)^{-s} \left( \log \left( 1 + \frac{t}{a} \right) \right)^m \right) x^n
\end{equation}

\[ = \frac{(-1)^m}{(1 - \lambda)^s} \sum_{l=0}^{n-m-s} \sum_{i=0}^{s} \frac{n(l)}{a^{n-l+i}} (1 - \lambda)^{s-l} (-\lambda)^i S_1 (n-l, m) PC_{l-i}^k (s : a). \]

Therefore, by (2.49) and (2.50), we obtain the following theorem.

**Theorem 9.** For \( n \geq 0, k \in \mathbb{Z} \) and \( a \neq 0 \), we have

\begin{equation}
PC_n^k (x : a)
= \frac{1}{(1 - \lambda)^s} \sum_{m=0}^{n} \left\{ (-1)^m \sum_{l=0}^{n-m-s} \sum_{i=0}^{s} \frac{n(l)}{a^{n-l+i}} \right\} \times (1 - \lambda)^{s-l} (-\lambda)^i S_1 (n-l, m) PC_{l-i}^k (s : a) \}
\end{equation}
By the same method as Theorem [9] we get

\[(2.51)\]

\[
P_C^{(k)}_n(x:a) = \frac{1}{(1-\lambda)} \sum_{m=0}^{n} \left\{ \sum_{i=0}^{n-m} \sum_{l=0}^{i} \frac{n}{a^{m-l}} (-\lambda)^{r-i} S_1(n-l,m) P_C^{(k)}(i:a) \right\} H_m^{(s)}(x|\lambda).
\]

For

\[
P_C^{(k)}_n(x:a) \sim \left( e^{a(e^{-t}-1)} \frac{1}{\text{Li}_k(-t)} , a \left( e^{-t} - 1 \right) \right),
\]

\[
x^{(n)} = x(x+1) \cdots (x+n-1) \sim (1, 1-e^{-t}),
\]

let us assume that

\[(2.52)\]

\[
P_C^{(k)}_n(x:a) = \sum_{m=0}^{n} C_{n,m} x^{(m)}.
\]

From (1.24), we have

\[(2.53)\]

\[
C_{n,m} = \frac{1}{m!(-a)^m} \left[ e^{-t} \text{Li}_k \left( \log \left( 1 + \frac{t}{a} \right) \right) \right] x^{(m)}
\]

\[
= \frac{1}{(-a)^m} \left( \frac{n}{m} \right) x^{(m)}
\]

\[
= \frac{1}{(-a)^m} \left( \frac{n}{m} \right) P_C^{(k)}_{n-m}(0:a).
\]

Therefore, by (2.52) and (2.53), we obtain the following theorem.

**Theorem 10.** For \(n \geq 0, k \in \mathbb{Z}\) and \(a \neq 0\), we have

\[
P_C^{(k)}_n(x:a) = \sum_{m=0}^{n} \left( \frac{n}{m} \right) (-a)^m P_C^{(k)}_{n-m}(0:a) x^{(m)},
\]

where \(x^{(m)} = x(x+1) \cdots (x+m-1)\).

**Remark.** By the same method as Theorem 10 we get

\[
P_C^{(k)}_n(x:a) = \sum_{m=0}^{n} \left( \frac{n}{m} \right) a^m P_C^{(k)}_{n-m}(0:a) (x)_m,
\]

where \((x)_m = x(x-1) \cdots (x-m+1)\).

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