The $D \to 2$ Limit of General Relativity

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Abstract

A method for taking the $D \to 2$ limit of $D$-dimensional general relativity is constructed, yielding a two-dimensional theory which couples gravitation to conserved stress-energy. We show how this theory is related to those obtained via an alternative dimensional reduction approach.
The study of gravity in two spacetime dimensions has been of considerable interest for several years [1, 2, 3, 4, 5, 6, 7, 8]. Such theories can have quite a rich and interesting structure which reproduces qualitatively much that is found in general relativity, (e.g. black holes, FRW-type cosmologies, gravitational collapse) even though they are mathematically much simpler. This simplicity makes them both a useful pedagogical tool and an interesting arena for the study of quantum gravitational effects.

The topological character of the Einstein-Hilbert action in $D = 2$ dimensions has led theorists to construct such gravitational theories in a manner which circumvents the Einstein equations [1, 7, 9]. Indeed, the triviality of Einstein’s equations in two spacetime dimensions seemingly indicates that $D = 2$ general relativity does not make sense. We wish to point out here that this is not the case. Specifically, we show that (at least formally), one can take the $D \to 2$ limit of Einstein’s equations. This yields a 2D theory of gravity previously considered in the literature [2, 3, 4, 5] in which gravitation is generated by stress-energy and stress-energy is in turn acted upon by gravitation, just as in $D > 2$ general relativity. We then elucidate via a dimensional-reduction argument the connections between this approach and other 2D theories [1, 4].

We begin with the gravitational action in $D$ dimensions

$$S = \frac{1}{\kappa_D} \int d^D x \sqrt{-g} R + \mathcal{L}_M^{(D)},$$

(1)

where $\mathcal{L}_M^{(D)}$ is the $D$-dimensional matter Lagrangian. Since the Einstein tensor goes to zero like $(1 - \frac{D}{2})$ when $D \to 2$, we can extract non-trivial dynamics in this limit by assuming that the gravitational coupling constant $\kappa_D$ also vanishes like $(1 - \frac{D}{2})$ in this limit [10].

Also, we may subtract a term $\int d^D x \sqrt{-\tilde{g}} \tilde{R}/\kappa_D$ without changing the result in two dimensions if it becomes a total derivative as $D \to 2$, that is, if

$$\lim_{D \to 2} \frac{1}{\kappa_D} \tilde{G}_{\mu\nu} = 0.$$  

(2)

We assume this is the case for some metric $\tilde{g}_{\mu\nu}$ conformal to $g_{\mu\nu}$ (i.e., $\tilde{g}_{\mu\nu} = e^{\Psi} g_{\mu\nu}$). Then [11]

$$\tilde{R} = e^{-\Psi} (R - (D - 1) g^{\mu\nu} \Psi_{;\mu ;\nu} - \frac{1}{4} (D - 2)(D - 1) g^{\mu\nu} \Psi_{;\mu} \Psi_{;\nu}).$$

(3)
Since $\sqrt{-g} = e^{D\Psi/2}\sqrt{-\tilde{g}}$, the action

$$S = \frac{1}{\kappa_D} \int d^Dx (\sqrt{-g}R - \sqrt{-\tilde{g}}\tilde{R}) + \mathcal{L}_M^{(D)}$$

becomes

$$S = \frac{1}{\kappa_D} \int d^Dx \sqrt{-g} \left( R - e^{(\frac{D}{2} - 1)\Psi} [R - (D - 1)g_{\mu\nu}\Psi_{;\mu\nu}]ight)$$

$$- \frac{1}{4}(D - 2)(D - 1)g_{\mu\nu}\Psi_{;\mu}\Psi_{;\nu} \right) + \mathcal{L}_M^{(D)}.$$  

The quantity of interest is the limit of this action as $D \to 2$, so we may expand the exponential and discard terms of order $(\frac{D}{2} - 1)^2$. The action in this limit is then

$$S = \frac{1}{\kappa_D} \int d^Dx \sqrt{-g} \left( (D - 1)g_{\mu\nu}\Psi_{;\mu\nu} + \frac{1}{4}(D - 2)(D - 1)g_{\mu\nu}\Psi_{;\mu}\Psi_{;\nu}ight)$$

$$- \left( \frac{D}{2} - 1 \right) \Psi R + \left( \frac{D}{2} - 1 \right)(D - 1)\Psi g_{\mu\nu}\Psi_{;\mu\nu} \right) + \mathcal{L}_M^{(2)} + O(\frac{D}{2} - 1)^2.$$  

We discard the leading term, as it is a total derivative, and integrate the final term by parts to obtain

$$S = \frac{1 - \frac{D}{2}}{\kappa_D} \int d^Dx \sqrt{-g} (\Psi R + \frac{1}{2}(D - 1)g_{\mu\nu}\Psi_{;\mu}\Psi_{;\nu}) + \mathcal{L}_M^{(2)} + O(\frac{D}{2} - 1)^2.$$  

We now wish to choose $\kappa_D$ so as to obtain agreement with the Newtonian theory in two dimensions. This may be done by taking $\lim_{D \to 2} \kappa_D = (1 - \frac{D}{2})8\pi G$, and we thus obtain the limit of the action (4) as $D \to 2$:

$$S = \frac{1}{8\pi G} \int d^Dx \sqrt{-g} (\Psi R + \frac{1}{2}g^{ab}\Psi_{;a}\Psi_{;b}) + \mathcal{L}_M^{(2)}.$$  

This action yields the field equation and conservation law

$$R = 8\pi GT, \quad T_{;b}^{ab} = 0$$

along with an auxiliary equation for $\Psi$

$$\frac{1}{2}(\Psi_{;a}\Psi_{;b} - \frac{1}{2}g_{ab}\Psi_{;c}\Psi^{;c}) + g_{ab}g^{cd}\Psi_{;cd} - \Psi_{;ab} = 8\pi GT_{ab}.$$
Note that the classical evolution \( \Psi \) of the gravity/matter system is independent of the evolution of \( \Psi \), although the converse is not true. This theory has already been studied in some detail, and its classical and semi-classical properties exhibit remarkably strong similarities to general relativity. It reduces to the lineal theory of gravity studied by Jackiw when the stress energy tensor is taken to be a constant (\( T_{ab} \sim \Lambda g_{ab} \)).

The condition (2) on the metric \( \tilde{g}_{\mu \nu} \) may be translated into a condition on \( \Psi \) by using the conformal relation \( \tilde{g}_{\mu \nu} = e^{\Psi} g_{\mu \nu} \), which implies

\[
\tilde{G}_{\mu \nu} = G_{\mu \nu} + (\frac{D}{2} - 1) \left( \frac{1}{2} (\Psi_{;\mu} \Psi_{;\nu} - \frac{1}{2} g_{\mu \nu} g^{\rho \sigma} \Psi_{;\rho} \Psi_{;\sigma}) + g_{\mu \nu} g^{\rho \sigma} \Psi_{;\rho\sigma} - \Psi_{;\mu\nu} \right),
\]

and \( G_{\mu \nu} = \kappa_D T_{\mu \nu} \), which follows from the action (4). Thus, (4) is equivalent to

\[
\lim_{D \to 2} T_{\mu \nu} - \frac{1}{8\pi G} \left( \frac{1}{2} (\Psi_{;\mu} \Psi_{;\nu} - \frac{1}{2} g_{\mu \nu} g^{\rho \sigma} \Psi_{;\rho} \Psi_{;\sigma}) + g_{\mu \nu} g^{\rho \sigma} \Psi_{;\rho\sigma} - \Psi_{;\mu\nu} \right) = 0.
\]

The \( \Psi \) field equation (11) guarantees that the quantity appearing in the limit (12) vanishes for \( D = 2 \), so we need only require that this quantity “vary” smoothly for \( D \to 2 \). That is, we require that \( \Psi \) as a function of \( D \) must be \( C^2 \) in some neighborhood of \( D = 2 \).

An alternate approach for obtaining 2D actions from 4D general relativity is via dimensional reduction [13]. Consider a spherically symmetric metric in four dimensions

\[
ds^2 = e^{\sigma(r)} g_{ab} dx^a dx^b + D(r)(d\theta^2 + \sin^2 \theta d\phi^2),
\]

where latin indices run from 0 to 1, \( g_{ab}(x^a) \) is the metric on the two-dimensional submanifold, and \( r = r(x^a) \) is a scalar field on this submanifold. The Ricci scalar for this metric may be written as

\[
\frac{4}{D} R = e^{-\sigma} [R - \nabla^2 \sigma - \frac{2}{D} \nabla^2 D + \frac{1}{2D^2} (\nabla D)^2] + \frac{2}{D},
\]

where \( \frac{4}{D} R \) is the 4D Ricci scalar, \( R \) is the 2D Ricci scalar associated with \( g_{ab} \), and \( \nabla \) is the covariant derivative associated with \( g_{ab} \). If we also note that...
\(\sqrt{-g}d^4x = \sqrt{-g}d^2xDe^\sigma \sin \theta d\theta d\phi\), we may rewrite the gravitational part of the action (1) for \(D = 4\) as

\[
S = 4\pi \int d^2x \sqrt{-g} \left( D(r)R + [\sigma'D' + \frac{D'^2}{2D}](\nabla r)^2 + 2\Lambda e^\sigma \right),
\]

(15)

where the \(\theta, \phi\) integration has been carried out, the prime denotes derivative with respect to \(r\), and \(\sigma \rightarrow \sigma + \ln(\Lambda)\) for convenience.

If we take \(V(r) = 2\Lambda e^\sigma(r)\) and \(H(r) = \sigma'D' + \frac{D'^2}{2D}\), (15) is a special case of the general vacuum action for dilaton gravity \([14, 15]\).

\[
S = \int d^2x \sqrt{-g}(H(r)g^{ab}\nabla_a r \nabla_b r + V(r) + D(r)R).
\]

(16)

In the absence of any matter action this model actually only depends upon the one function \(V\), since reparametrizations of the field \(r\) accompanied by \(r\)-dependent Weyl rescalings of the metric allow one to relate models with different \(H\)'s and \(D\)'s \([14, 15]\). The action (16) reduces to the gravitational part of (8) for \(H = \frac{1}{2}, D = r\) and \(\Lambda \rightarrow 0\), and to the effective target space action for non-critical string theory \([6, 7]\) for \(D = e^2r = \frac{1}{4}H\).

We may see that this action contains the same information as (1) by noting that the field equations \(4G_{\mu\nu} = 0\) follow from it. \(4G_{ab} = 0\) follows from variation with respect to \(g_{ab}\), and

\[
4G_{\phi\phi} = \sin^2(\theta)4G_{\theta\theta} = 0
\]

(17)

follows from variation with respect to \(r\).

To summarize, we have shown that a rescaling of the gravitational coupling constant by a factor of \((\frac{D}{2} - 1)\) permits one to take a \(D \rightarrow 2\) limit of general relativity. The resultant action, (8), yields a 2D theory in which stress-energy generates gravity and gravity acts in turn upon stress-energy in a manner quite analogous to general relativity \([9]\). This theory is also found as a special case of 2D dilaton gravity as derived by more tradition dimensional reduction methods. It stands in contrast to the wide class of other dilaton gravity theories in that the classical evolution of the gravity/matter system is unaffected by the evolution of any Brans-Dicke scalar. In this sense the theory based on (8) may be said to be the closest thing there is to general relativity in two dimensions.
Acknowledgments

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