Tensor product algebras, Grassmannians and Khovanov homology

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Abstract. We discuss a new perspective on Khovanov homology, using categorifications of tensor products. While in many ways more technically demanding than Khovanov’s approach (and its extension by Bar-Natan), this has distinct advantage of directly connecting Khovanov homology to a categorification of \((C^2)\otimes T\), and admitting a direct generalization to other Lie algebras.

While the construction discussed is a special case of that given in [Webb], this paper contains new results about the special case of \(sl_2\) showing an explicit connection to Bar-Natan’s approach to Khovanov homology, to the geometry of Grassmannians, and to the categorified Jones-Wenzl projectors of Cooper and Krushkal. In particular, we show that the colored Jones homology defined by our approach coincides with that of Cooper and Krushkal.

1. Introduction

“Man is a knot, a web, a mesh into which relationships are tied.”

–Antoine Saint-Exupery (1942)

Khovanov homology has proven one of the most remarkable constructions of recent years, and has stimulated a great deal of work in the field of knot homology. One natural question, which has attracted a great deal of attention, is whether the Reshetikhin-Turaev invariants for other Lie algebras and representations have categorifications like Khovanov homology; a general construction of such invariants was given by the author in [Webb], building on a decade’s worth of work by many authors.

From the original construction of Khovanov homology, it’s not easy to see why this should be possible; after all, the early definitions of Khovanov homology had no clear connection to tensor products of representations of \(sl_2\). Our intent in this note is to sketch out a new construction of Khovanov homology which can be generalized to other representations of other Lie algebras.

This construction is a special case of that given in [Webb]; following that paper, it will first be described in Section 2 in purely algebraic language, introducing certain algebras \(T\ell\) whose representation categories categorify the vector space \((C^2)\otimes T\) as a \(sl_2\) module (in a sense that we will make more precise). The results of that section are with a few exceptions special cases of those of [Webb], and many of the proofs will be farmed out.

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Another part of our aim is also to describe the relationship of this construction with geometry. In the case of \( \mathfrak{sl}_2 \), this underlying geometry is that of Grassmannians; for higher rank groups, it is the geometry of Nakajima quiver varieties (see [W eb a, W eb c]). More specifically, the algebra \( T^\ell \) is isomorphic to a convolution algebra defined using the Grassmannian and certain related varieties. This geometry provides a motivation for understanding these algebras, and a more systematic way of thinking about their definition, as well as relating this work to more traditional geometric representation theory. In particular, it shows how our approach is Koszul dual to that of Khovanov [K ho 02] and Stroppel [Str 05]. While a number of related geometric results have appeared in the literature (for example in [W eb d]), this precise connection seems not to have been written before.

Finally, in the last section, we will give a short account of how to precisely match up the construction we have given with Bar-Natan’s construction of Khovanov homology using a quotient of the cobordism category. As shown by Chatav [Cha12], Bar-Natan’s construction applied to cobordisms between flat tangles (what is often called the Temperley-Lieb 2-category) can be interpreted as a 2-category which acts on the derived categories of modules over \( T^\ell \) (for all \( \ell \)). This allows us to show directly that:

**Theorem 1.1.** The knot invariants defined in [W eb b] for the representation \( \mathbb{C}^2 \) of \( \mathfrak{sl}_2 \) agree with Khovanov homology.

We can also interpret the categorified Jones-Wenzl projector of Cooper and Krushkal [C K12] as projection onto a natural subcategory in our picture. This ultimately shows that:

**Theorem 1.2.** The knot invariants defined in [W eb b] for the higher dimensional representations of \( \mathfrak{sl}_2 \) agree with those of [C K12] based on the categorified Jones-Wenzl projector.

2. **Tensor product algebras of \( \mathfrak{sl}_2 \)**

I see but one rule: to be clear. If I am not clear, all my world crumbles to nothing.

– Stendhal (1840)

2.1. **Stendhal diagrams.** We wish to define an algebra as discussed in the introduction.

**Definition 2.1.** A **Stendhal diagram** is an arbitrary number of smooth red and black curves in \( \mathbb{R} \times [0, 1] \) subject to the rules:

- these curves must be oriented downward at each point. In particular, they have no local minima or maxima;
- black curves can intersect other black curves and red curves, but pairs of red curves are not allowed to intersect;
- this collection of curves has no tangencies or triple intersection points.

Each black strand can additionally carry dots that don’t occur at crossing points; we’ll represent a group of a number of dots as a single dot with that number next to it. We’ll consider these up to isotopy that doesn’t change any of these conditions.
Here are two examples of Stendhal diagrams:

![Stendhal Diagrams](image)

Stendhal diagrams have a product structure given by letting $ab$ be given by stacking $a$ on top of $b$, and attempting to attach strands while preserving colors. If this is not possible, then we simply say that the composition is 0.

$ab = \cdot$ \quad $ba = 0$

To make this a bit more explicit, if we have $\ell$ red strands and $k$ black strands, then we can define a map $\kappa: [1, \ell] \to [0, k]$ attached to any generic horizontal slice of a Stendhal diagram sending $h$ to the number of black strands left of the $h$th red strand (counted from left). We must have that the function attached to the top of $b$ ($y = 1$) coincides with that attached to the bottom of $a$ ($y = 0$), or the product is 0.

Fix a field $\mathbb{k}$.

**Definition 2.2.** Let $T_{\ell}^n$ be the algebra spanned over $\mathbb{k}$ by Stendhal diagrams with $\ell$ red strands and $(\ell - n)/2$ black strands modulo the local relations:

(2.1a) $\bigotimes = \bigotimes$  

(2.1b) $\bigotimes = \bigotimes$  

(2.1c) $\bigotimes = 0$  

(2.1d) $\bigotimes = \bigotimes$  

\[\text{By convention, if } \ell \not\equiv n \pmod{2}, \text{ then } T_{\ell}^n = \{0\}.\]
This last equation perhaps requires a little explanation. It should be interpreted as saying that we set to 0 any Stendhal diagram with a slice $y = a$ where the left most strand is black, that is, where $\kappa(1) > 0$.

The degree of a Stendhal diagram is an integer assigned to each diagram, given by the sum of the number of red/black crossings plus twice the number of dots, minus twice the number of black/black crossings. Note that the relations given above are homogeneous for the grading induced by this degree, so we get a grading on $T_{\ell}^\kappa_n$. Since grading adds under composition, this makes $T_{\ell}^\kappa_n$ into a graded algebra.

It will be convenient for us to name several elements of $T_{\ell}^\kappa_n$, which form a generating set:

- Let $y_{i,k}$ denote the degree 2 diagram $\epsilon_k$ with a single dot added on the $i$th strand.
• Let $\psi_{i,\kappa}$ be the diagram that adds a single crossing of the $i$ and $i+1$st strands to $e_\kappa$; if they are separated by a red strand, the crossing should occur to the right of it. The degree of this element is $-2$ if there is no intervening red strand.

• $\iota^+_{i,\kappa}$ denote the element which creates a single crossing between the $i$th black strand of $e_\kappa$ with a red strand to its left if this is possible without creating black crossings (i.e. if $i - 1$ is in the image of $\kappa$); similarly, $\iota^-_{i,\kappa}$ creates crossing with the red strand to the right, if this is possible. These diagrams have degree 1.

Note that $\iota_{\pm i,\kappa}$ is the only one of these elements which has different sequences at top and bottom; if $\kappa$ is the bottom, we let $\kappa_{\pm i}$ be the corresponding top, that is, $\kappa$ with its value at the last/first place it takes value $i$ increased/decreased to $i \pm 1$.

A family of closely related algebras is the cyclotomic nilHecke algebra $\mathcal{R}^\ell$, as discussed in [Lau12, §5.1]. These algebras are indexed by positive integers $\ell$ which consist of diagrams like those above with no red strands, and only the relations (2.1a–2.1c); in place of (2.1h), we have the relation that $y_1^\ell e_0 = 0$.

**Proposition 2.3.** The map $\mathcal{R}^\ell \to T^\ell$ which places a nilHecke diagram to the left of $\ell$ red strands is a homomorphism.

**Proof.** Obviously, the relations (2.1a–2.1c) are unchanged and thus hold. We need only check that the image $y_1 e_0$ of $y_1$ under this homomorphism satisfies $y_1^\ell e_0 = 0$.

This is an immediate consequence of (2.1g) and (2.1h):

\[
\cdots \ell = \begin{array}{c}
\text{\includegraphics[width=0.3\textwidth]{diagram}}
\end{array} = 0.
\]

In fact, this map is injective, as we will show below.

2.2. A faithful representation. These relations may seem strange, but actually, they arise naturally from a faithful representation.

**Definition 2.4.** For each $\kappa$, we define an ideal $I_\kappa \subset R = \mathbb{Z}[Y_1, \ldots, Y_k]$ generated by the complete symmetric functions $h_p(Y_1, \ldots, Y_{\kappa(q)})$ for $p > q - \kappa(q) - 1$, and $h_p(Y_1, \ldots, Y_k)$ for $p > \ell - k$.

By the usual inclusion-exclusion argument, we have that

\[
h_p(Y_1, \ldots, Y_j) = h_p(Y_1, \ldots, Y_m) - e_1(Y_{j+1}, \ldots, Y_m)h_{p-1}(Y_1, \ldots, Y_m) + e_2(Y_{j+1}, \ldots, Y_m)h_{p-2}(Y_1, \ldots, Y_m) - \cdots
\]

so $h_p(Y_1, \ldots, Y_i) \in I_\kappa$ if $p > q - j - 1$ and $j \leq \kappa(q)$. Another useful observation is that if $\kappa(1) > 0$, then $1 \in I_\kappa$.

**Remark 2.5.** As we’ll discuss in Section 3, this quotient ring is the cohomology ring of a particular smooth Schubert cell in its Borel presentation. Thus, this foreshadows a geometric construction of our algebra as discussed in that section.
Lemma 2.6. The algebra $T^\ell_{(i-2)_k}$ acts on the sum $\oplus_k R/I_k$ sending $e_k$ to the obvious projections, and the other elements acting by the formulae:

$$y_{i,k}(f(Y)) = Y_if(Y)$$

$$\psi_{i,k}(f(Y)) = \frac{f(Y) - f(s_i\cdot Y)}{Y_{i+1} - Y_i} \quad (i \notin \text{im } \kappa)$$

$$t^+_{i,k}(f(Y)) = f(Y) \quad (i - 1 \in \text{im } \kappa)$$

$$t^-_{i,k}(f(Y)) = Y_if(Y) \quad (i \in \text{im } \kappa)$$

Since these elements generate the algebra, these formulae determine the representation. In particular, one can work out that for $\psi_{i,k}$ general, one should multiply the Demazure operator above by $Y_i$ raised to the number of red strands in the middle.

Proof. In [Webb, 3.10], it is shown that these operators on sums of copies of the polynomial rings satisfy all the relations of $T^\ell$ except the violating relation (2.1h), that is, they define an action of the algebra $\hat{T}^\ell$ introduced in [Webb].

Next, we wish to check that $\hat{T}^\ell$ preserves the ideals $I_\kappa$, so that the action on the quotients is well-defined. This is essentially tautological for $e_k$ and $y_{i,k}$. The action of $\psi_i$ commutes with multiplication by any polynomial which is symmetric in the variables $Y_i$ and $Y_{i+1}$. Thus, if $i \notin \text{im } \kappa$, we have that the defining polynomials for the ideal $I_\kappa$ are indeed symmetric in these variables, so this ideal is invariant.

Thus, we have reduced to showing this invariance for $t^+_{i,k}$. It’s clear that if $\kappa' \geq \kappa$, then $I_\kappa \subset I_{\kappa'}$. Since $\kappa_i' \geq \kappa$, we have that $t^+_{i,k}$ induces a map. For $t^-_{i,k}$, we have no such inclusion, but we are not trying to check that the identity induces a map. We must instead show that $Y_i h_p(Y_1, \ldots, Y_{\kappa(q)}) \in I_{\kappa(q)}$ for $p > q - \kappa(q) - 1$.

If $\kappa(q) \neq i$, then this is clear from the definition. Now assume $\kappa(q) = i$. As discussed above, if $\kappa(q+1) \geq i+1$, then we have that $h_p(Y_1, \ldots, Y_{\kappa(q)})$ already; the multiplication by $Y_i$ is not even necessary. Thus, we need only consider the case where $\kappa(q+1) = i$. In this case, we have the desired inclusion when $p > q - \kappa(q)$, so we need only consider the case $p = q - i$. Then we are considering

$$Y_i h_{q-i}(Y_1, \ldots, Y_i) = h_{q-i+1}(Y_1, \ldots, Y_i) - h_{q-i+1}(Y_1, \ldots, Y_{i-1}).$$

We have just seen that the former term lies in $I_{\kappa(q)}$, and the latter does by definition.

Finally, it remains to check that this action factors through $T^\ell$. As we observed, $R/I_\kappa = 0$ if $\kappa(1) > 0$, so the relation (2.1h) is immediate modulo $I_\kappa$. □

This action allows one to show, as indicated earlier, that the cyclotomic nilHecke algebra appears as a subalgebra in $T^\ell$.

Theorem 2.7. The map $R^\ell \to T^\ell$ of Proposition 2.1 induces an isomorphism $R^\ell \simeq e_0 T^\ell e_0$.

Proof. Consider an element in the image of $R^\ell$; this is obtained by starting with the idempotent $e_0$, and multiplying it by elements $\psi_i$ and $y_i$. The formulae of Lemma 2.6 show that the action of these elements are given by Demazure operators and multiplication on $R/I_0$. Thus, our representation is compatible with the usual action
on the nilHecke algebra on polynomials. In [Lau12 5.3], Lauda shows that this action induces an isomorphism between the cyclotomic nilHecke algebra and a matrix ring over the cohomology of the Grassmannian. In particular, this action is faithful.

However, any element of kernel of the map to \(T^\ell\) must necessarily be in the kernel of the polynomial action. This shows that the former kernel is trivial.

This shows that we have an injective map \(R^\ell \to e_0 T^\ell e_0\). In order to see that it is surjective as well, we must show that any diagram with \(\kappa = 0\) at both top and bottom can be written as a sum of diagrams where all black strands stay right of all red ones. This is easily achieved using the relations (2.1d) and (2.1g). \(\square\)

2.3. Decategorification. This algebra appears in a number of different ways. Perhaps most significant for us is that it categorifies the tensor product representation of \(\mathfrak{sl}_2\).

**Definition 2.8.** We let \(T^\ell_n\)-\(\text{mod}\) be the category of finitely generated graded left \(T^\ell_n\)-modules.

We have a natural map \(\phi: T^\ell_n \to T^\ell_{n-2}\) given by adding a black strand at far right. This map is a homomorphism but not unital; instead it sends the identity to an idempotent \(e_\phi\) given by the sum of the idempotents \(e_\kappa\) where the rightmost strand is black.

**Definition 2.9.** We let \(\overline{\mathcal{I}}(M) = T^\ell_{n-2} \otimes_{T^\ell_n} M: T^\ell_n\)-\(\text{mod}\) \(\to T^\ell_{n-2}\)-\(\text{mod}\) be the induction functor for this map.

This functor has a biadjoint up to grading shift given by the functor \(\mathcal{E}(M) = e_M \mathcal{I}\). Similarly, we have an inclusion \(i: T^\ell_n \to T^\ell_{n+1}\) by simply adding a new red strand at the far right, and we let \(\mathcal{S}\) be the extension of scalars functor for this map. Let \(I: (\mathbb{C}^2)^{\otimes \ell} \to (\mathbb{C}^2)^{\otimes \ell+1}\) be the inclusion \(v \mapsto v \otimes [\frac{1}{\ell}]\). This has a right adjoint restriction functor that we will not need to consider.

There is a natural collection of left modules \(T^\ell e_\kappa\) over the algebra \(T^\ell\), given by the idempotents defined above. These are projective since they are summands of the left regular module. In terms of pictures, elements of \(P_\kappa = T^\ell e_\kappa\) are diagrams where we have fixed the strands at the bottom to be the sequence associated \(\kappa\), and where we let the elements of \(T^\ell\) act by attaching them at the top. Note that this module can also be built with the functors \(\overline{\mathcal{I}}\) and \(\mathcal{S}\) as follows: if we use \(P_\emptyset\) to denote the unique irreducible module over \(\mathcal{O}^0 \cong \mathcal{E}\), then

\[
(2.3) \quad P_\kappa \cong \mathcal{S}^{k-\kappa(0)} \mathcal{S}^{k-\kappa(1)} \mathcal{S}^{k-\kappa(1)} \mathcal{S}^{k-\kappa(1)} P_\emptyset.
\]

Now, we’ll relate this picture to the tensor product \((\mathbb{C}^2)^{\otimes \ell}\); for notational reasons, it will be easier to think of this as a \(\ell + 1\)-term tensor product with a trivial module spanned by \(\mathbb{C}\) as the first term.

**Theorem 2.10 ([Webb 3.32]).** The sum \(\oplus_n K^0(T^\ell_n) \otimes \mathcal{C}\) is canonically isomorphic to \((\mathbb{C}^2)^{\otimes \ell}\) intertwining the action of the functors \(\mathcal{E}, \overline{\mathcal{I}}, \mathcal{S}\) with the actions of \(E, F, I\). The functors \(\mathcal{E}\) and \(\overline{\mathcal{I}}\) define a categorical action of \(\mathfrak{sl}_2\), in the sense of Chuang and Rouquier [CR08].

It immediately follows from this theorem and (2.3) that this isomorphism sends

\[
(2.4) \quad [P_\kappa] \mapsto p_\kappa := F^{k-\kappa(0)}(F^{k-\kappa(1)} \cdots F^{k-\kappa(1)}(F^{k-\kappa(1)} \mathbb{C} \otimes [\frac{1}{\ell}] \cdots \otimes [\frac{1}{\ell}])).
\]
Proof (sketch). While we do not intend to give all the details, it seems worthwhile to give a sketch of this proof proceeds. If we can establish that (2.4) gives an isomorphism of vector spaces, then it will be clear that this map matches $F$ and $I$. If we establish that this is an isomorphism of vector spaces with bilinear forms where $\oplus_n K^0(T^\ell_n) \otimes \mathbb{C}$ is endowed with the Euler form, and $(\mathbb{C}^2)^{\otimes \ell}$ with the form $\langle - , - \rangle$ induced by the unique invariant form on $\mathbb{C}^2$, then it will follow that the adjoint to $\mathfrak{F}$ is sent to the adjoint of $F$, that is $\mathfrak{C}$ will be sent to $E$.

Thus, we need only establish this match. The key is to check that 

(2.5) \[ \dim \text{Hom}(P_\kappa, P_{\kappa'}) = \langle p_\kappa, p_{\kappa'} \rangle. \]

In [Webb, 3.32], this is done by induction on the number of tensor factors. In the particular case we consider here, the left hand side can also be computed using the cellular basis of Section 2.4 of this paper.

Once we know this, we immediately see that we have a map $a: \oplus_n K^0(T^\ell_n) \otimes \mathbb{C} \to (\mathbb{C}^2)^{\otimes \ell}$, since any linear combination of projectives that vanishes in the Grothendieck group is killed by the Euler form. Thus, the corresponding linear combination of $p_\kappa$'s is in the kernel of the form on $(\mathbb{C}^2)^{\otimes \ell}$ by (2.5); since this form is non-degenerate, this shows that this combination is 0.

The map $a$ is obviously surjective, so we only need to check that $\oplus_n K^0(T^\ell_n) \otimes \mathbb{C}$ has dimension $2^\ell$. This again would follow from the cellular basis in Section 2.4 (since the classes of the cell modules span the Grothendieck group, and $T^\ell_n$ has $(\ell + n) / 2$ cells). An argument which is closer to the proof in [Webb, 3.32] is to show that every projective over $T^\ell_n$ is a summand of $P_\kappa$ where at each step $\kappa$ increases by no more than one. □

Obviously, it’s natural to wonder where the more standard basis given by pure tensors can be found: let 

$$ v_\kappa = F^{\kappa(1)} \otimes F^{\kappa(2) - \kappa(1)} \left[ \begin{array}{c} 1 \\ 0 \\ \vdots \\ 0 \end{array} \right] \otimes \cdots \otimes F^{\kappa(\ell) - \kappa(\ell - 1)} \left[ \begin{array}{c} 1 \\ 0 \end{array} \right]. $$

Note that this vector is 0 if $\kappa(1) > 0$ or if $\kappa(k) - \kappa(k - 1) > 1$ for any $k$.

We write $\kappa \geq \kappa'$ if this inequality holds pointwise. There is another class of modules that corresponds to these vectors, which we call standard modules.

Definition 2.11. The standard module $S_\kappa$ is the quotient of $P_\kappa$ by the submodule spanned by all diagrams with a slice that corresponds to $\kappa' > \kappa$. This is the same the quotient by the submodule spanned by the image of every homomorphism $P_\kappa \to P_\kappa$ for $\kappa' > \kappa$.

Theorem 2.12 ([Webb, 4.5]). The isomorphism $K^0(T^\ell) \otimes \mathbb{C} \cong (\mathbb{C}^2)^{\otimes \ell}$ sends $[S_\kappa] \mapsto s_\kappa$.

Remark 2.13. The decategorification results of this section can be “upgraded” to take into account the grading on $T^\ell$. If we consider the abelian category of finitely generated projective graded $T^\ell$-modules, then the Grothendieck group of this category, which we denote $K^0_q(T^\ell)$ is naturally a $\mathbb{Z}[q, q^{-1}]$-module where $q$ acts by decreasing the grading of the module. The action induced by $\mathfrak{E}$ and $\mathfrak{F}$ on this category doesn’t satisfy the relations of $\mathfrak{sl}_2$, but rather of the quantum group $U_q(\mathfrak{sl}_2)$. See [KL10, Webb] for more details.
2.4. **A natural basis.** When faced with an unfamiliar algebra, one naturally looks for comforting points of familiarity. For the algebras we have introduced, one of these is provided by a basis. The basis vectors are indexed by pairs of certain diagrams: each diagram is based on a Young diagram which fits inside a $k \times (\ell - k)$ box. We'll draw partitions in the French style, with the shortest part at the top; we'll also always give the partition $k$ parts, adding 0’s as necessary, and index these smallest first $\lambda = (\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k)$.

**Definition 2.14.** A backdrop for this diagram is an association of a number between 1 and $\ell$ at the end to each row (which we’ll write in its first boxes), even those whose corresponding parts are 0. In addition, if we use the same number twice, part of the data of a backdrop is to choose an order on the rows with the same number; we’ll use the notation $i_1, \ldots, i_p$ to denote the $p$ instances of $i$. The number of the $j$th row from the top must be $\geq j + \lambda_j$.

To a backdrop $S$, we have an associated function $\kappa$, where $\kappa(p)$ is the number of rows with label $p$. Let $S$ be a backdrop on a Young diagram; we define an element $B_S$ of the algebra $T^\ell$ as the diagram with

- the bottom having a single black line to the right of the $(j + \lambda_j)$th red line corresponding to the $j$th row (the partition condition guarantees that there are no more than one black line between red lines; note that this is independent of the labels on rows) and
- the top with the number of black strands between the $j$th and $(j + 1)$st red strands given by the number of rows with label $j$; the order on rows with the same label allows us to match up rows with black strands at the top.
- The top and bottom both have black strands labeled by rows of the Young diagram; the diagram connects the black strands at the top and the bottom labeled by the same row. This diagram isn’t unique, but we choose one of them arbitrarily.

For example, the partition with $(1, 1, 3, 4)$ and $\ell = 8, k = 4$ with the labels $(4, 1, 7, 1, 8)$ has the associated diagram $B_S$ is

```
    ||||X
    ||X|
    ||XX
```

Often in this combinatorics of partitions, it is useful to think of a partition in a $k \times (\ell - k)$ box with a sign sequence that describes the boundary of the partition. Reading from NW to SE, we write a + when move southwards and a − when we move east. In more explicit terms, we put a $+$ in the $(j + \lambda_j)$th position for each $j$, and fill the rest with −’s.

For two different backdrops $S$ and $T$ of the same Young diagram, we have a vector $C_{S,T} = B_S B_T^*$ (note that if $S$ and $T$ are labelings on different Young diagrams, this product is 0). Note also that the number $k$ of black strands needs to specified beforehand.
Theorem 2.15 ([SW, 5.15]). The vectors $C_{S,T}$ where $S$ and $T$ range over all pairs of backdrops on Young diagrams in a $k \times (\ell - k)$ box form a basis of $T_{\ell-2k}$. In fact, they are a cellular basis of this algebra in the sense of Graham and Lehrer [GL96].

A cellular basis of an algebra, amongst other things, supplies a natural class of modules, the cell modules, which coincide with the standard modules $S_\kappa$.

The cellular basis also makes it easier to check an important faithfulness property for this algebra:

Lemma 2.16. The action of $T_\ell$ on its polynomial representation is faithful.

Proof. Assume that we have an element $k$ of its kernel. Since the kernel is a two-sided ideal, we can multiply at the bottom and top by elements which sweep all strands to the far right, and obtain an element of the kernel $k'$ where both top and bottom have $\kappa = 0$.

This sweeping operation sends the cellular basis vectors with a fixed top and bottom to a linearly independent set, so if $k \neq 0$, then $k' \neq 0$.

The resulting element can be straightened using the relations to be a usual nilHecke diagram to the right of all red strands. This diagram must act trivially on $R/I_0$, which is what we obtain for the polynomial representation when $\kappa = 0$. However, we’ve already seen that $e_0 T_\ell e_0 \cong R_\ell$ acts faithfully on this space, so all of $T_\ell$ acts faithfully. □

2.5. An example. The first interesting example is when $\ell = 2$ and $k = 1$; this corresponds to the weight 0 subspace of $\mathbb{C}^2 \otimes \mathbb{C}^2$.

The algebra $T_0^2$ is 5 dimensional: there are 2 Young diagrams that fit in a $1 \times 1$ box, corresponding to the partitions (0) and (1). Using the label 1 or 2 for (0) is an acceptable backdrop, and for (1), only 2 is an acceptable label.

Thus, we have 5 basis vectors, which are given by:

$$
\begin{array}{cccc}
\boxfill & \boxfill & | & | \\
\boxfill & \boxfill & | & | \\
\end{array}
$$

In the representation of this algebra defined by Lemma 2.6, we have $R = \mathbb{K}[y]$ and $I_{0,0} = (y^2), I_{0,1} = (y), I_{1,1} = R$, so the space on which they act is $\mathbb{K}[y]/(y^2) \oplus \mathbb{K}$; the algebra $T_0^2$ is precisely the endomorphisms of this module as a module over $\mathbb{K}[y]/(y^2)$. Experts will recognize this as Soergel’s description of the principal block of category $O$ for $\mathfrak{sl}_2$.

3. The geometry of Grassmannians

*In these days the angel of topology and the devil of abstract algebra fight for the soul of each individual mathematical domain.*

Hermann Weyl (1939)
3.1. **Definitions.** Let $\text{Gr}(k, \ell)$ be the Grassmannian of $k$-planes in $\mathbb{C}^\ell$. This projective variety has a well-known decomposition into Schubert cells. We first fix a flag $\mathbb{C}^1 \subset \mathbb{C}^2 \subset \cdots \subset \mathbb{C}^{\ell-1} \subset \mathbb{C}^\ell$. For each weakly increasing function $\kappa: [1, \ell] \to [0, k]$, we let

$$X_\kappa = \{ V \in \text{Gr}(k, \ell) \mid \dim(V \cap \mathbb{C}^m) = \kappa(m) \}$$

and also consider its closure, the Schubert variety

$$\bar{X}_\kappa = \{ V \in \text{Gr}(k, \ell) \mid \dim(V \cap \mathbb{C}^m) \geq \kappa(m) \}.$$ 

The function $\kappa(m)$ is often encoded in a partition; the standard choice seems to be the transpose of $(\kappa(\ell), \kappa(\ell-1), \ldots)$. Each Schubert variety has a resolution of singularities of the form

$$\tilde{X}_\kappa = \{ V_0 = \{0\} \subset \cdots \subset V_\ell \mid V_m \subset \mathbb{C}^m, \dim V_m = \kappa(m) \}.$$ 

This has a natural map $\tilde{X}_\kappa \to \bar{X}_\kappa$ forgetting all entries of the flag except for $V_m$. This map is a resolution of singularities since $\tilde{X}_\kappa$ is clearly smooth (it is a tower of Grassmannian fibrations), and it is an isomorphism over the locus $X_\kappa$ (since we are forced to take $V_m = V \cap \mathbb{C}^m$).

Now, let me introduce a closely related collection of varieties whose import will not be immediately clear. We introduce a fibration $p_\kappa: Y_\kappa \to X_\kappa$ where we choose a complete flag on $V_i/V_{i-1}$. That is,

$$Y_\kappa = \{ W_0 = \{0\} \subset \cdots \subset W_k \mid W_{\kappa(m)} \subset \mathbb{C}^m, \dim W_m = m \}.$$ 

Note that this space is actually a smooth Schubert variety in the full flag variety.

3.2. **Convolution.** We now want to use this geometry to define an algebra, using the method of convolution in homology. This method is discussed in much greater detail in [CG97, §2.7]. Whenever we have an algebraic map between smooth projective varieties $Y \to X$, the homology of the fiber product $A = H_*(Y \times_X Y; \mathbb{C})$ inherits a product structure. Informally, this product $a \star b$ is defined by pulling back $a$ and $b$ by the projections $p_{12}$ and $p_{23}$, then forgetting the last and first terms of $Y \times_X Y \times_X Y$, intersecting the resulting classes and pushing forward to $Y \times_X Y$.

For the reader unfamiliar with this technique, we’ll only need to directly apply the definition for a few calculations. First note that pushforward by the diagonal map on the homology of $Y$ induces an inclusion of algebras $\Delta_*: H_*(Y; \mathbb{C}) \to A$; the product structure on homology is intersection product. More general elements can be induced by a space $Z$ with two maps $h_1, h_2$, such that both induce the same map $Z \to X$; in this case, we consider the pushforward $(h_1 \times h_2)[Z] \in H_*(Y \times_X Y)$.

We’ll let $X = \text{Gr}(k, \ell)$ and $Y = \bigsqcup \kappa Y_\kappa$ with $p: Y \to X$ the usual projection. As before, we define $n$ by $\ell - n = 2k$, and denote the resulting convolution algebra by $A'_n$.

We’ll abuse notation, and let $W_m/W_{m-1}$ denote the line bundle on $Y$ whose fiber at each point is given by this line, and let $e(W_m/W_{m-1})$ be the homology class given by the divisor of this line bundle, that is, the Poincaré dual of its Euler class.

Let

$$Z_{i,k,k'} = \{ (W_i, W_j) \in Y_\kappa \times Y_{\kappa'} \mid W_k = W'_k \text{ for all } k \neq i \};$$
this variety is endowed with maps $h_1, h_2: Z_i \to Y$ forgetting the second and first entry of the pair respectively. We’ll also use $Z_0$ to denote the space where we require the flags to be equal.

The primary theorem of this note is that:

**Theorem 3.1.** The algebras $T_n^\ell$ and $A_n^\ell$ are isomorphic via the map

\[
e_{\kappa} \mapsto \Delta_{\kappa}[Y_\kappa] \quad y_m \mapsto e(W_m/W_{m-1}) \quad \psi_{i,\kappa} \mapsto (h_1 \times h_2)[Z_{i,\kappa}] \quad \iota_{i,\kappa}^\pm \mapsto \pm (h_1 \times h_2)[Z_{0,\kappa}^\pm]
\]

How is one to think about this theorem? While I would argue that this is really the correct definition of $T_n^\ell$, and that one should then derive the diagrammatic description, this is just moving the problem around. The key property of $A_n^\ell$ is that it acts on the homology $H_\ast(Y; \mathbb{k})$. Let $x_i = e(W_m/W_{m-1})$.

**Lemma 3.2.** The action of $A_n^\ell$ on $H_\ast(Y; \mathbb{k})$ is faithful.

*Proof.* As noted in Ginzburg and Chriss [CG97], the algebra $A_n^\ell$ is the self-Ext algebra of $p_\ast C_\gamma$, so it suffices to show that any Ext between summands of this pushforward induces a non-zero map on hypercohomology. By the Decomposition theorem, this complex is a sum of shifts of IC sheaves, and the desired faithfulness for these is [BCS96] 3.4.2. □

**Lemma 3.3.** The homology $H_\ast(Y_\kappa; \mathbb{k})$ is isomorphic as an algebra under intersection product to $\mathbb{k}[x_1, \ldots, x_k]$ modulo the ideal $I_\kappa$ generated by $h_p(x_1, \ldots, x_{\kappa(q)})$ if $p \geq q - \kappa(q) - 1$.

*Proof.* This follows from the main theorem of [GR02]. □

Let us abuse notation, and use $e_\kappa$ to denote $\Delta_{\kappa}[Y_\kappa]$; this acts on $H_\ast(Y)$ by projection to $H_\ast(Y_\kappa)$.

**Lemma 3.4.** The homology classes, etc. act on $H_\ast(Y, \mathbb{k})$ by

\[
(h_1 \times h_2)[Z_{i,\kappa}] \star f(x_1, \ldots, x_k) = \frac{f(x_1, \ldots, x_k) - f(x_1, \ldots, x_{i+1}, x_i, \ldots, x_k)}{x_{i+1} - x_i}
\]

\[
(h_1 \times h_2)[Z_{0,\kappa}^\pm] \star f(x_1, \ldots, x_k) = f(x_1, \ldots, x_k)
\]

\[
(h_1 \times h_2)[Z_{0,\kappa}^\pm] \star f(x_1, \ldots, x_k) = -x_i f(x_1, \ldots, x_k)
\]

*Proof.* The correspondence $Z_{i,\kappa}$ is a $\mathbb{P}^1$ bundle under both projections given by base change of the space of pairs of flags that have relative position $\leq s_i$. Thus, the formula (3.6) follows from [BCG73] 5.7. If we have functions $\kappa \leq \kappa'$, then $\kappa'$ imposes a strictly weaker condition on flags; thus the correspondence $Z_{0,\kappa'}$ projects isomorphically to the first factor and $Z_{0,\kappa}^\pm$ to the second. Thus, the first correspondence induces a pullback and the second a pushforward in Borel-Moore homology. The formula (3.7) follows from the fact that pullback sends fundamental classes to fundamental classes and commutes with cap product. The formula (3.8) follows from the adjunction formula: the space $Z_{0,\kappa}^\pm$ inside $X_{\kappa_i}$ is the zero set of the induced map $W_i/W_{i-1} \to \mathbb{C}^\ell/\mathbb{C}^{q-1}$, so it is given by the Euler class of the line bundle $(W_i/W_{i-1})^\ast$, which is $-x_i$. □

*Proof of Theorem 3.1.* First, note that we have a map $T_n^\ell \to A_n^\ell$ defined by the equations given in Theorem 3.1. Both these can be identified with their image in the polynomial
representations by Lemmata 2.16 and 3.2. The polynomial representations can be matched by Lemma 3.3, and this intertwines the actions by Lemma 3.4. This map is thus also injective. We only need to prove surjectivity. We can do this by putting an upper bound on the dimension of $A^\ell_n$. We can filter the variety $Y_{\kappa_1} \times_X Y_{\kappa_2}$ according the preimages of the Schubert cells in $X$. The Schubert cell has a free action by a unipotent subgroup of $GL_\ell$ (depending on the cell), and is thus an affine bundle over a single fiber. Each Schubert cell contains a unique $T$-fixed point (here, $T$ is the torus of diagonal matrices), which is a coordinate subspace, spanned by the $(j + \lambda_j)$th coordinate vectors for $j = 1, \ldots, k$. If we consider the fiber over this point, then it inherits an action of $T$, and the fixed points are given by pairs of flags of coordinate spaces on this space, with compatibility conditions with the standard flag specified by $\kappa_1$ and $\kappa_2$. These are actually in bijection with pairs of backdrops whose associated functions are $\kappa_1, \kappa_2$. The flag is given by adding coordinate vectors corresponding to the rows by reading them in the order specified by the backdrop.

Thus, the $T$-fixed points of $Y \times_X Y$ are in bijection with pairs of backdrops on the same Young diagram; this gives an upper bound on the sum of the Betti numbers, that is on the dimension of $A^\ell_n$. However, this is the dimension of $T^\ell_n$ as computed by the basis, so the map $T^\ell_n \to A^\ell_n$ must be surjective.

\[ \square \]

3.3. Relationship to sheaves. While this is not necessary for understanding the overall construction, the discussion of convolution algebras would be incomplete without covering one of the prime motivations for introducing them: their connection to the category of sheaves. As shown in [CG97, 8.6.7], the convolution algebra $A^\ell_n$ can also be interpreted as an Ext algebra in the category of constructible sheaves (or equivalently, D-modules) on the Grassmannian itself. More precisely,

**Proposition 3.5.** $A^\ell_n \cong \text{Ext}^\bullet(p^*C_Y, p^*C_Y)$.

This Ext algebra completely controls the category of sheaves generated by $p^*C_Y$; there is a quasi-equivalence of dg-categories between the dg-modules over $A^\ell_n$ and the dg-category of sheaves generated by $p^*C_Y$.

This is a bit abstract, but we can actually make a stronger statement; by the Decomposition theorem, the sheaf $p^*C_Y$ is a sum of shifts of simple perverse sheaves. Replacing this sum with one copy of each simple perverse constituent, we obtain an object $G$ with the property that $A^\ell_n := \text{Ext}^\bullet(G, G)$ is a positively graded algebra with its degree 0 part commutative and semi-simple. The algebras $A^\ell_n$ and $A^\ell_n$ are Morita equivalent since they are Ext-algebras of objects with the same indecomposable constituents.

**Proposition 3.6.** The category of regular holonomic D-modules/perverse sheaves on the Grassmannian $\text{Gr}(k, \ell)$ which are smooth along the Schubert stratification is equivalent to the category of representations of the Koszul dual of $A^\ell_{\ell-2k}$ (the abelian category of linear complexes of projectives $A^\ell_{\ell-2k}$-modules).

For a thorough discussion of Koszul duality, its relationship to linear complexes, etc. see [MOS09]. This result is particularly interesting in view of the fact that this category already has an algebraic description related to Khovanov homology. The
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category of Schubert smooth perverse sheaves/D-modules on the Grassmannian is equivalent to the parabolic category $O$ for the corresponding maximal parabolic by [BCS96, 3.5.1] (interestingly, this equivalence is not simply taking sections of the D-module; see [Web11] for a more detailed discussion). Of course, those familiar with parabolic-singular duality for category $O$ (as proven in [BCS96]) will recognize that this implies that the category of $A^\ell_{\ell}$-modules is equivalent to a certain block of category $O$ of $\mathfrak{g}l_\ell$. This is proven in [Webb, §9] by other methods (since the one used here is much harder to generalize past $\mathfrak{sl}_2$), but this will perhaps not be too meaningful to topologists.

However, this parabolic category $O$ (denoted $O_{\ell-k,k}$ in [Str05]) played an important role in the original definition of Khovanov’s arc algebra. The most important case for understanding invariants is the 0 weight space, i.e. when $\ell = 2k$; in this case, Stroppel [Str05] has shown that an extension of Khovanov’s arc algebra has representation category equivalent to $O^{k,k}$, and is thus Koszul dual to $A^{\ell-2k}$. That is:

**Theorem 3.7.** The algebra $A^{2k}_0$ is Morita equivalent to the Koszul dual of Stroppel’s extended arc algebra $K^k$.

This is the sense in which our approach is Koszul dual to Stroppel’s. A similar theorem holds for other weight spaces, using further generalizations of the arc algebra given in [BS08].

4. **Khovanov homology**

In order to define a knot homology, we need to define functors between the categories of modules over $T^\ell$ for different choices of $\ell$ corresponding to tangles. These are defined explicitly using bimodules over the algebras $T^\ell$.

4.1. **Braiding.**

Spengler: [hesitates] We’ll cross the streams.
Venkman: Excuse me, Egon? You said crossing the streams was bad!
Spengler: Not necessarily. There’s definitely a very slim chance we’ll survive.

—Ghostbusters (1984)

The braiding bimodules are based on a simple principle used very successfully in the movie “Ghostbusters:” even if you were told not to do so earlier, you should “cross the streams.”

**Definition 4.1.** A $s_i$-Stendhal diagram is collection of oriented curves which is a Stendhal diagram except that there is a single crossing between the $i$ and $i+1$ strands.

Let $B_i$ be the $T^\ell - T^{\ell'}$-bimodule given by the quotient of the formal span of $s_i$-Stendhal diagrams by the same local relations (2.1a–2.1g) as well as

\[
\begin{align*}
\lambda_k - \lambda_{k-1} &= \lambda_{k-1} - \lambda_k \\
(4.9a)
\end{align*}
\]
More generally, one can fix a permutation for the red lines to carry out; the resulting bimodule $B_{\sigma}$ in this case will be the corresponding tensor product of $B_i$'s for a reduced expression of the permutation.

This bimodule has some beautiful properties:

- It has a cellular basis much like that of the algebra, indexed by pairs of backdrops on possibly different Young diagrams; in terms of sign sequences, we switch the $i$th and $(i+1)$st terms of the sign sequence. This corresponds to adding or removing a box in the diagonal $\{(i-j, j) | 0 < j < i\}$ if either of these is possible, and leaving the partition unchanged otherwise.
- In particular, as both a left and right module, it has a filtration whose successive quotients are standard modules.
- This bimodule has a geometric incarnation. We constructed the varieties $Y$ using a chosen standard flag; let $Y'$ be the same variety, but defined using a different flag $W_{\sigma}$ such that $W_i \neq C^i$ and $W_j = C^j$ for $j \neq i$. In this case, we can canonically identify $H_*(Y' \times_X Y') \cong T^{\ell}$, so $H^*(Y \times_X Y')$ is a natural bimodule over $T^{\ell}$; it turns out this is isomorphic to $B_i$. More generally, if we define $Y$ and $Y'$ using flags of relative position $\sigma$, the homology group $H^*(Y \times_X Y')$ will be isomorphic to the bimodule $B_{\sigma}$.

Given a bimodule $B$ over an algebra $A$, one can construct a functor $A\text{-mod} \to A\text{-mod}$ from $B$ in two different ways. You can consider the tensor product $B \otimes_A -,\text{ and the Hom space } \text{Hom}_A(B, -),\text{ which form an adjoint pair. The same is true of their derived functors}$

$$B \otimes - : D^b(A\text{-mod}) \to D^b(A\text{-mod}) \quad \text{and } \text{Hom}_A(B, -) : D^b(A\text{-mod}) \to D^b(A\text{-mod}).$$

If either one of these functors is an equivalence, the other one is its inverse (up to isomorphism of functors). Let $B_i = B_i \otimes -$.

**Theorem 4.2** ([Webb, 5.17]). The functors $B_i$ generate a strong action of the braid group on $\ell$ strands on the derived category $D^b(T^{\ell}\text{-mod})$.

It’s this braid action which is “responsible” for Khovanov homology. For lovers of category $O$, we can identify this with natural representation-theoretic functors: if we identify with a block of category $O$ which is “submaximally singular” then they match with twisting functors and if we use the Koszul dual identification with a regular block of parabolic category $O$, they match with shuffling functors (this is proven in [Webb]).

4.2. Cups and caps: $\ell = 2$.

“We are cups, constantly and quietly being filled. The trick is, knowing how to tip ourselves over and let the beautiful stuff out.”

–Ray Bradbury (1990)
In order to construct knot invariants, we need not just a braid group action, but also a way of closing up our braids. This is achieved by defining functors corresponding to cups and caps. Just as with the braiding, these are fairly simple minded functors easily guessed by drawing the appropriate pictures.

As preparation, let’s consider the case of a cup going from 0 strands to 2. In this case, we’ll simply want a left module over \( T^2 \) which categorifies the invariant vector in \( \mathbb{C}^2 \otimes \mathbb{C}^2 \). Since the functors \( E \) and \( F \) are exact, a module is killed by both of them if and only if the same holds for all its composition factors.

Both simples are one dimensional; thus, all but one of the idempotents \( e_\kappa \) acts by 0. Let \( L_0 \) be the simple quotient of \( P(0,0) \); the idempotent \( e_{(0,0)} \) acts by the identity on \( L_0 \). Let \( L_1 \) be the simple quotient of \( P_{(0,1)} \); the idempotent \( e_{(0,1)} \) acts by the identity on this module.

One can easily check that \( E L_0 \) is a simple module over \( T^2 \) and \( F L_1 \) is a simple module over \( T \). Let \( L_1 \) be the desired invariant representation.

**Proposition 4.3.** The class in the Grothendieck group of \([L_1]\) spans the invariants of \( \mathbb{C}^2 \otimes \mathbb{C}^2 \).

Let \( ⟨n⟩ \) be the “Tate twist” which increases the internal grading of a module by \( n \), and decreases its homological grading by \( n \). The cap corresponds to the functor \( R \text{Hom}(L_1, −) ⟨1⟩ \equiv Λ_1 \otimes ⟨−1⟩ \). Here \( Λ_1 \) refers to the right module obtained by letting \( T^l \) act on \( L_1 \) via the mirror image (through the \( x \)-axis) of diagrams.

The functors above are (up to shift) the right and left adjoints of the functor sending a \( k \)-vector space \( V \) to \( V \otimes_k L_1 \). The fact that \( R \text{Hom} \) and \( \otimes \) give the same functor, and many other special properties of these cap and cup functors comes from the special structure of a projective resolution of \( L_1 \).

**Proposition 4.4.** The minimal projective resolutions of the simples \( L_0, L_1 \) are given by

\[
P_{(0,1)} \xrightarrow{e_{(0,1)}} P_{(0,0)} \rightarrow L_0 \quad P_{(0,1)} \xrightarrow{e_{(0,1)}} P_{(0,0)} \xrightarrow{e_{(0,0)}} P_{(0,1)} \rightarrow L_1.
\]

In order to understand how the functors \( R \text{Hom}(L_1, −) \) and \( L_1 \otimes − \) are related, we can try applying them to projectives. Applying a right exact functor to a projective just gives a vector space in degree 0: thus, the projective \( Λ_1 \otimes P_{(0,0)} \) is sent to 0, and \( L_1 \otimes P_{(0,1)} \equiv \mathbb{C} \). On the other hand, \( R \text{Hom}(L_1, −) \) is left exact, so we require the full projective resolution. The result for any module \( P \) is the complex

\[
R \text{Hom}(L_1, P) = e_{(0,1)}P \xrightarrow{e_{(0,0)}} e_{(0,0)}P \xrightarrow{e_{(0,1)}} e_{(0,1)}P
\]

where leftmost term is degree 0 (so the rightmost is degree 2). This sends to \( P_{(0,0)} \) to 0 and \( P_{(0,1)} \) to \( \mathbb{C} \). We want to emphasize that there is a symmetry being used here: for example

\[
R \text{Hom}(L_0, P) = e_{(0,0)}P \xrightarrow{e_{(0,1)}} e_{(0,1)}P.
\]

Phrased differently, we have shown that:

**Proposition 4.5.** The cup and cap functors are biadjoint up to shift.
Another way of expressing this symmetry is that the Nakayama functor $S$ of $T_0^2$, given by derived tensor product with the bimodule $(T_0^2)^*$, sends the projective resolution of $L_1$ to an injective resolution of $L_1$ (shifted so that the cohomology is in degree $-2$), whereas $L_0$ is sent to a complex of injectives with cohomology in degrees $0$ and $-1$.

Since the algebra $T_0^2$ has finite global dimension (since it is quasi-hereditary), its Nakayama functor is actually a right Serre functor. Thus, for any simple the relation-ship between $\mathcal{R}$Hom and $\otimes$ is encoded by the fact that $\mathcal{R}$Hom($-, L^*$) $\cong \mathcal{L} \otimes -, and properties of a Serre functor guarantee

\[
\mathcal{L} \otimes - \cong \mathcal{R}$Hom($-, L^*$) $\cong \mathcal{R}$Hom($S^{-1}L, -$).
\]

Since $S^{-1}L_1 \cong L_1(-2)$, we obtain that

\[
L_1 \otimes - \cong \mathcal{R}$Hom($-, L^*$) $\cong \mathcal{R}$Hom($L_1, -$)$(-2)$.
\]

One important consequence of this is that the coalgebra $L_1 \otimes L_1$ and the algebra $\text{Ext}^\bullet(L_1, L_1)$ are identified with each other, giving a Frobenius structure on the resulting space. Of course, those familiar with Khovanov homology will know what Frobenius structure to expect:

**Proposition 4.6.** The Ext-bialgebra $\text{Ext}^\bullet(L_1, L_1)$ is isomorphic to $H^\bullet(S^2; \mathbb{k})$ with its usual Poincaré Frobenius structure.

This theorem holds over all fields, including those of characteristic 2; however, over characteristic 2, the element of this Ext-space attached to an open torus is 0 since this has Frobenius trace 2.

4.3. **Cups and caps:** $\ell > 2$. Now, let us turn to the more general case. Now, we have $\ell$ red strands, and expect to find functors either adding two more or capping off two existing ones. Furthermore, we expect it to be sufficient to consider the cup functors, and that the caps will make their appearance as adjoints.

What we would like to find is a bimodule which “inserts” a copy of $L_1$ with two new red strands attached to it. The beauty of using a pictorial approach is that we can literally do exactly that; the ugliness of a pictorial approach is that we then have to check a bunch of relations to make sure we didn’t just set everything to 0.

More formally, let a $+\mathcal{R}$-Stendhal diagram be a diagram which follows the Stendhal rules except that one of the red strands is a cup connected to the top in the $i + 1$st and $i + 2$nd position at $y = 1$; this cup must have a unique minimum, and there is a black strand which connects $y = 1$ to this minimum. One example of a $+\mathcal{R}$-Stendhal diagram with $\ell = 1$ is
Definition 4.7. Let $\mathcal{R}_i$ be the $T^{\ell+2} - T^\ell$-bimodule spanned over $\mathbb{K}$ by $+\tau$-Stendhal diagrams modulo the local relations of $T^\ell$ and the additional relations:

\[(4.10a)\]
\[
\begin{align*}
\begin{array}{c}
\text{Diagram 1}\n\end{array}
\end{align*}
\]
\[
\begin{align*}
\begin{array}{c}
\text{Diagram 2}\n\end{array}
\end{align*}
\]
\[
\begin{align*}
\begin{array}{c}
\text{Diagram 3}\n\end{array}
\end{align*}
\]

\[(4.10b)\]

The coevaluation functor $\mathcal{K}_i$ is given by $\mathcal{K}_i \otimes -$.

Of course, if $\ell = 0$, then the resulting bimodule is just $L_1$. Just as in the $\ell = 0$ case, the left and right adjoints of $\mathcal{K}_i$ differ by same shift. Let

$$
\mathcal{E}_i := \mathbb{R} \text{Hom}(\mathcal{K}_i, -)(1) \cong \mathcal{K}_i \otimes (-1).
$$

As the case of $\ell = 2$ shows, this is not an exact functor, but we can do calculations with it by taking a projective resolution of $\mathcal{R}_i$ as a left module. This can be done schematically as follows:

Here the boxes are there to fix the sequence at their top and impose no other relations. This is a complex of projective left modules; there is no right action that commutes with the differentials, though by general nonsense there is one “up to homotopy.”

What compatibility do we expect between these functors? For any composition of cups and caps, we have an associated functors, and obviously, we expect that any two ways of factoring a flat $(p, q)$-tangle (that is, one with no crossings) as a composition of functors will give isomorphic functors. However, we expect much more than this: the flat tangles form a 2-category, with morphisms given by cobordisms.

In order to connect this construction to Khovanov homology, we use a construction of Bar-Natan which defines a quotient of this category by imposing additional relations.

Definition 4.8. We let $\mathcal{BN}$ be the 2-category dotted cobordism category with the relations given in $[BN05, \S11.2]$. The objects of this category are non-negative integers, its 1-morphisms are flat tangles and its 2-morphisms are cobordisms decorated with dots modulo Bar-Natan’s “sphere,” “torus” and “neck cutting” relations.

Theorem 4.9 (Chatav $[Cha12, \S4.1]$). The functors $\mathcal{K}_i$ and $\mathcal{E}_i$ define a strict 2-representation $\gamma$ of the Bar-Natan 2-category $\mathcal{BN}$ such that $\gamma(\ell) = D^b(T^\ell \text{-mod})$, and the cup and cap functors are given by $\mathcal{K}_i$ and $\mathcal{E}_i$. 

Note that in this context, Bar-Natan’s relations actually follow immediately from Proposition 4.6, since these relations just express the structure of the cohomology ring $H^*(S^2; \mathbb{k})$. Bar-Natan’s relations then just specify that if $t$ is the unique element of degree 2 with trace 1, then this element has square 0, and that the dual basis to $\{t, 1\}$ under the Frobenius trace is $\{1, t\}$. 

4.4. Comparison with Khovanov homology. The calculations we have done thus far suggest an approach to finding a knot invariant, or more generally a tangle invariant. As usual, we should cut a tangle projection up into simple pieces consisting of a cup, cap or a single crossing, and use functors that correspond to these simple pieces. For any $(p, q)$-tangle $\mathcal{T}$, we choose a generic projection, cut into these pieces and let $\mathcal{K}(\mathcal{T}) : T^p -\text{mod} \to T^q -\text{mod}$ denote the composition of functors reading from bottom to top, associating

- $B_i$ to a positive crossing of the $i$th and $i + 1$st strands,
- $B_i^{-1}$ to a negative crossing of the $i$th and $i + 1$st strands,
- $K_i$ to a cup appearing between the $i$th and $i + 1$st strands, and
- $E_i^{-1}$ to a cap joining the $i$th and $i + 1$st strands.

Note, we are using unoriented knots; “positive” and “negative” as used above are relative to the $y$-coordinate in the plane (either both strands upward or downward oriented). For the moment, ignore that this depended on a choice of projection.

While what we have written thus far points naturally to this definition, it’s not completely satisfactory. It doesn’t have an obvious connection to Khovanov homology, nor have we checked that it defines a tangle invariant (that it doesn’t depend on the choice of projection).

However, we have an alternate definition of a knot invariant which fixes both these problems: we could simply transport structure from Bar-Natan’s paper. In this case, the cup and cap functors will be the same as ours, but we’ll have braiding functors which could potentially be different. These will be obtained by taking the image under the 2-functor $\gamma$ of a particular complex in Bar-Natan’s cobordism category, given by the saddle cobordism from the identity to the composition of a cap and cup.

Luckily, we can prove that this is actually the same crossing:

**Theorem 4.10.** The 2-functor $\gamma$ sends the cone of the crossing complex in $BN$ to the functor $B_i$.

There is an isomorphism between the knot invariant $\mathcal{K}(L)$ for a link $L$ by the braiding, cup and cap functors $B_i$, $K_i$, and $E_i$ and Khovanov homology $Kh(L)$.

Consider the action of Bar-Natan’s positive crossing: this is the cone of a map between two functors, the identity functor and $E_K(1)$. In fact, both of these correspond to derived tensor product with honest bimodules, given by the algebra $T^\ell$ itself, and the second by $\mathfrak{S}_i \otimes_{T^\ell} \mathfrak{S}_i$. Thus, the image of the crossing under $\gamma$ is the cone of the unit $\phi$ of the adjunction $(E, K(1))$.

This unit is given as usual by sending the identity $1 \in T^\ell$ to the canonical element of the pairing of $\mathfrak{S}_i$ with itself over $T^{\ell-2}$ given by matching the diagrams along the side with $\ell$ red strands, and simplifying. This is given by the sum of all diagrams
with no crossings, and a single pair of red cups and caps with a black strand inside each cup and cap. We can evaluate any other element of the algebra by multiplying the image of the identity on the left or right. Note that any idempotent which does not have exactly 1 black strand between these two reds will kill this element and thus be sent to zero.

In general, this evaluation can proceed by fixing some horizontal slice \( y = a \) and pinching the \( i + 1 \)st and \( i + 2 \)nd red strands the together to make a cup and cap; if at \( y = a \) there is not exactly 1 black strand between these two reds, we get 0.

On the other hand, we have a natural map \( \psi : B_i \to T_\ell \) given by using the “0-smoothing” of the red crossing, that is slicing vertically through the red crossing in order to produce two strands with no crossing. This is obviously compatible with the relations, injective, and has image killed by \( \phi \). Thus, we will complete the proof of Theorem 4.10 by showing:

**Lemma 4.11.** The map \( \psi \) induces an isomorphism \( B_i \cong \ker \phi \).

**Proof.** We can reduce to the case where \( \ell = 2 \) using [W 5.8 & 6.19]. Assuming \( \ell = 2 \), this is a simple calculation; one simply notes that both \( B_i \) and \( \ker \phi \) are 4 dimensional. The left cellular basis of \( B_i \) is given by

\[
\begin{align*}
\begin{array}{cc}
\hline
2 & \downarrow \\
\hline
\end{array} & \begin{array}{cc}
\hline
\hline
\end{array} & \begin{array}{cc}
\hline
\hline
\end{array} \\
\begin{array}{cc}
\hline
1 & \downarrow \\
\hline
\end{array} & \begin{array}{cc}
\hline
\hline
\end{array} & \begin{array}{cc}
\hline
\hline
\end{array} \\
\end{align*}
\]

These are sent under the map breaking open the crossing to 4 of the 5 basis vectors shown in Section 2.5 and indeed, the last basis vector is sent to a generator of \( \Omega_i \otimes_{T_\ell} \Omega_i \).

**Proof of Theorem 1.1** Since Khovanov homology can be identified as a complex of sums of the empty tangle in \( \mathcal{B}N \), in any 2-representation of \( \mathcal{B}N \) this complex will act as tensor product with the Khovanov homology of the link. In particular, each link will give an endofunctor of \( D^b(\mathcal{O}_0\text{-mod}) \cong D^b(\mathcal{O}_\mathbb{K}\text{-mod}) \) of the category of complexes of graded vector spaces up to quasi-isomorphism given by Khovanov homology.

The readers familiar with the literature on Khovanov homology might get a bit nervous around this point, since Bar-Natan’s construction while beautiful, was a well-known flaw: it only allows one to define functoriality maps on Khovanov homology up to sign. However, a fix for this issue was found by Clark, Morrison and Walker [CMW09] and is very easily transported into our picture. Recall that our identification with Khovanov homology involved considering a map \( B_i \to T_\ell \) and identifying its cokernel with \( \Omega_i \otimes_{T_\ell} \Omega_i \). While these modules are isomorphic, they are not canonically isomorphic. Rather than taking the obvious identification, one should insert factors of \( i \) or \( -i \) to account for orientations. We leave to the reader the details of transporting the disoriented Bar-Natan category into this picture.
4.5. **Jones-Wenzl projectors.** Another construction in the category $\mathcal{BN}$ which we would like to understand in terms of $T^\ell$ is the categorified Jones-Wenzl projector $P_\ell$ of Cooper and Krushkal [CK12]. Much like the crossing, we can easily transport this structure to an endofunctor using the 2-functor $\gamma$; however, since this complex is unbounded, it induces a autofunctor on the bounded above derived category $D^-(T^\ell\text{-mod})$ of graded $T^\ell$ modules.

**Definition 4.12.** We let $S_0$ be the subcategory of $D^-(T^\ell\text{-mod})$ consisting of complexes of projective-injectives. Each algebra $T^\ell_{-2k}\text{-mod}$ has a single indecomposable projective-injective; this is given by a divided power functor $\mathcal{B}(k)P_\emptyset$.

Note, in particular, that the category $S_0$ is equivalent to the bounded above derived category of modules over $\text{End}(\mathcal{B}(k)P_\emptyset)$ which is isomorphic to the cohomology ring of $H^*(\text{Gr}(k,\ell))$. This subcategory has an orthogonal $S_0^\perp$, given by the objects whose composition factors are all killed by $\mathcal{E}^k$. Typically, one has to specify left or right orthogonals in a categorical setting, but in this case, these coincide.

Since the left and right orthogonals coincide, there is a unique projection $\pi_\ell$ to $S_0$ killing this orthogonal. This may sound like an abstract operation, but in terms of algebras, it’s really very concrete. Consider the bimodule $T^\ell e_0T^\ell \subset T^\ell$. Essentially by definition, this is the bimodule of diagrams as in $T^\ell$ which have all black strands to the right of all red at $y = 1/2$.

**Lemma 4.13.** The projection functor $\pi_\ell$ coincides with the derived tensor product $T^\ell e_0T^\ell \otimes -$

**Proof.** This follows immediately from the fact that the category $S_0$ is generated by the summands of $T^\ell e_0$. □

One can think of this as the composition of two adjoint functors. Recall that $R^\ell = e_0T^\ell e_0$ is isomorphic to the cyclotomic nilHecke algebra with a degree $\ell$ relation, via the map that puts a nilHecke diagram to the right of $\ell$ red lines. We thus have a functor $M \mapsto e_0M$ which sends $T^\ell \text{-mod}$ to $R^\ell\text{-mod}$, and its left adjoint $T^\ell e_0 \otimes_{R^\ell} -$; taking derived tensor product is necessary since $T^\ell e_0$ is not projective as a right $R^\ell$-module.

**Lemma 4.14.** The category $S_0^\perp$ is the smallest triangulated subcategory of $D^-(T^\ell\text{-mod})$ which is closed under categorification functors and contains all highest weight simples of weight $< \ell$.

**Proof.** If $M$ is a module of weight $m$, we can identify $e_0M = (\mathcal{E}^{(\ell-m)/2}M$; thus all highest weight simples of weight $< \ell$ lie in $S_0^\perp$. On the other hand, $S_0^\perp$ is equivalent to the quotient $D^-(T^\ell\text{-mod})/S_0^\perp$, which is concentrated in weights strictly between $\ell$ and $-\ell$. Thus, it is generated by its highest weight simples, which all necessarily of weight $< \ell$. Thus, the same is true of $S_0^\perp$. □

---

3 Actually, there are dual categorical Jones-Wenzl projectors, one bounded above and one bounded below as complexes. We’ll always use the bounded above one.
Obviously, $T^m_m$ has a unique highest weight simple, which we denote $P_0$.

**Lemma 4.15.** The images of $P_0$ under the different flat $(\ell, m)$ tangles with no caps are a complete, irredundant list of highest weight simples of weight $m$.

**Proof.** Since the cup functors intertwine the categorification functors, the image of $P_0$ under any flat tangle is highest weight. In particular, any composition factor of such a module is highest weight.

We attach a sign sequence to one of the tangles $T$ above by putting a $+$ above each stand which goes from the bottom to the top and over the right end of each cup, and a $-$ over the left end of each cup. We can consider this sequence as an element of the tensor product of $\ell$ copies of the two-element crystal $\{+, -\}$ of $\mathbb{C}^2$. In this crystal, the sequence is highest weight, as there is no $-$ sign not canceled by a $+$ to its right. The action of the Kashiwara operator $e_i$ on the weight string generated by this element changes the rightmost $-$ on top of a through-strand to a $+$, leaving the cups unchanged.

We can associate an idempotent to $e_T$ as in our basis; we replace each $-$ by a red strand with a black to its right, and each $+$ by just a red strand. We order these sign sequences by the rule that $-+ > +-$; if we convert sign sequences to partitions in a box, then $\lambda \geq \mu$ if the diagram of $\lambda$ fits inside that of $\mu$. By [Webb], there is a unique highest weight simple such that $\dim e_T L_T = 1$ and $e_T L_T = 0$ for $T' > T$.

Consider the module $K_T := \mathcal{X}(T)(P_0)$. We can easily calculate that $\dim e_T K_T \leq 1$, since this space is spanned by the diagrams where the black strand from each cup follows the left side up to the top. In one example, this is the resulting diagram:

(4.11)

Any other diagram $d$ in $e_T K_T$ must have a black strand which passes through the left side of its cup. Using the relations, we can push this crossing lower, until it is the first crossing on this black strand. Correction terms will appear from (2.1d), but these will have fewer red/black crossings. The relations (4.10a) imply that the diagram where the black strand passes through the left side of the cup is 0, so we can write $d$ as a sum of diagrams with fewer red/black crossings. By induction, we may assume that there are no such crossings, and indeed the diagram we indicated in (4.11) spans.

Furthermore, this diagram generates the module $K_T$; in order to see this, pull the bottom of each cup toward the bottom of the diagram, making sure its minimum ends up to the right of the black strand for any cup in which it is nested. Eventually you will reach a Stendhal diagram applied to $e_T K_T$. Since the module $K_T$ is not zero (it categorifies a non-zero vector), this shows that $\dim e_T K_T = 1$.

An argument like that above shows that $e_T$ with $T' > T$ kills this module, since there is no diagram with the correct top which doesn’t have a black strand passed through the left side of its cup. Thus, $L_T$ must be a quotient of $K_T$.

The module $K_T$ is self-dual, so $L_T$ also appears as a submodule. Since $\dim e_T K_T = 1$, this is only possible if $K_T = L_T$. This is a complete irredundant list of highest weight simples, so we are done. □
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**Theorem 4.16.** The categorified Jones-Wenzl projector $P_\ell$ is sent by $\gamma$ to the projection $\pi_\ell = T_\ell e_0 T_\ell L \otimes_{T_\ell} -$ to the subcategory $S_0$.

**Proof.** The projection is distinguished by the fact that it is isomorphic to the identity functor on $S_0$ and kills all objects in $S_0^\perp$. Thus, we need only check that $P_\ell$ also has these properties.

The images of all 1-morphisms in $\mathcal{BN}$ commute with the functors $E$ and $F$. Since $S_0$ is generated by $\gamma^k P_0$ and $P_\ell$ acts by the identity on $T_\ell \text{-mod}$, it also acts by the identity on $S_0$.

On the other hand, $P_\ell$ kills the image of any cup functor, since it is invertible under turn-backs. Thus, by Lemma 4.15 it kills all highest weight simples of weight $< \ell$. Since it commutes with categorification functors, it kills the triangulated category generated by categorification functors applied to these simples. In turn, by Lemma 4.14 this category is $S_0^\perp$. This completes the proof. □

In [Webb, §7], we define a homology theory categorifying the colored Jones polynomial which uses generalizations of the algebras $T_\ell$. For each sequence of positive integers $n = (n_1, \ldots, n_m)$ with $\ell = \sum n_i$, we have an idempotent $e_n$ which is the sum of all idempotents where there is a group $n_1$ red strands, then some number of black strands, $n_2$ red strands, etc. In terms of $\kappa$, this means that the first $n_1$ values of $\kappa$ are the same, then the next $n_2$, etc. The algebra $T^n = e_n T^\ell e_n$ can represented using Stendhal diagrams as well, where we compress each group of $n_i$ strands between which no blacks are allowed into a single strand, labeled with $n_i$. This algebra naturally appears in the construction of categorified colored Jones polynomials since its Grothendieck group is a tensor product of simple $\mathfrak{sl}_2$ modules.

**Proposition 4.17.** The horizontal composition $P_{n_1} \otimes \cdots \otimes P_{n_m}$ of 1-morphisms in $\mathcal{BN}$ is sent by $\gamma$ to the projection $T_\ell e_n T_\ell L \otimes -$.

**Proof.** Much like that of Theorem 4.16 above, the proof is by checking that both functors act by the identity on the subcategory generated by $T_\ell e_n$ and trivially on its orthogonal.

The action on the subcategory generated by $T_\ell e_n$ can be understood by studying the actions on standardizations of projective-injectives of $T_\ell^{n_1} \otimes \cdots \otimes T_\ell^{n_m}$; this is the identity since

$$P_{n_1} \otimes \cdots \otimes P_{n_m} \circ S^n \equiv S^n \circ P_{n_1} \otimes \cdots \otimes P_{n_m}$$

where $S^n$ is the standardization functor from [Webb, §4]. Since the projection on the right-hand side sends each projective-injective to itself, $P_{n_1} \otimes \cdots \otimes P_{n_m}$ must act by the identity on the category generated by these standardizations.

On the other hand, the orthogonal to this category is generated by the images of cup diagrams with no cups that go between different groups of red strands. These are killed by $P_{n_1} \otimes \cdots \otimes P_{n_m}$ by contractibility under turnbacks. □

The colored Jones homology theory in [Webb, §7] is defined using tensor product with certain bimodules corresponding to the cups, caps and crossings. In fact their definition is essentially exactly like that of the functors $B_i$, $K_i$, and $E_i$ above.
Let \( \tau \) be a tangle with components labeled by integers, and \( \tau' \) its cabling, with each strand replaced by as many strands as its label. Let \((n_1, \ldots, n_m)\) and \((n'_1, \ldots, n'_m)\) be the sequence of labels at the top and bottom of the tangle.

Then we have the functor attached to this tangle by the homology theory of [Webb], which we denote \( \mathcal{H}^n(\tau) \), and the functor \( \mathcal{H}(\tau') \) attached to the cabling by the theory we have discussed in Sections 4.1–4.4. Assume now that \( \tau \) is a single crossing, cup or cap.

**Lemma 4.18.** As bimodules over \( T^n \) and \( T'^n \), \( \mathcal{H}^n(\tau) \) and \( e_n \mathcal{H}(\tau')e_n \) are isomorphic.

**Proof.** For the braiding map, this follows from the same argument as in [Webb 3.17]. For the cup and cap, these are equivalent so we need only consider one. The cabling of the cup is \( n \) nested cups. As usual, by considering the action on standardizations, we can reduce to the case where there are not any other red strands.

In this case, we need to show that these nested cups give us the unique invariant simple for \( T^l e_n \) after being multiplied by \( e_n e_n \). Multiplying by this idempotent is an exact functor, and it categorifies the projection \( (\mathbb{C}^2)^{\otimes 2l} \to \text{Sym}^l(\mathbb{C}^2) \otimes \text{Sym}^l(\mathbb{C}^2) \). In particular, it sends the image of nested cups to an invariant vector in \( \text{Sym}^l(\mathbb{C}^2) \otimes \text{Sym}^l(\mathbb{C}^2) \) which is the class of invariant simple. We can check this by looking at the coefficient of any monomial in the class, for example that of

\[
\left[ \begin{array}{cc}
0 \\
1
\end{array} \right] \otimes \cdots \left[ \begin{array}{cc}
0 \\
1
\end{array} \right] \otimes \left[ \begin{array}{cc}
1 \\
1
\end{array} \right] \otimes \cdots \otimes \left[ \begin{array}{cc}
1 \\
0
\end{array} \right]
\]

and checking that it is 1.

Thus \( e_n e_n \) is an honest module whose class in the Grothendieck group coincides with the correct simple. This is only possible if it is the desired simple itself. \( \square \)

**Corollary 4.19.** The colored Jones homology theories defined in [Webb] and [CK12] agree.

**Proof.** By its definition, the homology theory from [CK12] can be obtained by taking a generic tangle projection, sliced into crossings, cups, and caps; we’ll use cuts to mean the horizontal lines where we cut, and slices to mean the regions between two successive ones. We let \( \tau_k \) be the tangle in the \( k \)th slice from the bottom, and \( n_k \) the labeling of the strands at the \( k \)th slice.

Now, we take this tangle’s cabling, and insert a copy of \( P_n \) at each point where a strand of label \( n \) crosses one of the horizontal cuts. The image of this 1-morphism in Bar-Natan’s category is obtained by applying \( \mathcal{H}(\tau_k) \) for the different slices \( \tau_k \) of the cabled tangle with \( T^l e_n T^l \otimes - \) inserted at the \( k \)th cut. We can do the factorization \( T^l e_n T^l \otimes - \equiv T^l e_n T^l \otimes - \) at each cut, and move the factor into the slice above the cut, and the second factor into the slice below it. Thus, for each slice \( \tau_k \), we obtain \( e_n e_n \mathcal{H}(\tau_k) \equiv \mathcal{H}^n(\tau_k) \). By definition, taking this successive derived tensor product gives the homology theory from [Webb]. \( \square \)

**References**

[BGG73] I. N. Bernštejn, I. M. Gel’fand, and S. I. Gel’fand, *Schubert cells, and the cohomology of the spaces G/P*, Uspehi Mat. Nauk 28 (1973), no. 3(171), 3–26. MR 0429933 (55 #2941)
Ben Webster

[BGS96] Alexander Beilinson, Victor Ginzburg, and Wolfgang Soergel, Koszul duality patterns in representation theory, J. Amer. Math. Soc. 9 (1996), no. 2, 473–527.

[BN05] Dror Bar-Natan, Khovanov's homology for tangles and cobordisms, Geom. Topol. 9 (2005), 1443–1499 (electronic). MR MR2174270 (2006g:57017)

[BS08] J. Brundan and C. Stroppel, Highest weight categories arising from Khovanov's diagram algebra I: Cellularity, 2008, preprint.

[CG97] Neil Chriss and Victor Ginzburg, Representation theory and complex geometry, Birkhäuser Boston Inc., Boston, MA, 1997. MR MR1376244 (97h:20016)

[Cha12] Eitan Chatav, Representation theory of categorified quantum $\mathfrak{sl}_2$, Ph.D. thesis, Stony Brook University, 2012.

[CK12] Benjamin Cooper and Vyacheslav Krushkal, Categorification of the Jones-Wenzl projectors, Quantum Topol. 3 (2012), no. 2, 139–180. MR 2901969

[CMW09] David Clark, Scott Morrison, and Kevin Walker, Fixing the functoriality of Khovanov homology, Geom. Topol. 13 (2009), no. 3, 1499–1582.

[CR08] Joseph Chuang and Raphaël Rouquier, Derived equivalences for symmetric groups and $\mathfrak{sl}_2$-categorification, Ann. of Math. (2) 167 (2008), no. 1, 245–298.

[GL96] J. J. Graham and G. I. Lehrer, Cellular algebras, Invent. Math. 123 (1996), no. 1, 1–34. MR 1376244 (97h:20016)

[GR02] V. Gasharov and V. Reiner, Cohomology of smooth Schubert varieties in partial flag manifolds, J. London Math. Soc. (2) 66 (2002), no. 3, 550–562. MR MR1934291 (2003i:14064)

[Kho02] Mikhail Khovanov, A functor-valued invariant of tangles, Algebr. Geom. Topol. 2 (2002), 665–741 (electronic). MR MR1928174 (2004d:57016)

[KL10] Mikhail Khovanov and Aaron D. Lauda, A categorification of quantum $\mathfrak{sl}(n)$, Quantum Topol. 1 (2010), no. 1, 1–92. MR 2628852 (2011g:17028)

[Lau12] Aaron D. Lauda, An introduction to diagrammatic algebra and categorified quantum $\mathfrak{sl}_2$, Bull. Inst. Math. Acad. Sin. (N.S.) 7 (2012), no. 2, 165–270. MR 3024893

[MOS09] Volodymyr Mazorchuk, Serge Ovsienko, and Catharina Stroppel, Quadratic duals, Koszul dual functors, and applications, Trans. Amer. Math. Soc. 361 (2009), no. 3, 1129–1172.

[Str05] Catharina Stroppel, Categorification of the Temperley-Lieb category, tangles, and cobordisms via projective functors, Duke Math. J. 126 (2005), no. 3, 547–596.

[SW] Catharina Stroppel and Ben Webster, Quiver Schur algebras and $q$-Fock space, [arXiv:1110.1115]

[Weba] Ben Webster, A categorical action on quantized quiver varieties, [arXiv:1208.5957]

[Webb] ______, Knot invariants and higher representation theory, [arXiv:1309.3796]

[Webc] ______, On generalized category $\mathcal{O}$ for a quiver variety, in preparation.

[Webd] ______, Weighted Khovanov-Lauda-Rouquier algebras, [arXiv:1209.2463]

[Web11] ______, Singular blocks of parabolic category $\mathcal{O}$ and finite $W$-algebras, J. Pure Appl. Algebra 215 (2011), no. 12, 2797–2804. MR 2811563