Strings as perturbations of evolving spin-networks

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ABSTRACT

A connection between non-perturbative formulations of quantum gravity and perturbative string theory is exhibited, based on a formulation of the non-perturbative dynamics due to Markopoulou. In this formulation the dynamics of spin network states and their generalizations is described in terms of histories which have discrete analogues of the causal structure and many fingered time of Lorentzian spacetimes. Perturbations of these histories turn out to be described in terms of spin systems defined on 2-dimensional time-like surfaces embedded in the discrete spacetime. When the history has a classical limit which is Minkowski spacetime, the action of the perturbation theory is given to leading order by the spacetime area of the surface, as in bosonic string theory. This map between a non-perturbative formulation of quantum gravity and a 1+1 dimensional theory generalizes to a large class of theories in which the group $SU(2)$ is extended to any quantum group or supergroup. It is argued that a necessary condition for the non-perturbative theory to have a good classical limit is that the resulting 1+1 dimensional theory defines a consistent and stable perturbative string theory.

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1 Introduction

There are two approaches to quantum gravity that have made significant progress in the last ten years. In the perturbative or, more precisely, the background dependent, regime, string theory provides the only consistent description of the interaction of gravitons with other degrees of freedom. Moreover, there are good reasons to believe that whatever the true quantum theory of gravity is, string theory will describe its perturbative limit. Among these is that it resolves a paradox, which is how to have a theory which has a physical cutoff at a fixed length scale while remaining Lorentz invariant [1].

At the same time, while there are many results that point to the existence of a non-perturbative, background independent theory that unifies the various perturbative string theories, we do not yet know the form of that theory (despite some very interesting proposals [2, 3, 4]). Given the fact that S duality [5] and mirror symmetry [6] strongly suggest that the non-perturbative theory cannot be expressed in terms of the embeddings of strings, membranes or anything else in a fixed manifold, from what mathematical elements could non-perturbative string theory be constructed? If we remove manifolds from the quantum theory, one is left only with representation theory and combinatorics.

Could $\mathcal{M}$ theory be constructed from only representation theory and combinatorics? While the answer is not known, we may note that quantum general relativity has been constructed [7, 8, 9, 10, 11, 12, 13] and it is almost of this form. The states of the theory are labeled by spin networks [14, 8], which are constructed from combinatorics and the representation theory of $SU(2)$. The local operators of the theory may be expressed in terms of finite combinatorial operations on these states [14, 8, 9]. The result is a combinatorial picture of quantum geometry in which areas [15, 8], volumes [15, 8, 16] and lengths [17] are discrete and have computable spectra.

At the Hamiltonian level, quantum general relativity exists as a sensible quantum theory [11]; the only problem with it is that, as is the case for example for random surface theories away from their critical points, it seems

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1 Quantum general relativity is not completely combinatorial because the states of those theories are labeled by the embeddings of the spin networks in a prior three manifold $\Sigma$, up to diffeomorphisms. However, the local operators that have so far been studied do not depend on the embedding information. To arrive at a purely combinatorial picture of quantum geometry the embedding should be reconstructed from combinatorial information. This may be possible, given an appropriate non-embedded extension of the spin network states, as discussed in [13, 14].
to lack a continuum limit which describes massless particles moving in a Lorentzian spacetime\cite{20}. The evidence is that what has been constructed, at least so far, is the ultralocal, or $c \to 0$ limit of the theory\cite{20, 21}.

It is then natural to ask whether there might be some extension of the spin network states of quantum general relativity that may describe a non-perturbative string theory. The kinds of extensions that might be explored include the following: 1) extend the group whose representation theory labels the spin networks from $SU(2)$, which is related to symmetries of $3+1$ dimensional spacetime to other groups relevant for string theory, 2) add supersymmetry, 3) add labeled surfaces corresponding to $p$-form gauge theories, 4) make the theory completely background independent and combinatorial by removing the embedding of the spin networks in fixed manifolds 5) extend the dynamics from that given by general relativity to a larger set, to be constrained only by some appropriate notion of local causality.

The study of all of these extensions is in progress. Supersymmetry seems to be naturally incorporated, as in \cite{22}, as are $p$-form fields\cite{23}. New general forms of the dynamics have been explored, in both the Euclidean\cite{24, 25, 26, 27, 28, 29} and Lorentzian\cite{30, 18, 19} cases. How to extend the group from $SU(2)$ to any quantum group or Hopf algebra $G_q$, while at the same time dropping the dependence on embedding, has been understood in \cite{19}.

The next question is how should one test whether such an extension might yield a non-perturbative formulation of $M$ theory. (Or to put it in the native dialect, if a completely compactified version of $M$ theory walked in the door, how would we recognize it?) One necessary condition is that the theory must have a classical limit in which some Lorentzian manifolds turn out to be a good approximate description and general relativity is approximately true. A second necessary condition is that perturbation theory around this limit must be described in terms of the interactions of strings and branes in that manifold.

The problem of the classical limit is hard as it is a problem in critical phenomena. As the causal structure itself is dynamical this problem seems to be more analogous to non-equilibrium critical phenomena such as directed percolation than it is to second order phase transitions\cite{31}. Here we will assume that this problem has been solved and tackle the second problem, that of the perturbation theory around a classical limit. We will study a class of theories which are characterized by the choice of a quantum group

\[2\] Quantum deformation to $SU(2)_q$ is required to incorporate a cosmological constant with $q = e^{2\pi i/k + 2}$ with $k = 6\pi/G^2\Lambda$\cite{31, 32}.
or superalgebra $G_q$ and a set of evolution amplitudes $A$ (to be characterized below.) We will show that these theories have the following properties:

1. Their perturbation theory is described in terms of a $1+1$ dimensional spin system defined on a timelike surface embedded in the Lorentzian spacetime $(M, g)$ that arises in the classical limit of the theory.

2. If that spacetime $(M, g)$ is Minkowski spacetime and the group $G = SU(2)_q$, the effective action for the perturbation associated with a surface $X^\mu(\sigma, \tau)$ is to leading order proportional to the Nambu action of bosonic string theory, which is its area.

3. A necessary condition that one of this class of theories has a good classical limit is that the induced $1+1$ dimensional theory describes a consistent string theory in $3+1$ dimensions.

To find these results we work with a form of the dynamics of spin networks proposed by Markopoulou, which has built in local causality. While we have recently extended that formalism to a large class of theories in which the spin networks are extended to states defined in terms of the conformal blocks of a rational conformal field theory, we will work here with a restricted set of the states of these theories which can be described in terms of labeled triangulations of some manifold. The reason is that in the absence of a real understanding of the classical limit this allows us to sidestep some questions associated with that limit. We will also work here with the group $SU(2)_q$, although the extension to any group or Hopf algebra $G_q$ is straightforward.

Before starting, we note that we are not claiming to have shown that the theories we discuss here are non-perturbative string theories. Nor have we given sufficient conditions for them to be. However, we may note that a dynamical theory of the $(p,q)$ strings, which are a large set of BPS states may be naturally formulated in this framework.

In the next two sections we summarize the kinematical and dynamical framework for quantum theories of gravity with which we will work. In section 4 we describe the perturbation theory and show how a $1+1$ theory is constructed to describe the perturbations.
2 Summary of quantum spatial geometry in the dual picture

In the dual picture described by Markopoulou\cite{18} the space of states of a non-perturbative quantum theory of gravity based on a group $G_q$ is constructed as follows.

We begin with a three manifold $\Sigma$, to which we will associate a space of states $H_{\Sigma}$ Consider a simplicial decomposition $T$ of $\Sigma$. $T$ is labelled as follows. Each 2-dimensional face, $f$, in the triangulation $T$ is labelled by a representation $j(f)$ of $G_q$. Each tetrahedron $\tau \in T$ is labelled by an intertwiner map $\mu(\tau) \in V_{ijkl}^{(\tau)}$ for the four representations on the faces of the tetrahedron.

We may then denote the labelled triangulations as states $|T, \{j\}, \{\mu\}\rangle$ which we will require to form an orthonormal basis of $H_{\Sigma}$ The inner product on $H_{\Sigma}$ is defined as

$$\langle T, \{j\}, \{\mu\}|T', \{j'\}, \{\nu\}\rangle = \delta_{TT'}\delta_{\{j\}\{j'\}}\prod_{\tau \in T}\langle \mu(\tau)|\nu(\tau)\rangle_{V(\tau)}$$

(1)

where $\mu(\tau), \nu(\tau) \in V_{ijkl}^{(\tau)}$, and $\langle \ | \rangle_{V(\tau)}$ is the natural inner product in $V_{ijkl}^{(\tau)}$. Thus, the states are orthogonal unless there is an isomorphism of one triangulation to the other that preserves the labels on the faces. If $I$ labels the tetrahedra $\tau \in T$, then given a basis $\{\mu_I^{(I)}\}$ in the space of intertwiners $V_{ijkl}^{(\tau)}$, the states $|T', \{j\}, \{\mu_I^{(I)}\}\rangle$ give an orthonormal basis of $H_{\Sigma}$.

In the $SU(2)$ case, area\cite{15,8} and volume\cite{15,8,16} operators may then be used to assign areas to the faces and volume operators to each space of intertwiners in each tetrahedron\cite{19}.

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\footnote{We may note that these state spaces $H_{\Sigma}$ are subspaces of the full set of non-embedded states $H_{G_q}$ defined in \cite{19}. The correspondence is constructed as follows. To each triangulation $T$ we may associate a dual 2-surface $S(T)$ by the following procedure. Each tetrahedron $\tau_I \in T$ is mapped to a 4-punctured sphere $B_I$, so that the surface of $B_I$ has the same orientation as the surface of $\tau_I$. Two spheres $B_I$ and $B_J$ are joined at a puncture when that puncture corresponds to a face shared between the tetrahedra $\tau_I$ and $\tau_J$. The punctured is labelled by the representation on that face. Given a state $|T, \{j\}, \{\mu\}\rangle$ we then get a state $|S(T), \{j\}, \{\mu\}\rangle$ by transferring the representations and intertwiners according to the construction. This establishes a map from $H_{\Sigma}$ into a subspace of $H_{G_q}$. In the case $G = SU(2)$ these labeled two surfaces are equivalent to labeled graphs, i.e. to spin networks.}
3 Summary of causal evolution in the dual picture

According to the proposal of [18], the states of $S$ evolve according to a distinct set of rules. Any state $\Gamma_0$ may evolve to one of a finite number of possible successor states $\Gamma'_0$. Each $\Gamma'_0$ is derived from $\Gamma_0$ by application of one of four possible moves, called Pachner moves [34]. These moves modify the state $\Gamma_0$ in a local region involving one to four adjacent tetrahedra.

Consider any subset of $\Gamma$ consisting of $n$ adjacent tetrahedra, where $n$ is between 1 and 4, which make up $n$ sides of a four-simplex $S_4$. Then there is an evolution rule by which those $n$ tetrahedra are removed, and replaced by the other $5-n$ tetrahedra in the $S_4$. This is called a Pachner move. The different possible moves are called $n \to (5-n)$ moves (Thus, there are $1 \to 4$, $2 \to 3$ etc. moves.) The new tetrahedra must be labeled, by new representations $j$ and intertwiners $k$. For each move there are 15 labels involved, 10 representations on the faces and 5 intertwiners on the tetrahedra. This is because the labels involved in the move are exactly those of the four simplex $S_4$. For each $n$ there is then an amplitude $A_{n\to5-n}$ that is a function of the 15 labels. A choice of these amplitudes for all possible labels, for the four cases $1 \to 4$, $..., 4 \to 1$, then constitutes a choice of the dynamics of the theory.

The application of one of the possible Pachner moves to $\Gamma_0$, together with a choice of the possible labelings on the new faces and tetrahedra the move creates, results in a new spin network state $\Gamma_1$. This differs from $\Gamma_0$ just in a region which consisted of between 1 and 4 adjacent tetrahedra. The process may be continued a finite number of times $N$, to yield successor states $\Gamma_2, ..., \Gamma_N$.

Any particular set of $N$ moves beginning with a state $\Gamma_0$ and ending with a state $\Gamma_N$ defines a four dimensional combinatorial structure, which we will call a history, $\mathcal{M}$ from $\Gamma_0$ to $\Gamma_N$. Each history consists of $N$ combinatorial four simplices. We will require that every tetrahedra in the real triangulation has been subject to at least one move. This means that the boundary of $\mathcal{M}$, which is a set of tetrahedra, falls into two connected sets so that $\partial \mathcal{M} = \Gamma_0 \cup \Gamma_N$. All tetrahedra not in the boundary of $\mathcal{M}$ are contained in exactly two four simplices of $\mathcal{M}$.

Each history $\mathcal{M}$ is a causal set, whose structure is determined as follows. The tetrahedra of each four simplex, $S_4$ of $\mathcal{M}$ are divided into two sets, which are called the past and the future set. This is possible because each four simplex contains tetrahedra in two states $\Gamma_i$ and $\Gamma_{i+1}$ for some $i$ between 0 and $N$. Those in $\Gamma_i$ were in the group that were wiped out by the Pachner
move, which were replaced by those in $\Gamma_{i+1}$. Those that were wiped out are called the past set of that four simplex, the new ones, those in $\Gamma_{i+1}$ are called the future set. With the exception of those in the boundary, every tetrahedron is in the future set of one four simplex and the past set of another.

The causal structure of $M$ is then defined as follows. The tetrahedra of $M$ make up a causal set defined as follows:\footnote{A causal set is a set with a partial order, with no closed timelike loops\cite{35}.}. Given two tetrahedra $T_1$ and $T_2$ in $M$, we say $T_2$ is to the future of $T_1$ (written $T_2 > T_1$) if there is a sequence of causal steps that begin on $T_1$ and end on $T_2$. A causal step is a step from a tetrahedron which is an element of the past set of some four simplex, $S_4$ to any tetrahedron which is an element of the future set of the same four simplex. By construction, there are no closed causal loops, so the partial ordering gives a causal set.

This theory then falls partly within the causal set formulation of discrete quantum spacetime proposed by Sorkin and collaborators \cite{35} and 't Hooft\cite{36}. However, they have additional structure. Among these is the fact that each history $M$ may also be foliated by a number of spacelike slices $\Gamma$. A spacelike slice of $M$ is a connected set of tetrahedra of $M$ that 1) constitute some $\Gamma \in S$ (so it is dual to a 4 valent spin network with no free ends) and 2) no two of whose elements are causally related.

Each $\Gamma_i$ in the original construction of $M$ constitutes a spacelike slice of $M$. But there are also many other spacelike slices in $M$ that are not one of the $\Gamma_i$. In fact, given any spacelike slice $\Gamma$ in $M$ there are a large, but finite, number of slices which are differ from it by the application of one Pachner move. Because of this, there is in this formulation a discrete analogue of the many fingered time of the canonical picture of general relativity.

The dynamics is then to be specified by the choice of the amplitudes $A[M]$. By locality these should be products of amplitudes for each evolution step,

$$A[M] = \prod_{I=1}^{N} A_I[j, k, c]$$

where we have indicated that an amplitude $A_I[j, k, c]$ is associated with each causal step, which is realized by the action of a four simplex. It may in general depend on the spins on the faces, $j$, the intertwiners on the tetrahedra, $k$ and the causal structure (division of the tetrahedra into a future and past set, $c$.)
The transition amplitude from an initial state $\Gamma_0$ to a final state $\Gamma_f$ is given by a sum over all histories $\mathcal{M}$ that evolve $\Gamma_0$ to $\Gamma_f$.

$$A[\Gamma_0 \to \Gamma_f] = \sum_{\mathcal{M} | \partial \mathcal{M} = \Gamma_0 \cup \Gamma_f} A[\mathcal{M}]$$

A theory of this form is then specified by the pair $(G_q, \mathcal{A})$ of quantum groups and sets of amplitudes on labeled four simplices.

4 Perturbations of histories in terms of an induced $1+1$ dimensional theory

We now study the problem of describing a perturbation theory for the class of theories $(G_q, \mathcal{A})$ based on a quantum group $G_q$ and amplitudes $A$ which we have defined. To simplify the presentation we will work first with the simplest case in which $G_q = SU(2)_q$ and then discuss the extension to a general $G_q$.

To define a perturbation of a history we must first define the perturbation of a state $|T, \{j\}, \{\mu\}\rangle \in \mathcal{H}^\Sigma$. There are two classes of perturbations: those that change only the labelings, $\{j', \mu'\}$ leaving the triangulation $T$ fixed, and those that change the triangulation. Changes in the triangulations are generated by the Pachner moves but these are also the evolution moves. To cleanly separate evolution along one history from perturbations that take us from a history to a distinct history we consider only the first class of perturbations.

To study these perturbations we need to know how, given an initial set of consistent labels $\{j, \mu\}$ new sets $\{j', \mu'\}$ may be chosen that are both consistent and differ from the original set by a small change. To solve this problem we should remember that the consistency conditions express the gauge invariance of the theory under the gauge group $SU(2)_q$. Consistent changes in the labeling correspond to addition of Wilson loops around closed loops on the graph. In the dual picture in which we are working consistent perturbations are defined by putting a closed loop of labels in the elementary representation 1 in the triangulation corresponding to the addition of a Wilson loop to a dual graph. This corresponds to a loop $\gamma \in T$ of labels, by which we mean a sequence of alternating faces $f_i$ and tetrahedra $\tau_i$,

$$\gamma = \{\tau_1, f_1, \tau_2, f_2, \tau_3, \ldots, \tau_n, f_n\}.$$
where each face is between two adjacent tetrahedra: \( f_i \in \tau_i \) and \( f_i \in \tau_{i+1} \).

We then define the new state to be

\[
|\gamma \ast \Psi \rangle = |T, \{j^\prime\}, \{\mu^\prime\}\rangle.
\] (5)

where the new representations and intertwiners are changed only for the faces and tetrahedra (4) in \( \gamma \).

The representations are changed by taking the product along each link and intertwiner of the old label with the spin 1/2 edge, giving us a superpositions of new labels coming from the decomposition of \( j^\prime = 1 \otimes j \). The change in the intertwiners is obtained by splitting the 4-valent node associated to each tetrahedron along the path of the loop \( \gamma \). To calculate the result, we use the edge addition formula of spin networks (addition of angular momentum) \([37]\). Let the loop \( \gamma \) cross \( n \) faces and hence \( n \) intertwiners, labelled by \( j_i \) and \( k_i \), respectively. Call \( |T, \{j + \delta\}, \{k + \rho\}\rangle \) the state where the \( i \)’th face that \( \gamma \) crosses has been changed from \( j_i \) to \( j_i + \delta_i \) and the \( i \)’th intertwiner has been changed from \( k_i \) to \( k_i + \rho_i \). Then the new perturbed state is

\[
|\gamma \ast \Psi \rangle = \sum_{\rho_i=\pm1} \sum_{\delta_i=\pm1} |T, \{j + \delta\}, \{k + \rho\}\rangle \prod_a C(a).
\] (6)

\( C(a) \) is a weight factor for each trivalent node \( a \) in the splitting of the 4-valent ones along \( \gamma \). Each \( C(a) \) depends on a face spin \( j \) (and the corresponding \( \delta \)), an intertwiner spin \( k \) (and the corresponding \( \rho \)) and a third spin \( r \), from an edge not crossed by \( \gamma \). Thus, \( C(a) = C_{\delta \rho}^{jkr} \). From recoupling theory we find that \([37]\),

\[
C_{++}^{jkr} = 1
\] (7)

\[
C_{+-}^{jkr} = C_{-+}^{jkr} = (-1)^k j \Theta(k + 1, k, 1) \left\{ \begin{array}{c} k, r, k + 1 \\ j - 1, 1, j \end{array} \right\}
\] (8)

\[
C_{- -}^{jkr} = (-1)^{k-1} j \Theta(k - 1, k, 1) \left\{ \begin{array}{c} k, r, k + 1 \\ j - 1, 1, j \end{array} \right\}.
\] (9)

where the symbols are the quantum deformed theta and 6j symbols defined in \([37]\). Now that we know what perturbations of states we are interested in, let us define the corresponding perturbations of histories.

A perturbation of a history \( M \) should lead to a perturbation of every state \( |T, \{j\}, \{k\}\rangle \), or spacelike slice \( \Gamma \), of the history \( M \). The many-finger-time structure of the histories \( M \) imposes a strong constraint on the form of the perturbation of a history, because whenever two spacelike slices overlap
the perturbations must agree on the overlap. This means that the perturbation of a history must be given by a two surface \( S \) embedded in the history \( \mathcal{M} \) in such a way that every slice through it is a loop of the form of (3) that gives a perturbation of the state corresponding to that slice.

We will also require our perturbations to be causal. Causal perturbations are those in which the support of the perturbation on any slice is in the causal future of the support of the perturbation of the previous slice. This is satisfied when (a) for any spacelike slice \( \Gamma \), \( S \cap \Gamma \) is a closed loop \( \gamma(S) \in \Gamma \), and (b) for any 4-simplex in \( \mathcal{M} \), if \( S \) includes at least one of its future tetrahedra it also includes at least one of its past tetrahedra (i.e. \( S \) is timelike).

Thus, we have two spin fields on \( S \). The \( \delta = \pm 1 \) live on the perturbed faces, while the \( \rho = \pm 1 \) live on the perturbed tetrahedra. Thus, the perturbation defines a rather complicated spin system on the 2-dimensional surface. The perturbation \( \mathcal{M}' \) of the history \( \mathcal{M} \), is the superposition of all the histories in which the \( \delta \)'s and \( \rho \)'s take all their possible values.

A history \( \mathcal{M} \) is an amplitude from an initial spin network state to a final one. This amplitude is given by the product of the amplitudes for the 4-simplices \( S_4 \) that make up the history \( \mathcal{M} \),

\[
A[\mathcal{M}] = \prod_I A_I[j, k, c].
\] (10)

The amplitude \( \Delta W \) of the perturbation \( S \) is then given by

\[
\Delta W[\mathcal{M}, S] = W[\mathcal{M}'] - W[\mathcal{M}].
\] (11)

\( \Delta W \) can be calculated as an induced amplitude from (10) for the 4-simplices that \( S \) intersects. It is found to be

\[
\Delta W[\mathcal{M}, S] = -i \ln \left\{ \sum_{\delta, \rho = \pm 1} \prod_{S_4 \in S \cap \mathcal{M}} \frac{A[S_4; j + \delta, k + \rho)(\prod C_{\delta \rho})}{A[S_4; j, k]} \right\}
\] (12)

with the products taken over a set of loops in a foliation of \( \mathcal{M} \).

Contributions from 4-simplices not in the surface cancel out. As a result, the action of the perturbation is given by an effective spin system on the surface \( S \) with couplings given by the spins, intertwiners and causal structure of the 4-simplices on \( S \). That is, we may compute the cost of the

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\(^5\)A perturbation of a history based on a surface \( S \) which is not causal cannot be assigned a well defined effective action by the procedure described here.
perturbation by finding the vacuum to vacuum amplitude of the spin system
whose classical action is given by
\[ S^{\text{eff}}[\delta, \rho] = \sum_{S_4 \in S \cap \mathcal{M}} \ln \left\{ \frac{\mathcal{A}[S_4; j + \delta, k + \rho] (\prod C_{\delta \rho})}{\mathcal{A}[S_4; j, k]} \right\} \] (13)

Thus, there is a 2-dimensional field theory associated with the perturbations
of a history. This is the main result of this paper. In the following sections
we discuss its implications.

5 The classical limit and string theory

So far we have not invoked any assumption about the classical limit. The
basic idea we will explore now is that if the full quantum theory of gravity
has a classical limit, its perturbations should also have a classical limit,
which can be studied by analyzing the 2-dimensional system just derived.
It is a nontrivial task to analyze the critical behavior of this 2-dimensional
system, but at the same time it is likely to be a far easier task than the
analysis of the full 4-dimensional theory. As a first step, one can argue to a
useful conclusion as follows.

Let us assume that a particular history \( \mathcal{M} \) and perturbation \( S \) is chosen
such that

1. There is a smooth 4-dimensional spacetime and metric, \((\hat{\mathcal{M}}, \hat{g}_{ab})\) and
an embedding map \( e \) that embeds \( \mathcal{M} \) in \( \hat{\mathcal{M}} \) by assigning to the tetra-
hedra \( \tau \) of \( \mathcal{M} \) events, \( e(\tau) \) in \( \hat{\mathcal{M}} \), such that (a) the causal structure of
the events \( e(\tau) \) agrees with the causal set structure of \( \mathcal{M} \), and (b) for
each spacelike slice \( \Sigma \) in \( \mathcal{M} \) there is a spacelike slice \( \hat{\Sigma} \) in \( \hat{\mathcal{M}} \) such that
the areas of large surfaces and volumes of large regions computed from
either the smooth euclidean metric \( h_{ab} \) on \( \hat{\Sigma} \) or the labelings of faces
tetrahedra in \( \Sigma \) agree up to small errors (where small and large
are defined in Planck units). We then say that the spacetime \((\hat{\mathcal{M}}, \hat{g}_{ab})\)
is the “classical limit” of the discrete causal history \( \mathcal{M} \).

2. \( g_{ab} = \eta_{ab} \), the Minkowski metric.

3. The causal structures and labelings of the 4-simplices of \( \mathcal{M} \) are dis-
tributed randomly with respect to \( g_{ab} \), so that the distribution is in-
vant under a Poincaré transformation.
4. Corresponding to the surface $S \in \mathcal{M}$ there is a timelike cylindrical surface $\tilde{S} \in \tilde{\mathcal{M}}$. $\tilde{S}$ has large area and small extrinsic curvature.

Under these assumptions the action of the perturbation becomes

$$\Delta W[\mathcal{M}, S] = w N_4[S \cap \mathcal{M}]$$

(14)

where $N_4[S \cap \mathcal{M}]$ is the number of 4-simplices $S$ crosses and $\mathcal{M}$ and $w$ is the average value contributed to $\Delta W$ by one 4-simplex, when averaged over the different types and labelings that may appear.

However, by Poincaré invariance, the number $N_4[S \cap \mathcal{M}]$ can only be proportional to the area $A[\tilde{S}, g_{ab}]$ of $\tilde{S}$ computed from the spacetime metric $g_{ab}$, as that is the unique additive Poincaré invariant measure of the surface. Thus,

$$N_4[S \cap \mathcal{M}] = \frac{c}{l_{Pl}^2} A[\tilde{S}, g_{ab}],$$

(15)

which gives $\Delta W$ as

$$\Delta W[\mathcal{M}, S] = \frac{cw}{l_{Pl}^2} A[\tilde{S}, g_{ab}].$$

(16)

Thus we find that the action of the perturbation is proportional to the area of a 2-dimensional timelike surface. It is intriguing that this agrees to leading order with the Nambu action for the bosonic string. We may note also that the string scale $\alpha' = l_{Pl}^2/cw$ is computable in terms of the fundamental theory.

Of course, this is not enough to allow us to conclude that the perturbation theory given by (13) in fact becomes a consistent string theory under the assumptions we have indicated. It may just be the case that the effective action (14) describes an effective, non-critical string theory as in QCD. In fact, this is likely to be true in the case we have just considered. There is no consistent bosonic string theory based on the Nambu action, without additional degrees of freedom, in $3 + 1$ dimensions. We may note that this is consistent with the evidence that quantum general relativity in $3 + 1$ dimensions has no massless particles [20, 21].

6 Critical behavior and string theory

The question is then whether there is some extension of the formalism we considered in the last section in which $SU(2)_q$ is replaced by a general group
or supergroup $G_q$ which has good critical behavior for some choice of amplitudes $A[j,k,c]$. This means that the theory will have a semi-classical limit in which histories $\mathcal{M}$ may be described by a classical spacetime $(\tilde{\mathcal{M}}, \tilde{g})$ with various quantum fields $\hat{\phi}$ living on it. These must include massless particles corresponding to the graviton, as well as possibly chiral fermions and gauge fields. In the case that the classical limit $(\tilde{\mathcal{M}}, \tilde{g})$ of the history $\mathcal{M}$ is Minkowski spacetime, the perturbation theory must also be stable, otherwise we do not have a good theory of quantum gravity.

The same technique used in section 4 can be used for a general $G_q$ to obtain a $1+1$ dimensional discrete field theory. The spin variables $\delta_i$ and $\rho_i$ are replaced by more complicated variables which describe the multiplication of the representations and intertwiners of $G_q$ by one of the elementary representations. The result will be a generalization of (13).

There is a simple argument that this extension of (13) must include the degrees of freedom of a consistent string theory. This depends only on the assumptions that the semi-classical limit of the theory $(G_q, A)$ has massless gravitons and is stable around Minkowski spacetime.

The degrees of freedom in the theory (13) describe the possible gauge invariant perturbations of a history $\mathcal{M}$ that has a good classical limit. Included in the perturbation theory must be the massless graviton and other massless degrees of freedom. This means that the $1+1$ theory defined by the extension of (13) must itself have massless modes which correspond to the graviton and other massless modes. This is because the $1+1$ theory has been defined so that it includes all the weakly coupled long ranged modes described by the full theory. These massless modes propagate along the lightcone of the two surface, which is induced from the causal structure of the history $\mathcal{M}$, which by construction must agree with the lightcone of the spacetime $(\tilde{\mathcal{M}}, \tilde{g})$. It follows that there must be a sector of the theory given by the extension of (13) which is conformally invariant, which means it describes a consistent perturbative string theory.

Moreover, this string theory must be stable when the background $(\tilde{\mathcal{M}}, \tilde{g})$ is Minkowski spacetime as this follows from the assumption that the perturbation theory must be stable.

On the other hand, suppose that, as is likely the case with pure general relativity, (13) or its extensions do not describe a consistent perturbative string theory in $3+1$ dimensions. Then the theory $(G_q, A)$ does not have a perturbation theory which contains massless particles. The same is true for the issue of stability; if the theory defined by the extension of (13) is not stable around Minkowski spacetime then there are small perturbations
of the full non-perturbative theory that destabilize Minkowski spacetime. This means that it fails as a possible quantum theory of gravity. So we see that a necessary condition for a non-perturbative quantum theory of gravity \((G_q, \mathcal{A})\) to have a good continuum limit that reproduces classical general relativity plus quantum field theory is that the 1 + 1 theory described by (13) has a sector that corresponds to a consistent stable perturbative string theory.

We may note that there are many stable, consistent perturbative string theories in 3+1 dimensions. These can be constructed by compactification or by adding degrees of freedom to the worldsheet. It is also known that discrete 1+1 models may give rise to consistent perturbative string theories [38, 2, 3]. It will then be interesting to investigate whether there exists a choice of \((G_q, \mathcal{A})\) such that the extension of (13) leads to one of these consistent 3 + 1 dimensional string theories. If so, this will establish a relationship between non-perturbative quantum gravity and perturbative string theory.

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