An Application of Viscosity Theory to Higher Order Differential Equations

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Abstract

We directly apply the viscosity theory for fully nonlinear second-order differential equations to higher order differential operators. We prove existence and uniqueness theorems for equations of the general form:

$$
\epsilon \Delta v + G(D(\Delta v), D^2(\Delta v)) = f(x)
$$

when we have appropriate conditions on the functional $G$. As an explicit application, we show that the inhomogeneous $\infty$–Bilaplacian equation on a ball $B_R \subset \mathbb{R}^n$:

$$
\Delta^2_{\infty} u = (\Delta u)^3|D(\Delta u)|^2 = f(x)
$$

with Navier Boundary Conditions ($u = g \in C(\partial B_R)$, $\Delta u = 0$ on $\partial B_R$) admits solutions in $W^{2,\infty}(B_R)$ under some mild conditions on $f(x)$ (e.g. Hölder continuity).

1 Introduction

The development of viscosity theory for solving fully nonlinear partial differential equations has lead to strikingly general existence, uniqueness, and regularity results for second order problems. On the other hand, since the definition of a viscosity solution depends on what is essentially a second order phenomenon (i.e. some version of the maximum principle), the direct application of viscosity theory has always been limited to second order differential equations. In this paper, we examine compositions

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of differential operators of order less than or equal to two and directly apply viscosity theory to these higher order differential equations that are in “potential form”.

Generally, we have differential operators $L_0, L_1, \ldots, L_k$ of order less than or equal to two, and we consider the equation

$$(L_k \circ \cdots \circ L_1 \circ L_0)u = 0.$$ 

Commonly, when $L_k$ consists only of pure derivatives, an equation of this form is said to be in divergence form. Thus, the differential equations we deal with are in a form related to, yet more general than equations in divergence form.

If we use viscosity theory to resolve at least one of the operators $L_i$, we shall call the resulting solution a **viscopotential** solution to the problem. We must also take care in applying viscosity theory to such a composition of operators, since viscosity solutions encode only “indirect” and local behavior of the respective equation. Indeed, we shall see that the use of viscosity theory for resolving $L_j$ requires the use of viscosity theory for all $L_{j+i}$, with $i \geq 0$ as well.

The most explicit application for viscopotential solutions that we shall examine here is for the inhomogeneous $\infty$-Bilaplacian equation,

$$\Delta^2_\infty u = (\Delta u)^2|D(\Delta u)|^2 = f(x),$$

which is the formal limit as $p \to \infty$ of the $p$-Bilaplacian equation,

$$\Delta^2_p u = \Delta(|\Delta u|^{p-2}\Delta u) = f(x).$$

The latter equation arises from the minimization problem for the $L^p$ norm of $\Delta u$. In other words, the $p$-Bilaplacian is the Euler-Lagrange equation for the functional

$$\int_\Omega |\Delta u|^p dx.$$ 

Minimizers are sought in the space $W^{2,p}(\Omega)$, and the weak formulation of the homogeneous Dirichlet Problem for the $p$—Bilaplacian is solvable in $W^{2,p}$ whenever $p > 2$. Likewise, the $\infty$-Bilaplacian is the Euler-Lagrange equation for the $L^\infty$ variational problem of minimizing $\|\Delta u\|_\infty$, and minimizers are sought in the space $W^{2,\infty}$. The existence of minimizers for the $p$—Bilaplacian (weak solutions of $\Delta^2_p$) with $2 \leq p < \infty$ follows relatively easily from the coercivity of the related semilinear form:

$$(u, v) \to \int_\Omega (|\Delta u|^{p-2}\Delta u)\Delta v.$$
Further information on the $p$–Bilaplacian may be found in [5]. The case $p = 2$ is the linear Bilaplacian equation $\Delta^2 u = 0$. Among the usual choices for Boundary Values (Dirichlet, Neumann, Mixed), the Bilaplacian admits existence and uniqueness results for its weak formulation with Navier Boundary Conditions:

$$u = g, \quad \Delta u = 0 \text{ on } \partial \Omega$$

Indeed, with these boundary conditions, the Bilaplacian $\Delta^2$ can be appropriately decomposed into two iterations of Poisson’s Equation, and the natural function space to search for solutions becomes $W^{2,2}$. A discussion about the Bilaplacian along these lines is presented in [8]. Navier Boundary Conditions are the best choice for the equations we study, since we would also like to decompose higher order problems into second order ones. In particular, we shall analyze the Navier Boundary Problem for the $\infty$-Bilaplacian and perturbations of the same, obtaining the following theorems:

**Theorem 1.1.** Let $\epsilon > 0$ be arbitrary. Let $B_R$ be the ball of radius $R$ around the origin of $\mathbb{R}^n$. Let $f \geq 0$ be Hölder continuous and $g$ be a continuous function on $B_R$. We assume in addition that

$$f(x) = O((R^2 - |x|^2)^3|x|^2).$$

Then the Boundary Value Problem

$$\begin{cases}
\epsilon \Delta v + (\Delta v)^3 \partial (\Delta v)|^2 = f(x) \\
v = g \text{ on } \partial B_R \\
\Delta v = 0 \text{ on } \partial B_R
\end{cases}$$

has a $C^{2,\alpha}$ viscopotential solution $v_\epsilon$ for some $\alpha > 0$.

**Theorem 1.2.** Let $f, g$ satisfy the hypotheses of the previous theorem. The Boundary Value Problem

$$\begin{cases}
(\Delta v)^3 \partial (\Delta v)|^2 = f(x) \\
v - g \in W^{1,2}_0(B_R) \\
\Delta v = 0 \text{ a.e. on } \partial B_R
\end{cases}$$

admits two (possibly equal) viscopotential solutions $v_0, v_1 \in W^{2,\infty}(B_R)$.

The $\infty$–Bilaplacian equation does not admit classical $C^3$ solutions for the Dirichlet Problem unless we are in the homogeneous case with very simple boundary values (e.g. quadratic functions when in $\mathbb{R}$) [6]. On the other hand, our perturbations of
the $\infty$–Bilaplacian equation admit viscopotential solutions to the Navier Problem with $C^{2,\alpha}$ regularity, which is as good as we might expect. The solutions to the perturbed problem then help us construct a viscopotential solution to the actual $\infty$–Bilaplacian equation in the regularity class $W^{2,\infty}$, which is more regular than the natural regularity class for Navier Boundary Value Problems and in the class desired for the associated variational problem. In this way, we improve on the existing theory detailed, for example, in [5], [6] by dealing with the inhomogeneous case and by the generality of our boundary values for $v$. For example, the authors of [6] have shown existence for the Dirichlet Problem with boundary values in $W^{2,\infty}$, while we deal with continuous boundary values. Before moving on, we remark that the homogeneous Navier Problem for the $\infty$–Bilaplacian admits very simple solutions. Indeed, we have classical solutions that coincide in this case with harmonic functions with appropriate boundary values. In particular:

**Theorem 1.3.** Let $\Omega$ be a $C^2$ bounded domain, and let $g$ be a continuous function on $\bar{\Omega}$. Then the Navier Boundary Value Problem:

\[
\begin{cases}
(\Delta v)^3|D(\Delta v)|^2 = 0 \\
v = g \text{ on } \partial \Omega \\
\Delta v = 0 \text{ on } \partial \Omega
\end{cases}
\]

admits a unique classical solution $v$ that is precisely the solution to the Dirichlet problem $\Delta v = 0$ in $\Omega$, $v = g$ on $\partial \Omega$.

The generality of existence theorems in viscosity theory allows us to obtain results that apply to a much wider class of differential equations. We include these general results to illustrate how the breadth of viscosity theory transposes to the situation of higher order differential equations, but their use always requires one to check the sometimes rather stringent hypotheses of the theorems. We count among these all the theorems we have in Section 3.

In the following and final portion of this introductory section, we collect an ensemble of well-known results in elliptic and viscosity theory that we apply in our study. In Section 2, we introduce the use of viscopotential solutions for a “toy model”: nonlinear third-order ordinary differential equations. In Section 3, we deal with general partial differential equations in potential form, and in Section 4 we analyze the $\infty$-Bilaplacian equation, obtaining our main theorems.

We would like to thank Camillo De Lellis for very helpful conversations about viscosity theory and about this paper. We also thank him for advising us to examine the Bilaplacian and related partial differential equations.
1.1 Some Results in Elliptic Theory

The following two theorems are standard and some version of them may be found, for example, in [3].

**Theorem 1.4.** Let $\Omega$ be a bounded $C^2$ domain and let $f, g \in C(\overline{\Omega})$. Then the Dirichlet Problem

\[
\begin{align*}
\Delta u &= f \text{ in } \Omega \\
u - g &\in W^{1,2}_0(\Omega)
\end{align*}
\]

has a unique weak solution $u \in W^{2,2}(\Omega)$ such that

\[\Delta u = f \text{ a.e. in } \Omega.\]

**Theorem 1.5.** Let $\Omega$ be a bounded $C^2$ domain and let $\varphi \in C(\overline{\Omega})$. Let $f \in C^{0,\alpha}_{\text{loc}}(\Omega)$. Then the Dirichlet Problem

\[
\begin{align*}
\Delta u &= f \text{ in } \Omega \\
u &= \varphi \text{ on } \partial \Omega
\end{align*}
\]

has a unique $C^2$ classical solution. Moreover, if $\Omega$ is a Euclidean ball, then $u \in C^{2,\alpha}$.

Now we give an overview of the results we shall need from viscosity theory.

1.2 Some Results in Viscosity Theory

The definitions and results listed here are an amalgamation of what are now standard results from [1] and [2]. The only exception in this section is the more recent Theorem 1.14, which is from [4].

Let $\Omega$ be a locally compact domain in $\mathbb{R}^n$. We denote the extended real numbers by $\bar{\mathbb{R}}$. We consider second-order partial differential equations of the most general form:

\[E(x, u, Du, D^2u) = 0,\]  

(1)

where $E$ is a function $E: \Omega' \times \mathbb{R} \times \mathbb{R} \times \text{Sym}^{n \times n} \to \mathbb{R}$.

**Definition 1.** Equation (1) is called proper if and only if

\[E(x, r, p, X) \leq E(x, s, p, X)\]

whenever $r \leq s$. 

5
**Definition 2.** Equation (1) is called **degenerate elliptic** if and only if
\[ E(x, r, p, X) \leq E(x, r, p, Y) \]
whenever \( X \leq Y \) (i.e. \( X - Y \) is negative definite).

**Definition 3.** Let \( \Omega' \subset \Omega \) be a dense subset and \( u : \Omega' \to \mathbb{R} \). The upper semi-continuous (u.s.c.) regularization \( u^* \) on \( \Omega \) of \( u \) is
\[ u^*(x) = \inf \{ v(x) | v \text{ is upper semi-continuous and } v \geq u \} \tag{2} \]
Likewise we define the lower semi-continuous regularization (l.s.c.) to be:
\[ u_*(x) = \sup \{ v(x) | v \text{ is lower semi-continuous and } v \leq u \} \tag{3} \]

Let \( E \) as in Equation (1) be defined on the dense set \( \Omega' \subset \Omega \).

**Definition 4.** A function \( u : \Omega \to \mathbb{R} \) is a viscosity subsolution of Equation (1) (\( E \leq 0 \)) if and only if
- \( u^* \) is a real-valued function
- for any \( x \in \Omega \), if \( \varphi : \mathbb{R}^n \to \mathbb{R} \) is \( C^\infty \) and \( u^* - \varphi|_\Omega \) has a local maximum at \( x \), then
\[ E^*_s(x, u^*, D\varphi, D^2\varphi)) \leq 0. \tag{4} \]
Likewise, we may define the notion of viscosity supersolution.

**Definition 5.** A function \( u : \Omega \to \mathbb{R} \) is a viscosity solution of Equation (1) (\( E = 0 \)) if and only if it is both viscosity subsolution and a viscosity supersolution to the same problem.

We now collect some basic results in viscosity theory.

**Theorem 1.6.** Suppose \( \Omega \subset \mathbb{R}^n \) is open, \( u \in C^2(\Omega) \), and \( E \) is continuous. It follows that if \( u \) is a viscosity solution of Equation (1), then \( u \) is also a classical solution of the same.

**Theorem 1.7.** If \( u \) is a classical subsolution (supersolution) of Equation (1), then it is also a viscosity subsolution (supersolution) if \( E^*_s \) (\( E^*_s \)) is non-increasing in \( \text{Sym}^{n \times n} \).

**Theorem 1.8** (Perron’s Method for Existence). Let \( f \) and \( g \) be respectively a viscosity subsolution and viscosity supersolution of Equation (1) such that \( f^*_s > -\infty \) and \( g^*_s < \infty \) on \( \Omega \). If \( f \leq g \) on \( \Omega \) and \( E^*_s, E^*_s \) are non-increasing in \( \text{Sym}^{n \times n} \), then there exists a viscosity solution \( u \) of Equation (1) such that \( f \leq u \leq g \).
**Theorem 1.9.** Let \( h \in C(\partial \Omega) \) be arbitrary. Consider functionals \( F(x, S) : \Omega \times \text{Sym}^{n \times n} \rightarrow \mathbb{R} \) such that \( F \) is continuous, satisfies the following monotonicity condition for some \( \lambda > 0 \)

\[
F(x, S + tI) \geq F(x, S) + \lambda t
\]

and such that \( \mathcal{F}_S = \{F(\cdot, S)\} \) is equicontinuous in \( \Omega \). In addition, assume that there exist \( f, g \) respectively a subsolution and a supersolution of

\[
-F(x, D^2 u) = 0
\]

in \( \Omega \) such that \( f_* > -\infty \), \( g^* < \infty \), \( f \leq g \) on \( \Omega \), and \( g^* \leq f_* \) on \( \partial \Omega \). Then there exists a unique continuous solution \( u \) to the boundary value problem

\[
\begin{cases}
-F(x, D^2 u(x)) = 0 & \text{in } \Omega \\
u = h & \text{on } \partial \Omega
\end{cases}
\]

**Theorem 1.10.** Let \( \lambda > 0 \) be arbitrary. Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^n \), let \( f \) be continuous and \( G \) continuous, proper, and degenerate elliptic. Let \( u, v \) be respectively a subsolution and supersolution of

\[
\lambda u + G(u, Du, D^2 u) - f(x) = 0
\]

in \( \Omega \) such that \( u \leq v \) on \( \partial \Omega \). Then \( u \leq v \) in \( \overline{\Omega} \).

**Corollary 1.11.** Assume the hypotheses of the previous theorem. If \( u_0, u_1 \) are two continuous viscosity solutions of

\[
\lambda u + G(u, Du, D^2 u) - f(x) = 0
\]

such that \( u_0 = u_1 \) on \( \partial \Omega \), then \( u_0 = u_1 \) on \( \Omega \).

**Proof.** Since \( u_i \) is continuous, \( u_i = u_i^* = u_i^* \) for \( i \in 0, 1 \), and \( u_0, u_1 \) are both viscosity subsolutions and supersolutions of the same problem. Using Theorem 1.10, the result follows immediately. \( \square \)

For the rest of this section, we fix a \( C^2 \) bounded domain \( \Omega \) in \( \mathbb{R}^n \). We let

\[
d(x) := d(x, \partial \Omega) = \inf_{y \in \partial \Omega} |y - x|
\]

and define:

\[
\Omega_\gamma = \left\{ x \in \Omega \mid d(x) < \frac{1}{\gamma} \right\}
\]
where $\gamma$ is chosen sufficiently large for $d(x)$ to be in $C^2(\Omega_\gamma)$. We say that the functional $G(p, S)$ satisfies **Property P** if and only if there exists some $\lambda > 0$, $M > 0$ such that

$$G(\lambda cDd(x), \lambda cD^2d(x) - \lambda^2 cDd(x) \otimes Dd(x)) - f(x) \geq 0, \quad \forall x \in \Omega_\gamma, \forall c \text{ with } Me^{-1} \leq c \leq M.$$  

(8)

**Theorem 1.12.** If the functional $G$ is degenerate elliptic, satisfies Property P, $f(x)$ is continuous, and $G(0,0) \leq f(x)$, then the Boundary Value Problem

$$\begin{aligned}
    u + G(Du, D^2u) - f(x) &= 0 \\
    u &= 0 \text{ on } \Omega
\end{aligned}$$

admits a continuous viscosity solution.

The following theorem provides another illustration of the versatility of viscosity solutions. We can consider the perturbation of a problem and take limits without fear.

**Theorem 1.13.** Let $F_n(x, r, p, X)$ be a continuous proper functional and let

$$G(x, r, p, x) = \liminf_{n \to \infty} F_n(x, r, p, X).$$

Let $u_n$ be a continuous viscosity subsolution of $F_n = 0$. Then

$$\bar{U}(x) = \limsup_{n \to \infty} u_n(x)$$

is a viscosity subsolution of $G = 0$. Likewise,

$$\underline{U}(x) = \liminf_{n \to \infty} u_n(x)$$

is a viscosity supersolution of $G = 0$.

**Theorem 1.14 (II).** Consider the problem:

$$- \text{tr}(\sigma(Du)\sigma(Du)^T D^2u) + F(u, Du) + \lambda u - f(x) = 0.$$  

(9)

Suppose that either $\Omega = \mathbb{R}^n$ or $\Omega$ is a bounded domain with the necessary Dirichlet condition $u = 0$ on $\partial \Omega$. Suppose also that $f$ is $\alpha$–Hölder continuous and bounded, that $F$ is proper, and that matrix-valued $\sigma$ and real-valued $F$ are uniformly continuous. Then, there exists a unique, bounded viscosity solution $u$ of Equation (9) such that

$$\|u\|_{C^0} \leq \frac{\|f\|_{C^0}}{\lambda}$$

and in fact $u \in C^{0,\mu}(\Omega)$ where $0 < \mu \leq \alpha$.  

8
2 A Toy Model

Before discussing our results for partial differential equations, we shall find it instructive to examine a “toy model”: fully nonlinear third-order ordinary differential equations. More precisely, we consider the following boundary value problem:

\[ F(x, u_{xxx}) = 0 \text{ on } [-\pi, \pi] \]  
\[ u(-\pi) = u(\pi), \quad u_x(-\pi) = u_x(\pi) \]  

We now make the (formal) differential transformation \( u_x = w \) to work with a second-order differential equation instead. More precisely, if \( w(x) \) is a classical \( C^2 \) solution of

\[ F(x, w_{xx}) = 0 \text{ on } [-\pi, \pi] \]  
\[ w(-\pi) = w(\pi) \]  

we define

\[ u(x) := u(-\pi) + \int_{-\pi}^{x} w(s)ds \]  

which shall be a classical \( C^3 \) solution of (10), (11) if, in addition to solving (12), (13), \( w \) also has zero mean:

\[ \int_{-\pi}^{\pi} w(s)ds = 0. \]  

The last condition (15) is satisfied, for example, if the function \( w(x) \) is odd.

We have the following definition:

**Definition 6.** The function \( u(x) \) is a \( C^1 \) **viscopotential** solution to

\[ F(x, u_{xxx}) = 0 \text{ on } (-\pi, \pi), \quad u(-\pi) = u(\pi), \quad u_x(-\pi) = u_x(\pi) \]

if and only if

\[ u(x) = u(-\pi) + \int_{-\pi}^{x} w(s)ds \]

where \( w(x) \) is a continuous viscosity solution to

\[ F(x, w_{xx}) = 0 \text{ on } (-\pi, \pi), \quad w(-\pi) = w(\pi). \]

In what follows, we determine sufficient conditions for the functional \( F \) so that the boundary value problem (10), (11) has a unique \( C^1 \) viscopotential solution. Namely:
Theorem 2.1. Let $F(x, s) : [−\pi, \pi] \times \mathbb{R} \to \mathbb{R}$ be a continuous functional satisfying the hypotheses of Theorem 1.9, where we have identified $\text{Sym}^{1 \times 1}$ with $\mathbb{R}$. Assume in addition that we have the following symmetry condition:

$$F(-x, -s) = -F(x, s), \text{ for all permissable } (x, s).$$

It follows that the boundary value problem

$$-F(x, u_{xxx}) = 0, \quad u(-\pi) = u(\pi), \quad u_x(-\pi) = u_x(\pi)$$

has a unique $C^1$ viscopotential solution.

Proof. By the assumptions on the functional $F(x, s)$, we know that the problem:

$$-F(x, w_{xx}) = 0, \quad w(-\pi) = w(\pi)$$

has a unique continuous viscosity solution. We now show this solution must also be an odd function on $[-\pi, \pi]$. Indeed, let $\hat{w}(x) = -w(-x)$. Our goal is to show that $\hat{w} = w$ by showing that $\hat{w}$ is also a continuous viscosity solution.

Suppose that $\hat{w} - \phi$ has a local maximum at $x$, which implies that $-w(-x) - \phi(x)$ has a local maximum at $x$. Then $w(-x) + \phi(x)$ has a local minimum at $x$. Let $\hat{\phi}(x) = -\phi(-x)$. Then we see that $w(-x) - \hat{\phi}(-x)$ has a local minimum at $x$, or equivalently, $w(x) - \hat{\phi}(x)$ has a local minimum at $-x$. It follows from the supersolution criterion analogous to (4) that $-F(-x, \hat{\phi}_{xx}) \geq 0$, which implies $-F(-x, -\phi_{xx}) = F(x, \phi_{xx}) \geq 0$, which in turn gets $-F(x, \phi_{xx}) \leq 0$. Since $\phi$ was arbitrary, $\hat{w}$ is a subsolution of the same equation. By the same argument, since $w$ is a subsolution, $\hat{w}$ is a supersolution. Lastly, by our assumptions on the functional $F$, continuous viscosity solutions of our equation are unique, which means that $\hat{w} = w$, i.e. $w$ is odd.

Since $w$ is odd, we have that

$$\int_{-\pi}^{\pi} w(s) ds = 0$$

and the function

$$u(x) = u(-\pi) + \int_{-\pi}^{x} w(s) ds$$

is our unique $C^1$ viscopotential solution to the boundary value problem. \qed

We finish the section with two theorems that relate viscopotential and classical solutions.
Theorem 2.2. Let $F(x, s)$ be as in the statement of Theorem 2.1. If the boundary value problem

$$F(x, u_{xxx}) = 0, \quad u(-\pi) = u(\pi), \quad u_x(-\pi) = u_x(\pi)$$

has a classical $C^3$ solution, then it is the unique $C^1$ viscopotential solution.

Proof. Let $u$ be a $C^3$ classical solution of the problem. Then, $w(x) = u_x(x)$ is a $C^2$ function solving

$$-F(x, w_{xx}) = 0$$

in the classical sense. Since $F(x, s)$ is continuous and monotonically increasing in $s$, it follows from Theorem 1.7 that $w$ is a viscosity solution. This in turn implies that $u$ is a viscopotential solution since, by construction, we may write $u$ as

$$u(x) = u(-\pi) + \int_{-\pi}^{x} w(s)ds.$$

Theorem 2.3. If a viscopotential solution of

$$F(x, u_{xxx}) = 0, \quad u(-\pi) = u(\pi), \quad u_x(-\pi) = u_x(\pi)$$

is in $C^3$, then it is a classical solution.

Proof. Let $u$ be a viscopotential solution. Then there is a continuous function $w$ that is a viscosity solution of

$$-F(x, w_{xx}) = 0$$

such that

$$u(x) = u(-\pi) + \int_{-\pi}^{x} w(s)ds.$$

Since $u$ is in $C^3$, it follows that $w$ is in $C^2$. Thus, by Theorem 1.6, $w$ is also a classical solution of the problem, which immediately implies the same for $u$.

3 Differential Equations in Potential Form

We begin with the following definition:
Definition 7. A partial differential equation satisfied by scalar-valued $u$ is said to be in $n^{th}$-order potential form if and only if it can be written as

$$-F(x, Lu, D(Lu), \ldots, D^n(Lu)) = 0$$

where $L$ is any differential operator.

In order to apply the usual viscosity theory, we henceforth restrict ourselves to partial differential equations in $2^{nd}$-order potential form, namely:

$$-F(x, Lu, D(Lu), D^2(Lu)) = 0.$$  

We may now properly define our notion of solution to such problems:

Definition 8. Let $\Omega$ be a locally compact domain in $\mathbb{R}^n$ and let $L : \mathcal{F}_1 \rightarrow \mathcal{F}_2$ be a differential operator. A function $u$ is said to be a viscopotential solution of

$$-F(x, Lu, D(Lu), D^2(Lu)) = 0 \text{ on } \Omega$$  

(16) if and only if there exists a function $w \in \mathcal{F}_2$ such that

$$-F(x, w, Dw, D^2w) = 0 \text{ on } \Omega$$  

(17) in the viscosity sense and such that $w = Lu$ (in a classical or weak sense) where $u \in \mathcal{F}_1$.

Remark 3.1. More generally, we say that $u$ is a viscopotential solution of

$$(L_k \circ \cdots \circ L_1 \circ L_0)u = 0.$$  

if we use viscosity theory to resolve one of the operators $L_i$.

It is clear that finding viscopotential solutions to (16) is tantamount to first finding a viscosity solution $w$ to (17) and then solving the problem $Lu = w$, if we can. As we saw in the toy model of the previous section, properties of the functional $F$ and operator $L$ are essential for determining the existence (and uniqueness) of viscopotential solutions. Moreover, in the case of a boundary value problem, we saw that uniqueness results for viscosity solutions might be necessary in order to show viscopotential existence results.
3.1 Higher Order Quasilinear Equations in Potential Form

In this section, we simply apply the existing viscosity theory twice in order to attack fourth-order problems. Let $L_1, L_2$ be two second-order degenerate elliptic operators:

$$L_1[u] = L_1(x, u, Du, D^2u)$$

$$L_2[u] = L_2(x, u, Du, D^2u)$$

We consider the following fourth-order problem, which is a composition of the two operators:

$$L_1 \circ L_2[u] = 0$$  \hspace{1cm} \text{(18)}$$

We have the following theorem, which shows the existence and uniqueness of a Hölder continuous viscopotential solution to the composition of two quasilinear operators.

**Theorem 3.2.** Assume that $L_1$ and $L_2$ satisfy the hypotheses of Theorem 1.14. Then Equation (18) has a unique Hölder continuous viscopotential solution on $\mathbb{R}^n$.

*Proof.* By our hypotheses, we know that there is a unique Hölder continuous viscosity solution $w$ to the problem

$$L_1w = 0$$

and then a unique Hölder continuous viscosity solution $u$ to

$$L_2u = w.$$  

The function $u$ is by definition the desired viscopotential solution to (18). \hfill \square

Similarly, we may obtain an existence and uniqueness result for a Hölder continuous viscopotential solution to a quasilinear equation of arbitrarily high order (in a potential form like (18)) by employing a chain of viscosity problems. Note that the resulting viscopotential solutions themselves may yield little information about possible classical solutions to the same problem. On the other hand, the existing theory of viscosity solutions suffices to provide us with some notion of solution to such a problem.

3.2 General Equations with Navier Boundary Conditions

Suppose we are given the following boundary value problem:

$$\begin{cases}
\epsilon \Delta v + G(D(\Delta v), D^2(\Delta v)) = f(x), \text{ on } \Omega \\
v = g \text{ on } \partial \Omega \\
\Delta v = 0 \text{ on } \partial \Omega 
\end{cases}$$  \hspace{1cm} \text{(19)}$$

13
In what follows, we give sufficient conditions on the functional $G$, the domain $\Omega$, and the functions $f, g$ so that Problem (19) has a unique viscopotential solution. Then we show that (in some cases) if the unperturbed problem ($\epsilon = 0$) also admits a unique viscopotential solution, we may approximate it uniformly with solutions of the perturbed problems. Henceforth, we fix a $C^2$ bounded domain $\Omega$ in $\mathbb{R}^n$.

**Theorem 3.3.** Let $g$ be continuous, and suppose that

$$\epsilon w + G(Dw, D^2w) - f(x)$$

satisfies the hypotheses of Theorem 1.14, then the Boundary Value Problem (19) admits a unique $C^{2,\alpha}$ viscopotential solution $v_\epsilon$.

**Proof.** By our hypotheses, the problem:

$$\begin{cases}
\epsilon w + G(Dw, D^2w) = f(x), & \text{on } \Omega \\
w = 0 & \text{on } \partial\Omega
\end{cases}$$

admits a unique viscosity solution $w_\epsilon \in C^{0,\alpha}(\Omega)$, for some $\alpha > 0$. We now solve the Dirichlet Problem using Theorem 1.5:

$$\Delta v = w_\epsilon \text{ in } \Omega, \quad v = g \text{ on } \partial\Omega$$

and get a unique $C^{2,\alpha}$ solution $v_\epsilon$ which is a viscopotential solution of (19) by definition. \qed

**Theorem 3.4.** Assume the hypotheses of Theorem 3.3. Suppose

$$\begin{cases}
G(D(\Delta v), D^2(\Delta v)) = f(x), & \text{on } \Omega \\
v = 0 & \text{on } \partial\Omega \\
\Delta v = 0 & \text{on } \partial\Omega
\end{cases}$$

admits a $C^2$ viscopotential solution $v_0$. Assume further that the boundary value problem admits a comparison principle as in Theorem 1.10. Then the $v_\epsilon$ found in Theorem 3.3 converge uniformly to $v_0$.

**Proof.** Let $\Psi$ be the fundamental solution of the Poisson equation in the domain $\Omega$ (i.e. Green’s Function). Let $w_\epsilon$ be the continuous viscosity solution we found in the course of proving Theorem 3.3. Then $v_\epsilon$ is precisely equal to $\Psi * w_\epsilon$ for $\epsilon > 0$. Likewise, $w_0 = \Delta v_0 \in C^0$ is (by definition) a viscosity solution of

$$G(Dw, D^2w) = f(x).$$
From Theorem 1.13 and the assumed comparison principle for (20), it is easy to see that \( \lim_{\epsilon \to 0} w_{\epsilon} \to w_0 \) uniformly as \( \epsilon \to 0 \), so, clearly from the convolution formula, we may see that \( v_{\epsilon} \to v_0 \) uniformly as well.

Our final theorem in this section loosens the requirements on the functional \( G \) further:

**Theorem 3.5.** Suppose \( g \) is continuous in \( \bar{\Omega} \) and that

\[
\epsilon w + G(Dw, D^2w) - f(x)
\]
satisfies the hypotheses of Theorem 1.12. Then the (weak) Boundary Value Problem:

\[
\begin{cases}
\epsilon \Delta v + G(D(\Delta v), D^2(\Delta v)) = f(x), & \text{on } \Omega \\
v - g \in W^{1,2}_0(\Omega) \\
\Delta v = 0 \text{ a.e. on } \partial \Omega
\end{cases}
\]

admits a unique \( W^{2,\infty} \) viscopotential solution \( v_{\epsilon} \).

**Proof.** By our hypotheses, the problem:

\[
\begin{cases}
\epsilon w + G(Dw, D^2w) = f(x), & \text{on } \Omega \\
w = 0 \text{ on } \partial \Omega
\end{cases}
\]

admits a unique continuous and bounded viscosity solution \( w_{\epsilon} \). We now solve the Dirichlet Problem using Theorem 1.4:

\[
\Delta v = w_{\epsilon} \text{ a.e. in } \Omega, \quad v - g \in W^{1,2}_0(\Omega)
\]

and get a unique \( W^{2,2} \) weak solution \( v_{\epsilon} \) which is a viscopotential solution of the boundary value problem by definition. Since the viscosity solution \( w_{\epsilon} \) is continuous and bounded, our function \( v_{\epsilon} \) is *a posteriori* found to be in \( W^{2,\infty} \).

\[\square\]

### 4 The \( \infty \)-Bilaplacian Equation

As a practical application of our previous work, we study the Dirichlet problem for the inhomogeneous \( \infty \)-Bilaplacian equation:

\[
\Delta^2_{\infty} u = (\Delta u)^3|D(\Delta u)|^2 = f(x)
\]

(21)
As discussed earlier, the $\infty-$Bilaplacian is the formal limit of the $p-$Bilaplacian equation as $p \to \infty$. The latter equation is a nonlinear generalization of the Bilaplacian (Biharmonic) equation $\Delta^2 u = 0$, which models elastic motion of a thin plate (used in geology, for example) [8]. The nonlinear $p-$Bilaplacian can model traveling waves in suspension bridges [7]. Also, the $\infty-$Bilaplacian is the Euler-Lagrange equation for minimizing $\|\Delta u\|_\infty$, which is the prototypical $L^\infty$ variational problem and has been studied in [5], [6].

Typically, a Dirichlet problem for $\Delta^2_\infty$ admits neither a classical $C^3$ solution nor a weak $C^2$ solution, even for domains in $\mathbb{R}$. However, in what follows we show that a $W^{2,\infty}$ solution to the Navier Problem may be found that is constructed from (and bounded by) a sequence of $C^{2,\alpha}$ viscopotential solutions to a perturbation of $\Delta^2_\infty$.

**Theorem 4.1.** Let $\epsilon > 0$ be arbitrary. Let $B_R$ be the ball of radius $R$ around the origin of $\mathbb{R}^n$. Let $f \geq 0$ be Hölder continuous and $g$ be a continuous function on $\overline{B_R}$. We assume in addition that

$$f(x) = O((R^2 - |x|^2)^3|x|^2).$$

Then the Boundary Value Problem

$$(22) \quad \begin{cases}
\epsilon \Delta v + (\Delta v)^3|D(\Delta v)|^2 = f(x) \\
v = g \text{ on } \partial B_R \\
\Delta v = 0 \text{ on } \partial B_R
\end{cases}$$

has a $C^{2,\alpha}$ viscopotential solution $v$, for some $\alpha > 0$.

**Proof.** The boundary value problem

$$(22) \quad \begin{cases}
\epsilon w + w^3|Dw|^2 = f(x) \\
w = 0 \text{ on } \partial B_R
\end{cases}$$

can be seen to admit a continuous viscosity solution by Perron’s Method. Indeed since $f \geq 0$, $w_0 = 0$ is an obvious subsolution to the problem. it remains to construct a supersolution $w_1$ such that $w_1 = 0$ on $\partial B_R$. By assumption, there exists a constant $C$ such that

$$0 \leq f(x) \leq C((R^2 - |x|^2)^3|x|^2)$$

for all $x \in \overline{B_R}$. Let $M$ be such that:

$$4M^5 > C.$$
The function \( w_1(x) = M(R^2 - |x|^2) \) is now our desired supersolution. Obviously, \( w_1 = 0 \) on the boundary of the sphere, \( \partial B_R \). It remains to show that it satisfies the supersolution condition. Let \( \varphi \in C^\infty_0 \) be arbitrary such that \( w_1 - \varphi \) has a local minimum at \( x \in \Omega \). Without loss of generality, the value of the local minimum is zero, and because \( w_1 \) is smooth as well, we necessarily have \( Dw_1(x) = D\varphi(x) \). Thus, it suffices to show that
\[
\epsilon w_1 + w_1^3 |Dw_1|^2 \geq f(x)
\]
since \( w_1 \geq 0 \) and
\[
\epsilon w_1 + w_1^3 |Dw_1|^2 \geq w_1^3 |Dw_1|^2.
\]
To that end, we calculate:
\[
|Dw_1(x)|^2 = |2Mx|^2 = 4M^2 |x|^2
\]
and
\[
w_1^3(x) = M^3 (R^2 - |x|^2)^3.
\]
Thus,
\[
w_1^3(x)|Dw_1(x)|^2 \geq 4M^5 |x|^2 (R^2 - |x|^2)^3 \geq C((R^2 - |x|^2)^3 |x|^2) \geq f(x).
\]
Now Perron’s method for viscosity solutions guarantees the existence of a continuous solution \( w(x) \). The same existence result may also have been concluded from Theorem 1.14, from which we also have Hölder regularity and uniqueness. Additionally, we have that in \( B_R \):
\[
0 \leq w(x) \leq M(R^2 - |x|^2).
\]
Then, since \( w_\epsilon \) is Hölder continuous (say of exponent \( \alpha \)), we can solve the Dirichlet problem:
\[
\begin{cases}
\Delta v = w_\epsilon \\
v = g \text{ on } \partial B_R
\end{cases}
\]
and get a (by definition) viscopotential solution \( v_\epsilon \in C^{2,\alpha} \) to our problem. \( \square \)

**Theorem 4.2.** Let \( f, g \) satisfy the hypotheses of the previous theorem, and let \( w_\epsilon \) be the continuous viscosity solution to
\[
\epsilon w + w^3 |Dw|^2 = f(x)
\]

17
found in the proof of the previous theorem. The Boundary Value Problem

\[
\begin{aligned}
&\begin{cases}
(\Delta v)^3|D(\Delta v)|^2 = f(x) \\
v - g \in W^{1,2}_0(B_R) \\
\Delta v = 0 \text{ a.e. on } \partial B_R
\end{cases}
\end{aligned}
\]

admits two viscopotential solution \( v_0, v_1 \in W^{2,\infty}(B_R) \) that are possibly equal and such that

\[0 \leq \Delta v_0(x) \leq \liminf_{\epsilon \to 0} w_\epsilon(x)\]

and

\[\limsup_{\epsilon \to 0} w_\epsilon(x) \leq \Delta v_1(x) \leq M(R^2 - |x|^2)\]

almost everywhere in \( B_R \). In particular, \( v_0, v_1 \in C^{1,1}(\overline{B_R}) \).

Proof. We show the existence of \( v_0 \), the proof is almost identical for \( v_1 \). It is clear from Theorem 1.13 that

\[w(x) := \liminf_{\epsilon \to 0} w_\epsilon(x)\]

is a continuous supersolution of the viscosity problem that is greater than or equal to the subsolution 0. By Perron’s method, we are guaranteed a continuous solution \( w_0 \) to the viscosity problem such that

\[0 \leq w_0 \leq w.\]

Since our domain is a ball \( B_R \), we can apply Theorem 1.4 to uniquely solve the weak Dirichlet problem \( \Delta v = w_0 \) and get a weak solution \( v_0 \in W^{2,2}(B_R) \) such that

\[\Delta v_0 = w_0 \text{ a.e. in } B_R, \quad v_0 - g \in W^{1,2}_0(B_R).\]

Finally, because \( w_0 \) is continuous, \( w_0 = 0 \) on \( \partial B_R \), and \( \Delta v_0 = w_0 \) a.e., we can modify \( v_0 \) on a null set so that \( \Delta v_0 = 0 \) a.e. on \( \partial B_R \), giving us our desired viscopotential solution in \( W^{2,2} \) to the Boundary Value Problem. The function is \textit{a posteriori} in \( W^{2,\infty} \) because the trace of \( D^2 v_0 \) is a.e. bounded by construction.

Finally, we deal with the homogeneous Navier Boundary Value Problem:

**Theorem 4.3.** Let \( \Omega \) be a \( C^2 \) bounded domain, and let \( g \) be a continuous function on \( \overline{\Omega} \). Then the Navier Boundary Value Problem:

\[
\begin{aligned}
&\begin{cases}
(\Delta v)^3|D(\Delta v)|^2 = 0 \\
v = g \text{ on } \partial \Omega \\
\Delta v = 0 \text{ on } \partial \Omega
\end{cases}
\end{aligned}
\]
admits a unique classical solution \( v \) that is precisely the solution to the Dirichlet problem \( \Delta v = 0 \) in \( \Omega \), \( v = g \) on \( \partial \Omega \).

**Proof.** The problem

\[
\begin{align*}
    w^3|Dw|^2 &= 0 \\
    w &= 0 \text{ on } \partial \Omega
\end{align*}
\]

is easily seen to admit \( w = 0 \) as its unique solution. Finding a solution to the original problem is now tantamount to solving the standard Dirichlet problem \( \Delta v = 0 \) in \( \Omega \), \( v = g \) on \( \partial \Omega \). The solution to this is the harmonic function \( v \) with boundary values \( g \), and \( v \) is in particular an absolute minimizer of \( \| \Delta u \|_\infty \), which can be forthrightly seen.

Theorems 4.1 and 4.2 can probably be extended to sufficiently smooth bounded and star-shaped domains while maintaining the same supersolution construction; however, the inhomogeneity \( f \) should have a similar behavior at the boundary of the domain. Otherwise, we can use Theorem 1.14 for general domains, but without having such explicit control on \( w = \Delta v \) at the boundary.

We believe that the most interesting possibility in the conclusion of Theorem 4.2 is for

\[
\limsup_{\epsilon \to 0} w_\epsilon = \liminf_{\epsilon \to 0} w_\epsilon > 0 \text{ and } v_0 = v_1.
\]

If this were the case, we’d have a contender \( v \) for a non-trivial solution to the \( L^\infty \) variational problem of minimizing \( \| \Delta u \|_\infty \) whose Laplacian \( \Delta v \) is a uniform limit of Hölder continuous functions. Likewise, it would be interesting to show that the viscopotential solution of the unperturbed problem is also in \( C^{2,\alpha} \) and unique. We have no evidence for or against this possibility, so an explicitly stated conjecture is omitted here. However, we remark that it appears improving the regularity of viscosity solutions of

\[
w^3|Dw|^2 = f(x)
\]

would be an important first step towards such a result.

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