On a Weak Type Estimate for Sparse Operators of Strong Type

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Abstract—We define sparse operators of strong type on abstract measure spaces with ball-bases. Weak and strong type inequalities for such operators are proved.

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1. INTRODUCTION

The sparse operators are very simple positive operators recently appeared in the study of weighted estimates of Calderón-Zygmund and other related operators. It was proved that some well-known operators (Calderón-Zygmund operators, martingale transforms, maximal function, Carleson operators, etc.) can be dominated by sparse operators, and this kind of dominations imply a series of deep results for the mentioned operators (see [1, 2, 4–7]). In particular, Lerner’s [6] norm domination of the Calderón-Zygmund operators by sparse operators gave a simple alternative proof to the $A_2$-conjecture solved by Hytönen [3]. Lacey [5] established a pointwise sparse domination for the Calderón-Zygmund operators with an optimal condition (Dini condition) on the modulus of continuity, getting a logarithmic gain to the result previously proved by Conde-Alonso and Rey [1]. The paper [5] also proves a pointwise sparse domination for the martingale transforms, providing a short approach to the $A_2$-theorem proved by Treil-Thiele-Volberg [8]. For the Carleson operators norms sparse domination was proved by Di Plinio and Lerner [2], while the pointwise domination follows from a general result proved later in [4].

In this paper we consider sparse operators based on ball-bases in abstract measure spaces. The concept of ball-basis was introduced by the first author in [4]. Based on ball-basis the paper [4] defines a wide class of operators (including, in particular, the above mentioned operators) that can be pointwisely dominated by sparse operators. Some estimates of sparse operators in abstract spaces were obtained in [4]. In this paper we define a stronger version of sparse operators, and prove weak and strong type estimates for such operators.

We first recall the definition of the ball-basis from [4].

Definition 1.1. Let $(X, \mathcal{M}, \mu)$ be a measure space. A family of sets $\mathcal{B} \subset \mathcal{M}$ is said to be a ball-basis if it satisfies the following conditions.

B1) $0 < \mu(B) < \infty$ for any ball $B \in \mathcal{B}$.

B2) For any two points $x, y \in X$ there exists a ball $B \ni x, y$.

B3) If $E \in \mathcal{M}$, then for any $\varepsilon > 0$ there exists a finite or infinite sequence of balls $B_k$, $k = 1, 2, \ldots$, such that $\mu(E \Delta \bigcup_k B_k) < \varepsilon$.

B4) For any $B \in \mathcal{B}$ there is a ball $B^* \in \mathcal{B}$ (called a hull of $B$) satisfying the conditions

$$\bigcup_{A \in \mathcal{B}: \mu(A) \leq 2\mu(B), A \cap B \neq \emptyset} A \subset B^*, \quad \mu(B^*) \leq K\mu(B),$$

where $K$ is a positive constant.
A ball-basis $\mathcal{B}$ is said to be doubling if there is a constant $\eta > 1$ such that for any $A \in \mathcal{B}$, $A^* \neq X$, one can find a ball $B \in \mathcal{B}$ to satisfy
\[ A \subset B, \quad \mu(B) \leq \eta \cdot \mu(A). \tag{1.1} \]
In [4], it was shown that the condition (1.1) in the definition can equivalently be replaced by a stronger condition $\eta_1 \leq \mu(B)/\mu(A) \leq \eta_2$, where $\eta_2 > \eta_1 > 1$. It is well-known the non-standard features of non-doubling bases in many problems of analysis.

One can easily check that the family of Euclidean balls in $\mathbb{R}^n$ forms a ball-basis and it is doubling. An example of non-doubling ball-basis can serve us the martingale-basis defined as follows. Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $\{ \mathcal{B}_n : n \in \mathbb{Z} \}$ be a collection of measurable sets such that 1) each $\mathcal{B}_n$ is a finite or countable partition of $X$, 2) for each $n$ and $A \in \mathcal{B}_n$, the set $A$ is a union of sets $A' \in \mathcal{B}_{n+1}$, 3) the collection $\mathcal{B} = \bigcup_{n\in \mathbb{Z}} \mathcal{B}_n$ generates the $\sigma$-algebra $\mathcal{M}$, 4) for any points $x, y \in X$ there is a set $A \in \mathcal{B}$ such that $x, y \in A$. One can easily check that $\mathcal{B}$ satisfies all the ball-basis conditions B1)-B4). On the other hand, it is not always doubling. Obviously, it is doubling if and only if $\mu(\partial(B)) \leq c\mu(B)$, $B \in \mathcal{B}$, where $\partial(B)$ (parent of $B$) denotes the minimal ball satisfying $B \subset \partial(B)$.

Let $\mathcal{B}$ be a ball-basis in a measure space $(X, \mathcal{M}, \mu)$. For $f \in L^r(X)$, $1 \leq r < \infty$, and a ball $B \in \mathcal{B}$ we set
\[ \langle f \rangle_{B,r} = \left( \frac{1}{\mu(B)} \int_B |f|^r \right)^{1/r}, \quad \langle f \rangle^*_{B,r} = \sup_{A \in \mathcal{B}, A \supset B} \langle f \rangle_{A,r}. \]
A collection of balls $S \subset \mathcal{B}$ is said to be sparse or $\gamma$-sparse if for any $B \in S$ there is a set $E_B \subset B$ such that $\mu(E_B) \geq \gamma \mu(B)$ and the sets $\{ E_B : B \in S \}$ are pairwise disjoint, where $0 < \gamma < 1$ is a constant. We associate with $S$ the operators:
\[ A_{S,r}f(x) = \sum_{A \in S} \langle f \rangle_{A,r} \cdot I_A(x), \quad A^*_{S,r}f(x) = \sum_{A \in S} \langle f \rangle^*_{A,r} \cdot I_A(x), \]
called sparse and strong type sparse operators, respectively. The weak-$L^1$ estimate of $A_{S,1}$ in $\mathbb{R}^n$ (case $r = 1$) as well as its boundedness on $L^p$ ($1 < p < \infty$) were proved by Lerner [6]. The $L^p$-boundedness of $A_{S,r}$ for general ball-bases was shown by the first author in [4].

We say that a constant is admissible if it depends only on $p$ and on the constants $K$ and $\gamma$ from the above definitions, and the notation $a \lesssim b$ will stand for the inequality $a \leq c \cdot b$, where $c > 0$ is an admissible constant. The main result of this paper is the weak-$L^r$ estimate of $A^*_{S,r}$ generated by general ball-bases. More precisely, we have the following result.

**Theorem 1.1.** A sparse operator of strong type $A^*_{S,r}, 1 \leq r < \infty$, corresponding to a general ball-basis, is a bounded operator on $L^p$ for $r < p < \infty$, and satisfies the weak-$L^r$ estimate, that is,
\[ \|A^*_{S,r}(f)\|_p \lesssim \|f\|_p, \quad r < p < \infty, \tag{1.2} \]
\[ \mu \{ A^*_{S,r}(f) > \lambda \} \lesssim \frac{\|f\|_{L^p}}{\lambda^{\frac{1}{r}}}, \quad \lambda > 0. \tag{1.3} \]

The proof of $L^p$-boundedness of $A^*_{S,r}$ is simple and uses the duality argument as in [6]. Lerner’s [6] proof of weak-$L^1$ estimate in $\mathbb{R}^n$ applies the standard Calderón–Zygmund decomposition argument. The Calderón–Zygmund decomposition may fail if the ball-basis is not doubling, so for the weak-$L^r$ estimate in the case of general ball-basis we apply the function flattening technique displayed in Lemma 2.7. That is, we reconstruct the function $f \in L^r$ around the big values to get a $\lambda$-bounded function $g \in L^2$, having ball averages of $f$ dominated by those of $g$. As a result we will have $\|A^*_{S,r}(f)\|_{r,\infty} \lesssim \|A^*_{S,r}(g)\|_{2r,\infty}$, reducing the weak-$L^r$ estimate of $A^*_{S,r}$ to weak-$L^{2r}$. 

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2. AUXILIARY LEMMAS

Recall some definitions and propositions from [4]. We say that a set $E \subset X$ is bounded if $E \subset B$ for a ball $B \in \mathcal{B}$.

**Lemma 2.1** ([4]). Let $(X, \mathcal{M}, \mu)$ be a measure space with a ball-basis $\mathcal{B}$. If $E \subset X$ is bounded and $\mathcal{G}$ is a family of balls with $E \subset \bigcup_{G \in \mathcal{G}} G$, then there exists a finite or infinite sequence of pairwise disjoint balls $G_k \in \mathcal{G}$ such that $E \subset \bigcup_k G_k^*$.

**Definition 2.1.** For a set $E \in \mathcal{M}$ a point $x \in E$ is said to be a density point if for any $\varepsilon > 0$ there exists a ball $B \ni x$ such that $\mu(B \cap E) > (1 - \varepsilon)\mu(B)$. We say that a measure space $(X, \mathcal{M}, \mu)$ satisfies the density property if almost all points of any measurable set are density points.

**Lemma 2.2** ([4]). Any ball-basis satisfies the density property.

The $L^r$ maximal function associated to the ball-basis $\mathcal{B}$ we denote by

$$M_r f(x) = \sup_{B \in \mathcal{B}, x \in B} \langle f \rangle_{B,r}.$$

**Lemma 2.3** ([4]). If $1 \leq r < p \leq \infty$, then the maximal function $M_r$ satisfies the strong $L^p$ and weak-$L^r$ inequalities.

**Definition 2.2.** We say that $B \in \mathcal{B}$ is a $\lambda$-ball for a function $f \in L^r(X)$ if $\langle f \rangle_{B,r} > \lambda$. If, in addition, there is no $\lambda$-ball $A \supset B$ satisfying $\mu(A) \geq 2\mu(B)$, then $B$ is said to be a maximal $\lambda$-ball for $f$.

**Lemma 2.4.** Let the function $f \in L^r(X)$ have bounded support, and let $\lambda > 0$. There exist pairwise disjoint maximal $\lambda$-balls $\{B_k\}$ such that

$$G_\lambda = \{x \in X : M_r f(x) > \lambda\} \subset \bigcup_k B_k^*.$$

**Proof.** Since $f$ has bounded support, one can easily check that the set $G_\lambda$ is also bounded. Besides, any $\lambda$-ball is in some maximal $\lambda$-ball. Thus we conclude that $G_\lambda = \bigcup_\alpha B_\alpha$, where each $B_\alpha$ is a maximal $\lambda$-ball. Applying the above covering lemma, we find a sequence of pairwise disjoint balls $B_k$ such that $G_\lambda \subset \bigcup_k B_k^*$ and so we have (2.1).

Let $B \subset (a, b)$ be a Lebesgue measurable set. For a given positive real $\kappa \leq |B|$ denote

$$a(\kappa, B) = \inf \{a' : |(a, a') \cap B| \geq \kappa\}, \quad L(\kappa, B) = (a, a(\kappa, B)) \cap B.$$

Observe that $L(\kappa, B)$ determines the "leftmost" set of measure $\kappa$ in $B$ and $a(\kappa, B)$ does not depend on the choice of $a$.

**Lemma 2.5.** Let $A \subset B \subset (a, b)$ be Lebesgue measurable sets on the real line, and let $0 < \kappa \leq |A|$. Then we have

$$|L(\kappa, B) \triangle L(\kappa, A)| \leq 2|B \setminus A|.$$

**Proof.** Obviously, we have $a \leq a(\kappa, B) \leq a(\kappa, A) \leq b$. Since $|L(\kappa, B)| = |L(\kappa, A)|$, the sets

$$L(\kappa, B) \setminus L(\kappa, A) = ((a, a(\kappa, B)) \cap (B \setminus A)),$$

$$L(\kappa, A) \setminus L(\kappa, B) = ((a(\kappa, B), a(\kappa, A)) \cap A).$$

have the same measure. So, we get

$$|L(\kappa, B) \triangle L(\kappa, A)| = 2|((a, a(\kappa, B)) \cap (B \setminus A))| \leq 2|B \setminus A|.$$
Lemma 2.6. Let \((X, \mathcal{M}, \mu)\) be a non-atomic measure space and \(G_k\) be a finite or infinite sequence of measurable sets in \(X\). If a sequence of numbers \(\xi_k \geq 0\) satisfies \(\sum_k \xi_k < \infty\) and the condition
\[
\sum_{j: \mu(G_j) \leq \mu(G_k), G_j \cap G_k \neq \emptyset} \xi_j \leq \mu(G_k), \quad k = 1, 2, \ldots,
\]
then there exist pairwise disjoint measurable sets \(\tilde{G}_k \subset G_k\) such that
\[
\mu(\tilde{G}_k) = \xi_k, \quad k = 1, 2, \ldots.
\]

Proof. Without loss of generality we can suppose that \(\mu(G_k)\) is decreasing. Since the measure space is non-atomic, we can also suppose that \(G_k\) are Lebesgue measurable sets in \(\mathbb{R}\). We first assume that the sequence \(G_k, k = 1, 2, \ldots, n\), is finite. We apply backward induction. The existence of \(\tilde{G}_n \subset G_n\) satisfying \(\mu(\tilde{G}_n) = \xi_n\) follows from (2.2), since the latter implies \(\xi_n \leq \mu(G_n)\) and we have that the measure is non-atomic. We define \(\tilde{G}_n\) to be the lefmost set in \(G_n\), that is, \(\tilde{G}_n = L(\xi_n, G_n)\). Suppose by induction we have defined pairwise disjoint sets \(\tilde{G}_k \subset G_k\) satisfying (2.3) for \(l \leq k \leq n\). From (2.2) it follows that
\[
\mu\left(\bigcap_{k=1}^n \tilde{G}_k\right) \geq \mu(\bigcap_{k=1}^n G_k) - \sum_{1 \leq j \leq n: G_j \cap G_{l-1} \neq \emptyset} \mu(\tilde{G}_j) \geq \xi_{l-1}.
\]
Hence we can define \(\tilde{G}_{l-1} = L(\xi_{l-1}, \bigcap_{k=1}^n \tilde{G}_k)\). To proceed the general case we apply the finite case that we have proved. Then for each \(n\) we find a family of pairwise disjoint sets \(G_k^{(n)}, k = 1, 2, \ldots, n\) such that \(\mu(G_k^{(n)}) = \xi_k\) for \(1 \leq k \leq n\). Applying Lemma 2.5 and analyzing once again the leftmost selection argument of the tilde sets, one can observe that
\[
\mu(G_k^{(n+1)} \Delta G_k^{(n)}) \leq \sum_{j=k}^n \mu(G_k^{(n+1)} \cap G_j^{(n)}) \leq \xi_{n+1}.
\]
So, we conclude that
\[
\mu(G_k^{(n+1)} \Delta G_k^{(n)}) \leq \sum_{k=n+1}^m \xi_k, \quad m > n \geq k.
\]
The last inequality implies that for a fixed \(k\) the sequence \(\mathbb{I}_{G_k^{(n+1)}}\) converges in \(L^1\)-norm as \(m \to \infty\). Moreover, one can see that the limiting function is again an indicator function of a set \(\tilde{G}_k\), and the sequence \(\tilde{G}_k\) satisfies the conditions of Lemma 2.6.

Lemma 2.7. Let \((X, \mathcal{M}, \mu)\) be a non-atomic measure space, and let \(f \in L^r(X), 1 \leq r < \infty\), be a boundedly supported positive function. Then for any \(\lambda > 0\) there exists a measurable set \(E_\lambda \subset X\) such that
\[
\mu(E_\lambda) \leq \|f\|_r^r / \lambda^r, \quad \{x \in X : M_r f(x) > \lambda\} \subset E_\lambda,
\]
and the function
\[
g(x) = f(x) \cdot \mathbb{I}_{X \setminus E_\lambda}(x) + \lambda \cdot \mathbb{I}_{E_\lambda}(x)
\]
satisfies the conditions
\[
g(x) \leq \lambda \text{ a.e. on } X, \quad \langle f \rangle_{B_r^*} \lesssim \langle g \rangle_{B_r^*} \text{ whenever } B \in \mathcal{B}, B \notin E_\lambda.
\]

Proof. Applying Lemma 2.4 we find a sequence of pairwise disjoint maximal \(\lambda\)-balls \(B_k\) satisfying (2.1). Thus, applying the density property (Lemma 2.2), one can conclude that
\[
f(x) \leq \lambda \text{ for a.a. } x \in X \setminus \bigcup_k B_k^*.
\]
Given $B_k$, we associate the family of balls

$$\mathfrak{B}_k = \{B \in \mathfrak{B} : B \cap B_k^* \neq \emptyset, \mu(B) > 2\mu(B_k^*)\}. \tag{2.8}$$

Observe that if one of these families, say $\mathfrak{B}_{k_0}$, is empty, then in view of conditions B2) and B4), one can easily check that $X \subset B_{k_0}^*$. Then defining $E_\lambda = X$, the claim of the lemma will be satisfied. Hence we can assume that each $\mathfrak{B}_k$ is nonempty, and so, there is a ball $G_k \in \mathfrak{B}_k$ such that

$$\mu(G_k) \leq 2 \inf_{B \in \mathfrak{B}_k} \mu(B). \tag{2.9}$$

From $\lambda$-maximality of $B_k$ and the inequality $\mu(G_k) > 2\mu(B_k^*)$, we get $B_k^* \subset G_k^*$. $\langle f \rangle_{G_k^*,r} \leq \lambda$. This implies

$$\frac{1}{\lambda^r} \int_{G_k^*} f^r \leq \mu(G_k)^r \leq c \cdot \mu(G_k), \tag{2.10}$$

where $c > 0$ is an admissible constant. Denote

$$D_1 = B_1^*, \quad D_k = B_k^* \setminus \bigcup_{1 \leq j \leq k-1} B_j^*, \quad k \geq 2,$$

and consider the numerical sequence $\xi_k = \frac{\lambda}{\lambda^r} \int_{D_k} f^r$, $k = 1, 2, \ldots$, for some constant $\delta > 0$. Taking into account (2.10), for a small admissible constant $\delta > 0$ we obtain

$$\bigcup_{j : \mu(G_j) \leq \mu(G_k), G_j \cap G_k \neq \emptyset} \xi_j = \frac{\lambda}{\lambda^r} \int_{\bigcup_{j : \mu(G_j) \leq \mu(G_k), G_j \cap G_k \neq \emptyset} D_j} f^r \leq \frac{\delta}{\lambda^r} \int_{G_k^*} f^r \leq c \delta \mu(G_k) \leq \mu(G_k),$$

which gives condition (2.2). Since our measure space in non-atomic, applying Lemma 2.6, we find pairwise disjoint subsets $\tilde{G}_k \subset G_k$ such that

$$\mu(\tilde{G}_k) = \frac{\delta}{\lambda^r} \int_{D_k} f^r, \quad k = 1, 2, \ldots. \tag{2.11}$$

The disjointness of the sets $D_k$ implies

$$\sum_k \mu(\tilde{G}_k) = \frac{\delta}{\lambda^r} \sum_k \int_{D_k} f^r \approx \frac{\|f\|_r}{\lambda^r}. \tag{2.12}$$

From the $\lambda$-maximality and disjointness property of $B_k$, we get

$$\mu \left( \bigcup_k B_{k^*}^* \right) \lesssim \sum_k \mu(B_k) \leq \frac{1}{\lambda^r} \sum_k \int_{B_k^*} f^r \leq \frac{\|f\|_r}{\lambda^r}. \tag{2.13}$$

Denote $E_\lambda = \left( \bigcup_k \tilde{G}_k \right) \cup \left( \bigcup_k B_{k^*}^* \right)$. From (2.12) and (2.13) we get $\mu(E_\lambda) \lesssim \|f\|_r / \lambda^r$, and (2.7) implies (2.6). Hence it remains to prove that the function $g$ satisfies (2.6). Take a ball $B \in \mathfrak{B}$ with $B \not\subset E_\lambda$. First of all observe that for each $B_k$ satisfying $B \cap B_k^* \neq \emptyset$ we have $\mu(B) > 2\mu(B_k^*)$, since otherwise we would have $B \subset B_{k^*}^* \subset E_\lambda$, which is not true. Thus, whenever $B \cap B_k^* \neq \emptyset$ we have $B \in \mathfrak{B}_k$, then we get $\mu(G_k) \leq 2\mu(B_k^*)$, and so $\tilde{G}_k \subset G_k \subset B_{k^*}^*$. Besides, from (2.7) and the definition of $g$ it follows that $f(x) \leq g(x)$ a.e. on $X \setminus \bigcup_k B_{k^*}^*$. Hence, using (2.11) and the disjointness of $\tilde{G}_k$, we can write

$$\langle f \rangle_{B^*,r} = \frac{1}{\mu(B)} \left( \int_{B \cap \bigcup_k B_k^*} f^r + \int_{B \setminus \bigcup_k B_k^*} f^r \right)$$

$$\leq \frac{1}{\mu(B)} \left( \sum_{k : B_k^* \cap B \neq \emptyset} \int_{B \cap D_k} f^r + \int_{B \setminus D_k} \frac{\lambda}{\delta} \mu(\tilde{G}_k) \right) \leq \frac{1}{\mu(B)} \left( \sum_{k : B_k^* \cap B \neq \emptyset} \int_{D_k} f^r + \int_{B_k^*} \frac{\lambda}{\delta} \mu(\tilde{G}_k) \right)$$

$$= \frac{1}{\mu(B)} \left( \sum_{k : B_k^* \cap B \neq \emptyset} \frac{\lambda}{\delta} \mu(\tilde{G}_k) + \int_{B_k^*} \frac{\lambda}{\delta} \mu(\tilde{G}_k) \right) \approx \frac{1}{\mu(B)} \left( \sum_{k : B_k^* \cap B \neq \emptyset} \int_{B_k^*} \frac{\lambda}{\delta} \mu(\tilde{G}_k) + \int_{B_k^*} g^r \right) \lesssim \langle g \rangle_{B^*,r}.$$

This implies (2.6), and completes the proof of Lemma 2.7.
3. PROOF OF THEOREM 1.1

Proof of $L^p$-boundedness. For any $B \in \mathcal{S}$ we have $\langle f \rangle_{B,r} \leq M_r f(x)$ for all $x \in B$, and therefore $\langle f \rangle_{B,r} \leq \langle M_r f \rangle_{B,r}$, $B \in \mathcal{B}$. Let $E_B$ be the disjoint portions of the sparse collection of balls satisfying $\mu(E_B) \geq \gamma \cdot \mu(B)$. Also, suppose that $r < p < \infty$ and $q = p/(p-1)$. Thus, for positive functions $f \in L^p$ and $g \in L^q(X)$, we can write

$$\int_X A^*_{r,s} f \cdot g d\mu \leq \sum_{B \in S} \langle M_r f \rangle_{B,r} \int_B g d\mu = \sum_{B \in S} \langle M_r f \rangle_{B,r} \cdot \langle g \rangle_{B,1} \cdot \mu(B)$$

$$\leq \gamma^{-1} \sum_{B \in S} \langle M_r f \rangle_{B,r} \cdot (\mu(E_B))^{1/p} \cdot \langle g \rangle_{B,1} \cdot (\mu(E_B))^{1/q}$$

$$\leq \gamma^{-1} \left( \sum_{B \in S} \langle M_r f \rangle_{B,r}^{p} \cdot \mu(E_B) \right)^{1/p} \cdot \left( \sum_{B \in S} \langle g \rangle_{B,1}^{q} \cdot \mu(E_B) \right)^{1/q}$$

$$\leq \gamma^{-1} \left\| M_r f \right\|_p \left\| M_1 (g) \right\|_q \lesssim \left\| M_r f \right\|_p \cdot \left\| g \right\|_q \lesssim \left\| f \right\|_p \cdot \left\| g \right\|_q,$$

which completes the proof of $L^p$-boundedness.

Proof of Weak-$L^r$ estimate. Without loss of generality, we can assume that our measure space $(X, \mathcal{M}, \mu)$ is non-atomic, since any measure space can be extended to a non-atomic measure space by splitting the atoms as follows. Suppose $A \subset \mathcal{M}$ is the family of atomic elements of the measure space $(X, \mathcal{M}, \mu)$, that is, for any $a \in A$ we have $\mu(a) > 0$ and there is no proper $\mathcal{M}$-measurable set in $a$. We can suppose that each atom is continuum and let $(a_0, \mathcal{M}_0, \mu_0)$ be a non-atomic measure space on $a_0 \in A$ such that $\mu(a) = \mu(a_0)$. Denote by $\mathcal{M}'$ the $\sigma$-algebra on $X$ generated by $\mathcal{M}$ and by all $\mathcal{M}_0$, $a \in A$. Let $\mu'$ be an extension of $\mu$ such that $\mu'(E) = \mu_a(E)$ for any $\mathcal{M}_0$-measurable set $E \subset a$. Hence, $(X, \mathcal{M}', \mu')$ provides a non-atomic extension of the measure space $(X, \mathcal{M}, \mu)$.

Now let $f$ be a $\mathcal{M}$-measurable function. The balls are $\mathcal{M}$-measurable, and so they can not contain an atom $a$ partially. Thus, the left and right sides of inequality (1.3) are not changed if we consider $(X, \mathcal{M}', \mu')$ instead of the initial measure space. Hence, we can suppose that $(X, \mathcal{M}, \mu)$ is itself non-atomic. Applying Lemma 2.7, we find a function $g$ satisfying the conditions of the lemma. From (2.6) we get $\langle f \rangle_{B,r} \leq \langle g \rangle_{B,r}$ for any $B \in \mathcal{S}$ with $B \not\subset E_\lambda$ and hence, $A^*_{r,s} f(x) \leq A^*_{r,s} g(x)$, $x \in X \setminus E_\lambda$. Therefore, using the $L^{2r}$ bound of $A^*_{r,s}$, we obtain

$$\mu\{x \in X : A^*_{r,s} f(x) > \lambda\} \leq \mu(E_\lambda) + \mu\{x \in X \setminus E_\lambda : A^*_{r,s} g(x) > \lambda\}$$

$$\leq \frac{\|f\|_{2r}}{\lambda^r} + \frac{1}{\lambda^r} \int_{X \setminus E_\lambda} |g|^2 \leq \frac{\|f\|_{2r}}{\lambda^r} + \frac{\lambda^r}{\lambda^r} \int_{X \setminus E_\lambda} f^r \leq \frac{2\|f\|_{2r}}{\lambda^r}.$$ 

This completes the proof of Theorem 1.1.

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