The quantum speed limit

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ABSTRACT

How fast can a quantum system evolve? We answer this question focusing on the role of entanglement and interactions among subsystems. In particular, we analyze how the order of the interactions shapes the dynamics.

Keywords: Entanglement, dynamics, evolution, interactions, composite systems.

1. INTRODUCTION

The Hamiltonian $H$ is the generator of the dynamics of physical systems. In this respect, the energy content of a system will determine its evolution time scales. For instance, the time-energy uncertainty relation\textsuperscript{1–3} states that the time it takes for a system to evolve to an orthogonal configuration in the Hilbert space is limited by the inverse of the energy spread of the system. Analogously, the Margolus-Levitin theorem\textsuperscript{4} relates the same quantity to the mean energy. More generally, one can consider the minimum time required for a system to “rotate” from its initial configuration by a predetermined amount. In Ref. 5 we have shown that also this time is limited by the energy and energy spread. More specifically, given a value of $\epsilon \in [0, 1]$, consider the time $t$ it takes for a system in the state $|\Psi\rangle$ to evolve to a state $|\Psi(t)\rangle$ such that

$$P(t) \equiv |\langle \Psi | \Psi(t) \rangle|^2 = \epsilon.$$  

One can show that $t$ is always greater than the quantum speed limit time,

$$T_\epsilon(E, \Delta E) \equiv \max \left( \alpha(\epsilon) \frac{\pi \hbar}{2(E - E_0)}, \beta(\epsilon) \frac{\pi \hbar}{2\Delta E} \right),$$

where $E_0$ is the ground state energy of the system, $E = \langle \Psi | H | \Psi \rangle$, $\Delta E = \sqrt{\langle \Psi | (H - E)^2 | \Psi \rangle}$, and $\alpha(\epsilon)$ and $\beta(\epsilon)$ are the decreasing functions plotted in Fig. 1. If $\epsilon = 0$, from Eq. (2) one reobtains the time-energy uncertainty relation and the Margolus-Levitin theorem which, united, give the following lower bound for the time it takes for a system to evolve to an orthogonal configuration,

$$t \geq T_0(E, \Delta E) \equiv \max \left( \frac{\pi \hbar}{2(E - E_0)}, \frac{\pi \hbar}{2\Delta E} \right).$$

In the rest of the paper we will focus on this case only as a followup of a previous paper of ours\textsuperscript{6}. It is worth mentioning that when the system is a mixed state $\varrho$, the quantum speed limit time $T_\epsilon$ is the minimum time it takes for the system to evolve to a configuration $\varrho(t)$ such that the fidelity\textsuperscript{7}

$$F(\varrho, \varrho(t)) \equiv \left\{ \text{Tr} \left[ \sqrt{\sqrt{\varrho(t)} \varrho \sqrt{\varrho(t)}} \right] \right\}^2$$

between these two states is equal to $\epsilon$ (see Ref. 5 for details).

In addition to giving upper bounds to the evolution “speed” of a system, the quantity $T_\epsilon$ is useful to analyze the role of the entanglement among subsystems in the dynamics of composite systems: in Sec. 2 we review some of the material presented in Ref. 6 pointing out the differences between entangled and separable pure states in achieving the quantum speed limit bound in the context of non-interacting subsystems; in Sec. 3, on the other hand, we address the case of interacting subsystems and analyze in detail how the order of interaction influences the dynamical “speed” of composite systems.
The functions $\alpha(\epsilon)$ (continuous line) and $\beta(\epsilon)$ (dashed line) of Eq. (3). The analytical expression of $\beta$ is $\beta(\epsilon) = 2\arccos(\sqrt{\epsilon})/\pi$; the explicit analytical expression for $\alpha(\epsilon)$ is not known\textsuperscript{5}, even though one can evaluate this function numerically showing that $\alpha(\epsilon) \simeq \beta^2(\epsilon)$.

2. ENTANGLEMENT AND DYNAMICS: NON-INTERACTING SUBSYSTEMS

We show that pure separable states cannot reach the quantum speed limit unless all energy resources are devoted to one of the subsystems. This is no longer true if the initial state is entangled: a cooperative behavior is induced such that the single subsystems cannot be regarded as independent entities.

Consider a system composed of $M$ parts which do not interact. The Hamiltonian for such a system is

$$H = \sum_{i=1}^{M} H_i,$$

(5)

where $H_i$ is the free Hamiltonian of the $i$th subsystem. Without loss of generality, we assume that each $H_i$ term (and hence $H$) has zero ground state energy. A separable pure state has the form

$$|\Psi_{sep}\rangle = |\psi_1\rangle_1 \otimes \cdots \otimes |\psi_M\rangle_M,$$

(6)

where $|\psi_i\rangle_i$ is the state of the $i$-th subsystem which has energy $E_i$ and spread $\Delta E_i$. Since there is no interaction, the vector $|\Psi_{sep}\rangle$ remains factorizable at all times, i.e.

$$|\Psi_{sep}(t)\rangle = |\psi_1(t)\rangle_1 \otimes \cdots \otimes |\psi_M(t)\rangle_M.$$

(7)

This state becomes orthogonal to its initial configuration (6) if at least one of the subsystems $|\psi_i\rangle_i$ evolves to an orthogonal state. The time $T_{\perp}$ employed by this process is limited by the energies and the energy spreads of each subsystem, through Eq. (3). By considering the time corresponding to the “fastest” subsystem, the quantity $T_{\perp}$ has to satisfy the inequality

$$T_{\perp} \geq \max \left( \frac{\pi \hbar}{2E_{max}}, \frac{\pi \hbar}{2\Delta E_{max}} \right),$$

(8)

where $E_{max}$ and $\Delta E_{max}$ are the maximum values of the energy and energy spreads of the $M$ subsystems. For the state $|\Psi_{sep}\rangle$, the total energy and the total energy spread are

$$E = \sum_i E_i \geq E_{max},$$

(9)
\[ \Delta E = \left[ \sum_i \Delta E_i^2 \right]^{\frac{1}{2}} \geq \Delta E_{\text{max}}. \] (10)

This implies that the bound imposed by Eq. (8) is always greater or equal than \( T_0(E, \Delta E) \) of Eq. (3), attaining equality only when \( E_{\text{max}} = E \) or \( \Delta E_{\text{max}} = \Delta E \), e.g. when one of the subsystems has all the energy or all the energy spread of the whole system. In both these cases, only one such subsystem evolves in time and the remaining \( M - 1 \) are stationary. The gap between the bound (8) for separable pure states and the bound (3) for arbitrary states reaches its maximum value for systems that are homogeneous in the energy distribution, i.e. such that \( E_{\text{max}} = E/M \) and \( \Delta E_{\text{max}} = \Delta E/\sqrt{M} \). In this case, Eq. (8) implies that, for factorizable states, \( T_{\perp}/T_0(E, \Delta E) \geq \sqrt{M} \).

### 2.1. Entangled subsystems.

In order to show that the bound is indeed achievable when entanglement is present, consider the following entangled state

\[ |\Psi_{\text{ent}}\rangle = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} |n\rangle_1 \otimes \cdots \otimes |n\rangle_M, \] (11)

where \( |n\rangle_i \) is the energy eigenstate (of energy \( n\hbar\omega_0 \)) of the \( i \)-th subsystem and \( N \geq 2 \). The state \( |\Psi_{\text{ent}}\rangle \) is homogeneous since each of the \( M \) subsystems has energy and spread

\[ E = \hbar\omega_0(N-1)/2, \] (12)

\[ \Delta E = \hbar\omega_0\sqrt{N^2 - 1/(2\sqrt{3})}. \] (13)

On the other hand, the total energy and energy spread are given by \( E = ME \) and \( \Delta E = M\Delta E \) respectively and the quantum speed limit time (8) becomes,

\[ T_0 = \frac{\sqrt{3} \pi}{M\sqrt{N^2 - 1}\omega_0}. \] (14)

The scalar product of \( |\Psi_{\text{ent}}\rangle \) with its time evolved \( |\Psi_{\text{ent}}(t)\rangle \) gives

\[ P(t) = \left| \frac{1}{N} \sum_{n=0}^{N-1} e^{-inM\omega_0 t} \right|^2, \] (15)

where the factor \( M \) in the exponential is a peculiar signature of the energy entanglement. The value of \( T_{\perp} \) for the state \( |\Psi_{\text{ent}}\rangle \) is given by the smallest time \( t \geq 0 \) for which \( P(t) = 0 \), i.e. \( T_{\perp} = 2\pi/(NM\omega_0) \). This quantity is smaller by a factor \( \sim \sqrt{M} \) than what it would be for homogeneous separable pure states with the same value of \( E \) and \( \Delta E \), as can be checked through Eq. (8). Moreover, \( T_{\perp}/T_0(E, \Delta E) \) does not depend on the number \( M \) of subsystems and, for any value of the number of energy levels \( N \), is always close to one (see Fig. 2). In particular, for \( N = 2 \), the system achieves the quantum speed limit, i.e. \( T_{\perp} = T_0(E, \Delta E) \).

The above discussion shows that, for non interacting subsystems, pure separable states can reach the quantum speed limit only in the case of highly asymmetric configurations where one of the subsystems evolves to an orthogonal configuration at the maximum speed allowed by its energetic resources, while the other subsystems do not evolve. In the other cases (for instance the homogeneous configuration discussed above) entanglement is a necessary condition to achieve the bound. This, of course, does not imply that all entangled states evolve faster than their unentangled counterparts. In Refs. 5, 6 these arguments have been extended, with some caveats, to the case of mixed states.
Figure 2. Plot of the ratio between the minimum time $T_\perp$ it takes for system to evolve to an orthogonal configuration starting from the state $|\Psi_{\text{ent}}\rangle$ of Eq. (11) and the quantum speed limit time $\tau_0(E, \Delta E)$ of Eq. (14) as a function of the number $N$ of eigenstates of each subsystem. This ratio does not depend on the number of subsystems $M$ and for all $N$ is upper-bounded by its asymptotic value $2/\sqrt{3}$ (dashed line).

3. INTERACTIONS AND ENTANGLEMENT

In the case in which there are interactions among different subsystems, the Hamiltonian is

$$H = \sum_{i=1}^{M} H_i + H_{\text{int}},$$

where $H_{\text{int}}$ acts on the Hilbert spaces of more than one subsystem. Two situations are possible: either $H_{\text{int}}$ does not introduce any entanglement in the initial state of the system or $H_{\text{int}}$ builds up entanglement among subsystems. In the first case, no correlations among the subsystems are created so that each subsystem evolves independently as in Eq. (7), unless entanglement was present initially. Since this type of evolution can always be described as determined by an interaction-free effective Hamiltonian, the results of the previous section apply: i.e. the system reaches the quantum speed limit only if entanglement is present initially or if all energy resources are devoted to a single subsystem. In the second case, when $H_{\text{int}}$ builds up entanglement, the system may reach the bound even though no entanglement was already present initially. In fact, one can tailor suitable entangling Hamiltonians that speed up the dynamical evolution even for initial homogeneous separable pure states. In order to reach the bound, however, the interaction must $i)$ connect all the qubits and $ii)$ be sufficiently strong$^6$. A recent proposal of one of us$^8$ uses this effect to increase the communication rate of a qubit channel by a factor $\sqrt{M}$ over a communication channel composed of $M$ independent parallel channels which uses the same resources.

The order $K$ of the interaction (i.e. the number of subsystems that are involved in a single vertex of interaction) also plays an important role. The case analyzed in Ref. 6 assumes an $M$th order interaction to show that a separable pure state can be brought to the quantum speed limit even when energy resources are equally distributed among subsystems. Here we want to discuss in some detail what happens for $K < M$. For any given $K$, a rich variety of configurations are possible depending on how the interaction is capable of constructing entanglement between subsystems. The simplest example is an Ising-like model where there is a $K$-th order coupling between neighbors in a chain of qubits. In this context a $\sqrt{K}$ improvement over the non-interacting case is achievable. We now work out this model in detail. Consider a system of $M$ qubits...
governed by the following Hamiltonians:

\[ H_i = \hbar \omega_0 (1 - \sigma_x^{(i)}) \quad (i = 1, \ldots, M) , \]  
\[ H_{int} = \hbar \omega \sum_{j=1}^{Q} (1 - S_j) , \]  

where \( \sigma_x^{(i)} \) is the Pauli operator \( |0\rangle\langle1| + |1\rangle\langle0| \) acting on the \( i \)th spin, \( Q \) is the number of interaction terms and

\[ S_j = \sigma_x^{(i_1)} \otimes \sigma_x^{(i_2)} \otimes \cdots \otimes \sigma_x^{(l_1)} \quad (j = 1, \ldots, Q) , \]

with \( i_{lj} = 1, \ldots, M \) for \( l = 1, \ldots, K \) and \( i_{lj} \neq i_{l'j} \) for \( l \neq l' \). The operator \( S_j \) is a \( K \)th order interaction Hamiltonian that connects the spins \( i_{lj} \) through \( i_{lKj} \). The free Hamiltonians \( H_i \) rotate each spin along the \( x \) axis at a frequency \( \omega_0 \), while the interaction \( H_{int} \) collectively rotates the \( Q \) blocks of \( K \) spins at a frequency \( \omega \). For ease of calculation, we assume that each spin belongs only to two of the \( Q \) interacting groups and that no spin is left out. This enforces a symmetry condition on the qubits (i.e. all qubits are subject to the same interactions) and implies that \( S_1S_2\cdots S_Q = 1 \). Consider an initial state in which all qubits are in \( \sigma_z = |0\rangle\langle0| - |1\rangle\langle1| \) eigenstates, such as \( |\Psi\rangle = |0\rangle_1 \otimes \cdots \otimes |0\rangle_M \). This state is clearly homogeneous and has energy characteristics

\[ E = \hbar \omega_0 M + \hbar \omega Q \]  
\[ \Delta E = \sqrt{(\hbar \omega_0)^2 M + (\hbar \omega)^2 Q} , \]

so that, in the strong interaction regime* \( \omega \gg \omega_0 \), its quantum speed limit time (3) is†

\[ T_0(E, \Delta E) \approx \frac{\pi}{2\omega\sqrt{Q}} . \]  

Since \([H_i, H_{int}] = 0\) for all \( i \) and \([S_j, S_] = 0\) for all \( j, j' \), the time-evolved state has the form

\[ |\Psi(t)\rangle = U_0(t)U_{int}(t)|\Psi\rangle , \]

with

\[ U_0(t) = e^{-i\omega_0 t} \prod_{i=1}^{M} (\cos \omega_0 t + i \sigma_x^{(i)} \sin \omega_0 t) , \quad U_{int}(t) = e^{-i\omega t} \prod_{j=1}^{Q} (\cos \omega t + i S_j \sin \omega t) . \]

For \( \omega \gg \omega_0 \), this yields

\[ P(t) = |\langle \Psi |\Psi(t)\rangle|^2 \approx |(\cos \omega t)^Q + (i \sin \omega t)^Q|^2 . \]  

If \( Q \) is an odd number or twice an even number, \( P(t) \) is never zero: in this case the interaction does not help in reaching the quantum speed limit‡. However, if \( Q \) is twice an odd number, then \( P(t) = 0 \) has solution and the minimum time \( t \) for which this condition is satisfied is \( T_\perp = \pi/(4\omega) \). The ratio between this quantity and the quantum speed limit time (22) is, then, \( T_\perp/T_0(E, \Delta E) = \sqrt{Q}/2 \). We now need to connect the quantity \( Q \) to the order \( K \) of the interaction and the number \( M \) of spins. A simple way to visualize the interactions among qubits is to arrange them into polygonal structures, as in Fig. 3, where each line represents one of the \( Q \) interactions \( S_j \). Using this representation, it is easy to verify that \( Q = L + L(K - 2)/K \), where \( L = M/(K - 1) \) is the number of sides of the polygon. This means that

\[ \frac{T_\perp}{T_0(E, \Delta E)} = \sqrt{\frac{M}{2K}} , \]

which gives a \( \sqrt{K} \) improvement over the case of non-interacting subsystems for homogeneous separable configurations.

*In Ref. 6 we have shown that the strong interaction regime is the only one that allows to reach the quantum speed limit, since only in this regime one can build strong enough correlations among all subsystems.

†Notice that the ground state energy \( E_0 \) associated with the Hamiltonians (17) and (18) is null.

‡This is an artifact of our model. One can devise different configurations (e.g. dropping the qubits symmetry requirement) where these limitations do not apply.
4. CONCLUSIONS

In summary, we have assembled all the dynamical limitations connected to the energy of a system in a *quantum speed limit* relation, Eq. (2), that establishes how fast any physical system can evolve given its energy characteristics. We have proved that homogeneous separable pure states cannot exhibit speedup, while at least one entangled case that exhibits speedup exists. This suggests a fundamental role of energy entanglement in the dynamical evolution of composite systems. Moreover, we analyzed the role of interactions with symmetric configurations in which each subsystem interacts directly with $K-1$ other subsystems. In this case, we have shown that a $\sqrt{K}$ enhancement is possible over the non-interacting unentangled case with energy shared among subsystems.

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**Figure 3.** Relation between the order $K$ of the interaction, the number $M$ of qubits (i.e. the gray dots) and the number $Q$ of interaction terms for the example discussed in the text. The qubits can be organized in regular polygons with $L$ sides, each containing $K-1$ qubits: here we show some of the possible configurations. The lines (continuous, dashed or dotted) represent the $Q$ interactions $S_j$ ($j = 1, \cdots, Q$): each qubit is involved in two interaction terms. For example, in the first structure (with six qubits), the qubit number 2 interacts with qubits 1,3 and with 4,6. Among the examples plotted here, only the case $K = 4$, $M = 12$, $Q = 6$ satisfies Eq. (26).