Cyclic Sieving and Cluster Duality for Grassmannian

Linhui Shen* and Daping Weng†

Abstract

We introduce a decorated configuration space $\mathcal{C}_{onf}^x(n)$ with a potential function $W$. We prove the cluster duality conjecture of Fock-Goncharov for Grassmannians, that is, the tropicalization of $(\mathcal{C}_{onf}^x(n), W)$ canonically parametrizes a linear basis of the homogeneous coordinate ring of the Grassmannian $Gr_v(n)$. We prove that $(\mathcal{C}_{onf}^x(n), W)$ is equivalent to the mirror Landau-Ginzburg model of Grassmannian considered in [MR13, RW17]. As an application, we show a cyclic sieving phenomenon involving plane partitions under a sequence of piecewise-linear toggles.

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*Department of Mathematics, Michigan State University, linhui@math.msu.edu
†Department of Mathematics, Yale University, daping.weng@yale.edu
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1 Introduction
Throughout this paper we fix three positive integers $a, b, c$ and set $n := a + b$.

1.1 Cyclic Sieving Phenomenon
Let $S$ be a finite set and let $g$ be a permutation of $S$ of order $n$. We are interested in the size of the fixed point set $S^{g^d}$ of $g^d$ for $d \geq 0$. Let $F(q)$ be a polynomial of positive integral coefficients and let $\zeta = e^{2\pi\sqrt{-1}/n}$. Following Reiner-Stanton-White [RSW04], we make the following definition.

Definition 1.1. We say that the triple $(S, g, F(q))$ exhibits the cyclic sieving phenomenon (CSP) if the fixed point set cardinality $\#S^{g^d}$ is equal to the polynomial evaluation $F(\zeta^d)$ for all $d \geq 0$.

Many combinatorial models have been found to exhibit cyclic sieving phenomenon. Meanwhile the proofs of them often involve deep results in representation theory. For example, Rhoades [Rho10] proved the CSP for rectangular Young tableaux under the action of promotion, by using Kazhdan-Lusztig theory on representation of Hecke algebra. In [FK14], Fontaine and Kamnizter studied the CSP for minuscule Littelmann paths under rotation using geometric Satake correspondence and intersection homology of quiver varieties. For more examples, we refer the interested readers to a survey on this topic by Sagan [Sag11].

In the present paper, we prove the CSP for plane partitions under piecewise-linear toggling operation. We situate our result in the context of cluster theory for Grassmannian.

1.2 Plane Partitions
A size $a \times b$ plane partition is an $a \times b$ matrix $\pi = (\pi_{i,j})$ of non-negative integer entries $\pi_{i,j}$ that is weakly decreasing in rows and columns, and for which we define

$$|\pi| := \sum_{i,j} \pi_{i,j}.$$
Denote by \( P(a,b,c) \) the set of size \( a \times b \) plane partitions with largest entry \( \pi_{1,1} \leq c \). Let \([m]_q\) denote the quantum integer \( \frac{1-q^m}{1-q} \). MacMahon’s formula asserts that

\[
M_{a,b,c}(q) := \sum_{\pi \in P(a,b,c)} q^{\mid \pi \mid} = \prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} [i + j + k - 1]_q [i + j + k - 2]_q.
\]

Following [Rob16, §4], we consider piecewise-linear toggles on \( P(a,b,c) \). Let \( \pi \in P(a,b,c) \). We make the convention that \( \pi_{0,j} = \pi_{i,0} = c \) and \( \pi_{a+1,j} = \pi_{i,b+1} = 0 \). The piecewise-linear toggle \( \tau_{i,j} \) at the \((i,j)\)-th entry is an involution that sends \( \pi \) to a new plane partition \( \tau_{i,j} \pi \) such that

\[
(\tau_{i,j} \pi)_{k,l} := \begin{cases} 
\pi_{k,l} & \text{if } (k,l) \neq (i,j), \\
\max \{ \pi_{i,j+1}, \pi_{i+1,j} \} + \min \{ \pi_{i-1,j}, \pi_{i,j-1} \} - \pi_{i,j} & \text{if } (k,l) = (i,j).
\end{cases}
\] (1.2)

Let \( \eta \) be the sequence of piecewise-linear toggles that hits each entry exactly once in the order from bottom to top and from left to right, that is,

\[
\eta = \nu \circ \nu_{b-1} \circ \cdots \circ \nu_1, \quad \text{where } \nu_j = \tau_{1,j} \circ \tau_{2,j} \circ \cdots \circ \tau_{a,j}.
\]

For example, applying \( \eta \) to \( \begin{pmatrix} 3 & 2 & 2 \\ 3 & 1 & 0 \end{pmatrix} \in P(2,3,6) \) yields

\[
\begin{array}{ccc}
3 & 2 & 2 \\
3 & 1 & 0 \\
\end{array} \xrightarrow{\tau_{2,1}} \begin{array}{ccc}
3 & 2 & 2 \\
1 & 1 & 0 \\
\end{array} \xrightarrow{\tau_{1,1}} \begin{array}{ccc}
5 & 2 & 2 \\
1 & 1 & 0 \\
\end{array} \xrightarrow{\tau_{2,2}} \begin{array}{ccc}
5 & 2 & 2 \\
1 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{ccc}
5 & 5 & 2 \\
1 & 0 & 0 \\
\end{array} \xrightarrow{\tau_{2,3}} \begin{array}{ccc}
5 & 5 & 2 \\
1 & 0 & 0 \\
\end{array} \xrightarrow{\tau_{1,3}} \begin{array}{ccc}
5 & 5 & 3 \\
1 & 0 & 0 \\
\end{array}
\]

Our first main result is as follows.

**Theorem 1.3.** The action \( \eta \) on \( P(a,b,c) \) is of order \( n = a + b \). The triple \( (P(a,b,c), \eta, M_{a,b,c}(q)) \) exhibits the cyclic sieving phenomenon.

The proof of Theorem 1.3 involves representations of \( \text{GL}_n \). Recall the \( a^{th} \) fundamental weight of \( \text{GL}_n \)

\[
\omega_a := (1,1,\ldots,1,0,\ldots,0).
\]

Let \( V_{\omega_a} \) be the irreducible representation of \( \text{GL}_n \) of highest weight \( c\omega_a \). Let

\[
C_a := \begin{pmatrix} 0 & (-1)^{a-1} \\ \text{Id}_{n-1} & 0 \end{pmatrix} \in \text{GL}_n.
\] (1.4)

Theorem 1.3 is an easy consequence of the following result.
Theorem 1.5. The representation $V_{cωa}$ admits a natural basis $Θ(a,b,c)$ which is

- permuted under the action of $C_a$, and
- compatible with the weight decomposition of $V_{cωa}$.

There is an equivariant bijection between $Θ(a,b,c)$ under $C_a$ and the set $P(a,b,c)$ under $η$.

1.3 Cluster Duality for Grassmannian

Cluster algebras are a class of commutative algebras introduced by Fomin and Zelevinsky [FZ02]. Their geometric counterparts form a family of log Calabi-Yau varieties called cluster varieties.

A cluster ensemble\(^1\) is a pair $(S, X)$ of cluster varieties associated to an equivalent class of skew-symmetric matrices (or quivers) introduced by Fock and Goncharov [FG09]. The variety $S$ is equipped with an exceptional class $\{α\}$ of coordinate charts called $K_2$ clusters. A rational function of $S$ is called a universal Laurent polynomial if it can be expressed as a Laurent polynomial in every $α$. The ring $up(S)$ of universal Laurent polynomials of $S$ coincides with the upper cluster algebra of [BFZ05]. The variety $X$ is equipped with an exceptional class $\{χ\}$ of coordinate charts called Poisson clusters. Let $up(X)$ be the ring of universal Laurent polynomials in every $χ$. The cluster modular group $G$ is a discrete group acting on $S$ and $X$ that respects the cluster structures.

The Duality Conjecture of Fock and Goncharov [FG09] asserts that the ring $up(S)$ admits a natural basis $G$-equivariantly parametrized by the $Z$-tropical points of $X$, and vice versa. Cluster duality can be viewed as a manifestation of mirror symmetry between $S$ and $X$. For example, the cluster duality for moduli spaces of local systems has been investigated in [GS15, GS18].

The present paper focuses on the cluster duality for Grassmannians.

In detail, we introduce a pair of spaces, i.e., the decorated Grassmannian $Gr_a(n)$ and the decorated configuration space $Conf_n(a)$, both of which are variants of the Grassmannian $Gr_a(n)$.

The decorated Grassmannian $Gr_a(n)$ is essentially the affine cone over $Gr_a(n)$, Its coordinate ring $O(Gr_a(n))$ coincides with the homogeneous coordinate ring of $Gr_a(n)$. The $Gr_a(n)$ admits a particular divisor, whose compliment is denoted by $Gr_a^X(n)$. A result of Scott [Sco06] implied that $Gr_a^X(n)$ is equipped with a cluster $K_2$ structure, and the coordinate ring $O(Gr_a^X(n))$ coincides with an upper cluster algebra $up(S)$.

The decorated configuration space $Conf_n(a)$ parametrizes PGL($V$)-orbits of $n$-many lines in an $a$-dimensional vector space $V$ together with a linear isomorphism between every pair of cyclic neighboring lines. After imposing a consecutive general position condition, we obtain a smooth subvariety $Conf_n^X(a)$. We prove that

Theorem 1.6 (Theorem 2.36). The variety $Conf_n^X(a)$ is equipped with a cluster Poisson structure. Its coordinate ring $O(Conf_n^X(a))$ coincides with the algebra $up(X)$ of universal Laurent polynomials.

Combining with work of Gross, Hacking, Keel, and Kontsevich [GHKK18], we prove the following Theorem in Section 5.1.

Theorem 1.7. The pair $(Gr_a^X(n), Conf_n^X(a))$ admits a natural structure of cluster ensemble. The Duality Conjecture of Fock-Goncharov is true in this case, that is, the coordinate ring $O(Gr_a^X(n))$ admits a cluster modular group equivariant parametrization by the $Z$-tropical set of $Conf_n^X(a)$, and vice versa.

\(^1\)See Appendix A for a brief review on cluster ensemble.
In Section 2, we introduce several natural functions and maps on $\mathcal{G}_a^\times(n)$ and $\mathcal{C}onf_n^\times(a)$, which are summarized as follows:

\[
\begin{array}{lcl}
\text{decorated Grassmannian } & \mathcal{G}_a^\times(n); & \\
\text{free recaling } & \mathbb{G}_m \text{ action;} & \\
\text{boundary divisor } & D = \bigcup_i D_i; & \\
\text{action by a maximal torus } & T \subset \mathbb{G}_n; & \\
\text{twisted cyclic rotation } & C_a. & \\
\end{array}
\quad\longleftrightarrow\quad
\begin{array}{lcl}
\text{decorated configuration space } & \mathcal{C}onf_n^\times(a); & \\
\text{twisted monodromy } & P; & \\
\text{potential function } & W = \sum_i \vartheta_i; & \\
\text{weight map } & M : \mathcal{C}onf_n^\times(a) \to T^\vee; & \\
\text{cyclic rotation } & R. & \\
\end{array}
\]

We investigate the natural cluster correspondence between the ingredients in the above dictionary. In particular, the potential $W$ exhibits an explicit cyclic symmetry. It is essentially equivalent to the potential of Grassmannian considered in [EHX97, MR13, RW17]. We identify $W$ with the sum of theta functions associated to frozen vertices under the framework of [GHKK18]. As a consequence, we prove Theorem 1.5.

Marsh and Rietsch constructed a $B$-model for the Grassmannian in [MR13], which is of the form $(\text{Gr}^\times_a(n) \times \mathbb{G}_m, W_q)$. Rietsch and Williams proved in [RW17] that this $B$-model is an example of a cluster dual space of the Grassmannian. In contrast to their approach, our approach is more geometric and is purely motivated by the associated cluster structures. We include a section in the appendix describing the connection between our version of cluster duality and the version considered by Rietsch and Williams.

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2 Main Definitions

2.1 Decorated Grassmannian

Let $V$ be an $n$-dimensional vector space and $V^*$ be its dual. The Grassmannian $\text{Gr}_a(V^*)$ parametrizes $a$-dimensional subspaces in $V^*$. The decorated Grassmannian $\mathcal{G}_a(V^*)$ parametrizes pairs $(W^*, f^*)$, where $W^*$ is an $a$-dimensional subspace of $V^*$ and $f^* \in \bigwedge^a W^*$ is a nonzero $a$-form of $W$.

There is a free right $\mathbb{G}_m$-action on $\mathcal{G}_a(V^*)$ defined via rescaling the $a$-forms

\[
(W^*, f^*) \cdot t := (W^*, tf^*).
\]

(2.1)

It induces a left $\mathbb{G}_m$-action on $\mathcal{O}(\mathcal{G}_a(V^*))$. Irreducible representations of $\mathbb{G}_m$ are 1-dimensional and are classified by integers. Denote by $\mathcal{O}(\mathcal{G}_a(V^*))_c$ the eigenspace in $\mathcal{O}(\mathcal{G}_a(V^*))$ that is of weight $c$ with respect to the $\mathbb{G}_m$-action.

By the natural pairing between $\bigwedge^a V^*$ and $\bigwedge^a V$, every $g \in \bigwedge^a V$ gives rise to a regular function

\[
\Delta_g : \mathcal{G}_a(V^*) \to \mathbb{A}^1, \quad \Delta_g(W^*, f^*) := \langle f^*, g \rangle.
\]

The ring $\mathcal{O}(\mathcal{G}_a(V^*))$ is generated by $\Delta_g$. Under the $\mathbb{G}_m$-action, we get

\[
(t, \Delta_g)(W^*, f^*) = \Delta_g(W^*, tf^*) = \Delta_{tg}(W^*, f^*) = t\Delta_g(W^*, f^*).
\]

(2.2)

Therefore we conclude the following statement.
Proposition 2.3. The map $g \mapsto \Delta_g$ is an isomorphism between $\bigwedge^a V$ and $O(\mathcal{G}_a(V^*))_1$, and
\[
O(\mathcal{G}_a(V^*)) = \bigoplus_{c \geq 0} O(\mathcal{G}_a(V^*))_c.
\] (2.4)

The group GL(V) acts on V as well as its dual space $V^*$, and hence on the decorated Grassmannian $\mathcal{G}_a(V^*)$ and on the ring $O(\mathcal{G}_a(V^*))$. The representation $V_{\omega_a}$ of GL(V) is by definition $\bigwedge^a V$ and therefore is isomorphic to $O(\mathcal{G}_a(V^*))_1$ by Proposition 2.3. In general, (2.4) decomposes $O(\mathcal{G}_a(V^*))$ into irreducible representations of GL(V) with
\[
O(\mathcal{G}_a(V^*))_c \cong V_{\omega_a}.
\] (2.5)

From now on, we fix a basis $\{e_1, \ldots, e_n\}$ of V and identify V with the vector space $\mathbb{K}^n$. We abbreviate $\mathcal{G}_a(V^*)$ to $\mathcal{G}_a(n)$ and $\mathcal{G}_a(V^*)$ to $\mathcal{G}_a(n)$. Every a-element subset $I = \{i_1, \ldots, i_a\} \in \binom{[n]}{a}$ gives rise to a regular function $\Delta_I := \Delta_{e_I}$ where $e_I$ denotes the wedge product of $e_{i_1}, \ldots, e_{i_a}$ taken in ascending order (e.g., $\Delta_{[5,7,4]} := \Delta_{e_5 \wedge e_7 \wedge e_4}$). The functions $\Delta_I$ are called Plücker coordinates.

Let $D_i$ be the vanishing locus of the Plücker coordinate $\Delta_{\{i, i+1, \ldots, i+a-1\}}$. Let $\mathcal{G}_a^\times(n)$ denote the complement of $D := \cup_{i=1}^n D_i$. Since $\mathcal{G}_a^\times(n) \subset \mathcal{G}_a(n)$, we have $O(\mathcal{G}_a(n)) \subset O(\mathcal{G}_a^\times(n))$. The image of $D$ under the projection from $\mathcal{G}_a(n)$ to $\mathcal{G}_a(n)$ is an anticanonical divisor of $\mathcal{G}_a(n)$, of which the complement is denoted by $\mathcal{G}_a^\times(n)$.

Let $\text{Mat}_{a,n}^\times$ be the space of $a \times n$ matrices with column vectors $v_i$ such that every collection $\{v_i, v_{i+1}, \ldots, v_{i+a-1}\}$ of $a$-many cyclic consecutive column vectors is linearly independent. The group SL$_a$ acts freely on $\text{Mat}_{a,n}^\times$ by matrix multiplication on the left.

Lemma 2.6. The space $\mathcal{G}_a^\times(n)$ is canonically isomorphic to the quotient space $\text{SL}_a \backslash \text{Mat}_{a,n}^\times$.

Remark 2.7. Under the above isomorphism, the coordinates $\Delta_I$ are identified with the minors of $I$-columns in an $a \times n$ matrix.

Proof. Let $(W^*, f^*) \in \mathcal{G}_a^\times(n)$. Every $W^* \subset (\mathbb{K}^n)^*$ naturally induces a surjection $\pi : \mathbb{K}^n \to W$. Let $v_i := \pi(e_i)$ be the image of the basis element $e_i$ under $\pi$. The coordinate $\Delta_{i+1, \ldots, i+a} \neq 0$ is equivalent to the linear independence of $\{v_{i+1}, \ldots, v_{i+a}\}$. Up to the action of SL$_a$, there is a unique choice of linear isomorphisms from $W$ to $\mathbb{K}^a$ mapping the a-form $f^*$ to the standard a-form on $\mathbb{K}^a$. Hence we get a configuration in $\text{SL}_a \backslash \text{Mat}_{a,n}^\times$. It is easy to see that such a map is bijective. \[\square\]

Let $T = (\mathbb{G}_m)^n$ be the maximal torus of GL$_n$ consisting of invertible diagonal matrices. It acts on the right of $\text{Mat}_{a,n}^\times$ by rescaling the column vectors $v_1, \ldots, v_n$. Since $\mathcal{G}_a^\times(n) \cong \text{SL}_a \backslash \text{Mat}_{a,n}^\times$, the $T$-action on $\text{Mat}_{a,n}^\times$ descends to a $T$-action on the decorated Grassmannian $\mathcal{G}_a^\times(n)$.

Define the linear transformation $C_a$ on $V$ such that
\[
C_a(e_i) := \begin{cases} 
  e_{i-1} & \text{if } i \neq 1; \\
  (-1)^{a-1} e_n & \text{if } i = 1.
\end{cases}
\]

It induces a twisted cyclic rotation on $\mathcal{G}_a(n)$ still denoted by $C_a$.

To summarize, we obtain the following data
\[
\left\{ \begin{array}{l}
  \text{the decorated grassmannian } \mathcal{G}_a^\times(n);
  \\
  \text{the free } \mathbb{G}_m \text{-action on } \mathcal{G}_a^\times(n) \text{ by rescaling the } a \text{-form};
  \\
  \text{the boundary divisor } D = \bigcup_i D_i;
  \\
  \text{the } T \text{-action on } \mathcal{G}_a^\times(n);
  \\
  \text{the twisted cyclic rotation } C_a \text{ on } \mathcal{G}_a^\times(n).
\end{array} \right. 
\] (2.8)
2.2 Decorated Configuration Space

Let $W$ be a vector space of dimension $a$. The *configuration space* $\text{Conf}_n(W)$ parametrizes the $\text{PGL}(W)$-orbits of $n$ many lines in $W$, i.e.,

$$\text{Conf}_n(W) := \text{PGL}(W) \backslash \left( \prod_n \mathbb{P}W \right).$$

As on Figure 1, a *decorated configuration* is a $\text{PGL}(W)$-orbit of $n$ lines in $W$ together with linear isomorphisms $\phi_i : l_i \to l_{i-1}$ for each pair of neighboring lines. The decorated configuration space is

$$\mathcal{Conf}_n(W) := \text{PGL}(W) \backslash \left\{ \begin{array}{l}
\text{1-dimensional subspaces } l_1, \ldots, l_n \subset W \\
\text{and linear isomorphisms } \phi_i : l_i \to l_{i-1}
\end{array} \right\}.$$

A decorated configuration is denoted as $[\phi_1, l_1, \ldots, \phi_n, l_n]$. We frequently omit the subscript of $\phi_i$.

![Figure 1: A decorated configuration in $\mathcal{Conf}_6(W)$.

Two vector spaces of the same dimension are isomorphic up to choices of bases. Since the action of $\text{PGL}(W)$ has been quotient out, the configuration spaces of $n$ lines in vector spaces of the same dimension are canonically isomorphic and so are the decorated configuration spaces. Therefore we may abbreviate $\text{Conf}_n(W)$ and $\mathcal{Conf}_n(W)$ to $\text{Conf}_n(a)$ and $\mathcal{Conf}_n(a)$ respectively.

Let $\mathcal{Conf}_n^\times(a)$ be the subspace of $\text{Conf}_n(a)$ consisting of configurations $[l_1, \ldots, l_n]$ such that every collection $\{l_{i+1}, \ldots, l_{i+a}\}$ of $a$-many cyclic consecutive lines is linearly independent. The subspace $\mathcal{Conf}_n^\times(a)$ of $\mathcal{Conf}_n^\times(a)$ is defined in the same way.

Let $[\phi_1, l_1, \ldots, \phi_n, l_n] \in \mathcal{Conf}_n^\times(a)$. Let us compose $\phi$ in anti-clockwise order as on Figure 1. Let $(-1)^{a-1} P$ be the rescaling factor of the isomorphism $\phi_i \circ \cdots \circ \phi_n \circ \phi_1 \circ \cdots \phi_i$ of $l_i$. Note that $P$ is independent of the initial index $i$ chosen. We get a regular projection called *twisted monodromy* $P : \mathcal{Conf}_n^\times(a) \to \mathbb{G}_m$. \hfill (2.9)

Pick a non-zero vector $v_i \in l_i$ for each $1 \leq i \leq n$. Let $\vartheta_i$ be the scalar such that

$$\phi(v_{i-a+1}) - \vartheta_i v_{i-a+1} \in \text{Span} \{ l_{i-a+2}, \ldots, l_i \}. \hfill (2.10)$$

Note that $\vartheta_i$ is independent of the choices of $v_i$. We define the *potential function* on $\mathcal{Conf}_n^\times(a)$ to be the regular function

$$W = \sum_{i=1}^n \vartheta_i : \mathcal{Conf}_n^\times(a) \to \mathbb{A}^1. \hfill (2.11)$$

---

The definition of decorated configuration is motivated by an idea of A.B. Goncharov on pinnings.
Again pick a non-zero vector $v_i \in l_i$ for each $1 \leq i \leq n$; fix a volume form $\omega$ of the vector space $W$. For each $1 \leq k \leq n$, we define
\[ M_k = \frac{\omega (\phi (v_{k-a+1}) \wedge \cdots \wedge \phi (v_k))}{\omega (v_{k-a+1} \wedge \cdots \wedge v_k)}. \quad (2.12) \]
It is not hard to see that $M_k$ does not depend on the choices of $\omega$ and $v_i$. Therefore we obtain a weight map
\[ M : \mathcal{C}onf_n^\times (a) \to T^\vee \]
\[ [\phi_1, l_1, \ldots, \phi_n, l_n] \mapsto (M_1, \ldots, M_n) \quad (2.13) \]
where $T^\vee \cong (\mathbb{G}_m)^n$ is the dual torus of the maximal torus $T \subset \text{GL}_n$.

Lastly, there is an order $n$ biregular map
\[ R : \mathcal{C}onf_n^\times (a) \to \mathcal{C}onf_n^\times (a) \]
\[ [\phi_1, l_1, \phi_2, l_2, \ldots, \phi_n, l_n] \mapsto [\phi_n, l_n, \phi_1, l_1, \ldots, \phi_{n-1}, l_{n-1}] \quad (2.14) \]

To summarize, we get the following data
\[
\begin{cases}
\text{the decorated configuration space } \mathcal{C}onf_n^\times (a); \\
\text{the twisted monodromy } P : \mathcal{C}onf_n^\times (a) \to \mathbb{G}_m; \\
\text{the potential function } W = \sum \psi_i; \\
\text{the weight map } M : \mathcal{C}onf_n^\times (a) \to T^\vee; \\
\text{the cyclic rotation } R \text{ on } \mathcal{C}onf_n^\times (n). 
\end{cases} \quad (2.15)
\]

### 2.3 Maps among the Decorated Spaces

Recall the $n$-dimensional vector space $V$ with a basis $\{e_1, \ldots, e_n\}$. Let $\hat{l}_i$ be the line containing $e_i$ and let $\hat{\phi}_i : \hat{l}_i \to l_{i-1}$ be the linear isomorphism such that
\[ \hat{\phi}_i (e_i) := \begin{cases} 
  e_{i-1} & \text{if } i \neq 1; \\
  (-1)^{a-1} e_n & \text{if } i = 1.
\end{cases} \]
Every $W^* \in \text{Gr}_a^\times (n) \cong \text{Gr}_a^\times (V^*)$ induces a projection $\pi$ from $V$ to $W$. The lines $l_i := \pi (\hat{l}_i)$ in $W$ satisfy the consecutive general position condition, and $\hat{\phi}_i$ descend to isomorphisms $\phi_i : l_i \to l_{i+1}$. Hence we obtain $[\phi_1, l_1, \ldots, \phi_n, l_n] \in \mathcal{C}onf_n^\times (W) \cong \mathcal{C}onf_n^\times (a)$. It defines a natural injective map
\[ \text{Gr}_a^\times (n) \to \mathcal{C}onf_n^\times (a), \quad (2.16) \]
whose image consists of decorated configurations of twisted monodromy $P = 1$.

There is a surjective map $\mathcal{F}r_\alpha^\times (n) \to \text{Gr}_a^\times (n)$ defined by forgetting the $a$-form.

**Definition 2.17.** Denote by $\overset{\sim}{\text{Con}}f_n^\times (a)$ the space of $\text{SL}_a$-orbits of
\[ [\phi_1, v_1, \phi_2, v_2, \ldots, \phi_n, v_n] \quad (2.18) \]
where $v_i$ are vectors in $\mathbb{K}^a$ satisfying consecutive general position condition, and each $\phi_i$ is a linear isomorphism from the line spanning $v_i$ to the line spanning $v_{i-1}$. 

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By Lemma 2.6, the space $\mathcal{G}_a \times (n)$ consist of configurations $[v_1, \ldots, v_n]$ satisfying cyclic general position condition. Define the following map

$$\mathcal{G}_a \times (n) \to \widetilde{Conf}_n^\times (a)$$

$$[v_1, \ldots, v_n] \mapsto [\phi_1(v_1), v_1, \ldots, \phi_n, v_n],$$

with the isomorphisms $\phi$ defined by $\phi_1(v_1) := -v_n$ and $\phi_i(v_i) := v_{i-1}$ for the other $i$'s. It is not hard to see that this map is injective.

There is a surjective map $\widetilde{Conf}_n^\times (a) \to \text{Conf}_n^\times (a)$ by replacing each vector $v_i$ by its spanning line $l_i$. There is also a surjective map $\text{Conf}_n^\times (a) \to \text{Conf}_n^\times (a)$ defined by forgetting the isomorphisms $\phi_i$.

Combining the aforementioned maps, we get the following commutative diagram.

\[
\begin{array}{ccc}
\mathcal{G}_a \times (n) & \longrightarrow & \widetilde{Conf}_n^\times (a) \\
\downarrow & & \downarrow \\
\text{Conf}_n^\times (a) & \longrightarrow & \text{Conf}_n^\times (a)
\end{array}
\] (2.19)

### 2.4 Cluster Structures

The pair $(\mathcal{G}_a \times (n), \text{Conf}_n^\times (a))$ admits a natural structure of cluster ensemble associated to minimal bipartite graphs of rank $a$ on a disk with $n$ marked points on the boundary. In this section we focus on one particular minimal bipartite graph $\Gamma_{a,n}$ for each pair of parameters $(a, n)$ as follows.

![Minimal Bipartite Graph](image)

From a minimal bipartite graph $\Gamma$ we obtain a quiver $Q_\Gamma$ by the following four-step procedure.

\[\text{A rapid review on minimal bipartite graphs has been included in the appendix.}\]
• Assign a vertex to each face of $\Gamma$.

• For each black vertex of $\Gamma$, draw a clockwise cycle of arrows as follows.

• Remove a maximal subset of 2-cycles.

• Freeze the vertices corresponding to boundary faces of $\Gamma$.

For example, the minimal bipartite graph $\Gamma_{a,n}$ gives rise to a quiver $Q_{a,n}$ as follows.

Here the vertex assigned to the top-left face is indexed by $(0,0)$. The other faces of $\Gamma_{a,n}$ form an $a \times b$ grid. Their corresponding vertices of $Q_{a,n}$ are indexed in the same way as matrix entries. The gray vertices are frozen. Denote by $I$ the set of vertices of $Q_{a,n}$ and by $I^{uf}$ the set of unfrozen vertices of $Q_{a,n}$. The exchange matrix $\varepsilon$ of $Q_{a,n}$ is defined to be an $I \times I$ matrix with entries

$$
\varepsilon_{fg} = \# \{ g \to f \} - \# \{ f \to g \}.
$$

For simplicity, we will also use an integer $i \in \{1, \ldots, n\}$ to denote the frozen vertex corresponding to the boundary face lying between $i$ and $i+1$. In other words,

$$
frozen \ vertex \ i = \begin{cases} 
(i, b) & \text{if } 1 \leq i \leq a; \\
(a, n-i) & \text{if } a \leq i < n; \\
(0, 0) & \text{if } i = n.
\end{cases}
$$

Let $(\mathcal{A}_{a,n}, \mathcal{X}_{a,n})$ be the cluster ensemble associated to $Q_{a,n}$. See (5.31) for its rigid definition. Let $\{A_{i,j}\}_{(i,j)\in I}$ be the $K_2$ cluster of $\mathcal{A}_{a,n}$ associated to the quiver $Q_{a,n}$ and let $\{X_{i,j}\}_{(i,j)\in I}$ be the Poisson cluster of $\mathcal{X}_{a,n}$ associated to $Q_{a,b}$. Abusing notation, we will frequently write $f$ instead of $(i,j) \in I$ with $f$ being the face of $\Gamma_{a,n}$ corresponding to the vertex $(i,j)$ of $Q_{a,n}$. 
Cluster $K_2$ structure on $\mathcal{G}_a(n)$. Associate to each vertex $(i,j)$ of $Q_{a,n}$ is an $a$-element set

$$I(i,j) := \{b - j + 1, \ldots, b - j + i, b + i + 1, \ldots, n\}.$$  \hfill (2.20)

Recall the Plücker coordinates $\Delta_I$ of $\mathcal{G}_a(n)$. By defining

$$A_{i,j} := \Delta_I(i,j),$$  \hfill (2.21)

we get a birational map

$$\psi : \mathcal{G}_a(n) \dashrightarrow \mathcal{A}_{a,n}.$$  \hfill (2.22)

Scott [Sco06, Theorem 3] showed that the pull-back map $\psi^*$ gives an algebra isomorphism between $\mathcal{O}(\mathcal{G}_a(n))$ and the ordinary cluster algebra defined by the quiver $Q_{a,n}$; by allowing ourselves to invert the frozen variables we generalize his result to the following Theorem.

**Theorem 2.23.** The pull-back map $\psi^*$ is an algebra isomorphism between the upper cluster algebra

$$\text{up}(\mathcal{A}_{a,n}) := \mathcal{O}(\mathcal{A}_{a,n})$$

and

$$\mathcal{O}(\mathcal{G}_a(n)).$$

Cluster Poisson structure on $\text{Conf}_{a,n}$. Let $Q_{a,n}^{\text{uf}}$ denote the full subquiver of $Q_{a,n}$ spanned by vertices in $I_{\text{uf}}$. Let $(\mathcal{A}_{a,n}^{\text{uf}}, \mathcal{X}_{a,n}^{\text{uf}})$ be the cluster varieties associated to $Q_{a,n}^{\text{uf}}$.

There is a canonical regular map $p : \mathcal{A}_{a,n} \to \mathcal{X}_{a,n}^{\text{uf}}$ defined on the cluster coordinate charts associated to $Q_{a,n}$ such that

$$p^* (X_g) = \prod_{f \in I} A_{f,g}^{\text{uf}}, \quad \forall g \in I_{\text{uf}}.$$  \hfill (2.24)

Recall the surjective map $\mathcal{G}_a(n) \to \text{Conf}_{a,n}$ in (2.19). We define a rational map

$$\psi : \text{Conf}_{a,n} \dashrightarrow \mathcal{X}_{a,n}^{\text{uf}}$$  \hfill (2.25)

by first taking a lift from $\text{Conf}_{a,n}$ to $\mathcal{G}_a(n)$, mapping over to $\mathcal{A}_{a,n}$ via the birational equivalence (2.22), and then mapping it down to $\mathcal{X}_{a,n}^{\text{uf}}$ by the canonical $p$ map. The map (2.25) is well-defined and does not depend on the lift, because $\psi^* (X_g)$ for $g \in I_{\text{uf}}$ is a ratio of Plücker coordinates with the same collection of indices (counted with multiplicity) in the numerator and in the denominator. It is known that (2.25) is a birational equivalence (see for example [Wen18, Cor. 5.1.9]).

**Lemma 2.26.** The restricted exchange matrix $\epsilon|_{I \times I_{\text{uf}}}$ of $Q_{a,n}$ is of full-rank.

*Proof.* It suffices to show that the pull-back map $p^*$ via (2.24) is injective, which is equivalent to proving that the corresponding map $p : \mathcal{A}_{a,n} \to \mathcal{X}_{a,n}^{\text{uf}}$ is surjective.

By definition, we have the following commutative diagram

$$\begin{array}{ccc}
\mathcal{G}_a(n) & \xrightarrow{\psi} & \mathcal{A}_{a,n} \\
p & & p \\
\text{Conf}_{a,n} & \xrightarrow{\psi} & \mathcal{X}_{a,n}^{\text{uf}}
\end{array}$$  \hfill (2.27)

---

4The set is determined by the zig-zag strands of the minimal bipartite graph $\Gamma_{a,n}$. See Appendix B for more details.
with the map \( p \) on the left as defined in (2.19). Since both of the horizontal maps \( \psi \) are birational and the map \( p \) on the left is surjective, we know that the map \( p \) on the right is dominant. Note that the restriction of the map \( p \) on the right to each seed torus is a dominant morphism induced by a linear map between their character lattices, which forces it to be surjective. Therefore the map \( p \) on the right is surjective.

**Proposition 2.28.** The birational equivalence \( \psi : \text{Conf}_n^X(a) \rightarrow \mathcal{X}_{a,n}^{uf} \) induces an algebra isomorphism \( \psi^* \) between \( \mathcal{X}_{a,n}^{uf} : = \mathcal{O}(\mathcal{X}_{a,n}^{uf}) \) and \( \mathcal{O}(\text{Conf}_n^X(a)) \).

**Proof.** Since \( \psi \) map is birational, its pull-back map \( \psi^* \) is an isomorphism between fields of rational functions on \( \text{Conf}_n^X(a) \) and \( \mathcal{X}_{a,n}^{uf} \). Let \( F \) be a rational function on \( \mathcal{X}_{a,n}^{uf} \). It suffices to show that

\[
F \text{ is regular on } \mathcal{X}_{a,n}^{uf} \iff \psi^*(F) \text{ is regular on } \text{Conf}_n^X(a).
\]

Let us make use of the commutative diagram (2.27) again. Note that both of vertical regular maps \( p \) are surjective. A rational function on a space downstairs is regular if and only if its pull-back is a regular function on the corresponding space upstairs. Therefore it suffices to show that

\[
p^*(F) \text{ is regular on } \mathcal{X}_{a,n} \iff p^* \circ \psi^*(F) = \psi^* \circ p^*(F) \text{ is regular on } \mathcal{X}_a^X(n),
\]

which is a direct consequence of Theorem 2.23. □

**Cluster \( K_2 \) structure on \( \text{Conf}_n^X(a) \).** The quiver \( \hat{\mathcal{Q}}_{a,n} \) is obtained from \( \mathcal{Q}_{a,n} \) by adding, for each frozen vertex \( i \), a new frozen vertex \( i' \) and a new arrow from \( i \) to \( i' \). Denote the set of vertices of \( \hat{\mathcal{Q}}_{a,n} \) by \( \hat{I} \) and the exchange matrix of \( \hat{\mathcal{Q}}_{a,n} \) by \( \hat{\varepsilon} \). For instance, the quiver \( \hat{\mathcal{Q}}_{3,6} \) is as follows:

\[
\begin{array}{cccccc}
6' & \square & \rightarrow & \bullet & \rightarrow & \bullet & \rightarrow & \square & \rightarrow & 1' \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\bullet & \rightarrow & \bullet & \rightarrow & \bullet & \rightarrow & \square & \rightarrow & 2' \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
5' & 4' & 3' & 2' & 1' & 6'
\end{array}
\]

(2.29)

Let \( \hat{\mathcal{X}}_{a,n} \) be the cluster \( K_2 \) variety associated to the quiver \( \hat{\mathcal{Q}}_{a,n} \).

Let \( [\phi_1, v_1, \ldots, \phi_n, v_n] \in \text{Conf}_n^X(a) \). Define the scaling factors \( \lambda_i \) such that

\[
\phi(v_{i+1}) = \begin{cases} 
\lambda_i v_i & \text{if } i \neq n \\
(-1)^{a-1} \lambda_n v_n & \text{if } i = n.
\end{cases}
\]

We define a rational map

\[
\tilde{\psi} : \text{Conf}_n^X(a) \rightarrow \hat{\mathcal{X}}_{a,n}
\]
by identifying cluster variables of \( \mathcal{A}_{a,n} \) with functions on \( \mathcal{C}_{n}^\times \) as follows

\[
A_f := \begin{cases} 
\Delta_{(1,1)} & \text{if } f \text{ is a vertex } (i,j) \text{ of } Q_{a,n}; \\
\lambda_i \Delta_{(i-1,i+1,\ldots,i)} & \text{if } f \text{ is a new added vertex } i'. 
\end{cases}
\] (2.31)

**Corollary 2.32.** The map \( \tilde{\psi} : \mathcal{C}_{n}^\times \to \mathcal{A}_{a,n} \) is birational. Its pull-back map \( \tilde{\psi}^\ast \) is an algebra isomorphism between the upper cluster algebra \( \text{up} \left( \mathcal{A}_{a,n} \right) := O \left( \mathcal{A}_{a,n} \right) \) and \( O \left( \mathcal{C}_{n}^\times \right) \).

**Proof.** Let \( H = (G_m)^n \) be the split algebraic torus with coordinates \((A_1',\ldots,A_{n'})\). Note that there is no arrow between the vertices \( i' \) and the unfrozen vertices in \( Q_{a,n} \). Therefore the variables \( A'_i \) will not affect the cluster mutations. Hence we get

\[
\mathcal{A}_{a,n} \cong \mathcal{A}_{a,n} \times H.
\]

Every configuration in \( \mathcal{C}_{n}^\times \) consists of two pieces of data: a SL\(_a\)-orbit of vectors \([v_1,\ldots,v_n]\) satisfying the cyclic general position condition, and the isomorphisms \( \phi_i \) between the lines \( l_i \) spanned by the vectors \( v_i \). By Lemma 2.6, the data \([v_1,\ldots,v_n]\) is captured by a point in \( \mathcal{F}_{a,n} \). The isomorphisms \( \phi_i \) is captured by the scaling factors \( \lambda_i \) which by (2.31) is computed from \( A'_i \) together with the frozen variables of the corresponding point in \( \mathcal{F}_{a,n} \). Therefore it is easy to see that

\[
\mathcal{C}_{n}^\times \cong \mathcal{F}_{a,n} \times H.
\]

After the above identifications, the map \( \tilde{\psi} = \psi \times \text{Id} \). The Corollary follows from Theorem 2.23. \( \square \)

**Cluster Poisson structure on \( \mathcal{C}_{n}^\times \).** Analogous to the map (2.24), there is a canonical map \( p : \mathcal{A}_{a,n} \to \mathcal{X}_{a,n} \) defined on the cluster coordinate charts associated to \( Q_{a,n} \) by

\[
p^\ast (X_g) := \prod_{f \in I} A_{f,g}, \quad \forall g \in I.
\] (2.33)

Let \([\phi_1,l_1,\ldots,\phi_n,l_n] \in \mathcal{C}_{n}^\times \). Let us lift it to a configuration in \( \mathcal{C}_{n}^\times \) via picking a nonzero vector \( v_i \in l_i \) for each \( i \). Composing with the map \( p \circ \tilde{\psi} : \mathcal{C}_{n}^\times \to \mathcal{A}_{a,n} \to \mathcal{X}_{a,n} \), we get a rational map

\[
\psi : \mathcal{C}_{n}^\times \to \mathcal{X}_{a,n}
\]

The map \( \psi \) does not depend on the choices of \( v_i \in l_i \). Indeed, if the vertex \((i,j)\) is unfrozen, then \( X_{i,j} \) coincides with the cluster Poisson coordinates \( X_{i,j} \) on \( \mathcal{X}_{a,n} \). For a frozen vertex \( i \), one gets

\[
X_i = \begin{cases} 
\lambda_i \Delta_{(i-1,i+1,\ldots,i-1)} & \text{if } i = 1, \\
\lambda_i \Delta_{(i-1,i+1,\ldots,i-1)} & \text{if } 1 < i < a, \\
\lambda_i \Delta_{(i-1,i+1,\ldots,i-1)} & \text{if } a \leq i \leq n - 1, \\
\lambda_i \Delta_{(i-1,i+1,\ldots,i-1)} & \text{if } i = n,
\end{cases}
\] (2.34)

from which one can verify that \( X_i \) are independent of the choices of \( v_i \in l_i \).
Proposition 2.35. The map $\psi : \text{Conf}_{n}^\times(a) \rightarrow \mathcal{X}_{a,n}$ is a birational equivalence.

Proof. Let $U$ be an open subset of $\text{Conf}_{n}^\times(a)$ consisting of $[\phi_1, l_1, \ldots, \phi_n, l_n]$ such that

- every collection $\{l_1, \ldots, l_n\}$ of $a$-many lines is linearly independent.

Note that the Plücker coordinates on any lift of $U$ are nonzero.

Let $T_{Q_{a,n}} \subset X_{a,n}$ be the algebraic torus corresponding to the cluster chart associated to $Q_{a,n}$. By definition, the map $\psi$ restricted on $U$ is a regular map to $T_{Q_{a,n}}$. Let $(X_f)_{f \in I}$ be a generic point in $T_{Q_{a,n}}$. It suffices to show that it has a unique pre-image in $U$.

Recall that the map $\psi : \text{Conf}_{n}^\times(a) \rightarrow \mathcal{X}_{a,n}$ is birational. After imposing the generic condition, by using the unfrozen part $(X_f)_{f \in I}$, one can uniquely reconstruct a configuration of lines $[l_1, \ldots, l_n]$ satisfying the • condition. Take a representative $v_i \in l_i$ for each $i$. One can use the frozen part $(X_i)_{i=1}^n$ to uniquely reconstruct the isomorphisms $\phi_i : l_i \rightarrow l_i - 1$, since the frozen variables $X_i$ contains the information of $\lambda_{i-1}$ in (2.30). It is easy to see that the isomorphisms $\phi_i$ are independent of $v_i$ chosen. Therefore we obtain a unique configuration in $U$.

Theorem 2.36. The birational equivalence $\psi : \text{Conf}_{n}^\times(a) \rightarrow \mathcal{X}_{a,n}$ induces an algebra isomorphism between $\text{up}(\mathcal{X}_{a,n}) := O(\mathcal{X}_{a,n})$ and $O(\text{Conf}_{n}^\times(a))$.

Proof. Note that we have the following commutative diagram

$$
\begin{array}{ccc}
\text{Conf}_{n}^\times(a) & \xrightarrow{\psi} & \mathcal{X}_{a,n} \\
\downarrow & & \downarrow \\
\text{Conf}_{n}^\times(a) & \xrightarrow{\psi} & \mathcal{X}_{a,n}
\end{array}
$$

The Theorem follows by running the same argument as in the proof of Proposition 2.28.

3 Cluster Nature of Decorated Configuration Space

In Section 2.4, we prove that $O(\mathcal{G}_a(n)) \cong \text{up}(\mathcal{X}_{a,n})$ and $O(\text{Conf}_{n}^\times(a)) \cong \text{up}(\mathcal{X}_{a,n})$ as algebras. From now on, we will identify these algebras of regular functions and their corresponding field of rational functions, i.e., we will think of $\{A_f\}$ as regular functions on $\mathcal{G}_a(n)$ and think of $\{X_f\}$ as rational functions on $\text{Conf}_{n}^\times(a)$. In particular, the cluster $\{X_f\}$ gives rise to a canonical log Poisson structure on $\text{Conf}_{n}^\times(a)$ such that

$$\{X_f, X_g\} = 2\varepsilon_{fg} X_f X_g. \quad (3.1)$$

This section is devoted to studying the cluster natural of three ingredients of $\text{Conf}_{n}^\times(a)$, i.e., the twisted monodromy $P$, the cyclic rotation $R$, and the potential function $W$, in terms of the cluster $\{X_f\}_{f \in I}$ associated to the quiver $Q_{a,n}$. 

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3.1 Twisted Monodromy and Casimir

Proposition 3.2. The twisted monodromy

\[ P = \prod_{f \in I} X_f. \]

The function \( P \) is a Casimir element with respect to the Poisson bracket (3.1), that is, \( \{P, F\} = 0 \) for any rational function \( F \) on \( \mathcal{C}onf_n^x(a) \).

Proof. Let \( [\phi_1, l_1, \ldots, \phi_n, l_n] \in \mathcal{C}onf_n^x(a) \). Let us lift it to a point \( [\phi_1, v_1, \ldots, \phi_n, v_n] \in \mathcal{C}onf_n^x(a) \). By (2.33), we get

\[ \prod_{f \in I} X_f = \prod_{(f,g) \in I \times \tilde{I}} A^{x_{gf}} = \prod_{g \in \tilde{I}} A^{\sum_{f \in I} x_{gf}} \quad (3.3) \]

Note that \( Q_{a,n} \) is made of cycles. Therefore for every \( f \in I \), one has

\[ \sum_{g \in I} x_{gf} = 0. \quad (3.4) \]

Hence the only factors contributing to the product (3.3) are from the extra frozen vertices that expand \( Q_{a,n} \) to \( \tilde{Q}_{a,n} \). Therefore

\[ \prod_f X_f = \prod_{i=1}^n A_i = \prod_{i=1}^n \lambda_i - a \Delta_{\{i-a+1,i-a+2,\ldots,i\}} = \prod_{i=1}^n \lambda_i = P. \]

The last equality follows by comparing (2.30) with the definition of \( P \) in (2.17).

The statement that \( P \) is Casimir is a direct consequence of (3.4) and (3.1). \( \square \)

3.2 Cyclic Rotation and Cluster Transformation

Recall the twisted cyclic rotation \( C_a \) on \( \mathcal{G}r_n^x(n) \). Gekhtman, Shapiro, and Vainshtein proved that that \( C_a \) is a cluster \( K_2 \) automorphism that can be realized by a mutation sequence \( \rho \) (see [GSV10, Page 90]). In this section, we briefly recall the definition of \( \rho \). We show that the cyclic rotation \( R \) on \( \mathcal{C}onf_n^x(a) \) is a cluster Poisson automorphism realized by the same mutation sequence \( \rho \).

The vertices of the quiver \( Q_{a,n} \) are indexed by \( (i,j) \in I \). Let \( \rho \) be a mutation sequence hitting in the unfrozen part of \( Q_{a,n} \) along every column from bottom to top, starting at the leftmost unfrozen column and go all the way to the rightmost unfrozen column, that is,

\[ \rho := \sigma_{b-1} \circ \cdots \circ \sigma_2 \circ \sigma_1, \quad \text{where} \quad \sigma_i := \mu_{(1,i)} \circ \mu_{(2,i)} \circ \cdots \circ \mu_{(a-1,i)}. \quad (3.5) \]

Equivalently, the sequence \( \rho \) may be realized by a sequence of 2-by-2 moves\(^5\) on the minimal bipartite graph \( \Gamma_{a,n} \). Its resulting minimal bipartite graph \( \Gamma'_{a,n} \) is identical to \( \Gamma_{a,n} \) with the non-boundary faces remain in the same places and with all the boundary faces rotated to the neighboring one.

\(^5\)An explicit realization of \( \rho \) as a sequence of 2-by-2 moves has been included in the appendix.
the clockwise direction.

\[ \begin{array}{c}
\Gamma_{a,n} \\
\begin{array}{cccc}
\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\cdot & \cdot & \cdot & \cdot \\
\end{array}
\end{array} \]

\[ \begin{array}{c}
\Gamma'_{a,n} \\
\begin{array}{cccc}
\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\cdot & \cdot & \cdot & \cdot \\
\end{array}
\end{array} \]

One advantage of using minimal bipartite graph is that one may make use of the zig-zag strands to assign an \( a \)-element subset of \( \{1, \ldots, n\} \) to each face of the graph (see Definition 5.36). For example, the \( a \)-element subsets assigned to faces of \( \Gamma_{a,n} \) in \((2.20)\) arise precisely in this way, which in turn determine Plücker coordinates to these faces. Moreover, it is not hard to see that a 2-by-2 move combined with the cluster \( K_2 \) mutation formula yields precisely a Plücker relation (see \((5.37)\) for more details); hence we can conclude that the Plücker coordinates on any bipartite graph obtained from \( \Gamma_{a,n} \) via a sequence of 2-by-2 moves can be computed by using zig-zag strands as well.

Let \( \{A'_{f}\} \) be the \( K_2 \) cluster associated to \( \Gamma'_{a,n} \) after applying the mutation sequence \( \rho \). According the above discussion, the cluster \( \{A'_{f}\} \) is defined by the \( a \)-element sets \((2.20)\) assigned to faces of the minimal bipartite graphs \( \Gamma'_{a,n} \). Using the zig-zag strands on \( \Gamma'_{a,n} \), we find that

\[
I'(i, j) = \{b - j, \ldots, b - j + i - 1, b + i, \ldots, n - 1\},
\]

which yields

\[ A'_{i,j} = \Delta_{I'(i,j)}. \]

As a consequence, we show that the twisted rotation \( C_a \) is realized by the mutation sequence \( \rho \).

Let us apply the mutation sequence \( \rho \) to the extended quiver \( \overrightarrow{Q_{a,n}} \). It is easy to see that the obtained quiver \( \rho \overrightarrow{Q_{a,n}} \) is the same as \( \overrightarrow{Q_{a,n}} \) up to rotations of frozen vertices. For example, if we
start with $\tilde{Q}_{3,6}$ as in (2.29), then $\rho \tilde{Q}_{3,6}$ is as follows.

Let $\tilde{\varepsilon}_{fg}$ be the exchange matrix of the quiver $\rho \tilde{Q}_{a,n}$. The cluster $K_2$ frozen variables remain intact under mutations. Therefore $A'_{ij} = A_{ij}$.

Let $\{X'_{ij}\}$ be the Poisson cluster associated to $\rho Q_{a,n}$. We get

$$X'_g := \prod_f (A'_f)^{\tilde{\varepsilon}_{fg}},$$

As a consequence, the cyclic rotation $R$ on $\mathcal{C}onf_n(a)$ is a cluster Poisson automorphism realized by the mutation sequence $\rho$.

### 3.3 Potential Function

**Proposition 3.8.** In terms of the Poisson cluster $\{X_{i,j}\}$ associated to $Q_{a,n}$, the theta functions $\vartheta_i$ in (2.10) are

$$\vartheta_n = X_{0,0}, \quad \vartheta_a = X_{a,b}, \quad \vartheta_i = \sum_{j=1}^{b} X_{i,b}X_{i,b-1} \cdots X_{i,j} \quad \text{for } 0 < i < a, \quad (3.10)$$

$$\vartheta_i = \sum_{j=1}^{a} X_{a,n-i}X_{a-1,n-i} \cdots X_{j,n-i} \quad \text{for } a < i < n, \quad (3.11)$$

**Remark 3.12.** For (3.10), the terms in $\vartheta_i$ are in bijection with rectangles of all possible lengths across the $i$th row of the quiver $Q_{a,n}$ that ends at the vertex $(i,b)$. For (3.11), the terms in $\vartheta_i$ are in bijection with rectangles of all possible heights across the $(n-i)$th column of $Q_{a,n}$ that rises from
the vertex \((a, n - i)\). For instance, the formulas of \(\vartheta_2\) and \(\vartheta_5\) with \(a = 3\) and \(n = 7\) are as follows:

\[
\begin{align*}
(0, 0) & \\
(1, 1) & \\
(1, 2) & \rightarrow (1, 3) \rightarrow (1, 4) \\
(2, 1) & \rightarrow (2, 2) \rightarrow (2, 3) \rightarrow (2, 4) \\
(3, 1) & (3, 2) \rightarrow (3, 3) \rightarrow (3, 4) \\
\end{align*}
\]

\[\vartheta_2 = X_{2,4} + X_{2,3}X_{2,4} + X_{2,2}X_{2,3}X_{2,4} + X_{2,1}X_{2,2}X_{2,3}X_{2,4},\]

\[\vartheta_5 = \vartheta_{7-2} = X_{3,2} + X_{2,2}X_{3,2} + X_{1,2}X_{2,2}X_{3,2}.\]

Therefore the potential function \(W\) in (2.11) can be expressed as

\[
W = X_{0,0} + \sum_{i=1}^{a-1} \sum_{j=1}^{b} \vartheta_{i,b}X_{i,b} - 1 \ldots X_{i,j} + \sum_{j=1}^{b-1} \sum_{i=1}^{a} \vartheta_{a,j}X_{a-1,j} \ldots X_{i,j}. \tag{3.13}
\]

**Proof.** By definition \(\phi(v_{b+1}) - \vartheta_nv_{b+1} \in \text{Span}(v_{b+2}, \ldots, v_n)\). Hence

\[
0 = \Delta(\phi(v_{b+1}) - \vartheta_nv_{b+1}) \land v_{b+2} \land \ldots \land v_n = \lambda_b\Delta\{b, b+2, \ldots, b+n\} - \vartheta_n\Delta\{b+1, \ldots, n\}.
\]

Therefore

\[
\vartheta_n = \lambda_b\frac{\Delta\{b, b+2, \ldots, n\}}{\Delta\{b+1, \ldots, n\}} = X_{0,0}. \tag{3.14}
\]

Similarly we prove that \(\vartheta_a = X_{a,b}\).

Now let us prove (3.11). The proof of (3.10) goes along the same line.

For \(a < i < n\), let \(k = n - i\). Then \(0 < k < b = n - a\). For \(1 \leq j \leq a\), define the \(a\)-element set

\[
J(j, k) := \{b - k, b - k + 2, \ldots, b - k + j, b + j + 1, \ldots, n\}. \tag{3.15}
\]

Recall the \(a\)-element set \(I(j, k)\). One has the Plücker relation

\[
\Delta I_{(j,k-1)}\Delta I_{(j+1,k)} = \Delta I_{(j,k)}\Delta I_{(j+1,k)} + \Delta I_{(j+1,k)}\Delta J_{(j,k)} + \Delta J_{(j,k)}\Delta I_{(j+1,k)}. \tag{3.16}
\]

Dividing by \(\Delta I_{(j,k-1)}\Delta I_{(j+1,k)}\) on both sides, we get

\[
1 + \frac{\Delta I_{(j+1,k)}\Delta J_{(j,k)}}{\Delta I_{(j,k-1)}\Delta I_{(j+1,k)}} = \frac{\Delta I_{(j,k)}\Delta J_{(j+1,k)}}{\Delta I_{(j,k-1)}\Delta I_{(j+1,k)}}, \tag{3.17}
\]

Let us set

\[
Y_{j,k} := 1 + X_{j,k}(1 + X_{j-1,k}(\ldots (1 + X_{1,k})\ldots))
\]
Let us fix $k$. We prove by induction on $j$ that

$$Y_{j,k} = \frac{\Delta_{I(j,k)} \Delta_{J(j+1,k)}}{\Delta_{I(j,k-1)} \Delta_{I(j+1,k+1)}}.$$  \hspace{1cm} (3.17)

Note that $I(1,k+1) = J(1,k)$. Therefore for $j = 1$, we have

$$Y_{1,k} = 1 + X_{1,k} = 1 + \frac{\Delta_{I(1,k+1)} \Delta_{I(2,k)}}{\Delta_{I(1,k-1)} \Delta_{I(2,k+1)}} = \frac{\Delta_{I(1,k)} \Delta_{J(2,k)}}{\Delta_{I(1,k-1)} \Delta_{I(2,k+1)}}.$$  \hspace{1cm} (3.18)

If $1 < j < a$ and (3.17) is true for $j - 1$, then it is true for $j$ because

$$Y_{j,k} = 1 + X_{j,k} Y_{j-1,k} = 1 + \frac{\Delta_{I(j+1,k)} \Delta_{J(j,k)}}{\Delta_{I(j,k-1)} \Delta_{I(j+1,k+1)}} = \frac{\Delta_{I(j,k)} \Delta_{J(j+1,k)}}{\Delta_{I(j,k-1)} \Delta_{I(j+1,k+1)}}.$$  \hspace{1cm} (3.19)

Let $j = a - 1$, we get

$$X_{a,k} Y_{a-1,k} = a_k \frac{\Delta_{\{b-k,b-k+1,\ldots,n-k-1\}} \Delta_{\{b-k,b-k+2,\ldots,n-k\}}}{\Delta_{\{b-k+1,\ldots,n-k\}} \Delta_{\{b-k,b-k+1,\ldots,n-k-1\}}} = a_k \frac{\Delta_{\{b-k,b-k+2,\ldots,n-k\}}}{\Delta_{\{b-k,b-k+2,\ldots,n-k\}}} = \vartheta_{n-k},$$

where the last equality is similar to (3.14). On the other hand, we have

$$X_{a,k} Y_{a-1,k} = \sum_{j=1}^{a} X_{a,k} Y_{a-1,k} \cdots X_{j,k}.$$  \hspace{1cm} $\Box$

### 4 Tropicalization and Plane Partitions

In this section, we relate $\text{Conf}_n^\times(a)$ to plane partitions via tropicalization. See [FG09] for more details on tropicalization of positive spaces.

#### 4.1 Tropicalization

Let $\mathcal{X}$ be a cluster variety (either $K_2$ type or Poisson type). A positive rational function on $\mathcal{X}$ is a nonzero function that can be expressed as a ratio of two polynomials with non-negative integer coefficients in one (and hence all) cluster(s) of $\mathcal{X}$. The set $\mathcal{P}(\mathcal{X})$ of positive rational functions is a semifield, i.e., a set closed under addition, multiplication, and division. The pair $(\mathcal{X}, \mathcal{P}(\mathcal{X}))$ is called a positive space.

Let $\mathbb{Z}^I = (\mathbb{Z}, \min, +)$ be the semifield of tropical integers. We define the tropicalization of $(\mathcal{X}, \mathcal{P}(\mathcal{X}))$ to be the set

$$\mathcal{X}(\mathbb{Z}^I) := \text{Hom}_{\text{semifield}} \left( \mathcal{P}(\mathcal{X}), \mathbb{Z}^I \right).$$

Let $(\mathcal{X}, \mathcal{P}(\mathcal{X}))$ and $(\mathcal{Y}, \mathcal{P}(\mathcal{Y}))$ be a pair of cluster varieties. A rational map $f : \mathcal{X} \rightarrow \mathcal{Y}$ is positive if the pull back $f^* \mathcal{P}(\mathcal{Y})$ is contained in $\mathcal{P}(\mathcal{X})$. Every positive rational map $f$ admits a tropicalization $f^t : \mathcal{X}(\mathbb{Z}^I) \rightarrow \mathcal{Y}(\mathbb{Z}^I)$ defined by precomposing with the pull-back map $f^*$. In particular cluster mutations are subtraction-free. Therefore every cluster automorphism is positive and can be tropicalized. It induces a natural action of the cluster modular group $\mathcal{G}$ on $\mathcal{X}(\mathbb{Z}^I)$.

Let $\chi := \{X_i \mid 1 \leq i \leq m\}$ be a cluster coordinate chart of $\mathcal{X}$. Its tropicalization is a bijection

$$\chi^t : \mathcal{X}(\mathbb{Z}^I) \xrightarrow{\sim} \mathbb{Z}^m, \quad l \mapsto (l(X_1), l(X_2), \ldots, l(X_m)) := (x_1, \ldots, x_m).$$
The $m$-tuple $(x_1, \ldots, x_m)$ is called the tropical coordinates on $\mathcal{X}(\mathbb{Z}^t)$ in terms of the cluster $\chi$.

Tautologically, every positive function $f \in P(\mathcal{X})$ gives rise to a $\mathbb{Z}$-valued function

$$f^t : \mathcal{X}(\mathbb{Z}^t) \to \mathbb{Z}, \quad l \mapsto l(f),$$

which can be expressed as a piecewise linear function in terms of the tropical coordinates $(x_1, \ldots, x_m)$ by the following procedure:

1. Change every addition into taking minimum.
2. Change every multiplication into addition and every division into subtraction.
3. Change the constants that are coefficients of the original expression into 0.
4. Change the variables $X_i$ into $x_i$.

For example, the tropicalization of the positive rational function $f = \frac{2x_1x_2 + x_2^2 + 1}{(x_1 + x_2)^3}$ is

$$f^t = \min\{x_1 + x_2, 2x_2, 0\} - 3\min\{x_1, x_2\}.$$

Let us tropicalize the cluster variety $\mathcal{C}onf_{a,n}(a)$. Recall that the potential $W$ and the twisted monodromy $P$ are positive functions on $\mathcal{C}onf_{a,n}(a)$. Define the subset

$$\left\{ q \in \mathcal{C}onf_{a,n}(a) (\mathbb{Z}^t) \bigg| W^t(q) \geq 0, \quad P^t(q) = c. \right\} \quad (4.1)$$

The rotation $R : \mathcal{C}onf_{a,n}(a) \to \mathcal{C}onf_{a,n}(a)$ is a cluster automorphism and therefore can be tropicalized into $R^t : \mathcal{C}onf_{a,n}(a)(\mathbb{Z}^t) \to \mathcal{C}onf_{a,n}(a)(\mathbb{Z}^t)$. By definition $W$ and $P$ are invariant under $R$. Hence $R^t$ acts on (4.1). The remainder of the Section is devoted to proving the following Theorem.

**Theorem 4.2.** There is a natural equivariant bijection between the set (4.1) under the rotation $R^t$ and the set $P(a,b,c)$ of plane partitions under the toggling sequence $\eta$.

### 4.2 Gelfand-Zetlin Coordinates

Recall the cluster coordinates $\{X_{i,j}\}$ on $\mathcal{C}onf_n(a)$ associated to $Q_{a,n}$. To each vertex $(i,j)$ of $Q_{a,n}$ is associated

$$L_{i,j} := \prod_{\substack{k \geq i \\ell \geq j}} X_{k,l}.$$

With the convention that $L_{i,j} = 1$ for $i > a$ or $j > b$, we have

$$X_{0,0} = \frac{L_{0,0}}{L_{1,1}} \quad \text{and} \quad X_{i,j} = \frac{L_{i,j}L_{i+1,j+1}}{L_{i+1,j}L_{i,j+1}} \quad \text{for} \ (i,j) \neq (0,0).$$

Therefore $\{L_{i,j}\}$ forms a coordinate system on $\mathcal{C}onf_n(a)$, called Gelfand-Zetlin coordinate system.

The following Lemma is a direct consequence of Proposition 3.2 and (3.13).
Lemma 4.3. The twisted monodromy \( P = L_{0,0} \). The potential

\[
\mathcal{W} = \frac{L_{0,0}}{L_{1,1}} + L_{a,b} + \sum_{i=1}^{a-1} \sum_{j=1}^{b} \frac{L_{i,j}}{L_{i+1,j}} + \sum_{j=1}^{b-1} \sum_{i=1}^{a} \frac{L_{i,j}}{L_{i,j+1}}.
\]  

(4.4)

By setting \( L_{0,0} = e^t \), (4.4) coincides with the Lax operator on Grassmannian in [EHX97, B.25].

Lemma 4.5. Recall \( \lambda_k \) in (2.30) and \( A_{i,j} \) in (2.21). Set \( A_{i,0} = A_{0,j} = A_{0,0} \). Then

\[
L_{i,j} = \frac{A_{i-1,j-1}}{A_{i,j}} \prod_{k=i-a}^{b-j} \lambda_k, \quad \forall 1 \leq i \leq a, \quad \forall 1 \leq j \leq b,
\]

(4.6)

where the indices of \( \lambda_k \) are defined modulo \( n \).

Proof. We prove (4.6) by a double induction on the indices \( (i,j) \) in the descending order. It is certainly true for \( (a,b) \) since

\[
L_{a,b} = X_{a,b} = \lambda_n \frac{\Delta\{2,\ldots,n\}}{\Delta\{1,2,\ldots,a\}} = \lambda_n \frac{A_{a-1,b-1}}{A_{a,b}}.
\]

Take \( (a,j) \) with \( j < b \). The cluster coordinate \( X_{a,j} \) is associated to a boundary face. By definition

\[
X_{a,j} = \lambda_{b-j} \frac{\Delta\{b-j,b-j+1,\ldots,n-j\}}{\Delta\{b-j+1,b-j+2,\ldots,n\}} A_{a-1,j-1} = \lambda_{b-j} \frac{A_{a,j+1}}{A_{a,j}} \frac{A_{a-1,j-1}}{A_{a-1,j}}.
\]

(4.7)

If (4.6) is true for \( (a,j+1) \), then

\[
L_{a,j} = X_{a,j} L_{a,j+1} = \left( \lambda_{b-j} \frac{A_{a,j+1}}{A_{a,j}} \frac{A_{a-1,j-1}}{A_{a-1,j}} \right) \cdot \left( \frac{A_{a-1,j}}{A_{a,j+1}} \prod_{k=i-a}^{b-j} \lambda_k \right) = \frac{A_{a-1,j-1}}{A_{a,j}} \prod_{k=i-a}^{b-j} \lambda_k.
\]

Similarly, for \( (i,b) \) with \( i < a \), by induction we get

\[
L_{i,b} = X_{i,b} L_{i+1,b} = \left( \lambda_{i-a} \frac{A_{i-1,b} A_{i+1,b} A_{i-1,b-1}}{A_{i,b} A_{i-1,b} A_{i-1,b}} \right) \cdot \left( \frac{A_{i,b-1}}{A_{i+1,b}} \prod_{k=i-a+1}^{n} \lambda_k \right) = \frac{A_{i-1,b-1}}{A_{i,b}} \prod_{k=i-a}^{n} \lambda_k.
\]

Take \( (i,j) \) with \( i < a \) and \( j < b \). If (4.6) is true for \( (i+1,j) \), \( (i,j+1) \) and \( (i+1,j+1) \), then

\[
L_{i,j} = \frac{X_{i,j} L_{i+1,j} L_{i,j+1}}{L_{i+1,j+1}} = \left( \frac{A_{i-1,j-1} A_{i,j} A_{i-1,j} A_{i,j+1}}{A_{i-1,j} A_{i,j} A_{i-1,j} A_{i,j+1}} \right) \left( \frac{A_{i,j-1}}{A_{i+1,j}} \prod_{k=i-a+1}^{b-j} \lambda_k \right) \left( \frac{A_{i,j}}{A_{i+1,j+1}} \prod_{k=i-a+1}^{b-j} \lambda_k \right)^{-1} = \frac{A_{i-1,j-1}}{A_{i,j}} \prod_{k=i-a}^{b-j} \lambda_k.
\]

Recall the rotation \( R \) on \( Conf_n^\times(a) \). Set \( L_{i,0} = L_{0,j} = P \) and \( L_{a+1,j} = L_{i,n-a+1} = 1 \). Let

\[
L'_{i,j} := R L_{i,j}.
\]

Note that \( R P = P \). Therefore \( L'_{i,0} = L'_{0,j} = P \) and \( L'_{a+1,j} = L'_{i,n-a+1} = 1 \).
Lemma 4.8. We have
\[
L'_{i,j} = \frac{\left(L'_{i,j-1} + L_{i-1,j}\right) L'_{i+1,j} L_{i,j+1}}{L_{i,j} \left(L'_{i+1,j} + L_{i,j+1}\right)}, \quad \forall 1 \leq i \leq a, \quad \forall 1 \leq j \leq b.
\]

Proof. Let \(B_{i,j} := R^* A_{i,j} = \Delta_I'(i,j)\), where
\[
I'(i,j) = \begin{cases} b - j, \ldots, b - j + i - 1, & \text{consecutive } i \text{ indices} \\ b + i, \ldots, n - 1, & \text{consecutive } a - i \text{ indices}
\end{cases}
\]

By Lemma 4.5 we get
\[
L'_{i,j} = \frac{B_{i-1,j-1}}{B_{i,j}} \prod_{k=i-a-1}^{b-j-1} \lambda_k.
\]

It is easy to check that the coordinates satisfy the Plücker relations
\[
B_{i-1,j-2} A_{i-1,j} + A_{i-2,j-1} B_{i,j-1} = B_{i-1,j-1} A_{i-1,j-1},
\]
\[
B_{i,j-1} A_{i,j+1} + A_{i-1,j} B_{i+1,j} = B_{i,j} A_{i,j}.
\]

Let us take the ratio of them and multiply by \(\prod_{k=i-a}^{b-j} \lambda_k \lambda_{k-1}\) on both sides. The right hand side is
\[
\text{RHS} = \left(\frac{B_{i-1,j-1}}{B_{i,j}} \prod_{k=i-a-1}^{b-j-1} \lambda_k\right) \left(\frac{A_{i-1,j-1}}{A_{i,j}} \prod_{k=i-a}^{b-j} \lambda_k\right) = L'_{i,j} L_{i,j}.
\]

The left hand side becomes
\[
\text{LHS} = \frac{(B_{i-1,j-2} A_{i-1,j} + A_{i-2,j-1} B_{i,j-1}) / (A_{i-1,j} B_{i,j-1})}{(B_{i,j-1} A_{i,j+1} + A_{i-1,j} B_{i+1,j}) / (A_{i-1,j} B_{i+1,j})} \cdot \prod_{k=i-a}^{b-j} \lambda_k \lambda_{k-1}
\]
\[
= \left(\frac{B_{i-1,j-2}}{B_{i,j-1}} + \frac{A_{i-2,j-1}}{A_{i-1,j}}\right) \cdot \left(\frac{A_{i,j+1}}{A_{i-1,j}} + \frac{B_{i+1,j}}{B_{i,j-1}}\right)^{-1} \prod_{k=i-a}^{b-j} \lambda_k \lambda_{k-1}
\]
\[
= \left(\frac{B_{i-1,j-2}}{B_{i,j-1}} \prod_{k=i-a}^{b-j} \lambda_k + \frac{A_{i-2,j-1}}{A_{i-1,j}} \prod_{k=i-a}^{b-j} \lambda_k\right) \cdot \left(\frac{A_{i,j+1}}{A_{i-1,j}} \prod_{k=i-a}^{b-j-1} \lambda_k + \frac{B_{i+1,j}}{B_{i,j-1}} \prod_{k=i-a}^{b-j-1} \lambda_k\right)^{-1}
\]
\[
= \left(L'_{i,j-1} + L_{i-1,j}\right) \left(L^{-1}_{i,j+1} + L'_{i+1,j}\right)^{-1}
\]
\[
= \left(L'_{i,j-1} + L_{i-1,j}\right) L'_{i+1,j} L_{i,j+1}\]
\[
= \left(L'_{i,j-1} + L_{i-1,j}\right) \left(L'_{i+1,j} + L_{i,j+1}\right),
\]

which is precisely the right hand side. \(\square\)

Lemma 4.8 allows us to compute \(L'_{i,j}\) recursively. Let \(\Pi = (\Pi_{i,j})\) be a matrix such that its rows are numbered 0, \ldots, \(a + 1\) and its columns are numbered 0, \ldots, \(b + 1\). For \(1 \leq i \leq a\) and \(1 \leq j \leq b\), we define a birational toggling action \(\tau_{i,j}\) sending \(\Pi\) to the matrix \(\tau_{i,j} \Pi\) such that
\[
(\tau_{i,j} \Pi)_{k,l} := \begin{cases} \Pi_{k,l} & \text{if } (k,l) \neq (i,j); \\ \frac{\Pi_{k-1,l} + \Pi_{k-1,j-1}}{\Pi_{i,j} \left(\Pi_{i+1,j} + \Pi_{i,j+1}\right)} \Pi_{i,j} & \text{if } (k,l) = (i,j).
\end{cases}
\]

Recall the toggling sequence \(\eta\) in (1.2).
Lemma 4.9. Let us apply the toggling sequence \( \eta \) to the initial \((a+2) \times (b+2)\) matrix

\[
\Pi = \begin{pmatrix}
P & P & P & \ldots & P & 1 \\
P & L_{1,1} & L_{1,2} & \ldots & L_{1,b} & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
P & L_{a,1} & L_{a,2} & \ldots & L_{a,b} & 1 \\
1 & 1 & 1 & \ldots & 1 & 1
\end{pmatrix}.
\]

Then the final matrix applying \( \eta \) becomes

\[
\eta \Pi = \begin{pmatrix}
P & P & P & \ldots & P & 1 \\
P & L'_{1,1} & L'_{1,2} & \ldots & L'_{1,n-a} & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
P & L'_{a,1} & L'_{a,2} & \ldots & L'_{a,n-a} & 1 \\
1 & 1 & 1 & \ldots & 1 & 1
\end{pmatrix}.
\]

In other words, the pull-back of the Gelfand-Zetlin coordinates via the rotation \( R \) is given by \( \eta \).

Proof. Note that the sequence \( \eta \) toggles at each (internal) entry exactly once. It suffices to prove that the step \( \tau_{i,j} \) within the sequence \( \eta \) changes \( L_{i,j} \) to \( L'_{i,j} \). This follows from the fact that the toggling sequence \( \eta \) goes from bottom to top within each column and from left to right through all columns. So when we toggle at the entry \((i,j)\), the matrix entry to the left and the matrix entry below have already been changed to \( L'_{i,j-1} \) and \( L'_{i+1,j} \) respectively. The rest of the proof is just a straightforward comparison between the toggling formula and Lemma 4.8. \( \square \)

4.3 Proof of Theorem 4.2

The Gelfand-Zetlin coordinates \( L_{i,j} \) are clearly positive functions on \( \mathcal{C}onf_n^\times(a) \). Denote by \( l_{i,j} \) the tropicalization of \( L_{i,j} \). By Lemma 4.3, we get

\[
P^t = l_{0,0},
\]

\[
W^t = \min \left\{ l_{0,0} - l_{1,1}, l_{a,b} \right\} \cup \left( \bigcup_{1 \leq k \leq a, 1 \leq j \leq b-1} \left\{ l_{i,j} - l_{i+1,j} \right\} \right) \cup \left( \bigcup_{1 \leq k \leq b, 1 \leq i \leq a-1} \left\{ l_{i,j} - l_{i,j+1} \right\} \right). \tag{4.11}
\]

The condition that \( W^t \geq 0 \) is equivalent to the conditions

\[
l_{0,0} \geq l_{1,1}, \quad l_{a,b} \geq 0, \quad l_{i,j-1} \geq l_{i,j} \geq l_{i,j+1}, \quad \forall (i,j).
\]

After imposing the conditions that \( P^t = l_{0,0} = c \), we obtain a natural bijection between (4.1) and \( P(a,b,c) \) by identifying \( l_{i,j} \) with entries \( \pi_{i,j} \) in plane partitions.

It remains to show that this bijection is equivariant with respect to the actions of \( R^t \) and \( \eta \). Let us denote \( l'_{i,j} := (L'_{i,j})^t \). By Lemma 4.9, with the convention that \( l_{0,j} = l_{i,0} = l_{0,0} = c \) and \( l_{a+1,j} = l_{i,b+1} = 0 \), the \( l'_{i,j} \) can be computed recursively:

\[
l'_{0,0} = l_{0,0} = c
\]

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and for \((i, j) \neq (0, 0),\)

\[
\begin{align*}
l'_{i,j} &= \min \{l'_{i-1,j}, l_{i-1,j} + l_{i,j} + l_{i,j+1} - l_{i,j} - 1, l_{i+1,j}, l_{i+1,j} - l_{i,j}\} \\
&= \min \{l'_{i-1,j}, l_{i-1,j} + \max \{l'_{i,j+1}, l_{i,j+1} + l_{i,j} - 1, l_{i,j}\}\} - l_{i,j}.
\end{align*}
\] (4.12)

In the process of computing \(l'_{i,j}\), the coordinates below and to the left of \(l_{i,j}\) has been toggled in the way the mutation sequence is constructed. Therefore the formula (4.12) recovers the toggling formula (1.2). It concludes the proof of Theorem 4.2.

5 Cluster Duality

5.1 Duality Conjecture and Canonical Basis

Let \((\mathcal{A}_Q, \mathcal{X}_Q)\) be the cluster ensemble associated to a quiver \(Q\). Recall the algebras \(\text{up}(\mathcal{A}_Q)\) and \(\text{up}(\mathcal{X}_Q)\) and the cluster modular group \(G_Q\).

Duality Conjecture of Fock-Goncharov asserts that

**Conjecture 5.1** ([FG09, Conjecture 4.1]). The algebra \(\text{up}(\mathcal{A}_Q)\) admits a canonical basis \(G_Q\)-equivariantly parametrized by \(\mathcal{X}_Q(Z)^t\). The algebra \(\text{up}(\mathcal{X}_Q)\) admits a canonical basis \(G_Q\)-equivariantly parametrized by \(\mathcal{A}_Q(Z)^t\).

Conjecture 5.1 has been proved in [GHKK18] under two combinatorial assumptions.

**Theorem 5.2** ([GHKK18]). Conjecture 5.1 holds if the following conditions are satisfied.

- The non-frozen part of the quiver \(Q\) exists a maximal green (or reddening) sequence \(^6\).
- The exchange matrix \(\varepsilon = (\varepsilon_{ij})\) of \(Q\), with \(i\) running through the non-frozen vertices and \(j\) running through all the vertices, is of full rank.

**Remark 5.3.** Let us assume that the quiver \(Q\) satisfy the above combinatorial conditions. Let us denote the bases of \(\text{up}(\mathcal{A}_Q)\) and \(\text{up}(\mathcal{X}_Q)\) as follows

\[
\Theta_{\mathcal{A}} := \{q \mid q \in \mathcal{X}_Q(Z)^t\} \subset \text{up}(\mathcal{A}_Q),
\]

\[
\Theta_{\mathcal{X}} := \{p \mid p \in \mathcal{A}_Q(Z)^t\} \subset \text{up}(\mathcal{X}_Q).
\]

The bases elements \(\theta_q\) and \(\vartheta_p\) satisfy many remarkable properties. One of them is that every \(\theta_q \in \text{up}(\mathcal{A}_Q)\) in terms of the \(K_2\) cluster \(\{A_j\}\) associated to \(Q\) is expressed as

\[
\theta_q = \prod_j A_j^{x_j} F \left( \left( \prod_j A_j^{\varepsilon_{kj}} \right)_{k \in I_{uf}} \right) \] (5.4)

where \((x_j)\) is the tropical coordinates of \(q \in \mathcal{X}(Z)^t\) in terms of the Poisson cluster associated to \(Q\), and \(F\) is a polynomial with constant term 1 and variables of the form \(\prod_j A_j^{\varepsilon_{kj}}\) as \(k\) ranges through all unfrozen vertices. The elements \(\vartheta_p\) admit similar formulas.

\(^6\)A maximal green (reddening) sequence is speical sequence of quiver mutations introduced by Keller in order to study Donaldson-Thomas transformations. See [Kel13].
One may notice that, on the one hand, we define $p^* (X_i) = \prod_j A_j^{x_{ij}}$ in the definition of the $p$ map, and on the other hand, the above polynomial $F$ depends on $\prod_j A_j^{x_{ij}} = p^* (X_i^{-1})$. The reason this happens is that the cluster Poisson variables used in this paper are inverses of those used by Gross, Hacking, Keel, and Kontsevich in [GHKK18]. Such a switch frees us from considering tropicalization with taking maximum.

**Proof of Theorem 1.7.** For the quiver $Q_{a,n}$, a maximal green sequence was found by Marsh and Scott in [MS16], and the existence of a cluster Donaldson-Thomas transformation (which is equivalent to a reddening sequence) was proved by Weng in [Wen16]. By Lemma 2.26, the second combinatorial condition holds. Hence Conjecture 5.1 holds for the cluster ensemble $(\mathcal{A}_{a,n}, X_{a,n})$. By Theorems 2.23 and 2.36 we know that $\text{up} (\mathcal{A}_{a,n}) \cong \mathcal{O} (\mathcal{F} \mathcal{r}^{x}_a (n))$ and $\text{up} (\mathcal{X}_{a,n}) \cong \mathcal{O} (\mathcal{C} \text{ont}^{x}_a (a))$, which concludes the proof Theorem 1.7. □

### 5.2 Partial Compactification, Optimized Quiver, and Potential Function

Let $Q = (I^u \subset I, \varepsilon)$ be a quiver satisfying the combinatorial conditions in Theorem 5.2. Let $I^0 := I - I^u$ be the set of frozen vertices. For $i \in I^0$, let $D_i$ denote the (irreducible) boundary divisor of $\mathcal{A}_Q$ defined by setting $A_i = 0$. Let us glue $\mathcal{A}_Q$ with these boundary divisors, obtaining the partial compactified space

$$\overline{\mathcal{A}}_Q := \mathcal{A}_Q \cup \left( \bigcup_{i \in I^0} D_i \right).$$

This section is devoted to studying the ring $\mathcal{O} (\overline{\mathcal{A}}_Q)$ of regular functions on $\overline{\mathcal{A}}_Q$.

Let $f \in \text{up} (\mathcal{A}_Q)$. Denote by $\text{ord}_{D_i}(f)$ the order of $f$ along the boundary divisor $D_i$. Note that $f$ can be extended to a regular function on $D_i$ if and only if $\text{ord}_{D_i}(f) \geq 0$. Therefore

$$\mathcal{O} (\overline{\mathcal{A}}_Q) = \{ f \in \text{up}(\mathcal{A}_Q) \mid \text{ord}_{D_i}(f) \geq 0, \forall i \in I^0 \}. $$

Recall the canonical basis $\Theta_{\mathcal{A}^0}$ of $\mathcal{O}(\mathcal{A}^0)$. Consider the intersection

$$\Theta_{\overline{\mathcal{A}}} := \Theta_{\mathcal{A}^0} \cap \mathcal{O} (\overline{\mathcal{A}}_Q) = \{ \theta_q \in \Theta_{\mathcal{A}^0} \mid \text{ord}_{D_i}(\theta_q) \geq 0, \forall i \in I^0 \}. \quad (5.5)$$

Conjecture 9.8 of [GHKK18] implies that the intersection $\Theta_{\overline{\mathcal{A}}}$ descends to a linear basis of $\mathcal{O}(\overline{\mathcal{A}}_Q)$. The paper loc.cit. provides a sufficient condition under which the aforementioned conjecture holds.

**Definition 5.6.** Let $i \in I^0$. If $\varepsilon_{ki} \geq 0$ for all unfrozen vertices $k$, then we say the quiver $Q$ is optimized for $i$. If there exists a mutation sequence $\tau$ such that the mutated quiver $\tau Q$ is optimized for $i$, then we say that $\tau$ optimizes $Q$ in the equivalence class $|Q|$.  

**Remark 5.7.** Because of different conventions used in this paper and in [GHKK18], here we say $Q$ is optimized for $i$ if all arrows between $i$ and unfrozen vertices point towards the unfrozen ones.

**Proposition 5.8.** If every frozen vertex $i$ of $Q$ admits an optimized quiver in $|Q|$, then the set $\Theta_{\overline{\mathcal{A}}}$ forms a linear basis of $\mathcal{O}(\overline{\mathcal{A}}_Q)$.

**Proof.** The linear independence of $\Theta_{\overline{\mathcal{A}}}$ is clear. Suppose that

$$f = \sum_{q \in \mathcal{X}(\mathbb{Z}^i)} \alpha_q \theta_q \in \mathcal{O}(\overline{\mathcal{A}}_Q) .$$

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Define $q$.

Assume that every frozen vertex of Proposition 5.10. where $\delta$

For the other frozen vertices, let us apply the rotation mutation sequence

$W := \sum_{i \in I^0} \vartheta_{p_i}$.  

(5.9)

Proposition 5.10. Assume that every frozen vertex of $Q$ admits an optimized quiver in $|Q|$. Let $q \in \mathcal{X}(\mathcal{Z})$. Then $\theta_q \in \Theta_{\mathcal{X}}$ if and only if $W^{\vartheta}(q) \geq 0$.

Proof. By the definition of tropicalization, we have

$$W^{\vartheta}(q) = \min_{i \in I^0} \{ \vartheta_{p_i}(q) \}.$$  

Therefore $W^{\vartheta}(q) \geq 0$ if and only if every $\vartheta_{p_i}(q) \geq 0$. It suffices to show that if $i \in I^0$ admits an optimized quiver, then

$$\text{ord}_{D_i}(\theta_q) = \vartheta_{p_i}(q).$$  

Without loss of generality, let us assume that $Q$ is optimized for $i$, i.e., $\varepsilon_{ki} \geq 0$ for all unfrozen vertices $k$. Let $\{A_j\}$ be the cluster of $\mathcal{X}_{|Q|}$ associated to $Q$. Note that $\theta_q$ is of the form (5.4). Therefore

$$\text{ord}_{D_i}(\theta_q) = x_i := X^\vartheta_i(q),$$  

where $X_i$ is the cluster variable of $\mathcal{X}$ associated to the vertex $i$ in $Q$. By [GHKK18, Lemma 9.3], if $Q$ is optimized for $i$, then $\vartheta_{p_i} = X_i$.

Let us apply the above results to the cases of Grassmannians. It boils down to finding optimized quivers for frozen vertices in the quiver $Q_{a,n}$. As observed by L. Williams and appeared in [GHKK18, Proposition 9.4], the quiver $Q_{a,n}$ is optimized for the vertices $(0,0)$ and $(a,b)$; since the mutation sequence $\rho$ in (3.5) rotates the frozen vertices of $Q_{a,n}$ clockwise to their neighbors, by applying $\rho$ repeatedly, each frozen vertex $i$ has a chance to be at the position of $(0,0)$ and therefore admits an optimized quiver. Indeed, the quiver $\rho^{n-i}Q_{a,n}$ is optimized for the frozen vertex $i$.

The next Proposition shows that in the Grassmannian case, the potential function $W = \sum_{i} \vartheta_i$ in (2.11) coincides with the function $W$ in (5.9).

Proposition 5.11. Under the algebra isomorphism $\mathcal{X}_{a,n} \cong \mathcal{O}(\mathcal{G}onf_{\mathcal{Z}}^\varepsilon(a))$, the theta function $\vartheta_{p_i}$ is identified with the function $\vartheta_i$ defined in (2.10).

Proof. Note that $Q_{a,n}$ is optimized for $(0,0)$. By (3.9) and [GHKK18, Lemma 9.3], we have

$$\vartheta_n = X_{0,0} = \vartheta_{p_n}.$$  

For the other frozen vertices, let us apply the rotation mutation sequence $\rho$. Then

$$\vartheta_{p_i} = X_n^{\rho^{n-i}} = (R^n)^{n-i} X_n = (R^n)^{n-i} \vartheta_n = \vartheta_i.$$  

□
By Theorem 2.23, or more precisely by the original version [Sco06, Theorem 3], the coordinate ring \( \mathcal{O}(G(a,n)) \) is isomorphic to \( \mathcal{O}(\mathcal{G}_{a,n}) \). Combining Propositions 5.8, 5.10, and 5.11, we get

**Theorem 5.12.** Under the isomorphisms \( \mathcal{O}(G(a,n)) \cong \mathcal{O}(\mathcal{G}_{a,n}) \) and \( \mathcal{O}(\text{Conf}^\times(a)) \cong \text{up}(\mathcal{G}_{a,n}) \), the coordinate ring \( \mathcal{O}(G(a,n)) \) admits a natural basis

\[
\{ \theta_q \mid q \in \text{Conf}^\times(a)(\mathbb{Z}^t), \ W^t(q) \geq 0 \}.
\]

### 5.3 \( \mathbb{G}_m \)-action

Recall the \( \mathbb{G}_m \)-action on \( G(a,n) \) in (2.1). Let us restrict the \( \mathbb{G}_m \)-action to the open subset \( G(a,n) \). It induces a \( \mathbb{G}_m \)-action on \( \mathcal{O}(G(a,n)) \) extending the one on \( \mathcal{O}(G(a,n)) \).

**Proposition 5.13.** Let \( q \in \text{Conf}^\times(a)(\mathbb{Z}^t) \). Its corresponding theta function \( \theta_q \in \mathcal{O}(G(a,n)) \) is an eigenvector of the \( \mathbb{G}_m \)-action with weight \( P^t(q) \):

\[
t \cdot \theta_q = t^{P^t(q)} \theta_q.
\]

**Proof.** Let \( \{ A_{i,j} \} \) be the \( K_2 \) cluster of \( G(a,n) \) associated to \( Q_{a,n} \). By (5.4), the function \( \theta_q \) can be expressed as a Laurent polynomial

\[
\theta_q = \prod_{(i,j)} A_{i,j}^{x_{i,j}} F \left( \prod_g A_g^{z_{fg}} \right)_{f \in I_{af}},
\]

where \( (x_{i,j}) \) is the tropical coordinates of \( q \) with respect to the quiver \( Q_{a,n} \), and \( F \) is a polynomial with constant term 1 and variables of the form \( \prod_g A_g^{z_{fg}} \) for \( f \in I_{af} \).

By definition, every \( A_f \) is a Plücker coordinate and therefore is of weight 1 with respect to the \( \mathbb{G}_m \)-action. Since \( \sum_f z_{fg} = 0 \) for all \( g \in I_{af} \) by construction, the whole factor \( F \) is invariant under the \( \mathbb{G}_m \)-action. It implies that \( \theta_q \) is eigenvector of weight \( \sum_{(i,j)} x_{i,j} \). By Proposition 3.2, we have

\[
P^t(q) = \sum_{(i,j)} x_{i,j},
\]

which concludes the proof.

As a direct consequence, we get

**Corollary 5.15.** The representation \( \mathcal{O}(G(a,n))_c = V_{\alpha_a} \) has a canonical basis

\[
\Theta(a,b,c) := \{ \theta_q \mid q \in \text{Conf}^\times(a)(\mathbb{Z}^t), W^t(q) \geq 0, P^t = c \}.
\]

Combining with Theorem 4.2 we deduce that the basis \( \Theta(a,b,c) \) is in natural bijection with the plane partitions \( P(a,b,c) \).
5.4 Torus Action and Weight Decomposition

By (2.6), $\mathcal{F}_{\alpha}(n) \cong \text{SL}_a \setminus \text{Mat}^x_{a,n}$. The group $\text{GL}_n$ acts on the right of $\text{Mat}^x_{a,n}$ by matrix multiplication. The maximal torus $T = (\mathbb{G}_m)^n \subset \text{GL}_n$ of diagonal matrices acts by rescaling the column vectors $v_i$ of the matrices in $\text{Mat}^x_{a,n}$

$$(v_1, \ldots, v_n)(t_1, \ldots, t_n) := (t_1 v_1, t_2 v_2, \ldots, t_n v_n).$$

Its induced left $(\mathbb{G}_m)^n$-action on $\mathcal{O}(\mathcal{F}_{\alpha}(n))$ gives rise to a weight decomposition

$$\mathcal{O}(\mathcal{F}_{\alpha}(n)) = \bigoplus_{\mu} \mathcal{O}(\mu),$$

where $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n$ and $\mathcal{O}(\mu)$ consists of the functions $F$ such that

$$(t_1, \ldots, t_n).F = t_1^{\mu_1} \cdots t_n^{\mu_n}F.$$

In particular, if we restrict to the representation $V_{c\omega_a} = \mathcal{O}(\mathcal{F}_a(n))_c$, then we get the weight decomposition

$$V_{c\omega_a} = \bigoplus_{\mu} V_{c\omega_a}(\mu),$$

where $V_{c\omega_a}(\mu) := V_{c\omega_a} \cap \mathcal{O}(\mu)$.

In this section, we show that the theta basis $\Theta(a, b, c)$ is compatible with the weight decomposition of the representation $V_{c\omega_a}$.

Recall the following dual torus projection defined in (2.13)

$$M = (M_1, \ldots, M_n) : \text{Conf}^x_n(\alpha) \to \mathcal{T}^\vee.$$

Let us tropicalize the map $M$, obtaining

$$M' = (M'_1, \ldots, M'_n) : \text{Conf}^x_n(\alpha)(\mathbb{Z}^n) \to \mathcal{T}^\vee(\mathbb{Z}^n) \simeq \mathbb{Z}^n$$

**Proposition 5.17.** Let $q \in \text{Conf}^x_n(\alpha)$. The theta function $\theta_q$ is an eigenvector of the $T$-action on $\mathcal{O}(\mathcal{F}_a(n))$ with weight $M'(q)$, i.e.,

$$(t_1, \ldots, t_n).\theta_q = t_1^{M'_1(q)} \cdots t_n^{M'_n(q)}\theta_q.$$
steps is 1, until its height becomes 0 or until it touches the rightmost column.

When \( b < k \leq n \), we get

Example 5.18. When \((a, n) = (2, 5)\), the sets \(F_k\) are depicted as follows.

\[
F_1 = \begin{array}{|c|c|}
\hline
(1, 3) & \hline
(2, 3) & \hline
\end{array} \quad F_2 = \begin{array}{|c|c|}
\hline
(1, 2) & \hline
(2, 2) & (2, 3) & \hline
\end{array} \quad F_3 = \begin{array}{|c|c|c|}
\hline
(1, 1) & & \hline
(2, 1) & (2, 2) & \hline
\end{array}
\]

\[
F_4 = \begin{array}{|c|c|}
\hline
(0, 0) & \hline
(2, 1) & \hline
\end{array} \quad F_5 = \begin{array}{|c|c|c|}
\hline
(0, 0) & & \hline
(1, 1) & (1, 2) & (1, 3) & \hline
\end{array}
\]
Lemma 5.19. Recall the clusters \( \{X_{i,j}\} \) of \( \mathcal{C}onf_n^\times(a) \) associated to \( Q_{a,n} \) in (2.33). The function

\[
M_k = \prod_{(i,j) \in F_k} X_{i,j}.
\]

(5.20)

Proof. Recall the definition of \( M_k \) in (2.12). We prove the Lemma for \( 1 \leq k \leq \min\{a, b\} \). The proof for the other cases goes along the same line. Recall the Gelfand-Zetlin coordinates of \( \mathcal{C}onf_n^\times(a) \).

We get

\[
\prod_{(i,j) \in F_k} X_{i,j} = \frac{L_{1,b-k+1} L_{2,b-k+2} \cdots L_{k-1,b-1}}{L_{1,b-k+2} L_{2,b-k+2} \cdots L_{k-1,b}} \frac{L_{k-1,b}}{L_{k-1,b+1}} = \prod_{k=1}^{i-1} \frac{L_{i,b-k+i}}{L_{i,b-k+i+1}}
\]

By Lemma 4.5, we get

\[
\prod_{k=1}^{i-1} \frac{L_{i,b-k+i}}{L_{i,b-k+i+1}} = \frac{\Delta_{\{k-a, \ldots, k-1\}}}{\Delta_{\{k-a+1, \ldots, k\}}} \prod_{j=1}^{i-1} \lambda_j = M_k. \quad \square
\]

Proof of Proposition 5.17. The proof makes use of the expression (5.14) of \( \theta_q \) again. For a nonfrozen vertex \( f \), the product \( \prod_q A_{q}^{\ell_f} = p^* \left( X_f^{-1} \right) \) is independent of the rescaling of the column vectors \( v_l \) due to the well-defined-ness of the unfrozen variable \( X_f \). Therefore the polynomial \( F \) in (5.14) is invariant under the rescaling \( T \)-action. For the Plücker coordinates \( A_{i,j} \), note that it is affected by the \( t_k \) component of \( T \) if and only if \( (i, j) \in F_k \). Therefore

\[
(t_1, \ldots, t_n) \theta_q = t_1^{\mu_1} t_2^{\mu_2} \cdots t_n^{\mu_n} \theta_q, \quad \text{where } \mu_k = \sum_{(i,j) \in F_k} x_{i,j}.
\]

By Lemma 5.19, we get \( \sum_{(i,j) \in F_k} x_{i,j} = M_k^t(q) \). \quad \square

Combining Corollary 5.15 with Proposition 5.17, we get

Corollary 5.22. The weight space \( V_{\omega_{a}}(\mu) \) has a canonical basis

\[
\Theta(a, b, c)(\mu) := \{ \theta_q \mid q \in \mathcal{C}onf_n^\times(a) (\mathbb{Z}^t), \ W^t(q) \geq 0, \ P^t(q) = c, \ M^t(q) = \mu \}.
\]

(5.23)

Note that \( \Theta(a, b, c) \) is parametrized by the set \( P(a, b, c) \) of plane partitions. In the rest of this section, we present a concrete decomposition of \( P(a, b, c) \) compatible with the above decomposition of \( \Theta(a, b, c) \).

Recall that the Gelfand-Zetlin patterns \([GZ50]\) for \( GL_n \) are triangular arrays of integers with non-increasing rows and columns as follows

\[
\Lambda = \begin{pmatrix}
\lambda_{1,1} & \lambda_{1,2} & \cdots & \lambda_{1,n} \\
\lambda_{2,2} & \lambda_{2,3} & \cdots & \lambda_{2,n} \\
\lambda_{3,3} & \cdots & \lambda_{3,n} \\
\vdots & \ddots & \ddots \\
\lambda_{n,n} \\
\end{pmatrix}
\]

Let \( \delta_i := \sum_{k=1}^{i} \lambda_{k,n-i+k} \) be the sums of entries along diagonals. Define

\[
\text{wt}(\Lambda) := (\delta_1, \delta_2 - \delta_1, \ldots, \delta_n - \delta_{n-1})
\]

(5.24)
Let \( \pi = (\pi_{ij}) \in P(a, b, c) \). Note that \( 0 \leq \pi_{ij} \leq c \). It uniquely determines a Gelfand-Zetlin pattern as follows

\[
\Lambda_\pi := \begin{pmatrix}
c & c & \cdots & c & \pi_{1,1} & \pi_{1,2} & \cdots & \pi_{1,b} \\
c & c & \cdots & c & \pi_{2,1} & \pi_{2,2} & \cdots & \pi_{2,b} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
c & \pi_{a,1} & \pi_{a,2} & \cdots & \pi_{a,b} & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & & & & \\
\vdots & \vdots & & & & & & \\
0 & & & & & & & \\
\end{pmatrix}
\]

Consider the decomposition

\[
P(a, b, c) = \bigsqcup_{\mu} P(a, b, c)(\mu)
\]

where \( P(a, b, c)(\mu) := \{ \pi \in P(a, b, c) \mid \text{wt}(\Lambda_\pi) = \mu \} \)

**Proposition 5.25.** The basis \( \Theta(a, b, c)(\mu) \) is in natural bijection with \( P(a, b, c)(\mu) \).

**Proof.** Recall the tropical Gelfand-Zetlin coordinates \( \{l_{i,j}\} \) of \( \mathcal{G} \operatorname{on} f^\wedge_n (a) (\mathbb{Z}^t) \). Let us arrange them in the following triangle pattern

\[
l_{0,0} \quad l_{0,0} \quad \cdots \quad l_{0,0} \quad l_{1,0} \quad l_{1,1} \quad l_{1,2} \quad \cdots \quad l_{1,b} \\
l_{0,0} \quad \cdots \quad l_{0,0} \quad l_{2,1} \quad l_{2,2} \quad \cdots \quad l_{2,b} \\
\vdots \quad \vdots \quad \vdots \quad \ddots \quad \vdots \quad \vdots \quad \ddots \quad \vdots \\
l_{0,0} \quad l_{a,1} \quad l_{a,2} \quad \cdots \quad l_{a,b} \\
0 \quad 0 \quad \cdots \quad 0 \\
\vdots \quad \vdots \quad \vdots \quad \ddots \quad \vdots \quad \vdots \quad \ddots \quad \vdots \\
0 & & & & & & & \\
\]

Consider the sums of \( l_{i,j} \) along each diagonal:

\[
\delta_1 = l_{1,b}, \quad \delta_2 = l_{1,b-1} + l_{2,b}, \quad \ldots, \quad \delta_n = a l_{0,0}
\]

Following the same argument as in (5.21), we get

\[
M^t = (M_1^t, M_2^t, \ldots, M_n^t) = (\delta_1, \delta_2 - \delta_1, \ldots, \delta_n - \delta_{n-1}). \quad (5.26)
\]

Comparing (5.26) with (5.24), the Proposition follows. \( \square \)

### 5.5 Proofs of Main Theorems

**Proof of Theorem 1.5.** Corollaries 5.15 and 5.22 show that \( V_{cw_a} \) admits a natural basis \( \Theta(a, b, c) \) which is in bijection with the set of plane partitions \( P(a, b, c) \) and is compatible with the weight decomposition. It remains to show that the bijection is equivariant under \( C_a \)- and the toggling \( \eta \)-actions.

As discussed in Section 3.2, the action \( C_a \) on \( \mathcal{G} r_a^\wedge (n) \) and the action \( R \) on \( \mathcal{G} \operatorname{on} f^\wedge_n (a) \) are cluster automorphisms realized by the same mutation sequence \( \rho \). In other words, the actions \( C_a \)
and \( R \) correspond to the same element in the cluster modular group \( \mathcal{G}_{Q_{a,n}} \). By Conjecture 5.1 (proved in [GHKK18, Proposition 3.6]), the bijection between the theta basis of \( \mathcal{O}(\mathcal{G}r^\times_a(n)) \) and \( \mathcal{C}onf_{n\times a}(\mathbb{Z}^1) \) is equivariant under \( C_a^- \) and the tropical \( R_t^- \) actions. By Theorem 4.2, the \( R_t^- \) action coincides with the toggling \( \eta \) action. Theorem 1.5 is proved.

**Proof of Theorem 1.3.** Let \( D(p, q) \) be the diagonal matrix \( \text{diag} \left(p q^{n-1}, \ldots, p q, p\right)\). Let \( \Lambda \) be a representation of \( \text{GL}_n \). The character formula asserts that

\[
\text{Trace}_{\Lambda}(D(p, q)) = p^{(\omega_n, \lambda)} \sum_{\mu} \dim \Lambda(\mu) q^{(\rho, \mu)}
\]

where \( \Lambda(\mu) \) is the \( \mu \)-weight subspace of \( \Lambda \), \( \rho \) is the dominant weight \((n - 1, \ldots, 2, 1, 0)\), and \( \omega_n = (1, \ldots, 1)\).

Let \( \zeta = e^{2\pi \sqrt{-1}/n} \). The characteristic polynomial of the matrix \( C_a \) in (1.4) is

\[
\det(\text{Id}_n - C_a) = \lambda^n - (-1)^{a-1}.
\]

It has \( n \)-distinct roots \( \zeta^{-\frac{a-1}{2}}, \zeta^{-\frac{a-1}{2}}, \ldots, \zeta^{-\frac{a-1}{2}}\zeta^{-1} \) over \( \mathbb{C} \). Therefore \( C_a \) is conjugate to \( D \left( \zeta^{-\frac{a-1}{2}}, \zeta \right) \) and \( C_a^k \) is conjugate to \( D \left( \zeta^{-\frac{(a-1)k}{2}}, \zeta^k \right) \). By Theorem 1.5, the number of plane partitions fixed by \( \eta^k \) is equal to the number of basis vectors in \( \{ \theta_{\Phi(\pi)} \}_{\pi \in P(a,b,c)} \) fixed by \( C_a^k \). Therefore

\[
\# \left\{ \pi \in P(a,b,c) \mid \eta^k(\pi) = \pi \right\} = \text{Trace}_{\omega^a} C_a^k = \text{Trace}_{\omega^a} D \left( \zeta^{-\frac{(a-1)k}{2}}, \zeta^k \right) = \left( \zeta^k \right)^{\frac{(a-1)ac}{2}} \sum_{\mu} \dim \Lambda(\mu) \left( \zeta^k \right)^{(\rho, \mu)}
\]

By Proposition 5.25, we have

\[
\dim \Lambda(\mu) = \# P(a,b,c)(\mu).
\]

Let \( \pi \in P(a,b,c) \). We see from (5.24) that

\[
\langle \rho, \text{wt}(\Lambda_\pi) \rangle = \sum_{k=1}^{n} (n - k) (\delta_k - \delta_{k-1}) = \sum_{k=1}^{n-1} \delta_k = \frac{(a-1)ac}{2} + \sum_{i,j} \pi_{i,j} = \frac{(a-1)ac}{2} + |\pi|.
\]

Therefore

\[
\left( \zeta^k \right)^{\frac{(a-1)ac}{2}} \sum_{\mu} \dim \Lambda(\mu) \left( \zeta^k \right)^{(\rho, \mu)} = \sum_{\mu} \sum_{\pi \in P(a,b,c)(\mu)} \left( \zeta^k \right)^{|\pi|} = \sum_{\pi \in P(a,b,c)} \left( \zeta^k \right)^{|\pi|} = M_{a,b,c} \left( \zeta^k \right).
\]

**Appendix**

**A. Generalities on Cluster Ensembles.**

We briefly recall the definition of cluster ensembles following [FG09].

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**Quiver mutations.** Let $Q = (I_{uf} \subset I, \varepsilon)$ be a quiver without loops or 2-cycles: $I$ is the set of vertices, $I_{uf}$ is the set of unfrozen vertices, and $\varepsilon$ is an $I \times I$ skew-symmetric matrix called the *exchange matrix* encoding the data of number of arrows between vertices

$$\varepsilon_{i,j} = \# \{ j \to i \} - \# \{ i \to j \}.$$

For the rest of this appendix we let $m = \# I$ and $l = \# I_{uf}$.

Given a quiver $Q$, the *quiver mutation* $\mu_k$ at a non-frozen vertex $k \in I_{uf}$ creates a new quiver $\mu_k(Q)$ by the following procedure

1. For each pair of arrows $i \to k \to j$, create a “composite” arrow $i \to j$.
2. Reverse all arrows incident to $k$.
3. Remove any maximal disjoint collection of oriented 2-cycles.

This quiver mutation is involutive: $\mu_2^k(Q) = Q$. Repeating the process at every non-frozen vertex for each new quiver obtained via quiver mutations, we get an infinite $l$-valent tree $T_l$ such that every vertex $t$ of $T_l$ is assigned a quiver $Q_t = (I_{uf} \subset I, \varepsilon_t)$.

**Cluster ensembles.** Now assign to each vertex $t$ two coordinate charts: the $K_2$ cluster $\alpha_t = \{ A_{i,t} \mid i \in I \}$ and the Poisson cluster $\chi_t = \{ X_{i,t} \mid i \in I \}$. Geometrically, they correspond to a pair of algebraic tori:

$$T_{t,\alpha} = \text{Spec} \left( k[A_{1,t}^\pm, \ldots, A_{m,t}^\pm] \right), \quad T_{t,\chi} = \text{Spec} \left( k[X_{1,t}^\pm, \ldots, X_{m,t}^\pm] \right).$$

(5.27)

There is a homomorphism $p$ relating them:

$$p^* X_{i,t} = \prod_{j \in I} A_{j,t}^{\varepsilon_{ij}}. \quad (5.28)$$

The transition maps between the pairs of tori assigned to $Q_t$ and $Q_{t'} = \mu_k(Q_t)$ are as follows:

$$\mu_k^* A_{i,t'} = \begin{cases} A_{k,t}^{-1} \left( \prod_{j \in I} A_{j,t}^{[\varepsilon_{jk},t]^+} + \prod_{j \in I} A_{j,t}^{[-\varepsilon_{jk},t]^+] \right) & \text{if } i = k, \\ A_{i,t} & \text{if } i \neq k; \end{cases} \quad (5.29)$$

$$\mu_k^* X_{i,t'} = \begin{cases} X_{k,t}^{-1} & \text{if } i = k, \\ X_{i,t}(1 + X_{k,t}^{\text{sgn}(\varepsilon_{ik},t)})^{\varepsilon_{ik}} & \text{if } i \neq k. \end{cases} \quad (5.30)$$

Here $[\varepsilon]^+ = \max\{ \varepsilon, 0 \}$.

Let us glue all the algebraic tori via the transitions (5.29)(5.30), obtaining a pair of varieties called cluster ensembles

$$\mathcal{A}_Q = \bigcup_t T_{t,\alpha}, \quad \mathcal{X}_Q = \bigcup_t T_{t,\chi}. \quad (5.31)$$

where $t$ runs through all the vertices of the tree $T_l$. The map $p$ in (5.28) is compatible with the transition maps (5.29)(5.30). Therefore we get a natural map

$$p : \mathcal{A}_Q \longrightarrow \mathcal{X}_Q. \quad (5.32)$$

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In general the map $p$ is neither injective nor surjective.

The coordinate rings of these varieties are the algebras of universal Laurent polynomials

$$O(\mathcal{A}_{|Q|}) = \text{up}(\mathcal{A}_{|Q|}) := \bigcap_t k[A_{1,t}, \ldots, A_{m,t}],$$

$$O(\mathcal{X}_{|Q|}) = \text{up}(\mathcal{X}_{|Q|}) := \bigcap_t k[X_{1,t}, \ldots, X_{m,t}].$$

**Cluster modular group.** To each torus $T_{t,\alpha}$ in (5.27) is associated a differential form

$$\Omega_t := \sum_{i,j} \varepsilon_{ij,t} \frac{dA_{i,t}}{A_{i,t}} \wedge \frac{dA_{j,t}}{A_{j,t}}.$$  

This $\Omega_t$ is compatible with the transition (5.29) and therefore can be lifted to a global differential form $\Omega$ on $\mathcal{A}_{|Q|}$. A *cluster automorphism* $\tau$ of $\mathcal{A}_{|Q|}$ is a birational isomorphism of $\mathcal{A}_{|Q|}$ such that

- It preserves the differential form: $\tau^*\Omega = \Omega$.
- For every $K_2$-cluster $\alpha_t = \{A_{1,t}\}$, the pullback $\tau^*(\alpha_t) := \{\tau^* A_{1,t}\}$ remains a $K_2$-cluster, with its indices $i$ possibly permuted.

Locally, $\tau$ can be realized by a sequence of mutations that sends a quiver $Q$ to itself up to permutations of vertices. The *cluster modular group* $\mathcal{G}_{\mathcal{A}_{|Q|}}$ consists of cluster automorphisms of $\mathcal{A}_{|Q|}$.

To each torus $T_{t,\chi}$ is associated a bi-vector

$$B_t := \sum_{i,j} \varepsilon_{ij,t} X_{i,t} \frac{\partial}{\partial X_{i,t}} \wedge X_{j,t} \frac{\partial}{\partial X_{j,t}},$$

which can be lifted to a global bi-vector $B$ on $\mathcal{X}_{|Q|}$. A cluster automorphism of $\mathcal{X}_{|Q|}$ is a birational isomorphism of $\mathcal{X}_{|Q|}$ that preserves the bi-vector $B$ and permute its Poisson clusters. Denote by $\mathcal{G}_{\mathcal{X}_{|Q|}}$ the group of cluster automorphisms of $\mathcal{X}_{|Q|}$.

From the tropical cluster duality proved by Nakanishi and Zelevinsky [NZ12] one can deduce that the cluster modular group $\mathcal{G}_{\mathcal{X}_{|Q|}} = \mathcal{G}_{\mathcal{A}_{|Q|}}$. Hence we will drop the subscripts $\mathcal{A}$ and $\mathcal{X}$ in the notation and denote it by $\mathcal{G}_{|Q|}$.

**Quiver extensions.** Let $Q = (I^\text{uf} \subset I, \varepsilon)$ be a quiver. Let $\tilde{Q} = \left( I^\text{uf} \subset \tilde{I}, \tilde{\varepsilon} \right)$ be a quiver obtained from $Q$ by adding frozen vertices labelled by $I' = \{1', \ldots, f'\}$ and arrows such that $\tilde{Q}$ contains $Q$ as a full subquiver. In other words, $\tilde{I} = I \cup I'$ and $\tilde{\varepsilon}$ contains $\varepsilon$ as a submatrix. Let $(\mathcal{A}_{|\tilde{Q}|}, \mathcal{X}_{|\tilde{Q}|})$ be the cluster ensemble associated to $\tilde{Q}$.

Following [She14, (3.5)], we define the following map

$$k : \mathcal{A}_{|\tilde{Q}|} \xrightarrow{\tilde{p}} \mathcal{X}_{|\tilde{Q}|} \xrightarrow{j} \mathcal{X}_{|Q|}.$$  

The map $\tilde{p}$ is a natural map as (5.32). The map $j$ is a surjective map such that $j^* X_{i,t} = X_{i,t}$ for all $i \in I$. The map $k$ is the composition of $\tilde{p}$ and $j$. It is surjective if and only if the submatrix $\varepsilon|_{\tilde{I} \times I}$ of the exchange matrix $\varepsilon$ is of full rank. In this case we get a natural injection

$$k^* : \text{up}(\mathcal{X}_{|Q|}) \rightarrow \text{up}(\mathcal{A}_{|\tilde{Q}|}).$$

The following easy Lemma generalizes one key part of the proof of Theorem 2.36.
**Lemma 5.35.** Assume \( k \) is surjective. Let \( F \in k(\mathcal{X}_{Q}) \) be a rational function on \( \mathcal{X}_{Q} \). Then \( F \in \text{up}(\mathcal{X}_{Q}) \) if and only if \( k^{*}(F) \in \text{up}(\mathcal{A}_{\tilde{Q}}) \).

**Proof.** Let \( t \) be a vertex of \( T_{l} \). Let \( \chi_t = \{ X_{i,t} \mid i \in I \} \) and \( \tilde{\alpha}_t := \{ A_{j,t} \mid j \in \tilde{I} \} \) be its corresponding clusters. Since \( k^{*} \) is injective and it maps Laurent monomials to Laurent monomials, we have

\[
F \in k \left[ X_{1,t}^{\pm}, \ldots, X_{m,t}^{\pm} \right] \iff k^{*}(F) \in k \left[ A_{1,t}^{\pm}, \ldots, A_{m,t}^{\pm}, A_{1',t}^{\pm}, \ldots, A_{f',t}^{\pm} \right].
\]

By definition \( \text{up}(\mathcal{X}_{Q}) \) and \( \text{up}(\mathcal{A}_{\tilde{Q}}) \) are the intersections of Laurent polynomial rings. The Lemma follows directly. \( \square \)

**B. Generalities on Minimal Bipartite Graphs.**

We briefly recall the definition and basic constructions of minimal bipartite graphs. We mainly follow the convention used in \([Wen18]\).

Let \( D_n \) be a disk with \( n \) marked points on its boundary labeled \( 1, \ldots, n \) in the clockwise direction. Let \( \Gamma \) be a bipartite graph embedded into \( D_n \) with a single edge connected to every boundary marked point. We draw zig-zag strands on \( \Gamma \) by drawing the following pattern around each vertex according to the color of the vertex

A rank \( a \) minimal bipartite graph on \( D_n \) is a bipartite graph whose zig-zag strands do not self-intersect or form parallel bigons and go from \( i \) to \( i + a \) for every boundary marked point \( i \).

**Definition 5.36.** Let \( \Gamma \) be a rank \( a \) minimal bipartite graph on \( D_n \). Connected components of the complement of \( \Gamma \) are called faces. A boundary face is a face that contains part of the boundary of \( D_n \). Let \( \zeta_i \) be the zig-zag strand going from the boundary marked point \( i \) to \( i + a \). A face \( f \) is said to be dominated by \( \zeta_i \) if it lies to the left of \( \zeta_i \) with respect to the orientation of \( \zeta_i \). The dominating set \( I(f) \) of \( f \) is a collection of the indices of zig-zag strands that dominate \( f \).

It is know that every dominating set is of size \( a \). The set \( \{ \Delta_I(f) \} \) of Plücker coordinates with \( f \) runs through faces of \( \Gamma \) forms a \( K_2 \) cluster chart on \( \mathcal{G}_{r_{a}}^{\times}(n) \) associated to \( Q_{\Gamma} \), where \( Q_{\Gamma} \) is the quiver determined by \( \Gamma \) as in Section 2.4. For example, the cluster \( \{ \Delta_{I(i,j)} \} \) associated to \( Q_{a,n} \) in (2.21) is defined in this way.

There are two types of transformations on minimal bipartite graphs \( \Gamma \) called 2-by-2 moves.

- **Type I**

![Type I Diagram]

A rank \( a \) minimal bipartite graph on \( D_n \) is a bipartite graph whose zig-zag strands do not self-intersect or form parallel bigons and go from \( i \) to \( i + a \) for every boundary marked point \( i \).

![Type I Diagram]

A rank \( a \) minimal bipartite graph on \( D_n \) is a bipartite graph whose zig-zag strands do not self-intersect or form parallel bigons and go from \( i \) to \( i + a \) for every boundary marked point \( i \).

![Type I Diagram]

A rank \( a \) minimal bipartite graph on \( D_n \) is a bipartite graph whose zig-zag strands do not self-intersect or form parallel bigons and go from \( i \) to \( i + a \) for every boundary marked point \( i \).

![Type I Diagram]

A rank \( a \) minimal bipartite graph on \( D_n \) is a bipartite graph whose zig-zag strands do not self-intersect or form parallel bigons and go from \( i \) to \( i + a \) for every boundary marked point \( i \).
The dominating sets associated to the 5 related faces on picture are of the forms

\[ I(f_c) = J \cup \{i, k\}, \quad I(f_s) = J \cup \{k, l\}, \quad I(f_w) = J \cup \{i, l\}, \]
\[ I(f_e) = J \cup \{i, j\}, \quad I(f_n) = J \cup \{j, k\} \]

where \(1 \leq i < j < k < l \leq n\) and \(J\) is an \((a - 2)\)-element subset of \(\{1, \ldots, n\} \setminus \{i, j, k, l\}\). The type I 2-by-2 move changes the dominating set associated to the central face to

\[ I(f'_c) = J \cup \{j, l\}, \]

and keeps the rest intact. Note that one has the Plücker relation

\[ \Delta_{I(f_c)} \Delta_{I(f'_c)} = \Delta_{I(f_s)} \Delta_{I(f_n)} + \Delta_{I(f_e)} \Delta_{I(f_w)}. \]  

(5.37)

On the quiver level, it corresponds to the quiver mutation on \(Q_\Gamma\) at the vertex assigned to the central face, which locally is as follows

\[ \bullet \rightarrow \bullet \quad \leftrightarrow \quad \bullet \rightarrow \bullet \]

Therefore the Plücker relation (5.37) is compatible with the cluster mutation (5.27).

- **Type II.**

A type II 2-by-2 move changes neither the quiver nor the dominating sets of faces.

A result of Thurston [Thu17, Theorem 6] can be restated in the minimal bipartite graph language as saying that any two rank \(a\) minimal bipartite graphs on \(D_n\) can be transformed into one another via a sequence of 2-by-2 moves of the above two types. In conclusion, all the \(K_2\) cluster structures on \(\mathcal{G}r_a^K(n)\) defined by rank \(a\) minimal bipartite graphs on \(D_n\) are equivalent.

**C. Example of a Minimal Bipartite Graph Transforming under \(\rho\).**

Below is an example showing how a minimal bipartite graph transforms under the rotation mutation sequence \(\rho\). This example can be easily generalized to other standard minimal bipartite graph \(\Gamma_{a,n}\) with arbitrary parameters \((a, n)\). The example we choose to do is with parameters \((a = 3, n = 7)\).
D. Connection to Rietsch-Williams’s Cluster Duality

In [RW17], Rietsch and Williams identified the cluster dual of \((\mathcal{Gr}^\times_a(n), D)\) as \((\mathcal{Gr}^\times_a(n) \times \mathbb{G}_m, W_q)\), where \(D = \bigcup_i D_i\) is the same boundary divisor as the one considered in this paper. On the other side, besides ratios of Plücker coordinates, their potential function \(W_q\) also carries an auxiliary variable \(q\) that is the coordinate of \(\mathbb{G}_m\). Below we construct a map to identify the Rietsch-Williams cluster dual with the one considered in this paper and explain the origin of this auxiliary variable \(q\) from our perspective.

Let \([\phi_1, l_1, \ldots, \phi_n, l_n] \in \mathcal{Conf}_n^\times(a)\). Pick a non-zero vector \(v\) in \(l_1\). Define

\[
\Phi : \mathcal{Conf}_n^\times(a) \to \mathcal{Gr}_a^\times(n) \times \mathbb{G}_m
\]

\([\phi_1, l_1, \ldots, \phi_n, l_n] \mapsto ([v, \phi(v), \ldots, \phi^{n-1}(v)], P)
\]
Here $P$ is the twisted monodromy, and $[v, \phi(v), \ldots, \phi^{n-1}(v)]$ denote the matrix representative of the Grassmannian point (we abbreviate the subscripts of $\phi$ to simplify the notation). It is easy to see that $\Phi$ is a biregular isomorphism.

Recall that the potential function constructed by Rietsch and Williams takes the form

$$W_q = \frac{\Delta_{\{b+1,\ldots,n-1,1\}}}{\Delta_{\{b+1,\ldots,n\}}} + \sum_{i=1}^{n-1} \frac{\Delta_{\{i-a+1,\ldots,i-1,i+1\}}}{\Delta_{\{i-a+1,\ldots,i\}}}$$

Fix a volume form $\omega$ on $\mathfrak{k}^a$. For $1 \leq i \leq n-1$, we see that

$$\omega\left(\left(\phi^i(v) - \Phi^*\left(\frac{\Delta_{\{i-a+1,\ldots,i-1,i+1\}}}{\Delta_{\{i-a+1,\ldots,i\}}} \phi^{i-1}(v)\right) \wedge \phi^{i-2}(v) \wedge \cdots \wedge \phi^{i-a}(v)\right)\right)$$

$$= \omega\left(\left(\phi^i(v) - \frac{\omega(\phi^{i-a}(v) \wedge \cdots \wedge \phi^{i-2}(v) \wedge \phi^i(v))}{\omega(\phi^{i-a}(v) \wedge \cdots \wedge \phi^{i-1}(v))} \phi^{i-1}(v) \wedge \phi^{i-2}(v) \wedge \cdots \wedge \phi^{i-a}(v)\right)\right)$$

$$= 0;$$

Comparing it with (2.10) we conclude that

$$\Phi^*\left(\frac{\Delta_{\{i-a+1,\ldots,i-1,i+1\}}}{\Delta_{\{i-a+1,\ldots,i\}}}\right) = \theta_{i-a}.$$

For the remaining term we also see that

$$\omega\left(\left(\phi^n(v) - \Phi^*\left(\frac{\Delta_{\{b+1,\ldots,n-1,1\}}}{\Delta_{\{b+1,\ldots,n\}}} \phi^{n-1}(v)\right) \wedge \phi^{n-2}(v) \wedge \cdots \wedge \phi^b(v)\right)\right)$$

$$= \omega\left(\left(\phi^n(v) - P \frac{\omega(v \wedge \phi^b(v) \wedge \cdots \wedge \phi^{n-2}(v))}{\omega(\phi^b(v) \wedge \cdots \wedge \phi^{n-1}(v))} \phi^{n-1}(v) \wedge \phi^{n-2}(v) \wedge \cdots \wedge \phi^b(v)\right)\right)$$

$$= 0,$$

which implies that

$$\Phi^*\left(\frac{\Delta_{\{b+1,\ldots,n-1,1\}}}{\Delta_{\{b+1,\ldots,n\}}}\right) = \theta_b.$$

Therefore we can conclude that our potential function $\mathcal{W}$ is precisely

$$\mathcal{W} = \Phi^* (W_q).$$

Hence the map $\Phi: (\text{Conf}^\times(a), \mathcal{W}) \rightarrow (\text{Gr}_a^\times(n) \times G_m, W_q)$ is an isomorphism between our version of cluster dual space and Rietsch-Williams’ cluster dual space.

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