Finite in All Directions

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Dedicated to the memory of Feza Gürsey

We study toroidal compactifications of string theories which include compactification of a timelike coordinate. Some new features in the theory of toroidal compactifications arise. Most notably, Narain moduli space does not exist as a manifold since the action of duality on background data is ergodic. For special compactifications certain infinite dimensional symmetries, analogous to the infinite dimensional symmetries of the 2D string are unbroken. We investigate the consequences of these symmetries and search for a universal symmetry which contains all unbroken gauge groups. We define a flat connection on the moduli space of toroidally compactified theories. Parallel transport by this connection leads to a formulation of broken symmetry Ward identities. In an appendix this parallel transport is related to a definition of conformal perturbation theory.

May 24, 1993
1. Introduction

Symmetry has often been an important guide in finding the fundamental formulation of physical theories. Recent progress in closed string field theory (CSFT) suggests that we are now much closer to defining a theory which deserves the name string theory \([1]\). Unfortunately, the current formulation is unwieldy and must be regarded as an existence proof that covariant CSFT exists. The present work is motivated by the hope that a renewed investigation of the large symmetry algebras appearing in string theory will lead to a better understanding of CSFT. Common sense suggests that the formulation of CSFT should drastically simplify around a very symmetric background.

If \(\mathcal{C}\) is a CFT background of \(c = 26\) the ghost number one BRST cohomology \(H^1(\mathcal{C})\) may be regarded as the Lie algebra of (inner) automorphisms of the bosonic string background. For example, in uncompactified Minkowski space \(\mathbb{R}^{1,25}\) the Lie algebra is \(\mathbb{R}^{26} \oplus \mathbb{R}^{26}\) corresponding to translations and dual translations. (The full Poincaré symmetry does not arise as inner automorphisms.) Recent investigations of two-dimensional target spaces have revealed that some 2D backgrounds have infinite-dimensional unbroken symmetry algebras, e.g., algebras of area-preserving and volume-preserving diffeomorphisms \([2]\).

An essential ingredient in the new constructions of infinite-dimensional unbroken symmetries is the presence of negative-dimension vertex operators from the Liouville sector. In general, the existence of large unbroken symmetries is connected to the strange nature of time in string theory.

In this paper we will present further examples of backgrounds with infinite-dimensional unbroken symmetries. Our technique is elementary: we consider toroidal compactification of all spacetime coordinates. We are regarding compactified time as an unphysical ground state, analogous to the ground state \(\langle \phi \rangle = 0\) in electroweak theory. Our philosophy is that underlying symmetries can become manifest in backgrounds described by Higgs vev’s which differ from those chosen by nature. Some of the unbroken symmetries we encounter were proposed as being of fundamental importance to string theory several years ago \([3]\) \([4]\).

In outline, the main line of development in the paper is the following: We review toroidal compactification and discuss some subtleties that arise upon compactification of a timelike coordinate. We briefly describe the behavior of the BRST cohomology and some enhanced symmetry points. We show that there is a distinguished point in the moduli
space of toroidal compactifications which has, in some sense, maximal symmetry. We show how our considerations naturally suggest a “universal symmetry algebra” for toroidal compactifications. In spontaneously broken gauge theory Ward identities continue to hold even in the broken phase. Therefore there must be broken Ward identities even at points where the enhanced symmetries are not present. In section nine we propose a formulation of the broken Ward identities. Along the way we are forced to clarify several points about conformal perturbation theory and about the nature of spontaneously broken symmetries in toroidally compactified string theories. We conclude with some far-fetched speculations.

This long paper is full of digressions, philosophical remarks, speculations, proposed future directions etc. For those in a hurry we provide a:

**Summary of new results**

Section 2.5: New features of toroidal compactification of timelike coordinates. Proposition 3 shows that duality does not always act on sigma-model data. Proposition 5 shows that the action of duality on the space of toroidally compactified theories is ergodic. These features are thoroughly studied in an example in section 2.6.

Section 4.3: Proposition 8 shows that there are uniquely distinguished string compactifications, depending only on the number of gauged worldsheet supersymmetries. They are distinguished by the requirement that the closed string exactly factorize as a product of open strings.

Section 6: Proposition 10 constructs a candidate universal symmetry for the enhanced symmetries in toroidal compactification.

Section 7.1: Defines globally a parallel transport for the statespace of conformal field theories over the moduli space of lattices.

Section 9.1: Propositions 16, 17 formulate a set of broken Ward identities for the enhanced symmetries of toroidal compactifications.

Appendix A: Gives a complete contact term prescription to make sense of conformal perturbation theory for relating two toroidal compactifications. This involves a preferred choice of coordinates on Narain moduli space identified in Proposition 13.

We also include some tangential results which seem interesting to us. These include:

Section 2.7: Proposition 7 gives an approximate fundamental domain for the action of duality in Euclidean compactifications.

Section 4.4: Proposition 9 shows that compactifications with Monster symmetry are not dense.

Section 8.2: Proposition 14 establishes a simple formula for the cohomology class of string densities for symmetry states.
2. Toroidal Compactifications

2.1. Lattices

Toroidal compactifications in $n + 1$ dimensions are based on lattices in $\mathbb{R}^{n+1} \oplus \mathbb{R}^{n+1}$ which are even and unimodular (= self-dual) with respect to

$$\tilde{D} \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (2.1)$$

In order to consider the lattices as sets of points in $\mathbb{R}^{2n+2}$ we introduce a space of generator-matrices. Let

$$D = \begin{pmatrix} \eta & 0 \\ 0 & -\eta \end{pmatrix} \quad (2.2)$$

$$\eta_M^{ab} = \text{Diag}\{-1, +1^n\} \quad \eta_E^{ab} = \text{Diag}\{+1^{n+1}\}$$

in the case of Minkowskian and Euclidean compactifications, respectively and consider

$$\mathcal{M} = \{ \mathcal{E} \in GL(2n + 2; \mathbb{R}) : \mathcal{E}^{tr} \cdot D \cdot \mathcal{E} = \tilde{D} \} \quad (2.3)$$

$\mathcal{E}$ defines a lattice with basis: $(\mathcal{E}^A_I)$ where $A$ parametrize components and $I$ label basis vectors.

Of course, $D$ and $\tilde{D}$ are similar:

$$SDS = \tilde{D} \quad S = \frac{1}{\sqrt{2}} \begin{pmatrix} \eta & 1 \\ 1 & -\eta \end{pmatrix} \quad (2.4)$$

$$S = S^{tr} = S^{-1}$$

When we need to distinguish Euclidean and Minkowskian cases we use the notation $S_E$, $S_M$. Using (2.4) we may identify $\mathcal{M}$ with orthogonal groups

$$\mathcal{M} \cdot S = O(D; \mathbb{R})$$

$$S \cdot \mathcal{M} = O(\tilde{D}; \mathbb{R}) \quad (2.5)$$

Hence $\mathcal{M}$ admits a left $O(D)$ and a right $O(\tilde{D})$ action.

\[^1\] We denote $O(Q; \mathbb{R})$ for the orthogonal group determined by the quadratic form $Q$:

$$O(Q; \mathbb{R}) = \{ g : g^{tr}Qg = Q \}$$
A central theorem of the subject states that the moduli space of even unimodular lattices is (See [5] ch. 15.)

\[ \mathbb{L} \equiv \mathcal{M}/O(\hat{D}; \mathbb{Z}) \cong O(\hat{D}; \mathbb{R})/O(\hat{D}; \mathbb{Z}) \]  

(2.6)

2.2. Abstract construction

We construct a bundle of CFT statespaces \( \mathcal{H} \rightarrow \mathbb{L} \) as follows.

We form left and right Heisenberg-algebras with respect to the quadratic form \( +\eta \) from the loop algebra \( h = \text{Map}(S^1, \mathbb{R}^{n+1}) \). Splitting \( h = h^+ \oplus h^0 \oplus h^- \) as usual we have the state space

\[ \mathcal{H}_\Gamma = S(h^-_L) \otimes S(h^-_R) \otimes \mathbb{C}[\Gamma] \]  

(2.7)

where \( S \) denotes the symmetric algebra, the last factor is the group algebra of \( \Gamma \in \mathbb{L} \).

More concretely, consider the dimension one currents:

\[ i\partial Y^a(z) = \sum \beta_n^a z^{-n-1} \]

\[ i\bar{\partial} Y^a(z) = \sum \bar{\beta}_n^a z^{-n-1} \]  

(2.8)

with

\[ [\beta_n^a, \beta_m^b] = \eta^{ab} n \delta_{n+m,0} \]

\[ [\bar{\beta}_n^a, \bar{\beta}_m^b] = \eta^{ab} n \delta_{n+m,0} \]  

(2.9)

If \( \Gamma \in \mathcal{M} \) we denote \( \Gamma_L = \pi_L(\Gamma) \), \( \Gamma_R = \pi_R(\Gamma) \), and vectors \( (p_L; p_R) \in \Gamma \). In general \( \Gamma_{L,R} \) are not lattices but quasicrystals. We denote the left and right Fock space built on the momentum vacuum \( |p_L\rangle, |p_R\rangle \) by \( \mathcal{F}_{p_L}, \mathcal{F}_{p_R} \). The state \( |p_L; p_R\rangle \) is created by \( e^{ip_L \cdot Y(z)} e^{ip_R \cdot Y(\bar{z})} \mathcal{C}[p_L; p_R] \) where \( \mathcal{C} \) is a cocycle operator. To any point \( E \in \mathcal{M} \) we associate the CFT with statespace:

\[ \mathcal{H}_\Gamma = \bigoplus_{(p_L; p_R) \in E} \mathcal{F}_{p_L} \otimes \bar{\mathcal{F}}_{p_R} \]  

(2.10)

which is identical to (2.7). Up to a rearrangement of terms in the direct sum this only depends on \( \Gamma = [E] \in \mathbb{L} \) so we can write \( \mathcal{H}_\Gamma \).

The above conformal field theories carry a natural action of \( O(\eta) \times O(\eta) \), left and right Lorentz transformations, which change the lattice but preserve the correlation functions. We wish to identify CFT’s related by such Lorentz transformations to obtain the moduli space of CFT’s. More formally, if \( \Gamma_1 = g \cdot \Gamma_2 \) are related by \( g \in O(\eta) \times O(\eta) \) there is a corresponding isomorphism of vertex operator algebras \( U[g] : \mathcal{H}_{\Gamma_1} \rightarrow \mathcal{H}_{\Gamma_2} \) such that the maps

\[ \mathcal{P}_{h,n} \rightarrow \mathcal{H}^{\otimes n}_\Gamma \]  

(2.11)
defined by the operator formalism (i.e., the Segal functor for this CFT) satisfy:

\[
\begin{array}{c}
\mathcal{H}_{\Gamma_1} \\
\mathcal{P}_{h,n} \\
\mathcal{H}_{\Gamma_2}
\end{array}
\]

Here \( \mathcal{P}_{h,n} \) is the moduli space of genus \( h \) surfaces with \( n \) coordinatized punctures.

With the equivalence (2.12) understood the Narain moduli space of conformal field theories is

\[
\mathcal{N} = (O(\eta) \times O(\eta)) \backslash \mathcal{M}/O(\tilde{D}; \mathbb{Z})
\]

**Remarks:**

1. If we also identify CFT’s related by an exchange of leftmovers for rightmovers (e.g. in the moduli space of bosonic string theories) then the above holds with \( O(\eta) \times O(\eta) \) replaced by \( O(\eta) \times O(\eta) \ltimes \mathbb{Z}_2 \).

2. If we work with lattices-with-basis defined by \( \mathcal{E} \in \mathcal{M} \) then the above identifications lead to the moduli space:

\[
\mathbb{I} \mathcal{H} \equiv O(\eta) \times O(\eta) \backslash \mathcal{M} \cong O(\eta) \times O(\eta) \backslash O(D; \mathbb{R})
\]

\( \mathbb{I} \mathcal{H} \) is the analog of the upper half plane in Riemann surface theory. The relation between theories from change of basis clearly arises from the right-action of \( O(\tilde{D}; \mathbb{Z}) \) on \( \mathbb{I} \mathcal{H} \). See the discussion of duality below.

3. We have not actually proved that \( \mathcal{N} \) is a true moduli space since we have not shown there aren’t other equivalences. Locally this is obvious. Globally it is not so evident.

**2.3. Sigma-Model construction**

The previous discussion is a slightly unconventional description of a very well-known and widely studied class of theories – the toroidally compactified theories. We review this to set up some notation.

Let \( X^\mu \) be coordinates on the \( n + 1 \)-dimensional torus \( X^\mu \sim X^\mu + 2\sqrt{2}\pi \). Discussions of toroidal compactification usually begin with the \( \sigma \)-model action

\[
S_{\text{mink}} = \frac{1}{2\pi} \int_0^{2\pi} d\sigma \int d\tau \partial_\tau X^\mu E_{\mu\nu} \partial_\tau X^\nu
\]

(2.15)
where $\partial_\pm = \frac{1}{2}(\partial_\tau \pm \partial_\sigma)$ and the worldsheet has Minkowskian signature. The matrix $E$, referred to as the compactification data, belongs to the space of $\sigma$-model backgrounds

$$\mathcal{B} = \{ E | E = G + B, \text{signature}(G) = \eta \}$$

where we always write the decomposition of $E$ into symmetric and antisymmetric parts as $E = G + B$. We assume $G$ is an invertible quadratic form of signature $\eta$.

Canonical quantization leads in the standard way to the oscillator expansions:

$$\partial_+ X^\nu = \sum \bar{\alpha}_n^\nu e^{-iN(\tau + \sigma)}$$
$$\partial_- X^\nu = \sum \alpha_n^\nu e^{-iN(\tau - \sigma)}$$

$$[\alpha^\nu_n, \alpha^\mu_m] = n\delta_{n+m,0}G^{\mu\nu}$$

$$\alpha^\mu_0 = \frac{1}{\sqrt{2}}G^{\mu\nu}(p_\nu + E_{\nu\rho}w^\rho)$$
$$\bar{\alpha}^\mu_0 = \frac{1}{\sqrt{2}}G^{\mu\nu}(p_\nu - E^{tr}_{\nu\rho}w^\rho)$$

where $p_\nu, w^\rho$ are integers.

The Virasoro algebra is constructed as

$$L_n = \frac{1}{2} \sum \alpha^\mu_{n-m}G_{\mu\nu}\alpha^\nu_m$$

The spectrum of the theory is therefore governed by the operator

$$L_0 + \bar{L}_0 = \frac{1}{2} (p \ w) \mathcal{R}(E) \begin{pmatrix} p \\ w \end{pmatrix} + N + \bar{N}$$

where

$$\mathcal{R}(E) = \begin{pmatrix} G^{-1} & G^{-1}B \\ -BG^{-1} & G - BG^{-1}B \end{pmatrix}$$

and therefore sigma models associated to $E \in \mathcal{B}$ with equivalent spectra are classified by matrices $\mathcal{R}(E)$ equivalent under conjugation by $O(\mathcal{D})$.

2.4. Relation between the formulations

We now relate the abstract to the $\sigma$-model construction. If $e_\mu^a$ is a vielbein for $G_{\mu\nu}$ then $\beta_n^a = e_\mu^a \alpha^\mu_n(E)$ is canonically normalized as in (2.9). In principle we could use different vielbeins for left- and right-movers. It follows from (2.17) that the lattice of zero-modes, that is, the lattice of eigenvalues of $(\beta_0^a, \beta_0^a)$ is obtained from a generator matrix

$$\mathcal{E}(e_1, e_2, E) \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} e_1^{a\mu} & e_1^{a\mu}E_{\mu\nu} \\ e_2^{a\mu} & -e_2^{a\mu}E_{\mu\nu} \end{pmatrix}$$
Here \((e_{1,2})^\mu_\nu\) are vielbeins for \(G_{\mu\nu}\), and \(e^{a\mu} = e^a_{\nu}G^{\nu\mu}\). We will denote the space of matrices of the form (2.19) by \(\mathcal{M}_\sigma\). A simple direct computation shows that \(\mathcal{M}_\sigma\) is contained within the space \(\mathcal{M}\) of (2.3). Therefore, by projection there is a well-defined map \(\psi : \mathcal{B} \to \mathcal{H}\).

The relation between all the spaces we have discussed may be summarized in the following diagram:

\[
\begin{array}{ccc}
\mathcal{B} & \xrightarrow{\psi} & \mathcal{H} \\
\uparrow \mathcal{M}_\sigma & & \downarrow \mathcal{L} \\
\downarrow \mathcal{N} & & \\
\end{array}
\] (2.20)

This formulation makes the action of duality on \(\mathcal{B}\) manifest. The CFT only depends on the choice of lattice, not of its basis, and lattices are parametrized by \(\mathcal{L} = \mathcal{M}/O(\hat{D}; \mathbb{Z})\). Therefore we study the right action of \(O(\hat{D}; \mathbb{Z})\) on \(\mathcal{H}\). We may attempt to find a right \(O(\hat{D}; \mathbb{Z})\) action on compatification data \(\mathcal{B}\) by setting:

\[
A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in O(\hat{D}; \mathbb{Z})
\]

\[
E(e_1, e_2, E) \cdot A = E(\tilde{e}_1, \tilde{e}_2, \tilde{E})
\] (2.21)

If \(\tilde{E}\) exists such that the (2.21) is satisfied then it is easily computed. Using (2.19) and equating the 11 and 12 blocks in (2.21) leads to the pair of equations:

\[
e_1G^{-1}(a + Ec) = \tilde{e}_1\tilde{G}^{-1}
\]

\[
e_1G^{-1}(b + Ed) = \tilde{e}_1\tilde{G}^{-1}\tilde{E}
\] (2.22)

Dividing the two equations we get:

\[
E \rightarrow \tilde{E} = (a + Ec)^{-1}(b + Ed)
\] (2.23)

From the abstract formulation we see that we are simply making a change of basis in \(\mathcal{H}_\Gamma\). Thus the Segal functor is manifestly invariant, that is, all correlators are duality covariant.

**Remark:** The full duality group was first identified in [6] [7] as \(O(d, d; \mathbb{Z})\). In these references emphasis is placed on the matrix \(R(E)\) appearing in (2.18). The matrix \(R(E)\) is related to \(E\) by

\[
R(E) = E(e_1, e_2, E)^t \cdot \begin{pmatrix} \eta & 0 \\ 0 & \eta \end{pmatrix} \cdot E(e_1, e_2, E) \in O(\hat{D}; \mathbb{R})
\] (2.24)

If (2.21) holds then \(R(E) \rightarrow A^t R(E) A\). We find it easier to work with the “squareroot” \(E(e_1, e_2, E)\).
Of course (2.23) will only make sense if \((a + Ec)\) is invertible. In the case of Euclidean signature compactification this is true since, as we will now show, \(\psi\) is a diffeomorphism and hence every matrix \(E \in \mathcal{M}\) is of the form \(E(e_1, e_2, E)\).

We first need a technical result which will be used several times below.

**Proposition 1:** The Iwasawa decomposition

\[
O(\tilde{D}; \mathbb{R}) = K \cdot A \cdot N
\]

into compact, abelian and nilpotent subgroups is given by

\[
g = \frac{1}{2} \begin{pmatrix} R_1 + R_2 & R_1 - R_2 \\ R_1 - R_2 & R_1 + R_2 \end{pmatrix} \cdot \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} \cdot \begin{pmatrix} N & NB \\ 0 & (N^{tr})^{-1} \end{pmatrix}
\]

(2.25)

where \(R_i \in O(n+1)\), \(A\) is diagonal with positive entries, \(N\) is upper triangular with 1 on the diagonal, and \(B\) is an arbitrary \((n+1) \times (n+1)\) antisymmetric real matrix.

**Proof:** The embedding of the maximal compact subgroup is obtained by conjugating \(\begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix} \in O(D)\) by \(SE\). The form of the abelian and nilpotent groups are dictated by the condition for upper block-triangular matrices to lie in \(O(\tilde{D}; \mathbb{R})\). The rest follows from counting dimensions. ♠

**Proposition 2:** In the case of Euclidean compactifications \(\psi : \mathcal{B} \rightarrow \mathbb{H}\) is a diffeomorphism.

**Proof:** We apply the Iwasawa decomposition above. Using \(\mathcal{M} = S \cdot O(\tilde{D}; \mathbb{R})\) it follows from Proposition 1 that any matrix \(E \in \mathcal{M}\) can be written uniquely as

\[
\mathcal{E} = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix} \cdot S \cdot \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} \cdot \begin{pmatrix} N & NB \\ 0 & (N^{tr})^{-1} \end{pmatrix}
\]

(2.26)

where \(R_i, A, N, \text{ and } B\) are as in Proposition 1. Using the decomposition (2.26) we may define a map \(\pi : \mathcal{M} \rightarrow \mathcal{B}\) by associating to \(\mathcal{E} \in \mathcal{M}\) the compactification data:

\[
\pi(\mathcal{E}) = E = N^{-1}A^{-2}(N^{t})^{-1} + B.
\]

(2.27)

One easily checks that \(\pi(\mathcal{E}(e_1, e_2, E)) = E\). Since the Iwasawa decomposition describes a diffeomorphism of \(O(\tilde{D}; \mathbb{R})\) with \(KAN\) it follows that \(\mathcal{B}\) is diffeomorphic to \(O(n + 1) \times O(n + 1) \backslash O(n + 1, n + 1; \mathbb{R})\). ♠
2.5. New features of timelike compactification

There are two radically new features of toroidal compactification of timelike coordinates. First, in the Minkowskian case there is no analogous Iwasawa decomposition for arbitrary $O(D; \mathbb{R})$ matrices. This leads to

**Proposition 3.** The space of $\sigma$-model data $\mathcal{B}_{1,n}$ maps to a proper (open) subset of $\mathbb{H}$.

**Proof.** Use topology. In the Minkowskian case we have

$$\mathcal{B}_{1,n} = \{ E | E = G + B, \text{signature}(G) = (-1,+1^n) \} \quad (2.28)$$

and $\mathcal{B}$ is a vector bundle over the space of constant Minkowskian metrics $GL(n+1, \mathbb{R})/O(1,n; \mathbb{R})$. By a theorem of Mostow \cite{Mostow} we know that $GL(n+1, \mathbb{R})$ and $O(1,n; \mathbb{R})$ can be compatibly decomposed as $K \cdot E$ where $K$ is a maximal compact subgroup and $E$ is a Euclidean space, so that $GL(n+1, \mathbb{R})/O(1,n; \mathbb{R}) \cong O(n+1) \times (O(1) \times O(n)) F$ where $F$ is an $n^2 + n + 1$ dimensional representation of $O(1) \times O(n)$. Thus $\mathcal{B}_{1,n}$ deformation retracts to $\mathbb{R}P^n$.

Now recall that

$$\mathbb{H} \equiv (O(\eta) \times O(\eta)) \backslash \mathcal{M} \quad . \quad (2.29)$$

Again applying Mostow’s theorem we see that $\mathbb{H}$ is topologically a vector bundle over $\mathbb{R}P^n \times \mathbb{R}P^n$. Therefore $\mathcal{B}_{1,n}$ is not diffeomorphic to $\mathbb{H}$. Since $\psi$ is locally $1-1$ it’s image must be a proper open subset of $\mathbb{H}$. ♠

In fact, it is easy to give a criterion for when $\mathcal{E} \in \mathcal{M}$ lies in $\mathcal{M}_\sigma$:

**Proposition 4.** $\mathcal{E} \in \mathcal{M}_\sigma$ iff $\mathcal{E}_{11}$ is invertible, where $\mathcal{E}_{11}$ is defined by the block decomposition:

$$\mathcal{E} = \begin{pmatrix} \mathcal{E}_{11} & \mathcal{E}_{12} \\ \mathcal{E}_{21} & \mathcal{E}_{22} \end{pmatrix} \quad (2.30)$$

In this case: $E = \mathcal{E}_{11}^{-1}\mathcal{E}_{12}$.

**Proof:** Writing the defining equations for $\mathcal{E} \in \mathcal{M}$ gives several conditions on the blocks $\mathcal{E}_{ij}$. One then defines candidate vielbeins from $e_1 = \frac{1}{\sqrt{2}} \eta(\mathcal{E}_{11}^{tr})^{-1}$, $e_2 = \frac{1}{\sqrt{2}} \eta(\mathcal{E}_{21}^{tr})^{-1}$. (The invertibility of $\mathcal{E}_{21}$ follows from that of $\mathcal{E}_{11}$.) The conditions that these are vielbeins for the same metric, and that $\mathcal{E} = \mathcal{E}(e_1, e_2, E)$ follows from the conditions on $\mathcal{E}_{ij}$. ♠

One striking consequence of Proposition 3 is that the duality group $O(D; \mathbb{Z})$ does not act on $\mathcal{B}$. The reason is that when $\mathcal{E}_{11}$ is not invertible, the images of $\mathcal{E}$ under $O(D; \mathbb{Z})$

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\footnote{Conversations with G.D. Mostow and G. Zuckerman were instrumental in arriving at the results in this section.}
will typically have $(E \cdot A)_{11}$ invertible. This observation also answers in the affirmative a question raised by Proposition 3, namely, whether the projection $B \to N$ is actually onto.

**Example:** Take

$$G = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$$

Then $E \to E^{-1}$ does not make sense. We study 2D compactifications more thoroughly in the next section.

The second new feature of timelike compactification is even stranger. The action of a noncompact semisimple Lie group on a space of the form $G/\Gamma$ for $\Gamma$ an arithmetic discrete group is typically *ergodic*. More precisely, if $\Gamma \subset G$ is a *lattice*, i.e., a discrete subgroup such that $\mu(G/\Gamma) < \infty$, where $\mu$ is the Haar measure, then the action of a noncompact subgroup $H \subset G$ is ergodic, i.e., the only $H$-invariant subsets are of measure zero or total measure. For reviews of this branch of mathematics see, e.g. [9] [10]. In the present situation $O(\tilde{D}; \mathbb{Z})$ is a lattice in $O(\tilde{D}; \mathbb{R})$. The left action of $O(\eta) \times O(\eta)$, indeed, of any subgroup containing boosts of arbitrarily large rapidity, on $\mathbb{L}$ and the right-action of the duality group $O(\tilde{D}; \mathbb{Z})$ on $\mathbb{H}$ are ergodic. In fact, as a corollary of some recent powerful theorems of M. Ratner we may say much more: [3]

**Proposition 5:** Let $O_\Gamma$ be an $O(\eta) \times O(\eta)$ orbit of $\Gamma \in \mathbb{L}$, and let us consider its closure $\bar{O}_\Gamma$ in $\mathbb{L}$. This set is either

a.) a closed $O(\eta) \times O(\eta)$ orbit

or

b.) $\mathbb{L}$. That is, the orbit $O_\Gamma$ is dense in $\mathbb{L}$.

Moreover, for almost every $\Gamma \in \mathbb{L}$ possibility (b) is realized.

**Proof:** Conjecture 2 of [10] proved as Corollary B of Ratner’s paper [11] says that the closure of the orbit of any unipotent group in $\mathbb{L}$ is the orbit of some connected group $H \subset O(\tilde{D}; \mathbb{R})$. Moreover, this is true for any group generated by unipotents.

In our case, $O(\eta)$ (and hence $O(\eta) \times O(\eta)$) is generated by unipotent flows. For example, for $O(\eta)$ we can choose lightcone directions $\pm$, and transverse directions $j = 1, \ldots, n - 2$. The Lorentz generators $M_{\pm j}$ have zero eigenvalues under the adjoint action and hence exponentiate to give unipotent flows. On the other hand, their commutators generate the full Lorentz group.

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3 This was explained to me by G.D. Mostow.
Applying Ratner’s theorem we conclude that $\bar{O}_\Gamma$ is the orbit of some closed subgroup $O(\eta) \times O(\eta) \subset H \subset O(\check{\mathcal{D}}; \mathbb{R})$. Again, by studying the adjoint representation we see that if there are any “off-diagonal” generators of $H$ not in $O(\eta) \times O(\eta)$, then commutators with $O(\eta) \times O(\eta)$ will generate the whole group. Thus $H = O(\eta) \times O(\eta)$ or $H = O(\check{\mathcal{D}}; \mathbb{R})$, yielding possibilities $a$ and $b$ above. That possibility $(b)$ is almost always realized follows from Theorem 11.1 of [12]. ♠

Remarks:

1. We will exhibit the ergodic action quite explicitly for the case of compactification of $1 + 1$ dimensions in the next section.

2. The statements in Proposition 5 for the orbit structure have analogs for the action of the arithmetic group $O(\check{\mathcal{D}}; \mathbb{Z})$ on $\mathbb{H}$.

3. It follows from the above that the space $\mathcal{N}$ does not exist as a reasonable space. This result contradicts some folklore. Several developments might restore harmony: (1) It might be that $\mathcal{N}$ is embedded in some larger space of theories, and it is just a pathological subset. (2) It might be that the theories under discussion are pathological, and should not really be considered as CFT’s. (See the discussion of loop amplitudes in section 8.4.) Perhaps when time enters nontrivially the moduli space of string theories is typically smooth but closed timelike loops are a source of pathology. (3) It might be that we have uncovered an important aspect of the space of time-dependent string backgrounds, and that we will be forced to think about noncommutative geometry on $\mathcal{N}$. Time will tell.

2.6. Example: Compactification of $1 + 1$ Dimensions

It is quite well-known in Euclidean compactification that one can be very explicit about two-dimensional targets by use of the isomorphism of Lie algebras $so(2, 2) \cong sl(2) \times sl(2)$ [13] [4]. In this subsection we apply the isomorphism to our case. We work out both Euclidean and Minkowskian cases because the comparison is interesting.

*Group isomorphism.*

Set

$$
\delta_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \delta_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},
$$

$$
\delta_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \delta_4 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix},
$$

so that

$$
2 \det(x^\mu \delta_\mu) = x^\mu \check{D}_{\mu\nu} x^\nu
$$
Therefore we may define a homomorphism

\[
\psi : SL(2, \mathbb{R}) \times SL(2, \mathbb{R}) \to O(D; \mathbb{R})
\]

\[
\psi(A, B)_{\mu}^{\nu} = \frac{1}{2} \text{tr}(A \delta_{\nu} B^{tr} \delta^{tr}_{\mu})
\]

(2.31)

The group \(O(\mathring{D}; \mathbb{R})\) has four components and may be written as a disjoint union:

\[
O(\mathring{D}; \mathbb{R}) = O_0(\mathring{D}; \mathbb{R}) \amalg (O_0(\mathring{D}; \mathbb{R}) \cdot P_L) \amalg (O_0(\mathring{D}; \mathbb{R}) \cdot T_L) \amalg (O_0(\mathring{D}; \mathbb{R}) \cdot P_L T_L)
\]

(2.32)

The kernel of \(\psi\) is \(\mathbb{Z}_2\), generated by \((-1, -1)\), and the image is the connected component of the identity: \(O_0(\mathring{D}; \mathbb{R})\), so it defines an isomorphism:

\[
\frac{SL(2, \mathbb{R}) \times SL(2, \mathbb{R})}{\mathbb{Z}_2} \cong O_0(\mathring{D}; \mathbb{R})
\]

A very useful property of the homomorphism \(\psi\) is that it is compatible with Iwasawa decompositions. That is, decomposing \(SL(2, \mathbb{R})\) as

\[
SL(2, \mathbb{R}) = KAN
\]

where \(K = SO(2)\), \(A\) is diagonal, and \(N\) is unipotent and upper triangular, the images give the KAN subspaces in \(O(\mathring{D}; \mathbb{R})\) described in Proposition 1.

**Duality group.**

Since \(P_L, T_L \in O(\mathring{D}; \mathbb{Z})\), with \(P_L^2 = T_L^2 = 1\), it follows that \(O_0(\mathring{D}; \mathbb{Z})\) is an index 4 subgroup of \(O(\mathring{D}; \mathbb{Z})\). Using an explicit set of generators \([14]\) one may examine the inverse images to conclude that \(\psi\) defines an isomorphism

\[
\frac{SL(2, \mathbb{Z}) \times SL(2, \mathbb{Z})}{\mathbb{Z}_2} \cong O_0(D; \mathbb{Z})
\]

(2.33)

The full duality group is then given by

\[
1 \to \mathbb{Z}_2 \times \mathbb{Z}_2 \to O(\mathring{D}; \mathbb{Z}) \to \frac{SL(2, \mathbb{Z}) \times SL(2, \mathbb{Z})}{\mathbb{Z}_2} \to 1
\]

(2.34)
The group extension is defined by the action of $T_L, P_L$ as outer automorphisms:

$$T_L : (A, A') \rightarrow (JA'J, JAJ)$$
$$P_L : (A, A') \rightarrow (A', A)$$

(2.35)

$I \mathbb{H} : Euclidean$ Compactification

The distinction between Euclidean and Minkowskian compactifications first enters at this point. We identify the space of Euclidean generator matrices by

$$\mathcal{M} = S_E \cdot O(\bar{D}; \mathbb{R})$$

$S_E$ conjugates $O(2) \times O(2)$ to the maximal compact subgroup of $O(\bar{D}; \mathbb{R})$. Moreover $K \backslash O(\bar{D}; \mathbb{R}) \cong K_0 \backslash O(\bar{D}; \mathbb{R})$. By the compatibility of Iwasawa decompositions we find that

$$\mathbb{H} \equiv (O(2) \times O(2)) \backslash \mathcal{M} \cong (SO(2) \backslash SL(2, \mathbb{R})) \times (SO(2) \backslash SL(2, \mathbb{R}))$$

(2.36)

may be identified with the product of upper half planes.

$I \mathbb{H} : Minkowskian$ Compactification

Now we have

$$\mathcal{M} = S_M \cdot O(\bar{D}; \mathbb{R})$$

The left and right Lorentz groups $O(\eta) \times O(\eta)$ do not embed into the diagonal subgroup $A$, but rather correspond (in $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$) to the conjugate group

$$SO(1, 1) \times SO(1, 1) \cong sAs^{-1} \times s^{-1}As$$

$$s = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

In this way we identify

$$\mathbb{H} \cong ((\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)) \ltimes SO(1, 1) \times SO(1, 1) \backslash SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$$

(2.37)

where the last $\mathbb{Z}_2$ is $(-1, -1)$ and the first two are generated by

$$\left( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right) \quad \left( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right)$$

(2.38)

These are present because, of the 16 components of the left and right Lorentz groups only 4 are killed by passing to the connected component.
Interpretation of $\mathbb{H}_{Mink}$

It will be useful below to have a clear geometrical interpretation of $\mathbb{H}_{Mink}$.

Conformal equivalence classes of flat Lorentzian tori in $1 + 1$ dimensions may be specified by constant metrics, which may in turn be obtained from the standard metric in $1 + 1$ dimensions by compactifying on a lattice with basis $e^1, e^2$. Letting $e$ denote the matrix of zweibeins we obtain the Lorentzian metric

$$G = e^{tr} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} e$$

on the torus $(x \mod 1, t \mod 1)$. Consequently “Teichmüller space” for flat Lorentzian signature tori may be regarded as the homogeneous space $\mathbb{Z}_2 \times SO(1, 1) \backslash SL(2, \mathbb{R})$. Consider a different Iwasawa decomposition:

$$g = \begin{pmatrix} \cosh \beta & \sinh \beta \\ \sinh \beta & \cosh \beta \end{pmatrix} \begin{pmatrix} 1 + n/2 & -n/2 \\ n/2 & 1 - n/2 \end{pmatrix} \begin{pmatrix} \cos \omega & \sin \omega \\ -\sin \omega & \cos \omega \end{pmatrix}$$

(2.39)

where $n \in \mathbb{R}$. Using (2.39) we identify the homogeneous space $\mathbb{Z}_2 \times SO(1, 1) \backslash SL(2, \mathbb{R})$ as $\mathbb{R} \times S^1$. Thus, $\mathbb{H}_{mink}$ is a product of cylinders with a $\mathbb{Z}_2$ identification. $\mathbb{H}_{Mink}$ can be interpreted as the Teichmüller space of two flat Lorentzian tori, which are separately unoriented, but carry a relative orientation.

We can relate this discussion to the standard discussion of Teichmüller space for tori by introducing two real “modular parameters”:

$$\tau_+ = - \left( \frac{G_{12}}{G_{11}} + \sqrt{|\det G|} \right) = \frac{e_2^2 - e_1^2}{e_1^2 - e_2^2} = e^{-1} \cdot 1$$

$$\tau_- = - \left( \frac{G_{12}}{G_{11}} - \sqrt{|\det G|} \right) = - \frac{e_2^2 + e_1^2}{e_1^2 + e_2^2} = e^{-1} \cdot (-1)$$

(2.40)

In the last equalities we regard $e \in SL(2, \mathbb{R})$ acting on the real line by Möbius transformations. It is manifest from this formula that global diffeomorphisms act on $\tau_\pm$ by integral Möbius transformations.

The modular parameters are not always good coordinates on Teichmüller space. For example, one of $\tau_\pm$ will be infinite when one of the basis vectors becomes a null vector. We may remedy this problem by compactifying the real line to $\mathbb{R} = \mathbb{R} \cup \infty$. A second problem is that the diagonal $\tau_+ = \tau_-$ leads to degenerate tori where both basis vectors are parallel null vectors. By adding such degenerate tori we can make the Teichmüller space itself a torus $S^1 \times S^1$. Excising the circle $\tau_+ = \tau_-$ of degenerate tori we restore the cylinder $\mathbb{R} \times S^1$. 

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Relation to $\mathcal{B}$: Euclidean Case

We now construct the compactification data associated to an element of $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ by setting

$$\mathcal{E} = S_E \psi(A, A')$$

and then calculating $E = \mathcal{E}_{11}^{-1} \mathcal{E}_{12}$ in accord with Proposition 4. We find:

$$E = G + B$$

$$B = \frac{ab + cd}{a^2 + c^2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$G = \frac{1}{a^2 + c^2} \begin{pmatrix} (b')^2 + (d')^2 & -a'b' - c'd' \\ -a'b' - c'd' & (a')^2 + (c')^2 \end{pmatrix}$$

From which we may form:

$$\rho = B_{12} + i \sqrt{\det G} = \frac{di - b}{-ci + a} = A^{-1} \cdot i$$

$$\tau = \frac{G_{12}}{G_{22}} + i \sqrt{\det G} = \frac{d'i - b'}{-c'i + a'} = (A')^{-1} \cdot i$$

which parametrize the upper half planes.

Relation to $\mathcal{B}$: Minkowskian Case

In an entirely similar way we set

$$\mathcal{E} = S_M \psi(A, A')$$

and calculate $E$. We find

$$E = G + B$$

$$B = \frac{ab - cd}{a^2 - c^2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$G = -\frac{1}{a^2 - c^2} \begin{pmatrix} -(b')^2 + (d')^2 & a'b' - c'd' \\ a'b' - c'd' & -(a')^2 + (c')^2 \end{pmatrix}$$

We may now explicitly verify Proposition 3 since the formulae in (2.41) only make sense when $a^2 - c^2 \neq 0$. Indeed,

$$\det \mathcal{E}_{11} = -\frac{1}{2} (c^2 - a^2)$$

for the component of the identity, and similarly for other components. Excising the submanifolds $a^2 - c^2 = 0$, we see that the projection of $\mathcal{M}_\sigma$ to $\mathbb{H}$ has four components, depending on the signs of $a \pm c$. 

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Forming modular parameters as before we now have
\[
\rho_\mp \equiv B_{21} \mp \sqrt{|\det G|}
\]
\[
\tau_\mp \equiv \frac{G_{12}}{G_{22}} \mp \sqrt{|\det G|}
\]
(2.45)

In terms of the \(SL(2, \mathbb{R})\) parameters we have:
\[
\rho_- = \frac{-b + d}{c + a} = A^{-1}(-1)
\]
\[
\rho_+ = \frac{d - b}{a - c} = A^{-1}(+1)
\]
\[
\tau_- = (A')^{-1}(-1)
\]
\[
\tau_+ = (A')^{-1}(+1)
\]
(2.46)

for \(a^2 > c^2\). For \(a^2 < c^2\) we exchange \(\rho_- \leftrightarrow \rho_+\) and \(\tau_- \leftrightarrow \tau_+\).

These are modular parameters for a pair of flat spacetime Lorentzian tori, in accord with our interpretation of \(\mathbb{H}_{\text{Mink}}\). The tori which are missed in \(\mathcal{M} - \mathcal{M}_\sigma\) are those for which the dual torus, parametrized by \(\rho_\pm\) has a null basis vector.

**Action of duality: Euclidean case**

From the expression in (2.42) in terms of Mobius transforms of \(i\) we see that the right action of the \(SL(2, \mathbb{Z}) \times SL(2, \mathbb{Z})\) subgroup of the duality group is manifestly the Mobius action on the upper half planes. Using (2.35) we see that the remaining generators act by \(T_L : (\rho, \tau) \to (1/\tau, 1/\rho)\) and \(P_L : (\rho, \tau) \to (\tau, \rho)\). If \(T_L\) is composed with other duality generators it becomes simply a spacetime parity transformation.

**Action of duality: Minkowskian case**

Similarly, using the representation (2.43) we see that a duality transformation \((A, A') \in SL(2, \mathbb{Z}) \times SL(2, \mathbb{Z})\) acts on \((\rho_\pm, \tau_\pm)\) by Mobius transformations:
\[
(\rho_\pm, \tau_\pm) \to (A^{-1} \rho_\pm, (A')^{-1} \tau_\pm)
\]
(2.47)
as long as the image lies in the sigma-model set. As discussed above, we should compactify the \(\rho_\pm, \tau_\pm\) real lines and also admit degenerate tori. We explicitly verify that \(\mathcal{B} \to \mathcal{N}\) is onto since one can always transform infinity to a finite point.

The generators of the two remaining \(\mathbb{Z}_2\)'s act as follows. The transform of \((\rho_-, \rho_+, \tau_-, \tau_+)\) is
\[
T_L : (1/\tau_-, 1/\tau_+, 1/\rho_-, 1/\rho_+) \quad \text{if} \quad a^2 > c^2, (d')^2 > (b')^2
\]
\[
P_L : (\tau_-, \tau_+, \rho_-, \rho_+) \quad \text{if} \quad a^2 > c^2, (a')^2 > (c')^2
\]
(2.48)
The other cases are similar.

**Ergodic duality**

We can now show that duality acts ergodically by following an analysis of E. Artin [15], who showed that the generic geodesic on the modular curve $SO(2) \backslash SL(2, \mathbb{R}) / SL(2, \mathbb{Z})$ is dense. 

It suffices to consider the action of $PGL(2, \mathbb{Z})$ on $(\rho_-, \rho_+)$. WLOG, take $(\rho_-, \rho_+)$ to lie in the domain $1 < \rho_+, -1 < \rho_- < 0$. For $\rho_\pm$ in this domain consider the continued fraction expansions:

$$\rho_+ = \langle a_0, a_1, a_2, \ldots \rangle \equiv a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \ldots}}$$

$$-\rho_- = \langle 0, a_{-1}, a_{-2}, \ldots \rangle \equiv \frac{1}{a_{-1} + \frac{1}{a_{-2} + \ldots}}$$

Together these expansions define a bi-infinite series of positive integers

$$S = \cdots a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots .$$

If we “cut” the series in two at some point $a_n$ we can define two new continued fractions. Set

$$\rho_+^{(n)} = \langle a_n, a_{n+1}, a_{n+2}, \ldots \rangle$$

$$-\rho_-^{(n)} = \langle 0, a_{n-1}, a_{n-2}, \ldots \rangle$$

**Theorem** (Artin [15]).

a.) For all $n \in \mathbb{Z}$ the pair $(\rho_-^{(n)}, \rho_+^{(n)})$ are $PGL(2, \mathbb{Z})$ transforms of $(\rho_-, \rho_+)$.  

b.) All $PGL(2, \mathbb{Z})$ transforms which lie in the domain $1 < \rho_+, -1 < \rho_- < 0$ are given by (2.50).

Statement (a) is a straightforward consequence of the usual recursion relations for continued fractions [10].

Therefore, for almost all compactification data, the $SL(2, \mathbb{Z})$ orbit will be dense. Indeed, if we wish to approximate any pair

$$(\bar{\rho}_+, \bar{\rho}_-) = (\langle c_0, c_1, c_2, \ldots \rangle, \langle b_1, b_2, \ldots \rangle)$$

we can look deep in the sequence $S$. For generic sequences, and for any $m$, we can find $n$ large enough so that $a_{n-m}, \ldots, a_{n+m}$ corresponds to $b_m, b_{m-1}, \ldots b_1, c_0, c_1, \ldots c_{m-1}, c_m$.

\footnote{See the geometrical interpretation below.}
In this case the image \((\rho_+^{(n)}, \rho_-^{(n)})\) approximates \((\tilde{\rho}_+, \tilde{\rho}_-)\) and the approximation can be made arbitrarily good by increasing \(m\).

If \(\mathcal{S}\) is not generic then it will have repeating subsequences if we look deep enough. In particular, if \(a_j\) eventually becomes zero for \(|j|\) large and \(\rho_\pm\) are rational then the orbit is not dense. Correspondingly, the \(O(1, 1) \times O(1, 1)\) orbit in \(\mathbb{L}\) is closed. This will be important later.

**Geometrical interpretation**

There is a lovely geometrical picture for the ergodic action we are describing. The double coset \(SO(2) \setminus SL(2, \mathbb{R}) / SL(2, \mathbb{Z})\) is a Riemann surface \(\Sigma\). It is a punctured sphere with three orbifold points. We may regard \(SL(2, \mathbb{R}) / SL(2, \mathbb{Z})\) as the unit tangent bundle \(T_1 \Sigma\) in the natural constant curvature metric. The left action of the subgroup

\[
\begin{pmatrix}
\cosh t & \sinh t \\
\sinh t & \cosh t
\end{pmatrix}
\]

on \(SL(2, \mathbb{R}) / SL(2, \mathbb{Z})\) induces geodesic flow on \(T_1 \Sigma\). The projection of the geodesics on \(\Sigma\) are obtained from the upper half plane \(SO(2) \setminus SL(2, \mathbb{R})\) by projecting semicircles perpendicular to the real axis, as in

---

**Fig. 1:** A geodesic on the modular curve is obtained by projecting a semicircle from the upper half plane. The geodesic is specified by two real numbers \(\rho_\pm\) up to simultaneous \(SL(2, \mathbb{Z})\) transformation.

---

\(^5\) See [17] for a recent review.
To obtain the complete geodesic we must look at all $SL(2, \mathbb{Z})$ transforms of the semicircle. The connection between geodesics and a pair of real numbers, $\rho_{\pm}$ is established by interpreting $\rho_{\pm}$ as the basepoints of a semicircle in the upper half plane. The significance of the domain $1 < \rho_{+}, -1 < \rho_{-} < 0$ is that for these circles there is a nonzero intersection with the standard fundamental domain. Artin’s analysis shows that almost all geodesics are dense and characterizes those which are not. Note, for example, that the geodesics with rational basepoints are closed non-dense orbits. Other geodesics whose basepoints correspond to special continued fraction expansions will also fail to be dense.

2.7. Decompactifying

Another major difference in the Euclidean and Minkowskian cases is how one describes decompactification.

Although $\mathbb{L}$ has finite measure it is a noncompact. Indeed, the boundary components at infinity describe the ways in which a Euclidean spacetime torus can be decompactified. There exists a beautiful mathematical theory which describes the structure of arithmetic quotients, such as $\mathbb{L}$, at infinity. In this section we recall some of that theory and apply it to our case. A useful reference for this subject is [18].

The main idea is easily explained using the arithmetic quotient $T_1 \Sigma = SL(2, \mathbb{R})/SL(2, \mathbb{Z})$ introduced above. Introduce an Iwasawa decomposition:

$$g = kan = k \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix}$$

so that, with the right action on the upper half-plane $SO(2) \backslash SL(2, \mathbb{R})$ we identify

$$z = g^{-1}i = x + i\lambda^{-2}. \quad (2.51)$$

Note that although the fundamental domain for the action of $SL(2, \mathbb{Z})$ on $SO(2) \backslash SL(2, \mathbb{R})$ is relatively complicated, we can easily describe an approximate fundamental domain in terms of “Siegel sets.” Let: $A_t = \{a \in A : \lambda^2 < t\}$ and $N_t = \{n \in N : |x| < t\}$. Then $\Sigma_{t, \frac{1}{2}} = K \cdot A_t \cdot N_{\frac{1}{2}}$ defines an approximate fundamental domain for $t \geq \frac{\sqrt{3}}{2}$. More formally, $SL(2, \mathbb{R}) = \Sigma_{t, \frac{1}{2}} \cdot \Gamma$, where $\Gamma = SL(2, \mathbb{Z})$.

---

6 I am indebted to Howard Garland for explaining compactification of arithmetic quotients to me.
We can define what it means to go to infinity using the Siegel sets. Clearly, from (2.51) we want to let \( \lambda^2 \to 0 \). Thus, a neighborhood of infinity is defined by \( \Sigma_{t, \frac{1}{2}} \) for \( t \) small.

The generalization of this idea to \( \mathbb{L} = O(\tilde{D}; \mathbb{R})/O(\tilde{D}; \mathbb{Z}) \) is the following. Returning to the Iwasawa decomposition of Proposition 1 we define a root system using the characters of the maximal abelian subgroup. If \( A = \text{Diag}\{\lambda_1, \ldots, \lambda_n\} \) then a set of simple roots is given by

\[
\alpha_i = \frac{\lambda_i}{\lambda_{i+1}}, \quad 1 \leq i \leq n - 1
\]

\[
\alpha_n = \lambda_{n-1}\lambda_n
\]

We first define an approximate fundamental domain. The Siegel sets are defined by:

\[
\Sigma_{t, \omega} = KA_tN_\omega
\]

where

\[
A_t = \{a \in A : \alpha_i(a) < t, \quad i = 1, \ldots n\}
\]

\[
N_\omega = \left\{g = \begin{pmatrix} N & NB \\ 0 & (Ntr)^{-1} \end{pmatrix} : |n_{ij}| \leq \frac{1}{2}, |B_{ij}| \leq \frac{1}{2}\right\}
\]

and \( N_\omega \) is a compact fundamental region for \( N/N \cap \Gamma \), where \( \Gamma \) is the arithmetic group (= \( O(\tilde{D}; \mathbb{Z}) \) in our example). For example, we can, and will, define

Proposition 6: If \( t \geq \sqrt{\frac{3}{2}} \) then for \( G = O(\tilde{D}; \mathbb{R}) \)

\[
G = KA_tN_\omega \cdot \Gamma
\]

Proof: This is a straightforward generalization of the discussion on pp. 13-15 of [18]. We refer to [18] for notation. One defines \( \Phi(g) \) in the same way with respect to the standard Euclidean norm on \( \mathbb{R}^{2n} \). \( K \) is orthogonal with respect to this norm, and Lemma 1.5 of [18] goes through unaltered. Likewise the inductive proof of 1.6 of [18] needs only two small alterations. The first part of the inductive step may be established by the explicit description of two-dimensional compactification of the previous subsection. Second, to embed \( O(\tilde{D}_{n-1}; \mathbb{R}) \) into \( O(\tilde{D}_n; \mathbb{R}) \) we conjugate with the permutation taking

\[
x_2 \to x_3 \to \cdots \to x_n \to x_{n+1} \to x_2
\]

and use the obvious block diagonal embedding. ⊠
We may apply this to Euclidean compactifications using (2.27) to show that there is a simultaneous lower bound on all the “radii” in Euclidean to roidal compactifications:

**Proposition 7**: The set of compactification data

\[ E = N^{-1} \text{Diag}\{R_1^2, \ldots R_n^2\}(N^t)^{-1} + B \]

\[ R_i \geq t^{-1}R_{i+1} \quad 1 \leq i \leq n-1 \]

\[ R_{n-1}R_n \geq t^{-1}, \quad R_n \geq 1 \]

\[ |n_{ij}| \leq \frac{1}{7}, \quad |B_{ij}| \leq \frac{1}{7} \quad (2.55) \]

where \( N \) is upper triangular and \( t \geq \sqrt{\frac{3}{2}} \) contains a fundamental domain for the action of the duality group on \( B \).

**Proof**: All the inequalities in (2.55) follow by combining Proposition 6 and (2.27), except for \( R_n \geq 1 \). To establish this, note that the fundamental domain described by \( \Sigma_{t,\omega} \) is invariant under the permutation of simple roots \( \alpha_n \leftrightarrow \alpha_{n-1} \). This is just an element of the Weyl group corresponding to the exchange \( x_n \leftrightarrow x_{2n} \) and right multiplication by it preserves the \( KAN \) decomposition. Since it takes \( R_n \leftrightarrow R_{n-1} \) we may fix it with \( R_n \geq 1 \).

\[ \blacksquare \]

**Remark**: Simultaneously bounding the radii in toroidal compactification is not a trivial generalization of the one-dimensional case. This appears to have been first noted by E. Silverstein [19]. She correctly noted that one could either bound all the radii or transform the volume to infinity by duality transformations. Proposition 7 settles the matter in favor of the first case.

Returning to the description of \( \mathbb{I} \) at infinity, we characterize these regions in terms of the eigenvalues \( R_1^2, \ldots R_n^2 \) of the metric \( G \). Mathematically, “regions of infinity” in \( \mathbb{I} \) are described by degenerations of sets of simple roots [20]. That is, a region of infinity is defined by specifying a subset \( \theta \) of simple roots and requiring \( \alpha_i < \epsilon \) for \( \alpha_i \in \theta \). Of course, the regions at infinity are connected in complicated ways. The maximal regions of infinity are defined by letting one root approach zero holding all others fixed. One may easily check that letting the spinor root \( \alpha_n = \lambda_{n-1}\lambda_n \) approach zero holding all other roots fixed is total decompactification, while the analogous sets for other roots corresponds to partial decompactification. Thus, in the Euclidean case, the “regions at infinity” are in complete accord with our common sense.

Although it is difficult to find a precise fundamental domain for the duality group, the problem becomes much more tractable near infinity. Roughly speaking, the theorem
of parabolic transformations [21] characterizes those duality transformations that are left unfixed at infinity. More precisely, \( \{ \gamma : \Sigma_{t,\omega} \cdot \gamma \cap \Sigma_{t,\omega} \neq \emptyset \} \) is finite and lies in a certain “maximal parabolic subgroup.” For the case of the degeneration by the spinor root (total decompactification) the maximal parabolic is easily found to be:

\[
P_{\Delta-{\alpha_n}} = \{ g : g = \begin{pmatrix} (A^{tr})^{-1} & 0 \\ 0 & A \end{pmatrix} \cdot \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} : A \in GL(n, \mathbb{Z}), B_{ij} \in \mathbb{Z} \} \quad (2.56)
\]

The situation is far more complicated in the Minkowskian case because we do not have the Iwasawa decomposition. No general theory exists. We will merely illustrate a few points with an example.

**Example : Decompacting two dimensions**

In both the Euclidean and Minkowskian cases we begin with the space:

\[
\mathbb{I}L = \left[ SL(2, \mathbb{R})/SL(2, \mathbb{Z}) \times SL(2, \mathbb{R})/SL(2, \mathbb{Z}) \right] / (\mathbb{Z}_2 \times \mathbb{Z}_2) \quad (2.57)
\]

In the Euclidean case compactifications are represented by \( SO(2) \times SO(2) \) orbits. The two ways to approach infinity are

\[
\alpha_1 \to 0 : R_1/R_2 \to \infty, (R_1 R_2) < \infty \\
\alpha_2 \to 0 : R_1/R_2 < \infty, (R_1 R_2) \to \infty
\]

One of the \( \mathbb{Z}_2 \) subgroups in \((2.57)\) corresponds to the action of the Weyl group and identifies these boundaries. Since the homomorphism \( \psi \) preserves Iwasawa decompositions the two ways to go to infinity correspond to \( \text{Im} \rho \to \infty, \tau \) fixed, and \( \text{Im} \tau \to \infty, \rho \) fixed. The Weyl group action is the exchange \( \rho \leftrightarrow \tau \).

In the Minkowskian case. Compactifications are represented by \( SO(1,1) \times SO(1,1) \) orbits in \((2.57)\). These in turn may be thought of as pairs of geodesics in \( T_1 \Sigma \times T_1 \Sigma \), up to \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) identifications. The orbits extend to infinity in \( \mathbb{I}L \) even for points which are - in some sense - far from the uncompactified limit. For example, the special compactification

\[
\Gamma_* = (II^{1,1}; 0) \oplus (0; II^{1,1})
\]

corresponds to the pair of geodesics specified by

\[
(\rho_-, \rho_+, \tau_-, \tau_+) = (\infty, 0, \infty, 0)
\]
which projects to the image of the imaginary axis in each modular curve. On the other hand, the compactification data

\[ E = \begin{pmatrix} -R_0^2 & 0 \\ 0 & R_1^2 \end{pmatrix} \]

corresponds to the pair of geodesics with basepoints

\[(\rho_-, \rho_+, \tau_, \tau_+) = (-R_0/R_1, +R_0/R_1, -R_0R_1, +R_0R_1)\]

In particular, what is naively a “very decompactified torus,” with both \( R_1R_0, R_0/R_1 \) large can have an orbit which is “close” to the orbit of \( \Gamma_* \), at least when projected to \( \Sigma \).

2.8. Generalization to superstrings

The above considerations can be generalized to strings with \( N = 1, 2 \) local supersymmetries.

For \((1, 1)\) supersymmetries we let \( h \) be the Loop algebra on \( \mathbb{R}^{1,9} \). We replace the symmetric algebra \( S(h^-) \) by the corresponding “Weil algebra” \( W(h^-) = S(h^-) \otimes \Lambda(h^-) \). Combining left and right movers we have for the type II superstring

\[ \mathcal{H}_\Gamma^{NS} = W(h_L^-)_+ \otimes W(h_R^-)_+ \otimes \mathfrak{C}[\Gamma] \] (2.58)

where the subscript refers to a GSO projection. The Ramond sector is given by

\[ \mathcal{H}_\Gamma^R = W(h_L^-)_+ \otimes W(h_R^-)_+ \otimes \mathfrak{C}[\Gamma] \otimes \mathcal{S}_+ \otimes \mathcal{S}_\pm \] (2.59)

where \( \mathcal{S}_\pm \) are the 16-dimensional Majorana-Weyl spinor representations of \( Spin(1,9) \). The \( \pm \) distinguishes \( IIA \) and \( IIB \) theories. The full matter statespace associated to a lattice \( \Gamma \in \mathcal{M} \) is \( \mathcal{H}_\Gamma = \mathcal{H}_\Gamma^{NS} \oplus \mathcal{H}_\Gamma^R \). One interesting subtlety arises in the discussion of duality generalizing (2.21) [22]. The spinor representation \( \mathcal{S}_+ \otimes \mathcal{S}_\pm \) of \( O(\eta) \times O(\eta) \) defines a homogeneous vector bundle on \( \mathbb{H} \). Right-multiplication by \( A \in O(\tilde{D}; \mathbb{Z}) \) which is not in the connected component \( O_0(\tilde{D}; \mathbb{Z}) \) (e.g. left and right-moving parity) may switch \( \mathcal{S}_+ \leftrightarrow \mathcal{S}_- \). In such cases these components of the duality group do not act as symmetries.

Similarly for \((2, 2)\) supersymmetry we take \( h \) to be the loop algebra on \( \mathbb{R}^{2,2} \) and form the Weil algebra as above [23] [24]. There is no need to distinguish NS from R or any other twisted sectors [25].

For heterotic strings with \((0, 1)\) supersymmetries the only important change is that we must define \( O(D; \mathbb{R}) \) and \( O(\tilde{D}; \mathbb{R}) \) by taking the direct sum of \( D \) in (2.2) with the metric tensor of an even self-dual Euclidean lattice of dimension 16. One can also extend the discussion to heterotic strings of type \((0, 2)\) or \((1, 2)\) using the ideas of [26].

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3. BRST Cohomology

3.1. Review

The string theories associated to the conformal field theories $\mathcal{H}_\Gamma$, $\Gamma \in \mathbb{I}_L$ are defined by the off-shell chain complex

$$C^*_\Gamma = \mathcal{F}_{b,c} \otimes \mathcal{F}_{b,\bar{c}} \otimes \mathcal{H}_\Gamma$$

(3.1)

with BRST differential $Q + \overline{Q}$. The on-shell space is the BRST cohomology $H^*(\mathcal{H}_\Gamma)$. Both $C^*$ and $H^*$ are bi-graded by left and right ghost number: $H^g\bar{g}$. We will work with absolute cohomology for simplicity. Modifications for relative and semirelative are straightforward.

The essential theorem we will use was established in the proofs of the chiral no-ghost theorem \[27–32\]. We will follow most closely the statement in \[31\].

**Theorem**\[27–32\]. The absolute chiral BRST cohomology, $H_A^*(\mathcal{F}_p)$ is nonzero only for $*= 0, 1, 2, 3$ and is given by

$$H_A^0(\mathcal{F}_p) = \mathbb{R} \cdot |1\rangle$$

$p = 0$

$= 0$ \quad $p \neq 0$

$$H_A^1(\mathcal{F}_p) = \mathbb{R}^{26}$$

$p = 0$

$$= \mathbb{R}^d \quad d = p_{24}(n) \quad p^2 = 2 - 2n, n \geq 0, p \neq 0$$

$= 0$ \quad otherwise

(3.2)

together with $H_A^2 \cong H_A^{3-g}$. Here $p_{24}(n)$ denotes the number of partitions into 24 colors.

We will have occasion to use the equivalence to the old physical state conditions. This establishes an isomorphism of $H^g=1$ with the space of dimension one Virasoro primaries modulo spurious states: the ghost number one cohomology is spanned by elements of the form $cV$ where $V$ is a representative of

$$\mathcal{F}_p[1]^{Vir^+} / (\mathcal{F}_p[1]^{Vir^+} \cap Vir^- \cdot \mathcal{F}_p)$$

(3.3)

Here $Vir^\pm$ are the positively/negatively moded Virasoro subalgebras and the notation $\mathcal{F}_p[1]$ means restriction to the space of dimension one operators.

Completely analogous results hold for the SuperVirasoro algebra \[32\] \[33\]. In particular, at ghost number 1, where the superfield $C = c + \theta \gamma$ has ghost number 1, there is a one-one correspondence between matter primary superfields of $h = 1/2$ modulo spurious states, and BRST invariant states. The discontinuity in (3.2) at $p = 0$, which plays an important role below, goes through with minor modifications. Analogous results hold in the $N = 2$ case \[34\].

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3.2. BRST Cohomology on $\mathbb{H},\mathbb{L},\mathbb{N}$

The absolute cohomology for (3.1) is trivially calculated:

$$H^\ast(\mathcal{H}_\Gamma) = \bigoplus_{(p_L:p_R) \in \Gamma} H_Q^\ast(\mathcal{F}_{p_L}) \otimes H_Q^\ast(\mathcal{F}_{p_R})$$  (3.4)

At a generic point $\vec{\mathcal{L}}$ in $\mathbb{L}$ the cohomology $H^\ast$ is finitedimensional, and localized at momentum zero. This follows since by (3.2) $H^\ast(\mathcal{F}_p)$ is zero unless $p^2$ is an even integer $\leq 2$, but for generic lattices $p_L^2, p_R^2$ are not integral unless $p_L = p_R = 0$. Thus, at generic points in $\mathbb{L}$:

$$H^1 = \bigoplus_{c} c\partial X^\mu \cdot \mathbb{R} \oplus \bar{c}\partial \bar{X}^\mu \cdot \mathbb{R} \cong \mathbb{R}^{26} \oplus \mathbb{R}^{26}$$
$$H^{1,1} = \bigoplus_{c\bar{c}} c\bar{c}\partial X^\mu \partial \bar{X}^\nu \cdot \mathbb{R} \cong \mathbb{R}^{26} \otimes \mathbb{R}^{26}$$  (3.5)

Remarks:

1. The vast reduction of physical states noted above is not a stringy phenomena. Consider the D’Alembertian equation in a $1 + 1$-dimensional field theory in which time and space dimensions are incommensurate. The only solutions to the equations of motion are constants.

2. It is natural to consider the asymmetric orbifold [35] obtained by dividing by the $\mathbb{Z}_2$ transformation $Y \rightarrow -Y$ on both left and right-movers. In this case, for generic lattices, there is no $H^{1,1}$ cohomology since the twisted sectors have nonintegral $L_0, \bar{L}_0$. The entire (absolute) cohomology is $(H^0 + H^3) \otimes (\bar{H}^0 + \bar{H}^3)$.

3.3. Reduced String theories

At less generic points in $\mathbb{L}$ we can produce interesting examples of string compactifications where the space of physical states is infinite-dimensional, but still far smaller than normally encountered in string theory. In this section we sketch one such example.

Consider the point $E = Diag(-R_0^2, R_1^2, \ldots R_{25}^2)$, and the lattice

$$\Gamma_{\vec{L}} = \text{Span}\{ \frac{1}{\sqrt{2}}(n_\mu R_\mu, m_\mu R_\mu, \frac{n_\mu}{R_\mu} - m_\mu R_\mu) \}$$  (3.6)

which has generator matrix

$$\mathcal{E} = \exp \left[ \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix} \right] \frac{1}{\sqrt{2}} \begin{pmatrix} \eta & 1 \\ \eta & -1 \end{pmatrix}$$  (3.7)

\[ \text{i.e., on a dense subset of total measure} \]
with $\lambda = \text{Diag}\{\log R^{-1}_\mu\}$.

If the $R^2_\mu$ are all irrational and incommensurate the cohomology is described by (3.3). If some spatial dimensions, say $R_i$, $i = 1, \ldots, d$ have a length commensurate with the time direction then there will be an infinite number of physical states. If $R^2_0$ is rational we have an enhanced symmetry point as described in the next section.

If $R^2_0$ is irrational there are still an infinite number of states but these will always occur at levels $(0, 2), (1, 1), (2, 0)$. To see this, let $R_i = (\frac{p_i}{q_i})R_0$ and suppose $R^2_0$ is irrational and incommensurate with $R^2_j$ for $j > d$. A state with momentum $(p_L; p_R)$ will be in the cohomology if $p^2_L, p^2_R$ are even integers less than or equal to two. Clearly this requires the components in the directions $j > d$ to be zero. Moreover since $R^2_0$ is irrational we must have

$$-n^2_0 + \sum_{i=1}^{d} n^2_i (\frac{p^i}{q^i})^2 = 0$$

$$-m^2_0 + \sum_{i=1}^{d} m^2_i (\frac{q^i}{p^i})^2 = 0$$

Since $p^2_L$ and $p^2_R$ are both $\leq 2$ we have $\pm n\eta m \leq 2 \Rightarrow n\eta m \in \{-2, 0, 2\}$.

We can produce an infinite set of solutions to (3.8) by generalizing the usual construction of Pythagorean triples. The states associated with these solutions clearly have levels $(0, 2), (1, 1), (2, 0)$. The $(1, 1)$ are the familiar massless particles of bosonic string. The $(0, 2), (2, 0)$ states have energies rapidly going to infinity as we decompactify. Curiously, there are no tachyons.

Remark: The chaotic nature of the physical states is reminiscent of the nature of the special states in the 2D string.

4. Symmetries in Toroidal Compactification

4.1. Unbroken symmetries in string theory

Gauge symmetries of CSFT are associated with $G=1$ BRST cohomology classes. It is a well-known fact that the chiral ghost number one cohomology defines a Lie algebra. It has been discussed, for example, in [31][36–38] and elsewhere. Similarly, for the superstrings we again look for $G = 1$ cohomology. The superline integrals $\oint dzd\theta J$ where $J$ is a weight 1/2 superfield form a Lie superalgebra.
4.2. Enhanced symmetries

**Definition.** A point $\Gamma \in \mathbb{L}$ is called an *enhanced symmetry point* (ESP) if $\Gamma$ contains a positive rank sublattice of the form

$$(\gamma_L; 0) \oplus (0; \gamma_R) \subset \Gamma$$

where $\gamma_L$ or $\gamma_R$ contains vectors of norm $\leq 2$.

At enhanced symmetry points there will be extra ghost number one cohomology classes. These will always be of the form $cJ(z)$ or $\bar{c}\bar{J}(\bar{z})$ and form a Lie algebra $\mathcal{L}_\Gamma = H^{0,1} \oplus H^{1,0}$. The set of enhanced symmetry points contains the $O(\eta) \times O(\eta)$ orbit of $O(\tilde{D}; \mathbb{Q})/O(\tilde{D}; \mathbb{Z})$ and is a dense set of measure zero.

**Example 1.** At a generic point $L \Gamma = \mathbb{R}^{26} \oplus \mathbb{R}^{26}$.

**Example 2.** $\gamma_L$ is a root lattice of a simply laced group. $H^{1,0}$ is a finite dimensional Lie algebra.

**Example 3.** Generalized Kac-Moody Algebras $[39]$. When the lattice $\gamma_L$ is hyperbolic we obtain infinite dimensional symmetries of the ground state. These are already quite nontrivial if the ESP lattice $\gamma_{L,R}$ is $II_{1,1}$ or $II_{1,1} \oplus \sqrt{2}\mathbb{Z}$. In the latter case a related hyperbolic algebra has been extensively studied $[40]$.

4.3. A distinguished compactification

**Proposition 8.** If a timelike dimension is compactified then there is a unique point $\Gamma_* \in \mathcal{N}$ in the Narain moduli space at which the closed string completely factorizes between left and right movers:

$$\mathcal{H}_{\Gamma_*} = \mathcal{C} \otimes \bar{\mathcal{C}}$$

This is true in any string theory with $N = 0, 1, 2$ supersymmetries.

**Proof.** The theory factorizes iff the lattice $\Gamma$ may be written as a direct sum of left- and right-moving lattices

$$\Gamma = (\Gamma_L; 0) \oplus (0; \Gamma_R)$$

Since $\Gamma$ is even integral so are $\Gamma_{L,R}$. Since $\Gamma$ is unimodular it follows that $\Gamma_{L,R}$ are too. We now apply the uniqueness of even unimodular lattices with noneuclidean signature. For example, for the totally compactified $N = 0, 1, 2$ strings we have:

$$\Gamma_* \equiv (II^{1,25}; 0) \oplus (0; II^{1,25})$$

$$\Gamma_* \equiv (II^{1,9}; 0) \oplus (0; II^{1,9})$$

$$\Gamma_* \equiv (II^{2,2}; 0) \oplus (0; II^{2,2})$$
respectively, where \(II^{p,q}\) is the unique even integral unimodular lattice in \(\mathbb{R}^{p,q}\). We thus obtain (4.1) with 
\[ C = S(h_{1,25}^-) \otimes \mathbb{C}[II^{1,25}] \] for the bosonic string, and similarly for the \(N = 1, 2\) strings. Similar remarks also apply to heterotic strings. ♠

**Remarks:**

1. The distinguished point \(\Gamma_\ast\) may be regarded as a point of maximal symmetry in the moduli space of toroidal compactifications in the following sense: At an enhanced symmetry point the Lie algebra of symmetries \(H^1 = H^{1,0} \oplus H^{0,1}\) organizes the rest of the statespace into representations. For example, the space of ghost number 1, 1 states may be decomposed as 
\[ H^{1,1} = H^{0,1} \otimes H^{1,0} \oplus H_i \otimes \bar{H}_i \] (4.5)
where \(H_i, \bar{H}_i\) are representations of \(H^{0,1}, H^{1,0}\). If the rank of \(\gamma_L, \gamma_R\) is maximal then the sum will be finite and the theory is rational, otherwise it is infinite and the theory is quasirational.

In CSFT zero-modes of “gauge bosons” are of the form \(J_1 \bar{J}_2 \in H^{0,1} \otimes H^{1,0}\), that is, we should regard gauge bosons as the part of the ghost number 2 cohomology whose existence is dictated by the symmetry \(H^{0,1} \oplus H^{1,0}\). At the point \(\Gamma_\ast\) only the adjoint representation appears: “All states are (zero-modes of) gauge bosons.” In general we will refer to states in \(H^{0,1} \otimes H^{1,0}\) as “symmetry states.”

2. Given a point of maximal symmetry it is natural to ask if \(L_\ast = L_{\Gamma_\ast}\) is a universal symmetry of string theory in the sense that all other unbroken symmetry algebras which arise in toroidal compactification are subalgebras of \(L_\ast\). Unfortunately, maximal symmetry does not imply that \(L_\ast\) is universal. To see that \(H^1\) is not universal note that the only way in which we could have a Lie algebra embedding would be to have \((\gamma_L; 0) \oplus (0; \gamma_R)\) be a sublattice of \(\Gamma_\ast\). Consider the case of maximal rank. Then \((\gamma_L; 0) \oplus (0; \gamma_R)\) is only a sublattice if \(\det G\) for its metric tensor \(G\) is a perfect square. It is easy, using the examples of section 3.3, to construct ESP’s which do not have this property.

3. For the bosonic string the Lie algebra \(L_\ast\) is related to the Monster group. \(L_\ast = A \times A\) where \(A\) is the “Fake Monster Lie algebra” studied by Borcherds [39]. For \((1, 1)\) supersymmetry \(A\) is related to the hyperbolic Kac-Moody algebra \(E_{10}\).

4. A natural geometric construction of the groups corresponding to \(L_\ast\) would be very interesting.
4.4. Lorentz orbits and duality orbits

The ergodic nature of the duality action on $\mathbb{H}$ and of the Lorentz action on $\mathbb{L}$ discussed in section 2.5 raises the issue of the nature of the orbits of ESP’s, especially of the distinguished orbit $\Gamma^*$. Mathematically this appears to be a difficult and deep problem, but physical common sense leads us to expect that an enhanced symmetry point cannot have a dense orbit. The reason is that ESP’s correspond to quantum symmetries arising from special configurations of the spacetime lattice. If the Lorentz orbit of an ESP were dense then we could decompactify the spacetime torus and, no matter how large we make all the spacetime radii, there would be a nearby ESP. This strongly contradicts the physical idea that as one decompactifies one must recover the physics of the uncompactified theory. Therefore we state a:

**Conjecture**: Let $\Gamma$ be an ESP. Then the $O(\eta) \times O(\eta)$ orbit of $\Gamma \in \mathbb{L}$ is closed in $\mathbb{L}$.

**Example**: Consider compactification of $1 + 1$ dimensions as in section 2.6 above. If $E$ is rational then $\rho \pm, \tau \pm$ solve a quadratic equation with rational coefficients. Therefore the continued fraction expansions are periodic [41]. From the considerations of section 2.6 we see that the orbit is closed and not dense.

We can also settle the question for the particular case of the orbit $\Gamma^*$.

**Proposition 9.** The Lorentz orbit $\Gamma^* \subset \mathbb{L}$ is a closed $O(\eta) \times O(\eta)$ orbit.

**Proof.** Consider the space $\hat{\mathcal{M}}$ of all generator matrices for lattices $\Gamma \in \mathbb{L}$. Thus, we consider matrices $\mathcal{E}$ such that $\mathcal{E}^{\mu} D \mathcal{E}$ is integral, unimodular, with even diagonal. We have $\mathbb{L} = \hat{\mathcal{M}} / GL(52; \mathbb{Z})$ and $\hat{\mathcal{M}}$ admits a left $O(D; \mathbb{R})$ action. For a point $\mathcal{E}^*$ in the orbit $\Gamma^*$ we can find an integral change of basis $A \in GL(52; \mathbb{Z})$ such that

$$\mathcal{E}^* \cdot A = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$$

(4.6)

Moreover, the left $O(\eta) \times O(\eta)$ action preserves this form. Thus, it suffices to show that there are generator matrices $\mathcal{E} \in \mathcal{M}$ which cannot be brought arbitrarily close to this form by right action of $GL(52; \mathbb{Z})$. Write

$$\mathcal{E} = \begin{pmatrix} \mathcal{E}_{11} & \mathcal{E}_{12} \\ \mathcal{E}_{21} & \mathcal{E}_{22} \end{pmatrix}$$

and assume $\mathcal{E}_{11}$ is invertible. Then the equations $(\mathcal{E} \cdot A)_{12} = (\mathcal{E} \cdot A)_{21} = 0$ are equivalent to

$$A_{12} + \mathcal{E}_{11}^{-1} \mathcal{E}_{12} A_{22} = 0$$
$$A_{11} + \mathcal{E}_{21}^{-1} \mathcal{E}_{22} A_{21} = 0$$

(4.7)
Choosing $\mathcal{E}_{11}^{-1} \mathcal{E}_{12} = E = \text{Diag}\{R^2_{\mu}\}$ with $R^2_{\mu} = p_{\mu}/q_{\mu}$ rational we may take $\mathcal{E}$ to be given by (3.7). It is then easy to see that the most general integral matrix $A$ solving (4.7) is

$$A = \left( \begin{array}{cc} \text{Diag}\{p_{\mu}\} & 0 \\ 0 & \text{Diag}\{q_{\mu}\} \end{array} \right) \left( \begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array} \right) \left( \begin{array}{cc} \ell_1 & 0 \\ 0 & \ell_2 \end{array} \right)$$

(4.8)

where $\ell_{1,2}$ are invertible, integral, $26 \times 26$ matrices. Then

$$|\det A| = 2^{26} \det \ell_1 \det \ell_2 \prod_{\mu} (p_{\mu} q_{\mu})$$

(4.9)

since any matrix $A \in GL(52; \mathbb{Z})$ always has $|\det A| = 1$, while (4.9) is bounded away from 1 we see that no $GL(52; \mathbb{Z})$ matrix can make both $(\mathcal{E} \cdot A)_{12}$ and $(\mathcal{E} \cdot A)_{21} = 0$ arbitrarily small. ♠

4.5. Symmetry Points and Orbifold Points

The action of the duality group on $\mathbb{H}$ sometimes has nontrivial fixed points. In the Euclidean case these correspond to orbifold points of

$$\mathcal{N} = O(n) \times O(n) \backslash \mathcal{M}/O(\tilde{D}, \mathbb{Z})$$

From the point of view of right orbits of the duality group on $\mathbb{H}$ this comes about when elements of the duality group leave $E$ fixed. From the point of view of left orbits of $\mathbb{L}$ these orbifold points correspond to lattices $\Gamma \in \mathbb{L}$ which have automorphism groups $\text{Aut}(\Gamma)$ containing elements other than the trivial automorphism $x \rightarrow -x$.

This discussion generalizes straightforwardly to the Minkowskian case. However, it often happens that the automorphism groups of hyperbolic lattices are in fact infinite discrete groups, so the singularities at the “orbifold points of $\mathcal{N}$” can be rather wild. A spectacular example of this is provided by the orbits corresponding to $\Gamma_+$. For example, the fixed subgroup of the duality group is the infinite discrete group $\text{Aut}(II^{1,25}) \times \text{Aut}(II^{1,25})$.

For an interesting description of this group see [5] ch. 27

Orbifold points and enhanced symmetry points appear to be closely connected. Indeed, if the automorphisms are of $\Gamma \in \mathbb{L}$ are of the form $(R, 1)$ or $(1, R)$ then the orbifold point is necessarily an enhanced symmetry point, since $(R p_L, p_R) - (p_L, p_R) = (p'_L, 0)$ is in $\Gamma$ with $p'_L \neq 0$. Conversely, enhanced symmetry points related to affine Lie algebras or points with vectors of square-length two automatically correspond to orbifold points. The
reason is that the Weyl reflection in a vector of length two defines a nontrivial automorphism of the lattice.

For these reasons orbifold points are often identified with ESP’s. We have not managed to prove this. Because of the possibility that the automorphism group is nontrivially embedded in \( O(\eta) \times O(\eta) \) we must ask:

*Question 1:* Are all orbifold points enhanced symmetry points?

Conversely, there are enhanced symmetry points containing no vectors of length two. In all cases we have examined these ESP’s are also orbifold points so we also ask:

*Question 2:* Are all enhanced symmetry points orbifold points?

5. Spontaneous Symmetry Breaking

5.1. Disappearing states

Toroidal compactification of all dimensions presents an interesting example of backgrounds in which there are large unbroken symmetries together with infinite-dimensional BRST cohomology. When we move away from these points in a generic direction the symmetries disappear. Indeed, there is a standard description of spontaneous symmetry breaking in toroidal compactification [32]. The values of the couplings to the exactly marginal operators \( \partial Y^a \bar{Y}^{\bar{a}} \) in the Lagrangian correspond (up to contact term subtleties) to the vacuum expectation values of exactly massless spacetime scalar fields \( \Psi^{a,\bar{a}} \). ESP’s correspond to points where the vacuum expectation values leave unbroken gauge symmetries larger than \( U(1)^d \times U(1)^d \).

A generic perturbation away from an ESP such as \( \Gamma_* \) has drastic consequences - an infinite number of states disappears! While initially surprising, this is quite in line with the standard interpretation of spontaneous symmetry breaking in toroidal backgrounds. Consider the string field theory action for in an ESP background \( \Gamma \). Calculation of the action of \( \Pi \) will result in an expression like

\[
S(\Psi) = \sum_{(p_L,p_R) \in \Gamma} \Psi_I(p_L,p_R) (Q^{IJ} - (m^2)^{IJ}) \Psi_J(p_L,p_R) + \sum_{p_1,p_2,p_3} K^{IJ}_{a,\bar{a}} \Psi^{a,\bar{a}} \Psi_I \Psi_J + \cdots \tag{5.1}
\]

where \( K \) is some typically nonvanishing coupling computed from the Witten three-string vertex. The ellipsis indicates the higher terms of the nonpolynomial theory. \( \Psi^{a,\bar{a}}(p_L,p_R) \) are spacetime fields corresponding to the exactly marginal directions \( \partial Y^a \partial \bar{Y}^{\bar{a}} \).
The masses \((m^2)^{IJ}\) in (5.1) are integrally quantized while the momenta are constrained to lie on a lattice. Therefore, if we shift the vacuum expectation value of the fields \(\Psi^a,\bar{\Psi}^{\bar{a}}(0,0)\) the shift in the three-point interactions modifies the quadratic terms and in general leaves no zero-modes for the perturbed quadratic form acting on the lattice.

Thus, although the absence of states at a generic point is perfectly sensible from the spacetime field theory point of view as explained in section 3.2 the main novelty in string theory is that the global dimensions of space and time can be modified by giving vacuum expectation values to the spacetime fields. That is, the disappearance of states is the result of spontaneous symmetry breaking.

5.2. Goldstone vs. Higgs

The Higgs mechanism seems to be at odds with the discontinuity in the number of on-shell degrees of freedom noted above. Let us clarify this point.

Suppose there are \(d\) compact Euclidean directions and \(26 - d\) noncompact directions. Furthermore consider a family of compactifications \(g(t) \cdot \Gamma\) where \(g(t)\) is a family of \(O(D)\) Lorentz transformations and \(\Gamma\) is an ESP. Consider the evolution of the cohomology groups which at \(t = 0\) correspond to the gauge bosons of the enhanced symmetry. Let

\[
d(t) = \dim H^{1,1}(\mathcal{F}_{pL(t)} \otimes \bar{\mathcal{F}}_{pR(t)} \otimes \mathcal{F}_q(t) \otimes \bar{\mathcal{F}}_q(t))
\]

where \(q(t)\) denotes the momenta in the noncompact directions, and \(p_L(t = 0)^2 = 2\), \(p_R(t = 0) = 0\). Note that \(q(0)^2 = 0\), appropriate to a massless particle.

Of course, \(d(t)\) is a constant function for \(t \neq 0\). This follows from section 3.1: time will dress up compact momenta appropriately.

If \(q(0) \neq 0\) then \(d(t)\) is continuous at \(t = 0\). This statement of continuity of the number of on-shell degrees of freedom is essentially a statement of the Higgs mechanism. The evolution to nonzero values of \(t\) of the would-be scalar Goldstone mode

\[
\delta_J(\partial Y^a \Delta E_{ab} \bar{\partial} Y^b)e^{ip_L(0)Y}p_L(0)^a \Delta E_{ab} \bar{\partial} Y^b e^{iq(0) \cdot Z},
\]

where \(Z\) denotes noncompact dimensions, is \(Q\)-equivalent to the longitudinal mode of a massive vector boson:

\[
\epsilon_L \cdot \bar{\partial} Ze^{ip_L(t)Y}e^{ip_R(t)\dot{Y}}e^{iq(t) \cdot Z}
\]

\(^8\) The “evolution” is defined precisely in the section seven in terms of Lorentz transport.
Put another way, the massless cohomology of the rightmoving sector is, of course, 24 dimensional. At $t = 0$ we account for the degrees of freedom this way:

$$24 = d + (24 - d),$$

for $d$ scalar modes and $24 - d$ massless vector modes in $26 - d$ dimensions. At $t > 0$, we write:

$$24 = (d - 1) + (25 - d)$$

since there are only $d - 1$ scalar modes and $25 - d$ massive vector modes in $26 - d$ dimensions.

If $q(0) = 0$ then $d(t)$ is discontinuous at $t = 0$. This statement of discontinuity of the number of on-shell degrees of freedom is essentially a statement of Goldstone’s theorem. The discontinuity comes from the massless right-moving cohomology. At $t = 0$ all rightmoving momenta vanish, so $H^1(\tilde{F}_0) = 26$. Thus, at $t = 0$ we have

$$26 = d + (26 - d)$$

for the zeromodes of $d$ massless scalars and the zero modes of $26 - d$ gauge bosons in $26 - d$ dimensions. At $t > 0$ $d \rightarrow d - 1$: we lose a scalar, and $26 - d \rightarrow 25 - d$, the vector boson gets massive and has nonzero momenta. This accounts for a loss of two degrees of freedom.

When we toroidally compactify time the infinite reduction in the number of on-shell degrees of freedom in the neighborhood of an ESP is merely an extreme example of the the loss of the on-shell degrees of freedom common to all theories of spontaneously broken gauge theory.

**Remarks:**

1. The above discussion is closely related to an elegant remark of E. Verlinde. If a change of $\sigma$-model action $\Delta S$ breaks a symmetry for a current $J$ then $\delta J \Delta S$, which would be the zero-mode of the Goldstone boson, is always $Q$-trivial, at least to first order in $\Delta S$. It is not true that $\delta J \Delta S$ remains $Q$-trivial for finite deformations $\Delta S$. In the case of toroidal compactifications one can interpret Proposition 16 below as an extension to all-orders in $\Delta S$ of Verlinde’s remark.

2. There is one other way in which the string example of SSB is strikingly different from its field-theoretic counterpart. In the latter case there is usually at least one point in the space of Higgs vevs where the entire gauge symmetry remains unbroken. Remark 2 of section 4.3 shows that this does not hold in the string case. Nevertheless, we can try to construct such a “universal symmetry.” This is the subject of the next section.
6. A universal symmetry

**Definition:** A universal symmetry $\mathcal{L}$ is a Lie algebra such that, for all ESP’s $\Gamma \in \mathbb{L}$ there is a Lie algebra embedding $H^1(\mathcal{H}_\Gamma) \hookrightarrow \mathcal{L}$.

In this section we attempt to give a *natural* construction of such a universal Lie algebra for the ESP’s of the toroidally compactified bosonic string.\footnote{It is easy to construct unnatural universal symmetries. For example, just take the direct sum over all ESP’s!}

We begin by unifying left- and right-moving oscillators into a single Lorentz multiplet. Note that instead of (2.9) we could define

$$\rho^A_m = \beta^A_m = \beta^A_{m-n} \quad A = 1, \ldots n + 1$$

and write the single equation

$$[\rho^A_n, \rho^B_m] = D^{AB} n \delta_{n+m,0}.$$

This suggests that we should consider the toroidal compactification of the *open string* in a 52 dimensional spacetime with the metric $D$. The torus is obtained from the unique even self-dual lattice $II^{26,26}$. We use the quadratic form $D$ to form a Heisenberg algebra from the loop algebra on $\mathbb{R}^{26,26}$ with polarization $h = h^- \oplus h^0 \oplus h^+$, and form

$$\mathcal{H} = S(h^-) \otimes \mathbb{C}[II^{26,26}]$$

Since $O(D)$ is transitive on $\mathbb{L}$ one might think the appropriate Lie algebra is:

$$\mathcal{L} = \mathcal{H}[1]^{Vir^+}/(\mathcal{H}[1]^{Vir^+} \cap Vir^- \mathcal{H})$$

Unfortunately, this Lie algebra does not quite work.

The problem is that the Lie algebra (6.4) only contains $H^{1,0} \times (H^{0,1})^{opp}$ where $(H^{0,1})^{opp}$ is the *opposite* Lie algebra to $H^{0,1}$ obtained by the following construction. If $\mathcal{L}_\Gamma$ is the Lie algebra associated to an even integral lattice we may form the opposite Lie algebra $\mathcal{L}_\Gamma^{opp}$ by changing $T \to -T$, $[\beta, \beta] \to -[\beta, \beta]$. The definition of the vacuum remains unchanged, and $\beta_{-n}$ still raises $L_0$ by $+n$ units for $n > 0$, but the sign of the zero-mode Hamiltonian has changed: $H = -\frac{1}{2}p^2 + N$ where $N$ is the level. Thus, the roots of the Lie
algebra in general lie outside the lightcone, rather than inside the lightcone. To obtain a universal Lie algebra we need an algebra which is invariant under the opposite mapping.

To remedy this problem we allow both purely real and purely imaginary momenta. Closure of the OPE then forces us to consider linear combinations, but only with integer coefficients. Therefore we consider an extension of scalars from the $\mathbb{Z}$ module $II^{26,26}$ to a module over the Gaussian integers:

$$\tilde{\Gamma}^{26,26} = \mathbb{Z}[i] \otimes \mathbb{Z} II^{26,26} \quad (6.5)$$

we now form

$$\tilde{\mathcal{H}} = S(h^-) \otimes \mathbb{C}[\tilde{\Gamma}^{26,26}] \quad (6.6)$$

and denote by $\mathcal{L}_U$ the corresponding Lie algebra of dimension $1$ primaries

$$\mathcal{L}_U \equiv \tilde{\mathcal{H}}[1]^{Vir^+} / (\tilde{\mathcal{H}}[1]^{Vir^+} \cap Vir^- \cdot \tilde{\mathcal{H}}) \quad (6.7)$$

**Proposition 10:** All enhanced symmetries $H^{1,0}(\mathcal{H}_\Gamma) \oplus H^{0,1}(\mathcal{H}_\Gamma)$ for $\Gamma \in \mathbb{L}$ are Lie subalgebras of $\mathcal{L}_U$.

**Proof:** Using the projection $(\pi_L, \pi_R) : \mathbb{R}^{52} \to \mathbb{R}^{26} \oplus \mathbb{R}^{26}$ any closed string theory statespace is naturally a subspace of (6.6) where leftmoving oscillators are interpreted as the first 26 dimensions

$$\prod (\beta^a_n)^k \prod (\bar{\beta}^\alpha_{\bar{n}})^{\bar{k}} \langle p_L; p_R \rangle \mapsto \prod (\rho^a_n)^k \prod (i \rho^{\bar{\alpha} + 26}_{\bar{n}})^{\bar{k}} \langle p_L; i p_R \rangle \quad (6.8)$$

Since $\mathcal{L}_U$ is $O(D; \mathbb{R})$ invariant we can “position” the lattice so that the purely holomorphic or antiholomorphic vectors of $\mathcal{H}_\Gamma$ map into the first or second 26-dimensional components. This mapping takes the ghost number one cohomology of a closed string state into the Lie algebra $\mathcal{L}_U$ of $\tilde{\mathcal{H}}$. Moreover, it preserves the operator product structure for purely holomorphic or anti-holomorphic states and hence defines a Lie algebra embedding. ♠

**Remarks:**

1. In a sense which is difficult to make precise all of the toroidally compactified closed strings should perhaps be viewed as broken phases of the single theory $\tilde{\mathcal{H}}$. In particular, any given background breaks most of the $\mathcal{L}_U$ symmetries.

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10 There is an analogy to including both microscopic and macroscopic states, in Liouville terminology.
2. The distinguished closed string and the “open string” defined by (6.3) are closely
related. Consider the conformal field theory $\mathcal{C} = S(h_{1,25}^-) \otimes \mathcal{O}[I^{1,25}]$, which we may refer
to as the “universal open string.” We may combine two copies of this theory in two distinct
ways. First, the holomorphic CFT $\mathcal{C}$ admits an automorphism $I$ coming from $z \to 1/\bar{z}$,
i.e., time reversal symmetry on the worldsheet. Alternatively, we may use the “opposite
map” described above. Then

$$\text{“52 − dimensional open string” } \mathcal{H} = \mathcal{C} \otimes C^{opp}$$
$$\text{distinguished closed string } = \mathcal{C} \otimes I[\mathcal{C}]$$ (6.9)

This observation leads to some obvious speculations. See the conclusions.

3. The above construction of a universal symmetry is closely related to some consid-
erations of Giveon and Porratti on duality-invariant effective actions $[43]$. These authors
proposed that similar large (generalized) Kac-Moody symmetries play a role as funda-
mental symmetries in toroidal compactification. Indeed, if our construction is generalized to
the heterotic string in the obvious way our algebra would contain theirs as a subalgebra.
More recently, Giveon and Shapere have suggested a candidate universal group for the case
of the $\mathcal{N} = 2$ string $[44]$.

4. A theorem of Lian and Zuckerman $[45]$ states that any space formulated in terms
of the “old physical state conditions” such as (5.7) can be formulated in terms of BRST
cohomology. Thus it might be fruitful to examine the (rather odd) corresponding $c = 26$
open string theory.

5. As with the distinguished compactification symmetries, it would be very interesting
to find a natural geometrical interpretation of the group corresponding to (5.7).

6. Duality symmetries have been interpreted as being part of some mysterious gauge
group. This was first suggested in $[22]$ and further elaborated upon in $[14]$. Indeed,
at ESP’s producing affine Lie algebras the fixed subgroup of the duality group may be
interpreted as the Weyl group of the gauge group. If remark 5 can be understood it would
be natural to expect that $O(\hat{D}; \mathbb{Z})$ is some kind of Weyl subgroup.

7. Relating points on $\mathbb{L}$ and $\mathbb{H}$

We would like to study symmetry-breaking as a function of the scalar field vev’s,
which define different backgrounds $E \in B$ and hence different points on $\mathcal{N}$. Therefore we
need a notion of parallel transport of theories.
7.1. Connection and transport for a Lorentz-rotated Family

In this section we will define a natural connection on the Hilbert bundle $\mathcal{H} \to \mathbb{I}_L$ with fiber $\mathcal{H}_\Gamma$. Indeed this connection exists on any family of conformal field theories with statespaces $\mathcal{H}_{g,\Gamma}$ of the form (2.7) or (2.10), where $\Gamma$ is a lattice and $g$ belongs to a family of orthogonal transformations.

Since a connection is not tensorial, to define it we must choose a local framing. Therefore consider a patch $U \subset \mathcal{O}(D; \mathbb{R})$ and the family of theories $\mathcal{H}_{g,\Gamma}$ for $g \in U$. We choose our local framing of $\mathcal{H}$ to be the set of states

$$\prod \rho^A_n |g \cdot (p_L; p_R)\rangle$$

where $(p_L; p_R)$ range over the lattice $\Gamma$, $\rho^A_n$ are defined in (6.1) and the products of creation operators have $n < 0$ for $A \leq 26$ and $n > 0$ for $A > 26$.

In terms of $\rho^A_n$ we can define a representation of $\text{Lie}(\mathcal{O}(D; \mathbb{R}))$:

$$m_{AB} \in \text{Lie}(\mathcal{O}(D; \mathbb{R})) \to \mathcal{O}(m) \equiv \sum_{n \neq 0} \frac{1}{2n} \rho^A_n m_{AB} \rho^B_n$$

For $A, B = 1, \ldots, 26$ or $27, \ldots, 52$ these are the standard generators of left and right Lorentz transformations. Here leftmovers and rightmovers get mixed by a generic transformation.

The operators $\mathcal{O}(m)$ are rather delicate. They are not defined on the whole statespace, so strictly speaking the Bogoliubov transformation does not make sense. For example, $\mathcal{O}(m)^2$ is an ill-defined operator, and we cannot exponentiate $\mathcal{O}(m)$. Nevertheless $\text{Ad}\mathcal{O}(m)$ is well defined acting on $S(h^-_L) \otimes S(h^-_R)$.

**Definition:** Let $m \in \text{Lie}(\mathcal{O}(D; \mathbb{R}))$ be a tangent vector to $\mathcal{M}$ then the connection $\nabla_m : \mathcal{H}_{g,\Gamma} \to \mathcal{H}_{g,\Gamma}$ is defined with respect to the frame (7.1) by

$$\nabla_m \prod \beta^A_n |g \cdot (p_L; p_R)\rangle \equiv [\mathcal{O}(m), \prod \beta^A_n] |g \cdot (p_L; p_R)\rangle$$

**Proposition 11.** The connection $\nabla$ is flat.

**Proof:** This follows because we have a representation: $[\mathcal{O}(m_1), \mathcal{O}(m_2)] - \mathcal{O}([m_1, m_2]) = 0$ implies $[\nabla_{m_1}, \nabla_{m_2}] - \nabla_{[m_1, m_2]} = 0$. ♠

The formula (7.3) is easily exponentiated to give the formula for parallel transport on $\mathbb{I}_L$ by any $g \in \mathcal{O}(D; \mathbb{R})$:

**Proposition 12:** On a Lorentz-rotated family the parallel transport under the connection $\nabla$

$$T^g : \mathcal{H}_\Gamma \to \mathcal{H}_{g,\Gamma}$$
is obtained from

\[ \rho^n_A \rightarrow \rho'^n_A (DgD)_A^A \]

\[ |p_L; p_R \rangle \rightarrow |g \cdot (p_L; p_R) \rangle \] (7.5)

**Proof:** One need only check that \( \text{Ad}[O(m)] \) has been properly exponentiated. Note that one must normal order the final result after rotating the oscillators. ♦

**Example:** Consider the one-dimensional case with

\[ g(\lambda) = \begin{pmatrix} \cosh \lambda & \sinh \lambda \\ \sinh \lambda & \cosh \lambda \end{pmatrix} \] (7.6)

then

\[ T^g(\beta_{-n} \bar{\beta}_{-n} |0\rangle) = \cosh^2 \lambda \beta_{-n} \bar{\beta}_{-n} |0\rangle + n \cosh \lambda \sinh \lambda |0\rangle \] (7.7)

In particular, note that the space of exactly marginal operators is *not* preserved by the transport.

**Remarks:**

1. The above construction gives a globally defined flat connection on the vector bundles \( \mathcal{H} \rightarrow \mathcal{M} \) and \( \mathcal{H} \rightarrow \mathcal{I} \). By choosing coordinates one can also use it to construct a flat connection on \( \mathcal{H} \rightarrow \mathcal{I} \). This will be globally defined for Euclidean compactifications, but only defined on an open subset of \( \mathcal{I} \) in the Minkowskian case.

2. Similar formulae to those above have already appeared in two papers by Kugo and Zwiebach [46] and by Ranganathan [47]. Indeed our treatment was in part motivated by these papers. Nevertheless, we believe our connection differs from that in [47], as well as from that in [48]. The main difference lies in the treatment of the zero-modes. Strictly speaking, (7.4)(7.5) is *not* a Bogoliubov transformation, because \( O(m) \) cannot be exponentiated. The parallel transport of [47] analogous to (7.7) would produce a state with arbitrarily high particle number [11] in contrast to our transport (7.7).

3. Cecotti and Vafa [49] have defined a notion of parallel transport on the vector bundle of chiral primary fields over moduli spaces of \( N = 2 \) theories. Since the bosonic string can be regarded as a twisted \( N = 2 \) theory [50], [51], one might therefore try to apply the results of [49]. The chaotic nature of the BRST cohomology shows that the transport of [49], if it exists in this case, must be distinct from Lorentz transport. Moreover, it indicates that unitarity is an important ingredient of [49].

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[11] Compare, e.g., eq. (3.37) or (4.27) of [46] or eq. (12,13) of [47].
7.2. Conformal perturbation theory

The $\sigma$-model formulation of toroidal compactifications of sec. 2.3 suggests an entirely different transport on $\mathbb{H}$ in terms of conformal perturbation theory. Naively this looks rather trivial. The operator

$$\partial X \cdot \Delta E \cdot \bar{\partial} X$$

is an exactly marginal operator. The main idea of conformal perturbation theory is that one can calculate correlators for a theory with action $S_0 + \int \Phi$ in terms of an exponential series of correlators for a theory with action $S_0$ involving successive insertions of the operators $\int \Phi$.

In terms of path integrals, we may expect to relate correlators at $E' = E + \Delta E$, where $E', E \in \mathcal{B}$, along the following lines:

$$\left\langle \prod V_i(z_i) \right\rangle_{E'} \equiv \int [DX(\sigma, \tau)] e^{-\frac{i}{\hbar} \int \partial X E' \partial X} \prod V_i(z_i)$$

$$= \int [DX(\sigma, \tau)] e^{-\frac{i}{\hbar} \int \partial X E \partial X} e^{-\frac{i}{\hbar} \int \partial X \Delta E \partial X} \prod V_i(z_i)$$

$$= \left\langle e^{-\frac{i}{\hbar} \int \partial X \Delta E \partial X} \prod V_i(z_i) \right\rangle_{E}$$

(7.8)

Of course (7.8) is terribly naive. For one thing, states and operators are isomorphic in CFT, but the statespaces $\mathcal{H}_E$ and $\mathcal{H}_{E'}$ are different. Therefore, we should certainly modify the LHS of (7.8) to read

$$\left\langle \prod T_{E', E} V_i(z_i) \right\rangle_{E'}$$

where $T_{E', E} : \mathcal{H}_E \rightarrow \mathcal{H}_{E'}$ is some linear transformation of the operators at $E$ to the operators at $E'$. Note that if (7.8) makes sense, then, since the path-integral is unambiguous, the resulting parallel transport on $\mathcal{B}$ must be flat.

Unfortunately, (7.8) is also too naive for several other reasons. The manipulations in (7.8) ignore the divergences and contact terms which make quantum field theory nontrivial. There are three kinds of divergences:

1. Singularities of the marginal perturbation with operator insertions $V_i$. These are responsible for the fact that there is a nontrivial transport $T$. 

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2. Singularities of the marginal perturbation with itself in disconnected diagrams. These are responsible for a relatively trivial renormalization of the free energy. There are also infrared singularities, but these only occur for the free energy.

3. Singularities of the marginal operator with itself and with $V_i$. These contact term singularities are responsible for the contact terms leading to different parametrizations of moduli space.

Handling the above divergences requires introduction of a cutoff, which typically destroys conformal invariance. It thus might seem hopeless to use conformal perturbation theory to define transport along a moduli space of CFT’s. We will describe two ways to get around this problem. In this section we define the “little-disks transport.” In appendix A we describe a more subtle way of defining the series.

**Definition.** The little-disk cutoff configuration space

$$F_m(z_1, \ldots, z_n; \epsilon) \equiv \{(w_1, \ldots, w_m) \in \mathbb{C}^m | i \neq j \rightarrow |w_i - w_j| > \epsilon; |w_i - z_j| > \epsilon, i = 1, \ldots m, j = 1, \ldots n\} \quad (7.9)$$

**Definition:** Let $\mathcal{O}$ be a $(1,1)$ operator in a CFT $\mathcal{C}$. Let $\Phi_i$ be operators in $\mathcal{C}$. The point-split perturbation series is defined by the sum of CFT correlators:

$$\langle e^{\int_{\Sigma} \mathcal{O}} \prod_{j=1}^n \Phi_j(z_j, \bar{z}_j) \rangle_{\mathcal{C}, \epsilon} \equiv \sum_{m \geq 0} \frac{1}{m!} \int_{F_m(z_1, \ldots, z_n; \epsilon)} \prod_{k=1}^m d^2 w_k$$

$$\langle \prod_{k=1}^m \mathcal{O}(w_k, \bar{w}_k) \prod_{j=1}^n \Phi_j(z_j, \bar{z}_j) \rangle_{\mathcal{C}} \quad (7.10)$$

In the above sum we must factor out the disconnected terms which renormalize the free energy. These require an infrared cutoff beyond the ultraviolet cutoff we have specified. We may do this by imposing the extra condition $|z| < 1/\epsilon$. The remaining integrals are absolutely convergent. Examples show that for appropriate $\mathcal{O}$ the series will typically be convergent.

“**Definition”**: Point-split transport:

$$\frac{\langle e^{\int_{\Sigma} \mathcal{O}} \prod_{j=1}^n \Phi_j(z_j, \bar{z}_j) \rangle_{\mathcal{C}, \epsilon}}{\langle e^{\int_{\Sigma} \mathcal{O}} \rangle_{\mathcal{C}, \epsilon}} \equiv \prod_{j=1}^n \langle T^\epsilon \Phi_j(z_j, \bar{z}_j) \rangle \quad (7.11)$$

This definition is a little optimistic since we have not shown that the lhs can be expressed as a correlation function of local operators in any theory. One may be even more optimistic and
**Conjecture:** Let \( \Phi_i \) be a basis of scaling operators in the theory \( \mathcal{C} \). Then there is a CFT \( \mathcal{C}' \) (depending on \( \mathcal{O} \)) such that

\[
T_\epsilon \Phi_i \sim \epsilon^{(\Delta' + \overline{\Delta}') - (\Delta + \overline{\Delta})} T \Phi_i (1 + \mathcal{O}(1/\log \epsilon))
\]  

(7.12)
as \( \epsilon \to 0 \). Where \( T : \mathcal{H}_C \to \mathcal{H}_{C'} \) is a mapping of conformal field theories. The power of \( \epsilon \) is required since the conformal weights change. (7.12) is to be understood in the weak sense, i.e., as a statement about correlators, since the \( \mathcal{O}(1/\log \epsilon) \) corrections depend on the positions of all operators.

The conjecture is true at first order in perturbation theory, but this is relatively trivial. Difficulties arise already at second order in perturbation theory from subleading logarithms. For example, there are nontrivial contributions from regions of configuration space where the \( \epsilon \)-disks overlap.\(^\text{12}\) By examining the transport of the correlator \( \langle \partial Y(z_1) \partial Y(z_2) \rangle \), for example, it is easy to see that at order \( \mathcal{O}^2 \) the strictly point split series cannot reproduce the Lorentz-rotation transport defined in section 7.1. The little disks CPT is useful because it is universally and rigorously defined, and will indeed be employed to write a version of broken Ward identities in section 9. Unfortunately we do not understand very well what kind of transport it defines.

One may ask instead if it is possible to define the integrals in CPT so that we produce the Lorentz-transport. This is possible, and the prescription is described in detail in appendix A.

### 7.3. Zamolodchikov metric

Suppose we have a family \( \mathcal{F} \) of CFT’s obtained, at least formally, by perturbing the action by \( \int \lambda^i \mathcal{O}_i \) where \( \mathcal{O}_i \) are exactly marginal operators. These may be thought of as tangent vectors to the family of CFT’s. In this situation Zamolodchikov has defined a natural metric for geometry on \( \mathcal{F} \):

\[
ds^2_{\mathcal{Z}|\mathcal{C}(\lambda)} \equiv |z - w|^4 \langle e^{\int \lambda^i \mathcal{O}_i (z, \bar{z}) \mathcal{O}_j (w, \bar{w})} d\lambda^i \otimes d\lambda^j \rangle
\]

(7.13)

Indeed, considering the family of \( \sigma \)-models parametrized by \( E \in \mathcal{B} \) we have the metric

\[
ds^2_{\mathcal{Z}} = |z - w|^4 \langle \partial X^\mu \bar{\partial} X^\nu (z, \bar{z}) \partial X^p \bar{\partial} X^\xi (w, \bar{w}) \rangle E dE_{\mu\nu} \otimes dE_{\rho\xi}
\]

(7.14)

\[
= tr[G^{-1} dE G^{-1} dE^t]
\]

One can verify that the Zamolodchikov metric (7.14) is the right-invariant metric under the right-Mobius action of \( O(D; \mathbb{R}) \) on \( \mathcal{B} \).\(^\text{47}\)

\(^{12}\) As more disks overlap the integrals for these regions become rather tiresome.
7.4. Preferred coordinates on $\mathbb{H}$

Suppose we have a Lorentz-rotated family which is, at least formally, obtained by exponentiation of the exactly marginal operator $\partial Y^a \Delta E_{ab} \bar{\partial} Y^b$. Let us determine $\Delta E$ such that the Zamolodchikov metric is the invariant metric on $\mathbb{H}$, that is, such that

$$|z-w|^4 \langle \Pi \circ T^g (\beta_{-1}^a \bar{\beta}_{-1}^a)(z, \bar{z}) \Pi \circ T^g (\beta_{-1}^b \bar{\beta}_{-1}^b)(w, \bar{w}) \rangle d(\Delta E_{aa}) d(\Delta E_{bb})$$

(7.15)

is the right-invariant Zamolodchikov metric (7.14). Here

$$\Pi : \mathcal{H} \rightarrow \mathcal{H}^{\text{exactly marginal}}$$

is the projection from $\mathcal{H}$ to the subspace of exactly marginal operators.

**Proposition 13:** The preferred coordinate system on $\mathbb{H}$ determined by the condition that (7.15) coincide with the right-invariant Zamolodchikov metric is given by:

$$\Delta E(g) = -2g_{21}(g_{22})^{-1} \eta = -2\eta(g_{11})^{-1} g_{12}$$

(7.16)

**Proof:** Use the example (7.7) to calculate the transport of exactly marginal operators. Next substitute (7.16) in (7.15). Now derive a right-invariant invariant metric on $O(D)$ which descends to the invariant metric on $\mathbb{H}$ as follows. Decompose $g \in O(D)$ as

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$$

A simple computation shows that the invariant metric on $\mathbb{H}$ is

$$ds_{inv}^2 = \sum_\alpha \text{tr}(t^\alpha g^{-1} dg) \otimes \text{tr}(t^\alpha g^{-1} dg)$$

$$= -4\text{tr} \left[ \eta(dg_{11} \eta g_{21}^t - dg_{12} \eta g_{22}^t) \eta(dg_{21} \eta g_{11}^t - dg_{22} \eta g_{12}^t) \right]$$

(7.17)

where the sum is over a set of broken generators

$$t^\alpha = D \begin{pmatrix} 0 & e_{ij}^t \\ -e_{ij}^t & 0 \end{pmatrix}$$

orthogonal in the Killing metric. Finally, use the relations of the orthogonal group to identify (7.17) with (7.13). ♠

**Remarks**

1. $\Delta E(g)$ and $\Delta E = E \cdot g - E$ only agree to lowest order in perturbation theory.

2. From the formal definition (7.13) of the Zamolodchikov metric we expect $\Delta E(g)$ to be a good set of coordinates for conformal perturbation theory. This is verified in appendix A.

3. Although the connection $\nabla$ is flat, its projection to the subspace of $(1, 1)$ operators need not be flat [47]. Indeed, the projection is the Zamolodchikov connection, which has nonzero curvature. This curvature can be understood to arise from contact terms between the exactly marginal operators [52].

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7.5. Orbifold points and holonomy

The existence of both orbifold points and a flat connection on $\mathcal{N}$ implies that the fixed subgroups of the duality group have holonomy representations in CFT statespaces. In the Minkowskian case this is somewhat formal. Nevertheless, parallel transport on $\mathbb{H}$ from $E$ to $E \cdot \gamma$ may be combined with a simple change of basis to provide a holonomy representation $U(\gamma)$ in $\mathcal{H}_\Gamma$ where $\Gamma$ is constructed from $E$.

Carrying out this program in the case of the fixed subgroup $\text{Aut}(II^{1.25}) \times \text{Aut}(II^{1.25})$ should be very interesting. Perhaps a similar procedure in the case of $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds will lead to a covariant and geometric construction of the Fischer-Greiss Monster group. Carrying this speculation further, we may ask if all the sporadic groups may be so represented as holonomy representations around orbifold points within a single moduli space. This would provide a unified approach to the study of finite simple groups.

8. Amplitudes at Enhanced Symmetry Points

8.1. A philosophical digression on compactified time

As we have stressed above, we are regarding the finite radius of time as an unphysical but symmetric vev of a scalar field. Nevertheless, the idea that time might really be periodic, albeit with an enormous period, has been a recurring theme in human thought in all ages and cultures [53]. We digress and comment briefly on compactified time from a physicist’s standpoint.

Classically, closed timelike loops imply that all motion is periodic with fundamental period $T/n$ where $T$ is the length of the loop and $n$ is a positive integer. The effects of closed timelike loops on a physical system depend strongly on the Hamiltonian in question. Hamiltonians which do not have bound orbits are considered unphysical. In general, among Hamiltonians with bound and unbound orbits initial conditions are restricted to correspond to some (possibly none) of the bound orbits. Phase space is reduced to a collection of subvarieties reflecting quantization of energy. For example, in one-dimensional particle mechanics with Hamiltonian $\frac{1}{2}p^2 + V(x)$ the period integral

$$\int \frac{dx}{\sqrt{E - V(x)}}$$

is quantized in units of $T/n$. The nature of the resulting quantization of $E$ depends on the potential $V(x)$. For a square-well potential there is a minimum energy and $E_n \sim n^2$. 

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For a linear confining potential there is a maximum energy and $E_n \sim 1/n^2$. One notable exception to these remarks is the harmonic oscillator with frequency $\omega = n/T$. This is the unique system which imposes no quantization at all on phase space.

It is sometimes said that closed timelike loops imply a lack of causality. If time is globally periodic this is not true. Closed timelike loops merely force a further retreat from the illusion of free-will and determinism. In classical mechanics one is only free to choose initial conditions. Subsequent history is completely deterministic. If time is periodic one is no longer free to choose initial conditions arbitrarily, but the choice might be large and apparently arbitrary.

Quantum mechanics introduces new problems with closed timelike loops. To be sure, energies are now quantized in units of $\hbar/T$ but this effect is in fact negligible even in the most sensitive tests of QED. On the other hand, serious philosophical problems emerge connected with measurement theory. One can, of course, simply restrict attention to periodic solutions of the Schrodinger equation. Nothing is measured and nothing gets done, but life goes on. If we wish to introduce measurements then we can have more or less arbitrary solutions for time period $T$, with periodic collapse of the wavefunction at some specified set of times, say, $t = nT$. In order to have true periodicity the result of collapsing the wavefunction must always be the same. This contradicts the probabilistic interpretation of the wavefunction, unless the $\psi$ is an eigenstate of the observable being measured. Something more interesting happens if we consider two measurements at different times of noncommuting observables. In this case $\psi$ cannot be simultaneously an eigenstate of both observables, the collapse of the wavefunction must be nontrivial and the probabilistic interpretation of $\psi$ is inconsistent with time-periodicity. Thus, among other things, the Stern-Gerlach experiment shows that time cannot be periodic.

8.2. Genus zero string amplitudes at ESP’s

String densities defined by the operator formalism may be thought of as differential forms on moduli space: $\omega \in \Omega^*(M_{h,n}; H^*(H_{\Gamma})^{\otimes n})$ where $\Omega^*$ is the DeRham complex. String amplitudes are, at least formally, integrals of top-forms $\omega$ over $M_{h,n}$. One naturally asks: what are the string amplitudes for the distinguished compactification $\Gamma_*$? The answer is that almost all amplitudes are ill-defined. The reason for this is that whenever there are two symmetry states $x_1 \otimes \bar{x}_1, x_2 \otimes \bar{x}_2 \in H^{1,1}$ in a string amplitude with $[x_1, x_2] \neq 0$ and $[\bar{x}_1, \bar{x}_2] \neq 0$, the string density will have poles on the boundary of moduli space where
the two operators collide. This is true because the OPE of symmetry states necessarily contains symmetry states:

\[ c\bar{c}J^a_1 J^{\bar{a}_1}(1) c\bar{c} J^a_2 J^{\bar{a}_2}(2) \sim c\partial c\partial \partial \partial J^{a_1 a_2} J^{\bar{a}_1 \bar{a}_2} J^{a_3} J^{\bar{a}_3}(2) \]

The spacetime interpretation of this result is that states made from products of symmetry currents have three-point interactions defined by the structure constants of a Lie algebra. In particular, any two on-shell states fuse to produce an on-shell state. There is no “violation of the Wheeler-DeWitt constraint.” Consequently, some internal line must be on-shell and the amplitude must have a pole.  

Although the amplitudes are ill-defined, the string densities for symmetry-states take a remarkably simple form. In the remainder of this subsection we digress to explain this point, which is probably of purely mathematical interest.

We first define some notation. The operator product of two symmetry currents takes the form:

\[ J^a(z_1) J^b(z_2) \sim \cdot \cdot \cdot + \frac{-g^{ab}}{(z_{12})^2} + \frac{f^{ab}_c J^c(z_2)}{z_{12}} + \cdot \cdot \cdot \]  

(8.1)

The first ellipsis is in general nonvanishing because we are working in theories where the conformal dimensions are not bounded below. If the only operator with a one-point function is the identity, and if the BPZ inner product is nondegenerate, then \( g^{ab} \) defines a nondegenerate invariant form on the Lie algebra \( H^1 \). Let us now consider the string density associated with a set of “symmetry states” \( x \otimes \bar{x} \), where \( x \in H^{1,0} \) and \( \bar{x} \in H^{0,1} \). The string density is an \((n - 3, n - 3)\) form on \( \mathcal{M}_{0,n} \). For symmetry states it factorizes holomorphically:

\[ \omega(x_1 \otimes \bar{x}_1, \ldots, x_n \otimes \bar{x}_n) = \tilde{\omega}(x_1, \ldots, x_n) \wedge \tilde{\omega}(\bar{x}_1, \ldots, \otimes \bar{x}_n) \]  

(8.2)

where \( \tilde{\omega} \) is an \((n - 3)\)-form. Let us choose coordinates \((z_1, \ldots, z_n)\) for configuration space. Fixing three points, say, \( z_1, z_2, z_3 \) we define corresponding coordinates \((z_4, \ldots, z_n)\) for moduli space \( \mathcal{M}_{0,n} \).

**Proposition 14:** A representative for the DeRham cohomology \( \tilde{\omega}(x_1, \ldots, x_n) \in H^*(\mathcal{M}_{0,n}) \) is given by

\[ \tilde{\omega}(x_1, \ldots, x_n) = \left[ z_{12} z_{23} z_{31} \right] \frac{1}{n} \sum_{\sigma \in S_n} \frac{\text{tr}(x_{\sigma(1)} \cdots x_{\sigma(n)})}{z_{\sigma(1)} \sigma(2) \cdots z_{\sigma(n)} \sigma(1)} dz_4 \wedge \cdots \wedge dz_n \]  

(8.3)

\(^{13}\) Special state operators in 2D string theory are ill-defined for the same reason.
where tr is the nondegenerate form on $H^1$ defined by the two-point function of currents.

**Proof:** This is easily proven by induction. First, using the operator product expansion we deduce that the four-point functions of currents takes the form:

$$
\langle J^{a_1}(z_1)J^{a_2}(z_2)J^{a_3}(z_3)J^{a_4}(z_4) \rangle =
- \left[ \frac{f^{a_1a_2}f^{ba_3a_4}}{z_1z_2z_3z_4} + \frac{f^{a_1a_3}f^{ba_2a_4}}{z_1z_2z_3z_4} + \frac{f^{a_1a_4}f^{ba_2a_3}}{z_1z_2z_3z_4} \right]
+ O\left(\frac{1}{z_4^2}\right) + O\left(\frac{1}{z_4^3}\right) + \cdots
$$

(8.4)

By direct calculation one may check that this agrees with (8.3) up to a total derivative in $\frac{\partial}{\partial z_4}$.

Similarly, using the operator product expansion we may establish the ward identity

$$
\tilde{\omega}_n(x_1, \ldots, x_n) = \sum_{j=1}^{n-1} \frac{1}{z_n - z_j} \tilde{\omega}_{n-1}(x_1, \ldots, x_{j-1}, [x_j, x_n], \ldots x_{n-1}) \wedge dz_n + d\xi
$$

(8.5)

for some $(n-4)$-form $\xi$. It is simple to check that (8.3) satisfies (8.5) and hence the proposition follows. ♠

**Remarks:**

1. The sum over the permutation group in (8.3) effectively antisymmetrizes the $x$’s so that the form $\tilde{\omega}$ may be expressed in terms of polynomials in the structure constants.

2. The form $\tilde{\omega}$ defines a map on the tensor product of ghost number one states: $(H^1_Q)^{\otimes n} \to H^{n-3}(\mathcal{M}_{0,n})$, or, equivalently, a map

$$
H_{n-2}(\mathcal{M}_{0,n+1}) \otimes (H^1_Q)^{\otimes n+1} \to \mathfrak{c}
$$

(8.6)

These maps are very interesting in view of recent developments in the theory of BV algebras and related structures [54] [37] [38] [36] [55]. The top-degree homology of moduli space is: $L_n = H_{n-2}(\mathcal{M}_{0,n+1}) \cong H_{n-1}(F_n(\mathfrak{c}))$, where $F_n$ is the configuration space. The collection of spaces $L_n$ together form a linear operad known as the Lie operad: $\text{Lie}(n) = L_n$.

If $V$ is any graded vector space then by taking invariants w.r.t. the symmetric group $S_n$ and summing:

$$
\bigoplus_{n \geq 0} (L_n \otimes V^{\otimes n+1}) S_n
$$

we obtain the free Lie algebra on $V$ [54] [38] [36] [55]. Put another way: one can give an exotic definition of a Lie algebra structure on $V$ as a collection of maps

$$
L_n \otimes V^n \to V
$$
which are compatible with the operad structure.

Using the invariant form on $H_Q^1$ (=two point function) we can write the period maps (8.3) as

$$L_n \otimes (H_Q^1)^{\otimes n} \rightarrow H_Q^1$$

Put this way, we see that the period maps can be taken as the defining equations of the Lie algebra $H_Q^1$. They are sure to be especially interesting for the holomorphic CFT $C = S(h_{1,25}^-) \otimes \mathbb{Q}[H^{1,25}].$

8.3. Transport of amplitudes

If we transport the symmetry states away from a symmetric point the amplitudes become finite because the terms in the operator product are generically off-shell, so the BV product of [37] is zero. One way to define some of the amplitudes of the symmetry states is to rotate away from the ESP and define the amplitude at the ESP by a limiting procedure.

Example: Consider vertex operators corresponding to simple roots: $V = e^{ip_L \cdot Y} e^{ip_R \cdot Y} C(p_L; p_R)$ where $C$ is a cocycle operator. Up to a sign the four-point function is

$$\pi \frac{\Gamma(p_L^{(1)} \cdot p_L^{(2)} + 1)}{\Gamma(-p_R^{(1)} \cdot p_R^{(2)})} \frac{\Gamma(p_L^{(1)} \cdot p_L^{(3)} + 1)}{\Gamma(-p_R^{(1)} \cdot p_R^{(3)})} \frac{\Gamma(p_L^{(1)} \cdot p_L^{(4)} + 1)}{\Gamma(-p_R^{(1)} \cdot p_R^{(4)})}$$

For simple roots there must be either two gamma-function poles in the numerator and one in the denominator or vice versa. Rotating away from the symmetry point by a group element $g$ we can define $\tilde{p} = g \cdot p$ and $x_{ij} = p_L^{(i)} \cdot p_L^{(j)} - p_L^{(i)} \cdot p_L^{(j)}$. Depending on the states at the ESP the limiting behavior for $g \rightarrow 1$ can be any of 36 possibilities: $x_{ij}, x_{ij} x_{ik}/x_{is}, x_{ij}/x_{is}, 1, 1/x_{ij}, x_{is}/x_{ij} x_{ik}$. Thus in some cases ($x_{ij}$ or 1) the limit $g \rightarrow 1$ is well-defined.

Remarks:

1. The string amplitudes for a fixed set of states can be considered as functions on $\mathbb{H}$ if we use the amplitudes associated to states all related by duality then we can construct interesting automorphic functions on $\mathbb{H}$ with prescribed singularities at the ESP’s.

2. Perhaps one could use this idea to define special state correlators in 2D string theory.

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8.4. Higher genus amplitudes

Any naive attempt to extend the amplitudes of the above string theories to higher loops will run into problems with divergences. These divergences are unrelated to the tachyon, which, after all, is usually not in the spectrum. Rather, the compactification of the timelike coordinate forces us to worry about the arbitrarily negative conformal weights that enter in off-shell loops. Of course, since we are working with gaussian models, specific conformal blocks will be well-defined, but we cannot sew them together.

In ordinary string theories this problem is addressed in an ad hoc way. One simply analytically continues the gaussian integral by hand. If time is compact one can still do this, but it essentially amounts to an ad hoc sign change in the metric. One loses all the interesting connections to BRST cohomology, modular invariance etc. This is not the right idea.

We believe the most promising direction is to give the worldsheet a Minkowskian signature, and to generalize G. Segal’s axiomatic framework for CFT to the category of surfaces with Minkowski signature conformal structure. This will require the solution to at least three problems. First, one must decide what singularities to allow in the metric. Second, as we have seen in sec. 2.6, the modular group will act ergodically on the analog of Teichmüller space and one must decide whether the theory should be formulated on moduli space or on Teichmüller space. Third, the amplitudes will have to be understood as some kind of generalized functions. The third point can easily be understood for the case of the torus. Choosing the metric to be

\[ ds^2 = \frac{1}{2} \left[ (d\sigma^1 - \tau_+ d\sigma^2) \otimes (d\sigma^1 - \tau_- d\sigma^2) + (d\sigma^1 - \tau_- d\sigma^2) \otimes (d\sigma^1 - \tau_+ d\sigma^2) \right] \]

where \((\tau_-, \tau_+) \in \mathbb{R}^2\backslash\text{diagonal}\), we easily calculate the path integral for a boson on radius \( R \) to be:

\[
\int d\phi e^{i\pi \tau_+ p^2} \sqrt{\det g} (\nabla \phi)^2 \frac{1}{|\eta|^2} \sqrt{\frac{i|\tau_- - \tau_+|}{2R^2}} \sum e^{i\pi \tau_+ p^2} e^{-i\pi \tau_- p^2 R^2}
\]

for \( \tau_- > \tau_+ \). The sum on momenta is a delicate function of \( \tau_\pm \). For special compactifications and rational \( \tau_\pm \) the sum can be defined by a limiting procedure in terms of Dedekind symbols \([56]\).
9. Broken Ward identities

Consider a symmetry current \( J \) at an enhanced symmetry point \( \Gamma \). Fields are organized into representations \( R \): \( V^{R,\lambda} \) where \( \lambda \) is a weight vector. (The same representation may appear with (infinite) multiplicity in the theory.) The symmetric Ward identity is:

\[
0 = \sum_i \langle \delta J V^{R_i,\lambda_i}(z_i, \bar{z}_i) \prod_{j \neq i} V^{R_j,\lambda_j} \rangle
\]

\( \delta J V^{R_i,\lambda_i}(z_i, \bar{z}_i) \equiv \oint_{z_i} dw J(w) V^{R_i,\lambda_i}(z_i, \bar{z}_i) \)

Now suppose the symmetry is spontaneously broken by exactly marginal perturbations. What do the Ward identities look like?

The first important point to note is that in the broken phase the Ward identities surely exist. In spontaneously broken gauge theories the Ward identities, written as Slavnov-Taylor identities, hold true irrespective of the vacuum chosen by the theory. Indeed this is the crucial observation behind the proofs of renormalizability of spontaneously broken gauge theory.

There is one very important difference in the Ward identities in the broken and unbroken phase: in the unbroken phase the WI’s relate Green’s functions that can be taken to be simultaneously on-shell. They are therefore identities for \( S \)-matrix elements. In the broken phase this is not the case.

**Example:** Consider an \( SU(2) \) gauge theory. Denote \( W^\pm(p, \epsilon) \) the spacetime gauge field of momentum \( p \) and polarization \( \epsilon \) associated with generators \( \tau^\pm \) and \( Z(p, \epsilon) \) the gauge field associated with generator \( \tau^3 \). A typical Ward identity is

\[
\langle 0|T^*[W^+(1)Z(2)W^-(3)Z(4)]|0\rangle + \langle 0|T^*[W^+(1)W^-(2)Z(3)Z(4)]|0\rangle
\]

\[+ \langle 0|T^*[W^+(1)W^-(2)W^-(3)W^+(4)]|0\rangle = 0
\]

(9.2)

where 1, 2, 3, 4 refer to polarization and momentum. In the unbroken phase this is true of Green’s functions for \( p, \epsilon \) on-shell as well as off-shell. It is, in particular, true of on-shell \( S \)-matrix elements. In a broken phase where, typically, \( m^2_Z \neq m^2_{W^\pm} \) it is impossible to put all three terms simultaneously on-shell.

This observation shows that it is difficult to write the Ward identities for the broken gauge symmetries in string theory, since we must go off-shell. The most direct approach to this problem is to use conformal perturbation theory with the little disks cutoff.

**Proposition 15.** (Broken Ward Identities: First version.)
Let $T_{\epsilon} = T_{\epsilon}^O$ the the little-disks transport operator for a $(1,1)$ operator $O$ in a conformal field theory $C$ defined in (7.11). If $J$ is symmetry current in $C$ then we have

$$0 = \int_{F_1(z_1, \ldots, z_n; \epsilon)} d^2w \langle T_{\epsilon}(\rho(J) \cdot O)(w\bar{w}) \prod T_{\epsilon}V^{R,\lambda}(z_i\bar{z}_i) \rangle$$

$$+ \sum_i \langle T_{\epsilon}(\rho(J) \cdot V^{R,\lambda}) \prod_{j\neq i} T_{\epsilon}(V^{R,\lambda}) \rangle$$

(9.3)

**Proof:** Expand all terms according to the definition of the point-split perturbation series. Use the Ward identities in the symmetric theory:

$$0 = \sum_{k=1}^m \langle \delta J O_k \prod_{l\neq k}^n O_l \prod_{j=1}^n V_j \rangle + \sum_{i=1}^n \langle \prod_l O_l \delta J V_i \prod_{j\neq i}^n V_j \rangle$$

(9.4)

Now use the fact that

$$\int_{F_m(z_1, \ldots, z_n; \epsilon)} \prod_{i=1}^m d^2w_i(\cdot \cdot \cdot) = \int_{F_1(z_1, \ldots, z_n; \epsilon)} d^2w_1 \int_{F_{m-1}(z_1, \ldots, z_n, w_1; \epsilon)} \prod_{i=2}^m d^2w_i(\cdot \cdot \cdot)$$

(9.5)

to get (9.3). ♠

The disadvantage of this approach is that the strict little-disk cutoff gives a transport that we do not understand very well. On the other hand, when we interpret the Lorentz transport in conformal perturbation theory as in appendix A, we cannot rerun the argument of (9.4) because of contact terms. Thus while it is clear from spacetime reasoning that the broken WI's must exist they appear to be inaccessible from the world-sheet point of view. We seem to be at an impasse. In the next subsection we propose an end-run around the problem.

9.1. Transport of currents

Consider an enhanced symmetry point $\Gamma \in \mathbb{L}$ with a holomorphic current $J$. When the operator

$$O = \partial Y \cdot \Delta \mathcal{E} \cdot \bar{Y}$$

with $\Delta \mathcal{E}$ given by (7.16) is added to the action the symmetry associated to $J$ is broken if

$$\delta J O = \oint w J(z)dzO(w, \bar{w}) \neq 0.$$ 

Let $T^g = T^g_{\Gamma,\Gamma}$ be the Lorentz-transport to some point at which $T^g J$ is no longer an enhanced symmetry.
Proposition 16

(a) \( T^g(\delta J \mathcal{O}) \) is a total derivative:
\[
T^g(\delta J \mathcal{O})(z, \bar{z}) = -\frac{\partial}{\partial \bar{z}} (T^g J(z, \bar{z})) - \frac{\partial}{\partial z} (T^g \bar{J}(z, \bar{z}))
\] (9.6)

(b) \( \bar{J} \) may be calculated explicitly as follows. Let \( J = \mathcal{P} e^{ipY} \), where \( \mathcal{P} \beta_{-1}, \beta_{-2}, \ldots \) is a polynomial in holomorphic oscillators. Let
\[
Q_b[\beta_{-1}, \ldots] = \sum_{s \geq 1} (-\partial_z)^{s-1} \left[ \frac{\partial \mathcal{P}}{\partial \beta_{-s}^a}(z)(\Delta \mathcal{E})_b^a e^{ipY}(z) \right] e^{-ipY}(w)
\]
(9.7)

then
\[
\bar{J} = Q_b[\beta_{-1}, \ldots] \bar{\beta} e^{ipY}(w) - i\bar{Q}_b[\beta_{-1}, \ldots] \eta g_{22}^r \eta g_{21}^t \bar{b} e^{ipY}(w)
\] (9.8)

Proof: Proceed by direct calculation. Treat separately the contractions of \( \partial Y(w) \) with the exponential and the oscillator pieces:
\[
\delta J \mathcal{O}(w, \bar{w}) = \int_w \mathcal{P} e^{ipY}(z) dz \partial Y \cdot \Delta \mathcal{E} \cdot \bar{Y}(w, \bar{w})
\]
\[
= \psi_1(w, \bar{w}) + \psi_2(w, \bar{w})
\]
\[
\psi_1(w, \bar{w}) = i \mathcal{P}(w)p \cdot \Delta \mathcal{E} \cdot \bar{\beta} e^{ipY}
\]
\[
\psi_2(w, \bar{w}) = \partial_w K
\]
\[
\partial_w K = -: \frac{\partial \mathcal{P}}{\partial \beta_{-1}^a}(w)(\Delta \mathcal{E})_b^a \bar{\beta} e^{ipY}(w) : + \partial_w \bar{I}
\]
\[
I = \sum_{s \geq 2} \int_w dz (-\partial_z)^{s-2} \left( \frac{1}{z-w} \right) : \left[ \frac{\partial \mathcal{P}}{\partial \beta_{-s}^a}(z)(\Delta \mathcal{E})_b^a \bar{\beta} e^{ipY}(z) \right] :
\]

Compute \( T\psi_1 \) and \( T\psi_2 \) separately, use the relations of \( O(D; \mathbb{R}) \), and take proper account of the difference \( T(\partial_w \bar{J}) - \partial_w (T\bar{J}) \) arising from normal ordering. This gives the two terms on the right hand side of (9.6) after a cancellation. ✽

Proposition 17: (Broken Ward identities: second version.)
Proof: Use proposition 16 and integrate by parts.

Remarks:
1. Instead of deriving identities on string amplitudes it might be more interesting to use Proposition 17 to derive symmetry constraints on the 1PI vertices of closed string field theory by integrating over \( V_{0,n} \) of \( \mathcal{M} \). We have not succeeded in doing this because of the difficulties of handling Strebel differentials.

2. Proposition 17 provides an answer to the question: “What happens to the affine \( SU(2)_1^{(1)} \times SU(2)_1^{(1)} \) symmetry of the \( c = 1 \) gaussian model when we move away from the self dual point?” The answer is that to have a good deformation theory one should look at the entire chiral algebra. The operator product algebra of the transported chiral currents \( T \) closes on itself and is of index two in the full operator algebra of the gaussian model at finite radius. This deformation of chiral algebras is probably related to \( q \)-deformed \( SU(2)_1^{(1)} \), but the details are obscure.

3. We also hope to apply the Ward identities to high energy symmetries of string theory. By a clever decompactification of time one might arrange that \( T^g(\delta_J \mathcal{O}) \) is \( Q \)-trivial and \( T^g J \) is holomorphic. If this can be done we can explain some of the high-energy symmetries of string theory (see the conclusions).

10. Conclusions

This paper was motivated by recent progress in CSFT and in 2D string theory. A comparison with the situation in two-dimensional string theory is in order. Here too the string backgrounds leave unbroken an infinite-dimensional Lie algebra of symmetries. It is sometimes said that 2D string theory is uniquely characterized among string backgrounds in possessing large unbroken symmetries. Clearly, this is not the case. It is also sometimes suggested that the vast reduction in the number of states in the 2D case is characteristic of the existence of large enhanced symmetries. The above examples again provide counterexamples to this remark. The discontinuous behavior of the the BRST cohomology and the relation to spontaneous symmetry breaking are similar in both cases. The unbroken symmetries in 2D are different from our case in that there is a large ghost number zero cohomology ring (the “ground ring”). Associated with this is the extraordinary fact that symmetries preserve the relation \( p_L = p_R \) on tachyon vertex operators. Thus they exist at the self-dual as well as at infinite radius, hence identities derived at the SD radius are true
at infinite radius (at least for relations between special tachyons). This is a much simpler situation than discussed above.

We conclude with two (wild) speculations.

First: In ordinary particle physics broken gauge symmetries still have consequences for physics. For example, in the standard model, at high energies the scattering of $W$’s and $Z$’s are related by symmetry. One may ask if there are consequences for broken hyperbolic symmetries when time is decompactified. The transport operator $T^g$ maps states at fixed quantum numbers to states with infinite energy as we decompactify. Thus the identities in Proposition 17 should be interpreted as high energy symmetries of string theory. An intriguing set of speculations by D. Gross [57] and E. Witten [58] seems to be closely related to these symmetries. It would be interesting to make these remarks more concrete. Perhaps one can use the ergodic action of $O(η) \times O(η)$ on nearly decompactified spacetimes.

Second: Many facts about closed string field theory strongly indicate that the closed string is in a spontaneously broken phase of some more symmetric theory (e.g., topological field theory). For example, the string field transforms inhomogeneously $Ψ \rightarrow Ψ + QA + \cdots$ characteristic of a broken symmetry phase. Moreover, the action is nonpolynomial, reminiscent of the chiral lagrangian for pion dynamics. Finally, as is clear from the existence of enhanced symmetry points in toroidal compactifications, Minkowski space is a very unsymmetric ground state of string. In view of these remarks the discussion of section 5.5 suggests that the Lie algebra $L_U$ and its supersymmetric analogs may be very fundamental in string theory. It further suggests that the closed string is some kind of broken symmetry state of a 52-dimensional open string moving in a spacetime with signature $(η; −η)$. Recently Witten has made an interesting proposal for a background independent formulation of open string field theory [59,61]. Rather than find an analogous formulation for closed string field theory perhaps we should concentrate on the 52-dimensional open string and understand the broken phase of this theory.

Notes on the text:

1. This paper is in final form and will not be published anywhere else but on hep-th. Therefore, if the reader has occasion to make reference to it, we would appreciate it if he or she would use the hepth number and not just the preprint number. The same remark applies to our paper [62].

2. Many of our results were announced at the SUSY93 conference at Northeastern University on April 1. There, we learned of related independent work by K. Ranganathan, H. Sonoda, and B. Zwiebach [48] on formulating a connection on a family of CFT’s.
Acknowledgements

I would like to thank several colleagues for important and useful conversations relevant to this material. Special thanks are due to R. Plesser for many interesting discussions, and for enduring many versions of appendix A below. I also thank I. Frenkel, H. Garland, E. Getzler, J. Horne, B. Lian, G.D. Mostow, S. Ramgoolam, G. Segal, N. Seiberg, G. Zuckerman, and B. Zwiebach. This work is supported by DOE grant DE-AC02-76ER03075, DOE grant DE-FG02-92ER25121, and by a Presidential Young Investigator Award.

Appendix A. Summing the Conformal Perturbation Series

A.1. Preferred coordinates and conformal perturbation theory

In this appendix we prove the following. Let $\Gamma \in \mathbb{L}$, $g \in O(D; \mathbb{R})$, and define

$$O = \partial Y \cdot \Delta \mathcal{E} \cdot \bar{\partial} Y$$

$$\Delta \mathcal{E}(g) \equiv -2g_{12}(g_{11})^{-1} \eta = -2\eta(g_{11})^{-1}g_{12}$$

as in Proposition 13 above. Then, for an appropriate definition of the integrals over configuration spaces we have:

$$\left\langle e^{-\frac{i}{\hbar} \int_\Sigma \mathcal{O}} \prod_{j=1}^n \Phi_j(z_j, \bar{z}_j) \right\rangle_{\Gamma} = \prod_{i=1}^n \epsilon^{(\Delta'_i + \bar{\Delta}'_i) - (\Delta_i + \bar{\Delta}_i)} \left\langle \prod_{j=1}^n T^g(\Phi_j)(z_j, \bar{z}_j) \right\rangle_{g, \Gamma}$$

where $\epsilon$ is a cutoff parameter described below and $T^g$ is the Lorentz transport defined in section 7. (We could absorb $\epsilon$ into $T^g$.)

Our calculation will strike many readers as absurdly pedantic, so we offer a few words in its defense. The essential point is that the natural parametrization of $\mathbb{H}$ follows from the contact-term prescription:

$$\left\langle \partial Y^a(z) \bar{\partial} Y^b(\bar{w}) \right\rangle = -\eta^{ab} \pi \delta^{(2)}(z - w)$$

One might worry about multiple-contact terms when several points coalesce. Also, (A.3) must be combined with some kind of point-splitting regularization to define the integral of $\mathcal{O}(w, \bar{w})$ in the neighborhood of $\Phi_i(z_i, \bar{z}_i)$. We would like to specify exactly how this should be done. Therefore, we define the necessary integrals and the combinatorics with some care. It may also seem silly to spend so much effort on summing the series when we can already calculate the answer at any point in $\mathbb{L}$. We use these known answers to learn about how the integrals in conformal perturbation series should be defined. The main point is: one cannot simply integrate over one fixed configuration space at each order; the integrals must be defined in a more subtle way. Roughly speaking, the treatment of the divergences on the boundaries of moduli space depends on how we contract operators, i.e., how the operator product expansion describes the boundaries of moduli space.
A.2. Proof in one-dimensional case

We begin by considering the case of one Euclidean signature scalar, so we may write $\Delta \mathcal{E} = -2 \tanh \lambda$ for a boost $g(\lambda) \in O(1, 1)$ as in (7.6). The corresponding operator is:

$$\mathcal{O} = -2 \tanh \lambda \partial Y \bar{\partial} Y$$  \hspace{1cm} (A.4)

The generalization to several scalars is straightforward, and indicated at the end.

Consider the numerator of (A.2) and expand the exponential series. The $m^{th}$ order term requires a definition of the integral:

$$\int_{F_m} \prod d^2 w_i \langle \prod_{i=1}^{m} \partial Y \bar{\partial} Y(w_i, \bar{w}_i) \prod_{i=1}^{n} \Phi_i(z_i, \bar{z}_i) \rangle$$  \hspace{1cm} (A.5)

where $F_m$ is the configuration space of the $w_i$.

WLOG we may assume each operator is of the form

$$\Phi_i(z_i, \bar{z}_i) = P_i[\beta_{-1}, \ldots; \bar{\beta}_{-1}, \ldots] e^{i p_L^{(i)} Y(z_i)} e^{i p_R^{(i)} Y(\bar{z}_i)}$$

where $P_i$ is a monomial formed from products of the Heisenberg modes. Cocycle operators may be easily included and factor out of the conformal perturbation series.

We first need to define the notion of a contraction scheme. Consider the correlator of the $\Phi_i$. If we evaluate it using Wick’s theorem we will use three kinds of contractions:

$A_1$. $\beta_{-n}$ in $P_i$ contracts with $\beta_{-m}$ in $P_j$, for $i \neq j$.

$\bar{A}_1$. Similarly for $\bar{\beta}$.

$B_1$. $\beta_{-n}$ in $P_i$ contracts with $e^{i p_L^{(j)} Y(z_j)}$ for $i \neq j$. It is useful to consider the sum on $j$ of such contractions for $j \neq i$ as a single term. This eliminates irrelevant boundary terms at infinity below.

$\bar{B}_1$. Similarly for antiholomorphic oscillators

$C_1$. (Anti-)Holomorphic exponentials with (anti-)holomorphic exponentials.

Consider factoring out the result of contractions of type $C$. The sum over the remaining terms coming from contractions of types $A$ and $B$ is a sum over contraction schemes. Each scheme is a combinatorial prescription for how to contract oscillators in $\Phi_i$ with other oscillators or with (all) the exponentials. If there are no exponentials the sum over contraction schemes is the same as the sum over Wick contractions.
**Example:** The holomorphic correlator \( \langle \beta_{-1} e^{ip_1 Y(z_1)} \beta_{-1} e^{ip_2 Y(z_2)} e^{ip_3 Y(z_3)} \rangle \) is a sum of two contraction schemes:

\[
\prod_{i<j} z_{ij}^{p_i p_j} \left( -\frac{1}{(z_{12})^2} \right) + \prod_{i<j} z_{ij}^{p_i p_j} \left( -\frac{ip_2}{z_{12}} - \frac{ip_3}{z_{13}} \right) \left( -\frac{ip_1}{z_{21}} - \frac{ip_3}{z_{23}} \right) \quad (A.6)
\]

Now let us return to the integrand of (A.5). The correlator may be written as a sum over contraction schemes. We now try to relate the sum over schemes to the sum over schemes in the original correlator. Each term in the sum may be visualized as a product of chains of contractions of the operator \( \mathcal{O} \) beginning and ending with a contraction of \( \mathcal{O} \) with an oscillator or exponential in the operators \( \Phi_i \):

\[
\prod_{i=1}^{m} \partial Y \bar{\partial} Y(w_i, \bar{w}_i) \prod_{i=1}^{n} \Phi_i(z_i, \bar{z}_i) \right) = \sum_{S} \prod_{\ell} f_{c_{\ell}}(w_{i_1}, \ldots w_{i_k}; z_1, \ldots z_n) \quad (A.7)
\]

where \( f_{c_{\ell}} \) is the contribution of a chain, and we sum over schemes.

Comparing (A.7) with the sum for \( \prod \Phi_i \) we see that chains in (A.7) may be identified with contractions in the unperturbed correlators if the beginning and end of the chain matches cases A-C above. We will view the insertion of chains and subsequent integrals over the \( w \)'s as a “dressing” of the original contraction. In addition there are new kinds of contractions:

- **A2:** \( \beta \) in \( P_i \) with \( \bar{\beta} \) in \( P_j \), \( i \neq j \), and the conjugate.
- **A3:** \( \beta \) in \( P_i \) with another \( \beta \) in \( P_i \), and conjugate.
A4: $\beta$ in $P_i$ with $\bar{\beta}$ in $P_i$, and conjugate.

B2: $\beta$ in $P_i$ with the exponential at $i$ or with an antiholomorphic exponential at $j \neq i$, and conjugate.

C2: holomorphic with antiholomorphic exponentials.

D: Chains not involving the $\Phi$. These are cycles of length two, corresponding to the disconnected part of the sum.

Finally, we are ready to define the integrals in (A.3). We must integrate over different spaces for different contraction schemes. Roughly speaking, if we have a scheme that decomposes the set of $w_i$ into disjoint chains $c_1, \ldots c_\ell$ then we integrate over $F_{|c_\ell|}$ deleting disks of radius $\epsilon$ around the points $z_i$ only:

$$\int_{F_{|c_\ell|}} \prod d^2w_i \left( \prod_{i=1}^{m} \partial Y \bar{\partial} Y (w_i, \bar{w}_i) \right) \prod_{i=1}^{n} \Phi_i (z_i, \bar{z}_i) \equiv \sum \sum \int_{F_{|c_\ell|}} \prod d^2w_i f_{c_\ell} \quad (A.8)$$

We still have to define the singular integrals over the chains. We do this with the following integration formulae. We always assume $\sum p^{(j)}_L = \sum p^{(j)}_R = 0$. The integration prescription is:

1.

$$\int d^2w \left( \sum_{j=1}^{n} \frac{p^{(j)}_L}{w - z_j} \right) \left( \sum_{j=1}^{n} \frac{p^{(j)}_R}{\bar{w} - \bar{z}_j} \right) \equiv -\pi \sum_{1 \leq j \neq k \leq n} p^{(j)}_L \cdot p^{(k)}_R \log(|z_{jk}|^2/\epsilon^2)$$

2a. For $w_1 \neq z_i$:

$$\int d^2w \left( \sum_{j=1}^{n} \frac{p^{(j)}_L}{w - z_j} \right) \frac{1}{(w - w_1)^2} \equiv -\pi \sum_{j=1}^{n} \frac{p^{(j)}_L}{\bar{w}_1 - \bar{z}_j}$$

2b. For $n \geq 0$:

$$\int d^2w \left( \sum_{j=1}^{n} \frac{p^{(j)}_L}{w - z_j} \right) \frac{1}{(w - \bar{z}_1)^{n+2}} \equiv -\pi \frac{1}{(n+1)!} (\frac{\partial}{\partial \bar{z}_1})^n \sum_{j \neq 1} \frac{p^{(j)}_L}{\bar{z}_1 - \bar{z}_j}$$

3a. For $z_1 \neq z_2$, $n \geq 0$, $m \geq 0$:

$$\int \prod_{i=1}^{2k+1} d^2w_i \frac{1}{(z_1 - w_1)^{n+2}} \frac{1}{(\bar{w}_1 - \bar{z}_2)^2} \frac{1}{(w_2 - \bar{w}_3)^2} \cdots \frac{1}{(\bar{w}_{2k+1} - \bar{z}_2)^{m+2}} \equiv 0$$
3b. For $n, m \geq 0$

$$
\int \prod_{1}^{2k+1} d^2 w_i \left( \frac{1}{(z - w_1)^{n+2}} \frac{1}{(\bar{w}_1 - \bar{w}_2)^2} \frac{1}{(w_2 - w_3)^2} \cdots \frac{1}{(\bar{w}_{2k+1} - \bar{z})^{m+2}} \right)
\equiv \delta_{n,m} \frac{(-1)^n \pi^{2k+1}}{n + 1 \epsilon^{2n+2}}
$$

4a. For $z_1 \neq z_2$

$$
\int \prod_{1}^{2k} d^2 w_i \left( \frac{1}{(z_1 - w_1)^{n+2}} \frac{1}{(\bar{w}_1 - \bar{w}_2)^2} \frac{1}{(w_2 - w_3)^2} \cdots \frac{1}{(\bar{w}_{2k} - z_2)^{m+2}} \right)
\equiv \pi^{2k} \frac{1}{n + m + 2} \left( \frac{n + m + 2}{n + 1} \right) \frac{1}{(z_1 - z_2)^{n+m+2}}
$$

4b. For $n, m \geq 0$

$$
\int \prod_{1}^{2k} d^2 w_i \left( \frac{1}{(z - w_1)^{2+n}} \frac{1}{(\bar{w}_1 - \bar{w}_2)^2} \frac{1}{(w_2 - w_3)^2} \cdots \frac{1}{(\bar{w}_{2k} - z)^{2+m}} \right) \equiv 0
$$

It is important that the integrals above are related to but differ from the corresponding point-split integrals. The point-split integrals are easily evaluated using

$$
\int_{\Sigma} d^2 w \frac{\partial}{\partial \bar{w}} (f(w, \bar{w})) = -\frac{i}{2} \int_{\partial \Sigma} df
$$

where on the RHS one traverses the boundary with the left hand pointing inward to the region $\Sigma$. Choosing cuts one can evaluate the point-split integrals of type 1 exactly. For example, for $n = 2$ the exact point-split expression is

$$
2\pi p_{LP} \log \left( \frac{|z_{12}|^2}{\epsilon^2} - 1 \right)
$$

for $|z_{12}| > \epsilon$. For $n > 2$ integral 1 differs from the point-split answer by terms of order $O(\epsilon^2)$. Similarly, the other formulae are all related to the exact point-split formula:

$$
\int d^2 w \left( \frac{1}{(\bar{w}_1 - \bar{w})^2} \frac{1}{w - z} \right) = -\pi \frac{1}{\bar{w}_1 - \bar{z}}
$$

where we integrate over $|w - w_1| > \epsilon, |w - z| > \epsilon$ and assume $|w_1 - z| > \epsilon$. If we try to use the point-split integrals blindly then we find the log’s to not exponentiate and we destroy conformal invariance. We now prove that, if we use the above definitions of chain integrals we can preserve conformal invariance.
First note that formula 3a shows that chains of type \( A_2 \) contribute zero. Likewise, formula 4b shows that chains of type \( A_3 \) contribute zero. The new contractions of type \( A_4 \) are not associated with unperturbed contractions, but are associated with contractions of a different unperturbed correlator. These \( A_4 \) contractions are a normal ordering effect. See case 3 below. The remaining new types of contractions: \( B_2, C_2 \) may be considered as “dressings” of unperturbed contractions.

Organizing the sum this way we can examine the combinatorics. Chains involve ordered tuples but the ordering clearly does not affect the integrals above. Therefore if there are \( 2n_d \) chains of type D, \( n_e \) chains of type C, \( n_1, \ldots n_l \) disjoint chains of types A,B the sum of terms contributing to a given scheme at \( m^{th} \) order in perturbation theory will be a sum over partitions \( 2n_d + n_e + \sum n_j = m \) weighted by a combinatorial factor:

\[
\frac{m!}{(2n_d! \prod_i n_i)! (2n_d)!} \sum_{n_j} n_1! n_2! \ldots n_l! \quad (A.9)
\]

The conformal perturbation series for the dressings of a given contraction scheme now factors as a product of distinct infinite series. We examine each type of series in turn:

1. The disconnected terms, of course, factor out as an exponential series:

\[
\exp\left[\frac{\Delta E^2}{8\pi^2} \int \frac{d^2w_1 d^2w_2}{|w_1 - w_2|^4}\right] \quad (A.10)
\]

These are the only terms where an infrared as well as an ultraviolet cutoff is necessary. The denominator in the LHS of (A.2) cancels (A.10).

2. Contractions of type \( A_1, \bar{A}_1 \) are dressed by a geometric series. Thus, the dressing of a contraction of type A1 may be calculated using formula 4a:

\[
\beta_{-n}(z_1) - \cdots - \beta_{-m}(z_2) = \sum_{k \geq 0} \left( \frac{-\Delta E}{2\pi} \right)^{2k} \pi^{2k} \langle \beta_{-n}(z_1) \beta_{-m}(z_2) \rangle = \frac{1}{1 - \Delta E^2 / 4} \langle \beta_{-n}(z_1) \beta_{-m}(z_2) \rangle = \cosh^2 \lambda \langle \beta_{-n}(z_1) \beta_{-m}(z_2) \rangle \quad (A.11)
\]

where we identify \( \Delta E = -2 \tanh \lambda \).

3. Contractions of type \( A_4 \) also form a geometric series and result in new contraction schemes. These correspond to the fact that after rotating \( \beta \) in a polynomial of \( \beta, \bar{\beta} \)’s
one must normal order, e.g., as in (7.7). For example, consider summing the chains that connect $\beta_{-n}$ to $\bar{\beta}_{-n}$ at the same point $z_i$. Using formula 3b this series becomes:

$$
\beta_{-n}(z_i) - - - \bar{\beta}_{-n}(z_i) = \sum_{k \geq 0} \left( -\frac{\Delta E}{2\pi} \right)^{2k+1} \pi^{2k+1} \frac{n^{2k+1}}{e^{2n}}
$$

$$
= -\frac{\Delta E}{2} \frac{1}{1 - \Delta E^2/4} \frac{1}{n^{2n}}
$$

(A.12)

This corresponds to the normal-ordering in the transport:

$$
T^g([\cdots \beta_{-n} \bar{\beta}_{-n} \cdots]) = [\cdots (\cosh \lambda \beta_{-n} + \sinh \lambda \bar{\beta}_{-n})(\cosh \lambda \bar{\beta}_{-n} + \sinh \lambda \beta_{-n}) \cdots]
$$

$$
= \cosh^2 \lambda \cdots \beta_{-n} \bar{\beta}_{-n} \cdots
$$

(A.13)

4. Contractions of type B. Again we get a geometric series. Using formulae 2a,2b we see that the contraction of $\beta_{-n}$ at $z_i$ with holomorphic exponentials at $z_j$ is dressed as

$$
\beta_{-n}(z_i) - - - \prod_{j=1}^n e^{ip(j)Y} = \sum_{k \geq 0} \left( -\frac{\Delta E}{2\pi} \right)^{2k} \pi^{2k} \sum_{j \neq i} \frac{-ip(j)}{z_i - z_j}
$$

$$
+ \sum_{k \geq 0} \left( -\frac{\Delta E}{2\pi} \right)^{2k+1} \pi^{2k+1} \sum_{j \neq i} \frac{-ip_R(j)}{z_i - z_j}
$$

(A.14)

where $p_L' = \cosh \lambda p_L + \sinh \lambda p_R$.

5. Finally we consider chains of type C. Chains of length $k$ connect (anti-)holomorphic to (anti-)holomorphic exponentials for $k$ even and vice versa for $k$ odd. Since we are working with exponentials we can have an arbitrary number of chains of length $k$ connect any two exponentials. Suppose there are $r_i$ chains of length $2i$ connecting holomorphic to holomorphic exponentials, $\bar{r}_i$ chains of length $2i$ connecting anti-holomorphic to anti-holomorphic exponentials, and $s_i$ chains of length $2i + 1$ connecting holomorphic to anti-
holomorphic exponentials. Then the conformal perturbation series becomes the sum:

\[
\sum_{r_i, \bar{r}_i, s_i \geq 0} \prod_{i \neq j} \frac{1}{2r_i r_j} \frac{1}{2\bar{r}_i \bar{r}_j} \frac{1}{s_i} \left( -\frac{\Delta \mathcal{E}}{2\pi} \right) \sum \frac{\pi^{2i+1} \sum \pi^i p_L p_R \log \frac{|z_{ij}|^2}{e^2}}{\pi^{2i} \sum \pi^i p_L p_R \log \frac{|z_{ij}|^2}{e^2}} \prod \frac{|z_{ij}|^2}{\epsilon^2} = \\
exp \left[ -\frac{\Delta \mathcal{E}/2}{1 - \epsilon^2/\Delta \mathcal{E}^2/4} \sum_{i \neq j} \pi^i p_L p_R \log \frac{|z_{ij}|^2}{e^2} + \frac{\Delta \mathcal{E}^2/8}{1 - \epsilon^2/\Delta \mathcal{E}^2/4} \sum_{i \neq j} (\pi^i p_L p_R + \pi^i p_R p_L) \log \frac{|z_{ij}|^2}{e^2} \right]
\]  

(A.15)

Putting all these effects together, we see that we are simply describing the sum over contraction schemes for the rotated operators, rotated according to the Lorentz transport of sec. 7.

A.3. Generalization to n-dimensional case

Finally, if we have several Gaussian fields \( \Delta \mathcal{E} \) given in (A.1) is the correct parametrization with our integral prescription. For example, the generalization of (A.11) with \( n = m = 1 \) is

\[
\beta^a_{-1}(z_1) - \beta^b_{-1}(z_2) = \frac{-1}{(z_{12})^2} \eta \sum_{k \geq 0} \left( \frac{-1}{2\pi} \right)^{2k} \pi^{2k} (\Delta \mathcal{E} \eta \Delta \epsilon \epsilon^r \eta)^k \\
= \frac{-1}{(z_{12})^2} \eta \left( 1 - \frac{1}{4} \Delta \mathcal{E} \eta \Delta \epsilon \epsilon^r \eta \right) \\
= \frac{-1}{(z_{12})^2} (\eta g_{11} \eta g_{11} \eta)^{ab}
\]  

(A.16)

where in the last line we have used the relations of \( O(D; \mathbb{R}) \). Similarly, one may check that the other cases in the previous section generalize correctly with the choice (A.1).

A.4. Reparametrizations

Let us recall a well-known result, namely, that parametrizations of a moduli space of CFT’s encodes a contact term prescription [52] [63]. Suppose we consider the conformal perturbation series for an exactly marginal operator \( \mathcal{O}(z, \bar{z}) \). If two different ways of defining the integral

\[
\langle \left( \int \mathcal{O} \right)^n \prod \Phi \rangle
\]
are related by:
\[
\langle \left( \int \mathcal{O} \right)^n \prod \Phi \rangle_1 = \sum_{m \leq n} C_{n,m} \langle \left( \int \mathcal{O} \right)^m \prod \Phi \rangle_2
\]  
(A.17)

we may say that the two definitions differ by a contact term prescription. The sum \( \sum_{m \leq n} \) is necessary because \( \delta \)-function singularities of correlators of \( \mathcal{O} \) will be handled differently in the two prescriptions. Using (A.17) we can express the conformal perturbation series
\[
\langle e^{-\delta_1} \int \mathcal{O} \cdots \rangle_1
\]
in terms of amplitudes from definition 2. However, the resulting series is not a new conformal perturbation series
\[
\langle e^{-\delta_2} \int \mathcal{O} \cdots \rangle_2
\]
unless the contact term prescription is \textit{local}. The condition to have local contact terms is that there exists a set of numbers \( D_j \) such that, for all \( n, m \)
\[
\frac{1}{n!} C_{n,m} = \frac{1}{m!} \sum_{j_1 + \cdots + j_m = n} D_{j_1} \cdots D_{j_m}
\]  
(A.18)

When this is satisfied \( D_n = \frac{1}{n!} C_{n,1} \) and the two conformal perturbation series are related by a reparametrization:
\[
\delta_2 = \delta_1 + D_2(\delta_1)^2 + D_3(\delta_1)^3 + \cdots
\]  
(A.19)

Since the point-split series breaks conformal invariance it cannot be related to the definitions of this appendix by a reparametrization of \( \lambda \) or \( g \). We believe that what is going on is that the two contact term prescriptions are not related locally, i.e., do not satisfy (A.18).
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