Spectral Representation of Correlation Functions in Two-dimensional Quantum Field Theories

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**Abstract**

The non-perturbative mapping between different Quantum Field Theories and other features of two-dimensional massive integrable models are discussed by using the Form Factor approach. The computation of ultraviolet data associated to the massive regime is illustrated by taking as an example the scattering theory of the Ising Model with boundary.

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1 Introduction

Many two-dimensional integrable statistical models with a finite correlation length can be elegantly discussed in terms of relativistic particles in bootstrap interaction \[1\]. In this formulation, the key object is the elastic $S$-matrix that describes the scattering processes of the massive excitations in the Minkowski space. Once we know the exact $S$-matrix of the model under analysis and the corresponding spectrum, we may proceed further and compute several quantities of theoretical interest, among them the central charge and the critical exponents of the conformal field theory arising in the ultraviolet regime \[2\]. Aim of this talk is to discuss some features of the structure of integrable QFT in terms of the properties of their correlation functions. The most promising method for the computation of the correlation functions results to be the Form Factor Approach, as originally proposed in \[3, 4\]. In the following I will try to point out the reasons of the successful application of this approach together with several interesting properties which come out as by-products of its theoretical formulation. As a significant example of the computation of ultraviolet data in terms of a resummation of the Form Factors, we will consider the exact critical exponents of the energy and disorder operators in the Ising model with boundary. The scattering theory for such system has been recently proposed in \[19\].

2 Computation of Correlation Functions

To fully appreciate the bootstrap approach to the computation of the correlation functions, let us discuss the most common difficulties which arise in the perturbative method. Let 
\[
\mathcal{A} = \mathcal{A}_0 + \lambda \mathcal{A}_{\text{int}}
\] 
be the action of the theory, where $\mathcal{A}_0$ corresponds to a solvable QFT (e.g. a free theory, CFT, etc.) whereas $\mathcal{A}_{\text{int}}$ defines the interactive part. For the perturbative definition of the Green functions we have the formal expressions 
\[
G^{(N)}(x_1, \ldots, x_N) = \sum_{k=0}^\infty \lambda^k G_k^{(N)}(x_1, \ldots, x_N),
\] 
where $G_k^{(N)}(x_1, \ldots, x_N)$ is the $k$-th perturbative term. The above expression usually suffers of several drawbacks:

- We may face, for instance, the renormalization problem, i.e. the presence of infinities which should correctly be handled. In this case, we need to express our final computation in terms of physical quantities by means of the renormalization procedure. This is usually a painful task.

- Assuming we were able to go through the renormalization program, the resulting expression may present a low rate of convergence, if any!
In the light of the above considerations, a more efficient way to compute correlation functions is needed. This is provided by the spectral representation methods, i.e. by the possibility to express the correlation functions as an infinite series over multi-particle intermediate states. For instance, in a QFT with a self-conjugate particle the two-point function of an operator $O(x)$ in real Euclidean space is given by

$$
\langle O(x)O(0) \rangle = \sum_{n=0}^{\infty} \frac{d\beta_1 \ldots d\beta_n}{n!(2\pi)^n} < 0|O(x)|\beta_1, \ldots, \beta_n >_{\text{in in}} < \beta_1, \ldots, \beta_n|O(0)|0 >
$$

$$= \sum_{n=0}^{\infty} \int \frac{d\beta_1 \ldots d\beta_n}{n!(2\pi)^n} |F_n(\beta_1 \ldots \beta_n)|^2 \exp \left(-mr \sum_{i=1}^{n} \cosh \beta_i \right) \tag{2.3}
$$

where $r$ denotes the radial distance, i.e. $r = \sqrt{x_0^2 + x_1^2}$ and $\beta$ the rapidity variable. Similar expressions are obtained for higher point correlators. The functions

$$F_n(\beta_1, \ldots, \beta_n) = < 0|O(0)|\beta_1, \ldots, \beta_n > \tag{2.4}
$$

are the so-called Form Factors. Since the spectral representations are based only on the completeness of the asymptotic states, they are general expressions for any QFT. However, for integrable models, they become quite effective because the exact computation of the form factors reduces to finding a solution of a finite set of functional equations, as we will discuss below. Other advantages of the spectral representation method for the correlation functions may be summarized as follows:

1. It deals with physical quantities, i.e. there is no need of renormalization and the coupling constant dependence is taken into account at all orders by a closed expression of the Form Factors.

2. The above representation (2.3) and similar expressions for other correlators present a very fast rate of convergence for all values of the scaling variable $(mr)$. This is quite expected for large values of $(mr)$ which are dominated by the lowest massive state but the surprising result is that in many cases the series is saturated by the lowest terms also for small values of $(mr)$. This is due to a threshold suppression phenomenon \[7\] which we will present below.

3. The two-point correlation functions (2.3) have the form of a Grand-Canonical Partition Function of a one-dimensional fictious gas (with coordinate position $\beta_i$)

$$\Xi(mr) = \sum_{N=0}^{\infty} z^N Z_N(mr) \tag{2.5}
$$

but with a coordinate-dependent activity

$$z_i(mr, \beta_i) = \frac{1}{2\pi} e^{-mr \cosh \beta_i} \tag{2.6}
$$

This observation is extremely useful to recover ultraviolet data of the theory, as the anomalous dimensions of the fields, in terms of massive quantities \[5, 6\].
Let us discuss now the main properties of the Form Factors for 2-D Integrable Massive Field Theories which are the crucial quantities entering the spectral representation of correlation functions. If not explicit said, we consider for simplicity the case of a theory with only one self-conjugate particle. For local scalar operators $O(x)$, relativistic invariance implies that the form factors $F_n$ are functions of the difference of the rapidities $\beta_{ij}$

$$F_n(\beta_1, \beta_2, \ldots, \beta_n) = F_n(\beta_{12}, \beta_{13}, \ldots, \beta_{ij}, \ldots), \quad i < j . \quad (2.7)$$

Except for the poles corresponding to the one-particle bound states in all sub-channels, we expect the form factors $F_n$ to be analytic inside the strip $0 < \text{Im} \beta_{ij} < 2\pi$.

The form factors of a hermitian local scalar operator $O(x)$ satisfy a set of equations, known as Watson’s equations, which for integrable systems assume a particularly simple form \[3, 4\]

$$F_n(\beta_1, \ldots, \beta_i, \beta_{i+1}, \ldots, \beta_n) = F_n(\beta_1, \ldots, \beta_{i+1}, \beta_i, \ldots, \beta_n) S(\beta_i - \beta_{i+1}) , \quad (2.8)$$

$$F_n(\beta_1 + 2\pi i, \ldots, \beta_{n-1}, \beta_n) = F_n(\beta_2, \ldots, \beta_n, \beta_1) = \prod_{i=2}^{n} S(\beta_i - \beta_1) F_n(\beta_1, \ldots, \beta_n) .$$

In the case $n = 2$, eqs. (2.8) reduce to

$$F_2(\beta) = F_2(-\beta) S_2(\beta) ,$$

$$F_2(i\pi - \beta) = F_2(i\pi + \beta) . \quad (2.9)$$

It has been shown in \[4\] that eqs. (2.8), together with the next eqs. (2.12) and (2.14), can be regarded as a system of axioms which defines the whole local operator content of the theory.

The general solution of Watson’s equations can always be brought into the form \[4\]

$$F_n(\beta_1, \ldots, \beta_n) = K_n(\beta_1, \ldots, \beta_n) \prod_{i<j} F_{\min}(\beta_{ij}) , \quad (2.10)$$

where $F_{\min}(\beta)$ has the properties that it satisfies (2.9), is analytic in $0 \leq \text{Im} \beta \leq \pi$, has no zeros in $0 < \text{Im} \beta < \pi$, and converges to a constant value for large values of $\beta$. These requirements uniquely determine this function, up to a normalization. The remaining factors $K_n$ then satisfy Watson’s equations with $S_2 = 1$, which implies that they are completely symmetric, $2\pi i$-periodic functions of the $\beta_i$. They must contain all the physical poles expected in the form factor under consideration and must satisfy a correct asymptotic behaviour for large value of $\beta_i$. Both requirements depend on the nature of the theory and on the operator $O$.

Notice that one condition on the asymptotic behaviour of the FF is dictated by relativistic invariance. In fact, a simultaneous shift in the rapidity variables gives

$$F_n^O(\beta_1 + \Lambda, \beta_2 + \Lambda, \ldots, \beta_n + \Lambda) = F_n^O(\beta_1, \beta_2, \ldots, \beta_n) , \quad (2.11)$$
Secondly, in order to have a power-law bounded ultraviolet behaviour of the two-point function of the operator $\mathcal{O}(x)$ (which is the case we will consider), we have to require that the form factors behave asymptotically at most as $\exp(k\beta_i)$ in the limit $\beta_i \to \infty$, with $k$ being a constant independent of $i$. This means that, once we extract from $K_n$ the denominator which gives rise to the poles, the remaining part has to be a symmetric function of the variables $x_i \equiv e^{\beta_i}$, with a finite number of terms, i.e. a symmetric polynomial in the $x_i$’s.

The pole structure of the form factors induces a set of recursive equations for the $F_n$ which are of fundamental importance for their explicit determination. As function of the rapidity differences $\beta_{ij}$, the form factors $F_n$ possess two kinds of simple poles.

The first kind of singularities (which do not depend on whether or not the model possesses bound states) arises from kinematical poles located at $\beta_{ij} = i\pi$. They are related to the one-particle pole in a subchannel of three-particle states which, in turn, corresponds to a crossing process of the elastic $S$-matrix. The corresponding residues are computed by the LSZ reduction \cite{4} and give rise to a recursive equation between the $n$-particle and the $(n+2)$-particle form factors

$$-i \lim_{\beta \to \tilde{\beta}} (\tilde{\beta} - \beta) F_{n+2}(\tilde{\beta} + i\pi, \beta, \beta_1, \beta_2, \ldots, \beta_n) = \left(1 - \prod_{i=1}^{n} S(\beta - \beta_i)\right) F_n(\beta_1, \ldots, \beta_n).$$

(2.12)

The second type of poles in the $F_n$ only arise when bound states are present in the model. These poles are located at the values of $\beta_{ij}$ in the physical strip which correspond to the resonance angles. Let $\beta_{ij} = i\nu_{ij}^k$ be one of such poles associated to the bound state $A_k$ in the channel $A_i \times A_j$. For the $S$-matrix we have

$$-i \lim_{\beta \to \nu_{ij}^k} (\beta - i\nu_{ij}^k) S_{ij}(\beta) = \left(\Gamma_{ij}^k\right)^2$$

(2.13)

where $\Gamma_{ij}^k$ is the three-particle vertex on mass-shell. The corresponding residue for the $F_n$ is given by \cite{4}

$$-i \lim_{\epsilon \to 0} e^{F_{n+1}(\beta + \nu_{ij}^k - \epsilon, \beta - i\nu_{ij}^k + \epsilon, \beta_1, \ldots, \beta_{n-1})} = \Gamma_{ij}^k F_n(\beta, \beta_1, \ldots, \beta_{n-1})$$

(2.14)

where $\nu_{ab} \equiv (\pi - u_{ab}^c)$. This equation establishes then a recursive structure between the $(n+1)$- and $n$-particle form factors.

Important properties of the FF are pointed out by the following observations:

1. Notice that the functional and recursive equations satisfied by the Form Factors do not refer to any operator of the theory! This opens the possibility to classify the operator content of a massive QFT by computing the independent solutions of these equations and by associating them to the corresponding operators, as suggested and investigated in \cite{6, 10, 11}. The structure of the local operators in integrable QFT has been analysed by other points of view in \cite{12, 13, 14}.
2. The computation of the Form Factors only depend on the S-matrix. This implies that if the S-matrix \(S(\lambda)\) interpolates between two (or several) scattering matrices relative to different QFT by varying the parameter \(\lambda\), there should be a corresponding mapping between the operator content of the QFT encountered in the flow. This correspondance may be difficult to establish at the perturbative level and therefore it completely relies on the non-perturbative effects encoded in the exact S-matrix. One of the most striking example relative to this observation is the correspondance between the Sinh-Gordon model (which is a \(Z_2\) invariant model) and the Bullough-Dodd model (which does not present any symmetry at the perturbative level) for some particular values of the coupling constants of these models [16]. Another example of the non-perturbative mapping of the operator content of two QFT has been established for the Sinh-Gordon and Ising models [15].

3. As it follows from eq. (2.9), if \(S(0) = -1\) the two-particle Form Factor necessarily vanishes at threshold

\[
F_{\text{min}}(\beta_{ij}) \simeq \beta_{ij}.
\]

This observation is quite important since it permits to understand the reason of the fast rate of convergence of the spectral series also at short distance scales, as shown in several significant examples discussed in the literature [3, 7, 8, 9]. In fact, the correlation functions are saturated by the first matrix elements and for any practical aim, their computation requires relatively little analytic work. The argument goes as follows [7]. Let us consider the two-point function in the momentum space

\[
G(p) \simeq \int ds \frac{\sigma(s)}{p^2 + s},
\]

where

\[
\sigma(s) = \sum_N \int \frac{\beta_i}{2\pi} \ldots \frac{d\beta_N}{2\pi} \delta(s - \sum_i m \cosh \beta_i) \delta(\sum_i m \sinh \beta_i) |F_N|^2.
\]

The spectral function \(\sigma(s)\) gets more contributions each time that it passes through a threshold. If the matrix elements \(|F_N|^2\) were constants, its discontinuity at the threshold \((N m)\) due to the phase space would be given by

\[
\sigma(s) \simeq \theta(s - (N m)^2)(\sqrt{s - (N m)^2})^{N-3}.
\]

However, as consequence of eqs. (2.10) and (2.15) the spectral function has a much softer behaviour at the different thresholds

\[
\sigma(s) \simeq \theta(s - (N m)^2)(\sqrt{s - (N m)^2})^{N^2-3},
\]

and therefore the values of the correlation functions are saturated by the first terms of the spectral representations even at large values of the momenta.
3 One-point Functions in the Ising Model with Boundary

A relevant aspect of a QFT is the interpolation between its infrared and ultraviolet regimes. In particular, it is extremely important to establish the relationship between the most significant parameters associated to the scaling behaviour in the ultraviolet regime to those which characterize the infrared properties. The example we choose to illustrate this relationship is the Ising model with a boundary. The conformal field theory relative to the scaling behaviour of the fixed point of this model has been discussed in [17, 18] whereas the breaking of conformal invariance due to the presence of finite correlation length has been recently formulated in [19]. To compute the scaling dimensions in the presence of the boundary, it is sufficient to consider the one-point function of the energy operator

\[ \epsilon_0(t) = <0 \mid E(y,t) \mid B> \]

and the one-point function of the disorder operator

\[ \mu_0(t) = <0 \mid \mu(y,t) \mid B>, \]

where \(|B>\) is the boundary state (see below). To fix the notation, \(t\) is the distance of the operators from the boundary whereas \(y\) is their parallel coordinate. By translation invariance the above one-point functions depend only on \(t\). In the high temperature phase (the only one discussed here), these operators share the important property to couple only to an even number of particles, which we may consider as massive Majorana fermions described by annihilation and creation operators \(A(\beta)\) and \(A^\dagger(\beta)\). The mass of the fermion field is linearly related to the difference of the temperature, \(m = 2\pi(T-T_c)\).

The important quantity we need for our computation is the wave function of the boundary state \(|B>\) relative to the fixed and free boundary conditions. This has been determined in [19] and its explicit expression is given by

\[ |B> = \exp \left[ \int_0^\infty d\beta \frac{d^2 K(\beta)}{2\pi} A^\dagger(-\beta)A^\dagger(\beta) \right] |0> . \]  

(3.1)

where

\[ K(\beta) = \begin{cases} 
  i \tanh \frac{\beta}{2} & \text{fixed b.c.} \\
  -i \coth \frac{\beta}{2} & \text{free b.c.}
\end{cases} \]  

(3.2)

The computation of the one-point functions can be done by using the form factors determined in [3, 4, 5]. The energy operator couples only to the two-particle state,

\[ <0 \mid E(0) \mid \beta_1, \beta_2> = 2\pi i m \sinh \frac{\beta_{12}}{2} \]  

(3.3)

and its one-point function is given by

\[ \epsilon_0(t) = i m \int_0^\infty d\beta \sinh \beta K(\beta) e^{-2mt \cosh \beta} . \]  

(3.4)

*There may also be other terms in \(|B>\) which come from the bound state. However they do not contribute to the one-point functions computed in the text.
With a simple integration we obtain

$$\frac{\epsilon_0(t)}{m} = \begin{cases} K_1(2mt) - K_0(2mt) & \text{fixed b.c.} \\ -K_1(2mt) - K_0(2mt) & \text{free b.c.} \end{cases}$$

(K_0 and K_1 are Bessel functions) and in the ultraviolet limit (mt → 0) we recover the critical exponent \( x = 1 \) of the energy operator and the universal amplitudes determined in [18].

Concerning the computation of the one-point function of the disorder operator \( \mu(x,t) \), it couples to all states with an even number of particles. Using the form factors determined in [3, 5, 6] and a simple algebraic identity, the relevant expression can be written in this case as

$$<0 | \mu(0,0) | -\beta, \beta, \ldots -\beta, \beta > = \prod_{i=1}^{n} \tanh \beta_i \times det \left( \frac{2 \sqrt{\cosh \beta_i \cosh \beta_j}}{\cosh \beta_i + \cosh \beta_j} \right).$$

(3.6)

The one-point function of \( \mu(x,t) \) can be then expressed as a Fredholm determinant

$$\mu_0(t) = <0 | \mu(x,t) | B > = Det \left( 1 - z_{\pm} W_{\pm} \right),$$

(3.7)

where \( z_{\pm} = \pm 1/2\pi \) and \( W_{\pm} \) is the kernel of a linear integral symmetric operator

$$W_{\pm} = \frac{E_{\pm}(\beta,mt)E_{\pm}(\beta,mt)}{\cosh \beta_i + \cosh \beta_j};$$

(3.8)

$$E_{\pm}(\beta,mt) = e^{-mt \cosh \beta} \sqrt{\cosh \beta \pm 1}.$$

The plus sign of the above quantities refers to the fixed b.c. whereas the minus sign to the free b.c.. In both cases, \( \mu_0(t) \) may be written in terms of the eigenvalues of the integral operator and their multiplicity as

$$\mu_0(t) = \prod_{i=1}^{\infty} \left( 1 - z_{\pm} \lambda^{(i)}_{\pm} \right)^{a_i}.$$

(3.9)

As far as \( (mt) \) is finite, the kernel is square integrable and therefore all results valid for bounded symmetric operators apply (see, for instance [21]). However, when \( (mt) \rightarrow 0 \), the operator becomes unbounded. The mathematical problem has been studied in the literature [21]: the eigenvalues becomes dense in the interval \((0, \infty)\) according to the distribution

$$\lambda(p) = \frac{2\pi}{\cosh \pi p},$$

(3.10)

whereas, from Mercer’s theorem, their multiplicity grows logarithmically as \( a_i \sim \frac{1}{\pi} \ln \frac{1}{mx} \). The critical exponents of the disorder operator relative to fixed and free boundary conditions are therefore given by

$$x(z_{\pm}) = -\frac{1}{\pi} \int_0^{\infty} dp \ln \left( 1 - \frac{2\pi z_{\pm}}{\cosh p} \right) = -\frac{1}{8} + \frac{1}{2\pi^2} \arccos^2(-2\pi z_{\pm}) .$$

(3.11)
Substituting the values of $z_\pm$ we obtain the results obtained in [18], i.e. $x = 3/8$ for the fixed b.c. and $x = -1/8$ for the free b.c.

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