Perturbations of Dirac Operators and A KKW Type Theorem for Five Dimensional Manifolds with Boundary

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Abstract

In this paper, we prove a Kastler-Kalau-Walze type theorem associated with perturbations of Dirac operators for five dimensional manifolds with boundary.

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1. Introduction

The noncommutative residue found in \cite{1,2} plays a prominent role in noncommutative geometry. For arbitrary closed compact $n$-dimensional manifolds, the noncommutative residue was introduced by Wodzicki in \cite{2} using the theory of zeta functions of elliptic pseudodifferential operators. In \cite{3}, Connes used the noncommutative residue to derive a conformal 4-dimensional Polyakov action analogue. Furthermore, Connes made a challenging observation that the noncommutative residue of the square of the inverse of the Dirac operator was proportional to the Einstein-Hilbert action in \cite{4}. In \cite{5}, Kastler gave a brute-force proof of this theorem. In \cite{6}, Kalau and Walze proved this theorem in the normal coordinates system simultaneously, which is called the Kastler-Kalau-Walze theorem now.

An important application of Riemannian geometry is to allow us to define the volume element of a Riemannian manifold $(M_n,g)$. The noncommutative residue of Wodzicki \cite{2} and Guillemin \cite{1} is a trace on the algebra of (integer order) \(\Psi DOs\) on \(M\). An important feature is that it allows us to extend to all \(\Psi DOs\) the Dixmier trace, which plays the role of the integral in the framework of noncommutative geometry. Fedosov etc. defined a noncommutative residue on Boutet de Monvel’s algebra and proved that it was a unique continuous trace in \cite{7}. In \cite{8}, Schrohe gave the relation between the Dixmier trace and the noncommutative residue for manifolds with boundary. For an oriented spin manifold \(M\) with boundary \(\partial M\), by the composition formula in Boutet de Monvel’s algebra and the definition of \(\tilde{Wres}\) \cite{9}, \(\tilde{Wres}[\pi^+D^{-1}]_2\) should be the sum of two terms from interior and boundary of \(M\), where \(\pi^+D^{-1}\) is an element in Boutet de Monvel’s algebra \cite{9}.

Recently, Sitarz and Zajac investigated the spectral action for scalar perturbations of Dirac operators in \cite{10}. Iochum and Levy computed the heat kernel coefficients for Dirac operators with one-form perturbations in \cite{11}. In \cite{12}, Wang proved a Kastler-Kalau-Walze type theorem for perturbations of Dirac operators on compact manifolds with or without boundary. Furthermore, using Dirac operators with perturbations, Wang defined a spectral triple and established an infinitesimal equivariant index formula in \cite{13,14}. In \cite{15}, we proved a Kastler-Kalau-Walze type theorem for 5-dimensional manifolds with boundary. Motivated by \cite{10,15}, we study Dirac operators with one-form perturbations. In the present paper, we shall restrict

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our attention to the case of $\widetilde{\text{Res}}[\pi^+(D + c(X))^{-1} \circ \pi^+(D + c(X))^{-1}]$ for five dimensional manifolds with boundary. Our main result is as follows.

**Main Theorem:** Let $X = X' + a_s dx_n$ near the boundary and $X'|_{\partial M}$ is a one form on $\partial M$, the following identity holds

$$\widetilde{\text{Res}}[\pi^+(D + c(X))^{-1} \circ \pi^+(D + c(X))^{-1}] = \int_{\partial M} \left[ \frac{1}{16} \left( \frac{225}{32} K^2 + \frac{29}{4} s_M \right) - \left( \frac{155}{12} + 5i \right) s_{\partial M} \right]$$

$$+ \frac{25}{32} a_n K - \frac{35}{64} \left( \frac{1}{2} \right) K - 3a_n^2 |\partial M|$$

$$- \frac{3}{2} \partial_n (a_n)|_{\partial M} + \frac{15}{8} C^1(\nabla^0 \partial M(X'|_{\partial M}))$$

where $s_M$, $s_{\partial M}$ are respectively scalar curvatures on $M$ and $\partial M$, and the vector field $(X'|_{\partial M})^*$ is the metric dual of $X'|_{\partial M}$. $\nabla^0 \partial M$ is the Levi-Civita connection on $\partial M$, $C^1_t$ is the contraction of $(1, 1)$ tensors.

This paper is organized as follows: In Section 2, we define lower dimensional volumes of compact Riemannian manifolds with boundary. In Section 3, for five dimensional spin manifolds with boundary and the associated Dirac operators with one-form perturbations, we compute $\widetilde{\text{Res}}[\pi^+(D + c(X))^{-1} \circ \pi^+(D + c(X))^{-1}]$ and get a Kastler-Kalau-Walze type theorem in this case.

2. Lower-Dimensional Volumes of Spin Manifolds with boundary

In this section we consider an $n$-dimensional oriented Riemannian manifold $(M, g^M)$ with boundary $\partial M$ equipped with a fixed spin structure. We assume that the metric $g^M$ on $M$ has the following form near the boundary

$$g^M = \frac{1}{h(x_n)} g^{\partial M} + dx_n^2, \quad (2.1)$$

where $g^{\partial M}$ is the metric on $\partial M$. Let $U \subset M$ be a collar neighborhood of $\partial M$ which is diffeomorphic $\partial M \times [0, 1]$. By the definition of $h(x_n) \in C^\infty([0, 1])$ and $h(x_n) > 0$, there exists $\hat{h} \in C^\infty((-\epsilon, 1))$ such that $\hat{h}|_{(0, 1)} = h$ and $\hat{h} > 0$ for some sufficiently small $\epsilon > 0$. Then there exists a metric $\hat{g}$ on $M = M \cup_{\partial M} \partial M \times (-\epsilon, 0]$ which has the form on $U \cup_{\partial M} \partial M \times (-\epsilon, 0]$

$$\hat{g} = \frac{1}{\hat{h}(x_n)} g^{\partial M} + dx_n^2 \quad (2.2)$$

such that $\hat{g}|_M = g$. We fix a metric $\hat{g}$ on the $\hat{M}$ such that $\hat{g}|_M = g$.

Let us give the expression of Dirac operators near the boundary. Set $\tilde{E}_n = \frac{\partial}{\partial x_n}$, $\tilde{E}_j = \sqrt{h(x_n)} E_j \quad (1 \leq j \leq n - 1)$, where $\{E_1, \cdots, E_{n-1}\}$ are orthonormal basis of $T\partial M$. Let $\nabla^L$ denote the Levi-Civita connection about $g^M$. In the local coordinates $\{x_i; 1 \leq i \leq n\}$ and the fixed orthonormal frame $\{\tilde{E}_1, \cdots, \tilde{E}_n\}$, the connection matrix $\omega_{s,i}$ is defined by

$$\nabla^L(\tilde{E}_1, \cdots, \tilde{E}_n)^t = (\omega_{s,i})(\tilde{E}_1, \cdots, \tilde{E}_n)^t \quad (2.3)$$

Usually, the Dirac operator on spin manifold is taken to be the operator, which comes from the Levi-Civita connection on the tangent bundle to $M$. So the classical Dirac operator is simply defined on $S$ by

$$D = \sum_i c(e_i) \nabla^S e_i = \sum_{j=1}^n c(\tilde{E}_j) \left[ \tilde{E}_j + \frac{1}{4} \sum_{s,t} \omega_{s,t}(\tilde{E}_j)c(\tilde{E}_s)c(\tilde{E}_t) \right]. \quad (2.4)$$

However, that a wide class of generalized operators which come from connections with perturbations. Set $c(X)$ a Clifford action on $M$ and $X$ is a one form, the Dirac operators with one-form perturbations denote by

$$\tilde{D} = \sum_i c(e_i) \nabla^S e_i + c(X), \quad (2.5)$$
where \(c(e_i)\) denotes the Clifford action.

The next step is to express the lower dimensional volumes \(\text{Vol}^{[p_1,p_2]}_M\) in a purely differential geometric way. The local Riemannian invariants make sense independent of the existence of a spin structure, we can use the geometric expression for \(\text{Vol}^{[p_1,p_2]}_M\) to extend its definition to general. Denote by \(B\) Boutet de Monvel’s algebra, we recall the main theorem in [17].

**Theorem 2.1. (Fedosov-Golse-Leichtnam-Schrohe)** Let \(X\) and \(\partial X\) be connected, \(\dim X = n \geq 3\), \(A = \begin{pmatrix} \pi^+ P + G & K \\ T & S \end{pmatrix} \in B\), and denote by \(p, b\) and \(s\) the local symbols of \(P, G\) and \(S\) respectively. Define:

\[
\tilde{\text{Res}}(A) = \int_X \int_S \text{tr}_E [p_{-n}(x, \xi)] \sigma(\xi) dx + 2\pi \int_{\partial X} \int_{S'} \{\text{tr}_E [(tr b_{-n})(x', \xi')] + \text{tr}_F [s_{1-n}(x', \xi')]\} \sigma(\xi') dx',
\]

Then

1. \(\tilde{\text{Res}}([A, B]) = 0\), for any \(A, B \in B\);  
2. It is a unique continuous trace on \(B/B^{-\infty}\).

Let \(p_1, p_2\) be nonnegative integers and \(p_1 + p_2 \leq n\). Then by Sec 2.1 of [17], we have

**Definition 2.2.** Lower-dimensional volumes of spin manifolds with boundary are defined by

\[
\text{Vol}^{[p_1,p_2]}(M) := \tilde{\text{Res}}[\pi^+ \tilde{D}^{-p_1} \circ \pi^+ \tilde{D}^{-p_2}].
\]

Denote by \(\sigma_I(\xi)\) the \(I\)-order symbol of an operator \(A\). An application of (2.1.4) in [9] shows that

\[
\tilde{\text{Res}}[\pi^+ \tilde{D}^{-p_1} \circ \pi^+ \tilde{D}^{-p_2}] = \int_M \int_{|\xi|=1} \text{trace}_{\text{S}(\mathcal{M})} [\sigma_{-n} (\tilde{D}^{-p_1} \circ \tilde{D}^{-p_2})] \sigma(\xi) dx + \int_{\partial M} \Phi,
\]

where

\[
\Phi = \int_{|\xi|=1} \int_{-\infty}^{+\infty} \sum_{j,k=0}^{\infty} \frac{(-i)^{|\alpha|+j+k+1}}{\alpha! (j+k+1)!} \text{trace}_{\text{S}(\mathcal{M})} \left[ \partial^j_{x_n} \partial^k_{\xi_n} \partial^k_{\xi_n} \partial^j_{x_n} \sigma_T (\tilde{D}^{-p_1}) (x', 0, \xi', \xi_n) \right] d\xi_a \sigma(\xi') dx',
\]

and the sum is taken over \(r - k + |\alpha| + \ell - j - 1 = -n, r \leq -p_1, \ell \leq -p_2\).

3. A KKW type theorem for five dimensional spin manifolds with boundary

In this section, we compute the lower dimensional volume for five dimensional compact manifolds with boundary and get a Kastler-Kalau-Walze type formula in this case. From now on we always assume that \(M\) carries a spin structure so that the spinor bundle and the Dirac operators with one-form perturbations are defined on \(M\).

The following proposition is the key of the computation of lower-dimensional volumes of spin manifolds with boundary.

**Proposition 3.1.** [13] The following identity holds:

1. When \(p_1 + p_2 = n\), then, \(\text{Vol}^{[p_1,p_2]}_n M = c_0 \text{Vol}_M\);
2. when \(p_1 + p_2 \equiv n \mod 1\), \(\text{Vol}^{[p_1,p_2]}_n M = \int_{\partial M} \Phi\).
Thus, we get
\[ x \partial \] field in \( \tilde{\omega} \).

By the composition formula of pseudodifferential operators, then we have
\[ D_0^0 = (-\sqrt{-1})^{|\alpha|} \partial_{\alpha}^0; \quad \sigma(\tilde{D}) = p_1 + p_0; \quad \sigma(\tilde{D}^{-1}) = \sum_{j=1}^{\infty} q_{-j}. \] (3.4)

By the composition formula of pseudodifferential operators, then we have
\[ 1 = \sigma(\tilde{D} \circ \tilde{D}^{-1}) = \sum_{\alpha} \frac{1}{\alpha!} \partial_{\alpha}^0[\sigma(\tilde{D})]D_0^0[\sigma(\tilde{D}^{-1})] \]
\[ = (p_1 + p_0)(q_1 + q_2 + q_3 + \cdots) \]
\[ + \sum_j (\partial_{\xi_j}p_1 + \partial_{\xi_j}p_0)(D_{\xi_j}q_{-1} + D_{\xi_j}q_1 + D_{\xi_j}q_2 + D_{\xi_j}q_3 + \cdots) \]
\[ = p_1q_1 + (p_1q_2 + p_0q_1 + \sum_j \partial_{\xi_j}p_1D_{\xi_j}q_{-1}) \]
\[ + (p_1q_3 + p_0q_2 + \sum_j \partial_{\xi_j}p_1D_{\xi_j}q_2 + \cdots) \] (3.5)

Thus, we get
\[ q_1 = p_1^{-1}; \]
\[ q_2 = -p_1^{-1} \left[ p_0p_1^{-1} + \sum_j \partial_{\xi_j}p_1D_{\xi_j}(p_1^{-1}) \right]; \]
\[ q_3 = -p_1^{-1} \left[ p_0q_2 + \sum_{j=1}^{n-1} c(dx_{i,j})\partial_{x_{i,j}}q_{-2} + c(dx_{n})\partial_{x_n}q_{-2} \right]. \] (3.8)

Since \( \Phi \) is a global form on \( \partial M \), so for any fixed point \( x_0 \in \partial M \), we can choose the normal coordinates \( U \) of \( x_0 \) in \( \partial M \) (not in \( M \)) and compute \( \Phi(x_0) \) in the coordinates \( \tilde{U} = U \times [0, 1] \) and the metric \( h(x_0)g_{\partial M} + dx_n^2 \).

The dual metric of \( g_{\partial M} \) on \( \tilde{U} \) is \( h(x_0)g_{\partial M} + \frac{\partial}{\partial x_n} \otimes \frac{\partial}{\partial x_n} \). Write \( g^M_{ij} = g^M(dx_{i,j}, dx_{j,i}) \) and \( g^M_{ij} = g^M(dx_{i,j}, dx_j) \), then
\[ [g^M_{ij}] = \begin{bmatrix} \frac{1}{h(x_0)}g^M_{ij} & 0 \\ 0 & 1 \end{bmatrix}; \quad [g^M_{ij}] = \begin{bmatrix} h(x_0)[g^M_{ij}] & 0 \\ 0 & 1 \end{bmatrix}, \] (3.9)
and
\[ \partial_{x_{i,j}}g^M_{ij}(x_0) = 0, \quad 1 \leq i, j \leq n - 1; \quad g^M_{ij}(x_0) = \delta_{ij}. \] (3.10)

Let \( \{E_1, \ldots, E_{n-1}\} \) be an orthonormal frame field in \( U \) about \( g_{\partial M} \) which is parallel along geodesics and \( E_i = \frac{\partial}{\partial x_i}(x_0) \), then \( \{E_1 = \sqrt{h(x_0)}E_1, \ldots, E_n-1 = \sqrt{h(x_0)}E_n-1, E_n = dx_n \} \) is the orthonormal frame field in \( \tilde{U} \) about \( g^M \). Locally \( S(TM)|\tilde{U} \cong \tilde{U} \times \wedge^*_C(\frac{n}{2}) \). Let \( \{f_1, \ldots, f_n\} \) be the orthonormal basis of \( \wedge^*_C(\frac{n}{2}) \).

Take a spin frame field \( \sigma : \tilde{U} \to Spin(\tilde{U}) \) such that \( \pi \sigma = \{E_1, \ldots, E_{n}\} \) where \( \pi : Spin(M) \to O(M) \) is a double covering, then \( \{[\sigma, f_i], 1 \leq i \leq 4 \} \) is an orthonormal frame of \( S(TM)|\tilde{U} \). In the following, since the global form \( \Phi \) is independent of the choice of the local frame, so we can compute \( \text{tr}_{S(TM)} \) in the frame \( \{[\sigma, f_i], 1 \leq i \leq 4 \} \). Let \( \{\tilde{E}_1, \ldots, \tilde{E}_n\} \) be the canonical basis of \( R^n \) and \( c(\tilde{E}_i) \in Hom(\wedge^*_C(\frac{n}{2}), \wedge^*_C(\frac{n}{2})) \) be the Clifford action. By \([13]\), then
\[ c(\tilde{E}_i) = [[\sigma, c(\tilde{E}_i)]]; \quad c(\tilde{E}_i)[[\sigma, f_i]] = [\sigma, (c(\tilde{E}_i))f_i]; \quad \frac{\partial}{\partial x_{e_i}} = [[\sigma, \frac{\partial}{\partial x_{e_i}}]], \] (3.11)
Lemma 3.2. With the metric \( g^M \) on \( M \) near the boundary

\[
\partial x_j(|\xi|^2 g^M)(x_0) = \begin{cases} 
0, & \text{if } j < n; \\
h'(0)|\xi|^2 g_{\partial M}, & \text{if } j = n.
\end{cases}
\]

(3.12)

\[
\partial x_j[c(\xi)](x_0) = \begin{cases} 
0, & \text{if } j < n; \\
\partial x_n(c(\xi'))(x_0), & \text{if } j = n,
\end{cases}
\]

(3.13)

where \( \xi = \xi' + \xi_n dx_n \).

By Lemma 2.1 in [16], we have

Lemma 3.3. The symbol of the Dirac operators with one-form perturbations

\[
\sigma_{-1}(\tilde{D}^{-1}) = q_{-1} = \frac{\sqrt{-1}c(\xi)}{|\xi|^2};
\]

(3.14)

\[
\sigma_{-2}(\tilde{D}^{-1}) = q_{-2} = \frac{c(\xi)p_0 c(\xi)}{|\xi|^4} + \frac{c(\xi)}{|\xi|^6} \sum_j c(dx_j) \left[ \partial x_j(c(\xi))|\xi|^2 - c(\xi)\partial x_j(|\xi|^2) \right] - \frac{c(X)}{|\xi|^4} - \frac{2g(X, \xi)c(\xi)}{|\xi|^4};
\]

(3.15)

\[
\sigma_{-3}(\tilde{D}^{-1}) = q_{-3} = -\frac{1}{p_1} \left[ p_0 q_{-2} + \sum_{j=1}^{n-1} c(dx_j)\partial x_j q_{-2} + c(dx_n)\partial x_n q_{-2} \right];
\]

(3.16)

where

\[
p_0(x_0) = -h'(0)c(dx_n) + c(X).
\]

(3.17)

Let us now consider the \( q_{-3} \) of the Dirac operators with one-form perturbations. From Lemma 3.7 in [15], we have
Lemma 3.4. [13] For Dirac operators, the following identity holds:

\[ \sigma_{-3}(D^{-1})(x_0) \bigg|_{|\xi'|=1} = \frac{-i(h')^2}{(1 + \xi_n^2)^3} c(\xi) c(dx_n) c(\xi) c(dx_n) c(\xi) + \frac{ih'}{(1 + \xi_n^2)^3} c(\xi) c(dx_n) c(\xi) c(dx_n) c(\xi) \theta_{x_n}[c(\xi')](0) + \frac{1}{6} \sum_{\beta < n} R_{\beta < n}^\beta \left( R_{\beta < n}^\beta (x_0) \right) \frac{1}{(1 + \xi_n^2)^4} c(\xi) c(dx_n) c(\xi) c(dx_n) c(\xi) \]

From (3.16) and Lemma 3.2-Lemma 3.4, we obtain

Lemma 3.5. For Dirac operators with one-form perturbations, the following identity holds:

\[ \sigma_{-3}(\tilde{D}^{-1})(x_0) \bigg|_{|\xi'|=1} = \sigma_{-3}(D^{-1})(x_0) \bigg|_{|\xi'|=1} + \left[ - q_1 c(X) \sigma_{-2}(D^{-1}) - q_1 \frac{c(X)}{|\xi|^2} + q_1 \frac{2g(X, \xi)}{|\xi|^4} \right] \]

\[ = - q_1 \sum_{j=1}^{n-1} (dx_j) \partial_{x_j} \left( \frac{c(X)}{|\xi|^2} \right) + q_1 \sum_{j=1}^{n-1} (dx_j) \partial_{x_j} \left( \frac{2g(X, \xi)}{|\xi|^4} \right) \]

\[ - q_1 c(dx_n) \partial_{x_n} \left( \frac{c(X)}{|\xi|^2} \right) + q_1 c(dx_n) \partial_{x_n} \left( \frac{2g(X, \xi)}{|\xi|^4} \right) \]

\[ \bigg|_{|\xi'|=1} \]

From the remark above, now we can compute \( \Phi \) (see the formula (2.9) for the definition of \( \Phi \)). Since the sum is taken over \(-r - l + k + j + |\alpha| - 1 = 5\), \( r, l \leq -1 \), then we have the \( \int_{\partial_{\alpha}} \Phi \) is the sum of the following fifteen cases. Such cases (1)-(6) have been studied, similar to Case (1)-Case (6) in [15], we obtain

Case (1): \( r = -1, \ l = -1, \ k = 0, \ j = 1, \ |\alpha| = 1 \)

From (2.9), we have

\[ \text{Case (1)} = \frac{i}{2} \int_{|\xi'|=1} \int_{|\xi|=-1}^{+\infty} \sum_{|\alpha|=1} \text{trace} \left[ \partial_{x_n} \partial_{x_n}^\alpha \pi_{x_n} q_{-1} \times \partial_{x_n}^\alpha \partial_{x_n} q_{-1} \right] (x_0) d\xi \sigma(\xi') dx'. \]
By Lemma 3.3, for $i < n$, we have

$$
\partial_{x_i} q_{-1}(x_0) = \partial_{x_i} \left( \frac{\sqrt{1-r} \xi}{|\xi|^2} \right)(x_0) = \frac{\sqrt{1-r} \partial_{x_i} [c(\xi)](x_0)}{|\xi|^2} - \frac{\sqrt{1-r} \xi \partial_{x_i} (|\xi|^2)(x_0)}{|\xi|^4} = 0.
$$

(3.21)

So Case (1) vanishes.

**Case (2):** $r = -1$, $\ell = -1$, $k = 0$, $j = 2$, $|\alpha| = 0$

From (2.9), we have

$$
\text{Case (2) } = \frac{i}{6} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{j=2}^n \text{trace} \left[ \partial^2_{x_n} \pi^+_{\xi_n} q_{-1} \times \partial^3_{\xi_i} q_{-1} \right](x_0) d\xi_n \sigma(\xi') dx'.
$$

(3.22)

By Lemma 3.2, a simple computation shows

$$
\partial^2_{\xi_i} q_{-1}(x_0) \big|_{|\xi'|=1} = \frac{24 \xi_n - 24 \xi_n^3}{(1 + \xi_n^2)^4} \sqrt{-1} c(\xi') + \frac{-6 \xi_n^4 + 36 \xi_n^2 - 6}{(1 + \xi_n^2)^4} \sqrt{-1} c(dx_n).
$$

(3.23)

From Lemma 3.2, Lemma 3.3 and Lemma 3.4, we obtain

$$
\partial^2_{x_n} \pi^+_{\xi_n} q_{-1}(x_0) \big|_{|\xi'|=1} = \left( \frac{3}{4} (h'(0))^2 - \frac{1}{2} h''(0) \right) c(\xi') + \frac{1}{2} \xi_n - 2i \xi_n \partial_{x_n} c(\xi')
$$

$$
- h''(0) \left[ \frac{\xi_n - 2i}{4(\xi_n - i)^2} c(\xi') + \frac{1}{4(\xi_n - i)^2} c(dx_n) \right]
$$

$$
+ 2i (h'(0))^2 \left[ -3 \xi_n^2 - 9 \xi_n + 8i \right] c(\xi') + \frac{-i \xi_n - 3}{16(\xi_n - i)^3} c(dx_n).
$$

(3.24)

Using the Clifford relations combined with the cyclicity of the trace and $\text{tr} AB = \text{tr} BA$, then

$$
\text{tr} [c(\xi') c(dx_n)] = 0; \quad \text{tr} [c(dx_n)^2] = -4; \quad \text{tr} [c(\xi')^2](x_0) \big|_{|\xi'|=1} = -4;
$$

$$
\text{tr} [\partial_{x_n} c(\xi') c(dx_n)] = 0; \quad \text{tr} [\partial_{x_n} c(\xi') \times c(\xi')](x_0) \big|_{|\xi'|=1} = -2h'(0).
$$

(3.25)

From (3.22)-(3.25) and direct computations, we obtain

$$
\text{trace} \left[ \partial^2_{x_n} \pi^+_{\xi_n} q_{-1} \times \partial^3_{\xi_i} q_{-1} \right](x_0) \big|_{|\xi'|=1} = i(h'(0)) \left[ \frac{24 \xi_n - 24 \xi_n^3}{(1 + \xi_n^2)^4} \right]
$$

$$
+ i h''(0) \left[ \frac{24 \xi_n - 24 \xi_n^3 + 36 \xi_n^2 - 6}{(1 + \xi_n^2)^4} \right].
$$

(3.26)
Therefore

\[
\text{Case (2)} = -\frac{1}{6} (h'(0))^2 \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \frac{3(33\xi_n^5 - 75i\xi_n^4 - 94\xi_n^3 + 90i\xi_n^2 + 57\xi_n - 3i)}{2(\xi_n - i)^3(1 + \xi_n^2)^4} \partial_{\xi_n} \sigma(\xi') \, d\xi_n \, d\xi' 
\]

\[
-\frac{1}{6} h''(0) \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \frac{6(-9\xi_n^4 + 12\xi_n^3 + 14\xi_n^2 - 12i\xi_n - 1)}{2(\xi_n - i)^2(1 + \xi_n^2)^4} \partial_{\xi_n} \sigma(\xi') \, d\xi_n \, d\xi' 
\]

\[
= -\frac{1}{6} (h'(0))^2 \Omega_3 \int_{+\infty}^{1} \frac{3(33\xi_n^5 - 75i\xi_n^4 - 94\xi_n^3 + 90i\xi_n^2 + 57\xi_n - 3i)}{2(\xi_n - i)^3(1 + \xi_n^2)^4} \partial_{\xi_n} \sigma(\xi') \, d\xi_n 
\]

\[
-\frac{1}{6} h''(0) \Omega_3 \int_{+\infty}^{1} \frac{6(-9\xi_n^4 + 12\xi_n^3 + 14\xi_n^2 - 12i\xi_n - 1)}{2(\xi_n - i)^2(1 + \xi_n^2)^4} \partial_{\xi_n} \sigma(\xi') \, d\xi_n 
\]

\[
= -\frac{1}{6} (h'(0))^2 \Omega_3 \int_{+\infty}^{1} \frac{3(33\xi_n^5 - 75i\xi_n^4 - 94\xi_n^3 + 90i\xi_n^2 + 57\xi_n - 3i)}{2(\xi_n - i)^3(1 + \xi_n^2)^4} \partial_{\xi_n} \sigma(\xi') \, d\xi_n 
\]

where \( \Omega_3 \) is the canonical volume of \( S^3 \).

Similarly, we have

\[
\text{Case (3)}: \quad r = -1, \quad \ell = -1, \quad k = 0, \quad j = 0, \quad |\alpha| = 2 
\]

\[
\text{Case (3)} = -\frac{i}{4} \sum_{|\alpha| = 2} \partial_{\xi_n} \sigma(\xi') \, d\xi_n \, d\xi', 
\]

where \( \sum_{l < n} R^{\alpha}_{l|l} (x_0) \) is the scalar curvature \( s_{\alpha\nu} \).

\[
\text{Case (4)}: \quad r = -1, \quad \ell = -1, \quad k = 1, \quad j = 1, \quad |\alpha| = 0 
\]

\[
\text{Case (4)} = -\frac{i}{6} \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \partial_{\xi_n} \partial_{\xi_k} \sigma(\xi') \, d\xi_n \, d\xi' 
\]

\[
= -\frac{5}{16} (h'(0))^2 \pi \Omega_3 \, d\xi' 
\]

\[
\text{Case (5)}: \quad r = -1, \quad \ell = -1, \quad k = 1, \quad j = 0, \quad |\alpha| = 1 
\]

\[
\text{Case (5)} = -\frac{1}{2} \sum_{|\alpha| = 1} \partial_{\xi_n} \partial_{\xi_k} \sigma(\xi') \, d\xi_n \, d\xi' 
\]

\[
= 0. 
\]

\[
\text{Case (6)}: \quad r = -1, \quad \ell = -1, \quad k = 2, \quad j = 0, \quad |\alpha| = 0 
\]

\[
\text{Case (6)} = -\frac{i}{6} \int_{|\xi'|=1}^{+\infty} \sum_{k=2}^{+\infty} \partial_{\xi_n} \partial_{\xi_k} \sigma(\xi') \, d\xi_n \, d\xi' 
\]

\[
= \left( \frac{29}{64} (h'(0))^2 - \frac{3}{8} h''(0) \right) \pi \Omega_3 \, d\xi'. 
\]
Now we discuss the cases (7)-(15).

**Case (7):** \( r = -1, \ell = -2, k = 0, j = 1, |\alpha| = 0 \)

From (2.9) and the Leibniz rule, we obtain

\[
\text{Case (7)} = \frac{1}{2} \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace} \left[ \partial_{\xi_n} \partial_{x_n} \pi_{\xi_n}^{+} q_{-1} \times \partial_{\xi_n} q_{-2} \right] (x_0) d\xi_0 \sigma(\xi') dx'.
\]

By Lemma 3.3, (2.2.22) in [16] and direct computations, we obtain

\[
\frac{\partial^2_{\xi_n} \partial_{x_n} \pi_{\xi_n}^{+} q_{-1} (x_0)}{|\xi'|} = \frac{1}{(\xi_n - 1)^3} \partial_{x_n} [c(\xi')](x_0) + h'(0) \left[ -\frac{4i - \xi_n}{2(\xi_n - 1)^4} c(\xi') - \frac{3}{2(\xi_n - 1)^4} c(dx_n) \right].
\]

From Lemma 3.2 and Lemma 3.3, we have

\[
q_{-2} = \frac{c(\xi') \partial_{x_n} (c(\xi))}{|\xi'|} + \frac{c(\xi)}{|\xi'|} \sum_j c(dx_j) \left[ \partial_{x_j} (c(\xi))|\xi|^2 - c(\xi) \partial_{x_j} (|\xi|^2) \right] = \sigma_{-2} (D^{-1}) + \frac{c(X) c(\xi)}{|\xi'|^2}.
\]

Then

\[
\text{Case (7)} = -\frac{1}{2} \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace} \left[ \partial^2_{\xi_n} \partial_{x_n} \pi_{\xi_n}^{+} q_{-1} \times \sigma_{-2} (D^{-1}) \right] (x_0) d\xi_0 \sigma(\xi') dx'.
\]

Let \( X = X' + a_n dx_n \), by the relation of the Clifford action and \( \text{tr} AB = \text{tr} BA \), then we have the equalities:

\[
\text{tr}[c(\xi') c(X)] = -4g(X, \xi'); \quad \text{tr}[c(dx_n) c(X)] = -4a_n.
\]

Considering for \( i < n, \int_{|\xi'|=1} \{ \xi_i, \xi_{i+1} \cdots \xi_{2d+1} \} \sigma(\xi') = 0 \). From (3.34), (3.35) and direct computations, we obtain

\[
\text{trace} \left[ \partial^2_{\xi_n} \partial_{x_n} \pi_{\xi_n}^{+} q_{-1} \times \left( \frac{c(X)}{|\xi'|^2} - \frac{2g(X, \xi) c(\xi)}{|\xi'|^4} \right) \right] (x_0) = a_n h'(0) \left[ -\frac{6\xi_n^2 + 8i\xi_n + 6}{(\xi_n - i)^6} \right].
\]

By Case (7) in [13], then

\[
\text{Case (7)} = \frac{39}{32} (h'(0))^2 \pi \Omega_{3} dx'.
\]

\[
= \frac{1}{2} \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace} \left[ \partial^2_{\xi_n} \partial_{x_n} \pi_{\xi_n}^{+} q_{-1} \times \left( \frac{c(X)}{|\xi'|^2} - \frac{2g(X, \xi) c(\xi)}{|\xi'|^4} \right) \right] (x_0) d\xi_0 \sigma(\xi')
\]

\[
= \frac{39}{32} (h'(0))^2 \pi \Omega_{3} dx' - \frac{1}{2} a_n h'(0) \frac{2\pi i}{5!} \left[ -\frac{6\xi_n^2 + 8i\xi_n + 6}{(\xi_n - i)^6} \right] \bigg|_{\xi_n=i} \Omega_{3} dx'.
\]

**Case (8):** \( r = -1, \ell = -2, k = 0, j = 0, |\alpha| = 1 \)
From (2.9) and the Leibniz rule, we obtain

\[
\text{Case (8)} = - \int_{|\xi'|=1}^{+\infty} \sum_{|\alpha|=1} \text{trace} \left[ \partial^{\alpha}_2 \pi^+_{\xi',q-1} \times \partial^{\alpha}_2 \pi_{\xi',q-2} \right] (x_0) d\xi_n \sigma(\xi') dx'
\]

\[
= \int_{|\xi'|=1}^{+\infty} \sum_{|\alpha|=1} \text{trace} \left[ \partial^{\alpha}_2 \pi^+_{\xi',q-1} \times \partial^{\alpha}_2 \pi_{\xi',q-2} \right] (x_0) d\xi_n \sigma(\xi') dx'
\]

\[
= \int_{|\xi'|=1}^{+\infty} \sum_{|\alpha|=1} \text{trace} \left[ \partial^{\alpha}_2 \pi^+_{\xi',q-1} \times \partial^{\alpha}_2 \pi_{\xi',q-2} (D^{-1}) \right] (x_0) d\xi_n \sigma(\xi') dx'
\]

\[
= \int_{|\xi'|=1}^{+\infty} \sum_{|\alpha|=1} \text{trace} \left[ \partial^{\alpha}_2 \pi^+_{\xi',q-1} \times \partial^{\alpha}_2 \pi_{\xi',q-2} (D^{-1}) \right] (x_0) d\xi_n \sigma(\xi') dx'.
\]

(3.38)

From Lemma 3.2 and Lemma 3.3, a simple computation shows

\[
\partial^{\alpha}_2 (x_0) = \frac{-1}{2(\xi_n - i)^2} c(dx_i) - \frac{3i - \xi_n}{2(\xi_n - i)^3} c(\xi') + \frac{1}{(\xi_n - i)^3} c(dx_n).
\]

(3.39)

From Lemma 3.2, we have

\[
\partial^{\alpha}_2 \left( \frac{c(X)}{|\xi|^2} \right)_{|\xi'|=1} = \frac{1}{|\xi|^2} \partial_{\xi'} [c(X)] - \frac{2c(\xi)}{|\xi|^4} \partial_{\xi'} [g(X, \xi)].
\]

(3.40)

By the relation of the Clifford action and $\text{tr} AB = \text{tr} BA$, then we have the equalities at a fixed point $x_0$:

\[
\text{tr} [c(\xi') \partial_{\xi'} [c(X)]] (x_0) = -4 \partial_{\xi'} [g(X, \xi')] (x_0).
\]

Combining (3.39), (3.40) and direct computations, we obtain

\[
\sum_{|\alpha|=1} \text{trace} \left[ \partial^{\alpha}_2 \pi^+_{\xi',q-1} \times \partial^{\alpha}_2 \pi_{\xi',q-2} (D^{-1}) \right] (x_0)
\]

\[
= \frac{2}{(\xi_n - i)^3} \sum_{j=1}^{n-1} \partial_{\xi_j} [g(X, dx_j)] + \frac{-2\xi_n^3 + 6i\xi_n^2 - 2\xi_n - 2i}{(\xi_n - i)^3} \sum_{j=1}^{n-1} \xi_j \partial_{\xi_j} [g(X, dx_j)].
\]

(3.41)

Then an application of (16) in \cite{3} shows

\[
\int_{S^3} \xi_n \xi_{i\nu} = \frac{\pi^2}{2} \delta^{i\nu}.
\]

(3.42)

Using the integration over $S^3$ and the shorthand $\int = \frac{1}{2\pi} \int_{S^3} d^3 \nu$, we obtain $\Omega_3 = 2\pi^2$. Considering for $i < n$, \[\xi_{i1} \xi_{i2} \cdots \xi_{i(n-1)} \sigma(\xi') = 0. \] Let $X = X' + a_n dx_n$ near the boundary and $X'$ is a one form on $\partial_M$, therefore

\[
\text{Case (8)} = \left( \frac{3}{16} - \frac{5}{32} \right) s_{\partial_M} \pi_3 \Omega_3 dx'
\]

\[
+ \int_{|\xi'|=1}^{+\infty} \sum_{|\alpha|=1} \text{trace} \left[ \partial^{\alpha}_2 \pi^+_{\xi',q-1} \times \partial^{\alpha}_2 \pi_{\xi',q-2} (D^{-1}) \right] (x_0) d\xi_n \sigma(\xi') dx'
\]

\[
= \left( \frac{3}{16} - \frac{5}{32} \right) s_{\partial_M} \pi_3 \Omega_3 dx' - \frac{9}{16} \pi \Omega_3 \sum_{j=1}^{n-1} \partial_{\xi_j} [g(X, dx_j)] dx'
\]

\[
= \left( \frac{3}{16} - \frac{5}{32} \right) s_{\partial_M} \pi_3 \Omega_3 dx' - \frac{9}{16} \pi \Omega_3 C_1(\nabla^{\partial M} (X'|_{\partial M}^*)) dx',
\]

(3.43)
where the vector field \((X'|\partial M)^* = g^{BM}(X'|\partial M), \cdot\) and \(\nabla^{BM}\) is the Levi-civita connection on \(\partial M\), \(C_1\) is the contraction of \((1,1)\) tensors.

**Case (9):** \(r = -1, \ell = -2, k = 1, j = 0, |\alpha| = 0\)

From (2.9) and the Leibniz rule, we obtain

\[
\text{Case (9)} = \frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{|\alpha|=1} \text{trace} \left[ \partial_{\xi_n} \pi^+_{\xi_n} q_{-1} \times \partial_{\xi_n} \partial_{x_n} q_{-2} \right] (x_0) d\xi_n \sigma(\xi') dx'
\]

\[
= \frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{|\alpha|=1} \text{trace} \left[ \partial^2_{\xi_n} \pi^+_{\xi_n} q_{-1} \times \partial_{x_n} q_{-2} \right](x_0) d\xi_n \sigma(\xi') dx'
\]

\[
= \frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{|\alpha|=1} \text{trace} \left[ \partial^2_{\xi_n} \pi^+_{\xi_n} q_{-1} \times \partial_{x_n} \sigma_{-2}(D^{-1}) \right](x_0) d\xi_n \sigma(\xi') dx'
\]

\[
+ \frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{|\alpha|=1} \text{trace} \left[ \partial^2_{\xi_n} \pi^+_{\xi_n} q_{-1} \times \partial_{x_n} \left( \frac{c(X)}{|\xi|^2} - \frac{2g(X, \xi)c(\xi)}{|\xi|^4} \right) \right](x_0) d\xi_n \sigma(\xi') dx'.
\]

(3.44)

By (2.2.29) in [16], we have

\[
\partial^2_{\xi_n} \pi^+_{\xi_n} q_{-1}(x_0)|_{|\xi'|=1} = \frac{1}{(\xi_n - i)^3}c(\xi') + \frac{i}{2(\xi_n - i)^3}c(dx_n).
\]

(3.45)

From Lemma 3.2, a simple computation shows

\[
\partial_{x_n} \left( \frac{c(X)}{|\xi|^2} - \frac{2g(X, \xi)c(\xi)}{|\xi|^4} \right) = \partial_{x_n} \left( \frac{c(X)}{|\xi|^2} \right) - \partial_{x_n} \left( \frac{2g(X, \xi)c(\xi)}{|\xi|^4} \right)
\]

\[
= \frac{1}{|\xi|^2} \partial_{x_n} \left( c(X) \right) - \frac{h'(0)|\xi'|^2 c(X)}{|\xi|^4} - \frac{2c(\xi)}{|\xi|^4} \partial_{x_n} \left( g(X, \xi) \right)
\]

\[
- \frac{2g(X, \xi)c(\xi)}{|\xi|^4} \partial_{x_n} \left( c(\xi) \right) + \frac{4g(X, \xi)c(\xi)}{|\xi|^4} \partial_{x_n} \left( |\xi'|^2 \right).
\]

(3.46)

By the relation of the Clifford action and \(\text{tr}AB = \text{tr}BA\), then we have the equalities:

\[
\text{tr} \left[ c(dx_n) \partial_{x_n} [c(X)] \right] = -4\partial_{x_n} [a_n].
\]

Combining (3.45), (3.46) and direct computations, we obtain

\[
\sum_{|\alpha|=1} \text{trace} \left[ \partial^2_{\xi_n} \pi^+_{\xi_n} q_{-1} \times \partial_{x_n} \left( \frac{c(X)}{|\xi|^2} - \frac{2g(X, \xi)c(\xi)}{|\xi|^4} \right) \right]
\]

\[
= \frac{1}{(\xi_n - i)^3(1 + i \xi_n^2)^2} \text{tr} \left[ c(\xi') \partial_{x_n} [c(X)] \right] - \frac{h'(0)}{(\xi_n - i)^3(1 + i \xi_n^2)^2} \text{tr} \left[ c(\xi')c(X) \right]
\]

\[
+ \frac{8}{(\xi_n - i)^3(1 + i \xi_n^2)^2} \partial_{x_n} \left( g(X, \xi') \right) + \frac{4 \xi_n^3 - 8}{(\xi_n - i)^3(1 + i \xi_n^2)^3} h'(0) g(X, \xi')
\]

\[
+ \frac{8 \xi_n}{(\xi_n - i)^3(\xi_n + i)^2} \partial_{x_n} (a_n) + \frac{4 \xi_n^3 - 8 \xi_n}{(\xi_n - i)^3(\xi_n + i)^2} a_n h'(0).
\]

(3.47)
Combining these assertions, we see

\[ \int_{|\xi'|=1} \langle \xi_1, \xi_2, \ldots, \xi_{12+k} \rangle \sigma(\xi') = 0. \]

Therefore

Case (9) \[ = \left( - \frac{367}{128} (h'(0))^2 + \frac{103}{64} h''(0) \right) \pi \Omega_3 \delta x' + \frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{|\alpha|=1}^{\infty} \text{tr} \left[ \partial_{\xi_n}^2 \pi_{\xi_n}^{+, q-1} \times \partial_{x_n} \left( \frac{c(X)}{|\xi'|^2} - \frac{2g(X, \xi)c(\xi)}{|\xi'|^4} \right) \right] \left( x_0 \right) d\xi_n \sigma(\xi') \delta x' \]

\[ = \left( - \frac{367}{128} (h'(0))^2 + \frac{103}{64} h''(0) \right) \pi \Omega_3 \delta x' - \frac{3}{8} \pi \Omega_3 \partial_{x_n}(a_n) \delta x' + \frac{15}{64} \pi a_n h''(0) \Omega_3 \delta x'. \] 

(3.48)

Case (10): \( r = -2, \ell = -1, k = 0, j = 1, |\alpha| = 0 \)

From (2.9), we have

\[ \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr} \left[ \partial_{\xi_n} \pi_{\xi_n}^{+, q-2} \times \partial_{\xi_n}^2 \pi_{\xi_n}^{+, q-1} \right] \left( x_0 \right) d\xi_n \sigma(\xi') \delta x'. \] 

(3.49)

By the Leibniz rule, trace property and "+" and "-" vanishing after the integration over \( \xi_n \) in (7), then

\[ \int_{-\infty}^{+\infty} \text{tr} \left[ \partial_{\xi_n} \pi_{\xi_n}^{+, q-2} \times \partial_{\xi_n}^2 \pi_{\xi_n}^{+, q-1} (D^{-1}) \right] d\xi_n \]

\[ = \int_{-\infty}^{+\infty} \text{tr} \left[ \partial_{\xi_n} \pi_{\xi_n}^{+, q-2} (D^{-1}) \times \partial_{\xi_n}^2 \pi_{\xi_n}^{+, q-1} (D^{-1}) \right] d\xi_n - \int_{-\infty}^{+\infty} \text{tr} \left[ \partial_{\xi_n} \pi_{\xi_n}^{+, q-2} (D^{-1}) \times \partial_{\xi_n}^2 \pi_{\xi_n}^{+, q-1} (D^{-1}) \right] d\xi_n. \] 

(3.50)

Combining these assertions, we see

Case (10) \[ = \text{Case (9)} - \frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr} \left[ \partial_{\xi_n} \pi_{\xi_n}^{+, q-2} \times \partial_{\xi_n}^2 \pi_{\xi_n}^{+, q-1} \right] d\xi_n \sigma(\xi') \delta x' \]

\[ = \text{Case (9)} - \frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr} \left[ \partial_{\xi_n} \pi_{\xi_n}^{+, q-2} (D^{-1}) \times \partial_{\xi_n}^2 \pi_{\xi_n}^{+, q-1} (D^{-1}) \right] d\xi_n \sigma(\xi') \delta x' \]

\[ - \frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr} \left[ \partial_{\xi_n} \left( \frac{c(X)}{|\xi'|^2} - \frac{2g(X, \xi)c(\xi)}{|\xi'|^4} \right) \times \partial_{\xi_n}^2 \pi_{\xi_n}^{+, q-1} \right] d\xi_n \sigma(\xi') \delta x'. \] 

(3.51)

By Lemma 3.2, a simple computation shows

\[ \partial_{\xi_n} (x_0) \bigg|_{|\xi'|=1} = \frac{6c_1 - 2}{(1 + c_2)^3} \sqrt{-1c(\xi')} + \frac{2c_3 - 6c_1}{(1 + c_2)^3} \sqrt{-1c(\delta x_n)}. \] 

(3.52)

From Lemma 3.2, a simple computation shows

\[ \partial_{\xi_n} \left( \frac{c(X)}{|\xi'|^2} - \frac{2g(X, \xi)c(\xi)}{|\xi'|^4} \right) \]

\[ = \partial_{x_n} \left( \frac{c(X)}{|\xi'|^2} \right) - \partial_{x_n} \left( \frac{2g(X, \xi)c(\xi)}{|\xi'|^4} \right) \]

\[ = \frac{1}{|\xi'|^2} \partial_{x_n} \left( c(X) \right) \frac{h'(0) |\xi'|^2}{|\xi'|^4} - \frac{2c(\xi)}{|\xi'|^2} \partial_{x_n} \left( g(X, \xi) \right) \]

\[ + \frac{2g(X, \xi)c(\xi)}{|\xi'|^4} \partial_{x_n} \left( c(\xi) \right) + \frac{4g(X, \xi)c(\xi)}{|\xi'|^6} \partial_{x_n} \left( |\xi'|^2 \right). \] 

(3.53)
Combining (3.52), (3.53) and direct computations, we obtain
\[
\text{trace}igg[ \partial_{\xi_n} \left( \frac{c(X)}{|\xi|^2} - \frac{2g(X, \xi)c(\xi)}{|\xi|^4} \right) \times \partial_{\xi_n} q - 1 \bigg] \\
= \frac{6i\xi_n^2 - 2i}{(1 + \xi_n^2)^3} \text{tr} \left[ c(\xi') \partial_{\xi_n} [c(\xi)] \right] - \frac{(6i\xi_n^2 - 2i)h'(0)}{(1 + \xi_n^2)^5} \text{tr} \left[ c(\xi') c(X) \right] \\
+ \frac{8(6i\xi_n^2 - 2i)}{(1 + \xi_n^2)^5} \partial_{\xi_n} \left( g(X, \xi') \right) + \frac{8\xi_n(2i\xi_n^3 - 6i\xi_n)}{(1 + \xi_n^2)^5} \partial_{\xi_n} \left( g(X, \xi') \right) \\
+ \frac{8i(\xi_n^3 - 2\xi_n^3 + 3\xi_n)}{(1 + \xi_n^2)^5} \partial_{\xi_n} \left( a_n h'(0) \right). 
\]
(3.54)

Considering for \( i < n, \sum_{|\xi'|=1} \{ \xi_1, \xi_2 \ldots \xi_2n \} \sigma(\xi') = 0 \). Similar to Case (9), we obtain
\[
\text{Case (10)} \left( -\frac{367}{128} h'(0)^2 + \frac{103}{64} h''(0) \right) \pi \Omega d\xi' = -\frac{3 \pi}{8} \Omega \partial_{\xi_n}(a_n) dx' + \frac{15}{64} \pi a_n h'(0) \Omega d\xi' . 
\]
(3.55)

Case (11): \( r = -2, \ell = -1, k = 0, j = 0, |\alpha| = 1 \)
From (2.9), we have
\[
\text{Case (11)} = -\int_{|\xi'|=1}^{+\infty} \sum_{|\alpha|=1} \text{trace} \left[ \partial_{\xi_n}^2 \frac{\pi_{\xi_n} q - 2 \times \partial_{\xi_n} q - 1}{(\xi_n^2 - 1)(\xi_n^2 + 1)} \right] (x_0) d\xi_n \sigma(\xi') dx'. 
\]
(3.56)

By Lemma3.2 and Lemma 3.3, for \( i < n \), we have
\[
\partial_{\xi_n} q - 1 (x_0) = \partial_{\xi_n} \left( \frac{\sqrt{-1}c(\xi')}{|\xi|^2} \right) (x_0) = \frac{\sqrt{-1}\Omega c(\xi) [c(\xi)] (x_0)}{|\xi|^2} - \frac{\sqrt{-1}c(\xi) \partial_{\xi_n}(\xi^2) (x_0)}{|\xi|^4} = 0. 
\]
(3.57)

So Case (11) vanishes.

Case (12): \( r = -2, \ell = -1, k = 1, j = 0, |\alpha| = 0 \)
From (2.9) and the Leibniz rule, trace property and "++" and "−−" vanishing after the integration over \( \xi_n \) in \( \mathbb{C} \), we have
\[
\text{Case (12)} = -\frac{1}{2} \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace} \left[ \partial_{\xi_n} \partial_{\xi_n}^2 q - 2 \times \partial_{\xi_n} q - 1 \right] (x_0) d\xi_n \sigma(\xi') dx' \\
= -\frac{1}{2} \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace} \left[ \partial_{\xi_n}^2 q - 2 \partial_{\xi_n}^2 q - 1 \right] (x_0) d\xi_n \sigma(\xi') dx' \\
= \text{Case (7)} + \frac{1}{2} \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace} \left[ \sigma_{-2}(D^{-1}) \times \partial_{\xi_n}^2 q - 1 \right] (x_0) d\xi_n \sigma(\xi') dx' \\
= \text{Case (7)} + \frac{1}{2} \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace} \left[ \left( \frac{c(X)}{|\xi|^2} - \frac{2g(X, \xi)c(\xi)}{|\xi|^4} \right) \times \partial_{\xi_n}^2 q - 1 \right] (x_0) d\xi_n \sigma(\xi') dx'. 
\]
(3.58)

From Lemma 3.2-Lemma 3.4 and direct computations, we obtain
\[
\partial_{\xi_n}^2 q - 1 (x_0) |_{|\xi'|=1} = \frac{6i\xi_n^2 - 2i}{(1 + \xi_n^2)^3} \partial_{\xi_n} [c(\xi')] (x_0) + \sqrt{-1}h'(0) \left[ \frac{4(1 - 5\xi_n^2)}{(1 + \xi_n^2)^4} c(\xi') \right. \\
- \frac{12\xi_n(\xi_n^2 - 1)}{(1 + \xi_n^2)^4} c(dx_n) \right]. 
\]
(3.59)
Combining (3.58), (3.59) and direct computations, we obtain

\[
\begin{align*}
\text{trace} \left[ \frac{c(X)}{|\xi|^2} \right] & \frac{2g(X, \xi)c(\xi)}{|\xi|^4} \times \partial_{\xi_n}^2 \partial_{x_n} q_{-1} \\
= & \frac{(6i\xi_n^2 - 2i)}{(1 + \xi_n^2)^4} \text{tr} \left[ c(X) \partial_{\xi_n} [c(\xi')] \right] + \frac{4i(1 - 5\xi_n^2)}{(1 + \xi_n^2)^6} h'(0) \text{tr} \left[ c(\xi') c(X) \right] + \frac{8(\xi_n^2 - 10\xi_n^4 - 3\xi_n)}{(1 + \xi_n^2)^6} a_n h'(0). \\
\end{align*}
\]

Therefore

\[ Case \ (12) = \frac{39}{32} (h'(0))^2 \pi \Omega_3 dx' - \frac{5}{8} a_n h'(0) \pi \Omega_3 dx'. \]

**Case (13):** \( r = -2, \ell = -2, k = 0, j = 0, |\alpha| = 0 \)
From (2.9) and the Leibniz rule, we have

\[ \begin{align*}
\text{Case (13)} & = -i \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace} \left[ \pi_{\xi_n} q_{-2} \times \partial_{\xi_n} q_{-2} \right] (x_0) d\xi_n \sigma(\xi') dx' \\
& = i \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace} \left[ \partial_{\xi_n} \pi_{\xi_n} (D^{-1}) \times \sigma_{-2} (D^{-1}) \right] (x_0) d\xi_n \sigma(\xi') dx' \\
& + i \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace} \left[ \partial_{\xi_n} \pi_{\xi_n} (\sigma_{-2}(D^{-1})) \times \frac{c(X)}{|\xi'|^2} \right] (x_0) d\xi_n \sigma(\xi') dx' \\
& + i \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace} \left[ \partial_{\xi_n} \pi_{\xi_n} (\sigma_{-2}(D^{-1})) \times \frac{2g(X, \xi)c(\xi)}{|\xi'|^4} \right] (x_0) d\xi_n \sigma(\xi') dx'. \\
\end{align*} \]

By (2.1) in [9] and the Cauchy integral formula, then

\[ \begin{align*}
\pi_{\xi_n}^\pm \left( \frac{c(\xi)}{|\xi'|^4} \right) (x_0) \bigg|_{|\xi'|=1} & = \pi_{\xi_n}^\pm \left( \frac{c(\xi') + \xi_n c(dx_n)}{(1 + \xi_n^2)^2} \right) = \frac{1}{2\pi i} \lim_{\eta_n \to 0} \frac{c(\xi') + \eta_n c(dx_n)}{(\eta_n + i n^2 - \eta_n)^{-2}} d\eta_n \\
= & \left[ \frac{c(\xi') + \eta_n c(dx_n)}{(\eta_n + i n^2 - \eta_n)^{-2}} \right]_{\eta_n = i} + \frac{ic(\xi') + \eta_n c(dx_n)}{4(\xi_n - i)^2} \\
= & g(X', \xi') \pi_{\xi_n}^\pm \left( \frac{c(\xi)}{|\xi'|^4} \right) (x_0) \bigg|_{|\xi'|=1} + a_n \pi_{\xi_n}^\pm \left( \frac{\xi_n c(\xi)}{|\xi'|^2} \right) (x_0) \bigg|_{|\xi'|=1} \\
& + \frac{-a_n i}{4(\xi_n - i)^2} c(\xi') + \frac{-a_n i \xi_n}{4(\xi_n - i)^2} c(dx_n). \\
\end{align*} \]

Hence in this case,

\[ \begin{align*}
\partial_{\xi_n} \pi_{\xi_n}^\pm (\sigma_{-2}(D^{-1}))(x_0) \bigg|_{|\xi'|=1} & = h'(0) \frac{-i \xi_n - 3}{4(\xi_n - i)^3} c(\xi') c(dx_n) c(\xi') + h'(0) \frac{i}{(\xi_n - i)^2} c(\xi') \\
& + h'(0) \frac{i \xi_n - 1}{4(\xi_n - i)^2} c(dx_n) + \frac{1}{2(\xi_n - i)^3} \partial_{x_n} [c(\xi')] (x_0) \\
& + \frac{i \xi_n - 3}{4(\xi_n - i)^3} c(\xi') c(dx_n) \partial_{x_n} [c(\xi')] (x_0) \\
& + h'(0) \frac{-2i \xi_n - 8}{8(\xi_n - i)^8} c(\xi') + h'(0) \frac{2i \xi_n^2 + 4 \xi_n - 9i}{8(\xi_n - i)^4} c(dx_n). \end{align*} \]
Similarly, we obtain
\[ i \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace} \left[ \partial_{\xi_n} \pi_{\xi_n}^+ (\sigma_{-2}(D^{-1})) \right] \left( \frac{c(X)}{|\xi|^2} - \frac{2g(X, \xi)c(\xi)}{|\xi|^4} \right) (x_0) d\xi_n \sigma(\xi') dx' = \frac{15}{16} \pi a_n h'(0) \Omega_3 dx'. \]  
(3.67)

Then
\[ i \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace} \left[ \partial_{\xi_n} \pi_{\xi_n}^+ (\sigma_{-2}(D^{-1})) \times \left( \frac{c(X)}{|\xi|^2} - \frac{2g(X, \xi)c(\xi)}{|\xi|^4} \right) \right] (x_0) d\xi_n \sigma(\xi') dx' = \frac{\pi}{2} |X|^2 \Omega_3 \Omega_3 dx' - \frac{35\pi}{64} |X'|^2 \Omega_3 \Omega_3 dx'. \]  
(3.68)

Therefore
\[ \text{Case (13)} = - \frac{821}{256} (h'(0))^2 \pi \Omega_3 dx' + \frac{15}{16} \pi a_n h'(0) \Omega_3 dx' + \frac{\pi}{2} |X|^2 \Omega_3 \Omega_3 dx' + \frac{35\pi}{64} |X'|^2 \Omega_3 \Omega_3 dx'. \]  
(3.69)

\textbf{Case (14)}: \( r = -1, \ell = -3, k = 0, j = 0, |\alpha| = 0 \)

From (2.9) and the Leibniz rule, we have
\begin{align*}
\text{Case (14)} &= -i \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace} \left[ \pi_{\xi_n}^+ q_{-1} \times \partial_{\xi_n} q_{-3} (x_0) \right] d\xi_n \sigma(\xi') dx' \\
&= i \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace} \left[ \partial_{\xi_n} \pi_{\xi_n}^+ (\sigma_{-2}(D^{-1})) \right] (x_0) d\xi_n \sigma(\xi') dx' \\
&\quad + i \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace} \left[ \partial_{\xi_n} \pi_{\xi_n}^+ q_{-1} \times R_{-1} \right] (x_0) d\xi_n \sigma(\xi') dx'. \\
&= -\frac{c(\xi') + ie(dx_n)}{2(\xi_n - i)^2}. \\
&= \frac{\partial_{\xi_n} q_{-1}(x_0)}{|\xi'|=1}.
\end{align*}
(3.70)

In the orthonormal frame field, we have
\[ R_{-3}(x_0)_{|\xi'|=1} = -\frac{i(\xi_n^2 + \xi_n^0 - 2)}{(1 + \xi_n^2)^2} h'(0)c(\xi)c(X)c(dx_n) + \frac{i(2\xi_n^3 + 4\xi_n)}{(1 + \xi_n^2)^2} h'(0)c(\xi)c(\xi') \]
\[- \frac{c(\xi)}{(1 + \xi_n^2)^2} c(X)c(\xi') + \frac{2i}{c(\xi)} g(X', \xi') c(\xi)c(\xi) + \frac{2i\alpha a_n \xi}{(1 + \xi_n^2)^2} c(\xi) \]
\[- i(1 + \xi_n^2)^2 c(\xi) \sum_{j=1}^{n} c(dx_j) c(\xi) + \frac{2i}{(1 + \xi_n^2)^3} \partial_{x_j} g(X', \xi') c(\xi)c(\xi) \]
\[- i(1 + \xi_n^2)^2 c(\xi) \sum_{j=1}^{n} c(dx_j) c(\xi) + \frac{2i}{(1 + \xi_n^2)^3} \partial_{x_j} g(X, \xi) c(\xi)c(\xi) \]
\[- \frac{i(1 + \xi_n^2)^2 c(\xi)}{(1 + \xi_n^2)^3} c(\xi) \sum_{j=1}^{n} c(dx_j) c(\xi) - \frac{i}{(1 + \xi_n^2)^2} c(\xi)c(dx_n) c(\xi) \]
\[- \frac{2i}{(1 + \xi_n^2)^3} g(X, \xi)c(\xi)c(dx_n) c(\xi) - \frac{4ih'(0)}{(1 + \xi_n^2)^3} g(X, \xi) c(\xi)c(dx_n) c(\xi). \]  
(3.72)
By the relation of the Clifford action and $\text{tr}AB = \text{tr}BA$, then we have the equalities:

\[
\text{tr} \left[ c(\xi)c(\xi)c(X)c(\xi)c(dx_n)\partial x_n c(\xi) \right] = \text{tr} \left[ c(\xi')(c(\xi') + \xi_n c(dx_n))c(X)c(\xi')c(dx_n)\partial x_n c(\xi') \right] = \text{tr} \left[ c(\xi')c(\xi')(X)c(\xi')c(dx_n)\partial x_n c(\xi') \right] + \text{tr} \left[ c(\xi')\xi_n c(dx_n)\partial x_n c(\xi') \right]
\]

\[
= -2a_n h'(0) - \text{tr} \left[ \partial x_n (c(\xi'))c(X) \right].
\] (3.73)

Considering for $i < n$, $\int_{|\xi'|=1} \{l_i, \xi_{i_2} \cdots \xi_{i_{n+1}}\} \sigma(\xi') = 0$. From Lemma 3.5, combining (3.71)-(3.73) and direct computations, we obtain

\[
i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace} \left[ \partial x_n \pi_{\xi_n}^{+} q_{-1} \times R_{-3} \right] dx' \left[ (x_0) \pi_{\xi_n}^{+} \sigma(\xi') \right] dx'.
\]

\[
= -\frac{5}{8} \pi_{\xi_n}^{+} h'(0) \Omega_3 dx' + \frac{3}{4} \pi \partial x_n (a_n) \Omega_3 dx' + \frac{3}{4} \pi \Omega_3 C_1^1 (DX^*) dx'.
\] (3.74)

Therefore

\[
\text{Case (14)} = \left( \begin{array}{c}
\frac{239}{64} \left( h'(0) \right)^2 - \frac{27}{16} h''(0) - \frac{11}{192} s_{\partial x'} \pi \Omega_3 dx' \\
-\frac{5}{8} \pi_{\xi_n}^{+} h'(0) \Omega_3 dx' + \frac{3}{4} \pi \partial x_n (a_n) \Omega_3 dx' + \frac{3}{4} \pi \Omega_3 C_1^1 (DX^*) dx'.
\end{array} \right)
\] (3.75)

\textbf{Case (15): } $r = -3$, $\ell = -1$, $k = 0, j = 0$, $|a| = 0$

From (2.9) we have

\[
\text{Case (15)} = -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace} \left[ \pi_{\xi_n}^{+} q_{-3} \times \partial x_n q_{-1} \right] dx' \left[ (x_0) \pi_{\xi_n}^{+} \sigma(\xi') \right] dx'.
\] (3.76)

By the Leibniz rule, trace property and "++" and "−−" vanishing after the integration over $\xi_n$ in (7), then

\[
\int_{-\infty}^{+\infty} \text{trace} \left[ \pi_{\xi_n}^{+} q_{-3} \times \partial x_n q_{-1} \right] dx_n = \int_{-\infty}^{+\infty} \text{trace} \left[ q_{-3} \times \partial x_n \pi_{\xi_n}^{+} q_{-1} \right] dx_n - \int_{-\infty}^{+\infty} \text{trace} \left[ q_{-3} \times \partial x_n q_{-1} \right] dx_n.
\] (3.77)

Combining these assertions, we see

\[
\text{Case (15)} = \text{Case (14)} - i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace} \left[ R_{-3} \times \partial x_n q_{-1} \right] dx' \left[ (x_0) \pi_{\xi_n}^{+} \sigma(\xi') \right] dx'.
\] (3.78)

By Lemma 3.2, a simple computation shows

\[
\partial x_n q_{-1}(x_0) \bigg|_{|\xi'|=1} = \frac{-2\xi^n}{(1 + \xi_n^2)^2} \sqrt{-1c(\xi')} + \frac{1 - \xi_n^2}{(1 + \xi_n^2)^2} \sqrt{-1c(dx_n)}.
\] (3.79)

Similar to Case (14), combining (3.72),(3.79) and direct computations, we obtain

\[
\text{Case (15)} = \left( \begin{array}{c}
\frac{239}{64} \left( h'(0) \right)^2 - \frac{27}{16} h''(0) - \frac{11}{192} s_{\partial M} \pi \Omega_3 dx' \\
-\frac{5}{8} \pi_{\xi_n}^{+} h'(0) \Omega_3 dx' + \frac{3}{4} \pi \partial x_n (a_n) \Omega_3 dx' + \frac{3}{4} \pi \Omega_3 C_1^1 (\nabla^0 M(X'|\partial M)^*) dx'.
\end{array} \right)
\] (3.80)
Now $\Phi$ is the sum of the case $(1, 2, \cdots, 15)$, then we obtain
\[
\sum_{i=1}^{15} \text{case I} = \left( \frac{399}{256} (h'(0))^2 - \frac{29}{32} h''(0) - \left( \frac{17}{96} + \frac{5}{32} i \right) s_{\partial M} \right) \pi \Omega_3 dx' \\
+ \frac{25}{32} a_n h'(0) \Omega_3 dx' - \frac{7}{2} X^i_{\partial M} \Omega_3 dx' + \frac{35\pi}{64} |X'|_{\partial M} h'(0) \Omega_3 dx' - \pi a^2_n \Omega_3 dx' \\
- \frac{3}{4} \pi \partial_{x_n} (a_n) \Omega_3 dx' + \frac{15}{16} \pi C^1_i (\nabla^\partial M (X'|_{\partial M})^*) \Omega_3 dx'.
\] (3.81)

Hence we conclude that,

**Theorem 3.6.** Let $M$ be a five dimensional compact manifold with the boundary $\partial M$, and the Dirac operators with one-form perturbations $\tilde{D} = D + c(X)$, then
\[
\text{Vol}_{5}^{(1,1)} = \int_{\partial M} \left( \frac{399}{256} (h'(0))^2 - \frac{29}{32} h''(0) - \left( \frac{17}{96} + \frac{5}{32} i \right) s_{\partial M} \right) \pi \Omega_3 dx' \\
+ \frac{25}{32} a_n h'(0) \Omega_3 dx' - \frac{7}{2} X^i_{\partial M} \Omega_3 dx' + \frac{35\pi}{64} |X'|_{\partial M} h'(0) - a^2_n \\
- \frac{3}{4} \pi \partial_{x_n} (a_n) + \frac{15}{16} \pi C^1_i (\nabla^\partial M (X'|_{\partial M})^*) \pi \Omega_3 dx'.
\] (3.82)

Next we recall the Einstein-Hilbert action for manifolds with boundary (see [16] or [17]),
\[
I_{Ge} = \frac{1}{16\pi} \int_M s d\text{vol}_M + 2 \int_{\partial M} K d\text{vol}_{\partial M} := I_{Gr,i} + I_{Gr,b},
\] (3.83)
where
\[
K = \sum_{1 \leq i, j \leq n-1} K_{i,j} g^{i,j}_{\partial M}, \quad K_{i,j} = -\Gamma^a_{i,j},
\] (3.84)
and $K_{i,j}$ is the second fundamental form, or extrinsic curvature. Take the metric in Section 2, then by Lemma A.2 in [16], $K_{i,j} (x_0) = -\Gamma^a_{i,j} (x_0) = -\frac{1}{2} h'(0)$, when $i < j < n$, otherwise is zero. For $n = 5$, then
\[
K (x_0) = \sum_{i,j} K_{i,j} (x_0) g^{i,j}_{\partial M} (x_0) = \sum_{i=1}^{4} K_{i,i} (x_0) = -2h'(0).
\] (3.85)

Then
\[
I_{Gr,b} = -4h'(0) \text{Vol}_{M}.
\] (3.86)

On the other hand, by Proposition 2.10 in [12] and Th 3.9 in [12], we have

**Lemma 3.7.** Let $M$ be a 5-dimensional compact manifold with the boundary $\partial M$, then
\[
s_M (x_0) = 3(h'(0))^2 - 4h''(0) + s_{\partial M} (x_0).
\] (3.87)

Hence from (3.82)-(3.87), we obtain

**Theorem 3.8.** Let $M$ be a five dimensional compact spin manifold with the boundary $\partial M$, and the Dirac operators with one-form perturbations $\tilde{D} = D + c(X)$. Let $X = X' + a_n dx_n$ near the boundary and $X'$ is a one form on $\partial M$, the following identity holds
\[
\overset{\sim}{\text{Wres}}[\pi^+(D + c(X))^{-1} o \pi^+(D + c(X))^{-1}] = \int_{\partial M} \left[ \frac{1}{16} \left( \frac{225}{32} K^2 + \frac{29}{4} s_M |_{\partial M} - \left( \frac{155}{12} + 5i \right) s_{\partial M} \right) \\
- \frac{25}{32} a_n K - |X'|_{\partial M}^2 - \frac{35}{64} |X'|_{\partial M} K - 3a^2_n |_{\partial M} \\
- \frac{3}{2} \partial_{x_n} (a_n) + \frac{15}{8} C^1_i (\nabla^\partial M (X'|_{\partial M})^*) \pi^3 d\text{vol}_{\partial M}
\] (3.88)

where $s_M$, $s_{\partial M}$ are respectively scalar curvatures on $M$ and $\partial M$, and the vector field $(X'|_{\partial M})^*$ is the metric dual of $X'|_{\partial M}$, $\nabla^\partial M$ is the Levi-civita connection on $\partial M$, $C^1_i$ is the contraction of (1,1) tensors.
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