Particle-in-cell simulation of the head-on collision of large amplitude ion-acoustic solitary waves in a collisionless plasma

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Abstract
The head-on collision of ion-acoustic solitary waves in a collisionless plasma with cold ions and Boltzmann electrons is studied using the particle-in-cell (PIC) simulation. It is shown that solitary waves of sufficiently large amplitudes do not retain their identity after a collision. Their amplitudes decrease and their forms change. In the collision of solitary waves, accelerated ions are formed. The ion velocities can reach three speeds of sound. Dependencies of amplitudes of the potential and densities of ions and electrons after a head-on collision of identical solitary waves on their initial amplitude are presented.

1. Introduction

The formation and propagation of ion-acoustic solitary waves (IASWs) in plasmas have been intensively studied over a period of more than fifty years. Of particular interest is the interaction (collision) of IASWs with each other. The investigations were carried out both theoretically and experimentally in two-component electron-ion plasmas [1–9] and in multicomponent plasmas, which, in addition to positive ions and electrons, contain other components, such as negative ions, positrons or dust particles [10–17]. Analytical and numerical fluid models and kinetic approaches were used. A considerable number of papers have been devoted to the application of numerical methods for the investigation of IASWs [2–8, 16, 17].

A very effective method for studying IASWs is the particle-in-cell (PIC) simulation. Using this method, the propagation of IASWs in a plasma with cold ions and hot electrons, distributed according to the Boltzmann law, was investigated in [3–5]. The results obtained in the simulations are compared with the analytical solutions, and the scope of applicability of the latter is discussed. A comparison of the results of fluid and PIC simulations of the propagation of IASWs was carried out in [6] in the case when the ions have a small temperature. The PIC simulation was also used to study dust acoustic solitary waves in dust plasmas [16, 17].

In PIC simulations, IASWs are initialized according to the solutions of Korteweg—de Vries equation, namely, according to the hyperbolic secant squared. In [6], a Gaussian distribution was used for this purpose. Such initial distributions are convenient for small amplitudes of IASWs. But in the case of large amplitudes, these distributions do not remain unchanged over time and experience noticeable transformations [5]. We note in this connection that a simple and convenient method for calculating the profile of an IASW of arbitrary amplitude was proposed in [18]. In this paper it was also shown that an IASW of an arbitrary amplitude with a properly calculated initial profile propagates without changing its amplitude, velocity, and shape under PIC simulation.

In most papers, the studies of collisions of IASWs were performed for the case of not very large amplitudes and were aimed at determining the main results of the collision: the phase shifts and the trajectories of IASWs after the collision. In this paper we are dealing with head-on collisions of two identical IASWs of large amplitudes and the main attention is focused on the very process of collision. The aim of the paper is to show that in the collision of two IASWs of large amplitudes, a group of strongly accelerated ions appears. This phenomenon has not been described previously. Accelerated ions take energy away from colliding IASWs. As a
result, the amplitudes and propagation speeds of IASWs decrease, and their forms change. IASWs of sufficiently large amplitude do not retain their identity after a mutual collision.

2. Formulation of the problem

We study the head-on collision of IASWs on the assumption that the plasma is collisionless. The plasma motion is described by the Vlasov system of equations. The process under consideration develops on a time scale comparable to an ion plasma period. We may suppose that the electrons are in equilibrium with the electrostatic field. Their density is given by the Boltzmann law. In the one-dimensional case the basic system of equations is

\[
\frac{\partial f_i}{\partial t} + v \frac{\partial f_i}{\partial x} + \frac{\partial \varphi}{\partial x} \frac{\partial f_i}{\partial v} = 0,
\]

\[
\frac{\partial^2 \varphi}{\partial x^2} = - (Z_i n_i - n_e),
\]

\[
n_i = \int_{-\infty}^{\infty} f_i(x, v, t) dv,
\]

\[
n_e = \exp \varphi,
\]

where \(f_i, n_i, \) and \(Z_i\) are the normalized ion distribution function, density and electrical charge respectively; \(n_e\) is the normalized electron density; \(\varphi\) is the normalized electrostatic potential; and \(x, t\) and \(v\) are the normalized coordinate, time and velocity, respectively. Here, the Debye length, the inverse of ion Langmuir angular frequency, the ion sound speed and the electron density in the unperturbed region

\[
(T_{e0}/4\pi e^2n_{e0})^{1/2}, \quad (m_i/4\pi e Z_i e^2n_{e0})^{1/2}, \quad (Z_i T_{e0}/m_i)^{1/2}, \quad n_{e0}
\]

have been used as units of length, time, velocity and density respectively. The ion distribution function is normalized to \(n_{e0}(m_i/Z_i T_{e0})^{1/2}\). Here \(\varepsilon\) is the absolute value of the electron charge; \(m_i\) is the ion mass; \(T_{e0}\) is the electron temperature (in energy units) in the unperturbed region. In this region the ion density and temperature are denoted by \(n_{i0}\) and \(T_{i0}\), respectively, and the electrostatic potential is assumed to be zero. Evidently, we must assume that the charge neutrality in the unperturbed plasma \(Z_i n_{i0} = n_{e0}\) is satisfied.

The system of equations (1) must be supplemented by the initial and boundary conditions characterizing the problem under consideration. The initial condition is formulated as follows. At the initial time \(t = 0\) in the region \([-L, L]\) there is a collisionless electron-ion plasma with singly charged cold ions \((Z_i = 1, \ T_{i0} = 0)\) and electrons obeying the Boltzmann distribution. The plasma has a uniform distribution of quantities in space everywhere, except for two small regions. In these regions there are two IASWs of equal amplitude: solitary wave 1 with its amplitude coordinate \(x = -a\) and propagation velocity \(D > 0\) and solitary wave 2 with its amplitude coordinate \(x = a\) and propagation velocity \(-D\). At the boundaries \(x = \pm L\), the specular reflection condition is set for particles. The Poisson equation is solved with zero electric field at the boundaries. The electric potential \(\varphi(x, t)\) is measured from the zero value in the unperturbed region. In this formulation of the problem, two solitary waves propagate towards each other and at some time \(t > 0\) collide with each other in the neighborhood of the point \(x = 0\) and then diverge.

We investigate this problem by simulating it numerically using a particle-in-cell (PIC) code. In our PIC simulations, electrons follow the Boltzmann distribution while the ion population is sampled by macro-particles whose phase-space trajectories are computed according to the Boris numerical scheme. The initial distributions of the quantities in the solitary waves are calculated using the technique described in [18, 19]. These distributions are implemented in the numerical model by specifying the coordinates and velocities of the macro-particles. In simulations we distinguish two species of the macro-ions by their initial positions. At the initial time \(t = 0\), macro-ions 1 are located in the region \(-L < x < 0\) and macro-ions 2 are located in the region \(0 < x < L\). We are limited to a total of \(~4 \times 10^7\) macro-ions to sample the whole ion population in a one-dimensional collisionless plasma PIC simulation. Numerical simulation is carried out for \(L = 100, \ a = 60\). The number of cells is 8000, the number of macro-ions per cell is \(5 \times 10^3\), the spatial step \(\Delta x = 0.025\), the time step \(\Delta t = 0.0025\).

To solve the nonlinear Poisson equation, we use the quasilinearization method, in which the solution of the nonlinear equation reduces to solving a sequence of linear equations. At each time step, the initial approximation for the potential is assumed to be the same as at the previous time step. In this case, only a few iterations are required to obtain a solution of the nonlinear equation with good accuracy. In the simulations, the solitary waves formed at the initial time propagate without any changes until they collide.
3. Numerical results and discussion

Consider the results of numerical simulations. Figure 1 shows the dependence of the potential maximum $\varphi_m$ on time in the region $x \leq 0$ during the propagation and the head-on collision of two identical solitary waves for two cases of the initial amplitude $\varphi_{m0}$. The dependence $\varphi_m(t)$ in the region $x > 0$ have exactly the same form due to the symmetry of the problem. In the case of $\varphi_{m0} = 0.85$, there are several narrow peaks on the background of the constant value $\varphi_m = \varphi_{m0}$. It is easy to understand that the time intervals during which $\varphi_m$ remains unchanged correspond to the stationary propagation of the solitary wave with amplitude $\varphi_{m0}$ and the first peak corresponds to the head-on collision of the solitary waves at the point $x = 0$. In this case after the collision the maximum of the potential is equal to the initial amplitude of the potential $\varphi_m = \varphi_{m0}$.

Since the colliding solitary waves are exactly the same, the result of their head-on collision can be interpreted either as the passage of solitary waves through each other or as their reflection from each other. In the first interpretation, solitary wave 1 propagating in the region $x < 0$ before the collision propagates in the region $x > 0$ after the collision, and conversely, solitary wave 2 propagating in the region $x > 0$ before the collision propagates in the region $x < 0$ after the collision. After the collision at the point $x = 0$, solitary wave 1 (solitary wave 2) then collides with the boundary $x = L(x = -L)$ that specularly reflects plasma particles. Reflection of particles at the boundary leads to the fact that the entire solitary wave is reflected from the boundary. The second peak in figure 1(a) corresponds to these reflections.

Then the solitary waves again propagate towards each other and collide a second time at the point $x = 0$ (the third peak in figure 1(a)). The fourth peak in figure 1(a) corresponds to the next reflection of the solitary waves from the boundaries. As we see, in the case of $\varphi_{m0} = 0.85$ the collision of solitary waves with each other or with the boundaries does not lead to a change in their amplitude and, accordingly, in their forms Note that the preservation of the identity of solitary waves of small amplitude, when they are reflected from the boundary on which the particles should be reflected, was observed in a numerical experiment also for a chain of solitary waves [20].

A completely different result is obtained in the case of identical solitary waves with $\varphi_{m0} = 1.175$ (figure 1(b)). Here, after the first head-on collision at the point $x = 0$, the amplitudes of both solitary waves drop abruptly. The amplitudes are even noticeably smaller ($\varphi_m \sim 0.42$) than the solitary wave amplitudes in the case considered in figure 1(a). Obviously, the speed of propagation decreases and the form of each solitary wave changes, in particular, its width increases. Due to the larger width of the solitary waves formed after the first collision, the subsequent collision of each solitary wave with the specularly reflecting boundary (the second peak in figure 1(b)) takes a longer time interval than in the case shown in figure 1(a). It can be seen that subsequent collisions of the solitary waves with boundaries or with each other at the point $x = 0$ (the second and the third peak in figure 1(b)) do not change their amplitudes because these amplitudes are small enough. Essential changes occur only at sufficiently large amplitudes.
Obviously, a head-on collision of identical solitary waves can be regarded as a reflection of the solitary waves from each other. In this interpretation, after the head-on collision, each solitary wave changes the propagation direction, but remains in the original region $x < 0$ (solitary wave 1) or in region $x > 0$ (solitary wave 2). Let us use this idea for a more detailed consideration of the head-on collision of two solitary waves with the same amplitudes. Figure 2 shows the phase planes of ions $(x, v_i)$ and the spatial distribution of the electric field $E(x)$ near the point $x = 0$ for those times that are close to the time corresponding to the first peak in figure 1.

Consider the left column of figure 2 ($\mu_{0i} = 0.85$). When the solitary waves approach each other, the front sides of the velocity profiles of ions 1 and ions 2 become steeper with time (frames (a), (b), ..., (c)) corresponding to the times $t = 43, 44, ..., 47$. This is due to the fact that the electric field $E(x)$ in each solitary wave accelerates the ions towards the point $x = 0$, but the ions initially located near this point remain practically immobile and $E(0) = 0$. At these times, the potential $\varphi(x, t)$ has 2 extreme points ($E(x, t)$ has 4 extreme points).

The closest contact of solitary waves occurs at $t \approx 44.9$ (frame (c) corresponds to $t = 45$). The solitary waves merge into a solitary structure with one large potential maximum (the electric field has maximum and minimum). At this time, the amplitude of the electric field reaches its largest value, and the ion velocity amplitude reaches its lowest value. Note that the situation when the velocities of all ions are simultaneously equal to zero does not arise. Instead, a gradual change in the signs of the velocities of all ions occurs. This change takes some time. At first, the ions distant from the collision point begin to move in the opposite direction, and then eventually this process reaches ions near the point $x = 0$. Gradually the maximum ion velocity in absolute value increases and reaches the same absolute value as before the collision (frames (d), (e)).

The solitary structure with one large potential maximum formed at $t = 45$ splits into two solitary waves moving in opposite directions. We can say that the solitary waves change the direction of propagation, that is, they are reflected from each other. In the region $x < 0$ reflected solitary wave 1 propagates to the left, and in the region $x > 0$ reflected solitary wave 2 propagates to the right.
Figure 3. The phase planes of ions 1 ($x, v_1$) (a) and ions 2 ($x, v_2$) (b) after the collision at $t = 50$ in the case of $\varphi_{\text{in}} = 1.0$. 

After the collision of the solitary waves all the ions are in their original regions. The peaks of the ion velocity are at points where the electric field is close to zero. Exactly the same situation occurs with the free propagation of solitary waves, which leads to the fact that at first the ion velocities increase with the passage of the wave and then drop to zero. In our case of $\varphi_{\text{in}} = 0.85$, the velocities of all plasma ions involved in the collision of the solitary waves return to the initial zero value after the passage of the solitary waves. The wave energy is conserved and the solitary wave propagates with the same amplitude.

A different nature of the ion motion can be seen at the larger initial solitary wave amplitude $\varphi_{\text{in}} = 1.175$ (the right column of figure 2). Here, when the waves approach each other, the ion velocity profile and the electric field distribution undergo almost the same changes as we saw in the case of $\varphi_{\text{in}} = 0.85$ (frames (f) and (g) in figure 2). At $t \sim 39.5$ the waves merge into one non-moving structure with one potential maximum. The electric field greatly increases, its amplitude is approximately twice as large as in the case of $\varphi_{\text{in}} = 0.85$ (frame (h) in figure 2). Unlike the previous case, here the amplitude of the ion velocity profile falls not to small values, but only to 1.245 at its initial value of 1.458. Besides, the ion velocity profile is greatly narrowed, and in figure 2 (frame (h)) it has the form of a line for both ions 1 and ions 2. It can be seen that these lines are inclined, that is, ions with higher velocities overtake ions with lower velocities. This situation can be characterized as a breaking of the ion velocity profile. As a result, not all ions change the sign of velocity after the collision, as in the case of $\varphi_{\text{in}} = 0.85$. Some of the fast ions continue to move in the same direction and form a bunch of fast ions. Such a bunch, consisting of ions 1, leaves the region $-L < x < 0$ and passes into the region of ions 2: $0 < x < L$.

Similarly, a bunch of the fast ions 2 moving in the opposite direction leaves the region $0 < x < L$ and passes into the region of ions 1: $-L < x < 0$. Thus, in the collision of solitary waves, a peculiar exchange of ions takes place between the two regions.

The bunches of ions formed in this way turn out to be in regions of large electric field. By the time $t = 39.5$ the bunch of ions formed in solitary wave 1 falls into the region $0 \leq x \leq 0.158$ where $E(x) > 0$, and the ions in the bunch are accelerated in the positive direction. On the other hand, the bunch of ions formed in solitary wave 2 falls into the region $-0.158 \leq x \leq 0$ where $E < 0$, and ions in the bunch are accelerated in the negative direction (frames (h), (i) and (j) in figure 2). The accelerated ions have enough energy to leave the region of the solitary wave and then they move freely along the unperturbed plasma.

Figure 3 illustrates the separation of a bunch of accelerated ions from the bulk of their ‘own’ ions, along which the reflected solitary wave of reduced amplitude propagates in the opposite direction. The velocities of the accelerated ions lie in the interval from 2.4 to 3.0. It can be seen that the accelerated ions overtake the ‘strange’ solitary wave propagating in the same direction. These ions take energy from the solitary waves. As a result the amplitudes of the solitary waves decrease and the spatial distributions of the quantities in each solitary wave change.

From numerical experiments performed for different amplitudes of the solitary waves $\varphi_{\text{in}}$, it is established that for $\varphi_{\text{in}} \leq 0.925$ the solitary waves formed after the head-on collision practically do not differ from the solitary waves before the collision. In the limiting case of $\varphi_{\text{in}} = 0.925$, in contrast to the case of $\varphi_{\text{in}} = 0.85$ (figure 2, the left column), the formation of a bunch of accelerated ions is observed, as well as the exchange of ions between regions $x < 0$ and $x > 0$ (figure 4). However, the velocities of these ions are not sufficient to
overtake the solitary wave formed after the collision. It can be seen from figure 4 that the maximum ion velocity is about 1.2, but the propagation velocity of the wave with \( jm_0 = 0.925 \) is higher and equal to 1.393. Therefore, the bunches of accelerated ions quickly fall into the region of the decelerating electric field and the ions are slowed down (curves e, f, g in figure 4). The solitary wave passes through these ions, leaving them immovable. The amplitudes of the solitary waves remain unchanged after this collision.

If we increase the initial amplitude \( \varphi_{m0} \), then the amplitude of the ion velocity profile becomes larger. The number of ions that can be involved in the above described wave breaking process also increases. Therefore, in contrast to the case considered in figure 4, at \( \varphi_{m0} > 0.925 \), the accelerated ions receive more energy already at the stage of the solitary waves restoration after the collision. A noticeable fraction of the energy is carried away by fast ions even before the solitary waves are completely restored. The loss of energy leads to the fact that the solitary waves can not be restored to their previous states and their amplitudes decrease. The larger the initial IASW amplitude are, the more they lose energy due to this ion acceleration process. Accordingly, with increasing \( \varphi_{m0} \) the solitary wave amplitudes should decrease after collisions. This conclusion is confirmed by the results of numerical simulation. Figure 5 shows the amplitudes of the potential \( \varphi_m \) and densities of ions \( n_{im} \) and electrons \( n_{em} \) after a collision of solitary waves as functions of \( \varphi_{m0} \). It is seen that the larger the amplitudes of solitary waves before their collision, the smaller the amplitudes are after the collision. The figure confirms the presence of a noticeable space charge in the ion-acoustic solitary wave, which was discussed in [18]. At \( \varphi_{m0} > 0.925 \) the value of \( n_{im} - n_{em} \) decreases with increasing \( \varphi_{m0} \).

Beginning with the work of Sagdeev [21], it is known that IASW in a plasma with cold ions and Boltzmann electrons can exist over a wide range of potential amplitudes up to the critical value \( \varphi_{c} \approx 1.256 \). At small amplitudes, the IASW can be described by the Korteweg–de Vries equation. From the analysis of numerical solutions of this equation it was established that solitary waves can retain their identity when interacting with each other. For such waves the name ‘soliton’ was proposed [22]. Solitons can be regarded as an important subclass of solitary waves. The proposed definition of a soliton is of a qualitative nature. The study of collisions of solitary waves will allow us to determine more precise conditions for the existence of solitons. In particular, the results presented here give the boundary of the amplitude of the potential \( \varphi_{m0} \approx 0.925 \), above which the IASWs in a plasma with cold ions and Boltzmann electrons can not be regarded as solitons. If the initial amplitudes of

![Figure 4](image1.png)

*Figure 4.* The phase planes of ions 1 \((x, v_{i1})\) in the case of \( \varphi_{m0} = 0.925 \). The curves \( a, b, \ldots, g \) correspond to the times \( t = 42, 43, \ldots, 48 \).

![Figure 5](image2.png)

*Figure 5.* The amplitudes of the potential \( \varphi_m \) and densities of ions \( n_{im} \) and electrons \( n_{em} \) (right y-axis) after a collision of solitary waves versus their initial amplitude \( \varphi_{m0} \).
the waves are different, the result of the interaction of the IASWs depends on both their amplitudes. Then the soliton region should be determined in the plane of two amplitudes.

4. Conclusions

The head-on collision of IASWs of large amplitudes in a plasma consisting of cold ions and electrons obeying the Boltzmann distribution is studied. Numerical experiments performed with the help of PIC simulation show that after the collision of two identical IASWs, the amplitudes and shapes of both IASWs do not change if their initial amplitudes do not exceed \( \varphi_{m0} = 0.925 \). However, at \( \varphi_{m0} > 0.925 \), the amplitudes of both IASWs decrease after the collision. In this case, the collision of IASWs is accompanied by a breaking of the velocity profile of plasma ions. In each ISAW there are fast ions that overtake the peak of the IASW and enter the region of the accelerating electric field. As a result, before each IASW, a bunch of ions accelerated by the electric field is formed. The velocities of the accelerated ions are in the range from 2.4 to 3.0. Acceleration of ions occurs due to the energy of the IASW, and the amplitude of the latter decreases.

Thus, IASWs of sufficiently large amplitude in a plasma with cold ions and Boltzmann electrons do not retain their identity in a mutual collision. Their amplitudes and propagation speeds decrease, and their forms change. In the collision, some of the ions are accelerated to considerable velocities and leave the solitary wave regions.

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