On the Smallest Sets Blocking Simple Perfect Matchings in a Convex Geometric Graph

Chaya Keller and Micha A. Perles
Einstein Institute of Mathematics, Hebrew University
Jerusalem 91904, Israel

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Abstract

In this paper we present a complete characterization of the smallest sets which block all the simple perfect matchings in a complete convex geometric graph on $2m$ vertices. In particular, we show that all these sets are caterpillar graphs with a special structure, and that their total number is $m \cdot 2^{m-1}$.

1 Introduction

In this paper we consider geometric graphs (i.e., graphs whose vertices are points in the plane, and whose edges are segments connecting pairs of vertices), and in particular, convex geometric graphs (i.e., geometric graphs whose vertices are in convex position in the plane).

Definition 1.1 A simple perfect matching (SPM) in a geometric graph on $2m$ vertices is a set of $m$ pairwise disjoint edges (i.e., edges that do not intersect, not even in an interior point).

A natural Turán-type question (considered, e.g., in [1]) is: what is the maximal possible number of edges in a geometric graph on $2m$ vertices with no simple perfect matching?

An equivalent way to state the question is to consider sets which “block” all the SPMs:

Definition 1.2 A set of edges in a geometric graph $G$ is called a blocking set if it intersects (i.e., contains an edge of) every SPM of the graph.

Using this formulation, the question above is equivalent to the question:

Question 1.3 What is the minimal size (i.e., number of edges) of a blocking set in a complete geometric graph on $2m$ vertices?

It appears that the answer depends on the position of the vertices of the graph in the plane. It is easy to show that there always exists a blocking set of size $2m - 1$, and there exists a configuration in the plane for which $2m - 1$ is the minimal possible size. On the other hand, in an unpublished work ([2]), Perles proved that for any placement of the vertices, the size of a blocking set is at least $m$. This lower bound is attained (among other cases) in the case of convex geometric graphs (CGG.) Indeed, consider a complete convex geometric graph on $2m$ vertices, denoted in the sequel $CK(2m)$. The vertices of the graph form a convex polygon. It is easy to see that any set of $m$ consecutive edges on the boundary of the polygon is a blocking
set. The set of all edges of odd order emanating from a single vertex is also clearly a blocking set of size $m$.\footnote{For the sake of clarity, we present proofs of these two straightforward claims in Section 2.1.}

In this paper we present a complete characterization of the blocking sets of size $m$ in $CK(2m)$, called in the sequel blockers. It turns out that all these blockers are simple subtrees of a special structure, called caterpillars (see, e.g., [2]).

**Definition 1.4** A tree $T$ is a caterpillar (or a fishbone) if the derived graph $T'$ (i.e., the graph obtained from $T$ by removing all leaves and their incident edges) is a path (or is empty). A geometric caterpillar is simple if it does not contain a pair of crossing edges. A longest (simple) path in a caterpillar $T$ is called a spine of $T$.

Our main result is the following:

**Theorem 1.5** Let $V$ be the set of vertices of a convex $2m$-gon $P$, labelled cyclically from $0$ to $2m-1$, and let $G$ be the complete convex geometric graph on $V$. Any blocker of $G$ is a simple caterpillar graph whose spine lies on the boundary of the polygon and is of length $t \geq 2$. If the spine “starts” with the vertex 0 and the edge $[0,1]$, then the edges of the blocker are:

$$\{[i-1,i] : 1 \leq i \leq t\} \cup \{[t+j-1-\epsilon_{t+j},t+j+\epsilon_{t+j}] : 1 \leq j \leq m-t\},$$

(1)

where the $\epsilon_i$ are natural numbers satisfying $1 \leq \epsilon_{t+1} < \epsilon_{t+2} < \ldots < \epsilon_m \leq m-2$.

Conversely, any set of $m$ edges of the described form is a blocker in $G$.

If the polygon is regular, then the direction of each consecutive edge of the blocker, as listed above, is obtained from the direction of the preceding edge by rotation by $\pi/m$ radians. In the first $t$ edges, the “back” endpoint of each edge is the “front” endpoint of the previous edge. Starting with the $t+1$-st edge, the “back” endpoint goes “back” (as reflected by subtraction of the corresponding $\epsilon_i$), and the length of the edge changes accordingly. An example of a blocker in $CK(12)$ is presented in Figure 1.

The proof of the theorem involves various techniques, including examination of several specific classes of SPMs, as well as inductive arguments.

As an easy corollary of the structure theorem, we enumerate the blockers in $CK(2m)$:

**Proposition 1.6** Let $G = CK(2m)$ be a complete convex geometric graph on $2m$ vertices. The number of blocking sets of size $m$ in $G$ is $m \cdot 2^{m-1}$.

2 Preliminaries

In this section we introduce several basic definitions and observations, and consider two specific classes of SPMs that will be used in the proof of Theorem 1.5.

2.1 Definitions and Observations

**Definition 2.1** Let $V$ be the set of vertices of a convex $2m$-gon $P$, $m \geq 2$, labelled cyclically from $0$ to $2m-1$.

- A half-boundary of $P$ is a set of $m$ consecutive boundary edges.
Figure 1: A blocker in $CK(12)$ with spine of length $t = 3$. The edges of the blocker are depicted by full (not dotted) lines. In the notation of Theorem 1.5, $\epsilon_4 = 1$, $\epsilon_5 = 2$, and $\epsilon_6 = 4$. The angle $\alpha$ is $\pi/6$ radians. The diagonal $[2, 9]$ is parallel to the diagonal $[1, 10]$, and helps to depict the angle between the diagonals $[2, 7]$ and $[1, 10]$.

- The order of an edge $[i, i+k]$ (where the addition is modulo $2m$) is $\min(k, 2m-k)$. The boundary edges of $P$ are, of course, of order 1. We call the non-boundary edges, i.e., the edges that are diagonals of $P$, interior edges.

- Let $[i-1, i]$ and $[i+k-1, i+k]$ be two boundary edges of $P$ (where $0 < k < 2m$, and the addition within the edges is modulo $2m$). The (directed) distance from $[i-1, i]$ to $[i+k-1, i+k]$ is $k$. In particular, the distance from a boundary edge to its immediate successor is 1. (Note that the distance from $[i+k-1, i+k]$ to $[i-1, i]$ is $2m-k$.)

**Observation 2.2** Let $V$ be the set of vertices of a convex $2m$-gon $P$, labelled cyclically from 0 to $2m-1$, and let $G$ be the complete convex geometric graph on $V$. Then:

1. Any blocker in $G$ contains at least two boundary edges.
2. The set of all edges of odd order emanating from a single vertex is a blocker in $G$.
3. Any set of $m$ consecutive boundary edges of $P$ is a blocker in $G$.

**Proof:**

1. The boundary of $P$ is the disjoint union of two SPMs: $\{[2i, 2i+1] : i = 0, 1, \ldots, m-1\}$, and $\{[2i+1, 2i+2] : i = 0, 1, \ldots, m-1\}$. In order to intersect these two SPMs, any blocker has to contain at least two boundary edges.

2. Note that all the edges in any SPM are of odd order. Indeed, an edge $[i, j]$ of an SPM $M$ divides the remaining vertices of $P$ into sets $V_1, V_2$ of sizes $j-i-1$ and $2m-2-(j-i-1)$. Since the edges of $M$ do not intersect, the two vertices of any other edge are in the same set (either both in $V_1$ or both in $V_2$). As $M$ “covers” all the vertices of $P$, it follows that each of the sets $V_1, V_2$ contains an even number of vertices. Hence, $j-i-1$ is even, and thus the order of the edge $[i, j]$ is odd.
Let $B$ be the set of odd-order edges emanating from the vertex $v$, and let $M$ be an SPM. Since $M$ is a perfect matching, it contains an edge emanating from $v$. By the explanation above, this edge is of odd order, so it is included in $B$. Thus, $B$ intersects $M$, as asserted.

3. Assume w.l.o.g. that the set is $B = \{[0,1], [1,2], \ldots, [m-1, m]\}$, and let $M$ be an SPM. By the pigeonhole principle, $M$ contains an edge with both vertices in $\{0, 1, 2, \ldots, m\}$. Let $[i_0, j_0]$ (for $i_0 < j_0$) be a “shortest” (i.e., having the smallest order) edge with this property. Since the edges of $M$ cover all the vertices and do not intersect, each of the vertices in the set $\{i_0 + 1, i_0 + 2, \ldots, j_0 - 1\}$ is “connected” by $M$ to another vertex in this set. However, an edge that connects two such vertices is shorter than $[i_0, j_0]$, contradicting the assumption above. Thus, the set $\{i_0 + 1, i_0 + 2, \ldots, j_0 - 1\}$ is empty, so $[i_0, j_0]$ is a boundary edge, which is contained in $B$. Therefore $B$ intersects any SPM, as asserted.

□

2.2 Parallel SPMs

We start with a combinatorial generalization of the notion of parallel edges.

If the polygon $P$ (that consists of the vertices and boundary edges of $\text{CK}(2m)$) is regular, then its edges and diagonals have $2m$ directions: $m$ directions of the boundary edges and the diagonals of odd order, and $m$ directions of the diagonals of even order. The directions define an equivalence relation, whose equivalence classes consist of all the boundary edges and diagonals of the same direction. The equivalence classes of the first type (odd order) contain two boundary edges and $m - 2$ diagonals, and the equivalence classes of the second type (even order) contain $m - 1$ diagonals. This equivalence relation can be defined in a combinatorial way, that extends naturally to the edges and diagonals of any convex polygon of even order.

**Definition 2.3** Let $[p,q]$ and $[p',q']$ be disjoint segments connecting four different vertices of a convex polygon $P$ on $2m$ vertices, such that the order of the vertices on the boundary of the polygon is $p, q, p', q'$. The segments are called “parallel” if the number of boundary edges in the arc $\langle q, p' \rangle$ is equal to the number of boundary edges in the arc $\langle q', p \rangle$.

A special class of SPMs we consider consists of full equivalence classes of the relation defined above.

**Definition 2.4** The set of all edges which are parallel to a given boundary edge is called a “parallel SPM”. The parallel SPMs are of the form $M_l = \{[i,j] : i + j \equiv 2l - 1 (\mod 2m)\}$, for all $1 \leq l \leq m$.

The sets $\{M_l\}_{l=1}^m$ are pairwise disjoint. Since a blocker has only $m$ edges and intersects each of the parallel SPMs (i.e., each of the sets $M_l$), it must intersect each of the $M_l$-s in exactly one edge. We thus get the following:

**Observation 2.5** Any blocker contains exactly one edge of each of the equivalence classes of odd order.
Figure 2: A triangular SPM in $CK(12)$, corresponding to the case $i_0 = 0$, $a = 3$, $b = 2$, $c = 1$ (in the above notations). The initial (innermost) diagonals are drawn thick. The boundary edges of the SPM are $[2, 3]$, $[7, 8]$, and $[10, 11]$, and the distances between them are $5, 3, \text{ and } 4$, respectively.

## 2.3 Triangular SPMs

For any triple of positive integers $(a, b, c)$ with $a + b + c = m$ and a “starting point” $i_0$, $0 \leq i_0 \leq 2m - 1$, consider the triple of segments

\[
\left( [i_0, i_0 + 2a - 1], [i_0 + 2a, i_0 + 2a + 2b - 1], [i_0 + 2a + 2b, i_0 - 1] \right),
\]

where the additions are taken modulo $2m$. Note that the segments are pairwise disjoint diagonals (or edges) of the polygon $P$. This triple of segments can be extended to an SPM by adding the following segments:

\[
[i_0 + \varepsilon, i_0 + 2a - 1 - \varepsilon], \quad \varepsilon = 1, 2, \ldots, a - 1,
\]
\[
[i_0 + 2a + \varepsilon, i_0 + 2a + 2b - 1 - \varepsilon], \quad \varepsilon = 1, 2, \ldots, b - 1,
\]
\[
[i_0 + 2a + 2b + \varepsilon, i_0 - 1 - \varepsilon], \quad \varepsilon = 1, 2, \ldots, c - 1.
\]

An SPM of this form is called a triangular SPM (see Figure 2).

The boundary edges of this triangular SPM are

\[
[i_0 + a - 1, i_0 + a], [i_0 + 2a + b - 1, i_0 + 2a + b], [i_0 + 2a + 2b + c - 1, i_0 + 2a + 2b + c].
\]

The distances between these boundary edges, in cyclical order, are $a + b, b + c,$ and $c + a$, and by assumption, all of them are less than $m$. In the following proposition, that will be used in the proof of our main theorem, we claim that the converse holds as well:

**Proposition 2.6** For any triple of boundary edges $\left( [i_1 - 1, i_1], [i_2 - 1, i_2], [i_3 - 1, i_3] \right)$, $1 \leq i_1 < i_2 < i_3 < 2m$, such that the distance from each one to the next (in cyclical order) is less than $m$, there exists a triangular SPM whose boundary edges are $[i_1 - 1, i_1], [i_2 - 1, i_2], [i_3 - 1, i_3]$. 

Figure 3: Construction of an SPM in $CK(12)$ given three boundary edges. In this figure, $i_1 = 3$, $i_2 = 8$, and $i_3 = 11$. The given boundary edges are drawn thick. The obtained values are $a = 3$, $b = 2$, and $c = 1$, and the obtained SPM is the same as in Figure 2.

Proof: Denote the distances from each edge to the next, in cyclical order, by $p, q, r$. That is,

$$p = i_2 - i_1, \quad q = i_3 - i_2, \quad r = i_1 + 2m - i_3.$$ 

By assumption, $0 < p, q, r < m$. Consider a set of edges that consists of $a$ consecutive edges parallel to $[i_1 - 1, i_1]$ (i.e., $[i_1 - 1 - \epsilon, i_1 + \epsilon]$, $\epsilon = 0, 1, \ldots, a - 1$), $b$ consecutive edges parallel to $[i_2 - 1, i_2]$, and $c$ consecutive edges parallel to $[i_3 - 1, i_3]$. It is easy to see that this set is an SPM if the following three equalities hold:

$$\begin{align*}
(1) \quad a + b &= p, \\
(2) \quad b + c &= q, \\
(3) \quad c + a &= r.
\end{align*}$$

Summing the equations we get $2(a + b + c) = p + q + r = 2m$. Subtracting equations (1),(2),(3) from the equation $a + b + c = m$, we get the solutions $(a = m - q, b = m - r, c = m - p)$, and these are indeed positive integers. Thus, the set of edges defined above with $a = m - q, b = m - r, c = m - p$ is an SPM whose boundary edges are $[i_1 - 1, i_1], [i_2 - 1, i_2], [i_3 - 1, i_3]$, as claimed. $\square$

The construction of a triangular SPM from three given boundary edges is exemplified in Figure 2.

3 Proof of Theorem 1.5

In this section we present the proof of our main theorem. We start with an outline of the proof.

3.1 Proof Outline

The key observation is that a characterization of the possible boundary edges in a blocker leads to a full characterization of the blockers. The main step in the proof is the following lemma, characterizing the boundary edges of a blocker:

Lemma 3.1 The boundary edges of a blocker form a path of length $t$ on the boundary of the polygon, $2 \leq t \leq m$. 
Lemma 3.1 in turn, is proved in two steps. First we prove:

**Lemma 3.2** The boundary edges of a blocker are included in a half-boundary.

We prove Lemma 3.2 by showing that if the boundary edges are not included in a half-boundary then there exists an SPM of one of the two special kinds mentioned above ("parallel" and "triangular") that misses the blocker. Then we deduce Lemma 3.1 from Lemma 3.2 by an inductive argument. Using Lemma 3.1, we show that if a set of $m$ edges is not a caterpillar with the specified properties, then there exists an SPM that misses it. This proves one direction of Theorem 1.5.

The other direction of the theorem (asserting that any caterpillar with the specified properties is a blocker) is proved by double induction: A primary induction on $m$, and a secondary (backward) induction on the number of boundary edges in the caterpillar.

### 3.2 Proof of Lemma 3.2

We use the following technical lemma:

**Proposition 3.3** Let $S=\{[i_1, i_1+1], [i_2, i_2+1], \ldots, [i_k, i_k+1]\}$ be a set of $k$ boundary edges of $CK(2m)$, where $0 \leq i_1 < i_2 < \ldots < i_k \leq 2m-1$. Then at least one of the following holds:

1. $S$ contains two opposite edges (i.e., $i_\nu = i_\mu + m$ for some $\mu, \nu$, $1 \leq \mu < \nu \leq k$).
2. $S$ is included in a half-boundary (i.e., there exists a $\mu$, $1 \leq \mu < k$, such that $i_\mu + m < i_{\mu+1}$, or $i_k < i_1 + m$).
3. $S$ contains three edges such that the distance from each one to the next (in cyclical order) is less than $m$ (i.e., there exist $1 \leq \mu < \nu < \tau \leq k$ such that $i_\tau < i_\mu < i_\mu + m$, $i_\tau < i_\nu + m$, and $i_\mu + 2m < i_\tau + m$).

**Proof:** For $k = 1, 2$, it is easy to see that either (1) or (2) holds. Let $k > 2$, and assume that both (1) and (2) fail. Define $\mu_0 = \max\{\mu : 2 \leq \mu \leq k, i_\mu < i_1 + m\}$. Since (2) fails, we have $2 \leq \mu_0 < k$. Consider the edges $[i_1, i_1+1], [i_{\mu_0}, i_{\mu_0}+1], [i_{\mu_0+1}, i_{\mu_0+1}+1]$. These edges satisfy the requirements of (3). Indeed, by the definition of $\mu_0$, $i_{\mu_0} < i_1 + m$. Furthermore, we have $i_{\mu_0+1} < i_{\mu_0} + m$, since otherwise either (1) or (2) are satisfied. Finally, by the definition of $\mu_0$, $i_{\mu_0+1} \geq i_1 + m$, and equality cannot hold since, by assumption, (1) fails. Hence, $i_{\mu_0+1} > i_1 + m$, and thus (3) holds. This completes the proof. □

Now we are ready to prove Lemma 3.2. The formal statement of the lemma is the following:

**Lemma 3.4** Let $B = \{[i_1, i_1+1], \ldots, [i_k, i_k+1], [i_{k+1}, i_{k+1}], \ldots, [i_m, i_m]\}$ be a blocker in $CK(2m)$, where for all $1 \leq \mu \leq k$, $[i_\mu, i_\mu + 1]$ is a boundary edge, and for all $k < \mu \leq m$, $[i_\mu, i_\mu]$ is not a boundary edge. Then the edges $[i_1, i_1+1], \ldots, [i_k, i_k+1]$ are included in a half-boundary.

**Proof:** Consider the set $E = \{[i_1, i_1+1], \ldots, [i_k, i_k+1]\}$ of boundary edges of the blocker. Assume, w.l.o.g., that $0 \leq i_1 < i_2 < \ldots < i_k \leq 2m-1$. By Proposition 3.3, at least one of the following holds:

1. $E$ contains two opposite edges (i.e., there exist $\mu, \nu$, $1 \leq \mu < \nu \leq k$, such that $i_\nu = i_\mu + m$).
2. \( E \) contains three edges such that the distance from each to the next (in cyclical order) is less than \( m \) (i.e., there exist \( 1 \leq \mu < \nu < \tau \leq k \) such that \( i_\nu < i_\mu + m \), \( i_\tau < i_\nu + m \), and \( i_\mu + m < i_\tau \).

3. \( E \) is included in a half-boundary (i.e., there exists a \( \mu, 1 \leq \mu < k \), such that \( i_\mu + m < i_{\mu+1} \), or \( i_k < i_1 + m \)).

(1) is impossible, since by Observation 2.5 \( B \) does not contain two parallel edges.

If \( E \) contains three edges \([i_\mu,i_\mu+1],[i_\nu,i_\nu+1],[i_\tau,i_\tau+1] \) such that the distance from each one to the next (in cyclical order) is less than \( m \), then the triple of opposite edges,

\[
T = \left( [i_\mu + m,i_\mu + 1 + m],[i_\nu + m,i_\nu + 1 + m],[i_\tau + m,i_\tau + 1 + m] \right),
\]

also has this property. By Proposition 2.6, the triple \( T \) can be extended to a triangular SPM \( \tilde{T} \). Each edge of \( \tilde{T} \) is parallel to an edge in \( T \), and the only boundary edges of \( \tilde{T} \) are the three edges of \( T \). It follows that our blocker \( B \) misses the SPM \( \tilde{T} \) entirely: the only edges of \( B \) that are parallel to edges in \( T \) are the boundary edges \([i_\mu,i_\mu+1],[i_\nu,i_\nu+1] \), and \([i_\tau,i_\tau+1] \), which are not in \( \tilde{T} \). Hence, (2) is also impossible. Therefore, we are left with (3), i.e., \( E \) is included in a half-boundary, as claimed. \( \square \)

### 3.3 Proof of Lemma 3.1

In the proof of the lemma we use the following inductive technique.

Let \( B \) be a blocker in \( CK(2m) \). Consider a pair of consecutive boundary edges \( e, f \), such that \( e \in B \) and \( f \not\in B \). Such a choice is possible, since by Observation 2.2 the number of boundary edges of \( B \) is between 2 and \( m \). Assume, w.l.o.g., that \( e = [2m - 3,2m - 2] \), and \( f = [2m - 2,2m - 1] \).

Denote by \( CK(2m - 2) \) the geometric subgraph of \( CK(2m) \) obtained by omitting the endpoints of \( f \). The boundary of \( CK(2m - 2) \) is thus \( \{0,1,2,\ldots,2m - 3,0\} \).

**Claim 3.5** The set \( B \setminus \{e\} \) is a blocking set of size \( m - 1 \) (i.e., a blocker) in \( CK(2m - 2) \).

**Proof:** Let \( B' = B \cap E(CK(2m - 2)) \). \( B' \) is obtained from \( B \) by omitting \( e \) and any other edge that uses one of the vertices \( 2m - 2,2m - 1 \) (see Figure 3). If \( B' \) is not a blocking set in \( CK(2m - 2) \) then there exists an SPM in \( CK(2m - 2) \) that misses \( B' \), and thus misses \( B \). Adding the edge \( f \) to that SPM yields an SPM in \( CK(2m) \) that misses \( B \), contradicting the assumption that \( B \) is a blocker. Hence, \( B' \) is a blocking set in \( CK(2m - 2) \). Clearly, \( B' \subseteq B \setminus \{e\} \). This inclusion must be an equality, i.e., \( B' = B \setminus \{e\} \), since a blocking set in \( CK(2m - 2) \) cannot have fewer than \( m - 1 \) edges. \( \square \)

The same argument yields immediately the following corollary.

**Corollary 3.6** In the notations above, \( B \) does not contain any edge that uses one of the vertices of \( f \) (i.e., \( 2m - 2 \) and \( 2m - 1 \), except \( e \)).

Now we are ready to prove Lemma 3.1. The formal statement is the following:

**Lemma 3.7** Let \( B \) be a blocker in \( G = CK(2m) \). The boundary edges of \( B \) are consecutive (i.e., if \( B \) contains \( t \) boundary edges \([i_1,i_1+1],\ldots,[i_t,i_t+1] \), then these edges can be arranged in such a way that \( i_{\mu+1} = i_\mu + 1(\mod 2m) \), for all \( 1 \leq \mu < t \)).
Figure 4: An illustration of the proof of Claim 3.5. In this figure, \( m = 6 \), \( e = [9, 10] \), and \( f = [10, 11] \). The vertices of the induced subgraph \( CK(10) \) are 0, 1, \ldots, 9, the edges of \( B' \) are drawn thick, and \( B = B' \cup \{e\} \).

Figure 5: An attempt to construct the boundary edges of a blocker in \( CK(6) \).

**Proof:** The proof is by induction on \( m \).

**The case** \( m = 2 \): Any blocker in \( CK(4) \) contains exactly one edge of each \( M_l = \{[i, j] : i + j = 2l - 1 \text{ (mod } 4)\} \) (\( l = 1, 2 \)). Thus, any such blocker consists of two consecutive boundary edges.

**The case** \( m = 3 \): If a blocker in \( CK(6) \) contains three boundary edges, then, by Lemma 3.2, these edges are contained in a half-boundary, and thus are consecutive. Since by Observation 2.2 any blocker contains at least two boundary edges, we are left with the case where the blocker contains exactly two boundary edges. Assume on the contrary that these edges are not consecutive. Since these edges are not parallel either, we may assume, w.l.o.g., that they are \([0, 1]\) and \([4, 5]\) (see Figure 5). However, in this case the blocker misses the SPM \(([1, 2], [3, 4], [5, 0])\).

**The inductive step:** We assume that the claim holds for any blocker in \( CK(2m) \), and prove it for a blocker \( B \) in \( CK(2m + 2) \). Denote the set of boundary edges of \( B \) by \( E \), \( E = \{[i_1, i_1 + 1], \ldots, [i_t, i_t + 1]\} \), and assume w.l.o.g. that \([i_1, i_1 + 1] = [0, 1]\), that the sequence \(\{i_1, i_2, \ldots, i_t\}\) is monotone increasing, and that \( E \subset \{[0, 1], [1, 2], \ldots, [m, m + 1]\} \). (The last
In the graph $CK(12) = CK(2m + 2)$, the boundary edges of the blocker are $[0, 1]$ and $[4, 5]$. In the induced subgraph $CK(10)$ (to the right of the dotted line), the boundary edges of the induced blocker are $[4, 5]$ and possibly also $[1, 10]$, and both options lead to a contradiction.

Assumption is valid since, by Lemma 3.2, $E$ is included in a half-boundary. We perform a case-by-case analysis.

**Case 1:** $[1, 2] \in E$. Since $[2m + 1, 0] \not\in E$ and $[0, 1] \in E$, by the inductive technique presented above (Claim 3.5), the set $B \setminus \{0, 1\}$ is a blocker in the subgraph spanned by $\{1, 2, \ldots, 2m\}$. Hence, by the inductive assumption, the boundary edges of $B \setminus \{0, 1\}$ form a connected set that contains the edge $[1, 2]$, and therefore $E$, the set of boundary edges of $B$, is also connected.

(Note that since, by assumption, $[2m - 1, 2m] \not\in B$, the boundary edges of $B \setminus \{0, 1\}$ in the induced subgraph are $[1, 2], [2, 3], \ldots, [t - 1, t]$ for some $t \leq m + 1$, and possibly also $[2m, 1]$, but not $[2m - 1, 2m]$. Since $[2m, 1]$ is not a boundary edge in the original graph, it follows that $E = \{0, 1\}, [1, 2], [2, 3], \ldots, [t - 1, t]\}$, which is indeed a set of consecutive edges.)

**Case 2:** $[i_t - 1, i_t] \in E$. In this case we can use an argument symmetric to that used in Case 1.

**Case 3:** $[i_t - 1, i_t] \not\in E$, and $|E| \geq 3$. Let $[j, j + 1] \in E \cap \{2, 3, [3, 4], \ldots, [i_t - 2, i_t - 1]\}$. As in Case 1, by the inductive technique presented above, $B \setminus \{0, 1\}$ is a blocker in the subgraph spanned by $\{1, 2, \ldots, 2m\}$. The boundary edges of this blocker contain the edges $[j, j + 1]$ and $[i_t, i_t + 1]$, but not the edges $[i_t - 1, i_t]$ and $[2m - 1, 2m]$. Hence, they are not consecutive on the boundary $\langle 1, 2, \ldots, 2m, 1\rangle$ of this subgraph, contradicting the inductive assumption.

**Case 4:** $E = \{0, 1\}, [i_t, i_t + 1\}$. If $i_t = 1$, we are done. If not, then $B \setminus \{0, 1\}$ is a blocker in the subgraph spanned by $\{1, 2, \ldots, 2m\}$, whose boundary edges are $[i_t, i_t + 1]$, and possibly $[2m, 1]$. If both belong to $B \setminus \{0, 1\}$, this contradicts the inductive assumption, since $i_t \neq 1$, and thus the boundary edges are not consecutive. Otherwise, $B \setminus \{0, 1\}$ contains only one boundary edge of the induced subgraph, and this is impossible for a blocker (see Figure 6).

**3.4 Characterization of the Blockers**

In this section we present a complete characterization of the blockers and prove one direction of Theorem 1.5. We start with a few observations.
Let \( B \) be a blocker in \( CK(2m) \), and let \( e = [i, j] \in B \) \((0 \leq i < j \leq 2m - 1)\) be an edge that separates the remaining vertices into sets of \( 2k \) and \( 2l \) vertices, respectively. Clearly, \( k, l \geq 0 \) and \( k + l = m - 1 \) (see Figure 7).

Denote by \( G_1^- \) the subgraph of \( CK(2m) \) of order \( 2k \) spanned by the vertices \{\( i + 1, i + 2, \ldots, j - 1 \}\), and by \( G_1^+ \) the subgraph of \( CK(2m) \) of order \( 2k + 2 \) spanned by the vertices \{\( i, i + 1, i + 2, \ldots, j - 1, j \}\). Similarly, denote by \( G_2^- \) the subgraph of \( CK(2m) \) of order \( 2l \) spanned by the vertices \{\( j + 1, j + 2, \ldots, 2m - 1, 0, 1, \ldots, i - 1 \}\), and by \( G_2^+ \) the subgraph of \( CK(2m) \) of order \( 2l + 2 \) spanned by the vertices \{\( j, j + 1, j + 2, \ldots, 2m - 1, 0, 1, \ldots, i - 1, i \}\).

We observe that in \( G_1^- \), there exists an SPM all whose edges are parallel to \( e\): \{\( [i+\nu,j-\nu]:\nu=1,2,\ldots,k \}\}. Hence, \( G_2^- \) does not include an SPM disjoint from \( B \), since otherwise the union of these two SPMs would form an SPM in \( CK(2m) \) that misses \( B \). In other words, this implies that \( B \cap E(G_2^+) \) is a blocking set in \( G_2^+ \). In particular, it follows that \( |B \cap E(G_2^+)| \geq l + 1 \) and thus \( |(B \setminus \{e\}) \cap E(G_2^+)| \geq l \).

Repeating the same argument with \( G_1^+ \) replaced by \( G_2^- \) and \( G_2^+ \) replaced by \( G_1^+ \), we find that \( B \cap E(G_1^+) \) is a blocking set in \( G_1^+ \), hence \( |B \cap E(G_1^+)| \geq k + 1 \) and \( |(B \setminus \{e\}) \cap E(G_1^+)| \geq k \).

Since \( |B| = m = k + l + 1 \) and the edge sets \((B \setminus \{e\}) \cap E(G_1^+), (B \setminus \{e\}) \cap E(G_2^+), \) and \( \{e\} \) are pairwise disjoint, it follows that

\[
|(B \setminus \{e\}) \cap E(G_1^+)| = k, \quad |(B \setminus \{e\}) \cap E(G_2^+)| = l,
\]

and that \( B \) does not contain an edge that crosses \( e \) in an interior point. Since this argument holds for any \( e \in B \), we conclude that \( B \) is a simple (i.e., crossing-free) set of edges of \( CK(2m) \).

Now we are ready to prove the main theorem. First, we recall its statement:

**Theorem 3.8** Any blocker in \( CK(2m) \) is a simple caterpillar graph whose spine lies on the boundary of \( CK(2m) \) and is of length \( t \geq 2 \). If the spine “starts” with the vertex \( 0 \) and the edge \([0,1]\), then the edges of the blocker are:

\[
\{[i-1,i]:1 \leq i \leq t\} \cup \{[t+j-1-\epsilon_{t+j},t+j+\epsilon_{t+j}]:1 \leq j \leq m-t\},
\]

where the \( \epsilon_i \) are integers that satisfy \( 1 \leq \epsilon_{t+1} < \epsilon_{t+2} < \ldots < \epsilon_m \leq m - 2 \).

**Proof:** Let \( B \) be a blocker in \( CK(2m) \). Denote the number of its boundary edges by \( t \), and assume that these edges “start” with the vertex \( 0 \) and the edge \([0,1]\). By Lemma 3.3 the boundary edges of \( B \) are \([0,1],[1,2],\ldots,[t-1,t]\). We make the following three observations:
Figure 8: An illustration of Observation 3 in the proof of Theorem 3.8. In this figure, $m = 9$, $t = 5$, $l_1 = 1$, $l_2 = 4$, $l'_1 = 12$, and $l'_2 = 11$. The SPM that misses $B$ is depicted by dotted lines.

1. $B$ cannot contain an edge of the form $[l_1, l_2]$ for $1 \leq l_1 + 1 < l_2 \leq t$, since such an edge is either of even order, or is parallel to one of the boundary edges of $B$.

2. $B$ cannot contain a non-boundary edge $e$ such that all boundary edges of $B$ lie on one side of $e$. Indeed, if this happens, then we can define the subgraphs $G^+_1$ and $G^+_2$ as in the beginning of this subsection, and find that $B \cap E(G^+_1)$ is a blocker in $G^+_1$ that contains only one boundary edge of $G^+_1$, which is impossible, since $|G^+_1| \geq 4$.

A combination of these two observations implies that any non-boundary edge of $B$ connects one of the vertices $1, 2, \ldots, t - 1$ with one of the vertices $t + 1, t + 2, \ldots, 2m - 1$. Furthermore, the only edge in $B$ that contains the vertex 0 is $[0, 1]$, and the only edge in $B$ that contains the vertex $t$ is $[t - 1, t]$.

3. Consider two non-boundary edges $[l_1, l'_1], [l_2, l'_2] \in B$, and assume that $0 < l_1 < l_2 < t$. (Thus, by the previous observation, $t < l'_1, l'_2 < 2m$). First note that $l'_2 \leq l'_1$, since the edges of $B$ do not cross. Secondly we show that $l'_1 - l'_2 > l_2 - l_1$. Indeed, if the differences are equal then the edges $[l_1, l'_1]$ and $[l_2, l'_2]$ are parallel, which is impossible for a blocker. If $l'_1 - l'_2 < l_2 - l_1$, then one can construct an SPM that misses $B$. The edges of the SPM are the following (see Figure 8):

   (a) All the edges parallel to $[l_2, l'_2]$ on its left side: $[l_2 - \epsilon, l'_2 + \epsilon, \text{for } 1 \leq \epsilon \leq m - \frac{1}{2}(l'_2 - l_2 + 1)$. (Addition and subtraction here are modulo $2m$.)

   (b) All the edges parallel to $[l_1, l'_1]$ that lie on the right side of the edge $[l_2, l'_2]$ (in the weak sense): $[l_2 + \epsilon, l'_1 - l_2 + l_1 - \epsilon, \text{for } 0 \leq \epsilon \leq (l_1 + l'_1 - 1)/2 - l_2$.

   (c) Alternating boundary edges (this is possible since the number of remaining vertices is $l'_2 - (l'_1 - l_2 + l_1) = (l'_2 + l_2) - (l'_1 + l_1)$, which is a positive even number).

The assertion of the theorem follows immediately from these three observations.

The exact formulation of the theorem is explained as follows. The theorem lists the $m$ edges of $B$. The first $t$ are the boundary edges, and the remaining $m - t$ are the non-boundary edges, arranged by decreasing order of the vertex in which they meet the “boundary path” of
indeed a blocker by Observation 2.2. If \( t < m \), satisfying the assumptions and having more than one boundary edge, we get:

\[
\begin{align*}
  l_1 &= t + j - \epsilon_{t+j+1}, & l'_1 &= t + j + \epsilon_{t+j+1}, \\
  l_2 &= t + j - \epsilon_{t+j}, & l'_2 &= t + j + \epsilon_{t+j},
\end{align*}
\]

and therefore,

\[
l'_1 - l'_2 = 1 + \epsilon_{t+j+1} - \epsilon_{t+j} > \epsilon_{t+j+1} - \epsilon_{t+j} - 1 = l_2 - l_1.
\]

\[ \square \]

### 3.5 Proof of the Inverse Direction

In this subsection we prove the inverse direction of Theorem 1.5, namely, that any caterpillar subgraph of \( CK(2m) \) that satisfies the conditions mentioned above is a blocker.

**Theorem 3.9** Let \( B \) be the following set of \( m \) edges of \( CK(2m) \):

\[
\{[i-1,i] : 1 \leq i \leq t\} \cup \{[t+j-1-\epsilon_{t+j}, t+j+\epsilon_{t+j}] : 1 \leq j \leq m-t\},
\]

where the \( \epsilon_i \) are integers that satisfy \( 1 \leq \epsilon_{t+1} < \epsilon_{t+2} < \ldots < \epsilon_m \leq m-2 \) (and hence, \( t \geq 2 \)). Then \( B \) is a blocker in \( CK(2m) \).

**Proof:** The proof uses double induction: A primary induction on \( m \), and a secondary (backward) induction on the number of boundary edges in \( B \).

For \( m = 2 \) the claim is clear. For \( m = 3 \) there are only two possible sets \( B \) that satisfy the conditions (up to isomorphism). The first is a path of three consecutive boundary edges, which is indeed a blocker in \( CK(6) \) by Observation 2.2. The second consists of all diagonals of odd order emanating from a single vertex, and therefore intersects every SPM in one edge.

For \( m \geq 4 \), we assume that the assertion holds for \( m-1 \) and prove it for \( m \). Let \( B \) be a set of \( m \) edges of \( CK(2m) \) satisfying the assumptions, and let \( t \) be the number of boundary edges in \( B \). If \( t = m \), then \( B \) is a path of \( m \) consecutive boundary edges of \( CK(2m) \), which is indeed a blocker by Observation 2.2. If \( t < m \), we assume that the assertion holds for all sets \( B \) satisfying the assumptions and having more than \( t \) boundary edges, and prove the assertion for \( B \). Assume on the contrary that \( B \) is not a blocker, and thus there exists an SPM \( M \) that does not meet \( B \). We distinguish two cases:

1. **Case A:** \([2m-1,0] \in M\). In this case we omit the vertices \( 2m-1 \) and 0 from \( CK(2m) \), and show that \( B' = B \setminus \{[0,1]\} \) satisfies the assumptions of the theorem for the induced subgraph \( CK(2m-2) \) spanned by the vertices \( 1, 2, \ldots, 2m-2 \). On the other hand, \( B' \) is not a blocker, since \( M' = M \setminus \{[2m-1,0]\} \) is an SPM in that graph that does not meet \( B' \). This contradicts the inductive assumption on \( m \).

2. **Case B:** \([2m-1,0] \notin M\). In this case we add to \( B \) the edge \([2m-1,0]\), and omit from \( B \) the edge parallel to \([2m-1,0]\). We obtain a new set \( B'' \) that also satisfies the assumptions of the theorem (for the same graph \( CK(2m) \)), and has \( t+1 \) boundary edges. On the other hand, \( B'' \) is not a blocker, since it does not meet \( M \), contradicting the inductive assumption on \( t \).

\[ \text{Recall that by the assumptions of Theorem 3.9 the boundary edges of } B \text{ are } [0,1],[1,2],\ldots,[t-1,t]. \]
Now we discuss the two cases in more detail:

1. **Case A.** If $t = 2$ then $B$ is the set of all edges of odd order emanating from the vertex $1$. In this case, $B'$ is the set of all edges of odd order in $CK(2m - 2)$ emanating from the vertex 1, and thus clearly satisfies the assumption of Theorem 3.9.

If $t > 2$ then $B'$ is a caterpillar whose spine lies on the boundary of $CK(2m - 2)$. The spine contains the edges $[1, 2], [2, 3], \ldots, [t - 1, t]$. Each of the other edges of $B$, i.e., the edges of the form $[t + j - 1 - \epsilon_{t+j}, t + j + \epsilon_{t+j}] : 1 \leq j \leq m - t$, is “parallel” to the respective boundary edge $[t + j - 1, t + j]$ also in $CK(2m - 2)$. Thus, $B'$ contains a representative of each of the $m - 1$ directions in $CK(2m - 2)$.

If $\epsilon_m = m - 2$ then the last edge of the form $[t + j - 1 - \epsilon_{t+j}, t + j + \epsilon_{t+j}]$ is $[1, 2m - 2]$, which is a boundary edge in $CK(2m - 2)$. This edge extends the path $(1, 2, \ldots, t)$ from the left, and thus the length of the spine of $B'$ is $t$. If $\epsilon_m < m - 2$, then all the non-boundary edges of $B$ are also non-boundary edges in $CK(2m - 2)$.

In addition, any non-boundary edge of $B'$ (with respect to $CK(2m - 2)$) connects one of the internal vertices of the spine with one of the internal vertices of the rest of the boundary of $CK(2m - 2)$, and if $e_1 = [p_1, q_1]$ and $e_2 = [p_2, q_2]$ are two non-boundary edges of $B'$, where $p_1, p_2$ are on the spine and $q_1, q_2$ are not on the spine, then the distance between $q_1$ and $q_2$ is greater than the distance between $p_1$ and $p_2$ with respect to $CK(2m - 2)$ (i.e., this property is also inherited from the properties of $B$ in $CK(2m)$). This shows that $B'$ satisfies the assumptions of Theorem 3.9 with respect to $CK(2m - 2)$.

2. **Case B.** It is clear from the construction that $B''$ contains a representative of each of the $m$ directions in $CK(2m)$. In addition, $B''$ is a caterpillar whose spine $(2m - 1, 0, 1, \ldots, t)$ is a path on the boundary of $CK(2m)$. We have to check that the non-boundary edges of $B''$ connect an internal vertex of the spine with an internal vertex of the rest of the boundary of $CK(2m)$. In order to verify this, it is sufficient to show that $B$ does not contain an edge of the form $[2m - 1, i]$ for $0 < i \leq t$. This is indeed true since such an edge is either of even order or parallel to one of the boundary edges of $B$, hence cannot be contained in $B$. The other condition checked in Case A (the comparison of the distances between $q_1, q_2$ and $p_1, p_2$ for two non-boundary edges $[p_1, q_1]$ and $[p_2, q_2]$) holds for $B$, hence also for $B''$. This shows that $B''$ satisfies the assumptions of Theorem 3.9 with respect to $CK(2m)$.

This completes the proof of Theorem 3.9 and with it the proof of Theorem 1.5. □

### 3.6 The Number of Blockers in $CK(2m)$

**Proposition 3.10** The number of blockers in $CK(2m)$ is $m \cdot 2^{m-1}$.

**Proof:** We partition the set of blockers in $CK(2m)$ into subsets according to the number $t$ of boundary edges, and count the number of blockers with exactly $t$ boundary edges. Consider the blockers whose spine (i.e., path of boundary edges) is $(0, 1, \ldots, t)$. Each such blocker is uniquely determined by the number of edges emanating from each of the vertices $1, 2, \ldots, t - 1$. Hence, the number of such blockers is equal to the number of nonnegative integer solutions of the equation $x_1 + x_2 + \ldots + x_{t-1} = m - t$, that is known to be $\binom{m-t+t-2}{t-2} = \binom{m-2}{t-2}$. Thus, the number of blockers whose spine is of the form $(0, 1, \ldots)$ is $\sum_{t=2}^{m} \binom{m-2}{t-2} = 2^{m-2}$. Therefore, by symmetry, the total number of blockers in $CK(2m)$ is $2m \cdot 2^{m-2} = m \cdot 2^{m-1}$, as asserted. □
References

[1] Y.S. Kupitz, Extremal Problems of Combinatorial Geometry, Aarhus University Lecture Notes Series 53 (1979).

[2] F. Harary and A.J. Schwenk, The Number of Caterpillars, Disc. Math. 6 (1973), pp. 359–365.

[3] Micha A. Perles, unpublished.