Interfering resonances in a quantum billiard

E. Persson†, K. Pichugin*, I. Rotter† and P. Šeba‡

†Max-Planck-Institut für Physik komplexer Systeme, D-01187 Dresden, Germany, and Technische Universität Dresden, Institut für Theoretische Physik, D-01062 Dresden, Germany
*Institute of Physics, Academy of Sciences 660036 Krasnoyarsk, Russia
†Nuclear Physics Institute, Czech Academy of Sciences, 250 68 Rež near Prague, Czech Republic
‡seba@kostelec.czech.cz

PACS: 03.65.-w; 03.65.Nk; 03.80.+r; 05.30.-d

We present a method for numerically obtaining the positions, widths and wavefunctions of resonance states in a two dimensional billiard connected to a waveguide. For a rectangular billiard, we study the dynamics of three resonance poles lying separated from the other ones. As a function of increasing coupling strength between the waveguide and the billiard two of the states become trapped while the width of the third one continues to increase for all coupling strengths. As a function of a parameter controlling the coupling to the continuum, the widths of all states increase as long as the states are isolated. At a certain critical value of the parameter where the resonance states start to overlap each other, the widths bifurcate: the width of one of the states increases further while the width of the other ones decrease, see e.g. [5]. In other words: one of the states aligns with the channel and becomes short lived while the other ones decouple from the channel and become long-lived in spite of the strong coupling to the continuum.

The phenomenon of resonance trapping is theoretically well established but not proven directly up to now in experimental studies. In this letter we investigate the behavior of three neighboring resonances in a two-dimensional billiard connected to a single waveguide. As a function of the coupling between the resonator and the waveguide, we calculate both the position of the corresponding resonance poles and the Wigner-Smith time delay function. The time delay function, containing unique information on the interference between the resonance states, can be studied experimentally.

The model used is as follows. We consider a two dimensional billiard coupled to a waveguide (for instance a flat electromagnetic resonator). We have to solve the equation

\[ (-\nabla^2 + \lambda V) \Psi = E \Psi \]  

where \( V \) is a potential barrier between the billiard and the attached lead. We use the Dirichlet boundary condition, \( \Psi = 0 \), on the border of the billiard and of the waveguide. The waveguide has a width \( W \) and the wavefunction inside it has the asymptotic form

\[ \Psi = (e^{ikx} - R(E)e^{-ikx}) u(y) \]  

where \( u(y) \) is the transversal mode in the waveguide, \( k \) is the wave number and \( R(E) \) is the reflection coefficient. It holds \( E = k^2 + (\pi/W)^2 \). We choose the energy of the incoming wave so that only the first transversal mode in the lead is open, i.e.

\[ u(y) = \sin(\frac{\pi}{W} y) \]

Since \( |R(E)| = 1 \), we write \( R(E) = \exp(i\Theta(E)) \) with \( \Theta(E) \) real. A measurable quantity derived from \( R \) is the Wigner-Smith time delay function,

\[ \tau_w = \frac{d\Theta}{dE} \]

\( \tau_w \) is the time the wave spends inside the billiard [3].

The energies and widths of the resonance states are given by the poles of the function \( R(E) \) analytically continued into the lower complex plane. To find the poles we use the exterior complex scaling method [3]. The general idea is to study the system after a scaling transformation applied to the \( x \) coordinate, see [3]: \( x \rightarrow \tilde{x} = g(x) \). The function \( g \) is chosen as

\[ x \rightarrow \tilde{x} = g(x) \]

\[ g(x) = \begin{cases} x, & x \leq x_0 \\ x_0 + x - x_0 - x, & x > x_0 \end{cases} \]

where \( x_0 \) is the radius of the billiard. The exterior complex scaling method is based on the idea that the resonances are localized near the boundary and that the function \( g \) maps the region outside the billiard to the interior of the billiard. The method is equivalent to using the exterior complex scaling method for the wavefunction, i.e.,

\[ \Psi(x) = \begin{cases} \tilde{\Psi}(\tilde{x}), & \tilde{x} \leq x_0 \\ \tilde{\Psi}(\tilde{x}) - \tilde{\Psi}(x_0 + x - x_0 - x), & \tilde{x} > x_0 \end{cases} \]

where \( \tilde{\Psi}(\tilde{x}) \) is the wavefunction in the exterior complex scaled region.

In present-day high-resolution experimental studies, the properties of individual resonance states can be investigated even when the level density is high. As an example, nuclear states have recently been identified and studied experimentally at very high excitation energy [1]. Collisional damping at such an energy is the same as at low excitation energy (ground-state domain). This experimental result, being in contradiction to the standard statistical theory of nuclear reactions, can be justified by taking into account the interferences between resonance states arising from their interaction via the continuum [3]. In atoms, the coherent coupling of autoionizing states is studied experimentally e.g. in [5]. The authors point to the numerous possible applications of such investigations. Thus, the interferences between individual resonance states play an important rôle and need to be considered in detail.

Theoretically, such interferences have been studied in different fields of physics, see e.g. [1,4]. One important result is the phenomenon of resonance trapping, which arises from the interference of resonance states coupled to the same decay channel. As a function of a parameter controlling the coupling to the continuum, the widths of all states increase as long as the states are isolated.
\[
g(x) = \begin{cases} x & x \geq x_0 \\ \theta f(x) & x < x_0 \end{cases} \quad (5)
\]

with \(f(x)\) such that \(g(x)\) is three times smoothly differentiable and the inverse transformation \(g^{-1}(\tilde{x})\) exists. The attached waveguide extends from \(-\infty\) parallel to the x-axis and we choose \(x_0\) to be localized inside it. The related transformation of the wavefunction reads

\[
\Psi(x, y) \rightarrow \frac{1}{\sqrt{g'(x)}} \tilde{\Psi}(\tilde{x}, y). \quad (6)
\]

Using it the equation (1) becomes

\[
\begin{align*}
&\left(-\frac{\partial}{\partial \tilde{x}} \left( \frac{1}{g'^2} \frac{\partial}{\partial \tilde{x}} \right) - \frac{\partial^2}{\partial y^2}\right) \tilde{\Psi}(\tilde{x}, y) + \\
&\left(\lambda V(\tilde{x}, y) + \frac{2g'g'' - 5g''^2}{4g'^4}\right) \tilde{\Psi}(\tilde{x}, y) = E\tilde{\Psi}(\tilde{x}, y) \quad (7)
\end{align*}
\]

For a real parameter \(\theta\), this equation is fully equivalent to (1) since the transformation \(\tilde{\Psi}\) is unitary. Moreover, the two equations are fully identical for \(x > x_0\). Since \(x_0\) lies inside waveguide the shape of the resonator is not changed by the transformation \(\tilde{\Psi}\) which only rescales a part of the x-axis related to the waveguide. Moreover, since the waveguide is oriented parallel to the x-axis, the transformation does not change the boundary of the system. For \(\theta\) complex, \(\tilde{\Psi}\) ceases to be unitary and the spectral properties of (1) and (3) are different. The continuous spectrum of (1) extends over \((\pi/W)^2, \infty)\), whereas the continuous spectrum of (3) is rotated into the complex plane and is equal to

\[
\bigcup_{n=1: \infty} \left\{(n\pi/W)^2 + \theta^{-2}(0, \infty)\right\} \quad (8)
\]

This is a union of half-lines representing the continuous spectrum starting out from the real axis at every threshold energy \((n\pi/W)^2\) with an angle \(-2 \arg \theta\). The rotated continuous spectrum uncovers additive complex eigenvalues of (3), the positions of which are independent of \(\theta\). These eigenvalues coincide with the resonance poles of (3).

In the following we study the time delay \(\tau_w\) and the resonance poles of a rectangular billiard of size \(\Delta x \times \Delta y = 2 \times 3.14\) connected to a single waveguide with width \(W = 0.6\). We choose \(V\) as a rectangular potential barrier with height 1 located at \(-0.3 \leq x \leq 0\). By changing the parameter \(\lambda\) we can tune the coupling between the waveguide and the resonator. We calculate \(\tau_w\) by solving Eq. (1) with the Dirichlet boundary condition \(\Psi = 0\), and the asymptotic boundary condition (3) imposed at \(x = -13\). The resonance poles are found by the method of exterior complex scaling described above using \(x_0 = -2\).

In Fig. 1 we show the calculated time delay \(\tau_w\) as a function of \(\lambda\) and energy (panel a) as well as the dependence of the resonance poles on \(\lambda\) (panel b). At large \(\lambda\) (weak coupling to the waveguide) we see three isolated resonance states. As \(\lambda\) decreases (the coupling to the waveguide increases) the lifetimes of all three states decrease. The states attract each other in energy. As the resonances start to overlap, two of them become trapped while the third one gets short lived. At further decreased \(\lambda\) the lifetimes of the two trapped resonance states increase. The motion of the poles is reflected in the time delay function. The lifetime of the short lived state is, at small \(\lambda\), so short that it practically disappears when plotting the time delay. The numerical errors in the distances among the resonances are so small that the details of energy attraction and resonance trapping are stable.

Using the method of complex scaling we can also study the wavefunctions of the resonance states (Gamow states). The interference between the resonances states leads to a mixing of their wave functions with respect to the eigenfunctions of the resonator \((\lambda \rightarrow \infty)\). We illustrate this phenomenon in Fig. 2, where the wavefunctions of the three resonance states are shown for \(\lambda = 44, 23.5\) and 0, i.e. under the condition when (a) they are isolated, (b) very near to one another and (c) two of them are trapped while one is short lived.

For \(\lambda \rightarrow 0\), the amplitude of the wavefunction related to the resonance state which finally evolves into a short lived state is very small inside the resonator. This corresponds to a very small time delay, i.e. to a small probability of staying inside the billiard. The trapping of the two long lived resonance states occurs in two different ways. Both wavefunctions correspond to (almost) pure bound states of the billiard as long as \(\lambda\) is large and become mixed when their distance in the complex energy plane to the other two states becomes small. At still smaller \(\lambda\) one of the resonance states demixes and approaches again its pure shape. In accordance to that, its energy for small \(\lambda\) approaches that for large \(\lambda\). The wavefunction of the other trapped resonance state remains however mixed. Accordingly, its position at small \(\lambda\) differs from that for large \(\lambda\) and its wavefunction is mixed with that of the short lived resonance state.

The analysis presented in this letter shows that open billiards provide an excellent possibility to study the dynamics of resonance poles in detail. A realization can be achieved by means of flat microwave resonators connected to a waveguide where the coupling strength to the channel can be varied by hand. Such systems allow therefore to investigate directly the formation of different time-scales (resonance trapping) by tracing the corresponding time delay function. In particular, it is possible to show the contraintuitive result that the lifetimes of certain resonance states increase with increasing coupling to the continuum. The results of such investigations will help in analysing high-resolution experimental data in various fields of physics.
Acknowledgement: Valuable discussions with T. Gorin, B. Mirbach, M. Müller, J. Nöckel and G. Soff are gratefully acknowledged. This work has been partially supported by DFG, SMWK, Czech Grant Agency GAAV 1048804 and by the Foundation for Theoretical Physics in Slemeno, Czech Republic.

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Figure 1
Contour and surface plot of \( \log(\tau_w) \) (1.a) for a rectangular billiard (for parameters see text). The darker the plot is, the longer is the time delay. The motion of the corresponding resonance poles with \( \lambda \) (1.b). The positions of the resonance poles for \( \lambda = 44 \) are denoted by squares and for \( \lambda = 23.5 \) and \( \lambda = 0 \) with large dots.

Figure 2
The wavefunctions of three resonance states for \( \lambda = 44 \) (case a)), \( \lambda = 23.5 \) (case b)) and \( \lambda = 0 \) (case c)). The energies of the states at \( \lambda = 44 \) are 38.6 (state No 1), 38.8 (No 2) and 39.8 (No 3). The location of the potential barrier is marked by a rectangular in the waveguide.
