RELATIVE $K$-CYCLES AND ELLIPTIC BOUNDARY CONDITIONS

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Dedicated to Professor Zejian Jiang on his seventieth birthday

Abstract. In this paper, we discuss the following conjecture raised by Baum-Douglas: For any first-order elliptic differential operator $D$ on smooth manifold $M$ with boundary $\partial M$, $D$ possesses an elliptic boundary condition if and only if $\partial[D] = 0$ in $K_1(\partial M)$, where $[D]$ is the relative $K$-cycle in $K_0(M, \partial M)$ corresponding to $D$. We prove the “if” part of this conjecture for $\dim(M) \neq 4, 5, 6, 7$ and the “only if” part of the conjecture for arbitrary dimension.

First we fix some notation. $M$ is a compact oriented smooth manifold with smooth boundary $\partial M$. We always suppose that $M$ is embedded in some compact smooth manifold $\tilde{M}$ without boundary of the same dimension (e.g., $\tilde{M}$ can be taken as double of $M$). We denote $\tilde{M} = M \setminus \partial M$. Furthermore, we assume that $E_0$ and $E_1$ (in fact, all the vector bundles in this paper) are smooth complex Hermitian vector bundles over $M$ and that $D : C^\infty(E_0) \to C^\infty(E_1)$ is a first-order elliptic differential operator from smooth sections of $E_0$ to that of $E_1$. By $H^s(M, E_i)$ and $H^s(\partial M, E_i)$ we shall denote the Sobolev spaces of sections of $E_i$ and $E_i|_{\partial M}$ with respect to fixed smooth measures on $M$ and $\partial M$, respectively.

The elliptic boundary value problem (an elliptic operator with an elliptic boundary condition) has been studied for a long time. As noted in [1, 5, 6] and other references, there exist topological obstructions to impose an elliptic boundary condition on the above $D$. A fundamental problem is to find all such obstructions. Baum, Douglas, and Taylor constructed a relative $K$-cycle $[D] \in K_0(M, \partial M) \cong KK(C_0(\tilde{M}), \mathbb{C})$ (here $C_0(\tilde{M})$ is the algebra of continuous functions on $M$ which vanish on $\partial M$) corresponding to $D$ (see [2–4] for details). From the definition of relative $K$-homology group $K_0(M, \partial M)$ given by Baum, Douglas, and Taylor, the boundary map $\partial : K_0(M, \partial M) \to K_1(\partial M)$ is very concrete [2–4]. Also Baum and Douglas conjectured that the only obstruction for $D$ possessing elliptic boundary conditions is that $\partial[D] \neq 0$. More precisely, the following conjecture first appeared in [2] in a closely related form.

Conjecture. There exist a vector bundle $E_2$ over $\partial M$ and a zeroth-order pseudo-
differential operator $B$ defined from $C^\infty(\partial M, E_0)$ to $C^\infty(\partial M, E_2)$ such that

$\begin{pmatrix} D \\ B \circ \gamma \end{pmatrix}: H^1(M, E_0) \longrightarrow \begin{pmatrix} H^0(M, E_1) \\ \oplus \\ H^{1/2}(\partial M, E_2) \end{pmatrix}$

is Fredholm if and only if $\partial[D] = 0$ in $K_1(\partial M)$, where $\gamma : H^1(M, E_0) \longrightarrow H^{1/2}(\partial M, E_0)$ is the trace map.

**Remark 1.** Let $D$ be as above with principal symbol $p(x, \xi)$. A zeroth-order pseudo-differential operator $B$ with principal symbol $b(x, \xi)$ from $C^\infty(\partial M, E_0)$ to $C^\infty(\partial M, E_2)$ is said to be elliptic to $D$ (see [6], p. 233) if, for every $x \in \partial M$ and $\xi \in T^*_x(\partial M)$, the map $M^+_{x, \xi} \ni u \longrightarrow b(x, \xi)u(0) \in (E_2)_x$ is bijective, where $T^*_x(\partial M)$ and $(E_2)_x$ are the fibres at the point $x$ of the cotangent bundle $T^*\partial M$ and the bundle $E_2$, and, furthermore, $M^+_{x, \xi}$ is the set of all $u \in C^\infty(\mathbb{R}, (E_0)_t)$ with $p(x, \xi - i \frac{\partial}{\partial t} \cdot n_x)u(t) = 0$ ($n_x$ is the interior conormal vector of $M$ at $x$) which are bounded on $\mathbb{R}^+$. If $B$ is elliptic to $D$, then the above $(B \circ \gamma)$ is Fredholm. Such a system $(B \circ \gamma)$ is often called an elliptic boundary value problem or an elliptic operator with an elliptic boundary condition; meanwhile, $D$ is also said to possess an elliptic boundary condition.

**Remark 2.** Although the above elliptic boundary condition is used in most references, the original form of the conjecture in [2] is in a slightly different form from the above. In [2] the operator for the boundary condition is of the form $\gamma \circ B$, where $B$ is a zeroth-order pseudo-differential operator from $E_0$ to a smooth vector bundle over a neighborhood of $M$. The reason we use a slightly different form of the conjecture is as follows: for general zeroth-order pseudo-differential operator $B$ defined on $M$, there is no canonical way to restrict $B$ to $M$ as an operator $B_M : H^s(M) \longrightarrow H^s(M)$ when $s > 0$. So one needs to put some restriction on $B$. One of the natural restrictions is that $B$ has the transmission property with respect to $\partial M$ (see [5]). We also prove our theorem for this kind of boundary condition (see Theorem 1). It must be pointed out that the existence of a boundary condition of type $B \circ \gamma$ implies the existence of that of type $\gamma \circ B$.

In this paper, we prove the “only if” part of the conjecture which can be thought of as a generalization of Corollary 4.2 in [4] (there $B$ is a differential operator). Conversely, we prove that if dim($M$) $\neq 4, 5, 6, 7$ and $\partial[D] = 0$, then $D$ possesses an elliptic boundary condition as in Remark 1. Hence the “if” part of the conjecture has been proved for dim($M$) $\neq 4, 5, 6, 7$. The cases of dim($M$) being equal to 4, 5, 6, or 7 are still open, but we prove a theorem which can be thought of as the “if” part of the conjecture in the sense of stabilization in K-homology group for arbitrary dimension. Our results will be useful for constructing absolute $K$-cycles in $K_0(M)$ which are preimages of $[D] \in K_0(M, \partial M)$ under the canonical map from $K_0(M)$ to $K_0(M, \partial M)$ when $\partial[D] = 0$.

Our main results are the following:

**Theorem 1.** (“only if” part) $\partial[D] = 0$ if one of the following is true:

(i) There exist a smooth vector bundle $E_2$ over $\partial M$ and a zeroth-order pseudo-differential operator $B$ from $E_0|_{\partial M}$ to $E_2$ such that $(B \circ \gamma)$ in the conjecture is Fredholm.
There exist a bundle $E_2$ over a neighborhood of $M$ in $\tilde{M}$ and a zeroth-order pseudo-differential operator $B$ with transmission property with respect to $\partial M$ from $E_0$ to $E_2$ such that

\[
\begin{pmatrix} D & \gamma \circ B \end{pmatrix} : H^1(M, E_0) \to H^0(M, E_1) \oplus H^{1/2}(\partial M, E_2)
\]
is Fredholm.

**Theorem 2.** ("if " part) If $\partial[D] = 0$, then there exists a first-order elliptic differential operator $D_1$ acting on smooth vector bundles over $M$ with $[D_1] = 0$ in $K_0(M, \partial M)$ such that $D \oplus D_1$ possesses an elliptic boundary condition as in Remark 1, and, furthermore, if $\dim(M) \neq 4, 5, 6, 7$, then $D$ itself possesses an elliptic boundary condition.

The main idea of the proof of Theorem 1 is to construct an intertwining between $\partial[D]$ and a trivial element in $K_1(\partial M)$. In the proof, we use Calderon projection, functional calculus of pseudo-differential operators (including Boutet de Monvel type operators), and the techniques in the proof of Proposition 4.5 of [4].

The proof of Theorem 2 makes use of two key lemmas (see below).

Let $ST^*\partial M$ be the unit sphere bundle of $T^*\partial M$ over $\partial M$ and $\pi : ST^*\partial M \to \partial M$ be the canonical projection map. Let $\tilde{E}_0 = \pi^*(E_0|_{\partial M})$ be the bundle over $ST^*\partial M$. We write the principal symbol of $D$, in a coordinate neighborhood $U$ of $x \in \partial M$, as

\[
p(x, x_n, \xi, \xi_n) = \sum_{j=1}^{n-1} p_j(x, x_n) \xi_j + p_n(x, x_n) \xi_n,
\]
where $x_n$ is the coordinate for the normal direction of $\partial M$. We define

\[
\tau(x, \xi) = ip_n^{-1}(x, 0) \sum_{j=1}^{n-1} p_j(x, 0) \xi_j
\]
for $x \in \partial M$ and $\xi \in ST^*\partial M$. Then $\tau(x, \xi)$ is a map from a fibre of $E_0$ into itself and has no purely imaginary eigenvalue. Let $V_{\pm}$ be the subbundle of $\tilde{E}_0$ over $ST^*\partial M$ corresponding to the span of the generalized eigenvectors of $\tau(x, \xi)$ corresponding to the eigenvalues with positive/negative real parts.

**Lemma 1.** $\partial[D] = 0$ if and only if $[V_+] \in \pi^*K^0(\partial M) \subset K^0(ST^*\partial M)$.

**Lemma 2.** If $E_0$ and $E_1$ are vector bundles over $M$ which allow a first-order elliptic differential operator $D$ to act from one to the other, and if $\dim(M) = n$, then

(i) $f \dim(E_0) = f \dim E_1 \geq 2^{(n-1)/2}$;

(ii) $f \dim(E_0) = f \dim E_1 \geq 2^{(n-1)/2} + 1$ provided $n$ is even and $\partial[D] = 0$,

where $f \dim$ denotes dimension of each fibre of the vector bundles.
The Proof of Theorem 2. By Lemma 1, if $\partial[D] = 0$, one has 
\[ [V_+] \in \pi^* K^0(\partial M) \subset K^0(ST^*\partial M). \]

By Lemma 2, $f \dim(V_+) = \frac{f \dim(E_0)}{2} \geq 2^{n/2} - 1$. Therefore, $f \dim(V_+) \geq n - 1 > \frac{\dim(ST^*\partial M)}{2} = \frac{2n-3}{2}$, whenever $\dim(M) \geq 8$. Hence there exists a complex vector bundle $E_2$ over $\partial M$, such that $V_+ \cong \pi^* E_2$. This is also true for $\dim(M) \leq 3$, since the collection of complex vector bundles over $ST^*\partial M$ ($\dim(ST^*\partial M) \leq 3$) has property of cancellation.

Let $\psi$ be the bundle isomorphism 
\[ V_+ \xrightarrow{\psi} \pi^* E_2 \]
\[ ST^*\partial M \xrightarrow{\psi} ST^*\partial M \]

For any $(x,\xi) \in ST^*\partial M$, let $b(x,\xi)$ be the bundle map defined by 
\[ E_0 \xrightarrow{\text{project to}} V_+ \xrightarrow{\psi} E_2 \]
\[ ST^*\partial M \xrightarrow{\text{project to}} ST^*\partial M \]

Furthermore, let $B : E_0|_{\partial M} \rightarrow E_2$ be the zeroth-order pseudo-differential operator with symbol $b(x,\xi)$. It follows that $B$ is elliptic to $D$ (see [6]).

Example 1. If $M$ is a $\text{spin}^c$ manifold with smooth boundary and $D$ is the Dirac operator over $M$, then it is computed in [4] that $\partial[D] \neq 0$. Hence $D$ possesses no elliptic boundary condition (even possesses no boundary condition as in the conjecture).

Example 2. For any $D$, let $D^*$ be the formal adjoint of $D$. It is easy to prove that $[D] = -[D^*]$ in $K_0(M, \partial M)$; hence, $\partial[D \oplus D^*] = 0$. It follows that $D \oplus D^*$ possesses an elliptic boundary condition provided $\dim(M) \neq 4$ or 6. (It should be noted that we only need to exclude the manifolds with dimension 4 and 6 here, since the dimension of the bundle on which $D \oplus D^*$ acts is twice the dimension of the bundle on which $D$ acts.)

The details of the proofs will appear elsewhere.

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