Radiation-Dominated Quantum Friedmann Models *

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Abstract

Radiation-filled Friedmann-Robertson-Walker universes are quantized according to the Arnowitt-Deser-Misner formalism in the conformal-time gauge. Unlike previous treatments of this problem, here both closed and open models are studied, only square-integrable wave functions are allowed, and the boundary conditions to ensure self-adjointness of the Hamiltonian operator are consistent with the space of admissible wave functions. It turns out that the tunneling boundary condition on the universal wave function is in conflict with self-adjointness of the Hamiltonian. The evolution of wave packets obeying different boundary conditions is studied and it is generally proven that all models are nonsingular. Given an initial condition on the probability density under which the classical regime prevails, it is found that a closed universe is certain to have an infinite radius, a density parameter $\Omega = 1$ becoming a prediction of the theory. Quantum stationary geometries are shown to exist for the closed universe model, but oscillating coherent states are forbidden by the boundary conditions that enforce self-adjointness of the Hamiltonian operator.

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1. INTRODUCTION

The lack of a consistent quantum theory of the full gravitational field and its sources has stimulated the development of quantum cosmology, a less complete but more tractable method to investigate the influence of quantum effects on the evolution of the universe. The primordial universe, when presumably curvatures and densities approach the Planck scale, is believed to be the privileged scenario in which the quantum aspects of gravity are expected to become important or even dominant. In its broadest sense, quantum cosmology consists in “freezing out” all but a finite number of degrees of freedom of the the gravitational field plus its sources (through imposition of symmetry requirements) and then quantizing the remaining ones. This procedure, initiated by DeWitt [1], is expected to provide some general insights on what an acceptable quantum theory of gravity should be like, although it cannot be strictly valid and is open to criticism [2]. Such a line of attack has been extensively explored to quantize Friedmann-Robertson-Walker (FRW) universes with varying matter content such as a scalar field [3-6], radiation [7-9], a spinor field [10], dust [9,11-14] or a Rarita-Schwinger field [15].

The present paper is dedicated to a further study of the quantum theory of a radiation-filled FRW universe. Differently from previous investigations of this system [7-9], here we discuss both closed and open models, deal only with normalizable wave functions, and pay full attention to the domain of self-adjointness of the Hamiltonian operator. We follow the Arnowitt-Deser-Misner (ADM) genuine canonical quantization method [16]. In this approach one has to solve the constraint equations at the classical level and go over to a reduced phase space spanned by independent canonical variables alone, and this process demands a definite choice of time. Although often leading to complicated and time-dependent Hamiltonians, this formalism has the great advantage of reducing the problem to one of standard quantum mechanics, enabling one to make full use of the powerful theory of linear operators in Hilbert space. In our treatment the time variable is chosen as conformal time, as this enormously simplifies the form of the Hamiltonian operator, making the quantum dynamics exactly soluble. Further reasons for choosing conformal time are given in [8].

After the ADM reduction of phase space, only one degree of freedom remains, which is taken to be the scale factor $R$. Since $R$ is restricted to positive values, it becomes necessary to impose boundary conditions on the wave functions belonging to the domain of the Hamiltonian operator to ensure its self-adjointness. For the simplest of such boundary conditions the time evolution of wave packets of Gaussian type is worked out, and it is shown in full generality for the first time that both the closed and open models are nonsingular. It is remarked that in the context of the ADM quantization the so-called tunneling boundary condition on the universal wave function is in conflict with self-adjointness of the Hamiltonian operator, or, equivalently, with unitarity of the quantum evolution. This is not an artifact of the particular quantization scheme adopted here, since the ADM and Wheeler-DeWitt descriptions are equivalent for the model at hand [7]. An
initial condition such that the probability density is sharply concentrated at \( R = 0 \) and under which the classical regime sets in is considered. In the closed case, under such an extreme initial condition the probability of finding any finite radius for the universe vanishes. Therefore a density parameter \( \Omega = 1 \) becomes a prediction of the model, without neither sacrificing the requirement of square-integrability on the wave functions nor imposing boundary conditions inconsistent with the space of admissible state vectors. A physically questionable aspect of the initial condition adopted is pointed out, however. Stationary quantum geometries are shown to exist for the closed model, but the existence of oscillating coherent states of the geometry is precluded by the boundary conditions required to enforce self-adjointness of the Hamiltonian operator, a result at variance with previous findings [9].

The layout of this paper is as follows. In Section 2 the classical model is specified and the ADM reduction of phase space [7] is briefly reviewed. In Section 3 the problem of the necessary boundary conditions to ensure self-adjointness of the Hamiltonian operator is considered, and the respective propagators are written down when the two simplest of such boundary conditions are adopted. In Section 4 the motion of wave packets obeying different boundary conditions is obtained in closed form, and it is verified that the singularity is avoided in all cases. A special initial condition under which the quantum model is forced into the classical regime is discussed, and the consequences for the case of a closed universe are considered. Stationary quantum geometries are taken up in Section 5, with particular emphasis on how the restricted domain of self-adjointness of the Hamiltonian operator influences such states. Section 6 is devoted to some final comments.

2. DYNAMICS OF THE CLASSICAL MODEL

The line element for a homogeneous and isotropic universe can be written in the FRW form (we take \( c = 1 \))

\[
ds^2 = g_{\nu\lambda}dx^\nu dx^\lambda = -N(t)^2dt^2 + R(t)^2 \sigma_{ij}dx^i dx^j ,
\]

(2.1)

where \( \sigma_{ij} \) denotes the metric for a 3-space of constant curvature \( k = +1, 0 \) or \( -1 \), corresponding to spherical, flat or hyperbolic spacelike sections, respectively.

The matter content will be taken to be a perfect fluid, and Schutz’s canonical formulation of the dynamics of a relativistic fluid in interaction with the gravitational field will be employed [17]. The degrees of freedom ascribed to the fluid are five scalar potentials \( \varphi, \alpha, \beta, \theta, S \) in terms of which the four-velocity of the fluid is written as

\[
U_\nu = \frac{1}{\mu}(\varphi_\nu + \alpha \beta_\nu + \theta S_\nu) ,
\]

(2.2)
where $\mu$ is the specific enthalpy. By means of the normalization condition

$$g_{\nu\lambda}U^\nu U^\lambda = -1 \quad (2.3)$$

one can express $\mu$ in terms of the velocity potentials. The action for the gravitational field plus perfect fluid is

$$S = \int_M d^4x \sqrt{-g} (\dot{R} + 2\dot{\theta}) R + 2 \int_{\partial M} d^3\sqrt{h} h_{ij} K^{ij} + \int_M d^4x \sqrt{-g} p \quad (2.4)$$

in units such that $c = 16\pi G = 1$. In Eq. (2.4) $p$ is the pressure of the fluid, $(\dot{R} + 2\dot{\theta})$ is the scalar curvature derived from the spacetime metric $g_{\nu\lambda}$, $h_{ij}$ is the 3-metric on the boundary $\partial M$ of the 4-manifold $M$, and $K^{ij}$ is the second fundamental form of the boundary [18]. The surface term is necessary in the path-integral formulation of quantum gravity in order to rid the Einstein-Hilbert Lagrangian of second-order derivatives. Variations of the pressure are computed from the first law of thermodynamics.

Compatibility with the homogeneous spacetime metric is guaranteed by taking all of the velocity potentials of the fluid as functions of $t$ only. We shall take $p = (\gamma - 1)\rho$ as equation of state for the fluid, where $\gamma$ is a constant and $\rho$ is the fluid’s energy density (we shall eventually put $\gamma = 4/3$). In the geometry characterized by (2.1) the appropriate boundary condition for the action principle is to fix the initial and final hypersurfaces of constant time. The second fundamental form of the boundary becomes $K_{ij} = -h_{ij}/2N$. As described in its full details in [7], after inserting the metric (2.1) into the action (2.4), using the equation of state, computing the canonical momenta and employing the constraint equations to eliminate the pair $(\theta, p_\theta)$, what remains is a reduced action in the Hamiltonian form

$$S_r = \int dt \left\{ \dot{R}(\dot{R} + 2\dot{\theta}) + \dot{\theta}p_\theta + \dot{S}p_S - NH \right\} \quad (2.5)$$

where an overall factor of the spatial integral of $(\det G)^{1/2}$ has been discarded, since it has no effect on the equations of motion. The super-Hamiltonian $H$ is given by

$$H = -\left( \frac{p_R^2}{24R} + 6kR \right) + \frac{p_\theta^2}{2} R^{-3(\gamma - 1)} e^S. \quad (2.6)$$

The lapse $N$ plays the role of a Lagrange multiplier, and upon its variation it is found that the super-Hamiltonian $H$ vanishes. This is a constraint, revealing that the phase-space contains redundant canonical variables.
According to the ADM prescription, in order to perform a bona fide canonical quantization one must go over to a reduced phase space spanned by independent canonical variables alone. This can be achieved by first making a choice of time and then solving the super-Hamiltonian constraint equation $H = 0$ for the canonical variable conjugate to the time chosen in the first step. This ensures that the final action preserves its canonical form, and the Hamiltonian in the reduced phase space is identical to the canonical variable whose Poisson bracket is unity with whatever was chosen as time, but now expressed as a function of the remaining independent canonical variables [16]. In the conformal-time gauge $N = R$ (that is, henceforward $t$ denotes conformal time) and for $\gamma = 4/3$ (radiation) this procedure leads to the very simple reduced action $[7]$

$$S_r = \int dt \left\{ \dot{R} p_R - \left( \frac{p_R^2}{24} + 6kR^2 \right) \right\} .$$

(2.7)

Only one degree of freedom is left, namely the scale factor $R$, and the Hamiltonian in the reduced phase space is

$$H = \frac{p_R^2}{24} + 6kR^2 .$$

(2.8)

Hamilton’s equations of motion lead immediately to

$$\ddot{R} + kR = 0 .$$

(2.9)

The solution for $R(t)$ can be written as

$$R(t) = R_0 \begin{cases} \sin t & \text{if } k = 1 \\ t & \text{if } k = 0 \\ \sinh t & \text{if } k = -1 \end{cases}$$

(2.10)

with a suitable choice for the origin of conformal time $t$. The standard cosmic time $\tau$ is related to conformal time by

$$d\tau = R \, dt ,$$

(2.11)

hence

$$\tau = R_0 \begin{cases} 1 - \cos t & \text{if } k = 1 \\ t^2/2 & \text{if } k = 0 \\ \cosh t - 1 & \text{if } k = -1 \end{cases}$$

(2.12)

with the convention that $\tau = 0$ when $t = 0$. In the spatially flat case $(k = 0)$, for instance, one recovers the usual behavior $R = C \tau^{1/2}$ for the scale factor [19]. It is seen that Hamilton’s principle
based on the reduced action (2.7) gives rise to the same equations of motion as those obtained by first varying the full action (2.4) and then simplifying them through the use of the spacetime symmetries of homogeneity and isotropy and of the equation of state $p = \rho/3$. Such a consistency check is indispensable if quantization in minisuperspace is to have any meaning at all.

3. QUANTIZATION, SELF-ADJOINTNESS AND BOUNDARY CONDITIONS

The remarkably simple form of the Hamiltonian (2.8) makes it possible to find exact results for the cosmic evolution at the quantum level. The quantum dynamics is not so straightforward as one might think at first sight because the scale factor $R$ is restricted to the domain $R > 0$, so that the minisuperspace quantization in the $R$–representation deals only with wave-functions defined on the half-line $(0, \infty)$. It is well-known that in such circumstances one usually has to impose boundary conditions on the allowed wave functions otherwise the relevant operators will not be self-adjoint, the most important of all operators being the Hamiltonian, that must be self-adjoint in order that the time evolution be unitary. The need to impose boundary conditions to ensure self-adjointness has been long recognized by practitioners of the ADM formalism as applied to quantum cosmology [3,11]. Very recently it has also been seen to have non-trivial cosmological implications in the Wheeler-DeWitt approach [20]. What does not appear to have been duly emphasized is that self-adjointness conditions depend on the set of allowed state vectors. It has been argued by adherents of the many-worlds interpretation [8] and of the pilot-wave formulation [21] of quantum theory that non-normalizable wave functions are unavoidable in quantum cosmology. Irrespective of whether their arguments are physically well founded or not, what we want to stress here is that the self-adjointness conditions must be consistent with the point of view adopted, if the results obtained are to be regarded as trustworthy.

As follows from the substitution $p_R \rightarrow -id/dR$, the Hamiltonian operator associated with the classical Hamiltonian function (2.8) is (we take $\hbar = 1$)

$$\hat{H} = -\frac{1}{24} \frac{d^2}{dR^2} + 6kR^2 \quad (3.1)$$

defined on the half-line $(0, \infty)$. The condition for $\hat{H}$ to be symmetric (which, in turn, is a necessary condition for $\hat{H}$ to be self-adjoint) is

$$(\psi_1, \hat{H}\psi_2) = (\hat{H}\psi_1, \psi_2) \quad (3.2)$$

or
\[
\int_0^\infty \psi_1^*(R) \frac{d^2 \psi_2}{dR^2} dR = \int_0^\infty \frac{d^2 \psi_1^*}{dR^2} \psi_2(R) dR ,
\]

where the asterisk stands for complex conjugate. Integrating by parts twice this leads to

\[
(\psi_1 \frac{d \psi_2}{dR} - \frac{d \psi_1^*}{dR} \psi_2)(\infty) = (\psi_1 \frac{d \psi_2}{dR} - \frac{d \psi_1^*}{dR} \psi_2)(0) .
\]

If both \(\psi\) and its derivative are square-integrable, the left-hand side of (3.4) vanishes and we are left with

\[
(\psi_1 \frac{d \psi_2}{dR} - \frac{d \psi_1^*}{dR} \psi_2)(0) = 0 .
\]

Then it can be shown [22] that to ensure the validity of this condition it is necessary and sufficient that the domain of \(\hat{H}\) be restricted to those wave functions such that

\[
\psi'(0) = \alpha \psi(0)
\]

with \(\alpha \in (-\infty, \infty]\). This generic boundary condition was explicitly taken into account in [11] and implicitly used in the simplest cases \(\alpha = 0\) and \(\alpha = \infty\) in [7], in which only square-integrable wave functions were considered acceptable.

If the potential is unbounded from below the Hamiltonian may possess eigenstates \(\psi \in L^2(0, \infty)\) such that \(\psi'\) is not square-integrable. In this case, or if non-normalizable wave functions are allowed, the correct condition for symmetry of the Hamiltonian is Eq.(3.4), and as such it has been recently employed in [20]. However, in [8] non-square-integrable wave functions were argued to be necessary in quantum cosmology but, inconsistently with this point of view, Eq.(3.6) was imposed on the allowed wave functions to allegedly enforce self-adjointness of the Hamiltonian operator. As a consequence of demanding that the initial wave function when \(t = 0\) be perfectly localized at the singularity \(R = 0\), Tipler [8] was led to a universal wave function of the form

\[
\psi(R,t) = \left[ \frac{3i}{4L_P \sin t} \right]^{1/2} \exp \left[ (3\pi/4i)(\cot t)(R/L_P)^2 \right] \equiv A(t) \exp[iB(t)R^2] ,
\]

where \(L_P\) denotes the Planck length, and which satisfies the boundary condition (3.6) for \(\alpha = 0\). Since this wave function is not square-integrable, it should obey Eq.(3.4), that is

\[
\lim_{R \to \infty} [4iA^*BR] = 0 ,
\]
which is not satisfied because both $A(t)$ and $B(t)$ are different from zero. Therefore, a wave function of the form (3.7) cannot represent a possible state of the universe. This means that Tipler’s initial condition on the wave function of the universe together with his imposition of boundary condition (3.6) with $\alpha = 0$ are in conflict with the requirement that $\hat{H}$ be a self-adjoint operator, and this renders the conclusions of [8] invalid.

In the present work we shall deal only with square integrable wave functions, so that the set of admissible states in the $R$-representation is the Hilbert space $L^2(0, \infty)$. Therefore, the domain of self-adjointness of the Hamiltonian operator is restricted to those wave functions that obey (3.6). For the sake of simplicity, here we shall address ourselves only to the cases $\alpha = 0$ and $\alpha = \infty$, that is, the boundary conditions we shall be concerned with are

$$\psi'(0, t) = 0 \quad (3.9a)$$

or

$$\psi(0, t) = 0 \quad (3.9b)$$

Both of these conditions refer to what happens to a wave packet when it hits the singularity $R = 0$. The boundary condition (3.9b) was advocated by DeWitt to keep wave packets away from the singularity, but in general it is not powerful enough to prevent wave functions from becoming concentrated in the neighborhood of $R = 0$ [23]. As a matter of fact, it has been argued [11] that DeWitt’s boundary condition is just not relevant to the issue of quantum gravitational collapse.

The time development of the models is fully determined once one is in possession of the propagator or Green’s function. Let $G(x, y, t)$ be the propagator for the problem in the usual Hilbert space $L^2(-\infty, \infty)$. Then the propagator for the problem in the restricted Hilbert space $L^2(0, \infty)$ is

$$G^{(a)}(R, R', t) = G(R, R', t) + G(R, -R', t) \quad (3.10)$$

if the boundary condition is (3.9a), or

$$G^{(b)}(R, R', t) = G(R, R', t) - G(R, -R', t) \quad (3.11)$$

if the boundary condition is (3.9b), as noted by several authors [24,8,25].
The general Green’s function for the Hamiltonian (3.1) on the usual Hilbert space $L^2(-\infty, \infty)$ is

$$G(x, y, t) = \left[ \frac{6\sqrt{k}}{\pi i \sin(\sqrt{k} t)} \right]^{1/2} \exp \left\{ \frac{6i\sqrt{k}}{\sin(\sqrt{k} t)} \left[ (x^2 + y^2) \cos(\sqrt{k} t) - 2xy \right] \right\},$$

as one immediately obtains from the expression of the propagator for the harmonic oscillator [26] by setting $m = 12$ and $\omega = \sqrt{k}$. In the limiting case $k = 0$ the well-known free-particle propagator is regained, whereas for $k = -1$ all one has to do is make use of the simple formulae $\cos(it) = \cosh t$ and $\sin(it) = i \sinh t$. The latter case corresponds to the quantum mechanics of a particle in an inverted oscillator potential, studied extensively in [27].

In the quantum cosmology à la Hartle-Hawking-Vilenkin-Linde an essential role is played by initial or boundary conditions. One of these is the so-called tunneling boundary condition [28,29], according to which the wave function of the universe must consist only of outgoing modes at singular boundaries of superspace. In our present context this amounts mathematically to

$$J = \frac{i}{2} \left( \psi^* \frac{\partial \psi}{\partial R} - \psi \frac{\partial \psi^*}{\partial R} \right)_{R=0} > 0,$$

where $J$ is the probability current density. However, from (3.6) it follows immediately that $J = 0$ because $\alpha$ is a real number. One is forced to conclude that, at least for this minisuperspace model, the tunneling boundary condition cannot be implemented because it is irreconcilable with self-adjointness of the Hamiltonian operator or, equivalently, unitarity of the time evolution. Furthermore, since the ADM and Wheeler-DeWitt descriptions are equivalent for the present model [7], this difficulty is not an artifact of the particular quantization scheme adopted here.

4. EVOLUTION OF THE QUANTUM MODELS

We shall now dedicate some paragraphs to the description of the main features of the dynamical evolution of our quantum cosmological models. This will be done by first following the time development of wave packets and then by studying the effect of imposing a very special initial condition.

4.1 Motion of Wave Packets

Let us start by working out the dynamical evolution of representative initial wave packets. The first initial state to be considered is the one described at $t = 0$ by the normalized wave function
\[ \psi_0^{(a)}(R) = \left( \frac{8\sigma}{\pi} \right)^{1/4} e^{-\beta R^2} \]  

(4.1)

where \( \beta = \sigma + ip \) with \( p \) real and \( \sigma > 0 \), corresponding to the boundary condition (3.9a).

The initial wave function (4.1) is an even function of \( R \), so that

\[ \psi^{(a)}(R, t) = \int_0^\infty G^{(a)}(R, R', t)\psi_0^{(a)}(R')dR' \]

(4.2)

Inserting the propagator (3.12) and the initial wave function (4.1) into (4.2) and performing the Gaussian integration one finds

\[
\psi^{(a)}(R, t) = \left( \frac{8\sigma}{\pi} \right)^{1/4} \left\{ \frac{6\sqrt{k}}{\cos(\sqrt{k}t)[\beta \tan(\sqrt{k}t) - 6i\sqrt{k}]} \right\}^{1/2} \\
\times \exp \left\{ \frac{6i\sqrt{k}}{\tan(\sqrt{k}t)} \left( 1 + \frac{6i\sqrt{k}}{\cos^2(\sqrt{k}t)[\beta \tan(\sqrt{k}t) - 6i\sqrt{k}]} \right) R^2 \right\}.
\]

(4.3)

An important quantity is the expectation value of the scale factor

\[ \langle \hat{R}^{(a)} \rangle_t = \int_0^\infty R |\psi^{(a)}(R, t)|^2 dR, \]

(4.4)

which can be readily computed from the wave function (4.3). We find

\[
\langle \hat{R}^{(a)} \rangle_t = \frac{1}{12} \sqrt{\frac{2}{\pi\sigma}} \left\{ \begin{array}{ll}
\sqrt{\sigma^2 \sin^2 t + (6 - p \tan t)^2 \cos^2 t} & \text{if } k = +1 \\
\sqrt{\sigma^2 t^2 + (6 - pt)^2} & \text{if } k = 0 \\
\sqrt{\sigma^2 \sinh^2 t + (6 - p \tanh t)^2 \cosh^2 t} & \text{if } k = -1
\end{array} \right. 
\]

(4.5)

Notice that, in all cases, \( \langle \hat{R}^{(a)} \rangle_t \) never vanishes. For \( k = 0 \) or \( k = -1 \) and \( p > 0 \) the expectation value of the scale factor initially decreases, reaches a minimum value and then grows steadily without limit, whereas if \( p < 0 \) there is a continuous expansion without bound. As expected, for large \( t \) the highest expansion rate belongs to the hyperbolic model (\( k = -1 \)). For \( k = 1 \) the universe oscillates between a minimum and a maximum radius. An interesting interpretation of
this behavior in terms of reflection of parts of the wave packet as they hit the origin $R = 0$ can be found in [11].

It should be stressed that the previous results establish that, at the quantum level, the singularity is avoided in all cases (that is, $k = 0, \pm 1$) according to the following reasonable criterion [11,30]: the quantum system is singular at a certain instant if $\langle \psi | \hat{f} | \psi \rangle = 0$ for any quantum observable $\hat{f}$ whose classical counterpart $f$ vanishes at the classical singularity, $\psi$ being any state of the system at the instant under consideration. For FRW models the relevant quantum observable is $\hat{f} = \hat{R}$, since $R = 0$ defines the classical singularity. This criterion is in consonance with the usage in quantum cosmology. Indeed, since $\hat{R}$ is a positive operator on $L^2(0, \infty)$, if $\langle \hat{R} \rangle_t = 0$ then $\psi(t)$ is sharply peaked at $R = 0$, and a strong peak in the wave function at a certain classical configuration is regarded in quantum cosmology as a prediction of the occurrence of such a configuration [29].

We now turn our attention to the boundary condition (3.9b). As initial wave function let us choose

$$\psi_0^{(b)}(R) = \left( \frac{128\sigma^3}{\pi} \right)^{1/4} R e^{-\beta R^2}. \quad (4.13)$$

Taking advantage of the odd character of this wave function, we can write

$$\psi^{(b)}(R, t) = \int_0^\infty G^{(b)}(R, R', t)\psi_0^{(b)}(R')dR' = \int_{-\infty}^\infty G(R, R', t)\psi_0^{(b)}(R')dR'. \quad (4.14)$$

Insertion of (4.13) and (3.12) into (4.14) yields

$$\psi^{(b)}(R, t) = \left( \frac{128\sigma^3}{\pi} \right)^{1/4} \left[ \frac{216i k^{3/2}}{\sin^3(\sqrt{k} t)} \right]^{1/2} \left[ \beta - \frac{6i\sqrt{k}}{\tan(\sqrt{k} t)} \right]^{-3/2} \times R \exp\left\{ \frac{6i\sqrt{k}}{\tan(\sqrt{k} t)} \left( 1 + \frac{6i\sqrt{k}}{\cos^2(\sqrt{k} t)[\beta\tan(\sqrt{k} t) - 6i\sqrt{k}]} \right) R^2 \right\}. \quad (4.15)$$

The expectation value of the scale factor is found to be

$$\langle \hat{R} \rangle_t^{(b)} = 2 \langle \hat{R} \rangle_t^{(a)}, \quad (4.16)$$

so that there is no singularity for boundary condition (3.9b) either.

As a matter of fact, we have shown only that the states evolving from (4.1) or (4.13) are such that $\langle \hat{R} \rangle_t$ never vanishes. Incomplete analyses like ours of the quantum gravitational collapse problem have been made before [7,9]. This is insufficient, however, because in order to establish that the quantum cosmological models are nonsingular one has to prove that $\langle \hat{R} \rangle_t \neq 0$ for any
evolving state $\psi(t)$ for which $\langle \hat{R} \rangle_t$ is defined. A somewhat indirect proof of this will be given below.

Classically the presence of the singularity at $t = 0$ makes it physically mandatory to restrict the conformal time $t$ to positive values. The absence of singularity at the quantum level makes such a restriction unnecessary, so that $-\infty < t < \infty$ and the quantum cosmological models are not naturally endowed with an origin of time. This kind of situation is also encountered in dust-filled FRW models in the cosmic-time gauge [11].

4.2 Special Initial Condition

Having a quantum dynamical framework to describe the evolution of the universe is not enough to explain its present state, one has to face the problem of initial conditions, the gist of modern quantum cosmology. In the path-integral approach to quantum cosmology both the Hartle-Hawking and the Vilenkin-Linde proposals appear to suffer from vagueness and lack of generality, and can hardly be said to lead unambiguously to a unique universal wave function [29]. In the present minisuperspace model the Vilenkin-Linde tunneling boundary condition is not even implementable, as we have seen in Section 3.

With this in mind, we proceed tentatively to examine the outcome of imposing initial conditions on the probability density associated with the universal wave function. We content ourselves with discussing the extreme situation $\sigma \to \infty$, in which case $|\psi_0|^2$ becomes sharply concentrated around $R = 0$:

$$\lim_{\sigma \to \infty} |\psi_0^{(a)}(R)|^2 = \delta(R) . \quad (4.17)$$

Under such circumstances the universe starts with certainty from the singularity $R = 0$, so that (4.17) may be regarded as the condition for a quantum explosive birth of the universe, or what might be called a quantum big bang. For $\sigma$ sufficiently large Eq.(4.5) reduces to

$$\langle \hat{R} \rangle_t \approx \frac{1}{12} \sqrt{\frac{2\sigma}{\pi}} \left\{ \begin{array}{ll} \sin t & \text{if } k = +1 \\ t & \text{if } k = 0 \\ \sinh t & \text{if } k = -1 \end{array} \right. \quad (4.18)$$

so that the classical regime sets in – compare the above equation with (2.10).

Now for the promised proof that $\langle \hat{R} \rangle_t \neq 0$ for any evolving state $\psi(t)$, implying that our quantum cosmological models are nonsingular. If $\langle \hat{R} \rangle_{t_1} = 0$ for some $t_1$ then $|\psi(R,t_1)|^2 = \delta(R)$. Suppose that $\psi(R,t)$ with $t > t_1$ is an state evolved from $\psi(R,t_1)$ taken as initial condition. By letting $\sigma \to \infty$ it follows from (4.18) that $\langle \hat{R} \rangle_t = \infty$. Therefore, no state with finite expectation value of the scale factor can arise from $\psi(R,t_1)$. Since quantum mechanics is time reversible, no
state $\psi(R, t_0)$ with $t_0 < t_1$ and finite $\langle \hat{R} \rangle_{t_0}$ can evolve to $\psi(R, t_1)$, which proves that $\langle \hat{R} \rangle_t \neq 0$ for all evolving states $\psi(t)$ for which the expectation value of the scale factor is finite.

Let us focus our attention particularly on the closed model ($k=+1$), the only one for which the following considerations are meaningful. In our present treatment, that deals only with normalized wave functions, the probability $P(R < R_1; t)$ that at time $t$ the radius of the universe is smaller than a given radius $R_1$ is given by

$$P(R < R_1; t) = \int_0^{R_1} |\psi^{(a)}(R, t)|^2 dR = \left( \frac{8\sigma}{\pi} \right)^{1/2} \left[ \frac{36}{\cos^2 t[\sigma^2 \tan^2 t + (6 - p \tan t)^2]} \right]^{1/2}$$

$$\times \int_0^{R_1} \exp \left\{ -\frac{72\sigma R^2}{\cos^2 t[\sigma^2 \tan^2 t + (6 - p \tan t)^2]} \right\} dR .$$  \hfill (4.19)

With the change of variable

$$x = \frac{\sqrt{72\sigma}}{\cos t[\sigma^2 \tan^2 t + (6 - p \tan t)^2]^{1/2}} R ,$$  \hfill (4.20)

one gets

$$P(R < R_1; t) = \frac{2}{\sqrt{\pi}} \int_0^{R_*} e^{-x^2} dx$$  \hfill (4.21)

where

$$R_* = \frac{\sqrt{72\sigma}}{\cos t[\sigma^2 \tan^2 t + (6 - p \tan t)^2]^{1/2}} R_1 .$$  \hfill (4.22)

Notice that $R_* \to 0$ as $\sigma \to \infty$, hence $P(R < R_1; t) = 0$ for an explosive quantum beginning of the universe. This leads to the prediction that the density parameter $\Omega$ equals unity in the limit of a truly explosive birth of the universe. Such a prediction was called “inflation without inflation” by Tipler [8], but here it is derived without having to resort to non-normalizable wave functions.

If the initial wave function is chosen as (4.13), corresponding to the boundary condition (3.9b), Eq.(4.16 ) shows that if $\sigma$ is sufficiently large the classical regime takes over. For the closed model one readily finds

$$P(R < R_1; t) = \frac{4}{\sqrt{\pi}} \int_0^{R_*} x^2 e^{-x^2} dx .$$  \hfill (4.23)

Again in the limit $\sigma \to \infty$ the probability that at time $t$ the radius of the universe is smaller than a given radius $R_1$ is zero. Note that as $\sigma \to \infty$ the wave packet (4.13) also satisfies (4.17), thus
providing a concrete illustration of the fact that an initial wave function whose associated probability density is concentrated entirely at $R = 0$ can be harmonized with the boundary condition (3.9b). Besides, the prediction $\Omega = 1$ does not appear to be sensitive to boundary conditions of the type (3.6) on the wave function itself, but to result exclusively from the condition (4.17) that its modulus squared be sharply concentrated at $R = 0$ when $t = 0$. An initial condition on the wave function itself that gives rise to $\Omega = 1$ is known [8], but then one has to give up the square integrability requirement, and the resulting universal wave function does not belong to the domain of self-adjointness of the Hamiltonian operator, as remarked in Section 3.

Unfortunately, this state of affairs is still physically dubious. The inevitable singularity makes the restriction $t \geq 0$ dynamically obligatory in classical cosmology. On the other hand, it is only the imposition of the initial condition $|\psi_0(R)|^2 = \delta(R)$ that makes the instant $t = 0$ so especially distinguished as to induce the restriction $t \geq 0$ upon the quantized model too. This is so because no unitary evolution from an earlier time could have led to such a perfectly localized state at $t = 0$. This questionable feature is also present in Tipler’s treatment [8], in which use is made of non-normalizable wave functions. Although inconclusive, our tentative considerations were intended to suggest that it may be physically reasonable to impose initial conditions on some probability distribution engendered by the wave function rather than on the universal wave function itself.

5. QUANTUM STATIONARY GEOMETRIES

In the case of the closed FRW model the Hamiltonian operator (3.1) possesses square-integrable eigenfunctions. Consider the normalized wave functions

$$\psi_n(R) = \left(\frac{\sqrt{48}}{\sqrt{\pi} 2^n n!}\right)^{1/2} H_n(\sqrt{12} R) \exp(-6R^2)$$  \hspace{1cm} (5.1)

where $H_n$ denotes the $n$-th Hermite polynomial. They satisfy

$$\hat{H} \psi_n = (n + 1/2) \psi_n$$  \hspace{1cm} (5.2)

where $n$ is a non-negative integer. For even $n$ the wave functions (5.1) satisfy both Eq.(5.2) and the boundary condition (3.9a), whereas for odd $n$ they obey both Eq.(5.2) and the boundary condition (3.9b). The effect of the boundary conditions is to exclude either even or odd eigenfunctions. This does not agree with [9], where no boundary conditions are imposed on the wave functions belonging to the domain of $\hat{H}$, with the result that all values for $n$ are allowed. For an universe in any of the stationary states (5.1) nothing changes with time. This is a purely quantum effect
since radiation-filled FRW universes do not possess classical static solutions.

A more significant effect of imposing self-adjointness boundary conditions is the preclusion of oscillating coherent states, that is, nondispersive Gaussian wave packets whose center oscillates just like a solution to the classical equations of motion. From the general form of such wave functions [31] one recognizes at once that the general boundary condition (3.6) cannot be satisfied even if the real parameter $\alpha$ is allowed to be time dependent. It is clear, therefore, that a coherent state such that the classical solution (2.10) emerges as the expectation value of the scale factor does not exist, in contradiction with the findings in [9].

6. CONCLUDING REMARKS

In this paper we dealt with square-integrable wave functions only, that is, we limited ourselves to the orthodox framework of quantum mechanics in Hilbert space. Interpretational controversies apart, it is in this arena that the requirement of self-adjointness on the quantum observables is most naturally justified and easily understood. Accordingly, being careful about domains of operators becomes a necessity in quantum cosmology, and in the case of radiation-filled FRW universes the simplest boundary conditions required to enforce self-adjointness of $\hat{H}$ were taken into account. Initial or boundary conditions introduced with the purpose of selecting a unique wave function are customarily unrelated to the former, although on occasion this has been object of confusion in the literature. Sometimes, however, these two types of boundary conditions interfere with each other, as we pointed out in Section 3.

An important distinction should be emphasized between the classical and quantum cosmologies discussed in this paper. An origin of time comes into being dynamically in classical gravity due to the inevitable singularity, while no origin of time occurs naturally in quantum gravity, except if induced by a choice of initial conditions. On the other hand, admitting the hypothesis that a suitable initial condition exists, it is open to doubt whether it should be imposed on the wave function itself or on some probability distribution derived thereof.

Our treatment differs from the one based upon consideration of conformal fluctuations about a given geometry. Apart from lack of application of boundary conditions to ensure self-adjointness, there is a physically more important difference. In [9] fluctuations are discussed about an specific FRW classical solution of Einstein’s equations, whereas the ADM approach considers fluctuations that encompass all possible universes of the FRW kind. Although these standpoints may be classically undistinguishable, they are not necessarily equivalent in the quantum realm. This raises the question what the relation between these approaches is and which, if any, is appropriate from the physical point of view, a study we reserve for the future.

We have circumscribed our analysis to radiation as cause of curvature, that is, to a matter content consisting of a perfect fluid with polytropic index $\gamma = 4/3$. It can be shown [32] that in the
conformal-time gauge the form of the classical equation of motion of all Friedmann models with any \(\gamma\)-fluid as source can be reduced to that of a harmonic oscillator after a suitable change of variables. This suggests the possibility of extending the previous quantum treatment to cosmological models whose matter content is a perfect fluid with an arbitrary polytropic index. This is presently under investigation.
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