Acceleration of particles by black holes as a result of deceleration: ultimate manifestation of kinematic nature of BSW effect

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The recently discovered so-called BSW effect consists in the unbound growth of the energy \( E_{c.m.} \) in the centre of mass frame of two colliding particles near the black hole horizon. We consider a new type of the corresponding scenario when one of two particles ("critical") remains at rest near the horizon of the charged near-extremal black hole due to balance between the attractive and repulsion forces. The other one hits it with a speed close to that of light. This scenario shows in a most pronounced way the kinematic nature of the BSW effect. In the extremal limit, one would gain formally infinite \( E_{c.m.} \) but this does not happen since it would have require the critical massive particle to remain at rest on the null horizon surface that is impossible. We also discuss the BSW effect in the metric of the extremal Reissner-Nordström black hole when the critical particle remains at rest near the horizon.

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I. INTRODUCTION

The recent finding of the effect of the unbound growth of the energy \( E_{c.m.} \)in the centre of mass frame due to collisions of particles near the black hole horizon \[1\] (BSW effect) attracts now much attention. Both manifestations of this effect in different situation are being studied in detail and also the very nature of the effect itself is under investigation. It was observed in \[2\] that the underlying physical reason of the BSW effect can be explained in kinematic terms. Namely, it turns out that, roughly speaking, a rapid particle collides with a slow one near the horizon, this leads to the growth of the relative velocity and, as a

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result, to the unbound growth of the corresponding Lorentz gamma-factor, so the energy $E_{c.m.}$ becomes unbound near the horizon. This general circumstance was also confirmed in thorough analysis of the BSW effect in the Kerr metric [3]. Nonetheless, some doubts remain concerning the possibility to give an alternative explanation. If something is being accelerated to unbound energies, one is tempted to ask, what source does this, and what is the "physical" underlying reason of such an effect.

The aim of the present work is to reveal the kinematic nature of the BSW effect in the most pronounced way. To this end, we consider the situation when one of two colliding particles is motionless while the other one moves (as usual) with a finite energy in the frame of a distant observer. In a sense, this is the ultimate and clear manifestation of the kinematic nature of the effect under discussion that does not require to search for further hidden dynamic factors. The model which we discuss shows the key issue as clear as possible: the role of gravitation in producing the BSW effect of the unbound growth of $E_{c.m.}$ ("acceleration of particles") consists not in acceleration but in deceleration of one of two particles (in the sense that its velocity is reduced to zero)!

To achieve our goal, we consider the spherically symmetric metric of a charged black hole that admits the equilibrium of a particle that remains motionless. In other words, we want to balance the gravitation force by electrical repulsion. Apart from this, it is important that such a point be located in the vicinity of the horizon. For definiteness, we consider the innermost stable equilibrium point which is the counterpart of the innermost stable circular orbit for the Kerr metric [4]. Such orbits were discussed recently due to their potential astrophysical significance [5], [6]. (See their generalization to "dirty" rotating black holes [7].) There exists also their analog in the magnetic field where the BSW effect was studied recently in [8].

The simplest choice would seem to be the Reissner-Nordström (RN) black hole but for this metric the "orbit" with the required properties exists for indifferent equilibrium only (see Sec. V below). Therefore, for the analog of inner stable orbits we take the charged black hole with nonzero cosmological constant $\Lambda$. It turns out that it is required that $\Lambda < 0$, so we mainly deal with the Reissner-Nordström - anti-de Sitter one (RN-AdS) which is sufficient for our purposes. It is also worth noting that interest to black holes with the cosmological constant $\Lambda < 0$ revived in recent years due to AdS/CFT correspondence [9]. In addition, we consider also another type of "orbit" – a particle in the state of indifferent equilibrium
in the metric of the extremal Reissner–Nordström black hole.

II. EQUATIONS OF MOTION

Consider the space-time describing a charged black hole with the cosmological constant. Its metric can be written as

\[ ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2 d\Omega^2, \]  
(1)

\[ f = 1 - \frac{2m}{r} + \frac{Q^2}{r^2} - \frac{\Lambda r^2}{3}. \]  
(2)

Throughout the Letter we assume that the fundamental constants \( G = c = \hbar = 1 \). The horizon lies at \( r = r_+ \) where \( f(r_+) = 0 \). The electric potential

\[ \varphi = \frac{Q}{r} + C \]  
(3)

where the \( C \) is the constant of integration. It is assumed that we work in the gauge where the only nonvanishing component \( A_0 = -\varphi \). For the asymptotically flat case, say, for the Reissner-Nordström or Kerr-Newman metric, it is usually chosen \( C = 0 \) to have \( \varphi = 0 \) at infinity (see, e.g., eq. 3.63 of [10]). In the absence of asymptotic flatness, its choice becomes conditional. It is worth stressing that physically relevant quantities contain not the potential itself but the difference with respect to some reference point (infinity or horizon). For example, in black hole thermodynamics, the potential enters the action in the form \( \frac{\varphi(r) - \varphi(r_+)}{\sqrt{f}} \) that is nonsingular at the horizon (see eq. 4.15 of [11]). In equations of motion (see below) only the combination \( E - q\varphi \) appears where \( q \) is the particle’s charge. If we change the potential according to \( \varphi \rightarrow \varphi + C \), the corresponding shift in the energy \( E \rightarrow E + qC \). For convenience, we choose \( C = 0 \) in [3].

We restrict ourselves by radial motion since this case is the most interesting in the context under discussion. As is known, under the presence of the electromagnetic field, dynamics of the system is described by the generalized momentum \( P_\mu \) related to the kinematic one \( p_\mu = mw^\mu \) by the relation \( p_\mu = P_\mu - qA_\mu \) where \( w^\mu = \frac{dx^\mu}{d\tau} \) is the four-velocity of a test massive particle, \( \tau \) is the proper time, \( A_\mu \) is the vector potential. Due to staticity, the energy \( E = -P_0 \) of a particle moving in this metric is conserved, \( P_0 \) is the time component of the generalized momentum \( P_\mu \). Then, using also the relation \( u^0 = g^{00}u_0 \), we obtain (dot
denotes the derivative with respect to the proper time $\tau$)
\begin{align}
\dot{t} &= u^0 = \frac{X}{mf}, \\
X &= E - q\varphi.
\end{align}
(4)

We assume that $\dot{t} > 0$, so that $E - q\varphi > 0$.
\begin{equation}
m^2\dot{r}^2 = -V_{eff} = X^2 - m^2 f.
\end{equation}
(5)

Now, we are interested in equilibrium solutions $r = r_0 = \text{const}$,
\begin{equation}
V_{eff}(r_0) = 0.
\end{equation}
(6)

Additionally, we require that they possess the following properties: (i) $r_0$ is a perpetual turning point, (ii) it lies near the horizon, $r_+ \to r_0$. Condition (i) means that, in addition to (7), equation
\begin{equation}
V'_{eff}(r_0) = 0
\end{equation}
(8)
should hold. Eqs. (7), (8) ensure that not only $\dot{r}$ but also all higher derivatives vanish.

It follows from (6), (7) that for a particle with $\dot{r} = 0$,
\begin{equation}
X(r_0) = m\sqrt{f(r_0)}.
\end{equation}
(9)

It is instructive to elucidate for which types of black holes equations (7) and (8) are self-consistent near the horizon, so that equilibrium points exist there in agreement with requirement (ii). Physical motivation for considering this requirement comes from our main goal - investigation of the BSW effect since this effect occurs just in the vicinity of the horizon.

If we take the derivative of the effective potential $V_{eff}$ in eq. (6) and take into account also the relation (7), we obtain
\begin{equation}
-\frac{1}{2}V'_{eff}(r_0) = m\sqrt{f(r_0)}\frac{qQ}{r_0^2} - \frac{m^2}{2}f'(r_0).
\end{equation}
(10)

Let us consider the limit $r_0 \to r_+$, so $f(r_0) \to 0$. Then, it follows from (10) that $V'_{eff}(r_0) \to -m^2\kappa$ where we used the fact for the metric (1) $\kappa = \frac{1}{2}f'(r_+)$. Thus if $\kappa \neq 0$, eq. (8) cannot be satisfied in the horizon limit. Therefore, for nonextremal black holes the equilibrium points cannot exist near the horizon (although they can exist elsewhere at a finite distance from the horizon). This generalizes previous observations [3], [7] made for rotating black holes. However, if $\kappa \to 0$, the equilibrium points close to the horizon do exist as will be shown below.
III. PROPERTIES OF EQUILIBRIUM POINT

For the Kerr metric [4] and, in general, for axially-symmetric rotating black holes [7], there are so-called innermost stable orbits (ISCO) which correspond to the threshold of stability. We consider now their analogs in our case, so we must add to (7) and (10), also equation

$$V''_{\text{eff}}(r_0) = 0.$$  \hspace{1cm} (11)

For brevity, we will call this an innermost stable equilibrium point (ISEP).

We are interested in the near-horizon region where we can expand $f$ in the Taylor series with respect to $x = r_0 - r_+$:

$$f = 2\kappa x + Dx^2 + Cx^3 .... \hspace{1cm} (12)$$

From now on, we assume that $\kappa$ is a small parameter, so a black hole is a near-extremal. Then, this leads to an interplay between two small quantities $\kappa$ and $x$. We assume the condition

$$\kappa \ll Dx \hspace{1cm} (13)$$

which one can check a posteriori that for the solutions obtained.

Then, the procedure for the description of the equilibrium points is mathematically similar to that for the description of circular orbits in the background of rotating black holes [7]. In both cases, we are interested in solutions for which $\dot{r} = 0$ and which are on the threshold of stability. Therefore, I omit technical details (which are connected with simple but rather cumbersome calculations) and give the main results of eqs. (7), (8), (11).

It turns out that

$$x^3 \approx H^3\kappa^2, \hspace{1cm} (14)$$

where

$$H^3 = \frac{3r_+^3}{4(-\Lambda)(1 - 2\Lambda r_+^2)} \hspace{1cm} (15)$$

and the constants in (12)

$$D = \frac{1}{r_+^2} - 2\Lambda, \hspace{1cm} (16)$$

$$C = -\frac{2}{r_+^3} + \frac{8}{3}\frac{\Lambda}{r_+}. \hspace{1cm} (17)$$
As in the extremal limit $\kappa \to 0$ we must have $f > 0$ in the vicinity of the horizon from the outside, the coefficient $D > 0$. Then, in combination with $H > 0$, this entails that $\Lambda < 0$.

By substitution of (14) into (12) we obtain

$$\sqrt{f} \approx \left(\frac{3}{4}\right)^{1/3} (\Lambda)^{-1/3} (1 - 2\Lambda r_+^2)^{1/6} \kappa^{2/3}. \tag{18}$$

Although $r_0 \to r_+$, the proper distance $l = \int \frac{dr}{\sqrt{f}}$ between the particle and the horizon does not vanish and, moreover, it grows unbound when $\kappa \to 0$:

$$l \approx \frac{1}{\sqrt{D}} \ln \frac{x_0}{\kappa} \approx \frac{1}{3} \ln \frac{1}{\kappa}. \tag{19}$$

This is in full analogy with the rotating case [4], [7].

IV. COLLISIONS WITH UNBOUND ENERGY

Now, we consider the collision of two particles. To avoid unnecessary complications due to possible Coulomb repulsion between particles having the charge of the same sign, we can assume that the particle falling towards a black hole is neutral. Assuming, for simplicity, that both particles have the same mass $m$, the energy is given by the formula [12]

$$\frac{E_{c.m.}^2}{2m^2} = 1 + \frac{X_1 X_2 - Z_1 Z_2}{f m^2} \tag{20}$$

where

$$Z_i = \sqrt{X_i^2 - m^2 f}, \; i = 1, 2. \tag{21}$$

Let, for definiteness, a motionless particle have $i = 1$. The unbound growth of the energy $E_{c.m.}$ occurs if one particle has on the horizon $(X_1)_+ = 0$ (we call it critical) whereas for the other one $(X_2)_+ \neq 0$ (we call it usual) - see [12] for details.

In our case, $Z_1 = 0$ according to eq. (9), so the formula simplifies:

$$\frac{E_{c.m.}^2}{2m^2} = 1 + \frac{X_2}{m \sqrt{f}}. \tag{22}$$

Particle 1 has $X_1 \sim \sqrt{f}$, so it is near-critical near the horizon $f \to 0$ which is just the case we are dealing with. Then, we obtain for the collision near the horizon that the energy has the form

$$E_{c.m.} \approx \sqrt{2mX_2A\kappa^{-\frac{1}{4}}} \tag{23}$$
where it follows from (18) that
\[ A = \left( \frac{4}{3} \right)^{1/6} (-\Lambda)^{1/6} (1 - 2\Lambda r^2_+)^{-1/12}. \tag{24} \]

The dependence \( E_{\text{c.m.}} \sim \kappa^{-1/3} \) is similar to that for rotating black holes (cf. eq. 5.1 of [5] and eq. 89 of [7]).

V. DEGENERATE CASE: BSW EFFECT FOR A PARTICLE AT REST IN THE EXTREMAL REISSNER-NORDSTRÖM METRIC

In investigation of the BSW effect, one is led to deal with different limiting transitions: \( r_0 \to r_+, (X_1)_+ \to 0 \) that requires certain care. It was demonstrated earlier (see eqs. 11 and 15 of [13] and eqs. 8, 10 of [12]) that these limits do commute and give \( E_{\text{c.m.}} \to \infty \) in both cases for collisions of particles moving towards the horizon of an extremal black hole. In our case there are two distinctions from the aforementioned situation: (i) a black hole is nonextremal with small but nonzero \( \kappa \), (ii) the point of collision cannot be considered as an independent parameter since it coincides with the equilibrium point of particle 1 whose location \( r_0 \) is controlled by \( \kappa \) according to (14), (15). Therefore, one cannot make permutation between \( r_0 \to r_+ \) and \( \kappa \to 0 \) that represents now a self-consistent indivisible procedure. Meanwhile, one may ask what happens to the points of equilibrium if the limit \( \kappa = 0 \) is taken from the very beginning that corresponds to an extremal black hole.

Formally, \( r_0 \to r_+ \) in this limit. However, on the horizon which is light-like surface, the time-like trajectories cannot exist, so the solution \( r_0 = r_+ \) for it is fake (see the detailed analysis in Sec. III C of [5]). The real trajectory is not strictly static and asymptotically approaches the horizon [14], [15], [13], [5]. It would seem that in such circumstances the questions about ISEP in the near-horizon region do not make sense at all. Nonetheless, there is an exceptional case when ISEP degenerates into points of indifferent equilibrium. This happens just for extremal Reinssner-Nordstrom black holes. Then, in eq. (2), \( Q = M = r_+ \), \( \Lambda = 0 \). By direct check, it is easy to see that eqs. (7), (8) lead to the consequences that \( E = m = q \). But for these values of particle’s parameters, the effective potential \( V_{\text{eff}} = 0 \) for any \( r_0 \). Actually, this means that a particle can be at rest at any position \( r_0 \) due to balance between gravitational attraction and electric repulsion, so equilibrium is indifferent. (More on the properties of equilibrium in the Reissner-Nordström space-time can be found in [16]).
Now, eq. (22) gives us an exact expression

\[
\frac{E_{c.m.}^2}{2m^2} = 1 + \frac{X_2}{1 - \frac{r_0}{r_+}}
\]

(25)

for an arbitrary \( r_0 \). Thus we can see that for \( r_0 \to r_+ \), \( E_{c.m.} \sim (1 - \frac{r_0}{r_+})^{-1/2} \). In other words, we place particle 1 in any point at rest and inject another particle from the outside (say, from infinity). When the location of particle 1 approaches the horizon, \( E_{c.m.} \to \infty \), so we again obtain the BSW effect.

VI. KINEMATIC CENSORSHIP

There is one more question concerning the possibility to take the limit \( \kappa \to 0 \). It follows from our formulas for the BSW effect at ISEPs that for any small but nonzero \( \kappa \) the energy \( E_{c.m.} \) is large but finite. Can one simply take the value \( \kappa = 0 \) from the very beginning and gain an infinite energy? In any real physical event the actual energy that can be released must be finite. With respect to collisions of particles, it can be named "kinematic censorship". Therefore, the energy \( E_{c.m.} \) can be as large as one likes but it cannot be literally infinite. To understand, how this kinematic censorship is realized in our case, one should take into account explanations from the previous Section. We would like to stress it once again that the "orbit" \( r_0 = r_+ \) to which formally tends the ISEP is not suitable since a trajectory of a massive particle cannot lie on the light-like horizon surface. In the example with indifferent equilibrium in the extremal Reissner-Nordstrom background, the situation is even more clear: we can place particle 1 at any position \( r_0 > r_+ \) which is as close to \( r_0 \) as one likes but it cannot coincide with \( r_0 \) nonetheless.

VII. ROLE OF GRAVITY IN BSW EFFECT

The results obtained concern charged black holes and represent a counterpart of those for the circular orbits in the background of rotating black holes \[3, 7\]. Meanwhile, the basic point does not have an analogue in the case of the Kerr metric. Indeed, the circular orbits in the near-horizon region of the Kerr black hole lie in the ergosphere, so equilibrium is not possible there. The perpetual turning point in \[4\] is related to circular orbit, so a particle necessarily has the nonzero angular velocity with respect to a distant observer. Meanwhile,
in the present work a particle located near the horizon has both zero velocity and zero angular momentum.

The BSW effect with participation of such a particle sharpens and reveals its kinematic nature. Naively, one could think that the particle is accelerated during the infall into the black hole and the problem seemed to be to explain why and how this happens. However, we see that the real picture is quite different. One of two colliding particles is kept fixed to remain at rest near the horizon while the other one starts its motion from the outside. As the all process is essentially nonlocal, one cannot attribute the growth of relative energy to the action of force that exerts on some united object. Moreover, in the collision under discussion particle 2 that comes from the outside is typical (”usual”) in that it has an arbitrary finite individual energy and zero charge, so the relation $X_+ = 0$ typical of the critical particle is not satisfied for it. As a result, its velocity in the static frame approaches that of light \[2\], and this becomes true for any such a particle. As a result, the relative velocity of two usual particles remains finite and the BSW effect is impossible (see \[2\] for details in general and analysis for the Kerr metric in \[3\]). To gain the BSW effect, one should select such a particle that approaches the velocity less than that of light near the horizon. In the present case, it is the particle with literally zero velocity that remains zero all the time before collision. In other words, the role of gravitation consists here not in acceleration of particle 1 but in keeping it in rest due to balancing electric repulsion, so the eventual outcome of the unbound energy $E_{c.m.}$ is obtained in a sense as a consequence of deceleration or, more precisely, arrest of one of particles! Moreover, in the case of the Reissner-Nordström-anti-de Sitter black hole the combined action of gravitation, electricity and the cosmological term is to arrest particle 1 in such a way that it remains there on the threshold of stability. (One can take a particle with $V''_{eff} > 0$ instead of (11) to have it strictly stable with respect to radial displacement.) Another example with the same qualitative features is indifferent equilibrium of a charged particle in the metric of the extremal Reissner-Nordström black hole. It is this particle that remains at rest is now a critical one. It is in accord with the general principle that for the BSW effect to occur, one of particles should be critical and the other one should be usual \[2\], \[13\].

Thus the case considered above shows in a most pronounced way that the kinematic nature of the BSW enables one to obtain unbound energies because of different action of gravity on essentially different (in a kinematic sense) kinds of particles.
It is worth noting that in [3] the collision between the infalling particle and the particle at rest was discussed (after eq. 34). Meanwhile, there is a qualitative difference between both situations. In [3], the particle was kept at rest in the Schwazrschild background that was possible ”by hand” only and required an almost infinite force near the horizon. Meanwhile, in the example considered in our work, one can check that the value of acceleration \( a \) if finite. Indeed, one can calculate the scalar \( a^2 = a_\mu a^\mu \) where \( a_\mu = u_\mu;\nu u^{\nu} \), semicolon denotes the covariant derivative. It is easy to find that for the metric (1), \( a^2 = \frac{(f')^2}{4f} \approx 1-\frac{2\Lambda r_+^2}{r^4} \) is finite due to the near-extremal character of a black hole. Thus gravity in combination with electric repulsion and the cosmological constant, ensures the BSW effect on a motionless particle in a self-consistent way. In doing so, the particle waiting at rest plays the role of a target which is hit by the infalling particle.

VIII. CONCLUSION AND PERSPECTIVES

The type of the BSW effect discussed in the present Letter is somewhat different from the original one considered in the pioneering paper [1]. It makes the kinematic nature of the BSW effect especially pronounced. Inclusion of backreaction and radiation into general scheme can change the details of the effect significantly [17], [14] but one can expect that the main qualitative features of the BSW effect still persist just due to its kinematic nature. Moreover, the fact that in the scenario discussed in the present work, a near-critical or critical particle remains at rest, suggests that for it gravitational radiation (mentioned in aforementioned papers as a factor acting against the BSW effect) is absent now at all. The role of backreaction on the metric is more subtle but, anyway, the scenario of collisions considered in the present paper simplifies the picture and can be useful for further analysis. More detailed study of the generic BSW effect with account for all these factors is a nontrivial problem that needs separate treatment.

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