Axial Monopoles, Quantization of Electric Charge and Dynamical Discreteness of Space-Time

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Abstract
In the present contribution we show that the introduction of a conserved axial current in electrodynamics can explain the quantization of electric charge, preserving parity conservation, and introduces a dynamical discreteness into space-time.

1 - Introduction
In 1931 Dirac proposes an electromagnetic theory with magnetic monopoles[1], whose appeal is mainly connected to the possibility of explaining the quantization of the electric charge. In spite of this undeniable theoretical appeal, in Dirac’s theory one is faced with a symmetry problem: the terms responsible for the magnetic monopole in the generalized Maxwell’s equations violate their symmetry under space and time reversal.

In this work we propose the introduction of a new current, namely an axial electromagnetic current which presents the following differences as compared to previously proposed vector magnetic current:

a) the resulting theory preserves space and time inversion invariance;

b) besides the usual conservation of the vector electromagnetic current, we also have the conservation of an axial current;

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c) besides the charge quantization, we can obtain a dynamical discreteness of space-time;

2 - Axial photons and the axial electromagnetic current

We start with the generalized definition of the electromagnetic field tensor

\[ F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} + \epsilon_{\mu\nu\alpha\beta} \partial^{\alpha} B^{\beta} \]  

(1)

where \( B^{\mu} \) represents a new gauge field\(^2\). Maxwell’s equations for the fields \( A^{\mu} \) and \( B^{\mu} \) in Lorenz’s gauge (\( \partial^{\mu} A_{\mu} = \partial^{\mu} B_{\mu} = 0 \)) become

\[ \partial^{\nu} F^{\nu\mu} = \square A_{\mu} = j_{\mu} \]  

(2)

\[ \partial^{\nu} F^{\dagger\nu\mu} = \square B_{\mu} = g_{\mu} \]  

(3)

where \( F^{\dagger\nu\mu} \) corresponds to \( F_{\nu\mu} \)’s dual tensor.

The quantity \( F_{\mu\nu} \) in (1) is a tensor; \( \epsilon_{\mu\nu\alpha\beta} \) is a pseudo-tensor and therefore the field \( B_{\mu} \) must be a pseudo-vector or an axial field. From the point of view of quantum theory the field \( B_{\mu} \) represents photon-like particles except for \( P, T \) and \( C \) parities. In other words, axial photons. From this it follows that (3) is not invariant under time and space reversal, unless \( g^{\mu} \) is also a pseudo-vector. For this reason, we shall introduce the axial electromagnetic current given by

\[ g_{\mu} = -g \bar{\psi} \gamma_{\mu} \gamma_{5} \psi \]  

(4)

where \( \psi \) represents a spin 1/2 particle (axial monopole) with axial charge \( g \).

Since \( F^{\dagger\nu\mu} \) is antisymmetric one gets from (3)

\[ \partial^{\mu} g_{\mu} = 0 \]  

(5)

which means axial current conservation and therefore massless axial monopoles.
3 - Charge quantization

Let us analyze the compatibility of the axial electromagnetic current with charge quantization.

We consider the gauge-invariant wave function of a charged particle\cite{2} moving in the presence of the axial monopole’s electromagnetic field,

\[
\Phi_e(x, P') = \Phi_e(x, P) \exp \left[ -\frac{ie}{2} \int_S F_{\mu\nu} d\sigma_{\mu\nu} \right] \quad (6)
\]

\( S \) being any surface with contour \( P' - P \). Due to the arbitrariness of the surface \( S \) we can write

\[
\Phi_e(x, P) \exp \left[ -\frac{ie}{2} \int_S F_{\mu\nu} d\sigma_{\mu\nu} \right] = \Phi_e(x, P) \exp \left[ -\frac{ie}{2} \int_{S'} F_{\mu\nu} d\sigma_{\mu\nu} \right] \quad (7)
\]

which leads to the condition

\[
\exp \left[ -\frac{ie}{2} \oint_{S-S'} F_{\mu\nu} d\sigma_{\mu\nu} \right] = 1 \quad (8)
\]

or equivalently to

\[
\exp \left[ -ie \int_V \partial^\nu F_{\nu\mu} dV^\mu \right] = 1 \quad (9)
\]

where \( V \) is the volume involved by the closed surface \( S - S' \).

Using (3), we have

\[
\exp \left[ -ie \int_V g_{\mu} dV^\mu \right] = 1 \quad (10)
\]

which gives

\[
Q_V \equiv \int_V g_{\mu} dV^\mu = \frac{2\pi n}{e} \quad (11)
\]

\( n \) being any integer.

Using our definition of \( g_{\mu} (4) \), we get
\[ Q_V = \int_V \left( -g \bar{\psi} \gamma_\mu \gamma_5 \psi \right) dV^\mu \] (12)

As \( Q_V \) is a Lorentz scalar, we can perform the calculation in a convenient reference frame. Taking the axial monopole’s frame (we can do that formally, even axial monopole being massless) and using the standard representation for Dirac’s spinor

\[ \psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix} = \begin{pmatrix} \phi \\ 0 \end{pmatrix} \] (13)

we obtain

\[ Q_V = g \int_V (\phi \sigma_i \phi) dV^i \] (14)

where \( \sigma_i \) corresponds to the \( i^{th} \) Pauli matrix.

Taking now the axial monopole polarization axis in the direction of charge’s velocity (positive z-axis, say) we get

\[ Q_V = g \int \phi \phi dxdy dt \] (15)

Since the axial monopole must be massless, the charge’s velocity relative to it has to be necessarily 1, the velocity of light. Thus \( dt = dz \) and (15) leads to

\[ Q_V = g \int \phi \phi dx dy dz = g \] (16)

Equations (11) and (16) give

\[ \frac{eg}{2\pi} = n \] (17)

The main feature of this condition is that it does not depend on the distance between the electric charge and the axial monopole. It implies in charge quantization, in the same way of Dirac’s charge quantization condition[1].
4 - Axial monopole’s mass generation

As we have shown (see (5)), in the present context the axial monopole is necessarily massless, oppositely to the massive gauge solitonic description of magnetic monopoles[3]. Let us see what happens if we circumvent such restriction.

We shall do that through a dynamical mass generation mechanism, by introducing a Higgs scalar field with nonvanishing vacuum expectation value[4]. The Yukawa coupling between the Higgs field and the axial monopole will generate a mass term for the latter, but preserving the axial current conservation (5).

The free lagrangian for the massless axial monopole is

\[ L_0 = i \bar{\psi} \partial_\mu \gamma^\mu \psi \]  

This lagrangian is invariant under the \( U_A(1) \) transformations defined by

\[ U_A(1) : \psi \rightarrow e^{i\alpha \gamma_5} \psi \]

and this invariance leads to the axial current conservation (5).

Now, we add to this free lagrangian the Higgs and Yukawa terms, to obtain

\[ L = L_0 + L_{\text{Higgs}} - G \bar{\psi} L \phi_H \psi_R - G \bar{\psi}_R \phi_H^\dagger \psi_L \]  

where \( \phi_H \) stands for the Higgs scalar field, \( \psi_L \) and \( \psi_R \) are, respectively, the left and right component of \( \psi \) and \( G \) is the Yukawa coupling constant.

The lagrangian (20) leads to a massive Dirac equation for the axial monopole wave function \( \psi \), with a mass term given by

\[ M = GV \]

with \( V \) standing for the vacuum expectation value of the Higgs field.
It is easy to see that $\mathcal{L}$ is invariant under $U_A(1)$ transformations, since they transform the Higgs field as

$$\phi_H \rightarrow e^{-i\alpha} \phi_H$$

(22)

Using Noether’s theorem we can obtain the conserved current associated to this invariance. It is precisely our axial current (4).

5 - Dynamical discreteness of space-time

Let us investigate the consequences of the mass generation on the electric charge quantization condition.

If the axial monopole is not massless, the charge’s velocity relative to it, $v$, is not necessarily 1. Now, $dt = dz/v$ and from (15) we obtain

$$Q_V = \frac{e}{v} \int \phi^\dagger \phi \, dx \, dy \, dz = \frac{g}{v}$$

(23)

Inserting (23) in (11) we have, rather than (17), the condition

$$\frac{eg}{2\pi v} = n$$

(24)

that, again, is independent on the distance between the electric charge and the axial monopole.

The above relation can be satisfied if we simultaneously fulfill

$$\frac{eg}{2\pi} = n_0$$

(25)

and

$$v = \frac{n_0}{n}$$

(26)

with

$$n = n_0, n_0 + 1, n_0 + 2...$$

(27)
$n_0$ being an integer fixed by the values of $e$ and $g$.

Equation (25) is the charge quantization condition (17), already derived in the massless case. It can be formally obtained from (24) if we consider the limit in which the mass of the particle carrying electric charge vanishes. Or, in another way, if we “switch off” the axial monopole mass, taking the false Higgs vacuum, in which $V = 0$. Physically we do not expect charge quantization to depend on the mass of the particles or on any mass generation mechanism. We shall therefore assume (25) -- (27) as the only physical solution of (24).

Condition (26) restricts the values of charge’s velocity to rational numbers, a result integrated in the theories of discrete space-time\textsuperscript{[5]}. Besides, these rational values form a discrete sequence given by (27). For sufficiently high $n_0$, this sequence tends to a continuum, except for velocities very near 1, the light’s velocity.

If we consider a massive charged particle, equations (26) and (27) lead to an upper limit for the particle’s velocity, given by

$$v_0 = \frac{n_0}{n_0 + 1} < 1$$

since for a massive particle it is impossible to have $v = 1$. For $n_0 \gg 1$ the limit $v_0$ is very near 1.

This limitation of $v$ leads to upper limits for $p$ and $E$, the momentum and energy of such particle. For $n_0 \gg 1$ these upper limits are

$$p_0 \approx E_0 \approx m(n_0/2)^{\frac{1}{2}}$$

which are proportional to the particle’s mass, $m$.

The limitation of the energy-momentum space of the particle leads, through the uncertainty principle, to a discreteness of its space-time, with a fundamental length given by
The discreteness of space-time here has a dynamical nature, opposed to usual theories of discrete space-time\cite{6} where it is purely kinematical in origin. Here a fundamental length arises owing to the interaction between the charged particle and the axial monopole. Besides, the larger the electric charge’s mass the smaller the fundamental length $a$, such that in the classical limit ($m \gg 0$) space-time will tend to a continuum. This kind of discreteness we shall call dynamical discreteness.

The experimental upper limit $a < 10^{-16} \text{cm}$ for the fundamental length of space-time gives a lower limit for $n_0$. Inserting such limit in (30) and using for $m$ the electron’s mass, we get

\[ n_0 > 2 \times 10^6 \gg 1 \]  \hspace{1cm} (31)

Now, from (25) and using for $e$ the electron’s charge, we obtain a very high lower limit for the axial charge $g$

\[ g > 10^9 \]  \hspace{1cm} (32)

6 - Conclusion

The introduction of the concept of axial monopoles opens up several interesting theoretical perspectives. Well known symmetries in nature, such as space and time reversal, are preserved and a new symmetry, namely the $U_A(1)$ symmetry, arises in the context of electrodynamics. Charge quantization can be obtained and, in the massive case, it is shown to be intimately connected to a dynamical discreteness of space-time.

Now an important remark is in order: what can we say about the observation of such entity? Firstly we can say nothing definite about its mass, since the values for $G$ and $V$ in (21) are unknown. However, what is known
is that producing massive pairs with large opposite coupling constants (see (32)) may be a very difficult experimental task. Moreover, the coupling of the axial monopole to the electromagnetic field is not of vector character, but axial (see (3) and (4)). This means in particular that its production and detection will be directly associated with polarization conditions, which will render its observation non trivial, even in the massless case.

References

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