A new rearrangement inequality and its application for $L^2$-constraint minimizing problems

Masataka Shibata

Received: 11 February 2016 / Accepted: 6 November 2016 / Published online: 19 December 2016
© Springer-Verlag Berlin Heidelberg 2016

Abstract In this paper, we introduce a new type rearrangement inequality based on the Steiner rearrangement. To show orbital stability of standing wave solutions with respect to a nonlinear Schrödinger equation, $H^1$-precompactness of minimizing sequence of $L^2$-constraint minimizing problem is very important. Usually, by using the concentration compactness principle and some scaling argument, $H^1$-precompactness is obtained. To prove $H^1$-precompactness without scaling arguments, we introduce the rearrangement. The Steiner rearrangement is defined as a map from $H^1$ to $H^1$, whereas our rearrangement $(\cdot \star \cdot)$ is defined as a map from $H^1 \times H^1$ to $H^1$. By using the rearrangement, we show a strict inequality $\|\nabla (u \star v)\|_{L^2}^2 < \|\nabla u\|_{L^2}^2 + \|\nabla v\|_{L^2}^2$ under simple assumptions.

Keywords Rearrangement inequalities · Minimizing sequences · Nonlinear Schrödinger equations

Mathematics Subject Classification 35A23 · 35J20 · 35J50

1 Introduction

In this paper, we show a new rearrangement inequality and give some applications to $L^2$-constraint minimizing problems. In order to explain, we consider the following variational problem.

$$E_\alpha = \inf_{u \in M_\alpha} I(u),$$

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{1}{p + 1} \int_{\mathbb{R}^N} |u|^{p+1} dx,$$

$$M_\alpha = \left\{ u \in H^1(\mathbb{R}^N); \|u\|_{L^2(\mathbb{R}^N)}^2 = \alpha \right\}.$$
where $\alpha > 0$ is a given constant and $N \geq 1$. In this problem, it is well-known that $E_\alpha > -\infty$ if $1 < p < 1 + 4/N$, and we can expect the existence of a global minimizer.

Here, we recall the Schwartz rearrangement. For $u \in H^1(\mathbb{R}^N)$, we denote by $u^*$ the Schwartz rearrangement of $u$. It is well known that $u$ and $u^*$ are equimeasurable, $\|u\|_{L^r(\mathbb{R}^N)} = \|u^*\|_{L^r(\mathbb{R}^N)}$ for any $r \geq 1$, and

$$\int_{\mathbb{R}^N} |\nabla u^*|^2 \, dx \leq \int_{\mathbb{R}^N} |\nabla u|^2 \, dx. \tag{1.1}$$

Thus $\{u_n^*\}_{n \in \mathbb{N}} \subset M_\alpha$ is a minimizing sequence for any minimizing sequence $\{u_n\}_{n \in \mathbb{N}} \subset M_\alpha$. Therefore we can use compactness of the embedding $H^1_{rad}(\mathbb{R}^N) \subset L^{2+4/N}(\mathbb{R}^N)$ to obtain a minimizer $u \in M_\alpha$.

In addition, precompactness of any given minimizing sequence is important. Let $u$ be a global minimizer then $u$ is a solution of

$$-\Delta u + \mu u = |u|^{p-1} u \text{ in } \mathbb{R}^N,$$

where $\mu$ is a Lagrange multiplier. Put $v(t, x) = e^{it\mu}u(x)$ then $v$ is a standing wave of the following nonlinear Schrödinger equation.

$$i v_t = \Delta v + |v|^{p-1} v.$$

In [2], by using $H^1$-precompactness of any minimizing sequences, they showed orbital stability of the set of global minimizers. For this purpose, the subadditivity condition

$$E_{\alpha+\beta} < E_\alpha + E_\beta \tag{1.2}$$

plays an important rule. The subadditivity condition exclude the dichotomy of minimizing sequences, and it implies $H^1$-precompactness. In addition, the scaling arguments has been used to show the subadditivity condition. In this paper, we give an another proof to obtain the subadditivity condition. Let $u \in M_\alpha$ and $v \in M_\beta$ be a minimizer of $E_\alpha$ and $E_\beta$. We construct a function $w$ satisfying the following inequality.

$$\|w\|_{L^r} = \|u\|_{L^r} + \|v\|_{L^r},$$

$$\int_{\mathbb{R}^N} |\nabla w|^2 \, dx < \int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \int_{\mathbb{R}^N} |\nabla v|^2 \, dx \tag{1.3}$$

for $r = 2$, $p + 1$. The strict inequality (1.3) is useful. Thus we can obtain $w \in M_{\alpha+\beta}$ and

$$E_{\alpha+\beta} \leq I(w) < I(u) + I(v) = E_\alpha + E_\beta.$$

Hence (1.2) holds. Our main result is to construct such $w$ by using a new rearrangement. Since it does not require scaling arguments, we can apply $L^2$-constraint minimizing problem related to nonlinear elliptic systems. We remark that our new rearrangement based on the Schwartz rearrangement has been applied in [5]. In [5, Appendix A.1], the definition and properties of the rearrangement were presented. In this paper, we introduce a new rearrangement based on the Steiner rearrangement and give another proof of [5, Appendix A.1].

This paper is organized as follows. In Sect. 2, we introduce a new rearrangement and state our main theorem. In Sect. 3, we state application to the subadditivity condition. In Sect. 4, we state application to nonlinear elliptic systems.
2 Rearrangement

In this section, we introduce a new rearrangement and show our main results. For the purpose, we recall the Steiner rearrangement. About the rearrangement, see, for example [6,7].

2.1 The Steiner rearrangement

In the following, we write $x = (x_1, x')$ with $x_1 \in \mathbb{R}$, $x' \in \mathbb{R}^{N-1}$ and we denote by $L^i$ the $i$-dimensional Lebesgue measure. Let $u$ be a function satisfies the following condition (A).

(A) $u : \mathbb{R}^N \rightarrow \mathbb{R}$: measurable, $\lim_{|x|\rightarrow \infty} u(x) = 0$.

We denote by $u^*$ the Steiner symmetric rearrangement of $u$. The Steiner symmetric rearrangement $u^*$ is a function which satisfies the following properties:

- $x_1 \mapsto u(x_1, x')$ is symmetric with respect to the origin and non-increasing with respect to $|x_1|$ for any $x' \in \mathbb{R}^{N-1}$.
- $u^*(\cdot, x')$ is equimeasurable with $u(\cdot, x')$ for any $x' \in \mathbb{R}^{N-1}$. That is, for any $t > 0$, $x' \in \mathbb{R}^{N-1}$,

$$L^1 \left( \{ x_1 \in \mathbb{R}; |u(x_1, x')| > t \} \right) = L^1 \left( \{ x_1 \in \mathbb{R}; u^*(x_1, x') > t \} \right).$$

More precisely, the Steiner rearrangement $u^*$ is defined by

$$u^*(x_1, x') = \int_0^\infty \chi_{\{|u(\cdot, x')| > t\}}(x_1) dt,$$

where $A^*$ is the Steiner rearrangement of $A$ defined by

$$A^* = (\frac{-L^1(A)}{2}, \frac{L^1(A)}{2}).$$

We remark that the Steiner rearrangement is defined under more general assumptions. However, for simplicity, we assume the condition (A). About the Steiner rearrangement, we summarize well-known facts as follows.

**Proposition 2.1** Assume $u$ satisfies (A) and let $u^*$ be the Steiner symmetric rearrangement of $u$. Then

(i) $u^*$ is measurable in $\mathbb{R}^N$. Moreover, $|u|$ and $u^*$ is equimeasurable in $\mathbb{R}^N$, that is,

$$L^N \left( \{ x \in \mathbb{R}^N; |u(x)| > t \} \right) = L^N \left( \{ x \in \mathbb{R}^N; u^*(x) > t \} \right) \text{ for any } t > 0.$$

(ii) Let $\Phi_1, \Phi_2 : [0, \infty) \rightarrow \mathbb{R}$ be monotone functions. For $\Phi = \Phi_1 + \Phi_2$,

$$\int_{\mathbb{R}^N} \Phi(u^*) dx = \int_{\mathbb{R}^N} \Phi(|u|) dx$$

holds if

$$\left| \int_{\mathbb{R}^N} \Phi_1(|u|) dx \right| < \infty \text{ or } \left| \int_{\mathbb{R}^N} \Phi_2(|u|) dx \right| < \infty.$$

In particular,

$$\int_{\mathbb{R}^N} |u^*|^p dx = \int_{\mathbb{R}^N} |u|^p dx$$

for $1 \leq p < \infty$.

(iii) Assume $1 \leq p < \infty$. If $u \in W^{1,p}(\mathbb{R}^N)$, it holds that $u^* \in W^{1,p}(\mathbb{R}^N)$. Moreover,

$$\int_{\mathbb{R}^N} |\partial_i u^*|^p dx \leq \int_{\mathbb{R}^N} |\partial_i u|^p dx \text{ for } i = 1, \ldots, N.$$
2.2 Coupled rearrangement

Now we introduce a new rearrangement which we call **coupled rearrangement**. Suppose \( u \) and \( v \) satisfy the condition (A). The coupled rearrangement \( u \star v \) of \( u \) and \( v \) is defined as follows. For any \( x' \in \mathbb{R}^{N-1}, x_1 \mapsto (u \star v)(x_1, x') \) is symmetric with respect to the origin and monotone with respect to \( |x_1| \). For any \( t > 0, x' \in \mathbb{R}^{N-1}, \)

\[
\mathcal{L}^1 \left( \{ x_1 \in \mathbb{R}; |u(x_1, x')| > t \} \right) + \mathcal{L}^1 \left( \{ x_1 \in \mathbb{R}; |v(x_1, x')| > t \} \right) = \mathcal{L}^1 \left( \{ x_1 \in \mathbb{R}; (u \star v)(x_1, x') > t \} \right).
\]

More precisely, \( u \star v \) is defined by

\[
(u \star v)(x_1, x') = \int_0^\infty \chi_{\{ |u(\cdot, x')| > t\}} \star \chi_{\{ |v(\cdot, x')| > t\}}(x_1) dt,
\]

where

\[
A \star B = \left( -\frac{\mathcal{L}^1(A) + \mathcal{L}^1(B)}{2} \right), \left( \frac{\mathcal{L}^1(A) + \mathcal{L}^1(B)}{2} \right).
\]

About the coupled rearrangement, we can show similar properties as follows. We give the proofs in the next subsection.

**Lemma 2.2** Assume \( u \) and \( v \) satisfy the condition (A) and let \( u \star v \) be the coupled rearrangement of \( u \) and \( v \). Then,

(i) \( u \star v \) is measurable in \( \mathbb{R}^N \). Moreover,

\[
\mathcal{L}^N \left( \{ x \in \mathbb{R}^N; |u(x)| > t \} \right) + \mathcal{L}^N \left( \{ x \in \mathbb{R}^N; |v(x)| > t \} \right) = \mathcal{L}^N \left( \{ x \in \mathbb{R}^N; (u \star v)(x) > t \} \right) \quad \text{for any } t > 0.
\]

(ii) Let \( \Phi_1, \Phi_2 : [0, \infty) \to \mathbb{R} \) be monotone functions. For \( \Phi = \Phi_1 + \Phi_2, \)

\[
\int_{\mathbb{R}^N} \Phi(u \star v) dx = \int_{\mathbb{R}^N} \Phi(|u|) dx + \int_{\mathbb{R}^N} \Phi(|v|) dx.
\]

If

\[
\left| \int_{\mathbb{R}^N} \Phi_1(|u|) dx \right|, \left| \int_{\mathbb{R}^N} \Phi_2(|v|) dx \right| < \infty \quad \text{or} \quad \left| \int_{\mathbb{R}^N} \Phi_1(|u|) dx \right|, \left| \int_{\mathbb{R}^N} \Phi_2(|v|) dx \right| < \infty
\]

holds. In particular,

\[
\int_{\mathbb{R}^N} |u \star v|^p dx = \int_{\mathbb{R}^N} |u|^p dx + \int_{\mathbb{R}^N} |v|^p dx
\]

holds for any \( p \geq 1 \).

**Lemma 2.3** Assume \( 1 \leq p < \infty \). \( u \) and \( v \) satisfy the condition (A) and \( u, v \in W^{1,p}(\mathbb{R}^N) \). Then it holds that

\[
\int_{\mathbb{R}^N} |\partial_i (u \star v)|^p dx \leq \int_{\mathbb{R}^N} |\partial_i u|^p dx + \int_{\mathbb{R}^N} |\partial_i v|^p dx \quad \text{for } i = 1, \ldots, N.
\]

Our main theorem is the following strict inequality.
Theorem 2.4 For $u, v \in W^{1,p}(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)$ satisfying that $u, v > 0$, $\lim_{|x| \to \infty} v(x) = \lim_{|x| \to \infty} u(x) = 0$, and $u(x, x')$ and $v(x, x')$ are monotone decreasing with respect to $|x_1|$. Then, the following strict inequality holds.

$$\int_{\mathbb{R}^N} |\nabla(u \ast v)|^p \, dx < \int_{\mathbb{R}^N} |\nabla u|^p \, dx + \int_{\mathbb{R}^N} |\nabla v|^p \, dx.$$  

2.3 Proof of Lemma 2.2 and 2.3

In this subsection, we give the proofs of Lemma 2.2 and 2.3. To our purpose, we prepare the following lemma.

Lemma 2.5 Assume that $u$ and $v$ satisfy the condition (A). Then the following properties hold.

(i) The Steiner rearrangement and the coupled rearrangement are invariant to translation of $x_1$ direction. That is, for $s, t \in \mathbb{R}$, $\tilde{u}^* = u^*$ and $\tilde{u} \ast \tilde{v} = u \ast v$ hold, where $\tilde{u}(x_1, x') = u(x_1 + s, x')$, $\tilde{v}(x_1, x') = v(x_1 + t, x')$.

(ii) If $\sup u \cap \sup v = \emptyset$, it holds that $u \ast v = (u + v)^*$.

(iii) For $s > 0$, it holds that $(|u| - s)_+ \ast (|v| - s)_+ = (u \ast v - s)_+$.

Proof (i) It is clear by the definition of the coupled rearrangement.

(ii) It is sufficient to show

$$\mathcal{L}^1 \left( \{ x_1 \in \mathbb{R}; (u \ast v)(x_1, x') > t \} \right) = \mathcal{L}^1 \left( \{ x_1 \in \mathbb{R}; (u + v)^*(x_1, x') > t \} \right) \quad \text{for } t > 0, x' \in \mathbb{R}^{N-1}. \tag{2.2}$$

Fix $t > 0, x' \in \mathbb{R}^{N-1}$. By the definition of the Steiner rearrangement, we have

$$\mathcal{L}^1 \left( \{ x_1 \in \mathbb{R}; (u + v)^*(x_1, x') > t \} \right) = \mathcal{L}^1 \left( \{ x_1 \in \mathbb{R}; |u + v|(x_1, x') > t \} \right).$$

Since $\sup u \cap \sup v = \emptyset$, we have

$$\mathcal{L}^1 \left( \{ x_1 \in \mathbb{R}; |u + v|(x_1, x') > t \} \right) = \mathcal{L}^1 \left( \{ x_1 \in \mathbb{R}; |u|(x_1, x') > t \} \right) + \mathcal{L}^1 \left( \{ x_1 \in \mathbb{R}; |v|(x_1, x') > t \} \right).$$

On the other hand, by the definition of the coupled rearrangement, we have

$$\mathcal{L}^1 \left( \{ x_1 \in \mathbb{R}; (u \ast v)(x_1, x') > t \} \right) = \mathcal{L}^1 \left( \{ x_1 \in \mathbb{R}; |u|(x_1, x') > t \} \right) + \mathcal{L}^1 \left( \{ x_1 \in \mathbb{R}; |v|(x_1, x') > t \} \right).$$

Consequently, (2.2) holds.

(iii) By the definition of the coupled rearrangement, we can obtain that

$$\mathcal{L}^1 \left( \{ x_1 \in \mathbb{R}; ((|u| - s)_+ \ast (|v| - s)_+)(x_1, x') > t \} \right) = \mathcal{L}^1 \left( \{ x_1 \in \mathbb{R}; (|u| - s)_+(x_1, x') > t \} \right) + \mathcal{L}^1 \left( \{ x_1 \in \mathbb{R}; (|v| - s)_+(x_1, x') > t \} \right)$$

$$\mathcal{L}^1 \left( \{ x_1 \in \mathbb{R}; |u|(x_1, x') > s + t \} \right) + \mathcal{L}^1 \left( \{ x_1 \in \mathbb{R}; |v|(x_1, x') > s + t \} \right)$$

$$\mathcal{L}^1 \left( \{ x_1 \in \mathbb{R}; (u \ast v)(x_1, x') > s + t \} \right).$$
Proof of Lemma 2.2 and 2.3 First, we show Lemma 2.2. Fix \( s > 0 \). Since \( \lim_{|x| \to \infty} u(x) = \lim_{|x| \to \infty} v(x) = 0 \), there exists a positive constant \( R = R(s) \) such that

\[
\{ x; |u(x)| > s \}, \{ x; |v(x)| > s \} \subset B(0, R).
\]

Putting \( x_s = (3R, 0) \) and \( v_s(x_1, x') = v(x_1 - 3R, x') \), we have

\[
\{ x; |v_s(x)| > s \} \subset B(x_s, R).
\]

Especially, we obtain

\[
supp(|u| - s)_+ \cap supp(|v_s| - s)_+ = \emptyset.
\] (2.3)

By using Lemma 2.5 (iii), (i), and (ii), we have

\[
(u \ast v - s)_+ = (|u| - s)_+ \ast (|v| - s)_+
= (|u| - s)_+ \ast (|v_s| - s)_+
= ([|u| - s]_+ + (|v_s| - s)_+)^*.
\]

Thus we get

\[
(u \ast v - s)_+ = ([|u| - s]_+ + (|v_s| - s)_+)^* \quad \text{for any } s > 0.
\] (2.4)

Therefore, \((u \ast v - s)_+\) is Lebesgue measurable for any \( s > 0 \). It means that \( u \ast v \) is Lebesgue measurable. Moreover, by using Proposition 2.1 (i), (2.3), and (2.4), we have

\[
\mathcal{L}^N \left( \left\{ x \in \mathbb{R}^N; (u \ast v)(x) > s \right\} \right)
= \mathcal{L}^N \left( \left\{ x \in \mathbb{R}^N; (u \ast v)(x) - s/2)_+ > s/2 \right\} \right)
= \mathcal{L}^N \left( \left\{ x \in \mathbb{R}^N; \{|u(x)| - s/2)_+ + (|v_s(x)| - s/2)_+^* > s/2 \right\} \right)
= \mathcal{L}^N \left( \left\{ x \in \mathbb{R}^N; (|u(x)| - s)_+ + (|v_s(x)| - s)_+ > s/2 \right\} \right)
= \mathcal{L}^N \left( \left\{ x \in \mathbb{R}^N; (|u(x)| - s/2)_+ > s/2 \right\} \right)
+ \mathcal{L}^N \left( \left\{ x \in \mathbb{R}^N; (|v_s(x)| - s/2)_+ > s/2 \right\} \right)
= \mathcal{L}^N \left( \left\{ x \in \mathbb{R}^N; |u(x)| > s \right\} \right) + \mathcal{L}^N \left( \left\{ x \in \mathbb{R}^N; |v(x)| > s \right\} \right).
\] (2.5)

Thus Lemma 2.2 (i) holds. Since \( \Phi_1 \) and \( \Phi_2 \) are monotone, \( \Phi = \Phi_1 + \Phi_2 \) are Borel measurable. Thus we can show Lemma 2.2 (i) by using (2.5). For details, see [7, Section 1.13 and 3.3].

Next, to show Lemma 2.3, we assume that \( u, v \in W^{1,p}(\mathbb{R}^N) \). Fix \( i \in \{1, \ldots, N\} \) and \( s > 0 \). By using Proposition 2.1 (iii),

\[
\int_{\mathbb{R}^N} |\partial_i ([|u| - s]_+ + (|v_s| - s)_+^*)|^p dx \leq \int_{\mathbb{R}^N} |\partial_i ([|u| - s]_+ + (|v_s| - s)_+)|^p
= \int_{\mathbb{R}^N} |\partial_i |u| - s_+|^p + \int_{\mathbb{R}^N} |\partial_i |v_s| - s_+|^p
= \int_{\{x; u(x) > s\}} |\partial_i u|^p + \int_{\{x; v_s(x) > s\}} |\partial_i v|^p
\leq \int_{\mathbb{R}^N} |\partial_i u|^p + \int_{\mathbb{R}^N} |\partial_i v|^p.
\]
On the other hand, we have
\[
\int_{\mathbb{R}^N} |\partial_i (u \star v - s)|^p dx = \int_{\{x; (u \star v) (x) > s\}} |\partial_i (u \star v)|^p dx
\]
\[
= \int_{\mathbb{R}^N} |\partial_i (u \star v)|^p \chi_{\{x; (u \star v) (x) > s\}} dx.
\]
Therefore, by (2.4), we obtain
\[
\int_{\mathbb{R}^N} |\partial_i (u \star v)|^p \chi_{\{x; (u \star v) (x) > s\}} dx \leq \int_{\mathbb{R}^N} |\partial_i u|^p + \int_{\mathbb{R}^N} |\partial_i v|^p
\]
for any \(s > 0\).

Since \(\{x; (u \star v) (x) > s\}\) converges to \(\mathbb{R}^N\) monotonically as \(s \to 0\), we can apply the monotone convergence theorem to obtain
\[
\lim_{s \to 0} \int_{\mathbb{R}^N} |\partial_i (u \star v)|^p \chi_{\{x; (u \star v) (x) > s\}} dx = \int_{\mathbb{R}^N} |\partial_i (u \star v)|^p dx.
\]
It means the conclusion. \(\square\)

2.4 Proof of Theorem 2.4

To prove Theorem 2.4, the next lemma is essential.

Lemma 2.6 Assume \(f, g \in C^1 (\mathbb{R}, \mathbb{R})\), \(f, g > 0\), \(\lim_{|x| \to \infty} f (x) = \lim_{|x| \to \infty} g (x) = 0\), and \(f\) and \(g\) are non-increasing with respect to \(|x|\). Then the strict inequality
\[
\int_{\mathbb{R}} |(f \star g)' (x)|^p dx < \int_{\mathbb{R}} |f' (x)|^p dx + \int_{\mathbb{R}} |g' (x)|^p dx
\]
holds for \(1 \leq p < \infty\).

The key ingredient of the proof of Lemma 2.6 is the quantitative version of the decreasing rearrangement inequality. Here we recall the decreasing rearrangement as follows. Let \(f \in PC^1 ([0, b])\) and let \(\mu (\lambda) = L^\infty \{x \in [0, b]; f (x) > \lambda\}\), \(\lambda \in \mathbb{R}\), where \(PC^1 ([0, b])\) is the set of piecewise \(C^1\) functions. \(f^\# (x) = \mu^{-1} (1) (x \in [0, b])\) is called the decreasing rearrangement of \(f\). \(N_f (\lambda)\) is the multiplicity of \(f\) at the level \(\lambda\), that is,
\[
N_f (\lambda) = \# \{y \in [a, b]; f (y) = \lambda\},
\]
where \(#A\) means the number of elements of the set \(A\). Then we have the following key results.

Theorem 2.7 ([3, Theorem 1]). Let \(f^\#\) be the decreasing rearrangement of \(f \in PC^1 ([0, b])\). For any \(p \geq 1\), the following inequality holds:
\[
\int_{0}^{b} |(f^\#)' (x)|^p dx \leq \int_{0}^{b} \left| \frac{f' (x)}{N_f (f (x))} \right|^p dx.
\]

In [3, Theorem 1], Duff showed Theorem 2.7 for \(f \in C^1 ([0, b])\), but his proof can be modified slightly even for \(f \in PC^1 ([0, b])\).

Proof of Lemma 2.6 First, we prepare the following claim.
Therefore, the claim holds.

Applying Theorem 2.7, we get

$$\int_{-L}^{L} |(f^*)(x)|^p \, dx \leq 2^p \int_{-L}^{L} \left| \frac{f'(y)}{N_f(f(y))} \right|^p \, dy.$$

for any \( f \in PC^1([-L, L]) \) with \( f(-L) = f(L) = 0 \).

Put \( g(x) = f(x - L) \). Then \( g \in PC^1([0, 2L]) \). Since \( f \) and \( g \) are equimeasurable, by using the definition of rearrangements, we can obtain

$$f^*(x) = f^*(-x) = g^#(2x) \quad \text{for} \ x \in [0, L].$$

Thus we have

$$\int_{-L}^{L} |(f^*)(x)|^p \, dx = 2^p \int_{0}^{2L} |(g^#(y))|^p \, dy.$$

Applying Theorem 2.7, we get

$$\int_{0}^{2L} |(g^#(y))|^p \, dy \leq \int_{0}^{2L} \left| \frac{g'(y)}{N_g(g(y))} \right|^p \, dy = \int_{-L}^{L} \left| \frac{f'(y)}{N_f(f(y))} \right|^p \, dy.$$

Therefore, the claim holds.

Next, let \( f \) and \( g \) satisfy the assumptions the lemma. For sufficiently small \( s > 0 \), we have that \( (f - s)_+ \neq 0 \) and \( (g - s)_+ \neq 0 \). Since each support of \( (f - s)_+ \) and \( (g - s)_+ \) is compact, there are large \( x_0 \) and \( L \) such that

$$\text{supp}(f - s)_+ \cap \text{supp}(g(\cdot - x_0) - s)_+ = \emptyset,$$

$$h = \text{supp}(f - s)_+ + \text{supp}(g(\cdot - x_0) - s)_+ \in PC^1([-L, L]),$$

$$h(-L) = h(L).$$

Thus, we can apply the above claim to obtain

$$\int_{-L}^{L} |(h^*)(x)|^p \, dx \leq 2^p \int_{-L}^{L} \left| \frac{h'(y)}{N_h(h(y))} \right|^p \, dy \quad (2.7)$$

By Lemma 2.5 (i) and (iii), we have

$$\int_{\{x: (f \ast g)(x) > s\}} |(f \ast g)'(x)|^p \, dx = \int_{\mathbb{R}} |(f \ast g - s)_+'(x)|^p \, dx = \int_{-L}^{L} |(h^*)(x)|^p \, dx. \quad (2.8)$$

On the other hand, since \( (f - s)_+ \in PC^1([-L, L]), (f - s)_+ \neq 0 \), and \( (f(-L) - s)_+ = (f(L) - s)_+ = 0 \), we have

$$N_f(\lambda) \geq 2 \quad \text{for} \ \lambda \in \left[ 0, \max_{\mathbb{R}} f - s \right].$$

Similarly about \( g \), we have

$$N_g(\lambda) \geq 2 \quad \text{for} \ \lambda \in \left[ 0, \max_{\mathbb{R}} g - s \right].$$

Therefore, we obtain

$$N_h(\lambda) \geq 2 \quad \text{for} \ \lambda \in \left[ 0, \max_{\mathbb{R}} \{ \max_{\mathbb{R}} f, \max_{\mathbb{R}} g \} - s \right].$$
\[
N_h(\lambda) \geq 4 \quad \text{for} \quad \lambda \in \left[0, \min_{\mathbb{R}} \{\max_{\mathbb{R}} f, \max_{\mathbb{R}} g\} - s\right).
\]

It asserts the strict inequality
\[
\left| \frac{h'(y)}{N_h(h(y))} \right|^p \int_{-L}^{L} dy < \frac{1}{2^p} \int_{-L}^{L} |h'(y)|^p dy.
\]

By the definition of \(g\), it is clear that
\[
\int_{\{x: f(x) > s\}} |f'(x)|^p dx + \int_{\{x: g(x) > s\}} |g'(x)|^p dx = \int_{-L}^{L} |h'(x)|^p dx.
\]

Combining (2.7), (2.8), (2.9), and (2.10), we get
\[
\int_{\{x: (f \ast g)(x) > s\}} |(f \ast g)'(x)|^p dx < \int_{\{x: f(x) > s\}} |f'(x)|^p dx + \int_{\{x: g(x) > s\}} |g'(x)|^p dx.
\]

Moreover, we can apply Lemma 2.3 for \(\min\{f, s\}\) and \(\min\{g, s\}\) to obtain
\[
\int_{\{x: (f \ast g)(x) \leq s\}} |(f \ast g)'(x)|^p dx = \int_{\mathbb{R}} |(\min\{f \ast g, s\})'(x)|^p dx
\]
\[
= \int_{\mathbb{R}} |(\min\{f, s\} \ast \min\{g, s\})'(x)|^p dx
\]
\[
\leq \int_{\{x: f(x) \leq s\}} |f'(x)|^p dx + \int_{\{x: g(x) \leq s\}} |g'(x)|^p dx.
\]

(2.11) and (2.12) complete the lemma.

Proof of Theorem 2.4 Let \(u\) and \(v\) be functions satisfying that \(u, v \in W^{1,p}(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)\), \(u, v > 0, \lim_{|x| \to \infty} u(x) = \lim_{|x| \to \infty} v(x) = 0\), and \(u(x_1, x'), v(x_1, x')\) are monotone decreasing with respect to \(|x_1|\). By using Lemma 2.6, we have
\[
\int_{\mathbb{R}} |\partial_1(u \ast v)(x_1, x')|^p dx_1 < \int_{\mathbb{R}} |\partial_1 u(x_1, x')|^p dx_1 + \int_{\mathbb{R}} |\partial_1 v(x_1, x')|^p dx_1
\]
for any \(x' \in \mathbb{R}^{N-1}\). Integrating with respect to \(x'\) over \(\mathbb{R}^{N-1}\), we get
\[
\int_{\mathbb{R}^N} |\partial_1(u \ast v)|^p dx < \int_{\mathbb{R}^N} |\partial_1 u|^p dx + \int_{\mathbb{R}^N} |\partial_1 v|^p dx.
\]

On the other hand, By Lemma 2.3, we have
\[
\int_{\mathbb{R}^N} |\partial_i(u \ast v)|^p dx \leq \int_{\mathbb{R}^N} |\partial_i u|^p dx + \int_{\mathbb{R}^N} |\partial_i v|^p dx \quad \text{for} \quad i = 2, \ldots, N.
\]

Therefore, we obtain the theorem.

3 Application: the subadditivity condition

For given \(\alpha > 0\), we consider the following \(L^2\)-constraint minimizing problem.
\[
E_\alpha = \inf_{u \in M_\alpha} I[u].
\]
\[ I[u] = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx - \int_{\mathbb{R}^N} F(u) dx, \]
\[ M_\alpha = \left\{ u \in H^1(\mathbb{R}^N); \|u\|_{L^2(\mathbb{R}^N)}^2 = \alpha \right\}, \]
where \( F \) satisfies the following assumptions.

(F1) \( f \in C(\mathbb{C}, \mathbb{C}) \), \( f(0) = 0 \).

(F2) \( f(r) \in \mathbb{R} \) for \( r \in \mathbb{R} \), \( f(e^{i\theta}z) = e^{i\theta} f(z) \) for \( \theta \in \mathbb{R}, z \in \mathbb{C} \), and \( F(s) = \int_0^s f(\tau) d\tau \).

(F3) \( \lim_{z \to 0} f(z)/|z| = 0 \).

(F4) \( \lim_{|z| \to \infty} f(z)/|z|^{l-1} = 0 \), where \( l = 2 + 4/N \).

Moreover, we assume that the energy \( E_\alpha \) is negative, that is,

\[ (E1) \quad E_\alpha < 0 \quad \text{for} \quad \alpha > 0. \]

We remark that the condition \((E1)\) is satisfied if \( \lim \inf_{s \to 0} F(s)/s^l = \infty \). (See [9].) In [9], \( H^1 \)-precompactness of minimizing sequences was studied under more general conditions. In this section, we give another proof by using Theorem 2.4.

Throughout this section, we assume \((F1)-(F4)\) and \((E1)\) always. About the energy \( E_\alpha \), the following conditions holds.

**Lemma 3.1** ([9, Lemma 2.3]).

(i) \( E_{\alpha + \beta} \leq E_\alpha + E_\beta \) for any \( \alpha, \beta > 0 \).

(ii) \( E_\alpha < E_\beta \) if \( \alpha > \beta \).

(iii) \( \alpha \mapsto E_\alpha \) is continuous on \([0, \infty)\).

**Lemma 3.2** For any \( \alpha > 0 \), there exists a global minimizer \( u \in M_\alpha \).

By using the Schwartz rearrangement, \((E1)\), and compactness of embedding \( H^1_{\text{rad}}(\mathbb{R}^N) \subset L^p(\mathbb{R}^N) \), we can obtain a global minimizer. We omit the proof of Lemma 3.2.

By using Lemma 3.2 and the coupled rearrangement, we can show the subadditivity condition. Thus we get the following Proposition 3.3.

**Proposition 3.3** Suppose that \((F1)-(F4)\) and \((E1)\). Then, the subadditivity condition \((1.2)\) holds. Moreover, any minimizing sequence \( \{u_n\}_{n \in \mathbb{N}} \subset M_\alpha \) with respect to \( E_\alpha \) is precompact. That is, taking a subsequence if necessary, there exist \( u \in M_\alpha \) and a family \( \{y_n\}_{n \in \mathbb{N}} \subset \mathbb{R}^N \) such that \( \lim_{n \to \infty} u_n(\cdot - y_n) = u \) in \( H^1(\mathbb{R}^N) \). In particular, \( u \) is a global minimizer.

**Proof of Proposition 3.3** By the results in [2], it is sufficient to show the subadditivity condition \((1.2)\). For \( \alpha, \beta > 0 \), Lemma 3.2 asserts that there exist global minimizers \( u \) and \( v \) with respect to \( E_\alpha \) and \( E_\beta \). By the elliptic regularity theory, \( u, v \in C^1(\mathbb{R}^N) \) satisfy the condition \((A)\). Thus we can apply Lemma 2.2 and Theorem 2.4 to obtain

\[ E_{\|u \ast v\|_{L^2(\mathbb{R}^N)}^2} \leq I[u \ast v] < I[u] + I[v] = E_\alpha + E_\beta, \quad \|u \ast v\|_{L^2(\mathbb{R}^N)}^2 = \alpha + \beta. \]

Hence \((1.2)\) holds. \( \square \)

**4 Application to \( L^2 \) constraint minimizing problems related to semi-linear elliptic systems**

In this section, we consider the following \( L^2 \)-constraint minimizing problem.

\[ E_{\alpha, \beta} = \inf_{(u, v) \in M_{\alpha, \beta}} J[u, v], \]
Assume (G1)–(G5), and (E2). For Theorem 4.1

\[ J[u, v] = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + |\nabla v|^2 \, dx - \int_{\mathbb{R}^N} G(|u|^2, |v|^2) \, dx, \]

\[ M_{\alpha, \beta} = \left\{ (u, v) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N); \|u\|_{L^2(\mathbb{R}^N)}^2 = \alpha, \|v\|_{L^2(\mathbb{R}^N)}^2 = \beta \right\}, \]

where \( \alpha \) and \( \beta \) are nonnegative given constants. We assume the nonlinear term \( G(s) = G(s_1, s_2) \) satisfies that

(G1) \( G \in C^1([0, \infty) \times [0, \infty), \mathbb{R}) \), \( G(0) = 0 \).

(G2) \( \lim_{|s| \to 0} g_j(s) = 0 \) (\( j = 1, 2 \)), where \( g_j(s) = \frac{\partial G}{\partial s_j}(s) \) (\( j = 1, 2 \)).

(G3) \( \lim_{|s| \to \infty} g_j(s)/|s|^{2/N} = 0 \) (\( j = 1, 2 \)).

(G4) \( g_j \) is nondecreasing, that is, \( g_j(s, t) \leq g_j(s + h, t + k) \) for \( s, t, h, k \geq 0 \) (\( j = 1, 2 \)).

(G5) There exists \( \sigma > 0 \) such that \( G(s_1, 0) + G(0, s_2) < G(s_1, s_2) \) for \( 0 < s_1, s_2 \leq \sigma \).

Moreover, we suppose that

(E2) \( E_{\alpha, 0}, E_{0, \beta} < 0 \) for any \( \alpha, \beta > 0 \).

This type problem was studied in [4]. In [4], they proved the existence of global minimizers. Our goal in this section is to show \( H^1 \)-precompactness of minimizing sequences as follows.

**Theorem 4.1** Assume (G1)–(G5), and (E2). For \( \alpha, \beta \geq 0 \), any minimizing sequence \( \{ (u_n, v_n) \}_{n \in \mathbb{N}} \subset H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) \) with respect to \( E_{\alpha, \beta} \) is pre-compact. That is, taking a subsequence, there exist \( (u, v) \in M_{\alpha, \beta} \) and \( \{ y_n \}_{n \in \mathbb{N}} \subset \mathbb{R}^N \) such that

\[ u_n \to u, \quad v_n \to v, \quad \text{in} \ H^1(\mathbb{R}^N) \quad \text{as} \ n \to \infty. \]

To prove Theorem 4.1, we prepare the following lemma. We state the proof of the lemma in Appendix.

**Lemma 4.2** The energy \( E_{\alpha, \beta} \) satisfies that

(i) \( E_{\alpha + \alpha', \beta + \beta'} \leq E_{\alpha, \beta} + E_{\alpha', \beta'} \) for \( \alpha, \beta \geq 0 \).

(ii) \( E_{\alpha, \beta} < 0 \) for \( \alpha, \beta \geq 0 \), \( (\alpha, \beta) \neq (0, 0) \).

(iii) \( (\alpha, \beta) \mapsto E_{\alpha, \beta} \) is continuous on \([0, \infty) \times [0, \infty) \) \( \setminus \{ (0, 0) \} \).

Proof of Theorem 4.1 In the case \( \alpha = 0 \) or \( \beta = 0 \), the results are included in Proposition 3.3. So we consider the case \( \alpha, \beta > 0 \). Let \( \{ (u_n, v_n) \}_{n \in \mathbb{N}} \) be a minimizing sequence in \( M_{\alpha, \beta} \). By using the Gagliardo-Nirenberg inequality, we have that \( \{ (u_n, v_n) \}_{n \in \mathbb{N}} \) is bounded in \( H^1(\mathbb{R}^N) \).

(For details, see Lemma 5.1)

**Claim** \( \{ u_n \}_{n \in \mathbb{N}} \) and \( \{ v_n \}_{n \in \mathbb{N}} \) do not both vanish, that is,

\[ \liminf_{n \to \infty} \left( \sup_{y \in \mathbb{R}^N} \int_{B(y, 1)} |u_n|^2 \, dx + \sup_{y \in \mathbb{R}^N} \int_{B(y, 1)} |v_n|^2 \, dx \right) > 0. \]

Suppose that both \( \{ u_n \}_{n \in \mathbb{N}} \) and \( \{ v_n \}_{n \in \mathbb{N}} \) vanish. Then we can apply the P.-L. Lions lemma [8, Lemma I.1] to obtain that \( \lim_{n \to \infty} u_n = \lim_{n \to \infty} v_n = 0 \) in \( L^l(\mathbb{R}^N) \), where \( l = 2 + 4/N \). On the other hand, by (G1)–(G3), for any \( \epsilon > 0 \), there exists a positive constant \( C(G, \epsilon) \) such that

\[ |G(s)| \leq \epsilon (|s_1| + |s_2|) + C(G, \epsilon) (|s_1|^{2/N+1} + |s_2|^{2/N+1}). \]

Therefore we have

\[ J[u_n, v_n] \geq -\epsilon \int_{\mathbb{R}^N} |u_n|^2 + |v_n|^2 \, dx - C(G, \epsilon) \int_{\mathbb{R}^N} |u_n|^l + |v_n|^l \, dx. \]
Since \( \{u_n, v_n\}_{n \in \mathbb{N}} \) is the minimizing sequence over \( M_{\alpha, \beta} \), taking \( n \to \infty \), we have \( E_{\alpha, \beta} \geq -\varepsilon (\alpha + \beta) \). Since \( \varepsilon > 0 \) is arbitrary, \( E_{\alpha, \beta} \geq 0 \). It contradicts to Lemma 4.2 (ii).

In the above claim, we can assume \( \{u_n\}_{n \in \mathbb{N}} \) does not vanish without loss of generality.

**Claim** \( \{v_n\}_{n \in \mathbb{N}} \) does not vanish.

Suppose that \( \{v_n\}_{n \in \mathbb{N}} \) vanish. Since \( \{v_n\}_{n \in \mathbb{N}} \) is bounded in \( H^1(\mathbb{R}^N) \), we can apply the P.-L. Lions lemma to obtain \( \lim_{n \to \infty} v_n = 0 \) in \( L^1(\mathbb{R}^N) \). By using

\[
G(s_1, s_2) - G(s_1, 0) = \int_0^1 \frac{d}{d\theta} G(s_1, \theta s_2) d\theta = \int_0^1 g_2(s_1, \theta s_2)s_2 d\theta
\]

and (G1)–(G3), we have

\[
|G(s_1, s_2) - G(s_1, 0)| \leq \left( \varepsilon + C(G, \varepsilon)(|s_1|^{2/N} + |s_2|^{2/N}) \right) |s_2|.
\]

Thus, we can estimate as

\[
\left| \int_{\mathbb{R}^N} G(|u_n|^2, |v_n|^2) dx - \int_{\mathbb{R}^N} G(|u_n|^2, 0) dx \right|
\leq \varepsilon \beta + C(G, \varepsilon) \int_{\mathbb{R}^N} (|u_n|^{4/N} + |v_n|^{4/N}) |v_n|^2 dx
\leq \varepsilon \beta + C(G, \varepsilon) \left( \|u_n\|_{L^1(\mathbb{R}^N)}^{2/(N+2)} \|v_n\|_{L^1(\mathbb{R}^N)}^{N/(N+2)} + \|v_n\|_{L^1(\mathbb{R}^N)}' \right).
\]

Since \( \lim_{n \to 0} v_n = 0 \) in \( L^1(\mathbb{R}^N) \) and \( \varepsilon > 0 \) is arbitrarily,

\[
\int_{\mathbb{R}^N} G(|u_n|^2, |v_n|^2) dx - \int_{\mathbb{R}^N} G(|u_n|^2, 0) dx = o(1) \quad \text{as } n \to \infty.
\]

Thus we obtain

\[
J[u_n, v_n] \geq J[u_n, 0] + o(1) \geq E_{\alpha, 0} + o(1) \quad \text{as } n \to \infty.
\]

It contradicts to the assumption (E2). Hence \( \{v_n\}_{n \in \mathbb{N}} \) does not vanish.

Since \( \{u_n\}_{n \in \mathbb{N}} \) and \( \{v_n\}_{n \in \mathbb{N}} \) are \( H^1 \)-bounded sequences, taking a subsequence, there exist \( \{y_n\}_{n \in \mathbb{N}} \subset \mathbb{R}^N \), \( u \in H^1(\mathbb{R}^N) \setminus \{0\} \), and \( v \in H^1(\mathbb{R}^N) \) such that

\[
\begin{aligned}
&\{u_n(\cdot - y_n) \rightharpoonup u, \quad v_n(\cdot - y_n) \rightharpoonup v \text{ weakly in } H^1(\mathbb{R}^N),
&u_n(\cdot - y_n) \to u, \quad v_n(\cdot - y_n) \to v \text{ in } L^p_{loc}(\mathbb{R}^N) \text{ for } p \in [1, 2^*),
&u_n(\cdot - y_n) \to u, \quad v_n(\cdot - y_n) \rightharpoonup v \text{ a.e. in } \mathbb{R}^N \text{ as } n \to \infty.
\end{aligned}
\]

Put \( \phi_n = u_n(\cdot - y_n) - u, \psi_n = v_n(\cdot - y_n) - v, \alpha' = \|u\|_{L^2(\mathbb{R}^N)}^2 \) and \( \beta' = \|v\|_{L^2(\mathbb{R}^N)}^2 \).

Then \( 0 < \alpha' \leq \alpha \) and \( 0 \leq \beta' \leq \beta \) hold.

**Claim** \( \alpha' = \alpha. \)

Suppose that the claim does not hold, then \( \alpha' < \alpha \). By (G1)–(G3), we can apply the Brezis-Lieb lemma [1] to obtain

\[
J[u_n, v_n] = J[u, v] + J[\phi_n, \psi_n] + o(1) \geq E_{\alpha', \beta'} + E_{\|\phi_n\|_{L^2(\mathbb{R}^N)}^2, \|\psi_n\|_{L^2(\mathbb{R}^N)}^2} + o(1).
\]

Since \( \lim_{n \to \infty} \|\phi_n\|_{L^2(\mathbb{R}^N)}^2 = \alpha - \alpha' \) and \( \lim_{n \to \infty} \|\psi_n\|_{L^2(\mathbb{R}^N)}^2 = \beta - \beta' \), by Lemma 4.2 (iii), taking \( n \to \infty \), we have

\[
E_{\alpha, \beta} \geq J[u, v] + E_{\alpha - \alpha', \beta - \beta'}.
\]
On the other hand, by Lemma 4.2 (i),

$$J[u, v] + E_{\alpha' - \alpha'', \beta'} \geq E_{\alpha', \beta'} + E_{\alpha' - \alpha'', \beta'} \geq E_{\alpha', \beta'}. \quad (4.3)$$

By (4.2) and (4.3), we obtain that $(u, v)$ is a global minimizer with respect to $E_{\alpha', \beta'}$. To obtain a contradiction, we consider two cases $\beta - \beta' > 0$ and $\beta - \beta' = 0$. In the case $\beta - \beta' > 0$, noting $\alpha - \alpha' > 0$, let $\{ (\xi_n, \zeta_n) \}_{n \in \mathbb{N}} \subset M_{\alpha - \alpha', \beta - \beta'}$ be a minimizing sequence with respect to $E_{\alpha - \alpha', \beta - \beta'}$. Then, as discussed before, Neither $\{ \xi_n \}_{n \in \mathbb{N}}$ nor $\{ \zeta_n \}_{n \in \mathbb{N}}$ vanish. Therefore, taking a subsequence, there exist $\{ z_n \}_{n \in \mathbb{N}} \subset \mathbb{R}^N, \xi \in H^1(\mathbb{R}^N) \setminus \{0\}$, and $\zeta \in H^1(\mathbb{R}^N)$ such that

$$\xi_n \cdot z_n \rightharpoonup \xi, \quad \zeta_n \cdot z_n \rightharpoonup \zeta \quad \text{weakly in } H^1(\mathbb{R}^N),$$

$$\xi_n \cdot z_n \rightharpoonup \xi, \quad \zeta_n \cdot z_n \rightharpoonup \zeta \quad \text{in } L^p_{\text{loc}}(\mathbb{R}^N),$$

$$\xi_n \cdot z_n \rightharpoonup \xi, \quad \zeta_n \cdot z_n \rightharpoonup \zeta \quad \text{a.e. in } \mathbb{R}^N \quad \text{as } n \to \infty.$$

Putting $\alpha'' = \| \xi \|^2_{L^2(\mathbb{R}^N)}$ and $\beta'' = \| \zeta \|^2_{L^2(\mathbb{R}^N)}$, we have

$$E_{\alpha - \alpha', \beta - \beta'} \geq E_{\alpha'', \beta''} + E_{\alpha - \alpha'', \beta - \beta''},$$

and $(\xi, \zeta)$ is a global minimizer with respect to $E_{\alpha'', \beta''}$. Hence $(\xi, \zeta)$ is a solution of

$$\Delta \xi + g_1(\xi, \zeta) = \mu \xi, \quad \Delta \zeta + g_2(\xi, \zeta) = v \zeta \quad \text{in } \mathbb{R}^N,$$

where $\mu$ and $v$ are Lagrange multipliers. By using the elliptic regularity theory, $\xi$ and $\zeta$ is of class $C^1$ and satisfy the condition (A). Now we can apply Theorem 2.4 and Lemma 5.2 to get

$$E_{\alpha' + \alpha'', \beta' + \beta''} \leq J[(u \ast \xi, v \ast \zeta)] < J[u, v] + J[\xi, \zeta] = E_{\alpha', \beta'} + E_{\alpha'', \beta''}. \quad (4.4)$$

It contradicts to (4.2) and (4.3). In the case $\beta - \beta' = 0$, we can obtain contradiction by the same argument.

Thus, we have that $\| u \|^2_{L^2(\mathbb{R}^N)} = \alpha$ holds in (4.1). On the other hand, repeating the same argument for $\{ v_n \}_{n \in \mathbb{N}}$ instead of $\{ u_n \}_{n \in \mathbb{N}}$, taking a subsequence, there exist $\{ z_n \}_{n \in \mathbb{N}}, \tilde{u} \in H^1(\mathbb{R}^N)$, and $\tilde{v} \in H^1(\mathbb{R}^N)$ such that

$$\begin{align*}
&\left\{ u_n \cdot z_n \rightharpoonup \tilde{u}, \quad v_n \cdot z_n \rightharpoonup \tilde{v} \quad \text{weakly in } H^1(\mathbb{R}^N), \\
&u_n \cdot z_n \rightharpoonup \tilde{u}, \quad v_n \cdot z_n \rightharpoonup \tilde{v} \quad \text{in } L^p_{\text{loc}}(\mathbb{R}^N), \\
&u_n \cdot z_n \rightharpoonup \tilde{u}, \quad v_n \cdot z_n \rightharpoonup \tilde{v} \quad \text{a.e. in } \mathbb{R}^N \quad \text{as } n \to \infty.
\end{align*}$$

Moreover we have $\| \tilde{v} \|^2_{L^2(\mathbb{R}^N)} = \beta$.

**Claim** \( \lim \sup_{n \to \infty} |y_n - z_n| < \infty \)

If not, taking a subsequence, we can assume \( \lim \sup_{n \to \infty} |y_n - z_n| = \infty \). Since \( \| u \|^2_{L^2(\mathbb{R}^N)} = \alpha \) and \( \| \tilde{v} \|^2_{L^2(\mathbb{R}^N)} = \beta \), we have $\tilde{u} = v = 0$ a.e. in $\mathbb{R}^N$. By the Brezis-Lieb lemma,

$$J[u_n, v_n] = J[u, 0] + J[0, \tilde{v}] + J[u_n - u(\cdot + y_n), v_n - \tilde{v}(\cdot + z_n)].$$

On the other hand, since \( \lim_{n \to \infty} \| u_n - u(\cdot + y_n) \|_{L^2(\mathbb{R}^N)} = \lim_{n \to \infty} \| v_n - \tilde{v}(\cdot + y_n) \|_{L^2(\mathbb{R}^N)} = 0 \), the P.-L. Lions lemma asserts that

$$\lim \inf_{n \to \infty} J[u_n - u(\cdot + y_n), v_n - \tilde{v}(\cdot + z_n)] \geq 0.$$
As \( n \to \infty \), we get
\[
E_{\alpha, \beta} \geq J[u, 0] + J[0, \tilde{v}] \geq E_{\alpha, 0} + E_{0, \beta} \geq E_{\alpha, \beta}.
\] (4.4)
It means that \((u, 0)\) and \((0, \tilde{v})\) are global minimizers with respect to \(E_{\alpha, 0}\) and \(E_{0, \beta}\). By using (G5), we have
\[
E_{\alpha, \beta} \leq J[u, \tilde{v}] < J[u, 0] + J[0, \tilde{v}].
\]
It contradicts to (4.4). Hence the claim holds.

Thus, taking a subsequence, there exists \( z \in \mathbb{R}^N \) such that \( z_n = y_n + z + o(1) \) in \( \mathbb{R}^N \) as \( n \to \infty \). Put \( v = \tilde{v}(\cdot + z) \) then (4.1) holds for \((u, v) \in M_{\alpha, \beta}\). For \( \phi_n = u_n(\cdot - y_n) - u \) and \( \psi_n = u_n(\cdot - y_n) - v, \phi_n, \psi_n \to 0 \) in \( L^2(\mathbb{R}^N) \). By using the P.-L. Lions lemma, \( \phi_n, \psi_n \to 0 \) in \( L^1(\mathbb{R}^N) \). Hence \( \int_{\mathbb{R}^N} G(|\phi_n|^2, |\psi_n|^2)\,dx \to 0 \). By the Brezis-Lieb lemma,
\[
J[u_n, v_n] = J[u, v] + J[\phi_n, \psi_n] + o(1)
=(E_{\alpha, \beta} + \frac{1}{2} \int_{\mathbb{R}^N} |\nabla \phi_n|^2 + |\nabla \psi_n|^2\,dx + o(1)) \quad \text{as} \quad n \to \infty.
\]
Taking \( n \to \infty \), we obtain
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} |\nabla \phi_n|^2 + |\nabla \psi_n|^2\,dx = 0.
\]
Thus we get \( \lim_{n \to \infty} \phi_n = \lim_{n \to \infty} \psi_n = 0 \) in \( H^1(\mathbb{R}^N) \). It means the conclusion.

5 Appendix

In this section, we give the proofs of lemmas used in the above section.

**Lemma 5.1** Assume (G1)–(G4). For \( R > 0 \), there exists a constant \( C(N, G, R) > 0 \) such that
\[
\frac{1}{4} \left( \|\nabla u\|_{L^2(\mathbb{R}^N)}^2 + \|\nabla v\|_{L^2(\mathbb{R}^N)}^2 \right) \leq J[u, v] + C(N, G, R)
\] (5.1)
for \((u, v) \in M_{\alpha, \beta}\) with \( \alpha, \beta \in [0, R] \). Moreover, for \( \alpha, \beta \geq 0 \), any minimizing sequence \( \{(u_n, v_n)\}_{n \in \mathbb{N}} \subset M_{\alpha, \beta} \) is \( H^1 \)-bounded.

**Proof** By (G1)–(G3), for any \( \epsilon > 0 \), there exists \( C(G, \epsilon) > 0 \) such that
\[
|G(s_1, s_2)| \leq C(G, \epsilon)(|s_1| + |s_2|) + \epsilon(|s_1|^{2/N+1} + |s_2|^{2/N+1}).
\]
Therefore, by using the Gagliardo-Nirenberg inequality, for \((u, v) \in M_{\alpha, \beta}\), we have
\[
J[u, v] \geq -C(G, \epsilon)(\alpha + \beta) + \frac{1}{2} \left( \|\nabla u\|_{L^2(\mathbb{R}^N)}^2 + \|\nabla v\|_{L^2(\mathbb{R}^N)}^2 \right)
- \epsilon \left( \|u\|_{L^1(\mathbb{R}^N)} + \|v\|_{L^1(\mathbb{R}^N)} \right)
\geq -C(G, \epsilon)(\alpha + \beta) + \frac{1}{2} \left( \|\nabla u\|_{L^2(\mathbb{R}^N)}^2 + \|\nabla v\|_{L^2(\mathbb{R}^N)}^2 \right)
- \epsilon C(N) \left( \alpha^{4/N} \|\nabla u\|_{L^2(\mathbb{R}^N)}^2 + \beta^{4/N} \|\nabla v\|_{L^2(\mathbb{R}^N)}^2 \right)
\geq -2RC(G, \epsilon) + \left( \frac{1}{2} - \epsilon C(N)R^{4/N} \right) \left( \|\nabla u\|^2_{L^2(\mathbb{R}^N)} + \|\nabla v\|^2_{L^2(\mathbb{R}^N)} \right)
\]
Choosing $\varepsilon > 0$ satisfying $\varepsilon C(N)R^{4/N} < 1/4$, we have (5.1).

Let $\{(u_n, v_n)\}_{n \in \mathbb{N}} \subset M_{\alpha, \beta}$ be a minimizing sequence. Since $\{(u_n, v_n)\}_{n \in \mathbb{N}} \subset M_{\alpha, \beta}$, $\{u_n\}_{n \in \mathbb{N}}$ and $\{v_n\}_{n \in \mathbb{N}}$ are bounded in $L^2(\mathbb{R}^N)$, (5.1) asserts that $H^1$-boundedness. \hfill $\square$

**Proof of Lemma 4.2**

(i) For $\varepsilon > 0$, there exists $(u, v) \in M_{\alpha, \beta} \cap C_0^\infty(\mathbb{R}^N)$ and $(\phi, \psi) \in M_{\alpha', \beta'} \cap C_0^\infty(\mathbb{R}^N)$. By using parallel transformation, we can assume that $(\text{supp } \phi \cup \text{supp } \psi) = \emptyset$. Therefore $(u + \phi, v + \psi) \in M_{\alpha + \alpha', \beta + \beta'}$ and

$$E_{\alpha + \alpha', \beta + \beta'} \leq I[u + \phi, v + \psi] = I[u, v] + I[\phi, \psi] \leq E_{\alpha, \beta} + E_{\alpha', \beta'} + 2\varepsilon.$$ 

Since $\varepsilon > 0$ is arbitrarily, it asserts (i).

(ii) (i) and (E2) asserts (ii) immediately.

(iii) First we show the following.

**Claim 1** For $\alpha, \beta > 0$, $\liminf_{(h,k) \to (0,0)} E_{\alpha + h, \beta + k} \geq E_{\alpha, \beta}$.

Put $R = \max\{\alpha + 1, \beta + 1\}$ and assume $|h|, |k| < \min\{\alpha, \beta, 1\}$. We note that $0 < \alpha + h \leq R$ and $0 < \beta + k \leq R$. For $\varepsilon > 0$, by the definition of $E_{\alpha + h, \beta + k}$, there exists $(u, v) \in M_{\alpha + h, \beta + k}$ such that

$$E_{\alpha + h, \beta + k} \leq J[u, v] \leq E_{\alpha + h, \beta + k} + \varepsilon.$$ 

Putting

$$t = t(h, k) = \left(\min\left\{\frac{\alpha}{\alpha + h}, \frac{\beta}{\beta + k}\right\}\right)^{1/N},$$

$u_t(x) = u(x/t)$, and $v_t(x) = v(x/t)$, we have

$$\lim_{(h,k) \to (0,0)} t = 1,$$ 

(5.2)

$\|u_t\|_{L^2(\mathbb{R}^N)}^2 = t^N(\alpha + h) \leq \alpha$, and $\|v_t\|_{L^2(\mathbb{R}^N)}^2 = t^N(\beta + k) \leq \beta$. Therefore, by using (i) and (ii), we obtain

$$J[u_t, v_t] \geq E_{t^N(\alpha + h), t^N(\beta + k)} \geq E_{\alpha, \beta}.$$ 

On the other hand,

$$J[u_t, v_t] = \frac{t^{N-2}}{2} \int_{\mathbb{R}^N} |\nabla u_t|^2 + |\nabla v_t|^2 dx - t^N \int_{\mathbb{R}^N} G(|u_t|^2, |v_t|^2) dx$$

$$\leq t^N J[u, v] + \frac{t^{N-2}}{2} \left|1 - t^2\right| \int_{\mathbb{R}^N} |\nabla u|^2 + |\nabla v|^2 dx.$$ 

By Lemma 5.1,

$$\int_{\mathbb{R}^N} |\nabla u|^2 + |\nabla v|^2 dx \leq J[u, v] + C(N, G, R)$$

$$\leq \varepsilon + C(N, G, R).$$ 

Thus, noting (5.2), we get

$$E_{\alpha, \beta} \leq \liminf_{(h,k) \to (0,0)} E_{\alpha + h, \beta + k} + \varepsilon.$$ 

Since we can take $\varepsilon > 0$ arbitrarily, the claim holds.
Claim 2 For $\alpha, \beta > 0$, \( \limsup_{(h,k) \to (0,0)} E_{\alpha + h, \beta + k} \leq E_{\alpha, \beta} \).

We can show the claim as before. Actually, for $\epsilon > 0$, there exists $(u, v) \in M_{\alpha, \beta}$ such that

\[
E_{\alpha, \beta} \leq J[u, v] \leq E_{\alpha, \beta} + \epsilon.
\]

Putting

\[
t = t(h, k) = \left( \min \left\{ \frac{\alpha + h}{\alpha}, \frac{\beta + k}{\beta} \right\} \right)^{1/N},
\]

we have $u_t(x) = u(x/t)$, and $v_t(x) = v(x/t)$, we have $\lim_{(h,k) \to (0,0)} t = 1$, $\|u_t\|_{L^2(\mathbb{R}^N)}^2 = t^N \alpha \leq \alpha + h$, and $\|v_t\|_{L^2(\mathbb{R}^N)}^2 = t^N \beta \leq \beta + k$. Therefore, we obtain

\[
J[u_t, v_t] \geq E_{\alpha, \beta} \geq E_{\alpha + h, \beta + k}.
\]

On the other hand,

\[
J[u_t, v_t] = \frac{t^{N-2}}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + |\nabla v|^2 \, dx - t^N \int_{\mathbb{R}^N} G(|u|^2, |v|^2) \, dx
\]

\[
\leq t^N J[u, v] + \frac{t^{N-2} |1 - t^2|}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + |\nabla v|^2 \, dx.
\]

Since $u$ and $v$ are independent of $h$ and $k$, by (5.2), we get

\[
\limsup_{(h,k) \to (0,0)} E_{\alpha + h, \beta + k} \leq E_{\alpha, \beta} + \epsilon.
\]

Since we can take $\epsilon > 0$ arbitrarily, the claim holds.

Next, we consider the case $\alpha = 0$ or $\beta = 0$. It is sufficient to consider the case $\beta = 0$. By the same argument as above, we can show $\alpha \mapsto E_{\alpha, 0}$ is continuous. Therefore, we show the following claim.

Claim 3 \( \lim_{k \to 0} E_{\alpha, k} = E_{\alpha, 0} \) uniformly with respect to $\alpha \in [0, R]$.

For $\epsilon > 0$, there exists $(u, v) \in M_{\alpha, k}$ such that

\[
J[u, v] \leq E_{\alpha, k} + \epsilon.
\]

On the other hand, we have

\[
J[u, v] \geq J[u, 0] + \int_{\mathbb{R}^N} G(|u|^2, 0) - G(|u|^2, |v|^2) \, dx
\]

\[
\geq E_{\alpha, 0} + \int_{\mathbb{R}^N} G(|u|^2, 0) - G(|u|^2, |v|^2) \, dx.
\]

By using

\[
G(s_1, s_2) - G(s_1, 0) = \int_0^1 \frac{d}{d\theta} G(s_1, \theta s_2) \, d\theta = \int_0^1 g_2(s_1, \theta s_2) s_2 \, d\theta
\]

and (G1)–(G3), we have

\[
|G(s_1, s_2) - G(s_1, 0)| \leq \left( C(G, \delta) + \delta (|s_1|^{2/N} + |s_2|^{2/N}) \right) |s_2|.
\]
Thus,
\[
\left| \int_{\mathbb{R}^N} G(|u|^2, |v|^2) dx - \int_{\mathbb{R}^N} G(|u|^2, 0) dx \right|
\leq kC(G, \delta) + \delta \int_{\mathbb{R}^N} (|u|^{4/N} + |v|^{4/N}) |v|^2 dx
\]
\[
\leq kC(G, \delta) + \delta \left( \|u\|_{L^2(\mathbb{R}^N)}^{2/(N+2)} \|v\|_{L^1(\mathbb{R}^N)}^{N/(N+2)} + \|v\|_{L^2(\mathbb{R}^N)}^2 \right)
\]
\[
\leq kC(G, \delta) + \delta C(N) \left( \|\nabla u\|_{L^2(\mathbb{R}^N)}^{2/(N+2)} \|\nabla v\|_{L^1(\mathbb{R}^N)}^{N/(N+2)} + \|\nabla v\|_{L^2(\mathbb{R}^N)}^2 \right)
\]
By Lemma 5.1,
\[
\|\nabla u\|_{L^2(\mathbb{R}^N)}^2 + \|\nabla v\|_{L^2(\mathbb{R}^N)}^2 \leq 4(E_{\alpha,k} + \epsilon) + C(N, G, R) \leq C(N, G, R)
\]
for \(\epsilon \leq 1\), because of \(E_{\alpha,k} \leq 0\). Consequently we have
\[
\limsup_{k \to 0} \left| \int_{\mathbb{R}^N} G(|u|^2, |v|^2) dx - \int_{\mathbb{R}^N} G(|u|^2, 0) dx \right| \leq \limsup_{k \to 0} \left( kC(G, \delta) + \delta C(N, G, R) \right)
\]
\[
\leq \delta C(N, G, R).
\]
Since \(\delta > 0\) is arbitrarily,
\[
\lim_{k \to 0} \left| \int_{\mathbb{R}^N} G(|u|^2, |v|^2) dx - \int_{\mathbb{R}^N} G(|u|^2, 0) dx \right| = 0 \text{ uniformly with respect to } \alpha.
\]
Thus we have
\[
E_{\alpha,0} \leq \liminf_{k \to 0} E_{\alpha,k} + \epsilon \text{ uniformly with respect to } \alpha.
\]
Since \(\epsilon > 0\) is arbitrarily,
\[
E_{\alpha,0} \leq \liminf_{k \to 0} E_{\alpha,k} \text{ uniformly with respect to } \alpha.
\]
On the other hand, by (i) and (ii), \(E_{\alpha,k} \leq E_{\alpha,0}\) holds. Thus we get the conclusion. \(\square\)

**Lemma 5.2** Assume (G1)–(G4). For \(u, v, \phi, \psi \in H^1(\mathbb{R}^N)\) satisfying the condition (A),
\[
\int_{\mathbb{R}^N} G\left( (u \ast \phi)^2, (v \ast \psi)^2 \right) dx \geq \int_{\mathbb{R}^N} G(|u|^2, |v|^2) dx + \int_{\mathbb{R}^N} G(|\phi|^2, |\psi|^2) dx
\]
**Proof** For simplicity, we use \(u, v, \phi, \psi\) instead of \(|u|^2, |v|^2, |\phi|^2, |\psi|^2\). Noting \((u \ast \phi)^2 = |u|^2 \ast |\phi|^2\) and \((v \ast \psi)^2 = |v|^2 \ast |\psi|^2\), we show
\[
\int_{\mathbb{R}^N} G(u \ast \phi, v \ast \psi) dx \geq \int_{\mathbb{R}^N} G(u, v) dx + \int_{\mathbb{R}^N} G(\phi, \psi) dx \tag{5.3}
\]
for \(u, v, \phi, \psi \geq 0\).

By (G2) and (G4), \(g_j(s, t) \geq g_j(0, 0) = 0\). By using mean value theorem, we have
\[
\int_{\mathbb{R}^N} G(u, v) dx
\]
\[
= \int_{\mathbb{R}^N} G(u(x), v(x)) - G(0, v(x)) + G(0, v(x)) - G(0, 0) dx
\]
\[
= \int_{\mathbb{R}^N} dx \int_0^{u(x)} g_2(s, v(x)) ds + \int_{\mathbb{R}^N} dx \int_0^{v(x)} g_1(0, t) dt
\]
\[
\begin{align*}
= \int_{\mathbb{R}^N} dx \int_0^\infty g_2(s, v(x)) \chi_{\{x; u(x) > s\}}(x) ds + \int_{\mathbb{R}^N} dx \int_0^\infty g_1(0, t) \chi_{\{x; v(x) > t\}}(x) dt \\
= \int_{\mathbb{R}^N} dx \int_0^\infty \int_0^\infty \chi_{\{x; g_2(s, v(x)) > r\}}(x) \chi_{\{x; u(x) > s\}}(x) dr ds \\
+ \int_{\mathbb{R}^N} dx \int_0^\infty g_1(0, t) \chi_{\{x; v(x) > t\}}(x) dt \\
= \int_0^\infty \int_0^\infty (|\{x; g_2(s, v(x)) > r\} \cap \{x; u(x) > s\}| dr ds + \int_0^\infty g_1(0, t) |\{x; v(x) > t\}| dt. 
\end{align*}
\]

For each \( r, s > 0 \), Put \( t(r, s) = \sup\{t; g_2(s, t) \leq r\} \) if \( \{t; g_2(s, t) \leq r\} \neq \emptyset \), \( t(r, s) = -\infty \) if \( \{t; g_2(s, t) \leq r\} = \emptyset \). Then, by (G4), \( g_2(s, v(x)) > r \) if and only if \( v(x) > t(r, s) \). So we have

\[
|\{x; g_2(s, v(x)) > r\} \cap \{x; u(x) > s\}| = |\{x; v(x) > t(r, s)\} \cap \{x; u(x) > s\}|
\]

Hence

\[
\int_{\mathbb{R}^N} G(u, v) dx = \int_0^\infty \int_0^\infty (|\{x; v(x) > t(r, s)\} \cap \{x; u(x) > s\}| dr ds + \int_0^\infty g_1(0, t) |\{x; v(x) > t\}| dt. \tag{5.4}
\]

Similarly, we can obtain

\[
\begin{align*}
\int_{\mathbb{R}^N} G(\phi, \psi) dx &= \int_0^\infty \int_0^\infty (|\{x; \psi(x) > t(r, s)\} \cap \{x; \phi(x) > s\}| dr ds \\
+ \int_0^\infty g_1(0, t) |\{x; \psi(x) > t\}| dt, \tag{5.5}
\end{align*}
\]

\[
\begin{align*}
\int_{\mathbb{R}^N} G(u \ast \phi, v \ast \psi) dx &= \int_0^\infty \int_0^\infty (|\{x; (v \ast \psi)(x) > t(r, s)\} \cap \{x; (u \ast \phi)(x) > s\}| dr ds \\
+ \int_0^\infty g_1(0, t) |\{x; (v \ast \psi)(x) > t\}| dt. \tag{5.6}
\end{align*}
\]

Here, by Lemma 2.2 (i), we have

\[
|\{x; v(x) > t(r, s)\} \cap \{x; u(x) > s\}| + |\{x; \psi(x) > t(r, s)\} \cap \{x; \phi(x) > s\}|
\leq \min\{|\{x; v(x) > t(r, s)\}|, |\{x; u(x) > s\}|\} + \min\{|\{x; \psi(x) > t(r, s)\}|, |\{x; \phi(x) > s\}|\}
\leq \min\{|\{x; v(x) > t(r, s)\}| + |\{x; \psi(x) > t(r, s)\}|, |\{x; u(x) > s\}| + |\{x; \phi(x) > s\}|\}
= \min\{|\{x; (v \ast \psi)(x) > t(r, s)\}|, |\{x; (u \ast \phi)(x) > s\}|\}.
\]

Since \( \{x; (v \ast \psi)(x) > t(r, s)\} \) and \( \{x; (u \ast \phi)(x) > s\} \) are balls centered at the origin, we have

\[
\min\{|\{x; (v \ast \psi)(x) > t(r, s)\}|, |\{x; (u \ast \phi)(x) > s\}|\}
= |\{x; (v \ast \psi)(x) > t(r, s)\}| - |\{x; (u \ast \phi)(x) > s\}|
\]

Hence,

\[
|\{x; v(x) > t(r, s)\} \cap \{x; u(x) > s\}| + |\{x; \psi(x) > t(r, s)\} \cap \{x; \phi(x) > s\}|
\leq |\{x; (v \ast \psi)(x) > t(r, s)\} \cap \{x; (u \ast \phi)(x) > s\}|. \tag{5.7}
\]

On the other hand,

\[
|\{x; v(x) > t\}| + |\{x; \psi(x) > t\}| = |\{x; (v \ast \psi)(x) > t\}| \tag{5.8}
\]
because of Lemma 2.2 (i). Consequently, (5.4), (5.5), (5.6), (5.7) and (5.8) assert that this lemma.

References

1. Brézis, H., Lieb, E.: A relation between pointwise convergence of functions and convergence of functionals. Proc. Am. Math. Soc. 88(3), 486–490 (1983)
2. Cazenave, T., Lions, P.-L.: Orbital stability of standing waves for some nonlinear Schrödinger equations. Commun. Math. Phys. 85(4), 549–561 (1982)
3. Duff, G.F.D.: Integral inequalities for equimeasurable rearrangements. Can. J. Math. 22, 408–430 (1970)
4. Hajaiej, H.: Symmetric ground state solutions of $m$-coupled nonlinear Schrödinger equations. Nonlinear Anal. 71(10), 4696–4704 (2009)
5. Ikoma, N.: Compactness of minimizing sequences in nonlinear Schrödinger systems under multiconstraint conditions. Adv. Nonlinear Stud. 14(1), 115–136 (2014)
6. Kawohl, B.: Rearrangements and Convexity of Level Sets in PDE. Lecture Notes in Mathematics, vol. 1150. Springer, Berlin (1985)
7. Lieb, E.H., Loss, M.: Analysis, 2nd ed. Graduate Studies in Mathematics, vol. 14. American Mathematical Society, Providence (2001)
8. Lions, P.-L.: The concentration-compactness principle in the calculus of variations. The locally compact case. II. Ann. Inst. H. Poincaré Anal. Non Linéaire 1(4), 223–283 (1984)
9. Shibata, M.: Stable standing waves of nonlinear Schrödinger equations with a general nonlinear term. Manuscr. Math. 143(1–2), 221–237 (2014)