Poisson algebras, Weyl algebras and Jacobi pairs
Yucai Su
Department of Mathematics, Tongji University, Shanghai 200092, China
Email: ycsu@tongji.edu.cn

Abstract. We study Jacobi pairs in details and obtained some properties. We also study the natural Poisson algebra structure \((\mathcal{P}, [[\cdot, \cdot], \cdot])\) on the space \(\mathcal{P} := \mathbb{C}[y][[(x^{-1})]]\) for some sufficient large \(N\), and introduce some automorphisms of \((\mathcal{P}, [[\cdot, \cdot], \cdot])\) which are (possibly infinite but well-defined) products of the automorphisms of forms \(e^{adH}\) for \(H \in x^{1-N}\mathbb{C}[y][[(x^{-1})]]\) and \(\tau_c : (x, y) \mapsto (x, y - cx^{-1})\) for some \(c \in \mathbb{C}\). These automorphisms are used as tools to study Jacobi pairs in \(\mathcal{P}\). In particular, starting from a Jacobi pair \((F, G)\) in \(\mathbb{C}[x, y]\) which violates the two-dimensional Jacobian conjecture, by applying some variable change \((x, y) \mapsto (x^b, x^{1-b}(y + a_1x^{-b_1} + \cdots + a_kx^{-b_k}))\) for some \(b, b_i \in \mathbb{Q}^+, a_i \in \mathbb{C}\) with \(b_i < 1 < b\), we obtain a Jacobi pair still denoted by \((F, G)\) in \(\mathbb{C}[x^\pm, y]\) with the form \(F = x^{m\frac{1}{N}}(f + F_0), G = x^{n\frac{1}{N}}(g + G_0)\) for some positive integers \(m, n\), and \(f, g \in \mathbb{C}[y], F_0, G_0 \in x^{-N}\mathbb{C}[x^-1, y]\), such that \(F, G\) satisfy some additional conditions. Then we generalize the results to the Weyl algebra \(W = \mathbb{C}[v][[(u^{-1})]]\) with relation \([u, v] = 1\), and obtain some properties of pairs \((F, G)\) satisfying \([F, G] = 1\), referred to as Dixmier pairs.

Key words: Poisson algebras, Weyl algebras, Jacobian conjecture, Dixmier conjecture

Mathematics Subject Classification (2000): 17B63, 14R15, 14E20, 13B10, 13B25

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1 Introduction

It is a well known fact that if \( n \) polynomials \( f_1, \ldots, f_n \) are generators of the polynomial ring \( \mathbb{C}[x_1, \ldots, x_n] \), then the Jacobian determinant \( J(f_1, \ldots, f_n) = \det A \in \mathbb{C}\{0\} \) is a nonzero constant, where \( A = (\partial_{x_j} f_i)_{i,j=1}^n \) is the \( n \times n \) Jacobian matrix of \( f_1, \ldots, f_n \). One of the major unsolved problems of mathematics [24] (see also [5, 9, 29]), viz. the Jacobian conjecture, states that the reverse of the above statement also holds, namely, if \( J(f_1, \ldots, f_n) \in \mathbb{C}\{0\} \), then \( f_1, \ldots, f_n \) are generators of \( \mathbb{C}[x_1, \ldots, x_n] \).

This conjecture relates to many aspects of mathematics [1, 12–14, 23–26] and has attracted great attention in mathematical and physical literatures during the past 60 years and there have been a various ways of approaches toward the proof or disproof of this conjecture (here we simply give a short random list of references [2, 6, 8, 11, 15–18, 20, 28–30]). Hundreds of papers have appeared in connection with this conjecture, even for the simplest case \( n = 2 \) [3, 21, 22]. However this conjecture remains unsolved even for the case \( n = 2 \).

Let \( W_n \) be the rank \( n \) Weyl algebra, which is the associative unital algebra generated by \( 2n \) generators \( u_1, \ldots, u_n, v_1, \ldots, v_n \) satisfying the relations \( [u_i, u_j] = [v_i, v_j] = 0, \ [v_i, u_j] = \delta_{ij}, \) where the commutator \([\cdot, \cdot]\) is defined by
\[
[a, b] = ab - ba \quad \text{for } a, b \in W_n. \tag{1.1}
\]

Under the commutator, \( W_n \) becomes a Lie algebra, denoted by \( W^L_n \), called the rank \( n \) Weyl Lie algebra. With a history of 40 years, the Dixmier conjecture [10] states that every nonzero endomorphism \( W_n \) is an automorphism. This conjecture remains open for \( n \geq 1 \). It is well known [4, 7] that the rank \( n \) Dixmier conjecture implies the \( n \)-dimensional Jacobi conjecture and the \( 2n \)-dimensional Jacobi conjecture implies the rank \( n \) Dixmier conjecture.

In this paper, we study Jacobi pairs in details and obtained some properties. We also study the natural Poisson algebra structure \((\mathcal{P}, [\cdot, \cdot], \cdot)\) on the space \( \mathcal{P} := \mathbb{C}[y][((x^{-\frac{1}{n}}))] \) for some sufficient large \( N \), and introduce some automorphisms of \((\mathcal{P}, [\cdot, \cdot], \cdot)\) which are (possibly infinite but well-defined) products of the automorphisms of forms \( e^{ad_H} \) for \( H \in x^{1+\frac{1}{n}}\mathbb{C}[y][[(x^{-\frac{1}{n}})]] \) and \( \tau_c : (x, y) \mapsto (x, y - cx^{-1}) \) for some \( c \in \mathbb{C} \). These automorphisms are used as tools to study Jacobi pairs in \( \mathcal{P} \). In particular, starting from a Jacobi pair \((F, G)\) in \( \mathbb{C}[x, y] \) which violates the two-dimensional Jacobian conjecture, by applying some variable change \( (x, y) \mapsto (x^b, x^{1-b}(y + a_1x^{-b_1} + \cdots + a_kx^{-b_k})) \) for some \( b, b_i \in \mathbb{Q}_+, a_i \in \mathbb{C} \) with \( b_i < 1 < b \), we obtain a Jacobi pair still denoted by \((F, G)\) in \( \mathbb{C}[x^{\pm \frac{1}{n}}, y] \) with the form \( F = x^{\frac{m}{m+n}}(f + F_0), \ G = x^{\frac{n}{m+n}}(g + G_0) \) for some positive integers \( m, n \), and \( f, g \in \mathbb{C}[y], \ F_0, G_0 \in x^{-\frac{1}{n}}\mathbb{C}[x^{-\frac{1}{n}}, y], \) such that \( F, G \) satisfy some additional conditions (see Theorem 3.30).

Then we generalize the results to the Weyl algebra \( W = \mathbb{C}[v][((u^{-\frac{1}{n}}))] \) with relation \([u, v] = 1\) and obtain some properties of pairs \((F, G)\) satisfying \([F, G] = 1\), referred to as Dixmier pairs. In particular, one can define the Newton polygon \( NP(F) \) of \( F \) as for the case of Jacobi pairs (cf. Subsection 3.5 and arguments after (5.17)). We can suppose \( NP(F) \) has a vertex \((m_0, m)\) with \( 0 < m_0 < m \). First (as in the case of Jacobi pairs), from the pair \((F, G)\), by applying some automorphism, we obtain a Dixmier pair still denoted by \((F, G)\) in \( \mathbb{C}[u^{\pm \frac{1}{n}}, v] \) with the form \( F = u^{\frac{m}{m+n}}(f + F_0), \ G = u^{\frac{n}{m+n}}(g + G_0) \) for some positive integers \( m, n \), and \( f, g \in \mathbb{C}[v], \ F_0, G_0 \in u^{-\frac{1}{n}}\mathbb{C}[u^{-\frac{1}{n}}, u], \) such that \( F, G \) satisfy some additional conditions (cf. Theorem 5.2).

By applying the automorphism \((u, v) \mapsto (v, -u)\), we can assume \( F \) has a vertex \((m_0, m)\) with \( m_0 > m > 0 \), and we can further assume that the slope of the edge located at the right bottom
side of Supp $F$ and with the top vertex being $(m_0, m)$ is positive (as for the case of Jacobi pairs), which turns out to be 1 (cf. (5.17) and Theorem 5.3). Furthermore, the coefficient of the term $u^{m_0-1}v^{m-1}$ is always $\frac{m_0m}{2}$ (if we assume $C_{\text{coeff}}(F, u^{m_0}v^{m}) = 1$). We remark that this is the place where the great difference between Newton polygons of Jacobi pairs and Dixmier pairs occurs; for the Jacobi pairs, an edge of the Newton polygon can never have slope 1 (cf. Theorem 3.25).

The main results in the present paper are summarized in Theorems 3.6, 3.25, 3.30, 4.1, Corollary 4.2 and Theorems 5.2, 5.3.

2 Definition of the prime degree $p$, notations and preliminaries

In this section, we first give some notations and definitions, then we present some preliminary results.

2.1 Notations and definitions

Denote by $\mathbb{Z}, \mathbb{Z}_+, \mathbb{Z}_-, \mathbb{N}, \mathbb{Q}, \mathbb{Q}_+, \mathbb{C}$ the sets of integers, non-negative integers, negative integers, positive integers, rational numbers, non-negative rational numbers, complex numbers respectively. Let $\mathbb{C}(x, y) = \{\frac{P}{Q} \mid P, Q \in \mathbb{C}[x, y]\}$ be the field of rational functions in two variables. We use $\mathcal{A}, \mathcal{B}, \mathcal{C}$ to denote the following rings (they are in fact fields and $\mathcal{A}, \mathcal{C}$ are algebraically closed fields):

$$\mathcal{A} = \{f = \sum_{i \in \mathbb{Q}} f_ix^i \mid f_i \in \mathbb{C}, \text{Supp}_x f \subset \alpha - \frac{1}{\beta}\mathbb{Z}_+ \text{ for some } \alpha, \beta \in \mathbb{Z}, \beta > 0\},$$

$$\mathcal{B} = \mathcal{A}((y^{-1})) = \{F = \sum_{j \in \mathbb{Z}} F_jy^j \mid F_j \in \mathcal{A}, \text{Supp}_y F \subset \alpha - \mathbb{Z}_+ \text{ for some } \alpha \in \mathbb{Z}\}, \quad (2.1)$$

$$\mathcal{C} = \{F = \sum_{j \in \mathbb{Q}} F_jy^j \mid F_j \in \mathcal{A}, \text{Supp}_y F \subset \alpha - \frac{1}{\beta}\mathbb{Z}_+ \text{ for some } \alpha, \beta \in \mathbb{Z}, \beta > 0\}, \quad (2.2)$$

where

$$\text{Supp}_xf = \{i \in \mathbb{Q} \mid f_i ≠ 0\} \quad (\text{the support of } f),$$

$$\text{Supp}_yF = \{j \in \mathbb{Z} \mid F_j ≠ 0\} \quad (\text{the support of } F \text{ with respect to } y).$$

Denote $\partial_x = \frac{\partial}{\partial x}$ or $\frac{d}{dx}$, $\partial_y = \frac{\partial}{\partial y}$. For $F = \sum_{i \in \mathbb{Q}, j \in \mathbb{Z}} f_{ij}x^iy^j \in \mathcal{B}$, we define

$$\text{Supp } F = \{(i, j) \mid f_{ij} ≠ 0\} \quad (\text{called the support of } F), \quad (2.3)$$

and define

$$\deg_x F = \max\{i \in \mathbb{Q} \mid f_{ij} ≠ 0 \text{ for some } j\} \quad (\text{called the } x\text{-degree of } F),$$

$$\deg_y F = \max\{j \in \mathbb{Z} \mid f_{ij} ≠ 0 \text{ for some } i\} \quad (\text{called the } y\text{-degree of } F),$$

$$\deg F = \max\{i + j \in \mathbb{Q} \mid f_{ij} ≠ 0\} \quad (\text{called the total degree of } F).$$

Note that a degree can be $\infty$ (for instance, $F = 0$), or $\infty$ (for instance, $\deg_x F = \deg F = +\infty$ if $F = \sum_{i=0}^{\infty} x^{2i}y^{-i}$). An element $F = \sum_{i=0}^{\infty} f_iy^{m-i}$ is monic if $f_0 = 1$.

For any $h = x^\alpha + \sum_{i=1}^{\infty} h_i x^{\alpha - \frac{i}{\beta}} \in \mathcal{A}$ with $\alpha \in \mathbb{Q}, \beta \in \mathbb{N}$, and for any $a \in \mathbb{Q}$, we define $h^a$ to be the unique element in $\mathcal{A}$:

$$h^a = x^{\alpha a} \left(1 + \sum_{i=1}^{\infty} h_i x^{\alpha - \frac{i}{\beta}}\right)^a = x^{\alpha a} \sum_{j=0}^{\infty} \binom{a}{j} \left(\sum_{i=1}^{\infty} h_i x^{\alpha - \frac{i}{\beta}}\right)^j = x^{\alpha a} + \sum_{i=1}^{\infty} h_{a,i} x^{\alpha a - \frac{i}{\beta}}, \quad (2.4)$$
Remark 2.2 It may be more proper to define the prime degree \( p \) to be

\[
\sup_{i \geq 1} \frac{\deg_x f_i - \deg_x f_0}{i}.
\]

Then in case \( F \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}] \), both definitions coincide. However, when \( F \notin \mathbb{C}[x^{\pm 1}, y^{\pm 1}] \), it is possible that the prime line \( L_F \) defined as above only passes through one point of \( \text{Supp} F \). Since we do not like such a case to happen when we consider Jacobi pairs in later sections, we use (2.7) to define \( p \) instead of (2.9). (For example, for \( F = y + \sum_{i=0}^{\infty} x^i y^{-i} \), if we use (2.9) to define \( p \), it would be 1; but if we use (2.7) to define \( p \), it is \(+\infty\).)
Let \( p \neq \pm \infty \) be a fixed rational number. We always assume all elements under consideration below have prime degrees \( \leq p \) (and in the next section, we always take \( p = p(F) \)).

**Definition 2.3**

(1) Let 
\[
F = \sum_{i=0}^{\infty} f_i y^{m-i} \in \mathcal{B} \quad \text{with} \quad f_0 \neq 0.
\]

In what follows, we always use \( m \) to denote \( m = \deg_y F \) and use \( m_0 \) to denote \( \deg_x f_0 = m_0 \) until (3.79). We call \( x^{m_0} y^m \) the first term of \( F \). Suppose \( p(F) \leq p \). For \( r \in \mathbb{Q} \), we define the \( p \)-type \( r \)-th component (or simply the \( r \)-th component) of \( F \) to be
\[
F[r] = \sum_{i=0}^{\infty} C_{\text{coeff}}(f_i, x^{m_0+r+ip}) x^{m_0+r+ip} y^{m-i}, \tag{2.10}
\]

which simply collects those terms \( f_i x^iy^j \) of \( F \) with \((i,j)\) located in a line parallel to the prime line (cf. (2.8)). One immediately sees that \( F[r] = 0 \) if \( r > 0 \) and \( F[0] \neq 0 \). We remark that if \( p(F) > p \) then it is possible that \( F[r] \neq 0 \) for \( r > 0 \).

(2) Suppose \( F = \sum_{i=0}^{m} f_i y^{m-i} \in \mathbb{C}((x^{-1})[y]) \) with \( f_0 \) being a monic Laurent polynomial of \( x \) and \( p(F) = p \).

(i) We call \( F[0] \) the leading polynomial of \( F \), and \( F[<0] := \sum_{r<0} F[r] \) the ignored polynomial of \( F \).

(ii) We always use the bold symbol \( F \in \mathbb{C}[x^{\pm 1}][y] \) to denote the unique monic polynomial of \( y \) (with coefficients being Laurent polynomials of \( x \)) such that \( F[0] = x^{m_0} F'[m'] \) with \( m' \in \mathbb{N} \) maximal (we always use \( m' \) to denote this integer). Then (a polynomial satisfying (2.11) is usually called a power free polynomial),
\[
F \neq H^k \quad \text{for any} \quad H \in \mathbb{C}[x^{\pm 1}][y] \quad \text{and} \quad k > 1. \tag{2.11}
\]

We call \( F \) the primary polynomial of \( F \). We always use \( d \) to denote
\[
d = \deg_y F, \quad \text{thus} \quad m' = \frac{m}{d}. \tag{2.12}
\]

(3) An element of the form
\[
H = \sum_{i=0}^{\infty} a_i x^{r_0+ip} y^{0-i} \in \mathcal{B} \quad \text{with} \quad a_i \in \mathbb{C}, \quad r_0 \in \mathbb{Q}, \quad a_0 \in \mathbb{Z}_+.
\]

(i.e., its support is located in a line) is called a \( p \)-type quasi-homogenous element (q.h.e.), and it is called a \( p \)-type quasi-homogenous polynomial (q.h.p.) if it is a polynomial.

**Lemma 2.4**

(1) \( p(F) = p(F^a) \) if \( a, b \neq 0 \).

(2) \( p(FG) \leq \max\{p(F), p(G)\} \) with equality holds if one of the following conditions holds:

(i) \( p(F) \neq p(G) \) or

(ii) \( F, G \) both are polynomials, or

(iii) \( F, G \) are \( p \)-type q.h.e. such that \( FG \) is not a monomial.

(3) Suppose \( p(F) \leq p \). Then \( p(F) = p \) if and only if \( F[0] \) is not a monomial.
Proof. (1) We immediately see from (2.6) that \( \deg_x f_{a,b,i} \leq \deg_x f_{0}^{\frac{a}{m}} + ip(F) \), i.e., \( p(F^{\frac{a}{m}}) \leq p(F) \). Thus also, \( p(F) = p(F^{\frac{a}{m}}) \leq p(F^{\frac{a}{m}}) \). Hence \( p(F^{\frac{a}{m}}) = p(F) \).

(2) and (3) are straightforward to verify. \( \square \)

Note that the equality in Lemma 2.4(2) does not necessarily hold in general; for instance,

\[
F = (y + x^2)(y + x^3) = y^2 + (x^2 + x^3)y + x^5, \quad G = (y + x^3)^{-1},
\]

with \( p(F) = p(G) = 3 \), but \( FG = y + x^2 \) with \( p(FG) = 2 \).

2.2 Some preliminary results

We remark that the requirement that any element under consideration has prime degree \( \leq p \) is necessary, otherwise it is possible that in (2.14), there exist infinitely many \( r > 0 \) with \( H[r] \neq 0 \) and the right-hand side becomes an infinite sum.

Lemma 2.5  (1) Suppose \( F = \sum_{i=0}^{\infty} F_i \), where \( F_i = \sum_{j=0}^{\infty} f_{ij}y^{m_i-j} \in \mathcal{B} \) (and all have prime degree \( \leq p \)) with \( f_{ij} \neq 0 \), \( \deg_x f_{ij} = \tilde{m}_{ij} \) such that \( \tilde{m}_0 > \tilde{m}_1 > ... \) (in this case \( \sum_{i=0}^{\infty} F_i \) is called summable). Then

\[
 F[r] = \sum_{i=0}^{\infty} (F_i)_{[r+\tilde{m}_0-\tilde{m}_0+p(\tilde{m}_0-\tilde{m}_1)]}.
\]

(2) An element is a \( p \)-type q.h.e. \( \iff \) it is a component of itself.

(3) If \( F \) is a \( p \)-type q.h.e. with \( \deg_y F = m \), then \( F^{\frac{a}{m}} \) is a \( p \)-type q.h.e. if \( b|am \).

(4) Let \( H, K \in \mathcal{B} \) with prime degrees \( \leq p \), and \( r \in \mathbb{Q} \). Then (where “\( [r] \)” is defined as in (2.10))

\[
 (HK)[r] = \sum_{r_1+r_2=r} H[r_1]K[r_2].
\]

(5) Let \( F \) be as in Definition 2.3(2). Let \( \ell \in \mathbb{Z} \) with \( d|\ell \). Then for all \( r \in \mathbb{Q} \), \( x^{-\frac{ma}{m}} (F^l)[r] \in \mathbb{C}(x,y) \) is a rational function of the form \( F^a P \) for some \( a \in \mathbb{Z} \) and \( P \in \mathbb{C}[x^{\pm 1}][y] \).

(6) Let \( P, Q \in \mathcal{B} \) with prime degrees \( \leq p \) and \( Q \neq 0 \). Then each \( p \)-type component of the rational function \( R = \frac{P}{Q} \) is a rational function. Furthermore, there exists some \( \alpha \in \mathbb{N} \) such that the \( p \)-type \( r \)-th component \( R[r] = 0 \) if \( r \notin \frac{1}{\alpha} \mathbb{Z} \).

Proof. Using (2.10), (2.5) and (2.6), it is straightforward to verify (1)--(3).

(4) Suppose \( H = \sum_{i=0}^{\infty} h_i y^{a_i} \), \( K = \sum_{i=0}^{\infty} k_i y^{b_i} \) with \( \deg_y h_0 = \alpha_0 \), \( \deg_x k_0 = \beta_0 \). Then

\[
 HK = \sum_{i=0}^{\infty} \chi_i y^{a_i+b_i} \text{ with } \chi_i = \sum_{s_1+s_2=i} h_{s_1} k_{s_2}. \text{ Thus}
\]

\[
 C_{\text{eff}}(\chi_i, x^{\alpha_0+\beta_0+r+ip}) = \sum_{r_1+r_2=r} \sum_{s_1+s_2=i} C_{\text{eff}}(h_{s_1}, x^{\alpha_0+r_1+s_1}) C_{\text{eff}}(k_{s_2}, x^{\beta_0+r_2+s_2}).
\]

Hence we have (4).

(5) We have \( F = \sum_{s \in \mathbb{Q}} F_{[s]} \) and \( F[0] = x^{ma} F^{m} \). Thus

\[
x^{-\frac{ma}{m}} F^{\ell} = F^{\ell} (1 + \sum_{s < 0} x^{-ma} F_{[s]} F^{-m'})^{\ell} = \sum_{\substack{0 \geq r \in \mathbb{Q} \\ s_1 t_1 + \ldots + s_k t_k = r \\ 0 > s_1 > \ldots > s_k \\ t_i \in \mathbb{Z}_+, k \geq 0}} \left( \ell_{m} \right)^{t_1 + \ldots + t_k} \prod_{i=1}^{k} (F_{[s_i]})^{t_i} F^{\frac{a}{m}}(t_1 + \ldots + t_k)m'.
\]
By (4), if the $j$-th component of $(F_{[s_i]}^{t_i})$ is nonzero, then $j = s_i t_i$. By (2) and (3), if the $j$-th component of $F_{[s_i]}^{t_i} - (t_1 + \cdots + t_k)m'$ is nonzero, then $j = 0$. Thus by (1) and (4), the $r$-th component of $x^{-\frac{mn}{m} F\frac{m}{m}}$ for $r \leq 0$ is

$$x^{-\frac{mn}{m}} (F\frac{m}{m})[r] = \sum_{s_1 t_1 + \cdots + s_k t_k = r \atop t_i \in \mathbb{Z}_+, k \geq 0} \left( \frac{m}{t} \right) x^{-(t_1 + \cdots + t_k)m_0} \prod_{i=1}^{k} (F_{[s_i]}^{t_i} F^{t_i} - (t_1 + \cdots + t_k)m'), \quad (2.15)$$

which is a finite sum of rational functions of $y$ with coefficients in $\mathbb{C}[x^{\pm 1}]$ by noting that every component $F_{[s_i]}$ is a polynomial of $y$ and that the powers of $F$ in (2.15) are integers and $\{ r \mid F[r] \neq 0 \}$ is a finite set.

(6) By (4) and (5),

$$R[r] = \sum_{r_1 + r_2 = r} P_{[r_1]} (Q^{-1})_{[r_2]}. \quad (2.16)$$

Since every component of the polynomial $P$ is a polynomial, and $P$ has only finite nonzero components, thus the sum in (2.16) is finite. By (5), $(Q^{-1})_{[r_2]}$ is a rational function. Thus we have the first statement of (6). The second statement follows from (2.15) and (2.16).

Equation (2.15) in particular gives

$$\left( F\frac{m}{m} \right)[0] = x^{-\frac{mn}{m}} F\frac{m}{m}. \quad (2.17)$$

Remark 2.6 Suppose $F = \frac{P}{Q}$ is a rational function such that $\deg_y F \neq 0$ and $P, Q$ have the same prime degree $p$ and the same primary polynomial $F$, then $P[0] = x^a F^b$, $Q[0] = x^c F^d$ for some $a, b, c, d \in \mathbb{Z}$, so $F[0] = x^{a-c} F^{b-d}$. In this case we also call $F$ the primary polynomial of $F$ if $b \neq d$.

The result in Lemma 2.5(5) can be extended to rational functions as follows.

**Lemma 2.7** Let $F, G \in \mathbb{C}((x^{-1}))[y]$ with prime degree $p$ and primary polynomial $F$. Let $x^{\hat{m}0} y^m$, $x^m y^n$ be the first terms of $F$ and $G$. Let $a, b \in \mathbb{Z}$, $\hat{F} = F^a G^b \in \mathbb{C}((x^{-1}))[y]$ with $\hat{m} := \deg_y \hat{F} = am + bn \neq 0$. Set $\hat{m}_0 = am_0 + bn_0$. Let $\ell \in \mathbb{Z}$ with $d | \ell$. Then for all $r \in \mathbb{Q}$, $x^{-\frac{mn}{m}} (\hat{F}\frac{\hat{m}}{\hat{m}})[r] \in \mathbb{C}(x, y)$ is a rational function of the form $F^a P$ for some $\alpha \in \mathbb{Z}$ and $P \in \mathbb{C}[x^{\pm 1}][y]$.

**Proof.** By Lemma 2.5(5) (by taking $\ell = am$ or $bm$), each component of $F^a$, $G^b$ is a rational function of the form $F^a P$. Thus the “” version of (2.15) (which is still a finite sum by Lemma 2.5(6)) shows that we have the result. \qed

The following result generalized from linear algebra will be used in the next section.

**Lemma 2.8** Suppose $H \in \mathbb{C}((x^{-1}))[y]$ such that $\deg_y H > 0$. Let $m \in \mathbb{N}$. Suppose there exists a finite nonzero combination

$$P := \sum_{i \in \mathbb{Z}} p_i H^\frac{m}{m} \in \mathbb{C}((x^{-1}))[y] \quad \text{for some } p_i \in \mathbb{C}((x^{-1})). \quad (2.18)$$

Then $H = H^\frac{m}{m}$ is the $\frac{m}{m}$-th power of some polynomial $H_1 \in \mathbb{C}((x^{-1}))[y]$, where $d = \gcd(m, i \mid p_i \neq 0)$ is the greatest common divisor of the integer set $\{ m, i \mid p_i \neq 0 \}$. 


Proof. We thank Dr. Victor Zurkowski who suggested the following simple proof. Applying the operator \( \frac{H}{n!H^0} \) to (2.18) iteratively, we obtain a system of equations (regarding \( p_i H^m \) as unknown variables). Solving the system shows \( p_i H^m \in \mathbb{C}((x^{-1}))(y) \) for all \( i \). Then induction on \( \#A \) (where \( A = \{ i \mid p_i \neq 0 \} \)) gives the result (when \( \#A = 1 \), it is a well-known result of linear algebra, one can also simply prove as follows: Suppose \( H^m = R \), then \( Q^m H^n = R^m \). Since \( \mathbb{C}((x^{-1}))[y] \) is a uniquely factorial domain, by decomposing each polynomial into the product of its irreducible polynomials, we see that \( H \) is the \( \frac{m}{n} \) -th power of some polynomial). \( \square \)

We shall also need the following lemma.

**Lemma 2.9** Let \( \beta \in \mathbb{C} \setminus \{0\} \). Let \( A_\infty = \mathbb{C}[z_1, z_2, \ldots] \) be the polynomial ring with \( \infty \) variables, and denote \( z_0 = 1 \). Set

\[
h_\beta(x) = \left(1 + \sum_{i=1}^{\infty} z_i x^i\right)^\beta = \sum_{j=0}^{\infty} h_{\beta,j} x^j \in A_\infty[[x]],
\]

where

\[
h_{\beta,i} = \text{coeff}(h_\beta(x), x^i) = \sum_{i_1+2i_2+\cdots+mi_m = i, m \in \mathbb{N}, i_1, \ldots, i_m \in \mathbb{Z}_{+}} \binom{\beta}{i_1, \ldots, i_m} z_1^{i_1} \cdots z_m^{i_m} \text{ for } i \in \mathbb{Z}_+.
\]

Then for all \( r \in \mathbb{Z}_+ \), we have

\[
\sum_{s=0}^{\infty} (s+1-r) z_s h_{\beta,r-s} = 0 \quad \text{and} \quad \sum_{s=0}^{\infty} z_s h_{\beta,r-s} = h_{\beta+1,r}.
\]

**Proof.** Taking derivative with respect to \( x \) in (2.19), we have

\[
\sum_{j=0}^{\infty} j h_{\beta,j} x^{j-1} = \beta \left( \sum_{i=0}^{\infty} z_i x^i \right)^{\beta-1} \left( \sum_{j=0}^{\infty} j z_j x^{j-1} \right).
\]

Multiplying it by \( \sum_{i=0}^{\infty} z_i x^i \) and comparing the coefficients of \( x^{r-1} \) in both sides, we obtain

\[
\sum_{s=0}^{\infty} (r-s) z_s h_{\beta,r-s} = \beta \sum_{s=0}^{\infty} z_s h_{\beta,r-s},
\]

i.e., the first equation of (2.20) holds. To prove the second, simply write \( h_{\beta+1}(x) \) as \( h_{\beta+1}(x) = h_\beta(x) h_1(x) \) and compare the coefficients of \( x^r \). \( \square \)

Let \( F, G \in \mathcal{B} \) such that

\[
F = \sum_{i=0}^{\infty} f_i y^{m-i}, \quad G = \sum_{j=0}^{\infty} g_j y^{n-j} \quad \text{for some } f_i, g_j \in \mathcal{A}, \quad f_0 g_0 \neq 0,
\]

with \( m = \text{deg}_y F > 0 \) and \( n = \text{deg}_y G \), we can express \( G \) as

\[
G = \sum_{i=0}^{\infty} b_i F^\frac{m}{m} \quad \text{for some } b_i \in \mathcal{A},
\]

where by comparing the coefficients of \( y^{n-i} \), \( b_i \) can be inductively determined by the following (cf. (2.6)):

\[
b_i = h^{-m_0(n-i)} \left( g_i - \sum_{j=0}^{i-1} b_j f_{n-j,m,i-j} \right), \quad \text{or more precisely,}
\]

\[
b_i = h^{-m_0(n-i)} \left( g_i - \sum_{j=0}^{i-1} b_j \sum_{r_1+2r_2+\cdots+mr_m = i} \binom{n-j}{m} f_1^{r_1} \cdots f_m^{r_m} h^{-m_0(n-j)} - (r_1+\cdots+r_m)m_0 \right). \quad (2.23)
\]
Similarly, we can express the polynomial \( y \) as

\[
y = \sum_{i=0}^{\infty} \bar{b}_i F_{\frac{m}{m}}^{-i},
\]  

where \( \bar{b}_i \in A \) is determined by

\[
\bar{b}_i = h^{-\frac{m'(1-i)}{m}} \left( \delta_{i,0} - \sum_{j=0}^{i-1} \bar{b}_j \sum_{r_1+2r_2+\cdots+r_m=i-j} \left( \frac{1-i}{m} \right)^{r_1} \cdots \left( \frac{1-i}{m} \right)^{r_m} \frac{m'(1-j)-(r_1+\cdots+r_m)m_0}{m} \right). 
\]  

We observe a simple fact that if \( F, G \in A[y] \) then \( F^{-1} \) does not appear in the expression of \( G \) in (2.22) (we would like to thank Professor Leonid Makar-Limanov, who told us the following more general fact and the suggestion for the simple proof).

**Lemma 2.10** Let \( F, G \in A[y] \) be any polynomials with \( \deg_y F = m, \deg_y G = n \). Express \( G \) as in (2.22). We always have \( b_{im+n} = 0 \) for \( i \geq 1 \), namely, all negative integral power of \( F \) cannot appear in the expression.

**Proof.** The proof can be obtained by regarding \( F \) and \( G \) as polynomials in \( y \), from an observation that \( \int G \, dF \) is a polynomial while the term with \( F^{-1} \) would require the logarithmic term in integration. Iteration of this observation of course shows that the coefficients with all negative integral powers of \( F \) are zeros. \( \square \)

More generally, we have

**Lemma 2.11** Let \( F \in B \) with \( \deg_y F = m > 0 \) and \( k, \ell \in \mathbb{Z} \) with \( m \neq -k \). Then in the expression

\[
y^\ell F_{\frac{m}{m}} = \sum_{i=0}^{\infty} c_{\ell,i} F_{\frac{m}{m}}^{\frac{k+\ell-1}{m}} \text{ with } c_{\ell,i} \in A,
\]  

the coefficient of \( F^{-1} \) is

\[
c_{\ell,m+k+\ell} = -\frac{\ell}{m+k} \text{Coeff}(F_{\frac{m+k}{m}}, y^{-\ell}).
\]  

In particular, the coefficient \( c_{\ell,i} \) in \( y^\ell = \sum_{i=0}^{\infty} c_{\ell,i} F_{\frac{m}{m}}^{i} \) is

\[
c_{\ell,i} = -\frac{\ell}{i-\ell} \text{Coeff}(F_{\frac{m}{m}}, y^{-\ell}) \quad \text{for all } \ell \neq i.
\]  

We also have \( c_{0,0} = 1 \) and

\[
c_{\ell,\ell} = -\frac{1}{m} \sum_{s=0}^{\infty} s \text{Coeff}(F, y^{m-s}) \text{Coeff}(F^{-1}, y^{-m-\ell+s}) = \frac{1}{m} \text{Coeff}(F^{-1} \partial_y F, y^{-\ell}) \quad \text{if } \ell > 0.
\]  

**Proof.** For any \( A \in B \) with \( \deg_y A = a \), we always use (until the end of this proof) \( A_r \) to denote the homogenous part of \( A \) of \( y \)-degree \( a-r \), i.e.,

\[
A_r = \text{Coeff}(A, y^{a-r})y^{a-r}, \quad r \in \mathbb{Z}.
\]  

\[ (2.31) \]
Comparing the homogenous parts of $y$-degree $k + \ell - (r - s)$ in (2.27), multiplying the result by $(s \frac{k + \ell}{m} + 1 - r)F_s$ and taking sum over all $s$, we obtain (note that both sides are homogenous polynomials of $y$ of degree $m + k + \ell - r$)
\[
\sum_{s \in \mathbb{Z}_+} \left( s \frac{k + \ell}{m} + 1 - r \right) F_s \left( \frac{y^\ell}{m} \right)_{r, s} = \sum_{i=1}^{\infty} c_{\ell, i} \sum_{s \in \mathbb{Z}_+} \left( s \frac{k + \ell}{m} + 1 - r \right) F_s \left( \frac{y^\ell}{m} \right)_{r - i, s},
\]
where on the right-hand side, the terms with $i = 0$ vanish by (2.20) (we take $z_r = \frac{F_r}{F_0}$, $\beta = \frac{k + \ell}{m}$ in (2.20)), and $(y^\ell \frac{F^k}{m})_{r, s}$, $(F^{\frac{k + \ell - i}{m}})_{r, i - s}$ are the notations as in (2.31). Note from definition (2.31) that $(y^\ell \frac{F^k}{m})_{r, s} = y^\ell (F^\frac{k}{m})_{r, s}$. For $i \neq m + k + \ell$, we write $s \frac{k + \ell}{m} + 1 - r$ in both sides of (2.32) respectively as
\[
s \frac{k + \ell}{m} + 1 - r = \frac{m + k + \ell}{m + k} \left( s \frac{k}{m} + 1 - r \right) + \frac{r - i}{m + k},
\]
and substitute them into (2.32) (using (2.20) again, the first terms in the right-hand sides of (2.33) and (2.34) then become vanishing), we obtain (where the first equality follows from the second equation of (2.20))
\[
\frac{r \ell}{m + k} y^\ell \left( \frac{F^{\frac{m + k}{m}}}{m} \right)_r = \frac{r \ell}{m + k} y^\ell \sum_{s \in \mathbb{Z}_+} F_s \left( \frac{y^\ell}{m} \right)_{r, s - \ell}
\]
\[
= \sum_{1 \leq i \neq m + k + \ell} c_{\ell, i} \sum_{s \in \mathbb{Z}_+} \frac{(r - m - k - \ell)i}{m + k + \ell - i} F_s \left( \frac{F^{\frac{k + \ell - i}{m}}}{m} \right)_{r - i, s}
\]
\[
+ c_{\ell, m + k + \ell} \sum_{s \in \mathbb{Z}_+} \left( s \frac{k + \ell}{m} + 1 - r \right) F_s \left( F^{-1} \right)_{r - m - k - \ell, s}.
\]
Take $r = m + k + \ell$ in (2.35), then the first summand in the right-hand side is 0. As for the last summand, if $s \neq 0$, then $r - m - k - \ell - s < 0$ and so $(F^{-1})_{r - m - k - \ell, s} = 0$ (the definition of (2.31) shows $A_r = 0$ if $r < 0$). Thus (2.35) gives
\[
\frac{(m + k + \ell) \ell}{m} y^\ell \left( \frac{F^{\frac{m + k}{m}}}{m} \right)_{m + k + \ell} = -c_{\ell, m + k + \ell} (m + k + \ell) F_0 (F^{-1})_0 = -(m + k + \ell) c_{\ell, m + k + \ell}.
\]

Since $m + k + \ell > 0$ by our assumption (cf. (2.35)), (2.36) together with definition (2.31) shows we have (2.28). To prove (2.30), assume $\ell > 0$. We take $k = -m$, $r = \ell$ in (2.32) (but we do not substitute (2.33) into the left-hand side of (2.32)), then the left-hand side of (2.32) becomes (using the fact $\sum_{s \in \mathbb{Z}_+} F_s (F^{-1})_{s - \ell} = (F^{1-1})_{\ell} = 0$ by the second equation of (2.20))
\[
y^\ell \sum_{s \in \mathbb{Z}_+} \frac{\ell}{m} F_s (F^{-1})_{s - \ell}.
\]
Thus the left-hand side of (2.36) should be (2.37). Hence we have (2.30).

\[\square\]

### 3 Jacobi pairs and Jacobian elements

In this section, we discuss properties of Jacobi pairs and Jacobian elements in details. The main results of this section are Theorems 3.6, 3.25 and 3.30.
3.1 General discussions on Jacobi pairs in $B$

**Definition 3.1** (1) Define the Lie bracket $[,]$ on $B$ by the Jacobian determinant

$$[F,G] = J(F,G) = (\partial_x F)(\partial_y G) - (\partial_y F)(\partial_x G) \text{ for } F,G \in \mathbb{C}[x,y].$$

(3.1)

It is well-known (e.g., [19, 27]) that the triple $(B,\cdot,\cdot)$ (where $\cdot$ is the usual product in $B$) is a *Poisson algebra*, namely $(B,\cdot)$ is a Lie algebra, $(B,\cdot)$ is a commutative associative algebra, and the following compatible Leibniz rule holds:

$$[F,GH] = [F,G]H + G[F,H] \text{ for } F,G,H \in B.$$  

(3.2)

(2) A pair $(F,G)$ is called a *quasi-Jacobi pair* (or simply a *Jacobi pair*), if the following conditions are satisfied:

(i) $F,G \in B$ are of the form (2.21) such that $[F,G] \in \mathbb{C}\backslash\{0\}$,

(ii) $F$ has prime degree $p \neq \pm \infty$,

(iii) $p(G) \leq p$.

**Remark 3.2** (1) We always denote (until (3.79))

$$m := \deg_y F, \quad m_0 := \deg_x f_0, \quad n := \deg_y G, \quad n_0 := \deg_x g_0,$$

(3.3)

and use notations $b_i, b_i$ in (2.22)–(2.26). We always assume $m \geq 1$ and $m_0 \geq 0$ (but not necessarily $m_0 \in \mathbb{Z}$). Note that we always have $m_0 \neq m$ (by Lemma 3.6 since $(-\frac{m_0}{m}, -1) \in \text{Supp } F^{-\frac{1}{m}}$). Also note that $n$ can be negative, but the nonzero Jacobian determinant requires that $m + n \geq 1$ (cf. Lemma 3.8).

(2) If $F,G \in \mathbb{C}[x,y]$ with $[F,G] \in \mathbb{C}\backslash\{0\}$, then $(F,G)$ is called a (usual) *Jacobi pair*. In this case if necessary by exchanging $F$ and $G$, we can always suppose $p(G) \leq p$ (in fact $p(G) = p$ if $m + n \geq 2$ by Lemma 3.14). Thus a usual Jacobi pair is necessarily a quasi-Jacobi pair.

(3) We always suppose $f_0, g_0$ are monic (i.e., the coefficients of the highest powers of $x$ in $f_0, g_0$ are 1). If $f_0 \in \mathbb{C}[x]$ and $f_0 \neq 1$, let $x = a$ be a root of $f_0$. By applying the automorphism $(x,y) \mapsto (x+a,y)$, we can suppose $f_0$ does not contain the constant term. In this case, we also let $h \in \mathbb{C}[x]$ be the unique monic polynomial such that $f_0 = h^{m_0}$ with $m_0 \in \mathbb{N}$ maximal, and set $d_0 = \frac{m_0}{m_0}$. Then

$$h \neq h^k \text{ for any } h_1 \in \mathbb{C}[x], k > 1.$$  

(3.4)

If $f_0 = 1$, we set $h = x, d_0 = 1, m_0 = m_0 = 0$. If $f_0 \notin \mathbb{C}[x]$, we simply set $h = f_0, d_0 = m_0, m_0 = 1$.

(4) We always fix notations $h, m_0, d_0$.

**Example 3.3** For any $m_0, m \in \mathbb{N}, a, b \in \mathbb{C}$ with $m_0 > m$ and $a, b \neq 0$, the pairs

$$(F,G) = (x^{m_0}y^m + ax^{m_0-1}y^{m-1}, (x^{m_0}y^m + ax^{m_0-1}y^{m-1})^2 + bx^{1-m_0}y^{1-m}),$$

$$(F_1,G_1) = (x^4(y^2 + x^{-1})^3, x^{-2}(y^2 + x^{-1})^{-2}y),$$

are Jacobi pairs with $p(F) = -1$ (but if $m_0 < m$, then $p(G) = -\frac{3m-1}{3m-1} > -1$), and $p(F_1) = p(G_1) = -\frac{1}{2}$. Note that $p(G)$ can be smaller than $p(F)$; for instance, if we replace $G$ by $x^{1-m_0}y^{1-m}$ then $p(G) = -\infty < p(F)$ (but in this case $\deg_y F + \deg_y G = 1$).
Definition 3.4 (1) We introduce a notion, referred to as the trace, of an element $F \in \mathcal{B}$ to be (cf. (5.27) and (5.28) for the reason why we call it “trace")

$$\text{tr } F = \text{Res}_x \text{Res}_y F = C_{\text{coeff}}(F,(xy)^{-1}) \text{ for } F \in \mathcal{B}. \quad (3.5)$$

If $\text{tr } F = 0$, we say $F$ has the vanishing trace property.

(2) An element $F \in \mathcal{B}$ is called a Jacobian element if there is $G \in \mathcal{B}$ such that $[F,G] \in \mathbb{C}\setminus\{0\}$.

Remark 3.5 Note that it is important to consider the residue of an element in a specific space, otherwise the residue may be different; for instance, if we regard the element $H = \frac{1}{xy+(xy)^2}$ as in $\mathbb{C}[x^{\pm 1}]/((y))$ then $H = -\sum_{i=1}^{\infty}(-xy)^i$ and its residue (with respect to $x$) is $y^{-1}$, but if we consider it as in the space $\mathbb{C}[x^{\pm 1}]/((y^{-1}))$ then $H = \sum_{i=2}^{\infty}(-xy)^{-i}$ and its residue is zero.

The following Theorem 3.6(2) characterizes Jacobian elements.

Theorem 3.6 (1) Assume $(F,G)$ is a Jacobi pair in $\mathcal{A}[y]$. Suppose

$$\text{Res}_x(F \partial_y G) := C_{\text{coeff}}(F \partial_y G,x^{-1}) = 0. \quad (3.6)$$

Let $x = u(t), y = v(t) \in \mathbb{C}((t^{-1}))$, then $\text{Res}_t(F \frac{dG}{dt}) = J \text{Res}_t(u \frac{dv}{dt})$.

(2) An element $F \in \mathcal{B}$ is a Jacobian element if and only if $F \notin \mathcal{C}$ and $(F-c)^a$ has the vanishing trace property for all $a \in \mathbb{Q}$ and $c \in \mathbb{C}$ (note that $(F-c)^a$ might not be in $\mathcal{B}$ but in $\mathcal{C}$).

(3) Let $F \in \mathcal{B}$ be a Jacobian element, and $H \in \mathcal{B}$.

There exists some $K \in \mathcal{B}$ such that $H = [K,F]$ if and only if $HF^a$ has the vanishing trace property for all $a \in \mathbb{Q}$.

(4) Let $(F,G)$ be a Jacobi pair in $\mathcal{B}$ with $[F,G] = 1$. One has

$$\text{tr}([H,F,G]) = \text{tr } H \text{ for any } H \in \mathcal{B}. \quad (3.7)$$

Proof. (1) Assume $F = \sum_{i,j \in \mathbb{Z}+} f_{ij} x^i y^j$ and $G = \sum_{k,l \in \mathbb{Z}+} g_{k,l} x^k y^l$. Then $[F,G] = J$ and (3.6) imply

$$\sum_{i+k-\ell = a,b} (i-\ell) f_{ij} g_{k,l} = \delta_{a,1} \delta_{b,1} J, \quad \sum_{i+k-\ell = a,b} k f_{ij} g_{k,l} = 0, \quad \forall a,b \in \mathbb{C}. \quad (3.8)$$

Denoting $\partial_t = \frac{d}{dt}$, and considering 4 cases (noting that $j = \ell = 0$ if $j + \ell = 0$): (i) $i + k = 0 = j + \ell$; (ii) $i + k = 0 \neq j + \ell$; (iii) $i + k \neq 0 = j + \ell$; and (iv) $i + k \neq 0 \neq j + \ell$, we obtain from (3.8),

$$\text{Res}_t(F \partial_y G) = \text{Res}_t \left( \sum_{i,j,k,\ell} f_{ij} g_{k,l} (k u^{i+k-\ell} v^{j+\ell} \partial_t u + \ell u^{i+k} v^{j+\ell-1} \partial_t v) \right)$$

$$= \text{Res}_t \left( \sum_{i+k \neq 0, j+\ell > 0} f_{ij} g_{k,l} \left( \frac{\ell}{j+\ell} \partial_t (u^{i+k} v^{j+\ell}) + \frac{jk-il}{j+\ell} u^{i+k-1} v^{j+\ell} \partial_t u \right) \right)$$

$$= J \text{Res}_x(v \partial_t u).$$

One can also give the following simple proof as suggested by Dr. Victor Zurkowski: (3.6) means $F \partial_y G = \partial_x U$ for some $U \in \mathcal{A}[y]$. Then $\partial_y \partial_x U = -J + \partial_x(F \partial_y G)$, so $\partial_y U + Jx - F \partial_y G$ is a polynomial in $y$, denoted by $g$, and we can write $F \partial_y G - \partial_y U = Jx + F \partial_y G$. Thus $F \partial_t G = F \partial_y G \partial_t u + F \partial_y G \partial_y v = \partial_t (U + g) + Ju \partial_v$. Taking the residue with respect to $t$, the term from the exact derivative drops and one gets the result.

(2) “$\Rightarrow$”: First note that by computing the coefficient of $x^{-1}y^{-1}$ in the Jacobian determinant $[H,K]$, one immediately sees

$$\text{tr}([H,K]) = 0 \text{ for any } H,K \in \mathcal{C}. \quad (3.9)$$
Suppose $F, G \in \mathcal{B}$ such that $[F, G] = 1$. Then $(F-c)^a = \left[\frac{(F-c)^a}{a}, (F-c)^{2-a}G\right]$ (assume $a \neq 0$), we obtain the necessity by (3.9).

$\iff$: First assume $\deg_y F \neq 0$. Replacing $F$ by $F^{-1}$ if necessary, we can suppose $m := \deg_y F \in \mathbb{N}$. Express $y$ as in (2.25), then in fact $\hat{b}_i$ is $c_{1,i}$ in (2.27). Thus (2.29) and the sufficiency condition show that $x^{-1}$ does not appear in $\hat{b}_i$ for all $i \neq 1$. Therefore there exist $b_i \in \mathcal{A}$ such that $\partial x b_i = \frac{1-m}{m} \hat{b}_i$ for all $i \geq 0$ (cf. (3.11)). Set $G = \sum_{i=0}^{\infty} b_i F^{\frac{1-m-i}{m}}$. Then the proof of Lemma 3.8 shows $[F, G] = 1$.

Now assume $\deg_y F = 0$. Let $f_i := \mathrm{Coeff}_x(F, y^i)$ for $i \leq 0$. By replacing $F$ by $F-c$ for some $c \in \mathbb{C}$ if necessary, we can suppose $f_0 \neq 0$ does not contain the constant term. We can write $f_{-1} = \beta \partial_x f_0 + h$ for some $\beta \in \mathbb{C}$ and $h \in \mathcal{A}$ such that either $h = 0$ or else $i := \deg_x f_0 - 1 \neq j := \deg_x h$. If $h \neq 0$, by taking $a := \frac{i-1}{i+1} \neq 0 (i + 1 = \deg_x f_0 \neq 0$ if $\deg_x f_0 = 0)$, we have $F^a = (f_0 + f_{-1} y^{-1} + \cdots)^a = f_0^a (1 + af_0^{-1} (\beta \partial_x f_0 + h) y^{-1} + \cdots) = f_0^a + (\beta \partial_x f_0^a + af_0^{a-1} h) y^{-1} + \cdots$, and $(-1, -1) \in \mathrm{Supp} F^a$ by noting that $\partial_x(f_0^a)$ does not contain the term $x^{-1}$ but $\deg_x f_0^{a-1} h = -1$, a contradiction. Thus $h = 0$. Now one can uniquely determine $g_1 = (\partial_x f_0)^{-1}$ and $g_i$ for $i \geq 1$ inductively such that $G = g_1 y + \sum_{i=1}^{\infty} g_i y^{-i}$ satisfying $[F, G] = 1$, by comparing the coefficients of $y^{-i}$ for $i \geq 0$.

(3) Suppose $H = [K, F]$. Then by (3.7), we see $HF^a = [KF^a, F]$ has the vanishing trace property for all $a \in \mathbb{Q}$. Conversely suppose $HF^a$ has the vanishing trace property for all $a \in \mathbb{Q}$. Let $G$ be as constructed in the proof of (1) such that $[F, G] = 1$ and $m + n = 1, m_0 + n_0 = 1$ (cf. notation (3.3)). Using (2.25) and the fact that $FG$ has the highest term $xy$, we see that any element $x^iy^j$ with $i, j \in \mathbb{Q}$ can be written as $F^kG^\ell + F_1$, where $k, \ell \in \mathbb{Q}$ satisfy $km + \ell n = j$, and some $F_1 \in \mathcal{C}$ with $\deg_y F_1 < j$. Thus by induction on $\deg_y H$, we see that $H$ can be expressed as $H = \sum_{i,j \in \mathbb{Q}} c_{ij} F^iG^j$ for some $c_{ij} \in \mathbb{C}$. Note that $\mathrm{tr}(F^kG^\ell) = 0$ if and only if $(k, \ell) \neq (-1, -1)$. Thus $G^{-1}$ cannot appear in the expression of $H$, otherwise $HF^{-1}$ would have nonzero trace if $c_{i,-1} \neq 0$. Taking $K = \sum_{i,j \in \mathbb{Q}} c_{ij} F^iG^{j+1}$, we obtain $H = [K, F]$.

(4) one has

$\mathrm{tr}([H, F]G) = \mathrm{tr}(\partial_x H \partial_y FG - \partial_y H \partial_x FG) \quad = \mathrm{tr} \left( -H(\partial_x \partial_y FG + \partial_y \partial_x G) + H(\partial_y \partial_x FG + \partial_x \partial_y G) \right) = \mathrm{tr} H.$

This completes the proof of the theorem.

As a by-product of the above theorem, one can easily obtain

**Corollary 3.7** (1) If $(F, G)$ is a Jacobi pair in $\mathbb{C}[x, y]$ such that $\mathrm{Supp} F$ has a vertex $(m_0, m)$ with $m_0, m > 0$ (cf. (3.64)), then $F, G$ are not generators of $\mathbb{C}[x, y]$ (thus in particular, the proof of the two-dimensional Jacobian conjecture is equivalent to proving that a Jacobi pair $(F, G)$ in $\mathbb{C}[x, y]$ with $\mathrm{Supp} F$ having a vertex $(m_0, m)$ with $m_0, m > 0$ does not exists).

(2) Any $\sigma$ in (3.84) regarded as an automorphism of $\mathbb{C}[x^{\pm \frac{1}{N}}](y^{-\frac{1}{N}})$ does not change the vanishing trace property.

**Proof.** (1) Say $[F, G] = 1$ and assume $F, G$ are generators of $\mathbb{C}[x, y]$. Then $H = x^{m_0-1}y^{m-1} = \sum_{i,j \in \mathbb{Z}^+} a_{ij} F^iG^j$ for some $a_{ij} \in \mathbb{C}$, and so $H = [K, F]$ for $K = \sum_{i,j \in \mathbb{Z}^+} a_{ij} F^iG^{i+1}$. However, $\mathrm{tr}(HF^{-1}) \neq 0$, a contradiction with Theorem 3.6(3).

(2) An automorphism $\sigma$ of the form (3.84) is a product of automorphisms of forms $\sigma_1 : (x, y) \mapsto (x, y + \lambda x^{-\frac{p}{q}})$ and $\sigma_2 : (x, y) \mapsto (x^{\frac{q}{p}}, x^{\frac{q}{p}}y)$ for some $\lambda \in \mathbb{C}$ and $p, q \in \mathbb{Z}$ with $p < q$. Thus we can
suppose \( \sigma = \sigma_1 \) or \( \sigma_2 \). In the first case, \( \sigma \) is simply the exponential operator \( e^{ad_h} \) for \( h = \frac{q \lambda}{q-p} x^{1-\frac{p}{q}} \) (cf. (4.3)), which does not change the vanishing trace property by (3.9). As for the later case, one can check directly that \( \sigma_2(x^i y^j) = x^{\frac{m-\sigma_1}{n-p}} y^{j} \neq (xy)^{-1} \) if \( (i,j) \neq (-1,-1) \). \( \square \)

Now let \((F,G)\) be a Jacobi pair in \( B \) and we use notations in (2.22)–(2.26). From (2.25), we obtain
\[
\frac{1}{\partial_y F} = \sum_{i=0}^\infty \frac{1 - i}{m} \bar{b}_i F_1^{1-m-1}. \tag{3.10}
\]

**Lemma 3.8** We have \( m + n \geq 1 \) and \( b_i \in \mathbb{C} \) if \( i < n + m - 1 \). Furthermore,
\[
\partial_x b_{i+m+n-1} = -\frac{(1-i)J}{m} \bar{b}_i \quad \text{if} \quad i \geq 0. \tag{3.11}
\]

In particular,
\[
\partial_x b_{m+n-1} = -\frac{J}{m} \bar{b}_0 = -\frac{J}{m} h \frac{m}{m} \quad \text{(by (2.26))}. \tag{3.12}
\]

**Proof.** By (2.22), \( \partial_x G = A + B \partial_x F \), \( \partial_y G = B \partial_y F \), where
\[
A = \sum_{i=0}^\infty (\partial_x b_i) F_1^{n-i}, \quad B = \sum_{i=0}^\infty \frac{n-i}{m} \bar{b}_1 F_1^{n-i-1}.
\]

We obtain
\[
J = -A \partial_y F, \quad \text{i.e.,} \quad \sum_{i=0}^\infty (\partial_x b_i) F_1^{n-i} = -\frac{J}{\partial_y F}. \tag{3.13}
\]

The lemma follows from (3.10) by comparing the coefficients of \( F_1^{n-i} \) in (3.13) for \( i \in \mathbb{Z}_+ \). \( \square \)

**Remark 3.9** From Lemma 3.8, we see that in case \( F,G \in \mathbb{C}[x,y] \) and \( m = 1 \) then we can replace \( G \) by \( G - \sum_{i=0}^{n-1} b_i F_1^{n-i} \) to reduce \( \deg_y G \) to zero, thus obtain \( F,G \) are generators of \( \mathbb{C}[x,y] \) and \( F \) is a monic polynomial of \( y \). Since eventually we shall consider a usual Jacobi pair, from now on we suppose \( m \geq 2 \). Furthermore, we can always suppose \( n \geq m \) and \( m \not| n \).

Using (2.26), we obtain \( \deg_x \bar{b}_0 = -\frac{m}{m} \neq -1 \) (by Theorem 3.6 since \( (-\frac{m}{m},-1) \notin \text{Supp } F_1^{-1} \)), and for \( i > 0 \),
\[
\deg_x \bar{b}_i \leq -\frac{m_0(1-i)}{m} + \max_{0 \leq j < i} \left\{ \deg_x \bar{b}_j + \frac{m_0}{m}(1-j) + (i-j)p \right\}
\leq \sigma_i, \quad \text{where} \quad \sigma_i := i(p + \frac{m_0}{m}) - \frac{m_0}{m}, \tag{3.14}
\]
where, the last inequality follows from induction on \( i \). Thus
\[
1 + \sigma_0 \leq \deg_x b_{m+n-1} \leq \max\{0,1+\sigma_0\}, \quad \deg_x b_{m+n-1+i} \leq \max\{0,1+\sigma_i\} \quad \text{for } i \geq 1. \tag{3.15}
\]

We denote
\[
\eta_i = \deg_x b_{m+n-1+i} \quad \text{for } i \geq 0. \tag{3.16}
\]

We shall use (2.13) and (2.22) to compute \( G_{[r]} \). Thus we set \( F_i = b_i F_1^{n-i} \). Then \( p(F_i) \leq p \) by Lemma 2.4(1) and (2). Assume \( m + n \geq 2 \). Then \( b_0 \in \mathbb{C} \), and we have the data \( (\bar{m}_i, \bar{m}_i) \) in
Lemma 2.5(1) being \( \tilde{m}_i = n - i \), and

\[
\tilde{m}_i = \frac{m_0(n-i)}{m} \quad \text{if} \quad i < m+n-1, \quad \text{and} \quad \tilde{m}_{m+n-1+i,0} = \frac{m_0(1-m-i)}{m} + \eta_i \quad \text{if} \quad i \geq 0.
\]

Then (2.13) gives

\[
G_v = \sum_{i=0}^{\infty} (F_i)[r+\tilde{m}_0-\tilde{m}_0+p(\tilde{m}_0-\tilde{m}_i)] = \sum_{i=0}^{m+n-2} (F_i)[r+(p+\frac{m_0}{m})i] + \sum_{i=0}^{\infty} (F_m+n-1+i)[\alpha_{r,i}], \quad \text{where}
\]

where \( \alpha_{r,i} := r + (p + \frac{m_0}{m})(m + n - 1) + \frac{m_0}{m} + \sigma_i - \eta_i. \quad \text{(3.17)} \]

**Lemma 3.10** Assume \( m+n \geq 2 \). If \( p < -\frac{m_0}{m} \), then \( b_i = 0 \) for \( 1 \leq i < m+n-1 \).

**Proof.** Suppose there exists the smallest \( i_0 \) with \( 1 \leq i_0 < m+n-1 \) such that \( b_{i_0} \neq 0 \) (i.e., \( F_{i_0} \neq 0 \)). Then setting \( r = -(p + \frac{m_0}{m})i_0 > 0 \) in (3.17) gives a contradiction (cf. statements after (2.10)):

\[
0 = G_v = b_{i_0}(F_{\frac{n-i_0}{m}})[0] + \cdots 
eq 0,
\]

where the last inequality follows from the fact that \( y^{n-i_0} \) appears in \( (F_{\frac{n-i_0}{m}})[0] \) but not in any omitted terms. \( \Box \)

We denote

\[
b'_i = \text{Coeff}(b_i, x^0) \quad \text{(which is} \quad b_i \quad \text{if} \quad i < m+n-1). \quad \text{(3.20)}
\]

We always suppose that our Jacobi pair \( (F,G) \) satisfies the condition (by Lemma 3.14, this condition will be automatically satisfied by Jacobi pairs in \( C[x,y] \))

\[
p \neq -\frac{m_0}{m}. \quad \text{(3.21)}
\]

We always denote \( \mu \) to be (until the end of this section)

\[
\mu = m + n - \frac{1 + p}{\frac{m_0}{m} + p} \in \mathbb{Q}. \quad \text{(3.22)}
\]

**Lemma 3.11** Assume \( m+n \geq 2 \) and \( p < -\frac{m_0}{m} \). Then \( b'_i = 0 \) for all \( i \geq 1 \) (thus in particular \( \eta_i = 1 + \deg \tilde{b}_i \leq 1 + \sigma_i \) for all \( i \geq 0 \)).

**Proof.** Suppose \( b'_{i_0} \neq 0 \) for smallest \( i_0 \geq 1 \). Then \( i_0 > m+n-1 \) by Lemma 3.10. Setting \( r = -(\frac{m_0}{m} + p)i_0 > 0 \) in (3.17) gives a contradiction as in (3.19). \( \Box \)

**Lemma 3.12** Assume \( m+n \geq 2 \). Then \( p \geq -\frac{m_0}{m} + \frac{m-m_0}{m(m+n-1)} = \frac{-m_0+n_0-1}{m+n-1} \), where \( n_0 := \deg x g_0 \).

**Proof.** First by letting \( i = 0 \) in (2.23), we have \( n_0 = \deg x g_0 = \frac{mn}{m} \), from which we have \( -\frac{m_0}{m} + \frac{m-m_0}{m(m+n-1)} = \frac{-m_0+n_0-1}{m+n-1}. \)
Assume conversely $p < -\frac{m_0}{m} + \frac{m - m_0}{m(m + n - 1)}$. Then

$$r := -\left((p + \frac{m_0}{m})(m + n - 1) + \frac{m_0}{m} + \sigma_0 - \eta_0\right) > -(1 + \sigma_0 - \eta_0) = -1 + \frac{m_0}{m} + \eta_0 = 0,$$

where the last equality follows by noting that either $m_0 > m$ (so in this case $p < -\frac{m_0}{m} + \frac{m - m_0}{m(m + n - 1)} < -\frac{m_0}{m}$ and $b'_{m+n-1} = 0$ by Lemma 3.11, thus $\eta_0 = 1 - \frac{m_0}{m}$ by (3.12)), or else $m_0 < m$ (in this case $\eta_0 = 1 - \frac{m_0}{m}$ again by (3.12)). By Lemma 3.10, we have either $p + \frac{m_0}{m} \geq 0$ or $b_i = 0$ for $0 < i < m + n - 1$. Thus (3.17) gives a contradiction:

$$0 = G \left[ r \right] = F \left[ r \right] + (F_{m+n-1})_{\left[ 0, r \right]} + \cdots = (F_{m+n-1})_{\left[ 0 \right]} + \cdots \neq 0. \quad \square$$

Note from Lemma 3.12 that $\left(\frac{m_0}{m} + p\right)(m + n) - (1 + p) \geq 0$. Thus we obtain

$$\mu = m + n - \frac{1 + p}{\frac{m_0}{m} + p} \leq 0 \quad \text{if} \quad p < -\frac{m_0}{m}, \quad (3.23)$$

**Remark 3.13**

1. It is very important to assume $m + n \geq 2$ (otherwise $b_0 \notin \mathbb{C}$) and assume $p(G) \leq p$ (otherwise $G \left[ r \right]$ can be nonzero for $r > 0$).
2. Note that if $p$ is sufficiently small, one cannot replace $G$ by $\tilde{G} := F^{\ell} + G$ for $\ell > 0$ without changing $p(G)$; for instance, if $p < -\frac{m_0}{m}$ then $p(\tilde{G}) = -\frac{m_0}{m} > p$.

**Lemma 3.14** If $F, G \in \mathbb{C}[x, y]$ with $m + n \geq 2$, then $p > -\frac{m_0}{m}$ and $p(G) = p$ (thus the positions of $F$ and $G$ are symmetric).

**Proof.** If $m_0 < m$ the result follows from Lemma 3.12. Thus suppose $m_0 > m$. Note that in order for $[F, G] \in \mathbb{C}\{0\}$, $x$ must appear as a term in $F$ or $G$. If $F$ contain the term $x$, then $\deg_x f_m \geq 1$ and

$$p \geq \frac{\deg_x f_m - \deg_x f_0}{m} \geq -\frac{m_0}{m} + \frac{1}{m} > -\frac{m_0}{m}. $$

If $G$ contains the term $x$, then

$$p \geq p(G) \geq \frac{\deg_x g_n - \deg_x g_0}{n} \geq \frac{1 - n_0}{n} = \frac{1}{n} - \frac{m_0}{m} > -\frac{m_0}{m},$$

where $n_0 = \deg_x g_0 = \frac{m_0}{m}n$. Thus $p > -\frac{m_0}{m}$. Now letting $r = 0$ in (3.17) shows $G_{\left[ 0 \right]} = (F_{\left[ m \right]})_{\left[ 0 \right]}$ which is not a monomial by (2.17). Thus $p(G) = p$ by Lemma 2.4(3). \quad \square

**Lemma 3.15** Suppose $F, G \in \mathbb{C}[x(y)]$ with $m + n \geq 2$. If $b_i \neq 0$ and $i < m + n - 1$, then $m | m_0' (n - i)$. In particular, $m | m_0'n$ and $m | m_0'i$.

**Proof.** The lemma is trivial if $m_0' = 0$. Suppose $i_0 < m + n - 1$ is minimal such that $b_{i_0} \neq 0$ and $m \nmid m_0'(n - i_0)$. Multiplying (2.24) by $h \frac{m_0'(n - i_0)}{m}$, we see that $h \frac{m_0'(n - i_0)}{m}$ is a rational function of $x$, which contradicts Lemma 2.8 and (3.4). Thus $m | m_0'(n - i_0)$. Since $b_0 \neq 0$ by (2.24), we have $m | m_0'n$. \quad \square

**Lemma 3.16** Suppose $F, G \in \mathbb{C}[x(y)]$ and $0 < m_0 \leq m$ or $m \nmid m_0$. Then $h = x$, i.e., $d_0 = 1$, $m_0' = m_0$. 
Proof. From (2.24) and Lemma 3.15, we see $b_{m+n-1}$ has the form $b_{m+n-1} = h^{-\frac{m'}{m} - a}g$ for some $a \in \mathbb{Z}$ and $0 \neq g \in \mathbb{C}[x]$ (since $b_{m+n-1} \neq 0$ by (3.12)) such that $h \not| g$. Since $\deg_x b_0 = -\frac{m_0}{m}$, we have $\deg_x b_{m+n-1} = 1 - \frac{m}{m}$ or 0. If $\deg_x b_{m+n-1} = 1 - \frac{m}{m}$, then
\[
d_g := \deg_x g = \deg_x b_{m+n-1} + \left(\frac{m'}{m} + a\right)\deg_x h = 1 + ad_0 \text{ (by (3.12))}. \tag{3.24}
\]
If $\deg_x b_{m+n-1} = 0$ and $m_0 \neq m$ (which can happen only when $m < m_0$ and $d_g = \frac{m}{m} + ad_0$), then $m|m_0$, a contradiction with our assumption in the lemma.

Thus we have (3.24), which implies either $a \geq 0$ or $a = -1, \ d_0 = 1$. In the latter case, we have $h = x$ (since $h$ is monic without the constant term). Thus suppose $a \geq 0$. Now (3.12) shows
\[
\alpha h^{-a-1}g\partial_x h + h^{-a}\partial_x g = h^{-\frac{m'}{m}}\partial_y b_{m+n-1} = -\frac{J}{m}h^{-\frac{m'}{m}}b_0 = -\frac{J}{m}, \text{ where } \alpha = -\frac{m'}{m} - a. \tag{3.25}
\]
Factorize $h, g$ as products of irreducible polynomials of $x$:
\[
h = h_1^{i_1} \cdots h_{\ell}^{i_\ell}, \quad g = g_0 h_1^{i_1} \cdots h_{\ell}^{i_{\ell}} g_{r+1}^{j_{r+1}} \cdots g_s^{j_s}, \tag{3.26}
\]
for some $\ell, s, i_1, \ldots, i_{\ell}, j_1, \ldots, j_s \in \mathbb{N}$ and $0 \leq r \leq \min\{s, \ell\}, 0 \neq g_0 \in \mathbb{C}$, where
\[
g_1 := h_1, \ldots, g_r := h_r, \ h_{r+1}, \ldots, h_{\ell}, \ g_{r+1}, \ldots, g_s \in \mathbb{C}[x],
\]
are different irreducible monic polynomials of $x$ (thus, of degree 1). Multiplying (3.25) by $h^{a+1}$, using (3.26), and canceling the common factor
\[
h_1^{i_1+1} \cdots h_{\ell+1}^{i_{\ell}-1} h_{r+1}^{i_{r+1}-1} \cdots h_s^{i_s-1},
\]
noting that $\partial_x h_\eta = \partial_x g_\eta = 1$ for all $\eta$, we obtain
\[
\left(- (m_0' + am)g_{r+1} \cdots g_s \sum_{\eta=1}^s h_\eta^{i_1+1} \cdots h_{\ell}^{i_{\ell}+1} g_\eta^{j_{r+1}+1} \cdots g_s^{j_s+1}\right) g_{r+1}^{j_{r+1}+1} \cdots g_s^{j_s+1} = -Jh_1^{i_1+1-1} \cdots h_r^{i_r+1} \cdots h_{\ell+1}^{i_{\ell+1}+1} \cdots h_s^{i_s+1}. \tag{3.27}
\]
If $\ell > r$, then $h_\ell$ divides all terms except one term corresponding to $\eta = \ell$ in (3.27), a contradiction. Thus $\ell = r$. Since $g_{r+1}, \ldots, g_s$ do not appear in the right-hand side of (3.27), we must have $j_{r+1} = \ldots = j_s = 1$, and since the left-hand side is a polynomial, we have
\[
i_k a + 1 - j_k \geq 0 \quad \text{for } k = 1, \ldots, \ell. \tag{3.28}
\]
If $i_k a + 1 - j_k > 0$ for some $k$, then $h_k$ divides all terms except two terms corresponding to $\eta = k$ and $\lambda = k$ in (3.27), and the sum of these two terms is a term (not divided by $h_k$) with coefficient $-(m_0' + am)i_k + m_j k$. This proves
\[
i_k a + 1 - j_k = 0 \quad \text{or} \quad -(m_0' + am)i_k + m_j k = 0 \quad \text{for } k = 1, \ldots, \ell. \tag{3.29}
\]
If $a \geq 1$, then either case of (3.29) shows $i_k \leq j_k$, and thus, $h \not| g$, a contradiction with our choice of $g$. Hence $a = 0$.

Now (3.24) shows $d_g = 1$. Write $g = g_0 x + g_1$ for some $g_1 \in \mathbb{C}$. Then (3.25) gives
\[
\int \frac{dh}{h} = \frac{1}{m_0'} \left(\frac{J}{m} + g_0\right) \int \frac{dx}{g_0 x + g_1}, \text{ thus } h = c(g_0 x + g_1)^{m_0'} \frac{(g_0 + \frac{J}{m})}{m_0' g_0} \text{ for some } c \neq 0.
\]
Since $h \in \mathbb{C}[x]$, we must have $\beta := \frac{m_0'}{m_0' g_0} (g_0 + \frac{J}{m}) \in \mathbb{N}$, then (3.4) shows $\beta = 1$ (and so $m \neq m_0'$ since $J \neq 0$), and $h = x$ (since $h$ is monic without the constant term). \qed
Remark 3.17 If \( F, G \in \mathbb{C}[x,y] \) and \( p \leq 0 \), then by (2.7) (cf. (2.8)), we have \( \deg_x F = m_0 > 0 \), and by exchanging \( x \) and \( y \) if necessary, we can sometimes suppose \( m_0 > m \) and \( p < 0 \) (i.e., \( \Coeff(F,x^m) = y^{m_0} \) by Lemma 3.16) or sometimes suppose \( m_0 < m \) and \( p \leq 0 \).

3.2 Jacobi pairs in \( \mathbb{C}((x^{-1}))[y] \)

From now on we shall suppose \( F, G \in \mathbb{C}((x^{-1}))[y] \) with \( m, m+n \geq 2 \). Then each component \( F_r \) is in \( \mathbb{C}[x^{\pm 1}, y] \). Let \( F \) be the primary polynomial as in Definition 2.3(ii) and \( \mu \) be as in (3.22).

Lemma 3.18 If \( \mu < 0 \) (so \( p < -\frac{m\mu}{m} \) by (3.22)) then \( \ell(n) \) (recall notation \( d \) in (2.12)), and \( p(G) = p \) (thus the positions of \( F \) and \( G \) are symmetric in this case).

Proof. Note that if the equality holds in Lemma 3.12 then one can obtain \( \mu = 0 \), thus inequality holds in Lemma 3.12. Setting \( r = 0 \) in (3.17) gives (cf. (2.17))

\[
G[0] = (F_{\frac{\mu}{m}})[0] = x^{\frac{m_0 n}{m}} F^{rac{\mu}{m}},
\]

by noting the following facts:

(i) \( b_i = 0 \) for \( 1 \leq i < m+n-1 \) by Lemma 3.10;

(ii) \( H[a] = 0 \) for \( H \in \mathbb{C}((x^{-1}))[y] \) and all \( a > 0 \);

(iii) by (3.18), \( \alpha_{0,i} = (p + \frac{m_0}{m})(m+n-1) + \frac{m_0}{m} + \sigma_i - \eta_i > \frac{m-m_0}{m} + \frac{m_0}{m} + \sigma_i - \eta_i = 1 + \sigma_i - \eta_i \geq 0 \) for \( i \geq 0 \), where the part “>” is obtained by Lemma 3.12, and the part “≥” is by Lemma 3.11.

Since \( G[0] \) is a polynomial on \( y \), we have \( d|n \) by (2.11), (3.30) and Lemma 2.8. By Lemma 2.4(3), \( p(G) = p \). \( \square \)

Lemma 3.19 If \( 0 \leq i < \mu \) and \( b'_i \neq 0 \), then \( d|(n-i) \).

Proof. By Lemma 3.11, we can suppose \( p > -\frac{m\mu}{m} \). Assume \( i_0 \in \mathbb{Z}_+ \) is smallest such that \( i_0 < \mu \) with \( b'_{i_0} \neq 0 \) and \( d\not\mid(n-i_0) \). We shall use (3.17) to compute \( G[r] \) for \( r = -(p + \frac{m_0}{m})i_0 \).

First suppose \( i_0 \geq m+n-1 \). Then \( \mu > m+n-1 \). Thus we must have \( m_0 > m \) by (3.22). Set \( iv_0 = i_0 - (m+n-1) \). Then using \( i_0 < \mu = m + n - \frac{1+p}{\frac{m_0}{m}+p} \), we obtain

\[
iv_0 < 1 - \frac{1+p}{\frac{m_0}{m}+p} = \frac{m_0}{m} - \frac{1}{\frac{m_0}{m}+p},
\]

so \( 1 + iv_0(m_0/m) + p - m_0/m < 0 \),

i.e., \( 1 + \sigma_{iv_0} < 0 \) (cf. (3.14)). Thus by (3.15), we have \( \eta_{iv_0} \leq 0 \). Since \( b'_{m+n-1+i_0} \neq 0 \) (recall notation \( b'_i \) in (3.20)), we have \( \eta_{iv_0} = 0 \). Note from (3.14) that when \( i > 0 \), we have \( 1 + \sigma_i > 0 \). Thus by (3.15), \( \eta_i \leq \sigma_i + 1 \), and so by (3.18),

\[
\alpha_{r,i} = (p + \frac{m_0}{m})(m+n-1-i_0) + \frac{m_0}{m} + \sigma_i - \eta_i
= (p + \frac{m_0}{m})(m+n-1-i_0) + \frac{m_0}{m} + \sigma_i - \eta_i - \frac{m_0}{m}
= \sigma_i - \eta_i - (\sigma_{iv_0} - \eta_{iv_0}) \geq -1 - \sigma_{iv_0} > 0 \quad \text{if} \quad i > 0.
\]
Say the above holds when $i > i_1$. Then (3.17) gives (using $(F_i)_[s] = 0$ for all $s > 0$)

$$G_{[-(p+\frac{m_0}{m})i_0]} = \sum_{i=0}^{m+n-2} b_i(F^\frac{n-i}{m})[(i-i_0)(p+\frac{m_0}{m})] + (F_m-n-i)(F^\frac{n-i}{m})_0] + \sum_{0 \leq i < i_1, \alpha_i = -a} (F_{m+n-1+i})[\alpha_{r,i}].$$  \hspace{1cm} (3.31)

Let $a \geq 0$ be any rational number. Similar to (3.31), we also have

$$G_{[-(p+\frac{m_0}{m})i_0+a]} = \sum_{i=0}^{m+n-2} b_i(F^\frac{n-i}{m})[(i-i_0)(p+\frac{m_0}{m})+a] + \sum_{0 \leq i < i_1, \alpha_i = -a} b_{m+n-1+i}(F^\frac{1-m-i}{m})_0] + \sum_{0 \leq i < i_1, \alpha_{r,i}+a < 0} (F_{m+n-1+i})[\alpha_{r,i}+a].$$  \hspace{1cm} (3.32)

Note that $G_{[-(p+\frac{m_0}{m})i_0+a]}$ is a polynomial on $y$. Set

$$A := \max\{-i-i_0)(p+\frac{m_0}{m}), -\alpha_{r,j} \mid 0 \leq i \leq m+n-2, 0 \leq j \leq i_1, \alpha_{r,j} \leq 0\}.$$

If we first take $a = A$ in (3.32), then the last summand vanishes and the first summand is summed over those $i$'s such that $(i-i_0)(p+\frac{m_0}{m})+a = 0$, so we can use Lemma 2.8 to obtain that all nonzero terms in (3.32) are rational. Now by taking $a < A$ (and $a \geq 0$) in (3.32) and by induction on $A-a$ (note that there are only finitely many $a$'s with $0 \leq a \leq A$ such that there exist nonzero terms in (3.32)), we can see that all nonzero terms in (3.32) (thus in (3.31)) are rational. In particular, $F^\frac{n-i}{m} = x^{-\frac{m_0(n-i)}{m}}(F^\frac{n-i}{m})_0]$ (cf. (2.17)) is rational, a contradiction with (2.11).

Now suppose $0 \leq i_0 < m+n-1$. As above, noting from (3.14) that when $i \gg 0$, we have $1+\sigma_i > 0$ and $\sigma_i - \eta_i \geq -1$ by (3.15). Thus

$$\alpha_{r,i} = r + (p + \frac{m_0}{m})(m+n-1) + \frac{m_0}{m} \sigma_i - \eta_i \geq (p + \frac{m_0}{m})(m+n-1-i_0) + \frac{m_0}{m} - 1 > 0,$$

where the part “$>$” is obtained by using $i_0 < \mu = m+n - \frac{1+p}{m}$. As in (3.31), say the above inequality holds for $i > i_1$. Then (3.17) shows (in this case $b_{i_0} = b_{i_0}$ by (2.20))

$$G_{[-(p+\frac{m_0}{m})i_0]} = \sum_{0 \leq i < i_0} b_i(F^\frac{\alpha_{r,i}}{m})[(i-i_0)(p+\frac{m_0}{m})] + b_{i_0}(F^\frac{\alpha_{r,i}}{m})_0] + \sum_{0 \leq i < i_1, \alpha_{r,i} < 0} (F_{m+n-1+i})[\alpha_{r,i}].$$

Thus as in (3.31), we obtain a contradiction. \hfill \square

We always set

$$\tilde{b}_\mu = \begin{cases} b_\mu & \text{if } \mu \in \mathbb{Z}_+ \text{ and } \mu < m+n-1, \\ 0 & \text{otherwise}. \end{cases} \hspace{1cm} (3.33)$$

Let (cf. notation $\sigma_i$ in (3.15))

$$R_0 := \tilde{b}_\mu(F^\frac{\alpha_{r,i}}{m})_0] + \sum_{i=m+n-1}^{\infty} C_{\text{coeff}}(b_i, x^{1+\sigma_i-(m+n-1)})x^{1+\sigma_i-(m+n-1)}(F^\frac{n-i}{m})_0]$$

$$= \tilde{b}_\mu x^{-\frac{m_0(n-i)}{m}}(F^\frac{n-i}{m})_0] - \frac{J}{m} \sum_{i=0}^{\infty} \frac{1-i}{1+\sigma_i} \tilde{b}_{i,0}x^{1-m_0+ip}(F^\frac{1-i-m}{a})_0, \text{ where } \tilde{b}_{i,0} = C_{\text{coeff}}(\tilde{b}_i, x^{\sigma_i}) \in \mathbb{C}, \hspace{1cm} (3.34)$$
where the last equality follows from (2.17) and Lemma 3.8 (if $1 + \sigma_i = 0$, i.e., $i = \mu - (m + n - 1) \geq 0$, then $\bar{b}_\mu = 0$ by (3.33), in this case we use the convention that $-\frac{f(i-1)}{m(1+\sigma_i)}\bar{b}_{i,0}$ is regarded as the limit $\lim_{i \to \mu - (m + n - 1)} -\frac{j(1-i)}{m(1+\sigma_i)}\bar{b}_{i,0}$ which is defined to be $\bar{b}_\mu$, cf. (3.20)). We claim that

$$R_0 \neq 0, \quad \text{and } \deg_y R_0 = \frac{1+p}{m} + p - m, \quad \text{or } 1 - m. \quad (3.35)$$

This is because: if $\bar{b}_\mu \neq 0$ (then $\mu \in \mathbb{Z}_+$) then $C_{\text{coeff}}(R_0; y^{m-\mu}) = \bar{b}_\mu \neq 0$; if $\bar{b}_\mu = 0$ then $C_{\text{coeff}}(R_0, y^{1-m}) = \frac{1}{1+\sigma_0}b_{0,0}x^{1-m_0} \neq 0$ ($\bar{b}_{0,0} = C_{\text{coeff}}(f_0, x^{m_0}) = 1$ by (2.26)).

**Lemma 3.20** We have

$$R_0 = G_{[-\mu(p+a/m)]} - \delta b_0(F^n_m)[\mu(p+a/m)] - \sum_{0 \leq i < \mu} b_i(F^{-n_i}_{m,i})[(i-\mu)(p+a/m)] = F^{-a-m_i}P, \quad (3.36)$$

for some $a \in \mathbb{Z}$, $P \in \mathbb{C}[x^{\pm 1}][y]$, such that $F \parallel P$, where $\delta = 1$ if $\mu < 0$ or 0 otherwise.

**Proof.** The last equality follows from Lemmas 2.5(5), 3.15 and 3.19. To prove the first equality of (3.36), set $r = -\mu(\frac{a}{m} + p)$ in (3.17), we obtain

$$G_{[-\mu(\frac{a}{m} + p)]} = \sum_{j=0}^{m+n-2} b_j(F^{-n_j}_{m})[(\mu(p+a/m)] + \sum_{i=0}^{\infty} (F_m^{n_i})_{1+\sigma_i} = \sum_{i=0}^{\infty} \eta_i F^{1+i}.$$  

First assume $p > -\frac{ma}{m}$. Compare the $i$-th term of (3.37) with corresponding terms of (3.4) and (3.6):

(1) If $1 + \sigma_i \geq 0$ (so $m + n - 1 + i \geq \mu$ and (3.36) does not have such a term), then either

(i) $\eta_i = 1 + \sigma_i$: the $i$-th term of (3.37) corresponds to the $i$-th term of (3.4), or

(ii) $\eta_i < 1 + \sigma_i$: the $i$-th terms of (3.37) and (3.4) are both zero.

(2) If $1 + \sigma_i < 0$ (so $m + n - 1 + i < \mu$ and (3.4) does not have such a term), then either

(i) $\eta_i = 0$: the $i$-th term of (3.37) corresponds to the $i$-th term of (3.6), or

(ii) $\eta_i < 0$: the $i$-th terms of (3.37) and (3.6) are both zero.

This proves the lemma in this case. Assume $p < -\frac{ma}{m}$. Then by Lemmas 3.11 and 3.18, the summand in (3.36) is empty and the first summand in (3.37) has only one term corresponding to $j = 0$. We again have the lemma.

By (3.35) and (3.36),

$$d_P := \deg_y P = da + \frac{1+p}{m} + p \quad \text{(in case } \bar{b}_\mu \neq 0) \quad \text{or } da + 1 \quad \text{(in case } \bar{b}_\mu = 0). \quad (3.38)$$

Computing the zero-th component of (2.25), using (2.17), similar as in (3.17), we obtain

$$y = \sum_{i=0}^{\infty} (\bar{b}_i F^{\frac{1+i}{d}})[0] = \sum_{i=0}^{\infty} \text{Coeff}(\bar{b}_i, x^{\sigma_i}) x^{\sigma_i}(F^{\frac{1+i}{d}})[0] = \sum_{i=0}^{\infty} \bar{b}_{i,0} x^{ip} F^{\frac{1+i}{d}}.$$  

Taking $\partial_y$ gives

$$1 = \sum_{i=0}^{\infty} \frac{1-i}{d} \bar{b}_{i,0} x^{ip} F^{\frac{1+i}{d}} - \partial_y F. \quad (3.39)$$
Lemma 3.21 \((F_0, R_0) = (x^{m_0} F^{m'}, F^{-a-m'} P)\) is a Jacobi pair.

Proof. First note that \(R_0\) is a \(p\)-type q.h.e., thus \(p(R_0) = p\) (or \(-\infty\) if it is a monomial). By (3.36), \(R_0 \in \mathbb{C}(x,y)\) has the form (2.21). From (3.34) and (3.39), we see \([F_0, R_0] = J \in \mathbb{C}\{0\}\) (cf. proof of Lemma 3.8), which can be also proved as follows: Denote \(G_1 = \sum_{i=0}^{\infty} b_i F^{n_i/m}\) and \(H = G - G_1\). Then \([F, H] = J\). Note that \([F_0, H_0] = J\) has the highest term 1 (up to a scalar) since \(F\) has the highest term \(x^{m_0} y^n\) and \(H\) has the highest term \(x^{1-m_0} y^{1-m}\). Thus \([F_0, H_0] = J\) is a \(p\)-type q.h.e. whose support lies in a line passing the point \((0,0)\). Comparing the 0-th components in \([F, H] = J\) we obtain \([F_0, H_0] = J\). Since \(H_0\) and \(R_0\) only differ by some \(\mathbb{C}\)-combinations of rational powers of \(F\), we have \([F_0, R_0] = [F_0, H_0] = J\). \(\square\)

Multiplying (3.34) by \(F^{\mu-n/d}\), taking \(\partial_y\) and using (3.36), (3.22) and (3.39), we have (cf. the remark after (3.44))

\[
( - a - m' + \frac{\mu - n}{d} ) F^{-a-m'+\mu-n/d} P \partial_y F + F^{-a-m'+\mu-n/d} \partial_y P
\]

\[
= \partial_y ( R_0 F^{\mu-n/d} ) = - \frac{J}{m} \sum_{i=0}^{\infty} \frac{1 - i}{\sigma_i} b_{i,0} x^{1-m_0+ip} \partial_y ( F^{\frac{1+p}{d+p+1/m}} )
\]

\[
= J x^{1-m_0} \sum_{i=0}^{\infty} \frac{1 - i}{d} b_{i,0} x^{ip} F^{\frac{1+p}{d+p+1/m}} \partial_y F \cdot F^{\frac{1+p}{d+p+1/m}} = J x^{1-m_0} \frac{m_0 + m_0 p}{m p + m_0} F^{\frac{1+p}{d+p+1/m}}. \tag{3.40}
\]

We write \(p = \frac{\mu'}{q}\) for some coprime integers \(p', q\) such that \(q > 0\). Note that \(F\) being a monic \(p\)-type q.h.e., has the form

\[
F = y^d + \sum_{i=1}^{d} c_i x^i y^{d-i} \in \mathbb{C}[x^{\pm 1}][y] \text{ for some } c_i \in \mathbb{C}. \tag{3.41}
\]

Noting that \(x^{m_0} F^{m'} = F_0 \neq x^{m_0} y^m\) since \(p \neq -\infty\), we have \(c_i \neq 0\) for some \(i\). Hence at least one of \(p, 2p, ..., dp\) is an integer. Thus

\[
1 \leq q \leq d. \tag{3.42}
\]

Multiplying (3.40) by \(F^{\frac{1+p}{d+p+1/m}}\), we obtain the following differential equation on \(F\) and \(P\),

\[
\alpha_1 F^{-a-1} \partial_y F + \alpha_2 F^{-a} \partial_y P = \alpha_3, \quad \text{or} \quad \alpha_1 P \partial_y F + \alpha_2 F \partial_y P = \alpha_3 F^{a+1}, \tag{3.43}
\]

where

\[
\alpha_1 = -a \alpha_2 - m' (p' + q), \quad \alpha_2 = \frac{p'}{m} + m_0 q, \quad \alpha_3 = q J x^{1-m_0}. \tag{3.44}
\]

We remark that the only purpose of (3.40) is to prove (3.43), which can be also directly proved by the following formal arguments: Noting that \(F, P\) are \(p\)-type q.h.e., so \(x \partial_x F, x \partial_x P\) are combinations of \(F, y \partial_y F, P, y \partial_y P\), using this in \([x^{m_0} F^{m'}, F^{-a-m'} P] = J\), one can easily deduce (3.43) for some \(\alpha_1, \alpha_2 \in \mathbb{C}\) and some \(0 \neq \alpha_3 \in \mathbb{C}[x^{\pm 1}]\). Note that the first term of \(R_0\) in (3.34) does not contribute to (3.43), and whether or not \(P\) is a polynomial on \(y\) does not affect (3.43) either, thus for the purpose of proving (3.43) with \(\alpha_1\) satisfying (3.44), if necessary by changing the coefficient \(\beta_{\mu}\), we may suppose that the \(y\)-degree of \(P\) is \(\deg_y P = ad + \frac{1+p}{m+p} = ad + \frac{(p'+q)m}{m_0 q + p'm}\). Then by comparing the coefficients of the highest degree of \(y\) in (3.43), we immediately obtain \(\alpha_1 d + \alpha_2 (ad + \frac{(p'+q)m}{m_0 q + p'm}) = 0\). Thus up to a scalar, we can suppose \(\alpha_1, \alpha_2\) have the forms as in (3.44) (what is \(\alpha_3\) is not important since later on we only need the fact that \(\alpha_3 \neq 0\)).
Lemma 3.22  (1) For every irreducible factor \( Q \in \mathbb{C}[x^\pm][y] \) of \( F \) or \( P \), we have \( \deg_y Q = 1 \) or \( q \mid \deg_y Q \). If \( Q \) is monic and \( q \not| \deg_y Q \), then \( Q = y \).

(2) If \( d = q > 1 \), then \( F \) is irreducible and \( F = y^d + x^\nu \) (up to rescaling \( x \)).

**Proof.**  (1) Note that \( R_0 \) is a \( p \)-type q.h.e., and \( F \) is a \( p \)-type q.h.e. (cf. the right-hand side of (3.41)). By Lemma 2.5(4), \( P = R_0 F^{\alpha + \mu'} \) must be a \( p \)-type q.h.e. By Lemma 2.5(4) again, every irreducible factor \( Q \) of \( F \) or \( P \) must be q.h.e. of the form \( \sum_{i=0}^\gamma u_i x^{pi} y^{\gamma - i} \), where \( \gamma = \deg_y Q \) and \( u_i \in \mathbb{C} \). If \( q \not| \gamma \), then \( p\gamma = \frac{\nu' + \mu'}{q} \) cannot be an integer, thus \( u_i = 0 \), and \( Q \) contains the factor \( y \).

(2) It follows from (1) and (2.11) (cf. (3.41)). \( \square \)

3.3 Some examples of Jacobi pairs with \( p \leq 0 \)

Example 3.23  Below we obtain some Jacobi pairs starting from a simple one by adding some powers of \( F \) (or \( G \)) to \( G \) (or \( F \)).

(1) \((xy^2, y^{-1}) \rightarrow (xy^2, xy^2 + y^{-1}) \rightarrow (x^2 y^4 + xy^2 + 2xy + y^{-2}, xy^2 + y^{-1}) := (F, G)\). In this case, 
\( p = -\frac{1}{3}, m_0 = 2, m = 4, F[0] = (xy^2 + y^{-1})^2 \) and \( n = 2 \).

(2) \((x^2 y, x^3) \rightarrow (x^2 y, xy^2 + x^3) \rightarrow (x^2 y^4 + 2x^2 y^2 + x + x^2 y^2 + x^2) := (F, G)\). In this case, 
\( p = -\frac{1}{3}, m_0 = 2, m = 4, F[0] = (xy^2 + x^2)^2 \) and \( n = 2 \).

In Example 3.23, we have \( m \mid n \) or \( n \mid m \). If this holds in general, then the two-dimensional Jacobian conjecture can be proved by induction on \( m \). However, the following example shows that this may not be the case.

Example 3.24  (1) \( F = xy^2 + 2x^2 y - y = F[0], \ p = -\frac{3}{8}, m_0 = 1, m = 2, \) and \( G = x^2 y^3 + 3x^2 y^2 + \frac{3}{2} x^2 y - \frac{1}{2} x = R_0, n = 3 \).

(2) \( F = x^2 y^{10} + 2x y^4 = F[0], \ p = -\frac{1}{6}, m_0 = 2, m = 10, \) and \( G = x^3 y^{15} + 3x^2 y^9 + \frac{1}{2} x y^3 - \frac{1}{2} y^3 = R_0, n = 15 \).

For the above examples, we observe that if \( m_0 < m \) then either \( F, G \) must contain some negative power of \( y \) or contain some non-integral power of \( x \).

In case \( F, G \in \mathbb{C}[x, y], \) by Remark 3.17, we can suppose \( m_0 < m \) if \( p \leq 0 \). To better understand Jacobi pairs, we first suppose \( m_0 > m \) and obtain the following two lemmas. We rewrite \( F, G \) as (where \( f_{00} = 1 \))

\[
F = \sum_{i=0}^m f_{ij} y^{m-i} = \sum_{(m_0-i, m-j) \in \text{Supp } F} f_{ij} x^{m_0-i} y^{m-j}, \quad (3.45)
\]

\[
G = G_1 + R, \quad G_1 = \sum_{0 \leq i < \mu} b_i^j F_{\frac{\alpha m}{m}}^n, \quad R = \sum_{\min(m+n-1, \mu) \leq j < \infty} b''_j F_{\frac{\alpha m}{m}}^n, \quad (3.46)
\]

where \( b_i^j \) is defined in (3.20), and \( b''_j = b_j - b_j' \) if \( j < \mu \) and \( b''_j = b_j' \) if \( j \geq \mu \) (recall notation \( \mu \) in (3.22)). Then \([F, G] = [F, R]\).

Theorem 3.25  (1) Suppose \( F, G \) is a Jacobi pair in \( \mathbb{C}[x, y] \). Then \( p \neq -1 \).

(2) Suppose \( F, G \in \mathbb{C}[x^{\pm}, y] \) with \( m_0 > m \) and \( p < 0 \). Then \( p \leq -1 \). Furthermore, if \( p = -1 \) then \( \text{Res}_x (FD_x G) \neq 0 \) and \( F[0] = x^{m_0} (y - \beta x^{-1})^m \) for some \( \beta \in \mathbb{C} \setminus \{0\} \). Thus by replacing \( y \) by \( y + \beta x^{-1} \) for some \( \beta \in \mathbb{C} \) if necessary, we can suppose \( p < -1 \).
Proof. (1) If \( m_0 < m \), then \( p > -\frac{m_0}{m} > -1 \) by Lemma 3.12. Thus assume \( m_0 > m > 0 \). Suppose \( p = -1 \). Then \( \mu = m + n \) and \( \tilde{b}_\mu = 0 \) (cf. (3.33)). So (3.35) shows \( \deg_y R_0 = 1 - m \), and (3.34) in fact shows \( R_0 = R_{[0]} \). Thus we can write

\[
R = \sum_{i=0}^{\infty} r_i y^{1-m-i} = \sum_{(1-m_0-i,1-m-j) \in \text{Supp } R} r_{i,j} x^{1-m_0-i} y^{1-m-j}, \quad r_i \in \mathbb{C}(x^{-1}), \ r_{i,j} \in \mathbb{C}. \quad (3.47)
\]

By Lemma 2.10 and (3.46), we see \( F^{-1} \) does not appear in \( R \), thus we can write

\[
R = r'_0 F^{1-m} + \sum_{i=1}^{\infty} r'_i F^{-m-i} \quad \text{for some } r'_i \in \mathcal{A}.
\]

Since \( F^{1-m} = f_0^{1-m} (y^{1-m} + \frac{1-m}{m} f_0^{-1} f_1 y^{-m} + \ldots) \), comparing the coefficients of \( y^{1-m} \) and \( y^{-m} \) in the above equation on \( R \), we obtain \( r_0 = r'_0 f_0^{1-m} \) and

\[
r_1 = \frac{1-m}{m} r'_0 f_0^{1-m} f_0^{-1} f_1 = \frac{1-m}{m} r_0 f_0^{-1} f_1. \quad (3.48)
\]

If necessary, by replacing \( x \) by \( x + \beta \) for some \( \beta \in \mathbb{C} \) (which does not affect \( p \)), we can suppose \( f_{01} = 0 \). Thus

\[
0 = f_{10} = f_{01}, \quad (3.49)
\]

where the second equality follows from \( p < 0 \). Computing the coefficients of \( x^{-1} y^0 \) and \( x^0 y^{-1} \) in \( J = [F,R] \), using (3.46), (3.47) and (3.49), we obtain \( r_{10} = r_{01} = 0 \). This and (3.48) give

\[
r_{11} = \frac{1-m}{m} r_{00} f_{11}. \quad (3.50)
\]

Note that we obtain (3.50) only by (3.49) (and is independent of \( p \)). Thus by symmetry (exchanging \( x \) and \( y \)), we also have \( r_{11} = \frac{1-m_0}{m_0} r_{00} f_{11} \). Therefore

\[
r_{11} = f_{11} = 0, \quad (3.51)
\]

which can be also obtained by using \( \text{Res}_x(F \partial_x G) = \text{Res}_y(F \partial_y G) = 0 \). Since \( F_{[0]} \) and \( R_0 \) are \( p \)-type q.h.e with \( p = -1 \), of the form \( F = \sum_{i=0}^{m} f_i x^{m_0-i} y^{m-i} \) and \( R_0 = \sum_{i=0}^{\infty} r_i x^{1-m-i} y^{1-m-i} \), we can easily deduce

\[
x \partial_x F_{[0]} - y \partial_y F_{[0]} = (m_0 - m) F_{[0]}, \quad x \partial_x R_0 - y \partial_y R_0 = -(m_0 - m) R_0.
\]

Using this in \( J = (\partial_x F_{[0]})(\partial_y R_0) - (\partial_y F_{[0]})(\partial_x R_0) \), we obtain \( xJ = (m_0 - m) \partial_y (F_{[0]} R_0) \). For convenience, we can suppose \( J = m_0 - m \). Therefore,

\[
F_{[0]} R_0 = x y + a_0 \quad \text{for some } a_0 \in \mathcal{A}. \quad (3.52)
\]

Since \( F_{[0]} R_0 \) is a \( p \)-type q.h.e., we have \( a_0 \in \mathbb{C} \). By (3.51), \( a_0 = 0 \). Using (3.46), we have (here and below, all equalities associated with integration mean that they hold up to some elements in \( \mathcal{A} \))

\[
\int G(\partial_y F) dy = \sum_{i=0}^{m+n-1} \frac{m}{m+n-i} b'_i F_{i}^{m+n-i} + Q, \quad \text{where } Q = \int R(\partial_y F) dy.
\]
As in (3.34) and (3.36), this shows $Q[0] = \int R_0(\partial_y F[0]) dy \in \mathcal{A}(y)$ by Lemma 3.19. Factorize $F[0] = a_0' \prod_{i=1}^{k} (y - a_i)^{\lambda_i}$ in $\mathcal{A}(y)$, where $a_i \in A$, $a_0' \neq 0$, $\lambda_i > 0$ and $a_1, ..., a_k$ are distinct. Then (3.51) and the fact that $F[0]$ is not a monomial show $k > 1$ and some $a_i \neq 0$. Then (3.52) gives

$$Q[0] = \sum_{i=1}^{k} \lambda_i x \int \frac{ydy}{y - a_i} = \sum_{i=1}^{k} \lambda_i x \left( y + a_i \ln(y - a_i) \right),$$

which cannot be a rational function on $y$. A contradiction.

(2) Consider (3.43) and (3.44). If $d = 1$ then (3.41) shows $p \in \mathbb{Z}$. Thus $p \leq -1$. So suppose $d \geq 2$. If $a \leq -2$, then $F$ divides the left-hand side of the first equation of (3.43), a contradiction. Thus $a \geq -1$.

If $a = -1$, then the second case of (3.38) cannot occur since $d_P \geq 0$ and $d \geq 2$. Thus $\bar{b}_m \neq 0$, i.e., $\mu < m + n - 1$, which implies $m_0 \leq m$, a contradiction with our assumption.

Now suppose $a > 0$. Similar to the proof of Lemma 3.16, we factorize $F, P$ as products of irreducible polynomials of $y$ in the ring $\mathcal{A}[y]$:

$$F = f_1^{\ell} \cdots f_s, \quad P = h_0 f_1^{\ell_1} \cdots f_r f_{r+1}^{j_{r+1}} \cdots h_s, \quad (3.53)$$

for some $\ell, s, i_1, ..., i_r, j_1, ..., j_s \in \mathbb{N}$ and $0 \leq r \leq \min\{s, \ell\}$, $0 \neq h_0 \in \mathcal{A}$, where

$$h_1 := f_1, \quad ..., \quad h_r := f_r, \quad f_{r+1}, ..., \quad f_{r+1}, ..., \quad h_s \in \mathcal{A}[y],$$

are different irreducible monic polynomials of $y$ of degree 1. As in (3.27), we obtain (where $\alpha_1, \alpha_2, \alpha_3$ are as in (3.44))

$$h_0 \left( \alpha_1 h_{r+1} \cdots h_s \sum_{\eta=1}^{\ell} \frac{f_1^{\ell_1} \cdots f_{\ell_1}}{f_{\eta}} + \alpha_2 f_{r+1} \cdots f_r \sum_{\lambda=1}^{s} \frac{h_1 \cdots h_{s}}{h_{\lambda}} \right) h_{r+1}^{j_{r+1}} \cdots h_s^{j_s-1} = \alpha_3 f_1^{i_1+1-j_1} \cdots f_r^{i_r+1-j_r} f_{r+1}^{i_{r+1}} \cdots f_{\ell}^{i_{\ell}+1}. \quad (3.54)$$

Similar to the arguments after (3.27), we have $r = \ell$ and

$$j_{r+1} = ... = j_s = 1. \quad (3.55)$$

Also for all $k \leq \ell$,

$$i_k + 1 - j_k \geq 0, \quad \text{and} \quad (3.56)$$

$$i_k + 1 - j_k = 0 \quad \text{or} \quad \alpha_1 i_k + \alpha_2 j_k = 0. \quad (3.57)$$

First assume $a = 0$. Then (3.56) shows $j_k = 1$ for all $k$. Thus $s = d_P$, and (3.54) is simplified to

$$h_{\ell+1} \cdots h_s \sum_{\eta=1}^{\ell} \frac{(\alpha_1 i_\eta + \alpha_2) f_1 \cdots f_{\ell}}{f_{\eta}} + \alpha_2 f_{1} \cdots f_{\ell} \sum_{\lambda=\ell+1}^{s} \frac{h_{\ell+1} \cdots h_s}{h_{\lambda}} = \alpha_3. \quad (3.58)$$

If $\ell = 1$, then (3.53) shows that $i_1 = d$ and $F = f_d$. Write $f_1 = y + f_{11}$ for some $0 \neq f_{11} \in \mathcal{A}$ since $F \neq y^d$. Comparing the coefficients of $y^{d-1}$ in both sides of (3.41) gives $df_{11} = c_{11} x^p$, we see $c_{11} \neq 0$, thus $p \in \mathbb{Z}$ (since $F \in \mathbb{C}[x^{\pm 1}, y]$), i.e., $p \leq -1$. Thus suppose $\ell \geq 2$. Comparing the coefficients of the terms with highest $y$-degree (i.e., degree $s-1$, which is $\geq \ell-1 \geq 1$) in (3.58) shows (using (3.44) and $a=0$)

$$0 = \sum_{\eta=1}^{\ell} (\alpha_1 i_\eta + \alpha_2) + \sum_{\lambda=\ell+1}^{s} \alpha_2 = \alpha_1 d + s \alpha_2 = -m(p' + q) + s(p'm + m_0 q).$$
i.e., \( d_P = s = \frac{m(p' + q)}{m_0q + p'm} \), and the first case of (3.38) occurs, a contradiction with \( m_0 > m \).

Now suppose \( a \geq 1 \). If \( p' + q > 0 \), then either case of (3.57) shows \( i_k < j_k \) for all \( k \) (note from (3.44) that \(-\alpha_1 > \alpha_2\)), i.e., \( F|P \), a contradiction with our choice of \( P \). Thus \( p' + q \leq 0 \), i.e., \( p = \frac{p'}{q} \leq -1 \). Now assume \( p = -1 \). If necessary, by replacing \( y \) by \( y + \beta x^{-1} \) for some \( \beta \in \mathbb{C} \), we can suppose \( f_{11} = 0 \) (cf. notations in (3.45)). Note that we still have (3.50) since its proof does not require \( F, G \) to be polynomials on \( x \). Thus we have (3.51). Then as in the proof of Theorem 3.25, we have \( p \neq -1 \). This completes the proof of the theorem. \( \square \)

### 3.4 Reducing the problem to the case \( p \leq 0 \)

In this subsection, we want to show that the proof of the two-dimensional Jacobian conjecture can be reduced to the case \( p \leq 0 \).

**Lemma 3.26** Suppose \((F,G)\) is a Jacobi pair in \( \mathbb{C}[x^{\pm 1}, y] \) with \( p = \frac{p'}{q} > 0 \) and \( m_0 \geq 0 \). Then the primary polynomial \( F \) satisfies one of the following (up to re-scaling variable \( x \)):

(i) \( F = y + x^{p'} \) and \( q = 1 \).

(ii) \( F = y^{p'} + x \) and \( q > 1 \), \( m_0 = 0 \).

(iii) \( F = (y^{p'} + x)^i(y + \delta q_1x)q^2 \) for some \( 1 \neq \alpha \in \mathbb{C} \), \( i_1, i_2 \neq 0 \), \( \gcd(i_1, i_2) = 1 \), \( (i_1, i_2) \neq (1, 1) \), \( m_0 = 0 \).

**Proof.** If \( d = 1 \), then we have (i) (cf. (3.41)). Thus suppose \( d \geq 2 \). If \( a \leq -2 \), then \( F \) divides the left-hand side of the first equation of (3.43), a contradiction. Thus \( a \geq -1 \).

Suppose \( a = -1 \). Then the second case of (3.38) cannot occur since \( d_P \geq 0 \) and \( d \geq 2 \). Thus \( d + d_P = \frac{m(p' + q)}{p'm + m_0q} \leq \frac{m(p' + q)}{mp'} = 1 + \frac{q}{p'} \leq 1 + q \leq 1 + d \). (3.59)

So \( d_P \leq 1 \). If \( d_P = 0 \), i.e., \( \partial_y P = 0 \), then the second equation of (3.43) shows \( d = 1 \), a contradiction. If \( d_P = 1 \), then all equalities must hold in (3.59), i.e., \( m_0 = 0 \), \( p' = 1 \), \( d = q \). Thus we have (ii) by Lemma 3.22(2).

Now suppose \( a \geq 0 \). As in the proof of Theorem 3.25, we have (3.53)–(3.58). If \( a > 0 \), then either case of (3.57) shows \( i_k < j_k \) for all \( k \) (note from (3.44) that \(-\alpha_1 > \alpha_2 \) since \( p' + q > 0 \)), i.e., \( F|P \), a contradiction with our choice of \( P \). Thus \( a = 0 \). If \( \ell = 1 \), as in the arguments after (3.58), we obtain \( d = 1 \) (since \( F \) is power free, cf. (2.11)). Thus \( \ell \geq 2 \) (and so \( s \geq \ell \geq 2 \)). Again, as in the arguments after (3.58), we have

\[
0 < d_P = s = \frac{m(p' + q)}{m_0q + p'm} \leq 1 + \frac{q}{p'}.
\] (3.60)

Thus we have the first case of (3.38). Note that any irreducible polynomial \( Q \) in the ring \( \mathbb{C}[x^{\pm 1}][y] \) does not contain an irreducible factor of multiplicity \( \geq 2 \) in \( A[y] \). If \( d_P < q \) (then \( q > d_P = s \geq \ell \geq 2 \)), by Lemma 3.22(1), \( P \) has only one different irreducible factor, i.e., \( s = 1 \), a contradiction. Thus \( d_P \geq q \). Assume \( q = d_P \), which is equal to \( s \geq 2 \). Then \( p' = 1 \) by (3.60) because: if \( p' \geq 3 \) then \( 1 + \frac{\mu}{p'} \leq 1 + \frac{\mu}{2} < q \); if \( p' = 2 \) then \( q > 2 \) (since \( p', q \) are coprime) and \( 1 + \frac{\mu}{p'} = 1 + \frac{\mu}{2} < q \). Hence (3.60) gives \( m = q^2m_0 \). However, since we have the first case of (3.38), we see \( b_{\mu} \neq 0 \) by (3.33), and \( \frac{m_0}{m_\mu} \mu \) is integral by Lemma 3.15. Then by (3.22), \( \frac{m_0}{m} \cdot \frac{m(p' + q)}{m_0q + p'm} \) (which, by (3.60), is equal to
\( \frac{m_0}{m} d_P = \frac{m_0}{m} q = \frac{1}{q} \) is integral, a contradiction with the fact that \( q = d_P \geq 2 \). Thus \( d_P > q \) and by (3.60),

\[ d_P = q + 1, \quad p' = 1, \quad m_0 = 0. \] (3.61)

First suppose \( d_P \geq 3 \). Let \( H \in \mathbb{C}[x^{\pm 1}][y] \), a monic polynomial of \( y \) of degree \( k \), be an irreducible factor of \( F \) (in the ring \( \mathbb{C}[x^{\pm 1}][y] \)). Then Lemma 3.22 shows that either \( \deg_y F = 1 \) (and \( H = y \) by noting that \( q \geq 2 \)), or \( q | k \) (in this case \( H \) has \( k \) different irreducible factors in \( A[y] \)). Since \( F \) has only \( \ell \) different irreducible factors in \( A[y] \) and \( \ell \leq d_P = q + 1 \), we see that \( F \) has to have the form (up to re-scaling \( x \))

\[ F = (y^q + x)^{i_1} y^{i_2} \] for some \( i_1, i_2 \in \mathbb{Z}_+ \) with \( qi_1 + i_2 = d \). (3.62)

If \( d_P = s = 2 \), then \( q = d_P - 1 = 1 \), \( p = \frac{q'}{q} = 1 \), and \( \ell = 2 \) since \( 2 \leq \ell \leq s \). Thus \( F = (y+x)^{i_1}(y+\alpha x)^{i_2} \) (up to re-scaling \( x \)) for some \( 1 \neq \alpha \in \mathbb{C} \), \( i_1, i_2 \in \mathbb{Z}_+ \) by noting that each irreducible factor (in \( A[y] \)) of the \( p \)-type q.h.e. \( F \) is a \( p \)-type q.h.e., hence, of the form \( y + \beta x^p = y + \beta x \) for some \( \beta \in \mathbb{C} \).

In any case, \( i_1, i_2 \) are coprime by (2.11). If \( i_1 = 0 \) (then \( i_2 = 1 \)) or \( i_2 = 0 \) (then \( i_1 = 1 \)), or \( i_1 = i_2 = 1 \), we see that \( F|P \), a contradiction. Thus we have (iii). (We can also prove \( i_1 \neq i_2 \) as follows: If \( i_1 = i_2 \), then using \( m_0 = 0 \) and Definition 2.3(2)(ii), we have \( F_0 = F^{m'} = (y^q+x)^{i_1m'}(y+\delta_{q,1}\alpha x)^{i_1m'} \), which is a Jacobian element by Lemma 3.21. By re-denoting \( y^{q'+\lambda}, y+\delta_{q,1}\alpha x \) to be \( x, y \) respectively, we obtain from \( F_0 \) that \( H := x^{iv'}y^{iv'} \) is a Jacobian element, but \( (-1, -1) \in \text{Supp} H^{\frac{-1}{i^{iv'}}}, \) a contradiction with Theorem 3.6.) This proves the lemma. \( \square \)

Now for any Jacobi pair \((F, G)\) in \( \mathbb{C}[x^{\pm 1}, y] \), if we have Lemma 3.26(i), then by applying the automorphism \((x, y) \mapsto (x, y - x^{\ell'}) \), \( F \) becomes an element with prime degree \( < \frac{q'}{q} \). If we have Lemma 3.26(ii) or (iii), then by applying the automorphism \((x, y) \mapsto (x - y^{q'}, y) \), \( F \) becomes an element with a lower \( y \)-degree. Thus by induction on \( \deg_y F \) and on the prime degree \( p \), after a finite steps (note that we have only finite possible choices of \( p = \frac{q'}{q} \) since \( 1 \leq q \leq \deg_y F \) by (3.42)), we can suppose either \( F \) has \( y \)-degree \( \leq 1 \), or else, \( p \leq 0 \) (this implies \( m_0 > 0 \), otherwise, by the definition of the prime degree \( p \) in Definition 2.1, we would obtain \( F, G \in \mathbb{C}[x^{-1}, y] \) and \( (F, G) \) cannot be a Jacobi pair). Thus from now on, we can suppose \( m_0 > 0 \) and \( p \leq 0 \). Note that if the original \( F, G \) are in \( \mathbb{C}[x, y] \), then after the above process, the resulting \( F, G \) are still in \( \mathbb{C}[x, y] \). In this case, by Remark 3.17, we can suppose \( 0 < m_0 < m \) and \( p \leq 0 \). The above have in fact proved

**Lemma 3.27** If there exists a Jacobi pair in \( \mathbb{C}[x, y] \) which violates the two-dimensional Jacobian conjecture, then there exists a Jacobi pair \((F, G)\) in \( \mathbb{C}[x, y] \) satisfying \( 0 < m_0 < m \) and \( p \leq 0 \).

For our purpose of discussions, from now on we assume

\[ F, G \in \mathbb{C}[x^{\pm 1}, y] \] with \( 0 < m_0 < m \), and \( p \leq 0 \). (3.63)

In case \( p < 0 \), we always write \( p = -\frac{m_0}{m} \) for some coprime positive integers \( p'' \), \( q \). We can suppose \( d \geq 2 \) because (cf. (3.41)): If \( p = 0 \), by replacing \( y \) by \( y + \lambda \), where \( y = \lambda \) is root of \( F \), we can suppose \( c_d = 0 \) (in case \( F \) becomes a monomial after the replacement, this means that \( p \) becomes \( < 0 \)); if \( p < 0 \), then since \( p > -\frac{m_0}{m} > -1 \) by Lemma 3.12 and (3.63), we have \( p \notin \mathbb{Z} \), and \( c_1 \) has to be zero.

We can use similar arguments as those in the proof of Lemma 3.26 to obtain \( a = -1, 0 \) in (3.43) (even though in our case here \( p \leq 0 \), some arguments in the proof of Lemma 3.26 does not depend on \( p \); for instance, we again have (3.56) and (3.57), so if \( a \geq 1 \), then either of (3.57) again
shows $i_k < j_k$ because by Lemma 3.12, $p > -\frac{m_0}{m} > -1$, i.e., $p' + q > 0$ still holds). If we drop the condition $F \not| P$, we can always suppose $a = 0$ because if $a = -1$ we can re-denote $P$ to be $FP$ so that $a$ become zero. Furthermore, if we consider $p \leq 0$ and use similar arguments as in the proof of Lemma 3.26 (cf. (3.54)–(3.57)), we can obtain

Lemma 3.28 Every irreducible factor of $F$ in $\mathbb{C}[x^{\pm 1}, y]$ is a factor of $P$, and every irreducible factor of $P$ has multiplicity 1. So $2 \leq d = \deg_y P = \frac{1+p}{m+p} \in \mathbb{Z}$ (thus $b_\mu \neq 0$, cf. (3.22)). Furthermore, if $F \not| P$, then every irreducible factor of $P$ in $\mathbb{C}[x^{\pm 1}, y]$ is also a factor of $F$.

Lemma 3.29 We can suppose $y|P$.

Proof. If $p = 0$, then since we can suppose $c_d = 0$, i.e., $y|P$. So assume $p = -\frac{q'}{q} < 0$. Since $b_\mu \neq 0$, we see $\mu, \frac{m_0}{m} \mu$ are integers (cf. Lemma 3.15). By (3.22), $\frac{m(q-p')}{{m_0}q'-m{p'}}$, $\frac{m_0(q-p')}{m_0q'-mp'}$ are integers. Subtracting them by 1, we see $\frac{(m-m_0)q}{m_0q'-mp'}, \frac{(m-m_0)p'}{m_0q'-mp'}$ are integers. Since $p', q$ are coprime, we see $r = \frac{m-m_0}{m_0q'-mp'}$ is an integer. Since $d_p = qr + 1$ (cf. (3.38)), by Lemma 3.22(1), we see $P$ must have a monic irreducible factor of degree 1, which has to be $y$. \[\square\]

3.5 Newton polygons

In this subsection, we start from a Jacobi pair $(F, G)$ satisfying (3.63), and $2 \leq m < n$, $m \not| n$, and regard $F, G$ as in $\mathbb{C}[x^{\pm 1}, y]$ for some sufficient large $N \in \mathbb{N}$ (at the beginning we can take $N = 1$, later on we may need to enlarge $N$ after we apply some variable changes). We define the Newton polygon of $F$ to be the minimal polygon such that the region surrounded by it contains $(\text{Supp} F) \cup \{0\}$. Thus the Newton polygon of $F$ may look as in (3.64), where, $m_0, \tilde{m}_0, \tilde{n}_0 \in \mathbb{Q}_+, m, \tilde{m}, \tilde{n} \in \mathbb{N}$, and points such as $\tau, \check{\tau}, \hat{\tau}, ...$ are called vertices of $\text{Supp} F$, and line segments such as $L, \tilde{L}, \hat{L}, ...$ are called edges of $\text{Supp} F$; we shall always use $(n_0, n)$, $(\tilde{n}_0, \tilde{n})$, $(\hat{n}_0, \hat{n})$, ... to denote corresponding vertices of $G$.

![Newton polygon](image)

We always assume $p, q$ together with their different versions (e.g., the ““ version, the ““ version, etc.) are in $\mathbb{Z}_+$, and $q$ together with its different versions are nonzero.

The next lemma says that starting from any (not necessarily the top most) vertex $\check{\tau} = (\tilde{n}_0, \tilde{m})$, we can find a lower vertex $(\tilde{n}_0, \tilde{m})$ as long as the edge $L$ satisfies some condition.

Theorem 3.30 Suppose $\check{\tau} = (\tilde{n}_0, \tilde{m}) \in \text{Supp} F$ is a vertex of $\text{Supp} F$ such that $0 < \tilde{n}_0 < \tilde{m}$ with $\tilde{m}_0 \in \mathbb{Q}_+, \tilde{m} \in \mathbb{N}$, and $\check{L}$ is the edge of $\text{Supp} F$ as shown in (3.64) such that its slope is either $\frac{\check{q}}{\check{p}} > 0$ (cf. the statement before (2.8)) or $\infty$ (in this case, we write its slope as $\frac{\check{q}}{\check{p}}$ with $\check{p} = 0, \check{q} = 1$). Suppose correspondingly $(\tilde{n}_0, \tilde{m})$ is a vertex of $\text{Supp} G$ such that $\frac{\tilde{n}}{\tilde{m}_0} = \frac{\tilde{m}}{\tilde{m}} = \frac{\tilde{m}}{\tilde{n}_0}$, and the edge of $\text{Supp} G$ whose top vertex is $(\tilde{n}_0, \tilde{m})$ also has slope $\frac{\check{q}}{\check{p}}$. Assume $\tilde{n} \geq 2$ and (cf. Lemma 3.12)

$$\frac{\check{q}}{\check{p}} > -\frac{\tilde{n}_0}{\tilde{m}} + \frac{\tilde{m} - \tilde{n}_0}{\tilde{m}(\tilde{m} + \tilde{n} - 1)} = -\frac{\tilde{n}_0 + \tilde{n}_0 - 1}{\tilde{m} + \tilde{n} - 1}. \tag{3.65}$$
Then either $\hat{F}_0$ (the part of $F$ with support being the edge $\hat{L}$) has only one irreducible factor, or else, if necessary by changing $(x, y)$ to $(x, y + \alpha_0 x^{-\frac{\mu}{q}})$ for some $\alpha_0 \in \mathbb{C}$, there is a vertex $\hat{\tau} = (\hat{m}_0, \hat{m}) \in \text{Supp} F$ and a corresponding vertex $(\check{m}_0, \check{m})$ of $\text{Supp} G$ such that $\frac{\hat{m}}{\hat{m}_0} = \frac{\check{m}}{\check{m}_0}$ and
\[0 < \check{m}_0 < \hat{m}_0 < \check{m} < \hat{m} \text{ with } \check{m}_0 \in \mathbb{Q}_+, \text{ and } 2 \leq \check{m} \in \mathbb{N}. \] (3.66)

**Proof.** Suppose $F = \sum_{i,j} f_{ij} x^i y^j \in \mathbb{C}[x^{\pm \frac{\mu}{q}}, y]$ for some $f_{ij} \in \mathbb{C}$, and $\hat{\tau}, \hat{L}$ are as shown in (3.64). Then $\frac{\hat{m}_0 - \hat{m}}{p} = \frac{\check{m}_0 - \check{m}}{q}$, and $\hat{F}_0 = \sum_{(i,j) \in L} f_{ij} x^i y^j$, which can be re-written as
\[\hat{F}_0 = \hat{m}_0 \sum_{j=0}^{\hat{m}} f_j \alpha x^j y^j, \text{ where } f_j = f_{\hat{m}_0-j, \hat{m}_0-j}. \] (3.67)

Thus $\hat{F}_0$ is a $-\frac{\mu}{q}$-type q.h.p (cf. Definition 2.3(3)). Now regard $-\frac{\mu}{q}$ as the “prime degree” of $F$ and $\hat{F}_0$ as the leading polynomial of $F$, and define $-\frac{\mu}{q}$-type $r$-th component $\hat{F}_{\langle r \rangle}$ of $F$ as in (2.10) (with the data $(p, m_0, m)$ being $(-\frac{\mu}{q}, \check{m}_0, \check{m})$). We can write $F$ and define $F^a \in \mathbb{C}$ for $a \in \mathbb{Q}$ as (where $\hat{F}_{\langle < 0 \rangle} = \sum_{r < 0} \hat{F}_{\langle r \rangle}$ is the ignored polynomial, cf. Definition 2.3(2)(i))
\[F = \hat{F}_0 + \hat{F}_{\langle < 0 \rangle}, \quad F^a = \sum_{s=0}^{\infty} \left(\frac{a}{s}\right) \hat{F}_{\langle 0 \rangle}^{a-s} \hat{F}_{\langle < 0 \rangle}^s, \] (3.68)
i.e., we expand $F^a$ according to its $-\frac{\mu}{q}$-type components. This is well-defined. In fact, if we let $z \in \mathbb{C} \setminus \{0\}$ be an indeterminate, and replace $x, y$ by $zx, z^{-\frac{\mu}{q}} y$, and regard elements as in the field (cf. (2.2))
\[\mathbb{C}_z = \{F = \sum_{j \in \mathbb{Q}} F_j z^j \mid F_j \in \mathbb{C}, \text{ Supp}_z F \subset a - \frac{1}{b} \mathbb{Z}_+ \text{ for some } a, b \in \mathbb{Z}, b > 0\}, \]
then the $-\frac{\mu}{q}$-type $r$-th component $\hat{F}_{\langle r \rangle}$ is simply a $z$-homogeneous element of $\mathbb{C}_z$, and (3.68) simply means that we expand $F^a$ as an element in $\mathbb{C}_z$. Then we can use all arguments before and define $\hat{R}_0$ as in (3.34) such that $(\hat{F}_0, \hat{R}_0)$ is a Jacobi pair (cf. Lemma 3.21), and in case $\hat{F}_0$ has at least two irreducible factors, we have (cf. (3.35), Lemmas 3.20 and 3.28)
\[\hat{R}_0 = \hat{F}_0^{-1} \hat{P} \text{ for some } \hat{P} \in \mathbb{C}[x^{\pm \frac{\mu}{q}}, y] \text{ of } y\text{-degree } d_p = \frac{(\hat{q} - \hat{p}) \hat{m}}{\hat{m}_0 \hat{q} - \hat{m} \hat{p}} \geq 2. \] (3.69)

To be more precise, we give detailed arguments below. Write $G = \hat{G}_0 + \hat{G}_{\langle < 0 \rangle}$ as in (3.68), and suppose $\hat{G}_0$ has the highest term $x^{\hat{n}_0} y^{\check{n}}$ for some $\hat{n}_0 \in \mathbb{Q}_+, \check{n} \in \mathbb{N}$ such that $\frac{\hat{n}_0}{\hat{m}_0} = \frac{\check{n}}{\check{m}}$. Denote
\[\bar{F} = z^{\frac{\hat{m}}{q}} y^{\hat{n}_0} F(xz, z^{-\frac{\mu}{q}} y), \quad \bar{G} = z^{\frac{\check{n}}{q}} y^{\check{n}} G(xz, z^{-\frac{\mu}{q}} y), \quad \bar{J} := [\bar{F}, \bar{G}] = z^{-\check{n}} \bar{J}, \] (3.70)
where $\check{\mu} = \check{n}_0 + \check{n} + 1 - \frac{\hat{m}}{q} (\hat{n}_0 + \check{n} + 1) > 0$ (cf. (3.65)), $J = [F, G]$. Then clearly, for any $r \in \mathbb{Q}_+$, the $-\frac{\mu}{q}$-type $r$-th components $\bar{F}_{\langle r \rangle}$ of $\bar{F}$ and $\bar{G}_{\langle r \rangle}$ of $\bar{G}$ are simply the coefficients of $z^r$ in $\bar{F}$ and $\bar{G}$ respectively. As in (2.22), we can write $\hat{G}_0 = \sum_{\check{r}=0}^{\infty} \check{b}_0 \hat{F}_{\langle 0 \rangle}^{\check{r}}$ for some $\check{b}_0 \in \mathcal{A}$ with $\check{b}_{00} = 1$. Denote
\( \tilde{G}^1 = \tilde{G} - \sum_{i=0}^{\infty} \tilde{b}_0 \tilde{F}^{\frac{k_{i-1}}{m}}. \) Then \( \deg_x \tilde{G}^1 \in \mathbb{Q} \) is \( <0 \), denoted by \(-r_1\). Write \( \tilde{G}^1 = z^{-r_1}(\tilde{G}^1_{(0)} + \tilde{G}^1_{(<0)}) \) as in (3.68), and suppose \( \deg_y \tilde{G}^1_{(0)} = k_{r_1} \). Then again we can write

\[
\tilde{G}^1_{(0)} = \sum_{i=0}^{\infty} \tilde{b}_{i,r_1} \tilde{F}^{\frac{k_{i-1}}{m}} \quad \text{for some } \tilde{b}_{i,r_1} \in \mathcal{A} \text{ with } \tilde{b}_{0,r_1} \neq 0. \tag{3.71}
\]

Continuing this way, we can write (note that if \( \mu = 0 \), the following still holds with the first summand vanishing)

\[
\tilde{G} = \sum_{r \in \mathbb{Q}_+, r < \mu} z^{-r} \sum_{i=0}^{\infty} \tilde{b}_{i,r} \tilde{F}^{\frac{k_{i-1}}{m}} + z^{-\mu} \tilde{R}, \tag{3.72}
\]

for some \( k_r \in \mathbb{Z}, \tilde{b}_{i,r} \in \mathcal{A} \) with \( k_0 = \tilde{n} \), and some \( \tilde{R} \in \mathcal{C}_2 \) with \( \deg_x \tilde{R} \leq 0 \). As in Lemma 2.5(6), there exists some \( \alpha \in \mathbb{N} \) such that \( \tilde{b}_{i,r} = 0 \) if \( r \notin \frac{1}{\alpha} \mathbb{Z}_+ \). Assume there exists \( \mu, i_0 \) such that \( \partial_2 \tilde{b}_{i_0, r_0} \neq 0 \) for some \( i_0 \in \mathbb{Z}_+ \) (and we take \( i_0 \) to be smallest). Comparing the coefficients of \( z^{-r_0} \) in the last equation of (3.70), we easily obtain a contradiction (we have an equation similar to the first equation of (3.13)). Thus \( \tilde{b}_{i,r} \in \mathcal{C} \). Since \( \tilde{G}^1_{(0)} \) in (3.71) is a \( -\frac{q}{q'} \)-type q.h.e. (cf. Definition 2.3(3)) and \( \text{Supp} \tilde{F}^{\frac{k_{i-1}}{m}} \) lies in a different line with slope \( \frac{q}{q'} \) for different \( i \), there is at most one \( i \) such that \( \tilde{b}_{i,r_1} \neq 0 \). Thus we can assume \( \tilde{b}_{i,r_1} = 0 \) for \( i > 0 \), and (3.72) can be rewritten as (although we do not need the actual value of \( k_r \), one can compute that \( k_r = \tilde{n} - \frac{\gamma q}{n_0 q - m p} \) if \( \tilde{b}_r \neq 0 \) by using the fact that \( \text{Supp} \tilde{F}^{\frac{k_{i-1}}{m}} \) lies in the line passing through \( (\tilde{n}_0 - r, \tilde{n}) \) with slope \( \frac{q}{q'} \), i.e., \( (\tilde{n}_0 k_r, k_r) = (\tilde{n}_0 - r - i_0 \frac{p}{q}, \tilde{n} - i_0) \) for some \( i_0 \in \mathbb{Z} \)

\[
\tilde{G} = \sum_{r \in \frac{1}{\alpha} \mathbb{Z}_+, r < \mu} z^{-r} \tilde{b}_r \tilde{F}^{\frac{k_{i-1}}{m}} + z^{-\mu} \tilde{R} \quad \text{for some } \tilde{b}_r \in \mathcal{C}. \tag{3.73}
\]

Hence, \( [\tilde{F}, \tilde{R}] = z^{\mu}[\tilde{F}, \tilde{G}] = z^{\mu} \tilde{J} = J \). Write \( \tilde{R} = \tilde{R}_{(0)} + \tilde{R}_{(<0)} \) as before. Then we obtain \([\tilde{F}_{(0)}, \tilde{R}_{(0)}] = J\). As before, we use \( \tilde{F} \) to denote the primary polynomial of \( \tilde{F}_{(0)} \). Then as in the proof of Lemma 3.19, we see from (3.73) that \( \tilde{F}^{\frac{k_{i-1}}{m}} \) is an integral power of \( \tilde{F} \) if \( \tilde{b}_r \neq 0 \). Hence \( \tilde{R}_{(0)} \) is a rational function on \( y \) of the form \( \tilde{F}^{b} \tilde{P} \) for some \( b \in \mathbb{Z}, \tilde{P} \in \mathbb{C}[x^\frac{p}{q}, y] \) (as in Lemma 3.20). Noting that \( (\deg_x \tilde{R}_{(0)}, \deg_y \tilde{R}_{(0)}) = (\tilde{n}_0 - \mu, \tilde{n}) + \beta (\frac{q}{q'}, 1) \) for some \( \beta \in \mathbb{Q} \) (by considering \( \text{Supp} \tilde{R}_{(0)} \)), which must be also equal to either \( (1 - \gamma \tilde{n}_0, 1 - \gamma \tilde{n}) \) or \( (\gamma \tilde{n}_0, \gamma \tilde{n}) \) for some \( \gamma \in \mathbb{Q} \) (by the fact that \( [\tilde{F}_{(0)}, \tilde{R}_{(0)}] = J \)), we can solve that \( \deg_y \tilde{R}_{(0)} = 1 - \tilde{n} \) or \( \gamma \tilde{n}_0 (\frac{q q' - m p}{n_0 q - m p} - \tilde{n}) \) (as in (3.35)). Thus, if \( \tilde{F} \) has at least two irreducible factors (in \( \mathcal{A}[y] \)), we can obtain (3.69) by taking \( \tilde{R}_0 = \tilde{R}_{(0)} \) (cf. Lemmas 3.21 and 3.28).

We may suppose \( \tilde{F} \) has at least 2 irreducible factors (in \( \mathcal{A}[y] \)). Otherwise

\[
\tilde{F}_{(0)} = x^{\tilde{n}_0} (y - \alpha_0 x^{\frac{\mu}{\tilde{n}_0}}) \quad \text{for some } \alpha_0 \neq 0 \in \mathbb{C}. \tag{3.74}
\]

By replacing \( y \) by \( y + \alpha_0 x^{\frac{\mu}{\tilde{n}_0}} \) (in this case since \( x^{\frac{\mu}{\tilde{n}_0}} \in \mathbb{C}[x^\frac{p}{q}, y] \) we must have \( \tilde{q}/N \) and we do not need to enlarge \( N \)), \( F \) becomes an element with the “prime degree”, denoted \( -\frac{\tilde{q}}{q'} \), being smaller than \( -\frac{\mu}{\tilde{n}_0} \). Since \( -\frac{\tilde{q}}{q'} \geq -\frac{\tilde{n}_0}{\tilde{m}} \) (as in the proof of Lemma 3.12) and \( 1 \leq q' < N \tilde{n} \) (cf. (3.42), noting that \( F \) is now in \( \mathbb{C}[x^\frac{p}{q}, y] \)), we only have finite possible choices of \( -\frac{\tilde{q}}{q'} \). Thus \( F \) can eventually
become an element such that either the “>” in (3.65) becomes equality (i.e., \( \hat{\mu} \) becomes zero), or else (3.65) still holds but \( \hat{F} \) has at least 2 irreducible factors. Let us assume the later case happens \( \text{remark:} \) if \( F,G \in \mathbb{C}[x^\pm,y] \) and (3.74) occurs, then \( \frac{\hat{p}}{\hat{q}} \in \mathbb{Z} \) and so \( \hat{p} = 0 \) since the right-hand side of (3.65) is \( > 1 \), and thus after the above replacement, \( \hat{p} \) becomes nonzero, and we still have the following three facts: (i) \( F,G \in \mathbb{C}[x^\pm,y] \) since in (3.67), \( \hat{m}_0 - j \frac{\hat{p}}{\hat{q}} \geq \hat{m}_0 - \hat{m} \frac{\hat{p}}{\hat{q}} > 0 \) (whether or not the “>” in (3.65) becomes equality, we always have \( -\frac{\hat{p}}{\hat{q}} > -\frac{\hat{m}_0}{\hat{m}} \)), analogously, \( G(\hat{m}) \in \mathbb{C}[x,y] \); (ii) \( \hat{m}_0 \) is the part of \( \hat{F}, \hat{G} \) where \( \hat{m} \hat{G} \) is a Jacobi pair (cf. (3.73) with \( \hat{\mu} \)). By (3.72) with \( \hat{\mu} = 0 \) shows \( \hat{G} = \hat{R} \) and so \( (\hat{F}(\hat{m}),\hat{G}(\hat{m})) = (\hat{F}(\hat{m}),\hat{R}(\hat{m})) = J \), a contradiction with fact (ii), thus fact (iii) shows that we always have the later case, and in particular after the variable change (3.79), we \( \textbf{always} \) have a side \( \hat{L} \) with negative slope in (3.82)).

If \( \hat{p} = 0 \), then Lemma 3.28 shows \( \hat{m}_0 |\hat{m} \) (in general, for \( a,b \in \mathbb{Q} \) with \( a \neq 0 \), notation \( a | b \) means \( b \in \mathbb{Z} \), \( d_\hat{p} = \frac{\hat{m}}{\hat{m}_0} \), and the leading polynomial \( \hat{F}(\hat{m}) \) has at most \( d_\hat{p} \) irreducible factors, thus at least an irreducible factor of \( \hat{F}(\hat{m}) \) has multiplicity \( \geq \frac{\hat{m}}{d_\hat{p}} = \hat{m}_0 \). If every irreducible factor of \( \hat{F}(\hat{m}) \) has multiplicity \( \hat{m}_0 \), then \( \hat{F} \) must be a power of \( \hat{P} \), which contradicts the dot version of (3.43). Thus at least one irreducible factor, say,

\[
\hat{f}_1 = y - \alpha_0 \text{ for some } \alpha_0 \in \mathbb{C}, \text{ has maximal multiplicity, say, } \hat{m} = \hat{m}_0. \tag{3.75}
\]

Now replacing \( y \) by \( y + \alpha_0 \), we obtain that \( (\hat{m}_0,\hat{m}) \) (with \( \hat{m}_0 = \hat{m}_0 \)) is a vertex of \( \hat{F} \) satisfying \( 0 < \hat{m}_0 < \hat{m} < \hat{m} \) (we \( \textbf{remark} \) here that if our starting pair \( (\hat{F},\hat{G}) \) is in \( \mathbb{C}[x,y] \), then the resulting pair after the variable change is still in \( \mathbb{C}[x,y] \)). Now assume \( \hat{p} > 0 \). As above, suppose the following irreducible factor (in \( \mathcal{A}[y] \)) of \( \hat{F} \),

\[
\hat{f}_1 = y - \alpha_0 x^{-\frac{\hat{p}}{\hat{q}}} \text{ has maximal multiplicity, denoted } \hat{m} \text{ (thus } \hat{m} < \hat{m}). \tag{3.76}
\]

Then \( \hat{m} = \frac{\hat{m}}{\hat{d}_\hat{p}} = \frac{\hat{m}_0 - \hat{\mu} \hat{m}}{\hat{q} - \hat{p}} \) (cf. (3.38)). Applying the variable change (if necessary we enlarge \( N \) by \( \hat{q} \) times to ensure that after the variable change, all elements under consideration are in \( \mathbb{C}[x^\pm, y] \))

\[
\sigma : (x,y) \mapsto (x,y + \alpha_0 x^{-\frac{\hat{p}}{\hat{q}}}), \tag{3.77}
\]

we obtain a vertex \( (\hat{m}_0,\hat{m}) \), where \( 0 < \hat{m}_0 = \hat{m}_0 - (\hat{m} - \hat{m}_0)\frac{\hat{p}}{\hat{q}} < \hat{m} < \hat{m} \).

In any case, we have (3.66) except that we have not yet proved \( 2 \leq \hat{m} \). Using (3.17), (3.65) and (3.73), as in the proof of Lemma 3.12, we have \( \hat{G}(\hat{m}) = \frac{\hat{F}(\hat{m})}{\hat{m}_0} \), and so \( \frac{\hat{m}_0}{\hat{m}} = \frac{\hat{m}}{\hat{m}_0} = \frac{\hat{m}}{\hat{m}} = \frac{\hat{m}}{\hat{m}} \). Finally, if \( \hat{m} = 1 \), we would obtain \( n = m \frac{\hat{m}}{\hat{m}} = m \hat{m} \), a contradiction with the assumption that \( m \not| \hat{m} \).

The proof of Theorem 3.30 shows that we can eventually obtain a vertex, denoted \( (\hat{m}_0,\hat{m}) \), such that (where \( \frac{\hat{p}}{\hat{q}} \) is the slope of the edge \( \hat{L} \) with top vertex \( (\hat{m}_0,\hat{m}) \))

\[
\hat{m} \geq 2, \quad \frac{\hat{p}}{\hat{q}} = \frac{\hat{m}_0}{\hat{m}} - \frac{\hat{m} - \hat{m}_0}{\hat{m}(\hat{m} + \hat{m} - 1)}, \tag{3.78}
\]

and if \( \hat{F} \) is the part of \( F \) with support being the edge \( \hat{L} \) and \( \hat{G} \) is the part of \( G \) analog to \( \hat{F} \), then \( (\hat{F},\hat{G}) \) is a Jacobi pair (cf. (3.73) with \( \hat{\mu} \) replaced by \( \hat{\mu} = 0 \) and statements after (3.73)).

Now we apply the variable change (as before, if necessary we enlarge \( N \) by \( \hat{q} - \hat{p} \) times)

\[
(x,y) \mapsto (x \frac{\hat{q}}{\hat{p}}, x^{-\frac{\hat{p}}{\hat{q}}} y). \tag{3.79}
\]
Note that any line with slope $\frac{n}{m}$ is mapped under (3.79) to a line parallel to the $y$-axis, so the above pair $(\tilde{F}, \tilde{G})$ determined by the edge $\tilde{L}$ in (3.64) are mapped to a pair $(x^a f, x^b g)$ for some $a, b \in \mathbb{Q}$ and $f, g \in \mathbb{C}[y]$ with $deg_y f = m$, $deg_y g = n$. Using $[x^a f, x^b g] \in \mathbb{C}\{0\}$, we obtain $a + b = 1, \frac{a}{m} = \frac{n}{m + n}$, i.e., $a = \frac{m}{m + n}, b = \frac{n}{m + n}$. Note also that any line with slope $\frac{n}{m}$ is mapped to a line with a negative slope, thus the edge $\tilde{L}$ with slope $\frac{4}{3} > \frac{2}{3}$ in (3.64) is mapped to an edge $\tilde{L}$ with a negative slope, denoted $-\frac{1}{\alpha}$ (cf. (3.82)). Thus, by exchanging symbols $\tilde{L}, \tilde{r}, \tilde{n}, \tilde{n}$ and $L, \tau, m, n$, we see that $F$ and $G$ become (up to nonzero scalars) elements of the forms in (3.80) and (3.81) (so from now on, $m$ and $n$ do not denote the $y$-degrees of $F$ and $G$), such that the Newton polygon of $F$ looks as in (3.82), where the existence of the edge $\tilde{L}$ with negative slope $-\frac{1}{\alpha}$ follows from the remark in a few lines after (3.74).

\[
F = x^{\frac{m}{m+n}}\left(f + \sum_{i=1}^{M_1} x^{-\frac{n}{m+n}} f_i\right), \text{ where, } f = f(y) = y^m + \sum_{i=1}^{m} c_i y^{m-i} \text{ with } c_1 = 0,
\]

\[
G = x^{\frac{n}{m+n}}\left(g + \sum_{i=1}^{M_2} x^{-\frac{n}{m+n}} g_i\right), \text{ where, } g = g(y) = y^n + \sum_{i=1}^{n} d_i y^{n-i},
\]  

for some $m, n \in \mathbb{N}$ (with $2 \leq m < n$), $M_1, M_2 \in \mathbb{Z}_+$, $c_i, d_i \in \mathbb{C}$ (we can always assume $c_1 = 0$ by replacing $y$ by $y - c$ for some $c \in \mathbb{C}$ if necessary) and $f_i, g_i \in \mathbb{C}[y]$ (note that $x^{\frac{m}{m+n}} f$ is the part of $F$ whose support is the edge $L$, and that the point $(\frac{m}{m+n}, 0)$ may not belong to the edge $L$, i.e., $c_m$ can be zero, cf. the last statement of this section).

Furthermore, the proof of Theorem 3.30 shows that the part $\hat{f}^0$ of $F$ whose support is the edge $\hat{L}$ with slope $-\frac{1}{\alpha}$ has the form

\[
\hat{f}^0 = x^{\frac{m}{m+n}} y^{m} \prod_{i=1}^{e} (a_i x^{-\alpha_i y} + 1)^{m_i},
\]  

for some $e, m_i \in \mathbb{N}$, $a_i \in \mathbb{C}\{0\}$ and $m_i \leq m$ with inequality holds for at least some $i$ (since $m$ is the maximal multiplicity among all irreducible factors of $f_0$, cf. (3.76)), furthermore, $m, m_i, i = 1, \ldots, e$, have at least a common divisor (otherwise, $n$ must be a multiple of $m$, and so, for the original $F$ and $G$, we also have $m|n$, a contradiction with the assumption).

From (3.75) (and the remark after it), (3.77), (3.79), and proof of Theorem 3.30, and Corollary 3.7(2), we obtain

**Corollary 3.31** (1) The pair $(F, G)$ is in fact obtained from a Jacobi pair, denoted by $(F, G)$, in $\mathbb{C}[x, y]$ by applying an automorphism of the form

\[
\sigma : (x, y) \mapsto (x^{\frac{q}{q-p}}, x^{-\frac{p}{q-p}} y + \lambda_1 x^{-\frac{p_1 q}{q_1 (q-p)}} + \cdots + \lambda_s x^{-\frac{p_s q}{q_s (q-p)}}),
\]  

for some $0 \neq \lambda_i \in \mathbb{C}, p, q, p_i, q_i \in \mathbb{N}$ with $\frac{p_i}{q_i} < \frac{p_{i+1}}{q_{i+1}} < \frac{p}{q} < 1$ for $1 \leq i < s$. 
(2) For any \( i, j \in \mathbb{Q} \), we have \( \text{tr} (x^i y^j \tilde{F}) = 0 \) if and only if
\[
\text{tr} H = 0, \quad \text{where} \quad H = x^{\frac{a_i - p_j}{q - p}} y^{\lambda_1 x^{\frac{a_{i1} - p_j}{q - p}}} + \cdots + \lambda_n x^{\frac{a_n x^{\frac{a_{in} - p_j}{q - p}}} + \cdots} + \lambda_n x^{\frac{a_n x^{\frac{a_{in} - p_j}{q - p}}}} \} F.
\] (3.85)

Note that since \( \tilde{F}, \tilde{G} \in \mathbb{C}[x, y] \), we clearly have \( \text{tr} (x^i y^j \tilde{F}) = 0 \) if \( (i, j) \not\in \mathbb{Z}^2 \) (we wish that one may obtain some condition on \( F \) from (3.85)).

We can decompose \( F \) and \( G \) as sums of \( \alpha \)-type q.h.e. (cf. Definition 2.3(3))
\[
F = \sum_{i \in \mathbb{Q}_+} \hat{f}_i, \quad \text{where} \quad \hat{f}_i = x^{\frac{m}{m+n}} y^{-i} \sum_{j=\max\{0, i-m\}}^{\infty} f_{ij} x^{-\alpha_j} y^j \quad \text{for some} \quad f_{ij} \in \mathbb{C}, \quad (3.86)
\]
\[
G = \sum_{i \in \mathbb{Q}_+} \hat{g}_i, \quad \text{where} \quad \hat{g}_i = x^{\frac{n}{m+n}} y^{-i} \sum_{j=\max\{0, i-n\}}^{\infty} g_{ij} x^{-\alpha_j} y^j \quad \text{for some} \quad g_{ij} \in \mathbb{C}, \quad (3.87)
\]
where all sums are finite. We call \( \alpha \) the leading degree of \( F \) and \( G \).

From (3.80) and (3.81), we see that \( (x^m x^n f, x^n x^{m+n} g) \), being obtained from the pair \( (\tilde{F}, \tilde{G}) \) under the mapping (3.79), is a Jacobi pair, i.e.,
\[
\frac{m}{m+n} f' g' - \frac{n}{m+n} f' g = J,
\] (3.88)
for some \( J \in \mathbb{C} \setminus \{0\} \), where the prime stands for the derivative \( \frac{d}{dy} \). Thus \( f \) and \( g \) do not have common irreducible factors and all irreducible factors of \( f \) and \( g \) have multiplicity 1.

4 Poisson algebras

In this section, we use the natural Poisson algebra structure on \( \mathbb{C}[y]((x^{-1})) \) to discuss Jacobi pairs. The main results of this section are Theorem 4.1 and Corollary 4.2.

4.1 Poisson algebra \( \mathbb{C}[y]((x^{-1})) \) and exponential operator \( e^{\text{ad}_H} \)

Let \( \mathcal{P} := \mathbb{C}[y]((x^{-1})) \) (cf. notation in (2.1)), where \( N \) is some fixed sufficient large integer (such that all elements considered below are in \( \mathcal{P} \)). By Definition 3.1(1), \( \mathcal{P} \) is a poisson algebra. For \( H \in \mathcal{P} \), we use \( \text{ad}_H \) to denote the operator on \( \mathcal{P} \) such that
\[
\text{ad}_H(P) = [H, P] \quad \text{for} \quad P \in \mathcal{P}. \quad (4.1)
\]
If \( H \) has the form
\[
H = x(a_0 y + a_1) + \bar{H}, \quad \text{where} \quad \deg_x \bar{H} < 1, a_0, a_1 \in \mathbb{C}, \quad (4.2)
\]
then we can define the following exponential operator on \( \mathcal{P} \):
\[
e^{\text{ad}_H} := \sum_{i=0}^{\infty} \frac{1}{i!} \text{ad}_H^i, \quad (4.3)
\]
which is well defined in \( \mathcal{P} \), and is in fact an automorphism of the Poisson algebra \( (\mathcal{P}, [\cdot, \cdot], \cdot) \) with inverse \( e^{\text{ad}_{-\text{ad}_H}} \), i.e.,
\[
(e^{\text{ad}_H})^{-1} = e^{\text{ad}_{-\text{ad}_H}}, \quad e^{\text{ad}_H} (PQ) = e^{\text{ad}_H(P)} e^{\text{ad}_H(Q)}, \quad e^{\text{ad}_H([P, Q])] = [e^{\text{ad}_H(P)}, e^{\text{ad}_H(Q)]}, \quad (4.4)
\]
for \( P, Q \in \mathcal{P} \). For any \( H_1, H_2 \in \mathcal{P} \) having the form (4.2), one can verify
\[
e^{\text{ad}H_1} \cdot e^{\text{ad}H_2} = e^{\text{ad}K} \cdot e^{\text{ad}H_1}, \quad \text{where } K = e^{\text{ad}H_1}(H_2).
\] (4.5)

Note that for any \( H_i \in x^{1-\frac{i}{N}} \mathbb{C}[y][[x^{-\frac{1}{N}}]], i = 1, 2, \ldots \), both operators
\[
\prod e^{\text{ad}H_i} := \ldots e^{\text{ad}H_2} e^{\text{ad}H_1}, \quad \prod e^{\text{ad}H_i} := e^{\text{ad}H_1} e^{\text{ad}H_2} \ldots
\] (4.6)

are well defined on \( \mathcal{P} \): For any giving \( P \in x^{a} \mathbb{C}[y][[x^{-\frac{1}{N}}]] \) for some \( a \in \mathbb{Z} \), to compute, say, \( Q = \prod e^{\text{ad}H_i}(P) \), we can write \( Q = \sum_{j=0}^{\infty} x^{a_j} Q_j \) with \( Q_j \in \mathbb{C}[y] \). Then the computation of \( Q_j \) for each \( j \) only involves finite numbers of operators: \( e^{\text{ad}H_i}, i \leq j \). This together with (4.5) also shows that for any \( H_i \) as above, we can find some \( K_i \in x^{1-\frac{i}{N}} \mathbb{C}[y][[x^{-\frac{1}{N}}]] \) such that
\[
\prod e^{\text{ad}H_i} = \prod e^{\text{ad}K_i}.
\] (4.7)

**Theorem 4.1** Suppose \((F, G)\) is a Jacobi pair in \( \mathcal{P} \) satisfying (3.80) and (3.81). There exist
\[
H = x + \sum_{i=1}^{\infty} x^{1-\frac{i}{N}} h_i, \quad K = y + \sum_{i=1}^{\infty} x^{-\frac{i}{N}} k_i \in \mathcal{P} \quad \text{with} \quad h_i, k_i \in \mathbb{C}[y],
\] (4.8)
such that \([H, K] = 1\) and
\[
F = H^{m \frac{m}{m+n}} f(K), \quad G = H^{m \frac{m}{m+n}} g(K).
\] (4.9)

**Proof.** Let \( F, G \in \mathcal{P} \) be as in (3.80) and (3.81). Let \( i \) be the smallest positive integer such that \((f_i, g_i) \neq (0, 0)\). By computing the terms with \( x \)-degree \(-\frac{i}{N}\) in \([F, G] = J\), we obtain
\[
\frac{m}{m+n} f g_i' - \left( \frac{n}{m+n} - \frac{i}{N} \right) f' g_i + \left( \frac{m}{m+n} - \frac{i}{N} \right) f g_i' - \frac{n}{m+n} f' g_i = 0,
\] (4.10)

where the prime stands for the derivative \( \frac{d}{dy} \). First assume \( i \neq N \). Taking \( \tilde{h}_i = \frac{1}{N} (f_i g_i' - f' g_i) \), \( \tilde{k}_i = \frac{1}{(m+n)J} (m f g_i - n f_i g) \), using (3.88) and (4.10), we have
\[
\left( \frac{1-\frac{i}{N}}{N} \right) \tilde{h}_i + \tilde{k}_i = 0, \quad f_i = \frac{m}{m+n} \tilde{h}_i f + f' \tilde{k}_i, \quad g_i = \frac{n}{m+n} \tilde{h}_i g + g' \tilde{k}_i.
\] (4.11)

Take \( Q_i = \frac{N}{N-i} x^{1-\frac{i}{N}} \tilde{k}_i \). Then
\[
x^{m \frac{m}{m+n}} f_i = -[Q_i, x^{m \frac{m}{m+n}} f], \quad x^{m \frac{m}{m+n}} g_i = -[Q_i, x^{m \frac{m}{m+n}} g].
\] (4.12)

Thus if we apply the automorphism \( e^{\text{ad}Q_i} \) to \( F \) and \( G \), we can suppose \( f_i = g_i = 0 \).

Now assume \( i = N \). Then (4.10) gives \( m f g_i - n f_i g = (m+n) c J \) for some \( c \in \mathbb{C} \). This together with (3.88) implies \( m (g_i - c g') f = n (f_i - c f') g \). Since \( f \) and \( g \) are coprime (cf. the statement after (3.88)), we have \( g | (g_i - c g') \) in \( \mathbb{C}[y] \), i.e., there exists \( \tilde{k}_i \in \mathbb{C}[y] \) such that
\[
g_i - c g' = \frac{n}{m+n} \tilde{k}_i, \quad \text{thus,} \quad f_i - c f' = \frac{m}{m+n} \tilde{k}_i.
\] (4.13)

Now if we first apply \( e^{\text{ad}Q_i} \) to \( F, G \) (with \( Q_i = \int \tilde{k}_i dy \)), and then applying the automorphism
\[
\tau_c : (x, y) \mapsto (x, y - c x^{-1}),
\] (4.14)
we can suppose \( f_i = g_i = 0 \).

The above shows that by applying infinite many automorphisms \( e^{\text{ad}_Q}, \tau_c, i \geq 1 \), the pair \((F,G)\) becomes \((x^{\frac{m+n}{n}} f, x^{\frac{m+n}{n}} g)\). Thus \( F = \sigma(x^{\frac{m+n}{n}} f) = H^{\frac{m+n}{n}} f(K), G = \sigma(x^{\frac{m+n}{n}} g) = H^{\frac{m+n}{n}} g(K) \) for some automorphism \( \sigma \) of the form (cf. (4.4) and (4.7))

\[
\sigma := \ldots e^{\text{ad}_P} \ldots e^{\text{ad}_{P_{N+1}}} \tau_c e^{\text{ad}_{P_N}} \ldots e^{\text{ad}_P} = (\ldots e^{\text{ad}_{Q_j}} \ldots e^{\text{ad}_{Q_{N+1}}} \tau_c e^{\text{ad}_{Q_N}} \ldots e^{\text{ad}_Q})^{-1},
\]

for some \( P_i \in x^{1-\frac{m}{n}} \mathbb{C}[y][[x^{-\frac{1}{n}}]], \) where \( H = \sigma(x), K = \sigma(y) \) have the forms as in the theorem since \( \text{deg}_{x'} Q_i < 1 \) for all \( i \geq 1 \).

\[\square\]

Theorem 4.1 can be generalized as follows (one may wish to obtain some information on \( F,G \) from this result).

**Corollary 4.2** For any Jacobi pair \((F,G)\) in \( \mathbb{C}[x^{\pm \frac{1}{n}}, y] \), and any line \( L \) which meet the border of \( \text{Supp} F \) (either meet an edge of \( \text{Supp} F \) or a vertex of \( \text{Supp} F \)) such that \( L \) does not pass the origin. Regarding \( L \) as the prime line, one can start from \( F[0] = F_L \), the part of \( F \) corresponding to \( L \), to obtain that there exists an automorphism \( \sigma \) of \( \mathcal{D} = \mathbb{C}[y][((x^{-\frac{1}{n}}))] \) of the form (4.15) (we need to change \( D \) to \( \mathcal{D} = \mathbb{C}[y][((x^{\frac{1}{n}}))] \) if \( L \) is an edge at the left side of \( \text{Supp} F \)) with \( P_i, Q_i \) in the localized ring \( \mathcal{D}[F[0]} \) such that \( \sigma(F) = F[0] \) and \( \sigma(G) = \phi(F[0]) + R_0 \), where \( R_0 \) is as in Lemma 3.21, and \( \phi(F[0]) \) is a function of \( F[0] \) of the form \( \phi(F[0]) = \sum_{i \in \mathbb{Q}_+} a_i F[0]^i \) with \( a_i \in \mathbb{C} \) and \( F[0]^i \) is a rational function if \( a_i \neq 0 \).

### 4.2 Jacobi pairs in \( \mathbb{C}[y][((x^{-\frac{1}{n}}))] \) satisfying (3.80)–(3.83)

Now let \( F,G \in \mathbb{C}[x^{\pm \frac{1}{n}}, y] \) be a Jacobi pair satisfying (3.80)–(3.83). Let \( H,K \in \mathcal{P} \) be as in Theorem 4.1 (note that \( H,K \) are not necessarily in \( \mathbb{C}[x^{\pm \frac{1}{n}}, y] \)).

**Lemma 4.3** There exist a unique \( \alpha \), called the leading degree of \( H \) and \( K \) (as in (3.86)), such that \( H = \sum_{i \in \mathbb{Z}_+} H_{(-i)}, K = \sum_{i \in \mathbb{Z}_+} K_{(-i)} \) for some \( \beta \in \mathbb{N} \), where \( H_{(-i)}, K_{(-i)} \) are respectively the \( \alpha \)-th \(-i\)-th components of \( H,K \) of the following forms (cf. Definition 2.3(1)),

\[
H_{(-i)} = xy^{-i} \sum_{j \geq 1} h_{ij} x^{-j} y^i, \quad K_{(-i)} = y^{1-i} \sum_{j \geq \max\{0,i-1\}} k_{ij} x^{-j} y^i,
\]

for some \( h_{ij}, k_{ij} \in \mathbb{C} \) with \( k_{10} = 0 \) (note that \( K \) does not contain the constant term by (4.8)) and at least some \( h_{0j} \) or \( k_{0j} \) is nonzero for some \( j \geq 1 \).

**Remark 4.4** We remark here that when some negative power of \( y \) appears in an expression (as in (4.16)), we always regard the element as in a proper space which is a subspace of the space

\[
\mathcal{P} := \mathbb{C}((y^{-1}))((x^{-\frac{1}{n}})) \quad (\text{cf. notation in (2.1)}).
\]

**Proof of Lemma 4.3.** For \( i \geq 0 \), we inductively define \( Q_i, F^{(i)} = x^{\frac{m+n}{n}} \left(f + \sum_{j=1}^{\infty} x^{-\frac{j}{n}} f_j^{(i)}\right) \) and \( G^{(i)} = x^{\frac{m+n}{n}} \left(g + \sum_{j=1}^{\infty} x^{-\frac{j}{n}} g_j^{(i)}\right) \) as follows: \( Q_0 = 0, F^{(0)} = F, G^{(0)} = G \) (in particular, \( f_j^{(0)} = f_j, g_j^{(0)} = g_j \)). For \( i \geq 1 \), \( Q_i \) is defined as in the proof of Theorem 4.1 with \((F,G)\) replaced by \((F^{(i-1)}, G^{(i-1)})\), and set

\[
F^{(i)} = e^{\text{ad}_Q_i}(F^{(i-1)}), \quad G^{(i)} = e^{\text{ad}_Q_i}(G^{(i-1)}).
\]
We remark that since our purpose here is to prove (4.16), from the discussions below, we see it does not matter whether or not \( \tau_c \) defined in (4.14) is involved in (4.15). Thus for convenience, we may assume \( \tau_c \) is not involved. Alternatively, one can also formally regard the automorphism \( \tau_e \) as \( e^{adQ_N} \) with \( Q_N = \ln x \) and regard \( Q_N \) as an element with \( x \)-degree and \( y \)-degree being zero, and \( \partial_x Q_N = x^{-1}, \partial_y Q_N = 0 \). In this way, \( Q_N \) can be regarded as another \( Q_N \), and we define \( F^{(N)} = e^{adQ_N}(F^{(N)}), G^{(N)} = e^{adQ_N}(G^{(N)}) \) and \( F^{(N+1)} = e^{adQ_{N+1}}(F^{(N)}), G^{(N+1)} = e^{adQ_{N+1}}(G^{(N)}) \), etc.

Now we choose \( \gamma \) to be the minimal rational number such that (thus at least one equality holds below)

\[
\deg_y Q_i \leq 1 + i\gamma, \quad \deg_y f_i \leq m + i\gamma, \quad \deg_y g_i \leq n + i\gamma \quad \text{for } 0 \leq i \leq N + M_1 + M_2, \tag{4.18}
\]

where \( M_1, M_2 \) are as in (3.80), (3.81). Note from the edge \( \hat{L} \) in (3.82), we see that \( \gamma \geq \frac{1}{N\alpha} > 0 \). We claim that for all \( i \geq 0 \), we have

\[
\deg_y Q_i \leq 1 + i\gamma, \quad \deg_y f_j^{(i)} \leq m + j\gamma, \quad \deg_y g_j^{(i)} \leq n + j\gamma \quad \text{for all } j \geq 0. \tag{4.19}
\]

By (4.18), the claim holds for \( i = 0 \) (note that \( f_j^{(0)} = g_j^{(0)} = 0 \) if \( j > M_1 + M_2 \) by (3.80) and (3.81)). Inductively assume that (4.19) holds for some \( i = i_0 - 1 \geq 0 \). Now assume \( i = i_0 \). We want to prove

\[
\deg_y Q_{i_0} \leq 1 + i_0\gamma. \tag{4.20}
\]

If \( i_0 \leq N \), we already have (4.20) by (4.18). Assume \( i_0 > N \). By the inductive assumption,

\[
\deg_y f_{i_0}^{(i_0 - 1)} \leq m + i_0\gamma. \quad \text{Then the first equation of (4.12) with } i = i_0 - 1 \text{ shows that either}
\]

\[
\deg_y f_{i_0}^{(i_0 - 1)} = \deg_y Q_{i_0} + \deg_y f - 1 \text{ or else } (1 - \frac{i_0}{N}, \deg_y Q_{i_0}) = \beta(\frac{m}{m+N}, \deg_y f) \text{ for some } \beta \in \mathbb{Q}. \quad \text{The later case cannot occur since } i_0 > N \quad (\text{i.e., } 1 - \frac{i_0}{N} < 0). \quad \text{The first case implies (4.20). This proves (4.20) in any case. Now by definition of } e^{adQ_{i_0}} \text{ (cf. (4.3)), we know}
\]

\[
x_{\frac{m}{m+N}}^{\frac{m}{m+N}} f_j^{(i_0)} = \text{a combination of elements of forms } ad_{Q_{i_0}}^{j_1} \left(x_{\frac{m}{m+N}}^{\frac{m}{m+N}} f_j^{(i_0 - 1)}\right), \tag{4.21}
\]

for \( j_1, j_2 \in \mathbb{Z}_+ \) satisfying \( j_1i_0 + j_2 = j \). Thus for all \( j \),

\[
\deg_y f_j^{(i_0)} \leq \max\{j_1(\deg_y Q_{i_0} - 1) + \deg_y f_j^{(i_0 - 1)} \mid j_1, j_2 \in \mathbb{Z}_+, j_1i_0 + j_2 = j\} \leq m + j\gamma,
\]

where the last inequality is obtained by the inductive assumption. Analogously, \( \deg_y g_j^{(i_0)} \leq n + j\gamma \). This completes the proof of (4.19). Now let \( i_1 \geq 1 \) be minimal such that when \( i = i_1 \), at least one equality holds in (4.18). We claim

\[
\deg_y Q_{i_1} = 1 + i_1\gamma. \tag{4.22}
\]

Otherwise \( \deg_y Q_{i_1} < 1 + i_1\gamma \) and, say, \( \deg_y f_{i_1} = m + i_1\gamma \). We want to prove by induction on \( \ell \),

\[
\deg_y f_{i_1}^{(\ell)} = m + i_1\gamma \quad \text{for } 0 \leq \ell \leq i_1 - 1. \tag{4.23}
\]

By definition, (4.23) holds for \( \ell = 0 \). Inductively assume (4.23) holds for \( \ell - 1 < i_1 - 1 \). Similar to (4.21), \( x_{\frac{m}{m+N}}^{\frac{m}{m+N}} f_{i_1}^{(\ell)} \) is a combination of \( ad_{Q_{i_1}}^{j_1} \left(x_{\frac{m}{m+N}}^{\frac{m}{m+N}} f_{j_2}^{(\ell - 1)}\right) \) (for \( j_1, j_2 \in \mathbb{Z}_+ \) with \( j_1\ell + j_2 = i_1 \)), whose \( y \)-degree is either \( < m + i_1\gamma \) if \( j_1 \neq 0 \), or else \( = m + i_1\gamma \) if \( j_1 = 0 \). Note that \( x_{\frac{m}{m+N}}^{\frac{m}{m+N}} f_{i_1}^{(\ell - 1)} \)
does indeed appear as a term in $x^{\frac{m}{m+n}} \cdot \frac{1}{\pi} f_1(\ell)$. Thus (4.23) holds. However, (4.12) with $f_i, Q_i$ replaced by $(f_{i_1}^{(i_1-1)}, Q_{i_1})$ implies that $\deg_y f_{i_1}^{(i_1-1)} \leq \deg_y Q_{i_1} - 1 + \deg_y f < m + i_1 \gamma$, a contradiction with (4.23) (with $\ell = i_1 - 1$). This proves (4.22).

Using (4.4) and (4.5), we can explicitly determine $P_i$ in terms of $Q_1, \ldots, Q_i, \tau_c$ from (4.15); for instance,

$$P_1 = -\tau_c(Q_1), \quad P_2 = -\tau_c(e^{adP_1}(Q_2)), \quad P_3 = -\tau_c(e^{adP_2}e^{adP_1}(Q_3)), \ldots$$

In particular, if we write $P_i = \sum_{j=0}^i x^{1-j/\gamma} p_{ij}$ for some $p_{ij} \in \mathbb{C}[y]$, using (4.19) and (4.22), one can show that $\deg_y p_{ij} \leq 1 + j \gamma$ for all $i, j$ and $\deg_y p_{ij} < 1 + j \gamma$ if $j < i_1$, and furthermore, $i_1$ is the minimal integer such that the equality $\deg_y p_{i_1 i_1} = 1 + i_1 \gamma$ holds when $i = i_1$. This implies (using $H = \sigma(x), K = \sigma(y)$ and definitions of $h_i, k_i$ in (4.8), as discussions above)

$$\deg_y h_i \leq i \gamma, \quad \deg_y k_i \leq 1 + i \gamma \quad \text{for } i \geq 1,$$

and the equalities hold when $i = i_1$.

Thus we have (4.16) by choosing $\alpha = \frac{1}{\gamma N}$. □

Clearly, we have $\alpha \leq \bar{\alpha}$ (otherwise from (4.9), we would obtain $\bar{j}^0 = x^{\frac{m}{m+n}} m^m$, a contradiction with (3.83)). We rewrite $F = \sum_{i=0}^\infty F_{(-i)}$ according to the leading degree $\alpha$ of $H$ and $K$ (but not according to the leading degree $\bar{\alpha}$ of $F$ and $G$; in particular if $\alpha = \bar{\alpha}$ then $F_{(-i)} = \bar{j}^i$, cf. (3.86)), and call $F_{(-i)}$ the $\alpha$-type $-i$-th component of $F$. Then (cf. (3.83))

$$H_{m+n}^{m+n} K_{m}^{m} = F_0(0) = \begin{cases} x^{\frac{m}{m+n}} y^m \prod_{i=1}^e (a_i x^{-\hat{\alpha}} y + 1)^{m_i} & \text{if } \alpha = \bar{\alpha}, \\ x^{\frac{m}{m+n}} y^m & \text{if } \alpha < \bar{\alpha}. \end{cases} \tag{4.24}$$

Note that $(H_{m+n}^{m+n} K_{m}^{m}, H_{m+n}^{m+n} K_{1-m}^{1-m})$ is a Jacobi pair, also $(F_0, F_0^{-1} P_0)$ is a Jacobi pair, where $P_0$ has the form (cf. (3.69), (3.83), Lemmas 3.21 and 3.28)

$$P_0 = \begin{cases} xy \prod_{i=1}^{e+e'} (a_i x^{-\hat{\alpha}} y + 1) & \text{if } \alpha = \bar{\alpha}, \\ xy & \text{if } \alpha < \bar{\alpha}, \end{cases} \tag{4.25}$$

for some $e' \geq 0$ and $a_i \in \mathbb{C}$. Thus up to a nonzero scalar, $H_{m+n}^{m+n} K_{1-m}^{1-m}$ must have the form $F_0^{-1} P_0 + \psi$ with $[F_0, \psi] = 0$, and so $\psi$ is a function on $F_0$ (noting from (2.22) that in case $[G, F] = 0$, we can obtain that each $b_i$ does not depend on $x$, i.e., $G$ is a function on $F$). Since $H_{m+n}^{m+n} K_{1-m}^{1-m}$ is an $\alpha$-type q.h.e. (cf. Definition 2.3(3)), $\psi$ must be of the form $\theta_0 F_{m+n}^{\mu}$ for $\theta_0 \in \mathbb{C}$ and $\mu \in \mathbb{Q}$ (we do not need to know the exact value of $\mu$, however by noting that $\text{Supp}(F_0^{-1} P_0 + \psi)$ is on a line with slope $-\frac{1}{\alpha}$, the term $x^{\frac{m}{m+n}} y^{\mu m}$ in $F_0^{-1} P_0 + \psi$ must be of the form $x^{\frac{m}{m+n} - i \alpha} y^{1-m+i_0}$ for some $i_0 \in \mathbb{N}$ since $F_0^{-1} P_0 + \psi$ has a term $x^{\frac{m}{m+n}} y^{1-m}$, one can compute $\mu = \frac{n+(m+n)(1-m)\alpha}{m(1+(m+n)\alpha)}$). Thus we can solve (up to nonzero scalars)

$$H_0 = F_0^{-1} P_0 + \theta_0 F_{m+n}^{\mu+1} \left(P_0 + \theta_0 F_{m+n}^{\mu+1} \right)^{-\frac{1}{m+n}}, \quad K_0 = F_0^{-1} P_0 + \theta_0 F_{m+n}^{\mu+1} \left(P_0 + \theta_0 F_{m+n}^{\mu+1} \right)^{-\frac{1}{m+n}}, \quad H_0 K_0 = P_0 + \theta_0 F_{m+n}^{\mu+1}. \tag{4.26}$$
In particular, if \( \alpha < \dot{\alpha} \), then \( \theta_0 \neq 0 \) (otherwise \( H_{(0)} = x, K_{(0)} = y \), a contradiction with the definition of \( \alpha \)). Applying \( \partial_x, \partial_y \) to (4.9), using (3.88), we have

\[
H \partial_x K = \frac{1}{(m+n)J}(mF \partial_x G - nG \partial_x F), \quad H \partial_y K = \frac{1}{(m+n)J}(mF \partial_y G - nG \partial_y F),
\]

which imply that they are polynomials (i.e., elements in \( \mathbb{C}[x^{\pm \frac{1}{\alpha}}, y] \)). In particular, by (4.16), \( H_{(0)} K_{(0)} = yH_{(0)} \partial_y K_{(0)} + \frac{1}{\alpha} xH_{(0)} \partial_y K_{(0)} \) is a polynomial. Thus the last equation of (4.26) shows that if \( \theta_0 \neq 0 \), then \( F_{(0)}^{\mu+1} \) is a polynomial (thus \( (\mu + 1)m' \in \mathbb{Z}_+ \), where \( m' \) is the greatest common divisor of \( m, m_i, i = 1, ..., e \), cf. (4.24)). Note that in any case (either \( \alpha = \dot{\alpha} \) or \( \alpha < \dot{\alpha} \)), each irreducible factor (in \( \mathbb{C}[x^{\pm \frac{1}{\alpha}}, y] \)) of \( F_{(0)} \) is an irreducible factor of \( H_{(0)} K_{(0)} \) with multiplicity 1 by (4.24) and (4.25). Also note that since \( H_{(-i)}, K_{(-i)} \) are \( \alpha \)-type q.h.e. (cf. Definition 2.3(3)) of the form (4.16), we always have

\[
x \partial_x H_{(-i)} + \alpha y \partial_y H_{(-i)} = (1 - i \alpha)H_{(-i)}, \quad x \partial_x K_{(-i)} + \alpha y \partial_y K_{(-i)} = (1 - i)\alpha K_{(-i)} \quad \text{for all} \quad i \in \frac{1}{\beta} \mathbb{Z}_+.
\]

In particular, we can obtain

\[
1 = [H_{(0)}, K_{(0)}] = \frac{1}{\alpha y} (\alpha K_{(0)} \partial_x H_{(0)} - H_{(0)} \partial_x K_{(0)}),
\]

where the first equality follows by comparing the \( \alpha \)-type 0-th components in \( 1 = [H, K] \). We always denote

\[
\nu_0 = 1 + \frac{1}{\alpha}.
\]

Now let

\[
\nu > 0 \quad \text{be smallest such that} \quad (H_{(-\nu)}, K_{(-\nu)}) \neq (0, 0).
\]

Then (4.27) gives that (using (2.14) and (4.28))

\[
R_1 := (1 - \nu)H_{(0)} K_{(-\nu)} + H_{(-\nu)} K_{(0)}
\]

\[
\quad = y(H_{(0)} \partial_y K_{(-\nu)} + H_{(-\nu)} \partial_y K_{(0)}) + \frac{x}{\alpha} (H_{(0)} \partial_x K_{(-\nu)} + H_{(-\nu)} \partial_x K_{(0)}) \quad \text{is a polynomial}
\]

**Lemma 4.5** There exists a unique element

\[
Q_\nu = \sum_{j \geq \nu-1} q_{\nu,j} x^{1-j \alpha} y^{1+j-\nu} \in \mathcal{P} \quad \text{for some} \quad q_{\nu,j} \in \mathbb{C}, \quad \text{and} \quad \deg_x Q_\nu < 1,
\]

with \( q_{10} = 0 \) (if \( \nu = 1 \)) and \( q_{\nu_0, \nu_0 -1} = 0 \) (if \( \nu_0 \in \mathbb{N} \)), such that

\[
H_{(-\nu)} = [Q_\nu, H_{(0)}] = \frac{1}{\alpha y} (H_{(0)} \partial_x Q_\nu - (1 + \alpha(1-\nu))Q_\nu \partial_x H_{(0)}),
\]

\[
K_{(-\nu)} = [Q_\nu, K_{(0)}] = \frac{1}{\alpha y} (\alpha K_{(0)} \partial_x Q_\nu - (1 + \alpha(1-\nu))Q_\nu \partial_x K_{(0)}).
\]

**Proof.** First we set \( q_{\nu, \nu-1} = 0 \) and inductively choose unique \( q_{\nu,j} \in \mathbb{C} \) for \( j \geq \nu \) to satisfy

\[
(1 + j - \nu)q_{\nu,j} + \sum_{i \geq 1} ((1 - i \alpha)(1 + j - i - \nu) - i(1 - (j - i)\alpha))h_0q_{\nu,j-i} = h_{\nu,j}, \quad j \geq \nu,
\]
which implies $[Q_\nu, H(0)] = H_{-\nu}$. Set $\overline{K}_\nu = K_{-\nu} - [Q_\nu, K(0)]$. Then by (4.31), we have

$$1 = [H, K] = [H(0), K(0)] + [[Q_\nu, H(0)], K(0)] + [H(0), [Q_\nu, K(0)] + \overline{K}_\nu + \cdots ,$$

(4.36)

which implies $[H(0), \overline{K}_\nu] = 0$ since the omitted terms are those components whose component indices are $< -\nu$. Thus as in the arguments after (4.25), $\overline{K}_\nu = \lambda_\nu H^{a_\nu}_0$ for some $\lambda_\nu \in \mathbb{C}$ and $a_\nu \in \mathbb{Q}$. Since $\text{Supp} \overline{K}_\nu$ and $\text{Supp} K_{-\nu}$ are on the same line, we have $x^{a_\nu} = x^{-i_0} y^{1-\nu+i_0}$ for some $i_0$, thus, $a_\nu = (1 - \nu)\alpha$. First assume $\nu \neq \nu_0$ (cf. (4.30)). Then $K_{(-\nu)} = [Q_\nu + \frac{\lambda_\nu}{1+\alpha(1-\nu)} H^{1+\alpha(1-\nu)}_0, K(0)]$. Thus by re-denoting $Q_\nu + \frac{\lambda_\nu}{1+\alpha(1-\nu)} H^{1+\alpha(1-\nu)}_0$ to be $Q_\nu$, we can suppose $\lambda_\nu = 0$ (note that we still have $\deg_x Q_\nu < 1$ since if $\nu > 1$ then $\deg_x H^{1+\alpha(1-\nu)}_0 < 1$, and if $\nu = 1$ then $H(0)$ contains the term $x$, but since $K$ does not contain the constant term, i.e., $\lambda_\nu = 0$).

Now assume $\nu = \nu_0$. Then $a_{\nu_0} = -1$. Noting from (4.16) that among all components of $K$, only $K_{(-\nu_0)}$ can possibly contain the term $x^{-1}$, also we can deduce from (4.16) and (4.33) that $\{Q_{\nu_0}, K(0)\} = \frac{1}{y} K(0), \partial_x Q_{\nu_0}$ cannot contain the term $x^{-1}$. Thus $\text{Coeff}(K, x^{-1}) = \text{Coeff}(\overline{K}_\nu, x^{-1}) = \text{Coeff}(\lambda_\nu H^{a_\nu}_0, x^{-1}) = \lambda_\nu$. If necessary by replacing $y$ by $y - \lambda x^{-1}$ for some $\lambda \in \mathbb{C}$, we can always suppose $\text{Coeff}(K, x^{-1}) = 0$, i.e., $\lambda_\nu = 0$ (noting that since the original pair $(F, G)$ satisfies (3.6), and (3.6) is also satisfied by the Jacobi pair $(x^{\frac{m}{m+n}} f, x^{\frac{m}{m+n}} g)$, by Theorem 3.6(1) and (4.9), we see $\text{Res}_x (H\partial_x K) = 0$, from this we in fact have $\text{Coeff}(K, x^{-1}) = 0$). Hence in any case, we have the first equalities of (4.34) and (4.35), and the above proof also shows that such $Q_\nu$ is unique. Now using (4.33) we obtain as in (4.28),

$$\partial_y Q_\nu = \frac{1}{\alpha y}((1 + \alpha(1 - \nu))Q_\nu - x \partial_x Q_\nu),$$

(4.37)

from this and (4.28), we have the second equalities of (4.34) and (4.35). \qed

Computing the $\alpha$-type $-\nu$-th component $F_{-\nu}$ of $F$ in (4.9) gives that (cf. (3.80) and (4.31))

$$R_2 := H^{\frac{m}{m+n}} K^{\frac{n}{m}} \left( \frac{m H_{-\nu}}{(m+n) H(0)} + \frac{m K_{-\nu}}{K(0)} + c_\nu K_{-\nu} \right)$$

is a polynomial. (4.38)

Now we first assume $\nu \neq \nu_0$. Using (4.24), (4.26), (4.34), (4.35) and (4.37), we obtain that

$$\begin{aligned}
R_3 & := \frac{1}{y} H(0) K(0) \partial_x Q_\nu - \frac{1}{y} ((1 - \nu) H(0) \partial_x K(0) + K(0) \partial_x H(0)) Q_\nu, \\
R_4 & := \frac{1}{y} \left( (1 + (m+n)\alpha) H(0) K(0) \partial_x Q_\nu - \frac{1}{y} (1 + \alpha(1-\nu)) \left( (m+n) H(0) \partial_x K(0) + K(0) \partial_x H(0) \right) Q_\nu \right)
+ \frac{(m+n)\alpha}{m} c_\nu H(0) K_{-\nu}.
\end{aligned}$$

(4.39) (4.40)

Multiplying the first equation by $-(1 + (m+n)\alpha)$ then adding it to the second equation, and using (4.29), we solve

$$Q_\nu = q_\nu + \beta H(0) K_{-\nu},$$

where $\beta = -\frac{m+n}{m(m+n-1+\nu)} c_\nu$, (4.41)
and \( q_\nu = \frac{(1 + (m + n)\alpha)R_2 - R_1}{\alpha(m + n - 1 + \nu)} \). Thus if \( c_\nu = 0 \), we obtain that \( Q_\nu \) is rational (i.e., an element of the form \( \frac{P}{Q} \) with \( P, Q \in \mathcal{P} \)). Assume \( c_\nu \neq 0 \) (thus \( \nu \geq 2 \) by (3.80)). Similar to (4.32), we obtain that (note from (4.31) that the \( \alpha \)-type \(-2\nu\)-th components of \( H\partial_y K \) and \( H\partial_x K \) only involve \( H(0), K(0), H(-\nu), K(-\nu), H(-2\nu), K(-2\nu) \))

\[
R_5 := (1 - 2\nu)H(0)K(-2\nu) + (1 - \nu)H(-\nu)K(-\nu) + H(-2\nu)K(0),
\]

\[
R_6 := H(0)\partial_x K(-2\nu) + H(-\nu)\partial_x K(-\nu) + H(-2\nu)\partial_x K(0),
\]

are rational. Our attempt was to use (4.27)–(4.41) to prove:

(i) Suppose \( \nu \neq \nu_0. \) Then \( Q_\nu \) is a rational function. Furthermore, \( K_{(\nu)}^\nu \) is rational if \( c_\nu \neq 0; \)

(ii) Suppose \( \nu = \nu_0 \) (then \( \frac{1}{\alpha} \in \mathbb{N} \)). Then \( \partial_y Q_{\nu_0} = -\frac{r}{\alpha y}\partial_x Q_{\nu_0}, \) and \( H(-\nu_0) = \frac{1}{\alpha y}H(0)\partial_x Q_{\nu_0}, \)

\( K(-\nu_0) = \frac{1}{y}K(0)\partial_x Q_{\nu_0} \). Furthermore, \( m(1 + (m + n)\alpha)\partial_x Q_{\nu_0} + c_{\nu_0}K(-\nu_0)^\nu \) is rational.

We claim that if (i) and (ii) hold, then it would imply that a Jacobi pair \((F, G)\) in \( \mathbb{C}[x^{\pm \frac{1}{\alpha}}, y] \) satisfying (3.80)–(3.83) does not exist (which implies the two-dimensional Jacobi conjecture). Although (i) or (ii) might not necessarily be true, one may get some information from this.

## 5 Weyl algebras

In this section, we first generalize results of the previous sections. All undefined notations can be found in the previous sections. The main results in this section are Theorems 5.2, 5.3.

We denote

\[
\mathcal{A}_u = \{ f = \sum_{i=0}^{\infty} f_i u^{i+\frac{1}{\alpha}} \mid f_i \in \mathbb{C} \text{ and some } \alpha, \beta \in \mathbb{Z}, \beta > 0 \},
\]

\[
\mathcal{B}_{uv} = \{ F = \sum_{i=0}^{\infty} u^{i+\frac{1}{\alpha}} F_i \mid F_i \in \mathbb{C}((v^{-1})) \text{ and some } \alpha, \beta \in \mathbb{Z}, \beta > 0 \},
\]

so that \( \mathcal{A}_u \) is a field, and \( \mathcal{B}_{uv} \) is an associative unital algebra (which is in fact a divisible ring) such that the product obeys the following law:

\[
v^i u^j = \sum_{s \in \mathbb{Z}_+} s! \binom{i}{s} \binom{j}{s} u^{i-s} v^{j-s} \quad \text{for } i \in \mathbb{Q}, j \in \mathbb{Z},
\]

or more generally,

\[
v^i f = \sum_{s \in \mathbb{Z}_+} \binom{i}{s} f_u^{(s)} v^{i-s}, \quad g u^j = \sum_{s \in \mathbb{Z}_+} \binom{j}{s} u^{j-s} g^{(s)}, \quad \text{where } f_u^{(s)} = \partial_u^s f, \quad g^{(s)} = \partial_v^s g.
\]

for \( i \in \mathbb{Z}, j \in \mathbb{Q}, f \in \mathcal{A}_u, g \in \mathcal{C}((v^{-1})) \). Thus, the Weyl algebra \( W_1 \) is the subalgebra of \( \mathcal{B}_{uv} \) generated by \( u, v \). We remark that an element \( F \) of \( \mathcal{B}_{uv} \) is a combination of rational powers of \( u \) with coefficients in \( \mathbb{C}((v^{-1})) \), and we always write an element \( F \) in its standard form, namely, \( u \) always appears before \( v \) in any term of \( F \). Then we can define a linear map

\[
\rightarrow : \mathcal{B}_{uv} \rightarrow \mathcal{B} \text{ such that } \overrightarrow{u} = -x \text{ and } \overrightarrow{v} = y.
\]
We also denote \( \overleftarrow{\cdot} \) the inverse map of \( \overrightarrow{\cdot} \). Then clearly, for any \( F, G \in B_{uv} \), we have
\[
FG = \overleftarrow{\overrightarrow{F G}} + \cdots, \quad [F, G] = \overleftarrow{\overrightarrow{[F, G]}} + \cdots,
\] (5.6)
where the omitted terms in the first (resp., second) equation have \( u \)-degrees \( < \deg_u FG \) (resp., \( \deg_u [F, G] \)), and the bracket in the left-hand side is the usual commutator in \( B_{uv} \) defined by (1.1), the bracket in the right-hand side is the bracket in \( B \) defined by the Jacobian determinant (3.1).

If we define the bracket \([\cdot, \cdot]_W\) in \( B \) as
\[
[F, G]_W = \sum_{i=1}^{\infty} \frac{1}{i!} \left( (\partial_i^x F)(\partial_i^y G) - (\partial_i^y F)(\partial_i^x G) \right)
\] for \( F, G \in \mathbb{C}[x, y] \),
then clearly, the two Lie algebras \((B_{uv}, [\cdot, \cdot])\) and \((B, [\cdot, \cdot]_W)\) are isomorphic under the map \( \overrightarrow{\cdot} \) in (5.5).

Now consider the Weyl algebra \( B_{uv} \). Any element \( F = \sum_{i=0}^{\infty} u^{-\alpha - \frac{j}{s}} f_i \in B_{uv} \) with \( f_0 \neq 0 \) being monic has the inverse \( F^{-1} \), which is defined to be the unique element \( H = \sum_{i=0}^{\infty} u^{-\alpha - \frac{j}{s}} h_i \) with \( h_0 = f_0^{-1} \) and
\[
1 = FH = \sum_{i,j,s \in \mathbb{Z}_+} \left( -\alpha - \frac{j}{s} \right) u^{-\frac{j}{s}} f_i^{(s)} h_j \quad (\text{cf. (5.4)}).
\] (5.8)

Note that \( h_i \) is uniquely determined for all \( i \). Further assume \( f_0 \) has degree \( \deg_u f_0 = m > 0 \), then for any \( a, b \in \mathbb{Z} \), \( b > 0 \) with \( b|am \), we can define \( \overrightarrow{F^a} \) to be the unique element \( E = \sum_{i=0}^{\infty} u^{am - \frac{j}{s}} e_i \) in \( B_{uv} \) such that \( e_0 = f_0^{\frac{a}{m}} \) (which is defined as in (2.4)) and
\[
F^a = E^b = u^{\alpha a} e_0^b + \sum_{s=0}^{b-1} (u^{\frac{\alpha a}{s}} e_0)^s (u^{\frac{am - \frac{j}{s}}{1}} e_1) (u^{\frac{am}{s}} e_0)^{b-1-s} + \cdots.
\] (5.9)

Using (5.4), we see that when writing the right-hand side of (5.9) as a standard form, the coefficient of \( u^{am - \frac{j}{s}} \) is (where we use \( \ast \) to denote some coefficients which can be determined but not needed for our purpose),
\[
C_{\text{cof}}(E^b, u^{am - \frac{j}{s}}) = e_0^{b-1} e_s + \sum_{0 \leq i_0 \leq 1 \leq \cdots \leq 1 \leq s, 0 \leq i_0 \leq i_1 \leq \cdots \leq i_s < s} \ast e_i^{(k_0)} e_i^{(k_1)} \cdots e_i^{(k_s)},
\] (5.10)

Thus for each \( s \geq 1 \), (5.9) has a unique solution for \( e_s \), which has the form
\[
e_s = e_0^k h = f^\frac{am}{s} h \quad \text{for some} \quad k \in \mathbb{Z}, \ h \in \mathbb{C}((v^{-1})).
\] (5.11)

Note that if \( F \neq 0 \), then we have \( 0 = [F, F^{-1}] = [F, F^1] \), which implies \( [F, F^{-1}] = 0 \). Then for any \( a, b \in \mathbb{Q} \), we can write \( a = \frac{c}{q}, b = \frac{d}{q} \in \mathbb{Q} \) with \( c, d, q \in \mathbb{Z}, q > 0 \), and if we write \( H = F^{\frac{1}{q}} \), then (say \( c, d > 0 \))
\[
[F^a, F^b] = [H^c, H^d] = \sum_{i=0}^{c-1} H^i [H, H^d] H^{c-i-1} = \sum_{i=0}^{c-1} \sum_{j=0}^{d-1} H^i (H^j [H, H] H^{d-j-1}) H^{c-i-1} = 0.
\] (5.12)

Now suppose \((F, G)\) is a Dixmier pair in \( W_1 \), i.e., \( F, G \in W_1 \) with \([F, G] = 1\). As before, we can express \( G \) as
\[
G = \sum_{i=0}^{\infty} b_i F^{\frac{n-i}{m}} \quad \text{for some} \quad b_i \in A_u.
\] (5.13)
We define \( p = p(F) \) as before. First we assume \( p > -1 \). Then from (5.3), we can easily observe

\[
HK = KH + \lfloor \text{ignored} \rfloor \quad \text{for any } H, K \in \mathcal{B}_{uv},
\]

(5.14)

where, we use \( \lfloor \text{ignored} \rfloor \) to denote terms whose component lines are located below the prime line of the proceeding term, cf. (2.8)). Similar as in Definition 2.3, we define \( F \in \mathbb{C}[u^{\pm 1}][v] \) to be the \( p \)-type q.h.e. such that

\[
F[0] = u^{m_0} F_{m'} + \lfloor \text{ignored} \rfloor,
\]

with \( m' \) maximal. Then from \( \sum_{i=0}^{\infty} [F, b_i] F_{m_i} = \sum_{i=0}^{\infty} [F, b_i F_{m_i}] = [F, G] \in \mathbb{C}^* \), we see,

\[
b_i \in \mathbb{C} \ (i \leq m + n - 2), \quad \deg_u b_{m+n-1} = 1 + \sigma_0, \quad \deg_u b_{m+n-1+i} \leq 1 + \sigma_i \ (i \geq 0),
\]

(5.16)

where \( \sigma_i \) is defined in (3.14), and the last equation follows by comparing the \( p \)-type component. Using (5.14), as in the proof of (5.11), we see that the analogous result of Lemma 2.5(5) also holds, i.e., for all \( \ell \in \mathbb{Z} \) with \( d|\ell \), where \( d = \deg_u F \), and all \( r \in \mathbb{Q} \), the element \( u^{-\frac{m-k}{m}} (F^m)^r \) of \( \mathcal{B}_{uv} \) is of the form \( F^a P \) for some \( a \in \mathbb{Z}, P \in \mathbb{C}[u^{\pm 1}][y] \). Using this and (5.14), we obtain \( d|(n-i) \) if \( i \leq m + n - 2 \) with \( b_i \neq 0 \) (cf. Lemma 3.19), and we have analogous results of Lemmas 3.21, 3.26 and Theorem 3.25. In particular, we can assume

\[
p \leq 0, \quad \text{and furthermore, } p \leq -1 \text{ if } m_0 > m > 0.
\]

(5.17)

We remark that the following discussions will be similar to Subsection 3.5. We define the Newton polygon of \( F \) as in Subsection 3.5. Let \( (m_0, m) \) be any (not necessarily the top most) vertex of \( \text{Supp} F \) such that \( m_0 \geq 0 \), and \( (n_0, n) \) is the corresponding vertex of \( \text{Supp} G \) (then \( \frac{m_0}{m} = \frac{n_0}{n} \)). As before, we always assume \( 2 \leq m < n \) and \( m \nmid n \). We can assume \( m_0 \leq m \) (if necessary by using the automorphism \( (u, v) \mapsto (v, -u) \), cf. Remark 3.17). Note that in our case here, we cannot assume \( m_0 \neq m \) (which will be clear later). Also note that the arguments below do not need to assume \( m_0 > 0 \). As in Subsection 3.5, we regard \( F, G \) as elements in \( \mathbb{C}[u^{\pm \frac{1}{m}}, v] \) for some sufficient large \( N \). We define the prime degree \( p \) as in Theorem 3.30, such that \( -\frac{1}{p} \) is the slope of the unique edge (denoted by \( L \)) of \( \text{Supp} F \) which is located at the right bottom side of \( \text{Supp} F \) with top vertex \( (m_0, m) \). Note that if \( F \) has the form \( F = \sum_{i=0}^{\infty} u^{\frac{i-1}{m}} f_i \in \mathcal{B}_{uv} \) with \( f_0 \neq 0 \), we can rescale \( F \) so that \( f_0 \) becomes a monic polynomial of \( v \) (however we can only rescale \( u \) by \( a u \), and in the meantime rescale \( v \) by \( a^{-1} v \) for some \( 0 \neq a \in \mathbb{C} \) in order for \( u, v \) to satisfy \( [v, u] = 1 \)). First we need the following.

**Lemma 5.1** \( p > -\frac{m_0}{m} \).

**Proof.** Suppose conversely, \( p \leq -\frac{m_0}{m} \). First assume \( m_0 < m \). Define the automorphism \( \sigma \) of \( \mathbb{C}[u^{\pm \frac{1}{m}}, v] \) to be

\[
\sigma : (u, v) \mapsto \left( \frac{m-m_0}{m} u^{m-m_0}, \frac{u^{m_0}}{u^{m_0} v} \right).
\]

Note that under \( \sigma \), the edge \( L \) is mapped to the \( v \)-axis, and that \( F, G \) are mapped to some elements in \( \mathbb{C}[u^{-\frac{1}{m}}, v] \). However, any pair of elements in \( \mathbb{C}[u^{-\frac{1}{m}}, v] \) cannot form a Jacobi pair, a contradiction. Now assume \( m_0 = m \). As in the proof of Theorem 3.30: Let \( z \in \mathbb{C} \setminus \{0\} \) be an indeterminate, and apply the automorphism \( (u, v) \mapsto (zu, z^{-1} v) \). As in (3.70), we have (note that if we use notation \( \frac{\hat{b}}{\hat{d}} \) as in (3.70), then here we define \( -\frac{\hat{d}}{\hat{q}} \) to be \(-1 = -\frac{m_0}{m} \), but not to be \( p \))

\[
\bar{F} = z^{m-m_0} F(zx, z^{-1} y), \quad \bar{G} = z^{n-m_0} G(zx, z^{-1} y), \quad \bar{J} := [\bar{F}, \bar{G}] = z^p J,
\]
where \( \mu = m + n - m_0 - n_0 > 0 \). Note that since \( p < -1 \), both \( \tilde{F} \) and \( \tilde{G} \) only contain non-positive (rational) powers of \( z \), thus the last equation cannot hold, a contradiction. This proves the lemma.

Now by Lemma 5.1, we have \( p > -\frac{m_0}{m} \geq -1 \), so we can always apply the automorphism (cf. (3.79), note that \( p \) is the \( -\frac{p}{q} \) there)

\[
(u, v) \mapsto ((1 + p)u^{1/p}, u^{1/p}v),
\]

(5.18)

so that the edge \( L \) becomes an edge in the first quadrant, which is parallel to the \( y \)-axis. Thus we can always assume \( 0 < m_0 \leq m \) and \( p = 0 \). Hence we can write \( F \) as \( F = \sum_{i=0}^{M_1} u^{m_0 - i} f_i \) for some \( f_i \in \mathbb{C}[v] \) such that \( f_0 \) is a polynomial (which can be assumed to be monic) of \( v \) of degree \( m > 0 \), which contains at least two terms. We can further suppose that \( f_0 \) has at least two different roots, otherwise by change \( v \) to \( v + \alpha \) (where \( \alpha \) is the root of \( f_0 \)), \( f_0 \) becomes \( v^m \), i.e., \( p \) becomes negative (and we repeat the above to use (5.18) to change \( p \) to zero, this repeating can only last finite times, as in the proof of Theorem 3.30, cf. statement after (3.74)). In particular, we can obtain \( m_0 \neq m \).

Analogously, we write \( G = \sum_{i=0}^{M_2} u^{m_0 - i} g_i \) with \( g_0 \) being a polynomial of \( v \)-degree \( n \). We always regard \( F, G \) as in \( B_{uv} \). Note from \( [F, G] = J \neq 0 \) that \( m_0 + n_0 > 1 \).

First assume \( m_0 + n_0 > 1 \). Using (5.6) and Comparing the coefficients of \( u^{n_0 + n_0 - 1} \) in \( [F, G] = J \), we obtain

\[
g_0 = b_0 f_0^{\frac{m_0}{m}} \text{ for some nonzero } b_0 \in \mathbb{C}.
\]

(5.19)

Write \( f = F^{m'} \) with \( m' \) maximal. Then (5.19) proves \( F^{\frac{nm'}{m}} \) is a polynomial. Denote \( G^1 = G - \sum_{i=0}^{\infty} b_0 F^{\frac{nm'}{m}} \). Discussing as above (with \( G \) replaced by \( G_1 \) and using (5.5)) and continuing (similar to the arguments before (3.72)), we can eventually write \( G \) as

\[
G = \sum_{s \in \mathbb{Z}_+, s < m_0 - 1} b_s F^{\frac{ks}{m}} + u^{1-n_0} R,
\]

(5.20)

for some \( k_s \in \mathbb{Z} \), \( b_s \in \mathbb{C} \) with \( k_0 = n \), and some \( R \in B_{uw} \) with \( \deg_u R \leq 0 \), so that \( R \) can be written as \( R_0 = \sum_{i=0}^{\infty} u^{-i} r_i \). Using (5.9), (5.11) and (5.19), as in the proof of Lemma 3.15, we see from (5.20) that \( F^{\frac{ks}{m}} \) is rational if \( k_s \neq 0 \) (here an element \( H = \sum_{i=0}^{\infty} u^{-i} h_i \) is rational if each \( h_i \) is a rational function of \( v \), and \( r_0 \) is a rational function of the form \( r_0 = F^{-m'-n} P \) for some \( a \in \mathbb{Z} \), \( P \in \mathbb{C}[v] \). Using (5.5), we have \( J = [F, G] = [F, u^{1-m_0} R] \), which implies that \( (x^{m_0} f_0, x^{1-m_0} f_0) \) is a Jacobi pair in \( B \) by comparing the coefficients of \( u^0 \) and using (5.12). Namely, we have (where the prime stands for \( \partial_v \))

\[
-(m' + am_0)F^{-1-a}P'F' + m_0 F^{-a}P' = m_0 f r_0' - (1 - m_0) f' r_0 = J \in \mathbb{C},
\]

(5.21)

which has exactly the same form of (3.43) (with \( p' = 0, q = 1 \)). Thus as the discussions after (3.75), we can find a lower vertex of \( \text{Supp} F \). Continuing the above process (from (5.18)), \( F, G \) can finally become elements such that \( m_0 + n_0 = 1 \), so we can write \( m_0, n_0 \) as \( m_0 = \frac{n}{m+n} \), \( n_0 = \frac{m}{m+n} \). Thus in fact we have proved the following

**Theorem 5.2** Suppose there exists a Dixmier pair \((F, G)\) in \( \mathbb{C}[u, v] \) such that the Newton polygon of \( F \) has a vertex \((m_0, m)\) with \( m_0, m > 0 \) (we can assume \( m_0 \leq m \) by using the automorphism \((u, v) \mapsto (v, -u)\) if necessary, cf. Remark 3.17), and \((n_0, n)\) is the corresponding vertex of \( \text{Supp} G \).
satisfying $\frac{m}{m} = \frac{n}{n}$ and $2 \leq m < n$ and $m \nmid n$. Then there exists an automorphism $\sigma$ of $\mathbb{C}[u^{\pm \frac{1}{2}}, v]$, such that the pair $(\sigma(F), \sigma(G))$, again denoted as $(F, G)$, having the form (3.80)–(3.83) (with $x, y$ replaced by $-u, v$) such that $2 \leq m < n$ and $m \nmid n$.

From now on, we assume that $(F, G)$ is a Dixmier pair in $\mathbb{C}[u, v]$ such that the Newton polygon of $F$ has the a vertex $(m_0, m)$ with $m_0 > m > 0$.

Regarding the edge at the right bottom side of $\text{Supp } F$ with top vertex $(m_0, m)$ as the prime line, we can expression $G$ as in (5.13). We rewrite it as

$$G = \sum_{i=0}^{m+n-1} b'_i F^{\frac{n-i}{m}} + R,$$ \hspace{1cm} (5.22)

where $b'_i \in \mathbb{C}$ is defined similarly as in (3.20). Then all elements $G, F^{\frac{n-i}{m}}, R$ are in $\mathbb{C}[u^\pm 1]((v^{-1}))$. We denote

$$w = uv.$$ \hspace{1cm} (5.23)

It is easy to verify that

$$w^i w^j = w^{i+j} \text{ for } i, j \in \mathbb{Z}. \hspace{1cm} (5.24)$$

Note that for $i, j \in \mathbb{Z}$,

$$u^i v^j = \begin{cases} u^{-j}(w-1)\cdots(w-j+1) & \text{if } j \geq 0, \\ u^{-j}((w+j')(w+j'-1)\cdots(w+1))^{-1} & \text{if } j = -j' < 0. \end{cases} \hspace{1cm} (5.25)$$

Thus we can regard the above elements as in $\mathbb{C}[u^\pm 1]((w^{-1}))$. Then

$$G\partial_w F = \sum_{i=0}^{m+n-1} b'_i F^{\frac{n-i}{m}} \partial_w F + R\partial_w F.$$ \hspace{1cm} (5.26)

For $F \in \mathbb{C}[u^\pm 1]((w^{-1}))$, as in (3.5), we define the trace of $F$ to be

$$\text{tr}(F) = C_{\text{coeff}}(F, u^0 w^{-1}) = C_{\text{coeff}}(F, u^{-1} v^{-1}),$$ \hspace{1cm} (5.27)

where the second equality follows from (5.25) by regarding $F$ as an element in $\mathbb{C}[u^\pm 1]((v^{-1}))$. Using (5.24), we see

$$\text{tr}(HK) = \text{tr}(KH) \text{ for } H, K \in \mathbb{C}[u^\pm 1]((w^{-1})).$$ \hspace{1cm} (5.28)

Because of this property, we call it “trace”. Now we shall compute the trace of (5.26). Obviously $\text{tr}(G\partial_w F) = 0$ since $F, G \in \mathbb{C}[u^\pm 1][w]$. Let $H = F^{\frac{1}{m}}$, then

$$\text{tr}(F^{\frac{n-i}{m}} \partial_w F) = \text{tr}(H^{n-1}\partial_w (H^m)) = \frac{m}{m+n-i} \text{tr}(\partial_w (H^{m+n-i})) = 0,$$

by noting that $\partial_w$ is a derivation of $\mathbb{C}[u^\pm 1]((w^{-1}))$ (using (5.24) to verify) and for $i \leq 0$, we have $\partial_w H^i = \sum_{s=0}^{i-1} H^s \partial_w (H) H^{i-1-s}$, and $\partial_w (H^{-1}) = -H^{-1}\partial_w (H) H^{-1}$ by using $HH^{-1} = 1$, and using (5.28). This proves

$$\text{tr}(R\partial_w F) = \text{tr}(G\partial_w F) = 0.$$ \hspace{1cm} (5.29)
Since we can assume $p \leq -1$ by (5.17), if we take $p' = -1 \geq p$, then we can consider the $p'$-type components. So up to a nonzero scalar, we can assume

\begin{align*}
F_0 &= u^{m_0} v^m + \alpha u^{m_0-1}v^{m-1} + \text{(lower terms)}, \\
R_0 &= r_0(u^{1-m_0}v^{1-m} + \beta u^{-m_0}v^{-m}) + \text{(lower terms)},
\end{align*}

(5.30)

for some $\alpha, \beta \in \mathbb{C}$ (if $p < p'$ then $F_0 = u^{m_0}v^m$ and so $\alpha = 0$) and $r_0 = \frac{1}{m-m_0}$ (by condition $[F_0, R_0] = 1$). Thus (using (5.24))

\begin{align*}
F_0 &= u^{m_0-m}w(w-1)\cdots(w-m+1+\alpha) + \ldots = u^{m_0-m}(w^m - \left(\frac{m}{2} - \alpha\right)w^{m-1}) + \ldots, \quad (5.31) \\
R_0 &= r_0 u^{m-m_0} \left( ((w + m - 1)\cdots(w + 1))^{-1} + \beta((w + m)\cdots(w + 1))^{-1} \right) + \ldots
\end{align*}

\begin{align*}
&= r_0 u^{m-m_0} \left( w^{1-m} - \left(\frac{m}{2} - \beta\right)w^{-m} \right) + \ldots, \quad (5.32)
\end{align*}

where the omitted terms do not contribute to the trace. Hence (using (5.24) and (5.28))

\begin{align*}
(m - m_0)R \partial_w F &= m \left( (w + m_0 - m)^{1-m} - \left(\frac{m}{2} - \beta\right)(w + m_0 - m)^{-m} \right)w^{m-1} \\
&\quad - (m - 1) \left( \left(\frac{m}{2} - \alpha\right)w^{-1} + \ldots \right) \\
&= - \left( (2m_0 - 1) \left(\frac{m}{2} - \beta -(m-1)\alpha \right)w^{-1} + \ldots \right)
\end{align*}

Note that for $r < 0$, $(R \partial_w F)^{(r)}$ does not contain the term $u^0w^{-1}$, thus

\begin{equation}
0 = (m_0-m) \text{tr}(R \partial_w F) = (m_0-m) \text{tr}(R_0 \partial_w F_0) = (2m_0 - 1) \left(\frac{m}{2}\right) - m\beta - (m-1)\alpha. \quad (5.33)
\end{equation}

Since we can also regard elements in (5.22) as in $\mathbb{C}[w^{\pm 1}](v^{-1})$ (to distinguish the difference, we use $\partial_w^v$ to denote the derivative $\partial_w$ in $\mathbb{C}[w^{\pm 1}](v^{-1})$). Computing as above, we obtain

\begin{equation}
0 = (m_0-m) \text{tr}(F \partial_w^v R) = (2m - 1) \left(\frac{m_0}{2}\right) - m_0\beta - (m_0-1)\alpha. \quad (5.34)
\end{equation}

This together with (5.33) proves Theorem 5.3(1) below.

**Theorem 5.3**

1. Assume $(F, G)$ is a Dixmier pair such that $\text{Supp } F$ has a vertex $(m_0, m)$ with $m_0 > m > 0$. Then the edge $L$ at the right bottom side of $\text{Supp } F$ with top vertex $(m_0, m)$ always has slope 1. Furthermore, if we denote $F_0$ to be the part of $F$ corresponding to $L$, then $F_0 = u^{m_0}v^m + \frac{m_0m}{2}u^{m_0-1}v^{m-1} + \ldots$. More precisely, in (5.30), $\alpha, \beta$ are equal to

\begin{equation}
\alpha = \frac{m_0m}{2}, \quad \beta = \frac{(1-m_0)(1-m)}{2}. \quad (5.35)
\end{equation}

2. If we write $F_0, R_0$ as $F_0 = u^{m_0-m}f(w), R_0 = r(w)u^{m-m_0}$. Then

\begin{equation}
f(w)r(w) = \frac{1}{m-m_0}(w + \frac{m_0 - m + 1}{2}). \quad (5.36)
\end{equation}
(3) Assume \([F,G] = 1\). An analogous result to Theorem 3.6(1) holds, namely, for any automorphism \(\sigma\) of \(B_{uv}\), we have
\[
\text{tr}(\sigma(G)\partial_u \sigma(F)) = \text{tr}(\sigma(v)\partial_u \sigma(u)).
\]

**Remark 5.4**

(1) We remark that Lemma 5.3 is the place where the great difference between Newton polygons of Jacobi pairs and Dixmier pairs occurs; for the Jacobi pairs, an edge of the Newton polygon can never have slope \(1\) (cf. Theorem 3.25).

(2) The operators \(\partial_u, \partial_v^u\) are in fact the unique derivatives in \(B_{uv}\) such that
\[
\partial_u(u) = 0, \quad \partial_v(u) = u^{-1}, \quad \partial_v^u(u) = v^{-1}, \quad \partial_v^u(v) = 0.
\]

To compute \(\partial_v^u(u^a)\) with \(\frac{p}{q} \notin \mathbb{Z}\), \(p, q > 0\), we set \(h = \frac{p}{q}\) and assume \(\partial_v^u(h) = \sum_{i=0}^{\infty} c_i u^i v^{-i}\) for some \(c_i \in \mathbb{C}\), and use \(\partial_v^u(u^p) = \partial_v(u^h) = \sum_{i=0}^{p-1} h^i (\partial_v u) h^{q-i-1}\) to determine \(c_i\). Similarly, we can use \(hh^{-1} = 1\) to determine \(\partial_v^u(h^{-1})\). One can obtain
\[
\partial_v^u(u^a) = au^{a-1}v^{-1} - \left(\frac{a}{2}\right)u^{a-2}v^{-2} + \cdots \quad \text{for any} \quad a \in \mathbb{Q}.
\]

**Proof of Theorem 5.3.**

(1) has been proved. To prove (2), using
\[
1 = [F_0, R_0] = u^{m_0-m}f(w)r(u)u^{m-m_0} - r(w)f(u) = f(w+m-m_0)r(w+m-m_0) - r(w)f(u),
\]
from this and (5.35), we obtain (5.36).

(3) Denote \(\bar{u} = \sigma(u), \bar{w} = \sigma(w)\) (then \(\bar{w}u = \bar{u}(w+1)\)), \(a = m_0 - m\), and \(\ell = \partial_u\). Using (5.35) and (5.36), one can see \(f'(\bar{w})r(\bar{w}) = 0\). Then (“\(\equiv\)” means equality under taking tr)
\[
\sigma(G)\partial_u \sigma(F) \equiv \partial_u(\sigma(F_0))\sigma(R_0) \equiv \left(\bar{u}^a f(\bar{w}) + \bar{u}^{a-1}f'(\bar{w})\right) r(\bar{w})\bar{u}^{-a}
\equiv \sum_{i=0}^{a-1} \bar{u}^a \bar{u}^{a-i-1} \bar{v}^i f(\bar{w}) r(\bar{w}) \bar{u}^{-a} + f'(\bar{w}) r(\bar{w})
\equiv \sum_{i=0}^{a-1} \bar{u}^a \bar{u}^i f(\bar{w}) r(\bar{w}) \bar{u}^{-i-1} \equiv \bar{u}^a \bar{u}^{-1} \sum_{i=0}^{a-1} f(\bar{w} - i - 1) r(\bar{w} - i - 1)
\equiv \bar{u}^a \bar{u}^{-1} \bar{w} \equiv \bar{u}^a \bar{w} \equiv \sigma(v)\partial_u \sigma(u).
\]

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