Harmonic Space, Self-Dual Yang Mills and the $N = 2$ String.

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Abstract

The geometrical structure and the quantum properties of the recently proposed harmonic space action describing self-dual Yang-Mills (SDYM) theory are analyzed. The geometrical structure that is revealed is closely related to the twistor construction of instanton solutions. The theory gets no quantum corrections and, despite having SDYM as its classical equation of motion, its S matrix is trivial. It is therefore not the theory of the $N = 2$ string. We also discuss the 5-dimensional actions that have been proposed for SDYM.

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1 Introduction

Self-Dual Yang-Mills (SDYM) theory appears to be a basic ingredient of various areas of research in Physics and Mathematics. The instanton solutions \[1, 2, 3\] provide a non-perturbative field theory information, and are powerful tool for classifying smooth and exotic four manifolds \[4\]. SDYM in spacetimes with signature \((2, 2)\) has recently appeared as the effective theory of the \(N = 2\) heterotic string \[5\] (after it has been reduced to two or three dimensions), as well as the \(N = 2\) open string \[6\]. In the \(N = 2\) theories, three-point tree-level S-matrix elements are nontrivial, but do not describe scattering, and the four-point S-matrix elements vanish \[7, 5, 6\]. If, as one would expect, this is true also for the higher-point classical S-matrix elements, this would prove the widely-held belief that SDYM (and self-dual gravity) has no classical scattering \[8\], and is a four-dimensional integrable system with infinite number of degrees of freedom \[9\]. It has been conjectured that all two-dimensional integrable systems are reductions of SDYM \[10\] and, indeed, many integrable systems in two dimensions have been derived via such reductions \[11\]. Thus, SDYM may be a unifying master system for these theories.

Until now work on SDYM has been concentrated mainly on studying the classical equations of motion. It would clearly be interesting to be able to also study the quantum behavior of the theory. This is needed in order to be able to compare SDYM to the \(N = 2\) string at the quantum level, and also may shed light on the poorly understood notion of quantum integrability. Evidently, in order to proceed further one needs an action for the theory. One such action is written in terms of scalars on a Kähler manifold \[12\]. This action has been shown to produce the tree-level scattering amplitudes of the \(N = 2\) heterotic string (at least up to the four-point function) \[5\], but does not successfully describe the loop amplitudes. It has been conjectured that this may be because the action does not preserve enough of the geometric structure and symmetries of SDYM.

Because of this, there has been much interest in a harmonic space action recently proposed by Kalitzin and Sokatchev \[13\]. In this paper we first study the geometrical structure of SDYM in harmonic space, and elucidate the connection between the harmonic and twistor formalisms. We see that the harmonic-space action does indeed possess many of the geometrical properties desired for a SDYM theory. We then look at the quantum properties of the action, and show that its S matrix is trivial. The theory therefore does not describe the \(N = 2\) string. In the conclusion we discuss why this may be the case, and also discuss the two five-dimensional formalisms for SDYM proposed by Nair and Schiff \[12\].
2 Actions and the geometrical structure of SDYM

As a preliminary to understanding the geometrical structure of the harmonic space action, we first briefly describe the twistor interpretation of instantons, i.e. gauge fields with self-dual field strengths $F$ [4, 5]. This geometrical description will then be explicitly carried out using spinor notation, in order to connect it to the more algebraic harmonic approach, following which we reach the action of ref [13].

Complex coordinates and twistors

Any 2-form $F$ on $R^4$ can be decomposed into self-dual and anti-self-dual forms $F^+$ and $F^-$ as
\[ F = F^+ + F^- , \] (2.1)
where $*F^\pm = \pm F^\pm$, with $*$ being the Hodge operator. Group theoretically, this corresponds to the decomposition of the reducible 6-dimensional representation of $SO(4)$ into a $3_L$ and $3_R$ of $SU(2)_L \times SU(2)_R \simeq SO(4)$. In the twistor approach, one needs to introduce complex coordinates on $R^4 \simeq C^2$. This means choosing a complex structure on $R^4$, so the “Lorentz” group is reduced from $SO(4)$ to $U(2)$. On a complex manifold the exterior derivative $d \rightarrow \partial + \bar{\partial}$, so the 2-form $F$ naturally decomposes into
\[ F = F^{(2,0)} + F^{(1,1)} + F^{(0,2)} , \] (2.2)
where $(\alpha, \beta)$ refer to the degree of the form with respect to $(\partial, \bar{\partial})$. If $F$ is self-dual then
\[ F^{(2,0)} = 0 = F^{(0,2)} , \] (2.3)
but the converse is not necessarily true. This is because $F^{(1,1)}$ is a reducible 4-dimensional representation of $U(2)$, and can be further decomposed as
\[ F^{(1,1)} = F^{(1,1)}_{(0)} + \mu . \] (2.4)
Now $F^{(1,1)}_{(0)}$, which is irreducible, corresponds to the self-dual piece $F^+$, so $\mu$, which corresponds to the metric, must vanish for $F$ to be self-dual.

There are now two approaches to using this result to get equations implying SDYM. The first is simply to require that $F$ be a $(1,1)$ form with $\mu = 0$. This can be achieved by noting that eq. (2.3) implies that
\[ A = g^{-1} \partial g \]
\[ \bar{A} = g^\dagger \bar{\partial} g^{\dagger -1} . \] (2.5)
If \( g \) is taken to be hermitian, which can be done by an appropriate gauge choice, then the equation \( \mu = 0 \) becomes

\[
\eta^{\mu\nu} \partial_{\mu}(g^{-2} \partial_{\nu}g^2) = 0 ,
\]

which is the equation for SDYM originally written by Yang \[14\]. If one defines \( g = e^{\phi/2} \), eq. (2.6) becomes the equation of motion of the \( N = 2 \) heterotic string \[3\], and also of the open \( N = 2 \) string \[4\]. This equation of motion can be derived from an action for \( \phi \) that consists of an infinite series of terms \[4\], but can be rewritten more elegantly in five dimensions as a “Kähler-Chern-Simons” theory \[12\]. One can also write a five-dimensional action directly in terms of \( A \) that implies both that \( F \) is a \((1, 1)\) form and that \( \mu = 0 \) \[12\]. We shall discuss these actions in the conclusion of the paper.

The second approach is that taken in twistor theory. Both the twistor and harmonic-space formalisms are naturally written in Euclidean spacetimes with signature \((4, 0)\), and the Wick rotation to a \((2, 2)\) signature is not straightforward. We shall therefore be in Euclidean space from now on. The twistor construction of instantons uses the fact that, instead of explicitly demanding that \( \mu \) vanishes, one can rather require that \( F \) be of type \((1, 1)\) for all the complex structures (with a fixed orientation) that one can introduce on \( R^4 \). This is true because changing the complex structure mixes \( F^{(2,0)} \), \( F^{(0,2)} \) and \( \mu \) so, if \( F^{(2,0)} \) and \( F^{(0,2)} \) always vanish, only \( F^{(1,1)} \) can remain and \( F \) must be self-dual. Since the set of complex structures on \( R^4 \) is \( SO(4)/U(2) \approx S^2 \), one can find instantons by looking for \((1, 1)\) forms on \( R^4 \times S^2 \), and projecting them back to \( R^4 \). Thus one has the following theorem (compactifying \( R^4 \) to \( S^4 \)): A 2-form \( F \) on \( S^4 \) is self-dual iff its lift to \( CP^3 \approx S^4 \times S^2 \) is of type \((1, 1)\) \[15\].

At this stage one would like to write an action to implement these ideas. However, this is quite nontrivial. One can find an action whose equation of motion leads to \( F \) being \((1, 1)\) on a complex 3 manifold, but only if it is a Calabi-Yau manifold \[16\]. This is not the case for the projective twistor space \((CP^3)\), since the first Chern class of the \( S^2 \) piece is not zero. The action of \[16\] therefore contains a singularity on \( CP^3 \). As we shall see in the following, the harmonic-space action proposed in \[13\] does result in \( F \) being a \((1, 1)\) form on \( CP^3 \), where each different complex structure on \( R^4 \) is parameterized by a point in the fiber. However, it does this in a rather indirect fashion, and the action does not contain singularities.

**Spinor notation and twistors**

*As is well known, one can not have SDYM in Minkowski spacetime with signature \((3, 1)\), unless one relaxes the hermiticity of the vector potential.*
In order to clarify the content of the previous section, and to continue to harmonic space, it is necessary to use spinor notation. The four coordinates of spacetime are thus written as $x^\mu_i$, with $\mu$ and $i$ indices of $SU(2)_L$ and $SU(2)_R$ respectively, and satisfy the reality condition\footnote{Our conventions are $\varepsilon_{12} = 1$, $\varepsilon^{12} = 1$, $x_i = \varepsilon_{ij}x^j$, $x^j = \varepsilon^{ij}x_i$.}

$$x^\mu_i = \varepsilon^{ij}\varepsilon^{\mu\nu}(x^\nu_j)^* . \quad \text{(2.7)}$$

Similarly, the covariant derivative is $D_\mu \equiv \partial_\mu + A_\mu$. The Yang-Mills field strength now becomes

$$F_{\mu\nu j} = [D_{\mu\nu}, D_{\nu j}] \equiv \varepsilon_{ij} F^+_{(\mu\nu)} + \varepsilon_{\mu\nu} F^-_{(ij)} , \quad \text{(2.8)}$$

where the last identity uses the fact that an antisymmetric 2 tensor of $SU(2)$ must be proportional to the $\varepsilon$ symbol. Since $F^+_{(\mu\nu)}$ and $F^-_{(ij)}$ clearly transform as the $(3, 1)$ and $(1, 3)$ representations of $SU(2)_L \times SU(2)_R$, respectively, they are the self-dual and anti-self-dual pieces of the curvature. We thus see explicitly the decomposition of eq. (2.1), and see that the self-duality equation is simply

$$F^-_{(ij)} = 0 . \quad \text{(2.9)}$$

To see the meaning of this on $C^2$, we now introduce complex coordinates:

$$z^\mu = x^\mu_1$$
$$\bar{z}_\mu = (z^\mu)^* = x^\mu_2 = \varepsilon_{\mu\nu}x^{\nu 2} . \quad \text{(2.10)}$$

Since this definition does not break $SU(2)_L$, we see that the complex Lorentz group $U(2)$ is $SU(2)_L \times U(1)_R$, where $U(1)_R$, the diagonal subgroup of $SU(2)_R$, multiplies $z^\mu$ by a phase. Using the definition of $F$ in eq. (2.8), we see that

$$F_{z^\mu, z^\nu}^{(2, 0)} = \varepsilon_{\mu\nu} F^-_{11}$$
$$F_{\bar{z}_\mu, \bar{z}_\nu}^{(0, 2)} = - \varepsilon^{\mu\nu} F^-_{22} , \quad \text{(2.11)}$$

and

$$F^-_{12} = - \frac{1}{2} F_{z^\mu, \bar{z}_\mu}^{(1, 1)} . \quad \text{(2.12)}$$

This last term is (proportional to) $\mu$ so, as promised, we see that self-duality is equivalent to $F$ being a $(1, 1)$ form with $\mu = 0$. Also, since changes of complex structure mix $F^-_{11}$, $F^-_{12}$ and $F^-_{22}$, one sees explicitly that if $F$ is a $(1, 1)$ form for all complex structures, it is self-dual.

For the twistor approach, we now want to work on a fiber bundle with base space $R^4$ and all complex structures (with fixed orientation) on $R^4$ as the fiber. The fiber is
$SU(2)_R/U(1)_R \simeq S^2$. This can be seen by considering the space as the different ways of defining $z^\mu$, modulo the complex Lorentz group. Then the space is covered by rotating $z^\mu$ by an $SU(2)_R$ matrix $u^i_j$ in the definition eq. (2.10):

$$z^\mu \rightarrow x^{\mu j} u^1_j = z^\mu u^1_1 - \varepsilon^{\mu \nu} \bar{z}^\nu u^1_2,$$

modulo the $U(1)_R$ of multiplication of $z^\mu$ by a phase. To work in twistor space, one now needs coordinates on the projective twistor space. Unfortunately, by choosing an explicit parameterization of $S^2$ one obscures the symmetries of the $S^2$ making it harder to understand the resulting physics.

**Harmonic space**

In the harmonic approach\[\footnote{\text{The harmonic space construction presented here is that of ref \[17\]. Only the geometric interpretation, and the relation to twistor theory are original work. For more details on harmonic theory, see ref \[17\] and references therein.}} S^2 is instead represented as $SU(2)/U(1)$, where a new $U(1)$ is introduced to achieve the coseting \[15\]. The coordinates of harmonic space are the four-dimensional coordinates $x^{\mu i}$, and the harmonic coordinates $u^{\pm i}$, defined as

$$\begin{pmatrix} u^{+i} \\ u^{-i} \end{pmatrix} \in SU(2).$$

There are only three independent $u^{\pm i}$, parameterizing $SU(2)$, since they satisfy the $SU(2)$ orthogonality condition

$$u^{+i} u^{-i} \equiv \varepsilon_{ij} u^{+i} u^{-j} = 1.$$

Note that, unlike the usual approach to harmonic space, it is now clear that the $SU(2)$ of the harmonic space is the same as the $SU(2)_R$ of spacetime, since the harmonic coordinates are designed to give the space of complex structures of spacetime. The $+$ and $-$ signs of the $u^{\pm i}$’s indicate their transformations under the new $U(1)$ transformation, which acts on the $u$ matrix in eq. (2.14) by left-multiplication.

The harmonic space is reduced to the fiber bundle $R^4 \times S^2$, by imposing the condition that all fields in the space have a fixed $U(1)$ charge. A function $f^{(q)}(u)$ with $U(1)$ charge $q$ can then be defined by its harmonic expansion:

$$f^{(q)}(u) = \sum_{n=0}^{\infty} f^{(i_1 \ldots i_{n+q} j_1 \ldots j_n)} u^+_{i_1} \ldots u^+_{i_{n+q}} u^-_{j_1} \ldots u^-_{j_n}.$$
Here the $f$'s are symmetric, since any antisymmetric piece can be reduced using eq. (2.15). They are therefore irreducible $SU(2)$ tensors. Harmonic space therefore describes $S^2$ and general functions on it without using an explicit parametrization of the sphere.

One now wants to define harmonic-space integration and differentiation. The only integration rule which is $SU(2)$ invariant is

$$\int d^2u 1 = 1 \quad \int d^2u u_{i_1}^{+} \ldots u_{i_m}^{+} u_{j_1}^{-} \ldots u_{j_n}^{-} = 0 \quad (m + n \neq 0) . \quad (2.17)$$

Differentiation on $S^2$ is given in terms of the three Lie-derivatives $D^{++}$, $D^{--}$ and $D^0$. They are defined as

$$D^{++} \equiv u^{+i} \frac{\partial}{\partial u^{-i}}$$
$$D^{--} \equiv u^{-i} \frac{\partial}{\partial u^{+i}}$$
$$D^0 \equiv u^{+i} \frac{\partial}{\partial u^{+i}} - u^{-i} \frac{\partial}{\partial u^{-i}} . \quad (2.18)$$

Note that $D^0$ is the generator of the new $U(1)$. Finally, derivatives in the spacetime directions are lifted into the fiber bundle $R^4 \times S^2$ by defining

$$\partial^{+}_\mu \equiv u^{+i} \partial^{i}_{\mu}$$
$$\partial^{-}_\mu \equiv u^{-i} \partial^{i}_{\mu} . \quad (2.19)$$

The crucial feature of harmonic space is that $\partial^{+}_\mu$ and $\partial^{-}_\mu$ are Lorentz-covariant descriptions of $\partial_\mu$ and $\partial_\mu$. This can be seen by examining the transformation changing the complex structure in eq. (2.13), and the definition of the $u$'s in eq. (2.14). The complex structure of the fiber bundle is thus built into the derivatives.

Now, to describe SDYM using the harmonic formalism, one starts with the desired solution: Consider an ordinary four-dimensional vector potential $A_{\mu i}(x)$ on $R^4$. The four-dimensional covariant derivative $D_{\mu i} \equiv \partial_{\mu} + A_{\mu i}(x)$ is lifted into covariant derivatives on the fiber bundle as

$$D^\pm_{\mu} \equiv \partial^\pm_{\mu} + A^\pm_{\mu} = u^{\mp i} D_{\mu i} . \quad (2.20)$$

Note that the lifted connection is purely horizontal, i.e. it has no components in the direction of the fiber. Also, $A^+_{\mu}$ is linear in $u^{+i}$ and does not depend on $u^{-i}$. Using the definition of $D^{++}$ in eq. (2.18), and comparing to the expansion of a general $A^+_{\mu}$ as in eq. (2.16), this condition can be written as $D^{++}A^+_{\mu} = 0$, or

$$[D^{++}, D^+_{\mu}] = 0 . \quad (2.21)$$
One can return to the four-dimensional gauge field by projecting $A^+_\mu(x,u)$ to $R^4$. The projection is performed by integration:

$$A_{\mu i}(x) = \int d^2 u \, 2 \, u^- A^+_\mu(x,u),$$

(2.22)
as can be seen using eqs. (2.20) and (2.17). (Note that, since one can find the four-dimensional gauge field from $A^+$ alone, the corresponding equations for $A^-_\mu$ are redundant.)

The curvature is lifted in the same way to give:

$$F^{++}_{\mu\nu} = \varepsilon_{\mu\nu} \, u^i u^j F_{(ij)}(x)$$
$$F^{--}_{\mu\nu} = \varepsilon_{\mu\nu} \, u^i u^{-j} F_{(ij)}(x)$$
$$F^{+-}_{\mu\nu} = \varepsilon_{\mu\nu} \, u^i u^{-j} F_{(ij)}(x) + F_{(\mu\nu)}(x).$$

(2.23)

$F^{++}_{\mu\nu}$, $F^{--}_{\mu\nu}$ and $F^{+-}_{\mu\nu}$ are the $(2, 0)$, $(0, 2)$ and the $(1, 1)$ components of the curvature, respectively. Note that the $(1, 1)$ component consists of two orthogonal pieces with dimensions one and three, as in eq. (2.4).

The condition for $F$ to be self-dual, $F_{(ij)} = 0$, again implies that only the irreducible $(1, 1)$ piece of $F$ survives. For the $(2, 0)$ component, we have:

$$F^{++}_{\mu\nu} = 0 \quad \Rightarrow \quad [D^+_\mu, D^+_\nu] = 0.$$  

(2.24)

(This equation is the integrability condition for the equation $D^+_\mu \phi = 0$.) As stated in the twistor discussion, the vanishing of this component for all $u$, corresponding to the vanishing for all complex structures, is sufficient to show the self-duality of $F$. This can be seen by recovering the complete anti-dual part of the four-dimensional field strength from $F^{++}$:

$$F_{ij}(x) = \int d^2 u \, \frac{3}{2} \, u_i \, u^-_j \, \varepsilon^{\mu\nu} F^{++}_{\mu\nu} = 0.$$  

(2.25)

Thus, eq. (2.24), together with the constraint (2.21) can be regarded as the equations of motion of the theory [17].

**The action for SDYM in harmonic space**

To get an action for SDYM, it is easiest to first solve eq. (2.24). The general solution is

$$D^+_\mu(x,u) = e^{-v(x,u)} \partial^+_\mu e^{v(x,u)}. $$

(2.26)

With this definition of $D^+_\mu$, the remaining equation of motion, eq. (2.21), becomes

$$[D^{++}, e^{-v} \partial^+_\mu e^v] = 0 \quad \iff \quad [e^v D^{++} e^{-v}, \partial^+_\mu] = 0.$$  

(2.27)
Kalitzin and Sokatchev proposed an action for SDYM theory that uses a lagrange-multiplier field $P^{(-3)\mu}(x, u)$ to enforce this condition \[3\]. $P^{(-3)\mu}$ has a $U(1)$ charge $-3$, and is in the adjoint of the gauge group. The complete action is

$$ S_0 = \int d^4x \, d^2u \, Tr \left( P^{(-3)\mu} \partial^+_\mu \left( e^v D^{++} e^{-v} \right) \right). $$

The action has the usual gauge transformation

$$ e^{v(x,u)} \to e^{v(x,u)} e^\tau(x), $$

$$ P^{(-3)\mu}(x, u) \to P^{(-3)\mu}(x, u), $$

with a parameter $\tau$ that is independent of the harmonic coordinates. Under this transformation, using eq. (2.26), one sees that the covariant derivative $D^+\mu$ transforms as

$$ D^+\mu \to e^{-\tau(x)} D^+\mu e^{\tau(x)}, $$

as expected. The harmonic-space derivative $D^{++}$ is unaffected. Note that $P^{(-3)\mu}$ does not transform under the gauge transformation, even though it appears to be in the adjoint of the group\[\] The action is also invariant under the $P^{(-3)\mu}$ gauge transformation

$$ P^{(-3)\mu} \to P^{(-3)\mu} + \partial^{\mu+} b^{(-4)}(x, u). $$

As expected, variation with respect to $P^{(-3)\mu}$ yields the SDYM constraints. The usual problem with lagrange-multiplier actions is that the equations of motion of the other fields lead to the lagrange-multiplier field propagating \[19\]. What is unusual in this theory is that the variation of the action with respect to $v(x, u)$ implies the equation

$$ \partial^+ \mu P^{(-3)\mu} = 0. $$

The $P^{(-3)}$ field is therefore completely decoupled from the Yang-Mills field. Furthermore, because of the invariance (2.31), Kalitzin and Sokatchev argue that there is no nontrivial $P^{(-3)}$ that satisfies eq. (2.32), so $P^{(-3)}$ does not describe a new degree of freedom. The action $S_0$ therefore should describe the pure SDYM system.

Notes.

The equations of motion derived from the harmonic space action (2.28) are not simply $F^{(2,0)} = F^{(0,2)} = 0$ in a 3-complex dimensional space. The constraint that $F^{(2,0)}$ vanishes is

\[\] One can define the field $\tilde{P}^{(-3)} = e^{-v} P^{(-3)} e^v$, which does transform in the adjoint of the group. We shall need this in the background field quantization of the theory.
implemented via the definition of $D_\mu^+$ in eq. (2.26). Once the equation of motion eq. (2.21) fixes the form of the lift to $R^4 \times S^2$, this implies the vanishing of $F_{(ij)}$. The structure of the theory is therefore different from that of ref. [16]. By using the harmonic expansions of eq. (2.16), one can verify that the action is well defined and does not suffer from the singularities of ref. [16].

In addition to the aforementioned gauge transformations, the action is invariant under the transformations

$$e^\nu \rightarrow e^\lambda e^\nu,$$
$$P^{(3)-\mu} \rightarrow e^\lambda P^{(3)-\mu} e^{-\lambda},$$

where $\partial_\nu^+ \lambda(x, u) = 0$. This implies that $\Box \lambda = 0$, so this transformation is not a gauge invariance. The significance of this transformation will be discussed later.

3 Quantization of SDYM theory

The most important aspect of the action $S_0$ can be seen without dealing with the details of the theory: While the action is nonpolynomial in $v$, it is linear in $P^{(3)-\mu}$. $P^{(3)-3}$, therefore, always appears in the path integral as $P^{(3)-3}/\hbar$, so there are $1 - l$ external $P^{(3)-3}$’s in any $l$-loop Green function. Therefore, all $S$-matrix elements at tree level have one $P^{(3)-3}$ with an arbitrary number of $v$’s, all one-loop elements have an arbitrary number of $v$’s and no $P^{(3)-3}$’s, and there are no higher loop diagrams. This statement will hold as long as the theory is one-loop finite, so that the structure of the action is not spoiled by counterterms.

Since the results of the loop calculations in this theory involve some subtleties, we shall perform the quantization in two different ways: first a straightforward quantization, and then a quantization using a background field approach. In the next section, we shall give a simple argument, modulo issues of the measure of the theory, to explain our result—that the theory is actually a free theory. The reader who does not wish to go through the details of the calculation may be reassured that few of them are necessary for the understanding of the basic issues.

Standard quantization

Because of the two gauge symmetries (2.29) and (2.31) of the action, one needs to
gauge fix $S_0$. Appropriate gauge fixing conditions for the symmetries are \[13\]:

\[
\int d^2 u \, v(x, u) = 0 \\
\partial_\mu P^{(-3)\mu} = 0 ,
\]
respectively. The total gauge-fixed action is then $S = S_0 + S_{gf} + S_{gh}$, where

\[
S_{gf} = - \int d^4 x \, d^2 u \, \text{Tr} \left( \rho(x) v(x, u) + \Lambda^{(4)}(x, u) \delta_{\mu} P^{(-3)\mu}(x, u) \right) ; \\
S_{gh} = \int d^4 x \, d^2 u \, \bar{C}^a(x) \left( \delta^{ab} - \frac{1}{2} f_{abc} v^c(x, u) + \frac{1}{12} f_{aef} f^{efb} v^d + \cdots \right) C^b(x) \\
- \int d^4 x \, d^2 u \, \text{Tr} \left( \frac{1}{2} \chi^{(+4)}(x, u) \Box \chi^{(-4)}(x, u) \right) .
\]

Here $\rho(x)$ and $\Lambda^{(4)}(x, u)$ are the Landau-gauge lagrange multipliers, $C(x)$ and $\chi^{(-4)}(x, u)$ are the ghosts and $\bar{C}(x)$ and $\bar{\chi}^{(+4)}(x, u)$ are the anti-ghosts.

The propagators are \[13\]:

\[
\langle v^a(-p, u_1) \, P^{(-3)\mu a}(p, u_2) \rangle = 2 i \delta^{ab} \frac{p^{\mu} - u_1^+ u_2^-}{p^2} u_1^+ u_2^+ \\
\langle P^{(-3)\mu a}(-p, u_1) \, \Lambda^{(+4)b}(p, u_2) \rangle = 2 i \delta^{ab} \frac{p_1^+}{p^2} \delta^{(-4,4)}(u_1, u_2) \\
\langle v^a(-p, u) \, \rho(p) \rangle = -\delta^{ab} \\
\langle \chi^{(-4)a}(-p, u_1) \chi^{(+4)b}(p, u_2) \rangle = 2 \frac{\delta^{ab}}{p^2} \delta^{(-4,4)}(u_1, u_2) \\
\langle C^a(-p) \, \bar{C}^b(p) \rangle = -\delta^{ab} .
\]

Here, $\delta^{(q,-q)}$ and $u_1^+ u_2^- / u_1^+ u_2^+$ are harmonic distributions that are singular when $u_1 = u_2$. They are related by

\[
D_1^{++} \frac{u_1^+ u_2^-}{u_1^+ u_2^+} = \delta^{(2,-2)}(u_1, u_2) ,
\]

and are defined by their series expansions. For example, the harmonic space delta-functions are defined as \[13\]:

\[
\delta^{(q,-q)}(u_1, u_2) = \sum_{n=0}^\infty (-1)^{n+q} \frac{(2n + q + 1)!}{n!(n + q)!} (u_1^+)^{(n+q)}(u_1^-)^n(u_2^+)^n(u_2^-)^{n+q} .
\]

The theory contains two types of vertices, as can be read from eqs. \eqref{2.28} and \eqref{3.2}:

\footnote{Here, $u_1^+ u_2^+$ denotes $u_1^+ u_2^+$, etc.} The first type contains an arbitrary number of $v$ fields coupled to a $P^{(-3)}$. The second
consists of a $C$, a $C$ and an arbitrary number of $v$ fields. As explained above, all physical
one-loop diagrams contain only external $v$'s, with something running around the loop.
Since there are no vertices involving the $\Lambda^{(+4)}$ or $\rho$ fields, and since $\chi^{(-4)}$ and $\bar{\chi}^{(+4)}$ are
free, none of them appear in diagrams. There are thus only diagrams involving a $C$-ghost
loop and diagrams with a loop of $\langle v P^{(-3)} \rangle$ propagators. Generic examples of the two
types of diagrams are shown in Fig. 1:

Consider first the ghost diagrams: Note that, since the $\tau$ gauge fixing did not involve
derivatives, the ghost propagator is trivial in momentum space and the ghost vertex has
no momenta. The integration over the loop momenta is therefore trivial, and simply gives
a factor of $\int \frac{d^4p}{(2\pi)^4} = \delta^4(0)$. As an example, the 2-point ghost-loop correction to the
effective action is

$$\delta S_2 = -\frac{c_v}{24} \int d^2u_1 d^2u_2 d^4x \ Tr (v(x, u_1)v(x, u_2)) \delta^4(0). \quad (3.6)$$

The effective action contains infinitely many such terms, with arbitrary numbers of $v$
fields. These terms are not very pretty geometrically, since they are nonlocal in harmonic
space. However, if one use dimensional regularization, which is the only gauge invariant
regularization available, $\delta^4(0) \to 0$ and all these diagrams vanish.

The second type of diagram vanishes for a similar reason: In this case, the $\langle v P^{(-3)} \rangle$
propagator does contain a $1/p^2$ factor. However, in the $P^{(-3)}v^n$ vertices, $P^{(-3)}$ always
appears in the form $\partial^+ P^{(-3)\mu}$ (see eq. (2.28)). One therefore only needs the effective
propagator

$$\langle v^a(-p, u_1) \ P^+ P^{(-3)\mu} (p, u_2) \rangle = -i \delta^{ab} \frac{u_1^+ u_2^-}{u_1^+ u_2^-}, \quad (3.7)$$

which is again trivial in momentum space. This is not surprising, since $p^+_\mu P^{(-3)\mu} = 0$ by
the equation of motion of $P^{(-3)}$ (eq. (2.32)). The remaining part of the vertex (with the
$p^+_\mu$ factor removed) also has no space-time momenta, so the diagrams again all contain a
factor of $\delta^4(0)$.

There is, however, one important difference between the ghost and the $P^{(-3)}v$ dia-
grams. Since the $\langle v P^{(-3)} \rangle$ propagator is nontrivial in harmonic space, the $Pv$
diagrams contain complicated, and singular, harmonic space factors, such as $\delta^{(0,0)}(u, u)$ and
$1/(u^+ u^+)$. If we again use dimensional regularization in spacetime, and assume that the
harmonic-space divergences can be regulated\footnote{For example, $\zeta$-function regularization implies that $\delta^{(0,0)}(u, u) = \sum_l (2l + 1) \to 1/12$. Alternatively, one can regularize by restricting the sum to $l \leq L$. However, it is not clear how to perform these regularizations consistently for the different harmonic-space factors.}—so that $0 \times \infty \to 0$—we can conclude that
the one-loop corrections to the effective action vanish. The theory then requires no counterterms, so our argument that higher-loop corrections do not exist is valid. We conclude that the tree-level theory is exact. However, since this conclusion is based on somewhat delicate arguments, it will be useful to consider the background-field quantization to support it.

Quantum corrections in the background-field formalism

In the background-field formalism, one splits the fields of the theory into classical and quantum pieces. Thus,

\[ e^v \rightarrow e^{v_{cl}} e^v \]
\[ P^\mu \rightarrow e^{v_{cl}} (P^\mu_{cl} + P^\mu) e^{-v_{cl}} . \]

These slightly nontrivial splittings have been chosen to get nice gauge-transformation properties of the fields. Substituting these definitions into the action of eq. (2.28), one obtains the background-field action

\[ S_0 = \int d^4x d^2u \text{Tr} \left( (P^{(-3)\mu} + P^{(-3)\mu}_{cl}) \left[ \nabla^+_{\mu}, (e^v D^{++} e^{-v}) \right] \right) , \]

where

\[ \nabla^+_{\mu} = e^{-v_{cl}} \partial_\mu e^{v_{cl}} \equiv \partial_\mu + A^+_{\mu cl} , \]

in analogy to eq. (2.26).

Note that \( v_{cl} \) appears in the action only in the combination \( A^+_{\mu cl}(v) \). This means that the geometrical meaning of the effective action—and the structure of possible counterterms—will be much more transparent in the background field approach. Indeed, \( S_0 \) is invariant under the classical \( \tau(x) \) transformation

\[
\begin{pmatrix}
e^v \\
\nabla^+_{\mu} \\
P^{(-3)\mu}_{cl}
\end{pmatrix}
\rightarrow
e^{-\tau_{cl}(x)}
\begin{pmatrix}
e^v \\
\nabla^+_{\mu} \\
P^{(-3)\mu}_{cl}
\end{pmatrix}
\]

under which \( A^+_{\mu cl} \) transforms as a connection. \( S_0 \) is also invariant under the classical \( b^{(-4)}_{cl}(x, u) \) transformation

\[ P^{(-3)\mu}_{cl} \rightarrow P^{(-3)\mu}_{cl} + \nabla^\mu + b^{(-4)}_{cl}(x, u) , \]

where we have used the fact that

\[ F^{++}_{\mu\nu cl} = [\nabla^+_{\mu}, \nabla^+_{\nu}] \equiv 0 . \]
The action is also invariant under the quantum gauge transformations:

\[ e^v \rightarrow e^v e^{\tau(x)} \]
\[ P^\mu \rightarrow P^\mu + \nabla^+_\mu b(x, u) , \tag{3.14} \]

which need to be gauge-fixed. Since we want the full effective action to be invariant under the classical gauge transformations, the gauge choice must be covariant under background transformations. The gauge-fixing conditions (3.1) therefore become

\[ \int d^2 u v = 0 \]
\[ \nabla^-_\mu P^\mu = 0 . \tag{3.15} \]

The second gauge fixing one requires the introduction of the classical field \( A^-_{\mu cl} \), which is not in the original theory. The only requirement that needs to be satisfied in defining \( A^-_{\mu cl} \) is that it transforms in the required way. Such a field can be defined but it is not unique. For example, two possible definitions are

\[ \nabla^-_\mu \equiv e^{-v_{\mu cl}} \partial_\mu e^{v_{\mu cl}} , \]
\[ A^-_{\mu cl}(x) = u^{-i} A_{\mu cl}(x) , \]

with \( A_{\mu cl}(x) \) defined by projection as in eq. (2.22). It is a check of the theory that a particular definition of \( A^-_{\mu cl} \) should not be needed, since \( \nabla^-_\mu \) appears only in the gauge-fixing part of the lagrangian. In fact, \( A^-_{\mu cl} \) should not appear at all in the effective action.

The gauge-fixing and ghost actions are now

\[ S_{gf} = - \int d^4 x d^2 u \text{ Tr} \left( \rho(x) v(x, u) + \Lambda^{(4)}(x, u) \nabla^-_\mu P^{(-3)\mu}(x, u) \right) ; \]
\[ S_{gh} = \int d^4 x d^2 u C^a(x) \left( \delta^{ab} - \frac{1}{2} f^{abc} v^c(x, u) + \frac{1}{12} f^{ace} f^{edb} v^e v^d + \cdots \right) C^b(x) \tag{3.16} \]
\[ - \int d^4 x d^2 u \text{ Tr} \left( \bar{\chi}^{(+4)}(x, u) \nabla^-_\mu \nabla^+ \chi^{(-4)}(x, u) \right) . \]

In order to do one-loop calculations, one needs only the part of the action quadratic in the quantum fields. This gives the vertices shown in Fig. 2. The propagators are still given by eq. (3.3). Note that the \( \chi^{(-4)} \) ghosts are no longer abelian, and do not decouple. However, since there are no vertices between \( \rho, C \) or \( \bar{C} \) and the quantum fields, these fields do not appear in the calculation.

Let us first consider the one-loop two-point functions. The diagrams for them are depicted in Fig. 3: Diagram (a), with the loop of \( \langle v P^{(-3)} \rangle \) propagators, gives a pure \( A^+ A^+ \) contribution to the effective action. Similarly, diagram (b), with the \( \langle P^{(-3)} \Lambda^{(+4)} \rangle \) loop, gives a pure \( A^- A^- \) contribution. These terms are exactly canceled by the \( A^+ A^+ \) and \( A^- A^- \) parts of the ghost diagram (d). This means that the effective action contains only \( A^+ A^- \) terms, from the mixed diagram (c) and the remains of the ghost diagrams.
resulting 2-point effective action is
\[ \delta S_2 = -\frac{1}{2} \int \frac{d^4 p}{(2\pi)^4} \int d^2 u \delta^{00}(u, u) \text{Tr} \left( F_{cl \text{quad}}^{++}(-p, u) F_{cl \text{quad}}^{--}(p, u) \int \frac{d^4 l}{(2\pi)^4} \frac{1}{l^2(l+p)^2} \right), \]
where “quad” indicates the part of $F_{cl}$ quadratic in $A_{cl}$. Since $F_{cl}^{++} \equiv 0$, $F_{cl \text{quad}}^{++}$ contributes only terms cubic in $A_{cl}$. The one loop 2-point correction to the effective action therefore vanishes\[.\]

We can now continue to the 3-point functions. The diagrams are similar to the 2-point case, and will not be shown explicitly. As before, the pure $(A_{cl}^+)^3$ and $(A_{cl}^-)^3$ terms are canceled between the ghost and nonghost diagrams. The remaining diagrams give two contributions. The first completes the $F_{cl \text{quad}}$’s in eq. (3.17) towards the full $F_{cl}$’s. The second gives terms schematically of the form
\[ \delta S_3 \sim \int d^2 u \delta^{00}(u, u) \text{Tr} \left( F_{cl \text{quad}}^{++} F_{cl \text{quad}}^{--} \left( k^- A_{cl}^+ + k^+ A_{cl}^- \right) \right) \int \frac{d^4 l}{(2\pi)^4} \frac{1}{l^2(l-p)^2(l+r)^2}, \]
where $k^\pm$ are some combinations of the momenta. As in the 2-point case the identity $F_{cl}^{++} \equiv 0$ leads to the vanishing of the 3-point one-loop effective action.

Note that, as desired, $A_{\mu \text{cl}}$ does not appear in the quadratic or cubic parts of the effective action, since they vanished using just the definition of $A_{\mu \text{cl}}$. It is not easy to show that this property persists to the $n$-point function, but it should by the general argument given above. The only remaining calculation, therefore, is that of the $(A_{cl}^+)^n$ piece of the effective action. As before, one sees that this cancels between the ghost and nonghost diagrams. The one-loop effective action therefore vanishes identically.

In fact, this result could have been foreseen using just the background gauge invariance. As discussed above, the one-loop action contains only $v$ fields, with no $P^{(-3)}$ fields, and the $v$ fields have to appear in the form $A_{\mu}^+$. Since the only gauge-invariant function of $A_{\mu}^+$ is $F^{++}$, which vanishes identically, the one-loop effective action must vanish in this gauge. The explicit calculation has shown that there are no anomalies to upset this argument.

The background-field calculation has therefore confirmed the result of the straightforward quantization, and has justified the use of the dimensional-regularization argument. We remind the reader that the vanishing of the one-loop S-matrix means that all higher loop contributions also vanish, so the tree-level action is exact.

\[\text{In principle, one again has to regularize the harmonic divergences. However, in this case the divergence multiplies something that vanishes identically. Also, there is only one divergence in the background field case, since } 1/(u^+ u^+) \text{ terms never appear. One therefore expects no subtleties in the regularization.}\]
4 The S-matrix and the spectrum of the harmonic-space theory

We now turn to the classical S-matrix which we have shown to be the full S-matrix of the theory. As was seen in the beginning of section 3, the S-matrix elements will have one external \( P^{(-3)} \) field and an arbitrary number of \( v \) fields. In order to calculate an S-matrix element, one needs to know what states to put on the external legs. At this point, one might wonder why the S-matrix elements have an external \( P^{(-3)} \) field, since classical SDYM is supposed to be described by the \( v \) field alone. (The \( P^{(-3)} \) field was introduced simply as a lagrange multiplier.) Indeed, the presence of this field as an external state implies the triviality of the S matrix:

Consider a generic S-matrix element, depicted in Fig. 4. All the \( P^{(-3)} \)'s in the diagram appear in the form \( \partial_\mu P^{(-3)} \). In the internal lines, this implies that the diagram is local in harmonic space. However, on the external lines, this factor of \( \partial_\mu \) acts on the \( P^{(-3)} \) wave function. Since the equation of motion of \( P^{(3)} \) is \( \partial_\mu P^{(-3)} = 0 \), the diagram vanishes. Therefore, the S matrix is trivial.

The triviality of the S matrix leads one to the conclusion that one should be able to describe the theory in terms of free fields. Indeed, in terms of the field

\[
V^{++}(v) \equiv e^v D^{++} e^{-v},
\]

the action of eq. (2.28)

\[
S_0 = \int d^4x d^2u \ Tr \left( P^{(-3)} \partial_\mu V^{++} \right)
\]

is quadratic. Therefore, if one could take \( V^{++} \) as the variable of the path integral, instead of the field \( v \), the theory would obviously be free. Note that the transformation \( v \leftrightarrow V^{++} \) is local in spacetime. Also, it is almost one to one. Different \( v \)'s that differ by gauge transformations lead to the same \( V^{++} \), since \( V^{++} \) is gauge invariant. However, at least perturbatively, this is the only degeneracy of the mapping.

In fact, as pointed out in ref. [20], it is in some sense more natural to take \( V^{++} \) as the basic field in harmonic-space SDYM. (Indeed, this representation has been used in attempts to construct multi-instanton solutions [21].) That this can be done is seen by performing a change of frame from the original \( \tau \) frame, where covariant derivatives transform under \( \tau(x) \) transformations, to the \( \lambda \) frame, where they transform only under the \( \lambda(x,u) \) transformations. In the \( \tau \) frame, one has the flat derivatives \( D^{++} \) and \( D^{-} \), and the curved derivatives \( D_{\mu}^{\tau} \), with \( D_{\mu}^{\tau} = e^{-v} \partial_\mu e^v \). The change of frame consists of
transforming all derivatives as $\partial^{(\lambda)} \equiv e^v \partial^{(r)} e^{-v}$. As a result of this, $D^+_{\mu}$ becomes flat in the $\lambda$ frame, while $D^{++}$ and $D^{--}$ pick up the connections $V^{++}$ and $V^{--}$.

$V^{++}$ can now be taken as the fundamental field, from which all other fields can be derived. The condition that $A^{+(r)}_{\mu}$ be the lift of a space-time connection, eq. (2.21), becomes

$$\left[ D^{++}, D^+_{\mu} \right] = 0 \Rightarrow (-) \partial^+_{\mu} V^{++} = 0 ,$$

which is the equation of motion of the action (4.2). $V^{--}$ is derived from the constraint

$$\left[ D^{++}, D^{--} \right] = D^0 \Rightarrow D^{++} V^{--} = D^{--} V^{++} .$$

Note that this equation has no spacetime derivatives. Similarly, $A^{-(\lambda)}_{\mu}$ is determined from the constraint

$$\left[ D^{--}, D^+_{\mu} \right] = D^0 \Rightarrow A^{-(\lambda)}_{\mu} = - \partial^+_{\mu} V^{--} .$$

The field $V^{++}$, satisfying the equation of motion of eq. (4.3), thus gives a free field realization of SDYM. As an example of this approach, the one-instanton $SU(2)$ solutions with scale $\rho$ centered at $x = 0$ are described by [21]

$$V^{++}_{ab} = x^+ a x^- b ,$$

where, as usual, the internal $SU(2)$ is now identified with $SU(2)_L$ of spacetime.

**The spectrum of the harmonic theory**

Since the S matrix of the harmonic theory is trivial, the only information in the theory is its spectrum. This is determined by the solutions to the equations of motion (4.3) and (2.32), which state that $\partial_{\mu} P^{(-3)\mu} = 0$ and that $V^{++}$ is analytic in harmonic space ($X$ is analytic if $\partial^+_\mu X = 0$). The theory also has the $b^{(-4)}$ gauge invariance of eq. (2.31): $\delta P^{(-3)\mu} = \partial^{\mu} b^{(-4)}$. The solutions to these equations should be that $P^{(-3)\mu}$ is trivial, and that $V^{++}$ describes the self-dual configurations.

One should now recall the $\lambda(x, u)$ transformations of eq. (2.33), under which $D^{++}$ transforms as a nonabelian connection, and $P^{(-3)\mu}$ transforms covariantly:

$$\begin{pmatrix} D^{++} \\ P^{(-3)\mu} \end{pmatrix} \rightarrow e^{\lambda(x, u)} \begin{pmatrix} D^{++} \\ P^{(-3)\mu} \end{pmatrix} e^{-\lambda(x, u)} .$$

Since $\lambda(x, u)$ is restricted to be analytic, and is not an arbitrary function, these transformations should not be considered as gauge invariances of the action. Rather, they should be regarded as symmetry transformations linking different solutions of the theory. Since
the four-dimensional gauge fields are invariant under these transformations, all configurations related by $\lambda$ transformations give the same four-dimensional SDYM configuration. We conclude that the harmonic space action describes an infinite number of copies of SDYM, and may also have other degrees of freedom.

A detailed study of the spectrum of the harmonic-space theory is difficult to carry out in Euclidean space, since on $R^4$ the existence of solutions involves the behaviour of the fields at infinity. It is easier to consider the theory in $(2, 2)$ Minkowski space, where SDYM describes one propagating degree of freedom. In order to do this we shall not be concerned with global questions, but shall simply regard the theory as being defined by its harmonic expansion. (See eq. (2.16) for the expansion of a typical field.) Going to Minkowski space then means changing the reality properties of the fields, so that tensors of $SU(2)$ become tensors of $SL(2, R)$. The spectrum of the theory can now be easily analyzed. One sees that in $(2, 2)$ space, the $V^{++}$ field describes three degrees of freedom at the first level of its expansion ($V^{++} \equiv V_{ij} u^i u^j + \cdots$) and one new degree of freedom at each further level. One of the lowest level fields gives the four-dimensional gauge field, while all the rest can be obtained by $\lambda$ transformations. One also finds that the $P^{(-3)\mu}$ field is not trivial, but describes one free massless degree of freedom at each level in its harmonic expansion. We conclude that the harmonic space theory is a free theory describing infinitely many copies of SDYM, together with extra free particles that are decoupled from the SDYM. It is not the theory of the $N = 2$ string.

5 Conclusions

The harmonic space description of SDYM has been seen to be closely related to the twistor construction of Yang-Mills instanton solutions [2, 3, 15]. In particular, the solutions to the harmonic equations of motion provide a lift of self-dual configurations on a four manifold to the projective twistor space. The theory therefore has many of the geometrical properties that would be desired in an action formulation of SDYM. The fundamental fields of the harmonic action, $v$ or $V^{++}$, are related to the four-dimensional gauge fields in a nontrivial way. (See eqs. (4.1), (2.26) and (2.22).) In particular, there are infinitely many $v$ or $V^{++}$ fields corresponding to a particular four-dimensional configuration, since the $\lambda(x,u)$ transformations of eqs. (2.33) leave the gauge field invariant. This means that the action for the theory describes not pure SDYM, but infinitely many

*Therefore, the theory as such is not a counterexample to the “no go theorem” of ref. [14].
decoupled copies of it. In addition, in a space with a $(2, 2)$ signature, the action contains an infinite number of free scalar fields.

The complicated redefinition of $A_{\mu i}(x) \rightarrow V^{++}(x, u)$ leads to a great simplification to the theory. Indeed, the action is quadratic in terms of the fields $V^{++}$ and $P^{(-3)\mu}$. The theory therefore has the remarkable property of being able to describe the nonlinear interacting SDYM system as a free theory. It can do this since, while the theory is free, the $\lambda(x, u)$ transformations that map equivalent solutions to each other are nonlinear. Unfortunately, while the theory does give a simple description of classical SDYM, its $S$ matrix is trivial. It therefore does not describe the $N = 2$ string, which contains a non-trivial 3-point $S$-matrix element at tree level.

Since the harmonic theory is not relevant to the $N = 2$ string, we shall briefly survey the remaining actions that have been proposed for SDYM. The first possibility—considered and dismissed in ref. [19]—is to simply enforce the self-duality of $F_{\mu\nu}(A)$ with a lagrange multiplier $\Lambda^{(-)\mu\nu}$. Thus

$$S = \int d^4x \text{Tr} \left( \Lambda^{(-)\mu\nu}(F - \tilde{F})_{\mu\nu} \right).$$

(5.1)

Here $\Lambda^{(-)}$ is an anti-self-dual 2-tensor in the adjoint of the group. The equation of motion of $\Lambda^{(-)}$ gives SDYM. However, the equation of motion of $A_{\mu}$ shows that $\Lambda^{(-)}$ itself also propagates. What is worse is that, unlike the harmonic case, $\Lambda^{(-)}$ is coupled to the gauge field since it is in the adjoint of the group. The theory therefore describes SDYM interacting with other fields, and it is not appropriate for quantum calculations in SDYM or the $N = 2$ string.

The two remaining actions that have been proposed, both by Nair and Schiff [12], are written in a Kähler four-dimensional space times a line segment parameterized by $t \in [0, 1]$. The boundary at $t = 0$ is taken to be spacetime. These theories have the disadvantage that when the four manifold is taken to be $R^4$ one does not have manifest Lorentz invariance, since a particular complex structure must be singled out. The first action is written in terms of a five-dimensional connection form $A$ and the lagrange multiplier fields $\Phi$ and $\bar{\Phi}$, which are $(2, 0)$ and $(0, 2)$ spacetime forms cross $dt$, respectively, and are both in the adjoint of the gauge group. The action is

$$S = \int d^4x \, dt \text{Tr} \left( -\frac{n}{4\pi} \left( AdA + \frac{2}{3} A^3 \right) k + \Phi F + \bar{\Phi} F \right).$$

(5.2)

Here $n$ is an integer, $F$ is the field strength of $A$ and $k$ is the Kähler form on the four manifold. The lagrange multiplier fields enforce the condition that $F$ is a $(1, 1)$ form, and
the equation of motion of $A_t$ implies that $F \wedge k = 0$, so that $F$ is self-dual. The theory therefore gives the equations of motion of SDYM. However, the other equations of motion describe the time components of $F$ in terms of $\Phi$ and $\bar{\Phi}$, and lead to the result that $\Phi$ and $\bar{\Phi}$ satisfy the four-dimensional equations of motion $\nabla^2 \Phi = \nabla^2 \bar{\Phi} = 0$. In [12] it is argued that, at least on appropriate euclidean four manifolds, there are no such scalars, so the classical theory describes pure SDYM. However, this is certainly not the case in (2,2) space, where $\Phi$ and $\bar{\Phi}$ are propagating fields coupled to the gauge field. This theory therefore has the same problem as the naive lagrange-multiplier action of eq. (5.1), and also is not suitable for quantum calculations.

The other action of Nair and Schiff is a Kähler version of the two-dimensional Wess-Zumino-Witten action. It is written in terms of $J = g^2 = e^\phi$, where $\phi$ is the scalar of the $N = 2$ string, so it contains only the fields of the string theory. Its equation of motion is eq. (2.6), which is the equation of motion of Yang [14], so it is a description of SDYM. The action is

$$S(J) = -\frac{n}{4\pi} \int_{M^4} \sqrt{g} g^{\mu\nu} \text{Tr} \left( J^{-1} \partial_\mu J J^{-1} \partial_\nu J \right) + \frac{in}{12\pi} \int_{M^5} \text{Tr} \left( J^{-1} dJ \right)^3 \wedge k.$$ (5.3)

Here $J$ is again defined on the five dimensional surface, and is fixed to some $J_0$ on the boundary $t = 1$. Since this action gives the classical equation of motion of the string [3, 4] (at least up to the four-point functions), it is the only successful candidate for a field theory of the heterotic or open $N = 2$ string. It is the analogue of the Plebanski action for self-dual gravity [22], which gives a scalar field theory for the closed $N = 2$ string [4] in terms of the Kähler potential of the space.

Unfortunately, these theories do not give the correct quantum amplitudes of the $N = 2$ strings. In the closed string case, this has been seen in calculations of the partition function [23] and of the one-loop three-point function [24]. Thus, the partition function of the string is $1/4\pi \int d\tau d\bar{\tau} / \tau_2^2$, which is the partition function expected from a scalar in two rather than four dimensions. Similarly, the string calculation of the three-point functions is equal to that of the field theory only if the loop integrations are carried out in two rather than four dimensions. There have been several suggestions proposed for resolving this issue, but none of them have been successfully implemented. This remains a basic problem in the present understanding of the $N = 2$ string.
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**Figure captions.**

**Fig. 1** Generic one-loop Feynman diagrams.

**Fig. 2** Background field vertices.

**Fig. 3** One-loop background field diagrams.

**Fig. 4** A typical S-matrix element.