Traveling Waves in Spatial SIRS Models

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Abstract We study traveling wavefront solutions for two reaction–diffusion systems, which are derived respectively as diffusion approximations to two nonlocal spatial SIRS models. These solutions characterize the propagating progress and speed of the spatial spread of underlying epidemic waves. For the first diffusion system, we find a lower bound for wave speeds and prove that the traveling waves exist for all speeds bigger than this bound. For the second diffusion system, we find the minimal wave speed and show that the traveling waves exist for all speeds bigger than or equal to the minimal speed. We further prove the uniqueness (up to translation) of these solutions for sufficiently large wave speeds. The existence of these solutions are proved by a shooting argument combining with LaSalle’s invariance principle, and their uniqueness by a geometric singular perturbation argument.

Keywords Spatial SIRS models · Traveling waves · Shooting argument · LaSalle’s invariance principle · Geometric singular perturbation

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1 Introduction

Throughout history the spread of infectious diseases has had devastating effects on humans and animal populations, which in many cases results in epidemics or even pandemics, and causes the deaths of millions of people over vast areas of the earth. Examples of such infectious diseases include the European Plague or Black Death, small pox, influenza, HIV/AIDS, avian flu, swine flu, SARS, West Nile virus [23, 26]. In order to prevent or limit the impact of
epidemics and provide effective control strategies and useful predictions, mathematical modelling of epidemics has become a major focus of research for understanding the underlying mechanisms that influence the spread and transmission dynamics of infectious diseases. Many epidemic models have been developed [2, 4, 6, 13, 27, 29, 30, 32, 33]. In particular, reaction–diffusion equation models and integro-differential equation models have been used to study the spatial spread of infectious diseases, and their traveling wavefront solutions have been used to investigate the question of whether an infectious disease could persist as a wave front of infectives that travels geographically across vast distances. Relevant problems include to determine the thresholds above which traveling waves exist, find the minimum speed and asymptotic speed of propagation (which are usually equal) [3], and determine the stability of the traveling wave to perturbations [20]. Due to the complexity of the models, these problems often present very challenging dynamical system problems, on which extensive research has been done since the pioneering works of Fisher [10] and Kolmogorov et al. [19]; consequently, many outstanding results have been obtained, and various methods and techniques have been developed to tackle these problems (see, e.g., [11, 13, 25, 27, 28, 31, 33] and the references therein). With the current development of more realistic and sophisticated epidemic models, the study on these problems remains very active in mathematical epidemiology.

In this paper, we study traveling wavefront solutions for two reaction–diffusion systems, both are derived as diffusion approximations to their integro-differential equations models. These models are spatial analogs of a basic SIRS endemic model in one spatial dimension. The first model is a distributed-contacts model with a kernel describing daily contacts of infectives with their neighbors or the influences, by any reason, of the infectives on their neighbors. This model, which was studied recently by Li et al. [20], extends a distributed-contacts model of Kendall [18] (a spatial analogue of a SI endemic model). The second model is a nonlocal diffusion model describing the mobility of individuals around the spatial domain. This model generalizes a distributed-infectives model considered by Medlock and Kot [24] and nonlocal dispersal models in [16, 22]. Following the approaches used in Bailey [4], Hoppensteadt [13] and Kendall [18], the aforementioned nonlocal models are approximated by reaction–diffusion systems when their kernels are local. The aim of this paper is to show the existence of traveling waves for these diffusion approximations. In this and the next two sections, we concentrate our study on the first model, and in Sect. 4 we establish corresponding results for the second model.

The basic SIRS model mentioned above is described by a system of ODEs for the evolution of an infectious disease in a well-mixed and closed population. Dividing the total population into susceptible, infective, and recovered classes, with $s(t)$, $i(t)$, and $r(t)$ denoting the fractions of their population sizes at time $t$ respectively (thus $s + i + r \equiv 1$), the governing equations of the model are:

$$\frac{ds}{dt} = -\beta si + \delta r,$$
$$\frac{di}{dt} = \beta si - \gamma i,$$
$$\frac{dr}{dt} = \gamma i - \delta r,$$  \hspace{1cm} (1.1)$$

where the infection rate $\beta$, recovery rate $\gamma$, and the immunity loss rate $\delta$ are positive constants. The basic reproductive number of (1.1) is $\sigma := \beta/\gamma$, which is the average number of infectives produced by a single infective introduced into a completely susceptible population. It has been shown [6, 12, 30] that if $\sigma < 1$, then every nonnegative solution $(s(t), i(t), r(t))$ of (1.1) lying on the plane $s + i + r = 1$ approaches the disease-free equilibrium $(1, 0, 0)$ as $t \to \infty$, implying that the disease is eventually eradicated; while if $\sigma > 1$, then $(s(t), i(t), r(t))$ of (1.1) with $i(0) > 0$ approaches the endemic equilibrium $(s^*, i^*, r^*)$, given by
\[ s^* = \frac{1}{\sigma}, \quad i^* = \frac{1 - 1/\sigma}{1 + \gamma/\delta}, \quad r^* = \frac{\gamma}{\delta} i^*, \]  

(1.2)
yielding that the disease becomes endemic.

Li et al. [20] incorporated the spatial heterogeneity of epidemics into the model (1.1). Assuming that the density \( N(x) \) of the population at every position \( x \in (-\infty, \infty) \) is a positive constant and letting \( s(x,t), \ i(x,t) \) and \( r(x,t) \) be respectively the fractions of the population densities of susceptibles, infectives, and recovered classes at \( x \) and time \( t \) (thus \( s + i + r = 1 \)), the governing equations of their model are

\[ s_t = -\beta \int_{-\infty}^{\infty} k(x - y) i(y,t) \, dy \, s(x,t) + \delta r(x,t), \]

\[ i_t = \beta \int_{-\infty}^{\infty} k(x - y) i(y,t) \, dy \, s(x,t) - \gamma i(x,t), \]

\[ r_t = \gamma i(x,t) - \delta r(x,t), \]  

(1.3)
where \( s_t = \partial s/\partial t, \ i_t = \partial i/\partial t, \ r_t = \partial r/\partial t, \ \beta, \ \delta, \) and \( \gamma \) are constant as in (1.1), and the kernel \( k(x) \) satisfying \( k(-x) = k(x), \ k(x) \geq 0, \ k \in L^1(-\infty, \infty), \) and \( \int_{-\infty}^{\infty} k(x) \, dx = 1 \) is the contact distribution [18], with \( k(x - y) \) accounting for the proportion of the infectives at position \( y \) that contacts the susceptibles at \( x \). When \( \delta = 0 \), the model (1.3) reduces to the Kendall model [18], which was studied by Aronson [3], Barbour [5], Brown and Carr [7], and Mollison [25]. In particular, Aronson showed that the minimal wave speed is the asymptotic speed of propagation of disturbances from the steady state of the model.

In order to investigate the infection wavefronts for (1.3), Li et al. [20] studied traveling wave solutions for a diffusion approximation of (1.3) when the contact kernel \( k \) is local, i.e., \( k(x) = 0 \) for \( |x| \geq \varepsilon \), where \( \varepsilon \) is a small number. Assume further that the function \( i(x,t) \) does not change very much over the set of radius \( \varepsilon \), so that the fourth derivatives of \( i \) with respect to \( x \) are assumed to be \( O(1) \) on such a set. Using \( \int (x-y,t) = i(x,t) - yi_x(x,t) + \frac{1}{2} y^2 i_{xx}(x,t) - \frac{1}{6} y^3 i_{xxx}(x,t) + \frac{1}{24} y^4 i_{xxxx}(x,t) \) by Taylor’s formula, where \( x^* \) is some point between \( x-y \) and \( x \), they obtained

\[ \int_{-\infty}^{\infty} k(x-y) i(y,t) \, dy = \int_{-\infty}^{\infty} k(y) i(x-y,t) \, dy = i(x,t) \]  

\[ + \frac{1}{2} \int_{-\infty}^{\infty} k(y) y^2 \, dy \, i_{xx}(x,t) + O(\varepsilon^4), \]  

and then neglecting the \( O(\varepsilon^4) \) term, setting

\[ D := (\beta/2) \int_{-\infty}^{\infty} k(y) y^2 \, dy \]  

and using \( s(x,t) = 1 - i(x,t) - r(x,t) \), they derived the following diffusion approximation to (1.3):

\[ i_t = (\beta i + Di_{xx})(1 - i - r) - \gamma i, \quad r_t = \gamma i - \delta r. \]  

(1.4)
A special feature of (1.4) is the presence of the factor \( 1 - i - r \) in front of diffusion term \( i_{xx} \). Though \( D = O(\varepsilon^2) \) is small from the above derivation, we do not assume this in the rest of the paper.

Assume the reproduction number \( \sigma > 1 \). This implies that the system (1.4) has two uniform steady states \((0, 0)\) and \((i^*, r^*)\). Li et al. [20] looked for traveling wavefront solutions of (1.4) of the form \((i(x,t), r(x,t)) = (I(z), R(z)), z = ct + t, \) that move with constant speed \( c > 0 \) and connect the disease free and endemic equilibria \((0, 0)\) and \((i^*, r^*)\) at
\( z = \pm \infty \), respectively. By letting \( v = I' = dI/dz \) they reduced the problem to finding the heteroclinic solutions \((I, v, R)\) of the ODE system

\[
I' = v, \quad v' = \frac{cv + \gamma I}{D(1 - I - R)} - \frac{\beta I}{D}, \quad cR' = \gamma I - \delta R,
\]

satisfying the conditions

\[
I(z) > 0, \quad R(z) > 0, \quad I(z) + R(z) < 1, \quad \forall z \in (-\infty, \infty),
\]

\((I, v, R)(-\infty) = O := (0, 0, 0), \quad (I, v, R)(\infty) = E := (i^*, 0, r^*)\). (1.6)

They studied these solutions by using formal arguments and numerical simulations. Our purpose is to establish rigorously the following:

**Theorem 1.1** Let \( D > 0, \gamma > 0, \beta > 0, \delta > 0, \) and \( \sigma = \beta/\gamma > 1 \).

(i) (Existence) If \( c \) satisfies

\[
\frac{c^2}{D} > \max \left\{ 4\gamma(\sigma - 1), \frac{1}{4\gamma}\left(i^* \left(\frac{M_1}{i^*} + M_2\right)^2 + r^* M_2^2\right)\right\},
\]

with \( M_1 := \max\{\gamma(2\sigma - 1), \delta\} \) and \( M_2 := \max\{2(\sigma - 1)\gamma, \gamma\} \), then there exists a solution to (1.5) and (1.6).

(ii) (Uniqueness) If \( c^2/D \) is sufficiently large, then the solution to (1.5) and (1.6) is unique.

**Remarks 1**

(i) The first lower bound \( 4\gamma(\sigma - 1) \) in (1.7) is a necessary condition for the solutions of (1.5) on the 2-dimensional unstable manifold \( W^u(O) \) to be nodal near \( O \). That is, the quantity \( \sqrt{4D\gamma(\sigma - 1)} \) is the least value for the minimal wave speed.

(ii) The second lower bound in (1.7) results from the nonconstant diffusion in (1.4) and the Lyapunov function used in the proof of Theorem 1.1. We believe that this bound could be improved via constructing a better Lyapunov function.

(iii) The heteroclinic solutions obtained in Theorem 1.1 may be nodal or spiral near \( E \). For sufficiently large \( c^2/D \), sufficient conditions on the parameters \( \beta, \gamma \) and \( \delta \) are given at the end of Sect. 3 to determine which of the cases occurs.

(iv) Comparing to the results in the literature (see, e.g. [8, 14, 15, 21]), our global uniqueness result in Theorem 1.1 is new.

(v) If \( c^2/D \) is sufficiently large, it can be shown by a perturbation argument [1] that there exists a traveling wave solution for the nonlocal system (1.3) near each of those obtained in Theorem 1.1. Thus, the solutions in Theorem 1.1 provide the approximations to traveling waves of (1.3).

(vi) An analogous theorem is proved for our second model in Sect. 4, for which we are able to show that traveling wave solutions exist for all speeds \( c \geq \sqrt{4D\gamma(\sigma - 1)} \) and \( \sqrt{4D\gamma(\sigma - 1)} \) is the minimal speed.

The paper is organized as follows. In Sect. 2, we first change the system (1.5) by introducing new independent variable \( \xi \) into an equivalent system (2.1), in which the denominator \( 1 - I - R \) in (1.5) is removed. We prove Theorem 2.2 for the new system (2.1), whose existence part is proved by a shooting argument in combination with LaSalle’s invariance principle. Roughly speaking, we first show that a portion of the 2-dimensional locally unstable manifold \( W^u(O) \) of (2.1) lies inside a triangular pyramid (see Figs. 1, 2), then by a shooting argument that there exists at least one solution of (2.1) lying on this portion of \( W^u(O) \) and
remaining in the pyramid on \((-\infty, \infty)\), and then by LaSalle’s invariance principle that this solution approaches \(E\) as \(\xi \to \infty\). Such an approach was first developed by Dunbar [8] and subsequently simplified and generalized by, e.g., Huang et al. [14], Lin et al. [21], and Huang [15]. Nevertheless, it is not an easy task to carry out this approach for (2.1) due to its particular feature. In Sect. 3, we prove the uniqueness part of Theorem 2.2 by a geometric singular perturbation argument [1]. This argument also gives a shorter existence proof for sufficiently large \(c^2/D\). In Sect. 4, we establish a similar theorem to Theorem 2.2 for the second model.
2 Existence of Traveling Waves

Hereafter we assume that $\sigma > 1$ and $c > \sqrt{4D\gamma(\sigma - 1)}$. We first establish the following lemma.

Lemma 2.1  
(i) Assume that $(I, v, R)$ is a solution of (1.5) and (1.6). Define the independent variable $\xi$ by $\xi := \frac{1}{1 - I(c\tau) - R(c\tau)} d\tau$, and let

$$f(\xi) := I(cz), \quad u(\xi) := cv(cz), \quad g(\xi) := R(cz), \quad h(\xi) := 1 - f(\xi) - g(\xi), \quad \kappa := \frac{c^2}{D}.$$ 

Then, $(f, u, g)$ is a heteroclinic solution of

$$f' = hu, \quad u' = \kappa [u + (\gamma - \beta h)f], \quad g' = h(\gamma f - \delta g), \quad (2.1)$$

satisfying

$$f(\xi) > 0, \quad g(\xi) > 0, \quad f(\xi) + g(\xi) < 1 \quad \forall \xi \in (-\infty, \infty), \quad (f, u, g)(-\infty) = O, \quad (f, u, g)(\infty) = E. \quad (2.2)$$

(ii) Assume that $(f, u, g)$ is a solution of (2.1) and (2.2). Define

$$z := \int_0^\xi h(\tau/c) d\tau, \quad I(z) := f(\xi/c), \quad v(z) := \frac{1}{c} u(\xi/c), \quad R(z) := g(\xi/c).$$

Then, $(I, v, R)$ is a solution of (1.5) and (1.6).

Proof The definition of $\xi$ in (i) implies $d\xi/dz = 1/[1 - I(cz) - R(cz)] > 1$, yielding that $\xi$ transforms $(-\infty, \infty)$ onto $(-\infty, \infty)$. The rest of assertions of (i) can be verified directly with the aid of chain rule.

The definition of $z$ in (ii) implies $dz/d\xi = h(\xi/c) > 0$, which together with $h(-\infty) = 1$ and $h(\infty) = 1/\sigma$ yielding that $z$ transforms $(-\infty, \infty)$ onto $(-\infty, \infty)$. The rest of assertions of (ii) follow directly with the aid of chain rule.

Comparing the system (1.5) with (2.1), the latter is a smooth system on the whole phase space $\mathbb{R}^3$, with an invariant plane $1 - f - g = 0$ and additional equilibria $(f_0, u_0, g_0)$ consisting of the line given by the intersection of the planes $1 - f - g = 0$ and $u + \gamma f = 0$. A linearization of (2.1) at these equilibria with $g_0 > 0$ shows that there are 2-dimensional unstable manifold and 1-dimensional center manifold. Roughly speaking, this excludes the existence of heteroclinic solutions of (2.1) connecting the origin and these equilibria.

It follows from Lemma 2.1 that Theorem 1.1 is equivalent to the following theorem.

Theorem 2.2  
(i) (Existence) If $\kappa$ satisfies (1.7), then there exists a solution to (2.1) and (2.2).

(ii) (Uniqueness) If $\kappa$ is sufficiently large, then the solution to (2.1) and (2.2) is unique (up to a translation).

In the rest of the section, we show Theorem 2.2 (i) via several lemmas. The solutions of (2.1) and (2.2) to be shown lie in the open triangular pyramid $\Omega$ (see Fig. 1):

$$\Omega := \{(f, u, g) : f > 0, -\gamma f < u < \lambda_1 f, \quad g > 0, \quad f + g < 1\},$$
where $\lambda_1$ is the smaller positive root of the equation $\lambda^2 - \kappa \lambda + \kappa \gamma (\sigma - 1) = 0$ given in (2.5) below. We thus start with our study of the solutions of (2.1) starting in $\Omega$. The lemma below shows that such a solution, if it exits $\Omega$, can only do so from its two boundary sides $\Delta OAC$ and $\Delta OBC$ transversely.

**Lemma 2.3**  
(i) Both the g-axis and the plane $h = 1 - f - g = 0$ are invariant sets of (2.1).
(ii) Except on their common edge $OC$, the vector field of (2.1) points to the exterior of $\Omega$ on its faces $\Delta OAC$ and $\Delta OBC$ which lie on the planes $u = \lambda_1 f$ and $u = -\gamma f$, respectively.
(iii) At the interior points of its bottom face $\Delta OAB$ (which lies on the plane $g = 0$), the vector field of (2.1) points to the interior of $\Omega$.

**Proof** Since the vector field of (2.1) at each point $(0, 0, g)$ of g-axis is $(0, 0, -\delta(1 - g)g)$ and $h$ satisfies the equation $h' = -f' - g = -h(u + \gamma f - \delta g)$, the assertions in (i) follow at once.

Let $F(Q)$ be the vector field of (2.1) at a point $Q \in cl(\Omega)$ (the closure of $\Omega$). If $Q$ is an arbitrary point on $\Delta OAC$ but not on its edges $OC$, $AC$, and $BC$, then since $n_1 := (-\lambda_1, 1, 0)$ is an outward normal vector of the plane $u = \lambda_1 f$, using $0 < h < 1$ and $f > 0$ we have

$$n_1 \cdot F(Q) = -f[\lambda_1^2 h - \kappa \lambda_1 + \kappa(\beta h - \gamma)] > -[\lambda_1^2 - \kappa \lambda_1 + \kappa \gamma (\sigma - 1)]f = 0.$$

If $Q$ is an arbitrary point on $\Delta OBC$ but not on its edges $OC$ and $BC$, then since $n_2 := (-\gamma, -1, 0)$ is an outward normal vector of the plane $u = -\gamma f$, using $f > 0$ and $h > 0$ we have

$$n_2 \cdot F(Q) = -\gamma hu - \kappa[u + (\gamma - \beta) h] = (\gamma^2 + \kappa \beta)fh > 0.$$  

Since $g' = \gamma hf > 0$ at every interior point of $\Delta OAB$, the assertion (iii) follows. \qed

In the next lemma we study the local dynamics of (2.1) at $O$ and $E$.

**Lemma 2.4**  
(i) The Jacobian matrix $A_0$ of the vector field of (2.1) at $O$ has two positive eigenvalues $\lambda_1 < \lambda_2$ and one negative eigenvalue, and eigenvectors $V_1$ and $V_2$ associated to $\lambda_1$ and $\lambda_2$ respectively, where

$$V_1 := \left(1, \lambda_1, \frac{\gamma}{\lambda_1 + \delta}\right)^T, \quad V_2 := \left(1, \lambda_2, \frac{\gamma}{\lambda_2 + \delta}\right)^T. \quad (2.3)$$

The unstable manifold $W^u(O)$ of (2.1) is 2-dimensional, which is tangent to the plane spanned by $V_1$ and $V_2$ at $O$, and can be written as, for a sufficiently small $r_0 > 0$,

$$W^u(0) = \left\{(f, u, g)^T = \alpha_1 V_1 + \alpha_2 V_2 + O(\alpha_1^2 + \alpha_2^2) : (\alpha_1, \alpha_2) \in \mathbb{R}^2, \quad \alpha_1^2 + \alpha_2^2 \leq r_0 \right\}.$$  

(ii) The Jacobian matrix $A_1$ of the vector field of (2.1) at $E$ has one positive eigenvalue and two other eigenvalues that are either both negative real numbers or a complex conjugate pair with negative real parts. The stable manifold $W^s(E)$ of (2.1) is 2-dimensional.

**Proof** A routine computation yields that the Jacobian matrices $A_0$ and $A_1$ of the vector field of (2.1) at $O$ and $E$ are, respectively

$$A_0 = \begin{pmatrix} 0 & 1 & 0 \\ \kappa(\gamma - \beta) & \kappa & 0 \\ \gamma & 0 & -\delta \end{pmatrix}, \quad A_1 := \begin{pmatrix} 0 & 1/\sigma & 0 \\ \kappa \gamma \sigma i^* & \kappa \gamma \sigma i^* & 0 \\ \gamma/\sigma & 0 & -\delta/\sigma \end{pmatrix}. \quad (2.4)$$
Using \( \gamma - \beta = \gamma (1 - \sigma) \), we obtain that the characteristic equation of \( A_0 \) is \( (\lambda + \delta) [(\lambda^2 - \kappa \lambda + \kappa \gamma (\sigma - 1)] = 0 \), and the eigenvalues of \( A_0 \) are \( \lambda_3 = -\delta \), and

\[
\lambda_1 = \frac{\sqrt{\kappa}}{2} \left( \sqrt{\kappa} - \sqrt{\kappa - 4 \gamma (\sigma - 1)} \right), \quad \lambda_2 = \frac{\sqrt{\kappa}}{2} \left( \sqrt{\kappa + \sqrt{\kappa - 4 \gamma (\sigma - 1)}} \right).
\]

Since \( \sigma > 1 \), it follows that both \( \lambda_1 \) and \( \lambda_2 \) are positive with \( \lambda_1 < \lambda_2 \). A direct verification shows that \( V_1 \) and \( V_2 \) defined in (2.3) are the eigenvectors of \( A_0 \) associated to \( \lambda_1 \) and \( \lambda_2 \), respectively. Applying the stable manifold theorem yields the assertions for \( W^u(O) \) in (i).

Next, the characteristic equation of \( A_1 \) is \( P(\lambda) = \sigma \lambda^3 - (\kappa \sigma - \delta) \lambda^2 - \kappa (\gamma i^* + \delta) \lambda - \kappa \gamma i^*(\gamma + \delta) = 0 \). Since \( P(0) < 0 \), it follows that \( P \) has at least one positive zero, which is denoted by \( \rho_1 \). Let \( \rho_2 \) and \( \rho_3 \) be the other two zeros of \( P \). Upon using the relations among these zeros and the coefficients of \( P \), we get the equalities \( \rho_1(\rho_2 + \rho_3) + \rho_2 \rho_3 = -\kappa (\gamma i^* + \delta/\sigma) \) and \( \rho_1 \rho_2 \rho_3 = \kappa \gamma i^*(\gamma + \delta)/\sigma \). The latter equality together with \( \rho_1 > 0 \) yields \( \rho_2 \rho_3 > 0 \), and the former then yields \( \rho_2 + \rho_3 < 0 \). These two inequalities imply the assertions in (ii) readily.

The following lemma shows that a portion of the 2-dimensional local unstable manifold \( W^u(O) \) of (2.1) lies in \( \mathcal{C}(\Omega) \).

**Lemma 2.5** Let \(-\frac{\pi}{4} < \theta_0 < 0 \) be defined by \( \tan \theta_0 = \frac{\lambda_1 + \gamma}{\lambda_2 + \gamma} \), where \( \lambda_1 \) and \( \lambda_2 \) are given in (2.5). Then, for sufficiently small \( r_0 > 0 \), there exist continuous functions \( \theta_1(r) \) and \( \theta_2(r) \) for \( r \in (0, r_0) \) with \( \theta_1(0) = \theta_0 + o(r) \) and \( \theta_2(r) = o(r) \) as \( r \to 0 \) such that all points of the subset \( \Sigma_0 \) (the sector \( OA'O'B' \) in Fig. 2) of \( W^u(O) \), with \( \Sigma_0 \) defined by

\[
\Sigma_0 = \left\{ (f, u, g) \right\} = (r \cos \theta)V_1 + (r \sin \theta)V_2 + O(r^2) \in W^u(O) : 0 < r \leq r_0, \; \theta_1(r) \leq \theta \leq \theta_2(r) \}
\]

lie entirely in \( \Omega \), except for those on two edges \( OB' \) and \( OA' \) which lie on \( \Delta OBC \) and \( \Delta OAC \) respectively, where \( OB' = \Sigma_0 \cap \{ \theta = \theta_1(r), 0 < r \leq r_0 \} \) and \( OA' = \Sigma_0 \cap \{ \theta = \theta_2(r), 0 < r \leq r_0 \} \).

**Proof** Let \( \varepsilon \in (0, \lambda_2 - \lambda_1) \) be sufficiently small and define \( \theta_3 \in (-\pi/4, 0) \) and \( \theta_4 \in (0, \pi/4) \) by \( \tan \theta_3 = -\frac{\lambda_1 + \gamma}{\lambda_2 + \gamma} - \varepsilon \) and \( \tan \theta_4 = \varepsilon \). Note that \( \theta_3 = \theta_0 + o(1) \) and \( \theta_4 = o(1) \) as \( \varepsilon \to 0 \).

Let \( r_0 > 0 \) be sufficiently small and define \( \Sigma \) to be the subset of \( W^u(O) \) by

\[
\Sigma := \left\{ (f, u, g) \right\} = (r \cos \theta)V_1 + (r \sin \theta)V_2 + O(r^2) \in W^u(O) : 0 < r \leq r_0, \; \theta_3 \leq \theta \leq \theta_4 \}
\]

It follows from Lemma 2.4 (i) that for any \( (f, u, g) \in \Sigma \), there exists a unique \( 0 < r \leq r_0 \) and \( \theta_3 \leq \theta \leq \theta_4 \) such that \( f = r(\cos \theta + \sin \theta) + O(r^2) \), \( u = r(\lambda_1 \cos \theta + \lambda_2 \sin \theta) + O(r^2) \), and \( g = r \left( \frac{\gamma}{\lambda_1 + \delta} \cos \theta + \frac{\gamma}{\lambda_2 + \delta} \sin \theta \right) + O(r^2) \). Therefore, using \( \cos \theta > \cos \pi/4 = 1/\sqrt{2} \), and \( \tan \theta \geq \tan \theta_3 > -1 \), we have

\[
f = r \cos \theta (1 + \tan \theta) + O(r^2) \geq \frac{r}{\sqrt{2}} [1 + \tan \theta_3 + O(r)] > 0,
\]

and

\[
g = r \cos \theta \left( \frac{\gamma}{\lambda_1 + \delta} + \frac{\gamma}{\lambda_2 + \delta} \tan \theta \right) + O(r^2) > \frac{r}{\sqrt{2}} \left( \frac{\gamma}{\lambda_1 + \delta} - \frac{\gamma}{\lambda_2 + \delta} + O(r) \right) > 0.
\]
Furthermore, if \( \theta = \theta_3 \), then
\[
\frac{u}{f} = \frac{\lambda_1 + \lambda_2 \tan \theta_3 + O(r)}{1 + \tan \theta_3 + O(r)} = -\gamma - \frac{(\lambda_2 + \gamma)^2 \epsilon + O(r)}{\lambda_2 - \lambda_1 - (\lambda_2 + \gamma) \epsilon + O(r)}
\]
\[
= -\gamma - \frac{(\lambda_2 + \gamma)^2 \epsilon}{\lambda_2 - \lambda_1} \left( 1 + O(\epsilon) + O(r) \right) < -\gamma;
\]
and if \( \theta = \theta_4 \), then
\[
\frac{u}{f} = \frac{\lambda_1 + \lambda_2 \tan \theta_4 + O(r)}{1 + \tan \theta_4 + O(r)} = \lambda_1 + \frac{\lambda_2 - \lambda_1}{1 + \epsilon} \epsilon + O(r) > \lambda_1.
\]

Applying the intermediate value theorem together with the fact that the vector field of (2.1) are nontangential at the interior of \( \triangle OAC \) and \( \triangle OBC \) yields the assertions of Lemma 2.5.

\( \square \)

We are now in a position to show the following by a shooting argument.

**Lemma 2.6** There exists a solution \((f, u, g)\) of (2.1) such that \((f, u, g)(\xi) \in \Omega\) for \(\xi \in (-\infty, \infty)\) and \((f, u, g)(-\infty) = O\).

**Proof** Fix sufficiently small \(r_0 > 0\) and let \(C\) be the curve lying on \(\Omega \cap \Sigma_0\) defined by
\[
C := \{(f, u, g)^T := (r_0 \cos \theta) V_1 + (r_0 \sin \theta) V_2 + O(r_0^2) \in \Omega \cap \Sigma_0 : \theta_1(r_0) < \theta < \theta_2(r_0)\}.
\]

Clearly, \(C\) is a continuous (open) curve, with its point lying in \(\Omega\), and its boundary points \(A'\) corresponding to \(\theta = \theta_2(r_0)\) and \(B'\) corresponding to \(\theta = \theta_1(r_0)\) lying on the interiors of the faces \(\triangle OAC\) and \(\triangle OBC\), respectively (see Fig. 2). Let \(\phi(f_0, u_0, g_0)\) be the solution of (2.1) through an arbitrary point \((f_0, u_0, g_0) \in \mathbb{R}^3\) at \(\xi = 0\). We define two subsets \(A\) and \(B\) of \(C\) by
\[
A := \left\{ (f_0, u_0, g_0) \in C : \exists \xi_0 > 0 \text{ such that } \phi(f_0, u_0, g_0)(\xi) \in \Omega \forall \xi \in [0, \xi_0], \text{ and } \phi(f_0, u_0, g_0)(\xi_0) \in \triangle OAC \right\},
\]
and
\[
B := \left\{ (f_0, u_0, g_0) \in C : \exists \xi_0 > 0 \text{ such that } \phi(f_0, u_0, g_0)(\xi) \in \Omega \forall \xi \in [0, \xi_0], \text{ and } \phi(f_0, u_0, g_0)(\xi_0) \in \triangle OBC \right\}.
\]

Since the vector field of (2.1) at \(A'\) and \(B'\) points to the exterior of \(\Omega\) transversally, it follows from the smoothness of the vector field of (2.1) that all points of \(C\) near \(A'\) and \(B'\) belong to the sets \(A\) and \(B\), respectively. Thus both sets \(A\) and \(B\) are nonempty. The vector field of (2.1) on the faces \(\triangle OAC\) and \(\triangle OBC\) from Lemma 2.3 implies that both sets \(A\) and \(B\) are open (relative to \(C\)). Thus, by the connectedness of \(C\), the set \(C \setminus (A \cup B)\) is nonempty. Let \((f_0, u_0, g_0)\) be an arbitrary point in this set. It follows from Lemma 2.3 that the solution \(\phi(\tilde{f}_0, \tilde{u}_0, \tilde{g}_0)(\xi)\) stays in \(\Omega\) for all \(\xi \geq 0\). Using the vector field of (2.1) on the faces \(\triangle OAC\) and \(\triangle OBC\) and the invariance of \(W^u(O)\) together with the definition of \(\Sigma_0\) we conclude that \(\phi(\tilde{f}_0, \tilde{u}_0, \tilde{g}_0)(\xi) \in \Sigma_0 \subset \Omega\) for all \(\xi < 0\) and \(\phi(f_0, u_0, g_0)(\xi) \to O\) as \(\xi \to -\infty\). This proves the lemma.

\( \square \)

To show the solution found in Lemma 2.6 approaching \(E\) as \(\xi \to \infty\), we prove the following lemma by using LaSalle’s invariance principle.

**Lemma 2.7** Assume that (1.7) holds. Then every solution of (2.1) staying in \(\Omega\) on \([0, \infty)\) approaches \(E\) as \(\xi \to \infty\).
Proof Step 1. Construct a Lyapunov function for (2.1) in $\Omega$. As in [8], we first define a Lyapunov function in the triangle region $\triangle := \{(f, g) : f > 0, g > 0, f + g < 1\}$

$$L(f, g) := \left( f \frac{i^*}{i^*} - 1 - \ln \frac{f}{i^*} \right) + \frac{\beta}{2\delta r^*}(g - r^*)^2$$

(2.6)

for the associated reaction system of (2.1):

\[ f' = hf(\beta h - \gamma) = -\beta hf[(f - i^*) + (g - r^*)], \]
\[ g' = h(\gamma f - \delta g) = h[\gamma(f - i^*) - \delta(g - r^*)]. \]

(2.7)

Along the solutions of (2.7) lying in $\triangle$, we have, using $\gamma i^* = \delta r^*$,

$$L' = -h\left[ \frac{\beta}{i^*}(f - i^*)^2 + \frac{\beta}{r^*}(g - r^*)^2 \right] \leq 0.$$

We now define a Lyapunov function for the full system (2.1) by:

$$V(f, u, g) := \kappa L(f, g) - h \frac{\partial L}{\partial f} u, \quad (f, u, g) \in \Omega.$$

Then, along solutions $(f, u, g)$ of (2.1) lying in $\Omega$, using $\frac{\partial L}{\partial f} \frac{1}{i^*} = \frac{1}{f}$, $\frac{\partial^2 L}{\partial f^2} = \frac{1}{f^2}$ and $\frac{\partial^2 L}{\partial f \partial g} = 0$ we have

$$V' = \kappa \frac{\partial L}{\partial f} f' + \kappa \frac{\partial L}{\partial g} g' - h \frac{\partial L}{\partial f} u' - h' \frac{\partial L}{\partial f} u - h \frac{1}{f^2} f'u$$

$$= h \frac{\partial L}{\partial f}(\kappa u - u') + \kappa \frac{\partial L}{\partial g} g' - h' \frac{\partial L}{\partial f} u - h^2 \frac{u^2}{f^2}

= \kappa h \frac{\partial L}{\partial f} (\beta h - \gamma) + \kappa h \frac{\partial L}{\partial g} (\gamma f - \delta g) - h' \frac{\partial L}{\partial f} u - h^2 \frac{u^2}{f^2}

= -h\left[ \frac{\kappa \beta}{i^*}(f - i^*)^2 + \frac{\kappa \beta}{r^*}(g - r^*)^2 \right] - h' \frac{\partial L}{\partial f} u - h^2 \frac{u^2}{f^2}.$$  

(2.8)

Step 2. Show that $V' < 0$ in $\Omega \setminus \{E\}$. To do so, we first estimate $|h'(\partial L/\partial f)u|$ and $h^2 u^2/\sigma^2$.

Since $h' = -(f' + g') = -h(u + \gamma f - \delta g), u + \gamma f \geq 0, g > 0$, and $f, g \leq 1$, we have $u + \gamma f \leq \gamma f \leq \gamma$ if $u \leq 0$, and $u + \gamma f \leq \lambda_1 f + \gamma f \leq \lambda_1 + \gamma$ if $0 < u \leq \lambda_1 f$, and subsequently $|h'| \leq h \max\{u + \gamma f, \delta g\} \leq h \max\{\lambda_1 + \gamma, \delta\}$. Since

$$\lambda_1 = \frac{2\sqrt{\kappa} \gamma (\sigma - 1)}{\sqrt{\kappa} + \sqrt{4\gamma(\sigma - 1)}} \leq 2\gamma(\sigma - 1),$$

it follows that $|h'| \leq h \max\{\gamma(2\sigma - 1), \delta\} = h M_1$, and then using the expression of $\frac{\partial L}{\partial f}$,

$$|h' \frac{\partial L}{\partial f} u| \leq h M_1 \frac{|u|}{i^*} |f - i^*| \frac{|u|}{f}.$$

On the other hand, using $1 = 1/\sigma + i^* + r^*$, we have

$$h \frac{u^2}{f^2} = (1 - f - g) \frac{u^2}{f^2} = \frac{1}{\sigma} \frac{u^2}{f^2} - (f - i^*) \frac{u^2}{f^2} - (g - r^*) \frac{u^2}{f^2}.$$

\(\square\) Springer
Since $|u| \leq \max\{\lambda, \gamma\} \leq \max\{2\gamma(\sigma - 1), \gamma\} = M_2$, it follows that

$$-h^2 \frac{u^2}{f^2} \leq -h \left[ \frac{1}{\sigma} \frac{u^2}{f^2} - M_2 |f - \xi^*| \frac{|u|}{f} - M_2 ||g - r^*|| \frac{|u|}{f} \right].$$

Inserting the above estimates into (2.8) we get

$$V' \leq -h \left[ \frac{\kappa \beta}{i^*} (f - \xi^*)^2 + \frac{\kappa \beta}{r^*} (g - r^*)^2 + \frac{1}{\sigma} \frac{u^2}{f^2} - \left( \frac{M_1}{i^*} + M_2 \right) |f - \xi^*| \frac{|u|}{f} - M_2 |g - r^*| \frac{|u|}{f} \right]$$

$$= -h \left( f - \xi^*, g - r^*, \frac{|u|}{f} \right) Q \left( f - \xi^*, g - r^*, \frac{|u|}{f} \right)^	op,$$

where $Q$ is the symmetric matrix given by

$$Q = \begin{pmatrix}
\frac{\kappa \beta}{i^*} & 0 & -(M_1/i^* + M_2)/2 \\
0 & \frac{\kappa \beta}{r^*} & -M_2/2 \\
-(M_1/i^* + M_2)/2 & -M_2/2 & 1/\sigma
\end{pmatrix}.$$

Note that the condition (1.7) on $\kappa$ is equivalent to $\det Q > 0$, yielding that $Q$ is positive definite. It follows that $V' < 0$ for all $(f, u, g) \in \Omega \setminus \{E\}$.

Step 3. Let $\tilde{\phi} = (\tilde{f}, \tilde{u}, \tilde{g})$ be an arbitrary solution of (2.1) lying in $\Omega$ on $[0, \infty)$ and $\omega(\tilde{\phi})$ its $\omega$-limit set. Clearly, $\omega(\tilde{\phi}) \subset cl(\Omega)$. We show that $\omega(\tilde{\phi}) = \{E\}$.

Claim 1. There exists $m \in (0, i^*)$ such that $\omega(\tilde{\phi}) \subset cl(\Omega) \cap \{f \geq m\}$. If not, there would be a sequence $\{\xi_n\}$ such that $\xi_n > 0$, $\xi_n \to \infty$, and $\tilde{f}(\xi_n) \to 0$ as $n \to \infty$. Since $|\tilde{u}(\xi_n)|/\tilde{f}(\xi_n)$ is bounded by $\lambda + \gamma$ and $\ln \tilde{f}(\xi_n)/i^* \to \infty$ as $n \to \infty$, it follows that $V(\tilde{\phi})(\xi_n) \to \infty$ as $n \to \infty$, contradicting $V(\tilde{\phi})$ is decreasing on $[0, \infty)$.

Claim 2. Let $\Omega_m := cl(\Omega) \cap \{f \geq m\}$. Then either $\omega(\tilde{\phi}) \subset \Omega_m \cap \{h = 0\}$ or $\omega(\tilde{\phi}) = \{E\}$. To show the claim, noticing that $V$ is defined on $\Omega_m$ and (2.9) holds on $\Omega_m$, it follows from LaSalle’s invariance principle that $\omega(\tilde{\phi}) \subset \Omega_m \cap \{V' = 0\} = (\Omega_m \cap \{h = 0\}) \cup \{E\}$. Since the two sets on the right-hand side are disconnected and $\omega(\tilde{\phi})$ is a connected set, Claim 2 follows.

Claim 3. $\omega(\tilde{\phi}) = \{E\}$. Assume that the claim is false. It follows from Claim 2 that $\tilde{h}(\xi) \to 0$ as $\xi \to \infty$, yielding that there exists $\xi_0 > 0$ such that $\gamma - \beta \tilde{h}(\xi) > 0$ for $\xi > \xi_0$. Using the second equation in (2.1) we derive that either (i) $\tilde{u} < 0$ on $[\xi_0, \infty)$ or (ii) there exists $\xi_1 \geq \xi_0$ such that $\tilde{u}(\xi) = 0$ and $\tilde{u}(\xi) > 0$ for $\xi > \xi_1$. If the case (ii) occurs, then since $\tilde{u}' > \kappa \tilde{u}$ on $[\xi_1, \infty)$ we have, for any fixed $\xi_2 > \xi_1$, $\tilde{u}(\xi) > \tilde{u}(\xi_2) e^{\kappa(\xi - \xi_2)}$ for $\xi \geq \xi_2$, yielding $\tilde{u}(\xi) \to \infty$ as $\xi \to \infty$. This contradicts the boundedness of $\tilde{u}$ on $[0, \infty)$. If the case (i) happens, we have $\tilde{f}' = \tilde{h} \tilde{u} < 0$ on $[\xi_0, \infty)$ and thus $\tilde{u}(\xi) = \tilde{f}_0$ for some $\tilde{f}_0 \in [m, 1)$. This implies that $\tilde{g}(\xi) = 1 - \tilde{f}(\xi) - \tilde{h}(\xi) = \tilde{g}_0 := 1 - \tilde{f}_0 > 0$ as $\xi \to \infty$, yielding that $\tilde{h} > \tilde{h} f > \frac{1}{\delta} \delta \tilde{g}_0$ on $[\xi_3, \infty)$ for a sufficiently large $\xi_3 > \xi_0$. Noticing that the equation for $\tilde{h}$ is $\tilde{h}' = -\tilde{f}' - \tilde{g}' = \tilde{h}[\delta \tilde{g} - \gamma \tilde{f} - \tilde{u}] = \tilde{h}[\delta \tilde{g} - \beta \tilde{f} \tilde{h} - \frac{1}{\kappa} \tilde{u}]$, we have $\tilde{h}' > \tilde{h}\{\frac{1}{\delta} \delta \tilde{g}_0 - \frac{1}{\kappa} \tilde{u}\}$ on $[\xi_3, \infty)$, yielding $\tilde{h}(\xi) > \tilde{h}(\xi_3) \exp\left(\frac{1}{\kappa} \delta \tilde{g}_0 (\xi - \xi_3) - \frac{1}{\kappa} \frac{1}{\kappa} (\tilde{u}(\xi) - \tilde{u}(\xi_3))\right) \to \infty$ as $\xi \to \infty$. This contradicts the fact that $\tilde{h} < 1$ on $[0, \infty)$. The above contradictions prove Claim 3, whence the assertion of Step 3, and Lemma 2.7.

**Proof of Theorem 2.2 (i)** It is clear from Lemmas 2.6 and 2.7 that the assertion of Theorem 2.2 (i) follows.
3 Uniqueness of Traveling Waves

Let \( \varepsilon = 1/\kappa \) and rewrite the system (2.1) as

\[
\begin{align*}
f' &= hu, \\
\varepsilon u' &= u + (\gamma - \beta h)f, \\
g' &= h(\gamma f - \delta g),
\end{align*}
\]

which is a singularly perturbed system for sufficiently small \( \varepsilon \), with the slow variables \( f \) and \( g \) and the fast variable \( u \). Noticing that (3.1) is a smooth system in the whole phase space \( \mathbb{R}^3 \), it follows from the Fenichel geometric singular perturbation theory \([9,17]\) that, for every \( \varepsilon \in [0, \varepsilon_0) \) with a sufficiently small \( \varepsilon_0 > 0 \), there exists a slow manifold \( M_\varepsilon \) (generally not unique) of (3.1) given by (see Fig. 3 for \( M_0 \))

\[
M_\varepsilon := \{(f, u, g) : u = U(f, g, \varepsilon) := f(\beta h - \gamma) + \varepsilon H(f, g, \varepsilon), (f, g) \in cl(\Delta)\},
\]

where \( \Delta := \{(f, g) : f > 0, g > 0, f + g < 1\} \) as defined in the proof of Lemma 2.7 and \( cl(\Delta) \) is the closure of \( \Delta \), and \( H \) is a smooth function on \( cl(\Delta) \times [0, \varepsilon_0) \). Then the slow variables \( (f, g) \) of the solutions of (3.1) on \( M_\varepsilon \) satisfies the planar system

\[
\begin{align*}
f' &= h[f(\beta h - \gamma) + \varepsilon H(f, g, \varepsilon)], \\
g' &= h[\gamma f - \delta g],
\end{align*}
\]

Lemma 3.1 For sufficiently small \( \varepsilon > 0 \), if \( (f, u, g) \) is a heteroclinic solution of (3.1) satisfying (2.2), then \( (f, u, g) \) lies on the slow manifold \( M_\varepsilon \); i.e., \( u(\xi) = U(f(\xi), g(\xi), \varepsilon) \) for all \( \xi \in (-\infty, \infty) \).

Proof Let \( \varepsilon > 0 \) be sufficiently small and \( (f, u, g) \) a solution of (3.1) satisfying (2.2). We use two methods to show that \( (f, u, g) \) lies on \( M_\varepsilon \).

Method 1. From the slow manifold theory, it suffices to show that \( (f, u, g) \) lies in a small neighborhood of the critical manifold \( M_0 \). To show this, we let \( w(\xi) = u(\xi) - U(f(\xi), g(\xi), 0) = u(\xi) + f(\xi)[\gamma - \beta h(\xi)] \) for \( \xi \in (-\infty, \infty) \), and have
\[ \varepsilon w' = \varepsilon u' + \varepsilon(\gamma - \beta h)f - \varepsilon h'f = w + \varepsilon(\gamma - \beta h)hu + \varepsilon f h(u + \gamma f - \delta) \]

\[ = w + \varepsilon(\gamma - \beta h)(w + (\beta - \gamma)f) + \varepsilon f (w + \beta f h - \delta) = [1 + \varepsilon A(\xi)]w + \varepsilon B(\xi), \]

where \( A(\xi) := h(\xi)[\gamma - \beta h(\xi) + f(\xi)] \) and \( B(\xi) := h(\xi)\left\{-[\gamma - \beta h(\xi)]^2 + f(\xi)[\beta f(\xi)h(\xi) - \delta g(\xi)]\right\}. \) Note that since \((f(\xi), g(\xi)) \in \Delta \) for \( \xi \in (-\infty, \infty), \) we have \(|A(\xi)| \leq \beta + 1 \) and \(|B(\xi)| \leq K := (\beta - \gamma)^2 + \beta + \delta \) for \( \xi \in (-\infty, \infty). \) Since \( w \) is bounded on \((-\infty, \infty), \) we have, for \( \varepsilon < 1/[2(\beta + 1)] \) and \( \xi \in (-\infty, \infty), \)

\[ |w(\xi)| = \left| \int_{\xi}^\infty B(\eta) \exp\left[-\int_{\xi}^{\eta} \left[1 + A(\tau)\right] d\tau\right] d\eta \right| \leq K \int_{\xi}^\infty e^{-(\eta - \xi)/(2\varepsilon)} d\eta = 2K\varepsilon, \]

yielding that \((f, u, g)\) lies in the \(2K\varepsilon\) neighborhood of \( M_0. \)

Method 2. We show that \( \chi(\xi) := u(\xi) - U(f(\xi), g(\xi), \varepsilon) = 0 \) for \( \xi \in (-\infty, \infty). \) To this end, we first derive an equation for \( \chi. \) Let \((F_1, F_2, F_3)\) be the right-hand sides of \((3.1).\) Noting that the local invariance of the slow manifold \( M_0 \) implies that \( \varepsilon U_f(f, g, \varepsilon)F_1(f, U(f, g, \varepsilon), g, \varepsilon) + U_g(f, g, \varepsilon)F_3(f, U(f, g, \varepsilon), g, \varepsilon) = F_2(f, U(f, g, \varepsilon), g, \varepsilon) \) for \((f, g) \in cl(\Delta), \) where \( U_f = \partial U/\partial f \) and \( U_g = \partial U/\partial g, \) we have

\[ \varepsilon \chi' = \varepsilon u' - \varepsilon(U_f(f, g, \varepsilon)f' + U_g(f, g, \varepsilon)g') \]

\[ = \varepsilon u' - \varepsilon(U_f(f, g, \varepsilon)F_1(f, u, g, \varepsilon) + U_g(f, g, \varepsilon)F_3(f, u, g, \varepsilon)) \]

\[ = \varepsilon u' - \varepsilon(U_f(f, g, \varepsilon)F_1(f, U(f, g, \varepsilon), g, \varepsilon) + U_g(f, g, \varepsilon)F_3(f, U(f, g, \varepsilon), g, \varepsilon)) \]

\[ - \varepsilon[U_f(f, g, \varepsilon)F_1(f, u, g, \varepsilon) - F_1(f, U(f, g, \varepsilon), g, \varepsilon)] \]

\[ - \varepsilon[U_g(f, g, \varepsilon)F_3(f, u, g, \varepsilon) - F_3(f, U(f, g, \varepsilon), g, \varepsilon))] \]

\[ = F_2(f, u, g, \varepsilon) - F_2(f, U(f, g, \varepsilon), g, \varepsilon) - \varepsilon h U_f(f, g, \varepsilon)[u - U(f, g, \varepsilon)] \]

\[ = u - U(f, g, \varepsilon) - \varepsilon h U_f(f, g, \varepsilon)[u - U(f, g, \varepsilon)] \]

\[ = \left(1 - \varepsilon h U_f(f, g, \varepsilon)\right)\chi, \]

where we used \( F_3(f, u, g, \varepsilon) - F_3(f, U, g, \varepsilon) = 0 \) and \( F_1(f, u, g, \varepsilon) - F_1(f, U, g, \varepsilon) = h[u - U]. \)

Assume that \(|\chi(\xi_0)| > 0 \) for some \( \xi_0 \). Without loss of generality we assume that \( \chi(\xi_0) > 0 \) (otherwise replacing \( \chi \) by \(-\chi\)). Since \( 1 - \varepsilon h U_f(f, g, \varepsilon) > 1/2 \) for sufficiently small \( \varepsilon > 0, \) we have \( \chi'(\xi) > \frac{1}{2} \chi(\xi) \) for \( \xi > \xi_0, \) so that \( \chi(\xi) > \chi(\xi_0) \exp[(\xi - \xi_0)/(2\varepsilon)] \to \infty \) as \( \xi \to \infty, \) which contradicts the boundedness of \( \chi \) over \((-\infty, \infty). \) Therefore, \( \chi \equiv 0 \) on \((-\infty, \infty), \) as desired. \( \square \)

In the next lemma we show that there is a unique heteroclinic solution of \((3.1)\) lying on \( M_\varepsilon. \) For this, we need to write \( H \) in \((3.2)\) a suitable form. First, since the segment \([0 \leq g \leq 1]\) of the \( g\)-axis is invariant set of \((3.1)\) and \( U(0, g, 0) = 0, \) it follows that this segment lies on \( M_\varepsilon. \) This yields \( U(0, g, \varepsilon) = 0 \) so that \( H(0, g, \varepsilon) = 0. \) Note that \( cl(\Delta) \) is a convex set in \( \mathbb{R}^2. \) We have by the fundamental theorem of calculus (FTOC) that \( H(f, g, \varepsilon) = f H_1(f, g, \varepsilon) \) with \( H_1(f, g, \varepsilon) = \int_0^1 \frac{\partial H_1}{\partial f}(\tau f, g, \varepsilon) d\tau. \) Since \( 0 = H(i^*, r^*, \varepsilon) = i^* H_1(i^*, r^*, \varepsilon), \) it follows that \( H_1(i^*, r^*, \varepsilon) = 0. \) Again applying FTOC to the function \( \phi(\tau) = H_1(f, g, \varepsilon) + (1 - \tau) r g + (1 - \tau) r^* g \) yields \( H_1(f, g, \varepsilon) = (f - i^*) H_{11}(f, g, \varepsilon) + (g - r^*) H_{12}(f, g, \varepsilon), \) where
Thus (3.2) becomes

\[ f' = hf \left[ \beta h - \gamma + \varepsilon \left( (f - i^*) H_{11}(f, g, \varepsilon) + (g - r^*) H_{12}(f, g, \varepsilon) \right) \right], \]
\[ g' = h[\gamma f - \delta g], \quad (f, g) \in cl(\triangle). \]  

(3.3)

Note that when \( \varepsilon = 0 \), this system reduces (2.7).

**Lemma 3.2** For sufficiently small \( \varepsilon > 0 \), there is a unique (up to translation) heteroclinic solution \((f, u, g)\) of (3.1) that satisfies (2.2) and lies on the slow manifold \( \mathcal{M}_\varepsilon \).

**Proof** To show the lemma, it suffices to show that, for sufficiently small \( \varepsilon \geq 0 \), there is a unique heteroclinic solution \((f, g)\) of (3.3) lying entirely in \( \triangle \) with \( (f, g)(-\infty) = (0, 0) \) and \( (f, g)(\infty) = (i^*, r^*) \). We prove this by several steps.

Step 1. We show that the unstable manifold of (3.3) at \((0, 0)\) is 1-dimensional. The Jacobian matrix of the vector field of (3.3) is

\[ J_0 = \begin{pmatrix} \beta - \gamma - \varepsilon [i^* H_{11}(0, 0, \varepsilon) + r^* H_{12}(0, 0, \varepsilon)] & 0 \\ \gamma & -\delta \end{pmatrix}, \]

which has a negative eigenvalue \(-\delta\), a positive eigenvalue \(\lambda_+ = \beta - \gamma + O(\varepsilon)\), and an eigenvector \([1, \gamma/(\delta + \lambda_+)]^T\) associated with \(\lambda_+\). Thus \((0, 0)\) is a saddle point of (3.3), with the unstable manifold given by \(r[1, \gamma/(\delta + \lambda_+)]^T + O(r^2)\) for sufficiently small \(r\), and lying in the positive quadrant for \(r > 0\).

Step 2. We show that the set \(\triangle\) is a positively invariant set of (3.3). This follows from the facts that the boundaries \(f = 0\) and \(f + g - 1 = 0\) of \(\triangle\) are invariant sets of (3.3), and \(g' = \gamma hf > 0\) on the remaining boundary \(\{g = 0\} \cap \{f > 0\}\).

Step 3. Let \(L\) be the Lyapunov function defined in (2.6). We show that if \(\varepsilon \geq 0\) sufficiently small, then \(L' < 0\) along any solution of (3.3) lying in \(\triangle \setminus \{(i^*, r^*)\}\) on \([0, \infty)\). This follows from a direct computation:

\[ L' = h \left[ -\frac{\beta}{i^*}(f - i^*)^2 - \frac{\beta}{r^*}(g - r^*)^2 + \varepsilon i^* \left( (f - i^*) H_{11}(f, g, \varepsilon) + (f - i^*)(g - r^*) H_{12}(f, g, \varepsilon) \right) \right] < 0. \]

Step 4. Let \(\psi = (\tilde{f}, \tilde{g})\) be an arbitrary solution of (3.3) lying in \(\triangle \setminus \{(i^*, r^*)\}\) on \([0, \infty)\). We show that \(\omega(\psi) = \{(i^*, r^*)\}\). It follows from LaSalle’s invariance principle that \(\omega(\psi)\) lies in the set \(\{L' = 0\} \cap cl(\triangle) = \{(i^*, r^*)\} \cup \{(f, g) \in cl(\triangle) : h = 0\}\). Since the latter two sets are disconnected, it follows that either \(\omega(\psi) = \{(i^*, r^*)\}\), or \(\omega(\psi)\) lies entirely in the set \(\{(f, g) \in cl(\triangle) : h = 0\}\). We claim that the latter case does not happen.

Assuming that the claim is false, we have \(h(\xi) \to 0\) as \(\xi \to \infty\). Note that there exists a constant \(M > 0\) such that \(|(f - i^*) H_{11}(f, g, \varepsilon) + (g - r^*) H_{12}(f, g, \varepsilon)| \leq M\) for all
(f, g) ∈ cl (∆) and sufficiently small ε. It follow that there exists ξ₀ > 0 such that βh − γ + Mε < −γ/2 for all ξ ≥ ξ₀ and ε < γ/(4M). Thus, assuming ε < γ/(4M), we have f′ < 0 on [ξ₀, ∞) so that \( f(ξ) \rightarrow f₀ \) as \( ξ \rightarrow ∞ \) for some \( f₀ \in [0, 1) \) and \( g(ξ) = 1 - f - h \rightarrow g₀ := 1 - f₀ > 0 \) as ξ → ∞. It then follows from (3.3) that there exists sufficiently large \( ξ₁ > ξ₀ \) such that \( h'(ξ) = h(ξ)[δ\bar{g}(ξ) - β\bar{h}(ξ)f(ξ) - Mε] > \frac{1}{2}δ\bar{g}h(ξ) \) for all \( ξ > ξ₁ \), yielding \( h(ξ) \rightarrow ∞ \) as \( ξ \rightarrow ∞ \). This contradicts \( 0 < h < 1 \) on \( (−∞, ∞) \). Whence the above claim holds, yielding that \( ω(ψ) = \{(i^*, r^*)\} \).

Step 5. Let (f, g) be a solution of (3.3) with \((f, g)(0)\) belonging to the unstable manifold of (3.3) at \( O \) and the set \( ∆ \). It follows from the results in Steps 1-4 that such a solution gives a heteroclinic solution of (3.3) as stated in the beginning of the proof. Its uniqueness follows from the fact that the unstable manifold of (3.3) at \((0, 0)\) is 1-dimensional. This completes the proof of Lemma 3.2.

\[ \square \]

**Proof of Theorem 2.2 (ii)** It is trivial to see that the assertion of Theorem 2.2 (ii) follows directly from Lemmas 3.1 and 3.2.

At the end of this section, we present conditions for the solutions found in Theorems 1.1 and 2.2 to be nodal (resp., spiral) near \( E \) when \( c²/D \) is sufficiently large.

**Theorem 3.3** Let \( c²/D \) be sufficiently large. If

\[
(δ + γ)² + (β - γ)² ≥ (2 + \frac{4γ}{δ})(β - γ)(δ + γ),
\]

then the solutions found in Theorem 1.1 are nodal near \( E \); otherwise, these solutions are spiral near \( E \).

**Proof** From the proof of Lemma 3.2 it suffices to study the local dynamics of the system (3.3) with \( ε = 0 \) near \((i^*, r^*)\). The resulting system is (2.7), whose Jacobian matrix at \((i^*, r^*)\) is

\[
\left( \begin{array}{cc}
-βi^*/σ & -βi^*/σ \\
γ/σ & -δ/σ
\end{array} \right),
\]

with the characteristic equation \( σ²λ² + σ(βi^* + δ)λ + βi^*(δ + γ) = 0 \), and the eigenvalues \( λ = \frac{1}{2σ} \left[ -(βi^* + δ) ± \sqrt{(βi^* - δ)² - 4βi^*γ} \right] \). In order for \((i^*, r^*)\) to be nodal, it is necessary and sufficient to require \( (βi^* - δ)² - 4βi^*γ > 0 \), which, after inserting the formulas for \( i^* \) and \( σ \), is found to be equivalent to the one in (3.4). Theorem 3.3 thus follows.

\[ \square \]

Clearly, (3.4) is true if the ratio \((β - γ)/(δ + γ)\) is sufficiently large, and false if this ratio is close to one.

**4 Nonlocal Diffusion SIRS Model**

As mentioned in the introduction, Medlock and Kot [24] considered a distributed infectives model, which is a nonlocal diffusion SI model. Here we consider a corresponding nonlocal diffusion SIRS model, under the assumptions that (i) contacts are local but individuals move on the real line with the constant rate \( α > 0 \), and (ii) the density of the population is constant on \((−∞, ∞) \). Still letting \( s(x, t) \), \( i(x, t) \) and \( r(x, t) \) denote the fractions of the population densities of susceptible, infective, and recovered classes respectively, the governing equations of the model are:
\[ s_t = -\beta s_i + \delta r - \alpha s + \alpha \int_{-\infty}^{\infty} k(x-y)s(y,t)\,dy, \]
\[ i_t = \beta s_i - \gamma i - \alpha i + \alpha \int_{-\infty}^{\infty} k(x-y)i(y,t)\,dy, \]
\[ r_t = \gamma i - \delta r - \alpha r + \alpha \int_{-\infty}^{\infty} k(x-y)r(y,t)\,dy, \]  

(4.1)

where \(\beta, \gamma, \delta\) are positive constants, and the kernel \(k(x)\) represents the dispersal distribution and satisfies the same conditions as in (1.3), with \(k(x-y)\) prescribing the proportion of individuals leaving place \(y\) and going to \(x\).

To derive a diffusion approximation for (4.1), we make the same assumptions for \(k\) to be local as in the model (1.3), so that we have the approximations for the convolution integrals:
\[ \int_{-\infty}^{\infty} k(y)i(x-y,t)\,dy \approx i(x,t) + \frac{1}{2} \int_{-\infty}^{\infty} k(y)^2\,dy \quad \text{and} \quad \int_{-\infty}^{\infty} k(y)r(x-y,t)\,dy \approx r(x,t). \]

We remark that we did not include the diffusion term in the second convolution integral, for otherwise we would have to study traveling waves for a system of two reaction–diffusion equations, leading to the study of heteroclinic solutions of a 4-dimensional ODE system, whose existence proof turns out to be much more difficult and will be given in a different paper. Letting \(D := \frac{1}{2}\alpha \int_{-\infty}^{\infty} k(y)^2\,dy\) and using the identity \(s(x,t) = 1 - i(x,t) - r(x,t)\) yields the following diffusion approximation to (4.1):
\[ i_t = \beta(1 - i - r)i - \gamma i + Di_{xx}, \quad r_t = \gamma i - \delta r. \]  

(4.2)

Assuming \(\sigma > 1\), the system (4.2) also has the uniform steady states \((0,0)\) and \((i^*, r^*)\) with the latter defined in (1.2). Like for (1.4), we are interested in traveling wavefronts of (4.2) of the form \((i(x,t), r(x,t)) := (f(\xi), g(\xi)), \ \xi = (x + ct)/c = x/c + t, \) that connect \((0,0)\) and \((i^*, r^*)\) at \(\xi = \pm \infty\) and move with the constant speed \(c > 0\). Letting \(u := f' = df/d\xi, \ h := 1 - f - g, \) and \(\kappa := c^2/D\) yields that \((f, u, g)\) are heteroclinic solutions of
\[ f' = u, \quad u' = \kappa[u + (\gamma - \beta h)f], \quad g' = \gamma f - \delta g, \]  

(4.3)

satisfying
\[ f(\xi) > 0, \quad g(\xi) > 0 \quad \forall \xi \in (-\infty, \infty), \quad (f, u, g)(-\infty) = O, \quad (f, u, g)(\infty) = E. \]  

(4.4)

The following is our main result in this section.

**Theorem 4.1** Assume that \(\sigma > 1\). Then,

(i) For any \(\kappa \geq \kappa_0 := 4\gamma(\sigma - 1)\), there exists a solution to (4.3) and (4.4) lying entirely in the bounded set \(\Omega_\kappa\) (see Fig. 5) defined by
\[ \Omega_\kappa := \{(f, u, g) : 0 < f < 1 - 1/\sigma, -M < u < \lambda_1 f, 0 < g < \gamma(1 - 1/\sigma)/\delta, \} \]

where \(\lambda_1\) is given in (2.5), and \(M := \beta(1 + \gamma/\delta)(1 - 1/\sigma)^2\).

(ii) For sufficiently large \(\kappa\), there is a unique (up to translation) solution to (4.3) and (4.4).

**Remark 2** From the formula for \(\lambda_1\) in (2.5) one can check that \(\lambda_1\) is a decreasing function of \(\kappa \in [\kappa_0, \infty)\). Since \(\lambda_1 = \kappa_0/2 = 2\gamma(\sigma - 1)\) for \(\kappa = \kappa_0\), it follows that \(\lambda_1 \leq 2\gamma(\sigma - 1)\) for \(\kappa \geq \kappa_0\). Thus, all \(\Omega_\kappa\) for \(\kappa \geq \kappa_0\) are the subsets of

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Fig. 4  Sketched are the set $\Omega$ (bounded by shaded planes), the set $\Sigma_0$ (the sector $O'A'B'$), and three solution orbits of (4.3) from $\Sigma_0$, with the middle one being a heteroclinic orbit claimed in Theorem 4.1 (i) in the case that $E$ is spiral

$$\Omega_{\kappa_0} = \{(f, u, g) : 0 < f < 1 - 1/\sigma, -M < u < 2\gamma(\sigma - 1)f, 0 < g < \gamma(1 - 1/\sigma)/\delta\}.$$  

Theorem 4.1 will be proved by similar arguments used in Sects. 2 and 3 for Theorem 2.2. We give the corresponding lemmas and the modifications of their proofs. The main distinction lies in the proof of Lemma 4.6 where, due to the unboundedness of the set $\Omega$ (defined below), extra work is required to show that the positive semi-orbits of solutions of (4.3) lying in $\Omega$ are bounded. For convenience, we use the same notations as those in Sects. 2 and 3. We assume hereafter that $\sigma > 1$ and $\kappa \geq \kappa_0$. Since some lemmas below hold only for $\kappa > \kappa_0$, we shall state this condition whenever required.

For $\kappa \geq \kappa_0$ we define the open set (see Fig. 4)

$$\Omega := \{(f, u, g) : f > 0, u < \lambda_1 f, g > 0\}.$$  

Note that $\Omega$ is unbounded, and depends on $\kappa$ implicitly (since $\lambda_1$ depends on $\kappa$).

**Lemma 4.2** If $(f, u, g)$ is a solution of (4.3) with $(f, u, g)(0) \in \Omega$ and defined on a forward maximal interval $[0, \xi^+]$, then either $(f, u, g)(\xi) \in \Omega$ for all $\xi \in [0, \xi^+]$, or leaves $\Omega$ only from its faces $f = 0$ or $u = \lambda_1 f$ transversally.

**Proof** The assertions of the lemma follow from the following facts: (i) the $g$-axis is an invariant set of (4.3) (via directly checking); (ii) the vector field of (4.3) at each interior point of the face $g = 0$ of $\Omega$ points to the interior of $\Omega$ (since $g' = \gamma f > 0$); (iii) the vector field of (4.3) at each point $(0, u, g) (u < 0)$ of the face $f = 0$ of $\Omega$ points to the exterior of $\Omega$ (since $f' = u < 0$); (iv) the vector field of (4.3) at every point of the face $u = \lambda_1 f$ of $\Omega$ with $f > 0$ points to the exterior of $\Omega$ (since $u' - \lambda_1 f' = f[\kappa \lambda_1 + \kappa(\gamma - \beta)h - \lambda_1^2] > f[\kappa \lambda_1 + \kappa(\gamma - \beta) - \lambda_1^2] = 0$ where we used $h \leq 1 - f < 1$).  

\[\square\]
Lemma 4.3 (i) The unstable manifold $W^u(O)$ of (4.3) is 2-dimensional; furthermore, if $\kappa > \kappa_0$, $W^u(O)$ is tangent to the plane spanned by $V_1$ and $V_2$ at $O$, and can be written as, for a sufficiently small $r_0 > 0$,

$$W^u(O) = \left\{ (f, u, g)^T = \alpha_1 V_1 + \alpha_2 V_2 + O(\alpha_1^2 + \alpha_2^2) : (\alpha_1, \alpha_2) \in \mathbb{R}^2, \ \alpha_1^2 + \alpha_2^2 \leq r_0 \right\},$$

where $V_1$ and $V_2$ are given in (2.3).

(ii) The stable manifold $W^s(E)$ of (4.3) is 2-dimensional.

Proof A direct computation shows that the Jacobian matrix $A_0$ of the vector field of (4.3) at $O$ is the same as that of (2.1) given in (2.4). Thus applying the proof for Lemma 2.4 (i) yields the assertions in (i).

The Jacobian matrix of the vector field of (4.3) at $E$ is given by the same matrix $A_1$ in (2.4) with $\sigma$ replaced by 1. Then applying the same argument used there in the proof of Lemma 2.4 (ii) yields the assertion in (ii). □

Lemma 4.4 Assume that $\kappa > \kappa_0$. Then, for sufficiently small $r_0 > 0$, there exist continuous functions $\theta_1(r)$ and $\theta_2(r)$ for $r \in (0, r_0]$ with $\theta_1(r) = -\pi/4 + o(r)$, and $\theta_2(r) = o(r)$ as $r \to 0$ such that all points of the subset $\Sigma_0$ (the sector $OA'B'$ in Fig. 4) of $W^u(O)$, with $\Sigma_0$ defined by

$$\Sigma_0 := \left\{ (f, u, g)^T = (r \cos \theta)V_1 + (r \sin \theta)V_2 + O(r^2) \in W^u(O) : 0 < r \leq r_0, \ \theta_1(r) \leq \theta \leq \theta_2(r) \right\},$$

lie entirely in $\Omega$, except for those on its two edges $OB'$ and $OA'$ which lie on the faces $f = 0$ and $u = \lambda_1 f$ of $\Omega$ respectively, where $OB' = \Sigma_0 \cap \{ \theta = \theta_1(r), 0 < r \leq r_0 \}$ and $OA' = \Sigma_0 \cap \{ \theta = \theta_2(r), 0 < r \leq r_0 \}$.

Proof The proof can be carried out by a similar argument in the proof of Lemma 2.5 and is omitted. □

Lemma 4.5 Assume that $\kappa > \kappa_0$. Then there exists a solution $(f, u, g)$ of (4.3) such that $(f, u, g)(\xi) \in \Omega$ for all $\xi$ in its maximal existence interval $(-\infty, \xi^+)$ and $(f, u, g)(-\infty) = O$.

Proof Let $r_0 > 0$ be sufficiently small and define an open continuous curve $C$ on $\Omega \cap \Sigma_0$ by

$$C := \{(f, u, g)^T = (r_0 \cos \theta)V_1 + (r_0 \sin \theta)V_2 + O(r_0^2) \in \Omega \cap \Sigma_0 : \theta_1(r_0) < \theta < \theta_2(r_0) \}.$$ 

The same shooting argument in the proof of Lemma 2.6 gives the existence of a point $(f_0, u_0, g_0) \in C$ such that the solution $(f, u, g)$ of (4.3) with $(f, u, g)(0) = (f_0, u_0, g_0)$ stays in $\Omega$ for all $\xi$ in its maximal existence interval $(-\infty, \xi^+)$ with $(f, u, g)(-\infty) = O$. Since $\Omega$ is unbounded, we do not claim here that $\xi^+ = \infty$, and instead prove this in the next lemma. □

Lemma 4.6 Assume that $\kappa > \kappa_0$. Let $(f, u, g)$ be a solution of (4.3) claimed in Lemma 4.5. Then $\xi^+ = \infty$, and $(f, u, g)$ lies entirely in the set $\Omega_\kappa$ with $(f, u, g)(\infty) = E$.

Proof Step 1. Let $L$ be the function given in (2.6), and define the Lyapunov function

$$V(f, u, g) = \kappa L(f, g) - \frac{\partial L}{\partial f} u = \kappa \left( \frac{f}{i^*} - 1 - \ln \frac{f}{i^*} \right) + \frac{\kappa \beta}{2 \delta r^*} (g - r^*)^2 - \frac{f - i^*}{i^*} \cdot \frac{u}{f}$$

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for $f > 0$. Then, along any solution $(f, u, g)$ of (4.3) with $f > 0$, 

$$V' = -\frac{\kappa \beta}{i^*} (f - i^*)^2 - \frac{\kappa \beta}{r^*} (g - r^*)^2 - \frac{u^2}{f^2} \leq 0.$$  

(4.5)

Step 2. For notational clarity, we let $(f, \bar{u}, \bar{g})$ be the solution of (4.3) claimed in Lemma 4.5. We prove that $\xi^+ = \infty$ and $(f, \bar{u}, \bar{g})$ lies entirely in $\Omega_\kappa$.

Claim 1. $\bar{f}(\xi) < 1 - 1/\sigma$ for all $\xi \in (-\infty, \xi^+)$. Assume the claim is false. Then there exists the smallest $\xi_0 \in (-\infty, \xi^+)$ such that $\bar{f}(\xi_0) = 1 - 1/\sigma$ and $\bar{u}(\xi_0) = \bar{f}(\xi_0) \geq 0$. We indeed have $\bar{u}'(\xi_0) > 0$ for otherwise we would have $\bar{u}'(\xi_0) = \kappa [\gamma - \beta + \bar{f}(\xi_0) + \beta \bar{g}(\xi_0)] \bar{f}(\xi_0) = \kappa \beta \bar{f}(\xi_0) \bar{g}(\xi_0) > 0$, yielding $\bar{f}(\xi_0)$ is a local minimum, which contradicts $\bar{f}(\xi) < \bar{f}(\xi_0)$ for all $\xi < \xi_0$. It then follows that $(\bar{f}, \bar{u}, \bar{g})$ enters the set $\Omega_1 := \{f > 1 - 1/\sigma, 0 < u < \lambda_1 f, g > 0\}$ immediately after $\xi_0$ (see Fig. 5). Note that, as long as $(\bar{f}, \bar{u}, \bar{g})$ remains in $\Omega_1$ on $(\xi_0, \xi)$, we would have $\bar{u}'(\xi) = \kappa [\bar{u}(\xi) + \gamma - \beta + \bar{f}(\xi) + \beta \bar{g}(\xi)] \bar{f}(\xi) \geq \kappa \beta \bar{f}(\xi) \bar{g}(\xi) > 0$, so does $\bar{f}'(\xi) = \bar{u}(\xi) > 0$. This implies that $(\bar{f}, \bar{u}, \bar{g})$ remains in $\Omega_1$ for all $\xi \in (\xi_0, \xi^+)$. Since $\bar{f}' = \bar{u} < \lambda_1 \bar{f}$ on $(-\infty, \xi^+)$, it follows that $\bar{f}(\xi) < e^{\lambda_1(\xi - \xi_0)}$ on $(\xi_0, \xi^+)$, so that $0 < \bar{u}(\xi) < \lambda_1 e^{\lambda_1(\xi - \xi_0)}$ and $0 < \bar{g}(\xi) = \bar{g}(\xi_0) + \gamma \int_{\xi_0}^{\xi} e^{-\delta(\xi - \eta)} \bar{f}(\eta) d\eta \leq \bar{g}(\xi_0) + \gamma \int_{\xi_0}^{\xi} e^{-\lambda_1(\xi - \xi_0)}(\xi - \xi_0).$ Using these estimates we conclude $\xi^+ = \infty$. Then using $\bar{u}' > 0$ on $[\xi_0, \infty)$ and (4.3) we derive that $(\bar{f}, \bar{u}, \bar{g})(\xi) \to \infty$ as $\xi \to \infty$. Then, for $\xi > \xi_0$, using $\bar{u}(\xi)/\bar{f}(\xi) \leq \lambda_1 < \kappa/2$ and $\bar{f}(\xi) - i^* > 0$ we have

$$V(\bar{f}, \bar{u}, \bar{g})(\xi) \geq \kappa \frac{\bar{f}(\xi) - i^*}{i^*} - \kappa \ln \frac{\bar{f}(\xi)}{i^*} \to \infty \quad (\xi \to \infty),$$

contradicting from Step 1 that $V(\bar{f}, \bar{u}, \bar{g})(\xi) < V(\bar{f}, \bar{u}, \bar{g})(\xi_0)$ for $\xi > \xi_0$. This shows Claim 1.

Claim 2. $\xi^+ = \infty$ and $\bar{g}(\xi) < \gamma(1 - 1/\sigma)/\delta$ for all $\xi \in (-\infty, \xi^+)$. This is because, for all $\xi \in (-\infty, \xi^+)$, $\bar{g}(\xi) = \gamma \int_{\xi_0}^{\xi} e^{-\delta(\xi - \eta)} \bar{f}(\eta) d\eta < \gamma(1 - 1/\sigma)/\delta$, and $\bar{u}' = \kappa [\bar{u} + (\gamma - \beta) \bar{f} + \beta(\bar{f} + \bar{g})] > \kappa [\bar{u} + (\gamma - \beta)(1 - 1/\sigma)]$ so that $\bar{u}(\xi) > (\beta - \gamma)(1 - 1/\sigma) + [\bar{u}(\xi_1) + \cdots$
\((\gamma - \beta)(1 - 1/\sigma)\) is a solution of (4.3) with \(\tilde{f} > 0, \tilde{g} > 0 \) and \(\tilde{u} < \lambda_1 \tilde{f}\) yields \(\xi^+ = \infty\) and the claimed estimate for \(\tilde{g}\).

Claim 3. \(\tilde{u} > -M\) on \((-\infty, \infty)\), where \(M\) is the constant defined in Theorem 4.1 (i).

Assume that the claim is false. Then there exists the smallest \(\xi_1 \in (-\infty, \infty)\) such that \(\tilde{u}(\xi_1) = -M\), and then, at \(\xi_1\),
\[
\tilde{u}' < \kappa [-M + \beta (\tilde{f} + \tilde{g}) \tilde{f}] = \kappa \left[-M + \beta \left(1 + \frac{\gamma}{\delta}\right) \left(1 - \frac{1}{\sigma}\right)^2\right] = 0.
\]

This yields that \((\tilde{f}, \tilde{u}, \tilde{g})\) enters the set \(\Omega_2 := \{0 < f < 1 - 1/\sigma, u \leq -M, 0 < g < \gamma(1 - 1/\sigma)/\delta\}\) immediately after \(\xi_1\), and remains in \(\Omega_2\) for all \(\xi > \xi_1\) (see Fig. 5). Then we have \(\tilde{f}(\xi) < \tilde{f}(\xi_1) - M(\xi - \xi_1)\) on \((\xi_1, \infty)\), yielding \(\tilde{f}(\xi) \to -\infty\), which contradicts the fact that \(\tilde{f} > 0\) on \((-\infty, \infty)\).

It is clear that the assertions stated at the beginning of Step 2 follow the above claims.

Step 3. Show that \(\tilde{f} := (\tilde{f}, \tilde{u}, \tilde{g})(\xi) \to E\) as \(\xi \to \infty\). From the assertions in Step 2, we conclude that the \(\omega\)-limit set \(\omega(\tilde{f})\) of \(\tilde{f}\) lies in \(cl(\Omega_\kappa)\).

Claim 1. If \(\omega(\tilde{f}) \cap \{f > 0\} \neq \emptyset\), then \(\omega(\tilde{f}) \cap \{f > 0\} = \{E\}\).

Let \(w = (f, u, g)\) be an arbitrary point in \(\omega(\tilde{f}) \cap \{f > 0\}\). Let \(\phi(w)\) be the solution of (4.3) through \(w\) at \(\xi = 0\). Then for sufficiently small \(\varepsilon > 0\) we have \(\phi(w)(\xi) \in cl(\Omega_\kappa) \cap \{f > 0\}\) for all \(|\xi| < \varepsilon\). Note that the Lyapunov function \(V\) is defined in this set and \(V' < 0\) in this set except the point \(E\). It follows from LaSalle’s invariance principle that \(\phi(w)(\xi) = E\) for \(|\xi| < \varepsilon\). This shows \(w = E\), and thus Claim 1.

Claim 2. If \(\omega(\tilde{f}) \cap \{(f, u, g) \in cl(\Omega_\kappa) : f = 0\} \neq \emptyset\), then \(\omega(\tilde{f}) \cap \{(f, u, g) \in cl(\Omega_\kappa) : f = 0\} = \{O\}\).

Let \(w = (0, \bar{u}, \bar{g}) \in \omega(\tilde{f})\). We must have \(\bar{u} = 0\), for otherwise the solution of (4.3) through \(w\) at \(\xi = 0\) immediately exists the face \(f = 0\) which contradicts that \(\omega(\tilde{f})\) lies in \(cl(\Omega_\kappa)\). We also must have \(\bar{g} = 0\) for otherwise the solution of (4.3) through \(w\) given by \(\phi(w) = (0, 0, \bar{g}e^{-\delta \xi}) \to (0, 0, \infty)\) as \(\xi \to -\infty\), which contradicts again that \(\omega(\tilde{f})\) lies in \(cl(\Omega_\kappa)\). This shows Claim 2.

It follows from above claims that \(\omega(\tilde{f}) = \{O, E\}\). The connectedness of \(\omega(\tilde{f})\) yields that either \(\omega(\tilde{f}) = \{O\}\) or \(\omega(\tilde{f}) = \{E\}\). If the former holds, then we have \(\tilde{f}\) lies on stable manifold \(W^s(O)\) of (4.3), which contradicts the fact that \(W^s(O)\) is the \(g\)-axis. Therefore, the latter holds. This shows the assertion in Step 3, and thus Lemma 4.6.

**Proof of Theorem 4.1 (i)** It is clear that the assertion in Theorem 4.1 (i) for \(\kappa > \kappa_0\) follows from Lemmas 4.5 and 4.6. It remains to show the assertion for \(\kappa = \kappa_0\).

Step 1. We take an arbitrary sequence \(\{\kappa_n\}\) with \(\kappa_n > \kappa_0\), and \(\kappa_n \to \kappa_0\) as \(n \to \infty\) and let \(\phi_n := (f_{\kappa_n}, u_{\kappa_n}, g_{\kappa_n})\) be a solution of (4.3) with \(\kappa = \kappa_n\) obtained above. By translation invariance we may assume that \(f_{\kappa_n}(0) = i^*/2\). From Remark 2 it follows that all \(\phi_n\) lie in \(\Omega_{\kappa_0}\) so that they are uniformly bounded on \((-\infty, \infty)\), which together with (4.3) yields that their derivatives \(\phi'_n\) are uniformly bounded on \((-\infty, \infty)\). Applying Arzela–Ascoli theorem and the diagonalisation argument yields the existence of a subsequence \(\{\phi_{n_j}\}\), convergent uniformly on any compact intervals of \((-\infty, \infty)\). Let \(\phi_0 = (f_{\kappa_0}, u_{\kappa_0}, g_{\kappa_0})\) be the limit function of \(\{\phi_{n_j}\}\). It is trivial to show that \(\phi_0\) lies entirely in \(cl(\Omega_{\kappa_0})\) with \(f_{\kappa_0}(0) = i^*/2\), and is a solution of (4.3) with \(\kappa = \kappa_0\).

Step 2. Let \(\alpha(\phi_0)\) and \(\omega(\phi_0)\) be the \(\alpha\)- and \(\omega\)-limit sets of \(\phi_0\), respectively. Clearly, both sets are in \(cl(\Omega_{\kappa_0})\). We show that \(\alpha(\phi_0) = \{O\}\) and \(\omega(\phi_0) = \{E\}\).

Since the Lyapunov function \(V\) defined in the proof of Lemma 4.6 is also defined for \(\kappa = \kappa_0\) and the formula for \(V'\) holds as well, employing the same argument used in Step 3 in the proof of Lemma 4.6 gives \(\omega(\phi_0) = \{E\}\).
To show $\alpha(\phi_0) = \{O\}$, we claim that $\alpha(\phi_0)$ lies on the face $f = 0$ of $cl(\Omega_{\kappa_0})$, i.e., $\alpha(\phi_0) \subset cl(\Omega_{\kappa_0}) \cap \{f = 0\}$. Assume this is false and let $w = (\bar{f}, \bar{u}, \bar{g}) \in \alpha(\phi_0)$ with $\bar{f} > 0$. Then there exists a sequence $\{\xi_j\}$ such that $\xi_j \to -\infty$ and $\phi_0(\xi_j) \to w$ as $j \to \infty$, implying $V(\phi_0)(\xi_j) \to V_0 := V(w)$ as $j \to \infty$ by the continuity of $V$ at $w$. On the other hand, $V(\phi_0)$ is decreasing, it follows that $V(\phi_0)(\xi) \to V_0$ as $\xi \to -\infty$. Let $\phi(w)$ be the solution of (4.3) through $w$ at $\xi = 0$. Then for small $\xi_0 > 0$ we have $\phi(w)(\xi) \in cl(\Omega_{\kappa_0}) \cap \{f > 0\}$ for all $|\xi| \leq \xi_0$. The same argument above shows that $V'(\phi(w))(\xi) = V_0$ for $|\xi| \leq \xi_0$. Then a differentiation gives that $V'(\phi(\bar{w}))(\xi) = 0$ for $|\xi| \leq \xi_0$, yielding $\phi(w)(\xi) = E$ for $|\xi| \leq \xi_0$. In particular we have $w = E$. Since $w$ is arbitrarily chosen, we conclude that $\alpha(\phi_0) \cap \{f > 0\} = \{E\}$. Then the connectedness of $\alpha(\phi_0)$ gives $\alpha(\phi_0) = \{E\}$. This together with $\omega(\phi_0) = \{E\}$ as showed above implies that $\phi_0$ is a homoclinic solution of (4.3) connecting $E$ at $\xi = \pm\infty$. Note that the vector field of (4.3) on the face $f = 0$ with $u \leq 0$ yields that $f_{\kappa_0}(\xi) > 0$ for all $\xi \in (-\infty, 0)$ so that $\phi_0$ lies entirely in $cl(\Omega_{\kappa_0}) \cap \{f > 0\}$. We then have $0 = V(\phi_0)(\infty) - V(\phi_0)(0) = \int_{-\infty}^{\infty} V'(\phi_0)(\xi) d\xi$, which together with the formula of $V'$ in (4.5) yields $V'(\phi_0)(\xi) = 0$ and $\phi_0(\xi) = E$ for $\xi \in (-\infty, \infty)$. This is impossible since $f_{\kappa_0}(0) = i^*\bar{s}/2$. This contradiction shows the above claim.

Therefore, $\alpha(\phi_0)$ is contained in the face $f = 0$ of $cl(\Omega_{\kappa_0})$. The vector field of (4.3) at this face yields $\alpha(\phi_0) = \{O\}$.

Finally, the vector field of (4.3) on the boundary of $\Omega_{\kappa_0}$ (see Lemma 4.2) ensures that $\phi_0$ cannot intersect this boundary. Thus, $\phi_0$ lies entirely in $\Omega_{\kappa_0}$. We have shown that $\phi_0$ has all the properties stated in Theorem 4.1 (i), and thus completed the proof of Theorem 4.1 (i).

**Proof of Theorem 4.1 (ii)** The proof can be carried out by the same arguments as those in the proof of Theorem 2.2 (ii), and is thus omitted.

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