On the types for supercuspidal representations of inner forms of $\text{GL}_N$

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Abstract

Let $F$ be a non-Archimedean local field, $A$ be a central simple $F$-algebra, and $G$ be the multiplicative group of $A$. It is known that for every irreducible supercuspidal representation $\pi$, there exists a $[G, \pi]_G$-type $(J, \lambda)$, called a (maximal) simple type. We will show that $[G, \pi]_G$-types defined over some maximal compact subgroup are unique up to $G$-conjugations under some unramifiedness assumption on a simple stratum.

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1 Introduction

All the representations considered in this paper are smooth and over \( \mathbb{C} \). Let \( F \) be a non-Archimedean local field with its integer ring \( \mathfrak{o}_F \), its radical \( \mathfrak{p}_F \) and its residue field \( k_F = \mathfrak{o}_F / \mathfrak{p}_F \). Let \( G \) be the multiplicative group of a central simple \( F \)-algebra. Let \( \text{Irr}(G) \) be the set of the isomorphism classes in the irreducible \( G \)-representations. Let \( \mathcal{M}(G) \) be the category of representations of \( G \) which decomposes into indecomposable subcategories, indexed by \( \mathcal{B}(G) = \text{Irr}(G) / \sim \). This decomposition of category is called the Bernstein decomposition.

**Definition 1.1** Let \( s \) be an element of \( \mathcal{B}(G) \) and \( J \) be a compact open subgroup of \( G \). An irreducible \( J \)-representation \( \lambda \) is called an \( s \)-type if the following condition holds: for \( \pi \in \text{Irr}(G) \), \( \pi \in s \) if and only if \( \pi |_{J} \supseteq \lambda \).

When \( \pi \in \text{Irr}(G) \) is supercuspidal, for \( \pi' \in \text{Irr}(G) \), we have \( \pi \sim \pi' \) if and only if there exists an unramified character \( \chi \) of \( F^\times \) such that \( \pi' \cong \pi \otimes (\chi \circ \text{Nrd}_{A/F}) \), where \( \text{Nrd}_{A/F} \) is the reduced norm of \( A \) over \( F \). For \( \pi \in \text{Irr}(G) \) which is supercuspidal, we define

\[
[G, \pi]_G = \left\{ \pi' \in \text{Irr}(G) \mid \pi' \cong \pi \otimes (\chi \circ \text{Nrd}_{A/F}) \text{ for some unramified character } \chi \text{ of } F^\times \right\} \in \mathcal{B}(G).
\]

Then \( [G, \pi]_G \) consists of supercuspidal representations.

Let \( G = \text{GL}_N(F) \) and \( K \) be a maximal compact subgroup of \( G \). For \( N = 2 \), Henniart proved in the Appendix to [2] that if \( \pi \) is an irreducible supercuspidal representation of \( G \), then there exists a unique representation \( \rho \) of \( K \) such that \( (K, \rho) \) is \( [G, \pi]_G \)-type. Henniart also determined the number of isomorphism classes of irreducible representation \( \rho \) of \( K \) such that \( (K, \rho) \) is an \( s \)-type for every \( s \in \mathcal{B}(G) \). Paskunas extended the Henniart’s result on supercuspidal representations for general \( N \) in his paper [10], using the concept of simple types for \( \text{GL}_N(F) \), established in [1]. However, for general \( G \), it is not necessarily so. When \( D \neq F \), a maximal compact subgroup \( K \) is properly contained in the normalizer \( N_G(K) \) of \( K \) in \( G \). If \( (K, \rho) \) is \( [G, \pi]_G \)-type and \( k \in N_G(K) \), then \( (K, k^\rho) \) is also \( [G, \pi]_G \)-type, where \( k^\rho \) is the \( K \)-representation obtained by twisting \( \rho \) by \( k \). For some \( k \in N_G(K) \setminus K \), it may happen that \( k^\rho \) is not isomorphic to \( \rho \).

Then, we define \( \mathcal{S}(\tau) \) to be the set of \( (K, \tau) \) such that \( K \) is a maximal compact subgroup of \( G \), \( \tau \) is a \( K \)-representation and \( (K, \tau) \) is a \( [G, \pi]_G \)-type. Every \( G \)-conjugacy class of \( \mathcal{S}(\tau) \) is called a \( [G, \pi]_G \)-archetype.

The main result of this paper is following:

**Theorem 1.2** Let \( G = \text{GL}_m(D) \) and \( \pi \) be an irreducible supercuspidal representation of \( G \). If \( \pi \) contains a simple type \( (J, \lambda) \) attached to the simple stratum \([\mathfrak{A}, n, 0, \beta]\) such that \( F[\beta] \) is an unramified extension of \( F \), then there exists a unique \( [G, \pi]_G \)-archetype.
Our approach is similar to that in [10]. However, the same approach as in [10] cannot be applied in our case.

In the proof of the main theorem, we consider depth-zero supercuspidal representations of $G$. One of the key points of our proof is that for a given depth-zero supercuspidal representation $\sigma$, there exists some representation $\sigma'$ which is 'similar to' $\sigma$ in some sense. For our purpose, we cannot take representations which intertwine with $\sigma$ as the candidate for $\sigma'$. If $D = F$, such a representation is nothing but $\sigma$. However, when the dimension of $D$ over $F$ increases, the number of such representations also increases. Therefore, to show the existence of the representation $\sigma'$ with desired condition, we need to examine depth-zero supercuspidal representations more closely than [10].

In fact, we will construct some irreducible supercuspidal representation $\pi$ of $G$ such that a $[G, \pi]_G$-archetype is not unique in some case. Then, this representation is a counterexample for the conjecture on the unicity of archetypes in [8, Conjecture 4.4]. This conjecture holds for the split case $G = \text{GL}_N(F)$ and the multiplicative group $G = D^\times$ of a central division algebra over $F$ (for $G = D^\times$, see, for example, [5]). It is interesting that for the multiplicative group $G = \text{GL}_m(D)$ of a general central simple algebra over $F$, which is similar to these groups, the conjecture does not hold any longer.

In [9], the unicity of types is discussed for tame, toral and positive regular supercuspidal representations of a simply connected and semisimple group. Latham–Nevins showed that if we assume some unramifiedness on the representation, then the conjecture in [8] holds. For $G = \text{GL}_m(D)$, a simple stratum $[\mathfrak{A}, n, 0, \beta]$ for a toral supercuspidal representation is considered to satisfy the condition that $F[\beta]$ is a maximal subfield of $A$. From this point of view, both of the unramified assumptions seem to be linked to each other. We note that the result in this paper covers some nontoral representations of $G$.

We sketch the outline of this paper. In §2, we introduce the concepts which we will need later. In §3, using the Mackey decomposition we consider the restriction of an irreducible supercuspidal representation $\pi$ to a maximal compact open subgroup $K$ of $G$. We need to consider the double cosets of $K\backslash G/\mathcal{R}(\mathfrak{A})$, where $\mathcal{R}(\mathfrak{A})$ is the normalizer of some hereditary $\mathfrak{O}_F$-order $\mathfrak{A}$ in $A$, defined in §2. In §4, for every $g' \in G \setminus K\mathcal{R}(\mathfrak{A})$ we take a nice representative $g \in Kg'\mathcal{R}(\mathfrak{A})$. In §5, we consider irreducible $K$-representations contained in the representations corresponding to a pair $(g, Kg'\mathcal{R}(\mathfrak{A}))$ in the Mackey decomposition of $\pi|_K$ when the pair $(g, Kg'\mathcal{R}(\mathfrak{A}))$ has 'Property A'. In §6, the main theorem of this paper is proved.

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2 Notation and Preliminary

Let $F$ be a non-Archimedean local field. Let $\mathfrak{o}_F$ be its integers, $p_F$ be the prime ideal of $\mathfrak{o}_F$ and $k_F = \mathfrak{o}_F/p_F$ be the residue field of $F$ with the cardinality $q_F$. If $D$ is a finite extension field of $F$ or a division $F$-algebra, we use the same notation $\mathfrak{o}_D, p_D, k_D, q_D$.

2.1 Intertwining

If $H$ is a subgroup of $G$ and $g$ is an element of $G$, then we denote $gH = ghg^{-1}$. Moreover, if $\sigma$ is a representation of $H$, we define the representation $\sigma^g$ of $gH$ as follows:

$$\sigma^g(x) = \sigma(g^{-1}xg), \quad x \in gH.$$ 

Let $H_1, H_2, G'$ be subgroups of $G$, $H$ be a subgroup of $H_1 \cap H_2$. Let for $i = 1, 2$, $\sigma_i$ be a representation of $H_i$ such that $\text{Hom}_H(\sigma_1, \sigma_2)$ is finite-dimensional. We define

$$\langle \sigma_1, \sigma_2 \rangle_H = \dim \text{Hom}_H(\sigma_1, \sigma_2),$$

$$I_g(\sigma_1, \sigma_2) = \text{Hom}_{H \cap gH}(\sigma_1, ^g\sigma_2) \quad \text{for} \quad g \in G,$$

$$I_{G'}(\sigma_1, \sigma_2) = \{ g \in G' \mid I_g(\sigma_1, \sigma_2) \neq 0 \}.$$ 

We say $\sigma_1$ and $\sigma_2$ intertwine in $G$ if $I_G(\sigma_1, \sigma_2) \neq \emptyset$. When $H_1$ and $H_2$ are compact, $\sigma_1$ and $\sigma_2$ intertwine in $G$ if and only if $\sigma_2$ and $\sigma_1$ intertwine.

2.2 Lattices, hereditary orders

We recall definitions on lattices and hereditary orders from [13 §1].

Let $A$ be a central simple $F$-algebra and $G$ be the multiplicative group of $A$. Let $V$ be a simple right $A$-module. Then, $\text{End}_A(V)$ is a central division $F$-algebra. Let $D$ denote the opposite of $\text{End}_A(V)$. Then, $V$ becomes a right $D$-vector space with $\dim_D V = m$ and the natural $F$-algebra isomorphism $A \cong \text{End}_D(V)$ exists. If we fix a right $D$-isomorphism $V \cong D^m$, this isomorphism induces $A \cong M_m(D)$ and $G \cong \text{GL}_m(D)$.

Let $E/F$ be a subfield in $A$. Then $V$ is equipped with an $E$-vector space structure which is compatible with the $D$-action, whence $V$ becomes a right $E \otimes_F D$-module. Let $W$ be a simple right $E \otimes_F D$-module. Then, there exists $m' \in \mathbb{N}$ and a right $E \otimes_F D$-module isomorphism $V \cong W^{\oplus m'}$. Put $A(E) = \text{End}_D(W)$. By the above isomorphism, we have $F$-algebra isomorphisms $\text{End}_D(V) \cong \text{End}_D(W^{\oplus m'}) \cong M_{m'}(\text{End}_D(W)) = M_{m'}(A(E))$. The centralizer $\text{Cent}_{A(E)}(V)$ of $E$ in $A(E)$ is the set of $D$-morphisms which are also $E$-linear maps, that is, $E \otimes_F D$-linear maps. Therefore, $\text{Cent}_{A(E)}(V) = \text{End}_{E \otimes_F D}(W)$ is a division $E$-algebra, whose opposite algebra is denoted by $D'$. Let $B$ be the centralizer of $E$ in $A$. Under the isomorphism $A \cong M_{m'}(A(E))$, $E$ is diagonally embedded in $M_{m'}(A(E))$. Therefore, the isomorphism maps $B$ to $M_{m'}(D')$.

An $\mathfrak{o}_D$-submodule $\Lambda$ in $V$ is called an $\mathfrak{o}_D$-lattice in $V$ if it is a compact open submodule.
The number e = e(Λ) is called the period of Λ. An \(\sigma_D\)-sequence \(\Lambda = (\Lambda_i)\) in \(V\) is called an \(\sigma_D\)-chain if \(\Lambda_i \supset \Lambda_{i+1}\) for every \(i\).

Let \(\mathfrak{A}\) be an \(\mathfrak{f}\)-order in \(A\). Then, \(\mathfrak{A}\) is hereditary if every left and right ideal in \(\mathfrak{A}\) is \(\mathfrak{A}\)-projective.

Let \(\Lambda = (\Lambda_i)\) be an \(\sigma_D\)-sequence in \(V\). Put \(\mathfrak{P}_i(\Lambda) = \{ x \in A \mid x \Lambda_j \subset \Lambda_{i+j}, j \in \mathbb{Z} \}\). Then, \(\mathfrak{A} = \mathfrak{P}_0(\Lambda)\) is a hereditary \(\mathfrak{f}\)-order in \(A\). The radical of \(\mathfrak{A}\) is \(\mathfrak{P}(\mathfrak{A}) = \mathfrak{P}_1(\Lambda)\). For every hereditary \(\mathfrak{f}\)-order \(\mathfrak{A}\), there exists an \(\sigma_D\)-chain \(\Lambda\) such that \(\mathfrak{A} = \mathfrak{A}(\Lambda)\). If \([\Lambda_i : \Lambda_{i+1}]\) is constant for any \(i\), then \(\mathfrak{A} = \mathfrak{A}(\Lambda)\) is called principal.

Let \(\Lambda = (\Lambda_i)\) be an \(\sigma_D\)-chain in \(V\). Let \(\mathfrak{K}(\Lambda)\) be the set of \(g \in A\) with the condition that there exists \(n \in \mathbb{N}\) such that \(g(\Lambda_i) = \Lambda_{n+i}\) for all \(i\). For the hereditary \(\mathfrak{f}\)-order \(\mathfrak{A} = \mathfrak{A}(\Lambda)\), we put \(\mathfrak{K}(\mathfrak{A}) = \{ g \in G \mid g\mathfrak{A}^{-1} = \mathfrak{A} \}\). Then \(\mathfrak{K}(\mathfrak{A})\) is equal to \(\mathfrak{K}(\Lambda)\) and the group homomorphism \(v_{\mathfrak{A}}: \mathfrak{K}(\mathfrak{A}) \to \mathbb{Z}\) is defined as \(v_{\mathfrak{A}}(g) = n\) for \(g \in \mathfrak{K}(\mathfrak{A})\) with \(g\mathfrak{A} = \mathfrak{P}(\mathfrak{A})^n\). Then, the short exact sequence

\[
1 \to U(\mathfrak{A}) \to \mathfrak{K}(\mathfrak{A}) \to \mathbb{Z}
\]

exists, where \(U(\mathfrak{A}) = U^0(\mathfrak{A}) = \mathfrak{A}^\times\) is the unique maximal compact open subgroup of \(\mathfrak{K}(\mathfrak{A})\). For \(n \in \mathbb{N}_{>0}\), put \(U^n(\mathfrak{A}) = 1 + \mathfrak{P}(\mathfrak{A})^n\).

A hereditary \(\mathfrak{f}\)-order \(\mathfrak{A}\) in \(A\) is \(E\)-pure if \(E^\times \subset \mathfrak{K}(\mathfrak{A})\).

**Theorem 2.2** ([4, Theorem 1.3]) Let \(\mathfrak{A}\) be an \(E\)-pure hereditary \(\mathfrak{f}\)-order in \(A\) with its radical \(\mathfrak{P}\). Then, \(\mathfrak{B} = \mathfrak{A} \cap B\) is a hereditary \(\mathfrak{f}\)-order in \(B\) with its radical \(\mathfrak{Q} = \mathfrak{P} \cap B\).

If \(A(E)\) is as above, there is a unique \(E\)-pure hereditary \(\mathfrak{f}\)-order \(\mathfrak{A}(E)\) with its radical \(\mathfrak{P}(E)\); see [13, §1.5.2]. The hereditary \(\mathfrak{f}\)-order \(\mathfrak{A}(E)\) is principal. The equations \(\mathfrak{A}(E) \cap D' = \sigma_{D'}\) and \(\mathfrak{P}(E) \cap D' = p_{D'}\) hold.

### 2.3 Proper hereditary orders

Let \(A, E, B\) be as above. If \(V\) is a simple left \(A\)-module and \(W\) is a simple right \(E \otimes_F D\)-module, then an \(E \otimes_F D\)-module isomorphism \(V \cong W^{\oplus m'}\) induces the \(F\)-algebra isomorphism \(A \cong M_{m'}(A(E))\).
**Definition 2.3** ([13, §4.3.1]) A hereditary \(\mathfrak{a}_F\)-order \(\mathfrak{a}\) in \(A\) is proper if there exists an \(E \otimes_F D\)-module isomorphism \(V \cong W^{\oplus m'}\) and integers \(r, s\) such that \(rs = m'\) holds and

\[
\mathfrak{a} = \begin{pmatrix}
M_s(\mathfrak{A}(E)) & \cdots & M_s(\mathfrak{A}(E)) \\
\vdots & \ddots & \vdots \\
M_s(\mathfrak{B}(E)) & \cdots & M_s(\mathfrak{A}(E))
\end{pmatrix}
\]

under the identification between \(A\) and \(M_{m'}(A(E)) = M_r(M_s(A(E)))\) given by the isomorphism \(V \cong W^{\oplus m'}\).

If \(\mathfrak{a}\) is as above, since \(\mathfrak{A}(E)\) is principal, \(\mathfrak{a}\) is also principal and

\[
\mathfrak{b} = \mathfrak{a} \cap B = \begin{pmatrix}
M_s(\mathfrak{o}_{D'}) & \cdots & M_s(\mathfrak{o}_{D'}) \\
\vdots & \ddots & \vdots \\
M_s(\mathfrak{p}_{D'}) & \cdots & M_s(\mathfrak{o}_{D'})
\end{pmatrix}
\]

holds, whence \(\mathfrak{b}\) is a principal hereditary \(\mathfrak{o}_E\)-order in \(B\) with its period \(r\).

**Remark 2.4** (i) If \(\mathfrak{a}\) is proper and \(\mathfrak{b}\) is maximal, then \(r = 1\) and we have an isomorphism \(\mathfrak{a} \cong M_s(\mathfrak{A}(E))\).

(ii) If \(\mathfrak{b}\) is a principal hereditary \(\mathfrak{o}_E\)-order in \(B\), then there exists a proper hereditary \(\mathfrak{o}_F\)-order \(\mathfrak{a}\) in \(A\) such that \(\mathfrak{a} \cap B = \mathfrak{b}\), see [13, Remarque 4.8]. It is proved in [12, Lemme 1.6] that if \(\mathfrak{b}\) is maximal, then there is a unique hereditary \(\mathfrak{o}_F\)-order \(\mathfrak{a}\) in \(A\) such that \(\mathfrak{a} \cap B = \mathfrak{b}\), whence \(\mathfrak{a}\) is proper.

**2.4 Strata, simple characters, simple types**

**Definition 2.5** ([13, §2.1]) A 4-tuple \([\mathfrak{a}, n, r, \beta]\) is called a stratum of \(A\) if the following conditions hold:

- \(\mathfrak{a}\) is a hereditary \(\mathfrak{o}_F\)-order in \(A\).
- \(n, r\) are integers with \(n > r\).
- \(\beta\) is an element of \(\mathfrak{P}(\mathfrak{A})^{-n}\).

A stratum \([\mathfrak{a}, n, r, \beta]\) is pure if

(i) \(E = F[\beta]\) is a subfield,

(ii) \(\mathfrak{a}\) is \(E\)-pure,

(iii) \(v_\mathfrak{A}(\beta) = -n\).
A pure stratum $[\mathfrak{A}, n, r, \beta]$ is simple if $r < -k_0(\beta, \mathfrak{A})$, where for $\beta \in A$ the element $k_0(\beta, \mathfrak{A}) \in \mathbb{Z} \cup \{-\infty\}$ is defined as in §2.1 and satisfies $v_\mathfrak{A}(\beta) \leq k_0(\beta, \mathfrak{A})$.

If $[\mathfrak{A}, n, r, \beta]$ is a simple stratum of $A$, we can define subgroups $J(\beta, \mathfrak{A})$ and $H(\beta, \mathfrak{A})$ of $U(\mathfrak{A})$ as in [11, §3]. For $i \in \mathbb{N}$, put $J^i(\beta, \mathfrak{A}) = J(\beta, \mathfrak{A}) \cap U^i(\mathfrak{A})$ and $H^i(\beta, \mathfrak{A}) = H(\beta, \mathfrak{A}) \cap U^i(\mathfrak{A})$. We have an equation $J(\beta, \mathfrak{A}) = U(\mathfrak{B})J^1(\beta, \mathfrak{A})$, where $\mathfrak{B} = \mathfrak{A} \cap B$.

We can also define the set of ‘simple characters’ $\mathcal{C}(\beta, m, \mathfrak{A})$, consisting of characters of $H^1(\beta, \mathfrak{A})$ satisfying some condition, initially defined in [11, Définition 3.45]. We put $\mathcal{C}(\beta, \mathfrak{A}) = \mathcal{C}(\beta, 0, \mathfrak{A})$.

We recall properties on $\mathcal{C}(\beta, \mathfrak{A})$ from [12]. For every $\theta \in \mathcal{C}(\beta, \mathfrak{A})$, there exists a unique irreducible representation $\eta_\theta$ of $J^1(\beta, \mathfrak{A})$ such that $\eta_\theta|_{H^1(\beta, \mathfrak{A})}$ is a direct sum of $\theta$, called the Heisenberg representation of $\theta$. Moreover, for every $\eta_\theta$ as above, there exist a number of extensions $\kappa$ to $J(\beta, \mathfrak{A})$ such that $B^\kappa \subset \mathcal{I}_G(\kappa, \kappa)$, called the $\beta$-extensions of $\eta_\theta$. If $\kappa$ is as above, then $\mathcal{I}_G(\eta_\theta, \eta_\theta) = \mathcal{I}_G(\kappa, \kappa) = JB^\kappa J$ and for $g \in JB^\kappa J$, we have $\dim I_g(\eta_\theta, \eta_\theta) = \dim I_g(\kappa, \kappa) = 1$.

**Definition 2.6** ([13, §2.5 and §4.1]) Let $J$ be a compact open subgroup of $G$ and $\lambda$ be an irreducible $J$-representation. A pair $(J, \lambda)$ is called a simple type of level 0 if

(i) for some principal hereditary $s_r$-order $\mathfrak{A}$, we have $J = U(\mathfrak{A})$,

(ii) there exists an irreducible cuspidal representation $\sigma$ of $GL_s(k_D)$ such that $\lambda$ is the lift of the irreducible representation $\sigma^{\otimes r}$ of $U(\mathfrak{A})/U^1(\mathfrak{A}) \cong GL_s(k_D)^r$ to $U(\mathfrak{A})$.

A pair $(J, \lambda)$ is called a simple type of level $> 0$ if

(i) there exists a simple stratum $[\mathfrak{A}, n, 0, \beta]$ with $n > 0$ such that $J = J(\beta, \mathfrak{A})$,

(ii) there exist irreducible representations $\kappa, \sigma$ of $J$ with $\lambda = \kappa \otimes \sigma$ such that

- $\kappa$ is a $\beta$-extension of the Heisenberg representation $\eta_\theta$ for some $\theta \in \mathcal{C}(\beta, \mathfrak{A})$, and

- $\sigma$ is a $J(\beta, \mathfrak{A})$-representation trivial on $J^1(\beta, \mathfrak{A})$, and when $J(\beta, \mathfrak{A})$-representations which are trivial on $J^1(\beta, \mathfrak{A})$ are regarded as $U(\mathfrak{B})$-representations under the group isomorphism $J(\beta, \mathfrak{A})/J^1(\beta, \mathfrak{A}) \cong U(\mathfrak{B})/U^1(\mathfrak{B})$.

$(U(\mathfrak{B}), \sigma)$ is a simple type of level 0, where $B = \text{Cent}_A(F[\beta]), \mathfrak{B} = \mathfrak{A} \cap B$.

A pair $(J, \lambda)$ is a simple type if it is a simple type of level 0 or level $> 0$. 

7
2.5 Maximal simple types

Definition 2.7 ([13, 5.1]) Let $(J, \lambda)$ be a simple type. The simple type $(J, \lambda)$ is called maximal if one of the following conditions holds:

(i) The simple type $(J, \lambda)$ is of level 0 and $J = U(A)$ for some maximal $\mathfrak{o}_F$-order $A$ in $G$.

(ii) The simple type $(J, \lambda)$ is of level $> 0$, attached to the simple stratum $[A, n, 0, \beta]$ and $\mathfrak{B} = \mathfrak{A} \cap \text{Cent}_A(F[\beta]) = \mathfrak{A} \cap B$ is a maximal $\mathfrak{o}_{F[\beta]}$-order in $B$.

Theorem 2.8 ([7, Theorem 5.5(ii)] and [14, Théorème 5.21]) Let $(J, \lambda)$ be a simple type. The simple type $(J, \lambda)$ is maximal if and only if there exists an irreducible supercuspidal representation $\pi$ of $G$ such that $\lambda$ is contained in $\text{Res}_J^G \pi$. If $(J, \lambda)$ is contained in some irreducible supercuspidal representation $\pi$ of $G$, then $(J, \lambda)$ is a $[G, \pi]_G$-type. If $\pi$ is an irreducible supercuspidal representation, then there exists a simple type $(J, \lambda)$ such that $\lambda \subset \pi|_J$.

For a maximal simple type $(J, \lambda)$, we define a subgroup $\tilde{J}$ of $G$, containing and normalizing $F^\times J$.

Let $\mathfrak{A}$ be a maximal $\mathfrak{o}_F$-order in $A$ in this paragraph. We fix an isomorphism $A \cong M_m(D)$ such that under the identification by the isomorphism, $\mathfrak{A} = M_m(\mathfrak{o}_D)$. This isomorphism induces $U(\mathfrak{A})/U^1(\mathfrak{A}) \cong GL_m(k_D)$. Let $\sigma$ be a representation of $U(\mathfrak{A})$, which is trivial on $U^1(\mathfrak{A})$. Then $\sigma$ is the lift of a representation $\tau$ of $GL_m(k_D)$ to $U(\mathfrak{A})$. For $\gamma \in \text{Gal}(k_D/k_F)$, we define

$$(\gamma \cdot \sigma)(g) = \tau \left( (\gamma^{-1}(g_{i,j} \mod p_D))_{i,j} \right), \quad g = (g_{i,j})_{i,j} \in U(\mathfrak{A}).$$

When $(J, \lambda)$ is of level 0, let $\mathfrak{A}$ be a hereditary $\mathfrak{o}_F$-order in $A$ such that $J = U(\mathfrak{A})$, which is a maximal compact subgroup in $G$, and fix an isomorphism $A \cong M_m(D)$ with $\mathfrak{A} = M_m(\mathfrak{o}_D)$. We put

$$l_0 = \min \left\{ n' \in \mathbb{N}_{>0} \mid \gamma^{n'} \cdot \sigma \cong \sigma \text{ for any } \gamma \in \text{Gal}(k_D/k_F) \right\}.$$

We also fix a uniformizer $\varpi_D$ of $D$. Let $y$ be the diagonal matrix of $M_m(D)$ with every diagonal entry $\varpi_D$. Then $J = I_G(\sigma, \sigma)$ is a subgroup of $G$ generated by $U(\mathfrak{A})$ and $y^{l_0}$.

When $(J, \lambda)$ is of level $> 0$, let $[\mathfrak{A}, n, 0, \beta]$ be a simple stratum giving $(J, \lambda)$. Then there exist $\kappa, \sigma$ as Definition 2.6. Since $(J, \lambda)$ is maximal, $\mathfrak{B}$ is a maximal hereditary $\mathfrak{o}_F$-order in $B$ and $(U(\mathfrak{B}), \sigma)$ is a maximal simple type of $B$ of level 0. Therefore we can choose $y \in B^\times$ and $l_0$ as well as the case of simple type of level 0. Then $J = I_G(\lambda, \lambda)$ is a subgroup of $G$, generated by $J, F^\times$ and $y^{l_0}$.

Theorem 2.9 ([13, Théorème 5.2]) For a maximal simple type $(J, \lambda)$, there exists an extension $\tilde{\lambda}$ of $\lambda$ to $J$. If $\lambda$ is such an extension, $\text{c-Ind}_{J}^{G} \tilde{\lambda}$ is irreducible and supercuspidal.
2.6 Some property for $\mathfrak{A}(E)$

In this subsection, we fix an arbitrary finite field extension $E/F$ and will consider some property for $\mathfrak{A}(E)$. We put $d = (\dim_F D)^{1/2}$.

**Proposition 2.10** Then we have $\varpi_F \mathfrak{A}(E) = \mathfrak{P}(E)^{de'/(d,e')}$, where $e' = e(E/F)$ is the ramification index of $E/F$.

**Proof.** Let $W$ be a simple right $E \otimes_F D$-module and $\Lambda = (\Lambda_i)$ be an $\mathfrak{o}_D$-chain in $W$ such that $\mathfrak{A}(E) = \mathfrak{A}(\Lambda)$. Applying [13, Théorème 1.7] with $\Gamma = (p_i^D)$, we obtain $\rho_0 \in \mathbb{N}$ such that $\mathfrak{P}(E)^j \cap D^j = p_{D^j}^{[j/\rho_0]}$ for any $j \in \mathbb{N}$, where $[\cdot]$ is the ceil function. We have $\rho_0 = d/(d,e'r_0)$ by the proof in [14, Théorème 1.7], where $r_0$ is the integer satisfying $p_{D^j}^{r_0} \varpi_E = p_{D^j}^{r_0}$, that is, $r_0 = d/[E:F])$. Then we obtain $\rho_0 = (d,[E:F])/(d,e')$. Therefore, $\nu_{\mathfrak{A}(E)}(\varpi_D) = \rho_0 = (d,[E:F])/(d,e')$ and

$$\nu_{\mathfrak{A}(E)}(\varpi_E) = e' \nu_{\mathfrak{A}(E)}(\varpi_E) = e' \cdot \frac{d}{(d,[E:F])} \nu_{\mathfrak{A}(E)}(\varpi_D) = \frac{de' }{(d,e')}.$$  

This implies that $\varpi_F \mathfrak{A}(E) = \mathfrak{P}(E)^{de'/(d,e')}$. $\blacksquare$

**Corollary 2.11** Let $[\mathfrak{A}, n, 0, \beta]$ is a simple stratum for some maximal simple type such that $E = F[\beta]$ is an unramified extension of $F$. Then $\mathfrak{A}$ is a maximal hereditary $\mathfrak{o}_F$-order.

**Proof.** Let $\Lambda = (\Lambda_i)$ be as in the proof of Proposition 2.10. Since $E/F$ is unramified, $\Lambda_i p_D^d = \Lambda_i \varpi_F = \Lambda_{i+d}$ by Proposition 2.10. Then $\Lambda_i p_D = \Lambda_{i+1}$. Therefore $\mathfrak{A}(E) = \mathfrak{A}(\Lambda)$ is simply equal to $\text{End}_{\mathfrak{o}_D} (\Lambda_0)$ and $\mathfrak{A}(E)$ is maximal. Since $[\mathfrak{A}, n, 0, \beta]$ is a simple stratum for some maximal simple type, $\mathfrak{B} = \mathfrak{A} \cap \text{Cent}_A(E)$ is a maximal $\mathfrak{o}_F$-order in $\text{Cent}_A(E)$. Then $\mathfrak{A}$ is proper by Remark 2.1(i), and we have $\mathfrak{A} \cong M_s(\mathfrak{A}(E))$ for some $s$ by Remark 2.1(ii). Therefore $\mathfrak{A}$ is also maximal. $\blacksquare$

For later use, we will show the following lemma.

**Lemma 2.12** Let $E/F$ be an unramified extension and let $E_1$ be the unique subextension of $E/F$ such that $\dim_F E_1 = (d, [E:F])$. We fix an inclusion $E_1 \hookrightarrow D$. Let $W$ be a simple right $E \otimes_F D$-module. Then there exists a right $D$-basis $\mathcal{B}$ of $W$ such that under the identification of $A(E)$ with $M_{m_0}(D)$ given by $\mathcal{B}$,

(i) we have $\mathfrak{A}(E) = M_{m_0}(\mathfrak{o}_D)$, and

(ii) the canonical inclusion $E_1 \subset E \hookrightarrow A(E)$ coincides with the composition of the fixed inclusion $E_1 \hookrightarrow D$ and the diagonal embedding of $D$ in $M_{m_0}(D)$, where $m_0 = \dim_D(W)$. 

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Proof. First, we give an isomorphism of $W$ with a more concrete module. There is a (non-canonical) $E_1$-algebra isomorphism $E_1 \otimes_F D \cong M_{m_1}(D_1)$ for some division $E_1$-algebra $D_1$ with $m_1 = [E_1 : F]$. We put $\dim_{E_1} D_1 = d_1^2$, and then $d_1 = (d/[E_1 : F])$. Since $d_1$ and $[E : E_1]$ are coprime by the assumption, $D' = E \otimes_{E_1} D_1$ is a division $E$-algebra and

$$E \otimes_F D \cong E \otimes_{E_1} (E_1 \otimes_F D) \cong E \otimes_{E_1} M_{m_1}(D_1) \cong M_{m_1}(E \otimes_{E_1} D_1) = M_{m_1}(D').$$

Then, we have $\dim_{D'} W = m_1 = [E_1 : F]$.

Put $W_0 = E \otimes_{E_1} D$. We show $W \cong W_0$. The module $W_0$ has a natural right $E \otimes_F D$-module structure. Then, there exists a right $E \otimes_F D$-module isomorphism $W_0 \cong W^\otimes i$ for some positive integer $i$. Since, in particular, this isomorphism is also an $E$-vector space isomorphism, the equation $\dim_{E_1} W_0 = i \dim_{E_1} W$ holds. We have $\dim_{E_1} W_0 = \dim_{E_1} D = (\dim_F D)/[E_1 : F] = d_1^2/[E_1 : F]$ and $\dim_{E_1} W = d_1^2 \dim_{D'} W = d_1^2/[E_1 : F]$. Therefore, we obtain $i = 1$ and $W_0 \cong W$. Then we may assume $W = W_0$.

We have $m_0 = \dim_{D'} W_0 = [E : E_1]$. If an $E_1$-basis of $E$ is fixed, the embedding $E = \End_{E_1}(E) \subset \End_{E_1}(E_1) \cong M_{m_0}(E_1)$ is defined. We fix an $\mathfrak{o}_{E_1}$-basis $B' = \{b_1, \ldots, b_{m_0}\}$ of $\mathfrak{o}_E$, and obtain a $D$-basis $B = \{b_1 \otimes 1, \ldots, b_{m_0} \otimes 1\}$ of $W_0$. Under the identification of $M_{m_0}(D)$ with $A(E)$ defined by this basis, $E_1$ is embedded diagonally and $\mathfrak{o}_E^\times \subset \GL_{m_0}(\mathfrak{o}_{E_1}) \subset \GL_{m_0}(\mathfrak{o}_D)$.

Put $\mathfrak{A}_0 = M_{m_0}(\mathfrak{o}_D)$. Then, $\mathfrak{o}_E^\times$ and $E_1^\times \hookrightarrow D^\times$ are contained in $\mathfrak{A}(\mathfrak{A}_0)$. Since $E/E_1$ is unramified, this shows that $\mathfrak{A}_0$ is $E$-pure. Since $\mathfrak{A}(E)$ is the unique $E$-pure hereditary $\mathfrak{o}_D$-order, we have $\mathfrak{A}(E) = \mathfrak{A}_0$ and we obtain the desired conditions. 

3 Mackey Decomposition

In this section, we will show that for $K = \GL_{m_0}(\mathfrak{o}_D)$ and an irreducible supercuspidal representation $\pi$ of $G$, there exists a $[G, \pi]_G$-type $(K, \tau)$. Any maximal compact subgroup $K'$ of $G$ is $G$-conjugate to $K$, so this fact implies that there also exists a representation $\tau'$ of $K'$ such that $(K', \tau')$ is also a $[G, \pi]_G$-type.

Lemma 3.1 Let $s \in \mathcal{B}(G)$, $(J, \lambda)$ be an $s$-type and $K$ be a compact open subgroup of $G$ containing $J$. Assume $s$ contains an irreducible supercuspidal representation $\pi_0$ of $G$. Then, the irreducible components of $\text{Ind}_J^K \lambda$ contained in $\pi_0|_K$ are $s$-types. In particular, if $\text{Ind}_J^K \lambda$ is irreducible, $\text{Ind}_J^K \lambda$ is an $s$-type.

Proof. Let $\tau$ be an irreducible component of $\text{Ind}_J^K \lambda$ contained in $\pi_0|_K$ and $\pi \in \text{Irr}(G)$. We show that $\tau \subset \pi|_K$ if and only if $\pi \in s$.

Let $\tau \subset \pi|_K$. Then, we have

$$\langle \text{Ind}_J^K \lambda, \pi \rangle_K \geq \langle \tau, \pi \rangle_K > 0,$$

and by the Frobenius reciprocity we obtain $\langle \lambda, \pi \rangle_J > 0$. Since $(J, \lambda)$ is an $s$-type, it follows that $\pi \in s$. 

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 Conversely, let \( \pi \in \mathfrak{s} \). Then, there exists an unramified character \( \chi \) of \( F^\times \) with 
\[ \pi \cong \pi_0 \otimes \chi \circ \text{Nrd}_{A/F} \]
and we have \( \pi|_K \cong \pi_0|_K \). Therefore \( \tau \) is a \( K \)-subrepresentation of 
\( \pi|_K \cong \pi_0|_K \).

Let \( \pi \) be an irreducible supercuspidal representations of \( G \). There exists a simple type 
\( (J, \lambda) \) and a unique extension \( (\tilde{J}, \tilde{\lambda}) \) such that 
\[ \pi = \text{c-Ind}_J^G \tilde{\lambda}. \]

If \( (J, \lambda) \) is of level 0, then \( J \) is the multiplicative group of some maximal hereditary 
\( \mathfrak{o}_F \)-order \( \mathfrak{A} \) in \( A \). If \( (J, \lambda) \) attached to the simple stratum \([\mathfrak{A}, n, 0, \beta]\) with \( n > 0 \). Since 
\( \mathfrak{A} \) is maximal, \( \mathfrak{A} \) is a proper \( F[\beta] \)-pure hereditary \( \mathfrak{o}_F \)-order. Anyway, by \( G \)-conjugation, 
we may assume \( \mathfrak{A} \) is contained in \( M_m(\mathfrak{o}_D) \) and \( U(\mathfrak{A}) \) is a group consisting of block upper 
triangular matrices modulo \( p_D \). Let \( \Lambda = (\Lambda_i) \) be an \( \mathfrak{o}_D \)-chain which \( \mathfrak{A} \) is the hereditary 
\( \mathfrak{o}_F \)-order associated with, and let \( e \) be the period of \( \Lambda \). Then there is a natural bijection 
\[ \mathfrak{A}/\mathfrak{B} \cong \prod_{i=0}^{e-1} \text{End}_{k_D}(\Lambda_i/\Lambda_{i+1}). \]
We can take an \( \mathfrak{o}_D \)-chain \( \Lambda \) such that \( \mathfrak{A} = \mathfrak{A}(\Lambda) \) and a basis \( \{v_1, \ldots, v_m\} \) of \( V \) as 
\[ \Lambda_0 = v_1 \mathfrak{o}_D + \cdots + v_m \mathfrak{o}_D, \]
\[ \Lambda_i = v_1 \mathfrak{o}_D + \cdots + v_{(m/e)(e-i)} \mathfrak{o}_D + v_{(m/e)(e-i)+1} p_D + \cdots + v_m p_D \]
for \( 0 < i < e \).

**Lemma 3.2** If \( (\tilde{J}, \tilde{\lambda}) \) is as above, then \( \tilde{J} \) is contained in \( \mathfrak{R}(\mathfrak{A}) \).

*Proof.** The group \( \tilde{J} \) is generated by \( J, F^\times \) and \( y^0 \) as in \([2,3]\). Since \( J \subset \mathfrak{A} \) and \( F^\times \) is the 
center of \( G \), it is enough to show that \( y \) normalizes \( \mathfrak{A} \). Moreover, if \( (J, \lambda) \) is of level 0, 
then \( y \in \mathfrak{R}(\mathfrak{A}) \) by definition of \( y \). Then we may assume \( (J, \lambda) \) is of level > 0.

Because \( \mathfrak{A} \) is proper and \( (J, \lambda) \) is maximal, we may assume \( \mathfrak{A} = M_m'(\mathfrak{A}(E)) \). We 
fix a uniformizer \( \varpi_{D'} \) of \( D' \), contained in \( A(E) \), and \( y \) is a diagonal matrix such that 
it diagonal coefficients are all \( \varpi_{D'} \). Then \( y \mathfrak{A} y^{-1} = M_m'(\varpi_{D'} \mathfrak{A}(E) \varpi_{D'}^{-1}) \). Therefore, it is 
enough to show \( \varpi_{D'} \mathfrak{A}(E) \varpi_{D'}^{-1} = \mathfrak{A}(E) \).

Since \( \varpi_{D'} \in D' = \text{Cent}_{A(E)}(E) \) and \( \mathfrak{A}(E) \) is \( E \)-pure, for every \( x \in E^\times \), we have 
\[ x (\varpi_{D'} \mathfrak{A}(E) \varpi_{D'}^{-1}) x^{-1} = \varpi_{D'} x \mathfrak{A}(E) x^{-1} \varpi_{D'}^{-1} = \varpi_{D'} \mathfrak{A}(E) \varpi_{D'}^{-1}. \]
Therefore it follows that \( \varpi_{D'} \mathfrak{A}(E) \varpi_{D'}^{-1} \) is \( E \)-pure. Since \( \mathfrak{A}(E) \) is the unique \( E \)-pure 
hereditary \( \mathfrak{o}_F \)-order in \( A(E) \), we obtain \( \varpi_{D'} \mathfrak{A}(E) \varpi_{D'}^{-1} = \mathfrak{A}(E) \).

**Lemma 3.3** If \( K_0 \) is a compact open subgroup of \( G \) containing \( J \), then \( \tilde{J} \cap K_0 = J \). In 
particular, \( J \) is the unique maximal compact subgroup of \( \tilde{J} \).
Proof. If \((J, \lambda)\) is of level 0, then \(J\) is a maximal compact subgroup of \(G\), whence \(J \subset K_0\) implies \(J = K_0\).

If \((J, \lambda)\) is of level > 0, let \(L\) be a subgroup of \(\mathcal{R}(\mathfrak{B})\), generated by \(F^x\) and \(y^0\). Since \(y\) normalizes \(J\), we have \(\bar{J} = LJ\) and \(\bar{J} \cap K_0 = (L \cap K_0)J\). The group \(L \cap K_0\) is compact, whence contained in \(U(\mathfrak{B})\). Since \(U(\mathfrak{B})\) is a subgroup of \(J\), we have \(\bar{J} \cap K_0 = J\).

Put \(\tilde{\rho} = c\text{-Ind}^\mathfrak{R}(\mathfrak{B})_j \lambda\) and \(\rho = \text{Ind}^U(\mathfrak{B})_\lambda\). By the transivity of compact induction, we have \(\pi \cong c\text{-Ind}^G(\mathfrak{B}, \mathfrak{B}) \tilde{\rho}\). By Lemma 3.3 we have \(I_G(\lambda, \lambda) \cap U(\mathfrak{B}) = J\) and \(\rho\) is irreducible.

By the Mackey decomposition, we have
\[
\text{Res}^G_K \pi \cong \bigoplus_{g \in K \setminus (G/\mathfrak{R}(\mathfrak{B}))} \text{Ind}_K^{G \cap \mathfrak{R}(\mathfrak{B})} \text{Res}_K^\mathfrak{R}(\mathfrak{B})^g \tilde{\rho}.
\]

By the term corresponding to \(g = 1\) in the above decomposition, we have \(\pi|_K \supset \text{Ind}_U(\mathfrak{B}) \text{Res}_U^\mathfrak{R}(\mathfrak{B}) \tilde{\rho}\) since \(K \cap \mathfrak{R}(\mathfrak{B}) = U(\mathfrak{B})\). By the Mackey decomposition, we obtain
\[
\text{Res}_U^\mathfrak{B}(\mathfrak{B}) \tilde{\rho} \cong \bigoplus_{h \in U(\mathfrak{B}) \setminus \mathfrak{R}(\mathfrak{B})/\bar{J}} \text{Ind}_{U(\mathfrak{B})_j}^{U(\mathfrak{B})} \text{Res}_{U(\mathfrak{B})_j}^{h J} h \tilde{\lambda}.
\]

The group \(U(\mathfrak{B})\) is a normal subgroup of \(\mathfrak{R}(\mathfrak{B})\), hence \(U(\mathfrak{B}) = h U(\mathfrak{B})\), \(U(\mathfrak{B}) \cap h \bar{J} = h (U(\mathfrak{B}) \cap \bar{J}) = h J\) and \(\text{Res}_{U(\mathfrak{B})_j}^{h J} h \tilde{\lambda} = h \lambda\). Therefore we obtain
\[
\text{Res}_U^\mathfrak{B}(\mathfrak{B}) \tilde{\rho} \cong \bigoplus_{h \in U(\mathfrak{B}) \setminus \mathfrak{R}(\mathfrak{B})/\bar{J}} h \left(\text{Ind}_{U(\mathfrak{B})_j}^{U(\mathfrak{B})} \lambda\right) \cong \bigoplus_{h \in U(\mathfrak{B}) \setminus \mathfrak{R}(\mathfrak{B})/\bar{J}} h \rho.
\]

Corollary 3.4 Let \(h \in \mathfrak{R}(\mathfrak{B})\). Then, \(\text{Ind}_{U(\mathfrak{B})_j}^{U(\mathfrak{B})} h \rho\) is irreducible and a \([G, \pi]\)-type.

Proof. We have \(\text{Ind}_{U(\mathfrak{B})_j}^{U(\mathfrak{B})} h \rho = \text{Ind}_{U(\mathfrak{B})_j}^{U(\mathfrak{B})} \text{Ind}_{K_j}^U h \lambda = \text{Ind}_{h J} K_j h \lambda\). Since \((h J, h \lambda)\) is a simple type, it is enough to show that \(\text{Ind}_{h J} K_j h \lambda\) is irreducible by Lemma 3.1. We have
\[
I_K(h \lambda, h \lambda) = I_G(h \lambda, h \lambda) \cap K = h \bar{J} \cap K = (h J) \cap K = h J,
\]
from \(K \supset U(\mathfrak{B}) = h U(\mathfrak{B}) \supset h J\) and Lemma 3.3. Therefore, \(\text{Ind}_{h J} K_j h \lambda\) is irreducible.

Proposition 3.5 For every \(h \in \mathfrak{R}(\mathfrak{B})\), there exists \(k \in N_G(K)\) such that \(\text{Ind}_{h J} K_j  h \lambda \cong k(\text{Ind}_{h J} K_j h \lambda)\).

Proof. Assume that we find \(k \in N_G(K)\) such that \(h k \in \bar{J}\). Then \(h k\) is the element of \(I_G(\lambda, \lambda)\) and
\[
0 \neq \text{Hom}_J \left(\text{Res}_{J \cap K/h J}^J \lambda, \text{Res}_{J \cap K/h J}^{h J} h k \lambda\right) \cong \text{Hom}_J \left(\lambda, \text{Ind}_{J \cap K/h J}^J \text{Res}_{J \cap K/h J}^{h J} h k \lambda\right)
\]
\[
\cong \text{Hom}_J \left(\lambda, \bigoplus_{j \in J \cap K/h J} \text{Ind}_{J \cap K/h J}^J \text{Res}_{J \cap K/h J}^{j h J} h \lambda\right)
\]
\[
\cong \text{Hom}_J \left(\lambda, \text{Res}_J^K \text{Ind}_{J \cap K/h J}^K h \lambda\right) \cong \text{Hom}_K \left(\text{Ind}_{h J}^J \lambda, k(\text{Ind}_{h J}^{h J} h \lambda)\right).
\]
Since $\text{Ind}_J^G \lambda$ and $\text{Ind}_J^K \lambda$ are irreducible by Corollary 3.4, they are isomorphic. Therefore, it is enough to show that there exists $k \in N_G(K)$ such that $kh \in \hat{J}$.

First, we fix a uniformizer $\varpi_D$ of $D$ and embed $\sigma_D$ in $M_n(\sigma_D)$ diagonally. Then $\varpi_D \in \mathfrak{R}(\mathfrak{A})$ and $N_G(K)$ is generated by $K$ and $\varpi_D$.

We will show that $v_\mathfrak{A}(N_G(K) \cap \mathfrak{R}(\mathfrak{A}))$ and $v_\mathfrak{A}(\hat{J})$ generate $\mathbb{Z}$. If $(J, \lambda)$ is of level 0, then $\mathfrak{A}$ is maximal and $v_\mathfrak{A}(\varpi_D) = 1$, whence $v_\mathfrak{A}((N_G(K) \cap \mathfrak{R}(\mathfrak{A})) = \mathbb{Z}$. Then we may assume that $(J, \lambda)$ is of level $> 0$ with a simple stratum $[\mathfrak{A}, n, 0, \beta]$. We put $E = F[\beta]$. Since $\mathfrak{A}$ is proper and $\mathfrak{B}$ is maximal, the period of $\mathfrak{A}$ is equal to the period of $\mathfrak{A}(E)$ and we have $v_\mathfrak{A}(\varpi_D) = (1/d)v_\mathfrak{A}(\varpi_F) = e/(d, e')$ by Proposition 2.10. Hence, we have

\[ v_\mathfrak{A}(N_G(K) \cap \mathfrak{R}(\mathfrak{A})) \supset v_\mathfrak{A}(\varpi_D) = \frac{e'}{(d, e')} \mathbb{Z}. \]

On the other hand, we choose $y \in B^\times$ and $l_0 \in \mathbb{N}_{>0}$ as in 2.5. Then we have $v_\mathfrak{A}(y) = v_\mathfrak{A}(E(\varpi_D')) = (d, [E : F])/(d, e')$ by the proof of Proposition 2.10. Moreover, $l_0$ is a divisor of $|\text{Gal}(k_{D'}/k_E)| = d/(d, [E : F])$ by definition of $l_0$. Therefore

\[ v_\mathfrak{A}(\hat{J}) \supset v_\mathfrak{A}(y^{l_0}) = \frac{(d, [E : F])l_0}{(d, e')} \mathbb{Z} \supset \frac{d}{(d, e')} \mathbb{Z} \]

and we obtain

\[ v_\mathfrak{A}(N_G(K) \cap \mathfrak{R}(\mathfrak{A})) + v_\mathfrak{A}(\hat{J}) \supset \frac{e'}{(d, e')} \mathbb{Z} + \frac{d}{(d, e')} \mathbb{Z} = \frac{(d, e')}{(d, e')} \mathbb{Z} = \mathbb{Z}. \]

For fixed $h$, by the above discussion, there exist $k_1 \in N_G(K) \cap \mathfrak{R}(\mathfrak{A})$ and $j \in \hat{J}$ such that $v_\mathfrak{A}(k_1) + v_\mathfrak{A}(j) = v_\mathfrak{A}(h^{-1})$. Then we have $v_\mathfrak{A}(k_1 h j) = 0$ and $k_2 = k_1 h j \in U(\mathfrak{A})$. We put $k = k_2^{-1} k_1 \in N_G(K)$, whence we obtain $k h = k_2^{-1} k_1 h = j^{-1} \in \hat{J}$ and complete the proof.

For a general $g \in G$,

\[ \text{Res}_{K/G}^g \varrho = \bigoplus_{h \in U(\mathfrak{A}) \setminus \mathfrak{R}(\mathfrak{A}) \setminus \hat{J}} \text{Res}_{K/G}^g U(\mathfrak{A}) \text{Res}_{U(\mathfrak{A})}^g \varrho = \bigoplus_{h \in U(\mathfrak{A}) \setminus \mathfrak{R}(\mathfrak{A}) \setminus \hat{J}} \text{Res}_{E/G}^g U(\mathfrak{A}) \text{Res}_{U(\mathfrak{A})}^g \varrho. \]

Hence, we obtain

\[ \text{Res}_K^G \pi \simeq \bigoplus_{g \in K \setminus G/\mathfrak{R}(\mathfrak{A})} \bigoplus_{h \in U(\mathfrak{A}) \setminus \mathfrak{R}(\mathfrak{A}) \setminus \hat{J}} \text{Ind}_{K/G}^U U(\mathfrak{A}) \text{Res}_{K/G}^g U(\mathfrak{A}) \text{Res}_{U(\mathfrak{A})}^g \varrho. \]

**Lemma 3.6** Let $g \in G$ and $h \in \mathfrak{R}(\mathfrak{A})$. Let $\tau$ be a representation of $K$ such that

\[ \langle \tau, \text{Ind}_{K/G}^U U(\mathfrak{A}) \varrho \rangle_K \neq 0. \]

Then there exists $h' \in \mathfrak{R}(\mathfrak{A})$ such that

\[ h' \varrho = \text{Ind}_{J'}^U \lambda', \quad \langle \tau, \text{Ind}_{K/G,J'}^U \lambda' \rangle_K \neq 0, \]

where $(J', \lambda') = (h', J, h' \lambda)$ is a simple type.
Proof. By the Mackey decomposition,
\[
\text{Res}^\mathcal{U(A)}_{y^{-1}K \cap \mathcal{U(A)}} h \rho \\
\cong \bigoplus_{u \in y^{-1}K \cap \mathcal{U(A)} \setminus \mathcal{U(A)}/uh} \text{Ind}^{y^{-1}K \cap \mathcal{U(A)}}_{y^{-1}K \cap \mathcal{U(A)} \cap uh} \text{Res}^{uh}_{y^{-1}K \cap \mathcal{U(A)} \cap uh} \lambda \\
= \bigoplus_{u \in y^{-1}K \cap \mathcal{U(A)} \setminus \mathcal{U(A)}/uh} \text{Ind}^{y^{-1}K \cap \mathcal{U(A)}}_{y^{-1}K \cap uh} \text{Res}^{uh}_{y^{-1}K \cap uh} \lambda.
\]

Then
\[
\text{Ind}^{y^{-1}K \cap \mathcal{U(A)}}_{y^{-1}K \cap \mathcal{U(A)}} h \rho \\
\cong \bigoplus_{u \in y^{-1}K \cap \mathcal{U(A)} \setminus \mathcal{U(A)}/uh} \text{Ind}^{y^{-1}K \cap \mathcal{U(A)}}_{y^{-1}K \cap uh} \text{Res}^{uh}_{y^{-1}K \cap uh} \lambda.
\]

Therefore there exists \( u \in \mathcal{U(A)} \) such that
\[
\langle \tau, \text{Ind}^{K \cap \mathcal{U(A)}}_{K \cap \mathcal{U(A)}} g^{uh} \lambda \rangle_K = \langle g^{-1} \tau, \text{Ind}^{y^{-1}K \cap \mathcal{U(A)}}_{y^{-1}K \cap uh} \lambda \rangle_{y^{-1}K} \neq 0.
\]

Then \( h' = uh \) and \( (J', \lambda') = (uh, uh \lambda) \) satisfies the conditions of Lemma.

Proposition 3.7 Let \( g \in G \) and \( h \in \mathfrak{R}(\mathfrak{A}) \). Let \( \tau \) be an irreducible representation of \( K \) such that
\[
\langle \tau, \text{Ind}^{K \cap \mathcal{U(A)}}_{K \cap \mathcal{U(A)}} g^{h} \rangle_K \neq 0.
\]
Let \( (J, \lambda) \) be a simple type such that
\[
h \rho = \text{Ind}^{\mathcal{U(A)}}_{J} \lambda \text{ and } \langle \tau, \text{Ind}^{K \cap \mathcal{U(A)}}_{K \cap \mathcal{U(A)}} g \lambda \rangle_K \neq 0.
\]

Suppose for every irreducible summand \( \xi \) of \( \lambda|_{J \cap y^{-1}K} \), there exists an irreducible representation \( \lambda' \) of \( J \) such that
\[
\langle \xi, \lambda' \rangle_{J \cap y^{-1}K} \neq 0 \text{ and } I_G(\lambda, \lambda') = \emptyset.
\]

Then, \( \tau \) cannot be a type.

Proof. Assume that \( \tau \) is a type. Let \( \tilde{\tau} \) be an extension of \( \tau \) to \( F^\times K \). Then, by \([2], (5.2)\)
there are finitely many unramified characters \( \chi_i \) of \( F^\times \) such that
\[
c-\text{Ind}_{F^\times K}^{G} \tilde{\tau} = \bigoplus_i \pi \otimes (\chi_i \circ \text{Nrd}_{A/F}).
\]
Hence every irreducible subrepresentation of
\[
\mathrm{Res}_J^G \ c\mathrm{-Ind}_F^G J \tilde{\tau} \cong \bigoplus_{g' \in J \backslash G / F} \mathrm{Ind}_{J \cap g' F / J}^J \mathrm{Res}_{J \cap g' F / J}^{g' F / J} \tilde{\tau}
\]
is contained in \( \pi |_J \). In particular, \( \pi |_J \) contains every irreducible subrepresentation of
\[
\mathrm{Ind}_{J \cap g^{-1} K}^J \mathrm{Res}_{J \cap g^{-1} K}^{g^{-1} \tau}. 
\]
Since
\[
\langle g^{-1} \tau, \lambda \rangle_{J \cap g^{-1} K} = \langle \tau, g \lambda \rangle_{K \cap g^{-1} K} = \langle \tau, \mathrm{Ind}_{K \cap g^{-1} K}^K g \lambda \rangle_K \neq 0,
\]
there exists an irreducible summand \( \xi \) of \( \lambda |_{J \cap g^{-1} K} \) such that \( \langle g^{-1} \tau, \xi \rangle_{J \cap g^{-1} K} \neq 0 \). By our assumption, there exists an irreducible representation \( \lambda' \) of \( J \) such that
\[
\langle \xi, \lambda' \rangle_{J \cap g^{-1} K} \neq 0, \text{ and } I_G(\lambda, \lambda') = \emptyset.
\]
Then
\[
\langle \lambda', \mathrm{Ind}_{J \cap g^{-1} K}^J \mathrm{Res}_{J \cap g^{-1} K}^{g^{-1} \tau} \rangle_J = \langle \lambda', \mathrm{Ind}_{J \cap g^{-1} K}^J \mathrm{Res}_{J \cap g^{-1} K}^{g^{-1} \tau} \rangle_{J \cap g^{-1} K} \geq \langle \xi, g^{-1} \tau \rangle_{J \cap g^{-1} K} > 0,
\]
whence \( \lambda' \) is a \( J \)-subrepresentation of \( \mathrm{Ind}_{J \cap g^{-1} K}^J \mathrm{Res}_{J \cap g^{-1} K}^{g^{-1} \tau} \), so \( \lambda' \) is contained in \( \pi |_J \).

On the other hand, there exists an extension \((\tilde{J}, \tilde{\lambda})\) of \((J, \lambda)\) such that \( \pi \cong \mathrm{c-Ind}_J^G \tilde{\lambda} \). Then
\[
\lambda' \subset \pi |_J \cong \bigoplus_{g' \in J \backslash G / J} \mathrm{Ind}_{J \cap g' J}^J \mathrm{Res}_{J \cap g' J}^{g' \tilde{\lambda}}
\]
and there is \( g' \in G \) such that
\[
\lambda' \subset \mathrm{Ind}_{J \cap g' J}^J \mathrm{Res}_{J \cap g' J}^{g' \tilde{\lambda}} = \mathrm{Ind}_{J \cap g' J}^J \mathrm{Res}_{J \cap g' J}^{g' \lambda},
\]
where since \( J \cap (g' J) \) is a compact subgroup of \((g' J)\), the group \( J \cap (g' J) \) is a subgroup in \( g' J \) by Lemma 4.3 and we have \( J \cap (g' J) = J \cap (g' J) \cap g' J = J \cap g' J \). Therefore \( \langle \lambda', g' \lambda \rangle_{J \cap g' J} \neq 0 \) and \( I_G(\lambda, \lambda') \neq \emptyset \), which is a contradiction.

4 Representatives of \( K \backslash G / \mathfrak{K}(\mathfrak{A}) \)

Fix a uniformizer \( \varpi_D \) of \( D \). Put
\[
R = \left\{ \left( \begin{array}{c} \varpi_D^{a_1} \\ \vdots \\ \varpi_D^{a_m} \end{array} \right) \middle| a_1, \ldots, a_m \in \mathbb{Z} \right\}.
\]
For $a_1, \ldots, a_m \in \mathbb{Z}$, the diagonal matrix whose entries are $\omega_D^{a_1}, \omega_D^{a_2}, \ldots, \omega_D^{a_m}$ is denoted by $a_R(a_1, \ldots, a_m)$.

Lemma 4.1 Let $g'$ be an element of $G$ such that $Kg' \not\in K \mathcal{R}(\mathfrak{A})$. Then, there exists an element $g$ of $R$ such that its entries are $\omega_D^{a_1}, \omega_D^{a_2}, \ldots, \omega_D^{a_m}$ ($a_1, \ldots, a_m \in \mathbb{N}$), $a_{(m/e)+1} \geq \cdots \geq a_{(m/e)(i+1)}$ for $0 \leq i < e$, and one of the following conditions holds:

(i) There exists $0 \leq i < e$ and $1 \leq j < m/e$ such that $a_{(m/e)i+j} > a_{(m/e)i+j+1}$.

(ii) We have $a_{(m/e)i+1} = \cdots = a_{(m/e)(i+1)}$ for $0 \leq i < e$ and

- $a_1 \geq 2$,
- there exists $2 \leq j \leq e$ such that for $1 \leq i < e$ we have $a_{(m/e)i} > 0$ if and only if $i < j$.

Proof. Let $\mathfrak{A}_0 \subset \mathfrak{A}$ be a minimal hereditary $\mathfrak{A}$-order which consists of upper triangular matrices modulo $p_D$. The groups of permutation matrices, isomorphic to $S_m$, is naturally embedded in $G$. Let $W$ be the semidirect product of $S_m$ and $R$ in $G$. Then we have $G = \prod_{w \in W} U(\mathfrak{A}_0)w U(\mathfrak{A}_0)$ by [7] \S[0.8]. Therefore we obtain $G = \bigcup_{w \in W} Kg(\mathfrak{A}) = \bigcup_{g \in R} Kg(\mathfrak{A})$. Then we can pick $g'' = a_R(a_1, \ldots, a_m) \in Kg(\mathfrak{A}) \cap R$. Since $a_R(1, \ldots, 1) \in \mathcal{R}(\mathfrak{A})$, we may assume $a_1, \ldots, a_m \in \mathbb{N}$ and there exists $1 \leq i \leq m$ such that $a_i = 0$. We have $S_m^{x \leq e} \subset U(\mathfrak{A})$, so we may assume $a_{(m/e)i+1} \geq \cdots \geq a_{(m/e)(i+1)}$ for $0 \leq i < e$.

If there exists $0 \leq i < e$ and $1 \leq j < m/e$ such that $a_{(m/e)i+j} > a_{(m/e)i+j+1}$, then $g = g''$ satisfies (i). Otherwise, we define

$$t = \begin{pmatrix} I_{m/e} & & & \\ & \ddots & & \\ & & I_{m/e} & \\ I_{m/e} & & & I_{m/e} \end{pmatrix}$$

and

$$b_R(b_1, \ldots, b_e) = \begin{pmatrix} \omega_D^{b_1} I_{m/e} & & \\ & \ddots & \\ & & \omega_D^{b_e} I_{m/e} \end{pmatrix} \in R,$$

where $b_1, \ldots, b_e \in \mathbb{Z}$. Then we may assume there exist $b_1, \ldots, b_e \in \mathbb{N}$ such that $g'' = b_R(b_1, \ldots, b_e)$ and $\min \{b_1, \ldots, b_m\} = 0$. We put

$$\Pi = \begin{pmatrix} I_{m/e} & & \\ & \ddots & \\ & & I_{m/e} \\ \omega_D I_{m/e} & & I_{m/e} \end{pmatrix} \in \mathcal{R}(\mathfrak{A}),$$

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where $I_{m/e}$ is the $(m/e) \times (m/e)$ identity matrix. We define a bijective map $f : \tilde{W} \to \tilde{W}$ as

$$f(w) = t^{-1}wI, \quad w \in \tilde{W}.$$  

We have $f(b_R(b_1, \ldots, b_e)) = b_R(b_e + 1, b_1, \ldots, b_{e-1})$ and $f(g^n) \in Kg'\mathcal{R}(\mathfrak{A})$.

We put $i_1 = \min \{ 1 \leq i \leq e \mid b_i = 0 \}$ and $b_R(b'_1, \ldots, b'_e) = f^{e-i_1}(g^n)$. If $b'_1 \geq 2$, then $g = f^{e-i_1}(g^n)$ satisfies (ii) with $j = e$. If $b'_1 = 1$, we put $i_2 = \max \{ i \mid b_i = 1 \text{ for } i' \leq i \}$. If $i_2 = e-1$, then $(f^{-1})^{e-i} \circ f^{e-i}(g^n) = I_m$, and $Kg'\mathcal{R}(\mathfrak{A}) = K\mathcal{R}(\mathfrak{A})$, which is a contradiction. Therefore we have $i_2 < e - 1$. Then $g = (f^{-1})^{i_2} \circ f^{e-i_1}(g^n)$ satisfies (ii) with $j = e - i_2$. ■

**Definition 4.2** Let $g, g' \in G$. We say that a pair $(g, Kg'\mathcal{R}(\mathfrak{A}))$ has Property A if $Kg'\mathcal{R}(\mathfrak{A}) \neq K\mathcal{R}(\mathfrak{A})$ and $g$ is a representative of $Kg'\mathcal{R}(\mathfrak{A})$ as in Lemma 4.1 and satisfying (i).

**Lemma 4.3** If $\mathfrak{A}$ is maximal, then for every double coset $Kg'\mathcal{R}(\mathfrak{A})$ not equal to $K\mathcal{R}(\mathfrak{A})$, there is $g \in Kg'\mathcal{R}(\mathfrak{A})$ such that the pair $(g, Kg'\mathcal{R}(\mathfrak{A}))$ has Property A.

**Proof.** Since $\mathfrak{A}$ is maximal, we have $e = 1$. We pick $g^n = a_R(a_1, \ldots, a_m) \in Kg'\mathcal{R}(\mathfrak{A}) \cap R$ as in the proof of Lemma 4.1. The equation $e = 1$ implies $a_1 \geq \cdots \geq a_m \geq 0$. If $a_1 = \cdots = a_m$, then $g^n \in \mathcal{R}(\mathfrak{A}) \subset K\mathcal{R}(\mathfrak{A})$, which is a contradiction. Therefore there exists $1 \leq j < m$ such that $a_j > a_{j+1}$, so $(g^n, Kg'\mathcal{R}(\mathfrak{A}))$ has Property A. ■

**Proposition 4.4** Assume that a pair $(g, Kg'\mathcal{R}(\mathfrak{A}))$ has Property A. Then there is $0 \leq i_0 < e$ such that the image of $U(\mathfrak{A}) \cap g^{-1}K$ under the map

$$U(\mathfrak{A}) \to U(\mathfrak{A})/U^1(\mathfrak{A}) \cong \prod_{i=0}^{e-1} \text{Aut}_{kD}(\Lambda_i/\Lambda_{i+1}) \to \text{Aut}_{kD}(\Lambda_{i_0}/\Lambda_{i_0+1})$$

is contained in some proper parabolic subgroup of $\text{Aut}_{kD}(\Lambda_{i_0}/\Lambda_{i_0+1})$.

**Proof.** For $0 < i \leq e$, the morphism $U(\mathfrak{A}) \to \text{Aut}_{kD}(\Lambda_{e-i}/\Lambda_{e-i+1}) \cong \text{GL}_{m/e}(k_D)$ is as follows:

$$(A_{1,1} \cdots A_{1,e}) \vdots \ddots \vdots \vdots \vdots \vdots \vdots A_{e,1} \cdots A_{e,e}) \mapsto A_{i,i} \text{ mod } p_D,$$

where $A_{j,j'} \in M_{m/e}(a_D)$ for $1 \leq j,j' \leq e$. Since $g$ is as in Lemma 4.1 and satisfying (i), there exist $0 \leq i_1 < e$ and $1 \leq j_1 < m/e$ such that $a_{(m/e)i_1+j_1} > a_{(m/e)i_1+j_1+1}$. If $h = (h_{ij}) \in U(\mathfrak{A}) \cap g^{-1}K$, then $h_{ij} \in p_D$ for $(m/e)i_1 + j_1 < i \leq (m/e)(i_1 + 1)$ and $(m/e)i_1 + 1 \leq j \leq (m/e)i_1 + j_1$. We put $i_0 = e - (i_1 + 1)$. Then the image of $h = (h_{ij}) \in U(\mathfrak{A}) \cap g^{-1}K$ under the map $U(\mathfrak{A}) \to \text{Aut}_{kD}(\Lambda_{i_0}/\Lambda_{i_0+1})$ is contained in a proper parabolic subgroup of $\text{Aut}_{kD}(\Lambda_{i_0}/\Lambda_{i_0+1})$. ■
5 Double Cosets with Property A

Definition 5.1 Suppose \( (g, Kg\mathbb{K}(\mathfrak{A})) \) has Property A. We define the subgroup \( \mathcal{K}(g) \) of \( U(\mathfrak{A}) \) by

\[ \mathcal{K}(g) = \left( U(\mathfrak{A}) \cap g^{-1}K \right) U^1(\mathfrak{A}). \]

Definition 5.2 Let \( H \) be a subgroup of \( GL_N(\mathbb{F}_q) \) and put

\[ S = \left\{ h \in GL_N(\mathbb{F}_q) \mid \chi_h(X) = f(X)^l \text{ for some irreducible polynomial } f \in \mathbb{F}_q[X] \text{ and } l \in \mathbb{N} \right\}, \]

where \( \chi_h \) is the characteristic polynomial of \( h \).

The group \( H \) is sufficiently small if there exists a proper subfield \( \mathbb{F} \) of \( \mathbb{F}_{q^N} \) such that for every \( h \in H \cap S \) the roots of \( \chi_h \) are the elements of \( \mathbb{F} \).

The lemma in the following are proved in [10] Lemma 6.5.

Lemma 5.3 Let \( \mathbb{F}_q/\mathbb{F}_{q'} \) be an extension of finite fields and \( V_0 \) be a finite dimensional \( \mathbb{F}_q \)-vector space. Let

\[ \iota: \text{End}_{\mathbb{F}_q}(V_0) \rightarrow \text{End}_{\mathbb{F}_{q'}}(V_0) \]

be the natural embedding by scalar restriction. If \( H \) is a subgroup of \( \text{Aut}_{\mathbb{F}_q}(V_0) \) such that \( \iota(H) \) is contained in a proper parabolic subgroup of \( \text{Aut}_{\mathbb{F}_{q'}}(V_0) \), then \( H \) is sufficiently small.

Proposition 5.4 Suppose that a pair \( (g, Kg\mathbb{K}(\mathfrak{A})) \) has Property A and that \( E/F \) is unramified. The image of \( \mathcal{K}(g) \cap U(\mathfrak{B}) \) in \( U(\mathfrak{B})/U^1(\mathfrak{B}) \) is sufficiently small.

Proof. Let \( E_1/F \) be the unique field extension in \( E \) as in Lemma 2.12. Put \( B_1 = \text{Cent}_{A}(E_1) \) and \( \mathfrak{B}_1 = \mathfrak{A} \cap B_1 \). We also fix an inclusion \( E_1 \in D \).

We will show that \( \mathfrak{B}_1 \) is a maximal hereditary \( \mathfrak{o}_{E_1} \)-order. Let \( W \) be a simple right \( E \otimes_F D \)-module. By Lemma 2.12 there is a right \( D \)-basis \( B \) of \( W \) such that \( \mathfrak{A}(E) \) is identified with \( M_{[E:E_1]}(\mathfrak{o}_D) \) under the identification \( A(E) \cong M_{[E:E_1]}(D) \) induced by \( \mathfrak{B} \). On the other hand, since \( \mathfrak{A} \) is proper, there is an \( E \otimes_F D \)-isomorphism \( V \cong W^{\oplus \mathfrak{m}'} \) which induces the identification between \( A \) and \( M_{m'}(A(E)) \) such that \( \mathfrak{A} = M_{m'}(\mathfrak{A}(E)) \).

We define a \( D \)-basis \( B_1 \) of \( V \) by \( \mathfrak{B} \) and the \( D \)-isomorphism \( V \cong W^{\oplus \mathfrak{m}'} \), and obtain the identification \( A \cong M_{\mathfrak{m}}(D) \). Under this identification, the subrings \( \mathfrak{A}, E \) and \( E_1 \) in \( A \) correspond to the subrings in \( M_{\mathfrak{m}}(D) \) as follows:

- \( \mathfrak{A} = M_{m'}(M_{[E:E_1]}(\mathfrak{o}_D)) \).
- \( E \rightarrow A(E) \rightarrow M_{m'}(A(E)) \), where \( E \rightarrow A(E) \) is the canonical embedding and \( A(E) \rightarrow M_{m'}(A(E)) \) is the diagonal embedding.
• $E_1 \hookrightarrow D \hookrightarrow M_m(D)$, where $E_1 \hookrightarrow D$ is the inclusion we fixed above and $D \hookrightarrow M_m(D)$ is the diagonal embedding.

We put $D_1 = \text{Cent}_D(E_1)$, and then $B_1 = \text{Cent}_A(E_1) = M_m(D_1)$, so we have $\mathfrak{A} \cap B_1 = M_m(\sigma_{D_1})$ and $\mathfrak{B}_1$ is maximal.

Since $E_1/F$ is unramified, we have $k_{D_1} = k_D$ and $U(\mathfrak{B}_1)/U^1(\mathfrak{B}_1) \cong U(\mathfrak{A})/U^1(\mathfrak{A})$. Since $K(g) \supset U^1(\mathfrak{A})$, under the identification $U(\mathfrak{B}_1)/U^1(\mathfrak{B}_1) \cong U(\mathfrak{A})/U^1(\mathfrak{A})$, the image of $U(\mathfrak{B}_1)/K(g)$ in $U(\mathfrak{B}_1)/U^1(\mathfrak{B}_1)$ is identified with $K(g)/U^1(\mathfrak{A})$, which is equal to the image of $U(\mathfrak{A})/U^1(\mathfrak{A})$ and contained in some proper parabolic subgroup of $U(\mathfrak{A})/U^1(\mathfrak{A})$ by Proposition 4.4, as $\mathfrak{A}$ is maximal. Therefore the image of $U(\mathfrak{B}_1)/K(g)$ in $U(\mathfrak{B}_1)/U^1(\mathfrak{B}_1)$ is contained in some proper parabolic subgroup of $U(\mathfrak{B}_1)/U^1(\mathfrak{B}_1)$. We also have $B = \text{Cent}_{B_1}(E) = B_1$ is $E$-pure, and that $\mathfrak{B} = \mathfrak{B}_1 \cap B$. Hence, we may assume that $E_1 = F$.

The hereditary $\sigma_F$-order $\mathfrak{A}$ is maximal, and we have $\Lambda_{i+1} = \Lambda_i \varpi_D$. Since $\mathfrak{A}$ is $E$-pure, $\Lambda$ is also an $\sigma_E \otimes \sigma_p \sigma_D$-chain, where we have $\sigma_E \otimes \sigma_p \sigma_D \cong \sigma_{D'}$ because $[E:F]$ and $d$ are coprime and $E/F$ is unramified. Let $\mathfrak{B}'$ be the hereditary $\sigma_E$-order in $B = \text{End}_{D'}(V)$ corresponding to $\Lambda$. Since $E/F$ is unramified, $\varpi_D$ is also a uniformizer of $D'$ and $\Lambda_{i+1} = \Lambda_i \varpi_{D'}$ holds. Therefore we have

$$U(\mathfrak{B}')/U^1(\mathfrak{B}') \cong \text{Aut}_{k_{D'}}(\Lambda_0/\Lambda_1).$$

On the other hand, $\mathfrak{B}' = \{x \in B \mid x \Lambda_i \subset \Lambda_i, i \in \mathbb{Z}\} = \mathfrak{A} \cap B = \mathfrak{B}$ also holds. By Lemma 5.3 with $V_0 = \Lambda_0/\Lambda_1$, we obtain this proposition. 

Next, we examine the restriction of some cuspidal representations of $\text{GL}_N(\mathbb{F}_q)$ to its sufficiently small subgroup. Let $p$ be the characteristic of $\mathbb{F}_q$.

**Lemma 5.5** For integers $q > 1$ and $N > 1$, there exists a prime number $r$ such that $r$ divides $q^N - 1$, but does not divide $q^s - 1$ for any $0 < s < N$, except when $N = 2$ and $q = 2^i - 1$ with $i \geq 2$, or $N = 6$ and $p = 2$.

**Proof.** This result is known as Zsigmondy’s Theorem in [15].

**Proposition 5.6** Let $\sigma$ be an irreducible cuspidal representation of $\text{GL}_N(\mathbb{F}_q)$ with character $\chi$, and $H$ be a sufficiently small subgroup of $\text{GL}_N(\mathbb{F}_q)$. Suppose $[\mathbb{F}_q : \mathbb{F}_p] > 1$, and if $p = 2$ and $[\mathbb{F}_q : \mathbb{F}_p] \leq 3$, we further assume that $H$ is contained in some proper parabolic subgroup of $\text{GL}_N(\mathbb{F}_q)$. If $\xi$ is an irreducible representation of $H$ such that $\langle \xi, \sigma \rangle_H \neq 0$, then there exists an irreducible representation $\sigma'$ of $\text{GL}_N(\mathbb{F}_q)$ such that $\langle \xi, \sigma' \rangle_H \neq 0$ and $\sigma' \neq \gamma \cdot \sigma$ for any $\gamma \in \text{Gal}(\mathbb{F}_q/\mathbb{F}_p)$.

**Proof.** Let $S$ be as in Definition 5.2. Since $H$ is sufficiently small, there exists a proper subfield $\mathbb{F}$ of $\mathbb{F}_q^N$ such that for every $h \in H \cap S$ the roots of $\chi_h$ are in $\mathbb{F}$. Let $\Gamma = \text{GL}_N(\mathbb{F}_q), a = [\mathbb{F}_q : \mathbb{F}_p]$ and $b = [\mathbb{F} : \mathbb{F}_p]$. 

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According to [6], for a character $\Psi$ of $\mathbb{F}_q^\times$, such that $\Psi^{q^s-1} \neq 1$ for any $s$ dividing but not equal to $N$, we can define a class function $X_{\Psi}$ of $\Gamma$, which is a character of an irreducible cuspidal representation of $\Gamma$. Every character of an irreducible cuspidal representation of $\Gamma$ is obtained from a character of $\mathbb{F}_q^\times$ as above. If $\Theta$ is another character of $\mathbb{F}_q^\times$ such that $\Theta^{q^s-1} \neq 1$ for any $s$ dividing but not equal to $N$, we have $X_{\Theta} = X_{\Psi}$ if and only if $\Theta = \Psi^{q^s}$ for some $s \geq 0$. Moreover, if $\Psi|_{\mathbb{F}^\times} = \Theta|_{\mathbb{F}^\times}$, then $X_{\Psi}|_H = X_{\Theta}|_H$.

For every $\gamma \in \text{Gal}(\mathbb{F}_q/\mathbb{F}_p)$, we take a character $\Psi_\gamma$ of $\mathbb{F}_q^\times$ corresponding to $\gamma \cdot \sigma$. Let $\Psi = \Psi_1$. Suppose there exists another character $\Theta$ of $\mathbb{F}_q^\times$ such that $\Theta^{q^s-1} \neq 1$ for any $s$ dividing but not equal to $N$, $\Psi|_{\mathbb{F}^\times} = \Theta|_{\mathbb{F}^\times}$, and that $\Theta \neq \Psi^{q^s}$ for any $s \geq 0$ and $\gamma \in \text{Gal}(\mathbb{F}_q/\mathbb{F}_p)$. Let $\sigma'$ be a cuspidal representation of $\Gamma$ with character $X_{\Theta}$. Then $\sigma' \neq \gamma \cdot \sigma$ for any $\gamma \in \text{Gal}(\mathbb{F}_q/\mathbb{F}_p)$ because $\Theta \neq \Psi_{\gamma}^{q^s}$ for any $s$. Moreover, we have $X_{\Psi}|_H = X_{\Theta}|_H$, so $\sigma|_H = \sigma'|_H$.

In almost all cases, we can pick such $\Theta$ by counting characters of $\mathbb{F}_q^\times$. By Lemma 5.5, there exists a prime number $r$ such that $r$ divides $p^{aN} - 1 = q^N - 1$, but does not divide $p^s - 1$ for any $0 < s < aN$, except when $aN = 2$ and $p = 2^i - 1$ with $i \geq 2$, or $aN = 6$ and $p = 2$.

Suppose such $r$ as above exists. If $\Theta$ is not an $r$th power, $r$ divides the order of $\Theta$ since $\mathbb{F}_q^\times$ is cyclic. Then $\Theta^{q^s-1} \neq 1$ for any $0 < s < N$ as $0 < as < aN$ and $p^{as} - 1 = q^s - 1$ is not divided by $r$. In particular, $\Theta^{q^s-1} \neq 1$ for any $s$ dividing but not equal to $N$.

We count the characters $\Theta$ such that $\Theta$ is an $r$th power and $\Psi|_{\mathbb{F}^\times} = \Theta|_{\mathbb{F}^\times}$. At first, there exist $(q^N - 1)/(p^b - 1)$ characters such that $\Psi|_{\mathbb{F}^\times} = \Theta|_{\mathbb{F}^\times}$. For any character $\chi$ of $\mathbb{F}^\times$, let $\mathcal{C}(\chi)$ be the set of characters $\Theta$ of $\mathbb{F}_q^\times$ such that $\Theta$ is an $r$th power and $\Theta|_{\mathbb{F}^\times} = \chi$. Since $r$ does not divide $p^b - 1 = |\mathbb{F}^\times|$, we can take a character $\Theta_0 \in \mathcal{C}(\Psi|_{\mathbb{F}^\times})$. By multiplying $\Theta_0^{-1}$, we obtain one-to-one correspondence between $\mathcal{C}(\Psi|_{\mathbb{F}^\times})$ and $\mathcal{C}(1)$. The characters of $\mathbb{F}_q^\times$ trivial on $\mathbb{F}^\times$ correspond to the characters of $\mathbb{F}_q^\times/\mathbb{F}^\times$. Under this correspondence, the elements of $\mathcal{C}(1)$ correspond to the characters that are $r$th power, since $r$ does not divide $|\mathbb{F}^\times|$. Therefore, there will be

\[
\left(1 - \frac{1}{r}\right) \frac{q^N - 1}{p^b - 1}
\]

characters $\Theta$ such that $\Theta$ is not an $r$th power and $\Psi|_{\mathbb{F}^\times} = \Theta|_{\mathbb{F}^\times}$. The cardinality of the set $\{\Psi_{\gamma}^{q^s} \mid \gamma \in \text{Gal}(\mathbb{F}_q/\mathbb{F}_p), s \in \mathbb{N}\}$ is at most $aN$, so if the inequation

\[
\left(1 - \frac{1}{r}\right) \frac{q^N - 1}{p^b - 1} > aN
\]

holds, then we can take the desired character $\Theta$. Since $q^N = p^{aN}$ and $p^b = |\mathbb{F}| \leq |\mathbb{F}_q^\times|^{1/2} = p^{aN/2}$, it is enough to show

\[
\left(1 - \frac{1}{r}\right) (p^{aN/2} + 1) > aN.
\]
By our assumption, we have $a \geq 2$ and $aN \geq 2$. By using induction on $aN$, we can examine when the inequation holds. When $p \geq 5$, for any prime $r$ the inequation holds.

When $p = 3$, we may assume $r \geq 5$ and then the inequation holds. When $p = 2$, since $3 = 2^2 - 1$, we may assume $r \geq 5$ and then the inequation holds for $aN = 3$ or $aN > 5$.

The remaining cases are following: $(aN, p) = (6, 2), (4, 2), (2, 2)$, or $(2, 2^i - 1)$ with $i \geq 2$.

When $aN = 2$, we have $a = 2$ and $N = 1$, so $q = p^2$ and $F = \mathbb{F}_p$. Since $N = 1$, every irreducible representation of $\Gamma$ is a character. Our assumption that $\text{Res}_{\mathbb{F}_2} \Gamma$ is sufficiently small implies $H < \mathbb{F}_p^\times$. If $\chi$ is a character of $\mathbb{F}_p^\times$, there exist $p + 1$ characters of $\mathbb{F}_q^\times$ extending $\chi$. Since $p + 1 > 2$, there exists a character $\sigma'$ of $\mathbb{F}_q^\times$ extending $\sigma|_H$ such that $\sigma' \neq \sigma, \gamma \cdot \sigma$, where $\gamma \in \text{Gal}(\mathbb{F}_q/\mathbb{F}_p)$ is the nontrivial element.

Suppose $(aN, p) = (6, 2)$ or $(4, 2)$.

If $N = 1$, then $b \leq a/2$ and $H < \mathbb{F}_p^\times$. If $\chi$ is a character of $\mathbb{F}_q^\times$, there exist at least $2^{a/2} + 1$ characters of $\mathbb{F}_q^\times$ extending $\chi$. Because $2^2 + 1 > 4$ and $2^2 + 1 > 6$, we can deduce the proposition for this case in the same way as in the case that $aN = 2$.

If $N > 1$, then $a \leq 3$ and $H$ is contained in some proper parabolic subgroup $P$ of $\Gamma$ by our assumption. Let $U$ be the unipotent radical of the parabolic subgroup opposite to $P$. Let $\xi$ be as in this proposition. Since $U \cap P = 1$, we have $U \cap H = 1$. Hence, $\text{Res}_U \text{Ind}_H^U \xi \supset \text{Ind}_1^U \text{Res}_1^U \xi \supset 1_U$. Therefore there exists some irreducible summand $\sigma'$ of $\text{Ind}_H^U \xi$ such that $\langle 1_U, \sigma'_U \rangle \neq 0$. The inclusion $\sigma' \subset \text{Ind}_H^U \xi$ implies that $\langle \xi, \sigma'_H \rangle \neq 0$. Moreover, $\sigma'$ is not cuspidal, whence $\sigma' \neq \gamma \cdot \sigma$ as every $\gamma \cdot \sigma$ is cuspidal.

**Proposition 5.7** Suppose $\mathfrak{A}$ is a maximal hereditary $\mathfrak{o}_F$-order of $A$. Let $\pi$ be an irreducible supercuspidal representation of $G$ with the $U^1(\mathfrak{A})$-fixed part $\pi^{U^1(\mathfrak{A})} \neq 0$. Then, there exists an irreducible cuspidal representation $\tau$ of $U(\mathfrak{A})/U^1(\mathfrak{A})$ such that $\pi|_{U(\mathfrak{A})}$ contains the lift $\sigma$ of $\tau$ to $U(\mathfrak{A})$ and for every irreducible $U(\mathfrak{A})$-subrepresentation $\sigma'$ of $\pi^{U^1(\mathfrak{A})}$, there exists $\gamma \in \text{Gal}(k_D/k_F)$ with $\sigma' \cong \gamma \cdot \sigma$.

**Proof.** This is the result from [7, Theorem 5.5 (ii)].

**Corollary 5.8** Let $\mathfrak{A}$ be a maximal hereditary $\mathfrak{o}_F$-order of $A$ and $\sigma, \sigma'$ be irreducible representations of $U(\mathfrak{A})$, trivial on $U^1(\mathfrak{A})$. Suppose that $\sigma$ is cuspidal. Then, $\sigma$ and $\sigma'$ intertwine in $G$ if and only if there exists $\gamma \in \text{Gal}(k_D/k_F)$ such that $\sigma' \cong \gamma \cdot \sigma$.

**Proof.** There exists $h \in \mathfrak{A}(\mathfrak{A})$ such that $h \sigma \cong \gamma \cdot \sigma$. Then, if $\sigma' \cong \gamma \cdot \sigma$, these representations $\sigma$ and $\sigma' \cong h \sigma$ intertwine.

Conversely, suppose that $\sigma$ and $\sigma'$ intertwine. Then, there exists $h \in G$ such that

$$\text{Hom}_{U(\mathfrak{A})/\gamma \cdot \sigma} U(\mathfrak{A}) (\sigma', h \sigma) \neq 0.$$
Then we obtain $U(\mathfrak{A}) \cap J(\mathfrak{A}) = U(\mathfrak{A})$ and $\bar{h} \sigma |_{U(\mathfrak{A}) \cap J(\mathfrak{A})} = \bar{h} \sigma |_{U(\mathfrak{A}) \cap h U(\mathfrak{A})}$, so we have

$$0 \neq \langle \sigma', h \sigma \rangle_{U(\mathfrak{A}) \cap h U(\mathfrak{A})} = \langle \bar{h} \sigma, \text{Res}^{U(\mathfrak{A})}_{U(\mathfrak{A}) \cap h U(\mathfrak{A})} \sigma' \rangle_{U(\mathfrak{A}) \cap h U(\mathfrak{A})} = \langle \text{Ind}^{U(\mathfrak{A})}_{U(\mathfrak{A}) \cap h U(\mathfrak{A})} \text{Res}^{h U(\mathfrak{A})}_{U(\mathfrak{A}) \cap h U(\mathfrak{A})} \bar{h} \sigma, \sigma' \rangle_{U(\mathfrak{A}) \cap h U(\mathfrak{A})} = \text{dim} \text{Hom}_{U(\mathfrak{A})} \left( \sigma', \text{Ind}^{U(\mathfrak{A})}_{U(\mathfrak{A}) \cap h U(\mathfrak{A})} \text{Res}^{h U(\mathfrak{A})}_{U(\mathfrak{A}) \cap h U(\mathfrak{A})} \bar{h} \sigma \right).$$

Since $\sigma'$ is irreducible, there exists an inclusion of $U(\mathfrak{A})$-representations

$$\sigma' \hookrightarrow \text{Ind}^{U(\mathfrak{A})}_{U(\mathfrak{A}) \cap h U(\mathfrak{A})} \text{Res}^{h U(\mathfrak{A})}_{U(\mathfrak{A}) \cap h U(\mathfrak{A})} \bar{h} \sigma \subset \text{Res}^{G}_{h U(\mathfrak{A})} \text{c-Ind}^{h U(\mathfrak{A})}_{h \mathfrak{A}} \bar{\sigma} = \pi|_{U(\mathfrak{A})}.$$

By Proposition 5.7 there exists $\gamma \in \text{Gal}(k_D/k_F)$ such that $\sigma' \cong \gamma \cdot \sigma$. 

**Corollary 5.9** Let $\mathfrak{A}$ be a maximal hereditary $\sigma_F$-order of $A$, and let $\mathcal{K}$ be a subgroup of $U(\mathfrak{A})$ such that the image of $\mathcal{K}$ in $U(\mathfrak{A})/U^1(\mathfrak{A})$ is sufficiently small, and $\sigma$ be a lift of an irreducible cuspidal representation $\tau$ of $U(\mathfrak{A})/U^1(\mathfrak{A})$ to $U(\mathfrak{A})$. Moreover, if $q_F = 2$ and $q_D \leq 8$, we further assume that the image of $\mathcal{K}$ in $U(\mathfrak{A})/U^1(\mathfrak{A})$ is contained in some proper parabolic subgroup of $U(\mathfrak{A})/U^1(\mathfrak{A})$. Then, for every $\mathcal{K}$-subrepresentation $\zeta$ of $\sigma$, there exists an irreducible representation $\sigma'$ of $U(\mathfrak{A})$ such that $\sigma'$ is trivial on $U^1(\mathfrak{A})$, and we have $\langle \zeta, \sigma' \rangle_{\mathcal{K}} \neq 0$ and $I_G(\sigma, \sigma') = \emptyset$.

**Proof.** Let $H$ be the image of $\mathcal{K}$ in $U(\mathfrak{A})/U^1(\mathfrak{A})$. Let $\omega$ be a $H$-representation such that $\zeta$ is the lift of $\omega$.

Suppose there is an irreducible representation $\tau'$ of $U(\mathfrak{A})/U^1(\mathfrak{A})$ such that $\langle \omega, \tau' \rangle_H \neq 0$ and $\tau' \neq \zeta \cdot \tau$ for any $\gamma \in \text{Gal}(k_D/k_F)$. Let $\sigma'$ be the lift of $\tau'$ to $U(\mathfrak{A})$. Then $\langle \zeta, \sigma' \rangle_{\mathcal{K}} = \langle \omega, \tau' \rangle_H \neq 0$ and $\sigma' \neq \gamma \cdot \sigma$ for any $\gamma$. By Corollary 5.8 the latter condition implies $I_G(\sigma, \sigma') = \emptyset$. Therefore, it is enough to show that there exists such $\tau'$.

Unless the characteristic of $k_F$ is equal to 2, or $q_D \leq 8$, by Proposition 5.6 such a representation $\tau'$ exists. When $q_F = 2$ and $q_D \leq 8$, since $H$ is contained in some proper parabolic subgroup, we can apply Proposition 5.6 and obtain $\tau'$ with desired conditions. If $q_F > 2$ and $q_D = 4, 8$, then $q_D = q_F$ and $D = F$. Since $q_F \geq 4$, this corollary for this case is already proved in [10, Corollary 6.14].

**Proposition 5.10** Let $[\mathfrak{A}, n, 0, \beta]$ be a simple stratum of $A$ with $n > 0$, and $\theta \in \mathcal{C}(\beta, \mathfrak{A})$. Let $\kappa$ be a $\beta$-extension of the Heisenberg representation $\eta$ of $\theta$. Let $\sigma$ and $\sigma'$ be irreducible representations of $J$, trivial on $J^1$. If $I_{B^1}(\kappa \otimes \sigma, \kappa \otimes \sigma') = \emptyset$, then $I_G(\kappa \otimes \sigma, \kappa \otimes \sigma') = \emptyset$.

**Proof.** Let $x$ be an element of $G$ and $\phi \in I_x(\kappa \otimes \sigma, \kappa \otimes \sigma')$. We will show that $\phi = 0$. Let $X$ and $Y'$ be the representation spaces of $\kappa, \sigma$ and $\sigma'$. There exist $S_j \in \text{End}_C(X)$...
and $T_j \in \text{Hom}_C(Y, Y')$ such that $\phi = \sum_j S_j \otimes T_j$ and $T_j$ are linearly independent. For $h \in J^1 \cap \mathfrak{x} J^1$, we have

$$\phi \circ (\eta(h) \otimes id_Y) = \phi \circ ((\kappa \otimes \sigma)(h)) = (\mathfrak{x}(\kappa \otimes \sigma')(h)) \circ \phi = (\mathfrak{x} \eta(h) \otimes id_Y) \circ \phi.$$ 

Since $\phi = \sum_j S_j \otimes T_j$, we also have

$$\phi \circ (\eta(h) \otimes id_Y) = \sum_j (S_j \circ \eta(h)) \otimes T_j,$$

$$(\mathfrak{x} \eta(h) \otimes id_Y) \circ \phi = \sum_j (\mathfrak{x} \eta(h) \circ S_j) \otimes T_j,$$

and $\sum_j (S_j \circ \eta(h) - \mathfrak{x} \eta(h) \circ S_j) \otimes T_j = 0$. Since $T_j$ are linearly independent, it follows that $S_j \circ \eta(h) = \mathfrak{x} \eta(h) \circ S_j$, that is, $S_j \in I_x(\eta, \eta)$. Now $\text{dim}_C I_x(\eta, \eta) \leq 1$. Then we may assume $\phi = S \otimes T$ for some $S \in I_x(\eta, \eta) = I_x(\kappa, \kappa)$ and $T \in \text{Hom}_C(Y, Y')$. We may assume $S \neq 0$. Then $x \in I^G(\kappa, \kappa) = JB^x J$, so we may assume $x \in B^x$. For $h \in J \cap \mathfrak{x} J$, we have

$$(\mathfrak{x} \kappa(h) \circ S) \otimes (T \circ \sigma(h)) = (S \circ \kappa(h)) \otimes (T \circ \sigma(h)) = (\mathfrak{x} \kappa(h) \circ S) \otimes (\mathfrak{x} \sigma'(h) \otimes T).$$

Since $\mathfrak{x} \kappa(h) \circ S \neq 0$, it follows that $T \circ \sigma(h) = \mathfrak{x} \sigma'(h) \circ T$, that is, $T \in I_x(\sigma, \sigma')$. The assumption $I_{B^x}(\sigma, \sigma') = \emptyset$ implies $I_x(\sigma, \sigma') = 0$, whence $T = 0$ and $\phi = 0$. 

**Lemma 5.11** Let $[\mathfrak{A}, n, 0, \beta]$ be a simple stratum such that $\beta \in F$ and $\mathfrak{A}$ is a maximal $\sigma_F$-order in $A$.

(i) We have $J(\beta, \mathfrak{A}) = \mathcal{U}(\mathfrak{A})$ and $J^1(\beta, \mathfrak{A}) = H^1(\beta, \mathfrak{A}) = \mathcal{U}^1(\mathfrak{A})$.

(ii) Let $\theta \in \mathcal{G}(\beta, \mathfrak{A})$ and let $\kappa$ be a $\beta$-extension of $\theta$. Then $\kappa$ is the restriction of a character of $G$ to $U(\mathfrak{A})$.

(iii) Let $(J, \lambda)$ be attached to $[\mathfrak{A}, n, 0, \beta]$. Then $(J, \lambda)$ is obtained from twisting a simple type of $G$ of level 0 by some character of $G$.

**Proof.** Since $\beta \in F$, we have $B = \text{Cent}_A(\beta) = A$. Then $\mathfrak{B} = \mathfrak{A} \cap B = \mathfrak{A}$.

To show (i), we recall some property for $J(\beta, \mathfrak{A})$ and $H(\beta, \mathfrak{A})$ from [11]. The subrings $\mathfrak{J}(\beta, \mathfrak{A})$ and $\mathfrak{H}(\beta, \mathfrak{A})$ of $A$ are defined in [11] §3.3.1. According to [11] §3.3, the group $J(\beta, \mathfrak{A})$ (resp. $H(\beta, \mathfrak{A})$) is the multiplicative group of $\mathfrak{J}(\beta, \mathfrak{A})$ (resp. $\mathfrak{H}(\beta, \mathfrak{A})$).

Therefore, for (i), it is enough to show that $\mathfrak{J}(\beta, \mathfrak{A}) = \mathfrak{H}(\beta, \mathfrak{A}) = \mathfrak{A}$. Since $\beta \in F$, the element $\beta$ is 'minimal' in the sense of [11] 2.3.3. Then we have $\mathfrak{J}(\beta, \mathfrak{A}) = \mathfrak{H}(\beta, \mathfrak{A}) = \mathfrak{A}$ by [11] Proposition 3.42(1).

The simple character $\theta$ factors through $\text{Nrd}_{A/F}$ by [11] Proposition 3.47(3)]. Then there exists a character $\chi_1$ of $F^\times$ such that $\theta = \chi_1 \circ (\text{Nrd}_{A/F}|_{U(\mathfrak{A})})$. We put $\kappa_0 = \chi_1 \circ (\text{Nrd}_{A/F}|_{U(\mathfrak{A})})$. Then $\kappa_0$ is an extension of $\theta$ to $U(\mathfrak{A})$. 

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We will show that $\kappa_0$ is a $\theta$-extension of $\theta$. To show this, it is enough to show that $I_G(\kappa_0, \kappa_0) = G$. Since $\kappa_0$ is the restriction of a character of $G$, we have $\kappa_0(x^{-1}gx) = \kappa_0(g)$ for any $x \in G$ and $g \in U(\mathfrak{A})$. Therefore $x \in I_G(\kappa_0, \kappa_0)$, that is, $I_G(\kappa_0, \kappa_0) = G$.

There exists a character $\chi_2$ of $F^\times$ such that $\kappa = \kappa_0 \otimes \left( (\chi_2 \circ \text{Nrd}_{A/F}) \right)_{|U(\mathfrak{A})}$ by Théorème 2.28 in [12]. Since $\kappa_0$ and $(\chi_2 \circ \text{Nrd}_{A/F})_{|U(\mathfrak{A})}$ are the restrictions of characters of $G$, $\kappa$ is also the restriction of a character of $G$.

To show (iii), let $\kappa$ and $\sigma$ be irreducible $J(\beta, \mathfrak{A})$-representations such that $\lambda = \kappa \otimes \sigma$, as in Definition 2.6. Since $\mathfrak{B} = \mathfrak{A}$, the representation $\sigma$ is a simple type of $G$ of level 0. Moreover, $\kappa$ is the restriction of a character of $G$ by (ii). Therefore, (iii) holds. ■

**Proposition 5.12** Suppose $(g, Kg^\mathfrak{A}(\mathfrak{A}))$ has Property A. Let $(J_0, \lambda_0)$ be a maximal simple type such that $J_0 \subset U(\mathfrak{A})$. If $(J_0, \lambda_0)$ is of level $> 0$, then we further assume that $(J_0, \lambda_0)$ is attached to a simple stratum $[\mathfrak{A}, n, 0, \beta_0]$ such that $F[\beta_0]/F$ is unramified. We put $\rho = \text{Ind}_{J_0}^{U(\mathfrak{A})} \lambda_0$. Let $\tau$ be an irreducible $K$-representation such that

$$\langle \tau, \text{Ind}_{K \cap \sigma}^{U(\mathfrak{A})} (gh) \rho \rangle_K \neq 0$$

for some $h \in \mathfrak{A}(\mathfrak{A})$. Then, $\tau$ cannot be a type.

**Proof.** By Lemma 3.9 there exists $h' \in \mathfrak{A}(\mathfrak{A})$ such that $(J, \lambda) = (h' J_0, h' \lambda_0)$ is a simple type,

$$h' \rho = \text{Ind}_J^{U(\mathfrak{A})} \lambda$$

and

$$\langle \tau, \text{Ind}_{K \cap \sigma}^{U(\mathfrak{A})} g \lambda \rangle_K \neq 0$$

hold.

Assume that $(J_0, \lambda_0)$ is level 0. Then $J_0 = U(\mathfrak{A})$ is maximal and $\lambda_0$ is trivial on $U(\mathfrak{A})^1$. Since $h' \in \mathfrak{A}(\mathfrak{A})$, we have $J = U(\mathfrak{A})$ and $\lambda$ is also trivial on $U(\mathfrak{A})^1$. Therefore, the image of $K(g)$ in $U(\mathfrak{A})^1 / U(\mathfrak{A})$ is contained in some proper parabolic subgroup by applying Proposition 4.4 with $e = 1$. Let $\xi$ be an irreducible $J \cap g^{-1} K$-subrepresentation of $\lambda$. Since $\lambda$ is trivial on $U(\mathfrak{A})^1$, $\xi$ can be extended to a unique $K(g)$-subrepresentation of $\lambda$ by $\xi|_{U(\mathfrak{A})^1}$ being trivial. Hence by Corollary 5.7, there exists an irreducible $U(\mathfrak{A})$-subrepresentation $\lambda'$ of $U(\mathfrak{A})$ such that $\langle \xi, \lambda' \rangle_{K(g)} = 0$ and $\text{Ind}_{K \cap \sigma}^{U(\mathfrak{A})} g \lambda = 0$. Since $J \cap g^{-1} K \subset K(g)$, we have $\langle \xi, \lambda' \rangle_{J \cap g^{-1} K} \geq \langle \xi, \lambda' \rangle_{K(g)} > 0$. Therefore, the representation $\tau$ cannot be a type by Proposition 3.7.

Assume that $(J_0, \lambda_0)$ is attached to the simple stratum $[\mathfrak{A}, n, 0, \beta_0]$ with $n > 0$.

At first, we will consider the case when $\beta \in F$. We have $J_0 = U(\mathfrak{A})$ by Lemma 5.11 (i). Then $\rho = \lambda_0$. There exists a simple type $(U(\mathfrak{A}), \lambda_1)$ of level 0 and a character $\mu$ of $G$ such that $\lambda_0 = \mu \otimes \lambda_1$ by Lemma 5.11 (iii). We put $\tau_1 = \tau \otimes \mu^{-1}$. Since $\tau_1$ is a type if and only if $\tau$ is a type, it is enough to show $\tau_1$ cannot be a type. We have

$$\langle \tau_1, \text{Ind}_{K \cap \sigma}^{U(\mathfrak{A})} (gh) \lambda_1 \rangle_K = \langle \tau, \text{Ind}_{K \cap \sigma}^{U(\mathfrak{A})} (gh) \lambda_0 \rangle_K \neq 0.$$  Therefore, by the proposition for simple types of level 0, $\tau_1$ cannot be a type.

Then we may assume $F[\beta_0]/F$ is nontrivial. Since $(J_0, \lambda_0)$ is attached to the simple stratum $[\mathfrak{A}, n, 0, \beta_0]$, the type $(J, \lambda)$ is a simple type attached to the simple stratum
[\mathfrak{A}, n, 0, \beta] = [\mathfrak{A}, n, 0, h' \beta_0 h^{-1}]. Therefore, there exist representations \kappa, \sigma as in Definition 2.6. By our assumption, \( E = F[\beta] \cong F[\beta_0] \) is an unramified extension of \( F \). By the isomorphism \( J/J^1 \cong U(\mathfrak{B})/U^1(\mathfrak{B}) \), we can regard representations of \( U(\mathfrak{B}) \), trivial on \( U^1(\mathfrak{B}) \), as representations of \( J \), trivial on \( J^1 \).

Let \( \xi \) be an irreducible summand of \( \lambda|_{J^{s^{-1}}K} \). Then there exists an irreducible summand \( \zeta \) of \( \sigma|_{J^{s^{-1}}K} \) such that \( \langle \xi, \kappa \otimes \zeta \rangle_{J^{s^{-1}}K} \neq 0 \). Let \( K_0(g) = (J \cap g^{-1}K)J^1 \), and then \( K_0(g) \subset K(g) \) and \( \zeta \) can be extended to a \( K_0(g) \)-representation which is trivial on \( J^1 \), since \( \zeta \subset \sigma \) and \( \sigma \) is trivial on \( J^1 \). The group \( K_0(g) \) contains \( J^1 \) and \( \zeta \) is trivial on \( J^1 \), so \( \zeta \) can be regarded as a \( K_0(g) \cap U(\mathfrak{B}) \)-representation which is trivial on \( U^1(\mathfrak{B}) \). Because \( E/F \) is unramified and \( K_0(g) \subset K(g) \), the image of \( K_0(g) \cap U(\mathfrak{B}) \) in \( U(\mathfrak{B})/U^1(\mathfrak{B}) \) is sufficiently small by Proposition 5.4. Since the field extension \( E/F \) is nontrivial and unramified, we have \( q_E \neq 2 \). Then by Corollary 5.9 there exists an irreducible representation \( \sigma' \) of \( U(\mathfrak{B}) \), trivial on \( U^1(\mathfrak{B}) \), such that \( \langle \zeta, \sigma' \rangle_{U(\mathfrak{B})} \neq 0 \) and \( I_{B^x}(\sigma, \sigma') = \emptyset \). Let \( \lambda' = \kappa \otimes \sigma' \).

In the same way as in the proof of Proposition 5.10, it follows that \( \lambda' \) is irreducible. Since \( J^1 \subset K_0(g) \subset J \), we have \( K_0(g)/J^1 \cong (U(\mathfrak{B}) \cap K_0(g))/U^1(\mathfrak{B}) \). Hence, we have \( \langle \zeta, \sigma' \rangle_{K_0(g)} = \langle \zeta, \sigma' \rangle_{U(\mathfrak{B})} \neq 0 \). Therefore \( \langle \zeta, \sigma' \rangle_{J^{s^{-1}}K} \geq \langle \zeta, \sigma' \rangle_{K_0(g)} \) and \( \langle \xi, \lambda' \rangle_{J^{s^{-1}}K} \geq \langle \xi, \kappa \otimes \zeta \rangle_{J^{s^{-1}}K} \).

On the other hand, by Proposition 5.10 we have \( I_G(\lambda, \lambda') = \emptyset \). Therefore, by Proposition 5.7, \( \tau \) cannot be a type.

6 Proof of main theorem

Theorem 6.1 Let \( \pi \) be an irreducible supercuspidal representation of \( G \). Suppose \( \pi \) contains a simple type \( (J, \lambda) \) satisfying one of the following:

- \( (J, \lambda) \) is level 0,
- \( (J, \lambda) \) is level > 0 and attached to a simple stratum \([\mathfrak{A}, n, 0, \beta]\) such that \( F[\beta] \) is an unramified extension of \( F \).

Then, there exists a unique \([G, \pi]_G\)-archetype.

Moreover, if \( K \) is a maximal compact subgroup of \( G \) and \( (K, \tau) \) is a \([G, \pi]_G\)-type, then there exists a simple type \((J', \lambda')\) such that \( J' \subset K \) and \( \tau \cong \text{Ind}_J^K \lambda' \).

Proof. Let \( K \) be a maximal compact subgroup of \( G \) and \( (K, \tau) \) be a \([G, \pi]_G\)-type. All maximal compact subgroups of \( G \) are \( G \)-conjugate, so there exists some \( x \in G \) such that \( xK = \text{GL}_m(\mathfrak{o}_D) \). Since \( (K, \tau) \) and \( (xK, x\tau) \) are in the same \([G, \pi]_G\)-archetype, we may assume \( K = \text{GL}_m(\mathfrak{o}_D) \) and \( \tau \) is defined over \( K \). On the other hand, if \((J, \lambda)\) is attached to a simple stratum \([\mathfrak{A}, n, 0, \beta]\) \((\ast J, \ast \lambda)\) is a simple type attached to the simple stratum \([\ast \mathfrak{A}, n, 0, x\beta x^{-1}]\) for any \( x \in G \), and \( F[x\beta x^{-1}] \cong F[\beta] \) is an unramified extension of \( F \).
Then we may assume $U(\mathfrak{A})$ is contained in $K$. By the definition of maximal simple types of level 0 and Corollary 2.11 $\mathfrak{A}$ is a maximal $\mathfrak{p}_F$-order in $A$ and $K = U(\mathfrak{A})$. Let $\rho = \text{Ind}_F^K \lambda$. Then $\rho$ is irreducible and a $[G, \pi]_G$-type by Corollary 3.3 (this also shows the existence of a $[G, \pi]_G$-archetype). Since $\tau$ is contained in $\pi|_K$, there exists $g \in G$ and $h \in \mathcal{R}(\mathfrak{A})$ such that $\tau \subset \text{Ind}_{K \cap gK}^K gh \rho$.

Assume $g \notin K \mathcal{R}(\mathfrak{A})$. Since $\mathfrak{A}$ is maximal, there exists $g' \in Kg\mathcal{R}(\mathfrak{A})$ such that $(g', Kg\mathcal{R}(\mathfrak{A}))$ has property A by Lemma 4.3. Because $Kg\mathcal{R}(\mathfrak{A}) = Kg'\mathcal{R}(\mathfrak{A})$ and $h \in \mathcal{R}(\mathfrak{A})$, we may assume $(g, Kg\mathcal{R}(\mathfrak{A}))$ has Property A. Therefore, by Proposition 5.12, $\tau$ cannot be a type, which is a contradiction. Hence $g \in K\mathcal{R}(\mathfrak{A}) = \mathcal{R}(\mathfrak{A})$ and $\tau \subset gh \rho$. These representations are irreducible, so $\tau \cong gh \rho$. Then the representations $\tau$ and $\rho$ are in the same $[G, \pi]_G$-archetype. Therefore every $[G, \pi]_G$-type defined over some maximal compact subgroup of $G$ is $G$-conjugate to $(K, \rho)$, whence $[G, \pi]_G$-archetype is unique.

Let $(K', \tau')$ be a $[G, \pi]_G$-type with a maximal compact subgroup $K'$ of $G$. Then there exists $g \in G$ such that $\tau' \cong gh \rho$, and then $\tau' \cong \text{Ind}_{J'}^g \rho \lambda$. Since $(J', \lambda') = (g J, g \lambda)$ is a simple type, the proof is completed.

**Remark 6.2** If $\pi$ is an irreducible depth-zero supercuspidal representation, there exists a simple type of level 0 for $\pi$ by Proposition 5.7. Then this theorem implies the uniqueness of archetypes does not hold in general. Therefore, our example in the following is a counterexample for Latham’s result [8, Theorem 6.2].

### 7 An example without the unramified assumption

In this section, we introduce a case where there exist $[G, \pi]_G$-types on a maximal compact open subgroup $K$, which are not $N_G(K)$-conjugate, for some irreducible supercuspidal representation $\pi$. By this case, we find that the uniqueness of archetypes does not hold in general. Therefore, we provide an example in the following is a counterexample for Latham’s conjecture in [8, Conjecture 4.4].

Let $D$ be a quaternion algebra over $F$ and $A = M_2(D)$. We take an irreducible supercuspidal representation $\pi$ of $G = GL_2(D)$ such that there exists a simple type $(J, \lambda)$ for $\pi$, attached to a simple stratum $[\mathfrak{A}, n, 0, \beta]$ with the following conditions:

- there exists a field extension $E = E/\beta/F_1/F$ such that $E_1/F$ is a ramified, quadratic extension and $E/E_1$ is unramified and quadratic.
- $E$ is $E_1$-subalgebra in $M_2(E_1)$ with $\mathfrak{o}_E \subset M_2(\mathfrak{o}_{E_1})$, where the embedding $M_2(E_1) \subset M_2(D)$ is induced from some embedding $E_1 \subset D$.

Then we have $B = \text{Cent}_A(E) = E$, $\mathfrak{A} = M_2(\mathfrak{o}_D)$ and $\mathfrak{B} = B \cap \mathfrak{A} = \mathfrak{o}_E$.

Since $k_D$ and $k_E$ are quadratic extensions of the finite field $k_F$, the field $k_E$ identifies with $k_D$. We fix such an identification $k_D \cong k_E$. Therefore, we obtain a $k_F$-algebra
Indeed, the above inclusion factors
\[ k_D = k_E = \mathfrak{B}/(\mathfrak{B} \cap \mathfrak{P}) \hookrightarrow \mathfrak{A}/\mathfrak{P} = M_2(k_D). \]

Note that this inclusion is not necessarily a \( k_D \)-algebra inclusion, so \( k_E \) is not necessarily contained in the subring of diagonal matrices in \( M_2(k_D) \). However, there exists an element \( h \) in \( U(\mathfrak{A}) \) such that the image of \( ^h\mathfrak{B} \) in \( \mathfrak{A}/\mathfrak{P} \) is contained in the ring of diagonal matrices. Indeed, the above inclusion factors
\[ M_2(k_F) = M_2(k_{E_1}) = M_2(\mathfrak{o}_{E_1})/\mathfrak{p}_{E_1} M_2(\mathfrak{o}_{E_1}) \hookrightarrow \mathfrak{A}/\mathfrak{P} = M_2(k_D), \]

that is, we have a \( k_F \)-algebra inclusion \( k_D \hookrightarrow M_2(k_F) \). Then, if \( \alpha \) generates \( k_E \) over \( k_F \), the characteristic polynomial \( \chi \) of \( \alpha \in M_2(k_F) \) is equal to the characteristic polynomial of \( \alpha \in M_2(k_F) \), which is the minimal polynomial of \( \alpha \) over \( k_F \). Since the polynomial \( \chi \) has two different roots in \( k_F \), \( \alpha \) is diagonalizable as an element in \( M_2(k_F) \). Therefore there exists an element \( x \) in \( \text{GL}_2(k_D) \cong U(\mathfrak{A})/U^1(\mathfrak{A}) \) such that \( ^x k_F \) is contained the ring of diagonal matrices and a lift \( h \) of \( x \) in \( U(\mathfrak{A}) \) satisfies the desired condition.

We put \( K = \text{GL}_2(\mathfrak{o}_D) \) and \( g_0 = a_R(1,0) \) as in (i). Then \( g_0 \not\in \mathfrak{R}(\mathfrak{A}) = K\mathfrak{R}(\mathfrak{A}) \), that is, \( Kg_0\mathfrak{R}(\mathfrak{A}) \) is a non-trivial double coset of \( K \backslash G/\mathfrak{R}(\mathfrak{A}) \). We will show \( g_0hJ \subset K \). First, \( g_0^{-1}K \) contains the subgroup \( K' \) in \( \text{GL}_2(\mathfrak{o}_D) \) of diagonal matrices modulo \( \mathfrak{p}_D \). Since \( J = U(\mathfrak{B}).J^1 \subset U(\mathfrak{B}) U^1(\mathfrak{A}) \), it suffices to show that \( ^h(U(\mathfrak{B}) U^1(\mathfrak{A})) \subset K' \). These groups \( K' \) and \( ^h(U(\mathfrak{B}) U^1(\mathfrak{A})) = ^hU(\mathfrak{B}) U^1(\mathfrak{A}) \) contain \( U^1(\mathfrak{A}) \), so it is enough to show that the image of \( ^hU(\mathfrak{B}) \) in \( U(\mathfrak{A})/U^1(\mathfrak{A}) \) is contained in the counterpart of \( K' \), which is already proved in the above discussion.

We put \( g = g_0h \). Then \( J \) and \( gJ \) are subgroups in \( K \) and we can consider \([G, \pi]_G\)-types \( \text{Ind}_J^K \lambda \) and \( \text{Ind}_{gJ}^K g\lambda \).

For every element \( k \) in \( N_G(K) \), the representation \( ^k \left( \text{Ind}_J^K \lambda \right) \) is not isomorphic to \( \text{Ind}_J^K \lambda \). Indeed, if such \( k \) exists, we have \( \text{Hom}_K(\text{Ind}_J^K \lambda, \text{Ind}_{gJ}^K g\lambda) \neq 0 \) and \( k'kg \in I_G(\lambda, \lambda) = J \) for some \( k' \in K \). Because \( J \subset \mathfrak{R}(\mathfrak{A}) \) by Lemma 3.2, we obtain \( g \in K\mathfrak{R}(\mathfrak{A}) \). On the other hand, \( Kg\mathfrak{R}(\mathfrak{A}) = Kg_0\mathfrak{R}(\mathfrak{A}) \) is a non-trivial double coset, which leads to a contradiction.

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