THE LIE ALGEBROID ASSOCIATED WITH A HYPERSURFACE

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ABSTRACT. In this note we motivate the definition and use of Lie algebroids by revisiting the problem of reconstructing a hypersurface in Euclidean space from infinitesimal data.

1. Introduction

Lie algebroids are the infinitesimal version of Ehresmann’s categories differentiables [14], now called Lie groupoids. They were first introduced and studied by J. Pradines in 1967 [23]. Twenty years later Alan Weinstein and others pointed out important connections with Poisson geometry [26, 8]. There was a resurgence of interest, deep results followed, and the subject remains very active. Applications include foliation theory [22], noncommutative geometry [6, 19], Lagrangian mechanics and control theory [7], elasticity theory [18], and complex analysis [15].

Lie algebroids also arise naturally in elementary differential geometry. The purpose of the present note is to give a simple example of this by relating the recent observations of [4, 5] in a special case. Our note is meant as an invitation to Lie algebroids, about which no prior knowledge is assumed. Proper introductions and further references can be found in [6, 11, 13, 20]. For detailed historical notes, prior to 2003, see [20]. We assume familiarity with basic calculus on manifolds and Lie groups.

2. On the Killing fields tangent to a hypersurface

The infinitesimal isometries of $\mathbb{R}^{n+1}$ are the Killing fields, those vector fields whose local flows are rigid motions. To characterise a hypersurface $\Sigma$ in $\mathbb{R}^{n+1}$ up to rigid motions, it suffices to understand the relationship between $\Sigma$ and the Lie algebra $\mathfrak{g}$ of Killing fields.\footnote{For simplicity we shall behave as if $\Sigma$ is embedded in $\mathbb{R}^{n+1}$. However, with obvious modifications, our constructions apply to hypersurfaces that are merely immersed.} To this end, consider those smooth functions $X : \Sigma \to \mathfrak{g}$ on $\Sigma$ that are tangent to $\Sigma$, in the sense that

$$X(x)(x) \in T_x\Sigma,$$

for all $x \in \Sigma$. Such functions amount to sections of the vector bundle $A$ over $\Sigma$ defined by

$$A = \{(X, x) \in \mathfrak{g} \times \Sigma \mid X(x) \in T_x\Sigma\}.$$ 

Evidently, $A$ is a subbundle of the trivial bundle $\mathfrak{g} \times \Sigma$ of rank $n(n + 3)/2$. 
By construction, we have a canonical vector bundle morphism $$\#: A \to T\Sigma$$ defined by $$\#(X,x) = X(x)$$. This map happens to be surjective, suggesting that we think of $$A$$ as a ‘thickening’ of $$T\Sigma$$. Also, if $$\mathfrak{h} \subset A$$ denotes the kernel of $$\#$$ then the fibre $$\mathfrak{h}|_x$$ can be identified with the subalgebra of Killing fields vanishing at $$x \in \Sigma$$. In particular, $$\mathfrak{h}$$ is a bundle of Lie algebras.

Now $$A$$ itself is not a bundle of Lie algebras. However, we have the following intriguing observation: Just as the Jacobi-Lie bracket on vector fields on $$\Sigma$$ makes the space of vector fields $$\Gamma(T\Sigma)$$ into a Lie algebra, and satisfies the Leibniz identity,

$$[X, fY] = f[X,Y] + df(X)Y, \quad X, Y \in \Gamma(T\Sigma),$$

for all smooth functions $$f : \Sigma \to \mathbb{R}$$, so the section space $$\Gamma(A)$$ has a god-given Lie algebra bracket satisfying an almost identical identity, namely

$$[X, fY] = f[X,Y] + df(\#X)Y, \quad X, Y \in \Gamma(A).$$

Note here that we need the map $$\#: A \to T\Sigma$$ for we do not otherwise have a way of differentiating the function $$f$$ along a section of $$A$$.

How is this bracket on $$A$$ defined? First, define a bracket on $$\mathfrak{g} \times \Sigma$$ in the most naive way:

$$\{X, Y\}(x) = [X(x), Y(x)]_\mathfrak{g}.$$ 

Here we are viewing sections of $$A \subset \mathfrak{g} \times \Sigma$$ as $$\mathfrak{g}$$-valued functions, and the bracket on the right is the bracket on $$\mathfrak{g}$$, the restriction of the Jacobi-Lie bracket to the subalgebra of Killing fields. This bracket does not satisfy a Leibniz-type identity; rather it is ‘algebraic’ (bilinear over the ring of all smooth functions on $$\Sigma$$). However, we have:

**Proposition.** Let $$\nabla$$ denote the canonical flat connection on the trivial bundle $$\mathfrak{g} \times \Sigma$$. Then the bracket on $$\Gamma(A)$$ given by

$$[X, Y] = \nabla_{\#X}Y - \nabla_{\#Y}X + \{X, Y\}$$

is well-defined and satisfies the Leibniz identity (1). Moreover, with respect to this bracket, the map $$\Gamma(A) \to \Gamma(T\Sigma)$$ induced by $$\#: A \to T\Sigma$$ is a Lie algebra homomorphism.

It is not immediately obvious that the bracket is well-defined as $$A \subset \mathfrak{g} \times \Sigma$$ is not $$\nabla$$-invariant, unless $$\Sigma$$ has lots of symmetry (e.g., is a hyperplane or hypersphere). Moreover, $$\Gamma(A)$$ is not generally closed under the algebraic bracket $$\{\cdot, \cdot\}$$. The reader will find it instructive to prove this proposition on her own. A proof is included in the appendix.

3. **Lie algebroids**

**Definition.** A *Lie algebroid* with base $$\Sigma$$ is a vector bundle $$A$$ over $$\Sigma$$, together with a vector bundle morphism $$\#: A \to T\Sigma$$ covering the identity and called the anchor, and a Lie bracket on its space of sections, such that the Leibniz identity (1) holds.
Every tangent bundle is a Lie algebroid with the identity map as anchor, and every ordinary Lie algebra is a Lie algebroid over a point. Every Lie groupoid (see Section 4) has a Lie algebroid as its infinitesimalization. Other important examples are recalled in the appendix.

To explain the significance of the Lie algebroid \( A \) constructed above for a hypersurface \( \Sigma \subset \mathbb{R}^{n+1} \) we must say something about morphisms in the category of Lie algebroids. If a vector bundle map \( \omega: A_1 \to A_2 \) between Lie algebroids covers the identity on a common base \( \Sigma \), then we declare \( \omega \) to be a Lie algebroid morphism if the corresponding map of section spaces \( \Gamma(A_1) \to \Gamma(A_2) \) is a Lie algebra homomorphism. In particular, the anchor #: \( A \to T\Sigma \) of a Lie algebroid is always a Lie algebroid morphism, a corollary of the Jacobi and Leibniz identities; for a proof see the appendix. When the base of \( A_1 \) and \( A_2 \) are different then the definition of a morphism is more complicated and we give it here in a special case only: Suppose \( A \) is a Lie algebroid over \( \Sigma \) and \( g \) a Lie algebroid over a point (a Lie algebra). Then a linear map \( \omega: A \to g \) is a Lie algebroid morphism if the obvious extension to a morphism of vector bundles \( \bar{\omega}: A \to g \times \Sigma \) satisfies

\[
\bar{\omega}([X, Y]) = \nabla_{\#X} \bar{\omega}(Y) - \nabla_{\#Y} \bar{\omega}(X) + \{\bar{\omega}(X), \bar{\omega}(Y)\}.
\]

Here \( \nabla \) and \( \{\cdot, \cdot\} \) have the meanings given in Section 1. (In particular, a Lie algebroid morphism \( \omega: T\Sigma \to g \) is the same thing as a Mauer-Cartan form on \( \Sigma \), and \[1\] generalizes the classical Mauer-Cartan equations \[24\].)

4. An infinitesimal characterization of hypersurfaces

According to the preceding definition, if \( A \) is the Lie algebroid associated with a hypersurface \( \Sigma \subset \mathbb{R}^{n+1} \), then the composite \( \omega: A \to g \) of the inclusion \( A \subset g \times \Sigma \) with the projection \( g \times \Sigma \to g \) is a Lie algebroid morphism into the Lie algebra \( g \) of Killing fields. We call the map \( \omega: A \to g \) the logarithmic derivative of the embedding \( \Sigma \subset \mathbb{R}^{n+1} \), for it may be viewed as a generalization of Élie Cartan’s logarithmic derivative of a smooth map into a Lie group (see, e.g., \[24\]) known also as its Darboux derivative. According to the following theorem, any hypersurface can be reconstructed from its logarithmic derivative. In the statement \( \text{rad} g \subset g \) denotes the subalgebra of constant vector fields (the radical of the Lie algebra of Killing fields).

**Theorem** (The abstract Bonnet theorem). Let \( A \) be any Lie algebroid over a smooth, orientable, simply-connected, \( n \)-dimensional manifold \( \Sigma \), and let \( \omega: A \to g \) be a Lie algebroid morphism, where \( g \) is the Lie algebra of Killing fields on \( \mathbb{R}^{n+1} \).

Assume:

1. \( \text{rank} A = n(n + 3)/2 \).
2. The anchor #: \( A \to T\Sigma \) is surjective (transitivity).
3. \( \omega: A \to g \) is injective on fibres.
4. For some (and consequently any) \( x_0 \in \Sigma \), \( \omega \) maps the intersection of the kernel of # with the fibre \( A|_{x_0} \) to a subspace of \( g \) transverse to \( \text{rad} g \) (transversality).
Then $\Sigma$ can be realised as an immersed hypersurface in $\mathbb{R}^{n+1}$ in such a way that its associated Lie algebroid is isomorphic to $A$, and such that its logarithmic derivative is $\omega: A \to \mathfrak{g}$.

**Remark.** If a submanifold $\Sigma \subset \mathbb{R}^{n+1}$ is replaced by its image $\phi(\Sigma)$ under some rigid motion $\phi: \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$, then it logarithmic derivative is altered, but in a predictable way: just compose the old logarithmic derivative with $\text{Ad}_\phi: \mathfrak{g} \to \mathfrak{g}$, where $\text{Ad}_\phi \xi$ is the pushforward of the vector field $\xi$ under $\phi$. In this way, we may regard the logarithmic derivative of an embedding as an invariant under rigid motions. Moreover, up to rigid motions, every embedding can be reconstructed from this invariant, by the preceding theorem.

Noting that the sufficient conditions (1)–(4) are also necessary, we now sketch a proof of the abstract Bonnet theorem under the additional global assumption that $A$ is *integrable*, by which we mean that $A$ is the infinitesimalization of some Lie groupoid $\mathcal{G}$. This extra assumption is also necessary but turns out to be redundant [4]. In the proof we describe Lie groupoids sufficiently that the magnanimous reader will grasp the main ideas, and appreciate how Lie theory can be applied to problems of differential geometry in novel ways. In particular, let us emphasise that the same proof more-or-less delivers a characterization of arbitrary smooth maps $f: \Sigma \to G/H$ of a simply-connected manifold $\Sigma$ into an arbitrary homogeneous space $G/H$ ("Klein geometry"); if $\Sigma$ is not simply-connected, then there is a global obstruction to reconstructing such maps from infinitesimal data called the *monodromy*, also naturally formulated using Lie algebroid language [4]. Familiarity with our proof is not required in the remainder of this note.

**Sketch of proof of the theorem.** A Lie groupoid is a Lie group $\mathcal{G}$ whose identity element has been ‘blown up’ into a smooth embedded submanifold $\Sigma$ (the *base*) and for which multiplication is consequently defined only partially. In more detail, we should understand that $\mathcal{G}$ is a smooth manifold whose points represent the arrows (morphisms) of a category, each arrow $g \in \mathcal{G}$ beginning at one identity element $\alpha(g) \in \Sigma$ and ending at another, $\beta(g) \in \Sigma$. That is, the identity elements represent the objects of the category, and at the same time represent the identity arrows. Two arrows $g_1, g_2 \in \mathcal{G}$ are *multipliable* if they can be composed as arrows of the category, and their product is, by definition, their composition in the category. The partial multiplication is assumed to be smooth, as are the *source* and *target maps* $\alpha, \beta: \mathcal{G} \to \Sigma$ of the groupoid, which are additionally required to be submersions. Every element of $\mathcal{G}$ is assumed to have a multiplicative inverse. A morphism $\Omega$ between Lie groupoids is a smooth functor of the underlying categories. In particular, this means

$$\Omega(gh) = \Omega(g)\Omega(h),$$

whenever the product $gh$ is defined.
For the geometer, the canonical example of a Lie groupoid is the orthonormal frame groupoid of a Riemannian manifold \( \Sigma \), whose arrows consist of the isometries between possibly different tangent spaces of \( \Sigma \), the base of the groupoid.

The infinitesimalization of a Lie groupoid is a Lie algebroid. Just as the Lie algebra of a Lie group is the tangent space to the identity, so more generally, the Lie algebroid of a Lie groupoid \( G \) is the normal vector bundle of the base \( \Sigma \) of identity elements in \( G \). We omit here the description of the bracket and anchor.

Now suppose the Lie algebroid \( A \) in the theorem is the infinitesimalization of a Lie groupoid \( G \) over \( \Sigma \). In analogy with the case of Lie groups, we may suppose that \( G \) is ‘simply-connected’ which here means the source projection \( \alpha : G \to \Sigma \), taking an arrow to its starting point, has simply-connected fibres. The theorem known as Lie II states that every morphism of Lie algebras \( g_1 \to g_2 \) is the infinitesimalization of a homomorphism \( G_1 \to G_2 \) between Lie groups having \( g_1 \) and \( g_2 \) as their infinitesimalizations, assuming \( G_1 \) is simply-connected. Lie II generalizes to Lie groupoids, so that the morphism \( \omega : A \to g \) lifts to Lie groupoid morphism \( \Omega : G \to G \), where \( G \) is the isometry group of \( \mathbb{R}^{n+1} \).

To construct an immersion \( f : \Sigma \to \mathbb{R}^{n+1} \) using \( \Omega \) we first choose an appropriate target \( m_0 \in \mathbb{R}^{n+1} \) for the point \( x_0 \in \Sigma \) appearing in the transversality condition (4). To do this, let \( h_{x_0} \subset A \) denote the intersection of the kernel of \( \# : A \to T\Sigma \) with \( A|_{x_0} \). Then, by transversality and a dimension count, \( \omega(h_{x_0}) \) is a complement of \( \text{rad } g \) in \( g \). Every such complement is the subalgebra \( g_{m_0} \subset g \) of all Killing fields vanishing at some point \( m_0 \in \mathbb{R}^{n+1} \), so that \( \omega(h_{x_0}) = g_{m_0} \) and, in particular,

\[
\omega(h_{x_0}) \subset g_{m_0}.
\]

We define

\[
f(x) = \Omega(g) \cdot m_0,
\]

where \( g \in G \) is any arrow from \( x_0 \) to \( x \). Such an arrow exists, by what is called the transitivity of \( G \), following from the transitivity of \( A \). The main issue is to show that \( f(x) \) is independent of the choice of arrow \( g \). From (5) it suffices to show that

\[
\Omega(H_{x_0}) \subset G_{m_0},
\]

where \( H_{x_0} \subset G \) is the group of all arrows beginning and ending at \( x_0 \), and \( G_{m_0} \) is the group of isometries fixing \( m_0 \). It will not surprise the reader to learn that \( H_{x_0} \) is a Lie group whose infinitesimalization is the Lie algebra \( h_{x_0} \), so that condition (6) is precisely the infinitesimalization of condition (7). To show that the former implies the latter, it accordingly suffices to show that \( H_{x_0} \) is connected. However, it turns out that the restriction of the target projection \( \beta : G \to \Sigma \) to the connected fibre \( \alpha^{-1}(x_0) \), is a principal \( H_{x_0} \)-bundle over \( \Sigma \). The simple-connectivity of \( \Sigma \) and the long exact sequence in homotopy for the principal bundle shows that \( H_{x_0} \) is indeed connected.

To prove that \( f : \Sigma \to \mathbb{R}^n \) is an immersion with the desired properties requires that we describe the infinitesimalization functor from Lie groupoids to Lie algebroids in more detail and this not attempted here.

\( \square \)
5. THE FIRST AND SECOND FUNDAMENTAL FORMS

We suppose the reader is already acquainted with the classical Bonnet theorem for hypersurfaces [25, 17], also known as the fundamental theorem for hypersurfaces, and is no-doubt wondering: Where is the second fundamental form? Where are the Gauss-Codazzi equations? Naturally, this information is encoded in the logarithmic derivative of the hypersurface.

Let \( \omega : A \rightarrow g \) be any Lie algebroid morphism satisfying the hypotheses of the abstract Bonnet theorem above. This data amounts to extra structure on the two-manifold \( \Sigma \). We now define two symmetric tensors \( g_\omega \) and \( I I_\omega \) which coincide with the first and second fundamental forms of the hypersurface when \( \omega : A \rightarrow g \) is the logarithmic derivative of some embedding \( \Sigma \hookrightarrow \mathbb{R}^{n+1} \).

On account of condition (3) of the theorem, we may regard \( A \) as a subbundle of \( E := g \times \Sigma \) and do so from now on. Let \( \mathfrak{h} \subset A \) denote the kernel of the anchor \( \# : A \to T\Sigma \). Notice that when \( A \) is the Lie algebroid associated with an embedded hypersurface \( \Sigma \), we may evidently identify the quotient \( E/\mathfrak{h} \) with \( T\Sigma_{\mathbb{R}^{n+1}} \), the pullback of the tangent bundle of \( \mathbb{R}^{n+1} \) to \( \Sigma \), which contains \( T\Sigma \) as a subbundle. This follows from the fact that every tangent space of \( \mathbb{R}^{n+1} \) is spanned by the Killing fields evaluated there. In the general case, we may continue to regard \( T\Sigma \) as a subbundle of \( E/\mathfrak{h} \), for the transitivity of \( A \), condition (2), means we may regard the injection \( A/\mathfrak{h} \hookrightarrow E/\mathfrak{h} \) as a map \( T\Sigma \hookrightarrow E/\mathfrak{h} \). However, conditions (1) and (4) of the theorem ensure that the projection \( E \to E/\mathfrak{h} \) maps \( \text{rad } g \times \Sigma \) isomorphically onto \( E/\mathfrak{h} \). So we in fact have an inclusion \( T\Sigma \hookrightarrow \text{rad } g \times \Sigma \). Making the obvious identification \( \text{rad } g \simeq \mathbb{R}^{n+1} \), we recover an inclusion \( \iota : T_x\Sigma \hookrightarrow \mathbb{R}^{n+1} \) (for each \( x \in \Sigma \)) as in the case of a bona fide immersion. We now mimic the usual construction of the first and second fundamental forms for immersions, defining

\[
g_\omega(X, Y) = \langle \iota X, \iota Y \rangle, \\
I I_\omega(X, Y) = -\langle \iota X, d\mathbf{n}(Y) \rangle,
\]

where the \( \langle \cdot, \cdot \rangle \) is the standard inner product on \( \mathbb{R}^{n+1} \). Here \( \mathbf{n} : \Sigma \to S^n \subset \mathbb{R}^{n+1} \) is the analogue of the usual Gauss map, defined by declaring \( \mathbf{n}(x) \) to be orthogonal to \( \iota(T_x\Sigma) \subset \mathbb{R}^{n+1} \) and have length one. The ambiguity in its sign is resolved by supposing \( \Sigma \) has been oriented. The reader will now readily establish the following:

**Addendum.** If \( \Sigma \) is oriented and \( \omega : A \to g \) is the logarithmic derivative of an immersion \( \Sigma \subset \mathbb{R}^{n+1} \), then \( g_\omega \) and \( I I_\omega \) coincide with the usual first and second fundamental forms. In particular, the first and second fundamental forms of the realisation in the abstract Bonnet theorem will be \( g_\omega \) and \( I I_\omega \).

The classical Bonnet theorem follows from the fact that for any metric \( g \) and symmetric tensor \( I I \) on \( \Sigma \) satisfying the Gauss-Codazzi equations, we can construct a Lie algebroid morphism \( \omega : A \to g \) satisfying the hypotheses of the abstract Bonnet theorem above, and such that \( g_\omega = g \) and \( I I_\omega = I I \). For details, see [5].

\[2\] A slightly different but equivalent definition is given in [5].
Examples of Lie algebroids. We now mention some of the more important examples of Lie algebroids. If a Lie algebra \( g \) acts on a manifold \( \Sigma \), then the trivial bundle \( g \times \Sigma \) becomes a Lie algebroid over \( \Sigma \) called an action algebroid, whose anchor is the action map (see below). The image of the anchor map is tangent to the orbits of the action in this case; more generally an anchor has as image an involutive \( n \)-plane field (possibly with singularities) whose integrating leaves are called the orbits of the Lie algebroid. The \( n \)-plane field tangent to a regular foliation on a manifold \( \Sigma \) is a subalgebroid of the tangent bundle of \( \Sigma \) whose orbits are the leaves of the foliation. If \( \Sigma \) is a Poisson manifold, then \( T^*\Sigma \) is a Lie algebroid whose orbits are the symplectic leaves and, conversely, for every Lie algebroid \( A \), the dual bundle \( A^* \) is a (linear) Poisson manifold.

If \( P \) is a principal \( H \)-bundle over \( \Sigma \), then the quotient \( TP/H \) is a transitive Lie algebroid over \( \Sigma \) known as the Atiyah Lie algebroid. In particular, every \( G \)-structure determines a Lie algebroid, and every classical Cartan geometry determines a Koszul connection on the associated Atiyah Lie algebroid.

If \( L \) is the canonical line bundle of a contact structure, then its first jet bundle \( J^1L \) is a Lie algebroid. If \( A \) is a Lie algebroid, then so is every jet bundle \( J^kA \).

For more details, more examples, and generalizations of the examples just given, we refer the reader to the introductory treatments already cited, the references contained therein, and the following incomplete list of works: [9, 16, 11, 3, 7, 2, 15, 12].

The global counterpart of a Lie algebroid is called a Lie groupoid. Much of the relationship between Lie groups and Lie algebras carries over to the ‘oid’ case, with one important exception: Not every Lie algebroid is the infinitesimalization of a Lie groupoid, and the obstructions to ‘integrating’ a Lie algebroid to a Lie groupoid are subtle [10].

Anchor maps are morphisms. Here is a proof of the fact that the anchor \( \#: A \to T\Sigma \) of a Lie algebroid is a Lie algebroid morphism. Let \( X,Y,Z \) be arbitrary sections of \( A \) and \( f: \Sigma \to \mathbb{R} \) a smooth function. Then the Jacobi identity for the Lie bracket on \( \Gamma(A) \) implies

\[
[[X,Y], fZ] + [[fZ,X], Y] + [[Y, fZ], X] = 0.
\]

On the other hand, the Leibniz identity implies

\[
[[X,Y], fZ] = f[[X,Y], Z] + \mathcal{L}_{\#[X,Y]}f Z,
\]
\[
[[fZ,X], Y] = f[Y, [X, Z]] + \mathcal{L}_{\#Y}f [X, Z] + \mathcal{L}_{\#Y}\mathcal{L}_{\#X} f Z,
\]
\[
[[Y, fZ], X] = -f[X, [Y, Z]] - \mathcal{L}_{\#X}f [Y, Z] - \mathcal{L}_{\#Y}f [X, Z] - \mathcal{L}_{\#X}\mathcal{L}_{\#Y} f Z,
\]

where \( \mathcal{L} \) denotes Lie derivative. Substituting these equations into (1) delivers

\[
\mathcal{L}_{\#X}f Z + \mathcal{L}_{\#Y}\mathcal{L}_{\#X} f Z - \mathcal{L}_{\#X}\mathcal{L}_{\#Y} f Z = 0
\]

i.e., 

\[
\mathcal{L}_{\#[X,Y]-[\#X,\#Y]} f Z = 0,
\]
the second line following from the definition of the Jacobi-Lie bracket. Since \( f \) and \( Z \) are arbitrary, we conclude
\[
\#[X,Y] = \#[X,#Y]; \quad X,Y \in \Gamma(A).
\]

**Action algebroids.** Now suppose a Lie algebra \( \mathfrak{g} \) acts on a manifold \( M \), and let \( \xi \mapsto \xi^\dagger: \mathfrak{g} \to \Gamma(TM) \) denote the corresponding Lie algebra homomorphism. For example, we may take \( M = \mathbb{R}^{n+1} \) and let \( \mathfrak{g} \) be the Killing fields on \( \mathbb{R}^{n+1} \). A bracket on sections of the trivial bundle \( \mathfrak{g} \times M \) is defined by
\[
[X,Y] = \nabla_{#X}Y - \nabla_{#Y}X + \{X,Y\},
\]
where \( \nabla \) is the canonical flat connection on \( \mathfrak{g} \times M \) and \( \{X,Y\}(m) := [X(m),Y(m)]_{\mathfrak{g}}, m \in M \). Since the algebraic bracket \( \{\cdot,\cdot\} \) is \( \nabla \)-parallel, it is easy to see that \( \{\cdot,\cdot\} \) satisfies the Jacobi identity. The vector bundle \( \mathfrak{g} \times \mathbb{R}^{n+1} \) becomes the action algebroid associated with the Lie algebra action if we define the anchor \( \#: \mathfrak{g} \times M \to TM \) to be the action map \( \#(\xi,m) = \xi^\dagger(m) \).

In Lie algebroid jargon, the Lie algebroid \( A \) associated with a hypersurface \( \Sigma \subset \mathbb{R}^{n+1} \) is the pullback, in the category of Lie algebroids, under the immersion \( \Sigma \hookrightarrow \mathbb{R}^{n+1} \), of the action algebroid \( \mathfrak{g} \times \mathbb{R}^{n+1} \). Pullbacks are defined in [21, §4.2].

**Proof of Proposition 2.** Consider the action of \( \mathfrak{g} \) on \( \mathbb{R}^{n+1} \). A simple proof of the proposition rests on the fact that the action map \( \#: \mathfrak{g} \times \mathbb{R}^{n+1} \to TR^{n+1} \) is a Lie algebroid morphism, since it is the anchor map of the action algebroid \( \mathfrak{g} \times \mathbb{R}^{n+1} \).

Whence,
\[
(2) \quad \#[\tilde{X},\tilde{Y}] = \#[\tilde{X},\tilde{Y}]; \quad \tilde{X},\tilde{Y} \in \Gamma(\mathfrak{g} \times \mathbb{R}^{n+1}).
\]

So let \( X,Y \) be sections of \( A \subset \mathfrak{g} \times \Sigma \). Extend \( X \) and \( Y \) to sections \( \tilde{X} \) and \( \tilde{Y} \) of \( \mathfrak{g} \times \mathbb{R}^{n+1} \). Since \( \tilde{X} \) and \( \tilde{Y} \) restricted to \( \Sigma \) are sections of \( A \), the vector fields \( \#\tilde{X} \) and \( \#\tilde{Y} \) on \( \mathbb{R}^{n+1} \) are tangent to \( \Sigma \). In particular, for any \( x \in \Sigma \), we have
\[
(3) \quad \#[\#\tilde{X},\#\tilde{Y}](x) \in T_x\Sigma.
\]

But, on account of (2), the left-hand side of (3) coincides with \( \#[\tilde{X},\tilde{Y}](x) = \#[X,Y](x) \), which shows \( [X,Y] \) is a section of \( A \).

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