Continuous-variable Werner state: separability, nonlocality, squeezing and teleportation

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We investigate the separability, nonlocality and squeezing of continuous-variable analogue of the Werner state: a mixture of pure two-mode squeezed vacuum state with local thermal radiations. Utilizing this Werner state, coherent-state teleportation in Braunstein-Kimble setup is discussed.

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I. INTRODUCTION

Quantum entanglement and nonlocality are fundamental resources for the quantum information processing such as quantum teleportation [1], entanglement swapping [2], dense coding [3], quantum cryptography [4] and quantum computation [5]. The efficiency of quantum information processing significantly depends on the degree of entanglement or nonlocality of the quantum state shared by the parties involved in a given protocol. This dependence may be particularly vividly illustrated with the Werner states [6], which are formed by a mixture of maximally entangled state and a separable maximally mixed state,

$$\rho = p|\Psi\rangle\langle\Psi| + \frac{(1-p)}{d^2} I_1 \otimes I_2, \quad 0 \leq p \leq 1, \quad (1)$$

where

$$|\Psi\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\rangle_1 |i\rangle_2 \quad (2)$$

is maximally entangled state in $d$-dimensional Hilbert space.

The Werner state is characterized by a single parameter: the probability $p$ of the maximally entangled state in the mixture and the Werner state is entangled iff $p > 1/(1+d)$. When the Werner state is used as a quantum channel for teleportation, then the average teleportation fidelity is given by $F = p + (1 - p)/d$ [7, 8]. This figure should be compared with the maximum fidelity achievable by means of classical communication and local operations $F_C = 2/(1 + d)$ [9]. Since this boundary is reached exactly for $p = 1/(1 + d)$ one concludes that all entangled Werner states are useful for the teleportation. Particularly interesting is the Werner state of two qubits, because for this system both the necessary and sufficient conditions on inseparability and nonlocality have been established by Horodeckis [10, 11, 12]. An important feature of the two-qubit Werner states is a non-empty gap $1/3 < p \leq 1/\sqrt{2}$ between separable and nonlocal states.

In recent years, great attention has been paid to the quantum information processing with continuous variables (CV). Most protocols developed originally for discrete quantum variables (qubits) have been extended to continuous variables, namely teleportation [13], dense coding [14], entanglement swapping [15] and quantum cloning [16].

In this paper, we introduce a natural analogue of the Werner state [7] for the CV systems: a mixture of pure two-mode squeezed vacuum state and mixed separable thermal state. We analyze in detail the separability, nonlocality and squeezing of the CV Werner state and we also discuss its usefulness in the teleportation of coherent states. Since there is no general method how to test the separability of a generic state in infinite-dimensional Hilbert space, one has to resort to some particular tests. We use the Peres-Horodecki (PH) criterion based on partial transposition [11]. Remarkably, non-positive partial transpose is necessary and sufficient condition for inseparability of two-mode bipartite Gaussian states [17]. However, the CV Werner state discussed in this paper is not Gaussian and hence in our case the PH criterion provides only a sufficient condition on the entanglement.

Testing of nonlocality for CV systems is based predominantly on the Banaszek-Wódkiewicz form of the Bell inequalities [18] working with Wigner function of a state. Here, we suggest alternative CHSH Bell inequalities for continuous quantum variables. By means of specific local transformations we map the two-mode CV Werner state onto state of two qubits and then we employ the necessary and sufficient conditions on nonlocality for two-qubit system.

After discussing the separability and nonlocality of the Werner state we analyze its performance in quantum information processing. We consider the standard Braunstein-Kimble (BK) scheme for teleportation of CV [19] where the Werner state serves as a quantum channel. Specifically, we focus on the teleportation of coherent states and we compare our findings with the results that have been obtained for qubit or qudit teleportation with Werner states [20, 21].

The paper is organized as follows. In Sec. II, the CV analogue of Werner state is introduced. The mapping from infinite-dimensional Hilbert space to Hilbert space of two qubits is described in Sec. III. In Sec. IV, we will analyse the separability of the Werner state from two different points of view: after and before mapping on two-qubit system. Sec. V and Sec. VI are devoted to the nonlocality and squeezing of the Werner state. In Sec. VII,
the coherent-state teleportation with the Werner state is discussed. Finally, Sec. VIII contains the conclusions.

II. CONTINUOUS-VARIABLE WERNER STATE

A common resource of the quantum entanglement in CV information processing is the two-mode squeezed vacuum state generated by means of spontaneous parametric downconversion in the non-degenerate optical parametric amplifier (NOPA),

$$\rho_{\text{NOPA}} = (1 - \lambda_1^2) \sum_{m,n=0}^{\infty} \lambda_1^{m+n} |m,m\rangle\langle n,n|.$$  \hspace{1cm} (3)

Here $\lambda_1 = \tanh r$, $r$ is the squeezing parameter, and $|m,n\rangle = |m\rangle_A |n\rangle_B$ denotes the Fock state of two modes $A$ and $B$. The NOPA state approaches the maximally entangled EPR state \[19\] in the strong squeezing limit $r \rightarrow \infty$. In practice, the EPR state is well approximated by the NOPA state if $r > 2$. Recently, squeezing as large as $r \approx 2$ has been achieved experimentally \[20\].

A natural extension of the NOPA state to the Werner state for CV is based on the following point of view: The thermal state of modes $A$ and $B$ in the mixture \[21\] is the strong squeezing limit $r \rightarrow \infty$. In practice, the thermal state is well approximated by the NOPA state if $r > 2$. Recently, squeezing as large as $r \approx 2$ has been achieved experimentally \[20\].

\[22\]

where $\lambda_2 = \tanh s$ and the mean number of thermal photons in each mode reads $\langle n\rangle_T = \sinh^2(s)$.

It is thus natural to define the CV analogue of the Werner state \[4\] as a mixture of the NOPA state \[2\] and factorized thermal state \[23\],

$$\rho_T = (1 - \lambda_2^2) \sum_{m,n=0}^{\infty} \lambda_2^{2(m+n)} |m\rangle\langle m| \otimes |n\rangle\langle n|.$$  \hspace{1cm} (4)

The Werner states $\rho_W$ form a three-parametric family of states. The simplest analogue of Werner state can be obtained assuming $r = s$. In this case the Werner state and $d$-dimensional Werner state \[23\] become manifestly analogous in the strong squeezing limit when $\rho_W$ approaches a mixture of maximally entangled EPR state and maximally mixed state in infinite-dimensional Hilbert space.

III. MAPPING ON TWO-QUBIT SYSTEM

The simplest way in which one can study the separability and nonlocality properties of the two-mode state \[2\] is to map it by means of local operations on the two-qubit system for which separability and nonlocality conditions are well known \[13\]. In what follows the qubits corresponding to modes $A$ and $B$ are denoted as 1 and 2, respectively.

We introduce the Hermitian “spin one-half” operators $S_\alpha$, $\alpha = A, B$,

$$S_1^\alpha + iS_2^\alpha = 2 \sum_{m=0}^{\infty} |2m\rangle\langle 2m+1|,$$

$$S_3^\alpha = \sum_{m=0}^{\infty} (-1)^m |m\rangle\langle m|,$$

which satisfy the Pauli matrix algebra

$$[S_\alpha^\alpha, S_\beta^\beta] = 2i\varepsilon_{\alpha\beta\gamma} S_\gamma^\alpha, \hspace{1cm} (S_\alpha^\alpha)^2 = 1,$$  \hspace{1cm} (7)

where $\varepsilon_{ijk}$ is the totally antisymmetric tensor with $\varepsilon_{123} = +1$ and $\delta_{\alpha\beta}$ is the Kronecker symbol.

Let us for a moment restrict our attention to the mode $A$ and qubit 1. With the help of the operators \[24\] one can assign the following qubit density matrix $\rho_1$ to the density matrix $\rho_A$,

$$\rho_1 = \frac{1}{2} (I_1 + S^A \cdot \sigma),$$  \hspace{1cm} (8)

where

$$S^A \cdot \sigma = \sum_{i=1}^{3} \text{Tr}(\rho_A S_i^A)\sigma_i.$$  \hspace{1cm} (9)

$\sigma_i$ are standard Pauli matrices and $I_1$ is the identity operator on the Hilbert space of qubit 1.

The transformation \[24\] is physical, because it can be, at least in principle, performed in the lab. Let us assume that the qubit is represented by a two-level atom resonantly interacting with a single mode of electromagnetic field. Suppose that the interaction is governed by the following Hamiltonian

$$H_{\text{int}} = i\hbar \Omega \left( |0\rangle\langle 1| \sqrt{n} a^+ - |1\rangle\langle 0| a \sqrt{n} \right),$$  \hspace{1cm} (10)

where $a$ ($a^+$) is annihilation (creation) operator of the mode $A$ and $n = a^+a$. The Hamiltonian \[10\] can be considered as a kind of nonlinear Jaynes-Cummings model. The specific feature of $H_{\text{int}}$ is that its eigenvalues (the Rabi frequencies) are linearly proportional to the number of photons $n$. If the two-level atom is initially in its ground state $|0\rangle$ and if the interaction time $t$ is adjusted in such a way that $\Omega t = \pi/2$ then the output state of the atom is exactly given by Eq. \[10\]. Although the Hamiltonian \[10\] may be hard to implement in practice, it provides a clear physical picture behind the mathematical transformation \[24\].

Formally, the transformation \[24\] is a trace-preserving completely positive (CP) map. Making use of the correspondence between CP maps and positive semidefinite...
operators we can express the transformation as follows,
\[
\rho_1 = \text{Tr}_A[\chi_{A1}\rho_A^T \otimes I_1],
\]
where
\[
\chi_{A1} = \sum_{m=0}^{\infty} \sum_{k,l=0}^1 (|2m+k\rangle_{AA} \langle 2m+l| \otimes |k\rangle_{11} \langle l|)
\]
is a positive-semidefinite operator acting on the direct product of the Hilbert spaces of the mode \(A\) and of the qubit 1; \(|1\rangle_1\) and \(|0\rangle_1\) are basis states of qubit 1; and \(T\) stands for the transposition. Mapping now the two-mode density matrix to two-qubit density matrix
\[
\rho'_W = \text{Tr}_{AB}[\chi_{A1}\chi_{B2}\rho_W^T \otimes I_1 \otimes I_2],
\]
where the \(\chi_{B2}\) is obtained from (12) by replacements
\[
A \rightarrow B \text{ and } 1 \rightarrow 2,
\]
one gets
\[
\rho'_W = \frac{1}{4}(I_1 \otimes I_2 + S^A \cdot \sigma \otimes I_2 + I_1 \otimes S^B \cdot \sigma + \sum_{i,j=1}^3 t_{ij} \sigma_i \otimes \sigma_j).
\]
The elements \(t_{ij} = \text{Tr}(\rho_W S_i^A S_j^B)\) of the correlation tensor \(T\) explicitly read
\[
t_{11} = -t_{22} = \frac{2\lambda_1 p}{1 + \lambda_1^2},
\]
\[
t_{33} = p + (1 - p) \left( \frac{1 - \lambda_2^2}{1 + \lambda_2^2} \right)^2,
\]
\[
t_{ij} = 0, \quad i \neq j.
\]
On calculating the matrices \(S^A \cdot \sigma\) and \(S^B \cdot \sigma\) and taking into account the expressions one obtains after some algebra
\[
\rho'_W = \begin{pmatrix}
p(1 + \lambda_1^2) & 0 & 0 & \frac{2\lambda_1 p}{1 + \lambda_1^2} \\
0 & \lambda_2^2(1 - p) & 0 & 0 \\
0 & 0 & \lambda_2^2(1 - p) & 0 \\
\frac{2\lambda_1 p}{1 + \lambda_1^2} & 0 & 0 & \lambda_2^2(1 - p)
\end{pmatrix}.
\]

Thus we have mapped the state \(\rho_W\) onto this two-qubit state. Note that the transformation (13) is local; it is carried out separately on each subsystem \((A, 1)\) and \((B, 2)\). The essential feature of local unconditional transformations is that they cannot increase the amount of entanglement or nonlocality present in any bipartite state. This ensures that the properties of the state \(\rho_W\) reflect the properties of the original Werner state \(\rho_W\). If we find that the state \(\rho_W\) is entangled or nonlocal, then the same holds true about the original state \(\rho_W\).

### IV. SEPARABILITY

According to the PH partial transposition criterion the state (14) is entangled iff its partial transpose
\[
(\rho'_W)^{T_1}_{m_\mu,n_\nu} \equiv (\rho'_W)_{m_\mu,n_\nu},
\]
has some negative eigenvalue. Due to the specific structure of the matrix (16), it is easy to see that its partial transposition has negative eigenvalue if the off-diagonal elements of \(\rho'_W\) are larger than the central diagonal elements,
\[
\frac{2\lambda_1 p}{1 + \lambda_1^2} > \frac{\lambda_2^2(1 - p)}{(1 + \lambda_2^2)^2}.
\]

It is instructive to rewrite this condition as an inequality for the probability \(p\). After some algebra, one finds that the state (14) is entangled iff
\[
p > \frac{1}{1 + \left(\frac{s}{\tanh(2r)}\right)^2},
\]
where we have used the relations \(\lambda_1 = \tanh r\) and \(\lambda_2 = \tanh s\). For direct analogue of the Werner state with \(r = s\), and in the large squeezing limit, the state (14) and hence also the state (14) for which it is only sufficient condition for entanglement. The partially transposed matrix \(\rho_W^{T_1}\) has a block diagonal form with \(1 \times 1\) blocks in one-dimensional subspaces spanned by vectors \(\{|m, m\}\), \(m = 0, 1, \ldots\) and \(2 \times 2\) blocks in two-dimensional subspaces spanned by vectors \(\{|m, n\}, |n, m\}, m \neq n\),
where $l = 0, 1, \ldots$ and $m \neq n = 0, 1, \ldots$. According to the above mentioned separability criterion \([13]\) the state \((\mathbf{3})\) is entangled if there are such $m$, $n$ for which $x_{2}^{(mn)} < 0$. If $\lambda_{2} = 0$ then $x_{2}^{(mn)} < 0$ for all $p > 0$ and the state \((\mathbf{3})\) is always entangled. If $\lambda_{2} \neq 0$ then the inseparability condition $x_{2}^{(mn)} < 0$ is equivalent with the inequality

$$x^{(l)} = p(1 - \lambda_{1}^{2})\lambda_{1}^{2l} + (1 - p)(1 - \lambda_{2}^{2})\lambda_{2}^{2l},$$

$$x_{1,2}^{(mn)} = (1 - p)(1 - \lambda_{2}^{2})\lambda_{2}^{2(m+n)} \pm p(1 - \lambda_{2}^{2})\lambda_{1}^{m+n+1},$$

(20)

Three different cases must be considered in dependence on the value of the ratio $q = \lambda_{1}/\lambda_{2}^{2}$.

(i) If $q > 1$ then the factor $q^{m+n}$ in the denominator of Eq. (21) increases with increasing $m+n$ and consequently the right-hand side (R.H.S.) of (21) decreases attaining zero value in the limit $m+n \to \infty$. Hence, the state \((\mathbf{3})\) is entangled for any $p > 0$. In particular, a direct analogue of the Werner state ($r = s$) is entangled for every $p > 0$.

(ii) If $q = 1$ then also $q^{m+n} = 1$. From that it follows that the inequality (21) is independent on $m$ and $n$ and the state \((\mathbf{3})\) is entangled if

$$p > \frac{1 - \lambda_{1}}{2} = \frac{1 - \tanh r}{2}.$$  

(22)

(iii) If $q < 1$ then the R.H.S. of inequality (21) increases with growing $m + n$. Since the R.H.S. attains its minimum value for $m + n = 1$ the state \((\mathbf{3})\) is entangled if

$$p > \frac{(1 - \lambda_{2}^{2})^{2}}{(1 - \lambda_{2}^{2})^{2} + (1 - \lambda_{1}^{2})\frac{2}{\lambda_{2}^{2}}} = p_{1}$$

(23)

or equivalently,

$$p > \frac{1}{1 + \frac{\cosh^{2}(s)}{\cosh^{2}(r) \tanh r}}.$$  

(24)

The partial transposition criterion applied to the original state $\rho_{W}$ is stronger than that applied to $\rho_{W}^{*}$, because any local transformation \([13]\) preserves the positivity of the partial transpose. For instance, if $\tanh r > \tanh^{2}(s)$ then the entangled states \((\mathbf{3})\) for which

$$\frac{\tanh^{2}(2s)}{\tanh^{2}(2s) + 2 \tanh(2r)} \geq p > 0$$

(25)

are mapped on the separable two-qubit states \((\mathbf{4})\).

The region of Werner state inseparability is depicted in Fig. 1. We can see that the Werner state is entangled almost for every $p$ if the squeezing is sufficiently large. We emphasize again that the negative partial transpose is only sufficient condition on the entanglement and there may exist PPT entangled Werner states. One may even ask whether there exist any nontrivial separable CW Werner states \((\mathbf{3})\). Although we do not have the sufficient separability condition for generic bipartite CV states at present, it is possible to find conditions under which the Werner state is separable, i.e., it can be written as a convex combination of product states.

The state \((\mathbf{3})\) can be rewritten in the following form

$$\rho_{W} = \sum_{m=0}^{\infty} P_{m}|mm\rangle\langle mm| + \sum_{m\neq n=0}^{\infty} \rho^{mn},$$

(26)

where

$$P_{m} = p(1 - \lambda_{1}^{2})\lambda_{1}^{2m} + (1 - p)(1 - \lambda_{2}^{2})(1 - \lambda_{2}^{2}) \lambda_{1}^{2m},$$

(27)

and $\rho^{mn}$ are matrices in four-dimensional Hilbert subspaces spanned by the basis vectors $\{|mm\rangle$, $|mn\rangle$, $|nm\rangle$, $|nn\rangle\}$,

$$\rho^{mn} = \frac{1}{2} \begin{pmatrix} \alpha_{mn} & 0 & 0 & \beta_{mn} \\ 0 & \gamma_{mn} & 0 & 0 \\ 0 & 0 & \gamma_{mn} & 0 \\ \beta_{mn} & 0 & 0 & \alpha_{mn} \end{pmatrix},$$

(28)

where

$$\alpha_{mn} = p(1 - \lambda_{2}^{2})\lambda_{1}^{2(m+n)} + (1 - p)(1 - \lambda_{2}^{2})(1 - \lambda_{2}^{2}) \lambda_{2}^{8m},$$

$$\beta_{mn} = p(1 - \lambda_{1}^{2})\lambda_{1}^{m+n},$$

$$\gamma_{mn} = (1 - p)(1 - \lambda_{2}^{2})\lambda_{2}^{2(m+n)}.$$  

(29)

Obviously, if all $\rho^{mn}$ are positively semidefinite matrices, then $\rho_{W}$ is separable. From the matrix form \((\mathbf{28})\) of $\rho^{mn}$ one easily obtains the positivity condition $\alpha_{mn} \geq \beta_{mn}$ and the separability condition $\gamma_{mn} \geq \beta_{mn}$. Consequently, the Werner state is separable if both these inequalities are satisfied for all $m \neq n$. The second condition is identical with the necessary condition on separability of the Werner state $p \leq p_{m+n}$, where $p_{m+n}$ is given in the inequality \((\mathbf{21})\). Further constraint on $p$ follows from the positivity condition $\alpha_{mn} \geq \beta_{mn}$,

$$p \leq \frac{1}{1 + \frac{\left(1 - \lambda_{2}^{2}\right)^{2} \lambda_{1}^{2m+n} - (1 - \lambda_{1}^{2}) \left(\frac{\lambda_{2}}{\lambda_{1}}\right)^{m+n}}{\left(1 - \lambda_{2}^{2}\right)^{2} \lambda_{2}^{2m+n}}},$$

(30)

where $m \neq n = 0, 1, \ldots$. Similarly as in the case of entanglement we have to distinguish three regions in dependence on the value of the ration $\bar{q} = \lambda_{1}/\lambda_{2}^{2}$.

(i) If $\bar{q} > 1$ then the R.H.S. of the inequality \((\mathbf{30})\) goes to zero in the limit $m + n \to \infty$. Hence, the state \((\mathbf{3})\) is separable only for $p = 0$.

(ii) If $\bar{q} = 1$ then the expression in square brackets in the R.H.S. of the inequality \((\mathbf{30})\) attains its maximum
equal to unity in the limit \( m + n \to \infty \) and the Werner state can be separable only if

\[
p \leq \frac{(1 - \lambda_2^2)^2}{2(1 - \lambda_2^2 + \lambda_2^2)}.
\] (31)

In this case the condition \( p \leq p_1 \) is weaker than the inequality (31) which is thus sufficient condition for separability of the Werner state.

(iii) If \( \tilde{q} < 1 \) then the R.H.S. of the inequality (31) attains its minimum for \( m + n = 1 \) and the state (3) can be separable only if

\[
p \leq \frac{1}{1 + \frac{(1 - \lambda_2^2)^2}{(1 - \lambda_2^2 + \lambda_2^2)}}.
\] (32)

Since in this case the inequality (32) is stronger than the condition \( p \leq p_1 \), the inequality (32) is sufficient condition for separability of the Werner state.

V. NONLOCALITY

Due to the commutation rules (27) the nonlocality of the Werner state (3) can be investigated employing the standard two-qubit CHSH Bell inequalities in which the Pauli matrices are replaced with the single-mode operators (8),

\[
2 \geq 2 \left| \langle (a \cdot S^A)(b \cdot S^B) \rangle + \langle (a' \cdot S^A)(b \cdot S^B) \rangle + \langle (a \cdot S^A)(b' \cdot S^B) \rangle - \langle (a' \cdot S^A)(b' \cdot S^B) \rangle \right|,
\] (33)

where \( a, a', b, b' \) are real three-dimensional unit vectors and the angle brackets denote the averaging over the density matrix \( \rho_W \). It is instructive to formulate this approach in terms of the mapping introduced in Sec. III. We map the Werner state \( \rho_W \) onto the state of two qubits \( \rho'_{AB} \) and then we analyze the nonlocality of the state \( \rho'_{AB} \) characterized by the correlation tensor \( T \) whose elements are given by Eq. (15).

Now, according to the Horodecki criterion (14), if the sum of the two largest eigenvalues of the matrix \( U = T^T T \) is greater than unity then the state (3) violates the inequalities (33) for some choice of vectors \( a, a', b, b' \). The matrix \( U \) has two-fold eigenvalue \( t_{11}^2 \) and single eigenvalue \( t_{33}^2 \). It can be shown that the inequality \( t_{11}^2 \leq \frac{2}{1 + \left( \lambda_1^2(1 - \lambda_2^2) \right)} \) is satisfied for any \( \lambda_1, \lambda_2 \) and \( p \). Thus the maximal Bell factor is given by

\[
B_{\text{max}} = 2\sqrt{t_{11}^2 + t_{33}^2}.
\] (34)

Hence, the Bell inequality (33) is violated if \( t_{11}^2 + t_{33}^2 > 1 \). Substituting here from the formulas (15) one obtains after some algebra that the state (3) violates the Bell inequalities (33) if

\[
p > \frac{a(a - 1) + \sqrt{a^2(a - ab^2 + 2b^2)}}{a^2 + b^2},
\] (35)

where \( a = \tanh^2(2s) \) and \( b = \tanh(2r) \). The region of nonlocality of the state (3) is depicted in Fig. 2. In the large squeezing limit the direct analogue of the Werner state (3) with \( r = s \) is nonlocal if \( p > \frac{1}{\sqrt{2}} \) as in the case of two-qubit Werner state. However, it was found in the previous Section, that the original state \( \rho_{AB} \) in infinite-dimensional Hilbert space may be entangled even if the two-qubit state \( \rho'_{AB} \) is separable. We can conjecture that the nonlocality has a similar behavior and that the state \( \rho'_W \) may violate some Bell inequalities although the state \( \rho'_W \) admits local realistic description.

VI. SQUEEZING

Apart from entanglement and nonlocality, the measure of a nonclassicality of the Werner state (3) can be judged by means of squeezing. Since there is not preferred direction in the phase-space distribution of the thermal component (4) of the Werner state (3) one can expect that the state attains its maximum squeezing in the same quadrature as the NOPA state (3), i.e. in the quadrature \( x_A - x_B \) (where \( x_A, x_B \) are “position” quadratures of modes \( A \) and \( B \), respectively). Calculating the variance \( \langle (\Delta(x_A - x_B))^2 \rangle \) in the Werner state (3) and employing the squeezing condition \( \langle (\Delta(x_A - x_B))^2 \rangle < 1 \) one finds, that the Werner state is squeezed if

\[
p > \frac{1}{1 + \frac{2a_1(1 - \lambda_2^2)}{(1 + \lambda_1^2)(1 + \lambda_2^2)}} = \frac{1}{1 + \frac{1 - p}{1 + 4(p - p_1)}},
\] (36)

Interestingly, since for \( r = s \) and in the large squeezing limit the R.H.S. of the inequality goes to unity, we arrive at the family of Werner states which are never squeezed. This is the counterintuitive example of the Werner states which are not squeezed, however, they can be entangled or even nonlocal.

VII. TELEPORTATION

An interesting application of two-qubit Werner state arises in the quantum teleportation [4, 5]. By analogy, the proposed Werner state (3) can be utilized in BK scheme of the CV coherent-state teleportation [6].

In this scheme, two-mode squeezed vacuum state is shared between Alice and Bob. The Wigner function \( W_{\text{in}}(x_1, p_1) \) of the input Alice’s state and the Wigner function \( W_{\text{out}}(x_2, p_2) \) of the Bob’s output state are related by the convolution (23)

\[
W_{\text{out}}(x_2, p_2) = \frac{1}{4} \int_{-\infty}^{\infty} W_{\text{in}}(x_1, p_1) dx_1 dp_1 \times K_{AB}(x_2 - x_1, p_2 - p_1).
\] (37)

The kernel function \( K_{AB}(x, p) \) reads

\[
K_{AB}(x, p) = \int_{-\infty}^{\infty} K_{AB}(-x, x, p, -p) dx dp,
\] (38)
where \( x_\pm = x_A \pm x_B \) and \( p_\pm = p_A \pm p_B \), and 

\[ W_{AB}(x_A, p_A, x_B, p_B) = W_{AB}(x_-, x_+, p_-, p_+) \]

is the Wigner function of the state shared by Alice and Bob (quantum channel). The fidelity between Alice’s and Bob’s states can be calculated as follows,

\[
F = \frac{\pi}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_{in}(x_2, p_2) K_{AB}(x_2 - x_1, p_2 - p_1) \times W_{in}(x_1, p_1) dx_1 dp_1 dx_2 dp_2.
\]

The BK scheme is designed in such a way, that the fidelity is invariant under displacement transformations. In particular, all coherent states are teleported with the same fidelity. If the quantum channel is in NOPA state then this fidelity reads

\[
F_{\text{NOPA}} = \frac{1}{1 + e^{-2r}}.
\]

This exceeds the maximal classical value \( F = 1/2 \) for every \( r > 0 \). Now we consider the Werner state \( \rho \) in symmetric form with \( r = s \). One can find that the fidelity of teleportation in standard BK scheme utilizing such a Werner state is changed as follows

\[
F_W = p F_{\text{NOPA}} + (1 - p) \frac{1}{d}.
\]

where \( d = 2/(1 - \lambda_2^2) = 2\cosh^2 r \). The dependence of \( F_W \) on probability \( p \) and squeezing parameter \( r \) is depicted in Fig. 3.

In the limit of large squeezing, the fidelity \( F_W \approx p \). In order to teleport the coherent state with fidelity \( F_W > 1/2 \), we need to employ CV Werner state with \( p > 1/2 \). Note that for every \( p \neq 0 \), this Werner state is entangled (recall that we assume \( r = s \) here). This should be contrasted with results obtained for teleportation with \( d \)-dimensional Werner state, where if the shared Werner state is entangled then it can be useful for BK teleportation. In our case, some of the entangled Werner states in infinite-dimensional Hilbert space are not useful for BK teleportation protocol. It is an open question whether the BK scheme can be modified in such a way that the coherent states would be teleported with fidelity higher than 1/2 even when using entangled Werner states with \( p < 1/2 \).

VIII. CONCLUSIONS

A natural extension of the Werner state into CV systems is presented and separability, nonlocality and squeezing of this state is analyzed. In certain sense, the CV Werner state can be considered as a counterpart of the two-qubit Werner state. This relationship is established by the mapping. On the other hand, some features of the CV Werner state correspond to those of \( d \)-dimensional Werner states when \( d \to \infty \). For instance, in the simplest case when \( r = s \), the CV Werner state is entangled for any \( p > 0 \). Since \( d \)-dimensional Werner state is entangled when \( p > 1/(1+d) \), the above behavior of CV Werner state corresponds to the limit \( d \to \infty \).

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FIG. 1: Minimal probability \( p_{\text{min}} \) characterizing the entanglement of the CV Werner state as a function of the squeezing parameter \( r \) and the thermal noise parameter \( s \). The state \( \rho_W \) is entangled if \( p > p_{\text{min}} \).

FIG. 2: Maximal probability \( p_{\text{max}} \) characterizing the separability of the CV Werner state as a function of the squeezing parameter \( r \) and the thermal noise parameter \( s \). The state \( \rho_W \) is separable if \( p \leq p_{\text{max}} \).

FIG. 3: Minimal probability \( p_{\text{min}} \) characterizing the nonlocality of the CV Werner state as a function of the squeezing parameter \( r \) and the thermal noise parameter \( s \). The state \( \rho_W \) is nonlocal if \( p > p_{\text{min}} \).
FIG. 4: The dependence of the fidelity $F_W$ in standard BK scheme employing shared CV Werner state on the squeezing parameter $r = s$ and the probability of the NOPA state $p$. 
