CONTINUOUS-TIME MEAN-VARIANCE PORTFOLIO SELECTION WITH NO-SHORTING CONSTRAINTS AND REGIME-SWITCHING

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Abstract. The present article investigates a continuous-time mean-variance portfolio selection problem with regime-switching under the constraint of no-shorting. The literature along this line is essentially dominated by the Hamilton-Jacobi-Bellman (HJB) equation approach. However, in the presence of switching regimes, a system of HJB equations rather than a single equation need to be tackled concurrently, which might not be solvable in terms of classical solutions, or even not in the weaker viscosity sense as well. Instead, we first introduce a general result on the sign of geometric Brownian motion with jumps, then derive the efficient portfolio and frontier via the maximum principle approach; in particular, we observe, under a mild technical assumption on the initial conditions, that the no-shorting constraint will consistently be satisfied over the whole finite time horizon. Further numerical illustrations will be provided.

1. Introduction. Mean-variance portfolio selection is pioneered by Markowitz in 1952 (see Markowitz (1952)). The main idea is to find the optimal portfolio weights among assets to achieve the optimal trade-off between the mean and the variance of the portfolio return. Thereafter, Markowitz’s work was extended in various aspects. To name a few, Samuelson (1969) extended the work of Markowitz to a dynamic model and considered a discrete-time consumption-investment model with the objective of maximizing the overall expected consumption. Merton (1969, 1971) adapted a continuous time stochastic optimal control to model and then obtain the optimal portfolio strategy which results in the two-fund separation theorem. By using embedding method, Li and Ng (2000) extended Markowitz’s model to a discrete dynamic setting, and both the optimal strategy and the efficient frontier were obtained explicitly.

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Our present work considers a continuous-time optimal mean-variance portfolio selection problem over the regime-switching setting, or Markov-modulated models. Generally, in a regime-switching model, the market mode can take values in one of finitely many regimes. The key parameters, such as the bank interest rate, or stock appreciation and volatility rates, will change according to the different market modes being taken. Regime switching models have been applied to portfolio selection problems by many authors. Sotomayor and Cadenillas (2009) gave the explicit solution for an optimal consumption-investment problem under the utility maximization through HJB equation approach. Çanakgoğlu and Özekici (2010) solved an optimal investment problem explicitly under HARA utility maximization. The regime-switching concepts was originally applied to mean-variance portfolio selection by Zhou and Yin (2003). Chen et al. (2008) extended their work by incorporating an uncontrollable liability process modelled by a Markov-modulated geometric Brownian motion.

However, all the models mentioned so far allow short-selling in their investment strategies. In finance, short-selling (also known as shorting or going short) is the practice of selling assets that are not currently owned, and subsequently repurchasing them (“covering”) if needed. In the event of an interim price decline, the short seller will profit since the cost of repurchase will be less than the received amount from the preceding sale. Conversely, the short position would be closed out at a loss in the event that the price of a shorted asset should rise prior to repurchase. In a nutshell, short-selling allows you to profit on the downside. While there could be a sharp growth for the price of the asset, the corresponding potential loss could be substantial, which makes short-selling a highly risky practice; therefore, many brokerage firms do not have a favor on it, and forbid to take it in general.

When short-selling of assets is not allowed, one will face at a continuous-time no-shorting constrained mean-variance portfolio selection problem. This type of problem can naturally be solved by one or a hybrid of the following two approaches: Maximum Principle (Riccati equation) and HJB equation approaches. The former approach involves a completion of squares for the optimal strategy, and then for the remaining terms, one lets the coefficients of the wealth process and its square be zero, which leads to a pair of ordinary differential equations (ODEs): one of which is called Riccati equation, while the other one is used to handle the nonhomogeneous terms. The Riccati equation approach is applied widely when short-selling is permitted. One of the merits is its neat computation, see Zhou and Yin (2003) and Chen et al. (2008). However, when the short-selling is prohibited, due to the stochastic nature of the wealth process, the resulting optimal feedback type strategy may not satisfy the no-shorting constraint for all the time. This difficulty can be tackled by adopting piecewise expression for the optimal strategy via the HJB equation approach, see Li et al. (2002), Bai and Zhang (2008), Bi et al. (2011), Chen and Zhang (2016) and Li and Xu (2016). That is why the literature along this line is dominated by the HJB equation approach.

Our paper distinguishes the existing literature mainly in two aspects. First of all, unlike Li et al. (2002) or Bai and Zhang (2008), in which the deterministic market parameters (such as stock appreciation rate) were considered, we incorporate regime-switching parameters. Note that in Zhang and Chen (2016), they also considered a regime-switching model, whereas the interest rate process was still left as deterministic. When the regime switching is involved, a system of HJB equations will be derived corresponding to different market states, which is very difficult to
solve. In particular, when the interest rate process is also regime-switching, the resulting HJB equations seems even unsolvable. This explains why many regime switching papers left the interest rate process as deterministic. For example, see Wei et al. (2013), Wu and Chen (2015) and Zhang and Chen (2016). In our paper, we adopt regime-switching parameters for both the risk-free asset and the risky assets. We tackle the problem by the Riccati equation approach, where the regime-switching interest process is no longer vexing in the completion of square procedure.

Secondly, we incorporate the prohibition of short-selling in our model. This paper is an extension of the continuous-time mean-variance portfolio selection model proposed by Zhou and Yin (2003). In our knowledge, this type of constrained problems is barely resolved by the Riccati equation approach in the existing literature. We tackle the difficulty by studying the stochastic feature of the resulting wealth process. It is interesting to find that the dynamics of the resulting stochastic process with respect to the optimal feedback strategy follows a geometric Brownian motion (GBM) with jumps. One of useful features of GBM is sign-preserving as with that of the initial value. This property can also be proved for GBM with jumps under some conditions. Hence, if we assume that all the initial values corresponding to different market states take the same sign, the resulting stochastic process also preserves the same sign over the whole finite time horizon. Then we derive a uniform expression for the investment strategy instead of a piecewise zero expression in Li et al. (2002). Comparing with the HJB equation approach, we may impose an assumption on the initial values under each market state. We shall show in Section 3 that the assumption is nothing special than a reasonable one, namely on the target expected wealth value, that should be not less than the accumulated value at the risk-free rate.

Our paper is organized as follows. Section 2 formulates the mean-variance portfolio selection problem over regime-switching setting. The effect of no-shorting constraint on the optimal portfolio is considered in Section 3. Section 4 gives the optimal strategy and efficient frontier. Further numerical examples are given in Section 5. Finally, we conclude in Section 6.

2. Problem formulation. Throughout the paper, let \((\Omega,\mathcal{F},P)\) be a filtered complete probability space on which are defined with a standard \(d\)-dimensional Brownian motion \(W(t) := (W_1(t),...,W_d(t))'\) and a continuous-time stationary Markov chain \(\alpha(t)\) taking values in a finite state space \(\mathcal{M} = \{1,2,...,l\}\), such that \(W(t)\) and \(\alpha(t)\) are independent of each other. The Markov chain \(\alpha(t)\) has a generator \(Q = (q_{ij})_{l \times l}\) (transition rate matrix) and stationary transition probabilities:

\[ p_{ij}(t) = \mathbb{P}(\alpha(t) = j|\alpha(0) = i), \quad t \geq 0, \quad i,j = 1,2,...,l. \]

Define \(\mathcal{F}_t := \sigma\{W(s),\alpha(s) : 0 \leq s \leq t\}\). We denote by \(L^2_{\mathcal{F}}(0,T;\mathbb{R}^m)\) the set of all \(\mathbb{R}^m\)-valued, measurable stochastic processes \(f(t)\) adapted to \(\{\mathcal{F}_t\}_{t \geq 0}\), such that \(\mathbb{E}\int_0^T \|f(t)\|^2 dt < \infty\). We shall also use the following notations: \(M'\): the transpose of any matrix or vector \(M\);

\[ ||M||: \sqrt{\sum_{ij} m_{ij}^2} \text{ for any matrix or vector } M = (m_{ij}); \]
\(\mathbb{R}^n_+\): the subset of \(\mathbb{R}^n\) consisting of elements with nonnegative components;
\(C^2([0,T] \times \mathbb{R}^n):\) the space of all twice continuously differentiable functions on \([0,T] \times \mathbb{R}^n\);
\(L^2_{\mathcal{F}}(0,T;X):\) the Hilbert space of \(X\)-valued integrable functions \(f\) on \([0,T]\), being
adapted
\{F_t\}_{t \geq 0}, endowed with the norm \( (\int_0^T \| f(t) \|^2_X dt)^{\frac{1}{2}} \) for a given Hilbert space \( X \).

We consider a financial market in which \( d + 1 \) assets (or securities) are traded continuously. One of the assets is a bank account whose price process \( P_0(t) \) is subject to the following stochastic ODE:
\[
\begin{aligned}
&\begin{cases}
    dP_0(t) = r(t, \alpha(t))P_0(t)dt, & t \in [0, T], \\
    P_0(0) = p_0 > 0,
\end{cases} \\
\end{aligned}
\]
where \( r(t,i) \geq 0 \), for \( i = 1, 2, \ldots, l \), are given as the interest rate processes of the bank account corresponding to each market mode \( i \). The other \( d \) assets are stocks whose price processes \( P_m(t) \), \( m = 1, 2, \ldots, d \), satisfy the system of SDEs:
\[
\begin{aligned}
&\begin{cases}
    dP_m(t) = P_m(t) \left\{ b_m(t, \alpha(t))dt + \sum_{n=1}^{d} \sigma_{mn}(t, \alpha(t))dW_n(t) \right\}, & t \in [0, T], \\
    P_m(0) = p_m > 0,
\end{cases} \\
\end{aligned}
\]
where for each \( i = 1, 2, \ldots, l \), \( b_m(t, i) \) is the appreciation rate process and \( \sigma_{m}(t, i) = (\sigma_{m1}(t, i), \ldots, \sigma_{mn}(t, i)) \) is the volatility or the dispersion rate process of the \( n \)th stock, corresponding to the state \( \alpha(t) = i \). The volatility matrix process of assets is defined as \( \sigma(t, i) := (\sigma_{mn}(t, i))_{d \times d} \). As widely adopted in the literature, we impose the following assumptions.

**Assumption 2.1.** The interest rate process \( r(t, i) \) is non-negative under all market modes.

**Assumption 2.2.** The appreciation processes \( b_m(t, i) > r(t, i) \) for \( i = 1, 2, \ldots, l \) and \( m = 1, 2, \ldots, d \).

**Assumption 2.3.** The nondegeneracy condition holds. That is, there is a \( \delta > 0 \) such that,
\[ \sigma(t, i)\sigma(t, i)' \geq \delta I_d \text{ for all } t \in [0, T] \text{ and } i = 1, 2, \ldots, l, \]
where \( I_d \) is a \( d \times d \) identity matrix.

**Assumption 2.4.** All the functions \( r(t, i), b_m(t, i), \) and \( \sigma_{mn}(t, i) \) are measurable and uniformly bounded for \( t \in [0, T] \), all \( i = 1, 2, \ldots, l \), and \( m, n = 1, 2, \ldots, d \).

Suppose that the initial market mode \( \alpha(0) = i_0 \). Consider an investor with initial wealth \( x_0 \). Denote the total wealth process of the investor by \( x(t) \) at time \( t \in [0, T] \). Let \( u_m(t) \) be the total market value of the investor’s wealth in the \( n \)th asset, \( m = 0, 1, \ldots, d \), and we have
\[ x(t) = \sum_{m=0}^{d} u_m(t), \text{ for all } t \in [0, T], \]
where \( u(\cdot) = (u_1(\cdot), \ldots, u_d(\cdot))' \) is called a portfolio of the investor. Assuming that the trading of shares takes place continuously and that neither are those transaction costs nor consumptions incurred, one has
\[
\begin{aligned}
dx(t) &= \left\{ r(t, \alpha(t))x(t) + \sum_{m=1}^{d} [b_m(t, \alpha(t)) - r(t, \alpha(t))]u_m(t) \right\}dt \\
&\quad + \sum_{n=1}^{d} \sum_{m=1}^{d} \sigma_{mn}(t, \alpha(t))u_m(t)dW_n(t), \\
&= [r(t, \alpha(t))x(t) + B(t, \alpha(t))u(t)]dt + u(t)'\sigma(t, \alpha(t))dW(t), \quad (3)
\end{aligned}
\]
where
\[ B(t, i) := (b_1(t, i) - r(t, i), ..., b_d(t, i) - r(t, i))^\prime, \quad i = 1, 2, ..., l. \] (4)

Note that once \( u \) is determined, \( u_0 \), the allocation in the bank account is uniquely determined by \( x(t) - \sum_{m=1}^{d} u_m(t) \). An important restriction imposed in this paper is the prohibition of short-selling of stocks, i.e. it must be satisfied that \( u_m(t) \geq 0, \quad m = 1, ..., d \). On the other hand, borrowing from the money market (at the interest rate \( r(t, \alpha(t)) \)) is still allowed; that is, \( u_0 \) is not explicitly constrained.

**Definition 2.1.** A portfolio \( u \) is said admissible if \( u(\cdot) \in L_2^f(0, T; \mathbb{R}^{d+}) \) and the SDE (3) has a unique solution \( x(t) \) corresponding to \( u(t) \). In this case, we refer to \((x, u)\) as an admissible pair.

The investor’s objective is to find an investment strategy \( u \) such that the expected terminal wealth is \( E[x(T) = z] \) for some given \( z \in \mathbb{R}^1 \), while the risk measured by the variance of the terminal wealth
\[ \text{Var}x(T) = E[(x(T) - E[x(T)])^2] = E[x(T) - z]^2, \] (5)
is minimized. Finding such a portfolio \( u \) is referred to as the mean-variance portfolio selection problem. Specifically, we have the following formulation.

**Definition 2.2.** The mean-variance portfolio selection is a constrained stochastic optimization problem, parameterized by \( z \in \mathbb{R}^1 \):
\[
\begin{align*}
\text{minimize} \quad & J(x_0, i_0, u) := E[(x(T) - z)^2], \\
\text{subject to} \quad & E[x(T) = z, \\
& (x, u) \text{ is admissible}. \quad (6)
\end{align*}
\]

Moreover, the problem is called feasible if there is at least one portfolio satisfying all the constraints. The problem is called finite if it is feasible and the infimum of \( J(x_0, i_0, u(\cdot)) \) is finite. Finally, an optimal portfolio of the problem (6), if it ever exists, is called an efficient portfolio corresponding to \( z \); while the corresponding \((\text{Var}(x(T)), z) \in \mathbb{R}^2 \) and \((\sigma_{x(T)}, z) \in \mathbb{R}^2 \) are interchangeably called an efficient point, where \( \sigma_{x(T)} \) denotes the standard deviation of \( x(T) \). The set of all the efficient points is called the efficient frontier.

For the convex optimization problem (6), the equality constraint \( E[x(T) = z] \) can be tackled by introducing a Lagrangian multiplier. Define
\[
\begin{align*}
J(x_0, i_0, u, \lambda) := & \quad E\{(x(T) - z)^2 + 2\lambda[x(T) - z]\} \\
= & \quad E[x(T) + \lambda - z]^2 - \lambda^2, \quad \lambda \in \mathbb{R}^1.
\end{align*}
\]
Clearly Problem (6) is equivalent to
\[
\begin{align*}
\text{minimize} \quad & J(x_0, i_0, u, \lambda) := E[x(T) + \lambda - z]^2 - \lambda^2, \\
\text{subject to} \quad & (x, u) \text{ is admissible}. \quad (7)
\end{align*}
\]
This is a Markov-modulated stochastic LQ problem, which will be solved in Section 4.
3. Preliminary results. This section introduces some preliminary results before we approach Problem (7). Throughout this paper, the generalized Itô lemma for Markov-modulated processes (see Björk (1980)) will play a key role.

Lemma 3.1. Given a n-dimensional process \( x \) satisfying

\[
dx(t) = b(t, x(t), \alpha(t))dt + \sigma(t, x(t), \alpha(t))dW(t),
\]

and a family of functions \( \varphi(\cdot, \cdot, i) \in C^2([0, T] \times \mathbb{R}^n), \) \( i = 1, 2, ..., l \), we have

\[
d\varphi(t, x(t), \alpha(t)) \equiv \Delta_1 \varphi(t, x(t), \alpha(t))dt + \varphi_x(t, x(t), \alpha(t))\sigma(t, x(t), \alpha(t))dW(t) + dM(t),
\]

where

\[
\Delta_1 \varphi(t, x, i) := \varphi_t(t, x, i) + b(t, x, i)'\varphi_x(t, x, i) + \frac{1}{2} \text{tr}[\sigma(t, x, i)'\varphi_{xx}(t, x, i)\sigma(t, x, i)]
\]

\[\quad + \sum_{j=1}^{l} q_{ij} \varphi(t, x, j),\]

and

\[
dM(t)
\]

\[= \varphi(t, x(t), \alpha(t)) - \varphi(t, x(t), \alpha(t)) - \sum_{j \neq \alpha(t)} q_{ij} [\varphi(t, x, j) - \varphi(t, x, \alpha(t))] dt\]

\[= \varphi(t, x(t), \alpha(t)) - \varphi(t, x(t), \alpha(t)) - \sum_{j=1}^{l} q_{ij} \varphi(t, x, j) dt,\]

which is a compensated (mean zero) martingale over \( t \in [0, T] \). Or equivalently, we have

\[
d\varphi(t, x(t), \alpha(t)) = \Delta_2 \varphi(t, x(t), \alpha(t))dt + \varphi_x(t, x(t), \alpha(t))\sigma(t, x(t), \alpha(t))dW(t)
\]

\[+ \underbrace{\varphi(t, x(t), \alpha(t)) - \varphi(t, x(t), \alpha(t))}_{\text{jump}},\]

where

\[
\Delta_2 \varphi(t, x, i) := \varphi_t(t, x, i) + b(t, x, i)'\varphi_x(t, x, i) + \frac{1}{2} \text{tr}[\sigma(t, x, i)'\varphi_{xx}(t, x, i)\sigma(t, x, i)]
\]

\[= \Delta_1 \varphi(t, x, i) - \sum_{j=1}^{l} q_{ij} \varphi(t, x, j).\]

Remark 3.1. Note that when taking expectation of the corresponding time integral, the \( dM(t) \) term in (8) is often vanished due to its proper martingale nature, see Zhou and Yin (2003). Whereas we investigate the dynamics of stochastic processes, we usually use the form (9), also see more in Section 3.3.

3.1. Feasibility. Before we proceed on studying the optimality of the problem, we first address the solvability issue. In comparison with Zhou and Yin (2003), we have enforced an additional constraint of short-selling prohibition. In Problem (7), one aims to establish a control process \( u(t) \) only taking non-negative values such that the resulting wealth process also satisfies with the target constraint \( \mathbb{E}x(T) = z \).

Firstly, we consider the following system of linear ODEs with bounded coefficients, for \( i = 1, 2, ..., d \),

\[
\begin{align*}
\dot{\psi}(t, i) &= -r(t, i)\psi(t, i) - \sum_{j=1}^{d} q_{ij} \psi(t, j); \\
\psi(T, i) &= 1,
\end{align*}
\]

(10)
By using Picard iteration argument together with induction, it is clear that \( \psi(t,i) > 0 \) for all \( t \in [0,T], \ i = 1,2,...,d \). For similar techniques used, see Zhou and Yin (2003), also see Chen et al (2008) or Bensoussan et al. (2012). Then we have,

**Theorem 3.1.** Problem (7) is feasible whenever \( z \geq \psi(0,i_0)x_0 \).

**Proof.** We aim to establish an admissible portfolio that satisfies the constraint \( \mathbb{E}_x(T) = z \). For any \( \beta \in \mathbb{R} \), we consider the portfolio \( u^\beta(t,i) = \beta \psi(t,i)B(t,i) \), for any \( t \in [0,T] \) and \( i = 1,2,...,l \). Note that by definition and Assumption 2.2, all the elements of \( B(t,i) \) are positive, and so \( u^\beta(t,i) \) clearly satisfies the no-shorting constraint whenever \( \beta \geq 0 \). Denote the resulting wealth process by \( x^\beta(t) \). By applying Lemma 3.1 on the product of \( \psi(t,\alpha(t)) \) and \( x^\beta(t) \), we have

\[
\begin{align*}
\mathbb{E} (\psi(t,\alpha(t))x^\beta(t)) &= \psi(t,\alpha(t))x^\beta(t) - \int \sum_{j=1}^{l} q_{\alpha(t)-j} x^\beta(t)[\psi(t,j) - \psi(t,\alpha(t))]dt \\
&= \psi(t,\alpha(t))x^\beta(t) - \psi(t,\alpha(t)-)x^\beta(t) - \int \sum_{j=1}^{l} q_{\alpha(t)-j} x^\beta(t)\psi(t,j)dt.
\end{align*}
\]

Integrating (11) from 0 to \( T \), and then taking expectation, by invoking \( \psi(T,i) = 1 \) for all \( i = 1,2,...,l \) assets in (10), we obtain

\[
\mathbb{E}x^\beta(T) = \mathbb{E} (\psi(T,\alpha(T))x^\beta(T)) \\
= \psi(0,i_0)x_0 + \beta \mathbb{E} \left\{ \int_0^T B(t,\alpha(t)-)B(t,\alpha(t)-)\psi^2(t,\alpha(t)-)dt \right\}.
\]

By \( B(t,i) > 0 \) and the fact that \( \psi(t,i) > 0 \) for all \( i = 1,2,...,l \), one only requires that

\[
\beta = \frac{z - \psi(0,i_0)x_0}{\mathbb{E} \left\{ \int_0^T B(t,\alpha(t)-)B(t,\alpha(t)-)\psi^2(t,\alpha(t)-)dt \right\}} \geq 0,
\]

so as to ensure \( u^\beta \) to fulfill the no-shorting constraint with \( \mathbb{E}x^\beta(T) = z \) as well. □

**Remark 3.2.** In the case of no regime switching, the solution of (10) is \( \psi(t,i) = e^{\int_0^t r(s)ds} \), while \( x_0e^{\int_0^T r(s)ds} \) is simply the amount the investor would earn if all the initial wealth was invested purely in the bond over the whole time horizon \([0,T] \). And the condition in Theorem 3.1 suggests that it is also natural to only consider when the target is set at a higher rate than simply putting money in the bank.
Remark 3.3. In Zhou and Yin (2003), they imposed a condition:

\[ \mathbb{E}\left\{ \int_0^T B(t, \alpha(t))' B(t, \alpha(t)) dt \right\} > 0, \]  

(13)

which certainly holds as long as there is one stock with an appreciation-rate process being different from the interest-rate process in every market mode. In our paper, the condition (13) is implied by Assumption 2.2.

As an immediate result from the proof of Theorem 3.1, we have:

Corollary 3.1. For any \( z \geq \psi(0, i_0)x_0 \), an admissible portfolio that satisfies \( \mathbb{E}x(T) = z \) can be taken by setting the portfolio weight \( u^\beta(t, i) = \beta \psi(t, i) B(t, i) \), where \( \beta \) is given by equation (12).

3.2. Convex analysis results. In the following we recall some results from convex analysis in Xu and Shreve (1992).

Lemma 3.2. Let \( s \) be a continuous, strictly convex quadratic function

\[ s(y) := \frac{1}{2} \|(D'^{-1} y + (D')^{-1} B')^2 \]  

(14)

over \( y \in [0, +\infty)^m \), where \( B' \in \mathbb{R}_+^m \), \( D \in \mathbb{R}^{m \times m} \), and \( D'D > 0 \). Then \( s(y) \) has a unique minimizer \( \bar{y} \in [0, +\infty)^m \), i.e.,

\[ \|(D'^{-1} \bar{y} + (D')^{-1} B')^2 \leq \|(D'^{-1} y + (D')^{-1} B')^2, \quad y \in [0, +\infty)^m. \]

The Kuhn-Tucker conditions for the minimization of \( s \) in (14) over \( [0, +\infty)^m \) lead to the Lagrange multiplier vector \( \bar{v} \in [0, +\infty)^m \) such that \( \bar{v} = (D'D)^{-1} \bar{y} + (D'D)^{-1} B' \) and \( \bar{v}' \bar{y} = 0 \). The explicit expression of the minimizer is given in Lemma 3.3 which is also studied by Li and Xu (2016), here we include it for self-completeness.

Lemma 3.3. Let \( h \) be a continuous, strictly convex quadratic function

\[ h(y) = \frac{1}{2} y'D'D y - \alpha By \]  

(16)

over \( y \in [0, +\infty)^m \), where \( B' \in \mathbb{R}_+^m \), \( D \in \mathbb{R}^{m \times m} \), and \( D'D > 0 \).

1. For every \( \alpha \geq 0 \), \( h \) has the unique minimizer \( \alpha D^{-1} \bar{\xi} \in [0, +\infty)^m \), where \( \bar{\xi} = (D'^{-1} \bar{y} + (D')^{-1} B' \). Here \( \bar{y} \) is the minimizer of \( s \) specified in Lemma 3.1. Furthermore, \( \bar{y}'D^{-1} \bar{\xi} = 0 \) and

\[ h(\alpha \bar{v}) = h(\alpha D^{-1} \bar{\xi}) = -\frac{1}{2} \alpha^2 \|\bar{\xi}\|^2. \]

(17)

2. For every \( \alpha < 0 \), \( h(y) \) has the unique minimizer 0.

Remark 3.4. Lemma 3.1 shows the existence of the minimizer \( \bar{y} \) with non-negative elements. For different \( D \) and \( B \), the solution for \( \bar{y} \) would be different. Generally speaking, there is no explicit expression of \( \bar{y} \) in terms of \( D \) and \( B \). Only in some special cases, we can find the expression for \( \bar{y} \). One can easily show that \( \bar{y} = 0 \) when \( m = 1 \). That is, in the case of scalar valued \( D \) and \( B \), which corresponds to the LQ problem (7) where there is only one risky asset, we have the minimizer over \( [0, +\infty) \) is \( \bar{y} = 0 \). Another special case is when all the elements of \( (D'D)^{-1} B' \) are...
non-negative, then the minimizer \( \bar{y} \) over \([0, +\infty)^m\) is the \( m \)-dimensional 0 vector. In fact, by simple calculation, the expression of (14) can be written as

\[
s(y) = \frac{1}{2}(y'D^{-1} + BD^{-1})[(D')^{-1}y + (D')B'] \\
= \frac{1}{2}y'(D'D)^{-1}y + 2y'(D'D)^{-1}B' + B(D'D)^{-1}B' \\
\geq \frac{1}{2}[0 + 0 + B(D'D)^{-1}B'],
\]

where the minimum can be achieved when \( \bar{y} \) is a \( m \)-dimensional zero vector.

Now let us come back to the unconstrained LQ problem (7). For any \( t \in [0, T] \) and \( i = 1, 2, ..., l \), let

\[
y(t, i) = \arg \min_{y(t, i) \in [0, +\infty)^d} \frac{1}{2}\|\sigma(t, i)^{-1}y(t, i) + \sigma(t, i)^{-1}B(t, i)\|^2
\]

and

\[
\xi(t, i) = (\sigma(t, i)^{-1}\bar{y}(t, i) + (\sigma(t, i)^{-1}B(t, i) \in [0, +\infty)^d.
\]

Note that \( \xi(t, i) \) is a column vector independent of the wealth process \( x(t) \).

### 3.3. A class of control with short-selling prohibition.

In this subsection, we investigate the property of a class of feedback controls. We show that the no-short-selling constraint can be satisfied under some assumptions on the initial conditions. We will use those properties to show admissibility of the optimal control in the next section. Given any vector process \( \nu(t, i) \in \mathbb{R}^d \) for \( t \in [0, T] \) and \( i = 1, 2, ..., l \), we define a pair of adjoint equations:

\[
\begin{cases}
\dot{P}(t, i) = [2B(t, i)(\sigma(t, i)'), -\nu(t, i)']P(t, i) - \sum_{j=1}^l q_{ij}P(t, j), \\
P(T, i) = 1, \quad i = 1, 2, ..., l,
\end{cases}
\]

and

\[
\begin{cases}
\dot{H}(t, i) = r(t, i)H(t, i) - \frac{1}{m} \sum_{j=1}^l q_{ij}P(t, j)[H(t, j) - H(t, i)], \\
H(T, i) = 1, \quad i = 1, 2, ..., l.
\end{cases}
\]

**Remark 3.5.** The existence and uniqueness of the respective solutions of the above two systems of equations are evident as both are linear with continuous coefficients. As shown by Zhou and Yin (2003), we can also use Picard converging sequence techniques to show that the solutions of (21) and (22) must be positive.

Based on (21) and (22), we investigate the following type of feedback control,

\[
u(t, x, i) = -[\sigma(t, i)']^{-1}\nu(t, i)[x + (\lambda - z)H(t, i)].
\]

Plugging the feedback strategy (23) into the dynamics of the wealth process (3), we have the resulting wealth process,

\[
dx(t) = \left\{r(t, \alpha(t))x(t) - B(t, \alpha(t))'(\sigma(t, \alpha(t))')^{-1}\nu(t, \alpha(t))
\right. \\
\left. [x(t) + (\lambda - z^*)H(t, \alpha(t))]\right)dt - [x(t) + (\lambda - z^*)H(t, \alpha(t))]\nu(t, \alpha(t))'dW(t).
\]
Applying the generalized Itô formula to $P(t, \alpha(t))[x(t) + (\lambda - z)H(t, \alpha(t))]$, we obtain
\[
d\{P(t, \alpha(t))[x(t) + (\lambda - z)H(t, \alpha(t))]
= P(t, \alpha(t-))[x(t) + (\lambda - z)H(t, \alpha(t-))]dt + P(t, \alpha(t-))\{r(t, \alpha(t-))x(t)
- [x(t) + (\lambda - z)H(t, \alpha(t-))]B(t, \alpha(t-))'(\sigma(t, \alpha(t-)))^{-1} \nu(t, \alpha(t-))\}dt
- P(t, \alpha(t-))[x(t) + (\lambda - z)H(t, \alpha(t-))]\nu(t, \alpha(t-))'dW(t)
+ (\lambda - z)P(t, \alpha(t-))\{r(t, \alpha(t-))H(t, \alpha(t-))
- \frac{1}{P(t, \alpha(t-))}\sum_{j=1}^{l}q_{a(t-)}P(t, j)[H(t, j) - H(t, \alpha(t-))]\}dt
+ P(t, \alpha(t-))[x(t) + (\lambda - z)H(t, \alpha(t-))] - P(t, \alpha(t-))[x(t) + (\lambda - z)H(t, \alpha(t-))]
\]
\]
\[
\]
\[
\]
\[
= \begin{cases}
\left\{2B(t, \alpha(t-))'(\sigma(t, \alpha(t-)))^{-1} \nu(t, \alpha(t-)) - \nu(t, \alpha(t-))'\xi(t, \alpha(t-))
- 2r(t, \alpha(t-))P(t, \alpha(t-))[x(t) + (\lambda - z)H(t, \alpha(t-))] + P(t, \alpha(t-))r(t, \alpha(t-))x(t)
- P(t, \alpha(t-))[x(t) + (\lambda - z)H(t, \alpha(t-))]B(t, \alpha(t-))'(\sigma(t, \alpha(t-)))^{-1} \nu(t, \alpha(t-))
+ (\lambda - z)P(t, \alpha(t-))r(t, \alpha(t-))H(t, \alpha(t-))
- \frac{1}{P(t, \alpha(t-))}\sum_{j=1}^{l}q_{a(t-)}P(t, j)[x(t) + (\lambda - z)H(t, \alpha(t-))] + (\lambda - z)H(t, \alpha(t-))\right\}dt
- P(t, \alpha(t-))[x(t) + (\lambda - z)H(t, \alpha(t-))]\nu(t, \alpha(t-))'dW(t)
+ P(t, \alpha(t-))[x(t) + (\lambda - z)H(t, \alpha(t-))] - P(t, \alpha(t-))[x(t) + (\lambda - z)H(t, \alpha(t-))]
\end{cases}
\]
\]
\[
\]
\[
\]
\[
= \left\{B(t, \alpha(t-))'(\sigma(t, \alpha(t-)))^{-1} \nu(t, \alpha(t-)) - \nu(t, \alpha(t-))'\right\}
\]
\[
\times P(t, \alpha(t-))[x(t) + (\lambda - z)H(t, \alpha(t-))] - \frac{1}{P(t, \alpha(t-))}\sum_{j=1}^{l}q_{a(t-)}P(t, j)[x(t) + (\lambda - z)H(t, j)]\right\}dt
- P(t, \alpha(t-))[x(t) + (\lambda - z)H(t, \alpha(t-))]\nu(t, \alpha(t-))'dW(t)
+ P(t, \alpha(t-))[x(t) + (\lambda - z)H(t, \alpha(t-))] - P(t, \alpha(t-))[x(t) + (\lambda - z)H(t, \alpha(t-))]
\right\}. \quad (25)
\]

Note that in (25) the dynamics of $P(t, \alpha(t))[x(t) + (\lambda - z)H(t, \alpha(t))]$ is in the form of geometric Brownian motion with regime-switching. Without the presence of regime-switching, the solution to the SDE (25) takes the exponential form which means that the sign of the process is determined by the initial value. In the following we show that the sign-preserving property still holds with the presence of regime-switching. This is included in a general result for multi-dimensional jump processes.

Let $Y(t)$ be a general stochastic process with jumps among $Y_1(t), Y_2(t), ..., Y_l(t)$, where $Y_i(t)$ satisfies the following SDE,
\[
dY_i(t) = [a_{ii}(t)Y_i(t) + \sum_{j \neq i} a_{ij}(t)Y_j(t)]dt + Y_i(t)b_i(t)dW(t). \quad (26)
\]

Then we have the following result.

**Theorem 3.2.** Let the coefficients $a_{ij}(t) \geq 0$ for $j \neq i$, and all the initial values of $Y_i(0) \geq 0$ for $i = 1, 2, ..., l$, then we have $Y(t) \geq 0$ for $t > 0$. (Conversely, if all the initial values $Y_i(0) \leq 0$ for $i = 1, 2, ..., l$, then we have $Y(t) \leq 0$ for $t > 0$.)

**Proof.** By Theorem 2.1 in Hu (2000), we have $Y_i(t) \geq 0$ for $i = 1, 2, ..., l$. Since $Y(t)$ jumps among $Y_1(t), Y_2(t), ..., Y_l(t)$, then we have $Y(t) \geq 0$ for $t > 0$. \qed
Applying this theorem to \( P(t, \alpha(t))[x(t) + (\lambda - z)H(t, \alpha(t))] \) which can be taken as a stochastic process with jumps among the values of \( P(t, i)[x(t) + (\lambda - z)H(t, i)] \) for \( i = 1, 2, ..., l \). For fixed \( i = 1, 2, ..., l \), by (25), the dynamics of \( P(t, i)[x(t) + (\lambda - z)H(t, i)] \) can be described by

\[
\begin{align*}
\frac{d\{P(t, i)[x(t) + (\lambda - z)H(t, i)]\}}{dt} &= \left\{ B(t, i)(\sigma(t, i)')^{-1}\nu(t, i) - \nu(t, i)'\xi(t, i) - r(t, i) \right\} P(t, i)[x(t) + (\lambda - z)H(t, i)] \\
&\quad - \sum_{j=1}^{l} q_{ij} P(t, j)[x(t) + (\lambda - z)H(t, j)] dt \\
&\quad - P(t, i)[x(t) + (\lambda - z)H(t, i)]\nu(t, i)'dW(t)
\end{align*}
\]

Note that in (21) and (22), the dynamics of \( P(t, i) \) and \( H(t, i) \) for \( i = 1, 2, ..., l \) are described backwardly. Besides, remember that \( P(T, i) = H(T, i) = 1 \) for any \( i = 1, 2, ..., l \), then at \( t = T \), the terminal values of \( P(t, i)[x(t) + (\lambda - z)H(t, i)] \) are the same, namely \( x(T) + \lambda - z \). And also, from the property of transition rate matrix we have \( q_{ij} \geq 0 \) for \( j \neq i \). Therefore \( P(t, i)[x(t) + (\lambda - z)H(t, i)] \) preserves the same sign if we assume all the original values \( P(0, i)[x(0) + (\lambda - z)H(0, i)] \) have the same sign for \( i = 1, 2, ..., l \) (non-positive or non-negative). Then the value of the stochastic process \( P(t, \alpha(t))[x(t) + (\lambda - z)H(t, \alpha(t))] \) keeps the same sign with its original value. By Proposition 4.1 in Zhou and Yin (2003), the solutions of (29) must satisfy \( P(t, i) > 0 \) for all \( t \in [0, T] \) and \( i = 1, 2, ..., l \). Thus, we only need to introduce the assumption on the initial condition:

\[
x_0 + (\lambda - z^*)H(0, i) \leq 0 \quad \text{for all} \quad i = 1, 2, ..., l.
\] (27)

We show in the next section that this assumption is equivalent to a mild restriction on the expected terminal wealth \( z \).

**Remark 3.6.** Intuitively, the value of the stochastic process \( P(t, \alpha(t))[x(t)(\lambda - z)H(t, \alpha(t))] \) jumps among different levels of \( P(t, i)[x(t) + (\lambda - z)H(t, i)] \) for \( i = 1, 2, ..., l \). If all the levels remain at the same sign, so does \( P(t, \alpha(t))[x(t) + (\lambda - z)H(t, \alpha(t))] \). The graph I in Figure 1 is an illustration of our idea. For simplicity, we only include two regimes \( i = 1, 2 \). Then the process jumps between two levels of values. Due to the stochastic nature of the wealth process, we can see the value of \( P(t, \alpha(t))[x(t) + (\lambda - z)H(t, \alpha(t))] \) bounces between the boundaries by letting \( \alpha(t) = 1 \) and \( \alpha(t) = 2 \) for \( t \in [0, 1] \). To get a closer look, an amplified version for \( t \in [0, 0.1] \) is given in graph II in Figure 1.

4. **The optimal strategy and efficient frontier.** Based on our preliminary results, this section gives the analytical expression of the optimal feedback control and efficient frontier. First we define

\[
\rho(t, i) := B(t, i)\sigma(t, i)\sigma(t, i)'^{-1}B(t, i), \quad i = 1, 2, ..., l.
\] (28)

Then based on the vector \( \xi(t, i) \in [0, +\infty)^d \) defined in (20), we introduce the adjoint equations as follows.

\[
\begin{aligned}
\dot{P}(t, i) &= \rho(t, i) - \bar{y}(t, i)[\sigma(t, i)\sigma(t, i)']^{-1}\bar{y}(t, i) - 2r(t, i)P(t, i) - \sum_{j=1}^{l} q_{ij} P(t, j), \\
P(T, i) &= 1, \quad i = 1, 2, ..., l,
\end{aligned}
\] (29)
Figure 1. The value of the stochastic process \( P(t,\alpha(t))[x(t) + (\lambda - z)H(t,\alpha(t))] \).

and

\[
\begin{align*}
\dot{H}(t,i) &= r(t,i)H(t,i) - \frac{1}{P(t,i)} \sum_{j=1}^{l} q_{ij} P(t,j)[H(t,j) - H(t,i)], \\
H(T,i) &= 1, \quad i = 1, 2, \ldots, l.
\end{align*}
\]  

(30)

Based on \( P(t,i) \) and \( H(t,i) \), we define

\[
\theta(i_0) := \sum_{i=1}^{l} \sum_{j=1}^{l} \int_{0}^{T} p_{\alpha i}(t)q_{ij} P(t,j)[H(t,j) - H(t,i)]^2 dt > 0.
\]  

(31)

Then the optimal investment strategy is included in the following theorem.

**Theorem 4.1.** Suppose that \( z \geq \max\{\psi(0,i_0)x_0, 1-\theta(i_0)/H(0,i_0)x_0, \text{for } i \in \mathcal{M}\} \). Then we have

\[
P(0,i_0)H(0,i_0)^2 + \theta(i_0) - 1 < 0.
\]  

(32)

Moreover, the efficient portfolio under no-shorting constraint, corresponding to \( z \), as a function of the time \( t \), the wealth level \( x \), and the market mode \( i \), is

\[
\mathbf{u}^*(t,x,i) = -[\sigma(t,i)]^{-1}\xi(t,i)[x + (\lambda^* - z)H(t,i)],
\]  

(33)
where \( \tilde{\xi} \) is given by (20) and

\[
\lambda^* - z = \frac{z - P(0, i_0)H(0, i_0)x_0}{P(0, i_0)H(0, i_0)^2 + \theta(i_0) - 1}.
\]  

(34)

Furthermore, the optimal value of \( \text{Var}(x(T)) \), among all the wealth processes \( x(\cdot) \) satisfying \( Ex(T) = z \), is

\[
\text{Var}(x^*(T)) = \frac{P(0, i_0)H(0, i_0)^2 + \theta(i_0)}{1 - \theta(i_0) - P(0, i_0)H(0, i_0)^2} \left[ z - \frac{P(0, i_0)H(0, i_0)x_0}{P(0, i_0)H(0, i_0)^2 + \theta(i_0)} \right]^2 + \frac{P(0, i_0)\theta(i_0)}{P(0, i_0)H(0, i_0)^2 + \theta(i_0)} x_0^2.
\]  

(35)

**Proof.** Let \( u(t) \) be any admissible control and \( x(t) \) be the corresponding wealth process as given by (3). For simplicity, in the following calculation, the parameters \( r(t, \alpha(t--)), b_{\alpha}(t, \alpha(t--)), \sigma(t, \alpha(t--)), z(t, \alpha(t--)), \xi(t, \alpha(t--)) \) are written as \( r \), \( b_{\alpha} \), \( \sigma \), \( B \), \( z \) and \( \xi \), respectively. Applying the generalized Itô formula to \( P(t, \alpha(t)) [x(t) + (\lambda - z)H(t, \alpha(t))]^2 \) we obtain:

\[
d\{ P(t, \alpha(t))[x(t) + (\lambda - z)H(t, \alpha(t))]^2 \} = \{ \dot{P}(t, \alpha(t--)) [x(t) + (\lambda - z)H(t, \alpha(t--))]^2 \\
+ 2P(t, \alpha(t--)) [x(t) + (\lambda - z)H(t, \alpha(t--))] [\ddot{x}(t) \\
+ B^t u(t) + (\lambda - z)(\dot{H}(t, \alpha(t--)) + P(t, \alpha(t--))u(t)\sigma' u(t) \\
+ \sum_{j=1}^l q_{\alpha(t--)} P(t, j) [x(t) + (\lambda - z)H(t, j)]^2 \} dt + \{ \ldots \} dW(t) + dM(t)
\]

\[
= \{ P(t, \alpha(t--)) [u(t) - u^*(t, x(t), \alpha(t--))] [\sigma' u(t) - u^*(t, x(t), \alpha(t--)) ] \\
+ [\xi - 2B^t (\sigma')^{-1} \xi] [x(t) + (\lambda - z)H(t, \alpha(t--))]^2 \} dt \\
+ \{ [\dot{P}(t, \alpha(t--)) + 2rP(t, \alpha(t--))] x(t) \\
+ 2P(t, \alpha(t--)) [x(t) + (\lambda - z)H(t, \alpha(t--))] [\ddot{H}(t, \alpha(t--)) \\
+ r(\lambda - z)P(t, \alpha(t--))H(t, \alpha(t--)) + \sum_{j=1}^l q_{\alpha(t--)} P(t, j)(\lambda - z)H(t, j)] x(t) \\
+ P(t, \alpha(t--))(\lambda - z)^2 H^2(t, \alpha(t--)) + 2(\lambda - z)^2 P(t, \alpha(t--))H(t, \alpha(t--))\dot{H}(t, \alpha(t--)) \\
+ \sum_{j=1}^l q_{\alpha(t--)} P(t, j)(\lambda - z)^2 H^2(t, j) \} dt + \{ \ldots \} dW(t) + dM(t)
\]

\[
= \{ P(t, \alpha(t--)) [u(t) - u^*(t, x(t), \alpha(t--))] [\sigma' u(t) - u^*(t, x(t), \alpha(t--)) ] \\
+ [\xi - 2B^t (\sigma')^{-1} \xi] (\lambda - z)^2 H^2(t, \alpha(t--)) + 2(\lambda - z)^2 P(t, \alpha(t--))H(t, \alpha(t--))\dot{H}(t, \alpha(t--))
\]

\[
+ \sum_{j=1}^l q_{\alpha(t--)} P(t, j)(\lambda - z)^2 H^2(t, j) \} dt + \{ \ldots \} dW(t) + dM(t)
\]
+ \sum_{j=1}^{l} q_{o(t)} \lambda z^2 \left(H(t,j)^2\right) dt + \{...\} dW(t) + dM(t) \\
= P(t, \alpha(t)) \left\{ [u(t) - u^*(t, x(t), \alpha(t))]' (\sigma \sigma') [u(t) - u^*(t, x(t), \alpha(t))] \right\} dt \\
+ \left( \lambda - z \right)^2 \sum_{j=1}^{l} q_{o(t)} \lambda \left[H(t,o(t)) - H(t,j)^2\right] dt + \{...\} dW(t) + dM(t),

where $M(t)$ is a zero expected value martingale and $u^*(t, x, i)$ is defined in (33).

Integrating from 0 to $T$ and taking expectations and using $P(t, i) > 0$ for $i = 1, 2, ..., n$, we have

$$J(x_0, i_0, u(t), \lambda) = \mathbb{E}[x(T) + \lambda - z]^2 - \lambda^2$$

$$= \mathbb{E} \int_0^T P(t, \alpha(t)) \left\{ [u(t) - u^*(t, x(t), \alpha(t))]' (\sigma \sigma') [u(t) - u^*(t, x(t), \alpha(t))] \right\} dt \\
+ \left( \lambda - z \right)^2 \sum_{j=1}^{l} q_{o(t)} \lambda \left[H(t,o(t)) - H(t,j)^2\right] dt + \mathbb{E} \int_0^T P(0, i_0)[x_0 + (\lambda - z)H(0, i_0)]^2 - \lambda^2$$

$$\geq \mathbb{E} \int_0^T \left\{ (\lambda - z)^2 \sum_{j=1}^{l} q_{o(t)} \lambda \left[H(t,o(t)) - H(t,j)^2\right] dt + \mathbb{E} \int_0^T P(0, i_0)[x_0 + (\lambda - z)H(0, i_0)]^2 - \lambda^2$$

$$= \theta(i_0)(\lambda - z)^2 + P(0, i_0)[x_0 + (\lambda - z)H(0, i_0)]^2 - \lambda^2. \quad (36)$$

By Theorem 3.1, the mean-variance problem (6) is feasible for any $z \in [\psi(0, i_0)x_0, \infty)$. Namely, for any $z \in [\psi(0, i_0)x_0, \infty)$, there is an admissible portfolio $u(t)$ such that the corresponding solution $x(t)$ to (3) satisfies $\mathbb{E}_T(x) = z$. Moreover, using exactly the same approach as in the proof of Theorem 4.1, one can show that Problem (7) without the constraint $\mathbb{E}[x(T)] = z$ must have a finite optimal value; hence so does the problem (6). Therefore, (6) is finite for any $z \in [\psi(0, i_0)x_0, \infty)$. Then we obtain

$$\infty > J^*(x_0, i_0, u(t)) = \min_{\mathbb{E}_T(x) = z, \text{admissible } u(t)} J(x_0, i_0, u(t))$$

$$= \min_{\mathbb{E}_T(x) = z, \text{admissible } u(t)} \mathbb{E}\left[ x(T) - z \right]^2$$

$$= \max_{\lambda \in \mathbb{R}} \min_{\text{admissible } u(t)} \left\{ \mathbb{E}_T(x) + \lambda - z^2 - \lambda^2 \right\}. \quad (37)$$

By (36), $min_{\text{admissible } u(t)} \mathbb{E}[x(T)] + \lambda - z^2 - \lambda^2$ is in a quadratic form of $\lambda - z$. It follows from the finiteness of the value of $J^*(x_0, i_0, u(t))$ that (32) is proved. On the other hand, in view of (37), we maximize the quadratic function (36) over $\lambda - z$ and conclude that the maximizer is given by (34). The corresponding maximum value of $\min_{\text{admissible } u(t)} \mathbb{E}_T(x) + \lambda - z^2 - \lambda^2$ is the right-hand side of (35). Finally, the optimal control is given by (33) with $\lambda = \lambda^*$. Finally, we need to show that $u^*$ in (33) satisfies the no-shorting constraint. Plugging the expression of $\lambda^* - z$ in (34) into assumption (27), and applying the result $P(0, i_0)H(0, i_0)^2 + \theta(i_0) - 1 < 0$ in (32), we have

$$x_0 + \frac{z - P(0, i_0)H(0, i_0)x_0}{P(0, i_0)H(0, i_0)^2 + \theta(i_0) - 1} H(0, i_0)$$
of the Maximum Principle. We start our calculation from all the terms relating to Remark 4.1.

Plugging into assumption (27) we have

\[
\begin{align*}
\Rightarrow x_0(\theta(i_0) - 1) + zH(0, i_0) & \leq 0 \\
\Leftrightarrow x_0(\theta(i_0) - 1) + zH(0, i_0) & \geq 0 \\
\Leftrightarrow z & \geq \frac{1 - \theta(i_0)}{H(0, i_0)} x_0,
\end{align*}
\]

which has been stated at the beginning of the Theorem. Then under the assumption (38) and Theorem 3.2, the stochastic process \( x(t) + (\lambda^* - z)H(t, \alpha(t)) \) resulted from the efficient portfolio (33) becomes a negative process, then \(-[x(t) + (\lambda^* - z)H(t, \alpha(t))]\) is positive. Based on Lemma 3.2 and Lemma 3.3, the notation \( \xi(t, i) \) defined in (20) is a vector such that \( (\sigma(t, i)'^{-1}\xi(t, i) \) belongs to \([0, +\infty)^d\). While \( u^*(t, x(t), \alpha(t)) \) is the product of a positive stochastic process and \( (\sigma(t, i)'^{-1}\xi(t, i), \) then we conclude that each element of \( u^*(t, x(t), \alpha(t)) \) is nonnegative, that is, the no-shorting constraint is satisfied.

\[\square\]

Remark 4.1. Similar to Zhou and Yin (2003), the proof is a direct application of the Maximum Principle. We start our calculation from all the terms relating to \( u(t) \). A completion of square formula can be easily obtained by adding appropriate terms. By gathering up all the coefficients of \( x^2(t) \) and \( x(t) \), we find that all these terms can be canceled. Then the terms left in the equation are independent of \( u(t) \) and the stochastic wealth process \( x(t) \). The procedure used here is just the main idea of the Maximum Principle, which is a commonly used approach in solving stochastic optimal control problems and can lead to an explicit expression of the optimal control.

However, our calculation is a bit different with Zhou and Yin (2003). Note that in the expression of \( u^*(t, x, i) \) in (33), we include \( \xi(t, i) \) to ensure the elements of \( [\sigma(t, i)'^{-1}\xi(t, i) \) be non-negative. Then the admissibility of \( u^*(t, x(t), \alpha(t)) \), that is, the no-shorting constraint depends on the sign of the stochastic process \( x(t) + (\lambda^* - z)H(t, \alpha(t)) \), which is determined by its initial value \( x_0 + (\lambda^* - z)H(0, i_0) \) with the incorporation of the optimal strategy \( u^*(t, x(t), \alpha(t)) \) in (33). Then we only need to apply simple assumptions on the initial conditions to achieve admissibility.

Remark 4.2. To get some insights of assumption (27) or (38), we look at a special case of no regime-switching, by the definition (31) we have \( \theta(i) = 0 \). Besides, \( P(t, i) \) and \( H(t, i) \) degenerate to \( P(t) \) and \( H(t) \) in this case. Then the dynamics of \( H(t) \) in (30) can be solved explicitly, which is \( H(t) = \exp(-\int_t^T r(s)ds) \). Hence \( H(0) = \exp(-\int_0^T r(s)ds) \). By (34), we have

\[
\lambda^* - z = \frac{z - P(0)H(0)x_0}{P(0)H(0)^2 - 1}.
\]

Plugging into assumption (27) we have

\[
x_0 + (\lambda^* - z)H(0) = x_0 + \frac{[z - P(0)H(0)x_0]H(0)}{P(0)H(0)^2 - 1} = -x_0 + \frac{zH(0)}{P(0)H(0)^2 - 1} \leq 0.
\]

By the sign of \( P(0)H(0)^2 - 1 \) implied by (32) we have

\[
x_0 \leq zH(0) \iff z \geq x_0 \exp(\int_0^T r(s)ds).
\]

Note that this assumption is consistent with the one in Li et al. (2002) which states that the investor's expected wealth terminal wealth \( z \) cannot be less than \( x_0 \exp(\int_0^T r(s)ds) \), which coincides with the amount that the investor would earn if
all of the initial wealth were invested in the bond for the entire period \([0, T]\). Clearly, this is a reasonable assumption. For the general case with the presence of regime-switching, similar to Zhou and Yin (2003), \(H(t, i)\) has a financial interpretation: for fixed \((t, i)\), \(H(t, i)\) is a deterministic quantity representing the risk-adjusted discount factor at time \(t\) when the market mode is \(i\). They pointed out in their Remark 11 that \(H(t, i)\) are nothing else than a generalization of the discount factor between the present time to the terminal time under different market modes. In fact, comparing the no regime-switching assumption in (41) and the general case assumption in (38), the discount factor from time 0 to time \(T\) should be \(H(0, i_0)\) other than \(H(0, i_0)\). By (32) we have \(\theta(i_0) < 1 - P(0, i_0)H(0, i_0)^2 \leq 1\). Then we have the range \(0 < \theta(i_0) < 1\), that is, a fraction of \(\frac{1}{1 - \theta(i_0)}\) more discount should be applied from time 0 to \(T\). Equivalently, we would earn \(\frac{1-\theta(i_0)}{H(0, i_0)}x_0\) if all of the initial wealth were invested in the bond for the entire period \([0, T]\).

**Remark 4.3.** The case of single risky asset. As pointed out in Remark 3.4, in this case, the value of \(\bar{y}(t, i)\) in (19) is 0. Then the value of \(\xi(t, i)\) in (20) degenerates to \(0 + (\sigma(t, i))^{-1}B(t, i)' = (b(t, i) - r(t, i))/\sigma(t, i)\). That is, without the introduction of \(\bar{y}\) and \(\xi\), the no-shorting constrained problem can still be solved. This has been verified by Bai and Zhang (2008) where only one risky asset was included in the portfolio panel. However, in the case of multiple risky assets, the value of \(\bar{y}\) may not be 0, then \(\xi\) should arise in the expression of the optimal control. Li et al. (2002) provided an example where the value of \(\bar{y}\) is not 0.

**Remark 4.4.** The results for the case of no regime-switching. In this case, all the processes do not respond to the change in the market mode, namely, \(r(t, i) = r(t)\), \(b_m(t, i) = b_m(t), \sigma_{mn}(t, i) = \sigma_{mn}(t), B(t, i) = B(t), \psi(t, i) = \psi(t), \bar{y}(t, i) = \bar{y}(t), \xi(t, i) = \xi(t)\) and \(\rho(t, i) = \rho(t)\). The solution to the two systems of equations (29) and (30) degenerate to

\[
P(t) = e^{-\int_0^t [\rho(s) - \bar{y}(s)(\sigma(s)\sigma(s)')^{-1}\bar{y}(s) - 2r(s)]ds},
\]

\[
H(t) = e^{-\int_0^t r(s)ds}.
\]

The expression in (31) turns out to be \(\theta(i_0) = 0\), and assumption (38) reduces to \(z \geq x_0 \exp(\int_0^T r(s)ds)\), which coincides with the feasibility condition in (13) as shown in Remark 3.2. As a result, the optimal investment policy in Theorem 3.2 reduces to the following simplified form.

\[
u^*(t, x) = -[\sigma(t)\sigma(t)']^{-1}[\bar{y}(t) + B(t)][x + (\lambda^* - z)e^{-\int_0^t r(s)ds}],
\]

where

\[
\lambda^* - z = \frac{z - P(0)H(0)x_0}{P(0)H(0)^2 - 1}.
\]

Correspondingly, the efficient frontier is expressed as

\[
\text{Var}x^*(T) = \frac{P(0)H(0)^2}{1 - P(0)H(0)^2}[z - \frac{P(0)H(0)x_0}{P(0)H(0)^2}]^2,
\]

which is a perfect square. It is interesting to compare our results with Li et al. (2002). In Li et al. (2002), the optimal investment strategy takes the form of piecewise zero expression, where the zero investment part is determined by the negative sign of the process \(-[x(t) + (\lambda^* - z)e^{-\int_0^t r(s)ds}]\), and the non-zero part is exactly the same as our result for \(u^*(t, x)\) in (44). Note that under assumption
form where the investment strategy and the volatility matrices are investigated. Then the zero investment part in Li et al. (2002) is unnecessary. Hence our investment strategy in (44) takes a uniform expression instead of a piecewise one.

Remark 4.5. The value of minimum variance. If there is no assumption on the value of \( z \), it is obvious to see from (35) that the minimum variance is achieved at 

\[
P(0, i_0) H(0, i_0) \frac{1 - \theta(i_0)}{H(0, i_0) + \theta(i_0)}. \]

However, under the assumption \( z \geq \max \{ \psi(0, i_0) x_0, \frac{1 - \theta(i_0)}{H(0, i_0)} x_0, \text{for } i \in \mathcal{M} \} \) in Theorem 4.1, and based on the result in (32), we have

\[
1 - \theta(i_0) x_0 \geq \frac{P(0, i_0) H(0, i_0)}{P(0, i_0) H(0, i_0)^2 + \theta(i_0)},
\]

then the minimum variance is achieved at \( z = \max \{ \psi(0, i_0) x_0, \frac{1 - \theta(i_0)}{H(0, i_0)} x_0, \text{for } i \in \mathcal{M} \} \).

5. Numerical example. In this section, a numerical example is presented to demonstrate the results in the previous section. Suppose that the market mood is divided as “bearish” and “bullish” two regimes, which in our example corresponds to regime 1 and regime 2, respectively. The transition intensities are \( q_{12} = 0.8 \) and \( q_{21} = 0.2 \). Suppose the initial market mode is \( i_0 = 1 \). Consider an investor with \( x_0 = 1 \) and \( T = 5 \). Let the number of stocks on the market \( d = 3 \). The interest rate of the bond and the appreciation rate of the stocks under different market mode are \( r_1 = 0.02 \), \( r_2 = 0.05 \), \( b_1 = (0.04, 0.05, 0.06)' \) and \( b_2 = (0.1, 0.11, 0.12)' \), respectively, and the volatility matrices are

\[
\sigma_1 = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{2}{3} \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{2}{3} \end{pmatrix}.
\]

Then we have

\[
\sigma_1^{-1} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{3}{2} \end{pmatrix}, \quad \sigma_2^{-1} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix},
\]

and \( b_1 - r_1 I_3 = (0.02, 0.03, 0.04)' \) and \( b_2 - r_2 I_3 = (0.05, 0.06, 0.07)' \). Hence \( \sigma_1^{-1}(b_1 - r_1 I_3) = (-0.02, 0.03, 0.06)' \) and \( \sigma_2^{-1}(b_2 - r_2 I_3) = (-0.02, 0.06, 0.21)' \). Obviously, the function \( s(y) := \frac{1}{2} ||s^{-1} y + \sigma^{-1} (b - r I_3)|| \) over \( y \in [0, +\infty)^3 \) has a unique minimizer \( \bar{y}_1 = (0.02, 0.0, 0)' \) under bearish regime, and \( \bar{y}_2 = (0.02, 0, 0)' \) under bullish regime. Then based on (20), we have \( \xi_1 = (0, 0.03, 0.00)' \) and \( \xi_2 = (0, 0.06, 0.21)' \).

By Theorem 4.1, we have solution for any \( z \geq \max \{ \psi(0, i_0) x_0, \frac{1 - \theta(i_0)}{H(0, i_0)} x_0, \text{for } i \in \mathcal{M} \} \). By plugging in all the above parameters into Matlab, we have \( \psi(0, i_0) x_0 = 1.2176 \) and \( \frac{1 - \theta(i_0)}{H(0, i_0)} x_0 = 1.2156 \) for \( i = 1 \), and \( 1.2161 \) for \( i = 2 \). Hence we consider solution for any \( z \geq 1.2176 \). As an example, let \( z = 5 \). In the following we investigate the efficient portfolio \( u^*(t, x, i) \) in (33). Note that (33) is in the feedback form where the investment strategy \( u^* = (u_1, u_2, u_3)' \) is driven by the wealth process \( x(t) \) and the market regime process \( \alpha(t) \). Due to the stochastic nature of \( x(t) \) and \( \alpha(t) \), the efficient portfolio \( u^* \) is also a stochastic process. To get an idea of the investment strategy along with time, we plot Figure 2. Note that in Figure 2 the investment on the first stock \( u_1 = 0 \), this is consistent with our calculation since in both \( [\sigma_1^{-1}]^{-1} \xi_1 = (0, 0.03, 0.09)' \) and \( [\sigma_2^{-1}]^{-1} \xi_2 = (0, 0.06, 0.63)' \), the first element is always zero. There is no surprise with the presence of the second stock which has
Figure 2. A sample path of the efficient portfolio $u^* = (u_1, u_2, u_3)'$. 

A higher rate of return than stock one but with the same variance. Under mean-variance criteria, we certainly invest in stock two rather than stock one. In Figure 2 we also notice a slightly increasing investment in stock two and three, this is driven by the increasing value of stochastic process $-\left[ x(t) + (\lambda^* - z)H(t, \alpha(t)) \right]$. According to Section 3.3 and Theorem 4.1, as long as $z \geq 1 - \theta(i_0)H(0, i_0)x_0$, the stochastic process $-\left[ x(t) + (\lambda^* - z)H(t, \alpha(t)) \right]$ is always nonnegative, which is illustrated in Figure 3.

Figure 4 gives the corresponding efficient frontier. Starting from $E(x(T)) = z = 1.2176$, the variance increases as a hyperbola from the minimum variance 0.0102.

In the following we investigate the effect of regime-switching on the efficient frontier. First it is straightforward to expect a higher expected return in a bullish market than a bearish one. By letting $q_{12} = 1$ and $q_{21} = 0$ to indicate a pure bearish market, and $q_{12} = 0$ and $q_{21} = 1$ to indicate a bullish market, we have Figure 5 where the bullish efficient frontier is higher than the bearish one, which illustrate our intuitive idea. Secondly, even in the same market environment, the expected rate of return given a fixed level of variance could be different if we enter the market at different market modes. Let $q_{12} = 0.8$ and $q_{21} = 0.2$, we plot the efficient frontiers corresponding to different initial market modes $i_0 = 1$ and $i_0 = 2$ in Figure 5. We observe a slight higher expected rate of return under a given level of variance if we enter the market at a bullish time. The slight difference is a bit invisible so we also provide an amplified version on the right hand side of Figure 5.

6. Conclusion. This paper investigates a mean-variance portfolio selection problem with regime-switching under the constraint that short-selling is prohibited. By introducing a general result on the sign of geometric Brownian motion with jumps,
Figure 3. The process $-\{x(t) + (\lambda^* - z)H(t, \alpha(t))\}$.

Figure 4. The corresponding efficient frontier.
the mean-variance portfolio selection problem is solved by using a modified Maximum principle approach; in particular, our key observation is that only under a mild technical assumption on the initial conditions, the no-shorting constraint will consistently be satisfied over the whole finite horizon. Our numerical examples illustrate the efficient portfolio, efficient frontier and the effects of regime-switching.

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