Supercritical $\mathcal{N} = 2$ string theory

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Abstract

The $\mathcal{N} = 2$ string is examined in dimensions above the critical dimension ($D = 4$) in a linear dilaton background. We demonstrate that string states in this background propagate in a single physical time dimension, as opposed to two such dimensions present when the dilaton gradient vanishes in $D = 4$. We also find exact solutions describing dynamical dimensional reduction and transitions from $\mathcal{N} = 2$ string theory to bosonic string theory via closed-string tachyon condensation.

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1 Introduction

In a series of recent papers [1,2,3,4], a number of exactly solvable string backgrounds were presented, describing novel dynamical transitions between various different string theories. These transitions are initiated by closed-string tachyon condensation in some parent theory. The tachyon condensate forms a bubble of new vacuum that expands outward from a nucleation point at the speed of light. The transition to new vacua is driven by the evolution of target-space time, terminating in the late-time limit in a distinct final theory. Because of the exact solvability of these models, these dynamical transitions can be studied in detail. These solutions provide precise connections between theories that were previously thought to be entirely disparate. In particular, we are now able to dynamically connect theories in different numbers of spacetime dimensions, with varying degrees of worldsheet and spacetime supersymmetry.

The mechanisms studied in [1,2,3,4] can be classified into two major categories. In one class of transitions, the parent theory is formulated in \( D > D_{\text{crit}} \) supercritical spacetime dimensions. Following condensation of a closed-string tachyon, an expanding bubble of lower-energy vacuum forms in which string states with excitations along a certain number of spatial dimensions are driven out from the bubble region. Through this mechanism, some number \( n \) of spatial dimensions of the parent theory are removed completely from the dynamics in the late-time limit, so that the final theory lives in a total of \( D - n \) dimensions. From the point of view of the worldsheet theory, \( n \) coordinate embedding fields feel a worldsheet potential in the region of tachyon condensate that becomes infinitely sharply peaked at the origin. These fields can be integrated out of the theory, with the consequence that the effective couplings in the final theory are renormalized by loop diagrams. By simple diagrammatic arguments, however, it can be shown that quantum corrections to the connected correlators are zero beyond one-loop order, and quantum renormalizations can be computed exactly. The result is that the dilaton gradient and string-frame metric receive quantum corrections that ultimately compensate for the change in central charge due to the loss of \( n \) spatial dimensions.

In the second class of transitions, which we call c-duality, the parent theory is type 0 superstring theory, where the tachyon (which remains in the spectrum for the type 0 GSO projection) couples to the worldsheet theory as a \((1,1)\) superpotential. In this case, tachyon condensation nucleates a phase bubble wherein worldsheet supersymmetry is broken...
spontaneously, signaled by the presence of a particular $F$ term in the supersymmetry algebra. In the late-time limit, worldsheet supersymmetry is broken completely, and the final phase of the theory is purely bosonic string theory (with an additional current algebra). In fact, the final state of this transition realizes a particular mechanism discovered by Berkovits and Vafa in 1993 \cite{Vafa:1993de}, in which the bosonic string can be embedded in the solution space of the $\mathcal{N} = 1$ superstring.

Taken together, these mechanisms serve to connect a much broader space of string theories to the well-known supersymmetric moduli space (or duality web) of critical superstring theory. The pictures that emerge are semi-infinite lattices of connected theories, living in any number of target-space dimensions and possessing varying amounts of supersymmetry. In this paper we expand this landscape even further to include the supercritical $\mathcal{N} = 2$ string as a parent theory. One interesting aspect of this theory is that, unlike the critical case, the theory exhibits only a single physical timelike direction. As will be shown below, the $R$-symmetry component of the super-Virasoro conditions in the linear dilaton background act to eliminate one of the two timelike directions that are present in the sigma model prior to the application of the constraints. In particular, we demonstrate that supercritical $\mathcal{N} = 2$ string theory exhibits both dimension-quenching and c-duality transitions. As a corollary, we show explicitly how the bosonic string is embedded in the solution space of $\mathcal{N} = 2$ string vacua.

The $\mathcal{N} = 2$ string is an interesting object in its own right. Historically, this theory was discovered by Ramond and Schwarz \cite{Ramond:1971gb} in classifying the set of gauge algebras that admit a Virasoro subalgebra. It comprises a consistent, self-dual string theory \cite{Zwiebach:1985uq, Sen:1985wa}, and is critical in four real dimensions (albeit with $2 + 2$ signature). While a large number of interesting properties of the $\mathcal{N} = 2$ string have been described, the theory as a whole has remained somewhat more removed from direct phenomenological considerations. We hope that by connecting the $\mathcal{N} = 2$ string with a large collection of distinct string theories, new light may be shed on the broader role of this theory in quantum gravity.

In Section 2 we establish conventions and notation by briefly reviewing the parent $\mathcal{N} = 2$ string theory in supercritical dimensions. In Section 3 we study generic aspects of tachyon condensation in the parent theory. Section 4 describes dimension-quenching transitions driven by tachyon condensation. In Section 5 we describe the c-duality transition from $\mathcal{N} = 2$ string theory to purely bosonic string theory. Section 6 provides in detail the explicit
variable redefinition that embeds the bosonic string in the space of vacua of the $\mathcal{N} = 2$ string. Section 7 describes the BRST quantization of this embedding at tree level in the string coupling. The final section contains a summary of our results and a brief discussion of future research directions. A number of useful equations and definitions are recorded in the Appendices.

2 $\mathcal{N} = 2$ string theory in a linear dilaton background

The worldsheet action of $\mathcal{N} = 2$ string theory admits the $\mathcal{N} = 2$ superconformal algebra as a gauge symmetry. On the cylinder, the right-moving $\mathcal{N} = 2$ algebra is characterized by the following commutation relations:

$$[L_m, L_n] = (n - m) L_{m+n} + \frac{c}{12} \delta_{m,-n} (m^3 - m),$$

$$\{G_r, G_s\} = \{\bar{G}_r, \bar{G}_s\} = 0,$$

$$\{G_r, \bar{G}_s\} = 2 L_{r+s} + (r - s) J_{r+s} + \frac{c}{12} (4r^2 - 1) \delta_{r,-s},$$

$$[L_m, J_n] = -n J_{m+n}, \quad [J_m, J_n] = \frac{c}{3} m \delta_{m,-n},$$

$$[L_m, G_r] = (\frac{1}{2} m - r) G_{n+r}, \quad [L_m, \bar{G}_r] = (\frac{1}{2} m - r) \bar{G}_{m+r},$$

$$[J_m, G_r] = G_{m+r}, \quad [J_m, \bar{G}_r] = -\bar{G}_{m+r}. \quad (2.1)$$

$G_m$ and $\bar{G}_m$ are modes of the complex $\mathcal{N} = 2$ supercurrent, while $J_m$ and $L_m$ are modes of the R-current and Virasoro generators, respectively. The simplest backgrounds of the $\mathcal{N} = 2$ string have flat string-frame metric and constant dilaton gradient. Relative to the linear dilaton backgrounds of bosonic, type 0, type II or heterotic string theory, the linear dilaton background of $\mathcal{N} = 2$ string theory has reduced spatial symmetry, due to the transformation properties of the spacetime embedding coordinates under worldsheet supersymmetry. It is natural to employ complex bosonic embedding coordinates $\phi^\mu$, with $\mu$ running from 0 to $D_c - 1 \equiv \frac{1}{2} D - 1$ ($D_c$ denotes the number of complex dimensions, or half the number real
dimensions $D$). Along with their left- and right-moving fermionic superpartners $\tilde{\psi}^\mu, \psi^\mu$, the bosons transform under worldsheet supersymmetry according to the algebra presented in Appendix A.

**Signatures**

The $\phi^\mu$ coordinates comprise the lowest components of $(2, 2)$ chiral multiplets. As such, they have an intrinsic complex structure defined by their supersymmetry transformations. Worldsheet supersymmetry restricts the sigma model metric to be Kähler with respect to this complex structure. Among other things, this means that the real and imaginary parts of a single complex dimension must have the same signature. Since we are formulating our theory in Lorentzian spacetime, we demand that the direction

$$X^0 \equiv \text{Re} \phi^0$$

be timelike. This would seem to indicate that the presence of a second timelike dimension

$$Y^0 \equiv \text{Im} \phi^0$$

is inevitable (along with the exotic phenomena that typically accompany theories with multiple timelike directions). In a background with constant dilaton, this is indeed the case: the critical $\mathcal{N} = 2$ string with vanishing dilaton gradient exhibits $2+2$ signature. We will show, however, that the presence of a timelike or lightlike dilaton gradient renders the second time direction pure gauge. In such a background, the second timelike direction does not constitute an independent degree of freedom in the physical Hilbert space.

**Symmetries**

We define the dilaton dependence on the bosonic directions by

$$\Phi = \frac{1}{2} ( W^\mu \phi^\mu + W^*_\mu \bar{\phi}^\mu ) .$$

(2.4)

The complex dilaton gradient $W^\mu = 2\partial_{\phi^\mu} \Phi$ contributes to the central charge as

$$c_{\text{dilaton}} = 3\alpha' W^\mu W^*_\mu .$$

(2.5)

To cancel the worldsheet Weyl anomaly, the dilaton contribution to the central charge must satisfy $c_{\text{dilaton}} = 6 - \frac{3D}{2} = 6 - 3D_c$, so we obtain the following condition:

$$W^\mu W^*_\mu = \frac{2}{\alpha'} \left( 1 - \frac{D}{4} \right) = \frac{2}{\alpha'} \left( 1 - \frac{D_c}{2} \right) .$$

(2.6)
For $D > 4$, this means $W_\mu$ must be timelike. We can use a $U(D_c - 1, 1)$ transformation to set $W_i = \text{Im} W_0 = 0$ for $i = 1, \cdots, D_c - 1$. Furthermore, without loss of generality, we are free to choose $W_0 < 0$. This fixes the following conditions:

$$W_0 = W_0^* = -s,$$

with

$$s = \sqrt{\frac{D - 4}{2\alpha'}} = \sqrt{\frac{D_c - 2}{\alpha'}},$$

so that the dilaton itself satisfies $\Phi = -s \text{ Re } \phi^0 = -s X^0$.

Ignoring the dilaton, the spatial symmetry comprises a semidirect product of the translation group with the rotation group $U(D_c)$. This group is broken by the dilaton to a semidirect product of residual translations, with the residual $U(D_c - 1)$-dimensional rotational group acting on the spatial directions. The $(2D - 1)$-dimensional group of residual translations includes $D - 2$ real translations in the spatial directions, as well as a translation in the $Y_0 = \text{Im } \phi^0$ direction. The latter will turn out to be unphysical, so the residual symmetry group will be generated by $2D - 2$ real translations, and $U(D_c - 1)$ rotations of the spatial directions.

### 2.1 States on the cylinder: NS states in standard picture

Let us now discuss the physical states and operators of the $\mathcal{N} = 2$ string in this background. We will begin by discussing the physical state conditions at the level of old covariant quantization (OCQ), omitting for the moment any reference to Fadeev-Popov ghosts or BRST quantization. One important set of physical states of the $\mathcal{N} = 2$ string is the set of Neveu-Schwarz (NS) states in standard picture. These correspond to superconformal primaries of weight 0. The requirement is that the nonnegative modes of $G$, $\bar{G}$, $J$ and $L$ must annihilate physical states.

To establish the connection between $W_\mu$ and the string coupling, we identify the $SL(2,\mathbb{C})$-invariant state. This state is particularly easy to determine in a supersymmetric theory, as it is annihilated by both $G_{-\frac{1}{2}}$ and $\bar{G}_{-\frac{1}{2}}$. To this end, and to establish basic conventions, we
define the usual Fourier mode expansions for fields on the cylinder:

\[
\phi_m = -\frac{1}{\pi \sqrt{2\alpha'}} \int d\sigma e^{-im\sigma^+} \partial_+ \phi(\sigma),
\]

\[
\bar{\phi}_m = -\frac{1}{\pi \sqrt{2\alpha'}} \int d\sigma e^{-im\sigma^+} \partial_+ \bar{\phi}(\sigma),
\]

\[(2.9)\]

(note that by \(\bar{\phi}_m\), we mean the \(m\)th mode of the conjugated field \(\phi\), and not the conjugate of the mode \(\phi_m\) itself). Lightcone coordinates are defined with the convention \(\sigma^\pm \equiv \frac{1}{\sqrt{2}} (-\sigma^0 \pm \sigma^1)\). The center-of-mass values for the \(\phi\) fields are defined to be

\[
\phi_{CM} = \frac{1}{2\pi} \int d\sigma \phi, \quad \bar{\phi}_{CM} = \frac{1}{2\pi} \int d\sigma \bar{\phi},
\]

\[(2.10)\]

with conjugate momenta given by

\[
p_\mu = P_{\phi_{CM}} = \frac{1}{\alpha'} \eta_{\mu\nu} \dot{\phi}_{\nu}^{CM}, \quad \bar{p}_\mu = p_{\bar{\phi}_{CM}} = \frac{1}{\alpha'} \eta_{\mu\nu} \dot{\bar{\phi}}_{\nu}^{CM}.
\]

\[(2.11)\]

The canonical commutation relations are

\[
[\dot{\phi}(\sigma), \bar{\phi}(\tau)] = -2\pi i\alpha' \delta(\sigma - \tau),
\]

\[(2.12)\]

which means

\[
[\partial_+ \phi(\sigma), \partial_+ \bar{\phi}(\tau)] = -\pi i\alpha' \delta'(\sigma - \tau).
\]

\[(2.13)\]

With these conventions, we recover the standard commutation relation for the modes \(\phi_m\) and \(\bar{\phi}_m\):

\[
[\phi_m, \bar{\phi}_n] = m \delta_{m,-n}.
\]

\[(2.14)\]

The Fourier modes of \(\psi\) and \(\bar{\psi}\) on the cylinder can be similarly defined, with the corresponding anticommutation relation

\[
\{\psi_r, \bar{\psi}_s\} = \delta_{r,-s}.
\]

\[(2.15)\]
We can therefore write a consistent mode expansion of the generators of the \( \mathcal{N} = 2 \) algebra:

\[
L_m = \sum_n \phi_n \bar{\phi}_{m-n} + \frac{1}{2} \sum_r (m - 2r) \psi_r \bar{\psi}_{m-r} \\
+ \frac{i\sqrt{2} \alpha'}{4} m \left( W_\mu \phi_\mu + W^*_\mu \bar{\phi}_\mu \right) + \frac{\alpha'}{8} W_\mu W^\mu \delta_{m,0},
\]

\[
G_r = \sqrt{2} \sum_n \phi_n \bar{\psi}_{r-n} + i r \sqrt{\alpha'} W_\mu \psi^\mu_r,
\]

\[
\bar{G}_r = \sqrt{2} \sum_n \bar{\phi}_n \psi_{r-n} + i r \sqrt{\alpha'} W^*_\mu \bar{\psi}^\mu_r,
\]

\[
J_m = \psi_r \bar{\psi}_{m-r} + \frac{i\sqrt{2} \alpha'}{2} \left( W_\mu \phi^\mu_m - W^*_\mu \bar{\phi}^\mu_m \right).
\]

(2.16)

At this point we note that all operators are assumed to be normal-ordered.

Using the basic commutators in Eqns. (2.14) and (2.15), it is straightforward to verify that this set of generators satisfies the \( \mathcal{N} = 2 \) algebra with central charge \( c = c^{\text{free}} + c^{\text{dilaton}} \), where

\[
c^{\text{dilaton}} = 3 \alpha' W^*_\mu W^\mu.
\]

(2.17)

(Of course, we will eventually restrict to the form of \( W_\mu \) in Eqns. (2.7-2.8) above.) The zero-modes \( \phi^\mu_0 \) and \( \bar{\phi}^\mu_0 \) are related to the translators for the overall center-of-mass values in Eqns. (2.10) by

\[
\phi^\mu_0 = \sqrt{\frac{\alpha'}{2}} \eta^{\mu\nu} p_{\phi^\nu}, \quad \bar{\phi}^\mu_0 = \sqrt{\frac{\alpha'}{2}} \eta^{\mu\nu} p_{\bar{\phi}^\nu},
\]

(2.18)

where the momenta \( p_{\phi^\nu} \) and \( p_{\bar{\phi}^\nu} \) are defined according to

\[
p_{\phi^\mu} \equiv -i \frac{\partial}{\partial \phi^\mu_{\text{CM}}}, \quad p_{\bar{\phi}^\mu} \equiv -i \frac{\partial}{\partial \bar{\phi}^\mu_{\text{CM}}}.
\]

(2.19)

The \( G_{-\frac{1}{2}} \) and \( \bar{G}_{-\frac{1}{2}} \) conditions yield the two independent constraints

\[
p_{\phi^\mu} = \frac{i W_\mu}{2} = -i \partial_{\phi^\mu}, \quad p_{\bar{\phi}^\mu} = \frac{i W^*_\mu}{2} = -i \partial_{\bar{\phi}^\mu},
\]

(2.20)

so the identity state has a wavefunction of the functional form \( \exp \left( -\frac{1}{2} W^*_\mu \phi^\mu - \frac{1}{2} W^*_\mu \bar{\phi}^\mu \right) \).
**R-symmetry constraint**

The presence of the dilaton background changes the nature of the R-symmetry constraints. As an example, let us consider the oscillator ground state. In a background with vanishing dilaton gradient, the $J_0$ condition is vacuous when acting on the ground state. In contrast, the $J_0$ condition in a background with timelike linear dilaton imposes the following condition on the ground state:

$$ p_{Y^0} = 0 \quad (2.21) $$

(where $Y^0$ is defined in Eqn. (2.23)). The R-symmetry constraint therefore eliminates one dimension as a degree of freedom in the string wavefunction. For a general state, we obtain the following condition:

$$ p_{Y^0} = \frac{2}{\alpha' S} (N_\psi - N_{\bar{\psi}}) = \frac{4}{\alpha' S} Q_\psi \quad (2.22) $$

where $N_\psi$, for example, counts the number of excited $\psi$ oscillators, and $Q_\psi \equiv (N_\psi - N_{\bar{\psi}})$.

We conclude that $Y^0$ does not constitute an independent degree of freedom in the Hilbert space of the $\mathcal{N} = 2$ string in a linear dilaton background. Although the multiplet structure of the $\mathcal{N} = 2$ supersymmetry forces us from the outset to adopt two timelike directions, the gauge constraints of the $\mathcal{N} = 2$ algebra in this background leave only a single time coordinate.

**On-shell conditions for normalizable states**

Let us now consider the on-shell condition for string states. In the old covariant quantization of the $\mathcal{N} = 2$ string, the $L_0$ physical state condition is just $L_0 = 0$, so the on-shell condition is

$$ \eta^{\mu \nu} p_{\phi^\mu} p_{\bar{\phi}^\nu} + \frac{2}{\alpha'} E^{\text{osc}} + \frac{1}{4} W_\mu W^\mu = 0 \quad (2.23) $$

where $E^{\text{osc}}$ is the total oscillator energy of the state. We will assume throughout this section that our states are normalizable in the spatial directions $\phi^1, \cdots, \phi^{D_c-1}$. The wavefunction of a physical state is therefore of the form

$$ \Psi \propto \exp \left( i \omega X^0 + i \omega' Y^0 + i K_a \phi^a + i K_a^* \bar{\phi}^a \right) \quad (2.24) $$
where \( \omega \) and \( \omega' \) are not assumed to be real. For such a wavefunction, the on-shell condition reads:

\[
-\frac{1}{4} (\omega^2 + \omega'^2) + K^a K^a + \frac{2}{\alpha'} E_{\text{osc}} + \frac{1}{4} W^* W = 0 .
\] (2.25)

If there are \( N_\psi \) excited \( \psi \) oscillators and \( N_{\bar{\psi}} \) excited \( \bar{\psi} \) oscillators, we obtain

\[
\omega' = p_{Y^0} = \frac{2}{\alpha'} (N_\psi - N_{\bar{\psi}}) = \frac{4}{\alpha'} Q_\psi ,
\] (2.26)

where we have substituted the form for \( W^\mu \) determined above in Eqn. (2.7). This implies

\[
\omega^2 = 4K^a K^a + \frac{8E_{\text{osc}}}{\alpha'} - \frac{16Q_\psi^2}{s^2 \alpha'^2} - s^2 ,
\] (2.27)

where the magnitude of \( W^\mu \) is given explicitly in Eqn. (2.8).

If \( \omega^2 \geq 0 \), the state is oscillatory and does not represent an instability. Let us consider the possibility that \( \omega^2 < 0 \), where the state could potentially represent an instability. The physically relevant criterion for stability [1,9] is whether or not the wavefunction of the state can grow more quickly than \( \exp (-\Phi) = \exp (sX^0) \). That is, if \( \omega^2 = -\Gamma^2 \), with \( \Gamma \) real and positive, then the state represents an instability if and only if \( \Gamma > s \). We will now see that, as in the examples studied in [1,9], a physical state that is normalizable in the spatial directions never represents a true instability.

Since we are considering NS states in the standard picture, each \( \psi \) or \( \bar{\psi} \) carries an energy of at least \( E_{\text{osc}} = \frac{1}{2} \), and there are \( D_c - 2 \) species of \( \psi \) fermions transverse to the lightcone. One attains the lowest energy per unit \( Q_\psi \)-charge by filling fermi surfaces of each species to equal height. We conclude that the inequality

\[
E_{\text{osc}} \geq \frac{2Q_\psi^2}{D_c - 2}
\] (2.28)
must hold. This, in turn, implies

\[
\omega^2 \geq 4K^a K^a + \frac{16Q_\psi^2}{\alpha'(D_c - 2)} - \frac{16Q_\psi^2}{s^2 \alpha'^2} - s^2 .
\] (2.29)

The two terms containing \( Q_\psi^2 \) cancel in the above equation, since \( s^2 = \frac{D_c - 2}{\alpha'} \). This leaves the condition

\[
\omega^2 \geq 4K^a K^a - s^2 .
\] (2.30)

\(^{1}\)Light-cone fermions \( \psi^\pm \) and their complex conjugates are removed by the fermionic constraints and null-state equivalences generated by \( G, \bar{G} \).
We have assumed the state is normalizable in the spatial directions, so the quantity $K^a K^{a*}$ is real and positive. We therefore have that

$$\omega^2 \geq -s^2,$$

so an exponentially growing state can grow no faster than $\exp(sX^0)$, precisely saturating the stability bound.

### 2.2 Tachyons and worldsheet superpotentials

We have seen that the second timelike coordinate $Y^0$ is purely a gauge artifact in supercritical $\mathcal{N} = 2$ string theory. Excitations of $Y^0$ never give rise to negative-norm states, a second independent timelike degree of freedom in the wavefunction, or physical instabilities. This stands in contrast to the $\mathcal{N} = 2$ string with vanishing dilaton gradient, in which there is a true second timelike direction that produces a spectrum with energy unbounded from below (despite the absence of negative-norm states).

We will now discuss a second important way in which the $\mathcal{N} = 2$ string in $D > 4$ differs from its critical counterpart. In the supercritical $\mathcal{N} = 2$ string there are additional physical states at non-standard picture that have no counterpart in the critical theory. These states are non-normalizable, and are easy to understand in the language of operators: they correspond to superpotential deformations of the Lagrangian that preserve the full $\mathcal{N} = 2$ superconformal symmetry.

**Operators**

Any Lagrangian perturbation preserving the full $\mathcal{N} = 2$ algebra corresponds to a BRST-invariant operator (when integrated over the worldsheet). All NS states in standard picture correspond to Lagrangian perturbations that come from $\mathcal{N} = 2$ superconformal primaries $\mathcal{O}$ of weight $(0,0)$ and chiral R-charges $(0,0)$, integrated over all four Grassmann coordinates:

$$\mathcal{L} = \int d\theta^+ d\theta^+ \mathcal{O}.$$

In other words, NS states in the standard picture represent full superspace perturbations.

In the presence of a linear dilaton background, it is easy to see that there are also half-superspace perturbations preserving the full $\mathcal{N} = 2$ superconformal algebra. Consider a matter operator $\mathcal{O}$ of weight $(h, \tilde{h}) = (\frac{1}{2}, \frac{1}{2})$ and chiral R-charges $(r, \tilde{r}) = (-1, -1)$ that
is annihilated by $\bar{G}_{-\frac{1}{2}}$ and $\tilde{G}_{-\frac{1}{2}}$, in addition to being primary under the full $\mathcal{N} = (2, 2)$ superconformal algebra. We would then expect the perturbation

$$\Delta L_{\text{chiral}} = \int d\theta_+ d\theta_- \mathcal{O} \quad (2.33)$$

to correspond to an allowed physical state, though not necessarily one that is normalizable. It is useful to understand what such a state looks like in integrated and fixed picture, verifying its BRST invariance explicitly. This will be the essential goal in the remainder of this section.

The BRST current can be conveniently defined by collecting expressions for the superconformal generators of the $\mathcal{N} = 2$ ghost sector. The ghost sector of the $\mathcal{N} = 2$ string in conformal gauge consists of the usual reparametrization $bc$ ghosts, a complex $\beta\gamma$ system, and a ghost system $\hat{b}\hat{c}$ of weights $(1, 0)$, corresponding to the Fadeev-Popov ghosts of the $R$-symmetry. The superconformal generators for these sectors are:

$$T_{\text{ghost}} = 2ib'c + ib'c + i\hat{b}\hat{c}' + \frac{3}{2}(\bar{\beta}\gamma' + \beta\bar{\gamma}') + \frac{1}{2}(\bar{\beta}'\gamma + \beta'\bar{\gamma}) ,$$

$$J_{\text{ghost}} = i\bar{\beta}\gamma - i\beta\bar{\gamma} - 2ic'\hat{b} - 2ic\hat{b}' ,$$

$$G_{\text{ghost}} = \sqrt{2}\left(2\hat{b}\gamma' - ib\gamma + ic\beta' + \hat{b}'\bar{\gamma}\right) + \frac{1}{\sqrt{2}}(3ic'\beta - \hat{c}\beta) ,$$

$$\tilde{G}_{\text{ghost}} = \sqrt{2}\left(ib\bar{\gamma} + 2\hat{b}\bar{\gamma}' - ic\bar{\beta}' + \hat{b}'\bar{\gamma}\right) - \frac{1}{\sqrt{2}}(3ic'\bar{\beta} + \hat{c}\bar{\beta}) . \quad (2.34)$$

Here and below, the prime notation on fields indicates the action of the lightcone derivative $\partial_+$. With these generators in hand, the BRST current can be written as

$$j_{\text{BRST}} = cT + \frac{1}{2}\hat{c}J + \frac{i}{\sqrt{2}}\bar{\gamma}G - \frac{i}{\sqrt{2}}\gamma\tilde{G} , \quad (2.35)$$

where the bold-faced quantities $T$, $J$, $G$ are defined to be the usual generators in the physical sector plus one half the corresponding quantity in the ghost sector:

$$T = T + \frac{1}{2}T_{\text{ghost}} ,$$

$$J = J + \frac{1}{2}J_{\text{ghost}} ,$$

$$G = G + \frac{1}{2}G_{\text{ghost}} . \quad (2.36)$$
The left-moving current $\tilde{j}_{\text{BRST}}$ can be defined similarly.

**Vertex operators for short multiplets**

The familiar physical states of the $\mathcal{N} = 2$ string are generic, non-BPS (or non-short) chiral primaries. These operators are not annihilated by $G_{-\frac{1}{2}}$ or $\bar{G}_{-\frac{1}{2}}$. (For brevity, we will generally suppress all left-moving fields and refer only to the right-moving ghost and matter structure; exceptions will be noted explicitly.) For non-BPS $\mathcal{N} = 2$ superconformal primary matter operators of weight 0 and R-charge 0, the correct ghost dressing is $\delta(\gamma)\delta(\bar{\gamma})c$. (I.e., the dressed operators are BRST invariant.) Tachyonic excitations do not correspond to generic physical states, but rather to BPS primaries, denoted here by $\mathcal{O}$. We now wish to find the appropriate ghost dressing for $\mathcal{O}$.

Since the operators of interest lie in the NS sector, it will be convenient to avoid bosonizing the superghosts and work directly with the $\beta\gamma$ and $\bar{\beta}\bar{\gamma}$ systems. In terms of a generic function $f$ of $\bar{\gamma}$, we have the following commutation relation involving the BRST charge $Q_{\text{BRST}}$:

$$\left[Q_{\text{BRST}}, cf(\bar{\gamma})\right] = \frac{i}{2} cb\bar{\gamma} f'(\bar{\gamma}) + i\bar{\gamma}\gamma f(\bar{\gamma}) + cc' \left(f(\bar{\gamma}) + \frac{1}{2}\bar{\gamma}f'(\bar{\gamma})\right).$$  \hspace{1cm} (2.37)

Combining the above with a general matter chiral primary $\mathcal{O}$ of weight $h$ and R-charge $r$, we obtain

$$\left[Q_{\text{BRST}}, cf(\bar{\gamma})\mathcal{O}\right] = cc' \left(\frac{1}{2}\bar{\gamma}f'(\bar{\gamma}) + (1-h)f(\bar{\gamma})\right)\mathcal{O} + i\bar{\gamma}\gamma f(\bar{\gamma})\mathcal{O}$$

$$-\frac{i}{\sqrt{2}}c\bar{\gamma}f(\bar{\gamma})\Delta \mathcal{L} + \frac{i}{2} cb(\bar{\gamma}f'(\bar{\gamma}) - rf(\bar{\gamma})))\mathcal{O},$$  \hspace{1cm} (2.38)

where

$$\Delta \mathcal{L} \equiv G_{-\frac{1}{2}}\mathcal{O}.$$  \hspace{1cm} (2.39)

Setting the weight $h = \frac{1}{2}$ and R-charge $r = -1$, we see that the ghost dressing with $f(\bar{\gamma}) = \delta(\bar{\gamma})$ renders the operator $\mathcal{O}$ BRST-invariant. Likewise, anti-BPS primaries annihilated by $G_{-\frac{1}{2}}$ with weight $h = \frac{1}{2}$ and R-charge $r = 1$ must be dressed with $c\delta(\gamma)$.

**Existence of short physical operators**

In the supercritical $\mathcal{N} = 2$ string, short operators exist that satisfy the physical state conditions. The simplest is $\mathcal{O}$, which we take to be of the form $w(\phi)$. An arbitrary holomorphic

\[\text{Here we have used the identities } x \delta(x) = 0 \text{ and } x \delta'(x) = -\delta(x).\]
function $w$ will be annihilated by $\bar{G}_{-\frac{1}{2}}$, and will be automatically primary with respect to the $\mathcal{N} = 2$ algebra. The sole conditions remaining to be satisfied are that $h = \frac{1}{2}$ and $r = -1$. A holomorphic function of scalars has no singularities with itself, so its anomalous and canonical dimensions both vanish. The only contribution to the weight is therefore due to the linear dilaton:

$$L_0 w(\phi) = \frac{\alpha'}{2} W^\mu \partial_\mu w(\phi) = \frac{\alpha' s}{4} \partial_{\phi^0} w(\phi) .$$

The R-charge is also determined by the dilaton term:

$$J_0 w(\phi) = r w(\phi) = -w(\phi) = -\frac{\alpha' s}{2} \partial_{\phi^0} w(\phi) .$$

These conditions are saturated by the form

$$w(\phi) = \exp \left( B \phi^0 / \sqrt{2} \right) \mathbf{W}(\phi^0) ,$$

where $a = 1, \cdots, D_c - 1$, and

$$B = \frac{2\sqrt{2}}{s\alpha'} .$$

Note that $B$ is positive: the holomorphic tachyon can only increase exponentially in the direction of weak string coupling. This condition is automatically enforced, and differs from the cases studied in [1, 2, 3, 4], where the condition was chosen to render the background solvable.

**States of short multiplets**

States corresponding to the BPS primaries we have specified above are in the Fock vacuum of the $\beta$, $\bar{\gamma}$ system, but in the identity sector of the conjugate system. In other words, the states are annihilated by all positive modes of the $\bar{\beta}_r$, $\gamma_r$ oscillators, except for $\gamma_{\frac{1}{2}}$. They are, however, annihilated by the raising operator $\bar{\beta}_{-\frac{1}{2}}$. The properties of the conjugate states are the same, with the roles of $\gamma$, $\bar{\beta}$ interchanged with the roles of $\bar{\gamma}$, $\beta$. There is a shift in the $\phi^0$ momentum of the state relative to that of the corresponding operator, by virtue of the Liouville term. Explicitly, the wavefunction takes the functional form

$$| \exp \left( -W^*_{\mu} \bar{\phi}^\mu / 2 \right) f(\phi) \rangle = | \exp \left( s \bar{\phi}^0 / 2 \right) f(\phi) \rangle$$

for a short operator, and

$$| \exp \left( -W_{\mu} \phi^\mu / 2 \right) f(\bar{\phi}) \rangle = | \exp \left( s \phi^0 / 2 \right) f(\bar{\phi}) \rangle$$

for a long operator.
for an antishort operator, where we have taken an arbitrary holomorphic function \(f(\phi)\). The on-shell condition then requires that \(f(\phi)\) be of the form
\[
f(\phi) = \exp \left[ \left( \frac{s}{2} + \frac{2}{\alpha'/s} \right) \phi^0 \right] W(\phi^0) .
\] (2.46)

**GSO projections and holomorphic tachyons**

The subject of consistent GSO projections for the \(\mathcal{N} = 2\) string is a rich and interesting one. Even in the critical dimension \(D = 2D_c = 4\) there are a large number of options from which one can choose. For a more detailed exposition on this subject, the reader is referred to [10], for example. In dimensions \(D > D_c\), the set of allowed GSO projections is undoubtedly even more intricate. Since we are interested in studying tachyon condensation, we will adopt the diagonal GSO projection \((-1)^F = 1\), which is the simplest GSO projection that leaves all tachyons present in the space of physical states. This projection acts with a \((-1)\) on all left- and right-moving fermions simultaneously. For modular invariance to hold, one needs to include in the spectrum a single Ramond-Ramond sector, in which all worldsheet fermions are periodic. With this inclusion, the diagonal GSO projection is always modular invariant for any 2D theory of free fermions.

### 3 Condensation of holomorphic tachyons

We aim to study the physics of holomorphic tachyon condensation in supercritical \(\mathcal{N} = 2\) string theory. As noted, we will adopt the diagonal GSO projection, leaving the holomorphic tachyons as allowed deformations. The off-shell supersymmetry transformations of the \(D_c\) chiral multiplets are given in Appendix A. From these rules, we can construct a supersymmetric kinetic action. Modulo total derivatives, we have
\[
L_{\text{kin}} = \frac{1}{\pi \alpha'} \bar{\phi}^\mu \phi_\mu
\] (3.1)
The OPE of the fundamental $\phi$ fields reads:

$$\phi^\mu(\sigma)\bar{\phi}^\nu(\tau) \sim -\frac{\alpha'}{2} \log |(\sigma^+ - \tau^+)(\sigma^- - \tau^-)| \eta^{\mu\nu},$$

where, as usual, $\sim$ indicates equivalence up to nonsingular terms. The $\psi$ fields admit the OPE

$$\psi(\sigma)^\mu \bar{\psi}(\tau)^\nu \sim \bar{\psi}(\sigma)^\mu \psi(\tau)^\nu \sim \frac{i}{\sigma^+ - \tau^+} \eta^{\mu\nu}.$$  

(3.3)

Using the above equations, it is straightforward to verify that the local superconformal currents in this theory satisfy the OPEs recorded in Appendix B.

To make contact with the notation employed in [1, 2, 3, 4], we introduce the usual dilaton gradient $V_\mu$, which is given in terms of $W_\mu$ by

$$V_\mu = \frac{W_\mu}{\sqrt{2}}.$$  

(3.4)

This theory admits the following stress tensor and complex supercurrent:

$$T = -\frac{2}{\alpha'} \partial_+ \phi^\mu \partial_+ \bar{\phi}_\mu + \frac{i}{2} \left( \psi^\mu \partial_+ \bar{\psi}_\mu + \bar{\psi}^\mu \partial_+ \psi_\mu \right) + \frac{1}{\sqrt{2}} \left( V_\mu \partial_+^2 \phi^\mu + V_\mu^* \partial_-^2 \bar{\phi}^\mu \right),$$

$$G = \frac{2}{\sqrt{\alpha'}} \psi^\mu \partial_+ \bar{\psi}_\mu - \sqrt{2\alpha'}V_\mu \partial_+ \psi^\mu.$$  

(3.5)

We have chosen a normalization of $V_\mu$ in the stress tensor such that the linear dilaton contribution to the central charge takes the form determined above (see, e.g., Eqn. (2.17)):

$$c^{\text{dilaton}} = 6\alpha' V_\mu^* V_\mu = 3\alpha' W_\mu^* W_\mu.$$  

(3.6)

There is also an R-current defined by

$$J = \psi^\mu \bar{\psi}_\mu - i\sqrt{2}V_\mu \partial_+ \phi^\mu + i\sqrt{2}V_\mu^* \partial_- \bar{\phi}^\mu.$$  

(3.7)

At this point, we restrict $V_\mu$ (and hence $W_\mu$) to lie entirely in the real timelike direction:

$$V_+ = V_- = V_+^* = V_-^* = -\frac{q}{\sqrt{2}},$$

(3.8)

with $q$ taken to be real and greater than zero. Here, $q$ is related to the quantity $s$ defined above (Eqn. (2.8)) by

$$q = \frac{s}{\sqrt{2}}.$$  

(3.9)

\[\text{In addition to } V_\mu, \text{ we have introduced } q \text{ to make contact with the notation in [1, 2, 3, 4].}\]
This sets the dilaton decreasing toward the future. For later convenience, we also note the component values $V_0 = V_0^* = -q$.

The critical central charge for the $\mathcal{N} = 2$ string is $c_{\text{total}} = 6$. In $D_c$ complex dimensions there is a contribution of $\Delta c = 2D_c$ from the scalars and $\Delta c = D_c$ from their fermionic superpartners. With the assignment in Eqn. (3.3), the total dilaton contribution to the central charge is $c_{\text{dilaton}} = -6\alpha'q^2$. The magnitude of the dilaton gradient must satisfy $2\alpha'V^2 = -(D_c - 2)$, so we set

$$q = \sqrt{\frac{D_c - 2}{2\alpha'}}.$$  \hspace{1cm} (3.10)

In what follows, we wish to consider interaction terms in the kinetic Lagrangian in Eqn. (3.1), obtained by perturbing with a superpotential (plus its Hermitian conjugate). For a general superpotential $w$, we have

$$\mathcal{L}_{\text{int}} = Q\tilde{Q}\left(-\frac{i}{2\pi}w\right) + \text{h.c.}$$

$$= \frac{i\alpha'}{2\pi}(\partial_{\mu}\partial_{\nu}w)\psi_{\mu}\bar{\psi}_{\nu} - \frac{\sqrt{\alpha'}}{2\pi}\partial_{\mu}wF^\mu + \text{h.c.}$$  \hspace{1cm} (3.11)

By analogy with [1,2,3,4], we will choose particular forms of the superpotential to obtain transitions from the $\mathcal{N} = 2$ parent theory in supercritical dimensions to theories with reduced spatial dimensions, reduced supersymmetry, or a combination of the two. We start by describing *dimension-quenching* transitions, where condensation of a holomorphic tachyon removes spatial degrees of freedom from the theory.

### 4 Dimension-reducing transitions

In this section we will describe dynamical dimensional reduction in supercritical $\mathcal{N} = 2$ string theory, driven by closed-string tachyon condensation. In particular, we will describe the condensation of a holomorphic tachyon with exponential dependence on a lightlike direction and quadratic dependence on some number of transverse dimensions. We therefore focus on tachyon perturbations of the general form

$$T = \exp \left(B\phi^+\right) q_{ab} \phi^a \phi^b,$$  \hspace{1cm} (4.1)

where $q_{ab}$ is a quadratic form in the holomorphic coordinates $\phi^a$, $a \in \{2, \cdots, D_c - 1\}$. 

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For instance, to reduce the number of spacetime dimensions by one complex dimension, we can choose \( q_{ab} \phi^a \phi^b \equiv \frac{1}{2} \mu' \phi_2^2 \). The resulting worldsheet superpotential is
\[
w = \mathcal{T} = \frac{\mu}{2\alpha'} \exp \left( B \phi^+ \right) \phi_2^2,
\]
which in components gives the following worldsheet F-term coupling:
\[
\mathcal{L}_{\text{int}} = -\frac{\sqrt{\alpha'}}{2\pi} F^{\mu} \partial_{\phi^\mu} w - \frac{\sqrt{\alpha'}}{2\pi} \bar{F}^{\bar{\mu}} \partial_{\bar{\phi}^{\bar{\mu}}} w^*.
\]
Defining \( M \equiv \mu \exp \left( B \phi^+ \right) \), the bosonic worldsheet potential takes the form
\[
V_{\text{ws}} = \frac{\mu^2}{2\pi\alpha'} \exp \left( B \phi^+ + \bar{\phi}^+ \right) |\phi_2|^2 = \frac{|M|^2}{2\pi\alpha'} |\phi_2|^2,
\]
and the Yukawa coupling is given by
\[
\mathcal{L}_{\text{Yuk}} = \frac{i\mu}{2\pi} \exp \left( B \phi^+ \right) \left( \psi_2 \bar{\psi}_2 + B \phi_2 \bar{\psi}_2 \bar{\psi}_2 + B \phi_2 \bar{\phi}_2 \bar{\psi}_2 \bar{\psi}_2 + \frac{B^2}{2} \phi_2^2 \bar{\psi}_2 \bar{\psi}_2 \bar{\psi}_2 \bar{\psi}_2 \right) + \text{h.c.}
\]
\[
= \frac{iM}{2\pi} \left( \psi_2 \bar{\psi}_2 + B \phi_2 \bar{\psi}_2 \bar{\psi}_2 + B \phi_2 \bar{\phi}_2 \bar{\psi}_2 \bar{\psi}_2 + \frac{B^2}{2} \phi_2^2 \bar{\psi}_2 \bar{\psi}_2 \bar{\psi}_2 \bar{\psi}_2 \right) + \text{h.c.}
\]
The elements of the \( \phi_2 \) multiplet can be integrated out, treating the elements of the \( \phi^+ \) multiplet as fixed source terms. The path integral over the \( \phi_2, \bar{\phi}_2 \) multiplets is purely Gaussian, so it can be computed exactly. The effect of integrating out the \( \phi_2 \) multiplet is to generate effective terms involving fields in the \( \phi^+ \) multiplet (and their conjugates). Note that these effective terms have no dependence on the fields in the \( \phi^-, \bar{\phi}^- \) multiplets. As a result, there are no operator ordering ambiguities in terms that are generated by integrating out the \( \phi_2, \bar{\phi}_2 \) multiplets.

The simplicity of the worldsheet theory can be understood at the level of Feynman diagrams. The \( \phi_2, \bar{\phi}_2 \) multiplets admit unoriented propagators. The \( \phi^\pm, \bar{\phi}^\pm \) multiplets have oriented propagators, however, directed from + to – fields. If we draw the \( \phi_2, \bar{\phi}_2 \) propagators with solid lines, and the \( \phi^\pm, \bar{\phi}^\pm \) propagators with dotted, oriented lines, all connected Feynman diagrams have the structure of a single solid line segment or loop, with outgoing dotted lines attached at a number of vertices. There can be no closed loops involving dotted lines, because the incoming end of the propagator can never join itself to a vertex. As a result, the 2D worldsheet theory is exactly solvable at the quantum level (for further details, see Ref. [2]). Notably, quantum corrections to the classical theory vanish beyond one-loop order.
The computation of effective interactions in the theory with $\phi_2$, $\bar{\phi}_2$ integrated out can be performed exactly. The result is that coefficients of effective scalar couplings of dimension $\Delta$ and R-charge $r$ scale as $\exp\left((2 - \Delta)BX^+ + \frac{1}{2}rBY^+\right)$, in the limit where $X^+ \to \infty$. Here we have defined

$$\phi^+ \equiv X^+ + iY^+ \ . \ (4.6)$$

In the limit of large $X^+$, only marginal and relevant couplings can survive. Since the path integral over $\phi_2$, $\bar{\phi}_2$ and their superpartners is Gaussian, there are no nonperturbative quantum corrections, so all effective operators generated by integrating out the massive degrees of freedom are necessarily D-terms. It follows that no relevant couplings are generated. Only marginal couplings with $\Delta = 2$ are the only terms generated that survive in the limit of late light-cone time, $X^+ \to \infty$.

Employing the techniques described in [11, 2], we calculate the renormalization of the string-frame metric $G_{\mu\nu}$ and dilaton gradient generated by integrating out $\phi_2$ and its superpartners. We obtain the following results:

$$\Delta G_{X^+X^+} = \Delta G_{Y^+Y^+} = \frac{B^2\alpha'}{2} ,$$

$$\Delta G_{X^+Y^+} = 0 ,$$

$$\Delta \Phi = \frac{1}{2}BX^+ + \text{const} . \ (4.7)$$

As in [11,2], the effect is to renormalize the dilaton contribution to the central charge by an amount

$$\Delta c_{\text{dilaton}} = 3 , \ (4.8)$$

which precisely compensates the loss of central charge due to integrating out the chiral multiplet containing $\phi_2$, $\psi_2$, $\tilde{\psi}_2$. Note that the renormalization of the worldsheet metric indeed respects $(2,2)$ supersymmetry. In particular, the renormalized kinetic term for the bosons can be written in a form that makes this manifest:

$$\Delta L_{\text{kin}} = \frac{1}{2\pi\alpha'} \Delta G_{X^+X^+}(\partial_+ \phi^+ \partial_- \phi^+ + \partial_- \phi^+ \partial_+ \phi^+) . \ (4.9)$$
This comes from the superspace integral
\[ \Delta L_{\text{kin}} = Q \bar{Q} \tilde{Q} \tilde{Q} \left( -\frac{1}{4\pi \alpha'} \Delta G_{X+X+\phi^+\bar{\phi}^+} \right). \] (4.10)

The process of dimension quenching described in [2,4] is therefore realized in a precisely analogous manner. As described in [2,4], however, reducing to the minimal dimension raises certain subtle issues that reach beyond the scope of the present paper. For example, there is a nonzero null linear dilaton in the late-time limit of our theory: it would be useful to understand the precise relationship between this limit and the corresponding theory with no linear dilaton, which exhibits spacetime signature (2, 2).

5 Transition from \( \mathcal{N} = 2 \) to bosonic string theory

We now turn to transitions (called c-dualities in [4]) that connect \( \mathcal{N} = 2 \) string theory dynamically with bosonic string theory. Similar to the transitions starting from \( \mathcal{N} = 1 \) parent theories [3], the basic ingredient is the condensation of a lightlike tachyon. In the system at hand, we let the holomorphic tachyon profile depend on a lightlike combination of \( \phi^0 \) and \( \phi^1 \). The result is that the worldsheet potential for the bosonic fields vanishes identically, as do loop and multivertex tree diagrams. As in the transitions described in [1,2,3,4], time evolution in the target space drives a renormalization group flow on the worldsheet, dressed with an exponential of the lightlike tachyon profile. This renders the perturbation as a whole strictly scale invariant: the deformed theory is exactly conformal, both perturbatively and nonperturbatively in \( \alpha' \).

Specializing to the superpotential
\[ w \equiv \tilde{\mu} \exp (B \phi^+) \] (5.1)
gives the interaction Lagrangian
\[ L_{\text{int}} = -\frac{i \mu}{2\pi} \psi^+ \bar{\psi}^+ \exp (B \phi^+) - \frac{i \bar{\mu}}{2\pi} \bar{\psi}^+ \tilde{\psi}^+ \exp (\tilde{B} \bar{\phi}^+) \]
\[ + \frac{\mu}{2\pi B \sqrt{\alpha'}} F^+ \exp (B \phi^+) + \frac{\bar{\mu}}{2\pi \tilde{B} \sqrt{\alpha'}} \tilde{F}^+ \exp (\tilde{B} \bar{\phi}^+) , \] (5.2)
where we have defined
\[ \mu \equiv B^2 \alpha' \tilde{\mu} , \quad \bar{\mu} \equiv \tilde{B}^2 \alpha' \tilde{\mu} . \] (5.3)
By further setting

\[ M \equiv \mu \exp(B\phi^+) \quad \text{and} \quad \bar{M} \equiv \bar{\mu} \exp(\bar{B}\phi^+) \], \quad (5.4)

we can write Eqn. (5.2) as

\[ L_{\text{int}} = -iM^2 \frac{\pi}{2} \psi^+ \bar{\psi}^+ - i\bar{M}^2 \frac{\pi}{2} \bar{\psi}^+ \psi^+ + \frac{M}{2\pi B \sqrt{\alpha'}} F^+ + \frac{\bar{M}}{2\pi B \sqrt{\alpha'}} \bar{F}^+ \]. \quad (5.5)

Solving the equations of motion for the auxiliary fields, we find

\[ F^+ = \bar{F}^+ = 0 \quad \text{and} \quad F^- = \frac{M}{B \sqrt{\alpha'}} , \quad \bar{F}^- = \frac{\bar{M}}{B \sqrt{\alpha'}} . \quad (5.6)\]

The worldsheet potential therefore vanishes, and we obtain the following Lagrangian, split into lightcone and transverse contributions:

\[ \mathcal{L} = \mathcal{L}^\perp + \frac{i}{\pi} \left[ \bar{\psi}^+ \partial_+ \bar{\phi}^- + \bar{\psi}^- \partial_+ \bar{\psi}^+ + \psi^+ \partial_- \bar{\psi}^- + \psi^- \partial_- \bar{\psi}^+ - \frac{M}{2} \psi^+ \bar{\psi}^+ - \frac{\bar{M}}{2} \bar{\psi}^+ \bar{\psi}^+ \right] \]

\[ -\frac{1}{\pi \alpha'} \left[ (\partial_+ \phi^+)(\partial_- \bar{\phi}^-) + (\partial_+ \phi^-)(\partial_- \bar{\phi}^+) + (\partial_+ \bar{\phi}^+)(\partial_- \phi^-) + (\partial_+ \bar{\phi}^-)(\partial_- \phi^+) \right] . \quad (5.7)\]

The transverse Lagrangian \( \mathcal{L}^\perp \) takes the form

\[ \mathcal{L}^\perp \equiv \frac{1}{\pi \alpha'} (\partial_+ \phi^a)(\partial_- \bar{\phi}^a) + \frac{1}{\pi \alpha'} (\partial_+ \bar{\phi}^a)(\partial_- \phi^a) \], \quad (5.8)\]

with \( a \) running from 2 to \( D_c - 1 \).

The marginality condition for the tachyon is obtained by demanding that \( \int d\theta_+ d\theta_- \exp(B\phi^+) \) is weight \((1, 1)\). This translates to the requirement that \( Bq = 2/\alpha' \), and we will henceforth assume this relationship. (In particular, note that \( B \), along with \( q \), is real and positive.) At this stage, the equations of motion appear as

\[ \partial_- \psi^- = \frac{i}{2} \bar{M} \bar{\psi}^+ , \quad \partial_+ \bar{\psi}^- = -\frac{i}{2} \bar{M} \bar{\psi}^+ , \]

\[ \partial_- \bar{\psi}^- = \frac{i}{2} M \psi^+ , \quad \partial_+ \bar{\psi}^+ = -\frac{i}{2} M \psi^+ , \]

\[ \partial_+ \partial_- \phi^- = \frac{i\alpha'}{4} \bar{M} \bar{\psi}^+ \bar{\psi}^+ , \quad \partial_+ \partial_- \bar{\phi}^- = \frac{i\alpha'}{4} M \psi^+ \bar{\psi}^- , \]

\[ \partial_- \psi^+ = \partial_- \bar{\psi}^+ = \partial_+ \psi^- = \partial_+ \bar{\psi}^- = 0 \quad \text{and} \quad \partial_- \bar{\phi}^+ = \partial_+ \bar{\phi}^- = 0 . \]
The stress tensor can also be decomposed into contributions from the lightcone and transverse sectors of the theory:

\[ T \equiv T^{\text{LC}} + T^\perp, \]  

(5.10)

where

\[ T^{\text{LC}} \equiv T^{\phi^\pm} + T^{\psi^\pm}. \]  

(5.11)

Explicitly, we obtain following:

\[ T^{\phi^\pm} \equiv \frac{2}{\alpha'} \left( \partial_+ \phi^+ \partial_+ \bar{\phi}^- + \partial_+ \phi^- \partial_+ \bar{\phi}^+ \right) \]

\[ -\frac{q}{2} \left( \partial_+^2 \phi^+ + \partial_+^2 \phi^- + \partial_+^2 \bar{\phi}^+ + \partial_+^2 \bar{\phi}^- \right), \]

\[ T^{\psi^\pm} \equiv -\frac{i}{2} \left( \psi^+ \partial_+ \bar{\psi}^- + \psi^- \partial_+ \bar{\psi}^+ + \bar{\psi}^+ \partial_+ \psi^- + \bar{\psi}^- \partial_+ \psi^+ \right), \]

\[ T^\perp \equiv -\frac{2}{\alpha'} \partial_+ \phi^i \partial_+ \bar{\phi}^i + \frac{i}{2} \left( \psi^i \partial_+ \bar{\psi}^j + \bar{\psi}^i \partial_+ \psi^j \right). \]

(5.12)

The stress tensor \( T^{\text{LC}} \) is conserved in the presence of the interaction, even if \( T^{\phi^\pm} \) and \( T^{\psi^\pm} \) are not separately conserved.

As with the c-duality transitions starting from type 0 string theory, the perturbing superpotential induces worldsheet interaction terms that become infinitely strong in the distant future. Working by analogy from [3], we perform a canonical variable redefinition to render
these interaction terms weakly coupled in the distant future:

\[
\psi^- \equiv \tilde{M} \tilde{c}_6 , \quad \tilde{\psi}^- \equiv -\bar{M} \bar{c}_6 , \\
\bar{\psi}^- \equiv M \tilde{c}_6 , \quad \bar{\tilde{\psi}}^- \equiv -Mc_6 ,
\]

\[
\psi^+ \equiv 2c'_6 - \frac{1}{M} \bar{b}_6 + 2B(\partial_+ \phi^+)c_6 , \quad \tilde{\psi}^+ \equiv 2\tilde{c}'_6 + \frac{1}{M} \bar{b}_6 + 2B(\partial_- \phi^+)\bar{c}_6 ,
\]

\[
\bar{\psi}^+ \equiv 2\bar{c}'_6 - \frac{1}{M} b_6 + 2\bar{B}(\partial_+ \bar{\phi}^+)\bar{c}_6 , \quad \bar{\tilde{\psi}}^+ \equiv 2\bar{\tilde{c}}'_6 + \frac{1}{M} b_6 + 2\bar{B}(\partial_- \bar{\phi}^+)\tilde{c}_6 ,
\]

\[
\phi^- \equiv \chi^- - iB\alpha'M\bar{c}_6 \bar{c}_6 , \quad \bar{\phi}^- = \bar{\chi}^- - iB\alpha'M\bar{c}_6 c_6 ,
\]

\[
\phi^+ = \chi^+ , \quad \bar{\phi}^+ = \bar{\chi}^+ .
\]

We have traded the light-cone fermions \(\psi^\pm, \psi^{\pm\dagger}\) for a complex \(bc\) ghost system labeled by \(b_6, c_6\) and \(\bar{b}_6, \bar{c}_6\) (and their left-moving counterparts, which remain suppressed). For lack of better terminology, we will refer to these objects as antighosts (though not to be confused with antighosts in the sense of BV quantization). We will also count the imaginary parts \(u^\pm\) of the lightcone bosons as antighosts, for reasons that will become clear.

To preserve Lorentz invariance, these antighost fields have spins that are shifted from the more familiar Fadeev-Popov ghosts associated with worldsheet reparametrization symmetry. For reference, we record in Tab. 1 the weights and R-charges of the various objects we have introduced thus far, including the shifted antighost system. Furthermore, in accordance with Ref. [3], we refer to the new set of variables as infrared (IR) variables (while the original variables are denoted as ultraviolet, or UV). In terms of IR variables, the lightcone action takes the form

\[
\mathcal{L}_{LC} = \frac{1}{\pi\alpha'} \left( (\partial_+ \chi^+)(\partial_- \bar{\chi}^-) - (\partial_+ \chi^-)(\partial_- \bar{\chi}^+) - (\partial_+ \bar{\chi}^+)(\partial_- \chi^-) - (\partial_+ \bar{\chi}^-)(\partial_- \chi^+) \right)
\]

\[
+ \frac{i}{\pi} \left( -\bar{b}_6 \partial_- c_6 - b_6 \partial_- \bar{c}_6 - \bar{b}_6 \partial_+ \bar{c}_6 - \bar{\bar{b}}_6 \partial_+ c_6 + \frac{1}{2M} \bar{b}_6 b_6 + \frac{1}{2M} \bar{\bar{b}}_6 \bar{b}_6 \right).
\]

As intended, the IR theory becomes free in the \(|M| \to \infty\) limit.
| object | $(\hat{h}, h)$ | $(\bar{r}, r)$ |
|--------|----------------|----------------|
| $Q$    | $([0], [+\frac{1}{2}])$ | $(0,1)$ |
| $\tilde{Q}$ | $([+\frac{1}{2}], [0])$ | $(1,0)$ |
| $\bar{Q}$ | $([0], [+\frac{1}{2}])$ | $(0,-1)$ |
| $\tilde{\bar{Q}}$ | $([+\frac{1}{2}], [0])$ | $(-1,0)$ |
| $W$    | $(\frac{1}{2}, \frac{1}{2})$ | $(-1,-1)$ |
| $\bar{W}$ | $(\frac{1}{2}, \frac{1}{2})$ | $(1,1)$ |
| $\phi, \tilde{\phi}$ | $(0,0)$ | $(0,0)$ |
| $\psi$ | $(0, \frac{1}{2})$ | $(0,1)$ |
| $\tilde{\psi}$ | $(0, \frac{1}{2})$ | $(0,-1)$ |
| $\bar{\psi}$ | $(\frac{1}{2}, 0)$ | $(1,0)$ |
| $\tilde{\bar{\psi}}$ | $(\frac{1}{2}, 0)$ | $(-1,0)$ |
| $\mu \exp (B\phi^+)$ | $ (+\frac{1}{2} Bq\alpha' , +\frac{1}{4} Bq\alpha' )$ | $ (-\frac{1}{2} Bq\alpha' , -\frac{1}{2} Bq\alpha' )$ |
| $b_6$  | $(0, \frac{3}{2})$ | $(0,1)$ |
| $\tilde{b}_6$ | $(\frac{3}{2}, 0)$ | $(1,0)$ |
| $c_6$  | $(0, -\frac{1}{2})$ | $(0,1)$ |
| $\tilde{c}_6$ | $(-\frac{1}{2}, 0)$ | $(1,0)$ |
| $\bar{b}_6$ | $(0, \frac{3}{2})$ | $(0,-1)$ |
| $\tilde{\bar{b}}_6$ | $(\frac{3}{2}, 0)$ | $(-1,0)$ |
| $\bar{c}_6$ | $(0, -\frac{1}{2})$ | $(0,-1)$ |
| $\tilde{\bar{c}}_6$ | $(-\frac{1}{2}, 0)$ | $(-1,0)$ |

Table 1: The weights and R-charges of fundamental fields.
Just as with the case described in [3], we will need to define a normal-ordering prescription that is appropriate for the IR system. In the UV, OPEs of field monomials involve subtractions of terms that are proportional to $|M|$, and hence infinite in the late-time limit. The natural normal-ordering prescription for fields in the IR regime differs from that in the UV, but only by terms that are independent of $M$ or $\bar{M}$. These finite differences amount to quantum corrections to the classical expressions for the superconformal generators in IR variables. In turn, these corrections lead to a finite renormalization of the dilaton gradient. As in [3], the theory can be seen to admit no nontrivial Feynman graphs, and yet quantum corrections arise in moving to the IR description.

Henceforth, operators and expressions in IR variables are assumed to be normal-ordered under the proper IR ordering scheme. In [3], different operator orderings were made explicit by using the usual :: normal-ordering symbols in the UV regime, and introducing the bubble notation $\circ \circ \circ$ for use with IR variables. Since we are keeping normal-ordering implicit, we will not need to revert to this strategy in the present paper. (I.e., we will rely entirely on the results presented in [3], with no need to compute OPEs directly in the IR regime.)

Performing the classical transformations alone, the lightcone stress tensors in the $b_6c_6$-antighost and bosonic sectors of the theory appear as

$$T_{b_6c_6}^{LC} = -\frac{3i}{2} ((\partial_+ \bar{c}_6)b_6 + (\partial_+ c_6)\bar{b}_6) - \frac{i}{2} (\bar{c}_6(\partial_+ b_6) + c_6(\partial_+ \bar{b}_6)) ,$$

$$T_{\chi}^{LC} = -\frac{q}{2} \left( \partial_+^2 \chi^+ + \partial_+^2 \chi^- + \partial_+^2 \chi^+ + \partial_+^2 \chi^- \right) + \frac{2}{\alpha'} \left( \partial_+ \chi^- \partial_+ \chi^+ + \partial_+ \chi^- \partial_+ \chi^+ \right) .$$

(5.15)

To capture quantum corrections arising from the transformation to IR variables, it is useful to move to two real bosonic coordinates $u_6$ and $y_6$ (these coordinates carry a subscript, since they will undergo further redefinitions in the following section):

$$\chi^\pm = \frac{1}{\sqrt{2}} (y_6^\pm + i u_6^\pm) .$$

(5.16)

This breaks the bosonic lightcone stress tensor into two pieces:

$$T_{u}^{LC} = \frac{2}{\alpha'} \partial_+ u_6^- \partial_+ u_6^+ ,$$

$$\left(T_{y}^{LC}\right)_{\text{classical}} = \frac{2}{\alpha'} \partial_+ y_6^- \partial_+ y_6^+ - \frac{q}{\sqrt{2}} \left( \partial_+^2 y_6^- + \partial_+^2 y_6^+ \right) .$$

(5.17)
Quantum corrections to the classical transformations can be computed by referring directly to the calculations in [3] (here we essentially have two real copies of the system in [3]). In the $y$ sector of the bosonic stress tensor, quantum effects contribute the following correction:

$$\Delta T^{	ext{LC}}_y = \frac{2\sqrt{2}}{\alpha'q} \partial^2_+ g^+_6. \quad (5.18)$$

Below we will collect the classical and quantum contributions in the bosonic $y$ sector into the single expression

$$T^{	ext{LC}}_{\text{bose}} \equiv T^{	ext{LC}}_y + \Delta T^{	ext{LC}}_y. \quad (5.19)$$

From the results in [3], it is straightforward to compute the renormalization of the dilaton gradient $\Delta V^\mu$ arising from the shift to IR variables. We find

$$\Delta V_+ = \sqrt{2} B, \quad \Delta V^+_+ = \sqrt{2} \bar{B},$$

$$\Delta V_- = \Delta V^- = \Delta V_i = \Delta V^*_i = 0. \quad (5.20)$$

For the sake of presentation, we compute the transformed lightcone supercurrent in complex bosonic $\chi$ coordinates:

$$G^{	ext{LC}}_6 = \frac{1}{2}q\sqrt{\alpha'} b_6 - \frac{8}{q\alpha' \sqrt{2}} c_6 \partial_+ \chi^+ \partial_+ \bar{\chi}^- + \frac{4i}{q\sqrt{\alpha'}} b_6 c_6 \partial_+ c_6$$

$$+ \frac{4}{\sqrt{\alpha'}} \left( c_6 \partial^2_+ \chi^+ + (\partial_+ c_6) \partial_+ \chi^+ - (\partial_+ c_6) \partial_+ \bar{\chi}^- \right) + 2q\sqrt{\alpha'} \partial^2_+ c_6$$

$$- B\sqrt{\alpha'} \partial^2_+ c_6 - 2B^2 \sqrt{\alpha'} c_6 \partial^2_+ \chi^+. \quad (5.21)$$

The two terms in the last line are quantum corrections. We note that neither $\bar{\chi}^+$ nor $\chi^-$ appears in $G^{	ext{LC}}_6$.

The R-current transforms entirely classically:

$$J_6 = b_6 \bar{c}_6 + c_6 \bar{b}_6 - iq \left( \partial_+ \chi^+ + \partial_+ \chi^- = \partial_+ \bar{\chi}^+ - \partial_+ \bar{\chi}^- \right). \quad (5.22)$$

In particular, the original dilaton gradient $V_\mu$ and its complex conjugate, rather than the renormalized dilaton gradient $V_\mu + \Delta V_\mu$, enter the R-current in the IR variables. This is necessary for consistency, since the complex $bc$ antighosts and their conjugates contribute
to the R-current central term in exactly the same fashion as the complex ψ± fermions. The dilaton contribution to the central term in J must therefore remain unchanged. We conclude that $V_\mu$, rather than $V_\mu + \Delta V_\mu$, is the appropriate quantity appearing in J.

At this stage we have established the existence of a consistent limiting description at late target-space time. The tachyon perturbation has added a nonzero $F$ term (5.6) to the supersymmetry algebra, $F^- = \frac{M}{B\sqrt{\alpha'}}$, and deep in the IR regime worldsheet supersymmetry is spontaneously broken by an infinite amount. We therefore expect that the late-time limit of the theory comprises an embedding of bosonic string theory in the solution space of the $\mathcal{N} = 2$ string. In the following section we will show that this is indeed the case: the deep IR is in fact described by purely bosonic string theory.

6 Embedding of bosonic string theory

Now that we have moved to infrared variables and have accounted for all possible quantum effects resulting from the canonical variable transformation, we would like to understand in detail the late-time limit of the theory. In this section we will motivate and introduce a series of further canonical transformations that will bring the theory into a recognizable form. As noted above, we will use numerical subscripts on the shifted $bc$ antighosts to keep track of sequential variable transformations, such that the final shifted $bc$ variables will be labeled as $b_1, c_1$, etc. We will also find it useful to decompose expressions into lightcone and transverse sectors: $G \equiv G^{\text{LC}} + G^\perp$, $J = J^{\text{LC}} + J^\perp$ and $T = T^{\text{LC}} + T^\perp$, where $T^{\text{LC}} \equiv T^{\chi \pm} + T^{\text{antighost}}$.

We also find it convenient to explicitly separate the bosonic sector of the theory into lightcone and transverse contributions:

$$T_{\text{bose}} \equiv T^{\text{LC}}_{\text{bose}} + T^{\perp}_{\text{bose}}, \quad (6.1)$$

where $T^{\text{LC}}_{\text{bose}}$ has been defined above in Eqn. (5.19), and includes the lightcone contributions from the bosonic $y$ sector, plus the quantum correction computed in Eqn. (5.18). The stress tensor $T_{\text{bose}}$ denotes the bosonic sector of the theory with central charge exactly

$$c_{\text{bose}} = c_{\text{bose}}^{\text{LC}} + c^{\perp} = 26. \quad (6.2)$$

The condition that the total central charge of the $\mathcal{N} = 2$ parent theory takes the critical value $c_{\text{total}} = 6$ sets the value of the transverse component to

$$c^{\perp} = \frac{24}{\alpha' B^2} = 3(D_c - 2) = -c^{\text{dilaton}}. \quad (6.3)$$
For purposes of illustration and for future reference, we will keep $c_{\text{bose}}$, $c_{\text{bose}}^\perp$, and $c^\perp$ explicit until after the final variable redefinition below.

**Rescaling**

To begin, we perform a rescaling of the $bc$ antighost system to simplify subsequent manipulations:

\[
\begin{align*}
b_6 &= \frac{2}{2q\sqrt{\alpha'}} b_5 = B\sqrt{\alpha'} b_5, \\
c_6 &= \frac{2}{2q\sqrt{\alpha'}} c_5 = \frac{1}{2B\sqrt{\alpha'}} c_5, \\
\bar{b}_6 &= \frac{2}{2q\sqrt{\alpha'}} \bar{b}_5 = B\sqrt{\alpha'} \bar{b}_5, \\
\bar{c}_6 &= \frac{2}{2q\sqrt{\alpha'}} \bar{c}_5 = \frac{1}{2B\sqrt{\alpha'}} \bar{c}_5.
\end{align*}
\]

(6.4)

Even though the bosonic directions remain unchanged at this stage, we choose to relabel the subscripts to keep the presentation uniform:

\[
\begin{align*}
u_6 &= u_5, \\
y_6 &= y_5.
\end{align*}
\]

(6.5)

Again, we are assuming real values for $B$ and $q$, satisfying $Bq = \frac{2}{\alpha'}$.

Under this rescaling, the stress tensor $T$ and R-current $J$ do not change. In terms of the real bosonic coordinates $u_5$ and $y_5$, the complex supercurrent takes the form:

\[
\begin{align*}
G_5 &= b_5 + \frac{2\sqrt{2}}{B\alpha'} (i c_5 u_5'' + c_5 y_5'' + i c_5' u_5' + i c_5' u_5' - c_5 y_5' + c_5' y_5') - 2ic_5 c_5 \bar{b}_5 \\
&\quad - \frac{2}{\alpha'} c_5 (iB\alpha' \sqrt{2} u_5'' + u_5' u_5' + y_5' y_5' + iy_5' u_5' - iy_5' u_5' + \frac{B\alpha'}{\sqrt{2}} y_5') \\
&\quad + \left(\frac{4}{B^2\alpha'^2} - 1\right) c_5''.
\end{align*}
\]

(6.6)
Likewise, the R-current and stress tensor are

\[ J_5 = -\frac{2\sqrt{2}}{B\alpha'} (u_5^- + u_5^+) + c_5 \bar{b}_5 - \bar{c}_5 b_5 , \]

\[ T_5 = -\frac{i}{2} (c_5 \bar{b}_5' + \bar{c}_5 b_5') - \frac{3i}{2} (c_5' \bar{b}_5 + \bar{c}_5' b_5) - \frac{\sqrt{2}}{B\alpha'} (y_5^- + y_5^+) \]

\[ + \frac{2}{\alpha'} (u_5^+ u_5^- + y_5^+ y_5^-) + \sqrt{2} B y_5^+ , \]

\[ = T_{\text{bose}}^{\text{LC}} - \frac{i}{2} (c_5 \bar{b}_5' + \bar{c}_5 b_5') - \frac{3i}{2} (c_5' \bar{b}_5 + \bar{c}_5' b_5) + \frac{2}{\alpha'} u_5^+ u_5^- . \quad (6.7) \]

Note that \( T_{\text{bose}}^{\text{LC}} \) now appears explicitly in the stress tensor.

**Reflection symmetry**

Now we wish to perform a canonical transformation to make the bosonic \((y, u)\) theory reflection symmetric about the little group of the renormalized linear dilaton. We therefore introduce the infinitesimal generator

\[ g_5 \equiv \int d\sigma_1 g_5(\sigma) . \quad (6.8) \]

The expression for \( g_5 \) is given by the sum of an antighost-number two piece and antighost-number four piece:

\[ g_5 = g_5^{(2)} + g_5^{(4)} , \quad (6.9) \]

where

\[ g_5^{(2)} = \frac{\sqrt{2}}{B\alpha'} \left( \bar{c}_5 c_5 u_5^- + 3\bar{c}_5 c_5 u_5^+ + \bar{c}_5 c'_5 u_5^- + \bar{c}_5 c'_5 u_5^+ + i\bar{c}_5 c'_5 y_5^- \right) \]

\[-i\bar{c}_5 c'_5 y_5^+ + \bar{c}_5 c_5 u_5^- + \bar{c}_5 c_5 u_5^+ - i\bar{c}_5 c_5 y_5^- + i\bar{c}_5 c_5 y_5^+ \right) \]

\[-\frac{2}{\alpha'} \left( \bar{c}_5 c_5 y_5^- u_5^+ - \bar{c}_5 c_5 y_5^+ u_5^- + \frac{B\alpha'}{\sqrt{2}} \bar{c}_5 c_5 u_5^+ \right) , \]

\[ g_5^{(4)} = \frac{\sqrt{2}}{B\alpha'} \left( \bar{c}_5 c'_5 c_5 y_5^- - \bar{c}_5 c'_5 c_5 y_5^+ \right) . \quad (6.10) \]
We generate finite transformations by taking
\[ U_5 \equiv \exp \left( \frac{i}{2\pi} g_5 \right). \tag{6.11} \]
We thereby obtain the following transformed supercurrent, now written in terms of "4" variables:
\[
G_4 = U_5 G_5 U_5^\dagger = b_4 - c_4 T_{\text{bos}}^{LC} + G_4^{\perp} - i T_{\text{bos}}^{LC} \tilde{c}_4 c_4' c_4 + 2i \tilde{c}_4 c_4'' c_4' c_4 - \frac{7i}{6} \tilde{c}_4 c_4'' c_4 c_4
- 2ic_4 c_4 b_4 - \frac{3i}{2} c_4' c_4'' c_4 - \frac{3i}{2} c_4'' c_4' c_4 - 4c_4' \tilde{c}_4 c_4' c_4 c_4 - \frac{2}{\alpha} (c_4 u_4^{+' u_4^-})
+ ic_4 c_4 u_4^{+' u_4^-} + \frac{1}{B^{1/2} \alpha} (ic_4 c_4'' c_4 + ic_4' c_4'' c_4 - 2ic_4 c_4'' c_4' c_4 + 5ic_4'' c_4' c_4 c_4)
+ 4c_4' \tilde{c}_4 c_4' c_4 + 4c_4'' c_4' c_4 c_4 + \frac{2\sqrt{2}}{B \alpha} (ic_4' u_4^{+ u_4^-} + ic_4' u_4^{+ u_4^-} + \tilde{c}_4 c_4 u_4^{+ u_4} + \tilde{c}_4 c_4' c_4' u_4^{+ u_4} + \tilde{c}_4 c_4 c_4' u_4^{+ u_4})
+ \tilde{c}_4 c_4 c_4 u_4^{+ u_4} + \tilde{c}_4 c_4 c_4 c_4 u_4^{+ u_4}) - c_4''.
\tag{6.12}
\]
Throughout these intermediate stages, we will record successive transformations of the stress tensor and the R-current in Appendix C.

**R-current rotation**

At this stage we want to formulate a transformation that puts the R-current into a universal form in which the worldsheet fermions do not appear (which are contained in \( J^{\perp} \)). We therefore define a second canonical transformation generated by \( g_4 \):
\[
g_4 \equiv \int d\sigma_1 g_4(\sigma), \quad g_4 \equiv \frac{B}{\sqrt{2}} J^{\perp} u^{+}. \tag{6.13}
\]
Forming the finite transformation
\[
U_4 \equiv \exp \left( \frac{i}{2\pi} g_4 \right), \tag{6.14}
\]
we obtain

\[ G_3 = U_4 G_1 U_4^\dagger \]

\[ = b_3 - c_3 T_{\text{bose}}^L e^{-i \frac{\alpha}{\sqrt{2}} u^+} + \frac{-J^L B}{\sqrt{2}} c_3 u_3^{+t} + \frac{i c^L B}{6 \sqrt{2}} c_3 u_3^{+t} - \frac{c^L B^2}{12} c_3 u_3^{+t} u_3^{+t} \]

\[-2i c_3 \bar{c}_3 \bar{b}_3 - \frac{2}{\alpha'} c_3 u_3^{+t} u_3^{+t} + \frac{2i \sqrt{2}}{B \alpha'} (c_3 u_3^{+t} + c_3 u_3^{+t}) + i J_3^{+} c_3 - c_3'' \]

\[ + \frac{4}{B^2 \alpha'} c_3'' - i T_{\text{bose}}^L \bar{c}_3 c_3' c_3 + J_3^{+} \bar{c}_3 c_3' c_3 + 2i \bar{c}_3 c_3'' c_3' - \frac{7i}{6} \bar{c}_3 c_3'' c_3 + J_3^{+} \bar{c}_3 c_3' c_3 \]

\[ - \frac{3i}{2} \bar{c}_3 c_3'' c_3 - \frac{3i}{2} c_3'' \bar{c}_3 c_3 + \frac{i J_3^{+} B}{\sqrt{2}} \bar{c}_3 c_3' c_3 u_3^{+t} + \frac{c^L B}{6 \sqrt{2}} (\bar{c}_3 c_3' c_3 u_3^{+t} + \bar{c}_3 c_3' c_3 u_3^{+t}) \]

\[ - \frac{1}{12} i c^L B^2 \bar{c}_3 c_3' c_3 u_3^{+t} u_3^{+t} - \frac{2i}{\alpha'} \bar{c}_3 c_3' c_3 u_3^{+t} u_3^{+t} + \frac{i}{B^2 \alpha'} (\bar{c}_3 c_3'' c_3 + \bar{c}_3 c_3'' c_3 - 2 \bar{c}_3 c_3'' c_3) \]

\[ + 5 \bar{c}_3 c_3' c_3 + \frac{2 \sqrt{2}}{B \alpha'} (\bar{c}_3 c_3' c_3 u_3^{+t} + \bar{c}_3 c_3' c_3 u_3^{+t} + \bar{c}_3 c_3' c_3 u_3^{+t} + \bar{c}_3 c_3' c_3 u_3^{+t}) \]

\[ + 4 \left( \frac{1}{B^2 \alpha'} - 1 \right) \bar{c}_3 c_3' c_3' c_3 \]  \hspace{1cm} (6.15)

To illustrate the action of this transformation on the R-current explicitly, we record the form of \( J_4 \), prior to the linear redefinition defined by \( g_4 \) above, decomposed by antighost number:

\[ J_4^{(0)} = J_4^+ + c_4 \bar{b}_4 - \bar{c}_4 b_4 - \frac{2 \sqrt{2}}{B \alpha'} (u_4^{+t} + u_4^{+t}) \]

\[ J_4^{(2)} = \frac{1}{8} \left( -42 + c_{\text{bose}} - c^+ \right) \bar{c}_4 c_4'' - 4 \bar{c}_4' c_4 - 2 \bar{c}_4'' c_4 + \frac{1}{B^2 \alpha'} \left( 7 \bar{c}_4 c_4'' + 4 \bar{c}_4'' c_4 + 8 \bar{c}_4 \bar{c}_4' \right) \]

\[ J_4^{(4)} = -\frac{i}{8 \alpha' B^2} \left( 24 + \alpha' B^2 (c_{\text{bose}} - c^+) - 26 \right) \left( 3 \bar{c}_4' \bar{c}_4 c_4 + \bar{c}_4' \bar{c}_4' c_4' \right) \]  \hspace{1cm} (6.16)

As intended, the transformed version of the R-current (under \( U_4 \)) is completely independent
of $J^\perp$:

$$
J_3 = \frac{e^\perp B}{6\sqrt{2}} u_3^\perp + c_3 \bar{b}_3 - \bar{c}_3 b_3 - \frac{21}{4} \bar{c}_3 c''_3 + \frac{c^\perp c''_3}{8} c_3 c''_3 - 4 \bar{c}'_3 c'_3 - 2 \bar{c}''_3 c_3 + \frac{4}{B^2\alpha'} \bar{c}''_3 c_3
$$

$$
+ \frac{7}{B^2\alpha'} \bar{c}''_3 c''_3 + \frac{8}{B^2\alpha'} \bar{c}'_3 c'_3 - \frac{2\sqrt{2}}{B\alpha'} (u_3' + u_3'').
$$

(6.17)

Hermitian universal form

At this stage we see that the lightcone component of the bosonic stress tensor $T^\perp_{\text{bose}}$ appears explicitly in the supercurrent. We now want to move to a universal formulation written strictly in terms of a generic bosonic stress tensor $T_{\text{bose}}$, describing a bosonic theory of critical central charge $c_{\text{bose}} = 26$. We therefore define the generating function $g_3 \equiv \int d\sigma_1 g_3(\sigma)$, with

$$
g_3 = \frac{J^\perp}{2} \partial_+ (\bar{c}_3 c_3) - \frac{1}{12} \left( \tilde{G}^\perp e^{\pm i \sqrt{2} u^+} \bar{c}_3 c'_3 c_3 + G^\perp e^{-i \sqrt{2} u^+} \bar{c}_3 c''_3 c_3 \right) - \bar{c}_3 c'_3 c_3 \bar{b}_3 + \bar{c}'_3 c_3 b_3 c_3
$$

$$
- i \tilde{G}^\perp e^{\pm i \sqrt{2} u^+} \bar{c}_3 - i G^\perp e^{-i \sqrt{2} u^+} \bar{c}_3 + \frac{2\sqrt{2}}{B\alpha'} \left( \bar{c}_3 c'_3 u_3'' + \bar{c}'_3 c_3 u_3' \right).
$$

(6.18)

Acting with the finite unitary transformation $U_3 \equiv \exp \left( \frac{\sqrt{2}}{2\pi} g_3 \right)$ puts the supercurrent into the following form:

$$
G_2 = U_3 G_3 U_3^\dagger
$$

$$
= b_2 - c_2 T_{\text{bose}} - \frac{ic^\perp B}{6\sqrt{2}} \partial_+ \left( c_2 u_2'' \right) + \frac{2i\sqrt{2}}{B\alpha'} c_2 c''_2 c''_2 u_2'' + i\bar{c}_2 b_2 c_2 + i\bar{c}_2 b'_2 c_2 - i\bar{c}''_2 c_2 \bar{b}_2
$$

$$
+ 2i\bar{c}''_2 b_2 c_2 - \frac{2}{\alpha'} c_2 u_2'' u_2' + \frac{2i\sqrt{2}}{B\alpha'} \left( c'_2 u_2'' + c'_2 u_2' \right) + \left( \frac{4}{B^2\alpha'} - \frac{c^\perp}{6} - 1 \right) c''_2
$$

$$
+ \frac{ic^\perp}{12} c'_2 c'_2 c_2 - \frac{2i}{B^2\alpha'} c'_2 c'_2 c'_2 + \frac{ic^\perp}{8} c'_2 c''_2 c_2 + \frac{7ic^\perp}{24} c''_2 c''_2 c_2 + \frac{ic^\perp}{8} c''_2 c'_2 c_2
$$

$$
- \frac{i}{B^2\alpha'} (3\bar{c}_2 c''_2 c_2 + 3\bar{c}''_2 c'_2 c_2 + 7\bar{c}''_2 c''_2 c_2) + \frac{2\sqrt{2}}{B\alpha'} \left( \bar{c}_2 c'_2 c_2 u_2'' + \bar{c}_2 c''_2 c_2 u_2' \right)
$$

$$
+ 2\bar{c}_2 c''_2 c_2 u_2' \right) - \frac{c^\perp}{24} \bar{c}_2 c''_2 c'_2 c_2 + \frac{1}{B^2\alpha'} \bar{c}_2 c''_2 c'_2 c_2.
$$

(6.19)

We see that $T_{\text{bose}}$ appears explicitly in the second term.
If we assign the appropriate value $c^\perp = 24/(\alpha' B^2)$ (which is set by the condition on the parent $\mathcal{N} = 2$ theory that $c_{\text{total}} = 6$; see Eqn. (6.3)) and define

$$v^{\pm} \equiv B^{\pm 1} u^{\pm},$$

we find that, written in the bosonic variables $v^{\pm}$, the tachyon gradient $B$ scales out of the supercurrent entirely:

$$G_2 \equiv b_2 - c_2 T_{\text{bose}} - c''_2 + i\bar{c}_2 b_2 c_2' + i\bar{c}_2 b'_2 c_2 - ic'_{2} c_2 \bar{b}_2 + 2i \bar{c}'_2 b_2 c_2$$

$$+ \frac{2\sqrt{2}}{\alpha'} \left(i c'_2 v'^{+} + \bar{c}_2 c'_2 c_2 v''^{-} + \bar{c}_2 c''_2 c_2 v'^{-} + 2 \bar{c}'_2 c'_2 c_2 v'^{-}\right) - \frac{2}{\alpha'} c_2 v'^{+} v'^{-}. \quad (6.21)$$

**Final transformation**

The physics of this system is not yet clear. Ultimately, we are not interested in the details of the supercurrent at all, but only in its role in defining physical states via the OCQ or BRST procedure. The OCQ or BRST cohomology is covariant not only under unitary transformations, but under all similarity transformations as well. At this stage we would like to place the conjugate supercurrent $\bar{G}$ in the simplest form possible, and this can be achieved via a particular similarity transformation. We therefore define a non-Hermitian generator $g_2 \equiv \int d\sigma_1 g_2(\sigma)$, where

$$g_2 = ic'_2 c'_2 + \frac{i c^\perp}{6} \bar{c}'_2 c_2' + \frac{c^\perp B}{6\sqrt{2}} \bar{c}_2 c'_2 c_2 u^{+}_2 + \bar{c}_2 c'_2 c_2 \bar{b}_2 + \bar{c}_2 c'_2 c_2 c_2 + \frac{13}{8} \bar{c}'_2 \bar{c}_2 c''_2$$

$$- \frac{c_{\text{bose}}}{16} \bar{c}'_2 \bar{c}_2 c''_2 c_2 + \frac{11 c^\perp}{48} \bar{c}'_2 \bar{c}_2 c''_2 c_2 + \frac{i c^\perp B}{12\sqrt{2}} \bar{c}_2 c'_2 c_2 u^{+}_2 - iT_{\text{bose}} \bar{c}_2 c_2 - \frac{2i}{\alpha'} \bar{c}_2 c_2 u^{+}_2 u^{-}_2$$

$$+ \frac{1}{B\alpha'} \left(2\sqrt{2} \bar{c}_2 c_2 u^{-}_2 - \frac{4i}{B} \bar{c}'_2 c_2' - \frac{11}{2B} \bar{c}'_2 \bar{c}_2 c'_2 c_2 - 2\sqrt{2} \bar{c}_2 c'_2 u^{+}_2ight.$$  

$$- i\sqrt{2} \bar{c}_2 c'_2 c_2 u^{+}_2 - 3i\sqrt{2} \bar{c}_2 c'_2 c_2 u^{-}_2) \right). \quad (6.22)$$

The finite similarity transformation of interest takes the form

$$S_2 \equiv \exp \left(\frac{i}{2\pi} g_2 \right),$$

$$33$$
So that conjugating the supercurrent with $S_2$ yields

$$G_1 = S_2 G_2 S_2^{-1}$$

$$= b_1 - 2c_1 T_{bose} - \frac{ic^B}{6\sqrt{2}} c_1 u_1^{+''} + \frac{2i\sqrt{2}}{B\alpha'} c_1 u_1^{+''} - \frac{ic^B}{3\sqrt{2}} c_1' u_1^{+''} + 2i\bar{c}_1 b_1' c_1 - 2i\bar{c}_1 b_1' c_1$$

$$- 2i\bar{c}_1' c_1' \bar{b}_1 + 4i\bar{c}_1' b_1 c_1 - \frac{4}{\alpha'} c_1 u_1^{+''} u_1^{+''} + \frac{2i\sqrt{2}}{B\alpha'} (c_1 u_1^{+''} u_1^{+''} + 2c_1' u_1^{+''} + 2c_1' u_1^{+''})$$

$$- 2c_1'' - \frac{c_1'}{3} c_1'' + \frac{8}{B^2\alpha' c_1''}$$

$$- \frac{i}{16B^2\alpha'} (24 + \alpha' B^2 (c_{bose} - c_1) - 26) \left( \bar{c}_1 c_1'' c_1' - \bar{c}_1 c_1' c_1 - 4c_1'' c_1' c_1 - 6c_1'' c_1' c_1 \right)$$

$$+ \frac{7}{24B^2\alpha'} (24 + \alpha' B^2 (c_{bose} - c_1) - 26) \bar{c}_1' \bar{c}_1'' c_1' c_1$$.  \hspace{1cm} (6.24)

Since $S_2$ is generated by a non-Hermitian function, $\bar{G}_1$ is no longer given simply by the conjugate of $G_1$. Specifically, we obtain

$$\bar{G}_1 = \bar{b}_1 - \frac{i}{48B^2\alpha'} (24 + \alpha' B^2 (c_{bose} - c_1) - 26) \left( 6\bar{c}_1' \bar{c}_1 c_1'' - 3\bar{c}_1'' c_1' c_1 - \bar{c}_1'' c_1 c_1 \right)$$

$$+ \frac{1}{12B^2\alpha'} (24 + \alpha' B^2 (c_{bose} - c_1) - 26) \bar{c}_1'' \bar{c}_1' \bar{c}_1 c_1$$.  \hspace{1cm} (6.25)

The final expressions simplify significantly when we replace $c_{bose}$ and $c_1$ with values appropriate to the system at hand (see Eqn. (6.3)), and move to the rescaled bosonic $v^\pm$ variables (with $u^\pm = B^{\frac{-1}{2}} v^\pm$). Once again, we obtain expressions that are independent of the tachyon gradient $B$:

$$G_1 = b_1 - 2c_1 T_{bose} + 2i \left( \bar{c}_1 b_1 c_1' + \bar{c}_1 b_1' c_1 - c_1' c_1 \bar{b}_1 + 2\bar{c}_1 b_1 c_1 \right)$$

$$- \frac{4}{\alpha'} c_1 v_1^{+''} v_1^{+''} + \frac{2i\sqrt{2}}{\alpha'} \left( c_1 v_1^{+''} v_1^{+''} + 2c_1' v_1^{+''} \right) - 2c_1'''$$,

$$\bar{G}_1 = \bar{b}_1$$.  \hspace{1cm} (6.26)
Likewise, the final R-current and stress tensor are as follows:

\[ J_1 = -\frac{2\sqrt{2}}{\alpha'} v_1^- + c_1 \bar{b}_1 - \bar{c}_1 b_1 , \]

\[ T_1 = T_{\text{bose}} + \frac{2}{\alpha'} v_1^+ v_1^- - \left( \frac{i}{2} c_1 \bar{b}_1' + \frac{3i}{2} c_1' \bar{b}_1 + \text{h.c.} \right) . \]  

(6.27)

In summary, we began with the \( \mathcal{N} = 2 \) string in flat space with timelike linear dilaton. Following condensation of the closed-string tachyon profile specified by the superpotential in Eqn. (5.1), the theory asymptotes at late times to a constant CFT with no \( X^+ \) dependence in the Lagrangian, stress tensor, R-current or supercurrents. The theory admits a simple late-time limit that is static (modulo a time-dependent dilaton). We would like to understand in detail what this limit actually describes. In the next section we show that the theory deep in the IR regime is precisely equivalent to a certain background of bosonic string theory, with \( D-2 = 2D_c-2 \) noncompact dimensions, a linear dilaton, and an \( \text{SO}(2D_c-4)_L \times \text{SO}(2D_c-4)_R \) current algebra. (In this paper, \( D \) and \( D_c \) will always refer to numbers of dimensions of the \textit{initial} string background.) The corresponding bosonic string is precisely described by the generic stress tensor \( T_{\text{bose}} \). This system stands as an analogue of the bosonic string embedding into \( \mathcal{N} = 1 \) superstring theory introduced by Berkovits and Vafa in \cite{5}.  

7 \textbf{BRST quantization}

In this Section we will establish the equivalence of the late-time limit of our worldsheet theory to the bosonic string in a particular background. To do this, we quantize the string in the BRST formalism. We then take the standard form of the BRST current and perform a similarity transformation that reduces the BRST charge to two anticommuting pieces involving disjoint sets of worldsheet fields. One of the two pieces will have a trivially computable cohomology consisting of a single state, the vacuum \( |0\rangle \). The second piece is precisely equal to the BRST charge of the bosonic string in a particular background. The cohomology of the full product theory thus corresponds one-to-one with that of the bosonic string\footnote{This statement is strictly true modulo a subtlety concerning the zero mode of the \( v^- \) antighost, which we will discuss below.}.

In total, the BRST current of our original theory takes the form

\[ j_{\text{BRST}} = c^T + \frac{1}{2} \hat{c} J + \frac{i}{\sqrt{2}} \hat{\gamma} G - \frac{i}{\sqrt{2}} \gamma \bar{G} , \]  

(7.1)
with the quantities $T$, $J$, $G$ and $\bar{G}$ defined above in Eqn. (2.36). We want to show that this form can be brought into a final universal form by the action of a similarity transformation. One way to find such a transformation is to create an exhaustive list of possible operator-valued terms that could appear in a candidate generating function. These consist of field monomials that have zero total R-charge, zero ghost number and total weight one. Given the field content of the theory, there are a large number of possible terms (several hundred) that satisfy these conditions. Upon constructing a candidate generating function $g_0$ from these terms, we can solve for free coefficients by demanding that the full transformation yield the desired final incarnation of the BRST current, up to a total derivative:

$$S_0 j_{\text{BRST}} S_0^{-1} = j_{\text{new}}^{\text{BRST}} + j_{\text{BRST}}^{\text{deriv}},$$  \hspace{1cm} (7.2)

where $j_{\text{new}}^{\text{BRST}}$ is described below. Here, $S_0$ comprises a similarity transformation, as opposed to a unitary transformation. In other words, it is generated by a non-Hermitian generator, according to

$$S_0 = \exp (ig_0),$$  \hspace{1cm} (7.3)

with $g_0 \neq g_0^\dagger$. In the end, the final transformation rules are still somewhat complicated. We present these rules, including the complete form of the generating function $g_0$, in Appendix D.

Under the full transformation in Eqn. (7.2), the BRST current becomes

$$j_{\text{BRST}}^{\text{new}} = j_{\text{BRST}}^{\text{triv}} + j_{\text{BRST}}^{\text{bose}}$$

$$= -\frac{i}{\sqrt{2} \omega} \gamma \bar{b}_1 + \frac{i \omega}{\sqrt{2}} \bar{\gamma} b_1 - \frac{\sqrt{2}}{\alpha'} \hat{c} v^{-'} + i b' c + \frac{3}{2} c'' + c T_{\text{bose}}. \hspace{1cm} (7.4)$$

Since the overall transformation is rather complicated, we have included a Mathematica notebook in the arXiv source package of this paper that can be used to verify the result.\footnote{This notebook can also be found online at the URL http://sns.ias.edu/~swanson/BRST.nb.}

The total derivative term $j_{\text{BRST}}^{\text{deriv}}$ does not contribute to the BRST charge, and we can ignore it for the purpose of computing the physical state spectrum. The first term $j_{\text{BRST}}^{\text{triv}}$ is a quadratic piece involving the variables $c_1$, $b_1$, $v^\pm$, and the Fadeev-Popov ghosts for the R-symmetry and local supersymmetry:

$$j_{\text{BRST}}^{\text{triv}} = -\frac{i}{\sqrt{2} \omega} \gamma \bar{b}_1 + \frac{i \omega}{\sqrt{2}} \bar{\gamma} b_1 - \frac{\sqrt{2}}{\alpha'} \hat{c} v^{-'}. \hspace{1cm} (7.5)$$
The only cohomology in this sector is the vacuum $|0\rangle$, corresponding to the operator
\begin{equation}
|0\rangle \rightleftharpoons c_1 \bar{c}_1 \delta(\gamma)\delta(\bar{\gamma}) .
\end{equation}

For oscillators with nonvanishing mode number, this statement follows from a straightforward application of the BRST quartet principle. In particular, one should consider the three field quartets
\begin{equation}
(b_1, \gamma, \bar{c}_1, \beta) , \quad (\bar{b}_1, \gamma, c_1, \bar{\beta}) , \quad \left(v^{-'}, \hat{c}, v^+, \hat{b}\right) .
\end{equation}

One subtlety in this approach is that the field $v^-$ never appears undifferentiated in the current $j^{\text{triv}}_{\text{BRST}}$. This implies that, while $v^{-'}$ is the $Q$-image of a local operator (namely $\hat{b}$), the $Q$-closed operator $v^-$ is not. This leads to a strange situation in which $v^- (\sigma^\pm)$ is an element of the BRST cohomology, but $v^{-'}$ is BRST-exact.

Since $v^- (\sigma)$ is independent of $\sigma$ in cohomology, operators constructed from undifferentiated $v^-$ fields behave analogously to picture-changing operators. That is, $v^-$ momentum labels an infinite number of disjoint copies of the physical state cohomology, between which the exponentials $X_P \equiv \exp (i P v^-)$ interpolate. The operators $X_P$ functioning as picture-changing operators for “$v^-$-picture.” All PCOs for $v^-$-picture are invertible and position-independent, so it is manifest that all $v^-$-pictures are completely equivalent. It seems likely that a careful treatment of picture- and instanton-number-changing operators in the $\mathcal{N} = 2$ string framework will shed light on the correct treatment of $v^-$-pictures at the level of interacting strings.

The third piece, $j_{\text{BRST}}^{\text{bose}}$, is a BRST operator in the standard form, describing a bosonic string with a $c = 26$ conformal stress tensor:
\begin{equation}
j_{\text{BRST}}^{\text{bose}} = c T^{\text{bose}} + ib c' c + \frac{3}{2} c'' .
\end{equation}

The stress tensor $T^{\text{bose}}$ describes $2D_c - 4$ flat transverse real coordinates $\phi^a$, $\bar{\phi}^a$, as well as a pair of light-cone coordinates $y^\pm$, also with flat metric $\eta_{+-} = \eta_{-+} = -1, \eta_{++} = \eta_{--} = 0$. There is a varying dilaton whose contribution to the central charge is given by
\begin{equation}
c_{\text{dilaton}} = -\frac{3}{2} (D - 20) ,
\end{equation}
where $D = 2D_c$ is the number of real noncompact dimensions of the initial configuration.

The physical state cohomology $\mathcal{V}$ in the bosonic sector is characterized, as usual, by the $bc$ ghost vacuum times a matter primary of weight one. Given a matter primary of weight
one, the corresponding vertex operator will be of the form $\bar{c} c_{1} \bar{c}_{1} \delta(\gamma) \delta(\bar{\gamma}) V$ (along with the appropriate left-moving operators).

Finally, there is a stress tensor present for the $\psi^{a}$, $\psi^{a\dagger}$ degrees of freedom. These variables no longer have any significance as superpartners of the $\phi$ variables. In particular, currents generated by $\psi^{a}$, $\psi^{a\dagger}$ do not give rise to gauge bosons in the spectrum of the initial $\mathcal{N} = 2$ theory. In the late-time theory, however, these variables are primary operators of weight one, and therefore enter the theory on the same footing as any other such operator, giving rise to gauge bosons propagating in spacetime. The same is true for the currents generated by $\tilde{\psi}^{a}$, $\tilde{\psi}^{a\dagger}$. Together, these variables generate an $SO(D - 4)_L \times SO(D - 4)_R$ current algebra, with one real fermion for each real dimension transverse to the light cone.

The complex structure of the $\phi$ and $\psi$ degrees of freedom has decoupled completely, along with the original supersymmetry. By decomposing $\phi^{a} \equiv \frac{1}{\sqrt{2}}(y^{2a} + iy^{2a+1})$ and $\psi^{a} \equiv \frac{1}{\sqrt{2}}(\lambda^{2a} + i\lambda^{2a+1})$ for $a = 1, \ldots, D_{c} - 2$, it is clear that the theory has an $SO(D - 4)$ symmetry rotating the $\lambda^{A}$ (with the index $A$ labeling real directions), as well as an unrelated $SO(D - 3, 1)$ spatial symmetry rotating the spatial directions $(y^{0}, y^{1}, y^{A})$, if one ignores the dilaton. The dilaton gradient breaks this spatial symmetry down to $SO(D - 3)$ if the final dilaton gradient is timelike, $SO(D - 4)$ if the final dilaton gradient is lightlike, and $SO(D - 4, 1)$ if the final dilaton gradient is spacelike. The total central charge of the bosonic stress tensor $T_{\text{bose}}$ is $D - 2$ from the scalars $(y^{0}, y^{1}, y^{A})$, $\frac{1}{2}(D - 4)$ from the fermions $\lambda^{A}$ and $-3(D/2 - 10)$ from the dilaton, for a total of $c = 26$. Of course, this is the correct central charge for a consistent bosonic string theory.

8 Summary and conclusions

The $\mathcal{N} = 2$ string exhibits a number of interesting properties in supercritical dimensions. In fact, because the presence of a linear dilaton background renders one of the two timelike directions unphysical, it is somewhat natural to study this string theory in dimensions above the critical value $D = 4$. In line with previous studies of supercritical strings, we have found that closed-string tachyon condensation leads to a number of interesting dynamical transitions among different string theories. In certain examples, these transitions connect the $\mathcal{N} = 2$ string in various spacetime dimensions via dynamical dimensional reduction. We have also shown that closed-string tachyon condensation can drive a dynamical transition.
directly to bosonic string theory.

Although the existence of consistent string theories in supercritical dimensions was established long ago [12,13,14], it is apparent that such theories harbor a number of novel and surprising features. It seems likely that further exploration of supercritical string backgrounds will yield additional connections among theories that were previously thought to be completely distinct.

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Appendix

A $\mathcal{N} = 2$ supersymmetry algebra

In this appendix we record the off-shell supersymmetry transformations of the $D_\text{ch}$ chiral multiplets of our theory. To begin, the $\mathcal{N} = 2$ superalgebra provides the following:

$$\{Q, \bar{Q}\} = -2P_+, \quad \{\bar{Q}, \bar{Q}\} = -2P_- , \quad \text{(A.1)}$$

along with the vanishing quantities

$$Q^2 = \bar{Q}^2 = Q^2 = \bar{Q}^2 = \{Q, \bar{Q}\} = \{Q, \bar{Q}\} = \{Q, \bar{Q}\} = 0 . \quad \text{(A.2)}$$

We recall that $P_\pm = \frac{1}{\sqrt{2}}(-P_0 \pm P_1)$, so both $P_\pm$ are negative-definite in a unitary theory. The transformation laws for a chiral multiplet $\phi, \psi$ are as follows:

$$Q \phi = i\sqrt{\alpha'} \psi , \quad \bar{Q} \phi = i\sqrt{\alpha'} \bar{\psi} ,$$

$$Q \bar{\phi} = 0 , \quad \bar{Q} \bar{\phi} = 0 ,$$

$$Q \psi = 0 , \quad \bar{Q} \psi = F ,$$

$$\bar{Q} \psi = \frac{2}{\sqrt{\alpha'}} \partial_+ \phi , \quad \bar{Q} \psi = 0 ,$$

$$Q \bar{\psi} = -F , \quad \bar{Q} \bar{\psi} = 0 ,$$

$$\bar{Q} \bar{\psi} = 0 , \quad \bar{Q} \bar{\psi} = \frac{2}{\sqrt{\alpha'}} \partial_- \phi ,$$

$$Q F = 0 , \quad \bar{Q} F = 0 ,$$

$$\bar{Q} F = -2i \partial_+ \bar{\psi} , \quad \bar{Q} F = +2i \partial_- \psi . \quad \text{(A.3)}$$
The transformations of the conjugate multiplet are

\[ Q \bar{\phi} = 0 , \quad \tilde{Q} \bar{\phi} = 0 , \]

\[ \bar{Q} \bar{\phi} = i \sqrt{\alpha'} \tilde{\psi} , \quad \bar{\tilde{Q}} \bar{\phi} = i \sqrt{\alpha'} \tilde{\psi} , \]

\[ Q \tilde{\psi} = \frac{2}{\sqrt{\alpha'}} \partial_+ \bar{\phi} , \quad \tilde{Q} \tilde{\psi} = 0 , \]

\[ Q \bar{\psi} = 0 , \quad \tilde{Q} \bar{\psi} = \bar{F} , \]

\[ \bar{Q} \bar{\psi} = 0 , \quad \bar{\tilde{Q}} \bar{\psi} = \frac{2}{\sqrt{\alpha'}} \partial_- \bar{\phi} , \]

\[ Q \bar{\tilde{\psi}} = -F , \quad \tilde{Q} \bar{\tilde{\psi}} = 0 , \]

\[ Q \bar{F} = -2i \partial_+ \bar{\psi} , \quad \tilde{Q} \bar{F} = +2i \partial_- \bar{\psi} , \]

\[ \bar{Q} \bar{F} = 0 , \quad \bar{\tilde{Q}} \bar{F} = 0 . \] (A.4)

\[ \]

**B  OPEs of basic objects**

In this appendix we collect the OPEs of both fundamental and composite operators in our theory. The OPE of the fundamental $\phi$ fields reads:

\[ \phi^\mu(\sigma) \bar{\phi}^\nu(\tau) \sim -\frac{\alpha'}{2} \log \left| (\sigma^+ - \tau^+)(\sigma^- - \tau^-) \right| \eta^{\mu\nu} , \]  

where, as usual, $\sim$ indicates equivalence up to nonsingular terms. Similarly, the $\psi$ fields admit the OPE

\[ \psi(\sigma)^\mu \bar{\psi}(\tau)^\nu \sim \bar{\psi}(\sigma)^\mu \psi(\tau)^\nu \sim \frac{i}{\sigma^+ - \tau^+} \eta^{\mu\nu} . \]  

(B.2)
Furthermore, the OPEs of the various composite operators in the theory (namely, the complex supercurrent, the R-current and the stress tensor) are

\[ J(\sigma)J(\tau) \sim \frac{c}{3(\sigma^+-\tau^+)^2}, \]

\[ T(\sigma)J(\tau) \sim \frac{1}{(\sigma^+-\tau^+)^2} J(\tau) + \frac{1}{\sigma^+-\tau^+} \partial_+ J(\tau), \]

\[ T(\sigma)G(\tau) \sim \frac{3}{2(\sigma^+-\tau^+)^2} G(\tau) + \frac{1}{\sigma^+-\tau^+} \partial_+ G(\tau), \]

\[ T(\sigma)\bar{G}(\tau) \sim \frac{3}{2(\sigma^+-\tau^+)^2} \bar{G}(\tau) + \frac{1}{\sigma^+-\tau^+} \partial_+ \bar{G}(\tau), \]

\[ J(\sigma)G(\tau) \sim \frac{i}{\sigma^+-\tau^+} G(\tau), \]

\[ J(\sigma)\bar{G}(\tau) \sim -\frac{i}{\sigma^+-\tau^+} \bar{G}(\tau), \]

\[ G(\sigma)\bar{G}(\tau) \sim -\frac{2ic}{3(\sigma^+-\tau^+)^3} - \frac{2}{(\sigma^+-\tau^+)^2} J(\tau) - \frac{2i}{\sigma^+-\tau^+} T(\tau) - \frac{1}{\sigma^+-\tau^+} \partial_+ J(\tau), \]

\[ T(\sigma)T(\tau) \sim \frac{c}{2(\sigma^+-\tau^+)^4} + \frac{2}{(\sigma^+-\tau^+)^2} T(\tau) + \frac{1}{\sigma^+-\tau^+} \partial_+ T(\tau). \]  

(C) Details of the bosonic string embedding

In this appendix we record in detail each step of the multi-part variable transformation that brings us to the final incarnation of the IR theory in Eqns. (6.26-6.27) above. We start just after the rescaling transformation that introduces the set of variables labeled by the subscript 5. The supercurrent at this stage is presented above in Eqn. (6.6) with bosonic coordinates expressed in \( \chi \) variables. It can be reached from the supercurrent \( G_6 \) in Eqn. (5.21) by performing the rescalings in Eqn. (6.4). For completeness, we present \( G_5 \) in real \( u \) and \( y \)
bosonic variables (as opposed to the complex $\chi$ system):

\[
G_5 = b_5 + \frac{2\sqrt{2}}{B\alpha'} \left( ic_5 u_5^{-''} + c_5 y_5^{-''} + ic'_5 u_5^{-'} + ic'_5 u_5^{+'} - c'_5 y_5^{-'} + c'_5 y_5^{+'} \right) - 2ic'_5 c_5 \bar{b}_5 \\
- \frac{2}{\alpha'} c_5 \left( \frac{iB\alpha'}{\sqrt{2}} u_5^{+''} + u_5^{+'} u_5^{-'} + y_5^{+'} y_5^{-'} + iy_5^{-'} u_5^{+'} - iy_5^{+'} u_5^{-'} \right) + \frac{B\alpha'}{\sqrt{2}} y_5^{+''} \\
- c'' + \frac{4}{B^2\alpha^2} \alpha' c'_5 .
\]  

(C.1)

For the sake of collecting all of the relevant quantities in one place, we will also record the R-current and stress tensor after each step. The R-current at this stage takes the form

\[
J_5 = J^+ + c_5 \bar{b}_5 - 1\bar{c}_5 b_5 - \frac{21}{4} \bar{c}_5 e_5'' + \frac{c_{\text{bose}}}{8} \bar{c}_5 e_5'' - \frac{c^\perp}{8} \bar{c}_5 e_5'' - 4\bar{c}'_5 e_5' - 2\bar{e}_5'' c_5 + \frac{39i}{4} \bar{c}'_5 e_5'' c_5 \\
- \frac{3ic_{\text{bose}}}{8} \bar{c}'_5 e_5'' c_5 + \frac{3ic^\perp}{8} \bar{c}'_5 e_5'' c_5 + \frac{13i}{4} \bar{c}'_5 e_5'' c_5 - \frac{ic_{\text{bose}}}{8} \bar{c}'_5 e_5'' c_5 + \frac{i}{8} \bar{c}'_5 e_5'' c_5 \\
+ \frac{1}{B^2\alpha'} \left( 4\bar{c}_5 e_5'' c_5 - 3ic_5 \bar{c}_5 e_5'' c_5 - 9ic_5 \bar{c}_5 e_5'' c_5 + 7\bar{c}_5 e_5'' + 8\bar{c}_5 e_5' \right) \\
- \frac{2\sqrt{2}}{B\alpha'} \left( u_5^{-'} + u_5^{+'} \right) ,
\]  

(C.2)
and the stress tensor is given by

\[ T_5 = -\frac{i}{2} c_5 b_5^\prime - \frac{i}{2} \tilde{c}_5 b_5^\prime + \frac{11i}{24} c_5 c_5^\prime + \frac{ic_{\text{bose}}}{48} \tilde{c}_5 c_5^\prime - \frac{ic_5}{48} \tilde{c}_5 c_5^\prime - \frac{3i}{2} c_5^\prime b_5^\prime - \frac{3i}{2} \tilde{c}_5 b_5 - \frac{5i}{8} \tilde{c}_5 c_5^\prime
\]

\[ + \frac{ic_{\text{bose}}}{16} c_5 c_5^\prime - \frac{i c_5}{16} c_5^\prime c_5^\prime - i \tilde{c}_5^\prime c_5^\prime - i \tilde{c}_5^\prime c_5 - \frac{3}{8} \tilde{c}_5 \tilde{c}_5 c_5 c_5^\prime + \frac{c_{\text{bose}}}{16} \tilde{c}_5 \tilde{c}_5 c_5 c_5^\prime - \frac{c_5}{16} \tilde{c}_5 c_5 c_5^\prime c_5^\prime
\]

\[ + \frac{35}{24} \tilde{c}_5^\prime c_5 c_5^\prime c_5 + \frac{c_{\text{bose}}}{48} \tilde{c}_5^\prime c_5 c_5^\prime c_5 - \frac{c_5^\prime c_5 c_5^\prime c_5 c_5}{8} + \frac{c_{\text{bose}}}{8} \tilde{c}_5 c_5 c_5 c_5^\prime
\]

\[ + \frac{c_{\text{bose}}}{48} \tilde{c}_5 c_5 c_5^\prime - \frac{c_5^\prime c_5 c_5^\prime c_5 c_5}{8} + \frac{c_5^\prime c_5 c_5^\prime c_5 c_5}{16} + T_{\text{bose}} + T_5^\perp + u_5^+ u_5^- \frac{2}{\alpha'}
\]

\[ - \frac{3}{2B^2 \alpha'} \left( \tilde{c}_5^\prime c_5 c_5^\prime + \frac{c_5^\prime c_5 c_5^\prime c_5}{48} \right) - \frac{1}{B^2 \alpha'} \left( \tilde{c}_5^\prime^2 c_5 c_5^\prime c_5 + \frac{1}{2} \tilde{c}_5 c_5 c_5^\prime c_5 + \frac{1}{2} \tilde{c}_5^\prime c_5 c_5^\prime c_5 c_5
\]

\[ - \frac{i}{2} \tilde{c}_5 c_5^\prime - \frac{3i}{2} \tilde{c}_5 c_5^\prime
\]

\[ = - \frac{2 \sqrt{2}}{B \alpha'} \left( \tilde{c}_5^\prime c_5 u_5^{-\prime} + \tilde{c}_5 c_5^\prime u_5^- + \tilde{c}_5 c_5 u_5^- c_5^\prime + \tilde{c}_5 c_5^\prime u_5^+ u_5^-
\]

\[ + \tilde{c}_5^\prime c_5 u_5^+ + \tilde{c}_5 c_5^\prime u_5^+ u_5^-ight) - \frac{\sqrt{2}}{B \alpha'} \left( \tilde{c}_5 c_5 u_5^+ u_5^- + \tilde{c}_5 c_5^\prime u_5^+ u_5^-ight.
\]

\[ + \tilde{c}_5^\prime c_5 u_5^+ + \tilde{c}_5 c_5^\prime u_5^+
\]

\[ \right) \). \quad (C.3)
\]

The next transformation renders the bosonic \( y, u \) theory reflection symmetric about the little group of the renormalized linear dilaton. This is achieved by acting with the unitary transformation \( U_5 \):

\[ U_5 \equiv \exp \left( \frac{i}{2\pi} g_5 \right), \quad (C.4)
\]

where

\[ g_5 \equiv \int d\sigma_1 g_5(\sigma). \quad (C.5)
\]

The explicit form of \( g_5 \) is given above in Eqn. (6.10). This transformation moves us to
variables marked with the subscript 4. The transformed supercurrent takes the form

\[ G_4 = U_5 G_5 U_5^\dagger = b_1 + G^\perp - i T_{\text{bos}} L^c c_4' c_4 + 2 i c_4 c_4' c_4 - \frac{7i}{6} c_4 c_4'' c_4 - 2 i c_4 c_4 b_4 \]

\[ - \frac{3i}{2} c_4 c_4' c_4' - \frac{3i}{2} c_4'' c_4 - 4 c_4' c_4' c_4' c_4 - T_{\text{bos}} L^c c_4 - \frac{2}{\alpha'} (c_4 u_4^+ u_4^-) \]

\[ + i c_4 c_4 c_4 u_4^+ u_4^- + \frac{1}{B_\alpha'} (i c_4 c_4'' c_4 + i c_4 c_4' c_4' - 2 i c_4'' c_4 c_4 + 5 i c_4'' c_4 c_4 c_4 \]

\[ + 4 c_4 c_4 c_4 c_4 + 4 c_4'' c_4) + \frac{2\sqrt{2}}{B_\alpha'} (i c_4 u_4^+ + i c_4 c_4 u_4^+ + c_4 c_4 u_4^+ + c_4 c_4 u_4^+ \]

\[ + c_4 c_4 c_4 u_4^+ + c_4 c_4 c_4 u_4^+) - c_4'' \]. \hspace{1cm} (C.6)

We now record the R-current, decomposed according to antighost number (denoted by the superscript index on the left-hand side):

\[ J_4^{(0)} = J^\perp + c_4 b_4 - c_4 b_4 - \frac{2\sqrt{2}}{B_\alpha'} (u_4^+ + u_4^-) \]

\[ J_4^{(2)} = \frac{1}{8} \left( -42 + c_\text{bos} - c_\perp \right) c_4 c_4' - 4 c_4 c_4' - 2 c_4'' c_4 + \frac{1}{B_\alpha'} \left( 7 c_4 c_4'' + 4 c_4'' c_4 + 8 c_4' c_4' \right) \]

\[ J_4^{(4)} = - \frac{i}{8\alpha' B^2} \left( 24 + \alpha' B^2 (c_\text{bos} + c_\perp - 26) \right) (3 c_4 c_4 c_4 + c_4' c_4' c_4) \hspace{1cm} (C.7) \]

Similarly, the stress tensor is given by

\[ T_4^{(0)} = T_{\text{bos}} + T^\perp - \frac{i}{2} \left( c_4 b_4 + c_4 b_4 \right) - \frac{3i}{2} \left( c_4 c_4 b_4 + c_4 c_4 b_4 \right) + \frac{2}{\alpha'} u_4^+ u_4^- \]

\[ T_4^{(2)} = \frac{11 i}{24} c_4 c_4'' + \frac{i c_{\text{bos}} L^c}{48} c_4 c_4'' + \frac{i}{2 B_\alpha'} c_4 c_4'' - \frac{5i}{8} c_4' c_4'' + \frac{i c_{\text{bos}} L^c}{16} c_4 c_4'' - i c_4' c_4' \]

\[ - i c_4'' c_4 c_4'' + \frac{3i}{2 B_\alpha'} c_4 c_4'' - \frac{2\sqrt{2}}{B_\alpha'} c_4 c_4 u_4^+ + c_4 c_4 u_4^+ + c_4 c_4 u_4^+ + c_4 c_4 u_4^+ \]

\[ + c_4 c_4 u_4^+ \right) - \frac{\sqrt{2}}{B_\alpha'} (c_4 c_4 u_4^+ + c_4 c_4 u_4^+ + c_4 c_4 u_4^+ + c_4 c_4 u_4^+) \]

\[ + c_4 c_4 u_4^+ + c_4 c_4 u_4^+ \], \hspace{1cm} (C.9)
Under the corresponding finite unitary transformation, we obtain the transformed supercurrent in "3" variables:

\[ T_4^{(4)} = \frac{3}{8} \tilde{c}_4 \bar{c}_4 c'' c' + \frac{c_{\text{bose}}}{16} \tilde{c}_4 \bar{c}_4 c'' c' - \frac{c_{\text{bose}}}{16} \tilde{c}_4 \bar{c}_4 c'' c' - \frac{1}{2B^2\alpha'} \tilde{c}_4 \bar{c}_4 c'' c' + \frac{35}{24} \tilde{c}_4 \bar{c}_4 c'' c' \]

\[ + \frac{c_{\text{bose}}}{48} \tilde{c}_4 \bar{c}_4 c'' c' - \frac{c_{\text{bose}}}{16} \tilde{c}_4 \bar{c}_4 c'' c' + \frac{3}{4} \tilde{c}_4 \bar{c}_4 c'' c' + \frac{c_{\text{bose}}}{8} \tilde{c}_4 \bar{c}_4 c'' c' - \frac{c_{\text{bose}}}{8} \tilde{c}_4 \bar{c}_4 c'' c' \]

\[ + \frac{3}{8} \tilde{c}_4 \bar{c}_4 c'' c' + \frac{c_{\text{bose}}}{16} \tilde{c}_4 \bar{c}_4 c'' c' - \frac{c_{\text{bose}}}{16} \tilde{c}_4 \bar{c}_4 c'' c' + \frac{35}{24} \tilde{c}_4 \bar{c}_4 c'' c' + \frac{c_{\text{bose}}}{48} \tilde{c}_4 \bar{c}_4 c'' c' \]

\[ - \frac{c_{\text{bose}}}{48} \tilde{c}_4 \bar{c}_4 c'' c' - \frac{1}{B^2\alpha'} \left( \frac{3}{2} \tilde{c}_4 \bar{c}_4 c'' c' + \frac{3}{2} \tilde{c}_4 \bar{c}_4 c'' c' \right) . \]  

(C.10)

The next step is to define a unitary transformation that rotates the R-current into a particular universal form, in which \( J^\perp \) is absent entirely. This is achieved by defining the generating function

\[ g_4 \equiv - \frac{B}{\sqrt{2}} J^\perp u^+ . \]  

(C.11)

Under the corresponding finite unitary transformation, we obtain the transformed supercurrent in "3" variables:

\[ G_3 = b + G^+ e^{-i\frac{B}{\sqrt{2}} u^+} - \frac{J^\perp}{\sqrt{2}} c_3 u^3' + \frac{ic^+ B}{6\sqrt{2}} c_3 u_{3'}^3 - \frac{c^+ B^2}{12} c_3 u_{3'}^3 u_{3'}^3 - 2i c_3 c_3 \bar{b}_3 \]

\[ - T_{\text{bose}}^\perp c_3 - \frac{2}{\alpha'} c_3 u_{3'}^3 u_{3'}^3 - \frac{2i\sqrt{2}}{B\alpha'} \left( c_3 u_{3'}^3 + c_3 u_{3'}^3 + iJ^\perp c_3' - c_3'' \right) \]

\[ + \frac{4}{B^2\alpha'} i c_3 c_{3'} c_3 + J^\perp i c_3 c_3 c_3 + 2i c_3 c_{3'} c_3 - \frac{7i}{6} c_3 c_{3'} c_3 + J^\perp c_3' c_3 c_3 \]

\[ - \frac{3i}{2} c_3 c_{3'} c_3 - \frac{3i}{2} c_{3'} c_3 c_3 + \frac{iJ^\perp B}{\sqrt{2}} c_3 c_{3'} c_3 u_{3'}^3 + \frac{c^+ B}{6\sqrt{2}} \left( c_3 c_{3'} c_3 u_{3'}^3 + c_{3'} c_3 c_3 u_{3'}^3 \right) \]

\[ - \frac{1}{12} i c^+ B^2 c_3 c_{3'} c_3 u_{3'}^3 u_{3'}^3 - \frac{2i}{\alpha'} c_3 c_3 c_3 u_{3'}^3 u_{3'}^3 + \frac{i}{B^2\alpha'} \left( c_3 c_{3'} c_3 + c_3' c_3 c_3 - 2c_3 c_{3'} c_3 \right) \]

\[ + 5 c_3 c_{3'} c_3 + \frac{2\sqrt{2}}{B\alpha'} c_3 c_3 c_3 u_{3'}^3 \right) \]

\[ + 4 \left( \frac{1}{B^2\alpha'} - 1 \right) c_3 c_{3'} c_{3'} c_3 . \]  

(C.12)
Decomposing by antighost number, the stress tensor appears as

\[ T_3^{(0)} = -\frac{i}{2} (\bar{c}_3 \bar{b}_3 + \bar{c}_3 b_3) - \frac{3i}{2} (c'_3 \bar{b}_3 + c'_3 b_3) + \frac{2}{\alpha'} u_3^{+} u_3^{-}, \]  

\[ T_3^{(2)} = -\frac{1}{2} \partial_+ (J_{\perp} \bar{c}_3 c_3) + \frac{11i}{24} \bar{c}_3 c''_3 + \frac{ic_{\text{bose}}}{48} \bar{c}_3 c''_3 - \frac{ic_{\perp}}{48} \bar{c}_3 c''_3 - \frac{5i}{8} \bar{c}_3 c''_3 + \frac{ic_{\text{bose}}}{16} \bar{c}_3 c''_3 \\
- \frac{e^B}{12\sqrt{2}} (\bar{c}_3 c''_3 u_3^{+} + 2\bar{c}_3 c'_3 u_3^{+}) + \bar{c}_3 c''_3 u_3^{+} + 2\bar{c}_3 c'_3 u_3^{+} + 2\bar{c}_3 c'_3 u_3^{+} + \bar{c}_3 c''_3 u_3^{+} + \bar{c}_3 c''_3 u_3^{+} + \bar{c}_3 c''_3 u_3^{+} + \bar{c}_3 c''_3 u_3^{+} \\
- \frac{2\sqrt{2}}{B\alpha'} (\bar{c}_3 c''_3 u_3^{+} + \bar{c}_3 c''_3 u_3^{+}) + \bar{c}_3 c''_3 u_3^{+} + \bar{c}_3 c''_3 u_3^{+} + \bar{c}_3 c''_3 u_3^{+} + \bar{c}_3 c''_3 u_3^{+} + \bar{c}_3 c''_3 u_3^{+} \\
+ \bar{c}_3 c''_3 u_3^{+} + \bar{c}_3 c''_3 u_3^{+} + \bar{c}_3 c''_3 u_3^{+} + \bar{c}_3 c''_3 u_3^{+} \\
+ \bar{c}_3 c''_3 u_3^{+} + \bar{c}_3 c''_3 u_3^{+} + \bar{c}_3 c''_3 u_3^{+} + \bar{c}_3 c''_3 u_3^{+} \right), 

\[ T_3^{(4)} = \frac{3}{8} \bar{c}_3 c''_3 c_3 + \frac{c_{\text{bose}}}{16} \bar{c}_3 c''_3 c_3 - \frac{c_{\perp}}{16} \bar{c}_3 c''_3 c_3 - \frac{1}{2B^2\alpha'} \bar{c}_3 c''_3 c_3 + \frac{35}{24} \bar{c}_3 c''_3 c_3 \\
+ \frac{c_{\text{bose}}}{48} \bar{c}_3 c''_3 c_3 - \frac{c_{\perp}}{48} \bar{c}_3 c''_3 c_3 + \frac{c_{\text{bose}}}{8} \bar{c}_3 c''_3 c_3 - \frac{c_{\perp}}{8} \bar{c}_3 c''_3 c_3 \\
+ \frac{3}{8} \bar{c}_3 c''_3 c_3 + \frac{c_{\text{bose}}}{16} \bar{c}_3 c''_3 c_3 - \frac{c_{\perp}}{16} \bar{c}_3 c''_3 c_3 + \frac{35}{24} \bar{c}_3 c''_3 c_3 + \frac{c_{\text{bose}}}{48} \bar{c}_3 c''_3 c_3 \\
- \frac{c_{\perp}}{48} \bar{c}_3 c''_3 c_3 - \frac{1}{2B^2\alpha'} (\bar{c}_3 c''_3 c_3 + \bar{c}_3 c''_3 c_3) \\
- \frac{1}{B^2\alpha'} (\bar{c}_3 c''_3 c_3 + \frac{1}{2} \bar{c}_3 c''_3 c_3) . \]
At this stage we need to bring the supercurrent itself into a universal form in which only $T_{bose}$ appears explicitly. To this end, we employ the generating function defined above in Eqn. (6.15). The corresponding finite transformation yields the supercurrent

$$G_2 = b - \frac{ic^{-1}B}{6\sqrt{2}} \partial_+ \left( c_2 u_2^+ \right) + \frac{2i\sqrt{2}}{B\alpha'} c_2 u_2^{''} + ic_2 b_2 c_2' + ic_2 b_2' c_2 - ic_2 b_2 c_2'$$

$$+ 2ic_2' b_2 c_2 - T_{bose} c_2 - \frac{2 \alpha'}{c_2} u_2^{+'} u_2^{+} + \frac{2i\sqrt{2}}{B\alpha'} \left( c_2' u_2^{+'} + c_2' u_2^{+} \right) + \left( \frac{4}{B^2\alpha'} - \frac{c^{-1}}{6} \right) c_2''$$

$$+ \frac{ic^{-1}}{12} \bar{c}_2 \bar{c}_2'' c_2' - \frac{2i}{B^2\alpha'} \bar{c}_2 \bar{c}_2'' c_2' + \frac{ic^{-1}}{8} \bar{c}_2 \bar{c}_2'' c_2 + \frac{7ic^{-1}}{24} \bar{c}_2 \bar{c}_2'' c_2 + \frac{ic^{-1}}{8} \bar{c}_2'' c_2 c_2$$

$$- \frac{i}{B^2\alpha'} \left( 3\bar{c}_2 \bar{c}_2'' c_2 + 3\bar{c}_2'' c_2 c_2 + 7\bar{c}_2'' c_2 c_2 \right) + \frac{2\sqrt{2}}{B\alpha'} \left( \bar{c}_2 c_2 c_2 u_2^{+''} + \bar{c}_2 c_2 c_2 u_2^{+'} \right)$$

$$+ 2\bar{c}_2' c_2' c_2 u_2^{+'} - \frac{c^{-1}}{24} \bar{c}_2 \bar{c}_2'' c_2 c_2 + \frac{1}{B^2\alpha'} \bar{c}_2 \bar{c}_2'' c_2 c_2 .$$

(C.17)

The R-current takes the form

$$J_2 = \frac{c^{-1}B}{6\sqrt{2}} u_2^{+'} + c_2 b_2 - \bar{c}_2 b_2 - \frac{13}{4} \bar{c}_2 c_2'' + \frac{c_{bose}}{8} \bar{c}_2 c_2'' - \frac{c^{-1}}{8} \bar{c}_2 c_2'' + \frac{3}{B^2\alpha'} \bar{c}_2 c_2''$$

$$- \frac{2\sqrt{2}}{B\alpha'} \left( u_2^{+'} + u_2^{+} \right) .$$

(C.18)

Broken up by antighost number, the stress tensor takes the form

$$T_2^{(0)} = T_{bose} - \frac{i}{2} \left( c_2 b_2' + \bar{c}_2 b_2' \right) - \frac{3i}{2} \left( c_2 b_2 + \bar{c}_2 b_2 \right) + u_2^{+'} u_2^{+} \frac{2}{\alpha'} ,$$

$$T_2^{(2)} = - \frac{13i}{24} \bar{c}_2 c_2'' + \frac{ic_{bose}}{48} \bar{c}_2 c_2'' - \frac{ic^{-1}}{48} \bar{c}_2 c_2'' + \frac{i}{2B^2\alpha'} \bar{c}_2 c_2'' - \frac{13i}{8} \bar{c}_2 c_2'' + \frac{ic_{bose}}{16} \bar{c}_2 c_2''$$

$$- \frac{ic^{-1}}{16} \bar{c}_2 c_2'' - \frac{c^{-1}B}{12\sqrt{2}} \left( \bar{c}_2 c_2 u_2^{+''} + \bar{c}_2 c_2' u_2^{+''} + \bar{c}_2 c_2' u_2^{+'} + 2\bar{c}_2 c_2 u_2^{+''} + 2\bar{c}_2 c_2' u_2^{+'} \right)$$

$$+ 2\bar{c}_2 c_2' u_2^{+'} + \frac{3i}{2B^2\alpha'} \bar{c}_2 c_2'' - \frac{\sqrt{2}}{B\alpha'} \left( 2\bar{c}_2 c_2' u_2^{+''} + 2\bar{c}_2 c_2 u_2^{+''} + 2\bar{c}_2 c_2' u_2^{+'} + \bar{c}_2 c_2 u_2^{+''} \right)$$

$$+ \bar{c}_2 c_2' u_2^{+'} + \bar{c}_2' c_2 u_2^{+} - \bar{c}_2 c_2' u_2^{+''} + \bar{c}_2' c_2 u_2^{+'} - \bar{c}_2' c_2 u_2^{+}$$

$$- 2\bar{c}_2 c_2' u_2^{+''} + 2\bar{c}_2 c_2 u_2^{+''} - 2\bar{c}_2 c_2' u_2^{+'} \right) ,$$

(C.20)
\[ T_2^{(4)} = \frac{-1}{12B^2\alpha'}(B^2\alpha'c^\perp - 24)\left(\bar{c}_2^2 c_2^\prime c_2^\prime + c_2^\prime c_2^\prime\right) \]

\[ + 2\bar{c}_2^2 c_2 + c_2^\prime c_2 + \bar{c}_2^\prime c_2^\prime \]  \hspace{1cm} (C.21)

The final step is to render the supercurrent in a complex universal form, in which the conjugate current \( \bar{G}_1 \) is given simply by

\[ \bar{G}_1 = \bar{b}_1 \] \hspace{1cm} (C.22)

This final transformation is not unitary. Instead, it comprises a similarity transformation \( S_2 \), generated by the function in Eqn. (6.22) above. We obtain

\[ G_1 = b_1 + \frac{\text{i} c^\perp}{6\sqrt{2}} c_1 u_1'' + 2\text{i} \frac{\sqrt{2}}{B\alpha'} c_1 u_1'' + \frac{\text{i} c^\perp}{3\sqrt{2}} c_1 u_1' + 2\text{i} \bar{c}_1 b_1' c_1 - 2\text{i} c_1 \bar{b}_1 + 4\text{i} c_1 b_1 c_1 - 2T_{\text{bose}}c_1 - 4 \frac{\text{i} c_1 c_1''}{\alpha'} (c_1 u_1'' + 2c_1 u_1' + 2c_1 u_1') + 2\text{i} \frac{\sqrt{2}}{B\alpha'} (c_1 u_1'' + 2c_1 u_1' + 2c_1 u_1') \]

\[ -2\text{i} c_1 \bar{b}_1 + 4\text{i} c_1 b_1 c_1 - 2T_{\text{bose}}c_1 \]

\[ -2\text{i} c_1' - \frac{c^\perp}{3c_1'} + \frac{8}{B^2\alpha'} c_1'' \]

\[ - \frac{i}{16B^2\alpha'}(24 + \alpha' B^2 (c_{\text{bose}} - c^\perp - 26)) \left( \bar{c}_1 c_1'' c_1' - \bar{c}_1 c_1'' c_1' - 4\text{i} c_1' c_1'' c_1 - 6\text{i} c_1' c_1'' c_1 \right) \]

\[ + \frac{7}{24B^2\alpha'}(24 + \alpha' B^2 (c_{\text{bose}} - c^\perp - 26)) \bar{c}_1 c_1'' c_1' c_1 \] \hspace{1cm} (C.23)

The conjugated supercurrent now appears as

\[ \bar{G}_1 = \bar{b}_1 - \frac{i}{48B^2\alpha'}(24 + \alpha' B^2 (c_{\text{bose}} - c^\perp - 26)) \left( 6\text{i} c_1' c_1'' c_1' - 3\text{i} c_1' c_1'' c_1' - \bar{c}_1 c_1' \right) \]

\[ + \frac{1}{12B^2\alpha'}(24 + \alpha' B^2 (c_{\text{bose}} - c^\perp - 26)) \bar{c}_1 c_1'' c_1' c_1 \] \hspace{1cm} (C.24)

Decomposed by antighost number, the R-current takes the form

\[ J_1^{(0)} = \frac{u_1''}{6\sqrt{2}} + c_1 \bar{b}_1 - \bar{c}_1 b_1 - \frac{2\sqrt{2}}{B\alpha'} \left( u_1'' + u_1' \right) \]

\[ J_1^{(2)} = \frac{1}{8B^2\alpha'}(24 + \alpha' B^2 (c_{\text{bose}} - c^\perp - 26)) \bar{c}_1 c_1'' \]

\[ J_1^{(4)} = -\frac{i}{4B^2\alpha'}(24 + \alpha' B^2 (c_{\text{bose}} - c^\perp - 26)) \left( \bar{c}_1 c_1'' c_1' + \bar{c}_1 c_1'' c_1' \right) \] \hspace{1cm} (C.25)
Similarly, the stress tensor becomes

\[ T^{(0)}_1 = -\frac{i}{2} c_1 \bar{b}_1 - \frac{i}{2} \bar{c}_1 b_1' - \frac{3i}{2} c'_1 \bar{b}_1 - \frac{3i}{2} \bar{c}'_1 b_1 + u_1^{+'} u_1^{-'} \frac{2}{\alpha'} , \quad (C.26) \]

\[ T^{(2)}_1 = -\frac{i}{48 B^2 \alpha'} (24 + \alpha' B^2 (c_{\text{bose}} - c^\perp - 26)) \times \left( 3 \bar{c}'_1 c'''_1 + 9 \bar{c}'_1 c''_1 + 12 \bar{c}''_1 c'_1 + 4 \bar{c}'''_1 c_1 \right) , \quad (C.27) \]

\[ T^{(4)}_1 = -\frac{1}{48 B^2 \alpha'} (24 + \alpha' B^2 (c_{\text{bose}} - c^\perp - 26)) \left( 9 \bar{c}'_1 c''_1 c_1 + 7 \bar{c}'_1 \bar{c}''_1 c_1 + 5 \bar{c}''_1 \bar{c}'_1 c_1 \right) . \quad (C.28) \]

As noted above, the final expressions simplify greatly upon assigning the values

\[ c_{\text{bose}} = 26 , \quad c^\perp = \frac{24}{B^2 \alpha'} . \quad (C.29) \]

We obtain the final supercurrent in the form

\[ G_1 = b_1 + 2i \left( \bar{c}_1 b_1 c'_1 + \bar{c}_1 b'_1 c_1 - c'_1 c_1 \bar{b}_1 + 2 \bar{c}'_1 b_1 c_1 \right) - 2T_{\text{bose}} b_1 + c_1 \]

\[ -\frac{4}{\alpha'} c_1 u_1^{+'} u_1^{-'} + \frac{2i \sqrt{2}}{B \alpha'} \left( c_1 u_1^{'''} + 2c'_1 u_1^{-} \right) - 2c''_1 , \]

\[ \bar{G}_1 = \bar{b}_1 . \quad (C.30) \]

The R-current and stress tensor take the final forms

\[ J = -\frac{2 \sqrt{2}}{B \alpha'} u_1^{-'} + c_1 \bar{b}_1 - \bar{c}_1 b_1 , \]

\[ T = T_{\text{bose}} + \frac{2}{\alpha'} u_1^{+'} u_1^{-'} - \left( \frac{i}{2} \bar{c}_1 \bar{b}_1' + \frac{3i}{2} c'_1 \bar{b}_1' + \text{h.c.} \right) . \quad (C.31) \]

### D Explicit similarity transformation of the BRST current

Here we present the explicit canonical variable redefinition that renders the BRST current in the universal form presented in Eqn. (7.4). As described above, because of the presence
of a number of currents that commute with the BRST current, there is a certain amount of freedom to choose a generating function that yields Eqn. (7.4). The following choice has the property that it admits particularly simple transformation rules for the reparametrization $c$ ghosts, and the shifted-spin $c_1$ antighosts:

$$g_0 = 2i\hat{c}_1 c_1'' - \sqrt{2}b \gamma c_1 + \frac{3}{\sqrt{2}}c \beta \hat{c}_1' - \frac{3}{\sqrt{2}}c \beta \hat{c}_1' - i\sqrt{2}\hat{c} \dot{b} v^{+'}$$

$$+ \frac{1}{\sqrt{2}}c \beta' \hat{c}_1 - \frac{1}{\sqrt{2}}c \beta' c_1 + 2i\sqrt{2}b \gamma \hat{c}_1' - i\sqrt{2}\hat{b} \gamma' c_1 + 3i\sqrt{2}\hat{b} \gamma' c_1$$

$$- i\sqrt{2}\hat{c} \beta \hat{c}_1 - i\sqrt{2}\hat{c} \beta c_1 - \frac{4\sqrt{2}}{\alpha'}\hat{c}_1 c_1 v^{-''} + \frac{1}{3\sqrt{2}}\hat{c}_1 c_1 v^{-''} - \frac{4\sqrt{2}}{\alpha'}\hat{c}_1 c_1 v^{-''}$$

$$- \frac{1}{3\sqrt{2}}\hat{c}_1 c_1 v^{+''} - \frac{1}{2}b \hat{c} c_1 c_1 + \frac{1}{2}b \hat{c} c_1 c_1 - i\sqrt{2}\hat{c} \beta \hat{c}_1 v^{+''}$$

$$- i\sqrt{2}\hat{c} \beta c_1 v^{+''} + ic \hat{b} \beta c_1 + ic \hat{b} \beta c_1 - i\sqrt{2}\hat{c} \beta \hat{c}_1 c_1 - i\sqrt{2}\hat{c} \beta \hat{c}_1 c_1$$

$$+ \frac{1}{2}b \hat{c} \hat{c}_1 c_1 + \frac{1}{2}b \hat{c} \hat{c}_1 c_1 + \frac{1}{2}b \hat{c} \hat{c}_1 c_1 - \frac{1}{2}b \hat{c} \hat{c}_1 c_1 + 2\hat{c}_{1'} c_1 b_1 c_1$$

$$+ \frac{5}{6}c_1' c_1 c_1'' c_1 + \frac{3}{\sqrt{2}}b \hat{c} c_1 c_1 v^{+''} - \frac{3}{\sqrt{2}}b \gamma \hat{c}_1' c_1 c_1 + \frac{3}{\sqrt{2}}c \beta \hat{c}_1' c_1 c_1 + \frac{1}{3\sqrt{2}}c \beta \hat{c}_1' c_1 c_1 + \frac{1}{3\sqrt{2}}c \beta \hat{c}_1' c_1 c_1$$

$$+ \frac{1}{3\sqrt{2}}c \beta \hat{c}_1' c_1 c_1 - \frac{1}{3\sqrt{2}}c \beta \hat{c}_1' c_1 c_1 v^{+''} + \frac{1}{3\sqrt{2}}c \beta \hat{c}_1' c_1 c_1 v^{+''} - \frac{5}{3\sqrt{2}}c \beta \hat{c}_1' c_1 c_1$$

$$+ \frac{1}{3\sqrt{2}}c \beta \hat{c}_1' c_1 c_1 - \frac{1}{3\sqrt{2}}c \beta \hat{c}_1' c_1 c_1 v^{+''} + \frac{1}{3\sqrt{2}}c \beta \hat{c}_1' c_1 c_1 v^{+''} - \frac{5\sqrt{2}}{3}\gamma \hat{c}_1' c_1 c_1$$

$$+ \sqrt{2}b \gamma c_1'' c_1 + \frac{4\sqrt{2}}{3}\hat{b} \gamma c_1 c_1 c_1' c_1 - \frac{1}{3\sqrt{2}}\hat{b} \gamma c_1 c_1 v^{+''} + \frac{1}{3\sqrt{2}}b \gamma c_1 c_1 v^{+''}$$

$$- \frac{4\sqrt{2}}{3}\hat{b} \gamma c_1 c_1 c_1' c_1 - \frac{\sqrt{2}}{3}\hat{b} \gamma c_1 c_1 c_1' c_1 + \frac{3}{\sqrt{2}}c \beta \hat{c}_1 c_1 c_1 + \frac{4\sqrt{2}}{\alpha'}c \beta \hat{c}_1 c_1 c_1 v^{+''}$$

$$+ \frac{1}{5\sqrt{2}}c_1 c_1 c_1 c_1 c_1 v^{+''} - \frac{1}{6}c \beta \hat{c}_1 c_1 c_1 c_1 v^{+''} + \frac{1}{3}c \beta \hat{c}_1 c_1 c_1 c_1 v^{+''} + \frac{1}{6}c \beta \hat{c}_1 c_1 c_1 v^{+''}$$

$$- \frac{1}{3}c \beta \hat{c}_1 c_1 c_1 c_1 - \frac{1}{3}c \beta \hat{c}_1 c_1 c_1 c_1 - \frac{1}{3}c \beta \hat{c}_1 c_1 c_1 c_1 - \frac{2}{3}\beta \gamma c_1 c_1 c_1 c_1$$

$$+ \frac{2}{3}\beta \gamma c_1 c_1 c_1 c_1 + \frac{1}{3}\hat{b} \gamma c_1 c_1 c_1 c_1 v^{+''} - \frac{1}{3}\hat{b} \gamma c_1 c_1 c_1 c_1 v^{+''} + \frac{1}{3}\hat{b} \gamma c_1 c_1 c_1 c_1 v^{+''} + \frac{1}{5\sqrt{2}}c \beta \hat{c}_1 c_1 c_1 c_1 c_1 v^{+''}.$$  

(D.1)

When this generating function is promoted to the finite similarity transformation $S$, we
obtain the following explicit transformation rules for each of the fields of interest:

\[
b \rightarrow b - \frac{3i}{\sqrt{2}} \beta \bar{c}'_1 + \frac{3i}{\sqrt{2}} \beta \bar{c}'_1 - \sqrt{2} b v'' - \frac{i}{\sqrt{2}} \beta' \bar{c}''_1 + \frac{i}{\sqrt{2}} \beta' \bar{c}''_1 - 2 \bar{b} \bar{c}''_1 - 4 \bar{b} \bar{c}''_1 c'_1 - 2 \bar{b} \bar{c}''_1 c'_1 ;
\]

(D.2)

\[
c \rightarrow c + i \sqrt{2} \gamma c_1 + (2i) c c'_1 c_1 + \bar{c} \bar{c}_1 c_1 - ic' \bar{c}_1 c_1 + \sqrt{2} c \bar{c}_1 c_1 v'' ;
\]

(D.3)

\[
b_1 \rightarrow b_1 - i \sqrt{2} c \beta' + 2 \sqrt{2} b' \gamma' + \frac{1}{\sqrt{2}} \hat{c} \beta - \frac{3i}{\sqrt{2}} c' \beta + 2 \sqrt{2} b' \gamma + c' \beta v'' + 4 \bar{b} \bar{c}''_1
\]

+8c \hat{b}' c'_1 + 4c \hat{b}'' c_1 - 2 \beta \gamma \hat{c}' c_1 - 2 \beta \gamma \hat{c}' c_1 - 2 \beta \gamma' \hat{c}_1 - 2 \beta \gamma' \hat{c}_1 - 2 \bar{b} \bar{c}' c_1

+2i \hat{c} b' c'_1 + 2i \hat{c} b'' c_1 + 2i \hat{c} b'' c_1 c_1 + 6c' \hat{b} c'_1 + 6c' b c_1 - 2 \beta' \gamma \hat{c}_1

-2 \beta' \gamma \hat{c}_1 - 2 \bar{b} \bar{c}' c_1 + 4ic' c_{1}' c_1 c_1 - 2 \sqrt{2} c \beta \hat{c}_1 c_1 - 4 \sqrt{2} c \beta \hat{c}_1 c_1

-2 \sqrt{2} c \beta \hat{c}' c_1 + 2i \sqrt{2} \hat{b} c'_1 v'' - 4 \sqrt{2} c \beta \hat{c}' c_1 - 6 \sqrt{2} c \beta \hat{c}' c_1 + 2i \sqrt{2} \hat{b} c'_1 v''

-2 \sqrt{2} c \beta \hat{c}' c_1 + 4i \sqrt{2} \hat{b} \gamma \hat{c}' c_1 + 4i \sqrt{2} \hat{b} \gamma \hat{c}_1 c_1 - 4i \sqrt{2} \hat{b} \gamma' \hat{c}' c_1

-4i \sqrt{2} \hat{b} \gamma' \hat{c}' c_1 c_1 - 4i \sqrt{2} \hat{b} \gamma'' \hat{c}_1 c_1 - i \sqrt{2} \hat{b} \gamma \hat{c}_1 c_1 - 2i \sqrt{2} \hat{b} \gamma \hat{c}_1 c_1 - i \sqrt{2} \hat{b} \gamma \hat{c}_1 c_1

-3 \sqrt{2} c' \beta \hat{c}' c_1 - 6 \sqrt{2} c' \beta \hat{c}' c_1 c_1 - 3 \sqrt{2} c' \beta' \hat{c}_1 c_1 + 4i \sqrt{2} \hat{b} \gamma \hat{c}' c_1 + 4i \sqrt{2} \hat{b} \gamma \hat{c}_1 c_1

-4i \sqrt{2} \hat{b} \gamma \hat{c}' c_1 c_1 - 4i \sqrt{2} \hat{b} \gamma' \hat{c}_1 c_1 - 2i \hat{c} \beta \hat{c}' c_1 c_1 v'' - 4i \hat{c} \beta \hat{c}' c_1 c_1 v'' + 8 \hat{b} \bar{c} c''_1 c_1

+24i \hat{b} \bar{c}''_1 c_1 c_1 + 8i \hat{b} \bar{c}''_1 c_1 c_1 - 2i \hat{c} \beta \hat{c}' c_1 c_1 v'' + 16i \hat{b} \bar{c}''_1 c_1 c_1 + 32 \hat{b} \bar{c}''_1 c_1 c_1

+8i \hat{b} c''_1 c_1 c_1 - 4i \hat{c} \gamma c''_1 c_1 c_1 - 4i \hat{c} \gamma c''_1 c_1 c_1 + 4 \bar{b} \bar{c} c''_1 c_1 c_1

+12i \hat{c} \bar{b} c''_1 c_1 c_1 + 24i \hat{c} \bar{b} c''_1 c_1 c_1 + 12i \hat{c} \bar{b} c''_1 c_1 c_1 - 4i \hat{c} \gamma \gamma c''_1 c_1 c_1 + 4 \bar{b} \bar{c} c''_1 c_1 c_1

-4 \hat{c} \bar{b} c''_1 c_1 c_1 - 4 \hat{c} \bar{b} c''_1 c_1 c_1 + 4i \sqrt{2} \hat{b} \hat{c} c''_1 c_1 c_1 - 4 \sqrt{2} \hat{b} \hat{c} c''_1 c_1 c_1 v''

-8 \sqrt{2} c \hat{b} c''_1 c_1 c_1 v'' + 8i \sqrt{2} c \beta' \hat{c}_1 c_1 c_1 c_1 - 4 \sqrt{2} \hat{b} \hat{c} c''_1 c_1 c_1 v'' + 8 \sqrt{2} \hat{b} \hat{c} c''_1 c_1 c_1

-2 \sqrt{2} c \beta \hat{c}_1 c_1 c_1 c_1 + 6i \sqrt{2} c \beta \hat{c}_1 c_1 c_1 c_1 + 8 \sqrt{2} \hat{b} \gamma \hat{c}' c_1 c_1 c_1 - 4 \beta \hat{c}_1 c_1 c_1 c_1 v''

-16 \hat{b} \bar{c} c''_1 c_1 c_1 c_1 - \frac{16 \sqrt{2}}{\alpha'} c c''_1 c_1 c_1 - \frac{8 \sqrt{2}}{\alpha'} \bar{c}_1 c_1 c_1 c_1 v'' - \frac{8 \sqrt{2}}{\alpha'} \bar{c}_1 c_1 c_1 c_1 v''

+ c_1 v'' \frac{4i \sqrt{2}}{\alpha'} + c_1 v'' \frac{4i \sqrt{2}}{\alpha'} + 8i \hat{b} \bar{c} c''_1 c_1 c_1 + c_{1}'' ;
\]

(D.4)
\[\bar{b}_1 \rightarrow \bar{b}_1 - i\sqrt{2}b \bar{\gamma} + i\sqrt{2}c \beta' - 4\sqrt{2}b \bar{\gamma}' + \frac{1}{\sqrt{2}} \hat{c} \beta + i\sqrt{2}c_1 v'^{''} + \frac{3i}{\sqrt{2}} \hat{c}' \bar{\beta} - \sqrt{2}b' \bar{\gamma} \]
\[-2ibc c_1' - b \hat{c} c_1 + icb c_1' + c \beta v'^{ '+} + 8c c_1 c_1' - 4\beta \bar{c} c_1 + 4\beta \bar{c}' c_1
\[+2\beta \bar{c} c_1 + 2ib \bar{\gamma} v'^{ '+} + 4ib \hat{c} c_1' + 2ib \hat{c} c_1 - 8c' \hat{b} c_1' + c \beta' \bar{c} c_1 + 2\bar{\gamma}' \beta c_1 + ib' \hat{c} c_1
\][+2i\hat{c}_1 \bar{c}_1 b_1 - 2i\hat{c}_1 \bar{c}_1 c_1'' - 2c'' \hat{b} \bar{c}_1 + 2ic\hat{c}_1 c_1' - 4ic\hat{c}_1 c_1' + \frac{2i}{3} \hat{c}_1 \bar{c}_1 c_1 - \sqrt{2}b c \bar{c}_1 v''
\[+2\sqrt{2}b \bar{\gamma} c_1' c_1 + 4\sqrt{2}c \beta c_1' c_1 + 2\sqrt{2}c \beta' c_1 c_1 + 2\sqrt{2}c \beta'' c_1 c_1 - 2i\sqrt{2}b \bar{c}_1 v''' + 6i\sqrt{2}c \bar{b}' c_1 v' + 8i\sqrt{2}b \bar{\gamma} c_1' c_1 + \sqrt{2}b \bar{c}_1 v''
\][+4i\sqrt{2}b \bar{\gamma}' \bar{c}_1 c_1 - 8i\sqrt{2}b \bar{\gamma}' \bar{c}_1 c_1 - 8i\sqrt{2}b \bar{\gamma}' \bar{c}_1 c_1 - 3i\sqrt{2}c \beta c_1 c_1 + 3\sqrt{2}c \beta' c_1 c_1
\[+i\sqrt{2}c' b \bar{c}_1 v'' + 4i\sqrt{2}b' \bar{\gamma} \bar{c}_1 c_1 + 6i\sqrt{2}b' \bar{\gamma} \bar{c}_1 c_1 - 8i\sqrt{2}b' \bar{\gamma}' \bar{c}_1 c_1
\[+2\sqrt{2}c' \bar{c}_1 c_1 v'' + 2ib \hat{c} c_1' c_1 + 2b \hat{c} c_1' c_1 - 6ic \beta \bar{c}_1 c_1 v'' + 2c \bar{c}_1 v'' v'''
\[+8ic \hat{b} c_1 c_1 + 16ic \hat{b} c_1 c_1' + 16ic \hat{b} c_1' c_1 + 16ic \hat{b} c_1' c_1 + 16ic \hat{b} c_1' c_1 + 16ic \hat{b} c_1' c_1
\[+8ic \hat{b} c_1' c_1 + 4i\beta \bar{c} \bar{c}_1 c_1' - 4i\beta \bar{c} c_1' c_1 - 4\beta \bar{c} c_1 c_1 - 4ib \bar{c}_1 c_1 + 12c \bar{c}_1 c_1 + 14ic \bar{b} c_1 c_1 + 4i\beta' \bar{c}_1 c_1 + 10c \bar{c}_1 c_1 c_1
\[+4ic \bar{b} c_1 c_1 c_1 + 4ic' b \bar{c}_1 c_1 c_1 + 2i\sqrt{2}b c \bar{c}_1 c_1 c_1 v'' + 4i\sqrt{2}c \beta c_1' c_1 c_1 c_1
\[+4\sqrt{2}b c \bar{c}_1 c_1 c_1 v'' + 12\sqrt{2}b c \bar{c}_1 c_1 c_1 v'' + 10\sqrt{2}b c' \bar{c}_1 c_1 c_1 v''
\[+8\sqrt{2}b \bar{\gamma} c_1 c_1 c_1 - 2i\sqrt{2}b \bar{c} \bar{c}_1 c_1 c_1 v'' + 8\sqrt{2}b \bar{\gamma}' c_1 c_1 c_1
\[+2\sqrt{2}b \bar{c}_1 c_1 c_1 v'' + 8\sqrt{2}b' \bar{\gamma} c_1 c_1 c_1 + 4ic \hat{b} c_1 c_1 c_1 v'' v'' + 16c \hat{b} c_1 c_1 c_1 c_1
\[+c' \bar{c}_1 c_1 v'' - \frac{8\sqrt{2}}{\alpha'} + c' \bar{c}_1 c_1 v'' - \frac{8\sqrt{2}}{\alpha'} + c' \bar{c}_1 c_1 v'' - \frac{8\sqrt{2}}{\alpha'} - 3c''^*; \quad (D.5)
\]
\[\bar{e}_1 \rightarrow \bar{e}_1 - 2ic\bar{c}_1 c_1; \quad (D.6)
\]
\[\gamma \rightarrow -\sqrt{2}c c_1' - \frac{i}{\sqrt{2}} \hat{c} c_1 + \frac{1}{\sqrt{2}} \hat{c} c_1' - ic \bar{c}_1 v'^{ '+} - 2i\gamma c_1 c_1 + 2i\gamma c_1 c_1
\[+\sqrt{2} \hat{c} c_1 c_1 + \left(-i\sqrt{2}\right) \hat{c} c_1 c_1' + 2c \hat{c} c_1 c_1' v''; \quad (D.7)
\]
\[\bar{\gamma} \rightarrow \bar{\gamma} + \left(\sqrt{2}c c_1' - \frac{i}{\sqrt{2}} \hat{c} c_1 - \frac{1}{\sqrt{2}} \hat{c} c_1' - ic \bar{c}_1 v'^{ '+} - \sqrt{2}c \bar{c}_1 c_1 c_1\right)
\[+i\sqrt{2}c \bar{c}_1 c_1 c_1 + 2c \bar{c}_1 c_1 c_1 v''; \quad (D.8)
\]
\[ \beta \rightarrow \beta + \sqrt{2}b c_1 + 4i\sqrt{2}b' c'_1 + 3i\sqrt{2}b' c'_1 - 4i\beta \bar{c}_1 c'_1 - 8i\beta \bar{c}_1 c_1 \]

\[ + 2i\beta \bar{c}_1 c_1 - 2\hat{b} c_1 v^{*'} - 4i\beta' \bar{c}_1 c_1 - 8\sqrt{2}b \bar{c}_1 c''_1 c_1 - 16\sqrt{2}b' \bar{c}_1 c'_1 c_1 \]

\[ - 8\sqrt{2}b' \bar{c}_1 c'_1 c_1 - 8\beta \bar{c}_1 c'_1 c_1 \; ; \quad (D.9) \]

\[ \bar{\beta} \rightarrow \bar{\beta} - 2i\sqrt{2}b \bar{c}_1 c'_1 - 2i\beta \bar{c}_1 \bar{c}_1 + 4\sqrt{2}b' \bar{c}_1 c'_1 c_1 + 4\sqrt{2}b' \bar{c}_1 c'_1 c_1 \; ; \quad (D.10) \]

\[ v^+ \rightarrow v^+ + 2\sqrt{2}\bar{c}_1 c_1 \; ; \quad (D.11) \]

\[ v^- \rightarrow v^- - \frac{i\alpha'}{\sqrt{2}}\bar{c} \hat{b} \; ; \quad (D.12) \]

\[ \hat{b} \rightarrow \hat{b} - \frac{1}{\sqrt{2}}\beta \bar{c}_1 c_1 - \frac{1}{\sqrt{2}}\bar{\beta} c_1 + 2i \left( \hat{b} \bar{c}_1 c'_1 + \hat{b} \bar{c}_1 c_1 + \hat{b}' \bar{c}_1 c_1 \right) \; ; \quad (D.13) \]

\[ \hat{c} \rightarrow \hat{c} + \sqrt{2}c v^{*'} + 2\sqrt{2}\gamma \bar{c}_1 c'_1 - \sqrt{2}\gamma c'_1 c_1 + 3\sqrt{2}\gamma' c_1 + 2c \bar{c}_1 c'_1 + 6c \bar{c}_1 c_1 + i\bar{c} \bar{c}_1 c'_1 \]

\[ - i\bar{c} c'_1 c_1 + \bar{c} c'_1 c_1 + c' \bar{c}_1 c_1 - i\bar{c} \bar{c}_1 c_1 - c'' c_1 c_1 - i\sqrt{2}c \bar{c}_1 c_1 v^{*''} + i\sqrt{2}c \bar{c}_1 c'_1 v^{*'} \]

\[ - i\sqrt{2}c' \bar{c}_1 c_1 v^{*'} + 4i\sqrt{2}\gamma \bar{c}_1 c'_1 c_1 - i\sqrt{2}c' \bar{c}_1 c_1 v^{*'} - 4\bar{c} \bar{c}_1 c'_1 c_1 \]

\[ + 4ic' \bar{c}_1 c'_1 c_1 - 4\sqrt{2}c' \bar{c}_1 c'_1 c_1 v^{*'} \; . \quad (D.14) \]

### E  Commuting currents

It turns out that there are a number of linearly independent currents that commute with the BRST current. Here, we present these currents explicitly after our final variable redefinition, in which the BRST current appears in the universal form in Eqn. (7.4). The existence of these currents implies that the procedure for finding the generating function \( g_0 \) leaves a number of free coefficients undetermined. In obtaining the final version of \( g_0 \) used to
generate the transformation in Eqn. 7.2, we can use this freedom to choose particularly simple transformations rules for a subset of the total field content. The currents appear as follows:

\[
J_1 = i\bar{\beta}\gamma + c_1 \bar{b}_1; \quad \text{(E.1)}
\]

\[
J_2 = \bar{c}_1 b_1 + i\beta \bar{\gamma}; \quad \text{(E.2)}
\]

\[
J_3 = ic\hat{b}\beta\gamma + c\hat{b}c_1 \bar{b}_1 - i\sqrt{2}\epsilon' c\hat{b}\beta c_1 - \frac{2}{\alpha'} c\beta c_1 v''; \quad \text{(E.3)}
\]

\[
J_4 = ic\beta \hat{b}\gamma \bar{c}_1 + ic\beta \gamma \bar{b}_1 + c\beta \bar{c}_1 c_1 \bar{b}_1
\]

\[
+ c\beta \bar{c}_1 b_1 c_1 - i\sqrt{2}\epsilon' c\beta \beta \bar{c}_1 c_1; \quad \text{(E.4)}
\]

\[
J_5 = \hat{b}\gamma \epsilon_1' + \hat{b}\gamma' \epsilon_1 - \frac{2}{\alpha'} \bar{c}_1 \epsilon_1' v''; \quad \text{(E.5)}
\]

\[
J_6 = \hat{b}\gamma \epsilon_1' + \hat{b}\gamma' \epsilon_1 + 2i \frac{\bar{\epsilon}_1 c_1 v''}{\alpha'} + \frac{2i}{\alpha'} \bar{c}_1 \epsilon_1' v''; \quad \text{(E.6)}
\]

\[
J_7 = \hat{b}\gamma \epsilon_1' - \hat{b}\gamma' \epsilon_1 + \hat{b}' \gamma \bar{c}_1 + \bar{c}_1 \epsilon_1' v'' - \frac{2i}{\alpha'}; \quad \text{(E.7)}
\]

\[
J_8 = -\hat{b}\gamma \epsilon_1' + \hat{b}\gamma' \epsilon_1 + \hat{b}' \gamma c_1 - \frac{2i}{\alpha'} \bar{c}_1 c_1 v'' - \frac{2i}{\alpha'} \bar{c}_1 \epsilon_1' v''; \quad \text{(E.8)}
\]

\[
J_9 = \hat{b}\gamma \epsilon_1 v'' + \hat{b}\gamma c_1 v'' - \frac{2i}{\alpha'} \bar{c}_1 c_1 v'' v''; \quad \text{(E.9)}
\]

\[
J_{10} = \hat{b}\gamma \bar{c}_1 v''' + \hat{b}\gamma c_1 v''' + \hat{b}' \bar{c}_1 c_1 - \frac{2i}{\alpha'} \bar{c}_1 c_1 v'' v''; \quad \text{(E.10)}
\]

\[
J_{11} = -i\dot{b}\gamma \bar{c}_1 b_1 + \dot{b}\beta \gamma \bar{c}_1 + \dot{b}\gamma \beta \gamma c_1 - \frac{2i}{\alpha'} \bar{c}_1 c_1 v''; \quad \text{(E.11)}
\]

\[
J_{12} = i\dot{b}\beta \gamma \gamma \bar{c}_1 + i\dot{b}\beta \gamma \gamma c_1 + \dot{b}\gamma \bar{c}_1 b_1 c_1 + \frac{2\beta \bar{c}_1 c_1 v''}{\alpha'}; \quad \text{(E.12)}
\]
\[ \mathcal{J}_{13} = \frac{1}{\sqrt{2}} \hat{c} \hat{b} \beta \gamma - \frac{i}{\sqrt{2}} \hat{b} \bar{c}_1 b_1 + \hat{c}' c \hat{b} \beta \bar{c}_1 - \frac{i \sqrt{2}}{\alpha'} c \beta \bar{c}_1 v'' ; \] (E.13)

\[ \mathcal{J}_{14} = -\bar{c}_1 c'' + \bar{c}_1' c_1 ; \] (E.14)

\[ \mathcal{J}_{15} = -\bar{\beta} \gamma \bar{c}_1 c_1' + \bar{\beta} \gamma \bar{c}_1 c_1 + \gamma \bar{\beta} c_1' c_1 + i e_1 c_1' c_1 \bar{b}_1 ; \] (E.15)

\[ \mathcal{J}_{16} = c_1' \bar{c}_1 b_1 c_1 + i \beta \gamma c_1' \bar{c}_1 + i \beta \bar{\gamma} c_1' c_1 - i \beta \bar{\gamma} c_1' c_1 ; \] (E.16)

\[ \mathcal{J}_{17} = c \hat{b}' \hat{b} \gamma \bar{c}_1 + c \hat{b}' \hat{b} \gamma c_1 - \sqrt{2} \hat{c} c \hat{b}' \hat{b} \bar{c}_1 c_1 \]
\[ -\frac{2i}{\alpha'} c \hat{b}' \bar{c}_1 c_1 v'' + \frac{2i}{\alpha'} c \hat{b} \bar{c}_1 c_1 v'' ; \] (E.17)

\[ \mathcal{J}_{18} = -c \hat{b} \beta \gamma \bar{c}_1 c_1' + c \hat{b} \beta \gamma \bar{c}_1 c_1 + c \hat{b} \gamma \beta c_1' c_1 + i c \hat{b} \bar{c}_1 c_1' c_1 \bar{b}_1 \]
\[ + \sqrt{2} \hat{c} c \hat{b} \beta \bar{c}_1 c_1' c_1 - \frac{2i}{\alpha'} c \hat{b} \bar{c}_1 c_1' c_1 v'' ; \] (E.18)

\[ \mathcal{J}_{19} = i c \hat{b} \beta \gamma \bar{c}_1 c_1' + i c \hat{b} \beta \gamma \bar{c}_1 c_1 - i c \hat{b} \beta \gamma \bar{c}_1 c_1 + c \hat{b} \bar{c}_1 c_1' c_1 \bar{b}_1 \]
\[ - i \sqrt{2} \hat{c} c \hat{b} \beta \bar{c}_1 c_1' c_1 - \frac{2}{\alpha'} c \hat{b} \beta \bar{c}_1 c_1 v'' ; \] (E.19)

\[ \mathcal{J}_{20} = \hat{b} \gamma \bar{c}_1 c_1' c_1' - \hat{b} \gamma \bar{c}_1 c_1' c_1 - \hat{b} \gamma \bar{c}_1 c_1' c_1 \]
\[ + \hat{b} \gamma \bar{c}_1 c_1' c_1 - \frac{2i}{\alpha'} \bar{c}_1 c_1' c_1 v'' ; \] (E.20)

\[ \mathcal{J}_{21} = -\hat{b} \gamma c_1' c_1' + \hat{b} \gamma c_1' c_1 + 2 \hat{b} \gamma c_1' c_1 + \hat{b} \gamma c_1' c_1 \]
\[ + \hat{b} \gamma \bar{c}_1 c_1' c_1 + \frac{2i}{\alpha'} \bar{c}_1 c_1' c_1 v'' . \] (E.21)
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