Connecting the very large and the very small: effective particle mass in curved-space cosmological models

Peter R. Phillips

Department of Physics, Washington University, St Louis, MO 63130, USA

Accepted 2009 January 30. Received 2009 January 7

ABSTRACT

We investigate the propagation of electromagnetic fields and potentials in the plasma of the early Universe, assuming a Friedmann–Robertson–Walker background with negative curvature. Taking over results from classical plasma physics, we show that charged particles will acquire an effective mass that has not only the expected thermal component but also a non-thermal component due to the influence of distant matter. Although this is a direct effect of the vector potential, we show the theory is nevertheless gauge invariant. This phenomenon is therefore in the same category as the Aharonov–Bohm effect. The non-thermal component becomes increasingly important with time, and in some cosmological models can prove to be of decisive importance in bringing about the phase transition that generates normal masses.

Key words: elementary particles – plasmas – cosmology: theory – early Universe.

1 INTRODUCTION

In this paper, we study the effective mass of charged particles in a plasma, with particular emphasis on the early Universe, before ordinary masses are generated. In Minkowski space, particles of mass $m$ acquire an additional effective thermal mass, $m_{th}$, of general order of the temperature of the plasma. This is normally derived using path-integral methods (Narlikar & Padmanabhan 1986). We will estimate the effective mass from a different point of view, which will allow a straightforward extension to an expanding Friedmann–Robertson–Walker (FRW) space.

Our main conclusion is that if the FRW space has negative curvature, the effective mass acquires a new non-thermal component, $m_{nt}$, in addition to the thermal one. This new component is due to the cumulative effect of distant matter; it is not constant, but increases with time relative to the thermal component. In present-day standard cosmology, where inflation leads to a flat space, $m_{nt}$ will not appear. But there are alternative cosmological models in which it could be of decisive importance.

$m_{nt}$ appears because in curved space the potentials (but not the fields) have a tail (DeWitt & Brehme 1960; Narlikar 1970; Haghhipour 2005), as shown in Sections 6 and 8. In the tail, we have potentials but no fields, so $m_{nt}$ is a direct effect of the potentials. This is a quantum mechanical result, though we have derived it in the context of classical gravity. As with the Aharonov–Bohm effect (Aharonov & Bohm 1959; Tomomura et al. 1982), the appearance of $m_{nt}$ at first seems to violate gauge invariance, and we have to show why this is actually not so.

2 NOTATION

We consider a model consisting of a single massless, charged scalar field, $\Phi$, interacting with the usual electromagnetic fields and potentials. Units are chosen so that $c = \hbar = 1$. Our sign conventions are those of Weinberg (1972), so the metric signature in Minkowski space is $(−+++)$. When gravitational effects are introduced, $g = −\text{Det}(g_{\mu\nu})$.

For the early Universe, we assume a FRW metric corresponding to negative curvature. (Positive curvature would give similar effects, but involves an awkward discrete distribution of wave numbers.) We use coordinates $t$, $\chi$, $\theta$, and $\phi$, labelled 0, 1, 2 and 3, respectively, so that

$$ds^2 = −dt^2 + R^2(t) \left(d\chi^2 + \sinh^2 \chi \, d\theta^2 + \sinh^2 \chi \sin^2 \theta \, d\phi^2\right),$$

(1)

$$g_{\mu\nu} = \text{diag} \left[−1, R^2(t), R^2(t) \sinh^2 \chi, R^2(t) \sinh^2 \chi \sin^2 \theta\right],$$

(2)

$$\sqrt{g} = R(t) \sin^2 \chi \sin \theta.$$ (3)

The expansion parameter, $R(t)$, has the dimension of length, and can be thought of as the radius of the Universe at time $t$.

The conformal time, $\eta$, is related to the ordinary time $t$ by $d\eta = dt/R(t)$. In terms of $\eta$, $\chi$, $\theta$, $\phi$ (all dimensionless) we have

$$ds^2 = R^2(\eta) \left(−d\eta^2 + d\chi^2 + \sinh^2 \chi \, d\theta^2 + \sinh^2 \chi \sin^2 \theta \, d\phi^2\right),$$

(4)

$$g_{\mu\nu} = R^2(\eta) \text{diag} \left[−1, 1, \sinh^2 \chi, \sinh^2 \chi \sin^2 \theta\right].$$

(5)

*E-mail: prp@wuphys.wustl.edu
\[ \sqrt{\kappa} = R_0^2 \sin^2 \chi \sin \theta. \] 

Maxwell’s equations in free space are

\[ \sqrt{\kappa} F^{\alpha\beta} \right|_\beta = 0, \]  

\[ F_{\alpha\beta} + F_{\beta\alpha} + F_{\gamma\alpha,\beta} = 0. \]

Potentials and a gauge condition will be introduced later.

### 3 EFFECTIVE THERMAL MASS IN A PLASMA IN MINKOWSKI SPACE

We start from the simple, and at first sight pointless, observation that the kinetic-energy term of the Klein–Gordon equation for a charged scalar field in Minkowski space, \( \Phi^*(p - qA)\Phi \), when expanded, gives the term \( \Phi^* q_i A^i \Phi \). This has the same sign as the mass term, \( \Phi^* m^2 \Phi \), suggesting that in some circumstances, such as in a plasma, \( (q_i A^i) \) might play the role of the square of an effective thermal mass. \( (A^2) \) is the sum of contributions from particles in the immediate neighbourhood, roughly, those within the Debye sphere (Sturrock 1994), and should therefore be written \( (\sum_i A_i)^2 \). When we sum over all contributions, the \( A_i \) potentials will be randomly oriented, and will add according to the theory of random flights (Hughes 1995). \( (\sum_i A_i)^2 \) will then be zero, but \((\sum_i A_i)^2 \) will not.

In a proper quantum mechanical treatment, we have to represent the source particles by means of a density matrix. This is not important for a plasma in Minkowski space, but will be essential when we come to curved space, so we introduce the idea here. The density matrix provides a way of including all possible configurations of source particles, which are not, of course, individually observed. Instead of \( A_i \), we should write \( A_{\mu i} \), where \( \mu \) specifies a particular configuration. \( (A^2) \) is then a double sum, over the particles within a configuration, and over the various possible configurations; up to a normalization factor, \( (\sum_\mu (\sum_i A_{\mu i})^2) \). Dimensional considerations indicate that in the conditions of the early Universe this should have a value of order \( T^2 \), where \( T \) is the temperature of the plasma.

We are free to make an overall gauge transformation with an arbitrary function \( \psi(x^\mu) \). The various \( A_i \) become \( A_i + \nabla_i \psi \), and the source particles acquire corresponding phases. But we should note that this freedom is not the same as freedom to choose any gauge we wish to describe the propagation of the potentials. For example, we might decide to switch from Lorentz gauge (Jackson 1999) to Coulomb gauge. This would introduce a separate gauge function for each \( A_{\mu i} \), and there would be no single gauge function that could be used to generate a compensating phase for any chosen interacting particle. If we are concerned with the propagation of potentials we have to choose a gauge from the beginning that is suitable for that purpose, as discussed in Section 7. This choice is not altered by an overall gauge transformation.

We can argue, of course, that in Minkowski space we are not really dealing with a direct effect of the vector potential because electric and magnetic fields are also present, and they are the actual physical agents. This is true, but, as we will see in Section 9, it is no longer true in a curved space, where we have potentials without fields.

We will need to know the potential due to a single particle in the plasma at distances larger than the Debye length. If the particle is stationary, the potential declines exponentially. Surprisingly, however, starting in the 1950’s several authors (Neufeld & Ritchie 1955; Tappert 1967; Montgomery, Joyce & Sugihara 1968) discovered that a test charge moving uniformly is not exponentially screened, but generates the field of a quadrupole in a collisionless plasma. The restriction to a collisionless plasma was lifted in later papers (Yu, Tegeback & Stenflo 1973; Schroeder 1975), where it was shown that the far field of a moving test charge is that of a dipole with strength of order \( q \tau V \), where \( q \) is the charge, \( \tau \) the collision time as measured by \( t \) and \( V \) the velocity. This dipole form ensures that in Minkowski space the contribution of particles to the effective mass falls off rapidly with distance beyond the Debye radius. We shall see, however, that this is no longer the case in a curved space.

### 4 SOURCE PARTICLES IN THE PLASMA

To study the propagation of fields and potentials in a curved FRW space, we first need a model of the motion of the source particles that produce them. We are concerned with the early Universe, before any symmetry-breaking transitions that may generate particle masses as we know them. The properties of the plasma are therefore determined by a single number, the temperature. The photons will have the familiar Planck distribution. For simplicity, we assume the charged particles are associated with a single massless scalar field. This field will have an effective thermal mass, \( m\eta \), due to its interaction with the photons, so that \( m\eta \simeq T \). The charged particles will have a thermal distribution appropriate to this mass and temperature. The plasma will therefore be weakly relativistic; for simplicity, however, we will carry over some results from non-relativistic plasmas. Other plasma parameters are determined by \( T \); for example, the plasma frequency, \( \omega_p \), will be of order \( T \).

A charge that is part of the plasma, rather than being a test charge constrained to move in a certain way, will itself experience collisions. We can picture such a particle as describing Brownian motion, with its screening cloud sometimes closer, sometimes farther, but never fully established. We will model the most important aspect of a single step of this process by setting up a local set of Cartesian coordinates and placing a stationary screening charge \(-q\) at the origin. A charge \( q \) is initially at rest on the \( z \)-axis at \( z = -V t_\tau \), then begins to move with uniform velocity \( V \) along the \( z \)-axis, starting at \( t = -t_\tau \) and ending at \( t = t_\tau \), when it again comes to rest. Note that this choice of origin for \( t \) differs from that of the cosmic time used in section 2; \( \eta \), defined below, will differ in a similar way. We reconcile these differences later, in Section 10.

We are now going to take the result for a uniformly moving test charge and use it to calculate the field around our model charge that is subject to Brownian motion. We justify this step as follows. Imagine that the test charge, instead of moving uniformly from an indefinite time in the past, is actually stationary until time \( t = 0 \), and then starts moving uniformly. There will be transient fields, but after a few collision times the final field of a moving dipole will be established. During the transient period, the test charge will be moving away from its screening charge, which only gradually picks up speed and trails along behind. This is similar to the motion of the charge in our model. Since the transient fields form a bridge between the initial and final fields they must have a similar form to the field of the test charge, and extend comparably far.

During the period that the charge is moving (the pulse), the dipole moment will be \( \mathbf{D}(t) = q V t_\tau \). Before and after the pulse the dipole moment remains at a constant value, but this is of little interest because a constant dipole moment generates no magnetic field or vector potential. It is well known that Maxwell’s equations separate in conformal time, so we will write the dipole moment in terms of \( \eta \).

Define \( \tau = \tau / R(t) \) (dimensionless), so that the pulse extends from \( \eta = -\tau \) to \( \tau \). The subscript ‘s’ on \( R \) indicates the source time. \( R_s(t) \) can be treated as constant during the pulse, and the dipole moment
can be expressed as a Fourier integral,
\[
\mathcal{T}(\eta) = \frac{1}{2\pi} \int D(n) \exp(-in\eta) \, dn,
\]
\[
D(n) = \frac{2i q V R_s \sin(\pi \tau)}{n^2}.
\]

We estimate the value of \( \tau \) in Section 10. The z-axis of the local system used in this section will be parallel to the polar axis of the polar coordinates used in the bulk of the paper.

5 DIPOLE FIELDS

We will need only the three fields \( F_{10}, F_{20} \) and \( F_{12} \). Since Maxwell’s equations separate in conformal time the elementary solutions can be written
\[
F_{10} = f_{10}(\chi, \theta, \phi) \exp(-i n \eta),
\]
\[
F_{20} = f_{20}(\chi, \theta, \phi) \exp(-i n \eta),
\]
\[
F_{12} = f_{12}(\chi, \theta, \phi) \exp(-i n \eta).
\]

For dipole fields we try the following forms:
\[
f_{10} = f_1(\chi) P_1(\theta),
\]
\[
f_{20} = f_2(\chi) N_1(\theta),
\]
\[
f_{12} = f_3(\chi) N_1(\theta),
\]
with angular functions \( P_1(\theta) = \cos \theta \) and \( N_1(\theta) \equiv dP_1(\theta)/d\theta = \sin \theta \).

Maxwell’s equations now give (with a prime meaning \( d/d\chi \))
\[
in \sinh^2 \chi f_1 - 2 f_3 = 0,
\]
\[
in f_2 - f_3' = 0,
\]
\[
f_1 - f_2 + in f_3 = 0.
\]

5.1 Dipole magnetic field

From (17)–(19), we obtain the equation for \( f_3 \) alone
\[
\frac{d^2 f_3}{d\chi^2} + \left( n^2 - \frac{2}{\sinh^2 \chi} \right) f_3 = 0.
\]

This is the analogue of equation (16) of Mashhoon (1973), which he derived for a space of positive curvature. Define \( f_4(\chi) = f_3(\chi)/\sinh \chi \). Then \( f_4 \) satisfies
\[
\frac{d^2 f_4}{d\chi^2} + 2 \cosh \chi \frac{df_4}{d\chi} + \left( n^2 + 1 - \frac{2}{\sinh^2 \chi} \right) f_4 = 0.
\]

This is a special case, for \( l = 1 \) or \(-2\), of the equation for hyperbolic spherical functions, with solution (Bucher, Goldhaber & Turok 1995; Bander & Itzykson 1966):
\[
f_4(l, \chi) = N_l(-1)^{l+1} \sinh \chi \frac{d^{l+1}}{d(\cosh \chi)^{l+1}} \cos(n \chi),
\]
where \( N_l \) is a normalization factor that is irrelevant here. Choosing \( l = 1 \) (\(-2\)), we get the solutions of (21) that are regular (singular) at \( \chi = 0 \). With \( l + 1 < 0 \) interpreted as integration:
\[
f_{4, reg} = f_4(1, \chi) = \frac{1}{\sinh \chi} \left[ n \sinh \chi \cos(n \chi) - \cosh \chi \sin(n \chi) \right]
\]
\[
f_{4, sing} = f_4(-2, \chi) = \frac{1}{\sinh \chi} \left[ n \sinh \chi \sin(n \chi) + \cosh \chi \cos(n \chi) \right],
\]
\[
f_3 \] is constructed from that combination of \( f_{4, reg} \) and \( f_{4, sing} \) that is proportional to \( \exp(i n \chi) \) because when combined with \( \exp(-i n \eta) \) this represents outgoing waves,
\[
f_3 = \frac{C_1(n) \exp(i n \chi)}{\sinh \chi} (\cosh \chi - i \sinh \chi),
\]
where \( C_1(n) \) is a normalizing factor to be determined.

It is convenient at this point to express \( f_3 \) in terms of \( u = \tanh(\chi/2) \),
\[
f_3 = \frac{C_1(n) \exp(i n \chi)}{2u} (1 - 2iu + u^2).
\]

5.2 Dipole electric fields

From (17) and (25), we derive the equation for the radial electric field,
\[
f_1 = -ic_1(n) \exp(i n \chi) (1 - u^2)^2 (1 - 2iu + u^2).
\]

Similarly, from (18) and (26), we derive the equation for the transverse electric field,
\[
f_2 = \frac{ic_1(n) \exp(i n \chi)}{4u^3} \times \left[ (1 - u^2)^2 - 2iu(1 + u^2) - 4u^2u^2 \right].
\]

5.3 Normalizing the fields

From (9) and (10), we can get the near-field expression for the radial component of \( E \), and by comparison with (27) arrive at the form of the normalizing factor \( C_1(n) \). On the polar axis, at small distances, the radial \( E \) field at frequency \( n \) is
\[
E_r = -\frac{\partial}{\partial r} \left[ \frac{D(n) \exp(-i n \eta)}{r^2} \right] = \frac{2D(n) \exp(-i n \eta)}{r^3},
\]
where \( r = R_s \chi \).

In the same limit (small \( \chi \)), (27) gives \( f_1(\chi) = -2iC_1(n)/(n\chi^3) \), and so, on the axis,
\[
F_{10}(\eta, \chi) = -2iC_1(n)R_s^3 \exp(-i n \eta)/(n\eta^3).
\]

We now transform to a local Minkowski frame with coordinates \( t, r \),
\[
F_{10}(t, r) = \frac{\partial \eta}{\partial \bar{r}} \frac{\partial \chi}{\partial \bar{r}} F_{10}(\eta, \chi)
\]
\[
= -2iC_1(n)R_s \exp(-i n \eta)/(n\eta^3).
\]
\[
F_{10} \] is just the conventional radial electric field as given in (29), so using (10),
\[
C_1(n) = \frac{-2q V \sin(\pi \tau)}{n}.
\]
6 PROPAGATION OF THE MAGNETIC FIELD

(26), (32), (16) and (13) give the Fourier transform of the magnetic field,

\[ F_{12} = \frac{-qV \sin(n \tau)N_1(\theta) \exp[i(\chi - \eta)]}{nu} \times (1 - 2inu + u^2). \] (33)

We transform back into \( \chi, \eta \) space by dividing by \( 2\pi \) and integrating over \( n \) along the real axis. For \( \chi > \eta + \tau \) both exponentials in \( \sin(n \tau) \) allow us to close in the upper half plane (UHP), using the contour of Fig. 1. \( \sin(n \tau)/n \) is regular at \( n = 0 \), so we get zero, as required by causality. Similarly, when \( \chi < \eta - \tau \), we can close in the lower half plane (LHP), using a contour that is the inverse of Fig. 1, and we again get zero.

For \( \eta - \tau < \chi < \eta + \tau \), we must write \( \sin(n \tau) = \frac{[\exp(i\tau) - \exp(-i\tau)]/2i} \), divide the integrand into two pieces, and close the first integral in the UHP and the second in the LHP. Combining these two integrals we get

\[ F_{12} = \frac{qV \sin(\theta(1 + u^2))}{2u}. \] (34)

The propagation of \( F_{12} \) is shown in Fig. 2. An unexpected feature of the pulse is that the amplitude does not tend to zero for large \( \chi \), but to a constant asymptotic value. The conventional \( H \) field, does, of course, tend to zero, in fact exponentially. We can see this at a particular instant by setting up local Cartesian coordinates at a particular observation point, \( P \), at rest in the FRW space. Choose a point in the equatorial plane, \( \theta = \pi/2 \), with azimuth \( \phi = 0 \). Choose \( dx \) parallel to \( d\chi \), \( dy \) parallel to \( d\theta \) and \( dz \) parallel to \( d\phi \). Then \( dx = R_\theta \, d\chi \), and \( dy = R_\phi \, \sin \chi \, d\phi \), where the subscript ‘\( p \)’ refers to the particular instant chosen. The conventional \( H \) field is \( F_{12} \), given by

\[ F_{12} = \frac{\partial \chi}{\partial \chi} \partial \theta \partial \chi \partial y F_{12} = F_{12}/(R_\theta \sinh \chi), \] (35)

showing the exponential decline for large \( \chi \).

7 DIPOLE POTENTIALS

The potentials are defined by the following equations,

\[ A_0 = h_0(\chi)P_1(\theta) \exp(-i\eta \chi), \] (36)

\[ A_1 = h_1(\chi)P_1(\theta) \exp(-i\eta \chi), \] (37)

\[ A_2 = h_2(\chi)N_1(\theta) \exp(-i\eta \chi), \] (38)

\[ F_{\mu \nu} = \frac{\partial A_\mu}{\partial x^\nu} - \frac{\partial A_\nu}{\partial x^\mu}. \] (39)

From (39) we derive

\[ h_0' - h_1 = f_3, \] (40)

\[ h_0' + i h_1 = f_1, \] (41)

\[ h_0 + i h_2 = f_2. \] (42)

(40), (41) and (42) are not independent. If we differentiate (42), subtract it from (41) and use (19), we obtain (40). These three equations therefore do not suffice to define the potentials uniquely; this is to be expected because \( A_\mu \) is only defined up to a scalar gauge function, \( \psi \), so that \( A_\mu = A_\mu + \partial \psi/\partial x^\mu \) defines the same fields as \( A_\mu \).

To fix the potentials uniquely, we need a gauge condition. Although, as we saw in Section 3, there is a real sense in which our final result is gauge invariant, we still have to specify a gauge condition at this point to determine how the potentials in our model propagate over cosmological distances. The natural choice is the Lorentz condition (LC), as given in Jackson (1999). This is covariant in form and respects causality; it also has the advantage that the current density generating the vector potential is the complete current density, not (as for the Coulomb gauge) just the transverse part. The curved-space generalization of the LC is \( A'_\mu = 0 \), which in our metric becomes

\[ \frac{\partial}{\partial x^\mu} \left( \sqrt{g} g^{\mu \nu} A_\nu \right) = 0. \] (43)

This has the disadvantage, in a FRW space, that factors of \( R(t) \) remain. This means that the equations describing the propagation of the potentials will involve the time derivative of \( R \), and so will be coupled to the gravitational field equations. It is certainly possible that Nature works this way. But to simplify the mathematics, we propose here to use the following modified Lorentz condition (MLC),

\[ \frac{\partial}{\partial x^\mu} \left( \sqrt{g} g^{\mu \nu} A_\nu / R^2 \right) = 0, \] (44)
which leads to
\[ \sinh^2 \chi h_0 + \frac{d}{dx} \left[ \sinh^2 \chi h_1 \right] - 2h_2 = 0. \quad (45) \]

(44) may seem unphysical because it depends explicitly on \( R(\eta) \), and so seems to be restricted to FRW spaces, rather than being universally applicable. But it can be given a physical meaning in the context of conformally invariant theories. The connection is outlined in Appendix B.

The potentials are normalized by comparing them with the expressions for the fields. Together with the MLC, this determines the potentials uniquely.

### 7.1 Scalar potential

By combining (17), (18), (41), (42) and (45) we can derive an equation for \( h_0 \) alone,
\[ \frac{d^2 h_0}{d\chi^2} + 2 \coth \chi \frac{dh_0}{d\chi} + \left( n^2 - \frac{2}{\sinh^2 \chi} \right) h_0 = 0. \quad (46) \]

Defining \( \alpha = \sqrt{n^2 - 1} \), this becomes the equation for hyperbolic spherical functions, (21). As before, we choose the solution that represents outgoing waves. With \( u = \tanh (\chi/2) \),
\[ A_0 = P_n(\chi)h_0(\chi) \exp(-in\eta) = C_2(n)P_n(\chi) \exp(i(\alpha\chi - n\eta)) \]
\[ \times \left( 1 - u^2 \right) \left( 1 - 2i\alpha u + u^2 \right), \quad (47) \]
where \( C_2(n) \) is a normalization factor that depends on the driving function. \( C_2(n) \) is, of course, proportional to the normalization factor \( C_1(n) \) defined in (32). For small \( \chi \), (47), (42) and (28) give
\[ C_2(n) = \frac{iC_1(n)}{4n}. \quad (48) \]

### 7.2 Vector potential

The interesting component of the vector potential is the transverse one, \( A_z \), or in our notation \( A_2 \). (42) gives
\[ h_2 = \frac{-i f_2}{n} + i h_0. \quad (49) \]

This can be used to get \( h_2 \), and from that \( A_2 \), since \( f_2 \) and \( h_0 \) are known from (28) and (47).

Equation (49) shows that \( A_2 \) is the sum of two parts, one involving \( \exp(i\eta x) \), the other involving \( \exp(i\alpha x) \),
\[ A_2 = -q V N_\alpha(\theta) \sin(n\pi) e^{-in\eta} \frac{(N + A)}{2n u^2}. \quad (50) \]
\[ N = e^{i\alpha \chi} \left[ (1 - u^2) - 2iu(1 + u^2) - 4n^2 u^2 \right], \quad (51) \]
\[ A = e^{i\alpha \chi} (1 - u^2) [-1 + 2iu - u^2]. \quad (52) \]

### 8 PROPOSITION OF THE VECTOR POTENTIAL

The Fourier synthesis of \( A_2 \) is carried out, as for the magnetic field, by dividing by \( 2\pi \) and then integrating over \( n \), along a contour chosen to respect causality. There is no vector potential before the pulse begins, so the correct contour is a line parallel to the real axis and slightly above it. The integral must be taken along the whole path A–D in Fig. 1, including the semicircle B–C. This ensures that for \( \chi > n + \tau \) the contour can be closed in the UHP and the integral will be zero. There are two regions of interest, \( n - \tau < \chi < n + \tau \) (the ‘main pulse’), and \( \chi < n - \tau \) (the ‘tail’).

We get a non-zero result only for that part of the integral that involves a contour that is closed in the LHP. For the part of equation (50) that is proportional to \( \exp(i\eta x) \), we can shrink this contour to a small circle about the point \( n = 0 \), and we just have to find the residue there. For the part that is proportional to \( \exp(i\alpha x) \) we have to remember the branch points at \( n = \pm 1 \), so our contour can only be shrunk to the form shown in Fig. 3.

\( A_2 \) is the sum of contributions from small circles (or semicircles) around \( n = 0 \) (the pole terms) and the integrals along the cut from \( n = -1 \) to 1. The pole terms have simple analytic expressions, but the integrals must be evaluated numerically for each \( (n, \chi) \) pair. The integrations are straightforward, and the propagation of \( A_2 \) is shown in Fig. 4.

An important difference between Figs 2 and 4 is that in the latter the pulse has a non-zero tail for \( \chi < n - \tau \). This feature of the propagation of potentials in a curved space has been noted before [DeWitt & Brehme (1960), Narlikar (1970); for more recent work see Haghighipour 2005]. For dipole propagation, as here, the tail rises linearly from \( \chi = 0 \), and approaches a constant value for large \( \chi \). This asymptotic value is proportional to \( \tau \). In ordinary
9 EFFECTIVE THERMAL MASS IN A PLASMA IN FRW SPACE

In Section 3, we discussed the thermal mass in Minkowski space; we now have to investigate any changes there may be when we go to a FRW space. The most important one is the appearance of the tail following the pulses of \( A \). Since the fields themselves do not have tails, the vector potential in this region is a pure potential, without accompanying fields. It can therefore be represented as the gradient of a scalar function, \( \delta S/\partial x^\mu \).

In the asymptotic region, where both \( \eta \gg 1 \) and \( \chi \gg 1 \), \( S \) has a simple form. Both \( h_0 \) and \( h_1 \) tend to zero exponentially as \( \chi \to \infty \). \( h_2 \) is given by the contour integral of the previous section, and in the asymptotic region the integral along the cut is negligible compared to the pole term. So we get \( A_2 \approx 2qV\tau N_1(\theta) \), and \( S \approx 2qV\tau P_1(\theta) \).

If the potential from source particle \( i \) is denoted by \( A_2 \), then the total from all the particles in any one configuration is \( S = \sum S_i \). We might think that we could use \( S \) as a gauge function and remove the effect of \( A_2 \) in the tail entirely. But this is not so once we introduce a density matrix and sum over all configurations of the distant source particles. Each configuration now produces its own gauge function, \( S_\mu = \sum S_\mu \). These functions are all distinct, and there is no single gauge function that can be used to eliminate the effect of \( A_2 \) in the tail.

This is not to say gauge invariance is violated; as discussed in Section 3, we can introduce an arbitrary overall gauge function \( \psi(x^\mu) \). But this is quite different from the separate functions \( S_\mu \), each of which is determined by the position and orientation of the corresponding source dipole.

10 MASS GENERATION: FIRST CALCULATION

We will show here that \( \langle A^\mu A_\mu \rangle \) is composed of a thermal part, \( \langle A^\mu A_\mu \rangle_{th} \), which will be of order \( T^2 \), and a non-thermal part, \( \langle A^\mu A_\mu \rangle_{nt} \). We are concerned with the ratio

\[
p = \langle A^\mu A_\mu \rangle_{th}/\langle A^\mu A_\mu \rangle_{nt}.
\]

and its variation with time.

Moving particles in the distant plasma will generate pulses of \( A_\mu \), each of which consists of a ‘main pulse’ and a ‘tail’. The main pulse contains electric and magnetic fields, which are distinguished from the thermal background by their spectrum. The associated vector potential will contribute to \( \langle A^\mu A_\mu \rangle_{th} \). We will not investigate this in detail because, as we will see later, it is the potentials in the tail, not the main pulse, that are responsible for the increase of \( p \) with time. \( A_\mu \) in the tail is a pure gauge potential that generates no fields, but it, too, will contribute to \( \langle A^\mu A_\mu \rangle_{nt} \).

To study the development of \( \langle A^\mu A_\mu \rangle_{nt} \) we need a model of how the Universe came into being. We will suppose it originated as a bubble in a metastable vacuum (Bucher et al. 1995), and for simplicity assume that all fields at temperature \( T_{\text{max}} \) appeared without appreciable delay at the surface of the bubble. The evolution of \( \langle A^\mu A_\mu \rangle_{nt} \) can be visualized as follows. As each pulse passes the observation point, it leaves a memory in the form of the tail. These tails will not cancel but will add according to the theory of random flights (Hughes 1995). The scalar potential, of course, will remain close to zero; it is only the vector potential that accumulates these additions. Consequently, from now on, we will write \( \langle A^\mu A_\mu \rangle_{th} = \langle A^\mu A_\mu \rangle_{nt} \), and similarly for the non-thermal part.

In (54), both the numerator and the denominator will decrease as the temperature falls, but the numerator falls more slowly, so the ratio will gradually increase from zero. In some cosmological models, as we will see in Section 16, we would like to know when \( p \) becomes of order unity, so we will estimate this here.

It is convenient at this point to change coordinates so the observation point is at \( \eta = \eta_s, \chi = 0 \), and a general source point is at \( \eta, \chi \). Our formulae involve differences in \( \eta \) and \( \chi \), which are unchanged by this shift of origin. We will consider \( \eta \) and \( \tau \) as starting from zero at the time of minimum radius, \( R_{\text{min}} \), and maximum temperature, \( T_{\text{max}} \). We can picture the buildup of \( \langle A^\mu A_\mu \rangle_{nt} \) with the help of Fig. 5, where the coordinates are \( \eta \) (upwards) and \( \chi \). The row of boxes at the bottom of the diagram is at \( \eta = 0 \); this represents the surface of the bubble. Each box has thickness \( d \chi \) and duration \( \tau \). Two observation points are shown, \( P_1 \) and \( P_2 \), at times \( \eta_s \) and \( \eta_2 \). The lines \( P_1-B \) and \( P_2-C \) represent the past light cones. All plasma particles within these past light cones contribute to \( \langle A^\mu A_\mu \rangle_{nt} \). The contributions add incoherently over times longer than the collision time, \( \tau \), so we can imagine the whole diagram divided into time slices of duration \( \tau \), just like the one shown for \( \eta = 0 \). A box at source time \( \eta_s \) represents a spherical shell of volume \( 4\pi R^2 \sin^2 \chi d\chi \).

The number density of particles in this shell, for a thermal distribution, is

\[
\nu_s = 0.24T^3,
\]

and the number of particles in the shell is then

\[
dN_s = 4\pi(0.24T^3)R^2\sin^2 \chi d\chi
\]

\[
= 4\pi(0.24)A^3_{\text{th}} \sin^2 \chi d\chi.
\]

Here, we have introduced the quantity \( A_{\text{th}} = R^2T^2 \), which (at our current level of calculation) will be constant during the expansion, and therefore does not need another subscript, ‘s’ or ‘t’.

Multiplying (56) and (53), we get the contribution to \( A^\mu A_\mu \) from the particles in the shell. We convert this to \( g_{\mu \nu}A^\mu A_\nu \) by multiplying...
by $g^2$. 

$$4\pi(0.24)A_3^{3/4} \sin^2 \chi \, d\chi \left(2q_\tau^2 \left(V^2/\sin^2 \theta \right) \tanh^2(\alpha \chi) \right)/R_p^2 \sin^2 \chi. \quad (57)$$

The average of $\sin^2 \theta$ over a sphere gives $2/3$. For the average of $V^2$, we reason as follows: if the effective $m^2$ of the particles were purely thermal, we could write $V^2 = 1/2$. But the effective mass actually increases with time as the ratio $p$ from (54) increases from zero towards one. So a better estimate of $V^2 = 1/(2p)$. 

$$\eta = 0 \text{ at the surface of the bubble, and } \eta = \eta_0 \text{ at the observation point.}$$

For some time slice at time $\eta$, between these two limits, we get the contribution of all spherical shells to $\langle A^2 \rangle$ by integrating from $\chi = 0$ to the light cone at $\chi = \eta_0 - \eta$.

$$8\pi(0.24)A_3^{3/4} \left(1 - p \right) / 2 \left(2q_\tau^2 \right) \int_0^\eta \tanh^2(\alpha \chi) \left(R_p^2 \sin^2 \chi \right). \quad (58)$$

We now have to sum over all time slices of duration $\tau$. The sum can be converted into an integral by multiplying by $d\eta/\tau$.

$$32\pi(0.24)A_3^{3/4} q_\tau^2 \int_0^\eta d\eta \left(1 - p \right) \tau \times \int_0^\eta d\chi \tanh^2(\alpha \chi) / R_p^2. \quad (59)$$

In Section 4, we defined $\tau = \tau_c/R_c(t)$. For $\tau_c$, we will simply use equation (3.24) from Montgomery & Tidman (1964). The factor $U^3$ in the numerator will be of order $2(2 + p)^{-3/2}$; the quantity in the bracket in the denominator is of order unity and will be omitted. We then have, in our notation, $\tau_c = m^2(2 + p)^{-3/2}/(8\pi v(q_\tau^2 ln \Lambda)$.

For $m^2$, we will use the effective mass at time $t$. The purely thermal value is $q_\tau^2 T_c$, but we should multiply this by $(1 + p)$ to include the non-thermal part also. We have $v_s = 0.24 T_c^3$, so

$$\tau = \frac{1 + p}{8\pi \ln \Lambda(0.24)q_\tau^2 R_c(2 + p)^{3/2}} = \frac{1 + p}{8\pi \ln \Lambda(0.24)q_\tau^2 A_3^{3/4}(2 + p)^{3/2}}. \quad (60)$$

We can now get an expression for $p$ by using (60) in (59) and dividing by $T_c^2$.

$$p = \frac{4}{3 \ln \Lambda} A_3^{1/4} q_\tau^2 \int_0^\eta d\eta \left[1 + p \right] \frac{1}{q_\tau^2 A_3^{1/2}(2 + p)^{3/2}} \times \int_0^\eta \tanh^2(\alpha \chi) / R_p^2. \quad (61)$$

This integral equation is surely not correct in detail, but it does provide a general picture of the buildup of $p$ from $p = 0$ at $\eta_0 = 0$. We seek $\eta_{cw}$, the value of $\eta_0$ for which $p = 1$. Use $a = 0.735$, as before, and for definiteness set in $\Lambda = 10$; numerical solution then gives $\eta_{cw} \approx 11$.

A remarkable feature of (61) is that not only $A_{3}$ has disappeared, but also the coupling constant, $q$. The long-range character of the Coulomb interaction, however, is still apparent in $\Lambda$. An equation of this sort might reasonably have been expected to yield a value of $\eta_{cw}$ that was either extremely small, so the CW transition takes place almost immediately after the formation of the bubble or extremely large, so the transition never takes place at all. Instead, we get a value of $\eta_{cw}$ that is within one or two orders of magnitude of unity.

### 11 Introduction of Conductivity

Up to this point, we have been mainly concerned with potentials in the tail. But we have now to look more closely at the electric and magnetic fields in the main pulse for dipole propagation. These combine to give a Poynting vector directed outwards. In flat space, this vector falls off like $1/\chi^4$, which tends to zero even when integrated over the whole sphere. When we go to curved space, however, the Poynting vector tends to $(qV \sin^2 \chi) ^2$. The surface area of a sphere is $4\pi R^2 \sin^2 \chi$, so the integral of the Poynting vector tends to a constant non-zero value. One effect of curvature is to continuously increase the energy of the plasma.

It seems unlikely this extra energy will remain in the form of pulses like those of Fig. 2. We know such pulses propagate unchanged in a flat space, but when they have travelled cosmological distances, so the curvature becomes notable, we should expect a gradual thermalization. We will model this thermalization by including a small, constant conductivity, $\sigma$, in our equations. Note that $\sigma$ is not directly related to the normal conductivity of the plasma, which (in our cosmological units) would be very high. $\sigma$ is just a device to represent the slow thermalization, and will have a value of order unity. For technical reasons, which we explain below, we choose $\sigma = 1/(2\pi)$.

There are two reasons why we have to investigate the effect of $\sigma$.

(i) The thermalization of the pulses raises the temperature of the plasma, and so tends to reduce the value of the ratio $p$.

(ii) The slow decline in the height of the pulses will probably reduce the value of $\Lambda$ in the tail, and this will also reduce the value of $p$.

The gradual transfer of energy to the plasma implies that the product $R(\eta) T(\eta)$ will not remain constant, as in a simple expansion, but will slowly increase. In this respect, $\sigma$ simulates inflation, but on a much longer time-scale.

The equations involved here are analogous to those used previously, but are considerably more complicated, so we just give the main results.

### 12 Inclusion of Conductivity: Fields

The wave number, $k$, the permittivity, $\epsilon(n)$, and the conductivity, $\sigma$, satisfy the dispersion relation,

$$k^2 = n^2 \epsilon(n) = n(n + 4\pi i \sigma). \quad (62)$$

By analogy with Maxwell’s equations in flat space, the equations (17)–(19) for our dipole fields become

$$i n e \sin \chi f_1 - 2f_3 = 0, \quad (63)$$

$$i n e f_2 - f_3 = 0, \quad (64)$$

$$f_1 - f_3 + imf_3 = 0. \quad (65)$$
12.1 Magnetic field

From these, we obtain an equation for \( f_3 \) alone,

\[
\frac{d^2 f_3}{dx^2} + \left( n^2 \epsilon - \frac{2}{\sinh^2 \chi} \right) f_3 = 0.
\]  

Writing \( k^2 \) for \( n^2 \epsilon \) this equation takes a familiar form; \( f_3 \) can be derived from the empty-space formula simply by writing \( k \) for \( n \) everywhere, except, of course, for the normalization function, which we denote now by \( C_1(n, \sigma) \),

\[
f_3 = \frac{C_1(n, \sigma)}{2u} \exp(ikx) (1 - 2iku + u^2). \tag{67}
\]

\( C_1(n, \sigma) \) is most easily determined in flat space; this is sufficient since we only have to consider small distances. The calculation is done in Appendix A. We find

\[
C_1(n, \sigma) = \frac{-6 \epsilon V(n + 47\pi\sigma) \sin(n\pi)}{n(3n + 87\pi\sigma)}. \tag{68}
\]

12.2 Propagation of the magnetic field

The Fourier transform of the magnetic field becomes

\[
F_{12} = \frac{-3qV(n + 47\pi\sigma)N_1(\theta) \sin(n\pi)}{3n + 87\pi\sigma} \exp(ikx - i\eta) iu u
d \times \left( 1 - 2iku + u^2 \right). \tag{69}
\]

We can integrate around the pole at \( n = 0 \) in the same way as before, except that we have to respect the branch points of \( k \), at \( n = 0 \) and \( -47\pi\sigma \). We have also to take account of the pole at \( n = -87\pi\sigma / 3 \). A suitable contour is shown in Fig. 6.

The integrals are straightforward, and the resulting propagation of \( F_{12} \) is shown in Fig. 7.

13 INCLUSION OF CONDUCTIVITY:

POTENTIALS

Here, we follow the prescription of Landau and Lifshitz (Landau, Abrikosov & Khalatnikov 1954): use the same equations relating potentials and fields as in empty space, that is (40)–(42), but modify the LC by including the permittivity in the \( h_0 \) term,

\[
in \sinh^2 \chi h_0 + \frac{d}{dx} \left[ \sinh^2 \chi h_1 \right] - 2h_2 = 0. \tag{70}
\]

13.1 Scalar potential

From these equations, as before, we can derive an equation for \( h_0 \) alone,

\[
\frac{d^2 h_0}{dx^2} + 2 \coth x \frac{dh_0}{dx} + \left( n^2 \epsilon - \frac{2}{\sinh^2 \chi} \right) h_0 = 0. \tag{71}
\]

This is the same equation as we obtained for empty space, except that, just as for the magnetic field, in place of \( n^2 \epsilon \) we must write \( k^2 \equiv n^2 \epsilon \). (47) applies as before, provided we write \( \alpha = \sqrt{k^2 - T} \), and use a normalization function \( C_4(n, \sigma) \) in place of \( C_2(n) \),

\[
C_4(n, \sigma) = \frac{i C_4(n, \sigma)}{4(n + 47\pi\sigma)}. \tag{72}
\]

13.2 Vector potential

Defining \( n_p = -87\pi\sigma / 3 \), we find

\[
A_2 = \frac{-qV N_1(\theta) \sin(n\pi) e^{-i\eta}}{2(n - n_p) \eta^2 u^2} (K_\sigma + A_\sigma). \tag{73}
\]

\[
K_\sigma = e^{ikx} \left[ (1 - u^2)^2 - 2iku(1 + u^2) - 4k^2 u^2 \right]. \tag{74}
\]

\[
A_\sigma = e^{i\sigma} (1 - u^2) (1 + 2i\alpha u - u^2). \tag{75}
\]

We note that going from (50) to (73) takes three simple steps.

(i) Rename \( N \) and \( A \); call them \( K_\sigma \) and \( A_\sigma \), respectively.

(ii) In the final parenthesis of (50), substitute \( k \) for \( n \) in both \( K_\sigma \) and \( A_\sigma \); this includes redefining \( \alpha \) to be \( \sqrt{k^2 - T} \) rather than \( \sqrt{n^2 - T} \).
and \( n \geq 4 \), this graph is indistinguishable from the one using integration around the contour.

14 THE ENERGY-TRANSFER EQUATION

We will now investigate the effect of a non-zero conductivity on the temperature of the plasma. We start from Weinberg (1972), equation 4.7.9,

\[
T_{\mu \nu}^{\eta} = \frac{1}{\sqrt{\xi}} \frac{\partial}{\partial x^\mu} \left( \sqrt{\xi} T^{\mu \nu} \right) + \Gamma_{\mu \nu}^{i} T^{\eta i}.
\]  

(78)

This expression is zero for any value of \( \nu \), but the most important one for us is \( \nu = 0 \). We treat the plasma as a perfect fluid at rest in FRW coordinates \( \eta, \chi, \theta, \phi \), so that the four-velocity \( (U^0, U^i) \) has \( U^0 = 0, i = 1, 2, 3 \). Since \( g_{\mu \nu} U^\mu U^\nu = -1 \) and \( g_{00} = R^i(\eta), U^{00} = 1/R \).

Multiply (78) by \( \sqrt{\xi} d^3x U_0 \), sum over \( \nu \), and set the resulting expression equals to zero; we get

\[
\sqrt{\xi} d^3x U_0 T_{\mu \nu}^{\eta 0} = 0.
\]  

(79)

The left-hand side of this equation is a coordinate scalar with \( \sqrt{\xi} d^4x \) an element of proper volume. Expanding (79),

\[
\sqrt{\xi} d^4x U_0 \left[ \frac{1}{\sqrt{\xi}} \frac{\partial}{\partial x^0} \left( \sqrt{\xi} T^{00} \right) + \frac{1}{\sqrt{\xi}} \frac{\partial}{\partial x^i} \left( \sqrt{\xi} T^{i0} \right) + \Gamma_{ij}^{0} T^{ij} + \Gamma_{00}^{0} T^{00} \right] = 0.
\]  

(80)

From this, we can derive an integrodifferential equation for \( T^{00} \). The resulting equation is most simply written in terms of \( A_{\mu}(\eta) = R^i(\eta) T^{i0}(\eta) eV^4 m^4 \). A similar quantity was introduced in Section 10, but there it could be treated simply as a constant. Here, we calculate its variation with time,

\[
A_{\mu}(\eta) = P_1 \int_0^\eta d\eta' A_{\mu}^{x/4}(\eta') \frac{\sigma(\eta - \eta')}{2 + p(\eta)} \times \exp \left[ -2\pi \sigma(\eta - \eta') \right] = 0,
\]  

(81)

\[
P_1 = 3.97 \times 10^{-7} \text{ eV m}.
\]  

(82)
Note that because of the appearance of the function \( p(\eta) \), this equation by itself is not sufficient to calculate \( A_M \). We obtain an independent equation connecting these two functions in the next section.

15 MASS GENERATION: SECOND CALCULATION

We will now obtain an equation for the ratio \( p(\eta) \) analogous to (61), but based on the asymptotic form (76) for the tail of the vector potential. In our previous calculation, where the tail of \( A_2 \) reached a constant asymptotic value, we could be sure that \( p \) would rise monotonically; the only question was whether there would be factors of \( A_M \) that would make the increase far too fast or far too slow. (This is just what happens, e.g., if we use quadrupole rather than dipole potentials.) In this second calculation, we can anticipate that there will not be any unwanted factors of \( A_M \), but only a detailed calculation will show whether \( p \) continues to rise with \( \eta \).

The analogue of (56) is

\[
dN_\eta = 4\pi(0.24)A_M^{3/2}(\eta_0)\sinh^2 \chi d\chi.
\]

(83)

To get the contribution to \( A_2 \) at \( \eta_0 \) from the particles in the shell, we multiply (83) by \( A_2[\sigma, \chi, (\eta_0 - \eta)] \), with \( A_2(\sigma, \chi, \eta) \) given by (76).

Proceeding as before, we get the analogue of (59),

\[
\frac{2\pi(0.24)q^2}{3R(\eta_0)} \int_{\eta_0}^{\eta_0} d\eta_1 \left[ \frac{A_M^{3/2}(\eta_1)}{2 + p(\eta_1)} \right] \tau 
\times \int_0^{\eta_0} d\chi F^2 \left[ \sigma, \chi, (\eta_0 - \eta) \right],
\]

(84)

with \( F_0 \) given by (77).

The collision time, \( \tau \), will be given by

\[
\tau = \frac{(1 + p)}{8\pi \ln \Lambda(0.24)q^2A_M^{3/2}(\eta_0)(2 + p)^{3/2}}.
\]

(85)

We can now get an expression for \( p \) by using (85) in (84) and dividing by \( T^2(\eta_0) \),

\[
p(\eta_0) = \frac{3}{256\pi^2 \ln \Lambda A_M^{1/2}(\eta_0)} \int_{\eta_0}^{\eta_0} d\eta_1 \times \left\{ \frac{1 + p(\eta_1)}{2 + p(\eta_1)^{3/2}} \right\} \int_0^{\eta_0} d\chi F^2 \left[ \sigma, \chi, (\eta_0 - \eta) \right].
\]

(86)

with \( F \) given by (77).

15.1 Calculation of \( p \) and \( A_M \)

As before, we set \( \ln \Lambda = 10 \) for definiteness. Simultaneous numerical solution of (81) and (86) is straightforward, but we need a starting value for \( A_M \). We choose \( A_M = 2.5 \times 10^{-3} \text{eV}^4 \text{m}^4 \). This is approximately equal to the observed value today. \( p \) becomes equal to unity around \( \eta_{cw} \approx 480 \).

This value of \( \eta_{cw} \) is significantly larger than the previous value of 11 when \( \sigma = 0 \), but is still within a few orders of magnitude of unity. The most important feature of this calculation is that it shows \( p(\eta) \) does continue to rise, even when \( \sigma \neq 0 \).

16 APPLICATION TO THE MANNHEIM MODEL

In a series of papers (see Mannheim 2006 and references given there), Mannheim has developed a cosmology based on conformal gravity. The underlying geometry is a FRW space with negative curvature. The expansion parameter, \( R(\eta) \), has a minimum, non-zero value; we will assume this represents the initial size of the bubble.

A distinguishing aspect of this theory is that it contains no intrinsic mass scale, unlike conventional cosmology where the Planck mass provides a fundamental scale. It is therefore natural to ask whether, in Mannheim’s model, the steady increase in the ratio \( p \) could eventually trigger a phase change that would generate particle masses. We are not advocating the Mannheim model in this paper; it must still meet some major challenges before it can be regarded as a serious rival to the conventional model. We are simply using it as an example of the way the non-thermal component of the effective mass can play a crucial role.

The mechanism we have in mind is the Coleman–Weinberg (CW) transition (Coleman & Weinberg 1973), as discussed in the context of cosmology in Narlikar & Padmanabhan (1986). Fig. 10 shows the effective potential, \( V_{eff} \), for a typical field theory at finite temperature. (The formulae used are taken from Narlikar & Padmanabhan (1986).) Initially, \( V_{eff} \) is described by the upper curve, but as the effective mass of the field quanta increases, the second minimum will move lower, and eventually, when \( p \approx 1 \), it will cross the horizontal axis. At this point, a CW transition can take place and normal masses will appear. These will have magnitude \( m^2 \approx (A^2)_{\text{eff}} \).

By adjusting the parameters of the model we can arrange for sufficient conformal time to elapse from the formation of the bubble to the CW transition, assumed to occur at the weak interaction scale. The initial temperature of the bubble, \( T_{\text{max}} \), is then found to be very high, and we have to look for an explanation of why particle masses are so small compared to this temperature (this problem is analogous to the hierarchy problem in ordinary cosmology, where the Planck mass is so much greater than particle masses).

In the Mannheim model, the explanation is straightforward. The temperature of the CW transition is found to depend exponentially
17 CONCLUSION

We have studied the propagation of the vector potential in the plasma of the early Universe, assuming a FRW space of negative curvature, and conclude that \( \langle A^2 \rangle \) will slowly acquire a non-thermal part, \( \langle A^2 \rangle_\text{in} \), due to distant matter. Even though this is a direct effect of the vector potential, and cannot be attributed to fields, the theory is none the less gauge invariant. This effect is therefore in the same category as the Aharonov–Bohm effect (Aharonov & Bohm 1959; Tonomura et al. 1982).

Further research will be needed to determine whether this increase of \( \langle A^2 \rangle_\text{in} \) is sufficient to cause a CW transition and generate a mass scale.

Several additional questions are raised in the present paper, among them the following:

(i) Will a CW transition necessarily take place as \( \langle A^2 \rangle_\text{in} \) approaches \( \langle A^2 \rangle_\text{in} \)?

(ii) Is the tail of \( A \) important only in the early Universe, or are there other places, regions of high gravitational fields, where its effects can be observed even now?

(iii) How does the real complicated early plasma determine such things as the collision time?

(iv) Can we expect \( A \) to propagate over cosmological distances so that the CW transition takes place within the available conformal time? We have taken findings from ordinary plasma theory and used them in the very different circumstances of the early Universe.

These considerations are beyond the scope of this paper, which is solely concerned with the interplay, in a simple model, of field theory (density matrix, the Lagrangian for a scalar field, the CW transition) and the classical equations of propagation of the ordinary vector potential in a FRW space of negative curvature.

A well-known text (Peskin & Schroeder 1995) suggests a connection between large and small scales similar to the one explored here. After surveying the difficulties faced by current theories of the mass scale, the authors write: ‘... it may be that the overall scale of energy momentum is genuinely ambiguous and is set by a cosmological boundary condition’. We have presented a mechanism for such a connection. It is based on the familiar electromagnetic interaction, and nothing radically new seems to be required.

ACKNOWLEDGMENTS

We acknowledge helpful correspondence with Bryce DeWitt, Leonard Parker, Stephen Fulling, Don Melrose, David Montgomery and Philip Mannheim. We also wish to thank the chairman and faculty of the department of physics at Washington University for providing an office and computer support for a retired colleague. Cosmic space may be infinite, but office space is at a premium.

REFERENCES

Abramowitz M., Stegun I. A., 1970, Handbook of Mathematical Functions. National Bureau of Standards, Washington, DC

Aharonov Y., Bohm D., 1959, Phys. Rev., 115, 485

Bander M., Itzykson C., 1966, Rev. Mod. Phys., 38, 346

Bucher M., Goldhaber A. S., Turok N., 1995, Phys. Rev. D, 52, 3314

Coleman S., Weinberg E., 1973, Phys. Rev. D, 7, 1888

DeWitt B. S., Brehme R. W., 1960, Ann. Phys., 9, 220

Haghighipour N., 2005, Gen. Relativ. Gravit., 37, 327

Hoyle F., Narlikar J. V., 1974, Action at a Distance in Physics and Cosmology. Freeman, San Francisco

Hughes B. D., 1995, Random Walks and Random Environments. Clarendon Press, Oxford

Jackson J. D., 1999, Classical Electrodynamics, 3rd edn. Wiley, New York

Landau L. D., Abrikosov A. A., Khalatnikov I. M., 1954, Dokl. Akad. Nauk SSSR, 95, 773

Mannheim P. D., 2006, Prog. Part. Nucl. Phys., 56, 340

Masahoon B., 1973, Phys. Rev. D, 8, 4297

Montgomery D., Joyce G., Sugihara R., 1968, Plasma Phys., 10, 681

Montgomery D. C., Tidman D. A., 1964, Plasma Kinetic Theory. McGraw Hill, New York

Narlikar J. V., 1970, Proc. Natl. Acad. Sci. USA, 65, 483

Narlikar J. V., Padmanabhan T., 1986, Gravity, Gauge Theories and Quantum Cosmology. Reidel, Dordrecht

Neufeld J., Ritchie R. H., 1955, Phys. Rev., 98, 1632

Peskin M., Schroeder D. V., 1995, Introduction to Quantum Field Theory. Addison-Wesley, Reading

Schroeder H., 1975, Plasma Phys. Control. Fusion, 17, 1135

Sturrock P. A., 1994, Plasma Physics. Cambridge Univ. Press, Cambridge

Tappert F. D., 1967, PhD thesis, Princeton University

Tonomura A. et al., 1982, Phys. Rev. Lett., 48, 1443

Weinberg S., 1972, Gravitation and Cosmology. Wiley, New York

Yu M. Y., Tegeteck R., Stenflo L., 1973, Z. Phys., 264, 341

APPENDIX A: NORMALIZATION WITH CONDUCTIVITY

In this appendix, we will work in the usual spherical polar coordinates, and make connection with Riemannian coordinates when necessary.

Suppose we have a dipole at the origin, oscillating with time dependence \( \exp(-i \omega t) \) in the \( z \) direction. The surrounding medium is of uniform conductivity, \( \sigma \), so that the current density, \( j \), is given by \( j = \sigma E \). We will analyse this system by imagining a small sphere of radius \( r_1 \) cut out of the medium surrounding the dipole. Induced currents flowing in the medium will cause surface charges to appear on the sphere, and the total dipole moment will be the sum of the original dipole moment and that due to the induced charges. We assume the permittivity and magnetic susceptibility are essentially unity, so \( D = E \) and \( B = H \). In such a system \( \nabla \cdot j = 0 \) follows from Maxwell’s equations, so there is no volume charge in the medium.

Denoted by \( D_{\text{true}} = D(\omega) \exp(-i\omega t) \) the dipole moment at the centre of the sphere, where \( D(\omega) \) is the true dipole strength at angular frequency \( \omega \). The induced dipole moment due to the surface charges is \( D_{\text{ind}} \), so the total dipole moment is \( D_{\text{tot}} = D_{\text{true}} + D_{\text{ind}} \).

Just outside the sphere the electrostatic potential is given by

\[
V = D_{\text{tot}} P_1(\cos \theta)/r_1^2, \tag{A1}
\]

where \( P_1(\cos \theta) = \cos \theta \). The radial component of \( E \) is given by

\[
E_r = 2D_{\text{tot}} P_1(\cos \theta)/r_1^2 \tag{A2}.
\]

The surface charge density, \( s \), obeys the relation

\[
\frac{d\sigma}{dr} = -j_r, \tag{A3}
\]

where \( j_r \) is evaluated just outside the sphere. This gives

\[
s = -\frac{\sigma E_r}{\omega} = s_0 P_1(\cos \theta), \tag{A4}
\]

where

\[
s_0 = \frac{-2i\sigma D_{\text{tot}}}{r_1 \omega}. \tag{A5}
\]
The induced dipole moment, $D_{\text{ind}}$, is then given by an integral over the surface of the sphere,

$$D_{\text{ind}} = 2\pi \int_0^\pi \! d\theta \sin \theta r^2 [n_0 P_1(\cos \theta)] [r_1^2 P_1(\cos \theta)]$$

$$= \left( -\frac{4\pi \sigma}{\omega} \right) \left( \frac{2}{3} \right) D_{\text{int}}, \quad (A6)$$

giving

$$D_{\text{tot}} = D_{\text{new}} + D_{\text{ind}}$$

$$= \frac{3\omega D_{\text{new}}}{3\omega + 8\pi \sigma}, \quad (A7)$$

We set $H_\phi = C_s(\omega, \sigma) N_1(\theta) h(r) \exp(-i\omega t)$, where $C_s(\omega, \sigma)$ is a normalizing function, $N_1 = dP_1/d\theta$, and $h(r)$ satisfies

$$\frac{d^2 h}{dr^2} + \frac{2}{r} \frac{dh}{dr} + \frac{2}{r^2} h - \frac{k^2}{r^2} h = 0,$$  \quad (A8)

with $k^2 = \omega^2 + 4\pi \sigma \omega$. We choose the solution that represents outgoing waves, so

$$h(r) = h_1^{(1)}(kr)$$

$$= \left[ -\frac{i}{(kr)^2} - \frac{1}{kr} \right] \exp(ikr). \quad \text{(A9)}$$

Here, $h_1^{(1)}(kr)$ is the spherical Bessel function defined in Abramowitz & Stegun (1970).

The Maxwell equation $\nabla \times \mathbf{H} = -i(\omega + 4\pi \sigma)\mathbf{E}$ then gives

$$E_r = \frac{-2iC_s(\omega, \sigma) P_1(\cos \theta) h_1^{(1)}(kr) \exp(-i\omega t)}{r(\omega + 4\pi \sigma)}.$$  \quad (A10)

For small $r$ this becomes

$$E_r = \frac{-2C_s(\omega, \sigma) P_1(\cos \theta) \exp(-i\omega t)}{kr^3(\omega + 4\pi \sigma)}.$$  \quad (A11)

But also, for small $r$, we have (A2), so

$$C_s(\omega, \sigma) = \frac{-3\omega \sqrt{2} \chi^2(\omega + 4\pi \sigma) D(\omega)}{(3\omega + 8\pi \sigma)}, \quad \text{(A12)}$$

$$H_\theta = \frac{-3\omega \sqrt{2} \chi^2(\omega + 4\pi \sigma) D(\omega) N_1(\theta) h_1^{(1)}(kr) \exp(-i\omega t)}{(3\omega + 8\pi \sigma)}, \quad \text{(A13)}$$

$$E_r = \frac{6i\omega \sqrt{2} D(\omega) P_1(\theta) h_1^{(1)}(kr) \exp(-i\omega t)}{r(3\omega + 8\pi \sigma)}.$$  \quad (A14)

In this appendix, we have used ordinary polar coordinates in flat space. We need now to transform to coordinates of a FRW flat space, with metric

$$ds^2 = R^2(\eta) \left( -d\eta^2 + d\chi^2 + \chi^2 d\theta^2 + \chi^2 \sin^2 \theta \, d\phi^2 \right). \quad \text{(A15)}$$

To get from $H_\phi$ of (A13) to $F_{12}$ of (13), in the flat FRW metric (A15), we first refer to Weinberg (1972), Section 4.8, and multiply by $\chi$. We then convert from $t$ to $\eta$ and $r$ to $\chi$ by multiplying by $R^2(\eta)$. We also convert $\omega$ to $\kappa$, $k$ to $k$ and $\sigma$ to $\sigma = R(\eta)/\sigma,$

$$F_{12} = \frac{-3\omega \sqrt{2} (\omega + 4\pi \sigma) D(\omega) N_1(\theta) \chi h_1^{(1)}(k\chi) \exp(-i\omega \eta)}{R(\eta)(3\omega + 8\pi \sigma)}, \quad \text{(A16)}$$

We can obtain the corresponding function in curved space by multiplying (67) by $N_1(\theta) \exp(-i\eta \theta)$,

$$F_{12} = \frac{C_s(n, \sigma) N_1(\theta) \exp[i(k\chi - n\eta)]}{2u} \left( 1 - 2i\kappa u + u^2 \right). \quad \text{(A17)}$$

We can now find $C_s(n, \sigma)$ by matching (A16) with (A17) for small $\chi$,

$$C_s(n, \sigma) = \frac{3\pi \sqrt{2} \chi^2 D(n)}{R(\eta)(3\omega + 8\pi \sigma)}. \quad \text{(A18)}$$

With $D(n)$ given by (10), this reduces to

$$C_s(n, \sigma) = \frac{-6\pi \sqrt{2} \chi^2 D(n)}{n(3\omega + 8\pi \sigma)}. \quad \text{(A19)}$$

In the limit $\sigma \to 0$, (A19) tends to (32), as it should.

**APPENDIX B: THE MODIFIED LORENZ CONDITION IN CONFORMALLY INVARIANT THEORIES**

It is well known that although Maxwell’s equations are conformally invariant in four-dimensional space, the LC is not (Hoyle & Narlikar 1974, chapter 2). So a theory that aspires to be completely conformally invariant, such as the theory of Mannheim discussed in Section 16, must include a modified gauge condition if it is to describe the propagation of potentials. We note that the conformal weights of $g^{\mu\nu}, \sqrt{\gamma}, \Phi$ and $A_\gamma$ are $-2, 4, -1$ and $0$, respectively. The equation

$$\frac{\partial}{\partial x^\mu} (\sqrt{\gamma} g^{\mu\nu} \Phi^* A_\gamma) = 0 \quad \text{(B1)}$$

is therefore conformally invariant, and is a candidate for a gauge condition. We can linearize it by setting $\Phi^* \Phi$ equal to its expectation value. In the conditions of the early Universe, this will be proportional to $T^2$, where $T$ is the temperature. Assuming adiabatic expansion, $T^2$ will be proportional to $1/R^2$, so our gauge condition reduces to

$$\frac{\partial}{\partial x^\mu} (\sqrt{\gamma} g^{\mu\nu} A_\gamma / R^2) = 0, \quad \text{(B2)}$$

which is the MLC, (44).

This paper has been typeset from a \TeX\ file prepared by the author.