The \( k \)-path vertex cover: General bounds and chordal graphs

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Abstract

For an integer \( k \geq 3 \), a \( k \)-path vertex cover of a graph \( G = (V, E) \) is a set \( T \subseteq V \) that shares a vertex with every path subgraph of order \( k \) in \( G \). The minimum cardinality of a \( k \)-path vertex cover is denoted by \( \psi_k(G) \). We give estimates—mostly upper bounds—on \( \psi_k(G) \) in terms of various parameters, including vertex degrees and the number of vertices and edges. The problem is also considered on chordal graphs and planar graphs.

KEYWORDS

average degree, chordal graph, \( k \)-path vertex cover, planar graph, transversal, wireless sensor network

1 | INTRODUCTION

A \( k \)-path vertex cover (or \( P_k \)-transversal) in a graph \( G \) is a set \( T \subseteq V(G) \) that shares at least one vertex with every path of order \( k \) in \( G \); that is, \( G - T \) does not contain any \( P_k \) subgraph. In this article, we study the minimum cardinality of a \( k \)-path vertex cover in \( G \), denoted by \( \psi_k(G) \). We remark that the case of \( k = 2 \) means just the vertex cover (transversal) that is the complement of an independent set. It immediately follows that \( \psi_2(G) = |V(G)| - \alpha(G) \), where \( \alpha(G) \) denotes the size of a maximum stable set and is called the independence number of \( G \). Since these are central notions in graph theory and the literature contains many related deep results, here we concentrate on the cases with \( k \geq 3 \). The case of \( k = 3 \) is in close relation to the so-called dissociation number, which was first studied by Yannakakis [31]. A subset of vertices in a graph \( G \) is called a dissociation set if it induces a subgraph with maximum degree at most 1. The size of a maximum dissociation set in \( G \) is called the dissociation number of \( G \) and is denoted by \( \text{diss}(G) \). Thus, \( \psi_3(G) = |V(G)| - \text{diss}(G) \). In a part of the literature, the dissociation number is referred to as the \( 1 \)-independence number. For an integer \( k \geq 0 \), a \( k \)-independent set \( S \subseteq V(G) \) is a set of vertices in a graph \( G \) such that the maximum degree of the subgraph induced by \( S \) is at most \( k \). The \( k \)-independence number of \( G \) is the maximum cardinality of a \( k \)-independent set of \( G \), and is denoted by \( \alpha_k(G) \). The equality \( \psi_{k+2}(G) = |V(G)| - \alpha_k(G) \) then holds for every graph if \( k \in \{0, 1\} \) but it is not true in general if \( k \geq 2 \). The parameter \( \alpha_k(G) \) has been studied in several papers; see, for instance, [7-9, 17, 29] for some important lower bounds on \( \alpha_k(G) \) which we will refer to in our paper (Section 3). Several other connections can be found; for instance in [3], \( k \)-path vertex cover was shown to be related to \( k \)-path partition.

1.1 | Application in information technology

The motivation for the invariant \( \psi_k(G) \), which was introduced in [5], arises from communications in wireless sensor networks, where the data integrity is ensured by using Novotný’s \( k \)-generalized Canvas scheme [23]. The topology of wireless sensor networks can be represented by a graph, in which vertices represent sensor devices and edges represent communication channels between them. We suppose that there are protected and unprotected sensor devices in the model. The attacker is unable to copy secrets from a protected device; however, an unprotected device can be captured by the attacker who can gain control over it.
During the deployment and initialization of a sensor network, it should be ensured that at least one protected vertex exists on each path of order $k$ in the communication graph. The placement of protected sensors in a network is usually expensive. Thus an important goal is to minimize the cost of the network by minimizing the number of protected vertices, which coincides with the calculation of a minimum $k$-path vertex cover in a graph that represents such a network.

### 1.2 Known results on algorithmic complexity

It was proved in the pilot paper that the computation of $\psi_k(G)$ is in general an NP-hard problem for every $k$, but was shown to be solvable in linear time over the class of trees.

The $k$-path vertex cover problem has gained much attention in the area of computational graph theory. Approximation algorithms for $\psi_k(G)$ were given by Tu and Yang in [27] and Tu and Zhou in [28], and an exact algorithm for computing $\psi_k(G)$ in running time $O(1.5171^n)$ for a graph of order $n$ was presented in [15] and was later improved to $O^*(1.366^n)$ in [30]. The parameterized version of the same problem was also considered. The goal is to decide whether there is a $k$-path vertex cover in $G$ of size at most $\ell$, where $k$ and $\ell$ are fixed positive integers. A fixed parameter tractable (FPT) algorithm for the $3$-path vertex cover whose time complexity is $O^*(1.713^k)$ was given by Tsur in [26]. Tsur recently presented a FPT algorithm for the $4$-path vertex cover which runs in time $O^*(2.619^k)$ [25].

The weighted version of $k$-path vertex cover was introduced by Brešar, Krivoš-Belluš, Semanišin, and Šparl in [6]. In this version, vertices are given weights and the problem is to find a minimum-weight set such that the graph obtained by deleting this set of vertices has no path $P_k$ as a subgraph. Some special classes of graphs were considered, for instance, complete graphs and cycles, and an algorithm that computes the weighted $k$-path vertex cover number of a tree with time complexity $O(k \cdot |V(G)|)$ was presented. Approximation algorithms for the weighted and connected versions of the $k$-path vertex cover were considered in [19], and later also in [24] where the emphasis was primarily on $k = 3$. In both cases, an extra assumption was present, namely that the subgraph induced by a $k$-path vertex cover was connected. The connected version of the $k$-path vertex cover was also studied in [10, 21].

### 1.3 Known results on graph products

The $k$-path vertex cover problem was heavily studied on several types of graph products. Already in [4], the Cartesian product of two paths, that is, grid graphs was considered. In [20], those results were extended to the Cartesian product of three paths. The results on grid graphs were later improved in [14], and the ideas of the proofs were used on the strong product of paths. In the same paper, an upper and a lower bound were given for the $k$-path vertex cover of the lexicographic product of arbitrary graphs. The upper bound was shown to be tight for any choice of factor graphs when $k = 3$. The $k$-path vertex cover of the rooted product of arbitrary graphs was considered in [13].

### 1.4 Some known lower and upper bounds

Throughout many articles on $k$-path vertex cover the following problem appeared to gain much attention.

**Problem 1.** For a given $k$, determine the set of pairs $(a, b)$ of nonnegative reals such that $\psi_k(G) \leq an + bm$ holds for every graph with $n$ vertices and $m$ edges. Study the same also for interesting classes of graphs.

In the next theorems, we cite some tight related results from the paper of Brešar, Kardoš, Katrenič, and Semaniš [5], which will be applied later.

**Theorem 1** ([5]). If $T$ is a tree of order $n$ and $k$ an integer with $k \geq 3$, then $\psi_k(T) \leq n/k$.

**Theorem 2** ([5]). Let $G$ be a graph of order $n$ and size $m$. Then

$$\psi_3(G) \leq \frac{2n + m}{6} \quad \text{and} \quad \psi_3(G) \leq \frac{m}{2}.$$  

Moreover, if $G$ is subcubic, then $\psi_3(G) \leq n/2$ also holds.

Note that $\psi_3(G) \leq m/2$ follows from the procedure in which we sequentially remove a vertex of degree at least $2$ from $G$ and put it into the $3$-path vertex cover. During this procedure, we delete at most $m/2$ vertices and what remains at the end is a

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1 Suppressing polynomially bounded factors by the $O^*$-notation.
graph with maximum degree at most 1, hence $P_3$-free. Further, if $\Delta(G) \geq 3$, then the strict inequality $\psi_3(G) < m/2$ holds as we may remove a vertex and at least three incident edges in the first step of the procedure.

The relation $\psi_3(G) \leq n/2$ can also be obtained via an iterated algorithmic process. Here, starting with any partition $(V_1, V_2)$ of the vertex set of a subcubic graph, a vertex which has more than one neighbor in its partition class is moved to the other class, hence increasing the size of the cut, until no such vertices remain. Then each of $V_1$ and $V_2$ is a 3-path vertex cover.

In [4], the upper bound

$$\psi_3(G) \leq \frac{\ell}{\ell + 2} n + \frac{1}{(\ell + 1)(\ell + 2)} m$$

is established, where $\ell = \left\lceil \frac{m}{n} \right\rceil - 1$. Also this bound is tight.

1.5 | Our results

In this article, we continue this track of research and prove upper bounds on $\psi_k(G)$ in terms of various parameters, including vertex degrees and the number of vertices and edges. We generalize the result of [4] for $d$-regular graphs,

$$\psi_k(G) \geq \frac{d - k + 2}{2d - k + 2} n,$$

(1)
to arbitrary graphs in terms of minimum and maximum degree. We also consider the $k$-path vertex cover problem on some famous classes of graphs, such as chordal graphs and planar graphs.

Our article is structured as follows. In Section 2, we study Problem 1 and prove general upper bounds on $\psi_3(G)$, $\psi_4(G)$, and $\psi_k(G)$ in terms of the order and size of $G$. We also present a method that can be applied to generate, from feasible pairs $(a, b)$ of Problem 1, further feasible pairs $(a', b')$. In Section 3, we give lower and upper bounds on $\psi_k(G)$ with functions of vertex degrees. Chordal graphs are considered in Section 4. Section 5 describes a way to improve the estimates further and contains more open problems. In particular, Section 5.2 discusses the problem on planar graphs. Some further problems are mentioned in the concluding section.

1.6 | Notation

We use standard notation as follows. For a simple undirected graph $G$, we usually denote the order $|V(G)|$ and the size $|E(G)|$ by $n$ and $m$. The notations $\delta(G)$, $\Delta(G)$, $\chi(G)$, and $\omega(G)$ stand for the minimum vertex degree, the maximum vertex degree, the chromatic number, and the clique number of $G$, respectively. We will simply write $\delta$, $\Delta$, $\chi$, and $\omega$ if $G$ is clear from the context. The degree of a vertex $v \in V(G)$ in graph $G$ is denoted by $d_G(v)$, and the set of its neighbors is denoted by $N_G(v)$ and called the open neighborhood of $v$. Moreover, we define $N_G(v) = N_G(v) \cup \{v\}$ as the closed neighborhood of $v$. Also here, if $G$ is understood, we will simply write $d(v)$, $N(v)$ and $N[v]$. The average vertex degree is $\bar{d} = \frac{1}{2} |V(G)| = 2m/n$. The subgraph induced by vertex set $Y \subseteq V(G)$ in $G$ is referred to as $G[Y]$. Notations $P_k$, $C_k$, and $K_k$ stand for the path, cycle, and complete graph of order $k$, respectively.

2 | UPPER BOUNDS RELATED TO PROBLEM 1

Here our goal is to make a modest step in the direction of solving Problem 1. Suppose that $an + bm$ is a general upper bound on $\psi_k(G)$ for an integer $k \geq 3$. If $b = 0$, the best estimation is obtained with $a = 1$, as shown by complete graphs which have $\psi_k(K_n) = n - k + 1$; and for $a = 0$ with $k = 3$ the best pair is $b = 1/2$. The corresponding upper bound is attained by graphs in which every component is a $P_3$. Also, $(a, b) = (1/3, 1/6)$ provides a valid upper bound, by Theorem 2.

Theorem 3. If $G$ is a graph of order $n$ and size $m$, then

$$\psi_3(G) \leq \frac{n + m}{4},$$

where equality holds if and only if all components of $G$ are isomorphic to $C_4$.

Proof. We proceed by induction on $n$ and suppose that $G$ is given on $n \geq 3$ vertices. If $\Delta(G) \leq 3$, the inequalities $\psi_3(G) \leq n/2$ and $\psi_3(G) \leq m/2$ from Theorem 2 imply the statement. Otherwise, there exists a vertex $v \in V(G)$ with $d_G(v) \geq 4$. By the induction hypothesis, $G' = G - v$ satisfies the inequality. Note that $G'$ has $n' = n - 1$ vertices and
m’ ≤ m – 4 edges and further, if T’ is a 3-path vertex cover in G’, then T’ ∪ {v} is a 3-path vertex cover in G and therefore ψ_3(G) ≤ ψ_3(G’) + 1. We may conclude that

ψ_3(G) ≤ ψ_3(G’) + 1 ≤ \frac{n’ + m’}{4} + 1 ≤ \frac{n + m - 5}{4} + 1 < \frac{n + m}{4}.

This completes the proof for the upper bound.

Now, we consider the graphs attaining the upper bound in the theorem. Let G be any graph with ψ_3(G) = \frac{n + m}{4}. It suffices to prove that if G is connected, then G ≅ C_4. By Theorem 2, it follows from the upper bounds (2n + m)/6 and m/2 that n = m, for otherwise, if n < m, then (2n + m)/6 = (4n + 2m)/12 < (3n + 3m)/12 = (n + m)/4, and if m < n, then m/2 = 2m/4 < (n + m)/4. As we discussed after Theorem 2, Δ(G) ≥ 3 would imply ψ_3(G) < m/2 and, therefore, ψ_3(G) < \frac{n + m}{4} which is a contradiction. Hence, G satisfies Δ(G) ≤ 2 and n = m. This implies G ≅ C_n. Then ψ_3(G) = \left\lceil \frac{n}{3} \right\rceil,

which is easily seen to be smaller than n/2 unless n = 4.

The result in Theorem 3 was also independently obtained by Zak [32, Theorem 7], when studying H-transversals in hypergraphs, where H denotes a family of hypergraphs. Namely, an H-transversal of a hypergraph G is a set of vertices that intersects the vertex set of every subgraph H of G for which H ∈ H. If H = {P_k}, then the H-transversal is our k-path vertex cover.

**Theorem 4.** Let G be a graph of order n and size m. Then

ψ_3(G) ≤ \frac{4n + m}{9}

and the bound is sharp.

**Proof.** By Theorem 2, every graph satisfies ψ_3(G) ≤ \frac{2n + m}{6}. To show that ψ_3(G) ≤ \frac{4n + m}{9} is also true, we first note that the inequality clearly holds for graphs of order at most 2. Then, we proceed by induction on the order n of G and consider two cases.

- If m ≤ 2n, the inequality chain

\frac{4n + m}{9} = \frac{n}{3} + \frac{m}{6} + \left( \frac{n}{9} - \frac{m}{18} \right) ≥ \frac{n}{3} + \frac{m}{6} ≥ ψ_3(G)

establishes the statement without using the induction hypothesis.

- If m > 2n, then Δ(G) ≥ 5 follows. Let v be a vertex of degree at least 5. Deleting v from G, we obtain the graph G’ which is of order n’ = n – 1 and of size m’ ≤ m – 5. By the induction hypothesis, G’ satisfies the inequality, that is, ψ_3(G’) ≤ (4n’ + m’)/9. Moreover, if T’ is a 3-path vertex cover in G’, then T’ ∪ {v} covers all P_3 subgraphs in G. This yields ψ_3(G) ≤ ψ_3(G’) + 1. Then, we have the desired result by the following relations:

ψ_3(G) ≤ ψ_3(G’) + 1 ≤ \frac{4n’ + m’}{9} + 1 ≤ \frac{4(n – 1) + (m – 5)}{9} + 1 = \frac{4n + m}{9}.

To prove sharpness, we take G = K_6 – M which is obtained from the complete graph K_6 by removing a perfect matching M. Then we have n = 6, m = 12, and ψ_3(G) = 4.

We also present a general result that shows how one can obtain new feasible pairs (a’, b’) if a feasible pair (a, b) is already given.

**Theorem 5.** Let a and b be nonnegative reals and k an integer with k ≥ 3 such that ψ_k(G) ≤ an + bm holds for every graph G. Moreover, let x be an arbitrary positive integer which satisfies

\frac{1 - a - b}{b} ≤ x ≤ \frac{2 - 2a}{b},

(2)

and define a’ and b’ as follows:

a’ = a + bx - \frac{x(1 + b - a)}{x + 2}, \quad b’ = \frac{2 - 2a - bx}{x + 2}.

Then, the inequality

ψ_k(G) ≤ a’n + b’m

holds for every graph G.
Proof. Let \( a, b, a', b' \), and \( x \) satisfy the conditions in the theorem. Note that we need the condition \( a - b \leq 1 \) in order for the inequality (2) to make sense. We introduce the notation

\[
y = \frac{x(a + b + bx - 1)}{x + 2}.
\]

By (2), we have \( a + b + bx - 1 \geq 0 \) which implies \( y \geq 0 \). One can check that \( a' = a + y \) and \( b' = b - \frac{2y}{x} \). The second inequality in (2) states that \( bx \leq 2 - 2a \). Now the following sequence of equivalent formulas is valid:

\[
\begin{align*}
bx & \geq 2y, \\
bx & \geq 2x(a + b + bx - 1)/(x + 2), \\
b & \geq 2(a + b + bx - 1)/(x + 2), \\
bx + 2b & \geq 2a + 2b + 2bx - 2, \\
0 & \geq 2a + bx - 2, \\
2 - 2a & \geq bx.
\end{align*}
\]

As the second inequality in (2) is equivalent to \( bx \geq 2y \), we have \( b' \geq 0 \) and \( a' \geq a \geq 0 \). Consequently, the right-hand side of (3) is always nonnegative and inequality (3) clearly holds for small graphs with \( n \leq k - 1 \). We proceed by induction on \( n \). Consider a graph \( G \) of order \( n \) and size \( m \).

- First, suppose that \( m \leq \frac{an}{x} \). This implies \( n - 2m/x \geq 0 \). Using this relation and rewriting the formula \( a'n + b'm \), we get

\[
a'n + b'm = (a + y)n + \left(b - \frac{2y}{x}\right)m = (an + bm) + y\left(n - \frac{2m}{x}\right) \geq an + bm \geq \psi_k(G),
\]

which verifies (3) for the first case.

- Next, suppose that the strict inequality \( m > \frac{an}{x} \) holds. It implies \( \Delta(G) > x \) and therefore \( \Delta(G) \geq x + 1 \) holds. Let \( v \) be a vertex with \( d_G(v) \geq x + 1 \). Deleting \( v \) from \( G \), we obtain a graph \( G' \) with \( n' = n - 1 \) vertices and \( m' \leq m - (x + 1) \) edges. Since every edge that was deleted from \( G \) is covered by \( v \), we have \( \psi_k(G) \leq \psi_k(G') + 1 \). By our hypothesis, \( G' \) satisfies (3). The following computation proves the same property for \( G \):

\[
a'n + b'm \geq \left(a'n' + b'm'\right) + a' + b'(x + 1) \\
\geq \psi_k(G') + (a + y) + \left(b - \frac{2y}{x}\right)(x + 1) \\
= \psi_k(G') + a + bx + b - \frac{x + 2}{x}y \\
= \psi_k(G') + a + bx + b - (a + b + bx - 1) \\
= \psi_k(G') + 1 \geq \psi_k(G).
\]

This finishes the proof of the theorem. \( \square \)

Since the formulation of the above theorem is quite technical, we present some upper bounds which can be obtained starting with Theorem 4 and applying Theorem 5 iteratively. We begin with the case of \( k = 3 \) and with the upper bound \( \frac{3}{9}n + \frac{5}{9}m \) and set \( a_0 = 4/9, b_0 = 1/9 \). Then for each \( 1 \leq i \leq 10 \), we apply Theorem 5 with \( x = i + 4, a = a_{i-1} \), and \( b = b_{i-1} \) to obtain \( a_i = a' \) and \( b_i = b' \).

**Corollary 1.** For every graph \( G \) of order \( n \) and size \( m \), the following inequalities hold:

\[
\begin{align*}
\psi_3(G) & \leq \frac{11}{27}n + \frac{5}{63}m, \\
\psi_3(G) & \leq \frac{7}{12}n + \frac{5}{84}m, \\
\psi_3(G) & \leq \frac{17}{27}n + \frac{5}{108}m, \\
\psi_3(G) & \leq \frac{2}{3}n + \frac{1}{27}m, \\
\psi_3(G) & \leq \frac{23}{33}n + \frac{1}{33}m.
\end{align*}
\]
\[
\begin{align*}
\psi_3(G) & \leq \frac{13}{18} n + \frac{5}{198} m, \\
\psi_3(G) & \leq \frac{29}{39} n + \frac{5}{234} m, \\
\psi_3(G) & \leq \frac{16}{21} n + \frac{5}{273} m, \\
\psi_3(G) & \leq \frac{7}{9} n + \frac{1}{63} m, \\
\psi_3(G) & \leq \frac{19}{24} n + \frac{1}{72} m.
\end{align*}
\]

**Proposition 1.** If \( G \) is a graph of maximum degree \( \Delta(G) \leq 2 \), then

\[
\psi_k(G) \leq \frac{2}{k+1} n \quad \text{and} \quad \psi_k(G) \leq \frac{2}{k+1} m
\]

hold for every \( k \geq 3 \). Further, the first inequality is tight if and only if each component of \( G \) is a \((k+1)\)-cycle; the second one is tight if and only if each component is either an isolated vertex or a \((k+1)\)-cycle.

**Proof.** It follows from \( \Delta(G) \leq 2 \) that every component of \( G \) is either a path or a cycle. Since \( \psi_k \) is an additive invariant and

\[
\psi_k(P_n) = \left\lfloor \frac{n}{k} \right\rfloor = \left\lfloor \frac{m(P_n) + 1}{k} \right\rfloor \leq \psi_k(C_n) = \left\lfloor \frac{n}{k} \right\rfloor = \left\lfloor \frac{m(C_n)}{k} \right\rfloor
\]

holds for every \( n \geq k \), moreover \( \psi_k(P_n) = \psi_k(C_n) = 0 \) holds if \( n < k \), we easily derive

\[
\psi_k(G) \leq \frac{2}{k+1} m \leq \frac{2}{k+1} n
\]

and identify the sharp cases.

**Theorem 6.** Let \( G \) be a graph of order \( n \) and size \( m \). Then

\[
\psi_4(G) \leq \frac{n + 3m}{10}.
\]

**Proof.** We proceed by induction on \( n \). If \( \Delta(G) \leq 2 \), then \( 5\psi_4(G) \leq 2n + 15\psi_4 \leq 6m \) follow by Proposition 1. Therefore, we have \( 20\psi_4(G) \leq 2n + 6m \) and the statement follows. If \( \Delta(G) \geq 3 \), there is a vertex \( v \) of degree at least 3 and we may consider the graph \( G' = G - v \). Moreover, we have \( \psi_4(G) \leq \psi_4(G') + 1 \). Since \( G' \) has \( n' = n - 1 \) vertices, \( m' \leq m - 3 \) edges and, by our hypothesis, it satisfies \( \psi_4(G') \leq \frac{n' + 3m'}{10} \),

\[
\psi_4(G) \leq \psi_4(G') + 1 \leq \frac{n' + 3m'}{10} + 1 \leq \frac{(n-1) + 3(m-3)}{10} + 1 = \frac{n + 3m}{10}
\]

is obtained which completes the proof.

Theorem 6 holds with equality if each component of \( G \) is a \( C_5 \). (Currently the \( 5\)-cycle is the only connected graph for which we know the equality.) Moreover, using Theorem 5, further upper bounds on \( \psi_4 \) can be generated. However, we do not list such bounds here. Instead, generalizing Theorem 5, we close this section with describing a general class of recursions showing how further feasible pairs \( (a', b') \) can be obtained from an already known feasible pair \( (a, b) \).

**Theorem 7.** Assume that \( \psi_3(G) \leq an + bm \) is a universally valid inequality for all graphs \( G \). Suppose further that the real numbers \( a, b > 0 \) and the integer \( x > 0 \) satisfy the following inequalities:

\[
w \leq 2q, \quad a + qx < 1, \quad w \leq \frac{(a + b - 1) + (q + b)x}{x + 1}.
\]

Then \( \psi_k(G) \leq a'n + b'm \) also holds for every graph \( G \) with the values

\[
a' = a + qx, \quad b' = b - w.
\]

**Proof.** Let us note first that under the given conditions we have \( b' > 0 \). Indeed, from the assumed inequalities we obtain

\[
1 \leq a + qx + bx + b - (x + 1)w < 1 + (b - w)(1 + x).
\]

If \( G \) has average degree at most \( x \), then \( m \leq xn/2 \) holds, thus

\[
\psi_k(G) \leq an + bm = (a' - qx) n + (b' + w) m = a'n + b'm - (qx - wm) \leq a'n + b'm - (qx - 2wm) \leq a'n + b'm.
\]
Otherwise there exists a vertex \( v \) whose degree is at least \( x + 1 \). Supplementing any \( k \)-path vertex cover of the graph \( G' = G - v \) with \( v \), we obtain a \( k \)-path vertex cover of \( G \). Here \( G' \) has order \( n' = n - 1 \) and size \( m' \leq m - x - 1 \). Hence, assuming by induction that \((a', b')\) is a feasible pair for the smaller graph \( G'\), we obtain:

\[
\psi_k(G) \leq 1 + \psi_k(G') \leq 1 + a'(n - 1) + b'(m - x - 1) \\
= a'n + b'm - (a + qx + (b - w)(x + 1) - 1) \\
\leq a'n + b'm.
\]

Based on this theorem, one can derive an infinite class of feasible pairs \((a', b')\) already from one single pair \((a, b)\). Nevertheless, we think that the most interesting case occurs when the two upper-bound conditions on \(w\) coincide. This special value is the one exposed in Theorem 5. More explicitly, it is obtained by substituting \( q = \frac{(a+b+hx-1)}{x+2} \) and \( w = 2q \).

### 3 | ESTIMATES IN TERMS OF VERTEX DEGREES

In this section, we give general estimates on \( \psi_k(G) \) in terms of the order and of the minimum, maximum, and average vertex degrees.

#### 3.1 | In terms of \( \delta \) and \( \Delta \)

In the proof of the first general upper bound, we will refer to the following decomposition theorem of Lovász.

**Theorem 8** ([22]). If \( a \) and \( b \) are nonnegative integers and \( G \) is a graph of maximum degree at most \( a + b + 1 \), then the vertex set of \( G \) can be partitioned into two sets which induce subgraphs of maximum degree at most \( a \) and \( b \).

The following theorem was proved by Brešar, Kardoš, Katrenič, and Semaniš [5] in 2011.

**Theorem 9** ([5]). Let \( G \) be a graph of order \( n \) and of maximum degree \( \Delta \). Then

\[
\psi_3(G) \leq \left\lfloor \frac{\Delta + 1}{2} \right\rfloor n.
\]

By definition, \( \psi_k(G) \leq \psi_3(G) \) if \( k \geq 3 \). Theorem 9 therefore implies \( \psi_k(G) \leq \frac{\Delta + 1}{k+1} n \) for every \( k \geq 3 \) if \( \Delta \) is odd and \( \psi_k(G) \leq \frac{\Delta - 1}{2} n \) if \( \Delta \) is even. For \( k = 3 \) and any odd \( \Delta \) the bound is tight, as shown by the complete graphs \( K_{\Delta+1} \). Similarly, omitting a perfect matching from \( K_{\Delta+2} \) we obtain tight examples for all even \( \Delta \) with \( k = 3 \).

Our next result shows that the upper bound can be improved for all \( k \geq 4 \) and all \( \Delta \geq 4 \). Note that part (i) of Theorem 10 gives the same bound as Theorem 9 for \( k = 3 \) and that the bound stated for \( \Delta = 2 \) here is exactly the same as the one in Proposition 1.

**Theorem 10.** Let \( k \) and \( \Delta \) be integers with \( k \geq 3 \) and \( \Delta = 2 \) or \( \Delta \geq 4 \), and let \( G \) be a graph of order \( n \) and of maximum degree at most \( \Delta \). Then the following hold.

(i) If \( \Delta \geq 2 \) is even, then

\[
\psi_k(G) \leq \frac{(k-1)(\Delta - 2) + 4}{(k-1)\Delta + 4} n.
\]

(ii) If \( \Delta \geq 5 \) is odd, then

\[
\psi_k(G) \leq \frac{(k-1)(\Delta - 3) + 8}{(k-1)(\Delta - 1) + 8} n.
\]

**Proof.** We fix an integer \( k \geq 3 \) and proceed by induction on \( \Delta \). If \( \Delta(G) \leq 2 \) then, by Proposition 1, \( \psi_k(G) \leq \frac{2}{k+1} n \) holds. This verifies the basic case of (i). Now, assume that \( \Delta \) is even, \( \Delta \geq 4 \), and that \( G \) has maximum degree at most \( \Delta \). By Theorem 8, \( V(G) \) can be partitioned into two sets \( A \) and \( B \), such that \( \Delta(G[A]) \leq \Delta - 2 \) and \( \Delta(G[B]) \leq 1 \). Since \( G[B] \) is \( P_k \)-free, \( \psi_k(G) \leq \lvert A \rvert \) holds. On the other hand, if \( T \) is a minimum \( k \)-path vertex cover in \( G[A] \), then \( B \cup T \) covers
all paths of order \( k \) in \( G \), and we have \( \psi_k(G) \leq |B| + |T| \). If \( |A| \leq \frac{(k - 1)(\Delta - 2) + 4}{(k - 1)\Delta + 4} \) \( n \) holds, then \( \psi_k(G) \leq |A| \) establishes the upper bound. In the other case, we observe that

\[
\psi_k(G) \leq |B| + |T| \leq n - |A| + \frac{(k - 1)(\Delta - 4) + 4}{(k - 1)(\Delta - 2) + 4} |A| = n - \frac{2(k - 1)}{(k - 1)(\Delta - 2) + 4} |A| < \frac{(k - 1)(\Delta - 2) + 4}{(k - 1)\Delta + 4} n.
\]

This proves (i).

Suppose next that \( \Delta \) is odd and \( \Delta \geq 5 \). Say, \( \Delta = 2a + 2b + 1 \) where \( a, b \) are positive integers. From Theorem 8, we know that there exists a vertex partition \( V(G) = A \cup B \) such that \( \Delta(G[A]) \leq 2a \) and \( \Delta(G[B]) \leq 2b \). Moreover, by (i), there exist \( k \)-path vertex covers \( T_A \) in \( G[A] \) and \( T_B \) in \( G[B] \) with

\[
|T_A| \leq \frac{(k - 1)(a - 1) + 2}{(k - 1)a + 2} |A|, \quad |T_B| \leq \frac{(k - 1)(b - 1) + 2}{(k - 1)b + 2} |B|.
\]

Both \( T_A \cup B \) and \( T_B \cup A \) are \( k \)-path vertex covers of \( G \). Hence, denoting \( x = |A|/n \) and \( y = |B|/n = 1 - x \), we have

\[
\frac{\psi_k(G)}{n} \leq \min \left( \frac{(k - 1)(a - 1) + 2}{(k - 1)a + 2} x + (1 - x), \frac{(k - 1)(b - 1) + 2}{(k - 1)b + 2} (1 - x) + x \right).
\]

The first term of this minimization is a decreasing function of \( x \) while the second term is an increasing function. Hence, the upper bound never exceeds the one obtained when the two numbers are equal, that is

\[
\frac{k - 1}{(k - 1)b + 2} \left( \frac{1}{(k - 1)a + 2} + \frac{1}{(k - 1)b + 2} \right) x, \quad 1 = \frac{(k - 1)b + 2}{(k - 1)a + 2} + 1 x.
\]

Recalling that \( 2a + 2b = \Delta - 1 \), the worst case occurs when

\[
x = \frac{(k - 1)(2a) + 4}{(k - 1)(\Delta - 1) + 8}, \quad y = \frac{(k - 1)(2b) + 4}{(k - 1)(\Delta - 1) + 8}.
\]

Thus, substituting this particular value of \( x \) into the left-hand side of (4), we obtain

\[
\frac{\psi_k(G)}{n} \leq \frac{[(k - 1)(2a - 2) + 4] + [(k - 1)(2b - 2) + 4]}{(k - 1)(\Delta - 1) + 8} = \frac{(k - 1)(\Delta - 3) + 8}{(k - 1)(\Delta - 1) + 8}.
\]

This proves (ii) and completes the proof of the theorem.

For \( \Delta = 2, 4, 5, 6, 7, 8, 9, \ldots \), the following sequence is obtained for the coefficients of \( n \), which is strictly increasing for every \( k \geq 4 \):

\[
\frac{1}{2} < \frac{k + 1}{2k} < \frac{k + 3}{2k + 2} < \frac{2k}{3k - 1} < \frac{2k + 2}{3k + 1} < \frac{3k - 1}{4k - 2} < \frac{3k + 1}{4k} < \cdots.
\]

To derive a general lower bound in terms of minimum and maximum degree, we need the following extremal result due to Erdős and Gallai.

**Theorem 11** ([11]). If \( G \) is a graph on \( n \) vertices that does not contain a path of order \( k \), then it cannot have more than \( \frac{n(k-2)}{2} \) edges. Moreover, the bound is achieved when the graph consists of vertex-disjoint cliques on \( k - 1 \) vertices.

Now we are in a position to extend the inequality (1) from regular graphs to arbitrary graphs, using the approach in [4].

**Theorem 12.** Let \( G \) be a graph of order \( n \) with minimum degree \( \delta \) and maximum degree \( \Delta \). If \( k \geq 3 \) and \( \delta \geq k - 1 \), then

\[
\psi_k(G) \geq \frac{\delta - k + 2}{\delta + \Delta - k + 2} n.
\]

**Proof.** Let \( T \subseteq V(G) \) be a minimum \( k \)-path vertex cover of \( G \), that is, \( |T| = \psi_k(G) \), and let \( T = V(G) - T \). Further, let \( E_T \) and \( E_T^\perp \) be the set of edges induced by \( T \) and by \( T \), respectively, and let \( E_T\) be the set of edges with one endvertex in \( T \) and the other in \( T \). We observe that
In terms of degree sequence and average degree

In this section, we improve the following upper bound that was proved in [5]:

$$\sum_{v \in V(G)} d(v) = 2|E(G)| = 2|E_T| + 2|E_{T'}| + 2|E_{T''}|$$

and also that $$\Delta \cdot |T| \geq 2|E_T| + |E_{T'}|$$ and $$\delta \cdot |T| \leq |E_{T'}| + 2|E_{T''}|$$. Since the subgraph induced by $$T$$ with edge set $$E_T$$ does not contain a path $$P_k$$, Theorem 11 implies $$|E_{T'}| \leq \frac{|T|}{k-2}$$. From these formulas, we get

$$|T| \geq \frac{1}{\Delta} (2|E_T| + |E_{T'}|) \geq \frac{1}{\Delta} |E_{T'}| \geq \frac{1}{\Delta} (\delta |T| - |E_{T'}|)$$

$$\geq \frac{1}{\Delta} (\delta |T| - |T|(k-2)) = \frac{\delta - k + 2}{\Delta} |T|.$$ 

It follows that

$$n = |T| + |T| \leq |T| + \frac{\Delta}{\delta - k + 2} |T| = \frac{\delta + \Delta - k + 2}{\delta - k + 2} |T|$$

and

$$\psi_k(G) = |T| \geq \frac{\delta - k + 2}{\delta + \Delta - k + 2} n.$$ 

\[ \blacksquare \]

3.2 In terms of degree sequence and average degree

In this section, we improve the following upper bound that was proved in [5]:

$$\psi_k(G) \leq n - \frac{2k-2}{k} \sum_{v \in V(G)} \frac{1}{1 + d(v)}.$$ 

**Theorem 13.** Let $$G$$ be a graph of order $$n$$, without isolated vertices. Then for every integer $$k \geq 3$$ we have

$$\psi_k(G) \leq n - \frac{2k-3}{k-1} \sum_{v \in V(G)} \frac{1}{1 + d(v)}.$$ 

**Proof.** We take an ordering $$v_1, v_2, \ldots, v_n$$ of the vertex set at random, each of the $$n$$! orderings being equally likely. Having an ordering, we define a weight function $$w : V(G) \to \left\{ 0, \frac{k-2}{k-1}, 1 \right\}$$ as follows. For each vertex $$v_i$$, let $$w(v_i) = 1$$ if $$v_i$$ has no neighbor among $$v_1, \ldots, v_{i-1}$$; let $$w(v_i) = (k-2)/(k-1)$$ if $$v_i$$ has exactly one neighbor with an index smaller than $$i$$; and let $$w(v_i) = 0$$ otherwise. Define $$Y$$ as the set of vertices assigned with positive weights. Hence, a vertex $$v$$ belongs to $$Y$$ if and only if it appears as first or second vertex when the order is restricted to the closed neighborhood $$N[v]$$ of $$v$$. Since every order has the same probability and the permutation of $$N[v]$$ is random, each of the $$d(v) + 1$$ positions is equally likely to be taken by $$v$$. Consequently the expected total weight is

$$E = \left( 1 + \frac{k-2}{k-1} \right) \sum_{v \in V(G)} \frac{1}{1 + d(v)}.$$ 

Expectation of a random variable means a certain average over all outcomes of the event, thus there must exist a vertex order under which the total weight on the vertices of $$Y$$ is at least $$E$$. We choose such an ordering and work with it from this point on. Observe that $$G[Y]$$ is a forest. Indeed, should a cycle occur, its vertex of largest index would have at least two neighbors preceding it, a contradiction to the definition of $$Y$$. We remark further that any component of $$G[Y]$$ may contain at most one vertex of weight 1, that is, the first vertex in every component according to the ordering we fixed. Otherwise adding an edge between two such vertices would result in the same set $$Y$$ and the subgraph induced by $$Y$$ would contain a cycle which is a contradiction.

Now, we prove that $$G[Y]$$ contains a set $$S$$ of vertices such that $$G[S]$$ is $$P_k$$-free and $$|S| \geq \sum_{v \in Y} w(v)$$. It suffices to prove the analogous statement for the components of $$G[Y]$$. If $$G[Y_i]$$ is a component that does not contain any $$P_k$$-subgraphs, we simply set $$S_i = Y_i$$ and $$|S_i| \geq \sum_{v \in Y_i} w(v)$$ clearly holds. Otherwise, consider a minimum $$k$$-path vertex cover $$T_i$$ and its complement $$S_i$$ in the tree $$G[Y_i]$$. By Theorem 1, we have $$|S_i| \geq (k-1)|T_i|$$ and under the present assumption $$|S_i| \geq k - 1$$ holds. Moreover, at most one vertex in $$Y_i$$ may have a weight of 1, say a vertex in $$T_i$$ (the proof is the same if this vertex is from $$S_i$$). We infer the following:

$$\sum_{v \in Y_i} w(v) = \sum_{v \in T_i} w(v) + \sum_{v \in S_i} w(v) \leq \frac{k-2}{k-1} (|T_i| - 1) + 1 + \frac{k-2}{k-1} |S_i|$$

$$= \frac{k-2}{k-1} |T_i| + \frac{k-2}{k-1} |S_i| + \frac{1}{k-1} \leq \left( \frac{k-2}{(k-1)^2} + \frac{k-2}{k-1} \right) |S_i| + \frac{1}{k-1}$$

$$= |S_i| - \frac{1}{(k-1)^2} |S_i| + \frac{1}{k-1} \leq |S_i|.$$
The set \( S = \bigcup S_i \) induces a \( P_k \)-free subgraph in \( G \) and satisfies
\[
|S| \geq \sum_{v \in V(G)} w(v) \geq E \geq \frac{2k - 3}{k - 1} \sum_{v \in V(G)} \frac{1}{1 + d(v)}.
\]
Since the complement \( T = V(G) \setminus S \) is a \( k \)-path vertex cover in \( G \), we conclude that
\[
\psi_k(G) \leq |T| \leq n - \frac{2k - 3}{k - 1} \sum_{v \in V(G)} \frac{1}{1 + d(v)}.
\]

In terms of the average degree the next corollary follows from Theorem 13 and the relationship between the harmonic and arithmetic means, that is
\[
\frac{n}{\sum_{v \in V(G)} \frac{1}{1 + d(v)}} \leq \frac{1}{n} \sum_{v \in V(G)} (1 + d(v)) = 1 + \bar{d}.
\]

**Corollary 2.** If \( G \) is an isolate-free graph of order \( n \) and average degree \( \bar{d} \), then
\[
\psi_k(G) \leq \left( 1 - \frac{1}{\bar{d} + 1} \cdot \frac{2k - 3}{k - 1} \right) n
\]
for every \( k \geq 3 \).

If \( k = 3 \), then Theorem 13 yields
\[
\text{diss}(G) = \alpha_1(G) = n - \psi_3(G) \geq \sum_{v \in V(G)} \frac{3}{2 + 2d(v)}
\]
which corresponds to the Caro–Tuza bound on \( \alpha_1(G) \) proved in [8, 9] if \( G \) is without isolated vertices. We further note that Kogan [17] proved \( \alpha_1(G) \geq \frac{2}{\bar{d}+2}n \) which is better than our lower bound
\[
\alpha_1(G) = n - \psi_3(G) \geq \frac{3}{2\bar{d} + 2}n
\]
derived from Corollary 2 for the particular case of \( k = 3 \).

## 4 | CHORDAL GRAPHS

By definition, a graph is chordal if it does not contain any induced cycles on more than three vertices. In this section, we consider the well-studied class of chordal graphs and prove upper bounds on \( \psi_k \) for its members, in terms of the chromatic number and the order.

**Theorem 14.** If \( k \geq 2 \) and \( G \) is a chordal graph of order \( n \) and chromatic number \( \chi \), then
\[
\psi_k(G) \leq \left( 1 - \frac{2}{\chi} \cdot \frac{k - 1}{k} \right) n.
\]
Moreover, this bound is tight for every \( k \) with \( \chi = 2 \) and also for every \( \chi \) with \( k = 3 \).

**Proof.** Consider a \( \chi \)-coloring of \( G \), and let \( X_1 \) and \( X_2 \) be the two largest color classes. Clearly, \(|X_1| + |X_2| \geq \frac{2}{\chi} n\), because their average size cannot be smaller than the average taken over all color classes. The subgraph induced by \( X_1 \cup X_2 \) is bipartite and chordal. So, \( G[V(G) \setminus (X_1 \cup X_2)] \) is a forest, and it has a \( k \)-path vertex cover \( T \) with \(|T| \leq (|X_1| + |X_2|)/k\). Since \((V(G) \setminus (X_1 \cup X_2)) \cup T \) is a \( k \)-path vertex cover in \( G \), we have
\[
\psi_k(G) \leq n - (|X_1| + |X_2|) + \frac{1}{k}(|X_1| + |X_2|)
\]
\[
\leq n - \frac{k - 1}{k}(|X_1| + |X_2|)
\]
\[
\leq \left( 1 - \frac{2}{\chi} \cdot \frac{k - 1}{k} \right) n.
\]

Let now \( n \) be a multiple of \( k \chi \). If \( G = P_n \) (hence \( \chi = 2 \)), then the equality \( \psi_k(G) = n/k \) holds for any \( k \) because a vertex has to be included in a \( k \)-path vertex cover from any \( k \) consecutive vertices of \( P_n \). Also, for \( k = 2 \) with arbitrary \( \chi \) let \( G = P_{\frac{n}{\chi} - 1} \), the graph obtained from \( P_n \) by joining any two vertices whose distance along the path is less than \( \chi \). Then
\[ \chi(G) = n - n / \chi \]  because \( \chi - 1 \) vertices have to be included in a 2-path vertex cover from any \( \chi \) consecutive vertices of \( P_{n-1} \).

The above inequality is not tight in general; an improvement for almost all pairs \( k, \chi \) (but not for all, cf. Table 1) is given by the following result. We state it in terms of the clique number, due to the nature of its proof. But every chordal graph \( G \) satisfies \( \chi(G) = \omega(G) \), so the two theorems might be formulated in the same way.

**Theorem 15.** If \( k \geq 3 \) and \( G \) is a chordal graph on \( n \) vertices with clique number \( \omega \), then

\[ \psi_k(G) \leq \frac{\omega}{\omega + k - 1} n. \]

**Proof.** The validity of the assertion is easy to see if \( n \leq \omega + k - 1 \) because by deleting \( n - k + 1 \) vertices no room remains for \( P_k \), hence \( \psi_k(G) \leq n - k + 1 \leq \frac{\omega}{\omega + k - 1} n \) if \( n \) is so small. In the rest of the proof, we assume \( n > \omega + k - 1 \) and apply induction on \( n \).

The following argument is inspired by the theory of treewidth, and in particular by the notion of nice tree decomposition, but the formalism here is simpler because the graph \( G \) is assumed to be chordal. A detailed general treatment of tree decompositions can be found for example in the monograph [16], therefore we shall adopt facts from it without further references.

Let \( v_1, \ldots, v_n \) be the vertices of \( G \). Then there exists a rooted binary tree, say \( F \), and subtrees \( F_1, \ldots, F_n \) in it, where subtree \( F_i \) will represent vertex \( v_i \) for \( i = 1, \ldots, n \), with the properties that are listed below. According to the standard terminology, the vertices of \( F \) will be referred to as nodes. For every node \( x \) of \( F \), let us denote \( S_x = \{ i | x \in F_i \} \).

- \( V (F_i) \cap V (F_j) \neq \emptyset \) if and only if \( v_iv_j \in E(G) \).
- \( |S_x| \leq \omega \) for all \( x \in V(F) \).
- If \( x \) has two children \( x' \) and \( x'' \), then \( S_x = S_{x'} = S_{x''} \).
- If \( x \) has one child \( x' \), then \( S_x \subseteq S_{x'} \) or \( S_x \subseteq S_{x''} \), moreover \( |S_x| = |S_{x'}| + 1 \) or \( |S_x| = |S_{x''}| - 1 \).

For every \( x \in V(F) \), let \( V_x \) denote the set of those vertices \( v_i \) for which there exists an \( x' \) in the subtree rooted at \( x \) in \( F \) such that \( i \in S_{x'} \). Then we have, in particular, that \( |V_x| = n > \omega + k - 1 \) if \( x \) is the root of \( F \), while \( |V_x| = |S_x| \leq \omega \) if \( x \) is a leaf. By the property expressed in the fourth bullet above, \( |V_x| \) can increase by at most 1 when we move from a node to its parent toward the root. Thus we can choose a node \( x \) such that \( |V_x| = \omega + k - 1 \). Now we put all \( v \in S_x \) into the \( k \)-path vertex cover \( T \) to be constructed, delete the entire \( V_x \) from \( G \), moreover omit those \( F_i \) which correspond to vertices \( v_i \in V_x \). After this, we keep only those nodes of \( F \) which are incident with at least one of the remaining subtrees.

Observe that the \( F_j \) not containing \( x \) either are entirely in the subtree rooted at \( x \) (hence they are omitted during this step) or are disjoint from the subtree rooted at \( x \). Consequently, the vertex set corresponding to \( S_x \) is a vertex cut in \( G \), and there is no edge from \( V_x \) to \( G - V_x \). It follows that \( G[V_x \setminus T] \) is \( P_k \)-free, and every \( k \)-path vertex cover of \( G - V_x \) is completed to a \( k \)-path vertex cover of \( G \) with the current vertices of \( T \). Thus, applying the induction hypothesis we obtain

\[ \psi_k(G) \leq |S_x| + \frac{\omega}{\omega + k - 1} (n - |V_x|) \leq \frac{\omega}{\omega + k - 1} n. \]

### 5 CONCLUDING REMARKS AND OPEN PROBLEMS

In this concluding section, we discuss a method which leads to further improved upper bounds, study the case of planar graphs, and raise some problems which remain open for future research.
5.1 | Bounds from maximum degree

An interesting aspect of Theorem 10 is that it offers a self-improving scheme. We illustrate this with the case of $\Delta = 11$.

**Theorem 16.** If $G$ is a graph on $n$ vertices and $\Delta(G) = 11$, then

$$\psi_k(G) \leq \frac{3k+5}{4k+4} n$$

for every $k \geq 6$.

**Proof.** By Theorem 8, $G$ admits a vertex partition $A \cup B$ such that both $G[A]$ and $G[B]$ have maximum degree at most 5. Assume without loss of generality that $|A| \leq n/2 \leq |B|$ holds, and let $T_B$ be a $k$-path vertex cover in $G[B]$. By Theorem 10, we can ensure

$$|T_B| \leq \frac{k+3}{2k+2} |B| = |B| - \frac{k-1}{2k+2} |B| \leq |B| - \frac{k-1}{4k+4} n.$$  

Since $A \cup T_B$ is a $k$-path vertex cover, and $|A| + |B| = n$, the claimed upper bound follows. ■

Certainly, the inequality in the theorem is valid also for $k \leq 5$, but to beat the earlier formula $\frac{4k}{5k-1} n$ we need $(k-1)(k-5) > 0$. It is not clear at the moment how strong is the upper bound that can be derived as a limit of this approach as $\Delta$ gets large, and how restricted it will be as regards to the value of $k$.

We should also note that it was necessary to exclude $\Delta = 3$ from the range of validity in Theorem 10. Indeed, the formula in (ii) would then yield $\frac{4}{k+3} n$, that is, the coefficient of $n$ would tend to zero as $k$ gets large. But this would not be a valid formula, as the next result shows.

**Theorem 17.** For every $k$, there are infinitely many 3-regular connected graphs $G$ such that $\psi_k(G) > \frac{1}{4} |V(G)|$.

**Proof.** Fix any $k \geq 3$. It follows from results of Erdős and Sachs [12] that for every even $n \geq 2^k$ there exists a 3-regular graph $G$ of order $n$ such that the length of the shortest cycle (i.e., the girth of $G$) is larger than $k$. Let now $T$ be any $k$-path vertex cover in $G$, and consider the induced subgraph $H = G \setminus T$. Say, $H$ has connected components $H_1, \ldots, H_s$. Each $H_i$ is a tree (of diameter at most $k-2$). Regularity of degree 3 implies that there are exactly $|V(H_i)| + 2$ edges—that is, more than the order of $H_i$—from $V(H_i)$ to $V(G) \setminus V(H_i)$. Altogether more than $|V(H)|$ edges go from $H$ to the vertices of $T$, while at most $3|T|$ edges go from $T$ to $H$. It follows that $|T| > \frac{1}{3} |V(H)|$ and hence $|T| > \frac{1}{4} n$. ■

5.2 | Planar graphs

In this subsection, we present some partial results related to a conjecture which was first mentioned by Brešar, Kardoš, Katrenič, and Semaniš [5] in 2011. Note that it is easy to construct planar graphs that satisfy the relation with equality, for example, the vertex-disjoint union of octahedron graphs, adjacent in any planar way.

**Conjecture 1** ([5]). If $G$ is a planar graph of order $n$, then $\psi_3(G) \leq \frac{2}{3} n$.

It is a basic fact that $\omega(G) \leq 4$ holds for every planar graph. Then, Theorem 15 allows us to conclude that Conjecture 1 is true for those planar graphs which are also chordal. We may also derive upper bounds on $\psi_k(G)$ for $k \geq 4$.

**Corollary 3.** If $G$ is a graph of order $n$ which is chordal and planar, then

$$\psi_3(G) \leq \frac{2}{3} n, \quad \psi_4(G) \leq \frac{4}{7} n, \quad \text{and} \quad \psi_5(G) \leq \frac{1}{2} n.$$  

On the class of $K_3$-free planar graphs, Theorem 4 and the inequality $m \leq 2n - 4$ together imply

$$\psi_3(G) \leq \frac{4n + m}{9} \leq \frac{4n + 2n - 4}{9} = \frac{2}{3} n - \frac{4}{9}.$$  

Therefore, Conjecture 1 is also true for $K_3$-free planar graphs. Moreover, Corollary 5 will show that the upper bound $2n/3$ can be considerably improved for this graph class.

The forest number $\alpha(G)$ of a graph $G$ is defined as the maximum number of vertices in an induced forest of $G$. As mentioned shortly in the concluding section of [5], there is an important relation between the $k$-path vertex cover number and the forest number. For the sake of completeness, we present a proof for this statement.
Proposition 2 ([5]). For every integer \( k \geq 3 \) and for every graph \( G \) of order \( n \),
\[
\psi_k(G) \leq n - \frac{k-1}{k} a(G).
\]

Proof. Consider a set \( S \) of vertices such that \( |S| = a(G) \) and \( G[S] \) is a forest. By Proposition 1, every \( k \)-path inside \( G[S] \) can be covered by a set \( T \) of at most \( |S|/k \) vertices. The set \( (V(G) \setminus S) \cup T \) is clearly a \( k \)-path vertex cover in \( G \). Hence, we have
\[
\psi_k(G) \leq n - |S| + \frac{|S|}{k} = n - \frac{k-1}{k} |S| = n - \frac{k-1}{k} a(G).
\]

The following is a famous unsolved problem on the forest number of planar graphs.

Albertson–Berman conjecture ([1]) If \( G \) is a planar graph of order \( n \), then \( a(G) \geq n/2 \).

Proposition 2 shows that the Albertson–Berman conjecture, if true, implies Conjecture 1. Combining Proposition 2 and the following two results we may also conclude some upper bounds on \( \psi_k(G) \).

Theorem 18 ([2]). If \( G \) is a planar graph of order \( n \), then \( a(G) \geq \frac{2}{3} n \).

Theorem 19 ([18]). If \( G \) is a planar graph of order \( n \) that is \( K_3 \)-free, then \( a(G) \geq \frac{71}{128} n + \frac{9}{16} \).

Corollary 4. If \( G \) is a planar graph of order \( n \), then
\[
\psi_3(G) \leq \frac{11}{15} n < 0.734 n
\]
and
\[
\psi_6(G) \leq \frac{2}{3} n.
\]

Corollary 5. If \( G \) is a planar and \( K_3 \)-free graph on \( n \) vertices, then
\[
\psi_3(G) \leq \frac{121}{192} n - \frac{3}{8}.
\]

5.3 Further conjectures and open problems

Some of the estimates proved in this article are tight, while some others probably are not. In particular, on the class of chordal graphs we expect that the following improvement can be made.

Conjecture 2. If \( k \geq 3 \) and \( G \) is a chordal graph with clique number \( \omega \) and \( n = |V(G)| \), then
\[
\psi_k(G) \leq \frac{\omega - 1}{\omega + k - 2} n.
\]

The analogous inequality for \( k = 2 \) can easily be seen for every graph in which the equality \( \chi = \omega \) is valid—hence for all perfect graphs and in particular for every chordal graph—by taking a proper vertex coloring with \( \omega \) colors and deleting the largest color class.

The following further problems arise in a natural way in connection with our results on planar graphs.

Problem 2. Determine the smallest constants \( c_{1,k} \) and \( c_{2,k} \) such that
(i) if \( G \) is planar, then \( \psi_3(G) \leq (c_{1,k} + o(1)) n \);
(ii) if \( G \) is planar and \( K_3 \)-free, then \( \psi_k(G) \leq (c_{2,k} + o(1)) n \),
as \( n \to \infty \).

Note that Conjecture 1 formulates the explicit guess \( c_{1,3} = 2/3 \).

Corollary 1 offers various upper bounds on \( \psi_3(G) \). The following problem naturally arises.

Problem 3. Are the upper bounds stated in Corollary 1 sharp?

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DATA AVAILABILITY STATEMENT

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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