Wilson fermions with chirally twisted mass

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Lattice formulations of QCD with Wilson fermions and a chirally twisted quark mass matrix provide an attractive framework for non-perturbative numerical studies. Owing to reparameterization invariance, the limiting continuum theory is just QCD. No spurious quark zero modes, which are responsible for the problem with exceptional configurations, can occur at finite values of the quark mass. Moreover, the details of the lattice formulation can be adjusted so as to simplify the renormalization and the O($a$) improvement of several quantities of phenomenological relevance. The first exploratory studies in the quenched approximation yield very encouraging results.

1. Introduction

In recent years a lot of progress has been achieved about lattice regularizations of gauge theories with fermions [1]. On one hand, local (but non–ultralocal) fermionic actions that enjoy a lattice form of chiral symmetry and (almost) ideal renormalization properties have been discovered and put at work [2]: for all of them the critical Dirac operator satisfies the celebrated Ginsparg-Wilson relation [3]. On the other hand, more traditional formulations of lattice QCD (LQCD) based on Wilson and staggered quarks have been refined and widely used in realistic computations of hadronic observables and matrix elements. The simplest of these computations are currently performed with dynamical quarks.

In this contribution, I report on recent developments about Wilson fermions, which remove practical obstructions in dealing with light quarks and simplify the renormalization and O($a$) improvement of a number of observables. Improvements in the formulations of LQCD with staggered quarks are discussed elsewhere [4].

After shortly recalling the status of LQCD with Wilson quarks and its problem with exceptional configurations (sect. 2), I introduce –for the case of $N_f=2$ flavours– the formulation with chirally twisted mass (sect. 3). Then I discuss its basic properties (sect. 4), the first non-perturbative studies (sect. 5) and analogous lattice formulations of QCD with $N_f>2$ quark flavours, which can simplify the renormalization of some operators of the effective weak Hamiltonian (sect. 6).

2. Wilson fermions

The well known lattice formulation introduced by Wilson [5] provides a gauge invariant regularization for QCD with any number of quark flavours: the action is ultralocal and respects the global flavour symmetries of QCD, but all axial symmetries are broken by the Wilson term. This is no principle problem, as the flavour chiral invariance $SU(N_f) \otimes SU(N_f)$ can be restored in the (quantum) continuum limit [6], while the axial $U(1)$ invariance of the classical continuum theory is broken by quantum fluctuations.

The lack of chiral symmetry entails however some practically important consequences. First, complicated patterns of operator mixings arise, so that in many cases several operator subtractions are necessary.

References:

[1] Lattice formulation...

[2] More recent works...

[3] Ginsparg-Wilson relation...

[4] Further developments...

[5] Wilson formulation...

[6] Continuum limit...

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[1] Lattice formulation...

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[4] Further developments...

[5] Wilson formulation...

[6] Continuum limit...
are needed in order to restore the chiral multiplet structure. Then, the leading deviations of renormalized quantities from their continuum limit values are linear in the lattice spacing $a$ and typically non-negligible. Last but not least, the absence of a lower bound on the norm of the eigenvalues of the massive Dirac matrix may lead to unphysical divergences in fermionic observables on non-trivial gauge backgrounds.

2.1. $O(a)$ improved Wilson fermions

The problem with the leading cutoff effects has found a clean solution via Symanzik’s improvement programme \cite{7, 8, 9, 10, 11}, which allows to define and compute on-shell renormalized quantities with leading cutoff effects of order $O(a^2)$, though at the price of tuning the coefficients of further counterterms\footnote{Implementing this programme may be non-trivial and CPU-time demanding, e.g. in the theory with non-degenerate and/or dynamical quarks as well as for operators with complicated mixings \cite{12}.}. The LQCD action for the $O(a)$ improved theory with $N_f$ quark flavours reads:

$$S_k[U; g_0^2] + a^4 \sum_x \bar{\psi}(x) \left[ (D^c_W[U] + M_q) \psi \right] (x)$$

where $S_k[U; g_0^2]$ is the pure gauge action with coupling $g_0^2$, $M_q = \text{diag}(m_q^u, m_q^d, m_q^c, \ldots)$ is the subtracted ($M_q = M_0 - m_c$) quark mass matrix and

$$D^c_W[U] = \left\{ \gamma \nabla - \frac{2}{g^2} \gamma \nabla^* \gamma + c_{sw} \frac{g^2}{4} i \sigma \tilde{F} \right\} [U] + m_c$$

represents the critical Wilson–Dirac operator. Omitting Lorentz indices, we denote by $\nabla$ ($\nabla^*$, $\tilde{\nabla}$) the forward (backward, symmetrized) covariant lattice derivative and by $\tilde{F}_{\mu\nu}$ the clover lattice discretization of $F_{\mu\nu}$. The coefficients $c_{sw}$ and $am_c$ are dimensionless functions\footnote{At the non-perturbative level, $c_{sw}$ and $am_c$ are uniquely defined only up to $O(a)$ and $O(a^3)$ corrections \cite{11}, respectively.} of $g_0^2$ and $N_f$.

2.2. Exceptional configurations

In quenched simulations of LQCD on relatively coarse lattices and for moderately light quark flavours, gauge configurations can be sampled where the Wilson–Dirac matrix has one or few eigenvalues with norm exceptionally close to zero, i.e. much smaller than on the other configurations. The corresponding eigenvectors of the Wilson–Dirac matrix are referred to as “spurious” quark zero modes, because in a chiral invariant formulation of QCD the Dirac matrix can have zero modes only if some quark flavour is massless.

On gauge configurations with spurious quark zero modes fermionic observables may undergo fluctuations that exceed in modulus the typical ones by orders of magnitudes: in this sense, these configurations appear to be “exceptional”. Moreover, increasing the statistics does not reduce in general the standard deviation of the observables, as further, even larger fluctuations may occur. A reliable statistical analysis of the fermionic observable becomes hence impossible \cite{11, 13}. When employing the non-perturbatively $O(a)$ improved \cite{11} Wilson action, eq. (1), and the plaquette gauge action, this problem is felt at values of $m_q$ of about one half the value of the strange quark’s mass if $a \simeq 0.1$ fm \cite{14}. With the improved Wilson action ($c_{sw} = 0$) quenched computations can be pushed to somewhat lower values of the quark mass, but cutoff effects are harder to control \cite{13}.

In unquenched simulations, the importance sampling must give spurious quark zero modes a vanishingly small probability. State-of-the-art algorithms, however, account for the effects of sea quarks in a stochastic way: it is hence possible that configurations with nearly zero-modes of the Dirac matrix are produced in the updating process (e.g. a HMC trajectory). This would then result in an exceptionally low acceptance rate\footnote{Exceptionally large values of the product ‘driving force times MD integration step” have been observed \cite{14} in simulations with the HMC algorithm. The eigenvalues of the Dirac matrix were not simultaneously evaluated.}, which can possibly be avoided at the price of slowing down the algorithm. In the case of partially (un)quenched simulations, the problem with spurious valence quark zero modes is only alleviated: exceptional fluctuations of hadron correlators have indeed been observed \cite{15}.

A typical example of exceptional configurations is given in Fig. 1. There we plot versus the number $N$ of independent measurements the relative

\begin{align*}
\text{Ward identities to hold up to cutoff effects} & \text{[1].} \\
\text{Optional note: Implementing this programme may be non-trivial and CPU-time demanding, e.g. in the theory with non-degenerate and/or dynamical quarks as well as for operators with complicated mixings [12].} \\
\text{Footnote: At the non-perturbative level, $c_{sw}$ and $am_c$ are uniquely defined only up to $O(a)$ and $O(a^3)$ corrections [11], respectively.} \\
\text{Footnote: Exceptionally large values of the product ‘driving force times MD integration step” have been observed [14] in simulations with the HMC algorithm. The eigenvalues of the Dirac matrix were not simultaneously evaluated.}
\end{align*}
standard deviation, multiplied by $N^{1/2}$, of a pion channel correlator at fixed time separation: for the statistical analysis to be reliable, this quantity should approach a constant value as $N \gg 1$. Our example refers to a quenched simulation at $\beta = 6$ ($a \simeq 0.1$ fm) on a $24^3 \cdot 48$ lattice, with degenerate quark masses such that $m_{PS}/m_{V} \simeq 0.47$; employing the non-perturbatively $O(a)$ improved action \footnote{The full statistics was of $N = 260$ measurements \cite{18}.} leads to unacceptably large and rare fluctuations of the observable (empty symbols). On the other hand, one can work at equivalent parameters in a formulation based on Wilson quarks with chirally twisted mass, see sect. 3: of course, the bare quark mass parameters have to be adjusted so as to match the value of the renormalized current quark mass ($m'_{q}$ in the notation of sect. 4). In this case the occurrence of spurious quark zero modes is avoided: as shown in Fig. 1 (filled symbols), there are no exceptional fluctuations in the MC history of the pion channel correlator, so that a reliable statistical result \footnote{In general, $a\bar{m}_{c}$ and $c_{SW}$ may differ from $a\bar{m}_{c}$ and $c_{SW}$.} can be obtained.

3. Chirally twisted “mass”

The presence of chirally twisted “mass” terms is a general feature of lattice gauge theories with Wilson fermions, as soon as the “physical” fermion mass term is not aligned in chiral space to the Wilson term. Of course, the latter generates itself a “mass” term, whose divergent part requires appropriate subtraction.

\footnote{The matrix $\tau^{3}$ in $D_{SW}^{c}$ acts in flavour space.}

3.1. Twisted mass and parity

To the best of my knowledge, Osterwalder and Seiler first proposed \cite{19} a lattice formulation where the fermionic action is of the form

\[ S_{f}^{(q)} = \sum_{x} \bar{\psi}(x) [ (D_{\phi}^{c} + \tilde{m}) \psi^{c} ](x), \]

\footnote{The matrix $\tau^{3}$ in $D_{SW}^{c}$ acts in flavour space.}

where the fermionic action is of the form

\[ D_{\phi}^{c} = \gamma \nabla + e^{-i\phi \gamma_{5}} [ \tilde{m}_{c} - \frac{2}{3} \gamma \nabla + c_{SW} \frac{2}{3} i \sigma \hat{F} ], \]　(3)

\[ D_{SW}^{c} = \gamma \nabla + e^{-i\phi \gamma_{5} \tau^{3}} [ m_{c} - \frac{2}{3} \gamma \nabla + c_{SW} \frac{2}{3} i \sigma \hat{F} ], \]　(5)

\[ q = e^{-i\omega \gamma_{5} \tau^{3}/2} q', \quad \bar{q} = q' e^{-i\omega \gamma_{5} \tau^{3}/2}, \]　(6)

where $q'$ is a quark doublet field, the quark mass term ($\propto m'_{q}$) takes the usual form, but the critical Wilson-Dirac matrix is chirally twisted\footnote{The matrix $\tau^{3}$ in $D_{SW}^{c}$ acts in flavour space.}

\[ D_{SW}^{c} = \gamma \nabla + e^{-i\phi \gamma_{5} \tau^{3}} [ m_{c} - \frac{2}{3} \gamma \nabla + c_{SW} \frac{2}{3} i \sigma \hat{F} ]. \]　(5)
parameters are $\mu_q$ and $m_0$, as the $m_c$–dependence cancels between $m_q$ and $D_W^c$.

The formulation with fermionic action (3) or (6) is referred to as lattice twisted mass QCD (LtmQCD). At the classical level, it obviously represents a regularization of QCD with $N_f = 2$ (mass–degenerate) flavours. This property holds true at the quantum level [24], see sect. 4.

3.2. Protection against exceptions

The lattice theory with fermionic action (6) has been studied (for $c_{sw} = 0$) to establish whether, at finite $g_s^2$ and for values of $am_0$ in a certain range, parity and isospin can be spontaneously broken as conjectured by Aoki [22]. It was noted [23] that the lattice Dirac matrix corresponding to eq. (6) cannot be singular on any gauge background as long as $\mu_q \neq 0$, since

$$0 < \det[D_W^c + m_q + i\mu_q\gamma^3]\gamma^3] =$$

$$= \det[(D_W^c + m_q)(D_W^c + m_q) + \mu_q^2].$$  \hspace{1cm} (8)

Here $\det[...]$ (det[...]) denotes the fermionic determinant in the two-flavour (one-flavour) space.

The one-flavour Dirac matrix corresponding to the action (3) is also not singular on any gauge configurations, as long as $\tilde{m}\sin\phi \neq 0$. This is because $|\det[D^c + \tilde{m}]|^2$ can be written in the same form as the r.h.s. of eq. (6) with $m_q \Rightarrow \tilde{m}\cos\phi$ and $\mu_q \Rightarrow \tilde{m}\sin\phi$. Based on this property, the authors of Ref. [23] proposed to employ the fermionic lattice action (3) with $\tilde{m}_c = m_c$, $\tilde{c}_{sw} = c_{sw}$, $\pi/2 \geq |\phi| > 0$ and $\tilde{m} > 0$ to avoid the occurrence of spurious quark zero modes in massive LQCD with Wilson quarks. They presented numerical evidence, see Fig. 3 of Ref. [24], that exceptional configurations are avoided in the quenched approximation and discussed on a semi-classical level the relation between their lattice formulation and QCD in the continuum limit. It should be noted that in the quenched model the parity breaking inherent to the formulation with action (3) reduces to a mere $O(a)$ effect on renormalized quantities, but this is no longer true with (partially) unquenched quarks.

4. Basic properties of lattice tmQCD

I discuss here the properties of LtmQCD as an ultraviolet regularization of QCD with $N_f = 2$ mass-degenerate quark flavours\(^9\).

4.1. Symmetries and renormalizability

The LtmQCD action, see eq. (4) or (7), is invariant under lattice gauge transformations and translations, axis permutations, charge conjugation and the global symmetry $U(1)$ corresponding to conservation of the fermionic number. At $\mu_q \neq 0 \Leftrightarrow \omega = \arctan(m_q/m_0) \neq 0$, isospin symmetry is reduced to a $U(1)$–invariance with generator $\tau^3/2$ and axis reflections, such as parity, are no longer a symmetry. It is important to note the residual invariance $P_k$,

$$U_0(x) \rightarrow U_0(x_P), \quad U_0(x) \rightarrow U_0^{-1}(x_P - a\hat{k}),$$

$$q(x) \rightarrow \tau^1\gamma_0q(x_P), \quad \bar{q}(x) \rightarrow \bar{q}(x_P)\gamma_0\tau^1,$$  \hspace{1cm} (9)

where $x_P = (x_0, -x)$, as this symmetry rules out a term $\propto \text{tr}(FF^\dagger)$ in the quantum effective action.

A standard analysis [21] based on lattice symmetries and power counting shows that the model is renormalizable. The relations between bare and renormalized action parameters take in general the form (I recall $m_a = m_0 - m_c$):

$$g_R^2 = Z_g(g_0^2, am_m, am_\mu; am_\mu) g_0^2,$$

$$\mu_R = Z_\mu(g_0^2, am_m, am_\mu; am_\mu) m_\mu,$$

$$m_R = Z_m(g_0^2, am_m, am_\mu; am_\mu) m_\mu,$$  \hspace{1cm} (10)

but the renormalization factors $Z$ can be chosen to be independent of $am_m$ and $am_\mu$ (mass independent schemes). The additive renormalization of $m_0$ is independent of the quark mass parameters [1, 22, 24]: $m_c = m_c(g_0^2)$, up to intrinsic $O(a)$ corrections in the case of non-perturbative determinations of $m_c$. The ratio $Z_m/Z_\mu$ does not depend on the subtraction scale $\mu$, consistently with recovery of flavour chiral symmetry in the continuum limit. Noting the bare lattice identity

$$\partial^a \bar{q}^b = -2\mu_q e^{3bc} P^c,$$  \hspace{1cm} (11)

\(^9\)In this context $\mu_q$ played the role of an external zero-momentum source, which can orient the vacuum in a certain direction (in flavour chiral space), and the main interest was on the properties of the vacuum after taking the thermodynamic and zero–source limits.

\(^{10}\)A satisfactory regularization of $N_f = 2$ QCD with bare quark masses $m'_q \pm \delta m'_q$ can be obtained e.g. by adding to the action density in eq. (4) a term $\bar{q}'(x)\tau^3\delta m'_q q'(x)$.
where $\tilde{V}_\mu^b$ denotes the one-point split vector current, it is easy to argue \cite{21} that $Z_\mu Z_T = 1$

4.2. Continuum limit and cutoff effects

Although the physical interpretation of the fermionic correlation functions is most transparent in the quark basis corresponding to eq. (3), the renormalization of gauge–invariant correlation functions, including those with insertions of local operators, looks simpler in the quark basis corresponding to eq. (4). In this basis, indeed, the critical Wilson–Dirac matrix is given by $D_W^c$, and the mixing properties of the operators in the massless theory ($\mu_q = 0, m_0 = m_c$) are usually well known: the construction of lattice fields that are multiplicatively renormalizable and respect, up to cutoff effects, the chiral multiplet structure is hence straightforward. Concerning the overall, possibly scale–dependent renormalization factors of the various operator multiplets, we assume for simplicity \cite{21} that they, as well as $Z_R$, $Z_m$ and $Z_\mu$, are chosen to be independent of $\omega$.

In the quark basis of choice, the Ward identities of flavour chiral symmetry read (for $b = 1, 2, 3$):

\[ \tilde{\partial}_\mu (A_R)_{\mu}^b \simeq 2m_R (P_R)^b + i\mu_R \delta^{3bc} (S_R)^0, \]

\[ \tilde{\partial}_\mu (V_R)_{\mu}^b \simeq -2\mu_R \epsilon^{3bc} (P_R)^c, \]

where the symbol $\simeq$ means equality up to $O(\alpha)$ corrections and renormalized operators are defined as usual with Wilson fermions\footnote{For $b = 3$ severe power-like divergences must be subtracted to define $(S_R)^6$, which is no problem in perturbation theory but is delicate at the non-perturbative level.} e.g.

\[ \{ V_R, A_R \}_{\mu}^b = \bar{q} \{ Z_{\gamma \mu}, Z_{A \gamma \mu} \gamma_5 \} \frac{1}{2} \gamma^b q. \]

(13)

The form of eq. (12) reminds us that appropriate linear combinations of renormalized operators, reflecting the formal change of variables \cite{2}, should be taken, in order to obtain in the continuum limit operators with well defined parity and isospin properties. Exploiting eq. (12) one can indeed extract $m_R$ and $\mu_R$ and introduce the renormalized counterparts of $m_q'$ and $\omega$:

\[ m_R' = \sqrt{m_R^2 + \mu_R^2}, \quad \alpha = \arctan \left( \frac{\mu_R}{m_R} \right). \]

(14)

I remark that $\tan(\alpha) = (Z_\mu/Z_m) \tan(\omega)$ and $m_R^T = Z_m (g_8^2 \gamma_5) m_q'$. Then, if one considers (with\footnote{The definitions \cite{2} are special cases of those in eq. (12).\footnote{Phenomena like a possible Aoki phase \cite{22, 28}, are im-}

$\bar{b} = 1, 2$) the renormalized operators

\[ (V_R')_{\mu}^b \equiv (V_R)^3, \quad (A_R')_{\mu}^b \equiv (A_R)^3, \]

\[ (V_R'')_{\mu}^b \equiv \cos(\alpha) (V_R)^b + \epsilon^{3bc} \sin(\alpha) (A_R)^c, \]

\[ (A_R'')_{\mu}^b \equiv \cos(\alpha) (A_R)^b + \epsilon^{3bc} \sin(\alpha) (V_R)^c, \]

\[ (P_R')^3 \equiv \cos(\alpha) (P_R)^3 + \frac{1}{2} \sin(\alpha) (S_R)^0, \]

\[ (P_R'')^b \equiv (P_R)^b, \]

the Ward identities of flavour chiral symmetry take the usual form \cite{21}:

\[ \tilde{\partial}_\mu (A_R')_{\mu}^b \simeq 2m_R' (P_R')^b, \quad \tilde{\partial}_\mu (V_R')_{\mu}^b \simeq 0. \]

(15)

This result reflects the existence of a linear mapping among renormalized correlation functions computed with action parameters that correspond to different values of $\alpha$ and identical values of $g_8^2$ and $m_R'$. The mapping between correlators of gauge invariant fields with unequal space–time arguments at $\alpha = \tilde{\alpha}$ and $\alpha = 0$ reads\footnote{Phenomena like a possible Aoki phase \cite{22, 28}, are im-}

\[ \langle \phi_{kR}^{(r)} (x) \ldots \rangle_{\mu_R^2, \mu_R, \tilde{\alpha}} = \langle \phi_{kR}^{(r)} (x) \ldots \rangle_{\mu_R^2, \mu_R, 0}, \quad \phi_{kR}^{(r)} (x) = (R_{kR}^{(l)} (\tilde{\alpha}) \phi_{kR}^{(r)} (x). \]

(17)

Here $\phi_{kR}^{(r)}$ denotes the $k$-th field component of the renormalized chiral multiplet $r$ and $R^{(r)}(\omega)$ is the (formal) multiplet transformation matrix under quark transformations of the type (4). The dots in (17) stand for further local fields, with those on the l.h.s. being related to those on the r.h.s. by the appropriate product of chiral transformation matrices $R$. The relations (17) are regularization–independent properties and imply the recovery of parity, isospin and chirality in the continuum limit of LtmQCD \cite{21}. Moreover, it is useful to note that they look just as one would guess from simple formal arguments.

The full information on QCD with two mass–degenerate quark flavours can hence be obtained by computing correlation functions with the action (4) or (6). It is understood that the continuum limit must be approached at fixed values of the renormalized parameters, including $\alpha$. In this way, the cutoff effects inherent to any determination of $\alpha$ at finite lattice spacing have no impact on the (extrapolated) continuum limit results.
4.2.1. Simplified operator mixings

The regularization of QCD with fermionic action \( \text{(4)} \) differs from Wilson’s (improved) one as long as the quarks are massive. In the massless theory, \( \mu_q = 0 \) and \( m_0 = m_c \), any difference disappears, implying that the pattern of leading operator mixings is globally the same as in the original formulation by Wilson.

However, given a certain fermionic field, its physical interpretation depends on how it transforms under the parity and isospin operations, whose form is in turn dictated by the parameterization of the quark mass term. On the other hand, the mixing properties of the fermionic field depend crucially on its chiral orientation with respect to the Wilson term. Hence, for a specific fermionic field with a given physical interpretation (e.g. the physical axial current), the mixing properties may change, even at the leading level, with \( \omega \) (or \( \alpha \)). For instance, eq. \( \text{(15)} \) shows that the component \( (a'_{\mu R})_\mu \) of the physical non-singlet axial current is given by \( (A_R)_\mu \) if \( \alpha = \omega = 0 \) and by \( (V_R)_\mu \) if \( \alpha = \omega = \pi/2 \). The currents \( (A_R)_\mu \) and \( (V_R)_\mu \), which are defined in eq. \( \text{(3)} \), involve different renormalization factors \( Z_A \) and \( Z_V \), with \( Z_V = 1 \) if the one–point split definition of the bare current \( V_\mu^2 \) is adopted.

One can then adjust the ultraviolet regularization so as to simplify as far as possible the renormalization properties of certain—but not all—physical observables of interest. E.g. working with \( \omega = \pi/2 \) entails remarkable simplifications \( \text{(21)} \) in the renormalization of the decay constant of the charged pseudoscalar mesons and the chiral condensate, as \( (A'_{\mu R})_\mu = (V_R)_\mu \) and \( (S_R^0) = 2i(P_R) \).

4.2.2. Hamiltonian formalism at fixed \( a \)

Unimproved \( (c_{sw} = 0) \) lattice tmQCD admits a positive and selfadjoint transfer matrix \( \text{(24)} \), with the constraint |8+2am_0| > 6 and no constraint on \( a\mu_q \). Lattice correlation functions can hence be represented as usual in terms of operator matrix elements with time–dependent coefficients.

From the symmetry properties of LtmQCD, see sect. 4.1, it follows that the set of lattice quantum numbers for \( \omega \neq 0 \) is reduced as compared to that of Wilson’s formulation \( (\omega = 0) \): parity and isospin are replaced by the quantum numbers \( p_F \) and \( q_I \) corresponding to the unphysical parity transformation \( P_F \), eq. \( \text{(1)} \), and the unbroken isospin generator. This implies, for instance, that in LtmQCD the physical vacuum and neutral pion states are labelled by the same set of lattice quantum numbers and can be interpo-

lated by the same lattice field. Moreover, lattice operator mixings are constrained by \( p_F \) and \( q_I \) rather than parity and isospin quantum numbers. For instance, the operator \( (A'_{\mu R})_\mu \), which yields, as \( a \to 0 \), the first component of the physical isoscalar axial current, mixes at order \( a\mu_R \) with the second component of the physical isoscalar vector current.

In the quantum mechanical analysis of tmQCD correlators \( (\omega \neq 0) \) at fixed \( a \), one must hence include \( \text{(23, 13)} \) the contributions of matrix elements that would vanish if parity and isospin were exact symmetries\(^14\). The \( O(a) \)–improvement is expected to reduce the size of these contributions.

4.3. \( O(a) \) improvement: the case \( \alpha = \pi/2 \)

The \( O(a) \) improvement of LtmQCD \( \text{(23, 24)} \) is most conveniently discussed in the unphysical quark basis corresponding to eq. \( \text{(4)} \). Without loss of generality, one can assume that an infrared cutoff is in place (thanks to some specific choice of external momenta or boundary conditions), so that all correlation functions admit a Taylor expansion in powers of \( \mu_q \) and \( m_q \) around the massless theory. Once the latter has been fully renormalized and \( O(a) \) improved in a mass independent scheme, the renormalized action parameters of the massive theory are consistently \( \text{(4)} \) defined by eq. \( \text{(10)} \), with \( g_0^2, m_q \) and \( \mu_q \) replaced by:

\[
\begin{align*}
g_0^2 &= g_0^2(1 + b_g a \mu_q), \\
m_a &= m_q + b_m a m_q^2 + b_{ma} \mu_q^2, \\
\mu_a &= \mu_q(1 + b_{ma} a \mu_q) \quad \text{(18)}
\end{align*}
\]

\(^{14}\)The renormalized counterparts of these matrix elements actually vanish like \( a\mu_R \), or faster, in the continuum limit, because of the relations \( \text{(17)} \).
and \( Z_g \), \( Z_a \) and \( Z_\mu \) depending only on \( \bar{g}_0^2 \). The absence of a term \( \propto a\mu_q \) in the expression for \( \bar{g}_0^2 \) reflects the fact that the partition function of LtmQCD is even in \( \mu_q \), see eq. (8) for the fermionic determinant: hence an action counterterm of the form \( a\mu_q \text{tr}(FF) \) can not be generated.

Improved operators can be obtained by appropriate subtraction of the operator mixings that come with powers of \( am_q \) or \( a\mu_q \). In absence of power divergent mixings, the construction of multiplicatively renormalizable improved operators is rather simple, e.g.

\[
\begin{align*}
(A_1)^b_\mu &= A_\mu^b + c_A a\bar{\mu}_\mu P^b + a\mu_q b_A \epsilon^{3\mu}V^c, \\
(P_1)^b &= P^b, \quad \bar{b} = 1, 2.
\end{align*}
\]

(19)

The corresponding renormalized and improved operators are obtained after rescaling by suitable factors \( Z_A(1 + b_A a m_q) \) and \( Z_P(1 + b_P a m_q) \). It is useful to remark that the set of \( O(a) \) counterterms that have been introduced is slightly redundant [26]; one of them can be chosen arbitrarily, e.g. by setting \( b_m = -1/2 \).

Adopting a formulation with \( |\bar{m}_q| \approx |m_q| \leq O(a) \) and \( \mu_q \neq 0 \), i.e.

\[ |\alpha| = \pi/2 + O(a) \]

(20)

we find that \( a_R^2 = Z_g \bar{g}_0^2 \) and \( m'_R = Z_a \mu_q \) up to \( O(a^2) \) corrections, see eq. (15), and all \( b \)-type improvement coefficients are not needed. One needs instead to know the \( b \)-type improvement coefficients, like \( \tilde{b}_A \) in eq. (14), which are associated to operator mixings that violate parity or isospin. This observation implies that one can determine the \( b \)-type coefficients by requiring the improved correlation functions to satisfy parity and isospin invariances.\(^5\) Implementing the condition (20) requires to know only \( c_{sw} \) and \( m_c \) (unimproved). Moreover, for \( \alpha \) as in eq. (20), eq. (11) is interpreted as the physical PCAC relation, see eq. (15): \( O(a) \) improved estimates of the mass, \( m_{PS} \), and the decay constant, \( F_{PS} \), of the charged pseudoscalar meson can hence be obtained [27] from the correlator

\[
\mu_q G_P(x_0) = a^3 \sum_x \mu_q \langle P^1(x)P^1(0) \rangle,
\]

(21)

if \( c_{sw} \) and \( m_c \) are known. If one wishes to work with \( |\alpha| = \pi/2 + O(a^2) \), an improved estimate of \( m_c \) (which may require to know \( c_A \)) is also needed.

5. The first non-perturbative studies

The above theoretical understanding of LtmQCD has been checked at the one-loop level [20], as far as the action and the operators of eq. (13) are concerned. At the non-perturbative level, test studies have been carried out in the quenched approximation\(^4\) using the non-perturbatively improved [11] Wilson action. The authors of Ref. [29] studied LtmQCD in a box of volume \( L^3 \times T \) with Schrödinger functional boundary conditions \([27]\), \( L = 0.75 \) fm, \( T = 2L \), \( Lm'_R = 0.154 \), \( \alpha = 1.44 \) and four lattice resolutions: \( a \in [0.093, 0.046] \) fm. The scaling behaviour of some renormalized and improved quantities (which in large volume yield the mass and the decay constant of pseudoscalar and vector mesons) was found to be consistent with \( O(a) \) improvement, with the residual cutoff effects at \( a = 0.093 \) fm ranging from 0.5% to 9%.

The same setup and observables have then been employed for a study in realistically large volumes [18]: \( L = 1.5 \) to 2.2 fm and \( T = (2 \div 3)L \), so to ensure \( m_{PS} L \geq 4.5 \). This study was restricted to two lattice resolutions, \( a = 0.093 \) and 0.068 fm, and, for each of them, three sets of quark mass parameters, which correspond to \( |\alpha| = \pi/2 + O(a) \) and pseudoscalar meson masses in the range 1.85 \( \geq (m_{PS}/m_{K^0})^2 \geq 0.85 \).

Another test study of quenched LtmQCD with non-perturbatively improved action has been performed at fixed spatial volume, \( L = 1.5 \) fm, with \( T = 2L \), lattice spacing \( a = 0.093 \) fm, periodic boundary conditions and several values of the quark mass [10]: the lowest value of \( m_{PS} \) is about 320 MeV, though with \( m_{PS} L \cong 2.4 \). In none of these studies exceptional configurations

---

\(^{15}\)In principle, it is also possible to get rid of the mixings of order \( a\mu_R \) without improving the operators: this is the case if in the quantum mechanical analysis of the correlators one can disentangle, e.g. by looking at the values of the effective masses, the contributions arising from intermediate states with different parity and isospin.

\(^{16}\)For all quenched studies I conventionally set the “physical” scale by employing Sommer’s scale: \( r_0 = 0.5 \) fm.
were found, while values of $m_{PS}$ well below $m_{K\pm}$ were reached with a moderate computational effort: results for $m_{PS}^2/m_{\rho}^2$ at $a = 0.093$ fm from Refs. [18, 30] are shown in Fig. 2. Moreover, in agreement with the indications of the scaling test at $\alpha = 0$ (stQCD: [14]) and $|\alpha| \simeq \pi/2$ (tmQCD: [18]), $m_{PS}^2/m_{\rho}^2$ at $a = 0.093$ fm from Refs. [18, 30] are shown in Fig. 3.

Figure 2. $r_0 m_{PS}^2/m_{\rho}^2$ vs. $r_0 m_{\rho}^2$ at $\beta = 6$. The dashed vertical line corresponds to $m_{PS} = m_{K\pm}$.

The scaling behaviour of $F_{PS}$ in large volume is presented in Fig. 3 for the $O(\alpha)$ improved Wilson formulations with $|\alpha| \simeq \pi/2$ [18] and $\alpha = 0$ [14]. In the latter case, where four lattice resolutions were considered to allow for continuum extrapolation, the simulation data have been reanalysed to precisely match the renormalization conditions adopted for LtmQCD. Following closely Ref. [14], an analogous comparison has been carried out, see Fig. 3 for the combination of renormalization group invariant quark masses $\bar{M}+\bar{M}_s$, where $\bar{M}$ is the average mass of the $u$ and $d$ quarks. The results for $F_{PS}$ at $\bar{M}+\bar{M}_s$ that are obtained from the two lattice formulations should agree in the continuum limit: this seems to be the case within the statistical errors shown in the figures. Moreover, in agreement with the indications of the scaling test at $L \simeq 0.75$ fm, the tmQCD estimates of these quantities show pretty small cutoff effects.

Figure 3. $F_{PS}$ vs. $\alpha^2$ at $m_{PS}\simeq1.2m_{K\pm}$. The continuum extrapolation by Ref. [14] is also shown.

6. $B_K$ and the $\Delta I = 1/2$ rule

The two-flavour formulation –LtmQCD– that I discussed above can be extended in a variety of ways to describe QCD with several non-degenerate flavours of Wilson quarks. The details of the (Wilson–like) fermionic regularization should be fixed, case by case, so to render as simple as possible the mixing pattern of the operators that are relevant for the physical problem of interest, while avoiding the occurrence of spurious quark zero modes. The general reasons underlying this possibility are discussed in sect. 4.2.1.

For the computation of $B_K$ ($K^0$–$\bar{K}^0$ mixing) it is convenient to adopt the lattice formulation with fermionic action density:

$$L_F = \bar{q}(D^\gamma_W + im'_q\gamma_5\tau^3)q + \bar{s}(D^\gamma_W + m'_q)s$$

which corresponds to take $\alpha = \pi/2$ for the two degenerate light flavours and $\alpha = 0$ for the strange quark. In this quark basis the operator for (physical) parity–even $\Delta S = 2$ transitions reads [21]:

$$O^{(\Delta S=2)}_{VV+AA} = -2i\langle \bar{s}\gamma_\mu d)(\bar{q}\gamma_\mu q) \rangle \equiv -2iV_{sd}A_{sd}$$

Unquenched simulations with the action [3] have been performed to study issues related to the Aoki phase [3].
and is multiplicatively renormalizable. A computation of $B_K$ based on this approach is currently in progress \cite{22}.

For the computation of CP conserving $\Delta S = 1$ matrix elements of the weak effective Hamiltonian with active charm flavour, it is convenient to adopt the following lattice formulation \cite{33}:

$$
\mathcal{L}_F = \bar{\psi} \left( D^W_{\mu} + \tilde{m}_q + i\tilde{\mu}_q \gamma_5 \tau^3 \right) \psi 
$$

(24)

with $\psi^T = (u, d; s, c)$ and quark mass matrices

$$
\tilde{m}_q = \text{diag}(m_{q_1}, m_{q_1}, m_{q_2}, m_{q_2})
$$

$$
\tilde{\mu}_q = \text{diag}(\mu_{q_1}, -\mu_{q_1}, \mu_{q_2}, -\mu_{q_2}).
$$

(25)

Thinking of $\psi$ as a pair of doublets and setting

$$
\frac{\mu_{q_1}}{m_{q_1}} \equiv \tan \omega_l, \quad \frac{\mu_{q_2}}{m_{q_2}} = -\frac{\mu_{q_2}}{m_{q_2}} \equiv \tan \omega_h,
$$

(26)

one can show \cite{33} –by arguments analogous to those of sect. 4.2– that the operator

$$
O^{\pm}_{VA+AV} = [(V_{sd}A_{uu} + A_{sd}V_{uu} \pm \ldots) - (u \rightarrow c)]
$$

must be interpreted as the parity–even operator

$iO^{\pm}_{V_{VA+AA}}$ (relevant for $K \rightarrow \pi$ transitions), if $\omega_l = \omega_h = \pi/2$, and as the parity–odd operator

$O^{\pm}_{VA+AV}$ (relevant for $K \rightarrow \pi \pi$ transitions), if $\omega_l = -\omega_h = \pi/2$. Symmetries and power counting imply that $O^{\pm}_{VA+AV}$ does not mix with dimension six operators, while the leading mixing with dimension three operators has a coefficient $\propto a^{-1}(\mu_{q_2} - \mu_{q_1})(\mu_{q_2} - \mu_{q_1})$. Quadratic divergences in $K \rightarrow \pi$ matrix elements and spurious quark zero modes in general are hence avoided.

7. Conclusions

Wilson fermions with chirally twisted mass can provide a variety of alternative regularizations for lattice QCD. Thanks to an up to now unexploited freedom in formulating lattice QCD with Wilson quarks, they allow to avoid some of the most serious problems due to lack of lattice chiral symmetry, while preserving ultralocality of the action and hence a moderate computational cost.

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