Dirichlet problem for harmonic maps from strongly rectifiable spaces into regular balls in CAT(1) spaces

Yohei Sakurai

Received: 22 June 2023 / Accepted: 31 August 2023 / Published online: 16 September 2023
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Abstract
In this note, we study the Dirichlet problem for harmonic maps from strongly rectifiable spaces into regular balls in CAT(1) space. Under the setting, we prove that the Korevaar–Schoen energy admits a unique minimizer.

Keywords
Harmonic map · Korevaar–Schoen energy · Strongly rectifiable space · RCD space · CAT(1) space · Regular ball

Mathematics Subject Classification Primary 53E23 · Secondly 58E20

1 Introduction
1.1 CAT(0) target

The study of harmonic maps between singular spaces is one of the central topics in geometric analysis since the pioneering work by Gromov and Schoen [24]. In an earlier stage, the theory has been developed by Korevaar and Schoen [30], Jost [26, 27] and Lin [32], independently. Korevaar and Schoen [30] have introduced an energy for $L^2$-maps from a Riemannian domain into a complete metric space and derived its basic properties, for instance, the existence of energy density, the consistency of Sobolev functions, and its lower semicontinuity. Furthermore, they have proven that if the target space is a CAT(0) space (i.e., non-positively curved space in the sense of A. D. Alexandrov), then the energy admits a unique minimizer for the Dirichlet problem.

One of the research directions is to generalize the theory for non-smooth source spaces. Such attempts have been done by Gregori [16], Eells and Fuglede [11], Kuwae and Shioya [31], and so on. Gregori [16] and Eells and Fuglede [11] have dealt with a domain of a Lipschitz manifold and that of a Riemannian polyhedron, respectively. Kuwae and Shioya [31] have examined a metric measure space satisfying the so-called strongly measure contraction.
property of Bishop-Gromov type, called SMCPBG space. A typical example of SMCPBG space is an Alexandrov space, which is a metric space equipped with the notion of a lower sectional curvature bound.

In recent years, the theory of metric measure spaces with a lower Ricci curvature bound has been vastly developed. In the literature, the main research object is the so-called RCD\((K,N)\) space, which has been introduced by Ambrosio et al. [1] and Gigli [14]. It is natural to ask whether the Korevaar and Schoen theory can be extended to RCD\((K,N)\) spaces. Here we notice that an RCD\((K,N)\) space is not necessarily a SMCPBG space (especially, in the collapsed case) since the Bishop-type inequality is required in the SMCPBG condition, and hence the framework of Kuwae and Shioya [31] does not cover RCD\((K,N)\) spaces. Gigli and Tyulenev [22] were able to work in a more general context, covering the RCD setting. In their framework, source spaces are assumed to be locally uniformly doubling, strongly rectifiable and to satisfy a Poincaré inequality (more precisely, see Subsection 2.2). Under this setting, they have defined a Korevaar–Schoen-type energy and deduced some fundamental results such as the existence of energy density and its lower semicontinuity. They have concluded the solvability of the Dirichlet problem when the target space is a CAT\((0)\) space (see [22, Theorem 6.4]).

**Theorem 1.1** ([22]) Let \((X,d,m)\) be a locally uniformly doubling, strongly rectifiable space satisfying a Poincaré inequality, and let \(\Omega\) be a bounded open subset of \(X\) with \(m(X\setminus \Omega) > 0\). Let \(Y_o = (Y,d_Y,o)\) be a pointed CAT\((0)\) space. Let \(\bar{u} \in K^{1,2}_2(\Omega, Y_o)\) be a Korevaar–Schoen-type Sobolev map which determines a boundary value. Then the Korevaar–Schoen-type energy \(E_{2,\bar{u}}: L^2(\Omega,Y_o) \to [0,\infty]\) admits a unique minimizer.

Thanks to Theorem 1.1, one can introduce the notion of harmonic map from an RCD\((K,N)\) space to a CAT\((0)\) space. Very recently, Gigli [16] has shown a quantitative Lipschitz estimate for such harmonic maps and produced a Cheng-type Liouville theorem ([7]) based on [10, 17, 21] (see also [40, 41]). Mondino and Semola [35] have also obtained a similar result, independently.

### 1.2 CAT\((1)\) target

The purpose of this note is to yield an analog of Theorem 1.1 for the case where the target space is a regular ball in a CAT\((1)\) space (i.e., geodesic ball whose radius is strictly less than \(\pi/2\)). In the case where the source space is a Riemannian domain, the solvability of the Dirichlet problem has been established by Serbinowski [37]. Similarly to the non-positively curved case, the result in [37] has been extended to non-smooth source spaces (see Eells and Fuglede [11], Fuglede [12, 13] for Riemannian polyhedra, and Huang and Zhang [25] for Alexandrov spaces). We now aim to generalize it for RCD\((K,N)\) spaces. Our main result is the following:

**Theorem 1.2** Let \((X,d,m)\) be an infinitesimally Hilbertian, locally uniformly doubling, strongly rectifiable space satisfying a Poincaré inequality, and let \(\Omega\) be a bounded open subset of \(X\) with \(m(X\setminus \Omega) > 0\). Let \(Y_o = (Y,d_Y,o)\) be a pointed CAT\((1)\) space, and let \(\bar{B}_\rho(o)\) be a regular ball (i.e., a closed ball of radius \(\rho \in (0,\pi/2)\) centered at \(o\)). Let \(\bar{u} \in K^{1,2}_2(\Omega, \bar{B}_\rho(o))\) be a Korevaar–Schoen-type Sobolev map which determines a boundary value. Then the Korevaar–Schoen-type energy \(E_{2,\bar{u}}: L^2(\Omega,\bar{B}_\rho(o)) \to [0,\infty]\) admits a unique minimizer.

Theorem 1.2 enables us to introduce the notion of harmonic map from an RCD\((K,N)\) space into a regular ball in a CAT\((1)\) space. When the source space is a Riemannian domain,
Zhang et al. [41] have obtained a quantitative Lipschitz estimate and a Choi-type Liouville theorem ([8]).

2 Preliminaries

We say that \((X, d, m)\) is a metric measure space if \((X, d)\) is a complete separable metric space, and \(m\) is a non-negative Borel measure, which is finite on bounded sets. This section is devoted to preliminaries for metric measure spaces.

2.1 Sobolev functions

We briefly recall the non-smooth differential calculus on metric measure spaces. The readers can refer to [15, 18] for the details.

Let \((X, d, m)\) be a metric measure space and let \(C([0, 1], X)\) be the set of all curves in \(X\) defined on \([0, 1]\) with the uniform topology. For \(t \in [0, 1]\) the evaluation map \(e_t : C([0, 1], X) \to X\) is defined as \(e_t(\gamma) := \gamma_t\). A curve \(\gamma \in C([0, 1], X)\) is said to be absolutely continuous if there is \(f \in L^1(0, 1)\) such that

\[
\int_0^1 |\dot{\gamma}_t|^2 \, d\pi(\gamma) \, dt < +\infty, \quad (e_t)_#\pi \leq C m
\]

for some \(C > 0\). The Sobolev class \(S^2(X)\) is the set of all Borel functions \(f : X \to \mathbb{R}\) such that there exists a non-negative \(G \in L^2(m)\) such that

\[
\int |f(\gamma_1) - f(\gamma_0)| \, d\pi(\gamma) \leq \int_0^1 G(\gamma_t) |\dot{\gamma}_t| \, dt \, d\pi(\gamma)
\]

for all test plans \(\pi\). For \(f \in S^2(X)\), a non-negative function \(G \in L^2(m)\) satisfying (2.2) is called a weak upper gradient, and the minimal one in the \(m\)-a.e. sense is called the minimal weak upper gradient and it is denoted by \(|Df|\). The space

\[
W^{1,2}(X) := L^2(m) \cap S^2(X)
\]

equipped with the norm

\[
\|f\|_{W^{1,2}(X)}^2 := \|f\|_{L^2(m)}^2 + \|Df\|_{L^2(m)}^2
\]

is called Sobolev space and it can be proved that it is a Banach space.

It is well-known that there exists a unique couple \((L^2(T^*X), d)\), where \(L^2(T^*X)\) is an \(L^2(m)\)-normed \(L^\infty(m)\)-module and \(d : S^2(X) \to L^2(T^*X)\) is a linear operator such that the following hold (see e.g., [15], [18, Theorem 4.1.1]):

1. \(L^2(T^*X)\) is generated by \(\{df \mid f \in S^2(X)\}\);
2. for every \(f \in S^2(X)\), it holds that

\[
|df| = |Df| \quad \text{m-a.e.}
\]
The space $L^2(T^*X)$ and the operator $d$ are called the cotangent module and the differential, respectively. The tangent module $L^2(TX)$ is defined as the dual module of $L^2(T^*X)$. For $f \in S^2(X)$, let $\text{Grad}(f)$ be the set of all $v \in L^2(TX)$ such that
\[
d f (v) = |d f|^2 = |v|^2 \quad \text{m-a.e.}
\]
Note that $\text{Grad}(f)$ is non-empty (see [18, Remark 4.2.10]). The following chain rule holds (see e.g., [18, Theorem 4.2.15]):

**Theorem 2.1** ([18]) Let $f \in S^2(X)$ and $v \in \text{Grad}(f)$. If $\varphi : \mathbb{R} \to \mathbb{R}$ is Lipschitz, then $\varphi' \circ f v \in \text{Grad}(\varphi \circ f)$, where $\varphi' \circ f$ is arbitrarily defined on the inverse image of the non-differentiability points of $\varphi$.

A metric measure space $(X, d, m)$ is said to be infinitesimally strictly convex when for every $f \in S^2(X)$ the set $\text{Grad}(f)$ is a singleton; in this case, the unique element is denoted by $\nabla f$. Also, $(X, d, m)$ is said to be infinitesimally Hilbertian if $W^{1,2}(X)$ is a Hilbert space.

We have the following characterization (see e.g., [18, Theorem 4.3.3]):

**Theorem 2.2** ([18]) The following are equivalent:

1. $(X, d, m)$ is infinitesimally Hilbertian;
2. $L^2(T^*X)$ and $L^2(TX)$ are Hilbert modules;
3. $(X, d, m)$ is infinitesimally strictly convex, and for all $f, g \in W^{1,2}(X)$,
\[
\nabla (f + g) = \nabla f + \nabla g \quad \text{m-a.e.};
\]
4. $(X, d, m)$ is infinitesimally strictly convex, and for all $f, g \in W^{1,2}(X) \cap L^\infty(m)$,
\[
\nabla (fg) = f \nabla g + g \nabla f \quad \text{m-a.e.}.
\]

In the case where $(X, d, m)$ is infinitesimally Hilbertian, we denote by $\langle \cdot, \cdot \rangle$ the pointwise scalar product on $L^2(TX)$.

### 2.2 Doubling, Poincaré and strong rectifiability

Here we review a volume doubling property, a Poincaré inequality and the strongly rectifiability on metric measure spaces.

Let $(X, d, m)$ be a metric measure space. For $r > 0$ and $x \in X$, we denote by $B_r(x)$ the open ball of radius $r$ centered at $x$. A metric measure space $(X, d, m)$ is said to be locally uniformly doubling if for every $R > 0$ there exists $C_R > 0$ such that
\[
m(B_{2r}(x)) \leq C_R m(B_r(x))
\]
for all $x \in X$ and $r \in (0, R)$. We also say that $(X, d, m)$ satisfies a Poincaré inequality if for every $R > 0$ there are $C_R, \lambda = \lambda_R > 0$ such that
\[
\int_{B_r(x)} |f - f_{B_r(x)}| \, dm \leq C_R r \int_{B_{3r}(x)} \text{lip}(f) \, dm
\]
holds for every Lipschitz function $f : X \to \mathbb{R}$ and for all $x \in X$ and $r \in (0, R)$, where $f_{B_r(x)} := \int_{B_r(x)} f \, dm$ and $\text{lip}(f)$ is the pointwise Lipschitz constant.

We say that a metric measure space $(X, d, m)$ is a $d$-dimensional strongly rectifiable space if for every $\varepsilon > 0$ there exists an $\varepsilon$-atlas $\mathcal{A}^\varepsilon := \{(U_i^\varepsilon, \varphi_i^\varepsilon)\}_i$ such that the following hold:

1. Borel subsets $\{U_i^\varepsilon\}_i$ form a partition of $X$ up to an $m$-negligible set;
(2) $\varphi^\varepsilon_i$ is a $(1 + \varepsilon)$-bi-Lipschitz map from $U^\varepsilon_i$ to $\varphi^\varepsilon_i(U^\varepsilon_i) \subset \mathbb{R}^d$;

(3) for some $c_i > 0$, we have
\[
    c_i \mathcal{L}^d |_{\varphi^\varepsilon_i(U^\varepsilon_i)} \leq (\varphi^\varepsilon_i)^\# \mathcal{L}^d |_{U^\varepsilon_i} \leq (1 + \varepsilon) c_i \mathcal{L}^d |_{\varphi^\varepsilon_i(U^\varepsilon_i)}.
\]

**Remark 2.3** For $K \in \mathbb{R}$ and $N \in [1, \infty)$, every $\text{RCD}(K, N)$ space is a locally uniformly doubling, strongly rectifiable space satisfying a Poincaré inequality. Actually, the doubling property is due to [33, 38, 39]; the Poincaré inequality is due to [36]. Moreover, as stated in [22, Theorem 2.19], the strong rectifiability is a consequence of the results by Brué and Semola [6], Gigli and Pasqualetto [19], Kell and Mondino [28], Mondino and Naber [34], De Philippis et al. [9].

**Remark 2.4** Let $(X, d, m)$ be a strongly rectifiable space, and let $\{\varepsilon_n\}_n$ be a sequence with $\varepsilon_n \downarrow 0$. A family $\{A^{\varepsilon_n}\}_n$ of atlases is said to be aligned if for all $n$, $m$ and $(U^\varepsilon_i, \varphi^\varepsilon_i) \in A^{\varepsilon_n}$, $(U^\varepsilon_j, \varphi^\varepsilon_j) \in A^{\varepsilon_m}$, the map $\varphi^\varepsilon_i - \varphi^\varepsilon_j$ is $(\varepsilon_n + \varepsilon_m)$-Lipschitz on $U^\varepsilon_i \cap U^\varepsilon_j$. It has been observed in [20, Theorem 3.9], [22, Subsection 2.4] that for any sequence $\{\varepsilon_n\}_n$, an aligned family of atlases $\{A^{\varepsilon_n}\}_n$ exists.

### 2.3 Approximate metric differentiability

We further review the notion of approximate metric differentiability of maps introduced by Kirchheim [29] and Gigli and Tyulenev [22].

Let $(X, d, m)$ be a metric measure space, and let $(Y, dy)$ be a metric space. Let $u : X \to Y$ be a Borel map. The **pointwise Lipschitz constant** of $u$ at $x \in X$ is defined as
\[
    \text{lip}(u)(x) := \lim_{y \to x} \frac{dy(u(x), u(y))}{d(x, y)}
\]
if $x$ is not an isolated point, and $\text{lip}(u)(x) := 0$ otherwise. For a Borel subset $U \subset X$, a point $x \in U$ is said to be a **density point of** $U$ if
\[
    \lim_{r \downarrow 0} \frac{m(B_r(x) \cap U)}{m(B_r(x))} = 1.
\]

The **approximate Lipschitz constant** is defined by
\[
    \text{ap-lip}(u)(x) := \text{ap-lim}_{y \to x} \frac{dy(u(x), u(y))}{d(x, y)},
\]
where the right-hand side means the approximate upper limit (see e.g., [22, Subsection 2.1] for the precise definition). We have the following (see [22, Proposition 2.5]):

**Proposition 2.5** ([22]) Let $(X, d, m)$ be a uniformly locally doubling space, and let $(Y, dy)$ be a complete metric space. Let $U$ be a Borel subset of $X$, and let $u : U \to X$ be Lipschitz. Then for every density point $x \in X$ of $U$ we have
\[
    \text{lip}(u)(x) = \text{ap-lip}(u)(x).
\]

Let $(X, d, m)$ be a $d$-dimensional strongly rectifiable space, and let $\{A^{\varepsilon_n}\}_n$ be an aligned family of atlases for a sequence $\{\varepsilon_n\}_n$ with $\varepsilon_n \downarrow 0$. Let $(Y, dy)$ be a metric space. We denote by $s\mathbb{R}^d$ the set of all semi-norms on $\mathbb{R}^d$ equipped with the complete separable metric
\[
    D(n_1, n_2) := \sup_z |n_1(z) - n_2(z)|.
\]
where the supremum is taken over all \( z \in \mathbb{R}^d \) with \( |z| \leq 1 \). We write \( \|n\| := D(n, 0) \). We say that a Borel map \( u : X \to Y \) is \emph{approximately metrically differentiable at} \( x \in X \) relatively to \( \{A^x_n\}_n \) if the following hold:

1. For every \( n \), there exists \( i = i_{x,n} \) such that \( x \) belongs to \( U_i^{x_n} \), it is a density point of \( U_i^{x_n} \) and \( \varphi_i^{x_n}(x) \) is a density point of \( \varphi_i^{x_n}(U_i^{x_n}) \);

2. there exists \( m_d(x)(u) \in s^d \), called the \emph{metric differential of} \( u \) \( x \), such that

\[
\lim_{n \to \infty} \lim_{y \to x, y \in U_i^{x_n}} \frac{|d_Y(u(y), u(x)) - m_d(x)(u)(\varphi_i^{x_n}(y) - \varphi_i^{x_n}(x))|}{d(y, x)} = 0.
\]

Moreover, Gigli and Tyulenev [22] have shown the following (see [22, Lemma 3.4]):

\textbf{Lemma 2.6} ([22]) Let \( (X, d, m) \) be a strongly rectifiable space, and let \( \{A^x_n\}_n \) be an aligned family of atlases for a sequence \( \{\varepsilon_n\}_n \) with \( \varepsilon_n \downarrow 0 \). Let \( (Y, d_Y) \) be a complete metric space. If \( u : X \to Y \) is approximately metrically differentiable at \( x \in X \) relatively to \( \{A^x_n\}_n \), then

\[
\text{ap-lip}(u)(x) = \|m_d(x)(u)\|.
\]

Gigli and Tyulenev [22] have further concluded the m-a.e. approximately metrical differentiability of Borel maps satisfying the Lusin–Lipschitz property (see [22, Proposition 3.6]).

\subsection*{2.4 Korevaar–Schoen space}

In this section we recall the formulation and basic results concerning the Korevaar–Schoen-type energy introduced in Gigli and Tyulenev [22]. Let \( (X, d, m) \) be a metric measure space, and let \( \Omega \) be an open subset of \( X \). Let \( Y_0 = (Y, d_Y, \omega) \) be a pointed complete metric space.

We denote by \( L^0(\Omega, Y) \) the set of all Borel maps (up to m-a.e. equality) from \( \Omega \) to \( Y \) with separable range. Let \( L^2(\Omega, Y_0) \) be the set of all \( u \in L^0(\Omega, Y) \) such that

\[
\int_\Omega d_Y^2(u(x), \omega) \, dm(x) < +\infty,
\]

which is endowed with the metric

\[
d_{L^2}(u, v) := \left( \int_\Omega d_Y^2(u(x), v(x)) \, dm(x) \right)^{1/2}.
\]

Since \( Y \) is complete, it can be proved that so is \( L^2(\Omega, Y_0) \).

Let \( u \in L^2(\Omega, Y_0) \). For \( r > 0 \) and \( x \in \Omega \), the \emph{approximate energy} is defined by

\[
\text{ks}_{2,r}[u, \Omega](x) := \left( \int_{B_r(x)} \frac{d_Y^2(u(x), u(y))}{r^2} \, dm(y) \right)^{1/2} \quad \text{if} \ B_r(x) \subset \Omega,
\]

\[
\text{otherwise}.
\]

The \emph{Korevaar–Schoen space} \( KS^{1,2}(\Omega, Y_0) \) is defined by the set of all \( u \in L^2(\Omega, Y_0) \) such that

\[
E_2^2(u) := \lim_{r \downarrow 0} \int_\Omega \text{ks}_{2,r}[u, \Omega] \, dm < +\infty. \tag{2.3}
\]
Remark 2.7 Let $W^{1,2}(\Omega)$ denote the Sobolev space provided by means of [22, Definition 5.2]. One can also introduce the associated Sobolev space $W^{1,2}(\Omega, Y_o)$ as the set of all $u \in L^2(\Omega, Y_o)$ such that there is $G \in L^2(\Omega)$ such that for all 1-Lipschitz functions $f : Y \to \mathbb{R}$ we have $f \circ u \in W^{1,2}(\Omega)$ and $|D(f \circ u)| \leq G$ m.a.e. on $\Omega$ (see [22, Definition 5.3]).

We now present some basic properties of the Korevaar–Schoen space $KS^{1,2}(\Omega, Y_o)$ (see [22, Theorem 5.7]):

Theorem 2.8 ([22]) Let $(X, d, m)$ be a locally uniformly doubling, strongly rectifiable space satisfying a Poincaré inequality, and let $\Omega$ be an open subset. Let $Y_o = (Y, d_Y, o)$ be a pointed complete metric space. Then the following hold:

(1) $KS^{1,2}(\Omega, Y_o) = W^{1,2}(\Omega, Y_o)$;

(2) for every $u \in KS^{1,2}(\Omega, Y_o)$ there exists a function $e_2[u] \in L^2(\Omega)$, called the energy density, such that

$$ks_{2,r}[u, \Omega] \to e_2[u] \text{ m-a.e. on } \Omega \text{ and in } L^2(\Omega) \text{ as } r \downarrow 0.$$

In particular $\liminf$ in (2.3) is a limit and $E^2(u)$ can be written as

$$E^2(u) = \begin{cases} \int_{\Omega} e_2^2[u] \, dm & \text{if } u \in KS^{1,2}(\Omega, Y_o), \\ +\infty & \text{otherwise}; \end{cases}$$

(3) $E^2 : L^2(\Omega, Y_o) \to [0, +\infty]$ is lower semicontinuous;

(4) any $u \in KS^{1,2}(\Omega, Y_o)$ is approximately metrically differentiable m.a.e. in $\Omega$, where we extend $u$ to the whole $X$ by setting it to be constant outside of $\Omega$.

We also need the following (see [22, Propositions 4.6 and 4.19]):

Proposition 2.9 ([22]) Let $(X, d, m)$ be a $d$-dimensional, locally uniformly doubling, strongly rectifiable space satisfying a Poincaré inequality, and let $\Omega$ be an open subset. Then there exists $c_d > 0$ depending only on $d$ such that for every $u \in KS^{1,2}(\Omega, \mathbb{R})$, 

$$e_2[u] = c_d |Du| \text{ m-a.e. on } \Omega. \tag{2.4}$$

Let $W^{1,2}_o(\Omega)$ be the $W^{1,2}(\Omega)$-closure of the set of all functions in $W^{1,2}(\Omega)$ whose support is contained in $\Omega$. For a fixed $\tilde{u} \in KS^{1,2}_o(\Omega, Y_o)$, we define

$$KS^{1,2}_{\tilde{u}}(\Omega, Y_o) := \{ u \in KS^{1,2}(\Omega, Y_o) \mid d_Y(u, \tilde{u}) \in W^{1,2}_o(\Omega) \},$$

and the associated energy functional $E^2_{\tilde{u}} : L^2(\Omega, Y_o) \to [0, +\infty]$ as

$$E^2_{\tilde{u}}(u) := \begin{cases} E^2(u) & \text{if } u \in KS^{1,2}_{\tilde{u}}(\Omega, Y_o), \\ +\infty & \text{otherwise.} \end{cases}$$

We close this section with the following (see [22, Proposition 5.10]):

Proposition 2.10 ([22]) Let $(X, d, m)$ be a locally uniformly doubling, strongly rectifiable space satisfying a Poincaré inequality, and let $\Omega$ be an open subset. Let $Y_o = (Y, d_Y, o)$ be a pointed complete metric space. Then for a fixed $\tilde{u} \in KS^{1,2}_o(\Omega, Y_o)$ the following hold:

(1) $E^2_{\tilde{u}}$ is lower semicontinuous;

(2) for any $u, v \in KS^{1,2}_{\tilde{u}}(\Omega, Y_o)$ we have $d_Y(u, v) \in W^{1,2}_o(\Omega)$. 

\cite{Springer}
3 Dirichlet problem

In the present section, we give a proof of Theorem 1.2. We proceed along the line of the proof of [37, Theorem 1.16], [22, Theorem 6.4].

3.1 Technical lemma

This subsection is devoted to the proof of a technical lemma. Let $(X, d, m)$ be a metric measure space, and let $\Omega$ be an open subset of $X$. Let $Y_o = (Y, d_Y, o)$ be a pointed complete metric space. Let $u, v, w \in L^2(\Omega, Y_o)$. For $\alpha > 0$ and $x \in \Omega$, we define the following (Borel) sets

\[
\begin{align*}
\Omega_{v, \alpha, x} := \{ y \in \Omega \mid d_Y(v(x), v(y)) \geq \alpha \}, \\
\Omega_{w, \alpha, x} := \{ y \in \Omega \mid d_Y(w(x), w(y)) \geq \alpha \}, \\
\Omega_{v, w, \alpha, x} := \Omega_{v, \alpha, x} \cup \Omega_{w, \alpha, x}.
\end{align*}
\]

We also define a modified approximate energy by

\[
k_{2, r}[u, \Omega; v, w, \alpha](x) := \begin{cases} \frac{1}{m(B_r(x))} \int_{B_r(x) \setminus \Omega_{v, w, \alpha}} \frac{d^2_Y(u(x), u(y))}{r^2} \, dm(y) & \text{if } B_r(x) \subset \Omega, \\
0 & \text{otherwise.}
\end{cases}
\]

We now state and prove our key lemma (cf. [37, Lemma 1.12], [25, Lemma 3.1]):

**Lemma 3.1** Let $(X, d, m)$ be a locally uniformly doubling, strongly rectifiable space satisfying a Poincaré inequality, and let $\Omega$ be an open subset. Let $Y_o = (Y, d_Y, o)$ be a pointed complete metric space. Let $u, v, w \in KS^{1,2}(\Omega, Y_o)$ and $\alpha > 0$ fixed. Then we have

\[
k_{2, r}[u, \Omega; v, w, \alpha] \to e_2[u] \text{ m.a.e. on } \Omega \text{ and in } L^2(\Omega) \text{ as } r \downarrow 0.
\]

**Proof** We first prove the m.a.e. convergence. Note that for m.a.e. $x \in \Omega$ the following hold:

1. $k_{2, r}[u, \Omega](x) \to e_2[u](x)$, $k_{2, r}[v, \Omega](x) \to e_2[v](x)$, $k_{2, r}[w, \Omega](x) \to e_2[w](x)$ as $r \downarrow 0$;
2. $\text{lip } u(x) < +\infty$.

Indeed, the first property is a consequence of Theorem 2.8. Concerning the second one, Theorem 2.8 implies that $u$ is approximately metrically differentiable m.a.e. $x \in \Omega$ relatively to an aligned family $\{A^R_n\}$ of atlases. For such an $x \in \Omega$, and for each $n$, there is $i = i_{x,n}$ such that $x$ belongs to a chart $U_i^{R_n}$ of $A^R_n$, and it is a density point of $U_i^{R_n}$. Thanks to [22, (2.15), Corollary 3.10], we may assume that $u|_{U_i^{R_n}}$ is Lipschitz. By Proposition 2.5 and Lemma 2.6, $\text{lip } u(x)$ coincides with $|\text{md}_i(x)||$, which is finite.

Let $x \in \Omega$ be such that two properties above hold: we shall prove (3.3) for such an $x$. Since

\[
k_{2, r}[u, \Omega; v, w, \alpha](x) = k_{2, r}[u, \Omega](x) - \frac{1}{m(B_r(x))} \int_{B_r(x) \cap \Omega_{v, w, \alpha}} \frac{d^2_Y(u(x), u(y))}{r^2} \, dm(y),
\]

it suffices to show that

\[
\frac{1}{m(B_r(x))} \int_{B_r(x) \cap \Omega_{v, w, \alpha}} \frac{d^2_Y(u(x), u(y))}{r^2} \, dm(y) \to 0
\]

(3.4)
as $r \downarrow 0$. We have
\[
\frac{1}{m(B_r(x))} \int_{B_r(x) \cap \Omega_{v,w},\kappa,x} \frac{d^2_{\Omega}(u(x), u(y))}{r^2} \, dm(y)
\leq \frac{m(B_r(x) \cap \Omega_{v,w},\kappa,x)}{m(B_r(x))} \sup_{y \in B_r(x) \setminus \{x\}} \frac{d^2_{\Omega}(u(x), u(y))}{d^2(x, y)}
\leq \left( \frac{m(B_r(x) \cap \Omega_{v,w},\kappa,x)}{m(B_r(x))} + \frac{m(B_r(x) \cap \Omega_{v,w},\kappa,x)}{m(B_r(x))} \right) \sup_{y \in B_r(x) \setminus \{x\}} \frac{d^2_{\Omega}(u(x), u(y))}{d^2(x, y)}
\leq \frac{r^2}{\alpha^2} \left( k_{2,r}^2[v, \Omega](x) + k_{2,r}^2[w, \Omega](x) \right) \sup_{y \in B_r(x) \setminus \{x\}} \frac{d^2_{\Omega}(u(x), u(y))}{d^2(x, y)}
\]
where the last inequality follows from
\[
\frac{\alpha^2}{r^2} \frac{m(B_r(x) \cap \Omega_{v,w},\kappa,x)}{m(B_r(x))} \leq \frac{1}{m(B_r(x))} \int_{B_r(x) \cap \Omega_{v,w},\kappa,x} \frac{d^2_{\Omega}(v(x), v(y))}{r^2} \, dm(y) \leq k_{2,r}^2[v, \Omega](x),
\]
\[
\frac{\alpha^2}{r^2} \frac{m(B_r(x) \cap \Omega_{v,w},\kappa,x)}{m(B_r(x))} \leq \frac{1}{m(B_r(x))} \int_{B_r(x) \cap \Omega_{v,w},\kappa,x} \frac{d^2_{\Omega}(w(x), w(y))}{r^2} \, dm(y) \leq k_{2,r}^2[w, \Omega](x).
\]
Therefore, by letting $r \downarrow 0$ we obtain (3.4) and the sought m.a.e. convergence. The $L^2$-convergence follows from $k_{2,r}^2[u, \Omega; v, w, \alpha] \leq k_{2,r}^2[u, \Omega]$ and [22, Lemma 3.14].

### 3.2 CAT($\kappa$) spaces

In this subsection we shall review some basic properties of CAT($\kappa$) spaces. We refer to [3], [5]. For $\kappa \in \mathbb{R}$, let $M_\kappa$ be the 2-dimensional space form of constant curvature $\kappa$. Let $D_\kappa$ be the diameter of $M_\kappa$; more precisely,

\[
D_\kappa := \begin{cases}
\pi/\sqrt{\kappa} & \text{if } \kappa > 0, \\
+\infty & \text{otherwise}.
\end{cases}
\]

Let $(Y, d_Y)$ be a metric space. A curve $\gamma : [0, 1] \rightarrow Y$ is said to be a geodesic if

\[
d_Y(\gamma_t, \gamma_s) = |s - t|d_Y(\gamma_0, \gamma_1)
\]

for all $t, s \in [0, 1]$. Further, $(Y, d)$ is called a geodesic space if for any pair of points $x, y \in Y$, there is a geodesic $\gamma : [0, 1] \rightarrow Y$ with $\gamma_0 = x$ and $\gamma_1 = y$. For $\kappa \in \mathbb{R}$, a complete geodesic space $(Y, d_Y)$ is said to be a CAT($\kappa$)-space if it satisfies the following $\kappa$-triangle comparison principle: for every geodesic triangle $\Delta_{pqr}$ in $Y$ whose perimeter is less than $2D_\kappa$, and for every point $x$ on the segment between $q$ and $r$, it holds that

\[
d_Y(p, x) \leq d_\kappa(\bar{p}, \bar{x}),
\]

where $d_\kappa$ is the Riemannian distance on $M_\kappa$, and $\bar{p}$ and $\bar{x}$ are comparison points in $M_\kappa$.

Let us recall the following basic properties (see e.g., [5, Proposition 1.4]):

**Proposition 3.2** ([5]) For $\kappa \in \mathbb{R}$, let $(Y, d_Y)$ be a CAT($\kappa$) space. Then we have:

1. For each pair of points $x, y \in Y$ with $d_Y(x, y) < D_\kappa$, there exists a unique geodesic from $x$ to $y$; moreover, it varies continuously with its end points;
2. any ball in $Y$ whose radius is less than $D_\kappa/2$ is convex; namely, for each pair of points in such a ball, the unique geodesic joining them is contained the ball.
Remark 3.3 In [5], Proposition 3.2 has been deduced from the following: let $\gamma, \eta : [0, 1] \rightarrow Y$ be geodesics such that

$$\gamma_0 = \eta_0, \quad d_Y(\gamma_0, \gamma_1) + d_Y(\eta_1, \eta_0) < 2D_\kappa.$$  

For $l \in (0, D_\kappa)$, we assume $d_Y(\gamma_0, \gamma_1) \leq l$ and $d_Y(\eta_0, \eta_1) \leq l$. Then there is a constant $c_{\kappa, l} > 0$ depending only on $\kappa, l$ such that for all $t \in [0, 1]$,

$$d_Y(\gamma_t, \eta_t) \leq c_{\kappa, l} d_Y(\gamma_1, \eta_1). \quad (3.5)$$

We now collect some useful estimates on CAT(1) spaces, which have been stated in [37] without proof. The detailed proofs can be found in [4, Appendix A]. The first one is the following (see [37, p. 11, ESTIMATE I], and also [4, Lemma A.2]):

Lemma 3.4 ([37]) Let $(Y, d_Y)$ be a CAT(1) space, and let $\gamma, \eta : [0, 1] \rightarrow Y$ be geodesics with

$$d_Y(\gamma_0, \eta_0) + d_Y(\eta_0, \eta_1) + d_Y(\eta_1, \gamma_1) + d_Y(\gamma_1, \gamma_0) < 2\pi.$$

Then we have

$$\cos^2 \frac{d_Y(\gamma_0, \gamma_1)}{2} \leq \frac{1}{4} (d_Y(\eta_0, \eta_1) - d_Y(\gamma_0, \gamma_1))^2$$

$$\leq \frac{1}{2} (d_Y(\gamma_0, \eta_0) + d_Y(\gamma_1, \eta_1))$$

$$\quad + \text{Cub}(d_Y(\gamma_0, \eta_0), d_Y(\gamma_1, \eta_1), d_Y(\eta_0, \eta_1) - d_Y(\gamma_0, \gamma_1), d_Y(\gamma_1/2, \eta_1/2)),$$

where Cub denotes cubic terms in the indicated variables.

The second one is the following (see [37, p. 13, ESTIMATE II], and also [4, Lemma A.4]):

Lemma 3.5 ([37]) Let $(Y, d_Y)$ be a CAT(1) space, and let $\gamma, \eta : [0, 1] \rightarrow Y$ be geodesics with $\gamma_1 = \eta_1$ and

$$d_Y(\gamma_0, \eta_0) + d_Y(\eta_0, \eta_1) + d_Y(\eta_1, \gamma_0) < 2\pi.$$

Then for every $t, s \in [0, 1]$ we have

$$d_Y^2(\gamma_t, \eta_s) \leq \frac{\sin^2(1-t)}{\sin^2 d_Y(\gamma_0, \gamma_1)} (d_Y(\gamma_0, \eta_0) - (d_Y(\eta_0, \eta_1) - d_Y(\gamma_0, \gamma_1))^2)$$

$$+ (1-t)^2(d_Y(\eta_0, \eta_1) - d_Y(\gamma_0, \gamma_1))^2$$

$$- 2(1-t)(s-t) d_Y(\gamma_0, \gamma_1)(d_Y(\eta_0, \eta_1) - d_Y(\gamma_0, \gamma_1))$$

$$\quad + \text{Cub}(s-t, d_Y(\gamma_0, \eta_0), d_Y(\eta_0, \eta_1) - d_Y(\gamma_0, \gamma_1), d_Y(\gamma_1, \eta_s)).$$

Remark 3.6 In Lemmas 3.4 and 3.5, we further see that in each term of Cub, at least one of the variables has an exponent which is greater than or equal to two.

3.3 Energy estimates

In this subsection we are going to obtain some energy estimates which will be exploited in the proof of our main theorem. Let $(X, d, m)$ be a locally uniformly doubling, strongly rectifiable space satisfying a Poincaré inequality, and let $\Omega$ be an open subset of $X$. Let $Y_o = (Y, d_Y, o)$ be a pointed CAT(1) space. For $\rho \in (0, \pi/2)$, we denote by $\overline{B}_\rho(o)$ the
closed ball in $Y$ of radius $\rho$ centered at $o$. In view of Proposition 3.2, $\tilde{B}_\rho(o) = (B_\rho(o), d_Y, o)$ is a pointed CAT(1) space with itself. We call $\tilde{B}_\rho(o)$ a regular ball.

For $u, v \in L^2(\Omega, \tilde{B}_\rho(o))$ and $t \in [0, 1]$, we define a map $G_{t}^{u,v} : \Omega \to \tilde{B}_\rho(o)$ by $x \mapsto G_{t}^{u(x),v(x)}$, where $G_{t}^{u(x),v(x)}$ denotes the unique geodesic from $u(x)$ to $v(x)$. By virtue of Proposition 3.2, $G_{t}^{u,v}$ belongs to $L^2(\Omega, \tilde{B}_\rho(o))$. By the same argument as in [22, Section 6], we see that if $u, v \in L^2(\Omega, \tilde{B}_\rho(o))$, then $G_{t}^{u,v} \in L^2(\Omega, \tilde{B}_\rho(o))$, and it is the unique geodesic from $u$ to $v$.

We begin with the following energy density estimate (cf. [37, Lemma 1.13]):

**Lemma 3.7** Let $u, v \in KS^{1,2}(\Omega, \tilde{B}_\rho(o)), m := G_{1/2}^{u,v}$ and $d := d_Y(u, v)$. Then we have $m \in KS^{1,2}(\Omega, \tilde{B}_\rho(o), d) \in KS^{1,2}(\Omega, \mathbb{R})$ and

$$
\cos^2 \frac{d}{2} e_g^2[m] + \frac{1}{4} e_g^2[d] \leq \frac{1}{2} (e_g^2[u] + e_g^2[v]) \quad m.a.e. \text{ on } \Omega. \tag{3.6}
$$

**Proof** We check $m \in KS^{1,2}(\Omega, \tilde{B}_\rho(o), d) \in KS^{1,2}(\Omega, \mathbb{R})$. The claim for $d$ immediately follows from the triangle inequality. For what concerns $m$ let us set $\alpha := 2(\pi - 2\rho) > 0$. For $x \in \Omega$, we define $\Omega_{u,v,x}, \Omega_{v,u,x}, \Omega_{u,v,x}$ as in (3.1). For all $y \in \Omega \setminus \Omega_{u,v,x}$, we have

$$
d_Y(u(x), v(x)) + d_Y(v(x), u(y)) + d_Y(u(y), u(x)) < 4\rho + \alpha = 2\pi,
$$

$$
d_Y(v(y), u(y)) + d_Y(u(y), v(x)) + d_Y(v(x), v(y)) < 4\rho + \alpha = 2\pi.
$$

Therefore, by using (3.5) for $l = 2\rho$ twice via the midpoint of $u(y)$ and $v(x)$, for all $y \in \Omega \setminus \Omega_{u,v,x}$,

$$
d_Y(m(x), m(y)) \leq c_\rho (d_Y(u(x), u(y)) + d_Y(v(x), v(y))
$$

for some constant $c_\rho > 0$ depending only on $\rho$. This implies

$$
\begin{align*}
ks^{2, r}_{2,r}[m, \Omega](x) &\leq \frac{2c_\rho^2}{m(B_r(x))} \int_{B_r(x) \setminus \Omega_{u,v,x}} \left( \frac{d_Y^2(u(x), u(y))}{r^2} + \frac{d_Y^2(v(x), v(y))}{r^2} \right) \, dm(y) \\
+ &\frac{1}{m(B_r(x))} \int_{B_r(x) \cap \Omega_{u,v,x}} \frac{d_Y^2(m(x), m(y))}{r^2} \, dm(y) \\
&\leq 2c_\rho^2 \left( \ks^{2, r}_{2,r}[u, \Omega](x) + \ks^{2, r}_{2,r}[v, \Omega](x) \right) + \frac{4\rho^2}{\alpha^2} \frac{m(B_r(x) \cap \Omega_{u,v,x,x})}{m(B_r(x))} \\
&\leq \left( 2c_\rho^2 + \frac{4\rho^2}{\alpha^2} \right) \left( \ks^{2, r}_{2,r}[u, \Omega](x) + \ks^{2, r}_{2,r}[v, \Omega](x) \right), \tag{3.7}
\end{align*}
$$

where we used

$$
\frac{\alpha^2}{r^2} \frac{m(B_r(x) \cap \Omega_{u,v,x,x})}{m(B_r(x))} \leq \frac{\alpha^2}{r^2} \left( \frac{m(B_r(x) \cap \Omega_{u,v,x,x})}{m(B_r(x))} + \frac{m(B_r(x) \cap \Omega_{v,u,x,x})}{m(B_r(x))} \right) \\
\leq \ks^{2, r}_{2,r}[u, \Omega](x) + \ks^{2, r}_{2,r}[v, \Omega](x).
$$

Integrating (3.7) over $\Omega$, and letting $r \downarrow 0$, we deduce $m \in KS^{1,2}(\Omega, \tilde{B}_\rho(o))$.

We prove (3.6). Now let us set $\beta := \pi - 2\rho > 0$ and for $x \in \Omega$ define $\Omega_{u,v,b,x}, \Omega_{u,b,x}, \Omega_{u,v,b,x}$ as in (3.1) again. For all $y \in \Omega \setminus \Omega_{u,v,b,x}$, there holds

$$
d_Y(u(x), v(x)) + d_Y(v(x), v(y)) + d_Y(v(y), u(y)) + d_Y(u(y), u(x)) < 2\beta + 4\rho = 2\pi.
$$
Lemma 3.4 tells us that for all \( y \in \Omega \setminus \Omega_{u,v,\beta,x} \),
\[
\cos^2 \frac{d(x)}{2} d_Y^2(m(x), m(y)) + \frac{1}{4}(d(x) - d(y))^2 \\
\leq \frac{1}{2} \left( d_Y^2(u(x), u(y)) + d_Y^2(v(x), v(y)) \right) \\
+ \text{Cub}(d_Y(u(x), u(y)), d_Y(v(x), v(y)), d(x) - d(y), d_Y(m(x), m(y))).
\]

We shall write the last term as \( \text{Cub}(x, y) \) for short. It follows that
\[
\cos^2 \frac{d(x)}{2} \text{ks}_{2,r}[m, \Omega; u, v, \beta](x) + \frac{1}{4}\text{ks}_{2,r}[d, \Omega; u, v, \beta](x) \\
\leq \frac{1}{2} \left( \text{ks}_{2,r}[u, \Omega](x) + \text{ks}_{2,r}[v, \Omega](x) \right) + \frac{1}{m(B_r(x))} \int_{B_r(x) \setminus \Omega_{u,v,\beta,x}} \frac{\text{Cub}(x, y)}{r^2} \, dm(y),
\]
where \( \text{ks}_{2,r}[m, \Omega; u, v, \beta](x) \) and \( \text{ks}_{2,r}[d, \Omega; u, v, \beta](x) \) are defined as (3.2).

By Theorem 2.8 and Lemma 3.1, and by the same argument in the proof of Lemma 3.1, m-a.e. \( x \in \Omega \) satisfies the following properties:

1. \( \text{ks}_{2,r}[m, \Omega; u, v, \beta](x) \rightarrow e_2[m](x), \text{ks}_{2,r}[d, \Omega; u, v, \beta](x) \rightarrow e_2[d](x) \) as \( r \downarrow 0 \);
2. \( \text{ks}_{2,r}[u, \Omega](x) \rightarrow e_2[u](x), \text{ks}_{2,r}[v, \Omega](x) \rightarrow e_2[v](x) \) as \( r \downarrow 0 \);
3. \( \text{lip } u(x), \text{lip } v(x), \text{lip } d(x), \text{lip } m(x) < +\infty \).

We now fix such a point \( x \in \Omega \). Let us verify
\[
\frac{1}{m(B_r(x))} \int_{B_r(x) \setminus \Omega_{u,v,\beta,x}} \frac{\text{Cub}(x, y)}{r^2} \, dm(y) \rightarrow 0
\]
as \( r \downarrow 0 \). Each term in (3.9) can be written as
\[
\frac{1}{m(B_r(x))} \int_{B_r(x) \setminus \Omega_{u,v,\beta,x}} \frac{d_Y^2(u(x), u(y))d_Y^2(v(x), v(y))(d(x) - d(y))^{\theta_1}d_Y^2(m(x), m(y))}{r^2} \, dm(y)
\]
(3.10)
for \( \theta_1, \theta_2, \theta_3, \theta_4 \geq 0 \) such that \( \theta_1 + \theta_2 + \theta_3 + \theta_4 \geq 3 \) and at least one of \( i = 1, \ldots, 4 \) has \( \theta_i \geq 2 \) (see Remark 3.6). Let us assume \( \theta_3 \geq 2 \) since the other cases can be handled in a similar way. For the absolute value of (3.10) we have
\[
\left| \frac{1}{m(B_r(x))} \int_{B_r(x) \setminus \Omega_{u,v,\beta,x}} \frac{d_Y^2(u(x), u(y))d_Y^2(v(x), v(y))(d(x) - d(y))^{\theta_1}d_Y^2(m(x), m(y))}{r^2} \, dm(y) \right|
\]
\[
\leq |f(x)|^{\theta_1+\theta_2+\theta_3+\theta_4-2} \text{ks}_{2,r}^2[d, \Omega; u, v, \beta](x) I,
\]
(3.11)
where
\[
I := \sup_{y \in B_r(x) \setminus \{x\}} \left( \frac{d_Y^2(u(x), u(y))}{d_Y^2(x, y)} \frac{d_Y^2(v(x), v(y))}{d_Y^2(x, y)} \frac{|d(x) - d(y)|^{\theta_3-2}}{d_Y^2(m(x), m(y))} \right).
\]
By \( \theta_1 + \theta_2 + \theta_3 + \theta_4 - 2 \geq 1 \) and by the choice of \( x \), the right-hand side of (3.11) goes to 0 as \( r \downarrow 0 \). Thus, we confirm the validity of (3.9). Letting \( r \downarrow 0 \) in (3.8), we can derive the desired assertion.

We next prove the following (cf. [37, Lemma 1.14]):
Lemma 3.8 Assume that \((X, d, m)\) is infinitesimally Hilbertian. For \(u \in \text{KS}^{1,2}(\Omega, \bar{B}_\rho(o)), \eta \in \text{KS}^{1,2}(\Omega, [0, 1]),\) we define \(u_\eta : \Omega \to \bar{B}_\rho(o)\) by \(u_\eta := G^{\eta}_{u,o}\) and \(d := d_Y(u, o).\) Then we have \(u_\eta \in \text{KS}^{1,2}(\Omega, \bar{B}_\rho(o)), d \in \text{KS}^{1,2}(\Omega, \mathbb{R})\) and

\[
e^2_2[u_\eta] \leq \frac{\sin^2(1 - \eta)d}{\sin^2 d} \left( e^2_2[u] - e^2_2[d] \right) + e^2_2[1 - \eta]d \quad \text{m-a.e. on } \Omega.
\]

**Proof** Let us verify \(u_\eta \in \text{KS}^{1,2}(\Omega, \bar{B}_\rho(o)), d \in \text{KS}^{1,2}(\Omega, \mathbb{R}).\) Similarly to the proof of Lemma 3.7, the thesis for \(d\) follows from the triangle inequality. By virtue of (3.5) for \(l = \rho,\) for all \(x, y \in \Omega,
\]

\[
d_Y(u_\eta(x), u_\eta(y)) \leq c_\rho d_Y(u(x), u(y)) + |\eta(x) - \eta(y)|d_Y(u(y), o) = c_\rho d_Y(u(x), u(y)) + \rho|\eta(x) - \eta(y)|
\]

for some constant \(c_\rho > 0\) depending only on \(\rho.\) Therefore,

\[
ks^2_2, r[u_\eta, \Omega](x) \leq 2c_\rho ks^2_2, r[u, \Omega](x) + 2\rho^2 ks^2_2, r[\eta, \Omega](x).
\]

Integrating this estimate over \(\Omega,\) and letting \(r \downarrow 0,\) we conclude \(u_\eta \in \text{KS}^{1,2}(\Omega, \bar{B}_\rho(o)).\)

Using Lemma 3.5, for all \(x, y \in \Omega\) we have

\[
d^2_Y(u_\eta(x), u_\eta(y)) \leq \frac{\sin^2(1 - \eta(x))d(x)}{\sin^2 d(x)} \left( d^2_Y(u(x), u(y)) - (d(y) - d(x))^2 \right) + (1 - \eta(x))^2(d(y) - d(x))^2 + d^2(x)(\eta(y) - \eta(x))^2 - 2(1 - \eta(x))d(x)(\eta(y) - \eta(x))(d(y) - d(x)) + \text{Cub}(\eta(y) - \eta(x), d_Y(u(x), u(y)), d(y) - d(x), d_Y(u_\eta(x), u_\eta(y))).
\]

Setting \(\xi := 1 - \eta,\) we also have

\[
d^2_Y(u_\eta(x), u_\eta(y)) \leq \frac{\sin^2 \xi(x)d(x)}{\sin^2 d(x)} \left( d^2_Y(u(x), u(y)) - (d(y) - d(x))^2 \right) + \xi^2(x)(d(y) - d(x))^2 + d^2(x)(\xi(y) - \xi(x))^2 + 2\xi(x)d(x)(\xi(y) - \xi(x))(d(y) - d(x)) + \text{Cub}(\xi(x) - \xi(y), d_Y(u(x), u(y)), d(y) - d(x), d_Y(u_\eta(x), u_\eta(y))) = \frac{\sin^2 \xi(x)d(x)}{\sin^2 d(x)} \left( d^2_Y(u(x), u(y)) - (d(y) - d(x))^2 \right) + \xi^2(x)(d(y) - d(x))^2 + d^2(x)(\xi(y) - \xi(x))^2 + \frac{\xi(x)d(x)}{2} \left[ ((\xi + d)(x) - (\xi + d)(y))^2 - ((\xi - d)(x) - (\xi - d)(y))^2 \right] + \text{Cub}(\xi(x) - \xi(y), d_Y(u(x), u(y)), d(y) - d(x), d_Y(u_\eta(x), u_\eta(y))).
\]
Dividing the later expression by \( r^2 \) and integrating in \( y \) over \( B_r(x) \) lead to

\[
\begin{align*}
k s_2^2[r, u_\eta, \Omega](x) & \leq \frac{\sin^2 \xi(x)d(x)}{\sin^2 d(x)} \left( k s_2^2[r, u, \Omega](x) - k s_2^2[r, d, \Omega](x) \right) \\
+ \xi^2(x) k s_2^2[r, d, \Omega](x) + d^2(x) k s_2^2[r, \xi, \Omega](x) & + \frac{\xi(x)d(x)}{2} \left( k s_2^2[r, \xi + d, \Omega](x) - k s_2^2[r, \xi - d, \Omega](x) \right) \\
+ \int_{B_r(x)} \frac{\text{Cub}(x, y)}{r^2} \, dm(y),
\end{align*}
\]

where \( \text{Cub}(x, y) \) denotes the cubic terms in (3.12). In the same manner as in the proof of Lemma 3.7, we see

\[
\int_{B_r(x)} \frac{\text{Cub}(x, y)}{r^2} \, dm(y) \to 0 \quad m\text{-a.e. } x \in \Omega \quad \text{as } r \downarrow 0.
\]

By virtue of Theorem 2.8, we obtain

\[
e_2^2[u_\eta] \leq \frac{\sin^2 \xi d}{\sin^2 d} \left( e_2^2[u] - e_2^2[d] \right) + \xi^2 e_2^2[d] + d^2 e_2^2[\xi] + \frac{\xi d}{2} \left( e_2^2[\xi + d] - e_2^2[\xi - d] \right) \quad m\text{-a.e. on } \Omega.
\]

Now, it suffices to prove that

\[
\xi^2 e_2^2[d] + d^2 e_2^2[\xi] + \frac{\xi d}{2} \left( e_2^2[\xi + d] - e_2^2[\xi - d] \right) = e_2^2[\xi d] \quad m\text{-a.e. on } \Omega.
\]

Thanks to Proposition 2.9, for any \( u \in \mathbb{K} S^{1,2}(\Omega, \mathbb{R}) \), we have \( |Du| = c_d e_2[u] \) \( m\text{-a.e. on } \Omega \), and hence it is enough to show that

\[
\xi^2 |Dd|^2 + d^2 |D\xi|^2 + \frac{\xi d}{2} \left( |D(\xi + d)|^2 - |D(\xi - d)|^2 \right) = |D(\xi d)|^2 \quad m\text{-a.e. on } \Omega.
\]

Since \( (X, d, m) \) is infinitesimally Hilbertian, Theorem 2.2 tells us that

\[
|D(\xi d)|^2 = \langle \nabla(\xi d), \nabla(\xi d) \rangle = \xi^2 |Dd|^2 + d^2 |\nabla\xi|^2 + 2\xi d(\nabla\xi, \nabla d) \quad m\text{-a.e. on } \Omega,
\]

where we used the Leibniz rule. Furthermore, by the linearity of the gradient,

\[
|D(\xi + d)|^2 - |D(\xi - d)|^2 = \langle \nabla(\xi + d), \nabla(\xi + d) \rangle - \langle \nabla(\xi - d), \nabla(\xi - d) \rangle
\]

\[
= 4\langle \nabla\xi, \nabla d \rangle \quad m\text{-a.e. on } \Omega.
\]

Combining (3.14) and (3.15) yields

\[
|D(\xi d)|^2 = \xi^2 |Dd|^2 + d^2 |D\xi|^2 + \frac{\xi d}{2} \left( |D(\xi + d)|^2 - |D(\xi - d)|^2 \right) \quad m\text{-a.e. on } \Omega,
\]

and this is nothing but (3.13), thus concluding the proof. \( \square \)

We now prove that the energy \( E^\Omega_2 \) satisfies a certain kind of convexity (cf. [22, Proposition 1.15]):

\( \text{Springer} \)
Proposition 3.9 Assume that\((X, \mathbf{d}, m)\) is infinitesimally Hilbertian. For \(u, v \in KS_{1,2}(\Omega, \tilde{B}_\rho(o))\), we set \(m := G_{1/2}^{u,v} \in KS_{1,2}(\Omega, \tilde{B}_\rho(o))\) and \(\mathbf{d} := \mathbf{d}_Y(u, v), \mathbf{d} := \mathbf{d}_Y(m, o) \in KS_{1,2}(\Omega, \mathbb{R})\). Let \(\eta \in KS_{1,2}(\Omega, [0, 1])\) be a function determined by solving
\[
\frac{\sin(1 - \eta)\partial}{\sin \partial} = \cos \frac{\mathbf{d}}{2},
\] (3.16)
and define \(m_\eta := G_{\eta}^{\mu, \nu} \in KS_{1,2}(\Omega, \tilde{B}_\rho(o))\). Then we have
\[
E^\Omega_2(m_\eta) + \cos^8 \rho E^\Omega_2\left(\tan \frac{\mathbf{d}}{2}\right) \leq \frac{1}{2} E^\Omega_2(u) + \frac{1}{2} E^\Omega_2(v).
\]

Proof Combining Lemmas 3.7 and 3.8, we have
\[
e_2^2[m_\eta] \leq \frac{\sin^2(1 - \eta)\partial}{\sin^2 \partial} (e_2^2[m] - e_2^2[\partial]) + e_2^2[(1 - \eta)\partial] = \cos^2 \frac{\mathbf{d}}{2} (e_2^2[m] - e_2^2[\partial]) + e_2^2[(1 - \eta)\partial]
\]
\[
\leq \frac{1}{2} (e_2^2[u] + e_2^2[v]) - \frac{1}{4} e_2^2[\mathbf{d}] - \cos^2 \frac{\mathbf{d}}{2} e_2^2[\partial] + e_2^2[(1 - \eta)\partial]\]
\]
\[
\text{m -a.e. on } \Omega.
\] (3.17)

We now observe that the following hold:
\[
e_2^2[(1 - \eta)\partial] + \frac{\cos^4 \frac{\mathbf{d}}{2} \cos^4 \partial}{1 - \cos^2 \frac{\mathbf{d}}{2} \sin^2 \partial} e_2^2\left[\tan \frac{\mathbf{d}}{2}\right]
\]
\[
= \cos^2 \frac{\mathbf{d}}{2} e_2^2[\partial] + \frac{1}{4} e_2^2[\mathbf{d}]\]
\]
\[
\text{m -a.e. on } \Omega.
\] (3.18)

In view of Proposition 2.9, this is equivalent to the following:
\[
|D((1 - \eta)\partial)|^2 + \frac{\cos^4 \frac{\mathbf{d}}{2} \cos^4 \partial}{1 - \cos^2 \frac{\mathbf{d}}{2} \sin^2 \partial} \left|D\left(\tan \frac{\mathbf{d}}{2}\right)\right|^2 = \cos^2 \frac{\mathbf{d}}{2} |\mathbf{d}|^2 + \frac{1}{4} |\mathbf{d}|^2\]
\]
\[
\text{m -a.e. on } \Omega.
\]

Since \((X, \mathbf{d}, m)\) is infinitesimally Hilbertian, it can be derived from Theorems 2.1, 2.2 (chain rule, Leibniz rule), the definition of \(\eta\), and a straightforward calculation. Combining (3.17) and (3.18), we obtain
\[
e_2^2[m_\eta] + \cos^8 \rho e_2^2\left[\tan \frac{\mathbf{d}}{2}\right] \leq e_2^2[m_\eta] + \cos^4 \frac{\mathbf{d}}{2} \cos^4 \partial e_2^2\left[\tan \frac{\mathbf{d}}{2}\right]
\]
\[
\leq e_2^2[m_\eta] + \frac{\cos^4 \frac{\mathbf{d}}{2} \cos^4 \partial}{1 - \cos^2 \frac{\mathbf{d}}{2} \sin^2 \partial} e_2^2\left[\tan \frac{\mathbf{d}}{2}\right]
\]
\[
\leq \frac{1}{2} (e_2^2[u] + e_2^2[v])\]
\]
\[
\text{m -a.e. on } \Omega.
\]

By integrating this inequality over \(\Omega\), we obtain the desired one. □

3.4 Proof of theorem 1.2

We are now in a position to prove Theorem 1.2.
Proof of Theorem 1.2 Let \((X, d, m)\) be an infinitesimally Hilbertian, locally uniformly doubling, strongly rectifiable space satisfying a Poincaré inequality, and let \(\Omega\) be a bounded open subset of \(X\) with \(m(X \setminus \Omega) > 0\). Let \(Y_0 = (Y, d_Y, o)\) be a pointed \(\text{CAT}(1)\) space, and let \(\bar{B}_p(o)\) be a regular ball. Let \((u_n)_n \subset KS^{1,2}_2(\Omega, \bar{B}_p(o))\) stand for a minimizing sequence of \(E^{\Omega}_{2,\bar{u}}\). By Proposition 2.10, the functional \(E^{\Omega}_{2,\bar{u}}\) is lower semicontinuous, and hence it is sufficient to show that \((u_n)_n\) is an \(L^2(\Omega, \bar{B}_p(o))\)-Cauchy sequence.

Set \(I := \lim_n E^{\Omega}_{2,\bar{u}}(u_n) = \inf E^{\Omega}_{2,\bar{u}}\). We also set
\[
m_{n,m} := G_{1/2}^{u_n,u_m}, \quad d_{n,m} := d_Y(u_n, u_m), \quad \delta_{n,m} := d_Y(m_{n,m}, o), \quad m_{n,m,\eta} := G_{m_{n,m},\eta},
\]
where \(\eta_{n,m} \in KS^{1,2}(\Omega, [0, 1])\) is defined as (3.16). Proposition 3.9 yields
\[
\cos^8 \rho \frac{\tan \frac{d_{n,m}}{2}}{\cos \delta_{n,m}} \leq \frac{1}{2} E^{\Omega}_{2,\bar{u}}(u_n) + \frac{1}{2} E^{\Omega}_{2,\bar{u}}(u_m) - E^{\Omega}_{2,\bar{u}}(m_{n,m,\eta}) \leq \frac{1}{2} E^{\Omega}_{2,\bar{u}}(u_n) + \frac{1}{2} E^{\Omega}_{2,\bar{u}}(u_m) - I;
\]
in particular,
\[
\lim_{n,m \to \infty} \int_{\Omega} \left| \frac{\tan \frac{d_{n,m}}{2}}{\cos \delta_{n,m}} \right|^2 \, dm = 0.
\]
Furthermore, a Poincaré inequality under Dirichlet boundary conditions (see e.g., [22, Lemma 6.3], [2, Subsection 5.5]) together with Proposition 2.10 leads us to
\[
\lim_{n,m \to \infty} \int_{\Omega} \left| \frac{\tan \frac{d_{n,m}}{2}}{\cos \delta_{n,m}} \right|^2 \, dm = 0.
\]
Thus, \(\lim_{n,m \to \infty} \int_{\Omega} |d_{n,m}|^2 \, dm = 0\) since \(|d_{n,m}|^2 \leq 4 |\tan(d_{n,m}/2)|^2\) and \(|\cos \delta_{n,m}|^2 \leq 1\) over \(\Omega\). This concludes the proof.

Acknowledgements The author is grateful to Keita Kunikawa for fruitful discussions during this work. The author expresses his gratitude to Shouhei Honda for valuable comments. The author would like to thank the anonymous referee for useful comments. The author was supported by JSPS KAKENHI (JP23K12967).

Author Contributions This paper is written by single author.

Declarations

Conflict of interest The authors declare no competing interests.

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