Muon-proton Scattering

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Abstract
A recent proposal to measure the proton form factor by means of muon-proton scattering will use muons which are not ultrarelativistic (and also not nonrelativistic). The usual equations describing the scattering cross section use the approximation that the scattered lepton (usually an electron) is ultrarelativistic, with \( v/c \) approximately equal to 1. Here the cross section is calculated for all values of the energy. It agrees with the standard result in the appropriate limit.

Introduction
A proposal for muon-proton scattering at PSI [1] has been made in an attempt to help resolve the proton radius puzzle. A measurement made on the basis of the Lamb Shift in muonic hydrogen [2] disagrees with the radius measured in atomic spectroscopy and electron scattering experiments [3, 4, 5]. The proposal will directly test whether or not \( \mu^-p \) and \( e^-p \) scattering are the same and will perform measurements with \( \mu^\pm \) and \( e^\pm \) at low \( Q^2 \) in order to study the two-photon exchange contributions in greater detail. Since the muon is about 206.7 times heavier than the electron [6], for the energies mentioned in the proposal, the muons are neither ultrarelativistic nor nonrelativistic. For the muon momenta given in the proposal the value of \( v/c \) for the incoming lepton is between 0.7 and 0.9, while the standard expressions for the scattering cross section are valid only for \( v/c \) very close to 1. The standard kinematics assumptions made in the analysis of e-p scattering will not all be valid in the case of an experiment on mu-p scattering. The cross sections have been calculated without these approximations [7, 8], but since these results seem to have been forgotten, it is worth presenting another calculation of the basic cross section without them.

According to the proposal, scattering of negative and positive muons (and electrons) will be studied. The muon momenta will be in the range (115-210) MeV/c with scattering angles in the range 20° to 100°, corresponding to \( Q^2 \) in the range (0.01-0.1) (GeV/c)^2. For comparison, \( m_\mu^2c^2 = 0.01116 \) (GeV/c)^2.
Numerical values for the muon energy \( E = \sqrt{p^2 + m^2} \) and velocity \( \beta = |\vec{p}/E| = v/c \) corresponding to the incoming momenta in the proposal are given by:

| \( p \) (MeV/c) | \( E \) (MeV) | \( \beta \) |
|----------------|--------------|----------|
| 115            | 156.17       | 0.7364   |
| 153            | 185.94       | 0.8229   |
| 210            | 235.08       | 0.8933   |

Obviously the approximation \( \beta \approx 1 \) is not valid for the energies considered in the proposal. The radiative corrections to the scattering cross section are functions of \( Q^2/m^2 \), which is in the range of approximately 0.9-9.0. The usual formulas \([10, 11, 12]\), which assume that \( Q^2/m^2 \gg 1 \), will not be accurate.

The convention of Bjorken and Drell \([9]\) will be used. The metric used is defined by \( p_i \cdot p_j = E_i E_j - \vec{p}_i \cdot \vec{p}_j \). Also \( m \) is the lepton rest mass, \( M \) is the target rest mass, and \( \alpha = e^2/4\pi \). Use \( p_1 \) and \( p_3 \) for the incoming and outgoing muon four-momenta, and \( p_2 \) and \( p_4 \) for the incoming and outgoing proton four-momenta, respectively. In the lab system we have \( p_1 = (E, \vec{p}) \), \( p_3 = (E', \vec{p}') \), \( p_2 = (M, 0) \), \( p_4 = (M + \omega, q') \).

Here \( q = p_1 - p_3 = p_4 - p_2 \), and \( \omega = q_0 = E - E' \). It is useful to observe that \( q^2 = 2m^2 - 2p_1 \cdot p_3 = 2M^2 - 2p_2 \cdot p_4 = -2M\omega \). This is simply a result of energy conservation. Since \( q^2 \) is negative with the metric used here, it is sometimes convenient to define \( Q^2 = -q^2 \). The proton current is taken to have the usual on-shell form, characterized by

\[
\Gamma_\mu = F_1(q^2)\gamma_\mu + \kappa F_2(q^2)\frac{i\sigma_\mu\nu q^\nu}{2M}
\]

Here \( \kappa \) is the anomalous magnetic moment of the proton. The so-called Sachs form factors are related to \( F_1 \) and \( F_2 \) by \( G_M = F_1 + \kappa F_2 \), \( G_E = F_1 - Q^2/(4M^2)\kappa F_2 \).

Or, \( F_1 = (G_E + Q^2/(4M^2)G_M)/(1 + Q^2/(4M^2)) \) and \( \kappa F_2 = (G_M - G_E)/(1 + Q^2/(4M^2)) \)

For the calculation of the matrix element, the Gordon decomposition \([9]\)

\[
\bar{u}(p_4)\frac{i\sigma_\mu\nu q^\nu}{2M}u(p_2) = \bar{u}(p_4)\left[\gamma_\mu - \frac{(p_2 + p_4)_\mu}{2M}\right]u(p_2)
\]

is very useful. As a result, one may write

\[
\bar{u}(p_4)\Gamma_\mu u(p_2) = \bar{u}(p_4)\left[(F_1 + \kappa F_2)\gamma_\mu - \kappa F_2\frac{(p_2 + p_4)_\mu}{2M}\right]u(p_2)
\]

**Scattering Cross Section**

Following Chap. 7 of Ref. \([9]\), the invariant matrix element for scattering of a charged lepton from a proton in Born approximation is given by

\[
\mathcal{M}_{fi} = \bar{u}(p_3)\gamma_\mu u(p_1)\frac{e^2}{q^2}\bar{u}(p_4)\Gamma_\mu u(p_2)
\]

The cross section for scattering of the charged lepton into a given solid angle \( d\Omega \) about an angle \( \theta \) is given by

\[
\frac{d\sigma}{d\Omega} = \frac{m^2 M}{4\pi^2} \frac{p'/p}{M + E - (pE'/p')\cos\theta} |\mathcal{M}_{fi}|^2
\]
In Eq. 3 it is assumed that $|\mathcal{M}_{fi}|^2$ has been averaged over initial spins and summed over final spins. The final result is given by

$$
\frac{d\sigma}{d\Omega'} = \frac{\alpha^2}{q^4} \frac{p'/p}{1 + (E - (pE'/p') \cos \theta)/M} \left[ G_E^2 \frac{4EE' + q^2}{1 - q^2/4M^2} + G_M^2 \left( (4EE' + q^2)(1 - \frac{1}{1 - q^2/4M^2}) + \frac{q^4}{2M^2} + \frac{q^2m^2}{M^2} \right) \right] \tag{4}
$$

Recall that generally $-q^2 = Q^2 = 2M(E - E') = 2(EE' - pp' \cos \theta - m^2)$.

If the limit of very high lepton energies, one has

$$p \approx E, \quad p' \approx E', \quad q^2 \approx -4EE' \sin^2(\theta/2)$$

In this case, the cross section given in Eq. 4 reduces to

$$
\frac{d\sigma}{d\Omega'} = \frac{\alpha^2 \cos^2(\theta/2)}{4E^2 \sin^4(\theta/2)} \frac{1}{1 + 2E \sin^2(\theta/2)/M} \left[ \frac{Q^2}{2M^2} G_M^2 \left( \frac{1}{1 + Q^2/4M^2} + 2 \tan^2(\theta/2) \right) + \frac{G_E^2}{1 + Q^2/4M^2} \right] \tag{5}
$$

This agrees with the expression given in Sec. 4 of ref. [13] (and with equivalent expressions in other work).

**Figure 1:** Ratio of the exact scattering cross section to the cross section calculated with the commonly used relativistic approximation.

Figure 1 shows the ratio of the cross section calculated with the exact formula (Eq. 4) to that calculated with Eq. 5 but with exact muon kinematics. The form factors were taken from a parametrization by [14]. The difference is significant, especially at lower values of incident momentum.
Radiative Corrections

Radiative corrections to the electromagnetic properties of a muon or an electron modify the lepton scattering cross sections (and also produce energy level shifts in atoms).

According to the standard theory, the cross section for scattering is altered by a factor $1 - \delta$ where, for potential scattering at high momentum transfer \cite{15},

$$\delta \approx \frac{2\alpha}{\pi} \left[ \ln(Q^2/m^2) - 1 \right] \ln(E/E) + \delta_{\text{vertex}} + \delta_{\text{VP}}$$

Here $\Delta E$ is the energy acceptance of the detector, and is related to the fact that the lepton can emit very low-energy photons that would not be detected. The cross section for this process (soft photon bremsstrahlung) is infrared divergent; the divergence is canceled by a similar infrared divergence in the vertex correction. Since then, the calculations have been extended by many authors \cite{10, 11, 12} to include target recoil, radiation from the target, the target vertex correction, and two-photon exchange.

The energy of the recoiling proton will be rather small ($< (50 - 60) \text{MeV}$) in this experiment, so that soft photon radiation from the proton will contribute very little to the radiative corrections. The contribution from two photon exchange is beyond the scope of this paper. For this reason, only the purely leptonic contributions to the radiative correction will be considered. In most of the literature \cite{10, 11, 12}, it has been assumed that $Q^2/m^2$ is very large. It turns out that when this is not the case (as in the proposed experiment), there will be an extra contribution to the radiatively corrected cross section arising from the fact that the lepton vertex correction gives rise to two form factors $F_1$ and $F_2$ ($F_2$ corresponds to the anomalous magnetic moment in the limit $Q^2 \to 0$). This does not simply multiply the uncorrected cross section. Exact expressions for these form factors have been given by Feynman \cite{16}, in two textbooks \cite{7, 17}, and more recently by Barbieri et al. \cite{18}. They all agree with each other, except for different notation. Here the notation used in Ref. \cite{18} will be used.

To lowest order in $\alpha$ the form factors are given by

$$F_1(t) \approx 1 + \frac{\alpha}{\pi} F_1^{(2)}(t)$$

$$F_2(t) \approx \frac{\alpha}{\pi} F_2^{(2)}(t)$$

As usual, $t = q^2 = (p_1 - p_3)^2$.

For spacelike $q^2 = -Q^2$ the form factors can be expressed in terms of a variable $\Theta$ that is related to the momentum transfer by $\sinh(\phi) = Q/(2m)$ and

$$\Theta = \frac{1 - \tanh(\phi)}{1 + \tanh(\phi)}$$

It is useful to note that $\ln(\Theta) = -2\phi$ and that $Q^2/m^2 = (1 - \Theta)^2/\Theta = 4 \sinh^2(\phi)$.
In terms of this variable, 

\[
F_1(t) - 1 = \frac{\alpha}{\pi} \left[ \ln \left( \frac{m^2}{\lambda^2} \right) \left( 1 + \frac{\Theta^2}{1 - \Theta^2} \ln(\Theta) \right) - 1 - \frac{3\Theta^2 + 2\Theta + 3}{4(1 - \Theta^2)} \ln(\Theta) + \frac{1 + \Theta^2}{1 - \Theta^2} \left( \frac{\pi^2}{12} + L_2(-\Theta) - \frac{1}{4} \ln^2(\Theta) + \ln(\Theta) \ln(1 + \Theta) \right) \right] 
\] (6)

(The infrared divergent part will be compensated by contributions from bremsstrahlung of soft photons.)

and

\[
F_2(t) = -\frac{\alpha}{\pi} \frac{\Theta}{1 - \Theta^2} \ln(\Theta)
\]

In the limit of very small \( Q^2 \), \( F_2(0) \rightarrow \frac{\alpha}{\pi} \left( \frac{1}{2} - \frac{Q^2}{12m^2} \right) \).

The value of \( F_2(0) \) is well known. At very high momentum transfers, \( F_2(t) \) becomes negligibly small:

\[
F_2(t) \rightarrow -\frac{\alpha m^2}{\pi Q^2} \ln(m^2/Q^2)
\]

In the limit of high momentum transfer, \( F_1(t) \) becomes 

\[
F_1(t) - 1 \rightarrow \frac{\alpha}{\pi} \left[ (1 - \ln(Q^2/m^2)) \ln(m/\lambda) - 1 + \frac{3}{4} \ln(Q^2/m^2) + \frac{\pi^2}{12} - \frac{1}{4} \ln^2(Q^2/m^2) \right]
\] (7)

For the momentum transfers of interest in this experiment, \( F_2(t) \) is comparable to the other radiative corrections, and it results in a nonmultiplicative contribution to the radiative corrections. A calculation of this contribution to the radiatively corrected cross section for lepton-proton scattering will be given further below. The muon current including (vertex) radiative corrections of order \( \alpha \) will have the form

\[
\bar{u}(p_3) \left[ (F_1 + F_2) \gamma_\mu - F_2 \frac{(p_1 + p_3)_\mu}{2m} \right] u(p_1)
\]

where the Gordon decomposition has been used. The radiative corrections also include contributions from vacuum polarization, so that in the matrix element for scattering the contributions from muon and electron loops must be added to the term proportional to \( \gamma_\mu \). These are:

\[
\frac{\alpha}{3\pi} \left[ \frac{1}{3} + \left( \coth^2(\phi) - 3 \right)(1 - \phi \cdot \coth(\phi)) \right] = \frac{\alpha}{\pi} U_{2\mu}
\]

for muon loops and

\[
\frac{\alpha}{3\pi} \left[ \ln \left( \frac{Q^2}{m_e^2} \right) - \frac{5}{3} \right] = \frac{\alpha}{\pi} U_{2e}
\]

for electron loops. Note that for the contribution due to electron loops, one may use the usual high momentum transfer approximation.

The lepton trace will be evaluated to leading order in \( \alpha/\pi \), so that \( F_2^2 \approx 0 \) and

\[
(F_1 + F_2)^2 \approx 1 + \frac{2\alpha}{\pi} (F_1^{(2)} + F_2^{(2)}), \quad F_1 F_2 \approx \frac{\alpha}{\pi} F_2^{(2)}
\]

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Inclusion of the vacuum polarization contributions replaces \( F_1^{(2)} + F_2^{(2)} \) by \( F_1^{(2)} + F_2^{(2)} + U_{2m} + U_{2e} \).

A possible contribution to the vacuum polarization from hadronic loops has not been included. The earliest calculations of this contribution to the energy levels of muonic atoms \([20, 21]\) have indicated that, at least in the limit of small \( Q^2 \), this contribution is approximately 0.66 times the contribution from muon loops. Generally, it would be equal to \((2\alpha/\pi)\Pi_H(Q^2)\), where \( \Pi_H(Q^2) \) is defined in Ref. \([20]\). Since these papers were written, much better data for the cross section for \( e^+e^- \rightarrow \text{hadrons} \) have become available, so that a recalculation of \( \Pi_H(Q^2) \) as a function of \( Q^2 \) would be desirable. The same cross section enters into the correction to the anomalous magnetic moment of the muon.

One can show that the infrared divergent contribution to the real soft-photon bremsstrahlung cross section \( d\sigma_b \) given in Eq.(4.7) of Ref. \([12]\) (the coefficient of \( \ln(2\Delta E/\lambda) \)) is

\[
-\frac{2\alpha}{\pi} \ln(2\Delta E/\lambda) \left[ 1 - 2\phi \coth(2\phi) \right]
\]

where \( d\sigma_0 \) is the uncorrected cross section and \( \Delta E \) is the energy resolution of the detector. Since

\[
1 + \frac{1-e^{\Theta^2}}{\Theta} \ln(\Theta) = 1 - 2\phi \coth(2\phi)
\]

one can combine this with the infrared divergent part of Eq.(6) to obtain a contribution \( \delta_{\text{rad}} \) to the radiative correction:

\[
\delta_{\text{rad}} = -\frac{2\alpha}{\pi} \left[ 1 - 2\phi \coth(2\phi) \right] \ln(2\Delta E/m)
\]

Other noninfrared divergent contributions will also contribute; in the limit of large values of \( Q^2/m^2 \), these were given in Eq.(4.14) of Ref. \([12]\) and it can be seen that the contribution proportional to \( \ln^2(Q^2/m^2) \) in Eq.(7) is also canceled by some of these terms. This will probably happen also for the present case.

The additional contribution to the radiatively corrected cross section due to the presence of \( F_2(Q^2) \) will now be given. For this the revised lepton trace (compare with Eq.(10)) is needed. It is given by

\[
\frac{(F_1 + F_2)^2}{m^2} \left[ p_3^\mu p_1^\nu + p_1^\mu p_3^\nu + g^{\mu\nu} q^2/2 \right] - \frac{F_1 F_2}{m^2} (p_1 + p_3)^\mu (p_1 + p_3)^\nu
\]

The term with \((F_1 + F_2)^2 \approx 1\) corresponds to the uncorrected cross section and the term of order \( \alpha/\pi \) will contribute a factor multiplying the uncorrected cross section, thus giving \( \delta_{\text{el}} = \delta_{\text{vertex}} + \delta_{\text{VP}} \). Here \( \delta_{\text{vertex}} = 2(F_1 + F_2 - 1) \) (Since the infrared divergent part of \( F_1 \) will be canceled by contributions from radiation of real soft photons by the muon, it is not included in any of the numerical examples.) and \( \delta_{\text{VP}} = (2\alpha/\pi)(U_{2m} + U_{2e}) \). For the limiting case of very high values of \( Q^2/m^2 \), the expression for \( \delta_{\text{el}} \) agrees with the corresponding expression in Eq.(3.36) of Ref. \([12]\) The contributions to \( \delta_{\text{el}} \) from the vertex correction and from vacuum polarization (without the contribution from radiation of real (soft) photons) were also given (with a different, but equivalent notation) in Chap.51 of Ref. \([17]\). The presence of an additional correction, not proportional to the uncorrected cross section, was also mentioned in that work, but only in connection with scattering from a Coulomb potential. Hence only the contribution of the last term in the lepton trace (proportional to \((p_1 + p_3)^\mu (p_1 + p_3)^\nu\)) will have to be calculated for the case of muon-proton scattering.
The additional term (which will be multiplied by \( F_2 \)) will now be calculated. The proton trace is evaluated in the Appendix and is equal to

\[
\frac{G_M^2}{m^2} [p_2 \cdot p_4 + p_2 p_4 \mu + g^\mu \nu q^2 / 2] \\
+ \frac{G_E^2 - G_M^2}{2M^2(1 - q^2/4M^2)} (p_2 + p_4)(p_2 + p_4)(\frac{1}{M^2}(p_2 \cdot p_4 + M^2)
\]

The additional term in the spin-averaged square of the matrix element \(|\mathfrak{M}_{fi}|^2\) is then (without the factor \(-F_2\))

\[
\frac{4\pi^2\alpha^2}{m^2M^2(q^2)^2} \left[ G_M^2 \left[ 2p_2 \cdot (p_1 + p_3)p_4 \cdot (p_1 + p_3) + (p_1 + p_3)^2q^2 / 2 \right] \\
+ \frac{G_E^2 - G_M^2}{2(1 - q^2/4M^2)} ((p_1 + p_3) \cdot (p_2 + p_4))^2 \right]
\]

Evaluation of the scalar products gives finally

\[
\frac{4\pi^2\alpha^2}{m^2(q^2)^2} \left[ G_M^2 \left[ 2(E + E')^2 + (4m^2 - q^2)q^2 / 2M^2 \right] + 2 \frac{G_E^2 - G_M^2}{(1 - q^2/4M^2)} (E + E')^2 \right]
\]

As in Eq. [3] to obtain the contribution to the radiatively corrected cross section, this will be multiplied by \( \frac{m^2}{4\pi^2} \frac{p/p'}{1 + (E - (pE'/p')\cos\theta)/M} \) to obtain the cross section corresponding to the additional term. This will be denoted by \( d\sigma_1 \) while the cross section given in Eq. [4] will be denoted by \( d\sigma_0 \).

Figure 2 shows the ratio of the additional contribution to the radiatively corrected cross section \((F_2 \times d\sigma_1)\) to the usual contribution to the radiatively corrected cross section from the noninfrared part of the vertex correction \((2(F_1 + F_2) \times d\sigma_0)\) (solid (red) curve) for an initial muon momentum of 153 MeV/c. The dashed (green) curve shows a similar ratio, but including the contribution due to vacuum polarization (muon and electron loops). The contribution from the electron loops is significantly larger than the other contributions. For the momentum transfers of interest, the nonmultiplicative contribution to the radiative corrections is not negligible. However, the calculation of the additional correction is straightforward.

It is possible to include the contribution of \( F_2 \) in a multiplicative contribution to \( \delta \), but this would depend on the nucleon form factors in a rather complicated manner. In this case \( \delta_{\text{vertex}} = 2(F_1 + F_2 - 1) \) would be replaced by

\[
\delta_{\text{vertex}} = 2(F_1 - 1) - 2F_2 Q^2 \frac{1 - Q^2/4M^2}{1 + Q^2/4M^2} (G_E^2 + \frac{Q^2}{4M^2} G_M^2)
\]

\[
\cdot \left[ (4EE' - Q^2)(G_M^2 + \frac{G_E^2 - G_M^2}{1 + Q^2/4M^2}) + \frac{Q^2}{2M^2}(Q^2 - 2m^2)G_M^2 \right]^{-1}
\]

in order to include the additional term in \( \delta_{\text{el}} \) rather than treating it separately.
Figure 2: Ratio of the additional contribution to the radiatively corrected cross section to contributions to the usual radiatively corrected cross section

The table below gives the relative magnitude of the different contributions for a few cases. Recall that by definition, the contributions to lowest order in $\alpha$ are: $F_1 - 1 = (\alpha/\pi)F_1^{(2)}$, $F_2 = (\alpha/\pi)F_2^{(2)}$ and $\delta_{VP} = (2\alpha/\pi)(U_{2m} + U_{2e})$. Only the noninfrared part of $F_1^{(2)}$ is given in this table. The vacuum polarization contribution from electron loops is significantly larger than the other contributions. To check whether higher order electron vacuum polarization (corresponding to two loops) might be important, the value of $(\alpha/\pi)U_{2e}$ was also calculated.

| $\theta$ (°) | $Q^2 (GeV/c)^2$ | $F_1^{(2)}$ | $F_2^{(2)}$ | $U_{2m}$ | $U_{2e}$ | $(\alpha/\pi)U_{4e}$ |
|--------------|----------------|-------------|-------------|---------|---------|-------------------|
| 30           | 0.00611        | 0.0564      | 0.4588      | 0.0345  | 2.7582  | 0.0266            |
| 60           | 0.02133        | 0.1256      | 0.3841      | 0.1069  | 3.2146  | 0.0337            |
| 90           | 0.03907        | 0.1394      | 0.3259      | 0.1746  | 3.4164  | 0.0377            |
| 120          | 0.05394        | 0.1240      | 0.2908      | 0.2221  | 3.5239  | 0.0401            |

It turned out to be comparable to the vertex contributions. This indicates that the contribution from fourth order electron vacuum polarization might be comparable to other contributions. The expression for the complete contribution to this order has been given in Eq. (140) of Ref. [6] (see also Ref. [19]). In the limit $Q^2 / m_e^2 \gg 1$, this results in an additional contribution to $\delta_{VP}$ of $2(\alpha/\pi)^2 U_{4e}$ where

$$U_{4e} = U_{2e}^2 + U_{2e}U_{2m} - \frac{5}{24} + \zeta(3) + \frac{1}{4} \ln\left(\frac{Q^2}{m_e^2}\right)$$

To compare this with the other contributions, the last column gives also $(\alpha/\pi)U_{4e}$. Other higher order contributions are probably negligible.
The radiative corrections calculated for this example (and hence for the proposed experiment) are somewhat smaller than the radiative corrections encountered in electron scattering at higher momentum transfer. For an incoming muon momentum of 153 MeV/c and scattering angle of 60° the lowest order electron vacuum polarization contributes 1.49% to the correction; including the noninfrared part of the vertex correction, vacuum polarization due to muon loops, and fourth order electron vacuum polarization gives a leptonic contribution to $\delta_{el}$ of 1.80%. The correction varies from 1.57% to 1.93% as the scattering angle varies from 30° to 120°.

The extra contribution to the radiatively corrected cross section ($-F_2 \times d\sigma_1/d\Omega$) for this example was $-0.000078 \times 10^{-24}\,\text{cm}^2/\text{steradian}$ while the usual contribution to the radiatively corrected cross section from the noninfrared part of the vertex correction ($\delta_{el} \times d\sigma_0/d\Omega$) is $-0.000671 \times 10^{-24}\,\text{cm}^2/\text{steradian}$. The extra contribution is about 11% of the standard contribution.

Of course, the contribution from soft photon radiation, as well as from the proton vertex correction, and two-photon corrections should also be included. Since these depend on the proton structure, they should probably be given separately.

In conclusion, the radiative corrections for the experiment proposed in Ref. [1] can be expected to be smaller (only a few percent) than is the case for electron scattering at higher momentum transfers. They are dominated by vacuum polarization with electron loops. The fourth order electron vacuum polarization is comparable to the second order muon vacuum polarization and vertex corrections, and should not be ignored. There is an extra, nonmultiplicative contribution to the radiative corrections. It can be as large as 15% of the standard (multiplicative) radiative correction.

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Appendix

Here a few details of the calculation of the average over initial spins and sum over final spins of $|\mathcal{M}_{fi}|^2$ are given. $\mathcal{M}_{fi}$ is given by Eq. 2.

Following Chap. 7 of Ref. [9] the spin sum and average of $|\mathcal{M}_{fi}|^2$ is given by

$$|\mathcal{M}_{fi}|^2 = \frac{e^4}{4(q^2)^2} \cdot Tr\left[\left(\frac{p_3 + m}{2m}\right)\gamma^\mu \left(\frac{p_4 + m}{2m}\right)\gamma^\nu\right] \cdot Tr\left[\left(\frac{p_1 + M}{2M}\right)\Gamma^\mu \left(\frac{p_2 + M}{2M}\right)\Gamma^\nu\right]. \quad (9)$$

The lepton trace is

$$\frac{1}{m^2}[p_3^\mu p_4^\nu + p_1^\mu p_2^\nu + g^\mu\nu q^2 / 2] \quad (10)$$

The trace for the proton is

$$G_M^2 \frac{1}{2M} \left[\left(\frac{p_1 + M}{2M}\right)\gamma^\mu \left(\frac{p_2 + M}{2M}\right)\gamma^\nu + \left(\frac{\kappa F_2}{2M}\right)^2 (p_2 + p_4)_{\mu}(p_2 + p_4)_{\nu} Tr\left(\frac{p_1 + M}{2M}\right)\gamma^\nu \right] \quad (11)$$

where Eq. 1 has been used.

Evaluation of the traces gives

$$\frac{G_M^2}{M^2} [p_{2\mu} p_{4\nu} + p_{2\nu} p_{4\mu} + g^\mu\nu q^2 / 2] + \left(\frac{\kappa F_2}{2M}\right)^2 (p_2 + p_4)_{\mu}(p_2 + p_4)_{\nu} \frac{1}{M^2}(p_2 \cdot p_4 + M^2) \quad (12)$$

The terms involving $\kappa F_2 = (G_M - G_E)/(1 - q^2/4M^2)$ are

$$\frac{1}{M^2}(p_2 + p_4)_{\mu}(p_2 + p_4)_{\nu}\left[-G_M \kappa F_2 + \left(\frac{\kappa F_2}{2M}\right)^2 (1 - q^2/4M^2)\right]$$

In terms of the Sachs form factors

$$\left(\frac{\kappa F_2}{2M}\right)^2 (1 - q^2/4M^2) - G_M \kappa F_2 = \frac{G_E^2 - G_M^2}{2(1 - q^2/4M^2)}$$

The square of the matrix element becomes

$$|\mathcal{M}_{fi}|^2 = \frac{4\pi^2 \alpha^2}{m^2 M^2(q^2)^2} \left[G_M^2 (2p_1 \cdot p_2 p_3 \cdot p_4 + 2p_1 \cdot p_4 p_3 \cdot p_2 + (m^2 + M^2)q^2)\right] + \frac{G_E^2 - G_M^2}{2(1 - q^2/4M^2)}(2p_1 \cdot (p_2 + p_4)p_3 \cdot (p_2 + p_4) + q^2(p_2 + p_4) \cdot (p_2 + p_4)/2)$$

$$= \frac{4\pi^2 \alpha^2}{m^2 M^2(q^2)^2} \left[G_M^2 (4M^2 E E' + (M^2 + m^2)q^2 + q^4 / 2) + \frac{G_E^2 - G_M^2}{1 - q^2/4M^2} (4M^2 E E' + M^2 q^2)\right] \quad (13)$$