On the directed tile assembly systems at temperature 1.*

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1 Introduction

We show here that a model called directed self-assembly at temperature 1 is unable to do complex computations like the ones of a Turing machine. Since this model can be seen as a generalization of finite automata to 2D languages, a logical approach is to proceed in two steps. The first one is to develop a 2D pumping lemma and the second one is to use this pumping lemma to classify the different types of possible computation.

In the directed tile assembly system at temperature 1, tiles are positioned on a 2D discrete grid. The tiles are squares with some glues on their four sides. Two tiles can bind together if they are neighbors and have matching glues on their abuttal sides. There is only a finite number of tile types. Several tiles bounded together are called an assembly. A sequence of tiles such that each tile binds to the previous one is called a path. We consider initially a specific assembly called the seed and we study how this assembly grows when some new tiles bind on it. An assembly is called terminal when no more tile can bind to it. In the directed case considered here, a given set of tile types and a seed will always produce the same terminal assembly.

In [4], the authors demonstrate a pumping lemma: for any path growing from the seed which is lengthy enough, some part of it can be copy-pasted in order to grow an infinite periodic path (note that this lemma states that the path could also be fragile which can occurs only in the non-directed case). The bound proven for this pumpable lemma is a tower of exponential according to the number of tile types and the size of the seed (a previous version of this result is still available on arXiv [3]). Beforehand, Doty et al [2], assuming the existence of a pumping lemma, classify the different types of terminal assembly. Thus the

*Supported by European Research Council (ERC) award number 772766 and Science foundation Ireland (SFI) grant 18/ERCS/5746 (this manuscript reflects only the authors’ view and the ERC is not responsible for any use that may be made of the information it contains).
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combination of these two papers solves the directed temperature 1 conjecture ... but in an imperfect way. Indeed, since \textcite{2} is anterior to \textcite{4}, the authors assumed a different and stronger pumping lemma where there is no mention of the seed; any path of a given length in the terminal assembly is pumpable. Nevertheless, all the demonstrations made in \textcite{2} still hold with the pumping lemma of \textcite{4}.

The classification of Doty et al \textcite{2} leads to four kinds of terminal assembly: finite, infinite with/without comb, periodic with/without comb and bi-periodic. A comb is a periodic path growing on another periodic path. Combs are the most complex structure of the classification and cannot be dealt directly with the pumping lemma of \textcite{4} since they seem to grow arbitrarily far away from the seed. Nevertheless, since a comb grows on a periodic path, there exists an infinity of copies of the comb growing on the same periodic path. Then, pumping one comb is enough to deal with all of its copies. Moreover, the authors of \textcite{2} have shown that these copies cannot intersect. Finally, if a comb grows far away enough from the seed, it will have to intersect one of its copy before intersecting with the seed which is not possible. Hence, the pumping lemma of \textcite{4} can be used on a comb close to the seed.

In this paper, we harmonize the notations between \textcite{2} and \textcite{4} in order to clearly solve the directed temperature 1 conjecture, the lemma 20 explains in details how to break combs with the reasoning of the previous paragraph. We are also able to show that the bi-periodic structures cannot reuse a tile type and thus we give an optimal description of these structures. For a more detailed presentation of this model and the implications of this result, see the introduction of \textcite{4}.

2 Definitions (from \textcite{4}) and main theorems

As usual, let \( \mathbb{R} \) be the set of real numbers, let \( \mathbb{Z} \) be the integers, and let \( \mathbb{N} \) be the natural numbers including 0. The domain of a function \( f \) is denoted \( \text{dom}(f) \), and its range (or image) is denoted \( f(\text{dom}(f)) \).

2.1 Abstract tile assembly model

The abstract tile assembly model was introduced by Winfree \textcite{8}. In this paper we study a restriction of the abstract tile assembly model called the directed temperature 1 abstract tile assembly model, or noncooperative abstract tile assembly model. For definitions of the full model, as well as intuitive explanations, see for example \textcite{7,6}.

A tile type is a unit square with four sides, each consisting of a glue type and a nonnegative integer strength. Let \( T \) be a a finite set of tile types. The sides of a tile type are respectively called north, east, south, and west, as shown in the following picture:
An assembly is a partial function $\alpha : \mathbb{Z}^2 \rightarrow T$ where $T$ is a set of tile types and the domain of $\alpha$ (denoted $\text{dom}(\alpha)$) is connected. We let $\mathcal{A}^T$ denote the set of all assemblies over the set of tile types $T$. In this paper, two tile types in an assembly are said to bind (or interact, or are stably attached), if the glue types on their abutting sides are equal, and have strength $\geq 1$. An assembly $\alpha$ induces an undirected weighted binding graph $G_\alpha = (V, E)$, where $V = \text{dom}(\alpha)$, and there is an edge $\{a, b\} \in E$ if and only if the tiles at positions $a$ and $b$ interact, and this edge is weighted by the glue strength of that interaction. The assembly is said to be $\tau$-stable if every cut of $G_\alpha$ has weight at least $\tau$.

A tile assembly system is a triple $\mathcal{T} = (T, \sigma, \tau)$, where $T$ is a finite set of tile types, $\sigma$ is a $\tau$-stable assembly called the seed, and $\tau \in \mathbb{N}$ is the temperature. Throughout this paper, $\tau = 1$.

Given two $\tau$-stable assemblies $\alpha$ and $\beta$, we say that $\alpha$ is a subassembly of $\beta$, and write $\alpha \sqsubseteq \beta$, if $\text{dom}(\alpha) \subseteq \text{dom}(\beta)$ and for all $p \in \text{dom}(\alpha)$, $\alpha(p) = \beta(p)$. We also write $\alpha \to^T_1 \beta$ if we can obtain $\beta$ from $\alpha$ by the binding of a single tile type, that is: $\alpha \sqsubseteq \beta$, $|\text{dom}(\beta) \setminus \text{dom}(\alpha)| = 1$ and the tile type at the position $\text{dom}(\beta) \setminus \text{dom}(\alpha)$ stably binds to $\alpha$ at that position. We say that $\gamma$ is productive from $\alpha$, and write $\alpha \to^T_\gamma$ if there is a (possibly empty) sequence $\alpha_1, \alpha_2, \ldots, \alpha_n$ where $n \in \mathbb{N} \cup \{\infty\}$, $\alpha = \alpha_1$ and $\alpha_n = \gamma$, such that $\alpha_1 \to^T_\gamma \alpha_2 \to^T_\gamma \ldots \to^T_\gamma \alpha_n$. A sequence of $n \in \mathbb{Z}^+ \cup \{\infty\}$ assemblies $\alpha_0, \alpha_1, \ldots$ over $\mathcal{A}^T$ is a $\mathcal{T}$-assembly sequence if, for all $1 \leq i < n$, $\alpha_{i-1} \to^T_\gamma \alpha_i$.

Given two $\tau$-stable assemblies $\alpha$ and $\beta$, the union of $\alpha$ and $\beta$, write $\alpha \sqcup \beta$, is an assembly defined if and only if and for all $p \in \text{dom}(\alpha) \cap \text{dom}(\beta)$, $\alpha(p) = \beta(p)$ and either at least one tile of $\alpha$ binds with a tile of $\beta$ or $\text{dom}(\alpha) \cap \text{dom}(\beta) \neq \emptyset$. Then, for all $p \in \text{dom}(\alpha)$, we have $(\alpha \sqcup \beta)(p) = \alpha(p)$ and for all $p \in \text{dom}(\beta)$, we have $(\alpha \sqcup \beta)(p) = \beta(p)$.

The set of productions, or producible assemblies, of a tile assembly system $\mathcal{T} = (T, \sigma, \tau)$ is the set of all assemblies producible from the seed assembly $\sigma$ and is written $\mathcal{A}[\mathcal{T}]$. An assembly $\alpha$ is called terminal if there is no $\beta$ such that $\alpha \to^T_1 \beta$. The set of all terminal assemblies of $\mathcal{T}$ is denoted $A[\mathcal{T}]$. If there is a unique terminal assembly, i.e. $|A[\mathcal{T}]| = 1$, then $\mathcal{T}$ is directed. Along the article, this unique terminal assembly is denoted $\alpha$. Note that we do not assume that the seed could be reduced to a single tile as explained in details in appendix A.

The translation of an assembly $\alpha$ by a vector $\vec{v}$, written $\alpha + \vec{v}$, is the assembly $\beta$ defined for all $(x, y) \in (\text{dom}(\alpha) + \vec{v})$ as $\beta(x, y) = \alpha((x, y) - \vec{v})$. An assembly $\alpha$ is $\vec{v}$-periodic if and only if $\vec{v}$ is not the null vector $\vec{0}$ and $\alpha + \vec{v} = \alpha$. The periodicity of the terminal assembly determine its complexity: the three

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1Intuitively, an assembly is a positioning of unit-sized tiles, each from some set of tile types $T$, so that their centers are placed on (some of) the elements of the discrete plane $\mathbb{Z}^2$ and such that those elements of $\mathbb{Z}^2$ form a connected set of points.
main kinds of terminal assembly can be identified according to this parameter. An assembly $\alpha$ is bi-periodic if there exist two non-collinear vectors $\vec{u}$ and $\vec{v}$ such that $\alpha$ is $\vec{u}$-periodic and $\vec{v}$-periodic. An assembly is simply periodic if it is not bi-periodic and if there exists a vector $\vec{v}$ such that $\alpha$ is $\vec{v}$-periodic. Otherwise, an assembly is aperiodic. We define formally the complexity of an assembly and state the three main theorems.

**Definition 1.** The complexity of a finite assembly is 0. For $i \geq 1$, the complexity of an assembly $\alpha$ is $i$ if $\alpha = \bigcup_{\ell \in \mathbb{N}} (\beta + \ell \vec{v})$ where the complexity of $\beta$ is $i - 1$, or if it is a finite union of assembly of level less than $i$.

**Theorem 1** (Description of the bi-periodic terminal assemblies). Consider a directed tile assembly system $T = (T, \sigma, 1)$ whose terminal assembly $\alpha$ is bi-periodic. Then there exist two non-collinear vectors $\vec{u}$ and $\vec{v}$ and an assembly $\beta$ of complexity 0 whose size is bounded by $|T|^2$ such that

$$\alpha = \bigcup_{\ell, \ell' \in \mathbb{Z}} (\beta + \ell \vec{u} + \ell' \vec{v}).$$

**Theorem 2** (Description of the simply periodic terminal assemblies). Consider a directed tile assembly system $T = (T, \sigma, 1)$ whose terminal assembly $\alpha$ is simply periodic. Then there exists a vector $\vec{v}$ and an assembly $\beta$ of complexity 1 such that

$$\alpha = \bigcup_{\ell \in \mathbb{Z}} (\beta + \ell \vec{v}).$$

**Theorem 3** (Description of the aperiodic terminal assemblies). Consider a directed tile assembly system $T = (T, \sigma, 1)$ whose terminal assembly $\alpha$ is aperiodic, then the complexity of $\alpha$ is 2.

An assembly of complexity 0 is finite while assemblies of higher complexity are infinite. According to these three theorems we only need to study assemblies of complexity less than 2. Such assemblies could be represented by semi-linear sets. This technique was introduced in [2] and the Observation 4.2 in the arXiv version of [2] argues that such assemblies are not able of complex computations.

**Definition 2** (linear and semilinear sets). A set $X \subset \mathbb{Z}^2$ is linear if there exists $p \in \mathbb{Z}^2$ and two vectors $\vec{u}$ and $\vec{v}$ such that

$$X = \bigcup_{\ell, \ell' \in \mathbb{N}} \{ p + \ell \vec{u} + \ell' \vec{v} \}.$$

A semilinear set is a finite union of linear sets.

**Observation 1.** The domain of an assembly of complexity less than 2 is a semilinear set.

**Proof.** An assembly $\alpha^{(0)}$ of complexity 0 is a finite assembly, then its domain can be described by a finite union of $|\text{dom}(\alpha^{(0)})|$ linear sets where $\vec{u}$ and $\vec{v}$ are
both the null vector $\vec{0}$. If an assembly $\alpha^{(1)}$ is defined by as $\bigcup_{t \in \mathbb{N}} (\alpha^{(0)} + t \vec{v})$ where the complexity of $\alpha^{(0)}$ is 0, then for any linear set in the description of $\alpha^{(0)}$ we can set $\vec{v} = \vec{v}'$ instead of the null vector. The size of the semilinear set used to describe $\text{dom}(\alpha^{(1)})$ is the same as the one used to describe $\text{dom}(\alpha^{(0)})$. If an assembly of complexity 1 is the finite union of assembly of complexity less than 1 then the domain of such an assembly is the finite union of the semilinear sets used to describe the domains of these assemblies. Similarly, If an assembly $\alpha^{(2)}$ is defined as $\bigcup_{t \in \mathbb{N}} (\alpha^{(1)} + t \vec{v})$ where the complexity of $\alpha^{(1)}$ is 1, then for any linear set in the description of $\alpha^{(1)}$ we can set $\vec{v} = \vec{v}'$ instead of the null vector. The size of the semilinear set used to described $\text{dom}(\alpha^{(1)})$ is the same as the one used to describe $\text{dom}(\alpha^{(1)})$. If an assembly of complexity 2 is the finite union of assemblies of complexity less than 2, then the domain of such an assembly is the finite union of the semilinear sets used to describe the domain of these assemblies.

\[\square\]

2.2 Paths

This section introduces quite a number of key definitions and concepts that will be used extensively throughout the paper.

Let $T$ be a set of tile types. A tile is a pair $((x,y),t)$ where $(x,y) \in \mathbb{Z}^2$ is a position and $t \in T$ is a tile type. Intuitively, a path is a finite or one-way-infinite simple (non-self-intersecting) sequence of tiles placed on points of $\mathbb{Z}^2$ so that each tile in the sequence interacts with the previous one, or more precisely:

**Definition 3 (Path).** A path is a (finite or infinite) sequence $P = P_0P_1P_2\ldots$ of tiles $P_i = ((x_i,y_i),t_i) \in \mathbb{Z}^2 \times T$, such that:

- for all $P_j$ and $P_{j+1}$ defined on $P$ it is the case that $t_j$ and $t_{j+1}$ interact, and

- for all $P_j, P_k$ such that $j \neq k$ it is the case that $(x_j,y_j) \neq (x_k,y_k)$.

By definition, paths are simple (or self-avoiding), and this fact will be repeatedly used throughout the paper. For a tile $P_i$ on some path $P$, its x-coordinate is denoted $x_{P_i}$ and its y-coordinate is denoted $y_{P_i}$. The concatenation of two paths $P$ and $Q$ is the concatenation $PQ$ of these two paths as sequences, and is a path if and only if (1) the last tile of $P$ interacts with the first tile of $Q$ and (2) $P$ and $Q$ do not intersect each other.

For a path $P = P_0P_1P_2\ldots$, we define the notation $P_{i,i+1,\ldots,j} = P_iP_{i+1}\ldots P_j$, i.e. “the subpath of $P$ between indices $i$ and $j$, inclusive”. Whenever $P$ is finite, i.e. $P = P_0P_1P_2\ldots P_{n-1}$ for some $n \in \mathbb{N}$, $n$ is termed the length of $P$ and denoted by $|P|$. In the special case of a subpath where $i = 0$, we say that $P_{0,1,\ldots,j}$ is a prefix of $P$ and when $j = |P| - 1$, we say that $P_{i,\ldots,|P|-1}$ is a suffix of $P$. For any path $P = P_0P_1P_2,\ldots$ and integer $i \geq 0$, we write $\text{pos}(P_i) \in \mathbb{Z}^2$, or $(x_{P_i},y_{P_i}) \in \mathbb{Z}^2$, for the position of $P_i$ and $\text{type}(P_i)$ for the
tile type of \( P_t \). Hence if \( P_t = ((x_i, y_i), t_i) \) then \( \text{pos}(P_t) = (x_{P_t}, y_{P_t}) = (x_i, y_i) \) and \( \text{type}(P_t) = t_i \). A “position of \( P \)” is an element of \( \mathbb{Z}^2 \) that appears in \( P \) (and therefore appears exactly once), and an index \( i \) of a path of length \( n \in \mathbb{N} \) is a natural number \( i \in \{0, 1, \ldots, n - 1\} \). For a path \( P = P_0P_1P_2 \ldots \) we write \( \text{pos}(P) \) to mean “the sequence of positions of tiles along \( P \)” which is \( \text{pos}(P) = \text{pos}(P_0)\text{pos}(P_1)\text{pos}(P_2) \ldots \). For a finite path \( P = P_0P_1P_2 \ldots P_{|P|-1} \), we define \( P^+ \) as the path \( P_{|P|-1}P_{|P|-2} \ldots P_0 \). The vertical height of a path \( P \) is defined as \( \max\{\{y_{P_i} - y_{P_{i+1}} : 0 \leq i \leq j \leq |P| - 1\} \) \) and its horizontal width is \( \max\{\{x_{P_i} - x_{P_{i+1}} : 0 \leq i \leq j \leq |P| - 1\} \)

Although a path is not an assembly, we know that each adjacent pair of tiles in the path sequence interact implying that the set of path positions forms a connected set in \( \mathbb{Z}^2 \) and hence every path uniquely represents an assembly containing exactly the tiles of the path, more formally: for a path \( P = P_0P_1P_2 \ldots \) we define the set of tiles \( \text{asm}(P) = \{P_0, P_1, P_2, \ldots \} \) which we observe is an assembly and we call \( \text{asm}(P) \) a path assembly. Given a tile assembly system \( \mathcal{T} = (T, \sigma, 1) \) the path \( P \) is a producible path of \( \mathcal{T} \) if \( \text{asm}(P) \) does not intersect the seed \( \sigma \) and the assembly \( \text{asm}(P) \cup \sigma \) is producible by \( \mathcal{T} \), i.e. \( \text{asm}(P) \cup \sigma \in \mathcal{A}[\mathcal{T}] \), and \( P_0 \) interacts with a tile of \( \sigma \). Consider an assembly \( \alpha \) (resp. a path \( Q \)), as a convenient abuse of notation we sometimes write \( \sigma \cup P \) (resp. \( P \cup Q \)) as a shorthand for \( \sigma \cup \text{asm}(P) \) (resp. \( \text{asm}(P) \cup \text{asm}(Q) \)). Given a tile assembly system \( \mathcal{T} = (T, \sigma, 1) \) we define the set of producible paths of \( \mathcal{T} \) to be:

\[
\mathcal{P}[\mathcal{T}] = \{P \mid P \text{ is a path that does not intersect } \sigma \text{ and } (\text{asm}(P) \cup \sigma) \subseteq \mathcal{A}[\mathcal{T}]\}
\]

Given a directed tile assembly system \( \mathcal{T} = (T, \sigma, 1) \) and its unique terminal assembly \( \alpha \), the path \( P \) is a path of \( \alpha \) if \( \text{asm}(P) \) is a subassembly of \( \alpha \). We define the set of paths of \( \alpha \) to be:

\[
\mathcal{P}[\alpha] = \{P \mid P \text{ is a path and } \text{asm}(P) \text{ is a subassembly of } \alpha\}
\]

So far, we have defined paths of tiles (Definition 3). In our proofs, we will also reason about (untiled) binding paths in the binding graph of an assembly.

**Definition 4 (Binding path).** Let \( G = (V, E) \) be a binding graph. A binding path \( q \) in \( G \) is a sequence \( q_0, q_1, \ldots, q_{|q|-1} \) of vertices from \( V \) such that for all \( i \in \{0, 1, \ldots, |q| - 2\} \), \( \{q_i, q_{i+1}\} \in E \) (\( q \) is connected) and no vertex appears twice in \( q \) (\( q \) is simple).

The following observation was proven in [4].

**Observation 2.** Let \( \mathcal{T} = (T, \sigma, 1) \) be a tile assembly system and let \( \alpha \in \mathcal{A}[\mathcal{T}] \). For any tile \((x, y), t \in \alpha\) either \((x, y), t \) is a tile of \( \sigma \) or else there is a finite producible path \( P \in \mathcal{P}[\mathcal{T}] \) such that for some \( j \in \mathbb{N} \) contains \( P_j = ((x, y), t) \).

\(^{2}\)I.e. \( \text{asm}(P) \) is a partial function from \( \mathbb{Z}^2 \) to tile types, and is defined on a connected set.

\(^{3}\)Formally, non-intersection of a path \( P = P_0P_1 \ldots \) and a seed assembly \( \sigma \) is defined as: \( \forall t \text{ such that } t \in \sigma, \exists i \text{ such that } \text{pos}(P_i) = \text{pos}(t) \).

\(^{4}\)Intuitively, although producible paths are not assemblies, any producible path \( P \) has the nice property that it encodes an unambiguous description of how to grow \( \text{asm}(P) \) from the seed \( \sigma \), in path \( (P) \) order, to produce the assembly \( \text{asm}(P) \cup \sigma \).
Observation 3. Let $T = (T, \sigma, 1)$ be a directed tile assembly system whose terminal assembly is $\alpha$. For any tile $A \in \alpha$ there exists a finite producible assembly $\beta$ such that $A$ is a tile of $\beta$.

**Proof.** By the previous observation, if $A$ is a tile of $\sigma$ then $\beta = \sigma$ satisfies the observation. Otherwise, there exists a finite producible path $P \in P[T]$ such that for some $j \in \mathbb{N}$ contains $P_j = A$. Then $P \cup \sigma$ satisfies the observation. $\square$

Observation 4. Let $T = (T, \sigma, 1)$ be a directed tile assembly system whose terminal assembly is $\alpha$. For any tiles $((x, y), t) \in \alpha$ and $((x', y'), t') \in \alpha$ there is a path $P \in P[\alpha]$ such that for some $P_0 = ((x, y), t)$ and $P_{|P|-1} = ((x', y'), t')$.

**Proof.** Since $dom(\alpha)$ is a connected subset of $\mathbb{Z}^2$ there is an integer $n \geq 0$ and a binding path $p_{0,1,...,n}$ in the binding graph of $\alpha$ where $p_0 = (x, y)$ and $p_n = (x', y')$. We can then define $P$ as the path:

$$P = P_{0,1,...,n} = (p_0, \alpha(p_0))(p_1, \alpha(p_1)) \cdots (p_n, \alpha(p_n))$$

By definition of binding graph, for all $i \in \{0, i, \ldots, n - 1\}$, the tiles $(p_i, \alpha(p_i))$ and $(p_{i+1}, \alpha(p_{i+1}))$ on $P$ are adjacent in $\mathbb{Z}^2$ and interact on their abutting sides, meaning that $P \in P[\alpha]$, thus proving the statement. $\square$

For $A, B \in \mathbb{Z}^2$, we define $\overrightarrow{AB} = B - A$ to be the vector from $A$ to $B$, and for two tiles $P_i = ((x_i, y_i), t_i)$ and $P_j = ((x_j, y_j), t_j)$ we define $P_iP_j = \text{pos}(P_j) - \text{pos}(P_i)$ to mean the vector from $\text{pos}(P_i) = (x_i, y_i)$ to $\text{pos}(P_j) = (x_j, y_j)$. The translation of a path $P$ by a vector $\overrightarrow{v} \in \mathbb{Z}^2$, written $P + \overrightarrow{v}$, is the path $Q$ such that $|P| = |Q|$ and for all indices $i \in \{0, 1, \ldots, |P| - 1\}$, $\text{pos}(Q_i) = \text{pos}(P_i) + \overrightarrow{v}$ and $\text{type}(Q_i) = \text{type}(P_i)$.

Let $P$ be a path, let $i \in \{1, 2, \ldots, |P| - 2\}$, and let $A \neq P_{i+1}$ be a tile such that $P_{0,1,...,i}A$ is a path. Let also $\rho$ be the clockwise rotation matrix defined as $\rho = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and let $\tau = (\rho P_1 P_{i-1}, \rho^2 P_1 P_{i-1}, \rho^3 P_1 P_{i-1})$ (intuitively, $\tau$ is the vector of possible steps after $P_i$, ordered clockwise). We say that $P_{0,1,...,i}A$ turns right (respectively turns left) from $P_{0,1,...,i+1}$ if $\overrightarrow{P_iA}$ appears after (respectively before) $\overrightarrow{P_{i+1}}$ in $\tau$.

### 2.3 Intersections

If two paths, or two assemblies, or a path and an assembly, share a common position we say that they *intersect* at that position. Furthermore, we say that two paths, or two assemblies, or a path and an assembly, *agree* on a position if they both place the same tile type at that position and *conflict* if they place a different tile type at that position. We say that a path $P$ is *fragile* to mean that there is a producible assembly $\alpha$ that conflicts with $P$. Intuitively, if we grow $\alpha$ first, then there is at least one tile that $P$ cannot place:

**Definition 5** (Fragile). Let $T = (T, \sigma, 1)$ be a tile assembly system and $P \in P[T]$. We say that $P$ is fragile if there exists a producible assembly $\alpha \in$
\[ A[T] \text{ and a position } (x, y) \in (\text{dom}(\alpha) \cap \text{dom}(\text{asm}(P))) \text{ such that } \alpha((x, y)) \neq \text{asm}(P)((x, y)) \tag{5} \]

In a directed tile assembly system, there exists not fragile path, thus if a path conflicts with a producible assembly then this path is not a path of the unique terminal assembly.

**Observation 5.** Let \( T = (T, \sigma, 1) \) be a directed tile assembly system whose terminal assembly is \( \alpha \), a producible assembly \( \beta \) and a path \( P \) which intersects with \( \beta \). If all intersections are agreements then \( P \in \text{P}[\alpha] \), otherwise \( P \notin \text{P}[\alpha] \).

Consider two paths \( P \) and \( Q \), we say that \( Q \) grows on \( P \) at index \( i \), if the only intersection between \( Q \) and \( P \) occurs at \( \text{pos}(Q_0) = \text{pos}(P_i) \) and is an agreement. Also, \( Q \) is an arc of \( P \) between indices \( i \neq j \) if and only if there are exactly two interactions between \( Q \) and \( P \) which occur at \( \text{pos}(Q_0) = \text{pos}(P_i) \) and \( \text{pos}(Q_{|Q|-1}) = \text{pos}(P_j) \) and both are agreement, moreover for the special case where \( |Q| = 2 \), we should also have \( j \neq i+1 \) and \( j \neq i-1 \). The width of an arc \( Q \) of \( P \) is defined by \( |j-i| \). Note that, if \( P \in \text{P}[\alpha] \), then the path \( Q \) or the arc \( A \) do not necessarily belong to \( \text{P}[\alpha] \) since they can conflict with the seed.

### 2.4 Pumping a path

Next, for a path \( P \), we define sequences of points and tile types (not necessarily a path) called the pumping of \( P \) or the bi-pumping of \( P \).

**Definition 6** (Pumpings of \( P \)). Let \( T = (T, \sigma, 1) \) be a tile assembly system and a path \( P \) such that \( \text{type}(P_0) = \text{type}(P_{|P|-1}) \). We say that the “pumping of \( P \), denoted by \( (P)^* \), is the infinite sequence \( \mathbf{q} \) of elements from \( \mathbb{Z}^2 \times T \) defined by:

\[
\mathbf{q}_k = P_k \mod (|P|-1) + \frac{k}{|P|-1} P_0 P_{|P|-1} \rightarrow \text{ for } k \in \mathbb{N}.
\]

Whereas, we say that the “bi-pumping of \( P \), denoted by \( \mathbf{q}^* \), is the bi-infinite sequence \( \mathbf{q} \) of elements from \( \mathbb{Z}^2 \times T \) defined by:

\[
\mathbf{q}_k = P_k \mod (|P|-1) + \frac{k}{|P|-1} P_0 P_{|P|-1} \rightarrow \text{ for } k \in \mathbb{Z}.
\]

We will always consider cases where \( \mathbf{q} \) is self-avoiding and that in particular, for any \( s < t \), if the path \( P + sP_0P_{|P|-1} \rightarrow \) intersects with the path \( P + tP_0P_{|P|-1} \rightarrow \), then \( t = s+1 \) and the only intersection is an agreement between \( P_0 + tP_0P_{|P|-1} \) and \( P_{|P|-1} + sP_0P_{|P|-1} \rightarrow \). A sufficient condition for this is that the only intersection between \( P \) and \( P + P_0P_{|P|-1} \rightarrow \) is an agreement between \( P_0 + P_0P_{|P|-1} \rightarrow \) and \( P_{|P|-1} \rightarrow \) (see Lemma 24). If this condition is satisfied then \( P \) is called a good candidate.

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5Here, it might be the case that \( \alpha \) and \( P \) conflict at only one position by placing two different tile types \( t \) and \( t' \), but that \( t \) and \( t' \) may place the same glues along \( P \). In this case \( P \) is not producible when starting from the assembly \( \alpha \) because one of the tiles along the positions of \( P \) is of the wrong type.
and \((P)^*\) and \(*((P))^*\) are both paths. Remark that, for all \(k \in \mathbb{N}\) (resp. \(k \in \mathbb{Z}\)), we have \((P)^*_{k+|P|-1} = (P)^*_k + P_0 P_{|P|-1}\) (resp. \(*((P))^*_{k+|P|-1} = *((P))^*_k + P_0 P_{|P|-1}\).

**Definition 7** (Pumpable path). Let \(T = (T, \sigma, 1)\) be a directed tile assembly system and let \(\alpha\) be its unique terminal assembly. We say that a good candidate \(P\) is pumpable if \((P)^* \in P[\alpha]\) and bi-pumpable if \(*((P))^* \in P[\alpha]\). A good candidate which is pumpable but not bi-pumpable is called simply pumpable.

This definition of pumping does not take into account the position of the seed. In order to use the pumping lemma of \([4]\), we need a special definition of pumpable for producible paths. For a producible path \(P\) and two indices \(i, j\) on \(P\), we define a sequence of points and tile types (not necessarily a path) called the pumping of \(P\) between \(i\) and \(j\):

**Definition 8** (Pumping of a producible path \(P\) between \(i\) and \(j\)). Let \(T = (T, \sigma, 1)\) be a tile assembly system and \(P \in P[T]\). We say that the “pumping of \(P\) between \(i\) and \(j\)” is the sequence \(\overline{q}\) of elements from \(\mathbb{Z}^2 \times T\) defined by:

\[
\overline{q}_k = \begin{cases} 
P_k & \text{for } 0 \leq k \leq i \\
P_{i+1+((k-i-1) \mod (j-i))} + \left[ \frac{k-i-1}{j-i-1} \right] P_i P_j & \text{for } i < k, 
\end{cases}
\]

Intuitively, \(\overline{q}\) is the concatenation of a finite path \(P_{0,1,...,i}\) and an infinite periodic sequence of tile types and positions (possibly intersecting \(\sigma \cup P_{0,1,...,i}\), and possibly intersecting itself). The following definition gives the notion of pumpable path used in the pumping lemma \([3]\) which follows.

**Definition 9** (Producible pumpable path). Let \(T = (T, \sigma, 1)\) be a tile assembly system. We say that a producible path \(P \in P[T]\), is infinitely pumpable, if there are two integers \(i < j\) such that the pumping of \(P\) between \(i\) and \(j\) is an infinite producible path, i.e. formally: \(\overline{q} \in P[T]\).

In this case, we say that the pumping vector of \(\overline{q}\) is \(\overline{P_i P_j}\), and that \(P\) is pumpable with pumping vector \(\overline{P_i P_j}\).

The following pumping lemma is the main result of \([4]\).

**Theorem 4.** Let \(T = (T, \sigma, 1)\) be any tile assembly system and let \(P\) be a path producible by \(T\). If \(P\) has vertical height or horizontal width at least \((8|T|)^{|T|+1}(5|\sigma| + 6)\), then \(P\) is infinitely pumpable or fragile.

In our context, we will use the following version of this theorem where there are no fragile path and the bound is replaced by a generic function \(f(x, y)\) where \(x\) is the number of tile types and \(y\) is the size of the seed (since the bound computed in \([4]\) is probably not optimal and could be improved independently of the results presented here). Note that, unlike in \([2]\), we do not consider that the size of the seed could be reduced 1 as explained in appendix \([2]\).

**Theorem 5.** Let \(T = (T, \sigma, 1)\) be any directed tile assembly system and let \(P\) be a path producible by \(T\). If \(P\) has vertical height or horizontal width at least \(f(|T|, |\sigma|)\), then \(P\) is infinitely pumpable.
2.5 2D plane

2.5.1 Curves

A curve \( c : I \to \mathbb{R}^2 \) is a function from an interval \( I \subset \mathbb{R} \) to \( \mathbb{R}^2 \), where \( I \) is one of a closed, open, or half-open. All the curves in this paper are polygonal, i.e. unions of line segments and rays.

For a finite path \( P \), we call the embedding \( \mathcal{E}[P] \) of \( P \) the curve defined for all \( t \in [0, |P| - 1] \subset \mathbb{R} \) by:

\[
\mathcal{E}[P](t) = \text{pos}(P(t)) + (t - |t|)\overrightarrow{P(t)P(t+1)}
\]

Similarly, for a finite path \( p \), the embedding \( \mathcal{E}[p] \) of \( p \) is the curve defined for all \( t \in [0, |p| - 1] \subset \mathbb{R} \) by:

\[
\mathcal{E}[p](t) = p(t) + (t - |t|)\overrightarrow{p(t)p(t+1)}
\]

The ray of vector \( \vec{v} \) from (or, that starts at) point \( A \in \mathbb{R} \) is defined as the curve \( r : [0, +\infty) \to \mathbb{R}^2 \) such that \( r(t) = A + t\vec{v} \).

If \( C \) is a curve defined on some real interval of the form \([a, b]\) or \([a, b]\), and \( D \) is a curve defined on some real interval of the form \([c, d]\) or \([c, d]\), and moreover \( C(b) = D(c) \), then the concatenation \( \text{concat}(C, D) \) of \( C \) and \( D \) is the curve defined on \( \text{dom}(C) \cup (\text{dom}(D) - (c - b)) \) by:

\[
\text{concat}(C, D)(t) = \begin{cases} 
C(t) & \text{if } t \leq b \\
D(t + (c - b)) & \text{otherwise}
\end{cases}
\]

A curve \( c \) is said to be simple or self-avoiding if all its points are distinct, i.e. if for all \( x, y \in \text{dom}(c) \), \( c(x) = c(y) \Rightarrow x = y \).

The reverse \( c^- \) of a curve \( c \) defined on some interval \([a, b]\) (respectively \([a, b]\), \([a, b]\), \([a, b]\)) is the curve defined on \([-b, -a]\) (respectively \([-b, -a]\), \([-b, -a]\), \([-b, -a]\)) as \( c^-(t) = c(-t) \).

If \( A = (x_a, y_a) \in \mathbb{R}^2 \) and \( B = (x_b, y_b) \in \mathbb{R}^2 \), the segment \([A, B]\) is defined to be the curve \( s : [0, 1] \to \mathbb{R}^2 \) such that for all \( t \in [0, 1] \), \( s(t) = ((1 - t)x_a + tx_b, (1 - t)y_a + ty_b) \). We sometimes abuse notation and write \([A, B]\) even if \( A \) or \( B \) (or both) is a tile, in which case we mean the position of that tile instead of the tile itself.

For a curve \( c : \mathbb{R} \to \mathbb{R}^2 \) we write \( c(\mathbb{R}) \) to denote the range of \( c \) (whenever we use this notation the curve \( c \) has all of \( \mathbb{R} \) as its domain). When it is clear from the context, we sometimes write \( c \) to mean \( c(\mathbb{R}) \), for example

2.5.2 Cutting the plane with curves; left and right turns

In this paper we use finite and periodic infinite polygonal curves to cut the \( \mathbb{R}^2 \) plane into two pieces. The finite polygonal curves we use consist of a finite

\[\text{dom}(D) - (c - b)\] means \([b, d - (c - b)]\) if \( \text{dom}(D) = [c, d] \), and \([b, d - (c - b)]\) if \( \text{dom}(D) = [c, d] \).
number of concatenations of vertical and horizontal segments of length 1. If the curve is simple and closed we may apply the Jordan Curve Theorem to cut the plane into connected components.

**Theorem 6** (Jordan Curve Theorem). Let \( c \) be a simple closed curve, then \( c \) cuts \( \mathbb{R}^2 \) into two connected components.

Here, we have stated the theorem in its general form, although for our results the (easier to prove) polygonal version suffices. Moreover, one of two components is finite and we call the restriction of \( Z \) to this finite area, the interior of the cycle. The interior also includes the positions of the curve which are in \( Z \).

The second kind of curve we use is periodic ones. Indeed, we will consider a good candidate path \( P \) and we cut the plan using the embedding of \( *(P)^* \).

For such infinite polygonal curves we also state and prove a slightly different version of the polygonal Jordan Curve Theorem, as Theorem 11.

**Theorem 7** (Jordan Curve Theorem). Let \( c \) be a periodic bi-infinite simple curve, then \( c \) cuts \( \mathbb{R}^2 \) into two connected components.

In Section B.1 we define what it means for one curve to turn left or right from another, as well as left hand side and right hand side of a cut of the real plane. We define the left side \( L(*(P)^*) \) and right side \( R(*(P)^*) \) of a good candidate path \( P \) as the restriction of \( Z \) to the left hand side and right side of the curve obtained by \( *(P)^* \). Note that both the left side and right side contains \( \text{pos}(*(P)^*) \). Consider two good candidate paths \( P \) and \( Q \), we say \( P \) is greater than \( Q \), denoted by \( P \geq Q \), if and only if \( *(P)^* \) is inside the left side of \( *(Q)^* \), i.e. \( \text{pos}(*(P)^*) \subseteq *(L(Q))^* \). Moreover, if \( *(P)^* \neq *(Q)^* \), we say that \( P \) is strictly greater than \( Q \), denoted by \( P > Q \).

If we consider a good candidate path \( P \) and a vector \( \overrightarrow{v} \) such that \( *(P)^* \) and \( *(P)^* + \overrightarrow{v} \) do not intersect and \( P > P + \overrightarrow{v} \) then there exists \( \ell \) such that for any \( (x, y) \in \mathbb{Z}^2 \), \( (x, y) + \ell \overrightarrow{v} \in R(*(P)^*) \cap L(*(P)^* + \overrightarrow{v}) \) (see B.1 for more details).

Consider an arc \( A \) or a path \( P \) which grow on \( *(P)^* \) then their positions are either all in the left side of \( P \) or all in the right side of \( P \). Moreover if the width of the arc \( A \) is as least \( |P| \) then the arc \( A \) must intersect with \( A + N_0 P_{\left|P\right|-1} \) and this intersect does not occurs at its extremities, i.e. it occurs at \( \text{pos}(A_j) \) with \( 0 < j < |A| - 1 \).

**Observation 6.** Consider a directed tile assembly system \( T = (T, \sigma, 1) \) whose terminal assembly is \( \alpha \), a simply pumpable path \( P \) in \( \mathbf{P}[\alpha] \) and a path \( Q \) which grows on \( *(P)^* \) at index \( i \geq |P| - 1 \) then either \( Q \) intersects with \( Q + N_0 P_{\left|P\right|-1} \) or for all \( \ell \in \mathbb{N} \), \( Q + \ell N_0 P_{\left|P\right|-1} \) grows on \( *(P)^* \).

**Proof.** If some \( \ell \in \mathbb{N} \), the path \( Q + \ell N_0 P_{\left|P\right|-1} \) does not grow on \( *(P)^* \) then there exists \( 0 < j \leq |Q| - 1 \) such that \( \text{pos}(Q_j) + \ell N_0 P_{\left|P\right|-1} \in \text{pos}((P)^*) \). Since \( Q \) grows on \( *(P)^* \) then \( \text{pos}(Q_j) \in *(P)^* \). The binding path of \( Q_{0,1,...,j} \) is the binding path of an arc of \( *(P)^* \) whose width is at least \( |P| \) (since \( i \geq |P| - 1 \)) and thus \( Q_{0,1,...,j} \) intersects with \( Q_{0,1,...,j} + N_0 P_{\left|P\right|-1} \).

\[ \square \]
3 Proof of the main theorems

From now on, all tile assembly systems considered here are directed.

3.1 Study of the bi-pumpable paths

3.1.1 Role of the seed

In this subsection, corollary 1 and lemma 2 show the equivalence between \( \alpha \) is \( \mathcal{V} \)-periodic and there exists a bi-pumpable path \( P \) where \( \mathcal{V} = P_0 P_{|P|-1} \). To prove this result, we show that the seed can be reduced to any single tile of \( *\langle P\rangle^* \) (the arguments used in this proof are the ones used in lemma 4.8 of [2]). Later, this lemma will allow us to grow and pump paths easily.

Lemma 1. Let \( T = (T, \sigma, 1) \) be a directed tile assembly system and \( \alpha \) its unique terminal assembly. If a path \( \alpha \in \mathcal{P}[\alpha] \) is bi-pumpable then for any \( i \in \mathbb{Z} \), \( (T, *\langle P\rangle^*_i, 1) \) is directed and its terminal assembly is \( \alpha \).

Proof. Since \( *(P)^* \) is in \( \mathcal{P}[\alpha] \) then we can consider a finite producible assembly \( \beta \) of \( (T, \sigma, 1) \) such that \( *(\langle P\rangle^*_\beta)^* \) is a tile of \( \beta \). By hypothesis, \( *(\langle P\rangle^* \cup \beta) \) is producible by \( (T, \sigma, 1) \). Consider a path \( P \) producible by \( (T, *\langle P\rangle^*_i, 1) \), if \( P \) does not conflict with \( \beta \) or \( *(\langle P\rangle^*_i) \) then \( (R \cup *\langle P\rangle^*_i \cup \beta) \) is producible by \( (T, \sigma, 1) \) and \( R \) is in \( \mathcal{P}[\alpha] \).

For the sake of contradiction suppose that such a conflict exists. We assume that \( R_{|R|-1} \) is the only conflict between \( R \) and either \( \beta \) or \( *(\langle P\rangle^*) \) (by eventually replacing \( R \) with one of its prefix), see Figure 3.1 for an illustration of the following cases:

- If this conflict occurs with \( *(\langle P\rangle^*_j)^* \) for \( j > i \) (the case \( j < i \) is symmetric), since \( \beta \) and \( R \) are finite and \( P_0 P_{|P|-1} \) is not null, there exists \( \ell \in \mathbb{N} \) such that \( R + \ell P_0 P_{|P|-1} \) does not intersect with \( \beta \). Remark that \( R_0 + \ell P_0 P_{|P|-1} = *(\langle P\rangle^*_j)^* = \ell - (i - \ell) \). By definition of \( \beta \) and \( P \), the assembly \( \gamma = \beta \cup *\langle P\rangle^* \) is producible by \( (T, \sigma, 1) \). By definition of \( \ell \), the tile \( R_{|R|-1} + \ell P_0 P_{|P|-1} \) is not a tile of \( \beta \) and since \( j > i \), we have \( i + \ell(|P| - 1) < j + \ell(|P| - 1) \) thus \( R_{|R|-1} + \ell P_0 P_{|P|-1} \) is not a tile of \( \gamma \). Then the assembly \( \gamma \cup (R + \ell P_0 P_{|P|-1}) \) is producible by \( (T, \sigma, 1) \) and is in conflict with \( *(\langle P\rangle^*) \) which is a contradiction.

- Otherwise, this conflict occurs with \( \beta \). Since \( \beta \) and \( R \) are finite and \( P_0 P_{|P|-1} \) is not null, there exists \( \ell \in \mathbb{N} \) such that \( \beta + \ell P_0 P_{|P|-1} \) and \( R + \ell P_0 P_{|P|-1} \) do not intersect with \( \beta \). Since \( *(\langle P\rangle^*) \) is \( P_0 P_{|P|-1} \)-periodic then \( *(\langle P\rangle^*) \) does not conflict with \( \beta + \ell P_0 P_{|P|-1} \) or \( R + \ell P_0 P_{|P|-1} \). Then the two assemblies \( \beta \cup *(\langle P\rangle^*) \cup (\beta + \ell P_0 P_{|P|-1}) \) and \( \beta \cup *(\langle P\rangle^*) \cup (R + \ell P_0 P_{|P|-1}) \) are both producible by \( (T, \sigma, 1) \), but these two assemblies are in conflict which contradicts the fact that \( (T, \sigma, 1) \) is directed.
Thus, any path producible by \((T, *(P)_1^\ast, 1)\) is producible by \((T, \sigma, 1)\). Then creating a conflict in \((T, *(P)_1^\ast, 1)\) would create a conflict in \((T, \sigma, 1)\) and since \((T, \sigma, 1)\) is directed then \((T, *(P)_1^\ast, 1)\) is directed. To conclude, since \((T, *(P)_1^\ast, 1)\) is directed and \(\alpha\) is a terminal assembly which contains \(* (P)_1^\ast\), then \(\alpha\) is the unique terminal assembly of \((T, *(P)_1^\ast, 1)\).

As a corollary of this result, any path \(Q\) which grows on \(* (P)\) is in \(P[\alpha]\).

Moreover, since for any \(\ell \in \mathbb{Z}\), \(* (P) + \ell P_0 P_{|P|--1} = * (P)\) then \(* (Q) + \ell P_0 P_{|P|--1}\) also grows in \(\alpha\) and is in \(P[\alpha]\) which leads to the following corollary:

**Corollary 1.** Let \(T = (T, \sigma, 1)\) be a directed tile assembly system and \(\alpha\) its unique terminal assembly. If \(P \in P[\alpha]\) is bi-pumpable then \(\alpha\) is \(P_0 P_{|P|--1}\) periodic.

**Lemma 2.** Let \(T = (T, \sigma, 1)\) be a directed tile assembly system and \(\alpha\) its unique terminal assembly. If \(\alpha\) is periodic then there exists a path \(P \in P[\alpha]\) which is bi-pumpable.

**Proof.** By hypothesis, there exists a non null vector \(\vec{v}\), such that \(\alpha\) is \(\vec{v}\)-periodic. Consider a tile \(A\) of \(\alpha\) then \(A + \vec{v}\) is also tile of \(\alpha\) and there exists a finite path \(P \in P[\alpha]\) such that \(P_0 = A\) and \(P_{|P|--1} = A + \vec{v}\). Consider the shortest path \(Q\) of \(P[\alpha]\) such that \(Q_0 + s \vec{v} = Q_{|Q|--1}\) for some \(s \in \{-1, 1\}\). The path \(Q\) is correctly defined since \(P\) satisfies this criterion. Since \(\alpha\) is \(\vec{v}\)-periodic, then for \(\ell \in \mathbb{Z}\), \(Q + \ell \vec{v}\) is in \(P[\alpha]\). If \(Q\) and \(Q + \vec{v}\) intersect only at their extremities then \(Q\) is a good candidate and \(* (Q)\) is correctly defined, is in \(P[\alpha]\) and thus \(Q\) is bi-pumpable. Otherwise, there exists another intersection between \(Q\) and \(Q + \vec{v}\), i.e. there exists \(0 \leq i, j \leq |Q| - 1\) such that \(Q_i = Q_j + \vec{v}\) and if \(j > i\) (resp. \(i > j\)) then \((i, j) \notin (0, |Q| - 1)\) (resp. \((j, i) \notin (0, |Q| - 1))\). Thus, \(Q_{i,...,j}\) (resp. \(Q_{j,...,i}\)) contradicts the minimality of \(Q\).

### 3.1.2 Paths without redundancy

Now, we show that we can restrict our study to bi-pumpable paths whose two extremities are the only tile to share a common tile type.

**Definition 10.** A path \(P\) is without redundancy if for all \(0 \leq i < j \leq |P| - 1\), type\((P_i) = \text{type}\((P_j)\) implies that \(i = 0\) and \(j = |P| - 1\).

Of course the length of a path without redundancy is bounded by \(|T| + 1\).

Moreover, if two tiles of a path shares the same tiles type then a path without redundancy can be extracted from it.

**Lemma 3.** For any path \(P\) where there exists \(0 \leq i < j \leq |P| - 1\) such that type\((P_i) = \text{type}\((P_j)\), there exists \(i \leq i' < j' \leq j\) such that \(P_{i', i'+1,...,j'}\) is without redundancy.

**Proof.** Consider \(j' = \min\{i < j' \leq j : \text{there exists } i \leq i' < j' \text{ such that type}\((P_{i'}) = \text{type}\((P_{j'})\)}\). Since type\((P_i) = \text{type}\((P_j)\), \(j'\) is correctly defined and by its definition there exists \(i \leq i' < j'\) such that \(P_{i', i'+1,...,j'}\) is without redundancy.
\( \star(P) \) is in white, the seed is in black and \( \beta \) is the union of the seed and the blue path.

b) Two paths \( R \) and \( R' \) which are producible by \( \langle T, \star(P)_i, 1 \rangle \).

c) The three different ways to create a conflict in \( (T, \sigma, 1) \) by using translations of \( R \) and \( R' \).

Figure 3.1: Illustrations of proof of lemma 1.
The two following lemmas show that paths without redundancy are easy to pump in both direction.

**Lemma 4.** Let $T = (T, \sigma, 1)$ be a directed tile assembly system and $\alpha$ its unique terminal assembly. For any path without redundancy $P \in P[\alpha]$ if $P + P_0P_{|P|-1}$ is in $P[\alpha]$ then $P$ is a good candidate.

**Proof.** If $P$ and $P + P_0P_{|P|-1}$ intersect then there exist $0 \leq i, j \leq |P| - 1$ such that $P_i = P_j + P_{0P_{|P|-1}}$. Since $P_0P_{|P|-1}$ is not null then $i \neq j$. Moreover, since $P$ and $P + P_0P_{|P|-1}$ are both in $P[\alpha]$ then type($P_i$) = type($P_j$) and thus $i = |P| - 1$ and $j = 0$. \hfill \Box

The intuition of the following lemma is that a bi-pumpable path without redundancy could be extracted from a path $Q$ if both extremities of $Q$ belong to bi-pumpable paths and have the same tile types.

**Lemma 5.** Let $T = (T, \sigma, 1)$ be a directed tile assembly system, $\alpha$ its unique terminal assembly and two paths $P, P'$ of $P[\alpha]$ which are bi-pumpable. If there exists a path $Q$ such that type($Q_0$) = type($Q_{|Q|-1}$) and $Q_0$ (resp. $Q_{|Q|-1}$) is a tile of $*(P)^*$ (resp. $*(P')^*$) then there exist $0 \leq i < j \leq |Q| - 1$ such that the path $Q i \ldots j$ is a bi-pumpable path without redundancy of $P[\alpha]$.

**Proof.** Since type($Q_0$) = type($Q_{|Q|-1}$) then by lemma 3, there exists $0 \leq i < j \leq |Q| - 1$ such that the path $R = Q i \ldots j$ is without redundancy. Note that we just need to prove that $R$ is in $P[\alpha]$ and pumpable because $R^*$ satisfies the same hypothesis as $R$ and would also be pumpable. If both $R$ and $R^\tau$ are pumpable then $R$ is bi-pumpable.

If there is no conflict between $Q$ and $R + \ell R_0R_{|R|-1}$ for any $\ell \in \mathbb{N}$, then in particular, there are no conflict between $Q$ and $R + R_0R_{|R|-1}$ and thus the assembly $Q \cup (R + R_0R_{|R|-1})$ is correctly defined. Then $Q \cup (R + R_0R_{|R|-1})$ is producible by $(T, Q_0, 1)$. By lemma 4 and since $Q_0$ is a tile of $*(P)^*$, the tile assembly system $(T, Q_0, 1)$ is directed and its terminal assembly is also $\alpha$, then $R$ and $R + R_0R_{|R|-1}$ are both in $P[\alpha]$ and by lemma 4, $R$ is a good candidate. Then $(R)^*$ is correctly defined. Remind that we are in a case where $(R)^*$ does not conflict with $Q$ then $Q \cup (R)^*$ is producible by $(T, Q_0, 1)$ and $(R)^*$ is in $P[\alpha]$.

Otherwise, suppose for the sake of contradiction that there is a conflict between $Q$ and $R + \ell R_0R_{|R|-1}$ for some $\ell \in \mathbb{N}$, let

$$\ell = \min\{\ell' \in \mathbb{N} : R + \ell' R_0R_{|R|-1} \text{ conflicts with } Q\}$$

and let

$$m = \min\{0 \leq k \leq |R| - 1 : R_k + \ell R_0R_{|R|-1} \text{ conflicts with } Q\}$$

and $0 \leq k \leq |Q| - 1$ such that $\text{pos}(Q_k) = \text{pos}(R_m)$. If $\ell = 1$, we defined $R'$ as $(R_0, \ldots, R_m + P_0P_{|P|-1})$ and since $R'_0 = R_0 + P_0P_{|P|-1} = R_{|R|-1} = Q_j$, we have $m > 0$ and the conflict between $R'$ and $Q$ does not occur at $\text{pos}(Q_j)$ since $R'$ is simple, i.e. $k \neq j$. If $\ell > 1$ as explained in the previous paragraph $(R)^*$
is correctly defined and we defined $R'$ as $(R)_{[R]_{-1},...,([R]_{-1})+(l)}^*$ in this case. We also have $R'_0 = Q_j$ and $R'_{|[R]_{-1}}$ conflicts with $Q$ and this conflicts does not occur at pos($Q_j$) since $R'$ is simple, i.e. $k \neq j$. In both cases, $k \neq j$ and there is exactly one conflict between $R'$ and $Q$ with $Q_j \neq R'_{|[R]_{-1}}$. If $k > j$ (resp. $k < j$) then the assembly $Q_0,\ldots,j \cup R'$ (resp. $Q_j,\ldots,|Q|-1 \cup R'$) is producible by $(T, Q_0, 1)$ (resp. $(T, Q_{|Q|-1}, 1)$). By lemma 4\footnote{lemma 4} and since $Q_0$ (resp. $Q_{|Q|-1}$) is a tile of $\ast(P)^*$ (resp. $\ast(P')^*$), the tile assembly system $(T, Q_0, 1)$ (resp. $(T, Q_{|Q|-1}, 1)$) is directed and its terminal assembly is also $\alpha$, then $R'$ is in $P[\alpha]$ but it conflicts with $Q$ which is producible by $(T, Q_0, 1)$ (resp. $(T, Q_{|Q|-1}, 1)$) and thus also in $P[\alpha]$.

Note that, we can use the previous lemma with $P = P'$ which implies that if there is a bi-pumpable path, there is a bi-pumpable path without redundancy.

**Corollary 2.** Let $T = (T, \sigma, 1)$ be a directed tile assembly system and $\alpha$ be its unique terminal assembly. If there is a bi-pumpable path $P \in \alpha$ then there is a bi-pumpable path $Q$ without redundancy in $\alpha$.

**Proof.** If a path $P$ of $P[\alpha]$ is bi-pumpable then by definition of bi-pumpable, we have type($P_0$) = type($P_{|P|-1}$). Then, by lemma 3\footnote{lemma 3} there exists $0 \leq i < j \leq |P| - 1$ such that $Q = P_{i},\ldots,j$ is without redundancy. Since both extremities of $Q$ agree with $P$ (and are in $\ast(P)^*$), then by lemma 5\footnote{lemma 5} $Q$ is bi-pumpable.

The following lemma will be useful to partitioned the discrete plan in the next section (as explained in section 2.5.2). Indeed, if a vector $\vec{v}$ is not collinear with $P_0P_{|P|-1}$ where $P$ is a bi-pumpable path without redundancy then $P$ and $P + \vec{v}$ cannot intersect.

**Lemma 6.** Let $T = (T, \sigma, 1)$ be a directed tile assembly system and $\alpha$ be its unique terminal assembly. Consider a path $P$ without redundancy and some vector $\vec{v} \in \mathbb{Z}^2$ such that both $P$ and $P + \vec{v}$ are in $P[\alpha]$ and bi-pumpable. If $\ast(P)^*$ and $\ast(P)^* + \vec{v}$ intersect, we have $\vec{v} = \ell P_0P_{|P|-1}$ for some $\ell \in \mathbb{Z}$ and $\ast(P)^* = \ast(P)^* + \vec{v}$.

**Proof.** If $\ast(P)^*$ and $\ast(P)^* + \vec{v}$ intersect then this intersection is not a conflict since $\ast(P)^*$ and $\ast(P)^* + \vec{v}$ are both in $P[\alpha]$. Thus there exists $i, j \in \mathbb{Z}$ such that $\ast(P)_{j}^* = \ast(P)^* + \vec{v}$. Since $P$ is without redundancy then for all $k \in \mathbb{Z}$ such that type($\ast(P)_{k}^*$) = type($\ast(P)^*$), there exists $\ell$ such that $P_k = P_i + \ell P_0P_{|P|-1}$ and thus there exists $\ell \in \mathbb{Z}$ such that $\vec{v} = \ell P_0P_{|P|-1}$ for some $\ell \in \mathbb{Z}$. Finally, since $\ast(P)^*$ is $P_0P_{|P|-1}$-periodic then $\ast(P)^* = \ast(P)^* + \ell P_0P_{|P|-1} = \ast(P)^* + \vec{v}$.

### 3.1.3 Reusing a tile type

Suppose that a tile $A$ of $\alpha$ shares a common tile type with a tile used in a bi-pumpable path $P$ then either $A$ is a tile of $\ast(P)^*$ or not. In the second case, we show in this section that $\alpha$ is bi-periodic. This remark will allow us to
describe efficiently the bi-periodic terminal assemblies and prove one of the main theorem. We proceed in four steps, first we show that \( \alpha \) can grow from \( A \) (lemma 7), then we show that \( P_i\tilde{A} \) cannot be collinear with \( P_0P_{|P|−1} \) (lemma 8) and afterwards we can find a bi-pumpable \( Q \) without redundancy such that \( Q_0Q_{|Q|−1} \) is not collinear with \( P_0P_{|P|−1} \) (see lemma 10) and we conclude by proving one of the main theorem 8 (using lemma 11) by assembling a simple cycle, doing any possible binding in the interior of this cycle and concluding that \( \alpha \) is obtained by tiling the 2D plane with translations of this assembly.

**Lemma 7.** Let \( \mathcal{T} = (T, \sigma, 1) \) be a directed tile assembly system, \( \alpha \) its unique terminal assembly and a bi-pumpable path \( P \in P[\alpha] \) without redundancy. If there exist \( i \in Z \) and a tile \( A \) in \( \alpha \) such that \( \text{type}(A) = \text{type}(P_i) \) then \( \alpha \) is in \( P[\alpha] \).

**Proof.** First, if there exists \( j \in Z \) such that \( A \) is the tile \( \alpha \) then \( \alpha \) is producible by \( T, \alpha, 1 \) and by lemma 7 \( T, \alpha, 1 \) is directed and its terminal assembly is also \( \alpha \). Then \( \alpha \) is in \( P[\alpha] \). Otherwise, since \( A \) is a tile of \( \alpha \) and \( \alpha \) is producible by \( T, \alpha, 1 \) there exists a path \( Q \in P[\alpha] \) and \( j \in Z \) such that \( Q \) grows on \( \alpha \) at index \( j \) and \( Q_{|Q|−1} = A \). If \( Q \) does not conflict with \( \alpha \) then \( \alpha \) is producible by \( T, \alpha, 1 \) and by lemma 7 \( T, \alpha, 1 \) is directed and its terminal assembly is also \( \alpha \). Thus \( \alpha \) is in \( P[\alpha] \). Otherwise, for the sake of contradiction suppose that \( Q \) conflicts with \( \alpha \) then \( Q − P_i\tilde{A} \) conflicts with \( \alpha \).

Moreover, since \( Q_{|Q|−1} = A \) then \( \alpha \) is producible by \( T, \alpha, 1 \) and by lemma 7 \( T, \alpha, 1 \) is directed and its terminal assembly is also \( \alpha \). In this case, \( Q − P_i\tilde{A} \) is in \( P[\alpha] \) which is a contradiction since it conflicts with \( \alpha \).

**Lemma 8.** Let \( \mathcal{T} = (T, \sigma, 1) \) be a directed tile assembly system, \( \alpha \) its unique terminal assembly and a bi-pumpable path \( P \in P[\alpha] \) without redundancy. If there exists \( 0 \leq i \leq |P|−1 \) and a tile \( A \) of \( \alpha \) such that \( \text{type}(A) = \text{type}(P_i) \) and \( P_i\tilde{A} \) is collinear with \( P_0P_{|P|−1} \) then \( A \) is a tile of \( \alpha \).

**Proof.** See Figure 3.2 for an illustration of this reasoning and appendix B.1 for more technical details. Consider the line \( \ell^+ \) (resp. \( \ell^- \)) of direction \( P_0P_{|P|−1} \) included into \( \mathcal{L}(\text{type}(A)) \) (resp. \( \mathcal{R}(\text{type}(A)) \)) which is tangent to \( \mathcal{E}(\text{type}(A)) \). Then there exists \( 0 \leq j \leq |P|−1 \) (resp. \( 0 \leq k \leq |P|−1 \)), such that \( \ell^+ \) (resp. \( \ell^- \)) passes by \( \text{pos}(P_j) \) (resp. \( \text{pos}(P_k) \)). Let \( v^+ \) (resp. \( v^- \)) be \( P_0P_{|P|−1} \) rotated by \( \pi/2 \) (resp. \( −\pi/2 \)). Then the ray \( r^+ \) (resp. \( r^- \)) starting in \( \text{pos}(P_j) \) (resp. \( \text{pos}(P_k) \)) of direction \( v^+ \) (resp. \( v^- \)) intersects \( \text{type}(A) \) only at \( \text{pos}(P_j) \) (resp. \( \text{pos}(P_k) \)) and is included into \( \mathcal{L}(\text{type}(A)) \) (resp. \( \mathcal{R}(\text{type}(A)) \)). By hypothesis, \( P_i + P_i\tilde{A} = A \) is a tile of \( \alpha \) and by lemma 7 \( \text{type}(A) \) is in \( P[\alpha] \). Thus \( P_j + P_i\tilde{A} \) and \( P_k + P_i\tilde{A} \) are both tile of \( \alpha \). Since \( P_0P_{|P|−1} \) and \( P_i\tilde{A} \) are collinear, the ray \( r^+ \) (resp. \( r^- \)) still belongs to \( \mathcal{L}(\text{type}(A)) \) (resp. \( \mathcal{R}(\text{type}(A)) \)) and may intersect \( \text{type}(A) \).
Figure 3.2: Illustration of the proof of lemma 8. We consider a path $P$ such that that $^*P^*$ cuts the 2D plane in two parts. The translation of $P_j$ by $P_j \overrightarrow A$ is in the left side (in red) and the translation of $P_k$ by $P_k \overrightarrow A$ is in the right side (in blue). Then $^*P^*$ intersects with its translation by $P_k \overrightarrow A$.

only at $pos(P_j) + P_j \overrightarrow A$ (resp. $pos(P_k) + P_k \overrightarrow A$) thus $pos(P_j) + P_j \overrightarrow A$ is in $L(^*P^*)$ (resp. $pos(P_k) + P_k \overrightarrow A$ is in $R(^*P^*)$). Then there is an intersection between $^*P^*$ and $^*P^* + P_k \overrightarrow A$ which means by lemma 6 that $^*P^* = (^*P^* + P_k \overrightarrow A$.

Finally, $A = P_i + P_i \overrightarrow A$ is thus a tile of $^*P^*$.

Lemma 9. Let $\mathcal{T} = (T, \sigma, 1)$ be a directed tile assembly system, $\alpha$ its unique terminal assembly and two bi-pumpable paths $P, Q \in \mathbf{P}[\alpha]$ without redundancy. If $P_0 P_{|P|-1} \overrightarrow{P_i A}$ is collinear with $Q_0 Q_{|Q|-1} \overrightarrow{P_i A}$ then $P_0 P_{|P|-1} = s Q_0 Q_{|Q|-1}$ for some $s \in \{-1, 1\}$.

Proof. By corollary 1, $\alpha$ is $Q_0 Q_{|Q|-1}$-periodic which means that $P_0 + Q_0 Q_{|Q|-1}$ is a tile of $\alpha$. By lemma 8, since $P_0 P_{|P|-1}$ and $Q_0 Q_{|Q|-1}$ are collinear and since $P$ is without redundancy then $P_0 + Q_0 Q_{|Q|-1}$ is in $^*P^*$. Since $P_0 P_{|P|-1}$ and $Q_0 Q_{|Q|-1}$ are not null then $Q_0 Q_{|Q|-1} = \ell P_0 P_{|P|-1}$ for some $\ell \in \mathbb{Z}^*$. Similarly, $\alpha$ is $P_0 P_{|P|-1}$-periodic and $P_0 P_{|P|-1} = \ell' Q_0 Q_{|Q|-1}$ for some $\ell' \in \mathbb{Z}^*$. Then either $P_0 P_{|P|-1} = Q_0 Q_{|Q|-1}$ or $P_0 P_{|P|-1} = -Q_0 Q_{|Q|-1}$.

Lemma 10. Let $\mathcal{T} = (T, \sigma, 1)$ be a directed tile assembly system, $\alpha$ its unique terminal assembly and a bi-pumpable path $P \in \mathbf{P}[\alpha]$ without redundancy. If there is a tile $A$ of $\alpha$ such that type$(A) = type(P_i)$ and $P_i \overrightarrow A$ is not collinear with
\( \overrightarrow{P_0P_{|P|-1}} \) then there exists a bi-pumpable path \( Q \) without redundancy in \( P[\alpha] \) such that \( \overrightarrow{P_0P_{|P|-1}} \) is not collinear with \( \overrightarrow{Q_0Q_{|Q|-1}} \).

**Proof.** See Figure 3.3 for an illustration of this reasoning. By lemma 7, \( * (P)^* + \overrightarrow{P_iA} \) is in \( P[\alpha] \). Since \( \overrightarrow{P_0P_{|P|-1}} \) is not collinear with \( \overrightarrow{Q_0Q_{|Q|-1}} \) then by lemma 6, \( * (P)^* + \overrightarrow{P_iA} \) do not intersect. Nevertheless, both paths are in \( P[\alpha] \) and thus there exists a path \( Q \) in \( P[\alpha] \) such that \( Q_0 = P_i \), type(\( Q_{|Q|-1} \)) = type(\( Q_0 \)) = type(\( A \)) and \( Q_{|Q|-1} \) is a tile of \( * (P)^* + \overrightarrow{P_iA} \). Without loss of generality, we suppose that \( Q \) is a path of minimal length which satisfies these three hypothesis. From lemma 5 there exist \( 0 \leq i' \leq j' \leq |Q| - 1 \) such that the path \( R = Q_{i'},...,j' \) is without redundancy, is in \( P[\alpha] \) and is bi-pumpable. If \( R_0R_{|R|-1} \) is not collinear with \( \overrightarrow{P_0P_{|P|-1}} \) then the lemma is true. Suppose otherwise for the sake of contradiction. By corollary 1, \( \alpha = \alpha - \overrightarrow{R_0R_{|R|-1}} \) and since \( Q_{i'} = R_0 = R_{|R|-1} - \overrightarrow{R_0R_{|R|-1}} = Q_{j'} = R_0R_{|R|-1} \), then the path \( Q' = Q_0,...,i'(Q_{j'+1},...,|Q|-1 - \overrightarrow{R_0R_{|R|-1}}) \) is in \( P[\alpha] \). By definition of \( Q \), we have \( Q'_{|Q'|} = P_i \), type(\( Q'_{|Q'|-1} \)) = type(\( Q_{|Q|-1} \)) = type(\( A \)). Moreover by lemma 9, \( R_0R_{|R|-1} = sR_0P_{|P|-1} \) with \( s \in \{-1,1\} \) and then \( Q'_{|Q'|-1} \) is in \( * (P)^* + \overrightarrow{P_iA} \). Since \( |Q'| < |Q| \), this is a contradiction of the hypothesis that \( Q \) is of minimal length.

\( \square \)

**Lemma 11.** Let \( T = (T, \sigma, 1) \) be a directed tile assembly system, \( \alpha \) its unique terminal assembly and two bi-pumpable paths \( P, Q \in P[\alpha] \) without redundancy such that \( \overrightarrow{P_0P_{|P|-1}} \) is not collinear with \( \overrightarrow{Q_0Q_{|Q|-1}} \) then there exists an assembly \( \beta \) and two vectors \( \overrightarrow{u} \) and \( \overrightarrow{v} \) such that \( |\beta| \leq |T|^2 \) and \( \alpha = \bigcup_{\ell, \ell' \in \mathbb{Z}} (\beta + \ell \overrightarrow{u} + \ell' \overrightarrow{v}) \).

**Proof.** See Figure 3.4 for an illustration of this reasoning. Since \( \overrightarrow{P_0P_{|P|-1}} \) and \( \overrightarrow{Q_0Q_{|Q|-1}} \) are not collinear then \( * (P)^* \) and \( * (Q)^* \) intersect at least one time and only a finite number of times. Without loss of generality, suppose that \( P_0 = Q_0 \) is the only intersection between \( * (P)^* \) and \( (Q)^* \), i.e. the last intersection between \( P \) and \( Q \) according to \( Q \). Let \( \overrightarrow{u} = \overrightarrow{P_0P_{|P|-1}} \). Now consider \( i = \min \{ j > 0 : \text{type}(Q_j) = \text{type}(P_k) \text{ for some } 0 \leq k \leq |P| - 1 \} \) (note that \( i \) is correctly defined since \( \text{type}(Q_{|Q|-1}) = \text{type}(Q_0) = \text{type}(P_0) \)). Let \( 0 \leq j < |P| - 1 \) such that the type of \( P_j \) is the same as \( Q_i \), and let \( \overrightarrow{v} = \overrightarrow{P_jQ_i} \). By definition of \( Q_0 \) and since \( i > 0 \), \( Q_i \) is not in \( * (P)^* \) and by lemma 8 \( \overrightarrow{u} \) and \( \overrightarrow{v} \) are not collinear. Moreover, by definition of \( i \) and \( j \), we have \( \text{type}(Q_i) = \text{type}(P_j) \) and by lemma 7 \( * (P)^* + \overrightarrow{v} \) is in \( P[\alpha] \) and by lemma 6 it does not intersect with \( * (P)^* \). Consider the path

\[ R = P_j, P_{j-1}, \ldots, P_0, Q_1, Q_2, \ldots, Q_i. \]

By definition of \( i \) and \( j \), the path \( R \) is without redundancy and satisfies the hypothesis of lemma 5 then \( R \in P[\alpha] \) is bi-pumpable. Remind that \( \overrightarrow{v} = \overrightarrow{P_jQ_i} \).
a) The bi-pumpable path without redundancy $P$ and the tile $A$.

b) The path $^*(P) + P_iA$ can be assembled and we can consider a path $Q$ from $P_i$ to a tile of the same tile type. This path contains a bi-pumpable path without redundancy $R$ (which satisfies the hypothesis of the lemma in this case).

c) If $P_0P_{|P|-1}$ is collinear with $R_0R_{|R|-1}$ then we can build a new shorter path with the same properties as $Q$.

Figure 3.3: Illustration of the proof of lemma 10. Consider a bi-pumpable path $P$ without redundancy such that a tile $A$ has the same tile type as a tile of $P$, then we can find another bi-pumpable path $R$ without redundancy such that $P_0P_{|P|-1}$ is not collinear with $R_0R_{|R|-1}$.
\( R_0 \rightarrow \bigcup_{|\ell| - 1} \) and then by corollary 1, \( \alpha \) is \( \overrightarrow{v} \)-periodic. Let

\[ Q' = Q_{0,1,\ldots,i}. \]

Let \( P' \) be the subpath of \( *(P)* \) such that \( P'_0 = P_0 = Q_i - \overrightarrow{v} \), \( P'_{|P|-1} = P_j + \overrightarrow{u} = Q_j - \overrightarrow{u} + \overrightarrow{u} \). Since \( *(P)* + \overrightarrow{u} \) does not intersect with \( *(P)* \), then \( P \) does not intersect with \( P' + \overrightarrow{v} \). Similarly, since \( \overrightarrow{u} \) and \( \overrightarrow{0}Q|Q|^{-1} \) are not collinear, \( Q \) does not intersect with \( Q + \overrightarrow{u} \) which implies that \( Q' \) does not intersect with \( Q' + \overrightarrow{u} \). By definition of \( Q' \), the only intersection between \( Q' \) and \( P \) is \( P'_0 = P_0 \). Similarly the only intersection between \( Q' + \overrightarrow{u} \) and \( P \) is \( Q'_0 + \overrightarrow{u} = P_{|P|-1} \). The only intersection between \( Q' \) and \( P' + \overrightarrow{v} \) is \( Q'_{|Q'|-1} = P'_0 + \overrightarrow{v} \) by definition of \( Q'_{|Q'|-1} = Q_i \) (and since \( P' \) and \( Q \) shares only one common tile type) and similarly, the only intersection between \( Q' + \overrightarrow{u} \) and \( P' + \overrightarrow{v} \) is \( Q'_{|Q'|-1} + \overrightarrow{u} = P'_{|P'|-1} + \overrightarrow{v} \). Then the binding paths of the four paths \( Q' \), \( P \), \( P' + \overrightarrow{u} \) and \( P' + \overrightarrow{v} \) form a simple cycle \( C \) and let \( \beta \) be the finite assembly which is the restriction of \( \alpha \) to the interior of this cycle. Since \( Q' \) and \( P \) are without redundancy and share only one common tile type then \( |Q'| + |P'| \) is bounded by \( |T| + 2 \) and thus the length of \( C \) is bounded by \( 2|T| \) and its area is bounded by \( |T|^2 \). Then the size of \( \beta \) is bounded by \( |T|^2 \). Consider the assembly \( \bigcup_{\ell, \ell' \in \mathbb{Z}} (\beta + \ell \overrightarrow{u} + \ell' \overrightarrow{v}) \), this assembly is correctly defined: there are no conflict for any \( \ell, \ell' \in \mathbb{Z} \) between \( \beta \) and \( \beta + \ell \overrightarrow{u} + \ell' \overrightarrow{v} \) since \( \alpha \) is \( \overrightarrow{u} \)-periodic and \( \overrightarrow{v} \)-periodic and this assembly is connected since \( Q + \overrightarrow{u} \) is in both \( \beta \) and \( \beta + \overrightarrow{u} \) and \( P'_0 + \overrightarrow{v} \) is in both \( \beta \) and \( \beta + \overrightarrow{v} \). Then \( \bigcup_{\ell, \ell' \in \mathbb{Z}} (\beta + \ell \overrightarrow{u} + \ell' \overrightarrow{v}) \) is a subassembly of \( \alpha \). Now, for any \( (x, y) \in \mathbb{Z}^2 \) there exist \( \ell \) and \( \ell' \) such that \( (x, y) + \ell \overrightarrow{u} + \ell' \overrightarrow{v} \) is in the interior of the simple cycle \( C \) and thus for any tile \( A \) of \( \alpha \), there are \( \ell \) and \( \ell' \) and a tile \( B \) of \( \beta \) such that \( B = A + \ell \overrightarrow{u} + \ell' \overrightarrow{v} \). Thus, \( \bigcup_{\ell, \ell' \in \mathbb{Z}} (\beta + \ell \overrightarrow{u} + \ell' \overrightarrow{v}) = \alpha \).

Now we can prove the main theorem which describes the bi-periodic terminal assemblies.

**Theorem 8** (Description of the bi-periodic terminal assemblies). Consider a tile assembly system \( T = (T, \sigma, 1) \) whose terminal assembly \( \alpha \) is bi-periodic. Then there exist two non-collinear vectors \( \overrightarrow{u} \) and \( \overrightarrow{v} \) and an assembly \( \beta \) of complexity 0 whose size is bounded by \( |T|^2 \) such that

\[
\alpha = \bigcup_{\ell, \ell' \in \mathbb{Z}} (\beta + \ell \overrightarrow{u} + \ell' \overrightarrow{v}).
\]

**Proof.** Since \( \alpha \) is bi-periodic by lemma 2 there exists \( P \in \mathbf{P}[\alpha] \) which is bi-pumpable. By corollary 2 there exists a path without redundancy \( P' \in \mathbf{P}[\alpha] \) which is bi-pumpable. Since \( \alpha \) is bi-pumpable there exists a vector \( \overrightarrow{w} \) such that \( \alpha \) is \( \overrightarrow{w} \)-periodic and \( \overrightarrow{w} \) is not collinear with \( \overrightarrow{P_0P_{|P|-1}} \). Thus \( P_0 + \overrightarrow{w} \) is a tile of \( \alpha \) and is not a tile of \( *(P)* \) and by lemma 10 there exists a path without redundancy \( Q \in \mathbf{P}[\alpha] \) which is bi-pumpable and such that \( \overrightarrow{P_0P_{|P|-1}} \)
a) The two bi-pumpable paths without redundancy $P$ and $Q$.

b) $Q'$ is a prefix of $Q$ which ends where a tile of $Q$ has the same tile type than a tile of $P$ for the first time.

c) We grow $P' + u$ from $Q'|^{-1}$ to $Q'|^{-1} + v$ and we obtain a path $P'$.

d) $Q' + v$ binds to $P'|^{-1}$ and we obtain a simple cycle which delimits a finite area of the 2D plane.

e) This cycle tiles the 2D plane.

We consider two bi-pumpable paths without redundancy $P$ and $Q$ such that $P_0 - P$ is not collinear with $Q_0 - Q$, and we show that the 2D plane can be filled by these paths.
and $Q_0Q_{|Q|-1}$ are not collinear. By lemma 11, there exists a finite assembly $\beta$ and two vectors $\vec{u}$ and $\vec{v}$ such that $\alpha = \bigcup_{i \leq \ell \in \mathbb{Z}} (\beta + \ell \vec{u} + \ell' \vec{v})$. Moreover the size of $\beta$ is bounded by $|T|^2$ and the complexity of a finite assembly is 0 by definition.

### 3.1.4 Ordering the bi-pumpable paths

**Lemma 12.** Consider a tile assembly system $\mathcal{T} = (T, \sigma, 1)$ whose terminal assembly $\alpha$ is simply periodic and a bi-pumpable path $P \in P[\alpha]$ without redundancy. If an an arc $Q$ grows in the left side of $P$ then there exists a bi-pumpable path $R \in P[\alpha]$ without redundancy such that $R > P$.

**Proof.** Firstly, if $Q$ satisfies the hypothesis of lemma 5 (note that by definition of an arc both extremities of $Q$ are in $(P)'$) then there exists $0 \leq i < j \leq |Q| - 1$ such that $R = Q_{i,j}$ is a path without redundancy which is bi-pumpable. If $P_0P_{|P|-1}$ and $R_0R_{|R|-1}$ are not collinear then $\alpha$ is bi-periodic (by corollary 1) which contradicts the hypothesis of the lemma. By lemma 9, $P_0P_{|P|-1} = sR_0R_{|R|-1}$ with $s \in \{-1, 1\}$ then since $Q$ grows in the left side of $(P)^*$ then $R$ and $(R)^*$ are in $L((P)^*)$ and then $R \geq P$ Moreover, $(P)^* = (R)^*$ would contradicts the definition of an arc and then $R > P$.

Otherwise, $Q$ does not contains a path without redundancy. Then $Q$ and $Q + P_0P_{|P|-1}$ cannot intersect otherwise then would exist $0 \leq i, j \leq |Q| - 1$ such that $Q_i = Q_j + P_0P_{|P|-1}$ with $i \neq j$ (since $P_0P_{|P|-1}$ is not null). Then, $\text{type}(Q_i) = \text{type}(Q_j)$ and $Q$ would satisfy the hypothesis of lemma 5 and we would be in the previous case. Since $Q$ does not intersect $Q + P_0P_{|P|-1}$ then the width of $Q$ is less than $|P| - 1$. Then, without loss of generality we can suppose that there exists $0 \leq i < j \leq |P| - 1$ such that $Q$ starts in $P_i$ and ends in $P_j$. Consider the path

$$R = P_{0,1,...,|Q|-2}Q_{1,2,...,|Q|-2}P_{j+1,...,|P|-1}.$$ 

By definition of an arc $R \neq P$. If $R$ is not without redundancy then there exist $0 \leq k \leq |P| - 1$ and $1 \leq k' \leq |Q| - 2$ such that $\text{type}(P_k) = \text{type}(Q_{k'})$. By definition of an arc $Q_{k'}$ is not a tile of $(P)^*$ and by lemma 10 and corollary 1 \alpha is bi-periodic which contradicts the hypothesis of the lemma. By lemma 9, $R$ is bi-pumpable and by definition of $R$ and $Q$, $R$ is in $L((P)^*)$ and $R_0R_{|R|-1} = P_0P_{|P|-1}$ then $(R)^*$ is in $L((P)^*)$ and thus $R \geq P$. Finally, since $R \neq P$ then $R > Q$. \qed

**Lemma 13.** Consider a tile assembly system $\mathcal{T} = (T, \sigma, 1)$ whose terminal assembly $\alpha$ is simply periodic and two bi-pumpable paths $P$ and $Q$ of $P[\alpha]$ such that $P$ is without redundancy then either:

- $P \geq Q$;
- there exists a bi-pumpable path $R \in P[\alpha]$ without redundancy such that $R > P$.

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Proof. If \(* (P)^*\) does not intersect \(* (Q)^*\) then either \(P \succ Q\) or \(P \prec Q\). In the second case by lemma 3 and 5 there exist \(0 \leq i < j \leq |Q|\) such that \(R = Q_{i,j+1,\ldots,j}\) is bi-pumpable without redundancy. By corollary 1 \(P_0P_{|P|-1}\) and \(Q_0Q_{|Q|-1}\) are collinear otherwise \(\alpha\) would not be simply periodic. Moreover, by lemma 9 \(P_0P_{|P|-1} = sQ_0Q_{|Q|-1}\) for some \(s \in \{-1, 1\}\). Since \(* (P)^*\) does not intersect \(* (Q)^*\) then \(* (R)^*\) is in the left side of \(* (P)^*\) (since \(P \prec Q\)) and the lemma is true.

Otherwise, \(* (P)^*\) and \(* (Q)^*\) intersect. By corollary 1 \(P_0P_{|P|-1}\) and \(Q_0Q_{|Q|-1}\) are collinear otherwise \(\alpha\) would not be simply periodic. Moreover, since \(P\) is without redundancy then by lemma 9 \(Q_0Q_{|Q|-1} = \ell P_0P_{|P|-1}\) for some \(\ell \in \mathbb{N}^*\). Thus, there are an infinity of intersections between \(* (P)^*\) and \(* (Q)^*\). Then either \(P \succ Q\) or then there exist \(i < j\) such that \(* (Q)_{i,j+1,\ldots,j}\) is an arc in the left side of \(* (P)^*\) and by lemma 12 there exists a bi-pumpable path \(R \succ P\) of \(P_0\).

\[\square\]

**Lemma 14.** Consider a tile assembly system \(T = (T, \sigma, 1)\) whose terminal assembly \(\alpha\) is simply periodic. Then, there exist a path \(P^+\) (resp. \(P^-\)) without redundancy which is bi-pumpable and maximum (resp. minimum). Moreover, \(P_0^+P_{|P^+|-1}^+ = P_0^-P_{|P^-|-1}^-\).

Proof. Since \(\alpha\) is simply periodic then by lemma 2 there exists a bi-pumpable path \(P^{(0)}\) in \(P_0\) and by corollary 2 we can assume that \(\alpha^{(0)}\) is without redundancy. By lemma 13 either \(\alpha^{(0)}\) is maximum or there exists another bi-pumpable path \(P^{(1)}\) in \(P_0\) without redundancy such that \(P^{(1)} > \alpha^{(0)}\).

Two cases may occur: either \(|P^{(1)}| \prec |P^{(0)}|\) or \(|P^{(1)}| \succ |P^{(0)}|\). In the second case, since \(* (P^{(1)})^* \neq * (P^{(0)})^*\) there exists \(0 \leq i_1 \leq |P^{(1)}| - 1\) such that \(P^{(1)}_{i_1}\) is not a tile of \(P^{(0)}\). For the sake of contradiction, suppose that there more than \(|T|^2\) bi-pumpable paths without redundancy of \(P_0\) such that \(P^{(0)} < P^{(1)} < P^{(2)} < \ldots < P^{(|T|)}\). Since the length of a path without redundancy is at least 2 and less than \(|T| + 1\) then the case where for some \(0 \leq i \leq |T|^2\), \(P_{i+1}^{(i)} \prec P^{(i)}\) occurs at most \(|T|^2\) times consecutively. Since we have \(|T|^2\) paths, there exist \(i < j\) such that \(P^{(i)} \prec P^{(j)}\) and there exists \(0 \leq k \leq |P^{(j)}|\) such that the tile \(P_{k}^{(j)}\) is in the left side of \(* (P^{(j-1)})^*\), is not a tile of \(* (P^{(j-1)})^*\) and has the same tile type as a tile of \(P^{(j)}\). Since \(* (P^{(i)})^*\) is in the right side of \(* (P^{(j-1)})^*\), then the tile \(P_{k}^{(j)}\) is also on the left side of \(* (P^{(i)})^*\) and is not a tile of \(* (P^{(i)})^*\). Thus by lemmas 8 and 10 and corollary 2 \(\alpha\) is bi-periodic which contradicts the hypothesis of the lemma. Thus there exists a maximal bi-pumpable path \(P^+\) without redundancy and by lemma 13 \(P^+\) is also maximum. A similar reasoning shows that there is a minimum path \(P^-\). By lemma 1 \(P_0^+P_{|P^+|-1}^+\) and \(P_0^-P_{|P^-|-1}^-\) are collinear otherwise \(\alpha\) would not be simply periodic. By lemma 9 \(P_0^+P_{|P^+|-1}^+ = sP_0^-P_{|P^-|-1}^-\) for some \(s \in \{-1, 1\}\). If \(s = -1\) then we can consider \(P^{+/-}\) and the lemma is true.

\[\square\]
We introduce now the notion of $\vec{v}$-self-avoiding path, such a path cannot intersect with its translation by $\ell \vec{v}$ for some $\ell \in \mathbb{N}$. This notion will be useful in the next section in order to study the simply pumpable path (these paths are called combs in [2]).

**Definition 11.** Consider a directed tile assembly system $\mathcal{T} = (T, \sigma, 1)$ whose terminal assembly is $\alpha$ and a non null vector $\vec{v}$. A path $P$ is $\vec{v}$-self-avoiding if for any $\ell \in \mathbb{N}^*$, $P$ does not intersect with $P + \ell \vec{v}$ and there exists $L \in \mathbb{N}$ such that for all $\ell \geq L$, $P + \ell \vec{v} \in P[\alpha]$.

**Definition 12.** Consider a directed tile assembly system $\mathcal{T} = (T, \sigma, 1)$ and two non null, non collinear vectors $\vec{u}$ and $\vec{v}$. A path $P$ is $(\vec{u}, \vec{v})$-self-avoiding if for any $\ell, \ell' \in \mathbb{Z}$ such that $(\ell, \ell') \neq (0,0)$, $P$ does not intersect with $P + \ell \vec{u} + \ell' \vec{v}$.

We conclude this section, with the first half of the second main theorem [3]

**Lemma 15.** Consider a tile assembly system $\mathcal{T} = (T, \sigma, 1)$ whose terminal assembly $\alpha$ is simply periodic. There exist a finite assembly $\beta$ and two bi-pumpable paths $P^+$ and $P^-$ such that $\vec{v} = P_0^+ P_{p^+|-1}^+ = P_0^- P_{p^-|-1}^-$ and the restriction of $\alpha$ to $R(\ast(P^+)) \cap L(\ast(P^-))$ is
\[ \bigcup_{\ell \in \mathbb{Z}} (\beta + \ell \vec{v}). \]

Moreover, $\text{asm}(P^+)$ and $\text{asm}(P^-)$ are subassembly of $\beta$, the size of $\beta$ is bounded by $|T|^2$ and no arc grows on the left (resp. right) side of $\ast(P^+)$ (resp. $\ast(P^-)$) and any path growing on the left (resp. right) side of $\ast(P^+)$ (resp. $\ast(P^-)$) is $\vec{v}$-self-avoiding.

**Proof.** Since $\alpha$ is simply periodic, consider the paths $P^+$ and $P^-$ which satisfy the hypothesis of lemma 14 and let $\vec{v} = P_0^+ P_{p^+|-1}^+ = P_0^- P_{p^-|-1}^-$. Consider a path $P$ which grows on $\ast(P^+)$ in its left side. By lemma 14 for all $\ell \in \mathbb{Z}$ the path $P + \ell \vec{v}$ is in $P[\alpha]$ (we can consider that $P_0 + \ell \vec{v}$ is the seed). If for some $\ell \in \mathbb{N}^*$ and $P + \ell \vec{v}$ intersect then they have to agree since they are both in $P[\alpha]$ and they assemble an arc in the left side of $\ast(P^+)$ which by lemma 12 contradicts the maximality of $P^+$. Then, $P$ is $\vec{v}$-self-avoiding. Similarly any path growing in the right side of $\ast(P^-)$ is $\vec{v}$-self-avoiding.

Now, we need analyze the restrictions of $\alpha$ to $R(\ast(P^+)) \cap L(\ast(P^-))$. First, if $\ast(P^+)$ = $\ast(P^-)$, then the lemma is true with $\beta = P^+$. Secondly, if $\ast(P^+)$ and $\ast(P^-)$ intersect then we define $\beta$ as the restriction of $\alpha$ to the interior of $\ast(P^+)$ and $\ast(P^-)$ (see appendix B.1 for this definition in this special case). Otherwise, since $\ast(P^+)$ and $\ast(P^-)$ are in $P[\alpha]$ then there exists a path $Q$ such that the only intersection between $P^+$ (resp. $P^-$) and $Q$ is $Q_0$ (resp. $Q_{|Q| - 1}$). Without loss of generality we suppose that $Q$ is the path of minimal length which satisfies these hypothesis and that $Q_0 = P_0^+$ and $Q_0 = P_0^-$. Note that $Q$ is in $R(\ast(P^+)) \cap L(\ast(P^-))$. For the sake of contradiction, suppose that there exists $0 \leq i < j \leq |Q| - 1$ such that type($Q_i$) = type($Q_j$), then
by lemma \[5\] the path \( R = Q_{1,i+1,\ldots,j} \) is in \( P[\alpha] \) and is bi-pumpable. If \( \vec{v} \) and \( R_0 R_{|R|-1} \) are not collinear then by corollary \[1\] \( \alpha \) is bi-periodic which is a contradiction of the hypothesis. By lemma \[9\] \( R_0 R_{|R|-1} \rightarrow s \vec{v} \) for some \( s \in \{-1,1\} \), then the path \( Q_{0,1,\ldots,i}(Q_{j+1,i+2,\ldots,|Q|-1} - s \vec{v}) \) satisfies the same hypothesis has \( Q \) contradicting its minimal length. If \( Q \) and \( Q + \vec{v} \) intersect then there exists \( i \neq j \) such that type(\( Q_i \)) = type(\( Q_j \)) which as previously leads to a contradiction. Then by definition of \( Q \) and since \( \alpha \) is \( \vec{v} \)-periodic then the binding paths of \( P^+, Q + \vec{v}, P^- \) and \( Q \) form a simple cycle \( C \). We define \( \beta \) as the restriction of \( \alpha \) to the interior of this cycle. Moreover since \( \alpha \) is not bi-periodic then by lemma \[10\] \( P^+ \) and \( P^- \) cannot share a common tile type and the only common tile type between \( P^+ \) (resp. \( P^- \)) and \( Q \) is \( P_0^+ = Q_0 \) (resp. \( P_0^- = Q_{|Q|-1} \)). Then the length of the cycle \( C \) is bounded by \( 2|T| \) and thus the size of \( \beta \) is bounded by \( |T|^2 \). Consider the assembly \( \bigcup_{\ell \in \mathbb{Z}} (\beta + \ell \vec{v}) \), this assembly is correctly defined: there are no conflict for any \( \ell \in \mathbb{Z} \) between \( \beta \) and \( \beta + \ell \vec{v} \) since \( \alpha \) is \( \vec{v} \)-periodic and \( \beta \) is connected since \( Q + \vec{v} \) is in both \( \beta \) and \( \beta + \vec{v} \). Then \( \bigcup_{\ell \in \mathbb{Z}} (\beta + \ell \vec{v}) \) is a subassembly of \( \alpha \). Finally, for any \((x,y) \in \mathbb{Z}^2 \) which is in \( R(*(P^+)) \cap L(*(P^-)) \), there exists \( \ell \) such that \((x,y) + \ell \vec{v} \) is in the interior of \( C \). Thus for any tile \( A \) of the restriction of \( \alpha \) to \( R(*(P^+)) \cap L(*(P^-)) \), there are \( \ell \) and a tile \( B \) of \( \beta \) such that \( B = A + \ell \vec{v} \) (see appendix B.1). Thus, \( \bigcup_{\ell \in \mathbb{Z}} (\beta + \ell \vec{v}) \) is equal to the restriction of \( \alpha \) to \( R(*(P^+)) \cap L(*(P^-)) \).

\[ \square \]

3.2 Simply pumpable path

This section follows the key ideas of the arXiv version of \[2\]. We only give more details in lemma \[20\] and theorem \[10\] on how to deal with the combs (called here \( \vec{v} \)-self avoiding paths).

3.2.1 Growing on a strictly pumpable path

The three following lemmas shows that an arc of width at least \( |P| \) cannot grow on a simply pumpable path \( P \) which implies that any path growing on a simply pumpable path \( P \) is \( P_0 P_{|P|-1} \)-self-avoiding. Figure 3.5 illustrates the reasonings of the three following lemmas.

**Lemma 16.** Consider a tile assembly system \( \mathcal{T} = (T, \sigma, 1) \) whose terminal assembly is \( \alpha \) and a simply pumpable path \( P \) in \( P[\alpha] \). If there exists a path \( Q \) growing on \( (P)^* \) at index \( i \geq |P| - 1 \) such that \( Q \) and \( Q + jP_0 P_{|P|-1} \rightarrow \) intersect for some \( j \in \mathbb{N} \), then there exists a path \( R \) growing on \( (P)^* \) such that \( R \) and \( R + kP_0 P_{|P|-1} \rightarrow \) intersect for some \( k \in \mathbb{N} \) and for all \( \ell \in \mathbb{N} \), \( R + \ell P_0 P_{|P|-1} \rightarrow \) is in \( P[\alpha] \).

**Proof.** Since \( P \) is in \( P[\alpha] \), there exists a finite producible subassembly \( \beta \) (\( \sigma \) is a subassembly of \( \beta \)) such that \( P_0 \) is a tile of \( \beta \). Since \( \beta \) is finite and \( P_0 P_{|P|-1} \rightarrow \) is not null, there exists \( L \in \mathbb{N} \) such that for all \( \ell \geq L \), \( Q + \ell P_0 P_{|P|-1} \rightarrow \) does not
a) The seed is in black, the simply pumpable path $P$ is in gray and its pumping is in white, a blue path binds with the seed and $P_0$ and a path $Q$ (in red) growing on $(P)^*$ intersects with $(P)^*$. Then, it is possible to grow some translations of a prefix of $Q$ (in light red) which intersect.

b) Using the tiles in light red, it is possible to assemble a path of width $2(|P| - 1)$.

c) We try to assemble a translation of the blue path and the seed (in light blue), the previous arc allows us to remove a tile of $(P)^*$ which creates a conflict and a contradiction.

Figure 3.5: Consider a simple pumpable path $P$, if a path grows on $(P)^*$ then it is $P_0P_{|P|-1}$-self-avoiding (see lemmas 16, 17 and 18).
intersect with $\beta$. If for all $\ell \geq L$, $Q + \ell \overrightarrow{P_0P_{|P|−1}}$ does not intersect with $(P)^*$ then $Q + \ell \overrightarrow{P_0P_{|P|−1}}$ is a path of $P[\alpha]$ which grows on $P$. In this case, we set $R = Q + \ell \overrightarrow{P_0P_{|P|−1}}$ and $k = j$ and the lemma is true. Otherwise, we can define $m = \min\{n > 0 : \text{there exists } \ell > L \text{ such that } \text{pos}(Q)_n + \ell \overrightarrow{P_0P_{|P|−1}} \text{ is in pos}((P)^*)\}$.

By definition of $m$, there exists $L' \geq L$ such that $Q_m + L' \overrightarrow{P_0P_{|P|−1}}$ intersects with a tile $P_n$ of $(P)^*$. In this case, we set $k = 1$ and $R = Q_{0,1,\ldots,m−1} + L' \overrightarrow{P_0P_{|P|−1}}$.

By definition of $L'$, $L$ and $m$, for all $\ell \geq 0$, $R + \ell \overrightarrow{P_0P_{|P|−1}}$ grows on $(P)^*$ on tile $P_i + (L' + n(|P|−1))$ and does not intersect with $\beta$ and thus is in $P[\alpha]$. Now, since by hypothesis $Q$ grows on $(P)^*$ then $Q_m$ is not a tile of $(P)^*$ and since $i \geq |P| − 1$, we have $(i + L'(|P| − 1)) − n > |P| − 1$. The binding graph of $R(Q_m + L' \overrightarrow{P_0P_{|P|−1}})$ is the binding graph of an arc of width at least $|P|$ thus $R$ intersects with $R + \ell \overrightarrow{P_0P_{|P|−1}}$.

\begin{lemma}
Consider a tile assembly system $T = (T, \sigma, 1)$ whose terminal assembly is $\alpha$ and a simply pumpable path $P$ in $P[\alpha]$. If there exists a path $Q$ growing on $(P)^*$ such that $Q$ and $Q + i \overrightarrow{P_0P_{|P|−1}}$ intersect for some $i \in \mathbb{N}^*$ and for all $\ell \in \mathbb{N}$, $Q + \ell \overrightarrow{P_0P_{|P|−1}}$ is in $P[\alpha]$, then there exists an arc $A$ of width at least $|P|$ growing on $(P)^*$ such that for all $\ell \in \mathbb{N}$, $A + \ell \overrightarrow{P_0P_{|P|−1}}$ is in $P[\alpha]$.
\end{lemma}

\begin{proof}
Since $Q + i \overrightarrow{P_0P_{|P|−1}}$ intersects with $Q$ and $Q + 2i \overrightarrow{P_0P_{|P|−1}}$ and since all intersections are agreement (these three paths all belong to $P[\alpha]$), then we can define the following assembly:

$$
\beta = Q \cup (Q_{1,2,\ldots,|Q|−1} + i \overrightarrow{P_0P_{|P|−1}}) \cup (Q + 2i \overrightarrow{P_0P_{|P|−1}}).
$$

Since for all $\ell \in \mathbb{N}$, $Q + \ell \overrightarrow{P_0P_{|P|−1}}$ is in $P[\alpha]$, then for all $\ell \in \mathbb{N}$, $\beta + \ell \overrightarrow{P_0P_{|P|−1}}$ is a subassembly of $\alpha$. By definition of growing, the only intersection between $(P)^*$ and $\beta$ is $Q_0$ and $Q_0 + 2i \overrightarrow{P_0P_{|P|−1}}$. Thus there exists an arc $A$ growing on $(P)^*$ of width $2i(|P| − 1) > |P|$ such that $\text{asm}(A)$ is a subassembly of $\beta$. Moreover, for all $\ell \in \mathbb{N}$, $A + \ell \overrightarrow{P_0P_{|P|−1}}$ is in $P[\alpha]$.
\end{proof}

\begin{lemma}
Consider a tile assembly system $T = (T, \sigma, 1)$ whose terminal assembly is $\alpha$ and a simply pumpable path $P$ in $P[\alpha]$ then there exists no arc $A$ of width at least $|P|$ growing on $(P)^*$.
\end{lemma}

\begin{proof}
For the sake of contradiction suppose that exists an arc $A$ of width at least $|P|$ growing on $(P)^*$. Since $P$ is in $P[\alpha]$, there exists a finite subassembly $\beta$ of $\alpha$ such that $\beta$ is producible ($\sigma$ is a subassembly of $\beta$) and $P_0$ is a tile of $\beta$. Since $\beta$ is finite and $P_0P_{|P|−1}$ is not null, there exists $L \in \mathbb{N}$ such that for all $\ell \geq L$, $A + \ell \overrightarrow{P_0P_{|P|−1}}$ does not intersect with $\beta$. If for all $\ell \geq L$, $A + \ell \overrightarrow{P_0P_{|P|−1}}$ does not intersect with $(P)^*$ then $A + \ell \overrightarrow{P_0P_{|P|−1}}$ is a path of $P[\alpha]$. In this case, we set $B = A + \ell \overrightarrow{P_0P_{|P|−1}}$. Otherwise, since the width of $A$ is at least $|P|
we can suppose that $A_0 = P_i$ with $i > |P| - 1$ and there is an intersection between $A$ and $A + P_0P_{|P|-1}$ which does not occur at $\text{pos}(A_0)$ or $\text{pos}(A_{|A|-1})$. Let $Q = A_{0,1,\ldots,|A|-2}$, this path grows on $(P)^*$ at index $i$ and intersects with $Q + P_0P_{|P|-1}$. Then by lemmas 16 and 17 there exists an arc $B$ of width at least $|P|$ growing on $(P)^*$ such that for all $\ell \in \mathbb{N}$, $B + \ell P_0P_{|P|-1}$ is in $P[\alpha]$.

Now, let $0 \leq m < n$ such that the arc $B$ grows between $P_m$ and $P_n$ (then $n - m \geq |P|$). If there is no conflict between $\beta$ and $*(P)^*$ then $P$ would be bi-pumpable which contradicts the hypothesis of the lemma. Then there exists a conflict between $*(P)^*$ and $\beta$. Moreover, there exists $j \in \mathbb{N}$ such that $\beta + j P_0P_{|P|-1}$ does not intersect with $\beta$ and there is at least one conflict between $\beta + j P_0P_{|P|-1}$ and $(P)^*$ at pos$(P_k)$ for some $k > m$. Let $t$ be the tile such that pos$(t) = \text{pos}(P_k)$ and type$(t) = (\beta + j P_0P_{|P|-1})($pos$(P_k))$ then type$(t) \neq \text{type}(P_k)$. By definition of $\beta$, there is at least one agreement between $\beta + j P_0P_{|P|-1}$ and $(P)^*$ at $P_0 + j P_0P_{|P|-1}$. Thus, there exists a path $R$ which grows on $P$ such that $\text{asm}(R)$ is a sub-assembly of $\beta + j P_0P_{|P|-1}$ and the tile $R_{|P|-1}$ interacts with $t$. Moreover, since $\beta + j P_0P_{|P|-1}$ does not intersect with $\beta$ then $R$ is in $P[\alpha]$. Now since $k > m$ and $n - m \geq |P|$, there exists $\ell$ such that $m + \ell(|P|-1) < k < n + \ell(|P|-1)$. We remind that $\beta$ interacts with $(P)^*$ by definition, that $B + \ell P_0P_{|P|-1}$ and $R$ grow on $(P)^*$ and are in $P[\alpha]$, then the following assembly is correctly defined:

$$
\gamma = \beta \cup (P)^* \cup (B + \ell P_0P_{|P|-1}) \cup R.
$$

Moreover since $\beta$ is producible then $\gamma$ is also producible. Removing the tile $P_k$ in $(P)^*$ disconnects the path in two parts but adding $B + \ell P_0P_{|P|-1}$ reconnects these two parts, then it is possible to remove the tile $P_k$ in $\gamma$ and since pos$(t) = \text{pos}(P_k)$ and $t$ interacts with $R_{|P|-1}$, it is possible to switch $P_k$ by $t$ in $\gamma$. Since type$(P_k) \neq \text{type}(t)$, $(T, \sigma, 1)$ is not directed which contradicts the hypothesis.

**Corollary 3.** Consider a tile assembly system $T = (T, \sigma, 1)$ whose terminal assembly is $\alpha$, a simply pumpable path $P$ in $P[\alpha]$ and a path $Q$ growing on $(P)^*$ at index $i \geq |P| - 1$ then $Q$ is $P_0P_{|P|-1}$-self-avoiding and for all $\ell \in \mathbb{N}$ $Q + \ell P_0P_{|P|-1}$ grows on $(P)^*$.

**Proof.** For the sake of contradiction, suppose that there exists a path $Q$ growing on $(P)^*$ on a tile $P_i$ with $i \geq |P| - 1$ which is not $P_0P_{|P|-1}$-self-avoiding. If for some $j \in \mathbb{N}^*$, $Q$ intersects with $Q + j P_0P_{|P|-1}$ then by lemmas 16 and 17 there exists an arc $A$ growing on $P$ of width at least $|P|$ which is a contradiction of lemma 18. By observation 6 for all $\ell \in \mathbb{N}$, $Q + \ell P_0P_{|P|-1}$ grows on $(P)^*$. Finally, consider a finite assembly $\beta$ producible by $(T, \sigma, 1)$ such that $P_0$ is a tile of $\beta$, then $\beta \cup (P)^*$ is producible by $(T, \sigma, 1)$. Since $\beta$ is finite there exists $L \in \mathbb{N}$ such that $Q + \ell P_0P_{|P|-1}$ does not intersect with $\beta$ and thus for all $\ell \geq L$,
\[ \beta \cup (P)^* \cup (Q + \ell P_0 P_{|P| - 1}) \text{ is producible by } (T, \sigma, 1) \text{ and then } Q + \ell P_0 P_{|P| - 1} \text{ is in } P[\alpha]. \]

The following lemmas shows that any path growing on a \( \mathcal{V} \)-self-avoiding path \( P \) is \((\mathcal{V}, P_0 P_{|P| - 1})\)-self-avoiding.

**Lemma 19.** Consider a tile assembly system \( T = (T, \sigma, 1) \) whose terminal assembly is \( \alpha \), a simply pumpable (resp. bi-pumpable) path \( P \) in \( P[\alpha] \) such that \((P)^* \) (resp. \( \ast(P)^* \)) is \( \mathcal{V} \)-self-avoiding (with \( \mathcal{V} \) not collinear with \( P_0 P_{|P| - 1} \)) then any path growing on \((P)^* \) (resp. \( \ast(P)^* \)) at index \( i \geq |P| - 1 \) (resp. \( i \in \mathbb{Z} \)) is \((\mathcal{V}, P_0 P_{|P| - 1})\)-self-avoiding.

**Proof.** Consider a path \( Q \) growing on \((P)^* \) (resp. \( \ast(P)^* \)) at index \( i \geq |P| - 1 \) (resp. \( i \in \mathbb{Z} \)). For the sake of contradiction suppose that \( Q \) is not \((\mathcal{V}, P_0 P_{|P| - 1})\)-self-avoiding, then there exists \( \ell, \ell' \in \mathbb{Z} \) such that \( Q \) intersect with \( R = Q + \ell \mathcal{V} + \ell' P_0 P_{|P| - 1} \). See Figure 3.6 for an illustration of the following reasoning. If \( \ell = 0 \) then \( Q \) is not \( P_0 P_{|P| - 1} \)-self-avoiding which contradicts corollary 3. Remark that \( Q \) cannot intersect with \((P)^* \) except at \( Q_0 \) because \( Q \) grows on \((P)^* \) and if it intersects with \((P)^* \) then there exists \( j \in \mathbb{N} \) such that \( Q + j \mathcal{V} \) intersects with \((P)^* \) and thus \( Q + j \mathcal{V} \) would not grow on \((P)^* \) which contradicts corollary 3 (resp. \( Q \) grows on \( \ast(P)^* \)). Without loss of generality we suppose that \( Q \) turns left of \((P)^* \) (resp. \( \ast(P)^* \)) and then \( Q \) is in the left side of \( \ast(P)^* \) which implies that \( R \) is in the left side of \( \ast(P)^* + \ell \mathcal{V} \). Paths \( \ast(P)^* + \ell \mathcal{V} \) and \( \ast(P)^* \) cannot intersect since \((P)^* \) (resp. \( \ast(P)^* \)) is \( \mathcal{V} \)-self-avoiding and if \( \ast(P)^* + \ell \mathcal{V} < \ast(P)^* \), remark that there is also an intersection between \( Q - \ell \mathcal{V} - \ell' P_0 P_{|P| - 1} \) and \( R - \ell \mathcal{V} - \ell' P_0 P_{|P| - 1} = Q \) and that \( \ast(P)^* - \ell \mathcal{V} > \ast(P)^* \) in this case. Then without loss of generality, we can suppose that \( \ast(P)^* + \ell \mathcal{V} > \ast(P)^* \) and then \( Q_0 \) is in the right side of \( \ast(P)^* + \ell \mathcal{V} \) thus the path \( Q \) must intersect with \( \ast(P)^* + \ell P_i P_{j} \) to intersect with \( R \). In this case, since \( Q \) is \( P_i P_{j} \)-self-avoiding (by corollary 3) there exists \( L \in \mathbb{Z} \) such that \( Q + L P_i P_{j} \) and \( Q + (L + 2) P_i P_{j} \) are in \( P[\alpha] \), grow on \((P)^* \) and intersect with \((P)^* + \ell \mathcal{V} \). Moreover, we can suppose that \((P)^* + \ell \mathcal{V} \) is in \( P[\alpha] \) (up to some translation since \((P)^* \) is \( \mathcal{V} \)-self-avoiding) and then there exists an arc of \( P \) of width \( 2(|P| - 1) > |P| \) which is a subassembly of \( Q + L P_i P_{j}, Q + (L + 2) P_i P_{j} \) and \((P)^* + \ell \mathcal{V} \). This arc contradicts lemma 18.

The following lemma shows that for any simple pumpable path \( P \) there is an index \( i \) such that any path growing on \((P)^* \) after this index belongs to \( P[\alpha] \). This lemma is useful to study the terminal assembly since it implies that ultimately we do not care about what happens near the seed when analyzing a periodic path.

**Lemma 20.** Consider a tile assembly system \( T = (T, \sigma, 1) \) whose terminal assembly is \( \alpha \), a simply pumpable path \( P \) in \( P[\alpha] \) and a producible finite assembly
Figure 3.6: Illustration of proof [19]. Consider a simply pumpable path $P$ in gray which is $\vec{v}$-self-avoiding. Its pumping and its translation by $\vec{v}$ are in white. A path $Q$ (in dark red) grows on $(P)^*$ and collides with $Q + \vec{w}$ (in red), then using two translations of $Q$ (in light red) by $P_0 P_{|P|-1}$ and $3P_0 P_{|P|-1}$, we assemble an arc of width $2(|P| - 1)$ on $(P)^*$ with the help of $(P)^* + \vec{w}$. 
β such that $P_0$ is a tile of β. There exists an index $i$ such that any path growing on $(P)^*$ at index $j \geq i$ is in $P[\alpha]$. Moreover, this index $i$ depends only of $|\beta|$, $P$ and $|T|$.

Proof. Without loss of generality we suppose that $P_0$ is the only intersection between $\beta$ and $(P)^*$. Let $j = (4|\beta| + 2)(|P| - 1) + 1$ and consider the assembly $\gamma = \beta \cup (P)^*_{i,i+1,...,j}$, by definition of $\beta$, this assembly intersects with $P$ and all intersections are agreement, then $\gamma$ is producible by $= (T, \sigma, 1)$. Any assembly producible by $(T, \gamma, 1)$ is also producible by $(T, \sigma, 1)$ then $(T, \gamma, 1)$ is directed and its terminal assembly is also $\alpha$ (since $\gamma$ is a sub-assembly of $\alpha$). Since $\beta$ is finite and $P_0P_{|P|-1}$ is not null there exists $i > j + (|P| - 1)$ such that the distance between any tile of $(P)^*_{i,i+1,...,j}$ and any tile of $\beta$ is at least $f(|[T], |\gamma|)| + 1$ (see lemma 5). Remark that $i$ depends only of $|\beta|$, $P$ and $|T|$. See Figure 3.7 for an illustration of the following reasoning.

Consider a path $Q$ which grows on $(P)^*$ at position $P_k$ with $k \geq i$ and for the sake of contradiction, suppose that this path is not in $P[\alpha]$ which implies that $Q$ conflicts with $\beta$ and by the definition of $i$ the vertical height or the horizontal width of $Q$ is at least $f(|[T], |\gamma|)| + 1$. Let $m = \min\{n : pos(Q_n) \in dom(\beta)|$ and by definition of $m$, $Q_{0,1,...,m-1}$ is in $P[\alpha]$. Consider a finite simple cycle C which is made of the binding path of $(P)^*_{0,1,...,k}$, the binding path of $Q_{0,1,...,m}$ and a path in the binding graph of $\beta$ which links $pos(P_0)$ to $pos(Q_m)$. Let $R$ be the translation of $Q$ by $\ell P_0P_{|P|-1}$ for some $\ell \in \mathbb{N}$ such that $R_0$ is a tile of $P + P_0P_{|P|-1}$. Remark that either for all $\ell \in \mathbb{N}$, $R + \ell$ turns left of $(P)^*$ or for all $\ell \in \mathbb{N}$, $R + \ell$ turns right of $(P)^*$. In both cases, for all $\ell \in \mathbb{N}$, the tile $R_1 + \ell P_0P_{|P|-1}$ is in the interior of the cycle $C$. Remark that, there are at most $4|\beta|$ positions which are neighbors of a tile of $\beta$ which implies that if for all $0 \leq \ell \leq 4|\beta|$, the path $R + \ell P_0P_{|P|-1}$ conflicts with $\beta$ then there exists $\ell \neq \ell' \in \mathbb{N}$ such that $R + \ell P_0P_{|P|-1}$ and $R + \ell' P_0P_{|P|-1}$ intersect before intersecting with $\beta$ which contradicts corollary 3. Thus, there exists $0 \leq \ell \leq 4|\beta|$ such that $R + \ell P_0P_{|P|-1}$ does not intersect with $\beta$. Moreover, by definition of $R$, $R_0 + \ell P_0P_{|P|-1}$ is a tile of $(P)^*_{|P|+1,|P|-1,..,(4|\beta|+1)(|P|-1)}$ and then the path $R_{2,3,...,|P|-1} + \ell P_0P_{|P|-1}$ is producible by $(T, \gamma, 1)$ and is in $P[\alpha]$. Since the vertical height or the horizontal width of $R$ is at least $f(|[T], |\gamma|)|$, by the pumping lemma 5 $R_{1,2,...,|P|-1} + \ell P_0P_{|P|-1}$ is infinitely pumpable and let $S$ be its pumping (see definition 8). By definition of a pumping $S_0 = R_1 + \ell P_0P_{|P|-1}$ and $S$ is producible by $(T, \gamma, 1)$ and thus cannot intersect with $\gamma$. Moreover, since $S$ is infinite, it cannot stay in the interior of $C$ and thus it must intersect with either $Q_{0,1,...,m-1}$ or $(P)^*_{j+1,j+1,...,k}$. In the first case, tile $R_0 + \ell P_0P_{|P|-1}$, path $Q_{0,1,...,m-1}$ and path $S$ assemble an arc of $(P)^*$ of width at least $i - j \geq |P|$ (since $R_0 + \ell P_0P_{|P|-1}$ is a tile of $(P)^*_{0,1,...,j}$ and $k \geq i > j + (|P| - 1)$ and in the second case, tile $R_0 + \ell P_0P_{|P|-1}$ and a prefix of $S$ assemble an arc of $(P)^*$ of width at least $|P|$ (since $R_0 + \ell P_0P_{|P|-1}$ is a tile of $(P)^*_{0,1,...,(4|\beta|+1)(|P|-1)}$ and $j - 4|\beta| + 1)(|P| - 1) > |P|$). Both cases contradict lemma 18.
a) The seed is in black, the simply pumpable path $P$ is in dark gray and its pumping is in light gray until index $j$ and in white afterwards, a blue path binds with the seed and $P_0$ and $Q$ (in red) intersects with the seed.

b) If a path growing on the pumping between indices $|P| - 1$ and $j - |P| - 1$ intersects with $Q$ or the white part of the pumping then we have an arc of width at least $|P|$.

c) All the translations of $Q$ growing on the light gray part of the pumping start in the red area of the grid. The seed and the blue path cannot block them all without creating intersections. One of the translations must fully grow and becomes pumpable. Its pumping must leave the finite red area.

Figure 3.7: Illustration of proof [20]
3.2.2 Proving the last main theorems

We focus on the terminal assembly $\alpha$ of an aperiodic deterministic tile assembly system $T = (T, \sigma, 1)$ (Appendix C shows the analysis of the tile assembly system described in Figure C.1 and its terminal assembly is shown in Figure C.2). Firstly, from the seed, we grow all the finite paths which are not infinitely pumpable and all the prefixes of the infinitely pumpable paths until the end of the first simply pumpable path which appears in these paths (excluding the last tile of the simply pumpable path) as explained in Theorem 10 and shown in Figure C.3. The pumping lemma provides a bound on the length of these paths and thus we obtain a finite assembly (the gray one in Figure C.3). The complexity of the assemblies growing at the end of the simply pumpable paths is less than 2 (the red, green, orange and blue ones in Figure C.3). Secondly, we study these assemblies of complexity 2 as explained in lemma 23. Lemma 20 allow us to find an index where there is no more intersection with the previous assembly (the gray one in Figure C.4) after this index. The analysis of the assemblies of complexity 2 is made in two parts according to this index: the assembly growing after this index is obtained by "pumping" an assembly of complexity less than 1; before this index, the previous assembly (the gray one in Figure C.4) may block the grow of some paths, then we use a finite assembly to memorize what is happening in the finite area where these interactions may occur and only some assemblies of complexity 1 may grow on this finite assembly (see the red and green assemblies in Figure C.4). We proceed similarly to analyze the assemblies of complexity 1 as explained in lemma 22 and shown in Figure C.5. No more pumpable paths can grow after this step as explained in lemma 21.

Note that the algorithm explained here is described in reverse order in the following lemmas where we start by analyzing the assemblies of complexity 0, then 1, then 2 and finally we analyze the terminal assembly in the two last remaining theorem.

**Lemma 21.** Consider a tile assembly system $T = (T, \sigma, 1)$. If a path $P$ is $(\vec{u}, \vec{v})$-self-avoiding, then $P$ is finite and its length is bounded by a function depending only of $||\vec{u}||$ and $||\vec{v}||$.

**Proof.** We consider $\mathbb{R}^2$, the continuous $2D$ plane. Consider the line $L$ (resp. $L'$) passing by $(0, 0)$ of direction $\vec{u}$ (resp. $\vec{v}$). Consider the finite cycle $C$ which is the polygon defined by the four points $(0, 0), (0, 0) + \vec{u}, (0, 0) + \vec{u} + \vec{v}, (0, 0) + \vec{v}$. Let $t \in \mathbb{N}$ be the number of positions of $\mathbb{Z}^2$ which are inside the interior of $C$, note that $t$ is correctly defined since the area of the interior of $C$ is finite. Moreover, $t$ is bounded by a function depending only of $||\vec{u}||$ and $||\vec{v}||$. Without loss of generality, we can suppose that $L(\mathbb{R})$ (resp. $L'(\mathbb{R})$) is included into the left hand side of $L + \vec{v}$ (resp. $L' + \vec{u}$). Then, the interior of this cycle is the intersection between the right hand side of $L$, the left hand side of $L + \vec{v}$, the right hand side of $L'$ and the left hand side of $L' + \vec{u}$. Thus any for any $(x, y) \in \mathbb{Z}^2$ there exists $\ell \in \mathbb{Z}$ and $\ell' \in \mathbb{Z}$ such that $(x, y) + \ell \vec{u} + \ell' \vec{v}$ is in the interior of $C$. Finally, if $|P|$ is greater than $t$ then there exists $i, j$ and $\ell, \ell' \in \mathbb{Z}$.
such that \( \text{pos}(P_i) = \text{pos}(P_j) + \ell \vec{u} + \ell \vec{v} \) and \( P \) is not \((\vec{u}, \vec{v})\)-self-avoiding. □

**Lemma 22.** Consider a tile assembly system \( T = (T, \sigma, 1) \) whose terminal assembly is \( \alpha \), a simply pumpable path \( P \) in \( P[\alpha] \) which is \( \vec{v} \)-self-avoiding (with \( \vec{v} \) not collinear with \( P_0P_{|P|-1} \)) and a producible finite assembly \( \beta \) such that \( P_0 \) is a tile of \( \beta \). Then the union of \( (P)^* \) and all paths of \( P[\alpha] \) growing on \( (P)^* \) on a tile \( (P)^*_i \) with \( i \geq |P| - 1 \) is an assembly of complexity 1 whose domain is a semilinear set whose size depends of \( |\beta|, \vec{v}, P \) and \( |T| \).

**Proof.** All paths growing on \( (P)^* \) on a tile \( (P)^*_i \) with \( i \geq |P| - 1 \) are \((\vec{v}, P_0P_{|P|-1})\)-self-avoiding (by lemma 19) and are thus finite and their length bounded by a function which depends only of \( \vec{v} \) and \( P \) (by lemma 21). We remind that a finite path has a complexity of 0. By lemma 20, there exists an index \( j \) such that all paths growing on \( (P)^* \) at index \( k > j \) are in \( P[\alpha] \), this index depends on \( |\beta|, P \) and \( |T| \). Remark, that there is only a finite number of different paths of \( P[\alpha] \) which can grow on \( (P)^* \) at index \( |P| - 1 \leq k \leq j - 1 \) since their length is bounded by a function which depends only of \( \vec{v} \) and \( P \). Thus, we consider the assembly \( \gamma_0 \) which is the union of \( (P)^*_0, \ldots, \gamma_0 \) and all paths of \( P[\alpha] \) growing on \( (P)^* \) at index \( |P| - 1 \leq k \leq j - 1 \), this assembly is a finite union of finite paths and is thus finite (and its complexity is 0), its size depends of \( |\beta|, \vec{v}, P \) and \(|T|\) and by its definition it is a subassembly of \( \alpha \). Similarly, consider the assembly \( \gamma_1 \) which is the union of \( (P)^*_j, \ldots, |P| - 1 \) and all paths of \( P[\alpha] \) growing on \( (P)^* \) at index \( j \leq k \leq j + |P| - 2 \), this assembly is a finite union of finite paths and is thus finite (and its complexity is 0), its size depends of \( |\beta|, \vec{v}, P \) and \(|T|\) and by its definition it is a subassembly of \( \alpha \). Remark that for all \( \ell \in \mathbb{N} \), \( \gamma_1 + \ell P_0P_{|P|+1} \) intersects with \( \gamma_1 + (\ell + 1)P_0P_{|P|+1} \) and that \( \gamma_1 + \ell P_0P_{|P|+1} \) is sub-assembly of \( \alpha \) by definition of \( j \) and by corollary 3 and then the assembly \( \bigcup_{\ell \in \mathbb{N}} (\gamma_1 + \ell P_0P_{|P|+1}) \) is correctly defined. Moreover, this assembly is of complexity 1. Consider the assembly:

\[
\gamma = \gamma_0 \cup \left( \bigcup_{\ell \in \mathbb{N}} (\gamma_1 + \ell P_0P_{|P|+1}) \right) .
\]

This assembly is a subassembly of \( \alpha \) and for any path of \( P[\alpha] \) growing on \( (P)^* \) on a tile \( (P)^*_i \) with \( i \geq |P| - 1 \) either this path is a subassembly of \( \gamma_0 \) or there is \( \ell \in \mathbb{N} \) such that the translation of this path by \( -\ell P_0P_{|P|+1} \) is a subassembly of \( \gamma_1 \). Thus the union of all paths of \( P[\alpha] \) growing on \( (P)^* \) at index \( i \geq |P| - 1 \) is \( \gamma \) which is the union of an assembly of complexity 0 with an assembly of complexity 1 and thus is an assembly of complexity 1 whose domain is a semilinear set whose size depends of \( |\beta|, \vec{v}, P \) and \( |T| \). □

**Lemma 23.** Consider a tile assembly system \( T = (T, \sigma, 1) \) whose terminal assembly is \( \alpha \), a simply pumpable path \( P \) in \( P[\alpha] \) and a producible finite assembly \( \beta \) such that \( P_0 \) is a tile of \( \beta \). Then the union of \( (P)^* \) and all paths of \( P[\alpha] \) growing on \( (P)^* \) on a tile \( (P)^*_i \) with \( i \geq |P| - 1 \) is an assembly of complexity 2 whose domain is a semilinear set whose size depends of \( |\beta|, P \) and \(|T|\).
Proof. By lemma \ref{lemma:existence_of_index}, there exists an index \( j \) such that all paths growing on \((P)^{\ast}\) at index \( k > j \) are in \( P[\alpha] \), this index depends on \(|\beta|\), \( P \) and \( |T| \). We define \( \beta' \) as \( \beta \cup (P)^{\ast}_{0,1,\ldots,j+(|P|-1)} \). Consider a path \( Q \) which grows on \((P)^{\ast}\) at index \( |P| - 1 \leq i \leq j + |P| - 1 \). By lemma \ref{lemma:pumping_lemma}, \( Q \) is \( P[P]_{[-1]} \)-self-avoiding. By the pumping lemma \ref{lemma:pumping_lemma}, either \( Q \) is finite and its vertical height or horizontal width is less then \( f(|T|, |\beta'|) \) or it is pumpable between \( m \) and \( n \) for some \( 1 \leq m < n \leq |Q| - 1 \). In the second case, without loss of generality, we can assume that the vertical height or horizontal width of \( Q_{0\ldots,n} \) is less then \( f(|T|, |\beta'|) \) (otherwise there exist some smaller indices with the same properties as \( m \) and \( n \)) and then the union of \( Q_{0\ldots,n} \) and all paths of \( P[\alpha] \) which grow on \( Q \) on index \( n \) is an assembly of complexity 1 (by lemma \ref{lemma:assembly_of_complexity}).

Remark that, there are only a finite number of different paths of \( P[\alpha] \) of vertical height or horizontal width bounded by \( f(|T|, |\beta'|) \) which can grow on \((P)^{\ast}_{|P|-1,\ldots,j+|P|-2}\). Thus, we consider the assembly \( \gamma_0 \) which is the union of \((P)^{\ast}_{0\ldots,j}\) and all paths of \( P[\alpha] \) growing on \((P)^{\ast}\) at index \( |P| - 1 \leq i \leq j - 1 \), this assembly is a finite union of assemblies of complexity 1 and is thus of complexity 1 and its domain is a semilinear set whose size is bounded by a function which depends only of \(|\beta'|\), \( P \) and \( |T| \). Moreover, by its definition it is a subassembly of \( \alpha \). Similarly, consider the assembly \( \gamma_1 \) which is of the union of \((P)^{\ast}_{j,j+1,\ldots,j+|P|-1}\) and all paths of \( P[\alpha] \) growing on \((P)^{\ast}\) at index \( j \leq i \leq j + |P| - 2 \), the complexity of this assembly is 1 and its domain is a semilinear set whose size is bounded by a function which depends only of \(|\beta'|\), \( P \) and \( |T| \). Moreover, by its definition it is a subassembly of \( \alpha \). Remark that for all \( \ell \in \mathbb{N}, \gamma_1 + \ell \overrightarrow{P_0P[P]}_{|P|+1} \) intersects with \( \gamma_1 + (\ell + 1) \overrightarrow{P_0P[P]}_{|P|+1} \) and that \( \gamma_1 + \ell \overrightarrow{P_0P[P]}_{|P|+1} \) is sub-assembly of \( \alpha \) by definition of \( j \) and by corollary \ref{corollary:union_of_paths} and then the assembly \( \bigcup_{\ell \in \mathbb{N}} (\gamma_1 + \ell \overrightarrow{P_0P[P]}_{|P|+1}) \) is correctly defined. Moreover, this assembly is of complexity 2. Consider the assembly:

\[
\gamma = \gamma_0 \cup \left( \bigcup_{\ell \in \mathbb{N}} (\gamma_1 + \ell \overrightarrow{P_0P[P]}_{|P|+1}) \right).
\]

This assembly is a subassembly of \( \alpha \) and for any path of \( P[\alpha] \) growing on \((P)^{\ast}\) at index \( i \geq |P| - 1 \) either this path is a subassembly of \( \gamma_0 \) or there is \( \ell \in \mathbb{N} \) such that the translation of this path by \( -\ell \overrightarrow{P_0P[P]}_{|P|+1} \) is a subassembly of \( \gamma_1 \). Thus the union of all paths of \( P[\alpha] \) growing on \((P)^{\ast}\) at index \( i \geq |P| - 1 \), is \( \gamma \) which is the union of an assembly of complexity 1 with an assembly of complexity 2 and thus is an assembly of complexity 2 whose domain is a semilinear set whose size depends on \(|\beta'|\), \( P \) and \( |T| \). Remark that \(|\beta'|\) depends on \(|\beta|\) and \( j \) and that \( j \) depends on \(|\beta|\), \( P \) and \( |T| \), hence the result.

We can now prove the second half of the main theorem

\textbf{Theorem 9.} Consider a directed tile assembly system \( T = (T, \sigma, 1) \) whose terminal assembly \( \alpha \) is simply periodic. Then there exists a vector \( \vec{v} \) and an
assembly $\beta$ of complexity 1 such that

$$\alpha = \bigcup_{\ell \in \mathbb{Z}} (\beta + \ell \overrightarrow{v}).$$

Proof. Consider the paths $P^+$ and $P^-$, the assembly $\beta$ and the vector $\overrightarrow{v} = P^+_0 P^+_{|P^+|-1} = P^-_0 P^-_{|P^-|-1}$ of lemma 15. Consider a path $Q$ which grows on the left side of $(P^+)^*$ and by lemma 1, $\alpha$ is the unique terminal assembly of $(T,Q_0,1)$. By lemma 15, $Q$ is $P^+_0 P^+_{|P^+|-1}$-self-avoiding. By the pumping lemma [5], either $Q$ is finite and its vertical height or horizontal width is less then $f(T,1)$ or it is pumpable between $i$ and $j$ for some $1 \leq i < j \leq |Q| - 1$. In the second case, all paths of $P[\alpha]$ which grow on tile $Q_j$ of $Q_{0,1,...,j}$ belongs to an assembly of complexity 1 whose domain is a semilinear set whose size depends of $P^+$, $Q$ and $|T|$. Moreover, without loss of generality we can suppose that the vertical height or horizontal width of $Q_{0,1,...,j}$ is less then $f(T,1)$. Since there exist only a finite number of paths of vertical height or horizontal width bounded by $f(T,1)$, we can consider the assembly $\beta^+$ which is the union of $P^+$ and all the paths growing on $(P^+)^*$ at index $0 \leq k < |P| - 1$. The complexity of $\beta^+$ is 1 since it is the finite union of assembly of complexity less than 1. Moreover, $\beta^-$ can be defined similarly for the right side of $(P^-)^*$ and let

$$\gamma = \beta^+ \cup \beta^-. $$

Since $\alpha$ is $\overrightarrow{v}$-periodic, we have $\alpha = \bigcup_{\ell \in \mathbb{Z}} (\gamma + \ell \overrightarrow{v}).$ \qed

We can now prove the last main theorem

**Theorem 10.** Consider a directed tile assembly system $T = (T,\sigma,1)$ whose terminal assembly $\alpha$ is aperiodic, then the complexity of $\alpha$ is 2.

Proof. Consider the assembly $\beta$ which the union of the seed and all producible paths of $(T,\sigma,1)$ whose vertical height and horizontal width are less than $f(T, |\sigma|)$. If this assembly is terminal then $\alpha = \beta$ is finite and its complexity is 0. Let $\gamma = \beta$ and for any producible path $P$ of $(T,\sigma,1)$ such that $P$ is infinitely pumpable and $\text{asm}(P)$ is a subassembly of $\beta$ do the following:

- find the smallest index $j$ such that there exists $0 \leq i < j$ such that $P$ is pumpable between $i$ and $j$;

- $\gamma = \gamma \cup (P_{i,i+1,...,j})^* \cup \{Q : Q \in P[\alpha] \text{ and Q grows on } (P_{i,i+1,...,j})^* \text{ at index } k \geq j\};$

Since $\beta$ is finite then the number of producible path which are subassembly of $\beta$ is finite. By lemma 23, the complexity of the assembly $(P_{i,i+1,...,j})^* \cup \{Q : Q \in P[\alpha] \text{ and Q grows on } (P_{i,i+1,...,j})^* \text{ at index } k \geq j\}$ is less than 2 and thus at the end of this algorithm the complexity of the assembly $\gamma$ is less than 2. Moreover, we claim that $\alpha = \gamma$ which would conclude this result. Since $\gamma$ is the union of paths of $P[\alpha]$ then $\gamma$ is a subassembly of $\alpha$. For any tile $A$ of $\alpha$ either $A$ is a tile of $\beta$ (and thus of $\gamma$) or there exists a producible path $P$ of
(T, σ, 1) such that $P_{|P| - 1} = A$. If $A$ is not a tile of $\beta$, then the vertical height or horizontal width of $P$ is more than $f(T, |σ|)$ by definition of $\beta$ and thus there exists a prefix of $P$ whose vertical height or horizontal width is $f(T, |σ|)$. Again by definition of $\beta$, this prefix is a subassembly of $\beta$. This prefix is infinitely pumpable by the pumping lemma. Then there exist $0 \leq i < j \leq |P| - 1$, such that $P$ is pumpable between $i$ and $j$ and $asm(P_{i, j})$ is a subassembly of $\beta$. Without loss of generality we can suppose that $j$ is minimal and then either $A$ is a tile of $(P_{i, j})^*$ or a suffix of $P_{j, j+1, \ldots, |P| - 1}$ grows on $(P_{i, i+1, \ldots, j})^*$ at index $k > j$. In both cases, $A$ is a tile of $\gamma$.

\square

A Discussion about the seed

In [2], it was assumed that the seed of a directed tile assembly system at temperature 1 could be reduced to a single tile without loss of generality. In lemma 1, we show that this statement is true when the terminal assembly is periodic but this statement is not true in the general case. Figure B.1 exhibits a counter-example.

Our counter example is made of four kind of tile types, see Figure B.1a. The first kind of tile types are the ten black tiles which constitute the seed. The second and third ones are the three blue tiles and the three green tiles which are used to assemble two simply pumpable paths which can bind on the seed (using glues $g_2$ and $g_1$). The fourth set of tile types are the eleven red tiles which are used to assemble a path whose extremities bind with the first iteration of the two simply pumpable paths (using glues $g_3$ and $g_4$).

Figure B.1b shows the unique terminal assembly of this tile assembly system. Remark that only the first iteration of the red link can fully grow because all of its translations on the simply pumpable paths quickly collide with a previous period. Nevertheless, this link creates a cycle and if the seed is reduced to a single tile, as shown in Figure B.1c where this single tile is pointed by a red arrow, then it is possible to grow one half of the seed, then the green pumpable path, the red path and finally the blue path in both direction which leads to another terminal assembly where there is a blue tile where a black one should be (pointed by a black arrow in the Figure). If another initial tile is chosen, either the same technique leads to a conflict or we can grow an assembly in order to pump the green pumpable path first which will lead to a conflict.

Figure B.1d shows how the result of section 3.2 describes this terminal assembly using only a finite amount of information:

- there is a finite assembly in gray made of the seed, the red path and the first iteration of the green and blue pumpable paths.
- a vector $\vec{v}$ and a purple finite assembly made of one period of the green pumpable path and some red tiles, for all $i \in \mathbb{N}$ the translation of this assembly by $i \vec{v}$ is in the terminal assembly.
• a vector $\vec{w}$ and an orange finite assembly made of one period of the blue pumpable path and some red tiles, for all $i \in \mathbb{N}$ the translation of this assembly by $i\vec{w}$ is in the terminal assembly.

The interest of this remark is to show that the first period of a pumpable path may still assemble some artifacts which do not appear on the rest of the pumpable path.

\section{Wee lemmas and left/right turns for curves}

\textbf{Lemma 24} (Lemma 6.3 of \textsuperscript{[1]}. Consider a two-dimensional, bounded, connected, regular closed set $S$, i.e. $S$ is equal to the topological closure of its interior points. Suppose $S$ is translated by a vector $v$ to obtain shape $S_v$, such that $S$ and $S_v$ do not overlap. Then the shape $S_{cv}$ obtained by translating $S$ by $c * v$ for any integer $c \neq 0$ also does not overlap $S$.

The following lemma \textsuperscript{[5]} formalizes the intuition behind Definition 8:

\textbf{Lemma 25} (Lemma 2.5 of \textsuperscript{[5]}). Let $P$ be a path with tiles from some tileset $T$, $i < j$ be two integers, and $q$ be the pumping of $P$ between $i$ and $j$. Then for all integers $k \geq i$, $q_{k + (j - i)} = q_k + \overrightarrow{P_i P_j}$.

\textbf{Proof.} By the definition of $q$ (and using the fact that $(j - i) \ mod \ (j - i) = 0$):

$$q_{k + (j - i)} = P_{k + ((k + (j - i) - i - 1) \mod (j - i) + 1)} + \left\lfloor \frac{k + (j - i) - i - 1}{j - i} \right\rfloor \overrightarrow{P_i P_j} = P_{k + ((k - i - 1) \mod (j - i) + 1)} + \left\lfloor \frac{k - i - 1}{j - i} \right\rfloor \overrightarrow{P_i P_j} + \overrightarrow{P_i P_j} = q_k + \overrightarrow{P_i P_j}$$

\hfill $\square$

\section{B.1 Jordan curve theorem for infinite polygonal curves and left/right turns}

Consider a vector $\vec{w} = (u, v)$, the height of $(x, y) \in \mathbb{R}^2$ is defined as $h((x, y)) = -vx + uy$. The line $\ell : (-\infty, +\infty) \rightarrow \mathbb{R}^2$ of vector $\vec{w}$ passing through a point $(a, b) \in \mathbb{R}^2$ is defined as $\{(x, y) \mid h((x, y)) = h((a, b))\}$. This line cuts the 2D plane $\mathbb{R}^2$ into two connected components: the right-hand side $R = \{(x, y) \mid h((x, y)) \leq h((a, b))\}$ and the left-hand side $L = \{(x, y) \mid h((x, y)) \geq h((a, b))\}$.

We say that a curve $c$ turns right (respectively left) of $\ell$ if there exist $\epsilon > 0$ and $t \in \mathbb{R}$ such that $c(t)$ is a point of $\ell$ and for all $t < z \leq t + \epsilon$, $c(z)$ belongs to $R$ (respectively $L$) and is not on $\ell$. Moreover, we say that $c$ crosses $\ell$ from left to right (respectively from right to left) if there exist $t$ and $\epsilon > 0$ such that $c$ turns right (respectively left) of $\ell$ at $c(t)$, for all $t - \epsilon \leq z \leq t$, $c(t) \in L$ (respectively

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a) the different tile types.

b) the terminal assembly.

c) if the seed is reduced to a single tile, the tile assembly system is not directed.

d) the description of the terminal assembly.

Figure B.1: The seed cannot be reduced to a single tile.
\( c(t) \in \mathbb{R} \) and \( c(t - \varepsilon) \) is not on \( \ell \) We say that \( c \) crosses \( \ell \) if \( c \) crosses \( \ell \) either from left to right or from right to left. Now, we generalise these notions to a specific class of polygonal curves:

**Definition 13.** A simple bi-infinite periodic polygonal curve is a simple curve that is a concatenation of all the translations of a finite curve \( c : [a, b] \to \mathbb{R}^2 \) which is made of horizontal and vertical segments of length 1, i.e.:

\[
\text{concat}_{i \in \mathbb{Z}} \left( c + i\vec{c}(b) \right) .
\]

Let \( w_0, w_1 \in \mathbb{R}^2 \) and let \( c \) be a curve. We say that \( w_0 \) is connected to \( w_1 \) while avoiding \( c \) if there is a curve \( d : [0, 1] \to \mathbb{R}^2 \) with \( d(0) = w_0 \) and \( d(1) = w_1 \) and \( d \) does not intersect \( c \).

Consider a finite curve \( c : [a, b] \to \mathbb{R}^2 \) which is made of horizontal and vertical segments of lengths 1, let \( \vec{w} = \vec{c}(a)\vec{c}(b) = (u, v) \) and such that \( c \) and \( c + \vec{w} \) intersect only at \( \vec{c}(b) = \vec{c}(a) + \vec{w} \). Then the curve \( d = \text{concat}_{i \in \mathbb{Z}}(c + i\vec{w}) \) is a simple bi-infinite periodic polygonal curve (see Figure B.2). We call \( \vec{w} \) the direction of \( c \) and \( d \). The curve \( c \) considered here are such that \( h^+ = \max\{h((x, y)) \mid (x, y) \in c([a, b])\} \) and \( h^- = \min\{h((x, y)) \mid (x, y) \in c([a, b])\} \) are bounded. Remark that for any \((x, y) \in \mathbb{R}^2\), the height of \((x, y)\) is the height of \((x, y) + \vec{w}\), i.e. \(-v(x + u) + u(y + v) = -vx + uy = h((x, y))\) and thus for all \((x, y) \in d(\mathbb{R})\), \( h^- \leq h((x, y)) \leq h^+ \). Intuitively, the curve \( c \) is between two lines of direction \( \vec{w} \) (the lines \( \ell^+ \) and \( \ell^- \) of Figure B.2). We say that \((x, y) \in \mathbb{R}^2\) is below (resp. over) \( c \) if \( h((x, y)) \leq h^- \) (resp. \( h((x, y)) \geq h^+ \)).

Consider a vector \( \vec{w'} \) which is not collinear to \( \vec{w} \) then for any \((x, y) \in \mathbb{R}^2\), the height of \((x, y) + \vec{w'}\) is not the same as the height of \((x, y)\). In particular, if \( \vec{w'} = (-v, u) \) (resp. \( \vec{w'} = (v, -u) \)), i.e. \( \vec{w} \) rotated by \( \pi/2 \) (resp. \( -\pi/2 \)), then \( h((x, y)) < h((x, y) + \vec{w'}) \) (resp. \( h((x, y)) > h((x, y) + \vec{w'}) \)).

**Theorem 11.** Let \( c : \mathbb{R} \to \mathbb{R}^2 \) be a simple bi-infinite periodic polygonal curve, and let \((x_{\min}, y_{\min})\) and \((x_{\max}, y_{\max})\) be respectively below and over \( c \). Then \( c \) cuts \( \mathbb{R}^2 \) into two connected components:

1. the left-hand side of \( c \):

\[
c(\mathbb{R}) \cup \{w \mid w \in \mathbb{R}^2 \text{ is connected to } (x_{\max}, y_{\max}) \text{ while avoiding } c\},
\]

2. the right-hand side of \( c \):

\[
c(\mathbb{R}) \cup \{w \mid w \in \mathbb{R}^2 \text{ is connected to } (x_{\min}, y_{\min}) \text{ while avoiding } c\}
\]

**Proof.** The proof is similar to the proof of the Jordan Curve Theorem for polygonal curves.

\footnote{The definition of “\( c \) crosses \( \ell \) from left to right” ensures that certain kinds of intersections between \( c \) and \( \ell \) are not counted as crossing, those include: coincident intersection between two straight segments of length \( > 0 \), and “glancing” of a line “tangentially” to a corner.}
Figure B.2: A curve $c : [a, b] \rightarrow \mathbb{R}^2$ of direction $\mathbf{w}$ which generates a simple bi-infinite periodic polygonal curve and cuts the plane in two parts. All points which are over the curve are in the left hand side (in red) and all points which are under the curve are in the right hand side of the curve (in blue).

Let $\mathbf{v}$ be the direction of $c$, let $\mathbf{v}^+$ (resp. $\mathbf{v}^-$) be the vector $\mathbf{v}$ rotated by $\pi/2$ (resp. $-\pi/2$). For each point $(x, y) \in \mathbb{R}^2$ we define $\ell(x,y)$ to be the line of vector $\mathbf{v}^+$ through $(x,y)$, $\ell^+ (x,y)$ to be the ray of vector $\mathbf{v}^+$ from $(x,y)$, and $\ell^- (x,y)$ to be the ray of vector $\mathbf{v}^-$ from $(x,y)$.

Since $\mathbf{v}^+$ and $\mathbf{v}^-$ are not collinear, for all $(x,y) \in \mathbb{R}^2$, $\ell^- (x,y)$ and $\ell^+ (x,y)$ intersects $c$ at a finite number of points. Moreover, because $\ell(x,y)$ cuts the plane into two connected components, $\ell(x,y)$ crosses $c$ an odd number of times.

Let $L$ be the subset of $\mathbb{R}^2$ such that for all $(x,y) \in L$, $c$ crosses $\ell^- (x,y)$ an odd number of times, and let $R$ be the subset of $\mathbb{R}^2$ such that for all $(x,y) \in R$, $c$ crosses $\ell^+ (x,y)$ an odd number of times. Note that $L \cap R = c(\mathbb{R})$ (where $c(\mathbb{R})$ is the range of $c$), or in other words the intersection of $L$ and $R$ is the set of all points of $c$.

We claim that $L$ (respectively $R$) is a connected component: indeed, since $c$ is connected, if $(x_0, y_0)$ and $(x_1, y_1)$ are both in $L$ (respectively both in $R$), let $t_0, t_1 \in \mathbb{R}$ be the smallest real numbers such that $\ell^- (x_0, y_0)(t_0)$ (respectively $\ell^+ (x_0, y_0)(t_0)$) is on $c$, and $\ell^- (x_1, y_1)(t_1)$ (respectively $\ell^+ (x_1, y_1)(t_1)$) is on $c$. We know there is at least one such intersection because $\ell^- (x_0, y_0)$ and $\ell^- (x_1, y_1)$ (respectively $\ell^+ (x_0, y_0)$ and $\ell^+ (x_1, y_1)$) cross $c$ an odd number of times, hence at least once. Without loss of generality we suppose that $\ell^- (x_0, y_0)(t_0)$ (respectively $\ell^+ (x_0, y_0)(t_0)$) is before $\ell^- (x_1, y_1)(t_1)$ (respectively $\ell^+ (x_1, y_1)(t_1)$) accord-

---

The term “crosses” was defined with respect to a line $\ell(x,y)$. This definition is easily generalised to the (coincident) ray $\ell^+ (x,y)$, by considering only those crossings at locations that are simultaneously on both $\ell$ and the ray. Likewise for the ray $\ell^- (x,y)$.
ing to the order of positions along $c$.

Let $d$ be the curve defined as the concatenation of:

- $\ell^-(x_0, y_0)$ (respectively, $\ell^+(x_0, y_0)$) from $(x_0, y_0)$ up to $\ell^-(x_0, y_0)(t_0)$ (respectively, $\ell^+(x_0, y_0)(t_0)$),
- $c$ from $\ell^-(x_0, y_0)(t_0)$ to $\ell^-(x_1, y_1)(t_1)$ (respectively, from $\ell^+(x_0, y_0)(t_0)$ to $\ell^+(x_1, y_1)(t_1)$),
- $\ell^-(x_1, y_1)^\rightarrow$ (respectively, $\ell^+(x_1, y_1)^\leftarrow$) from $\ell^-(x_1, y_1)(t_1)$ (respectively, $\ell^+(x_1, y_1)(t_1)$) to $(x_1, y_1)$.

The curve $d$ is entirely in $L$ (respectively in $R$), starts at $(x_0, y_0)$ and ends at $(x_1, y_1)$, which proves that $L$ (respectively $R$) is indeed a connected component.

We claim that $L \cup R = \mathbb{R}^2$: indeed, for all $(x, y) \in \mathbb{R}^2$, since $\ell(x, y)$ crosses $c$ an odd number of times, $\ell^+(x, y)$ crosses $c$ an even number of times if and only if $\ell^-(x, y)$ crosses $c$ an odd number of times (and vice-versa), hence $(x, y)$ is in at least one of $L$ and $R$, and only the points of $c$ are in both.

As a conclusion, $c$ cuts the plane into two disjoint connected components: $L \setminus c(\mathbb{R})$ which we call the strict left-hand side of $c$ and $R \setminus c(\mathbb{R})$ which we call the strict right-hand side of $c$.

Let $(x, y)$ be a point over $c$ then $\ell^+(x, y)$ does not intersect $c$, and thus $\ell^-(x, y)$ intersects $c$ an odd number of times. $(x, y)$ is in $L$, and since $L$ is connected, we get Conclusion $[1]$. Let $(x, y)$ be a point under $c$ then $\ell^-(x, y)$ does not intersect $c$, and thus $\ell^+(x, y)$ intersects $c$ an odd number of times. $(x, y)$ is in $R$, and since $R$ is connected, we get Conclusion $[2]$. \hfill \Box

Using the technique in the proof of Theorem $[11]$ the definitions of turning right and left can be extended from a line to an infinite polygonal curve.

**Definition 14** (left-hand and right-hand side of a curve). The conclusion of Theorem $[11]$ defines the left-hand side $L \subseteq \mathbb{R}^2$, and right-hand side $R \subseteq \mathbb{R}^2$ of a simple bi-infinite periodic polygonal curve $c$. The strict left-hand side of a simple bi-infinite periodic polygonal curve $c : \mathbb{R} \to \mathbb{R}^2$ is the set $L \setminus c(\mathbb{R})$. Likewise the strict right-hand side of a simple bi-infinite periodic polygonal curve $c$ is the set $R \setminus c(\mathbb{R})$, where $R$ is the right-hand side of $c$.

We have already defined what this means for a curve to cross a line. Theorem $[11]$ enables us to generalise that definition to one curve turning from another simple bi-infinite periodic polygonal curve.

**Definition 15** (One curve turning left or right from another). Let $d$ be a curve and let $c : \mathbb{R} \to \mathbb{R}^2$ be a simple bi-infinite periodic polygonal curve. We say that $d$ turns left (respectively, right) from curve $c$ at the point $d(z) = c(w)$, for some $z, w \in \mathbb{R}$, if there is an $\epsilon > 0$ such that $d(z + \epsilon)$ is in the strict left-hand (respectively, right-hand) side of $d$ and $d(z')$ is not on $c$, for all $z'$ where $z < z' < z + \epsilon$.  

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Definition [15] is consistent with the definition of one path turning left/right from another (Section [2.2]) in the following sense. Consider two paths $P$ and $Q$ such that $Q$ turns right (respectively, left) of $P$ and consider a curve $c$ which contains $\mathcal{E}[P]$ (with the same orientation) then $\mathcal{E}[Q]$ turns right (respectively, left) of $c$. However, not all curves are the embedding of some path, hence the reverse implication does not hold. Also, the curve turning definition has no requirement analogous to the orientation requirement for path turns (implied by $i > 0$ and the definition of $\tau$ in the path turning definition).

Throughout the article, we will use the following fact several times. Consider a simple bi-infinite periodic polygonal curve $c$ of direction $\overrightarrow{v}$ and a vector $\overrightarrow{w}$ which is not collinear to $\overrightarrow{v}$ and such that $c + \overrightarrow{w}$ do not intersect. Without loss of generality, we suppose that $c([R]$ is included into $\mathcal{L}(c + \overrightarrow{w})$. In this case, $\mathcal{L}(c) \cap \mathcal{L}(c + \overrightarrow{w}) \subset \mathbb{R}^2$ is connected and we claim that for any $(x, y) \in \mathbb{R}^2$, there exists $i \in \mathbb{Z}$ such that $(x, y) + i\overrightarrow{w}$ is in $\mathcal{R}(c) \cap \mathcal{L}(c + \overrightarrow{w}) \subset \mathbb{R}^2$. Indeed, for all $t \in \mathbb{Z}$, let

$$h_i^+ = \max\{h((x, y) + t\overrightarrow{w}) \mid (x, y) \in c([a, b])\} \quad \text{and} \quad h_i^- = \min\{h((x, y) + t\overrightarrow{w}) \mid (x, y) \in c([a, b])\}.$$ 

Since, $c([R]$ is included into $\mathcal{L}(c + \overrightarrow{w})$ then for all $t \in \mathbb{Z}$, $h_i^+ > h_{i+1}^+$ and $h_i^- > h_{i+1}^-$. Thus, for any $(x, y) \in \mathbb{R}^2$ there exist $j < k$ such that $(x, y)$ is under $c + j\overrightarrow{w}$ (resp. over $c + k\overrightarrow{w}$) and thus in $\mathcal{R}(c + t\overrightarrow{w})$ (in $\mathcal{L}(c + k\overrightarrow{w})$). Then, $i = \max\{t : (x, y) \in \mathcal{R}(c + t\overrightarrow{w})\}$ is correctly defined and $(x, y)$ is in $\mathcal{R}(c + i\overrightarrow{w}) \cap \mathcal{L}(c + (i + 1)\overrightarrow{w})$. Thus, $(x, y) - i\overrightarrow{w}$ is in $\mathcal{R}(c) \cap \mathcal{L}(c + \overrightarrow{w})$.

Also consider two simple bi-infinite periodic polygonal curves $c$ and $c'$ both of direction $\overrightarrow{v}$ with $c(0) = c'(0)$ and such that $c'(\mathbb{R})$ is a subset of $\mathcal{R}(c)$. Then $\mathcal{R}(c) \cap \mathcal{L}(c')$ is connected and there exists $A \subset \mathbb{Z}^2$ such that $A$ is finite and $\mathbb{Z} \cap (\mathcal{R}(c) \cap \mathcal{L}(c')) = \bigcup_{i \in \mathbb{Z}} (A + i\overrightarrow{v})$. Indeed, consider $\ell$ (resp. $\ell'$) which is the restriction of $c$ (resp. $c'$) which starts in $c(0)$ (resp. $c'(0)$) and ends in $c(0) + \overrightarrow{v}$ then $\text{concat}(\ell, \ell')$ is a cycle but it may not be simple. Let $\overrightarrow{v'}$ be the rotation of $\overrightarrow{v}$ by $\pi/2$. Consider the translation of $\ell$ by $\epsilon \overrightarrow{v''}$ with $\epsilon > 0$, then

$$\text{concat}\left(\left[c(0), c(0) + \epsilon\overrightarrow{v''}, \ell + \epsilon\overrightarrow{v''}, \left[c(0) + \overrightarrow{v''} + \epsilon\overrightarrow{v''}, c(0) + \overrightarrow{v''}\right], \ell'\right]\right)$$

is now a simple cycle and let $I$ be its interior and let $A = I \cap \mathbb{Z}$. Moreover, $\epsilon$ could chosen small enough such that for any $(x, y) \in \mathbb{Z}^2$ such that $(x, y) \in A$, we have $(x, y) \in \mathcal{R}(c) \cap \mathcal{L}(c')$. Then $\mathbb{Z} \cap (\mathcal{R}(c) \cap \mathcal{L}(c')) = \bigcup_{i \in \mathbb{Z}} (A + i\overrightarrow{v})$. We call $A$, the interior of $c$ and $c'$.

### C Analysis of an aperiodic tile assembly system.

We thank Damien Woods for his support and advices.
Figure C.1: A tile set: the seed is in black and the colored paths are simply pumpable. Each tile drawn here is unique.
Figure C.2: the terminal assembly of the tile set described in Figure C.1.
Figure C.3: the terminal assembly is composed of a finite assembly in gray and four assemblies of complexity less than 2 (in blue, red, orange, and green).
Figure C.4: An assembly of complexity less than 2 can be decomposed as the union of a finite assembly (dark colors), some assemblies of complexity 1 (medium colors) and all the translations of an assembly of complexity less than 1 by a given vector (light color).
Figure C.5: An assembly of complexity less than 1 can be decomposed as the union of a finite assembly (light color) and all the translations of a finite assembly by a given vector (very light color).
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