GENERALIZED CLARKE EPIDERIVATIVES OF THE EXTREMUM MULTIFUNCTION TO A MULTI-OBJECTIVE PARAMETRIC DISCRETE OPTIMAL CONTROL PROBLEM

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Abstract. This paper deals with the generalized Clarke epiderivative of the extremum multifunction of a multi-objective parametric convex discrete optimal control problem with linear state equations and control constraints. By establishing an abstract result on the generalized epiderivative of the extremum multifunction of a multi-objective parametric convex mathematical programming problem, we derive a formula for computing the generalized Clarke epiderivative of the extremum multifunction to a multi-objective parametric convex discrete optimal control problem. Examples are given to illustrate the obtained results.

1. Introduction. Sensitivity analysis in parametric mathematical programming problems is the study of the behavior of value functions for the single-objective case or extremum multifunctions for the multi-objective case. There have been many papers dealing with the subdifferentials and differentiability properties of value functions in parametric single-objective mathematical programming problems (see [12, 13, 26, 30, 31]). Clarke [13, Theorem 6.52] gave a formula for the generalized gradient of value function by the Lipschitzian conditions and the assumptions that the solution set of perturbed problem is nonempty in a neighborhood of an unperturbed problem. Penot [30] proved that the value functions are Fréchet differentiable under a set of assumptions which involves a kind of coherence property. The results of Penot derived sufficient conditions under which value functions are Fréchet differentiable rather than formulas computing their derivatives.

Recently, multi-objective parametric mathematical programming problems have been studied by several mathematicians. We refer the reader to [11, 22, 23, 33, 34, 35, 37, 38] for papers which have a close connection to the present work. Tanino’s papers [37, 38] are among the first results in this direction. In those papers, via the concept of contingent derivative introduced by Aubin [2], the author has studied the behavior of the extremum multifunctions. Jahn and Rauh [20] have introduced the notion of contingent epiderivative in multi-objective mathematical programming.

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with multifunctions. Shi [33, 34], Kuk, Tanino, and Tanaka [22, 23] have obtained various sensitivity analysis results in this field. Using the concepts of generalized contingent epiderivatives introduced in [3, 9], Song and Wan have given some sensitivity results on parametric multi-objective mathematical programming problems. Namely, the authors have derived a formula for computing the generalized contingent epiderivative of the extremum multifunctions in terms of the derivative of the objective function and the contingent derivative of the constraint mapping.

By using the concept of generalized Clarke epiderivative of a multifunction introduced by Chen [8], Chuong [11] gave the formula for computing the generalized Clarke epiderivative of the extremum multifunction in terms of the Clarke tangent cone to the graph of a multifunction or the constraint mapping and/or the Fréchet derivative of the objective function. Note that in [11], Chuong considered the multi-objective parametric mathematical programming problem with functional constraints. However, in this direction, we did not see formulas for computing the generalized Clarke epiderivative of the extremum multifunction in a multi-objective parametric mathematical programming problem with geometrical and functional constraints.

Besides the study of sensitivity analysis in mathematical programming, the study of sensitivity analysis in optimal control is also of interest to many researchers (see [1, 10, 27, 31, 41, 40, 42]). Recently, Chieu and Yao in [10] have given a formula for an upper evaluation of the Fréchet subdifferential of the value function in a single-objective parametric discrete optimal control problem for the case where solution set admits a locally upper Lipschitzian selection. We in [41] have proved that if the solution map is V -inner semicontinuous, then the upper evaluation of the Fréchet subdifferential in [10] is also the upper evaluation of the Mordukhovich subdifferential of the value function.

Motivated by the work of Chieu and Yao [10] and our recent work [41], we [1] derived formulas for computing the subdifferential and the singular subdifferential of the optimal value function to a single-objective parametric convex discrete optimal control problem with convex cost functions and control constraints. Using virtue of the convexity, we proved that all the upper estimates in the above papers by Chieu and Yao and us become equalities without assumptions, like the nonemptiness of the Fréchet upper subdifferential of the objective function, as well as the existence of a local upper Lipschitzian selection of the solution map and the V -inner semicontinuity of the solution map. However, to the best of our knowledge, we did not see the sensitivity analysis results for smulti-objective parametric discrete optimal control problems.

In this paper, we first establish new result on the generalized epiderivative of the extremum multifunction of a multi-objective parametric convex mathematical programming problem with geometrical and functional constraints. Namely, the formulas for computing the generalized Clarke epiderivative of the extremum multifunction are presented in terms of the graph of the constraint mapping, the Clarke tangent cone to the constraint sets and the Fréchet derivative of the objective function. We then derive a formula for computing the generalized Clarke epiderivative of the extremum multifunction in a multi-objective parametric convex discrete optimal control problem with linear state equations and control constraints via the solution of state equations, the Clarke tangent cone to the constraint sets and the Fréchet derivative of the objective functions.
The paper is organized as follows. Section 2 formulates the control problem and recalls some auxiliary results from [3, 8, 9, 19, 25, 32, 33], and [38]. The generalized epiderivative of the efficient point multifunction to a specific convex mathematical programming problem is studied in Section 3 by invoking tools from functional analysis and finite-dimensional analysis. The last section establishes one theorem on estimating/computing the generalized Clarke epiderivative of the extremum multifunction to the multi-objective parametric convex discrete optimal control problem. Section 4 also presents an example to illustrate the main result of this paper.

2. Problem formulation and auxiliary results. This section is divided into three subsections. The first one presents the multi-objective parametric convex discrete optimal control problem which is of interest to us. The second one transforms the problem to a multi-objective parametric convex mathematical programming problem under geometrical and functional constraints. The last one recalls some notions on generalized differentiation, which are related to our problem.

2.1. Control problem. A wide variety of the problems in multi-objective discrete optimal control can be posed in the following form.

Determine a pair \((x, u)\) of a path

\[ x = (x_0, x_1, \ldots, x_N) \in X_0 \times X_1 \times \cdots \times X_N \]

and a control vector

\[ u = (u_0, u_1, \ldots, u_{N-1}) \in U_0 \times U_1 \times \cdots \times U_{N-1}, \]

which solve

\[ \min_{x,u,w} f(x, u, w), \tag{1} \]

and satisfy the state equation

\[ x_{k+1} = A_k x_k + B_k u_k + T_k w_k, \quad k = 0, 1, \ldots, N-1, \tag{2} \]

the initial, final conditions

\[ x_0 \in C_0, \quad x_N \in C_N, \tag{3} \]

and the constraints

\[ u_k \in \Omega_k \subset U_k, \quad k = 0, 1, \ldots, N-1. \tag{4} \]

The notations in (1)–(4) have the following meanings:

- \(k\) indexes the discrete time, \(N\) is the horizon or number of times control applied,
- \(x_k\) is the state of the system which summarizes past information that is relevant to future optimization, \(u_k\) is the control variable to be selected at time \(k\) with the knowledge of the state \(x_k\),
- \(w = (w_0, w_1, \ldots, w_{N-1}) \in W_0 \times W_1 \times \cdots \times W_{N-1}\) is a random parameter (also called disturbance or noise),
- \(f(x, u, w) = (f^1(x, u, w), f^2(x, u, w), \ldots, f^s(x, u, w))\), where

\[ f^i(x, u, w) = \sum_{k=0}^{N-1} h^i_k(x_k, u_k, w_k) + h^i_N(x_N), \quad i = 1, 2, \ldots, s, \]

- \(h^i_k : X_k \times U_k \times W_k \to \mathbb{R}\) is a convex function on \(X_k \times U_k \times W_k\); \(h^i_N : X_N \to \mathbb{R}\) is a convex function on \(X_N\),
- \(A_k : X_k \to X_{k+1}; B_k : U_k \to X_{k+1}; T_k : W_k \to X_{k+1}\) are linear mappings,
- \(X_k\) is a finite-dimensional space of state variables at stage \(k\), \(U_k\) is a finite-dimensional space of control variables at stage \(k\), \(W_k\) is a finite-dimensional space of random parameters at stage \(k\),
where

\[ \Omega = \Omega \times X = X \times \Omega \]

Reduction to a parametric mathematical programming problem. Put

\[ X = X_0 \times X_1 \times \cdots \times X_N, U = U_0 \times U_1 \times \cdots \times U_{N-1}, \text{ and } W = W_0 \times W_1 \times \cdots \times W_{N-1}. \]

Let \( \Omega = \Omega_0 \times \Omega_1 \times \cdots \times \Omega_{N-1} \) and \( \tilde{X} = X_1 \times X_2 \times \cdots \times X_{N-1} \). Then the problem (1) - (4) can be written as the following form:

\[
\min_{\mathbb{R}_+^*} f(w, x, u), \quad \text{subject to } (x, u) \in G(w) \cap (C_0 \times \tilde{X} \times C_N \times \Omega),
\]

where \( x = (x_0, x_1, \ldots, x_N) \), \( u = (u_0, u_1, \ldots, u_{N-1}) \), \( w = (w_0, w_1, \ldots, w_{N-1}) \),

\[
G(w) = \{(x, u) \in X \times U : x_{k+1} = A_k x_k + B_k u_k + T_k w_k, \ k = 0, 1, \ldots, N - 1 \}. \]

We say that \( y \in A \) is an efficient point of \( A \subset \mathbb{R}^s \) with respect to \( \mathbb{R}_+^s \) and write \( y \in \min_{\mathbb{R}_+^s} A \) if and only if \((y - \mathbb{R}_+^*) \cap A = \{y\}\). When \( A = \emptyset \), we stipulate that \( \min_{\mathbb{R}_+^s} A = \emptyset \).

Let \( F : W \rightrightarrows \mathbb{R}^s \) be a multifunction given by

\[
F(w) = (f \circ G_K)(w) := \{f(w, x, u) : (x, u) \in G_K(w)\}, \quad (5)
\]

where \( K = C_0 \times \tilde{X} \times C_N \times \Omega \) and \( G_K(w) = G(w) \cap K \), for all \( w \in W \). We put

\[
\mathcal{F}(w) = \min_{\mathbb{R}_+^s} F(w), \quad w \in W
\]

and call \( \mathcal{F} : W \rightrightarrows \mathbb{R}^s \) the extremum multifunction of the problem (1) - (4).

2.3. Some facts from variational analysis and generalized differentiation.

In this subsection, we recall some notions and facts from variational analysis and generalized differentiation which will be used in the sequel. These notations and facts can be found in [3, 8, 9, 19, 25, 32, 33], and [38].

Let \( F \) be a multifunction defined in (5). The effective domain, the graph and the epigraph of \( F \) are defined, respectively, by

\[
\text{dom } F := \{w \in W : F(z) \neq \emptyset\},
\]

\[
\text{gph } F := \{(w, y) \in W \times \mathbb{R}^s : y \in F(w)\},
\]

and

\[
\text{epi } F := \{(w, y) \in W \times \mathbb{R}^s : w \in \text{dom } F, \ y \in F(w) + \mathbb{R}_+^s\}.
\]

Definition 2.1. (i) \( F \) is said to be convex if

\[
\alpha F(w) + (1 - \alpha) F(w') \subset F(\alpha w + (1 - \alpha) w'), \ \forall w, w' \in W, \ \forall \alpha \in [0, 1].
\]

(ii) \( F \) is said to be \( \mathbb{R}_+^s \)-convex if

\[
\alpha F(w) + (1 - \alpha) F(w') \subset F(\alpha w + (1 - \alpha) w') + \mathbb{R}_+^s, \ \forall w, w' \in W, \ \forall \alpha \in [0, 1].
\]
Note (see e.g. [38]) that $F$ is convex if and only if \( \text{gph } F \) is a convex set in \( W \times \mathbb{R}^s \). From [19, Lemma 14.8], $F$ is \( \mathbb{R}_+^s \)-convex if and only if $\text{epi } F$ is a convex set in $W \times \mathbb{R}^s$.

Suppose that $D \subset X$, we denote the interior and the closure of $D$ by $\text{int } D$ and $\text{cl } D$, respectively. Given a point $\bar{x} \in \text{cl } D$. The Bouligand tangent cone (or contingent cone) and the Clarke tangent cone to $D$ at $\bar{x}$ are defined by

$$T^B(D; \bar{x}) = \liminf_{t \to 0^+} \frac{D - \bar{x}}{t} = \{ h \in X : \exists t_n \to 0^+, \exists h_n \to h, \bar{x} + t_nh_n \in D, \forall h \} ,$$

$$T^C(D; \bar{x}) = \liminf_{t \to 0^+} \frac{D - \bar{x}}{t} = \{ h \in X : \forall t_n \to 0^+, \forall \bar{x}_n \to \bar{x}, \exists h_n \to h, \bar{x}_n + t_nh_n \in D, \forall h \} .$$

Note that these cones are closed and $T^C(D; \bar{x})$ is convex. Moreover,

$$T^C(D; \bar{x}) \subset T^B(D; \bar{x})$$

and

$$T^C(D; \bar{x}) = T^B(D; \bar{x}) = T(D; \bar{x}) = \text{cl } (D(\bar{x})) = \{ \text{cone}(D - \bar{x}) \}$$

when $D$ is a convex set. It is easy to see that for $\bar{x} = (\bar{x}_1, \bar{x}_2) \in \text{cl } (D_1 \times D_2)$,

$$T^C(D_1 \times D_2; \bar{x}) = T^B(D_1 \times D_2; \bar{x}) = T(D_1 \times D_2; \bar{x}) = T(D_1; \bar{x}_1) \times T(D_2; \bar{x}_2) \quad (6)$$

when $D_1, D_2$ are convex subsets of $X$.

Given a subset $\tilde{D} \subset W \times \mathbb{R}^s$, for $(\bar{w}, \bar{y}) \in \text{cl } \tilde{D}$, we define the corresponding projections of the Bouligand tangent cone and the Clarke tangent cone to sets in the product space $W \times \mathbb{R}^s$ on the space $\mathbb{R}^s$ at $u \in W$ by

$$\Pi_u T^B(\tilde{D}; (\bar{w}, \bar{z})) := \{ y \in \mathbb{R}^s : (u, y) \in T^B(\tilde{D}; (\bar{w}, \bar{y})) \}$$

and

$$\Pi_u T^C(\tilde{D}; (\bar{w}, \bar{z})) := \{ y \in \mathbb{R}^s : (u, y) \in T^C(\tilde{D}; (\bar{w}, \bar{y})) \} .$$

**Definition 2.2.** Let $F : W \rightrightarrows \mathbb{R}^s$ be a multifunction and let $(\bar{w}, \bar{y}) \in \text{gph } F$. A multifunction $D^C F(\bar{w}, \bar{y}) : W \rightrightarrows \mathbb{R}^s$ is said to be the generalized Clarke epiderivative of $F$ at $(\bar{w}, \bar{y})$ if

$$D^C F(\bar{w}, \bar{y})(u) = \text{Min}_{\mathbb{R}^s_+} \Pi_u T^C(\text{epi } F; (\bar{w}, \bar{y})), \quad u \in W. \quad (7)$$

Chen [8] called this notion that generalized tangent epiderivative. It is easy to see that $\Pi_u T^C(\text{epi } F; (\bar{w}, \bar{y}))$ is a closed convex set and

$$\Pi_u T^C(\text{epi } F; (\bar{w}, \bar{y})) = \Pi_u T^C(\text{epi } F; (\bar{w}, \bar{y})) + \mathbb{R}^s_+ . \quad (8)$$

Note that if we use the Bouligand tangent cone to the epigraph of $F$ at $(\bar{w}, \bar{y})$ to replace the Clarke tangent cone to the epigraph of $F$ at $(\bar{w}, \bar{y})$ in (7), then we have the generalized contingent epiderivative of $F$ at $(\bar{w}, \bar{y})$, which was independently defined by Bednarczuk and Song [3], and by Chen and Jahn [9].

The following $\text{TP-cone}$ to graph $F$ at $(\bar{w}, \bar{y})$ was defined by Shi [33]

$$\text{TP}(\text{gph } F; (\bar{w}, \bar{y})) := \{(w, y) \in W \times \mathbb{R}^s : \exists \{t_n\} \subset (0, +\infty), \exists \{w_n\} \subset W, y_n \in F(w_n)$$

such that $w_n \to \bar{w}, t_n(w_n - \bar{w}, y_n - \bar{y}) \to (w, z) \} .$$

We can easily check that $T^B(\text{gph } F; (\bar{w}, \bar{y})) \subset \text{TP}(\text{gph } F; (\bar{w}, \bar{y}))$ and

$$T^B(\text{gph } F; (\bar{w}, \bar{y})) = T^C(\text{gph } F; (\bar{w}, \bar{y})) = \text{TP}(\text{gph } F; (\bar{w}, \bar{y})) = T(\text{gph } F; (\bar{w}, \bar{y}))$$

if gph $F$ is a convex set.
Definition 2.3. $F$ is called directionally compact at $(\bar{w}, \bar{y}) \in \text{gph } F$ if, for every sequence $t_n \to 0^+$ and for every sequence $\{h_n\} \subset W, h_n \to h \in W,$ any sequence $y_n$ with $\bar{y} + t_n y_n \in F(\bar{w} + t_n h_n)$ for each $n,$ contains a convergent subsequence.

Some sufficient conditions for the directionally compactness of $F$ at a given point in its graph can be found in [3].

Definition 2.4. (i) The set $D \subset \mathbb{R}^s$ is said to satisfy the domination property if

$$D \subset \text{Min}_{\mathbb{R}^s_+} D + \mathbb{R}^s_+.$$

(ii) We say that the domination property holds for $F$ around $\bar{w} \in W$ if there exists a neighborhood $V$ of $\bar{w}$ such that

$$F(w) \subset \text{Min}_{\mathbb{R}^s_+} F(w) + \mathbb{R}^s_+, \quad \text{for all } w \in V.$$

3. Generalized Clarke epiderivatives of the extremum multifunction in a multi-objective parametric mathematical programming problem.

This section gives a theorem, which is the main tool for our subsequent studies on the multi-objective parametric convex discrete optimal control problem. Suppose that $X, W$ and $Z$ are finite-dimensional spaces. Assume that $M : Z \to X$ and $T : W \to X$ are continuous linear mappings. Let $f : W \times Z \to \mathbb{R}^s$ be a $\mathbb{R}^s_+$-convex vector function and $\Omega$ be a closed convex subset of $Z.$ For each $w \in W,$ we put

$$H(w) := \{z \in Z : Mz = Tw\}.$$

Consider the problem

$$\text{Min}_{\mathbb{R}^s_+} f(w, z), \quad \text{subject to } z \in H(w) \cap \Omega. \quad (9)$$

Let $\tilde{F} : W \rightrightarrows \mathbb{R}^s$ be a multifunction given by

$$\tilde{F}(w) = (f \circ H_{\Omega})(w) := \{f(w, z) : z \in H_{\Omega}(w)\},$$

where $H_{\Omega}(w) = H(w) \cap \Omega,$ for all $w \in W.$ We put

$$\tilde{F}(w) = \text{Min}_{\mathbb{R}^s_+} \tilde{F}(w), \quad w \in W$$

and call $\tilde{F} : W \rightrightarrows \mathbb{R}^s$ the efficient point multifunction of the problem (9).

Thus, $H_{\Omega} : W \rightrightarrows Z$ be a multifunction with the domain and the graph dom $H_{\Omega} = \{w \in W : H(w) \cap \Omega \neq \emptyset\}, \quad \text{gph } H_{\Omega} = \{(w, y) \in W \times Y : y \in H(w) \cap \Omega\}.$

It is well known that $Q := \text{gph } H$ is a vector space. So,

$$T^C(Q; (\bar{w}, \bar{z})) = T^B(Q; (\bar{w}, \bar{z})) = T(Q; (\bar{w}, \bar{z})) = Q. \quad (10)$$

We can check that $\tilde{F}$ is $\mathbb{R}^s_+$-convex. Indeed, take any $w, w' \in W$ and $\alpha \in [0, 1],$ we need to show that

$$\alpha \tilde{F}(w) + (1 - \alpha) \tilde{F}(w') \subset \tilde{F}(\alpha w + (1 - \alpha) w') + \mathbb{R}^s_+.$$

By the definition of $\tilde{F}$ and the $\mathbb{R}^s_+$-convex property of $f,$ take any $z \in H(w) \cap \Omega,$ $z' \in H(w') \cap \Omega,$ we only need check the condition

$$\alpha z + (1 - \alpha) z' \in H(\alpha w + (1 - \alpha) w') \cap \Omega.$$

On the one hand, from the convex property of $\Omega$ and $z, z' \in \Omega$ we have

$$\alpha z + (1 - \alpha) z' \in (\alpha z + (1 - \alpha) z') + \mathbb{R}^s_+.$$

By the definition of $\tilde{F}$ and the $\mathbb{R}^s_+$-convex property of $f,$ take any $z \in H(w) \cap \Omega,$ $z' \in H(w') \cap \Omega,$ we only need check the condition

$$\alpha z + (1 - \alpha) z' \in H(\alpha w + (1 - \alpha) w') \cap \Omega.$$
On the other hand, from the convex property of $H$

$$\alpha z + (1 - \alpha)z' \in \alpha H(w) + (1 - \alpha)H(w') \subset H(\alpha w + (1 - \alpha)w').$$  \hspace{1cm} (12)$$

By (11) and (12), we obtain

$$\alpha z + (1 - \alpha)z' \in H(\alpha w + (1 - \alpha)w') \cap \Omega.$$  

Recall that for a convex subset $\Omega$ of $Z$,

$$H_\Omega : W \rightrightarrows Z$$

$$w \mapsto H(w) \cap \Omega.$$  

Define $\tilde{H}_\Omega : W \times \mathbb{R}^s \rightrightarrows Z$ as follows

$$\tilde{H}_\Omega(w, y) = \left\{ z \in H_\Omega(w) : y - f(w, z) \in \mathbb{R}^s_+ \right\}.$$  \hspace{1cm} (13)$$

This section is devoted to derive a formula for computing the generalized Clarke epiderivatives of the extremum multifunction $\tilde{F}$. First, we establish a property of the constraint mapping $\tilde{H}_\Omega$, which is needed in the rest of this paper.

**Lemma 3.1.** Assume that $(\bar{w}, \bar{z}) \in \text{gph } H_\Omega = \text{gph } H \cap (W \times \Omega), \bar{y} = f(\bar{w}, \bar{z})$. If

$$\Pi_0 T(\text{gph } H_\Omega; (\bar{w}, \bar{z})) = \left\{ z \in Z : (0, z) \in T(\text{gph } H_\Omega; (\bar{w}, \bar{z})) \right\} = \{0\},$$

then $\tilde{H}_\Omega$ is directionally compact at $((\bar{w}, \bar{y}), \bar{z})$.

**Proof.** We first prove that $H_\Omega$ is directionally compact at $(\bar{w}, \bar{z})$. Take arbitrary sequences $\{t_n\} \subset (0, +\infty), t_n \to 0, \{w_n\} \subset W, w_n \to w \in W, \{z_n\} \subset Z$ with $\bar{z} + t_n z_n \in H_\Omega(\bar{w} + t_n w_n)$ for all $n$. From $Z$ is finite-dimensional, we only need to prove that the sequence $\{z_n\}$ is bounded. On the contrary, assume that $\lim_{n \to \infty} z_n = \infty$. Putting

$$\hat{z}_n = \frac{z_n}{\|z_n\|}, \quad \hat{w}_n = \frac{w_n}{\|z_n\|}, \quad \hat{t}_n = t_n \|z_n\|.$$  

Then $\hat{t}_n \hat{z}_n = t_n z_n, \hat{t}_n \hat{w}_n = t_n w_n, \|\hat{z}_n\| = 1$ for all $n$ and $\hat{w}_n \to 0, \hat{t}_n \hat{w}_n \to 0$. We can suppose that $\lim_{n \to \infty} \hat{z}_n = \bar{z}$ with $\|\bar{z}\| = 1$ by taking a subsequence if necessary. Put

$$\hat{w}_n = \hat{w} + \hat{t}_n \hat{w}_n, \quad \hat{z}_n = \bar{z} + \hat{t}_n \hat{z}_n, \quad \hat{t}_n = \frac{1}{t_n}.$$  

We get $\hat{z}_n \in H_\Omega(\hat{w}_n)$ for all $n$ and $\hat{w}_n \to \hat{w}, \hat{t}_n (\hat{w}_n - \hat{w}, \hat{z}_n - \bar{z}) \to (0, \bar{z})$. This implies that

$$(0, \bar{z}) \in TP(\text{gph } H_\Omega; (\bar{w}, \bar{z})) = T(\text{gph } H_\Omega; (\bar{w}, \bar{z})), $$

counter to the assumption $\Pi_0 T(\text{gph } H_\Omega; (\bar{w}, \bar{z})) = \{0\}$. Thus, $H_\Omega$ is directionally compact at $(\bar{w}, \bar{z})$.

We now take any

$$\{t_n\} \subset (0, +\infty), t_n \to 0, \quad \{(w_n, y_n)\} \subset W \times \mathbb{R}^s, \quad (w_n, y_n) \to (w, y), \quad \{z_n\} \subset Z$$

with $\bar{z} + t_n z_n \in H_\Omega((\bar{w}, \bar{y}) + t_n (w_n, y_n))$. By the definition of $\tilde{H}_\Omega$, we have

$$\bar{z} + t_n z_n \in H_\Omega(\bar{w} + t_n w_n) \text{ and } \bar{y} + t_n y_n - f(\bar{w} + t_n w_n, \bar{z} + t_n z_n) \in \mathbb{R}_+^s.$$  

From the above proof, we obtain that $\{z_n\}$ contains a convergent subsequence. Thus, $\tilde{H}_\Omega$ is directionally compact at $((\bar{w}, \bar{y}), \bar{z})$. The proof of the lemma is complete.  \hfill $\square$
The following theorem gives a formula for computing the generalized Clarke epiderivatives of the extremum multifunction $\bar{F}$, which is the main result of this section.

**Theorem 3.2.** Let $\bar{w} \in W$ and let $\bar{z} \in H(\bar{w}) \cap \Omega$ be such that $\bar{y} = f(\bar{w}, \bar{z}) \in \bar{F}(\bar{w})$. Let the domination property holds for $\bar{F}$ around $\bar{w}$ and let $f$ be Fréchet differentiable at $(\bar{w}, \bar{z})$. Assume further that $\text{gph } H \cap [W \times \text{int } T(\Omega; \bar{z})] \neq \emptyset$ and $\text{ker } M \cap T(\Omega; \bar{z}) = \{0\}$. Then

$$D^C \bar{F}(\bar{w}, \bar{y})(w) = \text{Min}_{\mathbb{R}^+} \left\{ \nabla f(\bar{w}, \bar{z})(w, z) : (w, z) \in \text{gph } H \cap [W \times T(\Omega; \bar{z})] \right\},$$

for all $w \in W$.

**Proof.** By the definition of the generalized Clarke epiderivative of $\bar{F}$ at $(\bar{w}, \bar{y})$, for all $w \in W$

$$D^C \bar{F}(\bar{w}, \bar{y})(w) = \text{Min}_{\mathbb{R}^+} \Pi_w T^C(\text{epi } \bar{F}; (\bar{w}, \bar{y})).$$

From the domination property of $\bar{F}$ around $\bar{w} \in W$ and from $\bar{F}(w) \subset \bar{F}(\bar{w})$ for all $w \in W$, there exists $V \in \mathcal{N}(\bar{w})$ such that

$$\bar{F}(w) + \mathbb{R}^*_+ = F(w) + \mathbb{R}^*_+, \forall w \in V.$$

So,

$$T^C(\text{epi } \bar{F}; (\bar{w}, \bar{y})) = T^C(\text{epi } \bar{F}; (\bar{w}, \bar{y})).$$

Hence, for each $w \in W$,

$$\Pi_w T^C(\text{epi } \bar{F}; (\bar{w}, \bar{y})) = \Pi_w T^C(\text{epi } \bar{F}; (\bar{w}, \bar{y})).$$

Thus,

$$D^C \bar{F}(\bar{w}, \bar{y})(w) = \text{Min}_{\mathbb{R}^+} \Pi_w T^C(\text{epi } \bar{F}; (\bar{w}, \bar{y})).$$

By (8) we need to prove that

$$\Pi_w T^C(\text{epi } \bar{F}; (\bar{w}, \bar{y})) = \left\{ \nabla f(\bar{w}, \bar{z})(w, z) : (w, z) \in \text{gph } H \cap [W \times T(\Omega; \bar{z})] \right\} + \mathbb{R}^*_+.$$ (14)

From (6),

$$T(W \times \Omega; (\bar{w}, \bar{z})) = T(W; \bar{w}) \times T(\Omega; \bar{z}) = W \times T(\Omega; \bar{z}).$$

Since the assumption of theorem,

$$\text{int } T(W \times \Omega; (\bar{w}, \bar{z})) = W \times \text{int } T(\Omega; \bar{z}) \neq \emptyset.$$ (15)

By (10) and [44, Theorem 2.1], we get

$$\text{gph } H \cap [W \times T(\Omega; \bar{z})] = T(\text{gph } H; (\bar{w}, \bar{z})) \cap T(W \times \Omega; (\bar{w}, \bar{z})) = T(\text{gph } H \cap [W \times \Omega]; (\bar{w}, \bar{z})).$$

We first prove that

$$\left\{ \nabla f(\bar{w}, \bar{z})(w, z) : (w, z) \in T(\text{gph } H \cap [W \times \Omega]; (\bar{w}, \bar{z})) \right\} \subset \Pi_w T^B(\text{gph } \bar{F}; (\bar{w}, \bar{y})),$$

for all $w \in W$. For each $w \in W$, let

$$(w, z) \in T(\text{gph } H \cap [W \times \Omega]; (\bar{w}, \bar{z})) = T^B(\text{gph } H \cap [W \times \Omega]; (\bar{w}, \bar{z})).$$

Then, there are sequences $\{t_n\} \subset (0, +\infty), t_n \to 0$ and $\{(w_n, z_n)\} \subset W \times Z$,

$$(w_n, z_n) \to (w, z) \text{ with } \bar{z} + t_n z_n \in H(\bar{w} + t_n w_n) \cap \Omega \text{ for all } n.$$
We get $f(\bar{w} + t_n w_n, \bar{z} + t_n z_n) \in \tilde{F}(\bar{w} + t_n w_n), \forall n$. This means that
\[
f(\bar{w}, \bar{z}) + t_n \frac{f(\bar{w} + t_n w_n, \bar{z} + t_n z_n) - f(\bar{w}, \bar{z})}{t_n} \in \tilde{F}(\bar{w} + t_n w_n), \forall n.
\]
From the Fréchet differentiability of $f$ at $(\bar{w}, \bar{z})$, we get
\[
\lim_{n \to \infty} \frac{f(\bar{w} + t_n w_n, \bar{z} + t_n z_n) - f(\bar{w}, \bar{z})}{t_n} = \nabla f(\bar{w}, \bar{z})(w, z).
\]
So,
\[
y + t_n \nabla f(\bar{w}, \bar{z})(w, z) \in \tilde{F}(\bar{w} + t_n w_n), \forall n.
\]
Hence,
\[
(w, \nabla f(\bar{w}, \bar{z})(w, z)) \in T^B(\text{gph} \tilde{F}; (\bar{w}, \bar{y}))
\]
and (17) has been proved. We now show that
\[
\left\{\nabla f(\bar{w}, \bar{z})(w, z) : (w, z) \in T(\text{gph} H \cap [W \times \Omega]; (\bar{w}, \bar{z}))\right\} + \mathbb{R}^+ \subset \Pi_w T^C(\text{epi} \tilde{F}; (\bar{w}, \bar{y})).
\]
Take any $(w, z) \in T(\text{gph} H \cap [W \times \Omega]; (\bar{w}, \bar{z}))$ and any $k \in \mathbb{R}^+$. Since (17)
\[
(w, \nabla f(\bar{w}, \bar{z})(w, z)) \in T^B(\text{gph} \tilde{F}; (\bar{w}, \bar{y})).
\]
So, there are sequences $\{t_n\} \subset (0, +\infty)$, $t_n \to 0$ and $\{(w_n, y_n)\} \subset W \times \mathbb{R}^s$,
\[
(w_n, y_n) \to (w, \nabla f(\bar{w}, \bar{z})(w, z)) \text{ such that } \bar{y} + t_n y_n \in \tilde{F}(\bar{w} + t_n w_n) \text{ for all } n.
\]
For each $n = 1, 2, \ldots$, let $\bar{y}_n = y_n + k$. Then $\lim_{n \to 0} \bar{y}_n = \nabla f(\bar{w}, \bar{z})(w, z) + k$ and
\[
\bar{y} + t_n \bar{y}_n = \bar{y} + t_n y_n + t_n k \in \tilde{F}(\bar{w} + t_n w_n) + \mathbb{R}^s, \forall n.
\]
So,
\[
(w, \nabla f(\bar{w}, \bar{z})(w, z) + k) \in T^B(\text{epi} \tilde{F}; (\bar{w}, \bar{y})).
\]
Note that epi $\tilde{F}$ is a convex set. Hence
\[
(w, \nabla f(\bar{w}, \bar{z})(w, z) + k) \in T^C(\text{epi} \tilde{F}; (\bar{w}, \bar{y}))
\]
and (18) has been established.

We now prove (15). For each $w \in W$, take any $y \in \Pi_w T^C(\text{epi} \tilde{F}; (\bar{w}, \bar{y}))$, i.e.,
\[
(w, y) \in T^C(\text{epi} \tilde{F}; (\bar{w}, \bar{y})) = T^B(\text{epi} \tilde{F}; (\bar{w}, \bar{y})).
\]
So, there are sequences $\{t_n\} \subset (0, +\infty)$, $t_n \to 0$ and $\{(w_n, y_n)\} \subset W \times \mathbb{R}^s$,
\[
(w_n, y_n) \to (w, y) \text{ with } \bar{y} + t_n y_n \in \tilde{F}(\bar{w} + t_n w_n) + \mathbb{R}^s \text{ for all } n.
\]
Hence, there exists $\{z_n\} \subset H(\bar{w} + t_n w_n) \cap \Omega = H_\Omega(\bar{w} + t_n w_n)$ and $\{k_n\} \subset \mathbb{R}^s$, such that
\[
\bar{y} + t_n y_n = f(\bar{w} + t_n w_n, z_n) + k_n, \forall n.
\]
Hence, \( y = \frac{z_n - \tilde{z}}{t_n} \). We have \( z_n = \tilde{z} + t_n \hat{z}_n \in \tilde{H}_\Omega(\bar{w} + t_n w_n, \bar{y} + t_n y_n), \ \forall n \). From the assumption of theorem, we get
\[
\left\{ z \in Z : Mz = 0, \ z \in T(\Omega; \bar{z}) \right\} = \{0\}
\]
\[
\Leftrightarrow \left\{ z \in Z : (0, z) \in gph \ H \cap [W \times T(\Omega; \bar{z})] \right\} = \{0\}
\]
\[
\Leftrightarrow \Pi_0 \left[ gph \ H \cap [W \times T(\Omega; \bar{z})] \right] = \{0\}
\]
\[
\Leftrightarrow \Pi_0 \left[ T(gph \ H \cap [W \times \Omega]; (\bar{w}, \bar{z})) \right] = \{0\}
\]
\[
\Leftrightarrow \Pi_0 \left[ T(gph \ H_\Omega; (\bar{w}, \bar{z})) \right] = \{0\}.
\]
By Lemma 3.1, \( \tilde{H}_\Omega \) is directionally compact at \((\bar{w}, \bar{y}, \bar{z})\), without loss of generality we may suppose that \( \hat{z}_n \) converges to some \( \bar{z} \in Z \). Thus, we have proved that there exist sequences \( \{t_n\} \subset (0, +\infty), t_n \to 0 \) and \( \{(w_n, \hat{z}_n)\} \subset W \times Z, (w_n, \hat{z}_n) \to (w, \bar{z}) \) with \( \bar{z} + t_n \hat{z}_n \in H_\Omega(\bar{w} + t_n w_n) \) for all \( n \). This is equivalent to
\[
(w, \hat{z}) \in T^B(gph \ H_\Omega; (\bar{w}, \bar{z})) = T(gph \ H_\Omega; (\bar{w}, \bar{z})).
\]
By (16),
\[
(w, \hat{z}) \in gph \ H \cap [W \times T(\Omega; \bar{z})].
\]
Since (19), we have
\[
y_n - \frac{f(\bar{w} + t_n w_n, \bar{z} + t_n \hat{z}_n) - f(\bar{w}, \bar{z})}{t_n} = \frac{k_n}{t_n} \in \mathbb{R}^*_+.
\]
Hence, \( y - \nabla f(\bar{w}, \bar{z})(w, \hat{z}) \in \mathbb{R}^*_+ \). Thus, \( y \in \nabla f(\bar{w}, \bar{z})(w, \hat{z}) + \mathbb{R}^*_+ \) and (15) has been established. The proof of the theorem is complete. \( \square \)

We close this section with the following example to illustrate for Theorem 3.2.

**Example 3.3.** Let \( W = \mathbb{R}^2, Z = \mathbb{R}^3, \Omega = [1, +\infty) \times [0, +\infty) \times \mathbb{R}, s = 2, \)
\[
f(w, z) = f(w_1, w_2, z_1, z_2, z_3) = \left( w_1 + z_2, w_2 + z_1 + 2z_2 \right)
\]
and
\[
H(w) = H(w_1, w_2) = \{z = (z_1, z_2, z_3) \in \mathbb{R}^3 : z_1 + z_2 = w_1, z_3 = w_2\}.
\]
Assume that \( \bar{w} = (1, 0) \). Then one has \( \bar{z} = (1, 0, 0), \ y = f(\bar{w}, \bar{z}) = (1, 1) \) and
\[
D^C \bar{F}(\bar{w}, \bar{y}) = \left\{ (w_1, w_1 + w_2) \right\}, \text{ for } w = (w_1, w_2) \in \mathbb{R}^2.
\]
Indeed, we have
\[
\bar{F}(w) = \left\{ f(w, z) : z \in H(w) \cap \Omega \right\}
\]
\[
= \left\{ (w_1 + z_2, w_2 + z_1 + 2z_2) : z_1 + z_2 = w_1, z_3 = w_2, z_1 \geq 1, z_2 \geq 0 \right\}
\]
\[
= \left\{ (w_1 + z_2, w_1 + w_2 + 2z_2) : z_3 = w_2, z_1 \geq 1, z_2 \geq 0 \right\}.
\]
So,
\[
\bar{F}(w) = \left\{ (y_1, y_2) \in \mathbb{R}^2 : y_1 \geq w_1, y_2 \geq w_1 + w_2 \right\},
\]
for all \( w = (w_1, w_2) \in \mathbb{R}^2 \). Therefore, we observe that the domination property holds for \( \bar{F} \), \( \forall w \in \mathbb{R}^2 \). Moreover, for \( \bar{w} = (1, 0) \)
\[
H(\bar{w}) = \left\{ z = (1, z_2, z_3) \in \mathbb{R}^3 : z_1 + z_2 = 1, z_3 = 0 \right\},
\]
and \( \bar{z} = (1, 0, 0) \in H(\bar{w}) \cap \Omega \) as well as \( \bar{y} = f(\bar{w}, \bar{z}) = (1, 1) \in \mathcal{F}(\bar{w}) \). We have \( T(\Omega; \bar{z}) = [0, +\infty) \times [0, +\infty) \times \mathbb{R} \). So, it is easy to see that \( f \) is \( \mathbb{R}_+^n \)-convex and \( \text{gph} \ H \cap [W \times \text{int} T(\Omega; \bar{z})] \neq \emptyset \). Let \( M : \mathbb{R}^3 \to \mathbb{R}^2 \) and \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) be the mappings defined, respectively, by \( Mz = (z_1 + z_2, z_3) \) and \( Tw = (w_1, w_2) \). So,

\[
\ker M = \{ z = (z_1, z_2, z_3) \in \mathbb{R}^3 : Mz = 0 \} = \{ z = (z_1, z_2, 0) \in \mathbb{R}^3 : z_1 + z_2 = 0 \}.
\]

Hence,

\[
\ker M \cap T(\Omega; \bar{z}) = \{ 0 \}.
\]

Thus, all conditions of Theorem 3.2 are satisfied. By (14)

\[
D^C \mathcal{F}(\bar{w}, \bar{y})(w) = \min_{\mathbb{R}_+^2} \left\{ \nabla f(\bar{w}, \bar{z})(w, z) : (w, z) \in \text{gph} H \cap [W \times T(\Omega; \bar{z})] \right\}
\]

\[
= \{ (w_1, w_1 + w_2) \}, \quad w = (w_1, w_2) \in W.
\]

4. Generalized Clarke epiderivatives of the extremum multifunction in a multi-objective parametric discrete optimal control problem. Based on Theorem 3.2, we can obtain formulas for computing generalized Clarke epiderivatives of the extremum multifunction \( \mathcal{F} \) in the multi-objective parametric convex discrete optimal control problem (1)–(4). In the notation of Subsections 2.1 and 2.2, put \( Z = X \times U \). Then, the problem (1)–(4) can be written as the following form:

\[
\min_{\mathbb{R}_+^n} f(w, z), \quad \text{subject to} \quad z \in G(w) \cap K, \quad (20)
\]

where \( G(w) = \{ z = (x, u) \in Z : Mz = Tw \} \), \( M : Z \to \bar{X} \times X_N \) and \( T : W \to \bar{X} \times X_N \) are defined, respectively, by

\[
Mz = \begin{bmatrix}
-A_0 & I & 0 & 0 & \cdots & 0 & 0 & -B_0 & 0 & 0 & \cdots & 0 \\
0 & -A_1 & I & 0 & \cdots & 0 & 0 & -B_1 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & -A_{N-1} & I & 0 & 0 & 0 & \cdots & -B_{N-1}
\end{bmatrix}z
\]

and

\[
Tw = [T_0 w_0, T_1 w_1, \ldots, T_{N-1} w_{N-1}]^T, \quad \text{with} \quad z = [x_0, x_1, \ldots, x_N, u_0, u_1, \ldots, u_{N-1}]^T.
\]

Let \( \bar{w} \in W \) and \( \bar{z} = (\bar{x}, \bar{u}) \in G(\bar{w}) \cap K \), symbols \( h^k_1(x_k, \bar{u}_k), \frac{\partial h^k_1}{\partial u_k}(\bar{x}_k, \bar{u}_k) \), etc., stand, respectively, for \( h^k_1(\bar{x}_k, \bar{u}_k), \frac{\partial h^k_1}{\partial u_k}(\bar{x}_k, \bar{u}_k) \), etc. We assume further that the following assumptions hold:

(A1) There exist \( \bar{u}_k \in \text{int} T(\Omega_k, \bar{u}_k) (k = 0, 1, \ldots, N - 1) \), \( \bar{x}_0 \in \text{int} T(C, \bar{x}_0) \), \( \bar{x}_N \in \text{int} T(C_N, \bar{x}_N) \) and \( \bar{x}_l \in X_l \), \( \bar{u}_k \in W_k (l = 1, 2, \ldots, N - 1, k = 0, 1, \ldots, N - 1) \) such that

\[
(\bar{w}, \bar{z}) = (\bar{w}_0, \bar{w}_1, \ldots, \bar{w}_{N-1}, \bar{x}_0, \bar{x}_1, \ldots, \bar{x}_N, \bar{u}_0, \bar{u}_1, \ldots, \bar{u}_{N-1})
\]

satisfy the state equation (2).

(A2) There exist

\[
\dot{x}_0 \in T(C_0, \bar{x}_0), \quad \dot{x}_N \in T(C_N, \bar{x}_N), \quad \dot{u}_k \in T(\Omega_k, \bar{u}_k) (k = 0, 1, \ldots, N - 1)
\]
and \( \hat{x}_{k+1} = A_k \hat{x}_k + B_k \hat{u}_k, \ k = 0, 1, \ldots, N - 1 \)\\
\[ \implies \begin{pmatrix} \ddot{x} = (\dot{x}, \ddot{u}) = (\dot{x}_0, \ddot{x}_1, \ldots, \ddot{x}_N, \ddot{u}_0, \ddot{u}_1, \ldots, \ddot{u}_{N-1}) = 0_z \end{pmatrix}. \]

We are now ready to state our main result.

**Theorem 4.1.** Suppose that \( \bar{w} = (\bar{w}_0, \bar{w}_1, \ldots, \bar{w}_{N-1}) \in W; \)
\( \bar{z} = (\bar{x}, \bar{u}) = (\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_N, \bar{u}_0, \bar{u}_1, \ldots, \bar{u}_{N-1}) \in G_K(\bar{w}) = G(\bar{w}) \cap K, \)
\( \bar{y} = (\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_s) = f(\bar{w}, \bar{z}) \in \mathcal{F}(\bar{w}), \) and the domination property holds for \( F \) around \( \bar{w}. \) Let \( h^i_N \) be Fréchet differentiable at \( x_N \) and \( h^i_k \) be Fréchet differentiable at \( (\bar{x}_k, \bar{u}_k, \bar{w}_k) \) for \( i = 1, 2, \ldots, s \) and \( k = 0, 1, \ldots, N-1. \) Assume that assumptions (A1) and (A2) are fulfilled. Then for any \( w \in W, \) the generalized Clarke epiderivative of the extremum multifunction \( \mathcal{F}(\bar{w}, \bar{y}) \) at \( w \) is defined by
\[
\begin{align*}
D^c \mathcal{F}(\bar{w}, \bar{y})(w) &= \text{Min}_{R^1} \left\{ \left( \sum_{k=0}^{N-1} \frac{\partial h^i_k}{\partial w_k} w_k + \sum_{k=0}^{N-1} \frac{\partial h^i_k}{\partial x_k} x_k + \sum_{k=0}^{N-1} \frac{\partial h^i_k}{\partial u_k} u_k + \sum_{k=0}^{N-1} \frac{\partial h^i_k}{\partial w_k} w_k \right) \right. \\
&\left. + \sum_{k=0}^{N-1} \frac{\partial h^i_k}{\partial x_k} x_k + \sum_{k=0}^{N-1} \frac{\partial h^i_k}{\partial u_k} u_k \right. \\
&\left. + \sum_{k=0}^{N-1} \frac{\partial h^i_k}{\partial w_k} w_k \right) : x_0 \in T(C_0; \bar{x}_0), \ x_N \in T(C_N; \bar{x}_N), \ u_k \in T(\Omega_k; \bar{u}_k), \\
&\left. w_k \in W_k, \ x_l \in X_l \ (k = 0, 1, \ldots, N-1, l = 1, 2, \ldots, N-1) \right\}. \ (21)
\end{align*}
\]

**Proof.** Note that the single-value function \( h^i_k : X_k \times U_k \times W_k \to R \) is convex if and only if the set-value function
\[
h^i_k : X_k \times U_k \times W_k \ni (x_k, u_k, w_k) \mapsto \{ h^i_k(x_k, u_k, w_k) \}
\]
is \( R^1 \)-convex. So, since \( h^i_N \) and \( h^i_k \) are convex for \( i = 1, 2, \ldots, s \) and \( k = 0, 1, \ldots, N-1, \) we have that \( f \) is \( R^1 \)-convex. By the assumption (A1), we get
\[
gph G \cap [W \times \text{int } T(K; \bar{z})] \neq \emptyset.
\]
From the assumption (A2), it is easy to check that
\[
\ker M \cap T(K; \bar{z}) = \{0\}.
\]
Thus, all conditions of Theorem 3.2 are satisfied. For any \( w \in W, \) by (14),
\[
D^c \mathcal{F}(\bar{w}, \bar{y})(w) = \text{Min}_{R^1} \left\{ \nabla f(\bar{w}, \bar{z})(w, z) : (w, z) \in gph \ G \cap [W \times T(K; \bar{z})] \right\}. \ (22)
\]
Since
\[
f(x, u, w) = \left( f^1(x, u, w), f^2(x, u, w), \ldots, f^s(x, u, w) \right),
\]
where
\[
f^i(x, u, w) = \sum_{k=0}^{N-1} h^i_k(x_k, u_k, w_k) + h^i_N(x_N), \ i = 1, 2, \ldots, s,
\]
we have
\[
\nabla_w f^i(\bar{x}, \bar{u}, \bar{w}) = \sum_{k=0}^{N-1} \nabla_w h^i_k(\bar{x}_k, \bar{u}_k, \bar{w}_k) = \left( \frac{\partial h^i_k}{\partial w_0}, \frac{\partial h^i_k}{\partial w_1}, \ldots, \frac{\partial h^i_N}{\partial w_{N-1}} \right).
\]
Example 4.2.

Also, we get

\[ \nabla_z f^i(\bar{x}, \bar{u}, \bar{w}) = \sum_{k=0}^{N} \left( \nabla_x h_k^i(\bar{x}_k, \bar{u}_k, \bar{w}_k), \nabla_u h_k^i(\bar{x}_k, \bar{u}_k, \bar{w}_k) \right) \]

\[ = \left( \frac{\partial h_k^i}{\partial x_0}, \frac{\partial h_k^i}{\partial x_1}, \ldots, \frac{\partial h_k^i}{\partial x_{N-1}}, \frac{\partial h_k^i}{\partial u_0}, \frac{\partial h_k^i}{\partial u_1}, \ldots, \frac{\partial h_k^i}{\partial u_{N-1}} \right). \]

So, for each

\[ w = (w_0, w_1, \ldots, w_{N-1}) \in W, \quad z = (x, u) = (x_0, x_1, \ldots, x_N, u_0, u_1, \ldots, u_{N-1}) \in X \times U, \]

we have

\[ \nabla f(\bar{w}, \bar{z})(w, z) = \left( \sum_{k=0}^{N-1} \frac{\partial h_k^i}{\partial w_k} w_k + \sum_{k=0}^{N-1} \frac{\partial h_k^i}{\partial x_k} x_k + \sum_{k=0}^{N-1} \frac{\partial h_k^i}{\partial u_k} u_k, \sum_{k=0}^{N-1} \frac{\partial h_k^i}{\partial w_k} w_k + \sum_{k=0}^{N-1} \frac{\partial h_k^i}{\partial x_k} x_k + \sum_{k=0}^{N-1} \frac{\partial h_k^i}{\partial u_k} u_k \right). \tag{23} \]

Besides, for each \((w, z) = (w_0, w_1, \ldots, w_{N-1}, x_0, x_1, \ldots, x_N, u_0, u_1, \ldots, u_{N-1})\), we have \((w, z) \in \text{gph \, G}\) if and only if \((w, z)\) satisfies the state equation (2). Moreover, if

\[ (w, z) = (w_0, w_1, \ldots, w_{N-1}, x_0, x_1, \ldots, x_N, u_0, u_1, \ldots, u_{N-1}) \in W \times T(K; \bar{z}), \]

then

\[ x_0 \in T(C_0; \bar{x}_0), \quad x_N \in T(C_N; \bar{x}_N), \quad u_k \in T(\Omega_k; \bar{u}_k), \quad w_k \in W_k, \quad x_l \in X_l \]

\((k = 0, 1, \ldots, N-1, \ l = 1, 2, \ldots, N-1)\). Hence, since (22), (23), we obtain (21). The proof of the theorem is complete.

To illustrate Theorem 4.1, we provide the following example.

Example 4.2. Let \(N = 2\), \(X_0 = X_1 = X_2 = \mathbb{R}, \quad U_0 = U_1 = \mathbb{R}, \quad W_0 = W_1 = \mathbb{R}\). We consider the problem of finding \(u = (u_0, u_1) \in \mathbb{R}^2\) and \(x = (x_0, x_1, x_2) \in \mathbb{R}^3\) such that

\[ \text{Min}_{\mathbb{R}^2} f(x, u, w), \]

\[ x_1 = x_0 + u_0 - w_0, \quad x_2 = x_1 + u_1, \quad x_0 \geq 0, \quad x_2 \leq 0, \quad u_0 \geq 0, \quad u_1 \geq 0, \tag{24} \]

where \(f(x, u, w) = (f^1(x, u, w), f^2(x, u, w))\), and

\[ f^1(x, u, w) = (x_0 + u_0 - w_0)^2 + (x_1 + u_1)^2 + w_1 + x_2, \]

\[ f^2(x, u, w) = (x_0 + u_0 - w_0)^2 + (x_1 + u_1)^2 + w_1 + x_2. \]

It is easy to see that \((\bar{x}, \bar{u})\) is a Pareto solution of the problem at \(w = (w_0, w_1)\) if and only if

\[ \bar{x} = (\alpha, 0, 0); \quad \bar{u} = (w_0 - \alpha, 0) \] with \(w_0 \geq \alpha \geq 0\).

Then the following condition is fulfilled:

• For \(\bar{w} = (0, 1), \ \bar{x} = (0, 0, 0), \ \bar{u} = (0, 0)\) and \(\bar{y} = f(\bar{w}, \bar{x}, \bar{u}) = (1, 1)\), we have

\[ D^C \mathcal{F}(\bar{w}, \bar{y})(w) = \{(w_1, w_1)\}, \quad w = (w_0, w_1) \in W = \mathbb{R}^2. \]

Indeed, when \(\bar{w} = (0, 1)\), the problem becomes

\[ \text{Min}_{\mathbb{R}^2} f(x, u, \bar{w}) \]

such that \(x_{k+1} = x_k + u_k \) \((k = 0, 1)\), \(x_0 \geq 0, \quad u_0 \geq 0, \quad u_1 \geq 0, \quad x_2 \leq 0, \quad x_3 \leq 0, \)
where \( f(x, u, \bar{w}) = (f^1(x, u, \bar{w}), f^2(x, u, \bar{w})) \),
\[
f^1(x, u, \bar{w}) = (x_0 + u_0)^2 + (x_1 + u_1)^2 + 1 + x_2^2
\]
and
\[
f^2(x, u, \bar{w}) = (x_0 + u_0)^2 + (x_1 + u_1)^2 + 1 + x_2^4.
\]
We have \( \bar{x} = (0, 0, 0); \bar{y} = (0, 0) \) is the unique Pareto solution of the problem at \( \bar{w} \)
and \( \bar{y} = (\bar{y}_1, \bar{y}_2) = f(\bar{x}, \bar{u}, \bar{w}) = (1, 1) \in F(\bar{w}) \). It is easy to see that
\[
F(w) \subset \text{Min}_{\mathbb{R}^2_+} F(w) + \mathbb{R}^2_+ = [w_1, +\infty) \times [w_1, +\infty), \ w = (w_0, w_1) \in W.
\]
So, the domination property holds for \( F \) around \( \bar{w} \).

For the problem \((P_w)\), one has
\[
C_0 = [0, +\infty), \ C_2 = (-\infty, 0], \ \Omega_0 = [0, +\infty), \ \Omega_1 = [0, +\infty).
\]

\( M : \mathbb{R}^5 \to \mathbb{R}^2 \) and \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) are defined, respectively, by
\[
Mz = [x_1 - x_0 - u_0, x_2 - x_1 - u_1]^T \ \text{and} \ T w = [-u_0, 0]^T,
\]
where \( z = [x_0, x_1, x_2, u_0, u_1]^T \), \( w = [w_0, w_1]^T \). We note that
\[
T (C_0; \bar{x}_0) = [0, +\infty), \ T (C_2; \bar{x}_2) = (\infty, 0]
\]
and
\[
T (\Omega_0; \bar{u}_0) = [0, +\infty), \ T (\Omega_1; \bar{u}_1) = (\infty, 0).
\]

There exist
\[
\tilde{x}_0 = 1 \in \text{int} T (C_0; \bar{x}_0), \ \tilde{u}_0 = 1 \in \text{int} T (\Omega_0; \bar{u}_0), \ \tilde{u}_1 = \frac{1}{2} \in \text{int} T (\Omega_1; \bar{u}_1),
\]
\[
\tilde{x}_2 = -\frac{1}{2} \in \text{int} T (C_2; \bar{x}_0), \ \tilde{w}_0 = 3 \in \mathbb{R}, \ \tilde{w}_1 = 3 \in \mathbb{R}, \ \text{and} \ \tilde{x}_1 = \tilde{x}_0 + \tilde{w}_0 - \bar{w}_0 = -1
\]
such that \( (\tilde{w}, \tilde{z}) = \left(3, 3, 1, -1, -\frac{1}{2}, 1, \frac{1}{2}\right) \) satisfies the state equation (24). Hence, the assumption \((A_1)\) is fulfilled. Moreover, take any
\[
\tilde{x}_0 \in T (C_0; \bar{x}_0) = [0, +\infty), \ \tilde{x}_2 \in T (C_2; \bar{x}_0) = (-\infty, 0], \ \tilde{u}_0 \in T (\Omega_0; \bar{u}_0) = [0, +\infty),
\]
\[
\tilde{u}_1 \in T (\Omega_1; \bar{u}_1) = [0, +\infty) \ \text{and} \ \tilde{x}_1 = \tilde{x}_0 + \tilde{u}_0 \geq 0.
\]

On the one hand, \( \tilde{x}_2 = \tilde{x}_1 + \tilde{u}_1 \geq 0 \). On the other hand, \( \tilde{x}_2 \in T (C_2; \bar{x}_0) = (-\infty, 0) \)
Therefore, \( \tilde{x}_2 = 0 \). This implies that
\[
\tilde{x}_1 = 0, \ \tilde{u}_1 = 0, \ \tilde{x}_0 = 0, \ \tilde{u}_0 = 0.
\]

Thus, the assumption \((A_2)\) is also fulfilled. By Theorem 4.1, we obtain the assertion of example.

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