BASE MANIFOLDS FOR LAGRANGIAN FIBRATIONS ON HYPERKÄHLER MANIFOLDS

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ABSTRACT. Let \( f : X \to B \) be a fibration from a hyperkähler manifold to a complex space \( B \). Assuming that \( B \) is smooth, we show that \( B \cong \mathbb{P}^n \). This generalises a theorem of J.-M. Hwang to the Kähler case.

1. INTRODUCTION

One of the most important and startling conjectures in the study of hyperkähler manifolds \( X \) says that the base space of any non-trivial fibration \( X \to B \) is the complex projective space \( \mathbb{P}^n \), where \( n = \dim X / 2 \). We refer the reader to [GHJ03, 21.4] for a discussion of this conjecture. Any such fibration is automatically Lagrangian with respect to the holomorphic symplectic form by works of Matsushita; see [Mat03] or Section 2 for a summary of his results.

With the additional hypothesis that \( X \) be projective and that \( B \) be smooth, the conjecture is known to hold by work of Jun-Muk Hwang [Hwa08]. In this short note, we remove Hwang’s projectivity assumption on \( X \), and prove the following result:

**Theorem 1.1.** Let \( X \) be a hyperkähler manifold, and let \( f : X \to B \) be a fibration onto a complex space \( B \). If \( B \) is smooth, then \( B \cong \mathbb{P}^n \).

Our proof is a simple combination of fundamental results due to A. Fujiki [Fuj83], J.-M. Hwang [Hwa08], D. Matsushita [Mat03, Mat09], and Y.-T. Siu [Siu91]: after noticing that the base manifold has to be projective, we pull back a very ample line bundle from \( B \) to \( X \) and use this line bundle to deform the given fibration to a sequence of projective ones. Then, we apply Hwang’s theorem to these projective deformations and use global deformation rigidity of \( \mathbb{P}^n \) to conclude the desired result for the central fibre.

2. PRELIMINARIES

We start by fixing our notation and by recalling definitions of the basic objects investigated in this note.

**Definition 2.1.** A compact Kähler manifold is called *hyperkähler* or *irreducible holomorphic symplectic* if it is simply-connected, and if \( H^0(X, \Omega_X^2) = C\sigma \), where \( \sigma \) is everywhere non-degenerate. A *fibration* on \( X \) is a (proper) surjective holomorphic map \( f : X \to B \) with \( f_*\mathcal{O}_X = \mathcal{O}_B \) from \( X \) to a complex space \( B \) with \( 0 < \dim B < \dim X \). In particular, the base \( B \) of a fibration is normal, and \( f \) has connected fibres. A *Lagrangian fibration* on \( X \) is a
fibration \( f : X \to B \) such that every irreducible component of every fibre of \( f \) is a Lagrangian subvariety with respect to the holomorphic symplectic form \( \sigma \).

To make this note more self-contained, we collect some known results concerning fibrations on hyperkähler manifolds, which we will use in the subsequent proof, in the following proposition.

**Proposition 2.2.** Let \( X \) be a hyperkähler manifold of dimension \( 2n \), and let \( f : X \to B \) be a fibration onto a normal complex space \( B \). Then, \( f \) is a Lagrangian fibration onto a normal projective variety. In particular, \( B \) has dimension \( n \).

**Proof.** As explained in [AC13, Thm. 1 and footnote], using results of Varouchas [Var86, Var89] and the fundamental results of Matsushita [Mat99, Mat01, Mat00], one shows without any a priori assumption on the base of the fibration that \( B \) is a normal Kähler space. Then, [Mat03, Thm. 2.1 and Thm. 3.1] imply the claim. \( \square \)

3. **Proof of Theorem 1.1**

Let \( X \) be a hyperkähler manifold of dimension \( 2n \), and let \( f : X \to B \) be a fibration onto a smooth complex space \( B \).

Since \( B \) is projective by Proposition 2.2, there exists a very ample line bundle on \( B \). Let \( L \) denote its pullback under \( f \). Furthermore, let \( \mathcal{X} \to (S, 0) \) be the (smooth) Kuranishi space of \( X \); in particular, \( \mathcal{X} \to (S, 0) \) is a smooth family of hyperkähler manifolds. By [Mat09, Thm. 1.1(1) and 1.1(2)] there exists a smooth hypersurface \((S_L, 0)\) in \((S, 0)\), and a line bundle \( \mathcal{L} \) on the pullback \( \mathcal{X}_L = S_L \times_S \mathcal{X} \) of \( \mathcal{X} \) to \( S_L \) such that the restriction of \( \mathcal{L} \) to the fibre over the reference point 0 is isomorphic to \( L \). We denote the natural projection \( \mathcal{X}_L \to S_L \) by \( p \), and we note that both \( \mathcal{X}_L \) as well as \( p \) are smooth. As usual we will take a representative of the germ \((S_L, 0)\) and shrink it if necessary (keeping the base point), usually without mentioning this explicitly.

By [Mat09, Thm. 1.1(3) and Cor. 1.2] the pushforward \( p_* \mathcal{L} \) is a vector bundle, the canonical map \( p^* p_* \mathcal{L} \to \mathcal{L} \) is surjective, and thus gives rise to a morphism \( F: \mathcal{X}_L \to \mathbb{P}_{S_L}(p_* \mathcal{L}^\vee) \) over \( S_L \) that extends \( f: X \to B \) to the whole family \( \mathcal{X}_L \).

**Lemma 3.1.** We have \( b_2(X) \geq 4 \). In particular, \( S_L \) has positive dimension.

**Proof.** Let \( \alpha \) be a Kähler class on \( X \). Since \( L \) is not ample, its Chern class is not a multiple of \( \alpha \), hence \( h^{1,1}(X) \geq 2 \), implying the first claim. For the second claim, note that the dimension of the Kuranishi space \( S \) of \( X \) is \( \dim H^1(X, T_X) = h^{1,1}(X) \geq 2 \), and that \( S_L \) is a hyperplane in \( S \). \( \square \)

So \( \mathcal{X}_L \to S_L \) is a positive-dimensional smooth family of hyperkähler manifolds. Restricting the family \( \mathcal{X} \) to a general smooth embedded disk \( \Delta \subset S_L \) through the origin, we obtain
a commutative diagram

\[
\begin{array}{ccc}
\mathcal{X}_\Delta & \xrightarrow{F} & \mathcal{B}_\Delta \\
p & \downarrow & \downarrow \pi \\
\Delta & & \Delta
\end{array}
\]

(3.1)

where \( p: \mathcal{X}_\Delta \to \Delta \) is a smooth family of hyperkähler manifolds with smooth total space, and \( \mathcal{B}_\Delta \) is the scheme-theoretic image of the fibration induced by a sufficiently high tensor power of \( \mathcal{L}|_{\mathcal{X}_\Delta} \). Note that \( \pi: \mathcal{B}_\Delta \to \Delta \) is a flat family with normal total space.

**Lemma 3.2.** The scheme-theoretic fibre \((\mathcal{B}_\Delta)_0 = \pi^{-1}(0)\) is reduced, hence smooth.

**Proof.** Since \( \mathcal{B}_\Delta \) is normal, it is non-singular at general points of the central fibre. Since \( p \) is a smooth morphism, it follows from diagram (3.1) that \((\mathcal{B}_\Delta)_0\) is generically reduced. Let \( t \) be a coordinate on \( \Delta \). We note that \( \pi^*t \) is not a zerodivisor in any of the local rings \( \mathcal{O}_{\mathcal{B}_\Delta, b} \) of points \( b \in \pi^{-1}(0) \). Consequently, as \( \mathcal{B}_\Delta \) satisfies Serre’s condition \( S_2 \), the scheme \((\mathcal{B}_\Delta)_0 = (\mathcal{O}^{-1}(0), \mathcal{O}_{\mathcal{B}_\Delta}/\pi^*t \cdot \mathcal{O}_{\mathcal{B}_\Delta})\) does not have any embedded components, and is therefore reduced, cf. [Mat80, p. 125]. Hence, \((\mathcal{B}_\Delta)_0 = ((\mathcal{B}_\Delta)_0)_{\text{red}} = B\), which is smooth by assumption. \( \square \)

Lemma 3.2 implies that \( \pi: \mathcal{B}_\Delta \to \Delta \) is flat with smooth central fibre, hence a smooth morphism. Moreover, [Fuj83, Thm. 4.8(2)] implies that there exists a dense subset \( T \subset \Delta \) such that \( X_t := p^{-1}(t) \) is projective for all \( t \in T \), see also [GHJ03, Prop. 26.6]. Therefore, by Hwang’s theorem [Hwa08] the fibre \( B_t := \pi^{-1}(t) \) is isomorphic to \( \mathbb{P}^n \) for all \( t \in T \). Hence, we find a sequence of points \( \{p_v\}_{v \in \mathbb{N}} \) in \( T \subset \Delta \) such that \( \lim_{v \to \infty} (p_v) = 0 \), and such that \( B_t \cong \mathbb{P}^n \). Hence, global deformation rigidity of \( \mathbb{P}^n \), see [Siu91, paragraph following the Main Theorem] implies that the central fibre is likewise isomorphic to \( \mathbb{P}^n \). This concludes the proof of Theorem 1.1.

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