ON THE MOTIVE OF CODIMENSION 2 LINEAR SECTIONS OF \( \text{Gr}(3, 6) \)

ROBERT LATERVEER

ABSTRACT. We consider Fano sevenfolds \( Y \) obtained by intersecting the Grassmannian \( \text{Gr}(3, 6) \) with a codimension 2 linear subspace (with respect to the Plücker embedding). We prove that the motive of \( Y \) is Kimura finite-dimensional. We also prove the generalized Hodge conjecture for all powers of \( Y \).

1. INTRODUCTION

Given a smooth projective variety \( Y \) over \( \mathbb{C} \), let \( A_i(Y) := \text{CH}_i(Y)_\mathbb{Q} \) denote the Chow groups of \( Y \) (i.e. the groups of \( i \)-dimensional algebraic cycles on \( Y \) with \( \mathbb{Q} \)-coefficients, modulo rational equivalence). Let \( A_i^{\text{hom}}(Y) \subset A_i(Y) \) denote the subgroup of homologically trivial cycles.

The famous Bloch–Beilinson conjectures [10], [33] predict that the Hodge level of the cohomology of \( Y \) should have an influence on the size of the Chow groups of \( Y \). In particular, there is the following conjecture:

**Conjecture 1.1.** Let \( Y \) be a smooth projective variety of Hodge coniveau \( \geq c \) (i.e. the Hodge numbers \( h^{p,q}(Y) \) vanish provided \( p + q \geq 2c \) and \( p < c \)). Then

\[
A_i^{\text{hom}}(Y) = 0 \quad \forall \ i < c .
\]

(This is known as the “generalized Bloch conjecture”; for motivation and background cf. [33, Section 1.2] or [10].)

Let \( \text{Gr}(3, 6) \) be the Grassmannian of 3-dimensional linear subspaces of a fixed 6-dimensional vector space. In this note, we consider smooth complete intersections

\[
Y = \text{Gr}(3, 6) \cap H_1 \cap H_2 \subset \mathbb{P}^{19}
\]

of the Grassmannian with 2 Plücker hyperplanes \( H_1, H_2 \). This \( Y \) is a 7-dimensional Fano variety of Hodge coniveau 3 (i.e. \( h^{p,7-p}(Y) = 0 \) for \( p < 3 \)). The Hodge theory of \( Y \) has been studied by Donagi in his thesis [6]. The derived category of \( Y \) has been studied by Deliu [5] (this derived category is not yet well-understood, because HPD for \( \text{Gr}(3, 6) \) is still conjectural, cf. Remarks 2.3 and 5.1 below).

Our main result is that Conjecture 1.1 is verified in this case:

**Theorem (=Theorem 3.1).** Let

\[
Y := \text{Gr}(3, 6) \cap H_1 \cap H_2 \subset \mathbb{P}^{19}
\]

2010 Mathematics Subject Classification. Primary 14C15, 14C25, 14C30.

Key words and phrases. Algebraic cycles, Chow groups, motive, Bloch-Beilinson conjectures, Kimura finite-dimensionality, generalized Hodge conjecture.
be a smooth dimensionally transverse intersection, where the $H_j$ are Plücker hyperplanes. Then

$$A_i^{\text{hom}}(Y) = 0 \quad \forall i \neq 3.$$ 

In particular, $Y$ has finite-dimensional motive (in the sense of [14]).

The argument proving Theorem 3.1 uses instances of the Franchetta property (cf. subsection 2.3 below). This is similar to, and inspired by, the seminal work of Voisin on Conjecture [11, 31, 32]. Theorem 3.1 has the following consequence:

**Corollary** (=Corollary 4.1). Let $Y$ be as in Theorem 3.1. The generalized Hodge conjecture is true for all powers of $Y$.

The argument of Corollary 4.1 is as follows: there is a certain elliptic curve $C$ naturally associated to $Y$ (this $C$ is called the Segre curve in honour of C. Segre who studied this curve more than a century ago [26]). An equivalent formulation of Theorem 3.1 is the relation of Chow motives

$$h(Y) \cong h(C)(-3) \oplus \bigoplus_1(\ast) \quad \text{in } M_{\text{rat}}.$$ 

Thus, to prove the generalized Hodge conjecture for powers of $Y$ one is reduced to powers of $C$, for which the generalized Hodge conjecture is known.

Another application of Theorem 3.1 concerns Voevodsky’s conjecture on smash-equivalence (Corollary 4.3).

It would be interesting to understand more generally the Chow groups of linear sections (of codimension $r > 2$) of $\text{Gr}(3, 6)$, cf. Remark 5.1.

**Conventions.** In this note, the word variety will refer to a reduced irreducible scheme of finite type over $\mathbb{C}$. A subvariety is a (possibly reducible) reduced subscheme which is equidimensional.

All Chow groups will be with rational coefficients: we denote by $A_j(Y) := CH_j(Y)_\mathbb{Q}$ the Chow group of $j$-dimensional cycles on $Y$ with $\mathbb{Q}$-coefficients; for $Y$ smooth of dimension $n$ the notations $A_j(Y)$ and $A^{n-j}(Y)$ are used interchangeably. The notations $A_i^{\text{hom}}(Y)$ and $A_i^{\text{AJ}}(X)$ will be used to indicate the subgroup of homologically trivial (resp. Abel–Jacobi trivial) cycles.

The contravariant category of Chow motives (i.e., pure motives with respect to rational equivalence as in [25], [22]) will be denoted $M_{\text{rat}}$.

2. Preliminaries

2.1. Codimension 1 linear sections. As a warm-up for the codimension 2 case, let us first consider codimension 1 linear sections of $\text{Gr}(3, 6)$.

**Theorem 2.1** (Donagi [6]). Let

$$Y := \text{Gr}(3, 6) \cap H \subset \mathbb{P}^{19}.$$ 

be a smooth dimensionally transverse intersection, where $H$ is a Plücker hyperplane. The interesting Hodge numbers of $Y$ are

$$h^{p,8-p}(Y) = \begin{cases} 4 & \text{if } p = 4, \\ 0 & \text{otherwise}. \end{cases}$$

Proof. This is contained in [6, Section 3.4].

Remark 2.2. Let $Y = \text{Gr}(3,6) \cap H$ be a hypersurface as in Theorem 2.1. As the Hodge coniveau of $Y$ is 4, Conjecture 1.1 (combined with the Bloch–Srinivas argument [4, 17]) predicts that

$$A_i^{\text{hom}}(Y) = 0 \quad \forall \ i.$$

In this case, this is readily verified using a construction of Donagi’s [6]: we choose a hyperplane $P_5 \subset V_6$, and we consider the incidence variety

$$\widetilde{\text{Gr}} := \left\{ (A, \ell) \in \text{Gr}(3,V_6) \times \text{Gr}(2,P_5) \mid \ell \subset A \right\} \subset \text{Gr}(3,V_6) \times \text{Gr}(2,P_5).$$

The first projection $\widetilde{\text{Gr}} \to \text{Gr}(3,V_6)$ is birational (it is actually a blow-up with center the locus $\sigma_{111}(P)$ of subspaces contained in $P$). The second projection

$$\Pi: \widetilde{\text{Gr}} \to \text{Gr}(2,P_5)$$

is a $\mathbb{P}^3$-fibration. We now consider the morphism

$$\Pi_Y: \widetilde{Y} \to \text{Gr}(2,P_5),$$

obtained by restricting $\Pi$ to the strict transform $\widetilde{Y}$ of $Y$ in $\widetilde{\text{Gr}}$. As explained in [6, Section 3.4], for $P_5$ generic with respect to $Y$, the morphism $\Pi_Y$ is a $\mathbb{P}^2$-fibration over $\text{Gr}(2,P_5) \setminus S$, and a $\mathbb{P}^3$-fibration over $S$, where $S \subset \text{Gr}(2,P_5)$ is a closed subvariety isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Applying the Chow-theoretic version of Cayley’s trick [11, Theorem 3.1] to this set-up, we find that $A_i(\widetilde{Y})$ is a direct sum of Chow groups of $\text{Gr}(2,5)$ and of $S$, hence $\widetilde{Y}$ has trivial Chow groups. It follows that $Y$ also has trivial Chow groups, i.e. the prediction (1) is verified. (Another proof of (1) is given in [19, Theorem 3.2].)

Remark 2.3. Let $Y = \text{Gr}(3,6) \cap H$ be a hypersurface as in Theorem 2.1. The theory of homological projective duality [15, 16, 27] suggests that the derived category of $Y$ should admit a full exceptional collection. The construction of the HPD dual of $\text{Gr}(3,6)$ appears to be an open problem (cf. [5, Conjecture 21], where a non-commutative resolution of the double cover of $\mathbb{P}^{10}$ branched along a certain quartic hypersurface is suggested as a candidate). Nevertheless, it seems likely the existence of a full exceptional collection for $Y$ can be proven by looking at the above construction and applying a categorical version of Cayley’s trick ([12, Theorem 2.6] or [3, Proposition 47]).

2.2. Codimension 2 linear sections.

Theorem 2.4 (Donagi [6]). Let

$$Y := \text{Gr}(3,6) \cap H_1 \cap H_2 \subset \mathbb{P}^{19}$$

be a smooth dimensionally transverse intersection, where $H$ is a Plücker hyperplane. The interesting Hodge numbers of $Y$ are

$$h^{p,8-p}(Y) = \begin{cases} 4 & \text{if } p = 4, \\ 0 & \text{otherwise}. \end{cases}$$

Proof. This is contained in [6, Section 3.4].

Remark 2.2. Let $Y = \text{Gr}(3,6) \cap H$ be a hypersurface as in Theorem 2.1. As the Hodge coniveau of $Y$ is 4, Conjecture 1.1 (combined with the Bloch–Srinivas argument [4, 17]) predicts that

$$A_i^{\text{hom}}(Y) = 0 \quad \forall \ i.$$
be a smooth dimensionally transverse intersection, where the $H_j$ are Plücker hyperplanes. The Hodge diamond of $Y$ is

\[
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 1 & & 0 & \\
0 & 0 & & 0 & & 0 & 0 \\
0 & 0 & & 2 & & 0 & 0 \\
0 & 0 & & 0 & & 0 & 0 & 0 \\
0 & 0 & & 3 & & 0 & 0 & 0 & 0 \\
0 & 0 & & 0 & & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & & 0 & & 3 & 0 & 0 & 0 & 0 \\
0 & 0 & & 0 & & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & & 2 & & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & & 0 & & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & & 0 & & 0 & 0 & 0 & 0 & 0 \\
 & & 0 & & 0 & & 0 & 0 & 0 & 0 & 0 \\
 & & & 1 & & & 
\end{array}
\]

**Proof.** This is contained in [6, Section 3.4]. (Alternatively, one could apply [7] to find the Hodge number $h^{4,3}(Y)$.) $\Box$

**Corollary 2.5.** Let $Y := \text{Gr}(3, 6) \cap H_1 \cap H_2$ be as in Theorem 2.4. There exist an elliptic curve $C$ and a correspondence $\Gamma \in A^4(C \times Y)$ inducing an isomorphism

$$
\Gamma_* : H^1(C, \mathbb{Q}) \cong H^7(Y, \mathbb{Q}).
$$

**Proof.** Let $J^4(Y)$ denote the intermediate Jacobian. Because the Hodge coniveau of $H^7(Y, \mathbb{Q})$ is 3, $J^4(Y)$ is an abelian variety. Because $h^{4,3}(Y) = 1$, the dimension of $J^4(Y)$ is 1, i.e. $J^4(Y)$ is an elliptic curve. As explained for instance in [21, Proof of Lemma 3], the general theory of abelian varieties guarantees the existence of an elliptic curve $C$ and a correspondence $\Gamma \in A^4(C \times Y)$ such that there is a commutative diagram

$$
\begin{array}{ccc}
A_\text{hom}^1(C) & \xrightarrow{\Gamma_*} & A_\text{hom}^4(Y) \\
\downarrow AJ & & \downarrow AJ \\
C \cong J^1(C) & \xrightarrow{\Gamma_*} & J^4(Y) \\
\end{array}
$$

Here $AJ$ is the Abel–Jacobi map, and $J^1(C)$ is the Jacobian of $C$. The left arrow is an isomorphism, and the lower horizontal arrow is an isogeny of elliptic curves. It follows that $\Gamma$ induces an isomorphism

$$
\Gamma_* : H^{0,1}(C) \cong H^{3,4}(Y),
$$

and hence (taking the complex conjugate) also

$$
\Gamma_* : H^{1,0}(C) \cong H^{4,3}(Y).
$$
Invoking the Hodge decomposition $H^7(Y, \mathbb{C}) = H^{4,3}(Y) \oplus H^{3,4}(Y)$ (and likewise $H^1(C, \mathbb{C}) = H^{1,0}(C) \oplus H^{0,1}(C)$), we find that $\Gamma$ induces an isomorphism

$$\Gamma_* : H^1(C, \mathbb{C}) \overset{\cong}{\rightarrow} H^7(Y, \mathbb{C}).$$

□

Remark 2.6. It is shown by Donagi [6] that one can actually find a geometric incarnation of the elliptic curve $C$ of Corollary 2.5. There is a certain elliptic curve in $\text{Gr}(3, 6)$ naturally associated to the pencil of hyperplanes defining $Y$, this is called the Segre curve in [6, Section 3.3] (in honour of C. Segre who had already studied this curve [26]). Donagi proves [6, Theorem 3.8] that there is a natural isomorphism (with a geometric interpretation) from the Segre elliptic curve to the intermediate Jacobian $\mathcal{J}^4(Y)$.

Remark 2.7. Let $Y := \text{Gr}(3, 6) \cap H_1 \cap H_2$ be as in Theorem 2.4. To understand the Chow groups of $Y$, it is natural to try and apply the method of Remark 2.2. That is, one would like to consider the morphism

$$\Pi_Y : \tilde{Y} \rightarrow \text{Gr}(2, 5)$$

obtained by restricting the morphism $\Pi : \tilde{\text{Gr}} \rightarrow \text{Gr}(2, 5)$ (of Remark 2.2) to the strict transform $\tilde{Y}$ of $Y$ in $\tilde{\text{Gr}}$. Unfortunately, this approach runs into trouble: the locus $S \subset \text{Gr}(2, 5)$ (where the fibers of $\Pi_Y$ have larger dimension) is now a quadric surface bundle with some singular fibers, and it seems difficult to handle the Chow groups of $S$.

For this reason, I have preferred to use the “spread” method (which means considering the universal family of sections $Y$) to prove Theorem 3.1.

Remark 2.8. Let $Y := \text{Gr}(3, 6) \cap H_1 \cap H_2$ be as in Theorem 2.4. Homological projective duality predicts that there is a semi-orthogonal decomposition

$$\mathcal{D}^b(Y) = \langle \mathcal{D}^b(C), A_1, \ldots, A_r \rangle,$$

where $C$ is the Segre elliptic curve and the $A_j$ are exceptional objects. As mentioned in Remark 2.3, a conjectural candidate for the HPD dual of $\text{Gr}(3, 6)$ is identified in [5, Conjecture 21].

2.3. The Franchetta property.

Definition 2.9. Let $\mathcal{Y} \rightarrow B$ be a smooth projective morphism, where $\mathcal{Y}, B$ are smooth quasi-projective varieties. We say that $\mathcal{Y} \rightarrow B$ has the Franchetta property in codimension $j$ if the following holds: for every $\Gamma \in A^j(\mathcal{Y})$ such that the restriction $\Gamma|_{Y_b}$ is homologically trivial for the very general $b \in B$, the restriction $\Gamma|_{Y_b}$ is rationally trivial, i.e. $\Gamma|_{Y_b}$ is zero in $A^j(Y_b)$ for all $b \in B$.

We say that $\mathcal{Y} \rightarrow B$ has the Franchetta property if $\mathcal{Y} \rightarrow B$ has the Franchetta property in codimension $j$ for all $j$.

This property is studied in [23], [2], [8], [9].

Definition 2.10. Given a family $\mathcal{Y} \rightarrow B$ as above, with $Y := Y_b$ a fiber, we write

$$GDA_B^j(Y) := \text{Im} \left( A^j(\mathcal{Y}) \rightarrow A^j(Y) \right)$$
for the subgroup of \textit{generically defined cycles}. In a context where it is clear to which family we are referring, the index \( B \) will often be suppressed from the notation.

With this notation, the Franchetta property amounts to saying that \( GDA_B(Y) \) injects into cohomology, under the cycle class map.

2.4. A Franchetta-type result.

\textbf{Proposition 2.11.} Let \( M \) be a smooth projective variety with \( A^\text{hyp}_1(M) = 0 \). Let \( L_1, \ldots, L_r \to M \) be very ample line bundles, and let \( \mathcal{Y} \to B \) be the universal family of smooth dimensionally transverse complete intersections of type

\[ Y = M \cap H_1 \cap \cdots \cap H_r, \quad H_j \in |L_j|. \]

Assume the fibers \( Y = Y_b \) have \( H^{\text{tr}}_{Y}(Y, \mathbb{Q}) \neq 0 \). There is an inclusion

\[ \ker \left( GDA_B(Y \times Y) \to H^2_{\text{tr}}(Y \times Y, \mathbb{Q}) \right) \subset \left( (p_1)^*GDA_B(Y), (p_2)^*GDA_B(Y) \right), \]

where \( p_1, p_2 \) denote the projection from \( Y \times Y \) to first resp. second factor.

\textit{Proof.} This is essentially Voisin’s “spread” result \cite[Proposition 1.6]{32} (cf. also \cite[Proposition 5.1]{20} for a reformulation of Voisin’s result). We give a different proof based on \cite{8}. Let \( B := \mathbb{P}H^0(M, L_1 \oplus \cdots \oplus L_r) \) (so \( B \subset B \) is a Zariski open), and let us consider the projection

\[ \pi: \mathcal{Y} \times_B \mathcal{Y} \to M \times M. \]

Using the very ampleness assumption, one finds that \( \pi \) is a \( \mathbb{P}^s \)-bundle over \((M \times M) \setminus \Delta_M\), and a \( \mathbb{P}^t \)-bundle over the diagonal \( \Delta_M \). That is, \( \pi \) is what is termed a \textit{stratified projective bundle} in \cite{8}. As such, \cite[Proposition 5.2]{8} implies the equality

\[ GDA_B(Y \times Y) = \text{Im} \left( A^*(M \times M) \to A^*(Y \times Y) \right) + \Delta_n GDA_B(Y), \]

where \( \Delta: Y \to Y \times Y \) is the inclusion along the diagonal. The assumption \( A^\text{hyp}_1(M) = 0 \) implies that \( M \) has the Chow–Künneth property, i.e. \( A^*(M \times M) \) is isomorphic to \( A^*(M) \otimes A^*(M) \) (this follows from \cite[Proposition 4.22]{33}). Base-point freeness of the \( L_j \) implies that

\[ GDA_B(Y) = \text{Im} \left( A^*(M) \to A^*(Y) \right). \]

The equality \eqref{2} thus reduces to

\[ GDA_B(Y \times Y) = \left( (p_1)^*GDA_B(Y), (p_2)^*GDA_B(Y), \Delta_Y \right) \]

(where \( p_1, p_2 \) denote the projection from \( S \times S \) to first resp. second factor). The assumption that \( Y \) has non-zero transcendental cohomology implies that the class of the diagonal \( \Delta_Y \) is not decomposable in cohomology (indeed, decomposable correspondences act as zero on the transcendental cohomology). It follows that

\[ \text{Im} \left( GDA_B(Y \times Y) \to H^2_{\text{tr}}(Y \times Y, \mathbb{Q}) \right) = \text{Im} \left( \text{Dec}^2(Y \times Y) \to H^2_{\text{tr}}(Y \times Y, \mathbb{Q}) \right) \oplus \mathbb{Q}[\Delta_Y], \]
where we use the shorthand

$$\text{Dec}^j(Y \times Y) := \left( (p_1)^*GDA_B^*(Y), (p_2)^*GDA_B^*(Y) \right) \cap A^j(Y \times Y)$$

for the decomposable cycles. We now see that if $\Gamma \in GDA_{\dim Y}(Y \times Y)$ is homologically trivial, then $\Gamma$ does not involve the diagonal and so $\Gamma \in \text{Dec}_{\dim Y}(Y \times Y)$. This proves the proposition. \(\square\)

**Remark 2.12.** Proposition 2.11 has the following consequence: if the family $Y \rightarrow B$ has the Franchetta property, then $Y \times_B Y \rightarrow B$ has the Franchetta property in codimension $\dim Y$.

### 2.5. A Chow–K"unneth decomposition.

**Lemma 2.13.** Let $M$ be a smooth projective variety with $A^*_{\text{hom}}(Y) = 0$. Let $Y \subset M$ be a smooth complete intersection as in Proposition 2.11 of dimension $\dim Y = d$. The variety $Y$ has a self-dual Chow–K"unneth decomposition $\{\pi_Y^j\}$ with the property that

$$h^j(Y) := (Y, \pi_Y^j, 0) = \oplus \mathbb{I}(*) \text{ in } M_{\text{rat}} \quad \forall \ j \neq d.$$  

Moreover, this decomposition is generically defined: writing $Y \rightarrow B$ for the universal family (of complete intersections of the type of $Y$), there exist relative projectors $\pi_Y^j \in A^d(Y \times_B Y)$ such that $\pi_Y^j = \pi_Y^j|_b$ (where $Y = Y_b$ for $b \in B$).

**Proof.** This is a standard construction, one can look for instance at [24] (in case $d$ is odd, which will be the case in this note, the “variable motive” $h(Y)^{\text{var}}$ of [24] Theorem 4.4 coincides with $h^d(Y)$). \(\square\)

### 3. Main result

**Theorem 3.1.** Let

$$Y := \text{Gr}(3, 6) \cap H_1 \cap H_2 \subset \mathbb{P}^{19}$$

be a smooth dimensionally transverse intersection with 2 hyperplanes $H_1, H_2$ (with respect to the Pl"ucker embedding). Let $C$ be the Segre curve associated to $Y$. There is an isomorphism of motives

$$h(Y) \cong h(C)(-3) \oplus \bigoplus \mathbb{I}(*) \text{ in } M_{\text{rat}}.$$  

In particular, $Y$ has finite-dimensional motive (in the sense of [14]), and

$$A^i_{\text{AJ}}(Y) = 0 \quad \forall i.$$  

**Proof.** Let $Y \rightarrow B$ denote the universal family of smooth dimensionally transverse intersections as in the theorem, where $B$ is a Zariski open in

$$\bar{B} := \mathbb{P} H^0(\text{Gr}(3, 6), \mathcal{O}_{\text{Gr}(3, 6)}(1)^{\oplus 2}).$$

Proposition 2.11 applies to this set-up (with $M = \text{Gr}(3, 6)$), and gives an inclusion

$$\ker \left( GDA_B^*(Y \times Y) \rightarrow H^{14}(Y \times Y, \mathbb{Q}) \right) \subset \left< (p_1)^*GDA_B^*(Y), (p_2)^*GDA_B^*(Y) \right>.$$
Let us construct an interesting cycle in $GDA_B^4(Y \times Y)$ to which we can apply (4). For any $Y = Y_b$ with $b \in B$, Corollary 2.5 gives us a smooth curve $C = C_b$ and a cycle $\Gamma \in A^4(C \times Y)$ inducing a surjection

$$\Gamma_*: H^1(C, \mathbb{Q}) \twoheadrightarrow H^7(Y, \mathbb{Q}).$$

Writing $C \to B$ for the universal family of Segre curves, the cycle $\Gamma$ naturally exists relatively, i.e. $\Gamma \in GDA_B^4(F \times Y)$. Since both $C$ and $Y$ verify the standard conjectures, the right-inverse to $\Gamma_*$ is correspondence-induced, i.e. there exists $\Psi \in A^4(Y \times C)$ such that

$$(\Gamma \circ \Psi)_* = id: H^7(Y, \mathbb{Q}) \to H^7(Y, \mathbb{Q}).$$

(This follows as in [28, Proof of Proposition 1.1]).

We now involve the (generically defined) Chow–Künneth decomposition $\pi^j_Y \in A^7(Y \times Y)$ given by Lemma 2.13. The above means that for $Y = Y_b$ for any $b \in B$, there is vanishing

$$(\Delta Y - \Gamma \circ \Psi) \circ \pi^7_Y = 0 \text{ in } H^{14}(Y \times Y, \mathbb{Q}).$$

Applying Voisin’s Hilbert schemes argument [31, Proposition 3.7], [33, Proposition 4.25] (cf. also [18, Proposition 2.10] for the precise form used here), we can assume that $\Psi$ is also generically defined, and hence

$$(\Delta Y - \Gamma \circ \Psi) \circ \pi^7_Y \in GDA^7(Y \times Y).$$

Now looking at (4), we learn that this cycle is decomposable, i.e.

$$(\Delta Y - \Gamma \circ \Psi) \circ \pi^7_Y \in \langle (p_1)^*GDA^*(Y), (p_2)^*GDA^*(Y) \rangle.$$

That is, for any $Y = Y_b$ with $b \in B$ we can write

$$(\Delta Y - \Gamma \circ \Psi) \circ \pi^7_Y = \gamma \text{ in } A^7(Y \times Y),$$

with $\gamma \in A^*(Y) \otimes A^*(Y)$. Since the $\pi^j_Y, j \neq 7$ of Lemma 2.13 are also decomposable (i.e. they are in $A^*(Y) \otimes A^*(Y)$), this implies that we can write

$$(\Delta Y - \Gamma \circ \Psi) = \gamma' \text{ in } A^7(Y \times Y),$$

with $\gamma' \in A^*(Y) \otimes A^*(Y)$. Being decomposable, $\gamma'$ does not act on Abel–Jacobi trivial cycles, and so

$$A^i_{AJ}(Y) \xrightarrow{\Psi_*} A^i_{AJ}(C) \xrightarrow{\Gamma_*} A^i_{AJ}(Y)$$

is the identity. But $C$ being a curve, the group in the middle vanishes for all $i$. This proves the vanishing

$$A^i_{AJ}(Y) = 0$$

for all $Y = Y_b$. The Kimura finite-dimensionality of $Y$ also follows, since submotives of sums of motives of curves are finite-dimensional.

We have now proven that there is a split injection

$$h^7(Y) \hookrightarrow h^1(C)(-3) \text{ in } M_{rat}.$$
Since the motive $h^1(C)$ is indecomposable and $h^7(Y)$ is non-zero, this injection is actually an isomorphism

$$h^7(Y) \cong h^1(C)(-3) \text{ in } \mathcal{M}_{\text{rat}}.$$  

Combining (5) with the equalities

$$h(Y) = h^7(Y) \oplus 1 \oplus 1(-1) \oplus 1(-2)^{\oplus 2} \oplus 1(-3)^{\oplus 3} \oplus 1(-4)^{\oplus 3} \oplus 1(-5)^{\oplus 2} \oplus 1(-6) \oplus 1(-7),$$  

$$h(C) = h^1(C) \oplus 1 \oplus 1(-1) \text{ in } \mathcal{M}_{\text{rat}}$$

(cf. Lemma 2.13), this gives the isomorphism (3).

4. TWO CONSEQUENCES

Corollary 4.1. Let $Y$ be as in Theorem 3.1. Then the generalized Hodge conjecture is true for $Y^m$ for all $m \in \mathbb{N}$.

Proof. The isomorphism of motives of Theorem 3.1 implies there is an isomorphism of Hodge structures

$$H^j(Y^m, \mathbb{Q}) \cong H^{j-6m}(C^m, \mathbb{Q})(-3m) \oplus \bigoplus H^*(C^{m-1}, \mathbb{Q})(\ast) \oplus \cdots \oplus \bigoplus \mathbb{Q}(\ast).$$

Since this isomorphism is also compatible with the coniveau filtration [28, Proposition 1.2], one is reduced to proving the generalized Hodge conjecture for powers of $C$. This is known thanks to work of Abdulali [1, Section 8.1] (cf. also [29, Corollary 3.13]).

Remark 4.2. Corollary 4.1 does not really use the full force of Theorem 3.1: to prove Corollary 4.1 it suffices to have an isomorphism of homological motives linking $Y$ and $C$; such an isomorphism follows readily from Corollary 2.5.

For the next consequence, we recall that a cycle $a \in A^i(Y)$ is called smash-nilpotent if $a^N$ is zero in $A^{Ni}(Y^N)$ for some $N \in \mathbb{N}$. Two cycles are smash-equivalent if their difference is smash-nilpotent. Voevodsky has conjectured that smash-equivalence coincides with numerical equivalence for all smooth projective varieties [30]. Using Theorem 3.1, we verify this in some cases:

Corollary 4.3. Let $Y$ be as in Theorem 3.1. Then smash-equivalence and numerical equivalence coincide on $Y^m$ for $m \leq 3$.

Proof. The isomorphism of motives of Theorem 3.1 implies there is an isomorphism of Chow groups

$$A^j(Y^m) \cong A^{j-3m}(C^m) \oplus \bigoplus A^*(C^{m-1}) \oplus \cdots \oplus \bigoplus \mathbb{Q},$$

respecting any adequate equivalence relation. The result follows, since smash-equivalence and numerical equivalence coincide (for zero-cycles and divisors [30] and hence) for all surfaces, and for abelian threefolds [13].
5. AND BEYOND?

**Remark 5.1.** What can one say about the Chow groups (or Chow motive) of smooth dimension-
ally transverse intersections

\[ Y := \text{Gr}(3, 6) \cap H_1 \cap \cdots \cap H_r \subseteq \mathbb{P}^{19} \]

for arbitrary \( r \)? The conjectural HPD picture drawn in [5, Section 5.3] suggests the following prediction: for \( r \leq 5 \), there is an isomorphism of motives

\[
\text{h}(Y) \oplus \bigoplus \mathbb{I}(\ast) \cong \text{h}(X)(r-5) \oplus \bigoplus \mathbb{I}(\ast) \quad \text{in } M_{\text{rat}},
\]

where \( X \) is a double cover of \( \mathbb{P}^{r-1} \) branched along a certain smooth quartic hypersurface. (For \( r > 5 \), \( X \) should be a certain resolution of singularities of a double cover of \( \mathbb{P}^{r-1} \) branched along a singular quartic.)

To prove this isomorphism for \( r > 2 \) using the method of this note, one is essentially reduced to proving this isomorphism holds modulo homological equivalence and is generically defined.

**Acknowledgments.** Thanks to the referee for helpful comments. Thanks to Mama-san of the izakaya in Schiltigheim.

**References**

[1] S. Abdulali, Tate twists of Hodge structures arising from abelian varieties, in: Recent Advances in Hodge Theory: Period Domains, Algebraic Cycles, and Arithmetic (M. Kerr and G. Pearlstein, eds.), London Math. Society Lecture Note Series 427, Cambridge University Press 2016, pp. 292—307.

[2] N. Bergeron and Z. Li, Tautological classes on moduli space of hyperkähler manifolds, Duke Math. J., arXiv:1703.04733.

[3] M. Bernardara, E. Fatighenti and L. Manivel, Nested varieties of K3 type, arXiv:1912.03144.

[4] S. Bloch and V. Srinivas, Remarks on correspondences and algebraic cycles, American Journal of Mathematics Vol. 105, No 5 (1983), 1235—1253.

[5] D. Deliu, Homological projective duality for \( \text{Gr}(3, 6) \), (2011) Publicly Accessible Penn Dissertations 316, [http://repository.upenn.edu/edissertations/316](http://repository.upenn.edu/edissertations/316).

[6] R. Donagi, On the geometry of Grassmannians, Duke Math. J. 44 no. 4 (1977), 795—837.

[7] E. Fatighenti and G. Mongardi, A note on a Griffiths-type ring for complete intersections in Grassmannians, arXiv:1801.09586.

[8] L. Fu, R. Laterveer and Ch. Vial, The generalized Franchetta conjecture for some hyper-Kähler varieties (with an appendix joint with M. Shen), Journal Math. Pures et Appliquées (9) 130 (2019), 1—35.

[9] L. Fu, R. Laterveer and Ch. Vial, The generalized Franchetta conjecture for some hyper-Kähler varieties, II, arXiv:2002.05490.

[10] U. Jannsen, Motivic sheaves and filtrations on Chow groups, in: Motives (U. Jannsen et alii, eds.), Proceedings of Symposia in Pure Mathematics Vol. 55 (1994), Part 1.

[11] Q. Jiang, On the Chow theory of projectivization, arXiv:1910.06730v1.

[12] Q. Jiang and N. Leung, Derived category of projectivization and flops, arXiv:1811.12525.

[13] B. Kahn and R. Sebastian, Smash-nilpotent cycles on abelian 3-folds, Math. Res. Letters 16 (2009), 1007—1010.
ON THE MOTIVE OF CODIMENSION 2 LINEAR SECTIONS OF $\text{Gr}(3, 6)$

[14] S.-I. Kimura, Chow groups are finite dimensional, in some sense, Math. Ann. 331 no 1 (2005), 173—201,
[15] A. Kuznetsov, Homological projective duality for Grassmannians of lines, [math.AG/0610957],
[16] A. Kuznetsov, Homological projective duality, Publications Mathématiques de l'I.H.E.S. 105 no. 1 (2007), 157—220,
[17] R. Laterveer, Algebraic varieties with small Chow groups, Journal Math. Kyoto Univ. Vol. 38 no 4 (1998), 673—694,
[18] R. Laterveer, A family of cubic fourfolds with finite-dimensional motive, Journal of the Mathematical Society of Japan 70 no. 4 (2018), 1453—1473,
[19] R. Laterveer, On the Chow groups of Plücker hypersurfaces in Grassmannians, preprint,
[20] R. Laterveer, J. Nagel and C. Peters, On complete intersections in varieties with finite-dimensional motive, Quarterly Journal of Math. 70 no. 1 (2019), 71—104,
[21] J. Murre, Abel–Jacobi equivalence versus incidence equivalence for algebraic cycles of codimension 2, Topology 24 no. 3 (1985), 361—367,
[22] J. Murre, J. Nagel and C. Peters, Lectures on the theory of pure motives, Amer. Math. Soc. University Lecture Series 61, Providence 2013,
[23] N. Pavic, J. Shen and Q. Yin, On O’Grady’s generalized Franchetta conjecture, Int. Math. Res. Notices (2016), 1—13,
[24] C. Peters, On a motivic interpretation of primitive, variable and fixed cohomology, Math. Nachrichten 292 no. 2 (2019), 402—408,
[25] T. Scholl, Classical motives, in: Motives (U. Jannsen et alii, eds.), AMS Proceedings of Symposia in Pure Mathematics Vol. 55 (1994), Part 1,
[26] C. Segre, Sui complessi lineari di piani nello spazio a cinque dimensioni, Annali di Mat. 27 (1918), 75—123,
[27] R. Thomas, Notes on homological projective duality, in: Algebraic Geometry, Salt Lake City 2015 (T. de Fernex et al., eds.), AMS Proceedings of Symposia in Pure Math. Vol. 97 Part 1, 585—610,
[28] Ch. Vial, Niveau and coniveau filtrations on cohomology groups and Chow groups, Proceedings of the LMS 106(2) (2013), 410—444,
[29] Ch. Vial, Generic cycles, Lefschetz representations and the generalized Hodge and Bloch conjectures for abelian varieties, [arXiv:1803.00857], to appear in Annali SNS Pisa,
[30] V. Voevodsky, A nilpotence theorem for cycles algebraically equivalent to zero, Internat. Math. Research Notices 4 (1995), 187—198,
[31] C. Voisin, The generalized Hodge and Bloch conjectures are equivalent for general complete intersections, Ann. Sci. Ecole Norm. Sup. 46 fascicule 3 (2013), 449—475,
[32] C. Voisin, The generalized Hodge and Bloch conjectures are equivalent for general complete intersections, II, J. Math. Sci. Univ. Tokyo 22 (2015), 491—517,
[33] C. Voisin, Chow Rings, Decomposition of the Diagonal, and the Topology of Families, Princeton University Press, Princeton and Oxford, 2014,