Graph connectivity based strong quantum nonlocality with genuine entanglement

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Strong nonlocality based on local distinguishability is a stronger form of quantum nonlocality recently introduced in multipartite quantum systems: an orthogonal set of multipartite quantum states is said to be of strong nonlocality if it is locally irreducible for every bipartition of the subsystems. Most of the known results are limited to sets with product states. Shi et al. presented the first result of strongly nonlocal entangled sets in [Phys. Rev. A 102, 042202 (2020)] and there they questioned the existence of strongly nonlocal set with genuine entanglement. In this work, we relate the strong nonlocality of some special set of genuine entanglement to the connectivities of some graphs. Using this relation, we successfully construct sets of genuinely entangled states with strong nonlocality. As a consequence, our constructions give a negative answer to Shi et al.’s question, which also provide another answer to the open problem raised by Halder et al. [Phys. Rev. Lett. 122, 040403 (2019)]. This work associates a physical quantity named strong nonlocality with a mathematical quantity called graph connectivity.

I. INTRODUCTION

Quantum nonlocality, one of the most surprising properties in quantum mechanics, is usually being detected with entangled states by their violations of Bell-type inequalities. In addition, the local indistinguishability of an orthogonal set of quantum states is also being widely used to illustrate the phenomenon of quantum nonlocality. It is well known that an orthogonal set of quantum states can be perfectly distinguished by positive operation value measurement (POVM)[1]. Bennett et al. [2] presented an example of orthogonal product states in $\mathbb{C}^3 \otimes \mathbb{C}^3$ that are locally indistinguishable (here only local operations and classical communications are allowed). Therefore, all the information of the given set can be inferred by using global measurement, but only partial information can be deduced when only local measurement are allowed. They named such a phenomenon as quantum nonlocality without entanglement. Since then, the quantum nonlocality based on local indistinguishability has been studied extensively (see Refs. [3–36] for an incomplete list). Moreover, the local indistinguishability of quantum states has also been practically applied in quantum cryptography primitives such as data hiding [37, 38] and secret sharing [39–41].

In each protocol that can perfectly distinguished the set of states, the states remain to be orthogonal to each other after every local measurement. Measurements with such property are called orthogonality preserving measurement. Based on this kind of measurement, Halder et al. [42] introduced the concept local irreducibility, a stronger form of local indistinguishability. A set of multiparticle orthogonal quantum states is said to be locally irreducible if it is not possible to locally eliminate one or more states from that set using orthogonality preserving measurement. They presented two sets of product states in $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ and $\mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$ that are locally irreducible for each bipartition of the corresponding tripartite systems. They named such phenomenon as strong nonlocality without entanglement. After that, Yuan et al. [43] constructed some strongly nonlocal sets without entanglement in higher dimensional systems $\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$ and even an example in $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$. Recently, Shi et al. [44] developed a strong method which helps them to construct some strongly nonlocal orthogonal product sets in general 3,4,5-parties systems. Based on the local irreducibility in some multipartitions, Zhang et al. [45] generalized the concept of strong nonlocality to more general settings.

Intuitively, the more entanglement of a given set, the easier it is to show the strong nonlocality. However, this may not be the case. In fact, Halder et al. found examples of strong nonlocality without entanglement but they proposed the open question: are there any orthogonal entangled bases that present the strong nonlocality (see Ref. [42])? Moreover, as a negative example, they showed that a special three-qubit GHZ basis is locally irreducible in all bipartitions. Soon after that, Shi et al. [46] provided a positive answer to the above question by constructing sets of strongly nonlocal entangled states (which are separable in some partition of the tripartite systems) in $\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$ for $d \geq 3$. But they doubted the existence of genuinely entangled set which is still strongly nonlocal. Therefore, it is interesting to consider whether genuinely entangled set presents this kind of strong nonlocality or not.

In this paper, we construct genuinely entangled sets with strong nonlocality in $\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$ for $d \geq 3$. First,
we give an example in $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$. Then by relating some graphs to some special genuinely entangled set, we prove that the connectivities of these graphs are sufficient to show the strong nonlocality of this given set. As a consequence, we construct a genuinely entangled set with $d^3-(d-2)^3$ (or $d^3-(d-2)^3$) elements in $\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$ when $d \geq 3$ is odd (or is even) that is of strong nonlocality.

The rest of this article is organized as follows. In Sec. II, we give some necessary notation and definitions. In Sec. III, we present a general method to construct strongly nonlocal set which is consisting of GHZ like states. In Sec. IV, we extend this construction to more general case and present an example. Finally, we draw a conclusion and present some interesting problems in section V.

II. PRELIMINARIES

For an integer $d \geq 2$, denote $\mathbb{Z}_d$ to be the group defined over $\{0, 1, \cdots, d-1\}$ with mod $d$ as its “+” operation. Let $\mathcal{H}$ be a Hilbert space of dimension $d$. We always assume that its computational basis is $\{|i\rangle \mid i \in \mathbb{Z}_d\}$.

In this section, we will introduce the definition of the graph and some related usages, the concept of genuine entanglement and the definition of strongest nonlocality.

Graph. A graph (see Ref. [47] for more details) is a pair $G = (V,E)$, where $V$ is a set whose elements are called vertices and $E$ is a set of paired vertices, whose elements are called edges.

In a graph, a pair of vertices $\{x, y\}$ (where $x, y \in V$) is called connected if there is a path with edges from $x$ to $y$; Otherwise, the pair is called disconnected. A connected graph is a graph in which every pair of vertices in the graph is connected. There is an important concept called connected component which is related to the connectivity of a graph. A connected component is a maximal connected subgraph of a graph. Then a graph is connected if and only if it has exactly one connected component.

Genuine entanglement. In a bipartite system $\mathcal{H}_A \otimes \mathcal{H}_B$, a pure $|\Psi\rangle_{AB}$ is called an entangled state if it can not be written as some tensor product of two local pure states, i.e., $|\Psi\rangle_{AB}$ is not of the form $|\phi\rangle_A \otimes |\theta\rangle_B$. For a pure state $|\Psi\rangle_{A_1 \cdots A_n}$ in multipartite systems $\otimes_{i=1}^n \mathcal{H}_{A_i}$, it is called a genuinely entangled state if it is entangled for each bipartition of $\{A_1, \cdots, A_n\}$ (see Ref. [48]).

The most well known genuinely entangled state is the Greenberger-Horne-Zeilinger (GHZ) state $(|000\rangle + |111\rangle)/\sqrt{2}$ and the W state $(|100\rangle + |010\rangle + |001\rangle)/\sqrt{3}$ in three qubits. For a higher dimensional systems $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \mathbb{C}^{d_3}$, the states $(|i_1j_1k_1\rangle \pm |i_2j_2k_2\rangle)/\sqrt{2}$ are also genuinely entangled if $i_1 \neq i_2, j_1 \neq j_2$, and $k_1 \neq k_2$. We call such states as GHZ like states under the computational basis. In addition, the state

$$\frac{(|i_1j_1k_1\rangle + w|i_2j_2k_2\rangle + w^2|i_3j_3k_3\rangle + \cdots + w^{d-1}|i_dj_dk_d\rangle)}{\sqrt{d}}$$

(where $w$ is any $d$-th root of unit and $i_m \in \mathbb{Z}_{d_1}$, $j_m \in \mathbb{Z}_{d_2}$, $k_m \in \mathbb{Z}_{d_3}$) is also genuinely entangled if $i_m \neq i_m', j_m \neq j_m'$ and $k_m \neq k_m'$ whenever $m \neq m'$. We call these states GHZ like states with weight $d$. And the set $\{|i_mj_mk_m\rangle\}_{m=1}^d$ with the property, $i_m \neq i_m'$, $j_m \neq j_m'$, and $k_m \neq k_m'$ whenever $m \neq m'$, is called coordinate different. One should note that the GHZ like states with weight $d$ in $\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$ are 1-uniform states [49] or absolute maximally entangled states [50–52]. In this paper, we mainly pay attention to the above form of genuinely entangled states.

Strongest nonlocality. A measurement is nontrivial if not all the POVM elements are proportional to the identity operator. Otherwise, the measurement is trivial. An set of orthogonal states is said to be of the strongest nonlocality if only trivial orthogonality-preserving POVM can perform for each bipartition of the subsystems.

Note that a set of the strongest nonlocality must also be of strong nonlocality by definition. In this paper, we mainly pay attention to the strongest nonlocality of some given orthogonal genuinely entangled set (OGES).

III. STRONGLY NONLOCAL SETS OF GHZ LIKE STATES

In this section, we show that genuinely entangled set that are of the strongest nonlocality do exist. Moreover, the strongest nonlocality of some special sets are rather simple as it is determined by the connectivity of some related graphs. First, we present an example to show that orthogonal set of genuinely entangled states with the strongest nonlocality do exist even in $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$.

FIG. 1. This shows the geometric structure of $S_t$ defined in Eq. (1).
Example 1 In $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$, the set $S := \bigcup_{i=1}^3 S_i$ given by Eq. (1) is an OGES of the strongest nonlocality. The size of this set is 26 (the geometric structure of $S$ can be seen in Fig. 1).

\begin{align}
S_1 & := \{(0|A|i|B|j|1+c\} \pm |A|i+1|B|j)c \mid (i, j) \in Z_2 \times Z_2\},
S_2 & := \{(i+1)|A|B|j|c \pm |A|B|j+1)c \mid (i, j) \in Z_2 \times Z_2\},
S_3 & := \{|i+1|A|B|j|c \pm |A|B|j+1)c \mid (i, j) \in Z_2 \times Z_2\},
S_4 & := \{|0|A|B|c \pm |2|A|B|c \}\}.
\end{align}

(1)

**Proof.** Without loss of generality, let $B$ and $C$ come together to perform a joint orthogonality-preserving POVM $E = M^\dagger M$, where $E = (a_{ijk,l})_{i,j,k,l \in Z_2}$. Then the postmeasurement states $\{|I \otimes M|\psi\rangle|\psi \rangle \in S\}$ should be mutually orthogonal, i.e.,

$$\langle \phi | I \otimes M |\psi\rangle = 0$$

(2)

for $|\psi\rangle, |\phi\rangle \in S$ and $|\psi\rangle \neq |\phi\rangle$.

First, we show that the matrix $E$ is diagonal. For any pair of non-equal coordinates ($i, j$) and ($k, l$) in $Z_3 \times Z_3$, one finds that there exist two pairs of genuinely entangled states $\{|\psi_{\pm}\rangle := |0|A|B|j|l|c \pm |a_1|A|B|j|l|c\}$ and $\{|\phi_{\pm}\rangle := |0|A|B|j|l|c \pm |a_2|A|B|j|l|c\}$ (maybe $|0|A|B|j|l|c$ or $|0|A|B|j|l|c$ in the second term in $S$).

Replacing $|\psi\rangle$ by one of $|\psi_{\pm}\rangle$ and $|\phi\rangle$ by one of $|\phi_{\pm}\rangle$ in Eq. (2), we get four equations. Using these four relations, one obtains that

$$a_{ij,kl} = A|0|B|j|c \langle I \otimes M |0|A|k|B|l|C \rangle = 0.$$

Therefore, the matrix $E$ is diagonal under the computational basis.

In the following, we show that $E$ is indeed proportional to the identity operator. Note the following observation: applying Eq. (2) to any pair of states in $S$ of the form $|i_1|A|j_1|B|k_1)c \pm |i_2|A|j_2|B|k_2)c$, as $i_1 \neq i_2$, one could deduce that

$$a_{i_1,j_1,k_1} = a_{i_2,j_2,k_2}.$$

(3)

Using this observation to the following pairs:

$$|0|A|B|0|c \pm |2|A|B|0|c,$$
$$|0|A|B|1|c \pm |2|A|B|1|c,$$
$$|0|A|B|0|c \pm |2|A|B|0|c,$$
$$|0|A|B|1|c \pm |2|A|B|1|c,$$
$$|0|A|B|0|c \pm |2|A|B|0|c,$$
$$|0|A|B|1|c \pm |2|A|B|1|c,$$

we have the equations

$$a_{00,00} = a_{21,21} = a_{12,12},$$
$$a_{12,12} = a_{20,20} = a_{20,20} = a_{11,11},$$
$$a_{11,11} = a_{02,02} = a_{02,02} = a_{10,10},$$
$$a_{10,10} = a_{01,01} = a_{01,01} = a_{22,22}.$$

Therefore, one concludes that $E$ is proportional to the identity operator.

To prove that the elements on the diagonal of $E$ are equal to each other, one can also attach the set $S$ with a graph $\mathcal{G}_A(S) = (V_A(S), E_A(S))$ defined as follows:

$$V_A(S) := Z_3 \times Z_3, \{ (j_1, k_1), (j_2, k_2) \} \in E_A(S) \text{ if and only if there exist some } i_1, i_2 \in Z_3 \text{ such that the pair of states } |i_1|A|j_1|B|k_1)|c \pm |i_2|A|j_2|B|k_2)c \text{ are in } S.$$
**S** is of the strongest nonlocality if and only if all the corresponding graphs of **S** are connected.

**Proof.**  
\( (\Leftarrow) \) Without loss of generality, suppose that \( B \) and \( C \) come together and perform a joint orthogonality-preserving POVM \( \{E = M^\dagger M\}, \) where \( E = (a_{ijk}, k) \in \mathbb{Z}_{d_2} \times \mathbb{Z}_{d_3}, \) The postmeasurement states \( \{I_A \otimes M|\psi\rangle |\psi\rangle \in \mathcal{S}\) should be mutually orthogonal, i.e.,

\[
\langle \phi | I_A \otimes E | \phi \rangle = 0 \quad (4)
\]

for \( |\psi\rangle, |\phi\rangle \in \mathcal{S} \) and \( |\psi\rangle \neq |\phi\rangle. \) First, we show that the matrix \( E \) is diagonal. For any pair of non-equal coor-dinates \((i, j)\) and \((k, l)\) in \( \mathbb{Z}_{d_2} \times \mathbb{Z}_{d_3}, \) as \( S \) is plane containing, there exist two pairs of GHZ like states \( \{|\psi_{\pm}\rangle := |\phi_0\rangle_A |i\rangle_B |j\rangle_C \pm |\phi_1\rangle_A |i\rangle_B |j\rangle_C \} \) and \( \{|\phi_{\pm}\rangle := |\phi_0\rangle_A |i\rangle_B |j\rangle_C \pm |\phi_1\rangle_A |i\rangle_B |j\rangle_C \} \) (maybe \( |\phi_0\rangle_A |i\rangle_B |j\rangle_C \) or \( |\phi_0\rangle_A |i\rangle_B |j\rangle_C \) is in the second term) in \( \mathcal{S}. \) Replacing \( |\psi\rangle \) by one of \( |\psi_{\pm}\rangle \) and \( |\phi\rangle \) by one of \( |\phi_{\pm}\rangle \) in Eq. (2), we get four equations. Using these four relations, one obtain

\[
a_{ijkl} = A_{ijkl} |i\rangle_B |j\rangle_C |k\rangle_B |l\rangle_C \quad (5)
\]

where \( (j_1, k_1) = (j_1, k_1) \) and \( (j_{N+1}, k_{N+1}) = (j_L, k_L). \) The definition of the edges of \( \mathcal{G}_A(S), \) for each edge \( e_l = \{(j_l, k_l), (j_{l+1}, k_{l+1})\}, \) there is a pair of states of the form \( |\psi_{\pm}\rangle := |\phi_0\rangle_A |i\rangle_B |j\rangle_C \pm |\phi_1\rangle_A |i\rangle_B |j\rangle_C \in \mathcal{S}. \) Applying Eq. (4) to each pair of genuinely entangled states \( |\phi_0\rangle_A |i\rangle_B |j\rangle_C \pm |\phi_1\rangle_A |i\rangle_B |j\rangle_C, \) one can obtain that

\[
a_{ijkl} = a_{ijkl} |i\rangle_B |j\rangle_C \quad (6)
\]

From the path in Eq. (5) and Eq. (6), one can easily deduce that

\[
a_{ijkl} = a_{ijkl} |i\rangle_B |j\rangle_C \quad (7)
\]

Therefore, one can conclude that the matrix \( E \) is indeed proportional to identity operator.

\( (\Rightarrow) \) Without loss of generality, we only need to show that \( \mathcal{G}_A(S) \) is connected. By the strongest nonlocality of \( S, \) there is only one solution (which is proportional to the identity operator) for \( E \) when it satisfies all the equations in Eq. (4). By applying Eq. (4) to two states \( |\phi\rangle \) and \( |\psi\rangle \) from different pair of GHZ like states, we can only obtain some linear equations for the diagonal elements of \( E. \) Therefore, the only following equations are related to the diagonal elements of \( E\)

\[
\langle \phi_+ | I_A \otimes E | \phi_- \rangle = 0 \quad |\phi_{\pm}\rangle \text{ a pair of GHZ like states in } \mathcal{S}.
\]

For each pair of GHZ like states \( |\phi_{\pm}\rangle = |i_1\rangle_A |j_1\rangle_B |k_1\rangle_C \pm |i_2\rangle_A |j_2\rangle_B |k_2\rangle_C \) in \( \mathcal{S}, \) the corresponding equation in the above set is just \( a_{j_1k_{1j}j_2k_{2j}}. \) This equation exactly corresponds to the edge connecting \((j_1, k_1)\) and \((j_2, k_2)\) in \( \mathcal{G}_A(S). \) From this correspondence, one can deduce that \( \mathcal{G}_A(S) \) is connected as the diagonal elements of \( E \) are equal to each other.

From the above theorem, the problem of the strongest nonlocality of some special orthogonal set with GHZ like states can be determined by the connectivity of its related graphs. Moreover, one can easily extend the above theorem to multipartite systems. Now we apply Theorem 1 to get some sets of the strong nonlocality with genuine entanglement.

**Example 2** In \( \mathbb{C}^3 \otimes \mathbb{C}^4 \otimes \mathbb{C}^5, \) the set \( S := \bigcup_{i=1}^{4} S_i \) is an OGES of the strongest nonlocality, where

\[
S_1 := \{(0)A |i\rangle A |j + 1\rangle C \pm |2)A |i + 1\rangle B |j\rangle C | (i, j) \in Z_3 \times Z_4\},
\]

\[
S_2 := \{(i + 1)A |0\rangle B |j\rangle C \pm |iA |3)B |j + 1\rangle C | (i, j) \in Z_2 \times Z_4\},
\]

\[
S_3 := \{(i + 1)A |0\rangle B |j\rangle C \pm |i + 1A |j\rangle B |4)C | (i, j) \in Z_2 \times Z_3\},
\]

\[
S_4 := \{(0)A |0\rangle B |0\rangle C \pm |2)A |3)B |4)C\}.
\]

In fact, \( S \) is a special set of GHZ like states that is plane containing. Moreover, its corresponding graphs \( \mathcal{G}_A(S), \mathcal{G}_B(S) \) and \( \mathcal{G}_C(S) \) are showed in the Fig. 3 and they are all connected. By Theorem 1, the set \( S \) is of the strongest nonlocality.

**Proposition 1** In \( \mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d, \) \( d \geq 3 \) and odd, the set \( \bigcup_{i=1}^{4} S_i \) given by Eq. (8) is an OGES of the strongest nonlocality. The size of this set is \( d^3 - (d - 2)^3. \) Here and the following we use the notation \( d := d - 1 \) for simplicity.
Proposition 2 In $\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$, $d \geq 4$ and even, the set $\bigcup_{i=1}^d S_i$ given by Eq. (9) is an OGES of the strongest nonlocality. The size of this set is $d^3 = (d-2)^3 + 2$.

The proof of Proposition 2 is given in Appendix. Moreover, as an example, we show the corresponding graphs of $S$ for $d = 4$ in figure 4.

IV. STRONGLY NONLOCAL SETS OF HIGH WEIGHTED GHZ LIKE STATES

In this section, we will consider the strong nonlocality of those sets of high weight GHZ like states. We consider a general tripartite systems $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \mathbb{C}^{d_3}$ where $d_i \geq 3$ for $1 \leq i \leq 3$. We also make some assumptions on the set $S$ to be distinguished. The first one is that the set $S$ can be divided into several $d$-tuples (maybe with different $d$ and here $d \leq \min\{d_1, d_2, d_3\}$) of GHZ like states with weight $d$ as the following form

$$\left\{ \sum_{m=1}^d w_{d m}^{(m-1) n} |i_m j_m k_m| \mid n \in \mathbb{Z}_d \right\}$$

where $w_d := \frac{1}{d^{d-2}}$. Such $d$-tuples can be also regarded as $d$ linear combinations of $d$ coordinate different vectors $\{ |i_m j_m k_m| \}_{m=1}^d$ of the computational basis where the combination coefficients of each state are the elements in the row of the $d$ dimensional Fourier transform $F_d := (w_d^{(m-1)(n-1)})_{m, n=1}^d$. We refer to such set $S$ as a special set of high weighted GHZ like states on the computational basis. The second one is that the set $C(S)$ defined by $\{ |i_m j_m k_m| \mid \sum_{m=1}^d w_{d m}^{(m-1) n} |i_m j_m k_m| \in S, n \in \mathbb{Z}_d \}$ contains the subsets $\{ i_0 \} \times \mathbb{Z}_d \times \mathbb{Z}_d \times \{ j_0 \} \times \mathbb{Z}_d \times \{ k_0 \}$ for some $i_0 \in \mathbb{Z}_d, j_0 \in \mathbb{Z}_d, k_0 \in \mathbb{Z}_d$. And we call the set $S$ is plane containing if it satisfies the second condition.

For each partition $A|BC, B|CA$, and $C|AB$, we attach the set $S$ with a graph $\mathcal{G}_A(S)$, $\mathcal{G}_B(S)$ and $\mathcal{G}_C(S)$ respectively. Here we give the exact description of $\mathcal{G}_B(S)$ as example, $\mathcal{G}_A(S)$ and $\mathcal{G}_C(S)$ can be defined similarly. The graph $\mathcal{G}_B(S)$ is determined by its vertexes $V_B(S)$ and edges $E_B(S)$. Here $V_B(S)$ is defined to be $\mathbb{Z}_d \times \mathbb{Z}_d$. For each $d$-tuples of GHZ like states with weight $d$, say $\{ \sum_{m=1}^d w_{d m}^{(m-1) n} |i_m j_m k_m| \mid n \in \mathbb{Z}_d \} \subseteq S$, it contributes to the edges $\{ (k_m, i_m), (k_{m+1}, i_{m+1}) \}$ whenever $1 \leq m \neq n \leq d$ (which is equivalent to that the subgraph with nodes $\{ (k_m, i_m) \}_{m=1}^d$ is a complete graph). The edges $E_B(S)$ of $\mathcal{G}_A(S)$ are completely determined by the edges obtained in this form.

Theorem 2 Let $S$ be a special orthogonal set of high weighted GHZ like states under computational basis that is plane containing in $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \mathbb{C}^{d_3}$. If all the corresponding graphs of $S$ are connected, then the set $S$ is of the strongest nonlocality.

The proof of Theorem 2 is given in Appendix. Note if a subgraph (which contains all vertexes) of a graph is connected then the graph itself is also connected. Now we define a subgraph $\hat{\mathcal{G}}_B(S)$ of $\mathcal{G}_B(S)$. The nodes of $\hat{\mathcal{G}}_B(S)$ are also $V_B(S)$. But the edges of $\hat{\mathcal{G}}_B(S)$ come from the following way: For each $d$-tuples of GHZ like states with weight $d$, say $\{ \sum_{m=1}^d w_{d m}^{(m-1) n} |i_m j_m k_m| \mid n \in \mathbb{Z}_d \} \subseteq S$ (by reordering, one can assume that $k_m$ is incremental), it contributes to the edges $\{ (k_m, i_m), (k_{m+1}, i_{m+1}) \}$ whenever $1 \leq m \neq n \leq d - 1$. $\hat{\mathcal{G}}_A(S)$ and $\hat{\mathcal{G}}_C(S)$ can be defined similarly. We present an construction of a basis of GHZ like states with weight 4 in $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \mathbb{C}^{d_3}$ whose corresponding graphs are all connected.

Example 3 In $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \mathbb{C}^{d_3}$, the set $S := \bigcup_{i=1}^{16} S_i$ given by Table I is an OGES of the strongest nonlocality. The size of this set is 64.

We show the corresponding graph $\hat{\mathcal{G}}_A(S)$ of $S$ in the Fig. 4. Clearly, it is connected. One can check that the graphs $\hat{\mathcal{G}}_B(S)$ and $\hat{\mathcal{G}}_C(S)$ share the same graph as $\hat{\mathcal{G}}_A(S)$. Hence $S$ is of the strongest nonlocality.

V. CONCLUSION AND DISCUSSION

We study the strongest nonlocality of set which is only containing genuinely entangled states. We show that the strongest nonlocality of some special set of genuinely entangled states is rather simple. In fact, it is equivalent to
check whether its corresponding graphs are connected or not. On the other hand, we can construct some special set of GHZ like states in $\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$ whose corresponding graphs are all connected. Therefore, we have successfully constructed set of GHZ like states that is of the strongest nonlocality. This give answers to a question asked by Shi et al. in reference [46]. This also provides another answer, which is different from that of Shi et al.’s, to an open question raised by Halder et al. in reference [42].

There are also some questions left to be considered. For example, whether we can construct some smaller set with the strongest nonlocality via the OGES than the OPS? Whether the absolutely entangled states can present the phenomenon of strong nonlocality for four or more parties systems?

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APPENDIX

Proof of Proposition 1: The structure of the set $S$ has similar geometric picture in Fig.1. Therefore, $S$ is plane containing. By the symmetricity of the construction of $S$, we only need to show that $\mathcal{G}_A(S) = (V_A(S), E_A(S))$ is connected.

By definition, $V_A(S) = \mathbb{Z}_d \times \mathbb{Z}_d$. From those states belong to $S_1$, we know that $\{(i, j+1), (i+1, j)\} \in E_A(S)$ for $i, j \in \mathbb{Z}_{d-1}$. From those states belong to $S_2$, we can obtain that $\{(1, j), (d, j+1)\} \in E_A(S)$ for $j \in \mathbb{Z}_d$. Using these results, one can obtain that for any nodes $(i, j)$ and $(k, l)$ in $V_A(S)$, if $i + j \equiv k + l \mod d$, then they must belong to some same connected component. As a result, there are at most $d$ connected components $C_i := \{(j, k) \in V_A(S) \mid j + k \equiv i \mod d\}$. On the other hand, the states in $S_3$ implies that $(j, d)$ and $(j+1, 0)$ also belong to some same connected component when $j \in \mathbb{Z}_d$. Therefore, $C_{[j-1]}$ and $C_{[j+1]}$ are in the same connected component for $j = 0, 1, \ldots, d-2$. With these at hand, one can show that these sets are connected in the following way:

$$C_{[0]} - C_{[2]} - \cdots - C_{[d-1]} = C_{[1]} = C_{[3]} = \cdots = C_{[d-2]}.$$

Therefore, there are only one connected component in total and $\mathcal{G}_A(S)$ is connected.

Proof of Proposition 2: For each $X = A, B, C$ and $i \in \mathbb{Z}_d$, denote $C^X_i := \{(j, k) \in V_X(S) \mid j + k \equiv i \mod d\}$. Using $S_1, S_2$ and $S_3$, with similar argument as above, one can show that $C^X_{[j-1]}$ and $C^X_{[j+1]}$ are in the same connected component for $j = 0, 1, \ldots, d-2$. As $d-2$ is even but $d-3$ is odd, we have

$$C^X_{[0]} = C^X_{[2]} = \cdots = C^X_{[d-2]},$$

$$C^X_{[1]} = C^X_{[3]} = \cdots = C^X_{[d-1]} - C^X_{[1]}.$$

Hence, there are at most two connected components for each $\mathcal{G}_A(S), \mathcal{G}_B(S), \mathcal{G}_C(S)$ and they are determined by the parity of $i + j$ for a node $(i, j) \in V$. We complete the proof into three cases as follows.

(a) $\mathcal{G}_A(S)$. By those states in $S_4$, we know that $(0, 0)$ and $(2, 3)$ belong to the same connected component. Therefore, the two connected components coincide with each other.

(b) $\mathcal{G}_B(S)$. By those states in $S_5$, we know that $(d, d)$ and $(3, 2)$ belong to the same connected component. Therefore, the two connected components coincide with each other.

| Set | $C_1$ | $C_2$ | $C_3$ | $C_4$ | Set | $C_1$ | $C_2$ | $C_3$ | $C_4$ |
|-----|-------|-------|-------|-------|-----|-------|-------|-------|-------|
| $B_1$ | 000 | 121 | 122 | 333 | $B_3$ | 013 | 132 | 201 | 320 |
| $B_2$ | 003 | 111 | 222 | 330 | $B_4$ | 021 | 133 | 200 | 312 |
| $B_3$ | 030 | 112 | 221 | 303 | $B_5$ | 022 | 101 | 233 | 310 |
| $B_4$ | 033 | 122 | 211 | 300 | $B_6$ | 012 | 230 | 322 | 323 |
| $B_5$ | 001 | 113 | 230 | 322 | $B_7$ | 011 | 103 | 220 | 332 |
| $B_6$ | 002 | 123 | 210 | 331 | $B_8$ | 012 | 130 | 203 | 321 |

TABLE I. This table shows the 1-uniform states in $\mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$. Each $B_i$ contains four coordinate differently elements of the computational basis. And it corresponds to four 1-uniform states $S_i$, which are linear combinations of the four vectors with coefficients in the rows of $F_4$.  

**FIG. 5.** This shows the graph $\mathcal{G}_A(S)$ corresponding to the OGES of $\mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$ defined by Table (I).
Proof of Theorem 2: Without loss of generality, suppose that $B$ and $C$ come together to perform a joint orthogonality-preserving POVM named \{$E = M^tM\}$, where $E = (a_{ij},k_{i})$, $k \in \mathbb{Z}_{d_2}, j, k \in \mathbb{Z}_{d_3}$. The postmeasurement states \{I_A \otimes M|\psi\rangle|\psi\rangle \in \mathcal{S}\} should be mutually orthogonal, i.e.,

$$\langle \phi | I_A \otimes E | \psi \rangle = 0$$

(10)

for $|\psi\rangle, |\phi\rangle \in \mathcal{S}$ and $|\psi\rangle \neq |\phi\rangle$. First, we show that the matrix $E$ is diagonal. For any pair of non-equal coordinates $(k,l)$ and $(k',l')$ in $\mathbb{Z}_{d_2} \times \mathbb{Z}_{d_3}$, as $\mathcal{S}$ is plane containing, there exist some $i_0 \in \mathbb{Z}_{d_1}$ and two sets of high weight GHZ like states

\begin{equation*}
\{|\psi_n\rangle := \sum_{m=1}^{d} w_{d}^{(m-1)n} |i_{m}, j_{m}, k_{m}\rangle | n \in \mathbb{Z}_{d_1} \} \subseteq \mathcal{S},
\end{equation*}

\begin{equation*}
\{|\phi_n\rangle := \sum_{m'=1}^{d} w_{d}^{(m'-1)n'} |i_{m'}, j_{m'}, k_{m'}\rangle | n' \in \mathbb{Z}_{d_1} \} \subseteq \mathcal{S}
\end{equation*}

such that $|i_0k_0\rangle$ and $|i_0k_0\rangle$ is one of the summation terms of $|\psi_n\rangle$ and $|\phi_n\rangle$, respectively (they cannot appear in just one GHZ like state by our definition). Therefore, we have

$$\langle \phi_n | I_A \otimes E | \psi_n \rangle = 0, \text{ for } n \in \mathbb{Z}_{d_1}, n' \in \mathbb{Z}_{d_1}.$$

(11)

Clearly, $|i_0k_0\rangle$ and $|i_0k_0\rangle$ can be written as some linear combinations of $\{|\psi_n\rangle\}_{n=1}^{d}$ and $\{|\phi_n\rangle\}_{n'=1}^{d}$ respectively. Therefore, from Eqs. (11), one obtains that

$$a_{k',l'} = A \langle i_0 | B \langle k' | C \langle l' | A \otimes E | i_0 \rangle A | k \rangle B | l \rangle C = 0.$$

Therefore, the matrix $E$ must be a diagonal matrix under the computational basis.

In the following, we prove that the elements on the diagonal of $E$ are equal. As the graph $\mathcal{G}_A(\mathcal{S})$ is connected, for any two different nodes $(j_1, k_1)$ and $(j_L, k_L)$ of $\mathcal{G}_A(\mathcal{S})$, there is a path connecting them, namely

$$(j_1, k_1) \rightarrow (j_2, k_2) \rightarrow \ldots \rightarrow (j_N, k_N) \rightarrow (j_{N+1}, k_{N+1})$$

(12)

here $(j_1, k_1) = (j_1, k_1)$ and $(j_{N+1}, k_{N+1}) = (j_L, k_L)$. By the definition of the edges of $\mathcal{G}_A(\mathcal{S})$, each edge $e_i = \{(j_1, k_1), (j_{i+1}, k_{i+1})\}$ must come from some $d(t)$-tuples of weight $d(t)$ GHZ like states

$$\{| \psi_n(t) \rangle := \sum_{m=1}^{d(t)} w_{d(t)}^{(m-1)n_1} w_{d(t)}^{(m-1)n_2} |i_m, j_m, k_m\rangle | n \in \mathbb{Z}_{d(t)} \} \subseteq \mathcal{S}.$$

More exactly, there exist $i_t, i_{t+1} \in \mathbb{Z}_{d_t}$ such that $|i_tj_1k_{i_1}\rangle$ and $|i_{t+1}j_{t+1}k_{i_{t+1}}\rangle$ are two summation terms in $|\phi_n(t)\rangle$, i.e.,

$$|i_tj_1k_{i_1}\rangle = |i_m, j_m, k_m\rangle \text{ and } |i_{t+1}j_{t+1}k_{i_{t+1}}\rangle = |i_m, j_m, k_m\rangle.$$

Applying Eq. (10) to $|\phi_n(t)\rangle$ and $|\phi_{n'}(t)\rangle$, one can obtain that

$$\langle \phi_{n'}(t) | I_A \otimes E | \psi_n(t) \rangle = 0, \text{ for } n \in \mathbb{Z}_{d(t)} \setminus \{0\}.$$

(13)

As $i_{m_t} \neq i_{n_1}$ whenever $m \neq n$, from the $(d(t) - 1)$ equations in Eqs. (13), one deduce that

$$a_{j_nk_1m_1j_nk_{m_1}} = a_{j_nk_2m_2j_nk_{m_2}}.$$

(12)

Hence, $a_{j_nk_1j_nk_{i_1}} = a_{j_{n+1}k_{i+1}j_{n+1}k_{i+1}}$. From the path in Eq. (12) one could easily deduce that

$$a_{j_1k_1j_Lk_L} = a_{j_Lk_Lj_Lk_L}.$$

Therefore, it can be concluded that the matrix $E$ is indeed proportional to identity operator.
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