A model reference adaptive system approach for nonlinear online parameter identification

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Abstract. Dynamical systems, for instance in model predictive control, often contain unknown parameters, which must be determined during system operation. Online or on-the-fly parameter identification methods are therefore necessary. The challenge of online methods is that one must continuously estimate parameters as experimental data becomes available. The existing techniques in the context of time-dependent partial differential equations exclude the case where the system depends nonlinearly on the parameters. Based on a model reference adaptive system approach, we present an online parameter identification method for nonlinear infinite-dimensional evolutionary systems.

Keyword. Online method, online estimation, reference adaptive system, parameter identification, infinite-dimensional systems, partial differential equations.

1 Introduction

Evolutionary processes in science and engineering that also spatial dependence are usually modeled by time dependent partial differential equations (PDEs). These models often contain finite or infinite dimensional parameters, such as spatially varying drift or diffusion coefficients as well as source terms, whose values are unknown and have to be recovered on the basis of indirect observations of the system. Whenever parameter identification needs to take place during the operation of the considered system – as is the case, e.g., in model predictive control – so-called online methods have to be employed. Model reference adaptive system (MRAS) do so by setting up a dynamic update law for both the parameter and the state, where the evolutionary system for the state is a modification of the original model, while the parameter evolution is driven by the observation mismatch. Additionally, stabilizing terms are introduced. Online parameter identification has been extensively studied in the finite dimensional setting, e.g. [8, 17, 21]; however, when it come to infinite dimensional models as arising in the context of PDEs, less is known so far. We refer to the literature review in [2, 14, 15] for MRAS based approaches in the PDE context.

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Following up on [3], which in its turn was strongly inspired by [2] and [15], we here extend the scope to problems that are nonlinear not only with respect to the state but also with respect to the parameter – a situation that could not be tackled by the MRAS approaches investigated so far. To this end, we first on all focus on the situation of complete state observations. This may extend to the case of partial observations similarly in principle to [3], however with additional difficulties and obstacles arising due to parameter nonlinearity. Thus investigations on partial observations are postponed to future work. In this context we wish to point to recent progress on ensemble Kalman filters for inverse problems, see, e.g., [1, 4, 22], which are clearly promising in the online identification setting. However, note that this approach, being based on statistical considerations, is quite different from the one we are following here.

The paper is structured as follows. In Section 1.1 we introduce the model class under consideration as well as the adaptive system that we propose for the online parameter identification task. Section 2 provides an error analysis as well as convergence results for both exact and noisy data. Moreover, well-posedness of the adaptive system and therewith well-definedness of the method is established there. Finally, in Section 3 we discuss two examples of coefficient identification problems in parabolic PDEs, where the convergence conditions from Section 2 can be verified.

1.1 Adaptive system

We consider the problem of identifying the stationary parameter \( q^* \in X \) in the evolution equation

\[
\begin{align*}
D_t u^*(t) + f(q^*, u^*(t)) &= g(t) \quad t > 0 \\
u^*(0) &= u_0
\end{align*}
\]  

(1)

from given full observations \( z \) of the state \( u^* \) over time

\[
z(t) = G u^*(t) = u^*(t) \quad t > 0.
\]  

(3)

We pose this problem on the function spaces

\[
X \subseteq V \subseteq Y \text{ with continuous embeddings}
\]

where \( V \hookrightarrow H \hookrightarrow V^* \), \( X \hookrightarrow H \hookrightarrow X^* \) are Gelfand triples

and additionally assume that

\[
f : H \times V \to V^* \quad \text{induces the Nemytskii operator} \ f \ (\text{same notation}) : \]

\[
[f(q, u)](t) = f(q(t), u(t)) \quad \text{and} \ \]

\[
f : L^2([0, \infty); H) \times L^2([0, \infty); V) \to L^2([0, \infty); V^*)
\]

g \in L^2([0, \infty); V^*)
\]

\[
u_0 \in H
\]

\[
G = \text{Id} : V \to Y (\supseteq V),
\]
i.e., the observation operator is the continuous embedding.

To derive an update law, we first consider the parameter to be a function of time, however, with zero time derivative. This yields the equivalent model system to (1)-(2) (where we skip the time variable) as

\[
\begin{align*}
D_t q &= 0 \quad (4) \\
D_t u^* + f(q, u^*) &= g \quad (5) \\
(q, u^*)(0) &= (q^*, u_0). \quad (6)
\end{align*}
\]

Hence, the parameter \( q \) behaves as a time constant function and is identical with the exact one \( q^* \). Online identification means that the parameter identification, the data collection process and the system operation are taking place simultaneously. During these processes, data \( z \) is fed into a model reference system to adapt the solution of this system to the data \( z \).

We propose in the following the reference model adaptive system in \((q, u)\)

\[
\begin{align*}
D_t q + \sigma(D_t z + f(q, z) - g) &- f'_q(q^0, z)^*(u - z) = 0 \quad (7) \\
D_t u + f(q, z) + C(\|q\|_H)(u - z) &= g \quad (8) \\
(q, u)(0) &= (q_0, u_0). \quad (9)
\end{align*}
\]

Here, \( z \) is the solution to the model system (1)-(2) measured by the observation process (3); \( q_0 \) is an initial guess for \( q^* \), the exact parameter in the model system; for a linear operator \( A : H \rightarrow V^* \), \( A^* : V \rightarrow H \) denotes its adjoint, and we assume

\[ f'_q(q^0, z) \in L^\infty([0, \infty); L(H, V^*)) \]

to be a linearization of \( f \) with respect to its first variable; typically just the Gâteaux derivative. Moreover, \( \sigma \in \{0, 1\} \) is a switching parameter, which is set to zero if \( f(q, z) - f(q^*, z) - f'_q(q^0, z)(q - q^*) = 0 \) (which is, e.g., the case if \( f \) is linear with respect to \( q \)) and to one, else.

The model reference adaptive system (7)-(9) is well-defined since

\[
\begin{align*}
g - D_t z &= f(q^*, z) \in L^2([0, \infty); V^*) \subset L^2([0, \infty); X^*) \\
f'_q(q^0, z) : L^2([0, \infty); H) &\rightarrow L^2([0, \infty); V^*) \\
f'_q(q^0, z)^* : L^2([0, \infty); V) &\rightarrow L^2([0, \infty); H) \subset L^2([0, \infty); X^*). \quad (7)-(9)
\end{align*}
\]

This results in \( D_t q \) living in the space \( L^2([0, \infty); X^*) \), which is a conventional choice for time-dependent PDEs. The linear operator \( C(\|q(t)\|_H) \in L(V, V^*) \) will be specified according to assumption (A4) later. We search for the solution of this system in the space

\[
\begin{align*}
u \in U := L^2([0, \infty); V) \cap H^1([0, \infty); V^*) \cap L^\infty([0, \infty); H) \\
q \in X := L^2([0, \infty); H) \cap H^1([0, \infty); X^*) \cap L^\infty([0, \infty); H). \quad (10)-(11)
\end{align*}
\]
The reference system (7)-(9) mimics the model system (4)-(6) in the sense that as time progresses, by matching \( u \) to \( z \) (8) adapts to (5), and (7) evolves to
\[
D_t q + f(q, z) - f(q^*, z) = 0,
\]
which is supposed to drive \( q \) towards \( q^* \) in (11). Hence, we expect to obtain convergence of the state and of the parameter as \( t \) tends to infinity
\[
r(t) := u(t) - z(t) \to 0 \quad \text{and} \quad e(t) := q(t) - q^* \to 0 \quad \text{as} \quad t \to \infty.
\]

In order to establish convergence (12), we make the following assumptions.

**Assumption 1.1.**

(A1) For the exact parameter \( q^* \in \mathcal{X} \), the exact state \( u^* \in \mathcal{U} \) exists, and \( z = u^* \) is the full measurement data.

(A2) \( f \) is differentiable at \( q^0 \in \mathcal{X} \) and \( f'_q(q^0, z) \in L^\infty([0, \infty); L(H, V^*)) \). Moreover, for any \( q \in H \) there exists a monotonically increasing function \( L : [0, \infty) \to [0, \infty) \) such that
\[
\|f(q, z) - f(q^*, z) - f'_q(q^0, z)(q - q^*)\|_{V^*} \leq L\|q\|_H \|q - q^*\|_H.
\]

(A3) If in (A2) \( L \equiv 0 \), i.e., \( \sigma = 1 \), then there exists a constant \( C_{coe} > 0 \) such that for all \( q \in \mathcal{X} \)
\[
(f(q, z) - f(q^*, z), q - q^*)_{X^*, X} \geq C_{coe} \|q - q^*\|_H^2.
\]

(A4) \( C \) is chosen such that for given \( q \in H \), and all \( v, w \in V \)
\[
\langle C\|q\|_H v, v \rangle_{V^*, V} \geq \left( \frac{L^2\|q\|_H^2}{2C_{coe}} + M \right) \|v\|_V^2 =: \tilde{M}(\|q\|_H)\|v\|_V^2
\]
\[
\langle C\|q\|_H v, w \rangle_{V^*, V} \leq \tilde{N}(\|q\|_H)\|v\|_V\|w\|_V
\]
for some constant \( M > 0 \) and some monotonically increasing function \( \tilde{N} : [0, \infty) \to [0, \infty) \).

Indeed, the model reference adaptive system (7)-(9) is an extension of the system devised in [2] for the case of \( f \) being linear with respect to \( q \), which in our notation reads as
\[
D_t q - f(q^*, z^*)(u - z) = 0
\]
\[
D_t u + f(q, z) + C(u - z) = g
\]
\[
(q, u)(0) = (q_0, u_0)
\]
to the case of nonlinear and possibly also time dependent \( f \), that is, of a nonautonomous system (cf. Remark 2.2 below). As shown in [2], the operator \( C \) can be chosen as being independent of \( q \) which can also be seen from (A4). In (7)-(9) as well as (15)-(17), the terms containing \( u - z \) take into account the residual in the observation equation and exploit it for driving the estimated parameter-state pair \((q, u)\) to the exact one. The additional term \( D_t z + f(q, z) - g \) in (11) is just the residual in the state equation; in case \( \sigma = 1 \) it is used, together with the assumed coercivity (A3) to control the nonlinearity of \( f \).
Remark 1.2. Assumption (A2) is satisfied with \( L \equiv 0 \) if \( f \) is linear with respect to \( q \). In this case assumption (A3) is not needed. For simplicity of exposition, and since the linear case has already been discussed in [2], in the following we only consider the nonlinear case \( L \not\equiv 0, \sigma = 1 \).

Remark 1.3. If (A3) holds for a stronger norm of \( q - q^* \), we also state (A2) with this stronger norm on \( q - q^* \). In this case, (A2) is more feasible to be satisfied, thus enabling higher nonlinearity with respect to \( q \) in the model. Also from (A2) another enabling of higher nonlinearity is that \( z \) belongs to a smoother space.

Remark 1.4. The estimate in assumption (A2) is fulfilled if, for example, \( f \) is locally Lipschitz continuous with respect to \( q \) in the sense that

\[
\forall M, \exists L(M) \geq 0 : \|f(q, z) - f(q^*, z)\|_{V^*} \leq L(M)\|q - q^*\|_H \quad \forall q, q^* \in X : \|q\|_H, \|q^*\|_H \leq M.
\]

In addition, \( f(\cdot, u) \) is Gateaux differentiable on \( L^\infty([0, \infty); H) \cap L^2([0, \infty); H) \). Let us consider

\[
\frac{1}{\epsilon} \|f(u, p + \epsilon \xi) - f(u, p) - \epsilon f'_p(u, p) \xi\|_{L^2([0, T], V^*)} = \left( \int_0^T r_\epsilon(t)^2 dt \right)^{\frac{1}{2}}.
\]

From local Lipschitz continuity, we deduce, by choosing \( M = \|q\|_{L^\infty([0, T], H)} + 1, \)

\[
\|f'_p(z(t), p(t))\xi(t)\|_{V^*} = \lim_{\epsilon \to 0} \left\| \frac{f(z(t), p(t) + \epsilon \xi(t)) - f(z(t), p(t))}{\epsilon} \right\|_{V^*} \leq L(M)\|\xi(t)\|_H.
\]

Thus for all \( \epsilon \in [0, \bar{\epsilon}], \lambda \in [0, 1], \)

\[
\|f'_p(u(t), p(t) + \lambda \epsilon \xi(t))\|_{H \to V^*} \leq L(M)
\]

implies

\[
r_\epsilon(t) = \left\| \int_0^1 \left( f'_p(u(t), p(t) + \lambda \epsilon \xi(t)) - f'_p(u(t), p(t)) \right) \xi d\lambda \right\|_{V^*} \leq 2L(M)\|\xi(t)\|_H, \quad (18)
\]

which is uniformly bounded as well as square integrable in time. Applying Lebesgue’s Dominated Convergence Theorem yields Gateaux differentiability of \( f \) on \( L^\infty([0, \infty), H) \cap L^2([0, \infty); H) \).

2 Error estimates

By subtracting the model system (4)-(6) from the reference system (7)-(9), we see that the error components \( (r, e) = (u - u^*, q - q^*) \) satisfy the following nonlinear differential
equations
\begin{align*}
D_t r + f(q, z) - f(q^*, z) + C(\|q\|_H)r &= 0 \\
D_t e + f(q, z) - f(q^*, z) - f'_q(q^0, z^*)r &= 0 \\
(r, e)(0) &= (0, q_0 - q^*)
\end{align*} 
(19) \quad (20) \quad (21)

We first derive an upper bound for \((r, e)\).

2.0.1 Boundedness of \((r, e)\)

We test (19) and (20) respectively by \(r(t) \in V\) and \(e(t) \in X\), then sum up the outcome
\begin{align*}
0 &= \frac{1}{2} \frac{d}{dt} \left[ \|r\|_H^2 + \|e\|_H^2 \right] (t) + \langle f(q(t), z(t)) - f(q^*(t), z(t)), e(t) \rangle_{X^*, X} \\
&\quad + \langle f(q(t), z(t)) - f(q^*(t), z(t)) - f'_q(q^0, z(t))e(t), r(t) \rangle_{V^*, V} + \langle C(\|q(t)\|_H) r(t), r(t) \rangle_{V^*, V} \\
&\geq \frac{1}{2} \frac{d}{dt} \left[ \|r(t)\|_H^2 + \|e(t)\|_H^2 \right] + C_{\text{coe}} \|e(t)\|_H^2 - L(\|q(t)\|_H) \|e(t)\|_H \|r(t)\|_V \\
&\quad + \bar{M}(\|q(t)\|_H) \|r(t)\|_{V^*}^2 \\
&\geq \frac{1}{2} \frac{d}{dt} \left[ \|r\|_H^2 + \|e\|_H^2 \right] (t) + C_{\text{coe}} \|e(t)\|_H^2 - \left( \frac{C_{\text{coe}}}{2} \|e(t)\|_H^2 + \frac{L^2(\|q(t)\|_H)}{2C_{\text{coe}}} \|r(t)\|_{V^*}^2 \right) \\
&\quad + \bar{M}(\|q(t)\|_H) \|r(t)\|_{V^*}^2 \\
&\geq \frac{1}{2} \frac{d}{dt} \left[ \|r\|_H^2 + \|e\|_H^2 \right] (t) + \frac{C_{\text{coe}}}{2} \|e(t)\|_H^2 + M \|r(t)\|_V^2. \\
\end{align*} 
(22)

Above, we make use of assumptions [(A3),(A4)]. This estimate reveals the first observation
\[ t \mapsto E(t) := \|r(t)\|_H^2 + \|e(t)\|_H^2 \] is decreasing.

We then have the inequality
\[ E'(t) + \min \{C_{\text{coe}}; 2MC_{V \rightarrow H}\} E(t) \leq 0, \]
where \(C_{V \rightarrow H}\) is the norm of the continuous embedding \(V \hookrightarrow H\). This means that the function defined by \(\tilde{E}(t) = \exp \left( \min \{C_{\text{coe}}; 2MC_{V \rightarrow H}\} t \right) E(t)\) satisfies \(\tilde{E}'(t) \leq 0\) and is therefore monotonically decreasing, in particular \(\tilde{E}(t) \leq \tilde{E}(0) = [\|r(0)\|_H^2 + \|e(0)\|_H^2]\), hence
\[ E(t) \leq \exp \left( - \min \{C_{\text{coe}}; 2MC_{V \rightarrow H}\} t \right) \left[ \|r(0)\|_H^2 + \|e(0)\|_H^2 \right], \]
(23)

that is, an exponential convergence rate of the total error.

Next, by integrating over \([0, T]\) we get
\begin{align*}
\frac{1}{2} \left[ \|r(T)\|_H^2 + \|e(T)\|_H^2 \right] + \frac{C_{\text{coe}}}{2} \int_0^T \|e(t)\|_H^2 dt + M \int_0^T \|r(t)\|_V^2 dt \\
\leq \frac{1}{2} \left[ \|r(0)\|_H^2 + \|e(0)\|_H^2 \right] \quad \text{for any } T > 0
\end{align*} 
(24)
thus

\[ \| D_t r \|_{L^2([0,\infty);V')}^2 = \int_0^\infty \left( \sup_{\| v \| \leq 1} \langle D_t r, v \rangle_{V', V} \right)^2 \, dt \]

\[ = \int_0^\infty \left( \sup_{\| v \| \leq 1} \langle -f(q, z) + f(q^*, z) - C(\| q \|_H) r, v \rangle_{V', V} \right)^2 \, dt \]

\[ \leq \int_0^\infty \left( \sup_{\| v \| \leq 1} \left[ \left( L(\| q(t) \|_H) + \| f'_q(q^0, z(t)) \|_{H \to V'} \right) \| e(t) \|_H \| v \|_V \\
\quad + \tilde{N}(\| q(t) \|_H) \| r(t) \|_V \| v \|_V \right] \right)^2 \, dt \]

\[ \leq \frac{\left( L(\| q \|_{L^\infty([0,\infty);H)}) + \| f'_q(q^0, z) \|_{L^\infty([0,\infty);L(H,V'))} + \tilde{N}(\| q \|_{L^\infty([0,\infty);H)}) \right)^2}{\min \{ C_{\text{coe}}; 2M \}} \times \left[ \| r(0) \|_H^2 + \| e(0) \|_H^2 \right] \ 	ext{ (25)} \]

also, due to \( X \subseteq V \),

\[ \| D_t e \|_{L^2([0,\infty);X')} \leq \| D_t e \|_{L^2([0,\infty);V')} \]

\[ = \int_0^\infty \left( \sup_{\| v \| \leq 1} \langle -f(q, z) + f(q^*, z) + f'_q(q^0, z) r, p \rangle_{V', V} \right)^2 \, dt \]

\[ \leq \int_0^\infty \left( \sup_{\| p \| \leq 1} \left[ \left( L(\| q(t) \|_H) + \| f'_q(q^0, z) \|_{H \to V'} \right) \| e(t) \|_H \| p \|_V \\
\quad + \| f'_q(q^0, z) \|_{H \to V'} \| p \|_H \| r(t) \|_V \right] \right)^2 \, dt \]

\[ \leq \frac{\left( L(\| q \|_{L^\infty([0,\infty);H)}) + 2 \| f'_q(q^0, z) \|_{L^\infty([0,\infty);L(H,V'))} \right)^2}{\min \{ C_{\text{coe}}; 2M \}} \times \left[ \| r(0) \|_H^2 + \| e(0) \|_H^2 \right] . \ 	ext{ (26)} \]

Summarizing (23)-(26), we obtain the result:

**Proposition 2.1.** Let Assumption 1.1 be fulfilled. Then the following statements on the parameter \( q \) and the state \( u \) as well as the corresponding errors \( e = q - q^* \), \( r = u - u^* \) hold true.

(i) \( u \in \mathcal{U} = L^2([0,\infty);V) \cap H^1([0,\infty);V') \cap L^\infty([0,\infty);H) \).
\( q \in \mathcal{X} = L^2([0,\infty);H) \cap H^1([0,\infty);X^*) \cap L^\infty([0,\infty);H) \).

(ii)

\[ \sup_{t \geq 0} \left[ \| r(t) \|_H^2 + \| e(t) \|_H^2 \right] + C_{\text{coe}} \int_0^\infty \| e(s) \|_H^2 \, ds + 2M \int_0^\infty \| r(s) \|_V^2 \, ds \]

\[ \leq \left[ \| r(0) \|_H^2 + \| e(0) \|_H^2 \right] . \]
Remark 2.2. The proposed method can be generalized to nonautonomous model systems with \( f : [0, \infty) \times H \times V \to V^* \). In this case, we assume that \( f \) meets the Caratheodory conditions and \([A2], [A3]\) hold uniformly for almost all \( t \in (0, \infty) \).

Remark 2.3. The coercivity assumption \([A3]\) (which we verify for an example in Section 3) considerably facilitates the proof of parameter convergence, as compared to the more general proof via persistence of excitation in \([2]\). To compare these two conditions, we recall that for \( f \) linear with respect to \( q \), the state \( u \) is called uniformly persistently excited iff

\[
\exists \ell > 0, \mu > 0 \forall h \in X \setminus \{0\} \forall t_0 \in [0, \infty) \exists t_1, t_2 \in [t_0, t_0 + \ell] \exists v \in V \setminus \{0\} : \int_{t_1}^{t_2} \langle f(h, u(t)), v \rangle_{V^*, V} ds \geq \mu \|h\|_X \|v\|_V.
\]

On the other hand, it is readily checked, that by choosing \( v := h \in X \subseteq V, t_1 = t_0, t_2 = t_0 + \ell, \) and setting \( \mu = lC_{\text{coe}} \), assumption \([A3]\) yields

\[
\exists \ell > 0, \mu > 0 \forall h \in X \setminus \{0\} \forall t_0 \in [0, \infty) \exists t_1, t_2 \in [t_0, t_0 + \ell] \exists v \in V \setminus \{0\} : \int_{t_1}^{t_2} \langle f(h, u(t)), v \rangle_{V^*, V} ds \geq \mu \|h\|_H \|v\|_H.
\]

Thus, indeed \([A3]\) implies a certain persistence of excitation, however, with respect to weaker norms.

### 2.1 Noisy data

We now turn to the practically relevant setting of noisy observations \( z_\delta \) in place of \( z = u^* \) with a certain noise level \( \delta > 0 \) that we assume to be given in the data space norm, that is,

\[
\|z_\delta - z\|_{L^p([0, \infty); Y)} \leq \delta. \tag{27}
\]

Since \( z_\delta \) does not satisfy the regularity requirements needed to be inserted into the model reference adaptive system, we smooth it by, e.g., filtering or local averaging, more abstractly, by applying regularizing operators \( R^{sp} : Y \to V \) (pointwise in time), \( R^{ti} : L^p([0, \infty); Y) \to W^{1,p}([0, \infty); H) \) (typically nonlocal in time) such that – by an appropriate choice of the regularization parameters contained in the definition of \( R^{sp}, R^{ti} \) – the estimates

\[
\|R^{sp}(z_\delta(t)) - z(t)\|_V \leq \delta^{sp}(t), \quad \|D_t(R^{ti}(z_\delta)) - z\|_{L^p([0, \infty); H)} \leq \delta^{ti} \tag{28}
\]
There exists a constant \( \tilde{\alpha} \).

There exist constants \( \tilde{\alpha} \).

Assumption 1.1 holds for some \( p \).

Proposition 2.5. Let Assumption 1.1 with the state (28) and noise bound (28) be satisfied. Then the following statements on the parameter \( q \) and the state \( u \) as well as the corresponding errors \( \epsilon = q - q^* \), \( r = u - u^* \) hold true.

(i) \( u \in U = L^2([0, \infty); V) \cap H^1([0, \infty); V^*) \cap L^\infty([0, \infty); H) \).

(ii) For any \( \omega < \min\{C_{\text{coe}}, 2MC_{\text{V-H}}\} \) there exists \( C > 0 \) such that for all \( t \geq 0 \)

\[
\|r(t)\|_H^2 + \|e(t)\|_H^2 \leq \exp \left( -\omega t \right) \left[ \|r(0)\|_H^2 + \|e(0)\|_H^2 \right] + C \left( \|\tilde{\delta}^s_p\|_{L^p(0,t)}^2 + (\delta_i^s)^2 \right)
\]

Proof. The crucial estimate follows analogously to the proof of Proposition 2.1, using the fact that with noisy data the error system becomes

\[
D_t r + f(q, z) - f(q^*, z) + C(\|q\|_H)r = d^n \tag{30}
\]

\[
D_t e + f(q, z) - f(q^*, z) - (f'(q^*, z) - d^0)^* r = d^\ell \tag{31}
\]

\[
(r, e)(0) = (0, q_0 - q^*) \tag{32}
\]
in place of (19)-(21), where
\[ d^0(t) = f'(q^0, z(t)) - f'(q^0, R^{sp}(z^\delta(t))) \in L(H, V^*) \]
\[ d^u(t) = f(q(t), z(t)) - f(q(t), R^{sp}(z^\delta(t))) - C(\|q(t)\|_H)(z(t) - R^{sp}(z^\delta(t))) \in V^* \]
\[ d^l(t) = f(q(t), z(t)) - f(q(t), R^{sp}(z^\delta(t))) + D_i(z(t) - R^{si}(z^\delta(t))) + f'_q(q^0, R^{sp}(z^\delta(t)))(z(t) - R^{sp}(z^\delta(t))) \in H \]

By testing with \( r(t) \) and \( e(t) \), respectively we get, in place of (22)
\[
\langle d^u(t), r(t) \rangle_{V^*, V} + \langle d^l(t), e(t) \rangle_{X^*, X} \\
\geq \frac{1}{2} \frac{d}{dt} \left[ \|e\|^2_H + \|r\|^2_V \right] (t) + C_{coe}\|e(t)\|^2_H \\
- \left( L(\|q(t)\|_H + \bar{L}_0\tilde{\delta}^{sp}(t)) \right) \|e(t)\|_H \|r(t)\|_V + \tilde{\omega}(\|q(t)\|_H) \|r(t)\|^2_V \\
\geq \frac{1}{2} \frac{d}{dt} \left[ \|r(t)\|^2_H + \|e(t)\|^2_H \right] + C_{coe}\|e(t)\|^2_H \\
- \left( \frac{C_{coe}}{2}\|e(t)\|^2_H + \frac{L(\|q(t)\|_H + \bar{L}_0\tilde{\delta}^{sp}(t))^2}{2C_{coe}} \|r(t)\|_V^2 \right) + \tilde{\omega}(\|q(t)\|_H) \|r(t)\|^2_V \\
\geq \frac{1}{2} \frac{d}{dt} \left[ \|r(t)\|^2_H + \|e(t)\|^2_H \right] + \frac{C_{coe}}{2}\|e(t)\|^2_H + M\|r(t)\|^2_V \\
(33) \]

An application of Young’s Inequality and Assumption 2.3 as well as multiplication by two yields
\[ E'(t) + \omega E(t) \leq \frac{1}{\epsilon_1} \|d^u(t)\|^2_{V^*} + \frac{1}{\epsilon_2} \|d^l(t)\|^2_H \]
\[ \leq C(\epsilon_0, \epsilon_1, \epsilon_2)\tilde{\delta}^{sp}(t)^2 + \frac{2}{\epsilon_2}\|D_i(z(t) - R^{si}(z^\delta(t)))\|^2_H := D(t), \]
where \( \omega = \min \{C_{coe} - \epsilon_2; 2MC_{V-H} - \epsilon_1\}, \)
\[ C(\epsilon_0, \epsilon_1, \epsilon_2) = \sup_{t \in [0, \infty)} \left( \frac{\bar{L}_1 + \tilde{N}(\|q(t)\|_H))^2}{\epsilon_1} + \frac{2}{\epsilon_2} \left( \bar{L}_2 + \|f'_q(q^0, z(t))\|_{L(H, V^*)} + \bar{L}_0\tilde{\delta}^{sp}(t) \right)^2. \]

For \( \bar{E}(t) := e^{\omega t} E(t) \) this means \( \bar{E}'(t) \leq e^{\omega t} D(t) \) and thus \( \bar{E}(t) \leq \bar{E}(0) + \int_0^t e^{\omega s} D(s) \, ds = E(0) + \int_0^t e^{\omega s} D(s) \, ds \), that is,
\[ E(t) \leq e^{-\omega t} E(0) + \int_0^t e^{-\omega(t-s)} D(s) \, ds \leq e^{-\omega t} E(0) + C_\omega \|D\|_{L^{p/2}(0, t)} \]
\[ \leq e^{-\omega t} E(0) + C_\omega \left( C(\epsilon_0, \epsilon_1, \epsilon_2)\tilde{\delta}^{sp}\|^{2}_{L^{p}(0, t)} + \frac{2}{\epsilon_2}(\tilde{\delta}^{si})^2 \right) \]

where \( C_\omega = 1 \) if \( p = 2 \) and \( C_\omega = \left( \frac{p}{p-2}\omega \right)^{-\frac{p-2}{p}} \) if \( p > 2 \). \[ \square \]
2.2 Unique solvability of the adaptive system

We now formulate the setting that the reference adaptive system is governed by the initial-value problem

\[
\frac{d}{dt}(u, q)(t) + F(u, q)(t) = G(t) \quad t > 0
\]

(34)

\[
(u, q)(0) = (u_0, q_0)
\]

(35)

with

\[
F : V \times H \to V^* \times X^*
\]

\[
F : \begin{pmatrix} u(t) \\ q(t) \end{pmatrix} \mapsto \begin{pmatrix} f(q, z)(t) + C(\|q(t)\|_H)(u - z)(t) \\ f(q, z)(t) - f(q^*, z)(t) - f'_z(q^0, z^*)(u - z)(t) \end{pmatrix} = \begin{pmatrix} F_1(u(t), q(t)) \\ F_2(u(t), q(t)) \end{pmatrix},
\]

(36)

\[
G(t) = \begin{pmatrix} g(t) \\ 0 \end{pmatrix} \in V^* \times X^*
\]

(37)

and define the pairing

\[
\langle F(u, q), (\tilde{u}, q^*) \rangle = \langle F_1(u, q), \tilde{u} \rangle_{V^*, V} + \langle F_2(u, q), q^* \rangle_{X^*, X}.
\]

In the following, we prepare some evaluations which will be used to prove pseudomonotonicity of \( F \).

Pseudomonotonicity with respect to \( u \)

Let us consider the function \( A := C(\| \cdot \|_H)(\cdot - z) \). Since \( A \) is bounded and demicontinuous on \( V \times H \), i.e., \((u_n, q_n) \to (u, q)\) then \( A(u_n, q_n) \to A(u, q), n \to \infty \), according to \([20, -Lemma 2.10]\) it is pseudomonotone if this statement holds

If \( (u_n, q_n) \xrightarrow{n \to \infty} (u, q) \) and \( \limsup_{n \to \infty} \langle A(u_n, q_n) - A(u, q), (u_n, q_n) - (u, q) \rangle \leq 0 \),

(38)

then \( (u_n, q_n) \xrightarrow{n \to \infty} (u, q) \).

(39)

It suffices to consider the simple form \( C(\| \cdot \|_H)(\cdot - z) = \| \cdot \|_H C(\cdot - z) =: A(e, q) \) (as the argument for the higher order of \( \| \cdot \|_H \) is similar)

\[
\langle A(u_n, q_n) - A(u, q), (u_n, q_n) - (u, q) \rangle_{V^*, V} = \langle A(e_n, q_n) - A(e, q), e_n - e \rangle_{V^*, V}
\]

\[
= \langle q_n \|H C(e_n) - \|q\|_H C(e), e_n - e \rangle_{V^*, V}
\]

\[
= \|q_n \|H \|e_n\|_{V'}^2 + \|q\|_H \|e\|_{V'}^2 - \|q_n \|H \langle C(e_n), e \rangle_{V^*, V} - \|q\|_H \langle C(e), e_n \rangle_{V^*, V}
\]

\[
\geq \|q_n \|H \|e_n\|_{V'}^2 + \|q\|_H \|e\|_{V'}^2 - \frac{\|q_n \|H}{2} (\|e_n\|_{V'}^2 + \|e\|_{V'}^2) - \frac{\|q\|_H}{2} (\|e_n\|_{V'}^2 + \|e\|_{V'}^2)
\]

\[
= \frac{1}{2} (\|q_n \|H - \|q\|_H) (\|e_n\|_{V'}^2 - \|e\|_{V'}^2).
\]
From the assumption (38) and weak lower semicontinuity of \( \| \cdot \|_H \) and \( \| \cdot \|_V \), we deduce
\[
0 \geq \limsup_{n \to \infty} \langle A(u_n, q_n) - A(u, q), (u_n, q_n) - (u, q) \rangle_{V^*, V}
\geq \limsup_{n \to \infty} \frac{1}{2} (\|q_n\|_H - \|q\|_H) (\|e_n\|_V^2 - \|e\|_V^2)
\geq \liminf_{n \to \infty} \frac{1}{2} (\|q_n\|_H - \|q\|_H) (\|e_n\|_V^2 - \|e\|_V^2) = 0,
\]
which implies the product \( (\|q_n\|_H - \|q\|_H) (\|e_n\|_V^2 - \|e\|_V^2) \to \infty , n \to \infty \). This tells us that as \( n \to \infty \), either \( e_n \overset{V}{\rightharpoonup} e \) or \( q_n \overset{H}{\rightharpoonup} q \).

In addition, we observe
\[
0 \overset{n \to \infty}{\leftarrow} \langle \|q_n\|_H C(e_n) - \|q\|_H C(e), e_n - e \rangle_{V^*, V}
= \langle \|q_n\|_H C(e_n) - \|q_n\|_H C(e) + \|q_n\|_H C(e) - \|q\|_H C(e), e_n - e \rangle_{V^*, V}
= \|q_n\|_H \|e_n - e\|_V^2 + (\|q_n\|_H - \|q\|_H) \langle e, e_n - e \rangle_{V^*, V}.
\]
Then, if \( q_n \overset{H}{\rightharpoonup} q \), the second term in this sum converges to 0. This constraints \( \|e_n - e\|_V \) to converge to 0 as well. As a consequence
\[
\text{As } n \to \infty : \quad e_n \overset{V}{\rightharpoonup} e \quad \text{always.}
\]
Comparing to (39), \( A \) is pseudomonotone with respect to variable \( u \).

**Pseudomonotonicity with respect to \( q \)**

Strong convergence of \( q_n \) in \( H \) is straightforwardly attainable by using (A3)
\[
0 \geq \limsup_{n \to \infty} \langle f(q_n, z) - f(q, z), q_n - q \rangle_{V^*, V} \geq C_{\text{coe}} \lim_{n \to \infty} \|q_n - q\|_H^2
\]
which implies
\[
\text{As } n \to \infty : \quad q_n \overset{H}{\rightharpoonup} q.
\]
Together with the fact that \( f(\cdot, z) \) is bounded and continuous by (A2), it yields \( f \) is pseudomonotone with respect to variable \( q \).

Combining the obtained results, we are now able to evaluate pseudomonotonicity of \( F \)

**Pseudomonotonicity with respect to \( (u, q) \)**

First of all, boundedness and continuity of \( F \) is deduced from these properties of \( f(\cdot, z) \) and \( A \).

We then begin the assumption (38)
\[
(u_n, q_n) \overset{n \to \infty}{\longrightarrow} (u, q) \quad \limsup_{n \to \infty} \langle F(u_n, q_n) - F(u, q), (u_n, q_n) - (u, q) \rangle \leq 0
\]
and see

\[
K := \langle F_1(u_n, q_n) - F_1(u, q), e_n - e \rangle_{V^*, V} + \langle F_2(u_n, q_n) - F_2(u, q), q_n - q \rangle_{X^*, X} \\
= \langle f(q_n, z) - f(q, z), q_n - q \rangle_{X^*, X} + \langle f(q_n, z) - f(q, z), f_q^0(q, z)(q_n - q), e_n - e \rangle_{V^*, V} \\
+ \langle C(\|q_n\|)e_n - C(\|q\|)e, e_n - e \rangle_{V^*, V} \\
=: K_1 + K_2 + K_3 \\
\geq \frac{1}{2} K_1 + \frac{1}{2} K_3
\]

if estimating in the same manner with (22). From the facts \( K_1 \geq 0 \) (see (A3)) and \( K_3 \geq 0 \) for sufficiently large \( n \) (see (40)), we get the implication

\[
\limsup_{n \to \infty} K \leq 0 \quad \text{implies} \quad \begin{cases} 
\limsup_{n \to \infty} K_1 = 0 \quad \text{(or} \leq 0) \\
\limsup_{n \to \infty} K_3 = 0 \quad \text{(or} \leq 0)
\end{cases}
\]

This allows us to apply the achieved result (42) for \( K_1 \) and (41) for \( K_3 \), thus, to conclude

\[
\text{As } n \to \infty : \quad (u_n, q_n) \xrightarrow{V\times H} (u, q).
\]  \( (43) \)

With that, pseudomonotonicity of \( F \) with respect to \((u, q)\) has been proven.

Now, we show unique existence of a solution to the adaptive system (7)-(8) through the three following steps

1. **Approximate solutions**
   Using Galerkin method, we can construct approximate solutions to (7)-(8) on the finite-dimensional subspaces.
   These approximate solutions are uniformly bounded according to Proposition \(2.1\).
   Thus, there exists a subsequence that weakly converges to some \((u, q)\) \(\in U \times X\).

2. **Weak limit of approximate solutions is a solution**
   This weak limit \((u, q)\) is indeed a weak solution to (7)-(8) since \( F \) is pseudomonotone by (43).
   
   **Proof.** [20, Theorem 8.27]

3. **Uniqueness of the solution**
   From (22), it shows that \( F \) satisfies the condition

\[
\exists \rho_1, \rho_2 = \text{const, } \forall (u, q), (\bar{u}, q^*) \in V \times H : \\
\langle F(u, q) - F(\bar{u}, q^*), (u, q) - (\bar{u}, q^*) \rangle \geq \rho_1 \|u - \bar{u}\|_H^2 + \rho_2 \|q - q^*\|_H^2,
\]

which ensures uniqueness of the solution to the adaptive system in \( U \times X \).

**Proof.** [20, Theorem 8.31]
Before studying concrete examples, we summarize the unique existence theory for evolution systems

\[ D_t u(t) + f(q, u(t)) = g(t) \quad \text{for a.e. } t \in (0, T), \quad u(0) = u_0 \quad (44) \]

in the same function space setting proposed in Section 1.1, except for the finite time interval.

**Theorem 2.6.** Assume that for fixed \( q \in X \),

1. \( f \) is pseudomonotone
2. \( f \) is semi-coercive:
   \[ \forall v \in V : \langle f(q, v), v \rangle_{V^*, V} \geq c_0 |v|^2_V - c_1 |v|_V - c_2 \|v\|^2_H \]
   with some \( c_0 > 0 \) and some seminorm \( |\cdot|_V \) satisfying \( \|\cdot\| \leq c_1 (|\cdot|_V + \|\cdot\|_H) \) for some \( c_1 > 0 \).
3. \( f \) satisfies the regularity condition:
   \[ g \in W^{1,\infty,2}([0, T); V^*, V^*) \]
   \[ u_0 \in V \text{ such that } f(u_0) - f(0) \in H \]
   \[ \langle f(q, u) - f(q, v), u - v \rangle_{V^*, V} \geq C_0 |u - v|^2_V - C_2 \|u - v\|^2_H \]
   with some \( C_0 > 0 \).

Then equation (44) has a unique solution \( u \in W^{1,\infty,\infty}([0, T); H, H) \cap W^{1,\infty,2}([0, T); V, V) \), where \( W^{1,p,q}([0, T); V_1, V_2) := \{u \in L^p([0, T); V_1) : D_t u \in L^q([0, T); V_2)\} \).

**Proof.** [20] Theorems 8.18, 8.31. \( \square \)

### 3 Examples

Let us examine unique existence of the solution to the linear parabolic problem perturbed by a nonlinear term

\[ D_t u - \nabla \cdot (a \nabla u) + cu + \phi(u)\psi(a, c) = g \text{ in } \Omega \times [0, T) \]
\[ u(0) = u_0 \text{ in } \Omega \times \{0\} \quad (45) \]

on a bounded smooth domain \( \Omega \subset \mathbb{R}^3 \) with

\[ c \in L^2(\Omega), \quad 0 < a \leq a(x) \leq \overline{a} \quad \forall x \in \Omega \]
\[ V = H^1(\Omega), \quad H = L^2(\Omega) \]
\[ g \in W^{1,\infty,2}([0, T); V^*, V^*) \]
\[ u_0 \in V. \quad (47) \]
Denoting
\[-\nabla \cdot (a \nabla u) + cu + \phi(u)\psi(a, c) := f_1(u) + f_2(u) + f_3(u),\]
it is evident that for the linear term,

1. \(f_1 : V \to V^*\) is monotone and continuous, thus pseudomonotone [6, Lemma 6.7] (one can also argue through coercivity).

\(f_2 : V \to V^*\) is strongly continuous, i.e., \(u_n \rightharpoonup u\) implies \(f_2(u_n) \to f_2(u)\). Indeed, for \(u_n \rightharpoonup u\) in \(V\)

\[
\|f_2(u_n) - f_2(u)\|_{V^*} = \sup_{\|v\|_{V^*} \leq 1} \int_\Omega c(u_n - u)v \, dx
\]

\[
\leq \sup_{\|v\|_{V^*} \leq 1} \|c\|_{L^2(\Omega)} \|u_n - u\|_{L^6(\Omega)}^{\frac{1}{2}} \|u_n - u\|_{L^2(\Omega)}^{\frac{1}{2}} \|v\|_{L^6(\Omega)}
\]

\[
\leq (CH^{1\to L^6})^{\frac{1}{2}} \|c\|_{L^2(\Omega)} \|u_n - u\|_{V}^{\frac{1}{2}} \|u_n - u\|_{H}^{\frac{1}{2}} \to 0 \quad \text{as } n \to \infty
\]
due to the fact that the weakly convergent sequence \(u_n\) is bounded in \(V\) and the embedding \(V = H^1(\Omega) \hookrightarrow L^2(\Omega) = H\) is compact. Thus \(f_2\) is pseudomonotone [6, Lemma 6.7].

Hence, \(f_{12} := f_1 + f_2\) is pseudomonotone [6, Lemma 6.8].

2. \(f_{12}\) is semi-coercive

\[
\langle f_{12}(u), u \rangle_{V^*, V} = \int_\Omega (-\nabla \cdot (a \nabla u) + cu)u \, dx
\]

\[
\geq q \|\nabla u\|_{L^2(\Omega)} - C_{H^{1\to L^6}} \|c\|_{L^2(\Omega)} \left( \frac{c_1}{4\epsilon} + \epsilon \right) \|u\|_{H^1(\Omega)}^2 - \frac{C_{H^{1\to L^6}}}{16\epsilon c_1} \|c\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)}^2,
\]

\[
:= c_0 \|u\|_{V}^2 - c_2 \|u\|_{H}^2
\]

with \(c_0 > 0\) as \(\epsilon, \epsilon_1\) are arbitrarily small [12, Section 3.1, (69)]

3. \(f_{12}\) satisfies the regularity condition with

\[
C_0 := c_0 > 0, \quad C_1 = 0, \quad C_2 := c_2
\]
due to its linearity. We also assume

\[
u_0 \in H^2(\Omega)
\]

such that \(f_{12}(u_0) - f_{12}(0) = f_{12}(u_0) \in H\).

Relying on this, unique existence of the solution to (45)-(46) then boils down the question of whether the nonlinear term \(f_3\) fulfills the conditions in Theorem 2.6.

We make an assumption that

\[
f_3 = \phi(\cdot)\psi(a, c) \text{ is monotone, and } |\phi(u)| \leq C_\phi(1 + |u|^\alpha)
\]
then \( f_3(u) \in V^* \) provided that
\[
\alpha \leq 5 \quad \text{if } \psi(a, c) \in L^\infty(\Omega), \quad \text{as } \|f_3(u)\|_{V^*} \leq C\|\psi(a, c)\|_{L^\infty(\Omega)}\|\phi(u)\|_{L^{5/3}}
\]
\[
\alpha \leq 2 \quad \text{if } \psi(a, c) \in L^2(\Omega), \quad \text{as } \|f_3(u)\|_{V^*} \leq C\|\psi(a, c)\|_{L^2(\Omega)}\|\phi(u)\|_{L^3}. \tag{50}
\]

We now observe

1. \( f_{13} := f_1 + f_3 \) is pseudomonotone according to [18 Arxiv version, Section 8.2, (S1)].

   Consequently, \( f \) is is pseudomonotone.

2. \( f \) is semi-coercive, since
   \[
   \langle f_3(u), u \rangle_{V^*, V} \geq \langle f_3(0), u \rangle_{V^*, V} \geq -C\|\psi(a, c)\|_{H}\|u\|_{H} \geq -C_{V \to H}\phi\|\psi(a, c)\|_{H}\|u\|_{V}
   \]
   meaning \( c_1 := C_{V \to H}\phi\|\psi(a, c)\|_{H}. \)

3. \( f \) fulfills the regularity condition, since
   \[
   \langle f_3(u) - f_3(v), u - v \rangle_{V^*, V} \geq 0.
   \]

As \( u_0 \in H^2(\Omega) \hookrightarrow L^\infty(\Omega) \), it follows that if \( \psi(a, c) \in H \), then
\[
\|f_3(u_0) - f_3(0)\|_H \leq C_{\phi}(2 + (C_{H^2 \to L^\infty})^\gamma/\|u_0\|_{L^\infty})\|\psi(a, c)\|_H.
\]

This verification proves unique existence of the solution
\[
u \in W^{1,\infty,\infty}([0, T); H, H) \cap W^{1,\infty,2}([0, T); V, V) \quad \forall T > 0
\]
to the problem (45)-(46).

Now we turn to lift the regularity of \( u \) to \( L^\infty([0, T); H^2(\Omega)) \hookrightarrow L^\infty([0, T) \times \Omega) \) to attain boundedness in time and space for the solution. This will allow justification of the examples considered in the next sections.

We observe from (45) that
\[
D_t u \in L^\infty(0, T; L^2(\Omega)) \quad \text{as } u \in W^{1,\infty,\infty}(0, T; H, H)
\]
\[
g \in L^\infty(0, T; L^2(\Omega)) \quad \text{if we assume, together with (47), } g \in W^{1,\infty,2}([0, T); H, V^*)
\]
\[
\phi(u) \psi(a, c) \in L^\infty(0, T; L^2(\Omega)) \quad \text{if } \psi(a, c) \in L^\infty(\Omega), \text{ and}
\]
\[
\|\phi(u)\|_{L^2(\Omega)} \leq \|u\|_{V}^{\gamma}, \gamma \geq 0 \text{ as } u \in W^{1,\infty,2}([0, T); V, V),
\]
thus constrain: \( \alpha \leq 3 \),

which yields the remaining term \(-\nabla(a\nabla u) + cu =: b \in L^\infty(L^2(\Omega)) \). If we assume \( u = 0 \) on \( \partial\Omega \) then \( b \) also has zero boundary. Suppose that there exists \( \hat{c} \in L^\infty(\Omega) \) close to \( c \), then for each \( t \in (0, T), \)
\[
-\Delta u + \frac{\hat{c}}{a} u = \frac{1}{a}(b + \nabla a \cdot \nabla u + (\hat{c} - c)u)
\]
\[
u - \left(-\Delta + \frac{\hat{c}}{a}\right)^{-1} \left(\frac{1}{a}(\nabla a \cdot \nabla u + (\hat{c} - c)u)\right) =: (\text{Id} - K)u = \left(-\Delta + \frac{\hat{c}}{a}\right)^{-1} \left(\frac{1}{a}b\right),
\]
with \((-\Delta + \frac{\tilde{c}}{a})^{-1} : L^2(\Omega) \to H^2(\Omega)\) for \(\tilde{c} \in L^\infty(\Omega)\) and \(\partial\Omega \in C^2\) being a bounded operator [5, Section 6.3, Theorem 4].

Then \(K : H^2(\Omega) \cap H_0^1(\Omega) \to H^2(\Omega) \cap H_0^1(\Omega)\) is a linear bounded operator with \(\|K\| < 1\) if

\[
\|Kv\|_{H^2(\Omega)} = \left\|(-\Delta + \frac{\tilde{c}}{a})^{-1}\left(\frac{1}{a}(\nabla a \cdot \nabla v + (\tilde{c} - c)v)\right)\right\|_{H^2(\Omega)} \leq \frac{C\tilde{c},a}{a}\|\nabla a \cdot \nabla v + (\tilde{c} - c)v\|_{L^2(\Omega)}
\]

\[
\leq \frac{C\tilde{c},a}{a} (C_{H^1 \to L^6} \|\nabla a\|_{L^3(\Omega)} \|v\|_{H^2(\Omega)} + C_{H^2 \to L^\infty} \|\tilde{c} - c\|_{L^2(\Omega)} \|v\|_{H^2(\Omega)}) < \|v\|_{H^2(\Omega)}
\]

for any \(v \in H^2(\Omega) \cap H_0^1(\Omega)\), provided that

\[
C_{H^1 \to L^6} \|\nabla a\|_{L^3(\Omega)} + C_{H^2 \to L^\infty} \|\tilde{c} - c\|_{L^2(\Omega)} < \frac{a}{C\tilde{c},a}.
\]

Applying the Neumann series for \(K\) with \(\|K\| < 1\), we have

\[
\|u\|_{H^2(\Omega)} = \left\|(\text{Id} - K)^{-1} (-\Delta + \frac{\tilde{c}}{a})^{-1} \left(\frac{1}{a}b\right)\right\|_{H^2(\Omega)} \leq \frac{1}{1 - \|K\|_{H^2(\Omega) \to H^2(\Omega)}} \left\|(-\Delta + \frac{\tilde{c}}{a})^{-1} \left(\frac{1}{a}b\right)\right\|_{H^2(\Omega)} \leq \frac{C\tilde{c},c/a}{1 - \|K\|_{H^2(\Omega) \to H^2(\Omega)}} \|b\|_{L^2(\Omega)}.
\]

Then we get

\[
u \in L^\infty([0, T); H^2(\Omega)) \hookrightarrow L^\infty([0, T) \times \Omega) \ \forall T > 0.
\]

We summarize these results in the following proposition

**Proposition 3.1.** In Problem ([15] - [46]), we assume

\[
\psi(c, a) \in L^\infty(\Omega),
\]

\(a \in L^\infty(\Omega) \cap W^{1,3}(\Omega), \ a > 0 \ \text{on} \ \Omega\)

\(c \in L^2(\Omega), \ \text{with} \ C_{H^1 \to L^6} \|\nabla a\|_{L^3(\Omega)} + C_{H^2 \to L^\infty} \|\tilde{c} - c\|_{L^2(\Omega)} < \frac{a}{C\tilde{c},a} \ \text{for some} \ \tilde{c} \in L^\infty(\Omega)\)

\(g \in W^{1,\infty,2}([0, T); H, V^*)\)

\(u_0 \in H^2(\Omega)\)

\(\phi(\cdot)\psi(a, c) \ \text{is monotone, and} \ \ |\phi(u)| \leq C_\phi(1 + |u|^3)\).

Then this evolution problem admits a unique solution

\[
u \in Z := W^{1,\infty,\infty}([0, T); H, H) \cap W^{1,\infty,2}([0, T); V, V) \cap L^\infty([0, T) \times \Omega) \ \forall T > 0.
\]

By default, these conditions will be imposed on the exact state equations in the upcoming examples. This is to ensure the exact state, that shall be in the following denoted by
z, exists in the proper space as stated in (A1), moreover is bounded in \((t, x)\) to facilitate the validation of assumptions (A2)-(A3) regarding the identified coefficients. This allows us to assume

\[
\text{for any } T > 0, \exists M_z > 0, \forall (x, t) \in \Omega \times (0, T): \quad |\phi(z)| \leq C_\phi(1 + |z|^3) \leq M_z. \tag{52}
\]

Coefficients \(c, a\) here will respectively play the role of the exact space-dependent coefficient \(q^*\) in the examined problems. Thus we can impose

\[
\text{exists } M_{q^*} > 0, \forall x \in \Omega: \quad |\psi(q^*)| \leq M_{q^*}. \tag{53}
\]

### 3.1 Identification of a potential \(-c\) problem with nonlinear perturbation

We consider the model

\[
\begin{align*}
D_t u - \Delta u + qu + \phi(u)\psi(q) &= g \quad \text{in } \Omega \times [0, \infty) \tag{54} \\
u(0) &= u_0 \quad \text{in } \Omega \times \{0\} \tag{55}
\end{align*}
\]

on a smooth bounded domain \(\Omega \subseteq \mathbb{R}^3\) under the assumptions

- the exact state \(z(t, x) \geq c > 0 \quad \forall x \in \Omega, t > 0,
- \phi(z)\psi(\cdot)\) is monotone, and \(|\psi'(q)| \leq C_\psi(1 + |q|^\beta-1)\)

To achieve boundedness away from zero of \(z\) by means of a maximum principle \([19, \text{Lemma 2.1, Chapter 2}]\) we impose inhomogeneous Dirichlet boundary conditions

\(z = h \geq c \quad \text{on } \partial \Omega \times [0, \infty)\)

and assume that also

\(u_0 \geq c, \quad g \geq q^*c + M_zM_{q^*}.\)

From this and the fact that then \(\hat{z} = z - c\) solves

\[
\begin{align*}
D_t \hat{z} - \Delta \hat{z} + q^*\hat{z} &= g - q^*c - \phi(z)\psi(q^*) \geq 0 \quad \text{in } \Omega \times [0, T) \\
\hat{z} &= h - c \geq 0 \quad \text{on } \partial \Omega \times [0, \infty) \\
\hat{z}(0) &= u_0 - c \geq 0 \quad \text{in } \Omega \times \{0\}
\end{align*}
\]

we conclude that \(\hat{z} \geq 0\), i.e., \(z \geq c\). Since \(z \in L^\infty([0, T) \times \Omega), \forall T > 0\) as claimed in Proposition 3.1, it makes sense to have \(z \geq c\) on \([0, T), \forall T\) hence \(z \geq c\) on \([0, \infty)\).

In order to work with homogeneous boundary conditions in the following, we assume that there exists an extension \(\tilde{h} \in \mathcal{Z}\) of \(h\) and replace \(u\) by \(\tilde{u} = u - \tilde{h}\), \(u_0\) by \(\tilde{u}_0 = u_0 - \tilde{h}(0)\), \(g\) by \(\tilde{g} := g - D_t\tilde{h} + \Delta\tilde{h}\), \(z\) by \(\tilde{z} = z - \tilde{h}\); after skipping the tildes again, the model becomes

\[
\begin{align*}
D_t u - \Delta u + q(u + \tilde{h}) + \phi(u + \tilde{h})\psi(q) &= g \quad \text{in } \Omega \times [0, \infty) \tag{56} \\
u &= 0 \quad \text{on } \partial \Omega \times [0, \infty) \tag{57} \\
u(0) &= u_0 \quad \text{in } \Omega \times \{0\} \tag{58}
\end{align*}
\]
and the positivity condition on the exact state and its value under \( \phi \) read as
\[
z + \bar{h} \geq \varepsilon > 0 , \quad \phi(z + \bar{h}) \geq 0
\]

Thus we can use the spaces
\[
V = H^1_0(\Omega), \quad X = H^1(\Omega), \quad H = L^2(\Omega)
\]
and set
\[
f(q, u) = -\Delta u + q(u + \bar{h}) + \phi(u + \bar{h})\psi(q)
\]
We verify Assumption [1.1]

(A3) **Coercivity of \( f \)**

\[
\langle f(q, z) - f(q^*, z), q - q^* \rangle_{X^*, X} \\
= \langle (z + \bar{h})(q - q^*), q - q^* \rangle_{X^*, X} + \langle \phi(z + \bar{h})(\psi(q) - \psi(q^*)), q - q^* \rangle_{X^*, X} \\
\geq \varepsilon \| q - q^* \|^2_H =: C_{\text{coec}} \| q - q^* \|^2_H.
\]
Here, we invoke positivity of \( z + \bar{h} \) and monotonicity of \( \phi(z + \bar{h})\psi \).

(A2) **Local Lipschitz continuity of \( f \)**
We observe that, with a constant \( C \) depending on \( M_z, C_\psi, \bar{h}, \) and \( \Omega \)
\[
\langle f(q, z) - f(q^*, z), v \rangle_{V^*, V} \\
= \langle \phi(z + \bar{h})(\psi(q) - \psi(q^*) - \psi'(q^*)(q - q^*)), v \rangle_{V^*, V} \\
= \left< \phi(z + \bar{h}) \left[ \int_0^1 (\psi'(q^* + \lambda(q - q^*)) - \psi'(q^*)) \, d\lambda (q - q^*) \right], v \right>_{V^*, V} \\
\leq CM_{z+h}\| v \|_{L^2} \| q - q^* \|_{L^2} \| 1 + |q|^\beta - 1 + |q^*|^\beta - 1 + |q^0|^\beta - 1 \|_{L^3} \\
\leq CH^{1-\delta}CM_{z+h}\| v \|_{H^1} \| q - q^* \|_{L^2} (1 + \| q \|_{L^\beta(\delta - 1)}^{\beta - 1} + \| q^* \|_{L^\beta(\delta - 1)}^{\beta - 1} + \| q^0 \|_{L^{3(2-\alpha)}}^{\beta - 1}) \\
=: L^{\alpha, z}(\| q \|_H) \| q - q^* \|_H \| v \|_V
\]
provided that \( 1 \leq \beta \leq 5/3 \).
An example for this is \( \alpha = 3, \beta = 5/3 \), see the remark below.

(A4) **Choosing the linear operator \( \mathcal{C} \)**
Taking \( \mathcal{C}(\| q \|_H)v = \left( L^{q^*, z}(\| q \|_H) + 1 \right)(-\Delta)v \), then (A4) is fulfilled.

**Remark 3.2.** One of the feasible choices is \( \phi(z)\psi(q^*) = z^3|q^*|^{\frac{3}{2}}q^* \) with the exact coefficient \( q^* \) and exact state \( z \) being nonnegative.
Remark 3.3. The Lipschitz constant $L^{q_0,z}$ here depends on $M_z$, where $M_z$ depends on $T$ as in shown in \[52\], thus we have $L^{q_0,z} = L^{q_0,z,T}$. This leads to, in assumption [(A4)], $\tilde{\mathcal{M}} = \tilde{\mathcal{M}}^T$, $\tilde{N} = \tilde{N}^T$, but still $C_{\text{co}} = C$ being independent of $T$ as $z \geq C$ on $[0, \infty)$. Therefore, comparing to Proposition 2.1 we obtain for this example

$$ u \in L^2([0, \infty); V) \cap H^1([0, T); V^*) \cap L^\infty([0, \infty); H) \quad \forall T > 0 $$

$$ q \in L^2([0, \infty); H) \cap H^1([0, T); X^*) \cap L^\infty([0, \infty); H) \quad \forall T > 0. $$

Remark 3.4. As visible from the estimate above for (A3) coercivity could as well be achieved by imposing strict positivity of $\phi(z + \bar{h})$ (bounded away from zero) and uniform monotonicity of $\psi$.

Remark 3.5. Smallness of $L^{q_0,z} (\|q\|_H)$ could be achieved by assuming closeness of $q(t)$ and $q^0$ to $q^*$ (as for $q(t)$, this can be bootstrapped from the exponential decay estimate (23)). This can be seen by rewriting $\psi'(q^* + \lambda(q - q^*)) - \psi'(q^0) = \int_0^1 \psi''(q^0 + s(q^* + \lambda(q - q^*)) - q^0)) (q^* + \lambda(q - q^*) - q^0) ds$, in the above estimate for (A2), which under a growth assumption $|\psi''(q)| \leq C_{\psi}(1 + |q|^{\beta-2})$ gives an estimate that is qualitatively similar to the above one (including the conditions on $\alpha, \beta$), but with a constant that can be made small for $q(t)$ and $q^0$ close to $q^*$. Preventing $L^{q_0,z} (\|q\|_H)$ and therewith the factor in the definition of $\mathcal{C}(\|q\|_H)$ from getting too large makes sense in order to avoid that the system becomes very stiff, which might lead to high computational costs in simulation over a large time horizon.

3.2 Identification of a diffusion coefficient – a problem with non-linear perturbation

Let us study the problem

\begin{align}
D_t u - \nabla \cdot (q \nabla u) + \bar{f}(u, q) &= g \text{ in } \Omega \times [0, \infty) \quad (59) \\
u(0) &= u_0 \text{ in } \Omega \times \{0\} \quad (60)
\end{align}

on a smooth bounded domain $\Omega \subseteq \mathbb{R}^3$ under the assumptions

\[ \bar{f}(u, q) := u \Delta q + \phi(u) \psi(q) \]

the exact state $z(t, x) \leq -C < 0 \quad \forall x \in \Omega, \ t > 0,$

$\phi(z) \psi(\cdot)$ is monotone, and $|\psi'(q)| \leq C_{\psi}(1 + |q|^{\beta-1})$

To achieve boundedness away from zero of $z$ by means of a maximum principle \[19\] Lemma 2.1, Chapter 2 we impose inhomogeneous Dirichlet boundary conditions

$$ u = h \leq 0 \text{ on } \partial\Omega \times [0, \infty) $$

and assume that also

$$ u_0 \leq -C, \quad g \leq -C \Delta q^* - M_z M_q^*, \quad \Delta q^* \in L^\infty(\Omega). $$
This and the fact that \( \hat{z} = z + \zeta \) solves
\[
D_t \hat{z} - \nabla \cdot (q^* \nabla z) + \hat{z} \Delta q^* = g + \zeta \Delta q^* - \phi(z)\psi(q^*) \leq 0 \text{ in } \Omega \times [0, \infty)
\]
\[
\hat{z} = h + \zeta \leq 0 \text{ on } \partial \Omega \times [0, \infty)
\]
\[
\hat{z}(0) = u_0 + \zeta \leq 0 \text{ in } \Omega \times \{0\}
\]
enable us to conclude \( \hat{z} \leq 0 \), i.e., \( z \leq -\zeta \leq 0 \). Similar to the c-problem, to work with homogeneous boundary conditions we assume existence of an extension \( \tilde{h} \in Z \) of \( h \) and replace \( u \) by \( \tilde{u} = u - \tilde{h} \), \( u_0 \) by \( \tilde{u}_0 = u_0 - \tilde{h}(0) \), \( g \) by \( \tilde{g} := g - D_t \tilde{h} + \nabla \cdot (q \nabla \tilde{h}) - \tilde{h} \Delta q, \) \( z \) by \( \tilde{z} = z - \tilde{h} \); after skipping the tildes, the model becomes
\[
D_t u - \nabla \cdot (q \nabla u) + u \Delta q + \phi(u + \bar{h}) \psi(q) = g \text{ in } \Omega \times [0, \infty) \quad (61)
\]
\[
u = 0 \text{ on } \partial \Omega \times [0, \infty) \quad (62)
\]
\[
u(0) = u_0 \text{ in } \Omega \times \{0\} \quad (63)
\]

According to the unique existence theory for linear parabolic PDEs, the diffusion coefficient must be positive; together with the continuous embedding \( H^2(\Omega) \hookrightarrow L^\infty(\Omega) \), we chose the spaces
\[
V = H^1_0(\Omega), \quad X = H^1_0(\Omega) \cap H^2(\Omega), \quad H = L^2(\Omega)
\]
and denote the intermediate space \( \bar{X} : = H^1_0(\Omega) \) then \( X \subset \bar{X} \subset V \).
We then set
\[
f(q, u) = -\nabla \cdot (q \nabla u) + u \Delta q + \phi(u + \bar{h}) \psi(q).
\]
\( f \) is well-defined, since is it shown in \( \text{(17)} \) that for \( q \in H^2(\Omega) \), thus \( \Delta q \in L^2(\Omega) \), we have \( f(q, u) \in V^* \).

We now verify Assumption \( \text{(A3)} \) Coercivity of \( f \)
\[
\langle f(q, z) - f(q^*, z), q - q^* \rangle_{X^*, X}
\]
\[
= \langle -\nabla \cdot ((q - q^*) \nabla z) + z \Delta (q - q^*), q - q^* \rangle_{X^*, X} + \langle \phi(z + \bar{h})(\psi(q) - \psi(q^*)), q - q^* \rangle_{X^*, X}
\]
\[
\geq \int_\Omega -\nabla \cdot ((q - q^*) \nabla z)(q - q^*) \, dx + \int_\Omega z\Delta(q - q^*)(q - q^*) \, dx \quad (64)
\]
\[
= \int_\Omega (q - q^*) \nabla z \nabla (q - q^*) \, dx + \int_\Omega z\Delta(q - q^*)(q - q^*) \, dx
\]
\[
= -\int_\Omega z \nabla \cdot ((q - q^*) \nabla (q - q^*)) \, dx + \int_\Omega z\Delta(q - q^*)(q - q^*) \, dx
\]
\[
= -\int_\Omega z |\nabla (q - q^*)|^2 \, dx
\]
\[
=: C_{coe} \| q - q^* \|^2_{X^*}.
\]
Above, we firstly invoke monotonicity of $\phi(z)\psi$ then apply integration by parts with taking into account $q, q^* \in H^1_0(\Omega)$.

As noticed in Remark 1.3, achieving the $\tilde{X}$-norm here allows us to estimate in Assumption (A2) the quantity $(q - q^*)$ with this strong norm. However, for the Lipschitz constant $L$ we need to stay with the weak norm $\|q\|_H$ since in (25)-(26), $L(\|q\|_{L^\infty([0,\infty);H)})$ is required; and this uniformly bounded in time property is only attainable for $\|q(t)\|_H$ as proven in (23) or (24).

(A2) Local Lipschitz continuity of $f$

With a constant $C$ depending on $C_\phi, C_\psi, \Omega$, we see

$$
\langle f(q, z) - f(q^*, z) - f_q(q^0, z)(q - q^*), v \rangle_{V^*, V} = \langle \phi(z + \bar{h})(\psi(q) - \psi(q^*) - \psi'(q^0)(q - q^*)), v \rangle_{V^*, V}
$$

$$
= \left\langle \phi(z + \bar{h}) \left[ \int_0^1 (\psi'(q^* + \lambda(q - q^*)) - \psi'(q^0)) d\lambda(q - q^*) \right], v \right\rangle_{V^*, V}
\leq CM_{z+\bar{h}}\|v\|_{L^6}\|q - q^*\|_{L^6} \|1 + |q|^\beta - 1 + |q^*|^\beta - 1 + |q_0|^\beta - 1\|_{L^{\frac{2}{\beta}}}
\leq (C_{H^1 \to L^6})^2 CM_{z+\bar{h}}\|v\|_{H^1}\|q - q^*\|_{H^0_0}(1 + \|z\|_{H^1}) \left( 1 + \|q\|_{L^{\frac{6(\beta-1)}{2(\beta-1)}}} + \|q^*\|_{L^{\frac{6(\beta-1)}{2(\beta-1)}}} + \|q_0\|_{L^{\frac{6(\beta-1)}{2(\beta-1)}}} \right)
=: L^{\frac{6}{\beta}, z}(\|q\|_H)\|q - q^*\|_{\tilde{X}}\|v\|_V,
$$

provided that $1 \leq \beta \leq 7/3$.

An example for this is $\alpha = 3, \beta = 7/3$, see the remark below.

(A4) Choosing the linear operator $C$

Taking $C(\|q\|_H)v = \left( L^{\frac{6}{\beta}, z}(\|q\|_H) + 1 \right) (-\Delta)v$, then (A4) is fulfilled.

Remark 3.6. One of the feasible choices is $\phi(z)\psi(q^*) = z^3 - z|q^*|^\frac{2}{3}q^*$ with the exact coefficient $q^*$ being nonnegative whose derivative $\Delta q^*$ being bounded, and the exact state $z$ being nonpositive. The term $z^3$ is monotone w.r.t to $z$, and the term $-z|q^*|^\frac{4}{3}q^*$ is monotone w.r.t $q^*$ as $z < 0$. In addition, $-z|q^*|^\frac{4}{3}q^*$ is linear in $z$ hence plays the role of $cz$ with $c := -|q^*|^\frac{4}{3}q^* \in L^2(\Omega)$ for $q^* \in X$, then monotonicity in $z$ is not required.

Remark 3.7. Similar to Remark 3.3 and together with the involvement of $\tilde{X}$, we obtain for this example

$$
u \in L^2([0, \infty); V) \cap H^1([0, T); V^* ) \cap L^\infty([0,\infty); H) \quad \forall T > 0
$$

$$
q \in L^2([0, \infty); \tilde{X}) \cap H^1([0, T); (\tilde{X})^* ) \cap L^\infty([0,\infty); H) \quad \forall T > 0.
$$
4 Conclusions and outlook

In this paper, we have proposed and analyzed an online parameter identification method for problems governed by nonlinear time-dependent PDEs. Our approach introduces a dynamic update law for both the state and the parameter via a model reference adaptive system. This system contains a linear PDE for the state and an auxiliary nonlinear one for the stationary parameter. Under suitable structural assumptions on operators, unique existence results for this nonlinear adaptive system were given. In addition, by evaluating the error system, we proved that the solution to the adaptive system converges to the exact state and parameter for both exact as well as noisy data. Our key contribution is that we tackle the case where the system of PDEs is not only nonlinear with respect to the state, but also nonlinear with respect to the parameters; this situation has not been investigated thus far.

There are several future directions which could extend the study we have presented. Numerical experiments to illustrate the performance of our scheme are yet to be provided, and would be a natural continuation of this work. In the present work, we restrict ourselves to the case where the full state is measured. A modified scheme for the practically relevant situation of partial state observation is highly desirable. On the other hand, an extension of our result in the framework of time dependent parameters is also an appealing topic.

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