Abstract—Hybrid codes simultaneously encode both quantum and classical information into physical qubits. We give several general results about hybrid codes, most notably that the quantum codes comprising a genuine hybrid code must be impure and that hybrid codes can always detect more errors than comparable quantum codes. We also introduce the weight enumerators for general hybrid codes, which we then use to derive linear programming bounds. Finally, inspired by the construction of some families of nonadditive codes, we construct several infinite families of genuine hybrid codes with minimum distance two and three.

Index Terms—Quantum error-correcting codes, hybrid codes, nonadditive codes, codeword stabilized codes, linear programming bound.

I. INTRODUCTION

Hybrid codes simultaneously encode classical and quantum information into quantum digits such that the information is protected against errors when transmitted through a quantum channel. The simultaneous transmission of classical and quantum information was first investigated by Devetak and Shor [2], who characterized the set of admissible rate pairs. Notably, they showed that, at least for certain small error rates, time-sharing a quantum channel is inferior to simultaneous transmission. Constructions of hybrid codes were first studied by Kremsky, Hsieh, and Brun [18] in the context of entanglement-assisted stabilizer codes and by Bény, Kempf, and Kribs [4] who outlined an operator-theoretic construction.

More recently, Grassl, Lu, and Zeng [14] gave linear programming bounds for a class of hybrid codes and constructed a number of hybrid stabilizer codes with parameters better than those of hybrid codes constructed from quantum stabilizer codes. In particular, these genuine hybrid codes outperform “trivial” hybrid codes regardless of the error rate of the channel. Additional work on hybrid codes has been done from both a coding theory approach [22] and from an operator-theoretic approach [21], as well as over fully correlated quantum channels [20]. While they are still relatively unstudied, multiple uses for hybrid codes have already become apparent, including protecting hybrid quantum memory [19] and constructing hybrid secret sharing schemes [35].

In this paper we give some general results regarding hybrid codes, most notably that at least one of the quantum codes comprising a genuine hybrid code must be impure, a fact hinted at in [14], [21], as well as show that a hybrid code can always detect more errors than a comparable quantum code. We also generalize the weight enumerators of Grassl et al. [14] from hybrid stabilizer codes to more general nonadditive hybrid codes and use them to derive linear programming bounds. Finally, we give multiple constructions for infinite families of hybrid codes with good parameters. The first of these families are single error-detecting hybrid stabilizer codes with parameters \([n, n - 3: 1, 2]\) where the length \(n\) is odd. The second is a collection of families of single error-correcting hybrid codes constructed using stabilizer pasting, where we paste together stabilizers from Gottesman’s \([2^j, 2^j - j - 2, 3]\) stabilizers codes [10] and the small distance 3 hybrid codes with \(n = 7, 9, 10, 11\) from [14]. Each of these families of hybrid codes were inspired by families of nonadditive quantum codes, especially those constructed by Rains [25] and Yu, Chen, and Oh [34].

II. HYBRID CODES

Suppose that we want to simultaneously transmit classical and quantum messages. Our goal will be to encode them into the state of \(n\) quantum digits that have \(q\)-levels each, so that the encoded message can be transmitted over a quantum channel. In other words, an encoded message is a unit vector in the Hilbert space

\[
H = \bigotimes_{k=1}^{n} C^q \cong C^{nq}.
\]

A hybrid code has the parameters \((n, K: M)\), if and only if it can simultaneously encode one of \(M\) different classical messages and a superposition of \(K\) orthogonal quantum states into \(n\) quantum digits with \(q\) levels.

We can understand the hybrid code as a collection of \(M\) orthogonal \(K\)-dimensional quantum codes \(C_m\) that are indexed by the classical messages \(m \in [M] := \{1, 2, \ldots, M\}\). If we want to transmit a classical message \(m \in [M]\) and a quantum state \(\varphi\), then we need to encode \(\varphi\) into the quantum code \(C_m\). We will refer to the each of the quantum codes \(C_m\) as inner codes and the collection \(\mathcal{C} = \{C_m \mid m \in [M]\}\) as the outer code.

A. Error Detection

The encoded states will be subject to errors when transmitted through a quantum channel. Our first task will be to characterize the errors that can be detected by the hybrid code. We will set up a projective measurement that either upon receipt of a state \(|\psi\rangle\) in \(H\) either (a) returns \(\epsilon\) to indicate
that an error happened or (b) claims that there is no error and returns a classical message \( m \) and a projection of \( |\psi\rangle \) onto \( C_m \).

Let \( P_m \) denote the orthogonal projector onto the quantum code \( C_m \) for all integers \( m \) in the range \( 1 \leq m \leq M \). For distinct integers \( a \) and \( b \) in the range \( 1 \leq a, b \leq M \), the quantum codes \( C_a \) and \( C_b \) are orthogonal, so \( P_a P_b = 0 \). It follows that the orthogonal projector onto \( C = \bigoplus_{m=1}^{M} C_m \) is given by

\[
P = P_1 + P_2 + \cdots + P_M.
\]

We define the orthogonal projection onto \( C^\perp \) by \( P_e = 1 - P \).

For the hybrid code \( \{C_m\ |\ m \in [M]\} \), we can define a projective measurement \( \mathcal{P} \) that corresponds to the the set \( \{P_1, P_2, \ldots, P_M, P_e\} \) of projection operators that partition unity.

We can now define the concept of a detectable error. An error \( E \) is called detectable by the hybrid code \( \{C_m\ |\ m \in [M]\} \) if and only if for each index \( a, b \) in the range \( 1 \leq a, b \leq M \), we have

\[
P_b E P_a = \begin{cases} 
\lambda_{E,a} P_a & \text{if } a = b, \\
0 & \text{if } a \neq b
\end{cases}
\]

for some scalar \( \lambda_{E,a} \).

The motivation for calling an error \( E \) detectable is the following simple protocol. Suppose that we encode a classical message \( m \) and a quantum state into a state \( \psi_m \) of \( C_m \), and transmit it through a quantum channel that imparts the error \( E \). If the error is detectable, then measurement of the state \( E \psi_m = E P_m \psi_m \) with the projective measurement \( \mathcal{P} \) either (E1) returns \( \epsilon \), which signals that an error happened, or (E2) returns \( m \) and corrects the error by projecting the state back onto a scalar multiple \( \lambda_{E,m} \psi_m = P_m E P_m \psi_m \) of the state \( \psi_m \).

The definition of a detectable error ensures that the measurement \( \mathcal{P} \) will never return an incorrect classical message \( d \), since \( P_d E P_m \psi_m = 0 \) for all \( d \neq m \), so the probability of detecting an incorrect message is zero. An error that is not detectable by the hybrid code can change the encoded classical, the encoded quantum information, or both.

The next proposition shows that hybrid codes can always detect more errors than a comparable quantum code that encodes both classical and quantum information. This is remarkable given that the advantages are much less apparent when one considers minimum distance, see [14].

**Proposition 1.** The subset \( D \) of detectable errors in \( B(H) \) of an \( \{(n, K : M)\}_q \) hybrid code form a vector space of dimension

\[
dim D = q^{2n} - (MK)^2 + M.
\]

In particular, a \( \{(n, K : M)\}_q \) hybrid code with \( M > 1 \) can detect more errors than an \( \{(n, KM)\}_q \) quantum code.

**Proof.** It is clear that any linear combination of detectable errors is detectable. If we choose a basis adapted to the orthogonal decomposition \( H = C \oplus C^\perp \) with

\[
C = C_1 \oplus C_2 \oplus \cdots \oplus C_M,
\]

then an error \( E \) is represented by a matrix of the form

\[
\begin{pmatrix} A & R \\ S & T \end{pmatrix}
\]

Since \( E \) is detectable, the \( MK \times MK \) matrix \( A \) must satisfy

\[
A = \lambda_{E,1} K \oplus \lambda_{E,2} K \oplus \cdots \oplus \lambda_{E,M} K,
\]

where \( 1_k \) denote a \( K \times K \) identity matrix, but \( R, S, \text{ and } T \) can be arbitrary. Therefore, the dimension of the vector space of detectable errors is given by \( q^{2n} - (MK)^2 + M \). The vector space of detectable errors of an \( \{(n, KM)\}_q \) quantum code has dimension \( q^{2n} - (KM)^2 + 1 \), which is strictly less than \( q^{2n} - (MK)^2 + M \).

**B. Error Correction**

We briefly recall the concept of a nice error basis (see [15], [16], [17] for further details), so that we can define a suitable notion of weight for the errors. Let \( G \) be a group of order \( q^2 \) with identity element \( 1 \). A nice error basis on \( \mathbb{C}^q \) is a set \( \mathcal{E} = \{\rho(g) \in U(q) | g \in G\} \) of unitary matrices such that

(i) \( \rho(1) \) is the identity matrix,

(ii) \( \text{Tr}(\rho(g)) = 0 \) for all \( g \in G \setminus \{1\} \),

(iii) \( \rho(g) \rho(h) = \omega(g,h) \rho(gh) \) for all \( g, h \in G \),

where \( \omega(g,h) \) is a nonzero complex number depending on \( (g,h) \in G \times G \); the function \( \omega: G \times G \to \mathbb{C}^\times \) is called the factor system of \( \rho \). We call \( G \) the index group of the error basis \( \mathcal{E} \). The nice error basis that we have introduced so far generalizes the Pauli basis to systems with \( q \geq 2 \).

We can obtain a nice error basis \( \mathcal{E}_n \) on \( H \cong \mathbb{C}^{nq} \) by tensoring \( n \) elements of \( \mathcal{E} \), so

\[
\mathcal{E}_n = \mathcal{E}^\otimes n = \{E_1 \otimes E_2 \otimes \cdots \otimes E_n | E_k \in \mathcal{E}, 1 \leq k \leq n\}.
\]

The weight of an element in \( \mathcal{E}_n \) are the number of non-identity tensor components. We write \( wt(E) = d \) to denote that the element \( E \) in \( \mathcal{E}_n \) has weight \( d \). A hybrid code with parameters \( (n, K : M, d)_q \) has minimum distance \( d \) if it can detect all errors of weight less than \( d \).

**Example 2.** To construct our nonadditive hybrid code \( C \) we will combine two known degenerate stabilizer codes. The first code \( C_a \) is the \([6,1,3]_2 \) code constructed by extending the \([5,1,3]_2 \) Hamming code, see [5], where the stabilizer is given by

\[
\langle XXZIIZ, ZXXZI, IZZXXI, ZIIXXI, IIIIXX \rangle.
\]

The second code \( C_b \) is a \([6,1,3]_2 \) code not equivalent to \( C_a \), see [28]. Its stabilizer is given by

\[
\langle YIIZXY, ZXIIXZ, IZXXXX, IIIIZI, ZZIZI \rangle.
\]

We can check that the resulting two codes are indeed orthogonal to each other. The resulting code \( \tilde{C} \) is a \([6,2:2,1]_2 \) nonadditive hybrid code, since there are several errors of weight one such that \( P_a E P_a \neq 0 \), for example \( E = IIIXXI \). This shows that even though \( C_a \) and \( C_b \) are optimal quantum codes on their own, together they make a hybrid code with an
extremely poor minimum distance. Later we will see how to construct hybrid codes with better minimum distances.

We conclude this subsection with a few remarks on sets of detectable and correctable errors. Detectable errors have many nice features. The set \( \mathcal{D} \) of all detectable errors of a hybrid code is a vector space that contains the identity operator, is closed under taking adjoints * , and is a closed subspace of \( B(H) \). Therefore, the set \( \mathcal{D} \) of detectable errors is an operator system of the the \( \mathbb{C}^* \)-algebra \( B(H) \). This means that we can express every detectable error in \( \mathcal{D} \) as a linear combination of detectable errors that are positive operators. Indeed, an operator \( E \) in \( \mathcal{D} \) can be expressed as linear combination \( E = A + iB \), where \( A = \frac{1}{2}(E + E^*) \) and \( B = \frac{i}{2}(E^* - E) \) are self-adjoint operators in \( \mathcal{D} \). A self-adjoint operator \( X \) in \( \mathcal{D} \) can be expressed as the difference of the positive operators \( \| X \| \mathbf{1} \) and \( \| X \| \mathbf{1} - X \). In short, the set of detectable errors of a hybrid code has a quite well-behaved structure.

On the other hand, whenever we consider the correctability of errors, we must consider an entire set of errors rather than a single error. Depending on the set of errors that we would like to correct, a given error operator \( E \) might or might not be correctable. It is not difficult to show that a unital set \( \mathcal{E} \) of errors is correctable if and only if the set \( \mathcal{E}^* \mathcal{E} = \{ F^*E | E,F \in \mathcal{E} \} \) of errors is detectable. In other words, all errors \( E,F \in \mathcal{E} \) must satisfy
\[
P_a F^*E P_a = \begin{cases} \lambda_{F^*E,a}P_a & \text{if } a = b, \\ 0 & \text{if } a \neq b \end{cases}
\]
for all \( a,b \in [M] \).

C. Genuine Hybrid Codes

In general, it is not difficult to construct hybrid codes using quantum stabilizer codes. As Grassl et al. \cite{14} pointed out, there are three simple constructions of hybrid codes that do not offer any real advantage over quantum error-correcting codes:

**Proposition 3** \cite{14}. Hybrid codes can be constructed using the following “trivial” constructions:

1) Given an \( ((n,K:M,d))_q \) quantum code of composite dimension \( KM \), there exists a hybrid code with parameters \( ((n,K:M,d))_q \).

2) Given an \( [[n,k,m,d]]_q \) hybrid code with \( k > 0 \), there exists a hybrid code with parameters \( [n,k - 1:m + 1,d]_q \).

3) Given an \( [[n_1,k_1,d]]_q \) quantum code and an \( [[n_2,m_2,d]]_q \) classical code, there exists a hybrid code with parameters \( [[n_1 + n_2,k_1:m_2,d]]_q \).

We say that a hybrid code is genuine if it cannot be constructed using one of the above constructions, following the work of Yu et al. on genuine nonadditive codes \cite{34}. We also refer to a hybrid stabilizer code that provides an advantage over quantum stabilizer codes as a genuine hybrid stabilizer code.

Multiple genuine hybrid stabilizer codes with small parameters were constructed by Grassl et al. in \cite{14}. One characteristic of each of these hybrid codes is that their inner codes are all degenerate. Here, we extend this to the case of general hybrid codes and show that for a genuine hybrid code, at least one of its inner codes must be degenerate. Recall that a quantum code is pure if distinct detectable errors map the code to orthogonal subspaces. A code that is not pure is called impure.

**Proposition 4.** Suppose \( C \) is a genuine \( ((n,K:M,d))_q \) hybrid code. Then at least one inner code \( C_m \) of the hybrid code \( C \) is impure. This implies that at least one inner code must be degenerate.

**Proof.** Seeking a contradiction, suppose that every inner code of the hybrid code \( C \) is pure. For \( m \in [M] \), let \( P_m \) denote the orthogonal projector onto the \( m \)-th inner code of the hybrid code \( C \). For every nonscalar error operator \( E \) of weight less than \( d \), we have
\[
P_a E P_b = 0,
\]
where \( a,b \in [M] \). Let \( P = P_1 + P_2 + \cdots + P_M \) denote the projector onto the \( KM \)-dimensional vector space spanned by the inner codes. Then
\[
PEP = 0,
\]
so the image of \( P \) is an \( ((n,K:M,d))_q \) quantum code, contradicting that the hybrid code \( C \) is genuine.

D. Hybrid Stabilizer Codes

All of the hybrid codes constructed by Grassl et al. \cite{14} were given using the codeword stabilizer (CWS)/union stabilizer framework, see \cite{6}, \cite{12}, which we will briefly describe here. Starting with a quantum code \( C_0 \), we choose a set of \( M \) coset representatives \( t_i \) from the normalizer of \( C_0 \) (we will always take \( t_1 \) to be \( 1 \)), and then construct the code
\[
C = \bigcup_{i \in [M]} t_iC_0.
\]
In the case of hybrid codes, \( t_iC_0 \) are our inner codes and \( C \) is our outer code. If both \( C_0 \) and \( C \) are stabilizer codes, we say that \( C \) is a hybrid stabilizer code.

The generators that define a hybrid code can be divided into those that generate the quantum stabilizer \( S_0 \) which stabilizes the outer code \( C \) and those that generate the classical stabilizer \( S_C \) which together with \( S_0 \) stabilizes the inner code \( C_0 \) \cite{18}. We often write the generators in the form of a generator matrix. Here we give the generator matrix for the \( [7,1:1,3]_2 \) hybrid stabilizer code given in \cite{14}:
The generators above the dotted line generate $S_Q$, and the generator between the dotted line gives $S_C$. The coset representative $t$ is given below the double line. In the CWS framework, $(S_Q, S_C)$ stabilizes the inner code $C_0$, and $S_Q$ stabilizes the outer code $C = C_0 \cup tC_0$. The normalizer of $C_0$ is given between the single and double line, and the normalizer of $C$ is everything below the dotted line.

We will often only include the generators of the stabilizers in our generator matrix, as they are sufficient to fully define the hybrid code, as shown in the following theorem:

**Proposition 5.** Let $C$ be an $[[n, k : m, d]]_q$ hybrid stabilizer code over a finite field of prime order $q$ with quantum stabilizer $S_Q$ and classical stabilizer $S_C = \langle g_1^c, \ldots, g_m^c \rangle$. Then the stabilizer code $C$, associated with classical message $c \in \mathbb{F}_q^n$ is given by the stabilizer

$$\langle S_Q, \omega^{c_1} g_1^c, \ldots, \omega^{c_m} g_m^c \rangle,$$

where $c_i$ is the $i$-th entry of $c$ and $\omega$ is a primitive $q$-th root of unity.

**Proof.** There are $q^{k+m}$ codewords stabilized by $S_Q$. Each of these codewords is an eigenvector of $g_i^c$, which naturally partitions the code into $q$ cosets based on eigenvalues. Repeating this with all of the classical generators, we get $q^m$ cosets of codewords each of size $q^k$. Since $v$ being an eigenvector of $g_i^c$ with eigenvalue $\omega^{-1}$ means that it is a $+1$ eigenvector of $\omega g_i^c$, therefore each coset is the $+1$ eigenspace of a stabilizer of the form $\langle S_Q, \omega^{c_1} g_1^c, \ldots, \omega^{c_m} g_m^c \rangle$, where the string $c \in \mathbb{F}_q^n$ can be used to index the stabilizer codes. \qed

### III. Weight Enumerators and Linear Programming Bounds

Weight enumerators for hybrid stabilizer codes were given by Grassl et al. \[14\], but those weight enumerators will not work for nonadditive hybrid codes such as the one given in Example 2. In this section, we define weight enumerators for general hybrid codes following the approach of Shor and Laflamme \[29\].

#### A. Weight Enumerators

For a $[[n, K : M, d]]_q$ hybrid code $C$ defined by the projector $P = P_1 + \cdots + P_M$, we define the two weight enumerators of the code following Shor and Laflamme \[29\]:

$$A(z) = \sum_{d=0}^{n} A_d z^d \quad \text{and} \quad B(z) = \sum_{d=0}^{n} B_d z^d,$$

where the coefficients are given by

$$A_d = \frac{1}{K^2 M^2} \sum_{\text{wt}(E)=d} \text{tr}(EP) \text{tr}(E^*P)$$

and

$$B_d = \frac{1}{KM} \sum_{\text{wt}(E)=d} \text{tr}(EPE^*P).$$

We can expand these weight enumerators and write them in terms of the weight enumerators of the inner code projectors $P_a$. Let

$$A^{(a,b)}(z) = \sum_{d=0}^{n} A_d^{(a,b)} z^d \quad \text{and} \quad B^{(a,b)}(z) = \sum_{d=0}^{n} B_d^{(a,b)} z^d,$$

where

$$A_d^{(a,b)} = \frac{1}{K^2} \sum_{\text{wt}(E)=d} \text{tr}(EP_a) \text{tr}(E^*P_b)$$

and

$$B_d^{(a,b)} = \frac{1}{K} \sum_{\text{wt}(E)=d} \text{tr}(EPE^*P_b).$$

Then the weight enumerators of $C$ can be written as

$$A(z) = \frac{1}{M^2} \sum_{a,b=1}^{M} A^{(a,b)}(z) \quad \text{and} \quad B(z) = \frac{1}{M} \sum_{a,b=1}^{M} B^{(a,b)}(z).$$

Note that $A^{(a,a)}(z)$ and $B^{(a,a)}(z)$ are the weight enumerators of the quantum code associated with projector $P_a$.

While the weight enumerator $B(z)$ is the same as the one introduced by the authors in \[22\], the weight enumerator $A(z)$ is different. There the $A^{(a,b)}(z)$ weight enumerators with $a \neq b$ were ignored, causing $A(z)$ and $B(z)$ to not satisfy the MacWilliams identity. The way presented here is more natural, as it treats both the inner and outer codes as quantum codes.

**Proposition 6.** Let $C$ be a $[[n, K : M]]_q$ hybrid code with weight distributions $A_d$ and $B_d$. Then for all integers $d$ in the range $0 \leq d \leq n$ and all $a \in [M]$ we have

1) $0 \leq A_d \leq B_d$
2) $0 \leq A_d^{(a,a)} \leq B_d^{(a,a)}$.

**Proof.** For every orthogonal projector $P : \mathbb{C}^n \to \mathbb{C}^n$ of rank $K$, we have

$$0 \leq \frac{1}{K^2} \sum_{\text{wt}(E)=d} \text{tr}(EP) \text{tr}(E^*P)$$

by the non-negativity of the trace inner product. Furthermore, we can write this inequality in the form

$$0 \leq \frac{1}{K^2} \sum_{\text{wt}(E)=d} \text{tr}(EP) \text{tr}(E^*P) = \frac{1}{K^2} \sum_{\text{wt}(E)=d} |\text{tr}(EP)|^2 = \frac{1}{K^2} \sum_{\text{wt}(E)=d} |\text{tr}((IIPEI)P)|^2.$$
Using the Cauchy-Schwarz inequality, we obtain

\[ 0 \leq \frac{1}{K^2} \sum_{\text{wt}(E) = d} \text{tr}(\langle\Pi E\Pi\rangle \langle\Pi E\Pi\rangle^*) \text{tr}(\Pi^*\Pi) \]

\[ = \frac{1}{K} \sum_{\text{wt}(E) = d} \text{tr}(E\Pi E^*\Pi) . \]

Substituting \( \Pi = P \) implies (1) and substituting \( P = P_a \) implies (2).

The main utility of weight enumerators for quantum codes is that they allow for a complete characterization of the error-correction capability of the code in terms of the minimum distance of the code. In the following proposition, we prove a similar result for the weight enumerators of hybrid codes.

**Proposition 7.** Let \( C \) be a \( (n, K:M) \) hybrid code with weight distributions \( A_d \) and \( B_d \). Then \( C \) can detect all errors in \( \mathcal{E}_n \) of weight \( d \) if and only if \( A_d^{(a,a)} = B_d^{(a,a)} \) for all \( a \in [M] \) and \( B_d^{(a,b)} = 0 \) for all \( a,b \in [M], a \neq b \).

**Proof.** Recall that an error is detectable by a code if and only if it satisfies the hybrid Knill-Laflamme conditions:

\[ \langle c_i^{(a)} | E | c_j^{(b)} \rangle = \alpha_E^{(a)} \delta_{i,j} \delta_{a,b}. \]

Suppose that all errors of weight \( d \) are detectable by \( C \). Then

\[ A_d^{(a,a)} = \frac{1}{K^2} \sum_{\text{wt}(E) = d} \left| \sum_{i=1}^{K} \langle c_i^{(a)} | E | c_i^{(a)} \rangle \right|^2 \]

\[ = \sum_{\text{wt}(E) = d} \left| \alpha_E^{(a)} \right|^2 . \]

Similarly, we have

\[ B_d^{(a,a)} = \frac{1}{K} \sum_{\text{wt}(E) = d} \left| \sum_{i=1}^{K} \langle c_i^{(a)} | E | c_i^{(a)} \rangle \right|^2 \]

\[ = \sum_{\text{wt}(E) = d} \left| \alpha_E^{(a)} \right|^2 . \]

Therefore, we have that \( A_d^{(a,a)} = B_d^{(a,a)} \). Additionally, if \( a \neq b \), then \( \langle c_i^{(a)} | E | c_j^{(b)} \rangle = 0 \), so \( B_d^{(a,b)} = 0 \).

Conversely, suppose that (a) \( A_d^{(a,a)} = B_d^{(a,a)} \) for all \( a \in [M] \) and (b) \( B_d^{(a,b)} = 0 \) for all \( a,b \in [M], a \neq b \). Condition (a) implies that equality holds for each \( E \) in the Cauchy-Schwarz inequality. Therefore, we have that \( P_a E P_a \) and \( P_a \) must be linearly dependent, so there must be a constant \( \alpha_E^{(a)} \in \mathbb{C} \) such that \( P_a E P_a = \alpha_E^{(a)} \), or equivalently, \( \langle c_i^{(a)} | E | c_j^{(a)} \rangle = \alpha_E^{(a)} \delta_{i,j} \), for all errors of weight \( d \). Condition (b) implies that \( \langle c_i^{(a)} | E | c_j^{(b)} \rangle = 0 \) if \( a \neq b \), for all errors of weight \( d \). Putting these together, we get the hybrid Knill-Laflamme conditions, so all errors of weight \( d \) are detectable.

**B. Linear Programming Bounds**

One of the more useful properties of weight enumerators is that they satisfy the MacWilliams identity [29]:

\[ B_d^{(a,b)}(z) = \frac{K}{q^n} \left( 1 - (q^2 - 1) \right)^n A_d^{(a,b)} \left( \frac{1 - z}{1 + (q^2 - 1) z} \right). \]

The MacWilliams identities, along with the results from Propositions 6 and 7 and the shadow inequalities for binary codes [24] allow us to define linear programming bounds on the parameters of general hybrid codes (see [3], [5], [24] for linear programming bounds on quantum codes). Let

\[ K_j(r) = \sum_{k=0}^{j} \binom{j}{k} (q^2 - 1)^{j-k} \binom{n-r}{k} . \]

denote the \( q^2 \)-ary Krawtchouk polynomials.

**Proposition 8.** The parameters of an \( (n, K:M, d) \) hybrid code must satisfy the following conditions:

1) \( A_j = \frac{1}{M} \sum_{a,b=1}^{M} A_j^{(a,b)} \)

2) \( B_j = \frac{1}{M} \sum_{a,b=1}^{M} B_j^{(a,b)} \)

3) \( A_0^{(a,b)} = 1 \)

4) \( B_0^{(a,b)} = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{if } a \neq b \end{cases} \)

5) \( A_j^{(a,a)} = B_j^{(a,a)}, \text{ for all } 0 \leq j < d \)

6) \( B_j^{(a,b)} = 0, \text{ for all } 0 \leq j < d, a \neq b \)

7) \( 0 \leq A_j^{(a,a)} \leq B_j^{(a,a)}, \text{ for all } 0 \leq j \leq n \)

8) \( 0 \leq A_j \leq B_j, \text{ for all } 0 \leq j \leq n \)

9) \( 0 \leq B_j^{(a,b)}, \text{ for all } 0 \leq j \leq n \)

10) \( B_j^{(a,b)} = \frac{K}{q^n} \sum_{r=0}^{j} K_j(r) A_r^{(a,b)}, \text{ for all } 0 \leq j \leq n \) (MacWilliams Identity)

11) \( 0 \leq \sum_{r=0}^{j} (-1)^r K_j(r) A_r^{(a,b)}, \text{ for all } 0 \leq j \leq n, \text{ for binary codes (Shadow Inequalities)} \)

**Proof.** Conditions (1) and (2) follow from the definition of \( A_j \) and \( B_j \). The constraints (3) and (4) respectively result from substituting \( E = I \) into the definition of \( A_0^{(a,b)} \) and \( B_0^{(a,b)} \).

The Knill-Laflamme error-detecting conditions of the hybrid codes shown in Proposition 7 imply the constraints (5) and (6).

The claims (7) and (8) are a consequence of Proposition 6. Essentially, these two conditions follow from the Cauchy-Schwarz inequalities when applied to the quantum and hybrid projectors, respectively.

The statement (9) is simply a consequence of the non-negativity of all \( B_j^{(a,b)} \). Conditions (10) and (11) follow from the MacWilliams identities [29] and shadow inequalities [24] respectively.

Notably missing in our linear programming bounds is part of the nested code condition found in the linear programming bounds for hybrid stabilizer codes, namely that \( A_d \leq A_d^{(a,a)} \) for all \( d \). In fact we can construct a nonadditive hybrid code that violates this condition, as shown in the example below.
Example 9. We return to our $((6, 2: 2, 1))_2$ nonadditive hybrid code from Example 2. The weight distributions for $C_a$, $C_b$, and $C$ are

$$A^{(a,a)} = [1, 1, 0, 0, 15, 15, 0]$$

$$A^{(b,b)} = [1, 0, 1, 0, 11, 16, 3]$$

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 6 & \frac{31}{4} & \frac{3}{4} \end{bmatrix},$$

where the weight distributions are the coefficients of the weight enumerators. These weight distributions clearly violate the inequality $A_d \leq A^{(a,a)}_d$.

If we consider only hybrid stabilizer codes, we have that all of the weight distributions for the inner quantum codes are identical. This along with our error-detecting condition from Theorem 7 give us that a stabilizer hybrid stabilizer code can detect all errors of weight $d$ if and only if $A^{(a,a)}_d = B^{(a,a)}_d = B_d$. Additionally, straightforward calculations give us the missing piece of the nested code condition, $A_d \leq A^{(a,a)}_d$ for all $d$. Thus we recover the linear programming bounds of Grassl et al. when we restrict our bounds to hybrid stabilizer codes.

IV. FAMILY OF SINGLE ERROR DETECTING CODES

In [23], Rains et al. constructed a $((5, 6, 2))_2$ nonadditive quantum code which was later extended to several families of $((n, q^{n-3} < K < q^{n-2}, 2))_2$ nonadditive codes with $n$ odd, see [1], [2], [8], [25], [27], [30]. Rains [25] also showed that for any $((n, K, 2))_2$ quantum code with odd $n$,

$$K \leq 2^{n-2} \left( 1 - \frac{1}{n-1} \right).$$

In particular, this disallows the existence of a $((n, 2^{n-2}, 2))_2$ quantum code.

Here we give a construction for a family of single-error-detecting hybrid stabilizer codes such that $n$ is odd and $KM = 2^{n-2}$, so these codes have the remarkable feature in that they allow one to squeeze in an additional classical bit. The structure of the generator matrices of these codes is similar to the structure of the family of even-length stabilizer codes with parameters $[n, n-2, 2]_q$, see [11], [25].

Theorem 10. For $n$ odd, there exists an $[n, n-3:1, 2]_2$ genuine hybrid code with generator matrix

$$\begin{pmatrix}
X^\otimes n-1 & X \\
Z^\otimes n-1 & I \\
\cdots & \cdots \\
I^{\otimes n-1} & X
\end{pmatrix}$$

Proof. Recall that a number is said to have even parity if it has an even number of $1$’s in its binary expansion. Let $J \subseteq F^{n-1}_2$ be the set of even integers with even parity. We define two codes $C_0$ and $C_1$ as follows:

$$C_0 = \left\{ \frac{1}{2} \left( |x⟩ + |\overline{x}\rangle \right) \left( |0⟩ + |1⟩ \right) | x \in J \right\},$$

$$C_1 = \left\{ \frac{1}{2} \left( |x⟩ - |\overline{x}\rangle \right) \left( |0⟩ - |1⟩ \right) | x \in J \right\}.$$

It is clear that the stabilizer of $C_0$ is $\langle X^\otimes n, Z^\otimes n-1 I, I^\otimes n-1 X \rangle$ and that the stabilizer of $C_1$ is $\langle X^\otimes n, Z^\otimes n-1 I, -I^\otimes n-1 X \rangle$, as given in the generator matrix above.

To show that our hybrid code has minimum distance 2, we note first that both $C_0$ and $C_1$ have minimum distance 2 when viewed as separate quantum codes. Thus we only need to look at how single-qubit Pauli errors affect the classical information. Consider two codewords $|c_i^{(0)}⟩$ and $|c_j^{(1)}⟩$, one from each quantum code. If $i \neq j$, it is clear that $\langle c_i^{(0)} | E | c_j^{(1)}⟩ = 0$ for any single-qubit Pauli error, since they will be linear combinations of disjoint sets of orthonormal basis vectors. Therefore, we can consider only the case when $i = j$.

Suppose that a single-qubit error occurs on the first $n-1$ qubits. Then

$$\langle |0⟩ + |1⟩ | (0⟩ - |1⟩) \rangle = \langle 0 | 0⟩ - ⟨1 | 1⟩ = 0,$$

so $\langle c_i^{(0)} | E | c_j^{(1)}⟩ = 0$. Similarly, if a single-qubit error occurs on the last qubit, then

$$\langle x⟩ + (\overline{x}) | (x⟩ - |\overline{x}\rangle) \rangle = \langle x | x⟩ - ⟨\overline{x} | \overline{x}⟩ = 0,$$

so $\langle c_i^{(0)} | E | c_j^{(1)}⟩ = 0$. Thus the hybrid code given by $C_0 \oplus C_1$ has minimum distance 2.

By a result of Rains [25, Theorem 2], for a general $((n, K, 2))_2$ quantum code with $n$ odd, we have

$$K \leq 2^{n-2} \left( 1 - \frac{1}{n-1} \right).$$

In particular, this precludes the possibility of an $((n, 2^{n-2}, 2))_2$ code for $n$ odd. Similarly, suppose that there we could construct a code in our family using an $[n_q, k, d]_2$, quantum code and an $[n_c, m, d]_q$ classical code. Then we would have $n_q + n_c = n, k = n - 3$, and $m = 1$, and in particular, we have an $[n_q, n_q + n_c - 3, 2]_2$ quantum code. By the quantum Singleton bound, we must have $n_c \leq 1$, forcing us to have a $[1, 1, 2]_2$ classical code, which of course does not exist. It follows that all of the codes in our family must be genuine hybrid codes.

V. FAMILIES OF HYBRID CODES FROM STABILIZER PASTING

In this section, we construct two families of single-error correcting hybrid codes that can encode one or two classical bits. An infinite family of nonadditive quantum codes was constructed by Yu et al. [34] by pasting together (see [9]) the stabilizers of Gottesman’s $[2^j, 2^j - j - 2, 3]_2$ codes with the observables of the $((9, 12, 3))_2$ and $((10, 24, 3))_2$ nonadditive codes [31], [32].

Below we give the generator matrices of the hybrid codes originally given by Grassl et al. [14] that we will use in the construction of our families. The generator matrix for the $[7, 1:1, 3]_2$ code was previously given in [1]. A $[9, 2:2, 3]_2$ hybrid code is given by the generator matrix [2].
where the parameter $a$ define a pure single-qubit Pauli error has a unique syndrome, allowing for the correction of any single-qubit error.

Theorem 11. Let $m$ be a nonnegative integer and $n$ a positive integer given by

$$n = \frac{2^{2m+5} - 32}{3} + a,$$

where the parameter $a$ is a small positive integer that is specified below. Then there exists

(a) an $\left[ n, n - 2m - 6 : 1, 3 \right]_2$ hybrid code for $a = 7$ and

(b) an $\left[ n, n - 2m - 7 : 2, 3 \right]_2$ hybrid code for $a = 9, 10, 11$.

Proof. Roughly speaking, we construct our code by partitioning the first $\left( 2^{2m+5} - 32 \right)/3$ qubits into disjoint sets, forming a perfect code on each partition, and use one of the four small hybrid codes on the remaining last $a$ qubits. These codes are then “glued” to one another by using stabilizer pasting. Other than a small number of degenerate errors introduced by the small hybrid code that must be handled individually, each single-qubit Pauli error has a unique syndrome, allowing for the correction of any single-qubit error.

We will now describe the code construction in more detail. We take the $n = \left( 2^{2m+5} - 32 \right)/3 + a$ qubits and partition them into disjoint sets

$$U_m \cup U_{m-1} \cup \cdots \cup U_1 \cup V_a,$$

where $|U_k| = 2^{2k+3}$ and $|V_a| = a$. The set $U_m$ contains the first $2^{2m+3}$ qubits, $U_{m-1}$ the next $2^{2m+1}$ qubits, and so forth. The final $a$ qubits are contained in $V_a$.

Let $k$ be an integer in the range $1 \leq k \leq m$. On the qubits in the set $U_k$, we can construct an optimal stabilizer code of length $2^{2k+3}$ with $2k + 5$ stabilizer generators, following Gottesmann [10]. The $2k + 5$ stabilizer generators are given as follows. Two of these generators are the tensor product of only Pauli-$X$ and $Z$ operators, which we call $X_{U_k}$ and $Z_{U_k}$ respectively. We define the other $2k + 3$ stabilizers by

$$S^{k}_j = X^{h_j} Z^{h_{j-1} + h_1 + h_{2k+3}},$$

for $j \in [2k+3]$. Here we let $h_j$ be the $j$-th row of the $(2k+3) \times 2^{2k+3}$ matrix $H_k$, whose $i$-th column is the binary representation of $i$, $h_0$ is defined to be the all-zero vector, and $X^{h_j} = X^{h_{j,0}} X^{h_{j,1}} \cdots X^{h_{j,2k+3}-1}$, with $Z^{h_j}$ defined similarly.

For the set $V_a$, let $H^\Omega$ be the generators of the quantum stabilizer $S^\Omega$ of the length $a$ hybrid code defined by generator matrix (1), (2), (3), or (4), and $H^\Omega$ be the generators of the classical stabilizer $S^\Omega$ (since the length 7 hybrid code with generator matrix (1) only has one generator in $S^\Omega$, we can remove $H^\Omega$). The stabilizer can be pasted together as in the generator matrix given below, where suitable identity operators should be inserted in the blank spaces:

$$\begin{pmatrix}
X_{U_m} \\
Z_{U_m} \\
S^m_1 X_{U_{m-1}} \\
S^m_2 Z_{U_{m-1}} \\
\vdots \\
S^m_{2m-6} X_{U_2} \\
S^m_{2m-5} Z_{U_2} \\
\vdots \\
S^m_{2m-3} S^m_{2m-5} \cdots S^m_1 X_{U_1} \\
S^m_{2m-2} S^m_{2m-4} \cdots S^m_2 Z_{U_1} \\
\vdots \\
S^m_{2m-1} S^m_{2m-3} \cdots S^m_3 S^m_1 H^\Omega \\
S^m_{2m} S^m_{2m-2} \cdots S^m_4 S^m_2 H^\Omega \\
S^m_{2m+1} S^m_{2m+1} \cdots S^m_5 S^m_3 H^\Omega \\
S^m_{2m+2} S^m_{2m+2} \cdots S^m_6 S^m_4 H^\Omega \\
\vdots \\
S^m_{2m+3} S^m_{2m+3} \cdots S^m_7 S^m_5 H^\Omega \\
\vdots \\
H^\Omega_1 \\
H^\Omega_2 \\
\end{pmatrix}$$

Suppose that we have an single-qubit Pauli error on the block $U_m$. Since the code is pure, the syndrome of each error
will be distinct and such that the Pauli-\(X\), \(Y\), and \(Z\) syndromes will start with 01, 11, and 10 respectively. However, this leaves all of the syndromes starting with 00 unused, so Pauli-\(X\), \(Y\), and \(Z\) errors on the block \(U_{m-1}\) will have distinct syndromes starting with 0001, 0011, and 0010 respectively. Continuing on, any single-qubit Pauli error occurring on the block \(U_k\) will have a distinct syndrome starting with \(2(m-k)\) 0s.

All of the syndromes of errors occurring on the block \(V_a\) start with \(2m\) 0s. Here our code is not pure, but it is almost pure, with the only degenerate errors being the weight 2 errors in \(C_e\). For example, when \(V_a\) has 11 qubits, it will have three weight 1 degenerate errors: \(Z_1\) (a Pauli-\(Z\) on the first qubit of the block), \(Z_2\), and \(X_9\), each with the syndrome 00011 (preceded by \(2m\) 0s). If we measure this syndrome, we apply the operator \(ZZIIIIXII\) to the state, which maps the original codeword to itself up to a global phase. Note, however, that while this global phase is the same for codewords of the same inner code for a given error, it may differ for codewords from different inner code. In fact, this is exactly what prevents the outer code from being a distance 3 quantum code rather than a distance 3 hybrid code. The argument for when \(V_a\) has 7, 9, and 10 qubits is similar.

Since we know how to correct any single-qubit Pauli error based on its syndrome, each of the codes must have minimum distance 3.

Here we show that these hybrid codes are better than optimal quantum stabilizer codes using a result of Yu et al. [33].

**Proposition 12.** Let \(m\) be a nonnegative integer and \(n\) a positive integer given by

\[
n = \frac{2^{2m+5} - 32}{3} + a,
\]

where \(a \in \{7, 9, 10, 11\}\). Then there does not exist an \([n, n-2m-5, 5]_2\) stabilizer code.

**Proof.** When \(a = 7, 9, 10\), we have

\[
n = \frac{2^{2m+5} - 32}{3} + a
= \frac{2^{2m+5} - 8}{3} + (a - 8)
= \frac{8}{3} \left(4^{m+1} - 1\right) + (a - 8).
\]

By a result of Yu et al. [33] Theorem 1], distance 3 stabilizer codes with lengths of the form

\[
\frac{8}{3} \left(4^k - 1\right) + b,
\]

where \(b \in \{-1, 1, 2\}\), can exist if and only if

\[2m + 5 \geq \left\lceil \log_2(3n + 1) \right\rceil + 1.
\]

But in this case we have

\[
\left\lceil \log_2(3n + 1) \right\rceil + 1 = \left\lceil \log_2\left(2^{2m+5} + 3a - 31\right) \right\rceil + 1
> \left\lceil \log_2\left(2^{2m+5} - 2^{2m+4}\right) \right\rceil + 1
= 2m + 5,
\]

so when \(a = 7, 9, 10\), there is no distance 3 stabilizer code of length \(n\).

When \(a = 11\), a different case of [33] Theorem 1] applies, so distance 3 stabilizer codes with lengths of this form can exist if and only if

\[2m + 5 \geq \left\lceil \log_2(3n + 1) \right\rceil.
\]

However, this gives us

\[
\left\lceil \log_2(3n + 1) \right\rceil = \left\lceil \log_2\left(2^{2m+5} + 2\right) \right\rceil
> \left\lceil \log_2\left(2^{2m+5}\right) \right\rceil
= 2m + 5,
\]

so when \(a = 11\), there is likewise no distance 3 stabilizer code of length \(n\).

\[\square\]

**VI. CONCLUSION AND DISCUSSION**

In this paper we have proven some general results about hybrid codes, showing that they can always detect more errors than comparable quantum codes. Furthermore we proved the necessity of impurity in the construction of hybrid codes. Additionally, we generalized weight enumerators for hybrid stabilizer codes to nonadditive hybrid codes, allowing us to develop linear programming bounds for nonadditive hybrid codes. Finally, we have constructed several infinite families of hybrid stabilizer codes that provide an advantage over optimal stabilizer codes.

Both of our families of hybrid codes were inspired by the construction of nonadditive quantum codes. In hindsight this is not very surprising, as the examples of hybrid codes with small parameters given by Grassl et al. [14] were constructed using a CWS/union stabilizer construction. Most interesting is that all known good nonadditive codes with small parameters have a hybrid code with similar parameters. This would suggest that looking at larger nonadditive codes such as the quantum Goethals-Preparata code [12] or generalized concatenated quantum codes [13] might be helpful in constructing larger hybrid codes. Alternatively, it may be possible to use the existence of hybrid codes to point to where nonadditive codes may be found. For instance the existence of an \([11, 4:2, 3]_2\) hybrid code suggests a nonadditive code with similar parameters might exist.

As previously suggested by Grassl et al. [14], one possible way to construct new hybrid codes with good parameters is to start with degenerate quantum codes with good parameters. Another possible approach to constructing new hybrid stabilizer codes is to find codes such that there are few small weight errors that are in the normalizer but not in the stabilizer, and then add those small weight errors to the generating set of the stabilizer to get a degenerate code. Here, the original code becomes the outer code of the hybrid code and the degenerate code the inner code.

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