1 Introduction

We start with a very brief description of the principal events in the history of our subject.

About 1870 M. Noether discovered [N] that the group of birational automorphisms of the projective plane \( \text{Bir} \mathbf{P}^2 \), which is known also as the Cremona group \( \text{Cr} \mathbf{P}^2 \), is generated by its subgroup \( \text{Aut} \mathbf{P}^2 = \text{Aut} \mathbf{C}^3 / \mathbf{C}^* \) and any quadratic Cremona transformation \( \tau \), which in a certain system of homogeneous coordinates can be written down as

\[
\tau: (x_0: x_1: x_2) \to (x_1 x_2: x_0 x_2: x_0 x_1).
\]

Noether’s arguments were as follows. Take any Cremona transformation

\[
\chi: \mathbf{P}^2 \dashrightarrow \mathbf{P}^2.
\]

Then either \( \chi \) is a projective isomorphism, or the proper inverse image of the linear system of lines in \( \mathbf{P}^2 \) is a linear system \( |\chi| \) of curves of the degree \( n(\chi) \geq 2 \) with prescribed base points \( a_1, \ldots, a_N \), including infinitely near ones. Let \( \nu_1, \ldots, \nu_N \) be their multiplicities with respect to the system \( |\chi| \), and assume that \( \nu_1 \geq \nu_2 \geq \ldots \geq \nu_N \). Then, as far as two lines intersect each other at one point, the free intersection of two curves of the system \( |\chi| \) (that is, the intersection outside the base locus) equals 1. So

\[
n^2(\chi) = \sum_{i=1}^{N} \nu_i^2 + 1.
\]

Moreover, the curves in \( |\chi| \) are rational, and nonsingular outside the base locus, and so, computing their geometric genus by their arithmetical one, we get

\[
(n(\chi) - 1)(n(\chi) - 2) = \sum_{i=1}^{N} \nu_i (\nu_i - 1).
\]
It is easy to derive from these two equalities that \( N \geq 3 \) and the first three greatest multiplicities satisfy the Noether’s inequality
\[
\nu_1 + \nu_2 + \nu_3 > n.
\]

Now, if \( a_1, a_2, a_3 \) lie directly on the plane \( \mathbf{P}^2 \) (i.e., among them there are no infinitely near points), then we can take the composition
\[
\chi \tau: \mathbf{P}^2 \dashrightarrow \mathbf{P}^2,
\]
where \( \tau \) is a quadratic transformation constructed with respect to the triplet \( (a_1, a_2, a_3) \).

Let us prove that \( n(\chi \tau) < n(\chi) \).

Indeed, the degree of a curve is the number of the points of intersection with a line. But the points of intersection of a generic line \( L \) and a curve \( C \) from \( |\chi \tau| \) are in 1-1 correspondence with the points of free intersection of their images \( \tau(L) \) and \( \tau(C) \). But \( \tau(L) \) is a conic passing through \( a_1, a_2, a_3 \), whereas \( \tau(C) \in |\chi| \). Thus the intersection number equals
\[
2n(\chi) - \nu_1 - \nu_2 - \nu_3 < n.
\]

Proceeding in this manner, we “untwist” the “maximal” triplets until we come to the case \( n(\chi) = 1 \). Noether’s theorem would have been proved but for the maximal triplets which include infinitely near points, when Noether’s arguments do not work. It took about 30 years to complete the proof.

The second part of our story begins in the first years of the present century, when G.Fano made his first attempts to extend two-dimensional birational methods to three-dimensional varieties \([F1,2]\).

He started with trying to describe birational correspondences of three-dimensional quartics \( V_4 \subset \mathbf{P}^4 \). His choice of the object of study was really the best: up to this day, the quartic is one of the principle touchstones for multi-dimensional birational constructions.

Reproducing Noether’s arguments, Fano considered a birational correspondence \( \chi: V \dashrightarrow V' \) between two smooth three-dimensional quartics and, taking the proper inverse image \( |\chi| \) of the linear system of hyperplane sections of \( V' \subset \mathbf{P}^4 \), came to the following conclusion. Either \( |\chi| \) is cut out on \( V \) by hyperplanes, and then \( \chi \) is a projective isomorphism, or \( |\chi| \) is cut out on \( V \) by hypersurfaces of the degree \( n(\chi) \geq 2 \), and then the base locus \( |\chi| \) satisfies certain conditions, similar to the Noether’s inequality. These conditions are now called the Fano’s inequalities.

Fano asserted that, if \( n(\chi) \geq 2 \), then something like one of the following two cases happens:

either there is a point \( x \in V \) such that \( \text{mult}_x |\chi| > 2n \),

or there is a curve \( C \subset V \) such that \( \text{mult}_C |\chi| > n \).
It should be admitted that Fano never asserted that only these two cases are possible. He gives the following example: a base point \( x \in V \) and a base line \( L \subset E \), where \( E \subset \tilde{V} \) is the exceptional divisor of the blowing up of \( x \), satisfying the inequality
\[
\text{mult}_x |\chi| + \text{mult}_L |\tilde{x}| > 3n.
\]
But the general level of understanding and technical weakness of his time prevented him from giving a rigorous and complete description of what happens when \( n(\chi) \geq 2 \).

Then Fano asserted that none of his conditions can hold. It is really amazing, that practically all of his arguments being absolutely invalid, this very assertion is true. (It is still more amazing that this was very often the case with Fano: wrong arguments almost always led him to deep and true conclusions.) For instance, to exclude the possibility of a curve \( C \) with \( \text{mult}_C |\chi| > n \), he employs the arguments of arithmetic genus of a general surface in \( |\chi| \). It seems that Fano tried to reproduce Noether’s arguments which used the genus of a curve in \( |\chi| \). However, Iskovskikh and Manin found out that these arguments actually lead to no conclusion [IM].

Being sure that the case \( n(\chi) \geq 2 \) is impossible, Fano formulated one of his most impressive claims: any birational correspondence between two non-singular quartics in \( \mathbb{P}^4 \) is a projective isomorphism. In particular, the group of birational automorphisms \( \text{Bir} V = \text{Aut} V \) is finite, generically trivial, and \( V \) is nonrational.

In this way Fano did a lot of work in three-dimensional birational geometry [F3]. He gave a description (however incomplete and unsubstantiated) of birational correspondences of three-dimensional cubics, complete intersections \( V_{2,3} \) in \( \mathbb{P}^5 \) and many other varieties. A lot of his results has not been completed up to this day.

However, because of the very style of Fano’s work, his numerous mistakes and, generally speaking, incompatibility of his geometry with the universally adopted standards of mathematical arguments, his ideas and computations had been abandoned for a long period of about twenty years.

In late sixties Yu.I.Manin and V.A.Iskovskikh in Moscow (after a series of papers on two-dimensional birational geometry) started their pioneer program in three-dimensional birational geometry. As a result, in 1970 they developed a method which was sufficiently strong to prove the Fano’s claim on the three-dimensional quartic [IM]. We shall refer to this method as to the \textit{method of maximal singularities}. By means of this method Iskovskikh proved later a few more Fano’s claims and corrected some of Fano’s mistakes [I]. In seventies-eighties several students of Iskovskikh — A.A.Zagorskii, V.G.Sarkisov [S1, S2], S.L.Tregub [T1, T2], S.I.Khashin [Kh] and the author [P1-P5] — had been working in this field, trying to describe birational correspondences of certain classes of algebraic varieties. Sometimes, although not very often, their work was successful: the method of maximal singularities was extended to a number of classes of varieties, of arbitrary dimension and possibly singular, including a big class of conic bundles. The well-known Sarkisov program [R1, C1] was born in the framework of this field, too. At the same time, the method
really works only for varieties of a very small degree. One must admit that at present we have no good method for studying birational geometry of multi-dimensional Fano varieties and Fano fibrations.

Nevertheless, the results obtained by means of the method of maximal singularities can not be proved at present in any other way (see [K] for an alternative approach in the spirit of $p$-characteristic tricks).

This paper is an extraction made from the lectures given by the author during his stay at the University of Warwick in September-December 1995. Since [IP] has been published, it makes no sense to reproduce all the details of excluding/untwisting procedures here. At the same time, [IP] was actually written in 1988. After that the real meaning of the “test class” construction became clearer, and some new methods of exclusion of maximal singularities appeared [P5,6]. The aim of the present paper is to give an easy introduction to the method of maximal singularities. We restrict ourselves by explanation of crucial points only. The principal and most difficult part of the method – that is, exclusion of infinitely near maximal singularities, – is presented here in the new form, simple and easy. This version of the method has never been published before.

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2 Maximal singularities of birational maps

Fix a projective $\mathbb{Q}$-factorial variety $V$ with at most terminal singularities over the field $\mathbb{C}$ of complex numbers.

2.1 Test pairs

Definition. A pair $(W, Y)$, where $W$ is a projective variety such that $\dim W = \dim V$, $\text{codim Sing} W \geq 2$ and $Y$ is a divisor on $W$ is said to be a test pair, if the following conditions hold:

a) $|Y|$ is free from fixed components,

b) there exists a number $\alpha \in \mathbb{R}_+$ such that for any $\beta > \alpha, \beta \in \mathbb{Q}$ the linear system

$$|M(Y + \beta K_W)|$$
is empty for $M \in \mathbb{Z}_+, M\beta \in \mathbb{Z}$ (the adjunction break condition).

The minimal $\alpha \in \mathbb{R}_+$ satisfying the condition b) is said to be the index (or threshold) of the pair $(W,Y)$. We shall denote it by $\alpha(W,Y)$.

Our aim is to study the maps

$$\chi : V \rightarrow W.$$ 

**Examples**

We list the types of varieties, which were among the principal objects of (more or less successful) study by means of the method of maximal singularities during the last twenty five years:

- a smooth quartic $V_4 \subset \mathbb{P}^4$;

- a complete intersection $V_{2,3} \subset \mathbb{P}^5$;

- a singular quartic $x \in V_4 \subset \mathbb{P}^4$;

- a smooth hypersurface $V_M \subset \mathbb{P}^M$;

- a double projective space $\sigma : V \rightarrow \mathbb{P}^n$ branched over a smooth hypersurface $Z_{2n} \subset \mathbb{P}^n$.

Now let us give the principal examples of test pairs, explaining briefly what do we need them for.

$$(\mathbb{P}^n, \text{hyperplane})$$ – to decide whether $V$ is rational; note that the index of this pair is equal to $1/(n + 1)$;

$$(\varphi : W \rightarrow S \text{ – a Fano fibration, } Y = \varphi^{-1}(\text{very ample divisor on } S))$$ – to decide whether there are structures of Fano fibrations on $V$; for instance, take a conic bundle or Del Pezzo fibration; the index here is obviously zero;
(\(V, (-MK_V)\)) – to describe the group Bir \(V\) and to give the birational classification within the same family of Fano varieties.

The first step
Assume that there exists a birational map
\[ \chi : V \to W. \]
Take the proper inverse image \(|\chi| \subset |D|\) of the linear system \(|Y|\) on \(V\). Denote by \(\text{Bs}|\chi|\) its base subscheme.
This system \(|\chi|\) and this subscheme \(\text{Bs}|\chi|\) are the principal objects of our study.

2.2 The language of discrete valuations
We shall remind briefly the necessary definitions and facts about discrete valuations. For more details see [P5,6].
Let \(X\) be a quasi-projective variety.
Denote by \(\mathcal{N}(X)\) the set of geometric discrete valuations
\[ \nu : \mathcal{C}(X)^* \to \mathbb{Z}, \]
which have a centre on \(X\). If \(X\) is complete, then \(\mathcal{N}(X)\) includes all the geometric discrete valuations. The centre of a discrete valuation \(\nu \in \mathcal{N}(X)\) is denoted by \(Z(X, \nu)\).

Examples.
1. Let \(D \subset X\) be a prime divisor, \(D \not\subset \text{Sing } X\). Then \(D\) determines a discrete valuation
   \[ \nu_D = \text{ord}_D. \]
2. Let \(B \subset X\) be an irreducible cycle, \(B \not\subset \text{Sing } X\). Then \(B\) determines a discrete valuation:
   \[ \nu_B(f) = \text{mult}_B(f)_0 - \text{mult}_B(f)_\infty. \]
   Note that if \(\sigma_B : X(B) \to X\) is the blowing up of \(B\), \(E(B) = \sigma_B^{-1}(B)\) is the exceptional divisor, then
   \[ \nu_B = \nu_{E(B)}, \]
where \(\mathcal{C}(X)\) and \(\mathcal{C}(X(B))\) are naturally identified.

Definition. Let \(\nu \in \mathcal{N}(X)\) be a discrete valuation. A triplet \((\tilde{X}, \varphi, H)\), where \(\varphi : \tilde{X} \to X\) is a birational morphism, \(H \not\subset \text{Sing } \tilde{X}\) is a prime divisor, is called a realization of \(\nu\), if \(\nu = \nu_H\).

Multiplicities
Let \(\nu \in \mathcal{N}(X)\), \(\mathcal{J} \subset \mathcal{O}_X\) be a sheaf of ideals.
Definition. The multiplicity of \(\mathcal{J}\) at \(\nu\) equals
\[ \nu(\mathcal{J}) = \text{mult}_H \varphi^* \mathcal{J}, \]
where \((\tilde{X}, \varphi, H)\) is a realization of \(\nu\).

Let \(|\lambda| \subset |D|\) be a linear system of Cartier divisors, \(\mathcal{L}(|\lambda|) \subset \mathcal{O}_X(D)\) be the subsheaf generated by the global sections in \(|\lambda|\). Set

\[
\mathcal{J}(|\lambda|) = \mathcal{L}(|\lambda|) \otimes \mathcal{O}_X(-D) \subset \mathcal{O}_X.
\]

Obviously, \(\mathcal{J}(|\lambda|)\) is the ideal sheaf of the base subscheme \(Bs|\lambda|\).

**Definition.** The multiplicity of \(|\lambda|\) at \(\nu\) equals

\[
\nu(|\lambda|) = \nu(\mathcal{J}(|\lambda|)).
\]

Now let \(X\) be \((\mathbb{Q}-)\)Gorenstein, \(\pi : X_1 \to X\) be a resolution. Then

\[
K_{X_1} = \pi^*K_X + \sum_i d_i E_i
\]

for some prime divisors \(E_i \subset X_1\). Take a realization \((\tilde{X}, \varphi, H)\), of \(\nu \in \mathcal{N}(X_1) = \mathcal{N}(X)\). Then we get an inclusion

\[
\varphi^*\omega_{X_1} \hookrightarrow \omega_{\tilde{X}},
\]

and the following ideal sheaf on \(\tilde{X}\):

\[
K(X_1, \varphi) = \omega_{\tilde{X}}^* \otimes \varphi^*\omega_{X_1} \hookrightarrow \mathcal{O}_{\tilde{X}}.
\]

**Definition.** The canonical multiplicity (discrepancy) of \(\nu\) is equal to \(d_i\), if \(\nu = \nu_{E_i}\), and to

\[
K(X, \nu) = \text{mult}_H K(X_1, \varphi) + \sum_i d_i \nu(E_i),
\]

otherwise.

**Example.** Let \(B \subset X, B \not\subset \text{Sing } X\) be an irreducible cycle of codimension \(\geq 2\). Then

\[
\nu_B(\mathcal{J}) = \text{mult}_B \mathcal{J},
\]

\[
\nu_B(|\lambda|) = \text{mult}_B |\lambda|,
\]

\[
K(X, \nu_B) = \text{codim } B - 1.
\]

### 2.3 Maximal singularities

Let us return to our variety \(V\) and birational map \(\chi : V \dashrightarrow W\). Denote by \(n(\chi)\) the index (threshold) of the pair \((V, D)\).

**Definition.** A discrete valuation \(\nu \in \mathcal{N}(V)\) is said to be a maximal singularity of \(\chi\), if the following inequality holds:

\[
\nu(|\chi|) > n(\chi)K(V, \nu).
\]

**Theorem 2.1** Either \(\alpha(V, D) \leq \alpha(W, Y)\), or \(\chi\) has a maximal singularity.
**Proof:** see [P5,6]. It is actually so easy that can be left to the reader as an exercise. The idea of the proof can be found in any paper concerned with these problems (for instance, [IM, I, P1, IP]). Just keep in mind that the proof should not depend upon resolution of singularities.

**Example.** Let $V$ be smooth with $\text{Pic} \cong \mathbb{Z}K_V$ and assume that the anticanonical system $|-K_V|$ is free. Then

$$|\chi| \subset |-n(\chi)K_V|,$$

and for a birational automorphism $\chi \in \text{Bir} V$ either $n(\chi) = 1$, or $\chi$ has a maximal singularity.

**Maximal cycles**

Let $V$ be non-singular.

**Definition.** An irreducible cycle $B \not\subset \text{Sing} V$ of codimension $\geq 2$ is said to be a maximal cycle, if $\nu_B$ is a maximal singularity. Explicitly:

$$\text{mult}_B |\chi| > n(\chi)(\text{codim } B - 1).$$

**Definition.** A maximal singularity $\nu \in \mathcal{N}(V)$ is said to be infinitely near, if it is not a maximal cycle.

**Remark.** The meaning of these two definitions is to separate “shallow” maximal singularities, which are not very far from the “ground” $V$, and “deep” ones, which take a lot of blow-ups to get to them. When $V$ is singular, these definitions should be modified slightly by adding some valuations sitting at singularities (see [P3,6]).

### 3 The untwisting scheme

Assume that $\alpha(v, D) > \alpha(W, Y)$. Then $\chi$ has a maximal singularity. The untwisting scheme gives an idea of simplifying $\chi$ according to its maximal singularities.

#### 3.1 Basic Conjecture

We say that $V$ satisfies the Basic conjecture, if for any $\chi : V \dashrightarrow W$ in the hypothesis of Theorem [2.1] we can replace the words “maximal singularity” by the words “maximal cycle”: if

$$\alpha(v, D) > \alpha(W, Y),$$

then $\chi$ has a maximal cycle.

#### 3.2 Excluding maximal cycles

Assume that $V$ satisfies the Basic conjecture. Then the first thing to be done is to describe all the cycles $B \subset V$ which can occur as maximal. In other words, all the cycles $B$ such that for some $D \in \text{Pic} V$

$$|D - \nu B|$$
is free from fixed components for some
\[ \nu > (\text{codim } B - 1)\alpha(V, D). \]

### 3.3 Untwisting maps

The second step of the scheme is to construct an automorphism \( \tau_B \in \text{Bir } V \) for each \( B \) singled out at the previous step. The cycle \( B \) should be maximal for \( \tau_B \).

### 3.4 Untwisting

If \( B \) is maximal for \( \chi : V \rightarrow W \), take the composition
\[ \chi \circ \tau_B : V \rightarrow W. \]
It should turn out that
\[ n(\chi \circ \tau_B) < n(\chi). \]
Iterating, we come to a sequence of cycles \( B_1, \ldots, B_k \) such that
\[ n(\chi \circ \tau_{B_1} \circ \cdots \circ \tau_{B_k}) \leq \alpha(W, Y). \]

### 3.5 Birationally rigid varieties

Informally speaking, \( V \) is birationally rigid, if the untwisting scheme works on it.

**Definition.** \( V \) is said to be **birationally rigid**, if for any test pair \((W, Y)\) and any map \( \chi : V \rightarrow W \) there exists \( \chi^* \in \text{Bir } V \) such that
\[ n(\chi \circ \chi^*) \leq \alpha(W, Y). \]
If, moreover, \( \text{Bir } V = \text{Aut } V \), then \( V \) is said to be **birationally superrigid**.

**Remark.** The untwisting scheme, when it works, gives not only the fact of birational rigidity, but also a set of natural generators of the group \( \text{Bir } V \) – that is, the maps \( \tau_B \).

**Proposition 3.1** Assume that \( V \) is birationally rigid and \( \text{Pic } V \cong \mathbb{Z} \). Then \( V \) has no structures of Fano fibrations.

**Proof.** Assume that there is a map
\[ \chi : V \rightarrow W, \]
where \( p : W \rightarrow S \) is a Fano fibration. Take \( Y \) to be \( p^{-1}(Q) \), where \( Q \subset S \) is a very ample divisor. Then
\[ n(\chi \circ \chi^*) = 0 \]
for some \( \chi^* \in \text{Bir } V \), so that
\[ |\chi \circ \chi^*| \subset |- n(\chi \circ \chi^*)K_V| = |0|. \]
Contradiction.

In particular, \( V \) has no structures of a conic bundle or a Del Pezzo fibration.
4 Excluding maximal cycles

We show by example how it is to be done. A lot of other examples can be found in the original papers [IM, I, P1-6, IP].

4.1 Double spaces

Let $\sigma : V \to \mathbb{P}^m \supset W_{2m}$ be a smooth double space of the index 1, $m \geq 3$, branched over a smooth hypersurface $W$ of the degree $2m$. Let $|\chi| \subset |-nK_V|$ be a system free from fixed components.

**Theorem 4.1** $|\chi|$ has no maximal cycles.

**Corollary 4.1** Modulo Basic conjecture $V$ is superrigid.

**Proof.** It breaks into two parts: we exclude maximal points and maximal cycles of positive dimension separately.

4.2 Points

Obviously, a point $x \in V$ cannot be maximal: take a plane $\bar{P} \ni \bar{x} = \sigma(x)$, then $P = \sigma^{-1}(\bar{P})$ is a nonsingular surface, $|\chi|_P$ has no fixed curves, so that for any $D_1, D_2 \in |\chi|_P$

\[(D_1 \cdot D_2) = 2n^2.\]

But $\text{mult}_x D_i > 2n$ : a contradiction.

4.3 Curves

**Proposition 4.1** For any curve $C \subset V$

\[\text{mult}_C |\chi| \leq n.\]

Obviously, our theorem is an immediate consequence of this fact.

**Proof of the Proposition.** Let us consider the following three cases:

1. $C = \sigma^{-1}(\bar{C}), \bar{C} \notin W$;
2. $C = \sigma(C) \subset W$;
3. $\sigma : C \to \bar{C}$ is birational, $\bar{C} \notin W$.

**The easy first case.**

Take a generic line $\bar{L}$ intersecting $\bar{C}$, $L = \sigma^{-1}(\bar{L})$ is a smooth curve. The linear series

\[|\chi|_L\]

is of the degree $2n$ and has $\geq 2$ points $\in \sigma^{-1}(\bar{L} \cap \bar{C})$ of the multiplicity $\text{mult}_C |\chi|$. 

10
The second case, not very difficult.

Take a generic point \( x \in \mathbb{P}^m \) and the cone \( Z(x) \) over \( \bar{C} \) with the vertex \( x \). Then \( Z(x) \cap W = \bar{C} \cup \bar{R}(x) \), where the residual curve \( \bar{R}(x) \) intersects \( \bar{C} \) at \( \deg \bar{R}(x) \) different points (see [P5]). Let \( \bar{R}(x) \) be the curve \( \sigma^{-1}(\bar{R}(x)) \), then \( \sigma : \bar{R}(x) \to \bar{R}(x) \) is an isomorphism, and

\[
|\chi|_{\bar{R}(x)}
\]

is a linear series of the degree \( n \deg \bar{R}(x) \) which has \( \deg \bar{R}(x) \) base points of the multiplicity \( \mult_C |\chi| \).

The non-trivial third case.

Again take a generic point \( x \in \mathbb{P}^m \) and the cone \( Z(x) \) over \( \bar{C} \) with the vertex \( x \). Let \( \varphi : X \to \mathbb{P}^m \) be the blowing up of \( x \) with the exceptional divisor \( E \), so that the projection

\[
\pi : X \to \mathbb{P}^m - 1 = E
\]

is a regular map, \( X \) being a \( \mathbb{P}^1 \)-bundle over \( E \). Let \( \alpha : Q \to \bar{C} \) be the desingularization of \( \bar{C} \),

\[
\bar{S} = Q \times_{\bar{C}} X
\]

be a \( \mathbb{P}^1 \)-bundle over \( Q \). Obviously, \( \text{Num} \bar{S} = A^1(S) = \mathbb{Z}f \oplus \mathbb{Z}e \), where \( f \) is the class of a fiber and \( e \) is the class of the exceptional section coming from the vertex of the cone. Obviously, \( f^2 = 0, (f \cdot e) = 1, e^2 = -d \), where \( d = \deg \bar{C} = \deg \pi(\bar{C}) \). Let \( h \) be the class of a hyperplane section, \( h^2 = d \), so that \( h = e + df \).

Denote by \( \bar{C} \) the inverse image of \( C \) on \( \bar{S} \). Obviously, its class \( \bar{c} \) equals \( h \).

For a generic point \( x \) the set \( \sigma^{-1}(Z(x)) \cap \text{Bs} |\chi| \) contains at most two curves: \( C \) itself and the other component of \( \sigma^{-1}(\bar{C}) \); moreover, the inverse image \( \bar{W} \) of \( W \) on \( \bar{S} \) is a non-singular curve.

Now let us take the surface \( S = \bar{S} \times_{Z(x)} V \), that is, the double cover of \( \bar{S} \) with the smooth branch divisor \( \bar{W} \). Denote the image of \( C \) on \( S \) by \( C^* \) again, the other component of \( \sigma^{-1}(\bar{C}) \) on \( S \) by \( C^* \). The inverse image of the linear system \( |\chi| \) on \( S \) has at most two fixed components \( C, C^* \) of the multiplicities \( \nu, \nu^* \), respectively. Therefore the system \( |nh - \nu c - \nu c^*| \) is free from fixed components, and we get the following inequalities:

\[
\left((nh - \nu c - \nu^* c^*) \cdot c, c^* \right) \geq 0,
\]

\[
\left((nh - \nu c - \nu^* c^*) \cdot c^*, c \right) \geq 0.
\]
It is easy to compute the multiplication table for the classes $h, c$ and $c^*$. The only necessary intersection number (the other are obvious) is
\[(c \cdot c^*)_S = \frac{1}{2}(\tilde{c} \cdot \tilde{w})_{\tilde{S}} = md.\]
Now we get the following system of linear inequalities:
\[
(n - \nu^*) + (m - 1)(\nu - \nu^*) \geq 0,
\]
\[
(n - \nu) + (m - 1)(\nu^* - \nu) \geq 0.
\]
If, for instance, $\nu \geq \nu^*$, then by the second inequality $\nu \leq n$. By symmetry we are done. Q.E.D.

4.4 The general idea of exclusion

It is very simple: to construct a sufficiently big family of curves or surfaces intersecting the cycle being excluded at as many points as possible (or containing it) and, at the same time, having as small “degree” as possible.

Then we restrict our linear system to such a curve or surface and get a contradiction.

4.5 What do we know about maximal cycles

They do not exist:

for smooth hypersurfaces of the degree $M$ in $\mathbb{P}^M, M \geq 4$ [P5];

for smooth double spaces $V_2 \to \mathbb{P}^M \supset W_{2M}, M \geq 3$: [I] for $M = 3$, [P2] for $M \geq 4$, see also [IP] (and for slightly singular as well [P6]);

for smooth double quadrics $V_4 \to Q_2 \subset \mathbb{P}^{M+1}$, branched over $Q_2 \cap W_{2M-2}, M \geq 4$ [P2], see also [IP].

For a singular quartic $V_4 \subset \mathbb{P}^4$ with a unique double singular point $x$ there can be only 25 maximal cycles: either $x$ itself, or one of 24 lines on $V$, containing $x$. Moreover, a maximal cycle is always unique [P3].

For a double quadric $\sigma : V_4 \to Q_2 \subset \mathbb{P}^4$, branched over $Q_2 \cap W_4$, there can be at most one maximal cycle, that is, a line $L \subset V, (L \cdot K_V) = -1, \sigma(L) \not\subset W_4$ [I, IP].

For a complete intersection $V = V_{2,3} = Q_2 \cap Q_3 \subset \mathbb{P}^5$ a maximal cycle $B$ is a curve: either a line $L$, or a smooth conic $Y$ such that the unique plane $P(Y) \supset Y$ is
contained in the quadric $Q_2$. Moreover, there can be at most two maximal curves, and if there are exactly two maximal curves, then they are lines $L_1$ and $L_2$ such that the unique plane $P(L_1 \cup L_2) \supset (L_1 \cup L_2)$ is contained in $Q_2$ [1,IP].

5 Untwisting maximal cycles

We give a simple example of untwisting (probably the simplest one): the untwisting procedure for the maximal singular point $x \in V_4 \subset P^4$ on a singular quartic $V$ [P3].

5.1 Construction of the untwisting map

Let $\pi : V \setminus \{x\} \rightarrow P^3$ be the projection from $x$, deg $\pi = 2$. Then the untwisting map $\tau : V \rightarrow V$ permutes the points in the fibers of $\pi$.

Let $\sigma : V_0 \rightarrow V$ be the blowing up of $x$, $E = \sigma^{-1}(x) \cong P^1 \times P^1$ be the exceptional divisor, $L_i, i = 1, \ldots, 24$, be the proper inverse images of lines on $V$, passing through $x$.

**Lemma 5.1** $\tau$ extends to an automorphism of

$$V_0 \setminus \bigcup_{1 \leq i \leq 24} L_i.$$

*Its action on Pic$V_0 = \mathbb{Z}h \oplus \mathbb{Z}e$ is given by the following relations:*

$$\tau^*h = 3h - 4e,$$

$$\tau^*e = 2h - 3e.$$

**Proof.** $\pi$ extends to a morphism $V \rightarrow P^3$ of the degree 2. It is not well-defined only on the one-dimensional fibers, which are exactly the 24 lines $L_i$.

Thus $\tau$ is an automorphism of the complement of a set of codimension 2, so that its action $\tau^*$ on Pic$V$ is well-defined.

Obviously, for any plane $P \subset P^3$ its inverse image $\pi^{-1}(P)$ represents an invariant class,

$$\tau^*(h - e) = h - e.$$

Furthermore, $\pi(E)$ is a quadric in $P^3$, $\pi(H)$ is a quartic in $P^3$, where $H \subset V$ is a hyperplane section disjoint from $E$. Thus

$$e + \tau^*e = 2(h - e),$$

$$h + \tau^*h = 4(h - e).$$

Q.E.D.
5.2 Untwisting

Let $\chi : V \to W$ be our birational map. We define the number $\nu_x(\chi) \in \mathbb{Z}_+$ in the following way: the class of the proper inverse image of the linear system $|\chi|$ on $V_0$ is

$$n(\chi)h - \nu_x(\chi)e.$$ 

The condition “the singular point $x$ is maximal for $|\chi|$” means that

$$\nu_x(\chi) > n(\chi).$$

Now consider the composition $\chi \circ \tau : V \to W$.

Lemma 5.2 (i) $n(\chi \circ \tau) = 3n(\chi) - 2\nu_x(\chi)$.

(ii) $\nu_x(\chi \circ \tau) = 4n(\chi) - 3\nu_x(\chi)$.

Proof. Since $\tau$ is an automorphism in codimension 1, we can write down

$$n(\chi \circ \tau)h - \nu_x(\chi \circ \tau)e = \tau^*(n(\chi)h - \nu_x(\chi)e).$$

Applying the formulae, obtained in the previous section, we get our Lemma.

Now if $x$ is a maximal point for $\chi$, then $\nu_x(\chi) > n(\chi)$, so that $n(\chi \circ \tau) < n(\chi)$ and $\nu_x(\chi \circ \tau) < n(\chi \circ \tau)$, and $x$ is no longer a maximal cycle.

The maximal cycle $x$ is untwisted.

6 Infinitely near maximal singularities. I

The techniques necessary to exclude infinitely near maximal singularities is developed.

6.1 Resolution

Let $X$ be any quasi-projective variety, $\nu \in \mathcal{N}(X)$ be a discrete valuation, $B = Z(X, \nu) \not\subset \text{Sing } X, \text{codim } B \geq 2$.

Proposition 6.1 Either $\nu = \nu_B$, or for the blow up

$$\sigma_B : X(B) \to X,$$

$$E(B) = \sigma_B^{-1}(B),$$

we get: $\nu \in \mathcal{N}(X(B)), Z(X(B), \nu) \subset E(B)$ is an irreducible cycle of codimension $\geq 2$ and

$$\sigma_B(Z(X(B), \nu)) = B.$$
Proof: easy.
Consider the sequence of blow ups
\[ \varphi_{i,i-1} : X_i \to X_{i-1}, \]
i \geq 1, where \( X_0 = X, \varphi_{i,i-1} \) blows up the cycle \( B_{i-1} = Z(X_{i-1}, \nu) \) of codimension \( \geq 2 \),
\[ E_i = \varphi_{i,i-1}^{-1}(B_{i-1}) \subset X_i. \]
Set also for \( i > j \)
\[ \varphi_{i,j} = \varphi_{j+1,j} \circ \ldots \circ \varphi_{i,i-1} : X_i \to X_j, \]
\[ \varphi_{i,i} = \text{id}_{X_i}. \]
For any cycle \((\ldots)\) we denote its proper inverse image on \( X_i \) by adding the upper index \( i \): \((\ldots)^i\).
Note that \( \varphi_{i,j}(B_i) = B_j \) for \( i \geq j \).
Note also that although all the \( X \)'s are possibly singular, \( B_i \not\subset \text{Sing} X_i \) for all \( i \).

Proposition 6.2 This sequence is finite: for some \( K \in \mathbb{Z}_+ \) the triplet \((X_K, \varphi_{K,0}, E_K)\) realizes \( \nu, \nu = \nu_{E_K} \).

Proof: see [P6] or prove it yourself (it is easy).

Definition. The sequence \( \varphi_{i,i-1}, i = 1, \ldots, K \), is said to be the resolution of the discrete valuation \( \nu \) (with respect to the model \( X \)).

6.2 The graph structure

Definition. For \( \mu, \nu \in \mathcal{N}(X) \) set
\[ \mu \leq \nu \]
if the resolution of \( \mu \) is a part of the resolution of \( \nu \).
In other words, for some \( L \leq K \)
\[ (X_L, \varphi_{L,0}, E_L) \]
is a realization of \( \mu \).

Definition. We define an oriented graph structure on \( \mathcal{N}(X) \), drawing an arrow from \( \nu \) to \( \mu \),
\[ \nu \xrightarrow{X} \mu, \]
if \( \mu \leq \nu \) and \( B_{K-1} \subset E_K^{K-1} \).
Denote by \( P(\nu, \mu) \) the set of all paths from \( \nu \) to \( \mu \) in \( \mathcal{N}(X) \), which is non-empty if and only if \( \nu \geq \mu \). Set
\[ p(\nu, \mu) = |P(\nu, \mu)|, \]
if \( \nu \neq \mu \), and \( p(\nu, \nu) = 1 \). Set \( \mathcal{N}(X, \nu) \) to be the subgraph of \( \mathcal{N}(X) \) with the set of vertices smaller (or equal) than \( \nu \).
6.3 Intersections, degrees and multiplicities

Let \( B \subset X, B \not\subset \text{Sing} X \) be an irreducible cycle of codimension \( \geq 2 \), \( \sigma_B : X(B) \to X \) be, as usual, its blowing up, \( E(B) = \sigma_B^{-1}(B) \) be the exceptional divisor. Let

\[
Z = \sum m_i Z_i, \\
Z_i \subset E(B)
\]

be a \( k \)-cycle, \( k \geq \dim B \). We define the degree of \( Z \) as

\[
\deg Z = \sum_i m_i \deg \left( Z_i \cap \sigma_B^{-1}(b) \right),
\]

where \( b \in B \) is a generic point, \( \sigma_B^{-1}(b) \cong \mathbb{P}^{\text{codim} B - 1} \) and the right-hand side degree is the ordinary degree in the projective space.

Note that \( \deg Z_i = 0 \) if and only if \( \sigma_B(Z_i) \) is a proper closed subset of \( B \).

Our computations will be based upon the following statement.

Let \( D \) and \( Q \) be two different prime Weyl divisors on \( X, D^B \) and \( Q^B \) be their proper inverse images on \( X(B) \).

Lemma 6.1

(i) Assume that \( \text{codim} B \geq 3 \). Then

\[
D^B \cdot Q^B = (D \cdot Q)^B + Z,
\]

where \( \cdot \) stands for the cycle of the scheme-theoretic intersection, \( \text{Supp} Z \subset E(B) \), and

\[
\text{mult}_B(D \cdot Q) = (\text{mult}_B D)(\text{mult}_B Q) + \deg Z.
\]

(ii) Assume that \( \text{codim} B = 2 \). Then

\[
D^B \cdot Q^B = Z + Z_1,
\]

where \( \text{Supp} Z \subset E(B), \text{Supp} \sigma_B(Z_1) \) does not contain \( B \), and

\[
D \cdot Q = [(\text{mult}_B D)(\text{mult}_B Q) + \deg Z] B + (\sigma_B)_* Z_1.
\]

Proof. Let \( b \in B \) be a generic point, \( S \ni b \) be a germ of a non-singular surface in general position with \( B, S^B \) its proper inverse image on \( X(B) \). We get an elementary two-dimensional problem: to compute the intersection number of two different irreducible curves at a smooth point on a surface in terms of its blowing up. This is easy. Q.E.D.

Multiplicities in terms of the resolution

We divide the resolution \( \varphi_{i,i-1} : X_i \to X_{i-1} \) into the lower part, \( i = 1, \ldots, L \leq K \), when \( \text{codim} B_{i-1} \geq 3 \), and the upper part, \( i = L + 1, \ldots, K \), when \( \text{codim} B_{i-1} = 2 \). It may occur that \( L = K \) and the upper part is empty.

Let \( |\lambda| \) be a linear system on \( X \) with no fixed components, \( |\lambda|^j \) its proper inverse image on \( X_j \). Set

\[
\nu_j = \text{mult}_{B_{j-1}} |\lambda|^j^{-1}.
\]
Obviously,
\[ \nu_{E_j}(|\lambda|) = \sum_{i=1}^{j} p(\nu_{E_j}, \nu_{E_i})\nu_i \]
and
\[ K(X, \nu_{E_j}) = \sum_{i=1}^{j} p(\nu_{E_j}, \nu_{E_i})(\text{codim } B_{i-1} - 1) \].

For simplicity of notations we write
\[ i \rightarrow j \]
instead of
\[ \nu_{E_i} \xrightarrow{X} \nu_{E_j} \]

Now everything is ready for the principal step of the theory.

7 Infinitely near maximal singularities. II.

The principal computation.

We prove the crucial inequalities which enable us to exclude infinitely near maximal singularities for the cases of low degree.

7.1 Counting multiplicities

Let \( D_1, D_2 \in |\lambda| \) be two different generic divisors. We define a sequence of codimension 2 cycles on \( X_i \'s \) setting
\[
D_1 \cdot D_2 = Z_0, \\
D_1^1 \cdot D_2^2 = Z_0^1 + Z_1, \\
\ldots, \\
D_1^i \cdot D_2^i = (D_1^{i-1} \cdot D_2^{i-1})^i + Z_i, \\
\ldots,
\]
where \( Z_i \subset E_i \). Thus for any \( i \leq L \) we get
\[ D_1^i \cdot D_2^i = Z_0^i + Z_1^i + \ldots + Z_{i-1}^i + Z_i. \]

For any \( j > i, j \leq L \) set
\[ m_{i,j} = \text{mult}_{B_{j-1}}(Z_{i}^{j-1}) \]
(the multiplicity of an irreducible cycle along a smaller cycle is understood in the usual sense; for an arbitrary cycle we extend the multiplicity by linearity).

The crucial point

Lemma 7.1 If \( m_{i,j} > 0 \), then \( i \rightarrow j \).

Proof. If \( m_{i,j} > 0 \), then some component of \( Z_{i}^{j-1} \) contains \( B_{j-1} \). But \( Z_{i}^{j-1} \subset E_{i}^{j-1} \). Q.E.D.
Degree and multiplicity
Set \( d_i = \deg Z_i \).

**Lemma 7.2** For any \( i \geq 1, j \leq L \) we have

\[
m_{i,j} \leq d_i.
\]

**Proof.** The cycles \( B_a \) are non-singular at their generic points. But since \( \varphi_{a,b} : B_a \to B_b \) is surjective, we can count multiplicities at generic points. Now the multiplicities are non-increasing with respect to blowing up of a non-singular cycle, so we are reduced to the obvious case of a hypersurface in a projective space. Q.E.D.

The very computation
We get the following system of equalities:

\[
\begin{align*}
\nu_1^2 + d_1 &= m_{0,1}, \\
\nu_2^2 + d_2 &= m_{0,2} + m_{1,2}, \\
\vdots \\
\nu_i^2 + d_i &= m_{0,i} + \ldots + m_{i,i-1}, \\
\vdots \\
\nu_L^2 + d_L &= m_{0,L} + \ldots + m_{L-1,L}.
\end{align*}
\]

Now

\[
d_L \geq \sum_{i=L+1}^{K} \nu_i^2 \deg(\varphi_{i-1,L}) \cdot B_{i-1} \geq \sum_{i=L+1}^{K} \nu_i^2.
\]

**Definition.** A function \( a : \{1, \ldots, L\} \to \mathbb{R}_+ \) is said to be compatible with the graph structure, if

\[
a(i) \geq \sum_{j \to i} a(j)
\]

for any \( i = 1, \ldots, L \).

**Examples:** \( a(i) = p(L, i) \), \( a(i) = p(K, i) \).

**Theorem 7.1** Let \( a(\cdot) \) be any compatible function. Then

\[
\sum_{i=1}^{L} a(i)m_{0,i} \geq \sum_{i=1}^{L} a(i)\nu_i^2 + a(L) \sum_{i=L+1}^{K} \nu_i^2.
\]

**Proof.** Multiply the \( i \)-th equality by \( a(i) \) and put them all together: in the right-hand part for any \( i \geq 1 \) we get the expression

\[
\sum_{j \geq i+1} a(j)m_{i,j} = \sum_{j \geq i+1} a(j)m_{i,j} \leq d_i \sum_{j \to i} a(j) \leq a(i)d_i.
\]
In the left-hand part for any $i \geq 1$ we get

$$a(i)d_i.$$ 

So we can throw away all the $m_{i,*}, i \geq 1$, from the right-hand part, and all the $d_i, i \geq 1$, from the left-hand part, replacing $=$ by $\leq$. Q.E.D.

**Corollary 7.1** Set $m = m_{0,1} = \text{mult}_{B_0}(D_1 \cdot D_2), D_i \in |\chi|$. Then

$$m \left( \sum_{i=1}^{L} a(i) \right) \geq \sum_{i=1}^{L} a(i)\nu_i^2 + a(L) \sum_{i=L+1}^{K} \nu_i^2.$$ 

### 7.2 Applications

**Corollary 7.2** Set $r_i = p(K,i)$. Then

$$m \left( \sum_{i=1}^{L} r_i \right) \geq \sum_{i=1}^{K} r_i\nu_i^2.$$ 

**Proof:** for $i \geq L + 1$ obviously $r_i \leq r_L$. Q.E.D.

**Corollary 7.3** (Iskovskikh and Manin [IM]). Let $\dim V = 3, \nu \in \mathcal{N}(V)$ be a maximal singularity such that $Z(V, \nu) = x$ - a smooth point, $m = \text{mult}_x C$, where the curve $C = (D_1 \cdot D_2)$ is the intersection of two generic divisors from $|\chi|$, $n = n(\chi)$ and assume $|-K_V|$ to be free. Then

$$m \left( \sum_{i=1}^{L} r_i \right) \left( \sum_{i=1}^{K} r_i \right) > n^2 \left( 2 \sum_{i=1}^{L} r_i + \sum_{i=L+1}^{K} r_i \right)^2.$$ 

In particular, $m > 4n^2$.

**Proof.** It follows immediately from the fact that $\nu$ is a maximal singularity and the previous Corollary. Let us prove the last statement. Denoting

$$\sum_{i=1}^{L} r_i, \sum_{i=L+1}^{K} r_i$$

by $\Sigma_0, \Sigma_1$, respectively, we get

$$4\Sigma_0(\Sigma_0 + \Sigma_1) \leq (\Sigma_1 + 2\Sigma_0)^2,$$

and that is exactly what we want.

**Corollary 7.4** (Iskovskikh and Manin [IM]). The Basic conjecture for a smooth quartic $V \subset \mathbb{P}^4$ is true.

**Proof.** Obviously, $m \leq 4n^2$. It is a contradiction with the previous corollary.

Since it is easy to show that on $V_4 |\chi|$ has no maximal cycles ([IM] or [P5]), we get

**Corollary 7.5** (Iskovskikh and Manin [IM]). A smooth three-dimensional quartic $V \subset \mathbb{P}^4$ is a birationally superrigid variety.
8 The Sarkisov theorem on conic bundles

We give an extremely short version of the proof of the Sarkisov theorem [S1,2]. The idea of the proof is essentially the same as in these well-known Sarkisov’s papers. At the same time our general viewpoint of working in codimension 1 makes the arguments brief and very clear.

8.1 Formulation of the theorem

Let $S$ be a smooth projective variety, $\dim S \geq 2$, $\mathcal{E}$ be a locally free sheaf on $S$, $\text{rk} \mathcal{E} = 3$. Let

$$X \subset \mathbb{P}(\mathcal{E}) \xrightarrow{\pi} S$$

be a standard conic bundle, that is, a smooth hypersurface with

$$\text{Pic} X = \mathbb{Z}K_X \oplus \pi^* \text{Pic} S.$$

Denote by $C \subset S$ the discriminant divisor. Note that $C$ has at most normal crossings, the fiber over any point outside $C$ is a smooth conic, the fiber over generic point of any component of $C$ is a pair of distinct lines, and the inverse image of any component of $C$ on $X$ is irreducible.

Let $\tau : V \to F$ be another conic bundle of the same dimension (not necessarily smooth).

**Theorem 8.1** If $|4K_S + C| \neq \emptyset$, then any birational map

$$\chi : X \dasharrow V$$

transfers fibers into fibers, that is, there exists a map $\bar{\chi} : S \dasharrow F$ such that

$$\tau \circ \chi = \bar{\chi} \circ \pi.$$

8.2 Start of the proof

Denote by

$$\mathcal{F} = \{C_u | u \in U\}$$

the proper inverse image of the family of conics $\tau^{-1}(q), q \in F$, and by

$$\bar{\mathcal{F}} = \{\bar{C}_u = \pi(C_u) | u \in U\}$$

its image on the “ground” $S$. When some birational operations are performed with respect to these families, the parametrizing set $U$ is to be replaced by some dense open subset; but for brevity we shall just keep it in mind and use the same symbol $U$, meaning it to be as small as necessary.

Let $\sigma : S^* \to S$ be a birational morphism such that:

1. $S^*$ is projective and non-singular in codimension 1;
2. the proper inverse image

$$\mathcal{F}^* = \{L_u | u \in U\}$$
of the family $\mathcal{F}$ on $S^*$ is free in the following sense: for any cycle $Z \subset S^*$ of codimension $\geq 2$ a general curve $L_u$ does not meet $Z$. Existence of such a morphism $\sigma$ can be proved quite elementary without use of the Hironaka theory (see [P6]). Set

$$P^* = P(\sigma^* E),$$

$$X^* = X \times_S S^* \subset P^*,$$

$X^*$ being a singular conic bundle over $S^*$. For simplicity of notations the natural morphisms of $X^*$ to $S^*$, $X$ will be denoted by $\pi, \sigma$ respectively, and the map $\chi \circ \sigma$ just by $\chi$.

**Proposition 8.1** There exist: a closed subset $Y \subset S^*$ of codimension $\geq 2$, a non-singular conic bundle $\pi : W \to S^* \setminus Y$

with the non-singular discriminant divisor

$$C^* \subset S^* \setminus Y$$

and

$$\text{Pic} W \cong \mathbb{Z}K_W \oplus \pi^* \text{Pic} S^*,$$

and a fiber-wise map

$$\lambda : X^* \to W,$$

$$\pi \circ \lambda = \pi.$$ Moreover,

$$\{4K_{S^*} + C^*\} \neq \emptyset.$$

**Proof.** We obtain $W$ by means of fiber-wise restructuring of $X^*$ over the prime divisors $T \subset S^*$ such that $\text{codim} \, \sigma(T) \geq 2$. If $t \in C(S^*)$ is a local equation of $T$ on $S^*$, then at the generic point of $T$ the variety $X^*$ is given by one of the two following types of equations:

- **case 1:** $x^2 + t^k ay^2 + t^lbz^2, k \leq l,$
- **case 2:** $x^2 + y^2 + t^k az^2,$

where $(x : y : z)$ are homogeneous coordinates on $\mathbb{P}^2$, and $a, b$ are regular and non-vanishing at a generic point of $T$. In the case 1 for $k \geq 2$ the variety $X^*$ has a whole divisor of singular points, that is, $\pi^{-1}(T)$. Blow it up $[k/2]$ times. Now in both cases the singularity of our variety over $T$ is either of the type $A_n$ or of the type $D_n$. Blowing up the singularities, covering $T$, and contracting afterwards $(-1)$-components in fibers, we get the Proposition. The last statement is easily obtained by computing the discrepancy of $\nu_T$ on $S$.

Denote $\chi \circ \lambda^{-1}$ by $\chi : W \to V$ again.

Let $Z \subset W \times V$ be the (closed) graph of $\chi, \varphi$ and $\psi$ be the projections (birational morphisms) onto $W$ and $V$, respectively. Obviously, $Z$ is projective over $W$. 21
**Proposition 8.2** For any closed set \( Y^* \supset Y \) of codimension \( \geq 2 \) there exists an open set \( U \subset F \) such that

\[
\varphi^{-1} \pi^{-1}(U) \subset \varphi^{-1} \pi^{-1}(S^* \setminus Y^*)
\]

and \( \psi^{-1} \tau^{-1}(U) \) is projective over \( \tau^{-1}(U) \subset V \).

**Proof:** it follows immediately from the fact that the family of curves \( F^* \) is free on \( S^* \).

### 8.3 The test surface construction

Now let \( |H^*| \) be any linear system, which is the inverse image of a very ample linear system on \( F \), and \( |\chi| \) be its proper inverse image on \( W \). Write down

\[
|\chi| \subset |-\mu K_W + \pi^* A|
\]

for some \( \mu \in \mathbb{Z}_+ \) and \( A \in \text{Pic } S^* \). If \( \mu = 0 \), we get the statement of the Theorem. So assume that \( \mu \geq 1 \).

Let us show that this is impossible.

In the notations of the last Proposition, set \( Q = \psi^{-1} \tau^{-1}(U) \). Obviously, we may assume that

\[
\psi : Q \rightarrow \tau^{-1}(U) \subset V
\]

is an isomorphism. For a generic conic \( R_u, u \in U \),

\[
(H^* \cdot R_u) = 0,
\]

\[
(K_V \cdot R_u) = -2.
\]

So the same is true on \( Q \). Thus for some prime divisors \( T_1, \ldots, T_m \subset Q \) we get

\[
\left( -\mu \varphi^* K_W + \varphi^* \pi^* A - \sum_{i=1}^m a_i T_i \right) \cdot \psi^{-1}(R_u) = 0,
\]

\[
\left( \varphi^* K_W + \sum_{i=1}^m d_i T_i \right) \cdot \psi^{-1}(R_u) = -2.
\]

Making the set \( U \) smaller if necessary, we may assume that

\[
(T_i \cdot \psi^{-1}(R_u)) \geq 1
\]

for all \( i \). Thus the cycles

\[
\pi \circ \varphi(T_i)
\]

have codimension 1 in \( S^* \) and \( T_i \)’s can be realized by the successions of blow-ups

\[
\varphi_{j,j-1}^{(i)} : X_j \to X_{j-1}^{(i)} \bigcup E_j \to B_{j-1}^{(i)}.
\]
where \( B_0^{(i)} = \varphi(T_i), B_{j+1}^{(i)} \) covers \( B_j^{(i)} \). \( E_{K(i)}^{(i)} = T_i \). Since \(|\chi|\) has no fixed components, \( \deg(B_{j+1}^{(i)} \rightarrow B_j^{(i)}) = 1 \) and the corresponding graph of discrete valuations is a chain. Taking the union of these blow-ups (that is, throwing away some more cycles of codimension 2 from \( S^* \)), we get on \( Q \) that

\[
|\tilde{\chi}| \subset -\mu \varphi^* K_W + \varphi^* \pi^* A - \sum_{i,j} \nu_{i,j} E_j^{(i)}
\]

whereas the canonical divisor on \( Q \) is equal to

\[
\varphi^* K_W + \sum_{i,j} E_j^{(i)}.
\]

Consequently, as far as \( \mu \geq 1 \), the divisor

\[
\varphi^* \pi^* A - \sum_{i,j} (\nu_{i,j} - \mu) E_j^{(i)}
\]

intersects \( \psi^{-1}(R_u) \) negatively. Of course, we may assume that

\[
\nu_{i,K(i)} \geq \mu + 1
\]

for all \( i = 1, \ldots, m \).

Now consider the surface \( \Lambda_u = \pi^{-1}(\pi \circ \varphi(\psi^{-1}(R_u))) \) (the test surface – see [P5,6]) and its proper inverse image \( \Lambda_u^* \) on \( Q \). These surfaces are projective and, since \( F^* \) is free, we get

\[
(D^2 \cdot \Lambda^*) \geq 0,
\]

where \( D \) is the class of \( \psi^{-1}(|H^*|) \). On the other hand, setting \( L = \psi^{-1}(R_u), \bar{L} = \pi(L) \), we can write down \( (D^2 \cdot \Lambda^*) \) as

\[
4\mu(A \cdot \bar{L}) - \mu^2 \left( (4K^*_S + C^*) \cdot \bar{L} \right) - \sum_{i,j} \nu_{i,j}^2 (E_j^{(i)} \cdot L)
\]

(since for a generic \( u \in U \) the curve \( \psi^{-1}(R_u) \) intersects all the \( T \)'s transversally). At the same time, according to the remark above,

\[
(A \cdot \bar{L}) < \sum_{i,j} (\nu_{i,j} - \mu)(E_j^{(i)} \cdot L),
\]

so that

\[
4\mu(A \cdot \bar{L}) < \sum_{i,j} 4\mu(\nu_{i,j} - \mu)(E_j^{(i)} \cdot L) \leq \sum_{i,j} \nu_{i,j}^2 (E_j^{(i)} \cdot L).
\]

Since the intersection

\[
(4K^*_S + C^*) \cdot \bar{L}
\]

23
is obviously nonnegative, we get a contradiction:

$$(D^2 \cdot A^*) < 0.$$ 

Q.E.D.

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