Existence results in the theory of fractional hybrid differential equations of fractional orders

Al-Issa, Sh. M

December 10, 2019

Abstract

In this paper we study existence results for initial value problems for hybrid fractional integro-differential equations. Our investigation is based on the Dhage hybrid fixed point theorem. Some fundamental fractional differential inequalities are also established which are utilized to prove the existence of extremal solutions. Necessary tools are considered and the comparison principle is proved which will be useful for further study of qualitative behavior of solutions.

Keywords: hybrid differential equations, Fractional differential inequalities, Existence theorem, Comparison principle

Mathematics Subject Classification: : 47H30; 46E10; 34A08; 46E15.

1 Introduction

The differential equations involving Riemann-Liouville differential operators of fractional order $0 < \alpha < 1$ are very important in modeling several physical phenomena and therefore seem to deserve an independent study of their theory parallel to the well-known theory of ordinary differential equations. It is worth noting that the perturbation techniques are useful in the nonlinear analysis for studying the dynamical systems described by nonlinear differential and integral equations, the perturbed differential equations are categorized into various types. An important type of these perturbations is called a hybrid differential equation (i.e. quadratic perturbation of a nonlinear differential equation), see for more details [2] and the references therein. Recently, this issue has received much attention [8][6]. We mention that the hybrid fixed point theory can be used to develop the existence theory for the hybrid equations. For more details we refer the reader to [7][8][9][10]. Dhage and Lakshmikantham [8] scrutinized. In this paper we study existence results for initial value problems for fractional hybrid differential equations (FHDEs) involving Riemann-Liouville differential
operators

\[
\begin{aligned}
D^\alpha \left(\frac{x(t) - \sum_{i=1}^{m} \frac{I^\beta g_i(t, x(t))}{g(t, x(t))}}{g(t, x(t))}\right) &= f_1(t, I^\beta f_2(t, x(t))) \quad t \in J = [0, T],
\end{aligned}
\]

where \(D^\alpha\) denotes the Riemann-Liouville fractional derivative of order \(\alpha\), \(0 < \alpha < 1\), \(I^\beta\) is the Riemann-Liouville fractional integral of order \(\beta > 0\), \(g(t, x(t)) \in C(J \times R, R \setminus \{0\})\), \(f_i(t, x(t)) \in C(J \times R, R)\), \(i = 1, 2\), and \(h_i(t, x(t)) \in C(J \times R, R)\) with \(h_i(0, 0) = 0 (i = 1, 2, \ldots, m)\). by using a hybrid fixed point theorem for three operators in a Banach algebra due to Dhage \[21\], and under mixed Lipschitz and Carathéodory conditions. Some fundamental fractional differential inequalities are also established which are utilized to prove the existence of extremal solutions. Necessary tools are considered and the comparison principle is proved which will be useful for further study of qualitative behavior of solutions.

2 Preliminaries

In this section, we introduce some basic definitions and preliminary facts which we need in the sequel \[1\].

Denote by \(L^1(I)\) be the class of Lebesgue integrable functions on the interval \(I = [0, T]\), and let \(\Gamma(.)\) denotes the gamma function.

**Definition 2.1.** \[22\] The Riemann-Liouville fractional derivative of order \(\alpha > 0\) of a continuous function \(f : (0, +\infty) \to R\) is given by

\[
D^\alpha g(t) = \frac{1}{\Gamma(n - \alpha)} \left(\frac{d}{dt}\right)^{(n)} \int_0^t \frac{f(s)}{(1 - s)^{\alpha-n+1}} \, ds, \quad t \in [a, b].
\]

where \(n = [\alpha] + 1\), \([\alpha]\) denotes the integer part of number \(\alpha\), provided that the right side is pointwise defined on \((0, +\infty)\).

**Definition 2.2.** \[22\] The Riemann-Liouville of a fractional integral of the function \(f \in L^1(I)\) of order \(\alpha \in R^+\) is defined by

\[
I^\alpha f(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s) \, ds.
\]

and when \(a = 0\), we have \(I^\alpha f(t) = I^\alpha_0 f(t)\).

**Definition 2.3.** An algebra \(X\) is a vector space endowed with an internal composition law noted by

\[
(\cdot) : X \times X \to X, \ (x, y) \to x.y,
\]

which is associative and bilinear. A normed algebra is an algebra endowed with a norm satisfying the following property:

For all \(x, y \in X\) we have

\[
\|x.y\| \leq \|x\| \cdot \|y\|.
\]

A complete normed algebra is called a Banach algebra.
Definition 2.4. Let $X$ be a normed vector space. A mapping $T : X \to X$ is called $D$–Lipschitzian, if there exists a continuous and nondecreasing function $\phi : R^+ \to R^+$, such that

$$\|Tx - Ty\| \leq \phi_D(\|x - y\|)$$

for all $x, y \in X$ where $\phi(0) = 0$.

Sometimes, we call the function $\phi_D$ to be a $D$–function of the mapping $T$ on $X$. Obviously, every Lipschitzian mapping is $D$–Lipschitzian. Further, if $\phi(r) < r$, then $T$ is called nonlinear contraction on $X$. An important fixed point theorem that has been commonly used in the theory of nonlinear integral equations is the generalization of Banach contraction mapping principle proved in [19].

Recently B.C. Dhage in [20] proved a fixed point theorem involving three operators in a Banach algebra by blending the Banach fixed point theorem with that Schauder fixed point principle.

In [21], B.C. Dhage gave a proof of the fixed point theorem in [20] in the case of Lipschitz maps.

Now we state a useful lemma which are helpful in transforming the fractional differential equation into an equivalent Riemann-Louville integral equation.

Lemma 2.1. ([22]) Let us consider $0 < \alpha < 1$ and $w \in L^1(0, 1)$.

(H1) $D^\alpha I^\alpha w(t) = w(t)$ holds.

(H2)

$$D^\alpha I^\alpha w(t) = w(t) - \frac{I^{1-\alpha}w(t)|_{t=0}}{\Gamma(\alpha)} t^{\alpha-1}$$

hold almost everywhere on $J$.

Let $X = C(J, R)$ be the vector of all real-valued continuous functions on $J = [0, T]$. We equip the space $X$ with the norm $\|x\| = \sup_{t \in J}|x(t)|$. Clearly, $C(J, R)$ is a complete normed algebra with respect to this supremum norm.

By a solution of the FHDE (1) we mean a function $x \in C(J, R)$ such that

(i) the function $t \to \frac{x(t) - \sum_{i=1}^m \int_{g_i(t, x(t))}^{t(t) - \sum_{i=1}^m \frac{g(t, x(t))}{y_i(t, x(t))}}{\int_{g_i(t, x(t))}^{t(t) - \sum_{i=1}^m \frac{g(t, x(t))}{y_i(t, x(t))}}}$ is continuous for each $x \in C(J, R)$ and

(ii) $x$ satisfies the equations in (1).

The theory of strict and nonstrict differential inequalities related to ODEs and hybrid differential equations is available in the literature (see [15, 16, 17, 18]).

It is known that differential inequalities are useful for proving the existence of extremal solutions of ODEs and hybrid differential equations defined on $J$. 

\[\text{Definition 2.4. Let } X \text{ be a normed vector space. A mapping } T : X \to X \text{ is}\]

\[\text{called } D\text{–Lipschitzian, if there exists a continuous and nondecreasing function }\]

\[\phi : R^+ \to R^+, \text{ such that}\]

\[\|Tx - Ty\| \leq \phi_D(\|x - y\|)\]

\[\text{for all } x, y \in X \text{ where } \phi(0) = 0.\]

\[\text{Sometimes, we call the function } \phi_D \text{ to be a } D\text{–function of the mapping } T \text{ on } X.\]

\[\text{Obviously, every Lipschitzian mapping is } D\text{–Lipschitzian. Further, if }\]

\[\phi(r) < r, \text{ then } T \text{ is called nonlinear contraction on } X.\]

\[\text{An important fixed point theorem that has been commonly used in the theory of nonlinear integral equations is the generalization of Banach contraction mapping principle proved in [19].}\]

\[\text{Recently B.C. Dhage in [20] proved a fixed point theorem involving three operators in a Banach algebra by blending the Banach fixed point theorem with that Schauder fixed point principle.}\]

\[\text{In [21], B.C. Dhage gave a proof of the fixed point theorem in [20] in the case of Lipschitz maps.}\]

\[\text{Now we state a useful lemma which are helpful in transforming the fractional differential equation into an equivalent Riemann-Louville integral equation.}\]

\[\text{Lemma 2.1. ([22]) Let us consider } 0 < \alpha < 1 \text{ and } w \in L^1(0, 1).\]

\[\text{(H1) } D^\alpha I^\alpha w(t) = w(t) \text{ holds.}\]

\[\text{(H2)}\]

\[D^\alpha I^\alpha w(t) = w(t) - \frac{I^{1-\alpha}w(t)|_{t=0}}{\Gamma(\alpha)} t^{\alpha-1}\]

\[\text{hold almost everywhere on } J.\]

\[\text{Let } X = C(J, R) \text{ be the vector of all real-valued continuous functions on } J = [0, T]. \text{ We equip the space } X \text{ with the norm } \|x\| = \sup_{t \in J}|x(t)|. \text{ Clearly, } C(J, R) \text{ is a complete normed algebra with respect to this supremum norm.}\]

\[\text{By a solution of the FHDE (1) we mean a function } x \in C(J, R) \text{ such that}\]

\[\text{(i) the function } t \to \frac{x(t) - \sum_{i=1}^m \int_{g_i(t, x(t))}^{t(t) - \sum_{i=1}^m \frac{g(t, x(t))}{y_i(t, x(t))}}{\int_{g_i(t, x(t))}^{t(t) - \sum_{i=1}^m \frac{g(t, x(t))}{y_i(t, x(t))}}}}\]

\[\text{is continuous for each } x \in C(J, R) \text{ and}\]

\[\text{(ii) } x \text{ satisfies the equations in (1).}\]

\[\text{The theory of strict and nonstrict differential inequalities related to ODEs and}\]

\[\text{hybrid differential equations is available in the literature (see [15, 16, 17, 18]).}\]

\[\text{It is known that differential inequalities are useful for proving the existence of}\]

\[\text{extremal solutions of ODEs and hybrid differential equations defined on } J.}\]
3 Fractional hybrid differential equation

In this section we consider the initial value problem \( I \). The following hybrid fixed point theorem for three operators in a Banach algebra \( B \), due to Dhage \cite{27}, will be used to prove the existence result for the initial value problem \( I \).

Lemma 3.1. Let \( S \) be a nonempty, closed convex and bounded subset of a Banach algebra \( X \) and let \( A, C : X \rightarrow X \) and \( B : S \rightarrow X \) be three operators such that:

(a) \( A \) and \( C \) are Lipschitzian with Lipschitz constants \( \delta \) and \( \rho \), respectively,

(b) \( B \) is compact and continuous, and,

(c) \( x = AxBy + Cx \Rightarrow x \in S \), for all \( y \in S \).

(d) \( \delta M + \rho < r \), for \( r > 0 \) where \( M = \| B(S) \| \).

Then the operator \( x = AxBx + Cx \) has a fixed point in \( S \).

Consider the following assumptions:

\( (A_0) \) The function \( x \rightarrow \frac{x(t) - \sum_{i=1}^{m} \frac{\int_{a}^{b} h_i(t,x(t))}{g(t,x(t))}, \; i = 1, 2, ..., m, \) is increasing in \( R \) for all \( t \in J \).

\( (A_1) \) The functions \( g : J \times R \rightarrow R \setminus \{0\} \), and \( h_i : J \times R \rightarrow R \), \( h(0,0) = 0 \), \( i = 1, 2, ..., m \), are continuous and there exist two constants \( k_i, L \), satisfying

\[
|h_i(x,y) - h_i(x,y)| \leq k_i|x - y|, \quad i = 1, 2, ..., m.
\]

\[
|g(t,x) - g(t,y)| \leq L|x - y|.
\]

for all \( t \in J \) and \( x, y \in R \).

\( (A_2) \) \( f_i : [0,T] \times R \rightarrow R \), \( i = 1, 2 \) satisfy Caratheodory condition i.e \( f_i \) are measurable in \( t \) for any \( x \in R \) and continuous in \( x \) for almost all \( t \in [0,T] \).

There exists four functions \( t \rightarrow a_i(t) \), \( t \rightarrow b_i(t) \) such that

\[
|f_1(t,x)| \leq a_1(t) + b_1(t)|x|, \quad \forall \; (t,x) \in J \times R,
\]

\[
|f_2(t,x)| \leq m(t), \quad \forall \; (t,x) \in J \times R,
\]

where \( a_1(\cdot), \; m(\cdot) \in L^1 \) and \( b_1(\cdot) \) are measurable and bounded.

And \( I^\gamma_2 \; a_1(\cdot) \leq M_1 \) and \( I^\gamma_2 \; m(\cdot) \leq M_2, \quad \forall \; \gamma \leq \alpha, \quad c \geq 0 \).

\( (A_3) \) There exists a number \( r > 0 \) such that

\[
r \geq \frac{\sum_{i=1}^{m} \frac{k_1 T_1}{\Gamma(\gamma_i + 1)} + \frac{G M_1 T^\alpha - \gamma}{\Gamma(\alpha - \gamma + 1)} + \frac{G b_1 M_2 T^{\alpha + \beta - \gamma}}{\Gamma(\alpha + \beta - \gamma + 1)}}{1 - \left[ \sum_{i=1}^{m} \frac{k_1 T_1}{\Gamma(\gamma_i + 1)} + \frac{L M_1 T^\alpha - \gamma}{\Gamma(\alpha - \gamma + 1)} + \frac{L b_1 M_2 T^{\alpha + \beta - \gamma}}{\Gamma(\alpha + \beta - \gamma + 1)} \right]}
\] (2)

where \( G = \sup_{t \in J} |g(t,0)| \), and

\[
\sum_{i=1}^{m} \frac{k_1 T_1}{\Gamma(\gamma_i + 1)} + \frac{L M_1 T^\alpha - \gamma}{\Gamma(\alpha - \gamma + 1)} + \frac{L b_1 M_2 T^{\alpha + \beta - \gamma}}{\Gamma(\alpha + \beta - \gamma + 1)} < 1.
\] (3)
to prove our main existence result for continuous solutions of the differential equations of fractional order \(1\). The following useful lemma is immediate and follows from the theory of fractional calculus.

**Lemma 3.2.** Assume that hypothesis \((A_0) - (A_3)\) holds, \(\alpha, \beta, \text{ and } \gamma_i \in (0, 1)\) \(i = 1, 2, ..., m\). If a function \(x \in C(J, R)\) is a solution of the FHDE \((1)\), then it satisfies the following quadratic fractional integral equation

\[
x = \sum_{i=1}^{m} \Gamma^\alpha h_i(t, x(t)) + g(t, x(t))I^\alpha f_1(t, I^\beta f_2(t, x(t)))
\]

(4)

**Proof.** Applying Riemann-Liouville fractional integral of order \(\alpha\) on both sides of \((1)\), we obtain

\[
I^\alpha D^\alpha \left( \frac{x(t) - \sum_{i=1}^{m} \Gamma^\alpha h_i(t, x(t))}{g(t, x(t))} \right) = I^\alpha f_1(t, I^\beta f_2(t, x(t)))
\]

so, from Lemma \(2.1\) we conclude that

\[
x(t) - \sum_{i=1}^{m} \Gamma^\alpha h_i(t, x(t)) = I^{1-\alpha} \left[ \frac{x(t) - \sum_{i=1}^{m} \Gamma^\alpha h_i(t, x(t))}{g(t, x(t))} \right]_{t=0}^{t=1} = I^\alpha f_1(t, I^\beta f_2(t, x(t))), \quad t \in J.
\]

Since \(\frac{x(t) - \sum_{i=1}^{m} \Gamma^\alpha h_i(t, x(t))}{g(t, x(t))}\) \(t\) \(=\) \(0\), we have

\[
x(t) = \sum_{i=1}^{m} \Gamma^\alpha h_i(t, x(t)) = I^\alpha f_1(t, I^\beta f_2(t, x(\phi(t)))) = \sum_{i=1}^{m} \Gamma^\alpha h_i(t, x(t)) \quad t \in J.
\]

Thus, \((4)\) holds.

Conversely, assume that \(x\) satisfies HIE \((4)\). Then dividing by \(g(t, x(t))\) and applying \(D^\alpha\) on both sides of \((4)\), so \((1)\) is satisfied. Again, substituting \(t = 0\) in \((1)\) yields

\[
\frac{x(0) - \sum_{i=1}^{m} \Gamma^\alpha h_i(0, x(0))}{g(0, x(0))} = 0 = \frac{0}{g(0, x(0))}, \quad i = 1, 2, ..., m.
\]

Since the map \(x \rightarrow \frac{x - \sum_{i=1}^{m} \Gamma^\alpha h_i(t, x)}{g(t, x)}\) is increasing in \(R\) for all \(t \in J\), the map \(x \rightarrow \frac{x - \sum_{i=1}^{m} \Gamma^\alpha h_i(0, x)}{g(0, x)}\) is injective in \(R\), hence \(x(0) = 0\). The proof is completed. \(\Box\)

At this stage, our target is to prove the following existence theorem.
Theorem 3.3. Assume that the hypotheses \((A_0) - (A_4)\) hold. Then the FHDE \((1)\) has at least one solution defined on \(J\).

**Proof.** By Lemma 3.2, problem \((1)\) is equivalent to the quadratic fractional integral equation \((4)\). Set \(X := C(J, \mathbb{R})\) and define a subset \(S\) of \(X\) as

\[
S := \{x \in X, \|x\| \leq r\}.
\]

where \(r\) satisfies inequality \((2)\). Clearly \(S\) is closed, convex, and bounded subset of the Banach space \(X\). Now we define three operators; Consider the operators \(A : X \to X\), \(B : S \to X\) and \(C : X \to X\) defined by:

\[
(Ax)(t) = g(t, x(t)), \quad t \in J
\]

\[
(Bx)(t) = I^\alpha f_1(t, I^\beta f_2(t, x(t))), \quad t \in J
\]

\[
(Cx)(t) = \sum_{i=1}^{m} I^{\gamma_i} h_i(t, x(t)) = \sum_{i=1}^{m} \int_0^t \frac{(t-s)^{\gamma_i-1}}{\Gamma(\gamma_i)} h_i(s, x(s)) \, ds, \quad t \in J, (i = 1, 2, ..., m)
\]

Then the integral equation \((4)\) is transformed into the operator equation as:

\[
x(t) = Ax(t) \cdot Bx(t) + Cx(t), \quad t \in J.
\]

We shall show that \(A\), \(B\) and \(C\) satisfy all the conditions of Lemma 3.1 This will be achieved in the following series of steps.

**Step 1.** We first show that \(A\) and \(C\) are Lipschitzian on \(X\). To see this, let \(x, y \in X\). So

\[
|Ax(t) - Ay(t)| = |g(t, x(t)) - g(t, y(t))| \\
\leq L |x(t) - y(t)| \leq L \|x - y\|
\]

which implies \(\|Ax - Ay\| \leq L \|x - y\|\) for all \(x, y \in X\). Therefore, \(A\) is a Lipschitzian on \(X\) with Lipschitz constant \(L\). Analogously, for any \(x, y \in X\), we have

\[
|Cx(t) - Cy(t)| = \left| \sum_{i=1}^{m} I^{\gamma_i} h_i(t, x(t)) - \sum_{i=1}^{m} I^{\gamma_i} h_i(t, y(t)) \right| \\
\leq \sum_{i=1}^{m} \int_0^t \frac{(t-s)^{\gamma_i-1}}{\Gamma(\gamma_i)} k_i |x(s) - y(s)| \, ds \\
\leq \|x - y\| \sum_{i=1}^{m} \frac{k_i T^{\gamma_i}}{\Gamma(\gamma_i + 1)}
\]

This means that

\[
\|Cx - Cy\| \leq \sum_{i=1}^{m} \frac{k_i T^{\gamma_i}}{\Gamma(\gamma_i + 1)} \|x - y\|.
\]
This shows that \( C \) is a Lipschitz mapping on \( X \) with the Lipschitz constant \( \sum_{i=1}^{n} \| L_i \| \).

**Step 2.** we show that \( B \) is a compact and continuous operator on \( S \) into \( X \).

First we show that \( B \) is continuous on \( X \). Let \( \{ x_n \} \) be a sequence in \( S \) converging to a point \( x \in S \). Then by the Lebesgue dominated convergence theorem, let us assume that \( t \in J \) and since \( f_2(t, x(t)) \) is continuous in \( X \), then \( f_2(t, x_n(t)) \) converges to \( f_2(t, x(t)) \), (see assumption \( (A_2) \)) applying Lebesgue Dominated Convergence Theorem, we get

\[
\lim_{n \to \infty} I^\beta f_2(s, x_n(s)) = I^\beta f_2(s, x(s)).
\]

Also, since \( f_1(t, x(t)) \) is continuous in \( x \), then using the properties of the fractional-order integral and applying Lebesgue Dominated Convergence Theorem, we get

\[
\lim_{n \to \infty} Bx_n(t) = \lim_{n \to \infty} I^\alpha f_1(s, I^\beta f_2(s, x_n(s)) = I^\alpha f_1(s, I^\beta f_2(s, x(s)) = Bx(t).
\]

Thus, \( Bx_n \to Bx \) as \( n \to \infty \) uniformly on \( R_+ \), and hence \( B \) is a continuous operator on \( S \) into \( X \).

Now, we show that \( B \) is a compact operator on \( S \). It is enough to show that \( B(S) \) is a uniformly bounded and equicontinuous set in \( X \). On the one hand, let \( x \in S \) be arbitrary. Then by hypothesis \( (A_2) \),

\[
|Bx(t)| \leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f_1(s, I^\beta f_2(s, x(s))|ds
\]

\[
\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} [a_1(s) + b_1(s)I^\beta |f_2(s, x(s))|]ds,
\]

\[
\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} a_1(s)ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} b_1(s)|I^\beta |f_2(s, x(s))||ds,
\]

\[
\leq I^\alpha a_1(t) + b_1 \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} I^\beta m(s)ds,
\]

\[
\leq I^\alpha a_1(t) + b_1 I^{\alpha+\beta} m(t)
\]

\[
\leq I^{\alpha-\gamma} I^\alpha a_1(t_2) + b_1 I^{\alpha+\beta-\gamma} I^\gamma m(t)
\]

\[
\leq M_1 \frac{(t-s)^{\alpha-\gamma-1}}{\Gamma(\alpha - \gamma)} ds + b_1 M_2 \int_0^t \frac{(t-s)^{\alpha+\beta-\gamma-1}}{\Gamma(\alpha + \beta - \gamma)} ds
\]

\[
\leq M_1 \frac{T^{\alpha-\gamma}}{\Gamma(\alpha - \gamma + 1)} + b_1 M_2 \frac{T^{\alpha+\beta-\gamma}}{\Gamma(\alpha + \beta - \gamma + 1)} = K.
\]

for all \( t \in J \). Taking supremum over \( t \),

\[
\|Bx(t)\| \leq K
\]
for all \( x \in S \). This shows that \( B \) is uniformly bounded on \( S \).

Now, we proceed to show that \( B(S) \) is also equicontinuous set in \( X \). Let \( t_1, t_2 \in J \), and \( x \in S \). (without loss of generality assume that \( t_1 < t_2 \)), then we have

\[
(Bx)(t_1) - (Bx)(t_2) \\
\leq \int_0^{t_2} \frac{(t_2 - s)^{\alpha - 1}}{\Gamma(\alpha)} f_1(s, I^\beta f_2(s, x(s))) ds - \int_0^{t_1} \frac{(t_1 - s)^{\alpha - 1}}{\Gamma(\alpha)} f_1(s, I^\beta f_2(s, x(s))) ds \\
\leq \int_0^{t_1} \frac{(t_2 - s)^{\alpha - 1}}{\Gamma(\alpha)} f_1(s, I^\beta f_2(s, x(s))) ds + \int_{t_1}^{t_2} \frac{(t_2 - s)^{\alpha - 1}}{\Gamma(\alpha)} f_1(s, I^\beta f_2(s, x(s))) ds \\
- \int_0^{t_1} \frac{(t_1 - s)^{\alpha - 1}}{\Gamma(\alpha)} f_1(s, I^\beta f_2(s, x(s))) ds \\
\leq \int_0^{t_1} \frac{(t_1 - s)^{\alpha - 1}}{\Gamma(\alpha)} f_1(s, I^\beta f_2(s, x(s))) ds + \int_{t_1}^{t_2} \frac{(t_2 - s)^{\alpha - 1}}{\Gamma(\alpha)} f_1(s, I^\beta f_2(s, x(s))) ds \\
- \int_0^{t_1} \frac{(t_1 - s)^{\alpha - 1}}{\Gamma(\alpha)} f_1(s, I^\beta f_2(s, x(s))) ds, \\
\leq \int_0^{t_1} \frac{(t_2 - s)^{\alpha - 1} - (t_1 - s)^{\alpha - 1}}{\Gamma(\alpha)} f_1(s, I^\beta f_2(s, x(s))) ds \\
+ \int_{t_1}^{t_2} \frac{(t_2 - s)^{\alpha - 1}}{\Gamma(\alpha)} f_1(s, I^\beta f_2(s, x(s)))
and

\[
\begin{align*}
& |(Bx)(t_1) - (Bx)(t_2)| \\
& \leq \int_0^{t_1} \frac{(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}}{\Gamma(\alpha)} |f_1(s, I^\beta f_2(s, x(s)))| ds \\
& + \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} |f_1(s, I^\beta f_2(s, x(s)))| ds \\
& \leq \int_0^{t_1} \frac{(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}}{\Gamma(\alpha)} |a(s) + b(s) I^\beta| f_2(s, x(s))| ds \\
& + \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} |a(s) + b(s) I^\beta| f_2(s, x(s))| ds \\
& \leq \frac{||a||}{\Gamma(\alpha)} \int_0^{t_1} (t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1} ds + \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} ds \\
& + \frac{||b||}{\Gamma(\alpha)} \int_0^{t_1} (t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1} I^\beta f_2(s, x(s)) ds + \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} I^\beta f_2(s, x(s)) ds \\
& \leq \frac{||a||}{\Gamma(\alpha+1)} \left( \frac{|t_2^\alpha - t_1^\alpha - 2(t_2-t_1)^\alpha|}{\Gamma(\alpha+1)} \right) \\
& + \frac{||b||}{\Gamma(\alpha)} \int_0^{t_1} (t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1} I^\beta m(s) ds + \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} I^\beta m(s) ds \\
& \leq a \left( \frac{|t_2^\alpha - t_1^\alpha - 2(t_2-t_1)^\alpha|}{\Gamma(\alpha+1)} \right) \\
& + bM_2 \left[ \int_0^{t_1} (t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1} \frac{(s-\tau)^{\beta-\gamma-1}}{\Gamma(\beta-\gamma)} ds + \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} \frac{(s-\tau)^{\beta-\gamma-1}}{\Gamma(\beta-\gamma)} ds \right] \\
& \leq a \left( \frac{|t_2^\alpha - t_1^\alpha - 2(t_2-t_1)^\alpha|}{\Gamma(\alpha+1)} \right) \\
& + bM_2 \left[ \int_0^{t_1} (t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1} \frac{(s-\tau)^{\beta-\gamma}}{\Gamma(\beta-\gamma+1)} ds + \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} \frac{(s-\tau)^{\beta-\gamma}}{\Gamma(\beta-\gamma+1)} ds \right] \\
& \leq a \left( \frac{|t_2^\alpha - t_1^\alpha - 2(t_2-t_1)^\alpha|}{\Gamma(\alpha+1)} \right) \\
& + bM_2 \left[ \int_0^{t_1} (t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1} \frac{(s-\tau)^{\beta-\gamma}}{\Gamma(\beta-\gamma+1)} ds + \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} \frac{(s-\tau)^{\beta-\gamma}}{\Gamma(\beta-\gamma+1)} ds \right] \\
& \leq a \left( \frac{|t_2^\alpha - t_1^\alpha - 2(t_2-t_1)^\alpha|}{\Gamma(\alpha+1)} \right) \\
& + bM_2 \left[ \int_0^{t_1} (t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1} \frac{(s-\tau)^{\beta-\gamma}}{\Gamma(\beta-\gamma+1)} ds + \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} \frac{(s-\tau)^{\beta-\gamma}}{\Gamma(\beta-\gamma+1)} ds \right] \\
& \leq a \left( \frac{|t_2^\alpha - t_1^\alpha - 2(t_2-t_1)^\alpha|}{\Gamma(\alpha+1)} \right) + bM_2 \left( \frac{|t_2^\alpha - t_1^\alpha - 2(t_2-t_1)^\alpha|}{\Gamma(\alpha+1)\Gamma(\beta-\gamma+1)} \right).
\end{align*}
\]
i.e.,
\[
| (Bx)(t_2) - (Bx)(t_1) | \
\leq a \left( \frac{|t_2^\alpha - t_1^\alpha - 2(t_2 - t_1)^\alpha|}{\Gamma(\alpha + 1)} \right) + bM_2 \left( \frac{|t_2^\alpha - t_1^\alpha - 2(t_2 - t_1)^\alpha| T^{\beta - \gamma}}{\Gamma(\alpha + 1)\Gamma(\beta - \gamma + 1)} \right)
\]
which is independent of \( x \) for all \( t \in X \). Let \( S \) be the set for all \( t, x \in S \). Then, we obtain. Hence, for \( \epsilon > 0 \), there exists a \( \delta > 0 \) such that
\[
|t_2 - t_1| < \delta \implies | (Bx)(t_2) - (Bx)(t_1) | < \epsilon
\]
for all \( t_2, t_1 \in J \) and for all \( x \in S \). This shows that \( B(S) \) is an equicontinuous set in \( X \). Now, the set \( B(S) \) is a uniformly bounded and equicontinuous set in \( X \), so it is compact by the Arzela-Ascoli theorem. As a result, \( B \) is a complete continuous operator on \( S \).

**Step 3.** The hypothesis (c) of Lemma 3.1 is satisfied. Let \( x \in X \) and \( y \in S \) be arbitrary elements such that \( x = AxB + Cx \). Then we have
\[
|x(t)| \leq |Arx||By(t)| + |Cx(t)|
\]
\[
\leq |g(t, x(t))| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f_1(s, I^\beta f_2(s, y(s)))| ds + \sum_{i=1}^m I^{\gamma_i}|h_i(t, x(t))|
\]
\[
\leq (|g(t, x(t)) - g(t, 0)| + |g(t, 0)|) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f_1(s, I^\beta f_2(s, y(s)))| ds
\]
\[
+ \sum_{i=1}^m \int_0^t \frac{(t-s)^{\gamma_i-1}}{\Gamma(\gamma_i + 1)} (|h_i(t, x(t))| - |h_i(t, 0)|) + |h_i(t, 0)|
\]
\[
\leq (L|x(t)| + G) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left( |a_1(s) + b_1(s)I^\beta f_2(s, y(s))| ds
\]
\[
+ \sum_{i=1}^m \frac{(k_i |x(t)| + H_i)T^{\gamma_i}}{\Gamma(\gamma_i + 1)} \right)
\]
\[
\leq (L|x(t)| + G) \left[ |f_1 a_1(t) + b_1 I^\alpha m(t)| + \sum_{i=1}^m \frac{(k_i |x(t)| + H_i)T^{\gamma_i}}{\Gamma(\gamma_i + 1)} \right]
\]
\[
\leq |Lr + G| [I^\alpha a_1(t) + b_1 I^\alpha m(t)] + \sum_{i=1}^m \frac{(k_i |x(t)| + H_i)T^{\gamma_i}}{\Gamma(\gamma_i + 1)}
\]
\[
\leq (Lr + G) \left[ M_1 \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha - \gamma)} ds + b_1 M_2 \left( M_0 \frac{S^{\alpha-\gamma}}{\Gamma(\alpha - \gamma + 1)} + b_1 M_2 \frac{S^{\alpha+\beta-\gamma}}{\Gamma(\alpha + \beta - \gamma + 1)} \right) + \sum_{i=1}^m \frac{(k_i r + H_i)T^{\gamma_i}}{\Gamma(\gamma_i + 1)} \right].
\]
Therefore
\[ |x(t)| \leq \sum_{i=1}^{m} \left( k_i r + H_i \Gamma(\gamma_i) \right) + (Lr + G) \left( \frac{M_1 T^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} + \frac{b_1 M_2 T^{\alpha+\beta-\gamma}}{\Gamma(\alpha+\beta-\gamma+1)} \right). \]

Taking supremum over \( t \),
\[ \|x(t)\| \leq \sum_{i=1}^{m} \left( k_i r + H_i \Gamma(\gamma_i) \right) + (Lr + G) \left( \frac{M_1 T^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} + \frac{b_1 M_2 T^{\alpha+\beta-\gamma}}{\Gamma(\alpha+\beta-\gamma+1)} \right) \leq r. \]

Therefore, \( x \in S \).

**Step 4.** Finally we show that \( \delta M + \rho < 1 \), that is, (d) of Lemma 3.1 holds. Since
\[ M = \|B(S)\| = \sup_{x \in S} \left\{ \sup_{t \in J} |Bx(t)| \right\} \leq \frac{M_1 T^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} + \frac{b_1 M_2 T^{\alpha+\beta-\gamma}}{\Gamma(\alpha+\beta-\gamma+1)} \]
and by \((A_3)\) we have
\[ LM + \sum_{i=1}^{m} \frac{k_i T^{\gamma_i}}{\Gamma(\gamma_i + 1)} < 1. \]

with \( \delta = L \) and \( \rho = \sum_{i=1}^{m} \frac{k_i T^{\gamma_i}}{\Gamma(\gamma_i + 1)} \). Thus all the conditions of Lemma 3.1 are satisfied and hence the operator equation \( x = ABx + Cx \) has a solution in \( S \). In consequence, problem \( \text{(1)} \) has a solution on \( J \). This completes the proof.

4 Fractional hybrid differential inequalities

We discuss a fundamental result relative to strict inequalities for the FHDE \( \text{(1)} \).

**Lemma 4.1.** ([22]). Let \( m : R_+ \to R \) be locally Hölder continuous such that for any \( t_1 \in (0, +\infty) \), we have
\[ m(t_1) = 0 \quad \text{and} \quad m(t) \leq 0 \quad \text{for} \quad 0 \leq t \leq t_1. \quad (9) \]
Then it follows that
\[ D^\alpha m(t_1) \geq 0. \quad (10) \]

**Theorem 4.2.** Assume that hypotheses \((A_0)\) holds. Suppose that there exist functions \( y, z : J \to R \) that are locally Hölder continuous such that
\[ D^\alpha \left( \frac{y(t) - \sum_{i=1}^{m} \Gamma_\gamma h(t, y(t))}{g(t, y(t))} \right) \leq f_1(t, I^\beta f_2(t, y(t))) \quad t \in J, \ i = 1, 2, \ldots, m, \quad (11) \]
and
\[ D^\alpha \left( \frac{z(t) - \sum_{i=1}^{m} I^\beta_i h_i(t, z(t))}{g(t, z(t))} \right) \geq f_1(t, I^\beta f_2(t, z(t))) \quad t \in J, \ i = 1, 2, ..., m, \] (12)
one of the inequalities being strict. Then
\[ y(0) < z(0) \] (13)
implies
\[ y(t) < z(t) \] (14)
for all \( t \in J. \)

**Proof.** Suppose that inequality (12) is strict. Assume that the claim is false. Then there exists a \( t_1 \in J, \ t_1 > 0 \) such that \( y(t_1) = z(t_1) \) and \( y(t) < z(t) \) for \( 0 \leq t < t_1. \)

Define
\[ Y(t) = \frac{y(t) - \sum_{i=1}^{m} I^\beta_i h_i(t, y(t))}{g(t, y(t))} \quad \text{and} \quad Z(t) = \frac{z(t) - \sum_{i=1}^{m} I^\beta_i h_i(t, z(t))}{g(t, z(t))} \]

Then we have \( Y(t_1) = Z(t_1) \) and by virtue of hypothesis \((A_0)\), we get \( Y(t) < Z(t) \) for all \( 0 \leq t < t_1. \)

Setting \( m(t) = Y(t) - Z(t), \ 0 \leq t \leq t_1 \), we find that \( m(t) \leq 0, \ 0 \leq t \leq t_1 \) and \( m(t_1) = 0 \). Then by Lemma 4.1 we have \( D^\alpha m(t_1) \geq 0. \) By (11) and (12), we obtain
\[ f_1(t_1, I^\beta f_2(t_1, y(t_1))) \geq D^\alpha Y(t_1) \geq D^\alpha Z(t_1) > f_1(t_1, I^\beta f_2(t_1, y(t_1))). \]

This is a contradiction with \( y(t_1) = z(t_1). \) Hence, the conclusion (14) is valid and the proof is complete. \( \square \)

The next result is concerned with nonstrict fractional differential inequalities which requires a kind of one sided Lipschitz condition.

**Theorem 4.3.** Assume that the conditions of Theorem 5.2 hold with inequalities (11) and (12). Suppose that there exists a real number \( M > 0 \) such that
\[ f_1(t, I^\beta f_2(t, x_1)) - f_1(t, I^\beta f_2(t, x_2)) \leq \frac{M}{1 + t^q \left( x_1 - \sum_{i=1}^{m} I^\beta_i h_i(t, x_1) - x_2 - \sum_{i=1}^{m} I^\beta_i h_i(t, x_2) \right) g_2(t, x_2)} \], \( t \in J \)
for all \( x_1, x_2 \in R \) with \( x_1 \geq x_2. \) Then \( y(0) \leq z(0) \) implies, provided \( MT^q \leq \frac{1}{1-\eta} \),
\[ y(t) \leq z(t) \] (15)
for all \( t \in J. \)
Proof. We set
\[
\frac{z_e(t) - \sum_{i=1}^{m} I^\beta_h_i(t, z_e(t))}{g_2(t, z_e(t))} = \frac{z(t) - \sum_{i=1}^{m} I^\beta_h_i(t, z(t))}{g_2(t, z(t))} + \epsilon(1 + t^q)
\]
for small \( \epsilon > 0 \), so that we have
\[
\frac{z_e(t) - \sum_{i=1}^{m} I^\beta_h_i(t, z_e(t))}{g_2(t, z_e(t))} \geq \frac{z(t) - \sum_{i=1}^{m} I^\beta_h_i(t, z(t))}{g_2(t, z(t))} \implies z_e(t) \geq z(t).
\]
Let \( Z_e(t) = \frac{z_e(t) - \sum_{i=1}^{m} I^\beta_h_i(t, z_e(t))}{g_2(t, z_e(t))} \) so that \( Z(t) = \frac{z(t) - \sum_{i=1}^{m} I^\beta_h_i(t, z(t))}{g_2(t, z(t))} \) for \( t \in J \).

Since
\[
f_1(t, I^\beta f_2(t, z_e(t))) - f_1(t, I^\beta f_2(t, z(t))) \leq \frac{M}{1 + t^q} \left( \frac{z_e(t) - \sum_{i=1}^{m} I^\beta h_i(t, z_e(t))}{g_2(t, z_e(t))} - \frac{z(t) - \sum_{i=1}^{m} I^\beta_h_i(t, z(t))}{g_2(t, z(t))} \right),
\]
for all \( t \in J \) and \( M T^q \leq \frac{1}{\Gamma(1-q)} \), one has
\[
D^\alpha Z_e(t) = D^\alpha Z(t) + \epsilon D^\alpha(1 + t^q)
\geq f_1(t, I^\beta f_2(t, z(t))) + \epsilon \left( \frac{1}{t^q \Gamma(1-q)} + \Gamma(1+q) \right)
\geq f_1(t, I^\beta f_2(t, z(t))) - M \epsilon + \epsilon \left( \frac{1}{t^q \Gamma(1-q)} + \Gamma(1+q) \right)
> f_1(t, I^\beta f_2(t, z_e(t)))
\]
Also, we have \( z_e(0) > z(0) \geq y(0) \). Hence, by an application of Theorem 5.2 with \( z = z_e \) yields that \( y(t) < z_e(t) \) for all \( t \in J \). By the arbitrariness of \( \epsilon > 0 \), taking the limits as \( \epsilon \to 0 \), we have \( y(t) \leq z(t) \) for all \( t \in J \). This completes the proof.

5 Existence of maximal and minimal solutions

In this section, we shall prove the existence of maximal and minimal solutions for the FHDE (1) on \( J = [0, T] \). We need the following definition in what follows.

Definition 5.1. A solution \( r \) of the FHDE (1) is said to be maximal if for any other solution \( x \) to the FHDE (1) one has \( x(t) \leq r(t) \), for all \( t \in J \). Similarly, a solution \( \rho \) of the FHDE (1) is said to be minimal if \( \rho(t) \leq x(t) \), for all \( t \in J \), where \( x \) is any solution of the FHDE (1) on \( J \).

We discuss the case of maximal solution only, as the case of minimal solution is similar and can be obtained with the same arguments with appropriate modifications. Given an arbitrary small real number \( \epsilon > 0 \), consider the following initial value problem of FHDE of order \( 0 < \alpha < 1 \),
\[
\begin{align*}
\left\{ D^\alpha \left( \frac{x(t) - \sum_{i=1}^{m} I^\gamma_i h_i(t, x(t))}{g(t, x(t))} \right) = f_1(t, I^\beta f_2(t, x(t))) + \epsilon & \quad \text{a.e. } t \in J, \\
x(0) = 0,
\end{align*}
\] (16)

where \( g(t, x(t)) \in C(J \times \mathbb{R} \setminus \{0\}) \), \( f_i(t, x(t)) \in C(J \times \mathbb{R}) \), \( i = 1, 2 \), and \( h_i(t, x(t)) \in C(J \times \mathbb{R}) \) with \( h_i(0,0) = 0 \) (\( i = 1, 2, \ldots, m \)).

An existence theorem for the FHDE (16) can be stated as follows.

**Theorem 5.1.** Assume that hypotheses (A_0)(A_3) hold. Suppose that inequality (3) holds. Then for every small number \( \epsilon > 0 \), the FHDE (16) has a solution defined on \( J \).

**Proof.** The proof is similar to Theorem 3.3 and we omit the details. \( \square \)

Our main existence theorem for maximal solution for the FHDE (1) is

**Theorem 5.2.** Assume that hypotheses (i)(vi) hold. Furthermore, if condition (3) holds, then the FHDE (1) has a maximal solution defined on \( J \).

**Proof.** Let \( \{ \epsilon_n \}_{n=0}^{\infty} \) be a decreasing sequence of positive real numbers such that \( \lim_{n \to \infty} \epsilon_n = 0 \). By Theorem 5.1, then there exists a solution \( r(t, \epsilon_n) \) defined on \( J \) of the FHDE,

\[
\begin{align*}
\left\{ D^\alpha \left( \frac{x(t) - \sum_{i=1}^{m} I^\gamma_i h_i(t, x(t))}{g(t, x(t))} \right) = f_1(t, I^\beta f_2(t, x(t))) + \epsilon_n \ & \quad t \in J \\
\right.
\]
(17)
\[
x(0) = 0,
\]

Then for any solution \( u \) of the FHDE (17) satisfies

\[
D^\alpha \left( \frac{u(t) - \sum_{i=1}^{m} I^\gamma_i h_i(t, u(t))}{g(t, u(t))} \right) \leq f_1(t, I^\beta f_2(t, u(t)))
\]

and any solution of auxiliary problem (17) satisfies

\[
D^\alpha \left( \frac{r(t, \epsilon_n) - \sum_{i=1}^{m} I^\gamma_i h_i(t, r(t, \epsilon_n))}{g(t, r(t, \epsilon_n))} \right) = f_1(t, I^\beta f_2(t, r(t, \epsilon_n))) + \epsilon_n \]

where \( u(0) = 0 \leq \epsilon_n = r(0, \epsilon_n) \). By Theorem 4.3 we infer that

\[
u(t) \leq r(t, \epsilon_n)
\]

for all \( t \in J \) and \( n \in N \cup \{0\} \).

Since \( \epsilon_2 = r(0, \epsilon_2) \leq r(0, \epsilon_1) = \epsilon_1 \), then by Theorem 4.3 we infer that \( r(t, \epsilon_2) \leq r(t, \epsilon_1) \). Therefore, \( r(t, \epsilon_n) \) is a decreasing sequence of positive real numbers, the limit

\[
r(t) = \lim_{n \to \infty} r(t, \epsilon_n)
\]
(19)
exists. We show that the convergence in (19) is uniform on $J$. To finish, it is enough to prove that the sequence $r(t, \epsilon_n)$ is equicontinuous in $C(J, R)$. Let $t_1, t_2 \in J$ with $t_1 < t_2$ be arbitrary. Then,

$$ |r(t_1, \epsilon_n) - r(t_2, \epsilon_n)|$$

$$ = \left| \sum_{i=1}^{m} I^\gamma_i h_i(t_1, r(t_1, \epsilon_n)) + g(t_1, r(t_1, \epsilon_n)) \int_0^{t_1} \frac{(t_1 - s)^{\alpha-1}}{\Gamma(\alpha)} (f_1(s, t^\beta f_2(s, r(s, \epsilon_n))) + \epsilon_n) ds \right|$$

$$ - \sum_{i=1}^{m} I^\gamma_i h_i(t_2, r(t_1, \epsilon_n)) + g(t_2, r(t_2, \epsilon_n)) \int_0^{t_2} \frac{(t_2 - s)^{\alpha-1}}{\Gamma(\alpha)} (f_1(s, t^\beta f_2(s, r(s, \epsilon_n))) + \epsilon_n) ds \right|$$

$$ \leq \left| \sum_{i=1}^{m} I^\gamma_i h_i(t_1, r(t_1, \epsilon_n)) - \sum_{i=1}^{m} I^\gamma_i h_i(t_2, r(t_2, \epsilon_n)) \right|$$

$$ + |g(t_1, r(t_1, \epsilon_n)) \int_0^{t_1} \frac{(t_1 - s)^{\alpha-1}}{\Gamma(\alpha)} (f_1(s, t^\beta f_2(s, r(s, \epsilon_n))) + \epsilon_n) ds$$

$$ - g(t_2, r(t_2, \epsilon_n)) \int_0^{t_1} \frac{(t_1 - s)^{\alpha-1}}{\Gamma(\alpha)} (f_1(s, t^\beta f_2(s, r(s, \epsilon_n))) + \epsilon_n) ds|$$

$$ + |g(t_2, r(t_2, \epsilon_n)) \int_0^{t_1} \frac{(t_1 - s)^{\alpha-1}}{\Gamma(\alpha)} (f_1(s, t^\beta f_2(s, r(s, \epsilon_n))) + \epsilon_n) ds$$

$$ - g(t_2, r(t_2, \epsilon_n)) \int_0^{t_2} \frac{(t_2 - s)^{\alpha-1}}{\Gamma(\alpha)} (f_1(s, t^\beta f_2(s, r(s, \epsilon_n))) + \epsilon_n) ds|$$

$$ \leq \sum_{i=1}^{m} \int_0^{t} \frac{(t - s)^{\gamma_i-1}}{\Gamma(\gamma_i)} k_i |r(t_1, \epsilon_n) - r(t_2, \epsilon_n)| ds$$

$$ + |g(t_1, r(t_1, \epsilon_n)) - g(t_2, r(t_2, \epsilon_n))| \int_0^{t_1} \frac{(t_1 - s)^{\alpha-1}}{\Gamma(\alpha)} (f_1(s, t^\beta f_2(s, r(s, \epsilon_n))) + \epsilon_n) ds$$

$$ + |g(t_2, r(t_2, \epsilon_n))| \int_0^{t_1} \frac{(t_1 - s)^{\alpha-1}}{\Gamma(\alpha)} |(f_1(s, t^\beta f_2(s, r(s, \epsilon_n))) + \epsilon_n)| ds$$

$$ \leq |r(t_1, \epsilon_n) - r(t_2, \epsilon_n)| \sum_{i=1}^{m} \frac{k_i T^{\gamma_i}}{\Gamma(\gamma_i + 1)}$$

$$ + |g(t_1, r(t_1, \epsilon_n)) - g(t_2, r(t_2, \epsilon_n))| [M_1 \frac{T^{\alpha-\gamma}}{\Gamma(\alpha - \gamma + 1)} + b_1 M_2 \frac{T^{\alpha+\beta-\gamma}}{\Gamma(\alpha + \beta - \gamma + 1)} + \epsilon_n \frac{T^\alpha}{\Gamma(\alpha + 1)}]$$

$$ + G_s [M_1 \frac{(t_2 - t_1)^{\alpha-\gamma}}{\Gamma(\alpha - \gamma + 1)} + b_1 M_2 \frac{(t_2 - t_1)^{\alpha+\beta-\gamma}}{\Gamma(\alpha + \beta - \gamma + 1)} + \epsilon_n \frac{(t_1 - t_2)^{\alpha}}{\Gamma(\alpha + 1)}]$$

where $G_s = \sup_{t \in J \times [-N, N]} |g(t, x)|$.

Since $g$ is continuous on compact set $J \times [-N, N]$, it is uniformly continuous there. Hence,

$$ |g(t_1, r(t_1, \epsilon_n)) - g(t_2, r(t_2, \epsilon_n))| \to 0 \quad \text{as} \quad t_1 \to t_2.$$
uniformly for all $n \in N$.

Therefore, from the above inequality, it follows that

$$|r(t_1, \epsilon_n) - r(t_2, \epsilon_n)| \to 0 \quad \text{as} \quad t_1 \to t_2$$

uniformly for all $n \in N$. Therefore,

$$r(t, \epsilon_n) \to r(t) \quad \text{as} \quad n \to \infty$$

for all $t \in J$.

Next, we show that the function $r(t)$ is a solution of the FHDE (11) defined on $J$. Now, since $r(t, \epsilon_n)$ is a solution of the FHDE (16), we have

$$r(t, \epsilon_n) = \sum_{i=1}^{m} I_{\gamma_i} h_i(t, r(t, \epsilon_n)) + g(t, r(t, \epsilon_n)) \int_{0}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left( f_1(s, I_{\beta} f_2(s, r(s, \epsilon_n))) + \epsilon_n \right) ds$$

for all $t \in J$. Taking the limit as $n \to \infty$ in the above Eq. (20) yields

$$r(t) = \sum_{i=1}^{m} I_{\gamma_i} h_i(t, r(t)) + g(t, r(t)) \int_{0}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s, I_{\beta} f_2(s, r(s))) ds$$

for all $t \in J$. Thus, the function $r$ is a solution of the FHDE (11) on $J$. Finally, from inequality (16), it follows that $u(t) \leq r(t)$ for all $t \in J$. Hence, the FHDE (11) has a maximal solution on $J$. This completes the proof. \qed

### 6 Comparison theorems

The main problem of the differential inequalities is to estimate a bound for the solution set for the differential inequality related to the FHDE (11). In this section, we prove that the maximal and minimal solutions serve as bounds for the solutions of the related differential inequality to FHDE (11) on $J = [0, T]$.

**Theorem 6.1.** Assume that hypotheses $(A_0) - (A_3)$ hold. Suppose that there exists a real number $M > 0$ such that

$$f_1(t, I_{\beta} f_2(t, x_1(t))) - f_1(t, I_{\beta} f_2(t, x_2(t)) \leq \frac{M}{1 + t^q} \left( x_1(t) - \sum_{i=1}^{m} I_{\gamma_i} h_i(t, x_1(t)) - x_2(t) - \sum_{i=1}^{m} I_{\gamma_i} h_i(t, x_2(t)) \right), \quad t \in J$$

for all $x_1, x_2 \in R$ with $x_1 \geq x_2$, here $MT^q \leq \frac{1}{\Gamma(1-q)}$. Furthermore, if there exists a function $u \in C(J, R)$ such that

$$D^n \left( u(t) - \sum_{i=1}^{m} I_{\gamma_i} h_i(t, u(t)) \right) \leq f_1(t, I_{\beta} f_2(t, u(t))) \quad \text{a.e.} \quad t \in J,$$

$$u(0) \leq 0,$$

then

$$u(t) \leq r(t)$$

for all $t \in J$. This completes the proof. \qed
Then
\[ u(t) \leq r(t) \] (22)
for all \( t \in J \), where \( r \) is a maximal solution of the FHDE (1) on \( J \).

**Proof.** Let \( \epsilon > 0 \) be arbitrary small. By Theorem 5.2, \( r(t, \epsilon) \) is a maximal solution of FHDE (16) and the limit
\[ r(t) = \lim_{\epsilon \to 0} r(t, \epsilon) \] (23)
is uniform on \( J \) and the function \( r \) is a maximal solution of FHDE (1) on \( J \). Hence, we obtain
\[
\begin{cases}
D^\alpha \left( \frac{r(t, \epsilon) - \sum_{i=1}^m I^{\gamma_i} h_i(t, r(t, \epsilon))}{g(t, r(t, \epsilon))} \right) = f_1(t, I^\beta f_2(t, r(t, \epsilon))) + \epsilon & t \in J \\
r(0, \epsilon) = 0,
\end{cases}
\]
From the above inequality, it follows that
\[
\begin{cases}
D^\alpha \left( \frac{r(t, \epsilon) - \sum_{i=1}^m I^{\gamma_i} h_i(t, r(t, \epsilon))}{g(t, r(t, \epsilon))} \right) > f_1(t, I^\beta f_2(t, r(t, \epsilon))) & t \in J \\
r(0, \epsilon) = 0,
\end{cases}
\] (24)

Now, we apply Theorem 4.3 to inequalities (21) and (24) and conclude that \( u(t) < r(t, \epsilon) \) for all \( t \in J \). This further, in view of limit (23), implies that inequality (22) holds on \( J \). This completes the proof.

**Theorem 6.2.** Assume that hypotheses \((A_0) - (A_3)\) hold. Suppose that there exists a real number \( M > 0 \) such that
\[
\begin{align*}
&f_1(t, I^\beta f_2(t, x_1(t))) - f_1(t, I^\beta f_2(t, x_2(t)) \leq \\
&\frac{M}{1 + t^q} \left( x_1(t) - \sum_{i=1}^m I^{\gamma_i} h_i(t, x_1(t)) - x_2(t) - \sum_{i=1}^m I^{\gamma_i} h_i(t, x_2(t)) \right), \quad t \in J
\end{align*}
\]
for all \( x_1, x_2 \in R \) with \( x_1 \geq x_2 \), where \( MT^q \leq \frac{1}{1-(1-q)} \). Furthermore, if there exists a function \( u \in C(J, R) \) such that
\[
\begin{cases}
D^\alpha \left( \frac{v(t) - \sum_{i=1}^m I^{\gamma_i} h_i(t, v(t))}{g(t, r(t, \epsilon))} \right) > f_1(t, I^\beta f_2(t, v(t))) & \text{a.e. } t \in J \\
v(0) > 0.
\end{cases}
\]
Then
\[ \rho(t) \leq v(t) \]
for all \( t \in J \), where \( \rho \) is a minimal solution of FHDE (1) on \( J \).

Note that Theorem 6.1 is useful to prove the boundedness and uniqueness of the solutions for FHDE (1) on \( J \). A result in this direction is as follows.
Theorem 6.3. Assume that hypotheses \((A_0) - (A_3)\) hold. Suppose that there exists a function \(G : J \times R_+ \rightarrow R_+\) such that

\[
f_1(t, I^\beta f_2(t, x_1) - f_1(t, I^\beta f_2(t, x_2) \leq G \left( \frac{x_1(s) - \sum_{i=1}^m I^\gamma_i h_i(t, x_1)}{g(t, x_1)} - \frac{x_2 - \sum_{i=1}^m I^\gamma_i h_i(t, x_2)}{g(t, x_2)} \right), \ t \in J
\]

for all \(x_1, x_2 \in R\). If an identically zero function is the only solution of the differential equation

\[
D^\alpha m(t) = G(t, m(t)) \quad a.e. \quad t \in J, \ m(0) = 0
\]

then FHDE (11) has a unique solution on \(J\).

Proof. By Theorem 3.3, the FHDE (11) has a solution defined on \(J\). Suppose that there are two solutions \(u_1\) and \(u_2\) of the FHDE (11) existing on \(J\) with \(u_1 > u_2\). Define a function \(m : J \rightarrow R\) by

\[
m(t) = \left| \frac{u_1(t) - \sum_{i=1}^m I^\gamma_i h_i(t, u_1(t))}{g(t, u_1(t))} - \frac{u_2(t) - \sum_{i=1}^m I^\gamma_i h_i(t, u_2(t))}{g(t, u_2(t))} \right|
\]

In view of hypothesis \((A_0)\), we conclude that \(m(t) > 0\), as \(D^\alpha (|x(t)|) \leq |D^\alpha x(t)|\) for \(t \in J\), we have

\[
D^\alpha m(t) = \left| D^\alpha \left( \frac{u_1(t) - \sum_{i=1}^m I^\gamma_i h_i(t, u_1(t))}{g(t, u_1(t))} \right) - D^\alpha \left( \frac{u_2(t) - \sum_{i=1}^m I^\gamma_i h_i(t, u_2(t))}{g(t, u_2(t))} \right) \right|
\]

\[
= \left| f_1(t, I^\beta f_2(t, u_1(t)) - f_1(t, I^\beta f_2(t, u_2(t)) | \right|
\]

\[
\leq G(t, \frac{u_1(t) - \sum_{i=1}^m I^\gamma_i h_i(t, u_1(t))}{g(t, u_1(t))} - \frac{u_2(t) - \sum_{i=1}^m I^\gamma_i h_i(t, u_2(t))}{g(t, u_2(t))})
\]

for almost everywhere \(t \in J\), and that \(m(0) = 0\).

Now, we apply Theorem 3.1 with \(g(t, x) \equiv 1\) and \(h_i(t, x) \equiv 0\) to get that \(m(t) = 0\) for all \(t \in J\). This give that

\[
\frac{u_1(t) - \sum_{i=1}^m I^\gamma_i h_i(t, u_1(t))}{g(t, u_1(t))} = \frac{u_2(t) - \sum_{i=1}^m I^\gamma_i h_i(t, u_2(t))}{g(t, u_2(t))}
\]

for all \(t \in J\). Finally, in view of hypothesis \((A_0)\) we conclude that \(u_1(t) = u_2(t)\) on \(J\). This completes the proof.

7 Existence of extremal solutions in vector segment

Sometimes it is desirable to have knowledge of the existence of extremal positive solutions for the FHDE (11) on \(J\). In this section, we shall prove the existence
of maximal and minimal positive solutions for the FHDE (1) between the given upper and lower solutions on \( J = [0, T] \). We use a hybrid fixed point theorem of Dhage [11] in ordered Banach spaces for establishing our results. We need the following preliminaries in what follows.

A non-empty closed set \( K \) in a Banach algebra \( X \) is called a cone with vertex 0, if

(i) \( K + K \subseteq K \),
(ii) \( \lambda K \subseteq K \) for \( \lambda \in R, \lambda \geq 0 \),
(iii) \( (-K) \cap K = 0 \), where 0 is the zero element of \( X \),
(iv) a cone \( K \) is called to be positive if \( K \circ K \subseteq K \), where \( \circ \) is a multiplication composition in \( X \).

We introduce an order relation \( \leq \) in \( X \) as follows. Let \( x, y \in X \). Then \( x \leq y \) if and only if \( y - x \in K \). A cone \( K \) is said to be normal if the norm \( \| \cdot \| \) is semi-monotone increasing on \( K \), that is, there is a constant \( N > 0 \) such that \( \| x \| \leq N \| y \| \) for all \( x, y \in K \) with \( x \leq y \). It is known that if the cone \( K \) is normal in \( X \), then every order-bounded set in \( X \) is norm-bounded. The details of cones and their properties appear in Heikkilä and Lakshmikantham [13].

Lemma 7.1. [11] Let \( K \) be a positive cone in a real Banach algebra \( X \) and let \( u_1, u_2, v_1, v_2 \in K \) be such that \( u_1 \leq v_1 \) and \( u_2 \leq v_2 \). Then \( u_1u_2 \leq v_1v_2 \).

For any \( a, b \in X \), the order interval \( [a, b] \) is a set in \( X \) given by

\[ [a, b] = \{ x \in X : a \leq x \leq b \}. \]

Definition 7.1. A mapping \( T : [a, b] \to X \) is said to be nondecreasing or monotone increasing if \( x \leq y \) implies \( Tx \leq Ty \) for all \( x, y \in [a, b] \).

We use the following fixed point theorems of Dhage [12] for proving the existence of extremal solutions for IVP (1) under certain monotonicity conditions.

Lemma 7.2. [12] Let \( K \) be a cone in a Banach algebra \( X \) and let \( u_1, u_2, v_1, v_2 \in K \) be such that \( u_1 \leq v_1 \) and \( u_2 \leq v_2 \). Then \( u_1u_2 \leq v_1v_2 \).

For any \( a, b \in X \), the order interval \( [a, b] \) is a set in \( X \) given by

\[ [a, b] = \{ x \in X : a \leq x \leq b \}. \]

Definition 7.1. A mapping \( T : [a, b] \to X \) is said to be nondecreasing or monotone increasing if \( x \leq y \) implies \( Tx \leq Ty \) for all \( x, y \in [a, b] \).

We use the following fixed point theorems of Dhage [12] for proving the existence of extremal solutions for IVP (1) under certain monotonicity conditions.

Lemma 7.2. [12] Let \( K \) be a cone in a Banach algebra \( X \) and let \( a, b \in X \) be such that \( a \leq b \). Suppose that \( A, B : [a, b] \to K \) are two nondecreasing operators such that

(a) \( A \) and \( C \) are Lipschitzian with a Lipschitz constant \( a \) and \( b \) respectively.
(b) \( B \) is completely continuous,
(c) \( AxBx + Cx \in [a, b] \) for each \( x \in [a, b] \).

Further, if the cone \( K \) is positive and normal, then the operator equation \( AxBx + Cx = x \) has a least and a greatest positive solution in \( [a, b] \), whenever \( aM + b < 1 \), where \( M = \| B([a, b]) \| := \sup \{ \| Bx \| : x \in [a, b] \} \).

We equip the space \( C(J, R) \) with the order relation \( \leq \) with the help of a cone \( K \) defined by

\[ K = \{ x \in C(J, R) : x(t) \geq 0, \forall t \in J \} \tag{26} \]

It is well known that the cone \( K \) is positive and normal in \( C(J, R) \). We need the following definitions in the sequel.
Definition 7.2. A function \( a \in C(J, R) \) is called a lower solution of FHDE (1) defined on \( J \) if it satisfies (11). Similarly, a function \( a \in C(J, R) \) is called an upper solution of FHDE (1) defined on \( J \) if it satisfies (12). A solution to FHDE (1) has a minimal and a maximal positive solution defined on \( J \) and vice versa.

We consider the following set of assumptions:

\((B_0)\) \( g : J \times R \to R_+ - \{0\}, f_i : J : R \to R_+ \) \( (i = 1, 2) \)

\((B_1)\) FHDE (1) has a lower solution \( a \) and an upper solution \( b \) defined on \( J \) with \( a \leq b \).

\((B_2)\) The function \( x \to \frac{z(t) - \sum_{i=1}^{m} f_i(t, x(t))}{g(t, x(t))} \) is increasing in the interval \( [\min_{t \in J} a(t), \max_{t \in J} b(t)] \) almost everywhere for \( t \in J \).

\((B_3)\) The functions \( h(t, x), g(t, x), f_1(t, x) \) and \( f_2(t, x) \) are nondecreasing in \( x \) almost everywhere for \( t \in J \).

\((B_4)\) There exists a function \( k \in L^1(J, R) \) such that \( f_1(t, I^\alpha f_2(t, x(t))) \leq k(t) \).

We remark that hypothesis \((B_4)\) holds in particular if \( g \) is continuous, \( f_1 \) and \( f_2 \) are \( L^1 \) Carathodory on \( J \times R \).

Theorem 7.3. Suppose that assumptions \((A_1)\) and \((B_0) - (B_4)\) hold. Then FHDE (1) has a minimal and a maximal positive solution defined on \( J \).

Proof. Now, FHDE (1) is equivalent to integral equation (4) defined on \( J \). Let \( X = C(J, R) \). Define three operators \( A, B \) and \( C \) on \( X \) by (5), (6) and (7) respectively. Then the integral equation (4) is transformed into an operator equation \( Ax(t)Bx(t) + Cx(t) = x(t) \) in the Banach algebra \( X \). Notice that hypothesis \((B_0)\) implies \( A, B : [a, b] \to K \) and \( C : [a, b] \to X \). Since the cone \( K \) in \( X \) is normal, \([a, b]\) is a norm bounded set in \( X \). Now it is shown, as in the proof of Theorem 3.3, that \( A \) and \( C \) are Lipschitzian with the Lipschtiz constant \( L \) and \( k_i \), \((i = 1, 2, ..., m)\). Similarly \( B \) is a completely continuous operator on \([a, b]\) into \( X \). Again, hypothesis \((B_3)\) implies that \( A, B \) and \( C \) are nondecreasing on \([a, b]\). To see this, let \( x, y \in [a, b] \) be such that \( x \leq y \). Then, by hypothesis \((B_3)\),

\[
Ax(t) = g(t, x(t)) \leq g(t, y(t)) = Ay(t)
\]

for all \( t \in J \). Similarly, we have

\[
Bx(t) = I^\alpha f_1(t, I^\beta f_2(t, x(t))) \leq I^\alpha f_1(t, I^\beta f_2(t, y(t))) = By(t)
\]

Also, for all \( t \in J \), we have

\[
Cx(t) = \sum_{i=1}^{m} I^{\gamma_i} h_i(t, x(t)) \leq \sum_{i=1}^{m} I^{\gamma_i} h_i(t, y(t)) = Cy(t)
\]
for all \( t \in J \). So, \( A, C \) and \( B \) are nondecreasing operators on \([a, b]\). Again Lemma 7.1 and hypothesis \((B₃)\) together imply that

\[
a(t) \leq \sum_{i=1}^{m} I^{h_i}(t, a(t)) + g(t, a(t)) \int_{0}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s, I^{\beta} f_2(s, x(s))) \, ds
\]

\[
\leq \sum_{i=1}^{m} I^{h_i}(t, x(t)) + g(t, x(t)) \int_{0}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s, I^{\beta} f_2(s, x(s))) \, ds
\]

\[
\leq \sum_{i=1}^{m} I^{h_i}(t, b(t)) + g(t, b(t)) \int_{0}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s, I^{\beta} f_2(s, x(s))) \, ds
\]

\[
\leq b(t)
\]

for all \( t \in J \) and \( x \in [a, b] \). As a result, \( a(t) \leq Ax(t) Bx(t) + Cx(t) \leq b(t) \) for all \( t \in J \) and \( x \in [a, b] \). Hence, \( Ax(t) Bx(t) + Cx(t) \in [a, b] \) for all \( x \in [a, b] \).

Now, we apply Lemma 7.2 to the operator equation \( Ax \cdot Bx + C(x) = x \) to yield that FHDE (1) has a minimal and a maximal positive solution in \([a, b]\) defined on \( J \). This completes the proof. \( \square \)

**Acknowledgment**

The authors wish to express their gratitude to the anonymous referees for their valuable Comments and suggestions, which allowed to improve an early version of this work.

**Competing interests**

The authors declare that they have no competing interests.

**Authors contribution**

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

**References**

[1] A. Jeribi, N. Kaddachi and B. Krichen, *Fixed point theorems of block operator matrices on Banach algebras and an application to functional integral equations*. Mathematical Methods in Applied Sciences 36(6), 621743 (2012).

[2] Dhage, BC, *Quadratic perturbations of periodic boundary value problems of second order ordinary differential equations*. Differ. Equ. Appl. 2, 465-486 (2010).
[3] Dhage, BC: Nonlinear quadratic first order functional integro-differential equations with periodic boundary conditions. Dyn. Syst. Appl. 18, 303-322 (2009).

[4] Dhage, BC, Karande, BD: First order integro-differential equations in Banach algebras involving Caratheodory and discontinuous nonlinearities. Electron. J. Qual. Theory Differ. Equ. 2005, 21, 16 pp (2005).

[5] Dhage, BC, ORegan, BD: A fixed point theorem in Banach algebras with applications to functional integral equations. Funct. Differ. Equ. 7, 259-267 (2000).

[6] Dhage, BC, Salunkhe, SN, Agarwal, RP, Zhang, W: A functional differential equation in Banach algebras. Math. Inequal. Appl. 8, 89-99 (2005).

[7] Dhage, BC: On α−condensing mappings in Banach algebras. Math. Stud. 63, 146-152 (1994).

[8] Dhage, BC, Lakshmikantham, V: Basic results on hybrid differential equations. Nonlinear Anal. Hybrid Syst. 4, 414-424 (2010).

[9] Dhage, BC: A nonlinear alternative in Banach algebras with applications to functional differential equations. Nonlinear Funct. Anal. Appl. 8, 563-575 (2004).

[10] Dhage, BC: Fixed point theorems in ordered Banach algebras and applications. Panam. Math. J. 9, 93-102 (1999).

[11] B.C. Dhage, A nonlinear alternative in Banach algebras with applications to functional differential equations, Nonlinear Funct. Anal. Appl. 8, 563575 (2004).

[12] B.C. Dhage, Fixed point theorems in ordered Banach algebras and applications, Panamer. Math. J. 9 (4), 93102 (1999).

[13] S. Heikkil, V. Lakshmikantham, Monotone Iterative Technique for Non-linear Discontinues Differential Equations, Marcel Dekker Inc., New York, (1994).

[14] A.A. Kilbas, H.H. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier Science B.V., Amsterdam, (2006).

[15] Dhage, BC, Lakshmikantham, V: Basic results on hybrid differential equations, Nonlinear Anal., Real World Appl. 4, 414-424 (2010).

[16] Dhage, BC, Jadhav, NS: Basic results in the theory of hybrid differential equations with linear perturbations of second type, Tamkang Journal of Mathematics 44.2, 171-186(2013).
[17] Lakshmikantham, V, Leela, S: *Differential and Integral Inequalities*. Academic Press, New York (1969)

[18] Zhao, Y, Sun, S, Han, Z, Li, Q: *Theory of fractional hybrid differential equations*. Comput. Math. Appl. 62, 1312-1324 (2011).

[19] F.E. Browder, *Nonlinear operators and nonlinear equations of evolution in Banach spaces*. Nonlinear Functional Analysis (Proc. Sympos. Pure Math.) XVIII, Part 2, Chicago, III, 1308 (1968).

[20] B.C. Dhage, *On some variants of Schauders fixed point principle and applications to nonlinear integral equations*. J. Math. Phys. Sci. 22(5), 603611 (1988).

[21] B.C. Dhage, *A fixed point theorem in Banach algebras involving three operators with applications*. Kyungpook Math. J. 44(1) 145155 (2004).

[22] A.A. Kilbas, H.H. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier Science B.V., Amsterdam, (2006).