Entanglement between two spatially separated ultracold interacting Fermi gases

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Multiparticle entangled states, essential ingredients for modern quantum technologies, are routinely generated in experiments of atomic Bose-Einstein condensates (BECs). However, the entanglement in ultracold interacting Fermi gases has not been yet exploited. In this work, by using an ansatz of composite bosons, we show that many-particle entanglement between two fermionic ensembles localized in spatially separated modes can be generated by splitting an ultracold interacting Fermi gas in the (molecular) BEC regime. This entanglement relies on the fundamental fermion exchange symmetry of molecular constituents and might be used for implementing Bell test of quantum nonlocality in oncoming experiments.

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The progress towards the generation and manipulation of large ensembles of ultracold entangled atoms has been mainly focused on bosonic particles. Indeed, most of the experiments aimed at generating multiparticle entangled states of matter, such as spin squeezing states 1, twin Fock states 2, 3, non-Gaussian states 4, 5 or Dicke states 6, deal with BECs. These states can exhibit full many-particle entanglement 7, 8 including Einstein-Podolsky-Rosen (EPR) 9 and Bell 10, 11 correlations.

Although entanglement-enhanced precision in atomic interferometry has been achieved with the aforementioned states 1, 8, further quantum information applications require individual addressing of the subsystems. In addition, the indistinguishability of the atoms makes the standard notion of entanglement more subtle, since the very notion of entangled subsystems makes sense when each of the entangled parties can be individually addressed. Nevertheless, the generation of entanglement in identical particle systems is strongly related to the correlations due to the fundamental particle-exchange symmetry of the wavefunction. Particularly, correlations appearing amongst inaccessible identical particles due entirely to symmetrization, can be extracted into an entangled state of independent modes in one-to-one correspondence 12, 13. An example of this would be the splitting of a two indistinguishable-particle state into two individually addressable modes. For instance, the state \( (|↑⟩_1 |↑⟩_2 ± |↓⟩_1 |↓⟩_2 ) / \sqrt{2} \) yields the state \( (|↑⟩_1 |↓⟩_2 ± |↓⟩_1 |↑⟩_2 ) / \sqrt{2} \), once each particle is fixed in one of the two modes. This entangled state (in the spin degrees of freedom) can be used for Bell measurements between two independent particle resources 14. Also, the generation of entanglement by splitting an ensemble of ultracold identical particles into two entangled twin-Fock states of atomic BEC was recently demonstrated 15. The above procedure, which entangles individually addressable subsystems, will allow to exploit correlations due to indistinguishability as a resource in several quantum information tasks.

The Pauli exclusion principle makes the physics of ultracold interacting fermions and bosons to dramatically differ 16. For instance, the crossover from BEC to BCS superfluidity 17, 18, a remarkable feature of strongly correlated fermion systems, is achieved with two-component ultracold interacting Fermi gases 19, 20. Since multiple occupation of the same single fermion state is forbidden, even the simplest state of identical fermions (a single Slater determinant) has correlations due to the fermion-exchange antisymmetry, which are extractable in the form of mode-entanglement 13. Then, it is natural to ask: Is it possible to generate multiparticle entangled states by splitting an ultracold interacting Fermi gas? Could this entanglement be used to perform test of quantum nonlocality?

In this letter we give an affirmative answer to these questions. In the regime where the scattering length characterizing the interaction between different fermion species is positive \( a > 0 \), a finite fraction of fermion pairs condenses to the same molecular bound state \( |ψ_{gs}⟩ \) forming a BEC of diatomic molecules 21. We faithfully compute the entanglement between two ensembles of ultracold fermionic atoms generated by splitting a molecular BEC and demonstrate that large ensembles of the order of \( 10^6 \) fully entangled fermionic atoms can be generated in the laboratory. We also show that fluctuations in the single-particle spectral density of these two individually accessible fermionic ensembles can be almost perfectly correlated, and that the entanglement generated could be used for implementing Bell tests of quantum nonlocality if the precision of the measurements reach single-particle resolution.

For an attractive short-range interaction between different fermion species \( (A \text{ and } B) \) the ground state of two-component Fermi gases at zero temperature can be
approximated by a Fock state of composite bosons, \(|N\rangle\), whenever two-fermion bound states exist. The pair-correlated state \(|N\rangle\) is given by successively of identical and independent coboson operators \(\hat{c}_j^\dagger\) on the vacuum \(|\psi_{\text{vac}}\rangle\). The interaction strength and the confining frequency \(\Omega\) is the ground state of a single-trapped-molecule, instead of the usual pair projection from a BCS state. Indeed, in the BEC regime, the universal dimer-dimer scattering length given by the coboson ansatz, \(\hat{a}_{dd}^{\text{cob}} = 0.64 a\), matches closely the well-established \(a_{dd} \approx 0.6 a\) \(\chi\), and the molecular condensate fraction \(\langle\hat{n}\rangle\) matches remarkably well the fixed-node diffusion Monte Carlo and Bogoliubov results.

Following \(\chi\), we compute the state \(|\psi_{\text{gs}}\rangle\) by solving the Schrödinger equation of a simple model of Feshbach molecule applied to the \(^6\text{Li}\) broad resonance. Then, by using a discretization technique \(\chi\), the state \(|\psi_{\text{gs}}\rangle\) is written in the particular basis of single fermion states of each fermion species, \(|\{a_j\}\rangle\) and \(|\{b_j\}\rangle\), which are given by Schmidt decomposition. \(|\psi_{\text{gs}}\rangle = \sum_j \sqrt{\lambda_j} \left|\begin{array}{l}a_j \\ b_j\end{array}\right\rangle\) with \(\sum_j \lambda_j = 1\). The computed Schmidt distribution, \(\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_S)\), depends on the ratio between the interaction strength and the confining frequency \(\Omega\), \(\Lambda(a/\omega^2)\), and has finite but large enough \(S\) \((\approx 10^6)\). We order \(\lambda_1 \geq \lambda_2 \geq \cdots \lambda_S\), in increasingly single-particle energy.

The coboson creation operator is naturally defined as \(\hat{c}_j = \sqrt{\lambda_j} \hat{a}_j + \delta j\), where \(\delta j\) creates a fermion \(A (B)\) in the single-fermion state \(|a_j\rangle\) \(|b_j\rangle\). Because of the Pauli principle, \((\hat{a}_j^\dagger)^2 = (\hat{b}_j^\dagger)^2 = 0\), and

\[
|N\rangle = \frac{1}{\sqrt{\chi_N}} \sum_{j_1 \neq j_2 \neq \cdots \neq j_N} \prod_{k=1}^{N} \sqrt{\lambda_{j_k}} \hat{a}_{j_1}^\dagger \hat{b}_{j_2}^\dagger |0\rangle,
\]

where the coboson normalization factor \(\chi_N\) is the elementary symmetric polynomial \(\chi_N = N! \sum_{1 \leq j_1 < j_2 < \cdots < j_N} \lambda_{j_1} \lambda_{j_2} \cdots \lambda_{j_N}\) and \(|0\rangle = \otimes_{j=1}^{S} |\text{vac}\rangle_{a_j} \otimes |\text{vac}\rangle_{b_j}\) is the vacuum. The ensemble of fermionic interactions \(|N\rangle\) is controlled by the universal interaction parameter \(k_F a\), where \(k_F = (6\pi^2 n)^{1/3}\) is the Fermi wave number of a non-interacting gas with atom-pair density \(n = N/V\). The volume of the system is \(V = 4\pi L^3/3\), with \(L = \sqrt{\hbar/m\Omega}\) being the characteristic length of the trap and \(m\), the atomic \(^6\text{Li}\) mass.

Beam-splitter-like dynamics in ultracold interacting Fermi gases can be very complicated to be theoretically address. However, in the strong attractive interaction regime, ultracold fermionic atoms co-tunnel between two separated traps as pairs \(\chi\). In the splitting of a molecular BEC, fluctuations between hyperfine states are negligible, keeping the fermionic ensembles of each trap unpolarized. Fermion pairs can therefore be described by a single bifermion creation operator \(\hat{d}_j^\dagger = \hat{a}_{j_1} \hat{b}_{j_2}^\dagger\), which simplifies the dynamics of these two-fermion composite boson systems \(\chi\). We consider a splitting dynamical governed by the evolution operator \(\hat{d}_j \rightarrow (\sqrt{T} \hat{d}_{j_{\text{L}}} + \sqrt{R} \hat{d}_{j_{\text{R}}})\), where \(R (T = 1 - R)\) is the reflection (transmission) coefficient, and \(\hat{d}_{j_{\text{L}}} |0\rangle_q = |d_{j_q}\rangle = |a_{j_q} b_{j_q}\rangle\). This unitary operation describes the experimental situation where a trapped fermionic gas is split into two identical traps of the same volume than the initial one, \(V\), keeping the magnetic field fixed in order to preserve the total particle correlation of the system and the value of global observables such as the total condensate fraction. Then, the \(N\)-coboson Fock state evolves as \(\chi\)

\[
|N\rangle \rightarrow |\Psi_N\rangle = \sum_{M=0}^{N} \sqrt{\binom{N}{M}} |\Phi_{M,N-M}\rangle,
\]

where the states

\[
|\Phi_{M,N-M}\rangle = (N!\chi_N)^{-\frac{1}{2}} \sum_{1 \leq j_1 \leq j_2 \leq \cdots \leq j_N} \prod_{l=1}^{M} \sqrt{\lambda_{j_{l}}} \hat{a}_{j_{l}}^\dagger \hat{b}_{j_{l}}^\dagger |0\rangle_2
\]

are orthonormal, \(|\Phi_{N_1,N_2} |\Phi_{N'_1,N'_2}\rangle = \delta N_1,N'_1 \delta N_2,N'_2\). For \((k_F a)^{-1} < 1\) the splitting dynamics of Eqs. \(\chi\) and \(\chi\) is jeopardized by molecular dissociations \(\chi\).

As for ideal bosons or distinguishable particles, fermion pairs are distributed binomially on the two modes of a perfect beam-splitter \((T = R = 1/2)\). However, the final state \(|\Psi_N\rangle\) is a multiparticle entangled state, \(|\Phi_{M,N-M}\rangle \neq |\Phi_1\rangle |N-M\rangle_2\). Pauli correlations are preserved in the splitting process; see the constraints on the \(j\)’s in Eqs. \(\chi\) and \(\chi\), and no more than a single fermion occupies the same single-fermion state, independently of the mode in which it is localized. Analogous to the EPR thought experiment \(\chi\), measurements on ensemble 1 yield predictions on the measurement results of ensemble 2. Specifically, Pauli correlations between \(M\) and \(N-M\) fermion pairs in the initial state \(|N\rangle\) are mapped onto multiparticle entanglement between two individual modes. This multiparticle entanglement between individual modes becomes operationally accessible when the system is projected onto the state \(|\Phi_{M,N-M}\rangle\) with fixed particle number \(\chi\) and therefore, even for states with nonzero particle fluctuation, the system can be well described by states with a defined particle number.

We quantify the amount of entanglement between two fermionic ensembles in the state \(|\Phi_{M,N-M}\rangle\) using the purity \(P_q\) of ensemble \(q = 1, 2\) \(P_1 = P_2\). Counting fermion states and their multiplicities, we show in the appendix \(\chi\) that the purity \(P_1\) is a symmetric polynomial which can be expanded as a linear combination.
of elementary symmetric polynomials, \( P_1 = \binom{N}{M}^{-1} + \sum_{m=0}^{N-2} \alpha_m \chi_N x^{N-2m}/\chi_N \), where \( \alpha_m = \alpha_m(N,M) > 0 \). Both \( \alpha_m \) and \( \chi_N \) can be evaluated by recursion formulas, allowing the computation of \( P_1 \) up to \( N = 10^3 \), with \( S \approx 10^9 \). The maximum entanglement that can be generated is the one given by the splitting of a single Slater state of \( N \) identical fermions [13], i.e. \( P_1 \geq \binom{N}{M}^{-1} \).

If molecular constituents are not perfectly bound, fermion exchange interactions become relevant yielding strongly correlated fermion ensembles [18]. According to this, the entanglement between modes (1,2) increases with \( k_F a \). We found that, for small \((k_F a)^{-1}\) and large \( N \), highly entangled mBECs are generated since the purity \( P_1 \) decreases many orders of magnitude (see Fig. 1c). Due to the universality of the normalization ratio with the interaction parameter, which holds \( \chi_{N+1}/\chi_N |k_F a| = \chi_{N+1}/\chi_N |k_F a|^{27} \), the entanglement is equally distributed with the population imbalance of the condensates, independently of the interaction parameter and the total number of particle. This is shown in Fig. 1c, where we plot the purity as a function of \( 1 - 2M/N \).

The number of effective (not negligible) Schmidt coefficients decreases as the value of \((k_F a)^{-1}\) diminishes (see Fig. 1d). In the BCS limit \((k_F a)^{-1} \ll -1\) the momentum distribution of the atoms vanishes for \( k > k_F \), and the single-particle spectral density exhibits therefore only one effective Schmidt coefficient being the others \( S - 1 \) infinitesimally small. Since we compute the wave function of a Feshbach molecule \( |\psi_{gs}\rangle \) in the strong binding regime our approach is only valid for \((k_F a)^{-1} \geq 0.5 \) [27]. However, according with the above two observations, we foresee that unitary \((k_F a)^{-1} = 0\) Fermi gases present a Schmidt distribution with just a few \( N \) effective coefficients. This leads to a state \(|N\rangle\) that can be handled by a single Slater state of \( N \) identical fermion pairs. Since, in this limit \( \chi_m = 0 \) for \( m > N \), a maximum entangled state with \( P_1 \approx \binom{N}{M}^{-1} \) (small grey dots in Fig. 1d) could be generated.

From a Quantum Information point of view, the advantage of coboson theory over mean-field and Bogoliubov theories [17, 18] is that the single-particle density matrix, \( \rho_a \) (or \( \rho_b \)), of the fermionic ensemble \(|N\rangle\) has the same eigenstates \(|a_j\rangle\) (\(|b_j\rangle\)) as the two-fermion state \(|\psi_{gs}\rangle\), which we have already obtained. The eigenvalues of \( \rho_a \) are \( (a_i|\rho_a|a_j) = (a_i|\rho_a|a_j)_N = ND_j \delta_{i,j} \), where \( \delta_{i,j} \) is the Kronecker delta, \( D_j = \lambda_j \chi_{N-1}/\chi_N \), and \( \lambda_j \) are the elementary symmetric polynomials of \( \Lambda_j = (\lambda_1, \lambda_2, \ldots, \lambda_j, \ldots, \lambda_S) \). Moreover, \( (a_i|\rho_a|a_i) \) are the populations of the single-fermion states of the gas (the exact single-particle spectral density) which can be experimentally measured with energy resolution of \( h \times 2.1 \) kHz [24, 25]. We note that the pair density fulfills \( (d_i|\rho_a|d_j) = (d_i|\rho_a|d_j)_{N} = 0 \) with \( i \neq j \) due to fermions exchanges.

Particle correlations can be read from the occupation probabilities of the single-fermion states. For instance, when \((k_F a)^{-1} \leq 0.5 \) (blue line), almost \( t \) fermion pairs populate the \( t \) lowest energetic states, \( \hat{\Lambda}_t = (\lambda_1, \lambda_2, \ldots, \lambda_t) \), see Fig. 2a. Since no more than \( t \) fermion pairs can populate this spectral region due to Pauli blocking, particle fluctuations are strongly suppressed. This is shown in Fig. 2b, where we plot the probability \( P(n) = \sum_{1 \leq j_1 < j_2 < \cdots < j_n \leq t} (\prod_{k=1}^n \gamma_k^{n_k} \tilde{d}_{j_k}) / N = (\binom{N}{n}) \chi_n \lambda_{N-n}^t / \chi_N \), with \( \lambda_{S-t} = (\lambda_{t+1}, \lambda_{t+2}, \ldots, \lambda_S) \), of finding \( n \) fermion pairs in \( \Lambda_t \). \( \chi_n \lambda_{S-t} \) prevents populations larger than \( t \). For \((k_F a)^{-1} = 2 \) (colored in green) the fermionic ensemble behaves as a perfect BEC of uncorrelated bosonic molecules which yields Poissonian distributions of \( P(n) \). Tuning to \((k_F a)^{-1} = 1 \) (orange) and then to 0.5 (blue), the probability distribution \( P(n) \) change from binomial to sub-binomial. The later resembles the typical particle distributions of spin squeezing states of entangled atomic BEC [1].

Splitting the interacting fermion ensemble, the probability of detecting \( n_1 \) and \( n_2 \) fermion pairs in modes 1 and 2 respectively, on \( \Lambda_t \) is given by \( P_{1,2}(n_1, n_2) = \binom{M}{n_1} \binom{M-n_1}{n_2} \chi_n \lambda_{N-n}^t / \chi_N \). If \((k_F a)^{-1}\) decreases, the entanglement between these \( n_1 \) and \( n_2 \) particles moves towards its maximum value. This is reflected by the sub-Poissonian probability distribution \( P_{1,2}(n_1, n_2) = \sum_{n_1=0}^{n_2} P_{1,2}(n_1, n_2) \) of finding \( n_1 \) fermion pairs in the spectral region \( \Lambda_t \) of mode 1 (Fig. 2c). For \((k_F a)^{-1} = 0.5 \) and \( t = 56 \), \( P_t(1) \) approaches the binomial distribution \( \binom{t}{n_1}/2^t \). Particle fluctuations in each individual mode are highly correlated in this spectrum range.

The entanglement resulting from this splitting guaran-
FIG. 2. a) Mean population of the t-lowest energetic states of the single-particle spectrum ($\langle N \rangle_N = N \sum_{j=1} B_j$) of an interacting Fermi gas with $N = 10^3$ fermion pairs, and interaction parameter $(k_F a)^{-1} = 2.1$, and $0.5$ (green, orange, and blue, respectively). Suppression of particle fluctuations in this spectral region $\Lambda_t$ is shown in b) where the probability $P(n)$ is plotted for $t = 56$. Dashed areas are Poissonian distributions and gray joined dots are binomial distributions. Panel c) shows the particle fluctuations in $\Lambda_t$ of the fermionic ensemble $P_1(n_1)$ after splitting the system into two balanced ensembles of $M = N/2$ fermion pairs. When decreasing $(k_F a)^{-1}$, $P_1(n_1)$ approaches the binomial distribution $\binom{n_1}{2} / 2^n$ of two maximally entangled fermionic ensembles with perfectly corrected fluctuations.

tees the existence of quantum correlations that could be used to test violations of local realism. For instance, when considering the set of measurements $Q = Z_1$, $R = X_1$, $S = (X_2 - Z_2)/\sqrt{2}$, and $T = (X_2 + Z_2)/\sqrt{2}$ based on projections of the single-particle spectral density with the single-fermion state $j$ of mode $q$ occupied $|\psi_j\rangle_q$ or empty $|\chi_j\rangle_q$, $j = 1, 2$, $q = 1, 2$, $j = 1, 2, 3$, $q = 1, 2, 3$, $j = 1, 2, 3$, $q = 1, 2, 3$, together with measurements in the rotated spectrum, $X_2 = |\psi_j\rangle_q a_j + |\chi_j\rangle_q a_j$. Then the CHSH inequality $M = Q S + R T + Q T$ can be violated. Figure 3 shows that for the lowest energetic states, with small $(k_F a)^{-1}$ and large $N$, the resulting $\langle M \rangle_{P,M,N-M} = \sqrt{2} (2 D_J (N + \sqrt{M(N-M) - 1})$ obtained from the projections $\langle e_j|2 |\psi_j\rangle_{\Phi_{M,N-M}} = \sqrt{MD_J}$, $\langle e_j|2 |\phi_j\rangle_{\Phi_{M,N-M}} = \sqrt{(N-M)D_J}$ and $\langle e_j|2 |\phi_j\rangle_{\Phi_{M,N-M}} = \sqrt{1-ND_J$ (see appendix B) reaches values above 2 (the classical limit).

Normally, the quantum state produced in experiments has a fluctuating number of particle. For a given volume of the trap $V$ and magnetic field (or equivalently a scattering length $a$) the single-fermion states $|\psi_j\rangle$ and $|\chi_j\rangle$ and the classical limit. Fluctuations in the total particle number can be implemented considering an initial state $\rho = \sum_N \zeta_N |N\rangle \langle N|$, with $\sum_N \zeta_N = 1$ and mean number $\sum_N \zeta_N N = \bar{N}$. Afterwards, the system is split into two ensembles $\rho \rightarrow \rho_{spl} = \sum_N \zeta_N |\varphi_N\rangle \langle \varphi_N|$ with arbitrary particle distribution $|\varphi_N\rangle = \sum_{M=0}^N \left( \sum_{M-N-M} \right) |\Phi_{M,N-M}\rangle$ (which includes the binomial case). Since the number of particle in both modes are measured during the detection we define the observables as $Z_q = \sum_q (|\epsilon_j\rangle_q q |\epsilon_j\rangle_{N_q} - |\epsilon_j\rangle_{N_q} q |\epsilon_j\rangle_{N_q})$ and $X_q = \sum_q (|\epsilon_j\rangle_q q |\epsilon_j\rangle_{N_q} q |\epsilon_j\rangle_{N_q})$ to take into account all particle number $N_q$ in mode $q$. The observable $Q S'$, fulfills $\langle Q S' \rangle_{\rho_{spl}} = \langle Q S \rangle_{\Phi_{\Lambda_t}}$ where $\Lambda_t$ is the mean number of fermion pairs in mode 1. The same relation is fulfilled for $RS$, $RT$ and $QT$ such that $\langle M \rangle_{\rho_{spl}} = \langle M \rangle_{\Phi_{\Lambda_t}}$ (appendix B). Since classically the observable $Q S$, $RS$, $RT$ and $QT$ can only take the values $\pm 1$, the same inequality $M \leq 2$ holds for fluctuating particle number. The inequality $\langle M \rangle_{\rho_{spl}} \leq 2$ is violated for nonzero particle invariance when $N$ is large and $M \approx N/2$. Moreover, in appendix B it is shown numerically that the spectral density $D_J(N)$ fits extremely well with the function $\lambda/(1 + \lambda(N-1))$. Therefore, issues concerning the fluctuation of the particle number are fully considered.

Interference with molecular wave matter was experimentally demonstrated in Ref. [20] by splitting a BEC of the order of $10^6$ fermion pairs. A double-well potential is generated by fast transforming a Gaussian optical dipole trap in order to keep the motional potential of the atoms. This splitting process is performed at large $(k_F a)^{-1} \approx 3$, where our splitting dynamics apply. Then, the external magnetic field, acting globally on both condensates, is adiabatically ramped down (in a time scale larger than $\omega^{-1}$ [30]) increasing the scattering length. Since the interaction parameter $k_F a$ can be tuned to arbitrary position, two entangled fermionic ensembles with almost perfectly correlated spectral densities $P_1 \approx (N^{-1})$ can be generated. The rotations of the spectral density might be achieved, for instance, by performing these two measurements in a time interval larger than $\omega^{-1}$, allowing to evolve freely one of the fermionic ensembles during a larger time interval. As occurs in BECs of bosonic atoms [13] [31], single-particle resolution are required for the violation of a Bell inequality. However, this requirement seems to be achievable in ultracold Fermi gases if the precision of the single-particle spectral

FIG. 3. Violation of the CHSH inequality $\langle M \rangle \leq 2$ for local theories. The represented states $j = 1, 5, 21, 57, 121$ are the first non-degenerated states (nl) of Fig. 1 with $(k_F a)^{-1} = 0.5$, $l = 0$ and $n = 1, 2, 3, 4, 5$. 

Quantum

Classical

$2 \sqrt{2}$

$\langle M \rangle \leq 2$

$-1$

$1$

$2$

$5$

$10$

$50$

$100$

$500$

$1000$

$N$
Alternative to the proposed Bell test, inequalities that involve only two-body correlations [11], experimentally accessible in Fermi gases [15], could be implemented to test the nonlocality. Also parity measurements based on a broad spectral range (instead of a single particle state \( j \)) could be used for that aim [14], with the disadvantage that all particles must be detected. Beyond the presented creation of spatial entanglement, Pauli correlations can be used to generate highly entangled fermionic ensembles in two spatially separated modes by using the interference between independent particle resources [40], or in many modes by separating the gas into single-molecules in an optical lattice. Moreover, the interference of two-fermion composites [40] could be used for Quantum Metrology. Last, considering that interacting fermion systems of up to ten pairs were deterministically prepared in a quasi one-dimensional dipole trap [36, 37], where the number of available single fermion states is fully controlled, we expect that deterministic entanglement can be generated by splitting these interacting few-fermion system.

Appendix A: Derivation of the purity \( P_1 \).

To quantify the amount of entanglement between two molecular BECs we compute the purity \( P_2 \) of the reduced density matrix of the particles localized in one of the modes, for instance mode \( q = 1 \). Let \( \rho_1 \) be the reduced density matrix of particles in mode 1 corresponding to the projected state \( |\Phi_{N,M}\rangle \) (Eq. (3) in the main text) with a fixed number of particles in each mode, i.e.

\[
\rho_1 = Tr_2(\rho) = \sum_{1 \leq k_{M+1} < \cdots < k_N \leq S} 2^{|\Phi_{N,M}|} |\Phi_{N,M}\rangle |\Phi_{N,M}\rangle_2,
\]

where \( \rho = |\Phi_{N,M}\rangle \langle \Phi_{N,M}| \), \( Tr_2 \) stands for the trace over all \( (N - M) \) particles in mode 2, and

\[
|k_{M+1}, \ldots, k_N\rangle_2 = |\Phi_{N-M}\rangle = \prod_{j=M+1}^{N} d_{2,k_j}^\dagger |0\rangle_2.
\]

In the main text, the set \( \zeta(N - M) \) gives the \( S!/(S - N + M)!/(N - M)! \) states \( |\Phi_{N-M}\rangle_2 \) of Eq. (A1), such that

\[
\rho_1^2 = \sum_{1 \leq k_{M+1} < \cdots < k_N \leq S, 1 \leq l_{M+1} < \cdots < l_N \leq S} 2^{|\Phi_{N,M}|} |\Phi_{N,M}\rangle |\Phi_{N,M}\rangle_2 2^{|\Phi_{N-M}|} |\Phi_{N-M}\rangle_2 |\Phi_{N-M}\rangle_2.
\]

The purity of the reduced density matrix is given by the trace of \( \rho_1^2 \),

\[
P_1 = Tr(\rho_1^2) = \sum_{1 \leq k_1 < \cdots < k_M \leq S} 1 |k_1, \ldots, k_M| \rho_1^2 |k_1, \ldots, k_M\rangle_1.
\]

Projections of the state \( |k_{M+1}, \ldots, k_N\rangle_2 \) onto \( |\Phi_{N,M}\rangle \) are straightforwardly obtained by counting the multiplicity of the state \( |k_{M+1}, \ldots, k_N\rangle_2 \), that is,

\[
2 |k_{M+1}, \ldots, k_N| \Phi_{N,M}\rangle = (N! \chi_N)^{-1/2} \sqrt{\frac{N!}{(M)!}} (N! - M)! \prod_{j=M+1}^{N} \sqrt{\lambda_{k_j}} \sum_{k_1, \ldots, k_M = 1}^{S} \prod_{k_1 \neq \cdots \neq k_N}^{M} \sqrt{\lambda_{k_i}} |0\rangle_1.
\]

From Eq. (A5) it follows that

\[
\langle \Phi_{N,M}|k_{M+1}, \ldots, k_N\rangle_2 2^{|\Phi_{N-M}|} |\Phi_{N-M}\rangle = \frac{(N - M)!}{\chi_N} \prod_{j=M+1}^{N} \sqrt{\lambda_{k_j}} \lambda_{l_j} \sum_{l_1, \ldots, l_M = 1}^{S} \prod_{l_1 \neq \cdots \neq l_M \neq k_{M+1} \neq \cdots \neq k_N}^{M} \lambda_{l_i},
\]

and

\[
1 |k_1, \ldots, k_M| |\Phi_{N-M}\rangle = (N! \chi_N)^{-1/2} \sqrt{\frac{N!}{M!}} (N! - M)! \prod_{i=1}^{N} \sqrt{\lambda_{l_i}} \prod_{j=M+1}^{N} \sqrt{\lambda_{l_j}}.
\]

density detection [23] is improved, because some of the lowest energetic states are non-degenerated.
For a large number of particles $N \gg 1$ and Schmidt coefficients $S \gg N$, the numerical evaluation of the sum in Eq. (A8) becomes infeasible. However, we can expand such equation as a linear combination of elementary symmetric polynomials \(\chi_N\), which can be evaluated for large $N$ and $S$ by using the recursion \(\chi_N = (N-1)! \sum_{m=1}^{N} \frac{(-1)^{m+1}}{(N-m)!} M(m)\chi_{N-m}, \tag{A9}\)

where $M(m) = \sum_{j=1}^{S} \lambda_j^m$ are the power sums \(\lambda_j\) of the Schmidt coefficient distribution $\mathbf{\Lambda} = (\lambda_1, \ldots, \lambda_S)$. Eq. (A9) allows us to evaluate the purity $P_1$, e.g., with $N \approx 10^3$ and $S \approx 10^6$. Since Eq. (A8) is a symmetric polynomial containing $2N$ coefficients with 2 as maximum multiplicity, we can perform the partial sum of indices $k_{M+1} \ldots k_N$ and $l_{M+1} \ldots l_N$. By counting the multiplicity of the coefficients, Eq. (A8) can be written as

\[
P_1 = \frac{1}{\chi_N^2} \sum_{L=0}^{M} \left( \frac{M!}{(M-L)!} \right)^2 \sum_{k_1, \ldots, k_{M-L} = 1}^{S} \left( \chi_{N-M}^{[k_1, \ldots, k_{M-L}]} \right)^2 \prod_{i=1}^{L} \lambda_{k_i}^2 \prod_{j=L+1}^{2M-L} \lambda_{k_j}, \tag{A10}\]

where $[k_1, \ldots, k_p]$ is the Schmidt coefficient distribution $\mathbf{\Lambda} = (\lambda_1, \ldots, \lambda_S)$ without considering the coefficients $\lambda_{k_1}, \ldots, \lambda_{k_p}$. If we consider the relation \(\chi_{N_1} \chi_{N_2} = \sum_{L=0}^{N_1} \frac{N_1!N_2!}{L!(N_1-L)!(N_2-L)!} \sum_{k_1, \ldots, k_{N_1+N_2-L} = 1}^{S} \prod_{i=1}^{L} \lambda_{k_i}^{N_1+N_2-L} \prod_{j=L+1}^{N_2} \lambda_{k_j}, \tag{A11}\)

with $N_2 = N_1 = N - M$, in order to expand $\left( \chi_{N-M}^{[k_1, \ldots, k_{M-L}]} \right)^2$, we obtain

\[
P_1 = \frac{1}{\chi_N^2} \sum_{L_1=0}^{N-M} \sum_{L_2=0}^{N-M} \frac{1}{L_1!(M-L_1)!} \left( \frac{M!}{(M-L_1)!} \right)^2 \frac{1}{L_2!(N-M-L_2)!} \left( \frac{(N-M)!}{(N-M-L_2)!} \right)^2 \times \sum_{k_1, \ldots, k_{2N-L_1-L_2} = 1}^{S} \prod_{i=1}^{L_1+L_2} \lambda_{k_i}^{2N-L_1-L_2} \prod_{j=L_1+L_2+1}^{2N-L_2} \lambda_{k_j}, \tag{A12}\]

\[
= \frac{1}{\chi_N^2} \sum_{L_T=0}^{N} \frac{1}{L_T!(M-L_T)!} \left( \frac{M!}{(M-L_T)!} \right)^2 \frac{1}{(N-M-L_T)!} \left( \frac{(N-M)!}{(N-M-L_T+L_T)!} \right)^2 \times \sum_{k_1, \ldots, k_{2N-L_T} = 1}^{S} \prod_{i=1}^{L_T} \lambda_{k_i}^{2N-L_T} \prod_{j=L_T+1}^{2N-L_T} \lambda_{k_j}, \tag{A13}\]

with $L_T = L_1 + L_2$. Finally, if we rearrange Eq. (A11) we find that the purity is given by the following lineal combination of elementary symmetric polynomials (Eq. (4) in the main text),

\[
P_1 = \left( \frac{N}{M} \right)^{-1} + \frac{1}{\chi_N} \sum_{L_T=0}^{N-2} \alpha_{L_T} \chi_{L_T} \chi_{2N-L_T}, \tag{A14}\]

where $\alpha_{L_T}$ are coefficients that depend on $L_T$. The expression (A14) provides an efficient way to compute the purity $P_1$ for large systems.
where $\alpha_{LT}$ is evaluated by recursion,

$$\alpha_{LT} = \frac{(2N - 2LT)!}{(2N - L_T)!} \sum_{L_1=\text{Max}[0,L_T-N+M]}^{\text{Min}[L_T,M]} \frac{1}{L_1!(L_T-L_1)!} \left( \frac{M!(N-M)!}{(M-L_1)!(N-M-L_T+L_1)!} \right)^2 \left( -\frac{M!(N-M)!}{L_T!((-L_T+N)!)^2} - \sum_{k=L+1}^{N-2} \frac{k!(2N-k)!}{(k-L_T)!L_T!(2N-k-L_T)!} \right).$$

Appendix B: Observables for the CHSH inequality and $\mathcal{M}_{\text{CHSH}}$

In the main text we show that the generated entanglement when splitting an interacting fermion ensemble into individually accessible modes can be strong enough to test the nonlocality of quantum mechanics. This nonlocal behaviour is due to quantum correlations which can be recognized in the state $\left| \Phi_{M,N-M} \right\rangle$ (Eq. (3) in the main text), when it is written as a superposition of the states $|d_j\rangle_1 (|\bar{d}_j\rangle_2)$ and $|0_j\rangle_1 (|\bar{0}_j\rangle_2)$ of having a fermion-pair in the $j$th states of mode 1 (2) or not having it, respectively. The resulting equation is:

$$|\Phi_{M,N-M}\rangle = \sqrt{MD_j} \left| d_j \right\rangle_1 |0_j\rangle_2 |\Phi_{M-1,N-M}\rangle + \sqrt{(N-M)D_j} \left| 0_j \right\rangle_1 \left| d_j \right\rangle_2 |\Phi_{\alpha_{\beta_{\gamma_{\delta}}},N-M-1}\rangle + \sqrt{1 - ND_j} \left| 0_j \right\rangle_1 \left| 0_j \right\rangle_2 |\Phi_{M,N-M}\rangle.$$  

(B1)

Notice that this state becomes a maximally entangled Bell-like state when the occupation probability of the state $j$ fulfill that $ND_j \rightarrow 1$. The proposed set of measurements which violate the CHSH inequality $\mathcal{M} = QS + RS + RT - QT$ are inspired on the aforementioned superposition terms of the state $|\Phi_{M,N-M}\rangle$.

The possible results of the measurements in the ensemble 1 (2) are $Q$, $R = \pm 1$ ($S$, $T = \pm 1$), such that for local realism $\mathcal{M} \leq 2$ [23]. In the main text we define the observables $Q = Z_1$, $R = X_1$, $S = (X_2 - Z_2)/\sqrt{2}$, and $T = (X_2 + Z_2)/\sqrt{2}$, based on projections of the single-particle spectral density with the single-fermion state $j$ of mode $q$ occupied,

$$|\alpha_{j,N_q}\rangle_q = |d_j\rangle_q \otimes \frac{1}{(N_q - 1)!} \sum_{j_1,j_2,\ldots,j_{N_q-1}=1}^{S} \sum_{j_1 \neq j_2 \neq \ldots \neq j_{N_q-1} \neq j} |d_{j_k}\rangle_q,$$

(B2)

or empty,

$$|\epsilon_{j,N_q}\rangle_q = \frac{1}{N_q!} \sum_{j_1,j_2,\ldots,j_{N_q}=1}^{S} \sum_{j_1 \neq j_2 \neq \ldots \neq j_{N_q} \neq j} |d_{j_k}\rangle_q,$$

(B3)

i.e. $Z_q = |\epsilon_{j,N_q}\rangle_q \langle \alpha_{j,N_q}| - |\alpha_{j,N_q}\rangle_q \langle \epsilon_{j,N_q}|$ and $X_q = |\alpha_{j,N_q}\rangle_q \langle \alpha_{j,N_q}| + |\alpha_{j,N_q}\rangle_q \langle \epsilon_{j,N_q}|$. The expected values of these observables on the projected state with $N_1 = M$ and $N_2 = N - M$ fermion pairs in mode 1 and 2 respectively, are given by

$$\langle QS \rangle_{\Phi_{M,N-M}} = \frac{(-A_{N,M}^{(j)})^2 + 2A_{N,M}^{(j)}B_{N,M}^{(j)} + (B_{N,M}^{(j)})^2 + (C_{N,M}^{(j)})^2}{\sqrt{2}} = -1 + 2D_jN + 2\sqrt{D_jM}\sqrt{1 - D_jN},$$

$$\langle RS \rangle_{\Phi_{M,N-M}} = \sqrt{2}(-A_{N,M}^{(j)} + B_{N,M}^{(j)})C_{N,M}^{(j)} = \sqrt{2}\sqrt{D_j(-M + N)}\left(\sqrt{D_jM} - \sqrt{1 - D_jN}\right),$$

$$\langle RT \rangle_{\Phi_{M,N-M}} = \sqrt{2}(A_{N,M}^{(j)} + B_{N,M}^{(j)})C_{N,M}^{(j)} = \sqrt{2}\sqrt{D_j(-M + N)}\left(\sqrt{D_jM} + \sqrt{1 - D_jN}\right),$$

$$\langle QT \rangle_{\Phi_{M,N-M}} = \frac{(-A_{N,M}^{(j)})^2 + 2A_{N,M}^{(j)}B_{N,M}^{(j)} - (B_{N,M}^{(j)})^2 - (C_{N,M}^{(j)})^2}{\sqrt{2}} = 1 - 2D_jN + 2\sqrt{D_jM}\sqrt{1 - D_jN},$$

(B4)
fluctuations in the total number of particle of the system can be considered with an initial state
If the magnetic field (and therefore the scattering length $M$)

$$A_{N,M}^{(j)} = \langle \sigma_j, M | 2 \langle e_j, N-M | \Phi_{M,N-M} \rangle = \sqrt{MD_j},$$

$$B_{N,M}^{(j)} = \langle \sigma_j, M | 2 \langle e_j, N-M | \Phi_{M,N-M} \rangle = \sqrt{(N-M)D_j},$$

$$C_{N,M}^{(j)} = \langle \sigma_j, M | 2 \langle e_j, N-M | \Phi_{M,N-M} \rangle = \sqrt{1-ND_j},$$

which yields to the CHSH observable given in the main text:

$$\langle \mathcal{M} \rangle_{\Phi_{M,N-M}} = \sqrt{2} \left( 2D_j \left( N + \sqrt{M(N-M)} \right) - 1 \right).$$

The quantum state produced in experiments of ultracold Fermi gases usually has a fluctuating number of particle. If the magnetic field (and therefore the scattering length $a$) and the volume of the trap $V$ are fixed in the experiment, fluctuations in the total number of particle of the system can be considered with an initial state

$$\rho_{in} = \sum_N \zeta_N |N\rangle \langle N|,$$

with $\sum_N \zeta_N = 1$, and mean particle number $\sum_{N} \zeta_N N = \bar{N}$. It is worth noting that the above observables are related to the occupation probability of the single-particle states $j = 1, \ldots, S$ as $\text{Tr}[\rho_{in} \sum_N |\sigma_j, N\rangle \langle \sigma_j, N|] = \langle \hat{a}_j^\dagger \hat{a}_j \rangle_{\rho_{in}} = \sum_N \zeta_N \langle \hat{a}_j^\dagger \hat{a}_j \rangle_N = \sum_N \zeta_N N D_j(N)$, where $D_j(N) = \lambda_j \chi_{N-1}/\chi_N$. Complementary, the non occupation probability of the single-fermion states are $\text{Tr}[\rho_{in} \sum_N |\sigma_j, N\rangle \langle \sigma_j, N|] = 1 - \langle \hat{a}_j^\dagger \hat{a}_j \rangle_{\rho_{in}}$.

The state $\rho_{in}$ ideally evolves under a perfect beam-splitter as $\rho_{in} \rightarrow \rho_{spl} = \sum_N \zeta_N |\Psi_N\rangle \langle \Psi_N|$, being $|\Psi_N\rangle$ the state of Eq. (2) in the main text. However one can consider non binomial beam-splitter $\rho_{spl} = \sum_N \zeta_N |\varphi_N\rangle \langle \varphi_N|$, with $|\varphi_N\rangle = \sum_{M=0}^N \sqrt{\delta_M^{(N)} \Phi_{M,N-M}}$ being a state superposition with arbitrary particle distribution, $\sum_{M=0}^N \delta_M^{(N)} = 1$, and with total mean number of fermion pairs in mode 1 given by $\bar{M} = \sum_N \zeta_N \sum_{M=0}^N \delta_M^{(N)} M$. To take into account all particle number $N_q$ in mode $q$ when measurements are performed we now define $Z_q = \sum_{N_q} \langle \beta_q, N_q| q \langle \beta_q, N_q| - |\beta_q, N_q\rangle q \langle \beta_q, N_q| \rangle$ and $X_q = \sum_{N_q} \langle \beta_q, N_q| q \langle \beta_q, N_q| + |\beta_q, N_q\rangle q \langle \beta_q, N_q| \rangle$. The mean value of any observable $\hat{O}$ for the state $\rho_{spl}$ is given by $\text{Tr}[\rho_{spl} \hat{O}] = \sum_N \zeta_N \langle \varphi_N| \hat{O} |\varphi_N\rangle$, such that mean values of $QS$, $RS$, $RT$ and $QT$ can be easily computed as

$$\langle QS \rangle_{\rho_{spl}} = \sum_N \sum_{M=0}^N \zeta_N \delta_M^{(N)} \langle QS \rangle_{\Phi_{M,N-M}} = \frac{-1 + 2D_j^{(N)} \bar{N} + 2\sqrt{D_j^{(N)}} \sqrt{1 - D_j^{(N)} \bar{N}}}{\sqrt{2}},$$

$$\langle RS \rangle_{\rho_{spl}} = \sum_N \sum_{M=0}^N \zeta_N \delta_M^{(N)} \langle RS \rangle_{\Phi_{M,N-M}} = \sqrt{2} \sqrt{D_j^{(N)} (-\bar{M} + \bar{N})} \left( \sqrt{D_j^{(N)}} \bar{M} - \sqrt{1 - D_j^{(N)} \bar{N}} \right),$$

$$\langle RT \rangle_{\rho_{spl}} = \sum_N \sum_{M=0}^N \zeta_N \delta_M^{(N)} \langle RT \rangle_{\Phi_{M,N-M}} = \sqrt{2} \sqrt{D_j^{(N)} (-\bar{M} + \bar{N})} \left( \sqrt{D_j^{(N)}} \bar{M} + \sqrt{1 - D_j^{(N)} \bar{N}} \right),$$

$$\langle QT \rangle_{\rho_{spl}} = \sum_N \sum_{M=0}^N \zeta_N \delta_M^{(N)} \langle QT \rangle_{\Phi_{M,N-M}} = \frac{1 - 2D_j^{(N)} \bar{N} + 2\sqrt{D_j^{(N)}} \sqrt{1 - D_j^{(N)} \bar{N}}}{\sqrt{2}}.$$
can be violated for nonzero particle invariance when \( \bar{N} \) is large and \( \bar{M} \approx \bar{N}/2 \). Moreover, in Fig. 4 it is shown numerically that the spectral density \( D_1(N) \) fits extremely well with the function \( \lambda_1/(1 + \lambda_1(N - 1)) \). Any error of the observables due to the particle counting can be taken into account just propagating the errors of the measured \( \bar{N} \pm \Delta \bar{N} \) and \( \bar{M} \pm \Delta \bar{M} \).

![Graph](image)

FIG. 4. Occupation probability of the lowest energetic state, \( N_r D^{(N_r)} \), normalized to \( N_r \lambda_1 \), for a Fermi gas with \( (k_F a)^{-1} = 0.5 \) and mean number of particle \( \bar{N} \). Red lines are the function \( 1/(1 + D(N_r - 1)) \)

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