Solving second-order conic systems with variable precision

Felipe Cucker · Javier Peña · Vera Roshchina

Abstract We describe and analyze an interior-point method to decide feasibility problems of second-order conic systems. A main feature of our algorithm is that arithmetic operations are performed with finite precision. Bounds for both the number of arithmetic operations and the finest precision required are exhibited.

Keywords Second-order cones · Interior-point methods · Variable precision

Mathematics Subject Classification 90C25 · 90C51 · 65G30 · 65G50

1 Introduction

It is now widely accepted that the most efficient algorithms for solving the general type of second-order conic problems are interior-point methods (IPMs). IPMs infallibly demonstrate very fast numerical convergence, by far outperforming their theoretical estimates.

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Second-order conic programming problems contain linear programming problems as a special case, and at the same time can be embedded into the class of semidefinite programming problems. It is, however, not advisable to solve SOCP problems by semidefinite programming methods (see \cite{1,18}) as IPMs that solve SOCP directly have a much better complexity (both in theory and in practice). SOCP problems have lately received considerable attention due to their many applications \cite{18}; they appear to be at the boundary of the problems for which interior-point methods can solve large instances, a fact that is linked to the implementation of commercial software for the solution of second-order programs such as MOSEK\(^1\) or CPLEX\(^2\).

We are interested in solving homogeneous second-order conic feasibility problems. That is, given a second-order cone \(K \subset \mathbb{R}^n\) and a matrix \(A \in \mathbb{R}^{m \times n}\), decide which one of the primal-dual pair of problems

\[
\begin{align*}
Ax &= 0, \quad (P) \\
x \succeq_K 0,
\end{align*}
\]

\[
\begin{align*}
A^T y + s &= 0, \quad (D) \\
s \succeq_K 0,
\end{align*}
\]

is strictly feasible (i.e., the relevant conic constraint is strict) and provide a solution to the feasible problem. It is well-known that each of (P) and (D) above has a strict solution if and only if the other one has no nonzero solutions.

Recall that a second-order cone is a direct product of a finite number of Lorentz cones. The Lorentz cone \(\mathcal{L}_p \subset \mathbb{R}^{p+1}\) is defined to be

\[
\mathcal{L}_p = \{ x \in \mathbb{R}^{p+1} \mid x_0 \geq \|x\| \},
\]

where for a vector \(x \in \mathbb{R}^{p+1}\) indexed from 0 to \(p\) we let \(\bar{x} = (x_1, x_2, \ldots, x_p) \in \mathbb{R}^p\). For our primal-dual pair of problems (P)-(D) we take

\[
K = \mathcal{L}_{n_1} \times \mathcal{L}_{n_2} \times \cdots \times \mathcal{L}_{n_r},
\]

where \(r\) is the number of Lorentz cones comprising \(K\), and \(\sum_{i=1}^{r} (n_i + 1) = n\) with \(n_i\) being positive integers for all \(i\) from 1 to \(r\).

We propose a finite-precision algorithm for solving the SOCP feasibility problem and provide rigorous bounds for the finest machine precision and the maximal number of iterations needed. The proposed algorithm is designed to work with variable precision, that is, the machine precision can be re-adjusted along the way.

Our bounds depend on Renegar’s condition number \([14,16]\), which is consistent with similar bounds obtained for the polyhedral case in \([6]\). Let \(\rho_P(A)\) and \(\rho_D(A)\) be the distance to infeasibility of (P) and (D) respectively defined by

\[
\rho_P(A) = \inf \{ \| \Delta A \| : (A + \Delta A)x = 0, x \succ_K 0 \text{ is infeasible} \}
\]

and

\[
\rho_D(A) = \inf \{ \| \Delta A \| : -(A + \Delta A)^T y \succ_K 0, y \in \mathbb{R}^m \text{ is infeasible} \}.
\]

\(^1\) http://www.mosek.com/.

\(^2\) http://www.ilog.com/products/cplex/.
Renegar’s condition number $C(A)$ is defined as the reciprocal of the relative distance to ill-posedness of the pair (P)–(D):

$$C(A) := \frac{\|A\|}{\max\{\rho_P(A), \rho_D(A)\}}.$$ 

Although any equivalent matrix norm can be used to define $C(A)$, in our analysis we choose to use the standard operator norm induced by the Euclidean scalar product. We say that the problem is ill-posed if both $\rho_P(A) = \rho_D(A) = 0$ and hence $C(A) = \infty$.

Our main result, Theorem 1, shows that there exists a finite precision interior-point method which, with input a matrix $A \in \mathbb{R}^{m \times n}$ and a second-order conic structure $K$ (consisting of $r$ Lorentz cones), decides which one of the two systems (P) or (D) is feasible. We estimate both the number of iterations of the algorithm and the precision required as functions of the size of the matrix, the number $r$ of Lorentz cones in $K$, and the condition number of the problem. The finest required precision is

$$u = \frac{1}{\mathcal{O}((m + n)^{5/2} r^{11.5} C(A)^{7/2})},$$

and the number of main interior-point iterations performed by the algorithm is bounded by

$$\mathcal{O}(r^{1/2} (\log r + \log C(A))).$$

Strictly speaking, our algorithm solves both the decision problem—decide which one of the problems (P) and (D) is feasible—and the function problem—if either one of the problems (P) or (D) is strictly feasible produce a (possibly approximate) solution for it. For a precise version of our main result, the reader should check the statement of Theorem 1.

Throughout the paper, we use standard notation wherever possible. We index our variables according to the second-order conic structure. That is, $x = (x_1, x_2, \ldots, x_r)$, $s = (s_1, s_2, \ldots, s_r)$, where $x_i, s_i \in \mathbb{R}^{n_i + 1}$ for all $i = 1, \ldots, r$. Throughout the paper we assume that $\|A_i\|_F = 1/\sqrt{r}$, and hence $\|A\|_F = 1$. Note that this assumption is trivial from a computational viewpoint; if $A_i \neq 0_{m \times (n_i + 1)}$, it takes a few operations to reduce the matrix to this form and it is easy to recover solutions of the original system from those for the reduced one. The condition number of the new matrix may change, however. But one can show as in [6, §11.2] that this change can not be large.

Finite precision analyses are pervasive in Numerical Linear Algebra; they are much less common in optimization. While the effects of finite precision when solving linear programming problems had been early noticed (e.g. [2,5,11,17,19,23]) there was no condition-based round-off analysis even for linear programming problems until recently. This was done for the feasibility problem for polyhedral conic systems [6], for the optimal value of linear programs [21], and for the computation of optimal basis and optimal solutions of linear programs [4]. To the best of our knowledge, our work is the first such analysis for nonlinear cones.
Our paper is organized as follows. In Sect. 2 we use a relaxation scheme introduced by Peña and Renegar [13] and Vera et al. [22] to reformulate the feasibility problem via an optimization one and recall the basic ideas of interior-point methods. Then we relax the standard results of IPM analysis to make room for computational errors. We do not deal with finite-precision issues directly until Sect. 3, where we describe our algorithm in detail and estimate errors arising on every step of floating-point computations. The last section is devoted to the proof of the main result, and essentially fits the error estimates obtained in Sect. 3 into the gaps made for this purpose in our extension of the IPM analysis done in Sect. 2.

2 Interior-point method for SOCP feasibility problem

We use a relaxation scheme introduced by Peña and Renegar in [13] and later extended in [22]. This relaxation scheme reformulates the feasibility problem (P)–(D), for the more general case when $K$ is a symmetric cone, as a pair of primal-dual optimization problems in higher dimension and solves this pair by a standard short-step interior-point method. We next summarize the main ingredients of this approach.

It was shown in [22] that the pair (P)–(D) is equivalent to the following primal-dual pair of optimization problems

\[
\begin{align*}
\min & \quad \bar{c}^T \bar{x} \\
\text{s.t.} & \quad \mathcal{A}\bar{x} = \bar{b} \\
& \quad \bar{x} \succeq_{\mathcal{K}} 0 \\
\end{align*}
\]

and

\[
\begin{align*}
\max & \quad \bar{b}^T \bar{y} \\
\text{s.t.} & \quad \mathcal{A}^T \bar{y} + \bar{s} = \bar{c} \\
& \quad \bar{s} \succeq_{\mathcal{K}} 0,
\end{align*}
\]

where $\mathcal{K} = K \times \mathcal{L}_n \times \mathcal{L}_m$, $\bar{x} = (x, t, x', \tau, x'')$, $\bar{s} = (s, t_s, s', \tau_s, s'') \in \mathbb{R}^{n+1+n+1+m}$, $\bar{y} = (y, y', -\eta) \in \mathbb{R}^{m+n+1}$,

\[
\mathcal{A} := \begin{bmatrix}
A & 0 & 0 & 0 & I_m \\
-I_n & 0 & I_n & 0 & 0 \\
0 & 1 & 0 & 0 & 0
\end{bmatrix}, \quad \bar{b} = \begin{bmatrix} 0 \end{bmatrix}, \quad \bar{c} = \begin{bmatrix} 0 \\
0 \\
0 \\
1 \\
0
\end{bmatrix}.
\]

This equivalence should be understood in the following sense: If $\rho(A) > 0$ then a primal-dual interior-point method applied to the pair $(P'')$–$(D'')$ yields a strict solution to whichever of (P) or (D) is strictly feasible. In particular, since the optimal value of the pair $(P'')$–$(D'')$ is zero, a corresponding strict solution to the original feasibility problem can be straightforwardly recovered from the first entries of the extended variables ($x$ in the case of (P) and $y$ and $s$ in the case of (D)).
Note that from \( \|A\|_F = 1 \) we get
\[
\|A\| = \max_{\|\bar{x}\| \leq 1} \sqrt{\|Ax - x''\|^2 + \|x' - x\|^2 + r^2} \leq \max_{\|\bar{x}\| \leq 1} \sqrt{2\|Ax\|^2 + 2\|\bar{x}\|^2} \leq 2.
\]

(2.1)

In the sequel, to simplify notation, we will denote \( m := m + n + 1 \) and \( n := 2n + m + 2 \) so that \( A \in \mathbb{R}^{m \times n}, \bar{b} \in \mathbb{R}^m \) and \( \bar{c} \in \mathbb{R}^n \). We also let \( r := r + 2 \).

The pair \((P')-(D')\) can be solved via a primal-dual interior-point algorithm. We refer the reader to [14] and the references therein for a detailed exposition of the theory of IPMs. We next recall the concepts and results from this basic theory that will be used in the paper. Consider the following self-scaled barrier function for the cone \( K \):
\[
f(\bar{x}) = -\left( \sum_{i=1}^{r} \ln(x_{0i}^2 - \|x_i\|^2) + \ln(t^2 - \|x'\|^2) + \ln(\tau^2 - \|x''\|^2) \right).
\]

(2.2)

Let \( g(\bar{x}) = \nabla f(\bar{x}) \) and \( H(\bar{x}) = \nabla^2 f(\bar{x}) \) denote respectively the gradient and the Hessian of \( f \). Let \( e \in K \) denote the unique point such that \( H(e) = I \), that is, \( e = (e_1, \ldots, e_r) \) where \( e_{i0} = 1 \) and \( \bar{e}_i = 0 \) for \( i = 1, \ldots, r \).

Sometimes we will also need to work with the self-scaled barrier function for the cone \( K \):
\[
\bar{f}(x) = -\sum_{i=1}^{r} \ln(x_{0i}^2 - \|x_i\|^2),
\]

and will let \( \bar{g}(x) = \nabla \bar{f}(x) \) and \( \bar{H}(x) = \nabla^2 \bar{f}(x) \). Notice that \( H(\bar{x}) = \begin{bmatrix} H(x) & 0 \\ 0 & \bar{H}(\bar{x}) \end{bmatrix} \), where \( \bar{H}(\bar{x}) \) denotes the Hessian of the function \( -\ln(t^2 - \|x'\|^2) - \ln(\tau^2 - \|x''\|^2) \).

Given \( \bar{x} \in K \), the local norm \( \| \cdot \|_{\bar{x}} \) in \( \mathbb{R}^n \) is defined as
\[
\| \bar{u} \|_{\bar{x}}^2 := \bar{u}^T H(\bar{x}) \bar{u}.
\]

Likewise for \( x \in K \).

The central path of \((P')-(D')\) is the set of solutions \( \{(\bar{x}(\mu), \bar{y}(\mu), \bar{s}(\mu)) \in \text{int}(K) \times \mathbb{R}^m \times \text{int}(K) : \mu > 0 \} \) to the system of equations
\[
\begin{align*}
\mathcal{A} \bar{x} & = \bar{b} \\
\mathcal{A}^T \bar{y} + \bar{s} & = \bar{c} \\
\bar{s} + \mu \bar{g}(\bar{x}) & = 0.
\end{align*}
\]

(2.3)

Given \( z = (\bar{x}, \bar{y}, \bar{s}) \in \text{int}(K) \times \mathbb{R}^m \times \text{int}(K) \) define
\[
\mu(z) := \frac{\bar{x}^T \bar{s}}{2r}.
\]
Note that if \( z \) belongs to the central path for a certain value of \( \mu \) then \( \mu(z) = \mu \). We may sometimes write \( \mu \) for \( \mu(z) \) when \( z \) is clear from the context.

The basic idea of a path-following interior-point method is to generate a sequence of points on a suitable neighborhood of the central path that converges to optimality. The suitable neighborhood is the following.

**Definition 1** Given \( \beta \in (0, 1/15) \), the central neighborhood \( N_\beta \) is defined as the set of points \( z = (\vec{x}, \vec{y}, \vec{s}) \in \text{int}(K) \times \mathbb{R}^m \times \text{int}(K) \), such that the following constraints hold:

\[
\begin{align*}
A\vec{x} &= \vec{b} \\
A^T\vec{y} + \vec{s} &= \vec{c} \\
\|\vec{s} + \mu(z)g(\vec{x})\| - \mu(z)g(\vec{x}) &\leq \beta.
\end{align*}
\]

The main computational step of each interior-point iteration is to solve a linearization of the central path Eq. (2.3) at the current iterate \( z \in N_\beta \). The linearization that we will rely on is as follows. We will see (cf. Proposition 1(f) below) that for all \( \vec{x}, \vec{s} \in \text{int}(K) \) there exists a unique scaling point \( w \in K \) such that

\[
H(w)\vec{x} = \vec{s}.
\]

Given \( z = (\vec{x}, \vec{y}, \vec{s}) \in N_\beta \), the Nesterov-Todd direction \( (\Delta\vec{x}, \Delta\vec{y}, \Delta\vec{s}) \) is the solution to the following linearization of (2.3):

\[
\begin{align*}
A\Delta\vec{x} &= 0 \\
A^T\Delta\vec{y} + \Delta\vec{s} &= 0 \\
\Delta\vec{x} + H(w)^{-1}\Delta\vec{s} &= -(\mu g(\vec{x}) + \vec{x}).
\end{align*}
\]

By [22, Proposition 4.6], the initial point in step (i) of Algorithm IP below is in the central neighborhood \( N_\beta \). Furthermore, by [22, Propositions 4.4 and 4.5] if the original pair \((P)–(D)\) is well-posed (i.e. \( C(A) < \infty \)), then a point \( z \in N_\beta \) with \( \mu(z) \) small enough yields a strict solution to either \((P)\) or \((D)\), whichever is feasible. Indeed, by [22, Theorem 3.1] Algorithm IP halts in at most \( O(\sqrt{r}(\log r + \log C(A))) \) iterations and yields a solution to either \((P)\) or \((D)\). (See [22] for details.)

**Algorithm IP(A)**

Let \( \beta, \delta \) be the following constants

\[
\beta = \frac{1}{15}, \quad \delta = \frac{1}{45}.
\]

(i) Let

\[
\alpha := \frac{1}{\sqrt{2r}}; \quad M := \frac{\alpha\|Ae\|}{\beta};
\]

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and
\[
\begin{align*}
\tilde{x} &= (\alpha e, 1, \alpha e, 2M, -\alpha Ae) \\
\tilde{y} &= (0, \frac{M}{\alpha} e, \frac{M}{\alpha^2}) \\
\tilde{s} &= (\frac{M}{\alpha} e, \frac{M}{\alpha^2}, -\frac{M}{\alpha} e, 1, 0).
\end{align*}
\]

(ii) If \( A^T y \prec_K 0 \), then HALT and return \( y \) as a strictly feasible solution for (D).

(iv) If \( \sigma_{\min}(A^T (x) - 1)^2 A T) > r \mu(z) \), then HALT and return \( x + \frac{\mu(z)}{2} g(z) + \tilde{x} \) as a strictly feasible solution for (P).

(v) Set \( \mu := (1 - \delta \sqrt{2}) \mu(z) \).

(vi) Compute \( \Delta z := (\Delta \tilde{x}, \Delta \tilde{y}, \Delta \tilde{s}) \) by solving (2.4) for \( \mu = \mu(z) \) and update \( z \) by setting
\[
z^+ := z + \Delta z
\]

(vii) Go to (ii).

It should be noted that the analysis in [22] assumes that all computations are performed with infinite precision. Our initial step for a finite-precision algorithm is to show that the results in [22] can be extended to make room for computational errors. In particular, Lemma 1 below shows that even if the system (2.4) is solved inexactly, we can still ensure that the iterates remain in the central neighborhood.

Lemma 1 Let \( z \in N_\beta \), \( \mu = (1 - \frac{\delta}{\sqrt{2}}) \mu(z) \) with \( |\delta' - \delta| \leq \frac{\delta}{24} \) and \( z^+ = z + \Delta z \) be such that
\[
\begin{align*}
A^T \Delta \tilde{x} &= 0 \\
A^T \Delta \tilde{y} + \Delta \tilde{s} &= 0 \\
\Delta \tilde{x} + H(w)^{-1} \Delta \tilde{s} &= -(\mu g(z) + \tilde{x}) + \varrho,
\end{align*}
\]
where \( \|\varrho\| \leq \frac{\mu(z)}{120r(2\mu(z) + 1)} \). Then \( z^+ \in N_\beta \) and \( |\mu(z^+) - \mu(z)| \leq \frac{\mu(z)}{120r^2} \).

Proof See Sect. 2.2. \( \square \)

We note that when the solution \( \Delta z \) to (2.4) is computed exactly, i.e., when \( \varrho = 0 \) in (2.5), the point \( z^+ := z + \Delta z \) satisfies \( \mu(z^+) = \mu(z) \). For details, see [14].

The following two lemmas are in the same spirit as [22, Propositions 4.4 and 4.5]. In particular, they guarantee that if either (P) or (D) is strictly feasible then a point \( z \in N_\beta \) with \( \mu(z) \) small enough yields a strict solution to either (P) or (D). Lemma 2 provides the relevant bound for \( \mu(z) \) in the case (P) is strictly feasible and Lemma 3 does so for a strictly feasible (D).

Lemma 2 Let \( z = (\tilde{x}, \tilde{y}, \tilde{s}) \in N_\beta \) and assume \( \rho_P(A) > 0 \). Then
\[
\sigma_{\min}(A^T (x) - 1)^2 A T) \geq \left( \frac{(1 - \beta) \rho_P(A)}{\beta + 2r} \right)^2 - (2r \mu(z))^2.
\]

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The latter in turn implies that if \( \mu(z) < \frac{(1-\beta)\rho_D(A)}{2r(r+\beta)} \) then the point

\[
x + \frac{1}{H(x)^{-1}} A^T (A H(x)^{-1} A^T)^{-1} x''
\]

is a strict solution to (P).

Proof This is an immediate consequence of the proof of Proposition 4.4 in [22, pages 259–260]. \( \square \)

**Lemma 3** Let \( z = (\bar{x}, \bar{y}, \bar{s}) \in N_\beta \) and assume \( \rho_D(A) > 0 \). Then

\[
\| H(s) \| \leq \frac{4r^2}{(1-\beta)^2 \rho_D(A)^2}.
\]

In particular, for \( i = 1, \ldots, r \)

\[
s_{i0} - \| \bar{s}_i \| \geq \frac{1-\beta}{2r \sqrt{r}} \rho_D(A). \tag{2.7}
\]

Furthermore, if \( \mu(z) < 4r^2 (1-\beta) \rho_D(A) \) then \( y \) is a strict solution to (D).

Proof This is an immediate consequence of the proof of Proposition 4.5 in [22, page 260]. \( \square \)

The rest of this section is devoted to proving Lemma 1 and a technical lemma related to the conditioning of the matrix arising at each interior-point iteration of Algorithm IP. In Sect. 2.1 we state and prove a few technical statements which will be employed in the subsequent proofs. Then in Sect. 2.2 we prove Lemma 1. The proof of Lemma 1 is a straightforward adaptation of the proof of Theorem 3.7.3 in [14]. Section 2.3 presents Lemma 11, which is similar in spirit to Lemma 2. This technical result will be crucial in our finite precision analysis in Sect. 3.

### 2.1 A few useful relations

The analysis of IPMs heavily relies on the properties of the barrier function. Here we briefly recall a few essentials that will be used later. More details can be found in [14]. The barrier function \( f \) gives rise, for each point \( x \) in the domain \( D_f \) of \( f \), to a local inner product \( \langle \cdot, \cdot \rangle_x \) induced by \( x \) and defined by

\[
\langle u, v \rangle_x = \langle u, H(x)v \rangle.
\]

The local norm \( \| \cdot \|_x \) is then given by \( \| v \|_x = \langle v, v \rangle_x^{1/2} \). In the local inner product \( \langle \cdot, \cdot \rangle_x \), the gradient at \( y \) is \( g_x(y) := H(x)^{-1} g(y) \) and the Hessian is \( H_x(y) := H(x)^{-1} H(y) \).

Our function \( f \) defined by (2.2) is a self-scaled barrier with the barrier parameter \( \nu = 2r \). We will also use single components of \( f \): \( f_i = -\ln(x_{0i}^2 - \| x_i \|^2) \) for all \( i = 1, \ldots, r \). For each \( f_i \) the barrier parameter is \( \nu = 2 \). Our development relies on the following key properties of self-scaled barrier functions [14].
Proposition 1 Let \( f \) be a \( v \)-self-scaled barrier function and \( x \in D_f \subseteq \mathbb{R}^n \).

(a) If \( \|y - x\|_x < 1 \) then \( y \in D_f \) and, for all \( v \neq 0 \),
\[
1 - \|y - x\|_x \leq \frac{\|v\|_y}{\|v\|_x} \leq \frac{1}{1 - \|y - x\|_x}.
\]

(b) \( \{z \in D_f : \langle z - x, g(x) \rangle \geq 0 \} \subseteq \{z : \|z - x\|_x \leq v \} \).

(c) \( -g(x) \in D_f, -g(-g(x)) = x, H(-g(x)) = H(x)^{-1}, \) and \( \|H(x)^{-1}\| \leq \|x\|^2 \).

(d) For \( t > 0 \)
\[
g(tx) = \frac{1}{t} g(x), \quad \text{and} \quad H(tx) = \frac{1}{t^2} H(x).
\]

(e) \( H(x)x = -g(x) \) and \( x, g(x) = -v \).

(f) Given another point \( s \in D_f \), there exist a unique “scaling point” \( w \in D_f \) such that
\[
H(w)x = s, \quad \text{and} \quad H(w)g(s) = g(x)
\]
and a unique “reverse scaling point” \( w^* \in D_f \) such that
\[
H(w^*)s = x, \quad \text{and} \quad H(w^*)g(x) = g(s).
\]

Furthermore, \( w^* := -g(w) \) and, for all \( \mu > 0 \), the points \( \overline{w} := \sqrt{\mu} w \) and \( \overline{w}^* := \sqrt{\mu} w^* \) satisfy
\[
\|\overline{w}^* - s\|_{\overline{w}^*} = \|x - \overline{w}\|_{\overline{w}},
\]
and
\[
\|s + \mu g(x)\|_{-\mu g(x)} \geq \min \left\{ \frac{1}{5}, \frac{4}{5} \|x - \overline{w}\|_{\overline{w}} \right\}.
\]

(g) If \( \|y - x\|_x \leq 1 \) then \( \|g_x(y) - g_x(x) - H_t(x)(y - x)\|_s \leq \frac{\|y - x\|_y^2}{1 - \|y - x\|_x} \).

(h) If \( \|x - y\|_y \leq 1 \) then \( \|v\|_{-g_y(x)} \leq (1 + \|x - y\|_y)\|v\|_y \) for all \( v \in \mathbb{R}^n \).

\[\square\]

Lemma 4 Let \( z \in \mathcal{N}_\beta \) and denote \( \mu = \mu(z) \). Then \( \|x\| = \|x'\| \leq 1, \|x''\| \leq \tau \leq 2\mu, \|s''\| \leq t_s \leq 2\mu, \) and \( \|s\| \leq 2\mu + 1 \).

\[\square\]

Proof The bounds on \( x, x' \) and \( s'' \) follow from the equalities \( A\bar{x} = \bar{b} \) and \( A^T\bar{y} + \bar{s} = \bar{c} \) together with \( \bar{x}, \bar{s} \in K \). The inequalities \( \|x''\| \leq \tau \leq 2\mu \) and \( \|s''\| \leq t_s = \eta \leq 2\mu \) follow from the equalities \( \tau + \eta = \bar{c}^T\bar{x} - \bar{b}^T\bar{y} = \bar{x}^T\bar{s} = 2\mu \). Since \( z \in \mathcal{N}_\beta \), we have \( s = As'' - s' \) and therefore \( \|s\| \leq \|s''\| + \|A\| \|s''\| \leq 2\mu + 1 \).

The next lemma bounds the norm of the scaling matrix using the results above.
Lemma 5 Assume \( z \in \mathcal{N}_\beta \), with \( \beta \leq 1/15 \). Then

\[
\| H(w) \|, \| H(w)^{-1} \| \leq \frac{4(2r \mu(z) + 1)^2}{\mu(z)}.
\]

Proof Let \( \mu = \mu(z) \), \( w = \sqrt{\mu}w \), and \( w^* = -\sqrt{\mu}g(w) \). By Proposition 1(f) we have \( \| \mathbf{x} - w \|_w \leq \frac{5}{4}\beta \), therefore, by Proposition 1(a) and (c), respectively,

\[
\| H(w)^{-1} \| \leq \frac{1}{(1 - \frac{5}{4}\beta)^2} \| H(\mathbf{x})^{-1} \| \leq \frac{1}{(1 - \frac{5}{4}\beta)^2} \| \mathbf{x} \|^2.
\]

From Lemma 4 we have \( \| \mathbf{x} \|^2 \leq 3 + 8\mu^2r^2 \). Hence,

\[
\| H(w)^{-1} \| = \frac{1}{\mu} \| H(w)^{-1} \| \leq \frac{3 + 8\mu^2r^2}{(1 - \frac{5}{4}\beta)^2\mu} \leq \frac{4 + 16\mu^2r^2}{\mu} \leq \frac{4(2r\mu + 1)^2}{\mu}.
\]

Similarly, applying Proposition 1(f) and Lemma 4 to \( H(w^*^{-1}) \) we have

\[
\| H(w) \| = \frac{1}{\mu} \| H(w^*)^{-1} \| \leq \frac{\| \mathbf{x} \|^2}{(1 - \frac{5}{4}\beta)\mu} \leq \frac{4(2r\mu + 1)^2}{\mu}.
\]

\( \square \)

Lemma 6 Assume \( z \in \mathcal{N}_\beta \) with \( \beta \leq 1/15 \), and \( \frac{|\mu - \mu(z)|}{\mu(z)} \leq \frac{1}{\frac{1}{4}\sqrt{2}} \). Then

\[
\| H(w)^{-1/2}(\mu g(\mathbf{x}) + \mathbf{s}) \| \leq \frac{\mu(z)^{1/2}}{2}.
\]

Proof Let \( \mu = \mu(z) \), \( w = \sqrt{\mu}w \), and \( w^* = -\sqrt{\mu}g(w) \). Since \( z \in \mathcal{N}_\beta \), by Proposition 1(f) we have

\[
\beta \geq \| \mu g(\mathbf{x}) + \mathbf{s} \|_{g(\mathbf{x})} = \frac{4}{5} \| \mathbf{x} - w \|_w = \frac{4}{5} \| w^* - \mathbf{s} \|_{w^*}.
\]

Then, applying Proposition 1 and (2.8) twice and using \( \beta \leq 1/15 \), we obtain

\[
\| \mu g(\mathbf{x}) + \mathbf{s} \|_{w^*} \leq \| \mu g(\mathbf{x}) + \mathbf{s} \| \leq \frac{5}{3} \| \mu g(\mathbf{x}) + \mathbf{s} \|_{g(\mathbf{x})} \leq \frac{5}{3} \| \mu g(\mathbf{x}) + \mathbf{s} \|_{\mu g(\mathbf{x})}.
\]

By the triangle inequality and (2.8)

\[
\| \mu g(\mathbf{x}) + \mathbf{s} \|_{\mu g(\mathbf{x})} \leq \frac{|\mu - \mu|}{\mu} \| g(\mathbf{x}) \|_{g(\mathbf{x})} + \| \mu g(\mathbf{x}) + \mathbf{s} \|_{\mu g(\mathbf{x})} \leq \frac{4}{15}.
\]
From (2.9) and (2.10) we have

\[ \| H(w)^{-1/2}(\mu g(\bar{x}) + \bar{s})\| = \mu^{1/2}\|\mu g(\bar{x}) + \bar{s}\|_{w^*} \leq \frac{4}{9}\mu^{1/2} \leq \frac{\mu^{1/2}}{2}. \]

Lemma 7 Let \( z \in \mathcal{N}_\beta \) for some \( \beta \leq \frac{1}{15} \). Then for \( i = 1, \ldots, r \)

\[ x_i^T s_i \geq 2(1 - \beta)\mu(z) \tag{2.11} \]

and

\[ (x_{i0}^2 - \|\bar{x}_i\|^2)(s_{i0}^2 - \|\bar{s}_i\|^2) \geq 4(1 - \beta)^2\mu(z)^2. \tag{2.12} \]

Proof Let \( \mu = \mu(z) \). From \( \|s_i + \mu g(x_i)\|^2_{-\mu g(x_i)} \leq \beta^2 \) and Proposition 1 we have

\[ \beta^2 \geq \|s_i\|^2_{-\mu g(x_i)} - \frac{2}{\mu} \langle x_i, s_i \rangle + 2. \tag{2.13} \]

From Proposition 1(a, e)

\[ \|s_i\|_{-\mu g(x_i)} \geq \|s_i\|_{s_i} \left(1 - \|s_i + \mu g(x_i)\|^2_{-\mu g(x_i)}\right) \geq \sqrt{2}(1 - \beta). \tag{2.14} \]

Now (2.11) follows from (2.13) and (2.14). By Proposition 1

\[ 0 \leq \|s_i + \mu g(x_i)\|^2_{s_i} = \|s_i\|^2_{s_i} - 2\mu \langle g(x), g(s) \rangle + \| - \mu g(x_i)\|^2_{s_i} \]

and by the definition of \( f \)

\[ \langle g(x), g(s) \rangle = \frac{4\mu(x_i, s_i)}{(x_{i0}^2 - \|\bar{x}_i\|^2)(s_{i0}^2 - \|\bar{s}_i\|^2)}. \]

Therefore,

\[ (x_{i0}^2 - \|\bar{x}_i\|^2)(s_{i0}^2 - \|\bar{s}_i\|^2) \geq \frac{8\mu(x_i, s_i)}{\|s_i\|^2_{s_i} + \mu^2\|g(x_i)\|^2_{s_i}}. \tag{2.15} \]

By Proposition 1

\[ \| - \mu g(x_i)\|_{s_i} \leq \frac{\| - \mu g(x_i)\|_{-\mu g(x_i)}}{1 - \|s_i + \mu g(x_i)\|_{-\mu g(x_i)}} \leq \frac{\sqrt{2}}{1 - \beta}. \] \tag{2.16} 

Now (2.12) follows from (2.11), (2.15) and (2.16). \( \square \)
Lemma 8 Let \( z \in \mathcal{N}_\beta \) for some \( \beta \leq \frac{1}{15} \). Then
\[
\tau \geq (1 - \beta) \mu(z). \tag{2.17}
\]

Proof By Lemma 7, taking \( i = r + 1 \) in (2.11), and using \( \|s''\| \leq 1 \) from Lemma 4
\[
2(1 - \beta) \mu \leq \tau \tau + x''^T s'' \leq \tau (1 + \|s''\|) \leq 2 \tau,
\]
which yields (2.17). \( \square \)

2.2 Proof of Lemma 1

Throughout this proof, let \( w \) be the scaling point of the pair \( \vec{x}, \vec{s} \) and \( \vec{w} = \sqrt{\mu}w \). By Lemma 5 and the assumptions on the norm of \( \varrho \) and on \( \mu \) we have the following bound
\[
\|\varrho\|_w \leq \frac{1}{\mu^{1/2}} \|H(w)^{1/2}\| \|\varrho\| \leq \frac{1}{\mu^{1/2}} \frac{2(2r\mu + 1)}{\mu^{1/2}} \leq \frac{\mu^{1/2}}{60\mu^{1/2}} \leq \frac{1}{50}. \tag{2.18}
\]

Since \( z \in \mathcal{N}_\beta \), and \( \delta' \leq \frac{1}{45} \cdot \frac{25}{24} < \frac{\sqrt{6}}{105} < \frac{\sqrt{2}}{105} \), we have
\[
\|\vec{s} + \vec{\mu}g(\vec{x})\|_{-g(\vec{x})} \leq \|\vec{s} + \mu g(\vec{x})\|_{-g(\vec{x})} + \|\mu - \vec{\mu}\| \|g(\vec{x})\|_{-g(\vec{x})} \leq \beta \mu + \delta' \mu < \frac{\mu}{13}.
\]

Therefore, by Proposition 1(f)
\[
\|\vec{x} - \vec{w}\|_w \leq \frac{5}{4} \|\vec{s} + \vec{\mu}g(\vec{x})\|_{-\vec{\mu}g(\vec{x})} \leq \frac{5}{4} \cdot \frac{1}{13} = \frac{5}{52}. \tag{2.19}
\]

Recall that by Proposition 1(e)
\[
\vec{w} = -H(w)^{-1}g(w) = -g_{\vec{w}}(\vec{w}), \tag{2.20}
\]
and that in the local inner product the Hessian \( H_{\vec{w}}(\vec{w}) \) is the identity
\[
H_{\vec{w}}(\vec{w}) = H^{-1}(\vec{w}) H(\vec{w}) = I. \tag{2.21}
\]

Define
\[
\begin{align*}
u &:= g_{\vec{w}}(\vec{x}) + 2\vec{w} - \vec{x} \\
&= g_{\vec{w}}(\vec{x}) - g_{\vec{w}}(\vec{w}) + H_{\vec{w}}(\vec{w})\vec{w} - H_{\vec{w}}(\vec{w})\vec{x} \quad \text{(by (2.20) and (2.21))} \\
&= g_{\vec{w}}(\vec{x}) - g_{\vec{w}}(\vec{w}) - H_{\vec{w}}(\vec{w})(\vec{x} - \vec{w}). \tag{2.22}
\end{align*}
\]
Since $\|\bar{x} - w\|_w < \frac{5}{52} < 1$, Proposition 1(g) yields

$$
\|u\|_w = \|g_w(\bar{x}) - g_w(w) - H_w(w)(\bar{x} - w)\|_w \\
\leq \frac{\|\bar{x} - w\|_w^2}{1 - \|\bar{x} - w\|_w^2} \\
= \frac{25}{1 - \frac{5}{52}} \\
= \frac{25}{52 \cdot 47},
$$

(2.23)

Observe that

$$
\Delta \bar{x} + H(w)^{-1} \Delta \bar{s} = -H(w)^{-1}(\bar{s} + \mu g(\bar{x})) + \varrho \\
= -\bar{x} - \mu H(w)^{-1} g(\bar{x}) + \varrho \quad \text{by Prop. 1(f)} \\
= -\bar{x} - H(w)^{-1} g(\bar{x}) + \varrho \quad \text{by Prop. 1(d)} \\
= -\bar{x} - g_w(\bar{x}) + \varrho \\
= 2(\bar{x} - \bar{x} - u + \varrho) \quad \text{(by (2.22))}.
$$

Hence $\bar{w} - \bar{x} = \frac{1}{2}(\Delta \bar{x} + H(w)^{-1} \Delta \bar{s} + u - \varrho)$ and so

$$
\bar{w} - \bar{x}^+ = \frac{1}{2}(-\Delta \bar{x} + H(w)^{-1} \Delta \bar{s} + u - \varrho), \quad \text{and} \\
\bar{w} - H(w)^{-1} \bar{s}^+ = \frac{1}{2}(\Delta \bar{x} - H(w)^{-1} \Delta \bar{s} + u - \varrho).
$$

Consequently,

$$
H(w)^{-1} \bar{s}^+ = 2\bar{w} - \bar{x}^+ - u + \varrho.
$$

Since $\Delta \bar{x} \perp_w H(w)^{-1} \Delta \bar{s}$, we have

$$
\|\bar{w} - \bar{x}^+\|_w \leq \frac{1}{2} \left( \| - \Delta \bar{x} + H(w)^{-1} \Delta \bar{s}\|_w + \|u\|_w + \|\varrho\|_w \right) \\
= \frac{1}{2} \left( \|\bar{w} - \bar{x} - \frac{1}{2} u + \frac{1}{2} \varrho\|_w + \frac{1}{2} \|u\|_w + \frac{1}{2} \|\varrho\|_w \right) \\
= \|\bar{w} - \bar{x}\|_w - \frac{1}{2} u + \frac{1}{2} \varrho + \frac{1}{2} \|u\|_w + \frac{1}{2} \|\varrho\|_w \\
\leq \|\bar{w} - \bar{x}\|_w + \|u\|_w + \|\varrho\|_w \\
= \frac{5}{52} + \frac{25}{52 \cdot 47} + \frac{1}{47} = \frac{6}{47} \quad \text{(by (2.18), (2.19), and (2.23))}.
$$

Thus
From Proposition 1(h) we have for all \( v \in \mathbb{R}^n \)

\[
\|v\|_{g_{\mathbf{w}}(\bar{x}^+)} \leq (1 + \|\mathbf{w} - \bar{x}^+\|_w) \|v\|_w \leq \left(1 + \frac{6}{47}\right) \|v\|_w = \frac{53}{47} \|v\|_w.
\]

Thus

\[
\|\hat{s}^+ + \mu g(\bar{x}^+)\|_{-g(\bar{x}^+)} = \frac{\mu}{\|H(w)^{-1}\bar{s}^+ + g_{\mathbf{w}}(\bar{x}^+)\|_{-g_{\mathbf{w}}(\bar{x}^+)}} < \frac{53}{47} \cdot \frac{107}{2132} < \left(1 - \frac{0.01}{15}\right) \mu.
\]

To finish we need to show that \( z^+ = (\bar{x}, \bar{y}, \bar{s}) \in \mathcal{N}_\beta \) and \( \mu(z^+) \) is close to \( \bar{\mu} \). Since \( \|\mathbf{w} - \bar{x}^+\|_w \leq \frac{6}{47} < 1 \), Proposition 1(g) yields \( \bar{x}^+ \in \text{int}(\mathcal{K}) \) and so \( -g(\bar{x}^+) \in \text{int}(\mathcal{K}) \). From (2.24) we get \( \|\hat{s}^+ + \mu g(\bar{x}^+)\|_{-\mu g(\bar{x}^+)} < 1 \) and thus Proposition 1(g) again yields \( \bar{s}^+ \in \text{int}(\mathcal{K}) \). Furthermore, by assumption we have

\[
\Delta \bar{x} + H(w)^{-1} \Delta \bar{s} + \bar{x} = -\mu g(\bar{s}) + \varrho.
\]

Taking inner product with \( \hat{s} \) and using Proposition 1(e,f) we get

\[
\langle \hat{s}, \Delta \bar{x} \rangle + \langle \bar{x}, \Delta \bar{s} \rangle + \langle \hat{s}, \bar{x} \rangle = -\mu \langle \hat{s}, g(\bar{s}) \rangle + \langle \hat{s}, \varrho \rangle = 2\mu \bar{\mu} + \langle \hat{s}, \varrho \rangle.
\]

Since \( \Delta \bar{x} \perp \Delta \bar{s} \), we have \( \langle \bar{x} + \Delta \bar{x}, \bar{s} + \Delta \bar{s} \rangle = \langle \bar{x}, \bar{s} \rangle + \langle \bar{x}, \Delta \bar{s} \rangle + \langle \Delta \bar{x}, \bar{s} \rangle \), so by (2.25)

\[
\mu(z^+) = \frac{1}{2\mu} \langle \bar{x} + \Delta \bar{x}, \bar{s} + \Delta \bar{s} \rangle = \frac{1}{2\mu} \left( \langle \bar{x}, \bar{s} \rangle + \langle \bar{x}, \Delta \bar{s} \rangle + \langle \Delta \bar{x}, \bar{s} \rangle \right) = \bar{\mu} + \frac{\langle \hat{s}, \varrho \rangle}{2\mu}.
\]

Using (2.26), Lemma 4 and the assumption on \( \varrho \), we get

\[
|\mu(z^+) - \bar{\mu}| = \left| \frac{\langle \hat{s}, \varrho \rangle}{2\mu} \right| \leq \frac{\mu}{120r^2} = \frac{\bar{\mu}}{120r^2} \left(1 - \frac{\delta}{\sqrt{2}r} \right) < \frac{\bar{\mu}}{120}.
\]

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Therefore, from (2.24) and (2.27) we get
\[ \|\vec{s}^+ + \mu (z^+ + g(\vec{x}^+ + g(\vec{x}^+ + 2r|\mu (z^+) - \bar{\mu}|) \leq \left( \frac{1}{15} - 0.01 \right) + \frac{1}{120} \bar{\mu} < \frac{1}{15} \mu (z^+). \] (2.28)

Observe that \( z^+ \) satisfies the linear equations \( A\vec{x}^+ = 0, A^T\vec{y}^+ + \vec{s}^+ = 0 \) by assumption. This together with \( \vec{x}, \vec{s} \in \text{int}(K) \) and (2.28) yields \( z^+ \in N_\beta \). \( \square \)

2.3 On the condition of the matrix \( AH(w)^{-1/2}A^T \)

The purpose of this section is to present Lemma 11 which will be crucial in our finite precision analysis in Sect. 3. This lemma is in the same spirit as Lemma 2. While it will not be used until Sect. 3, we choose to place it here to facilitate understanding of the proof.

We will rely on the following key characterization of the distance to ill-posedness due to Renegar [16, Theorem 3.5]. For a detailed discussion of this and related results, see also [9,10,12].

**Proposition 2** For any given linear operator \( A : \mathbb{R}^n \to \mathbb{R}^m \) and any cone \( K \)
\[ \rho_P(A) = \sup \{ \delta : \|v\| \leq \delta \Rightarrow v \in \{ Ax : \|x\| \leq 1, x \in K \} \} \]
and
\[ \rho_D(A) = \sup \{ \delta : \|u\| \leq \delta \Rightarrow u \in \{ A^T y : \|y\| \leq 1 \} + K^* \} \].
\( \square \)

We will also rely on the following perturbation result, an extension of [22, Theorem 5.1].

**Lemma 9** Let \( \beta \leq \frac{1}{15}, z = (\vec{x}, \vec{y}, \vec{s}) \in N_\beta \). Assume \( \vec{b} \in \mathbb{R}^m \) is such that \( (\vec{y}, \vec{b}) \leq 0 \) and
\[ A\vec{v} = \vec{b} \Rightarrow \frac{1}{15} H(w)^{1/2} \vec{v} \geq 1 \] (2.29)
where \( w \) is the scaling point of the pair \( \vec{x}, \vec{s} \).

If \( \alpha > \frac{\beta + 2r}{1 - \frac{9}{4} \beta} \), then the optimal value of the perturbed problem
\[ \min \vec{c}^T \vec{u} \]
\[ \text{s.t. } A\vec{u} = \vec{b} + \alpha \vec{b}, \]
\[ \vec{u} \succeq_\kappa \vec{0}, \] (2.30)
is at least \( \vec{c}^T \vec{x} \).
Proof This follows by putting together [22, Theorem 5.1] and Proposition 1(a, f) as we next explain. Since \( z \in N_\beta \) Proposition 1(f) yields
\[
\| \vec{x} - \vec{w} \|_w \leq \frac{5}{4} \beta < 1,
\]
where \( \vec{w} = \sqrt{\mu} \vec{w} \). Applying Proposition 1(a) twice we obtain
\[
\| \vec{v} \| - \mu g(\vec{s}) \geq (1 - \beta)\| \vec{v} \| \geq (1 - \beta) \frac{1}{1 - \frac{9}{4} \beta} \| \vec{v} \|_w
\]
for all \( \vec{v} \). Furthermore, observe that
\[
\| \vec{v} \|_w = \frac{1}{\mu} \| H(w)^{1/2} \vec{v} \|
\]
for all \( \vec{v} \). Hence (2.29) implies that
\[
A \vec{v} = \frac{1}{1 - \frac{9}{4} \beta} \vec{b} \Rightarrow \| \vec{v} \|_{-\mu g(\vec{s})} \geq 1.
\]
Therefore, by [22, Theorem 5.1] it follows that the optimal value of (2.30) is at least \( \vec{c}^T \vec{x} \).

\[\Box\]

Lemma 10 Let \( \Delta b = (\Delta b^I, \Delta b^{II}, \Delta b^{III}) \in \mathbb{R}^{m+n+1} = \mathbb{R}^m \). Then there exists \( \vec{\alpha} \) such that
\[
A \vec{\alpha} = \vec{b} + \Delta b, \quad \vec{\alpha} \succeq 0, \quad \text{and} \quad \vec{c}^T \vec{\alpha} \leq \max\{0, (1 + 2\rho^2 P(A))^{1/2} \| \Delta b \| - \rho P(A)\}.
\]

Proof Let
\[
\lambda = \min \left\{ 1, \frac{1 - \| \Delta b^{II} \| - |\Delta b^{III}|}{\| \Delta b^I \|} \right\}.
\]
From Proposition 2 it follows that there exists a \( u \) such that
\[
Au = \lambda \Delta b^I, \quad u \succeq 0, \quad \| u \| \leq \frac{\lambda \| \Delta b^I \|}{\rho P(A)}.
\]
Let \( \vec{u} = (u, 1 + \Delta b^{III}, u + \Delta b^{II}, (1 - \lambda)\| \Delta b^I \|, (1 - \lambda)\Delta b^I) \). Observe that by construction
\[
A^T \vec{u} = \vec{b} - \Delta b, \quad \vec{u} \succeq 0.
\]
Finally,
\[
\vec{c}^T \vec{u} = (1 - \lambda)\| \Delta b^I \|
\]
\[
= \max\{0, \| \Delta b^I \| + \rho P(A)(\| \Delta b^{II} \| + |\Delta b^{III}| - 1)\}
\]
\[
\leq \max\{0, (1 + 2\rho^2 P(A))^{1/2} \| \Delta b \| - \rho P(A)\},
\]
where the last inequality can be obtained by elementary analysis. 
\[\Box\]
Lemma 11 Let \( z \in \mathcal{N}_\beta \). Then if \( \rho_P(A) > 0 \),

\[
\sigma_{\min}(\mu^{1/2}H(w)^{-1/2}A^T) \geq \frac{\mu(z)}{6r};
\]
\[
\sigma_{\max}(\mu^{1/2}H(w)^{-1/2}A^T) \leq \mu^{1/2}\|H(w)^{-1/2}\|\|A^T\| \leq 4(1 + 2r\mu(z)).
\] (2.31) (2.32)

Proof Lemma 9 implies that there exists \( \Delta b \in \mathbb{R}^m \) satisfying

\[
\|\Delta b\| \leq \frac{\beta + 2r}{1 - \frac{9\beta}{4}} \sigma_{\min}(\mu^{1/2}H(w)^{-1/2}A^T)
\] (2.33)

and such that the optimal value of the following problem

\[
\min \tilde{c}^T \tilde{u} \\
\text{s.t. } \tilde{A} \tilde{u} = \tilde{b} + \Delta b, \\
\tilde{u} \succeq K 0
\] (2.34)

is at least \( \tilde{c}^T \tilde{x} \). Assume that \( \|\Delta b\| < (1 + 2\rho_P^2(A))^{-1/2}(\tilde{c}^T \tilde{x} + \rho_P(A)) \). Then by Lemma 10 the optimal value of (2.34) is \( 0 < \tilde{c}^T \tilde{x} \), which contradicts the earlier conclusion. Therefore, \( \|\Delta b\| \) must satisfy

\[
\|\Delta b\| \geq (1 + 2\rho_P^2(A))^{-1/2}(\tilde{c}^T \tilde{x} + \rho_P(A)).
\] (2.35)

Putting (2.33) and (2.35) together, we get

\[
\sigma_{\min}(\mu^{1/2}H(w)^{-1/2}A^T) \geq \frac{(1 - \frac{5\beta}{3})(\tilde{c}^T \tilde{x} + \rho_P(A))}{(\beta + 2r)(1 + 2\rho_P^2(A))^{1/2}}.
\]

Since \( \langle \tilde{c}, \tilde{x} \rangle = \tau \geq (1 - \beta)\mu(z) \) by Lemma 8, \( 0 < \rho_P(A) \leq \|A\| \leq 1 \) and \( \beta < \frac{1}{\tau} \), we have (2.31). Inequality (2.32) follows from the bound \( \|A\| \leq 2 \) and Lemma 5:

\[
\sigma_{\max}(\mu(z)^{1/2}H(w)^{-1/2}A^T) \leq \mu(z)^{1/2}\|H(w)^{-1/2}\|\|A^T\| \leq 4(1 + 2r\mu).
\]

\[\square\]

3 Finite precision analysis

3.1 Floating-point arithmetic

Here we briefly recall the basics of floating-point arithmetic which we will use in this paper. A slightly more extensive introduction is in [6, §7]. Detailed treatments can be found in books on numerical linear algebra such as [7].

We call floating-point numbers a set \( \mathbb{F} \subset \mathbb{R} \) containing 0, rounding map a transformation \( \text{round} : \mathbb{R} \to \mathbb{F} \) and round-off unit a constant \( u \in \mathbb{R} \) satisfying \( 0 < u < 1 \). We require for such a triple that the following properties hold:
(i) For any $x \in \mathbb{F}$, $\text{round}(x) = x$. In particular $\text{round}(0) = 0$.

(ii) For any $x \in \mathbb{R}$, $\text{round}(x) = x(1 + \delta)$ with $|\delta| \leq u$.

We also define on $\mathbb{F}$ arithmetic operations following the scheme

$$x \tilde{\circ} y = \text{round}(x \circ y)$$

for any $x, y \in \mathbb{F}$ and $\circ \in \{+, -, \times, /\}$ so that

$$\tilde{\circ} : \mathbb{F} \times \mathbb{F} \to \mathbb{F}.$$ 

It follows from (ii) above that, for any $x, y \in \mathbb{F}$ we have

$$x \tilde{\circ} y = (x \circ y)(1 + \delta), \quad |\delta| \leq u.$$ 

We will also use a floating-point version $\tilde{\sqrt{\cdot}}$ of the square root which, similarly, satisfies

$$\tilde{\sqrt{x}} = \sqrt{x}(1 + \delta), \quad |\delta| \leq u.$$ 

When combining many operations in floating-point arithmetic, quantities such as $\prod_{i=1}^{n}(1 + \delta_i)^{\rho_i}$ naturally appear. The proof of the following propositions can be found in Chapter 3 of [7]. The notation they introduce, the quantities $\gamma_n$ and $\theta_n$, and the relations showed therein, will be widely used in our round-off analysis.

**Proposition 3** If $|\delta_i| \leq u$, $\rho_i \in \{-1, 1\}$ and $nu < 1$ then

$$\prod_{i=1}^{n}(1 + \delta_i)^{\rho_i} = 1 + \theta_n$$

where

$$|\theta_n| \leq \gamma_n = \frac{nu}{1 - nu}.$$ 

\[\Box\]

**Proposition 4** For any positive integer $k$ such that $ku < 1$ let $\theta_k$ be any quantity satisfying

$$|\theta_k| \leq \gamma_k = \frac{ku}{1 - ku}.$$ 

The following relations hold.

(1) $(1 + \theta_k)(1 + \theta_j) = 1 + \theta_{k+j}$,

(2) \[
\frac{1 + \theta_k}{1 + \theta_j} = \begin{cases} 
1 + \theta_{k+j} & \text{if } j \leq k \\
1 + \theta_{k+2j} & \text{if } j > k,
\end{cases}
\]
(3) If $ku, jv \leq 1/2$ then $\gamma_k \gamma_j \leq \gamma_{\min\{k, j\}}$.

(4) $i \gamma_k \leq \gamma_{ik}$.

(5) $\gamma_k + u \leq \gamma_{k+1}$.

(6) $\gamma_k + \gamma_j + \gamma_k \gamma_j \leq \gamma_{k+j}$.

When computing an arithmetic expression $q$ with a round-off algorithm, errors will accumulate and we will obtain another quantity which, we recall, we denote by $\text{fl}(q)$. We will also write $\text{Error}(q) = |q - \text{fl}(q)|$.

An example of round-off analysis which will be useful in the sequel is given in the next proposition whose proof can be found in Section 3.1 of [7].

**Proposition 5** There is a round-off algorithm which, with input $x, y \in \mathbb{R}^n$, computes the dot product of $x$ and $y$. The computed value $\text{fl}(\langle x, y \rangle)$ satisfies

$$\text{fl}(\langle x, y \rangle) = \langle x, y \rangle + \theta_{\lfloor \log_2 n \rfloor + 1} \langle |x|, |y| \rangle$$

where $|x| = (|x_1|, \ldots, |x_n|)$. In particular, if $x = y$ the algorithm computes $\text{fl}(\|x\|^2)$ satisfying

$$\text{fl}(\|x\|^2) = \|x\|^2 (1 + \theta_{\lfloor \log_2 n \rfloor + 1}).$$

The following result deals with summation errors. The proof can be found in [7], Section 4.2.

**Proposition 6** There is a round-off algorithm which, with input $x \in \mathbb{R}^n$, computes the sum of $x_i$. The computed value $\text{fl}(\sum_{i=1}^{n} x_i)$ satisfies

$$\text{fl}\left(\sum_{i=1}^{n} x_i\right) = \sum_{i=1}^{n} x_i + \theta_{\lfloor \log_2 n \rfloor} \sum_{i=1}^{n} |x_i|.$$

In the next section we will have to deal with square roots. The following result will help us to do so.

**Proposition 7** Let $\theta \in \mathbb{R}$ such that $|\theta| \leq 1/2$. Then, $\sqrt{1 + \theta} = 1 + \theta'$ with $|\theta'| \leq |\theta|$. In particular, for $a \geq 0$

$$\text{fl}\left(\sqrt{a(1 + \theta)}\right) = \sqrt{a(1 + \theta_{k+1})}. \quad (3.1)$$

**Proof** By the intermediate value theorem we have that $\sqrt{1 + \theta} - 1 = |\theta|(\sqrt{\xi})'$ with $\xi \in (1 - |\theta|, 1)$ if $\theta < 0$, $\xi \in (1, 1 + \theta)$ otherwise. But

$$|\sqrt{\xi}'| = \left| \frac{1}{2\sqrt{\xi}} \right| \leq \frac{1}{\sqrt{2}}$$
the last since $|\xi| \geq 1/2$.

Then (3.1) follows from the above. \hfill $\square$

Our choice of $u = \phi(\mu(w))$, for the function $\phi$ in (3.2) below, guarantees that $ku < 1/2$ holds whenever we encounter $\theta_k$, and consequently, $\theta_k \leq 2ku$. We will therefore not bother the reader by repeating this fact each time we use it.

3.2 The finite precision algorithm

In this section we present a finite precision algorithm that determines which one of (P) or (D) is strictly feasible and provides a solution. In the case when the dual problem (D) is feasible, after sufficiently refining the precision we will be able to obtain an exact feasible solution to (D), however, for the primal problem only an approximation to a feasible solution is possible due to the structure of the problem: we cannot compute a point on the linear subspace $Ax = 0$ exactly with finite precision. However, we can obtain a forward-approximate primal solution of any desired accuracy. To describe this in more detail, we need the following definition of a $\gamma$-approximate solution.

**Definition 2** Let $\gamma \in (0, 1)$. A point $\hat{x} \in \mathbb{R}^n$ is a $\gamma$-forward solution of the system $Ax = 0$, $x \succeq_K 0$, if $\hat{x} \succ_K 0$, and there exists $\tilde{x} \in \mathbb{R}^n$ such that

$$A\tilde{x} = 0, \quad \tilde{x} \succ_K 0$$

and

$$\|\hat{x} - \tilde{x}\| \leq \gamma \|\hat{x}\|.$$ 

The point $\tilde{x}$ is said to be an associated solution for $\hat{x}$. Observe that by definition, the existence of a $\gamma$-forward solution automatically guarantees the existence of a strict solution.

We are now ready to present our main result and give a precise description of the related algorithm. The proof of Theorem 1 is deferred to Sect. 4.

**Theorem 1** There exists a finite precision algorithm which, with input a matrix $A \in \mathbb{R}^{m \times n}$ and a number $\gamma \in (0, 1)$, finds either a strict $\gamma$-forward solution $x \in \mathbb{R}^n$ of $Ax = 0$, $x \succeq_K 0$, or a strict solution $y \in \mathbb{R}^m$ of the system $A^Ty \preceq_K 0$. The machine precision varies during the execution of the algorithm. If (P) is strictly feasible, the finest required precision is

$$u^* = \left( c(n + m)^{5/2} r^{8} C(A)^{7/2} \left( 1 + \frac{1}{\gamma} \right)^{7/2} \right)^{-1},$$

and in the case when (D) is strictly feasible,

$$u^* = \left( c(n + m)^{5/2} r^{11.5} C(A)^{7/2} \right)^{-1}.$$
where \( c \) is a universal constant. The number of main (interior-point) iterations of the algorithm is bounded by

\[
O\left( r^{1/2}(\log(r) + \log(C(A)) + |\log \gamma|) \right)
\]

if (P) is strictly feasible and by the same expression without the \(|\log \gamma|\) term if (D) is.

**Remark 1** In the numerical analysis literature, fixed precision is used more commonly than variable precision. We note here that from our variable precision analysis we can obtain a fixed precision one. Indeed, assume the precision \( u \) is fixed. Then our algorithm could run until the point at which it should get a precision finer than \( u \). If it found the answer before this point it could return it (and this answer would be guaranteed to be correct). If not, it could halt and return a failure message. Furthermore, the only reason for \( u \) to be insufficient is that \( C(A) \) is too large. Solving the bound for \( u \) in Theorem 1 we obtain a lower bound \( C_u \) for \( C(A) \). Thus, the failure message could be something like “The condition of the data is larger than \( C_u \). To solve the problem I need more precision.” We note that although the statement of Theorem 1 depends on the condition number \( C(A) \), Algorithm FP described below does not require any information on \( C(A) \) as input. The only required input are the matrix \( A \) and a constant \( \gamma \in (0, 1) \).

We are now ready to describe our primal-dual algorithm. This is essentially an extension of Algorithm IP from Sect. 2 with some additional features. One of these features is the stopping criteria and the other one is the presence of finite precision and the adjustment of this precision as the algorithm progresses. To ensure the correctness of the algorithm, the precision will be set to

\[
\phi(\mu(z)) := \frac{\mu(z)^{7/2}}{c\sqrt{2r}\|A\|^{5/2}(2r\mu + 1)^{11/2}}
\]  

(3.2)

at each iteration. Here \( c \) is a universal constant.

Let \( \beta = \frac{1}{15} \) and \( \delta = \frac{1}{45} \).

**Algorithm FP** \((A, \gamma)\)

(i) Set the machine precision to \( u := \frac{1}{c\sqrt{(m+n)^5/2}} \)

\[
\alpha := \frac{1}{\sqrt{2r}}, M = \frac{\alpha\|Ae\|}{\beta},
\]

\[
z := \left( \alpha e, 1, \alpha e, 2M, -\alpha A e, 0, \frac{M}{\alpha} e, -\frac{M}{\alpha^2} e, \frac{M}{\alpha} e, \frac{M}{\alpha^2} e, -\frac{M}{\alpha} e, 1, 0 \right)
\]

(ii) Set the machine precision to \( u := \phi(\mu(z)) \).

(iii) If for \( i = 1, \ldots, r \) \( s_i - \|z_i\| - 6\mu(z)r > 0 \) then HALT and return \( y \) as a strict solution for \( A^T y \preceq_K 0 \).
(iv) If $\sigma_{\min}(\overline{H}(x)^{-1/2}A^T) \geq \frac{3r\mu(z)}{\gamma}$, then HALT and return $x$ as a $\gamma$-forward solution for $Ax = 0$, $x \succeq K 0$.

(v) Set $\overline{\mu} := \left(1 - \frac{\delta}{\sqrt{2}r}\right)\mu(z)$.

(vi) Update $z$ by solving the linearization (2.4) of (2.3) for $\mu = \overline{\mu}$.

(vii) Go to (ii).

The matrix $\overline{H}(w)$ used in step (iv) is the upper-left $n \times n$ block of $H(w)$, where $w$ is the scaling point of $(\vec{x}, \vec{s})$.

The precise way we solve the system in (vi) is as follows:

(a) Compute a solution $\Delta \vec{y}$ of

$$(AH(w)^{-1}A^T)\Delta \vec{y} = AH(w)^{-1}(\vec{s} + \overline{\mu}g(\vec{x})). \tag{3.3}$$

(b) Let $\vec{y} := \vec{y} + \Delta \vec{y}$,

$$\begin{pmatrix} \Delta x \\ \Delta \tau \end{pmatrix} := \begin{pmatrix} I_n & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} (H(w)^{-1}A^T \Delta \vec{y} - (\overline{\mu}g(\vec{s}) + \vec{x})).$$

Then set $x := x + \Delta x$ and $\tau := \tau + \Delta \tau$.

(c) Let

$$x' := x, \quad s' := -y', \quad t := 1, \quad t_s := \eta, \quad x'' := -Ax, \quad s'' := -y, \quad s := y' - A^T y, \quad \tau_s = 1.$$

Remark 2: The finite-precision errors in the computations in (b) and (c) are negligible compared to the errors involved in solving the linear system on step (a). Therefore, for ease of exposition, we will assume that the computations in (b) and (c) in step (vi) are exact. We also assume that the initial point $z$ in step (i) and the value of $\mu$ in step (v) of Algorithm FP are computed exactly. We stress that these assumptions have no consequences in the complexity or accuracy bounds. By making them we can greatly reduce the length of our exposition and focus our analysis on the critical stages of the algorithm. We assume that the smallest singular value in step (iv) above is computed using a backward stable algorithm (e.g., QR factorization). This guarantees that the computed $\overline{\mu}(\sigma_{\min}(\overline{H}(x)^{-1/2}A^T))$ is the exact $\sigma_{\min}((\overline{H}(x)^{-1/2}A^T) + E)$ for a matrix $E$ with $\|E\| \leq c_1 n^{\delta/2} u\|H(x)^{-1/2}A^T\|$ for some universal constant $c_1$ (see, e.g., [3, Chapter 2]).

Under the assumption of infinite precision on steps (b) and (c) the next point $z^+$ thus defined lies in the linear subspace $\{A\vec{x} = \vec{b}, A^T \vec{y} - \vec{s} = \vec{c}\}$. Moreover, $\Delta z = (\Delta x, \Delta y, \Delta z)$ satisfies system (2.5) for some (possibly large) $r$.

The crux of our finite precision analysis is the estimation of the floating-point errors in step (a) above, which we present in Sect. 3.3. That analysis relies on the following technical lemma.
Lemma 12 Assume $z \in \mathcal{N}_\beta$ and let $w = w(z)$ be its scaling point. With precision $u = \phi(\mu(z))$ we can compute $B = H(w)^{-1/2}A^T$ and $D = H(w)^{-1/2}$ (where $H(w)$ is the upper-left $n \times n$ block of $H(w)$) satisfying

$$\|f_1(B) - B\|, \|f_1(D) - D\| \leq \frac{1}{336 \cdot 240} \cdot \frac{\mu^2}{r(2r\mu + 1)^2}$$

(3.4)

as well as $q = -H(w)^{-1/2}(\tilde{g}(\tilde{x}) + \tilde{s})$ satisfying

$$\|f_1(q) - q\| \leq \frac{1}{16 \cdot 240} \cdot \frac{\mu^{3/2}}{(2r\mu + 1)^2}.$$ 

(3.5)

The proof of Lemma 12 in turn relies on the following technical result.

Lemma 13 Let $z \in \mathcal{N}_\beta$, and the finite-precision computations are performed with $u = \phi(\mu(z))$. Then

$$\text{Error} \left( \frac{\det s_i}{\det x_i} \right) \leq \frac{\det s_i}{\det x_i} \gamma M, \quad \text{Error} \left( \frac{\det s_i \det x_i}{\det x_i} \right) \leq \det s_i \det x_i \gamma M,$$

(3.6)

where

$$M = \frac{(2r\mu + 1)^2(\log_2(n + m) + 2)}{\mu^2(1 - \beta)^2}.$$ 

Proof Observe that from Proposition 5

$$\text{Error}(\det s_i) = \|s_i\|^2 \theta_{\lfloor \log_2 n_i \rfloor + 1}, \quad \text{Error}(\det x_i) = \|x_i\|^2 \theta_{\lfloor \log_2 n_i \rfloor + 1}$$

(3.7)

for all $i \in \{1, \ldots, r\}$. Let $\kappa_i := \lfloor \log_2 n_i \rfloor + 1$. From (3.7) we have

$$\text{Error} \left( \frac{\det s_i}{\det x_i} \right) \leq \frac{\det s_i}{\det x_i} + \|s_i\|^2 \gamma_{\kappa_i} (1 + \gamma_1) - \frac{\det s_i}{\det x_i}$$

$$= \frac{\det s_i}{\det x_i} \cdot \gamma_1 \det s_i \det x_i + \|s_i\|^2 \gamma_{\kappa_i+1} \det x_i + \|x_i\|^2 \gamma_{\kappa_i} \det s_i$$

$$\leq \frac{\det s_i}{\det x_i} \cdot \left( \frac{\|x_i\|^2}{\det x_i} \gamma_{\kappa_i} + \frac{\|s_i\|^2}{\det s_i} \gamma_{\kappa_i+1} + \gamma_1 \right),$$

and from Lemma 7

$$\det x_i \geq \frac{4\mu^2(1 - \beta)^2}{\|s_i\|^2}, \quad \det s_i \geq \frac{4\mu^2(1 - \beta)^2}{\|x_i\|^2}.$$ 

Therefore, using Lemma 4

$$\text{Error} \left( \frac{\det s_i}{\det x_i} \right) \leq \frac{\det s_i}{\det x_i} \cdot \frac{\|x_i\|^2 \|s_i\|^2 \gamma_{\kappa_i} + \|x_i\|^2 \|s_i\|^2 \gamma_{\kappa_i+1} + \gamma_1}{2\mu^2(1 - \beta)^2}$$
\[
\frac{\det s_i}{\det x_i} \leq \frac{(2r\mu + 1)^2\gamma_{k_i} + (2r\mu + 1)^2\gamma_{k_i+1} + \gamma_1}{2\mu^2(1 - \beta)^2}
\]
\[
\leq \frac{\det s_i}{\det x_i} \cdot \frac{(2r\mu + 1)^2}{\mu^2(1 - \beta)^2\gamma_{k_i+1}},
\]
which yields the first inequality in (3.6). The second relation is obtained analogously. □

**Proof of Lemma 12** It is well-known (see [20, §3.2]) that the scaling matrix \( H \) has a block-diagonal structure, where each block corresponds to a Lorentz cone; moreover, each individual block can be represented as follows

\[
H_i(w(x_i, s_i))^{1/2} = i^{-1} \left( \begin{array}{cc} \alpha & -\xi^T \\ -\zeta & I + \xi\xi^T/\alpha \end{array} \right),
\]

where \( i = \left[ \frac{s_i^2 - \|x_i\|^2}{x_i^2 - \|x_i\|^2} \right]^{1/4}, \alpha = \frac{\bar{\zeta}}{\det(\bar{\zeta})^{1/2}}, \) and \( \zeta = \frac{\xi}{\det(\xi)^{1/2}} \) with \( \xi = (\xi_0, \bar{\xi}) = (i^{-1}s_0 + ix_0, i^{-1}\bar{s}_i - i\bar{x}_i) \) for all \( i = 1, \ldots, r. \)

From Proposition 5 for all \( i = 1, \ldots, r, \) \( \text{Error}((x_i, s_i)) \leq \|x_i\|\|s_i\|\gamma_{\log_2(n + m) + 1}, \) and using Lemmas 4 and 7

\[
\text{Error}((x_i, s_i)) \leq (x_i, s_i) \frac{2r\mu + 1}{2(1 - \beta)}\gamma_{\log_2(n + m) + 1} \leq (x_i, s_i)\gamma_M. \tag{3.8}
\]

By Proposition 7 and Lemma 13

\[
\text{fl}(\sqrt{\det x_i\det s_i}) = \sqrt{\det x_i\det s_i}(1 + \gamma_{M+1}). \tag{3.9}
\]

Therefore, from (3.8) and (3.9)

\[
\text{Error}(\det \xi) = \text{Error} \left( 2 \left( \sqrt{\det x_i\det s_i} + (s_i, x_i) \right) \right) = \left| 2 \left( \sqrt{\det x_i\det s_i}(1 + \theta_{M+1}) + (x_i, s_i)(1 + \theta_M) \right) (1 + \theta_2) - \text{det} \xi \right|
\]
\[
= \left| 2 \left( \sqrt{\det x_i\det s_i}\theta_{M+1} + (x_i, s_i)\theta_M \right) (1 + \theta_2) + \text{det} \xi \theta_2 \right|
\]
\[
= \text{det} \xi |\theta_{M+1} + \theta_2 + \theta_{M+1}\theta_2| \leq \text{det} \xi \gamma_{M+3}.
\]

The last inequality due to Proposition 4).

Further

\[
\text{Error}(i^{-1}\alpha) = \text{Error} \left( \frac{i^{-2}s_i + x_i}{\sqrt{\det \xi}} \right) = i^{-1}\alpha\theta_{3M+4}; \tag{3.10}
\]
\[
\text{fl}(i^{-1}\zeta)_j = \text{fl} \left( \frac{i^{-2}s_{ij} + x_{ij}}{\sqrt{\det \xi}} \right) = (i^{-1}\zeta)_j \frac{i^{-2}|s_{ij}| + |x_{ij}|}{\sqrt{\det \xi}} \theta_{3M+4}. \tag{3.11}
\]
hence
\[
\text{Error}(i^{-1}\zeta_j) \leq i^{-1} \alpha \gamma_{3M+4}.
\]

It remains to evaluate the errors in the bottom-left block of \(H_i(w(x_i, s_i))\). We have
\[
\text{Error}(\xi_k) \leq \xi_0 \gamma_{2M+4},
\]
then
\[
\text{Error}\left(i^{-1} \xi_k \xi_l \frac{1}{1 + \alpha}\right) \leq i^{-1} \frac{\xi_0^2}{1 + \alpha} \gamma_{11M+33};
\]
\[
\text{Error}\left(i^{-1} + i^{-1} \xi_k \xi_l \frac{1}{1 + \alpha}\right) \leq i^{-1} \left(1 + \frac{\xi_0^2}{1 + \alpha}\right) \gamma_{11M+33}.
\]

Finally, we have
\[
\|f_1(H(w)^{-1/2}) - H(w)^{-1/2}\| \leq (n_i + 1)i^{-1} \max \left\{\alpha \gamma_{3M+4}, \left(1 + \frac{\xi_0^2}{1 + \alpha}\right) \gamma_{11M+34}\right\}.
\]

Observe that by Lemma 7
\[
\det \xi = 2(\sqrt{\det x_i \det \bar{s}_i + (x_i, s_i)}) \geq 8(1 - \beta) \mu;
\]
\[
\xi_0 = i^{-1} s_0 + i x_0 \leq \frac{x_0^2 + s_0^2}{\sqrt{2\mu}(1 - \beta)}; \quad i^{-1} \leq \frac{s_0}{\sqrt{2\mu}(1 - \beta)}.
\]

Then
\[
\alpha = \frac{\xi_0}{\det \xi^{1/2}} \leq \frac{s_0^2 + x_0^2}{4\mu(1 - \beta)}; \quad (3.12)
\]
\[
1 + \frac{\xi_0^2}{1 + \alpha} = 1 + \frac{\xi_0^2}{\det \xi + \xi_0} \leq 1 + \xi_0 \leq 1 + \frac{x_0^2 + s_0^2}{4\mu(1 - \beta)}.
\]

Observe that \(x_0^2 + s_0^2 < 2(2\mu + 1)^2\). Hence we have
\[
i^{-1} \max \left\{\alpha \gamma_{3M+4}, \left(1 + \frac{\xi_0^2}{1 + \alpha}\right) \gamma_{11M+34}\right\} \leq \frac{5(2\mu + 1)^{7/2}}{\mu^{3/2}} \gamma_{11M+34}.
\]
Therefore,
\[
\| \mathcal{E}(H(w)^{-1/2}) - H(w)^{-1/2} \| \leq \frac{5(n + m)(2r\mu + 1)^{7/2}}{\mu^{3/2}} \gamma_{11M+34} \quad (3.13)
\]
and
\[
\| \mathcal{E}(\mathcal{H}(w)^{-1/2}) - \mathcal{H}(w)^{-1/2} \| \leq \frac{5n(2r\mu + 1)^{7/2}}{\mu^{3/2}} \gamma_{11M+34}.
\]

Now we estimate the error in computing \( B = H(w)^{-1/2} \mathcal{A}^T \). First, observe that the error for multiplication of \( \mathcal{E}(H(w)^{-1/2}) \) by \( \mathcal{A}^T \) can be estimated as follows (see \[7, Chapter 22\])
\[
\| \mathcal{E}(H(w)^{-1/2}) \mathcal{A}^T - \mathcal{E}(\mathcal{E}(H(w)^{-1/2}) \mathcal{A}^T) \| \leq n^2u \| \mathcal{E}(H(w)^{-1/2}) \| \| \mathcal{A}^T \| + c_2u^2
\]
for some universal constant \( c_2 \). Therefore, recalling that \( \| \mathcal{A} \| \leq 2 \) (see \[2.1\]),
\[
\| B - \mathcal{E}(B) \| \leq \| H(w)^{-1/2} \mathcal{A}^T - \mathcal{E}(H(w)^{-1/2}) \mathcal{A}^T \|
+ \| \mathcal{E}(H(w)^{-1/2}) \mathcal{A}^T - \mathcal{E}(\mathcal{E}(H(w)^{-1/2}) \mathcal{A}^T) \|
\leq (\| H(w)^{-1/2} - \mathcal{E}(H(w)^{-1/2}) \| + n^2u \| \mathcal{E}(H(w)^{-1/2}) \|) \| \mathcal{A}^T \| + c_2u^2
\leq 2\| H(w)^{-1/2} - \mathcal{E}(H(w)^{-1/2}) \|
+ 2n^2u \| (H(w)^{-1/2} - \mathcal{E}(H(w)^{-1/2})) \| + \| (H(w)^{-1/2}) \| + c_2u^2
= 2(1 + n^2u) \| H(w)^{-1/2} - \mathcal{E}(H(w)^{-1/2}) \| + n^2u \| (H(w)^{-1/2}) \| + c_2u^2
< \frac{40(n + m)^2(2r\mu + 1)^{7/2}}{\mu^{3/2}} \gamma_{12M+34} + c_2u^2,
\]
where the last bound follows from (3.13) and Lemma 5. Since our precision \( u \) satisfies (3.2), this yields (3.4) for \( B \). Observe that here we only care about the order of the problem-driven parameters, not the constants, as \( c \) in the precision update formula (3.2) can be adjusted to accommodate any multiplicative constants. The corresponding bound for \( D \) is obtained analogously.

It remains to evaluate the errors in computing \( q \). By straightforward computation we obtain for each ‘block’
\[
q_{i0} = -\frac{1}{2} \det \xi_i^{1/2} \left( 1 - \mu^{-2} \frac{i_i^{-2}}{\det x_i} \right);
\]
\[
\bar{q}_i = -\left( \frac{1 - \mu^{-2} \frac{i_i^{-2}}{\det x_i}}{\det \xi_i^{1/2} + \xi_i} \right) \left[ \left( s_{i0} + \frac{1}{2} i_i \det \xi_i^{1/2} \right) \bar{x}_i + \left( \frac{1}{2} i_i^{-1} \det \xi_i^{1/2} + x_{i0} \right) \bar{s}_i \right].
\]

We obtain (3.5) by a similar argument as when evaluating \( \| \mathcal{E}(H(w)^{-1/2}) - H(w)^{-1/2} \| \). We omit this tedious exercise for the sake of brevity.
\[\square\]
3.3 Finite-precision analysis of solving the Newton system

The main result of this section is Lemma 14, which bounds the round-off error in the computation in the reduced Eq. (3.3) in (a), when it is performed with finite precision.

Observe that the reduced system of Eq. (3.3) is equivalent to the least-squares problem

$$\min_v \| Bv + q \|^2$$

for $B = H(w)^{-1/2}A^T$ and $q = -H(w)^{-1/2}((\mu g(\vec{x})) + \vec{s})$. We rely on this equivalence to obtain the bound in Lemma 14. More precisely, we apply a known round-off error result for the least-squares problem, namely Proposition 8. Lemma 14 follows from Proposition 8 and suitable bounds on the norm of $q$ and singular values of $B$ obtained earlier in Lemmas 6 and 11 respectively.

Recall the following stability property of Golub’s method for linear least-squares (cf. [8, Chapter 16]).

**Proposition 8** Let $B \in \mathbb{R}^{p \times l}$ ($p \geq l$) have full rank. Let $u$ denote the machine precision. If Golub’s method is applied to

$$\min_{v \in \mathbb{R}^l} \| Bv + f \|$$

the computed solution is the exact solution to a problem

$$\min_{v \in \mathbb{R}^l} \| (B + \delta B)v + (f + \delta f) \|$$

where

$$\| \delta B \| \leq c_3ulp^{3/2} \| B \|, \quad \| \delta f \| \leq c_3ulp \| f \|$$

and $c_3$ is a universal constant independent of $p$ and $l$. \hfill \Box

**Lemma 14** Let $z \in N_\beta$. With precision $u = \phi(\mu)$ in all arithmetic operations, we can compute a vector $\ell_1(\Delta \vec{y})$ such that

$$\| (AH(w)^{-1}A^T)\ell_1(\Delta \vec{y}) - AH(w)^{-1}((\mu g(\vec{x})) + \vec{s})) \| \leq \frac{\mu(z)}{120r(2r\mu(z) + 1)}.$$

In addition, $\ell_1(\Delta \vec{y}) \leq 12r\mu^{-1/2}$.

**Proof** Let $B = H(w)^{-1/2}A^T$ and $q = H(w)^{-1/2}((\mu g(\vec{x})) + \vec{s})$, then the system of equations

$$(AH(w)^{-1}A^T)\Delta \vec{y} = AH(w)^{-1}((\mu g(\vec{x})) + \vec{s})$$
can be written as $B^T B \Delta \vec{y} = -B^T q$ and its solutions are those of the least squares problem

$$\min_{v \in \mathbb{R}^m} \| Bv + q \|. \quad (3.14)$$

Hence we can apply Golub’s method to (3.14) to compute a solution to the original equation. Let $\Delta \vec{y}$ be the vector actually computed by Golub’s method when solving (3.14). Then $\Delta \vec{y}$ is the exact solution of

$$\min_{v \in \mathbb{R}^m} \| \tilde{B}v + \tilde{q} \|. \quad (3.15)$$

for some $\tilde{B}$ and $\tilde{q}$ satisfying

$$\| \tilde{B} - \hat{1}(B) \| \leq c_3 u m^{3/2} n \| \hat{1}(B) \|, \quad \| \tilde{q} - \hat{1}(q) \| \leq c_3 u m n \| \hat{1}(q) \|. \quad (3.16)$$

where $c_3$ is a universal constant. Let $\Delta B = \tilde{B} - B$, $\Delta q = \tilde{q} - q$. Since $\Delta \vec{y}$ is an exact solution of the least squares problem (3.15), we have $B^T \Delta \vec{y} + B^T \Delta q = 0$, and thus

$$B^T B \Delta \vec{y} + B^T q = -(\tilde{B}^T \Delta B + \Delta B^T B) \Delta \vec{y} - (\tilde{B}^T \Delta q + \Delta B^T q). \quad (3.17)$$

From Lemmas 5 and 12 and $\| A^T \| \leq 2$ we have

$$\| \hat{1}(B) \| \leq \| \hat{H}(w)^{-1/2} \| A^T \| + \| \hat{1}(B) - B \| \leq \frac{4(2r\mu + 1)}{\mu^{1/2}}. \quad (3.18)$$

Analogously, Lemmas 6 and 12 yield

$$\| \hat{1}(q) \| \leq \| q \| + \| \hat{1}(q) - q \| \leq \frac{\mu^{1/2}}{2} + \frac{\mu^{3/2}}{(2r\mu + 1)^2}; \quad (3.19)$$

Then from (3.16), (3.18), (3.19) and our choice of $u$ we have

$$\| \tilde{B} - \hat{1}(B) \| \leq 4c_3 u m^{3/2} n \frac{2r\mu + 1}{\mu^{1/2}} \leq \frac{1}{336 \cdot 240} \cdot \frac{\mu^2}{r(2r\mu + 1)^2}; \quad (3.20)$$

$$\| \tilde{q} - \hat{1}(q) \| \leq 4c_3 u m n \mu^{1/2} \leq \frac{1}{16 \cdot 240} \cdot \frac{\mu^{3/2}}{r(2r\mu + 1)^2}. \quad (3.21)$$

Here we assume that the constant $c$ in (3.2) is chosen so that inequalities (3.20) and (3.21) hold. This ensures that the rest of the proof goes through. Applying Lemma 12 again and using (3.20) and (3.21),

$$\| \Delta B \| \leq \| \hat{1}(B) - B \| + \| \tilde{B} - \hat{1}(B) \| \leq \frac{1}{168 \cdot 240} \cdot \frac{\mu^2}{r(2r\mu + 1)^2}; \quad (3.22)$$
\[ \| \Delta q \| \leq \| \mathbb{1}(q) - q \| + \| \widetilde{q} - \mathbb{1}(q) \| \leq \frac{1}{8} \cdot \frac{\mu^{3/2}}{240} \cdot \frac{1}{(2r\mu + 1)^2}. \] (3.23)

Then from (3.22), (3.23) and using the bounds on \( \| B \| \) and \( \| q \| \) discussed above,
\[ \| \widetilde{B} \| \leq \| B \| + \| \Delta B \| \leq 4 \cdot \frac{2r\mu + 1}{\mu^{1/2}}; \quad \| \widetilde{q} \| \leq \| q \| + \| \Delta q \| \leq \mu^{1/2}. \] (3.24)

From (3.22) and (3.24) we have
\[ \| \widetilde{B}^T \Delta B + \Delta B^T B \| \leq (\| \widetilde{B} \| + \| \Delta B \|) \cdot \Delta B \| \leq \frac{1}{240} \cdot \frac{\mu^{3/2}}{12r(2r\mu + 1)}. \] (3.25)

From (3.22), (3.23) and (3.24)
\[ \| \widetilde{B}^T \Delta q + \Delta B^T q \| \leq \| B \| \cdot \| \Delta q \| + \| \Delta B \| \cdot \| q \| \leq \frac{1}{240} \cdot \frac{\mu}{2r\mu + 1}. \] (3.26)

It remains to bound \( \| \Delta \tilde{y} \| \). Using Lemma 11 and (3.22) we have
\[ \sigma_{\min}(\tilde{B}) \geq \sigma_{\min}(B) - \| \Delta B \| \geq \frac{\mu}{6r} - \frac{\mu}{12r} = \frac{\mu}{12r}. \]

From (3.24) we have \( \| \widetilde{q} \| \leq \mu^{1/2} \). Observe that since \( \| \widetilde{B} \Delta \tilde{y} + \tilde{q} \| = \min_v \| \widetilde{B} v + \tilde{q} \|,
\[ \| \Delta \tilde{y} \| \leq \frac{\| \tilde{q} \|}{\sigma_{\min}(B)} \leq 12 \mu^{-1/2}. \] (3.27)

Finally, we have from (3.17), (3.25), (3.26) and (3.27)
\[ \| B^T B \Delta \tilde{y} + B^T q \| \leq \| \widetilde{B}^T \Delta B + \Delta B^T B \| \cdot \| \Delta \tilde{y} \| + \| \widetilde{B}^T \Delta q + \Delta B^T q \| \leq \frac{1}{240} \cdot \frac{\mu^{3/2}}{12r(2r\mu + 1)} \cdot 12r\mu^{-1/2} + \frac{1}{240} \cdot \frac{\mu}{2r\mu + 1} \leq \frac{\mu}{120r(2r\mu + 1)}. \]

\[ \square \]

3.4 Finite-precision analysis of termination conditions

**Lemma 15** (Dual termination) Let \( \rho_D(A) > 0 \) and \( z \in \mathcal{N}_\beta \) with \( \mu(z) \leq \frac{\rho_D(A)}{40r^2} \). Then
\[ \mathbb{1}(s_{i0} - \| s_{\tilde{y}} \|) > \mathbb{1}(6\mu(z)r) \text{ for } i \in 1 : r. \] (3.28)

Moreover, if \( z \in \mathcal{N}_\beta \) satisfies (3.28), then the subcomponent \( y \) of \( z \) is a strict feasible solution to (D); in other words, \( A^T y \prec_K 0 \).
Proof From our choice of precision \( u = \phi(\mu(z)) \) and the fact that \( z \in N_{\beta} \) it readily follows that

\[
\text{Error}(s_{i0} - \|\overline{s}_i\|) \leq r\mu(z) \tag{3.29}
\]

On the other hand, by Lemma 3 we have

\[
s_{i0} - \|\overline{s}_i\| \geq \frac{1 - \beta}{2r\sqrt{r}} \rho_D(A) \geq \frac{1 - \beta}{2r\sqrt{r}} (40r^3\mu(z)) \geq 10\mu(z)r.
\]

Thus

\[
\ell_1(s_{i0} - \|\overline{s}_i\|) \geq \ell_1(8r\mu(z)) > \ell_1(6r\mu(z)).
\]

Now assume (3.28) holds. Again, by (3.29) we get

\[
s_{i0} - \|\overline{s}_i\| \geq 5r\mu(z).
\]

Since \( \bar{A}y + \bar{s} = \bar{c} \), in particular \( A^Ty - y' + s = 0 \). So \( -A^Ty = s - y' \). Since \( \|y'\| \leq \eta \leq \tau + \eta = c^T\bar{x} - \bar{b}^T\bar{y} = 2r\mu(z) \), it follows that for \( i = 1 : r \) we have

\[
s_{i0} - y'_{i0} - \|\overline{s}_i - y_i'\| \geq s_{i0} - \|\overline{s}_i\| - y'_{i0} - \|y_i'\| \geq 5r\mu(z) - 2\|y'\| \geq r\mu(z) > 0.
\]

Therefore, \( s - y' \succ_{K} 0 \) and consequently \( A^Ty = y' - s \prec_{K} 0 \). \( \Box \)

Lemma 16 (Primal termination) Assume \( z \in N_{\beta} \). If \( \mu(z) \leq \frac{\rho_p(A)}{10r^2} \left( 1 + \frac{1}{\gamma} \right)^{-1} \) then in step (iv) the algorithm yields

\[
\ell_1 \left( \sigma_{\min} \left( H(x)^{-1/2} A^T \right) \right) \geq \ell_1 \left( \frac{3r\mu(z)}{\gamma} \right). \tag{3.30}
\]

Moreover, if \( z \in N_{\beta} \) satisfies (3.30) then the subcomponent \( x \) of \( z \) is a \( \gamma \)-forward solution of \( Ax = 0, x \geq_{K} 0 \), and

\[
\tilde{x} = x - H(x)^{-1}A^T(AH(x)^{-1}A^T)^{-1}Ax
\]

is an associated solution for \( x \).

Proof Let \( D = H(x)^{-1/2}A^T \) and assume that we compute \( \sigma_{\min}(D) \) using a backward stable algorithm (e.g., QR factorization). Then the computed \( \ell_1(\sigma_{\min}(D)) \) is the exact \( \sigma_{\min}(\ell_1(D) + E) \) for a matrix \( E \) with \( \|E\| \leq c_1n^{5/2}u\ell_1(D) \) for some universal constant \( c_1 \) (see [3, Chapter 2]). We have

\[
\text{Error}(\ell_1(\sigma_{\min}(D))) \leq \|\ell_1(D) - D\| + c_1n^2u(\|D\| + \|\ell_1(D) - D\|)
\]

\[
\leq (1 + c_1n^2u)\|\ell_1(D) - D\| + c_1n^2u\|H^{-1/2}(x)\|A\|. \tag{3.32}
\]
where the last inequality follows from Lemmas 5 and 12. By our choice of $u$ we have

$$\text{Error}(\sigma_{\min}(D)) \leq r\mu(z).$$

Therefore,

$$f_{\bar{l}}(\sigma_{\min}(D)) \geq \sigma_{\min}(D) - r\mu(z).$$

Using the bound from Lemma 2 we have

$$f_{\bar{l}}(\sigma_{\min}(D)) > 4r\mu(z) \left( \frac{\rho_P(A)}{10r^2\mu(z)} - 1 \right).$$

By our choice of $\mu(z)$ this yields

$$f_{\bar{l}}(\sigma_{\min}(D)) > \frac{4r\mu(z)}{\gamma} \geq f_{\bar{l}} \left( \frac{3r\mu(z)}{\gamma} \right).$$

Now assume (3.30) holds. Again by (3.31) we get

$$\sigma_{\min}(D) \geq f_{\bar{l}}(\sigma_{\min}(D)) - r\mu(z) \geq \frac{2r\mu(z)}{\gamma}.$$

Denote $\Delta x = -H(x)^{-1}A^T(AH(x)^{-1}A^T)^{-1}Ax$. From Lemma 4 and $\|Ax\| = \|x''\|$ we have

$$\|\Delta x\|_x^2 = (Ax)^T(AH(x)^{-1}A^T)^{-1}Ax \leq \frac{\|x''\|_x^2}{\sigma_{\min}(D)^2} \leq \frac{(2r\mu(z))^2}{(2r\mu(z)/\gamma)^2} \leq \gamma^2.$$

Since $x >_K 0$ and $\gamma < 1$, Proposition 1(a) implies that $x + \Delta x >_K 0$.

Furthermore, by Proposition 1(a,c) we have

$$\frac{\|\Delta x\|}{\|x\|} \leq \frac{\|\Delta x\|}{\|H(x)^{-1/2}\|} \leq \|H(x)^{1/2}\Delta x\| = \|\Delta x\|_x \leq \gamma < 1.$$

Therefore, $\bar{x} = x + \Delta x$ is a $\gamma$-forward solution of $Ax = 0, x \geq_K 0$. \qed

4 Proof of the main result

We are finally in a position to prove our main result (Theorem 1). We first prove that on every step the algorithm keeps up with the central path, and at the same time the value of $\bar{\mu}$ decreases by a fixed factor. Then we show that once $\mu$ is small enough to satisfy either dual or primal termination conditions (Steps (iii) and (iv) of the algorithm),
the algorithm terminates and yields a correct answer. Then the bound on the number of iterations follows trivially from the termination bounds on $\mu$ and the factor of the decrease of $\mu$. Similarly we obtain bounds for the finest precision based on the precision update function $\phi$ and the aforementioned bounds on $\mu$.

Before we go ahead with the proof, we need to guarantee that the initial point satisfies all the necessary bounds. The following result is an immediate consequence of [22, Proposition 4.6].

**Lemma 17** (Computation of the initial point) The initial point

$$z := \left( \alpha e, 1, \alpha e, 2M, -\alpha Ae, 0, \frac{M}{\alpha} e, -\frac{M}{\alpha^2}, \frac{M}{\alpha} e, -\frac{M}{\alpha} e, 1, 0 \right)$$

where $\alpha = \frac{1}{\sqrt{2}r}$ and $M = \frac{\alpha \|Ae\|}{\beta}$, satisfies $z \in \mathcal{N}_\beta$ and $\mu(z) = \frac{\alpha \|Ae\|}{\beta} = O(1)$. \hfill \Box

**Proof of Theorem 1** We first disregard the halting steps (iii) and (iv) of the algorithm and prove that, no matter how many iterations we have performed, all our iterates stay close enough to the central path. We use an induction argument.

The induction base is given by Lemma 17. We now assume that at the start of Step (ii) of the algorithm the value of $z$ satisfies $z \in \mathcal{N}_\beta$. We need to show that the vector $z^+$ computed in step (vi) is also in $\mathcal{N}_\beta$.

From Lemma 14 it follows that the point $z^+$ computed with finite precision in (a), and infinite precision in (b) and (c), satisfies

$$\mathcal{A}^T \Delta \bar{x} = 0$$
$$\mathcal{A}^T \Delta \bar{y} + \Delta \bar{s} = 0$$
$$\Delta \bar{x} + H(w)^{-1} \Delta \bar{s} = -(\mu g(\bar{s}) + \bar{x}) + \varrho,$$

for some $\varrho$ with $\|\varrho\| \leq \frac{\mu(z)}{120r^2(1+\mu(z))}$. Hence from Lemma 1 we have $z^+ \in \mathcal{N}_\beta$. Hence, $z \in \mathcal{N}_\beta$ on every iteration.

Now we show the bounds for the number of iterations and the finest precision.

From Lemma 16 we know that once $\mu(z)$ reaches the lower bound of $\frac{\rho_p(A)}{10r^2} \left( 1 + \frac{1}{\gamma} \right)^{-1}$, the algorithm yields a correct $\gamma$-approximate solution to the primal problem.

It follows from Lemma 1 that

$$\mu(z^+) \leq \mu + \frac{\mu(z)}{120r^2} = \left( 1 - \frac{\delta}{\sqrt{2}r} + \frac{1}{120r^2} \right) \mu(z).$$

Therefore, taking into account that $r^2 \geq \sqrt{2r} \cdot 3 \sqrt[3]{\frac{\gamma}{2}} > 3\sqrt{2r}$ and that $\delta = 1/45$, we have

$$\mu(z^+) < \left( 1 - \frac{1}{60\sqrt{2r}} \right) \mu(z).$$
Given an initial value of $\mu(z_0)$, after $k$ iterations we have

$$
\mu(z_k) \leq \left( 1 - \frac{1}{60\sqrt{2}r} \right)^k \mu(z_0).
$$

Since we want $\mu(z) \leq \frac{\rho P(A)}{10r^2} \left( 1 + \frac{1}{\gamma} \right)^{-1}$, we have the condition

$$
\left( 1 - \frac{1}{60\sqrt{2}r} \right)^k \mu(z_0) \leq \frac{\rho P(A)}{10r^2} \left( 1 + \frac{1}{\gamma} \right)^{-1}.
$$

By taking logarithms on both sides, and using $\mu(z_0) = O(1)$, we get the desired relation

$$
k = O \left( \frac{1}{r^{1/2}} (\log(r) + \log(C(A)) + |\log \gamma|) \right).
$$

Since the algorithm halts once $\mu(z) \leq \frac{\rho P(A)}{40r^2} \left( 1 + \frac{1}{\gamma} \right)^{-1}$, we deduce from our precision update formula (3.2) that the finest required precision $u^*$ satisfies

$$
u^* \geq \left( c(m + n)^{5/2} r^{8} C(A)^{7/2} \left( 1 + \frac{1}{\gamma} \right)^{7/2} \right)^{-1}.
$$

Similarly, for the dual feasible case from Lemma 15 we know that $\mu(z) \leq \frac{\rho P(A)}{40r^2}$ guarantees successful termination of the algorithm. Hence we similarly get the bound

$$
k = O \left( \frac{1}{r^{1/2}} (\log(r) + \log(C(A))) \right),
$$

and for the finest precision we have

$$
u^* \geq \left( c(m + n)^{5/2} r^{11.5} C(A)^{7/2} \right)^{-1}.
$$

□

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