Boundary Magnetization of the Generalized McCoy-Wu type Random Ising Models

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Abstract. This paper investigates the generalized McCoy-Wu type random Ising model. It is found that boundary magnetization behaves as \((T_c - T)^{\beta'}\), \(\beta' = 1\) just below the critical temperature. From numerical analyses it is revealed that this type of critical behavior, \(\beta' = 1\), is universal for any type of randomness. This universal behavior is explained in terms of the destruction of the devil’s staircase structure.

1. Introduction
In the past thirty years, much effort has been done to reach some understandings of an ordered phase in the disordered magnetic materials. [1] In the research of the random spin systems, exactly solvable models are convincing. In one such model, McCoy and Wu(MW) discovered the two-dimensional exactly solvable model in 1968. [2] Their results are summarized in an excellent review and book. [3] [4] The extension of the MW model has been tried by several authors. In this extension, the horizontal bonds also vary row to row randomly, but the same in each row. Hence, this model is called the generalized MW(GMW) model in this paper. This extension is so attractive, because it may give insight into the spin glass problem. That is, the GMW model can include the frustration effect. [5] In 1980, Longa obtained an exact expression of the ‘critical temperature’ of the GMW model. [6] In our previous work, we derived the explicit expression of free energy in a exact manner. [7] In 1987, Shanker and Murthy studied the critical singularity of the GMW model in detail, i.e. the GMW model exhibits the Griffith singularity. [8] In spite of above the detailed studies, the nature of the GMW model have not been clarified yet. We would like to discuss the relation between the devil’s staircase (DS) structure and the GMW type random spin system.

2. The GMW type random Ising model and its free energy
In this paper, we consider the two-dimensional random Ising model on the square lattice. This lattice has \(M(\text{vertical}) \times N(\text{horizontal})\) lattice points, and is periodic in the horizontal direction. We set the boundary condition to be free for the vertical direction. Ising spin on this lattice interacts with the coupling strength \(J^V(j)\) (vertical) and \(J^H(j)\) (horizontal), where \(j\) denotes
the row. Thus, we can write the model Hamiltonian of the GMW model as follows,
\[ H = -\sum_{i,j} J^V(j) \sigma_{i,j} \cdot \sigma_{i,j+1} - \sum_{i,j} J^H(j) \sigma_{i,j} \cdot \sigma_{i+1,j}, \]

where the spin variable at the site \( i, j \) takes the value \( \sigma_{i,j} = \pm 1 \). When the vertical bonds are random, we call this model V-type which is the original MW model itself. In the same manner, we refer to H-type and VH-type according to their type of randomness.

Below, we derive the exact solution of the GMW model following the original work of MW. Using the translational symmetry of the system, the square of the partition function is expressed as
\[ Z^2 = \prod_\theta \text{det} B(\theta), \]

where the \( 4M \times 4M \) matrices \( B(\theta) \) are given by
\[
B(j, j : \theta) = \begin{pmatrix}
0 & 1 + z^H e^{i\theta} & -1 & -1 \\
-1 - z^V e^{-i\theta} & 0 & 1 & -1 \\
1 & -1 & 0 & 1 \\
1 & 1 & -1 & 0
\end{pmatrix},
\]
\[
B(j, j + 1 : \theta) = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & z^V \\
0 & 0 & 0 & 0
\end{pmatrix},
\]

with \( \theta = (2j - 1)/N (j = 1, 2, \cdots, N) \). In the above,\n\[
z^H_j = \tanh \beta J^H(j), \quad z^V_j = \tanh \beta J^V(j),
\]

where \( \beta = 1/T, T \) is the temperature, and we set \( k_B = 1 \). We can reduce the matrix \( B(\theta) \) to the product of \( 2 \times 2 \) random matrices for each \( \theta \). Thus, we can evaluate Eq.(2) using random variable \( y_j(\theta) \) which is defined in the following recursion relation,
\[
y_{j+1}(\theta) = \frac{a_j(\theta) + z^V_j y_j(\theta)}{a_j^2(\theta) + b_j^2(\theta) + a_j(\theta) z^V_j y_j(\theta)},
\]

where
\[
a_j(\theta) = -2 z^H_j \sin \theta \left| 1 + z^H_j e^{i\theta} \right|^{-2}, \quad b_j(\theta) = (1 - z^H_j^2) \left| 1 + z^H_j e^{i\theta} \right|^{-2},
\]

with \( y_0(\theta) = 0 \). Finally, the free energy of the GMW model is obtained as
\[
-\beta F = \lim_{M \to \infty} \frac{1}{M} \left[ \sum \ln \{ 2 \cosh (\beta J^V(j)) \} + \sum \ln \{ 2 \cosh (\beta J^H(j)) \} \right] \\
+ \lim_{M \to \infty} \frac{1}{4\pi M} \left[ \sum \int d\theta \ln \{ 1 + z^H_j^2 + 2 z^H_j \cos \theta \} \right] \\
+ \sum \int d\theta \ln \{ a_j^2(\theta) + b_j^2(\theta) + a_j(\theta) z^V_j y_j(\theta) \}. \]
3. Boundary magnetization

3.1. Derivation of boundary magnetization formula

By the use of the same method which is developed in §2, we can also obtain boundary magnetization. To obtain boundary magnetization, a magnetic field \( h \) is applied to the \( M \)-th row. The partition function of such a system (\( M \times N \) lattice with boundary magnetic field \( h \)) is equivalent to one of a new lattice which includes an additional row which is connected to the original lattice by bonds of strength \( z \), where \( z = \tanh(\beta h) \). For convenience we define the variable \( c = -2 \sin \theta \mid 1 + e^{i \theta} \mid^{-2} \). Therefore, to obtain boundary magnetization the problem is reduced to calculating the exact expression of the free energy for the new \( (M + 1) \times N \) lattice. Such free energy is obtained in the same manner in §2.

In this way, boundary magnetization is given by the derivative with respect to the applied magnetic field \( h \),

\[
M = z + \frac{1}{2\pi} \int_{-\pi}^{\pi} s(y_M) d\theta,
\]

where

\[
s(x) = \frac{z(1 - z^2)x}{c + z^2x}.
\]

Here, in this integral an important contribution is the part of near \( \theta = 0 \). Then, we can expand \( s(y_M) \) at \( \theta = 0 \), and extend the integral limits to infinity. Thus, we obtain the compact formula of boundary magnetization,

\[
M = \sqrt{\prod_{j=1}^{N} (1 + z_j^H)^2 \prod_{j=2}^{N} w_j^+ \prod_{j=1}^{N} w_j^- (1 + z_j^H)^2 (1 + (1 - z_j^H)^2)}
\]

where

\[
w_j^+ = (1 + z_j^H)^2, \quad w_j^- = (1 - z_j^H)^2.
\]

3.2. Numerical analysis for the binary type randomness

In the following, we restrict ourselves to the ferromagnetic random spin system to clarify the issue. We can determine the ‘critical temperature’ \( T_c \) at which the boundary magnetization becomes non-zero. Boundary magnetization just below the critical temperature is described as \( M \sim (T_c - T)^{\beta'} \), where \( \beta' \) is the critical exponent for boundary magnetization. For the pure system, we can derive from Eq.(11) \( (T_c - T)^{1/2} \), i.e. \( \beta' = 1/2 \). On the other hand, for the GMW model, \( \beta' = 1 \) is concluded as follows. We must take the sample average over the many sample realizations. Here, we consider the V-type GMW model for simplicity. This sample average is carried out numerically by the random bonds which are generated by the distribution function,

\[
P(J^V(j)) = (1 - p)\delta(J^V(j) - J^V_1) + p\delta(J^V(j) - J^V_2),
\]

where \( p \) is the concentration of \( J^V_2 \) (\( J^V_1 > 0 \)). From the numerical analysis, we found that boundary magnetization is well plotted by a straight line just below the critical temperature, i.e. \( \beta' = 1 \). For the ferromagnetic Gaussian randomness, which distribution function is given by

\[
P(J^V(j)) = \sqrt{\frac{\pi \Delta}{2}} \exp\left(-\frac{J^V(j)^2}{2\Delta}\right),
\]

with \( J^V(j) > 0 \), we also meet \( \beta' = 1 \). Hence, it seems that \( \beta' = 1 \) is universal for any type of ferromagnetic bond randomness. In the next section, we will give evidence of this universality, that is \( \beta' = 1 \) may be independent of the bond distribution function.
Figure 1. Phase diagram of $\theta - T$: The phase boundary which separates the DS phase and non-DS phase is shown. Boundary fields $h$ are (a) 0.128 and (b) 0.002, respectively. Bond distribution is binary type: $\frac{1}{2}(\delta(J^V(j) - 0.5) + \delta(J^V(j) - 1.0))$.

4. The devil’s staircase structure

As in the preceding section, we can calculate boundary magnetization for ferromagnetic binary type bond randomness using Eq.(11) and Eq.(13). By the sample average, we can guess $\beta' = 1$ for any type of randomness. In this section, this universal nature is explained in terms of the DS structure for $s(y_M)$. We can develop the argument of Bruntuma and Aeppli for an integrated distribution function of $s(y_M)$. [9] We define the integrated distribution function $N(s)$ by

$$N(s) = \int_{-\infty}^{s} d's' << \delta(s' - s(y_M)) >>,$$

where $<< \cdots >>$ denotes the sample average. Here, we take the binary type distribution function Eq.(13) for simplicity. Hence, we can show that this function $N(s)$ satisfies the following recursion relation,

$$N(s) = (1-c)N(g_1(s)) + cN(g_2(s)), \quad g_1(x) = s^{-1}(x; J_1^V), \quad g_2(x) = s^{-1}(x; J_2^V). \quad (16)$$

In the above, $s^{-1}(x; J_1^V)$ and $s^{-1}(x; J_2^V)$ are the inverse functions of $s(x)$ with $J_1^V$ and $J_2^V$, respectively. It can be shown that this recursion relation leads to the DS structure for $N(s)$. The DS structure destroys and becomes a smooth function in a region of $\theta - T$, however. We can determine this phase boundary by $g_1(\frac{1}{2}) = g_2(\frac{1}{2})$. These results are shown in figures 1(a) and (b) for typical binary bond randomness with some $h$.

We will illuminate the reason for the universal critical behavior. At first it should be stressed that the criticality of spontaneous boundary magnetization comes from the contribution around $\theta = 0$ and $h = 0$. As is shown in figure 1(b) the shaded region (the non-DS phase) would collapse to the $\theta = 0$ line in the limit $h \to 0$. From these two facts, near the critical temperature the dominant contribution of the integral of boundary magnetization comes from the one of the destroyed distribution functions. This would induce the universal critical behavior $\beta' = 1$.

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