Complete Wiener-Hopf solution of the x-ray edge problem

V. Janiš
Institute of Physics, Academy of Sciences of the Czech Republic,
CZ-18040 Praha 8, Czech Republic

We present a complete solution of the soft x-ray edge problem within a field-theoretic approach based on the Wiener-Hopf infinite-time technique. We derive for the first time within this approach critical asymptotics of all the relevant quantities for the x-ray problem as well as their nonuniversal prefactors. Thereby we obtain the most complete field-theoretic solution of the problem with a number of new experimentally relevant results. We make thorough comparison of the proposed Wiener-Hopf technique with other approaches based on finite-time methods. It is proven that the Fredholm, finite-time solution converges smoothly to the Wiener-Hopf one and that the latter is stable with respect to perturbations in the long-time limit. Further on we disclose a wide interval of intermediate times showing quasicritical behavior deviating from the Wiener-Hopf one. The quasicritical behavior of the core-hole Green function is derived exactly from the Wiener-Hopf solution and the quasicritical exponent is shown to match the result of Nozières and De Dominicis. The reasons for the quasicritical behavior and the way of a crossover to the infinite-time solution are expounded and the physical relevance of the Nozières and De Dominicis as well as of the Wiener-Hopf results are discussed.

I. INTRODUCTION

The problem of x-ray absorption and emission in metals has been widely studied experimentally as well as theoretically in recent years. Various approaches have confirmed that both absorption and emission spectra display a sharp threshold reflecting the edge of the Fermi distribution of conduction electrons. Foundation for a microscopic description of the singular edge behavior was laid by the pioneering papers of Anderson, Mahan, and Nozières and co-workers. They demonstrated that the electron-electron Coulomb repulsion and a collective many-electron response to a sudden generation of a transient core-hole - conduction electron pair by absorption (emission) of light are responsible for the existence of the x-ray edge singularity. Fundamentals of the threshold behavior are captured by a microscopic, tight-binding model proposed by Mahan and solved nonperturbatively for the first time by Nozières and De Dominicis. It is now called the Mahan, Nozières and De Dominicis (MND) model.

There are two separable processes contributing to the singular edge behavior within the MND model. First, it is a rearrangement of the ground state of \( N \) noninteracting electrons due to a sudden change of a local, core-hole potential known as Anderson orthogonality catastrophe. Second, it is motion of an electron excited by absorption of light from a localized core level to an extended state away from equilibrium and subjected to the so-called "final-state interactions". The calculation of soft x-ray spectra at the edge represents hence one of the simplest, real many-body problems allowing for an exact solution.

Although the model MND Hamiltonian is rather simple and its exact equilibrium thermodynamics can be found in closed form for all temperatures and interaction strengths, the x-ray absorption and emission represent nonequilibrium processes to which sophisticated techniques have to be applied. Several methods have been proposed and developed in an effort to determine the edge structure of soft x-ray spectra. They can be divided into two classes: field-theoretic approach using many-body Green functions and Kubo formula on the one side and quantum-mechanical calculations based on the Fermi golden rule and transition amplitudes between one-electron states (Slater determinants) on the other side.
A breakthrough in the description of the x-ray problem was achieved in the paper of Nozières and De Dominicis. They succeeded to separate the dynamics of the core and the band particles and thereby to reformulate the problem as an effective one-body theory. The calculation of a two-particle susceptibility needed for the absorption (emission) amplitude was reduced to evaluation of two one-particle Green functions. These two Green functions are related to the two separable processes contributing to the singular edge behavior of x-ray spectra. The rearrangement of the ground state of \( N \) noninteracting electrons due to a sudden change of a local, core-hole potential is described by the core-hole Green function. Motion of an electron excited by absorption of light is determined by a nonequilibrium Green function of conduction electrons. Separation of the dynamics of the local hole and the transient conduction electron is the bedrock of the approaches aiming at an exact solution of the critical threshold asymptotics of x-ray spectra.

Quantum mechanical approaches treat only the conduction electrons as live particles. The local core states act as an external potential on which noninteracting conduction electrons scatter. A transition between the ground state of free electrons and eigenstates of an electron gas in a local potential determines the x-ray absorption and emission amplitudes. Combescot and Nozières proposed a determinantal formulation in which the transition amplitude is a sum of determinants. Mahan and co-workers used analytic properties of the scattering T-matrix and represented the transition amplitude as a sum of dispersion integrals. The most comprehensive approach using quantum mechanical transition amplitudes was achieved by Ohtaka and Tanabe who used the Fermi golden rule and the Fredholm method to sum up a series of Slater determinants. These quantum mechanical approaches have a great advantage that they are applicable to the evaluation of the Anderson orthogonality catastrophe as well as to the final state interaction. The effect of bound states due to strong electron hole attraction can be estimated within this approach too. However, the final result is a sum of an infinite series the convergence of which cannot be strictly proven, at least with rigorous mathematical methods. Moreover the eigenstates of the electron gas in a local potential are not known explicitly and an approximate scheme to sum over the electron momenta using the formula of Friedel relating the eigenenergies with the phase shifts is used. Since the quantum mechanical approach of Ohtaka and Tanabe is able to determine not only the critical exponents of the edge behavior but also the nonuniversal prefactors, it is now considered as the most complete and exact solution of the MND problem.

Simultaneously to the quantum mechanical approach to the x-ray problem a field-theoretic scheme has been developed. Contrary to the former method the latter treats both types of particles of the MND problem as dynamical objects subjected to a dynamics governed by the MND Hamiltonian. The absorption and emission amplitudes are then represented via a Kubo formula using many-body Green functions. In analogy to other many-body problems such as Kondo, Anderson or Hubbard models, various field theoretic techniques have been applied. Weak-coupling perturbation theory was used in the early papers on the MND problem. First nonperturbative solution within the field-theoretic approach was found by Nozières and De Dominicis. An asymptotic approximation of the authors and an exact solution of the resulting singular integral equation led to the confidence that the exact solution to the MND problem had been found. However, mathematical justification of the exactness of the applied asymptotic transformation is missing and only universal critical exponents can be obtained from it. The full exact solution would demand to calculate also the nonuniversal prefactors shaping the edge of the spectra. The construction of Nozières and De Dominicis is up to now the only field theoretic solution capable to evaluate the critical exponents for both the Green functions in the x-ray problem. Later approaches concentrated mostly on the Anderson orthogonality catastrophe, i. e. on the evaluation of the Green function of the local quasi-particle.

Rivier and Simanek and Hamann circumvented the asymptotic transformation of Nozières and De Dominicis and used the formalism of the Hilbert problem. They found nonperturbative critical exponents for the overlap between the Fermi sea and a new ground state of the electron gas in a local Coulomb potential. It became clear from these papers how a delicate problem it is to find the proper limit of infinite time (lattice volume). The long time asymptotics of the Green function in the MND and related Falicov-Kimball model was analyzed recently by means of the Wiener-Hopf method in. There is up to now no analogous analysis of the nonequilibrium Green function of the conduction electrons reflecting the final-state interaction in the MND problem. The field-theoretic approach is considered at present as less complete than the quantum mechanical one. Moreover the critical exponent of the Green function of the
local, core electron (hole) calculated with the nonperturbative Hilbert-problem (Wiener-Hopf) trick does not match the quantum mechanical result. While quantum-mechanical and Nozières and De Dominicis solutions predict a value $\delta(0)^2/\pi^2$, the Wiener-Hopf and Hilbert boundary-problem techniques produce a value $\delta(0)^2/2\pi^2$. Recently the author analyzed the difference between the Wiener-Hopf and the Nozières and De Dominicis exponents and concluded that the value of the critical exponent depends on the way how the effective lifetime of the transient electron-hole pair and the relaxation time are limited to infinity.

In this paper we complete the field-theoretic solution of the x-ray problem based on the Wiener-Hopf trick and derive asymptotic limits for the Green function of the transient conduction electron and of the x-ray absorption amplitude. Thereby not only the critical exponents are obtained but also their nonuniversal prefactors revealing a number of new physically relevant features. Further on we discuss and explain in detail the differences between the finite-time and infinite-time approaches to description of the critical edge behavior. We show that the finite-time solution smoothly converges to the Wiener-Hopf one, but that there is a wide interval of intermediate times where deviations to the Wiener-Hopf asymptotics are apparent. We derive an exact formula for a long-time asymptotics for the Green function of the core hole including critical as well as quasicritical time regimes. We show that the quasicritical edge exponent is that of Nozières and De Dominicis, i.e. twice the value of the Wiener-Hopf solution. The reasons for a quasicritical behavior and the way of a crossover to the infinite-time Wiener-Hopf solution are expounded and the physical relevance of the Nozières and De Dominicis as well as of the Winer-Hopf approaches are discussed.

The layout of the paper is as follows. Section II summarizes the basic mathematics of the x-ray problem, i.e. the fundamental integral equations and available methods for their solutions. In Sec. III we hint to an inconsistency between the critical exponents obtained from the orthogonality catastrophe and from the MND problem. In Sec. IV we present the full Wiener-Hopf long-time asymptotics for the relevant quantities in the x-ray problem. Section V analyzes the relevant long-time scales and shows the way how the critical or quasicritical behaviors with different exponents may arise and how various quantities reach their infinite-time limits. In section VI we discuss physical and practical consequences of the obtained results.

II. MATHEMATICAL BACKGROUND OF THE X-RAY PROBLEM

The aim of the paper is to discuss the source of the differences in the predictions of the critical exponents of the edge singularity. To avoid any ambiguities due to different definitions or formulas and to be able to discuss exact solutions we first summarize the basic mathematics of the problem to be studied. To determine the genuine long-time asymptotics of the x-ray absorption or emission amplitude, it is necessary to specify first the model within which we will work. We hence define the model and fundamental equations to be solved. Then we characterize mathematical methods for solving the fundamental equations that may lead to exact solutions, at least in particular limiting cases.

A. Model and fundamental quantities for the x-ray problem

The canonical model of the x-ray problem is that of Mahan, Nozières and De Dominicis, the dynamics of which is governed by a tight-binding Hamiltonian

$$\hat{H}_{MND} = \sum_k \epsilon_k a_k^\dagger a_k - E_c b^\dagger b - N^{-1} \sum_{k,k'} V_{kk'} a_k^\dagger a_{k'} b b^\dagger$$

where $b, b^\dagger$, and $a_k, a_k^\dagger$ are annihilation and creation operators of the core and conduction electrons, respectively. Although it is not necessary we confine the analysis only to the case of a contact potential $V_{kk'} = V$ (lowest x-ray channel). Hamiltonian (1) is a many-body formulation used in field-theoretic approaches. Since the local electrons have no kinetic energy they can be excluded from the problem. We then distinguish two Hamiltonians with conduction electrons only, the initial one
\[ \hat{H}_i = \sum_k \epsilon_k a_k^\dagger a_k \]  

(2a)

and the final one

\[ \hat{H}_f = \sum_k \epsilon_k a_k^\dagger a_k - N^{-1} \sum_{k,k'} V_{kk'} a_k^\dagger a_{k'} . \]  

(2b)

The advantage of Hamiltonian (2a) is that it equally describes the dynamics before and after absorption (emission) of light.

Absorption of light is described by a transition amplitude obtained from the Fermi golden rule

\[ I(\omega) = 2 \text{Re} \sum_f \int_0^\infty dt e^{-i(E_f-E_0^N-\omega) t} |\langle \Psi_f(N+1)|\hat{W}^{\prime}\rangle| \langle \Phi(N) \rangle |^2 , \]  

(3)

where \( \Phi(N) \) is the ground state of \( N \) noninteracting electrons from the conduction band, \( E_0^N = E_F^0 - E_c \) is the ground-state energy of the noninteracting system of \( N \) extended electrons and one local hole, and \( \Psi_f(N+1) \) are eigenstates of the Hamiltonian (2b) with \( N+1 \) electrons labeled by the subscript \( f \). The operator \( \hat{W} = 1/N \sum_k w(k) a_k^\dagger \) represents the process of creation of a new conduction electron due to absorption of light. Determinantal formulation uses this expression to evaluate the threshold behavior of x-ray spectra.

In this paper we will work within the field-theoretic approach. Instead of (3) we use an equivalent Kubo formula with the conduction as well as with the local electrons

\[ I(\omega) = \frac{1}{N^2} \sum_{k,k'} w(k) w(k')^* \int_0^\infty dt \left[ e^{i\omega t} \chi_{k,k'}(t) + e^{-i\omega t} \chi_{k,k'}^*(t) \right] , \]  

(4)

where \( \chi_{k,k'}(t) = \langle \Phi(N) | b_T \left[ \hat{S}_{MND} b^\dagger(t) a_{k'}(t) a_k^\dagger(0) b(0) \right] b^\dagger |\Phi(N) \rangle \) and \( \omega = \omega \pm i\eta \). We denoted the S-matrix of the MND model in the interaction picture as \( \hat{S}_{MND} = T \exp \{ -i \int_{-\infty}^\infty dt \hat{V}(t) \} \), with the unperturbed Hamiltonian \( \hat{H}_0 = \hat{H}_i - E_c^0 b^\dagger b \).

Fermi golden rule (3) uses quantum mechanical matrix elements that enable a direct access to the absorption amplitude. However the eigenstates of the Hamiltonian \( \hat{H}_f \) are not known in closed form. On the other hand the Kubo formula (4) does not require knowledge of the eigenstates of \( \hat{H}_f \), but it demands Green functions to be known off the mass shell. The quantum-mechanical approach was exhaustively described in [4], and we do not deal with it any longer. We will follow the quantum-field approach with the Kubo formula (4).

Nozières and De Dominicis [4] made a fundamental observation that the two-particle susceptibility factorizes due to the locality of the core particle into a product of two one-particle Green functions

\[ \chi_{k,k'}(t) = \left\langle \Phi(N) | b_T \left[ \hat{S}_{MND} b^\dagger(t) b(0) \right] b^\dagger |\Phi(N) \rangle \right\rangle \left\langle \Phi(N) | b_T \left[ \hat{S}_{MND} (0,t) a_{k'}(t) a_k^\dagger(0) \right] b^\dagger |\Phi(N) \rangle \right\rangle , \]  

(5)

where \( \hat{S}_{MND} (0,t) \) denotes the S-matrix on a time interval \([0,t]\). The former function expresses an overlap between the ground states of \( \hat{H}_i \) and \( \hat{H}_f \) and is responsible for the orthogonality catastrophe. The latter one is a nonequilibrium Green function of the conduction electrons containing the final-state interactions. While the Green function of the local core hole was calculated by several field-theoretic techniques there is no nonperturbative, field-theoretic solution for the nonequilibrium Green function of the conduction electrons. We construct it in the next sections.

We now determine fundamental dynamical equations for auxiliary Green functions from which both the averages on the right-hand side of (3) can be determined. The average of the core particle can be represented as
Equations (6c) and (7b) are the desired exact Dyson equations to be solved. They are identical with closed loops in the core-hole Green function and the transient (non equilibrium) form of the band-electron known contrary to other many-body problems. The only evidence of many-body effects here are the (6c) and (7b). They can be written in a generic form

\[ \text{same integral kernel}. \]

Exact solution of the x-ray problem then equals an exact solution of equations analogous to (6c). We use a representation

\[ F_{1}: \text{Integral kernel element of the causal Green function of the conduction electrons}, \]

\[ \text{where} \ \lambda \ \text{analogous to (6c)}. \]

The Dyson equation for the Green function \( \Gamma_{T}(t_{1}, t_{2}; \lambda) \) looks like:

\[ \Gamma_{T}(t_{1}, t_{2}; \lambda) = G_{c}(t_{1} - t_{2}) - \lambda \int_{0}^{T} dt' G_{c}(t_{1} - t') \Gamma_{T}(t', t_{2}; \lambda) , \]

where \( G_{c}(t) = N^{-1} \sum_{k} G_{c}(t, k) = (2\pi N)^{-1} \sum_{k} \int d\omega e^{-i\omega t} [\omega - \epsilon_{k} + i\eta \text{sgn} \omega]^{-1} \) is the diagonal (local) element of the causal Green function of the conduction electrons, \( \lambda \) is an intermediate interaction \( (\lambda \in [0, V]) \).

The Dyson equation for the nonequilibrium Green function of the conduction electrons has a structure analogous to (6c). We use a representation

\[ \Gamma_{T}(T, 0; k', k) \equiv \langle \Phi(N) | b_{T} \left[ S_{MND}(0, T) \ a_{k'}(T) a_{k}^{\dagger}(0) \right] b_{T}^{\dagger} | \Phi(N) \rangle \]

\[ = G_{c}(T, k) \delta(k - k') - V \int_{0}^{T} dt' G_{c}(T - t', k') \Gamma_{T}(t', 0; k) , \]

where the function \( \Gamma_{T}(t_{1}, t_{2}; k), T \geq t_{1}, t_{2} \geq 0 \) fulfills a closed equation

\[ \Gamma_{T}(t_{1}, t_{2}; k) = G_{c}(t_{1} - t_{2}; k) - V \int_{0}^{T} dt' G_{c}(t_{1} - t') \Gamma_{T}(t', t_{2}; k) . \]

Equations (6c) and (7b) are the desired exact Dyson equations to be solved. They are identical with equations (17) and (22) of the Nozières and De Dominicis paper. The kernel of these integral equations is the diagonal element of the Green function of noninteracting conduction electrons and is explicitly known contrary to other many-body problems. The only evidence of many-body effects here are the closed loops in the core-hole Green function and the transient (nonequilibrium) form of the band-electron propagator.

B. Basic techniques for solving Dyson equations in the MND problem

The above field-theoretic approach reduced the MND problem to two separate integral equations with the same integral kernel. Exact solution of the x-ray problem then equals an exact solution of equations (6c) and (7b). They can be written in a generic form

\[ \Gamma_{T}(t_{1}, t_{2}; \kappa) = \Gamma(0)(t_{1} - t_{2}; \kappa) - \lambda \int_{0}^{T} dt' G_{c}(t_{1} - t') \Gamma_{T}(t', t_{2}; \kappa) , \]

where \( \lambda \in [0, V] \) is interaction strength and \( \kappa \) is a parameter depending on the type of the Green function to determine.[4] For \( T < \infty \) it is a Fredholm integral equation and can be solved using the Fredholm method. With it we can find a solution in the Hilbert space of square integrable functions on the interval \([-T, T], L_{2}(T, T)\). To obtain a unique solution to (8) we have to fulfill the following two conditions:

**F1:** Integral kernel \( G_{c}(t_{1} - t_{2}) \) is a Hilbert-Schmidt operator in variables \( t_{1}, t_{2} \), i.e.

\[ \| \hat{G}_{c} \|_{HS}^{2} := \int_{0}^{T} dt_{1} dt_{2} |G_{c}(t_{1} - t_{2})|^{2} = T/2 \int_{-T}^{T} dt |G_{c}(t)|^{2} < \infty \]

and
The Wiener-Hopf technique solves (8) exactly and at weak coupling uniquely under two assumptions:

- The integral kernel reflecting an effective half-time of the exponential decay of the Coulomb potential between the core hole and conduction electrons defined in our units, $\bar{T}_2$, has the same meaning as in the Fredholm, $\bar{T}_1(0)$.

- The small imaginary part $\eta$ in the end of the calculations. Physically a small imaginary part $i\eta$ to it, i.e. $\lambda \rightarrow \lambda + i\eta$. Otherwise it is impossible to perform the limit $T \rightarrow \infty$.

Mathematically this imaginary part secures the convergence of the formal limit $T \rightarrow \infty$. To keep the theory unitary it is necessary to push $\eta \rightarrow 0$ in the end of the calculations.

Another technique how to solve the MND problem, i.e. the integral equation (8) in the asymptotic limit $T \rightarrow \infty$ is the Wiener-Hopf trick. If the interaction strength $-\lambda$ becomes a characteristic number of the spectrum of the integral kernel $G_c(t)$ is continuous, then (8) and (9) become divergent. We have to replace the Fredholm with the Wiener-Hopf method based on the Hilbert boundary problem.

The Wiener-Hopf technique solves (8) exactly and at weak coupling uniquely under two assumptions:

**WH1:** $L_2(-T,T) = L_2^- \oplus L_2^+ := L_2(-T,0) \oplus L_2(0,T)$;

**WH2:** if $f, g \in L_2^-(L_2^+)$ then $(f * g)(t) := \int_{-T}^{T} dt' f(t-t')g(t')$ is orthogonal to $L_2^-(L_2^+)$.

These conditions are strictly fulfilled only in the limit $T = \infty$. At $T < \infty$ the spectrum of $G_c(t)$ is discrete and WH2 is violated. The spectral representation (10) is in the Wiener-Hopf solution replaced with another decomposition

$$\bar{\Gamma}_+(\omega_1,\omega_2; \kappa) = \frac{\tilde{\Gamma}^{(0)}(\omega_2; \kappa)}{i(\omega_2 - \omega_1 - i\eta)} \tilde{\Phi}_-(\omega_2) \tilde{\Phi}_+(\omega_1),$$

where the “spectral sum” (integral) appears in the exponential functions

$$\Phi_{\pm}(x) = \exp \left\{ -\frac{1}{2\pi i} \int_{-\infty}^{\infty} dy e^{-i(x-y)\eta} \ln[1 + \lambda \tilde{G}_c(y)] \right\}.$$  

Here

$$\bar{\Gamma}_+(\omega_1,\omega_2; \kappa) := \int_{-\infty}^{\infty} dt_1 dt_2 \theta(\pm t_1) e^{i\omega_1 t_1} e^{-i\omega_2 t_2} e^{-\eta(|t_1|+|t_2|)} \bar{\Gamma}(t_1,t_2; \kappa)$$

are Fourier transforms. Contrary to the Fredholm method the Wiener-Hopf trick does not anticipate knowledge of the eigenstates and eigenvalues of the integral kernel. The small imaginary part $\eta$ has the same meaning as in the Fredholm, finite-time approach. It secures the convergence of the Fourier integrals used in the Wiener-Hopf solution.
However, the parameter \( \eta \) has quite a different relation to the relaxation time. The relaxation time \( T \) is strictly infinity in formulas (11) whereby the parameter \( \eta \) must remain finite. We hence have \( \eta T \to \infty \).

Leading long-time asymptotics \( T \to \infty \) within the Wiener-Hopf solution is obtained by replacing the infinite time interval by a finite one in the solution (11). I.e. the Fourier integrals come over into Fourier series and we obtain in leading order of \( T^{-1} \) for the Matsubara frequencies \( \omega_n = (2n + 1) \pi T \) and \( \nu_m = 2m \pi T \).

\[
\Gamma^+_{T}(\nu_m, \nu_m'; \kappa) = \frac{T}{4} \delta_{m,m'} \frac{\bar{G}^{(0)}(\nu_m'; \kappa)}{1 + \lambda \bar{G}_c(\nu_m)} \quad \text{(12a)}
\]

\[
\Gamma^+_{T}(\nu_m, \nu_n; \kappa) = \frac{-1}{2i(\nu_m - \nu_n)} \frac{\bar{G}^{(0)}(\nu_n; \kappa)}{1 + \lambda \bar{G}_c(\nu_n)} \Phi_+(\nu_m), \quad \text{(12b)}
\]

where \( \Gamma^+_{T} \) is a projection of the Wiener-Hopf solution \( \Gamma_+ \) onto the interval \([0, T]\). The exponential function \( \Phi_+ \) on a finite interval splits into two,

\[
\Phi_+(\nu_m) = \exp \left\{ \frac{1}{2} \ln \left( 1 + \lambda \bar{G}_c(\nu_m) \right) - \frac{1}{T} \sum_n \frac{e^{i(\nu_m - \omega_n) \eta}}{i(\nu_m - \omega_n)} \ln \left( 1 + \lambda \bar{G}_c(\omega_n) \right) \right\}
\]

and \( \Phi_+(\omega_n) \), where fermionic and bosonic frequencies interchange their roles.

### III. INCONSISTENCY IN THE EXISTING INTERPRETATION OF THE RESULTS FOR THE CRITICAL EDGE EXPONENTS

It is generally accepted that the various existing approaches to the MND problem confirm the pioneering result of Nozières and De Dominicis on the critical edge exponents. We now show that it is not quite the case and that there is a discrepancy if the results obtained for the Anderson orthogonality catastrophe and for the critical behavior of the Green function of the core hole are compared. The discrepancy between the two existing results becomes apparent if we systematically use only one and the same construction scheme on both quantities.

In the orthogonality catastrophe one has to evaluate a matrix element \( \langle \Phi(N) | \Psi_N(t) \rangle \) in the limit \( t \to \infty \). Here \( | \Psi_N(t) \rangle = \exp \left\{ -i \hat{H}_{MND} t \right\} | \Phi(N) \rangle \). A simple calculation yields

\[
\langle \Phi(N) | \Psi_N(t) \rangle = e^{-iE_0^N t} \left\langle \Phi(N) | \hat{S}_{MND}(t, 0) | \Phi(N) \right\rangle
\]

\[
= e^{-iE_0^N t} \left\langle \Phi(N) | \hat{T} \exp \left\{ -i \int_0^t dt' \hat{V}_1(t') \right\} | \Phi(N) \right\rangle = -ie^{-iE_{0}^{N}t} \hat{G}_h(t). \quad \text{(14)}
\]

The last equality follows from (6a) and the fact that the core electron does not propagate in the MND model. From the other side, if we start from the Fermi golden rule (3) and consider only the vacuum contribution, we obtain

\[
\sum_f e^{-i(E_f - E_0^N t)} | \langle \Psi_f(N) | \Phi(N) \rangle |^2 = \sum_f e^{-iE_c t} | \langle \Phi(N) | \hat{H}_0 t \right\} | \Psi_f(N) \rangle
\]

\[
\times | \langle \Psi_f(N) | \exp \left\{ -i \hat{H}_{MND} t \right\} | \Phi(N) \rangle = e^{-iE_{c}t} | \langle \Phi(N) | \exp \left\{ -i \hat{H}_0 t \right\} \exp \left\{ -i \hat{H}_{MND} t \right\} | \Phi(N) \rangle
\]

\[
e^{-iE_c t} | \langle \Phi(N) | \hat{S}_{MND}(t, 0) | \Phi(N) \rangle = -i \hat{G}_h(t). \quad \text{(15)}
\]

It hence means that the overlap calculated in the orthogonality catastrophe (15) is identical with the Green function of the local, core hole used in the Nozières and De Dominicis and other field-theoretic approaches starting with Kubo formula (1). Now the calculation of the transition amplitude in the orthogonality catastrophe based on the Hilbert boundary problem (2) leads to the following asymptotics.
\[ |\langle \Phi(N) | \Psi_N(t) \rangle| \to C \exp \left\{ -\frac{\delta(0)^2}{2\pi^2} \ln \xi t \right\} \]  

(16a)

with \( \delta(0) = \text{Im} \ln \left[ 1 + V \tilde{G}(-i\eta) \right] > 0 \). The result of Nozières and De Dominicis and others for the Green function of the core hole reads

\[ |G_h(t)| \to C \exp \left\{ -\frac{\delta(0)^2}{\pi^2} \ln \xi t \right\}. \]  

(16b)

If we accept the result (16b) as correct, we then get in conflict with the critical exponent of the Anderson orthogonality catastrophe. If we accept the result for the overlap amplitude we do not recover the result (16b), since (14) and (15) are simple exact relations derived without any special assumptions.

It is worth commenting on the determinantal approach using the eigenstates of the final Hamiltonian \( \hat{H}_f \). Their construction is based on two assumptions. First, they assume that \( \langle \Phi(N) | \Psi_0(N) \rangle \) if \( t \to \infty \), where \( |\Psi_0(N)\rangle \) is the ground state of the final Hamiltonian \( \hat{H}_f \) with N conduction electrons. It means that adiabatic theorem is assumed to be valid. Second, it is shown that each term in the sum over the eigenstates of \( \hat{H}_f \) from (15) contains a factor \( |\langle \Phi(N) | \Psi_0(N) \rangle|^2 \). It is tacitly assumed (but not proven) that the infinite sum converges in the infinite-time limit and hence the square of the overlap between the two ground states appears also in the Green function of the core hole. This, however, would be in conflict with the equivalence of the Fermi golden rule (3) and the Kubo formula (4) proven in various books on condensed matter. The explanation why the determinantal approach failed to disclose the discrepancy between the orthogonality catastrophe and the x-ray results is that the second assumption about the convergence of the sum over the eigenstates of \( \hat{H}_f \) in the long-time limit does not hold. The interchange of the summation and the limit to infinite times is not justified. We see that our present knowledge of peculiarities of the edge behavior is insufficient and the problem deserves further analysis.

IV. LONG-TIME ASYMPTOTICS USING THE WIENER-HOPF TECHNIQUE

In this section we show how to use the Wiener-Hopf solution from the infinite-time limit (11) to evaluate long-time asymptotics of the relevant quantities in the MND problem. They are the Green function of the core hole, the nonequilibrium Green function of the conduction electrons, and the x-ray amplitude. Since the Green functions are singular only at zero temperature, we give an explicit derivation for this case.

As we already mentioned, the long-time asymptotics is derived within the Wiener-Hopf method if we take the infinite-time solution (13) and project it onto a finite interval \([-T, T]\). The functions are periodically continued beyond this interval. Strictly speaking the Wiener-Hopf formulas (12), (13) are no longer exact on a finite interval, but we show in the next section that the exact solution deviates from the Wiener-Hopf one only in higher orders of the inverse length of the interval, \( T^{-1} \) and that the finite-time solution converges to the Wiener-Hopf one at sufficiently long times.

A. Green function of the core hole

Green function of the core hole was extensively studied in the framework of the Falicov-Kimball model in Ref. It differs from the MND one only in that the impurities are densely distributed on the lattice. The method of solution is otherwise the same. Representation (15) is used together with the auxiliary Green function \( \Gamma_T(t', t'_+, \lambda) \) and defining equations (12) and (13). These are exact equations that, however, do not possess exact analytic solution in the whole (infinite) time range. The long-time asymptotics calculated with the Wiener-Hopf solution has the following form
Since we are interested only in leading-order long-time behavior, we replace the discrete sums with continuous integrals whenever possible, i.e., when the integrals converge. The discrete sum is a regularization needed to make the infinite-time limit sensible. First of all the exponential function $\Phi_+$ can be represented as

$$\Phi_+(\omega) = \exp \left\{ \frac{1}{\pi} \int_{0}^{\infty} \frac{d\omega'}{\omega' - \omega - i\eta} \text{Im} \ln \left[ 1 + \lambda \tilde{G}(\omega' + i\eta) \right] \right\}$$

and the sums in (17) can be rewritten into

$$C(T) = \frac{iT}{\pi} \int_{-\infty}^{0} d\omega \delta(\omega) + \frac{U}{2\pi} \int_{-\infty}^{0} \frac{d\omega}{\omega^2} I_+(\omega)$$

where $I_+(\omega) := \theta(\omega)(I(\omega) - I(0^+)) + \theta(\omega)(I(\omega) - I(0^-))$ and $\delta(\omega) = \text{Im} \ln \left[ 1 + V \tilde{G}(\omega - i\eta) \right]$ with

$$I(\omega) = \frac{1}{\pi i} \int_{-\infty}^{0} d\omega' \Phi_+(\omega') \left[ \frac{1}{\Phi_+(\omega' + \omega)} + \frac{1}{\Phi_+(\omega' - \omega)} \right] \text{Im} \frac{\tilde{G}(\omega' + i\eta)}{1 + \lambda \tilde{G}(\omega' + i\eta)}.$$

The first term on the left-hand side of (19) is a shift of the ground-state energy of the conduction electrons due to the core-hole potential matching the Fumi theorem. The second one is a logarithmically singular contribution causing a critical algebraic long-time decay of the Green function of the core hole. The logarithmic divergence of the integral with the function $I_+(\omega)$ appears due to a jump in the derivative of this function at the origin. When evaluating this jump we obtain

$$C(T) = \frac{iT}{\pi} \int_{-\infty}^{0} d\omega \delta(\omega) - \frac{1}{2\pi^2} \delta(0)\ln \xi T,$$

where $\xi$ is an effective bandwidth of the conduction electrons. This result is in accord with the calculation of Hamann done for the Anderson orthogonality catastrophe.

**B. Nonequilibrium Green function of the conduction electrons**

Although there are various approaches to the orthogonality catastrophe or the Green function of the core hole, there are only a few techniques to calculate the nonequilibrium Green function of the conduction electrons. It is surprising but a full, field-theoretic solution for this function has not yet been published. Except for the series of papers by Nozières and co-workers, the solution for this Green function are based on the quantum-mechanical golden rule or Slater determinants. We present now a field theoretic, Wiener-Hopf solution to this nonequilibrium function in the long-time limit.

To reach the long-time asymptotics with the Wiener-Hopf method we must solve equation (14) for finite time intervals, then put $t_1 = T$ and $t_2 = 0$ and only after this perform the limiting process $T \to \infty$. The Wiener-Hopf solution as given in (12) applied to eq. (14) reads
\[ f(\nu_m = 0) \equiv \frac{1}{2} [f(0^+) + f(0^-)] . \] (23)

The Wiener-Hopf solution to the full nonequilibrium Green function of the conduction electrons reads

\[ \Gamma_T(T, 0; k_1, k_2) = \frac{1}{T^2} \sum_{m} e^{i\nu_m} \eta \tilde{G}_T(\nu_m; k_1, k_2) - \frac{1}{T^2} \sum_{n} e^{i\omega_n} \eta \tilde{G}_T(\omega_n; k_1, k_2) + \frac{V}{2T^2} \sum_{m, n} \eta (\nu_m - \omega_n) \left[ \frac{\tilde{G}_c(\nu_m; k_1) \tilde{G}_c(\omega_n; k_2)}{1 + V \tilde{G}_c(\omega_n)} \frac{\Phi_+(\omega_n)}{\Phi_+(\nu_m)} + (\nu_m \leftarrow \omega_n) \right] . \] (24)

Here we denoted the equilibrium Green function of the conduction electrons with the core-hole potential \( V \) as

\[ \tilde{G}_c(\omega; k_1, k_2) = \frac{T}{2} \tilde{G}_c(\omega; k_1) \left[ \delta(k_1 - k_2) - \frac{V}{1 + V \tilde{G}_c(\omega)} \tilde{G}_c(\omega; k_2) \right] \] (25)

Relations (22) – (25) complete the Wiener-Hopf solution for the nonequilibrium Green function of the conduction electrons in the MND problem with a contact core-hole potential. It is worth noting that this solution is unique for a sufficiently weak interaction \( V \), i.e., when no bound states interfere.

It is easy to confirm that \( \Gamma_\infty(\infty, 0; k_1, k_2) \), obtained from (24) in the limit \( T \to \infty \), vanishes, since the fermionic and bosonic contributions exactly cancel each other. Only the terms proportional to the powers \( T^{-1} \) and higher do not compensate and render a finite contribution. They contain singular terms dominant in the long time limit. To obtain the most singular expressions in the long time limit, it is sufficient to consider only contributions from the lowest, both fermionic \((n = \pm 1)\) and bosonic \((m = 0)\), frequencies with the function \( \Phi_+ \) in the denominator. Namely only then we obtain a logarithmically divergent contribution to the sum. The contribution from \( m = 0 \) (c.f. Eq. (24)) is one half of the sum over the lowest fermionic frequencies with \( n = \pm 1 \), while the higher bosonic frequencies \( m = \pm 1, 2, ... \) are exactly compensated in the long-time limit by the fermionic ones \( n = \pm 2, 3, ... \), respectively.

We assess the long-time limit of the sum in the exponent of the function \( \Phi_+ \) from (22). The sum turns to a principal value integral. The causal Green function \( \tilde{G}_c(\omega) \) has a jump in its imaginary part at \( \omega = 0 \) where the integrand linearly diverges. These two coincidences lead to a logarithmic singularity in the limit \( T \to \infty \). A straightforward calculation yields an asymptotic form of the functions \( \Phi_+ \):

\[ \Phi_+(\omega = \pm \pi/T) \to \Phi_+(\nu = 0^\pm) = e^{\mp i\delta(0)/2} \exp \left\{ \frac{\delta(0)}{2} \ln |\xi T| \right\}, \] where \( \xi \) is a suitable (nonuniversal) cutoff for the asymptotics \( \omega \to 0^\pm \). We assumed \( \eta T \to \infty \), the Wiener-Hopf regime. Inserting these singular contributions to (22) we obtain

\[ \Gamma_T(T, 0; k_1, k_2) = \frac{V}{2T} \left( \xi T \right) \pi \frac{\delta(0)}{\epsilon_{k_1} + i\eta} \left[ \frac{e^{i\delta(0)/2}}{\epsilon_{k_1} + i\eta} \right] P \int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} \frac{\Phi_-(\omega)}{\omega} \tilde{G}_c(\omega; k_2) \] (26)

as leading, singular term in the limit \( T \to \infty \). We used
\( \Phi_-(\omega) = \frac{\Phi_+(\omega)}{1 + V G_c(\omega)} = \exp \left\{ \frac{1}{\pi} \int_{-\infty}^{0} d\omega' \frac{\delta(\omega')}{\omega' - \omega + i\eta} \right\} . \) \hspace{1cm} (27)

The critical exponent is again one half of that of Nozières and De Dominicis and other finite-time (determinantal) approaches.

Expression (26) can still be simplified. The integral with the function \( \Phi \) can explicitly be evaluated using analytical properties of \( \Phi_- \) and \( \tilde{G}_c \). We have

\[
P \int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} \frac{\Phi_-^{*}(\omega) \tilde{G}_c^{*}(\omega; k_2)}{\omega} = P \int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} \frac{\Phi_-^{*}(\omega)}{\omega(\omega - \epsilon_{k_2} - i\eta)}
\]

\[
= P \int_{0}^{\infty} \frac{d\omega}{\pi} \frac{\Phi_-^{*}(\omega)}{\omega} \text{Im} \left[ \frac{1}{\omega - \epsilon_{k_2} + i\eta} \right] = -\frac{\theta(\epsilon_{k_2})}{\epsilon_{k_2}} \Phi_-^{*}(\epsilon_{k_2}) . \hspace{1cm} (28)
\]

Using this result we can write a final expression

\[
\Gamma_T(T, 0; k_1, k_2) = \frac{V}{2T} (\xi T) \frac{\delta(0)}{\pi} \frac{\theta(\epsilon_{k_2})}{\epsilon_{k_2}} \Phi_-^{*}(\epsilon_{k_2}) \Re \left[ \frac{e^{i\delta(0)/2}}{\epsilon_{k_2} + i\eta} \right] . \hspace{1cm} (29)
\]

Note that momenta \( k_1 \) and \( k_2 \) do not enter the singular part of the nonequilibrium Green function symmetrically. The reason for this lies in asymmetric treatment of the ends of the integration domain in the fundamental integral equation (7b). While at \( t = 0 \) the system undergoes a sudden change, i.e., the core potential is immediately switched on, the other end is pushed to infinity, \( T \to \infty \), and the core potential is being adiabatically damped by a small exponential factor \( e^{-\eta T} \) to make the long-time limit meaningful. There is no abrupt change in the long-time tail. A sharp edge (represented by a step function) and a long-time tail dominated by the long-time asymptotics of the integral kernel \( G_c(T) \) are equally important for the existence of the edge singularity. If the sharp edge at \( t = 0 \) were removed, the function \( \Gamma_T(T, 0; k_1, k_2) \) would reduce to the equilibrium function \( \Gamma_T(T; k_1, k_2) \). If the integration interval were kept finite we could not get to the Fermi energy closer than to a distance \( \pi T \) and no logarithmic divergences would show up.

C. Absorption amplitude

Having derived expressions for both the Green functions we can now construct the absorption amplitude (4). We first write down the two-particle susceptibility \( \chi_{k,k'}(t) \) in the asymptotic limit of long times. Using formulas (5), (6), (21), and (26) we explicitly obtain for leading (singular) long-time asymptotics

\[
\chi_{k,k'}(t) = \frac{-iV}{2T} (\xi T) \alpha \exp \left\{ -i(E_c - \xi T) \right\} \frac{\theta(\epsilon_{k})}{\epsilon_{k'}} \Phi_-^{*}(\epsilon_{k}) \Re \left[ \frac{e^{i\delta(0)/2}}{\epsilon_{k} + i\eta} \right] . \hspace{1cm} (30)
\]

where \( \xi = \int_{-\infty}^{0} d\omega \delta(\omega) \) and \( \alpha = \delta(0)/\pi - \delta(0)^2/2\pi^2 \). The factor \( i \) appears due to the definition of the Green function of the core hole (6a). This expression is exact (within the Wiener-Hopf infinite-time approach) not only for the critical exponent of the algebraic decay but also for the nonuniversal prefactor, the time-independent part of (6a). To obtain the absorption amplitude \( I(\omega) \) we need to know the susceptibility \( \chi \) not only in the long-time limit but on the whole positive frequency axis. However, if the critical exponent \( \alpha \) from (4) is positive, i.e., the absorption amplitude diverges at the Fermi level, then the long-time behavior of the susceptibility also determines the edge behavior of the absorption amplitude. In the isotropic case with contact core potential we can derive an exact expression for the absorption amplitude in the vicinity of the edge. We insert (30) into (4) and realize that from the real part in (30) only the Fermi energy \( (\epsilon = 0) \) survives. Then the absorption amplitude is
\[ I_{\text{sing}}(\omega) = 2\text{Im} \left[ \pi V \rho(0) \sin(\frac{\delta(0)}{2}) w(0) \int_0^\infty d\omega(\epsilon) \rho(\epsilon) \frac{\Phi_-(\epsilon)}{\epsilon} \int_{T_0}^\infty dt e^{i(\omega - E_c + \mathcal{E}) t} \right], \]  

(31)

where \( \rho(\epsilon) \) is the DOS of the conduction electrons and \( w(\epsilon) = w(\epsilon)^* \equiv w(\epsilon_k) \). We stress that \( I_{\text{sing}} \)

is only the singular part of the absorption amplitude, i.e. \( \alpha > 0 \). We used a new parameter \( T_0 \) as a lower bound for the time integration, since only there the long-time asymptotics of the electron-hole susceptibility \( \chi \) dominates. The value of this parameter depends on the range of applicability of the Wiener-Hopf solution. We will discuss the problem in the next section. At present we know that at least \( T_0 \approx 1/\eta \). In the Wiener-Hopf regime then \( T_0/T \to 0 \) and \( \Delta \omega T_0 \ll 1 \), with \( \Delta \omega = \omega - E_c + \mathcal{E} \). We can extend the time integration onto the whole positive axis and perform the integral explicitly. It yields \( \exp\{i\alpha \pi/2\} [\xi/\omega - E_c + \mathcal{E}]^\alpha \Gamma(\alpha) \).

The absorption amplitude must be positive, otherwise it is unphysical. It is the case, as expected, only for \( \omega \geq E_c - \mathcal{E} > 0 \). We explicitly have

\[ I_{\text{sing}}(\omega) = 2\pi V \rho(0) \sin(\frac{\delta(0)}{2}) \sin(\frac{\alpha \pi}{2}) \Gamma(\alpha) w(0) \int_0^\infty d\omega(\epsilon) \rho(\epsilon) \frac{\Phi_-(\epsilon)}{\epsilon} \left[ \frac{\xi}{\Delta \omega} \right]^\alpha. \]  

(32)

The integral in (32) is finite, i.e. does not bring any new divergence into the absorption amplitude. The critical edge exponent in the Wiener-Hopf solution is one half of the critical exponent obtained from the Nozières and De Dominicis and finite-time approaches. Formula (32) is the Wiener-Hopf expression for the asymptotic limit \( \omega \to 0 \) of the absorption amplitude provided the leading low-frequency term is singular, i.e. \( \alpha > 0 \), and the frequency is sufficiently close to the threshold, i.e. \( \Delta \omega T_0 \ll 1 \).

We can make a few general statements about the threshold behavior of the absorption amplitude (32). First of all the singular part of the absorption amplitude is very weak at weak coupling, namely it is proportional to \( (V \rho(0))^2 \). Second, initial and final states are differently treated in the absorption process. The initial core electron can be excited to any empty state \( (\epsilon > 0) \) whereby the energy is not conserved due to instantaneous absorption. The excited state decays in the long-time limit towards a final state where the conservation of energy is restored and the additive conduction electron settles onto the Fermi surface \( (\epsilon = 0) \). A new ground state is reached. This is a very different result from the conjecture that the diagonal part of the two-particle susceptibility, \( \chi_{kk} \), yields the principal contribution to the absorption amplitude, i.e. the initial and finite states enter the absorption amplitude symmetrically.

V. RELEVANT TIME SCALES IN THE X-RAY PROBLEM AND TRANSITION FROM FREDHOLM TO WIENER-HOPF SOLUTIONS

In the preceding section we derived the Wiener-Hopf long-time solution for the relevant Green functions of the MND problem and also the edge behavior of the absorption amplitude. We obtained a critical behavior with a power law but with critical exponents that equal one half the values derived by Nozières and De Dominicis and most other approaches. To understand the difference it is necessary to distinguish carefully relevant time scales in the x-ray problem and to determine in which limiting cases either solution is applicable.

The MND Hamiltonian (1) contains two relevant finite energy or time scales. It is an effective bandwidth \( \xi \) related to the kinetic energy and the core hole - conduction electron interaction strength \( V \). In our notation \( V > 0 \). Apart from this model-dependent energy scales there are a few "infinite-time" scales reflecting experimental realization of the process. First of all it is the relaxation time \( T \) determined by the waiting time at which a new equilibrium after the absorption of light is reached. This time determines the length of the time interval on which we solve the fundamental Dyson equation (8). There is also an effective life-time of the transient pair of a core hole and a conduction electron, \( \tau = 1/\eta \). Apart from these two infinite times there is also an infinite scale connected with volume of the sample, or with the number of lattice sites \( N \). The last very large scale is the inverse temperature \( \beta \). All the scales, not defined from the Hamiltonian, must be limited to infinity to reach an ideal critical behavior, the edge singularity. The actual critical asymptotics may be "unstable" and may depend on the order in which the limiting infinite values are reached. We compare only the two general approaches discussed in Sec. II. B,
i.e. the Fredholm and the Wiener-Hopf one, since they show the differences in the critical exponents we want to understand.

We know from the discussion in Sec. II. B that the Wiener-Hopf solution is exact if $\tau/T \rightarrow 0$, i.e. if the relaxation time is much larger than the effective life-time of the core hole. This is, however, only a sufficient condition for the Wiener-Hopf infinite-time regime to set on. We would like to find necessary conditions distinguishing the Wiener-Hopf solution from the Fredholm one.

### A. Small parameters for the Fredholm and Wiener-Hopf solutions of the Dyson equation

Restrictions on applicability of the Fredholm solution to the equation (33) in the long-time limit come from the assumptions F1 and F2 of the Fredholm method. The Fredholm method is based on a (Neumann) perturbation expansion that must have a finite convergence radius. Hence using (33) we obtain an upper bound

$$\lambda^2 = V^2 ||\tilde{G}||^2_{HS} = T/2 \int_{-T}^{T} dt |G_\epsilon(t)|^2 = \pi^2 V^2 \rho(0)^2 \xi T < M,$$

where $M$ is a constant of order unity. Further on the spectrum of the kernel of a Fredholm integral equation must be discrete. It means that the distance between the nearest energy levels must be greater than the resolution of the measuring instruments. Combining this result with the restriction due to the effective life-time of the core hole we obtain

$$T \ll \mathcal{N}^{2/3}, \quad T \ll \beta, \quad T \ll \tau = \frac{1}{\eta}.$$

The above conditions are sufficient to secure the Fredholm solution is reliable and valid. However bound (33) is very restrictive in long times. It tells us that we cannot reach the infinite-time limit for a fixed interaction strength $V$. We can assume $\lambda$ from (33) to be a small parameter for the Fredholm solution. It means that the smaller $\lambda$ the better the Fredholm solution reproduces an exact one. The latter three bounds say that the edge exponent for the absorption amplitude governing the infinite-volume limit $\mathcal{N} \rightarrow \infty$ must be that of the Wiener-Hopf solution. The same holds also for the limit $\beta \rightarrow \infty$ for $\Delta \omega \approx \eta$.

It is more elaborate to find a small parameter for the Wiener-Hopf solution, i.e. an analogue to $\lambda$ from (33). The Wiener-Hopf trick leads to a nonperturbative solution and the small parameter will not depend on the interaction strength explicitly. Deviations of an exact solution on a finite interval appear due to violation of the assumption WH2 from Sec. II. B. A small parameter for the Wiener-Hopf solution is a measure of these deviations. The problem with the application of the Wiener-Hopf trick onto a finite-interval integral equation lies in different decompositions of Fourier transforms diagonalizing convolution. For a semi-infinite interval we decompose any function into its positive and negative parts, i.e. functions the Fourier transforms of which are analytic in the upper (lower) half-planes of complex energies, respectively. For a finite interval we have to decompose a function into its fermionic and bosonic parts in order to factorize (diagonalize) convolution in time. We relate the two decompositions and quantify deviations from the Wiener-Hopf solution in the long-time limit.

Let us have a function $F(t) \in L_2(-\infty, \infty)$. Let $F_\pm$ be its projections onto positive and negative time half-axis, respectively. We now cut off the function $F$ at $\pm T$ and continue the function periodically onto the rest of the real axis. We denote such a function as $F(T; t)$. It is sufficient to consider the argument $t \in (-T, T]$. We use a variable $\zeta = \pm 1$ for bosonic and fermionic projections, respectively. Then

$$F_\zeta^+(T; t) = [F_+(T; t) + \zeta e^{-\eta T} \theta(t) F_-(T; -T + t)],$$
$$F_\zeta^-(T; t) = [F_-(T; t) + \zeta e^{-\eta T} \theta(-t) F_+(T; T + t)],$$

where again $t \in (-T, T]$. Note that the functions $F_\zeta^\pm(T; t)$ have two contributions. Each is centered around the either edge of the interval $[0, T]$. The Wiener-Hopf solution does not treat the edges symmetrically and the contribution from the upper edge is suppressed by a small exponential term $e^{-\eta T}$. Using (33) we define a general difference function
\[
\Delta F(T; t) = e^{-\eta T} [\theta(t) F_-(T; -T + t) + \theta(-t) F_+(T; T + t)]
\]  

useful for building up a small parameter for an expansion around the Wiener-Hopf solution.

The Fourier transform of the above functions and its analytic structure in the plane of complex energies is essential for the Wiener-Hopf solution. Using the fact that \( \tilde{F}_\pm(z) \) are analytic in the upper (lower) half-plane and supposing that they vanish at the respective infinity, we obtain from (33)

\[
\tilde{F}_\pm (T; \omega) = \frac{1}{2} \left[ 1 + \zeta e^{-\eta T \mp i\omega T} \right] \tilde{F}_\pm (\omega),
\]

where \( \tilde{F}_\pm \) are Fourier transforms on the infinite interval \((-\infty, \infty)\). The difference function, when specified to the integral kernel of the interal equation (8) has a Fourier transform

\[
\Delta \tilde{G}_T(\omega) = e^{-\eta T} \left[ e^{i\omega T} \tilde{G}_-(\omega) + e^{-i\omega T} \tilde{G}_+(\omega) \right],
\]

where the projections \( \tilde{G}_\pm \) are defined analogously to (33) and (27)

\[
\begin{align*}
\tilde{G}_+(\omega) &= p_+ [\tilde{G}_e](\omega) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega' \frac{e^{i(\omega' - \omega)\eta}}{\omega' - \omega - i\eta} \tilde{G}_e(\omega') = \frac{1}{\pi} \int_{0}^{\infty} d\omega' \text{Im} \tilde{G}(\omega' + i\eta), \\
\tilde{G}_-(\omega) &= p_- [\tilde{G}_e](\omega) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega' \frac{e^{-i(\omega' - \omega)\eta}}{\omega' - \omega + i\eta} \tilde{G}_e(\omega') = \frac{1}{\pi} \int_{\infty}^{0} d\omega' \text{Im} \tilde{G}(\omega' + i\eta).
\end{align*}
\]

To find a small parameter for an expansion around the Wiener-Hopf solution we have to consider the fundamental Dyson equation (8) in frequency representation. The equation splits into a couple mixing the bosonic and fermionic parts. We can formally write the pair of equations as

\[
\tilde{\Gamma}_f = \tilde{\Gamma}_f^{(0)} - 2V \tilde{G}_c, \tilde{\Gamma}_f^+, \quad \tilde{\Gamma}_b = \tilde{\Gamma}_b^{(0)} - 2V \tilde{G}_c, \tilde{\Gamma}_b^+,
\]

where, if we assume only the active variable,

\[
\tilde{\Gamma}_f^\pm(\omega_n) = p^\pm_f [\tilde{\Gamma}_f](\omega_m) = \frac{1}{2} \left[ \tilde{\Gamma}_f \mp \frac{2}{T} \sum_{m=-\infty}^{\infty} \frac{e^{\pm i(\omega_n - \nu_m)\eta}}{(\omega_n - \nu_m)} \tilde{\Gamma}_b(\nu_m) \right]
\]

are projections onto positive (negative) time axis within the interval \([-T, T]\). From the solution (40) again only the positive projection \( \tilde{\Gamma}_f^+ \) has physical meaning. We see that the bosonic and fermionic functions are mixed. The difference between bosonic and fermionic functions in the plane of complex energies is given by the difference function (38). The bosonic function vanishes at fermionic frequencies and vice versa. If the functions \( \tilde{\Gamma}_f(\omega) \) and \( \tilde{\Gamma}_f(\omega) \) were identical, the Wiener-Hopf solution would be exact.

In the long-time limit (in leading order) the projectors \( p^\pm_f \) can again be expressed as integrals

\[
\begin{align*}
p_f^\pm \left[ \tilde{G}_e \right] (\omega) &= \pm \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega' \frac{e^{\pm i(\omega' - \omega)\eta}}{\omega' - \omega \mp i\eta} \left[ \tilde{G}_e(\omega') - \zeta \Delta \tilde{G}_T(\omega') \right] + \zeta \Delta \tilde{G}_T(\omega) \\
&= p_+ \left[ \tilde{G}_e \right] (\omega) + \zeta p_- \left[ \Delta \tilde{G}_T \right] (\omega).
\end{align*}
\]

The difference between (32) and the infinite-time expression is only in the function \( \Delta \tilde{G}_T(\omega) \).

As we already mentioned the Wiener-Hopf solution on finite intervals breaks down because the assumption WH2 is violated. It means that convolution does not remain within positive or negative subspaces \( L^\pm_2 \). A measure of this “non-orthogonality” can be defined as

\[
\kappa_\zeta(\omega) = p_\zeta^\pm \left[ p^\pm_\zeta \left[ \tilde{G}_e \right] p^\pm_\zeta \left[ \tilde{G}_e \right] \right] (\omega).
\]

It is evident that \( \kappa_\zeta(\omega) = 0 \) at \( T = \infty \). The bosonic and fermionic functions (labeled with subscript \( \zeta \)) lose their meaning when going over to infinite time. We have to come to positive and negative projections.
on the whole real axis. We sum over the index $\zeta$ to obtain the Wiener-Hopf solution $\Gamma_+$ independent of $\zeta$. We can introduce a new $\zeta$-independent small parameter $\kappa_T = \sum \kappa_{\zeta}$. After a few simple manipulations we obtain

$$\kappa_T(\omega) = -\frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega' \frac{p- \left[ \Delta \tilde{G}_T(\omega') \right]^2}{\omega' - \omega + i\eta}. \quad (44)$$

We use explicit representation (38) for the function $\Delta \tilde{G}_T(\omega)$ and perform the integral over $x$ analytically. In the remaining expression we substitute $x = \omega T$ and realize that only small values of $x$ contribute in the long-time limit. The contributions from higher frequencies are canceled due to negative interference. If $\xi$ is an effective bandwidth we finally end up with

$$\kappa_T(\omega) = \frac{\rho(0)^2}{\pi} \left[ \int_{-\xi T}^{\xi T} \frac{dx}{x-\omega T} \sin x \right]^2 \left[ \int_{-\xi T}^{\xi T} \frac{dx}{x-\omega T} \right]^2, \quad (45)$$

where $\rho(0) = -1/\pi \text{Im} \tilde{G}(i\eta)$. Since we are interested in the limit $T \to \infty$ with a fixed frequency $\omega$, we assume that $\omega T \sim \xi T \gg 1$. The integral is independent of frequency and can be bounded from above by

$$\kappa_T = \frac{\rho(0)^2}{\pi} \left( \ln \xi T \right)^2 \frac{4e^{-2\eta T}}{\pi^2 (\xi T)^2}. \quad (46)$$

A dimensionless small parameter is obtained if we devide the norm of $\kappa_T(\omega)$ on an interval $[-\xi T, \xi T]$ by the norm of the function $\tilde{G}_+(\omega)\tilde{G}_+(\omega)$ on the same interval. The desired small parameter then is

$$\Delta = \frac{2}{\pi \xi T} e^{-\eta T}. \quad (47)$$

We see that the finite-time solution $\Gamma_+^T$ approaches the Wiener-Hopf one $\Gamma_+$ algebraically and the small parameter is proportional to $(\xi T)^{-1}$. The Winer-Hopf solution is hence stable in the long-time limit and we can systematically expand around it.

Having found small parameters for both the Fredholm and Wiener-Hopf solutions we can distinguish their domains of validity. Comparing (33) and (41) we see that the Fredholm, perturbative solution is applicable for $\xi T \ll 1$ while the Wiener-Hopf one in the opposite limit $\xi T \gg 1$. There is hence a (smooth) transition from the Fredholm to the Wiener-Hopf solution if both the small parameters are approximately equal. Comparison of the parameters $\lambda$ and $\Delta$ as obtained in (33) and (41) yields an estimate for the transition time $T_c$ and frequency $\omega_c$

$$T_c \approx \frac{1}{\xi} \sqrt{\frac{4}{\pi^4 V^2 \rho(0)^2}}, \quad (48a)$$

$$\Delta \omega_c \approx \frac{\sqrt{\pi^4 V^2 \rho(0)^2}}{4}. \quad (48b)$$

The long-time Wiener-Hopf solution to the Dyson equation (8) becomes asymptotically exact if the relaxation time is sufficiently larger than the crossover value $T_c$ or when the frequency is correspondingly smaller than $\Delta \omega_c$.

### B. Critical edge exponents

We determined the domains of validity of the Fredholm (short-time) and the Wiener-Hopf (long-time) solutions to the equation (8). They are opposite edges of the parameters $\xi T$ and $\eta T$. The former is characterized by $\xi T \ll 1$ and $\eta T \ll 1$, while the latter is valid if $\xi T \gg 1$ and $\eta T \gg 1$. The two methods are hence, in some sense, complementary. Formally it is possible to continue weak-coupling perturbation
theory term by term to the long-time regime and one obtains an estimate for the critical edge behavior. This estimate coincides with the weak-coupling expansion of the solution of Nozières and De Dominicis. However, perturbation theory in the long-time regime is very sensitive to the order in which the earlier discussed various large scales are limited to infinity. One has to be careful in using perturbation expansion in the long-time limit. On the other hand the stability of the Wiener-Hopf solution does not yet mean that all the long-time asymptotic results are determined from the Wiener-Hopf solution \( \Gamma_+ \) only. To derive the algebraic decay of the Fredholm solution to the Wiener-Hopf one, we summed over the subscript \( \zeta \). Such a symmetric sum is well defined at infinite time even without a damping factor. The antisymmetric part, namely the difference between the bosonic and fermionic functions, rapidly fluctuates between \( \Gamma_+ \) and 0 and allows for an infinite-time (continuous) limit only if the damping factor is applied. It now depends on a specific quantity how it approaches its infinite-time limiting value. The transition from the Fredholm to the Wiener-Hopf solution will then be algebraic for symmetric quantities and weakly exponential for antisymmetric functions. Since we are interested rather in the critical exponents than in the solution of the Dyson equation \( \mathcal{G} \) itself, we construct long-time limits of the specific combinations of the Green functions \( \Gamma_\zeta^\pm \) necessary for the edge behavior.

### 1. Orthogonality catastrophe

To determine the critical edge exponent of the Green function of the core hole we have to evaluate the function \( C(T) \) from (12). This representation can be rewritten with the aid of the projectors \( p_\zeta^\pm \) from (14). We introduce double indices, one for each independent variable. We can write

\[
C(T) = \frac{T}{2} \int_0^U d\lambda \sum_{\zeta} \frac{1}{T} \sum_{i=-\infty}^{\infty} e^{i\omega_i \eta} \Gamma^{\pm}_{\zeta\zeta} (w_l, w_\ell; \lambda), \tag{49}
\]

where \( w_l \) is either bosonic, \( \nu_m \), or fermionic, \( \omega_n \), frequency according to the subscript \( \zeta \). Replacing the finite-time projectors with their representations using the infinite-time ones (15) we obtain

\[
C(T) = T \int_0^U d\lambda \frac{1}{T} \sum_{n} e^{i\omega_n \eta} \left\{ \Gamma^{\pm}_{+\pm}(\omega_n, \omega_n; \lambda) + e^{-2\eta T} \Gamma^{\pm}_{-\pm}(\omega_n, \omega_n; \lambda) \right\}. \tag{50}
\]

This expression is exact at any time if we insert exact expressions for the functions \( \Gamma^{\pm}_{+-} \) and \( \Gamma^{\pm}_{-+} \). The function \( \Gamma^{\pm}_{-+} \) expresses a contribution from the emission of light, i.e., from filling of the core hole by a conduction electron.

We do not know the exact solution for arbitrary finite times. However, according to the preceding section we can use the Wiener-Hopf solution in the long-time regime, i.e., for \( T \gg T_c \) from (13), to obtain an exact long-time asymptotics. In this regime we know the exact asymptotic form of these functions to the Wiener-Hopf solution projected onto the finite interval. Although only the symmetric functions contribute to the function \( C(T) \), we nevertheless have two significant terms in the long-time limit. The former one, not damped, is the Wiener-Hopf solution from Section IV. The damped term goes beyond the infinite-time Wiener-Hopf solution and makes the difference between the Fredholm and Wiener-Hopf solutions. It is at intermediate time scale \( (T < 1/\eta) \) of the same order as the former one and that is why the finite-time methods predict a different critical exponent for the Green function of the core hole.

We can now evaluate nonperturbatively the correction to the Wiener-Hopf critical exponent at intermediate (mesoscopic) times. To this end we have to evaluate the negative function \( \Gamma_- \) within the Wiener-Hopf trick. This function is not a solution of \( \mathcal{G} \) but of a “conjugate” equation where the integration interval \([0, T]\) is replaced by \([-T, 0]\). Then a Wiener-Hopf solution to such an equation is formally identical with (12), only the exponential functions have inverse meaning. We have to use the negative projection

\[
\Phi_-(\nu_m) = \exp \left\{ -\frac{1}{2} \ln \left( 1 + \lambda \mathcal{G}_c(\nu_m) \right) - \frac{1}{T} \sum_n \frac{e^{i(\nu_m - \omega_n) \eta}}{\nu_m - \omega_n} \ln \left( 1 + \lambda \tilde{\mathcal{G}}_c(\omega_n) \right) \right\} \tag{51}
\]
and similarly for the fermionic frequencies.

We do the same calculation as with the Wiener-Hopf solution in Sec. IV. A. It yields the function $C(T)$ in the following form

$$C(T) = \left(1 + e^{-2\eta T}\right) \left[\frac{iT}{\pi} \int_{-\infty}^{0} d\omega \delta(\omega) - \frac{\delta(0)^2}{2\pi^2} \ln \xi T \right].$$

(52)

This is an exact long-time asymptotic form of the Green function of the core hole at times much greater than typical times of the MND Hamiltonian, i.e. $\xi T \gg 1$. Function (52) shows a quasicritical behavior in the Fredholm regime, i.e. when $\eta T \ll 1$, with a critical exponent of the Nozières and De Dominicis solution. The Wiener-Hopf critical exponent is realized only for very long times of order of the effective lifetime of the transient, excited electron, $T \sim \tau$. Nevertheless the Wiener-Hopf solution is able to produce an exact quasicritical exponent and nonuniversal prefactor of the core-hole Green function. It is necessary to remind that (52) was derived exactly only in the weak-coupling regime where no bound states interfere. The same applies to the Nozières and De Dominicis solution as well.

2. Final-state interaction and absorption amplitude

A fundamental quantity for determination of the critical behavior of the excited conduction electron is the nonequilibrium Green function (24). We can generally write it as

$$\Gamma_T(T, 0; k_1, k_2) = \frac{1}{2} \sum_{\zeta, \zeta'} \frac{1}{T^2} \sum_{w_j, w_l} \tilde{G}_{\zeta}(w_j, k_1) \tilde{\Gamma}_{\zeta'}^+(w_j, w_l; k_2).$$

(53)

We see that this expression does contain antisymmetric combination of the finite-time fermionic and bosonic functions. It consists of two contributions. First one is the Wiener-Hopf solution from Sec. IV. It contributes due to the difference between arguments of fermionic and bosonic functions, i.e. due to the difference $\omega_n - \nu_m$. The second one arises from the functional difference between the fermionic and bosonic projectors, i.e. due to the difference function $\Delta \tilde{G}_T(\omega)$ from (38). There is no Wiener-Hopf solution for this function. We hence cannot derive analogous asymptotic formula for the nonequilibrium Green function of the conduction electrons as we did for the core hole in (52). One has to go beyond the infinite-time Wiener-Hopf approach to derive an asymptotically exact behavior of the function $\Gamma_T(T, 0; k_1, k_2)$ in the long-time regime. To this end it is necessary to construct a dynamical equation for the difference function $\tilde{\Gamma}_b - \tilde{\Gamma}_f$ which has the infinite-time limit only with the damping factor $e^{-\eta T}$. Since we are unable to find the genuine long-time asymptotics of the Green function of the transient conduction electron (53) at intermediate times, we are also unable to make a conclusion about asymptotic behavior of the absorption amplitude for frequencies not too close to the threshold, i.e. for $\Delta \omega \gg \eta$.

We do not have a formula for the absorption amplitude at experimentally relevant frequencies and hence we can ask what the physical meaning the Wiener-Hopf solution has. The proper meaning of the Wiener-Hopf solution for the absorption amplitude is that it defines the limiting threshold value of the edge singularity. It means that it determines the height of the threshold peak at finite temperatures and for finite volumes. Behavior of the edge peak observed when limiting the temperature (volume) to zero (infinity) is determined by the Wiener-Hopf solution. Also the shift of the ground-state energy is identical with the prediction of the Fummi theorem only within the Wiener-Hopf infinite-time solution. Different results for different limiting processes and a crossover from intermediate long to very long times give explanation to the inconsistency in the interpretation of the critical exponents from Sec. III. The limit to the infinite volume is at the threshold governed by the Wiener-Hopf solution, while the long-time or low-frequency limits are dominated by the quasicritical, finite-time solution of the basic one-body integral equation (8).
VI. CONCLUSIONS

In this paper we analyzed in detail the MND problem within the one-body formulation of Nozières and De Dominicis with field-theoretic Green functions and Kubo formula for the absorption amplitude. We discussed the differences between two possible ways to solve the fundamental Dyson equation \( \frac{d}{dt} \Gamma + \Delta \Gamma = \frac{1}{2} \Delta \Gamma \) in the long-time limit, the finite-time (Fredholm) and infinite-time (Wiener-Hopf) methods. We showed that each method is based on a different decomposition of the full function in order to diagonalize (factorize) convolution in time. For the finite-time method it is a decomposition into fermionic and bosonic parts, the Fourier transforms of which are defined on odd, even frequencies, respectively. A decomposition into projections onto retarded (positive-time) and advanced (negative-time) functions is for the Wiener-Hopf method appropriate. We determined respective ranges of validity and reliability of both the methods and found corresponding small parameters for each of them. We showed that they are in some sense complementary. We proved that the solution of Wiener and Hopf for \( \Gamma \) and found corresponding small parameters for each of them. We showed that they are in some sense appropriate. We determined respective ranges of validity and reliability of both the methods and found corresponding small parameters for each of them. We showed that they are in some sense complementary. We proved that the solution of Wiener and Hopf for \( \Gamma + \Delta \Gamma \in L_2^\infty \) is obtained as a long-time limit of the Fredholm solution if \( \xi T \gg 1 \), where \( \xi \) is an effective bandwidth of the conduction electrons. It means that the Wiener-Hopf solution is stable to perturbations in the long-time limit and one can systematically expand around it.

We derived the long-time asymptotic limits of all the relevant quantities of the x-ray problem within the Wiener-Hopf solution. We calculated the Green function of the core hole, of the transient conduction electron excited by absorption of light in the long-time limit and determined not only the critical exponents but also the noncritical prefactors. In case of the divergent edge we also obtained the Wiener-Hopf expression for the x-ray absorption amplitude. The critical exponents come out in the Wiener-Hopf solution as one half of the Nozières and De Dominicis result.

To understand the difference we analyzed the role various very large time scales play for the critical edge behavior. We showed that it is necessary to introduce an effective lifetime, \( \tau \), of the transient electron-hole pair and compare the relaxation time, \( T \), with this scale in both of the approaches. The Fredholm, finite-time approach is characterized by the ratio \( T/\tau \to 0 \) while the Wiener-Hopf solution by \( T/\tau \to \infty \). Unitarity of the theory demands to limit the lifetime \( \tau \) to infinity in the end of the calculations. We hence confirmed that the Wiener-Hopf results for the relevant quantities of the x-ray problem are exact for times comparable with \( \tau \) or frequencies \( \Delta \omega \sim \eta = 1/\tau \). We showed that the Wiener-Hopf theory is exact for any quantity if the parameter driving the system to the edge singularity is of order of the lifetime of the transient electron-hole pair.

Because of various large scales in the x-ray problem the results we obtain for the critical behavior are dependent on the trajectory along which we approach the critical point in the space of relevant parameters. Most importantly we found a wide interval of intermediate, experimentally relevant times \( T/\tau \ll 1 \) or frequencies \( \Delta \omega \gg \eta \) showing a quasicritical behavior. This intermediate-time asymptotics deviates from the long-time Wiener-Hopf one, which causes the differences between the critical exponents calculated from the Fredholm or the Wiener-Hopf solutions. We derived general representations for the quasicritical long-time behavior of the Green functions of the core hole and of the transient conduction electron. It is clear from them that the quasicritical behavior deviates macroscopically from the infinite-time solution.

Based on the proof of asymptotic exactness of the Wiener-Hopf solution for the functions \( \Gamma \pm \tau \) we were able to derive an exact long-time asymptotic formula for the Green function of the core hole. This formula covers the Wiener-Hopf as well as the quasicritical Fredholm long-time asymptotics. The critical exponent for the quasicritical behavior, coincides with the result of Nozières and De Dominicis in the finite-time, Fredholm regime, \( T/\tau \to 0 \).

With our analysis we demonstrated usefulness and effectivity of the Wiener-Hopf approach to obtain the critical edge behavior within the MND model with all its details. We determined the range of validity of the Wiener-Hopf solution and explained the differences between its critical exponents and the quasicritical exponents from the finite-time approaches at least at weak coupling. Although the Wiener-Hopf critical exponents have less experimental relevance, we showed that only the Wiener-Hopf method is able to produce an exact, physically relevant long-time asymptotics (exponent and prefactor) of the core-hole Green function. We also obtained new results for the nonequilibrium Green function of the transient conduction electron and for the absorption amplitude very close to the edge. What remains to do is to extend the Wiener-Hopf approach to be able to determine quasicritical behavior of the absorption amplitude and to assess effects of bound states to continue the results to strong coupling.
ACKNOWLEDGMENTS

The author thanks Professors Kazuo Ohtaka for useful discussion and Kazuo Ueda for hospitality at ISSP of the University of Tokyo where the paper was finished. Financial support of the Japan Society for the Promotion of Science under The JSPS Invitation Fellowship Program for Research in Japan is acknowledged. The paper was supported in part by the grant No. 202/95/0008 of the Grant Agency of the Czech Republic.

1 P. W. Anderson, Phys. Rev. Lett. 18, 1049 (1967)
2 G. D. Mahan, Phys. Rev. 163, 612 (1967)
3 B. Roulet, J. Gavoret, and P. Nozières, Phys. Rev. 178, 1072 (1969), P. Nozières, J. Gavoret, and B. Roulet, ibid 178, 1084 (1969)
4 P. Nozières and C. De Dominicis, Phys. Rev. 178, 1097 (1969)
5 K. Ohtaka and Y. Tanabe, Rev. Mod. Phys. 62, 929 (1990)
6 M. Combescot and P. Nozières, J. Phys. (Paris) 32, 913 (1971)
7 W. J. Pardee and G. D. Mahan, Phys. Lett. 45A, 117 (1973)
8 G. D. Mahan, Phys. Rev. B21, 1421 (1980)
9 D. R. Penn, M. S. Girvin and G. D. Mahan, Phys. Rev. B24, 6971 (1981),
10 G. D. Mahan, Phys. Rev. B25, 5021 (1982)
11 K. Ohtaka and Y. Tanabe, Phys. Rev. B28, 6833 (1983)
12 K. Ohtaka and Y. Tanabe, Phys. Rev. B30, 4235 (1984)
13 Y. Tanabe and K. Ohtaka, Phys. Rev. B29, 1653 (1984)
14 J. Friedel, Comments on Solid State Phys. 2, 21 (1969)
15 N. Rivier and E. Simanek, Phys. Rev. Lett. 26, 435 (1970)
16 D. R. Hamann, Phys. Rev. Lett. 26, 1030 (1970)
17 V. Janiš, Phys. Rev. B49, 1612 (1994)
18 V. Janiš, Acta Physica Slovaca 44, 391 (1994)
19 V. Janiš, Physica B, (1996), to appear

It is necessary to stress that Eq. (8) in this form holds only for times \( t_1, t_2 \in [0, T] \). Although mathematically we can obtain a solution for negative times too, this solution has no physical meaning.

21 V. I. Smirnov, A Course of Higher Mathematics, (Addison-Wesley, Reading, Mass. 1965), Vol. IV
22 H. Hochstadt, Integral Equations, (Wiley, New York 1973), Chap. 5
23 G. Rickayzen, Green's Functions and Condensed Matter, (Academic Press, London 1984)
24 Y. Tanabe and K. Ohtaka, Phys. Rev. B32, 2036 (1985)
25 F. G. Fumi, Philos. Mag. 46, 1007 (1955)

The exact expression for \( \xi \) is \( \delta(0) \ln(\pi \xi) = \lim_{\rho \to 0} \left\{ \int_0^\infty \frac{d\omega}{\omega(\delta(\omega) + \delta(0) \ln(\rho}} \right\} \). This effective bandwidth enables to define a small parameter \((\xi T)^{-1}\) controlling an expansion around the Wiener-Hopf solution at \( T = \infty \).