Quantum Mechanical Formulation Of Quantum Cosmology For Brane-World Effective Action

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Abstract

Canonical quantization of the Brane-World effective action presented by Kanno and Soda containing higher order curvature invariant terms, has been performed. It requires introduction of an auxiliary variable. As observed in a series of publications by Sanyal and Modak, here again we infer that properly chosen auxiliary variable leads to a Schrödinger like equation where the kinetic part of a canonical variable disentangles from the rest of the variables giving a natural quantum mechanical flavour of time. Further, the effective Hamiltonian turns out to be hermitian, leading to the continuity equation. Thus, a quantum mechanical probability interpretation is plausible. Finally, the extremization of the effective potential leads to Einstein’s equation and a well behaved classical solution, which is a desirable feature of the gravitational action containing higher order curvature invariant terms.

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1 Introduction

The aim of quantum cosmology is to give a quantum mechanical meaning to the existing classical cosmological models. Despite intense research in this field all the attempts in this regard went in vain. In a series of recent publications, [1], [2] and [3], it has been shown that, there still exists the possibility of giving a quantum mechanical interpretation to quantum cosmology at least in an isotropic and homogeneous minisuperspace model.

Under a $(3 + 1)$ decomposition it is always possible to express the space-time metric locally in the form,

$$ds^2 = -(n^2 - n^i n_i) dt^2 + 2 n^i dx^i dt + h_{ij} dx^i dx^j.$$  

(1)

where $n$ and $n_i$ are the lapse function and the shift vector respectively and $h_{ij}$ is the induced metric on the submanifold. In view of (1) the Einstein-Hilbert action

$$S = \frac{1}{16\pi G} \int \sqrt{-g} \ R \ d^4 x - \frac{1}{8\pi G} \int \sqrt{h} \ k \ d^3 x$$  

(2)

leads to Hamiltonian constraint equation $H = 0$ rather than the Hamiltonian. In action (2) $k$ is the trace of the extrinsic curvature and $h$ is the determinant of the metric on the submanifold. The second term on the right hand side is the well-known Gibbons-Hawking term. The canonical quantization of the Hamiltonian constraint equation yields zero energy Schrödinger equation, known as the Wheeler-deWitt equation.

$$-\hbar^2 G_{ijkl} \frac{\partial}{\partial h_{ij}} \frac{\partial}{\partial h_{kl}} + \sqrt{h} \ 3 R \Psi(3G) = 0$$  

(3)

where, $G_{ijkl}$ is the infinite dimensional metric of the so called superspace. Since classical cosmological models are described by minisuperspace models, therefore, the complication arising out of infinite dimensional superspace can be avoided by considering $h_{ij}$’s to be functions of time alone, reducing the problem to the finite dimensional
minisuperspace. Effectively, it reduces the problem of quantum cosmology to that of quantum mechanics that can be solved with appropriate boundary condition. Such a boundary condition was proposed by Hartle-Hawking [4] by going over to the euclidean functional integral.

Now, the Euclidean functional integral for the Einstein-Hilbert action (2) is not bounded from below, as it diverges badly and the programme of reducing the problem of quantum cosmology to that of quantum mechanics, where one can construct the Hilbert space and give a probabilistic interpretation fails. This is one of the reasons to modify action (2) by the introduction of higher order curvature invariant terms in a way such that the Euclidean functional integral converges. Horowitz [5] has shown that introduction of \( R^2 \) term in the action leads to Schrödinger like equation in the sense that one of the parameters of the theory acts as time parameter. Stell [6] claimed that the action \( \int d^4x \sqrt{-g}[AC_{ijkl}^2 + B^4 R + C^4 R^2] \) is perturbatively renormalizable in 4-dimensions. Starobinsky [7] considered Einstein’s equation with quantum one loop correction which contains \( R^2 \) term and obtained Inflationary solution. Hawking and Luttrell [8] have shown that \( R^2 \) action under a conformal transformation leads to Einstein-Hilbert action minimally coupled with a Scalar field. Recently, the importance of considering \( R^2 \) term in the action has further been increased as it has been observed that the 4-dimensional Brane world effective action contains such term.

The most attractive feature of higher order theory of gravity has been demonstrated in our recent works [1], [2] and [3]. The Hamiltonian formulation of the action containing higher order curvature invariant terms requires the introduction of an auxiliary variable. In terms of this variable, as the action is expressed in canonical form, one of the true degrees of freedom disentangles from the kinetic part, giving rise to a quantum mechanical flavour of time. However, in [1], it has been shown that one can even introduce auxiliary variable in Einstein-Hilbert action without Gibbons-Hawking term. The classical field equations remain unchanged as expected, since total derivative terms in no way affect them. However, it leads to totally wrong Wheeler-deWitt equation. Now, it is not possible to find a general Gibbons-Hawking type surface term like one above, for gravitational actions containing higher order curvature invariant terms. As a consequence, one can introduce the auxiliary variable straight into the action giving birth to wrong quantum dynamics. This result has been illustrated earlier [1], [2] and [3]. Total derivative terms thus play important role in quantum physics and so we conclude that all such removable terms appearing in the minisuperspace model under consideration should be removed prior to the introduction of auxiliary variable to get the correct and unique quantum description of the theory. This important fact has not been taken care of by the earlier workers [8], [5] in this field. This proposal when taken up for quantization, yields Schrödinger like Wheeler-deWitt equation with excellent features. The effective Hamiltonian turns out to be hermitian and the continuity equation now identifies the nature of space and time like variables in the Robertson-Walker minisuperspace model under consideration. This establishes a quantum mechanical idea of probability and current densities in quantum cosmology. Further, an effective potential emerges in the process, whose extremization yields vacuum Einstein’s equation, which is of course a desirable feature in the weak energy limit of higher order gravity theory. Thus we conclude that the quantum mechanical formulation of quantum cosmology requires the modification of the Einstein-Hilbert action by introducing higher order curvature invariant terms at least in the Robertson-Walker minisuperspace model.

In view of such excellent results it seems interesting to take up the proposal to quantize the 4-dimensional brane world effective action [9] which contains higher order curvature invariant terms as well. In the last few years there has been lot of investigation on the brane world scenario particularly due to an interesting model proposed by Randall and Sundrum [10] based on the following action,

\[
S = \frac{1}{16\pi G_N \ell} \int d^4 x (\dot{\phi} \dot{R} + \frac{12}{\ell^2}) - \sigma \int d^4 x \sqrt{-h} + \int d^4 x \sqrt{-h} L_{\text{matter.}}
\]

(4)

which dictates that gravity can propagate everywhere in all the dimensions while the standard model matter is restricted in the 4-dimensional space time only called brane. In the above action the 4-dimensional brane with tension \( \sigma \) is assumed to be embedded in the 5-dimensional asymptotically anti-de-Sitter bulk with a curvature scale \( \ell \). However, for the understanding of a lot of fundamental problems we need only the low energy 4-dimensional effective action, which is expected to contain higher order curvature invariant terms. There are different avenues to find such action, which is not our present deal. However, in a recent interesting paper Kanno and Soda [9] have derived the following effective action by the use of low energy iteration scheme,

\[
S_e = \frac{1}{16\pi G_N} \int d^4 x \sqrt{-g} [R + \frac{\beta l^2}{3} R^2 + (\delta - 1/4) l^2 (R_{\mu\nu} R^{\mu\nu} - \frac{R^2}{3})] + S_{\text{matter}} + S_{\text{eft}},
\]

(5)

where, \( S_{\text{eft}} \) is the action corresponding to some appropriate conformal field theory. Our aim is to canonically quantize this 4-dimensional effective action (5). Now one can raise the very fundamental question of the viability of quantising an already effective action. Here we present a favourable argument. The above effective action (5)
reduces to the Einstein-Hilbert action being coupled to some matter field only in even lower energy limit. We know that our present universe is well explained by the standard model. So, such an effective action has nothing to do with the present day universe. Further, the low energy iteration scheme that has been carried out to find the above action (5) may not be low enough to turn the visible 4-dimensional world classical. Rather, such an effective action (5) can play some important roles only in the quantum era of the 4 dimensional world and one might obtain the standard model results through semiclassical approximation.

The following section has been devoted in quantising the above action (5) and to extremize the effective potential to present some standard classical solutions which are desirable features of higher order gravity theory. In section (3) we present the continuity equation to give the quantum mechanical probability interpretation. A brief summary of our work has been presented in section (4).

2 Wheeler-deWitt equation and extremization of the effective potential

We consider the 4-dimensional effective Brane-World action (5) given by Kanno and Soda [9], in the following form,

\[ S = \frac{1}{16\pi G_N} \int d^4 x \sqrt{-\eta} [R - 3\phi,\mu\phi^\mu - 3e^{2\phi} \chi,\mu\chi^\mu + \frac{\beta_2^2}{3} R^2 + (\delta - \frac{1}{4}l^2)(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3} R^2)], \]  

(6)

where, in the matter sector we have taken axion(\chi)-dilaton(\phi) field. In the closed Robertson-Walker metric

\[ ds^2 = -dt^2 + a(t)^2 [d\chi^2 - \sin^2(\chi)(d\theta^2 + \sin^2\theta d\phi^2)] \]

(7)

the Ricci scalar is

\[ R = 6(\ddot{a} + \dot{a}^2 + \frac{1}{a^2}) \]

(8)

It is interesting to note that the last term in the action (6), viz. \((\delta - 1/4)^2 \int d^4 x \sqrt{-\eta} (R_{\mu\nu}R^{\mu\nu} - R^2/3)\) does not contribute to the field equation, since it can be integrated out by parts to yield a total derivative term, viz. \(\Sigma_2 = -2M(\delta - 1/4)^2(\frac{2}{3} \dot{a}^3 + a)\), where \(M = \frac{3\gamma}{4}\). So we are left with the following form of the action (6),

\[ S = \frac{1}{16\pi G_N} \int d^4 x \sqrt{-\eta} [R - 3\phi,\mu\phi^\mu - 3e^{2\phi} \chi,\mu\chi^\mu + \frac{\beta_2^2}{3} R^2] + \Sigma_2, \]

(9)

The quantum cosmology of this form of action, although not in the context of brane-world scenario, has been studied earlier by Sanyal and modak [2] in the conformal form of the above metric (7). We proceed here to do the same work in proper time co-ordinate with a different type of matter field and in a different context of brane-world scenario. Substitution of the form of the Ricci scalar \(R\) from (8) in the above form of the action (9) leads to,

\[ S = M \int [(a^2\ddot{a} + a\dot{a}^2 + a + \gamma \dot{a}^2 + 2\gamma \dot{a}^2 + 2\gamma a \dot{a} + 2\gamma \dot{a} + \gamma (\dot{a}^2 + 1)^2 + \frac{1}{2} a^3 \dot{a}^2 + \frac{1}{2} a^3 e^{2\phi} \chi^2] + \Sigma_2, \]

(10)

where, dot stands for \(\frac{d}{dt}\) and \(\gamma = 2\beta_2^2\). According to our proposal, we remove all removable total derivative terms from the action (10), so that only lowest order terms appear in it. In the process we get,

\[ S = M \int [a - a\dot{a}^2 - \gamma (a\dot{a}^2 + (\dot{a}^2 + k)^2) + \frac{1}{2}(\dot{\phi}^2 + e^{2\phi} \chi^2) a^3] + \Sigma_1, \]

(11)

where, \(\Sigma_1 = M(\dot{a}a^2 + \frac{2}{3} \gamma \dot{a}^3 + 2\gamma a \dot{a} \dot{a} + \Sigma_2).\)

Since the action (11) can not any further be made free from the second derivartive terms of the field variable, so for a Hamiltonian formulation of such an action we need to define an auxiliary variable at this stage, which is

\[ Q = -\frac{\partial S}{\partial \dot{a}} = -2M\gamma a \dot{a}. \]

(12)

Introducing the auxiliary variable \(Q\) in action (11) and expressing the action in the canonical form, after removing the remaining total derivative terms, we obtain,

\[ S = \int [\dot{Q} \dot{a} - \frac{Q^2}{4M\gamma a} + M(a - a\dot{a}^2 + \gamma (\dot{a}^2 + 1)^2 + a^3 (\dot{\phi}^2 + e^{2\phi} \chi^2)) dt + \Sigma, \]

(13)
where, $\Sigma = \Sigma_1 - \dot{Q}/2\dot{a}$. The classical field equations are,

$$Q = -2M\gamma a\dot{a}$$  \hspace{1cm} (14)

which is the definition of $Q$ given in (12).

$$\dot{\phi} + 3\frac{\dot{a}}{a} - e^{2\phi} \dot{\chi}^2 = 0$$  \hspace{1cm} (15)

$$\dot{\chi}a^3 e^{2\phi} = c$$  \hspace{1cm} (16)

where, $c$ is a constant.

$$\dot{Q} + \frac{M\gamma}{a^2}[(\dot{a}^2 + 1)(4a\ddot{a} - 3\dot{a}^2 + 1) + 8a\dot{a}^2\ddot{a}] - \frac{Q^2}{4M\gamma a^2} - Ma^2[\frac{\dot{\phi}}{a} + \frac{\dot{a}^2}{a^2} + \frac{1}{2} (\dot{\phi}^2 + e^{2\phi}\dot{\chi}^2)] = 0$$  \hspace{1cm} (17)

and

$$\dot{Q}\dot{a} + \frac{Q^2}{4M\gamma a} + \frac{M\gamma}{a}[3\dot{a}^4 + 2\dot{a} - 1] - Ma^2[\frac{\dot{\phi}}{a} + \frac{1}{2} (\dot{\phi}^2 + e^{2\phi}\dot{\chi}^2)] = 0$$  \hspace{1cm} (18)

Now equation (18) is essentially the Hamiltonian constraint equation which we now express in the phase space variables as,

$$H = P_aP_Q - M[\gamma (P_\phi^2 + 1)^2/a - aP_Q^2 + a] + \frac{1}{2Ma^3}(P_\phi^2 + e^{-2\phi}P_\chi^2) = 0,$$  \hspace{1cm} (19)

where, $P_a, P_Q, P_\phi$ and $P_\chi$ are the canonical momenta corresponding to $a, Q, \phi$ and $\chi$ respectively. The Wheeler-deWitt equation is obtained through the quantization of the Hamiltonian constraint equation (19). Note that canonical quantization should be performed with the basic variables, which are $a, \dot{a}, \phi$ and $\chi$ here.. Let us choose $\dot{a} = x$, and hence replace $P_Q$ by $x$ and $Q$ by $-P_x$. It is evident that $P_x$ is the canonical momentum corresponding to $x = \dot{a}$. Hence equation (19) turns out to be,

$$H = xP_a + \frac{P_x^2}{4M\gamma a} + \frac{1}{2Ma^3}(P_\phi^2 + e^{-2\phi}P_\chi^2) - M[\gamma (\frac{x^2 + 1}{a})^2 - ax^2 + a].$$  \hspace{1cm} (20)

As the Hamiltonian $H$ is constrained to vanish, we get the Wheeler-deWitt equation as,

$$i\hbar \frac{\partial \psi}{\partial a} = -\frac{\hbar^2}{2M\gamma x}(\frac{\partial^2 \psi}{\partial x^2} + \frac{n \partial \psi}{x \partial x}) - \frac{\hbar^2}{2M a^2}(\frac{\partial^2 \psi}{\partial \phi^2} + e^{-2\phi}(\frac{\partial^2 \psi}{\partial \chi^2})) - M[\gamma (\frac{x^2 + 1}{a})^2 - ax^2 + a^2],$$  \hspace{1cm} (21)

where $n$ is the operator ordering index. With the choice $a = e^{\alpha}$, the above eqn. reduces to

$$i\hbar \frac{\partial \psi}{\partial \alpha} = -\frac{\hbar^2}{2M\gamma x}(\frac{\partial^2 \psi}{\partial x^2} + \frac{n \partial \psi}{x \partial x}) - \frac{\hbar^2}{2M x}(\frac{\partial^2 \psi}{\partial \phi^2} + e^{-2\phi}(\frac{\partial^2 \psi}{\partial \chi^2})) e^{-2\alpha} - M[\gamma (\frac{x^2 + 1}{a})^2 - (\frac{x^2 - 1}{a})] e^{2\alpha}(\frac{\partial \psi}{\partial \alpha}) = 0.$$  \hspace{1cm} (22)

Which essentially is of the same form for $\gamma = -\beta$ as obtained in our previous work [2]. Now, one can write,

$$i\hbar \frac{\partial \psi}{\partial \alpha} = \hat{H}_0 \psi.$$  \hspace{1cm} (23)

where, $\hat{H}_0 \psi$ is given by,

$$\hat{H}_0 \psi = -\frac{\hbar^2}{2M\gamma x}(\frac{\partial^2 \psi}{\partial x^2} + \frac{n \partial \psi}{x \partial x}) - \frac{\hbar^2}{2M x}(\frac{\partial^2 \psi}{\partial \phi^2} + e^{-2\phi}(\frac{\partial^2 \psi}{\partial \chi^2})) e^{-2\alpha} + V_e \psi.$$  \hspace{1cm} (24)

The effective potential $V_e$ is,

$$V_e = -M[\gamma (\frac{x^2 + 1}{a})^2 - \frac{x^2 - 1}{a}] e^{2\alpha}. $$  \hspace{1cm} (25)

It is evident that The Wheeler-deWitt equation (23) takes the Schrodinger-like form, where $\alpha$ acts as the time parameter and $\hat{H}_0$ is the effective Hamiltonian given by equation (24). This feature is also consistent with the intrinsic concept of general theory of relativity, as time there has no independent existence from geometry in describing gravitation, rather it is imbued in the theory, unlike situations encountered in the conventional classical and quantum mechanics, where time is an external parameter.
The effective potential can be extremized at the energy scale where potential energy dominates over the kinetic energy, to obtain the following equation
\[(x^2 + 1)[\gamma(3x^2 - 1) - e^{2\alpha}] = 0.\] (26)

The above equation yields either vacuum Einstein’s equation
\[x^2 + 1 = 0,\] (27)
or
\[\gamma(3x^2 - 1) - e^{2\alpha} = 0\] (28)
which admits a solution for \(\gamma > 0\)
\[a = \sqrt{\gamma \sinh \left(\frac{t - t_0}{\sqrt{3\gamma}}\right)},\] (29)

This is evidently a wonderful result showing that the extremum of the effective action gives well-behaved classical solutions. This is a desirable feature of higher order theory of gravity proving the merit of our proposal.

3 Probability and current density

One of the most important features observed in quantum mechanics is that, the state of a system is described by a wavevector belonging to an abstract Hilbert space and the norm of the wavevector must be positive definite or zero. This idea emerged from the interpretation of the probability density to describe a given state from the continuity equation which is obtained by using the Schrödinger equation. Probability interpretation follows from the simple mathematical appearance of the Schrödinger equation. No such interpretation of the probability density in general exists in quantum cosmology, when the action contains terms linear in the Ricci scalar coupled with some matter field. This is due to the fact that there is no time a priori in the Wheeler-deWitt equation in a gravitational theory described by Einstein-Hilbert action.

It is to be noted that the continuity equation along with the conventional notion of the probability density can only be introduced with the proper choice of the canonical variables in such a way that the Hamiltonian constraint is quadratic in the canonical momenta along with a term linear in momentum whose canonical coordinate acts as an intrinsic time variable. This type of canonical quantization is possible only when the action contains at least a quadratic curvature term. It turns out that at least in the homogeneous and isotropic minisuperspace quantum cosmological model, \(R^2\) has a generic feature to yield a modified Wheeler-deWitt equation that looks like usual time-dependent Schrödinger equation, where an intrinsic geometric variable \(\alpha\), related to the scale factor plays the role of time. This further gives rise to a quantum mechanical probability interpretation of quantum cosmology, as we shall show in the following.

It is to be noted that \(H_0\) operator given by equation (24) is hermitian and as a consequence we obtain the continuity equation, viz.
\[\frac{\partial \rho}{\partial \alpha} + \nabla \cdot \mathbf{J} = 0,\] (30)
where \(\rho\) and \(\mathbf{J}\) are the probability and the current densities respectively for the choice of the operator ordering index \(n = -1\). \(\rho\) and \(\mathbf{J}\) are given by \(\rho = \psi^* \psi\) and \(\mathbf{J} = (J_x, J_\phi, J_\chi)\), where
\[J_x = \frac{i\hbar}{4M\gamma x}(\psi^* \psi_x - \psi \psi_x^*),\quad J_\phi = \frac{i\hbar e^{-2\alpha}}{2Mx}(\psi^* \psi_\phi - \psi \psi_\phi^*),\]
and
\[J_\chi = \frac{i\hbar e^{-2(\alpha + \phi)}}{2Mx}(\psi^* \psi_\chi - \psi \psi_\chi^*),\] (31)
with,
\[\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial \phi}, \frac{\partial}{\partial \chi}).\] (32)

One can also find the continuity equation for other values of the operator ordering index but with respect to a new variable \(y\) which is functionally related to \(x\) only. Since the above probability and the current densities
are dynamical quantities, therefore the wavefunction and its derivatives should remain finite at all epoch of the evolution of the Universe, if and only if there are no singularities in the domain of quantum cosmology.

Further in analogy with quantum mechanics it is noted that the variable $\alpha$ in equation can be identified as the time parameter in quantum cosmology. The variables $x(=\dot{\alpha}e^\alpha), \phi$ and $\chi$ act as spatial coordinate variables in this context.

4 Conclusion

Quantum cosmology for an action containing $R + R^2$ term has been performed and published earlier [2]. It is noted that the effective action for Brane-World-Cosmology contains similar geometric terms, apart from $(R_{\mu\nu}R^{\mu\nu} - \frac{R^2}{2})$, which does not contribute to the field equation in the Robertson-Walker minisuperspace model. Previously, the work containing $R + R^2$ term was performed in the conformal time coordinate. Here we perform it for the 4-dimensional brane-world effective action coupled to axion-dilaton field in the proper time co-ordinate. The most important feature of the present work is that a possibility has been explored to give a quantum mechanical interpretation of a minisuperspace cosmological model. This came out due to the fact that an internal time parameter has automatically been picked up to express the Wheeler-deWitt equation in the standard form of time dependent Schrödinger equation with an effective Hamiltonian that is hermitian. Hence the continuity equation can be written and the probabilistic interpretation follows automatically. Further, the extremum of the effective potential produces some standard classical solutions, which are no less important. We observe that the 4-dimensional effective brane-world action which is obtained through low energy iteration scheme, does in no way leads to the standard model, if the action is treated classically. We conclude that the effective action still remains in the quantum domain of the 4-dimensional world.

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