GROTHENDIECK GROUP INVARIANTS FOR PARTLY SELF-ADJOINT OPERATOR ALGEBRAS

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Abstract. Partially ordered Grothendieck group invariants are introduced for general operator algebras and these are used in the classification of direct systems and direct limits of finite-dimensional complex incidence algebras with common reduced digraph $H$ (systems of $H$-algebras). In particular the dimension distribution group $G(A; C)$, defined for an operator algebra $A$ and a self-adjoint subalgebra $C$, generalises both the $K_0$ group of a $\sigma$–unital $C^*$–algebra $B$ and the spectrum (fundamental relation) $R(A)$ of a regular limit $A$ of triangular digraph algebras. This invariant is more economical and computable than the regular Grothendieck group $G^r_{H}(-)$ which nevertheless forms the basis for a complete classification of regular systems of $H$-algebras.

1. Introduction

Two invariants have proven to be fundamental for the classification of approximately finite-dimensional operator algebras. For self-adjoint algebras the scaled ordered $K_0$ group provides a complete invariant for $C^*$-algebra isomorphism. See Elliott [Ell]. On the other hand for triangular limit algebras determined by regular embeddings the so called spectrum, or fundamental relation, is a topological binary relation which provides a complete invariant for isometric isomorphism. See Power [Po4]. In what follows we introduce various Grothendieck group invariants, and in particular dimension distribution groups, which generalise these and we obtain classifications of systems and limits in terms of them.

An operator algebra $A$ has associated with it a specific inclusion $A \to \mathcal{L}(H)$ for some Hilbert space $H$ and so comes equipped with a $C^*$-algebra inclusion $A \to C^*(A)$. Accordingly the natural homomorphisms between operator algebras $A, A'$ are the star-extendible homomorphisms, that is, those homomorphisms that are restrictions of $C^*$-algebra homomorphisms $C^*(A) \to C^*(A')$. The set $Pisom(A)$ of partial isometries in $A$ whose initial and final projections belong to $A$ and the set $Pi(A) = Pisom(A \otimes M_\infty)$ associated with

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the stable algebra $A \otimes M_\infty$ of $A$, are, as we shall see, a rich source of invariants for star-extendible isomorphism. The dimension distribution groups given here are derived in this way and in the case of approximately finite algebras they provide information on the "rank distribution" of decompositions of partial isometries relative to the block structure of finite-dimensional superalgebras. This is in parallel to the view that the $K_0$ class of a projection determines its rank distribution.

Recall that a (complex) digraph algebra, or finite-dimensional incidence algebra, is a subalgebra $A$ of $M_n = M_n(\mathbb{C})$ which contains a maximal abelian self-adjoint subalgebra (a masa) which, without loss of generality, we may take to be the diagonal subalgebra. In Power [Po3] (see also Heffernan [He]) we considered the nonselfadjoint limit algebras $A = \lim\lim\lim\ldots\lim A_k, \alpha_k$ for which the embeddings $\alpha_k$ are star-extendible maps between direct sums of $2 \times 2$ block upper triangular matrix algebras. This is the simplest family of partly self-adjoint algebras which are limits of digraph algebras and the classification may be effected by the scaled $K_0$ group together with a binary relation on the scale of $K_0$. This binary relation, called the algebraic order, is determined directly by the set $\text{Pisom}(A)$.

The case of limits of $r \times r$ block upper triangular matrices, for $r > 2$, is much more complicated. Star-extendible embeddings between such matrix algebras are no longer automatically regular in the sense of being decomposable into a sum of multiplicity one embeddings and even in the case of regular embeddings, the multiplicity signature increases rapidly with $r$. Nevertheless, we obtain classifications of families of regular direct systems $\mathcal{A} = \{A_k, \alpha_k\}$ up to regular isomorphism. For small values of $r$, or for restricted families of embeddings, the induced group homomorphism $G(\alpha_k) : G(A_k) \to G(A_{k+1})$ determines the inner unitary equivalence class of $\alpha_k$ and we show how the limit group $G(\mathcal{A})$ may be used in classification. For more general families, such as the regular systems of $H$-algebras, more complicated Grothendieck group invariants, such as the scaled regular Grothendieck group $G_r^H(A)$ are effective. This perspective is developed further in [Po7] where we use metrized variants of such ordered groups in the classification of general limit algebras of irregular systems.

For regular systems of digraph algebras we may view the partial isometry dimension distribution group quite naturally as the Elliott dimension group determined by the Bratteli diagram whose nodes represent the edges of the reduced digraphs of the building block algebras. The improper edges, or loops, are in correspondence with the vertices of the reduced digraphs and so the sub-Bratteli diagram determined by these improper edges gives rise to $K_0(\mathcal{A})$ and its inclusion in $G(\mathcal{A})$.

The purely algebraic problem of classifying locally finite associative algebras has been emphasised recently by Zaleskii [Zal]. The invariants given here seem to be particularly useful in this consideration, at least for complex locally finite associative algebras that are determined by regular embeddings. In fact it seems likely that for most families of regular digraph algebra systems, and possibly for all such systems, the star-extendible isomorphism class of the locally finite algebra

$$A = \lim\lim\lim\ldots\lim A$$
determines the system $\mathcal{A}$ up to regular star-extendible isomorphism. This is known for a number of special cases; self-adjoint systems, triangular systems, $T_2$-algebra systems, systems of cycle algebras and various order preserving systems. See Donsig [Do] and Hopenwasser and Power [HoP2] for example.

The paper is organised as follows. In Section 2 we define the dimension distribution group $G(\mathcal{A})$ for a regular system of digraph algebras as well as the regular Grothendieck group invariants. The dimension distribution group $G(\mathcal{A}; C)$ is defined for a nonselfadjoint operator algebra $A$ containing a regular maximal abelian self-adjoint subalgebra $C$. In the case of triangular AF algebras the group $G(\mathcal{A}; C)$ coincides with the group of continuous integer valued functions on the spectrum which vanish at infinity, and both AF C*-algebras and TAF algebras are classified by the same invariant. In Section 3 the middle ground between these extremes is considered. Regular direct systems of $T_2$-algebras and $T_3$-algebras are shown to be classified in terms of the dimension distribution group and a natural matrix unit scale. Similar results hold for restricted classes of systems of $T_4$ algebras and for systems of cycle algebras. In section 4 we indicate how regular Grothendieck groups are used in the classification of certain regular systems of $H$-algebras and cycle algebras.

2. Grothendieck group invariants

2.1. Regular Grothendieck groups. First we introduce the general Grothendieck group invariants.

Let $\mathcal{A} = \{A_k, \alpha_k\}$ be a direct system of digraph algebras for which the algebra homomorphisms $\alpha_k : A_k \rightarrow A_{k+1}$ are star-extendible and regular. Such systems will be the fundamental objects of the sequel. Two systems $\mathcal{A}, \mathcal{A}'$ are said to be regularly isomorphic if there exist star-extendible regular maps $\phi_k, \psi_k$ forming a commuting diagram

\[
\begin{array}{cccccc}
A_1 & \longrightarrow & A_{n_1} & \longrightarrow & A_{n_2} & \longrightarrow & \cdots \\
\phi_1 \downarrow & & \psi_1 \downarrow & & \phi_2 & \downarrow & \psi_2 & \cdots \\
A_1' & \longrightarrow & A_{m_1}' & \longrightarrow & A_{m_2}' & \longrightarrow & \cdots 
\end{array}
\]

where the horizontal maps are those of the subsystems $\{A_{n_k}\}, \{A_{m_k}'\}$ of $\mathcal{A}, \mathcal{A}'$ respectively.

The commuting diagram above generally leads to many natural invariants and the main aim in classifying a given family is to find a minimal or economical set of invariants which give complete invariants for regular isomorphism. Thus, if $\mathcal{F}$ is the family of all triangular systems then the spectrum $R(\mathcal{A})$ provides a complete invariant. On the other hand, if $\mathcal{A}$ is the subfamily of triangular alternation algebras then $R(\mathcal{A})$ may be replaced by an equivalence class of a pair of generalised integers. (See [HoP1], [Po].) Likewise in the partly self-adjoint setting of rigid systems of 4-cycle algebras, which is indicated in section 3.4, a certain jointly scaled group $K_0(\mathcal{A}) \oplus H_1(\mathcal{A})$ provides complete invariants, whereas for the subfamily of even stationary matroid systems the abelian group $H_1(\mathcal{A})$ together with the ordered $K_0$ group, suffices. These facts may be found in [DoP2].

Let $\mathcal{F}_r, r = 1, 2, \ldots$, be the family of regular star-extendible systems with digraph algebras which are $T_r$-algebras, where by a $T_r$-algebra we mean a block $r \times r$ upper triangular...
matrix algebra. Also, for an antisymmetric transitive digraph $H$, let $\mathcal{F}_H$ denote the systems with digraph algebras $A_k$ whose reduced digraph is $H$, for all $k$. We refer to such digraph algebras as $H$-algebras.

For a reduced digraph $H$ define $V_H^r(A)$ to be the abelian semigroup of equivalence classes $[\phi]$ of star-extendible regular embeddings

$$\phi : A(H) \to A \otimes M_\infty.$$ 

That is we require $\phi$ to be a regular star-extendible embedding from $A(H)$ to $A_k \otimes M_n$ for some $n$ and some algebra $A_k$ of the direct system $\mathcal{A}$, and such embeddings $\phi, \psi$ are equivalent if there exists a unitary $u$ in $A_j \otimes M_N$ for some $j, N$ such that $u\phi(a)u^* = \psi(a)$ and $\phi(a) = u^*\psi(a)u$ for all $a$ in $A(H)$. The additive semigroup operation is given by $[\phi] + [\psi] = [\phi \oplus \psi]$. Define $G_H^r(A)$ to be the Grothendieck group of $V_H^r(A)$. It can be shown that $V_H^r(A)$ has cancellation and so embeds injectively in $G_H^r(A)$. We refer to $G_H^r(A)$ as the regular Grothendieck group invariant of $A$, for the digraph $H$, the terminology reflecting the fact that there are no restrictions on the embeddings beyond regularity and star extendibility.

Note that if $H$ is the trivial digraph with one vertex and edge then $G_H^r(A)$ is naturally isomorphic to $K_0(A)$ through the correspondence of $[\phi]$ with the $K_0$ class of the projection $\phi(1)$.

On the other hand if $A(H)$ is the upper triangular matrix algebra $T_4$ then, as we see later, there are 35 classes of multiplicity one embeddings from $T_4$ to $T_4 \otimes M_\infty$. It follows that the regular Grothendieck group of $T_4 \otimes M_k$ for the digraph for $T_4$ is isomorphic to $\mathbb{Z}^{35}$. Thus if $\mathcal{A}$ is a regular system of $T_4$-algebras then $G_H^r(\mathcal{A})$ is a dimension group of the form $\lim \to (\mathbb{Z}^{35}, \hat{\alpha}_k)$ where each $\hat{\alpha}_k$ is realised by a nonnegative integral matrix.

The regular Grothendieck group $G_H^r(A)$ admits a natural ordering which is induced, as is the case for $K_0$, by a scale $\Sigma_H^r(\mathcal{A}) \subseteq G_H^r(\mathcal{A})$. This scale is defined to be the image in $G_H^r(\mathcal{A})$ of the classes of those embeddings $\phi$ which map into $\mathcal{A}$ rather than the stable system $\mathcal{A} \otimes M_\infty$. The partially ordered scaled abelian group $(G_H^r(\mathcal{A}), \Sigma_H^r(\mathcal{A}))$ is clearly an invariant for regular isomorphism and in Section 4 we indicate how the pair $(G_H^r(\mathcal{A}), K_0(\mathcal{A}))$ may be used in the classification of the regular systems for $\mathcal{F}_H$.

First we turn to the consideration of a more computable invariant.

2.2. The dimension distribution group $G(A)$. For an AF $C^*$-algebra $B = \lim \to B_k$ the $K_0$ class of a projection $p$ in $B_l$ determines how the rank of $p$ is distributed in the summands of $B_k$ for $k > l$. The idea behind the formulation of the group $G(A; C)$ is to capture in a similar way how the rank of a partial isometry in a non-self-adjoint limit algebra $A = \lim \to A_k$ is distributed in the summands and in the block subspaces of $A_k$ for large $k$.

Let $\alpha : T_3 \otimes M_m \to T_3 \otimes M_n$, with $n$ larger than $m$, be a star-extendible algebra homomorphism. Suppose, furthermore, that the image of each standard matrix unit $e_{ij} \otimes f_{kl}$, $1 \leq i \leq j \leq 3, \ 1 \leq k, l \leq m$, is a sum of standard matrix units. Then to each image

$$\alpha(e_{ij} \otimes f_{kl}) = \begin{bmatrix} v_{11} & v_{12} & v_{13} \\ 0 & v_{22} & v_{23} \\ 0 & 0 & v_{33} \end{bmatrix}$$

we require

$$v_{11} + v_{13} + v_{22} + v_{23} + v_{33} = 1.$$
we may associate the rank distribution matrix in $T_3(\mathbb{Z})$ defined by

$$rk(\alpha(e_{ij} \otimes f_{kl})) = \begin{bmatrix}
\text{rank}(v_{11}) & \text{rank}(v_{12}) & \text{rank}(v_{13}) \\
0 & \text{rank}(v_{22}) & \text{rank}(v_{23}) \\
0 & 0 & \text{rank}(v_{33})
\end{bmatrix}$$

and this matrix is independent of $k,l$. The correspondence

$$rk(e_{ij} \otimes f_{11}) \rightarrow rk(\alpha(e_{ij} \otimes f_{11}))$$

extends to an abelian group homomorphism $\hat{\alpha} : T_3(\mathbb{Z}) \rightarrow T_3(\mathbb{Z})$. Moreover, under the natural identification $K_0(T_3 \otimes M_m)$ is equal to $D_3(\mathbb{Z})$, the diagonal subgroup, and the restriction $\hat{\alpha}|D_3(\mathbb{Z})$ coincides with $K_0\alpha$. Plainly, $(\beta \circ \alpha) = \hat{\beta} \circ \hat{\alpha}$.

More generally we may define $\hat{\alpha}$ in a similar way when $\alpha : A_1 \rightarrow A_2$ is a regular star-extendible homomorphism between digraph algebras, that is, one which is a direct sum of multiplicity one embeddings. In this case matrix units can be chosen for $A_1$ and $A_2$ so that matrix units map to sums of matrix units.

Thus, given a direct system $\mathcal{A} = \{A_k, \alpha_k\}$, we can introduce the abelian group $G(\mathcal{A}) = \lim G(A_k), \hat{\alpha}_k)$. together with, as we see below, a natural ordering and a scale $\Sigma(\mathcal{A})$ which augments the $K_0$ ordering and scale. We call $G(\mathcal{A})$ the (scaled ordered) dimension distribution group, or the partial isometry dimension distribution group, of the system $\mathcal{A}$. In fact there are scaled ordered group surjectons

$$\pi_f : G(\mathcal{A}) \rightarrow K_0(\mathcal{A}), \pi_i : G(\mathcal{A}) \rightarrow K_0(\mathcal{A})$$

and the quadruple

$$((G(\mathcal{A}), \Sigma(\mathcal{A})), (K_0(\mathcal{A}), \Sigma_0(\mathcal{A}), \pi_f, \pi_i)$$

may be viewed as a generalisation of a dual form of the spectrum of a triangular limit algebra.

2.3. The dimension distribution group $G(A; C)$. Fix a general operator algebra $A$ and a self-adjoint subalgebra $C$. We now give a general formulation of a dimension distribution group invariant $G(A; C)$. However our principal concern is for a pair $(A, C)$ where

$$A = \lim A = \lim (A_k, \alpha_k), \quad C = \lim C = \lim (C_k, \alpha_k)$$

are associated with a regular star-extendible digraph algebra system $\mathcal{A}$ and subsystem $\mathcal{C}$ of masas $C_1, C_2, \ldots$ such that, for all $k, \alpha_k$ maps the partial isometry normaliser of $C_k$ into the partial isometry normaliser of $C_{k+1}$. Recall that an element $v$ in a star algebra $B$ normalises a subalgebra $E$ of $B$ if $vEv^* \subseteq E$ and $v^*Ev \subseteq E$.

Write $M_\infty(A)$ for the usual stable algebra $M_\infty(C) \otimes A$ of $A$, write $D_\infty(A)$ for its diagonal subalgebra and define the following sets of projections and partial isometries.

$$Pr(A) = \{p \in M_\infty(A) : p = p^2 = p^*\},$$
$$Pi(A) = \{v \in M_\infty(A) : v^*v, vv^* \in Pr(A)\},$$
$$\tilde{Pr}(A; C) = \{p \in Pr(A) : p \text{ normalises } D_\infty(C)\},$$
\[
Pr(A; C) = \{ p \in Pr(A) : p \text{ normalises } M_\infty(C) \},
\]
\[
\tilde{Pi}(A ; C) = \{ v \in Pi(A) : v \text{ normalises } D_\infty(C) \},
\]
\[
Pi(A; C) = \{ v \in Pi(A) : v \text{ normalises } M_\infty(C) \}.
\]

Also write \(Proj(A), \text{Pisom}(A), Proj(A; C), \text{Pisom}(A; C)\) for the appropriate sets in \(A\), rather than the stable algebra of \(A\). In particular \(\text{Pisom}(A)\) is the set of partial isometries in \(A\) with initial and final projections in \(A\), and \(\text{Pisom}(A; C)\) is the subset of elements which normalise the masa \(C\).

For \(v, w\) in \(Pi(A)\) write \(v \simeq w\) if \(zvz^* = w\) for some unitary \(z\) in \(M_\infty(A)\). More precisely \(zvz^* = w\) with \(z\) a unitary in some sufficiently large matrix algebra containing \(v\) and \(w\). Write \(v \sim w\) if \(zvy = w\) for some unitaries \(z, y\) in \(M_\infty(A)\). Similarly for the four normalising sets above, define unitary equivalence and equivalence with respect to the appropriate normalising unitaries. The resulting equivalence classes provide the twelve monoids,

\[
Pr(A)/ \simeq, \ Pr(A)/ \approx, \ldots, \ Pi(A; C)/ \approx
\]

and twelve associated Grothendieck groups.

In the case of finite-dimensional digraph algebras one may also consider the following subset of \(\text{Pisom}(A)\) consisting of (let us say) regular partial isometries:

\[
\text{Pisom}_{\text{reg}}(A) = \{ v \in \text{Pisom}(A) : v = v_1 + \cdots + v_i, v_i \in \text{Pisom}(A), v_i \text{ rank } 1 \}.
\]

Then it is elementary to obtain the identifications

\[
\text{Pisom}_{\text{reg}}(A)/ \sim = \text{Pisom}(A; C)/ \sim, \quad \text{Pi}_{\text{reg}}(A)/ \sim = \text{Pi}(A; C)/ \sim,
\]

in this setting.

Returning to the general pair \(A, C\), note that for projections \(p, q\) if \(zpy = q\), with \(z, y\) unitary, then \(y^*py = q\). Thus,

\[
Pr(A)/ \sim = Pr(A)/ \approx,
\]

with similar equalities for \(Pr(A; C)\) and \(\tilde{Pr}(A; C)\). Since, in general, \(\sim\) equivalence of partial isometries extends unitary equivalence of projections it is natural to restrict attention to the following Grothendieck groups:

\[
K_0(A) = \text{Gr}(Pr(A)/ \approx),
\]
\[
K_0(A; C) := \text{Gr}(Pr(A; C)/ \approx),
\]
\[
\tilde{K}_0(A; C) := \text{Gr}(\tilde{Pr}(A; C)/ \approx),
\]
\[
G_1(A) := \text{Gr}(Pi(A)/ \sim),
\]
\[
G(A; C) := \text{Gr}(Pi(A; C)/ \sim),
\]
\[
\tilde{G}(A; C) := \text{Gr}(\tilde{Pi}(A; C)/ \sim).
\]

These groups are appropriate in the context of \(\sigma\)-unital local operator algebras, where by local we mean, as in Blackadar [13], that \(A\) (and \(C\)) are closed under holomorphic functional calculus. In particular, an unclosed union of closed subalgebras is a local operator
algebra, as are algebraic direct limits of norm-closed operator algebras. In this context
\( Gr(Pr(A)/ \approx) \) is the usual \( K_0 \) group of \( A \). Also if \( B \) is a \( \sigma \)-unital \( C^* \)-algebra then
\[
G(B; B) = K_0(B).
\]

The group \( G_t(A) \), which we refer to as the "total partial isometry" dimension distribution group, does not yield a convenient invariant for digraph algebras and their limit algebras. At least, \( G_t(A) \) seems inappropriate for the class of regular limit algebras. This is indicated below in Example 2. Accordingly we make the following definition of \( G(A) \) in this case, and with this definition we have the natural identification \( G(A) = \lim \rightarrow G(A_k) \) for a star-extendible regular system of digraph algebras.

**Definition 2.1.** The dimension distribution group \( G(A) \) of a digraph algebra \( A \) is defined to be the group \( G(A; C) \), where \( C \) is a masa in \( A \).

**Example 1.** \( G_t(T_2(\mathbb{C})) \) is the abelian group \( T_2(\mathbb{Z}) \). In fact if \( v \in Pisom(T_2 \otimes M_n) \) then
\[
v = \begin{bmatrix} v_1 & v_2 \\ 0 & v_3 \end{bmatrix}
\]
with respect to the block upper triangular decomposition. From the definition of \( Pisom( ) \) it follows that \( v^*v \) and \( vv^* \) are block diagonal. Thus \( v_1 \) and \( v_3 \), and hence \( v_2 \) are partial isometries. The correspondence \( [v] \rightarrow rk(v) \) leads to the stated identification.

**Example 2.** If \( n \geq 3 \) then \( G_t(T_n(\mathbb{C})) \) contains \( \mathbb{Z}^c \), where \( c \) is the cardinality of the continuum. Indeed, there are uncountably many \( \sim \) equivalence classes in \( T_3(\mathbb{C}) \) of partial isometries of the form
\[
\begin{bmatrix} 0 & v_1 & v_2 \\ 0 & v_4 & v_3 \\ 0 & 0 & 0 \end{bmatrix}
\]
where the \( 2 \times 2 \) submatrix is a unitary matrix in \( M_2 \). These classes lead to an injection \( \mathbb{Z}_+ \rightarrow Pi(T_3(\mathbb{C}))/ \sim. \) The proliferation of such equivalence classes is one reason for the consideration of normalising partial isometries.

**Example 3.** For \( n \geq 0 \), \( G(T_n; D_n) = \tilde{G}(T_n; D_n) = T_n(\mathbb{Z}) \). More generally, if \( X \) is a totally disconnected locally compact Hausdorff space, then
\[
G(T_n \otimes C(X); D_n \otimes C(X)) = \tilde{G}(T_n \otimes C(X); D_n \otimes C(X))
\]
\[
= T_n(\mathbb{Z}) \otimes K_0(C(X))
\]
\[
= \mathbb{Z}^{n(n+1)/2} \otimes C(X, \mathbb{Z})
\].

One can establish this identification and that of the next example, by a simple direct argument, or by appealing to the more general discussion of the next section.

**Example 4.** Let \( A = \lim(T_{2^k}, \rho_k) \) be the \( 2^\infty \) refinement algebra with spectrum \( R(A) \). (See [Po4] and below.) Let \( B \) be a UHF \( C^* \)-algebra with \( K_0 \) group \( \mathbb{Q}_B \) realised as an
ordered unital subgroup of $\mathbb{Q}$. Also let $C$ be a masa of the form \((A \cap A^*) \otimes D\) with $D$ a regular canonical masa of $B$. Then
\[
G(A \otimes B, C) = C(R(A), \mathbb{Q}_B),
\]
where the right hand side is the group of continuous $\mathbb{Q}_B$-valued functions on the spectrum which vanish at infinity. In fact the masa $C$ is intrinsic to the algebra $A \otimes B$ as the unique (up to star-extendible automorphism) standard AF diagonal. See Theorem 4.1 of [Po4]. Thus we may write $G(A \otimes B)$ for $G(A \otimes B; C)$.

2.4. $G(A; C)$ and the spectrum $R(A)$. Let $A = \lim \rightarrow (A_k, \alpha_k)$ be a limit of digraph algebras with respect to regular star-extendible embeddings. Then for $k = 1, 2, \ldots$ there are matrix unit systems \(\{e_{ij}^k\}\) for the finite-dimensional $C^*$-algebras $B_k = C^*(A_k)$ such that $A_k$ is spanned by some of these matrix units, including the diagonal matrix units, and such that $\alpha_k$ maps matrix units in $\{e_{ij}^k\}$ to sums of matrix units in $\{e_{ij}^{k+1}\}$. Thus if $C_k$ is the masa of $A_k$ (and $B_k$) spanned by $\{e_{ii}^k\}$ then we have the triple $C \subseteq A \subseteq B$, where
\[
B = C^*(A) = \lim \rightarrow (B_k, \tilde{\alpha}_k),
\]
with $\tilde{\alpha}_k$ the unique extension of $\alpha_k$ and where $C$ is the regular canonical masa
\[
C = \lim \rightarrow (C_k, \alpha_k).
\]

We now recall the definition of the fundamental relation or topological binary relation, or spectrum, $R(A; C)$ for the pair $(A, C)$. See [Po1], [Po2] and Chapter 7 of [Po4].

Each matrix unit $e = e_{ij}^k$ induces a partial homeomorphism $\alpha$ of the Gelfand space $X$ of $C$ such that
\[
\alpha(y)(e e^* c) = y(c)
\]
for all characters $y$ with $y(e^* e) = 1$. The set of such $y$ is the domain $d(\alpha)$ of $\alpha$ and the graph of $\alpha$ is the subset $E$ of $X \times X$ given by $E = \{(x, y) : y \in d(\alpha), x = \alpha(y)\}$.

The binary relation $R(A; C)$ is then defined as
\[
R(A; C) = \bigcup_{i,j,k} \{E_{ij}^k : E_{ij}^k \text{ is the graph for } e_{ij}^k \in A_k\}
\]
and the topology on $R(A; C)$ is the smallest topology for which each $E_{ij}^k$ is both closed and open. In particular $R(A; C)$ is a locally compact topological space. It can be shown that $R(A; C)$ is well-defined and independent of the particular choice of direct system or matrix units for the pair $(A, C)$. See [Po4].

When $A$ is triangular, that is, when $A \cap A^* = C$, then $A$ is referred to as a TAF algebra and we write $R(A)$ for $R(A; C)$. This topological binary relation is a complete invariant for isometric isomorphism and has been a useful tool for classification and analysis. This can be seen, for example, in various articles which have appeared since [Po4], such as [HoLe], [HoPo], [PoWa], [Po5], [DoHo], [DoHu], [PePo].

Let us now consider the dimension distribution group $G(A; C)$ with $A$ as above and assume that $A$ is a TAF algebra.
Proposition 2.2. If \( A \) is a TAF algebra then \( G(A; C) = C(R(A), \mathbb{Z}) \).

Proof. Note first that the abelian semigroup \( C(R(A), \mathbb{Z}_+) \) of positive integer-valued continuous functions on the spectrum \( R(A) \) (vanishing at infinity will be understood) is generated by the family of characteristic functions \( \chi_E \) for closed-open sets \( E \) associated, as above, with the matrix units \( e \) of the fixed matrix unit system \( \{e_{ij}^k\} \). It is a fundamental fact, which follows quickly from Lemma 5.5 of [Po4], that if \( v \in \text{Pisom}(A; C) \) then \( v \) is unitarily equivalent by a unitary in \( C \) to a partial isometry which is a finite sum of matrix units. Furthermore, note that if \( v, w \in \text{Pisom}(A; C) \) then, since \( A \) is triangular, \( v \sim w \) if and only if \( v = cw \) for some element \( c \) in \( C \). Thus to the class \([v]_\sim \) in \( \text{Pisom}(A; C)/\sim \) there is a unique element \( z \) in \( \text{Pisom}(A; C) \) which is a sum \( z = e_1 + \cdots + e_r \) of the given matrix units, such that \([v]_\sim = [z]_\sim \). Although the decomposition \( z = e_1 + \cdots + e_r \) is not unique the associated correspondence

\[ \kappa : [v] \rightarrow \chi_{E_1} + \cdots + \chi_{E_r} \]

gives a well-defined semigroup injection

\[ \text{Pisom}(A; C)/\sim \rightarrow C(R(A), \mathbb{Z}_+) \].

Consider now \( v \in \tilde{\text{Pi}}(A; C) \) so that \( v \in M_n(A) \subseteq M_\infty(A) \), for some \( n \), and \( v \) normalises \( D_n(C) \). By the reasoning above, with \( D_n(C) \) playing the role of \( C \), \( v \) is unitarily equivalent to a partial isometry \( e \in M_n(A) \) of the form \( e = (g_{ij})_{i,j=1}^n \) where each \( g_{ij} \) is a sum of matrix units for the given matrix unit system for \( A \). Since \( e \) normalises \( D_n(C) \) the partial isometries \( \{g_{ij}\}_{i,j=1}^n \) are orthogonal, that is, they have orthogonal range projections and orthogonal initial projections. It follows that in \( M_{n^2}(A) \) we have \( e \sim w \) where \( w \) is the block diagonal partial isometry

\[ w = \sum_{i=1}^n \sum_{j=1}^n g_{ij}. \]

If we define

\[ \kappa : \text{Pisom}(A; C)/\sim \rightarrow C(R(A), \mathbb{Z}_+) \]

by

\[ \kappa([v]_\sim) = \sum_{i=1}^n \sum_{j=1}^n \kappa([g_{ij}]_\sim). \]

then \( \kappa \) is a well defined semigroup surjection. We claim that \( \kappa \) is injective. We have, by orthogonality again, \( e \sim f \) where \( f \) is a block diagonal sum

\[ f = \sum_{j=1}^N \oplus f_j \]

in \( D_N(A) \) with each \( f_j \) a matrix unit in \( A_M \), for some sufficiently large \( M, N \). If

\[ f' = \sum_{j=1}^N \oplus f'_j \]

is another such sum, for the same $M, N$, but with, perhaps, some summands equal to zero, then $f \sim f'$ if and only if each $f'_j$ in the sum for $f'$ appears in the sum for $f$, with the same multiplicity, and vice versa. However, this is precisely the condition that

$$\sum_{j=1}^{N} \chi_{F_j} = \sum_{j=1}^{N} \chi_{F'_j},$$

and so it follows that $\kappa$ is injective.

Thus

$$\tilde{G}(A; C) = Gr(\tilde{P}i(A; C)/\sim) = Gr(C(R(A), \mathbb{Z}_+)) = C(R(A), \mathbb{Z}).$$

We have shown, in Lemma 2.4 of [Po6], that a partial isometry $w$ in $M_n \otimes B$ which normalises $M_n \otimes C$ has the form $dw_1$ where $d$ is a partial isometry in $M_n \otimes C$ and $w_1$ is a partial isometry in $B_k$, for some $k$, which normalises $C_k$. From this one can deduce that there is a natural identification of $G(A; C)$ with $\tilde{G}(A; C)$. Thus

$$G(A; C) = C(R(A), \mathbb{Z}).$$

At the other extreme we have

**Proposition 2.3.** Let $C$ be a standard AF diagonal subalgebra in the AF $C^*$-algebra $B$ (as above). Then

$$G(B; C) = K_0(B).$$

**Proof.** The identification follows from the fact that normalising partial isometries in a finite-dimensional $C^*$-algebra are $\sim$ equivalent, by a normalising unitary, if and only if they have the same rank distribution in the summands of the $C^*$-algebra. \qed

2.5. Orders and Scales.

**Definition 2.4.** The positive cone $G(A; C)_+$ of $G(A; C)$ is defined to be the semigroup generated by the (images of) classes $[v]$ where $v \in Pisom(A; C)$ and the scale of $G(A; C)$ denoted $\Sigma(A; C)$ is defined to be the set of these (images of) classes.

In particular $G(T_n; D_n)_+ = T_n(\mathbb{Z}_+)$ and $\Sigma(T_n; D_n)$ is the set of zero-one elements with at most one nonzero entry in each row and column. This follows from the next proposition.

Let $A$ be an algebra with $D_n \subseteq A \subseteq M_n$, that is, let $A$ be a digraph algebra. Then there are natural scaled group homomorphisms

$$\pi_f : G(A) \to K_0A,$$

$$\pi_i : G(A) \to K_0A$$

which extend the correspondences

$$[v] \to [vv^*], \quad [v] \to [v^*v]$$
for $v \in Pi(A; D_n)$. We may assume without loss of generality that

$$A = \text{span}\{e_{ij} : e_{ij} \in E(G)\}$$

where $\{e_{ij}\}$ is the standard matrix unit system, and $E(G)$ is the set of edges of a directed graph with vertices $\{1, \ldots, n\}$. Let $G_r$ be the reduced graph of $G$ obtained from $G$ by identifying vertices which are equivalent. (Thus, vertex $i$ is equivalent to vertex $j$ if the matrix units $e_{ii}, e_{jj}$ lie in the same matrix algebra summand of $A \cap A^\ast$.) Then $G(A) = \mathbb{Z}^{|E(G_r)|}$. The digraph algebra $A(G_r)$ is a triangular matrix algebra, with matrix units $f_{kl}$ say, and there is a natural identification of $G(A)$ with the abelian (additive) subgroup generated by the matrix units $\{f_{kl} : (k, l) \in E(G_r)\}$. A straightforward induction proof yields the following connection between scales.

**Proposition 2.5.** Let $A$ be a digraph algebra and let $g \in G(A)_+$. Then $g \in \Sigma(A; D_n)_+$ if and only if $\pi_i(g)$ and $\pi_f(g)$ belong to $\Sigma_0 A$, the scale of $K_0 A$.

**Definition 2.6.** (i) Let $A = \lim \rightarrow (A_k, \alpha_k)$ be a regular star-extendible direct system of digraph algebras with regular canonical masa $C = \lim \rightarrow (C_k, \alpha_k)$. Then the dimension distribution group invariant, or the *dimension distribution group quadruple*, for the pair $A, C$ is the quadruple

$$G(A; C) = (G(A; C), K_0(A), \pi_f, \pi_i).$$

(ii) Two such quadruples, for $A, C$ and $A', C'$, respectively, are said to be isomorphic if there is a $\pi$-respecting scaled group isomorphism $\theta : G(A; C) \to G(A'; C')$, that is, if there are two commuting diagrams

$$
\begin{array}{ccc}
G(A; C) & \xrightarrow{\theta} & G(A'; C') \\
\pi \downarrow & & \pi' \downarrow \\
K_0 A & \xrightarrow{\theta | K_0(A)} & K_0 A'
\end{array}
$$

where the horizontal maps are scaled group isomorphisms and where $\pi = \pi_i$ or $\pi_f$.

Of course the distribution group quadruple is indeed an invariant for the pair $(A, C)$, with respect to masa preserving star-extendible isomorphism. In the self-adjoint case it degenerates to the scaled $K_0$ group, and so, by Elliott’s theorem $G(A; C)$ is a well defined complete invariant for AF $C^*$-algebras. At the other extreme, in the triangular case, where there is only one distinguished masa, the distribution group quadruple is an invariant for star-extendible isomorphism. In fact it is a complete invariant in this case also.

**Theorem 2.7.** Let $A, A'$ be limit algebras of regular star-extendible systems of digraph algebras and suppose that $A, A'$ are either self-adjoint or triangular. Then $A, A'$ are star extendibly isomorphic if and only if their dimension distribution group quadruples are isomorphic.
Proof. We have already commented on the self-adjoint case so assume that \( A, A' \) are triangular and their quadruple invariants are isomorphic. By Proposition 2.3 there are two commuting diagrams

\[
\begin{array}{ccc}
C(R(A), \mathbb{Z}) & \overset{\theta}{\longrightarrow} & C(R(A'), \mathbb{Z}) \\
\pi \downarrow & & \pi' \downarrow \\
C(X, \mathbb{Z}) & \overset{\theta|K_0(A)}{\longrightarrow} & C(X', \mathbb{Z})
\end{array}
\]

with \( \pi = \pi_f \) for one diagram and \( \pi = \pi_i \) for the other. Thus the isomorphisms \( \theta \) and \( \theta|K_0A \) are induced by homeomorphisms \( \nu : R(A) \to R(A') \) and \( \gamma : X \to X' \), respectively. In view of the commuting diagrams, it follows that \( \nu = \gamma \times \gamma \). Thus \( \nu \) is a topological binary relation isomorphism. Since the spectrum is a complete invariant (by \([Po3]\)) it follows that \( A \) and \( A' \) are star-extendibly isomorphic. 

Note that the scale of \( G(A; C) \) plays a crucial role in the self-adjoint context and a rather redundant role in the triangular context.

Recall that the algebraic order \( S(A; C) \) for the pair \((A, C)\) is the transitive binary relation in \( \Sigma_0(A) \times \Sigma_0(A) \) which is the range of the correspondence

\[
[v] \to ([vv^*], [v^*v])
\]

for \( v \in \text{Pisom}(A; C) \). In the next section we shall see that it is also fruitful to consider more elaborate "matrix unit scales" in higher products \( \Sigma_0(A) \times \cdots \times \Sigma_0(A) \).

3. Classifications with \( G(A) \)

We now turn to the classification of specific regular systems.

Given an isomorphism of invariants there are two fundamental issues to address when constructing a regular isomorphism between systems \( \mathcal{A} \) and \( \mathcal{A}' \). These are, firstly, the existence of digraph algebra homomorphisms, with a given correspondence of invariants, and, secondly, the uniqueness up to inner conjugacy of these same homomorphisms. Resolving the existence issue enables the construction of \( \phi_1 \) and an initial \( \psi_1 \), as in the diagram in Section 1, whilst uniqueness enables the correction of \( \psi_1 \) to obtain a commuting triangle. It is usually the existence question which is a more subtle issue.

The dimension distribution group \( G(\mathcal{A}) \) can equally well be defined for systems of digraph spaces and regular linear maps. This suggests that the lifting of a restriction of a dimension distribution group isomorphism to digraph algebra homomorphism is conditional on its preservation of an invariant which reflects the multiplication in some way.

3.1. Matrix unit scales. Consider a \( \pi_i, \pi_f \) respecting ordered group homomorphism

\[
\theta : G(T_3) \to G(B)
\]

where \( B \) is a digraph algebra. Plainly \( \theta \) is liftable to a regular star-extendible homomorphism if and only if the triple

\[
\{\theta(e_{12}), \theta(e_{23}), \theta(e_{13})\}
\]
belongs to a "matrix unit scale" in $(\Sigma(B) \times \Sigma(B) \times \Sigma(B)$) given by
\[(\{[\phi(e_{12})], [\phi(e_{23})], [\phi(e_{13})] \} : \phi : T_3 \to B)\]
where $\phi$ runs over star-extendible regular homomorphisms. In fact the property of $\pi_i, \pi_f$
respecting could be incorporated into matrix unit scale preservation by enlarging the scale
to the set
\[\{\Pi_{1 \leq i \leq 3}[\phi(e_{ij})] : \phi : T_3 \to B\} \subseteq (\Sigma(B))^{(6)}\].

We refer to this ordered set as the matrix unit scale or, more precisely, as the $T_3$-matrix
unit scale. Thus, as we see in the existence lemma below, one way to ensure existence, that
is, to ensure the liftability of a restriction $\Phi|G(A_1)$ of a given scaled group isomorphism
$\Phi : G(A) \to G(A')$, is to demand that $\Phi$ preserve the appropriate matrix unit scale.

**Definition 3.1.** (i) Let $H$ be an antisymmetric digraph and let $B$ be a digraph algebra.
Then the $H$-matrix unit scale $S_H(B)$ for $B$ is the subset of
\[\Pi_{(i,j) \in E(H)} \Sigma(B)\]
given by the set of elements of the form
\[\Pi_{(i,j) \in E(H)}[\phi(e_{ij})]\]
where $(i, j)$ ranges through the edges of $H$ and where $\phi$ ranges over regular star-extendible
homomorphisms from $A(H)$ into $B$.

(ii) If $A$ is a regular star-extendible system $\{A_k, \phi_k\}$ of digraph algebras then the $H$-
matrix unit scale $S_H(A)$ is defined to be the union of the images of the scales $S_H(A_k)$ in
the ordered product $\Sigma(A) \nu$, where $\nu$ is the number of edges of $H$.

Note that the inclusion $S_H(A_k) \to S_H(A_{k+1})$ is simply the direct product of the coordinate
inclusions.

**Lemma 3.2.** (Existence lemma.) Let $A_1, A_2$ be $H$-algebras and let $\theta : G(A_1) \to G(A_2)$
be a group homomorphism. Then the following conditions are equivalent.

(i) There is a regular star-extendible homomorphism $\phi : A_1 \to A_2$ with $G(\phi) = \theta$.

(ii) $\theta(S_H(A_1)) \subseteq S_H(A_2)$.

**Proof.** Assume that $H_1$ and $H_2$ are the digraphs of $A_1, A_2$ so that the reduced digraphs of
$H_1$ and $H_2$ are equal to $H$. Let $i : H \to H_1$ be a digraph injection with associated (multi-
PLICity one) injection $\alpha : A(H) \to A(H_1)$ for which the induced group homomorphism
$G(\alpha)$ is the identity. With this injection we identify $A(H)$ as a subalgebra of $A_1$. Plainly
each matrix unit $e$ in $A_1$ admits a unique representation $e = e_1 f e_2$ with $f$ a matrix unit
for $A(H)$ and $e_1, e_2$ matrix units for $A_1 \cap A_1^*$. Let
\[\beta_0 : A_1 \cap A_1^* \to A_2\]
\[\beta_1 : A(H) \to A_2\]
be homomorphisms which map matrix units to sums of matrix units, and for which $K_0 \beta_0 = \theta|\Sigma_0 A_1$, and $G(\beta_1) = \theta$. The existence of the C*-algebra homomorphism $\beta_0$ follows from
the inclusion $\theta(\Sigma_0 A_1) \subseteq \Sigma_0 A_2$, which is implied by condition (ii). The existence of the
star-extendible homomorphism $\beta_1$ also follows from the condition (ii). In order to define $\beta_1$ we only need limited information from the second condition, namely that

$$\Pi_{(i,j)\in E(H)} \theta(g_{ij}) = \theta(\Pi_{(i,j)\in E(H)}(g_{ij})) \in S_H(A_2),$$

where here we write $g_{ij}$ for the natural basis elements of $G(A(H))$, indexed by edges of $H$. Having specified $\beta_1$ we may choose $\beta_0$ in such a way that these maps agree on $A_1(H) \cap A_2(H)$. By the unique factorisation of matrix units mentioned above there is a well-defined linear map $\phi : A_1 \to A_2$ which extends both $\beta_1$ and $\beta_0$ and which satisfies $\phi(e_1f) = \phi(e_1)\phi(f)\phi(e_2)$. It follows that $\phi$ is an algebra homomorphism, and, since $\beta_1$ is star-extendible, it follows that $\phi$ is star-extendible.

One can also obtain an analogue of the lemma for a restricted family of embeddings.

**Definition 3.3.** Let $\Omega$ be a family of regular star-extendible embeddings between $H$-algebras which is closed under compositions and direct sums, and let $\mathcal{F}(\Omega)$ be the corresponding family of systems $A$.

(i) If $A$ is a digraph algebra then the $\Omega$-scale, $S_\Omega(A)$, is the subset of $S_H(A)$ corresponding to homomorphisms $\phi : A(H) \to A$ which belong to $\Omega$.

(ii) If $A \in \mathcal{F}(\Omega)$ then the $\Omega$-scale, $S_\Omega(A)$, is defined to be the union of the images of the scales $S_\Omega(A_k)$ for all $k$.

As in the proof above it follows that a group homomorphism $\theta : G(A_1) \to G(A_2)$ lifts to a homomorphism in $\Omega$ if and only if $\theta(S_\Omega(A_1)) \subseteq S_\Omega(A_2)$.

**Definition 3.4.** Let $\Omega$ be as in the last definition. Then $\Omega$ is said to have the uniqueness property if whenever $\phi, \psi : A(H) \to A$ are two regular star-extendible embeddings from $\Omega$, with the same induced maps between the dimension distribution groups, $\phi, \tilde{\psi} : G(A(H)) \to G(A)$, then $\phi$ and $\tilde{\psi}$ are inner conjugate.

We can now obtain the following abstract theorem.

**Theorem 3.5.** Let $\Omega$ be a family of star-extendible regular embeddings between $H$-algebras and suppose that $\Omega$ has the uniqueness property. If $A, A'$ are two systems in $\mathcal{F}(\Omega)$ then $A$ and $A'$ are regularly isomorphic if and only if there exists a dimension distribution group isomorphism $\Phi : G(A) \to G(A')$ such that $\Phi(S_\Omega(A)) \subseteq S_\Omega(A')$.

**Proof.** Since $G(A')$ and $S_\Omega(A')$ are finitely generated the composition

$$G(A_1) \to G(A) \overset{\Phi}{\to} G(A')$$

factors as

$$G(A_1) \overset{\Phi_1}{\to} G(A'_{m_1}) \to G(A'),$$

for some $m_1$, where $\Phi_1$ is a matrix unit scale preserving abelian group homomorphism. By the last lemma $\Phi_1$ lifts to a star-extendible regular algebra homomorphism $\phi_1$ in $\Omega$. Similarly the composition

$$G(A'_{m_1}) \to G(A') \overset{\phi_1^{-1}}{\to} G(A)$$
factors through $G(A_n)$, for some $n_1$, and this map lifts to $\psi_1$. Since $G(\psi_1 \circ \phi_1)$ agrees with the given group homomorphism $i$ from $G(A_1)$ to $G(A_n)$, it follows from the uniqueness property that we may replace $\psi_1$ by an inner conjugate to obtain

$$\psi_1 \circ \phi_1 = \alpha_{n_1-1} \circ \cdots \circ \alpha_2 \circ \alpha_1.$$ 

Continue in this way to obtain the desired regular isomorphism. 

### 3.2. Classification of $T_2$ and $T_3$ systems

We now obtain an abstract dimension distribution group classification of the families $F_2$, $F_3$ of regular star-extendible systems of $T_2$-algebras and $T_3$-algebras. This will follow immediately from Theorem 3.3 once we demonstrate that the family $\Omega_3$ of star-extendible regular embeddings between $T_3$-algebras has the uniqueness property above. This is the assertion of Lemma 3.4.

**Theorem 3.6.** The direct systems $A$ in $F_2$ and $F_3$ are classified up to regular isomorphism by the scaled ordered group invariant $(G(A), S_{mu}(A))$ where $S_{mu}(A))$ is the matrix unit scale.

Observe first that a multiplicity one embedding $\theta : A_1 \rightarrow A_2$ between $T_r$-algebras has inner conjugacy class determined by the inner conjugacy class of the restriction $\theta|A_1 \cap A_1^\ast$. Also note that this restriction necessarily respects the ordering of the $r$ summands of $A_1 \cap A_1^\ast$ and $A_2 \cap A_2^\ast$. Conversely, to any assignment of $r$ ordered objects to $r$ ordered boxes (in which subsequent objects must not be assigned to preceding boxes) there is an associated multiplicity one embedding $\theta$, at least if the summands of $A_2 \cap A_2^\ast$ are sufficiently large. In the case of $T_3$-algebras it follows that there are ten classes of multiplicity one embeddings, $\theta_1, \theta_2, \ldots, \theta_{10}$ in which the element $a \oplus b \oplus c$ of $A_1 \cap A_1^\ast$ is mapped to $A_2 \cap A_2^\ast$ according to the ten ordered assignments (or partitionings) indicated below:

- $\theta_1 : (a \ b \ c | 0 \ 0 \ 0 | 0 \ 0 \ 0)$
- $\theta_2 : (a \ b \ 0 | c \ 0 \ 0 | 0 \ 0 \ 0)$
- $\theta_3 : (a \ b \ 0 | 0 \ 0 \ | c \ 0 \ 0)$
- $\theta_4 : (a \ 0 \ 0 | b \ c \ 0 | 0 \ 0 \ 0)$
- $\theta_5 : (a \ 0 \ 0 | b \ 0 \ | c \ 0 \ 0)$
- $\theta_6 : (a \ 0 \ 0 | 0 \ 0 \ 0 | b \ c)$
- $\theta_7 : (0 \ 0 \ 0 | a \ b \ c | 0 \ 0 \ 0)$
- $\theta_8 : (0 \ 0 \ 0 | a \ b \ 0 | c \ 0 \ 0)$
- $\theta_9 : (0 \ 0 \ 0 | a \ 0 \ 0 | b \ c \ 0)$
- $\theta_{10} : (0 \ 0 \ 0 | 0 \ 0 \ 0 | a \ b \ c)$

In general one can show, by a simple recursion argument, that there are $R = \binom{2r-1}{r}$ ordered assignments of $r$ objects to $r$ boxes.

With the ordered $R$-tuple $\theta_1, \theta_2, \ldots, \theta_R$ specified we can now define the multiplicity signature of a regular star-extendible embedding $\phi$ between $T_r$ algebras to be the (unique) $R$-tuple $\text{sig}(\phi) = \{r_1, \ldots, r_R\}$ for which $\phi$ is inner conjugate to $r_1 \theta_1 \oplus \cdots \oplus r_R \theta_R$, where $r_k \theta_k$ is shorthand for $\theta_k \oplus \cdots \oplus \theta_k$, $r_k$ times. It is essentially a tautology that $\phi$ that $\phi'$ are inner conjugate if and only if they have the same signature. It follows that unital $T_r$-systems $A$ and $A'$ with $A_k = A'_k$ for all $k$, are regularly isomorphic if $\text{sig}(\alpha_k) = \text{sig}(\alpha'_k)$ for all $k$.

**Lemma 3.4.** (Uniqueness) Let $\phi, \psi : A_1 \rightarrow A_2$ be regular star-extendible embeddings between $T_3$-algebras. If the induced scaled group homomorphisms $G\phi, G\psi : G(A_1) \rightarrow G(A_2)$ agree then $\phi, \psi$ are inner conjugate.
Proof. The general case follows in a straightforward way from the case $A_1 = T_3$. Let $x = e_{12}, y = e_{23}, z = e_{13}$ be the off-diagonal matrix units of $A_1$, and for notational economy let $x, y, z$ also denote the corresponding elements of $G(A_1) = T_3(Z)$. Note that $G\theta_k : G A_1 \rightarrow G A_2$ is determined as a $\pi$-respecting group homomorphism by the triple $G\theta_k(x), G\theta_k(y), G\theta_k(z)$ and, more generally, $G\phi$ is determined by the triple $G\phi(x), G\phi(y), G\phi(z)$. Each matrix $G\phi(x)$ has up to six nonzero entries, and so the lemma will follow if we can show that the eighteen entries for the triple $G\phi(x), G\phi(y), G\phi(z)$ determine the multiplicity signature $r = \{r_1, r_2, \ldots, r_10\}$ of $\phi$. Writing these eighteen entries as a row vector $w$ we have $w = rX$ where $X$ is the 10 by 18 matrix

One can check that this matrix has has rank 10 and so $r$ may be obtained from $w$, as required. \hfill \Box

One should note that although $x$ and $y$ determine $z$ the rank distributions of $\phi(x)$ and $\phi(y)$ need not determine the rank distribution of $\phi(z)$. And indeed, the $10 \times 12$ matrix resulting from overlooking the distribution of $\phi(z)$ only has rank 9. It follows then that there are nonconjugate $\phi, \phi'$ for which $\phi(x)$ has the same rank distribution as $\phi'(x)$ and $\phi(y)$ has the same rank distribution as $\phi'(y)$.

3.3. Classifying $T_4$ systems. Arguing as above, there are 35 conjugacy classes of multiplicity one algebra homomorphisms from $T_4$ to $T_4 \otimes M_n$, for $n \geq 4$, and so the multiplicity signature of a regular star-extendible embedding between $T_4$-algebras is a 35-tuple. Arguing just as in the proof of Lemma 3.4 the family $\Omega_4$ of such embeddings has the uniqueness property if and only if the (ordered) sextet of integral matrices

$$G\phi(e_{12}), G\phi(e_{23}), G\phi(e_{34}), G\phi(e_{13}), G\phi(e_{24}), G\phi(e_{14})$$

determines the multiplicity signature of a regular star-extendible algebra homomorphism $\phi : T_4 \rightarrow A$, into a $T_4$-algebra $A$. Each image $G\phi(e_{ij})$ is a $4 \times 4$ upper triangular matrix, with up to ten nonzero entries, and so the sextet provides 60 linear equations for the 35-tuple. However, (computer) calculation shows the rank of the associated $35 \times 60$ coefficient matrix to be 31 and so $\Omega_4$ fails to have the uniqueness property.

On the other hand, let $\Omega_4^-$ be the family of embeddings which are regular and star-extendible and which have no multiplicity one summand $\theta$, of degenerate type, with range
in the self-adjoint subalgebra. Clearly there are four such degenerate embeddings \( \theta \). Then (computer) calculation shows that the associated \( 31 \times 60 \) coefficient matrix has rank 31. Thus \( \Omega_4^- \) does have the uniqueness property and the family \( \mathcal{F}_4^- = \mathcal{F}(\Omega_4^-) \) admits classification in terms of \( G(\mathcal{A}) \);

**Theorem 3.7.** The direct systems \( \mathcal{A} \) of \( \mathcal{F}_4^- = \mathcal{F}(\Omega_4^-) \) are classified by the scaled ordered dimension group \( (G(\mathcal{A}), S_{\Omega_4^-}(\mathcal{A})) \).

We note that for \( \mathcal{A} \) in \( \mathcal{F}_4^- \) the group \( G(\mathcal{A}) \) is of the form \( \lim_{\to} \mathbb{Z}^{10}, G(\alpha_k) \) and so somewhat more computable, in principle, than \( \Gamma_H^* = \lim_{\to} (\mathbb{Z}^{35}, G_H^*(\alpha_k)) \).

### 3.4. Systems of cycle algebras

A 4-cycle algebra is a digraph algebra \( A \) whose reduced digraph is isomorphic to the 4-cycle digraph \( D_4 \). The simplest of these is \( A(D_4) \) which is the subalgebra of \( M_4(\mathbb{C}) \) spanned by the matrix units \( e_{11}, e_{22}, e_{33}, e_{44}, e_{13}, e_{14}, e_{24}, e_{23} \). The so-called rigid embeddings between 4-cycle algebras are those embeddings \( \phi : A_1 \to A_2 \) for which \( \phi \) is inner conjugate to \( r_1 \theta_1 \oplus r_2 \theta_2 \oplus r_3 \theta_3 \oplus r_4 \theta_4 \), where \( \theta_1, \ldots, \theta_4 \) are multiplicity one star-extendible embeddings whose \( K_0 \) matrices have the form

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

These embeddings are those associated with the digraph automorphisms of \( D_4 \), rather than the digraph endomorphisms. The dimension distribution group \( G(A(D_4)) \) is naturally identifiable with \( \mathbb{Z}^4 \), and we may order the generators so that the induced morphism

\[
G(\phi) = \begin{bmatrix} K_0 \phi & 0 \\ 0 & \Gamma_\phi \end{bmatrix} : \mathbb{Z}^4 \to \mathbb{Z}^4
\]

is realised by the integral matrix

\[
\begin{bmatrix}
1 + r_4 & r_2 + r_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 + r_3 & r_1 + r_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 + r_2 & r_3 + r_4 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & r_3 + r_4 & r_1 + r_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & r_1 & r_4 & r_3 & r_2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & r_4 & r_1 & r_2 & r_3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & r_3 & r_2 & r_1 & r_4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & r_2 & r_3 & r_4 & r_1
\end{bmatrix}
\]

In view of the form of \( \Gamma_\phi \), in which the multiplicity signature for \( \phi \) is evident, the family of such embeddings has the uniqueness property. Likewise the family of rigid embeddings between \( 2n \)-cycle algebras, has the uniqueness property and it follows from Theorem 3.6 that these systems admit a classification in terms of a scaled dimension distribution group.

However, for these systems there is a more economical and revealing classification scheme which has been developed in Power [Po4] and in Donsig and Power [DoP2, DoP3]. Associated with a regular embedding \( \phi \) between digraph algebras there is a natural induced
homology group homomorphism $H_1\phi$, which, in the case of 4-cycle algebras, is a group homomorphism from $\mathbb{Z}$ to $\mathbb{Z}$. For a rigid embedding $\phi$, as above, this map is realised by the $1 \times 1$ integral matrix

$$[r_1 - r_2 + r_3 - r_4].$$

The fact that $H_1\phi$ and $K_0\phi$ determine the inner conjugacy class of $\phi$ is at the heart of the classification of rigid 4-cycle systems $A$ in terms of $K_0H_1(A)$ and various scales.

We return to these systems in the next section.

4. Classifications with $G^r_H(A)$

Let $H$ be a connected transitive antisymmetric digraph, so that $A(H)$ is a triangular digraph algebra with digraph $H$. Let $G^r_H(A)$ be the regular Grothendieck group of the regular star-extendible system $\mathcal{A} = \{A_k, \alpha_k\}$, with order induced by the scale $\Sigma_H^r(\mathcal{A})$. Note that if $\alpha : A \to A'$ is a regular star-extendible homomorphism of digraph algebras then there is a natural induced scaled ordered group homomorphism

$$\hat{\alpha} : G^r_H(A) \to G^r_H(A')$$

which is induced by the correspondence $V^r_H(A) \to V^r_H(A')$ given by $[\phi] \to [\alpha \circ \phi]$. Also it follows from the definition that $G^r_H(A) = \lim\limits_{\to}(G^r_H(A_k), \hat{\alpha})$ so that $G^r_H(A)$ is a scaled ordered dimension group.

Let $\mathcal{F}_H$ be the family of regular star-extendible systems $\{A_k, \alpha_k\}$ of $H$-algebras with the convention that $A_1 = A(H)$. We say, simply, that $\mathcal{A}, \mathcal{A}'$ in $\mathcal{F}_H$ are isomorphic if there exists a commuting diagram as in section 2.1, and that $\mathcal{A}, \mathcal{A}'$ are stably isomorphic if there is such a diagram for the stable systems $\{A_k \otimes M_k, \alpha_k \otimes i_k\}$, $\{A'_k \otimes M_k, \alpha'_k \otimes i_k\}$ where $\{M_k, i_k\}$ is the system for the stable algebra $M_\infty$.

Recall that the inner unitary equivalence class $[\phi]$ of a regular star-extendible homomorphism $\phi : A(H) \to A$ is determined by its multiplicity signature, which is an $r$-tuple of nonnegative integers, where $r$ is the number of multiplicity one embeddings from $A(H)$ to $A$. It is this basic fact that ensures that the natural inclusion $V^r_H(A) \to G^r_H(A)$ is an injection, identifiable with the injection $\mathbb{Z}_r^+ \to \mathbb{Z}^r$, and that $G^r_H(A)$ is a dimension group. This basic fact also leads to the following uniqueness lemma.

**Lemma 4.1.** *(Uniqueness.)* Let $\phi, \psi : A_1 \to A_2$ be regular star-extendible homomorphisms between $H$-algebras with $\hat{\phi} = \hat{\psi}$. Then $\phi$ and $\psi$ are inner conjugate.

**Proof.** Let $i : A(H) \to A_1$ be a proper injection. Since $\phi, \psi$ are inner conjugate if and only if their restrictions to $i(A(H))$ are inner conjugate we may as well assume that $A_1 = A(H)$. The injection $i$ determines the element $[i]$ of $V^r_H(A_1)$, which we may identify with the element $[\hat{i}]$ of $G^r_H(A_1)$. By definition $\hat{\phi}([i]) = [\phi \circ i]$, the class in $G^r_H(A_1)$. Thus the hypotheses imply that $[\phi] = [\psi]$ as classes in $G^r_H(A_2)$ and the lemma follows.  \[\Box\]
For a given ordered group homomorphism \( \gamma : G^r_H(A_1) \to G^r_H(A_2) \) there need not exist a lifting \( \phi : A_1 \to A_2 \otimes M_k \) with \( \hat{\phi} = \theta \). However we can guarantee this existence in two ways.

Let \( \theta_1, \ldots, \theta_R \) be the distinct multiplicity one injections \( \theta_i : A(H) \to A(H) \otimes M_k \) with \( k \geq R \), corresponding to the \( R \) endomorphisms of the digraph \( H \). In analogy with the definition of the "multi-scale" \( S_H(B) \) given in section 3, define the multi-cone

\[
C^*_H(A_1) \subseteq G^r_H(A_1)_+ \times \cdots \times G^r_H(A_1)_+
\]

as the subset of the \( R \)-fold product consisting of the elements

\[
([\phi \circ \theta_1], \ldots, [\phi \circ \theta_R])
\]

associated with regular star extendible embeddings \( \phi : A(H) \to A_1 \otimes M_k \), for arbitrary \( k \).

Now, if \( \gamma : G^r_H(A_1) \to G^r_H(A_2) \) preserves the multi-cone then

\[
(\gamma([\theta_1], \ldots, \gamma([\theta_R])) = ([\phi \circ \theta_1], \ldots, [\phi \circ \theta_R])
\]

for some embedding \( \phi : A(H) \to A_2 \otimes M_k \). Moreover, since \( \hat{\phi} \) agrees with \( \theta \) on the generators of \( G^r_H(A(H)) \), it follows that \( \hat{\phi} = \theta \).

Naturally, the multiscale \( \hat{\Sigma}_H(A_1) \) is defined as the subset of the multi-cone associated with the classes of embeddings of \( A(H) \) into the algebra \( A_1 \) itself.

An alternative equivalent requirement that guarantees the existence of a lifting for \( \gamma \) is to take into account the fact that an induced ordered group homomorphism \( \hat{\phi} \) respects the natural right action of the semigroup \( \text{End}(H) = \{ \theta_1, \ldots, \theta_k \} \) and to require that \( \gamma \) respects this right action. We discuss this important perspective in [Po7].

**Lemma 4.2.** (Existence.) Let \( A_1, A_2 \) be \( H \)-algebras and let \( \theta : G^r_H(A_1) \to G^r_H(A_2) \) be an ordered group homomorphism which preserves the multi-cone. Then for sufficiently large \( k \) there exists a regular star-extendible algebra homomorphism \( \phi : A_1 \to A_2 \otimes M_k \) such that \( \hat{\phi} = \theta \).

In view of the lemmas above we can obtain the following theorem in the usual fashion.

**Theorem 4.3.** Systems \( \mathcal{A}, \mathcal{A}' \) in \( \mathcal{F}_H \) are stably regularly isomorphic if and only if the scaled ordered dimension groups \( G^r_H(\mathcal{A}) \) and \( G^r_H(\mathcal{A}') \) are isomorphic by an isomorphism which preserves the multiscale.

**Examples.**

For \( H = T_3 \) the regular Grothendieck group invariant, which has the form \( \lim(Z^{10}, \alpha_k) \), and the integral matrix for \( \alpha_k \) can be computed from the multiplicity signature and the \( 10 \times 10 \) multiplication table for the semigroup of order preserving partitions.

For subfamilies of \( T_3 \)-algebra systems it becomes appropriate to consider a reduced Grothendieck group invariant as in the following example.

Let \( \mathcal{G} \) be the family of embeddings \( \phi \) between \( T_3 \)-algebras which are unitarily equivalent to a direct sum \( \rho_r \oplus \sigma_3 \oplus \delta_1 \) where \( \rho_r \) is a refinement embedding of multiplicity \( r \),
\[ \rho_r : \begin{bmatrix} a & e & g \\ b & f & c \end{bmatrix} \rightarrow \begin{bmatrix} a \otimes I_r & e \otimes I_r & g \otimes I_r \\ b \otimes I_r & f \otimes I_r & c \otimes I_r \end{bmatrix} \]

where \( \sigma_{3s} \) is a standard embedding of multiplicity 3s,

\[ \sigma_{3s} : [a] \rightarrow \begin{bmatrix} \sigma_s(a) & 0 & 0 \\ \sigma_s(a) & 0 & \sigma_s(a) \end{bmatrix}, \]

and where \( \delta_t \) is a degenerate embedding of multiplicity \( t \), with

\[ \delta_t : a \rightarrow \begin{bmatrix} \sigma_t(a) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \]

View the embedding \( \phi \) as having reduced multiplicity signature \( \{r, s, t\} \) and view \( \phi \) itself as a direct sum of the basic embeddings \( \rho_1, \sigma_3 \) and \( \delta_1 \). The family \( \mathcal{G} \) is closed under compositions and the corresponding reduced Grothendieck group \( G_\mathcal{G}(A) \) for the system \( A = \{A_k, \alpha_k\} \) with \( \text{sig}(\alpha_k) = \{r_k, s_k, t_k\} \) may be identified with the dimension group

\[ \lim_{\rightarrow} (\mathbb{Z}^3, \begin{bmatrix} r_k & 0 & 0 \\ s_k & r_k + s_k & 0 \\ t_k & t_k & r_k + s_k + t_k \end{bmatrix}). \]

These dimension groups and their multiscales classify the stable systems in the family up to regular isomorphism.

Let us also illustrate these methods with an application to the locally finite algebras \( A \) of the towers \( A = \{A_k, \alpha_k\} \) of 2m-cycle algebras where the inclusions \( \alpha_k : A_k \rightarrow A_{k+1} \) are of rigid type, as indicated in section 3.4. Let \( V_m(A) \) be the semigroup in \( V_{2m}(A) \) corresponding to classes \( [\phi] \) associated with rigid embeddings and let \( G_m(A) \) be the associated Grothendieck group. Plainly \( G_m(A_k) \) is identifiable with \( \mathbb{Z}^{2m} \) with generators corresponding to the 2m symmetries of the digraph \( D_{2m} \). Furthermore, \( G_m(\alpha_k) \), the induced ordered group homomorphism, is effected by viewing \( \mathbb{Z}^{2m} \) as the group ring, \( \mathbb{Z}[D_{2m}] \) say, for the automorphism group of \( D_{2m} \) and by regarding \( G_m(\alpha_k) \) as left multiplication by \( g(\alpha_k) \in \mathbb{Z}[D_{2m}] \), the element \( r_1 \theta_1 + \cdots + r_{2m} \theta_{2m} \) of \( \mathbb{Z}[D_{2m}] \) for the multiplicity signature of \( \alpha_k \). In case \( m = 3 \) this is given by multiplication by the matrix

\[ \begin{bmatrix} r_1 & r_2 & r_5 & r_4 & r_3 & r_6 \\ r_2 & r_1 & r_4 & r_3 & r_6 & r_5 \\ r_3 & r_6 & r_5 & r_2 & r_1 & r_4 \\ r_4 & r_5 & r_6 & r_1 & r_2 & r_3 \\ r_5 & r_4 & r_1 & r_6 & r_3 & r_2 \\ r_6 & r_3 & r_2 & r_5 & r_4 & r_1 \end{bmatrix}. \]

In this way one obtains an identification of the reduced Grothendieck group \( G_m(A) \) as
$G_m(A) = \lim_{\rightarrow} (\mathbb{Z}[D_{2m}], g(\alpha_k))$.

We remark that one can see from Section 3.4 in the case of 4-cycle algebras, that the group $G_m(A)$ appears naturally as a summand of the dimension distribution group $G(A)$.

A key result in Donsig and Power \cite{DoP3} is that for $m \geq 3$ the locally finite algebras $A = \lim_{\rightarrow} (A_k, \alpha_k)$, $A' = \lim_{\rightarrow} (A'_k, \alpha'_k)$ of such towers $A, A'$ are star-extendibly isomorphic if and only if the towers are regularly isomorphic. In view of this we may define $G_m(A)$ in a well-defined way as the group $G_m(A)$. Combining these fact with Theorem 4.3 (adapted to the reduced Grothendieck group) leads to the following classification. This classification may also be extended to the operator algebras of the systems.

**Theorem 4.4.** Let $m \geq 3$ and let $A$ and $A'$ be the locally finite complex algebras associated with the towers $A = \{A_k, \alpha_k\}$, $A' = \{A'_k, \alpha'_k\}$ consisting of $2m$-cycle algebras, for $m \geq 3$, and rigid embeddings. Then $A$ and $A'$ are stably isomorphic if and only if the ordered groups $G_m(A), G_m(A')$ are isomorphic by an isomorphism which preserves the multicone.

Although we have focused on systems $\{A_k, \alpha_k\}$ for which the reduced digraph of $A_k$ is connected, with bounded diameter, the arguments and results of the paper also extend in a routine way to more general systems, and in particular, to systems of direct sums of $H$-algebras.

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