Dual gluons and monopoles in 2+1 dimensional Yang-Mills theory

Ramesh Anishetty *, Pushan Majumdar †, H.S.Sharatchandra ‡
Institute of Mathematical Sciences, C.I.T campus Taramani. Madras 600-113

Abstract

2+1-dimensional Yang-Mills theory is reinterpreted in terms of metrics on 3-manifolds. The dual gluons are related to diffeomorphisms of the 3-manifolds. Monopoles are identified with points where the Ricci tensor has triply degenerate eigenvalues. The dual gluons have the desired interaction with these monopoles. This would give a mass for the dual gluons resulting in confinement.

PACS No.(s) 11.15-q, 11.15 Tk

*e-mail:ramesha@imsc.ernet.in
†e-mail:pushan@imsc.ernet.in
‡e-mail:sharat@imsc.ernet.in
I. INTRODUCTION

Quark confinement is well understood in 2+1 dimensional compact U(1) gauge theory. It is a consequence of the existence of a monopole plasma \cite{1,2}. Duality transformation \cite{3} turned out to be very useful in this context. It is of interest to know how far these ideas can be extended to non-abelian gauge theories. For this reason, duality transformation for 2+1-dimensional Yang-Mills theory was obtained in lattice gauge theory in both hamiltonian \cite{4} and partition function \cite{5} formulations. The dual theory exhibits close relationship to 2+1-dimensional gravity, but without diffeomorphism invariance. This also indicates a way of describing the dynamics using local gauge invariant variables.

In this paper, we consider duality transformation for 2+1-dimensional (continuum) Yang-Mills theory in close analogy to the case of compact U(1) lattice gauge theory \cite{3}. We reinterpret the Yang-Mills theory as a theory of 3-manifolds, as in gravity, but without diffeomorphism invariance. We use this relation for identifying the dual gluons and their interactions. The dual gluons are related to diffeomorphisms of the 3-manifold. We also identify the monopoles in the dual theory. 't Hooft \cite{6} has advocated the use of a composite Higgs to locate the monopoles. Here we propose to use the orthogonal set of eigenfunctions of a gauge invariant, (symmetric) local, matrix-valued field for this purpose. Isolated points where the eigenvalues are triply degenerate have topological significance and they locate the monopoles. We use the Ricci tensor to construct a new coordinate system for the 3-manifold. The monopoles are located at the singular points of this coordinate system and they have the expected interactions with the dual gluons. We expect that these interactions lead to a mass for the dual gluons and result in confinement as in the U(1) case.

Lunev \cite{7} has pointed out the relationship of 2+1-dimensional Yang-Mills theory with gravity. He uses a gauge invariant composite $B^a_i B^a_j$ as a metric, and rewrites the classical Yang-Mills dynamics for it. The corresponding formulation of the quantum theory is somewhat involved. Our metric is in a sense dual of Lunev’s choice. As we make formal transformations in the functional integral, the quantum theory is simpler and has a nicer interpretation.

There are also approaches to relate 3+1-dimensional Yang-Mills theory to a theory of a metric \cite{8}. On the other hand, the dual theory in 3+1-dimensions can also be related to a new SO(3) gauge theory \cite{9}.

In section 2 we briefly review duality transformation and confinement in 2+1-dimensional compact U(1) lattice gauge theory. In section 3 we obtain the dual description of 2+1-dimensional Yang-Mills theory in close analogy to section 2. We point out the close relationship to gravity and identify the dual gluons and their interactions. In section 4 we provide a new characterization of monopoles using eigenfunctions of the symmetric matrix $B^a_i B^a_i$. In section 5 we use the Ricci tensor to construct a preferred coordinate system for 3-manifolds. We relate the monopoles to singularities of this coordinate system. We also identify their interactions with the dual gluons. Section 6 contains our conclusions.
II. REVIEW OF CONFINEMENT IN 2+1-DIMENSIONAL COMPACT U(1) LATTICE GAUGE THEORY

In this section we briefly review duality transformation [3] and confinement [1] in 2+1 dimensional compact U(1) lattice gauge theory. This provides a paradigm for our analysis of 2+1 dimensional Yang-Mills case.

The Euclidean partition function in the Villain formulation is given by

\[ Z = \sum_{h_{ij}} \prod_{n} \int_{-\infty}^{\infty} dA_i(n) \exp \left( -\frac{1}{4\kappa^2} \sum_{nij} [\Delta_i A_j(n) - \Delta_j A_i(n) + h_{ij}(n)]^2 \right). \]  

(1)

Here \( A_i(n) \in (-\infty, \infty) \) are non-compact link variables on links joining the sites \( n \) and \( n + \hat{i} \). \( h_{ij}(n) = 0, \pm 1, \pm 2 \ldots \) are integer variables corresponding to the monopole degrees of freedom and are associated with the plaquette \( (n\hat{i}\hat{j}) \). \( \Delta_i \) is the difference operator, \( \Delta_i \phi(n) = \phi(n + \hat{i}) - \phi(n) \). We may introduce an auxiliary variable \( e_i(n) \) to rewrite \( Z \) as

\[ Z = \sum_{h_{ij}} \prod_{n} \int_{-\infty}^{\infty} dA_i(n) \int_{-\infty}^{\infty} d\phi(n) \exp \left( -\sum_{n} [e_i(n)]^2 + \frac{2i}{\kappa} \sum_{nij} \epsilon_{ijk} e_j(n) [\Delta_i A_j(n) + \frac{1}{2} h_{ij}(n)] \right). \]  

(2)

Integration over \( A_j(n) \) gives the \( \delta \) function constraint

\[ \epsilon_{ijk} \Delta_j e_k(n) = 0 \]  

(3)

for each \( n \) and \( \hat{i} \). The solution is \( e_i(n) = \Delta_i \phi(n) \). Thus we get the dual form of the partition function

\[ Z = \sum_{h_{ij}} \prod_{n} \int_{-\infty}^{\infty} d\phi(n) \exp \sum_{n} \left( -[\Delta_i \phi(n)]^2 + \frac{i}{2\kappa} \phi(n) \rho(n) \right), \]  

(4)

where \( \rho(n) = \frac{1}{2} \epsilon_{ijk} \Delta_i h_{jk}(n) \). This has the following interpretation. The field \( \phi \) describes the dual photon. (In 2+1 dimensions, the photon has only one transverse degree of freedom and this is captured by the scalar field \( \phi(n) \)). The monopole number at site \( n \) is given by \( \rho(n) \). It takes integer values and the dual photon couples locally to it with strength \( 1/\kappa \).

If we sum over the monopole degrees of freedom, we get a mass term for \( \phi(n) \) [13]. The reason for this is that the monopole plasma is screening the long range interactions between the monopoles. A Wilson loop for the electric charges in this system would correspond to a dipole sheet in this plasma. This gives an area law and hence a linear confining potential between static electric charges.

The advantage of this formal duality transformation is that it gives a precise separation of the ‘spin wave’ and the ‘topological’ degrees of freedom. Therefore it provides a stepping stone for going beyond semi-classical approximations.

We use this approach for 2+1 dimensional Yang-Mills theory in the next section.
III. DUAL GLUONS IN 2+1-DIMENSIONAL YANG-MILLS THEORY

In this section we point out the close relationship between Yang-Mills theory and Einstein-Cartan formulation of gravity in 2+1 (or 3 Euclidean) dimensional space. We use this analogy extensively throughout the paper.

The Euclidean partition function of 2+1 dimensional Yang-Mills theory is

\[ Z = \int \mathcal{D}A_i^a(x) \exp \left( -\frac{1}{2\kappa^2} \int d^3x B_i^a(x) B_i^a(x) \right) \]  

(5)

where \( \{ A_i^a(x), \ (i, a = 1, 2, 3) \} \) is the Yang-Mills potential and

\[ B_i^a = \frac{1}{2} \varepsilon_{ijk} (\partial_j A_k^a - \partial_k A_j^a + \varepsilon^{abc} A_j^b A_k^c) \]  

(6)

is the field strength. As in section 2, we rewrite \( Z \) as

\[ Z = \int \mathcal{D}A_i^a(x) \mathcal{D}e_i^a(x) \exp \left\{ \int d^3x \left( -\frac{1}{2} [e_i^a(x)]^2 + \frac{i}{\kappa} e_i^a(x) B_i^a(x) \right) \right\} . \]

(7)

The second term in the exponent is precisely the Einstein-Cartan action for gravity in 3 (Euclidean) dimensions. \( e_i^a(x) \) is the driebein and \( \omega_i^{ab} = e^{abc} A_i^c \) the connection 1-form.

In contrast to section 2, we do not get a \( \delta \) function constraint on integrating over \( A_i^a \) in this case. Since \( A \) appears at most quadratically in the exponent, the integration over \( A \) may be explicitly performed. This integration is equivalent to solving the classical equations of motion for \( A \) as a functional of \( e \) and replacing \( A \) by this solution :

\[ \varepsilon_{ijk} (\partial_j \delta^{ac} + \varepsilon^{abc} A_j^b[e]) e_k^c(x) = 0. \]  

(8)

Now (8) is precisely the condition for a driebein \( e \) to be torsion free with respect to the connection 1-form \( A_i^a \).

If we assume the 3 \times 3 matrix \( e_i^a \) to be non-singular, then this solution \( A[e] \) can be explicitly given \([10]\). In this case, no information is lost by multiplying (8) by \( e_i^a \) and summing over \( a \). We get, \( \varepsilon_{ijk} e_i^a \partial_j e_k^c + |e|(e^{-1})^m_b \varepsilon_{klm} \varepsilon_{ijk} A_j^b[e] = 0 \). Defining \( A_j^b(e^{-1})^m_b = A_{jm} \), we get, \( A_{ii}[e] - \delta_{ii} A_{mm}[e] = (1/|e|) \varepsilon_{ijk} e_i^a \partial_j e_k^c \). Taking the trace on both sides, \( A_{mm}[e] = -(1/2|e|) \varepsilon_{ijk} e_i^a \partial_j e_k^c \). Then, \( A_i^b[e] = \frac{1}{|e|} \left( \varepsilon_{ijk} e_i^a \partial_j e_k^c - \frac{1}{2} \delta_{ii} \varepsilon_{klm} e_m^a \partial_j e_k^c \right) \).

By a shift of \( A, \ A = A[e] + A' \), the integration over \( A \) reduces to

\[ \int \mathcal{D}A' \exp \left( \frac{i}{\kappa} \int A'_{ia} e_{in, jb} A'_j \right) = \frac{1}{\det^{1/2}(e_{ia,jb})} = \frac{1}{\det^{3/2}(e_i^a)}, \]  

(9)

where \( e_{ia,jb} = \varepsilon_{ijk} e^{abc} e_c^e \).

\( B_i^a \) is related to the Ricci tensor \( R_{ik} \) as follows:

\[ R_{ik} = F_{ij}^a e_k^a (e^{-1})_j^i \]  

(10)

where \( F_{ij}^a = \varepsilon_{ijk} e^{abc} B_c^k \). Thus an integration over \( A \) gives,
\[ Z = \int \mathcal{D}g \exp \left( -\frac{1}{2} g_{ii} + \frac{i}{\kappa} \sqrt{g} R \right) \]  

(11)

where the metric \( g_{ij} = e^a_i e^a_j \) and \( R = R_{ik} g^{ki} \). Note that \( \mathcal{D}g = \mathcal{D}e \det^{-3/2}(e^a_i) \), as required. The configurations where \( e \) is singular is naively a set of measure zero, so that the assumption \( |e| \neq 0 \) is reasonable.

Equation (11) provides a reformulation of 2+1-dimensional Yang-Mills theory (classical or quantum) in terms of gauge invariant degrees of freedom. It is now a theory of metrics on 3-manifolds; which however is not diffeomorphism invariant because of the term \( g_{ii} \) in the action. As a result, not only the geometry of the 3-manifold, but also the metric \( g_{ij} \) of any coordinate system chosen on the manifold is relevant.

For 3 dimensional (Euclidean) gravity, an integration over \( e \) would give the \( \delta \)-function constraint \( R_{ij} = 0 \), resulting in a topological field theory \([12]\). There are no massless gravitons as a consequence. Now however, the diffeomorphisms provide massless degrees of freedom corresponding to gluons. They may be described as follows. The 3 manifolds are described by the metric \( g_{ij} \) in the coordinate system \( x \). We may choose a new coordinate system \( \phi^A(x) \) (\( A = 1, 2, 3 \)), with a standard form of the metric \( G_{AB}[^\phi] \). We have

\[ g_{ij}(x) = \frac{\partial \phi^A}{\partial x^i} G_{AB}[\phi] \frac{\partial \phi^B}{\partial x^j}. \]  

(12)

This gives the form of the action as,

\[ S = \int d^3 x \left[ -\left( \frac{\partial \phi^A}{\partial x^i} G_{AB}[\phi] \frac{\partial \phi^B}{\partial x^j} \right) + \frac{i}{2\kappa} \left| \frac{\partial \phi^A}{\partial x^i} \right| \sqrt{G[\phi]} R[\phi] \right], \]  

(13)

where \( \left| \frac{\partial \phi^A}{\partial x^i} \right| = \det \left( \frac{\partial \phi^A}{\partial x^i} \right) \). We identify \( \phi^A(x) \) (\( A = 1, 2, 3 \)) as the dual gluons. A simple way of seeing this is as follows. Note that the second term comes with a factor \( i = \sqrt{-1} \), whereas the first term does not. In this sense it is analogous to the \( \theta \)-term in QCD which continues to have the factor \( i = \sqrt{-1} \) in the Euclidean version.. Consider a random phase approximation to \( Z \). The extrema of the phase factor correspond to solutions of the the vacuum Einstein equations. In this case (3 dimensions), this means that the space is flat. Now we may choose the standard form \( G_{AB} = \delta_{AB} \). \( \phi^A \) now represent arbitrary curvilinear coordinates for that manifold. Then the first term in (13) is just \((\nabla \phi^A)^2 \). This describes three massless scalars. As in section 2 they represent the one transverse degree of freedom for each color. Thus the gluons are now described in terms of gauge invariant, local, scalar degrees of freedom.

In the general case \( R \neq 0 \), consider normal coordinates \( \phi^A(x) \) at a given point. The metric has the standard form,

\[ G_{AB}[\phi] = \delta_{AB} + R_{ABCD}[\phi] \phi^C \phi^D + \ldots. \]  

(14)

\( \phi^A \) represents the dual gluons and \( R \) the geometric aspects of the manifold. Both are degrees of freedom of 2+1 dimensional Yang-Mills theory. \( \phi^A \) are invariant under the Yang-Mills gauge transformations. Thus equation (13) describes Yang-Mills dynamics in terms of gauge invariant degrees of freedom.
IV. MONOPOLES

We now identify the monopoles of Yang-Mills theory in terms of the dual variables. Monopoles are related to Yang-Mills configurations \( \{ A^a_i(x) \} \) with a non-trivial \( U(1) \) fibre bundle structure. In such configurations, the monopoles are characterized by points with the following property. Consider a surface enclosing a point and a set of based loops spanning it. Consider eigenvalues of the corresponding Wilson loop operator. As one spans the sphere, the eigenvalue changes continuously from zero to \( 2\pi \) instead of coming back to zero. Thus such points have topological meaning. Moreover a small change in their position can produce a large change in the expectation value of the Wilson loop. Therefore we may expect that such points are relevant for confinement, even though a semi-classical or dilute gas approximation may not be available. Therefore it is important to provide a characterization of these monopoles and their interactions with the dual gluons.

In case of 't Hooft-Polyakov monopole, the location of the monopoles is given by the zeroes of the Higgs field. In pure gauge theory we do not have such an explicit Higgs field. 't Hooft has proposed use of a composite Higgs for this case.

We follow a different procedure here. Consider the eigenvalue equation of the positive symmetric matrix \( B^a_i(x)B^b_i(x) = I^{ab}(x) \) for each \( x \).

The eigenvalues \( \lambda^A(x), (A = 1, 2, 3) \) are real and the corresponding three eigenfunctions \( \chi^A_a(x), (A = 1, 2, 3) \) form an orthonormal set. The monopoles in any Yang-Mills configuration \( A^a_i(x) \) can be located in terms of \( \chi^A_a(x) \). We will illustrate this explicitly in case of the Prasad-Sommerfield solution. For this \( I^{ab} \) has the tensorial form,

\[
I^{ab}(x) = P(r)\delta^{ab} + Q(r)x^a x^b
\]

with \( P(0) \neq 0 \) and finite. At \( r = 0 \), the eigenvalues are triply degenerate. Away from \( r = 0 \), two eigenvalues are still degenerate, but the third one is distinct from them. The corresponding eigenfunction (labelled \( A=1 \), say) is \( \chi^1_a(x) = \hat{x}^a \). This precisely has the required behaviour for the composite Higgs at the center of the monopole.

We may regard \( \chi^A_a(x) \) as providing three independent triplets of (normalized) Higgs fields. Using them, we may construct three abelian gauge fields,

\[
b^A_i(x) = \chi^A_a(x)B^a_i(x) - \frac{1}{3}\epsilon_{ijk}\epsilon^{abc}\chi^A_a\partial_j\chi^B_b\partial_k\chi^C_c
\]

We have

\[
b^A_i(x) = \epsilon_{ijk}\partial_ja^A_k - \frac{1}{3}\epsilon_{ijk}\epsilon^{abc}\chi^A_a\partial_j\chi^B_b\partial_k\chi^C_c
\]

where the three abelian gauge potentials are given by \( a^A_i(x) = \chi^A_a(x)A^a_i(x) \). For each \( A = 1, 2, 3 \), the second part of the right hand side is the topological current for the Poincare-Hopf index. It is the contribution of the magnetic fields due to the monopoles. These monopoles are located at points where this index is non-zero.

Since, \( \chi^a_A = \epsilon_{ABC}\epsilon^{abc}\chi^B_b\chi^C_c \), we may rewrite our abelian fields as
\[ b_i^A(x) = \epsilon_{ijk} (\partial_j a_k^A(x) + \epsilon^{ABC} c_j^B c_k^C) \] (19)

where \( c_i^A = \epsilon^{ABC} x^B \partial_i x^C \) has the form of a ‘pure gauge’ potential, but is not, because of the singularity in \( (\chi^A) \).

Thus for any configuration \( A_i^a(x) \) of the Yang-Mills potential, monopoles may be characterized as the points where the eigenvalues of the symmetric matrix \( B_i^a(x) B_i^b(x) \) become triply degenerate. We may use the corresponding eigenfunctions to construct three abelian gauge fields with respective monopole sources. Instead of \( I^{ab} \), we may also use the gauge invariant symmetric tensor field \( B_i^a(x) B_i^b(x) \) and it’s eigenfunctions \( \chi_i^A(x) \). This provides a gauge invariant description of the monopoles.

We may also use the Ricci tensor \( R_i^j = R_{ijk}(x) g^{kj}(x) \) for this purpose. The three eigenfunctions \( \chi_i^A(x), (A = 1, 2, 3) \) (Ricci principal directions [16]) provide three orthogonal vector fields for the 3-manifold. In regions where eigenvalues of \( R_i^j \) are degenerate, the choice of the vector fields is not unique. One can make any choice requiring continuity. However isolated points where \( R_i^j \) is triply degenerate are special, and have topological significance. At such points the vector fields are singular. Thus the monopoles correspond to the singular points of these vector fields. The index of the singular point is the monopole number.

We emphasize that the centers have a topological interpretation which is independent of the way we construct them.

V. INTERACTION OF DUAL GLUONS WITH MONOPOLES

Dual gluons are identified with a coordinate system \( \phi^A(x) \ A = 1, 2, 3 \) on the 3-manifold eqns. [13][14]. We now consider special coordinate systems which are singular at the location of the monopole. In case of the Prasad-Sommerfield monopole, the correspond to the spherical coordinates \( (r, \theta, \phi) \) with the monopole at the origin. In the general case, we may construct the coordinate system as follows. At the site of the monopole, one of the eigenfunctions \( \chi_1^A(x) \) say, has the radial behaviour. Then we may construct the integral curves of this vector field by solving the equations,

\[
\frac{dx^1}{\chi_1^1(x)} = \frac{dx^2}{\chi_2^2(x)} = \frac{dx^3}{\chi_3^3(x)}. \quad (20)
\]

We may choose these integral curves to be the equivalent of the \( r \)-coordinate, i.e. we identify these curves with \( \theta = \text{constant}, \phi = \text{constant} \) curves of the new coordinate system. Consider closed surfaces surrounding the monopole which are nowhere tangential to these integral curves. A simple choice is just the spherical surfaces. We may identify them with the surfaces \( r = \text{constant} \). (We have not specified the \( \theta, \phi \) coordinates completely, but this is not required for our purpose.) We thus have a coordinate system \( \chi^A(x) \) whose coordinate singularities correspond to the monopoles. In this coordinate system, \( \int d^3x \sqrt{g} R = \int d^3x (\epsilon_{ijk} \epsilon^{ABC} \partial_i \chi^A \partial_j \chi^B \partial_k \chi^C) \sqrt{g}(x) R(x) \) where \( g_{ij} \) is the metric in this coordinate system.

Now \( \partial_i(\epsilon_{ijk} \partial_j \chi^2 \partial_k \chi^3) \) is non-zero at \( x = x_0 \) and is related to the monopole charge at \( x_0 \) as follows. Let \( \chi^A(x) - \chi^A(x_0) = \rho(x) \hat{\chi}^A(x) \) where \( \hat{\chi}^A(x) \hat{\chi}^A(x) = 1 \). We see that there is a coupling of the field combination \( \sqrt{g}(x) R(x) \rho^3(x) \) to the monopole charge density.
\[ \partial_i k_i(x) = m_i \delta^3(x_0), \] where \( k_i(x) = \epsilon_{ijk} \epsilon^{ABC} \hat{\chi}_A \partial_j \hat{\chi}_B \partial_k \hat{\chi}_C. \) Thus a certain combination of the dual gluon \( \phi^A(x) \) and the geometric degree of freedom \( R(x) \) couples to the monopoles. In analogy to the compact U(1) lattice gauge theory (sec. 2), this may be expected to give a mass for the dual gluon and hence confinement. There are other interactions which are not of topological origin and these are to be interpreted as self interactions.

VI. CONCLUSION

We have argued that the duality transformation for 2+1 dimensional Yang-Mills theory can be carried out in close analogy to the abelian case. The dual theory has geometric interpretation in terms of 3-manifolds. We identified the dual gluons with the coordinates of the 3-manifolds and monopoles with the coordinate singularities. We expect that this will provide a new approach for understanding quark confinement.

VII. ACKNOWLEDGEMENT

One of us PM wishes to thank Dr. Elizabeth Gasparim and Dr. Mahan Mitra for explanation of several mathematical concepts.
REFERENCES

[1] A. M. Polyakov, Phys. Lett. B59, 82 (1975)
[2] A. M. Polyakov, Nucl. Phys. B120, 429 (1977)
[3] T. Banks, R. Myerson, and J. Kogut, Nucl. Phys. B129, 493 (1977)
[4] R. Anishetty and H. S. Sharatchandra, Phys. Rev. Lett 65, 813 (1990)
[5] R. Anishetty, S. Cheluvraja, H. S. Sharatchandra and M. Mathur, Phys. Lett. B314, 387 (1993)
[6] G. ’t Hooft, Nucl. Phys. B190, 455 (1981)
[7] F. A. Lunev, Phys. Lett. B295, 99 (1992)
[8] M. Bauer, D. Z. Freedman and P. E. Haagensen, Nucl. Phys. B428, 147 (1994); P. E. Haagensen and K. Johnson, Nucl. Phys. B439, 597 (1995).
[9] Pushan Majumdar and H. S. Sharatchandra, imsc/98/05/22, hep-th/9805102, (1998).
[10] Pushan Majumdar and H. S. Sharatchandra, imsc/98/04/14, hep-th/9804128, (1998)
[11] T. T. Wu and C. N. Yang, Phys. Rev. D12, 3845 (1975)
[12] E. Witten Nucl. Phys. B311, 46 (1988)
[13] P. Goddard and D. I. Olive, Rep. Prog. Phys. 41 1357 (1978)
[14] J. Arafune, P. G. O. Freund and C. J. Goebel, Journal of Math. Phys. 16, 433 (1975)
[15] M. K. Prasad and C. M. Sommerfield, Phys. Rev. Lett 35, 760 (1975)
[16] L. P. Eisenhart, Riemannian Geometry, (Princeton University Press), (1949).