ON THE UNRAMIFIED PRINCIPAL SERIES OF GL(3) OVER NON-ARCHIMEDEAN LOCAL FIELDS

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Abstract. Let $F$ be a non-archimedean local field and let $\mathcal{O}$ be its ring of integers. We give a complete description of the irreducible constituents of the restriction of the unramified principal series representations of $GL_3(F)$ to $GL_3(\mathcal{O})$.

1. Introduction

1.1. Overview. Let $F$ be a non-archimedean local field with ring of integers $\mathcal{O}$, maximal ideal $\mathcal{P}$ and let $\pi$ be a fixed uniformizer. Our interest in this paper is in the restriction of unramified principal series representations of $GL_n(F)$ to its maximal compact subgroup $GL_n(\mathcal{O})$. More precisely, let $B$ be a Borel subgroup of $GL_n$ and let $T$ be a maximal torus contained in $B$. Concretely, we may take the group of upper triangular matrices and the group of diagonal matrices, respectively. Any linear character $\chi : T(F) \to \mathbb{C}^\times$ can be inflated to a linear character of $B(F)$, still denoted $\chi$. The principal series representation corresponding to $\chi$ is the induced representation $\text{Ind}_{B(F)}^{GL_n(F)}(\chi)$ of $GL_n(F)$ on the space of continuous functions

$$V_\chi = \{ f \in C(GL_n(F)) \mid f(gb) = \chi(b)\|b\|^{1/2}f(g), \text{ for all } g \in GL_n(F), b \in B(F) \},$$

where $b \mapsto \|b\|$ is the modular character of $B(F)$. We assume that the representation is unramified, namely, that the restriction of $\chi$ to $T(\mathcal{O})$ is trivial, and focus on the decomposition to irreducible constituents of its restriction to $GL_n(\mathcal{O})$. By Frobenius reciprocity this restriction is isomorphic to $\text{Ind}_{B(\mathcal{O})}^{GL_n(\mathcal{O})}(1)$. The case $n = 2$ was fully treated in [2]. Partial results for the case $n = 3$ were obtained in [1] and it is the goal of this paper to give a complete description in this case. Further results on the restriction of principal series representations of $GL_n$ to the maximal compact subgroup can be found in [3].

1.2. The unramified principal series of $GL(3)$. We first describe and reformulate several results concerning $GL_3$ which were obtained by Campbell and Nevins in [1]. Let $G = GL_3(\mathcal{O})$, let $B$ be its subgroup of upper triangular matrices and let $V = \text{Ind}_B^G(\mathbf{1})$. For $\ell \in \mathbb{N}$, let $K^\ell$ denote the $\ell$th principal congruence subgroup of $G$, namely, the kernel of the canonical map from $G$ to $GL_3(\mathcal{O}/\mathcal{P}^\ell)$. Being normal subgroups of $G$, the groups $K^\ell$ give rise to a coarse decomposition of $V$ into $G$-invariant spaces $V \cong \oplus_{\ell=1}^\infty V^{K^\ell}/V^{K^{\ell-1}}$. A finer decomposition of these spaces, yet not to irreducible ones, is obtained by considering certain ‘parabolic’ subgroups of $G$ which are defined as follows. Let

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\(\mathbb{N}_0\) stand for \(\mathbb{N} \cup \{0\}\) endowed with the natural ordering and consider \(\mathbb{N}_0^3\) with the product ordering, i.e. \(c \preceq d\) if and only if \(c_i \leq d_i\) for \(i = 1, 2, 3\). Let \(C\) be the cone
\[
C = \{(c_1, c_2, c_3) \in \mathbb{N}_0^3 \mid c_1, c_2 \leq c_3 \leq c_1 + c_2\}.
\]
To an element \(c = (c_1, c_2, c_3) \in C\) we associate the compact open subgroup of \(G\)
\[
P_c = \begin{bmatrix}
0 & 0 & 0 \\
p_{c_1} & 0 & 0 \\
p_{c_3} & p_{c_2} & 0
\end{bmatrix} \cap G,
\]
which contains \(B\). The defining inequalities of \(C\) ensure that \(P_c\) is indeed a group. Let \(C\) be the cone
\[
C = \{(c_1, c_2, c_3) \in \mathbb{N}_0^3 \mid c_1, c_2 \leq c_3 \leq c_1 + c_2\}.
\]
We have
\[
V_c = U_c/\sum_{d< c} U_d, \quad (c \in C).
\]
The mutual relations between the representations \(V_c\) are completely determined in [1]. To describe them we introduce the auxiliary functions \(\lambda, \kappa, \mu : C \to \mathbb{N}_0\). For \(c \in C\) let
\[
\lambda(c) = c_3, \\
\kappa(c) = c_1 + c_2 - c_3, \\
\mu(c) = \min\{c_1 + c_2 - c_3, c_3 - c_1, c_3 - c_2\}.
\]
With these we have
\[(CN1)\] Let \(c, d \in C\). Then
\[
V_c \simeq V_d \text{ if and only if } \lambda(c) = \lambda(d), \kappa(c) = \kappa(d), \mu(c) = \mu(d) \text{ and } c = d \text{ if } \kappa(c) > \mu(c).
\]
Statement \((CN1)\) is a reformulation of several results from [1] which requires a proof (see Proposition 6.2 below). Let \(C^\circ\) and \(\partial C\) denote the interior and boundary of \(C\), respectively. That is
\[
C^\circ = \{c = (c_1, c_2, c_3) \mid c_1, c_2 < c_3 < c_1 + c_2\}, \\
\partial C = C \setminus C^\circ.
\]
Concerning the irreducibility of the representations \(V_c\) one has
\[(CN2)\] The representation \(V_c\) is irreducible if and only if \(c \in \partial C\).
Statement \((CN2)\) is proved in [1, Theorems 6.1, 7.1 and 8.1]. The complete description of the double coset spaces \(P_c \setminus G/P_d\) \((c, d \in C)\), obtained in [1], does not lend itself to decompose the representations \(V_c\) \((c \in C^\circ)\), which comprise most of the constituents of \(V\). However, crucial for us
is the following observation made in [1, §8]. Let \( \rho = (1, 1, 1) \). Let \( m \in \mathbb{N} \) and \( c \in C^o \cap C + m \rho \). For all \( d \in [c - m \rho, c] \) let

\[
\begin{align*}
U^m_d &= \text{Ind}_{F_e}^{P_c \cdot m \rho}(1), \\
V^m_c &= U^m_c / \sum_{c - m \rho < d < c} U^m_d.
\end{align*}
\]

(1.2)

It follows that \( V_c = \text{Ind}_{F_e}^{G}(V^m_c) \). For the special case \( m = \mu(c) \) the relations between \( V_c \) and \( V^m_c \) become very tight.

\((\text{CN3})\) Let \( c \in C^o \). Then \( \dim \text{End}_G(V_c) = \dim \text{End}_{P_c \cdot m(c)}(V^\mu_c) \). In particular, the irreducible constituents of \( V_c \) are induced from the irreducible constituents of \( V^\mu_c \).

We remark that any linear combination of the invariants \( \lambda, \kappa \) and \( \mu \) above could be used throughout. We made this particular choices as they have natural interpretations: \( \lambda(c) \) is the level of the representation \( V_c \), and both \( \kappa(c), \mu(c) \) turn out to be the parameters which control the decomposition of \( V_c \) \( (c \in C^o) \).

1.3. Description of results and organization of the paper. The main result of this paper is a complete description of the irreducible constituents of the representations \( V_c \) \( (c \in C^o) \). These are described in terms of induced linear characters of certain subquotients of the groups \( P_c \). Consequently we obtain a complete description of the irreducibles in \( V \). In Section §2 we define twisted Heisenberg groups and their toral extensions. As it turns out, these are quotients of \( P_c \) \( (c \in C^o) \) which carry the complete information on the representations \( V_c \) but are much more accessible. In Section §3 we define and analyze in detail specific multiplicity free representations of toral extensions of twisted Heisenberg groups and in Section §4 we show that the \( V^m_c \)'s for \( m \in \mathbb{N} \) and \( \mu(c) \geq m \) are subrepresentations of these multiplicity free representations. It remains to identify what are the components of \( V^m_c \) and this is achieved in Section §5. We end the paper by computing the multiplicities and dimensions of the irreducible constituents in \( V \). Notably, these depend only on the residue field \( \mathbb{O}/\mathfrak{P} \), and the dependence is through substitution in universal polynomials defined over \( \mathbb{Z} \).

1.4. Notation, conventions and tools. For \( m \in \mathbb{N} \) we write \( \mathbb{O}_m \) for the finite quotient \( \mathbb{O}/\mathfrak{P}^m \).

We use \( \text{val}(\cdot) \) to denote the valuation on \( \mathbb{O} \). It is convenient to use the same symbol for the finite quotients with the convention that \( \text{val}(0) = m \) for \( 0 \in \mathbb{O}_m \). When chances for confusion are slim we use the same notation for elements or subsets of \( \mathbb{O} \) and their respective images in \( \mathbb{O}_m \). For a group \( G \) and elements \( g, x \in G \) we write \( g x = g x g^{-1} \) for the left conjugation action and \( x^g = g^{-1} x g \) for right conjugation. If \( H \subset G \) we write \( g H = g H g^{-1} \). If \( \chi \) is a character of \( H \) we write \( g \chi \) for the character of \( g H \) defined by \( g \chi(x) = \chi(g^{-1} x g) \) for all \( x \in g H \). Throughout we use fairly standard tools from representation theory such as Mackey and Clifford theories and the following criterion for the existence of non-trivial intertwining operators. If \( \chi_i \) are linear characters of subgroups \( H_i \) of \( G \) we have the following realization of the intertwining operators

\[
\text{Hom}_G(\text{Ind}_{H_1}^{G}(\chi_1), \text{Ind}_{H_2}^{G}(\chi_2)) = \{f : G \to \mathbb{C} \mid f(h_2 g h_1) = \chi_2(h_2) f(g) \chi_1(h_1), \forall h_i \in H_i, \forall g \in G\},
\]

and an element \( g \in G \) supports a non-zero intertwining function if and only if \( \chi_1 = g \chi_2 \) on \( H_1 \cap g H_2 \).
2. Twisted Heisenberg groups and their toral extensions

To study the representation \( V^m_c \) we first need to analyze \( U^m_c \), which is the permutation representation of \( P_{c-m\rho} \) arising from its action on the right cosets \( P_{c-m\rho}/P_c \) for \( m \in \mathbb{N}, c \in C^o \cap C + m\rho \).

**Definition 2.1.** Let \( R \) be a commutative ring with identity and \( \delta \in R \). The \( \delta \)-twisted Heisenberg groups over \( R \), denoted \( H^\delta_R \), is the set of triples in \( R^3 \) endowed with the multiplication
\[
(x, y, z) \cdot (x', y', z') = (x + x', y + y', z + z' + \delta yx').
\]

A **toral extension of a \( \delta \)-twisted Heisenberg group** is the semidirect product \( H^\delta_R \rtimes (R^\times)^3 \) of \( (R^\times)^3 \) and \( H^\delta_R \), with the action
\[
(t_1, t_2, t_3) \cdot (x, y, z) = t_1^{-1}t_2x, t_2^{-1}t_3y, t_1^{-1}t_3z),
\]
for \( t = (t_1, t_2, t_3) \in (R^\times)^3 \) and \( (x, y, z) \in H^\delta_R \). We denote this semidirect product by \( B^\delta_R \).

The family \( \{H^\delta_R \mid \delta \in R\} \) interpolate between the standard Heisenberg group \( (\delta = 1) \) and the abelian group \( R^3 \) with addition \( (\delta = 0) \). The group \( B^1_R \) is isomorphic to the group of invertible triangular matrices over \( R \). It is convenient to keep the analogy with this special case and use a matrix presentation for \( B^1_R \):

\[
[(x, y, z), (t_1, t_2, t_3)] \mapsto \begin{bmatrix} 1 & x & 1 \\ z & y & 1 \\ \end{bmatrix} \circ \delta \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix},
\]

with multiplication defined by
\[
\begin{bmatrix} t_1 \\ x & t_2 \\ z & y & t_3 \end{bmatrix} \circ \delta \begin{bmatrix} t'_1 \\ x' & t'_2 \\ z' & y' & t'_3 \end{bmatrix} = \begin{bmatrix} t_1t'_1 \\ xt'_1 + t_2x' \\ zt_1 + \delta yx' + t_3z' \\ t_2t'_2 \\ yt'_2 + t_3y' \\ t_3t'_3 \end{bmatrix},
\]
which is almost the usual matrix multiplication except for the twist by \( \delta \) at the \((3,1)\)-entry. It will be useful in the sequel to have the following description of \( B^1_R \) modulo its center.

**Proposition 2.2.** Let \( R \) be a commutative ring with identity and \( \delta \in R \). Let
\[
E^\delta_R = R^2 \rtimes_\delta \begin{bmatrix} R^\times & R^\times \\ R & R^\times \end{bmatrix},
\]
with the action
\[
\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \circ \delta \begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} \alpha x \\ \beta x\delta + \gamma z \end{bmatrix}.
\]

Then \( B^\delta_R / Z_{B^\delta_R} \simeq E^\delta_R \).

**Proof.** The explicit isomorphism is
\[
\phi : \begin{bmatrix} t_1 \\ x & t_2 \\ z & y & t_3 \end{bmatrix} \mapsto \begin{bmatrix} t_1^{-1}x \\ z \end{bmatrix} \rtimes \begin{bmatrix} t_2 \\ y \\ t_3 \end{bmatrix}.
\]
\[\square\]
Let \( R = \mathcal{O}/\mathfrak{p}^m = \mathcal{O}_m \) and in this case denote the groups above \( H_m^\delta, B_m^\delta \) and \( E^\delta \). Let \( T_m \) stand for the subgroup of diagonal matrices in \( B_m^\delta \). The relevance of the groups \( B_m^\delta \) and \( E^\delta \) to the permutation representations \( U^m_c \) comes from Theorem 2.4.

For \( c \in \mathcal{C} \) let

\[
N_c = \begin{bmatrix} 1 & \mathfrak{p}^{c_1} & 1 \\ \mathfrak{p}^{c_2} & 1 & 1 \end{bmatrix}, \quad N^+ = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad T^m = T \cap K^m.
\]

**Lemma 2.3.** Let \( c = (c_1, c_2, c_3) \in \mathcal{C} \) and let \( m \in \mathbb{N} \) such that \( \mu(c) \geq m \). Let \( \delta = \pi^{\kappa(c-mp)} = \pi^{c_1+c_2-c_3-m} \). Then the map \( \eta : P_{c-mp} \rightarrow (B_m^\delta, o_\delta) \) defined by

\[
\eta : \begin{bmatrix} \pi^{c_1-m}x & * & * \\ \pi^{c_3-m}z \pi^{c_2-m}y & t_2 & t_3 \end{bmatrix} \mapsto \begin{bmatrix} t_1 \\ x \\ z \end{bmatrix} \mod \mathfrak{p}^m,
\]

is a surjective homomorphism with kernel \( N_cT^mN^+ \).

**Proof.** First, note that the condition \( \mu(c) \geq m \) implies that \( c - mp \in \mathcal{C} \), hence \( P_{c-mp} \) is indeed a group. Clearly, \( \eta \) is surjective and a straightforward matrix multiplication shows that \( \eta \) is a homomorphism if and only if

(a) \( c_1 - m, c_2 - m, c_3 - m \geq m \), and
(b) \( c_3 - c_1, c_3 - c_2, c_1 + c_2 - c_3 \geq m \).

The condition (b) is equivalent to \( \mu(c) \geq m \) and (a) follows from (b). Finally, it is clear that \( N_cT^mN^+ \) is contained in the kernel of \( \eta \) and a short matrix computation shows equality. \( \square \)

**Theorem 2.4.** Let \( c = (c_1, c_2, c_3) \in \mathcal{C} \) and let \( m \in \mathbb{N} \) such that \( \mu(c) \geq m \). Let \( \delta = \pi^{\kappa(c-mp)} = \pi^{c_1+c_2-c_3-m} \). Then

(a) The action of \( P_{c-mp} \) on \( P_{c-mp}/P_c \) factors through \( B_m^\delta \) via \( \eta \).
(b) As \( B_m^\delta \)-spaces \( P_{c-mp}/P_c \simeq B_m^\delta/T_m \). In particular, we have

\[
U_c^m \simeq \text{Ind}_{T_m}^{B_m^\delta}(1),
\]

as \( B_m^\delta \)-representations.

Before proving Theorem 2.4 we set up notation and make some preparation. Let \( e_{ij} \) denote the matrix whose \((i,j)\)th entry equal one and the rest of entries are zero. For \( i \neq j \) and \( x \in R \) (a commutative ring) let

\[
u_{ij}(x) = I + xe_{ij}, \quad \text{(elementary unipotent)},
\]

\[
u_i(x) = I + (x - 1)e_{ii}, \quad \text{(elementary semisimple)}.
\]

The following are well known (and easily verified).

\[
u_{ij}(x)\nu_{kl}(y)\nu_{ij}(x)^{-1} = I + ye_{kl} + \begin{cases} 0, & \text{if } i \neq l, j \neq k; \\
-xye_{kj}, & \text{if } i = l, j \neq k; \\
ye_{ii}, & \text{if } i \neq l, j = k; \\
x(\mathbf{e}_{ll} - \mathbf{e}_{kk}) - x^2ye_{ij}, & \text{if } i = l, j = k.
\end{cases}
\]

for all \( x, y \in R, i \neq j \) and \( k \neq l \).
For all $x \in R$, $y \in R^\times$.

**Lemma 2.5.** For all $c \in C$ and $m \in \mathbb{N}$ with $\mu(c) \geq m$ the following hold.

1. $P_{c, m} = N_{c, m}TN^+$.
2. $N_c \triangleleft N_{c, m}$.
3. $[T^m, N_{c, m}] \subset N_c$.

**Proof.**

1. The map $N_{c, m} \times T \times N^+ \to P_{c, m}$ given by $(n, t, n^+) \mapsto ntn^+$ is a bijection.
2. Since both groups are generated by elementary unipotents \[2.7\] it is enough to check that $gng^{-1} \in N_c$ for elementary unipotents $g \in N_{c, m}$, $n \in N_c$. There are only two pairs of such elements which do not commute and for them we verify using \[2.8\] that

$$u_{21}(\pi^{c_1-m}x)u_{32}(\pi^{c_2}y)u_{21}(\pi^{c_1-m}x)^{-1} = I + \pi^{c_2}ye_{32} - \pi^{c_1+c_2-m}xye_{31} \in N_c,$$

$$u_{32}(\pi^{c_2-m}x)u_{21}(\pi^{c_1}y)u_{32}(\pi^{c_2-m}x)^{-1} = I + \pi^{c_1}ye_{21} + \pi^{c_1+c_2-m}xye_{31} \in N_c.$$

3. Follows immediately from \[2.9\].

**Proposition 2.6.** For all $c \in C$ and $m \in \mathbb{N}$ with $\mu(c) \geq m$ the following hold.

1. As $N_{c, m}$-spaces $P_{c, m}/P_c \simeq N_{c, m}TN^+/N_cT \simeq N_{c, m}/N_c$.
2. The group $N_cT^mN^+$ is contained in the kernel of the $P_{c, m}$-action on $P_{c, m}/P_c$.

**Proof.** (1) By Lemma \[2.5\], the group $N_{c, m}$ acts transitively on the left cosets of $P_c = N_cTN^+$ in $P_{c, m} = N_{c, m}TN^+$. The stabilizer of $P_c$ under this action is $N_c$. It follows that, as $N_{c, m}$-spaces, $P_{c, m}/P_c \simeq N_{c, m}/N_c$. This in particular implies that every element in $P_{c, m}/P_c$ can be represented as $nN_cTN^+$ for some $n \in N_{c, m}/N_c$.

(2) It is clear from Lemma \[2.5\] that both $T^m$ and $N_c$ act trivially on $P_{c, m}/P_c$. We show that $N^+$ acts trivially as well. We need to show that $h(nN_cTN^+) = nN_cTN^+$ for $h \in N^+$ and $n \in N_c$, or equivalently that $hn^{-1} \in N_cTN^+$. Since both groups $N_{c, m}$ and $N^+$ are generated by unipotents it is enough to check that

$$u_{ij}(x)u_{kl}(y)u_{ij}(x)^{-1} \in N_cTN^+,$$

for all $u_{ij}(x) \in N_{c, m}$ and $u_{kl}(y) \in N^+$. The possible indices are $(i, j) \in \{(2, 1), (3, 1), (3, 2)\}$ and $(k, l) \in \{(1, 2), (1, 3), (2, 3)\}$. This is straightforward using \[2.8\]. Thus $N_cT^mN^+$ is contained in the kernel.
Proof of Theorem 2.4. Using Lemma 2.5 and Proposition 2.6 we have a commutative diagram

\[
\begin{array}{cccccc}
N_c \times T^m \times N^+ & \longrightarrow & N_c T^m N^+ \\
\downarrow & & \downarrow \\
N_{c-m\rho} \times T \times N^+ & \longrightarrow & P_{c-m\rho} \\
\downarrow & & \downarrow \\
N_{c-m\rho}/N_c \times T_m \times \{1\} & \longrightarrow & P_{c-m\rho}/N_c T^m N^+ \\
\downarrow \downarrow & & \downarrow \downarrow \\
H^\delta_m \times T_m \times \{1\} & \longrightarrow & B^\delta_m,
\end{array}
\]

with vertical maps being embeddings or canonical epimorphisms and horizontal bijective maps given by multiplication. The identification of the last two rows is induced by the map \(\eta: P_{c-m\rho} \rightarrow (B^\delta_m, \circ_{\delta})\). Assertions (a) and (b) follow.

□

For \(r = (r_1, r_2, r_3) \in N_0^3\) with \(r_3 \leq r_1 + r_2\) let

\[
(2.10) \quad N^\delta_r = \begin{bmatrix} 1 & P^{r_1} & 1 \\ P^{r_3} & P^{r_2} & 1 \end{bmatrix} \mod P^m, \quad Q^\delta_r = \begin{bmatrix} O^\times & P^{r_1} & O^\times \\ O^{r_1} & P^{r_2} & O^\times \end{bmatrix} \mod P^m
\]

considered as a subgroup of \(B^\delta_m\). As \(N^\delta_r\) does not intersect the center of \(B^\delta_m\) we also let \(N^\delta_r\) denote its image in \(E^\delta\). Let \(\Theta\) denote the image of \(T_m\) in \(E^\delta\).

Corollary 2.7. For all \(c \in C, m \in \mathbb{N}\) with \(\mu(c) \geq m\) and \(d\) such that \(c - m\rho \leq d \leq c\) we have

\[
U^m_d \simeq \text{Ind}_{Q^\delta_d-c+m\rho}^{B^\delta_m} (1),
\]
as \(B^\delta_m\)-representations.

\[
U^m_d \simeq \text{Ind}_{N^\delta_d-c+m\rho}^{E^\delta} \Theta (1),
\]
as \(E^\delta\)-representations. In particular, \(U^m_c \simeq \text{Ind}_{\Theta}^{E^\delta} (1)\).

Proof. The first part follows from the proof of Theorem 2.4 observing that \(\eta(P_d) = Q^\delta_d-c+m\rho\). The second part follows by observing that the center of \(B^\delta_m\) acts trivially.

□

3. On the permutation representation \(\mathbb{C}[E^\delta/\Theta]\)

Let \(E^\delta = A \rtimes_{\delta} \Gamma\), with

\[
A = \begin{bmatrix} O^\times_m \\ O^\times_m \end{bmatrix},
\]

\[
\Gamma = \begin{bmatrix} O^\times_m & O^\times_m \\ O_m^\times & O^\times_m \end{bmatrix},
\]

and the action given by (2.4). Let \(\Theta\) stand for diagonal matrices in \(\Gamma\)

\[
\Theta = \begin{bmatrix} O^\times_m \\ O^\times_m \end{bmatrix}.
\]
The aim of this section is to construct and analyze certain subrepresentations of the permutation representation \( \text{Ind}_{\tilde{A}}^{E^\delta} (1) = \mathbb{C}[E^\delta / \Theta] \). To simplify notation we identify \( A \) and \( \Gamma \) with their images in \( E^\delta \). We start with the permutation representation of \( A \Theta \) on \( \mathbb{C}[A \Theta / \Theta] \). The group \( A \Theta \) is isomorphic to \( a \in A, \theta \in \Theta \). Let \( \psi : \mathcal{O}_m \to \mathbb{C}^\times \) be a character which establishes the self duality \( \mathcal{O}_m \simeq \hat{\mathcal{O}}_m \), that is, every character of \( \mathcal{O}_m \) is of the form \( \psi_\xi : x \mapsto \psi(\xi x) \) for some \( \xi \in \mathcal{O}_m \). The characters of \( A \) are parameterized by pairs \( a = (\xi, \zeta) \in \mathcal{O}_m^2 \) which correspond to the character \( \varphi_{\xi, \zeta} : (x, z) \mapsto \psi(\xi x + \zeta z) \).

Using the induced action of \( \Theta \) on \( A \) we start with the permutation representation of \( \hat{\Theta} \) on \( A \). We start with the permutation representation of \( A \Theta \). We now induce the representations \( W \mathcal{O} \) to irreducible \( \hat{\Theta} \)-fixed vector and whose restriction to \( A \) is the sum of the characters in the orbit \( \Omega_{ij} \). In particular, \( W \mathcal{O} \simeq \text{Ind}_{A \Theta_{ij}} (\varphi') \) for every \( \varphi \in \Omega_{ij} \), and

\[
\text{dim } W \mathcal{O} = |\Omega_{ij}| = |\pi^i \mathcal{O}_m^\times| |\pi^j \mathcal{O}_m^\times| = q^{2m-2-i-j}(q-1)^2.
\]

We now induce the representations \( W \mathcal{O} \) further to \( E^\delta \), and define

\[
\hat{W} \mathcal{O} = \text{Ind}_{A \Theta}^{E^\delta} (W \mathcal{O}).
\]

Our next goal is to find the irreducible components of \( \hat{W} \mathcal{O} \).

**Lemma 3.1.** There exists decomposition to irreducible \( A \Theta \)-representations

\[
\text{Ind}_{A \Theta}^{E^\delta} (1) = \bigoplus_{0 \leq i, j \leq m} W_{ij},
\]

where \( W_{ij} \) is the unique irreducible representation of \( A \Theta \) which contains a \( \Theta \)-fixed vector and whose restriction to \( A \) is the sum of the characters in the orbit \( \Omega_{ij} \). In particular, \( W_{ij} \simeq \text{Ind}_{A \Theta_{ij}} (\varphi') \) for every \( \varphi \in \Omega_{ij} \), and

\[
(3.2)
\]

We now induce the representations \( W \mathcal{O} \) further to \( E^\delta \), and define

\[
\hat{W} \mathcal{O} = \text{Ind}_{A \Theta}^{E^\delta} (W \mathcal{O}).
\]

Our next goal is to find the irreducible components of \( \hat{W} \).

**Lemma 3.2.** Let \( \varphi_{\xi, \zeta} \in \Omega_{00} \) and denote \( \epsilon = \delta \xi^{-1} \zeta \). The stabilizer of \( \varphi_{\xi, \zeta} \) in \( \Gamma \) is equal to \( \Delta_m \), where

(a) If \( \delta \in \mathcal{O}_m^\times \)

\[
\Delta_m = \{ [\alpha, \beta] | \alpha \in \mathcal{O}_m^\times, \beta = \epsilon^{-1}(1 - \alpha) \} \simeq \mathcal{O}_m^\times.
\]

(b) If \( \delta \in \pi \mathcal{O}_m \)

\[
\Delta_m = \{ [\alpha, \beta] | \beta \in \mathcal{O}_m, \alpha = 1 - \epsilon \beta \} \simeq \mathcal{O}_m.
\]
Proof. Using the action (2.4) we have
\[
\begin{bmatrix} \alpha & \gamma \\ \beta & \gamma \end{bmatrix} \varphi_{\xi, \zeta} = \varphi_{\alpha \xi + \delta \beta \zeta, \gamma \zeta},
\]
hence the stabilizer consists of elements \([\alpha \beta \gamma]\) whose entries solve the equations
\[
\xi = \alpha \xi + \delta \beta \zeta,
\]
\[
\zeta = \gamma \zeta.
\]
The solution depends on \(\delta\) being a unit or not and is given by cases (a) and (b).

Theorem 3.3. The representation \(\tilde{W}_{00}\) of \(E^\delta\) is multiplicity free with equidimensional irreducible constituents. More precisely, \(\tilde{W}_{00}\) has a decomposition
\[
\tilde{W}_{00} \simeq \bigoplus_{\sigma \in \Sigma} L_{\sigma},
\]
where
\[
\Sigma = \begin{cases} 
\hat{\mathcal{O}}_{m}, & \text{if } \delta \in \mathcal{O}_{m}; \\
\hat{\mathcal{O}}_{m}, & \text{if } \delta \in \pi \mathcal{O}_{m}.
\end{cases}
\]
The representations \(L_{\sigma}\) are irreducible, non-equivalent and are induced from the one-dimensional extensions of \(\varphi = \varphi_{1,1}\) to \(\text{Stab}_{E^\delta}(\varphi) = A \Delta^\delta_m\). In particular, their dimension is
\[
\dim L_{\sigma} = [E^\delta : \text{Stab}_{E^\delta}(\varphi)] = \begin{cases} 
q^{2m-1}(q-1), & \text{if } \delta \in \mathcal{O}_{m}; \\
q^{2m-2}(q-1)^2, & \text{if } \delta \in \pi \mathcal{O}_{m}.
\end{cases}
\]

Proof. Using Lemma 3.1 we have \(\tilde{W}_{00} \simeq \text{Ind}_{E^\delta}^\mathcal{O}_{m}(\varphi)\) for any \(\varphi = \varphi_{\xi, \zeta}\) with \(\xi, \zeta \in \mathcal{O}_{m}\) which we may specify as \(\xi = \zeta = 1\). By Lemma 3.2 we have \(\text{Stab}_{E^\delta}(\varphi) = A \Delta^\delta_m\). Being a semidirect product, the characters of \(A \Delta^\delta_m\) which extend \(\varphi\) are of the form \(\varphi_{\sigma}\) where \(\sigma \in \Sigma = \Delta^\delta_m\). It follows that

\[
L_{\sigma} = \text{Ind}_{A \Delta^\delta_m}^{E^\delta}(\varphi \sigma), \quad (\sigma \in \Sigma),
\]
are irreducible and distinct. By Clifford’s theorem, the representations \(L_{\sigma}\) are precisely the irreducible constituents of \(\text{Ind}_{A}^{E^\delta}(\varphi)\), perhaps with multiplicities. However, their direct sum is of dimension \([E^\delta : A]\), therefore, they occur with multiplicity one.

4. AN EMBEDDING

The main result in this section is the following.

Theorem 4.1. For all \(c \in C\) and \(m \in \mathbb{N}\) with \(\mu(c) \geq m\), the representation \(V_c^m\) is a subrepresentation of \(\tilde{W}_{00}\). In particular, \(V_c^m\) is multiplicity free.

Later on we shall prove a finer result (Theorem 5.1) in which the precise decomposition of \(V_c^m\) is given. Nevertheless, the arguments needed to prove Theorem 4.1 are softer and seem to be in a better position towards generalization to \(\text{GL}_n\) with \(n > 3\).

Proposition 4.2. Let \(c \in C\) and let \(m \in \mathbb{N}\) such that \(\mu(c) \geq m\). Then, for \(0 \leq i, j \leq m\), we have \(\tilde{W}_{ij} \subset U_d^m\) if one (or both) of the following hold.
(a) $i > 0$ and $d = (c_1 - 1, c_2, c_3)$.
(b) $j > 0$ and $d = (c_1, c_2, c_3 - 1)$.

Proof. Let $i, j$ and $d$ satisfy (a) or (b). Note that the assumptions on $c$ and $m$ imply that $d \in \mathbb{C}$ and that $c - mp < d < c$. Let $r = d - (c - mp)$. By Lemma 3.1 and Corollary 2.7 we have

$$\widetilde{W}_{ij} \cong \text{Ind}_{A^0}^{E^0}(W_{ij}) \cong \text{Ind}_{A^0}^{E^0} \left(\text{Ind}_{A^0}^{A^0} (\varphi'_\xi, \zeta)\right);$$

$$U^m_d \cong \text{Ind}_{A^0}^{E^0} \left(\text{Ind}_{N^0_f\Theta}^{A^0} (1)\right),$$

for some $(\xi, \zeta) \in \pi^i \mathcal{O}_m^\times \times \pi^j \mathcal{O}_m^\times$. Therefore, it is enough to show that $W_{ij} \subset \text{Ind}_{N^0_f\Theta}^{A^0} (1)$. Since $W_{ij}$ is irreducible the latter is equivalent to

$$1 = \dim_{\mathbb{C}} \text{Hom}_A \left(\text{Ind}_{A^0}^{A^0} (\varphi'_\xi, \zeta), \text{Ind}_{N^0_f\Theta}^{A^0} (1)\right)$$

$$= \dim_{\mathbb{C}} \left\{ f : A\Theta \to \mathbb{C} \mid f(h_1 g h_2) = 1(h_1) f(g) \varphi'_\xi, \zeta (h_2), \text{ for all } h_1 \in N^0_f\Theta, g \in A\Theta, h_2 \in A\Theta_{ij}\right\}.$$ 

Since $(A\Theta_{ij}) (N^0_f\Theta) = A\Theta$, the only candidate for the support of a non-zero intertwining function is the identity element. To see that such non-zero function exists we need to check that $1 = \varphi'_\xi, \zeta$ on $A\Theta_{ij} \cap N^0_f\Theta$. Indeed, for a general element in this intersection we have

$$\varphi'_\xi, \zeta \left(\begin{bmatrix} x \\ 0 \end{bmatrix} \right) \times \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \psi(\xi x) = 1,$$

for case (a) of the proposition, since $x \in \pi^{m-1} \mathcal{O}_m$ and $\xi \in \pi \mathcal{O}_m$. Similarly

$$\varphi'_\xi, \zeta \left(\begin{bmatrix} 0 \\ z \end{bmatrix} \right) \times \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \psi(\zeta z) = 1,$$

for case (b) of the proposition, since $z \in \pi^{m-1} \mathcal{O}_m$ and $\zeta \in \pi \mathcal{O}_m$. \hfill \Box

Proof of Theorem 4.1. By Lemma 3.1 and (3.2) we have

$$U^m_c = \bigoplus_{0 \leq i, j \leq m} \tilde{W}_{ij},$$

and by Proposition 4.2

$$\sum_{c - mp < d < c} U^m_d \supset \sum_{(i, j) \neq (0, 0)} \tilde{W}_{i, j}.$$ 

These together imply that $V^m_c = U^m_c / \sum_{c - mp < d < c} U^m_d$ embeds in $\tilde{W}_{0, 0}$. \hfill \Box

5. A Decomposition

Combining Theorems 3.3 and 4.1 we obtain a list of candidate irreducible subrepresentations of $V^m_c$ for $c \in \mathbb{C}$ and $m \in \mathbb{N}$ with $\mu(c) \geq m$. In this section we pin down those irreducible representations that indeed occur. For $m \in \mathbb{N}$ let $\Delta^\delta_m$ be as in Lemma 3.2, namely, isomorphic to $\mathcal{O}_m^\times$ or $\mathcal{O}_m$ depending on whether $\delta$ is invertible or not, respectively. To simplify notation we choose such isomorphisms and identify $\Delta^\delta_m$ with either $\mathcal{O}_m^\times$ or $\mathcal{O}_m$. Set $\Delta^0_0$ to be the trivial group. Let $\iota : \Delta^\delta_m \to \Delta^\delta_{m-1}$ denote reduction modulo $\pi^{m-1}$. The map $\iota$ induces an embedding of characters
\[\iota^*: \hat{\Delta}^\delta_{m-1} \hookrightarrow \hat{\Delta}^\delta_m.\] The image consists of characters of \(\Delta^\delta_m\) which factor through \(\Delta^\delta_{m-1}\). Recall that for \(\varphi = \varphi_{1,1}\) we have \(\text{Stab}_{E^\delta}(\varphi) \simeq A \rtimes \Delta^\delta_m\), and that for \(\sigma \in \hat{\Delta}^\delta_m\)

\[L_\sigma = \text{Ind}^{E^\delta_{A\Delta^\delta_m}}_{A\Delta^\delta_m}(\varphi\sigma).\]

We also recall that

\[U^m_d \simeq \text{Ind}^{E^\delta}_{Q_{d-c+m\rho}}(1),\]

and that

\[Q^\delta_r = \begin{bmatrix} \mathcal{P}^r_1 & \mathcal{P}^r_3 \\ \mathcal{P}^r_2 & \mathcal{O}_m^{\times} \end{bmatrix}.\]

**Theorem 5.1.** For \(c \in C, m \in \mathbb{N}\) with \(\mu(c) \geq m\) and \(\sigma \in \hat{\Delta}^\delta_m\) we have

\[\text{Hom}_{E^\delta}(L_\sigma, V^m_c) = \begin{cases} 0, & \text{if } \sigma \in \iota^*(\hat{\Delta}^\delta_{m-1}); \\ 1, & \text{otherwise.} \end{cases}\]

**Proof.** We show that \(L_\sigma \leq \sum_{c-m\rho \leq d < c} U^m_d\) if and only if \(\sigma \in \iota^*(\hat{\Delta}^\delta_{m-1})\). This is equivalent to the validity of

\[J(S): \]\n
\[\begin{align*}
\text{Hom}_{E^\delta}(L_\sigma, U^m_d) & \neq \{0\} \text{ for some } d \in S \text{ if } \sigma \in \iota^*(\hat{\Delta}^\delta_{m-1}) \\
\text{Hom}_{E^\delta}(L_\sigma, U^m_d) & = \{0\} \text{ for all } d \in S \text{ if } \sigma \not\in \iota^*(\hat{\Delta}^\delta_{m-1})
\end{align*}\]

for \(S = \{d \mid c - m\rho \geq d < c\}\). Since each of the representations \(U^m_d\) \((d \in S)\) is contained in one of the three maximal representations indexed by \(d \in S' = \{c - e_1, c - e_2, c - e_3\}\), with \(\{e_i\}\) denoting the standard basis, it is enough to prove that \(J(S')\) holds. Using Corollary 2.7 and the definition of \(L_\sigma\) we have that for \(d \in S'\) and \(r = d - (c - m\rho)\)

\[
\text{Hom}_{E^\delta}(L_\sigma, U^m_d) = \text{Hom}_{E^\delta}\left(\text{Ind}^{E^\delta_{A\Delta^\delta_m}}_{A\Delta^\delta_m}(\varphi\sigma), \text{Ind}^{E^\delta}_{Q^\delta_r}(1)\right)
\]

\[= \begin{cases} f: E^\delta \to \mathbb{C} | f(h_1g_2h_2) = f(g)\varphi(\sigma), & \forall h_1 \in Q^\delta_r, \forall g \in E^\delta, \forall h_2 \in A\Delta^\delta_m \end{cases},
\]

and this space is null if and only if \(\varphi\sigma \not\simeq g1 = 1\) on \(A\Delta^\delta_m \cap gQ^\delta_r\) for all \(g \in E^\delta\). We first observe that for \(r = m\rho - e_3 = (m, m, m - 1)\) this is indeed the case since the subgroup

\[N^\delta_{m\rho - e_3} = \begin{bmatrix} 0 & 0 \\ \pi^{m-1} \mathcal{O}_m & 1 \\ 0 & 1 \end{bmatrix} \subset A\Delta^\delta_m \cap N^\delta_{m\rho - e_3} \Theta
\]

is a characteristic subgroup in \(E^\delta\), and \(\varphi\) is nontrivial on it. Since this is true for any \(\sigma\) the proof is reduced to the validity of \(J(S'')\) with \(S'' = \{c - e_1, c - e_2\}\). A straightforward computation shows that for the corresponding values of \(r\), and regardless of the value of \(\delta\), the elements

\[g(y) = \begin{bmatrix} 0 & 0 \\ 0 & y \end{bmatrix}, y \in \mathcal{O}_m,
\]
form an exhaustive set of representatives for the double coset space $Q_\delta^r \backslash E^\delta / A\Delta^\delta_m$. Conjugating $Q_\delta^r$ with \( g(y) \) gives

\[
\begin{cases}
\begin{bmatrix} x \\ \delta xy \end{bmatrix} \times \begin{bmatrix} \theta_1 \\ (\theta_1 - \theta_2)y \theta_2 \end{bmatrix} | \theta_1, \theta_2 \in O_m^\infty, x \in \mathcal{P}^{m-1} \}, & \text{if } r = m\rho - e_1;
\end{cases}
\]

(5.1)

\[
\begin{cases}
\begin{bmatrix} 0 \\ \theta \end{bmatrix} \times \begin{bmatrix} u + (\theta - \theta_2)y \\ \theta_2 \end{bmatrix} | \theta_1, \theta_2 \in O_m^\infty, u \in \mathcal{P}^{m-1} \}, & \text{if } r = m\rho - e_2.
\end{cases}
\]

We now proceed according to the value of \( \delta \).

(a) \( \delta \in O_m^\infty \).

The intersection \( \varphi(y)Q_\delta^r \cap A\Delta^\delta_m \) is given by

\[
\begin{cases}
\begin{bmatrix} x \\ \delta xy \end{bmatrix} \times \begin{bmatrix} \theta \\ (\theta - 1)y \theta \end{bmatrix} | \theta \in 1 + \mathcal{P}^{m-\text{val}(\delta+1)}, x \in \mathcal{P}^{m-1} \}, & \text{if } r = m\rho - e_1;
\end{cases}
\]

\[
\begin{cases}
\begin{bmatrix} 0 \\ u \end{bmatrix} \times \begin{bmatrix} \theta \\ (\theta - 1)y \theta \end{bmatrix} | \theta \in 1 + \mathcal{P}^{m-\text{val}(\delta+1)-1}, u = (1 - \theta)(1 + \delta y)\delta^{-1} \}, & \text{if } r = m\rho - e_2.
\end{cases}
\]

If \( r = m\rho - e_1 \), then for a given \( \sigma \) the element \( g(y) \) supports a nonzero intertwiner if and only if

\[
\varphi \left[ \begin{bmatrix} x \\ \delta xy \end{bmatrix} \sigma(\theta) = \psi(x(1 + y\delta)) \sigma(\theta) = 1, \forall x \in \mathcal{P}^{m-1}, \forall \theta \in 1 + \mathcal{P}^{m-\text{val}(\delta+1)}
\right],
\]

that is, \( \text{val}(1 + y\delta) > 0 \) and \( \sigma|_{1+\mathcal{P}^{m-\text{val}(\delta+1)}} = 1 \). Therefore, \( L_\sigma \subseteq U_{c-e_1}^m \) if and only if \( \sigma|_{1+\mathcal{P}^{m-1}} = 1 \).

If \( r = m\rho - e_2 \), then for a given \( \sigma \) the element \( g(y) \) supports a nonzero intertwiner if and only if

\[
\varphi \left[ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \sigma(\theta) = 1, \forall \theta \in 1 + \mathcal{P}^{m-\text{val}(\delta+1)-1},
\right],
\]

that is, \( \sigma|_{1+\mathcal{P}^{m-\text{val}(\delta+1)-1}} = 1 \). Therefore, \( L_\sigma \subseteq U_{c-e_2}^m \) if and only if \( \sigma|_{1+\mathcal{P}^{m-1}} = 1 \).

(b) \( \delta \in \pi O_m \).

The intersection \( \varphi(y)Q_\delta^r \cap A\Delta^\delta_m \) is given by

\[
\begin{cases}
\begin{bmatrix} x \\ 0 \end{bmatrix} \times \begin{bmatrix} \theta \\ (\theta - 1)y \theta \end{bmatrix} | \theta \in 1 + \mathcal{P}^{m-\text{val}(\delta+1)}, x \in \mathcal{P}^{m-1} \}, & \text{if } r = m\rho - e_1;
\end{cases}
\]

\[
\begin{cases}
\begin{bmatrix} 0 \\ u \end{bmatrix} \times \begin{bmatrix} 1 \\ u \theta \end{bmatrix} | u \in \mathcal{P}^{m-1}, \text{if } r = m\rho - e_2.
\end{cases}
\]

If \( r = m\rho - e_1 \), then

\[
\varphi \left[ \begin{bmatrix} x \\ 0 \end{bmatrix} \neq 1,
\right],
\]

and we conclude that \( L_\sigma \not\subseteq U_{c-e_1}^m \) for all \( \sigma \)'s.

If \( r = m\rho - e_2 \), then if \( \sigma|_{\mathcal{P}^{m-1}} = 1 \) in fact all the elements \( g(y) \) support a non-zero intertwining function and none of them otherwise. Therefore, \( L_\sigma \subseteq U_{c-e_2}^m \) if and only if \( \sigma|_{\mathcal{P}^{m-1}} = 1 \).
The theorem is proved. \qed

**Definition 5.2.** Let \( c \in C^0 \) and let \( \delta = \pi^{\kappa(c) - \mu(c)} \). For \( \sigma \in \hat{\Delta}_{\mu(c)} - t^* \left( \hat{\Delta}_{\mu(c)} - 1 \right) \) let

\[
\widetilde{L}_{c,\sigma} = \text{Ind}^G_{P_{c-\mu(c)\rho}} \left( \text{Ind}^{P_{c-\mu(c)\rho}}_{E^c_{\delta}} (L_\sigma) \right),
\]

where \( \text{Ind}^{P_{c-\mu(c)\rho}}_{E^c_{\delta}} (L_\sigma) \) stands for the pullback of \( L_\sigma \) along the composition of the quotient maps (2.5) and (2.6).

**Corollary 5.3.** For \( c \in C^0 \) the decomposition of \( V_c \) to irreducible representations is multiplicity free and given by

\[
V_c = \bigoplus_{\sigma \in \hat{\Delta}_{\mu(c)} - t^* \left( \hat{\Delta}_{\mu(c)} - 1 \right)} \widetilde{L}_{c,\sigma}.
\]

**Proof.** Follows from Theorem 5.1 and statement (CN3). \qed

6. MULTIPlicITIES AND DEgrees

In this section we compute explicitly the dimensions and multiplicities of the irreducible constituents in \( V = \text{Ind}^G_B(1) \).

6.1. MULTIplicITIES. We first settle a small debt from subsection 1.2. we shall need the following lemma.

**Lemma 6.1.** Let \( x, y, z \in \mathbb{N} \) be such that \( 0 < x + y - z < \min\{x, y\} \). Then the following are equivalent

1. \( x + y - z \leq \lfloor \min\{x, y\}/2 \rfloor \).
2. \( \mu(x, y, z) = \min\{x + y - z, z - x, z - y\} = x + y - z = \kappa(x, y, z) \).

**Proof.** Without loss of generality assume that \( x \leq y \). Then (1) is equivalent to \( 2(x + y - z) \leq x \leq y \), which in turn is equivalent to \( x + y - z \leq z - y \leq z - x \), and the latter is equivalent to (2). \qed

**Proposition 6.2.** Let \( c, d \in C \). Then \( V_c \simeq V_d \) if and only if \( \lambda(c) = \lambda(d), \kappa(c) = \kappa(d), \mu(c) = \mu(d) \) and \( c = d \) if \( \kappa(c) > \mu(c) \).

**Proof.** We go over the various possibilities for \( c \).

1. If \( c \in C \) satisfies \( c_3 = c_1 + c_2 \), then by Theorem 6.1 and Proposition 6.2 in [1], for any \( d \in C \) we have \( V_c \simeq V_d \) if and only if \( c_3 = d_3 \) and \( c_1 + c_2 = d_1 + d_2 \). These two equations are equivalent to \( \lambda(c) = \lambda(d) \) and \( \kappa(c) = \kappa(d) \), and in such case \( 0 \leq \mu(c) \leq \kappa(c) = 0 \).
2. If \( c \in C \) satisfies \( c_3 = \max\{c_1, c_2\} \geq 1 \), then by [1, Theorem 7.1] and the discussion preceding it, for any \( d \in C \) we have \( V_c \simeq V_d \) if and only if \( c = d \). The conditions on \( c \) imply that \( \kappa(c) > \mu(c) = 0 \) and the assertion follow.
3. It remains to treat \( c, d \in C^0 \). In particular this means that \( \kappa(c), \kappa(d) > 0 \). By [1, Theorem 8.1], we have that \( V_c \simeq V_d \) if and only if
   i. \( \lambda(c) = c_3 = d_3 = \lambda(d) \).
(ii) \( \kappa(c) = c_1 + c_2 - c_3 = d_1 + d_2 - d_3 = \kappa(d) \).

(iii) \( c_1 + c_2 - c_3 \leq \lfloor \min\{c_1, c_2, d_1, d_2\}/2 \rfloor \).

In the presence of (i) and (ii), condition (iii) is equivalent to \( \mu(c) = \kappa(c) = \kappa(d) = \mu(d) \), by applying Lemma 6.1 to \( c \) and to \( d \).

Define an equivalence relation on \( C \) by setting \( c \sim d \) if and only if \( V_c \simeq V_d \). Let \( a : \mathbb{N}_0^3 \to \mathbb{N}_0 \) be the function defined by

\[
a(m, k, \ell) = \begin{cases} \ell - 3k + 1, & \text{if } \ell \geq 3k = 3m \geq 0 ; \\ 1, & \text{if } \ell \geq 2m + k > 3m \geq 0 , \end{cases}
\]

and zero otherwise.

**Proposition 6.3.** \( |[c]| = a(\mu(c), \kappa(c), \lambda(c)) \) for every \( c \in C \).

**Proof.** If \( c \in C \) satisfies \( \kappa(c) = k = m = \mu(c) \) and \( \lambda(c) = c_3 = \ell \) then \( k = c_1 + c_2 - \ell \leq \ell - c_1, \ell - c_2 \). It follows that \( 2k \leq c_1, c_2 \leq \ell - k \), in particular \( \ell \geq 3k \), and that the \( c \)'s satisfying these conditions are precisely of the form

\[
\{(2k + i, \ell - k - i, \ell) \mid i = 0, \ldots, \ell - 3k\},
\]

and their number is \( \ell - 3k + 1 \).

If \( c \in C \) satisfies \( \kappa(c) = k > m = \mu(c) \) and \( \lambda(c) = c_3 = \ell \), then Proposition 6.2 implies that \( V_c \simeq V_d \) if and only if \( c = d \) hence \( [c] \) is a singleton. We also have that

\[
\ell = (c_1 + c_2 - c_3) + (c_3 - c_1) + (c_3 - c_2) \geq k + 2m.
\]

**Corollary 6.4.** A complete decomposition of \( V = \text{Ind}_H^G(1) \) to irreducible representations is given by

\[
V \simeq \bigoplus_{c \in \partial C} V_c \bigoplus \left( \bigoplus_{[c] \in C^0/\sim} \bigoplus_{\sigma \in \Delta_{m-1}^+ \Delta_{m-1}} \mathcal{L}_{c, \sigma}^{\oplus a(m, k, \ell)} \right).
\]

**Proof.** Follows from (1.1), (CN2), Corollary 5.3 and Proposition 6.3.

6.2. **Dimensions.** The equidimensionality of the irreducible constituents of \( V_c \) implies that the dimension of an irreducible subrepresentation \( W \) of \( V \) is uniquely determined by the elements \( c \in C \) with \( \text{Hom}_G(W, V_c) \neq (0) \). The dimension of an irreducible representation occurring in \( V_c \) (\( c \in C \)) is

\[
\begin{cases}
1, & \text{if } c = (0, 0, 0); \\
q^2 + q, & \text{if } c = (1, 0, 1), (0, 1, 1); \\
q^3, & \text{if } c = (1, 1, 1); \\
(1 + q^{-1})(1 - q^{-3})q^{2\lambda(c)}, & \text{if } \kappa(c) = \mu(c), \lambda(c) \geq 2; \\
(1 - q^{-2})(1 - q^{-3})q^{2\lambda(c) + \kappa(c) - \mu(c)}, & \text{if } \kappa(c) > \mu(c), \lambda(c) \geq 2.
\end{cases}
\]
Indeed, the formulae for $\lambda(c) \leq 1$ are well known, see e.g. [1]. For $c \in C^0$ with $\lambda(c) \geq 2$ they follow from the construction of $\tilde{L}_{c,\sigma}$:

$$\dim \tilde{L}_{c,\sigma} = [GL_3(\mathcal{O}) : P_{c-\mu(c)}] \cdot \dim L_{\sigma}$$

$$= (1 + q^{-1})(1 + q^{-1} + q^{-2}) \cdot q^{c_1 + c_2 + c_3 - 3\mu(c)} \cdot \begin{cases} q^{2\mu(c)}(1 - q^{-1}), & \text{if } \kappa(c) = \mu(c); \\
q^{2\mu(c)}(1 - q^{-1})^2, & \text{if } \kappa(c) > \mu(c); \end{cases}$$

$$= \begin{cases} (1 + q^{-1})(1 - q^{-3})q^{2\lambda(c)}, & \text{if } \kappa(c) = \mu(c); \\
(1 - q^{-2})(1 - q^{-3})q^{2\lambda(c) + \kappa(c) - \mu(c)}, & \text{if } \kappa(c) > \mu(c). \end{cases}$$

For $c \in \partial C$ with $\lambda(c) \geq 2$ the formulae are given by Theorems 6.1 and 7.1 in [1].

6.3. Uniform representation growth. Let $\zeta_{G,V}(s) = \sum_{n=1}^{\infty} r_n(G,V)n^{-s}$ with $r_n(G,V)$ the number of irreducible $n$-dimensional subrepresentations of $V$ of dimension $n$. The generating function $\zeta_{G,V}(s)$ enumerates irreducible representations in $V$ according to their dimensions and regardless of their isomorphism type. For every $n \in \mathbb{N}$, let $p_n \in \{0, 1, 2\}$ be the residue of $n$ modulo 3 and let $f_n(x), g_n(x) \in \mathbb{Z}[x]$ be the polynomials

$$f_n(x) = \begin{cases} x^{\lfloor n/6 \rfloor - 1}(p_{\frac{x}{3}} + 1)x + (2 - p_{\frac{x}{3}}), & \text{if } n \in 2\mathbb{N}_0 + 4; \\
0, & \text{otherwise.} \end{cases}$$

$$g_n(x) = \frac{x^{\lfloor n/2 \rfloor} - 1}{x - 1} + \frac{x^{\lfloor n/2 \rfloor - 1} - 1}{x - 1} + \min\{p_n, 1\}x^{\lfloor n/2 \rfloor - 1}, \quad (n \geq 5).$$

Denote $\eta_1(q) = (1 + q^{-1})(1 - q^{-3})$ and $\eta_2(q) = (1 - q^{-2})(1 - q^{-3})$.

**Theorem 6.5.** For every non-archimedean local field $F$ with ring of integers $\mathcal{O}$ and residue field of cardinality $q$

$$\zeta_{G,V}(s) = 1 + 2(q + q^2)^{-s} + q^{-3s} + \sum_{n=4}^{\infty} f_n(q) (\eta_1(q)q^n)^{-s} + \sum_{n=5}^{\infty} g_n(q) (\eta_2(q)q^n)^{-s}.$$

**Proof.** The first three terms in $\zeta_{G,V}(s)$ correspond to the subrepresentations of $V_c$ with $\lambda(c) \in \{0, 1\}$. For the remaining, we observe that the dimension of an irreducible representation in $V$ with $\lambda(c) \geq 2$, as listed in (6.2), determines whether the representation $V_c$ which contains it has $\kappa(c) = \mu(c)$ or $\kappa(c) > \mu(c)$, hence determines its construction. We formally set $\mathcal{O}_0^\times = \{1\}$ and $\mathcal{O}_0 = \{0\}$ to be the trivial groups and $\mathcal{O}_{-1}^\times = \mathcal{O}_{-1}$ to be the empty set. For the irreducible representations of dimension $\eta_1(q)q^n$ we have

$$r_{\eta_1(q)q^n}(G,V) = \sum_{m: 0 \leq 6m \leq 2\ell = n} a(m, m, \ell) |\mathcal{O}_m^\times \cdot \mathcal{O}_{m-1}^\times|$$

$$= \sum_{0 \leq m \leq \lfloor \ell/3 \rfloor} (\ell - 3m + 1) |\mathcal{O}_m^\times| - \sum_{-1 \leq m \leq \lfloor \ell/3 \rfloor - 1} (\ell - 3m - 4) |\mathcal{O}_m^\times|$$

$$= (\ell - 3 \lfloor \ell/3 \rfloor + 1) |\mathcal{O}_{\lfloor \ell/3 \rfloor}^\times| + 3 \sum_{0 \leq m \leq \lfloor \ell/3 \rfloor - 1} |\mathcal{O}_m^\times|$$

$$= q^{\lfloor n/6 \rfloor - 1} \left((p_{\frac{x}{3}} + 1)q + (2 - p_{\frac{x}{3}})\right),$$

for $n \in 2\mathbb{N}_0 + 4$ and zero otherwise.
It remains to consider the irreducible representations of dimension \( \eta_2(q)q^n \). By (6.2), an irreducible subrepresentation of \( V \) of dimension \( \eta_2(q)q^n \) is contained in some \( V_c \) with \( \kappa(c) > \mu(c) \).

For \( m, n \in \mathbb{N}_0 \) let \( S(m, n) \) be the (possibly empty) subset of \( C \)

\[
S(m, n) = \{ c \in C \mid 2\lambda(c) + \kappa(c) - \mu(c) = n, \kappa(c) > \mu(c) = m \}.
\]

If \( S(m, n) \) is non-empty and \( c \in S(m, n) \) we have \( a(\mu(c), \kappa(c), \lambda(c)) = 1 \), and by Corollary 5.3 the number of irreducible constituents of \( V_c \) is equal to \( |O_m \setminus \pi O_m| \). It follows that the number of irreducible constituents of \( V \) of dimension \( \eta_2(q)q^n \) is equal to

\[
\sum_{m, k: n = 2\ell + k - m, k > m} a(m, k, \ell)|O_m \setminus \pi O_m| = \sum_{m \in \mathbb{N}_0} |S(m, n)||O_m \setminus \pi O_m|.
\]

Note that by the definition of \( S(m, n) \) this is a finite sum for every \( n \in \mathbb{N} \). Unraveling definitions and carrying the necessary book keeping yields, for \( n \geq 5 \),

\[
|S(m, n)| = \begin{cases} 
2 \left\lfloor \frac{n}{2} \right\rfloor - 2m, & \text{if } 0 \leq m \leq \left\lfloor \frac{n}{2} \right\rfloor - 1, n \not\equiv 0 \mod 3; \\
2 \left\lfloor \frac{n}{2} \right\rfloor - 2m + 1, & \text{if } 0 \leq m \leq \left\lfloor \frac{n}{2} \right\rfloor, n \equiv 0 \mod 3,
\end{cases}
\]

and zero otherwise. By substituting these values in (6.3), we see that for \( n \equiv 1, 2 \mod 3 \),

\[
\sum_{m=0}^{\left\lfloor \frac{n}{2} \right\rfloor - 1} 2\left( \left\lfloor \frac{n}{2} \right\rfloor - m \right)(q - 1)q^{m-2} = 2q^{\left\lfloor \frac{n}{2} \right\rfloor} - 1 \quad q - 1,
\]

and for \( n \equiv 0 \mod 3 \),

\[
\sum_{m=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (2\left\lfloor \frac{n}{2} \right\rfloor - 2m + 1)(q - 1)q^{m-2} = \frac{q^{\left\lfloor \frac{n}{2} \right\rfloor + 1} - 1}{q - 1} + \frac{q^{\left\lfloor \frac{n}{2} \right\rfloor} - 1}{q - 1}.
\]

Combining these expressions, we obtain the result. \( \square \)

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