WARPS AND BARS FROM THE EXTERNAL TIDAL TORQUES OF TUMBLING DARK HALOS

JOHN DUBINSKI 1 AND DALIA CHAKRABARTY 2

1 Department of Astronomy and Astrophysics, University of Toronto, 50 St. George Street, Toronto, ON M5S 3H4, Canada; dubinski@astro.utoronto.ca
2 School of Physics & Astronomy, University of Nottingham, Nottingham NG7 2RD, UK; dalia.chakrabarty@nottingham.ac.uk

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ABSTRACT

The dark matter halos in Λ cold dark matter cosmological simulations are triaxial and highly flattened. In many cases, these triaxial equilibria are also tumbling slowly, typically about their short axes, with periods of order a Hubble time. Halos may therefore exert a slowly changing external torque on spiral galaxies that can affect their dynamical evolution in interesting ways. We examine the effect of the external torques exerted by a tumbling quadrupolar tidal field on the evolution of spiral galaxies using N-body simulations with realistic, disk galaxy models. We measure the amplitude of the external quadrupole moments of dark halos in cosmological simulations and use these to force disk galaxy models in a series of N-body experiments for a range of pattern speeds. We find that the torques are strong enough to induce long-lived transient warps in disks similar to those observed in real spirals and also induce the bar instability at later times in some galaxy models that are otherwise stable for long periods of time in isolation. We also observe forced spiral structure near the edge of the disk where normally self gravity is too weak to be responsible for such a structure. This overlooked influence of dark halos may well be responsible for many of the peculiar aspects of disk galaxy dynamics.

Key words: galaxies: kinematics and dynamics – methods: N-body simulations – stellar dynamics

Online-only material: animations

1. INTRODUCTION

The nearly flat rotation curves of spiral galaxies are direct evidence for the existence of massive dark halos around the galaxies, within the paradigm of Newtonian–Einsteinian gravity (Roberts & Whitehurst 1975; Rubin et al. 1979, 1982, 1985; van Albada et al. 1985; Salucci & Burkert 2000; Sofue & Rubin 2001). The dark halos forming in cold dark matter (CDM) cosmological models have been identified with the inferred dark halos of galaxies; a substantial amount of work has gone into trying to test for the consistency between reality and simulation (e.g., Barnes & Efstathiou 1987; Frenk et al. 1988; Dubinski & Carlberg 1991; Navarro et al. 1996; Bullock et al. 2001; Neto et al. 2007). Most work has focused on the spherically averaged density profile and the implications of this profile for galactic rotation curves. Although debate still continues on the value of the slope of the inner cusp (Gentile et al. 2004; Ferreras et al. 2007) and whether or not the inferred rotation curves are consistent with the real ones (Moore 1994; de Blok & McGaugh 1998; de Blok et al. 2003), it seems that the consistency between the real and simulated dark halos is quite strong and that the CDM paradigm is in reasonable shape in this sense (Primack 2007).

Other work has focused on secondary dynamical effects of dark halos that arise from their triaxial nature and their implication for other properties in disks, namely, oval distortions, warps, and perhaps bars (Binney 1978; Sparke 1984; Franx & de Zeeuw 1992; Sackett et al. 1994; Kuijken & Tremaine 1994; Weinberg 1998; Sellwood 2003; Berentzen et al. 2006). The flattened potential of triaxial halos has been recognized in some work to induce oval distortions to otherwise circular disks, as well as vary the velocity along the orbit (Hayashi et al. 2007). The orbits of stars in the disk become flattened in a direction orthogonal to the halo density. However, the detection of such oval distortions and non-circular velocities is marginal at best, suggesting that halos may be nearly axisymmetric in the plane of an ordinary, high surface-brightness spiral galaxy. On the other hand, the effects of non-circularity in dwarfs and low-surface brightness galaxies may be stronger (Valenzuela et al. 2007; Hayashi & Navarro 2006).

Another source of complex dynamical effects is the probable misalignment between a disk and a triaxial dark halo. Disk–halo misalignment has been argued to be the origin for warps seen in most edge-on spiral galaxies. A misalignment implies that the disk will feel a torque from the halo and the nearly circular orbits will therefore precess. If the disk has self-gravity and the halo potential is static, warped modes can arise (e.g., Toomre 1983; Sparke & Casertano 1988; Kuijken 1991). A fatal flaw in this hypothesis is that gravitational interactions between the disk and the halo are strong, with the precessing disk experiencing dynamical friction from the halo (e.g., Nelson & Tremaine 1995; Binney et al. 1998). One therefore expects the disk and halo to become aligned with one another within a few dynamical times and this is indeed borne out in experiments (Dubinski & Kuijken 1995). Thus, it has been suggested that it is the outer halo that is misaligned with the disk, while a tight coupling exists between the inner halo and the disk (Bailin et al. 2005; Binney 2007).

The origin of galactic warps then remains a perplexing issue but alternative scenarios suggest a transient origin. Other ideas suggest that the cosmic infall of gas and dark matter alters the relative orientation the disk and dark halo due to addition of angular momentum (e.g., Ostriker & Binney 1989; Debattista & Sellwood 1999; Jiang & Binney 1999) and the resulting torques on the disk lead to transient warping. van der Kruit (2007) also considers the possibility of the onset of a late warped outer disk, due to gas infall, in a configuration that is independent of the relatively younger inner truncated stellar disk. The infall picture has also been recently used by Shen & Sellwood (2006) to reproduce the warping of a simulated disk galaxy; in fact, their
Tidal fields from satellite galaxies are known to excite dramatic behavior in spirals; Kalirai et al. (2006) invoke the possibility of a warp or overdensity induced by satellite interaction to explain the origin of the secondary cold population that they identify in M31, while the grand design spiral in M51 is inferred to be the result of its interaction with its companion NGC 5194 (Toomre & Toomre 1972; Salo & Laurikainen 2000). The Magellanic origin of the Galactic warp (Weinberg 1998; García-Ruiz et al. 2002) is revisited by Weinberg & Blitz (2006) in which they suggest that the warp is formed as a joint handiwork of the tidal field of the Magellanic Clouds as well as the effect of the distortions that such has on the Galactic halo. The overall picture that emerges from these studies is that the source of the warping torque lies with asymmetries in the halo, at radii beyond the gravitational influence of the disk.

Dark halos formed in simulations performed within the CDM scenario suggest that not only do they often settle into triaxial figures of equilibrium but that they can also tumble, much like a rigid body (Dubinski 1992; Bureau et al. 1999; Bailin & Steinmetz 2004). Static and tumbling equilibrium are both perfectly valid in analogy to the two solutions for the homogeneous Jacobi ellipsoids (Chandrasekhar 1969) which include a tumbling and static solution with internal circulation. Bailin & Steinmetz (2004) find a log normal distribution of halo pattern speeds with a mean value of about $\Omega_p = 0.15 \text{ km s}^{-1} \text{ kpc}^{-1}$, corresponding to a tumbling period of 44 Gyr. A typical halo will then tumble through $120^\circ$ over a Hubble time. The tumbling periods of dark halos are rather long but nevertheless, a picture emerges in which disk galaxies are embedded within slowly tumbling, triaxial halos with rotation axes that are probably slightly misaligned with the disk rotation axes. It is likely that the inner halo and disk are aligned with each other but the outer halo (say beyond 100 kpc) is tumbling slowly and thereby affecting the disk with its tidal field. A pertinent question to ask then is how strong are the torques that a tumbling dark halo exerts on the disk at the center and whether or not these torques can have a significant influence on the evolution of a spiral galaxy over a Hubble time?

In this paper, we carry out a series of experiments to study the effect of a slowly tumbling, external quadrupolar potential, on the evolution of a disk, with the primary goal of inducing galactic warps. We first measure the expected strength of the quadrupolar tidal field directly from dark matter halos within CDM cosmological simulations and subsequently estimate the torques expected on exponential disks. We find that the typical torques from cosmological dark halos are generally an order of magnitude larger than the effect of the Large Magellanic Cloud (LMC), rendering them more effective in triggering something dynamically interesting.

Guided by such experimentally motivated numbers, we set up simulations of ideal galactic models embedded in nearly spherical halos, forced slowly by an external quadrupolar tidal field. We discover in these experiments that it is fairly simple to make warps of amplitudes similar to the observed ones.

We also find that the gradual tidal forcing of tumbling dark halos has the effect of pushing an otherwise stable disk into a region where it becomes subject to the bar instability at late times. This is a new bar triggering mechanism and may be an additional factor that affects the evolution of the fraction of barred galaxies over cosmic history (e.g., Curir et al. 2008; Sheth et al. 2008).

The plan of the paper is as follows. In Section 2, we calculate the amplitude of the torque on a galactic disk, expected from the external tidal field of the triaxial dark halo, as extracted from cosmological N-body simulations. In Section 3, we present N-body models of M31 and the Milky Way (MW) that are forced by the external quadrupolar field with the same strength expected from cosmological simulations. In Section 4, we present the results of these simulations while in Section 5 we present analytical models of rigid disks forced by external tidal fields, to quantify these effects. In Section 6, we conclude with a discussion of the implications of these results for warps, bars, and spirals in real galaxies.

2. TIDAL TORQUES ON DISKS FROM COSMOLOGICAL DARK HALOS

The dark halos in CDM cosmological models are highly flattened and triaxial with typical axis ratios $c/a = 0.4$ and $b/a = 0.6$ (e.g., Barnes & Efstathiou 1987; Frenk et al. 1988; Dubinski & Carlberg 1991; Warren et al. 1992; Jing & Suto 2002). Analysis of the halo spin parameter shows a tendency for alignment with the minor axis, suggesting that a disk that forms within a dark halo is likely to have its own spin angular momentum closely aligned with the halo minor axis (e.g., Dubinski 1992; Warren et al. 1992; Dubinski & Kuijken 1995; Binney 2007). The angular momentum within halos is distributed between internal streaming and tumbling motion. Early studies showed that halos are slowly tumbling through space with long periods. More recent quantitative analyses in standard CDM cosmological models show that the distribution of tumbling frequencies is roughly log-normal (Bailin & Steinmetz 2004) and peaked at about $0.15 \text{ km s}^{-1} \text{ kpc}^{-1}$ with $\sigma = 0.83$. The dissipative infall of the baryonic matter during galaxy formation can also modify the halo, both increasing its central concentration (e.g., Blumenthal et al. 1984) as well as rounding out the shape by increasing $b/a$ within the region of influence of the forming disk while leaving $c/a$ about the same (Dubinski 1994; Kazantzidis et al. 2004). The axis ratios of the volumetric density contours at radii beyond the influence of the disk are less affected and remain about the same as the initial values.

Disks and halos are also found to be tightly coupled through dynamical friction in the inner regions of galaxies (Dubinski & Kuijken 1995), so one would expect the disk to lie in the principal plane of the halo, probably with the spin vectors of the disk and halo aligned (Bailin et al. 2005; Libeskind et al. 2007). However, misalignment may persist at larger radii and the slow tumbling of a halo could lead to an external torque on the disk. Although current self-consistent simulations of galaxy formation have not fully addressed this point, we make the hypothesis that the disks are aligned with the principal planes of their inner halos through dynamical friction while the outer halos remain misaligned and may slowly be changing orientation with the tumbling frequencies measured in cosmological dark matter simulations.

We estimate this torque through the analysis of more than 2000 dark halos extracted from a cosmological dark matter simulation with near standard parameters $\Omega_m = 0.7$, $\Omega_{\Lambda} = 0.3$, $σ_8 = 0.9$, and $h = 0.7$ (Spergel et al. 2007). The simulation contains $512^3$ particles in a cube of dimension $L = 70 h^{-1}$ Mpc and was run using the GOTPM cosmological N-body code (Dubinski et al. 2004). The simulation was run from $z = 70$ using 2800 equal time steps. The comoving particle softening length was set to $3.5 h^{-1}$ kpc. Halos are extracted in a two stage
roughly equal to the extent of the disk, though such alignment may extend further (Dubinski & Kuijken 1995; Binney et al. 1998). Any torque acting on the disk from the halo probably then arises from misalignments beyond this radius. It is useful to quantify these effects more concretely. According to the NFW model, dark matter halos extend to the virial radius \( r_{200} = rv_c^2 \), where \( c \) is the concentration, \( r_c \) is the scale radius of the density profile, and the mean enclosed density is 200 times the critical density \( \rho_c = 3H_0^2/8\pi G \). At \( r_{200} \), halos have characteristic mass \( M_{200} \) and circular velocity \( v_{c,200} = GM_{200}/r_{200} \). Typical concentrations for galactic scale halos are \( c \approx 15 \) (e.g., Bullock et al. 2001) though they can vary considerably with a scatter \( \Delta(\log c) = 0.18 \). A galaxy halo is usually characterized by the peak value of the rotation curve \( v_{c,\text{max}} \) rather than \( v_{200} \). Analysis of the NFW rotation curve shows that the maximum value of the rotation curve \( v_{c,\text{max}} \) occurs at \( r = 2.16r_c \) and is related to \( v_{200} \):

\[
v_{200} = 2.15v_{c,\text{max}} f(c) \]

with

\[
f(c) = c^{-1/2} \left( \ln(1 + c) - \frac{c}{1 + c} \right)^{1/2}.
\]

Given the parameters, \( c \) and \( v_{c,\text{max}} \), we find \( v_{200} \) and then can determine \( f(r_{200}) \) and \( M_{200} \) through the usual identities (Navarro et al. 1996):

\[
r_{200} = 100 \left( \frac{v_{200}}{100 \text{ km s}^{-1}} \right) h^{-1} \text{ kpc}, \quad M_{200} = 2.325 \times 10^{11} \left( \frac{r_{200}}{100 \text{ kpc}} \right)^2 \left( \frac{v_{200}}{100 \text{ km s}^{-1}} \right)^2 M_\odot.
\]

As a specific example, consider a MW sized NFW dark halo with a \( v_{c,\text{max}} = 220 \text{ km s}^{-1} \) and concentrations with the range \( c = 10–20 \). Assuming \( h = 0.7 \), these models will have virial masses in the range \( M_{200} = 1.1–2.0 \times 10^{12} M_\odot \) and scale radii from \( r_s = 11–26 \text{ kpc} \). The scale radius \( r_s \) is roughly the size of the disk and since the disk and halo are tightly coupled within this region, we expect alignment. We can expect misalignment and tidal torques from the halo mass distribution for halo mass beyond the edge of the disk and so we need to calculate the external component of the potential beyond \( r > r_s \) to estimate the tidal torques.

For a halo particle distribution, the external potential from matter with \( r > r_0 \) at a point \((r, \theta, \phi)\) with \( r < r_0 \) is given by the expression (e.g., Binney & Tremaine 2008),

\[
\Phi_{\text{ext}}(r'; r < r_0) = -4\pi G \sum_{l=\infty}^{l=\infty} \sum_{m=-l}^{m=l} c_{lm} r' Y_l^m(\theta, \phi),
\]

where \( Y_l^m \) is the usual spherical harmonic function and the coefficients \( c_{lm} \) are evaluated from the matter beyond \( r > r_0 \) through

\[
c_{lm} = \frac{N}{4\pi (2l+1)} \int Y_l^m(\theta, \phi) Y_l^m(\theta_o, \phi_o) d\theta d\phi, \quad r_o > r_0
\]

with \( \alpha \) indexing a list of the \( N \) particles that have \( r > r_0 \).

For a dark halo, it is natural to compute these coefficients for the particles with \( r_0 = r_s \) and \( 2r_s \) as a reasonable measure of the external torquing potential of a dark halo.
Taking the lead from cosmic microwave background (CMB) analysis, a useful way of quantifying the strength of the halo tidal potential is through the parameter $c_l$ defined as

$$c_l = \left( \sum_{l \geq 2} \frac{|c_{lm}|^2}{2l+1} \right)^{1/2}.$$  

(7)

The monopole term will lead to zero torque and the dipole terms are probably unimportant since dark halos tend to be ellipsoidal and do not show a significant lopsided mode. The most important terms for torquing are the quadrupole terms $l = 2$; one expects successive even $l$ terms to have some effect but with increasingly lower strengths. We therefore focus on the quadrupole terms that have a strength given by the parameter $c_2(r > r_s)$ as the main source of torque on a misaligned disk. We now calculate this value for our sample of halos and determine its importance for disk torques.

Since the value of $c_2$ depends on the mass of the halo, we can rescale all halos to some reference mass. We are interested mainly in the distribution of the relative strength of the external tidal potential in typical spiral galaxies; so we first renormalize the halos to have the same scale radius and scale mass. The following formalism is then undertaken with these halos.

1. We first fit the scale parameters $r_s$ and $M_s$ to the NFW density profile in $\log \rho - \log r$ space with the potential–density pair of the spherical NFW profile given by

$$\Phi(r) = -\frac{GM_s}{r} \log (1 + r/r_s),$$

(8)

$$\rho(r) = \frac{M_s}{4\pi r^2 (r + r_s)^2}.$$  

(9)

2. Next, we determine $r_{200}$ as the radius containing the mean density 200 times the critical density of the universe. We compute $v_{c,max}$, $v_{200}$, and $M_{200} = M_s \left[ \ln(1+r_s/r) - c^2/(1+c) \right]$ from the potential.

3. We rescale all halos to a putative MW model for our determination and call the rescaled tidal parameter $c_{2,MW}$. We use the parameters $v_{c,max} = 220 \text{ km s}^{-1}$, a concentration of $c = 10$ which implies that $r_s = 26 \text{ kpc}$ and $M_{200} = 2 \times 10^{12} M_\odot$ from the equations above.

To correlate this parameter to a specific example, we compute the expected value of the quadrupole tidal parameter for a satellite of the type of LMC, (mass $M = 10^{10} M_\odot$ at a distance of $R = 50 \text{ kpc}$). We refer to this tidal parameter as $c_{2,LMC}$. The LMC has a minor tidal effect on the dynamics of the Galaxy at its current distance (e.g., Hunter & Toomre 1969; García–Ruiz et al. 2002) but simulations of disk galaxy satellite encounters indicate that interactions in which the orbit of the LMC intersects the disk during a close encounter in the past may excite a strong tidally generated spiral (Weinberg & Blitz 2006).

A close encounter with the LMC that brings it to half its current distance is expected to correspond to a quadrupolar tidal field that is about 8 times stronger since $c \sim r^{-3}$. Thus, the ratio $q_{t\text{idal}} = c_{2,MW}/c_{2,LMC}$ is a good indicator of the active strength of the halo tidal field on Galactic dynamical evolution. If $q_{t\text{idal}} \approx 1$, then the tidal field is comparable to the effect of the LMC on the Galaxy and so is quite weak while if $q_{t\text{idal}} \approx 8$, we might expect significant tidal torques on the Galaxy that may cause interesting evolution in the form of spirals, bars, and warps.

Figure 2 shows the distribution of values of the parameter $q_{t\text{idal}}$ measured for the sample of halos using two different outer radii, $r_0 = r_s$, and $r_0 = 2r_s$, plotted against the axis ratio $c/a$. The halo axis ratios are determined using the normalized inertia tensor of halo particles within $r < r_s$ using an iterative method that gives a measure of the best-fit ellipsoid of the distribution (Dubinski & Carlberg 1991). There appears to be no strong correlation between the value of $c_2$ and halo shape since the scatter is quite large. The mean value of $q_{t\text{idal}}$ is about 5 for $r_0 = 2r_s$ compared to about 25 for a cutoff at $r_0 = r_s$. The implication then is that a MW disk that is misaligned with the outer halo will experience a tidal field that is perhaps 5 to 25 times the strength of the tidal field of an LMC-type satellite, depending on the radial location where the misalignment begins. For halos with a larger concentration $c$, both $r_s$ and $M_{200}$ are smaller for the MW model but the values of $c_2$ vary within a factor of a few. This result is somewhat surprising for it implies that the external tidal field acting on a disk galaxy from a triaxial halo is comparable in strength to the tidal field of a large satellite having a close encounter with a galactic disk.

Finally, the disk will only be torqued if there is a significant misalignment with the halo independent of the strength of $q_{t\text{idal}}$. One might imagine that more flattened halos with the largest $q_{t\text{idal}}$ are more closely aligned and thus have a weaker potential for torquing. We therefore looked for a correlation between $q_{t\text{idal}}$ and the relative alignment of the inner and outer halo $\theta$. We see no such effect (Figure 3) and conclude that significant perturbations can arise from external torquing due to this misalignment.

We now go on to demonstrate some of these effects in controlled N-body simulations.
3. SIMULATIONS

We perform a series of N-body simulations using a modified version of a parallelized tree code (Dubinski 1996). The code has been changed to include an external quadrupolar potential field that is tumbling at a fixed pattern speed \( \Omega_p \). Forces on particles are the sum of the N-body force and the external potential field. If particles stray beyond a radius \( r_0 \), then the field is shut off. For the simulations described here, the radius is \( r_0 = 60 \) kpc. A model of a galactic disk plus surrounding dark halo is simulated within this tidal field. We also make sure that the model remains centered on the quadrupole field. This prevents the outer dark matter halo in the N-body model from being significantly distorted.

The model galaxies in question represent the MW and Andromeda (M31), as given by Widrow & Dubinski (2005) (WD herein). These models both have exponential disks with Hernquist-type bulges embedded within a cuspy dark halo similar to an NFW profile. We use the most stable versions of these models called MWb and M31a in WD, but we refer to them simply as MW and M31 in this paper. The M31 model has been tested at a variety of resolutions and is known to be stable against bar formation after run times of 10 Gyr (Gauthier et al. 2006). The MW model on the other hand is mildly unstable and develops a bar later on after \( t = 6 \) Gyr as we shall see below. The particles are distributed with the following numbers in both the M31 and MW runs: \( N_{\text{disk}} = 1500,000 \), \( N_{\text{bulge}} = 500,000 \), and \( N_{\text{halo}} = 2000,000 \). The model parameters are provided in Table 1.

We also populate the disks with particles out to large radii to better investigate edge effects. The MW model extends to 10 radial scale lengths while the M31 model goes out to 7 scale lengths. We note that the scale radii of the MW and M31 halos are \( r_s \approx 10 \) kpc and somewhat smaller than our putative MW halo in the discussion above. The parameters are fitted values for observed rotation curves and inferred brightness profiles of the two galaxies. The \textit{real} inner halos of M31 and MW are likely contracted compared to the pure dark matter models. The mass model fits to the real data have scale radii that are smaller by a factor of 2 in accordance with expectations of the contraction of halos by dissipative processes during galaxy formation.

We have found it convenient to select a set of simulation units that set the scale lengths and rotation velocities to values near unity. We scale the simulation units to the physical units in the following way:

1. gravitational constant \( G = 1 \),
2. 1 simulation mass unit \( = 4 \) kpc,
3. 1 simulation velocity unit \( = 220 \) km s\(^{-1} \),
4. 1 simulation mass unit \( = 4.5 \times 10^{10} M_{\odot} \),
5. 1 simulation time unit \( = 17.7 \) Myr.

We express the general external tidal field as

\[
\Phi_{\text{ext}}(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} P_{lm}(\cos \theta) (a_{lm} \cos m \phi + b_{lm} \sin m \phi) r^l. \tag{10}
\]

The coefficients \( a_{lm} \text{ and } b_{lm} \text{ describe the strength of the field and are derived from Equation (6). If we assume that the quadrupole is aligned with the principle axes and is due to a plane-symmetric distribution, then the } b_{lm} \text{ terms are zero. For these simulations, we are only interested in the } l = 2 \text{ quadrupole terms. The pure external quadrupole field then becomes}

\[
\Phi_{\text{ext}}(r, \theta, \phi) = \sum_{m=0}^{2} a_{2m} P_{2m}(\cos \theta) \cos(m \phi) r^2. \tag{11}
\]

We estimate a suitable tidal field using the following procedure. We first take an NFW profile that has been flattened into a perfect ellipsoidal shape with axis ratios \( q_1 = b/a \) and \( q_2 = c/a \). We then determine the spherically averaged density profile of this distribution and fit a spherical NFW profile to determine the scale radius, \( r_s \) and mass \( M_s \) as is done in cosmological simulations. We are interested in the tidal field due to material beyond a radius, \( r_0 = 2 r_s \), so we compute the coefficients of the quadrupolar tidal field for the material beyond this radius.

As an example, we find that for a flattened NFW halo with \( q_1 = 0.6 \) and \( q_2 = 0.5 \) with a scale length \( r_s = 26 \) kpc and \( M_s = 2 \times 10^{12} M_{\odot} \) that the quadrupolar tidal field coefficients in simulation units are

1. \( a_{20} = 0.00050 \),
2. \( a_{21} = 0.0 \),

| Model | \( M_d \) | \( R_d \) | \( z_d \) | \( r_s \) | \( v_s \) | \( v_R \) | \( R_0 \) | \( \delta R_0 \) | \( Q \) |
|-------|--------|------|------|------|------|------|------|------|--------|
| MW    | 3.3    | 2.81 | 0.44 | 8.8  | 345  | 435  | 0.88 | 30    | 1.3 at 2.5\( R_d \) |
| M31   | 7.7    | 5.58 | 0.60 | 12.9 | 337  | 461  | 1.83 | 30    | 1.25 at 2\( R_d \) |

\textbf{Notes.} Column 1: model for galaxy; Column 2: disk mass \( (10^{10} M_{\odot}) \); Column 3: disk scale length (kpc); Column 4: disk scale height (kpc); Column 5: NFW-halo scale radius (kpc); Column 6: NFW-halo scale velocity (km s\(^{-1} \)); Column 7: characteristic bulge velocity (km s\(^{-1} \)); Column 8: bulge scale length; Column 9: disk truncation radius (kpc); Column 10: truncation width (kpc); Column 11: Toomre \( Q \)-parameter.
distribution of halos for material with No. 2, 2009 Warps and Bars from the External Tidal Torques 2073

To tumbling periods of \( r \) that Bailin & Steinmetz (2004) find a log-normal distribution
4.0, 2.0, 1.0, and 0.5 times the age of the universe. We recall \( \Omega \) peaking at \( r \)
that the corotation radii are mainly larger than \( r \), and so we model the effect of the triaxial halo by this quadrupolar

\[ \frac{\Omega}{c^2} = \frac{1}{4} \cdot 15 \times 10^{-11} \] km s\(^{-1}\) kpc\(^{-1}\) corresponding to tumbling periods of \( T = 56, 28, 14, \) and 7 Gyr, respectively.

We note that these are very slow compared to the pattern speeds of barred galaxies (\( \Omega_p \approx 20 \) km s\(^{-1}\) kpc\(^{-1}\)) and so the corotation radii are mainly larger than \( r_{200} \) (The circular frequency at \( r_{200} \) in the NFW model is \( \Omega_{200} = v_{200}/r_{200} = H_0/100 \approx 0.7 \) km s\(^{-1}\) kpc\(^{-1}\)). These periods are approximately 4.0, 2.0, 1.0, and 0.5 times the age of the universe. We recall that Bailin & Steinmetz (2004) find a log-normal distribution peaking at \( \Omega_p = 0.15 \) km s\(^{-1}\) kpc\(^{-1}\), corresponding to a tumbling period of 44 Gyr. Thus, our simulations probe the fast tumbling tail of this distribution—the half of the distribution

To introduce a misalignment, the disk is tilted by 30° with respect to the field and the field is made to rotate about the \( z \)-axis with different pattern speeds. This choice of the tilt angle is arbitrary but is approximately the expected angle for at least 30% of dark halos, according to the distribution shown in Figure 1. The 4 pattern speeds that we choose are \( \Omega_p = 0.11, 0.22, 0.44, \) and 0.88 km s\(^{-1}\) kpc\(^{-1}\) corresponding to tumbling periods of \( T = 56, 28, 14, \) and 7 Gyr, respectively.

We simulate each run for 8000 equal time-steps with \( \delta t = 0.05 \) units, corresponding to about 7.1 Gyr in physical units. We soften gravity with a Plummer softening kernel with radius of 40 pc for the stars and 100 pc for the dark matter. Energy is conserved typically to within 0.5% and total angular momentum to within 1%.

3. \( a_{22} = -0.00015 \),
4. the quadrupole amplitude is \( c_2 = 0.00051 \).

\[ \frac{\Omega}{c^2} = \frac{1}{4} \cdot 10^{20} M_\odot \] s\(^{-1}\).

\[ \frac{\Omega}{c^2} = \frac{1}{4} \cdot 10^{10} M_\odot \] km s\(^{-1}\) kpc\(^{-1}\) corresponding to tumbling periods of \( T = 56, 28, 14, \) and 7 Gyr, respectively.

We assume that the inner halo has aligned with the disk and is essentially axisymmetric and remains aligned for the course of the simulation. As a comparison, the quadrupole coefficients for a LMC-like satellite of \( M = 10^{10} M_\odot \) at a distance of 50 kpc placed on the \( x \)-axis, is \( c_2 = 8.12 \times 10^{-5} \) in simulation units so that \( q_{\text{tidal}} = 6.3 \) close to expectation value for the cosmological distribution of halos for material with \( r > 2r_s \).

3 Animations are available at the Web site http://www.cita.utoronto.ca/~dubinski/warpmovies

### 4. RESULTS

Before exploring the results for externally torqued disks, we first examine the evolution of control models. Figure 4 shows the final state for the MW and M31 models in the absence of any external torquing potential. The disks remain planar and there are minimal signs of disk thickening and warping resulting from the amplification of the Poisson noise in the \( N \)-body disk. There are some remnants of spiral structure in the disks which arise from swing amplification of the noise in the disk. This transient spiral structure heats the disk azimuthally and slowly fades away as the simulations progress.

When we turn on the external quadrupole potential with the expected amplitudes from cosmological dark halos, we clearly see the various effects predicted for disk torquing. Figures 5 and 6 show the evolution of representative models of the MW and M31 runs from three perpendicular views. Associated videos also show how the external torque causes the disk to precess and nutate like a tilted gyroscope on a table top. The evolution of the direction of the spin vector of the disks described by the normalized \( x \) and \( y \) components of the angular momentum quantifies this behavior (Figure 7) and the nutational frequency depends on the pattern speed of the forcing quadrupole potential. All runs indicate that the model disk precesses through an angle of 30°–60° over 7 Gyr as it responds to the external quadrupolar field implying precession periods of more than 40 Gyr or precession rates less than 0.15 rad Gyr\(^{-1}\). We see below that these precession rates are in accord with analytical estimates from a rigid disk model forced by an external potential. These precession rates are about an order of magnitude smaller than those found in models with a disk misaligned within a triaxial potential (e.g., Kuijken 1991; Jeon et al. 2009) used in models of warping behavior that postulate that the main source of torque on a disk is due to the potential of the inner part of the triaxial halo. The phenomenology that we are exploring is therefore different and the resulting effects are slower and more subtle. So while the precession periods are long compared to the Hubble time, a disk can slew through a significant angle over its lifetime and this dynamical evolution leads to a more gradual transient warping at the disk edge in a mechanism similar to those invoking the accretion of angular momentum (Ostriker & Binney 1989). We conclude that this must be an important process in disk dynamical evolution over the age of the universe.

| Model      | \( a_{20} \) \(10^{-3}\) | \( a_{22} \) \(10^{-3}\) | \( \Omega_p \) (km s\(^{-1}\) kpc\(^{-1}\)) | \( T \) (Gyr) |
|------------|----------------|----------------|----------------|-------------|
| MW-con     | 0.0            | 0.0            | 0.0            | \( \infty \) |
| MW-0       | 5.0            | -1.5           | 0.11           | 56          |
| MW-1       | 5.0            | -1.5           | 0.22           | 28          |
| MW-2       | 5.0            | -1.5           | 0.44           | 14          |
| MW-3       | 5.0            | -1.5           | 0.88           | 7           |
| MW-2a      | 2.5            | -0.75          | 0.44           | 28          |
| MW-2b      | 1.3            | -0.38          | 0.44           | 28          |
| MW-2c      | 0.63           | -0.19          | 0.44           | 28          |
| M31-con    | 0.0            | 0.0            | 0.0            | \( \infty \) |
| M31-0      | 5.0            | -1.5           | 0.11           | 56          |
| M31-1      | 5.0            | -1.5           | 0.22           | 28          |
| M31-2      | 5.0            | -1.5           | 0.44           | 14          |
| M31-3      | 5.0            | -1.5           | 0.88           | 7           |

**Notes.** Column 1: model name; Column 2: component \( a_{20} \) of the external quadrupolar field \((10^{-4} M_\odot \) s\(^{-2}\)); Column 3: component \( a_{22} \) of the external quadrupolar field \((10^{-4} M_\odot \) s\(^{-2}\)); Column 4: halo pattern speed \( \Omega_p \) (km s\(^{-1}\) kpc\(^{-1}\)); Column 5: tumbling period (Gyr).
Figure 5. Snapshots of the Milky Way disk, at three different times from the four different runs near the beginning, middle, and end of the simulation. The three views of the disk are presented from the point of views that are initially face-on (x–y top-left) and the two perpendicular edge-on views (x–z bottom-left; y–z top-right). The disk precesses in all cases and the tilting induces a transient warp in the outer regions of the disk that persists for about 7 Gyr. The tidal forcing also excites spiral structure in the outer disk.

(An animation of this figure is available in the online journal.)

Figure 6. Final snapshots of two representative M31 runs shown at $t = 7.1$ Gyr. The left panel corresponds to the three views of the M31 disk (with the same layout as Figure 5) at the end of the run M31-3 with the fastest forcing pattern speed. The panel on the right shows the end of run M31-0 with the slowest pattern speed. The warping of the disk is more subtle in this case but is still apparent.

(An animation of this figure is available in the online journal.)

For real galaxies, we expect a range of halo tumbling pattern speeds, tidal field strengths, and initial tilt angles, so there should be also be a range of warping behavior in the population at large.

4.1. Warps

While the disk stars within approximately 5 radial scale lengths stay within the plane and act like a rigid body because of their self-gravity, stars near the edge of the disk begin to precess at different frequencies and begin to warp away from the disk. Our simulations indicate that the model disks develop strong warping early on in the runs and that such warping is sustained through the length of the run (about 7 Gyr; Figure 8).

The disk demonstrates strong flocculent spiral pattern, to even the disk-edge where self-gravity is considered too weak to excite such structure. This is advanced as the handiwork of the external quadrupole since we do not notice prominent outer spiral patterns in the control runs. This is particularly noticeable in the MW runs that have a more extended disk.

Following Levine et al. (2006), we quantify the disk warps that develop in the simulations by analyzing the vertical deviation of the disk from a plane at different radii through the function

$$W(\phi) = \sum_m W_m e^{im\phi},$$

(12)

where $\phi$ is the particle azimuth.
Figure 7. Evolution of spin axis of the MW and M31 models—MW-0, M31-0 (red); MW-1, M31-1 (green); MW-2, MW-3 (blue); and MW-3, M31-3 (black). This time evolution is represented here by the normalized components of the disk angular momentum vector $L_x/L$ and $L_y/L$. The circle corresponds to a fixed inclination of $\theta = 30^\circ$. The tumbling axis of the quadrupole potential is perpendicular to the plane of this plot. The disk responds by precessing in the counterclockwise direction as expected for this quadrupole potential. Nutations are clearly seen as the disk responds to the triaxial halo. The nutational period is generally smaller than the precession frequency. Even after the 7 Gyr time of the simulation the disk only precesses through about 60$^\circ$.

Figure 8. Prominent warp develops by the end of the simulation in model M31-3 with a quadrupole field pattern speed of $\Omega_p = 0.88$ km s$^{-1}$ kpc$^{-1}$.

The inner, nearly coplanar disk is considered to be in the $x$–$y$ plane. The particles inside the $j^{th}$ ring—of radius $R_j$ and width $\delta R$—are assigned the height $z_j$ from the midplane with $\phi_j$ defined as the corresponding particle azimuth. We first rotate into the frame of the inner disk before the analysis. We determine the midplane of the disk by computing the moment of inertia of the disk particles within 3 scale lengths where the disk is nearly planar and then diagonalizing the inertia tensor to give the orientation of the disk. We determine the coefficients $W_m$ using a Fourier analysis of the particle distribution through

$$ W_m = \frac{1}{N_k} \sum_j z_j e^{-im\phi_j}. \tag{13} $$

The amplitude of the deviation is given by $|W_m|$ and the phase angle through $\phi = \tan^{-1}(Im(W_m)/Re(W_m))/m$. We can then construct functions of $W_m$ versus $R$ to examine disk warping. The function $W_0(R)$ corresponds to $m = 0$ bowl-shaped deviations, $W_1(R)$ corresponds to the usual $m = 1$ integral-sign shaped warps while $W_2(R)$ are second order “scalloped” warps.

Figure 9 shows the results of this analysis for the four models of both the MW and M31 system. The dominant vertical deviation is the $m = 1$ integral-signed warp. For the MW models, the warp turns upward at $R \approx 15$ kpc and continues to $R = 30$ kpc reaching about 5 kpc above the plane. The M31 models shows a qualitatively similar behavior though the warping begins at the larger radius of $R \approx 20$ kpc but only reaches about 3 kpc above the plane at a radius of $R = 40$ kpc.

The M31 model is more extended and massive than the MW model while being forced by the same external tidal field and so shows a smaller degree of warping behavior.

We also examined the alignment of the warps by measuring the LON from the phase angle determined from the vertical Fourier analysis. We present the tip-LON plots for our models following Briggs (1990; Figure 10). The warps begin with a relatively straight LON when they begin to move out of the plane at the disk edge but generally show a leading spiral curving out to $\sim 6R_d$ following the observations from Briggs’s analysis. At
Figure 10. Inclination (tip) vs. longitude or tip-LON plots for the warps in the MW disk models at the last snapshot of the simulations at $t = 7$ Gyr. The four models with tumbling periods of $P = 56, 28, 14,$ and $7$ Gyr are given by the black, red, green, and blue lines respectively. The dashed circles correspond to $3$ and $6^\circ$ of tip respectively while the longitude is measured around the circles. The LON of the warp is approximately straight when it begins to move out of the plane but becomes more erratic at higher inclinations. The LON tends to curve in a leading spiral form similar to what is observed in real warps. The warps are less pronounced in the M31 models but still behave in a similar manner.

larger radii, the warping becomes less coherent resulting mainly by the onset of differential precession of the stellar disk orbits and the modulated forcing of the tumbling external quadrupole potential. While we are only modeling a pure $N$-body system one expects that the mechanism will generate coherent warping in the gas as well at least out to 6 scale lengths.

4.2. Bars

The tidal distortion due to a galaxy interaction in a flyby can trigger the bar instability under some conditions (e.g., Noguchi 1987; Gerin et al. 1990). The effect is due to a transient external quadrupole that disturbs the stellar orbits in the center of the galaxy. Under our hypothesis of a tumbling misaligned halo, there is also a changing external quadrupole potential that can perturb the disk and potentially trigger the bar. The physical situation is somewhat different than a impulsive galactic flyby, however, in that rate of change of the orientation is slow and the amplitude of the tidal potential is constant. Nevertheless, the MW disk is found to develop a fast bar at about 3 Gyr. Once formed, the bar is sustained and its pattern speed declines from an initial value of 50 to 35 km s$^{-1}$ kpc$^{-1}$ by the end of the run in accord with typical expectations for models like this (Figure 11) (e.g., Debattista & Sellwood 1998; O’Neill & Dubinski 2003; Dubinski et al. 2009).

Figure 12 shows the evolution of the strength of the bar that is triggered in the MW for different models by the torquing provided by the external halo. We find a well-developed bar, of nearly similar strengths, forming by about 3 Gyr in all the runs, indicating that the rate of tumbling of the halo is immaterial to the bar strength. However, $\Omega_p$ of the quadrupole appears to have a weak effect on the time of onset of the bar instability—this is hastened with increasing tumbling period $T$, i.e., falling pattern speed of the external halo, though this is not marked; the fractional difference in $\Omega_p$ appears to be about 16%. This small difference in the bar onset time is also noticed in the snapshots from these two runs, shown in Figure 5.

Instead, we find that it is the strength of the quadrupole of the external halo that is the crucial factor in controlling the bar instability. In fact, a control run performed with a null external quadrupole leads to a much later ($\approx 5.8$ Gyr) onset of a very weak bar in the MW disk (maximum bar strength is less than 0.4 times the bar strength reached in the other runs); see Figure 4. Thus, we assure ourselves that the inherent bar unstable nature of the MW model is very weak, albeit non-zero. Additionally, the control runs for both models result in significantly weaker spirality, especially at the edges of the disk and no warp.

The M31 disk is found to be unaffected by the halo torquing, in regard to bar formation. This is apparent from the pictures of the M31 disk at the end of the simulations (Figure 6); the result is found independent of the halo pattern speed used in the run.
5. ANALYTIC RIGID DISK MODEL

It is interesting to gauge the effects of the external quadrupolar term in the potential of a tumbling triaxial dark halo in an analytical model using a rigid, thin exponential disk for comparison to the phenomenology in the simulations. We emphasize again that the inner disk and halo are tightly coupled through dynamical friction and so remain aligned. The only possible source of torquing on the disk is a misaligned and possibly tumbling outer halo. The assumption of a rigid disk is a reasonable assumption since the simulations show that the disk maintains coherence because of self-gravity within $R \lesssim 5R_d$, even while it tips in response to the external tidal torque. We use the classical Euler equations of motions for a rigid spinning disk. The dynamic variables are the standard Euler angle ($\theta$, $\phi$, $\psi$) where $\theta$ is the inclination of the disk, $\phi$ is the longitude of the disk, and $\psi$ is the angle about the disk’s symmetry axis. In the presence of a time-dependent torque, the disk will precess and nutate and so $\theta$ and $\phi$ will evolve in time.

In these calculations, we set up the disk with an initial inclination of $\theta = 30^\circ$ and set the halo quadrupole potential tumbling about the $z$-axis with $\theta = 0$ for direct comparison to the results of the simulations.

5.1. Euler Equations of Motion

After Goldstein et al. (2002, see p. 210 and Equation (5.52)), the Lagrangian of the disk is

$$\mathcal{L} = \frac{I_1}{2}(\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) + \frac{I_2}{2}(\dot{\phi} \cos \theta + \dot{\psi})^2 - V(\theta, \phi, t),$$

(14)

where $I_1$ is the moment of inertia about the symmetry axis $I_1 = I_2$ are the moments measured about an axis in the plane of the disk. The potential energy $V$ is determined from the interaction between a rigid thin exponential disk and a tumbling external quadrupole potential. The equations of motion can then be written as

$$I_1 \ddot{\theta} - I_1 \dot{\phi}^2 \sin \theta \cos \theta + S \dot{\phi} \sin \theta + \frac{\partial V}{\partial \theta} = 0,$$

(15)

$$I_1 \dot{\phi} \sin^2 \theta + 2I_1 \dot{\phi} \dot{\theta} \sin \theta \cos \theta - S \dot{\theta} \sin \theta + \frac{\partial V}{\partial \phi} = 0,$$

(16)

where $S = I_1 (\dot{\phi} \cos \theta + \dot{\psi})$ (the disk angular momentum) is a constant of motion and $V$ is the potential energy between the disk and the tumbling halo. A tilted disk will naturally precess. For example, with an axisymmetric potential $V(\theta)$, the expected precession frequency can be derived from the first Euler equation by assuming that $\theta$ is constant. For small $\dot{\phi}$ we expect

$$\dot{\phi}_{\text{prec}} \approx -\frac{1}{S \sin \theta} \frac{\partial V}{\partial \theta}$$

(17)

(cf. Goldstein et al. 2002). If the potential also depends on $\phi$ and is rotating about the $z$-axis, we can expect a more complex evolution combining both precession and nutation.

5.1.1. Direct Solution of the Euler Equations for a Rigid Exponential Disk

We first consider the interaction of a rigid exponential disk with an external quadrupolar tidal field for comparison to the results of the $N$-body simulations. We point out that this process is quite different from the disk precession examined in work that has treated a triaxial halo as a rigid background potential (Kuijken 1991; Jeon et al. 2009). In those treatments, the precession rate is an order of magnitude larger due to the much larger torques felt by the inner halo. However, the large torque in these simulations is artificially high since we expect dynamical friction to align the disk and inner halo within a few disk dynamical times (Dubinski & Kuijken 1995). In this study, we assume that only the outer halo can exert any torque and subsequently the strength of the torque is considerably smaller, typically by an order of magnitude.

In Appendix A, we compute the time-dependent interaction potential between a rigid exponential disk and an external quadrupole potential to plug into the Euler equations. The potential is

$$V(\theta, \phi; t) = 3M_d R_d^2 \left\{ -\frac{a_{20}}{2}(3 \cos^2 \theta - 1) + 3a_{22} \cos[2(\Omega_p t - \phi)] \sin^2 \theta \right\},$$

(18)

where $M_d$ and $R_d$ are the mass and exponential scale radius of the disk, $a_{20}$ and $a_{22}$ are the quadrupole terms for the external terms of a triaxial potential in Equation (11), $\Omega_p$ is the tumbling frequency of the quadrupole potential assumed to be about the $z$-axis with $\theta = 0$ and we assume that $\phi = 0$ at $t = 0$. We can compute the moments of inertia $I_1$ and $I_3$ and the spin $S$ directly from the $N$-body model and so fully specify the Euler equations for our case.

For the M31 model, we solve the Euler equations of motion numerically using the same initial state as the $N$-body model along with four different pattern speeds for the quadrupole potential. We use the values of $I_1 = 3M_d R_d^2$ and $I_3 = 6M_d R_d^2$ expected for an exponential disk as well as the spin $S$ (disk angular momentum) measured from the $N$-body models. The specific parameters for the M31 model in dimensionless units are $M = 1.75$, $R_d = 1.39$ so that $I_1 = 10.17$, $I_3 = 20.34$, and $S = 5.14$. Using Equation (17), we can estimate the precession period assuming that $a_{22}$ time averages to zero for sufficiently large $\Omega_p$. We find a value of $\theta_{\text{prec}} = -0.14$ km s$^{-1}$ kpc$^{-1}$ which is comparable in magnitude to our chosen pattern speeds but opposite in sign. This halo corresponds to our choice of $\theta_{\text{total}}$ of about 6.3. We then integrate the Euler equations for approximately 53 Gyr corresponding to a little more than a typical precession period to get a complete picture of the evolution of the disk for comparison to the $N$-body simulations. Figure 13 shows the evolution of the spin axis of the disk for comparison to the similar plots for the $N$-body simulation (Figure 7). As in Figure 7, Figure 13 presents this evolution by tracing the normalized, $x$ component of the disk angular momentum against the $y$ component of the same.

The agreement in behavior for the four pattern speeds is excellent suggesting that the $N$-body disk is behaving in a similar manner to a rigid body over this time period. The nutation induced by the triaxiality is readily apparent again in this plot. We note that the nutation period is shorter for a higher pattern speed. Thus, for larger pattern speeds, as the nutation period is small, we might expect a spiral galaxy to wobble several times over its lifetime if it is embedded in a rapidly tumbling dark halo. For the slower pattern speeds, the nutation period becomes comparable to the precession period and the resulting effect is simply a steady change of the disk orientation. Stars and gas on the outer edge of the disk that have weaker self-gravity
are prone to precess differentially and this may be an additional origin of galactic disk warps.

5.2. Resonant Interactions

For the flattened halo discussed here the precession rate of disk is negative while the tumbling rate is positive and in the same sense as the disk. While this case is the most reasonable, it is possible that the sign of the precession rate and halo tumbling rate might be the same and so resonances can occur when $\Omega_p = \Omega_{\text{prec}}$. In the first investigations of orbits in a rotating triaxial potential, Binney (1978, 1981) discovered that under certain conditions retrograde orbits in the rotating frame can be unstable leading to large $z$-motions due to resonant interactions with the potential and suggested that this might be a mechanism to produce galactic warps. Heisler et al. (1982) demonstrated the existence of Binney’s resonance in orbital studies using an analytic triaxial potential. Tremaine & Yu (2000) have even suggested that this resonance may provide a mechanism for creating polar ring galaxies. Accreted gas and forming stars counter rotating with respect to main disk may be levitated out of the plane if there is an underlying rotating triaxial potential.

So far the study in this paper has elucidated how self-consistent $N$-body disks evolve in tumbling triaxial potential rotating in the same direction as the disk. In the other case of a counter-rotating disk, it is interesting to see if Binney’s resonance manifests itself in disk evolution in our models.

Let us first consider the simple rigid disk model. Consider the Euler equations again for the case where $\phi = 0$ and $\theta = 0$ corresponding to steady precession at a fixed angle $\theta$ at a constant rate. For this to be the case in general, we need to find the condition such that $\partial V/\partial \phi = 0$ at all times. Differentiating Equation (18) with respect to $\phi$ and setting it to zero implies that $\sin[2(\Omega_p t - \phi)] = 0$. If $\phi = \phi_{\text{prec}}$ then this implies a resonance when $\Omega_p = \phi_{\text{prec}}$. With this resonant condition, we can use the first Euler equation to find $\phi$ from the quadratic equation

$$- I_1 \dot{\phi}^2 \sin \theta \cos \theta + S \dot{\phi} \sin \theta + \frac{\partial V}{\partial \theta} = 0, \quad (19)$$

where $\frac{\partial V}{\partial \theta}$ is known from Equation (A25, see Appendix). For a flat rotation curve with speed $v_c$, the disk angular momentum is given by $S = 2 M_d R_d v_c$. If we assume that $\phi_{\text{prec}}$ is small then to a good approximation it is given by

$$\phi_{\text{prec}} \approx - \frac{9}{2} \frac{R_d}{v_c} (\alpha_{20} + 2 \alpha_{22}) \cos \theta. \quad (20)$$

For our model of M31, with our estimate of $\alpha_{20}$ and $\alpha_{22}$ for the coefficients of an external quadrupole potential and an initial disk inclination of $\theta = 30^\circ$, the resonance occurs when $\Omega_{\text{p, res}} = \phi_{\text{prec}} = -0.055 \text{ km s}^{-1} \text{ kpc}^{-1}$. Thus, we see that the size of this typical resonant precession frequency is comparable to the expected tumbling frequencies of dark halos.

We can examine the behavior of the disk precessional evolution near this resonance. In Figure 14, we represent the disk spin axis evolution to bring out the unstable behavior of rigid models with $\Omega_p$ set to $\Omega_{\text{p, res}}/2$, $\Omega_{\text{p, res}}$, $2 \Omega_{\text{p, res}}$, and $4 \Omega_{\text{p, res}}$. The radically different near-resonance disk behavior noted in this figure, marks the position of a region of instability. Here the disk inclination can change by large amount and the precession and nutational behavior are quite irregular.

We also consider an additional M31 $N$-body model for comparison to the rigid disk model. We evolve it in the same way as previous models but here examine a counter-rotating external quadrupole potential with $\Omega_p = -0.11 \text{ km s}^{-1} \text{ kpc}^{-1}$ to see if the irregular behavior that is seen in the simple rigid disk model is reproduced in a self-consistent $N$-body model. Figure 15 shows a comparison of the time evolution of the polar angles ($\theta$, $\phi$) representing the direction of the disk angular momentum vector for the rigid disk and $N$-body disk models. (Here $\phi$ is offset by $90^\circ$ from the usual definition as the angle of the LON in the Euler equation.) For the rigid disk, we adjust the initial inclination slightly to $\theta = 34^\circ$ and find excellent agreement with the $N$-body disk evolution. We have integrated for the unusually long time of $25 \text{ Gyr}$ to see the full tilt evolution. However, the amplitude of the quadrupole potential could easily be twice as large based on the observed variance of $q_{\text{tidal}}$ in cosmological halos and so this behavior would occur in half the time since from Equation (20) we see that the precession rate is...
directly proportional to the external quadrupole amplitude. It is remarkable that in this counter-rotating case, the disk inclination grows by 90° before reversing course.

While the Binney resonance is derived in the context of closed collisionless orbits in rotating triaxial potentials, a version of it also appears to be operating collectively in a self-gravitating disk in the same context. Our idealized model of an N-body disk forced by a rotating external quadrupole potential tips through 90° when counter-rotating tumbling frequencies are near the resonant value. The case of counter-rotation is probably rare but not impossible given the existence of some disks with counter-rotating components (e.g., Rubin et al. 1992). It may also help explain the phenomenon of polar rings as suggested by Tremaine & Yu (2000). Accreted gas that is counter-rotating with respect to the tumbling halo might rise out of the halo principal plane within a Hubble time given a large enough quadrupole amplitude and suitable tumbling frequency. One caveat to be aware of is that given the long timescales of this process, dynamical friction may begin to align even the outer parts of the halo so the resulting torque may diminish with time and so weaken the effect.

6. CONCLUSIONS

In this paper, we discuss the effect of the tidal field of a tumbling and triaxial external halo on the dynamical evolution of a galactic disk. We have shown that the disk precesses in response to its misalignment with the external halo and the gradual re-alignment of the disk with the external tumbling halo causes it to warp. The disk is also seen to develop strong spirality to its edge in the potential of the external halo that it sits in. It is also suggested that if a disk feels a significant external quadrupole that is essentially static or at most slowly tumbling, it might trigger a bar instability, late in the disk.

The conclusions achieved in this work are based on analytical calculations that are backed up by N-body simulations of the MW and M31 models, subject to the quadrupolar tug of an external tumbling and triaxial halo. The quadrupolar strength is judged via cosmological simulations, while a range of halo tumbling periods are scanned through in the simulations. We advance this dynamical mechanism as an important cause of non-axisymmetric structures to set in galactic disks. An important result of our work is that the prodding of the disk by a external quadrupole may not always produce bars in the disks. Thus, the M31 disk was found to be robust to the bar instability, though the MW disk was found to be much more susceptible to the bar being triggered by this mechanism. However, both disks were found to warp. The dispersion in cosmological halo properties imply that the external quadrupole is rather weak in some systems and so may not be effective in driving disk evolution.

Our work elucidates the importance of the quadrupolar term in the potential of the external halo, particularly, in terms of the triggering of the bar and warp instabilities. Given the viability of such tidal prodding of the disk by the external halo, detailed investigation of this mechanism in galaxy formation simulations is suggested.

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APPENDIX

We calculate the interaction potential between an exponential disk and a time-dependent external quadrupole potential rotating with pattern speed \( \Omega_p \). If we first assume the potential is static and aligned with the body axes \((x', y', z')\), then we can write the potential from Equation (11) in Cartesian coordinates as

\[
\Phi_{\text{ext}}(x', y', z') = b_1 x'^2 + b_2 y'^2 + b_3 z'^2
\]  
(A1)

with

\[
x' = r \cos \phi \sin \theta,
\]  
(A2)
\[
y' = r \sin \phi \sin \theta,
\]  
(A3)
\[
z' = r \cos \theta,
\]  
(A4)

where one can show easily that the coefficients \(b_1, b_2, \) and \(b_3\) are defined in terms of the harmonic expansion in Equation (11) through

\[
b_1 = -a_{20}/2 + 3a_{22},
\]  
(A5)
\[
b_2 = -a_{20}/2 - 3a_{22},
\]  
(A6)
\[
b_3 = a_{20}.
\]  
(A7)

If the potential is rotating counterclockwise about the \(z'\)-axis with pattern speed \( \Omega_p \) then we can introduce the time dependence of the potential into the coordinates \((x', y', z')\) using inertial frame coordinates \((x, y, z)\) where

\[
x' = x \cos \Omega_p t - y \sin \Omega_p t,
\]  
(A8)
\[
y' = x \sin \Omega_p t + y \cos \Omega_p t,
\]  
(A9)
\[
z' = z,
\]  
(A10)

so that potential becomes

\[
\Phi_{\text{ext}}(x, y, z; t) = d_1(t)x^2 + d_2(t)y^2 + d_3(t)xy + d_4 z^2
\]  
(A11)
with the coefficients
\[
d_1(t) = -a_{20}/2 + 3a_{22} \cos 2\Omega_p t, \quad (A12)
\]
\[
d_2(t) = -a_{20}/2 - 3a_{22} \cos 2\Omega_p t, \quad (A13)
\]
\[
d_3(t) = 6a_{22} \sin 2\Omega_p t, \quad (A14)
\]
\[
d_4 = a_{20}. \quad (A15)
\]

We can thus derive the time-dependent interaction potential between a tilted exponential disk and this tumbling quadrupole potential. The parametric equation of a ring of radius \( R \) with inclination angle \( \theta \) and LON is given by
\[
x = R(\cos \phi \cos \psi - \sin \phi \cos \theta \sin \psi), \quad (A16)
\]
\[
y = R(\sin \phi \cos \psi + \cos \phi \cos \theta \sin \psi), \quad (A17)
\]
\[
z = R \sin \theta \sin \psi, \quad (A18)
\]
where \( \psi \) is the angle measured along the ring. The mass element of the ring is given by
\[
dM = \Sigma(R)R \, dR \, d\psi, \quad (A19)
\]
so we can derive the interaction potential by integrating over \( \psi \) and \( R \) through
\[
V(\theta, \phi; t) = \int d\psi \int \Sigma(R)R \, dR \, \Phi_{\text{ext}}(x, y, z). \quad (A20)
\]

If we assume an exponential disk with
\[
\Sigma(R) = \frac{M_d}{2\pi R_d^2} e^{-R/R_d} \quad (A21)
\]
and substitute the expressions for \((x, y, z)\) in Equation (A18) and do the integrals we arrive at the intermediate form of the interaction potential
\[
V(\theta, \phi; t) = 3M_d R_d^2 \left\{ d_1(t)(1 - \sin^2 \phi \sin^2 \theta) + d_2(t) \times \left( 1 - \cos^2 \phi \sin^2 \theta \right) + d_3(t)(\sin 2\phi \sin 2\theta)/2 + d_4 \sin^2 \theta \right\}. \quad (A22)
\]
This can be reduced to the following simpler form in terms of the original quadrupole coefficients \(a_{20}\) and \(a_{22}\):
\[
V(\theta, \phi; t) = 3M_d R_d^2 \left\{ -a_{20}/2 \left( 3 \cos^2 \theta - 1 \right) + 3a_{22} \cos\left[2(\Omega_p t - \phi)\right] \sin^2 \theta \right\}. \quad (A23)
\]

The derivatives of the interaction potential used in the Euler equations are then
\[
\frac{\partial V}{\partial \theta} = 9M_d R_d^2 \sin \theta \cos \theta \left\{ a_{20} + 2a_{22} \cos\left[2(\Omega_p t - \phi)\right] \right\}, \quad (A24)
\]
\[
\frac{\partial V}{\partial \phi} = 18M_d R_d^2 a_{22} \sin\left[2(\Omega_p t - \phi)\right] \sin^2 \theta. \quad (A25)
\]
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