On non-supersymmetric generalizations of the Wilson–Maldacena loops in $\mathcal{N} = 4$ SYM

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Abstract

Building on our previous work arXiv:1712.06874 we consider one-parameter Polchinski–Sully generalization of the Wilson–Maldacena (WM) loops in planar $\mathcal{N} = 4$ SYM theory. This breaks local supersymmetry of WM loop and leads to running of the deformation parameter $\zeta$. At three-loop level, we compute the ladder diagram contribution to the expectation value of the circular loop which is dominant for large $\zeta$. The limit $\zeta \gg 1$, $\lambda \zeta^2 = \text{fixed}$ in which the expectation value is determined by the Gaussian adjoint scalar path integral might be exactly solvable despite the lack of global supersymmetry. We study similar generalization of the $\frac{1}{4}$-BPS “latitude” WM loop which depends on two parameters (in addition to the ’t Hooft coupling $\lambda$). One may also introduce another supersymmetry-breaking parameter – the winding number of the scalar coupling circle. We find the two-loop expression for the expectation value of the associated loop by combining the ladder diagram contribution with an indirect determination of the non-ladder contribution using 1d defect CFT perturbation theory.

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1. Introduction

As proposed in [1] and recently discussed in [2], it is interesting to consider a one-parameter family of Wilson loop operators defined in $\mathcal{N} = 4$ SYM by

$$ W^{(\zeta)}(C) = \frac{1}{N} \text{Tr} \mathcal{P} \exp \oint_{C} d\tau \left[ i A_\mu(x) \dot{x}^\mu + \zeta \left| \Phi_m(x) n^m \right| \right]. \quad (1.1) $$

Here $n_m(\tau)$ is a unit 6-vector and $\zeta$ is a real parameter. The operator $W^{(\zeta)}(C)$ interpolates between the standard Wilson loop $W^{(0)}$ and the “locally-supersymmetric” Wilson–Maldacena (WM) loop $W^{(1)}$ [3,4]. In particular, one may consider the case of a circular loop with $n^m = \text{const}$ for which (1.1) with $\zeta = 1$ is 1/2-BPS (preserves 8 out of the 8 + 8 superconformal symmetries of $\mathcal{N} = 4$ SYM) but has no global supersymmetry if $\zeta \neq 1$.

The scale dependence of the (renormalized) coupling $\zeta$ is controlled at one-loop by the beta function [1]

$$ \beta(\zeta) = \mu \frac{d\zeta}{d\mu} = \frac{\lambda}{8 \pi^2} \zeta (\zeta^2 - 1) + \mathcal{O}(\lambda^2), \quad (1.2) $$

where $\lambda = g^2 N$ is the 't Hooft coupling and we consider the planar limit. $\zeta = 0, 1$ are expected to be conformal points to all orders in $\lambda$.

In the case of the circular loop with constant $n^m$ the two-loop weak-coupling expression for $\langle W^{(\zeta)} \rangle$ is found to be [2]

$$ \langle W^{(\zeta)} \rangle = 1 + \frac{\lambda}{8} + \lambda^2 \left[ \frac{1}{192} + \frac{1}{128 \pi^2} (\zeta^2 - 1)^2 \right] + \mathcal{O}(\lambda^3). \quad (1.3) $$

Starting at $\lambda^3$ order, the UV divergences in (1.3) do not cancel but can be absorbed into a renormalization of $\zeta$, as we will see below.

As discussed in [2], the expectation value $\langle W^{(\zeta)} \rangle$ may be interpreted as the partition function of an effective “defect” 1d QFT which becomes conformal at $\zeta = 0, 1$. Expectation values with insertions of suitable local operators along the loop at the conformal points obey the general properties of CFT correlators [5]. For simple scalar operators that are coupled to $\zeta$, their insertions are controlled by the dependence of $\langle W^{(\zeta)} \rangle$ on $\zeta$.

A strong-coupling counterpart of the RG flow (1.2) was discussed in [1,2]. In particular, AdS/CFT predicts that the locally BPS Wilson loop and the standard Wilson loop should be dual to the string partition function on the disc with the standard (Dirichlet) and alternate (Neumann) boundary conditions in $S^5$ [10,1]. This allows to formulate a prediction for the strong-coupling limit of (1.1), as discussed in [2]. To connect the weak/strong coupling expansions of (1.1) requires finding the weak-coupling series (1.3) to all orders in $\lambda$. Optimistically, this might be possible (despite the absence of global supersymmetry for $\zeta \neq 1$) due to underlying integrability of the $\mathcal{N} = 4$ SYM theory.

The higher loop contributions to the expectation value of (1.1) simplify if one takes the large $\zeta$, small $\lambda$ limit with $\lambda \zeta^2 \ll 1$. In this case the scalar coupling in (1.1) dominates over the vector one and also the planar SYM theory becomes effectively free (with only the kinetic term for the scalar field $\Phi_m$ surviving). As a result, the planar scalar ladder diagrams give dominant contribution to

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2 For recent examples of application of the defect CFT approach to Wilson loop computations in $\mathcal{N} = 4$ SYM see [6–9].
\( \langle W^{(\zeta)} \rangle \) in this limit. Still, it is not clear how to find the exact (all-order) ladder-diagram expression for this expectation value due to non-trivial effect of the path ordering in (1.1).\(^3\)

One of our aims below will be to investigate how such ladder correlators are organized. In [2], it was shown that there is a rather simple computational procedure to compute ladder diagrams. We shall extend the discussion in [2] to the \( \lambda^3 \) level. In general, considering only ladder diagrams is a major limitation, but one may attempt to reconstruct the full 3-loop \( \lambda^5 \) term in (1.3) by completing the ladder contribution using some extra physical constraints.

We shall also explore more general loops with non-constant \( n^m \) in (1.1). In particular, one may consider a 2-parameter \( \zeta = (\zeta_1, \zeta_2) \) family of loop operators \( W^{(\zeta_1, \zeta_2)} \) interpolating between the standard circular Wilson loop and the \( \frac{1}{2} \)-BPS WM loop considered in [12–14]. This extension is defined by a circle in \( x^\mu \)-space and \( n^m \) corresponding to a latitude of \( S^2 \subset S^5 \):\(^4\)

\[
\begin{align*}
x^\mu &= R (\cos \tau, \sin \tau, 0, 0), & n^m &= (\sin \theta_0 \cos \tau, \sin \theta_0 \sin \tau, \cos \theta_0, 0, 0, 0), \
W^{(\zeta_1, \zeta_2)} &= \zeta \Phi_m n^m = \zeta_1 \Phi^3 + \zeta_2 (\cos \tau \Phi^1 + \sin \tau \Phi^2), \\
\zeta_1 &= \zeta \cos \theta_0, & \zeta_2 &= \zeta \sin \theta_0.
\end{align*}
\]

For \( \zeta = 1 \) this is the \( \frac{1}{2} \)-BPS latitude WM loop for which the exact expression is known – given by the \( \frac{1}{2} \)-BPS WM loop with the replacement \( \lambda \rightarrow \lambda \cos^2 \theta_0 \) [14,15]. \( W^{(\zeta_1, \zeta_2)} \) may be viewed as a 2-parameter deformation of the \( \frac{1}{2} \)-supersymmetric (preserving 2 out of 8 \( Q \)-supercharges) loop of [12] that has trivial expectation value which corresponds to the special case of \( \zeta_1 = 0, \zeta_2 = 1 \) or \( \zeta = 1, \theta_0 = \frac{\pi}{2} \). Here we shall analyze the renormalization of the couplings \( \zeta_1 \) and \( \zeta_2 \) at two loops, and at three loops in ladder limit. This will provide interesting information about the two \( \beta \)-functions associated to \( \zeta_1 \) and \( \zeta_2 \) generalizing (1.2).

Another non-supersymmetric Wilson loop generalization we shall study is the introduction of a non-trivial winding of the \( n^m(\tau) \) contour in the auxiliary \( S^5 \) space. In particular, one may make the replacement \( \tau \rightarrow \nu \tau \) in the expression for \( n^m \) in (1.4). For \( \zeta = 1 \) the corresponding WM loop will no longer be \( \frac{1}{2} \)-BPS and thus \( \langle W \rangle \) will be a non-trivial function of the winding \( \nu \) and the coupling \( \lambda \) yet to be determined.\(^5\) We shall compute the two-loop order demonstrating its finiteness (which is expected to hold to all orders as this is a special case of a locally supersymmetric WM loop).

We shall start in Sec. 2 with a review of an efficient regularization procedure suitable for the analysis of the ladder diagrams contributing to the expectation value of generalized Wilson loops. We shall apply it at three loops in the case of the circular contour and discuss renormalization properties of the resulting expectation value.

In Sec. 3 we shall discuss the two-parameter loop (1.4). We shall present its complete two loop expression by adapting the results of [2] to this case. The three-loop contribution from ladder diagrams will also be found, extracting information about the \( \beta \)-functions for the two couplings \( \zeta_1, \zeta_2 \).

Sec. 4 will be devoted to the analysis of the effect of winding of the scalar-space contour. We will present the two-loop expression of the wound loop by combining a direct computation of the ladder diagram contribution with an indirect determination of the non-ladder contribution.

\(^3\) The loop equation governing the large \( N \) adjoint scalar loop expectation value was considered in [11].

\(^4\) In what follows we shall often set the radius of the circle \( R \) to 1.

\(^5\) Note that this deformation is different from multiply wound supersymmetric generalizations of the circular WM loop where winding in space–time is correlated with that in the scalar coupling term, see [16].
inferred by exploiting the 1d defect CFT perturbation theory. Several Appendices will contain some technical details.

2. Ladder diagram contribution to ζ-deformed $\frac{1}{2}$-BPS circular WM loop

In this section we shall consider the expectation of the operator (1.1) for the standard circular loop with $n^n = \text{const}$ and discuss the evaluation of the planar ladder diagram contributions. We shall use a particular regularization scheme based on mode expansion and convenient point splitting. From a computational point of view, this approach is superior to the dimensional regularization [2]. This claim does not apply to non-ladder diagrams that will not be considered in this section.6

2.1. Mode regularization

Let us briefly recall the mode regularization method proposed in [2]. In the case of the ζ-deformed circular WM loop contributions from $\ell$-loop planar ladder diagrams containing $\ell$ scalar and vector propagators attached to the circular loop lead to expressions like

$$I_G \equiv I(i_{i1}i_2)...(i_{2\ell-1}i_{2\ell}) = \int_{\tau_1 > ... > \tau_{2\ell}} d^{2\ell} \tau \ G(\tau_1)...G(\tau_{2\ell}),$$

(2.1)

where $G(\tau)$ is a connected diagram.

$$G(\tau) = (\zeta^2 - \cos \tau) \mathcal{D}(\tau), \quad \mathcal{D}(\tau) = \frac{1}{4 \sin^2 \frac{\tau}{2}}, \quad \tau_{ij} = \tau_i - \tau_j.$$ (2.2)

Here $\{i_1, \ldots, i_{2\ell}\}$ is a permutation of $\{1, \ldots, 2\ell\}$ associated with a planar diagram $G_\ell = (i_{i1}i_2)...(i_{2\ell-1}i_{2\ell})$. The diagram is built by taking points $\tau_1 > \tau_2 > ... > \tau_{2\ell}$ on the circular loop and connecting the pairs $(i_1, i_2)...(i_{2\ell-1}, i_{2\ell})$. Clearly, the planarity constraint allows only certain permutations.7 In what follows we shall use the following notation for the path-ordered integral in (2.1):

$$\int [d^{2\ell} \tau] F(\tau) \equiv \int \prod_{\tau_1 > ... > \tau_{2\ell}} d^{2\ell} \tau \ F(\tau) = \int_{0}^{2\pi} d\tau_1 \int_{0}^{\tau_{1}} d\tau_2 ... \int_{0}^{\tau_{2\ell} - 1} d\tau_{2\ell} F(\tau).$$ (2.3)

The mode regularization procedure is based on using the formal Fourier mode expansion of8

$$\mathcal{D}(\tau) = \frac{\hat{f}}{2(1 - \cos \tau)} = -\sum_{n=1}^{\infty} n \cos(n \tau)$$

with a particular short-distance cutoff $\varepsilon \to 0^9$

$$\mathcal{D}(\tau) \to \mathcal{D}_{\varepsilon}(\tau) = \frac{1 - \cos \tau \cosh \varepsilon}{2(\cosh \varepsilon - \cos \tau)^2} = \sum_{n=1}^{\infty} e^{-n\varepsilon} n \cos(n \tau).$$ (2.4)

6 In this case the dimensional regularization (or, more precisely, regularization by dimensional reduction appropriate in a supersymmetric theory) is the most convenient one.

7 Notice that there may be different pairings leading to the same diagram topology due to periodicity on the circle. An example is the two-loop equivalence (12)(34) ≅ (14)(23). Nevertheless, summing over all planar terms as in (2.1) gives the correct contribution without possible over- or under-counting. This is because (2.1) is nothing but enumeration of all possible contractions after the expansion of the exponential in the loop operator.

8 $\mathcal{D}(\tau)$ is the scalar propagator (which is also the same as the vector field propagator in the Feynman gauge that we shall assume) restricted to the circle.

9 Here $\varepsilon$ is dimensionless parameter, i.e. $\varepsilon = \varepsilon' \mu$, where $\varepsilon' \to 0$ is small-scale cutoff and $\mu$ is a normalization mass scale (e.g., the inverse radius of the circle which was set to 1 in 2.2).
Then we get the regularized expression for (2.1)

\[ I_{G_1}(\varepsilon) = (-1)^{\ell} \sum_{n_1,\ldots,n_\ell=1}^{\infty} e^{-\varepsilon(n_1+\cdots+n_\ell)} n_1 \cdots n_\ell \]

\[ \times \int \left[ d^2 \tau \right] (\tau^2 - \cos \tau_{i_1i_2}) \cdots (\tau^2 - \cos \tau_{i_{2\ell-1}i_2}) \cos(n_1 \tau_{i_1i_2}) \cdots \cos(n_\ell \tau_{i_{2\ell-1}i_2}) \]  

(2.5)

Expanding in \( \varepsilon \to 0 \) and discarding poles and terms \( \mathcal{O}(\varepsilon) \), we are left with an expression that may contain powers of \( \log \varepsilon \). These should be the counterparts of the dimensional regularization poles.

Writing first

\[ (\tau^2 - \cos \tau) \mathcal{D}(\tau) = (\tau^2 - 1) \mathcal{D}(\tau) + \frac{1}{2}, \]  

(2.6)

and then using (2.4) one may represent the contribution of each diagram as a power series in \( \tau^2 - 1 \)

\[ I_{G_1} = \sum_{r=0}^{\ell} (\tau^2 - 1)^r I_{G_1}^{(r)}. \]  

(2.7)

Here the \( r = 0 \) term corresponds to the \( \frac{1}{2} \)-BPS WM loop, i.e. [17]

\[ I_{G_1}^{(0)} = \frac{1}{2^\ell} \frac{(2\pi)^{2\ell}}{(2\ell)!} = \left\{ 1, \pi^2, \frac{\pi^4}{6}, \frac{\pi^6}{90}, \ldots \right\}, \quad \ell = 0, 1, 2, 3, \ldots \]  

(2.8)

Then

\[ \langle W(\xi) \rangle = \sum_{\ell=0}^{\infty} \frac{\lambda^\ell}{(8\pi^2)^\ell} \sum_{G_1} I_{G_1}, \]  

(2.9)

where the \( \ell \)-loop normalization prefactor \( \frac{1}{(8\pi^2)^\ell} \) is the product of \( \frac{1}{2^\ell} \) coming from the color generator \( t^a t^a \) contractions and \( \frac{1}{(4\pi)^\ell} \) from the normalization of the scalar field propagator.

2.2. Three-loop contributions to \( \langle W(\xi) \rangle \)

The two-loop analysis can be found in [2] and is briefly summarized in Appendix A.1. We have only the two diagrams (12)(34) and (14)(23) and they give the contributions (dropping the power-divergent \( \frac{1}{x^k} \) terms)

\[ I_{(14)(23)} = \frac{\pi^2}{2} (\xi^2 - 1)^2 + \frac{\pi^4}{6}, \quad I_{(12)(34)} = 2\pi^2 (\xi^2 - 1) \log \varepsilon + \frac{\pi^4}{6}. \]  

(2.10)

Including also the one-loop term \( \frac{\lambda^2}{8} \) we obtain

\[ \langle W(\xi) \rangle_{\text{ladder}} = 1 + \frac{\lambda}{8} + \lambda^2 \left[ \frac{1}{192} + \frac{(1 - \xi^2)^2}{128 \pi^2} - \frac{1 - \xi^2}{32 \pi^2} \log \varepsilon \right] + \mathcal{O}(\lambda^3). \]  

(2.11)

The logarithmically divergent term is canceled by the contribution of other non-ladder diagrams with internal vertices; this contribution does not, however, change the finite part (see [2]). Consistently with the fact that ladder diagrams should dominate in the large \( \xi \) limit, the divergent
term in (2.11) is subleading at large \( \zeta \), i.e. it is \( O(\zeta^2\lambda^2) \) as compared to the finite contribution \( O(\zeta^2\lambda^2) \) that comes purely from the ladder diagrams.

The novel three-loop contributions are derived in detail in Appendices A.2, A.3. The three-loop ladder diagrams \( G_3 = (12)(34)(56) \), etc., give the following contributions \( I_{G_3}^{(r)} \) to (2.1), (2.7) (we omit \( O(\epsilon) \) terms)

\[
\begin{align*}
I_{(16)(25)(34)}^{(3)} &= \frac{1}{\epsilon} \left( \pi^2 \log \epsilon + \pi^2 \log 2 + \frac{\pi^2}{2} \right) - \frac{3\pi^2}{2}, \\
I_{(16)(25)(34)}^{(2)} &= -\frac{\pi^4}{6\epsilon} + \frac{\pi^4}{2}, \\
I_{(16)(25)(34)}^{(1)} &= 0, \\
I_{(12)(34)(56)}^{(3)} &= \frac{\pi^2}{2} \log \epsilon + \frac{\pi^2}{2} \log 3, \\
I_{(12)(34)(56)}^{(2)} &= 3\pi^2 \log^2 \epsilon, \\
I_{(12)(34)(56)}^{(1)} &= \frac{1}{2} \pi^4 \log \epsilon + \frac{3\pi^2}{2} \zeta(3), \\
I_{(12)(36)(45)}^{(3)} &= I_{(14)(23)(56)}^{(3)} = -\frac{\pi^2 \log \epsilon}{2\epsilon} + \frac{\pi^2}{2} \log \epsilon + \frac{\pi^2}{4}, \\
I_{(12)(36)(45)}^{(2)} &= I_{(14)(23)(56)}^{(2)} = -\frac{\pi^4}{6\epsilon} + \pi^2 \log^2 \epsilon - \pi^2 \log \epsilon + \frac{\pi^2}{2}, \\
I_{(12)(36)(45)}^{(1)} &= I_{(14)(23)(56)}^{(1)} = \frac{1}{3} \pi^4 \log \epsilon + 2\pi^2 \zeta(3), \\
I_{(16)(23)(45)}^{(3)} &= 4\pi^2 \log 2 + \frac{\pi^2}{2} \log \epsilon - \frac{\pi^2}{2} \log 3 - \frac{2\pi^2}{3}, \\
I_{(16)(23)(45)}^{(2)} &= -\pi^2 \log^2 \epsilon + \frac{\pi^4}{6}, \\
I_{(16)(23)(45)}^{(1)} &= \frac{\pi^4}{6} \log \epsilon - \frac{\pi^2}{2} \zeta(3).
\end{align*}
\]

Here \( \zeta(n) \) is the Riemann zeta-function. The three-loop ladder contributions to the Wilson loop (1.1), (2.11) are then (dropping power divergences)

\[
\langle W^{(C)} \rangle_{\text{ladder}} = 1 + \frac{\lambda}{8} + \lambda^2 \left[ \frac{1}{192} + \frac{(1 - \zeta^2)^2}{128 \pi^2} - \frac{1 - \zeta^2}{32 \pi^2} \log \epsilon \right] \\
+ \lambda^3 \left[ \frac{1}{9216} - \frac{5(1 - \zeta^2)}{512 \pi^4} \zeta(3) + \frac{(1 - \zeta^2)^2}{768 \pi^2} \left( 1 + \frac{1}{2\pi^2} (8 - 5 \zeta^2) \right) \right] \\
- \left( \frac{1 - \zeta^2}{384 \pi^2} + \frac{1 - \zeta^2}{256 \pi^4} (2 - \zeta^2) \right) \log \epsilon + \frac{(1 - \zeta^2)^2}{128 \pi^4} \log^2 \epsilon \right] + O(\lambda^4). 
\]
2.3. Discussion

If we pretend for a moment that the ladder approximation is consistent by itself, we find that the logarithmically divergent terms in (2.13) may be formally absorbed into a redefinition of both $\zeta$ and $\lambda$ \(^\text{10}\) i.e. assuming that the parameters appearing in (2.13) are the “bare” ones

$$\zeta_b(\epsilon) = \zeta + \frac{\lambda}{8\pi^2} \zeta (\zeta^2 - 1) \log \epsilon + \ldots,$$

$$\lambda_b(\epsilon) = \lambda + \frac{\lambda^2}{4\pi^2} (1 - \zeta^2) \log \epsilon + \frac{\lambda^3}{32\pi^4} \left[ 2 (1 - \zeta^2) \log^2 \epsilon + (1 - \zeta^2)^3 \log \epsilon \right] + \ldots. \quad (2.14) \quad (2.15)$$

Here one may make the dependence of the renormalized parameters $\zeta, \lambda$ on the renormalization mass scale $\mu$ explicit by redefining $\epsilon \to \epsilon \mu$ and setting $\frac{d\zeta_b}{d\mu} = 0, \frac{d\lambda_b}{d\mu} = 0$. Note that the sign of $\frac{d\zeta}{d\mu}$ here appears to be opposite to the one in the $\beta$-function (1.2).

The ’t Hooft coupling $\lambda$ should not of course run in the full SYM theory (which includes interactions and thus also diagrams with internal vertices) so it should not run in (2.13). In fact, all logarithms should be cancelled by (i) the expected 1-loop renormalization of $\zeta$ consistent with (1.2), and (ii) all remaining divergences should be cancelled by the contributions of other diagrams. A major simplification occurs if we decide to keep only the highest power of $\zeta$ at each order in expansion in $\lambda$. These leading terms may get contributions only from the scalar ladders graphs (ladder graphs with vector propagators give terms subleading in $\zeta$ which also receive contributions from other non-ladder diagrams). Thus, it should be captured exactly by the expression in (2.13), i.e.

$$W^{(\zeta)}_{\zeta \gg 1} = 1 + \frac{\lambda}{8} + \frac{\lambda^2 \zeta^4}{128\pi^2} + \frac{\lambda^3 \zeta^6}{1536\pi^4} (-5 + 6 \log \epsilon) + \mathcal{O}(\lambda^4). \quad (2.16)$$

The coupling in (2.16) is the bare one $\zeta_b$ and the divergence is canceled by the redefinition in terms of the renormalized $\zeta$, consistent with the beta function in (1.2) (cf. (2.14)). $\zeta_b = \zeta - \frac{\lambda \zeta^3}{8\pi^2} \log \epsilon + \mathcal{O}(\lambda^2). \quad (2.11)$ This is of course not unexpected as the $\zeta^3$ term in the $\beta$-function (1.2) comes from the ladder graph with the scalar propagator [1]. After the renormalization, the divergent $\log \epsilon$ factor is replaced by $\log \mu$, i.e. the renormalized expression reads

$$W^{(\zeta)}_{\zeta \gg 1} = 1 + \frac{\lambda}{8} + \frac{\lambda^2 \zeta^4}{128\pi^2} + \frac{\lambda^3 \zeta^6}{1536\pi^4} (-5 + 6 \log \mu) + \mathcal{O}(\lambda^4). \quad (2.17)$$

If one includes also the 4-loop ladder graph contributions and keeps only the leading $\lambda^n \zeta^{2n}$ terms one expects to find

$$W^{(\zeta)}_{\zeta \gg 1} = 1 + \frac{\lambda}{8} + \frac{\lambda^2 \zeta^4}{128\pi^2} + \frac{\lambda^3 \zeta^6}{1536\pi^4} (-5 + 6 \log \epsilon) + \frac{\lambda^4 \zeta^8}{4096\pi^6} \left[ w_4 + 2 (-5 + b_2) \log \epsilon + 9 \log^2 \epsilon \right] + \mathcal{O}(\lambda^5). \quad (2.18)$$

\(^{10}\) Some motivation for doing this may be as follows: if we consider just a free theory of adjoint vectors and scalars and then compute the expectation of the generalized WL (1.1) one may expect renormalizability – then one may be allowed to formally renormalize both $\zeta$ and $\lambda$.

\(^{11}\) To simplify the expressions, in the following we shall usually suppress the “b” (bare) and “r” (renormalized) subscripts on $\zeta$ in respective expectation values. In general, the bare couplings will always be accompanied by $\log \epsilon$ terms, while the renormalized couplings will come together with $\log \mu$ terms.
Here $w_4$ is a finite constant and the $\log^2 \epsilon$ term is fixed by consistency with the 1-loop $\beta$-function (1.2), i.e. it should be possible to eliminate all divergences by the following redefinition $\zeta \rightarrow \zeta_b(\epsilon, \zeta)$ in (2.18) (here $\zeta$ is the renormalized coupling)

$$
\zeta_b = \zeta - \left[ \frac{\lambda \xi^3}{8 \pi^2} + b_2 \frac{\lambda^2 \xi^5}{(8 \pi^2)^2} + \ldots \right] \log \epsilon + \frac{3 \lambda \xi^5}{128 \pi^4} \log^2 \epsilon + \ldots ,
$$

where $b_2$ is the 2-loop coefficient in the $\beta$-function. In the large $\zeta$ limit

$$
\beta(\zeta \gg 1) = \frac{\lambda \xi^3}{8 \pi^2} + \frac{\lambda^2 \xi^5}{(8 \pi^2)^2} b_2 + O(\lambda^3) .
$$

Direct computation of the 4-loop term in (2.18) remains a challenge. In the next section we will attempt to indirectly infer additional information about the value of $b_2$ by considering a more general Wilson loop.

3. $\zeta$-deformation of $\frac{1}{4}$-BPS latitude WM loop

Let us now consider a generalization of the $\frac{1}{4}$-BPS supersymmetric Wilson–Maldacena loop corresponding to latitude in $S^5$ [12–14] defined by (1.4) where $\theta_0$ is a constant parameter. Due to $\frac{1}{4}$-BPS property of this WM loop, the dependence of its expectation value on the latitude angle $\theta_0$ can be found just by the redefinition $\lambda \rightarrow \lambda \cos^2 \theta_0$ in the expectation value for the $\frac{1}{2}$-BPS circular WM loop [14,18]. As in the circle case [17] all non-vanishing contributions to the latitude WM loop come from ladder diagrams while contributions of non-ladder diagrams mutually cancel. For $\theta_0 = 0$ we get back the $\frac{1}{2}$-BPS circle where $\langle W \rangle = \frac{2}{\sqrt{\lambda}} J_1(\sqrt{\lambda})$ while for $\theta_0 = \frac{\pi}{2}$ we get the $\frac{1}{2}$-supersymmetric loop of [12] for which $\langle W \rangle = 1$.

To generalize the latitude loop we add a coefficient $\zeta$ in front of the scalar coupling term as in (1.1), (1.5). This is also equivalent to introducing the two couplings $\zeta_1, \zeta_2$ as in (1.6). Below we shall first present the full two-loop expression for $\langle W^{(\zeta_1, \zeta_2)} \rangle$ defined by (1.1) and (1.5) and then discuss the three-loop contributions from ladder diagrams only.

Let us note that for $\tau$-dependent direction $n^\mu$, there is no 1d reparametrization and, in particular, scale invariance in the WM loop (1.1). Thus for $\theta_0 \neq 0$ there will be no 1d conformal invariance even for $\zeta = 0$ or $\zeta = 1$.

The explicit classical breaking of scale invariance is not in conflict with UV finiteness that still holds due to local supersymmetry of the WM loop ($\zeta = 1$) case. Conformal perturbation theory will of course apply if we expand near the $\zeta = 0$ point, i.e. in powers of $\zeta_1$ and $\zeta_2$.

3.1. Complete two-loop contribution

At two-loop level, it is possible to compute $\langle W^{(\zeta_1, \zeta_2)} \rangle$ by building on the analysis of the $\zeta$-deformation of the circular loop in [2]. In this case the one and two-loop diagrams contributing to $\langle W^{(\zeta)} \rangle$ contain the $\zeta$-coupling in the integrand factors like

$$
\zeta^2 |\dot{x}(\tau_1)| |\dot{x}(\tau_2)| - \dot{x}(\tau_1) \cdot \dot{x}(\tau_2) = \zeta^2 - \cos \tau_{12} ,
$$

\[12\] In particular, computing correlation functions of scalar fields along the loop, i.e. $\langle \text{Tr}[\Phi \cdots \Phi \exp \int (i A \cdot \dot{x} + |\dot{x}|[\Phi \cdot n])] \rangle$, we cannot interpret them as 1d CFT correlators because of the explicit $\tau$ dependence in the scalar coupling in the exponent.
where the first term is from the coupling of the scalars to the loop, while the second term corresponds to the vector coupling. There is one such factor in the one loop diagram and in the two-loop self-energy and internal vertex diagrams, and two such factors in two-loop ladder diagrams.

To see this in detail, let us adopt the same labeling of diagrams as in [2] and review each contribution separately. In dimensional regularization with space–time dimension \(d = 2 \omega \equiv 4 - 2 \epsilon\), the only one-loop diagram is

\[
W_1(\zeta) = \frac{\Gamma(\omega - 1)}{16 \pi^{\omega}} \oint_{\mathcal{C}} d^{2} \tau \frac{\xi^2 |\dot{x}(\tau_1)| |\dot{x}(\tau_2)| - \dot{x}(\tau_1) \cdot \dot{x}(\tau_2)}{|x(\tau_1) - x(\tau_2)|^{2 \omega - 2}} \tag{3.2}
\]

which indeed contains the explicit factor (3.1). At two loops, we have ladder, self-energy, and internal vertex contributions. The ladder diagrams are

\[
W_{2,1a}(\zeta) = \frac{\Gamma(\omega - 1)}{64 \pi^{2 \omega}} \int [d^4 \tau] \frac{\xi^2 |\dot{x}(\tau_1)| |\dot{x}(\tau_2)| - \dot{x}(\tau_1) \cdot \dot{x}(\tau_2)}{(|x(\tau_1) - x(\tau_2)|^2 |x^3 - x^4|^2)^{\omega - 1}},
\]

\[
W_{2,1b}(\zeta) = \frac{\Gamma(\omega - 1)}{64 \pi^{2 \omega}} \int [d^4 \tau] \frac{\xi^2 |\dot{x}(\tau_1)| |\dot{x}(\tau_2)| - \dot{x}(\tau_1) \cdot \dot{x}(\tau_2)}{(|x(\tau_1) - x(\tau_2)|^2 |x^2 - x^3|^2)^{\omega - 1}}, \tag{3.3}
\]

i.e. both have two factors of (3.1). The self-energy contribution has one factor of (3.1)

\[
W_{2,2}(\zeta) = -\frac{\Gamma(\omega - 1)}{128 \pi^{2 \omega} (2 - \omega)(2\omega - 3)} \int d\tau_1 d\tau_2 \xi^2 [\dot{x}(\tau_1)| \dot{x}(\tau_2)| - \dot{x}(\tau_1) \cdot \dot{x}(\tau_2)}{|x(\tau_1) - x(\tau_2)|^{2 \omega - 3}} \tag{3.4}
\]

Finally, the sum of the internal vertex diagrams, one with mixed scalar–vector \(\Phi \Phi A\) vertex and the other with a triple vector \(A^3\) vertex, reads (here \(\Delta(x) = (\partial^2)^{-1} = \frac{\Gamma(\omega - 1)}{4\pi^\omega} \frac{1}{|x|^{2\omega - 2}}\))

\[
W_{2,3}(\zeta) = -\frac{1}{4} \oint d^3 \tau \varepsilon(\tau_1, \tau_2, \tau_3) \left[ \xi^2 |\dot{x}(\tau_1)| |\dot{x}(\tau_3)| - \dot{x}(\tau_1) \cdot \dot{x}(\tau_3) \right] \times \dot{x}(\tau_2) \cdot \frac{\partial}{\partial x(\tau_1)} \int d^{2\omega} y \Delta(x(\tau_1) - y) \Delta(x(\tau_2) - y) \Delta(x(\tau_3) - y), \tag{3.5}
\]

and thus also has one factor of (3.1).

Turning now to the case of the latitude loop (1.5), the combination (3.1) is to be replaced by

\[
\zeta_1^2 + \zeta_2^2 \cos \tau_{12} - \cos \tau_{12} = (1 - \zeta_2^2) \left( \zeta_2^2 - \cos \tau_{12} \right), \quad \hat{\zeta} = \frac{\zeta_1}{\sqrt{1 - \zeta_2^2}}. \tag{3.6}
\]

Thus, we obtain

\[
(W_{\zeta_1,\zeta_2}) = 1 + \lambda (1 - \zeta_2^2) W_1(\hat{\zeta})
\]

\[
+ \lambda^2 \left[ (1 - \zeta_2^2)^2 W_{2,1}(\hat{\zeta}) + (1 - \zeta_2^2) W_{2,2}(\hat{\zeta}) + (1 - \zeta_2^2) W_{2,3}(\hat{\zeta}) \right] + \ldots \tag{3.7}
\]

Here the \(W\)-functions are the ones for the circle case (3.3),(3.4),(3.5) computed already in [2].
Introducing the shortcut
\[
\frac{1}{\varepsilon'} \equiv \frac{1}{\varepsilon} + 2 \log \pi + 2 \gamma_E ,
\]
they read\(^\text{13}\)
\[
\begin{align*}
W_1(\zeta) &= \frac{1}{8} + \frac{1}{8} (1 - \zeta^2) \varepsilon + O(\varepsilon^2), \\
W_{2,1}(\zeta) &= \frac{1}{192} + (1 - \zeta^2) \left[ \frac{1}{64 \pi^2 \varepsilon'} + \frac{1}{128 \pi^2} (7 - 3 \zeta^2) \right] + O(\varepsilon), \\
W_{2,2}(\zeta) &= \zeta^2 W_{2,2}(1) + (1 - \zeta^2) \left[ - \frac{1}{64 \pi^2 \varepsilon'} - \frac{1}{16 \pi^2} \right] + O(\varepsilon), \\
W_{2,3}(\zeta) &= -W_{2,2}(1) + (1 - \zeta^2) \left[ - \frac{1}{64 \pi^2 \varepsilon'} - \frac{1}{64 \pi^2} \right] + O(\varepsilon), \\
W_{2,2}(1) &= - \frac{1}{64 \pi^2 \varepsilon'} - \frac{1}{32 \pi^2} + O(\varepsilon).
\end{align*}
\]
Using these expressions in (3.7), we obtain
\[
\langle W^{(\zeta_1,\zeta_2)} \rangle = 1 + \frac{\lambda}{8} \left[ 1 - \zeta_2^2 + \varepsilon (1 - \zeta_1^2 - \zeta_2^2) + O(\varepsilon^2) \right] + \lambda^2 \left[ \frac{1}{192} (1 - \zeta_2^2)^2 + \frac{(\zeta_1^2 + \zeta_2^2 - 1)(3 \zeta_1^2 + 7 \zeta_2^2 - 1)}{128 \pi^2} + \frac{\zeta_2^2 (\zeta_1^2 + \zeta_2^2 - 1)}{64 \pi^2 \varepsilon'} + O(\varepsilon) \right] + O(\lambda^3).
\]
(3.10)
The couplings in (3.10) are the bare ones, i.e. they should be replaced by,\(^\text{14}\) cf. (3.12),
\[
\begin{align*}
\lambda_b &= \mu^2 \varepsilon \lambda, \\
\zeta_{1b} &= \zeta_1 + \frac{1}{2 \varepsilon} \beta_1(\zeta_1, \zeta_2) + \ldots, \\
\zeta_{2b} &= \zeta_2 + \frac{1}{2 \varepsilon} \beta_2(\zeta_1, \zeta_2) + \ldots,
\end{align*}
\]
where \(\zeta\) are the renormalized couplings. The resulting expression for (3.10) expressed in terms of renormalized couplings should be finite and that determines the leading contribution to the \(\beta_2\)-function to be \(\frac{\lambda}{8 \pi^2} \zeta_2 (\zeta_1^2 + \zeta_2^2 - 1)\). We shall assume that \(\beta_1\) has a similar structure, i.e.
\[
\begin{align*}
\beta_1(\zeta_1, \zeta_2) &= \frac{\lambda}{8 \pi^2} \zeta_1 (\zeta_1^2 + \zeta_2^2 - 1) + \ldots, \\
\beta_2(\zeta_1, \zeta_2) &= \frac{\lambda}{8 \pi^2} \zeta_2 (\zeta_1^2 + \zeta_2^2 - 1) + \ldots.
\end{align*}
\]
(3.12)
This is a natural generalization of the expression for the \(\beta\)-function for \(\zeta\) in [1] found for the \(\zeta_1 = \zeta, \zeta_2 = 0\) case. Indeed, assuming that the diagrams with internal vertices do not contribute to the 1-loop beta function we then need to add just the ladder graph with the vector propagator and that leads to the \(-1\) terms in (3.12). Then the \(\beta\)-function for \(\zeta = \sqrt{\zeta_1^2 + \zeta_2^2}\) is the same as in (1.2) while \(\zeta_1 / \zeta_2\) or \(\theta_b\) in (1.6) is not renormalized at one-loop order.

\(^{13}\) Notice that the sum of the non-ladder two-loop diagrams is proportional to \(1 - \zeta^2\) as it should be since for \(\zeta = 1\) (i.e. the \(\frac{1}{4}\)-BPS loop) it is known that non-ladder diagrams mutually cancel at all orders.

\(^{14}\) The coupling \(\lambda\) does not renormalize, but, as discussed in [2], the explicit factor \(\mu^2 \varepsilon\) is nevertheless necessary to fix dimensions at generic \(\varepsilon\).
From (3.11) and (3.12) we obtain the finite expression ($\zeta^2 = \zeta_1^2 + \zeta_2^2$, see (1.6))

$$
\langle W^{(\zeta_1, \zeta_2)} \rangle = 1 + \frac{\lambda}{8}(1 - \zeta_2^2) + \lambda^2 \left[ \frac{1}{192} (1 - \zeta_2^2) + \frac{(\zeta_1^2 + \zeta_2^2 - 1)(\zeta_1^2 + a\zeta_2^2 - 1)}{128 \pi^2} \right] + O(\lambda^3),
$$

$$
a \equiv 5 + 4 \log(\mu_{\text{DR}} R), \\
\mu_{\text{DR}} \equiv \pi e^{\gamma_E} \mu,
$$

where the presence of the scheme-dependent constant $a$ reflects the running of $\zeta_2$ (DR refers to the dimensional regularization scheme). Notice also that we have reintroduced the implicit length scale $R$. Of course, for $\zeta = 1$, we recover the expected modification $\lambda \to \lambda(1 - \zeta_2^2) = \lambda \cos^2 \theta_0$ of the $1/2$-BPS loop. The small $\theta_0$ expansion of (3.13) is briefly discussed in Appendix B.

### 3.2. Three-loop ladder diagram contribution and the large $\zeta$ limit

Restricting to ladder diagram contributions, the expectation value of the loop with generic couplings $\zeta_1$ and $\zeta_2$ can be effectively obtained from the knowledge of expectation value of $\zeta$-deformation of the $1/2$-BPS circular loop. According to the previous discussion (cf. (3.6)) to get the expectation value $\langle W^{(\zeta_1, \zeta_2)} \rangle_{\text{ladder}}$ in the latitude case it is enough to make the following replacements in (2.13)

$$
\lambda \to \lambda(1 - \zeta_2^2), \quad \zeta^2 \to \frac{\zeta_1^2}{1 - \zeta_2^2}. \quad (3.14)
$$

Keeping only the contributions with the highest power of $\zeta_1$ and $\zeta_2$ at each order in $\lambda$ isolates the terms that can only come from the scalar ladder graphs and thus are completely determined using the replacement (3.14). Some discussion of the complete ladder contribution may be found in Appendix C. We thus get the following generalization of the circular loop expression (2.16) (i.e. of the case when $\zeta_1 = \zeta$, $\zeta_2 = 0$)

$$
\langle W^{(\zeta_1, \zeta_2)} \rangle_{\zeta \gg 1} = 1 - \frac{1}{8} \frac{\lambda}{\zeta_2^2} + \lambda^2 \left[ \frac{\zeta_2^4}{192} + \frac{(\zeta_1^2 + \zeta_2^2)^2}{128 \pi^2} - \frac{\zeta_2^2 (\zeta_1^2 + \zeta_2^2)}{32 \pi^2} \log \epsilon \right]
$$

$$
+ \lambda^3 \left[ - \frac{\zeta_2^6}{9216} - \frac{\zeta_2^2 (\zeta_1^2 + \zeta_2^2)}{768 \pi^2} - \frac{(\zeta_1^2 + \zeta_2^2)^2 (5 \zeta_1^2 + 8 \zeta_2^2)}{1536 \pi^4} + \frac{5 \zeta_2^4 (\zeta_1^2 + \zeta_2^2)^2 \zeta_2 R \gamma_E}{512 \pi^4}
$$

$$
+ \left( \frac{\zeta_2^4 (\zeta_1^2 + \zeta_2^2)}{384 \pi^2} + \frac{(\zeta_1^2 + \zeta_2^2)^2 (\zeta_1^2 + 2 \zeta_2^2)}{256 \pi^4} \right) \log \epsilon - \frac{\zeta_2^2 (\zeta_1^2 + \zeta_2^2)^2}{128 \pi^4} \log^2 \epsilon \right] + \ldots \quad (3.15)
$$

In the special case of $\zeta_1 = 0$ (or $\theta_0 = \pi/2$ in (1.4), (1.6)) corresponding to the $\zeta = \zeta_2$-deformation of the $1/4$-supersymmetric loop we find from (3.14)

$$
\langle W^{(0, \zeta_2)} \rangle_{\zeta_2 \gg 1} = 1 - \frac{1}{8} \frac{\lambda}{\zeta_2^2} + \lambda^2 \zeta_2^4 \left[ \frac{1}{192} + \frac{1}{128 \pi^2} - \frac{1}{32 \pi^2} \log \epsilon \right]
$$

---

15. The scheme where $a = 1$ has the special property that $\beta_\lambda (W^{(\zeta_1, -\zeta_2)})$ gives the $\beta$-function for the coupling $\zeta_1$. In general, this is what happens for a perturbation of the free energy $F$ around a conformal fixed point, i.e. we expect to have relations $\beta_\lambda F(g) = C_{ij}(g) \beta_j(g)$ expressing stationarity of the free energy. Here, such a relation appears to be accidental as $\langle W^{(\zeta_1, -\zeta_2)} \rangle = e^{-F}$ does not, in general, have such an interpretation unless we are on the critical line $\zeta = 1$.

16. $x(x) \sim R$ implies that each loop comes with a factor $\lambda_0 R^{2\epsilon}$ from ($||x||^2 - x \cdot x'$$)/x - x'^2\omega^{-2}$. In terms of the renormalized coupling this is $\lambda (\mu R)^{2\epsilon}$ producing logs of the dimensionful quantity $\mu R$. 

\[- \lambda^2 \frac{\lambda^2}{\xi} \left( \frac{1}{9216} + \frac{1}{768 \pi^2} - \frac{5 \xi}{512 \pi^4} + \frac{1}{192 \pi^4} - \left( \frac{1}{128 \pi^4} + \frac{1}{384 \pi^2} \right) \log \varepsilon \right.
\left. + \frac{1}{128 \pi^4} \log^2 \varepsilon \right) + \mathcal{O}(\lambda^4) \, . \tag{3.16}\]

This is equal to (2.13) evaluated at \( \xi = 0 \) and with \( \lambda \to -\lambda \xi^2 \) as required by (3.14) (taking into account that in (3.14) we kept only the highest powers of \( \xi \) at each order).

We may now attempt to absorb the divergences in (3.14) by a renormalization of the \( \xi_i \) couplings, \textit{i.e.} by replacing \( \xi_i \) by their bare values\(^\ref{footnote:17}\)

\[
\xi_{1,b} = \xi_1 + \left[ \frac{\lambda}{8 \pi^2} F_1(\xi_1, \xi_2) + \ldots \right] \log \varepsilon + \ldots, \\
\xi_{2,b} = \xi_2 + \left[ \frac{\lambda}{8 \pi^2} \xi_2 (\xi_1^2 + \xi_2^2) \\
+ \frac{\lambda^2}{(8 \pi^2)^2} (\xi_1^2 + \xi_2^2) \left( \frac{\xi_1^2 + \xi_2^2}{\xi_2} + \frac{\xi_1}{\xi_2} F_1(\xi_1, \xi_2) + \ldots \right) \log \varepsilon \\
+ \left[ \frac{\lambda^2}{(8 \pi^2)^2} \left( -\frac{1}{2} \xi_2 (\xi_1^2 + \xi_2^2) (\xi_1 - 3 \xi_2^2) - 2 \xi_1 \xi_2 F_1(\xi_1, \xi_2) + \ldots \right) \right] \log^2 \varepsilon + \ldots \, . \tag{3.17}\]

For such redefinition to represent the solution of the RG equations \( \beta_i = \mu \frac{d \xi_i}{d \mu} = -\frac{\lambda}{8 \pi^2} F_i(\xi_1, \xi_2) + \mathcal{O}(\lambda^2) \) we need to require that

\[
F_1(\xi_1, \xi_2) = -\xi_1 (\xi_1^2 + \xi_2^2), \quad F_2(\xi_1, \xi_2) = -\xi_2 (\xi_1^2 + \xi_2^2) \, . \tag{3.18}\]

This leads to the following expressions for the \( \beta \)-functions

\[
\beta_1(\xi_1, \xi_2) = \frac{\lambda}{8 \pi^2} \xi_1 (\xi_1^2 + \xi_2^2) + \ldots, \tag{3.19}
\]

\[
\beta_2(\xi_1, \xi_2) = \frac{\lambda}{8 \pi^2} \xi_2 (\xi_1^2 + \xi_2^2) - \frac{\lambda^2}{64 \pi^4} \xi_2 (\xi_1^2 + \xi_2^2)^2 + \ldots \, . \tag{3.20}\]

For \( \lambda \gg 1 \) (\textit{i.e.} \( \xi_1, \xi_2 \gg 1 \)) the one-loop part of these expressions is clearly consistent with the previous result (3.12). Inspired by (3.20), (3.12) a natural expectation for the structure of the two-loop \( \beta \)-functions is

---

\(^{\ref{footnote:17}}\) In a theory with dimensionless running couplings \( g_i \) and loop counting parameter \( \lambda \) the \( \beta \)-functions have the form \( \mu \frac{d \xi_i}{d \mu} = \beta_i(g) = \lambda \beta^{(1)}_i(g) + \lambda^2 \beta^{(2)}_i(g) + \lambda^3 \beta^{(3)}_i(g) + \ldots \). In general, the bare couplings \( g_{i,b}(\varepsilon) \) depending on dimension-length cutoff \( \varepsilon \to 0 \) will be related to renormalized couplings \( g_i(\mu) \) depending on renormalization (dimension-mass) scale \( \mu \) will have the general structure

\[
g_{i,b} = g_i + \left[ \lambda G^{(1)}_{1,i}(g) + \lambda^2 G^{(2)}_{1,i}(g) + \ldots \right] \log(\mu \varepsilon) + \left[ \lambda^2 G^{(2)}_{2,i}(g) + \lambda^3 G^{(3)}_{2,i}(g) + \ldots \right] \log^2(\mu \varepsilon) + \ldots \, .
\]

The functions \( G^{(n)}_{k,i}(g) \) can be expressed in terms of \( \beta_i(g) \) and its derivatives using the condition \( \mu \frac{d g_{i,b}}{d \mu} = 0 \). For example, in the case of one coupling \( g = g(\mu) \), one finds the familiar relation

\[
g_b = g - \beta(g) \log(\mu \varepsilon) + \frac{1}{2} \left[ \lambda^2 \beta(g) \beta'(g) + \lambda^3 \left( \beta(g) \beta(g) \right)' + \ldots \right] \log^2(\mu \varepsilon) \\
- \frac{1}{6} \left( \lambda^3 \beta(g)(\beta'(g))^2 + \beta(g) \beta''(g) + \ldots \right) \log^3(\mu \varepsilon) + \ldots \, .
\]
\[
\begin{align*}
\beta_1(\zeta_1, \zeta_2) &= \frac{\lambda}{8\pi^2} \zeta_1 (\zeta_1^2 + \zeta_2^2 - 1) - \frac{\lambda^2}{64\pi^4} \zeta_1 (\zeta_1^2 + \zeta_2^2 - 1)(\zeta_1^2 + \zeta_2^2 + c_1) + O(\lambda^3), \\
\beta_2(\zeta_1, \zeta_2) &= \frac{\lambda}{8\pi^2} \zeta_2 (\zeta_1^2 + \zeta_2^2 - 1) - \frac{\lambda^2}{64\pi^4} \zeta_2 (\zeta_1^2 + \zeta_2^2 - 1)(\zeta_1^2 + \zeta_2^2 + c_2) + O(\lambda^3),
\end{align*}
\]

(3.21)

where \(c_i\) are some constants to be determined. Then the circle \(\zeta_1^2 + \zeta_2^2 = 1\) (i.e. the latitude WM loop with \(\zeta = 1\) and arbitrary \(\theta_0\)) is still a line of fixed points. If it turns out that \(c_1 \neq c_2\) then for \(\zeta \neq 1\) the ratio \(\frac{\zeta_1}{\zeta_2}\) or \(\theta_0\) starts running at two loops.

4. Winding generalization of deformed \(\frac{1}{4}\)-BPS WM loop

Another generalization of the latitude WM loop is obtained by introducing windings of the \(x^\mu\) and \(n^m\) contours in (1.4):

\[
\begin{align*}
x^\mu(\tau) &\rightarrow x^\mu(\mathbf{v}_1 \tau), \\
n^m(\tau) &\rightarrow n^m(\mathbf{v}_2 \tau),
\end{align*}
\]

(4.1)

where \(v_1, v_2\) are integers. Let us denote by \(\langle W^{(\zeta_1, \zeta_2)}(\mathbf{v}_1, \mathbf{v}_2; \lambda)\rangle\) the expectation value of the resulting \(\zeta\)-deformed loop. Local supersymmetry requires \(\zeta = 1\) as a necessary condition, i.e. any loop with \(n^m n_m = 1\) is locally supersymmetric for generic values of \(v_1\) and \(v_2\). Global \(\frac{1}{4}\)-supersymmetry is present only if \((\zeta_1, \zeta_2) = (0, 1)\) (\(\zeta = 1, \theta_0 = \frac{\pi}{2}\)) and \(v_1 = v_2\). At one loop order we have

\[
\begin{align*}
\langle W^{(\zeta_1, \zeta_2)}(\mathbf{v}_1, \mathbf{v}_2)\rangle &= 1 + \lambda \langle W^{(\zeta_1, \zeta_2)}(\mathbf{v}_1, \mathbf{v}_2)\rangle + O(\lambda^2), \\
\langle W^{(\zeta_1, \zeta_2)}(\mathbf{v}_1, \mathbf{v}_2)\rangle &= \frac{v_1^2}{8\pi^2} \int_{\tau_1 > \tau_2} d\tau_1 d\tau_2 \frac{\zeta_1^2 + \zeta_2^2 \cos(\nu_2 \tau_{12}) - \cos(\nu_2 \tau_1 \tau_2)}{4 \sin^2 \frac{\nu_2 \tau_{12}}{2}}.
\end{align*}
\]

(4.2)

(4.3)

Expanding in modes, we have

\[
\begin{align*}
W_{1}^{(\zeta_1, \zeta_2)}(\mathbf{v}_1, \mathbf{v}_2) &= \frac{v_1^2}{8\pi^2} \sum_{n=1}^{\infty} (-n) \int_{\tau_1 > \tau_2} d\tau_1 d\tau_2 \left[\zeta_1^2 + \zeta_2^2 \cos(\nu_2 \tau_{12}) - \cos(\nu_1 \nu_2 \tau_1 \tau_2)\right] \cos(n \nu_1 \tau_1). \\
\end{align*}
\]

(4.4)

Assuming \(v_1 \neq v_2\), the integral is non-zero for \(n = 1, \frac{v_2}{v_1}\). The second contribution is present only if \(\frac{v_2}{v_1} \notin \mathbb{N}\). Thus

\[
\begin{align*}
W_{1}^{(\zeta_1, \zeta_2)}(\mathbf{v}_1, \mathbf{v}_2) &= \begin{cases} \\
\frac{v_1^2}{8}, & \frac{v_2}{v_1} \notin \mathbb{N}, \\
\frac{v_1^2}{8} \left(1 - \zeta_2^2 \frac{v_2}{v_1}\right), & \frac{v_2}{v_1} \in \mathbb{N}.
\end{cases}
\end{align*}
\]

(4.5)

This result can be cross checked with the more conventional dimensional regularization approach, see Appendix E. Notice that (4.5) can be written in the form

\[
\begin{align*}
W_{1}^{(\zeta_1, \zeta_2)}(\mathbf{v}_1, \mathbf{v}_2) &= \frac{v_1^2}{8} F\left(\frac{v_2}{v_1}\right), \\
F(x) &= \begin{cases} \\
\frac{1}{8}, & x \text{ irrational,} \\
\frac{1}{8} \left(1 - \zeta_2^2 x\right), & x \text{ rational.}
\end{cases}
\end{align*}
\]

(4.6)

---

18 The result turns out to be finite so we do not need to introduce an exponential damping factor in the sum.

19 Inspection of the special case \(v_1 = v_2\) reveals that the same expression is still valid.
If $\xi_1 = 0, \xi_2 = 1$, the correction vanishes for $\nu_1 = \nu_2$ which corresponds to multiply wound $\frac{1}{4}$-supersymmetric loop. The opposite case of $\xi_1 = 1, \xi_2 = 0$ describes the multiply wound $\frac{1}{2}$-BPS WM loop (with no dependence on $\nu_2$): here the one-loop contribution is simply multiplied by $\nu_1^2$. This replacement rule $\lambda \rightarrow \nu \lambda_1^2$ extends to all orders as follows from the matrix model solution [19,20].

In the remaining part of this section, we shall restrict consideration to the case of

$$\nu_1 = 1, \quad \nu_2 \equiv \nu \in \mathbb{N}^+, \quad (\xi_1, \xi_2) = (0, \xi). \quad (4.7)$$

The choice of $(\xi_1, \xi_2) = (0, \xi)$ corresponds to 1-parameter deformation of the $\frac{1}{4}$-supersymmetric WM loop [12] (with trivial expectation value for $\xi = 1$). According to (4.5), at one-loop order we get

$$(W^{(0,\xi)}(1, \nu)) = 1 + \frac{\lambda}{8} (1 - \xi^2 \nu) + O(\lambda^2). \quad (4.8)$$

In the winding generalization of the supersymmetric case ($\xi = 1$) we get a non-trivial 1-loop contribution $\frac{\lambda}{8} (1 - \nu)$: the choice of unequal winding numbers $\nu_1 = 1$ and $\nu_2 = \nu > 1$ breaks space–time supersymmetry and thus $W(\nu) \equiv W^{(0,1)}(1, \nu)$ is then a non-trivial function of $\nu$ and $\lambda$.

To determine what happens at two loops, we shall first consider the ladder diagrams and then include all other contributions indirectly using information from defect CFT$_1$ corresponding to perturbations near the $\xi = 0$ conformal point.

4.1. Two-loop ladder contribution

The two ladder diagrams contributing at two loops are (14)(23) and (12)(34). From the results of Appendix D, we obtain (here $\xi$ is the bare coupling)

$$(W^{(0,\xi)}(1, \nu))_{\text{ladder}} = 1 + \frac{\lambda}{8} (1 - \xi^2 \nu) + \lambda^2 \left[ \frac{(1 - \xi^2 \nu)^2}{192} + \frac{1 + \xi^2 (2 - \nu)(2 \nu - 1)}{128 \pi^2} \right.$$

$$+ \frac{\xi^2 (1 + \nu - \xi^2 \nu)}{32 \pi^2} \left[ \psi(\nu) - \psi(1) \right] + \frac{\xi^2 \nu (2 - \xi^2 \nu)}{64 \pi^2} \left[ \psi'(\nu) - \psi'(1) \right]$$

$$- \frac{(\xi^2 - 1)(\xi^2 \nu - 1)}{32 \pi^2} \log \varepsilon \bigg] + O(\lambda^3). \quad (4.9)$$

where $\xi$ stands for bare coupling and $\psi(\nu) = \Gamma'(\nu)/\Gamma(\nu)$.

The two-loop terms $\sim \lambda^2 \xi^0$ and $\sim \lambda^2 \xi^2$ potentially receive contributions from non-ladder diagrams too. Instead, the terms $\sim \lambda^2 \xi^4$ come only from ladder diagrams. A special limit where non-ladder diagrams may be neglected is that of $\xi \gg 1$ with $\lambda \xi^2$–fixed. The coefficient of the UV logarithm is linear in $\lambda^2 \xi^4 \nu$ and is absorbed by a shift in the one-loop term with $\beta \sim \lambda \xi^3$ having no dependence on $\nu$, i.e. being the same as the previous one for $\nu = 1$. Introducing the

\[\text{Footnote 20: Let us add a technical remark. For } \nu_1 \neq 1, \text{ the treatment of the planar restriction on diagrams becomes quite involved at higher loops. The reason is that we are computing integrals like } \int_0^{2\pi} dx_1 \ldots dx_n (\Phi(x_1) \ldots \Phi(x_n))_{\text{planar}} \text{ and, if } x(\tau) \text{ describes a multiply wound circle, then the domain of integration has to be properly split in order to select the correct planar contractions. A detailed discussion of this issue can be found in [21].}\]
renormalization scale $\mu_{\text{MR}}$ (corresponding to the mode regularization scheme, see discussion after (2.15)) we find for the renormalized expression\(^{21}\) (here $\zeta$ is the renormalized coupling)

\[
(W^{(0,\zeta)}(1, \nu)) \overset{\zeta \gg 1}{=} 1 - \frac{\lambda}{8} \zeta^2 \nu + \lambda^2 \zeta^4 \left[ \frac{\nu^2}{192} + \frac{2 \nu - 1}{128 \pi^2} \right] - \frac{\nu \left[ \psi(\nu) - \psi(1) \right]}{32 \pi^2} - \frac{\nu^2 \left[ \psi'(\nu) - \psi'(1) \right]}{64 \pi^2} - \frac{\nu}{32 \pi^2} \log(\mu_{\text{MR}} R) + \ldots.
\]

(4.10)

At $\zeta = 1$, the two-loop extension of (4.8) due to ladder diagrams only reads

\[
\mathcal{W}(\nu)_{\text{ladder}} = (W^{(0,1)}(1, \nu))_{\text{ladder}} = 1 + \frac{\lambda}{8} (1 - \nu) + \lambda^2 W_2(\nu) + O(\lambda^3),
\]

(4.11)

where $W_2(\nu)$ is the following non-trivial function\(^{22}\)

\[
W_2(\nu) = \frac{1}{64 \pi^2} \left[ (1 - \nu) + \frac{\pi^2}{3} (1 - \nu)^2 + 2 [\psi(\nu) - \psi(1)] + \nu (2 - \nu) [\psi'(\nu) - \psi'(1)] \right].
\]

(4.12)

Its explicit values for $\nu = 1, 2, 3, \ldots$ are $W_2(\nu) = 0, 1 \frac{1}{192}, 1 \frac{1}{48} + \frac{19}{256 \pi^2}, \ldots$.

\subsection*{4.2. Extracting non-ladder contribution from $\zeta = 0$ defect $\text{CFT}_1$}

In (3.13), we have obtained the complete two-loop contribution to $\langle W^{(\zeta_1, \zeta_2)} \rangle$ without winding evaluated in dimensional regularization. The scheme dependence is contained in the constant $a = 5 + 4 \log(\mu_{\text{DR}} R)$, where $\mu_{\text{DR}}$ corresponds to dimensional regularization. The extension to a non trivial winding $(\nu_1, \nu_2) = (1, \nu)$ would require the evaluation from scratch of all (ladder, self-energy and internal vertex) two-loop diagrams within the same scheme.

Nevertheless, there is a shortcut that bypasses the actual evaluation of the non-ladder diagrams. In general, $(W^{(0,\zeta)}(1, \nu))$ has $\zeta^0$, $\zeta^2$, and $\zeta^4$ contributions. The $\zeta^0$ contribution can not depend on the $\nu_1$-winding (as for $\zeta = 0$ the scalar coupling in (1.5) vanishes) and thus can be computed at $\nu = 1$. The $\zeta^4$ contribution comes only from ladder diagrams and may be found using mode regularization, \textit{i.e.} from (4.10). Finally, the $\zeta^2$ contribution can be extracted by a $\zeta = 0$ defect $\text{CFT}_1$ where all the necessary data does not depend on $\nu$ and can be fixed by comparison with the complete result (3.13) evaluated in dimensional regularization. As a last step, the relation between the dimensional regularization and the mode regularization schemes, \textit{i.e.}

\footnote{Another interesting regime is $\nu \gg 1$ where we have from (4.9)

\[
(W^{(0,\zeta)}(1, \nu))_{\text{ladder}} \overset{\nu \gg 1}{=} 1 - \frac{\lambda}{8} \zeta^2 \nu + \frac{\lambda^2}{128} \zeta^4 \nu^2 + O(\lambda \nu^{n-1}).
\]

Possible additional contributions from non-ladder diagrams involve smaller powers of $\zeta$ at each loop.}

\footnote{The large $\nu$ expansion of (4.12) is

\[
W_2(\nu) = \frac{\lambda^2}{128} \left[ \nu^2 - 2 \left( 1 + \frac{2}{\pi^2} \right) \nu + \frac{4 (\log \nu + \gamma_E)}{\pi^2} + \frac{5}{3} + \frac{2}{3} + O(\nu^{-1}) \right].
\]
between $\mu_{\text{DR}}$ and $\mu_{\text{MR}}$ can be found via a suitable analysis of finite renormalizations in the two schemes.

4.2.1. Using CFT data

We begin with the detailed determination of the $\zeta^2$ contribution. Starting with the winding generalization of the scalar coupling in (1.5)\n\[ \zeta_1 \Phi^3 + \zeta_2 [\cos(\nu \tau) \Phi^1 + \sin(\nu \tau) \Phi^2], \quad (4.13) \]
we may view the expansion in powers of $\zeta_1, \zeta_2$ as perturbation theory near the conformal point $\zeta_1 = \zeta_2 = 0$ $\zeta_1 = \zeta_2 = 0$ where all 6 scalar fields $\{\Phi^m\}$ are equivalent [2]. The scalars restricted to the circular loop correspond to operators in 1d CFT (which we will also denote as $\Phi$). In the products of couplings and operators may involve the bare or renormalized quantities, $\zeta_b \Phi_1 = \zeta_1 \Phi_1$. Here it is convenient to work with the renormalized operators and couplings (suppressing the label “r”). In particular, the two-point function of the CFT operators corresponding to the scalar fields restricted to the circular loop of radius $R$ and renormalized at scale $\mu_{\text{DR}}$, reads (here $|s_{12}| = |x(\tau_1) - x(\tau_2)| = 2R \sin \frac{s_{12}}{2}$)\n\[ \langle \Phi^m(\tau_1) \Phi^m(\tau_2) \rangle \delta^{mn} \frac{\mu_{\text{DR}}^2 C_0(\lambda)}{2 \mu_{\text{DR}} R \sin \frac{s_{12}}{2} \Delta} = \delta^{mn} \frac{C_0(\lambda)}{|s_{12}|^2} \frac{1}{|\mu_{\text{DR}} s_{12}|^2 (\Delta - 1)}, \quad (4.14) \]
\[ \Delta = 1 - \frac{\lambda}{8 \pi^2} + O(\lambda^2), \quad C_0(\lambda) = \frac{\lambda}{8 \pi^2} - C_0^{(1)} \frac{\lambda^2}{\pi^2} + O(\lambda^3). \quad (4.15) \]
Expanding the expectation value $\langle W^{(0,\zeta)}(1, \nu) \rangle$ at small $\zeta$ gives\n\[ \log \langle W^{(0,\zeta)}(1, \nu) \rangle = \log \langle W^{(0,0)} \rangle + \zeta^2 D_2(\nu) + O(\zeta^4), \quad (4.16) \]
where the first term is independent on $\nu$ and was already computed in [2] at two loops\n\[ \langle W^{(0,0)} \rangle = 1 + \frac{\lambda}{8} + \lambda^2 \left( \frac{1}{192} + \frac{1}{128 \pi^2} \right) + O(\lambda^3). \quad (4.17) \]
The second term in (4.16) is expressed in terms of the CFT $D_2$ 2-point function $D_2(\nu) = \frac{1}{2} 2\pi \int_0^{2\pi} d\tau_1 d\tau_2 \langle \Phi(\tau_1) \Phi(\tau_2) \rangle$, $\Phi(\tau) = \cos(\nu \tau) \Phi^1 + \sin(\nu \tau) \Phi^2$. (4.18)
Using (4.15), we can write it as follows\n\[ D_2(\nu) = \frac{1}{2} \mu_{\text{DR}}^{-2 (\Delta - 1)} C_0(\lambda) \int_0^{2\pi} d\tau_1 d\tau_2 \cos(\nu \tau_1) \cos(\nu \tau_2) + \sin(\nu \tau_1) \sin(\nu \tau_2) \quad (4.19) \]
\[ = \mu_{\text{DR}}^{-2 (\Delta - 1)} 2^{-2 \Delta(\lambda)} \pi C_0(\lambda) \int_0^{2\pi} d\tau \cos(\nu \tau) \left( \sin \frac{\tau}{2} \right)^{-\Delta(\lambda)}. \quad (4.19) \]
\[ ^{23} \text{We omit the trivial overall} \ 1/R^2 \text{ factor, but will keep an explicit} \ R \text{ in the log}(\mu_{\text{DR}} R) \text{ below.} \]
This integral can be computed using the method discussed in Appendix E:\textsuperscript{24}
\[
D_2(\nu) = -\frac{\lambda}{8} \nu + \lambda^2 \left[ -\nu \log(\mu_{\text{DR}} R) - \nu C_0^{(1)} + \frac{1 - 2 \nu + 2 \nu [\psi(\nu) - \psi(1)]}{64 \pi^2} \right] + O(\lambda^3).
\]
(4.20)

To determine $C_0^{(1)}$, \textit{i.e.} the second order correction to the 2-point function normalization in (4.15), let us specialize (4.20) to $\nu = 1$
\[
D_2(1) = -\frac{\lambda}{8} + \lambda^2 \left[ -\log(\mu_{\text{DR}} R) - C_0^{(1)} - \frac{1}{64 \pi^2} \right] + O(\lambda^3).
\]
(4.21)

This can be compared to the coefficient of $\zeta_2^2$ in the expansion of $\log(W^{(0,\zeta_2)})$, \textit{i.e.} using the DR result (3.13)
\[
\log(W^{(0,\zeta_2)}) = \log(W^{(0,0)}) + \zeta_2^2 \left[ -\frac{\lambda}{8} + \lambda^2 \left( \frac{1}{192} - \frac{3}{64 \pi^2} - \frac{\log(\mu_{\text{DR}} R)}{32 \pi^2} \right) \right] + O(\lambda^3)
\]
(4.22)

Comparing (4.21) and (4.22) gives
\[
C_0^{(1)} = -\frac{1}{192} + \frac{1}{32 \pi^2}.
\]
(4.23)

4.2.2. Matching regularization schemes

Next, we need to find a relation between the mode regularization scale $\mu_{\text{MR}}$ appearing in (4.10), and the dimensional regularization scale $\mu_{\text{DR}}$ appearing in (4.20). This can be achieved by comparing the $\zeta_4^4$ term of the DR expression (3.13) at $\zeta_1 = 0$ with the $\nu = 1$ values of the MR result (4.10). One gets
\[
\log(\mu_{\text{DR}} R) + \log(\mu_{\text{MR}} R) + 1 = 0.
\]
(4.24)

4.2.3. Reconstruction of full $\langle W^{(0,\zeta)}(1, \nu) \rangle$

As a final step, collecting all terms that make up the full expression of $\langle W^{(0,\zeta)}(1, \nu) \rangle$ and exponentiating (4.16) we obtain
\[
\langle W^{(0,\zeta)}(1, \nu) \rangle = \zeta^0 \left[ 1 + \frac{\lambda}{8} + \left( \frac{1}{192} + \frac{1}{128 \pi^2} \right) + O(\lambda^3) \right]
\]
\[
+ \zeta^2 \left\{ -\frac{\lambda}{8} \nu - \frac{\lambda^2}{64} + \lambda^2 \left[ -\nu \log(\mu_{\text{DR}} R) \right] - \nu C_0^{(1)} + \frac{1 - 2 \nu + 2 \nu [\psi(\nu) - \psi(1)]}{64 \pi^2} \right\} + O(\lambda^3)
\]
\[
+ \zeta^4 \left\{ \frac{\lambda^2}{192} + \frac{2 \nu - 1}{128 \pi^2} - \frac{\nu [\psi(\nu) - \psi(1)]}{32 \pi^2} - \frac{\nu^2 [\psi'(\nu) - \psi'(1)]}{64 \pi^2} \right. \]
\[
- \frac{\nu}{32 \pi^2} \log(\mu_{\text{MR}} R) \right\} + O(\lambda^3)
\]
(4.25)

where we have taken the $\zeta^4$ term from the mode regularization calculation (4.10). After some regrouping, and replacing $\mu_{\text{MR}}$ by $\mu_{\text{DR}}$ using (4.24), we finally obtain

\textsuperscript{24} Notice that the second order correction to $\Delta$ does not appear at this order since at weak coupling $C_0(\lambda)$ starts at order $\lambda$. 
\[ (W^{(0,\zeta)}(1, \nu)) = 1 + \frac{\lambda}{8} (1 - \zeta^2 \nu) \]
\[ + \frac{\lambda^2}{192} \left[ -\frac{1}{192} + \frac{\zeta^2 \nu (1 + \zeta^2 \nu)}{192} - \frac{\zeta^4 - 2\zeta^2 - 1}{128 \pi^2} + \frac{\zeta^2 (3\zeta^2 - 4) \nu}{64 \pi^2} \right. \]
\[ - \frac{\nu \zeta^2 (\zeta^2 - 1) [\psi'(\nu) - \psi(1)]}{32 \pi^2} - \frac{\zeta^4 \nu^2 [\psi'(\nu) - \psi'(1)]}{64 \pi^2} \]
\[ + \frac{\nu \zeta^2 (\zeta^2 - 1)}{32 \pi^2} \log(\mu_{\text{DR}} R) \bigg] + \mathcal{O}(\lambda^3). \quad (4.26) \]

4.3. Winding deformation of the $\frac{1}{4}$-supersymmetric loop

Setting $\zeta = 1$ gives the expectation value of the $\frac{1}{4}$-supersymmetric WM loop generalized to winding number $\nu$ along the big circle of $S^5$ ($\theta_0 = \frac{\pi}{2}$). For $\nu > 1$ global supersymmetry is broken, so we expect to get a non-trivial function of $\lambda$. We find from (4.26) the following finite two-loop expression

\[ \mathcal{W}(\nu) \equiv \langle W^{(0,1)}(1, \nu) \rangle = 1 + \frac{\lambda}{8} (1 - \nu) + \lambda^2 \left[ -\frac{(1 - \nu)(\nu + 2)}{192} - \frac{1 - \nu}{64 \pi^2} \right. \]
\[ - \frac{\nu^2 [\psi'(\nu) - \psi'(1)]}{64 \pi^2} \bigg] + \mathcal{O}(\lambda^3). \quad (4.27) \]

Then $\mathcal{W}(0)$ is the standard Wilson loop expectation value (4.17), $\mathcal{W}(1) = 1$ while, e.g., for $\nu = 2$ and $3$ we get

\[ \mathcal{W}(2) = 1 - \frac{\lambda}{8} + \frac{\lambda^2}{48} \left( \frac{1}{48} + \frac{3}{64 \pi^2} \right) + \mathcal{O}(\lambda^3), \]
\[ \mathcal{W}(3) = 1 - \frac{\lambda}{4} + \frac{\lambda^2}{96} \left( \frac{5}{96} + \frac{37}{256 \pi^2} \right) + \mathcal{O}(\lambda^3). \quad (4.28) \]

Since this case ($\zeta = 1$) corresponds to locally supersymmetric WM loop in finite $\mathcal{N} = 4$ SYM theory, the function $\mathcal{W}(\nu)$ should be finite at all orders of expansion in $\lambda$. It would be very interesting to compute it exactly (possibly using integrability). To recall, for the circular WM loop (corresponding to contour in $S^5$ shrinking to a point, or $\theta_0 = 0$) the winding of the space–time circle can be completely absorbed into a rescaling of $\lambda$ (as seen, e.g., from the matrix model representation [19]). On the other hand, the unwound $\frac{1}{4}$-supersymmetric loop is trivial, $\mathcal{W}(1) = 1$ [12].

Another interesting question is to try to understand the behaviour of (4.27) at strong coupling using $AdS_5 \times S^5$ string theory picture. At leading order in inverse string tension, one may generalize the discussion in [12] to any winding $\nu$ in the big circle of $S^5$. In this case the induced world-sheet metric is (here $\tau \in (0, 2\pi)$, $\sigma \in (0, \infty)$)

\[ ds^2 = \left[ \frac{1}{\sinh^2 \sigma} + \frac{\nu^2}{\cosh^2 (\nu \sigma)} \right] (d\tau^2 + d\sigma^2), \quad (4.29) \]
interpolating between the $AdS_2$ metric (for $\nu = 0$, corresponding to the standard circular loop in $AdS_5$) and the $\nu = 1$ case when the world sheet ends also on a big circle of $S^5$ [12]. The string action proportional to the regularized area is then ($T = \frac{\sqrt{\lambda}}{2\pi}$)

$$I = 2\pi T \int_{\varepsilon}^{\infty} d\sigma \left[ \frac{1}{\sinh^2\sigma} + \frac{\nu^2}{\cosh^2(\nu\sigma)} \right] = \frac{\sqrt{\lambda}}{\varepsilon} - \sqrt{\lambda}(1 - |\nu|) + O(\varepsilon). \quad (4.30)$$

The renormalized action leads to the following prediction for the behaviour of (4.27) at strong coupling:

$$\mathcal{W}(\nu) \stackrel{\lambda \gg 1}{=} e^{\sqrt{\lambda}(1-|\nu|)}. \quad (4.31)$$

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Appendix A. Calculation of three-loop ladder diagrams in mode regularization

Below we will summarize some details of computation of higher loop ladder graphs in the case of the circular Wilson loop. These diagrams correspond to “rainbow” diagrams in the case of a straight line and we shall plot them this way below for simplicity. By “scalar ladders” we will mean the ladder diagrams with only scalar propagators (i.e. coming from the first term in $\mathcal{G}(\tau)$ in (2.2)); they give the only relevant contributions in the large $\zeta$ limit.

A.1. Two-loop scalar ladders

Let us first recall a simple computation of 2-loop scalar ladders from [2]. Let us consider the $\zeta^2$ term in the (14)(23) ladder diagram (for notation see (2.3))

$$I = \int [d^4 \tau] \mathcal{D}(\tau_{14}) \mathcal{D}(\tau_{23}), \quad \mathcal{D}(\tau) = \frac{1}{4 \sin^2 \frac{\tau}{2}}. \quad (A.1)$$

Using the regularized expression for $\mathcal{D}(\tau)$ in (2.4) we get (following the notation in (2.7))

$$I_{(14)(23)}^{(2)} = \sum_{n,m=1}^{\infty} e^{-\varepsilon (n+m)} S_{n,m}, \quad S_{n,m} = nm \int [d^4 \tau] \cos(n \tau_{14}) \cos(m \tau_{23}). \quad (A.2)$$

After integration, we find

---

25 In general, in the case of two non-trivial windings $v_1$ and $v_2$ the euclidean solution in conformal gauge has the $AdS_5$ part (in Poincare coordinates) as $x^\mu = \frac{1}{\cosh(v_1 \sigma)}(\cos(v_1 \tau), \sin(v_1 \tau), z = \tanh(v_1 \sigma)$ and the $S^5$ part as $\varphi = v_2 \tau, \cos \theta = \tanh(v_2 \sigma)$. Then the induced metric is $ds^2 = \left(\frac{v_1^2}{\sinh^2(v_1 \sigma)} + \frac{v_2^2}{\cosh^2(v_2 \sigma)}\right)(d\tau^2 + d\sigma^2)$. The resulting finite part of the string action is $I_{\text{fin}} = -\sqrt{\lambda}(|v_1| - |v_2|)$ which vanishes in the supersymmetric case $v_1 = v_2$ (cf. [16]).
\[ S_{n,m} = \frac{1}{2mn(m^2 - n^2)^2} \left[ 2m^2(m^2 - 3n^2) - 2(m^2 - n^2)^2 \cos(2\pi n) \\
+ n^2(m + n)^2 \cos(2\pi(m - n)) + n^2(m - n)^2 \cos(2\pi(m + n)) \right]. \] (A.3)

For generic integer \( n, m \) one checks that \( S_{n,m} = 0 \). The removable singularity occurring for \( n = m \) must be considered separately. In this case, one finds \( S_{n,n} = -\frac{\pi}{2} \). Thus

\[ I(\frac{2}{3})_{(14)(23)} = -\frac{1}{2} \pi^2 (\coth \varepsilon - 1) = -\frac{\pi^2}{2\varepsilon} + \frac{\pi^2}{2} + \mathcal{O}(\varepsilon). \] (A.4)

Dropping the power divergence (which is equivalent to \( \zeta \)-function regularization) we reproduce the coefficient of the \( \zeta^4 \) term in \( I(\frac{2}{3})_{(2.10)} \) (other terms in (2.10) get contributions also from ladder diagrams with vector propagators).

### A.2. Three-loop scalar ladders

- \((16)(25)(34)\) (Fig. 1)

![Ladder Diagram](image)

Fig. 1. Ladder diagram associated with the \((16)(25)(34)\) contraction. Here and in the following pictures, the horizontal thick line represents the loop parametrized by \( \tau \in [0, 2\pi] \). Scalar propagators are half-circles attached to the loop.

At three loops we have similarly

\[ I(\frac{3}{2})_{(16)(25)(34)} = \int [d^6 \tau] D(\tau_{16}) D(\tau_{25}) D(\tau_{34}). \] (A.5)

Using again (2.4), we find

\[ I(\frac{3}{2})_{(16)(25)(34)} = \sum_{n,m,s=1}^\infty e^{-\varepsilon(n+m+s)} S_{n,m,s}, \] (A.6)

\[ S_{n,m,s} = -nms \int [d^6 \tau] \cos(n \tau_{16}) \cos(m \tau_{25}) \cos(s \tau_{34}). \] (A.7)

The integral in (A.7) is non zero in the following cases

\[
S_{n,m,s} = \begin{cases} 
\frac{\pi^2}{s}, & m = n, m \neq s, s \neq 2m, \\
-\frac{\pi^2}{n}, & m = s, m \neq n, n \neq 2m, \\
-\frac{m^2}{2n(m+n)}, & s = m + n, m \neq n, \\
\frac{\pi^2}{4m}, & s = 2m, m = n, \\
-\frac{m^2}{2n(m-n)}, & m = s + n, \\
\frac{m^2}{2n(m-n)}, & n = m + s, m \neq s, \\
-\frac{3\pi^2}{4m}, & n = 2m, m = s, \\
0, & \text{else}.
\end{cases}
\] (A.8)
Due to symmetry of the sum in (A.6) the contributions of the first two cases in (A.8) mutually cancel. The other cases give

\[
I^{(3)}_{(16)(25)(34)}(\varepsilon) = -\frac{\pi^2}{2} \sum_{n,m=1}^{\infty} e^{-2\varepsilon (n+m)} \frac{m}{n (n+m)} + \sum_{m=1}^{\infty} e^{-4\varepsilon m} \left( \frac{\pi^2}{4m} + \frac{\pi^2}{4m} \right) \\
- \frac{\pi^2}{2} \sum_{n,s=1}^{\infty} e^{-2\varepsilon (n+s) s + n} \frac{s+n}{n s} - \frac{\pi^2}{2} \sum_{m,s=1}^{\infty} e^{-2\varepsilon (m+s) m + n} \frac{m}{(m+s) s} \\
+ \sum_{s=1}^{\infty} e^{-4\varepsilon s} \left( -\frac{3\pi^2}{4s} + \frac{\pi^2}{4s} \right) \\
= \frac{1}{2} \pi^2 \left( \left[ 1 + 2 \log(1 - e^{-2\varepsilon}) \right] \coth \varepsilon - 1 \right) \\
= \frac{1}{\varepsilon} \left( \pi^2 \log \varepsilon + \pi^2 \log 2 + \frac{\pi^2}{2} \right) - \frac{3\pi^2}{2} + O(\varepsilon). \tag{A.9}
\]

\(\bullet (12)(34)(56)\) (Fig. 2)

\[
\sum_{n,m,s=1}^{\infty} e^{-3\varepsilon s} \frac{s+n}{n s} = \frac{1}{2} \pi^2 \log(1 - e^{-3\varepsilon}) = \frac{\pi^2}{2} \log \varepsilon + \frac{\pi^2}{2} \log 3 + O(\varepsilon). \tag{A.11}
\]

The computational procedure should now be clear. In the following, we shall just present the relevant expressions without detailed comments.

\(\bullet (12)(36)(45)\) (Fig. 3)

\[
S_{n,m,s} = \begin{cases} 
\frac{\pi^2}{2s}, & m = n = s, \\
0, & \text{else}.
\end{cases} \tag{A.12}
\]
\[ I^{(3)}_{(12)(36)(45)}(\varepsilon) = \pi^2 \sum_{n,m=1}^{\infty} \frac{e^{-\varepsilon (n+2m)}}{n} = -\frac{\pi^2 \log (1 - e^{-\varepsilon})}{e^{2\varepsilon} - 1} \]
\[ = -\frac{\pi^2 \log \varepsilon}{2\varepsilon} + \frac{\pi^2}{2} \log \varepsilon + \frac{\pi^2}{4} + \mathcal{O}(\varepsilon). \]  

(A.13)

- \((14)(23)(56)\)
\[ S_{n,m,s} = \begin{cases} \frac{\pi^2}{s}, & m = n, \\ 0, & \text{else}. \end{cases} \]  

(A.14)

\[ I^{(3)}_{(14)(23)(56)}(\varepsilon) = \pi^2 \sum_{m,s=1}^{\infty} \frac{e^{-\varepsilon (s+2m)}}{s} = -\frac{\pi^2 \log (1 - e^{-\varepsilon})}{e^{2\varepsilon} - 1} \]
\[ = -\frac{\pi^2 \log \varepsilon}{2\varepsilon} + \frac{\pi^2}{2} \log \varepsilon + \frac{\pi^2}{4} + \mathcal{O}(\varepsilon). \]  

(A.15)

This is equal to the \((12)(36)(45)\) expression \((A.13)\). Indeed, the two diagrams are mirror images under the reflection around \(\tau = \pi\), and the measure in the integral in \((2.3)\) is also invariant under this reflection.

- \((16)(23)(45)\) (Fig. 4)
\[ S_{n,m,s} = \begin{cases} \frac{\pi^2}{s} \frac{s}{s^2 - n^2}, & m = n \neq s, \\ \frac{\pi^2 m}{m^2 - s^2}, & s = n \neq m, \\ \frac{\pi^2}{s}, & n = m = s, \\ 0, & \text{else}. \end{cases} \]  

(A.16)

\[ I^{(3)}_{(16)(23)(45)}(\varepsilon) = \pi^2 \sum_{s=1}^{\infty} \frac{e^{-3\varepsilon s}}{s} + 2 \pi^2 \sum_{m,n=1}^{\infty} \frac{e^{-\varepsilon (m+2n)}}{m^2 - n^2} \]
\[ = \frac{\pi^2}{2} \sum_{s=1}^{\infty} \frac{e^{-3\varepsilon s}}{s^2} + \pi^2 \sum_{m,n=1}^{\infty} \frac{e^{-\varepsilon (m+2n)}}{m^2 - n^2} \]
\[ = \frac{\pi^2}{2(e^{3\varepsilon} - 1)} \left[ - (e^{3\varepsilon} - 1) \log(1 - e^{-3\varepsilon}) + 2(e^{3\varepsilon} + 1) \log(1 - e^{-2\varepsilon}) \right. \]
\[ + 4e^{\varepsilon} (e^{\varepsilon} + 1) \coth^{-1}(2e^{\varepsilon} + 1) - 4 \log(1 - e^{-\varepsilon}) \right] \]
\[ = \frac{4\pi^2 \log 2}{3\varepsilon} + \frac{\pi^2}{2} \log \varepsilon - \frac{\pi^2}{2} \log 3 - \frac{2\pi^2}{3} + \mathcal{O}(\varepsilon). \]  

(A.17)
• **Total three-loop scalar ladder contribution:**

\[
I_{\text{total}}^{(3)}(\varepsilon) = \frac{\pi^2}{\varepsilon} \left( \frac{1}{2} + \frac{7}{3} \log 2 \right) + 2\pi^2 \log \varepsilon - \frac{5\pi^2}{3} + O(\varepsilon).
\]  

(A.18)

A.3. Full \(\zeta\)-dependent two-loop and three-loop ladder diagram contributions

To determine the terms with subleading powers of \(\zeta\) we need also to include contributions with vector exchanges in (2.1), (2.2). This is straightforward to do using the relation (2.6) which allows one to reduce the result to a combination of lower-loop scalar ladder contributions. For example, at two loops we have

\[
(\zeta^2 - \cos \tau) \mathcal{D}(\tau) (\zeta^2 - \cos \tau') \mathcal{D}(\tau') = \left[ (\zeta^2 - 1) \mathcal{D}(\tau) + \frac{1}{2} \right] \left[ (\zeta^2 - 1) \mathcal{D}(\tau') + \frac{1}{2} \right] 
\]

\[
= (\zeta^2 - 1)^2 \mathcal{D}(\tau) \mathcal{D}(\tau') + \frac{1}{2} (\zeta^2 - 1) [\mathcal{D}(\tau) + \mathcal{D}(\tau')] + \frac{1}{4}.
\]  

(A.19)

The first term is equivalent to the one computed in the previous section while the term linear in \(\zeta^2 - 1\) is proportional to the one loop scalar ladder.

Below we will present the results for the full two-loop and three-loop ladder contributions using the notation introduced in (2.7), (2.9).

• \((14)(23)\)

  The expression for the scalar ladder \(I_{(14)(23)}^{(2)}\) was found already in (A.4). The contribution \(I_{(14)(23)}^{(1)}\), linear in \(\zeta^2 - 1\) vanishes and thus dropping power divergences we get for the total two-loop term in (2.7), (2.9) [2]

\[
I_{(14)(23)} = \frac{\pi^2}{2} (\zeta^2 - 1)^2 + \frac{\pi^4}{6}.
\]  

(A.20)

• \((12)(34)\)

  For \(I_{(12)(34)}^{(2)}\) we find

\[
S_{n,m} = nm \int [d^4 \tau] \cos(n \tau_{12}) \cos(m \tau_{34}) = \frac{m^2 (1 - \cos 2\pi n) - n^2 (1 - \cos 2\pi m)}{mn (m^2 - n^2)}.
\]  

(A.21)

For generic integer \(n, m\) one checks that \(S_{n,m} = 0\). Considering separately the removable singularity in \(n = m\), we find that \(S_{n,n} = 0\), i.e. \(I_{(12)(34)}^{(1)} = 0\). For \(I_{(12)(34)}^{(2)}\) we get from (A.19) a single sum

\[
S_n = \int [d^4 \tau] \frac{1}{2} \left[ - n \cos(n \tau_{12}) - n \cos(n \tau_{34}) \right] = \frac{1}{n^3} - \frac{\cos(2\pi n)}{n^3} - \frac{2\pi^2}{n}.
\]  

(A.22)

There are no special limits to be considered and, replacing \(\cos(2\pi n) \rightarrow 1\), we get

\[
I_{(12)(34)}^{(1)} = -2\pi^2 \sum_{n=1}^{\infty} e^{-n \varepsilon} \frac{1}{n} = 2\pi^2 \log (1 - e^{-\varepsilon}) = 2\pi^2 \log \varepsilon + O(\varepsilon).
\]  

(A.23)

As a result,

\[
I_{(12)(34)} = 2\pi^2 (\zeta^2 - 1) \log \varepsilon + \frac{\pi^4}{6},
\]  

(A.24)
which is same as the expression in eq. (B.23) in [2].

• \((16)(25)(34)\)
  Here we need to add to the scalar ladder \(I^{(3)}_{(16)(25)(34)}\) found in (A.9) also the contributions with two propagators

\[
I^{(2)}_{(16)(25)(34)}(\varepsilon) = \sum_{n,m=1}^{\infty} e^{-\varepsilon(n+m)} S_{n,m}, \quad S_{n,m} = \begin{cases} -\frac{\pi^4}{4} + \frac{2\pi^2}{n^2}, & m = n, \\ 0, & \text{else}. \end{cases} \tag{A.25}
\]

Doing the sum gives

\[
I^{(2)}_{(16)(25)(34)}(\varepsilon) = 2\pi^2 \text{Li}_2(e^{-2\varepsilon}) - \frac{1}{6}\pi^4 (\coth \varepsilon - 1) = -\frac{\pi^4}{6\varepsilon} + \frac{\pi^4}{2} + O(\varepsilon) \tag{A.26}
\]

\(I^{(1)}_{(16)(25)(34)}\) turns out to vanish.

• \((12)(34)(56)\)
  \(I^{(2)}_{(12)(34)(56)}\) has

\[
S_{n,m} = \frac{3\pi^2}{nm}, \tag{A.27}
\]

and thus after summation

\[
I^{(2)}_{(12)(34)(56)}(\varepsilon) = 3\pi^2 \log^2(\sinh \varepsilon - \cosh \varepsilon + 1) = 3\pi^2 \log^2 \varepsilon + O(\varepsilon). \tag{A.28}
\]

We also find

\[
I^{(1)}_{(12)(34)(56)}(\varepsilon) = \sum_{n=1}^{\infty} e^{-n\varepsilon} \left(\frac{3\pi^2}{2n^3} - \frac{\pi^4}{2n}\right)
= \frac{1}{2} \left[3\pi^2 \text{Li}_3(e^{-\varepsilon}) + \pi^4 \log(\sinh \varepsilon - \cosh \varepsilon + 1)\right]
= \frac{3}{2} \pi^2 \zeta_R(3) + \frac{1}{2} \pi^4 \log \varepsilon + O(\varepsilon). \tag{A.29}
\]

• \((12)(36)(45) = (14)(23)(56)\)
  Here in \(I^{(2)}\) we have

\[
S_{n,m} = \begin{cases} \frac{2\pi^2(m^3 - 2mn^2)}{n(m^2 - n^2)^2}, & m \neq n, \\ \frac{\pi^4}{8n^2}, & m = n. \end{cases} \tag{A.30}
\]

To sum this, we symmetrize the \(m \neq n\) case and replace

\[
\frac{2\pi^2(m^3 - 2mn^2)}{n(m^2 - n^2)^2} \rightarrow \frac{\pi^2(m^4 - 4m^2n^2 + n^4)}{mn(m^2 - n^2)^2} = -\frac{\pi^2}{2(n-m)^2} + \frac{\pi^2}{2(m+n)^2} + \frac{\pi^2}{mn}. \tag{A.31}
\]

The last two terms are computed by summing over unconstrained \(n, m\) and subtracting the \(n = m\) part. The first term is twice the \(n > m\) contribution and can be found by setting \(n = m + k\) and summing over unconstrained \(k, m\). Finally, we add the \(m = n\) case of (A.30). The result is
We also find

\[
I^{(1)}_{(12)(36)(45)}(\varepsilon) = \sum_{n=1}^{\infty} e^{-n\varepsilon} \left( -\frac{\pi^2}{2n^3} - \frac{\pi^4}{6n} \right) = 2\pi^2 \text{Li}_3(e^{-\varepsilon}) + \frac{1}{3}\pi^4 \log(1 - e^{-\varepsilon})
\]

\[
= \frac{1}{3}\pi^4 \log \varepsilon + 2\pi^2 \zeta_R(3) + O(\varepsilon).
\]  

(A.33)

**• (16)(23)(45)**

Here, we have for \( I^{(2)} \)

\[
S_{n,m} = \left\{ \begin{array}{ll}
-\frac{\pi^2}{nm}, & m \neq n, \\
0, & \text{else.}
\end{array} \right.
\]

(A.34)

\[
I^{(2)}_{(16)(23)(45)}(\varepsilon) = \pi^2 \left[ \text{Li}_2(e^{-2\varepsilon}) - \log^2(1 - e^{-\varepsilon}) \right] = \frac{\pi^4}{6} - \pi^2 \log^2 \varepsilon + O(\varepsilon).
\]  

(A.35)

\( I^{(1)} \) is given by

\[
I^{(1)}_{(16)(23)(45)}(\varepsilon) = \sum_{n=1}^{\infty} e^{-n\varepsilon} \left( -\frac{\pi^2}{2n^3} - \frac{\pi^4}{6n} \right)
\]

\[
= \frac{\pi^4}{6} \log(1 - e^{-\varepsilon}) - \frac{\pi^2}{2} \text{Li}_3(e^{-\varepsilon}) = \frac{1}{6}\pi^4 \log \varepsilon - \frac{\pi^2}{2} \zeta_R(3) + O(\varepsilon).
\]  

(A.36)

Summing up the above results for different diagrams we find the full ladder diagram contribution in (2.12), (2.13).

**Appendix B. Small \( \theta_0 \) expansion of \( \langle W(\zeta_1,\zeta_2) \rangle \)**

We can write the two-loop expression of \( \langle W(\zeta_1,\zeta_2) \rangle \) in (3.13) in terms of \( \zeta \) and \( \theta_0 \) using (1.6) and expand in small \( \theta_0 \). At \( \theta_0^2 \) order we get

\[
\frac{\partial^2}{\partial \theta_0^2} \log(\langle W(\zeta_1,\zeta_2) \rangle) \bigg|_{\theta_0=0} = \frac{\zeta^2}{4} \left[ -\lambda + \lambda^2 \left( \frac{1}{24} + \frac{(\zeta^2 - 1) [1 + \log(\mu_{\text{DR}} R)]}{4\pi^2} \right) \right] + O(\lambda^3).
\]  

(B.1)

For \( \zeta = 1 \) this is just the small \( \theta_0 \) limit of the \( \frac{1}{4} \)-BPS latitude loop and in this case we know that (B.1) is proportional to the Bremsstrahlung function [22]^{26}

^{26} If \( \zeta = 1 \) in (3.13), we simply have the \( \frac{1}{4} \)-BPS latitude result which is same as the \( \frac{1}{4} \)-BPS circular loop expression with \( \lambda \rightarrow \lambda \cos^2 \theta_0 \).
\[
\frac{\partial^2}{\partial \theta_0^2} \log \langle W^{(\xi_1, \xi_2)} \rangle \bigg|_{\theta_0=0} = -\frac{\lambda}{4} + \frac{\lambda^2}{96} + \cdots = -4\pi^2 B(\lambda), \quad B(\lambda) = \frac{\sqrt{\lambda}}{4\pi^2} \frac{I_2(\sqrt{\lambda})}{I_1(\sqrt{\lambda})}.
\]

(B.2)

As is well known, this relation allows one to determine the two-point function coefficient for the conformal operators corresponding to \(\Phi^1\) and \(\Phi^2\) in the defect CFT. Eq. (B.2) is equivalent to the following correlator of the scalar fields restricted to the loop

\[
\int_0^{2\pi} d\tau \int_0^{2\pi} d\tau' \langle \Phi(\tau)\Phi(\tau') \rangle - 4\pi^2 B(\lambda),
\]

\[\Phi(\tau) = \cos \tau \Phi^1(x(\tau)) + \sin \tau \Phi^2(x(\tau)).\]

(B.3)

Conformal symmetry predicts that for the scalars \(\Phi^1\) and \(\Phi^2\) that are not coupled to the loop for \(\theta_0 = 0\) (and thus have protected dimension \(\Delta = 1\))

\[
\langle \Phi^a(\tau_1) \Phi^b(\tau_2) \rangle \delta^{ab} \frac{C_0(\lambda)}{2 \sin \frac{\xi_1^2}{2}} \Delta = 1.
\]

(B.4)

Then (B.3) gives the known result [22]

\[
C_0(\lambda) = 2 B(\lambda) = \frac{\lambda}{8\pi^2} - \frac{\lambda^2}{192\pi^2} + O(\lambda^3).
\]

(B.5)

This follows from\(^{27}\)

\[
\int_0^{2\pi} d\tau \int_0^{2\pi} d\tau' \langle \Phi(\tau)\Phi(\tau') \rangle 2C_0(\lambda) \int_0^{2\pi} d\tau \int_0^{2\pi} d\tau' \frac{\cos \tau \cos \tau'}{(4 \sin^2 \frac{\xi_1^2}{2})^{\Delta}} =
\]

\[
= 2C_0(\lambda) \int_0^{2\pi} d\tau \int_0^{2\pi} d\tau' \frac{\cos(\tau + \tau') \cos \tau'}{(4 \sin^2 \frac{\xi_1^2}{2})^{\Delta}} = 2C_0(\lambda) \int_0^{2\pi} d\tau \int_0^{2\pi} d\tau' \frac{\cos \tau \cos^2 \tau'}{(4 \sin^2 \frac{\xi_1^2}{2})^{\Delta}}
\]

\[
= 2\pi C_0(\lambda) \int_0^{2\pi} d\tau \frac{\cos \tau}{(4 \sin^2 \frac{\xi_1^2}{2})^{\Delta}} = \pi^{3/2} C_0 \frac{4^1-\Delta \Delta (\frac{1}{2} - \Delta)}{\Gamma(2 - \Delta)} \Delta \to 1 - 2\pi^2 C_0,
\]

(B.6)

and comparing with (B.3).

**Appendix C. Three-loop ladder diagram contribution to \(\langle W^{(\xi_1, \xi_2)} \rangle\)**

Let us present the full ladder diagram three-loop contribution to the \((\xi_1, \xi_2)\) generalized circular loop. Its large \(\xi_i\) limit was given in (3.14). The subleading terms in this expression (i.e. terms with non-maximal powers of \(\xi_i\)) will of course get contributions also from other non-ladder diagrams and thus their ladder expressions will be incomplete. Explicitly,

\(^{27}\) Here we need to keep generic \(\Delta\) first and send it to 1 at the end.
\begin{equation}
(W^{(\zeta_1, \zeta_2)})_{\text{ladder}} = 1 + \frac{\lambda}{8} (1 - \zeta_2^2) + \lambda^2 \left[ \frac{1}{192} (1 - \zeta_2^2)^2 + \frac{1}{128\pi^2} (\zeta_1^2 + \zeta_2^2 - 1)^2 + \frac{1}{32\pi^2} (1 - \zeta_2^2)(\zeta_1^2 + \zeta_2^2 - 1) \log \epsilon \right]
+ \lambda^3 \left[ \frac{1}{9216} (1 - \zeta_2^2)^3 + \frac{1}{768\pi^2} (1 - \zeta_2^2)(\zeta_1^2 + \zeta_2^2 - 1)^2 - \frac{1}{1536\pi^4} (\zeta_1^2 + \zeta_2^2 - 1)^2(5\zeta_1^2 + 8\zeta_2^2 - 8) + \frac{5}{512\pi^4} (1 - \zeta_2^2)^2(\zeta_1^2 + \zeta_2^2 - 1) \log \epsilon + \frac{1}{384\pi^2} (1 - \zeta_2^2)(\zeta_1^2 + \zeta_2^2 - 1)^2 \log^2 \epsilon \right] + \ldots . \tag{C.1}
\end{equation}

Assuming the minimal subtraction scheme, \textit{i.e.} removing only the logarithms of \(\epsilon\), one has the two special cases: \(\zeta_2 = 0\) and \(\zeta_1 = 0\). In the first case we get (2.13) with divergent terms omitted

\begin{equation}
(W^{(\zeta_1, 0)})_{\text{ladder}} = 1 + \frac{\lambda}{8} + \lambda^2 \left[ \frac{1}{192} + \frac{1}{128\pi^2} (1 - \zeta_1^2)^2 \right]
+ \lambda^3 \left[ \frac{1}{9216} + \frac{1}{768\pi^2} (1 - \zeta_1^2)^2 - \frac{1}{1536\pi^4} (1 - \zeta_1^2)^2(5\zeta_1^2 - 8) + \frac{5}{512\pi^4} (1 - \zeta_1^2) \zeta_1(3) \right] + \ldots , \tag{C.2}
\end{equation}

where the \(\zeta_1 \to 1\) limit is the circular loop expression (all non-ladder diagrams are known to cancel for \(\zeta_1 = 1\)). For \(\zeta_1 = 0\) we get

\begin{equation}
(W^{(0, \zeta_2)})_{\text{ladder}} = 1 + \frac{\lambda}{8} (1 - \zeta_2^2) + \lambda^2 (1 - \zeta_2^2)^2 \left( \frac{1}{192} + \frac{1}{128\pi^2} \right)
+ \lambda^3 (1 - \zeta_2^2)^3 \left[ \frac{1}{9216} + \frac{1}{768\pi^2} + \frac{1}{192\pi^4} - \frac{5}{512\pi^4} \zeta_2(3) \right] + \ldots . \tag{C.3}
\end{equation}

and again for \(\zeta_2 = 1\) we reproduce the known exact result \(W = 1\) for the \(\frac{1}{4}\)-supersymmetric loop of [12].

**Appendix D. Winding generalization of deformed \(\frac{1}{4}\)-BPS WM loop: two-loop order**

We can split the calculation of the two-loop diagrams (12)(34) and (14)(23) with winding separating the scalar and vector exchanges and the interference term appearing at two loops.

**D.1. Scalar exchanges**

This is the contribution that comes purely from the coupling \(\sim \Phi_m n^m\).

**Diagram (14)(23)**

With the same notation as in Appendix A, we find for generic integer \(\nu > 0\) (we explicitly symmetrize with respect to the exchange \(n \leftrightarrow m\))
\[ S_{n,m} = \begin{cases} \frac{\pi^2}{48} (-3 + 8 \pi^2 v^2), & n = m = v \\ \frac{-\pi^2 m^2 (m^2 + v^2)}{2 (m^2 - v^2)^2}, & n = m \neq v \\ \frac{-\pi^2 m (m + 2v)}{4(m + v)^2}, & n = m + 2v \\ \frac{-\pi^2 m (m + 2v)}{4(n + v)^2}, & m = n + 2v \\ \frac{\pi^2 m (m - 2v)}{4(m - v)^2}, & m + n = 2v, n \neq m \\ 0, & \text{else.} \end{cases} \] (D.1)

The regularized sum is (we directly put \( \varepsilon \to 0 \) in finite sums)

\[
I_{(14)(23)}^{\text{scalar}}(\varepsilon) = \frac{\pi^2}{48} (-3 + 8 \pi^2 v^2) + \sum_{n+m=2v, n \neq m} \frac{\pi^2 m (m - 2v)}{4(m - v)^2} - \sum_{m \neq v} e^{-2m \varepsilon} \frac{\pi^2 m^2 (m^2 + v^2)}{2 (m^2 - v^2)^2} - 2 \sum_{m} e^{-2(m + v) \varepsilon} \frac{\pi^2 m (m + 2v)}{4(m + v)^2}. \] (D.2)

Let us evaluate the relevant infinite sums

\[
- \sum_{m \neq v} e^{-2m \varepsilon} \frac{\pi^2 m^2 (m^2 + v^2)}{2 (m^2 - v^2)^2} = - \sum_{m=1}^{v-1} \frac{\pi^2 m^2 (m^2 + v^2)}{2 (m^2 - v^2)^2} - \sum_{m=v+1}^{\infty} e^{-2m \varepsilon} \frac{\pi^2 m^2 (m^2 + v^2)}{2 (m^2 - v^2)^2}. \] (D.3)

The first term here is finite and can be computed in closed form. The second contribution can be simplified by expressing it in terms of the basic sum

\[
S_1(\varepsilon) = \sum_{m=v+1}^{\infty} \frac{e^{-2m \varepsilon}}{(m^2 - v^2)^2} = \frac{e^{-2(v+1) \varepsilon}}{4v^3} \left[ v \Phi(e^{-2 \varepsilon}, 2, 2v + 1) + v e^{2 \varepsilon} \text{Li}_2(e^{-2 \varepsilon}) + e^{2(2v+1) \varepsilon} B_2(e^{-2 \varepsilon}, 2v + 1, 0) + e^{2 \varepsilon} \log(1 - e^{-2 \varepsilon}) \right]. \] (D.4)

Here \( \Phi(z, s, a) = \sum_{k=0}^{\infty} \frac{z^k}{|k+a|^s} \) is the Lerch transcendent function and \( B_2(a, b) = \int_0^a dt t^a (1 - t)^{b-1} \) is the Beta-function. The other remaining sum can be put into the form

\[
-2 \sum_{m} e^{-2(m + v) \varepsilon} \frac{\pi^2 m (m + 2v)}{4(m + v)^2} = -\frac{1}{4} \pi^2 e^{-2(v+1) \varepsilon} \left[ -2v^2 \Phi(e^{-2 \varepsilon}, 2, v + 1) + \coth \varepsilon + 1 \right]. \] (D.5)

Collecting everything, we find

\[
I_{(14)(23)}^{\text{scalar}}(\varepsilon) = -\frac{\pi^2}{2\varepsilon} + \frac{1}{2} \pi^2 [2v^2 \psi^{(1)}(v) + 2v - 1] + \mathcal{O}(\varepsilon). \] (D.6)
Diagram (12)(34)
In this case, the $\tau$ integral gives, for generic (positive integer) $\nu$,

$$S_{n,m} = \begin{cases} 
\frac{\pi^2}{12} \left( 3 + 2 \pi^2 v^2 \right), & n = m = \nu \\
\frac{\pi^2 m v (m^2 + v^2)}{(m^2 - v^2)^2}, & n = \nu, m \neq \nu \\
\frac{\pi^2 n v (n^2 + v^2)}{(n^2 - v^2)^2}, & m = \nu, n \neq \nu \\
0, & \text{else.}
\end{cases} \quad (D.7)$$

The regularized sum is (we set again $\varepsilon \to 0$ in finite or convergent sums)

$$I_{(12)(34)}^{\text{scalar}}(\varepsilon) = \frac{\pi^2}{12} \left( 3 + 2 \pi^2 v^2 \right) + 2 \sum_{m \neq \nu} e^{-(m+\nu) \varepsilon} \frac{\pi^2 m v (m^2 + v^2)}{(m^2 - v^2)^2}. \quad (D.8)$$

The infinite sum can be written in terms of $S_1(\varepsilon)$ defined in (D.4):

$$I_{(12)(34)}^{\text{scalar}}(\varepsilon) = \frac{1}{2} \pi^2 v \left[ \pi^2 v - 4 v \psi^{(1)}(\nu) - 4 \psi^{(0)}(\nu) - 4 \log \varepsilon - 4 \gamma_E \right] + \mathcal{O}(\varepsilon). \quad (D.9)$$

Total two-loop contribution from scalar coupling
Adding the two diagrams, and including the overall factor $\frac{\lambda^2}{(8\pi^2)^2}$, we find

$$W_{2\text{-loop}}^{\text{scalar}} = -\frac{\lambda^2}{32 \pi^2} v \log \varepsilon + \frac{\lambda^2}{128 \pi^2} \left[ \pi^2 v^2 - 2 v^2 \psi^{(1)}(\nu) - 4 \gamma_E v + 2 v - 4 v \psi^{(0)}(\nu) - 1 \right] \quad (D.10)$$

D.2. Vector exchanges and scalar–vector interference contributions

The vector contribution is independent of $v$ and may be found from the $\zeta$-independent part of (2.13)

$$W_{2\text{-loops}}^{\text{vector}} = \frac{\lambda^2}{192} + \frac{1}{128 \pi^2} - \frac{1}{32 \pi^2} \log \varepsilon. \quad (D.11)$$

The interference term coming from

$$\frac{\cos(\nu \tau) - \cos \tau}{4 \sin^2 \frac{\tau}{2}} \frac{\cos(\nu \tau') - \cos \tau'}{4 \sin^2 \frac{\tau'}{2}}, \quad (D.12)$$

may be written as

$$-\frac{\cos(\nu \tau) \cos \tau' + \cos \tau \cos(\nu \tau')}{4 \sin^2 \frac{\tau}{2} 4 \sin^2 \frac{\tau'}{2}} = -\frac{\cos(\nu \tau) + \cos(\nu \tau')}{4 \sin^2 \frac{\tau}{2} 4 \sin^2 \frac{\tau'}{2}} + \frac{1}{2} \left[ \frac{\cos(\nu \tau)}{4 \sin^2 \frac{\tau}{2}} + \frac{\cos(\nu \tau')}{4 \sin^2 \frac{\tau'}{2}} \right].$$

The first term is a simplified double integral contribution. The second one has only one denominator and involves a single sum only.\textsuperscript{28}

\textsuperscript{28} This cumbersome splitting may appear as a minor simplification, but we found it to be a useful improvement from a computational point of view.
Diagram (14)(23): Interference I

\[ S_{n,m} = \begin{cases} 
\frac{\pi^2(2m^2+2n\nu+\nu^2)}{2m(m+n)}, & n = m + \nu \\
\frac{\pi^2(2n^2+2m\nu+\nu^2)}{2n(n+m)}, & m = n + \nu \\
-\frac{\pi^2(2n^2-2m\nu+\nu^2)}{2n(n-\nu)}, & n + m = \nu \\
0, & \text{else}.
\end{cases} \]  

(D.13)

\[ I_{\text{int}}^{\text{I}}(14)(23)(\varepsilon) = -\sum_{n+m=\nu} \frac{\pi^2(2n^2 - 2n\nu + \nu^2)}{2n(n-\nu)} + 2\sum_{m} e^{-(m+\nu)e} \frac{\pi^2(2m^2 + 2m\nu + \nu^2)}{2m(m+\nu)} \]

\[ = \frac{\pi^2}{\varepsilon} + \pi^2 \left[-2\nu + 2\nu(\psi^{(0)}(\nu) + \gamma) + 1 \right] + O(\varepsilon). \]  

(D.14)

Diagram (14)(23): Interference II

\[ S_n = \begin{cases} 
-\frac{\pi^4}{3} \nu, & n = \nu \\
0, & \text{else}.
\end{cases} \]  

(D.15)

\[ I_{\text{int}}^{\text{II}}(14)(23)(\varepsilon) = -\frac{\pi^4}{3} \nu + O(\varepsilon). \]  

(D.16)

Diagram (12)(34): Interference I

\[ S_{n,m} = \begin{cases} 
-2\pi^2, & n = m = \nu \\
-\frac{\pi^2}{m}, & n = \nu, m \neq \nu \\
-\frac{\pi^2}{n}, & m = \nu, n \neq \nu \\
0, & \text{else}.
\end{cases} \]  

(D.17)

\[ I_{\text{int}}^{\text{I}}(12)(34)(\varepsilon) = -2\pi^2 - 2\sum_{m \neq \nu} e^{-(m+\nu)e} \frac{\pi^2 \nu}{m} = 2\pi^2 \nu \log \varepsilon + O(\varepsilon). \]  

(D.18)

Diagram (12)(34): Interference II

\[ S_n = \begin{cases} 
-\frac{\pi^2(3+4\pi^2\nu^2)}{12\nu}, & n = \nu \\
-\frac{\pi^2}{2n\pi^2(n^2-\nu^2)^2}, & \text{else}.
\end{cases} \]  

(D.19)

\[ I_{\text{int}}^{\text{II}}(12)(34)(\varepsilon) = -\frac{\pi^2(3 + 4\pi^2\nu^2)}{12\nu} - 2\pi^2 \sum_{n \neq \nu} e^{-n\varepsilon} \frac{n(n^2 + \nu^2)}{(n^2 - \nu^2)^2} \]

\[ = \frac{2}{3} \pi^2 \left[-\pi^2 \nu + 3\nu \psi^{(1)}(\nu) + 3\psi^{(0)}(\nu) + 3\log(\varepsilon) + 3\gamma_E \right] + O(\varepsilon). \]  

(D.20)

Total two-loop contribution

Adding the expressions (D.10), (D.11), (D.14), (D.16), (D.18) and (D.20) we find the total in (4.12). This is the contribution to the $\frac{1}{4}$-supersymmetric loop $\zeta_1 = 0$, $\zeta_2 = 1$ with winding in $S^5$. The similar result with $\zeta_1 = 0$ and generic $\zeta_2$, $\nu$ is obtained by combining the same pieces. Indeed, purely scalar contributions will have a $\zeta_2^4$ factor while the scalar–vector interference will have $\zeta_2^2$. This leads to the expression (4.9).
Appendix E. One-loop contribution in dimensional regularization

As a consistency check of the mode regularization, let us derive (4.5) in dimensional regularization. For the purposes of illustration we specialize to the case \( \nu_1 = 1 \) and \( \nu_2 = \nu \in \mathbb{N} \). We start with

\[
W_1^{(\zeta_1, \zeta_2)}(1, \nu) = \frac{\Gamma(\omega - 1)}{16 \pi^\omega} \int_0^{2\pi} d\tau_1 d\tau_2 \frac{\zeta_1^2 + \zeta_2^2 \cos(\nu \tau_1 \zeta_2) - \cos(\tau_1 \zeta_2)}{(4 \sin^2 \frac{\tau_1 \zeta_2}{2})^{\omega - 1}}. \tag{E.1}
\]

The result will be finite, so we may ignore the replacements \( \pi^\omega \rightarrow \pi^2 \), etc. We first write

\[
W_1^{(\zeta_1, \zeta_2)}(1, \nu) = \frac{1}{64 \pi^2} \int_0^{2\pi} d\tau_1 d\tau_2 \left[ \zeta_1^2 + \zeta_2^2 \cos(\nu \tau_1 \zeta_2) - \cos(\tau_1 \zeta_2) \right] (\sin^2 \frac{\tau_1 \zeta_2}{2})^{1-\omega}.
\]

Then, we use the following Chebyshev polynomial expansion valid for integer \( \nu \geq 1 \)

\[
\cos(\nu \tau z) = (-1)^\nu T_\nu((\sin \frac{\tau}{z})
= (-1)^\nu \left[ (2^\nu - 1)/(\nu!)(2z^2)^{\nu-1} + 2^{\nu-1}(\sin^2 \frac{\tau}{z})^\nu \right], \tag{E.3}
\]

and

\[
\int_0^{2\pi} d\tau_1 d\tau_2 (\sin^2 \frac{\tau_1 \zeta_2}{2})^{\nu - 1} = \frac{4\pi^{3/2} \Gamma(\omega + \frac{1}{2})}{\Gamma(1 + \omega)}. \tag{E.4}
\]

After evaluating the finite summation over \( k \), we find

\[
W_1^{(\zeta_1, \zeta_2)}(1, \nu) = \frac{1}{8} - \frac{4\nu^{-3} \Gamma(3 - 2\omega) \Gamma(\omega + \nu - 1) \sin(\pi \omega)}{\pi \Gamma(2 + \nu - \omega)} \zeta_2^2. \tag{E.5}
\]

Finally, in the four dimensional \( \omega \rightarrow 2 \) limit, we get

\[
W_1^{(\zeta_1, \zeta_2)}(1, \nu) = \frac{1}{8}(1 - \nu \zeta_2^2), \tag{E.6}
\]
in agreement with (4.5).

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