Stability of equilibria for the SO(4) free rigid body

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Abstract

It is shown that for the generalized rigid body certain Cartan subalgebras (called of coordinate type) of $\mathfrak{so}(n)$ are equilibrium points for the rigid body dynamics. In the case of $\mathfrak{so}(4)$ there are three coordinate type Cartan subalgebras which on a regular adjoint orbit give three Weyl group orbits of equilibria. These coordinate type Cartan subalgebras are the analogues of the three axes of equilibria for the classical rigid body in $\mathfrak{so}(3)$. The nonlinear stability and instability of these equilibria is determined. In addition to these equilibria there are others that come in curves. It is shown that these curves of equilibria are nonlinearly stable in the sense that the only possible drift is in the direction of the curve itself.

1 Introduction

The goal of the present work is to find the analogue of the long axis–short axis stability theorem for the SO(4)-free rigid body. To do this, one needs to determine first what are the analogues the usual three axes of equilibria in the three dimensional case. It will be shown that they are replaced by special Cartan subalgebras that we shall call coordinate type Cartan subalgebras. For the general case of SO(n) it is proved that these coordinate type Cartan subalgebras are equilibria.

If $n = 4$ then, on a regular adjoint orbit, all the Cartan type equilibria are organized in three Weyl group orbits. The nonlinear stability and instability for these equilibria is determined taking into account the symplectic geometry of the orbit and the complete integrability of the system. The results in this paper complete and extend some previous work of Fehér and Marshall [5] and Spiegler [17].

In addition to the Cartan type equilibria, there are, on every regular orbit, curves of equilibria. It is shown that these are nonlinearly stable as a family, that is, if a solution of the SO(4)-free rigid body equation starts near an equilibrium on such a curve, at any later time it will stay close to this curve but in the direction of the curve itself it may drift.

The implication of the topological structure of the energy-momentum level sets on bifurcation phenomena in the dynamics was extensively studied by Oshemkov [12], [13], Bolsinov and Fomenko [4].
2 Equilibria for the generalized rigid body

The equations of the rigid body on \( \mathfrak{so}(n) \) are given by

\[
\dot{M} = [M, \Omega],
\]

where \( \Omega \in \mathfrak{so}(n) \), \( M = \Omega J + J\Omega \in \mathfrak{so}(n) \) with \( J = \text{diag}(\lambda_i) \), a real constant diagonal matrix satisfying \( \lambda_i + \lambda_j \geq 0 \), for all \( i, j = 1, \ldots, n, i \neq j \) (see, for example, [10]). Note that \( M = [m_{ij}] \) and \( \Omega = [\omega_{ij}] \) determine each other if and only if \( \lambda_i + \lambda_j > 0 \) since \( m_{ij} = (\lambda_i + \lambda_j)\omega_{ij} \) which physically means that the rigid body is not concentrated on a lower dimensional subspace of \( \mathbb{R}^n \).

It is well known and easy to verify that equations (2.1) are Hamiltonian relative to the minus Lie-Poisson bracket

\[
\{F, G\}(M) := \frac{1}{2} \text{Trace}(M[\nabla F(M), \nabla G(M)]),
\]

and the Hamiltonian function

\[
H(M) := -\frac{1}{4} \text{Trace}(M\Omega).
\]

Here \( F, G, H \in C^\infty(\mathfrak{so}(n)) \) and the gradient is taken relative to the Ad-invariant inner product

\[
\langle X, Y \rangle := -\frac{1}{2} \text{Trace}(XY), \quad X, Y \in \mathfrak{so}(n)
\]

which identifies \( (\mathfrak{so}(n))^* \) with \( \mathfrak{so}(n) \). This means that \( \dot{F} = \{F, H\} \) for all \( F \in C^\infty(\mathfrak{so}(4)) \), where \( \{\cdot, \cdot\} \) is given by (2.2) and \( H \) by (2.3), if and only if (2.1) holds. Note that the linear isomorphism \( X \in \mathfrak{so}(n) \mapsto XJ + JX \in \mathfrak{so}(n) \) is self-adjoint relative to the inner product (2.4) and thus \( \nabla H(M) = \Omega \).

Let \( E_{ij} \) be the constant antisymmetric matrix with 1 on line \( i \) and column \( j \) when \( i < j \), that is, the \((k,l)\)-entry of \( E_{ij} \) equals \( (E_{ij})_{kl} = \delta_{kl}\delta_{ij} - \delta_{kj}\delta_{il} \). Then \( \{E_{ij} | i < j\} \) is a basis for the Lie algebra \( \mathfrak{so}(n) \). We have

\[
(E_{ij} E_{ks})_{ab} = \delta_{ai}\delta_{jk}\delta_{bs} - \delta_{aj}\delta_{ik}\delta_{bs} - \delta_{ai}\delta_{js}\delta_{bk} + \delta_{aj}\delta_{is}\delta_{bk}
\]

and hence \( E_{ij}^2 \) is the diagonal matrix whose only non-zero entries \(-1\) occur on the \( i \)th and \( j \)th place. In addition, if \( i < j \) and \( k < s \), we get

\[
[E_{ij}, E_{ks}] = \delta_{jk}E_{is} + \delta_{is}E_{jk} - \delta_{ik}E_{js} - \delta_{js}E_{ik}
\]

where \( E_{rp} := -E_{pr} \), if \( r > p \).

Since

\[
[M, \Omega] = [\Omega J + J\Omega, \Omega] = \Omega J\Omega + J\Omega^2 - \Omega^2 J - \Omega J\Omega = [J, \Omega^2],
\]

we see that \( M \) is an equilibrium if and only if \( [J, \Omega^2] = 0 \). Assuming that all \( \lambda_i \) are distinct, this condition is equivalent to the statement that \( \Omega^2 \) is a diagonal matrix.

**Theorem 2.1.** Let \( \mathfrak{h} \subset \mathfrak{so}(n) \) be a Cartan subalgebra whose basis is a subset of \( \{E_{ij} | i < j\} \). Then any element of \( \mathfrak{h} \) is an equilibrium point of the rigid body equations (2.1).

**Proof.** We have to prove that for any \( M \in \mathfrak{h} \) the matrix \( \Omega^2 \) is diagonal. Since the linear isomorphism \( \Omega \leftrightarrow M \) is given by a diagonal matrix in the basis \( \{E_{ij} | i < j\} \) of \( \mathfrak{so}(n) \) it follows that \( M \in \mathfrak{h} \) if and only if \( \Omega \in \mathfrak{h} \).

So let \( \Omega \in \mathfrak{h} \) with \( \Omega = \sum_{s=1}^k \alpha_s E_s \), where \( k := [n/2] = \dim \mathfrak{h} \) and \( \{E_1, \ldots, E_k\} \subset \{E_{ij} | i < j\} \) is the basis of \( \mathfrak{h} \). Then

\[
\Omega^2 = \left( \sum_{s=1}^k \alpha_s E_s \right)^2 = \sum_{s=1}^k \alpha_s^2 E_s^2 + \sum_{l \neq p} \alpha_l \alpha_p (E_l E_p + E_p E_l).
\]

Since \( \mathfrak{h} \) is a Cartan subalgebra we have \( [E_l, E_p] = 0 \) which is equivalent to \( E_l E_p = E_p E_l \) for any \( l, p \in \{1, \ldots, k\} \). Then \( (E_l E_p)^l = E_l^l E_p^l = (-1)^2 E_p E_l = E_l E_p \). Consequently, the matrix \( E_l E_p \) is symmetric. Since \( E_l, E_p \in \{E_{ij} | i < j\} \), we distinguish the following cases for \( l \neq p \):
(a) $E_i = E_{ij}, E_p = E_{js}, i < j < s$, in which case the product $E_iE_p$ is not symmetric because the $(i, s)$-entry equals 1 and the $(s, i)$-entry vanishes. So this case cannot occur.

(b) $E_i = E_{ij}, E_p = E_{sj}, i < j, s < j, i \neq s$. Then, $E_iE_{sj}$ is not symmetric because the $(i, s)$-entry equals $-1$ and the $(s, i)$-entry vanishes. So this case cannot occur.

(c) $E_i = E_{ij}, E_p = E_{is}, i < j, i < s, j \neq s$. Then $E_{ij}E_{is}$ is not symmetric because the $(j, s)$-entry equals $-1$ and the $(s, j)$-entry vanishes. So this case cannot occur.

(d) $E_i = E_{ij}, E_p = E_{ks}, i < j, k < s, \{i, j\} \cap \{k, s\} = \emptyset$. In this case $E_iE_p = O_n$.

Thus, the only possible case in (2.6) is (d) which implies that

$$\Omega^2 = \sum_{s=1}^{k} \alpha_s^2 E_s^2$$

which is a diagonal matrix. ■

We shall call a Cartan subalgebra as in Theorem 1 a coordinate type Cartan subalgebra. The dynamics of (2.1) leaves the adjoint orbits of $SO(n)$ invariant. Since the intersection of a regular orbit (that is, one passing through a regular semisimple element of $so(n)$) with a Cartan subalgebra is a Weyl group orbit (see, e.g. [8]), we conclude that the union of the Weyl group orbits determined by the coordinate type Cartan subalgebras of $so(n)$ are equilibria of (2.1).

3 The adjoint orbits of $so(4)$

The Lie algebra of the compact subgroup $SO(4) = \{A \in gl(4,\mathbb{R}) \mid A^tA = I_4, \det(A) = 1\}$ of the special linear Lie group $SL(4,\mathbb{R})$ is $so(4)$. In this section we present the geometry of the (co)adjoint orbits of $SO(4)$ in $so(4)$.

We choose as basis of $so(4)$ the matrices

$$E_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \quad E_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \quad E_3 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix};
$$

$$E_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}; \quad E_5 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}; \quad E_6 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix};$$

and hence we represent $so(4)$ as

$$so(4) = \left\{ M = \begin{bmatrix} 0 & -x_3 & x_2 & y_1 \\ x_3 & 0 & -x_1 & y_2 \\ -x_2 & x_1 & 0 & y_3 \\ -y_1 & -y_2 & -y_3 & 0 \end{bmatrix} \mid x_1, x_2, x_3, y_1, y_2, y_3 \in \mathbb{R} \right\}. \quad (3.1)$$

This choice of basis was made for computational convenience as we shall see below. Note that $E_1 = -E_{23}$, $E_2 = E_{13}$, $E_3 = -E_{12}$, $E_4 = E_{14}$, $E_5 = E_{24}$, $E_6 = E_{34}$. From (2.5), it follows that the multiplication for this basis of $so(4)$ is given by the following table (the convention is to calculate [row, column]):

| $\cdot$ | $E_1$ | $E_2$ | $E_3$ | $E_4$ | $E_5$ | $E_6$ |
|--------|--------|--------|--------|--------|--------|--------|
| $E_1$  | 0      | $E_3$  | $-E_2$ | 0      | $E_6$  | $-E_5$ |
| $E_2$  | $-E_3$ | 0      | $E_1$  | $-E_6$ | 0      | $E_4$  |
| $E_3$  | $E_2$  | $-E_1$ | 0      | $E_5$  | $-E_4$ | 0      |
| $E_4$  | 0      | $E_6$  | $-E_5$ | 0      | $E_3$  | $-E_2$ |
| $E_5$  | $-E_6$ | 0      | $E_4$  | $-E_3$ | 0      | $E_1$  |
| $E_6$  | $E_5$  | $-E_4$ | 0      | $E_2$  | $-E_1$ | 0      |

3
In the basis \{E_1, \ldots, E_6\}, the matrix of the Lie-Poisson structure (2.2) is
\[
\Gamma_- = \begin{bmatrix}
0 & -x_3 & x_2 & 0 & -y_3 & y_2 \\
x_3 & 0 & -x_1 & y_3 & 0 & -y_1 \\
-x_2 & x_1 & 0 & -y_2 & y_1 & 0 \\
0 & -y_3 & y_2 & 0 & -x_3 & x_2 \\
y_3 & 0 & -y_1 & x_3 & 0 & -x_1 \\
y_2 & y_1 & 0 & -x_2 & x_1 & 0
\end{bmatrix}.
\] (3.2)

Since \text{rank} \, \mathfrak{so}(4) = 2, there are two functionally independent Casimir functions which are given respectively by
\[
C_1(M) := -\frac{1}{4} \text{Trace}(M^2) = \frac{1}{2} \left( \sum_{i=1}^{3} x_i^2 + \sum_{i=1}^{3} y_i^2 \right)
\]
and
\[
C_2(M) := -\text{Pf}(M) = \sum_{i=1}^{3} x_i y_i.
\]

Thus the generic adjoint orbits are the level sets
\[
\text{Orb}_{c_1,c_2}(M) = (C_1 \times C_2)^{-1}(c_1, c_2), \quad (c_1, c_2) \in \mathbb{R}^2.
\]

Note that if \(M \neq 0\), then \(\text{d}C_j(M) \neq 0\) for \(j = 1, 2\).

The Lie algebra \(\mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3)\) is of type \(A_1 \times A_1\) and, consequently, the positive Weyl chamber, which is the moduli space of (co)adjoint orbits, is isomorphic to the positive quadrant in \(\mathbb{R}^2\). In the basis of \(\mathfrak{so}(4)\) that we have chosen above, the positive Weyl chamber is given by the set \(\{(c_1, c_2) \in \mathbb{R}^2 \mid c_1 \geq |c_2|\}\).

To characterize the adjoint orbits of \(\text{SO}(4)\) it is convenient to split \(\mathfrak{so}(4) = V_u \oplus V_v\), where \(V_u := \{u := (u_1, u_2, u_3) \mid u_i := x_i + y_i, i = 1, 2, 3\} \cong \mathbb{R}^3\), \(V_v := \{v := (v_1, v_2, v_3) \mid v_i := x_i - y_i, i = 1, 2, 3\} \cong \mathbb{R}^3\).

Instead of the independent Casimir functions \(C_1, C_2\) we consider the following two independent Casimir functions
\[
D_u(M) := 2C_1(M) + 2C_2(M) = \|u\|^2, \quad D_v(M) := 2C_1(M) - 2C_2(M) = \|v\|^2.
\]

Note that
\[
\text{Orb}_{c_1,c_2}(M) = (C_1 \times C_2)^{-1}(c_1, c_2) = (D_u \times D_v)^{-1}(d_1, d_2), \quad \text{where} \quad d_1 = 2c_1 + 2c_2, \quad d_2 = 2c_1 - 2c_2.
\]

These considerations yield the following characterization of the \(\text{SO}(4)\)-adjoint orbits.

**Theorem 3.1.** Denote by \(S^2_r\) the sphere in \(\mathbb{R}^3\) of radius \(r\). If \(c_1 > 0\) and \(c_1 > |c_2|\), then the adjoint orbit \(\text{Orb}_{c_1,c_2}(M)\) equals \(S^2_{\sqrt{2c_1+2c_2}} \times S^2_{\sqrt{2c_1-2c_2}}\), where \(S^2_{\sqrt{2c_1+2c_2}} \subset V_u\), \(S^2_{\sqrt{2c_1-2c_2}} \subset V_v\), and hence it is regular. If \(c_1 = |c_2| > 0\), then the adjoint orbit \(\text{Orb}_{c_1,c_2}(M)\) is either \(S^2_{2\sqrt{c_1}} \times \{0\}\), with \(S^2_{2\sqrt{c_1}} \subset V_u\), or \(\{0\} \times S^2_{2\sqrt{c_1}}\), with \(S^2_{2\sqrt{c_1}} \subset V_v\), and so it is singular. If \(c_1 = c_2 = 0\), then the adjoint orbit \(\text{Orb}_{c_1,c_2}\) is the origin of \(\mathfrak{so}(4)\) and so it is singular.

In all that follows we shall denote by \(\text{Orb}_{c_1,c_2}\) the regular adjoint orbit \(\text{Orb}_{c_1,c_2}\), where \(c_1 > 0\) and \(c_1 > |c_2|\), which is equivalent to \(d_1, d_2 > 0\).

Using the Lie bracket table in the chosen basis given above, it is immediately seen that the coordinate type Cartan subalgebras of \(\mathfrak{so}(4)\) are \(t_1, t_2, t_3\), where
\[
t_1 := \text{span}(E_1, E_4) = \left\{ M^1_{a,b} := \begin{bmatrix} 0 & 0 & 0 & b \\ 0 & 0 & -a & 0 \\ 0 & a & 0 & 0 \\ -b & 0 & 0 & 0 \end{bmatrix} \mid a, b \in \mathbb{R} \right\},
\]
where

\[
\begin{align*}
\mathfrak{t}_2 := \text{span}(E_2, E_5) &= \left\{ M_{a, b}^2 := \begin{bmatrix}
0 & 0 & a & 0 \\
0 & 0 & 0 & b \\
-a & 0 & 0 & 0 \\
0 & -b & 0 & 0
\end{bmatrix} \mid a, b \in \mathbb{R} \right\}, \\
\mathfrak{t}_3 := \text{span}(E_3, E_6) &= \left\{ M_{a, b}^3 := \begin{bmatrix}
0 & -a & 0 & 0 \\
-a & 0 & 0 & 0 \\
0 & 0 & 0 & b \\
0 & 0 & -b & 0
\end{bmatrix} \mid a, b \in \mathbb{R} \right\}.
\end{align*}
\]

The intersection of a regular adjoint orbit and a coordinate type Cartan subalgebra has four elements which represents a Weyl group orbit. Thus we expect at least twelve equilibria for the rigid body equations (2.1) in the case of \(\mathfrak{so}(4)\). Specifically, we have the following result.

**Theorem 3.2.** The following equalities hold:

(i) \(\mathfrak{t}_1 \cap \text{Orb}_{c_1; c_2} = \{ M_{a, b}^1, M_{-a, -b}^1, M_{b, a}^1, M_{-b, -a}^1 \} \),

(ii) \(\mathfrak{t}_2 \cap \text{Orb}_{c_1; c_2} = \{ M_{a, b}^2, M_{-a, -b}^2, M_{b, a}^2, M_{-b, -a}^2 \} \),

(iii) \(\mathfrak{t}_3 \cap \text{Orb}_{c_1; c_2} = \{ M_{a, b}^3, M_{-a, -b}^3, M_{b, a}^3, M_{-b, -a}^3 \} \),

where

\[
\begin{align*}
\begin{cases}
 a = \frac{1}{\sqrt{2}} \left( \sqrt{c_1 + c_2} + \sqrt{c_1 - c_2} \right) \\
 b = \frac{1}{\sqrt{2}} \left( \sqrt{c_1 + c_2} - \sqrt{c_1 - c_2} \right)
\end{cases}
\end{align*}
\]

\( (3.3) \)

**Proof.** Let \( M_{a, b}^1 \in \mathfrak{t}_1 \cap \text{Orb}_{c_1; c_2} \). Then \( M_{a, b}^1 \in \mathfrak{t}_1 \), \( \mathcal{C}_1 = 2 \mathcal{C}_1 (M_{a, b}^1) = \alpha^2 + \beta^2 \), and \( c_2 = \mathcal{C}_2 (M_{a, b}^1) = \alpha \beta \). This system of equations has the solutions

\[
(a, \beta) \in \{(a, b), (-a, -b), (b, a), (-b - a)\},
\]

where

\[
\begin{align*}
\begin{cases}
 a = \frac{1}{\sqrt{2}} \left( \sqrt{c_1 + c_2} + \sqrt{c_1 - c_2} \right) \\
 b = \frac{1}{\sqrt{2}} \left( \sqrt{c_1 + c_2} - \sqrt{c_1 - c_2} \right)
\end{cases}
\end{align*}
\]

Similar arguments with obvious modifications prove assertions (ii) and (iii). \( \blacksquare \)

The intersections \(\mathfrak{t}_1 \cap \text{Orb}_{c_1; c_2}, \mathfrak{t}_2 \cap \text{Orb}_{c_1; c_2}, \mathfrak{t}_3 \cap \text{Orb}_{c_1; c_2}\) are Weyl group orbits.

4 **The \(\mathfrak{so}(4)\)-rigid body**

We shall work from now on with a generic \(\mathfrak{so}(4)\)-rigid body, that is, \(\lambda_i + \lambda_j > 0\) for \(i \neq j\) and all \(\lambda_i\) are distinct. The equations of motion are hence \(\mathbf{M} = [\mathbf{M}, \Omega]\), where \(\mathbf{M} = J \Omega + \Omega J, \Omega \in \mathfrak{so}(4)\), \(J = \text{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)\), \(\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R}\). The relationship between \(\Omega = [\omega_{ij}] \in \mathfrak{so}(4)\) and the matrix \(\mathbf{M} \in \mathfrak{so}(4)\) in the representation (3.1) is hence given by

\[
\begin{align*}
(\lambda_3 + \lambda_2)\omega_{32} &= x_1 & (\lambda_1 + \lambda_3)\omega_{13} &= x_2 & (\lambda_2 + \lambda_1)\omega_{21} &= x_3 \\
(\lambda_1 + \lambda_4)\omega_{14} &= y_1 & (\lambda_2 + \lambda_4)\omega_{24} &= y_2 & (\lambda_3 + \lambda_4)\omega_{34} &= y_3
\end{align*}
\]
and thus the equations of motion (2.1) are equivalent for \( n = 4 \) to the system

\[
\begin{align*}
\dot{x}_1 &= \left( \frac{1}{\lambda_1 + \lambda_2} - \frac{1}{\lambda_1 + \lambda_3} \right) x_2 x_3 + \left( \frac{1}{\lambda_1 + \lambda_2} - \frac{1}{\lambda_3 + \lambda_4} \right) y_2 y_3 \\
\dot{x}_2 &= \left( \frac{1}{\lambda_2 + \lambda_3} - \frac{1}{\lambda_1 + \lambda_3} \right) x_1 x_3 + \left( \frac{1}{\lambda_1 + \lambda_2} - \frac{1}{\lambda_3 + \lambda_4} \right) y_1 y_3 \\
\dot{x}_3 &= \left( \frac{1}{\lambda_1 + \lambda_3} - \frac{1}{\lambda_2 + \lambda_3} \right) x_1 x_2 + \left( \frac{1}{\lambda_2 + \lambda_3} - \frac{1}{\lambda_1 + \lambda_4} \right) y_1 y_2 \\
\dot{y}_1 &= \left( \frac{1}{\lambda_3 + \lambda_4} - \frac{1}{\lambda_1 + \lambda_3} \right) x_2 y_3 + \left( \frac{1}{\lambda_1 + \lambda_2} - \frac{1}{\lambda_3 + \lambda_4} \right) x_3 y_2 \\
\dot{y}_2 &= \left( \frac{1}{\lambda_2 + \lambda_3} - \frac{1}{\lambda_3 + \lambda_4} \right) x_1 y_3 + \left( \frac{1}{\lambda_1 + \lambda_2} - \frac{1}{\lambda_3 + \lambda_4} \right) x_3 y_1 \\
\dot{y}_3 &= \left( \frac{1}{\lambda_2 + \lambda_3} - \frac{1}{\lambda_1 + \lambda_4} \right) x_1 y_2 + \left( \frac{1}{\lambda_1 + \lambda_3} - \frac{1}{\lambda_1 + \lambda_4} \right) x_2 y_1.
\end{align*}
\]

(4.1)

The Hamiltonian (2.3) has in this case the expression

\[
H(M) = -\frac{1}{4} \text{Trace}(M\Omega) = \frac{1}{2} \left( \frac{1}{\lambda_2 + \lambda_3} x_2^2 + \frac{1}{\lambda_1 + \lambda_3} x_3^2 + \frac{1}{\lambda_1 + \lambda_2} x_1^2 + \frac{1}{\lambda_1 + \lambda_4} y_1^2 + \frac{1}{\lambda_2 + \lambda_4} y_2^2 + \frac{1}{\lambda_3 + \lambda_4} y_3^2 \right).
\]

The Hamiltonian nature of system (4.1) can be checked in this case directly, writing \((\dot{x}_1, \dot{x}_2, \dot{x}_3, \dot{y}_1, \dot{y}_2, \dot{y}_3)^T = \Gamma_-(\nabla H)^T\), where the Poisson structure \(\Gamma_\pm\) is given by (3.2) and

\[
(\nabla H)^T = \left( \frac{x_1}{\lambda_2 + \lambda_3}, \frac{x_2}{\lambda_1 + \lambda_3}, \frac{x_3}{\lambda_1 + \lambda_2}, \frac{y_1}{\lambda_1 + \lambda_4}, \frac{y_2}{\lambda_2 + \lambda_4}, \frac{y_3}{\lambda_3 + \lambda_4} \right)^T.
\]

Theorem 4.1. If \( \mathcal{E} \) denotes the set of the equilibrium points of (4.1), then \( \mathcal{E} = \mathbf{t}_1 \cup \mathbf{t}_2 \cup \mathbf{t}_3 \cup \mathbf{s}_+ \cup \mathbf{s}_- \), where \( \mathbf{s}_\pm \) are the three dimensional vector subspaces given by

\[
\mathbf{s}_\pm := \text{span}_R \left\{ \left( \frac{1}{\lambda_1 + \lambda_4} E_1 \pm \frac{1}{\lambda_2 + \lambda_3} E_4 \right), \left( \frac{1}{\lambda_2 + \lambda_4} E_2 \pm \frac{1}{\lambda_1 + \lambda_3} E_5 \right), \left( \frac{1}{\lambda_3 + \lambda_4} E_3 \pm \frac{1}{\lambda_1 + \lambda_2} E_6 \right) \right\}.
\]

Proof. Setting the right hand side of system (4.1) equal to zero (which is equivalent to \( \Omega^2 \) being diagonal) yields the system

\[
\begin{align*}
(\lambda_2 + \lambda_4)(\lambda_3 + \lambda_4)x_2x_3 - (\lambda_1 + \lambda_2)(\lambda_1 + \lambda_3)y_2y_3 &= 0 \\
(\lambda_1 + \lambda_4)(\lambda_3 + \lambda_4)x_1x_3 - (\lambda_1 + \lambda_2)(\lambda_2 + \lambda_3)y_1y_3 &= 0 \\
(\lambda_1 + \lambda_4)(\lambda_2 + \lambda_4)x_1x_2 - (\lambda_1 + \lambda_2)(\lambda_2 + \lambda_3)y_1y_2 &= 0 \\
(\lambda_1 + \lambda_2)(\lambda_2 + \lambda_4)x_2y_3 - (\lambda_1 + \lambda_3)(\lambda_2 + \lambda_4)x_3y_2 &= 0 \\
(\lambda_1 + \lambda_2)(\lambda_2 + \lambda_3)x_1y_3 - (\lambda_2 + \lambda_3)(\lambda_2 + \lambda_4)x_3y_1 &= 0 \\
(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_4)x_1y_2 - (\lambda_2 + \lambda_3)(\lambda_2 + \lambda_4)x_2y_1 &= 0.
\end{align*}
\]

(4.2)

Assume that \( x_1 \neq 0 \) and solve for \( x_2, x_3, y_2, y_3 \):

\[
\begin{align*}
x_2 &= \frac{(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_4)y_1y_2}{(\lambda_1 + \lambda_4)(\lambda_2 + \lambda_4)x_1} \\
x_3 &= \frac{(\lambda_1 + \lambda_2)(\lambda_2 + \lambda_4)y_1y_3}{(\lambda_1 + \lambda_4)(\lambda_2 + \lambda_4)x_1} \\
y_2 &= \frac{(\lambda_2 + \lambda_3)(\lambda_2 + \lambda_4)x_2y_1}{(\lambda_1 + \lambda_4)(\lambda_2 + \lambda_4)x_1} = \frac{(\lambda_2 + \lambda_3)^2 y_1^2 y_2}{(\lambda_1 + \lambda_4)^2 x_1^2} \\
y_3 &= \frac{(\lambda_2 + \lambda_3)(\lambda_3 + \lambda_4)x_3y_1}{(\lambda_1 + \lambda_4)(\lambda_2 + \lambda_4)x_1} = \frac{(\lambda_2 + \lambda_3)^2 y_1^2 y_3}{(\lambda_1 + \lambda_4)^2 x_1^2}.
\end{align*}
\]

If \( y_2 = y_3 = 0 \) then the last two equations above hold. The first two imply \( x_2 = x_3 = 0 \). Therefore, all equations in the system (4.2) hold for any \( y_1 \). Obviously the system (4.2) is satisfied even if \( x_1 = 0 \). This shows that any element of \( \mathbf{t}_1 := \text{span}(E_1, E_4) \) is a solution of the system (4.2).
Assume that at least one of $y_2, y_3$ does not vanish. For example, if $y_2 \neq 0$ then the third equation in the system above yields
\[ x_1 = \pm \frac{\lambda_2 + \lambda_3}{\lambda_1 + \lambda_4} y_1 \]
and thus the first equation becomes
\[ x_2 = \pm \frac{\lambda_1 + \lambda_3}{\lambda_2 + \lambda_4} y_2. \]
If also $y_3 \neq 0$, then the last equation gives the same value for $x_1$ and so the second equation implies
\[ x_3 = \pm \frac{\lambda_1 + \lambda_2}{\lambda_3 + \lambda_4} y_3. \]
It is clear that one can let $y_2$ and $y_3$ be zero in these relations which shows that the solution of the system is in this case given by the vector
\[
x_1 E_1 + x_2 E_2 + x_3 E_3 + y_1 E_4 + y_2 E_5 + y_3 E_6
\]
\[
= \pm \frac{\lambda_2 + \lambda_3}{\lambda_1 + \lambda_4} y_1 E_1 \pm \frac{\lambda_1 + \lambda_3}{\lambda_2 + \lambda_4} y_2 E_2 \pm \frac{\lambda_1 + \lambda_2}{\lambda_3 + \lambda_4} y_3 E_3 + y_1 E_4 + y_2 E_5 + y_3 E_6
\]
\[
= \pm y_1 (\lambda_2 + \lambda_3) \left( \frac{1}{\lambda_1 + \lambda_4} E_1 \pm \frac{1}{\lambda_2 + \lambda_3} E_4 \right) \pm y_2 (\lambda_1 + \lambda_3) \left( \frac{1}{\lambda_2 + \lambda_4} E_2 \pm \frac{1}{\lambda_1 + \lambda_3} E_5 \right)
\]
\[
\pm y_3 (\lambda_1 + \lambda_2) \left( \frac{1}{\lambda_3 + \lambda_4} E_3 \pm \frac{1}{\lambda_1 + \lambda_2} E_6 \right)
\]
for $y_1, y_2, y_3 \in \mathbb{R}$ arbitrary. This shows that any vector in the three dimensional subspaces $s_{\pm}$ is a solution of the system (4.2).

One repeats this argument for the pairs $(y_1, y_3)$ and $(y_1, y_2)$ and concludes that the solutions of the system (4.2) are in the sets $t_2 \cup s_+ \cup s_-$ and $t_3 \cup s_+ \cup s_-$, respectively. Thus any solution of the system (4.2) is in the set $t_1 \cup t_2 \cup t_3 \cup s_+ \cup s_-$. The converse is an easy verification: one checks that any element of $t_1 \cup t_2 \cup t_3 \cup s_+ \cup s_-$ is a solution of the system (4.2).

Note that $s_{\pm}$ are not Lie subalgebras of $\mathfrak{so}(4)$ and that $s_{\pm} \cap t_i \neq \emptyset$, $i = 1, 2, 3$. Let us compute $s_{\pm} \cap \text{Orb}_{c_1,c_2}$ for $c_1 > 0$ and $c_1 \geq |c_2|$. For an arbitrary element of $s_{\pm}$, we have
\[
x_1 = \pm \frac{a_1}{\lambda_1 + \lambda_4}, \quad x_2 = \pm \frac{a_2}{\lambda_2 + \lambda_4}, \quad x_3 = \pm \frac{a_3}{\lambda_3 + \lambda_4}, \quad y_1 = \pm \frac{a_1}{\lambda_2 + \lambda_3}, \quad y_2 = \pm \frac{a_2}{\lambda_1 + \lambda_3}, \quad y_3 = \pm \frac{a_3}{\lambda_1 + \lambda_2}.
\]
Note that if $a_i = a_j = 0$, $i \neq j$, $i, j \in \{1, 2, 3\}$, then the equilibrium lies in $t_k$, where $k \in \{1, 2, 3\} \setminus \{i, j\}$. Thus the equilibria in $s_{\pm}$ that are not in $t_1 \cup t_2 \cup t_3$ must have at least two of $a_1, a_2, a_3$ different from zero. From (4.3) we deduce
\[
0 < c_1 = C_1(M) = \frac{1}{2} \left( \sum_{i=1}^{3} x_i^2 + \sum_{i=1}^{3} y_i^2 \right)
\]
\[
= \frac{a_2^2}{2} \left( \frac{1}{(\lambda_1 + \lambda_4)^2} + \frac{1}{(\lambda_2 + \lambda_3)^2} \right) + \frac{a_3^2}{2} \left( \frac{1}{(\lambda_2 + \lambda_4)^2} + \frac{1}{(\lambda_1 + \lambda_3)^2} \right)
\]
\[
+ \frac{a_2^2}{2} \left( \frac{1}{(\lambda_3 + \lambda_4)^2} + \frac{1}{(\lambda_1 + \lambda_2)^2} \right)
\]
and
\[
c_2 = C_2(M) = \sum_{i=1}^{3} x_i y_i = \pm \left( \frac{a_1^2}{(\lambda_1 + \lambda_4)(\lambda_2 + \lambda_3)} + \frac{a_2^2}{(\lambda_2 + \lambda_4)(\lambda_1 + \lambda_3)} + \frac{a_3^2}{(\lambda_3 + \lambda_4)(\lambda_1 + \lambda_2)} \right)
\]
which shows that $c_1 \geq |c_2| \geq 0$. In the generic case when at least one of the inequalities

$$\lambda_1 + \lambda_4 \neq \lambda_2 + \lambda_3, \quad \lambda_2 + \lambda_4 \neq \lambda_1 + \lambda_3, \quad \lambda_3 + \lambda_4 \neq \lambda_1 + \lambda_2$$

(4.6)
hold, we have $c_1 > |c_2|$. If, in addition, at least one of the $a_i \neq 0$, that is, the equilibrium is not at the origin, then $|c_2| > 0$. Assume that we deal only with such generic rigid bodies in $\mathfrak{so}(4)$; then $c_1 > |c_2| > 0$, that is, all equilibria on $\mathfrak{g}_\pm$ necessarily lie on a regular adjoint orbit $\text{Orb}_{c_1;c_2}$ (see Theorem 3.1). These equilibria are not isolated. To describe them, express conditions (4.4) and (4.5) in the new variables

$$b_1 := \frac{a_1}{\sqrt{(\lambda_1 + \lambda_4)(\lambda_2 + \lambda_3)}}, \quad b_2 := \frac{a_2}{\sqrt{(\lambda_2 + \lambda_4)(\lambda_1 + \lambda_3)}}, \quad b_3 := \frac{a_3}{\sqrt{(\lambda_3 + \lambda_4)(\lambda_1 + \lambda_2)}},$$

to get intersections of ellipsoids with spheres in $\text{Orb}_{c_1;c_2} \cap \mathfrak{g}_\pm$ given by

$$0 < c_1 = \frac{b_1^2}{2} \left( \lambda_2 + \lambda_3 + \lambda_1 + \lambda_4 \right) + \frac{b_2^2}{2} \left( \lambda_1 + \lambda_3 + \lambda_2 + \lambda_4 \right) + \frac{b_3^2}{2} \left( \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \right)$$

$$c_2 = \pm (b_1^2 + b_2^2 + b_3^2).$$

(4.7)

(4.8)

This shows that if $c_2 > 0$ then $\mathfrak{g}_- \cap \text{Orb}_{c_1;c_2} = \emptyset$ and that if $c_2 < 0$, then $\mathfrak{g}_+ \cap \text{Orb}_{c_1;c_2} = \emptyset$. The considerations above prove the following.

**Corollary 4.2.** On a generic adjoint orbit $\text{Orb}_{c_1;c_2}$, $c_1 > |c_2| > 0$, the equilibria of (4.1) are given by the twelve points in Theorem 3.2 forming three Weil group orbits in $t_1$, $t_2$, $t_3$, and the subsets in $\mathfrak{g}_\pm$ described by (4.7) and (4.8).

Figure 1: The Weyl chamber. The shaded domains represent the two disjoint connected components corresponding to regular orbits. On a regular orbit corresponding to the upper domain we find equilibria of type $t_1$, $t_2$, $t_3$, $\mathfrak{g}_+$. On a regular orbit corresponding to the lower domain we find equilibria of type $t_1$, $t_2$, $t_3$, $\mathfrak{g}_-$.

The equations of the rigid body immersed in a fluid is the case of Clebsch system. It is a Hamilton-Poisson system on the Lie algebra $\mathfrak{e}(3)$ and was proved by Bobenko [3], see also [14], [15], [18], that this system is also a Hamilton-Poisson system on the Lie algebra $\mathfrak{so}(4)$. Nevertheless, our rigid body (4.1) is different from the Clebsch system.
5 Nonlinear stability

In this section we study the nonlinear stability of the equilibrium states $E \cap \text{Orb}_{c_1;c_2}$ for the dynamics (4.1) on a generic adjoint orbit.

Since the system (4.1) on a generic adjoint orbit is completely integrable ([11], [16], [9], [10]), for the $so(4)$ case we have a supplementary constant of motion. Using Mishchenko’s method [9], [16], we obtain the following additional constant of the motion for the equations (4.1) commuting with $H$:

$$I(M) = (\lambda_2^2 + \lambda_3^2)x_1^2 + (\lambda_1^2 + \lambda_3^2)x_2^2 + (\lambda_1^2 + \lambda_2^2)x_3^2 + (\lambda_1^2 + \lambda_2^2)x_4^2 + (\lambda_2^2 + \lambda_3^2)y_1^2 + (\lambda_2^2 + \lambda_3^2)y_2^2 + (\lambda_3^2 + \lambda_4^2)y_3^2.$$

Without loss of generality, we can choose an ordering for $\lambda_i$’s, namely

$$\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4.$$

The restriction of the dynamics $\dot{M} = [M, \Omega]$ to the regular adjoint orbit $\text{Orb}_{c_1;c_2}$ is thus a completely integrable Hamiltonian system

$$(\text{Orb}_{c_1;c_2}, \omega_{\text{Orb}_{c_1;c_2}}, H|_{\text{Orb}_{c_1;c_2}}),$$

where $\omega_{\text{Orb}_{c_1;c_2}}$ is the orbit symplectic structure on $\text{Orb}_{c_1;c_2}$. The Hamiltonian system (5.1) has all equilibria given by Corollary 4.2. These are of two types:

$$\mathcal{K}_0 := \left\{ M \in \text{Orb}_{c_1;c_2} \mid d \left( H|_{\text{Orb}_{c_1;c_2}} \right)(M) = 0, \quad d \left( I|_{\text{Orb}_{c_1;c_2}} \right)(M) = 0 \right\}$$

$$\mathcal{K}_1 := \left\{ M \in \text{Orb}_{c_1;c_2} \mid d \left( H|_{\text{Orb}_{c_1;c_2}} \right)(M) = 0, \quad d \left( I|_{\text{Orb}_{c_1;c_2}} \right)(M) \neq 0 \right\}.$$

**Proposition 5.1.** $\mathcal{K}_0 = \text{Orb}_{c_1;c_2} \cap (t_1 \cup t_2 \cup t_3)$ and $\mathcal{K}_1 = \text{Orb}_{c_1;c_2} \cap [(s_+ \cup s_-) \setminus (t_1 \cup t_2 \cup t_3)]$.

**Proof.** The proof is a direct verification. One checks that if $M \in t_1 \cup t_2 \cup t_3$ then $d \left( I|_{\text{Orb}_{c_1;c_2}} \right)(M) = 0$ and that if $M \in (s_+ \cup s_-) \setminus (t_1 \cup t_2 \cup t_3)$ then $d \left( I|_{\text{Orb}_{c_1;c_2}} \right)(M) \neq 0$, which proves the proposition by Theorem 3.2.

This is seen in the following way. By Theorems 3.2 and 4.1 all equilibria of (5.1) are formed by the set $\text{Orb}_{c_1;c_2} \cap (t_1 \cup t_2 \cup t_3) \cup [\text{Orb}_{c_1;c_2} \cap (s_+ \cup s_-)]$, where

$$\text{Orb}_{c_1;c_2} \cap (t_1 \cup t_2 \cup t_3) = \left\{ (x, y) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid (x_1, y_1) = \pm(a, b), \pm(b, a), x_2 = y_2 = x_3 = y_3 = 0 \right\}$$

$$\cup \left\{ (x, y) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid (x_2, y_2) = \pm(a, b), \pm(b, a), x_1 = y_1 = x_3 = y_3 = 0 \right\}$$

$$\cup \left\{ (x, y) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid (x_3, y_3) = \pm(a, b), \pm(b, a), x_1 = y_1 = x_2 = y_2 = 0 \right\}$$

$$= \left\{ (u, v) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid (u_1, v_1) = \pm(a + b, a - b), \pm(a + b, b - a), u_2 = v_2 = u_3 = v_3 = 0 \right\}$$

$$\cup \left\{ (u, v) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid (u_2, v_2) = \pm(a + b, a - b), \pm(a + b, b - a), u_1 = v_1 = u_3 = v_3 = 0 \right\}$$

$$\cup \left\{ (u, v) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid (u_3, v_3) = \pm(a + b, a - b), \pm(a + b, b - a), u_1 = v_1 = u_2 = v_2 = 0 \right\}$$

(5.2)

(since $x + y = u$ and $x - y = v$) and

$$\text{Orb}_{c_1;c_2} \cap (s_+ \cup s_-) = \{ \text{intersection of the ellipsoids 4.7 with the spheres 4.8} \}.$$
Now one can immediately see that for any equilibrium $M$ in $\mathcal{K}$ we have $\mathbf{d} \left( I_{\text{Orb}_{c_1;c_2}} \right)(M) = 0$.

Next, take an equilibrium in $\text{Orb}_{c_1;c_2} \cap \left( \mathcal{K}_{+} \cup \mathcal{K}_{-} \setminus (\mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3) \right)$. For example, assume that $c_2 > 0$ so that $\text{Orb}_{c_1;c_2} \cap \mathcal{K}_{-} = \varnothing$. Therefore by (4.3) we have

$$u_1 = a_1 \left( \frac{1}{\lambda_1 + \lambda_4} + \frac{1}{\lambda_2 + \lambda_3} \right), \quad u_2 = a_2 \left( \frac{1}{\lambda_2 + \lambda_4} + \frac{1}{\lambda_1 + \lambda_3} \right), \quad u_3 = a_3 \left( \frac{1}{\lambda_3 + \lambda_4} + \frac{2}{\lambda_1 + \lambda_2} \right),$$

$$v_1 = a_1 \left( \frac{1}{\lambda_1 + \lambda_4} - \frac{1}{\lambda_2 + \lambda_3} \right), \quad v_2 = a_2 \left( \frac{1}{\lambda_2 + \lambda_4} - \frac{1}{\lambda_1 + \lambda_3} \right), \quad v_3 = a_3 \left( \frac{1}{\lambda_3 + \lambda_4} - \frac{1}{\lambda_1 + \lambda_2} \right),$$

and at least two of $a_1, a_2, a_3$ are not zero since the equilibrium is not in $\mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$. Then

$$\mathbf{d} \left( I_{\text{Orb}_{c_1;c_2}} \right)(M) \cdot \delta M$$

$$= -2 a_2 a_3 \left( \frac{\lambda_1 + \lambda_3 + \lambda_2 + \lambda_4}{\lambda_2 + \lambda_4} \right)^2 \left( \frac{\lambda_2 - \lambda_3}{\lambda_1} \right) \left( \frac{\lambda_1 - \lambda_4}{\lambda_2 + \lambda_1} \right) m_1$$

$$+ a_2 a_3 \left( \frac{\lambda_1 + \lambda_3 + \lambda_2 + \lambda_4}{\lambda_2 + \lambda_4} \right)^2 \left( \frac{\lambda_2 - \lambda_4}{\lambda_1} \right) \left( \frac{\lambda_1 - \lambda_3}{\lambda_2 + \lambda_1} \right) m_2$$

$$- a_2 a_3 \left( \frac{\lambda_1 + \lambda_3 + \lambda_2 + \lambda_4}{\lambda_2 + \lambda_4} \right)^2 \left( \frac{\lambda_3 - \lambda_4}{\lambda_1} \right) \left( \frac{\lambda_1 - \lambda_2}{\lambda_2 + \lambda_1} \right) m_3$$

$$+ \frac{a_2 a_3 \left( \lambda_2 - \lambda_3 \right) \left( \frac{\lambda_1 - \lambda_4}{\lambda_2 + \lambda_4} \right) \left( \frac{\lambda_1 + \lambda_3 + \lambda_2 + \lambda_4}{\lambda_2 + \lambda_4} \right) \left( \frac{-\lambda_2 - \lambda_3 + \lambda_1 + \lambda_4}{\lambda_2 + \lambda_1} \right) n_1}{\left( \frac{\lambda_2 + \lambda_4}{\lambda_1} \right) \left( \frac{\lambda_1 + \lambda_3 + \lambda_2 + \lambda_4}{\lambda_2 + \lambda_4} \right) \left( \frac{\lambda_1 + \lambda_3 + \lambda_2 + \lambda_4}{\lambda_2 + \lambda_1} \right) \left( \frac{-\lambda_2 - \lambda_3 + \lambda_1 + \lambda_4}{\lambda_2 + \lambda_1} \right) n_2}$$

$$+ \frac{a_2 a_3 \left( \lambda_2 - \lambda_4 \right) \left( \lambda_1 - \lambda_3 \right) \left( \frac{\lambda_1 + \lambda_3 + \lambda_2 + \lambda_4}{\lambda_2 + \lambda_4} \right) \left( \frac{-\lambda_2 - \lambda_4 + \lambda_1 + \lambda_3}{\lambda_2 + \lambda_1} \right) n_3}{\left( \frac{\lambda_2 + \lambda_4}{\lambda_1} \right) \left( \frac{\lambda_1 + \lambda_3 + \lambda_2 + \lambda_4}{\lambda_2 + \lambda_4} \right) \left( \frac{\lambda_1 + \lambda_3 + \lambda_2 + \lambda_4}{\lambda_2 + \lambda_1} \right) \left( \frac{-\lambda_2 - \lambda_4 + \lambda_1 + \lambda_3}{\lambda_2 + \lambda_1} \right) n_3}.$$
where \( A, B \in \mathbb{R} \) (see, e.g., [4], Theorems 1.3 and 1.4).

Equilibria of type 1 are called *center-center* with the corresponding eigenvalues for the linearized system: \( iA, -iA, iB, -iB \).

Equilibria of type 2 are called *center-saddle* with the corresponding eigenvalues for the linearized system: \( A, -A, iB, -iB \).

Equilibria of type 3 are called *saddle-saddle* with the corresponding eigenvalues for the linearized system: \( A, -A, B, -B \).

Equilibria of type 4 are called *focus-focus* with the corresponding eigenvalues for the linearized system: \( A + iB, A - iB, -A + iB, -A - iB \).

The main result of the paper is the following.

**Theorem 5.2.** All equilibria in \( t_1 \cap \text{Orb}_{c_1;e_2} \) and \( t_3 \cap \text{Orb}_{c_1;e_2} \) are of center-center type and therefore nonlinearly stable. All equilibria in \( t_2 \cap \text{Orb}_{c_2;e_2} \) are of center-saddle type and therefore unstable. Since \( \text{Orb}_{t_1;e_2} \) is a generic adjoint orbit, if the initial condition is close to the given equilibrium but on a nearby adjoint orbit, it will stay close to it for all time.

The proof consists of analyzing each case separately.

**The equilibria in \( t_1 \).** We begin with the study of stability and non-degeneracy for the equilibria \( M_{a,b}^1 \in t_1 \cap \text{Orb}_{c_1;e_2} \) (see Theorem 5.2). To determine \( DX_{H_{\text{Orb}_{c_1;e_2}}}(M_{a,b}^1) : \mathbb{R}^4 \to \mathbb{R}^4 \), we compute from (4.1) the linearized equations at \( M_{a,b}^1 \) in the variable \( \delta M \in T_{M_{a,b}^1} \text{Orb}_{c_1;e_2} \):

\[
\begin{align*}
\frac{d}{dt} \delta x_1 &= 0 \\
\frac{d}{dt} \delta x_2 &= \left( \frac{1}{x_2 + \lambda_3} - \frac{1}{\lambda_1 + \lambda_2} \right) a \delta x_3 + \left( \frac{1}{\lambda_1 + \lambda_4} - \frac{1}{\lambda_3 + \lambda_4} \right) b \delta y_3 \\
\frac{d}{dt} \delta x_3 &= \left( \frac{1}{x_1 + \lambda_3} - \frac{1}{\lambda_2 + \lambda_3} \right) a \delta x_2 + \left( \frac{1}{\lambda_2 + \lambda_4} - \frac{1}{\lambda_1 + \lambda_4} \right) b \delta y_2 \\
\frac{d}{dt} \delta y_1 &= 0 \\
\frac{d}{dt} \delta y_2 &= \left( \frac{1}{x_2 + \lambda_3} - \frac{1}{\lambda_1 + \lambda_2} \right) a \delta y_3 + \left( \frac{1}{\lambda_1 + \lambda_4} - \frac{1}{\lambda_3 + \lambda_4} \right) b \delta x_3 \\
\frac{d}{dt} \delta y_3 &= \left( \frac{1}{x_2 + \lambda_3} - \frac{1}{\lambda_1 + \lambda_2} \right) a \delta y_2 + \left( \frac{1}{\lambda_2 + \lambda_4} - \frac{1}{\lambda_1 + \lambda_4} \right) b \delta x_2.
\end{align*}
\]

However, since

\[
T_M \text{Orb}_{c_1;e_2}(M) = \ker T_M(C_1 \times C_2) = \left\{ \delta M \in \mathfrak{so}(4) \left| \sum_{i=1}^3 (x_i \delta x_i + y_i \delta y_i) = \sum_{i=1}^3 (x_i \delta y_i + y_i \delta x_i) = 0 \right. \right\},
\]

we conclude that in the case of the equilibrium \( M_{a,b}^1 \) we have

\[
T_{M_{a,b}^1} \text{Orb}_{c_1;e_2}(M) = \left\{ \delta M \in \mathfrak{so}(4) \left| a \delta x_1 + b \delta y_1 = 0, a \delta y_1 + b \delta x_1 = 0 \right. \right\}.
\]

This means that in (5.4) the four variations \( \delta x_2, \delta x_3, \delta y_2, \delta y_3 \) are independent and thus the \( 4 \times 4 \) infinitesimal symplectic matrix of the linearized equations on the tangent space to the orbit is

\[
\begin{bmatrix}
0 & \left( \frac{1}{x_2 + \lambda_3} - \frac{1}{\lambda_1 + \lambda_2} \right) a & 0 & \left( \frac{1}{\lambda_1 + \lambda_4} - \frac{1}{\lambda_3 + \lambda_4} \right) b \\
\left( \frac{1}{x_1 + \lambda_3} - \frac{1}{\lambda_2 + \lambda_3} \right) a & 0 & \left( \frac{1}{\lambda_2 + \lambda_4} - \frac{1}{\lambda_1 + \lambda_4} \right) b & 0 \\
0 & \left( \frac{1}{x_2 + \lambda_3} - \frac{1}{\lambda_1 + \lambda_2} \right) b & 0 & \left( \frac{1}{\lambda_2 + \lambda_3} - \frac{1}{\lambda_3 + \lambda_4} \right) a \\
\left( \frac{1}{x_1 + \lambda_3} - \frac{1}{\lambda_2 + \lambda_3} \right) b & 0 & \left( \frac{1}{x_2 + \lambda_3} - \frac{1}{\lambda_1 + \lambda_2} \right) a & 0
\end{bmatrix}.
\]
though it is infinitesimally symplectic relative to $[\omega(M^1_{a,b})]$, does not have the usual expression of an element in $\mathfrak{sp}(4,\mathbb{R})$. The characteristic polynomial of this matrix is

$$z^4 + \left[ \frac{a^2}{(\lambda_2 + \lambda_3)^2} \left( \frac{(\lambda_2 - \lambda_4)(\lambda_3 - \lambda_4)}{(\lambda_3 + \lambda_4)(\lambda_2 + \lambda_3)} + \frac{(\lambda_1 - \lambda_3)(\lambda_1 - \lambda_2)}{(\lambda_1 + \lambda_2)(\lambda_1 + \lambda_3)} \right) \right. + \left. - \frac{b^2}{(\lambda_1 + \lambda_4)^2} \left( \frac{(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_4)}{(\lambda_2 + \lambda_4)(\lambda_1 + \lambda_2)} + \frac{(\lambda_1 - \lambda_3)(\lambda_3 - \lambda_4)}{(\lambda_3 + \lambda_4)(\lambda_1 + \lambda_3)} \right) \right] z^2 + \frac{1}{(\lambda_1 + \lambda_4)^2(\lambda_2 + \lambda_3)^2(\lambda_3 + \lambda_4)(\lambda_1 + \lambda_3)} \left[ \frac{1}{(\lambda_3 + \lambda_4)(\lambda_1 + \lambda_2)} + \frac{1}{(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_4)} \right] a^2 b^2 + \frac{1}{(\lambda_1 + \lambda_4)^2(\lambda_2 + \lambda_3)^2(\lambda_3 + \lambda_4)(\lambda_2 + \lambda_3)} \left[ \frac{a^4}{(\lambda_2 + \lambda_3)^4} + \frac{b^4}{(\lambda_1 + \lambda_4)^4} \right] .$$

The discussion of the position of the four roots in the complex plane is very complicated since the signs of the coefficients of $z^2$ and $z^0$ vary and depend on the relative size of the real numbers $a$ and $b$ which are arbitrary.

Thus, we proceed in a different way. We have already seen that the linear operators $DX_{H|_{\text{Orb}_{x_1,x_2}}}(M^1_{a,b})$ and $DX_{I|_{\text{Orb}_{x_1,x_2}}}(M^1_{a,b})$ are commuting, where $M^1_{a,b}$ is an equilibrium. We shall prove that these operators are also linearly independent. Indeed, since the equations of motion for $I$ are

$$\begin{align*}
\dot{x}_1 &= 2(\lambda_2^2 - \lambda_3^2)(x_2x_3 - y_2y_3) \\
\dot{x}_2 &= 2(\lambda_2^2 - \lambda_1^2)(x_3x_1 - y_3y_1) \\
\dot{x}_3 &= 2(\lambda_2^2 - \lambda_3^2)(x_1x_2 - y_1y_2) \\
\dot{y}_1 &= 2(\lambda_2^2 - \lambda_1^2)(x_3y_3 - x_3x_2) \\
\dot{y}_2 &= 2(\lambda_2^2 - \lambda_3^2)(x_3x_2 - x_3x_1) \\
\dot{y}_3 &= 2(\lambda_2^2 - \lambda_3^2)(x_1y_2 - x_2y_1),
\end{align*}$$

(5.6)

it follows that the linear equations given by $DX_{I|_{\text{Orb}_{x_1,x_2}}}(M^1_{a,b})$ are

$$\begin{align*}
\frac{d}{dt}\delta x_1 &= 0 \\
\frac{d}{dt}\delta x_2 &= 2(\lambda_2^2 - \lambda_1^2)(a\delta x_2 - b\delta y_3) \\
\frac{d}{dt}\delta x_3 &= 2(\lambda_2^2 - \lambda_3^2)(a\delta x_2 - b\delta y_2) \\
\frac{d}{dt}\delta y_1 &= 0 \\
\frac{d}{dt}\delta y_2 &= 2(\lambda_2^2 - \lambda_3^2)(b\delta x_3 - a\delta y_1) \\
\frac{d}{dt}\delta y_3 &= 2(\lambda_2^2 - \lambda_3^2)(a\delta y_2 - b\delta x_2),
\end{align*}$$

(5.7)

so that the infinitesimally symplectic matrix of the linearized equations on the tangent space to the orbit relative to the matrix $[\omega(M^1_{a,b})]$ of the orbit symplectic form is

$$\begin{pmatrix}
0 & 2a(\lambda_2^2 - \lambda_3^2) & 2a(\lambda_2^2 - \lambda_1^2) & 0 & -2b(\lambda_2^2 - \lambda_1^2) \\
2a(\lambda_2^2 - \lambda_3^2) & 0 & -2b(\lambda_2^2 - \lambda_3^2) & 0 & -2a(\lambda_2^2 - \lambda_3^2) \\
0 & 2b(\lambda_2^2 - \lambda_3^2) & 0 & -2a(\lambda_2^2 - \lambda_3^2) & 0 \\
-2b(\lambda_2^2 - \lambda_3^2) & 0 & 2a(\lambda_2^2 - \lambda_3^2) & 0 & 0 \\
0 & 2a(\lambda_2^2 - \lambda_3^2) & 0 & -2b(\lambda_2^2 - \lambda_3^2) & 0.
\end{pmatrix}$$

(5.8)

The matrices (5.5) and (5.8) are linearly independent. This is seen in the following way. A linear combination of these matrices yields a $2 \times 8$ homogeneous linear system and we need to show that the rank of its matrix is 2. It turns out that three pairs among these equations are identical. Now, using appropriate pairs one sees that the genericity hypothesis (4.6) implies that there always exists a $2 \times 2$ minor with non-vanishing determinant.

Therefore, the span of $DX_{H|_{\text{Orb}_{x_1,x_2}}}(M^1_{a,b})$ and $DX_{I|_{\text{Orb}_{x_1,x_2}}}(M^1_{a,b})$ forms a two dimensional Abelian subalgebra of the infinitesimally symplectic linear maps on $\left( T_{M^1_{a,b}} \text{Orb}_{x_1,x_2}, \omega|_{\text{Orb}_{x_1,x_2}} \right)$. We want to show that it is a Cartan subalgebra in order to conclude that that $M^1_{a,b}$ is a non-degenerate equilibrium.
This is the case if and only if \( \text{span}_X \left\{ DX_{H_{(\text{orb}^1,\text{c}_2)}}(M^1_{a,b}), DX_{I_{(\text{orb}^1,\text{c}_2)}}(M^1_{a,b}) \right\} \) contains an element all of whose eigenvalues are distinct (see, e.g. \cite{4}, \S 1.8.2).

To show the existence of such an element we begin with the study of the characteristic polynomial

\[ z^4 + v_1 z^2 + w_1 = 0, \]

of \((5.8)\), where

\[ w_1 = 16(\lambda_1^2 - \lambda_2^2)(\lambda_1^2 - \lambda_3^2)(\lambda_2^2 - \lambda_1^2)(\lambda_2^2 - \lambda_3^2)(a^2 - b^2)^2 > 0 \quad \text{since} \quad a \neq b \]

\[ v_1 = S_1 a^2 + T_1 b^2 \]

\[ S_1 = 4 \left( 2\lambda_1^2 \lambda_2^2 - \lambda_1^2 \lambda_3^2 + \lambda_1^4 + \lambda_2^4 - \lambda_2^2 \lambda_3^2 - \lambda_3^2 \lambda_1^2 - \lambda_3^2 \lambda_2^2 \right) \]

\[ T_1 = 4 \left( \lambda_1^2 - \lambda_1^2 \lambda_2^2 + 2\lambda_1^2 \lambda_3^2 - \lambda_2^2 \lambda_3^2 - \lambda_2^2 \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \right). \]

Using \((3.3)\) we have

\[ v_1 = c_1 (S_1 + T_1) + \sqrt{c_1^2 - c_2^2} (S_1 - T_1) \]

and

\[ S_1 + T_1 = 4(\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_2^2)^2 \geq 0 \]

\[ S_1 - T_1 = 4(\lambda_1^2 - \lambda_2^2 + \lambda_2^2 - \lambda_3^2)(\lambda_1^2 - \lambda_2^2 + \lambda_3^2 - \lambda_2^2) > 0 \]

which shows that \( v_1 > 0 \).

The discriminant of the quadratic equation in \( z^2 \) is

\[ \Delta_1 = v_1^2 - 4w_1 = 16(\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_2^2)^2 \left( S_1^4 a^4 + T_1^4 a^2 b^2 + U_1^4 b^4 \right), \]

where

\[ S_1' = (\lambda_2^2 - \lambda_3^2)^2 > 0, \quad U_1' = (\lambda_2^2 - \lambda_3^2)^2 > 0 \]

\[ T_1' = 2 \left( -\lambda_3^2 \lambda_2^2 - \lambda_1^2 \lambda_3^2 + 2\lambda_1^2 \lambda_2^2 + 2\lambda_2^2 \lambda_3^2 - \lambda_1^2 \lambda_2^2 - \lambda_2^2 \lambda_1^2 \right). \]

Furthermore,

\[ S_1^4 a^4 + T_1^4 a^2 b^2 + U_1^4 b^4 = 2c_1^2 (S_1' + U_1') + c_2^2 (T_1' - S_1' - U_1') + 2c_1 \sqrt{c_1^2 - c_2^2} (S_1' - U_1') \]

\[ = 2c_1^2 (S_1' + U_1') + c_2^2 (S_1' + T_1' + U_1') + 2c_1 \sqrt{c_1^4 - c_2^2 (S_1' - U_1')}. \]

Since \( \lambda_1 > \lambda_2 > \lambda_3 > \lambda_4 \) we have

\[ S_1' + U_1' = (\lambda_2^2 - \lambda_2^2)^2 + (\lambda_2^2 - \lambda_3^2)^2 > 0 \]

\[ S_1' - U_1' = (\lambda_2^2 + \lambda_2^2 - \lambda_3^2 - \lambda_2^2)(\lambda_2^2 - \lambda_2^2 + \lambda_2^2 + \lambda_3^2) > 0 \]

\[ S_1' + U_1' + T_1' = (\lambda_2^2 - \lambda_2^2 - \lambda_2^2)^2 \geq 0. \]

Thus, if \( \lambda_2^2 + \lambda_2^2 \neq \lambda_2^2 + \lambda_2^2 \) we have \( \Delta_1 > 0 \) (recall \( c_1 > |c_2| \)). Since \( v_1 > 0, w_1 > 0 \), the equation \( t^2 + v_1 t + w_1 = 0 \) has two non-zero distinct negative real roots and, therefore, the equation \( z^4 + v_1 z^2 + w_1 = 0 \) has two distinct pairs of purely imaginary roots different from zero. Thus \( \text{span}_X \left\{ DX_{H_{(\text{orb}^1,\text{c}_2)}}(M^1_{a,b}), DX_{I_{(\text{orb}^1,\text{c}_2)}}(M^1_{a,b}) \right\} \) is a Cartan subalgebra an it is of the first type in \((5.3)\). It follows that the equilibrium \( M^1_{a,b} \) is non-degenerate and nonlinearly stable because it is of center-center type (see \cite{4}, Theorem 1.5).

The above computations being independent of the sign and permutation of \( a \) and \( b \), by an analogous reasoning we obtain nonlinear stability for the other three equilibria \( M^1_{-a,-b}, M^1_{b,a}, M^1_{-b,-a} \) in the Weyl orbit of \( M^1_{a,b} \).
If $\lambda_1^2 + \lambda_2^2 = \lambda_2^2 + \lambda_3^2$ the eigenvalues of $DX_{t_1|orb_{c_1;c_2}}(M_{a,b}^1)$ are conjugate purely imaginary of multiplicity two. In order to determine non-degeneracy of the equilibrium $M_{a,b}^1$, we have to find a linear combination $DX_{t_1|orb_{c_1;c_2}}(M_{a,b}^1) + \alpha DX_{t_1|orb_{c_1;c_2}}(M_{a,b}^1)$, where $\alpha$ is a non-zero real number, that has distinct eigenvalues. The eigenvalues of this linear combination are the roots of the equation

$$u'_1 z^4 + u'_1 z^2 + w'_1 = 0,$$

where

$$u'_1 = (\lambda_1 + \lambda_2)(\lambda_1 + \lambda_3)(\lambda_1 + \lambda_4)^4(\lambda_2 + \lambda_3)^4(\lambda_2 + \lambda_4)(\lambda_3 + \lambda_4) > 0$$

$$w'_1 = (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)(\lambda_3 - \lambda_4)(X_1 \alpha^2 + Y_1 \alpha + Z_1)^2 \geq 0,$$

with

$$X_1 = 4(\lambda_1 + \lambda_2)(\lambda_1 + \lambda_3)(\lambda_1 + \lambda_4)^2(\lambda_2 + \lambda_3)^2(\lambda_2 + \lambda_4)(\lambda_3 + \lambda_4)(a^2 - b^2) \neq 0$$

and $Y_1, Z_1$ are also expressions of $\lambda_1, \lambda_2, \lambda_3, \lambda_4, a, b$.

The discriminant of the quadratic equation $u'_1 t^2 + v'_1 t + w'_1 = 0$ obtained by denoting $z^2 = t$ is

$$\Delta'_1 = 4(\lambda_1 + \lambda_4)^6(\lambda_2 + \lambda_3)^6(Y_2 \alpha + Z_2)^2 D,$$

where

$$D = 2 \left((c_1^2 - c_2^2) ((\lambda_1^2 - \lambda_4^2)^2 + (\lambda_2^2 - \lambda_3^2)^2) + c_1 \sqrt{c_1^2 - c_2^2} (\lambda_1^2 - \lambda_4^2)(\lambda_2^2 - \lambda_3^2)(\lambda_3^2 - \lambda_2^2 + \lambda_3^2 - \lambda_1^2) \right) > 0$$

$$Y_2 = -2(\lambda_1 + \lambda_2)(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_4)(\lambda_3 + \lambda_4)(\lambda_1 + \lambda_4 - \lambda_2 - \lambda_3)$$

$$Z_2 = \lambda_1 \lambda_4 - \lambda_2 \lambda_3.$$

Note that $Y_2 = 0$ if and only if $\lambda_1 + \lambda_4 = \lambda_2 + \lambda_3$, because $\lambda_i + \lambda_j > 0$ for any $i \neq j$. Then $(\lambda_1 + \lambda_4)^2 = (\lambda_2 + \lambda_3)^2$ and $\lambda_1^2 + \lambda_3^2 = \lambda_2^2 + \lambda_3^2$ imply $Z_2 = 0$. Conversely, if $Z_2 = 0$, then $\lambda_1^2 + \lambda_3^2 = \lambda_2^2 + \lambda_3^2$, that is $\lambda_1 + \lambda_4 = \pm(\lambda_2 + \lambda_3)$. Since the solution with minus is not possible because $\lambda_i + \lambda_j > 0$ for any $i \neq j$, we conclude that $Y_2 = 0$. Thus, $Y_2 = 0$ if and only if $Z_2 = 0$.

However, if $Y_2 = 0$, so $Z_2 = 0$ which means that $\lambda_1\lambda_2 = \lambda_3\lambda_4$, then we also have $(\lambda_1 - \lambda_4)^2 = (\lambda_2 - \lambda_3)^2$ and consequently $\lambda_1 - \lambda_4 = \pm(\lambda_2 - \lambda_3)$. The solution with minus is impossible because $\lambda_1 + \lambda_2 > \lambda_3 + \lambda_4$ since, by hypothesis, $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4$. Hence we must have $\lambda_1 - \lambda_4 = \lambda_2 - \lambda_3$ which together with $\lambda_1 + \lambda_4 = \lambda_2 + \lambda_3$ implies that $\lambda_1 = \lambda_2$ which is also impossible since $\lambda_1 > \lambda_2$.

Therefore $Y_2 \neq 0$ and hence $\Delta'_1 > 0$ if we choose $\alpha \neq -Z_2/Y_2$.

Furthermore, $u'_1$ has the expression

$$v'_1 = 2(\lambda_1 + \lambda_4)^2(\lambda_2 + \lambda_3)^2 \sqrt{c_1^2 - c_2^2}(X_3 \alpha^2 + Y_3 \alpha + Z_3),$$

where

$$X_3 = 2(\lambda_1 + \lambda_2)(\lambda_1 + \lambda_3)(\lambda_1 + \lambda_4)^2(\lambda_2 + \lambda_3)^2(\lambda_2 + \lambda_4)(\lambda_3 + \lambda_4)(\lambda_1^2 - \lambda_3^2 + \lambda_2^2 - \lambda_4^2)(\lambda_1^2 - \lambda_2^2 + \lambda_3^2 - \lambda_4^2) > 0.$$
The equilibria in $t_2$. We proceed as in the previous case. It is easy to see that $\mathbf{D}X_{I|\text{Orb}_{1};c_2}(M_{a,b}^2)$ and $\mathbf{D}X_{H|\text{Orb}_{1},c_2}(M_{a,b}^2)$ generate a 2-dimensional subspace. The eigenvalues of $\mathbf{D}X_{I|\text{Orb}_{1};c_2}(M_{a,b}^2)$ are the roots of the equation

$$z^4 + v_2 z^2 + w_2 = 0,$$

where

$$w_2 = -16(\lambda_1^2 - \lambda_2^2)(\lambda_1^2 - \lambda_3^2)(\lambda_2^2 - \lambda_3^2)(a^2 - b^2)^2 < 0$$

$$v_2 = 4 \left( -\lambda_3^2 \lambda_4^2 - \lambda_2^2 \lambda_3^2 - \lambda_1^2 \lambda_4^2 + \lambda_1^2 \lambda_2^2 + \lambda_4^4 + 2 \lambda_1^2 \lambda_3^2 \right) a^2$$

$$+ 4 \left(- \lambda_3^2 \lambda_4^2 + 2 \lambda_2^2 \lambda_4^2 - \lambda_2^2 \lambda_2^2 - \lambda_1^2 \lambda_4^2 + \lambda_1^4 + \lambda_3^4 - \lambda_2^4 \lambda_2^2 \right) b^2.$$

The quadratic equation $t^2 + v_2 t + w_2 = 0$ has discriminant

$$\Delta_2 = v_2^2 - 4 w_2 = 16(\lambda_1^2 - \lambda_2^2 + \lambda_2^3 - \lambda_3^2)^2 \left(S_2 a^4 + T_2 a^2 b^2 + U_2 b^4 \right),$$

where

$$S_2 = (\lambda_2^2 - \lambda_3^2)^2 > 0,$$

$$T_2 = 2 \left(- \lambda_3^2 \lambda_2^2 - \lambda_1^2 \lambda_4^2 + 2 \lambda_1^2 \lambda_3^2 + 2 \lambda_2^2 \lambda_4^2 - \lambda_3^2 \lambda_1^2 - \lambda_2^2 \lambda_3^2 \right).$$

Moreover, the discriminant of the quadratic expression $S_2 a^4 + T_2 a^2 b^2 + U_2 b^4$ is

$$T_2^2 - 4 S_2 U_2 = -16(\lambda_1^2 - \lambda_2^2)(\lambda_1^2 - \lambda_3^2)(\lambda_2^2 - \lambda_3^2)(\lambda_3^2 - \lambda_2^2) < 0,$$

which implies that $S_2 a^4 + T_2 a^2 b^2 + U_2 b^4 > 0$ and consequently $\Delta_2 > 0$. Therefore, the equation $t^2 + v_2 t + w_2 = 0$ has two non-zero distinct real roots of opposite signs and thus the equation $z^4 + v_2 z^2 + w_2 = 0$ has two distinct real roots and two distinct purely imaginary roots. Thus $M_{a,b}^{2,0}$ is a non-degenerate equilibrium and span$_R \left\{ \mathbf{D}X_{H|\text{Orb}_{1},c_2}(M_{a,b}^2), \mathbf{D}X_{I|\text{Orb}_{1};c_2}(M_{a,b}^2) \right\}$ is a Cartan subalgebra of the second type in $(5,3)$. Thus, $M_{a,b}^{2,0}$ is an unstable equilibrium of center-saddle type.

As before, the above computations being independent of the sign and permutation of $a$ and $b$, by an analogous reasoning we obtain non-degeneracy and instability for the other three equilibria $M_{-a,-b}^2, M_{b,a}^2, M_{b,-a}^2$ in the Weyl orbit of $M_{a,b}^2$ and hence all equilibria in $t_2 \cap \text{Orb}_{c_1,c_2}$ are of center-saddle type and therefore unstable.

The equilibria in $t_3$. We proceed as in case of equilibria in $t_1$. It is easy to see that $\mathbf{D}X_{I|\text{Orb}_{1};c_2}(M_{a,b}^3)$ and $\mathbf{D}X_{H|\text{Orb}_{1},c_2}(M_{a,b}^3)$ generate a 2-dimensional subspace. The eigenvalues of $\mathbf{D}X_{I|\text{Orb}_{1};c_2}(M_{a,b}^3)$ are the roots of the equation

$$z^4 + v_3 z^2 + w_3 = 0,$$

where

$$w_3 = 16(\lambda_1^2 - \lambda_2^2)(\lambda_1^2 - \lambda_3^2)(\lambda_2^2 - \lambda_3^2)(a^2 - b^2)^2 > 0$$

$$v_3 = 4 \left( \left( \lambda_1^2 - \lambda_3^2 \right) \left( \lambda_2^2 - \lambda_3^2 \right) + \left( \lambda_1^2 - \lambda_2^2 \right) \left( \lambda_3^2 - \lambda_2^2 \right) \right) a^2 +$$

$$+ 4 \left( \left( \lambda_1^2 - \lambda_3^2 \right) \left( \lambda_2^2 - \lambda_3^2 \right) + \left( \lambda_2^2 - \lambda_2^2 \right) \left( \lambda_3^2 - \lambda_2^2 \right) \right) b^2 > 0.$$

The quadratic equation $t^2 + v_3 t + w_3 = 0$ has discriminant

$$\Delta_3 = v_3^2 - 4 w_3 = 16(\lambda_1^2 - \lambda_2^2 + \lambda_2^3 - \lambda_3^2)^2 \left(S_3 a^4 + T_3 a^2 b^2 + U_3 b^4 \right),$$

where

$$S_3 = (\lambda_3^2 - \lambda_4^2)^2 > 0,$$

$$T_3 = 2 \left[ (\lambda_1^2 - \lambda_3^2) (\lambda_2^2 - \lambda_3^2) + (\lambda_2^2 - \lambda_2^2) (\lambda_3^2 - \lambda_2^2) \right] > 0.$$

which implies that $S_3 a^4 + T_3 a^2 b^2 + U_3 b^4 > 0$ and consequently $\Delta_3 > 0$. 

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Since \(v_3 > 0, w_3 > 0\), the equation \(t^2 + v_3 t + w_3 = 0\) has two non-zero distinct negative real roots and therefore equation \(z^2 + v_3 z^2 + w_3 = 0\) has two distinct pairs of purely imaginary roots and zero is not a root. Thus span \(\{X_{H|\text{orb}_{1,2}}(M_{a,b}^3), X_{H|\text{orb}_{1,2}}(M_{a,b}^3)\}\) is a Cartan subalgebra an it is of the first type in (5.3). It follows that the equilibrium \(M_{a,b}^3\) is non-degenerate and nonlinearly stable because it is of center-center type (see [4], Theorem 1.5).

The same holds for the other three equilibria \(M_{a,-b}^3, M_{b,a}^3, M_{b,-a}^3\) in the Weyl group orbit of \(M_{a,b}^3\), that is, all equilibria in \(t_3 \cap \text{Orb}_{1,2}\) are of center-center type and therefore nonlinearly stable.

This proves Theorem 5.2. Next, we begin the analysis of the remaining equilibria.

The equilibria in \(s_\pm\). The equilibria from the families \(s_+\) and \(s_-\) are not isolated on the adjoint orbits. In fact, they come in curves or points described by intersecting the ellipsoids (4.7) with the spheres (4.8), both families having the center at the origin.

The linearized equations at such an equilibrium \(M_e \in \text{Orb}_{\epsilon_1,\epsilon_2} \cap [(s_+ \cup s_-) \setminus (t_1 \cup t_2 \cup t_3)]\) are

\[
\begin{align*}
\frac{d}{dt} \delta x_1 &= \frac{(\lambda_2 - \lambda_3)(-\lambda_2 + \lambda_4) - (\lambda_3 - \lambda_4)\lambda_3 \delta x_2}{(\lambda_2 + \lambda_4)(\lambda_3 + \lambda_4)} \delta x_1 + (\lambda_1 + \lambda_2)\lambda_3 \delta y_2, \\
\frac{d}{dt} \delta x_2 &= \frac{(\lambda_2 - \lambda_3)(\lambda_3 - \lambda_4) - (\lambda_3 - \lambda_4)\lambda_2 \delta x_1}{(\lambda_2 + \lambda_4)(\lambda_3 + \lambda_4)} \delta x_2 + (\lambda_1 + \lambda_2)\lambda_3 \delta y_1, \\
\frac{d}{dt} \delta x_3 &= \frac{(\lambda_2 - \lambda_3)(\lambda_3 - \lambda_4) - (\lambda_3 - \lambda_4)\lambda_3 \delta x_2}{(\lambda_2 + \lambda_4)(\lambda_3 + \lambda_4)} \delta x_3 + (\lambda_1 + \lambda_2)\lambda_3 \delta y_3, \\
\frac{d}{dt} \delta y_1 &= \frac{(\lambda_2 - \lambda_4)(\lambda_2 - \lambda_4) - (\lambda_2 - \lambda_4)\lambda_3 \delta x_2}{(\lambda_2 + \lambda_4)(\lambda_3 + \lambda_4)} \delta y_1 + (\lambda_1 + \lambda_2)\lambda_3 \delta y_3, \\
\frac{d}{dt} \delta y_2 &= \frac{(\lambda_2 - \lambda_4)(\lambda_2 - \lambda_4) - (\lambda_2 - \lambda_4)\lambda_2 \delta x_1}{(\lambda_2 + \lambda_4)(\lambda_3 + \lambda_4)} \delta y_2 + (\lambda_1 + \lambda_2)\lambda_3 \delta y_3, \\
\frac{d}{dt} \delta y_3 &= \frac{(\lambda_2 - \lambda_4)(\lambda_2 - \lambda_4) - (\lambda_2 - \lambda_4)\lambda_3 \delta x_2}{(\lambda_2 + \lambda_4)(\lambda_3 + \lambda_4)} \delta y_3 + (\lambda_1 + \lambda_2)\lambda_3 \delta y_3.
\end{align*}
\]

(see (4.1), (4.3)). The characteristic equation of the associated 6 \times 6 matrix is

\[t^4(k_4 t^2 + k_1 a_1^2 + k_2 a_2^2 + k_3 a_3^2) = 0,\]

where

\[k_4 = (\lambda_2 + \lambda_3)^2 (\lambda_1 + \lambda_3)^2 (\lambda_3 + \lambda_4)^2 (\lambda_2 + \lambda_4)^2 (\lambda_3 + \lambda_4)^2 (\lambda_1 + \lambda_2)^2 > 0\]
\[k_1 = 4 (\lambda_3 + \lambda_4)(\lambda_3 + \lambda_4)(\lambda_3 + \lambda_4)(\lambda_3 + \lambda_4)(\lambda_3 + \lambda_4)(\lambda_3 + \lambda_4)(\lambda_3 + \lambda_4)^2 \geq 0\]
\[k_2 = 4 (\lambda_3 + \lambda_4)(\lambda_3 + \lambda_4)(\lambda_3 + \lambda_4)(\lambda_3 + \lambda_4)(\lambda_3 + \lambda_4)(\lambda_3 + \lambda_4)(\lambda_3 + \lambda_4)^2 \geq 0\]
\[k_3 = 4 (\lambda_3 + \lambda_4)(\lambda_3 + \lambda_4)(\lambda_3 + \lambda_4)(\lambda_3 + \lambda_4)(\lambda_3 + \lambda_4)(\lambda_3 + \lambda_4)(\lambda_3 + \lambda_4)^2 \geq 0\]

and thus there are four zero eigenvalues; recall \(\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4\). A double zero eigenvalue is expected since the generic orbit is four dimensional. Restricting the linearized system to the tangent space to the orbit (which equals \(\text{ker } dC_1(M) \cap \text{ker } dC_2(M)\)) yields a linear system whose eigenvalues are the roots of the polynomial \(t^2(k_4 t^2 + k_1 a_1^2 + k_2 a_2^2 + k_3 a_3^2) = 0\). Therefore, the linearization of the integrable system (5.1) on the four dimensional adjoint orbit \(\text{Orb}_{\epsilon_1,\epsilon_2} \cap [(s_+ \cup s_-) \setminus (t_1 \cup t_2 \cup t_3)]\) has the following eigenvalues: 0 is a double eigenvalue and there are two other purely imaginary conjugate eigenvalues which can also degenerate to 0. Consequently, these equilibria can only be degenerate cases of type 1 or type 2 in (5.3). Thus, we cannot infer any stability conclusion from the linearized system.

Note that the only time that 0 can be a quadruple eigenvalue is when \(a_2 = a_3 = k_1 = 0\).

As before, we use the additional constant of motion \(I_{\text{Orb}_{\epsilon_1,\epsilon_2}} := I_{\text{Orb}_{\epsilon_1,\epsilon_2}}\) that commutes with \(H_{\text{Orb}_{\epsilon_1,\epsilon_2}} := H_{\text{Orb}_{\epsilon_1,\epsilon_2}}\). However, by Proposition 5.1 \(dI_{\text{Orb}_{\epsilon_1,\epsilon_2}}(M_e) \neq 0\), so we can not apply the method used for studying the stability for the equilibria in \(K_0 = \text{Orb}_{\epsilon_1,\epsilon_2} \cap (t_1 \cup t_2 \cup t_3)\).
We shall use energy methods (see [1], [7], [11], [2]). If

\[ m_0 = -\frac{1}{\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4}, \quad n_0 = -\frac{1}{\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4} \]

then \( d(H + m_0 C_1 + n_0 C_2)(M_e) = 0 \) and the Hessian \( D^2(H + m_0 C_1 + n_0 C_2)(M_e) \) has characteristic polynomial

\[ t^3(t - \alpha_1)(t - \alpha_2)(t - \alpha_3) = 0, \]

where

\[ \alpha_1 = \frac{(\lambda_1 + \lambda_3)^2 + (\lambda_2 + \lambda_4)^2}{(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_4)(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)} > 0 \]
\[ \alpha_2 = \frac{(\lambda_1 + \lambda_4)^2 + (\lambda_2 + \lambda_3)^2}{(\lambda_1 + \lambda_4)(\lambda_2 + \lambda_3)(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)} > 0 \]
\[ \alpha_3 = \frac{(\lambda_1 + \lambda_2)^2 + (\lambda_3 + \lambda_4)^2}{(\lambda_1 + \lambda_2)(\lambda_3 + \lambda_4)(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)} > 0. \]

We suppose, without lose of generality, that \( \alpha_1 \neq 0, \alpha_2 \neq 0 \). The Hessian \( D^2 H_{\text{Orb}_{c_1,c_2}}(M_e) = D^2(H + m_0 C_1 + n_0 C_2)(M_e)|_{T_{M_e} \text{Orb}_{c_1,c_2}} \), where \( T_{M_e} \text{Orb}_{c_1,c_2} = \ker dC_1(M_e) \cap \ker dC_2(M_e) \) has eigenvalues \( 0, \beta_1, \beta_2, \beta_3 \) computed in a conveniently chosen basis for \( T_{M_e} \text{Orb}_{c_1,c_2} \). Using Viète’s relations for the characteristic polynomial of \( D^2 H_{\text{Orb}_{c_1,c_2}}(M_e) \) computed in the above basis we have:

\[ \beta_1 \beta_2 \beta_3 = 4 \frac{A_1 + A_2 \left( \frac{a_3}{a_1} \right)^2 + A_3 \left( \frac{a_3}{a_2} \right)^2}{B}, \]

where:

\[ A_1 = (\lambda_1^2 - \lambda_2^2)(\lambda_3^2 - \lambda_4^2)[(\lambda_1 + \lambda_2)^2 + (\lambda_3 + \lambda_4)^2] > 0; \]
\[ A_2 = (\lambda_1^2 - \lambda_4^2)(\lambda_2^2 - \lambda_3^2)(\lambda_1 + \lambda_4)^2 > 0; \]
\[ A_3 = (\lambda_1^2 - \lambda_2^2)(\lambda_2^2 - \lambda_3^2)(\lambda_2 + \lambda_4)^2 > 0; \]
\[ B = (\lambda_1 + \lambda_2)^3(\lambda_1 + \lambda_3)(\lambda_1 + \lambda_4)(\lambda_2 + \lambda_3)(\lambda_2 + \lambda_4)(\lambda_3 + \lambda_4)^3. \]

This shows that \( \beta_1, \beta_2, \beta_3 \) are all non-zero and positive since \( D^2 H_{\text{Orb}_{c_1,c_2}}(M_e) \) is positive semi-definite as it is a restriction of the positive semi-definite bilinear form \( D^2(H + m_0 C_1 + n_0 C_2)(M_e) \).

Now we can conclude the following theorem.

**Theorem 5.3.** Each curve of equilibria in \( \text{Orb}_{c_1,c_2} \cap [(s_+ \cup s_-) \setminus (t_1 \cup t_2 \cup t_3)] \) is nonlinearly stable. That is, if a solution of \( \{5.1\} \) starts near an equilibrium on such a curve, at any later time it will stay close to the curve in \( \text{Orb}_{c_1,c_2} \cap [(s_+ \cup s_-) \setminus (t_1 \cup t_2 \cup t_3)] \) containing this equilibrium, but in the direction of this curve it may drift. Since \( \text{Orb}_{c_1,c_2} \) is a generic adjoint orbit, if the perturbation is close to the given equilibrium but on a neighboring adjoint orbit the same situation occurs.

**Remark 5.1.** One can pose the legitimate question if the statement of the theorem above could be strengthened in the sense that the drift in the neutral direction is impossible, at least for some equilibria. This would then prove the nonlinear stability of such an equilibrium on these curves of equilibria. To achieve this, one would have to show that \( D^2 H|_{\text{Orb}_{c_1,c_2}}(M_e) \) is definite when restricted to the leaf \( L_{M_e} \) (to be defined below), which would give nonlinear stability by Arnold’s method (which is proved to be equivalent with the other energy methods, see [2]). We shall show below that the method is inconclusive so we do not know which, if any, of the equilibria on these curves are nonlinearly stable.

So let’s try to apply the Arnold stability method to such an equilibrium \( M_e \). We need to study the definiteness of the Hessian of the constant of the motion \( H_{\text{Orb}_{c_1,c_2}} + \alpha I_{\text{Orb}_{c_1,c_2}} \) evaluated at \( M_e \).
restricted to the invariant level set 

\[ L_{M_e} := I_{Orb_{1;2}}^{-1} \left( I_{Orb_{1;2}} \left( M_e \right) \right) \]

of the dynamics (5.1). The conditions

\[ d(H_{Orb_{1;2}} + \alpha I_{Orb_{1;2}})(M_e) = 0, \quad dI_{Orb_{1;2}}(M_e) = 0, \quad \text{and} \quad dI_{Orb_{1;2}}(M_e) \neq 0 \]

(since \( M_e \in X_1 \)) imply \( \alpha = 0 \). Consequently, we have to study the definiteness of the Hessian of \( H_{Orb_{1;2}} \) at \( M_e \) restricted to the tangent space at \( M_e \) of \( L_{M_e} \). We shall prove below that this definiteness does not hold.

Let \( c_{M_e}(t) \) be the curve of equilibria for \( X_{H_{Orb_{1;2}}} \) with \( c_{M_e}(0) = M_e \). Then \( X_{H_{Orb_{1;2}}}(c_{M_e}(t)) = 0 \) and by differentiation \( DX_{H_{Orb_{1;2}}}(0) \cdot \dot{c}_{M_e}(0) = 0 \). Equivalently, using the formula of the linearization of a Hamiltonian vector field at a critical point on a symplectic manifold, we have \( \Lambda^{-1}D^2 H_{Orb_{1;2}}(M_e) \cdot \dot{c}_{M_e}(0) = 0 \), where \( \Lambda \) is the 4 \( \times \) 4 matrix associated to the symplectic form on the adjoint orbit \( Orb_{1;2} \). As \( \Lambda \) is nondegenerate, we obtain that \( D^2 H_{Orb_{1;2}}(M_e) \cdot \dot{c}_{M_e}(0) = 0 \), which shows that \( \dot{c}_{M_e}(0) \) is in the eigendirection corresponding to the eigenvalue 0 for \( D^2 H_{Orb_{1;2}}(M_e) \).

We shall prove that \( X_{I_{Orb_{1;2}}}(M_e) \) is collinear with \( \dot{c}_{M_e}(0) \). Suppose not; then \( X_{I_{Orb_{1;2}}}(M_e) \) is not tangent to the curve \( c_{M_e}(t) \) at the point \( M_e \). Consequently, for a small \( s \in \mathbb{R} \), along the integral curve of \( X_{I_{Orb_{1;2}}} \), we find an \( s \in \mathbb{R} \) such that \( \Phi_{s,I_{Orb_{1;2}}}(M_e) = x_1 \), where \( x_1 \notin c_{M_e}(t) \). Since \( \{ H_{Orb_{1;2}}, I_{Orb_{1;2}} \} = 0 \), the flows \( \Phi_{t,H_{Orb_{1;2}}} \) of \( X_{H_{Orb_{1;2}}} \) and \( \Phi_{s,I_{Orb_{1;2}}} \) of \( X_{I_{Orb_{1;2}}} \) commute and hence

\[
\Phi_{-s,I_{Orb_{1;2}}} \circ \Phi_{t,H_{Orb_{1;2}}} \circ \Phi_{s,I_{Orb_{1;2}}} \circ \Phi_{t,H_{Orb_{1;2}}}(M_e) = M_e.
\]

Because in a neighborhood of \( M_e \) the only equilibria for the vector field \( X_{H_{Orb_{1;2}}} \) are on the curve \( c_{M_e}(t) \), we conclude that \( x_2 := \Phi_{-t,I_{Orb_{1;2}}}(x_1) \neq x_1 \). But then

\[
\Phi_{-s,I_{Orb_{1;2}}}(x_2) = \Phi_{-s,I_{Orb_{1;2}}}(\Phi_{t,H_{Orb_{1;2}}}(x_1)) = (\Phi_{-s,I_{Orb_{1;2}}} \circ \Phi_{t,H_{Orb_{1;2}}})(\Phi_{s,I_{Orb_{1;2}}}(M_e)) = (\Phi_{-s,I_{Orb_{1;2}}} \circ \Phi_{t,H_{Orb_{1;2}}} \circ \Phi_{s,I_{Orb_{1;2}}} \circ \Phi_{t,H_{Orb_{1;2}}})(M_e) = M_e.
\]

Thus the Hessian of \( H_{Orb_{1;2}} \) restricted to the level manifold \( L_{M_e} \) has a 0 eigenvalue in the direction \( X_{I_{Orb_{1;2}}}(M_e) \).

This shows that the use of the constant of the motion \( I \) does not improve the nonlinear stability result of equilibrium points of type \( X_1 \) in Theorem 5.3.

**Remark 5.2.** The eigenvalue 0 for the Hessian is expected since the equilibrium \( M_e \) lies on the curve obtained by intersecting the ellipsoids \( 4.5 \) and \( 4.4 \), both having centers at the origin; the 0-eigenspace is tangent to this curve of equilibria, as proved above. So the only stability we can expect is the stability transversal to the direction of the curve of equilibria.

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