HOMOGENIZATION OF A LOCALLY PERIODIC TIME-DEPENDENT DOMAIN

MORTEZA FOTOUHIST

Department of Mathematical Sciences, Sharif University of Technology
Tehran, Iran

MOHSEN YOUSEFNEZHAD

Department of Mathematical Sciences, Sharif University of Technology
Tehran, Iran

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Abstract. We consider the homogenization of a Robin boundary value problem in a locally periodic perforated domain which is also time-dependent. We aim at justifying the homogenization limit, that we derive through asymptotic expansion technique. More exactly, we obtain the so-called corrector homogenization estimate that specifies the convergence rate. The major challenge is that the media is not cylindrical and changes over time. We also show the existence and uniqueness of solutions of the microscopic problem.

1. Introduction. Several mathematical models arising from systems biology, material sciences, and other applied sciences, give rise to partial differential equations with a complex structure because of impurities and inhomogeneity in these models, see [8, 13]. Some kind of these problems involve spatial inhomogeneities of the underlying material at microscopic scales and hence a detailed analytical or even numerical approach becomes infeasible due to these heterogeneities [36]. A natural and well-established methodology is to average out these impurities in the medium through some mechanism, appropriate for the model equation. Such mechanism is commonly referred to as finding the effective, homogenized, model of the equation. There are two general strategies to achieve this goal. One strategy is the phenomenological approach, in which one directly establishes the equations governing the macroscopic behavior without inquiring the detailed structure of the microscale. In this case, the parameters of the macroscopic model are fitted by the experiment. Another strategy is to start from the microscale model of heterogeneities and deduce the effective equations based on averaging methods such as asymptotic expansion technique, and the effective coefficients are obtained mathematically [8, 13]. From the analytical perspective, we ask for a rigorous justification of the effective model and, if available, error estimates describing the difference to the original microscopic model [22].

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* Corresponding author.
In recent years there has been a renewed interest in mathematical models where the bulk domain changes over time, giving rise to free and moving boundary value problems \[4, 6, 9, 12, 24, 32, 37\]. A concrete example of such a model is the diffusion in the brain extracellular space. The effective diffusion of the PDE model of the brain cell microenvironment describes the average behavior of the physiology of neuronal populations \[19, 20\]. The extracellular space of the brain is a heterogeneous complex medium in which several factors impose constraints on the diffusion process \[20, 37\]. The primary constraints are mainly consequences of the geometrical structure of the medium, that changes in time due to specialized features and physiological conditions in the brain, such as transport of water in brain ischemia \[37\]. In such cases, the microscopic problem is a diffusion equation in bounded micro-domain that evolves in time \[37\]. From the mathematical point of view, we deal with a Robin boundary problem that allows boundary to evolve with time \[37\].

As a first step before handling the homogenization of a moving boundary problem, we consider in this paper a parabolic problem in a non-cylindrical domain. We are also concerned with a problem that arises in the locally periodic microstructure. Indeed, the geometry of the heterogeneities is such that the perforation is periodic in space, but not the shape and size of them, and they may as well vary in time. For more details, the reader is referred to Section 2.1, where we explain our concept of time-dependent locally periodic domain. Although various homogenization results have been achieved for periodic and locally periodic fixed microstructures over time \[2, 7, 18, 26, 27\], we study the asymptotic behavior of the model in a non-cylindrical domain.

To capture the macrostructure in the homogenized form, an extension of the formal two-scale asymptotic expansion to the level set framework was introduced in \[34\]. This method was recently applied to mathematical models of evolving brain extracellular space, colloid dynamics, drug delivery systems, and biofilm growth \[28, 29, 31, 37\]. In these works the corresponding upscaled form is derived, leaving out the convergence aspects of the model. Our main goal in this paper is to discuss this convergence problem, by a new and simple constructive technique. Indeed, we obtain corrector estimates which show the speed of convergence as \(\epsilon \to 0\) based on a suitable norm (See Theorem 4.1). Such estimates justify the homogenization limit which in the literature are usually called error and corrector estimates and provide a method to evaluate the accuracy of the upscaled model.

The main idea is that we approximate the microscopic problem while considering time slices of the time-dependent domain. Next, we solve a family of approximating problems corresponding to cylindrical domains. The corrector estimates for each cylindrical problem can be obtained with classical methods. Finally, we show that these estimates give us the desired corrector estimates for the original problem. Such estimates obtained in this framework are especially interesting from a computational point of view. Indeed, in the context of Multiscale Finite Element Method, these estimates are needed to ensure the convergence of the method \[3, 14\].

This paper is organized as follows: Section 2 provides a detailed description of the model at the microscale level in the time and space dependent microstructure. In Section 3 we show the existence and uniqueness results for microscale equations in non-cylindrical domains. In Section 4, we introduce a macroscopic model in two-scale limit form and prove the convergence rate theorem for the homogenization problem.
2. **Statement of the problem.** In this section we introduce the notations, definitions, and the microscopic model. Firstly, we want to construct the geometry of the problem under locally periodic micro-medium structure.

2.1. **Definition of a time-dependent periodic micro-medium.** We define a perforated domain consisting of area related to the micro-medium. To define time and space-dependent micro-medium, we follow the defined geometry structure in [7]. Let $\Omega \subset \mathbb{R}^d, d \geq 2$, be a smooth bounded domain. Denote:

$$J_\epsilon = \{ j \in \mathbb{Z}^d | \text{dist}(\epsilon j, \partial \Omega) \geq \epsilon \sqrt{d} \},$$

$$Y = \{ y \in \mathbb{R}^d | -1/2 \leq y_i \leq 1/2 \text{ for } i = 1, 2, \cdots, d \}.$$

A proper way to parameterize the interface $\Gamma_\epsilon(t)$ for interior boundary areas is applying a level set function, which is named $G_\epsilon(t, x)$:

$$x \in \Gamma_\epsilon(t) \iff G_\epsilon(t, x, x_\epsilon) = 0.$$  

Since we expect the perforations to follow the locally periodic structure, we consider $G_\epsilon$ to depends on a microscopic variable and be periodic with respect to that, i.e.:

$$G_\epsilon(t, x) := G(t, x, \frac{x}{\epsilon}).$$

Throughout the paper, the function $G(t, x, y)$ satisfies the following assumption.

**Assumption 1.** Consider a function $G \in C^2([0, T] \times \overline{\Omega} \times Y)$ and extend it in the third variable to be $Y$-periodic. Moreover, assume that

(i) $G(t, x, 0) < \text{cons.} < 0$, and $G(t, x, y)|_{y \in \partial Y} > \text{cons.} > 0$ for all $x \in \Omega, t \in (0, T)$,

(ii) $\partial_y G(t, x, y) \neq 0$, for all $(t, x, y) \in [0, T] \times \overline{\Omega} \times Y$,

(iii) $\partial_t G(t, x, y) \leq 0$, for all $(t, x, y) \in (0, T) \times \Omega \times Y$.

The perforated domain will be defined by

$$\Omega_\epsilon(t) := \Omega \setminus \left( \bigcup_{j \in J'} Q_{\epsilon, j}(t) \right),$$
where
\[ Q_{\epsilon,j}(t) := \left\{ x \in \epsilon(Y + j) \mid G(t, x, \frac{x}{\epsilon}) < 0 \right\}, \]
are holes in the domain. The Assumption 1 insures that the holes \( Q_{\epsilon,j} \) do not intersect. Furthermore, we study the case that the holes are getting bigger (condition \((iii)\)) during the diffusion process. In Figure 1, a schematic picture of the domain \( \Omega_{\epsilon}(t) \) is depicted for a fixed time \( t \).

In this study, the following system is considered

\[
\begin{aligned}
\frac{\partial u_{\epsilon}}{\partial t} - \Delta u_{\epsilon} &= 0, & x &\in \Omega_{\epsilon}(t), \\
\nabla u_{\epsilon} \cdot n_{\epsilon}(t, x) + \epsilon q_{\epsilon}(x) u_{\epsilon} &= 0, & x &\in \Gamma_{\epsilon}(t), \\
\nabla u_{\epsilon} \cdot N &= 0, & x &\in \partial \Omega, \\
u_{\epsilon}(x, 0) &= f_{\epsilon}(x), & x &\in \Omega_{\epsilon}(0),
\end{aligned}
\tag{2.1}
\]

where, \( n_{\epsilon}(t, x) \) is the outward normal vector on \( \Gamma_{\epsilon}(t) \) and \( N \) is the normal vector on \( \partial \Omega \). Furthermore, we assume that \( q_{\epsilon}(x) := q(x, \frac{x}{\epsilon}) \) in which \( q(x, y) \) is a \( Y \)-periodic function in the second variable.

**Assumption 2.** We assume the following conditions on initial data and parameters of the problem (2.1):

\[
\begin{aligned}
f_{\epsilon} &\in L^\infty(\Omega_{\epsilon}(0)) \cap H^2(\Omega_{\epsilon}(0)), \\
q(x, y) &\in C^1(\Omega \times Y).
\end{aligned}
\]

Furthermore, we assume that there is a function \( f \in L^\infty(\Omega) \cap H^2(\Omega) \) such that

\[ \| f_{\epsilon} - f \|_{H^2(\Omega_{\epsilon}(0))} = O(\sqrt{\epsilon}). \]

Here, we present an expansion of the normal vector \( n_{\epsilon} \) in a power series in \( \epsilon \). It will be necessary to prove our main result in Section 4. This expansion can be done in terms of the level set function \( G_{\epsilon} \), which we assume to be sufficiently regular so that all the following computations make sense. We have

\[
n_{\epsilon}(t, x) = \frac{\nabla G_{\epsilon}(t, x)}{|
abla G_{\epsilon}(t, x)|} = \frac{\nabla_x G + \epsilon^{-1} \nabla_y G}{|\nabla_x G + \epsilon^{-1} \nabla_y G|}, \text{ at } x \in \Gamma_{\epsilon}(t).
\]

In the same fashion, we get

\[
n_{\epsilon}(t, x) = n_0(t, x, \frac{x}{\epsilon}) + \epsilon n_1(t, x, \frac{x}{\epsilon}) + O(\epsilon^2),
\]

where

\[
n_0(x, y, t) = \frac{\nabla_y G(t, x, y)}{|\nabla_y G(t, x, y)|}, \tag{2.2}
\]

and

\[
n_1(t, x, y) = \frac{\nabla_x G}{|\nabla y G|} - \frac{(\nabla_x G \cdot \nabla y G)}{|\nabla y G|^2} \frac{\nabla_y G}{|\nabla y G|}. \tag{2.3}
\]
3. Existence and uniqueness of microscopic problem. The purpose of this section is to prove the existence and uniqueness results for the microscale problem with the time dependent domain. More precisely, given the open sets \( \Omega(t) \subset \subset O \) with the boundary \( \Gamma(t) \), such that \( O \) is a fixed open subset of \( \mathbb{R}^d \). We shall consider the following problem:

\[
\begin{aligned}
& u_t - \Delta u = 0, & x \in \Omega(t), \\
& \nabla u \cdot n + q(x)u = 0 & x \in \Gamma(t), \\
& \nabla u \cdot N = 0 & x \in \partial \Omega, \\
& u(x,0) = u_I(x) & x \in \Omega(0).
\end{aligned}
\]  

(3.1)

In this section, we suppose that the time-dependent domain, \( \Omega(t) \), defined by

\[ \Omega(t) = \{ x \in \Omega : G(t,x) > 0 \}, \]

in which the smooth function \( G \) satisfies:

- **H1.** \( G \in C^2([0,T] \times O) \) and \( \partial_r G(t,x) \neq 0 \), for all \( (t,x) \in [0,T] \times O \).

- **H2.** \( \partial_t G(t,x) \leq 0 \), for all \( (t,x) \in [0,T] \times O \).

Moreover, we shall use standing conditions on the functions \( q \) and \( u_I \) as follow:

- **H3.** \( q \in C^1(O) \) and \( u_I \in H^1(\Omega(0)) \).

It is noteworthy that \( H1 \) will be concluded by the condition (ii) in Assumption 1 for small enough parameter \( \epsilon \). Then the result of this section can be applied to the locally periodic perforated domain defined in the previous section. It is also clear that \( \Omega(s) \subset \Omega(t) \) for \( s > t \) in our case according to the assumption \( H2 \). Before going to the existence result, we state some spaces of functions and define the weak solution of the problem.

3.1. Spaces of functions. Here, we construct Lebesgue and Sobolev spaces of functions defined on non-cylindrical domains. These function spaces are needed in order to state the variational formulation of problem (3.1). We refer the reader to [16] for similar definitions, further references and more details on the construction of spaces.

3.1.1. The Lebesgue space \( L^2(0,T;L^2(\Omega(t))) \). Consider \( O \) to be an open subset of \( \mathbb{R}^d \) and let \( \Omega(t) \subset O \), for each \( t \in [0,T] \). The space \( L^2(0,T;L^2(\Omega(t))) \) which is defined in the following:

\[ L^2(0,T;L^2(\Omega(t))) := \{ u \in L^2(0,T;L^2(O)) : u(t) \in L^2(\Omega(t)), \]

\( u(.,t) = 0 \) on \( O \setminus \Omega(t) \) for a.e. \( t \in (0,T) \}, \]

is a Banach space with the norm

\[ \| u \|_{L^2(0,T;L^2(\Omega(t)))} := \left( \int_0^T \| u(t) \|_{L^2(\Omega(t))}^2 \, dt \right)^{1/2}. \]

3.1.2. The Sobolev space \( L^2(0,T;H^1(\Omega(t))) \). Similar to the classical case in cylindrical domain, we can define the Sobolev space \( L^2(0,T;H^1(\Omega(t))) \) by

\[ L^2(0,T;H^1(\Omega(t))) := \{ u \in L^2(0,T;L^2(\Omega(t))) : \nabla u \in L^2(0,T;L^2(\Omega(t))) \}, \]

with respect to the norm

\[ \| u \|_{L^2(0,T;H^1(\Omega(t)))} := \left( \int_0^T \| u(t) \|_{H^1(\Omega(t))}^2 \, dt \right)^{1/2}. \]  

(3.2)
Remark 1. It should be noted that the space $L^2(0, T; H^1(\Omega(t)))$ is the closure of the space $C^1(\widetilde{\Omega})$ with respect to the norm (3.2), where

$$\widetilde{\Omega} = \{(t, x) : x \in \Omega(t), 0 < t < T\}.$$  

For details on such statement, the readers can see [23].

3.1.3. The trace space of $L^2(0, T; L^2(\Gamma(t)))$. From now on, we assume that the Lebesgue measure $|\Omega(t)|$ and $(d-1)$-dimensional Hausdorff measure $|\Gamma(t)|$ are bounded away from zero (uniformly in $t$). This condition can be easily deduced from the assumption H1. Now, let us introduce

$$\Gamma = [0, T] \times \Gamma(t) := \bigcup_{t \in [0, T]} (\{t\} \times \Gamma(t)).$$

and define the trace space

$$L^2((0, T); L^2(\Gamma(t))) := \left\{ u \in L^2(\Gamma) : u(t) \in L^2(\Gamma(t)) \text{ for a.e. } t \in (0, T) \right\},$$

where

$$\|u\|_{L^2(0, T; L^2(\Gamma(t)))} := \left( \int_0^T \|u(t)\|^2_{L^2(\Gamma(t))} \, dt \right)^{1/2},$$

is a norm for this space. The next proposition defines the trace operator on $L^2(0, T; H^1(\Omega(t)))$.

**Proposition 1** (Theorem 2.2.18, [16]). Let $\gamma_t : H^1(\Omega(t)) \rightarrow L^2(\Gamma(t))$ be the continuous trace operator for each $t \in (0; T)$. Then under assumption H1, $\|\gamma_t\|$ is uniformly bounded in $t \in (0, T)$. Furthermore, there exists the bounded linear operator, which is called the distributed trace,

$$\gamma : H^1(\Omega) \rightarrow L^2(\Gamma),$$

such that $\gamma(u)(t) := \gamma_t(u(t))$ for each $t \in (0, T)$.

We are not going to prove here the uniform boundedness of the trace operator. However, the next lemma and the forthcoming discussion will show why Proposition 1 must be correct. (see Remark 3.) Indeed, we need the result of the following lemma later to prove the existence and uniqueness of the solution for problem (3.1).

**Lemma 3.1** (Lemma 1, [1]). Let $D \subset \mathbb{R}^d$ be a bounded Lipschitz domain. For every $\beta > 0$, there exists a constant $C(\beta, D) > 0$ such that

$$\int_{\partial D} u^2 \, dx \leq \beta \int_D |\nabla u|^2 \, dx + C \int_D u^2 \, dx$$

for all $u \in H^1(D)$.

**Remark 2.** For $\beta > 0$ the constant $C$ in Lemma 3.1 has the form of $C = 1/\beta \lambda_1(1/\beta)$ in which for a parameter $\alpha > 0$, value $\lambda_1(\alpha)$ is the first eigenvalue of the problem:

$$\begin{cases}
\Delta u = \lambda(\alpha)u, & \text{in } D, \\
\frac{\partial u}{\partial n} = \alpha u, & \text{on } \partial D.
\end{cases}$$
Definition 3.2. We say that a bounded domain $D \subset \mathbb{R}^d$ satisfies a uniform interior sphere condition, if there exists an $r_0 > 0$, such that for each point $x \in \partial D$ one can find a ball $B$ of radius $r_0$, satisfying $B \subset D$ and $x \in \partial B$.

Lemma 3.3 (Theorem 2.4, [11]). If $D \subset \mathbb{R}^d$, satisfies a uniform interior sphere condition of radius $2r_0$, then

$$\lambda^D_1(\alpha) \leq \lambda^{B(0,r_0)}_1(\alpha).$$

Remark 3. There exists a fixed value of $r_0$ such that for all time $0 \leq t \leq T$, the time-dependent domain $\Omega(t)$ with the assumption $H_1$ satisfies the uniform interior sphere condition of radius $r_0$. This fact proves Proposition 1. Furthermore, we can conclude that the constant $C$ in Lemma 3.1 is uniformly bounded when $D = \Omega(t)$ and $0 \leq t \leq T$.

3.2. Weak form of solution. In the following, we want to construct the weak form of the solution of (3.1). To this end, we provide some remarks and lemmas which will be needed to obtain the weak formulation.

Definition 3.4. Let $V$ be $L^2(0, T; H^1(\Omega(t)))$, then $V' = L^2(0, T; (H^1(\Omega(t)))')$ will be the dual space of $V$ (see [23]). We define the Banach space

$$W := \{ u \in V | u_t \in V' \}, \quad \| u \|_W := \| u \|_V + \| u_t \|_{V'},$$

in which $u_t$ represents the distributional derivative of $u$ with respect to time.

Lemma 3.5 (see [25]). Suppose assumption $H_1$ and consider a function $\rho := \rho(t, x) \in W$. If $v$ denote the moving velocity of the interface $\Gamma(t)$, then

$$\frac{d}{dt} \int_{\Omega(t)} \rho \, dx = \int_{\Omega(t)} \frac{\partial \rho}{\partial t} \, dx + \int_{\Gamma(t)} \rho v_n \, dx,$$

in which $v_n = v \cdot n$ is the normal velocity of the boundary.

We can conclude obviously the following proposition from Lemma 3.5.

Proposition 2. Suppose assumption $H_1$ and let $u, v \in W$, then the following integration by parts formula holds

$$\int_{t_1}^{t_2} \langle u, v \rangle_s + \langle v_t, u \rangle_s \, ds = \int_{\Omega(t_2)} u(t_2)v(t_2) \, dx - \int_{\Omega(t_1)} u(t_1)v(t_1) \, dx$$

$$- \int_{t_1}^{t_2} \int_{\Gamma(s)} v_n(u(s)v(s)) \, dxdx,$$

for any $0 \leq t_1 \leq t_2 \leq T$, where $\langle \cdot, \cdot \rangle_s$ indicates the pairing between $(H^1(\Omega(t)))'$ and $H^1(\Omega(t))$.

To get the weak formulation, multiply (3.1) by a test function $\phi \in C^\infty(\mathbb{R} \times \mathbb{R}^d)$, we will have:

$$\int_0^T \int_{\Omega(t)} \partial_t u \phi \, dxdt + \int_0^T \int_{\Omega(t)} \nabla u \cdot \nabla \phi \, dx \, dt + \int_0^T \int_{\Gamma(t)} qu \phi \, dx \, dt = 0. \quad (3.3)$$

By using Proposition 2 for the functions $u$ and $\phi$, we get:

$$\int_{\Omega(T)} u(T)\phi(T) \, dx - \int_{\Omega(0)} u(0)\phi(0) \, dx$$

$$= \int_0^T \int_{\Omega(t)} \phi \partial_t u \, dxdt + \int_0^T \int_{\Omega(t)} u\partial_t \phi \, dx \, dt + \int_0^T \int_{\Gamma(t)} v_nu \phi \, d\sigma. \quad (3.4)$$
Now substitute (3.4) in (3.3), we can get the following weak formulation:

\[-\int_0^T \int_{\Omega(t)} u\partial_t \phi \, dx \, dt + \int_0^T \int_{\Omega(t)} \nabla u \cdot \nabla \phi \, dx \, dt \]

\[+ \int_0^T \int_{\Gamma(t)} (q - v_n)u \phi \, dx \, dt = \int_{\Omega(0)} u(0)\phi(0) \, dx - \int_{\Omega(T)} u(T)\phi(T) \, dx.\]

Therefore, our concept of weak solution will be as follow.

**Definition 3.6.** A weak solution of (3.1) is a function \( u \) that satisfies the following conditions:

(i) \( u \in W \cap C(0, T; L^2(\Omega(t))) \),

(ii) \( \lim_{t \to 0} u(t) \big|_K = u_I \) in \( L^2(K) \) for all compact set \( K \subset \subset \Omega(0) \),

(iii) For all \( \phi \in C^\infty(\mathbb{R} \times \mathbb{R}^d) \) that \( \phi(T, \cdot) = 0 \), we have:

\[-\int_0^T \int_{\Omega(t)} u\partial_t \phi \, dx \, dt + \int_0^T \int_{\Omega(t)} \nabla u \cdot \nabla \phi \, dx \, dt \]

\[+ \int_0^T \int_{\Gamma(t)} (q - v_n)u \phi \, dx \, dt = \int_{\Omega(0)} u_I(x)\phi(0, x) \, dx. \tag{3.5}\]

### 3.3. Construction of the approximate microscopic solutions

In this section, we construct an approximation of the solution to (3.1) which consists in performing a time slicing of the domain and then solve a family of approximating equations in the cylindrical domain.

Let us divide the interval \( I = [0, T] \), into subintervals \( 0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T \), (let \( I_k = [t_k, t_{k+1}] \), \( k = 0, 1, 2, \cdots, N - 1 \) so that \( P := \max_{k=0, \ldots, N-1} |t_k - t_{k+1}| \to 0 \) as \( N \to \infty \). We iteratively solve a parabolic problem in the cylindrical domain \( \Omega(t_k) \times I_k \):

\[
\begin{cases}
  u^k_t = \Delta u^k, & t \in I_k, \ x \in \Omega(t_k), \\
  \nabla u^k \cdot n + qu^k = 0, & t \in I_k, \ x \in \partial \Omega(t_k), \\
  \nabla u^k \cdot N = 0, & t \in I_k, x \in \partial \Omega, \\
  u^k(t_k, x) = \lim_{t \to t_k^-} u^{k-1}(t, x), & x \in \Omega(t_k).
\end{cases}
\tag{3.6}
\]

For \( k = 0 \), let \( u^0(0, x) = u_I(x) \in L^2(\Omega(0)) \) as initial condition. Notice that the iterative initial condition for \( t = t_k \) makes sense thanks to the continuity properties of \( u^{k-1} \), (see Proposition 3).

**Definition 3.7.** We say that a function

\[ u^k \in L^2(t_k, t_{k+1}; H^1(\Omega(t_k))) \bigcap C(t_k, t_{k+1}; L^2(\Omega(t_k))) \]

is a weak solution of (3.6) on \( [t_k, t_{k+1}] \) that satisfies the following condition for all \( \phi \in C^\infty(\mathbb{R} \times \mathbb{R}^d) \),

\[-\int_{t_k}^{t_{k+1}} \int_{\Omega(t_k)} u^k\partial_t \phi \, dx \, dt + \int_{t_k}^{t_{k+1}} \int_{\Omega(t_k)} \nabla u^k \cdot \nabla \phi \, dx \, dt \]

\[+ \int_{t_k}^{t_{k+1}} \int_{\partial \Omega(t_k)} qu^k \phi \, dx \, dt = \int_{\Omega(t_k)} u^k(t_k)\phi(t_k) \, dx - \int_{\Omega(t_k)} u^k(t_{k+1})\phi(t_{k+1}) \, dx. \]

**Proposition 3** (see [21]). Problem (3.6) admits a unique global in time weak solution in the space \( L^2(t_k, t_{k+1}; H^1(\Omega(t_k))) \bigcap C(t_k, t_{k+1}; L^2(\Omega(t_k))) \).
Since \( u^k \in C(t_k, t_{k+1}; L^2(\Omega(t_k))) \), we can define the traces
\[
\begin{align*}
  u^k(t_k) := \lim_{t \to t_k^+} u^k(t), \\
  u^k(t_{k+1}) := \lim_{t \to t_{k+1}^-} u^k(t),
\end{align*}
\]
where the limit is taken in \( L^2(\Omega(t_k)) \). This is the reason why the initial condition makes sense in (3.6). Now we are ready to define the approximate solution
\[
u^P(t, x) = \sum_{k=0}^{N-1} \chi_{[t_k, t_{k+1})} \left( u^k(t, x) \chi\Omega(t_k)(x) \right),
\]
(3.7)
which is defined in the domain
\[
\Omega^P := \bigcup_{k=0}^{N-1} [t_k, t_{k+1}) \times \Omega(t_k).
\]
(3.8)
We denote the boundary of \( \Omega^P \) with
\[
\Gamma^P = \bigcup_{k=0}^{N-1} [t_k, t_{k+1}) \times \partial\Omega(t_k).
\]
Indeed, \( \Omega^P \) makes an approximation of the domain \( \tilde{\Omega} \) such that \( \tilde{\Omega} \subset \Omega^P \). See Figure 2 for a schematic picture.

**Definition 3.8.** Let \( A \) and \( B \) be two non-empty subsets of \( \mathbb{R}^d \). Hausdorff metric \( d_H(A, B) \) is defined by
\[
\begin{align*}
  d_H(A, B) &= \max \left\{ \sup_{a \in A} \inf_{b \in B} d(a, b), \sup_{b \in B} \inf_{a \in A} d(a, b) \right\}.
\end{align*}
\]

**Lemma 3.9** (see [5]). Assume \( H1 \). The domain \( \Omega^P \) converges to \( \tilde{\Omega} \), and \( \Gamma^P \) converges to \( \Gamma \) in the Hausdorff metric. As a consequence, \( \chi\Omega^P \rightharpoonup \chi\tilde{\Omega} \) strongly in norm \( L^p(\mathbb{R} \times \mathbb{R}^d) \) for all \( p < \infty \).

In the two next lemmas, we can see some estimates on the approximate solution \( u^P \).
Lemma 3.10. There holds

(i) \[ \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} \int_{\Omega(t_k)} |\nabla u^P(t)|^2 \, dx \, dt + \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} \int_{\partial\Omega(t_k)} |u^P(t)|^2 \, dx \, dt \leq C, \]

(ii) \[ \sup_{t \in (0,T)} \int_{\Omega(t)} (u^P(t))^2 \, dx \leq C, \]

for some constant C > 0 depending only on \( \tilde{\Omega} \), \( u_I \) and \( T \).

Proof. Fix \( k \) and notice that the pairing \( u^P \) with \( u^P_I \) on \((t_k, t_{k+1}) \times \Omega(t_k)\) makes sense. After integration by parts we get:

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega(t_k)} (u^P)^2 \, dx = \int_{\Omega(t_k)} u^P u^P \, dx = \int_{\Omega(t_k)} (\Delta u^k) u^k \, dx
\]

\[
= \int_{\Omega(t_k)} -|\nabla u^k|^2 \, dx - \int_{\partial\Omega(t_k)} q(u^k)^2 \, d\sigma.
\]

Integrating the former equality on \([t_k, t_{k+1}]\), we obtain:

\[
\frac{1}{2} \int_{\Omega(t_k)} (u^P(t_{k+1}))^2 \, dx - \frac{1}{2} \int_{\Omega(t_k)} (u^P(t_k))^2 \, dx
\]

\[
= - \int_{t_k}^{t_{k+1}} \int_{\Omega(t_k)} |\nabla u^k|^2 \, dx \, ds - \int_{t_k}^{t_{k+1}} \int_{\partial\Omega(t_k)} q(u^k)^2 \, d\sigma \, ds.
\]

Now consider arbitrary \( t \in [t_j, t_{j+1}] \), and sum integrals on the intervals \( I_0, I_1, \ldots, I_j \), and \([t_j, t)\).

\[
\frac{1}{2} \int_{\Omega(t_j)} (u^P(t))^2 \, dx + \sum_{k=0}^{j-1} \int_{t_k}^{t_{k+1}} \int_{\Omega(t_k)} |\nabla u^k|^2 \, dx \, ds + \int_{t_j}^{t} \int_{\Omega(t_j)} |\nabla u^k|^2 \, dx \, ds
\]

\[
+ \sum_{k=0}^{j-1} \int_{t_k}^{t_{k+1}} \int_{\partial\Omega(t_k)} q(u^k)^2 \, d\sigma \, ds + \int_{t_j}^{t} \int_{\partial\Omega(t_j)} q(u^l)^2 \, d\sigma \, ds
\]

\[
\leq \frac{1}{2} \int_{\Omega(0)} (u^P(0))^2 \, dx,
\]

where we have used the relation \( u^k(t_k, \cdot) = u^{k-1}(t_k, \cdot) \chi_{\Omega(t_k)} \). According to Lemma 3.1 for suitable parameter \( \beta \), (see also Remark 3) there is a constant \( C > 0 \) such that

\[
\int_{\Omega(t_j)} (u^P(t))^2 \, dx + \int_{0}^{t} \int_{\Omega^P(t_j)} |\nabla u^P|^2 \, dx \, ds
\]

\[
\leq C \int_{0}^{t} \int_{\Omega^P(t)} |u^P|^2 \, dx \, ds + \int_{\Omega(0)} |u_I|^2 \, dx,
\]

(3.9)

where \( \Omega^P(t) = \{ x : (x, t) \in \Omega^P \} \), and \( \Omega^P \) defines in (3.8). Let

\[
\psi(t) = \int_{0}^{t} \int_{\Omega^P(t)} |u^P|^2 \, dx \, ds,
\]

which is a piecewise \( C^1 \) function. Then by (3.9), we have \( \psi'(t) \leq C \psi(t) + \| u_I \|_{L^2} \) which implies that \( \psi(t) \) is bounded for \( 0 \leq t \leq T \). Again use (3.9) to prove statement (ii). The proof of statement (i) will also accomplish by (3.9) and the trace inequality in Lemma 3.1. \( \square \)
Lemma 3.11. There exists constant $C > 0$ depending only on $\bar{\Omega}$, $u_I$ and $T$, such that
\[
\sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} \int_{\Omega(t_k)} |u^P_I|^2 \, dx \, dt \leq C.
\]

Proof. Due to the classical result in the regularity theory for parabolic PDE, we know that $u^k_I \in L^2(t_k, t_{k+1}; L^2(\Omega(t_k)))$ and we have
\[
\int_{\Omega(t_k)} |u^k_I|^2 \, dx = \int_{\Omega(t_k)} u^k \Delta u^k \, dx = -\frac{1}{2} \int_{\Omega(t_k)} \frac{d}{dt}|\nabla u^k|^2 \, dx - \frac{1}{2} \int_{\partial \Omega(t_k)} \frac{d}{dt} q|u^k|^2 \, d\sigma.
\]
Let us define
\[
\eta^k(t) = \int_{\Omega(t_k)} |\nabla u^k(t)|^2 \, dx + \int_{\partial \Omega(t_k)} q|u^k(t)|^2 \, d\sigma,
\]
then
\[
\int_0^T \int_{\Omega^P(t)} |u^P|^2 \, dx \, dt \leq \eta^0(0) - \eta^N(T) + \sum_{k=1}^{N-1} (\eta^k(t_k) - \eta^{k-1}(t_k)). \tag{3.10}
\]
On the other hand by the trace inequality in Lemma 3.1, we conclude that
\[
\eta^0(0) - \eta^N(T) \leq C(\|u_I\|_{H^1(\Omega(0))} + \|u^P(T)\|_{L^2(\Omega(T))}). \tag{3.11}
\]
Now consider the solution of the first-order linear PDE, $\nabla \psi \cdot \nabla G = q|\nabla G|$ with the Cauchy data $\psi = 0$ on $\partial \Gamma(0)$. The condition $H1$ insures that $\psi$ can obtain by integral
\[
\psi(X(t)) = \int_0^t q(X(s))|\nabla G(s, X(s))| \, ds,
\]
where $X(t)$ is a characteristic solving $X' = \nabla G(t, X)$. Reminding that $n = \frac{\nabla G}{|\nabla G|}$, the solution $\psi$ satisfies the condition
\[
\partial_n \psi = q, \text{ on } \Gamma(t).
\]
Now denote $\Sigma = \Omega^P \setminus \bar{\Omega}$ and notice that $\Sigma(t_k) = \Omega(t_{k-1}) \setminus \overline{\Omega(t_k)}$, although $\Sigma(t)$ converges to $\partial \Omega(t_k)$, when $t \to t_k^+$. Then
\[
\eta^k(t_k) - \eta^{k-1}(t_k) = -\int_{\Sigma(t_k)} |\nabla u^{k-1}(t_k)|^2 \, dx - \int_{\partial \Omega(t_k)} (\partial_n \psi)(u^{k-1})^2 \, d\sigma
\]
\[+ \int_{\partial \Omega(t_k)} |\nabla u^P|^2 \, d\sigma + \int_{\Sigma(t_k)} |\nabla u^P + u^P \nabla \psi|^2 \, dx\]
\[+ \int_{\Sigma(t_k)} |u^P|^2 |\nabla \psi|^2 \, dx \leq C \int_{\Sigma(t_k)} (u^P(t_k))^2 \, dx,
\]
where constant $C$ is an upper bound for $\Delta \psi$ which depends only on $G$ and $q$. We also have used this fact that $u^k = u^{k-1}$ in $\Omega(t_k)$ and so their traces on $\partial \Omega(t_k)$ are equal. Then for simplicity, we denote them by $u^P$. Remind that $\Sigma(t_{k-1}) = \partial \Omega(t_{k-1})$ and $\Sigma(t_k) = \Omega(t_{k-1}) \setminus \overline{\Omega(t_k)}$, we obtain,
\[
\int_{\Sigma(t_k)} (u^P(t_k))^2 \, dx = \int_{t_{k-1}}^{t_k} \frac{d}{dt} \int_{\Sigma(t)} (u^P(t))^2 \, dx
\]
Lemma 3.13. Let

\[
\int_{t_{k-1}}^{t_k} \left( \int_{\Omega(t)} 2u^P(t)u^P_t(t) \, dx - \int_{\Gamma(t)} (u^P(t))^2 \nu_\sigma \, d\sigma \right) \, dt
\]

\[
\leq \int_{t_{k-1}}^{t_k} \int_{\Omega(t)} \frac{1}{2}(u^P_t(t))^2 + 2(u^P(t))^2 \, dx \, dt
\]

\[
+C\|u^P\|_{L^2(t_{k-1}, t_k; H^1(\Omega(t)))}.
\]

Combine the result with (3.10) and (3.11), to get

\[
\int_0^T \int_{\Omega(t)} |u^P_t|^2 \, dx \, dt \leq C(\|u_I\|_{H^1(\Omega(0))} + \|u^P(T)\|_{L^2(\Omega(T))} + \|u^P\|_{L^2(0,T; H^1(\Omega(t))))}.
\]

Now apply Lemma 3.10 to complete the proof. □

Recalling the definition of $u^P$, we obtain the following result from Lemma 3.10 and Lemma 3.11.

Corollary 1. There exists a constant $C > 0$ depending only on $\tilde{\Omega}$, $u_I$ and $T$ and independent of any partition in $\mathcal{P}$ such that

\[
\|u^P\|_{L^2(0,T; H^1(\Omega(t))))} \leq C,
\]

\[
\|u^P\|_{L^2(0,T; L^2(\Omega(t))))} \leq C.
\]

3.4. Existence of solutions. In this section we prove the existence of weak solutions for (3.1).

Lemma 3.12. There is a function $u$ such that the following statement holds (up to extracting a subsequence) when $\mathcal{P} \to 0$,

\[
u^P|_{\tilde{\Omega}} \rightharpoonup u \text{ weakly in } L^2(0,T; H^1(\Omega(t))).
\]

Proof. Since $\tilde{\Omega} \subset \Omega^P$, the statement follows from Corollary 1. □

In the next lemma, we state convergence on the approximated boundary.

Lemma 3.13. Let $u$ be the function defined in Lemma 3.12. Then the following convergence holds for every $\phi \in C^\infty(\mathbb{R} \times \mathbb{R}^d)$, when $\mathcal{P} \to 0$,

\[
\int_0^T \int_{\Gamma^P(t)} u^P \phi \, d\sigma \, dt \to \int_0^T \int_{\Gamma(t)} u \phi \, d\sigma \, dt.
\]

Proof. Let $\Sigma^P = \Omega^P \setminus \tilde{\Omega}$, then similar to the proof of Lemma 3.11, we can find a function $\psi \in H^2(\Sigma^P(t))$ such that $\Delta \psi = \phi$ on $\Gamma(t)$ and $\Gamma^P(t)$. This is equivalent to solve the first-order linear equation $\nabla \psi \cdot \nabla G = \phi |\nabla G|$ in the domain $\Sigma^P(t)$. Furthermore, the estimate $\|\psi\|_{H^2(\Sigma^P(t))} \leq C\|\phi\|_{H^2(\Sigma^P(t))}$ is valid where constant $C$ depends only on $G$. Now we can write

\[
\left| \int_{\Gamma^P(t)} u^P \partial_\nu \psi \, d\sigma - \int_{\Gamma(t)} u^P \partial_\nu \psi \, d\sigma \right| = \left| \int_{\Sigma^P(t)} \nabla u^P \cdot \nabla \psi + u^P \Delta \psi \, d\sigma \right|
\]

\[
\leq \|u^P\|_{H^1(\Omega^P(t))} \|\psi\|_{H^2(\Sigma^P(t))}.
\]

Hence

\[
\left| \int_0^T \left( \int_{\Gamma^P(t)} u^P \phi \, d\sigma - \int_{\Gamma(t)} u^P \phi \, d\sigma \right) \, dt \right| \leq C\|u^P\|_{L^2(0,T; H^1(\Omega^P(t)))} \|\phi\|_{L^2(0,T; H^1(\Sigma^P(t)))}
\]

tends to zero because of Corollary 1 and Lemma 3.9. On the other hand, according to the compactness of the trace operator, we know that $u^P \to u$ strongly in $L^2(0,T; \Gamma(t))$. This completes the proof of the lemma. □
Now, we are ready to state and prove the existence theorem.

**Theorem 3.14** (Existence). The function \( u \) defined in Lemma 3.12 is a weak solution of problem (3.1) in the sense of Definition 3.6.

**Proof.** Let \( \phi \in C^\infty(\mathbb{R} \times \mathbb{R}^d) \) such that \( \phi(T, \cdot) = 0 \). By Proposition 2, we have

\[
- \int_{\Omega(0)} u_I(x) \phi(0, x) \, dx = \int_0^T \frac{d}{dt} \int_{\Omega(s)} u^P \phi \, dx ds
\]

\[
= \int_0^T \int_{\Omega(s)} \Delta u^P \phi + u^P \partial_t \phi \, dx ds + \int_0^T \int_{\Gamma(s)} u^P \phi v_n \, d\sigma ds
\]

\[
= \int_0^T \int_{\Omega(s)} -\nabla u^P \cdot \nabla \phi + u^P \partial_t \phi \, dx ds + \int_0^T \int_{\Gamma(s)} (\partial_n u^P + u^P v_n) \phi \, d\sigma ds.
\]

In order to obtain the statement (iii) in Definition 3.6, we must show that the following term vanishes when \( P \to 0 \), (remind the weak convergence of \( u^P \))

\[
\int_0^T \int_{\Gamma(s)} (\partial_n u^P + qu^P) \phi \, d\sigma ds.
\]

To prove this, note that for a fixed value \( s \), we have

\[
\left| \int_{\Gamma^P(s)} \partial_n u^P \phi \, d\sigma - \int_{\Gamma(s)} \partial_n u^P \phi \, d\sigma \right| \leq \int_{\Sigma^P(s)} \left| \nabla u^P \cdot \nabla \phi + \Delta u^P \phi \right| \, dx
\]

\[
\leq ||u^P||_{H^1(\Sigma^P(s))} \||\phi||_{H^1(\Sigma^P(s))} + \||\Delta u^P||_{L^2(\Sigma^P(s))} \||\phi||_{L^2(\Sigma^P(s))}
\]

\[
\leq (||u^P||_{H^1(\Omega^P(s))} + ||u^P||_{L^2(\Omega^P(s))}) \||\phi||_{H^1(\Sigma^P(s))}.
\]

From Corollary 1 and Lemma 3.9, we conclude that

\[
\int_0^T \int_{\Gamma^P(s)} \partial_n u^P \phi \, d\sigma - \int_{\Gamma(s)} \partial_n u^P \phi \, d\sigma \, ds \leq C \||\phi||_{L^2(0,T;H^1(\Sigma^P(t)))},
\]

converges to zero. Now apply the boundary condition \( \partial_n u^P + qu^P = 0 \) on \( \Gamma^P \) and Lemma 3.13 to show the convergence of (3.12) to zero.

According to Definition 3.6, we need to verify the initial condition. Consider a cylinder \( K \times [0, t_1] \subset \subset \Omega \), so the regularity theory of parabolic equations in cylindrical domains prove that \( u \in C(0, t_1; L^2(K)) \). Obviously, the statement (ii) in Definition 3.6 is valid.

3.5. **Uniqueness of solutions.** In this subsection, the uniqueness of solution will be proven.

**Theorem 3.15** (Uniqueness). The weak solution of the problem (3.1) is uniquely determined in the class of weak solutions.

**Proof.** Assume that \( \hat{u}(t) \), and \( \tilde{u}(t) \) are two solutions to the problem (3.1) and denote \( u(t) = \hat{u}(t) - \tilde{u}(t) \). Let \( a \) be an arbitrary point from the interval \((0, T)\) and \( \delta \in (a, T-a) \). Choose a function \( \phi \) which is defined as follows:

\[
\phi(t) = \begin{cases} u(t) \eta_a(t), & 0 \leq t \leq a + \delta, \\ 0, & a + \delta < t \leq T, \end{cases}
\]

(3.13)

where \( \eta_a(t) \in C^\infty(0, T) \) satisfying \( \eta_a = 1 \) for \( t \in (0, a) \) and \( \eta_a = 0 \) for \( t \in (a, a+T) \). We pick the sequence \( \phi_m \in C^\infty(\mathbb{R} \times \mathbb{R}^d) \), such that

\[
\phi_m \rightarrow \phi \quad \text{strongly in} \quad L^2(0,T;H^1(\Omega(t))).
\]
Substitute $\phi_m$ in (3.5), and apply Proposition 2, it follows that

$$
\int_0^T \int_{\Omega(t)} \phi_m \partial_t u \, dx \, dt = - \int_0^T \int_{\Omega(t)} \nabla u \cdot \nabla \phi_m \, dx \, dt - \int_0^T \int_{\Gamma(t)} qu \phi_m \, dx \, dt,
$$

and then passing to the limit as $m \to \infty$,

$$
\int_0^T \int_{\Omega(t)} \phi \partial_t u \, dx \, dt = - \int_0^T \int_{\Omega(t)} \nabla u \cdot \nabla \phi \, dx \, dt - \int_0^T \int_{\Gamma(t)} qu \phi \, dx \, dt.
$$

Now we substitute $\phi(t)$ from (3.13),

$$
\int_0^a \int_{\Omega(t)} u \partial_t u \, dx \, dt = - \int_0^a \int_{\Omega(t)} |\nabla u|^2 \, dx \, dt - \int_0^a \int_{\Gamma(t)} qu^2 \, dx \, dt - J_\delta, \quad (3.14)
$$

where

$$
J_\delta = \int_a^{a+\delta} \int_{\Omega(t)} \eta u \partial_t u \, dx \, dt + \int_a^{a+\delta} \int_{\Omega(t)} \eta |\nabla u|^2 \, dx \, dt
$$

$$
+ \int_a^{a+\delta} \int_{\Gamma(t)} \eta qu^2 \, dx \, dt = O(\delta).
$$

Again apply Proposition 2, it yields that

$$
\frac{1}{2} \int_0^a \frac{d}{dt} \|u(t)\|_{L_2(\Omega(t))}^2 \, dt = \int_0^a \int_{\Omega(t)} u \partial_t u \, dx \, dt + \frac{1}{2} \int_0^a \int_{\Gamma(t)} v_n u^2 \, dx \, dt
$$

$$
= - \int_0^a \int_{\Omega(t)} |\nabla u|^2 \, dx \, dt + \int_0^a \int_{\Gamma(t)} \left( \frac{1}{2} v_n - q \right) u^2 \, dx \, dt - J_\delta,
$$

where we have used the relation (3.14) in the last equality. Note that by the trace inequality in Lemma 3.1 (see also Remark 3), there is a constant $C > 0$ such that

$$
\int_0^a \frac{d}{dt} \|u(t)\|_{L_2(\Omega(t))}^2 \, dt \leq C \int_0^a \int_{\Omega(t)} u^2 \, dx \, dt - J_\delta,
$$

and then

$$
\|u(a)\|_{L_2(\Omega(a))}^2 \leq C \int_0^a \int_{\Omega(t)} u^2 \, dx \, dt - J_\delta.
$$

From Gronwall’s inequality, we obtain

$$
\|u(a)\|_{L_2(\Omega(a))}^2 \leq - J_\delta e^{Ca}
$$

Now, let $\delta \to 0$ to get $u(a) = 0$.

4. **Convergence rate of homogenization.** In this section, we study heterogeneous problem (2.1). First, we introduce some mathematical notation for effective and two-scale model, and then make a formulate for the homogenized model. Finally, we find a suitable corrector estimate in the time-dependent locally periodic micro-domain.
4.1. **Macroscopic model (upscale form).** A formal locally periodic asymptotic method is applied to derive the upscale model. In order to formulate the upscaled equations and to obtain a closed formula for the effective transport coefficients, we use the notation:

\[
B(t, x) := \{ y \in Y : G(t, x, y) < 0 \},
\]

and

\[
Y(t, x) := Y - B(t, x).
\]

We will prove that the solution of (2.1) converges to the solution of the following upscaled model as \( \epsilon \to 0 \),

\[
\begin{aligned}
\theta(t, x) \partial_t u_0 &= \nabla_x \cdot (D(t, x) \nabla_x u_0) - Q(t, x) u_0, & x \in \Omega, t \in (0, T), \\
\partial_n u_0 &= (D(t, x) \nabla_x u_0) \cdot N = 0, & x \in \Gamma, t \in (0, T), \\
u_0(0, x) &= f(x), & x \in \Omega,
\end{aligned}
\]  

(4.1)

where the porosity \( \theta(t, x) \) of the medium is given by

\[
\theta(t, x) := |Y(t, x)| = 1 - |B(t, x)|,
\]

and

\[
Q(t, x) = \int_{\partial B(t, x)} q(x, y) \, d\sigma(y).
\]

Moreover, the effective diffusivity tensor \( D(t, x) \) is defined by

\[
D(t, x) := \int_{Y(t, x)} \left( I + \nabla_y M(t, x, y) \right) \, dy.
\]

Here \( M := (M_1, M_2, M_3, \cdots, M_d) \) is a vector function which is obtained from the \( x \) and \( t \)-dependent cell problem

\[
\begin{aligned}
\Delta_y M_j(t, x, y) &= 0, & x \in \Omega, y \in Y(t, x), \\
\nabla_y M_j(t, x, y) \cdot n_0 &= -e_j \cdot n_0, & x \in \Omega, y \in \partial B(t, x), \\
M_j(t, x, y) &\text{ is } Y - \text{periodic.}
\end{aligned}
\]

(4.2)

We should remark that \( n_0 \) is the outward normal vector on the boundary \( \partial B(t, x) \) and satisfies the relation (2.2).

**Remark 4.** The solution of problem (4.2) has at least the regularity \( M_j \in C^2((0, T) \times \Omega; W^{2,s}(Y(t, x))) \) for \( 1 \leq s < \infty \). (see Lemma 6.2 in Appendix).

We define the corrector term

\[
u_1(t, x, y) = M(t, x, y) \cdot \nabla_x u_0(t, x).
\]

Obviously, we can see that it satisfies the following problem

\[
\begin{aligned}
\Delta_y u_1(t, x, y) &= 0, & x \in \Omega, y \in Y(t, x), \\
\nabla_y u_1(t, x, y) \cdot n_0 &= -\nabla_x u_0 \cdot n_0, & x \in \Omega, y \in \partial B(t, x), \\
u_1(t, x, y) &\text{ is } Y - \text{periodic.}
\end{aligned}
\]

(4.3)
Remark 5. The averaged diffusion tensor $\mathcal{D}$ is symmetric ($\mathcal{D}_{ij} = \mathcal{D}_{ji}$) and (uniformly in $x$ and $t$) positive definite, i.e. there is some $\theta > 0$ such that: (See Appendix in [28] for the proof)

$$\sum_{i,j=1}^{d} \zeta_i \mathcal{D}_{ij}(t,x) \zeta_j \geq \theta \|\zeta\|^2$$

for all $t \in (0, T)$, $x \in \Omega$, $\zeta \in \mathbb{R}^d$.

Remark 6. By asymptotic expansion technique, one can find the macroscopic model (4.1) from the heterogeneous problem (2.1) similar to what has been shown in [37, 34]. Here, we will show the convergence rate and present a corrector estimate (see Theorem 4.1 for details).

Remark 7. Problem (4.1) is a linear parabolic problem, then the proof of existence and uniqueness of the global in time solution follows in the standard techniques (see Theorem 1.1.2 in [30]).

Here, we state some regularity results for the solutions of problem (4.1) which need to the proof of main results.

Proposition 4. Let $\mathbf{H1}$ and Assumption 2 hold. Then for $u_0$, the solution of (4.1), the following estimates hold:

$$\|\nabla \partial_t u_0\|_{L^2((0,T) \times \Omega)} \leq C,$$

$$\|u_0\|_{L^2((0,T); H^3(\Omega))} \leq C.$$ (4.4)

Proof. We transform the problem (4.1) into a classical non-divergence form

$$\partial_t u_0 - \sum_{j,k=1}^{n} \frac{D_{jk}}{\theta} \frac{\partial^2 u_0}{\partial x_j \partial x_k} - \frac{1}{\theta} (\nabla_x \cdot \mathcal{D}^T) \cdot \nabla u_0 + \frac{Q}{\theta} u_0 = 0.$$ (4.5)

By defining

$$a_{jk} := \frac{D_{jk}}{\theta}, \quad b := \frac{1}{\theta} (\nabla_x \cdot \mathcal{D}^T), \quad c := \frac{Q}{\theta},$$

we have

$$\partial_t u_0 + Lu_0 = 0.$$ (4.6)

By regularity theory for parabolic PDE (see for example Theorem 5.4 in [33], or Theorem B.4.3 in [16]), we know that

$$u_0 \in H^1(0,T; L^2(\Omega)) \cap L^2(0,T; H^3(\Omega)).$$

To have this regularity, we just need the conditions $f \in H^1(\Omega)$, $a_{jk} \in C^0((0,T) \times \Omega)$, $b, c \in L^{s_1}(0,T; L^{s_2}(\Omega))$ where $s_1, s_2 \geq 2$ satisfy $\frac{2}{s_1} + \frac{d}{s_2} < 1$.

Now to obtain the higher regularity (4.4), let $v := \partial_t u_0$ and differentiate PDE (4.5) with respect to $t$. We can easily check that $v$ is the unique weak solution of

$$\begin{cases}
\partial_t v + Lv = L_t u_0, & \text{in } (0,T) \times \Omega \\
\partial_{\nu} v = 0, & \text{on } [0,T] \times \partial \Omega \\
v = \bar{f}, & \text{on } \{t = 0\} \times \Omega,
\end{cases}$$ (4.6)
where

\[ L_t u_0 := - \sum_{j,k=1}^d (\partial_t a_{jk}) \frac{\partial^2 u_0}{\partial x_j \partial x_k} - (\partial_t b) \cdot \nabla u_0 + (\partial_t c) u_0, \]

\[ \tilde{f} := -Lf. \]

Considering $H_1$, Assumption 2 and Remark 4 with

\[ L_t u_0 \in L^2((0,T) \times \Omega), \quad \tilde{f} \in L^2(\Omega), \]

the classical existence theorem (see for example Theorem 1.1.2 in [30]) implies that

\[ v \in L^2(0,T; H^1(\Omega)) \], \quad \partial_t v \in L^2(0,T; (H^1(\Omega))'). \]

It deduces the first relation in (4.4). The second can obtain from the relation $L u_0 = -\partial_t u_0 = -v \in L^2(0,T; H^1(\Omega))$ and the regularity results for the elliptic operator $L$.

4.2. Main result. Now we are ready to state the main result of our paper. Recall that $u_\epsilon$ is the solution of microscopic problem (2.1), $u_0$ is the solution for the macroscopic problem (4.1) and $u_1$ is the solution of the problem (4.3) as well. We are going to prove the following theorem.

**Theorem 4.1.** Under assumptions 1 and 2 for any sufficiently small $\epsilon$ the following estimate holds

\[ \|u_0 + \epsilon u_{1,\epsilon} - u_\epsilon\|_{L^2(0,T; L^2(\Omega_\epsilon(t)))} \leq C\sqrt{\epsilon}, \quad (4.7) \]

where $u_{1,\epsilon} = u_1|_{y = \frac{x}{\epsilon}}$, and the constant $C$ is independent of $\epsilon$.

4.3. Construction of the approximate solution. The main idea for the proof of Theorem 4.1 is that we approximate at the same time all three problems (2.1), (4.1) and (4.3), by considering time slices with a fixed partition $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T$. Then, we show that these approximate solutions satisfy the similar corrector estimates (4.7) uniformly, (i.e. there exists a constant $C > 0$ for all partitions). So, after passing to the limit, the desired estimate will be obtained for the original problem.

To follow this idea, let us define the locally periodic medium on time slices to approximate the microscale model (2.1). As we mentioned in the Section 2.1, denote

\[ J_\epsilon = \left\{ j \in \mathbb{Z}^d \mid \text{dist}(\epsilon j, \partial\Omega) \geq \epsilon \sqrt{d} \right\}. \]

We set also

\[ Q_{\epsilon,j}^k = Q_{\epsilon,j}(t_k) := \left\{ x \in \epsilon(Y + j) \mid G(t_k, x, \frac{x}{\epsilon}) < 0 \right\}, \]

and introduce the perforated domain as follows:

\[ \Omega_\epsilon^k = \Omega_\epsilon(t_k) := \Omega \setminus \Omega_{\epsilon,l}(t_k), \]

where

\[ \Omega_{\epsilon,l}^k = \Omega_{\epsilon,l}(t_k) := \bigcup_{j \in J_\epsilon} Q_{\epsilon,j}(t_k), \]
We refer to [7, 18] for existence and more insight on the role of such cut-off functions.

We start with the macroscopic problem (4.1) and the corrector problem (4.3) on the time slice.

Also, for each $\Omega$ and the effective diffusivity tensor $D$, where the porosity $\theta$ where the homogenization procedure in the form of the following two-scale model

$$\begin{aligned}
\partial_t u^k \cdot \nabla u^k = \nabla \cdot (D(x) \nabla u^k) + Q^k(x)u^k, & \quad x \in \Omega, t \in I_k, \\
\nabla u^k \cdot \mathbf{N} = 0, & \quad x \in \Gamma, t \in I_k, \\
u^k(x, t_k) = \lim_{t \to t_k^-} u^k(x, t), & \quad x \in \Omega.
\end{aligned}$$

For $t_0 = 0$, we let $u^0(x, 0) = f(x)$. Furthermore, we use the notation $\Omega_{\epsilon,b} = \Omega \setminus \bigcup_{j \in J} (\epsilon(Y + j))$, and the smooth cut-off function $\chi_\epsilon(x)$ satisfying $0 \leq \chi_\epsilon(x) \leq 1$, $\chi_\epsilon(x) = 0$ if $x \in \Omega_{\epsilon,b}$ and $\chi_\epsilon(x) = 1$ if dist$(x, \Omega_{\epsilon,b}) \geq$ dist$(\Omega_{\epsilon,l}, \Omega_{\epsilon,b})$ for all $k$. Moreover, $\epsilon |\nabla \chi_\epsilon| \leq C$ and $\epsilon^2 |\Delta \chi_\epsilon| \leq C$, with the constant $C$ independent of $\epsilon$ such that

$$\begin{aligned}
\|1 - \chi_\epsilon\|_{L^2(\Omega)} & \leq \epsilon^{1/2} C, \\
\|\nabla \chi_\epsilon\|_{L^2(\Omega)} & \leq \epsilon^{-1/2} C, \\
\|\Delta \chi_\epsilon\|_{L^2(\Omega)} & \leq \epsilon^{-3/2} C.
\end{aligned}$$

We refer to [7, 18] for existence and more insight on the role of such cut-off functions.

In order to analyze the convergence rate of the homogenization, we also discretize the macroscopic problem (4.1) and the corrector problem (4.3) on the time slice. We start with

$$B^k(x) := \{ y \in Y : G(t_k, x, y) < 0 \},$$

and

$$Y^k(x) := Y - B^k(x).$$

Also, for each $j \in \{1, 2, 3, \ldots, d\}$, consider the following $x$ and $k$-dependent cell problem

$$\begin{aligned}
\Delta_y M^k_j(x, y) = 0, & \quad x \in \Omega, y \in Y^k(x), \\
\nabla_y M^k_j(x, y) \cdot n_0 = -e_j \cdot n_0, & \quad x \in \Omega, y \in \partial B^k(x), \\
M^k_j(x, y) \text{ is } Y \text{- periodic.}
\end{aligned}$$

The solution to this cell problem allows us to write the results of the formal homogenization procedure in the form of the following two-scale model

$$\begin{aligned}
\partial_t \theta^k(x)u^k_0 = \nabla \cdot (D^k(x) \nabla u^k_0) + Q^k(x)u^k_0, & \quad x \in \Omega, t \in I_k, \\
\nabla u^k_0 \cdot \mathbf{N} = 0, & \quad x \in \Gamma, t \in I_k, \\
u^k_0(x, t_k) = \lim_{t \to t_k^-} u^k_0(x, t), & \quad x \in \Omega,
\end{aligned}$$

where the porosity $\theta^k(x)$ of the medium is given by

$$\theta^k(x) := |Y^k(x)| = 1 - |B^k(x)|,$$

and the effective diffusivity tensor $D^k(x)$ is define by

$$D^k(x) := \int_{Y^k(x)} (I + \nabla_y M^k_j(x, y)) \, dy,$$

where $M^k := (M^k_1, M^k_2, M^k_3, \ldots, M^k_d)$, and

$$Q^k(x) = \int_{\partial B^k(x)} q(x, y) \, d\sigma(y).$$

(4.8)
Remark 8. Similar to the definition of approximate solution in (3.7), we can define $u_\epsilon^P$, $u_0^P$ and $u_1^P$. Moreover, same as the Corollary (1), one can obtain
\[
\|u_\epsilon^P\|_{L^2(0,T;H^1(\Omega^\epsilon(t)))} \leq C,
\|u_0^P\|_{L^2(0,T;H^3(\Omega))} \leq C,
\|u_1^P\|_{L^2((0,T)\times\Omega;W^{1,\infty}(Y^P(t),x))} \leq C.
\]

4.4. Proof of Theorem 4.1. In this section, we prove the main result of the paper. The auxiliary results which are stated in the following lemmas are essential in the proof. They mainly concern integral estimates for rapidly oscillating functions with prescribed average. We postpone the proof to the Appendix.

Lemma 4.2. Let $P^k(x, y) \in H^1(\Omega; L^s(Y^k(x)))$ and $p^k(x, y) \in H^1(\Omega; L^s(\partial B^k(x)))$ be uniformly bounded with respect to $k$ for some $s > d$ (remind that $d$ is the dimension of the space) such that $\int_{Y^k(x)} P^k(x, y) \, dy = \int_{\partial B^k(x)} p^k(x, y) \, d\sigma$. Then the inequality
\[
\left| \int_{\Omega_k^c} P^k(x, \frac{x}{\epsilon}) \phi \, dx - \epsilon \int_{\Gamma_k^c} p^k(x, \frac{x}{\epsilon}) \phi \, d\sigma \right| \leq \epsilon C \|\phi\|_{H^1(\Omega_\epsilon(t_k))},
\]
holds for every $\phi \in H^1(\Omega_\epsilon(t_k))$. The constant $C$ does not depend on $\epsilon$ and $k$.

Lemma 4.3. Let $\Pi_\epsilon$ be a subset of $\{x \in \Omega : \text{dist}(x, \partial \Omega) \leq C_0 t\}$. Then the following inequality
\[
\left| \int_{\Pi_\epsilon} \nabla_x u^k_0 \phi \, dx \right| \leq C\epsilon^{3/2} \|\phi\|_{H^1(\Omega_\epsilon(t_k))}
\]
holds for all $\phi \in H^1(\Omega_\epsilon(t_k))$. The constant $C$ does not depend on $\epsilon$ and $k$.

The next lemma provides a relation which is necessary to apply Lemma 4.2. We bring it here without proof.

Lemma 4.4 [(Lemma 5.1, [18])]. The following relation holds
\[
\int_{Y^k(x)} \nabla_x \cdot ((I + \nabla_y M^k) \nabla_x u^k_0) \, dy - \nabla_x \cdot \int_{Y^k(x)} (I + \nabla_y M^k) \nabla_x u^k_0 \, dy
= - \int_{\partial B^k(x)} n_1 \cdot (I + \nabla_y M^k) \nabla_x u^k_0 \, d\sigma,
\]
where $n_1$ is the tangential vector obtained by (2.3).

Proof of Theorem 4.1. We start with a partition $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T$ and let $P := \max_{k=0, \ldots, N} |t_k - t_{k+1}| \to 0$ as $N \to \infty$. Choose a subsequence for which all three sequences $u_\epsilon^P$, $u_0^P$ and $u_1^P$ are weakly convergent as $P \to 0$, (see Remark 8). Then, it is sufficient to prove the following estimate:
\[
\|u_0^P + \epsilon u_1^P - u_\epsilon^P\|_{L^2(0,T;H^1(\Omega^\epsilon(t)))} \leq C\sqrt{\epsilon},
\]
where the constant $C$ is independent of $\epsilon$ and $P$. We define
\[
z_\epsilon^P(t, x) := u_0^P(t, x) + \epsilon \chi_\epsilon(x) u_1^P(t, x, x/\epsilon) - u_\epsilon^P(t, x).
\]
By Remark 8, we know that there exists a constant $C > 0$ independent of $\epsilon$ and $P$ such that (Indeed we need also to apply Lemma 6.3 in Appendix for $u_1^P$)
\[
\|z_\epsilon^P\|_{L^2(0,T;H^1(\Omega^\epsilon(t)))} \leq C. \tag{4.9}
\]
Then
\[
\Delta z^k = \Delta_x u^k(t, x) + \epsilon \chi(x) \Delta_x u^k_1(t, x, y)|_{y=z} + 2\chi(x) \nabla_x \cdot \nabla_y u^k(t, x, y)|_{y=z} \\
\quad + \frac{1}{\epsilon} \chi(x) \Delta_y u^k_1(t, x, y)|_{y=z} + \epsilon \Delta_x \chi(x) u^k_1(t, x, y)|_{y=z} - \Delta_x u^k(t, x) \\
\quad + 2\epsilon \nabla_x \chi(x) \cdot \nabla_x u^k_1(t, x, y)|_{y=z} + 2\nabla_x \chi(x) \cdot \nabla_y u^k_1(t, x, y)|_{y=z}.
\]

Now, apply the equations
\[
\theta^k(x) \partial_t u^k_0 = \nabla_x \cdot (D^k(x) \nabla_x u^k_0) - Q(x) u^k_0, \\
\Delta_y u^k_1(t, x, y) = 0, \\
\partial_t u^k_0 = \Delta_x u^k,
\]
to obtain
\[
\partial_t z^k - \Delta z^k = \frac{1}{\theta^k(x)} \left( \nabla_x \cdot (D^k(x) \nabla_x u^k_0) - Q(x) u^k_0 \right) + \epsilon \chi(x) \partial_t u^k_1|_{y=z} \\
\quad - \Delta_x u^k_0 - \epsilon \chi(x) \Delta_x u^k_1|_{y=z} - 2\chi(x) \nabla_x \cdot \nabla_y u^k_1|_{y=z} \\
\quad - \epsilon \Delta_x \chi(x) u^k_1|_{y=z} - 2\epsilon \nabla_x \chi(x) \cdot \nabla_x u^k_1|_{y=z} \\
\quad - 2\nabla_x \chi(x) \cdot \nabla_y u^k_1|_{y=z}. \tag{4.10}
\]

On the boundary \( \Gamma_{\epsilon}(t_k) \), we have
\[
\frac{\partial z^k}{\partial n\epsilon} = \nabla_x z^k \cdot n\epsilon \\
\quad = \nabla_x u^k_0 \cdot n\epsilon + \epsilon \nabla_x u^k_1 \cdot n\epsilon + \nabla_y u^k \cdot n\epsilon - \nabla_x u^k \cdot n\epsilon \\
\quad = \nabla_x u^k_0 \cdot n\epsilon + \epsilon \nabla_x u^k_1 \cdot n\epsilon + \nabla_y u^k \cdot n\epsilon + \epsilon q(x, \frac{y}{\epsilon}) u^k_1. \tag{4.11}
\]

Now multiply (4.10) by \( \phi \) and use the relation (4.11) to get
\[
\int_{\Omega(t_k)} \partial_t z^k \phi \, dx + \int_{\Omega(t_k)} \nabla_x z^k \cdot \nabla_x \phi \, dx \\
= \int_{\Omega(t_k)} \frac{1}{\theta^k(x)} \left( \nabla_x \cdot (D(x) \nabla_x u^k_0) - Q(x) u^k_0 \right) \phi \, dx \\
\quad + \epsilon \int_{\Omega(t_k)} \chi(x) \partial_t u^k_1|_{y=z} \phi \, dx - \int_{\Omega(t_k)} \Delta_x u^k_0 \phi \, dx \\
\quad - \epsilon \int_{\Omega(t_k)} \chi(x) \Delta_x u^k_1|_{y=z} \phi \, dx - 2\int_{\Omega(t_k)} \chi(x) \nabla_x \cdot \nabla_y u^k_1|_{y=z} \phi \, dx \\
\quad - \int_{\Gamma_{\epsilon}(t_k)} \nabla_x u^k_0 \cdot n\epsilon \phi \, dx - \epsilon \int_{\Gamma_{\epsilon}(t_k)} \nabla_x u^k_1|_{y=z} \cdot n\epsilon \phi \, dx \\
\quad - \int_{\Gamma_{\epsilon}(t_k)} \nabla_y u^k_1|_{y=z} \cdot n\epsilon \phi \, dx + \epsilon \int_{\Gamma_{\epsilon}(t_k)} q(x, \frac{y}{\epsilon}) u^k_1 \phi \, dx \\
\quad - \int_{\Omega(t_k)} \epsilon \Delta_x \chi(x) u^k_1|_{y=z} \phi - 2\epsilon \nabla_x \chi(x) \cdot \nabla_x u^k_1|_{y=z} \phi \\
\quad - 2\int_{\Omega(t_k)} \nabla_x \chi(x) \cdot \nabla_y u^k_1|_{y=z} \phi \, dx.
\]
By applying the boundary condition \( \nabla_y u_1 \cdot n_0 = -\nabla_x u_0 \cdot n_1 \) (see (4.3)), and considering the relation \( n_z = n_0 + \epsilon n_1 + O(\epsilon^2) \), we can write

\[
\int_{\Omega_{\epsilon}(t_k)} \partial_t z^k \phi \, dx + \int_{\Omega_{\epsilon}(t_k)} \nabla_x z^k \cdot \nabla \phi \, dx 
= \int_{\Omega_{\epsilon}(t_k)} \frac{1}{\theta^k(x)} (\nabla_x \cdot (D(x) \nabla_x u^k_0) - Q(x) u^k_0) \phi \, dx 
+ \epsilon \int_{\Omega_{\epsilon}(t_k)} \chi(x) \nabla_{\epsilon} u_1 \big|_{y = \bar{z}} \phi \, dx - \int_{\Omega_{\epsilon}(t_k)} \Delta_x u^k_0 \phi \, dx 
- \epsilon \int_{\Omega_{\epsilon}(t_k)} \chi(x) \Delta_x u^k_1 \big|_{y = \bar{z}} \phi \, dx - 2 \int_{\Omega_{\epsilon}(t_k)} \chi(x) \nabla_x \cdot \nabla_y u^k_1 \big|_{y = \bar{z}} \phi \, dx 
- \epsilon \int_{\Gamma_{\epsilon}(t_k)} \nabla_x u^k_0 \cdot n_1 \phi \, dx - \epsilon \int_{\Gamma_{\epsilon}(t_k)} \nabla_x u^k_1 \big|_{y = \bar{z}} \phi \, dx + \epsilon \int_{\Gamma_{\epsilon}(t_k)} q(x, \frac{x}{\epsilon}) u^k \phi \, dx 
- \int_{\Omega_{\epsilon}(t_k)} \epsilon \Delta_x \chi^{k}_{\epsilon}(x) u^k_1 \big|_{y = \bar{z}} \phi - 2\epsilon \nabla_x \chi^{k}_{\epsilon}(x) \cdot \nabla_x u^k_1 \big|_{y = \bar{z}} \phi 
- 2 \int_{\Omega_{\epsilon}(t_k)} \nabla_x \chi^{k}_{\epsilon}(x) \cdot \nabla_y u^k_1 \big|_{y = \bar{z}} \phi + O(\epsilon^2) ||\phi||_{L^2(\Omega_{\epsilon}(t_k); H^1(\Omega_{\epsilon}(t_k)))}. \tag{4.12}
\]

We take into account the identity

\[
(\nabla_y \cdot \nabla_x u^k_1) \big|_{y = \bar{z}} = \epsilon \nabla_x \cdot \left( \nabla_x u^k_1(x, \frac{x}{\epsilon}) \right) - \epsilon (\Delta_x u^k_1) \big|_{y = \bar{z}},
\]

which gives

\[
\epsilon \int_{\Gamma_{\epsilon}(t_k)} \nabla_x u^k_1 \big|_{y = \bar{z}} \cdot n_z z^k \, d\sigma = \epsilon \int_{\Omega_{\epsilon}(t_k)} \nabla_x u^k_1 \big|_{y = \bar{z}} \cdot \nabla(\chi_z z^k) \, dx 
+ \int_{\Omega_{\epsilon}(t_k)} \chi_z z^k \nabla_x \cdot (\nabla_x u^k_1 \big|_{y = \bar{z}}) \, dx 
= \epsilon \int_{\Omega_{\epsilon}(t_k)} \nabla_x u^k_1 \big|_{y = \bar{z}} \cdot \nabla(\chi_z z^k) + \chi_z z^k \Delta_x u^k_1 \big|_{y = \bar{z}} \, dx 
+ \int_{\Omega_{\epsilon}(t_k)} \chi_z z^k \nabla_y \cdot \nabla_x u^k_1 \big|_{y = \bar{z}} \, dx. \tag{4.13}
\]

Form now on, put \( u^k_{1,\epsilon} = u^k_1 \big|_{y = \bar{z}} \). By substituting \( \phi = z^k \) in (4.12) and using (4.9) and (4.13), we obtain:

\[
\frac{1}{2} \partial_t ||z^k||_{L^2(\Omega_{\epsilon}(t_k))}^2 + \int_{\Omega_{\epsilon}(t_k)} |\nabla_x z^k|^2 \, dx + O(\epsilon^2) 
= - \int_{\Omega_{\epsilon}(t_k)} \nabla_x \cdot (D^k(x) \nabla_x z^k) \, dx + \int_{\Omega_{\epsilon}(t_k)} \frac{1}{\theta^k(x)} (\nabla_x \cdot (D^k(x) \nabla_x u^k_0)) \, z^k \, dx 
- \epsilon \int_{\Gamma_{\epsilon}(t_k)} (\nabla_x u^k_0 + \nabla_x u^k_{1,\epsilon}) \cdot n_z z^k \, d\sigma + \int_{\Omega_{\epsilon}(t_k)} \nabla_x \cdot \nabla_y u^k_{1,\epsilon} \, z^k \, dx 
- \epsilon \int_{\Gamma_{\epsilon}(t_k)} \frac{1}{\theta^k(x)} Q(x) u^k_0 z^k \, dx + \epsilon \int_{\Gamma_{\epsilon}(t_k)} q(x, \frac{x}{\epsilon}) u^k \phi \, d\sigma 
+ \epsilon \int_{\partial\Omega_{\epsilon}(t_k)} \nabla_x u^k_{1,\epsilon} \cdot n_z z^k \, d\sigma 
\]
\[+ \epsilon \int_{\Omega_{t(k)}} \chi_\epsilon(x) \partial_t u_{1,\epsilon}^k z_{\epsilon}^k \, dx\]
\[- \int_{\Omega_{t(k)}} \chi_\epsilon(x) \nabla_x \cdot \nabla_y u_{1,\epsilon}^k z_{\epsilon}^k \, dx + \epsilon \int_{\Omega_{t(k)}} \nabla_x u_{1,\epsilon}^k \cdot \nabla(\chi_\epsilon z_{\epsilon}^k) \, dx\]
\[- \int_{\Omega_{t(k)}} (\epsilon \Delta_x \chi_\epsilon(x) u_{1,\epsilon}^k z_{\epsilon}^k + 2 \epsilon \nabla_x \chi_\epsilon(x) \cdot \nabla_x u_{1,\epsilon}^k + 2 \nabla_x \chi_\epsilon(x) \cdot \nabla_y u_{1,\epsilon}^k) z_{\epsilon}^k \, dx.\]

Then
\[
\frac{1}{2} \partial_t \|z_{\epsilon}^k\|^2_{L^2(\Omega_{t(k)})} + \int_{\Omega_{t(k)}} |\nabla_x z_{\epsilon}^k|^2 \, dx + O(\epsilon^2)
= - \int_{\Omega_{t(k)}} \nabla_x \cdot (\nabla_x u_{0}^k + \nabla_y u_{1,\epsilon}^k) z_{\epsilon}^k \, dx + \int_{\Omega_{t(k)}} \frac{1}{\vartheta_k(x)} (\nabla_x \cdot (D^k(x) \nabla_x u_{0}^k)) z_{\epsilon}^k \, dx
- \epsilon \int_{\Gamma_{t(k)}} (\nabla_x u_{0}^k + \nabla_y u_{1,\epsilon}^k) \cdot n_1 z_{\epsilon}^k \, d\sigma
- \int_{\Omega_{t(k)}} \frac{1}{\vartheta_k(x)} Q(x) u_{0}^k z_{\epsilon}^k \, dx + \epsilon \int_{\Gamma_{t(k)}} q(x, \frac{\vartheta_k(x)}{\epsilon}) u_{0}^k z_{\epsilon}^k \, d\sigma + \epsilon^2 \int_{\Gamma_{t(k)}} q(x, \frac{\vartheta_k(x)}{\epsilon}) u_{1,\epsilon}^k z_{\epsilon}^k \, d\sigma
+ \epsilon \int_{\Omega_{t(k)}} \chi_\epsilon \partial_t u_{1,\epsilon}^k z_{\epsilon}^k \, dx + \epsilon \int_{\Omega_{t(k)}} \nabla_x u_{1,\epsilon}^k \cdot \nabla(\chi_\epsilon z_{\epsilon}^k) \, dx
+ \int_{\Omega_{t(k)}} \nabla_x \cdot \nabla_y u_{1,\epsilon}^k z_{\epsilon}^k \, dx - \int_{\Omega_{t(k)}} \chi_\epsilon \nabla_x \cdot \nabla_y u_{1,\epsilon}^k z_{\epsilon}^k \, dx
- \int_{\Omega_{t(k)}} (\epsilon \Delta_x \chi_\epsilon(x) u_{1,\epsilon}^k z_{\epsilon}^k + 2 \epsilon \nabla_x \chi_\epsilon(x) \cdot \nabla_x u_{1,\epsilon}^k + 2 \nabla_x \chi_\epsilon(x) \cdot \nabla_y u_{1,\epsilon}^k) z_{\epsilon}^k \, dx,
\]
and then, we have the following estimate:
\[
\frac{1}{2} \partial_t \|z_{\epsilon}^k\|^2_{L^2(\Omega_{t(k)})} + \int_{\Omega_{t(k)}} |\nabla_x z_{\epsilon}^k|^2 \, dx + O(\epsilon^2) \leq I_1 + I_2 + \cdots + I_7,
\]
where
\[
I_1 = \int_{\Omega_{t(k)}} \nabla_x \cdot (\nabla_x u_{0}^k + \nabla_y u_{1,\epsilon}^k) z_{\epsilon}^k \, dx - \int_{\Omega_{t(k)}} \frac{1}{\vartheta_k(x)} (\nabla_x \cdot (D^k(x) \nabla_x u_{0}^k)) z_{\epsilon}^k \, dx
+ \epsilon \int_{\Gamma_{t(k)}} (\nabla_x u_{0}^k + \nabla_y u_{1,\epsilon}^k) \cdot n_1 z_{\epsilon}^k \, d\sigma,
\]
\[
I_2 = \int_{\Omega_{t(k)}} \frac{1}{\vartheta_k(x)} Q(x) u_{0}^k z_{\epsilon}^k \, dx - \epsilon \int_{\Gamma_{t(k)}} q(x, \frac{\vartheta_k(x)}{\epsilon}) u_{0}^k z_{\epsilon}^k \, d\sigma,
\]
\[
I_3 = \epsilon^2 \int_{\partial\Omega_{t(k)}} q(x, \frac{\vartheta_k(x)}{\epsilon}) u_{1,\epsilon}^k z_{\epsilon}^k \, d\sigma,
\]
\[
I_4 = \epsilon \int_{\Omega_{t(k)}} \nabla_x u_{1,\epsilon}^k \cdot \nabla(\chi_\epsilon z_{\epsilon}^k) \, dx,
\]
\[
I_5 = \int_{\Omega_{t(k)}} (1 - \chi_\epsilon)(\nabla_x \cdot \nabla_y u_{1,\epsilon}^k) z_{\epsilon}^k \, dx,
\]
\[
I_6 = \epsilon \int_{\Omega_{t(k)}} \chi_\epsilon(x) \partial_t u_{1,\epsilon}^k z_{\epsilon}^k \, dx.
\]
The relation (4.8) helps us to apply Lemma 4.2 and obtain the estimate
\[ I_7 = \left| \int_{\Omega_k} \left( \epsilon \Delta x \chi(x) u_0 \right) dx + 2\epsilon \nabla_x \chi(x) \cdot \nabla_x u_0^k + 2\nabla_x \chi(x) \cdot \nabla_y u_0^k \right| dz. \]
For \( I_7 \), we use \( u^k = M^k \nabla x u_0 \) and \( D^k(x) := \int_{Y^k(x)} (I + \nabla_y M^k) \nabla x u_0^k dy \) and set
\[ P^k(x, y) = \nabla_x \cdot ((I + \nabla_y M^k) \nabla x u_0^k) - \frac{1}{\theta^k(x)} \frac{1}{\nabla_x \cdot \int_{Y^k(x)} (I + \nabla_y M^k) \nabla x u_0^k dy,} \]
\[ p^k(x, y) = -(I + \nabla_y M^k) \cdot n_1. \]
According to Lemma 4.2 and Lemma 4.4, we have \( I_7 \leq \epsilon C \| z_\epsilon^k \|_{H^1(\Omega_k)}. \) For \( I_2 \), let
\[ P^k(x, y) = \frac{1}{\theta^k(x)} Q^k(x) u_0^k, \]
\[ p^k(x, y) = q(x, y) u_0^k. \]
The relation (4.8) helps us to apply Lemma 4.2 and obtain the estimate \( I_2 \leq \epsilon C \| z_\epsilon^k \|_{H^1(\Omega_k)}. \) The term \( I_3 \) clearly satisfies the estimate \( I_3 \leq \epsilon C \| z_\epsilon^k \|_{H^1(\Omega_k)}. \)
By Lemma 4.3 and the properties of \( \chi \), we estimate \( I_4 \leq \epsilon C \| z_\epsilon^k \|_{H^1(\Omega_k)}. \) Application of the regularity results in Proposition 4, \( I_6 \leq \epsilon C \| z_\epsilon^k \|_{H^1(\Omega_k)}. \) For \( I_5 \), by using the properties of \( \chi \) we have \( I_5 \leq \epsilon C \| z_\epsilon^k \|_{H^1(\Omega_k)}. \) Finally, we obtain
\[ \frac{1}{2} \int_{\Omega_k} \left| \nabla_z z_\epsilon^k \right|^2 \ dx \leq \epsilon C \| z_\epsilon^k \|_{H^1(\Omega_k)}. \]
Therefore,
\[ \frac{1}{2} \int_{\Omega_k} \left( z_\epsilon^k (t_{k+1}) \right) dx + \int_{t_k}^{t_{k+1}} \int_{\Omega_k} \left| \nabla_z z_\epsilon^k \right|^2 \ dx \leq \frac{1}{2} \int_{\Omega_k} \left( z_\epsilon^k (t_k) \right) dx + \epsilon C \int_{t_k}^{t_{k+1}} \| z_\epsilon^k \|_{H^1(\Omega_k)}. \]
On the other hand, we have
\[ \int_{t_k}^{t_{k+1}} \| z_\epsilon^k \|_{H^1(\Omega_k)} \leq (t_{k+1} - t_k)^{1/2} \left( \int_{t_k}^{t_{k+1}} \| z_\epsilon^k \|_{H^1(\Omega_k)}^2 \right)^{1/2}, \]
which implies that by summing up the previous inequality from \( k = 0 \) to \( k = N - 1 \), we have
\[ \frac{1}{2} \int_{\Omega_k} \left( z_\epsilon^p (T) \right) dx + \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} \int_{\Omega_k} \left| \nabla_z z_\epsilon^p \right|^2 \ dx \leq \frac{1}{2} \int_{\Omega_k} \left( z_\epsilon^p (0) \right) dx + \epsilon C \sum_{k=0}^{N-1} \left( \int_{t_k}^{t_{k+1}} \| z_\epsilon^p \|_{H^1(\Omega_k)}^2 \right)^{1/2} \leq \frac{1}{2} \int_{\Omega_k} \left( z_\epsilon^p (0) \right) dx + \epsilon C \left( \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} \| z_\epsilon^p \|_{H^1(\Omega_k)}^2 \right)^{1/2}. \]
Then
\[ \| z_\epsilon^p \|_{L^2(0, T; L^2(\Omega^p_\epsilon(t)))}^2 + \| z_\epsilon^p \|_{L^2(0, T; H^1(\Omega^p_\epsilon(t)))}^2 \leq \epsilon C_1 + \epsilon C_2 \| z_\epsilon^p \|_{L^2(0, T; H^1(\Omega^p_\epsilon(t)))}, \]
which in particular, implies
\[ \| z_\epsilon^p \|_{L^2(0, T; H^1(\Omega^p_\epsilon(t)))} \leq \epsilon C_1 + \epsilon C_2 \| z_\epsilon^p \|_{L^2(0, T; H^1(\Omega^p_\epsilon(t)))}. \]
Then, by (4.9) and (4.14) we have

\[ \|z^P\|_{L^2(0,T;H^1(\Omega^P(t)))} \leq C\sqrt{\epsilon}. \]  \hfill (4.15)

By substituting \( z^P = u^P + \epsilon\chi u^P_{\epsilon} - u^P \) in (4.15), and taking the evidence relation \( \|\epsilon u^P_{\epsilon}(1 - \chi)\|_{H^1(\Omega^P(t))} \leq \sqrt{\epsilon}C \), the proof will be complete.

\[ \square \]

5. Conclusions. We studied the concept of homogenization for a parabolic problem with a Robin boundary condition in a time-dependent microstructure. More precisely, we dealt with a locally periodic microstructure in a formulation that allowed the domain to dependent on time and use a level set function to define it. First, we presented a simple approach to construct solutions of the microscopic problem which consist in performing a time slicing of the domain and solving a family of approximating problems in the cylindrical domain. Similar to the periodic case, we applied the idea of oscillating function, which is synchronous with oscillations in either microstructure or coefficients of microscopic problem. However, we focused on locally periodic media. Finally, we provided a family of the approximated macroscopic problem and corresponding corrector results. Next, using a limiting procedure we obtained the desired corrector estimate between the original microscopic problem and its macroscopic approximation.

6. Appendix. In the last part of the appendix, we will prove Lemma 4.2 and Lemma 4.3. Although similar results have been proved in other literature, the main challenge here is that we have a sequence of domains \( \Omega_k(t_k) \) and we desire to obtain some estimation independent of \( k \). First of all, we introduce the method which can help to overcome this challenge. We recall a basic definition of the coordinate transformations in continuum mechanics, see e.g. [10, 15, 17].

**Definition 6.1 (Regular \( C^2 \)-motion).** A function \( \Psi : (0, T) \times \Omega \times Y(0) \rightarrow (0, T) \times \Omega \times Y(t, x) \) is called a regular \( C^2 \)-motion, if

1. \( \Psi \in C^2((0, T) \times \Omega \times Y) \).
2. For each \( x \in \Omega \) and \( t \in (0, T) \), the function \( \Psi(t, x, \cdot) : Y(0) \rightarrow Y(t, x) := \Psi(t, x, Y(0)) \) is bijective.
3. There exist positive constants \( C_1 \) and \( C_2 \) such that

\[ C_1 \leq \det \nabla_x \Psi(t, x, y) \leq C_2, \]

for all \( (t, x, y) \in (0, T) \times \Omega \times Y(0) \).

**Remark 9.** The existence of regular \( C^2 \)-motion mapping is ensured by Assumption 1, the fact that \( G \in C^2([0, T] \times \bar{\Omega} \times Y) \) and \( \partial_y G(t, x, y) \neq 0 \). (See [35]).

**Lemma 6.2.** Let \( P(x, y) \in H^1(\Omega; L^s(Y^k(x))) \) and \( p(x, y) \in H^1(\Omega; L^s(\partial B^k(x))) \) be \( Y \)-periodic functions satisfying \( \int_{Y^k(x)} P(x, y) \, dy = \int_{\partial B^k(x)} p(x, y) \, d\sigma \). Then there exists the \( Y \)-periodic solution \( v \in H^1(\Omega; W^{1,s}(Y^k(x))) \) for the problem

\[ \begin{cases} 
\Delta_y v(x, y) = P(x, y), & y \in Y^k(x), \\
\nabla_y v(x, y) \cdot n_0 = p(x, y), & y \in \partial B^k(x), 
\end{cases} \]  \hfill (6.1)

which satisfies the following estimate for a universal constant \( C > 0 \) independent of \( k \),

\[ \|v\|_{H^1(\Omega; W^{1,s}(Y^k(x)))} \leq C(\|P\|_{H^1(\Omega; L^s(Y^k(x)))} + \|p\|_{H^1(\Omega; L^s(\partial B^k(x)))}). \]
Lemma 5.2, [2]

Lemma 6.3 is necessary for the proof of Lemma 4.2.

Proof. First, consider the regular and satisfies
\( L \) in the problem (6.1) by this map to obtain a new equation for \( \tilde{v} = v \circ \Psi \) in the referential domain \( Y(0) \). The transformed function \( \tilde{v} \) satisfies in a second-order linear PDE \( L_y \tilde{v} = P \circ \Psi \), in which the coefficients of \( L_y \) are \( C^2 \) regular with respect to the \( x \) parameter and continuous with respect to the \( y \) variable. Then \( \tilde{v} \in H^1(\Omega; W^{2,\alpha}(Y(0))) \) and satisfies

\[
\| \tilde{v} \|_{H^1(\Omega; W^{2,\alpha}(Y(0)))} \leq C(\| P \circ \Psi \|_{H^1(\Omega; L^1(\Omega, (Y(0))))} + \| P \circ \Psi \|_{H^1(\Omega; L^1(\Omega, (\partial B(\Omega))))}),
\]

where the constant \( C \) depends only on \( L_y \) and the domain \( Y(0) \). Now use the mapping \( \Psi \) to come back to the original domain and complete the proof. \( \square \)

The next lemma is one of the preliminary tools in two-scale convergence which is necessary for the proof of Lemma 4.2.

Lemma 6.3 (Lemma 5.2, [2]). Let \( \psi(x, y) \in L^2(\Omega; C(Y_c)) \). Then for any positive value of \( \epsilon, \psi(x, \frac{x}{\epsilon}) \) is a measurable function on \( \Omega \) such that

\[
\| \psi(x, \frac{x}{\epsilon}) \|_{L^2(\Omega)} \leq \| \psi(x, y) \|_{L^2(\Omega; C(Y_c))}.
\]

Proof of Lemma 4.2. The proof is inspired by the proof of Lemma 5.3 in [18]. The problem

\[
\begin{align*}
\Delta_y v^k(x, y) &= P^k(x, y) & y \in Y^k(x) \\
\nabla_y v^k(x, y) \cdot n_0 &= p^k(x, y) & y \in \partial B^k(x)
\end{align*}
\]

(6.2)

has a solution which is \( Y \)-periodic in \( y \) and unique up to an additive constant. Let multiply the equation (6.2) by the function \( \phi \in H^1(\Omega_k(t_k)) \) and integrate it over the domain \( \Omega_k(t_k) \),

\[
\begin{align*}
&\left| \int_{\Omega_k} p^k(x, \frac{x}{\epsilon}) \phi \ dx - \epsilon \int_{\Gamma_k^+} p^k(x, \frac{x}{\epsilon}) \phi \ d\sigma \right| \\
&= \left| \int_{\Omega_k^+} \Delta_y v^k(x, y)|_{y=\frac{z}{\epsilon}} \phi \ dx - \epsilon \int_{\Gamma_k^+} p^k(x, \frac{x}{\epsilon}) \phi \ d\sigma \right| \\
&= \epsilon \int_{\Omega_k} \left( \nabla_x [\nabla_y v^k(x, y)|_{y=\frac{z}{\epsilon}}] - \nabla_x \nabla_y v^k(x, y)|_{y=\frac{z}{\epsilon}} \right) \phi \ dx - \epsilon \int_{\Gamma_k^+} p^k(x, \frac{x}{\epsilon}) \phi \ d\sigma \\
&= \epsilon \int_{\Omega_k} (n_0 + cn^k) \nabla_y v^k(x, y)|_{y=\frac{z}{\epsilon}} \phi \ dx - \epsilon \int_{\Omega_k} \nabla_y v^k(x, y)|_{y=\frac{z}{\epsilon}} \nabla_x \phi \ dx \\
&\quad - \epsilon \int_{\Omega_k} \nabla_x \nabla_y v^k(x, y)|_{y=\frac{z}{\epsilon}} \phi \ dx - \epsilon \int_{\Gamma_k^+} p^k(x, \frac{x}{\epsilon}) \phi \ d\sigma \\
&\leq \epsilon^2 \left( \int_{\Omega_k^+} n_0 \cdot \nabla_y v^k(x, y)|_{y=\frac{z}{\epsilon}} \phi \ dx + \epsilon \int_{\Omega_k^+} \nabla_y v^k(x, y)|_{y=\frac{z}{\epsilon}} \nabla_y \phi \ dx \right) \\
&\quad + \epsilon \int_{\Gamma_k^+} \nabla_x \nabla_y v^k(x, y)|_{y=\frac{z}{\epsilon}} \phi \ dx \leq \epsilon \| \nabla_y v^k \|_{H^1(\Omega_k(t_k); C(Y^k(x)))} \| \phi \|_{H^1(\Omega_k(t_k))}
\end{align*}
\]

In the last inequality, Lemma 6.3 is applied. Notice that according to Lemma 6.2, we know that \( \| v^k \|_{H^1(\Omega_k(t_k); C^{1,\alpha}(Y^k(x)))} \) is independent of \( \epsilon \) and \( k \). \( \square \)
Proof of Lemma 4.3. The result is similar to Lemma 4 in [7]. Although, the proof show that the constant $C$ is independent of $\epsilon$, by the existence of $C^2$-motion mapping (Remark 9) we will get the related constant does not depend on $k$ as well. □

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E-mail address: fotouhi@sharif.edu

E-mail address: m.yousefnezhad96@student.sharif.ir; mohsen.yousefnezhad@gmail.com