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SPIN-INDEPENDENT EFFECTIVE MASS
IN A VALLEY-DEGENERATE ELECTRON SYSTEM

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Abstract

In a generic spin-polarized Fermi liquid, the masses of spin-up and spin-down electrons are expected to be different and to depend on the degree of polarization. This expectation is not confirmed by the experiments on two-dimensional heterostructures. We consider a model of an $N$-fold degenerate electron gas. It is shown that in the large-$N$ limit, the mass is enhanced via a polaronic mechanism of emission/absorption of virtual plasmons. As plasmons are classical collective excitations, the resulting mass does not depend on $N$, and thus on polarization, to the leading order in $1/N$. We evaluate the $1/N$ corrections and show that they are small even for $N = 2$.

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The observation of an apparent metal-insulator transition in high-mobility Si metal-oxide-semiconductor-field-effect-transistors (MOSFET’s) [1] challenged the scaling theory of localization [2], which predicts that a two-dimensional (2D) system undergoes only a continuous crossover between weak and strong localization regimes. Although there has been a substantial progress in understanding of transport and thermodynamic properties of MOSFET’s and other heterostructures [3, 4], the origin of the observed phenomena is still a subject of discussion. Although a conventional (dirty) Fermi-liquid (FL) theory [5, 6] can account for many observed effects at least qualitatively and, in some cases, quantitatively, there is also a number of non-FL scenarios for the anomalous metallic state [7, 8]. On the experimental side, the main argument for the FL-nature of the metallic state is the observation of quite conventional Shubnikov-de Haas (ShdH) oscillations [3, 4], which implies an existence of well-defined quasiparticles albeit with the renormalized effective mass $m^*$ and spin susceptibility $\chi^s$. The ShdH and magneto-resistance experiments show that at low densities both $m^*$ and $\chi^s$ are significantly enhanced compared to their band values [4] and, according to some studies [9, 10], even diverge at the resistive transition point.

Although none drastically non-FL features of the metallic state have been found in ShdH measurements as of now, there is one very intriguing observation which does seem to present a challenge for the FL theory, at least in its conventional formulation. Namely, in all studies when the spin and orbital degrees of freedom were controlled independently by applying a tilted magnetic field, the effective masses, $m^+_s$ and $m^-_s$, and Dingle temperatures (impurity scattering rates), $T_{D^+_s}$ and $T_{D^-_s}$, of spin-up and -down electrons, were found to be almost the same. Moreover, $m^*$ in MOSFETs [11, 12] was found to be independent of the spin polarization, whereas $T_D$ was shown to depend on the polarization only weakly. In n-GaAs, the effective mass was found to depend on the parallel magnetic field [13]; however, this behavior was attributed to the coupling between the in- and out-of-plane degrees of freedom (Stern effect [14]), which is to be expected in systems with wider quantum wells. Given that the Stern effect is subtracted off, the resulting dependence of $m^*$ on the polarization is likely to be weak.

Why is this strange? Polarization is expected to lead to two effects: the spin-splitting of the effective mass, i.e., $m^+_s \neq m^-_s$, and dependences of both $m^+_s$ and $m^-_s$ on the polarization. The first effect can be understood by considering a partially spin-polarized FL as a two-component system. As the densities of the components are different, the corresponding couplings describing the interactions between the same and opposite spins are also different; hence a priori the mass renormalizations should also be different. That the masses should depend on polarization can be seen from considering two limiting cases: of zero- and full polarization. At fixed density $n$, the Fermi energy is doubled by fully polarizing the 2D system, hence the ratio of the Coulomb to Fermi energy $g \equiv e^2 \sqrt{\pi n}/E_F$ differs by a factor of 2 between the cases of zero and full polarization. The experiment shows that the mass does depend on the density; however, if $g$ is the only dimensionless parameter that determines the mass renormalization, the same effect can
be achieved by either varying $n$ or by varying $E_F$ via polarization at fixed $n$. Also, different Fermi velocities should result in different impurity scattering times for spin-up and -down electrons; hence the Dingle temperatures are also expected to be different. However, this is not what the experiment shows.

The qualitative arguments given above can be verified in a number of ways. Back in 1971, Overhauser predicted the spin-splitting and polarization dependence of $m^*$ within the RPA approximation for the 3D case [15]. Repeating the calculation in 2D gives a similar result:

$$m^*_i/m = 1 + \left(\frac{r_s}{\sqrt{2}}\right) \ln r_s \mp \left(\frac{r_s \xi}{2\sqrt{2}}\right) \ln r_s,$$

where $\xi = (n_\uparrow - n_\downarrow) / (n_\uparrow + n_\downarrow) \ll 1$ is the polarization and $r_s = m e^2 / \sqrt{\pi n}$. In the fully-polarized regime ($\xi = 1$), the spin-down electrons disappear, whereas the renormalization of $m^*_\uparrow$ is by a factor of $\sqrt{2}$ smaller than for $\xi = 0$. This argument can be generalized for a (partially) spin-polarized FL [16], where the Landau interaction function has three independent components: $f^\uparrow\uparrow$, $f^\downarrow\downarrow$, and $f^\uparrow\downarrow = f^\downarrow\uparrow$. The Galilean invariance then gives

$$m/m^*_\uparrow = 1 - F_1^\uparrow - (k_F^\uparrow/k_F^\downarrow) F_1^\downarrow;$$

$$m/m^*_\downarrow = 1 - F_1^\downarrow - (k_F^\downarrow/k_F^\uparrow) F_1^\uparrow,$$

where $F_1^{ij} = m \int d\theta \cos \theta f^{ij}(\theta)/(2\pi)^2$, with $i, j = \uparrow, \downarrow$. Again, in general, $m^*_\uparrow \neq m^*_\downarrow$. In addition, the spin-splitting and polarization dependence of $m^*$ are also obtained within the Gutzwiller approximation for the Hubbard model [17] (in this case, mass-splitting disappears at half-filling but the polarization dependence survives).

Absence of the polarization dependence of the effective mass suggests that $m^*$ is renormalized via the interaction with some classical degree of freedom, which is not affected by the quantum degeneracy of the electron states. In this paper, we show such a mechanism may be provided by the interaction with (virtual) plasmons which dominate the mass renormalization beyond the weak-coupling regime. To this end, we turn to a model of a Coulomb gas with large degeneracy $N$, considered previously in Refs. [18], [19]. This model is relevant, first of all, to valley-degenerate systems, such as the (001) surface of a Si MOSFET, where $N = 4$ (two valleys and two spin projections). As the valley degeneracy plays a very important role in the dirty FL theory [5, 6] it is important to elucidate its role for the properties of a clean FL. However, the $1/N$ expansion turns out to be converging reasonably fast even for a non-valley degenerate system ($N = 2$) and, as such, it provides a simple yet non-trivial way of going beyond the weak-coupling limit for not too strong Coulomb interaction.

For a 2D $N$-fold degenerate Coulomb gas, the Fermi momentum is scaled down by a factor of $N^{-1/2}$ (since one has to distribute the same number of electrons among $N$ isospin flavors), whereas the inverse screening radius ($\kappa$), proportional to the density of states, is scaled up by a factor of $N$. The ratio $\alpha \equiv \kappa/k_F = r_s N^{3/2}/2$ controls the crossover between the regimes of weak ($\alpha \ll 1$) and strong ($\alpha \gg 1$) screening. For $N \gg 1$, both of these regimes are compatible with
the condition $r_s \ll 1$ which guarantees that the screening cloud includes many electrons, so that the mean-field theory is applicable. For $\alpha \ll 1$, the screening radius $\kappa^{-1} = \alpha^{-1}k_F^{-1}$ is larger than the Fermi wavelength. [This case also includes the usual RPA scheme for $N = 2$ – see Eq. (1).] The mass renormalization is mostly due to elastic scattering within the particle-hole continuum with momentum transfers $q \sim \kappa$, whereas the interaction with plasmons is small. In this regime, the mass depends on total degeneracy ($N$) and is thus strongly affected by polarization. Also, as scattering is mostly by small angles, $m^* < m$. For $\alpha \gg 1$, the effective screening radius $q_0^{-1} = (2\alpha)^{-1/3}k_F^{-1}$ is smaller than the Fermi wavelength (but still larger than the distance between electrons); hence, scattering is isotropic ($s$-wave). The particle-hole continuum contribution to $m^*$ is greatly reduced for $s$-wave scattering, whereas the interaction with virtual plasmons now plays a dominant role. As the plasmon is a classical collective mode, it is not affected by a change in $N$. Consequently, the leading term in the $N^{-1}$ expansion for $m^*$ does not depend on $N$, whereas the next-to-leading term happens to be numerically small.

The effective mass is found from the self-energy via the usual relation (valid for a small renormalization)

$$m^*/m = 1 - \left. \left( \frac{\partial \Delta \Sigma_k(\epsilon)}{\partial \epsilon_k} + \frac{\partial \Delta \Sigma_k(\epsilon)}{\partial (i\epsilon)} \right) \right|_{k \to k_F, \epsilon \to 0},$$

where $\Delta \Sigma_k(\epsilon) = \Sigma_k(\epsilon) - \Sigma_{k_F}(0)$. It is convenient to separate $\Delta \Sigma_k(\epsilon)$ into the static and dynamic parts as

$$\Delta \Sigma_k(\epsilon) = \Delta \Sigma_k^s(\epsilon) + \Delta \Sigma_k^{\text{dyn}}(\epsilon), \quad (2)$$

where the static part for $\epsilon_k \equiv (k^2 - k_F^2)/2m \to 0$ is

$$\Delta \Sigma_k^s(\epsilon) = \int \frac{d\omega}{2\pi} \frac{d^2q}{(2\pi)^2} V_q(0) \left[ G_{k+q}(\epsilon + \omega) - G_{k+q}(\epsilon) \right]$$

$$= \frac{m}{(2\pi)^2} \int_0^{2\pi} d\theta \cos \theta V_{2kF \sin \theta/2}(0) \quad (3)$$

\[\text{FIG. 1: Excitation spectrum for an } N\text{-fold degenerate 2D Coulomb gas in the strong-screening regime} \quad (r_s N^{3/2} \gg 1). \text{ The plasmon dispersion crosses over from the } \sqrt{q} \text{ to } q^2 \text{ form at } q \sim q_0 \sim r_s^{1/3} n^{1/2} \gg k_F. \text{ Processes with momentum and energy transfers in the shaded oval } (q \sim q_0 \text{ and } \omega \sim q_0^2/m) \text{ dominate the mass enhancement. The plasmon spectrum merges with the continuum at } q = q_1 \sim r_s^{1/2} N^{1/4} n^{1/2} \gg q_0.\]
with $G_k^{-1}(\varepsilon) = i\varepsilon - \epsilon_k$ and

$$V_q(\omega) = \left[ q/2\pi e^2 - \Pi_q(\omega) \right]^{-1}. \quad (4)$$

The dynamic part is

$$\Delta \Sigma^\text{dyn}_k(\varepsilon) = \int \frac{d\omega}{2\pi} \frac{d^2q}{(2\pi)^2} \left[ V_q(\omega) - V_q(0) \right] \times [G_{k+q}(\varepsilon + \omega) - G_{kF+q}(\varepsilon)]. \quad (5)$$

In what follows, we will need the following two forms of the polarization bubble

$$\Pi_q(\omega) = N \int \frac{d\varepsilon}{2\pi} \int \frac{d^2k}{(2\pi)^2} G_k(\varepsilon) G_{k+q}(\varepsilon + \omega)$$

$$= - \left\{ \begin{array}{ll}
(mN/2\pi) \left( 1 - |\omega|/\sqrt{\omega^2 + v_F^2 q^2} \right), & \text{for } q \ll k_F; \\
2n\varepsilon_q/(\varepsilon_q^2 + \omega^2), & \text{for } q \gg k_F,
\end{array} \right. \quad (6)$$

where $v_F = \sqrt{4\pi n/m^2 N}$ and $\varepsilon_q = q^2/2m$.

In the weak-screening regime, $\Delta \Sigma^\text{st}_k(\varepsilon)$ [Eq. (2)]/$m^*$ gives the main contribution to $m^*$. To logarithmic accuracy, $m^*/m = 1 + \left( r_s \sqrt{N}/2\pi \right) \ln (r_s N^{3/2}) + O(r_s)$ in this regime. [For $N = 2$ and $\xi = 0$, this reduces back to Eq. (1)]. In this regime, the plasmon contribution to $m^*$ is a subleading, $O(r_s)$-term.

Now we turn to the strong-screening regime. The static screened potential in Eq. (3) is evaluated for $q = 2k_F \sin \theta/2 \leq 2k_F$. In this range, $V_q(0) = 2\pi e^2/(q + \kappa)$ is of the same form as in the weak-screening regime but now $V_q(0)$ depends on $q$ only weakly because $q \ll \kappa$. Consequently, the angular averaging in Eq. (3) renders the static contribution to $m^*$ small: $(m^*/m - 1)^{\text{st}} = 8/3\pi N\alpha$. Using the large-$q$ form of $\Pi$ in Eq. (6), one obtains $V_q(0) = 2\pi e^2 q^2/(q^2 + \varepsilon_q^2)$ for $q \gg k_F$, where $q_0 = (2\alpha)^{1/3} k_F \gg k_F$ is the inverse screening radius in this regime [19]. The main contribution to $m^*$ comes from the region of large $q$ and $\omega$ in Eq. (5), i.e., from the plasmon region. In the strong-screening regime, the plasmon dispersion is given by $\omega_p = \sqrt{\varepsilon_q^2 + 2\pi e^2 nq/m}$. The crossover between the $\sqrt{q}$ and $q^2$ behaviors occurs at $q \sim q_0$ (cf. Fig. 1). The plasmon runs into the continuum at $q \sim q_1 = k_F(\alpha/2)^{1/2} \gg q_0$. Most importantly, being the classical collective mode, plasmon is not affected by a change in $N$. The mass renormalization can be estimated as follows. Typical momenta and energy transfers are of the order of $q_0$ and $\varepsilon_{q_0}$, respectively; thus $V_{q_0}(\varepsilon_{q_0}) \sim \varepsilon_{q_0}^2/q_0$, and $G \sim \omega^{-1} \sim \varepsilon_{q_0}^{-1}$. Combining these estimates together, one finds that $(m^*/m - 1)^{\text{dyn}} \sim \int d^2q \int d\omega V_q G^2 \sim r_s^{2/3}$, which is larger than the static contribution by $\alpha^{5/3} \gg 1$. To perform an actual calculation, we notice that the plasmon contribution from the region of large $q$ to the effective mass can be written as

$$m^*/m = 1 + \frac{i}{\pi} \int_0^\infty d\varepsilon_q \text{Res}_{\omega} \frac{V_q(\omega)}{(i\omega - \varepsilon_q)}|_{\omega = i\omega_p}, \quad (7)$$

where only the poles of $V_q(\omega)$ were taken into account, and where we have used the expansion $\epsilon_{k+q} = \epsilon_k + v_F q \cos \theta (1 + \epsilon_k/2E_F) + \varepsilon_q$. Substituting the large-$q$ form of $\Pi$ [Eq. (6)] into $V_q(\omega)$
in Eq. (7), one arrives at the result of Ref. [19] for the leading $1/N$ term in $m^*$

$$m^*/m = 1 + Cr_s^{2/3},$$

where $C = \Gamma(1/3)\Gamma(1/6)/60\sqrt{\pi} \approx 0.14$.

Corrections to the leading term are obtained by including (a) interaction corrections to the bubble [Fig. 2(a)], (b) vertex correction to the self-energy [Fig. 2(b)], and (c) corrections to the polarization bubble from the small-$q$ region. Estimating the diagrams in Fig. 2(a,b) in the same way as for the leading term, we find that both (a) and (b) contribute $N$-independent, $r_s^{4/3}$ corrections to Eq. (8). We have verified by an explicit calculation that these estimates do hold. Next, we consider correction (c) and show that it gives the next-to-leading term in the $1/N$ expansion.

The $1/q$ correction to the large-$q$ form of the bubble [Eq. (6)] is

$$\delta \Pi_q(\omega) = \frac{4n^2 \pi (3\omega^2 - \varepsilon_q^2 r_s^2)}{mN} \frac{\omega^2 + \varepsilon_q^2}{(\omega^2 + \varepsilon_q^2)^3}. (9)$$

At the plasmon pole ($\omega^2 = -\omega_p^2$) and for $q \sim q_0$, the relative correction $|\delta \Pi_q(\omega)/\Pi_q(\omega)| \sim 1/\alpha^{2/3}$, hence one can expect the next-to-leading term in the mass to be of order $r_s^{2/3}/\alpha^{2/3} \sim 1/N$. Indeed, a correction to the bubble (9) shifts the position of the plasmon-pole from $\omega_p^2$ to $\omega_p^2 + \Delta^2$, where $\Delta^2 = 8\pi^2 ne^2 (3r^2 + 1)/Nmqr^4$ and $r = \sqrt{1 + (q_0/q)^3}$. Substituting this result into Eq. (7), and evaluating the $q$-integral to log-accuracy (the upper limit is determined by $q \sim q_1$, corresponding to the region where the plasmon runs into the continuum), we obtain $m^*$ within the next-to-leading order in $1/N$ as

$$m^*/m = 1 + 0.14r_s^{2/3} + \frac{1}{12N} \log \left( r_sN^{3/2} \right) + \mathcal{O}\left( \frac{1}{r_sN^{5/2}} \right), (10)$$

where the last term is the static contribution of the continuum. We see that the $1/N$ expansion generates the series in powers of $(r_sN^{3/2})^{-1}$.

Now we apply our main result, Eq. (10), to real systems. [In what follows, we neglect the last term in Eq. (10).] First of all, due to a small numerical coefficient in the leading term in Eq. (10), the actual constraint on $r_s$ being small is rather soft: a two-fold enhancement of the mass occurs only for $r_s \approx 20$, hence smaller values of $r_s$ still allows for a reasonable description within the mean-field theory. Eq. (10) agrees well with the observed dependence of $m^*(r_s)$ for Si MOSFETs in the range $r_s = 2 - 6$; for larger $r_s$, the theoretical value of $m^*$ falls below the

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**FIG. 2:** a: corrections to the bubble; b: vertex correction to the self-energy.
experimental one. In the interval $2 \leq r_s \leq 6$, the $1/N$ term in Eq. (10) is not that small: it constitutes 18-26 % and 26-32% of the leading term for $N = 4$ and $N = 2$, correspondingly. However, the relative change in $m^*$ due to full spin polarization ($N \rightarrow N/2$)

$$\frac{\Delta m}{m_{avg}} = 2 \times \frac{m^*(N/2) - m^*(N)}{m^*(N/2) + m^*(N)} \times 100\%, \tag{11}$$

is small. $\Delta m/m_{avg}$ as a function of $r_s$ is shown in Fig. 3 for $N = 4$ and $N = 2$. In both cases, these changes are less than 3 %, which is likely to be below the experimental error in the measured mass. At finite polarization, the result in Eq. (10) changes to

$$\frac{m_{^*+}}{m} = 1 + 0.14r_s^{2/3} + \frac{1 + \xi^2}{12N^*} \log \left[ \frac{r_s N^{3/2}}{1 + \xi^2} \right]. \tag{12}$$

Notice that although an explicit polarization dependence does occur in the second term, there is no spin-splitting of the masses to this order in $1/N$. Eq. (12) is valid as long as there are still many spin-down electrons within the screening radius or, equivalently, $1-\xi \gg r_s^{2/3} \sim (m^*/m-1)$. Fig. 3 shows that the effective mass remains essentially constant in the whole range of $\xi$, which is in agreement with the experiment [12].

To leading order in $1/N$, the renormalization of $\chi^*$ is entirely due to that in $m^*$, so that $g^* = \chi^*/m^*$ remains unrenormalized [19]. We found that this remains true up to the next-to-leading term in $1/N$. This result is in qualitative agreement with the experiments on Si MOSFETs. However, recent experiment on AlAs system shows that the $g^*$ factor is affected by lifting the valley degeneracy [20]. More work is required to attribute this behavior to a many-body effect.

Now, we comment briefly on the impurity scattering rate in the large-$N$ limit. In the strong-screening regime, the screening radius ($\xi_0^{-1}$) is much shorter than the Fermi wavelength. Therefore, scattering even on charged impurities is in the $s$-wave regime. We assume that the main role is played by impurities within the 2D layer. Due to a peculiarity of 2D scattering [21], the
scale of the scattering cross-section section is set by the wavelength (rather than by the impurity size \( a \sim q_0^{-1} \)) and depends on \( a \) weakly: \( A \sim k_F^{-1} / \ln^2 (k_Fa) \). Consequently, the scattering rate \( 1/\tau = n_i v_F A \), where \( n_i \) is the concentration of impurities, has only a weak dependence on the polarization (via \( k_F \) under the logarithm). Thus \( 1/\tau \) (Dingle temperature) for spin-up and down-electrons are close to each other. Notice that both ShdH and weak-field Hall effect [22] show that \( 1/\tau \), while being the same for spin-up and spin-down electrons, increases strongly with \( r_s \). Within our model, this can only be explained by an increase in the number of scatterers \( n_i \) with decreasing electron density—not an improbable scenario for Si MOSFETs.

Finally, we observe that as the mass is renormalized by plasmons with large \( q \), the behavior of the plasmon spectrum at small \( q \) (gapped or gapless) is irrelevant. Consequently, in 3D the mass is renormalized in a similar way: \( m^* / m = 1 + C_3 D r_s^{3/4} \), where \( C_3D \) does not depend on \( N \). Therefore, a finite thickness of the 2D layer should not affect the (approximate) spin-independence of the mass.

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