NORMALITY OF ORTHOGONAL AND SYMPLECTIC
NILPOTENT ORBIT CLOSURES IN POSITIVE
CHARACTERISTIC

HUSILENG XIAO AND BIN SHU

Abstract. In this note we investigate the normality of closures of orthogonal and symplectic nilpotent orbits in positive characteristic. We prove that the closure of such a nilpotent orbit is normal provided that neither type $d$ nor type $e$ minimal irreducible degeneration occurs in the closure, and conversely if the closure is normal, then any type $e$ minimal irreducible degeneration does not occur in it. Here, the minimal irreducible degenerations of a nilpotent orbit are introduced by W. Hesselink in [6] (or see [11] from which we take Table 1 for the complete list of all minimal irreducible degenerations). Our result is a weak version in positive characteristic of [11, Theorem 16.2(ii)], one of the main results of [11] over complex numbers.

1. Preliminaries

1.1. Let $G$ be a connected algebraic group over an algebraically closed field $\mathbb{K}$ and $\mathfrak{g} = \text{Lie}(G)$. A nilpotent orbits of $G$ is an orbit of a nilpotent element in $\mathfrak{g}$ under the adjoint action of $G$. It is well known that any connected reductive group has only finitely many nilpotent orbits. For all classical groups, the nilpotent orbits were parameterized in terms of partitions. The normality of closures of nilpotent orbit of classical group have been studied by several authors. However, there is still an open question to decide the normality of the closures of nilpotent orbits when $\text{char}(\mathbb{K})$, the characteristic $\mathbb{K}$, is positive. Our purpose is to investigate such a problem.

1.2. In 1979 and 1980’s, Kraft-Procesi in [10] and [11] determined the normality of orbit closures for all complex classical groups by using smooth equivalent arguments (with few exceptions of the very even orbits in the special orthogonal group $D_{2l}$ remaining, which was completed by Sommers in [13]). For other types, A. Broer in [1] finished the work on the normality of nilpotent orbit closures (corresponding to two pairwise orthogonal short root) by vanishing result of cohomology of line bundles of flag variety.

In the case when $\text{char}(\mathbb{K})$ is positive, J. F. Thomsen in [14] proved that A. Broer’s result holds in good characteristic and decided the normality of some nilpotent orbit closures (corresponding to two pairwise orthogonal short roots). Generally, the difficulty of extending Kraft-Procesi method for linear general groups over complex numbers to the case of positive characteristic fields is the failure of the statement over $\mathbb{C}$ “if reductive algebraic $G$ acts on affine variety $V$. If $\pi : V \to V_0$ is quotient map, then so is the restriction of $\pi$ to any $G$-stable subvariety”. Donkin
in [2] overcome this difficulty by means of representation theory. He viewed the coordinate ring $\mathbb{K}[V]$ as a $G$-module, then considered certain module filtration of $\mathbb{K}[V]$ (called good filtrations). This enables him to prove that all closures of nilpotent orbits of general linear groups in positive characteristic are normal.

1.3. Recently, E. Goldstein in his doctoral thesis [4] investigated the normality of the closures of nilpotent orbits of orthogonal and symplectic groups. There Goldstein exploited Donkin’s method to the orthogonal and symplectic groups in positive characteristic. He finally obtained Proposition 5.2 of [4] which is crucial to decide the normality of some nilpotent orbit closures. Let us introduce our main result in the next subsections.

1.4. Throughout the paper, we always assume $\mathbb{K}$ is an algebraically closed filed of characteristic $p > 2$. Let $V$ be finite dimensional vector space over $\mathbb{K}$, $G$ be an algebraic group of $O(V)$ or $Sp(V)$ which is determined by a nondegenerate form $(\cdot, \cdot)$ with $(u, v) = \varepsilon(v, u)$ where $\varepsilon \in \{1, -1\}$. Call $V$ a quadratic space of type $\varepsilon$ (shortly an orthogonal space in case $\varepsilon = 1$, a symplectic space in case $\varepsilon = -1$).

Let $g = \mathfrak{so}(V, \mathbb{K}), \mathfrak{sp}(V, \mathbb{K})$ be their Lie algebras. Then the nilpotent orbits $O_{\varepsilon, \sigma}$ under the adjoint action of $G$ in $g$ is completely determined by a partition $\sigma$ of $n = \dim(V)$. The corresponding young diagram is called $\varepsilon$-diagram. We denote by $|\sigma|$ the size of $\sigma$, which is equal to $\sum_{i=1}^{t} ir_i$ for $\sigma = [1^{r_1}2^{r_2}3^{r_3} \ldots t^{r_t}]$.

There is a well-known classification result on nilpotent orbits of $g = \mathfrak{so}(V, \mathbb{K}), \mathfrak{sp}(V, \mathbb{K})$.

**Lemma 1.1.** Let $\sigma = [1^{r_1}2^{r_2}3^{r_3} \ldots t^{r_t}]$. Then the following statements hold.

1. The partition $\sigma$ is 1-diagram if and only if $r_i$ is even for even $i$.
2. The partition $\sigma$ is -1-diagram if and only if $r_i$ is even for old $i$.

**Definition 1.2.** Let $\eta$ be an $\varepsilon$-diagram.

1. An $\varepsilon$-diagram $\sigma$ is called $\varepsilon$-degeneration of $\eta$ if $|\sigma| = |\eta|$ and $O_{\varepsilon, \sigma} \in \overline{O_{\varepsilon, \eta}}$, which is denoted by $\sigma \leq \eta$. This gives an ordering for $\varepsilon$-diagrams.
2. An $\varepsilon$-degeneration $\sigma$ of $\eta$ is called minimal if $\sigma \neq \eta$ and there is no $\varepsilon$-diagram $\nu$ such that $\sigma < \nu < \eta$. Geometrically this means $O_{\varepsilon, \sigma}$ is open in complement of $O_{\varepsilon, \eta}$ in $\overline{O_{\varepsilon, \eta}}$.

Then we have the following observations on $\varepsilon$-degeneration.

**Lemma 1.3.** Let $\sigma = (\sigma_1 \geq \sigma_2 \geq \cdots)$ and $\eta = (\eta_1 \geq \eta_2 \geq \cdots)$ be two $\varepsilon$-diagrams and $|\sigma| = |\eta|$. Then the following statements hold.

1. The inclusion $O_{\varepsilon, \sigma} \subset \overline{O_{\varepsilon, \eta}}$ happens if and only if $\sum_{i=1}^{j} \sigma_i \leq \sum_{i=1}^{j} \eta_i$ for all $j$.
2. Assume that $\sigma \leq \eta$ be an $\varepsilon$-degeneration with the first $r$ rows and the first $s$ columns of both $\sigma$ and $\eta$ coinciding respectively. Denote by $\sigma', \eta'$ the new diagrams obtained by erasing these $r$ rows and $s$ columns from $\sigma$ and $\eta$ respectively, and put $\varepsilon' = (-1)^s$. Then $\sigma' < \eta'$ is an $\varepsilon'$-degeneration and $\text{codim}_{\overline{O_{\varepsilon, \eta}}} O_{\varepsilon, \sigma} = \text{codim}_{\overline{O_{\varepsilon', \eta'}}} O_{\varepsilon', \sigma'}$. 
Table 1. Classification of minimal irreducible degenerations

| type  | a                      | b                      | c                      | d                      |
|-------|------------------------|------------------------|------------------------|------------------------|
| Lie algebra | $\mathfrak{sp}_2$ | $\mathfrak{sp}_{2n}$ | $\mathfrak{so}_{2n+1}$ | $\mathfrak{sp}_{4n+2}$ |
| $\varepsilon$ | $-1$       | $-1$       | $1$                    | $-1$                    |
| $\eta$    | $(2)$       | $(2n)$      | $(2n + 1)$             | $(2n + 1, 2n + 1)$     |
| $\sigma$  | $(1, 1)$    | $(2n - 2, 2)$ | $(2n - 1, 1, 1)$      | $(2n, 2n, 2)$          |
| codim $O_{\varepsilon, \eta}$ | 2     | 2          | 2                      | 2                      |
| $e$       | $\mathfrak{so}_{4n}$  | $\mathfrak{so}_{2n-1}$ | $\mathfrak{sp}_{2n}$ | $\mathfrak{so}_{2n}$        |
| $f$       | 1          | 1          | -1                     | 1                      |
| $g$       | $(2, 2, 1^{2n-4})$    | $(2, 1^{2n-2})$        | $(2, 2, 1^{2n-4})$     |
| $h$       | $(2n - 1, 2n - 1, 1, 1)$ | $(1^{2n+1})$ | $1^{2n}$          | $1^{2n}$          |
|          | 2          | $(4n - 2)$  | $(2n)$                 | $(4n - 2)$             |

Proof. For (1), one can refer to [9]. For (2), it can be proved by the same arguments as the complex number case given in [4]. □

In the setup of the above lemma (2), we say that the $\varepsilon$-degeneration $\sigma \leq \eta$ is obtained from the $\varepsilon'$-degeneration $\sigma' < \eta'$ by adding rows and columns. An $\varepsilon$-degeneration $\sigma \leq \eta$ is called irreducible if it is not obtained by adding rows and columns in a non-trivial way. In [11], all minimal irreducible degenerations are classified as listed in Table I (see also [4, §4.3]). This classification was given in [6] without restriction to character of field.

1.5. The main result of the present paper.

Theorem 1.4. Let $V$ be a vector space over an algebraically closed field $\mathbb{K}$ of $\text{char}(\mathbb{K}) \neq 2$, $G = O(V)$ (resp. $Sp(V)$) be the orthogonal (resp. symplectic) group corresponding to the defining non-degenerate bilinear form $(\cdot, \cdot)$ of type $\varepsilon = 1$ (resp. $\varepsilon = -1$). Then the following statements hold.

(1) For any nilpotent orbit $O_{\varepsilon, \eta}$, its closure $\overline{O_{\varepsilon, \eta}}$ is normal if $\eta$ has neither degeneration of type $d$ nor degeneration of type $e$.

(2) Conversely, for a given nilpotent orbit $O_{\varepsilon, \eta}$, if its closure $\overline{O_{\varepsilon, \eta}}$ is normal then $\eta$ does not contain a degeneration of type $e$.

Here, the types $d$ and $e$ are as listed as in Table I.

The main text of the paper will be devoted to the proof of the above theorem. In the concluding subsection, some examples are presented for demonstrating the main theorem. Our approach is based on Theorem 2.1 by Goldstein (cf. [4]) along with some crucial observations on the separability and the decomposability arising from quadratic spaces under action of reductive groups (see Lemma 2.8 and Claim 3.7), which make Kraft-Procesi’s arguments on smoothly equivalent singularities over complex numbers in [11] revivified in our case.
2. ON SMOOTH PROPERTY OF TWO CANONICAL MAPS

2.1. Let us first recall the main result Proposition 5.2 in [4] which is actually a modular version of [11, Theorem 9.2(ii)], one of the main results of [11] over complex numbers. This result will be important to the proof of Theorem 1.4.

**Theorem 2.1.** (Goldstein [4]) Let $\mathcal{O}$ be a nilpotent orbit of the symplectic or orthogonal group. Then $\mathcal{O}$ is normal if and only if it is normal at all points contained in the orbits $\mathcal{O}_i$ of codimension 2.

**Lemma 2.2.** Assume that the pair $(\eta, \sigma)$ is type $a, b, c$ as in Table 1. Then $\overline{\mathcal{O}}_{\epsilon, \eta}$ is normal at $\mathcal{O}_{\epsilon, \sigma}$.

**Proof.** It is well known that $\overline{\mathcal{O}}_{\epsilon, \eta} = N$, the nilpotent cone of the corresponding Lie algebra, which is normal (cf. [9, Corollary 8.5]). So $\overline{\mathcal{O}}_{\epsilon, \eta}$ is normal at $\mathcal{O}_{\epsilon, \sigma}$. □

**Lemma 2.3.** The closure $\overline{\mathcal{O}}_{\epsilon, \eta}$ is not normal at $\mathcal{O}_{\epsilon, \sigma}$ in the case of type $e$.

**Proof.** The closure $\overline{\mathcal{O}}_{\epsilon, \eta}$ is reducible, which is a non-trivial union of two closures of equal-dimensional SO($V$)-orbits. So it is not normal. Note that $\mathcal{O}_{\epsilon, \sigma}$ is of codimension 2 in $\overline{\mathcal{O}}_{\epsilon, \eta}$. By Theorem 2.1, $\overline{\mathcal{O}}_{\epsilon, \eta}$ is not normal at $\mathcal{O}_{\epsilon, \sigma}$. □

**Remark 2.4.** The closure $\overline{\mathcal{O}}_{\epsilon, \eta}$ is normal at $\mathcal{O}_{\epsilon, \sigma}$ in the case of type $f, g, h$, which comes from the fact that it is the closure of the orbit of a highest weight vector in the adjoint representation (cf. [9, Proposition 8.13] and the Remark following it). We are only concerned with those codimension 2 orbits inside. So we will not use this fact in this paper.

2.2. The canonical maps $\pi$ and $\rho$. Let $V$ and $U$ be two quadratic spaces of type $\epsilon$ and $-\epsilon$ respectively, and $G(U)$ and $G(V)$ the orthogonal or sympletic groups defined by the given quadratic forms on $U$ and $V$ respectively, depending on the values of $\epsilon$ and of $-\epsilon$. Denote by $\mathfrak{g}(U)$ and $\mathfrak{g}(V)$ the Lie algebras of $G(U)$ and of $G(V)$ respectively. Denote by $L(V, U)$ the linear space $K$-spanned by all linear maps between $V$ and $U$. For a given $X \in L(V, U)$, define $X^*$ to be the adjoint map of $X$, this is to say, the unique element in $L(U, V)$ satisfying $(Xv, u) = (v, X^*u)$ for all $v \in V, u \in U$. We consider the following canonical maps:

$$L(V, U) \xrightarrow{\pi} \mathfrak{g}(U) \quad \rho \downarrow \quad \mathfrak{g}(V)$$

where $\pi(X) = X \circ X^*$, $\rho(X) = X^* \circ X$. Define naturally an action of the group $G(U) \times G(V)$ on $L(V, U)$ via $(g, h)X = gXh^{-1}$. Then both $\pi$ and $\rho$ are $G(U)$- and $G(V)$-equivariant. (Note that $G(U)$ and $G(V)$ act on $\mathfrak{g}(U)$ and $\mathfrak{g}(V)$ by adjoint.) Let $L'(V, U) := \{Y \in L(V, U) \mid Y$ is surjective$, simply written as $L'$. We immediately have the following lemma by the same arguments as in the proof of [11, Lemma 4.2].
Lemma 2.5. For any \( Y \in L' \) the stabilizer of \( Y \) in \( G(U) \) is trivial and \( \rho^{-1}(\rho(Y)) \) is an orbit under \( G(U) \).

we also need following lemma in positive character case

2.3. Now for a given quadratic space \( V \), take a nilpotent element \( D \in V \) with conjugacy class \( O_{\varepsilon,\eta} \). Consider the new form on \( V \) given by \( |v, u| := (u, Dv) \). Clearly it is of type \(-\varepsilon\) and its kener is exactly \( \text{Ker}D \). Set \( U = \text{Im}D \), and take \( X : V \to U \) which is defined by the canonical decomposition \( D = I \circ X : V \to U \hookrightarrow V \), where \( I : U \hookrightarrow V \) is the inclusion map. Then we can canonically define a non-degenerate form on \( U \) of type \(-\varepsilon\). Note that \( X^* = I \). We have \( D = I \circ X = X^* \circ X \in g(V) \). Then \( D' := X \circ X^* \) is also a nilpotent element in \( g(U) \). The corresponding young diagram \( \eta' \) of \( D' \) is obtained from the young diagram of \( \eta \) by erasing the first column (see [11, §2.2 and §2.3] and [11, §4.1]). Set \( N_{\varepsilon,\eta} := \pi^{-1}(O_{\varepsilon,\eta'}) \). Note that \( N_{\varepsilon,\eta} \) is stable under \( G(U) \times G(V) \). We then have the following observation by the same arguments as in the proof of [11, Lemma 4.3].

Lemma 2.6. The following statements hold.

1. \( \rho(N_{\varepsilon,\eta}) = \overline{O_{\varepsilon,\eta}} \)
2. \( \rho^{-1}(O_{\varepsilon,\eta}) \) is a single orbit under \( G(U) \times G(V) \) contained in \( N_{\varepsilon,\eta} \cap L' \)
3. \( \pi(\rho^{-1}(O_{\varepsilon,\eta})) = O_{-\varepsilon,\eta'} \)
4. The above three statements are still valid to other degenerations of \( \eta \) by erasing the first column.

2.4. Keep the setup in §2.2. We have the following proposition which is somewhat a weaker version in positive characteristic of [11, Proposition 11.1]. This proposition will play a role similar to [11, Proposition 11.1] and the followed remark in the complex number case.

Proposition 2.7. Maintain the notations as above. Then the following statements hold.

1. The map \( \pi \) is smooth in \( L' \), and \( \pi(L') = \{ d \in g(U) \mid \text{Rank}(D) \geq 2m - n \} \).
2. \( \rho(L') = \{ D \in g(V) \mid \text{Rank}(D) = m \} \) and \( \rho|_{L'} : L' \to \rho(L') \) is a smooth morphism, admitting the fibers isomorphic to \( G(U) \)

Proof. The description here on the images of \( \pi \) and \( \rho \) is the same thing as what has been done in the case of characteristic 0 given in [11, Proposition 11.1]. So we only need prove the remaining assertions.

1. In the proof of [11, Proposition 11.1(1)], the authors calculate explicitly the tangent map \( (d\pi)_X \) of \( \pi \) at all \( X \in L'(V, U) \), showing that \( (d\pi)_X \) is surjective. Their arguments still valid for the algebraically closed field \( \mathbb{K} \) (note that \( \text{char}(\mathbb{K}) > 2 \)). This is to say, in our case we still have that the tangent map \( (d\pi)_X \) is surjective for all \( X \in L'(V, U) \). By [5, Proposition 10.4] \( \pi \) is smooth at all points in \( L'(U, V) \). We complete the proof of (1).

2. First notice that \( L' = L'(V, U) \) is one orbit under \( GL(V) \) by right multiplication and so is \( \rho(L') \) under action \( D \mapsto (g^{-1})^* \circ D \circ g^{-1} \). And \( \rho \) is a \( GL(V) \)-equivariant
morphism. Fix \( X \in L' \) and let \( q \) and \( q' \) be two orbit mappings associated with \( X \) as below

\[
\begin{array}{ccc}
GL(V) & \xrightarrow{q} & GL(V).X \\
\downarrow & & \\
GL(V),\rho(X)
\end{array}
\]

We first prove \( L' \cong GL(V)/H, \rho(L') \cong GL(V)/H' \). Here \( H \) (resp. \( H' \)) is the centralizer of \( X \in L' \) (resp. \( \rho(X) \in \mathfrak{g}(U) \)). So by [7, 12.4] we only need to prove that both \( q \) and \( q' \) are separable. For this by [7, 5.5] we only need to prove that \( q \) and \( q' \) are smooth. In fact, \( L' \) is an open subset of the vector space \( L(V, U) \) and \( J_X(L') = L(V, U) \). Then we have \( \text{Im}((dq)_{\text{Id}}) = \mathfrak{gl}(V) \cdot X \). Here the dot action 
\( \cdot " \) is left multiplication, and \( \text{"Id"} \) stands for the identity element. On the other side, the surjective property of \( X \) implies that \( \mathfrak{gl}(V) \cdot X = L(V, U) \). Hence \( (dq)_{\text{Id}} \) is surjective. Note that \( q \) is an orbit map, thereby \( GL(V) \)-equivariant. By [3, Proposition 10.4], \( q \) is smooth.

Next we prove \( q' \) is smooth. As \( \rho(L') \) is an orbit of \( GL(V) \), by the same arguments as in the last paragraph it suffices for us to prove \( (dq')_{\text{Id}} \) is surjective at the identity element. We have \( (dq')_{\text{Id}}(Z) = -Z^*D - DZ \) with \( D = \rho(X) \in \rho(L') \). We will further show that \( \dim \text{Im}((dq')_{\text{Id}}) = \dim L' - \dim G(U) = \dim \rho(L') \) in the forthcoming Lemma 2.8 (the second equality is due to the fact that \( \rho^{-1}(D) \) is isomorphic to \( G(U) \)). Thus, \( dq' \) is surjective at \( \text{Id} \), thereby surjective at all points in \( GL(V) \). So \( q' \) is smooth. Thus we have \( L' \cong GL(V)/H \) and \( \rho(L') \cong GL(V)/H' \). Note that under the above identification, \( q' = \rho|_{L'} \circ q \). Hence \( \rho|_{L'} \) is smooth. Therefore the proof is completed modulo Lemma 2.8.

**Lemma 2.8.** Keep the notations and assumptions in the above proposition. Then \( \dim \text{Im}(dq')_{\text{Id}} = \dim(L') - \dim(G(U)) = \dim(\rho(L')) \).

**Proof.** We only need to prove the lemma in the case of \( \mathfrak{g}(V) = \mathfrak{sp}(V) \), and omit the arguments for the case of \( \mathfrak{g}(V) = \mathfrak{o}(V) = \mathfrak{so}(V) \) because the latter is the same. So we assume \( \dim(V) = 2l \geq 4 \) (it is trivial when \( \dim(V) = 2 \)), and further take \( \{e_1, \cdots, e_l, e_{1+l}, \cdots, e_{2l}\} \), a basis of \( V \) compatible with the quadratic form \( (\cdot, \cdot) \). This is to say \( (e_i, e_j) = \text{sgn}(i - j)\delta_{\mid i-j \mid} \). Then by a direct computation we have

\[
E^*_{i,j} = \begin{cases} 
E_{j+l,i+l}, & \text{for } i, j \leq l; \\
E_{j-l,i-l}, & \text{for } l < i, j \leq 2l; \\
E_{j-l,i+l}, & \text{for } i \leq l < j; \\
E_{j+l,i-l}, & \text{for } j \leq l < i.
\end{cases}
\]

Now we view \( (dq')_{\text{Id}} \) (composing with \( \mathfrak{g}(V) \hookrightarrow \mathfrak{gl}(V) \)) as an element of \( \text{End}(\mathfrak{gl}(V)) \). We proceed the arguments by different cases.

(1) Suppose \( m \) is even, say \( 2k \). By description of \( \rho(L') \) which is an orbit in the above proposition, we may take \( D \) to be a diagonal matrix of rank \( m \) as below

\[
D = \text{diag}\{1, \cdots, 1, 0, \cdots, 0, -1, \cdots -1, 0, \cdots, 0\},
\]
where \( k \)-times 1 and \( k \)-times \(-1\) foremost and continuously occur in the first \( l \)-block and the last \( l \)-block respectively. It is elementary to check that subspace \( V(i, j) := \mathbb{K}\text{-span}\{E_{i,j}, E_{i,j}^*\} \) are stable under the map \((dq')_{\text{Id}}(Z) = -Z^*D - DZ\).

\[
\mathfrak{gl}(V) = \bigoplus_{i,j \leq l} V(i, j) \bigoplus \bigoplus_{i \leq l < j - 1 \leq i} V(i, j) \bigoplus \bigoplus_{j \leq l < i - l \geq j} V(i, j).
\]

We find the image of restriction of \((dq')_{\text{Id}}\) on each \( V(i, j) \) are independent to \text{char}(\mathbb{K}) \text{ (under assumption char}(\mathbb{K}) \neq 2 \text{ ). and the lemma holds in case of char}(\mathbb{K}) = 0, so complete the proof of the lemma.

(2) Suppose \( m \) is odd, say \( 2k + 1 \). By the previous reason we may take \( D \) to be a matrix of rank \( m \) as below

\[
D = \text{diag}\{1, \ldots, 1, 0, \ldots, 0, -1, \ldots -1, 0, \ldots, 0\} + E_{1,l+1},
\]

where for the diagonal matrix, \( k \)-times 1 and \( k \)-times \(-1\) foremost and continuously occur in the first \( l \)-block and the last \( l \)-block respectively. We construct \( V(i, j) \) (it is enough to do for some pairs of \((i, j)\)) as below

\[
V(i, j) = \begin{cases} 
\mathbb{K}\text{-span}\{E_{i,j}, E_{i,j}^*\}, & \text{for } (i, j) \neq (1, k), (l + 1, k), (k, 1)(k, l + 1), \forall k \leq 2l; \\
\mathbb{K}\text{-span}\{E_{i,j}, E_{j-l,1}, E_{1,j}, E_{j-l,l+1}\}, & \text{for } i = l + 1, l + 1 < j < 2l; \\
\mathbb{K}\text{-span}\{E_{l+1,j}, E_{1,j}, E_{l+j+1}, E_{l+j,l+1}\}, & \text{for } i = l + 1, 1 < j < l; \\
\mathbb{K}\text{-span}\{E_{1,l+1}E_{1,l+1}, E_{i+1,l+1}, E_{i+1,l+1}\}, & \text{for } (i, j) = (1, 1).
\end{cases}
\]

It is easy to check that those subspaces of \( \text{End}(\mathfrak{gl}(V)) \) are stable under \((dq')_{\text{Id}}\) and \( \text{End}(\mathfrak{gl}(V)) \) can be written as direct sum of some of those \( V(i, j) \)’s. By the same argument above we can prove our lemma in this case. So we complete the proof of lemma.

\[\square\]

3. Smoothly Equivalent Singularities

**Definition 3.1.** Let \( U, V \) be two varieties, and \( u \in U, v \in V \). We call singularity of \( U \) at \( u \) **smoothly equivalent** to singularity of \( V \) at \( v \) if there exist a pair of \((W, w)\) with \( W \) a variety containing a point \( w \) and two morphisms \( \varphi, \psi:\)

\[
\begin{array}{ccc}
W & \xrightarrow{\varphi} & U \\
\downarrow{\psi} & & \downarrow{} \\
V & & \\
\end{array}
\]

with \( \varphi(w) = u, \psi(w) = v \) and \( \varphi, \psi \) are smooth at \( w \). This defines an equivalent relation between pointed varieties and denote the equivalence class of \((U, u)\) by \( \text{Sing}(U, u) \).

**Lemma 3.2.** Assume \( \text{Sing}(U, u) = \text{Sing}(V, v) \). Then \( U \) is normal at \( u \) if and only if \( V \) is normal at \( v \).
Proof. As in [11] Remark 12.2, by the structure theorem of smooth map which is valid in arbitrary characteristic (cf. [12] Proposition 3.24c), any smooth map locally can be expressed as $\mathbb{K}^n \times U \rightarrow U$. The lemma follows. (or see [12] Proposition 3.17b)

In spirit of the arguments in [11] for the complex classical groups, we will manage to make Propositions “induction by canceling rows” and “induction by canceling columns” revived in our case.

Theorem 3.3. Assume that $\varepsilon$-degeneration $\sigma \leq \eta$ is obtained from $\varepsilon'$-degeneration $\sigma' < \eta'$ by adding rows and columns and $\operatorname{codim}_{O_{\varepsilon,\eta}} O_{\varepsilon,\sigma} = \operatorname{codim}_{O_{\varepsilon',\eta'}} O_{\varepsilon',\sigma'} = 2$. Then

$$\operatorname{Sing}(\overline{O_{\varepsilon,\eta}}, O_{\varepsilon,\sigma}) = \operatorname{Sing}(\overline{O_{\varepsilon',\eta'}}, O_{\varepsilon',\sigma'}).$$

For this, we will make some necessary preparations. The proof will follow the forthcoming Propositions 3.4 and 3.9.

Proposition 3.4. (Induction by canceling rows) Assume that $\varepsilon$-degeneration $\sigma \leq \eta$ is obtained from $\varepsilon$-degeneration $\sigma < \eta$ by adding rows, then

$$\operatorname{Sing}(\overline{O_{\varepsilon,\eta}}, O_{\varepsilon,\sigma}) = \operatorname{Sing}(\overline{O_{\varepsilon',\eta'}}, O_{\varepsilon',\sigma'}).$$

This proposition corresponds to [11] Proposition 13.4. The proof of [11] Proposition 13.4 is dependent on [11] Proposition 13.1. However, the arguments for the latter can not carry out in positive characteristic, because it seems impossible to find $H'$-stable decomposition used there for arbitrary reductive group. Fortunately, we observe that some desirable decompositions still exist for the groups we are concerned with (we will write out it explicitly, see Lemma 3.6). Before that, we also need the notation of cross section as in [11] 12.4.

Definition 3.5. Let $X$ be a variety with a regular action of an algebraic group $G$. A cross section at a point $x \in X$ is defined to be a locally closed subvariety $S \subseteq X$ such that $x \in S$ and the map $G \times S \rightarrow X, (g, s) \mapsto g \cdot s$, is smooth at the point $(e, x)$. Of course we have $\operatorname{Sing}(S, x) = \operatorname{Sing}(X, x)$.

Proof of Proposition 3.4 Let $V$ be a quadratic space of type $\varepsilon$ of dimension $|\eta|$, and $x \in O_{\varepsilon,\eta} \subset g(V)$. By assumption the diagrams $\eta$ and $\sigma$ are decomposed into $\eta = \nu + \eta'$, $\sigma = \nu + \sigma'$ respectively where $\nu$ is a common row of $(\sigma, \eta)$ and $(\sigma', \eta')$ is obtained from $(\sigma', \eta')$ by adding the diagram $\nu$. Write the decomposition $V = W \oplus V'$ and such that $x = (z, x') \in g(W) \oplus g(V')$, $y = (z, y') \in g(W) \oplus g(V')$ where $y \in O_{\varepsilon,\sigma'}; y' \in O_{\varepsilon',\eta'} \subseteq g(V') \ y' \in O_{\varepsilon',\sigma'} \subseteq g(V')$ and $z \in g(W)$. Now let us consider the following reductive groups $G := GL(V), G' := GL(W) \times GL(V')$, $H := G(V), H' = G(W) \times G(V')$. Then we have $H' = G' \cap H$ and satisfies the assumption of [11] Proposition 13.1, namely

1. $\operatorname{codim}_{O_{\varepsilon,\eta}} G' y = \operatorname{codim}_{O_{\varepsilon,\eta}} G y$.
2. $G' x \cap \eta' = H' x$.
3. $G' x$ is normal in $y$.

The above (1) is due to the property of nilpotent orbits in the type $A$. As to (3), it is the main result of [2]. By the same arguments as in the proof of [11] Proposition 3.4.
we only need to prove that the statement of Proposition 13.1 is still valid to our situation. Hence the proposition will follow Lemma 3.6 presented below. The proof is completed modulo the forthcoming lemma.

**Lemma 3.6.** Maintain the notations as in the above proof. Then \( \text{Sing}(H'x, y) = \text{Sing}(\overline{H'}x, y) \)

**Proof.** Firstly let make the following claim

**Claim 3.7.** There is an \( H' \)-stable decomposition of \( g \) as below

\[
g = h' \oplus M' \oplus M_0 \oplus D
\]

where \( g' = h' \oplus M' \), \( h = h' \oplus M_0 \).

We certify the claim. Suppose that \((\cdot, \cdot)\) is the defining quadratic form of \( G \), i.e. \( G := \{ g \in GL(V) \mid (gv, gu) = (u, v) \} \). For a given \( f \in \text{End}(V) \), define \( f^* \) via \((fu, v) = (u, f^*v) \). Then we have \( G = \{ g \in GL(V) \mid g^{-1} = g^* \} \), and \( g = \{ x \in \text{gl}(V) \mid -x = x^* \} \). The we have \( \text{End}(V) = g + M \), where \( M = \{ x \in \text{End}(V) \mid x = x^* \} \). Set \( M' = \{ x \in g' \mid x = x^* \} \) and \( M_0 = \{ x \in h \mid x(v) \in W \text{ for all } v \in V'; x(w) \in V' \text{ for all } w \in W \} \), \( D = \{ x \in g \mid x(v) \in W \text{ for all } v \in V'; x(w) \in V' \text{ for all } w \in W; x = x^* \} \). It is easy to verified that those spaces are \( H' \)-stable and

\[
g = h' \oplus M' \oplus M_0 \oplus D.
\]

So the claim follows.

Secondly, by the same argument as in the proof of Proposition 13.1, we can complete the proof of the lemma. For readers’ convenience, we give the detailed arguments.

Now we have vector space decomposition

\[
h' = [h', y] \oplus N'_0, \quad M' = [M', y] \oplus \overline{N'}
\]

and

\[
M_0 = [M_0, y] \oplus \overline{N}_0, \quad D = [D, y] \oplus \overline{D}.
\]

Set \( N := N'_0 \oplus \overline{N'} \oplus \overline{N}_0 \oplus \overline{D} \). Then we have

\[
g = [g, y] \oplus N
\]

\[
g' = [g', y] \oplus N', \text{ where } N' := N \cap g',
\]

\[
h = [h, y] \oplus N_0, \text{ where } N_0 := N \cap h,
\]

\[
h' = [h', y] \oplus N'_0, \text{ where } N'_0 := N \cap h'.
\]

So we can define

\[
S := (N + y) \cap Gx,
\]

\[
S' = (N' + y) \cap G'x,
\]

\[
S_0 := (N_0 + y) \cap \overline{G}x,
\]

\[
S'_0 = (N'_0 + y) \cap \overline{G'}x.
\]

From the forthcoming Lemma 3.8 it follows that those \( S, S', S_0, S'_0 \) are cross sections of \( Gx, G'x, \overline{G}x, \overline{G'}x \) respectively under the adjoin action of corresponding groups.
So we have \( \text{Sing}(\overline{H'x}, y) = \text{Sing}(S'_0, y) \), \( \text{Sing}(\overline{Hx}, y) = \text{Sing}(S_0, y) \). Hence it is enough to prove \( \text{Sing}(S'_0, y) = \text{Sing}(S_0, y) \). By the same arguments as in \cite{10}, we have \( S'_0 = S' \cap \mathfrak{h} \). From Property (1) listed in the proof of Proposition \( 3.4 \) and the forthcoming Lemma \( 3.8 \), we have \( \dim S = \dim S' \). From Property (3) listed in the proof of Proposition \( 3.4 \), \( S \) is normal in \( y \). So there exists an open neighborhood \( F \) of \( y \) in \( S \) such that \( F \subset S' \) and \( F \) open in \( S' \). Since we have
\[
S \cap \mathfrak{h} \supseteq S_0 \supseteq S'_0 = S' \cap \mathfrak{h}.
\]
Hence \( F \cap \mathfrak{h} \) is a common open neighborhood of \( y \) in \( S_0 \) and \( S' \). Thus we have \( \text{Sing}(S'_0, y) = \text{Sing}(S_0, y) \). The proof is completed.

The following lemma was a modular version of the arguments \cite{11, 12.4}. It is valid in our situation because the actions of groups are separable. For general construction see \cite[17.11.1]{3}

**Lemma 3.8.** Maintain the notations and assumptions as above. Then \( S, S', S_0, S'_0 \) are cross sections of \( Gx, G'y, \overline{Hx}, \overline{H'y} \) at \( y \) respectively under the adjoin action of corresponding groups. Furthermore, \( \text{codim}_{Gx} Gx = \dim_y S, \text{codim}_{G'y} G'y = \dim_y S' \), \( \text{codim}_{\overline{Hx}} \overline{Hx} = \dim_y S_0 \) and \( \text{codim}_{\overline{H'y}} \overline{H'y} = \dim_y S'_0 \).

**Proof.** We only need to prove the claim in the case of \( S'_0 \) at \( y \). The remaining case is the same. In fact we already have \( \mathfrak{h}' = [\mathfrak{h}', y] \oplus N'_0 \) with \( N'_0 := N \cap \mathfrak{h}' \). Consider the following diagram

\[
\begin{array}{ccc}
H' \times S'_0 & \xrightarrow{j'} & H' \times \{N'_0 + y\} \\
\downarrow \text{Ad} & & \downarrow \text{Ad} \\
\overline{H'x} & \xrightarrow{j} & \mathfrak{h}',
\end{array}
\]

where \( \text{Ad}, \text{Ad}' \) are adjoint actions, and \( j' \) and \( j \) are the inclusion maps. It is easy to calculate the image of tangent map of \( \text{Ad} \) at \( (e, y) \) is \([\mathfrak{h}', y] \oplus N'_0 \) which is equal to \( \mathfrak{h}' \). So \( \text{Ad} \) is smooth at \( (e, y) \). Since base change preserves smoothness, \( \text{Ad}' \) is smooth at \( (e, y) \). So the first part of the lemma follows.

As \( \text{char}(\mathbb{K}) = p > 2 \) is good for \( G(V') \) and \( G(W) \), it is also good for \( H' = G(V') \times G(W) \). So we have \( T_y(H'y) = [\mathfrak{h}', y] \), thereby \( \dim(H'y) = \dim([\mathfrak{h}', y]) \). Combining with the fact that \( T_y(H'y) \oplus T_y(y + N'_0) = [\mathfrak{h}', y] \oplus N'_0 = \mathfrak{h}' \), we have that \( N'_0 \) and \( H'y \) intersect only at \( y \) in a neighborhood \( \widehat{O} \) of \( y \) in \( H'y \), thereby so do \( S' \) and \( H'y \). In this neighborhood, \((\text{Ad}')^{-1}(\widehat{O}) = \widehat{H'} \times y \) where \( \widehat{H'} \) is an open subset of \( H' \) such that \( \widehat{H'} = \tau^{-1}(\widehat{O}) \) with \( \tau \) being an orbit map of \( y \) under adjoint action of \( H' \). The second part of the lemma follows from the following base change

\[
\begin{array}{ccc}
\widehat{H'} \times y & \xrightarrow{j} & H' \times S'_0 \\
\downarrow \text{Ad}' & & \downarrow \text{Ad} \\
\widehat{O} & \xrightarrow{j} & \overline{H'x},
\end{array}
\]
and the fact that base change preserves the relative dimension.

**Proposition 3.9.** (Induction by canceling columns) assume that $\varepsilon$-degeneration $\sigma \leq \eta$ arises from the $\varepsilon'$-degeneration $\sigma' \leq \eta'$ by adding columns and

$$\text{codim}_{\mathcal{O}_{\varepsilon,\eta}} \mathcal{O}_{\varepsilon,\sigma} = \text{codim}_{\mathcal{O}_{\varepsilon',\eta'}} \mathcal{O}_{\varepsilon',\sigma'} = 2.$$  

Then $\text{Sing}(\mathcal{O}_{\varepsilon,\eta}, \mathcal{O}_{\varepsilon,\sigma}) = \text{Sing}(\mathcal{O}_{\varepsilon',\eta'}, \mathcal{O}_{\varepsilon',\sigma'})$.

We will prove the above proposition by the same arguments as in the proof of [11, Proposition 13.1]. Actually, we need to check some details in the case of positive characteristic with some modification.

**Proof.** It is enough to treat the case when the $\varepsilon$-degeneration $\sigma \leq \eta$ arises from the $\varepsilon'$-degeneration $\sigma' \leq \eta'$ by adding a single column. Let $V$ be a quadratic space of type $\varepsilon$ and of dimension $|\eta|$, $U$ be a quadratic space of type $\varepsilon'$ and of dimension $|\eta'|$. By Lemma 2.6, we consider following restriction of map $\pi$ and $\rho$ on locally closed set

$$N_{\varepsilon,\eta} := \pi^{-1}(\overline{\mathcal{O}_{-\varepsilon,\eta'}}) \xrightarrow{\pi} \overline{\mathcal{O}_{-\varepsilon,\eta'}} \xrightarrow{\rho} \mathcal{O}_{\varepsilon,\eta'}.$$  

We claim that there exists $x \in N_{\varepsilon,\eta}$ such that $\pi(x) \in \mathcal{O}_{-\varepsilon',\sigma'}$, $\rho(x) \in \mathcal{O}_{-\varepsilon,\sigma}$ and that $x, \pi, \rho$ satisfy the assumption of Definition 3.1. The arguments will proceed by steps.

(1) We assert that the map $\pi$ is smooth on the open subset set

$$M'_{\varepsilon,\eta} := \pi^{-1}(\mathcal{O}_{-\varepsilon,\eta'} \cup \mathcal{O}_{-\varepsilon,\sigma'}) \cap L'$$

of $N_{\varepsilon,\eta}$. Indeed, $\mathcal{O}_{-\varepsilon,\eta'} \cup \mathcal{O}_{-\varepsilon,\sigma'}$ is open subset of $\overline{\mathcal{O}_{-\varepsilon,\eta'}}$ (note that $\text{codim}_{\mathcal{O}_{-\varepsilon,\eta'}} \mathcal{O}_{-\varepsilon,\sigma'} = 2$). So $M'_{\varepsilon,\eta}$ is an open subset of $N_{\varepsilon,\eta}$. By Lemma 2.6, $\pi(M'_{\varepsilon,\eta}) = \mathcal{O}_{-\varepsilon,\eta'} \cup \mathcal{O}_{-\varepsilon,\sigma'}$. This comes from the following observation. We actually have base changes as below:

$$\begin{array}{ccc}
M'_{\varepsilon,\eta} & \xrightarrow{j} & L' \\
\pi & & \pi \\
\mathcal{O}_{-\varepsilon,\eta'} \cup \mathcal{O}_{-\varepsilon,\sigma'} & \xrightarrow{j} & \pi(L')
\end{array}$$

where $j$'s are inclusions locally closed subvarieties. As base change preserves smoothness, we have $\pi$ is smooth on $M'_{\varepsilon,\eta}$ by Proposition 2.7.

(2) By the same argument as the above (1), we can assert that the map $\rho$ is smooth on the open subset set $M_{\varepsilon,\eta} = \rho^{-1}(\mathcal{O}_{\varepsilon,\eta} \cup \mathcal{O}_{\varepsilon,\sigma})$ of $N_{\varepsilon,\eta}$.

(3) By Lemma 2.6(4), we have that there exist $x \in M_{\varepsilon,\eta} \cap M'_{\varepsilon,\eta}$ such that $\pi(x) \in \mathcal{O}_{-\varepsilon,\sigma'}$ and $\rho(x) \in \mathcal{O}_{\varepsilon,\sigma}$.

Summing up, we complete the proof. □
Remark 3.10. It is possible to omit the assumption codim_{\mathcal{O}_{\varepsilon,\eta}} \mathcal{O}_{\varepsilon,\sigma} = codim_{\mathcal{O}'_{\varepsilon,\eta'}} \mathcal{O}'_{\varepsilon,\sigma'} = 2 in the above proposition by generalizing [11, Proposition 11.1(ii)] to the positive characteristic case.

4. Proof of the main result

4.1. Proof of Theorem 1.4. Owing to Theorem 2.1, we only need to consider all \mathcal{O}_{\varepsilon,\sigma} of codimenion 2 in \mathcal{O}_{\varepsilon,\eta}, which is of course a minimal degeneration. By canceling “rows” and “columns” we obtain a minimal irreducible \varepsilon'-degeneration \sigma' \leq \eta’ (see [11, §3.4]), which is among types a, b, c, e in Table 1. By Lemmas 2.2 and 2.3 the normality of \mathcal{O}_{\varepsilon',\sigma'} at \mathcal{O}'_{\varepsilon',\eta'} is known. From Theorem 3.3 we can clearly determine as the theorem says, the normality of \mathcal{O}_{\varepsilon,\eta} at \mathcal{O}_{\varepsilon,\sigma}. The proof is completed.

Remark 4.1. (1) The reason why we exclude type d is that we are not able to prove Proposition 10.2 of [11] in positive characteristic. The important tool in [11] is Grauert-Riemenschneider vanishing theorem which does not hold ever in positive characteristic. Thus we are not able to obtain the modular version of Proposition 15.4 of [11].

(2) By the arguments in the above proof, one easily has the following observation. If the normality of \mathcal{O}_{\varepsilon,\eta} at \mathcal{O}_{\varepsilon,\sigma} is in type e is known, then we can determine normality of all nilpotent orbits of the orthogonal and symplectic groups. which is under investigation.

(3) In comparison with J. Thomsen’s results on the normality of nilpotent orbit closures in positive characteristic (cf. [14]), our results are more extensively applicable. This is easily seen by some plain examples listed in §4.2 (1) and (2) below.

4.2. Examples. In the concluding subsection with the same notations and assumptions as in the main theorem, we illustrate our main result by some examples.

(1) Let \mathfrak{g} = so_{11} and \eta = [7, 2, 2]). Then the nilpotent variety \mathcal{O}_{1,\eta} is not normal because it has codimension 2 degeneration \mathcal{O}_{1,\sigma} with \sigma = [7, 1, 1, 1, 1, 1]. By erasing first row, the pair (\eta, \sigma) corresponds to the minimal irreducible degeneration ([2, 2], [1, 1, 1, 1, 1]) (by erasing first row ) of type e in Table 1.

(2) Let \mathfrak{g} = sp_{8} and \eta = [6, 1, 1]. Then the nilpotent variety \mathcal{O}_{1,\eta} has only one codimension 2 degeneration \mathcal{O}_{1,\sigma} with \sigma = [4, 2, 2] and by erasing first column the pair (\eta, \sigma)corresponds to minimal irreducible degeneration([5],[3,1,1]) of type c in Table 1.

(3) Let \mathfrak{g} = sp_{14} and \eta = [4, 4, 3, 3]. In this case, we are not able to judge if the nilpotent variety \mathcal{O}_{1,\eta} is normal. It has a codimension 2 degeneration of \mathcal{O}_{1,\sigma} with \sigma = [4, 4, 2, 2, 2]. This case corresponds to the minimal degeneration of type d in Table 1.

References

[1] A. Broer, Line bundles on the cotangent bundle of the flag variety, Invent. Math. 113(1991), 1-20.
[2] S. Donkin, *The normality of closures of conjugacy classes of matrices*, Invent. Math. 101 (1990), 717-736.
[3] Grothendiek, A. et Dieudonné, J., *Elements de géométrie algébrique O-IV*. Publ. Math. de l'I.H.E.S. 11, 20, 24, 32, Paris (1961-67).
[4] E. Goldstein, *Nilpotent Orbits in the Symplectic and Orthogonal Groups*, Doctoral Thesis, Tufts University, 2011.
[5] R. Hartshorne, *Algebraic geometry*, Springer-Verlag, New York, 1977. (GTM 52)
[6] W. Hesselink, * Singularities in the nilpotent scheme of a classical group*, Trans. Amer. Math. Soc., 222 (1976), 1-32.
[7] J. E. Humphreys, *Linear algebraic groups*, Springer-Verlag, New York, 1975. (GTM 21)
[8] J. C. Jantzen, *Representations of algebraic groups*, 2nd ed., American Mathematical Society, Providence, RI, 2003. (MSM 107)
[9] J. C. Jantzen, *Nilpotent orbits in representation theory*, in "Lie Theory (PM 228)," Birkhäuser Boston, 2004, 1-206.
[10] H. Kraft and C. Procesi, *Closures of conjugacy classes of matrices are normal*, Invent. Math. 53 (1979), 227-247.
[11] H. Kraft and C. Procesi, *On the geometry of conjugacy classes in classical groups*, Comment. Math. Helv. 57 (1982), 539-602.
[12] J. S. Milne, *Étale cohomology*, Princeton University Press, Princeton N.J., 1980. (Princeton mathematical series)
[13] E. Sommers, *Normality of very even nilpotent varieties in D2l*, Bull. London Math. Soc. 37 (2005), 351C360.
[14] J. Thomsen, *Normality of certain nilpotent varieties in positive characteristic*, J. Algebra 227 (2000), no. 2, 595-613

DEPARTMENT OF MATHEMATICS, EAST CHINA NORMAL UNIVERSITY, SHANGHAI 200241, CHINA.

E-mail address: hs1523@163.com

DEPARTMENT OF MATHEMATICS, EAST CHINA NORMAL UNIVERSITY, SHANGHAI 200241, CHINA.

E-mail address: bshu@math.ecnu.edu.cn