Critical QCD in Nuclear Collisions

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Abstract

A detailed study of correlated scalars, produced in collisions of nuclei and associated with the \( \sigma \)-field fluctuations, \((\delta \sigma)^2 = \langle \sigma^2 \rangle\), at the QCD critical point (critical fluctuations), is performed on the basis of a critical event generator (Critical Monte-Carlo) developed in our previous work. The aim of this analysis is to reveal suitable observables of critical QCD in the multiparticle environment of simulated events and select appropriate signatures of the critical point, associated with new and strong effects in nuclear collisions.

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I. INTRODUCTION

The existence of a critical point in the phase diagram of QCD, for nonzero baryonic density, is of fundamental significance for our understanding of strong interactions and so its experimental verification has become an issue of high priority [1]. For this purpose an extensive programme of event-by-event searches for critical fluctuations in the pion sector is in progress in experiments with heavy ions from SPS to RHIC energies [2]. In [3] we have emphasized, however, that in order to reveal critical density fluctuations in multiparticle environment, one has to look for unconventional properties in the momentum distribution of reconstructed dipions ($\pi^+\pi^-$-pairs) [9] with invariant mass just above the two-pion threshold. In fact, the QCD critical point, if it exists, communicates with a zero mass scalar field ($\sigma$-field) which at lower temperatures ($T < T_c$) may reach the two-pion threshold and decay in very short time scales owing to the fact that its coupling to the two-pion system is strong. Obviously, the fundamental, underlying pattern of $\sigma$-field fluctuations, built-up near the critical point by the universal critical exponents of QCD [3], is phenomenologically within reach if and only if the study of correlated sigmas, reconstructed near the two-pion threshold, becomes feasible. In the present work we perform a detailed feasibility study of the observables related to the detection of the QCD critical point in nuclear collisions. In order to proceed we summarize, first, the principles on which the behaviour of a critical system of sigmas is based [1,3].

(a) The geometrical structure of the critical system in transverse space (after integrating in rapidity) consists of $\sigma$-clusters with a fractal dimension $d_F = \frac{2(\delta - 1)}{\delta + 1}$ leading to a power law, $<\sigma^2> \sim |\vec{x}|^{-\frac{2(\delta - 1)}{\delta + 1}}$, for the $\sigma$-field fluctuations, within each cluster ($\delta$ : isotherm critical exponent)

(b) In transverse momentum space the $\sigma$-fluctuations obey a power law $<\sigma^2> \sim |\vec{p}_\perp|^{-\frac{2(\delta - 1)}{\delta + 1}}$ leading to observable intermittent behaviour of factorial moments: $F_2(M) \sim (M^2)^{\frac{\delta - 1}{2(\delta - 1)}}$ where $M^2$ is the number of 2D cells [4].

(c) The density fluctuations of dipions ($\pi^+\pi^-$-pairs) with invariant mass close to two-pion threshold ($2m_\pi$) incorporate the sigma-field fluctuations, $(\delta\sigma)^2 \approx <\sigma^2>$, at the critical point, under the assumption that the sigma mass reaches the two-pion threshold ($m_\sigma \approx 2m_\pi$) in a time scale shorter than the relaxation time of critical fluctuations.

(d) The QCD critical point belongs to the universality class of the 3D Ising system in
which $\delta \approx 5$.

On the basis of these principles and the fact that critical clusters in the above universality class interact weakly [5], one may construct a Monte-Carlo generator (Critical Monte-Carlo: CMC) able to simulate events of critical sigmas, correlated according to the above prescription [3]. Then, if the corresponding invariant mass distribution is known, we can also simulate the decay of the critical sigmas to pions which are experimentally observable. To complete the numerical experiment we use the momenta of the charged pions, obtained from the decay of the critical sigmas, to form, in an event-by-event basis, neutral dipions ($\pi^+\pi^-$ pairs) and to look for fingerprints of the original critical sigma fluctuations in their momentum distribution. To check the applicability of our approach we perform also a comparative study between the revealed critical correlations-fluctuations in CMC and the corresponding behaviour of a conventional Monte-Carlo (HIJING).

The input parameters of the simulation are the size of the system in rapidity ($\Delta$) and transverse space ($R_\perp$) as well as the proper time scale ($\tau_c$) characteristic for the formation of the critical system. In what follows we will use exclusively $\Delta = 6$ for the rapidity size, adapted to the SPS-NA49 conditions ($E_{\text{beam}} \approx 158 \text{ GeV}/n$). The parameters $R_\perp$, $\tau_c$ can then be tuned in order to fit the mean charged pion multiplicity of systems of different size ($CC$, $SiSi$ and $PbPb$) studied experimentally at this energy [6]. For each choice of input parameters we produce, using the CMC generator, a large set of critical sigma events. In order to generate the corresponding pion sector we assign to the sigmas the invariant mass distribution $\rho(m_{\sigma})$ and then we let them decay into pions with a branching ratio 1 : 2 for neutral to charged. The choice of $\rho(m_{\sigma})$ is determined by the requirement that the resulting inclusive neutral dipion invariant mass distribution $\rho(m_{\pi^+\pi^-})$ resembles to a large extent the corresponding distribution obtained in experiments with heavy ion collisions at high energies. In order to perform our analysis in terms of observable quantities we choose the mean multiplicity of positive charged pions per event $< n_{\pi^+} >$ as the basic parameter (instead of $R_\perp$, $\tau_c$) characterizing different $A + A$ systems. We note that within the CMC approach the property $< n_{\pi^+} > = < n_{\pi^-} >$ is exactly fulfilled. We will investigate three different cases: low ($< n_{\pi^+} > \approx 10$), intermediate ($< n_{\pi^+} > \approx 30$) and high ($< n_{\pi^+} > \approx 220$) positive charged pion multiplicity resembling the $C + C$, $Si + Si$ and $Pb + Pb$ system at 158 $\text{GeV}/n$ respectively.
FIG. 1: The critical sigma invariant mass distribution \( \rho(m_\sigma) \) (solid line) as well as the corresponding neutral dipion distribution \( \rho(m_{\pi^+\pi^-}) \) (full circles) for 30000 CMC events with \(< n_{\pi^+} >= 11.31\).

II. THE CRITICAL SIGMA SECTOR

First we consider the case of low pion multiplicity. Before going on with the analysis it is worth to emphasize that, in general, the determination of the critical sigma sector using the observed momenta of charged pions, is a very difficult task. This is due to the absence of a characteristic pattern in the inclusive dipion invariant mass distribution attributed to the presence of critical sigmas. To illustrate this property we show in Fig. 1 the inclusive distribution of the sigma invariant mass calculated before the decay of the sigmas into pions (solid line) as well as the corresponding distribution for neutral dipions reconstructed from the final pion momenta (full circles) for 30000 CMC events with \(< n_{\pi^+} >= 11.31\) (dataset I). Both distributions are equally normalized. Apart from a peak at \( m_{\pi^+\pi^-} \approx 450 \, MeV \) due to kinematics there is no other structure in the invariant mass profile. Therefore the detection of the critical sigma sector has to go through the, more subtle, study of density fluctuations in momentum space.

In order to reveal the underlying critical fluctuations, at the level of observation, one has first to perform factorial moment analysis in small cells of the momentum space \([4]\). We have chosen transverse momenta for this analysis in order to avoid additional assumptions about
FIG. 2: The second factorial moment $F_2$ in transverse momentum space of both the negative pions as well as the critical sigmas. We use the same dataset as in Fig. 1.

the role of longitudinal rapidity in the description of the statistical mechanics of the system [3]. Applying factorial moment analysis to the transverse momenta $(p_x, p_y)$ of the negative pions in the sample of the 30000 critical events we obtain for the second moment a weak intermittency effect: $F_2 \sim M^2 s^2$ with $s^2 \approx 0.077$ much smaller than the expected to occur in the critical system ($s^2 = 2/3$). We note with $M$ the number of bins in each momentum space component. The corresponding factorial moment for the sigmas, before their decay, follows closely the theoretical prediction: $s_2 \approx 0.66$. This behaviour is displayed in Fig. 2.

The reason for the suppression of fluctuations in the pionic sector is the kinematical distortion of the self-similar pattern formed in the sigma sector due to the sigma-decay. The strength of this distortion increases with momentum transfer: $Q = \sqrt{m_\sigma^2 - 4m_\pi^2}$ and becomes negligible near the two-pion threshold ($m_\sigma \approx 2m_\pi$). Here $m_\sigma^2 = (p_{\pi^+} + p_{\pi^-})^2$ where $p_{\pi^\pm}$ are the four momenta of the charged pions produced through the sigma-decay. Thus the search for critical fluctuations is based on an accurate reconstruction of the momenta of the decaying sigmas. In particular $(\pi^+, \pi^-)$ pairs with invariant mass close to the two-pion threshold are the best candidates to carry potentially the geometrical features of the critical isoscalars. The great advantage of our approach is that it allows a self-consistency test for
the reconstruction of the sigma sector at the level of density fluctuations due to our exact knowledge of the sigma momenta and the corresponding fluctuation pattern.

III. RECONSTRUCTION OF POWER-LAWS

In practice the reconstruction of the critical momentum fluctuations is performed by looking, event by event, for \((\pi^+, \pi^-)\) pairs fulfilling the criterion \(A:\)

\[
A = \{(\pi^+, \pi^-) | 4m^2 \leq (p_{\pi^+} + p_{\pi^-})^2 \leq (2m_\pi + \epsilon)^2 \}
\]

(1)

with \(\epsilon \ll 2m_\pi\). The momentum of the corresponding neutral dipion is then obtained as:

\[
\vec{p}_{\pi\pi} = \vec{p}_{\pi^+} + \vec{p}_{\pi^-}.
\]

In order to ensure that all the available critical sigmas within the above kinematical region are recovered in the reconstruction we have to use full pairing forming all possible pairs \((\pi^+, \pi^-)\) fulfilling (1) for a given \(\pi^+\). However the full pairing introduces as a side effect a combinatorial background which has to be treated appropriately. We will come back to this point later on. Thus for each value of \(\epsilon\) we obtain a set of events including dipion momenta. We perform factorial moment analysis in transverse momentum space for each such dataset. According to the previous discussion, for decreasing values of \(\epsilon\) the fluctuations measured by the intermittency exponent \(s_2\) of the corresponding factorial moment should increase leading to \(s_2 \to \frac{2}{3}\) as \(\epsilon \to 0^+\). In Fig. 3a we show the second moment in transverse momentum space obtained from data sets of reconstructed dipions for three different values of \(\epsilon\) (5, 50 and 500 MeV). We observe an increase of the slope \(s_2\) with decreasing \(\epsilon\). However this increase has two different origins: (i) the presence of nontrivial fluctuations of dynamical origin and (ii) kinematically induced fluctuations through the constraint (1) which become of the same order as the dynamical ones for \(\epsilon \to 0\). It must be noted that the functional forms of \(F_2\) for the different \(\epsilon\) values are not exact power-laws and the exponent \(s_2\) is an effective one. Increasing \(\epsilon\) the deviations from a power-law description increase too. In order to control the kinematical fluctuations we form datasets with mixed events through appropriate shuffling of the momenta in the original CMC events. Then we calculate the second factorial moments for the datasets consisting of mixed events. We show in Fig. 3b the results of this analysis. Also in the case of mixed events we observe a clear increase of the effective slope \(s_2^{(m)}\) for decreasing \(\epsilon\). Due to the absence of dynamical fluctuations in the mixed events it is natural to conclude that the observed behaviour in \(F_2^{(m)}\) is attributed
FIG. 3: (a) The second factorial moment $F_2$ in transverse momentum space of reconstructed dipions for $\epsilon = 5, 50$ and $500$ $MeV$ (log-log plot). We use the same 30000 CMC events as in Fig. 1. In (b) we show for comparison the corresponding factorial moments for datasets consisting of mixed events.

to the kinematical fluctuations.

These can be removed if we introduce the difference $\Delta F_q = F_q - F_q^{(m)}$ between the factorial moments of real and mixed events (for any order $q$). For $q = 2$ the quantity $\Delta F_2$ represents the correlator of the corresponding dipions. Thus in the difference $F_2 - F_2^{(m)}$ the combinatorial background representing the uncorrelated part of the reconstructed dipions is suppressed and the critical fluctuations associated with the correlated part of the dipion sector are recovered. We illustrate how this subtraction succeeds in practice in Fig. 4 where we present $\Delta F_2$ for $\epsilon = 5$ $MeV$ and $< n_{\pi+} >= 11.31$. Although the moments $F_2$ and $F_2^{(m)}$ are not exact power-laws, as indicated above, the difference $\Delta F_2$ is significantly better fitted by a power-law and the corresponding slope $\phi_2 = 0.65(02)$ is very close to the slope obtained from the dataset of the sigmas before their decay (see Fig. 1).

The fluctuations in the CMC events, as described by $\Delta F_2$, are induced by strongly correlated critical QCD dynamics and are expected to be at least one order of magnitude greater than fluctuations originating from conventional hadronic dynamics. To check the validity of this statement we have calculated $\Delta F_2$ using 33176 events obtained from the HIJING Monte-Carlo generator simulating the SPS $C + C$ system at 158 $GeV/n$ and involving only noncritical QCD dynamics. In Figs. 5a-d we show the results of our calculations for
FIG. 4: (a) The log-log plot of the correlator $\Delta F_2$ in transverse momentum space of reconstructed dipions for $\epsilon = 5$ MeV obtained using 30000 CMC events (the same as in Fig. 1) and the corresponding mixed events.

four different values of $\epsilon$. For comparison we plot in the same figure the correlator for the 30000 CMC events with $< n_{\pi^+} >= 11.31$. We observe that for all values of $\epsilon$ the correlator $\Delta F_2$ for the HIJING system remains flat fluctuating around zero, while for the CMC events as $\epsilon$ decreases the characteristic, for the critical system, power-law behaviour is setting on according to the previous discussions.

IV. THE CRITICAL INDEX

It is worth investigating in more detail the role of the parameter $\epsilon$ in our analysis. Practically the desired limit $\epsilon \rightarrow 0^+$ is not accessible for several reasons. First of all looking at Fig. 1 we observe that in this limit the number of initial critical sigmas as well as the number of reconstructed dipions vanishes. We practically need infinite statistics in order to extend our analysis in this kinematical region. In addition, in the range of very small $\epsilon$ values the fluctuations induced by the kinematical constraint $\mathbf{1}$ become of the same order as the dynamical fluctuations due to the critical sigmas. Our entire treatment is based on
FIG. 5: The correlator $\Delta F_2$ for the CMC and the HIJING system using four different values of $\epsilon$: (a) 5 MeV, (b) 20 MeV, (c) 50 MeV and (d) 120 MeV.

an accurate cancellation of the kinematical fluctuations in the difference $\Delta F_2$ and this requires again very high statistics if $\epsilon \to 0$. In fact, using $\epsilon$ values smaller than 4 $MeV$ in the analysis of the 30000 CMC events considered so far, the statistical errors increase rapidly and prohibit a reliable reveal of the critical fluctuations. On the other hand, if we increase $\epsilon$, the combinatorial background due to the reconstructed dipions is increasing rapidly and the relative measure of the critical sector goes to zero. Therefore we must search for an optimal region of $\epsilon$ values to perform our analysis. In order to achieve a more quantitative criterion for the determination of the appropriate region of $\epsilon$-values we consider more carefully the sector of reconstructed dipions. In the CMC case the reconstructed dipions are divided into two subsets: the set of real sigmas and the set of fake sigmas. Let us denote by $< n_{r,\sigma} >_{\epsilon}$ the mean number of real sigmas per event with invariant mass in the kinematical window $\Pi$ for a given value of $\epsilon$. Within our approach we investigate only the part of the sigmas which decays into a pair of opposite charged pions. Therefore we have:

$$< n_{r,\pi^+} >_{\epsilon} = < n_{r,\pi^-} >_{\epsilon} = < n_{r,\sigma} >_{\epsilon}$$

where $< n_{r,\pi^+(-)} >_{\epsilon}$ is the mean number of positive (negative) charged pions per event produced through the decay of the real sigmas with invariant mass in the region $\Pi$. Obviously
to a good approximation the corresponding number of fake sigmas is given by:

\[ \langle n_{f,\sigma} \rangle \approx \langle n_{r,\pi^+} + n_{r,\pi^-} \rangle \epsilon - \langle n_{r,\pi^+} \rangle \epsilon \]  

(3)

The first term on the right hand side of eq.(3) is dominated by uncorrelated \((\pi^+, \pi^-)\) pairs. Therefore we can write: \[ \langle n_{r,\pi^+} + n_{r,\pi^-} \rangle \epsilon \approx \langle n_{r,\pi^+} \rangle \epsilon \langle n_{r,\pi^-} \rangle \epsilon \] und using the property:

\[ \langle n_{r,\pi^+} \rangle \epsilon = \langle n_{r,\pi^-} \rangle \epsilon \]

we finally obtain:

\[ \langle n_{f,\sigma} \rangle \epsilon \approx \langle n_{r,\pi^+} \rangle \epsilon^2 - \langle n_{r,\pi^+} \rangle \epsilon \]

(4)

A natural constraint ensuring the dominance of the real sigmas over the fake ones is to use in our analysis \(\epsilon\) values fulfilling the condition:

\[ \langle n_{r,\pi^+} \rangle \epsilon \approx \langle n_{r,\pi^+} \rangle \epsilon^2 - \langle n_{r,\pi^+} \rangle \epsilon \]  

(5)

which simplifies to: \(\langle n_{r,\pi^+} \rangle \epsilon < 2\). This is fullfiled for example for the value \(\epsilon = 5\) \(\text{MeV}\) which we have used so far in our analysis since in this case \(\langle n_{\pi^+} \rangle \approx (5\text{ MeV}) = 1.12\). We also observe that \(\langle n_{\pi^+} \rangle \epsilon \approx \langle n_{r,\pi^+} \rangle \epsilon\) for any value of \(\epsilon\), where \(\langle n_{\pi^+} \rangle \epsilon\) is the mean number of dipions per event obtained through the reconstruction in the kinematical window \([1\text{]}\). The upper bound in \(\langle n_{r,\pi^+} \rangle \epsilon\) corresponds to an upper bound for \(\epsilon\). The lower bound both for \(\epsilon\) as well as \(\langle n_{r,\pi^+} \rangle \epsilon\) is determined by the statistics according to the discussion in the previous paragraph. For practical purposes one applies the stronger bound \(\langle n_{\pi^+} \rangle \epsilon < 2\) for the estimation of the appropriate region of \(\epsilon\)-values to be used in the data analysis. The great advantage of restricting \(\langle n_{\pi^+} \rangle \epsilon\) instead of \(\epsilon\) becomes more clear when we compare the analysis in CMC datasets simulating \(A + A\) processes of different size. In this case the dependence of \(\langle n_{\pi^+} \rangle \epsilon\) on \(\epsilon\) varies from system to system. To illustrate this we have produced two additional CMC datasets, with \(\langle n_{\pi^+} \rangle = 29.69\) (dataset II) and \(\langle n_{\pi^+} \rangle = 213.96\) (dataset III) respectively, each consisting of 30000 events. In Fig. 6 we show the function \(\langle n_{\pi^+} \rangle \epsilon\) for the CMC datasets I, II and III. The horizontal lines correspond to the values \(\langle n_{\pi^+} \rangle \epsilon = 5, \langle n_{\pi^+} \rangle \epsilon = 1.5\) and \(\langle n_{\pi^+} \rangle \epsilon = 1.1\).

For increasing size of the system (increasing \(\langle n_{\pi^+} \rangle\)) the \(\epsilon\) values leading to a given \(\langle n_{\pi^+} \rangle \epsilon\) decrease. The complete analysis within our approach for these datasets involves the reconstruction of the isoscalar sector as well as the calculation of the corresponding correlator \(\Delta F_2\). However a comparative study between the different datasets in terms of factorial moments is possible only for classes of events characterized by almost the same
FIG. 6: The function $< n_{\pi^+\pi^-} > (\epsilon)$ for the three different CMC datasets described in text.

multiplicity since in the opposite case artificial fluctuations are induced [8]. Thus, to compare the systems of different size, we have to calculate the correlator for fixed multiplicity of reconstructed dipions $< n_{\pi^+\pi^-} >$ choosing a suitable value of $\epsilon$ in each case. As in the case of dataset I we use the effective slope (or critical index) $\phi_2$ of the correlator ($\Delta F_2 \sim M^{2\phi_2}$) as a measure of the fluctuations in the corresponding dataset. In Fig. 7 we show the results of our calculations for the three datasets I, II and III using $< n_{\pi^+\pi^-} > \epsilon = 5$ (full circles), $< n_{\pi^+\pi^-} > \epsilon = 1.5$ (open stars) and $< n_{\pi^+\pi^-} > \epsilon = 1.1$ (crosses). We plot $\phi_2$ as a function of the mean number of positive pions per event $< n_{\pi^+} >$ respectively. It is clearly seen in Fig. 7 that the critical fluctuations, within the limitations of our analysis due to statistics, are at best recovered for CMC systems involving medium to small size nuclei and using in the analysis $\epsilon$ values fulfilling the constraint $< n_{\pi^+\pi^-} > \epsilon < 2$. The critical index $\phi_2$ approaches the theoretically expected value $\frac{2}{3}$. In fact we see that for $< n_{\pi^+\pi^-} > \epsilon = 1.1$ and $< n_{\pi^+} > = 11.31$ we obtain exactly the critical QCD prediction. This is the main result of the present work providing us with a useful guide for the search of QCD critical fluctuations in relativistic ion collisions.
V. CONCLUSIONS

In conclusion, we have shown that a set of well prescribed observables (factorial moments, correlators, intermittency exponents) associated with the existence of a critical point in quark matter, can be established in nuclear collisions. These observables belong to the reconstructed isoscalar sector describing massive dipions ($\pi^+\pi^-$ pairs) near the two-pion threshold and their behaviour reveals strong critical effects suggested by $\sigma$-field fluctuations near the critical point. We claim that a search for such a critical behaviour in heavy ion experiments is feasible within the framework of a reconstruction procedure of the momenta of massive dipions, discussed in this work. The critical effects in this sector although independent of the system size can be at best recovered, through the proposed reconstruction algorithm, in collisions of relatively small nuclei. The appropriate kinematical window to look for these effects is also determined. Our study has also shown that although it is not possible to reveal any conventional sign of the sigma itself in experiments with nuclei at high energies, nevertheless its density fluctuations associated with the critical point are observable and can be measured and studied in a systematic way. Therefore our proposal is to study, using the above observables, different processes at the SPS and RHIC with the aim to scan a substantial area of the phase diagram, in a systematic search for the QCD critical point in collisions of nuclei.
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[9] We use the term dipions to indicate exclusively a pair of opposite charged pions. This will be the case throughout in the present work.
[10] The superscript \((m)\) is used to indicate mixed events