Ricci curvatures on Hermitian manifolds

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Abstract. In this paper, we introduce the first Aeppli-Chern class for complex manifolds and show that the $(1,1)$-component of the curvature 2-form of the Levi-Civita connection on the anti-canonical line bundle represents this class. We also derive curvature relations on Hermitian manifolds and the background Riemannian manifolds. Moreover, we study non-Kähler Calabi-Yau manifolds by using the first Aeppli-Chern class and the Levi-Civita Ricci-flat metrics. In particular, we construct explicit Levi-Civita Ricci-flat metrics on Hopf manifolds $S^{2n-1} \times S^1$. We also construct a smooth family of Hermitian metrics on general Hopf manifolds, such that their Riemannian scalar curvature are constant and vary smoothly between negative infinity and a positive number. In particular, it shows that Hermitian manifolds with nonnegative first Chern class can admit Hermitian metrics with strictly negative Riemannian scalar curvature.

Contents

1. Introduction 2
2. Background materials 8
  2.1. Almost Hermitian manifolds 8
  2.2. Ricci curvature on almost Hermitian manifolds 9
  2.3. Curvatures on Hermitian manifolds 11
3. Geometry of the Levi-Civita Ricci curvature $R^{(1)}$ 12
  3.1. The Chern connection and Levi-Civita connection on $(T^{1,0}M, h)$ 12
  3.2. Elementary computations on Hermitian manifolds 14
  3.3. Geometry of the first Levi-Civita Ricci curvature 15
  3.4. Hermitian manifolds with nonnegative $R^{(1)}$ 18
  3.5. Hypothetical complex structures on $S^6$ 20
4. Curvature relations on Hermitian manifolds 20
  4.1. Ricci curvature relations 20
  4.2. Scalar curvature relations 23
5. Special metrics on Hermitian manifolds 24
6. Levi-Civita Ricci-flat and constant negative scalar curvature metrics on Hopf manifolds 27
  6.1. Levi-Civita Ricci-flat metrics on Hopf manifolds 27
  6.2. Hermitian metrics of constant negative scalar curvatures on Hopf manifolds 29
  6.3. Pluriclosed metrics on the projective bundles over Hopf manifolds 30
7. Appendix: the Riemannian Ricci curvature and $\ast$-Ricci curvature 31
References 34
1. Introduction

In this paper, we study the relationship between Riemannian geometry and Hermitian geometry. More precisely, we investigate the curvature relations on Hermitian manifolds and their background Riemannian manifolds. In particular, we study non-Kähler Calabi-Yau manifolds by different Ricci-flat metrics.

Let \((M, h)\) be a Hermitian manifold and \(g\) the background Riemannian metric. It is well-known that, when \((M, h)\) is not Kähler, the complexification of the real curvature tensor \(R\) is extremely complicated. Moreover, on the Hermitian holomorphic vector bundle \((T^{1,0}M, h)\), there are two typical connections: the Levi-Civita connection and the Chern connection. The curvature tensors of them are denoted by \(R\) and \(\Theta\) respectively. It is well-known that the complexified Riemannian curvature tensor \(R\) of \((M, h)\) is well-known that, when \((M, h)\) is well-known that, the complexified Riemannian curvature tensor \(R\) is closely related to the Riemannian geometry of \(M\), and \(\Theta\) can characterize many complex geometric properties of \(M\) whereas \(R\) can be viewed as a bridge between \(R\) and \(\Theta\), i.e. a bridge between Riemannian geometry and Hermitian geometry.

Let \(\{z^i\}_{i=1}^n\) be the local holomorphic coordinates centered at a point \(p \in M\). The Chern curvature \(\Theta\) is an \(End(T^{1,0}M)\)-valued \((1,1)\)-form, i.e. \(\Theta \in \Gamma(M, \Lambda^{1,1}T^*M \otimes End(T^{1,0}M))\) and \(R \in \Gamma(M, \Lambda^2T^*M \otimes End(T^{1,0}M))\). However, \(R \in \Gamma(M, \Lambda^2T^*M \otimes End(TM))\). Therefore, we can compare them when restricted on the subspace \(\Gamma(M, \Lambda^{1,1}T^*M \otimes End(T^{1,0}M))\), that is, we can find relations between \(R_{ij\bar{k}\ell}^{(1)}\), \(R_{ij\bar{k}\ell}^{(2)}\) and \(\Theta_{ij\bar{k}\ell}^{(1)}\). The significant difference is that the symmetry \(R_{ij\bar{k}\ell} = R_{k\bar{i}j\ell}\) does not hold for \(R\) or \(\Theta\). We denoted by

\[
R^{(1)} = \sqrt{-1}R_{ij}^{(1)}dz^i \wedge d\bar{z}^j \quad \text{with} \quad R^{(1)}_{ij} = h^{k\bar{l}}R_{ij\bar{k}\ell},
\]

and

\[
R^{(2)} = \sqrt{-1}R_{ij}^{(2)}dz^i \wedge d\bar{z}^j \quad \text{with} \quad R^{(2)}_{ij} = h^{k\bar{l}}R_{ij\bar{k}\ell}.
\]

\(R^{(1)}\) and \(R^{(2)}\) are called the first Levi-Civita Ricci curvature and the second Levi-Civita Ricci curvature of \((T^{1,0}M, h)\) respectively. Similarly, we can define the first Chern-Ricci curvature \(\Theta^{(1)}\) and the second Chern-Ricci curvature \(\Theta^{(2)}\). As shown in [39], \(R^{(2)}\) and \(\Theta^{(2)}\) are closely related to the geometry of \(M\), for example, we can use it to study the cohomology groups and plurigenera of compact Hermitian manifolds. On the other hand, it is well-known that \(\Theta^{(1)}\) represents the first Chern class \(c_1(M) \in H^{2,1}_\partial(M)\). However, in general, the first Levi-Civita Ricci form \(R^{(1)}\) is not \(d\)-closed, and so it can not represent a class in \(H^{1,1}_\partial(M)\).

We introduce two cohomologies to study the geometry of compact complex (especially, non-Kähler) manifolds, the Bott-Chern cohomology and the Aeppli cohomology:

\[
H^{p,q}_{BC}(M) := \frac{\text{Ker}d \cap \Omega^{p,q}(M)}{\text{Im}\partial d \cap \Omega^{p,q}(M)} \quad \text{and} \quad H^{p,q}_{A}(M) := \frac{\text{Ker}\overline{\partial} \cap \Omega^{p,q}(M)}{\text{Im}\partial \cap \Omega^{p,q}(M) + \text{Im}\overline{\partial} \cap \Omega^{p,q}(M)}.
\]

Let \(\text{Pic}(M)\) be the set of holomorphic line bundles over \(M\). As similar as the first Chern class map \(c_1: \text{Pic}(M) \to H^{1,1}_\partial(M)\), there is a first Aeppli-Chern class map

\[
c_1^{AC}: \text{Pic}(M) \to H^{1,1}_{A}(M),
\]
which can be described as follows. Given any holomorphic line bundle \( L \to M \) and any Hermitian metric \( h \) on \( L \), its curvature form \( \Theta_h \) is locally given by \(-\sqrt{-1}\partial\bar{\partial}\log h\). Then \( \Theta_h \) is a \( \partial\bar{\partial} \)-closed real \((1,1)\)-form and if we choose a different metric \( h' \) then \( \Theta_{h'} - \Theta_h = \sqrt{-1}\partial\bar{\partial}\log \left( \frac{h'}{h} \right) \) is globally \( \partial\bar{\partial} \)-exact, so we can defined \( c_1^{AC}(L) \) to be the class of \( \Theta_h \) in \( H^{1,1}_{A}(M) \) (modulo a constant \( 2\pi \)). For a complex manifold \( M \), \( c_1^{AC}(M) \) is defined to be \( c_1^{AC}(K_M^{-1}) \) where \( K_M^{-1} \) is the anti-canonical line bundle \( \wedge^n T^{1,0} M \). We can define the **first Bott-Chern class** map \( c_1^{BC} : \text{Pic}(M) \to H^{1,1}_{BC}(M) \) in a similar way.

In this paper, we study non-Kähler Hermitian manifolds by using the first Aeppli-Chern class \( c_1^{AC}(M) \) and the first Levi-Civita Ricci form \( \mathcal{R}^{(1)} \). At first we observe that the first Levi-Civita Ricci form \( \mathcal{R}^{(1)} \) represents the first Aeppli-Chern class \( c_1^{AC}(M) \). More precisely,

**Theorem 1.1.** Let \((M,\omega)\) be a compact Hermitian manifold, then we have

\[
\mathcal{R}^{(1)} = \Theta^{(1)} - \frac{1}{2}(\partial\bar{\partial}^*\omega + \bar{\partial}\partial^*\omega).
\]

In particular, \( \mathcal{R}^{(1)} \) represents the first Aeppli-Chern class \( c_1^{AC}(M) \in H^{1,1}_{A}(M) \). Moreover,

1. \( \mathcal{R}^{(1)} \) is d-closed if and only if \( \partial\bar{\partial}^*\omega = 0 \);
2. if \( \partial\bar{\partial}^*\omega = 0 \), then \( \mathcal{R}^{(1)} \) represents the real first Chern class \( c_1(M) \in H^2_{dR}(M) \);
3. if \( \omega \) is conformally balanced, then \( \mathcal{R}^{(1)} \) represents the first Chern class \( c_1(M) \in H^1_{\Omega}(M) \) and also the first Bott-Chern class \( c_1^{BC}(M) \in H^{1,1}_{BC}(M) \);
4. \( \mathcal{R}^{(1)} = \Theta^{(1)} \) if and only if \( d^*\omega = 0 \), i.e. \((M,\omega)\) is a balanced manifold.

We also show that, on complex manifolds supporting \( \partial\bar{\partial} \)-lemma (e.g. manifolds in Fujiki class \( \mathcal{C} \) and in particular, Moishezon manifolds), the converse of (2) and (3) hold. Note also that the \( \partial\bar{\partial} \)-lemma is necessary for the converse statement of (3).

It is obvious that, on a compact Kähler manifold \((M,\omega)\), \( c_1(M) = c_1^{BC}(M) = c_1^{AC}(M) \) and the first Chern class plays a key role in Kähler geometry. In particular, Yau’s celebrated solution to Calabi’s conjecture is a fundamental tool to study Kähler geometry.

**Theorem 1.2** ([50]). Let \((M,\omega)\) be a compact Kähler manifold. If the real \((1,1)\) form \( \eta \) represents the first Chern class \( c_1(M) \), then there exists a smooth function \( \varphi \in C^\infty(M) \) such that the Kähler metric \( \tilde{\omega} = \omega + \sqrt{-1}\partial\bar{\partial}\varphi \) has Ricci curvature \( \eta \), i.e.

\[
\text{Ric}(\tilde{\omega}) = \eta.
\]

Recently, Tosatti and Weinkove obtained a Hermitian analogue of Yau’s fundamental result:

**Theorem 1.3** ([50]). Let \((M,\omega)\) be a compact Hermitian manifold. If the real \((1,1)\) form \( \eta \) represents the first Bott-Chern class \( c_1^{BC}(M) \), then there exists a smooth function \( \varphi \in C^\infty(M) \) such that the Hermitian metric \( \tilde{\omega} = \omega + \sqrt{-1}\partial\bar{\partial}\varphi \) has Chern-Ricci curvature \( \eta \), i.e.

\[
\Theta^{(1)}(\tilde{\omega}) = \eta.
\]
There is an important class of manifolds, so called Calabi-Yau manifolds which are extensively studied by mathematicians and also physicians. In this paper, a Calabi-Yau manifold is a complex manifold with $c_1(M) = 0$ and we will focus on non-Kähler Calabi-Yau manifolds. It is obvious that if $c_1^{BC}(M) = 0$, then $c_1(M) = 0$. (The converse is not valid in general.) There are many nice results on non-Kähler Hermitian manifolds with vanishing first Bott-Chern class. They are always characterized by using the first Chern Ricci curvature $Θ^{(1)}$ and also the related Monge-Ampère type equations (e.g. [17, 24, 50, 51, 52, 53, 54]). For more details on this subject, we refer the reader to the nice survey paper [49].

Firstly, we make the following simple observation:

**Corollary 1.4.** Let $M$ be a complex manifold. If $c_1(M) = 0$, then $c_1^{AC}(M) = 0$.

That means, it is very natural to study non-Kähler Calabi-Yau manifolds by using the first Aeppli-Chern class and the first Levi-Civita Ricci curvature $R^{(1)}$. A Hermitian metric $ω$ on $M$ is called Levi-Civita Ricci-flat if $R^{(1)}(ω) = 0$. It is well-known that Hopf manifolds $M = S^{2n-1} \times S^1$ are all non-Kähler Calabi-Yau manifolds ($n \geq 2$), i.e. $c_1(M) = 0$; however, there is no Chern Ricci-flat Hermitian metrics on $M$. On the contrary, we can construct explicit Levi-Civita Ricci-flat metrics on them.

**Theorem 1.5.** Let $M = S^{2n-1} \times S^1$ with $n \geq 2$ and $ω_0$ the canonical metric on $M$.

1. The perturbed Hermitian metric

   $ω = ω_0 - \frac{4}{n} R^{(1)}(ω_0)$

   is Levi-Civita Ricci-flat, i.e. $R^{(1)}(ω) = 0$.

2. Moreover, the smooth family of Hermitian metrics

   $ω_λ := ω_0 + 4λ R^{(1)}(ω_0)$, $λ > −1$,

   has constant Riemannian scalar curvature

   $s_λ = \frac{n(n-1)}{2(1+λ)^2} \left[ λ - 1 - 2n \right]$, $n \geq 2$.

   which varies between $−∞$ and $\frac{n^2(n-1)}{4}$. In particular, there exist Hermitian manifolds with $c_1 \geq 0$ which can admit Hermitian metrics with negative constant Riemannian scalar curvature.

Note that, by a recent result of Lebrun ([34]), one can see that the background Riemannian metric of the Levi-Civita Ricci-flat metric on $S^3 \times S^1$ is not Einstein. To our knowledge, this is the first example to show that Hermitian manifolds with nonnegative first Chern class can admit Hermitian metrics with strictly negative Riemannian scalar curvature.

As motivated by the celebrated Theorem 1.2 and Theorem 1.3, the following question is interesting.

**Problem 1.6.** Let $M$ be a compact complex manifold. For a fixed Hermitian metric $ω_0$ on $M$, and a real $(1, 1)$-form $η$ representing $c_1(M)$, does there exist a $(0, 1)$-form $γ$ such that the Hermitian metric $ω = ω_0 + ∂γ + \overline{∂γ}$ has

   $R^{(1)}(ω) = η$?
Or, in particular, if $c_1(M) = 0$, does there exist a $(0, 1)$-form $\gamma$ such that the Hermitian metric $\omega = \omega_0 + \partial\gamma + \overline{\partial}\gamma$ is Levi-Civita Ricci-flat, i.e. $\mathfrak{R}^{(1)}(\omega) = 0$?

The first Levi-Civita Ricci curvature $\mathfrak{R}^{(1)}$ is also closely related to possible symplectic structures on Hermitian manifolds. It is easy to see that, on a Hermitian manifold $(M, \omega)$, $\mathfrak{R}^{(1)}$ is the $(1, 1)$-component of the curvature 2-form of the Levi-Civita connection on $K^{-1}_M$. Hence, if $\mathfrak{R}^{(1)}$ is strictly positive, it can induce a symplectic structure on $M$ (see Theorem 3.15 and also [35]). Moreover, the symplectic structures thus obtained are not necessarily Kähler.

As applications of Theorem 1.1, we can characterize Hermitian manifolds by the first Levi-Civita Ricci curvature $\mathfrak{R}^{(1)}$.

**Theorem 1.7.** Let $(M, h)$ be a compact Hermitian manifold. If the first Levi-Civita Ricci curvature $\mathfrak{R}^{(1)}$ is quasi-positive, then $c_1^A(M) > 0$ and the Kodaira dimension $\kappa(M) = -\infty$. In particular, $H^2_{dR}(M), H^{1,1}_{BC}(M)$ and $H^1_A(M)$ are all non-zero.

If $\dim \mathbb{C}M = 2$, by the Enriques-Kodaira classification of compact complex surfaces, it is easy to see that if $c_1^A(M) > 0$ and $\kappa(M) = -\infty$, $M$ is birational to a minimal rational surface (e.g. [62, p.151]), and in particular, $M$ is Kähler. On the other hand, on a projective manifold $M$, $c_1(M)$ and $c_1^{AC}(M)$ are “numerically” equivalent, i.e. for any irreducible curve $\gamma$ in $M$,

$$c_1^{AC}(M) \cdot \gamma = c_1(M) \cdot \gamma.$$ 

For example, if $\mathfrak{R}^{(1)}$ is semi-positive, we can see from (1.2) that the anti-canonical line bundle $K^{-1}_M$ is numerically effective. Moreover, if $\mathfrak{R}^{(1)}$ is quasi-positive, then $K^{-1}_M$ is a big line bundle. It is interesting to find a higher dimensional non-Kähler manifold with positive first Levi-Civita Ricci curvature $\mathfrak{R}^{(1)}$.

**Question 1.8.** On a Riemannian manifold $(M, g)$, which kinds of Riemannian curvature conditions on $g$ can imply the quasi-positivity of $\mathfrak{R}^{(1)}$.

It is easy to see that if the Hermitian manifold $(M, g)$ has positive constant Riemannian sectional curvature, then $\mathfrak{R}^{(1)}$ is positive. It is also hopeful that weakly quarter pinched Riemannian sectional curvature could imply the quasi-positivity of $\mathfrak{R}^{(1)}$. On the other hand, since the positivity condition is an open condition, $\mathfrak{R}^{(1)}$ is still positive in a small neighborhood of a positive constant sectional curvature metric. As an application of this simple observation, one can see the following result of Lebrun which is also observed in [5] and [48].

**Corollary 1.9 ([33]).** On $\mathbb{S}^6$, there is no orthogonal complex structure compatible with metrics in some small neighborhood of the round metric.

Next, we investigate the relations between various Ricci curvatures on Hermitian manifolds. As introduced above, on a Hermitian manifold $(M, \omega)$ there are six different types of Ricci curvatures:

1. the $(1, 1)$-component of the complexified Riemannian Ricci curvature, $\mathfrak{Ric}$;
2. the Hermitian Ricci curvature $\text{Ric}_H = \sqrt{-1} R_{\overline{\gamma}} dz^i \wedge d\overline{z}^j$ where $R_{\overline{\gamma}} = h_{\overline{\gamma}} R_{\overline{\gamma}}$;
3. the first Levi-Civita Ricci curvature $\mathfrak{R}^{(1)}$;
the second Levi-Civita Ricci curvature $R^{(2)}$;
(5) the first Chern Ricci curvature $\Theta^{(1)}$;
(6) the second Chern Ricci curvature $\Theta^{(2)}$.

If $(M, \omega)$ is Kähler, all Ricci curvatures are the same, but it is not true on general Hermitian manifolds. We shall explore explicit relations between them by using the Hermitian metric $\omega$ and its torsion $T$. We write them down with a reference curvature, e.g. $\Theta^{(1)}$, to the reader’s convenience.

**Theorem 1.10.** Let $(M, \omega)$ be a compact Hermitian manifold.

1. The first Levi-Civita Ricci curvature is
$$R^{(1)} = \Theta^{(1)} - \frac{1}{2}(\partial\partial^* \omega + \overline{\partial}\overline{\partial}^* \omega),$$
and the second Levi-Civita Ricci curvature is
$$R^{(2)} = \Theta^{(1)} - \frac{1}{2}(\partial\partial^* \omega + \overline{\partial}\overline{\partial}^* \omega) - \sqrt{-1} T \circ T + \sqrt{-1} T \boxdot T.$$

2. The second Chern-Ricci curvature can be written as
$$\Theta^{(2)} = \Theta^{(1)} - \sqrt{-1} \Lambda (\partial\overline{\partial} \omega) - (\partial\partial^* \omega + \overline{\partial}\overline{\partial}^* \omega) + \sqrt{-1} \left( T \circ T + 3T \boxdot T \right);$$

3. The Hermitian Ricci curvature is
$$\text{Ric}_H = \Theta^{(1)} - \frac{1}{2}(\partial\partial^* \omega + \overline{\partial}\overline{\partial}^* \omega) - \sqrt{-1} T \circ T.$$

4. The $(1,1)$-component of the Riemannian Ricci curvature is
$$\text{Ric} = \Theta^{(1)} - \sqrt{-1} (\Lambda \partial\overline{\partial} \omega) - \frac{1}{2}(\partial\partial^* \omega + \overline{\partial}\overline{\partial}^* \omega) + \sqrt{-1} \left( 2T \boxdot T + T \circ T \right)$$
$$+ \frac{1}{2} \left( T([\partial^* \omega]^*) + T([\partial^* \omega]^*) \right),$$
where $(\partial^* \omega)^*$ is the dual vector of the $(0,1)$-form $\partial^* \omega$.

From these curvature relations, one can see clearly the geometry of many Hermitian manifolds with special metrics (e.g. $d^* \omega = 0$, $\partial\overline{\partial} \omega = 0$). In particular, these curvature relations may enlighten the study of various Hermitian Ricci flows (e.g. [45, 46, 47], [39], [24, 51, 52, 55, 25, 26]) by using the well-studied Hamilton’s Ricci flow. Of course, it is also natural to define new Hermitian Ricci flows by certain Ricci curvatures with significant geometric meanings, for example, the first Levi-Civita Ricci curvature $R^{(1)}$.

As straightforward consequences, from the Ricci curvature relations, we can also obtain relations between the corresponding scalar curvatures. It is also well-known that the positive scalar curvature can characterize the geometry of the manifolds. In [60], Yau proved that, on a compact Kähler manifold $(M, \omega)$, if the total scalar curvature is positive, then all plurigenera $p_m(M)$ vanish, and so the Kodaira dimension of $M$ is $-\infty$. Based on Yau’s result, Heier-Wong ([29]) observed that on a projective manifold, if the total scalar curvature is positive, the manifold is uniruled, i.e. it is covered by rational curves. On a Hermitian manifold $M$, Gauduchon showed ([22]) that if the total Chern scalar curvature
is positive, then $p_m(M) = 0$ and $\kappa(M) = -\infty$. On the other hand, the Riemannian scalar curvature on Riemannian manifolds is extensively studied. In particular, by Trudinger, Aubin and Schoen’s solution to the Yamabe problem, it is well-known that every Riemannian metric is conformal to a metric with constant scalar curvature. To understand the relations between Riemannian geometry and Hermitian geometry, the following relation is of particular interest.

**Corollary 1.11.** On a compact Hermitian manifold $(M, \omega)$, the Riemannian scalar curvature $s$ and the Chern scalar curvature $s_C$ are related by

$$s = 2s_C + \left( \langle \partial \bar{\partial}^* \omega + \bar{\partial} \partial^* \omega, \omega \rangle - 2|\partial^* \omega|^2 \right) - \frac{1}{2} |T|^2. \tag{1.5}$$

Moreover, according to the different types of Ricci curvatures, there are five different scalar curvatures and the following statements are equivalent:

1. $(M, \omega)$ is Kähler;
2. $\int s \cdot \omega^n = \int 2s_C \cdot \omega^n$;
3. $\int s_C \cdot \omega^n = \int s_R \cdot \omega^n$;
4. $\int s_C \cdot \omega^n = \int s_H \cdot \omega^n$;
5. $\int s_H \cdot \omega^n = \int s_{LC} \cdot \omega^n$.

A similar formulation as (1.5) by using “Lee forms” is also observed by Gauduchon ([23]). See also [56, 57, 3, 9, 10, 1, 31, 21, 4, 40, 32] for some curvature relations on complex surfaces. For a complete list of various scalar curvature relations, see Corollary 4.4, Corollary 4.5 and Corollary 4.6.

Finally, we study special metrics on Hermitian manifolds. At first, we observe the following identity on general compact Hermitian manifolds.

**Proposition 1.12.** On a compact Hermitian manifold $(M, \omega)$, for any $1 \leq k \leq n - 1$, we have

$$\int \sqrt{-1} \partial \omega \wedge \bar{\partial} \omega \cdot \frac{\omega^{n-3}}{(n-3)!} = \|\partial^* \omega\|^2 - \|\partial \omega\|^2, \tag{1.6}$$

and

$$\int \sqrt{-1} \omega^{n-k-1} \wedge \partial \bar{\partial} \omega^k = (n-3)!k(n-k-1) \left( \|\partial \omega\|^2 - \|\partial^* \omega\|^2 \right). \tag{1.7}$$

The form

$$\sqrt{-1} \omega^{n-k-1} \wedge \partial \bar{\partial} \omega^k$$

was firstly introduced in [17] by Fu-Wang-Wu to define a generalized Gauduchon’s metric. More precisely, a metric $\omega$ satisfying $\sqrt{-1} \omega^{n-k-1} \wedge \partial \bar{\partial} \omega^k = 0$ for $1 \leq k \leq n - 1$ is called a $k$-Gauduchon metric. The $(n-1)$-Gauduchon metric is the original Gauduchon metric. It is well-known that, the Hopf manifold $S^{2n+1} \times S^1$ can not support a metric with $\partial \bar{\partial} \omega = 0$ (SKT) or $d^* \omega = 0$ (balanced metric). However, they showed in [17] that on $S^5 \times S^1$, there exists a 1-Gauduchon metric $\omega$, i.e. $\omega \wedge \partial \bar{\partial} \omega = 0$.  

A straightforward application of Proposition 1.12 is the following interesting fact:

**Corollary 1.13.** If \((M, \omega)\) is \(k\)-Gauduchon for \(1 \leq k \leq n - 2\) and also balanced, then \((M, \omega)\) is Kähler.

One can also get the following analogue in the “conformal” setting:

**Corollary 1.14.** On a compact complex manifold, the following are equivalent:

1. \((M, \omega)\) is conformally Kähler;
2. \((M, \omega)\) is conformally \(k\)-Gauduchon for \(1 \leq k \leq n - 2\), and conformally balanced.

In particular, the following are equivalent:

1. \((M, \omega)\) is Kähler;
2. \((M, \omega)\) is \(k\)-Gauduchon for \(1 \leq k \leq n - 2\), and conformally balanced;
3. \((M, \omega)\) is conformally balanced and \(\Lambda^2(\partial \bar{\partial} \omega) = 0\).

We need to point out that the equivalence of (3) and (4) is also proved by Ivanov and Papadopoulos in [32, Theorem 1.3] (see also [11, Proposition 2.4]). However, in [18], Fu-Wang-Wu proved that there exists a non-Kähler 3-fold which can support a 1-Gauduchon metric and a balanced metric simultaneously. By Corollary 1.13, they must be different Hermitian metrics. For more works on Hermitian manifolds with specials metrics, we refer the reader to [1, 2, 4, 9, 10, 12, 13, 14, 15, 16, 17, 18, 19, 21, 36, 39, 38, 41, 49, 42, 43, 44] and the references therein.

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2. Background materials

2.1. Almost Hermitian manifolds. Let \((M, g, \nabla)\) be a \(2n\)-dimensional Riemannian manifold with Levi-Civita connection \(\nabla\). The tangent bundle of \(M\) is denoted by \(T_R M\). The curvature tensor of \((M, g, \nabla)\) is defined as

\[
R(X, Y, Z, W) = g\left(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z,Z,W\right)
\]

for any \(X, Y, Z, W \in T_R M\). Let \(T_C M = T_R M \otimes \mathbb{C}\) be the complexification of the tangent bundle \(T_R M\). We can extend the metric \(g\), the Levi-Civita connection \(\nabla\) to \(T_C M\) in the \(\mathbb{C}\)-linear way. For instance, for any \(a, b \in \mathbb{C}\) and \(X, Y \in T_C M\),

\[
g(aX, bY) := ab \cdot g(X, Y).
\]

Hence for any \(a, b, c, d \in \mathbb{C}\) and \(X, Y, Z, W \in T_C M\),

\[
R(aX, bY, cZ, dW) = abcd \cdot R(X, Y, Z, W).
\]

Let \((M, g, J)\) be an almost Hermitian manifold, i.e., \(J : T_R M \to T_R M\) with \(J^2 = -1\), and for any \(X, Y \in T_R M\), \(g(JX, JY) = g(X, Y)\). We can also extend \(J\) to \(T_C M\) in the \(\mathbb{C}\)-linear way. Hence for any \(X, Y \in T_C M\), we still have

\[
g(JX, JY) = g(X, Y).
\]
2.2. Ricci curvature on almost Hermitian manifolds. Let \( \{x^1, \cdots, x^n, x^{n+1}, \cdots, x^n\} \) be the local real coordinates on the almost Hermitian manifold \((M, J, g)\). In order to use Einstein summations, we use the following convention:

\[
\begin{align*}
\{x^i\} & \text{ for } 1 \leq i \leq n; & \{x^I\} & \text{ for } n + 1 \leq I \leq 2n \text{ and } I = i + n.
\end{align*}
\]

Moreover, we assume,

\[
\begin{align*}
J \left( \frac{\partial}{\partial x^i} \right) &= \frac{\partial}{\partial x^I} \quad \text{and} \quad J \left( \frac{\partial}{\partial x^I} \right) = -\frac{\partial}{\partial x^i}.
\end{align*}
\]

By using real coordinates \( \{x^i, x^I\} \), the Riemannian metric is represented by

\[
ds_g^2 = g_{i\ell} \, dx^i \otimes dx^\ell + g_{i\ell} \, dx^i \otimes dx^L + g_{I\ell} \, dx^I \otimes dx^\ell + g_{IJ} \, dx^I \otimes dx^J,
\]

where the metric components \( g_{i\ell}, g_{i\ell}, g_{I\ell} \) and \( g_{IJ} \) are defined in the obvious way by using \( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^\ell}, \frac{\partial}{\partial x^I}, \frac{\partial}{\partial x^L} \). By the \( J \)-invariant property of the metric \( g \), we have

\[
g_{i\ell} = g_{IL}, \quad \text{and} \quad g_{i\ell} = g_{Li} = -g_{\ell I} = -g_{\ell I}.
\]

We also use complex coordinates \( \{z^i, \bar{z}^i\}_{i=1}^n \) on \( M \):

\[
\begin{align*}
z^i & := x^i + \sqrt{-1} x^I, & \bar{z}^i & := x^i - \sqrt{-1} x^I.
\end{align*}
\]

(Note that if the almost complex structure \( J \) is integrable, \( \{z^i\}_{i=1}^n \) are the local holomorphic coordinates.) We define, for \( 1 \leq i \leq n \),

\[
\begin{align*}
dz^i & := dx^i + \sqrt{-1} dx^I, & d\bar{z}^i & := dx^i - \sqrt{-1} dx^I
\end{align*}
\]

and

\[
\begin{align*}
\frac{\partial}{\partial z^i} := \frac{1}{2} \left( \frac{\partial}{\partial x^i} - \sqrt{-1} \frac{\partial}{\partial x^I} \right), & \quad \frac{\partial}{\partial \bar{z}^i} := \frac{1}{2} \left( \frac{\partial}{\partial x^i} + \sqrt{-1} \frac{\partial}{\partial x^I} \right).
\end{align*}
\]

Therefore,

\[
\begin{align*}
\frac{\partial}{\partial x^i} &= \frac{\partial}{\partial z^i} + \frac{\partial}{\partial \bar{z}^i}, & \frac{\partial}{\partial x^I} &= \sqrt{-1} \left( \frac{\partial}{\partial z^i} - \frac{\partial}{\partial \bar{z}^i} \right).
\end{align*}
\]

By the \( \mathbb{C} \)-linear extension,

\[
\begin{align*}
J \left( \frac{\partial}{\partial z^i} \right) &= \sqrt{-1} \frac{\partial}{\partial z^i}, & J \left( \frac{\partial}{\partial \bar{z}^i} \right) &= -\sqrt{-1} \frac{\partial}{\partial \bar{z}^i}.
\end{align*}
\]

Let’s define a Hermitian form \( h : T_C M \times T_C M \to \mathbb{C} \) by

\[
(2.13) \quad h(X, Y) := g(X, Y), \quad X, Y \in T_C M.
\]

By \( J \)-invariant property of \( g \),

\[
(2.14) \quad h_{ij} := h \left( \frac{\partial}{\partial z^j}, \frac{\partial}{\partial z^i} \right) = 0, \quad \text{and} \quad h_{IJ} := h \left( \frac{\partial}{\partial \bar{z}^j}, \frac{\partial}{\partial \bar{z}^i} \right) = 0
\]

and

\[
(2.15) \quad h_{ij} := h \left( \frac{\partial}{\partial z^j}, \frac{\partial}{\partial \bar{z}^i} \right) = \frac{1}{2} \left( g_{ij} + \sqrt{-1} g_{ij} \right).
\]
It is obvious that $(h_{ij})$ is a positive Hermitian matrix. Here, we always use the convention $h_{ji} = h_{ij}$ since $g$ is symmetric over $T_C M$. The (transposed) inverse matrix of $(h_{ij})$ is denoted by $(h^{ij})$, i.e. $h_{ij} h^{jk} = \delta^i_k$. One can also show that,

\[(2.16) \quad h^{ij} = 2 \left( g^{ij} - \sqrt{-1} g^{iJ} \right) \]

From the definition equation (2.13), we see the following well-known relation on $T_C M$:

\[(2.17) \quad ds_h^2 = \frac{1}{2} ds_g^2 - \frac{\sqrt{-1}}{2} \omega \]

where $\omega$ is the fundamental 2-form associated to the $J$-invariant metric $g$:

\[(2.18) \quad \omega(X,Y) = g(JX,Y). \]

In local complex coordinates,

\[(2.19) \quad \omega = \sqrt{-1} h_{ij} dz^i \wedge d\bar{z}^j. \]

In the following, we shall use the components of the complexified curvature tensor $R$, for example,

\[(2.20) \quad R_{\bar{i}jk\bar{t}} := R \left( \frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j}, \frac{\partial}{\partial z^k}, \frac{\partial}{\partial \bar{z}^\ell} \right), \]

and in particular we use the following notation for the complexified curvature tensor:

\[(2.21) \quad R_{ij\bar{k}l} := R \left( \frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j}, \frac{\partial}{\partial z^k}, \frac{\partial}{\partial \bar{z}^l} \right). \]

It is obvious that, the components of (C-linear) complexified curvature tensor have the same properties as the components of the real curvature tensor. We list some properties of $R_{\bar{i}jk\bar{t}}$ for examples:

\[(2.22) \quad R_{\bar{i}j\bar{k}t} = -R_{\bar{j}i\bar{k}t}, \quad R_{\bar{i}j\bar{k}t} = R_{\bar{k}i\bar{j}t}, \]

and in particular, the (first) Bianchi identity holds:

\[(2.23) \quad R_{\bar{j}i\bar{k}t} + R_{\bar{k}i\bar{j}t} + R_{\bar{i}t\bar{j}k} = 0. \]

**Definition 2.1.** Let $\{e_i\}_{i=1}^{2n}$ be a local orthonormal basis of $(T_R M, g)$, the Riemannian Ricci curvature of $(M,g)$ is

\[(2.24) \quad Ric(X,Y) := \sum_{i=1}^{2n} R(e_i, X, Y, e_i), \]

and the corresponding Riemannian scalar curvature is

\[(2.25) \quad s = \sum_{i=1}^{2n} Ric(e_i, e_i). \]

**Lemma 2.2.** On an almost Hermitian manifold $(M,h)$, the Riemannian Ricci curvature of the Riemannian manifold $(M,g)$ satisfies

\[(2.26) \quad Ric(X,Y) = h^{ij} \left[ R \left( \frac{\partial}{\partial z^i}, X, Y, \frac{\partial}{\partial z^j} \right) + R \left( \frac{\partial}{\partial \bar{z}^i}, Y, X, \frac{\partial}{\partial \bar{z}^j} \right) \right] \]
for any $X,Y \in T_R M$. The scalar curvature is
\begin{equation}
(2.27) \quad s = 2 \bar{h}^j h^{k\bar{\ell}} \left( 2 R_{\bar{i}k\bar{j}} - R_{\bar{i}jk} \right).
\end{equation}

**Proof.** See Lemma 7.1 in the Appendix. \qed

In order to formulate the curvature relations more effectively, we introduce new curvature notations as following:

**Definition 2.3.** The Riemannian Ricci tensor can also be extended to $T_C M$, and we denote the associated $\langle 1,1 \rangle$-form component by\
\begin{equation}
(2.28) \quad \mathcal{R}ic = \sqrt{-1} \mathcal{R}_{i\bar{j}} dz^i \wedge d\bar{z}^j \quad \text{with} \quad \mathcal{R}_{i\bar{j}} := Ric \left( \frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j} \right).
\end{equation}

The **Hermitian Ricci curvature** of the complexified curvature tensor is\
\begin{equation}
(2.29) \quad Ric_H = \sqrt{-1} R_{i\bar{j}} dz^i \wedge d\bar{z}^j, \quad \text{with} \quad R_{i\bar{j}} := h^{k\bar{\ell}} R_{ij\bar{k}\bar{\ell}}.
\end{equation}

The corresponding **Hermitian scalar curvature** of $h$ is given by\
\begin{equation}
(2.30) \quad s_H = h^{i\bar{j}} R_{i\bar{j}}.
\end{equation}

We also define the **Riemannian type scalar curvature** as\
\begin{equation}
(2.31) \quad s_R = h^{i\bar{j}} h^{k\bar{\ell}} R_{ij\bar{k}\bar{\ell}}.
\end{equation}

It is obvious, $s_H \neq s_R$ in general.

**Corollary 2.4.** On an almost Hermitian manifold $(M,h)$, we have\
\begin{equation}
(2.32) \quad \mathcal{R}_{i\bar{j}} = 2 \left( h^{k\bar{\ell}} R_{k\bar{j}i\bar{\ell}} \right) - R_{i\bar{j}}
\end{equation}

and\
\begin{equation}
(2.33) \quad s = 2 h^{i\bar{j}} \mathcal{R}_{i\bar{j}} = 4 s_R - 2 s_H.
\end{equation}

**Proof.** (2.32) follows from (2.26) and the Bianchi identity (2.23). (2.33) follows from (2.27). \qed

### 2.3. Curvatures on Hermitian manifolds.

Let $(M,h,J)$ be an almost Hermitian manifold. The Nijenhuis tensor $N_J : \Gamma(M,T_R M) \times \Gamma(M,T_R M) \to \Gamma(M,T_R M)$ is defined as\
\begin{equation}
(2.34) \quad N_J(X,Y) = [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY].
\end{equation}

The almost complex structure $J$ is called **integrable** if $N_J \equiv 0$ and then we call $(M,g,J)$ a Hermitian manifold. By Newlander-Nirenberg’s theorem, there exists a real coordinate system $\{x^i,x^I\}$ such that $z^i = x^i + \sqrt{-1}x^I$ are local holomorphic coordinates on $M$. Moreover, we have $T_C M = T^{1,0} M \oplus T^{0,1} M$ where\
\begin{equation}
T^{1,0} M = \text{span}_\mathbb{C} \left\{ \frac{\partial}{\partial z^1}, \ldots, \frac{\partial}{\partial z^n} \right\} \quad \text{and} \quad T^{0,1} M = \text{span}_\mathbb{C} \left\{ \frac{\partial}{\partial \bar{z}^1}, \ldots, \frac{\partial}{\partial \bar{z}^n} \right\}.
\end{equation}

Let $\varphi$ be a $(p,q)$-form on $(M,g)$, and\
\begin{equation}
(2.35) \quad \varphi = \frac{1}{p!q!} \sum_{i_1,\ldots,i_p,j_1,\ldots,j_q} \varphi_{i_1,\ldots,i_p,j_1,\ldots,j_q} dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q},
\end{equation}
where $\varphi_1, \ldots, \varphi_1, \ldots, \varphi_p$ is skew symmetric in $i_1, \ldots, i_p$ and also skew symmetric in $j_1, \ldots, j_q$.

The local inner product is defined as

$$
|\varphi|^2 = \langle \varphi, \varphi \rangle = \frac{1}{p!q!} \sum_{i_1, \ldots, i_p; j_1, \ldots, j_q} h_{i_1 i_2 \cdots i_p} h_{j_1 j_2 \cdots j_q} \varphi_{i_1 \cdots i_p} \overline{\varphi_{j_1 \cdots j_q}}.
$$

The norm on $\Omega^{p,q}(M)$ is

$$
\|\varphi\|^2 = \langle \varphi, \varphi \rangle = \int \omega^n = n!
$$

It is well-known that there exists a real isometry $* : \Omega^{p,q}(M) \to \Omega^{n-q,n-p}(M)$ such that

$$
\langle \varphi, \psi \rangle = \int \varphi \wedge *\psi,
$$

for $\varphi, \psi \in \Omega^{p,q}(M)$.

3. Geometry of the Levi-Civita Ricci curvature $\mathfrak{R}^{(1)}$

3.1. The Chern connection and Levi-Civita connection on $(T^{1,0}M, h)$

3.1.1. Curvature of the Chern connection on $(T^{1,0}M, h)$. On the Hermitian holomorphic vector bundle $(T^{1,0}M, h)$, the Chern connection $\nabla^Ch$ is the unique connection which is compatible with the complex structure and also the Hermitian metric. The curvature tensor of $\nabla^Ch$ is denoted by $\Theta$ and its curvature components are

$$
\Theta_{i\bar{j}k\bar{l}} = -\frac{\partial^2 h_{k\bar{l}}}{\partial z^i \partial \bar{z}^j} + h_{k\bar{l}} \frac{\partial h_{i\bar{j}}}{\partial \bar{z}^j} \frac{\partial h_{k\bar{l}}}{\partial z^i}.
$$

It is well-known that the (first) Chern Ricci curvature

$$
\Theta^{(1)} := \sqrt{-1} \Theta_{i\bar{j}} d z^i \wedge d \bar{z}^j
$$

represents the first Chern class $c_1(M)$ of $M$ where

$$
\Theta^{(1)}_{i\bar{j}} = h^{k\bar{l}} \Theta_{i\bar{j}k\bar{l}} = -\frac{\partial^2 \log \det (h_{k\bar{l}})}{\partial z^i \partial \bar{z}^j}.
$$

The second Chern Ricci curvature $\Theta^{(2)} = \sqrt{-1} \Theta^{(2)}_{i\bar{j}} d z^i \wedge d \bar{z}^j$ with components

$$
\Theta^{(2)}_{i\bar{j}} = h^{k\bar{l}} \Theta_{k\bar{i}j\bar{l}}.
$$

The Chern scalar curvature $s_C$ of the Chern curvature tensor $\Theta$ is defined by

$$
s_C = h^{i\bar{j}} h^{k\bar{l}} \Theta_{i\bar{j}k\bar{l}}.
$$
3.1.2. The induced Levi-Civita connection on \((T^{1,0}M, h)\). Since \(T^{1,0}M\) is a subbundle of \(T_C M\), there is an induced connection \(\hat{\nabla}\) on \(T^{1,0}M\) given by
\[
\hat{\nabla} = \pi \circ \nabla : \Gamma(M, T^{1,0}M) \to \Gamma(M, T_C M \otimes T_C M) \rightarrow \Gamma(M, T_C M \otimes T^{1,0}M).
\]
Moreover, \(\hat{\nabla}\) is a metric connection on the Hermitian holomorphic vector bundle \((T^{1,0}M, h)\) and it is determined by the relations
\[
\hat{\nabla} \left( \frac{\partial}{\partial z^i} \right) := \Gamma^p_{ik} \frac{\partial}{\partial z^p} \quad \text{and} \quad \hat{\nabla} \left( \frac{\partial}{\partial z^k} \right) := \Gamma^p_{jk} \frac{\partial}{\partial z^p}
\]
where
\[
\Gamma^k_{ij} = \frac{1}{2} h^{k\ell} \left( \frac{\partial h_{ij}}{\partial z^\ell} + \frac{\partial h_{ij}}{\partial z^\ell} - \frac{\partial h_{ij}}{\partial z^\ell} \right), \quad \text{and} \quad \Gamma^k_{ij} = \frac{1}{2} h^{k\ell} \left( \frac{\partial h_{ij}}{\partial z^\ell} - \frac{\partial h_{ij}}{\partial z^\ell} \right).
\]
The curvature tensor \(\mathcal{R} \in \Gamma(M, \Lambda^2 T_C M \otimes T^{1,0}M \otimes T^{1,0}M)\) of \(\hat{\nabla}\) is given by
\[
\mathcal{R}(X, Y)s = \hat{\nabla}_X \hat{\nabla}_Y s - \hat{\nabla}_{[X,Y]} s - \hat{\nabla}_s [X,Y]
\]
for any \(X, Y \in T_C M\) and \(s \in T^{1,0}M\). The curvature tensor \(\mathcal{R}\) has components
\[
\mathcal{R}^\ell_{ijk} = - \left( \frac{\partial \Gamma^\ell_{jk}}{\partial z^i} - \frac{\partial \Gamma^\ell_{ik}}{\partial z^j} + \Gamma^s_{ik} \Gamma^\ell_{js} - \Gamma^s_{jk} \Gamma^\ell_{si} \right) \quad \text{and} \quad \mathcal{R}^\ell_{ijk} = - \mathcal{R}^\ell_{ijk};
\]
\[
\mathcal{R}^\ell_{ijk} = - \left( \frac{\partial \Gamma^\ell_{jk}}{\partial z^i} - \frac{\partial \Gamma^\ell_{ik}}{\partial z^j} + \Gamma^s_{ik} \Gamma^\ell_{js} - \Gamma^s_{jk} \Gamma^\ell_{si} \right);
\]
and
\[
\mathcal{R}^\ell_{ijk} = - \left( \frac{\partial \Gamma^\ell_{jk}}{\partial z^i} - \frac{\partial \Gamma^\ell_{ik}}{\partial z^j} + \Gamma^s_{ik} \Gamma^\ell_{js} - \Gamma^s_{jk} \Gamma^\ell_{si} \right).
\]
With respect to the Hermitian metric \(h\) on \(T^{1,0}M\), we use the convention
\[
\mathcal{R}^{\bullet\bullet\ell}_{ijk} := \sum_{s=1}^n \mathcal{R}^s_{\bullet\bullet\ell} h_{s\bar{s}}.
\]

**Corollary 3.1** ([39, Proposition 2.1]). We have the following relations:
\[
R^\ell_{ijk} = \mathcal{R}^\ell_{ijk}, \quad R^\ell_{ijk} = \mathcal{R}^\ell_{ijk},
\]
and
\[
R^\ell_{ijk} = - \left( \frac{\partial \Gamma^\ell_{jk}}{\partial z^i} - \frac{\partial \Gamma^\ell_{ik}}{\partial z^j} + \Gamma^s_{ik} \Gamma^\ell_{js} - \Gamma^s_{jk} \Gamma^\ell_{si} \right) = \mathcal{R}^\ell_{ijk} + \Gamma^\ell_{\pi i} \cdot \Gamma^\ell_{\pi j},
\]
Next, we define Ricci curvatures and scalar curvatures for \((T^{1,0}M, h, \hat{\nabla})\).

**Definition 3.2.** The first Levi-Civita Ricci curvature of the Hermitian vector bundle \((T^{1,0}M, h, \hat{\nabla})\) is
\[
\mathcal{R}^{(1)} = \sqrt{-1} \mathcal{R}^{(1)}_{ij} \, dz^i \wedge dz^j \quad \text{with} \quad \mathcal{R}^{(1)}_{ij} = h^{\ell \ell} \mathcal{R}^{\ell}_{jk\ell}.
\]
and the second Levi-Civita Ricci curvature of it is
\begin{equation}
\mathfrak{R}^{(2)} = \sqrt{-1} \mathfrak{R}^{(2)} \ dz^i \wedge d\bar{z}^j \quad \text{with} \quad \mathfrak{R}^{(2)}_{ij} = h^{k\bar{r}} \mathfrak{R}_{k\bar{r}ij}.
\end{equation}

The Levi-Civita scalar curvature of $\bar{\nabla}$ on $T^{1,0}M$ is denoted by
\begin{equation}
\mathcal{S}_{\text{LC}} = h^{k\bar{r}} h^{l\bar{s}} \mathfrak{R}_{k\bar{r}lj}. \tag{3.17}
\end{equation}

The (first) Chern-Ricci curvature $\Theta^{(1)}$ represents the first Chern class $c_1(M)$, but in general, the first Levi-Civita-Ricci curvature $\mathfrak{R}^{(1)}$ does not represent a class in $H^2_{dR}(M)$ or $H_{dR}^1(M)$, since it is not $d$-closed. We shall explore more geometric properties of $\mathfrak{R}^{(1)}$ in the next subsection.

### 3.2. Elementary computations on Hermitian manifolds

In this subsection, we recall some elementary and well-known computational lemmas on Hermitian manifolds.

#### Lemma 3.3.

Let $(M, h)$ be a compact Hermitian manifold and $\omega = \sqrt{-1} \ h^{i\bar{j}} \ dz_i \wedge d\bar{z}^j$.

\begin{equation}
\partial^* \omega = -\sqrt{-1} \Lambda (\partial \omega) = 2 \sqrt{-1} \Gamma^k_{ik} dz^i.
\end{equation}

Proof. By the well-known Bochner formula (e.g. [39]),

\begin{equation}
[\bar{\partial}^*, L] = \sqrt{-1} (\partial + \tau)
\end{equation}

where $\tau = [\Lambda, \partial \omega]$, we see $\bar{\partial}^* \omega = \sqrt{-1} \Lambda (\partial \omega) = 2 \sqrt{-1} \Gamma^k_{ik} dz^i$. \hfill $\Box$

#### Lemma 3.4.

Let $(M, h, \omega)$ be a Hermitian manifold. For any $p \in M$, there exist local holomorphic “normal coordinates” $\{z^i\}$ centered at $p$ such that
\begin{equation}
h^{i\bar{j}}(p) = \delta_{ij} \quad \text{and} \quad \Gamma^k_{ij}(p) = 0.
\end{equation}

In particular, at $p$, we have
\begin{equation}
\Gamma^k_{ji} = \frac{\partial h^k_{i\bar{r}}}{\partial z^j} - \frac{\partial h^k_{j\bar{r}}}{\partial z^i}.
\end{equation}

Let $T$ be the torsion tensor of the Hermitian metric $\omega$, i.e.
\begin{equation}
T^k_{ij} = h^{k\bar{r}} \left( \frac{\partial h^i_{j\bar{r}}}{\partial z^k} - \frac{\partial h^i_{k\bar{r}}}{\partial z^j} \right).
\end{equation}

In the following, we shall use the conventions:
\begin{equation}
T \square T := h^{p\bar{q}} h^{k\bar{r}} T^k_{jp} \cdot T^j_{iq} \ dz^i \wedge d\bar{z}^j, \quad \text{and} \quad T \circ T := h^{p\bar{q}} h^{k\bar{r}} h^{i\bar{s}} h^{l\bar{t}} T^k_{ip} \cdot T^l_{jq} \ dz^i \wedge d\bar{z}^j.
\end{equation}

It is obvious that the $(1,1)$-forms $T \circ T$ and $T \square T$ are not the same.

#### Lemma 3.5.

At a fixed point $p$ with “normal coordinates” (3.19), we have
\begin{equation}
(T \square T)_{ij} = T^k_{ip} \cdot T^j_{kp} = 4 \frac{\partial h^i_{k\bar{r}}}{\partial z^j} \cdot \frac{\partial h^k_{j\bar{r}}}{\partial z^j}
\end{equation}

and
\begin{equation}
(T \circ T)_{ij} = T^j_{pq} \cdot T^i_{pq} = 4 \frac{\partial h^i_{j\bar{r}}}{\partial z^p} \cdot \frac{\partial h^j_{i\bar{r}}}{\partial z^p}.
\end{equation}

Moreover

\begin{equation}
\text{tr}_\omega \left( \sqrt{-1} T \square T \right) = \text{tr}_\omega \left( \sqrt{-1} T \circ T \right) = |T|^2.
\end{equation}
3.3. Geometry of the first Levi-Civita Ricci curvature.

**Definition 3.6.** Let $M$ be a compact complex manifold. A Hermitian metric $\omega$ on $M$ is called balanced, if $d^*\omega = 0$. $\omega$ is called conformally balanced, if there exists a smooth function $\varphi : M \to \mathbb{R}$ and a balanced metric $\omega_B$ such that $\omega = e^{\varphi} \omega_B$.

**Theorem 3.7.** Let $(M, \omega)$ be a compact Hermitian manifold, then we have

\begin{align}
(3.25) \quad R^{(1)} &= \Theta^{(1)} - \frac{1}{2} (\partial \partial^* \omega + \partial^* \partial \omega), \\
(3.26) \quad R^{(2)} &= \Theta^{(1)} - \frac{1}{2} (\partial \partial^* \omega + \partial^* \partial \omega) - \frac{\sqrt{-1}}{4} T \circ \overline{T} + \frac{\sqrt{-1}}{4} T \square \overline{T}.
\end{align}

In particular, $R^{(1)}$ represents the first Aeppli-Chern class $c_1^{AC}(M) \in H^{1,1}_A(M)$. Moreover,

1. $R^{(1)}$ is $d$-closed if and only if $\partial \partial^* \omega = 0$;
2. if $\partial \partial^* \omega = 0$, then $R^{(1)}$ represents the real first Chern class $c_1(M) \in H^2_{dR}(M)$;
3. if $\omega$ is conformally balanced, then $R^{(1)}$ represents the first Chern class $c_1(M) \in H^2_{BC}(M)$ and also the first Bott-Chern class $c_1^{BC}(M) \in H^{1,1}_{BC}(M)$;
4. $R^{(1)} = \Theta^{(1)}$ if and only if $d^* \omega = 0$, i.e. $(M, \omega)$ is a balanced manifold.

**Proof.** By formula (3.19), one can see

\begin{align}
\partial \partial^* \omega &= -2\sqrt{-1} \frac{\partial \Gamma^k_{jk}}{\partial z^i} dz^i \wedge d\overline{z}^j \\
\text{and} \quad \partial^* \partial \omega &= -2\sqrt{-1} \frac{\partial \Gamma^k_{jk}}{\partial \overline{z}^i} d\overline{z}^j \wedge dz^i.
\end{align}

On the other hand, we have

\begin{align}
(3.28) \quad \Gamma^k_{ik} + \Gamma^k_{jk} = h^{k\ell} \frac{\partial h_{ik}}{\partial z^\ell} = \frac{\partial \log \det(h)}{\partial z^i}.
\end{align}

Hence by (3.10), at a fixed point $p$ with “normal coordinates” (3.19), we obtain

\begin{align}
(3.29) \quad R^{(1)}_{ij} = -\frac{\partial \Gamma^k_{jk}}{\partial z^i} + \frac{\partial \Gamma^k_{ij}}{\partial z^k} = \Theta^{(1)}_{ij} + \frac{\partial \Gamma^k_{jk}}{\partial z^i} + \frac{\partial \Gamma^k_{ij}}{\partial z^k}.
\end{align}

Therefore by (3.27) and (3.29), we obtain (3.25). On the other hand, by (3.14),

\begin{align}
(3.30) \quad R_{ij} - R^{(1)}_{ij} = h^{k\ell} h_{i\ell} \Gamma^i_{k\ell} \Gamma^k_{j\ell} = \frac{\partial h_{i\ell}}{\partial z^j} \cdot \frac{\partial h_{k\ell}}{\partial z^i} = -\frac{1}{4} (T \circ T)_{ij}
\end{align}

and similarly

\begin{align}
(3.31) \quad R_{ij} - R^{(2)}_{ij} = h^{k\ell} h_{i\ell} \Gamma^i_{k\ell} \Gamma^k_{j\ell} = \frac{\partial h_{i\ell}}{\partial z^j} \cdot \frac{\partial h_{k\ell}}{\partial z^i} = -\frac{1}{4} (T \square T)_{ij}.
\end{align}

Hence, by using (3.25), we obtain (3.26). By (3.25), we see $R^{(1)}$ represents the first Aeppli-Chern class $c_1^{AC}(M)$, i.e. $[R^{(1)}] = [\Theta^{(1)}]$ as classes in $H^{1,1}_A(M)$.

Next we prove the properties of $R^{(1)}$,

1. By (3.25) again, $dR^{(1)} = -\frac{1}{2} (\partial \partial^* \omega + \partial^* \partial \omega)$. By degree reasons, $dR^{(1)} = 0$ if and only if $\partial \partial^* \omega = 0$. 

15
If $\partial \bar{\partial}^* \omega = 0$, we have

$$\mathcal{R}^{(1)} = \Theta^{(1)} - \frac{1}{2}(\partial \bar{\partial}^* \omega + \bar{\partial} \partial^* \omega) = \Theta^{(1)} - \frac{1}{2}dd^* \omega.$$ 

Hence $[\mathcal{R}^{(1)}] = [\Theta^{(1)}] \in H^{2}_{\partial \bar{\partial}}(M)$.

If $\omega$ is conformally balanced, there exists a smooth function $f$ and a balanced metric $\omega_f$ such that $\omega_f = e^f \omega$. We denote by an extra index $f$ the corresponding quantities with respect to the new metric $\omega_f$. The Christoffel symbols of $\omega_f$ are

$$(\Gamma_f^k_{ij}) = \frac{1}{2} e^{-f} g^{k\ell} \left( \frac{\partial(e^f g_{j\ell})}{\partial z^i} - \frac{\partial(e^f g_{j\ell})}{\partial z^j} \right) = \Gamma_{ij}^k + \frac{1}{2} \left( \delta_{jk} f_i - g_{k\ell} g_{ji} f_{\ell} \right).$$

In particular,

$$(\Gamma_f^k_{ij}) = \Gamma_{ij}^k + \frac{n - 1}{2} f_i.$$ 

By (3.25), we obtain

$$\partial \bar{\partial}^* f \omega_f = \partial \bar{\partial}^* \omega + \sqrt{-1}(n - 1) \partial f.$$ 

Therefore,

$$\bar{\partial} \partial^* \omega_f = \bar{\partial} \partial^* \omega - (n - 1) \sqrt{-1} \partial \bar{\partial} f \quad \text{and} \quad \partial \partial^* \omega_f = \partial \partial^* \omega.$$ 

Since $\omega_f$ is balanced, i.e. $\bar{\partial} \partial^* \omega_f = 0$, we obtain

$$\partial \partial^* \omega + \bar{\partial} \partial^* \omega = 2(n - 1) \sqrt{-1} \partial \bar{\partial} f.$$ 

Hence, $\mathcal{R}^{(1)} = \Theta^{(1)} - (n - 1) \sqrt{-1} \partial \bar{\partial} f$, i.e. $[\mathcal{R}^{(1)}] = [\Theta^{(1)}] \in H^{1,1}_{\bar{\partial} \partial}(M)$. Hence, $\mathcal{R}^{(1)}$ represents the first Chern class $c_1(M) \in H^{1,1}_{\bar{\partial} \partial}(M)$ and also the first Bott-Chern class $c^{BC}_1(M) \in H^{1,1}_{\bar{\partial} \partial}(M)$.

If $\mathcal{R}^{(1)} = \Theta^{(1)}$ if and only if $\partial \partial^* \omega + \bar{\partial} \partial^* \omega = 0$. By pairing with $\omega$, we see the latter is equivalent to $d^* \omega = 0$.

The proof of Theorem 3.7 is complete. \qed

**Example 3.8.** In this example, we shall construct a Hermitian metric with strictly positive $\mathcal{R}^{(1)}$, but not $\Theta^{(1)}$. Let $M$ be a Fano manifold with complex dimension $n \geq 2$. By Yau’s theorem, there exists a Kähler metric $\omega$ on $M$ such that $\mathcal{R}^{(1)} = \Theta^{(1)}$. For any smooth function $\varphi$, we define $\omega_t = e^{t\varphi} \omega$. By (3.3), one can see

$$\Theta^{(1)}_{\omega_t} = \Theta^{(1)} - nt\sqrt{-1} \partial \bar{\partial} \varphi.$$ 

Hence, by (3.25) and (3.33),

$$\mathcal{R}^{(1)}_{\omega_t} = \mathcal{R}^{(1)} - \sqrt{-1} t \partial \bar{\partial} \varphi.$$ 

Let $t_0 = \sup \{ t > 0 \mid \Theta^{(1)}_{\omega_t} = \Theta^{(1)} - nt\sqrt{-1} \partial \bar{\partial} \varphi \geq 0 \}$, and $t_1 := \frac{3}{2} t_0$, then

$$\mathcal{R}^{(1)}_{\omega_{t_1}} = \mathcal{R}^{(1)} - t_1 \sqrt{-1} \partial \bar{\partial} \varphi > 0$$ 

but $\Theta^{(1)}_{\omega_{t_1}}$ is not positive definite.
Definition 3.9 ([7]). A compact complex manifold \( M \) is said to satisfy the \( \partial\overline{\partial} \)-lemma if the following statement holds: if \( \eta \) is \( d \)-exact, \( \partial \)-closed and \( \overline{\partial} \)-closed, it must be \( \partial\overline{\partial} \)-exact. In particular, on such manifolds, for any pure-type form \( \psi \in \Omega^{p,q}(M) \), if \( \psi \) is \( d \)-closed and \( \partial \)-exact, then it is \( \partial\overline{\partial} \)-exact.

It is well-known that all compact Kähler manifolds satisfy the \( \partial\overline{\partial} \)-lemma. Moreover, if \( \mu : \widetilde{M} \to M \) is a modification between compact complex manifolds and if the \( \partial\overline{\partial} \)-lemma holds for \( \widetilde{M} \), then the \( \partial\overline{\partial} \)-lemma also holds for \( M \). In particular, Moishezon manifolds and also manifolds in Fujiki class \( \mathcal{E} \) satisfy the \( \partial\overline{\partial} \)-lemma. For more details, we refer to [7, 2, 43, 44] and also the references therein.

In the following, we show on complex manifolds with \( \partial\overline{\partial} \)-lemma, the converse of (2) and (3) in Theorem 3.7 are also true.

Proposition 3.10. Let \( M \) be a compact complex manifold on which the \( \partial\overline{\partial} \)-lemma holds. Let \( \omega \) be a Hermitian metric on \( M \).

1. \( \mathcal{R}(1) \) represents the real first Chern class \( c_1(M) \in H^2_{dR}(M) \) if and only if \( \overline{\partial}\partial^*\omega = 0 \);
2. \( \mathcal{R}(1) \) represents the first Chern class \( c_1(M) \in H^2_{\overline{\partial}}(M) \) if and only if \( \omega \) is conformally balanced.

Proof. (1) If \( [\mathcal{R}(1)] = [\mathcal{Q}(1)] \in H^2_{dR}(M) \), by (3.25), \( \mathcal{R}(1) = \mathcal{Q}(1) - \frac{1}{2} dd^*\omega + \frac{i}{2} (\overline{\partial}\partial^*\omega + \overline{\partial}\overline{\partial}\omega) \), there exists a 1-form \( \gamma \) such that \( d\gamma = \overline{\partial}\partial^*\omega + \overline{\partial}\overline{\partial}\omega \). The compatibility condition \( d^2\gamma = 0 \) implies \( \partial\overline{\partial}\partial^*\omega + \overline{\partial}\overline{\partial}\overline{\partial}\omega = 0 \). Hence \( \partial\overline{\partial}\omega = 0 \). If we set \( \gamma = \gamma^{1,0} + \gamma^{0,1} \), then \( \partial\gamma^{1,0} = \overline{\partial}\omega \). Therefore, \( \partial\gamma^{1,0} \) is both \( d \)-closed and \( \partial \)-exact. By \( \partial\overline{\partial} \)-lemma, there exists some \( \eta \) such that \( \partial\eta = \partial\overline{\partial}\eta \). By degree reasons, \( \partial\gamma^{1,0} = 0 \), that is \( \partial\overline{\partial}\omega = 0 \).

(2) If \( [\mathcal{R}(1)] = [\mathcal{Q}(1)] \in H^2_{\overline{\partial}}(M) \), there exists \( (1,0) \)-form \( \tau \) such that \( \overline{\partial}\tau = \partial\partial^*\omega \). So \( \overline{\partial}\tau \) is both \( d \)-closed and \( \overline{\partial} \)-exact, hence by \( \partial\overline{\partial} \)-lemma, there exists a smooth function \( \varphi \) such that \( \partial\overline{\partial}\omega = \overline{\partial}\tau = \sqrt{-1}\partial\overline{\partial}\varphi \). By [23], there exists a smooth function \( f \) such that \( \omega_f := e^{\varphi - f} \omega \) is a Gauduchon metric, i.e. \( \partial\overline{\partial}\omega_f^{-1} = 0 \). We use the index \( f \) to denote the operations with respect to the new metric \( \omega_f \). For example,

\[
\|\partial_f^*\omega_f\|^2_f = (\partial\overline{\partial}_f^*\omega_f, \omega_f).f.
\]

We can see \( \partial\overline{\partial}_f^*\omega_f = \partial\overline{\partial}\omega - \sqrt{-1}\partial\overline{\partial}f = \sqrt{-1}\partial\overline{\partial}(\varphi - f) \). Moreover,

\[
(\partial\overline{\partial}_f^*\omega_f, \omega_f) = (\partial\overline{\partial}\omega, \omega_f) = \int \sqrt{-1}\partial\overline{\partial}(\varphi - f) \wedge \omega_{n-1} \frac{1}{(n-1)!} = 0,
\]

since \( \omega_f \) is a Gauduchon metric. Therefore, \( \partial\overline{\partial}_f\omega_f = 0 \), i.e. \( \omega_f \) is balanced and so \( \omega \) is conformally balanced.

Remark 3.11. By (3.33), a conformally balanced metric \( \omega \) satisfies \( \partial\overline{\partial}\omega = 0 \). On the other hand, if \( H^2_{\overline{\partial}}(M) = 0 \), then \( \partial\overline{\partial}\omega = 0 \) if and only if \( \omega \) is conformally balanced. On the Hopf manifold \( S^{2n-1} \times S^1 \) with \( n \geq 2 \), the canonical metric \( \omega_0 \) satisfies \( \partial\overline{\partial}\omega_0 = 0 \), but it is not conformally balanced.

Corollary 3.12. Let \( M \) be a complex manifold. Then

\[
c_1^{BC}(M) = 0 \implies c_1(M) = 0 \implies c_1^{AC}(M) = 0.
\]
Moreover, on a complex manifold satisfying the $\partial \overline{\partial}$-lemma,
\[ c_1^{BC}(M) = 0 \iff c_1(M) = 0 \iff c_1^{AC}(M) = 0. \]

**Proof.** The first statement is obvious. For the second statement, we only need to show that, on a complex manifold $M$ with $\partial \overline{\partial}$-lemma if $c_1^{AC}(M) = 0$, then $c_1^{BC}(M) = 0$. Indeed, if $c_1^{AC}(M) = 0$, for any Hermitian metric $\omega$ on $M$, $\Theta^{(1)} = \partial A + \overline{\partial}A$ where $A$ is a $(0,1)$ form. It is obvious that $\partial A$ is $d$-closed and $\partial$-exact, and so there exists a smooth function $f \in C^\infty(M, \mathbb{R})$ such that $\partial A = \sqrt{-1} \partial \overline{\partial}f$, i.e. $\Theta^{(1)} = 2\sqrt{-1} \partial \overline{\partial}f$. \[ \square \]

**Remark 3.13.** It is well-known that $S^{2n-1} \times S^1$ $(n \geq 2)$ has $c_1(M) = c_1^{AC}(M) = 0$, but $c_1^{BC}(M) \neq 0$. It is also interesting to find a complex manifold $N$ with $c_1^{AC}(N) = 0$ but $c_1(N) \neq 0$.

### 3.4. Hermitian manifolds with nonnegative $\mathfrak{R}^{(1)}$.
In this subsection, we study the geometry of Hermitian manifolds with nonnegative first Levi-Civita Ricci curvature $\mathfrak{R}^{(1)}$.

**Definition 3.14.** Let $M$ be a compact complex manifold. The Kodaira dimension $\kappa(M)$ of $M$ is defined to be
\[ (3.37) \quad \kappa(M) := \limsup_{m \to \infty} \frac{\log \dim_{\mathbb{C}} H^0(M, mK_M)}{\log m}. \]

The following result is well-known (e.g. \cite{22} or \cite{39}).

**Theorem 3.15.** Let $(M, h)$ be a compact Hermitian manifold. If the first Levi-Civita Ricci curvature $\mathfrak{R}^{(1)}$ is quasi-positive, then $c_1^0(M) > 0$ and the Kodaira dimension $\kappa(M) = -\infty$. In particular, $H^2_{\text{dR}}(M), H^{1,1}_{\mathfrak{R}}(M), H^{1,1}_{BC}(M)$ and $H^{1,1}_A(M)$ are all non-zero.

**Proof.** At first, let’s recall the general theory for vector bundles. Let $\nabla^E$ be a connection on the holomorphic vector bundle $E$. Let $r$ be the rank of $E$, then there is a naturally induced connection $\nabla^{\det(E)}$ on the determine line bundle $\det(E) = \Lambda^r E$,
\[ (3.38) \quad \nabla^{\det(E)}(s_1 \wedge \cdots \wedge s_r) = \sum_{i=1}^r s_1 \wedge \cdots \wedge \nabla^E s_i \wedge \cdots \wedge s_r. \]

The curvature tensor of $(E, \nabla^E)$ is denoted by $R^E \in \Gamma(M, \Lambda^2 T^* M \otimes \text{End}(E))$ and the curvature tensor of $(\det E, \nabla^{\det(E)})$ is denoted by $R^{\det(E)} \in \Gamma(M, \Lambda^2 T^* M)$. We have the relation that
\[ (3.39) \quad \text{tr} R^E = R^{\det(E)} \in \Gamma(M, \Lambda^2 T^* M). \]

Note that the trace operator is well-defined without using the metric on $E$. Moreover, $\text{tr} R^E = R^{\det E}$ is a $d$-closed 2-form. By Bianchi identity, we know, for any vector bundle $(F, \nabla^F)$
\[ \nabla^{F \otimes F^*} R^F = 0. \]

In particular, if $F$ is a line bundle, $F \otimes F^* = \mathbb{C}$ and $\nabla^{F \otimes F^*} = d$. Hence $d (R^{\det E}) = 0$. On the other hand, by Chern-Weil theory (e.g. \cite[Theorem 1.9]{61}), $R^{\det E}$ represents the real first Chern class $c_1(E) \in H^2(M, \mathbb{Z})$. In fact, let $\nabla^{\text{Ch}}$ be the Chern connection on the
Hermitian holomorphic line bundle \((\det E, h)\), and \(\Theta_{\det E}\) be the Chern curvature, then by Chern-Weil theory,

\[ R_{\det E} - \Theta_{\det E} = d\beta \]

for some 1-form \(\beta\). It is well known that the Chern curvature \(\Theta_{\det E}\) of the Hermitian line bundle \((\det E, h)\) represents the first Chern class \(c_1(M) \in H^{1,1}_\partial(M)\).

Now we go back to the setting on the Hermitian manifold \((M, \omega)\). Let \(E = T^{1,0}M\) with Hermitian metric \(h\) induced by \(\omega\). With respect to the Levi-Civita connection \(\bar{\nabla}\) on \(E\), we have a decomposition

\[ R_{\det E} = \eta^{2,0} + \eta^{0,2} + \eta^{1,1}. \]

It is obvious that

\[ \eta^{1,1} = \sqrt{-1} R^{k}_{j\bar{k}} dz^i \wedge d\bar{z}^j = \mathcal{R}^{(1)}, \quad \text{and} \quad \eta^{0,2} = \overline{\eta}^{2,0}. \]

It is also easy to see that \(\eta^{2,0} = -\frac{1}{2} \partial \overline{\partial} \omega\). Hence

\[ \int \left( R_{\det E} \right)^n = \sum_{\ell=0}^{\left[ \frac{n}{2} \right]} \binom{n}{2\ell} \binom{2\ell}{\ell} \int (\eta^{2,0} \wedge \overline{\eta}^{2,0})^\ell \wedge (\eta^{1,1})^{n-2\ell}. \]

It is obvious that, if \(\eta^{1,1}\) is quasi-positive,

\[ \int (\eta^{2,0} \wedge \overline{\eta}^{2,0})^\ell \wedge (\eta^{1,1})^{n-2\ell} \geq 0. \]

for \(1 \leq \ell \leq \left[ \frac{n}{2} \right]\) and \(\int (\eta^{1,1})^n > 0\). That is

\[ \int \left( R_{\det E} \right)^n > 0. \]

We obtain

\[ \int c_1^n(M) = \int \left( R_{\det E} \right)^n > 0. \]

Moreover, if \(\mathcal{R}^{(1)}\) is quasi-positive, by (3.25), the total Chern scalar curvature \(s_C\) is positive. Hence \(H^0(M, mK_M) = 0\) for all \(m > 0\) and in particular, \(\kappa(M) = -\infty\). On the other hand, if \(H^{1,1}_A(M) = 0\), we obtain \(\Theta^{(1)} = \partial B + \overline{\partial} C\) for 1-forms \(B\) and \(C\), and so

\[ \int_M \left( \Theta^{(1)} \right)^n = \int_M (\partial B + \overline{\partial} C) \wedge \left( \Theta^{(1)} \right)^{n-1} = 0, \]

which is a contradiction to \(c_1^n(M) > 0\). One can prove the non-vanishing of other cohomologies groups similarly. \(\square\)

**Remark 3.16.**

1. Note that, in general,

\[ \int_M \left( \Theta^{(1)} \right)^n \neq \int_M \left( \mathcal{R}^{(1)} \right)^n. \]

2. When \(\dim_{\mathbb{C}} M = 2\) and \(\mathcal{R}^{(1)}\) is quasi-positive, then \(M\) is Kähler by the Enriques-Kodaira classification of compact complex surfaces.

3. When \(M\) is in the Fujiki class \(\mathcal{C}\) and \(\mathcal{R}^{(1)}\) is strictly positive, then \(M\) is a Kähler manifold ([6, Theorem 0.2]).
3.5. **Hypothetical complex structures on** $S^6$. Let $(M, h)$ be a Hermitian manifold with constant Riemannian sectional curvature $K$, i.e., for any $X, Y, Z, W \in T^\mathbb{R}M$,

$$R(X, Y, Z, W) = K \cdot (g(X, W)g(Y, Z) - g(X, Z)g(Y, W)).$$

Therefore, by the complexification process,

$$R_{i\overline{j}k\overline{l}} = K \cdot h_{i\overline{j}}h_{k\overline{l}}, \quad R_{i\overline{j}} = K \cdot h_{i\overline{j}} , \quad R_{ijkl} = R_{i\overline{j}k\overline{l}} = 0. $$

In particular, $Ric_H = K \cdot \omega$. If $K > 0$, we see from (3.14) that

$$\mathfrak{R}^{(1)} \geq Ric_H = K \cdot \omega_h > 0.$$ 

By Theorem 3.15, $c_1^b(M) > 0$. Now we get Lebrun’s result that

**Corollary 3.17** ([33]). On $S^6$, there is no orthogonal complex structure compatible with metrics in some small neighborhood of the round metric.

We consider a compact complex manifold $M$ diffeomorphic to the six-sphere $S^6$, assuming one exists. It is a well-known open problem to determine whether such a manifold $M$ exists. It is conjectured in [30, Section 10] that $\kappa(M) = -\infty$. Now we can see from Corollary 1.11, if $M$ has a Hermitian metric with positive total Riemannian scalar curvature, then $\kappa(M) = -\infty$.

4. Curvature relations on Hermitian manifolds

4.1. **Ricci curvature relations.** As introduced in the previous sections, on a compact Hermitian manifold $(M, \omega)$ there are six different types of Ricci curvatures:

1. the $(1, 1)$ component of the complexified Riemannian Ricci curvature $\mathfrak{Ric}$;
2. the Hermitian Ricci curvature $Ric_H$;
3. the first Levi-Civita Ricci curvature $\mathfrak{R}^{(1)}$;
4. the second Levi-Civita Ricci curvature $\mathfrak{R}^{(2)}$;
5. the first Chern Ricci curvature $\Theta^{(1)}$;
6. the second Chern Ricci curvature $\Theta^{(2)}$.

In this subsection, we shall explore explicit relations between them by using $\omega$ and its torsion $T$. The main result of this subsection is as follows.

**Theorem 4.1.** Let $(M, \omega)$ be a compact Hermitian manifold.

1. The first Levi-Civita Ricci curvature is

$$\mathfrak{R}^{(1)} = \Theta^{(1)} - \frac{1}{2} (\partial \partial^* \omega + \overline{\partial \partial^*} \omega),$$

and the second Levi-Civita Ricci curvature is

$$\mathfrak{R}^{(2)} = \Theta^{(1)} - \frac{1}{2} \left( \partial \overline{\partial^*} \omega + \overline{\partial \partial^*} \omega \right) - \frac{\sqrt{-1}}{4} T \circ T + \frac{\sqrt{-1}}{4} T \Box T.$$

2. The Hermitian Ricci curvature is

$$Ric_H = \Theta^{(1)} - \frac{1}{2} \left( \partial \overline{\partial^*} \omega + \overline{\partial \partial^*} \omega \right) - \frac{\sqrt{-1}}{4} T \circ T.$$
(3) The second Chern-Ricci curvature is
\[ \Theta^{(2)} = \Theta^{(1)} - \sqrt{-1} \Lambda (\partial \bar{\partial} \omega) - (\partial \partial^* \omega + \bar{\partial} \bar{\partial}^* \omega) + \frac{\sqrt{-1}}{4} (T \circ T + 3T \square T). \]

(4) The (1, 1)-component of the Riemannian Ricci curvature is
\[ \mathcal{R}ic = \Theta^{(1)} - \sqrt{-1} (\Lambda \partial \bar{\partial} \omega) - \frac{1}{2} (\partial \partial^* \omega + \bar{\partial} \bar{\partial}^* \omega) + \frac{\sqrt{-1}}{4} (2T \square T + T \circ T) \]
\[ + \frac{1}{2} \left( (\partial^* \omega)^{\#} + T((\partial^* \omega)^{\#}) \right), \]
where \((\partial^* \omega)^{\#}\) is the dual vector of the \((0, 1)\)-form \(\partial^* \omega\).

Note also that the proofs of (4.1), (4.2) and (4.3) are already contained in Theorem 3.7.

**Theorem 4.2.** Let \((M, \omega)\) be a Hermitian manifold, we have
\[ \Theta^{(2)} = \Theta^{(1)} - \sqrt{-1} \Lambda (\partial \bar{\partial} \omega) - (\partial \partial^* \omega + \bar{\partial} \bar{\partial}^* \omega) + \frac{\sqrt{-1}}{4} (T \circ T + 3T \square T). \]

In particular,
\[ |T|^2 = \langle \sqrt{-1} \Lambda \partial \bar{\partial} \omega, \omega \rangle + \langle \partial \partial^* \omega + \bar{\partial} \bar{\partial}^* \omega, \omega \rangle. \]

**Proof.** At point \(p\) with “normal coordinates” (3.19), we have
\[ \mathfrak{g}^{(1)}_{ij} = \frac{1}{2} \sum_k \left( \frac{\partial^2 h_{ik}}{\partial z^k \partial \bar{z}^j} + \frac{\partial^2 h_{jk}}{\partial z^i \partial \bar{z}^k} \right) - \sum_{k,q} \frac{\partial h_{ik}}{\partial z^q} \frac{\partial h_{jq}}{\partial \bar{z}^j}, \]
\[ = \frac{1}{2} \sum_k \left( \frac{\partial^2 h_{ik}}{\partial z^k \partial \bar{z}^j} + \frac{\partial^2 h_{jk}}{\partial z^i \partial \bar{z}^k} \right) - \frac{1}{4} (T \square T)_{ij}, \]
and
\[ \mathfrak{g}^{(2)}_{ij} = \frac{1}{2} \sum_k \left( \frac{\partial^2 h_{ik}}{\partial z^k \partial \bar{z}^j} + \frac{\partial^2 h_{jk}}{\partial z^i \partial \bar{z}^k} \right) - \sum_{k,q} \frac{\partial h_{ik}}{\partial z^q} \frac{\partial h_{jq}}{\partial \bar{z}^j}, \]
\[ = \frac{1}{2} \sum_k \left( \frac{\partial^2 h_{ik}}{\partial z^k \partial \bar{z}^j} + \frac{\partial^2 h_{jk}}{\partial z^i \partial \bar{z}^k} \right) - \frac{1}{4} (T \circ T)_{ij}. \]

Similarly,
\[ \Theta^{(1)}_{ij} = - \frac{\partial^2 h_{ik}}{\partial z^i \partial \bar{z}^j} + \frac{\partial h_{ik}}{\partial \bar{z}^j} \frac{\partial h_{kj}}{\partial \bar{z}^i} = - \frac{\partial^2 h_{ik}}{\partial z^i \partial \bar{z}^j} + \frac{1}{4} (T \square T)_{ij}, \]
and
\[ \Theta^{(2)}_{ij} = - \frac{\partial^2 h_{ij}}{\partial z^k \partial \bar{z}^k} + \frac{\partial h_{ij}}{\partial \bar{z}^k} \frac{\partial h_{kj}}{\partial \bar{z}^i} = - \frac{\partial^2 h_{ij}}{\partial z^k \partial \bar{z}^k} + \frac{1}{4} (T \circ T)_{ij}. \]

A straightforward computation shows
\[ \sqrt{-1} \Lambda (\partial \bar{\partial} \omega) = \sqrt{-1} \left[ \left( \frac{\partial h_{ij}}{\partial z^k \partial \bar{z}^k} + \frac{\partial h_{ik}}{\partial z^j \partial \bar{z}^i} \right) - \left( \frac{\partial h_{ik}}{\partial z^j \partial \bar{z}^k} + \frac{\partial^2 h_{jk}}{\partial z^i \partial \bar{z}^k} \right) \right] dz^i \wedge d\bar{z}^j. \]
Similarly, \( (4.12) \)

\[
\mathcal{R}(1) + \mathcal{R}(2) = \Theta^{(1)} + \Theta^{(2)} + \sqrt{-1} \Lambda (\partial \partial \omega) - \frac{\sqrt{-1}}{2} (T \circ T + T \square T).
\]

Now \((4.5)\) follows from \((4.12), (4.1)\) and \((4.2)\). Since \(s_C = Tr_{\omega} \Theta^{(1)} = Tr_{\omega} \Theta^{(2)}, \) \((4.6)\) follows from \((4.5)\).

**Theorem 4.3.** On the compact Hermitian manifold \((M, \omega)\), the \((1,1)\) component \(\mathcal{Ric}\) of the Riemannian Ricci curvature is

\[
\mathcal{Ric} = \Theta^{(1)} - \sqrt{-1} (\Lambda \partial \partial \omega) - \frac{1}{2} (\partial \partial^* \omega + \partial \partial \omega) + \frac{\sqrt{-1}}{4} (2T \square T + T \circ T)
\]

\[
(4.13)
\]

where \((\partial^* \omega)^\#\) is the dual vector of the \((0,1)\)-form \(\partial^* \omega\).

**Proof.** By \((2.32)\), we see

\[
\mathcal{R}_{1j} = 2R_{1j}^\ell - R_{1j}^\ell
\]

At a fixed point \(p\) with “normal coordinates” \((3.19)\), we see from \((3.14)\) that

\[
R_{1j}^\ell = - \left( \frac{\partial \Gamma_{\ell}}{\partial z^j} - \frac{\partial \Gamma_{\ell}}{\partial z^i} + \Gamma^s_{ij} \Gamma_{\ell}^s - \Gamma^s_{ji} \Gamma_{\ell}^s - \Gamma_{\ell}^s \Gamma^s_{ij} \right)
\]

\[
= - \left( \frac{\partial \Gamma_{\ell}}{\partial z^j} - \frac{\partial \Gamma_{\ell}}{\partial z^i} - \Gamma_{\ell}^s \Gamma^s_{ij} \right)
\]

\[
= - \left( \frac{\partial \Gamma_{\ell}}{\partial z^j} - \frac{\partial \Gamma_{\ell}}{\partial z^i} \right) - \frac{\partial h_{i\ell}}{\partial y^i} \cdot \frac{\partial h_{j\ell}}{\partial y^j}.
\]

From \((3.28)\), we obtain

\[
(4.15)
\]

\[
- \frac{\partial \Gamma_{\ell}}{\partial z^j} = \theta^{(1)}_{ij} + \frac{\partial \Gamma^\ell_{ij}}{\partial z^j}.
\]

On the other hand,

\[
\frac{\partial \Gamma_{\ell}}{\partial z^j} = \frac{\partial}{\partial z^j} \left( \frac{1}{2} h_{i\ell} \left( \frac{\partial h_{i\ell}}{\partial z^j} - \frac{\partial h_{j\ell}}{\partial z^i} \right) \right)
\]

\[
= \frac{1}{2} \left( \frac{\partial^2 h_{i\ell}}{\partial z^i \partial z^j} - \frac{\partial^2 h_{j\ell}}{\partial z^i \partial z^j} \right) - \frac{\partial h_{i\ell}}{\partial z^i} \cdot \frac{\partial h_{j\ell}}{\partial z^j}.
\]

Similarly,

\[
\frac{\partial \Gamma_{\ell}}{\partial z^j} = \frac{\partial}{\partial z^j} \left( \frac{1}{2} h_{i\ell} \left( \frac{\partial h_{i\ell}}{\partial z^j} - \frac{\partial h_{j\ell}}{\partial z^i} \right) \right)
\]

\[
= \frac{1}{2} \left( \frac{\partial^2 h_{i\ell}}{\partial z^i \partial z^j} - \frac{\partial^2 h_{j\ell}}{\partial z^i \partial z^j} \right) - \frac{\partial h_{i\ell}}{\partial z^i} \cdot \frac{\partial h_{j\ell}}{\partial z^j}.
\]

By \((4.11)\), we see

\[
(4.16)
\]

\[
\frac{\partial \Gamma_{\ell}}{\partial z^j} = \frac{\partial \Gamma_{\ell}}{\partial z^j} - \frac{1}{2} \Lambda (\partial \partial \omega) \circ \partial^* \omega - \frac{\partial h_{i\ell}}{\partial z^i} \cdot \frac{\partial h_{j\ell}}{\partial z^j} + \frac{\partial \Gamma_{\ell}}{\partial z^j} \frac{(T \square T)_{ij}}{4}.
\]
From (4.15) and (4.16), we obtain

\[ R^t_{\ell i} = \Theta^{(1)} + \frac{\partial T^t_{\ell i}}{\partial z^j} + \frac{\partial T^t_{\ell j}}{\partial z^i} - \frac{1}{2} \left( \Lambda \partial \overline{\partial \omega} \right)_{ij} - \frac{\partial h^t_{ij}}{\partial z^i} \cdot \frac{\partial h^t_{ij}}{\partial z^j} - \frac{\partial h^t_{ji}}{\partial z^j} \cdot \frac{\partial h^t_{ji}}{\partial z^i} + \frac{(T \Box T)_{ij}}{4}. \]

Since \( \partial^* \omega = -2\sqrt{-1}R^t_{\ell i} dz^i \), the corresponding \((1,0)\) type vector field is

\[ (\partial^* \omega)^\# = -2\sqrt{-1}h^{t i} \Gamma^t_{ij} \frac{\partial}{\partial z^i}. \]

The \((1,1)\) form

\[ T((\partial^* \omega)^\#) = -2\sqrt{-1}h^{t i} T^t_{jk} \left( h^{\pi t} \Gamma^t_{ik} \right) dz^i \wedge d\overline{z}^j = -4\sqrt{-1} \frac{\partial h^t_{ij}}{\partial z^i} \frac{\partial h^t_{ij}}{\partial z^j} dz^i \wedge d\overline{z}^j. \]

Finally, we see

\[ \sqrt{-1} R^t_{ij} dz^i \wedge d\overline{z}^j = \Theta^{(1)} - \frac{\partial \partial^* \omega + \overline{\partial} \partial^* \omega}{2} - \frac{\sqrt{-1}}{2} \Lambda(\partial \overline{\partial \omega}) + \frac{\sqrt{-1} T \Box T}{4} \]

\[ + T([\partial^* \omega]^\#) + T([\partial^* \omega]^\#). \]

By (4.3), (4.14) and (4.20), we get (4.13). \( \square \)

4.2. Scalar curvature relations. On a Hermitian manifold \((M, \omega)\), we can define five different types of scalar curvatures:

1. \( s \), the scalar curvature of the background Riemannian metric;
2. \( s_R = h^{t i} h^{k j} R^t_{ijkt} \), the Riemannian type scalar curvature;
3. \( s_H = h^{t i} h^{k j} R^t_{ijkt} \), the scalar curvature of the Hermitian curvature;
4. \( s_{LC} = h^{t i} h^{k j} \Theta^t_{ijkt} \), the scalar curvature of the Levi-Civita connection;
5. \( s_C = h^{t i} h^{k j} \Theta^t_{ijkt} \), the scalar curvature of the Chern connection.

By Theorem 4.1, we get the corresponding scalar curvature relations:

**Corollary 4.4.** Let \((M, \omega)\) be a compact Hermitian manifold, then

\[ s = 2s_C + \left( \langle \partial \partial^* \omega + \overline{\partial} \partial^* \omega, \omega \rangle - 2|\partial^* \omega|^2 \right) - \frac{1}{2} |T|^2, \]

\[ s_{LC} = s_C - \frac{1}{2} \langle \partial \partial^* \omega + \overline{\partial} \partial^* \omega, \omega \rangle, \]

\[ s_H = s_C - \frac{1}{2} \langle \partial \partial^* \omega + \overline{\partial} \partial^* \omega, \omega \rangle - \frac{|T|^2}{4}, \]

and

\[ s_R = s_C - \frac{1}{2} |\partial^* \omega|^2 - \frac{1}{4} |T|^2. \]

**Proof.** By (2.33), we know \( s = 2h^{t i} \partial^* \overline{\partial} \partial \omega \). By (4.13),

\[ s = 2s_C - \langle 2\sqrt{-1}(\Lambda \partial \overline{\partial \omega})_i, \omega \rangle - \langle \partial \partial^* \omega + \overline{\partial} \partial^* \omega, \omega \rangle + \frac{3|T|^2}{2} - 2|\partial^* \omega|^2, \]

where we use the fact that \( tr \omega T([\partial^* \omega]^\#) = -|\partial^* \omega|^2 \) (see (4.19)). By (4.6), i.e.,

\[ |T|^2 = \langle \sqrt{-1} \Lambda \partial \overline{\partial \omega}, \omega \rangle + \langle \partial \partial^* \omega + \overline{\partial} \partial^* \omega, \omega \rangle, \]

23
Corollary 4.5. Let \((M, \omega)\) be a compact Hermitian manifold. Then the following are equivalent:

1. \((M, \omega)\) is Kähler;
2. \(\int s \cdot \omega^n = \int 2s_C \cdot \omega^n;\)
3. \(\int s_C \cdot \omega^n = \int s_R \cdot \omega^n;\)
4. \(\int s_C \cdot \omega^n = \int s_H \cdot \omega^n;\)
5. \(\int s_H \cdot \omega^n = \int s_{LC} \cdot \omega^n.\)

Corollary 4.6. Let \((M, \omega)\) be a compact Hermitian manifold. Then the following are equivalent:

1. \((M, \omega)\) is balanced;
2. \(\int s \cdot \omega^n = \int 2s_R \cdot \omega^n;\)
3. \(\int s \cdot \omega^n = \int 2s_H \cdot \omega^n;\)
4. \(\int s_C \cdot \omega^n = \int s_{LC} \cdot \omega^n;\)
5. \(\int s_R \cdot \omega^n = \int s_H \cdot \omega^n.\)

5. Special metrics on Hermitian manifolds

Before discussing special metrics on Hermitian manifolds, we need the following observation which is the integral version of (4.6). We assume \(\dim \mathbb{C} M = n \geq 3.\)

Proposition 5.1. On a compact Hermitian manifold \((M, \omega)\), for any 1 \(\leq k \leq n - 1\), we have

\[
\int \sqrt{-1} \partial \omega \wedge \overline{\partial} \omega \cdot \frac{\omega^{n-k-1}}{(n-3)!} = \|\partial^* \omega\|^2 - \|\partial \omega\|^2,
\]

and

\[
\int \sqrt{-1} \omega^{n-k-1} \wedge \partial \overline{\partial} \omega = (n-3)!k(n-k-1) \left(\|\partial \omega\|^2 - \|\partial^* \omega\|^2\right).
\]

Proof. At first, it is easy to see

\[
\partial \overline{\partial} \omega^k = k \omega^{k-1} \partial \overline{\partial} \omega + k(k-1) \omega^{k-2} \partial \omega \wedge \overline{\partial} \omega.
\]

On the other hand, the \((1, 2)\)-form \(\alpha := \overline{\partial} \omega - \frac{k \Lambda_\omega}{n-3}\) is primitive, i.e. \(\Lambda \alpha = 0.\) Hence, by [58, Proposition 6.29],

\[
\ast (\alpha) = (-1)^{3(n+1)/2} \left(\sqrt{-1}\right)^{1-2} \frac{L^{n-3}}{(n-3)!} \frac{L^{n-3}}{(n-3)!} \alpha = -\sqrt{-1} \omega^{n-3} \wedge \alpha.
\]
Therefore
\[
(\partial \omega, \partial \omega - \frac{L \Lambda \partial \omega}{n-1}) = (\partial \omega, \alpha) = \int \partial \omega \wedge * (\alpha) = - \int \sqrt{-1} \partial \omega \wedge \alpha \cdot \frac{\omega^{n-3}}{(n-3)!}.
\]
In particular, we have
\[
\int \sqrt{-1} \partial \omega \wedge \overline{\partial} \omega \cdot \frac{\omega^{n-3}}{(n-3)!} = - \left( \partial \omega, \partial \omega - \frac{L \Lambda \partial \omega}{n-1} \right) + \int \sqrt{-1} \partial \omega \wedge \frac{L \Lambda \overline{\partial} \omega}{n-1} \cdot \frac{\omega^{n-3}}{(n-3)!}.
\]
\[
= -||\partial \omega||^2 + ||\Lambda \partial \omega||^2 + \int \sqrt{-1} \partial \omega \cdot \frac{L \Lambda \overline{\partial} \omega}{n-1} \cdot \frac{\omega^{n-2}}{(n-3)!}.
\]
Form (3.18), \( \partial^* \omega = -\sqrt{-1} \Lambda \overline{\partial} \omega \), we have
\[
\int \sqrt{-1} \partial \omega \wedge \overline{\partial} \omega \cdot \frac{\omega^{n-3}}{(n-3)!} = ||\partial^* \omega||^2 - ||\partial \omega||^2.
\]
From integration by parts, we also get
\[
\int \sqrt{-1} \partial \partial \omega \cdot \frac{\omega^{n-3}}{(n-2)!} = ||\partial \omega||^2 - ||\partial^* \omega||^2.
\]
Hence (5.2) follows from (5.3), (5.6) and (5.7).
\[
\square
\]
Corollary 5.2. We have the follow relation on a compact Hermitian manifold \((M, \omega)\)
\[
\int s_R \cdot \frac{\omega^n}{n!} = \int s_{LC} \cdot \frac{\omega^n}{n!} - \frac{1}{(n-3)!2k(n-k-1)} \int \sqrt{-1} \omega^{n-k-1} \wedge \overline{\partial} \omega^k.
\]
Proof. It follows from Corollary 4.4 and formula (5.2).
\[
\square
\]
Fu-Wang-Wu defined in [17] that a Hermitian metric \(\omega\) satisfying
\[
(\partial \omega, \partial \omega - \frac{L \Lambda \partial \omega}{n-1}) = 0, \quad 1 \leq k \leq n-1
\]
is called a \(k\)-Gauduchon metric. It is obvious that \((n-1)\)-Gauduchon metric, i.e. \(\overline{\partial} \omega^{n-1} = 0\) is the original Gauduchon metric. It is well-known that, Hopf manifolds \(\mathbb{S}^{2n+1} \times \mathbb{S}^1\) cannot support Hermitian metrics with \(\partial \omega = 0\) (SKT) or \(d^* \omega = 0\) (balanced). They showed in [17] that on \(\mathbb{S}^5 \times \mathbb{S}^1\), there exists a 1-Gauduchon metric \(\omega\), i.e. \(\omega \wedge \overline{\partial} \omega = 0\).
As a straightforward application of Proposition 5.1, we obtain:
Corollary 5.3. If \((M, \omega)\) is \(k\)-Gauduchon \((1 \leq k \leq n-2)\) and also balanced, then \((M, \omega)\) is Kähler.

This is also true in the "conformal" setting:
Corollary 5.4. On a compact complex manifold, the following are equivalent:
(1) \((M, \omega)\) is conformally Kähler;
(2) \((M, \omega)\) is conformally \(k\)-Gauduchon for \(1 \leq k \leq n-2\), and conformally balanced;
In particular, the following are also equivalent:

(3) \((M, \omega)\) is Kähler;
(4) \((M, \omega)\) is \(k\)-Gauduchon for \(1 \leq k \leq n-2\), and conformally balanced;
(5) \((M, \omega)\) is conformally balanced and \(\Lambda^2(\partial\overline{\partial}\omega) = 0\).

\textbf{Proof.} We first show (2) implies (1). Since \(\omega\) is conformally balanced, \(\omega = e^F \omega_B\) for a balanced metric \(\omega_B\) and a smooth function \(F \in C^\infty(M, \mathbb{R})\). By the conformally \(k\)-Gauduchon condition, we know there exists \(\tilde{F} \in C^\infty(M, \mathbb{R})\) and a \(k\)-Gauduchon metric \(\omega_G\) such that \(\omega = e^F \omega_B\). Let \(f = F - \tilde{F}\), then \(\omega_G = e^f \omega_B\). Since \(\omega_G\) is \(k\)-Gauduchon,
\[
\left(e^f \omega_B\right)^{n-k-1} \wedge \partial\overline{\partial} \left(e^f \omega_B\right)^k = 0,
\]
and we obtain
\[
(5.10) \quad \omega_B^{n-k-1} \wedge \partial\overline{\partial}(e^f \omega_B)^k = 0.
\]

**Claim:** If a balanced metric \(\omega_B\) satisfies (5.10), then \(f\) is a constant and \(\omega_B\) is Kähler.

Since \(\omega_B\) is balanced, i.e., \(\partial\omega_B^{n-1} = \overline{\partial}\omega_B^{n-1} = 0\), we see \(\omega_B^{n-k-1} \wedge \partial\omega_B^k = 0\). From (5.10), we get
\[
(5.11) \quad e^k f^n(k-1) \wedge \partial\overline{\partial} \omega_B^n + \omega_B^n \wedge \partial\overline{\partial} \left(e^k f^n\right) = 0.
\]
Hence,
\[
(5.12) \quad \int_M e^k f^n \omega_B^n-k-1 \wedge \partial\overline{\partial} \omega_B^n = -\int_M \omega_B^n \wedge \partial\overline{\partial} \left(e^k f^n\right) = 0.
\]
Using integration by parts and the balanced condition \(\partial\omega_B^{n-k-1} \wedge \partial\omega_B^k = 0\), we obtain
\[
0 = -\sqrt{-1} \int_M e^k f \omega_B^{n-k-1} \wedge \partial\overline{\partial} \omega_B^k = \sqrt{-1} k(n-k-1) \int_M e^k f \omega_B^{n-3} \partial\omega_B \wedge \overline{\partial} \omega_B \\
= \sqrt{-1} k(n-k-1)(n-3) ! \int_M e^k f \partial\omega_B \wedge \left(\mathrm{e}^{\frac{k}{2}} \overline{\partial} \omega_B \cdot \frac{\omega_B^{n-3}}{(n-3)!}\right) \\
= -k(n-k-1)(n-3) ! \left\| e^{\frac{k}{2}} \overline{\partial} \omega_B \right\|_B^2,
\]
and so \(\partial\omega_B = 0\), i.e. \(\omega_B\) is Kähler. Note that, in the last step we use the fact that \(e^{\frac{k}{2}} \overline{\partial} \omega_B\) is primitive, i.e.
\[
\Lambda \left(e^{\frac{k}{2}} \overline{\partial} \omega_B\right) = e^{\frac{k}{2}} \Lambda \overline{\partial} \omega_B = -\sqrt{-1} e^{\frac{k}{2}} \partial^* \overline{\partial} \omega_B = 0,
\]
where the norm \(\|\cdot\|_B\), \(\partial^*\) and the contraction \(\Lambda\) are taken with respect to \(\omega_B\). From (5.11), we see that \(f\) must be constant if \(\omega_B\) is Kähler. The proof of the claim is complete. We know \(\omega\) is conformally Kähler.

The equivalence of (4) and (3) follows from the proof of the claim in the last paragraph under the condition \(\omega = \omega_G\), i.e. \(\tilde{F} = 0\) and \(f = F\). Next we show (5) implies (3). In
fact, if ω is conformally balanced, i.e. ω = e^f ω_B for some balanced metric ω_B and smooth function f ∈ C^∞(M, R), the condition Λ^2 Ω = 0 implies

\[ \Lambda^2_B \partial \bar{\partial} (e^f \omega_B) = 0 \]

where Λ_B is the contraction operator with respect to ω_B. By duality, we have

\[ 0 = \int_M \partial \bar{\partial} (e^f \omega_B) \wedge \omega_B^{-2} = \int_M e^f \omega_B \wedge \partial \bar{\partial} \omega_B^{-2}. \]

As similar as the proof in the last paragraph, we obtain both ω and ω_B are Kähler. □

6. Levi-Civita Ricci-flat and constant negative scalar curvature metrics on Hopf manifolds

In this section, we construct special Hermitian metrics on non-Kähler manifolds related to Hopf manifolds. More precisely,

(1) We construct explicit Levi-Civita Ricci-flat metrics on S^{2n-1} × S^1;
(2) We construct a smooth family of explicit Hermitian metrics h_λ with λ ∈ (-1, +∞) on S^{2n-1} × S^1 such that their Riemannian scalar curvature are constants and vary from a positive constant to -∞. In particular, we obtain Hermitian metrics with negative constant Riemannian scalar curvature on Hermitian manifolds with c_1 ≥ 0;
(3) We construct pluriclosed metrics on the projective bundles over S^{2n-1} × S^1.

6.1. Levi-Civita Ricci-flat metrics on Hopf manifolds. Let’s recall an example in [39, Section 6]. Let M = S^{2n-1} × S^1 be the standard n-dimensional (n ≥ 2) Hopf manifold. It is diffeomorphic to C^n \{0\}/G where G is cyclic group generated by the transformation z → 1/z. It has an induced complex structure from C^n \{0\}. On M, there is a natural metric

\[ h = \sum_{i=1}^{n} \frac{4}{|z|^2} dz^i \otimes d\bar{z}^i. \]

The complexified curvature components are

\[ R_{i\bar{k}j\bar{l}} = \frac{2 \delta_{i\bar{d}} \delta_{j\bar{k}}}{|z|^4} - \frac{\delta_{i\bar{d}z^j \bar{z}^k} + \delta_{j\bar{k}z^i \bar{z}^d}}{|z|^6}. \]

The curvature components of R are

\[ R_{i\bar{j}k\bar{l}} = \frac{3 \delta_{i\bar{d}} \delta_{j\bar{k}}}{|z|^4} - \frac{2 \delta_{i\bar{d}z^j \bar{z}^k} + 2 \delta_{j\bar{k}z^i \bar{z}^d} - \delta_{ij} \bar{z}^k \bar{z}^d}{|z|^6}. \]

The Chern curvature components are

\[ \Theta_{i\bar{j}k\bar{l}} = -\frac{\partial^2 h_{i\bar{k}}}{\partial z^i \partial \bar{z}^j} + h_{i\bar{q}} \frac{\partial h_{k\bar{l}}}{\partial z^q} \frac{\partial h_{p\bar{r}}}{\partial \bar{z}^r} = \frac{4 \delta_{kl} (\delta_{ij} |z|^2 - z^j \bar{z}^i)}{|z|^6}. \]

As computed in [39],

\[ \Theta^{(1)} = n \cdot \sqrt{-1} \partial \bar{\partial} \log |z|^2, \quad \Theta^{(2)} = \frac{n-1}{4} \omega_h; \]
(6.6) \( \mathcal{R}^{(1)} = \sqrt{-1} \partial \bar{\partial} \log |z|^2 \), \( \mathcal{R}^{(2)} = \frac{4-n}{4} \cdot \sqrt{-1} \partial \bar{\partial} \log |z|^2 + \frac{n-1}{16} \omega_h \);

(6.7) \( \text{Ric}_H = \frac{\sqrt{-1}}{2} \partial \bar{\partial} \log |z|^2 \);

(6.8) \( \text{Ric} = \frac{n-1}{2} \cdot \sqrt{-1} \partial \bar{\partial} \log |z|^2 + \frac{n-1}{8} \omega_h \)

and

(6.9) \( s_R = \frac{n^2-n}{8}, \ s_H = \frac{n-1}{8}, \ s = \frac{(2n-1)(n-1)}{4}, \ s_{LC} = \frac{n-1}{4}, \ s_C = \frac{n(n-1)}{4} \).

**Remark 6.1.** Let \( M = S^{2n-1} \times S^1 \) with \( n \geq 2 \).

1. Since \( \partial \log |z|^2 \) is a well-defined function on \( M \), one can see from (6.5) that \( c_1(M) = 0, \ c_1^{BC}(M) = 0, \) but \( c_1^{BC}(M) \neq 0 \). In particular, the canonical line bundle is topologically trivial but not holomorphically trivial;

2. Since the Chern scalar curvature \( s_C = \frac{n(n-1)}{4} \), one can see that, \( H^0(M, mK_M) = 0 \) for any integer \( m \geq 1 \). In particular, the Kodaira dimension of \( M \) is \(-\infty\), and so \( K_M \) is not a torsion line bundle;

3. The canonical metric \( \omega_h \) satisfies \( \partial \bar{\partial}^* \omega_h = 0 \), but it is well-known that \( M \) cannot support any balanced metric;

4. \( \partial \bar{\partial} \)-lemma does not hold on \( M \);

5. From the semi-positive \((1,1)\) form \( \Theta^{(1)} \), we get \( (\Theta^{(1)})^n = 0 \) and so \( c^n_1(M) = 0 \). By Theorem 3.15, \( M \) cannot admit a Hermitian metric with quasi-positive Hermitian-Ricci curvature. Moreover, the quasi-positive curvature condition in Theorem 3.15 cannot be replaced by nonnegative curvature condition.

Next we construct explicit Levi-Civita Ricci-flat Hermitian metrics on all Hopf manifolds.

**Theorem 6.2.** Let

\[
(6.10) \quad \tilde{\omega} = \omega_h - \frac{4}{n} \mathcal{R}^{(1)}(\omega_h),
\]

then the first Levi-Civita Ricci curvature of \( \tilde{\omega} \) is zero, i.e.

\[
(6.11) \quad \mathcal{R}^{(1)}(\tilde{\omega}) = 0.
\]

**Proof.** We consider the perturbed metric

\[
(6.12) \quad \tilde{\omega} = \omega_h + 4\lambda \mathcal{R}^{(1)}(\omega_h), \quad \text{with} \quad \lambda > -1,
\]

where \( \mathcal{R}^{(1)}(\omega_h) = \sqrt{-1} \partial \bar{\partial} \log |z|^2 \). That is

\[
\tilde{h}_{ij} = \frac{4}{|z|^2} \left( (1 + \lambda) \delta_{ij} - \frac{\lambda z_i z_j}{|z|^2} \right), \quad \text{and} \quad \tilde{h}^{ij} = \frac{1}{4} \left( \frac{\delta_{ij} - \lambda z_i z_j}{1 + \lambda} \right).
\]

Moreover,

\[
\frac{\partial \tilde{h}_{ij}}{\partial \bar{z}^k} = \frac{8 \lambda z_j z^k z_i}{|z|^6} - \frac{4(1 + \lambda) \delta_{ij} z^k + 4 \delta_{ik} z^j}{|z|^4} \quad \text{and} \quad \frac{\partial \tilde{h}^{ij}}{\partial z^k} = \frac{4 \delta_{ik} z^j - \delta_{ij} z^k}{|z|^4}. \]
The Christoffel symbols of $\tilde{h}$ are
\[
\tilde{\Gamma}^\lambda_{\ ji} = \frac{1}{2} \tilde{h}_{\ i} \left( \frac{\partial \tilde{h}_{\ i}^{\ \lambda}}{\partial \sigma^j} - \frac{\partial \tilde{h}_{\ i}^{\ \lambda}}{\partial \sigma^j} \right) = - \frac{(n - 1) z^j}{2 |z|^2 (1 + \lambda)},
\]
and
\[
\partial^* \tilde{\omega} = -2\sqrt{-1} \tilde{h}_{\ ji} \sigma^j = \sqrt{-1} \frac{n - 1}{1 + \lambda} \mathbf{J} \log |z|^2, \quad \text{and} \quad \frac{\partial \partial^* \tilde{\omega} + \sqrt{-1} \mathbf{J} \partial^* \tilde{\omega}}{2} = \frac{n - 1}{1 + \lambda} \cdot \sqrt{-1} \mathbf{J} \partial \log |z|^2,
\]
where the adjoint operators $\mathbf{J}^*$ and $\partial^*$ are taken with respect to the new metric $\tilde{\omega}$. Finally, since $\det(\tilde{h}) = (1 + \lambda)^{n-1} 4^n |z|^{-2n}$, we obtain
\[
\Theta^{(1)}(\tilde{\omega}) = -\sqrt{-1} \partial \log \det \tilde{h} = n \cdot \sqrt{-1} \partial \log |z|^2,
\]
and
\[
\mathfrak{R}^{(1)}(\tilde{\omega}) = \Theta^{(1)}(\tilde{\omega}) - \frac{\partial \partial^* \tilde{\omega} + \sqrt{-1} \mathbf{J} \partial^* \tilde{\omega}}{2} = \left( n - \frac{n - 1}{1 + \lambda} \right) \cdot \sqrt{-1} \partial \log |z|^2.
\]
Now it is obvious that when $\lambda = -\frac{1}{n}$, $\mathfrak{R}^{(1)}(\tilde{\omega}) = 0$. \hfill \Box

6.2. **Hermitian metrics of constant negative scalar curvatures on Hopf manifolds.** Using the same setting as in Theorem 6.2, we let
\[
\tilde{\omega} = \omega_h + 4\lambda \mathfrak{R}^{(1)}(\omega_h), \quad \text{with} \quad \lambda > -1.
\]
Hence, the Chern scalar curvature of $\tilde{\omega}$ is
\[
\tilde{s}_C = \langle \Theta^{(1)}(\tilde{\omega}), \tilde{\omega} \rangle = \langle n \cdot \sqrt{-1} \partial \log |z|^2, \tilde{\omega} \rangle = \frac{n(n - 1)}{4(1 + \lambda)}.
\]
On the other hand, as computed in Theorem 6.2,
\[
\langle \partial \partial^* \tilde{\omega} + \sqrt{-1} \mathbf{J} \partial^* \tilde{\omega}, \tilde{\omega} \rangle = 2 \left( \frac{n - 1}{1 + \lambda} \cdot \sqrt{-1} \partial \log |z|^2, \tilde{\omega} \right) = \frac{(n - 1)^2}{2(1 + \lambda)^2}.
\]
Similarly,
\[
|\partial^* \tilde{\omega}|^2 = \frac{(n - 1)^2}{4(1 + \lambda)^2}.
\]
Moreover, for the torsion $\tilde{T}$ of $\tilde{\omega}$, it is
\[
\tilde{T}^m_{\ ij} = \tilde{h}^{m\sigma} \left( \frac{\partial \tilde{h}_{\ i}^{\ \sigma}}{\partial z^j} - \frac{\partial \tilde{h}_{\ i}^{\ \sigma}}{\partial z^j} \right) = \frac{\delta_{m\ell} z^j - \delta_{mj} z^j}{(1 + \lambda)} \frac{1}{|z|^2}
\]
and so
\[
|\tilde{T}|^2 = \tilde{h}_{m\sigma} \cdot \tilde{h}^{j\eta} \cdot \tilde{h}^{m\sigma} \cdot \tilde{T}^m_{\ ij} \cdot \tilde{T}^m_{\ k} = \frac{n - 1}{2(1 + \lambda)^2}.
\]
Finally, by the scalar curvature relation formula (4.21), we see the Riemannian scalar curvature $\tilde{s}$ of $\tilde{\omega}$ is
\[
\tilde{s} = 2\tilde{s}_C + \left( \langle \partial \partial^* \tilde{\omega} + \sqrt{-1} \mathbf{J} \partial^* \tilde{\omega}, \tilde{\omega} \rangle - 2|\partial^* \tilde{\omega}|^2 \right) - \frac{1}{2} |\tilde{T}|^2 = 2\tilde{s}_C - \frac{1}{2} |\tilde{T}|^2 = \frac{n(n - 1)}{2(1 + \lambda)^2} \left[ \lambda - \frac{1 - 2n}{2n} \right].
\]
It is easy to see that when \( \lambda \in (-1, \infty) \),
\[
(6.18) \quad \tilde{s} \in (-\infty, \frac{n^2(n-1)}{4}] .
\]

More precisely, when \( \lambda = \frac{1-n}{n} \), \( \tilde{s} = \frac{n^2(n-1)}{4} \). Hence, for the smooth family of Hermitian metrics \( \tilde{\omega} = \omega_h + 4\Lambda^2 \omega(\omega_h) \), with \( \lambda > -1 \),
\begin{itemize}
  
  \begin{enumerate}
  
  \item \( \lambda > \frac{1-2n}{2n} \), \( \tilde{\omega} \) has positive constant Riemannian scalar curvature \( \tilde{s} \);
  
  \item \( \lambda = \frac{1-2n}{2n} \), \( \tilde{\omega} \) has constant zero Riemannian scalar curvature \( \tilde{s} \);
  
  \item \( \lambda < \frac{1-2n}{2n} \), \( \tilde{\omega} \) has negative constant Riemannian scalar curvature \( \tilde{s} \).
  
  Note also that when \( \lambda \to -1 \), \( \tilde{s} \to -\infty \).
  
\end{enumerate}
\end{itemize}

To our knowledge, this is the first example to show that Hermitian manifolds with non-negative first Chern class can admit Hermitian metrics with strictly negative Riemannian scalar curvature.

**Remark 6.3.** By using the same ideas and also the Ricci curvature relations in Section 4, one can construct various “Einstein metrics” by using different Ricci curvatures introduced in the previous sections.

### 6.3. Pluriclosed metrics on the projective bundles over Hopf manifolds

Let \( M = S^{2n-1} \times S^1 \) with \( n \geq 2 \), and \( E = T^{1,0}M \). Suppose \( X = \mathbb{P}(E^*) \) and \( L = \mathcal{O}(E^*)(1) \) is the tautological line bundle of the fiber bundle \( \pi : X \to M \). By the adjunction formula,
\[
K_X = L^{-n} \otimes \pi^* (K_M \otimes \det E)
\]
we see
\[
(6.20) \quad K_X = L^{-n}.
\]

It is obvious, when restricted to the fiber \( X_s := \mathbb{P}(E^*_s) \cong \mathbb{P}^{n-1} \), \( K_X|_{X_s} \cong \mathcal{O}_{\mathbb{P}^{n-1}}(-n) \). Hence, \( K_X \) is not topologically trivial and moreover, \( c_1(X) \geq 0 \). However, by a straightforward calculation we see \( c_1^{2n-1}(X) = 0 \) where \( \dim \mathbb{C}X = 2n - 1 \). In fact, as described in [37], there is a natural Hermitian metric \( g \) on \( L \) induced by the Hermitian metric \( \omega_h \) on \( E = T^{1,0}M \). Since the Chern curvature tensor of \( (E = T^{1,0}M, \omega_h) \) is Griffiths-semi-positive (see formula (6.4), or [39, Proposition 6.1]), the curvature \( (1,1) \)-form \( \Theta_g \) of \( (L, g) \) is semi-positive and strictly positive on each fiber. We can see that \( (\Theta_{\omega_h})^{2n-1} = 0 \) and so \( c_1^{2n-1}(X) = 0 \). It is easy to see that the Chern scalar curvature of \( X \) is strictly positive, i.e.
\[
n \cdot \text{tr}_g \Theta_g > 0.
\]

Hence, \( H^0(X, mK_X) = 0 \) for any integer \( m > 0 \). Now we summarize the discussion as follows.

**Proposition 6.4.** Let \( M = S^{2n-1} \times S^1 \) with \( n \geq 2 \), and \( E = T^{1,0}M \). Suppose \( X = \mathbb{P}(E^*) \).
\begin{itemize}

  \begin{enumerate}

  \item \( K_X^{-1} \) admits a Hermitian metric with semi-positive Chern curvature;

  \item \( K_X^{-1} \) is not topologically trivial, i.e. \( c_1(X) \neq 0 \). Moreover, \( c_1(X) \geq 0 \), but \( c_1^{2n-1}(X) = 0 \);

  \item \( H^0(X, mK_X) = 0 \) for any integer \( m > 0 \).

\end{enumerate}

30
Next we consider a more general setting. Suppose \( n \geq 2 \) and \( k \geq 1 \). Let \( M_n = \mathbb{S}^{2n-1} \times \mathbb{S}^1 \). \( E_k = T^{1,0}M_n \oplus \cdots \oplus T^{1,0}M_n \) and \( X_{n,k} = \mathbb{P}(E_k^*) \rightarrow M_n \). Since \( \pi : X_{n,k} \rightarrow M_n \) is a proper holomorphic submersion and \( M_n \) can not admit any balanced metric, by [41, Proposition 1.9], \( X_{n,k} \) can not support balanced metrics. In particular, \( X_{n,k} \) is not in the Fujiki class \( \mathcal{C} \). On the other hand, we consider a special case \( \pi : X_{2,k} \rightarrow M_2 \). It is obvious that \( E_k \) has an induced Hermitian metric with Griffiths-semi-positive curvature. Let \( F \) be the tautological line bundle of \( X_{2,k} \). The induced Hermitian metric on \( F \) has semi-positive curvature tensor \( \Theta_F \) which is also strictly positive on each fiber. Now we can construct a family of Hermitian metrics \( \omega \) with \( \partial \bar{\partial} \omega = 0 \) on \( X_{2,k} \). Let \( \omega_0 \) be the canonical metric (6.1) on \( M_2 = \mathbb{S}^3 \times \mathbb{S}^1 \), and it is obvious that \( \partial \bar{\partial} \omega_0 = 0 \). Then for any \( \lambda > 0 \),

\[
\omega := \pi^*(\omega_0) + \lambda \Theta_F
\]

is a Hermitian metric on \( X_{2,k} \). Moreover, it satisfies \( \partial \bar{\partial} \omega = 0 \).

**Proposition 6.5.** Suppose \( n \geq 2 \) and \( k \geq 1 \). Let \( M_n = \mathbb{S}^{2n-1} \times \mathbb{S}^1 \). \( E_k = T^{1,0}M_n \oplus \cdots \oplus T^{1,0}M_n \) and \( X_{n,k} = \mathbb{P}(E_k^*) \). Then

1. \( X_{n,k} \) can not support any balanced metric;
2. \( X_{2,k} \) admits a Hermitian metric \( \omega \) with \( \partial \bar{\partial} \omega = 0 \).

7. Appendix: the Riemannian Ricci curvature and \( * \)-Ricci curvature

In this appendix, we provide more details on the complexification of Riemannian Ricci curvatures on almost Hermitian manifolds.

**7.1. The Riemannian Ricci curvature.**

**Lemma 7.1.** On an almost Hermitian manifold \((M, h)\), the Riemannian Ricci curvature of the background Riemannian manifold \((M, g)\) satisfies

\[
\text{Ric}(X,Y) = h^{\overline{i}} \left[ R \left( \frac{\partial}{\partial \overline{z}^i}, X, Y, \frac{\partial}{\partial \overline{z}^j} \right) + R \left( \frac{\partial}{\partial \overline{z}^j}, Y, X, \frac{\partial}{\partial \overline{z}^i} \right) \right]
\]

for any \( X,Y \in T_{\overline{\theta}}M \). The Riemannian scalar curvature of \((M, g)\) is

\[
s = 2h^{\overline{i}}h^{\overline{j}} \left( 2R_{\overline{\theta}i\overline{j}k\overline{l}} - R_{\overline{\theta}i\overline{k}\overline{l}} \right).
\]
Proof. For any \( X,Y \in T_\mathbb{R}M \), by using real coordinates \( \{ x^i, x^I \} \) and the relations (2.7), (2.11), (2.15) and (2.16), one can see:

\[
Ric(X,Y) = g_{i\ell} R\left( \frac{\partial}{\partial x^i}, X, Y, \frac{\partial}{\partial x^\ell} \right) + g_{iL} R\left( \frac{\partial}{\partial x^i}, X, Y, \frac{\partial}{\partial x^L} \right) \\
+ g^{i\ell} R\left( \frac{\partial}{\partial x^i}, X, Y, \frac{\partial}{\partial x^\ell} \right) + g^{iL} R\left( \frac{\partial}{\partial x^i}, X, Y, \frac{\partial}{\partial x^L} \right) \\
= g^{i\ell} R\left( \frac{\partial}{\partial z^i}, X, Y, \frac{\partial}{\partial z^\ell} \right) + \sqrt{-1} g^{iL} R\left( \frac{\partial}{\partial z^i}, X, Y, \frac{\partial}{\partial z^L} \right) \\
+ \sqrt{-1} g^{i\ell} R\left( \frac{\partial}{\partial z^i}, X, Y, \frac{\partial}{\partial z^\ell} \right) - g^{iL} R\left( \frac{\partial}{\partial z^i}, X, Y, \frac{\partial}{\partial z^L} \right) \\
= h^{i\ell} R\left( \frac{\partial}{\partial z^i}, X, Y, \frac{\partial}{\partial z^\ell} \right) + h^{iL} R\left( \frac{\partial}{\partial z^i}, X, Y, \frac{\partial}{\partial z^L} \right) \\
= h^T R\left( \frac{\partial}{\partial z^i}, X, Y, \frac{\partial}{\partial z^\ell} \right) + h^T R\left( \frac{\partial}{\partial z^i}, X, Y, \frac{\partial}{\partial z^L} \right).
\]

By using the symmetry \( R(X,Y,Z,W) = R(Y,X,W,Z) \) for \( X,Y,Z,W \in T_\mathbb{C}M \), we get the following formula for the Riemannian scalar curvature,

\[
s = h_T h^T \left( R_{ik\ell j} + R_{i\ell k j} + R_{j k i \ell} + R_{j \ell i k} \right) \\
= 2 h_T h^T \left( R_{ik\ell j} + R_{i\ell k j} \right) \\
= 2 h_T h^T \left( 2 R_{ik\ell j} - R_{j \ell i k} \right).
\]

The proof of Lemma 7.1 is complete. \( \square \)

7.2. \(*\)-Ricci curvature and \(*\)-scalar curvature.

**Definition 7.2.** Let \( \{ e_i \}_{i=1}^{2n} \) be an orthonormal basis of \( (T_\mathbb{R}M, g) \), the (real) \(*\)-Ricci curvature of \( (M, g) \) is defined to be (e.g. [56])

\[
(7.3) \quad Ric^*(X,Y) := \sum_{i=1}^{2n} R(e_i, X, JY, Je_i),
\]

for any \( X,Y \in T_\mathbb{R}M \). The \(*\)-scalar curvature (with respect to the Riemannian metric) is defined to be

\[
(7.4) \quad s^* = \sum_{j=1}^{2n} Ric^*(e_j, e_j).
\]

It is easy to see that, for any \( X,Y \in T_\mathbb{R}M \),

\[
(7.5) \quad Ric^*(X,Y) = Ric^*(JY, JX).
\]
Lemma 7.3. We have the following formula for $*$-Ricci curvature,

\begin{align}
(7.6) \quad \text{Ric}^*(X,Y) &= \sqrt{-1}h^\mathcal{G}R\left(\frac{\partial}{\partial \bar{z}^i}, X, JY, \frac{\partial}{\partial z^l}\right) - \sqrt{-1}h^\mathcal{T}R\left(\frac{\partial}{\partial z^i}, X, JY, \frac{\partial}{\partial \bar{z}^l}\right) \\
(7.7) &= \sqrt{-1}h^\mathcal{G}R\left(\frac{\partial}{\partial \bar{z}^i}, \frac{\partial}{\partial \bar{z}^k}, X, JY\right).
\end{align}

The $*$-scalar curvature is

\begin{equation}
(7.8) \quad s^* = 2h^\mathcal{G}h^\mathcal{T}R_{\ell jk\ell} = 2s_H.
\end{equation}

Proof. As similar as the computations in Lemma 7.1, we can write down the $*$-Ricci curvature in real coordinates $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^k}\}$ and show:

\begin{align*}
\text{Ric}^*(X,Y) &= \sqrt{-1}h^\mathcal{G}R\left(\frac{\partial}{\partial \bar{z}^i}, X, JY, \frac{\partial}{\partial z^l}\right) - \sqrt{-1}h^\mathcal{T}R\left(\frac{\partial}{\partial z^i}, X, JY, \frac{\partial}{\partial \bar{z}^l}\right) \\
&= 2\sqrt{-1}h^\mathcal{G}R\left(\frac{\partial}{\partial \bar{z}^k}, \frac{\partial}{\partial \bar{z}^j}, X, JY\right),
\end{align*}

where the last step follows by Bianchi identity. Therefore (7.7) follows by the relation (7.5). For the scalar curvature $s^*$, by definition, it is

\begin{align*}
s^* &= g^{jk}\text{Ric}^*\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right) + g^{jK}\text{Ric}^*\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^K}\right) \\
&\quad + g^{jk}\text{Ric}^*\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right) + g^{jK}\text{Ric}^*\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^K}\right).
\end{align*}

By using the symmetry $R(X,Y,Z,W) = R(Y,X,W,Z)$ and (7.6), we see

\begin{align*}
s^* &= (\sqrt{-1}h^\mathcal{G})\left[\sqrt{-1}h^\mathcal{T}R\left(\frac{\partial}{\partial \bar{z}^i}, \frac{\partial}{\partial \bar{z}^j}, \frac{\partial}{\partial z^k}, \frac{\partial}{\partial z^l}\right) - \sqrt{-1}h^\mathcal{G}R\left(\frac{\partial}{\partial \bar{z}^i}, \frac{\partial}{\partial \bar{z}^j}, \frac{\partial}{\partial \bar{z}^k}, \frac{\partial}{\partial \bar{z}^l}\right)\right] \\
&\quad - (\sqrt{-1}h^\mathcal{T})\left[\sqrt{-1}h^\mathcal{G}R\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}, \frac{\partial}{\partial \bar{z}^k}, \frac{\partial}{\partial \bar{z}^l}\right) - \sqrt{-1}h^\mathcal{T}R\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}, \frac{\partial}{\partial \bar{z}^k}, \frac{\partial}{\partial \bar{z}^l}\right)\right] \\
&= -h^\mathcal{G}h^\mathcal{T}R_{\ell jk\ell} + h^\mathcal{G}h^\mathcal{T}R_{\ell jk\ell} + h^\mathcal{G}h^\mathcal{T}R_{\ell jk\ell} - h^\mathcal{G}h^\mathcal{T}R_{\ell jk\ell} \\
&= 2h^\mathcal{G}h^\mathcal{T}R_{\ell jk\ell} = 2s_H,
\end{align*}

where the last step follows by Bianchi identity. $\square$
Remark 7.4. It is easy to see that ∗-Ricci curvature is neither symmetric nor skew-symmetric. For example, by using the Hermitian Ricci tensor and (7.7), we see the following submatrix of the real matrix representation of ∗-Ricci curvature:

\[
Ric^* \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) = \sqrt{-1} h^{k\ell} R \left( \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}, \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right)
\]

\[
= -h^{k\ell} R \left( \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}, \frac{\partial}{\partial z^i} + \frac{\partial}{\partial \bar{z}^i}, \frac{\partial}{\partial z^j} - \frac{\partial}{\partial \bar{z}^j} \right)
\]

\[
= \left( R_{ij} - R_{ij} \right) + \left( R_{ij} - R_{ij} \right)
\]

(7.9)

The first part in (7.9) is skew symmetric whereas the second part is symmetric. Hence, as a real (0, 2) tensor, it is impossible to define the positivity or negativity for the ∗-Ricci curvature. That is \( Ric^*(X, X) > 0 \) for all nonzero vector \( X \in T \mathbb{R} M \) can not happen on any almost Hermitian manifold. To make the ∗-Ricci tensor a symmetric tensor, an extra condition as

(7.10) \[ R(X, Y, Z, W) = R(X, Y, JW, JZ) \]

is sufficient (see, e.g. [28, 56]).
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