Injectivity almost everywhere and mappings with finite distortion in nonlinear elasticity

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Abstract
We show that a sufficient condition for the weak limit of a sequence of $W^{1,q}_n$-homeomorphisms with finite distortion to be almost everywhere injective for $q \geq n - 1$, can be stated by means of composition operators. Applying this result, we study nonlinear elasticity problems with respect to these new classes of mappings. Furthermore, we impose loose growth conditions on the stored-energy function for the class of $W^{1,n}_n$-homeomorphisms with finite distortion and integrable inner as well as outer distortion coefficients.

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1 Introduction

Some problems in nonlinear elasticity (including, for instance, those involving hyperelastic materials) reduce to that of minimizing the total energy functional. In this situation, and in contrast to the case of linear elasticity, the integrand is almost always nonconvex, while the functional is nonquadratic. This renders the standard variational methods inapplicable. Nevertheless, for a sufficiently large class of applied nonlinear problems, we may replace convexity with certain weaker conditions, i.e. polyconvexity [3].

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Denote by $\mathbb{M}^{m \times n}$ the set of $m \times n$ matrices. Recall that a function $W: \Omega \times \mathbb{M}^{3 \times 3} \to \mathbb{R}$, $\Omega \subset \mathbb{R}^3$, is called polyconvex if there exists a convex function $G(x, \cdot): \mathbb{M}^{3 \times 3} \times \mathbb{M}^{3 \times 3} \times \mathbb{R}_+ \to \mathbb{R}$ such that

$$G(x, F, \text{Adj} F, \det F) = W(x, F)$$

for all $F \in \mathbb{M}^{3 \times 3}$ with $\det F > 0$, almost everywhere (henceforth abbreviated as a.e.) in $\Omega$.

Let $\Omega$ be a bounded domain in $\mathbb{R}^3$ which boundary $\partial \Omega$ satisfies the Lipschitz condition. Ball’s method [3] is to consider a sequence $\{\varphi_k\}_{k \in \mathbb{N}}$ minimizing the total energy functional

$$I(\varphi) = \int_{\Omega} W(x, D\varphi) \, dx$$

over the set of admissible deformations

$$\mathcal{A}_B = \{ \varphi \in W^1_1(\Omega), \ I(\varphi) < \infty, \ J(x, \varphi) > 0 \text{ a.e. in } \Omega, \ \varphi|_{\partial \Omega} = \overline{\varphi}|_{\partial \Omega} \},$$

where $\overline{\varphi}$ are Dirichlet boundary conditions and $J(x, \varphi)$ stands for the Jacobian of $\varphi$, $J(x, \varphi) = \det D\varphi(x)$. Furthermore, it is assumed that the coercivity inequality

$$W(x, F) \geq \alpha(|F|^p + |\text{Adj} F|^q + (\det F)^r) + g(x)$$

holds for almost all $x \in \Omega$ and all $F \in \mathbb{M}^{3 \times 3}$, $\det F > 0$, where $p \geq 2$, $q \geq \frac{p}{p-1}$, $r > 1$ and $g \in L^1(\Omega)$. Adj $F$ denotes the adjoint matrix, i.e. a transposed matrix of $(2 \times 2)$-subdeterminants of $F$. Moreover, the stored-energy function $W$ is polyconvex. By coercivity, it follows that the sequence $(\varphi_k, \text{Adj} D\varphi_k, \det D\varphi_k)$ is bounded in the reflexive Banach space $W^1_p(\Omega) \times L^q(\Omega) \times L^r(\Omega)$. Relying on the relation between $p$ and $q$, one can conclude that there exists a subsequence converging weakly to an element $(\varphi_0, \text{Adj} D\varphi_0, \det D\varphi_0)$. For the limit $\varphi_0$ to belong to the class $\mathcal{A}_B$ of admissible deformations, we need to impose the additional condition:

$$W(x, F) \to \infty \text{ as } \det F \to 0_+$$

(see [7] for more details). This condition is quite reasonable since it fits in with the principle that “extreme stress must accompany extreme strains”. Another important property of this approach is the sequentially weakly lower semicontinuity of the total energy functional,

$$I(\varphi) \leq \lim_{k \to \infty} I(\varphi_k),$$

which holds because the stored-energy function is polyconvex. It is also worth noting that Ball’s approach admits the nonuniqueness of solutions observed experimentally, see [3] for more details.

One of the most important requirements of continuum mechanics is that interpenetration of matter does not occur, from which it follows that any deformation has to be injective. Global injectivity of deformations has been established by Ball [4] within the existence theory based on minimization of the energy [3]. More precisely, if $\varphi: \overline{\Omega} \to \mathbb{R}^n$, $\Omega \subset \mathbb{R}^n$, is a mapping in $W^1_p(\Omega)$, $p > n$, coinciding on the boundary $\partial \Omega$ with a homeomorphism $\overline{\varphi}$ and $J(x, \varphi) > 0$ a.e. in $\Omega$, $\overline{\varphi}(\Omega)$ is Lipschitz, and if for some $\sigma > n$

$$\int_{\Omega} |J(\varphi(x))|^{-\sigma} \, dx = \int_{\Omega} \frac{|\text{Adj} D\varphi(x)|\sigma}{J(x, \varphi)^{\sigma-1}} \, dx < \infty,$$

then $\varphi$ is a homeomorphism of $\Omega$ on $\varphi(\Omega)$ and $\varphi^{-1} \in W^1_\sigma(\overline{\varphi}(\Omega))$.

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1 Müller et al. [44] improved the assumption $q \geq \frac{p}{p-1}$ to $q \geq \frac{n}{n-1}$.  

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To apply this result to nonlinear elasticity it is required that some additional conditions on the stored-energy function be imposed in order to obtain invertibility of deformations. Thus, in [4] (see also [13, Exercise 7.13]), it is considered a domain $\Omega \subset \mathbb{R}^3$ with a Lipschitz boundary $\partial \Omega$ and a polyconvex stored-energy function $W$. Suppose that there exist constants $\alpha > 0$, $p > 3$, $q > 3$, $r > 1$, and $m > \frac{2q}{q-3}$, as well as a function $g \in L_1(\Omega)$ such that

$$W(x, F) \geq \alpha (|F|^p + |\text{Adj} F|^q + (\det F)^r + (\det F)^{-m}) + g(x)$$  \hspace{1cm} (6)

for almost all $x \in \Omega$ and all $F \in \mathbb{M}^{3 \times 3}$, $\det F > 0$. Take a homeomorphism $\overline{\varphi}: \overline{\Omega} \rightarrow \overline{\Omega}'$ in $W^{1,q}_s(\Omega)$ with $J(x, \overline{\varphi}) > 0$ a.e. in $\Omega$. Then there exists a mapping $\varphi: \Omega \rightarrow \Omega'$ minimizing the total energy functional (1) over the set of admissible deformations (2), which is a homeomorphism due to (5) with $q^{-1} \in W^1_s(\overline{\Omega})$, $\sigma = \frac{q(1+m)}{q+m} > 3$.

In this article we obtain the injectivity property (Theorem 8) based on the boundedness of the composition operator $\varphi^*: L^1_p(\Omega') \rightarrow L^1_q(\Omega)$. Boundedness of these operators is intimately related to a condition of finite distortion. Recall that a $W^{1,\text{loc}}$-mappng $\varphi: \Omega \rightarrow \mathbb{R}^n$ with nonnegative Jacobian, $J(x, \varphi) \geq 0$ a.e., is called a mapping with finite distortion if

$$|D\varphi(x)|^n \leq K(x) J(x, \varphi) \quad \text{for almost all } x \in \Omega,$$

where $1 \leq K(x) < \infty$ a.e. in $\Omega$. Basically, it means that $D\varphi(x) = 0$ for almost all $x \in \Omega$, such that $J(x, \varphi) = 0$. Similarly, the finite codistortion condition means that $\text{Adj} D\varphi(x) = 0$ a.e. on the zero set of the Jacobian.

The function

$$K_O(x, \varphi) = \frac{|D\varphi(x)|^n}{J(x, \varphi)}$$

is called the outer distortion coefficient.\(^2\) It is worth noting that mappings of finite distortion (with some regularity of the distortion coefficient) arise in nonlinear elasticity from geometric considerations: it would be desirable that the deformation is continuous, maps sets of measure zero to sets of measure zero, is a one-to-one mapping and that the inverse map has “good” properties. Hence, many research groups all over the word have worked on this issue (see [2,11,12,22,25–28,32–35,37–39,48,66] and a lot more). It is known that in the planar case ($\Omega, \Omega' \subset \mathbb{R}^2$) a homeomorphism $\varphi \in W^{1,\text{loc}}(\Omega)$ has an inverse homeomorphism $\varphi^{-1} \in W^{1,\text{loc}}(\Omega')$ if and only if $\varphi$ is a mapping with finite distortion [26,27]. In the spatial case $W^{1,\text{loc}}$-regularity of the inverse mapping was shown for $W^{1,\text{q,loc}}$-homeomorphism with finite codistortion, $q > n - 1$, with the integrable inner distortion\(^3\)

$$K_I(x, \varphi) = \frac{|\text{Adj} D\varphi(x)|^n}{J(x, \varphi)^{n-1}}.$$

Moreover, the relaxation of (5) on the case $\sigma = n$,

$$\int_{\Omega'} |D\varphi^{-1}(y)|^n \, dy = \int_{\Omega} \frac{|\text{Adj} D\varphi(x)|^n}{J(x, \varphi)^{n-1}} \, dx = \int_{\Omega} K_I(x, \varphi) \, dx,$$

holds [48,61].

In [32–35] the authors study $W^{1}_n$-homeomorphisms $\varphi: \Omega \rightarrow \Omega'$ between two bounded domains in $\mathbb{R}^n$ with finite energy and consider the behavior of such mappings. In general, the weak $W^{1}_n$-limit of a sequence of homeomorphisms may lose injectivity. However, if there

\(^2\) Considering that $D\varphi = 0$ a.e. if $J(x, \varphi) = 0$, it is assumed that $K_O(x, \varphi) = 1$ if $J(x, \varphi) > 0$.

\(^3\) Considering that $\text{Adj} D\varphi = 0$ a.e. if $J(x, \varphi) = 0$, it is assumed that $K_I(x, \varphi) = 1$ if $J(x, \varphi) > 0$.
is a condition on totally boundedness of norms of the inner distortion \(\|K_{f}(\cdot, \varphi)\|_{L_{1}(\Omega)}\) and some additional requirements, then the limit map is a homeomorphism.\(^4\) The main idea behind the proof of existence and global invertibility is to investigate admissible deformations \(\varphi_{k}\) in parallel with its inverse \(\varphi_{k}^{-1}\) along to a minimizing sequence \(\{\varphi_{k}\}_{k \in \mathbb{N}}\). This is possible\(^5\) due to integrability of the inner distortion as this ensures the existence and regularity of an inverse map belonging to \(W_{n}^{1}\). Note that the authors of these papers include requirements of integrability of the inner distortion coefficient in the coercivity inequality. The authors of the current paper prefer to include this condition to the class of admissible deformation, so as to obtain “finer graduation” of deformations.

We also emphasize that the aforementioned regularity properties of an inverse homeomorphism (including the case \(q = n - 1\)) can be obtained using a technique of the theory of bounded operators of Sobolev spaces. Putting \(p = \frac{\sigma(n-1)}{\sigma-1}\), \(p' = \sigma\), \(q = n - 1\), \(q' = \infty\), \(\varrho = \sigma\) in Theorem 3 [61, Theorem 3] we derive the aforementioned result from [4]. By taking \(p = p' = q = n\), \(q = n - 1\), \(q' = \infty\) in the same theorem one can obtain the regularity of an inverse mapping from [48].

Whenever we deal with \(W_{n}^{1}\)-mappings with finite distortion in this article, we reduce coercivity conditions on the stored-energy function to

\[
W(x, F) \geq \alpha|F|^{n} + g(x). \tag{7}
\]

For given constants \(p, q \geq 1\) and \(M > 0\), and the total energy \(I\), identified by (1), we define the class of admissible deformations

\[
\mathcal{H}(p, q, M) = \{\varphi : \Omega \to \Omega' \text{ is a homeomorphism with finite distortion,} \varphi \in W_{1}^{1}(\Omega), \ I(\varphi) < \infty, \ J(x, \varphi) \geq 0 \ a.e. \ in \ \Omega, \ K_{O}(\cdot, \varphi) \in L_{P}(\Omega), \ \|K_{f}(\cdot, \varphi)\|_{L_{q}(\Omega)} \leq M\},
\]

where \(K_{O}(x, \varphi)\) and \(K_{I}(x, \varphi)\) are the outer and the inner distortion coefficients. We prove an existence theorem in the following formulation (see precise requirements in Sect. 4.2).

**Theorem 1** (Theorem 11 below) Let \(\Omega, \Omega' \subset \mathbb{R}^{n}\) be bounded domains with Lipschitz boundaries. Given a polyconvex function \(W(x, F)\), satisfying the coercivity inequality (7), and assuming \(\mathcal{H}(n - 1, s, M)\) is not empty with \(M > 0\), \(s > 1\), then there exists at least one homeomorphic mapping

\[
\varphi_{0} \in \mathcal{H}(n - 1, s, M) \text{ such that } I(\varphi_{0}) = \inf\{I(\varphi), \varphi \in \mathcal{H}(n - 1, s, M)\}.
\]

The existence theorem is also obtained for classes of mappings with prescribed boundary values and the same homotopy class as a given one, and covers the case \(s = 1\) in some cases (Sect. 4.2). Note that the class of admissible deformations from the paper [32] is related to those considered in the present paper (see Remark 9). For the same reason, the elasticity result of [4] can be derived from the result of the present paper. Indeed, the integrability of the distortion coefficient follows from the Hölder inequality and (6) by \(s = \frac{\sigma r}{r n + \sigma - n}\) where \(\sigma = \frac{q(1+m)}{a+m}\) (see Sect. 5).

Some important properties of mappings of these classes can be found in [62,63]. Note also that the property of mapping to be sense preserving in the topological way follows from

\(^4\) In this paper we use a notation \(\|F\|_{L_{P}(\Omega)}\) for the norm of \(F(\cdot)\) in \(L_{P}(\Omega)\). In some another texts the same norm is denoted by \(\|F\|_{L_{P}(\Omega)}\).

\(^5\) See [61] for another proof of this property under weaker assumptions.
the property that the required deformation is a mapping with bounded \((n, q)\)-distortion if \(q > n - 1\) [62, Lemma 16].

Additionally, there is a different approach to injectivity which was proposed by Ciarlet and Nečas in [15]. This approach rests upon the additional injectivity condition

\[
\int_{\Omega} J(x, \varphi) \, dx \leq |\varphi(\Omega)|
\]  

(8)

on the admissible deformations if \(\Omega \subset \mathbb{R}^n\) is a bounded open set with \(C^1\)-smooth boundary, \(\varphi \in W^1_p(\Omega), \, p > n, \) and \(J(x, \varphi) > 0\) a.e. in \(\Omega\). Under these assumptions, the minimization problem of the energy functional can be constrained to a.e. injective deformations. In the three-dimensional case the relation (8) under the weaker hypothesis \(p > n - 1\) was studied in [58]. In this case, \(\varphi\) may no longer be continuous and the inverse mapping \(\varphi^{-1}\) has only regularity \(BV_{\text{loc}}(\varphi(\Omega), \mathbb{R}^n)\). Local invertibility properties of the mapping \(\varphi \in W^1_p(\Omega), \, p \geq n, \) under the condition \(J(x, \varphi) > 0\) a.e., can be found in [18]. The case \(p > n - 1\) is considered in the recent paper [8], the approach of which uses the topological degree as an essential tool and based on some ideas of [45]. Some other studies of local and global invertibility in the context of elasticity can be found in [9,14,16,23,24,35,45,46,55–57]. Also, see [5,6] for a general review of research in the elasticity theory.

We will now give an outline of the paper. The first section contains general auxiliary facts and some facts about mappings with finite (co-)distortion. The second section is devoted to the injectivity almost everywhere property (Theorems 8 and 10). This property follows from jointly boundedness of pullback operators defined by a sequence of homeomorphisms \(\varphi_k\) and the uniform convergence of inverse homeomorphisms \(\psi_k\) (Lemma 4). Moreover, as a consequence, we obtain the strict inequality \(J(x, \varphi_0) > 0\) a.e. (Lemma 10). The third section is dedicated to the existence theorem. In the fourth section we give two examples to illustrate advantages of our method. The appendix contains some discussion about geometry of domains that does not directly bear on the subject of this paper but is of independent sense.

Some ideas of this article were announced in the note [66].

### 2 Mappings with finite (co-)distortion

Mappings with finite distortion is a natural generalization of mappings with bounded distortion. The reader not familiar with mappings with bounded distortion is referred to [53,54]. To take a close look at the theory of mapping with bounded distortion, the reader can study monographs [27,31].

In this section we present some important concepts and statements necessary to proceed. On a bounded domain \(\Omega \subset \mathbb{R}^n\), i.e. a nonempty, connected, and open set, we define in the standard way (see [40] for instance) the space \(C^\infty_0(\Omega)\) of smooth functions with compact support, the Lebesgue spaces \(L^p(\Omega)\) and \(L^p_{\text{loc}}(\Omega)\) of integrable functions, and Sobolev spaces \(W^1_p(\Omega)\) and \(W^1_{p,\text{loc}}(\Omega)\), \(1 \leq p \leq \infty\). A mapping \(f \in L^1_{\text{loc}}(\Omega)\) belongs to homogenous Sobolev class \(L^1_p(\Omega)\), \(p \geq 1\), if it has the weak derivatives of the first order and its differential \(Df(x)\) belongs to \(L^p_p(\Omega)\).

**Definition 1** We say that a bounded domain \(\Omega \subset \mathbb{R}^n\) has a Lipschitz boundary if for each \(x \in \partial \Omega\) there exists a neighborhood \(U\) such that the set \(\Omega \cap U\) is represented by the inequality \(\xi_n < f(\xi_1, \ldots, \xi_{n-1})\) in some Cartesian coordinate system \(\xi\) with Lipschitz continuous function \(f : \mathbb{R}^{n-1} \to \mathbb{R}\).
Domains with Lipschitz boundary are sometimes called domains having the strong Lipschitz property.

Recall that for topological spaces $X$ and $Y$, a continuous mapping $f : X \to Y$ is discrete if $f^{-1}(y)$ is a discrete set for all $y \in Y$ and $f$ is open if it takes open sets onto open sets.

**Definition 2** [36,64] Given an open set $\Omega \subset \mathbb{R}^n$ and a mapping $f : \Omega \to \mathbb{R}^n$ with $f \in W_{1,\text{loc}}^{1}(\Omega)$ is called a mapping with finite distortion, whenever

$$|Df(x)|^n \leq K(x)|J(x, f)|$$

for almost all $x \in \Omega$,

where $1 \leq K(x) < \infty$ a.e. in $\Omega$.\(^6\)

In other words, the finite distortion condition amounts to the vanishing of the partial derivatives of $f \in W_{1,\text{loc}}^{1}(\Omega)$ almost everywhere on the zero set of the Jacobian $Z = \{x \in \Omega : J(x, f) = 0\}$. Similarly, the finite codistortion condition means that $\text{Adj}Df(x) = 0$ a.e. on the the set $Z$. If $K \in L_{\infty}(\Omega)$, a mapping $f$ is called a mapping with bounded distortion (or a quasiregular mapping).

For a mapping with finite distortion with $J(x, f) \geq 0$ a.e. the functions

$$K_O(x, f) = \frac{|Df(x)|^n}{J(x, f)} \quad \text{and} \quad K_I(x, f) = \frac{|\text{Adj}Df(x)|^n}{J(x, f)^{n-1}}$$

(9)

when $0 < J(x, f) < \infty$ and $K_O(x, f) = K_I(x, f) = 1$ otherwise are called the outer and the inner distortion coefficients of $f$ at the point $x$. It is easy to see that

$$K_I^{\frac{1}{n-1}}(x, f) \leq K_O(x, f) \leq K_I^{n-1}(x, f)$$

for a.e. $x \in \Omega$.

In [52] Reshetnyak proved strong topological properties of mappings with bounded distortion: continuity, openness, and discreteness. Theorem 2.3 of [64] shows that $W_{n,\text{loc}}^1$-mapping with finite distortion and nonnegative Jacobian, $J(x, f) \geq 0$ a.e., is continuous.

In recent years, a lot of research has been done in order to find the sharp assumptions for these topological properties in the class of mappings with finite distortion, for example, [22,25,28,36,38].

**Theorem 2** [49] Let $f : \Omega \to \mathbb{R}^n$, $n \geq 2$, be a non-constant mapping with finite distortion satisfying $J(x, f) \geq 0$ a.e., $f \in W_{n,\text{loc}}^1(\Omega)$, $K_O(\cdot, f) \in L_{n-1,\text{loc}}(\Omega)$ and $K_I(\cdot, f) \in L_{s,\text{loc}}(\Omega)$ for some $s > 1$. Then $f$ is discrete and open.

**Remark 1** In [30] the counterexample to discreteness is presented. Namely, it is proved that there exists, in every dimension $n \geq 3$, a Lipschitz mapping of finite distortion such that the distortion satisfies $K_O \in L_{n-1}$ (and $K_I \in L_1$) but it maps a line segment to a point. However, it remains an open problem to find out whether openness follows under aforementioned assumptions. Also, it is not known if the condition $K_I \in L_p$ for some $p > 1$ implies openness and/or discreteness. Some positive results can be found in [50].

On the other hand, mappings with finite distortion are closely related to boundedness of composition operators of Sobolev spaces. Recall that a measurable mapping $\varphi : \Omega \to \Omega'$ induces a bounded operator $\varphi^* : L_p^1(\Omega') \to L_q^1(\Omega)$ by the composition rule, $1 \leq q \leq p < \infty$, if the operator $\varphi^* : L_p^1(\Omega') \cap \text{Lip}_{\text{loc}}(\Omega') \to L_q^1(\Omega)$ with $\varphi^*(f) = f \circ \varphi$, $f \in L_p^1(\Omega') \cap \text{Lip}_{\text{loc}}(\Omega')$, is bounded.

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\(^6\) Some authors include condition $J(x, f) \geq 0$ in Definition 2. We do not use the condition for the Jacobian to be non-negative as it is unnecessary in the context of the theory of composition operators, see details in [61].
Lemma 1 [69] If a measurable mapping \( \varphi \) induces a bounded composition operator

\[
\varphi^*: L^1_p(\Omega') \to L^1_q(\Omega), \quad 1 \leq q \leq p < \infty,
\]

then \( \varphi \) has finite distortion.

Now we consider a generalization of inner and outer distortion functions, which is more conducive to dealing with composition and pullback operators. Following [61], for a mapping \( f : \Omega \to \Omega' \) of class \( W^1_{1,\text{loc}}(\Omega) \) define the (outer) distortion operator function

\[
K_{f,p}(x) = \begin{cases} \frac{|Df(x)|}{|J(x, f)|^{1/p}} & \text{for } x \in \Omega \setminus Z, \\ 0 & \text{otherwise}, \end{cases}
\]

and the (inner) distortion operator function

\[
K_{f,p}(x) = \begin{cases} \frac{\left| \text{Adj} Df(x) \right|}{|J(x, f)|^{(n-1)/p}} & \text{for } x \in \Omega \setminus Z, \\ 0 & \text{otherwise}, \end{cases}
\]

where \( Z \) is a zero set of the Jacobian \( J(x, f) \).

Remark 2 Note that \( K_{\Omega}(x, f) = K_{f,p}(x) \) and \( K_{f}(x, f) = K_{f,p}(x) \) if \( x \in \Omega \setminus Z \). Hence \( K_{\Omega}(\cdot, f) \in L^r_{n-1}(\Omega) \) results in \( K_{f,p}(\cdot) \in L^r(n-1)(\Omega) \), and \( \|K_{f}(\cdot, f)\ | L^s_{1}(\Omega)\| \leq M \) implies \( \|K_{f,p}(\cdot)\ | L^s_{1}(\Omega)\| \leq M^{1/n} \) for \( q = ns \).

The following theorem shows the regularity properties which ensure that the direct and the inverse homeomorphisms belong to corresponding Sobolev classes.

Theorem 3 [61, Theorem 3] Let \( \varphi : \Omega \to \Omega' \) be a homeomorphism with the following properties:

1. \( \varphi \in W^1_{q,\text{loc}}(\Omega) \), \( n-1 \leq q \leq \infty \);
2. The mapping \( \varphi \) has finite codistortion;
3. \( K_{\varphi,p} \in L^q(\Omega) \), where \( \frac{1}{q} = \frac{n-1}{q} - \frac{n-1}{p} \), \( n-1 \leq q \leq p \leq \infty \) (\( \varphi = \infty \) for \( q = p \)).

Then the inverse homeomorphism \( \varphi^{-1} \) has the following properties:

1. \( \varphi^{-1} \in W^1_{p',\text{loc}}(\Omega') \), where \( p' = \frac{p}{p-n+1} \) (\( p' = 1 \) for \( p = \infty \));
2. \( \varphi^{-1} \) has finite distortion \( (J(y, \varphi^{-1}) > 0 \text{ a.e. for } n \leq q) \);
3. \( K_{\varphi^{-1},q'} \in L^q(\Omega') \), where \( q' = \frac{q}{q-n+1} \) (\( q' = \infty \) for \( q = n - 1 \)).

Moreover,

\[
\|K_{\varphi^{-1},q'}(\cdot)\ | L^q(\Omega')\| = \|K_{\varphi,p}(\cdot)\ | L^q(\Omega)\|.
\]

Remark 3 If replace the condition 2 with “the mapping \( \varphi \) has finite distortion” and the condition 3 with one with the outer distortion operator function: “\( K_{\varphi,p} \in L^s_{\infty}(\Omega) \) where \( \frac{1}{s} = \frac{1}{q} - \frac{1}{p}, n-1 \leq q \leq p \leq \infty \) (\( s = \infty \) for \( q = p \))”. Then the conclusion of this theorem is valid with the next estimate

\[
\|K_{\varphi^{-1},q'}(\cdot)\ | L^q(\Omega')\| \leq \|K_{\varphi,p}(\cdot)\ | L^s_{\infty}(\Omega)\|^{n-1}
\]

(see [61, Theorem 4]).
Theorem 4 [61,68,69] A homeomorphism $\varphi: \Omega \to \Omega'$ induces a bounded composition operator

$$\varphi^*: L^q_p(\Omega') \to L^q_p(\Omega), \quad 1 \leq q \leq p < \infty,$$

where $\varphi^*(f) = f \circ \varphi$ for $f \in L^q_p(\Omega')$, if and only if the following conditions hold:

1. $\varphi \in W^1_{q,\text{loc}}(\Omega)$;
2. The mapping $\varphi$ has finite distortion;
3. $K_{\varphi,p}(\cdot) \in L_p(\Omega)$, where $\frac{1}{q} = \frac{1}{p} - \frac{1}{q}$, $1 \leq q \leq p < \infty$ (and $\vartheta = \infty$ for $q = p$).

Moreover,

$$\|\varphi^*\| \leq \|K_{\varphi,p}(\cdot) \mid L_p(\Omega)\| \leq C\|\varphi^*\|$$

for some constant $C$.

Theorem 5 [61, Theorem 6] Assume that a homeomorphism $\varphi: \Omega \to \Omega'$ induces a bounded composition operator $\varphi^*: L^q_p(\Omega') \to L^q_p(\Omega)$ for $n - 1 \leq q \leq p < \infty$, where $\varphi^*(f) = f \circ \varphi$ for $f \in L^q_p(\Omega')$ (and in the case $p = \infty$ the mapping $\varphi$ has finite codistortion).

Then the inverse mapping $\varphi^{-1*}$ induces a bounded composition operator $\varphi^{-1*} : L^q_1(\Omega) \to L^q_1(\Omega')$, where $q' = \frac{q}{q-n+1}$ and $p' = \frac{p}{p-n+1}$, and has finite distortion.

Moreover,

$$\|\varphi^{-1*}\| \leq \|K_{\varphi^{-1},q'}(\cdot) \mid L_1(\Omega')\| \leq \|K_{\varphi,p}(\cdot) \mid L_p(\Omega)\|^{n-1}, \quad \text{where } \frac{1}{p'} = \frac{1}{p'} - \frac{1}{q'}.$$

Recall that a differential $(n - 1)$-form $\omega$ on $\Omega'$ is defined as

$$\omega(y) = \sum_{k=1}^n a_k(y) \, dy_1 \wedge dy_2 \wedge \cdots \wedge dy_k \wedge \cdots \wedge dy_n.$$

A form $\omega$, with measurable coefficients $a_k$, belongs to $L_p(\Omega', \Lambda^{n-1})$ if

$$\|\omega \mid L_p(\Omega', \Lambda^{n-1})\| = \left( \int_{\Omega} \left( \sum_{k=1}^n a_k(y)^2 \right)^{p/2} \, dy \right)^{1/p} < \infty.$$

Let $f = (f_1, \ldots, f_n) : \Omega \to \Omega'$ belong to $W^1_{q(n-1),\text{loc}}(\Omega)$ and $\omega$ be a smooth $(n - 1)$-form. Then the pullback $\varphi^*\omega$ can be written as

$$f^*\omega(x) = \sum_{k=1}^n a_k(f(x)) \, df_1 \wedge \cdots \wedge df_k \wedge \cdots \wedge df_n.$$

For any $\omega \in L_p(\Omega', \Lambda^{n-1})$ the pullback operator $\tilde{f}^*\omega(x)$ is defined by continuity [60, Corollary 1.1]:

$$\tilde{f}^*\omega(x) = \begin{cases} f^*\omega(x), & \text{if } x \in \Omega \setminus (Z \cup \Sigma), \\ 0, & \text{otherwise}. \end{cases} \quad (10)$$

As consequence of [60, Theorem 1.1] we can obtain

Theorem 6 A homeomorphism $f : \Omega \to \Omega'$ induces a bounded pullback operator $\tilde{f}^*: L_p(\Omega', \Lambda^{n-1}) \to Lq(\Omega, \Lambda^{n-1}), 1 \leq q \leq p < \infty$, if and only if:

---

7 Necessity is proved in [68,69] (see also earlier work [59]), and sufficiency, in Theorem 6 of [61].
1. $f : \Omega \to \Omega'$ has finite codistortion;
2. $K_{f, p(n-1)} \in L_\infty(\Omega)$ where $\frac{1}{p} = \frac{1}{q} - \frac{1}{p}$.

Moreover, the norm of the operator $\tilde{f}^*$ is comparable with $\|K_{f, p(n-1)} \mid L_\infty(\Omega)\|$. 

**Theorem 7** [60, Theorem 1.3] Assume that a homeomorphism $\varphi : \Omega \to \Omega'$ belongs to $W^{1, \infty}_{-1, loc}(\Omega)$ and induces a bounded pullback operator

$$\tilde{\varphi}^* : L_p(\Omega', \Lambda^{n-1}) \to L_q(\Omega, \Lambda^{n-1}), \quad 1 \leq q \leq p \leq \infty.$$ 

Then the inverse mapping $\varphi^{-1} \in W_{1, loc}^1(\Omega)$ induces a bounded pullback operator

$$\tilde{\varphi}^{-1*} : L_{q'}(\Omega, \Lambda^1) \to L_{p'}(\Omega, \Lambda^1),$$

where $q' = \frac{n}{n-1}$ and $p' = \frac{p}{p-1}$. Moreover, the norm of the operator $\tilde{\varphi}^{-1*}$ is comparable with the norm of $\tilde{\varphi}^*$.

### 3 Almost-everywhere injectivity

It is well known that the limit of homeomorphisms need not be homeomorphism or even an injective mapping. It is illustrated by the simple example of mappings $\varphi_k(x) = |x|^{k-1}x$ on the punctured unit ball. Here we have the limit mapping $\varphi_0(x) \equiv 0$ and injectivity is lost.

In the celebrated paper of Ciarlet and Nečas [15] the conception of injectivity almost everywhere “in image” of $\varphi$ was introduced in the sense that for almost all $y \in \varphi(\Omega)$ preimage $\varphi^{-1}(y)$ consists of exactly one point.

In this paper we consider more delicate definition—injectivity almost everywhere “in preimage”, i.e. we will say that a mapping $\varphi : \Omega \to \mathbb{R}^n$ is injective almost everywhere whenever there exists a negligible set $S \subset \Omega$ outside which $\varphi$ is injective.

The sequence of homeomorphisms $\varphi_k = (\varphi_{k,1}, \varphi_{k,2}) : [-1, 1]^2 \to [-1, 1]^2$ of the class $W^1_2([-1, 1]^2)$ with integrable distortion, such that

$$\varphi_{k,1}(x_1, x_2) = \begin{cases} 2x_1\xi_k(x_2) & \text{if } x_1 \in [0, \frac{1}{2}], \\ 2(1 - \xi_k(x_2))x_1 - (1 - 2\xi_k(x_2)) & \text{if } x_1 \in (\frac{1}{2}, 1], \end{cases}$$

and $\varphi_{k,2}(x_1, x_2) = x_2$, with $\xi_k(t) = \frac{1+(k-1)t}{2k}$, shows that injectivity almost everywhere can also take place (Fig. 1).

**Theorem 8** Let $\Omega, \Omega' \subset \mathbb{R}^n$ be bounded domains with Lipschitz boundaries. Consider a sequence of homeomorphisms $\varphi_k$, which maps $\Omega$ onto $\Omega'$, with $\varphi_k \in W^{1}_{n-1, loc}(\Omega)$, $J(x, \varphi_k) \geq 0$ a.e. and $J(x, \varphi_k)$ is not a.e.-zero, such that:

1. $\varphi_k \to \varphi_0$ weakly in $W^{1}_{n-1, loc}(\Omega)$;
2. Every mapping $\varphi_k$ induces a bounded pullback operator

$$\tilde{\varphi}_k : L^{\frac{n}{n-1}}(\Omega', \Lambda^{n-1}) \to L^{\frac{r}{r-1}}(\Omega, \Lambda^{n-1})$$

for some $r \in [n-1, n]$;
3. The norms of the operators $\|\tilde{\varphi}_k^*\|$ are totally bounded.

Then $\varphi_0$ is an a.e.-injective mapping with finite distortion.

**Remark 4** Note that the orientation preserving condition plays a key role in the theory of mapping with finite distortion. Namely, we need the condition $J(x, \varphi_0) \geq 0$ a.e. in $\Omega$ to be
fulfilled. It could be easily proved in the case $W_p^1$. Indeed, it follows directly from the weak convergence of $J(\cdot, \varphi_k)$ in $L_1(K)$, for every $K \in \Omega$. Among other things, we can establish the nonnegativity of the Jacobian by using weak convergence (see [53, Sect.4.5]).

More general result was obtained by Hencl and Onninen in [29], where $J(x, \varphi_0) \geq 0$ is established for $\varphi_0$ being a $W^{1,p}_\text{loc}(\Omega)$-weak limit of homeomorphisms $\varphi_k$, with $p \geq 1$ for $n = 2$ or 3, and $p > \left[\frac{n}{2}\right]$ for $n \geq 4$; if $J(x, \varphi_k) \geq 0$ a.e. and $J(x, \varphi_k)$ is not a.e.-zero.

By Theorem 6 conditions 2 and 3 of Theorem 8 can be replaced by totally boundedness of inner distortion operator functions $K_{\varphi_k,n}$ in $L_\varrho$ with $\varrho = \frac{rn}{(n-1)(n-r)} \geq n$.

**Corollary 1** Let $\Omega, \Omega' \subset \mathbb{R}^n$ be bounded domains with Lipschitz boundaries. Consider a sequence of homeomorphisms of finite codistortion $\varphi_k$, which maps $\Omega$ onto $\Omega'$, with $\varphi_k \in W^{1,n-1}_\text{loc}(\Omega)$, $J(x, \varphi_k) \geq 0$ a.e. and $J(x, \varphi_k)$ is not a.e.-zero, such that:

1. $\varphi_k \to \varphi_0$ weakly in $W^{1,n-1}_\text{loc}(\Omega)$;
2. The norms of inner distortion operator functions $\|K_{\varphi_k,n} \|_{L_\varrho(\Omega)}$ are totally bounded for some $\varrho \geq n$.

Then $\varphi_0$ is an a.e.-injective mapping with finite distortion.

Taking into account $\|K_{\varphi,n} \|_{L_{ns}(\Omega)} = \|K_{I, \varphi} \|_{L_{s}(\Omega)}\|^{1/n}$ by Remark 2 we derive the next assertion.

**Corollary 2** Let $\Omega, \Omega' \subset \mathbb{R}^n$ be bounded domains with Lipschitz boundaries. Consider a sequence of homeomorphisms of finite codistortion $\varphi_k$, which maps $\Omega$ onto $\Omega'$, with $\varphi_k \in W^{1,n-1}_\text{loc}(\Omega)$, $J(x, \varphi_k) \geq 0$ a.e. and $J(x, \varphi_k)$ is not a.e.-zero, such that:

1. $\varphi_k \to \varphi_0$ weakly in $W^{1,n-1}_\text{loc}(\Omega)$ with $J(x, \varphi_0) \geq 0$ a.e. in $\Omega$;
2. The norms of inner distortion functions $\|K_{I, \varphi_k} \|_{L_{ns}(\Omega)}$ are totally bounded for some $s \geq 1$.

Then $\varphi_0$ is an a.e.-injective mapping with finite distortion.

**Remark 5** As it will be clear from the subsequent, the theorem is valid provided that composition operators $\varphi_k^\ast: L^1_{\rho}(\Omega') \to L^1_{\rho}(\Omega)$, $1 \leq \rho < n$, and $\psi_k^\ast: L^1_{\rho'}(\Omega) \to L^1_{\rho'}(\Omega')$,

8 The exponent $r$ from Theorem 8 can be expressed as $r = \frac{n(n-1)s}{ns+1-s} \geq n-1$. 

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Proof Consider a subsequence which converges almost everywhere in $\Omega$. If conditions of Theorem 7 the inverse mapping $\psi = \varphi^{-1}$ has finite distortion and induces a bounded pullback operator $\psi^*: L_{n/(n-1)}(\Omega', A^{n-1}) \to L_{r/(n-1)}(\Omega, A^{n-1})$ then by Theorem 8 the inverse mapping $\psi = \varphi^{-1}$ has finite distortion and induces a bounded pullback operator $\sim: L_r(\Omega, A^1) \to L_n(\Omega', A^1)$ for $r' = r/(n-1) \geq n$. Moreover, $\|\psi^*\| \sim \|\psi^*\|$.

Further, in accordance with Theorem 5 an inverse homeomorphism $\varphi = \psi^{-1}$ has finite distortion and induces a bounded composition operator $\varphi^*: L_1^n(\Omega') \to L_1^1(\Omega)$ for $\rho = \frac{1}{(n-1)^2 - r(n-2)} \geq 1$ and $\|\varphi^*\| \sim \|\psi^*\|^{-n-1} \sim \|\sim\|^{-n-1}$.

The proof of Theorem 8 spans the rest of the section and is split in a series of Lemmas.

Lemma 2 If conditions of Theorem 8 are fulfilled, then the mapping $\varphi_0$ induces a bounded composition operator $\varphi_0^*: L_1^n(\Omega') \cap \text{Lip}(\Omega') \to L_1^1(\Omega)$, $\rho = \frac{r}{(n-1)^2 - r(n-2)} \geq 1$.

Proof Consider $u \in L_1^n(\Omega') \cap \text{Lip}(\Omega')$. Since $\|\varphi_k^*\| \leq C$ by Remark 5, the sequence $w_k = \varphi_k^* u = u \circ \varphi_k$ is bounded in $L_1^n(\Omega)$. Using the Poincaré inequality and a compact embedding of Sobolev spaces (see [1, Theorems 6.2, 6.30] for instance), we obtain a subsequence with $w_k \to w_0$ in $L_1(\Omega)$ where $1 < t < \frac{n}{n-\rho}$. From this sequence, in turn, we can extract a subsequence which converges almost everywhere in $\Omega$. The same arguments ensure that $\varphi_k \to \varphi_0$ a.e. Then $w_0(x) = u \circ \varphi_0(x)$ for almost all $x \in \Omega$.

On the other hand, since $w_k$ converges weakly to $w_0$ in $L_1^1(\Omega)$, we have

$$\|u \circ \varphi_0 | L_1^1(\Omega)\| = \|w_0 | L_1^1(\Omega)\| \leq \lim_{k \to \infty} \|w_k | L_1^1(\Omega)\| = \lim_{k \to \infty} \|\varphi_k^*(u) | L_1^1(\Omega)\| \leq \lim_{k \to \infty} \|\varphi_k^*\| \cdot \|u | L_1^n(\Omega')\| \leq C \cdot \|u | L_1^n(\Omega')\|.$$}

Thus, $\varphi_0$ induces a bounded composition operator $\varphi_0^*: L_1^n(\Omega') \cap \text{Lip}(\Omega') \to L_1^1(\Omega)$, and moreover, $\|\varphi_0^*\| \leq C$. \hfill $\Box$

Similar we can obtain boundedness of pullback operator

$$\varphi_0^* : L_{n-1}(\Omega', A^{n-1}) \to L_{r-1}(\Omega, A^{n-1}).$$

Lemma 3 If conditions of Theorem 8 are fulfilled, then the mapping $\varphi_0$ induces a bounded pullback operator $\varphi_0^*: L_{n-1}(\Omega', A^{n-1}) \to L_{r-1}(\Omega, A^{n-1})$.

Note that Lemmas 1 and 2 provide that $\varphi_0$ has finite distortion.

Now we need to consider some regularity properties of the sequence $\{\varphi_k\}_{k \in \mathbb{N}}$ which meet the requirements of Theorem 8.

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9 $a \sim b$ means that there exist constants $C_1, C_2 > 0$, such that $C_1 a \leq b \leq C_2 a$. 

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Inj ectivity almost everywhere and mappings with finite...
Lemma 4  Let conditions of Theorem 8 be fulfilled, define a sequence of continuous mappings \( \psi_k : \Omega' \to \Omega \) as \( \psi_k = \varphi_k^{-1} \). Then there exists a subsequence \( \{\psi_{k_l}\}_{l \in \mathbb{N}} \) and a continuous mapping \( \psi_0 : \Omega' \to \Omega \) such that \( \psi_{k_l} \to \psi_0 \) locally uniformly.

Proof  Notice that the sequence \( \psi_k \) is uniformly bounded since the domain \( \Omega \) is bounded.

On the other hand, since \( \psi_k \in W^{1,\infty}_r(\Omega') \) (by Remark 5 and Theorem 4) we obtain the estimate (corollary of [42, Lemma 4.1])

\[
\text{osc}(\psi_k, S(y', r)) \leq L \left( \ln \frac{r_0}{r} \right)^{-\frac{1}{n}} \left( \int_{B(y', r_0)} |D\psi_k(y)|^n \, dy \right)^{\frac{1}{n}},
\]

where \( S(y', r) \) is the sphere of radius \( r < \frac{r_0}{2} \) centered at \( y' \) and \( B(y', r_0) \subset \Omega' \) is the ball of radius \( r_0 \) centered at \( y' \). It follows the equicontinuity of the family of functions \( \{\psi_k\}_{k \in \mathbb{N}} \) on any compact part of \( \Omega' \).

Hölder’s inequality, Theorem 4 and Theorem 5 yield

\[
\int_{B(y', r_0)} |D\psi_k(y)|^n \, dy \leq \int_{B(y', r_0)} \frac{|D\psi_k(y)|^n}{(J(y, \psi_k))^{\frac{n}{n'}}} \frac{J(y, \psi_k)^{\frac{n}{n'}}}{\varphi''(y)} \, dy
\]

\[
\leq \left( \int_{\Omega'} \frac{|D\psi_k(y)|^n}{(J(y, \psi_k))^{\frac{n}{n'}}} \, dy \right)^{\frac{n'}{n'}} \left( \int_{B(y', r_0)} J(y, \psi_k)^{\frac{n}{n'}} \varphi''(y) \, dy \right)^{\frac{n'}{n}}
\]

\[
\leq \|K_{\psi_k, r'}(\cdot)\| L_{\varphi''}(\Omega') \|\psi_k(B(y', r_0))\| \varphi''(y') \varphi''(y) \varphi''(y)
\]

\[
\leq C^n |\Omega|^{\frac{n}{n'}}
\]

where \( r' = \frac{r}{r-n+1}, \frac{1}{\varphi'} = \frac{1}{n} - \frac{1}{r' n}, \) and since \( \frac{n}{r'} \cdot \frac{\varphi'(y')}{\varphi''(y)} = 1 \).

Thus, we see that the family \( \{\psi_k\}_{k \in \mathbb{N}} \) is equicontinuous and uniformly bounded. By the Arzelà–Ascoli theorem there exists a subsequence \( \{\psi_{k_l}\} \) converging uniformly to a mapping \( \psi_0 \) as \( k_l \to \infty \). \( \qed \)

Lemma 5  Let conditions of Theorem 8 be fulfilled, then \( \varphi_0 \) is a.e.-injective in \( \Omega \setminus \varphi_0^{-1}(\partial \Omega') \).

Proof  The sequence \( \{\psi_k\}_{k \in \mathbb{N}} \) converges weakly in \( W^{1,\infty}_r(\Omega) \). Therefore, by the embedding theorem taking a subsequence if necessary, We know that \( \varphi_0 \) is an almost everywhere pointwise limit of the homeomorphisms \( \varphi_k : \Omega \to \Omega' \). In this case the images of some points \( x \in \Omega \) may belong to the boundary \( \partial \Omega' \).

Denote by \( S \subset \Omega \) a negligible set on which the convergence \( \varphi_k(x) \to \varphi_0(x) \) as \( k \to \infty \) fails. If \( x \in \Omega \setminus S \) with \( \varphi(x) \in \Omega' \) then the injectivity follows from the uniform convergence of \( \psi_k = \varphi_k^{-1} \) on \( \Omega' \) (see Lemma 4) and the identity

\[
\psi_k \circ \varphi_k(x) = x, \quad x \in \Omega \setminus S.
\]

Passing to the limit as \( k \to \infty \), we infer that

\[
\psi_0 \circ \varphi_0(x) = x, \quad x \in \Omega \setminus S.
\]

Hence, we deduce that if \( \varphi_0(x_1) = \varphi_0(x_2) \in \Omega' \) for two points \( x_1, x_2 \in \Omega \setminus S \) then \( x_1 = x_2 \). \( \qed \)
Now we verify that the set of points $x \in \Omega$ with $\varphi(x) \in \partial\Omega'$ is negligible. The proof of this statement is based on some properties of an additive function $\Phi$ defined on open bounded sets. For proving Lemma 6 below we modify the method of proof of [69, Theorem 4].

Given a bounded open set $A' \subset \mathbb{R}^n$, define the class of functions $\overset{\circ}{\dot{L}}_p^1(A')$ as the closure of the subspace $C_0^\infty(A')$ in the semi-norm of $L_p^1(A')$. In general, a function $f \in \overset{\circ}{\dot{L}}_p^1(A')$ is defined only on the set $A'$, but, extending it by zero, we may assume that $f \in L_p^1(\mathbb{R}^n)$.

Let us recall that a mapping $\Phi$ defined on open subsets from $\mathbb{R}^n$ and taking nonnegative finite values is called a monotone if $\Phi(V) \leq \Phi(U)$ for $V \subset U$ and a countably additive function of set (see [69]) if for any countable set $U_i \subset U \subset \mathbb{R}^n$, $i = 1, 2, \ldots, \infty$, of pairwise disjoint open sets the following inequality takes place.

**Lemma 6** (cf. Lemma 1 of [69]) Assume that the mapping $\varphi: \Omega \rightarrow \overline{\Omega}'$ induces a bounded composition operator

$$\varphi^*: L_p^1(\Omega') \cap \text{Lip}(\Omega') \rightarrow L_q^1(\Omega), \quad 1 \leq q < p \leq \infty.$$  

Then

$$\Phi(A') = \sup_{f \in \overset{\circ}{\dot{L}}_p^1(A') \cap \text{Lip}(A')} \left( \frac{\|\varphi^* f | L_q^1(\Omega)\|}{\|f | L_p^1(A' \cap \Omega')\|} \right)^{\sigma}, \quad \sigma = \begin{cases} \frac{pq}{p-q} & \text{for } p < \infty, \\ q & \text{for } p = \infty, \end{cases}$$

is a bounded monotone countably additive function defined on the open bounded sets $A'$ with $A' \cap \Omega' \neq \emptyset$.

**Remark 6** If $f \in \overset{\circ}{\dot{L}}_p^1(A') \cap \text{Lip}(A')$ and $A' \not\subset \overline{\Omega}'$, we consider a composition $\varphi^* f$ where it is well defined.

**Proof of Lemma 6** It is obvious that $\Phi(A'_1) \leq \Phi(A'_2)$ whenever $A'_1 \subset A'_2$.

Take disjoint sets $A'_i, i \in \mathbb{N}$ in $\Omega'$ and put $A'_0 = \bigcup_{i=1}^{\infty} A'_i$. Consider a function $f_i \in \overset{\circ}{\dot{L}}_p^1(A'_i) \cap \text{Lip}(A'_i)$ such that the conditions

$$\|\varphi^* f_i | L_q^1(\Omega)\| \geq \left( \Phi(A'_i) \left( 1 - \frac{\varepsilon}{2^i} \right) \right)^{1/\sigma} \|f_i | \overset{\circ}{\dot{L}}_p^1(A'_i)\|$$

and

$$\|f_i | \overset{\circ}{\dot{L}}_p^1(A'_i)\|^p = \Phi(A'_i) \left( 1 - \frac{\varepsilon}{2^i} \right) \quad \text{for } p < \infty$$

$$\|f_i | \overset{\circ}{\dot{L}}_p^1(A'_i)\|^p = 1 \quad \text{for } p = \infty$$

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hold simultaneously where \(0 < \varepsilon < 1\). Putting \(f_N = \sum_{i=1}^{N} f_i \in L^1_p(\Omega') \cap \text{Lip}(\Omega')\), and applying Hölder’s inequality in the case of equality,\(^{10}\) we obtain

\[
\|\varphi^* f_N \|_{L^1_q(\Omega)} \geq \left( \sum_{i=1}^{N} \left( \Phi(A_i') - \varepsilon \Phi(A_0') \right)^{\frac{1}{\sigma}} \| f_i \|_{L^1_p(N \bigcup A_i')} \right)^{\frac{1}{\sigma}}
\]

since the sets \(A_i\), on which the functions \(\nabla \varphi^* f_i\) are nonvanishing, are disjoint. This implies that

\[
\Phi(A_0')^{\frac{1}{2}} \geq \sup \left( \frac{\|\varphi^* f_N \|_{L^1_q(\Omega)}}{\| f_N \|_{L^1_p(N \bigcup A_i')}} \right)^{\frac{1}{2}} \Phi \left( \bigcup_{i=1}^{\infty} A_i' \right)
\]

where we take the sharp upper bound over all functions

\[
f_N \in L^1_p\left( \bigcup_{i=1}^{N} A_i' \right) \cap \text{Lip}\left( \bigcup_{i=1}^{N} A_i' \right), \quad f_N = \sum_{i=1}^{N} f_i,
\]

and \(f_i\) are of the form indicated above. Since \(N\) and \(\varepsilon\) are arbitrary,

\[
\sum_{i=1}^{\infty} \Phi(A_i') \leq \Phi \left( \bigcup_{i=1}^{\infty} A_i' \right).
\]

We can verify the inverse inequality directly by using the definition of \(\Phi\). \(\square\)

For estimating \(\Phi\) through multiplicity of covering, we need the following corollary to the Bezikovich theorem (see [20, Theorem 1.1] for instance).

**Lemma 7** For every open set \(U \subset \mathbb{R}^n\) with \(U \neq \mathbb{R}^n\), there exists a countable family \(\mathcal{B} = \{B_j\}\) of balls such that

1. \(\bigcup_j B_j = U;\)
2. if \(B_j = B_j(x_j, r_j) \in \mathcal{B}\) then \(\text{dist}(x_j, \partial U) = 12r_j;\)
3. the families \(\mathcal{B} = \{B_j\}\) and \(2\mathcal{B} = \{2B_j\}\), where the symbol \(2\mathcal{B}\) stands for the ball of doubled radius centered at the same point, constitute a covering of finite multiplicity of \(U;\)
4. if the balls \(2B_j = B_j(x_j, 2r_j), j = 1, 2, \) intersect then \(\frac{5}{2}r_1 \leq r_2 \leq \frac{7}{5}r_1;\)
5. we can subdivide the family \(\{2B_j\}\) into finitely many tuples so that in each tuple the balls are disjoint and the number of tuples depends only on the dimension \(n.\)

\(^{10}\) Let us remind that for \(a_i, b_i \geq 0, \frac{1}{k} + \frac{1}{k'} = 1, |\sum a_i b_i| = (\sum a_i^k)^{1/k} (\sum b_i^{k'})^{1/k'}\) if and only if \(a_i^k\) and \(b_i^{k'}\) are proportional.
Lemma 8 Take a monotone countably additive function \( \Phi \) defined on the bounded open sets \( A' \) with \( A' \cap \Omega' \neq \emptyset \). For every set \( A' \) there exists a sequence of balls \( \{ B_j \}_{j \in \mathbb{N}} \) such that

1. The families of \( \{ B_j \}_{j \in \mathbb{N}} \) and \( \{ 2B_j \}_{j \in \mathbb{N}} \) constitute a covering of finite multiplicity of \( U \);
2. \( \sum_{j=1}^{\infty} \Phi(2B_j) \leq \zeta_n \Phi(U) \) where the constant \( \zeta_n \) depends only on the dimension \( n \).

Proof In accordance with Lemma 7, construct two sequences \( \{ B_j \}_{j \in \mathbb{N}} \) and \( \{ 2B_j \}_{j \in \mathbb{N}} \) of balls and subdivide the latter into \( \zeta_n \) subfamilies \( \{ 2B_{1j} \}_{j \in \mathbb{N}}, \ldots, \{ 2B_{\zeta_n j} \}_{j \in \mathbb{N}} \) so that in each tuple the balls are disjoint: \( 2B_{ki} \cap 2B_{kj} = \emptyset \) for \( i \neq j \) and \( k = 1, \ldots, \zeta_n \). Consequently,

\[
\sum_{j=1}^{\infty} \Phi(2B_j) = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \Phi(2B_{kj}) \leq \sum_{k=1}^{\infty} \Phi(U) = \zeta_n \Phi(U).
\]

\[ \square \]

Mappings inducing a bounded composition operator are known to satisfy the Luzin \( N^{1-1} \)-property [69, Theorem 4].

Theorem 9 [69, Theorem 4] Take two open sets \( \Omega \) and \( \Omega' \) in \( \mathbb{R}^n \) with \( n \geq 1 \). If a measurable mapping \( \varphi : \Omega \to \Omega' \) induces a bounded composition operator

\[
\varphi^* : L^1_p(\Omega') \cap C^\infty(\Omega') \to L^1_q(\Omega), \quad 1 \leq q \leq p \leq n,
\]

then \( \varphi \) has the Luzin \( N^{1-1} \)-property, i.e. \( |\varphi^{-1}(A)| = 0 \) if \( |A| = 0, A \subset \Omega' \).

Remark 7 Theorem 4 of [69] is stated for a mapping \( \varphi : \Omega \to \Omega' \) generating a bounded composition operator \( \varphi^* : L^1_p(\Omega') \to L^1_q(\Omega) \) with \( 1 \leq q \leq p \leq n \). Observe that only smooth test functions are used in its proof, which therefore also justifies Theorem 9.

Here we obtain the next generalization of Theorem 9.

Lemma 9 If a measurable mapping \( \varphi : \Omega \to \Omega' \) induces a bounded composition operator

\[
\varphi^* : L^1_p(\Omega') \cap \text{Lip}(\Omega') \to L^1_q(\Omega), \quad 1 \leq q \leq p \leq n,
\]

then \( |\varphi^{-1}(E)| = 0 \) if \( |E| = 0, E \subset \Omega' \).

Proof If \( E \subset \Omega' \) then the statement of the theorem follows by Theorem 9. Consider the cut-off \( \eta \in C^\infty_0(\mathbb{R}^n) \) equal to 1 on \( B(0, 1) \) and vanishing outside \( B(0, 2) \). By Lemma 6 the function \( f(y) = \eta(\frac{y-\Omega}{r}) \) satisfies

\[
\|\varphi^* f \mid L^1_q(\Omega)\| \leq C_1 \Phi(2B)^{\frac{1}{p}} |B|^{\frac{1}{p'} - \frac{1}{q}}
\]

where \( B \cap \Omega' \neq \emptyset \) (let \( \Phi(2B)^{\frac{1}{p}} = \|\varphi^* f \| \) for any ball \( B \) if \( p = q \)). Take a set \( E \subset \partial \Omega' \) with \( |E| = 0 \). Since \( \varphi \) is a mapping with finite distortion [69], \( \varphi^{-1}(E) \neq \Omega \) (otherwise, \( J(x, \varphi) = 0 \) and, consequently, \( D\varphi(x) = 0 \), that is, \( \varphi \) is a constant mapping). Hence, there is a cube \( Q \subset \Omega \) such that \( 2Q \subset \Omega \) and \( |Q \setminus \varphi^{-1}(E)| > 0 \) (here \( 2Q \) is a cube with the same center as \( Q \) and the edges stretched by a factor of two compared to \( Q \)). Since \( \varphi \) is a measurable mapping, by Luzin’s theorem there is a compact set \( T \subset Q \setminus \varphi^{-1}(E) \) of positive measure such that \( \varphi : T \to \Omega' \) is continuous. Then, the image \( \varphi(T) \subset \Omega' \) is compact and \( \varphi(T) \cap E = \emptyset \). Consider an open set \( U \supset E \) with \( \varphi(T) \cap U = \emptyset \) and \( U \cap \Omega' \neq \emptyset \). Choose a tuple \( \{ B(y_i, r_i) \}_{i \in \mathbb{N}} \) of balls in accordance with Lemma 7: \( \{ B(y_i, r_i) \}_{i \in \mathbb{N}} \) and \( \{ B(y_i, 2r_i) \}_{i \in \mathbb{N}} \) are coverings of \( U \), and the multiplicity of the covering \( \{ B(y_i, 2r_i) \}_{i \in \mathbb{N}} \) is
finite \((B(y_i, 2r_i) \subset U\) for all \(i \in \mathbb{N}\). Then the function \(f_i\) associated to the ball \(B(y_i, r_i)\) enjoys \(\varphi^* f_i = 1\) on \(\varphi^{-1}(B(y_i, r_i))\) and \(\varphi^* f = 0\) outside \(\varphi^{-1}(B(y_i, 2r_i))\), in particular \(\varphi^* f_i = 0\) on \(T\). In addition, we have the estimate
\[
\|\varphi^* f_i \|_{L^q(2Q)} \leq \|\varphi^* f_i \|_{L^q(Q)} \leq C_1 \Phi(B(y_i, 2r_i))^\frac{1}{\sigma} |B(y_i, r_i)|^{\frac{1}{p} - \frac{1}{n}}.
\]

By the Poincaré inequality (see [40] for instance), for every function \(g \in W^{1,q}_{q,\text{loc}}(Q)\) with \(q < n\) vanishing on \(T\), we have
\[
\left(\int_Q |g|^{q^*} \, dx\right)^{1/q^*} \leq C_2 l(Q)^{n/q^*} \left(\int_{2Q} |\nabla g|^q \, dx\right)^{1/q}
\]
where \(q^* = \frac{na}{n-q}\) and \(l(Q)\) is the edge length of \(Q\).

Applying the Poincaré inequality to the function \(\varphi^* f_i\) and using the last two estimates, we obtain
\[
|\varphi^{-1}(B(y_i, r_i)) \cap Q|^{\frac{1}{q} - \frac{1}{n}} \leq C_3 \Phi(B(y_i, 2r_i))^\frac{1}{\sigma} |B(y_i, r_i)|^{\frac{1}{p} - \frac{1}{n}}.
\]

Note, that the constant \(C_3\) can depend on the cube \(Q\). In turn, Hölder’s inequality guarantees that
\[
\left(\sum_{i=1}^{\infty} |\varphi^{-1}(B(y_i, r_i)) \cap Q|\right)^{\frac{1}{q} - \frac{1}{n}} \leq C_3 \left(\sum_{i=1}^{\infty} \Phi(B(y_i, 2r_i))^\frac{1}{\sigma} \left(\sum_{i=1}^{\infty} |B(y_i, r_i)|\right)^{\frac{1}{p} - \frac{1}{n}}\right).
\]

As the open set \(U\) is arbitrary, this estimate yields \(|\varphi^{-1}(E) \cap Q| = 0\). Since the cube \(Q \subset \Omega\) is arbitrary, it follows that \(|\varphi^{-1}(E)| = 0\).

For the domain \(\Omega'\) with Lipschitz boundary we have \(|\partial \Omega'| = 0\). Note also that \(\varphi_0\) satisfies the conditions of Lemma 9 due to Lemma 2. Then Lemmas 5 and 9 imply Theorem 8.

Let us mention another interesting corollary of Theorem 9. Recall that a mapping \(f : \Omega \to \Omega'\) is said to be approximative differentiable at \(x \in \Omega\) with approximative derivative \(Df(x)\) if there is a set \(A \subset \Omega\) of density one at \(x^{11}\) such that
\[
\lim_{y \to x, \ y \in A} \frac{f(y) - f(x) - Df(x)(y-x)}{\|y - x\|} = 0.
\]

It is well known that Sobolev functions are approximative differentiable a.e. (see [17,27] for more details).

**Lemma 10** If an almost everywhere injective mapping \(\varphi : \Omega \to \Omega'\) with \(\varphi \in W^{1,1}_1(\Omega)\) and \(J(x, \varphi) \geq 0\) a.e. in \(\Omega\) has the Luzin \(N^{-1}\)-property then \(J(x, \varphi) > 0\) for almost all \(x \in \Omega\).

**Proof** Let \(E\) be a set outside which the mapping \(\varphi\) is approximatively differentiable and has the Luzin \(N^{-1}\)-property. Since \(\varphi \in W^{1,1}_1(\Omega)\), then \(|E| = 0\) (see [21,70]). In addition, we may assume that \(\{x \in \Omega \setminus E \ | \ J(x, \varphi) = 0\}\) is contained in a Borel set \(Z\). Put \(\sigma = \varphi(Z)\). By the change-of-variable formula [21, Theorem 2], taking the injectivity of \(\varphi\) into account, we obtain
\[
\int_{\Omega \setminus \Sigma} \chi_Z(x) J(x, \varphi) \, dx = \int_{\Omega \setminus \Sigma} (\chi_\sigma \circ \varphi)(x) J(x, \varphi) \, dx = \int_{\Omega'} \chi_\sigma(y) \, dy.
\]

\(^{11}\) i.e. \(\lim_{r \to 0} \frac{|A \cap B(x, r)|}{|B(x, r)|} = 1\).

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By construction, the integral in the left-hand side vanishes; consequently, \(|\sigma| = 0\). On the other hand, since \(\varphi\) has the Luzin \(N^{-1}\)-property, we have \(|Z| = 0\).

\[\text{Corollary 3} \quad \text{Let conditions of Theorem 8 be fulfilled, then } J(x, \varphi_0) > 0 \text{ for almost all } x \in \Omega.\]

\[\text{Theorem 10} \quad \text{Let } \Omega, \Omega' \subset \mathbb{R}^n \text{ be bounded domains with Lipschitz boundaries. Consider a sequence of homeomorphisms of finite distortion } \varphi_k, \text{ which maps } \Omega \text{ onto } \Omega', \text{ with } \varphi_k \in W_{n, \text{loc}}(\Omega), \text{ and } J(x, \varphi_k) \geq 0 \text{ a.e., such that:}\]

1. \(\varphi_k \to \varphi_0\) weakly in \(W_{n, \text{loc}}(\Omega)\);
2. The norms of inner distortion functions \(\|K_I(\cdot, \varphi_k)\|_{L^s}\) are totally bounded for some \(s > 1\);\(^{12}\)
3. The norms of outer distortion functions \(K_O(\cdot, \varphi_k) \in L_{n-1}(\Omega)\).

Then the mapping \(\varphi_0\) is a homeomorphism of finite distortion.

\[\text{Proof} \quad \text{By Corollary 2, in the light of Remark 4, the mapping } \varphi_0 \text{ is almost-everywhere injective, moreover, injectivity can be lost only if points go to the boundary. Moreover, } \varphi_0 \in W^1_n(\Omega) \text{ has finite distortion by Lemmas 1 and 2 (see also Remark 5), } K_O(\cdot, \varphi_0) \in L_{n-1}(\Omega) \text{ and } K_I(\cdot, \varphi_0) \in L_s(\Omega), s > 1, \text{ then the mapping } \varphi_0 \text{ is continuous and open by Theorem 2. Therefore so } \varphi_0 \text{ is a homeomorphism. Indeed, let } y \in \partial \Omega' \text{ be a point where injectivity is lost. Then there are } x_1, x_2 \in \Omega \text{ such that } \varphi_0(x_1) = \varphi_0(x_2) = y. \text{ Since mapping is continuous than neighborhoods } U_1 \subset \Omega \text{ and } U_2 \subset \Omega, \text{ of points } x_1 \text{ and } x_2 \text{ respectively, map to neighborhoods } V_1 \subset \Omega', V_2 \subset \Omega' \text{ of } y. \text{ This leads us to contradiction with openness of } \varphi_0. \]

\[\text{4 Elasticity}\]

The goal of this section is to prove the existence theorem for minimizing problem of energy functional in the classes \(\mathcal{H}(n - 1, s, M; \overline{\varphi})\) where \(s \in [1, \infty]\). Our prove works for all values of parameter \(s\). It is worth to note that at \(s = 1\) some results of this section look like some statements of paper \(\text{[32]}\). In our proof we use different arguments, such as the boundedness of composition operators. It gives an opportunity to apply them to new classes of deformations. Naturally, the proof of our main result differs substantially from previous works and is based crucially on the results and methods of \(\text{[61]}\).

For a comparison of our results with those in another papers see in Remark 9 and Sect. 5.

\[\text{4.1 Polyconvexity}\]

Let \(F = [f_{ij}]_{i, j=1, \ldots, n}\) be a \((n \times n)\)-matrix. For every pair of ordered tuples \(I = (i_1, i_2, \ldots, i_l), 1 \leq i_1 < \cdots < i_l \leq n, \text{ and } J = (j_1, j_2, \ldots, j_l), 1 \leq j_1 < \cdots < j_l \leq n, \text{ define the } l \times l\)-minor of the matrix \(F\)

\(^{12}\) The exponent \(r\) from Theorem 8 can be expressed as \(r = \frac{n(n-1)}{ns+1-s} \geq n - 1.\)
Notice that the $n \times n$-minor is the determinant of $F$. Let $F_\#$ be an ordered list of all minors of $F$. Let $F_\# \in D \subset \mathbb{R}^N$ for sufficiently large $N$ ($N = \binom{2n}{n}$), where $D$ be a convex set with nonnegative $n \times n$-minor.

**Definition 3** [3] A function $W : \mathbb{M}^{n \times n} \to \mathbb{R}$ is polyconvex if there exists a convex function $G : D \to \mathbb{R}$, such that

$$G(F_\#) = W(F).$$

Examples of polyconvex but not convex functions are

$$W(F) = \det F$$

and

$$W(F) = \text{tr \, Adj } F^T F = \| \text{Adj } F^T F \|^2$$

(see, for example, [13]). It is known that for a hyperelastic material with experimentally known Lamé coefficients it can be constructed a stored-energy function of an Ogden material (see [13,47] for more details). On the other hand, a well-known Saint-Venant–Kirchhoff material, is not polyconvex [13, Theorem 4.10].

### 4.2 Existence theorem

Let $\Omega, \Omega' \subset \mathbb{R}^n$ be two bounded domains with Lipschitz boundaries. Recall that a mapping $G : \Omega \times \mathbb{R}^m \to \mathbb{R}$ enjoys the Carathéodory conditions whenever $G(x, \cdot)$ is continuous on $\mathbb{R}^m$ for almost all $x \in \Omega$; and $G(\cdot, a)$ is measurable on $\Omega$ for all $a \in \mathbb{R}^m$.

Consider a functional

$$I(\varphi) = \int_\Omega W(x, D\varphi(x)) \, dx,$$

where $W : \Omega \times \mathbb{M}^{n \times n} \to \mathbb{R}$ is a stored-energy function with the following properties:

(a) **Polyconvexity** there exists a convex function $G : \Omega \times D \to \mathbb{R}$, $D \subset \mathbb{R}^N$, meeting Carathéodory conditions such that for all $F \in \mathbb{M}^{n \times n}$, $\det F \geq 0$, the equality

$$G(x, F_\#) = W(x, F)$$

holds almost everywhere in $\Omega$;

(b) **Coercivity** there exists a constant $\alpha > 0$ and a function $g \in L_1(\Omega)$ such that

$$W(x, F) \geq \alpha |F|^n + g(x)$$

(11)

for almost all $x \in \Omega$ and all $F \in \mathbb{M}^{n \times n}$, $\det F \geq 0$. 

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Given constants \( p, q \geq 1, M > 0 \) define the class of admissible deformations

\[
\mathcal{H}(p, q, M) = \{ \varphi : \Omega \to \Omega' \text{ is a homeomorphism with finite distortion,} \\
\varphi \in W^1_1(\Omega), I(\varphi) < \infty, J(x, \varphi) \geq 0 \text{ a.e. in } \Omega, \\
K_O(\cdot, \varphi) \in L^p(\Omega), \|K_I(\cdot, \varphi)\|_{L^q(\Omega)} \leq M \},
\] (12)

where \( K_O(x, \varphi) \) and \( K_I(x, \varphi) \) are the outer and the inner distortion functions defined by (9).

For these families of admissible deformations we have natural embeddings

\[
\mathcal{H}(p, q_2, M_2) \subset \mathcal{H}(p, q_1, M_1)
\]

if \( q_1 \leq q_2 \) and \( M_2|\Omega|^\frac{1}{p_2} - \frac{1}{q_2} \leq M_1 \). If \( p_1 \leq p_2 \) then

\[
\mathcal{H}(p_2, q, M) \subset \mathcal{H}(p_1, q, M)
\]

also holds.

**Theorem 11** (Existence theorem) Suppose that conditions (a) and (b) on the function \( W(x, F) \) are fulfilled and the set \( \mathcal{H}(n - 1, s, M) \) is nonempty, \( M > 0, s > 1 \). Then there exists at least one homeomorphic mapping

\[
\varphi_0 \in \mathcal{H}(n - 1, s, M) \quad \text{such that} \quad I(\varphi_0) = \inf \{I(\varphi), \varphi \in \mathcal{H}(n - 1, s, M)\}.
\]

If there is a homeomorphic Dirichlet data \( \overline{\varphi} : \overline{\Omega} \to \overline{\Omega}', \overline{\varphi} \in W^1_{n}(\Omega), J(x, \overline{\varphi}) > 0 \text{ a.e. in } \Omega, \|K_I(\cdot, \overline{\varphi})\|_{L^q(\Omega)} \leq M, \) and \( I(\overline{\varphi}) < \infty \), then we can define the classes of admissible deformations

\[
\mathcal{H}(p, q, M; \overline{\varphi}) = \{ \varphi \in \mathcal{H}(p, q, M), \varphi|_{\partial \Omega} = \overline{\varphi}|_{\partial \Omega} \text{ a.e. on } \partial \Omega \}.
\]

Because of Theorem 11 and compactness of the trace operator (see [40, Sect. 1.4.5–1.4.6] for instance) it can be easily obtained the next existence theorem with respect to a Dirichlet boundary condition \( \varphi|_{\partial \Omega} = \overline{\varphi}|_{\partial \Omega} \text{ a.e. on } \partial \Omega \).

**Corollary 4** Suppose that conditions (a) and (b) on the function \( W(x, F) \) are fulfilled and the set \( \mathcal{H}(n - 1, s, M; \overline{\varphi}) \) is nonempty, \( M > 0, s \geq 1 \). Then there exists at least one mapping \( \varphi_0 \in \mathcal{H}(n - 1, s, M; \overline{\varphi}) \) such that

\[
I(\varphi_0) = \inf \{I(\varphi), \varphi \in \mathcal{H}(n - 1, s, M; \overline{\varphi})\}.
\]

In some cases it is more convenient to consider deformations of the same homotopy class as a given homeomorphism \( \overline{\varphi} \) instead of deformations with prescribed boundary values.

In this case we can define the next class of admissible deformations

\[
\mathcal{H}(p, q, M; \overline{\varphi}, \text{hom}) = \{ \varphi \in \mathcal{H}(p, q, M), \varphi \text{ belongs to the same homotopy class as } \overline{\varphi} \}.
\]

**Corollary 5** Suppose that conditions (a) and (b) on the function \( W(x, F) \) are fulfilled and the set \( \mathcal{H}(n - 1, s, M; \overline{\varphi}, \text{hom}) \) is nonempty, \( M > 0, s \geq 1 \). Then there exists at least one mapping \( \varphi_0 \in \mathcal{H}(n - 1, s, M; \overline{\varphi}, \text{hom}) \) such that

\[
I(\varphi_0) = \inf \{I(\varphi), \varphi \in \mathcal{H}(n - 1, s, M; \overline{\varphi}, \text{hom})\}.
\]
Remark 8 Note that we can omit the condition that $\varphi$ is a homeomorphism in the definition of $\mathcal{H}(n-1, s, M; \varphi, hom)$ (and $\mathcal{H}(n-1, s, M; \overline{\varphi})$) if $s > 1$. Since $\varphi \in \mathcal{H}(n-1, s, M; \varphi)$ belongs to $W^n_1(\Omega), K_O(\cdot, \varphi) \in L_{n-1}(\Omega)$ and $K_1(\cdot, \varphi) \in L_1(\Omega), s > 1$, the mapping $\varphi$ is continuous, open and discrete (Theorem 2 and [49]). Also, it is known that continuous open discrete mapping $\varphi$, with the same homotopy class as a given homeomorphism $\overline{\varphi} \in W^n_1(\Omega), \varphi \in A(n, M; \overline{\varphi}, hom)$ belongs to the same homotopy class as $\overline{\varphi}$, with the same homotopy class as a given homeomorphism $\overline{\varphi} \in W^n_1(\Omega), \varphi \in A(n, M; \overline{\varphi}, hom)$.

Remark 9 Added to this is the fact that if we have boundary conditions, we do not need restriction on $K_O(x, \varphi)$ (see Remark 10 for details). Thereafter for $s \geq 1$ instead of $\mathcal{H}(n-1, s, M; \varphi)$ and $\mathcal{H}(n-1, s, M; \overline{\varphi}, hom)$ we can consider classes

$$A(s, M; \varphi) = \{ \varphi \in A(s, M), \varphi|_{\partial \Omega} = \overline{\varphi}|_{\partial \Omega} \text{ a.e. on } \partial \Omega \} \quad \text{and}$$

$$A(s, M; \overline{\varphi}, hom) = \{ \varphi \in A(s, M), \varphi \text{ belongs to the same homotopy class as } \overline{\varphi} \},$$

where

$$A(s, M) = \{ \varphi: \Omega \rightarrow \Omega' \text{ is a homeomorphism with finite codistortion, } \varphi \in W^n_1(\Omega), I(\varphi) < \infty, J(x, \varphi) \geq 0 \text{ a.e. on } \Omega, \|K_1(\cdot, \varphi)|_{L_1(\Omega)}\| \leq M \}. \quad \text{(13)}$$

Note that by Theorem 8 mappings of class $A$ have also finite distortion. Moreover, for a mapping being of the class $A(1, M)$ we ask the same requirements as those in the paper [32].

4.3 Proof of the existence theorem

In this section we prove the existence of a minimizing mapping for the functional

$$\overline{T}(\varphi) = I(\varphi) - \int_{\Omega} g(x) \, dx.$$ 

Observe now that the coercivity (11) of the function $W$ and the corollary of the Poincaré inequality (see [13, Theorem 6.1–8] for instance) ensure the existence of constants $c > 0$ and $d \in \mathbb{R}$ such that

$$\overline{T}(\varphi) = I(\varphi) - \int_{\Omega} g(x) \, dx \geq c \|\varphi|_{W^n_1(\Omega)}\|^n + d \quad \text{(15)}$$

for every mapping $\varphi \in \mathcal{H} = \mathcal{H}(n-1, s, M)$, where $\mathcal{H}$ is defined by (12).

Take a minimizing sequence $\{\varphi_k\}$ for the functional $\overline{T}$. Then

$$\lim_{k \to \infty} \overline{T}(\varphi_k) = \inf_{\varphi \in \mathcal{H}} \overline{T}(\varphi).$$

By (15) and the assumption $\inf_{\varphi \in \mathcal{H}} \overline{T}(\varphi) < \infty$, the sequence $\{\varphi_k\}_{k \in \mathbb{N}}$ is bounded in $W^n_1(\Omega)$.

Remind that Sobolev space $W^n_1$ has the “continuity” property of minors —rank-$l$ minors of $D\varphi_k$ are weakly converging if $\varphi_k$ belongs to $W^n_1$ with $p \geq l, 1 \leq l < n$ [3, 41, 51, 53]. In the case $l = n$ there is no weak convergence but something close to it [53, Sect. 4.5]. For achieving weak convergence of Jacobians, it is necessary to impose some additional conditions, for instance, nonnegativity of Jacobians almost everywhere [43]. Here it will be convenient for us the next formulation of this assertion, which can be found in [19].
Lemma 11 (Weak continuity of minors) Let Ω be a domain in \( \mathbb{R}^n \) and a sequence \( f_k : \Omega \to \mathbb{R}^n \), \( k = 1, 2, \ldots \), converge weakly in \( W_{1,\text{loc}}^1(\Omega) \) to a mapping \( f_0 \). For \( l \)-tuples \( 1 \leq i_1 < \cdots < i_l \leq n \) and \( 1 \leq j_1 < \cdots < j_l \leq n \) the equality

\[
\lim_{k \to \infty} \int_{\Omega} \frac{\partial (f_{i_1}^k, \ldots, f_{i_l}^k)}{\partial (x_{j_1}, \ldots, x_{j_l})} \, dx = \int_{\Omega} \frac{\partial (f_0^{i_1}, \ldots, f_0^{i_l})}{\partial (x_{j_1}, \ldots, x_{j_l})} \, dx
\]

holds for every \( \theta \) in \( L_{n/(n-l)}(\Omega) \), the space of functions in \( L_{n/(n-l)}(\Omega) \) with compact support in \( \Omega \), and corresponding \( l \times l \) minors\(^\ast\) of \( D_{f_k} \) and \( D_{f_0} \), \( l = 1, 2, \ldots, n - 1 \).

Moreover, if in addition \( J(x, f_k) \geq 0 \) a.e. in \( \Omega \), the equality (16) holds for \( l = n \).

Hence there exists a minimizing sequence fulfilling the conditions

\[
\begin{align*}
\varphi_k &\longrightarrow \varphi_0 \quad \text{weakly in } W_{1}^{1}(\Omega), \\
\text{Adj } D\varphi_k &\longrightarrow \text{Adj } D\varphi_0 \quad \text{weakly in } L_{\frac{n}{n-1},\text{loc}}(\Omega), \\
\vdots &
\end{align*}
\]

\[ J(\cdot, \varphi_k) \longrightarrow J(\cdot, \varphi_0) \quad \text{weakly in } L_{1,\text{loc}}(\Omega) \]

as \( k \to \infty \), where \( \varphi_0 \) guarantees the sharp lower bound \( \overline{T}(\varphi_0) = \inf_{\varphi \in \mathcal{H}} T(\varphi) \). It remains to verify that \( \varphi_0 \in \mathcal{H} \). To this end, we need the properties of mappings in \( \mathcal{H} \).

Theorem 10 ensures that \( \varphi_0 \) is a homeomorphism. Moreover, Corollary 3 implies the limit mapping \( \varphi_0 \) satisfies the strict inequality \( J(x, \varphi_0) > 0 \) a.e. in \( \Omega \).

Remark 10 Theorem 2 is not known if \( s = 1 \) (see Remark 1). However, we include the case \( s = 1 \) for classes \( \mathcal{H}(n - 1, s; M; \varphi) \) and \( \mathcal{H}(n - 1, s; M; \varphi, \text{hom}) \) (\( A(s; M; \varphi) \) and \( A(s; M; \varphi, \text{hom}) \)). Indeed, whereas both \( \varphi_k \) and \( \psi_k \) belong to Sobolev spaces \( W_{1}^{1}(\Omega) \) and \( W_{1}^{n}(\Omega') \), the same arguments as in Lemma 4 ensure that there are a sequence of homeomorphisms \( \{\varphi_k\}_{k \in \mathbb{N}} \) and a sequence of inverse homeomorphisms \( \{\psi_k\}_{k \in \mathbb{N}} \), which converge locally uniformly to \( \varphi_0 \) and \( \psi_0 \) respectively.

Then \( \varphi_0 \) and \( \psi_0 \) are continuous and

\[
\psi_0 \circ \varphi_0(x) = x, \quad \varphi_0 \circ \psi_0(y) = y,
\]

if \( \varphi_0(x) \notin \partial \Omega' \) and \( \psi_0(y) \notin \partial \Omega' \).

Since \( \varphi_0 \) coincides with the given homeomorphism \( \overline{\varphi} \) on the boundary (or is in the same homotopy class), \( \deg(y, \Omega, \varphi_0) = 1 \) for \( y \notin \varphi_0(\partial \Omega) \). Therefore for \( y \in \Omega' \) there is \( x \in \Omega \) such that \( \varphi_0(x) = y \in \Omega' \). Passing to the limit in \( \psi_k \circ \varphi_k(x) = x \), we obtain \( \psi_0(y) = x \in \Omega \).

Similar we obtain \( \varphi_0(x) = y \in \Omega' \) for \( x \in \Omega \).

In order to make sure that \( \varphi_0 \in \mathcal{H} \) it remains to verify

\[
K_O(\cdot, \varphi_0) \in L_{n-1}(\Omega) \quad \text{and} \quad \|K_I(\cdot, \varphi_0) \mid L_s(\Omega)\| \leq M.
\]

It follows from the semicontinuity property of distortion coefficient [19], [31, Theorem 8.10.1] (see this property under weaker assumption and some generalization in [65,67]).

\(^{\ast}\) i.e. determinants of the matrix that is formed by taking the elements of the original matrix from the rows whose indexes are in \((i_1, i_2, \ldots, i_l)\) and columns whose indexes are in \((j_1, j_2, \ldots, j_l)\).
In order to complete the proof, it remains to verify lower semicontinuity of the functional
\[
\int_{\Omega} W(x, D\varphi_0) \, dx \leq \liminf_{k \to \infty} \int_{\Omega} W(x, D\varphi_k) \, dx,
\]
using conventional technique for polyconvex case (see, for example, [43, Sect. 5]).

5 Examples

In all this section we consider \( n = 3 \).

As our first example consider an Ogden material with the stored-energy function \( W_1 \) of the form
\[
W_1(F) = a \operatorname{tr}(F^T F)^{\frac{p}{2}} + b \operatorname{tr} \operatorname{Adj}(F^T F)^{\frac{q}{2}} + c(\det F)^p + d(\det F)^{-m},
\]
where \( a > 0, b > 0, c > 0, d > 0, p > 3, q > 3, r > 1, \) and \( m > \frac{2q}{q-3} \). Then \( W_1(F) \) is polyconvex and the coercivity inequality holds [13, Theorem 4.9-2]:
\[
W_1(F) \geq \alpha (|F|^p + |\operatorname{Adj} F|^q) + c(\det F)^p + d(\det F)^{-m}.
\]

We have to solve the minimization problem
\[
I_1(\varphi_B) = \inf \{ I_1(\varphi) : \varphi \in A_B \},
\]
where \( I_1(\varphi) = \int_{\Omega} W_1(D\varphi(x)) \, dx \) and the class of admissible deformations
\[
A_B = \{ \varphi \in W^1_0(\Omega), \ I_1(\varphi) < \infty, \ J(x, \varphi) > 0 \text{ a.e. in } \Omega, \quad \varphi|_{\partial \Omega} = \varphi|_{\partial \Omega} \text{ a.e. on } \partial \Omega \}
\]
is defined by (2) for homeomorphic boundary conditions \( \varphi : \Omega \to \Omega', \varphi \in W^1_p(\Omega), J(x, \varphi) > 0 \text{ a.e. in } \Omega \) and \( I_1(\varphi) < \infty \). The result of Ball [4] ensures that there exists at least one solution \( \varphi_B \in A_B \) to this problem, which is a homeomorphism in addition.

Denote \( \inf_{\varphi \in A_B} I_1(\varphi) + m = M \) for any \( m > 0 \) and consider a class, defined by (13),
\[
A(s, M; \varphi) = \{ \varphi : \Omega \to \Omega' \text{ is a homeomorphism with finite codistortion}, \quad \varphi \in W^1_0(\Omega), \ I_1(\varphi) < \infty, \ J(x, \varphi) \geq 0 \text{ a.e. in } \Omega, \quad \|K_1(\cdot, \varphi)\ | L_s(\Omega)\| \leq M, \ \varphi|_{\partial \Omega} = \varphi|_{\partial \Omega} \text{ a.e. on } \partial \Omega \}.
\]

It is easy to check that \( \varphi \in A_B \) is a homeomorphism (by [4, Theorem 2]), has finite distortion (as \( J(x, \varphi) \geq 0 \text{ a.e.} \) and \( \|K_1(\cdot, \varphi)\ | L_s(\Omega)\| \leq M \) by the Hölder inequality for \( s = \frac{\alpha r}{r + \alpha - n} > 1 \) where \( \alpha = \frac{q(1+m)}{q-m} > n \). It means that \( A_B \cap A(s, M; \varphi) \neq \emptyset \). Moreover, a minimizing sequence \( \{\varphi_k\} \subset A_B \) of the problem (18) belongs to \( A(s, M; \varphi) \) as well.

On the other hand, for the functions of the form (17) Theorem 11 holds. Indeed, \( W_1(F) \) is polyconvex and satisfies
\[
W_1(F) \geq \alpha |F|^3 - \alpha,
\]
where \( \alpha \) plays the role of the function \( h(x) \) of (11). When we consider the same boundary conditions \( \varphi : \Omega \to \Omega' \) and solve the minimization problem
\[
I_1(\varphi_0) = \inf \{ I_1(\varphi) : \varphi \in A(s, M; \varphi) \}
\]
Lemma 4 and Remark 9 yields a solution \( \varphi_0 \in A(s, M; \varphi) \) which is a homeomorphism.
Let us discuss another example. Here the stored-energy function is of the form

\[ W_2(F) = a \text{tr}(F^T F)^{\frac{3}{2}}. \]

This function is polyconvex and satisfies

\[ W_2(F) \geq \alpha \| F \|^3, \]

but violates the inequality of the form (3). Moreover, \( W_2(F) \) violates the asymptotic condition

\[ W_2(x, F) \to \infty \text{ as } \det F \to 0_+, \]

which plays an important role in [4,7] and other articles.

Nevertheless, for the stored-energy function \( W_2 \) there exists a solution to the minimization problem

\[ I_2(\varphi_0) = \inf_{\varphi} I_2(\varphi) \]

in the class of homeomorphisms \( \varphi \in \mathcal{H}(n - 1, s, M) \), \( s > 1 \), where

\[ I_2(\varphi) = \int_{\Omega} W_2(D\varphi(x)) \, dx. \]

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References

1. Adams, R.A.: Sobolev Spaces. Academic Press, New York (1975)
2. Astala, K., Iwaniec, T., Martin, G.J., Onninen, J.: Extremal mappings of finite distortion. Proc. Lond. Math. Soc. (3) 91(3), 655–702 (2005)
3. Ball, J.M.: Convexity conditions and existence theorems in nonlinear elasticity. Arch. Ration. Mech. Anal. 63, 337–403 (1977)
4. Ball, J.M.: Global invertibility of Sobolev functions and the interpretation of matter. Proc. R. Soc. Edinb. Sect. A 88, 315–328 (1981)
5. Ball, J.M.: Some open problems in elasticity. In: P.N. et al. (ed.) Geometry, Mechanics, and Dynamics, pp. 3–59. Springer, New York (2002)
6. Ball, J.M.: Progress and puzzles in nonlinear elasticity, poly-, quasi- and rank-one convexity in applied mechanics. CISM Int. Centre Mech. Sci. 516, 1–15 (2010)
7. Ball, J.M., Currie, J.C., Olver, P.J.: Null Lagrangians, weak continuity, and variational problems of arbitrary order. J. Funct. Anal. 41, 135–174 (1981)
8. Barchiesi, M., Henao, D., Mora-Corral, C.: Local invertibility in Sobolev spaces with applications to nematic elastomers and magnetoelasticity. Arch. Ration. Mech. Anal. 224, 743–816 (2017)
9. Bauman, P., Phillips, D.: Univalent minimizers of polyconvex functionals in 2 dimensions. Arch. Ration. Mech. Anal. 126, 161–181 (1994)
10. Baykin, A.N., Vodopyanov, S.K.: Capacity estimates, Liouville’s theorem, and singularity removal for mappings with bounded \((p, q)\)-distortion. Sib. Math. J. 56(2), 237–261 (2015)
11. Benešová, B., Kampschulte, M.: Gradient Young measures generated by quasiconformal maps the plane. SIAM J. Math. Anal. 47, 4404–4435 (2015)
12. Benešová, B., Kružík, M.: Characterization of gradient Young measures generated by homeomorphisms in the plane. ESAIM Control Optim. Calc. Var. 22, 267–288 (2016)
13. Ciarlet, P.G.: Mathematical Elasticity Vol. I: Three-Dimensional Elasticity, Series “Studies in Mathematics and its Applications”. North-Holland, Amsterdam (1988)
14. Ciarlet, P.G., Nečas, J.: Unilateral problems in nonlinear three-dimensional elasticity. Arch. Ration. Mech. Anal. 87(4), 319–338 (1985)
15. Ciarlet, P.G., Nečas, J.: Injectivity and self-contact in nonlinear elasticity. Arch. Ration. Mech. Anal. 97(3), 171–188 (1987)
16. Conti, S., De Lellis, C.: Some remarks on the theory of elasticity for compressible Neohookean materials. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 2, 521–549 (2003)
17. Federer, H.: Geometric Measure Theory. Springer, Berlin (1969)
18. Fonseca, I., Gangbo, W.: Local invertibility of Sobolev functions. SIAM J. Math. Anal. 26, 280–304 (1995)
19. Gehring, F.W., Iwaniec, T.: The limit of mappings with finite distortion. Ann. Acad. Sci. Fenn. Math. 24, 253–264 (1999)
20. Guzmán, M.: Differentiation of integrals in $\mathbb{R}^n$. In: Lecture Notes in Mathematics, Vol. 481. Springer (1975)
21. Hajłasz, P.: Change of variables formula under minimal assumptions. Colloq. Math. 64(1), 93–101 (1993)
22. Heinonen, J., Koskela, P.: Sobolev mappings with integrable dilatation. Arch. Ration. Mech. Anal. 125(1), 81–97 (1993)
23. Henao, D., Mora-Corral, C.: Invertibility and weak continuity of the determinant for the modelling of cavitation and fracture in nonlinear elasticity. Arch. Ration. Mech. Anal. 197, 619–655 (2010)
24. Henao, D., Mora-Corral, C.: Regularity of inverses of Sobolev deformations with finite surface energy. J. Funct. Anal. 268, 2356–2378 (2015)
25. Hencl, S., Koskela, P.: Mappings of finite distortion: discreteness and openness for quasilight mappings. Ann. Inst. H. Poincaré Anal. Non Linéaire 22(3), 331–342 (2005)
26. Hencl, S., Koskela, P.: Regularity of the inverse of a planar Sobolev homeomorphism. Arch. Ration. Mech. Anal. 180(1), 75–95 (2006)
27. Hencl, S., Koskela, P.: Lectures on mappings of finite distortion. In: Lecture Notes in Mathematics, Vol. 2096. Springer (2014)
28. Hencl, S., Malý, J.: Mappings of finite distortion: Hausdorff measure of zero sets. Math. Ann. 324(3), 451–464 (2002)
29. Hencl, S., Onninen, J.: Jacobian of weak limits of Sobolev homeomorphisms. Adv. Calc. Var. 11(1), 65–73 (2018)
30. Hencl, S., Rajala, K.: Optimal assumptions for discreteness. Arch. Ration. Mech. Anal. 3, 775–783 (2013)
31. Iwaniec, T., Martin, G.: Geometric Function Theory and Non-linear Analysis. Oxford Mathematical Monographs. Clarendon Press, Oxford (2001)
32. Iwaniec, T., Onninen, J.: Hyperelastic deformations of smallest total energy. Arch. Ration. Mech. Anal. 194(3), 927–986 (2009)
33. Iwaniec, T., Onninen, J.: Deformations of finite conformal energy: existence, and removability of singularities. Proc. Lond. Math. Soc. (3) 100(1), 1–23 (2010)
34. Iwaniec, T., Onninen, J.: Deformations of finite conformal energy: boundary behavior and limits theorems. Trans. Am. Math. Soc. 363(11), 5605–5648 (2011)
35. Iwaniec, T., Onninen, J.: $n$-Harmonic mappings between annuli. Mem. Am. Math. Soc. 218, 1023 (2012)
36. Iwaniec, T., Šverák, V.: On mappings with integrable dilatation. Proc. Am. Math. Soc. 118, 185–188 (1993)
37. Koskela, P., Malý, J.: Mappings of finite distortion: the zero set of the Jacobian. J. Eur. Math. Soc. 5, 95–105 (2003)
38. Manfredi, J., Villamor, E.: An extension of Reshetnyak’s theorem. Indiana Univ. Math. J. 47(3), 1131–1145 (1998)
39. Martio, O., Malý, J.: Lusin’s condition (N) and mappings of the class $W^1_1$. J. Reine Angew. Math. 485, 19–36 (1995)
40. Maz’ya, V.: Sobolev spaces: with applications to elliptic partial differential equations. In: Grundlehren der mathematischen Wissenschaften, vol. 342. Springer, Berlin (2011)
41. Morrey, C.B.: Multiple Integrals in the Calculus of Variations. Springer, Berlin (1966)
42. Mostow, G.D.: Quasi-conformal mappings in $n$-space and the rigidity of the hyperbolic space forms. Publ. Math. Inst. Hautes Études Sci. 34, 53–104 (1968)
43. Müller, S.: Higher integrability of determinants and weak convergence in $L^1$. J. Reine Angew. Math. 412, 20–34 (1990)
44. Müller, S., Qi, T., Yan, B.S.: On a new class of elastic deformations not allowing for cavitation. Ann. Inst. H. Poincaré Anal. Non Linéaire 11, 217–243 (1994)
45. Müller, S., Spector, S.: An existence theory for nonlinear elasticity that allows for cavitation. Arch. Ration. Mech. Anal. 131(1), 1–66 (1995)
46. Müller, S., Spector, S., Tang, Q.: Invertibility and a topological property of Sobolev maps. SIAM J. Math. Anal. 27, 959–976 (1996)
47. Ogden, R.W.: Large deformation isotropic elasticity: on the correlation of theory and experiment for compressible rubber-like solids. Proc. Roy. Soc. Lond. A 328, 567–583 (1972)
48. Onninen, J.: Regularity of the inverse of spatial mappings with finite distortion. Calc. Var. Partial Differ. Equ. 26(3), 331–341 (2006)
49. Rajala, K.: Remarks on the Iwaniec–Šverák conjecture. Indiana Univ. Math. J. 59(6), 2027–2039 (2010)
50. Rajala, K.: Reshetnyak’s theorem and the inner distortion. Pure Appl. Math. Q. 7, 411–424 (2011)
51. Reshetnyak, Y.G.: On the stability of conformal mappings in multidimensional spaces. Siberian Math. J. 8(1), 69–85 (1967)
52. Reshetnyak, Y.G.: Space mappings with bounded distortion. Sib. Math. J. 8(3), 466–487 (1967)
53. Reshetnyak, Y.G.: Space mappings with bounded distortion. In: Translations of Mathematical Monographs, 73. AMS, New York (1989)
54. Rickman, S.: Quasiregular Mappings. Springer, Berlin (1993)
55. Šverák, V.: Regularity properties of deformations with finite energy. Arch. Ration. Mech. Anal. 100(2), 105–127 (1988)
56. Swanson, D., Ziemer, W.P.: A topological aspect of Sobolev mappings. Calc. Var. Partial Differ. Equ. 14(1), 69–84 (2002)
57. Swanson, D., Ziemer, W.P.: The image of a weakly differentiable mapping. SIAM J. Math. Anal. 35(5), 1099–1109 (2004)
58. Tang, Q.: Almost-everywhere injectivity in nonlinear elasticity. Proc. Roy. Soc. Edinb. Sect. A 109(1–2), 79–95 (1988)
59. Ukhlov, A.D.O.: On mappings generating the embeddings of Sobolev spaces. Sib. Math. J. 34(1), 185–192 (1993)
60. Vodopyanov, S.K.: Spaces of differential forms and maps with controlled distortion. Izv. Math. 74(4), 5–32 (2010)
61. Vodopyanov, S.K.: Regularity of mappings inverse to Sobolev mappings. Mat. Sb. 203(10), 1383–1410 (2012)
62. Vodopyanov, S.K.: Basics of the quasiconformal analysis of a two-index scale of spatial mappings. Sib. Math. J. 59(5), 805–834 (2018)
63. Vodopyanov, S.K.: Differentiability of mappings of the sobolev space $W^{1}_{n-1}$ with conditions on the distortion function. Sib. Math. J. 59(6), 983–1005 (2018)
64. Vodopyanov, S.K., Gol’dshtein, V.M.: Quasiconformal mappings and spaces of functions with generalized first derivatives. Sib. Math. J. 17(3), 399–411 (1976)
65. Vodopyanov, S.K., Kudyavtseva, N.A.: On the convergence of mappings with $k$-finite distortion. Math. Notes 102(6), 878–883 (2017)
66. Vodopyanov, S.K., Molchanova, A.O.: Variational problems of the nonlinear elasticity theory in certain classes of mappings with finite distortion. Dokl. Math. 92(3), 739–742 (2015)
67. Vodopyanov, S.K., Molchanova, A.O.: Lower semicontinuity of distortion coefficient of mappings with bounded $(\theta, 1)$-weighted $(p, q)$-distortion. Sib. Math. J. 57(5), 999–1011 (2016)
68. Vodopyanov, S.K., Ukhlov, A.D.O.: Sobolev spaces and $(P, Q)$-quasiconformal mappings of Carnot groups. Sib. Math. J. 39(4), 665–682 (1998)
69. Vodopyanov, S.K., Ukhlov, A.D.O.: Superposition operators in Sobolev spaces. Russian Math. (Iz. VUZ) 46(10), 9–31 (2002)
70. Whitney, H.: On totally differentiable and smooth functions. Pac. J. Math. 5(1), 143–159 (1951)

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