Quadrics and normal generation of line bundles on multiple coverings

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\textbf{ABSTRACT}

In this work, a \textit{minimally FIIQ} divisor on a smooth curve $X$ in $\mathbb{P}^d$ is defined by an effective divisor which fails to impose independent conditions on quadrics in $\mathbb{P}^d$ and all of whose proper subdivisors impose independent conditions on quadrics. Using this notion, we explicitly describe very ample non-special line bundles which fail to be normally generated on a simple $k$-fold covering $\phi : X \to Y$ in terms of line bundles on $Y$ under some numerical constraints. Furthermore, in case $g_Y = 1, 2$ we obtain a concrete classification of such line bundles on $X$.

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\section{1. Introduction}

Let $X$ be a smooth irreducible algebraic curve over an algebraically closed field of characteristic zero. A line bundle $\mathcal{L}$ on $X$ is said to be normally generated if it is very ample and $\varphi_{\mathcal{L}}(X) \cong \mathbb{P}H^0(X, \mathcal{L})$, i.e., $h^1(I_X(n)) = 0$ for every integer $n \geq 0$. It is a classical result that any line bundle $\mathcal{L}$ with $\deg \mathcal{L} \geq 2g + 1$ is normally generated. In 1986, Green and Lazarsfeld showed a breakthrough theorem that any very ample line bundle $\mathcal{L}$ with $\text{Cliff}(\mathcal{L}) < \text{Cliff}(X)$ is normally generated ([6, Theorem 1]). Here, $\text{Cliff}(\mathcal{L}) := \deg \mathcal{L} - 2h^0(X, \mathcal{L}) + 2$ and $\text{Cliff}(X) := \min \{ \text{Cliff}(\mathcal{L}) | h^0(X, \mathcal{L}) \geq 2, h^1(X, \mathcal{L}) \geq 2 \}$. These Clifford indices measure how special they are in their moduli spaces. Further, there exist very ample line bundles $\mathcal{L}$ on a smooth curve $X$ failing to be normally generated with $\text{Cliff}(\mathcal{L}) = \text{Cliff}(X)$ ([5, 6]).

We notice that [6, Theorem 1] follows from [6, Theorem 3] which determines the failure of the normal generation of a very ample line bundle $\mathcal{L}$ by the existence of an effective divisor $R$ on $X$ with $2\dim \langle R \rangle_{\mathcal{L}} + 2 \leq \deg R$ and $h^1(X, \mathcal{L}^2(-R)) = 0$ which fails to impose independent conditions in $\langle R \rangle_{\mathcal{L}}$, where $\langle R \rangle_{\mathcal{L}}$ is the space spanned by $R$. This theorem has played a big role in characterizing very ample line bundles which fail to be normally generated on smooth curves. In such works, it is important to analyze what properties $R$ or $\mathcal{L}(-R)$ may have ([2–4, 6, 8–10]).

The aim of the present work is to characterize very ample non-special line bundles on multiple coverings $\phi : X \to Y$ which fail to be normally generated by using simpler properties of line
bundles on $Y$ comparing to those on $X$. To prove our results, we need to clarify that some line bundles $\mathcal{M}$ on $X$ are composed with $\phi$, which means $\mathcal{M} = \phi^* \mathcal{N}$ for a line bundle $\mathcal{N}$ on $Y$ and $h^0(X, \mathcal{M}) = h^0(Y, \mathcal{N})$. It enables us to get an explicit description of a very ample line bundle $\mathcal{L}$ failing to be normally generated on $X$ in terms of line bundles on the base curve $Y$ by combined with the notion “minimally FIIQ” defined in what follows.

**Definition 1.1.** An effective divisor $R$ on a smooth curve $X \subset \mathbb{P}^r$ is said to be minimally FIIQ in $\mathbb{P}^r$ if $R$ fails to impose independent conditions on quadrics in $\mathbb{P}^r$ and any proper subdivisor $R'$ of $R$ imposes independent conditions on quadrics in $\mathbb{P}^r$.

We notice in the definition that $R$ is minimally FIIQ in $\mathbb{P}^r$ if and only if $R$ is in its spanned space $\Lambda$ since $h^1(\mathbb{P}^r, \mathcal{I}_{R/\mathbb{P}^r}(2)) = h^1(\Lambda, \mathcal{I}_{R/\Lambda}(2))$.

According to [6, Theorem 3] a minimally FIIQ divisor $R$ is associated to a very ample line bundle $\mathcal{L}$ failing to be normally generated. Further we prove that its minimally FIIQ property of $R$ on $X$ is passed to that property of a divisor on the base curve $Y$ when $g_Y = 1, 2$ (see Theorem A($a_2$) and Theorem B($b_1$)). In this regards, the composedness of line bundles is an essential ingredient in the present work (e.g. Figure 1). Concerning composedness, we have a result (Lemma 4.2 modified from [8, Lemma 5]) that a line bundle $\mathcal{M}$ on a simple $k$-fold covering is composed with $\phi$ if $\deg \mathcal{M} \leq g - 1$ and $\text{Cliff} \mathcal{M} < \frac{g+2}{2k+4} - 1$. Recall that $\phi : X \rightarrow Y$ is said to be simple if $\phi$ does not factor through a non-trivial morphism.

In this context, we investigate the existence of the minimally FIIQ divisor $R$ on $X$ with focusing on whether $R$ keeps the inequality $2 \dim \langle R \rangle_{\mathcal{L}} + 2 \leq \deg R$ in [6, Theorem 3], since it is equivalent to $\text{Cliff} \langle \mathcal{L}(-R) \rangle \leq \text{Cliff} \langle \mathcal{L} \rangle$ by the Riemann-Roch theorem (see Proposition 2.4). Through this work, we see that for a non-special line bundle $\mathcal{L}$ failing to be normally generated with $\text{Cliff} \mathcal{L} \leq \frac{g+2}{2k+4} - 1$ on a simple multiple covering $\phi : X \rightarrow Y$ there exists a minimally FIIQ divisor $R$ on $X$ such that the base point free part of $K_X \otimes \mathcal{L}^{-1}(R)$ is composed with $\phi$. The minimally FIIQ property and composedness give rise to the conclusion of Theorem A, in which the hypothesis $\deg \mathcal{L} \geq 2g + 1 - \frac{g}{2k+4}$ is equivalent to $\text{Cliff} \mathcal{L} \leq \frac{g+2}{2k+4} - 1$ and the condition ($a_2$) shows the keeping of the inequality $2 \dim \langle R \rangle_{\mathcal{L}} + 2 \leq \deg R$.

**Theorem A.** Assume that a smooth curve $X$ of genus $g$ admits a simple $k$-fold covering $\phi : X \rightarrow Y$ for a smooth irrational curve $Y$ of genus $\gamma$ with $k \geq 3$ and $g > k(2k - 1)\gamma$. Let $\mathcal{L}$ be a non-special very ample line bundle on $X$ with $\deg \mathcal{L} \geq 2g + 1 - \frac{g}{2k+4}$. Then $\mathcal{L}$ fails to be normally generated if and only if

$$\mathcal{L} \cong K_X - (\phi^* g^\phi_{\mathcal{L}} + B) + R$$

for some effective divisors $R, B$ on $X$ and a $g^\phi_{\mathcal{L}}$ with $n \geq 1$ on $Y$ satisfying

1. $\text{supp}(R \cap B) = \emptyset$, $\deg(R \cap \phi^*(\psi)) \leq 1$ and $\deg(B \cap \phi^*(\psi)) \leq k - 1$ for any $\psi \in Y$,
2. $R$ is minimally FIIQ in $\langle R \rangle_{\mathcal{L}}$ with $n + 3 \leq \deg R = n + \dim \langle R \rangle_{\mathcal{L}} + 2 \leq 2n + 2$,
3. $|\phi^* g^\phi_{\mathcal{L}} + B| = \phi^* g^\phi_{\mathcal{L}} + B$, i.e., $h^0(X, O_X((\phi^* D) + B)) = h^0(Y, O_Y(D))$ for $D \in g^\phi_{\mathcal{L}}$.

Let $\mathcal{L} \cong K_X - (\phi^* g^\phi_{\mathcal{L}} + B) + R$ for a minimally FIIQ divisor $R$ on $X$ as in Theorem A. If $\phi(R) := \sum n_i \phi(p_i)$ is also minimally FIIQ in $\mathbb{P}h^0(Y, D)$ for an appropriate line bundle $\mathcal{D}$ on $Y$, then $\phi(R)$ could be more precisely characterized comparing to $R$ since the base curve $Y$ is much simpler than the covering $X$. This enables us to get precise descriptions of such line bundles $\mathcal{L}$. Along this line, we work on a simple multiple covering $\phi : X \rightarrow Y$ of a smooth curve $Y$ of genus $\gamma = 1, 2$ under some constraints. We define $\mathcal{D} := O_Y(\phi(R) + K_Y - g^\phi_{\mathcal{L}})$ and show that $\phi_{\mathcal{L}}(R)$ and $\phi_{\mathcal{D}}(\phi(R))$ have the same geometric properties including the minimally FIIQ property in each corresponding spanned space (see Theorem B). Therefore we in Sec. 3 carry out an analysis of
minimally FIIQ divisors on smooth projective curves of genera 1, 2 before going to the proof of the Theorem B. We remark that the linear series \( g^n_{\ell} \) in Theorem A is non-special, i.e., \( g^n_{\ell} = g^n_{n+\gamma} \) for \( n \geq 2 \) and \( \gamma \leq 2 \) which are constraints of Theorem B.

**Theorem B.** Let \( X, Y \) and \( L \) be the same as in Theorem A with \( \gamma = 1, 2 \). Assume that

\[
L \cong K_X - (\phi^*_R g^n_{n+\gamma} + B) + R
\]

for some effective divisors \( R, B \) on \( X \) and a \( g^n_{n+\gamma} \) with \( n \geq 2 \) on \( Y \) satisfying (a1), (a2) and (a3) in Theorem A. Then for the line bundle \( D := O_Y(\phi(R) + K_Y - g^n_{n+\gamma}) \) on \( Y \) we have:

1. \( D \) is non-special with \( h^0(Y, D) = h^0(X, \phi^*D) = s + 1 \geq 3 \),
2. \( \varphi_L(R) \) and \( \varphi_D(\phi(R)) \) have the same geometric properties in each corresponding spanned space,
3. for \( s \geq \gamma + 1 \), \( D \) is very ample and \( \phi(R) \) is minimally FIIQ in \( \mathbb{P}H^0(Y, D) \),
4. \( 2n + 2 - \gamma \leq \deg R \leq 2n + 2 \)

where \( s := \dim(R)_L \).

As applications of Theorems A and B, we establish concrete classifications of very ample non-special line bundles \( L \) which fail to be normally generated on a simple \( k \)-fold covering \( \phi : X \to Y \) of genus \( \gamma = 1, 2 \) (see Theorem 5.1 for \( \gamma = 1 \) and Theorem 5.5, for \( \gamma = 2 \)). In addition, we deal with the very ampleness of non-special line bundles satisfying necessary conditions for the failure of normal generation in Theorems 5.1, 5.5 (see Propositions 5.3, 5.6).

In the present work, we study \( k \)-fold coverings with \( k \geq 3 \), since H. Lange and G. Martens [11] showed that a double covering of genus \( g \) of a smooth curve \( Y \) of genus \( \gamma \) admits no non-special normally generated line bundle of degree \( 2g + 1 - j \), where \( j > 3\gamma \) and such that \( g \geq j + 4 \) if \( \gamma \geq 2 \) resp. \( g \geq j + 3 \) if \( \gamma = 1 \).

**Notations and conventions**

1. If \( L \) is a very ample line bundle on \( X \), \( \varphi_L(X) \) is written as \( X \) if there is no confusing.
2. \( M(-g^n_{\ell}) := M(-D) \) for a line bundle \( M \) and a linear series \( g^n_{\ell} \) with \( D \in g^n_{\ell} \).
3. \( \phi(\sum n_i p_i) := \sum n_i \phi(p_i) \) for a multiple covering \( \phi : X \to Y \).
4. \( (R)_L \) : linear space spanned by an effective divisor \( R \) for a very ample line bundle \( L \) on \( X \).

For standard terminologies, we refer the readers to [1, 7].

**2. Quadrics and normal generation**

In this section we investigate a relation between the failure of the normal generation of a very ample line bundle \( L \) on a smooth curve \( X \) and the existence of minimally FIIQ divisor \( R \) on \( X \) with \( \text{Cliff}(L(-R)) \leq \text{Cliff}(L) \). This is based on the following theorem, whose statement is modified according to its proof for convenience sake.

**Theorem 2.1** ([6], Theorem 3). Let \( X \) be a smooth curve of genus \( g \) and \( L \) a very ample line bundle on \( X \) with \( \deg L \geq \frac{2g-3}{2} + \varepsilon \); \( \varepsilon = 0 \) if \( L \) is special, \( \varepsilon = 2 \) if \( L \) is non-special. Then, \( L \) fails to be normally generated if and only if there exist an integer \( 1 \leq s \leq h^0(X, L) - 3 \) and an effective divisor \( R \) on \( X \) with \( 2s + 2 \leq \deg R \leq \deg L - \frac{2}{2} \) such that \( R \) spans an \( s \)-plane \( \Lambda \) in \( \mathbb{P}H^0(X, L) \) in which \( R \) fails to impose independent conditions on quadrics.

**Remark 2.2.** The Riemann-Roch theorem tells that the condition \( \deg R \geq 2s + 2 \) in Theorem 2.1 is equivalent to \( \text{Cliff}(L(-R)) \leq \text{Cliff}(L) \).
According to [6, Theorem 3] we can associate a minimally FIIQ divisor $R$ associated to a very ample line bundle $\mathcal{L}$ failing be normally generated. As stated in the introduction, in order to pass the minimally FIIQ property of $R$ on a multiple covering $X$ to its base curve we focus on the keeping of the inequality $2\dim \langle R \rangle \mathcal{L} + 2 \leq \deg R$ in the way of finding a minimally FIIQ divisor from the divisor $R$. Thus we give the following lemma which implies the keeping of such an inequality.

**Lemma 2.3.** Let $\mathcal{L}$ be a very ample line bundle on a smooth curve $X$ of genus $g$. Assume that an effective divisor $R$ on $X$ fails to impose independent conditions on quadrics in the space $\langle R \rangle \mathcal{L}$ with $h^1(\mathcal{L}^2(-R)) = 0$. Then, there exists a sub-divisor $R'$ of $R$ and a non-trivial extension $0 \to \mathcal{K}_X \otimes \mathcal{L}^{-1} \to \mathcal{F} \to \mathcal{L} \to 0$ such that

(i) the extension is exact on global sections,
(ii) $\mathcal{L}(-R')$ is a subbundle of $\mathcal{F}$.

Further, $R'$ fails to impose independent conditions on quadrics in $\langle R' \rangle \mathcal{L}$.

**Proof.** Since $R$ fails to impose independent conditions on quadrics in $\langle R \rangle \mathcal{L}$, the map $Sym^2(H^0(\mathcal{L})) \to H^0(\mathcal{L}^2 \otimes \mathcal{O}_R)$ is not surjective by the exact sequence $0 \to I_R(2) \to \mathcal{O}_{\mathcal{F}}(2) \to \mathcal{O}_R(2) \to 0$. Following [6, p.79], we have the commutative diagram:

\[
\begin{array}{ccc}
0 & \to & 0 \\
\downarrow & & \downarrow \\
H^0(\mathcal{L}^2 \otimes \mathcal{O}_R)^* & \xrightarrow{\rho_R^*} & H^0(\mathcal{L})^* \otimes W_R^* \\
\downarrow & & \downarrow \\
H^0(\mathcal{L})^* & \xrightarrow{\rho'} & H^0(\mathcal{L})^* \otimes H^0(\mathcal{L})^* \\
\downarrow & & \downarrow \\
H^0(\mathcal{L}^2(-R))^* & \to & H^0(\mathcal{L})^* \otimes H^0(\mathcal{L}(-R))^* \\
\downarrow & & \downarrow \\
0 & \to & 0
\end{array}
\] (2.1)

where $W_R := H^0(\mathcal{L})/H^0(\mathcal{L}(-R))$ and $\rho_R : H^0(\mathcal{L}) \otimes W_R \to H^0(\mathcal{L}^2 \otimes \mathcal{O}_R)$ is given by the composition map:

$$H^0(\mathcal{L}) \otimes H^0(\mathcal{L}) \xrightarrow{\mu} H^0(\mathcal{L}^2) \xrightarrow{\text{evaluation}} H^0(\mathcal{L}^2 \otimes \mathcal{O}_R).$$

We notice that $\rho_R^*$ is not injective since the map $Sym^2(H^0(\mathcal{L})) \to H^0(\mathcal{L}^2 \otimes \mathcal{O}_R)$ is not surjective and its image is the same as that of $\rho_R$. Hence there exists a non-zero element $e \in \ker(\mu^*)$ which comes from a non-zero element $e' \in \ker(\rho_R^*)$. Thus $e$ maps to zero in $H^0(\mathcal{L}^2(-R))^*$ via the first vertical map in the diagram (2.1).

The non-zero element $e \in H^0(\mathcal{L}^2)^* \cong \text{Ext}^1(\mathcal{L}, \mathcal{K}_X \otimes \mathcal{L}^{-1})$ gives a non-trivial extension which splits on global sections:

$$0 \to \mathcal{K}_X \otimes \mathcal{L}^{-1} \to \mathcal{F} \to \mathcal{L} \to 0.$$

Since $e$ maps to zero in $H^0(\mathcal{L}^2(-R))^* \cong \text{Ext}^1(\mathcal{L}(-R), \mathcal{K}_X \otimes \mathcal{L}^{-1})$, we have the commutative diagram:

\[
\begin{array}{ccc}
\mathcal{L}(-R) & \xrightarrow{\epsilon} & 0 \\
\downarrow & & \downarrow \\
0 & \to & \mathcal{K}_X \otimes \mathcal{L}^{-1} \to \mathcal{F} \to \mathcal{L} \to 0.
\end{array}
\] (2.2)
Let $\tilde{R}$ be the vanishing locus of $\epsilon$. Assume that there is a point $p \in \tilde{R} - R$. Let $s$ be a section of $\mathcal{L}(-R)$ around $p$ such that $s(p) \neq 0$. However $\epsilon(s(p)) = 0$ which is contrary to the commutative diagram (2.2). Hence we have $\tilde{R} \leq R$. Then $R' := R - \tilde{R}$ is an effective sub-divisor of $R$ and $\mathcal{L}(-R')$ is a subbundle of $\mathcal{F}$. Thus we get the conclusion (i), (ii). Further, the divisor $R'$ fails to impose independent conditions on quadrics in $\langle R' \rangle_\mathcal{L}$ by [6, p.79] since $\mathcal{L}(-R')$ is a subbundle of $\mathcal{F}$. Therefore the proof of the lemma is completed. □

From Theorem 2.1 and Lemma 2.3 we obtain the following proposition.

**Proposition 2.4.** Let $X$ be a smooth curve of genus $g$ and $\mathcal{L}$ a very ample line bundle on $X$ with $\deg \mathcal{L} > \frac{3g-1}{2} + \varepsilon$; $\varepsilon = 0$ if $\mathcal{L}$ is special, $\varepsilon = 2$ if $\mathcal{L}$ is non-special. Then, $\mathcal{L}$ fails to be normally generated if and only if there is a minimally FIIQ divisor $R$ on $X$ such that $1 \leq s \leq h^0(X, \mathcal{L}) - 3$ and $2s + 2 \leq \deg R \leq \deg \mathcal{L} - \frac{g-1}{2}$, where $s := \dim(R)_{\mathcal{L}}$.

**Proof.** Assume that $\mathcal{L}$ is not normally generated. By Theorem 2.1 including its proof, there exist a non-trivial extension $\mathcal{F}$ of $\mathcal{L}$ by $K_\mathcal{L}^{-1}$ which is exact on global sections and a divisor $D \geq 0$ such that $D$ fails to impose independent conditions on quadrics in $\langle D \rangle_{\mathcal{L}}$ and $\mathcal{L}(-D)$ is a subbundle of $\mathcal{F}$. Further, $D$ satisfies $2\dim(D)_{\mathcal{L}} + 2 \leq \deg D \leq \deg \mathcal{L} - \frac{g-1}{2}$ and $1 \leq \dim(D)_{\mathcal{L}} \leq h^0(X, \mathcal{L}) - 3$.

For the divisor $D$ we define the following

$$S := \{(\tilde{\mathcal{F}}, \tilde{D}) \mid 0 \leq \tilde{D} \leq D \text{ and } \tilde{\mathcal{F}} \text{ is a non-trivial extension of } \mathcal{L} \text{ by } K_X \otimes \mathcal{L}^{-1}$$

which is exact on global sections and $\mathcal{L}(-\tilde{D})$ is a subbundle of $\tilde{\mathcal{F}}$.

Note that $S \neq \emptyset$. Let $(\tilde{\mathcal{F}}, \tilde{D})$ be an element of $S$. Since $\mathcal{L}(-\tilde{D})$ is a subbundle of $\mathcal{L}(-\tilde{D})$, the divisor $\tilde{D}$ fails to impose independent conditions on quadrics in $\langle \tilde{D} \rangle_{\mathcal{L}}$ by [6, p.79]. According to the conditions that the extension $\tilde{\mathcal{F}}$ is exact on global sections and $\mathcal{L}(-\tilde{D})$ is a subbundle of $\tilde{\mathcal{F}}$, we get $\text{Cliff}(\mathcal{L}(-\tilde{D})) \leq \text{Cliff}(\mathcal{L})$, which is equivalent to $\deg \tilde{D} \geq 2\dim(\tilde{D})_{\mathcal{L}} + 2$ by Remark 2.2.

Let $R$ be an effective divisor on $X$ such that $R$ has the minimal degree among all the effective divisors $D$ such that $(\tilde{\mathcal{F}}, \tilde{D}) \in S$. We claim that $R$ is a minimally FIIQ divisor on $X$. Assume that there exists a proper sub-divisor $R'$ of $R$ such that $R'$ fails to impose independent conditions on quadrics in $\langle R' \rangle_{\mathcal{L}}$. By Lemma 2.3, there exists a sub-divisor $R''$ of $R'$ and a non-trivial extension $\mathcal{F}'$ of $\mathcal{L}$ by $K_X \otimes \mathcal{L}^{-1}$ such that $(\mathcal{F}', R'') \in S$. This is contrary to the choice of $R$. Thus the divisor $R$ is minimally FIIQ. The converse follows from Theorem 2.1, which completes the proof of the proposition. □

### 3. Minimally FIIQ

This section is devoted to characterizing minimally FIIQ divisors on a smooth curve $Y \subset \mathbb{P}^s$. This in Sec. 5 enables us to obtain a concrete classification of non-normally generated line bundles on a simple multiple covering $X$ of the curve $Y$ in case $g_Y = 1, 2$.

**Lemma 3.1.** Let $Y \subset \mathbb{P}^s$ be a smooth curve embedded by a very ample line bundle $\mathcal{L}$. Assume that $h^i(I_{Y/P^s}(2)) = 0$ for $i = 1, 2$. An effective divisor $R$ on $Y$ is minimally FIIQ in $\mathbb{P}^s$ if and only if $h^0(K_Y - 2H + R) = 1$ and $h^0(K_Y - 2H + R - \varphi) = 0$ for any $\varphi \in \text{Supp}(R)$ where $H$ is a hyper-plane section of $Y$.

**Proof.** Consider the canonical exact sequence:

$$0 \rightarrow I_{Y/P^s}(2) \rightarrow I_{R/P^s}(2) \rightarrow I_{R/Y}(2) \rightarrow 0.$$  

Since $h^i(I_{Y/P^s}(2)) = 0$ for $i = 1, 2$, it follows that
\[ h^1(I_{R/P^2}(2)) = h^1(I_{R/Y}(2)) = h^1(O_Y(-R + 2H)) = h^0(K_Y - 2H + R). \]  \hspace{1cm} (3.1)

Hence an effective divisor \( R \) is minimally FIIQ in \( \mathbb{P}^s \) if and only if \( h^0(K_Y - 2H + R) = 1 \) and \( h^0(K_Y - 2H + R - \varphi) = 0 \) for any point \( \varphi \in \text{Supp}(R) \). \hfill \Box

**Proposition 3.2.** Let \( Y \) be a smooth curve of genus \( \gamma \geq 0 \) in \( \mathbb{P}^s \), \( s \geq 2 \) such that \( \text{deg}Y \geq \gamma \) and \( h^1(I_{Y/P^2}(j)) = 0 \) for \( j = 1, 2 \). An effective divisor \( R \) on \( Y \) is minimally FIIQ in \( \mathbb{P}^s \) if and only if \( R \) satisfies one of the following

(i) \[ 2s + 2 - 2\varepsilon \leq \text{deg}R < 2s + \gamma + 2 - 2\varepsilon; \]  
there is an effective divisor \( G \) and a quadric hypersurface \( Q \) such that \( R = (Y \cap Q) - G \), \( h^1(G) = 1 \) and \( h^1(G + \varphi) = 0 \) for any \( \varphi \in \text{Supp}(R) \). In particular, \( G \in |K_Y| \) for \( \text{deg}R = 2s + 2 - 2\varepsilon \).

(ii) \[ \text{deg}R = 2s + \gamma + 2 - 2\varepsilon; \]  
for any \( \varphi \in \text{Supp}(R) \) there is no quadric hypersurface \( Q \) satisfying \( R - \varphi \leq Y \cap Q \).

where \( \varepsilon := h^1(O_Y(1)) \).

**Proof.** Let \( Y \) be a rational normal curve in \( \mathbb{P}^s \), \( s \geq 1 \). An effective divisor \( R \) on \( Y \) is minimally FIIQ in \( \mathbb{P}^s \) if and only if \( \text{deg}R = 2s + 2 \). Hence it suffices to prove the proposition in case \( \gamma \geq 1 \).

\( (\Rightarrow) \) Assume that \( R \) is minimally FIIQ in \( \mathbb{P}^s \). We remark that \( h^2(I_{Y/P^2}(2)) = 0 \) for \( \text{deg}Y \geq \gamma \). Hence by Lemma 3.1, we have

\[ h^0(K_Y - 2H + R) = 1. \]

This forces that \( 0 \leq \text{deg}(K_Y - 2H + R) \leq \gamma \), which yields

\[ 2s + 2 - 2\varepsilon \leq \text{deg}R \leq 2s + \gamma + 2 - 2\varepsilon, \]

for \( \text{deg}Y = \text{deg}H = s + \gamma - h^1(O_Y(1)) \) and \( h^1(O_Y(1)) = \varepsilon \). On the one hand, the equation \( h^0(K_Y - 2H + R) = 1 \) means that there is an effective divisor \( \tilde{G} \in |K_Y - 2H + R| \).

Assume that \( \text{deg}R < 2s + \gamma + 2 - 2\varepsilon \). Then we get \( \text{deg} \tilde{G} \leq \gamma - 1 \), which combined with \( \tilde{G} \geq 0 \) implies that there is a canonical divisor \( K_Y \) such that \( G := K_Y - \tilde{G} \geq 0 \). This means \( R \in |O_Y(2)(-G)| \), whence \( R = (Y \cap Q) - G \) for some quadric hypersurface \( \mathcal{Q} \) in \( \mathbb{P}^s \) for \( h^1(I_Y(2)) = 0 \).

Note that \( h^1(G) = h^0(K_Y - 2H + R) = 1 \).

Assume that \( h^1(G + \varphi) = h^0(K_Y - 2H + R - \varphi) = 1 \) for some \( \varphi \in \text{Supp}(R) \). By the equalities as in (3.1) it follows that

\[ h^1(I_{R - \varphi/P^2}(2)) = h^0(K_Y - 2H + R - \varphi) = 1, \]

which implies that \( R - \varphi \) fails to impose independent conditions on quadrics in \( \mathbb{P}^s \). This is contrary to the assumption that \( R \) is minimally FIIQ. Therefore, we get \( h^1(G + \varphi) = 0 \). In particular, if \( \text{deg}R = 2s + 2 - 2\varepsilon \) then we have \( G = K_Y \) for some \( K_Y \in |K_Y| \) since \( \text{deg}G = 2\text{deg}Y - \text{deg}R = 2(s + \gamma - \varepsilon) - (2s + 2 - 2\varepsilon) = 2\gamma - 2 \) and \( h^1(G) = 1 \). Thus the result (i) is proved.

Assume that \( \text{deg}R = 2s + \gamma + 2 - 2\varepsilon \). If there is a quadric hypersurface \( \mathcal{Q} \) satisfying \( R - \varphi \leq Y \cap \mathcal{Q} \), then \( Y \cap \mathcal{Q} - R + \varphi \) is an effective divisor of degree \( \gamma - 1 \). This yields \( h^0(K_Y - (2H - R - \varphi)) \geq 1 \), whence by applying the equalities (3.1) the divisor \( R - \varphi \) fails to impose independent conditions on quadrics in \( \mathbb{P}^s \), which is a contradiction to the minimally FIIQ property of \( R \). Therefore there is no quadric hypersurface \( \mathcal{Q} \) such that \( R - \varphi \leq Y \cap \mathcal{Q} \). In sum, the necessary conditions are verified.

\( (\Leftarrow) \) (i) Let \( R \) be an effective divisor on \( Y \) satisfying the condition (i). Since \( h^1(G) = h^0(K_Y - 2H + R) = 1 \), Eq. (3.1) tells that the divisor \( R \) fails to impose independent conditions on quadrics in \( \mathbb{P}^s \). Let \( \varphi \) be arbitrary point in \( \text{Supp}(R) \). Then \( R - \varphi \) impose independent conditions on quadrics since \( h^0(K_Y - 2H + R - \varphi) = h^1(G + \varphi) = 0 \). Therefore \( R \) is minimally FIIQ in \( \mathbb{P}^s \).
(ii) Let $R$ be an effective divisor on $Y$ satisfying the condition (ii). Then the hypothesis $h^1(\mathcal{I}_Y/P(2)) = 0$ implies $h^0(2H - R + \varphi) = 0$ since there is no quadric hypersurface $\mathcal{Q}$ satisfying $R - \varphi \leq Y \cap \mathcal{Q}$. Therefore the Riemann-Roch theorem combined with $\deg(\mathcal{K}_Y - 2H + R) = \gamma$ yields $h^0(\mathcal{K}_Y - 2H + R - \varphi) = 0$, whence $R - \varphi$ imposes independent conditions on quadrics due to (3.1). And we get $h^0(\mathcal{K}_Y - 2H + R) = 1$ since $\deg(\mathcal{K}_Y - 2H + R) = \gamma$. Hence $R$ fails to impose independent conditions on quadrics in $\mathbb{P}^s$. Therefore $R$ is minimally FIIQ in $\mathbb{P}^s$. In sum, we complete the proof of the proposition.

The minimally FIIQ property plays a crucial role in our work on normal generation. Besides being minimally FIIQ, we sometimes need conditions for an effective divisor $R$ to fail to impose independent conditions on quadrics as in the following remark.

**Remark 3.3.** Let $Y$ be the same as in Proposition 3.2.

1. If $\deg R \geq 2s + \gamma + 2 - 2\epsilon$ then it follows that $h^0(\mathcal{K}_Y - 2H + R) = h^1(\mathcal{I}_R/P(2)) \neq 0$ for $\deg(\mathcal{K}_Y - 2H + R) \geq \gamma$. Thus $R$ always fails to impose independent conditions on quadrics in $\mathbb{P}^s$.
2. If $\deg R = 2s + \gamma + 1 - 2\epsilon$ then we have $\deg(2H - R) = \gamma - 1$. Hence the condition $h^0(\mathcal{K}_Y - 2H + R) \geq 1$ is equivalent to $h^0(2H - R) \geq 1$ by the Riemann-Roch theorem. Therefore, $R$ fails to impose independent conditions on quadrics in $\mathbb{P}^s$ if and only if $R \leq \mathcal{Q} \cap Y$ for a quadric hypersurface $\mathcal{Q}$ in $\mathbb{P}^s$ since $h^1(\mathcal{I}_Y/P(2)) = 0$.

**Corollary 3.4.** Let $E$ be a non-degenerate smooth elliptic curve of degree $s + 1$ in $\mathbb{P}^s$, $s \geq 2$. An effective divisor $R$ on $E$ is minimally FIIQ in $\mathbb{P}^s$ if and only if $R$ falls into one of the following conditions

- (i) $\deg R = 2s + 2$ and $R = E \cap \mathcal{Q}$ for a quadric hypersurface $\mathcal{Q}$ in $\mathbb{P}^s$ with $E \not\subseteq \mathcal{Q}$;
- (ii) $\deg R = 2s + 3$ and for any $\varphi \in \text{Supp}(R)$ there is no quadric hypersurface $\mathcal{Q}$ such that $R - \varphi \leq E \cap \mathcal{Q}$.

**Proof.** Since $\deg\mathcal{O}_E(1) = s + 1 \geq 3$, the line bundle $\mathcal{O}_E(1)$ is normally generated. Hence by Proposition 3.2, we have either $\deg R = 2s + 2$ or $\deg R = 2s + 3$ and $R$ satisfies the condition (i) or (ii), respectively.

**Corollary 3.5.** Let $Y$ be a non-degenerate smooth curve of genus 2 and degree $s + 2$ in $\mathbb{P}^s$, $s \geq 3$. An effective divisor $R$ on $Y$ is minimally FIIQ in $\mathbb{P}^s$ if and only if $R$ falls into one of the following conditions

- (i) $\deg R = 2s + 2$ and $R = (Y \cap \mathcal{Q}) - \mathcal{K}_Y$ for a quadric hypersurface $\mathcal{Q}$ in $\mathbb{P}^s$ and for some $K_Y \in |\mathcal{K}_Y|$;
- (ii) $\deg R = 2s + 3$ the divisor $R + \varphi$ on $E$ is cut out by a quadric hypersurface in $\mathbb{P} H^0(Y, h^n_{n+2})$ for some point $\varphi \in Y$ and $\varphi + \varphi' \not\in |\mathcal{K}_Y|$ for any point $\varphi' \in R$;
- (iii) $\deg R = 2s + 4$ and for any $\varphi \in \text{Supp}(R)$ there is no quadric hypersurface $\mathcal{Q}$ such that $R - \varphi \leq Y \cap \mathcal{Q}$.

**Proof.** Since $\deg\mathcal{O}_Y(1) = s + 2 \geq 5$, the line bundle $\mathcal{O}_Y(1)$ is normally generated. Hence Proposition 3.2 yields the conclusion of the corollary.

**4. Proofs of Theorems A, B**

**Proposition 4.1 (8, Lemma 5).** Assume that a smooth curve $X$ of genus $g$ admits a simple $k$-fold covering $\phi: X \to Y$ for a smooth curve $Y$ of genus $\gamma$ with $g > k\gamma$. Let $\mathcal{M}$ be a globally generated
line bundle on $X$ with $\deg M \leq g - 1$ and $h^0(X,\mathcal{M}) \geq 2$. Then $\mathcal{M}$ is composed with the morphism $\phi$ if

$$\Cliff(\mathcal{M}) < \min\left\{ \frac{g - k\gamma}{k - 1} - 3, \frac{2(k + \mu - 3)}{(2k + \mu - 1)^2}g - 1, \frac{\deg(K_X \otimes \mathcal{M}^{-1})}{3} \right\},$$

where $\mu := \lceil \frac{2k(1-k)}{g-k\gamma} \rceil$.

Recall that $\mathcal{M}$ is composed with $\phi$ if $\mathcal{M} = \phi^*\mathcal{N}$ and $h^0(X,\mathcal{M}) = h^0(Y,\mathcal{N})$ for some $\mathcal{N}$ on $Y$. Under specific conditions we get a simplified result regarding the composedness of line bundles on $X$ as follows.

**Lemma 4.2.** Let a smooth curve $X$ of genus $g$ admit a simple $k$-fold covering $\phi : X \to Y$ for a smooth irrational curve $Y$ of genus $\gamma$ with $k \geq 3$ and $g > k(2k - 1)\gamma$. If $\mathcal{M}$ is a line bundle on $X$ with $\deg \mathcal{M} \leq g - 1$ and $\Cliff(\mathcal{M}) < \frac{g}{k+2} - 1$, then $\mathcal{M}(-B)$ is composed with $\phi$ where $B$ is the base locus of $|\mathcal{M}|$.

**Proof.** The hypothesis $g > k(2k - 1)\gamma$ gives $\mu := \lceil \frac{2k(1-k)}{g-k\gamma} \rceil = 0$. We first notice that

$$\frac{2(2k - 3)}{(2k - 1)^2}g - 1 > \frac{1}{k + 2}g - 1$$

since $\frac{2(2k - 3)}{(2k - 1)^2} - \frac{1}{k + 2} = \frac{6k - 13}{(2k - 1)(k + 2)} > 0$ for $k \geq 3$. The hypotheses $k \geq 3$ and $g > k(2k - 1)\gamma$ yield

$$\frac{g - k\gamma}{k - 1} - \frac{g}{k + 2} = \frac{3g - k(k + 2)\gamma}{(k - 1)(k + 2)} > \frac{3(k(2k - 1)\gamma - k(k + 2)\gamma)}{(k - 1)(k + 2)} > 2,$$

which means $\frac{g - k\gamma}{k - 1} - 3 > \frac{g}{k + 2} - 1$. Also, we have

$$\frac{\deg(K_X \otimes \mathcal{M}^{-1})}{3} \geq \frac{g}{3} - 1 \geq \frac{g}{k + 2} - 1 \text{ for } k \geq 3.$$

Therefore Proposition 4.1 gives the conclusion of the lemma. $\square$

**Proof of Theorem A:** ($\Leftarrow$) We first show $h^1(\mathcal{L}^2(-R)) = 0$ in what follows. Since $\mathcal{L} \cong K_X - (\phi^*g^2 + B) + R$, we have

$$2g + 1 - \frac{g}{2k + 4} \leq \deg \mathcal{L} \leq 2g - 2 - k(n + 1) + 2n + 2 \leq 2g - (k - 2)n - k.$$

Hence it follows that $n \leq (k - 2)n < \frac{g}{2k+4} - 1$ for $k \geq 3$, which combined with $\deg R \leq 2n + 2$ gives $\deg R < \frac{g}{k+2}$. Thus the hypothesis $\deg \mathcal{L} \geq 2g + 1 - \frac{g}{2k+4}$ yields $\deg(\mathcal{L}^2(-R)) > 2g$. Therefore $h^1(\mathcal{L}^2(-R)) = 0$.

Since $R$ fails to impose independent conditions on quadrics in $\langle R \rangle_{\mathcal{L}}$, we get $h^1(\mathcal{I}_X(2)) \neq 0$ by $h^1(\mathcal{L}^2(-R)) = 0$ combined with the exact sequence:

$$0 \to \mathcal{I}_X(2) \to \mathcal{I}_R(2) \to \mathcal{L}^2(-R) \to 0.$$

As a consequence, $\mathcal{L}$ fails to be normally generated.

($\Rightarrow$) Since $\mathcal{L}$ is not normally generated and $\deg \mathcal{L} > \frac{3g+1}{2}$, by Proposition 2.4 there is a minimally FIIO divisor $R$ satisfying $4 \leq 2\dim(\mathcal{R})_{\mathcal{L}} + 2 \leq \deg R \leq \deg \mathcal{L} - \frac{g - 1}{2}$. According to Remark 2.2, the equation $2\dim(\mathcal{R})_{\mathcal{L}} + 2 \leq \deg R$ gives $\Cliff(\mathcal{L}(-R)) \leq \Cliff(\mathcal{L})$, whence

$$\Cliff(\mathcal{L}(-R)) \leq \frac{g}{2k + 4} - 1,$$

(4.1)

since the hypothesis $\deg \mathcal{L} \geq 2g + 1 - \frac{g}{2k+4}$ implies $\Cliff(\mathcal{L}) \leq \frac{g}{2k+4} - 1$. 
Assume that $\deg \mathcal{L}(-R) \leq g - 1$. By Lemma 4.2 together with (4.1) we get
$$|\mathcal{L}(-R)| = \phi^* g_e^\mu + \text{(base locus)}.$$Equation (4.1) combined with $\deg R \leq \deg \mathcal{L} - \frac{k + 1}{2}$ gives
$$\frac{g - 1}{2} - 2\tau \leq \text{Cliff} (\mathcal{L}(-R)) \leq \frac{g}{2k + 4} - 1,$$whence $\tau \geq \frac{k + 1}{2(2k + 4)} g$. Thus,
$$\frac{g}{2k + 4} - 1 \geq \text{Cliff} (\mathcal{L}(-R)) \geq k\epsilon - 2\tau \geq (k - 2)\epsilon \geq \frac{(k + 1)(k - 2)}{2(2k + 4)} g,$$which yields $k^2 - k - 4 \leq 0$. This cannot occur for $k \geq 3$. Therefore $\deg (K_X \otimes \mathcal{L}^{-1}(R)) \leq g - 1$.

By Lemma 4.2 combined with (4.1) there is a $g_e^n$ on $Y$ such that
$$|K_X \otimes \mathcal{L}^{-1}(R)(-\tilde{B})| = \phi^* g_e^n,$$where $\tilde{B}$ is base locus of $K_X \otimes \mathcal{L}^{-1}(R)$ and $n := h^1(X, \mathcal{L}(-R)) - 1$. We obtain $n \geq 1$ since $\deg R \geq 4$ and $2g - \deg (\mathcal{L}(-R)) - 2(n + 1) = \text{Cliff} (\mathcal{L}(-R)) \leq \text{Cliff} (\mathcal{L}) = 2g - \deg \mathcal{L}$.

Assume that $R \cap \tilde{B} > 0$. This implies $h^1(X, \mathcal{L}(\mathcal{L}(-R) \cap \tilde{B})) = h^1(X, \mathcal{L}(-R))$ since $\mathcal{L} \cong K_X - (\phi^* g_e^n + \tilde{B}) + R$ and $\tilde{B}$ is the base locus of $K_X \otimes \mathcal{L}^{-1}(R)$. Thus the Riemann-Roch theorem yields
$$\dim (R) = \dim (R - R \cap \tilde{B}) + \deg (R \cap \tilde{B}),$$whence $R - R \cap \tilde{B}$ also fails to impose independent conditions on quadrics in $\langle R - R \cap \tilde{B} \rangle$. This cannot occur since $R$ is minimally FIIQ. Thus we conclude
$$\text{supp} (R \cap \tilde{B}) = \emptyset \quad (4.2).$$Assume that $\deg (R \cap \phi^* (\varphi)) \geq 2$ for some $\varphi \in Y$. Since $|K_X \otimes \mathcal{L}^{-1}(R)| = \phi^* g_e^n + \tilde{B}$ and $\text{supp} (R \cap \tilde{B}) = \emptyset$, we have
$$|K_X \otimes \mathcal{L}^{-1}(R - R \cap \phi^* (\varphi))| = |\phi^* (g_e^n (-\varphi)) + (\phi^* (\varphi) - \phi^* (\varphi) \cap R) + \tilde{B}|,$$which gives $h^0(X, K_X \otimes \mathcal{L}^{-1}(R - R \cap \phi^* (\varphi))) = n$. Thus the Riemann-Roch theorem yields
$$h^0(X, \mathcal{L}(-R + R \cap \phi^* (\varphi))) - h^0(X, \mathcal{L}(-R)) = \deg (R \cap \phi^* (\varphi)) - 1,$$whence
$$\dim (R) = \dim (R - R \cap \phi^* (\varphi)) + \deg (R \cap \phi^* (\varphi)) - 1.$$This implies that if $R \cap \phi^* (\varphi)$ imposes independent conditions on quadrics in $\langle R - R \cap \phi^* (\varphi) \rangle$ then $R$ does the same in the space $\langle R \rangle$. Therefore $R - \phi^* (\varphi) \cap R$ fails to impose independent conditions on quadrics, which cannot occur since $R$ is minimally FIIQ. Consequently we get
$$\deg (R \cap \phi^* (\varphi)) \leq 1 \quad \text{for any } \varphi \in Y. \quad (4.3)$$Let $G$ be an effective divisor on $Y$ such that
$$B := \tilde{B} - \phi^*(G) \geq 0 \text{ and } \deg (B \cap \phi^* (\varphi)) \leq k - 1 \text{ for any } \varphi \in Y. \quad (4.4)$$Then
$$\mathcal{L} \cong K_X - (\phi^* g_e^n + B) + R, \quad g_e^n := |g_e^n + G|.$$In sum, by (4.2), (4.3) and (4.4) we get the results $(a_1)$ and $(a_3)$ of the theorem. Set $s := \dim (R)$. The Riemann-Roch theorem yields
$$s + 1 = h^0(X, \mathcal{L}) - h^0(X, \mathcal{L}(-R)) = \deg R - (n + 1),$$
for $h^1(X, \mathcal{L}) = 0$ and $h^1(X, \mathcal{L}(-R)) = n + 1$. By this combined with $\deg R \geq 2s + 2 \geq 4$ it forces that

$$n + 3 \leq \deg R = n + s + 2 \leq 2n + 2.$$  \hfill (4.5)

Hence we get the result $(a_2)$ of the theorem. Finally the proof of Theorem A is completed. \hfill $\blacksquare$

**Proof of Theorem B:** Let $\mathcal{D} := \mathcal{O}_Y(\phi(R) + K_Y - g^*_n \gamma)$. By the condition Theorem A$(a_2)$, it follows that $\deg \mathcal{D} = s + \gamma \geq 2\gamma - 1$ for $\gamma \leq 2$. The inequality $s \geq 2$ holds since $\deg R = n + s + 2 \geq 5$ and $R$ is minimally FIIQ in the $s$-dimensional space $\langle R \rangle_{\mathcal{L}}$. Thus $\mathcal{D}$ is non-special and $h^0(Y, \mathcal{D}) = s + 1$ for $\gamma \leq 2$. And the line bundle $\phi^* \mathcal{D}$ is composed with the covering map $\phi$, since Theorem A$(a_3)$ yields

$$\text{Cliff}(\phi^* \mathcal{D}) \leq k(s + \gamma) - 2s \leq \text{Cliff}(\phi^* g^*_n \gamma + B) \leq \text{Cliff}(\mathcal{L}) \leq \frac{g}{2k + 4} - 1,$$ \hfill (4.6)

for $s \leq n$. Therefore we get $h^0(X, \phi^* \mathcal{D}) = s + 1$, which completes the proof of the result $(b_1)$ of the theorem.

To prove the result $(b_2)$, we want to understand the morphism $\varphi_{\phi^* \mathcal{D}}$ on $X$ as a projection of $\varphi_{\mathcal{L}}(X)$ to $\mathbb{P}^s := \mathbb{P}h^0(Y, \mathcal{D})$ which preserves the scheme theoretic properties of $\varphi_{\mathcal{L}}(R)$ in $\mathbb{P}h^0(X, \mathcal{L})$ (cf. Figure 1). For convenience sake, we set

$$h^*_{s+\gamma} := |\mathcal{D}|.$$  

In order to figure out such a projection mentioned above, we need $h^0(X, \mathcal{L} \otimes (\phi^* \mathcal{D})^{-1}) \geq 1$. For this, we will show the inequality $\deg(K_X \otimes \mathcal{L}^{-1} \otimes \phi^* \mathcal{D}) \leq g - 2$ in what follows. From (4.6) we get $(k - 2)(s + \gamma) \leq \frac{g}{2k + 4}$, whence

$$\deg \phi^* \mathcal{D} = k(s + \gamma) \leq \frac{kg}{(k-2)(2k+4)} \leq \frac{g}{2} \text{ for } k \geq 3.$$ 

This combined with the hypothesis $\deg \mathcal{L} \geq 2g + 1 - \frac{g}{2k+4}$ yields

$$\deg(K_X \otimes \mathcal{L}^{-1} \otimes \phi^* \mathcal{D}) = \deg(K_X \otimes \mathcal{L}^{-1}(\phi^* h^*_{s+\gamma})) \leq \left(\frac{g}{2k + 4} - 3\right) + \frac{g}{2} \leq g - 2.$$ \hfill (4.7)

Hence the Riemann-Roch theorem gives $h^0(X, \mathcal{L} \otimes (\phi^* \mathcal{D})^{-1}) \geq 1$.

Set $\mathcal{M} := \mathcal{L} \otimes (\phi^* \mathcal{D})^{-1}$. For an effective divisor $M \in |\mathcal{M}| = |\mathcal{L}(-\phi^* h^*_{s+\gamma})|$, we have a commutative diagram in Figure 1, where $\pi_{|\mathcal{M}|\mathcal{L}}$ is a projection from $\langle M \rangle_{\mathcal{L}}$.

Since $|\phi^* \mathcal{D}| = |\phi^* h^*_{s+\gamma}|$ is composed with $\phi$, $\deg(R \cap \phi^*(\mathcal{V})) \leq 1$ for any $\mathcal{V} \in Y$ and $\deg R = n + s + 2 > s + \gamma$ for $\gamma \leq 2$, we have

$$h^0(\mathcal{L}(-M - R)) = h^0(\phi^* h^*_{s+\gamma} - R) = 0.$$ 

This combined with $h^0(\mathcal{L}(-M)) = \dim(\phi^* h^*_{s+\gamma}) + 1 = s + 1$ yields

\[\begin{array}{ccc}
X & \xrightarrow{\varphi_{\mathcal{L}}} & \varphi_{\mathcal{L}}(X) \subset \mathbb{P}^r := \mathbb{P}h^0(X, \mathcal{L}) \\
\phi & \downarrow & \varphi_{\phi^* \mathcal{D}} \\
\phi & \downarrow & \varphi_{\phi^* \mathcal{D}} \\
Y & \xrightarrow{\varphi_{\mathcal{D}} = \phi_{h^*_{s+\gamma}}} & \varphi_{\mathcal{D}}(Y) \subset \mathbb{P}^s := \mathbb{P}h^0(Y, \mathcal{D})
\end{array}\]

**Figure 1.** Relationship between $\varphi_\mathcal{L}$ and $\varphi_\mathcal{D}$.
dim(\langle M + R \rangle_L) = dim(\langle R \rangle_L) + \langle M \rangle_L + 1.

Therefore, the projection center \langle M \rangle_L is disjoint from \langle R \rangle_L by the following claim.

**Claim.** \( R \cap (\text{base locus of } M) = \emptyset. \)

**Proof of claim.** We observe the equivalence \(|K_X \otimes M^{-1}| = |(\phi^*g^{n+\gamma}_{s+\gamma} + \phi^*h^{s}_{s+\gamma} + B)(-R)|\) given by \( L \cong K_X - (\phi^*g^{n+\gamma}_{s+\gamma} + B) + R \) and \( M = L(-\phi^*h^{s}_{s+\gamma}). \) Thus the Riemann-Roch theorem tells the claim once we show the following equation: for any \( p \in \text{Supp}(R), \)

\[
h^0(X, (\phi^*g^{n+\gamma}_{s+\gamma} + \phi^*h^{s}_{s+\gamma} + B)(-R + p)) = h^0(X, (\phi^*g^{n+\gamma}_{s+\gamma} + \phi^*h^{s}_{s+\gamma} + B)(-R)).
\] (4.8)

From Eq. (4.6) it follows that

\[
\text{Cliff}(\phi^*g^{n+\gamma}_{s+\gamma} + \phi^*h^{s}_{s+\gamma} + B) \leq 2 \left( \frac{g}{2k + 4} - 1 \right) < \frac{g}{k + 2} - 1,
\] (4.9)

whence \(|\phi^*g^{n+\gamma}_{s+\gamma} + \phi^*h^{s}_{s+\gamma} + B|\) is composed with \( \phi. \) Further \( B \) is the base locus due to the condition \( \text{deg}(B \cap \phi^*(\varphi)) \leq k - 1 \) for any \( \varphi \in Y. \) Thus we obtain

\[
|\phi^*g^{n+\gamma}_{s+\gamma} + \phi^*h^{s}_{s+\gamma} + B| = |\phi^*(K_X + \phi(R)) + B,
\]
since \( h^{s}_{s+\gamma} = |O_Y(\phi(R) + K_Y - g^{n+\gamma}_{s+\gamma})|. \) The condition \( \text{supp}(R \cap B) = \emptyset \) implies that

\[
|\phi^*g^{n+\gamma}_{s+\gamma} + \phi^*h^{s}_{s+\gamma} + B)(-R)| = |\phi^*K_Y| + (B + \phi^*(\phi(R)) - R).
\]

Further, for any \( p \in \text{supp}(R) \)

\[
|\phi^*g^{n+\gamma}_{s+\gamma} + \phi^*h^{s}_{s+\gamma} + B)(-R + p)| = |\phi^*(K_Y + \phi(p))| + (B + \phi^*(\phi(R) - p) - (R - p)),
\]
since \( \text{deg}(R \cap \phi^*(\varphi)) \leq 1 \) for any \( \varphi \in Y. \) Thus we get Eq. (4.8) since \( h^0(Y, K_Y) = h^0(Y, K_Y(\phi(p))). \) Therefore the claim is verified.

By the claim, there is an effective divisor \( M \in |M| \) such that \( \langle R \rangle_L \cap \langle M \rangle_L = \emptyset. \) Then, \( \pi_M L \) is a projection into \( \mathbb{P}^s := \mathbb{P}H^0(Y, h^{s}_{s+\gamma}) \) and hence the scheme theoretic properties of \( \phi_L(R) \) are invariant via the projection. This implies that the scheme theoretic properties of \( \phi_L(R) \) in \( \langle R \rangle_L \) and \( \phi_L(\phi(R)) \) in \( \langle \phi(R) \rangle_L \) are the same. Especially, for \( s \geq \gamma + 1 \) the line bundle \( D \) is very ample and hence \( \phi(R) \) on \( Y \) is minimally FIIQ in \( \mathbb{P}^s \). Therefore \( (b_2) \) and \( (b_3) \) of the theorem are verified.

According to \( (b_3) \) combined with \( \text{deg}R = n + s + 2, \) Corollaries 3.4, 3.5 yield \( 2n + 2 - \gamma \leq \text{deg}R \leq 2n + 2 \) unless \( s = \gamma = 2. \) Assume that \( s = \gamma = 2. \) Then the minimally FIIQ property of \( R \) in the plane \( \langle R \rangle_L \) implies that either \( \text{deg}R = 6 \) or \( \text{deg}R = 7. \) Further if \( \text{deg}R = 7 \) then the equation \( \text{deg}R = n + s + 2 \) gives \( n = 3. \) Thus \( \text{deg}R \) also satisfies \( (b_4) \) in case \( s = \gamma = 2. \) which completes the proof of \( (b_4). \) In sum, we get the conclusion of **Theorem B.**

### 5. Applications

#### 5.1. Covering of an elliptic curve

In this subsection, we present a classification of non-special line bundles failing to be normally generated on a simple multiple covering of an elliptic curve.

**Theorem 5.1.** Let \( \phi : X \rightarrow E \) be a simple \( k \)-fold covering of an elliptic curve \( E \) such that \( k \geq 3 \) and \( g > k(2k - 1) \) and let \( M \) be a non-special very ample line bundle on \( X \) with \( \text{deg}M \geq 2g + 1 - \frac{g}{2k + 4}. \) Then \( M \) fails to be normally generated if and only if...
\( L \cong \mathcal{K}_X - (\phi^*g^n_{n+1} + B) + R, n \geq 1 \)

for some divisors \( B \) and \( R \) on \( X \) satisfying

1. \( \text{supp}(R \cap B) = \emptyset, \deg(R \cap \phi^*(\psi)) \leq 1 \) and \( \deg(B \cap \phi^*(\psi)) \leq k - 1 \) for any \( \psi \in E \);
2. \( \deg R = 2n + 2 \) or \( \deg R = 2n + 1 \geq 7 \);
3. in case \( \deg R = 2n + 2 \geq 6, \phi(R) = E \cap \mathcal{Q} \) for some quadric hypersurface \( \mathcal{Q} \) in \( \mathbb{P}H^0(E,g^n_{n+1}) \).

**Proof.** (\( \Rightarrow \)) Assume that \( L \) fails to be normally generated. Theorem A tells that

\[
L \cong \mathcal{K}_X - (\phi^*g^n_{n+1} + B) + R \tag{5.1}
\]

for some divisors \( R, B \) on \( X \) and a \( g^n_{n+1} \) on \( E \) satisfying the conditions (1), (2) and (3). Hence, it is enough to show that \( R \) satisfies the conditions (2) and (3) in the theorem. In case \( n = 1 \), the divisor \( R \) is of degree 4 and \( \varphi \subset (R) \) spans a line. Thus the minimally FIIQ property of \( \varphi \subset (R) \) does not give any constraints on \( R \).

Now we assume that \( n \geq 2 \). Considering the line bundle \( D \) defined in Theorem B, we have \( D \cong \mathcal{O}_E(\phi(R) - g^n_{n+1}) \) since \( \mathcal{K}_E \cong \mathcal{O}_E \). Theorem B combined with \( \gamma = 1 \) tells that \( |D| \) is a very ample linear series of degree \( s + 1 \) and dimension \( s \geq 2 \), which is denoted by \( h^s_{s+1} \). Recall that \( s = \dim (R)_L \). Since \( \phi(R) \) is minimally FIIQ in \( \mathbb{P}H^0(Y,h^s_{s+1}) \) and \( \deg(R \cap \phi^*(\psi)) \leq 1 \) for any \( \psi \in Y \), Corollary 3.4 yields that either \( \deg R = 2s + 2 \geq 6 \) or \( \deg R = 2s + 3 \geq 7 \). According to Theorem A(a), if \( \deg R = 2s + 2 \) (resp. \( \deg R = 2s + 3 \)), then \( s = n \) (resp. \( s = n - 1 \)). It turns out that

\[
\deg R = 2n + 2 \geq 6 \text{ or } \deg R = 2n + 1 \geq 7,
\]

which is the condition (2) in the theorem.

It remains to show that the divisor \( \phi(R) \) satisfies the condition (3) in the theorem. If \( \deg R = 2n + 2 \geq 6 \), Corollary 3.4 tells that \( \phi(R) = E \cap \mathcal{Q} \) for a quadric hypersurface \( \mathcal{Q} \) in \( \mathbb{P}H^0(E,g^n_{n+1}) \). Further this combined with \( h_{n+1}^n = |\phi(R) - g^n_{n+1}| \) implies \( \phi(R) \sim 2H \sim H + F \) for some \( F \in g^n_{n+1} \), which means \( h_{n+1}^n = g^n_{n+1} \). In sum, all the necessary conditions are verified.

(\( \Leftarrow \)) Let \( L \) be a very ample line bundle on \( X \) such that \( L \cong \mathcal{K}_X - (\phi^*g^n_{n+1} + B) + R \) for some divisors \( B, R \) on \( X \) and a \( g^n_{n+1} \) on \( E \) satisfying the conditions (1), (2) and (3). For \( n = 1 \), we have \( \deg R = 4 \) and hence \( \varphi \subset (R) \) spans a line. Thus Theorem 2.1 tells \( L \) fails to be normally generated.

Assume that \( n \geq 2 \). By \( \deg R = \dim (R)_L + n + 2 \) together with the condition (2), the dimension of \( |\phi(R) - g^n_{n+1}| \) is equal to \( s := \dim (R)_L \geq 2 \). As in the above, we set

\[
h_{s+1}^s := |\phi(R) - g^n_{n+1}|,
\]

which is the same linear series \(|D|\) as in Theorem B. By the similar arguments as in the proof of Theorem B, we get the commutative diagram in Figure 1 satisfying \( R \cap (\text{base locus of } \mathcal{M}) = \emptyset \) where \( \mathcal{M} := L(\phi^!h_{s+1}^s) \). Therefore the scheme theoretic properties of \( \varphi \subset (R)_L \) are invariant via the projection \( \pi_{(\mathcal{M})_L} \).

If \( \deg R = 2n + 1 \), we get \( s = n - 1 \) and thus \( \deg(\phi(R)) = 2s + 3 \). Remark 3.3(1) tells that the divisor \( \phi(R) \) fails to impose independent conditions on quadrics in \( \mathbb{P}H^0(E,h_{s+1}^s) \). Thus \( \varphi \subset (R) \) carries the same property in \( \mathbb{P}H^0(X,L) \). By Theorem 2.1, \( L \) fails to be normally generated.

Finally, we consider the case \( \deg R = 2n + 2 \geq 6 \). The condition (3) implies that the divisor \( \phi(R) \) fails to impose independent conditions on quadrics in \( \mathbb{P}H^0(E,h_{s+1}^s) \) by Corollary 3.4. Accordingly, \( \varphi \subset (R)_L \), whence \( L \) fails to be normally generated. In sum, the theorem is proved.

\[\square\]

**Remark 5.2.** We take a look at how the minimally FIIQ property of \( R \) or \( \phi(R) \) contributes to get Theorem 5.1. Basically, the minimally FIIQ property of \( R \) plays a key role in driving the conditions (1) and (2) in Theorem 5.1. In the proof of Theorem 5.1, it seems to have a loss in fully using the minimally FIIQ property of \( \phi(R) \). For instance, the part where we apply Remark 3.3 is...
carried out without considering the minimality condition that any proper subvulator of \( \phi(R) \) imposes independent conditions on quadrics in \( \mathbb{P}^r \). Regarding this loss, we first consider the case that \( L \cong K_X - (\phi^* g^n_{n+1} + B) + R \) with \( \deg R = 2n + 2 \). The Riemann-Roch theorem yields \( \langle R \rangle_L = (R - p)_L \) for any \( p \in \text{supp}(R) \) and hence by Corollary 3.4 \( \phi(R - p) \) imposes independent conditions on quadrics in the projective space spanned by \( \phi(R - p) \). Therefore \( \phi(R) \) is minimally \( FIIQ \) and hence there is not such a loss. On the other hand, in case \( L \cong K_X - (\phi^* g^n_{n+1} + B) + R \) with \( \deg R = 2n + 1 \geq 7 \), the minimality condition is not automatically established in itself unlike the case \( \deg R = 2n + 2 \). However the failure of normal generation of the line bundle \( L \) is verified regardless of the minimality condition.

Theorem 5.1 is obtained under the hypothesis that \( L \) is very ample. In the following proposition, we examine the very ampleness of the line bundle \( L \) in Theorem 5.1.

**Proposition 5.3.** Let \( \phi: X \to E \) be the same as in Theorem 5.1 with \( g > k(2k - 1) + 5 \) and let \( L \cong K_X - (\phi^* g^n_{n+1} + B) + R \) with \( \deg L \geq 2g + 1 - \frac{k}{2k+4} \) be given by effective divisors \( R, B \) on \( X \) and a \( g^n_{n+1} \) on \( E \) satisfying (1), (2) and (3) in Theorem 5.1. Then \( L \) is very ample if and only if it does not correspond to the case that \( n = 1 \) and \( \deg(R \cap F) \geq 2 \) for some \( F \in \phi^1 g^n_1 \).

**Proof.** \((\Rightarrow)\) Assume that \( L \cong K_X - (\phi^* g^n_2 + B) + R \) with \( \deg(R \cap F) \geq 2 \) for some \( F \in \phi^1 g^n_2 \). By the condition (1) in Theorem 5.1 the line bundle \( L \) is non-special and \( \deg(R \cap F) = 2 \). Then we have \( L \cong K_X - (F - R \cap F + B) + (R - R \cap F) \) with \( \deg(R - R \cap F) = 2 \) for \( \deg R = 4 \). According to the Riemann-Roch theorem it follows that \( h^0(L) = h^0(L(-(R - R \cap F))) \leq 1 \), which means that \( L \) is not very ample.

\((\Leftarrow)\) Assume that \( L \) is not very ample. Then there exists a divisor \( D_2 \in X^{(2)} \) such that \( h^0(L(-D_2)) \geq h^0(L) - 1 \). The Riemann-Roch theorem yields \( h^1(X, L(-D_2)) \geq 1 \), whence there is an effective divisor \( G \in |K_X \otimes L^{-1}(D_2)| = |\phi^* g^n_{n+1} + B - R + D_2| \). This means that \( |\phi^* g^n_{n+1} + B + D_2| = |R + G| \). Observing the equation \( \deg R \geq 2a + 2 \) in the proof of Theorem 5.1, we have \( \text{Cliff}(L(-R)) \leq \text{Cliff}(L) \leq \frac{g}{2k+4} - 1 \) for \( \deg L \geq 2g + 1 - \frac{g}{2k+4} \). It follows that

\[ \text{Cliff}(\phi^* g^n_{n+1} + B + D_2) \leq \text{Cliff}(\phi^* g^n_{n+1} + B) + 2 = \text{Cliff}(L(-R)) + 2 < \frac{g}{k+2} - 1, \]

where the last inequality is given by \( g > k(2k - 1) + 5 \geq 4(k + 2) \). Thus Lemma 4.2 gives

\[ |R + G| = |\phi^* g^n_{n+1} + B + D_2| = |\phi^* g^{n+a}_{n+a+1} + B', B' : \text{base locus}, 0 \leq a \leq 2. \quad (5.2) \]

It follows that \( \deg R - (n + a + 1) \leq \deg(R \cap B') \leq 2 - a \) since \( \text{supp}(R \cap B) = \emptyset \) and \( \deg(R \cap g^*(\psi)) \leq 1 \) for any \( \psi \in E \). This forces \( n = 1 \) since \( \deg R = 2n + 2 \geq 4 \) or \( 2n + 1 \geq 7 \). Hence \( L \cong K_X - (\phi^* g^1_2 + B) + R \) with \( \deg R = 4 \). Assume that \( \deg(R \cap F) \leq 1 \) for any \( F \in \phi^1 g^1_2 \). Then Eq. (5.2) yields \( \deg R - (1 + a) \leq \deg(R \cap B') \leq 2 - a \), which cannot occur for \( \deg R = 4 \). Therefore \( \deg(R \cap F) \geq 2 \) for some \( F \in \phi^1 g^1_2 \). In sum, we get the conclusion of the proposition. \( \square \)

Theorem 5.1 gives rise to the following result on non-special normally generated line bundles on a simple multiple covering of a smooth curve of an elliptic curve.

**Corollary 5.4.** Let \( \phi: X \to E \) be the same as in Theorem 5.1. And let \( \eta_d \in X^{(d)} \) and \( \xi_e \in X^{(e)} \) satisfy that \( h^0(X, \eta_d) = h^0(X, \xi_e) = 1 \), \( \text{supp}(\eta_d \cap \xi_e) = \emptyset \) and \( \deg(\eta_d \cap g^*(\psi)) \leq k - 1 \) for any \( \psi \in E \). If the number of points \( \psi \in E \) having \( \deg(\eta_d \cap g^*(\psi)) = k - 1 \) is at most one and \( 3 \leq e < d \leq \frac{g}{2k+4} - 1 \), then \( L \cong K_X - \eta_d + \xi_e \) is a non-special normally generated line bundle.

**Proof.** Since \( h^0(X, \eta_d) = h^0(X, \xi_e) = 1 \) and \( \text{supp}(\eta_d \cap \xi_e) = \emptyset \), the line bundle \( L \) is non-special. The hypotheses \( d \leq \frac{g}{2k+4} - 1 \) and \( e \geq 3 \) give \( \deg L \geq 2g + 1 - \frac{g}{2k+4} \). Suppose that \( L \cong K_X - \eta_d + \xi_e \) is not very ample. Then there are \( h_t \in X^{(1)} \) and \( \xi_2 \in X^{(2)} \) such that
This combined with supp(η_d ∩ ζ_e) = ∅ and e ≥ 3 implies η_d + ζ_e ∼ η_1 + ζ_2 and η_d + ζ_2 ≠ h_t + ζ_e as divisors, hence dim|η_d + ζ_2| = 1. Since Cliff(η_d + ζ_2) = d ≤ \( \frac{g}{2k+4} - 1 \), the line bundle \( \mathcal{O}_X(η_d + ζ_2) \) is composed with φ. It cannot occur the number of points \( φ \in E \) with \( \text{deg}(η_d ∩ φ^*(ψ)) = k - 1 \) is at most one. Therefore \( \mathcal{L} \) is very ample.

Assume that \( \mathcal{L} \) fails to be normally generated. Since \( \text{deg}\mathcal{L} > 2g + 1 - \frac{g}{2k+4} \), Theorem 5.1 gives \( \mathcal{L} \cong \mathcal{K}_X - (φ^*g^n_{n+1} + B) + R \) for some \( R \) and \( B \) satisfying the conditions (1), (2) and (3). This equivalence combined with \( \mathcal{L} \cong \mathcal{K}_X - η_d + ζ_e \) yields

\[
φ^*g^n_{n+1} + B + ζ_e \cong η_d + R,
\]

whence

\[
\text{Cliff}(η_d + R) \leq (d + 2n + 2) - 2n \leq \frac{g}{2(k + 2)} + 1 < \frac{g}{k + 2} - 1.
\]

for \( \text{deg}R \leq 2n + 2 \) and \( d \leq \frac{g}{2k+4} - 1 \). According to Lemma 4.2, the linear series \( |φ^*g^n_{n+1} + B + ζ_e| = |η_d + R| \) is composed with \( φ \) and \( \text{dim}|η_d + R| \geq n ≥ 1 \). It cannot occur by the hypothesis on the number of points \( φ \in E \) having \( \text{deg}(η_d ∩ φ^*(ψ)) = k - 1 \) and the condition \( \text{deg}(R ∩ φ^*(ψ)) ≤ 1 \) for any \( φ \in E \). Therefore \( \mathcal{L} \) is normally generated, which completes the proof of the corollary.

\[\square\]

### 5.2. Covering of a smooth curve of genus 2

In this subsection, we obtain a precise description of non-special line bundles failing to be normally generated on a simple multiple covering of a smooth curve of genus 2.

**Theorem 5.5.** Let \( φ : X \rightarrow Y \) be a simple \( k \)-fold covering of a smooth curve \( Y \) of genus 2 such that \( k ≥ 3 \) and \( g > 2k(2k - 1) \) and let \( \mathcal{L} \) be a non-special very ample line bundle on \( X \) with \( \text{deg}\mathcal{L} ≥ 2g + 1 - \frac{g}{2k+4} \). Then \( \mathcal{L} \) fails to be normally generated if and only if either

\[
\mathcal{L} \cong \mathcal{K}_X - (φ^*g^1_2 + B) + R \quad \text{with} \quad \text{deg}R = 4 \quad \text{or}
\]

\[
\mathcal{L} \cong \mathcal{K}_X - (φ^*g^n_{n+2} + B) + R, \quad n ≥ 1
\]

for some divisors \( B \) and \( R \) on \( X \) satisfying

1. \( \text{supp}(R ∩ B) = ∅ \), \( \text{deg}(R ∩ φ^*(ψ)) ≤ 1 \) and \( \text{deg}(B ∩ φ^*(ψ)) ≤ k - 1 \) for any \( ψ \in Y \);
2. for the case (\( \ddagger \)), \( \text{deg}R = 2n + 2 \), \( \text{deg}R = 2n + 1 ≥ 7 \) or \( \text{deg}R = 2n ≥ 10 \); further
   (2-a) in case \( \text{deg}R = 2n + 2 = 6 \), \( φ(R) \) lies on a conic in \( \mathbb{P}H^0(Y, h^1_Y) \),
   (2-b) in case \( \text{deg}R = 2n + 2 ≥ 8 \), \( φ(R) + K_Y = Y ∩ \mathcal{Q} \) for some \( K_Y ∈ |K_Y| \) and quadric
   hypersurface \( \mathcal{Q} \) in \( \mathbb{P}H^0(Y, h^1_{n+2}) \),
   (2-c) in case \( \text{deg}R = 2n + 1 ≥ 9 \), \( φ(R) ≤ Y ∩ \mathcal{Q} \) for some quadric
   hypersurface \( \mathcal{Q} \) in \( \mathbb{P}H^0(Y, h^1_{n+1}) \),

where \( h^1_{n+2} := |φ(R) + K_Y - g^n_{n+2}| \). Further, \( h^n_{n+2} = g^n_{n+2} \) in the case (2-b).

**Proof.** (\( \Rightarrow \)) Assume that \( \mathcal{L} \) fails to be normally generated. Theorem A tells that

\[
\mathcal{L} \cong \mathcal{K}_X - (φ^*g^1_2 + B) + R, \quad n ≥ 1
\]

for some divisors \( B, R \) on \( X \) and \( g^1_2 \) on \( X \) satisfying the conditions (\( a_1 \)), (\( a_2 \)) and (\( a_3 \)). Hence, the divisors \( B, R \) satisfy the condition (1) in the theorem. If \( g^1_2 \) is special then \( g^1_2 = g^2_2 \), otherwise
It remains to show the properties of \( \text{P} \), dependent conditions on quadrics in \( \text{P} \), the scheme theoretic properties of \( \dim \), degree \( \deg \) conic in \( \text{h} \). Therefore \( \text{R} \), the necessary condition for \( \text{R} \) fails to impose independent conditions on quadrics in \( \text{P} \). Moreover, observing the case \( \text{2-2} \), \( \text{R} \) fails to be normally generated. Finally, the sufficient conditions are verified.

Assume \( \text{R} \) is of degree \( \deg \) a double cover of a plane conic \( \text{h} \). By Theorem B, the geometric properties of \( \text{R} \) in the plane \( \text{P} \) is a double cover of a plane conic \( \text{h} \).

In sum, the theorem is proved.

\( \Rightarrow \) Let \( \text{L} \) be a very ample line bundle on \( \text{X} \) such that \( \text{L} \simeq \text{KX} - (\phi^* g^1_b + B) + R \) or \( \text{L} \simeq \text{KX} - (\phi^* g^n_{n+2} + B) + R \) for some divisors \( \text{R}, \text{B} \) on \( \text{X} \) and a \( g^n_{n+2} \) on \( \text{Y} \) satisfying the conditions (1) and (2). For the case \( n = 1 \), \( \phi_{\text{L}}(\text{R}) \) spans a line. Thus Theorem 2.1 tells \( \text{L} \) fails to be normally generated.

Now we assume \( n \geq 2 \). Consider \( |\text{D}| = |\phi(\text{R}) + K_{\text{Y}} - g^n_{n+2}| = h^s_{n+2} \), which is the same as in Theorem B with \( s = \dim(\langle \text{R} \rangle)_\text{L} \). The similar arguments as in the proof of Theorem B, we get the commutative diagram in Figure 1 with \( \langle \text{R} \rangle_\text{L} \cap (\text{M})_\text{L} = \emptyset \) where \( \text{M} \simeq \langle \text{L} \rangle_\text{L} - (\phi^* h^s_{n+2}) \). Therefore the scheme theoretic properties of \( \phi_{\text{L}}(\text{R}) \) in \( \text{P} \) are invariant via the projection \( \pi_{(\text{M})_\text{L}} \).

If \( \deg \text{R} \geq 10 \), we get \( s = n - 2 \geq 3 \) and thus \( \deg(\phi(\text{R})) = 2s + 4 \). Remark 3.3(1) tells that the divisor \( \phi(\text{R}) \) fails to impose independent conditions on quadrics in \( \text{P} \). Thus \( \text{R} \) carries the same property in \( \text{P} \). By Theorem 2.1, \( \text{L} \) fails to be normally generated.

Assume that \( \deg \text{R} = 2n + 1 \), which gives \( s = n - 1 \). In case \( \deg \text{R} = 7 \) it follows that \( \dim(\langle \text{R} \rangle)_\text{L} \geq 2 \) and hence \( \text{R} \) fails to impose independent conditions on quadrics in \( \text{P} \). If \( \deg \text{R} \geq 9 \) and \( \phi(\text{R}) \) satisfies (2-c) in the theorem, then Remark 3.3(2) tells that \( \phi(\text{R}) \) fails to impose independent conditions on quadrics in \( \text{P} \). It follows that \( \text{R} \) fails to impose independent conditions on quadrics in \( \text{P} \). By Theorem 2.1, \( \text{L} \) fails to be normally generated.

Assume that \( \deg \text{R} = 2n + 2 \), for which \( n \) equals \( s \). In both cases (2-a) and (2-b), Corollary 3.5 tells that \( \phi(\text{R}) \) fails to impose independent conditions on quadrics in the plane \( \text{P} \). Therefore \( \text{R} \) fails to impose independent conditions on quadrics in \( \text{P} \). According to Theorem 2.1, \( \text{L} \) fails to be normally generated. Finally, the sufficient conditions are verified.

Furthermore, observing the case (2-b) in which \( |\phi(\text{R}) + K_{\text{Y}}| = 2h^s_{n+2} \), we obtain that \( \phi(\text{R}) + K_{\text{Y}} \sim 2H \sim H + F \) for some \( F \subseteq g^n_{n+2} \) due to \( h^s_{n+2} = |\phi(\text{R}) + K_{\text{Y}} - g^n_{n+2}| \). This yields \( h^s_{n+2} = g^n_{n+2} \). In sum, the theorem is proved.
As in Remark 5.2, the proof of Theorem 5.5 has a loss in fully using the minimally FIIQ property of $\phi(R)$ as well. For instance, we carry out the proof in the parts where Remark 3.3 is applied without considering the minimality condition that any proper subdivisor of $\phi(R)$ imposes independent conditions on quadrics in $\mathbb{P}^r$.

Theorem 5.5 gives criteria for a very ample line bundle on a smooth curve of genus 2 to fail to be normally generated. In the following proposition, we show the very ampleness of $L$ satisfying the conditions in Theorem 5.5.

**Proposition 5.6.** Let $\phi : X \to Y$ be the same as in Theorem 5.5. And let $L$ be a non-special line bundle with $\deg L \geq 2g + 1 - \frac{\ell}{g+1}$ such that

\[
L \cong \mathcal{K}_X - (\phi^* g^1_2 + B) + R \quad \text{with} \quad \deg R = 4 \quad \text{or}
\]

\[
L \cong \mathcal{K}_X - (\phi^* g^n_{n+2} + B) + R, \quad n \geq 1
\]

for some effective divisors $R, B$ on $X$ and a $g^n_{n+2}$ on $Y$ satisfying the properties (1) and (2) in Theorem 5.5. Then the line bundle $L$ is very ample if and only if $R$ does not fall into any of the following:

(i) $\deg R = 4$ and $\deg (R \cap F) \geq 2$ for some $F \in \phi^* g^1_2$, $\ell = 2, 3$;

(ii) $\deg R = 6$ or $\deg R = 7$,

- $\deg \varphi_{k_1} = 1$, $\varphi_1 + \varphi_2 \leq \phi(R)$;
- $\deg \varphi_{k_2} = 2$, $\deg(\phi(R) \cap K_Y) \geq 2$ for some $K_Y \in |\mathcal{K}_Y|$, i.e., $\deg(\phi(R) \cap \varphi_{k_2}^*(q)) \geq 2$ for some $q$ on the conic $\varphi_{k_1}(Y)$,

where $h^1_{n+2} := |\phi(R) + K_Y - g^n_{n+2}|$ and $\varphi_1, \varphi_2$ on $Y$ are sent to the singular point of the plane curve $\varphi_{k_1}(Y)$ in case $\deg \varphi_{k_1} = 1$.

**Proof.** ($\Leftarrow$) Assume that $L$ is not very ample. We first deal with the case

\[
L \cong \mathcal{K}_X - (\phi^* g^n_{n+2} + B) + R, \quad n \geq 1.
\]

Then there is a divisor $D_2 \in X^{(2)}$ such that $h^0(\mathcal{L}(-D_2)) \geq h^0(L) - 1$. The Riemann-Roch theorem yields $h^1(X, \mathcal{L}(-D_2)) \geq 1$, whence there exist an effective divisor $G \in |\mathcal{K}_X \otimes \mathcal{L}^{-1}(D_2)| = |\phi^* g^n_{n+1} + B - R + D_2|$. Therefore, we get $|\phi^* g^n_{n+2} + B + D_2| = |R + G|$. As in the proof of Proposition 5.3, the hypothesis $\deg L > 2g + 1 - \frac{\ell}{g+1}$ combined with $g > 2k(2k-1)$ implies

\[
\text{Cliff}(\phi^* g^n_{n+2} + B + D_2) \leq \frac{\ell}{g+1} - 1 \quad \text{and hence Lemma 4.2 gives}
\]

\[
|R + G| = |\phi^* g^n_{n+2} + B + D_2| = \phi^* g^n_{n+2}^a + B', B' : \text{base locus, } 0 \leq a \leq 2.
\]

Since $\text{supp}(R \cap B) = \emptyset$ and $\deg(\phi \cap \varphi^*(\varphi)) \leq 1$ for any $\varphi \in Y$, it follows

\[
\deg R - (t + a) \leq \deg(\phi \cap \varphi) \leq 2 - a,
\]

where $t := \max\{\deg(R \cap F) \mid F \in \phi^* g^n_{n+2}\}$. We notice that $t \leq n + 2$ since $\deg(\phi \cap \varphi^*(\varphi)) \leq 1$ for any $\varphi \in Y$. Thus Eq. (5.4) implies $\deg R \leq n + 4$. Therefore we encounter only the following cases:

\[
(n, \deg R) = (1, 4), (2, 6), \text{ or } (3, 7)
\]

since $\deg R = 2n + 2 \geq 4$, $2n + 1 \geq 7$ or $2n \geq 10$ which is the condition (2) in Theorem 5.5. If $(n, \deg R) = (1, 4)$ then (5.4) yields $\deg(R \cap F) \geq 2$ for some $F \in \phi^* g^1_2$. It is the case (i).

Assume that $(n, \deg R) = (2, 6)$. Equation (5.4) gives $F \in \phi^* g^2_4$ having $\deg(R \cap F) = 4$ since $\deg(R \cap \varphi^*(\varphi)) \leq 1$ for any $\varphi \in Y$. Thus we get $-\phi(R) - H \geq 0$ where $F = \phi^* H$ for $H \in g^2_4$. Assume that $\deg \varphi_{k_2} = 1$. Then we have $h^2_4 = |K_Y + \varphi_1 + \varphi_2|$ where $\varphi_{k_2}(\varphi_1) = \varphi_{k_2}(\varphi_2)$ is the double point in $\varphi_{k_2}(Y)$. Thus it follows that
\[|K_Y + \varphi_1 + \varphi_2| = |\phi(R) + K_Y - g_4^2| = |K_Y + (\phi(R) - H)|, \]

since \( h_4^2 = |\phi(R) + K_Y - g_4^2| \). Since \( \varphi_{h_4^2}(Y) \) has the unique double point as its singularities, we have \( \phi(R) \geq \varphi_1 + \varphi_2 \). It is the first case of (ii) in the proposition. If \( \text{deg} \varphi_{h_4^2} = 2 \), then \( h_4^2 = |2K_Y| = |\phi(R) + K_Y - g_4^2| = |K_Y + (\phi(R) - H)| \), whence \( \phi(R) \geq K_Y \) for some \( K_Y \in |K_Y| \). This is the second case of (ii) in the proposition.

In case \((n, \text{deg}R) = (3, 7)\), Eq. (5.4) tells that \( \text{deg}(R \cap F) = 5 \) for some \( F \in \phi^* g_3^1 \), since \( \text{deg}(R \cap \phi'((\varphi))) \leq 1 \) for any \( \varphi \in Y \). By the similar arguments as in the above, we have either \( h_4^2 = |K_Y + \varphi_1 + \varphi_2| = |\phi(R) + K_Y - g_3^2| = |K_Y + (\phi(R) - H)| \) or \( h_4^2 = |2K_Y| = |\phi(R) + K_Y - g_3^2| = |K_Y + (\phi(R) - H)| \), where \( F = \phi^* H \) for \( H \in g_3^1 \). This yields to the cases of (ii) in the proposition.

If \( L \cong K_X - (\phi^* g_3^1 + B) + R \) with \( \text{deg}R = 4 \) then Eq. (5.4) yields that \( \text{deg}(R \cap F) \geq 2 \) for some \( F \in \phi^* g_3^1 \). In sum, we prove that \( L \) is very ample unless \( R \) falls into the cases (i) and (ii).

\[
\text{(i)} \quad \text{If} \ \text{deg}(R \cap F) \geq 2 \ \text{for some} \ F \in \phi^* g_3^1, \ \ell = 2, 3, \ \text{then} \ L \cong K_X - ((\phi^* g_3^1(-R \cap F) + B) + (R - R \cap F)) \ \text{with} \ h^0(X, \phi^* g_3^1(-R \cap F)) \geq 1 \ \text{and} \ \text{deg}(R - R \cap F) \leq 2. \ \text{It tells that} \ L \ \text{is not very ample.}
\]

\[
\text{(ii)} \quad \text{Let us consider the first case deg} \ \text{deg} \ varphi_{h_4^2} = 1 \ \text{with} \ \varphi_1 + \varphi_2 \leq \phi(R) \ \text{and} \ \varphi_{h_4^2}(\varphi_1) = \varphi_{h_4^2}(\varphi_2). \ \text{Assume that} \ L \ \text{is very ample. Then the very ampleness of} \ L \ \text{enables us to get the commutative diagram in Figure 1 such that} \ \varphi_{L}(R) \ \text{and} \ \varphi_{h_4^2}(\phi(R)) \ \text{have the same geometric properties in each corresponding spanned space. Thus it follows} \ \varphi_{h_4^2}(\varphi_1) \neq \varphi_{h_4^2}(\varphi_2) \ \text{which is a contradiction. Therefore} \ L \ \text{is not very ample in the first case of (ii).}
\]

Now we consider the second case that \( \text{deg} \ varphi_{h_4^2} = 2 \) and \( \text{deg}(\phi(R) \cap K_Y) = 2 \) for some \( K_Y \in |K_Y| \). Set \( \varphi + \varphi' := (\phi(R) \cap K_Y) \). Since \( \varphi + \varphi' \in K_Y \), it follows that \( \varphi_{h_4^2}(\varphi) = \varphi_{h_4^2}(\varphi') \) since \( \text{deg} \ varphi_{h_4^2} = 2 \) implies \( h_4^2 = |2K_Y| \). Assume that \( L \) is very ample. As in the first case of (ii), the commutative diagram in Figure 1 yields \( \varphi_{h_4^2}(\varphi) \neq \varphi_{h_4^2}(\varphi') \), which cannot occur. Hence \( L \) is not very ample in the second case of (ii). Therefore we get the conclusion of the proposition. \( \square \)

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References

[1] Arbarello, E., Cornalba, M., Griffiths, P., Harris, J. (1985). Geometry of Algebraic Curves. Berlin; Heidelberg; New York; Tokyo: Springer Verlag.
[2] Akahori, K. (2019). Normal generation and covers of small degree. Commun. Algebra. 47(2):611–623. DOI: 10.1080/00927872.2018.1492584.
[3] Ballico, E., Fontanari, C. (2010). Normally generated line bundles on general curves. II. J. Pure Appl. Alg. 214(8):1450–1455. DOI: 10.1016/j.jpaa.2009.08.002.
[4] Ballico, E., Keem, C., Kim, S. (2003). Normal generation of line bundles on a general k-gonal algebraic curve. Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. 6(8):557–562.

[5] Choi, Y., Kim, S., Kim, Y. (2007). Normal generation and Clifford index on algebraic curves. Math. Z. 257(1):23–31. DOI: 10.1007/s00209-007-0112-9.

[6] Green, M., Lazarsfeld, R. (1986). On the projective normality of complete linear series on an algebraic curve. Invent Math. 83(1):73–90. DOI: 10.1007/BF01388754.

[7] Hartshorne, R. (1977). Algebraic Geometry. New York; Heidelberg: Springer-Verlag.

[8] Kim, S. (2010). Normal generation of line bundles on multiple coverings. J. Alg. 323(9):2337–2352. DOI: 10.1016/j.jalgebra.2010.02.017.

[9] Kim, S., Kim, Y. (2004). Normal generation of line bundles on algebraic curves. J. Pure Appl. Alg. 192(1–3): 173–186. DOI: 10.1016/j.jpaa.2004.01.008.

[10] Kato, T., Keem, C., Ohbuchi, A. (1999). Normal generation of line bundles of high degrees on smooth algebraic curves. Abh. Math. Semin. Univ. Hambg. 69(1):319–333. DOI: 10.1007/BF02940883.

[11] Lange, H., Martens, G. (1985). Normal generation and presentation of line bundles of low degree on curves. J. Reine Angew. Math. 356:1–18. DOI: 10.1515/crll.1985.356.1.