ABSTRACT. A groupoid satisfying the left invertive law: \( ab \cdot c = cb \cdot a \) is called an AG-groupoid and is a generalization of commutative semigroups. We consider the concept of bi-commutativity in AG-groupoids and thus introduce left commutative AG-groupoids, right commutative AG-groupoids and bi-commutative AG-groupoids. We provide a method to test an arbitrary Cayley’s table for these AG-groupoids and explore some of the general properties of these AG-groupoids.

1. INTRODUCTION

An AG-groupoid \( S \) is in general a nonassociative groupoid that satisfies the left invertive law, \( ab \cdot c = cb \cdot a \). \( S \) is called medial if it satisfies the medial property, \( ab \cdot cd = ac \cdot bd \). It is easy to prove that every AG-groupoid is medial. \( S \) is called an AG-monoid if it contains the left identity element. Every AG-monoid is paramedial \[^{10}\] \(^{1}\), i.e., it satisfies the identity, \( ab \cdot cd = db \cdot ca \). Some of the well-known AG-groupoids with their identities are listed in Table 2. Recently many new classes of AG-groupoids have been introduced by various researchers\[^{9, 11, 12, 13}\]. These new classes are studied in a variety of papers like for instance\[^{4, 5, 6, 7, 14}\]. A groupoid \( G \) is called left (resp. right) commutative groupoid if \( G \) satisfies the identity \((ab)c = (ba)c \) (resp. \( a(bc) = a(cb) \)) \( \forall a, b, c \in G \). In this article we extend the concept of these groupoids to introduce some new classes of AG-groupoids, that we call a left commutative AG-groupoid or shortly an LC-AG-groupoid, a right commutative AG-groupoid or an RC-AG-groupoid, and a bi-commutative AG-groupoid or BC-AG-groupoid. We use the GAP \[^{2}\] software and the relevant data of \[^{3}\] to enumerate our new classes of AG-groupoids up to order 6. The following table illustrates nonassociative enumeration of these classes up to order 6.

**Key words and phrases.** AG-groupoids, AG-test, nuclear square AG-groupoids, transitive groupoids, alternative AG-groupoids.

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An AG-groupoid is a groupoid satisfying the left invertive law,
\[(1.1) \quad ab \cdot c = cb \cdot a.\]

We investigate our newly introduced classes of AG-groupoids and find some relations of these classes with already known classes of AG-groupoids\[8,15,16\]. We list some of the already known classes of AG-groupoids with their defining identities that will be used in the rest of this article.

| AG-groupoid                          | satisfying identity         |
|-------------------------------------|-----------------------------|
| Left nuclear square AG-groupoid     | \(a^2(bc) = (a^2b)c\)       |
| Middle nuclear square AG-groupoid   | \((ab^2)c = a(b^2c)\)       |
| Right nuclear square AG-groupoid    | \((ab)c^2 = a(bc^2)\)       |
| TL-AG-groupoid                      | \(ab = cd \Rightarrow ba = dc\) |
| Medial AG-groupoid                  | \(ab \cdot cd = ac \cdot bd\) |
| Paramedial AG-groupoid              | \(ab \cdot cd = db \cdot ca\) |
| Flexible -AG-groupoid               | \(ab \cdot a = a \cdot ba\) |
| AG-3-band-AG-groupoid               | \(a(aa) = (aa)a = a\)       |
| Left alternative AG-groupoid        | \(aa \cdot b = a \cdot ab\)  |
| Self-dual AG-groupoid               | \(a(bc) = c(ba)\)           |

| Order | 3  | 4  | 5  | 6  |
|-------|----|----|----|----|
| Nonassociative AG-groupoids         | 8  | 269| 31467| 40097003|
| LC-AG-groupoids                     | 6  | 194| 22276| 34845724|
| RC-AG-groupoids                     | 2  | 52 | 1800 | 170977  |
| BC-AG-groupoid                      | 2  | 47 | 1558 | 150977  |

Table 1. Enumeration of BC-AG-groupoids up to order 6.

2. Bi-commutative-AG-groupoids

**Definition 1.** An AG-groupoid \(S\) is called —
(a) — a left commutative AG-groupoid (LC-AG-groupoid) if \(\forall a,b,c \in S,\)
\[(2.1) \quad (ab)c = (ba)c\]
(b) — a right commutative AG-groupoid (RC-AG-groupoid) if \(\forall a,b,c \in S,\)
\[(2.2) \quad a(bc) = a(cb)\]
(c) — a bi-commutative AG-groupoid (BC-AG-groupoid) if it is both LC-AG-groupoid and an RC-AG-groupoid.
Example 1. We list some examples of smallest order of these AG-groupoids to show their existence. (i) LC-AG-groupoid of order $3$. (ii) RC-AG-groupoid of order $3$. (iii) BC-AG-groupoid of order $3$.

$\begin{array}{ccc}
* & 1 & 2 & 3 \\
1 & 1 & 1 & 1 \\
2 & 1 & 1 & 1 \\
3 & 2 & 1 & 1 \\
\end{array}$  

(i)

$\begin{array}{ccc}
* & 1 & 2 & 3 \\
1 & 1 & 1 & 1 \\
2 & 1 & 1 & 1 \\
3 & 2 & 2 & 1 \\
\end{array}$  

(ii)

$\begin{array}{ccc}
* & 1 & 2 & 3 \\
1 & 1 & 1 & 1 \\
2 & 1 & 1 & 1 \\
3 & 2 & 2 & 2 \\
\end{array}$  

(iii)

3. Left Commutative AG-Test

Testing a groupoid for an AG-groupoid has been well explained by P. V. Protic and N. Stevanovic[1]. In this section we discuss a procedure to test whether an arbitrary AG-groupoid $(G, \cdot)$ is an LC-AG-groupoid or not. For this we define the following binary operations;

$\begin{align*}
(3.1) & \hspace{1cm} a \circ b = ab \cdot x \\
(3.2) & \hspace{1cm} a \star b = ba \cdot x 
\end{align*}$

Now (2.1) holds if,

$\begin{align*}
(3.3) & \hspace{1cm} a \circ b = a \star b \\
\end{align*}$

or

$\begin{align*}
(3.4) & \hspace{1cm} a \circ b = b \circ a \\
\end{align*}$

To test whether an arbitrary AG-groupoid is an LC-AG-groupoid, it is necessary and sufficient to check if the operation “$\circ$” and “$\star$” coincide $\forall x \in G$. To this end we check the validity of equation (2.1) or $a \circ b = a \star b$. In other words it is enough to check whether the operation $\circ$ is commutative i.e. $a \circ b = b \circ a$. The tables of the operation “$\circ$” for any fixed $x \in G$ is obtained by multiplying a fixed element $x \in G$ by the elements of the “$\cdot$” table row-wise. It further gives the tables of the operation “$\star$” if these are symmetric along the main diagonal. Hence it could be easily checked whether an arbitrary AG-groupoid is left commutative AG-groupoid or not.

Example 2. Check the following AG-groupoid $(G, \cdot)$ for an LC-AG-groupoid.
We can extend Table 1 in the way as described above. It is obvious that the tables constructed for the operation “◦” on the right of the original table are symmetric about the main diagonal and thus coincide with the “⋆” tables as required. Hence \((G, ·)\) is an LC-AG-groupoid.

\[
\begin{array}{ccc}
1 & 2 & 3 \\
1 & 1 & 1 \\
2 & 1 & 1 \\
3 & 2 & 2 \\
\end{array}
\]

Table 1

4. right commutative AG-Test

In this section we discuss a procedure to check an AG-groupoid \((G, ·)\) for RC-AG-groupoid, for this we define the following binary operations;

\[
\begin{array}{c}
\text{(4.1)} \\
\text{a} \heartsuit \text{b} = \text{a} \cdot \text{bx} \\
\text{(4.2)} \\
\text{a} \lozenge \text{b} = \text{a} \cdot \text{xb} \\
\end{array}
\]

(2.2) holds if,

\[
\text{(4.3)} \\
\text{a} \heartsuit \text{b} = \text{a} \lozenge \text{b}
\]

For any fixed \(x \in G\), re-writing \(x\)-row of the “·” table as an index row of the new table and multiplying it by the elements of the index column to construct table of operation “◊”. These extended tables are given to the right of the original table in the following example.

Similarly the table for the operation “◇” for any fixed \(x \in G\) is obtained by taking the elements of \(x\)-column of the “·” table as an index row of the new table and multiplying it by the elements of the index column of the original table to construct tables for the operation “◇”, which are given downward in the extended table of the following example. If the tables for the operation “◇” and “◊” coincides for all \(x \in G\), then \((\text{4.3})\) holds and the AG-groupoid is left commutative-AG-groupoid in this case.

Example 3. Check the following AG-groupoid for BC-AG-groupoid.
Extend the above table in the way as described above we get the extended form as follow:

\[
\begin{array}{ccc|cccc}
\cdot & 1 & 2 & 3 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 1 & 1 & 1 & 1 & 1 & 1 \\
3 & 2 & 2 & 2 & 2 & 2 & 2 \\
\end{array}
\]

It is clear from the extended table that the tables for the operations "\(\heartsuit\)" and "\(\diamondsuit\)" coincide for every \(x \in G\), so \((G, \cdot)\) is an RC-AG-groupoid.

5. ideals in LC-AG-groupoids and RC-AG-groupoids

A subset \(A\) of the AG-groupoid \(S\) is a left (right) ideal of \(S\) if,

\[
(5.1) \quad SA \subseteq A(AS \subseteq A)
\]

\(A\) is a two sided ideal or simply an ideal of \(S\) if it is both left and right ideal of \(S\).

**Remark 1.** If \(S\) is an AG-groupoid and \(a \in S\), then by the identity \(1.1\), it follows that:

\[
(aS)S = \bigcup_{x,y \in S} (ax)y = \bigcup_{x,y \in S} (yx)a \subseteq Sa.
\]

From this we conclude that \((AS)S \subseteq SA\).

Further we have the following remarks for our new classes of AG-groupoids.

**Remark 2.** If \(S\) is an AG-groupoid with left identity \(e\) and \(a \in S\), then by the medial property and \(2.2\), it follows that:

\[
S(aS) = \bigcup_{x,y \in S} x(ay) = \bigcup_{x,y \in S} (ex)(ay) = \bigcup_{x,y \in S} (ea)(xy) \subseteq aS.
\]

In general for any \(A \subseteq S\) we conclude that \(S(AS) \subseteq AS\).
Remark 3. If $S$ is an LC-AG-groupoid and $a \in S$, then by (1.1 and 2.1), it follows that:

$$(Sa)S = \bigcup_{x,y \in S} (xa)y = \bigcup_{x,y \in S} (ax)y = \bigcup_{x,y \in S} (yx)a \subseteq Sa.$$  

Thus for any $A \subseteq S$ we conclude that $(SA)S \subseteq SA$.

Remark 4. If $S$ is an RC-AG-groupoid with left identity $e$ and $a \in S$, then by 1.1 and 2.2, it follows that:

$$S(Sa) = \bigcup_{x,y \in S} x(ya) = \bigcup_{x,y \in S} (ex)(ya) = \bigcup_{x,y \in S} (ax)(ey) = \bigcup_{x,y \in S} (ea)(xy) \subseteq aS.$$  

Hence in general $S(SA) \subseteq AS$ for $A \subseteq S$.

Remark 5. If $S$ is an RC-AG-groupoid with left identity $e$ and $a \in S$, then by medial property and 2.1, it follows that:

$$(Sa)S = \bigcup_{x,y \in S} (xa)y = \bigcup_{x,y \in S} (xa)(ey) = \bigcup_{x,y \in S} (ex)(ye)$$

$$= \bigcup_{x,y \in S} (xy)(ae) = \bigcup_{x,y \in S} (xy)(ea) \subseteq Sa.$$  

Thus $(SA)S \subseteq SA$ for $A \subseteq S$.

Definition 2. Let $S$ be an AG-groupoid and $A, B \subseteq S$, then $A$ and $B$ are right (left) connected sets if $AS \subseteq B$ and $BS \subseteq A$ ($SA \subseteq B$ & $SB \subseteq A$).

Remark 6. If $L$ is a left and $R$ is a right ideal of an LC-AG-groupoid $S$ then by (2.1 and 5.1) and left invertive law, we have

$$(LR)S = (SR)L = (RS)L \subseteq RL \text{ and } (RL)S = (SL)R \subseteq LR.$$  

It follows that $LR$ and $RL$ are right connected sets.

Proposition 1. Let $S$ be an LC-AG-groupoid then for each $a \in S$ the set $a \cup aS$ and $aS \cup Sa$ are right connected sets.

Proof. If $a \in S$, then by Remarks 1 and 3, we have $(a \cup aS)S = aS \cup (Sa)S \subseteq aS \cup Sa$, also,

$$(aS \cup Sa)S = (aS)S \cup (Sa)S \subseteq Sa \cup Sa \subseteq a \cup Sa.$$  

Hence the result follows.

Proposition 2. Let $S$ be an AG-groupoid and let $A$ and $B$ are right connected sets, then $A \cap B$ (if $A \cap B \neq \emptyset$) and $A \cup B$ are right ideals of $S$. 

Proof. We have,
\[(A \cap B) S \subseteq AS \cap BS \subseteq B \cap A\]
\[(A \cup B) S = AS \cup BS \subseteq B \cup A.\]
Hence the result follows. ■

**Theorem 1.** Let \(S\) be an LC-AG-groupoid then for each \(a \in S\) the set \(a \cup aS \cup Sa\) is right ideal of \(S\).

**Proof.** Let \(a \in S\), then by Remarks (1-3), we have
\[(a \cup aS \cup Sa) S = aS \cup (aS)S \cup (Sa)S \subseteq aS \cup Sa \cup Sa \subseteq aS \cup Sa \subseteq a \cup aS \cup Sa.\]
Hence the result follows. ■

**Theorem 2.** Let \(S\) be an LC-AG-groupoid then for each \(a \in S\) the set \(R(a) = (aS)a\) is the minimal right ideal of \(S\) containing \(a\).

**Proof.** Let \(a \in S\), then by medial property and (1.1 and 2.1), we have
\[
((aS)a) S = \bigcup_{x,y \in S} ((ax)a) y = \bigcup_{x,y \in S} (ya)(ax) = \bigcup_{x,y \in S} (ay)(ax) = \bigcup_{x,y \in S} (a \cdot yx)a = (aS)a.
\]
Hence, \(R(a)\) is a right ideal of \(S\). Now, let \(R\) be a right ideal of \(S\) such that \(a \in R\). Then we have
\[R(a) = (aS)a \subseteq (RS)R \subseteq RR \subseteq R.\]
So \(R(a)\) is a minimal right ideal of \(S\) containing \(a\). In general, if \(A \subseteq S\). Then \((AS)A\) is a minimal right ideal generated by \(A\). ■

**Theorem 3.** Let \(S\) be an RC-AG-groupoid with left identity \(e\) then for each \(a \in S\) the set \(J(a) = a \cup aS \cup Sa\) is the minimal (two sided) ideal of \(S\) containing \(a\).

**Proof.** By Proposition 2 and Remarks (2 and 4), we have
\[S(a \cup aS \cup Sa) = Sa \cup (aS) \cup S(Sa) \subseteq Sa \cup aS \cup aS \subseteq aS \cup Sa \subseteq a \cup aS \cup Sa.\]
Thus \(J(a)\) is a left ideal. Now again by Proposition 2 and Remarks (1 and 3), we have
\[(a \cup aS \cup Sa) S = aS \cup (aS)S \cup (Sa)S \subseteq aS \cup Sa \cup Sa \subseteq aS \cup Sa \subseteq a \cup aS \cup Sa.\]
Thus \(J(a)\) is a right ideal, and hence it is a two sided ideal or simply an ideal of \(S\). If \(J\) is an ideal of \(S\) and \(a \in J\) then
\[J(a) = a \cup (aS \cup Sa) \subseteq J \cup (JS \cup SJ) \subseteq J \cup (J \cup J) \subseteq J \Rightarrow J(a) \subseteq J.\]
Hence the result follows. ■
Theorem 4. If \( S \) is an RC-AG-groupoid with left identity \( e \) then for \( a \in S \), the sets \( a(Sa) \) and \( (aS)a \) are ideals of \( S \). If \( a \in a(Sa) \) (resp. \( a \in (aS)a \)) then \( a(Sa) \) (resp. \( (aS)a \)) is a minimal ideal generated by \( a \). Further if \( a \in a(Sa) \cap (aS)a \), then \( (aS)a = a(Sa) \) and it is minimal ideal generated by \( a \).

Proof. If \( a \in S \), then by the paramedial property \((1.1)\) and \((2.2)\) we have
\[
S(a(Sa)) = \bigcup_{x,y \in S} x(a(ya)) = \bigcup_{x,y \in S} (ex)(a \cdot ya) = \bigcup_{x,y \in S} (ea)(x \cdot ya) = \\
= \bigcup_{x,y \in S} a(x \cdot ay) = \bigcup_{x,y \in S} a(ay \cdot x) = \bigcup_{x,y \in S} a(xy \cdot a) \subseteq a(Sa).
\]
Similarly,
\[
(a(Sa))S = \bigcup_{x,y \in S} (a(xa))y = \bigcup_{x,y \in S} (y \cdot xa)a = \bigcup_{x,y \in S} (y \cdot ax)a = \\
= \bigcup_{x,y \in S} (ya)(ea) = \bigcup_{x,y \in S} (e \cdot ax)(ay) = \\
= \bigcup_{x,y \in S} (ea)(ax \cdot y) = \bigcup_{x,y \in S} a(xy \cdot a) \subseteq a(Sa)
\]
Hence, \( a(Sa) \) is an ideal of \( S \).

Now, again using the paramedial law, and \((1.1)\) and \((2.2)\) we have
\[
S((aS)a) = \bigcup_{x,y \in S} x(ay \cdot a) = \bigcup_{x,y \in S} (ex)(ay \cdot a) = \bigcup_{x,y \in S} (e \cdot ay)(xa) = \\
= \bigcup_{x,y \in S} (ya)(ea) = \bigcup_{x,y \in S} (x \cdot ay)(ea) = \\
= \bigcup_{x,y \in S} (ea \cdot xa)a = \bigcup_{x,y \in S} a(xy \cdot a) \subseteq (aS)a
\]
Similarly,
\[
S((aS)a) = \bigcup_{x,y \in S} (ax \cdot ay) = \bigcup_{x,y \in S} ya \cdot ax = \bigcup_{x,y \in S} yx \cdot aa = \\
= \bigcup_{x,y \in S} (y \cdot yx)(aa) = \bigcup_{x,y \in S} (a \cdot yx)(ae) = \bigcup_{x,y \in S} (a \cdot yx)(ea) \subseteq (aS)a.
\]
Hence \( (aS)a \) and \( a(Sa) \) are ideals of \( S \). If \( A \) is an ideal on \( S \), then for every \( a \in A \) we have \( (aS)a \subseteq A \) and \( a(Sa) \subseteq A \), clearly. If \( a \in A \cap (aS)a \) (resp. \( a \in A \cap a(Sa) \)), then \( (aS)a \) (resp. \( a(Sa) \)) is a minimal ideal generated by \( a \). If \( a \in A \cap (aS)a \cap a(Sa) \), then by minimality, it follows that \( (aS)a = a(Sa) \). Clearly, for each \( a \in S \) it holds that \( (aS)a \subseteq Sa \) and \( a(Sa) \subseteq Sa \).

\[\boxed{\text{6. characterization of BC-AG-groupoid}}\]

In this section we discuss the relations of BC-AG-groupoid with already known classes of AG-groupoids. We start with the following results which proves that every AG\(^*\)-groupoid is RC-AG-groupoid, but not an LC-AG-groupoid as in Example 4.
Theorem 5. Every AG*-groupoid is RC-AG-groupoid.

Proof. Let $S$ be an AG*-groupoid, and $a, b, c \in S$. Then $a(bc) = (ba)c = (ca)b = a(cb) \Rightarrow a(bc) = a(cb)$. Hence $S$ is RC-AG-groupoid.

The following example shows that every AG*-groupoid is not an LC-AG-groupoid.

Example 4. Let $S = \{1, 2, 3, 4, 5, 6\}$ then it is easy to verify that $S$ is an AG*-groupoid, but not an LC-AG-groupoid as, $(1 \ast 2) \ast 1 \neq (2 \ast 1) \ast 1$.

\[
\begin{array}{c|ccccccc}
\ast & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
1 & 3 & 4 & 5 & 5 & 5 & 5 \\
2 & 3 & 4 & 6 & 6 & 5 & 5 \\
3 & 5 & 5 & 5 & 5 & 5 & 5 \\
4 & 6 & 6 & 5 & 5 & 5 & 5 \\
5 & 5 & 5 & 5 & 5 & 5 & 5 \\
6 & 5 & 5 & 5 & 5 & 5 & 5 \\
\end{array}
\]

Now we prove the following:

Theorem 6. Every LC-AG*-groupoid is a semigroup.

Proof. Let $S$ be an LC-AG-groupoid satisfying AG*-groupoid property, and $a, b, c \in S$. Then $(ab)c = (ba)c = a(bc) \Rightarrow (ab)c = a(bc)$. Hence $S$ is a semigroup.

Theorem 7. Every BC-AG-groupoid is paramedial AG-groupoid.

Proof. Let $S$ be a BC-AG-groupoid, and $a, b, c, d \in S$. Then $ab \cdot cd = ab \cdot dc = ba \cdot dc = bd \cdot ac = db \cdot ac = db \cdot ca \Rightarrow ab \cdot cd = db \cdot ca$. Hence $S$ is paramedial AG-groupoid.

It follows that:

Corollary 1. Every BC-AG-groupoid is left nuclear square AG-groupoid.

The following counterexamples shows that neither AG**-groupoid nor BC-AG-groupoid is a nuclear Square AG-groupoid. However, both these properties jointly gives the desired relation as given in the following theorem.

Example 5. (i). AG**-groupoid of order 3 that is not a nuclear square AG-groupoid. (ii). BC-AG-groupoid that is not nuclear square AG-groupoid.
Theorem 8. Let $S$ be a BC-AG\textsuperscript{**}-groupoid. Then the following conditions are equivalent.

(i) $S$ is middle nuclear square AG-groupoid;
(ii) $S$ is right nuclear square AG-groupoid;
(iii) $S$ is nuclear square AG-groupoid.

Proof. Let $S$ be a BC-AG\textsuperscript{**}-groupoid. Then

(i) $\Rightarrow$ (iii). Assume (i) holds, let $a, b, c \in S$. Then $a(bc^2) = b(ac^2) = b(c^2a) = (bc^2)a = (c^2b)a = (ab)c^2 \Rightarrow a(bc^2) = (ab)c^2$. Thus $S$ is right nuclear square AG-groupoid. Now, by lemma (ii) $S$ is left nuclear square AG-groupoid as well. Hence $S$ is nuclear square AG-groupoid which is (iii).

(iii) $\Rightarrow$ (ii) Obvious. Finally we show,

(ii) $\Rightarrow$ (i). Assume (ii) holds, and let $a, b, c \in S$. Then $a(b^2c) = b^2(ac) = b^2(ca) = c(b^2a) = c(ab^2) = (ca)b^2 = (b^2a)c = (ab^2)c \Rightarrow a(b^2c) = (ab^2)c$. Which proves (i).

Hence the theorem is proved. \qed

Now we give an example of left alternative AG-groupoid and BC-AG-groupoid that are not flexible AG-groupoid.

Example 6. (i). left alternative AG-groupoid, which is not flexible AG-groupoid.
(ii). BC- AG-groupoid, which is not flexible AG-groupoid.

\begin{tabular}{|c|c|c|c|}
\hline
\cdot & 1 & 2 & 3 \\
\hline
1 & 1 & 1 & 1 \\
2 & 1 & 1 & 1 \\
3 & 1 & 2 & 2 \\
\hline
\end{tabular}

\begin{tabular}{|c|c|c|c|c|}
\hline
1 & 1 & 2 & 3 & 4 \\
\hline
1 & 3 & 3 & 3 & 2 \\
2 & 4 & 3 & 3 & 3 \\
3 & 3 & 3 & 3 & 3 \\
4 & 3 & 1 & 3 & 3 \\
\hline
\end{tabular}

\begin{tabular}{|c|c|c|c|}
\hline
\cdot & 1 & 2 & 3 \\
\hline
1 & 1 & 1 & 1 \\
2 & 1 & 1 & 1 \\
3 & 2 & 2 & 1 \\
\hline
\end{tabular}

\begin{tabular}{|c|c|c|c|}
\hline
1 & 1 & 2 & 3 \\
\hline
1 & 2 & 2 & 2 \\
2 & 3 & 3 & 3 \\
3 & 3 & 3 & 3 \\
\hline
\end{tabular}

However, we have the following:

Theorem 9. Every left alternative BC-AG-groupoid is flexible AG-groupoid.
Proof. Let $S$ be a BC-AG-groupoid satisfying also the left alternative AG-groupoid property, and let $a, b \in S$. Then $ab \cdot a = ba \cdot a = a \cdot ab = a \cdot ba \Rightarrow ab \cdot a = a \cdot ba$. Hence $S$ is flexible AG-groupoid. ■

**Theorem 10.** Every LC-AG-groupoid with left identity $e$ is commutative semigroup.

Proof. Let $S$ be a RC-AG-groupoid with left identity $e$, and let $a, b \in S$. Then $ab = ea \cdot eb = ae \cdot eb = (eb \cdot e)a = (be \cdot e)a = (ee \cdot b)a = ba \Rightarrow ab = ba$. Hence $S$ is commutative, but commutativity implies associativity in AG-groupoids, hence $S$ is commutative semigroup. ■

One can easily verify that neither $T^1$-AG-groupoid nor BC-AG-groupoid is self dual AG-groupoid.

**Example 7.** (i). $T^1$-AG-groupoid of order 3 which is self dual AG-groupoid. (ii). BC-AG-groupoid of order 3 which is not self dual AG-groupoid.

\[
\begin{array}{c|ccc}
\cdot & 1 & 2 & 3 \\
\hline
1 & 1 & 1 & 1 \\
2 & 1 & 1 & 3 \\
3 & 1 & 2 & 1 \\
\end{array} \quad \begin{array}{c|ccc}
\cdot & 1 & 2 & 3 \\
\hline
1 & 1 & 1 & 1 \\
2 & 1 & 1 & 1 \\
3 & 2 & 2 & 2 \\
\end{array}
\]

(i) (ii)

However, we prove the following:

**Theorem 11.** Every $T^1$-BC-AG-groupoid is self dual AG-groupoid.

Proof. Let $S$ be a $T^1$-BC-AG-groupoid, and let $a, b, c \in S$. Then $a(bc) = a(cb) \Rightarrow (bc)a = (cb)a = (ab)c = (ba)c \Rightarrow (bc)a = (ba)c \Rightarrow a(bc) = c(ba)$. Hence $S$ is self dual AG-groupoid. ■

**Theorem 12.** Every BC-AG-3-band is commutative semigroup.

Proof. Let $S$ be BC-AG-3-band, and let $a, b \in S$. Then $ab = (a \cdot aa)(b \cdot bb) = (ab)(aa \cdot bb) = (ba)(ab \cdot ab) = (ba)(ba \cdot ba) = (ba)(bb \cdot aa) = (b \cdot bb)(a \cdot aa) = ba \Rightarrow ab = ba$. Thus $S$ is commutative and hence is associative. Equivalently $S$ is commutative semigroup. ■

An element $a$ of an AG-groupoid $S$ is called left cancellative if $ab = ac \Rightarrow b = c$, right cancellative and cancellative elements are defined analogously.

**Theorem 13.** Every LC-AG-groupoid with a cancellative element is a commutative semigroup.
Proof. Let $S$ be an LC-AG-groupoid with a cancellative element $x$, and let $a, b \in S$. Then $ab \cdot x = ba \cdot x \Rightarrow ab = ba$. Thus $S$ is commutative, but commutativity implies associativity in AG-groupoids. Hence $S$ commutative semigroup. 

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