EXISTENCE AND UNIQUENESS FOR NONLINEAR ANISOTROPIC ELLIPTIC EQUATIONS

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ABSTRACT. We study the existence and uniqueness for weak solutions to some classes of anisotropic elliptic Dirichlet problems with data belonging to the natural dual space.

1. INTRODUCTION

In the present paper we study the existence and uniqueness of weak solutions to some classes of anisotropic elliptic equations with homogeneous Dirichlet boundary conditions.

Let us consider the following model problem

\[
\begin{aligned}
-\partial_x (a_i(x,u)(\varepsilon + |\partial_x u|^2)^{p_i-2}\partial_x u) &= f - \partial_x g_i \quad \text{in } \Omega \\
\partial_n u &= 0 \quad \text{on } \partial\Omega,
\end{aligned}
\]

where \( \Omega \) is a bounded open subset of \( \mathbb{R}^N \) with Lipschitz continuous boundary, \( N \geq 2, 1 < p_1, \ldots, p_N < +\infty, \varepsilon \geq 0 \), \( a_1, \ldots, a_N \) are Carathéodory functions, \( g_1, \ldots, g_N \) and \( f \) are functions belonging to suitable Lebesgue spaces.

The anisotropy of the problem is due to the growth in each partial derivative controlled by different powers. The interest in studying such operators is motivated by their applications to the mathematical modeling of physical and mechanical processes in anisotropic continuous medium.

The existence and regularity of weak solutions or solutions in the distributional sense to Problem (1.1) with \( g_i \equiv 0 \) have been studied in [22] when \( f \in L^m(\Omega) \) with \( m > 1 \), in [14] when datum \( f \) belongs to Marcinkiewicz spaces and in [6] for measure data. In [11] a comparison theorem and the derived a priori estimates are established via symmetrization methods. Some uniqueness results for Problem (1.1) have been obtained in [2] for weak solutions and data belonging to the dual of the anisotropic Sobolev space. Moreover when the datum \( f \) is only integrable the uniqueness of a renormalized solution is proved in [17].

As far as the uniqueness of a weak solution to Problem (1.1) is concerned, when \( \varepsilon = 0 \) and at least one \( p_i \) is less or equals to 2 in this paper we obtain the same uniqueness result of [2] by a different method. Instead we improve the result of [2] when \( \varepsilon > 0 \) and every \( p_i \) is greater than 2. The main tools in our proofs are Poincaré inequality and the embedding for the anisotropic Sobolev spaces.

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We also consider a class of anisotropic equations with a first order term, whose prototype is
\[
\begin{cases}
-\partial_x(a(x, u)(\varepsilon + |\partial_x u|^2)\partial_x u) + \sum_{i=1}^N b_i |\partial_x u|^{p_i - 1} = f - \partial_x g_i & \text{in } \Omega \\
u = 0 & \text{on } \partial\Omega,
\end{cases}
\]
where \(b_i\) belong to suitable Lebesgue spaces for \(i = 1, \ldots, N\).

To our knowledge, such a problem is not still deeply studied. As far as the existence of a weak solution is concerned, the presence of a lower order term produces a lack of coerciveness, which does not allow to use the classical methods. Here we prove the existence of a weak solution to Problem (1.2). As usual the main step in the proof is an a priori estimate. In order to avoid the assumption on smallness of the norm of the coefficients \(b_i\) (that implies the coerciveness of the operator), we adapt the method used in [8] (see also [3], [15], [19] and [20]), which consists in splitting the domain \(\Omega\) in a finite number of small domains \(\Omega_1, \ldots, \Omega_t\) in such a way to have small norms of the coefficients on \(\Omega_\sigma\) for \(\sigma = 1, \ldots, t\). Finally we consider a different class of anisotropic operator, whose lower order term satisfy a Lipschitz condition in order to obtain same uniqueness results following the idea of [1].

Problems (1.1) and (1.2) have been studied in the isotropic case by many authors. We just mention some of these papers: [4], [10], [12] and [13] for existence and regularity of weak solutions and [1], [9], [21] and [24] for the uniqueness (see also the references therein).

The paper is organized as follows. In § 2 we recall the standard framework of anisotropic Sobolev spaces, we detail the assumptions and we give the notion of weak solution. In § 3 we study the case of strongly monotone operator. Finally we investigate Problem (1.2): in § 4.1 we prove the existence of at least a weak solution and in § 4.2 we prove some uniqueness results.

2. Definitions, Assumptions and Preliminaries Results

2.1. Anisotropic Sobolev spaces. Let \(\Omega\) be a bounded open subset of \(\mathbb{R}^N\) \((N \geq 2)\) with Lipschitz continuous boundary and let \(1 < p_1, \ldots, p_N < \infty\) be \(N\) real numbers. The anisotropic space (see e.g. [25])
\[W^{1, \overline{p}}(\Omega) = \{ u \in W^{1,1}(\Omega) : \partial_x u \in L^{p_i}(\Omega), i = 1, \ldots, N \}\]
is a Banach space with respect to norm \(\|u\|_{W^{1, \overline{p}}(\Omega)} = \|u\|_{L^1(\Omega)} + \sum_{i=1}^N \|\partial_x u\|_{L^{p_i}(\Omega)}\).

The space \(W^{1, \overline{p}}_0(\Omega)\) is the closure of \(C_c^\infty(\Omega)\) with respect to this norm.

We recall a Poincaré-type inequality. Let \(u \in W^{1, \overline{p}}_0(\Omega)\), then for every \(q \geq 1\) there exists a constant \(C_P\) (depending on \(q\) and \(i\)) such that (see [18])
\[\|u\|_{L^q(\Omega)} \leq C_P \|\partial_x u\|_{L^{q_i}(\Omega)} \quad \text{for } i = 1, \ldots, N. \tag{2.1}\]
Moreover a Sobolev-type inequality holds. Let us denote by \(\overline{p}\) the harmonic mean of these numbers, i.e. \(\frac{1}{\overline{p}} = \frac{1}{N} \sum_{i=1}^N \frac{1}{p_i}\). Let \(u \in W^{1, \overline{p}}_0(\Omega)\), then there exists (see [25]) a constant \(C_S\) such that
\[\|u\|_{L^{\overline{p}}(\Omega)} \leq C_S \prod_{i=1}^N \|\partial_x u\|_{L^{p_i}(\Omega)} \tag{2.2},\]
where \( q = \frac{N}{p^*} \) if \( p < N \) or \( q \in [1, +\infty[ \) if \( p \geq N \). On the right-hand side of (2.2) it is possible to replace the geometric mean by the arithmetic mean: let \( a_1, \ldots, a_N \) be positive numbers, it holds

\[
\prod_{i=1}^{N} a_i^\frac{1}{p_i} \leq \frac{1}{N} \sum_{i=1}^{N} a_i, \tag{2.3}
\]

which implies by (2.2)

\[
\|u\|_{L^q(\Omega)} \leq C_N \sum_{i=1}^{N} \|\partial_{x_i} u\|_{L^{p_i}(\Omega)}. \tag{2.4}
\]

When

\[
\bar{p} < N \tag{2.5}
\]

hold, inequality (2.4) implies the continuous embedding of the space \( W^{1,\bar{p}}_0(\Omega) \) into \( L^q(\Omega) \) for every \( q \in [1, \bar{p}] \). On the other hand the continuity of the embedding \( W^{1,\bar{p}}_0(\Omega) \subset L^{p_+}(\Omega) \) with \( p_+ := \max\{p_1, \ldots, p_N\} \) relies on inequality (2.3).

It may happen that \( p^* < p_+ \) if the exponents \( p_i \) are not closed enough, then \( p_\infty := \max\{p^*, p_+\} \) turns out to be the critical exponent in the anisotropic Sobolev embedding (see [18]).

Proposition 2.1. If condition (2.5) holds, then for \( q \in [1, p_\infty] \) there is a continuous embedding \( W^{1,\bar{p}}_0(\Omega) \subset L^q(\Omega) \). For \( q < p_\infty \) the embedding is compact.

2.2. Assumptions and Definitions. We consider the following class of nonlinear anisotropic elliptic homogeneous Dirichlet problems

\[
\begin{align*}
-\partial_{x_i} a_i(x, u, \nabla u) + \sum_{i=1}^{N} H_i(x, \nabla u) &= f - \partial_{x_i} g_i & \text{in } \Omega \\
 u &= 0 & \text{on } \partial\Omega,
\end{align*}
\tag{2.6}
\]

where \( \Omega \) is a bounded open subset of \( \mathbb{R}^N \) with Lipschitz continuous boundary \( \partial\Omega \), \( N \geq 2, 1 < p_1, \ldots, p_N < \infty \) and (2.5) holds.

We assume that \( a_i : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R} \) and \( H_i : \Omega \times \mathbb{R}^N \to \mathbb{R} \) are Carathéodory functions such that

\[
\sum_{i=1}^{N} a_i(x, s, \xi) \xi_i \geq \lambda \sum_{i=1}^{N} |\xi_i|^{p_i} \quad \forall s \in \mathbb{R}, \xi \in \mathbb{R}^N \text{ and a.e. in } \Omega, \tag{2.7}
\]

\[
|a_i(x, s, \xi)| \leq \gamma \left[ |s|^{\frac{1}{p_i}} + |\xi|^2 + |\xi_i|^{p_i} \right], \tag{2.8}
\]

\[
(a_i(x, s, \xi) - a_i(x, s, \xi')) (\xi_i - \xi_i') > 0 \quad \text{for } \xi_i \neq \xi_i', \tag{2.9}
\]

\[
|H_i(x, \xi)| \leq b_i |\xi_i|^{p_i - 1} \tag{2.10}
\]

where \( b_i, \lambda, \gamma \) are some positive constants for \( i = 1, \ldots, N \).

Moreover we suppose that

\[
f \in L^{p_\infty}(\Omega) \tag{2.11}
\]

and

\[
g_i \in L^{p_i}(\Omega) \quad \text{for } i = 1, \ldots, N. \tag{2.12}
\]

We observe that in (2.10) we can also assume that \( b_i \in L^{r_i}(\Omega) \) with \( \frac{1}{r_i} = \frac{1}{p_i} - \frac{1}{p_\infty} \) for \( i = 1, \ldots, N \).
Finally we recall the definition of weak solution. A function \( u \in W^{1,2}_0(\Omega) \) is a weak solution to Problem (2.6) if
\[
\sum_{i=1}^{N} \int_{\Omega} [a_i(x,u,\nabla u)\varphi_{x_i} + H_i(x,\nabla u)] = \int_{\Omega} [f\varphi + \sum_{i=1}^{N} g_i\varphi_{x_i}] \quad \forall \varphi \in W^{1,2}_0(\Omega).
\]

3. Strongly monotone operators

In this section we consider Problem (2.6) with \( H_i \equiv 0 \) under the assumptions of strongly monotonicity of the operator and Lipschitz continuity of \( a_i \). More precisely we study the following class of nonlinear anisotropic elliptic homogeneous Dirichlet problems
\[
\begin{cases}
-\partial_{x_i} a_i(x,u,\nabla u) = f - \partial_{x_i} g_i & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\]

Let us assume that (2.7)-(2.9), (2.11) and (2.12) hold and that functions \( a_i \) satisfy
\[
(a_i(x,s,\xi) - a_i(x,s,\xi'))(\xi_i - \xi_i') \geq \alpha(\varepsilon + |\xi_i| + |\xi_i'|^{p_i-2})|\xi_i - \xi_i'|^2
\]
with \( \alpha > 0 \) and \( \varepsilon \geq 0 \) and the following Lipschitz condition
\[
|a_i(x,s,\xi) - a_i(x,s',\xi)| \leq \beta(\theta + |\xi_i|^{p_i-1})|s - s'|
\]
with \( \beta > 0, \theta \geq 0 \) for \( i = 1, \ldots, N \).

By the classical Leray-Lions theorem (see [23]) there exists at least a weak solution (see also [22]) to Problem (3.1) as in the isotropic case.

As far as the uniqueness is concerned, we will investigate separately the case when at least one \( p_i \leq 2 \) and the case when every \( p_i > 2 \) for \( i = 1, \ldots, N \). In this last case as for \( p \)-Laplace with \( p > 2 \), there is no uniqueness in general (see the counterexample in [1]). Then assumption (3.2) with \( \varepsilon > 0 \) seems to be necessary to get a uniqueness result.

**Theorem 3.1.** Let us assume \( p_i > 2 \) for \( i = 1, \ldots, N \), (2.7), (2.9), (2.11), (2.12), (3.2) with \( \varepsilon > 0 \) and (3.3). Then there exists a unique weak solution to Problem (3.1).

**Proof.** Let \( u \) and \( v \) be two weak solutions to Problem (3.1). Let us denote \( w = (u - v)^{+}, D = \{ x \in \Omega : w > 0 \} \), \( D_t = \{ x \in D : w < t \} \) for \( t \in [0,\sup w] \) and \( T_t \) the truncation function at height \( t \). Suppose that \( D \) has positive measure. Using \( \varphi = \frac{T_t(w)}{t} \) as test function in the difference of the equations, we obtain
\[
\sum_{i=1}^{N} \int_{D_t} [a_i(x,u,\nabla u) - a_i(x,v,\nabla v)] \partial_{x_i} \varphi \leq 0.
\]

For the convenience of the reader we are explicitly writing the sum sign. By (3.2) and (3.3) we get
\[
\sum_{i=1}^{N} \int_{D_t} (\varepsilon + |\partial_{x_i} u| + |\partial_{x_i} v|)^{p_i-2} |\partial_{x_i} \varphi|^2 \leq \frac{\beta}{\alpha} \sum_{i=1}^{N} \int_{D_t} \left( \theta + |\partial_{x_i} v|^{p_i-1} \right) |\partial_{x_i} \varphi|. \quad (3.4)
\]
Theorem 3.2. Let us assume (2.12), (3.7) and (3.8) with

Using (3.7) and (3.8) instead of (3.2) and (3.3) respectively, we obtain

and hypothesis (3.3) by

Using Young inequality with some \( \delta > 0 \) we have

\[
\int_{D_t} \left( \theta + |\partial_x v|^{p_i - 1} \right) |\partial_x \varphi|
\leq \theta \left( \frac{\delta}{2} \int_{D_t} |\partial_x \varphi|^2 + \frac{1}{4\delta} |D_t| \right) + \frac{\delta}{2} \int_{D_t} \left( |\partial_x \varphi|^2 |\partial_x v|^{p_i - 2} \right) + \frac{1}{4\delta} \int_{D_t} |\partial_x v|^{p_i},
\]

then by (3.4), choosing \( \delta \) small enough, we obtain

\[
\sum_{i=1}^{N} \int_{D_t} \left( \varepsilon + |\partial_x u| + |\partial_x v| \right)^{p_i - 2} |\partial_x \varphi|^2 \leq c \left( N |D_t| + \sum_{i=1}^{N} \int_{D_t} |\partial_x v|^{p_i} \right)
\]

for some positive constant \( c \) independent on \( t \). By Young inequality, (2.4) and (3.5) we obtain

\[
NC_S |D \setminus D_t|^{1 - \frac{q}{2}} \leq NC_S \|\varphi\|_{L_{\infty}^{p_i}(D)}
\leq \sum_{i=1}^{N} \int_{D_t} |\partial_x \varphi| \leq \frac{N}{2} |D_t| + \frac{1}{2} \sum_{i=1}^{N} \int_{D_t} |\partial_x \varphi|^2
\leq \frac{N}{2} |D_t| + \frac{1}{2} \sum_{i=1}^{N} \int_{D_t} |\partial_x \varphi|^2 \left( \varepsilon + |\partial_x u| + |\partial_x v| \right)^{p_i - 2}
\leq \frac{N}{2} |D_t| + \frac{1}{2} c \left( N |D_t| + \sum_{i=1}^{N} \int_{D_t} |\partial_x v|^{p_i} \right).
\]

The last term tends to zero when \( t \) goes to zero; this implies

\[
|D| = \lim_{t \to 0} |D \setminus D_t| = 0,
\]

from which the conclusion follows.

This approach also works if we replace hypothesis (3.2) by

\[
(a_i (x, s, \xi) - a_i (x, s, \xi')) (\xi_i - \xi_i') \geq \alpha |\xi_i - \xi_i'|^{p_i} \quad \alpha > 0
\]

and hypothesis (3.3) by

\[
|a_i (x, s, \xi) - a_i (x, s', \xi)| \leq \beta \left( \theta + |\xi_i|^{p_i - 1} + (|s| + |s'|)^{q_i} \right) |s - s'| \quad \beta > 0, \theta > 0 \quad \text{for some } q_i > 0, \beta > 0, \theta > 0 \text{ for } i = 1, \ldots, N.
\]

Theorem 3.2. Let us assume \( p_i > 2 \) for \( i = 1, \ldots, N \), (2.12), (2.7), (2.8), (2.11), (2.12), (3.7) and (3.8) with \( 0 < q_i \leq \frac{p_i}{p_i - 2} \). Then there exists a unique weak solution to Problem (3.1).

Proof. We argue as in the proof of Theorem 3.1 taking into account the following extra term in (3.4)

\[
\sum_{i=1}^{N} \int_{D_t} (|u| + |v|)^{q_i} |\partial_x \varphi|.
\]

Using (3.7) and (3.8) instead of (3.2) and (3.3) respectively, we obtain

\[
\alpha \sum_{i=1}^{N} \int_{D_t} |\partial_x \varphi|^{p_i} \leq \beta \sum_{i=1}^{N} \int_{D_t} \left( \theta + |\partial_x v|^{p_i - 1} \right) |\partial_x \varphi| + \sum_{i=1}^{N} \int_{D_t} (|u| + |v|)^{q_i} |\partial_x \varphi|.
\]
Using Young inequality with some \( \delta > 0 \) and choosing \( \delta \) small enough, we obtain the analogue of (3.9)

\[
\sum_{i=1}^{N} \int_{D_t} |\partial_{x_i} \varphi|^{p_i} \leq c \left( N |D_t| + \sum_{i=1}^{N} \int_{D_t} |\partial_{x_i} v|^{p_i} + \sum_{i=1}^{N} \int_{D_t} (|u| + |v|)^{p_i q_i} \right),
\]

for some positive constant \( c \) independent on \( t \). Since \( 0 < q_i \leq \frac{p_i}{p_i - 1} \), (3.11) allows us to conclude.

\[ \square \]

**Remark 3.3.** If in Theorem 3.2 we assume (3.12) holds instead of (3.7), we can take \( 0 < q_i \leq \frac{p_i}{p_i - 1} \). Moreover Theorem 3.2 holds if we replace (3.8) by

\[
|a_i (x, s, \xi) − a_i (x, s', \xi)| \leq \beta \left( \theta + |\xi_i|^{p_i - 1} + (|s| + |s'|)^q \right) \omega(|s − s'|)
\]

for \( i = 1, .., N \), where \( \omega : [0, +\infty) \to [0, +\infty] \) is such that \( \omega(s) \leq s \) for \( 0 \leq s \leq \rho \) for some \( \rho > 0 \).

Now we study Problem (3.1) when at least one \( p_i \) is less or equal to 2 and \( \varepsilon = 0 \) in (3.2). We argue as in Theorem 3.1 by using Poincaré inequality (2.1) instead of inequality (2.3). The following result is obtained by a different proof in [2] (see Theorem 2.1).

**Theorem 3.4.** Let us assume (2.5), (2.7), (2.8), (2.11), (2.12), (3.2) with \( \varepsilon = 0 \) and (3.3) with \( \theta = 0 \). If at least one \( p_i \) is less or equal to 2, then there exists a unique weak solution to Problem (3.1).

**Proof.** Arguing as in the proof of Theorem 3.1 we have

\[
\alpha \sum_{i=1}^{N} \int_{D_t} |\partial_{x_i} \varphi|^2 (|\partial_{x_i} u| + |\partial_{x_i} v|)^{p_i - 2} \leq \beta \sum_{i=1}^{N} \int_{D_t} |\partial_{x_i} v|^{p_i - 1} |\partial_{x_i} \varphi|.
\]

By Young inequality with some \( \delta > 0 \) we get

\[
\int_{D_t} |\partial_{x_i} v|^{p_i - 1} |\partial_{x_i} \varphi| \leq \frac{\delta}{2} \int_{D_t} |\partial_{x_i} \varphi|^2 (|\partial_{x_i} u| + |\partial_{x_i} v|)^{p_i - 2} + \frac{1}{4\delta} \int_{D_t} (|\partial_{x_i} u| + |\partial_{x_i} v|)^{p_i}.
\]

Putting (3.11) in (3.10) and choosing \( \delta \) small enough we obtain

\[
\sum_{i=1}^{N} \int_{D_t} |\partial_{x_i} \varphi|^2 (|\partial_{x_i} u| + |\partial_{x_i} v|)^{p_i - 2} \leq c_1 \sum_{i=1}^{N} \int_{D_t} (|\partial_{x_i} u| + |\partial_{x_i} v|)^{p_i}
\]

for some positive constant \( c_1 \) independent on \( t \). Let \( p_j \leq 2 \). Using Poincaré inequality (2.4), Young inequality and (3.11) we get

\[
C_p |D \setminus D_t| \leq \frac{c_2}{2} \left[ \int_{D_t} \frac{|\partial_{x_j} \varphi|^2}{(|\partial_{x_j} u| + |\partial_{x_j} v|)^2} + \int_{D_t} (|\partial_{x_j} u| + |\partial_{x_j} v|)^{2 - p_j} \right]
\]

\[
\leq \frac{c_1 c_2}{2} \sum_{i=1}^{N} \int_{D_t} (|\partial_{x_i} u| + |\partial_{x_i} v|)^{p_i} + \frac{c_2}{2} \int_{D_t} (|\partial_{x_j} u| + |\partial_{x_j} v|)^{2 - p_j}
\]

for some positive constant \( c_2 \) independent on \( t \). As in Theorem 3.1 condition (3.6) follows, then the assertion holds. \[ \square \]
Remark 3.5. Theorem 3.4 also holds if we replace (3.3) by
\[ |a_i(x,s,\xi) - a_i(x,s',\xi)| \leq \beta \left( |\xi|^{p_i-1} \right) \omega(|s - s'|) \quad \beta > 0 \]
for \( i = 1, \ldots, N \), where is like in Remark 3.3. Moreover if we suppose that every \( p_i < 2 \) Theorem 3.4 holds with \( \theta = 0 \) in (3.3). Finally we stress that Theorems 3.2 and 3.4 hold if in Problem (3.1) we add the term \( c(x,u) \) with suitable hypotheses (for example \( c \) is an increasing function with respect to \( u \)).

4. Operators with a first order term

In this section we consider Problem (2.6) when the functions \( a_i \) do not depend on \( u \). More precisely under the assumptions (2.7)-(2.12) we consider the following class of nonlinear anisotropic homogeneous Dirichlet problems:
\[
\begin{align*}
-\partial_i a_i(x,\nabla u) + \sum_{i=1}^{N} H_i(x,\nabla u) &= f - \partial_i g_i \quad \text{in} \quad \Omega \\
u &= 0 \quad \text{on} \quad \partial\Omega.
\end{align*}
\]

\[ (4.1) \]

4.1. An existence result for Problem (4.1). In this section we prove the existence of at least a weak solution to Problem (4.1). To our knowledge this result could not be found in literature.

The coercivity of the operator is guaranteed only if the norms of \( b_i \) are small enough. As usual we consider the approximate problems. Let \( H_i(x,\nabla u) \) be the truncation at levels \( \pm c \) of \( H_i \). It is well known (see e.g. [23]) that there exists a weak solution \( u_n \in W_0^{1,\overline{p}}(\Omega) \) to problem
\[
\begin{align*}
-\partial_i a_i(x,\nabla u_n) + \sum_{i=1}^{N} H_i(x,\nabla u_n) &= f - \partial_i g_i \quad \text{in} \quad \Omega \\
u &= 0 \quad \text{on} \quad \partial\Omega.
\end{align*}
\]

\[ (4.2) \]

The first and crucial step is an a priori estimate of \( u_n \). For the convenience of the reader we are writing explicitly the sum sign.

Lemma 4.1. Assume that (2.9), (2.7)-(2.10), (2.11) and (2.12) hold and let \( u_n \in W_0^{1,\overline{p}}(\Omega) \) be a solution to Problem (4.2). Then we have
\[
\sum_{i=1}^{N} \int_{\Omega} |\partial_i u_n|^{p_i} \leq C,
\]

\[ (4.3) \]

for some positive constant \( C \) depending on \( N, \Omega, A \), \( \gamma, p_i, b_i, \|f\|_{L^p(\Omega)}, \|g\|_{L^p(\Omega)} \) for \( i = 1, \ldots, N \).

Proof. In what follows we do not explicitly write the dependence on \( n \). The technique developed in [8] allows us to avoid the assumption on smallness of \( \|b_i\|_{L^\infty(\Omega)} \).

Let \( A \) be a positive real number, that will be chosen later. Then there exists \( t \) measurable subsets \( \Omega_1, \ldots, \Omega_t \) of \( \Omega \) and \( t \) functions \( u_1, \ldots, u_t \) such that \( \Omega_i \cap \Omega_j = \emptyset \) for \( i \neq j \) and \( |\Omega_i| = A \) for \( s \in \{1, \ldots, t-1\} \), \( \{x \in \Omega : |\nabla u_s| \neq 0\} \subset \Omega_s \), \( \nabla u = \nabla u_s \) a.e. in \( \Omega_s \), \( \nabla (u_1 + \ldots + u_s) u_s = (\nabla u) u_s, u_1 + \ldots + u_s = u \) in \( \Omega \) and \( \text{sign}(u) = \text{sign}(u_s) \) if \( u_s \neq 0 \) for \( s \in \{1, \ldots, t\} \).
Let us fix \( s \in \{1, \ldots, t \} \) and let us use \( u_s \) as test function in Problem (4.2). Using (2.7), Young and Hölder inequalities and Proposition 2.4 we obtain
\[
\sum_{i=1}^{N} \int_{\Omega} |\partial_{x_i} u_s|^p_i \leq c_1 \left( \|f\|_{L^{p_i}} \, d_x^+ + \sum_{i=1}^{N} \int_{\Omega} |H'(x, \nabla u)| \, |u_s| + \sum_{i=1}^{N} \|g_i\|_{L^{p_i}} \right) \tag{4.4}
\]
for some constant \( c_1 > 0 \), where \( d_x = \prod_{i=1}^{N} \left( \int_{\Omega} |\partial_{x_i} u_s|^p_i \right)^{\frac{1}{p_i}} \). Here and in what follows the constants depend on the data but not on the function \( u \).

Using condition (2.10), Hölder and Young inequalities and Proposition 2.4 we get
\[
\left| \sum_{i=1}^{N} \int_{\Omega} H^i(x, \nabla u) u_s \right| \leq \sum_{i=1}^{N} b_i \int_{\Omega} |\partial_{x_i} u|^{p_i-1} |u_s| \tag{4.5}
\]
\[
\leq \sum_{i=1}^{N} b_i \sum_{s=1}^{s} \int_{\Omega_s} |\partial_{x_i} u_s|^{p_i-1} |u_s| \leq 2 \sum_{i=1}^{N} A_{\frac{1}{p_i} - \frac{1}{\infty}}^{\frac{1}{p_i} - \frac{1}{\infty}} \left[ \int_{\Omega_s} |\partial_{x_i} u_s| |\Omega_s|^{\frac{1}{p_i} - \frac{1}{\infty}} + d_x^+ |\Omega_s|^{\frac{1}{p_i} - \frac{1}{\infty}} \right]
\]
\[
\leq c_2 \sum_{i=1}^{N} A_{\frac{1}{p_i} - \frac{1}{\infty}}^{\frac{1}{p_i} - \frac{1}{\infty}} \left[ \int_{\Omega_s} |\partial_{x_i} u_s|^{p_i} + \sum_{s=1}^{s} \int_{\Omega_s} \left| \partial_{x_i} u_s \right|^{p_i} + \sum_{i=1}^{N} d_x^+ \right]
\]
for some constant \( c_2 > 0 \). Putting (4.5) in (4.4) we obtain
\[
\sum_{i=1}^{N} \int_{\Omega} |\partial_{x_i} u_s|^{p_i} \leq c_1 \left( \|f\|_{L^{p_i}} \, d_x^+ + \sum_{i=1}^{N} \|g_i\|_{L^{p_i}} \right) + c_2 \sum_{i=1}^{N} A_{\frac{1}{p_i} - \frac{1}{\infty}}^{\frac{1}{p_i} - \frac{1}{\infty}} \left[ \int_{\Omega_s} |\partial_{x_i} u_s|^{p_i} + \sum_{s=1}^{s} \int_{\Omega_s} \left| \partial_{x_i} u_s \right|^{p_i} + \sum_{i=1}^{N} d_x^+ \right]
\]
(4.6)
If \( A \) is such that
\[
1 - c_1 c_2 \sum_{i=1}^{N} A_{\frac{1}{p_i} - \frac{1}{\infty}}^{\frac{1}{p_i} - \frac{1}{\infty}} > 0,
\]
inequality (4.6) becomes
\[
\sum_{i=1}^{N} \int_{\Omega} |\partial_{x_i} u_s|^{p_i} \leq c_3 \left( \|f\|_{L^{p_i}} \, d_x^+ + \sum_{i=1}^{N} \|g_i\|_{L^{p_i}} \right) + \sum_{s=1}^{s} \sum_{i=1}^{N} A_{\frac{1}{p_i} - \frac{1}{\infty}}^{\frac{1}{p_i} - \frac{1}{\infty}} \left( \sum_{j=1}^{N} \int_{\Omega_s} |\partial_{x_j} u_{\sigma_j}|^{p_i} \right) + \sum_{i=1}^{N} A_{\frac{1}{p_i} - \frac{1}{\infty}}^{\frac{1}{p_i} - \frac{1}{\infty}} d_x^+ \right]
\]
(4.8)
for some constant \( c_3 > 0 \). For \( s = 1 \) we get
\[
\int_{\Omega} |\partial_{x_i} u_1|^{p_i} \leq \sum_{i=1}^{N} \int_{\Omega} |\partial_{x_i} u_1|^{p_i} \leq c_3 \left[ \|f\|_{L^{p_i}} \, d_x^+ + \sum_{i=1}^{N} \|g_i\|_{L^{p_i}} + \sum_{i=1}^{N} A_{\frac{1}{p_i} - \frac{1}{\infty}}^{\frac{1}{p_i} - \frac{1}{\infty}} d_x^+ \right].
\]
(4.9)

Let us choose \( A \) such that (4.7) and
\[
1 - c_3 \sum_{i=1}^{N} A_{\frac{1}{p_i} - \frac{1}{\infty}}^{\frac{1}{p_i} - \frac{1}{\infty}} > 0
\]
Moreover by (2.8) and (2.10) for any \( i \)

Using convergence (4.14) we have for \( u \) for some \( c \)

Assume that (2.5), (2.7)-(2.10), (2.11) and (2.12) hold, then there exists at least a weak solution to Problem (4.1). Theorem 4.2.

Proof. We give only a sketch of the proof, because it is standard. By (4.3) we have

By this choice we obtain

Then there exists a constant \( c_5 > 0 \) such that \( d_1 \leq c_5 \) and by (4.9) we obtain

for some constant \( c_6 > 0 \). Moreover using (4.10) in (4.8) and iterating on \( s \) we have

then arguing as before we obtain

for some constant \( c_8 > 0 \). The assertion follows immediately since \( \|u\|_{W_{p-}^1(\Omega)} \leq k \sum_{i=1}^N \int_\Omega |\partial_i u|^p \leq c_1 \) for some positive \( k > 0 \).

Now we are able to prove the following existence result.

**Theorem 4.2.** Assume that (2.5), (2.7), (2.10), (2.11) and (2.12) hold, then there exists at least a weak solution to Problem (4.1).

Proof. We give only a sketch of the proof, because it is standard. By (4.3) the sequence \( \partial_i u_n \) is bounded in \( L^{p_i}(\Omega) \) so we have that

for some \( u \) and for some subsequence, which we still denote by \( u_n \). We can argue as in [7] to prove

Using convergence (4.14) we have for \( i = 1, \ldots, N \)

Moreover by (2.8) and (2.10) for any \( q_i \in [1, p_i'] \) we have

and

and

and
for some positive constant \( c \), for \( i = 1, \ldots, N \) and for any measurable subset \( E \). Then Vitali Theorem assures
\[
a_i(x, \nabla u_n) \to a_i(x, \nabla u) \quad \text{and} \quad H^*_i(x, \nabla u_n) \to H_i(x, \nabla u) \quad \text{strongly in } L^q(\Omega)
\]
for \( q_i \in [1, p'_i] \), that allow us to pass to the limit in the approximate problems. \( \square \)

**Remark 4.3.** The last theorem still holds if in (2.10) we assume \( b_i \in L^{1'}(\Omega) \) with
\[
\frac{1}{r_i} = \frac{1}{p_i} - \frac{1}{p_{i\infty}} \quad \text{for } i = 1, \ldots, N.
\]

### 4.2. Some uniqueness results for Problem (4.1)

The first uniqueness result is obtained when every \( p_i \) is no greater than 2 assuming the following Lipschitz condition on \( H_i \)
\[
|H_i(x, \xi) - H_i(x, \xi')| \leq h \frac{|\xi_i - \xi'_i|}{(\eta + |\xi_i| + |\xi'_i|)^{\sigma_i}}
\]
for some constants \( h > 0, \eta > 0 \) and \( \sigma_i > 0 \) for \( i = 1, \ldots, N \).

**Theorem 4.4.** Let \( 1 < p_i \leq 2 \) if \( N = 2, \frac{2N}{N+2} < p_i < 2 \) if \( N \geq 3 \) and \( \sigma_i > 1 - \frac{N}{2} \)
for \( i = 1, \ldots, N \). Let us assume (2.5), (2.7)-(2.12), (3.2) with \( \varepsilon = 0 \) and (4.19) with \( \eta > 0 \). Then there exists a unique weak solution to Problem (4.1).

**Proof.** Let us suppose \( u \) and \( v \) are two weak solutions to Problem (4.1) and denote \( w = (u - v)^+ \) and \( E_t = \{ x \in \Omega : t < w < \sup w \} \) for \( t \in [0, \sup w] \). We use
\[
w_t = \begin{cases} 
  w(x) - t & \text{if } w(x) > t \\
  0 & \text{otherwise}
\end{cases}
\]
as test function in the difference of the equations. Strong monotonicity (5.2) with \( \varepsilon = 0 \) and the Lipschitz condition (4.14) with \( \eta > 0 \) give
\[
\sum_{i=1}^{N} \int_{E_t} \frac{|\partial_x w_i|^2}{(\partial_x u + |\partial_x v|)^{2-p_i}} \leq \frac{h}{\alpha} \sum_{i=1}^{N} \int_{E_t} \frac{|\partial_x w_i|}{\eta + |\partial_x u + |\partial_x v||}^2
\]
Since \( \sigma_i > 1 - \frac{N}{2} \) by Young inequality and some easy computations we have
\[
\sum_{i=1}^{N} \int_{E_t} \frac{|\partial_x w_i|^2}{(\partial_x u + |\partial_x v|)^{2-p_i}} \leq c \int_{E_t} w_t^2
\]
for some positive constant \( c \) independent on \( t \). Moreover by (2.2) and Hölder inequality we get
\[
\frac{1}{C_S} \left( \int_{E_t} w_t^2 \right)^\frac{1}{2} \leq \prod_{i=1}^{N} \left( \int_{E_t} |\partial_x w_t|^{\frac{2N}{N+2}} \right)^{\frac{N+2}{2N}} \leq \prod_{i=1}^{N} \left( \int_{E_t} |\partial_x w_t|^2 \right)^{\frac{N}{2}} \left( \int_{E_t} (\partial_x u + |\partial_x v|)^{2-p_i} \right)^{\frac{2}{2-p_i}},
\]
then by (2.3) we obtain
\[
\frac{1}{C_S^2} \int_{E_t} w_t^2 \leq N \sum_{i=1}^{N} \int_{E_t} |\partial_x w_t|^2 \left( \int_{E_t} (\partial_x u + |\partial_x v|)^{2-p_i} \right)^{\frac{2}{2-p_i}} \sum_{i=1}^{N} \left( \int_{E_t} (\partial_x u + |\partial_x v|)^{2-p_i} \right)^{\frac{2}{2-p_i}}.
\]
Finally using (4.10) we get
\[ \frac{1}{C_S} \leq CN^2 \sum_{i=1}^{N} \left( \int_{E_i} (|\partial_{x_i} u| + |\partial_{x_i} v|)^{(2-p_i)\frac{N}{2}} \right)^{\frac{2}{N}}. \]
Since \((2 - p_i) \frac{N}{2} \leq p_i\) we have
\[ \lim_{t \to \sup w} \int_{E_t} (|\partial_{x_i} u| + |\partial_{x_i} v|)^{(2-p_i)\frac{N}{2}} = 0, \]
that gives a contradiction. \(\square\)

The second result is obtained when every \(p_i\) is greater than 2 but \(\varepsilon > 0\) in \((3.2)\) and we assume the following Lipschitz condition on \(H_i\)
\[ |H_i(x, \xi) - H_i(x, \xi')| \leq h_i(x) (|\xi| + |\xi'|)^{\sigma_i - 1} |\xi - \xi'| \] \hspace{1cm} (4.17)
with \(\sigma_i \geq 0, h_i \in L^{s_i}(\Omega)\) and \(s_i \geq \frac{p_{\infty} - p_i}{p_{\infty} - 2}\).

**Theorem 4.5.** Let us suppose
\[ N \geq 3, \ 2 \leq p_i \leq \frac{2Ns_i}{Ns_i - 2s_i - 2N}, \]
\[ s_i \geq \max \left\{ N; \frac{p_{\infty}p_i}{p_{\infty} - p_i} \right\} \text{ and } 0 \leq \sigma_i \leq \frac{p_i}{N} - \frac{p_i - 2}{2s_i} \]
for \(i = 1, \ldots, N\). Let us assume \((2.9), (2.11) - (3.3), (3.4)\) with \(\varepsilon > 0\) and \((4.17)\). Then there exists a unique weak solution to Problem \((4.1)\).

**Proof.** Arguing as in the proof of Theorem 4.4 we get
\[ \sum_{i=1}^{N} \int_{E_i} (|\partial_{x_i} u| + |\partial_{x_i} v|)^{p_i - 2} \leq \frac{1}{s_i} \sum_{i=1}^{N} \int_{E_i} h_i (|\partial_{x_i} u| + |\partial_{x_i} v|)^{\sigma_i} |\partial_{x_i} w_t| w_t. \] \hspace{1cm} (4.18)
If \(\sigma_i \geq \frac{p_{\infty} - 2}{2}\) by Young inequality we have
\[ \sum_{i=1}^{N} \int_{E_i} (|\partial_{x_i} u| + |\partial_{x_i} v|)^{p_{\infty} - 2} \leq c_1 \sum_{i=1}^{N} \int_{E_i} h_i^2 (|\partial_{x_i} u| + |\partial_{x_i} v|)^{2\sigma_i - (p_i - 2)} w_t^2 \] \hspace{1cm} (4.19)
for some positive constant \(c_1\) independent on \(t\). Using inequalities \((2.2)\) and \((2.3)\), Hölder inequality and \((4.19)\) we obtain
\[ \frac{1}{C_S} \left( \int_{E_t} |w_t^o|^2 \right)^{\frac{2}{N}} \leq \prod_{i=1}^{N} \left( \int_{E_i} (|\partial_{x_i} w_t^o|)^{p_i} \right)^{\frac{2}{p_i}} \leq c_2 \prod_{i=1}^{N} \left( \int_{E_i} (|\partial_{x_i} w_t|)^{2\sigma_i - (p_i - 2)} \right)^{\frac{2}{p_i} \sigma_i} \]
\[ \leq c_2 \sum_{i=1}^{N} \int_{E_i} h_i^2 (|\partial_{x_i} u| + |\partial_{x_i} v|)^{2\sigma_i - (p_i - 2)} w_t^2 \]
\[ \leq c_2 \left( \int_{E_t} w_t^2 \right)^{\frac{2}{N}} \sum_{i=1}^{N} \left( \int_{E_i} h_i^N (|\partial_{x_i} u| + |\partial_{x_i} v|)^{(2\sigma_i - p_i + 2)} \right)^{\frac{2}{N}}, \]
i.e.
\[ \frac{1}{C_S} \leq c_2 \sum_{i=1}^{N} \left( \int_{E_i} h_i^N (|\partial_{x_i} u| + |\partial_{x_i} v|)^{(2\sigma_i - p_i + 2)} \right)^{\frac{2}{N}} \]
for some positive constant $c_2$ (independent on $t$), that can be vary from line to line. Since $\frac{N}{s_i} + \frac{(2s_i-p_i+2)N}{p_i} \leq 1$ we have
\[
\lim_{t \to \sup w} \int_{E_t} h_i^N (|\partial_x u| + |\partial_x v|)^{2s_i-p_i+2} = 0,
\]
that gives a contradiction.

Conversely if $\sigma_i < \frac{p_i-2}{2}$ by (4.18) and Young inequality we have
\[
\sum_{i=1}^N \int_{E_t} |\partial_x w_i|^p \leq C \sum_{i=1}^N \int_{E_t} h_i^N (|\partial_x u| + |\partial_x v|)^{2\sigma_i} w_i^2
\]
for some positive constant $c_3$ independent on $t$. Using inequalities (2.2) and (2.3), Hölder inequality and (4.20) as before we obtain
\[
\frac{1}{C_S} \leq C_4 \sum_{i=1}^N \left( \int_{E_t} h_i^N (|\partial_x u| + |\partial_x v|)^{N\sigma_i} \right)^{\frac{1}{N}}
\]
for some positive constant $c_4$ independent on $t$. Since $\frac{N}{s_i} + \sigma_i \leq \frac{p_i}{p_i} - \frac{2}{p_i}$ for some positive constant $c_4$ independent on $t$. Since $\frac{N}{s_i} + \sigma_i \leq \frac{p_i}{p_i} - \frac{2}{p_i}$ the assert. □

Remark 4.6. We observe that for $p_i > \frac{2N}{N s_i - 2 s_i - 2 N}$, Theorem 4.5 holds for $0 \leq \sigma_i \leq \frac{p_i}{p_i} - \frac{2}{p_i}$. Moreover Theorems 4.4 and 4.5 hold if in (4.7) we add the term $c(x,u)$ as in Remark 3.3.

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