TRUNCATED PROJECTIVE SPACES, BROWN-GITLER SPECTRA AND INDECOMPOSABLE $\mathcal{A}(1)$-MODULES

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Abstract. A structure theorem for bounded-below modules over the subalgebra $\mathcal{A}(1)$ of the mod 2 Steenrod algebra generated by $Sq^1, Sq^2$ is proved; this is applied to prove a classification theorem for a family of indecomposable $\mathcal{A}(1)$-modules. The action of the $\mathcal{A}(1)$-Picard group on this family is described, as is the behaviour of duality.

The cohomology of dual Brown-Gitler spectra is identified within this family and the relation with members of the $\mathcal{A}(1)$-Picard group is made explicit. Similarly, the cohomology of truncated projective spaces is considered within this classification; this leads to a conceptual understanding of various results within the literature. In particular, a unified approach to Ext-groups relevant to Adams spectral sequence calculations is obtained, englobing earlier results of Davis (for truncated projective spaces) and recent work of Pearson (for Brown-Gitler spectrum).

1. Introduction

The study of modules over $\mathcal{A}(1)$, the finite sub-Hopf algebra of the mod 2 Steenrod algebra generated by $Sq^1$ and $Sq^2$, has topological significance through the use of the Adams spectral sequence to calculate cohomology or homology for connective orthogonal theory, $ko$: a change of rings reduces to calculating the $E_2$-term as $\text{Ext}_{\mathcal{A}(1)}$ in the category of $\mathcal{A}(1)$-modules. This is a classical topic, with significant applications; for example, calculations using truncated projective spaces have applications to non-immersion results, occurring in work of Davis, Mahowald and many others (see [DGM81, DGM83], for example).

The aim of this paper is to provide a unified approach to a number of related questions, for example elucidating the relationship between truncated projective spaces and dual Brown-Gitler spectra, as seen through the eyes of $ko$ (or, more prosaically, representations of $\mathcal{A}(1)$). These and related modules are of significant interest; for example, Mahowald’s theory of $bo$-resolutions [Mah81, Mah84] relies upon an understanding of such modules.

From the algebraic viewpoint, the cornerstone of this work is provided by the results of Section 3; they contain the essence of a number of classification results, including the calculation of the Picard group $\text{Pic}_{\mathcal{A}(1)}$, due to Adams and Priddy [AP76], and the classification of the local Picard groups, due to Yu [Yu93]. The main algebraic result is Theorem 3.5 which is applied in Section 4 to obtain new proofs of the above results. For the purposes of this introduction, the algebraic result can be paraphrased informally as follows, using the Margolis cohomology groups $H^j(M, Q_1)$, $j \in \{0, 1\}$ of an $\mathcal{A}(1)$-module $M$: if $M$ is bounded below and has an isolated lowest dimensional Margolis cohomology class, then there exists an

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inclusion of a small member of the Picard group $\text{Pic}_{A}(1)$ into $M$ which carries the lowest dimensional Margolis cohomology class.

Using these techniques, in Section 6 a classification is given of the reduced finite $\mathcal{A}(1)$-modules such that the underlying module over $E(1) = \Lambda(Q_{0}, Q_{1})$ is indecomposable (up to free factors) - see Theorem 6.12. This of significant interest, since the cohomology of truncated projective spaces and of Brown-Gitler spectra and their duals (up to free modules) fit into this family.

Whereas the situation for $E(1)$ is easy, there is currently no full classification of the finite, indecomposable $\mathcal{A}(1)$-modules: the hypothesis that the module remains indecomposable (up to free factors) after restriction to $E(1)$ is far from anodyne. For example, tensor products of certain members of the above family throw up an infinite family of non-trivial examples where indecomposability is not preserved.

In Section 7 it is shown that, up to action of the Picard group, there are natural choices of representatives for stable isomorphism classes in the above family: the algebraic results of the paper give a natural choice of orbit representatives under the Picard group. It turns out that existing results in the literature can be understood more conceptually using these representatives.

Sections 8 and 9 are devoted to the analysis of the cohomology of truncated projective spaces and of dual Brown-Gitler spectra. One of the key conclusions is the identification of the cases where the cohomology of a truncated projective space is stably equivalent as an $\mathcal{A}(1)$-module to that of a dual Brown-Gitler spectrum (up to suspension). A further point of significant interest is the relationship between the family of finite indecomposable modules considered here, elements of the Picard group and the cohomology of infinite real projective space.

The theory is applied in Section 10 to obtain (see Theorem 10.16) the $E_{\mathcal{A}(1)}$-groups for the modules covered by the above classification theorem. For simplicity, the calculations are presented in terms of stable ext, $\mathcal{E}_{\mathcal{A}(1)}$, defined in terms of the stable category of $\mathcal{A}(1)$-modules; the usual Ext-groups can easily be deduced, the only additional work required being for $\text{Hom}_{\mathcal{A}(1)}$, since $\mathcal{E}_{\mathcal{A}(1)}$ does not see the morphisms factoring across projectives. This recovers calculations by Davis [Dav74] and, for dual Brown-Gitler spectra, by Pearson [Pea14]. The method used here is simplified significantly by using the presentation of the modules in terms of elements of the $\mathcal{A}(1)$-Picard group, $\text{Pic}_{A}(1)$: a further leitmotif is that all calculations should be carried out graded over $\text{Pic}_{A}(1)$.

The above methods are algebraic: truncated projective spaces and dual Brown-Gitler spectra only occur via their cohomology with $\mathbb{F}_{2}$-coefficients. Section 11 sketches the relationship between Brown-Gitler spectra and classifying spaces of elementary abelian 2-groups as seen in the stable $\mathcal{A}(1)$-module category. This illustrates the utility of the above $\mathcal{E}_{\mathcal{A}(1)}$ calculations. These are also applied in Section 12 to consider the decomposition of tensor products of modules of the form appearing in Theorem 6.12. This has been addressed for truncated projective spaces by Davis in [Dav74] Theorem 3.9]; studying the question in the light of the classification of families obtained in Theorem 6.12 and the subsequent cohomological calculations renders the situation more transparent.

The original motivation for this work came from a desire to understand the relationship between the results of the author’s paper [Pow14], which considered the $ko(n)$-cohomology of elementary abelian 2-groups, and Pearson’s work [Pea14] on the $ko$-homology of Brown-Gitler spectra. This is explained in Section 13 in particular, this gives a second order approximation (as compared to the results of [Pow14]) to a detection property which is implicitly established by Cowen-Morton [Mor07] for the Hopf modules (for mod-2 homology) of the theories $ko(n)$ over the
Hopf ring of $ko$, using the relationship between homology of Brown-Gitler spectra and Hopf rings [Goe99].

2. Algebraic background

The underlying prime is taken to be 2 and $\mathbb{F}$ denotes the prime field $\mathbb{F}_2$; $\mathcal{A}(1)$ is the sub Hopf algebra of the Steenrod algebra generated by $Sq^1, Sq^2$ and $E(1)$ that generated by $Q_0 = Sq^1$ and $Q_1 = [Sq^2, Sq^1]$. Thus $E(1)$ is the exterior, primitively-generated Hopf algebra $\Lambda(Q_0, Q_1)$ and there is a short exact sequence of Hopf algebras:

$$F \to E(1) \to \mathcal{A}(1) \to \Lambda(Sq^2) \to F,$$

where $\overline{Sq^2}$ denotes the image of $Sq^2$.

An $\mathcal{A}(1)$-module is reduced if and only if $Q_0Q_1$ acts trivially. A bounded-below $\mathcal{A}(1)$-module $M$ can be written as $M \cong F \oplus M^{red}$, where $F$ is a free $\mathcal{A}(1)$-module and $M^{red}$ is reduced (see [Bru12 Proposition 2.1]).

**Notation 2.1.** Write $\simeq$ to denote stable isomorphism of $\mathcal{A}(1)$-modules.

An essential ingredient is the Adams and Margolis [AM71] criterion for a morphism of $\mathcal{A}(1)$-modules to be a stable isomorphism: namely, a morphism $f : M \to N$ between bounded-below $\mathcal{A}(1)$-modules is a stable isomorphism if and only if it induces an isomorphism $H^j(f, Q_i)$ on Margolis cohomology groups, for $j \in \{0, 1\}$.

The stable $\mathcal{A}(1)$-module category has for objects $\mathcal{A}(1)$-modules and morphisms $[M, N] := \text{Hom}_{\mathcal{A}(1)}(M, N)/\text{ProjHom}_{\mathcal{A}(1)}(M, N)$ where ProjHom$_{\mathcal{A}(1)}(M, N)$ is the space of morphisms which factor through a projective module. For further details on the category of $\mathcal{A}(1)$-modules and the associated stable module categories, the reader is referred to [Bru12, Section 2].

The following observation is useful:

**Lemma 2.2.** For $M$ a reduced $\mathcal{A}(1)$-module, the quotient map $\text{Hom}_{\mathcal{A}(1)}(F, M) \to [F, M]$ is an isomorphism.

**Notation 2.3.** The duality functor on the category of $\mathcal{A}(1)$- (respectively $E(1)$-) modules (induced by vector space duality) is denoted by $D$.

**Remark 2.4.** The conjugation $\chi$ of $\mathcal{A}(1)$ acts by $\chi(Sq^i) = Sq^i$, for $i \in \{1, 2\}$, since the characteristic is two, $Sq^1$ is primitive, and $(Sq^1)^2 = 0$. Hence, a distinction between right and left modules over $\mathcal{A}(1)$ is unnecessary and, in considering the dual of a module represented by a diagram giving the action of $Sq^1$ and $Sq^2$, it suffices to reverse the direction of the arrows and interpret the degrees correctly.

The structure of the category of $E(1)$-modules is well understood. For instance, in [Ada74, Theorem III.16.11], Adams gives a classification of the finite-dimensional indecomposable $E(1)$-modules, given by the free modules of rank one together with the family of finite lightning-flash modules (recall that a lightning-flash module is a sub-quotient of a bi-infinite module as represented in Figure 1). Moreover, he shows that every finite $E(1)$-module is a finite direct sum of modules of this type.

There is no analogous classification of indecomposable $\mathcal{A}(1)$-modules in the literature. An important point is that a reduced $\mathcal{A}(1)$-module is not in general reduced when restricted to $E(1)$; indeed, for $M$ an $\mathcal{A}(1)$-module, $M|_{E(1)}$ is reduced if and only if $Sq^2Sq^3$ acts trivially on $M$.

**Notation 2.5.** Write $\Omega F$ for the augmentation ideal of $\mathcal{A}(1)$ and $\Omega^{-1} F$ for its dual (which can be identified with the coaugmentation ideal, with appropriate degree
shift) and, for $M$ an $\mathcal{A}(1)$-module and $n \in \mathbb{Z}$,

$$\Omega^n M := \begin{cases} 
(\Omega^1 F)^{\otimes n} \otimes M & n \geq 0 \\
(\Omega^{-1} F)^{\otimes |n|} \otimes M & n < 0.
\end{cases}$$

**Remark 2.6.** The functor $\Omega^F \otimes -$ on $\mathcal{A}(1)$-modules induces an equivalence of the stable module category with inverse induced by $\Omega^{-1} F \otimes -$, since $\Omega^n \Omega^{-n} M$ is non-canonically isomorphic to $M$. (Note that $\Omega^n \Omega^{-n} M$ is non-canonically isomorphic to $M$.)

**Lemma 2.7.** For $M$ an $\mathcal{A}(1)$-module, there are natural isomorphisms

$$H^*(M, Q_0) \cong H^{*+1}(\Omega M, Q_0) \quad H^*(M, Q_1) \cong H^{*+3}(\Omega M, Q_1).$$

**Proof.** An application of the long exact sequences for $H^*(-, Q_j)$, $j \in \{0, 1\}$. \(\square\)

Adams and Priddy [AP76] calculated the structure of the Picard group, $\text{Pic}_{\mathcal{A}(1)}$, namely the group of stable isomorphism classes of stably invertible $\mathcal{A}(1)$-modules with respect to $\otimes$:

$$\text{Pic}_{\mathcal{A}(1)} \cong \mathbb{Z}^2 \oplus \mathbb{Z}/2,$$

generated by $\Sigma F$, $\Omega F$ and $J$, the Joker, the unique element of $\text{Pic}_{\mathcal{A}(1)}$ of order two (that is $J \otimes 2 \cong F$). The Margolis cohomology groups of $J$ are necessarily one-dimensional concentrated in degree 0, $H^0(J, Q_0) = H^0(J, Q_1) = F$, hence do not distinguish between $J$ and $F$. However, there is no non-trivial morphism between the two, so they are not stably isomorphic; indeed, the structure of $J$ is represented by the diagram

in which the generator $\Diamond$ represents the non-trivial Margolis cohomology classes.

**Remark 2.8.** Elements of the Picard group arise in the work of Milgram [Mil75]; his results were used by Davis, Gitler and Mahowald [DGM81], where the notation $Q_{j,n}$ was introduced. This obscures the group structure of $\text{Pic}_{\mathcal{A}(1)}$ (cf. [DGM81 Lemma 3.11]), hence is not adopted here.

**Example 2.9.** The modules $Q_{1,j}$, for $j \in \mathbb{N}$, can be identified as follows:

$$Q_{1,j} \cong \begin{cases} 
(\Sigma^{-1} \Omega)^{j+1} J & j = 2l; \\
(\Sigma^{-1} \Omega)^{j+3} F & j = 2l + 1.
\end{cases}$$

A module from this family is uniquely determined (up to stable isomorphism) by its Margolis cohomology groups. (These modules are illustrated, along with the other families $Q_{i,j}$, in [Bai10 Figure 1]; the reader should compare these diagrams with [Bru12 Figure 1].)
Figure 2. Diagrammatic representations of certain \( \mathcal{A}(1) \)-modules

\[
\begin{array}{cccccc}
J &\Omega^{-1}J &\Omega^{-2}J &\mathbb{F} &\Omega^{-1}\mathbb{F} \\
2 & & & & \\
1 & & & & \\
0 & & & & \\
-1 & & & & \\
-2 & & & & \\
-3 & & & & \\
-4 & & & & \\
-5 & & & & \\
-6 & & & & \\
-7 & & & & \\
\end{array}
\]

The dual modules identify as:

\[
DQ_{1,j} \simeq \begin{cases}
(\Sigma\Omega^{-1})^{d+1}J & j = 2l; \\
(\Sigma\Omega^{-1})^{d+3}\mathbb{F} & j = 2l + 1.
\end{cases}
\]

Note that these do not fit into the same families under the action of \( (\Sigma^{-1}\Omega)^d \), \( l \in \mathbb{Z} \).

Remark 2.10. The module \( \Omega^{-1}J \) illustrated in Figure 2 is the question-mark complex (with the appropriate degrees) and \( \Omega^{-1}\mathbb{F} \) is the quotient of \( \Sigma^{-6}\mathcal{A}(1) \) by its socle, \( \mathbb{F} \).

Remark 2.11. The \( \mathcal{A}(1) \)-module \( \mathcal{A}(1) \otimes_{\mathcal{A}(0)} \mathbb{F} \) is also of importance; it is represented by the following diagram:

(1)

3. Low dimensional behaviour of \( \mathcal{A}(1) \)-modules

This section addresses the fundamental question: given a bounded-below \( \mathcal{A}(1) \)-module \( M \) and knowledge of the structure of \( (M|_{E(1)})^{\text{red}} \), what can be said about the structure of \( M \), in particular in low dimensions. The information on \( (M|_{E(1)})^{\text{red}} \) is provided by the Margolis cohomology groups.

Propositions 3.1 and 3.4 have a technical appearance but are the foundations for many key structural results for \( \mathcal{A}(1) \)-modules. More-user-friendly statements are given in Theorem 3.5 and Corollary 3.8. First applications of these results are given in Section 4 and the main application in Section 6.

Proposition 3.1. Let \( M \) be a reduced \( \mathcal{A}(1) \)-module and \( d \in \mathbb{Z} \) such that \( M = M^{\geq d} \) and \( M^d \neq 0 \).

1. If \( H^d(M, Q_0) = 0 = H^{d+1}(M, Q_1) \), then
   (a) \( H^d(M, Q_1) = 0 \);
   (b) \( Sq^2 Sq^2 : M^d \rightarrow M^{d+4} \) is injective, in particular \( Sq^2 : M^d \rightarrow M^{d+2} \) is injective;
(c) the operation $Q_0 = Sq^1$ on $M^{d+2}$ induces a short exact sequence

$$0 \to V \to \text{image}(Sq^2) \oplus W \to 0,$$

where $V := \text{image}(Sq^2) \cap \ker(Q^0)$ and $W := \text{image}(Sq^1 Sq^2 : M^d \to M^{d+3})$;

(d) $W \subset \ker(Q_0) \cap \ker(Q_1)$ and the natural map $W \to H^{d+3}(M, Q_1)$ is injective;

(e) $V \subset \ker(Q_0) \cap \ker(Q_1)$ and

(i) the natural map $V \to H^{d+2}(M, Q_1)$ is injective;

(ii) the kernel of the natural map $V \to H^{d+2}(M, Q_0)$ lies in the image of $Q_0 : M^{d+1} \to M^{d+2}$;

(f) setting $V = (\text{image}(Sq^2) \cap \ker(Sq^2 Sq^1))^{d+2}$, $V \subset \tilde{V} \subset \ker(Q_1)$ and the natural map $\tilde{V} \to H^{d+2}(M, Q_1)$ is injective; in particular, $H^{d+2}(M, Q_1) \oplus H^{d+3}(M, Q_1)$ is non-trivial.

(2) If $H^d(M, Q_0) = 0 = H^{d+1}(M, Q_1)$ (as above) and $H^{d+2}(M, Q_1) = 0$, then

(a) $Sq^1 Sq^2$ induces an injection $M^d \to H^{d+3}(M, Q_1)$;

(b) the operation $Sq^2 Sq^1Sq^2 : M^d \to M^{d+5}$ is injective.

\textbf{Proof.} Suppose that $M = M^{d,d}$ is reduced, $M^d \neq 0$ and $H^d(M, Q_0) = 0 = H^{d+1}(M, Q_1)$; the $Q_0$ hypothesis implies that $Q_0 : M^d \to M^{d+1}$ is injective (since $M^{<d} = 0$) and, similarly, $Q_1 : M^{d+1} \to M^{d+4}$ is injective, by the $Q_1$ hypothesis. It follows that $Q_0 Q_1 = Q_1 Q_0 : M^d \to M^{d+4}$ is injective; this shows that $H^d(M, Q_1) = 0$ and $Sq^2Sq^2 = Q_0 Q_1$ acts injectively on $M^d$.

The statements of part (2) are consequences of the above; if $H^{d+2}(M, Q_1) = 0$, then $V = 0$, which is equivalent to the injectivity of $Sq^1 Sq^2 : M^d \to M^{d+3}$ and the injection corresponds to $W \hookrightarrow H^{d+3}(M, Q_1)$. Similarly, $V \subset H^{d+2}(M, Q_1) = 0$ implies that $Sq^2 Sq^1 Sq^2$ acts injectively on $M^d$. \hfill \square

\textbf{Corollary 3.2.} Let $M$ be a reduced $\mathcal{A}(1)$-module and $d \in \mathbb{Z}$ such that $M = M^{d,d}$, $M^d \neq 0$, and $H^d(M, Q_0) = 0$, then $H^{d+1}(M, Q_1) \neq 0$ for some $t \in \{1, 2, 3\}$.

\textbf{Notation 3.3.} For $M$ a graded vector space and $n \in \mathbb{Z}$, as usual $M^n$ denotes the component of degree $n$; when forming the tensor product with a graded module $[M^n]$ will be written to emphasize that this is considered as a vector space placed in degree zero.

The reader may wish to recall the diagrammatic representation of $\mathcal{A}(1) \otimes \mathcal{A}(0) \otimes F$ (see Remark 2.11) in relation to the following.

\textbf{Proposition 3.4.} Let $M$ be a reduced $\mathcal{A}(1)$-module and $d \in \mathbb{Z}$ such that $M = M^{d,d}$, $H^d(M, Q_0) \neq 0$ and $H^d(M, Q_1) = 0$.

Then

(1) $H^d(M; Q_0) \cong \ker\{M^d \xrightarrow{Q_0} M^{d+1}\} \neq 0$;
Theorem 3.5. Let $M$ be a reduced $\mathcal{A}(1)$-module which is bounded-below and such that $H^{<0}(M, Q_j) = 0$ for $j \in \{0, 1\}$ (so that $M = M^{\geq -3}$).

1. If $M^{-3} \neq 0$, then there is a canonical inclusion:
   
   \[ [M^{-3}] \otimes \Sigma^3 \Omega^{-1} \mathbb{F} \rightarrow M, \]

   the image $ Sq^1 Sq^2 (M^{-3}) $ lies in $\ker(Q_1)$ and the composite

   \[ M^{-3} \rightarrow Sq^1 Sq^2 \rightarrow M^0 \rightarrow H^0(M, Q_1) \]

   is a monomorphism, in particular is non-trivial.

2. If $M^{-3} = 0$, $M^{-2} \neq 0$, set $V^{-2} := \ker(Sq^1 Sq^2)^{-2}$ and $W^1 := \image(Sq^1 Sq^2)^{1}$; then there is a canonical inclusion

   \[ [V^{-2}] \otimes J \rightarrow M, \]

   the image $ Sq^2 (V^{-2}) $ lies in $\ker(Q_1)$ and the composite

   \[ V^{-2} \rightarrow Sq^2 \rightarrow M^0 \rightarrow H^0(M, Q_1) \]

   is a monomorphism.

   Similarly, there is an injection $ W^1 \rightarrow H^1(M, Q_1) $; in particular, if $H^1(M, Q_1) = 0$, then $V^{-2} = M^{-2}$.

3. If $M^{-3} = 0 = M^{-2}$ and $M^{-1} \neq 0$, set $X := \ker(Sq^2)^{-1}$ and $Y := \ker(Sq^2)^{-1}$ so that $X \subset Y \subset M^{-1}$:

   a. $Q_0 Y $ lies in $\ker(Q_1)$ and yields a subspace $Q_0 Y \subset H^0(M, Q_1)$;
   b. $Q_1$ acts injectively on $X$ by $Sq^2 Sq^1$, hence induces a monomorphism

   \[ [X] \otimes \Sigma^3 \Omega^{-1} J \rightarrow M. \]

   c. if $X = 0$ and $H^1(M, Q_1) = 0$, then $Sq^2 Sq^1 : M^{-1} \rightarrow M^4$ is injective. If, in addition, $H^2(M, Q_1) = 0$ then $Y = M^{-1}$ and there is a monomorphism

   \[ [M^{-1}] \otimes \Sigma^2 \Omega^{-2} J \rightarrow M. \]

   This restricts to an inclusion $ [M^{-1}] \otimes \mathbb{F} \rightarrow M $.

Proof. The result follows from Proposition 5.1.

In the case $M^{-3} \neq 0$, $H^{-3}(M, Q_0) = 0 = H^{-2}(M, Q_1) = H^{-1}(M, Q_1)$ hence $V = 0$ (in the notation of the Proposition), both $Sq^2 Sq^2$ and $Sq^2 Sq^1 Sq^2$ act injectively on $M^{-3}$, and $Sq^1 Sq^2$ embeds $M^{-3}$ into $H^0(M, Q_1)$. This corresponds to the stated embedding of copies of $\Sigma^3 \Omega^{-1} \mathbb{F}$ into $M$.

In the case $M^{-3} = 0$ and $M^{-2} \neq 0$, with $H^{-2}(M, Q_0) = 0 = H^{-1}(M, Q_1)$ by hypothesis, $Sq^2 Sq^2$ acts injectively on $M^{-2}$; by definition, $Sq^1 Sq^2$ acts trivially on $V^{-2}$ and $Sq^2$ embeds $V^{-2}$ into $H^0(M, Q_1)$. This corresponds to the embedding of copies of $J$ into $M$. The embedding of $W^1$ into $H^1(M, Q_1)$ is given in Proposition 3.1; in particular, if $H^1(M, Q_1) = 0$, then $W^1 = 0$ and $V^{-2} = M^{-2}$.

The remaining case $M^{-3} = 0 = M^{-2}$ and $M^{-1} \neq 0$ is slightly more delicate. The statement concerning $Q_0 Y$ is clear, from the relation $Sq^2 Sq^2 = Sq^1 Sq^2 Sq^1$, and the embedding corresponding to $X$ is obtained as above.
If $X = 0$, then $Sq^2 : M^{-1} \to M$ is injective; $Sq^2(M^{-1}) \cap \ker(Sq^2 S^1)$ lies in the kernel of $Q_1$ hence embeds into $H^1(M, Q_1)$. If the latter is trivial, it follows that $Sq^2 S^1 S^2$ acts injectively on $M^{-1}$.

The argument is concluded by using the hypothesis $H^2(M, Q_1) = 0$. By the above, $Sq^2$ acts injectively on $Sq^1 S^2(M^{-1})$, whereas $Sq^2$ acts trivially upon the image of $Sq^2 S^1$, so that $Sq^1 S^2(M^{-1}) \cap Sq^2 S^1(M^{-1}) = 0$ and hence $Sq^1 S^2(M^{-1}) \oplus Sq^2 S^1(M^{-1}) \subset (\ker(Q_1))^2$. Consider the map

$$Q_1 : M^{-1} \to Sq^1 S^2(M^{-1}) \oplus Sq^2 S^1(M^{-1}) \subset (\ker(Q_1))^2.$$ 

The component to $Sq^1 S^2(M^{-1})$ is injective, hence $Sq^2 S^1(M^{-1}) \cap Q_1(M^{-1}) = 0$. Thus, if $H^2(M, Q_1) = 0$, $Sq^2 S^1(M^{-1}) = 0$. This implies that $Sq^2 S^1(M^{-1}) = Sq^1 S^2 S^1(M^{-1}) = 0$, so that $Y = M^{-1}$ and, moreover, $Sq^1(M^{-1})$ lies in the socle of $M$. The conclusion follows.

**Remark 3.6.** The modules which appear in the above statement, $\Omega^{-1} F, J, \Omega^{-1} J, \Omega^{-2} J$ (up to suspensions) all lie in $\text{Pic}_{\mathcal{A}(1)}$. (See Figure 2 for a schematic representation of these modules.)

The conclusion of Theorem 3.5 is made more concrete (in a special case) in Corollary 3.8 below. Recall that the module $\Sigma^3 \Omega^{-1} J$ is the question-mark complex, suspended so that the $Q_1$-Margolis cohomology is in degree zero; the following generalizes Lemma 2.2.

**Lemma 3.7.** For $M$ a reduced $\mathcal{A}(1)$-module, the quotient $\text{Hom}_{\mathcal{A}(1)}(\Sigma^3 \Omega^{-1} J, M) \to [\Sigma^3 \Omega^{-1} J, M]$ is an isomorphism.

**Proof.** This follows from the fact that each non-trivial quotient of $\Sigma^3 \Omega^{-1} J$ has simple socle. □

**Corollary 3.8.** Let $M$ be a bounded-below $\mathcal{A}(1)$-module such that $H^{\geq 0}(M, Q_0) = 0 = H^{< 0}(M, Q_1), H^1(M, Q_1) = 0$ and $H^0(M, Q_1) \neq 0$. Then, for at least one of $N \in \{M^{\text{red}}, (J \otimes M)^{\text{red}}\}$, the following statements hold:

1. There exists a monomorphism of one of the following forms:
   (a) $F \hookrightarrow N$
   (b) $\Sigma^3 \Omega^{-1} J \hookrightarrow N$
   which induces an injection on $H^*(-, Q_1)$.
   In particular, one of the following is non-trivial:
   $[F, M], [F, M \otimes J], [\Sigma^3 \Omega^{-1} J, M], [\Sigma^3 \Omega^{-1} J, M \otimes J] = [\Sigma^3 \Omega^{-1} F, M]$.

2. If $H^0(M, Q_1) = F$ then $N = N^{\geq -1}$.

**Proof.** The hypotheses rule out the possibility that $M = M^{\geq 0}$. For suppose $M = M^{\geq 0}$, then $Sq^2 S^1 = Q_0 Q_1$ does not act injectively on $M^0$, since $H^0(M, Q_1) \neq 0$. Now $Q_0 : (\ker(Sq^2 S^1))^0 \to M^1$ is injective, since $H^0(M, Q_0) = 0$ and induces an injection to $H^1(M, Q_1)$. By hypothesis $H^1(M, Q_1) = 0$, hence this provides a contradiction.

Theorem 3.5 now can be applied in each of the cases $M = M^{\geq -t}$, $t \in \{1, 2, 3\}$, giving a morphism of the required form; the injectivity of the morphism is a simple consequence of the arguments used in the proof of Theorem 3.5.

For instance, an injection $\Sigma^3 \Omega^{-1} F \hookrightarrow M^{\text{red}}$ which is an isomorphism on $H^0(-, Q_1)$ induces a map

$$\Sigma^3 \Omega^{-1} J \to (M \otimes J)^{\text{red}}$$

which is an isomorphism on $H^0(-, Q_1)$. It is straightforward to check that this is necessarily injective.
The conclusion on the non-triviality of one of the stable homomorphism spaces follows from Lemmas 2.2 and 3.7.

Finally, the connectivity statement follows by using the hypothesis \( H^0(M, Q_1) = F \) to exclude additional classes in degrees \(-3\) and \(-2\). 

Remark 3.9. The hypothesis that \( H^1(M, Q_1) = 0 \) in Corollary 3.8 is necessary so as to apply Theorem 3.5. There are analogous statements without this hypothesis, but at the price of elegance. This is addressed in Section 6, notably in the proof of Theorem 6.12.

When the lowest Margolis cohomology group is \( Q_0 \)-cohomology, Theorem 3.5 gives:

**Corollary 3.10.** For \( M \) a reduced, bounded-below \( \mathcal{A}(1) \)-module, if \( H^{<0}(M, Q_0) = 0 = H^{\leq 1}(M, Q_1) \), then \( M = M^{\geq 0} \) and \( M^0 \neq 0 \) if and only if \( H^0(M, Q_0) \neq 0 \).

**Proof.** Theorem 3.5 shows that the vanishing of \( H^0(M, Q_1) \) and \( H^1(M, Q_1) \) implies that \( M^{-3} = M^{-2} = M^{-1} = 0 \). The final point follows as in Corollary 3.8. \( \square \)

### 4. First applications

Theorem 3.5 can be seen as the key ingredient in the proof of a number of results, as explained below, and is applied in Section 6 to prove a new classification result.

**Example 4.1.** Theorem 3.5 includes the main step used by Adams-Priddy [AP76] in their calculation of the Picard group, \( \text{Pic}_{\mathcal{A}(1)} \). Namely, as in [AP76], one reduces to the case where the Margolis cohomology groups are concentrated in degree zero (and are both one-dimensional). Then, if the module is not stably isomorphic to \( F \), Theorem 3.5 provides an embedding of \( J \), which induces an isomorphism of Margolis cohomology groups, hence is a stable isomorphism.

**Notation 4.2.** [Bru12] Let \( P_0 \) denote the unique (up to isomorphism) reduced \( \mathcal{A}(1) \)-module satisfying the following properties:

1. \( P_0 \) is bounded-below;
2. \( P_0 \) is \( Q_0 \)-acyclic and \( H^*(P_0, Q_1) \) is one-dimensional, concentrated in degree zero;
3. there is an inclusion \( \mathbb{F} \hookrightarrow P_0 \) which induces an isomorphism on \( H^*(-, Q_1) \).

**Remark 4.3.** The module \( P_0 \) is realized topologically by the mod 2 cohomology of the Thom spectrum \( \mathbb{R}P^\infty \).

From the above characterization it follows that \( P_0 \) is stably idempotent (\( P_0 \otimes P_0 \) is stably isomorphic to \( P_0 \)) and, more generally, if \( M \) is a bounded-below \( \mathcal{A}(1) \)-module which is \( Q_0 \)-acyclic, then \( F \to P_0 \) induces a stable isomorphism

\[ M \cong P_0 \otimes M. \]

**Proposition 4.4.** Let \( M \) be a bounded-below \( \mathcal{A}(1) \)-module which is \( Q_0 \)-acyclic and \( N \) a bounded-below \( \mathcal{A}(1) \)-module equipped with an \( \mathcal{A}(1) \)-linear morphism \( f : N \to M \) which induces an isomorphism \( H^*(f, Q_1) \) in \( Q_1 \)-Margolis cohomology. Then \( f \) induces a stable isomorphism between \( P_0 \otimes N \) and \( M \).

**Proof.** By the Adams and Margolis [AM71] criterion, it suffices to show that the morphism induces an isomorphism on \( H^*(-, Q_j) \) for \( j \in \{0,1\} \). Applying the Künneth isomorphism for Margolis cohomology, it follows that the following are stable isomorphisms

\[ P_0 \otimes N \xrightarrow{\cong} f \circ \text{Id} \otimes f \circ P_0 \otimes M \cong M. \]
The result follows.

As a consequence, one obtains a proof of a result first proven (by an intricate calculational method) in Yu’s thesis [Yu95]; a much simpler proof is given in [Bru12].

**Theorem 4.5.** Let \( P \) be a bounded-below \( \mathcal{A}(1) \)-module which satisfies the following:

1. \( P \) is \( Q_0 \)-acyclic;
2. \( H^*(P, Q_1) \) is one-dimensional, concentrated in degree zero.

Then \( P \) is stably isomorphic to one of the following:

\[
P_0, \quad P_0 \otimes \Sigma^3 \Omega^{-1} F, \quad P_0 \otimes J, \quad P_0 \otimes \Sigma^3 \Omega^{-1} F.
\]

Moreover, the inclusion \( F \to \Sigma^6 \Omega^{-2} J \) induces a stable isomorphism

\[
\Omega^2 P_0 \cong \Sigma^6 J \otimes P_0,
\]

hence, the above are stably isomorphic to

\[
P_0, \quad \Sigma^{-3} \Omega P_0, \quad \Sigma^{-6} \Omega^2 P_0 \cong J \otimes P_0, \quad \Sigma^{-9} \Omega^3 P_0 \cong \Sigma^{-3} \Omega J \otimes P_0.
\]

**Proof.** Apply Proposition 4.4 to the morphisms provided by Theorem 3.5. \( \square \)

**Remark 4.6.**

1. In the notation of [Bru12, Section 4], these modules are \( P_0, \Sigma^{-2} P_1, \Sigma^{-4} P_2, \Sigma^{-6} P_3 \). See [Bru12, Figure 1] for diagrammatic representations.
2. In Bruner’s notation, the module \( P_1 \) is realized as the reduced mod 2-cohomology of \( RP_\infty \) and, for \( n \in \{1, 2, 3\} \), there is an isomorphism of \( \mathcal{A}(1) \)-modules \( (P_1^\otimes n)^{\text{red}} \cong P_n \); thus, up to stable isomorphism of \( \mathcal{A}(1) \)-modules, \( P_n \) is realized as the reduced mod 2 cohomology of \( (RP_\infty)^\wedge n \).

For later use, the following standard result is recalled:

**Proposition 4.7.** Let \( M \) be a bounded-below \( \mathcal{A}(1) \)-module which is \( Q_0 \)-acyclic, then there are stable isomorphisms:

\[
\Omega^4 M \cong \Sigma^{12} M
\]

\[
\Omega^2 M \cong \Sigma^6 J \otimes M.
\]

**Proof.** These properties hold for \( P_0 \), by Theorem 4.5 whence the result via the stable isomorphism \( P_0 \otimes M \cong M \). \( \square \)

5. A Margolis-type killing construction for \( Q_1 \)-cohomology

Motivated by Corollary 3.8 and for use in the proof of Theorem 6.12 an explicit version for \( \mathcal{A}(1) \) of Margolis’s construction for killing \( H^*(-, Q_1) \)-cohomology classes is introduced. In the application, it will be convenient to work with modules which are bounded-above; this leads to the consideration of \( DP_0 \), the dual of \( P_0 \), and of related modules.

**Definition 5.1.** (Cf. [Bru12].) Let \( R \) denote the unique (up to isomorphism) bounded-below, reduced \( \mathcal{A}(1) \)-module which is \( Q_1 \)-acyclic and has \( H^*(R, Q_0) \) one-dimensional, concentrated in degree \(-1\).

**Remark 5.2.** The module \( R \) can be realized topologically as the mod 2 cohomology of the fibre of the transfer \( \Sigma^\infty RP_\infty \to S^0 \) (recall that the transfer is trivial in mod 2 cohomology) and also as the fibre of a map \( \mathbb{R} P_\infty^{-1} \to S^0 \) which is non-trivial in mod 2 cohomology (for instance, the composite \( \mathbb{R} P_\infty^{-1} \to \mathbb{R} P_\infty^0 \cong \mathbb{R} P_\infty \vee S^0 \to S^0 \), where the last map is the projection).

The characterization of \( R \) given in Definition 5.1 implies the following:
Lemma 5.3. There are stable equivalences:
\[ \Sigma^{-1}\Omega R \simeq R \]
\[ J \otimes R \simeq R. \]

There are non-split short exact sequences of \( \mathcal{A}(1) \)-modules (cf. [Bru12]):
\[
0 \to F \to P \to R \to 0
\]
\[
0 \to \Omega P_0 \to SR \to F \to 0.
\]

Remark 5.4. These short exact sequences can be realized topologically by applying mod 2 cohomology to the (stable) fibre sequences of Remark 5.2.

By inspection (and as implied by Theorems 3.10 and 4.5), there is a unique non-trivial morphism
\[ \Sigma^3\Omega^{-1}J \to \Sigma^{-3}\Omega P_0, \]
and this is a monomorphism.

Similarly for the dual \( DR \) of \( R \), there are unique non-trivial morphisms:
\[ F \to \Sigma^{-1}DR \]
\[ \Sigma^3\Omega^{-1}J \to \Sigma DR \]
and these fit into short exact sequences:
\[
0 \to F \to \Sigma^{-1}DR \to \Omega^{-1}DP_0 \to 0
\]
\[
0 \to \Sigma^3\Omega^{-1}J \to \Sigma DR \to \Sigma^{-3}DP_0 \to 0.
\]

Notation 5.5. For \( M \) an \( \mathcal{A}(1) \)-module equipped with a monomorphism \( F \hookrightarrow M \), let \( \tilde{M} \) be the module defined by the pushout of short exact sequences:
\[
\begin{array}{ccc}
F & \to & \Sigma^{-1}DR \\
\downarrow & & \downarrow \\
M & \to & \tilde{M}
\end{array}
\]
\[
\begin{array}{ccc}
& & \Omega^{-1}DP_0 \\
\Sigma^3\Omega^{-1}J & \to & \Sigma DR \\
\downarrow & & \downarrow \\
M & \to & \tilde{M}
\end{array}
\]

Lemma 5.6. For \( M \) an \( \mathcal{A}(1) \)-module equipped with a monomorphism \( F \hookrightarrow M \), there is a short exact sequence
\[
0 \to F \to M \oplus \Sigma^{-1}DR \to \tilde{M} \to 0
\]
and
(1) if \( M \) is reduced, then so is \( \tilde{M} \);
(2) if \( M = M \geq -1 \), then \( M \cong \tilde{M} \geq -1 \);
(3) if \( M \) is \( Q_0 \)-acyclic, then so is \( \tilde{M} \);
(4) if \( F \hookrightarrow M \) induces a non-trivial map in \( H^*(-, Q_1) \), then \( H^*(\tilde{M}, Q_1) \cong H^*(M, Q_1)/F \).

Proof. Straightforward. \( \square \)

Notation 5.7. For \( M \) an \( \mathcal{A}(1) \)-module equipped with a monomorphism \( \Sigma^3\Omega^{-1}J \hookrightarrow M \), let \( \tilde{M} \) be the module defined by the pushout of short exact sequences:
\[
\begin{array}{ccc}
\Sigma^3\Omega^{-1}J & \to & \Sigma DR \\
\downarrow & & \downarrow \\
M & \to & \tilde{M}
\end{array}
\]
\[
\begin{array}{ccc}
& & \Sigma^{-3}DP_0 \\
\Sigma^3\Omega^{-1}J & \to & \Sigma DR \\
\downarrow & & \downarrow \\
M & \to & \tilde{M}
\end{array}
\]

Lemma 5.6 has the following counterpart:
Lemma 5.8. For $M$ an $\mathfrak{a}(1)$-module equipped with a monomorphism $\Sigma^3\Omega^{-1}J \hookrightarrow M$, there is a short exact sequence

$$0 \rightarrow \Sigma^3\Omega^{-1}J \rightarrow M \oplus \Sigma DR \rightarrow \tilde{M} \rightarrow 0$$

and

(1) if $M$ is reduced, then so is $\tilde{M}$;
(2) if $M = \Sigma^2\Omega^{-1}$, then $M \cong \Sigma^2\Omega^{-1}$;
(3) if $M$ is $Q_0$-acyclic, then so is $\tilde{M}$;
(4) if $\Sigma^3\Omega^{-1}J \hookrightarrow M$ induces a non-trivial map in $H^*(-,Q_1)$, then $H^*(\tilde{M},Q_1) \cong H^*(M,Q_1)/F$.

Remark 5.9. In Section 6, where this result is applied, $M$ is a finite $\mathfrak{a}(1)$-module, hence the respective modules $\tilde{M}$, $\tilde{M}$ are bounded-above.

6. A FAMILY OF FINITE, INDECOMPOSABLE $\mathfrak{a}(1)$-MODULES

The aim of this section is to classify the isomorphism classes of finite indecomposable $\mathfrak{a}(1)$-modules $M$ such that $(M|_{E(1)})^{\text{red}}$ is indecomposable. The guiding principle is provided by the following consequence of [Ada74, Theorem III.16.11].

Proposition 6.1. The following conditions on a finite, reduced $E(1)$-module $L$ are equivalent:

(1) the total dimension of $H^*(L,Q_0) \oplus H^*(L,Q_1)$ is two;
(2) $L$ is a reduced indecomposable module.

Hence, for $M$ as above, there are three cases to consider:

(1) $\dim H^*(M,Q_0) = 1 = \dim H^*(M,Q_1)$: by the results of Adams and Priddy [AP76], this implies that $M$ is in the Picard group, $\text{Pic}(\mathfrak{a}(1))$, hence these modules are classified by the Picard group;
(2) $H^*(M,Q_1) = 0$ and $\dim H^*(M,Q_0) = 2$: this case is easily understood (note the relative simplicity of Proposition 3.4 compared to Proposition 5.1) - see Proposition 5.2;
(3) $H^*(M,Q_0) = 0$ and $\dim H^*(M,Q_1) = 2$: the most interesting case.

Recall the module $R$ of Definition 5.1 and the form of the module $\mathfrak{a}(1) \otimes_{\mathfrak{a}(0)} F$ (see Remark 2.11). By the results of Section 3, there is a canonical inclusion of $\Sigma^{-1}\mathfrak{a}(1) \otimes_{\mathfrak{a}(0)} F$ and, moreover, this fits into a non-split short exact sequence:

$$0 \rightarrow \Sigma^{-1}\mathfrak{a}(1) \otimes_{\mathfrak{a}(0)} F \rightarrow R \rightarrow \Sigma^4R \rightarrow 0$$

(cf. [Bru12]). Recursively one obtains an increasing filtration

$$0 = f_0R \subset f_1R = \Sigma^{-1}\mathfrak{a}(1) \otimes_{\mathfrak{a}(0)} F \subset f_2R \subset \ldots \subset f_iR \subset \ldots \subset R,$$

where $f_{i+1}R/f_iR \cong \Sigma^{4i-1}\mathfrak{a}(1) \otimes_{\mathfrak{a}(0)} F$, each $f_iR$ is $Q_1$-acyclic and, for $i \geq 1$,

$$H^*(f_iR,Q_0) \cong \begin{cases} F & i = -1,4i \\ 0 & \text{otherwise.} \end{cases}$$

The classification in the $Q_1$-acyclic case is as follows:

Proposition 6.2. Let $M \neq 0$ be a reduced, finite $\mathfrak{a}(1)$-module. The following conditions are equivalent:

(1) $M$ is $Q_1$-acyclic and $\dim H^*(M,Q_0) = 2$;
(2) $M \cong \Sigma^{d+1} f_iR$, where $H^*(M,Q_0)$ is non-zero in degrees $d$, $4i + d + 1$ for integers $d,1 \leq i$. 

Proof. The implication (2)⇒(1) is straightforward, hence consider the converse.

The hypotheses imply that $M|_{E(1)}^{red}$ is indecomposable (by Proposition 6.1) with the form of a $Q_1$-acyclic lightning flash, which gives that $H^*(M, Q_0)$ is non-zero precisely in degrees of the form $d, 4i + d + 1$, for $d \in \mathbb{Z}$ and $i \in \mathbb{N}$. The case $i = 0$ is excluded by Proposition 3.4.

The result can be proved by induction upon $i$. For $i = 1$, Corollary 3.10 and its dual imply that $M$ is concentrated in degrees $[d, d + 3]$. Proposition 3.4 provides an embedding $\Sigma^{d+1}A^{(1)} \otimes A^{(0)} F \cong M$ and it is straightforward to see that this is a stable isomorphism.

For the inductive step, $i > 1$, a similar argument provides a short exact sequence

$$0 \rightarrow \Sigma^{-1}A^{(1)} \otimes A^{(0)} F \rightarrow M \rightarrow M' \rightarrow 0$$

where $M'$ is reduced and has $Q_0$-Margolis cohomology in degrees $d + 4i + d + 1$, hence is isomorphic to $\Sigma^{d+4i+1}f_iR$, by the inductive hypothesis. There is a unique non-trivial extension of this form, which corresponds to $\Sigma^{d+1}f_iR$, as required. □

Now consider the $Q_0$-acyclic case; the associated $E(1)$-module $(M|_{E(1)})^{red}$ therefore has the following form:

\[ \begin{array}{cccccc}
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
\end{array} \]

The $Q_i$-Margolis cohomology classes are represented by the top left hand generator and the bottom right hand generator, indicated above by $\blacklozenge$; the case where the total dimension of $(M|_{E(1)})^{red}$ is two is exceptional (in this case, the bottom generator represents the $Q_1$-cohomology class of lowest degree). This gives:

**Lemma 6.3.** Let $M$ be a finite $Q_0$-acyclic $\mathcal{A}(1)$-module such that $(M|_{E(1)})^{red} \neq 0$ and is indecomposable, then $H^*(M, Q_1)$ has total dimension 2 and is concentrated in degrees $a, a + 2s - 3$, for $a \in \mathbb{Z}$ where $1 \leq s \in \mathbb{N}$ is the total dimension of the socle of $(M|_{E(1)})^{red}$.

First consider the family of modules which occurs in the exceptional case $s = 1$.

**Notation 6.4.** Write $Z$ for the $\mathcal{A}(1)$-module defined by the non-split extension

$$0 \rightarrow F \rightarrow \Sigma^{-1}Z \rightarrow \Sigma^{-1}F \rightarrow 0.$$

Note that $Z$ is not self dual, since $DZ \cong \Sigma Z$.

**Remark 6.5.** The $\mathcal{A}(1)$-module $Z$ is the mod 2 cohomology of the desuspended Moore spectrum $\Sigma^{-1}S^0/2$. The behaviour of duality on $Z$ corresponds to that of Spanier-Whitehead duality on $\Sigma^{-1}S^0/2$.

**Lemma 6.6.** There are stable isomorphisms:

$$\Omega^2 Z \cong \Sigma^{12} Z,$$

$$\Omega^2 Z \cong \Sigma^6 J \otimes Z,$$

$\Omega Z$ occurs in the non-split short exact sequence

$$0 \rightarrow \Sigma^3 J \rightarrow \Omega Z \rightarrow \Sigma^2 F \rightarrow 0,$$

and $\Omega^{-1} Z \cong \Sigma^{-1} D(\Omega Z)$.

**Proof.** The first statement is an application of Proposition 1.7. The structure of $\Omega Z$ is a straightforward computation and that of $\Omega^{-1} Z$ follows by duality. □
The structure of \( \Omega Z \) is:

\[
\begin{array}{c}
\bullet \\
\text{↗} \quad \text{↗} \\
\rightarrow \\
\downarrow \\
\text{↘} \quad \text{↘} \\
\text{↗} \\
\text{↗} \\
\end{array}
\]

with the generators placed in the appropriate degrees.

**Remark 6.7.** Where a Joker occurs as a subquotient of an \( \mathcal{A}(1) \)-module, as in the case \( \Omega Z \) above, the diagram will be simplified by representing the Joker by \( \circ \). In the above case, this gives \( \bullet \rightarrow \circ \).

**Remark 6.8.** The module \( Z \) is a quotient of \( P_0 \): there is a short exact sequence of \( \mathcal{A}(1) \)-modules:

\[ 0 \rightarrow \Sigma^{-1} \Omega P_0 \rightarrow P_0 \rightarrow Z \rightarrow 0. \]

Similarly, \( \Omega Z \) occurs as a quotient of \( \Omega^2 P_0 \) (up to suitable suspension) and \( \Omega^{-1} Z \) as a quotient of \( \Omega^{-1} P_0 \).

**Proposition 6.9.** Let \( P \) be a \( Q_0 \)-acyclic, bounded-below, reduced \( \mathcal{A}(1) \)-module such that \( H^*(P,Q_1) \) is one-dimensional, concentrated in degree zero, and let \( 2 \leq s \) be an integer. Then

1. the quotient \( P^{\leq 2(s-1)} \) is a reduced, finite \( \mathcal{A}(1) \)-module which is \( Q_0 \)-acyclic and has \( Q_1 \)-Margolis cohomology of total dimension 2, concentrated in degrees \( \{0, 2s - 3\} \);
2. there is a short exact sequence of \( \mathcal{A}(1) \)-modules:
   \[ 0 \rightarrow \Sigma^2(\Sigma^{-3} \Omega)^i P_0 \rightarrow P \rightarrow P^{\leq 2(s-1)} \rightarrow 0 \]
   where, for \( P \cong (\Sigma^{-3} \Omega)^i P_0, i \in \{0, 1, 2, 3\} \) and \( i + s \in \{0, 1\} \) is the residue of \( i + s \) modulo 2.
3. the stable isomorphism classes of the modules
   \[ \{ J^\varepsilon \otimes ((\Sigma^{-3} \Omega)^i P_0)^{\leq 2(s-1)} | \varepsilon \in \{0, 1\}, i \in \{0, 1, 2, 3\} \} \]
   are pairwise distinct;
4. under the action of \( \Sigma^{-3} \Omega \), the stable isomorphism classes of \( \{J\} \) form two distinct orbits of cardinal four, generated by \((P_0)^{\leq 2(s-1)}\) and \(((P_0 \otimes J)^{\text{red}})^{\leq 2(s-1)}\).

**Proof.** By Yu’s theorem (Theorem 4.5), the first two parts of the result can be read off by inspection from the structure of the representatives of the isomorphism classes of such modules (using the details provided by Bruner [Bru12], in particular [Bru12, Theorem 4.6]). Similarly, the proof that the stable isomorphism classes are pairwise distinct is straightforward.

The cardinality of the orbits of the stable isomorphism classes under \( \Sigma^{-3} \Omega \) divides four, by Proposition 4.7, since each of the modules considered is \( Q_0 \)-acyclic; moreover, since \( \Sigma^{-6} \Omega^2 P_0 \simeq J \otimes P_0 \not\simeq P_0 \), the orbits have cardinal four.

Finally, it is straightforward to check that the given elements lie in distinct orbits. \( \square \)

**Remark 6.10.**

1. The condition \( s \geq 2 \) is required due to the possible presence of a subquotient isomorphic to (a suspension of) the Joker in low degree. The appropriate quotient in the case \( s = 1 \) given in Remark 6.8 cannot be defined simply by truncation.
2. \( \mathcal{A}(1) \)-modules of the form considered in Proposition 6.9 already appear in the literature. First examples are given by the cohomology of truncated projective spaces; in his thesis [Dav74], Don Davis considered a related
family of \(\mathfrak{A}(1)\)-modules (see [Dav74, Definition 3.6]) and [Dav74, Lemma 3.8] can be interpreted as describing the action of the Picard group.

Example 6.11. The orbits under \(\Sigma^{-3}\Omega\) are easily understood. For example, in the case \(s = 2\), one of the orbits is illustrated in Figure 3 (See Remark 6.7 for the notation for attaching a Joker.)

The pattern in lowest degrees corresponds to that of the orbit \((\Sigma^{-3}\Omega)^{i}P_{0}\), whereas that in highest degrees is dual, hence cycles in the opposite order. The description of the second orbit is similar.

In both cases, \(\Sigma^{-6}\Omega^{2}\) operates as \(J \otimes -\), switching \(\bullet \leftrightarrow \circ\) in the degrees which correspond to the \(H^{*}(-, Q_{1})\) cohomology classes; this gives a form of symmetry across the diagonal. Thus, to understand the orbit, it is sufficient to calculate the action of \(\Sigma^{-3}\Omega\).

Theorem 6.12. Let \(M\) be a finite, reduced \(\mathfrak{A}(1)\)-module such that \(H^{*}(M, Q_{0}) = 0\) and \(H^{*}(M, Q_{1})\) has total dimension two, concentrated in degrees \(0, 2s - 3\), for \(1 \leq s \in \mathbb{N}\) the total dimension of the socle of \((M|_{E(1)})^{\text{red}}\);

(1) if \(s = 1\), then \(M \simeq (\Sigma^{-3}\Omega)^{t}Z\), for some \(t \in \{0, 1, 2, 3\}\);
(2) if \(s \geq 2\), then \(M \simeq J^{\otimes \varepsilon} \otimes P^{\leq 2(s-1)}\), a module of the form given in Proposition 6.9.

Proof. The cases \(s = 1\) and \(s = 2\) require separate treatment: \(s = 1\) is the exceptional case and the condition \(H^{1}(M, Q_{1}) = 0\) in Corollary 3.8 has to be worked around for \(s = 2\).

First consider the case \(s > 2\); by Corollary 3.8 (replacing \(M\) by \((M \otimes J)^{\text{red}}\) if necessary, which leads to the factor \(J^{\otimes \varepsilon} \otimes -\) with \(\varepsilon = 1\)), there exists a monomorphism of one of the following forms

(1) \(F \hookrightarrow M\)
(2) $\Sigma^3\Omega^{-1}J \hookrightarrow M$
and $M = M^{\geq -1}$.

Consider the first case; forming the finite-type, bounded-above module $M$ of Notation 5.8. Lemma 5.6 shows that $M$ is finite-type, reduced, $Q_0$-acyclic, with $H^*(M, Q_1)$ one-dimensional, concentrated in degree $2s - 3$. Thus, the dual of Theorem 4.5 identifies the isomorphism type of the module $M$ and, again by Lemma 5.6, $M$ is recovered as the submodule $M^{\geq -1}$. The required conclusion follows by dualizing.

The second case is analogous, mutatis mutandis, using $\tilde{M}$ of Notation 5.7 together with Lemma 5.8.

In the case $s = 2$, the same strategy can be applied, once the appropriate result corresponding to Corollary 5.8 has been established. In this case, the underlying $E(1)$-module $(M|_{E(1)})^{\text{red}}$ is of the following form:

$$\bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet$$

with $Q_1$-Margolis cohomology in degrees 0, 1. Proposition 3.1 implies that $M = M^{\geq -3}$; if $M^{-3} \neq 0$, then the argument of the proof of Proposition 3.1 provides a monomorphism $\Sigma^3\Omega^{-1} \mathbb{F} \hookrightarrow M$ inducing a monomorphism on $H^0(\cdot, Q_1)$; replacing $M$ by $M \otimes J$, one obtains a monomorphism $\Sigma^4\Omega^{-1} J \hookrightarrow (M \otimes J)$ inducing a monomorphism on $H^0(\cdot, Q_1)$. The argument now proceeds as above.

Now consider the case $M^{-3} = 0$ and $M^{-2} \neq 0$ then, as in Proposition 3.1, $Sq^2 Sq^2$ acts injectively on $M^{-2}$. Consider $Sq^3 Sq^2(M^{-2}) \subset M^1$, which clearly lies in $\ker(Q_0) \cap \ker(Q_1)$, since $M$ is reduced; by hypothesis on the form of $(M|_{E(1)})^{\text{red}}$, this is only possible if $Sq^3 Sq^2(M^{-2})$ lies in the image of $Q_0 Q_1$, which is excluded by the hypothesis that $M^{-3} = 0$. Hence there exists a monomorphism $J \hookrightarrow M$ and one proceeds as before.

In the remaining case, $M = M^{\geq -1}$; by duality, one can further reduce to the case where $M$ is concentrated in degrees $[-1, 2]$. In this case $M|_{E(1)}$ is necessarily $E(1)$-reduced, of total dimension four, and there are the two possibilities:

$$\bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet$$

These cases provide the isomorphism classes given in the statement.

It remains to consider the case $s = 1$, so that $(M|_{E(1)})^{\text{red}} \cong Z|_{E(1)}$. A similar strategy applies; Proposition 3.1 implies that $M$ is concentrated in degrees $[-4, 3]$; the first step is to show that $M^{-4} = 0$ (so that $M^3 = 0$ by a dual argument). This follows as above, since $Sq^4 Sq^2(M^{-4})$ lies in $\ker(Q_0) \cap \ker(Q_1)$, which is zero. Thus $M$ is concentrated in degrees $[-3, 2]$.

Suppose $M^{-3} \neq 0$, then $Sq^2 Sq^2$ acts injectively on $M^{-3}$ and $Sq^2(M^{-3}) \cap \ker(Q_0) \subset M^{-1}$ lies in $\ker(Q_0) \cap \ker(Q_1)$, which is zero (as above), hence $Sq^3 Sq^2 : M^{-3} \hookrightarrow M^0$. There are two cases:

1. If $Sq^2 Sq^1 Sq^2$ is not injective on $M^{-3}$, then one obtains an inclusion
   $$\Sigma^3\Omega^{-1}Z \hookrightarrow M$$
   which induces an isomorphism on Margolis cohomology groups, hence is an isomorphism.

2. Otherwise, by also invoking the dual argument, we may assume that $Sq^2 Sq^1 Sq^2$
   induces an isomorphism $M^{-3} \xrightarrow{\cong} M^2$ and, counting Margolis cohomology
classes, $\dim M^{-3} = 1$. From this it is straightforward to prove that $J \otimes Z$
and $M$ have the same underlying graded vector space; to check that they are isomorphic, it suffices to show that $M$ fits into a non-trivial extension of the form

$$0 \to J \to M \to \Sigma^{-1}J \to 0,$$

since there is a unique non-trivial extension of this form.

To exhibit the submodule $J$, using the above arguments, one can show that $\ker(Sq^1 Sq^2)$ is one-dimensional in degree $-2$ and the non-trivial class generates a submodule isomorphic to $J$. The quotient module is checked to be isomorphic to $\Sigma^{-1}J$, as required.

In the remaining cases, $M$ is concentrated in degrees $[-2,1]$, hence is isomorphic to $Z$, by consideration of $M|_{E(1)}$. □

7. An Interpretation of the Families Using the Picard Group

The presentation of the families arising in Section 6 was guided by the method of proof of Theorem 6.12. A convenient choice of generators for the orbits under the action of the Picard group (see Proposition 6.9) can be given by removing the lowest dimensional class of suitable elements of the Picard group.

For $1 \leq k \in \mathbb{N}$, the Picard group elements $\Sigma^{-(k+1)}Q^{k+1}F$ and $\Sigma^{-(k+1)}Q^{k+1}J$ have isomorphic Margolis cohomology groups, with $H^*(\cdot,Q_0)$ concentrated in degree zero and $H^*(\cdot,Q_1)$ in degree $2(k+1)$.

Remark 7.1. The case which would correspond to $s = 1$ in Theorem 6.12 is exceptional, hence is excluded in this section.

Lemma 7.2. If $k \geq 1$, for $N = N_{k,\varepsilon} := \Sigma^{-(k+1)}Q^{k+1}J^\otimes\varepsilon$, $\varepsilon \in \{0,1\}$,

1. $N = N^{\geq0}$ and $N^0 = F$;
2. the cyclic submodule generated by $N^0$ is isomorphic to $\mathcal{A}(1) \otimes\mathcal{A}(0) F$;
3. there is a unique non-trivial morphism $N_{k,\varepsilon} \to F$ which represents a non-trivial class in $[\Sigma^{-(k+1)}Q^{k+1}J^\otimes\varepsilon,F] \cong \text{Ext}_{\mathcal{A}(1)}^{k+1,1}(F,J^\otimes\varepsilon)$ and this induces an isomorphism in $H^*(-,Q_0)$;
4. for $A_{k,\varepsilon}$ defined by the short exact sequence

$$0 \to A_{k,\varepsilon} \to \Sigma^{-(k+1)}Q^{k+1}J^\otimes\varepsilon \to F \to 0,$$

$A_{k,\varepsilon}$ is a reduced, finite $\mathcal{A}(1)$-module which is $Q_0$-acyclic and has Margolis cohomology groups $H^*(-,Q_1)$ isomorphic to $F$ concentrated in degrees $\{3,2(k+1)\}$.

Proof. The embedding of $\mathcal{A}(1) \otimes\mathcal{A}(0) F$ in $\Sigma^{-(k+1)}Q^{k+1}J^\otimes\varepsilon$ is provided by Proposition 3.4 which, together with Proposition 3.1, also suffices to show the one-dimensionality of $N^0$. The non-triviality of the stable class of the projection onto degree zero follows by the dual of Lemma 2.2 since $N$ is reduced (by definition of $\Omega$); the relevant Ext group is $F$ (see Section 10). That the projection induces an isomorphism in $H^*(-,Q_0)$ is clear.

The final statement follows from the fact that $N_{k,\varepsilon}$ is finite and from the calculation of the Margolis cohomology from the associated long exact sequences. □

For $1 \leq k \in \mathbb{N}$, the above Lemma provides two reduced $\mathcal{A}(1)$-modules $A_{k,0}$, $A_{k,1}$ which are not stably isomorphic. After applying $\Sigma^{-3}$, so that the $Q_1$-Margolis cohomology classes are in degrees $\{0,2k-1\}$, these fit into the family of modules classified by Theorem 6.12 for $s = k + 1$ ($s \geq 2$).

The modules $A_{k,\varepsilon}$ can be viewed as ‘normalized’ choices of orbit representatives using the existence of an inclusion of the question mark complex to provide the normalization; this fits into the philosophy espoused by Corollary 5.8.
Proposition 7.3. For $k \geq 1$ and $\varepsilon \in \{0, 1\}$,

\begin{itemize}
  \item[(1)] there is a monomorphism $\Sigma^6 \Omega^{-1} J \hookrightarrow A_{k, \varepsilon}$ and
  \[ \text{Hom}_{\mathscr{A}(1)}(\Sigma^6 \Omega^{-1} J, (\Omega \Sigma^{-3})^t A_{k, \varepsilon}) = \begin{cases} 
    \mathbb{F} & t \equiv 0 \mod (4) \\
    0 & \text{otherwise};
  \end{cases} \]
  \item[(2)] $A_{k, 0} \not\cong (\Omega \Sigma^{-3})^t A_{k, 1}$ for $t \in \mathbb{Z}$;
  \item[(3)] under the action of $\Sigma^3 \Omega$, the modules $\Sigma^{-3} A_{k, 0}$, $\Sigma^{-3} A_{k, 1}$ generate the set of stable isomorphism classes of modules classified by Theorem 6.12, for $s = k + 1$.
\end{itemize}

Proof. The embedding $\mathscr{A}(1) \otimes_{\mathscr{A}(0)} \mathbb{F} \hookrightarrow \Sigma^{-(k+1)} \Omega^{k+1} J \otimes \varepsilon$ of Lemma 7.2 restricts to the monomorphism $\Sigma^6 \Omega^{-1} J \hookrightarrow A_{k, \varepsilon}$ and this is the unique non-trivial such morphism. The triviality of $\text{Hom}_{\mathscr{A}(1)}(\Sigma^6 \Omega^{-1} J, (\Omega \Sigma^{-3})^t A_{k, \varepsilon})$ in other cases can either be proved by the methods of Section 3 or by using the calculations of Section 10 as follows. Namely, by Lemma 8.1 one can work with stable groups $\varepsilon \mathfrak{x}_{\mathscr{A}(1)}$ (see Section 10); the triviality of the relevant groups follows from Theorem 10.16 except in the case of the fundamental class ($\mu$ in the notation of Section 10), corresponding to $t \equiv 0 \mod (4)$.

The second statement follows as an immediate consequence; in order to obtain a non-trivial stable map in $[\Sigma^6 \Omega^{-1} J, (\Omega \Sigma^{-3})^t A_{k, 1}]$, one must take $t \equiv 0 \mod (4)$, so that $(\Omega \Sigma^{-3})^t A_{k, 0} \cong A_{k, 1}$. However, $A_{k, 0}$ and $A_{k, 1}$ are not stably isomorphic.

The final statement then follows from Theorem 6.12 and Proposition 6.9.

Proposition 7.4. For $k \geq 1$ and $\varepsilon \in \{0, 1\}$, there are isomorphisms:

\[ A_{k, \varepsilon} \cong \left\{ \Sigma^4 \varepsilon D((\Omega^{-1} \Sigma)^{k+1-2\varepsilon} P_0) \right\}^{\geq 2} \cong \left\{ \Sigma^4 \varepsilon (\Omega^{-1} \Sigma)^{k+1-2\varepsilon} D P_0 \right\}^{\geq 2}. \]

Proof. For $\varepsilon \in \{0, 1\}$, since $A_{k, \varepsilon}$ and $(\Omega \Sigma^{-1})^{k+1} J \otimes \varepsilon$ are finite $\mathscr{A}(1)$-modules, results dual to those of Section 4 imply that the injection $A_{k, \varepsilon} \hookrightarrow (\Omega^{-1})^{k+1} J \otimes \varepsilon$

\[ A_{k, \varepsilon} \cong A_{k, \varepsilon} \otimes D P_0 \rightarrow (\Omega^{-1})^{k+1} J \otimes \varepsilon \otimes D P_0 \cong \Sigma^4 \varepsilon (\Omega^{-1})^{k+1-2\varepsilon} D P_0 \cong \Sigma^4 \varepsilon D((\Omega^{-1} \Sigma)^{k+1-2\varepsilon} P_0), \]

which gives an isomorphism on the top $Q_1$-Margolis cohomology class.

Since $A_{k, \varepsilon} = (A_{k, \varepsilon})^{\geq 2}$, this yields a morphism:

\[ A_{k, \varepsilon} \rightarrow \left\{ \Sigma^4 \varepsilon D((\Omega^{-1} \Sigma)^{k+1-2\varepsilon} P_0) \right\}^{\geq 2}. \]

By inspection of the underlying $E(1)$-modules and the fact that it is non-trivial in $Q_1$-Margolis homology, one sees that the morphism induces an isomorphism on both $Q_0$- and $Q_1$-Margolis cohomology groups, hence is a stable isomorphism. Since both modules are reduced (by construction), it is an isomorphism.

Example 7.5. The behaviour of the families $A_{k, 1}$ and $A_{k, 0}$ can be understood by considering the cases $1 \leq k \leq 4$ (see Figure 5 for $A_{k, 1}$ and Figure 6 for $A_{k, 0}$). Namely, the module $A_{k+4, \varepsilon}$ is obtained from $A_{k, \varepsilon}$ obtained by extending (and shifting degrees) by

\[ \cdots \cdots \cdots \cdots \cdots [A_{k, \varepsilon}], \]

where the dotted $S^2$ hits the lowest-dimensional class.
Similarly, observe that $A_{1,1}$ occurs as a subobject of $A_{3,0}$, as predicted by Proposition 7.4; likewise $A_{1,0}$ occurs as a subobject of $A_{3,1}$. This behaviour is general and corresponds to (a shift of) an extension by

\[ A_k,\varepsilon \to A_k + 4,\varepsilon \]

The composition of two such extensions assures the passage $A_k,\varepsilon \to A_{k+4,\varepsilon}$, as is evident from

\[ \Omega DA_k,\varepsilon \to F \to \Sigma^{(k+1)}\Omega^{-(k+1)}f^{\otimes \varepsilon} \to DA_k,\varepsilon \to 0 \]

The behaviour of duality is straightforward:

**Proposition 7.6.** For $k \in \mathbb{N}$ and $\varepsilon \in \{0, 1\}$, there is a stable isomorphism:

\[ A_k,\varepsilon \simeq \Sigma^{-(k+1+6\varepsilon)}\Omega^{k+2+2\varepsilon}DA_k,\varepsilon \]

*Proof.* This is a standard argument, using duality at the level of the stable module category, which is a tensor triangulated category. The dual of the defining exact sequence is of the form

\[ 0 \to F \to \Sigma^{(k+1)}\Omega^{-(k+1)}f^{\otimes \varepsilon} \to DA_k,\varepsilon \to 0 \]

which gives a distinguished triangle:

\[ \Omega DA_k,\varepsilon \to F \to \Sigma^{(k+1)}\Omega^{-(k+1)}f^{\otimes \varepsilon} \to . \]
Forming the tensor product with $\Sigma^{-(k+1)}Q^{(k+1)}J^{\otimes \varepsilon}$ then yields the result, since the resulting distinguished triangle is isomorphic to the original one, by the uniqueness of the non-trivial map in $[\Sigma^{-(k+1)}Q^{(k+1)}J^{\otimes \varepsilon}, F]$, as stated in Lemma 7.2.

Namely, this implies that $\Sigma^{-(k+1)}\Omega^{(k+1)}J^{\otimes \varepsilon} \otimes \Omega DA_{k,\varepsilon} \simeq A_{k,\varepsilon}$, from which the result follows by using $A_{k,\varepsilon} \otimes J \simeq \Omega^2 \Sigma^{-\varepsilon} A_{k,\varepsilon}$, by Proposition 7.7. □

Notation 7.7. For $\varepsilon \in \{0, 1\}$ and $k \in \mathbb{N}$, let $\mathcal{O}_{k,\varepsilon}$ denote the set of stable isomorphism classes represented by $\{(\Sigma^{-3}\Omega)^tA_{k,\varepsilon}|0 \leq t \leq 3\}$.

Proposition 7.8 implies that, for $k \geq 1$, $\mathcal{O}_{k,0} \coprod \mathcal{O}_{k,1}$ is the set of stable isomorphism classes of $Q_\varepsilon$-acyclic, finite $\mathcal{O}(1)$-modules with $Q_1$-Margolis cohomology $F$ in degrees $\{0, 2k-1\}$ and zero otherwise.

Notation 7.8. Denote by $D_8$ the dihedral group of order 8, the non-trivial semi-direct product $\mathbb{Z}/4 \rtimes \mathbb{Z}/2$.

Corollary 7.9. For $\varepsilon \in \{0, 1\}$ and $1 \leq k \in \mathbb{N}$, the associations

\[
A \mapsto d_k A := \Sigma^{2k+5} DA
\]

\[
A \mapsto \Sigma^{-3}\Omega A,
\]

for $A \in \mathcal{O}_{k,\varepsilon}$, induce an action of the group $D_8$ on $\mathcal{O}_{k,\varepsilon}$. The action of the centre $\mathbb{Z}/2 \subseteq D_8$ corresponds to $A \mapsto A \otimes J$.

Moreover, the fixed points $\mathcal{O}_{k,\varepsilon}^{d_k}$ under the duality operator have cardinal:

\[
|\mathcal{O}_{k,\varepsilon}^{d_k}| = \begin{cases} 2 \quad k \equiv 0 \mod (2) \\ 0 \quad k \equiv 1 \mod (2) \end{cases}
\]

and $(\Sigma^{-3}\Omega)^tA_{k,\varepsilon}$ is a $d_k$-fixed point if and only if $(\Sigma^{-3}\Omega)^tA_{k,\varepsilon} \otimes J \simeq (\Sigma^{-3}\Omega)^{t+2}A_{k,\varepsilon}$ is.

Proof. By construction (and Proposition 7.7), the group $\mathbb{Z}/4$ acts on $\mathcal{O}_{k,\varepsilon}$ via $\Sigma^{-3}\Omega$. Moreover, Proposition 7.6 gives the stable isomorphism

\[
A_{k,\varepsilon} \simeq (\Sigma^{-3}\Omega)^{k+2+2\varepsilon}d_kA_{k,\varepsilon},
\]

which shows that $A \mapsto d_k A$ induces an action of $\mathbb{Z}/2$ on $\mathcal{O}_{k,\varepsilon}$. It is straightforward to verify that this defines an action of $D_8$ on $\mathcal{O}_{k,\varepsilon}$; moreover, since $J$ is self-dual, the action $A \mapsto A \otimes J \simeq \Sigma^{-6}\Omega^2 A$ (again applying Proposition 4.7) corresponds to the centre.

Equation (2) implies that

\[
d_k \left((\Sigma^{-3}\Omega)^tA_{k,\varepsilon}\right) \simeq (\Sigma^{-3}\Omega)^{-(t+k+2+2\varepsilon)}A_{k,\varepsilon},
\]

thus $(\Sigma^{-3}\Omega)^tA_{k,\varepsilon}$ is a fixed point for $d_k$ if and only if $2t + k + 2 + 2\varepsilon \equiv 0 \mod (4)$. This establishes the final statement. □

Example 7.10. This behaviour is illustrated by Example 6.11 in the case $s = 2$ (so that $k = 1$), there are no fixed points under the duality operator (see Figure 3) whereas there are two fixed points in the case $s = 3$ ($k = 2$) (see Figure 4). To indicate that the behaviour is independent of $\varepsilon$, the second orbit is illustrated in Figure 7.

8. Truncated projective spaces

This section shows how the cohomology of certain truncated projective spaces fits into the classification of Section 7. This leads to a conceptual understanding of the cohomological behaviour of these modules.

The mod 2 cohomology $H^*(\mathbb{R}P^\infty) \cong F[u]$ is well understood as a module over the Steenrod algebra, with structure entirely determined by the fact that it is an
unstable algebra. Since \( u^4 \) is annihilated by \( Sq^1, Sq^2 \), multiplication by \( u^4 \) induces a monomorphism of \( \mathcal{A}(1) \)-modules, leading to a form of periodicity of the \( \mathcal{A}(1) \)-structure. As indicated in Remark 4.6, the reduced cohomology \( P := \tilde{H}^*(RP^\infty) \) identifies as an \( \mathcal{A}(1) \)-module with \( \Sigma \Omega P_0 \).

The \( \mathcal{A}(1) \)-module \( P \) embeds in \( P_{-\infty} \), which corresponds to \( F[u^{\pm 1}] \); this works over the full Steenrod algebra \( \mathcal{A} \) but is simpler over \( \mathcal{A}(1) \), since this corresponds to inverting \( u^4 \) and imposing periodicity.

**Notation 8.1.** For \( a \leq b \in \mathbb{Z} \), let \( P^b_a \) denote the subquotient of \( P_{-\infty} \) of elements in degrees \([a, b] \).

**Remark 8.2.**

1. The \( \mathcal{A}(1) \)-modules \( P^b_a \) can be realized as the mod 2 cohomology of \( RP^b_a \), namely, for \( a > 0 \), truncated projective space and, for \( a \leq 0 \), the appropriate Thom spectrum.

2. Periodicity as \( \mathcal{A}(1) \)-modules gives \( \Sigma^4 P^b_a \cong P^{b+4}_{a+4} \) hence, up to suspension, it is sufficient to consider the cohomology of truncated projective spaces.

Duality in this context is straightforward:

**Lemma 8.3.** For \( a \leq b \in \mathbb{Z} \), there is an isomorphism:

\[
DP^b_a \cong \Sigma^{1-4b} P^{3b-a-1}_{3b-1}.
\]

**Proof.** It is a basic fact (see [Bru12] for example) that the dual of \( P_{-\infty} \) is \( \Sigma P_{-\infty} \). The dual of \( P^b_a \) is \((DP_{-\infty})^{-a}_b \) which is isomorphic to \((\Sigma P_{-\infty})^{-a}_b = \Sigma((P_{-\infty})^{-a-1}_{-b-1}) \). Using periodicity, shifting by 4b gives the statement.

Here only the \( Q_\nu \)-acyclic cases are considered, namely those of the form \( P^b_a \) with \( 1 \leq m \leq n \). The case \( m - n = 0 \) is easily understood, so henceforth suppose that \( m - n > 0 \).

**Theorem 8.4.** For natural numbers \( 1 \leq m < n \),

1. if \( m \equiv 1 \mod (2) \),

\[
P^{2n}_{2m-1} \cong \begin{cases} 
\Sigma^{2m-3}A_{n-m,0} & n - m \equiv 2, 3 \mod (4) \\
\Sigma^{2m-3}A_{n-m,1} & n - m \equiv 0, 1 \mod (4); 
\end{cases}
\]

2. if \( m \equiv 0 \mod (2) \),

\[
P^{2n}_{2m-1} \cong \begin{cases} 
\Sigma^{2m-3}(\Omega^{-1}\Sigma^3)A_{n-m,0} & n - m \equiv 0, 3 \mod (4) \\
\Sigma^{2m-3}(\Omega^{-1}\Sigma^3)A_{n-m,1} & n - m \equiv 1, 2 \mod (4); 
\end{cases}
\]
Proof. All the modules considered are reduced, hence it suffices to establish that there are stable isomorphisms of the above form.

Since the $A_{k,\varepsilon}$ generate the respective orbits (see Proposition 7.3), there exists a stable isomorphism of the form:

$$P_{2m-1}^{2n} \simeq \Sigma^{2m-3}(\Omega^{-1}A_{k,n-m,\varepsilon}^\infty),$$

where the suspension and the value $k = m - n$ are determined by the $Q_1$-Margolis cohomology groups. It remains to determine $t$ and $\varepsilon$ as functions of $m, n$.

In the case $m \equiv 1 \mod (2)$, by inspection $Sq^2$ acts trivially on the bottom class and there is an inclusion of the question mark complex (suitably suspended) into $P_{2m-1}^{2n}$ (compare Section 3). It follows that $t = 0$, by the characterization of $A_{k,\varepsilon}$ given in Proposition 7.3. To conclude in this case, it suffices to select $\varepsilon$ so that $A_{n-m,\varepsilon}$ does not have a Joker at the top; the choice is dictated by Proposition 7.4 (see also Figures 6 and 5).

If $m \equiv 0 \mod (2)$, consider $\Omega \Sigma^{-3} P_{2m-1}^{2n}$; there is once again an inclusion of the question mark complex (suitably suspended), so the normalization of Proposition 7.3 now gives:

$$P_{2m-1}^{2n} \simeq \Sigma^{2m-3}(\Omega^{-1}A_{k,n-m,\varepsilon}^\infty),$$

The argument proceeds as above, but taking into account the cyclic shift induced by $(\Omega^{-1}\Sigma^3)$.

□

Remark 8.5. This result explains the intricacy of some of the statements concerning the modules $P_{2m-1}^{2n}$ in [Dav74]. It is much simpler to work with the modules $A_{k,\varepsilon}$.

9. Dual Brown-Gitler modules

This section gives a description of the dual Brown-Gitler modules (more precisely, the associated reduced $\mathcal{A}(1)$-modules) in terms of the families defined in Section 7. Here the conclusion is simpler than for truncated projective spaces, although the structure of the modules is richer (see Theorem 9.14).

Remark 9.1. With topological applications in mind and to fix conventions, recall the families $B(k), B_0(k)$ indexed by $\mathbb{N}$, where $B(k)$ is the $k$th Brown-Gitler spectrum [BG73, GLM93] and $B_0(k)$ is used to denote the $k$th integral Brown-Gitler spectrum (cf. [Shi84], where a different indexing is used); the notation $B_0$ follows [Pea14]. The indexing below follows that of [GLM93], in particular there are homotopy equivalences $B(2k) \simeq B(2k + 1)$ and $B_0(2k) \simeq B_0(2k + 1)$, so consideration is limited to the even-indexed spectra.

There is a map $i_k : B_0(k) \to B(k)$ which induces the surjection on mod 2 cohomology

$$H^*(B(k)) \cong \mathcal{A}/\mathcal{A}\{\chi(S^q)|2i > k\} \to H^*(B_0(k)) \cong \mathcal{A}/\mathcal{A}\{Sq^1, \chi(S^q)|2i > k\},$$

where $\chi$ is the conjugation of $\mathcal{A}$.

The relationship between these families is made clearer by the following facts for $k \in \mathbb{N}$:

1. $B_0(4k + 2) \simeq B_0(4k);
2. $i_{4k+2}$ induces an equivalence

$$(S^0/2) \wedge B_0(4k) \simeq (S^0/2) \wedge B_0(4k + 2) \simeq B(4k + 2);$$

3. there is a cofibre sequence (see [Shi84])

$$B_0(4k + 4) \xrightarrow{i_{4k+4}} B(4k + 4) \to \Sigma B_0(4k)$$

(this choice of triangle is for later convenience).
Here the dual Brown-Gitler spectra \( DB(k) \) and \( DB_0(k) \) are of greater relevance, where \( D \) denotes Spanier-Whitehead duality. The mod 2 cohomology of \( DB(k) \) is related to the family of injective Brown-Gitler modules in unstable module theory \( [\text{Sch}94] \) by

\[
H^*(DB(k)) \cong \Sigma^{-k} J(k)
\]

where \( J(k) \) is the injective envelope in unstable modules of \( \Sigma^k \mathbb{F} \). In particular, the top dimensional cohomology class of \( DB(k) \) is in degree zero. The families of modules considered below correspond respectively to \( H^*(DB(k)) \) and \( H^*(DB_0(k)) \).

Recall that the dual Steenrod algebra \( \mathcal{A}^* \) is a polynomial algebra \( \mathbb{F}[\xi_i | i \geq 1] \) on generators of cohomological degree \( |\xi_i| = 1 - 2^i \); the generators can be replaced by their conjugates \( \xi_i := \chi(\xi_i) \).

The natural action of \( \mathcal{A} \) on its dual \( \mathcal{A}^* \) is on the right; since \( \mathcal{A} \) is a Hopf algebra (in particular, with conjugation) the category of right \( \mathcal{A} \)-modules is equivalent to that of left \( \mathcal{A} \)-modules. As noted in Remark 2.4 over \( \mathcal{A}(1) \) the identity \( \chi(Sq^i) = Sq^i \) for \( i \in \{1, 2\} \) renders the translation simple.

With respect to the algebra generators \( \xi_i \), the above action (on the right) is given by the action of the Steenrod total power

\[
\xi_n Sq = \sum_{i=0}^{n} \xi_{n-i}^{2^i},
\]

where \( \xi_0 \) here is interpreted as 1.

For current purposes, \( \xi_0 \) is considered as an independent generator of degree zero (with \( \xi_0 Sq = \xi_0 \)) and equip the generators \( \xi_i \) (for \( i \geq 0 \)) with weights:

\[
\text{wt}(\xi_i) = 2^i
\]

so that the action of the Steenrod algebra preserves the weights.

In particular, considering the left \( \mathcal{A}(1) \)-action, for \( n > 0 \):

\[
\begin{align*}
Sq^1 \xi_n &= \xi_{n-1}^2 \\
Sq^2 \xi_n &= 0 \\
Sq^2 (\xi_n)^2 &= \xi_{n-1}^4.
\end{align*}
\]

The sub-algebras:

\[
\mathcal{F}_0 := \mathbb{F}[\xi^0_0, \xi^1_1, \xi^2_2, \ldots] \hookrightarrow \mathcal{F} := \mathbb{F}[\xi^2_0, \xi^1_1, \xi^2_2, \ldots] \hookrightarrow \mathbb{F}[\xi_0, \xi_1, \xi_2, \ldots]
\]

are stable under the action of the Steenrod algebra. More precisely one has:

**Lemma 9.2.** There are weight decompositions in the category of \( \mathcal{A} \)-modules:

\[
\mathcal{F}_0 \cong \bigoplus_{i \geq 0} \mathcal{F}_0(4i)
\]

\[
\mathcal{F} \cong \bigoplus_{j \geq 0} \mathcal{F}(2j)
\]

where \( \mathcal{F}_0(n) \) (respectively \( \mathcal{F}(n) \)) is the subspace of \( \mathcal{F}_0 \) (resp. \( \mathcal{F} \)) of elements of weight \( n \). Moreover, each \( \mathcal{F}_0(n) \) (respectively \( \mathcal{F}(n) \)) is of finite total dimension.

**Remark 9.3.** There are isomorphisms of \( \mathcal{A}(1) \)-modules:

\[
\mathcal{F}(2n) \cong H^*(DB(2n))
\]

\[
\mathcal{F}_0(4n) \cong H^*(DB_0(4n)).
\]

The notation reflects the fact that the spectrum \( \Sigma^k DB(k) \) is usually written \( T(k) \), so that \( H^*(T(k)) \cong J(k) \). Thus \( \mathcal{F}(2n) \cong \Sigma^{-2n} J(2n) \).

**Example 9.4.** There is an isomorphism \( \mathcal{F}(2) \cong \mathbb{Z} \), with basis \( \{\xi_1, \xi_0^2\} \).
Using Remark 9.3, the following Lemma gives algebraic versions of the Shimamota exact sequences of Remark 9.1.

**Lemma 9.5.** As an $\mathcal{A}$-module, $\mathcal{F}$ is free on $\{1, \zeta_0^2, \zeta_1, \zeta_0^2 \zeta_1\}$, which are of weights 0, 2, 2, 4 respectively. In particular, for $n \in \mathbb{N}$:

$$\mathcal{F}(4n + 2) \simeq \mathcal{A}_0(4n) \otimes \mathcal{F}(2)$$

as $\mathcal{A}$-modules and there is a short exact sequence of $\mathcal{A}$-modules:

$$0 \to \mathcal{A}_0(4n + 4) \to \mathcal{F}(4n + 4) \to \Sigma^{-1} \mathcal{A}_0(4n) \to 0.$$

**Proof.** Straightforward (the desuspension in the short exact sequence arises since $\zeta_0^2 \zeta_1$ has degree $-1$).

The aim here is to understand the structure of the $\mathcal{A}_0(n)$ (respectively $\mathcal{F}(n)$) as $\mathcal{A}(1)$-modules; a first step is understanding the action of the Milnor primitives. The following is clear (cf. [Rav93], for example).

**Lemma 9.6.** For $i, n \in \mathbb{N}$

$$Q_i \zeta_n = \begin{cases} 0 & i \geq n \\ \zeta_n^{2^{i+1}} & i < n. \end{cases}$$

Since the Milnor operations acts as derivations, this allows the calculation of the Margolis cohomology groups. For $n \in \mathbb{N}$, write $\alpha(n)$ for the sum of the digits of its binary expansion.

**Lemma 9.7.** (Cf. [Ada74, Part III], [DGMS71] Lemma 3.12.) There are isomorphisms of algebras:

$$H^*(\mathcal{F}, Q_0) \cong \mathbb{F}$$
$$H^*(\mathcal{F}, Q_1) \cong \mathbb{F}[\zeta_1]/(\zeta_1^2) \otimes \bigotimes_{i \not\equiv 1} \mathbb{F}[\zeta_i]/(\zeta_i)^4$$
$$H^*(\mathcal{A}_0, Q_0) \cong \mathbb{F}[\zeta_0]$$
$$H^*(\mathcal{A}_0, Q_1) \cong \bigotimes_{i \geq 1} \mathbb{F}[\zeta_i]/(\zeta_i)^4.$$

In particular, for $n \in \mathbb{N}$, $\dim H^*(\mathcal{A}_0(4n), Q_0) = \dim H^*(\mathcal{A}_0(4n), Q_1) = 1$, so that the modules $\mathcal{A}_0(4n)$ represent elements of the Picard group $\text{Pic}_\mathcal{A}(1)$, with $H^0(\mathcal{A}_0(4n), Q_0) = \mathbb{F}$ and $H^0(\mathcal{A}_0(4n), Q_1)$ concentrated in degree

$$\sum_{j \in \mathcal{F}} 2(1 - 2^{j+1}) = 2(\alpha(n) - 2n),$$

where $n = \sum_{j \in \mathcal{F}} 2^j$ is the dyadic decomposition.

The modules $\mathcal{F}(2n)$ are $Q_0$-acyclic for $n > 0$ and there are isomorphisms:

$$H^*(\mathcal{F}(4n + 4), Q_1) \cong H^*(\mathcal{A}_0(4n + 4), Q_1) \oplus \Sigma^{-1} H^*(\mathcal{A}_0(4n), Q_1)$$
$$H^*(\mathcal{F}(4n + 2), Q_1) \cong H^*(\mathcal{A}_0(4n), Q_1) \otimes \mathcal{F}(2).$$

In particular, the modules $\mathcal{F}(2n)$, for $n > 0$ are stably isomorphic to $Q_0$-acyclic, indecomposable $\mathcal{A}(1)$-modules.

**Proof.** Straightforward.

The result of [DGMS71] Lemma 3.12 is more precise: the modules $\mathcal{A}_0(4.2^i)$, $i \in \mathbb{N}$, are identified explicitly in $\text{Pic}_\mathcal{A}(1)$ and it is observed that the multiplicative structure of $\mathcal{A}_0$ induces stable isomorphisms

$$\bigotimes_{j \in \mathcal{F}} \mathcal{A}_0(4.2^i) \simeq \mathcal{A}_0(4n),$$

for $n = \sum_{j \in \mathcal{F}} 2^j$. This is used in [Ada74, Part III] to understand the structure of the modules $\mathcal{A}_0(n)$.
where \( n = \sum_{j \in I} 2^{v_j} \) is the dyadic decomposition. The result can be restated as follows (cf. Example 9.5).

**Lemma 9.8.** For \( i \in \mathbb{N} \), there is a stable isomorphism of \( \mathfrak{A}(1) \)-modules:

\[
\mathcal{T}_0(4^i) \simeq \begin{cases} 
\Sigma \Omega^{-1} J & i = 0 \\
(\Sigma \Omega^{-1})^{2^{i+1}-1} F & i > 0.
\end{cases}
\]

Hence, for \( 0 < n \in \mathbb{N} \), \( \mathcal{T}_0(4n) \simeq (\Sigma \Omega^{-1})^{2n-\alpha(n)} \otimes J^{\otimes \overline{n}} \), where \( \overline{n} \) is the residue of \( n \) modulo 2.

A similar argument applies to the modules \( \mathcal{T}(2n) \):

**Lemma 9.9.** For \( 0 < n \in \mathbb{N} \) with \( 2 \)-adic valuation \( \nu = \nu(n) \), the structure of \( \mathcal{T} \) as a \( \mathcal{T}_0 \)-module induces a stable isomorphism

\[
\mathcal{T}(2n) \simeq \mathcal{T}_0(2n - 2^{\nu+1}) \otimes \mathcal{T}(2^{\nu+1}).
\]

Hence \( \Sigma^{2n} \mathcal{T}(2n) \simeq (\Sigma \Omega^{-1})^{1-\alpha(n)} (\Sigma^{2^{\nu+1}} \mathcal{T}(2^{\nu+1})) \).

**Proof.** For notational simplicity, write \( n = 2m + 2^\nu \) (so that \( m = 2^\nu t \) for some \( t \in \mathbb{N} \)); observe that \( \alpha(m) = \alpha(n) - 1 \).

In the case \( \nu = 0 \), the stable isomorphism \( \mathcal{T}(2n) \simeq \mathcal{T}_0(4m) \otimes \mathcal{T}(2^{\nu+1}) \) follows immediately from Lemma 9.8, hence consider the case \( \nu > 0 \) (so that \( 2n \equiv 0 \pmod{4} \)). Both sides of the expression are \( Q_0 \)-acyclic and Lemma 9.7 shows that they have isomorphic \( Q_1 \)-Margolis cohomology groups, hence it remains to check that the multiplication induces an isomorphism in \( H^*(\mathcal{T}, Q_1) \); this can be seen more explicitly as follows.

The multiplication together with the short exact sequences given by Lemma 9.5 yield a commutative diagram

\[
\begin{array}{ccc}
\mathcal{T}_0(4m) \otimes \mathcal{T}_0(2^{\nu+1}) & \longrightarrow & \mathcal{T}_0(4m) \otimes \mathcal{T}(2^{\nu+1}) \\
\downarrow & & \downarrow \\
\mathcal{T}_0(2n) & \longrightarrow & \Sigma^{-1} \mathcal{T}_0(2n - 4)
\end{array}
\]

in which the rows are short exact. The outer vertical morphisms are stable isomorphisms by [DGM81, Lemma 3.12], hence so is the middle one.

Lemma 9.8 identifies \( \mathcal{T}_0(4m) \) up to stable isomorphism, which gives:

\[
\mathcal{T}(2n) \simeq (\Sigma \Omega^{-1})^{2m-\alpha(m)} \mathcal{T}(2^{\nu+1}) \otimes J^{\otimes \overline{m}}
\]

where \( \overline{m} \in \{0, 1\} \) is the residue of \( m \) modulo 2.

The factor \( J^{\otimes \overline{m}} \) can be removed using the fact that \( \mathcal{T}(2^{\nu+1}) \) is \( Q_0 \)-acyclic, hence Proposition 4.7 implies that \( \mathcal{T}(2^{\nu+1}) \otimes J^{\otimes \overline{m}} \simeq \Sigma^{-\overline{m}} \Omega^{-1} \mathcal{T}(2^{\nu+1}) \). It follows that

\[
\mathcal{T}(2n) \simeq \Sigma^{-\overline{m}} (\Sigma \Omega^{-1})^{2(m-\overline{m})-\alpha(m)} \mathcal{T}(2^{\nu+1}).
\]

The result is then rewritten in terms of \( \Sigma^k \mathcal{T}(k) \) for \( k \in \{2n, 2^{\nu+1}\} \), which gives

\[
\Sigma^{2n} \mathcal{T}(2n) \simeq (\Sigma \Omega^{-1})^{2(m-\overline{m})} (\Sigma \Omega^{-1})^{1-\alpha(m)} (\Sigma^{2^{\nu+1}} \mathcal{T}(2^{\nu+1})).
\]

Now, \( m - \overline{m} \equiv 0 \pmod{2} \), so \( 2(m - \overline{m}) \equiv 0 \pmod{4} \). Thus Corollary 7.9 implies that

\[
(\Sigma \Omega^{-1})^{2(m-\overline{m})} \mathcal{T}(2^{\nu+1}) \simeq \mathcal{T}(2^{\nu+1});
\]

this yields the final statement. \( \square \)

It remains to identify the stable isomorphism classes of the \( \mathcal{T}(2^{\nu+1}) \). The case \( \nu = 0 \) is already known (see Example 9.4). The general case is addressed using the algebraic Mahowald exact sequences [Mah77].
Lemma 9.10. For $n \in \mathbb{N}$, there is a short exact sequence of $A(1)$-modules:

$$0 \to T(2(2n-1)) \to T(4n) \to \Sigma^{-2n}T(2n) \to 0.$$ 

Proof. The submodule $T(2(2n-1))$ corresponds to those terms which are divisible by $\zeta_2^n$. The quotient is identified by re-indexing by $\zeta_i \mapsto \zeta_i - 1$ (which divides weights by two); this gives the suspension, since the highest dimensional term in the quotient is $\zeta_2^n$ (which has degree $-2n$), which re-indexes to $\zeta_0^{2n}$ (of degree zero). \qed

This is applied in the case $n = 2^{\nu-1}$ (for $\nu > 0$):

Lemma 9.11. For $0 < \nu \in \mathbb{N}$, there is a short exact sequence

$$0 \to T(4(2^{\nu-1} - 1) + 2) \to T(2^{\nu+1}) \to \Sigma^{-2\nu}T(2^\nu) \to 0.$$ 

Moreover, there is a stable isomorphism $T(4(2^{\nu-1} - 1) + 2) \simeq \Sigma^{-2\nu-2} \Omega^{\nu-1}Z$.

For $\nu = 1$, this gives a non-trivial short exact sequence:

$$0 \to Z \to T(4) \to \Sigma^{-2}Z \to 0.$$ 

Proof. The short exact sequences are provided by Lemma 9.10.

By Lemma 9.10 there is a stable isomorphism

$$\Sigma^{2^{\nu+1}-2}T(4(2^{\nu-1} - 1) + 2) \simeq (\Sigma\Omega^{-1})^{1-\nu}(\Sigma^2T(2)),$$

since $\alpha(2^{\nu-1} - 1) = \nu - 1$. Since $T(2) \simeq Z$, rewriting provides the stated stable isomorphism. \qed

Lemma 9.12. For $0 < \nu \in \mathbb{N}$, there is a unique non-trivial morphism

$$\Sigma^{5-2^\nu+1}\Omega^{-1}J \to T(2^{\nu+1})$$

and this is a monomorphism.

Proof. The inclusion of the desuspension of the question-mark complex is given by inspection of the low-dimensional structure of $T(2^{\nu+1})$, which is represented by:

$$\begin{array}{c}
\zeta_0^{4-1} \\
\zeta_0^{2} \zeta_{\nu-1} \\
\zeta_2 \\
\zeta_{\nu+1}
\end{array}$$

where the element $\zeta_2^2$ of degree $2(1 - 2^\nu)$ represents the lowest dimensional $Q_1$ Margolis cohomology class. \qed

Remark 9.13. For $\nu = 1$, the diagram in the proof of Lemma 9.12 describes the structure of $T(4)$, which corresponds (up to suspensions) with the reduced cohomology of $\mathbb{R}P^4$.

Theorem 9.14. For $0 < n \in \mathbb{N}$ with 2-adic valuation $\nu := \nu(n)$, there is a stable isomorphism

$$\Sigma^{2\nu}T(2n) \simeq (\Sigma\Omega^{-1})^{1-\alpha(n)}(\Sigma^{-1}A_{\nu,1}),$$

where one takes $A_{0,1} := \Sigma^3Z$. 

Proof. By Lemma 9.9 it suffices to show that $\Sigma^{2^\nu+1} \mathcal{F}(2^{\nu+1}) \simeq \Sigma^{-1}A_{\nu,1}$, for $\nu \in \mathbb{N}$.

The case $\nu = 0$ follows by the identification $\mathcal{F}(2) \simeq \mathbb{Z}$, hence suppose that $\nu > 0$. Lemma 9.12 provides an embedding $\Sigma^{-2\nu} \Omega^{-1}J \to \mathcal{F}(2^{\nu+1})$. Thus, by the normalization result of Proposition 7.3 one has

$$\mathcal{F}(2^{\nu+1}) \simeq \Sigma^{-(1+2^{\nu+1})}A_{k,\varepsilon}$$

for some $\varepsilon \in \{0, 1\}$ and $k \in \mathbb{N}$. The degrees of the $Q_1$-Margolis cohomology classes of $\mathcal{F}(2^{\nu+1})$ imply that $k = \nu$.

To show that $\varepsilon = 1$, one uses the Mahowald extension of Lemma 9.11. By considering the behaviour of the underlying $E(1)$-modules, the monomorphism

$$\Sigma^{5-\nu-2^{\nu+1}} \Omega^{-1}Z \hookrightarrow \mathcal{F}(2^{\nu+1})$$

must correspond to a non-trivial class in

$$[\Sigma^{5-\nu-2^{\nu+1}} \Omega^{-1}Z, \Sigma^{-(1+2^{\nu+1})}A_{\nu,\varepsilon}] = [\Sigma^5(\Sigma^{-1}\Omega)^{-1}Z, A_{\nu,\varepsilon}] = [Z, \Sigma^{-5}(\Sigma^{-1}\Omega)^{1-\nu}A_{\nu,\varepsilon}].$$

Now, by Proposition 7.3 $A_{\nu,\varepsilon}$ is a submodule of $\Sigma^{4\varepsilon}(\Omega^{-1})^{\nu+1-2s}D_P$. Moreover, as in Lemma 8.7, one can replace stable morphisms by $\text{Hom}_{\mathcal{A}(1)}$:

$$[Z, \Sigma^{-5}(\Sigma^{-1}\Omega)^{1-\nu}A_{\nu,\varepsilon}] \cong \text{Hom}_{\mathcal{A}(1)}(Z, \Sigma^{-5}(\Sigma^{-1}\Omega)^{1-\nu}A_{\nu,\varepsilon}) \cong \text{Hom}_{\mathcal{A}(1)}(Z, \Sigma^{4\varepsilon-5}(\Sigma^{-1}\Omega)^{2-2s}D_P),$$

where the second isomorphism is proved by a simple connectivity argument that shows that the truncation in Proposition 7.3 does not affect the value.

Finally, inspection shows that the latter group is non-trivial if $\varepsilon = 1$ but zero if $\varepsilon = 0$ (in which case, the presence of a Joker provides the obstruction). \qed

10. Ext calculations

This section exploits the presentation of the $\mathcal{A}(1)$-modules $A_{k,\varepsilon}$ in terms of the elements of the Picard group $\text{Pic}_{\mathcal{A}(1)}$ to give a uniform approach to $\text{Ext}_{\mathcal{A}(1)}$-calculations for these families; these become particularly transparent when graded over $\text{Pic}_{\mathcal{A}(1)}$. The main result is Theorem 10.16 which encompasses earlier results of Davis [Dav74] and Pearson [Pea14]. For an explanation of the ext charts which illustrate these results, see Convention 10.3.

Remark 10.1. These calculations have immediate applications via the Adams spectral sequence, as in the work of Davis and Pearson cited above.

It is useful to work with the stable form of $\text{Ext}$ defined using the stable $\mathcal{A}(1)$-module category and to grade by the Picard group $\text{Pic}_{\mathcal{A}(1)} \cong \mathbb{Z}^{\oplus 2} \oplus \mathbb{Z}/2$ (cf. Section 4), taking $(s, t, \varepsilon) \in \mathbb{Z}^{\oplus 2} \oplus \mathbb{Z}/2$ to correspond to $\Omega^{-s}\Sigma^t J^{\varepsilon} \in \text{Pic}_{\mathcal{A}(1)}$.

Definition 10.2. For $\mathcal{A}(1)$-modules $M, N$ and $(s, t, \varepsilon) \in \text{Pic}_{\mathcal{A}(1)}$,

$$\text{Ext}_{\mathcal{A}(1)}^{s, t, \varepsilon}(M, N) := [M, \Omega^{-s}\Sigma^t J^{\varepsilon}N].$$

By convention, $\varepsilon$ is taken to be zero where only a bigrading $(s, t)$ is specified.

Notation 10.3. For $\mathcal{A}(1)$-modules $M, N$, write $\mathcal{E}\text{xt}^{s, t}_{\mathcal{A}(1)}(M, N)$ for the $\text{Pic}_{\mathcal{A}(1)}$-graded stable ext groups.

Similarly, $\text{Ext}$ in the category of $\mathcal{A}(1)$-modules is bigraded $\text{Ext}_{\mathcal{A}(1)}^{s, t}(M, N) = \text{Ext}_{\mathcal{A}(1)}^{s}(M, \Sigma^t N)$, for $s \in \mathbb{N}$ and $t \in \mathbb{Z}$. There is a natural morphism

$$\text{Ext}_{\mathcal{A}(1)}^{s, t}(M, N) \to \mathcal{E}\text{xt}_{\mathcal{A}(1)}^{s, t}(M, N)$$
which is an isomorphism for $s > 0$ and is surjective for $s = 0$; Lemma 2.22 gives a
criterion for an isomorphism in degree zero. Whereas $\text{Ext}^{s,t}_{\mathcal{A}(1)}(M,N)$ is trivial
for $s < 0$, $\text{Ext}^{s,t}_{\mathcal{A}(1)}(M,N)$ is highly non-trivial in general.

**Remark 10.4.** With topological applications via the Adams spectral sequence in
view, Adams indexing $(t-s,s)$ is frequently used. Observe that $\text{Ext}^{s,t}_{\mathcal{A}(1)}(M,N) \cong
[(\Sigma^{-1}\Omega)^s M, \Sigma^{t-s} N]$, so it can be useful to use $\Sigma^{-1}\Omega$ in place of $\Omega$.

There is an exterior product:
$$\mathcal{E}xt_{\mathcal{A}(1)}(\mathbb{F}, M_1) \otimes \mathcal{E}xt_{\mathcal{A}(1)}(\mathbb{F}, M_2) \to \mathcal{E}xt_{\mathcal{A}(1)}(\mathbb{F}, M_1 \otimes M_2)$$
induced by the tensor structure of the stable module category. In particular
$\mathcal{E}xt_{\mathcal{A}(1)}(\mathbb{F}, \mathbb{F})$ has the structure of a commutative, Pic$_{\mathcal{A}(1)}$-graded $\mathbb{F}$-algebra and,
for $M$ an $\mathcal{A}(1)$-module, $\mathcal{E}xt_{\mathcal{A}(1)}(\mathbb{F}, M)$ is naturally a Pic$_{\mathcal{A}(1)}$-graded module over
this algebra.

**Lemma 10.5.** (Cf. [Dav74] Lemma 3.3 and [BG10], for example). The sub
Pic$_{\mathcal{A}(1)}$-graded algebra of $\mathcal{E}xt_{\mathcal{A}(1)}(\mathbb{F}, \mathbb{F})$ of classes with $t-s \geq 0$ is isomorphic to
$$\mathbb{F}[h_0, h_1, \kappa, \alpha]/(h_0 h_1, h_1^3, h_1 \kappa, \kappa^2 - h_0^2),$$
where, with respect to Adams indexing $(t-s,s,e)$:
$$|h_0| = (0, 1, 0), \quad |h_1| = (1, 1, 0), \quad |\kappa| = (0, 1, 1), \quad |\alpha| = (4, 2, 1).$$
For $a, b$ the generators of Adams indexing $|a| = (4, 3, 0)$ and $|b| = (8, 4, 0)$, one has:
$$a = \kappa \alpha, \quad b = \alpha^2.$$  
Hence there is an isomorphism of Pic$_{\mathcal{A}(1)}$-graded algebras
$$\mathcal{E}xt_{\mathcal{A}(1)}(\mathbb{F}, \mathbb{F})[\alpha^{-1}] \cong \mathbb{F}[h_0, h_1, \kappa, \alpha^\pm 1]/(h_0 h_1, h_1^3, h_1 \kappa, \kappa^2 - h_0^2),$$

**Remark 10.6.**
(1) The class $b$ is the usual algebraic Bott class. The relations $b = \alpha^2$, $a = \kappa \alpha$
and $\kappa^2 = h_0^2$ imply the standard relation $a^2 = h_2 b$. Inverting $\alpha$ is equivalent
to inverting $b$.

(2) There is a non-trivial class in $\mathcal{E}xt_{\mathcal{A}(1)}^*(\mathbb{F}, J)$ of Adams index $(t-s,s) =
(-2,0)$; multiplication by $b$ acts injectively on this class. Hence the subalgebra
of Lemma 10.5 of elements with $t-s \geq 0$ does not contain $\text{Ext}^t(J, J)$.

(3) For $t-s \geq 0$, the above exhausts $\mathcal{E}xt_{\mathcal{A}(1)}$; however, as indicated in Figures
11 and 12, the third quadrant corresponds to the first by rotational symmetry about
$(-2.5, -0.5)$ hence is highly non-trivial. (See Example 10.11 below for indications on how to derive this symmetry.)

**Notation 10.7.** Write $\mathcal{X}$ for the Pic$_{\mathcal{A}(1)}$-graded algebra
$$\mathcal{X} := \mathcal{E}xt_{\mathcal{A}(1)}(\mathbb{F}, \mathbb{F})[b^{-1}],$$
corresponding to the Bott-periodic version of stable ext.

**Convention 10.8.** As usual, $\mathcal{E}xt_{\mathcal{A}(1)}$ charts are indicated in the $(t-s,s)$-plane and
multiplication by $h_0$ is indicated by a vertical line and by $h_1$ by a diagonal line,
where $h_0, h_1$ are the usual generators in $\text{Ext}^{1,*}_{\mathcal{A}(1)}(\mathbb{F}, \mathbb{F})$.

Where the chart is $b$-periodic, only a portion of the chart which generates under
$b$-periodicity is given; in the remaining cases considered here, the charts can be
completed in the left and right hand half planes by understanding the action of $b$. 
The chart for $\mathcal{E}xt^{*,*}_{\mathfrak{A}(1)}(F,F)$ is given in Figure 8 and for $\mathcal{E}xt^{*,*}_{\mathfrak{A}(1)}(F,J)$ in Figure 9.

The following is a consequence of Proposition 4.7, an algebraic version of the existence of a $v_1$-self map.

**Lemma 10.9.** Let $M$ be a bounded-below $\mathfrak{A}(1)$-module which is $Q_0$-acyclic. Then

1. multiplication by $b$ on $\mathcal{E}xt^{*,*}_{\mathfrak{A}(1)}(F,M)$ is an isomorphism, hence $\mathcal{E}xt^{*,*}_{\mathfrak{A}(1)}(F,M)$ is a $\mathcal{K}$-module;
2. the stable isomorphism $M \otimes J \simeq \Sigma^{-6}\Omega^2 M$ induces an isomorphism

$$\mathcal{E}xt^{s,t,\epsilon}_{\mathfrak{A}(1)}(F,M) \cong \mathcal{E}xt^{s+2, t+6, \epsilon+1}_{\mathfrak{A}(1)}(F,M)$$

of $\mathcal{K}$-modules, which corresponds to multiplication by $\alpha$.

In particular, if $M$ is reduced, then $\text{Ext}^{*,*}_{\mathfrak{A}(1)}(F,M)$ is determined by $\mathcal{E}xt^{s \geq 0, t \geq 0}_{\mathfrak{A}(1)}(F,M)$ via the periodicity isomorphism

$$\mathcal{E}xt^{s,1}_{\mathfrak{A}(1)}(F,M) \cong \mathcal{E}xt^{s+4N, t+12N}_{\mathfrak{A}(1)}(F,M)$$
Figure 10. The chart for $\text{Ext}_{\mathcal{A}(1)}^{*,*}(\mathcal{F}, Z)$

for $N \in \mathbb{N}$ such that $t - s + 8N \geq 0$.

**Proof.** The first two statements follow from Proposition 4.7, using the more precise formulation given in Theorem 4.5 that the stable isomorphism $\Sigma^{-6}\Omega^2 P_0 \to P_0 \otimes J$ is induced by $\alpha$ so that the stable isomorphism $\Sigma^{-12}\Omega^4 P_0 \to P_0$ is induced by $b$.

The final statement follows by using periodicity with respect to multiplication by $b$ to extend to the upper half plane $s \geq 0$. The hypothesis that $M$ is reduced is required to identify $\text{Ext}_{\mathcal{A}(1)}^{0}(F, Z)$ with $\text{Hom}_{\mathcal{A}(1)}^{1}(F, Z)$ by Lemma 2.2. □

**Example 10.10.** Lemma 10.9 applies to:

1. $M = P_0$;
2. the reduced cohomology of the truncated projective spaces, the modules $P_{2n-1}^m$ of Section 8 (cf. [Dav74, Theorem 3.4]);
3. the modules $T(2n)^2$ of Section 9.

**Example 10.11.** The case $M = Z$ in Lemma 10.9 is of particular interest. There is a short exact sequence of $\mathcal{A}(1)$-modules:

$$0 \to F \to Z \to \Sigma^{-1}F \to 0.$$  

Lemma 10.9 implies that $\text{Ext}_{\mathcal{A}(1)}^{0}(F, Z)$ is $b$-periodic, hence is determined by the chart given in Figure 10, which indicates that $\text{Ext}_{\mathcal{A}(1)}^{0}(F, Z)$ is annihilated by $h_0$.

The long exact sequence for $\text{Ext}_{\mathcal{A}(1)}^{0}$ associated to the short exact sequence (3) has connecting morphism corresponding to multiplication by $h_0$. This leads to an understanding of the symmetry for $\text{Ext}_{\mathcal{A}(1)}^{0}(F, F)$ as indicated in Figures 8 and 9.

A further useful fact is provided by the results of Section 3, which yield vanishing lines for $\text{Ext}_{\mathcal{A}(1)}^{0}(F, M)$.

**Proposition 10.12.** Let $M$ be a bounded-below, $Q_0$-acyclic $\mathcal{A}(1)$-module such that $H^*(M, Q_1) = 0$ for $* \notin [d_1, d_2]$, $d_1 \leq d_2 \leq \infty$. Then $\text{Ext}_{\mathcal{A}(1)}^{s,t}(\mathcal{F}, M) = 0$ for either of the following:

$$t - s < 2s - d_2 - 3$$

$$t - s > 2s - d_1.$$

In particular, $h^m_0 \text{Ext}_{\mathcal{A}(1)}^{s,t}(\mathcal{F}, M) = 0$, where $m = 1 + \lfloor \frac{d_2 - d_1 + 3}{2} \rfloor$.

**Proof.** First consider the case $s = 0$; we may assume that $M$ is reduced, so that the results of Section 3 imply that $\text{Hom}_{\mathcal{A}(1)}(\Sigma^k\mathcal{F}, M) = 0$ for $k < d_1$ and $k > d_2 + 3$, which gives the bounds in this case.

For the general case, by Lemma 2.7, the Margolis cohomology $H^*((\Sigma^{-1}\Omega)^{-s}M, Q_1)$ is concentrated in $[d_1 - 2s, d_2 - 2s]$. The result now follows from the above. □

The groups $\text{Ext}_{\mathcal{A}(1)}^{s,t}(\mathcal{F}, A_{k,\varepsilon})$ for $k \geq 1$ and $\varepsilon \in \{0, 1\}$ can be calculated using the presentation given in Section 7

$$0 \to A_{k,\varepsilon} \to (\Omega \Sigma^{-1})^{k+1} J^{\otimes \varepsilon} \to \mathcal{F} \to 0.$$

(4)
Lemma 10.13. For $k \geq 1$ and $\varepsilon \in \{0,1\}$, the morphism induced by $(\Omega\Sigma^{-1})^{k+1} J^{\otimes \varepsilon} \to F$, 
\[ \mathcal{E}xt_{\mathcal{A}'}(F, (\Omega\Sigma^{-1})^{k+1} J^{\otimes \varepsilon}) \to \mathcal{E}xt_{\mathcal{A}'}(F, F), \]
is multiplication by $h_0^{k+1-\varepsilon} \kappa^\varepsilon$.

Proof. The element of $[F, (\Omega\Sigma^{-1})^{-(k+1)} J^{\otimes \varepsilon}]$ corresponding to the morphism is non-zero, hence corresponds to the unique non-zero class given. \(\square\)

The short exact sequence \(\mathcal{E}xt_{\mathcal{A}'}(F, F)\) corresponds to a distinguished triangle in the stable module category:

\[ \Omega F \to A_{k,\varepsilon} \to (\Omega\Sigma^{-1})^{k+1} J^{\otimes \varepsilon} \to . \]

Notation 10.14. For $k \geq 1$ and $\varepsilon \in \{0,1\}$, denote by

1. $\mu_{k,\varepsilon} \in \mathcal{E}xt_{\mathcal{A}'}(F, A_{k,\varepsilon})$ the image of $1 \in \mathcal{E}xt_{\mathcal{A}'}(F, F) = \mathcal{E}xt_{\mathcal{A}'}(F, \Omega F)$ under the morphism induced by $\Omega F \to A_{k,\varepsilon}$;
2. $\lambda_k \in \mathcal{E}xt_{\mathcal{A}'}(F, A_{k,1})$ the class such that $h_0 \lambda_k$ is the pre-image of the class $\alpha h_1 \in \mathcal{E}xt_{\mathcal{A}'}(F, J)$ under the morphism $A_{k,1} \to (\Omega\Sigma^{-1})^{k+1} J$;
3. $\nu_k \in \mathcal{E}xt_{\mathcal{A}'}(F, A_{k,0})$ the pre-image of the class $h_1 \in \mathcal{E}xt_{\mathcal{A}'}(F, F)$ under the morphism $A_{k,0} \to (\Omega\Sigma^{-1})^{k+1} F$.

The Adams indexing of these classes are $|\mu_{k,\varepsilon}| = (-1, 1)$, $|\lambda_k| = (-3, k)$ and $|\nu_k| = (1, k+2)$.

Remark 10.15. The classes $\lambda_k$ and $\nu_k$ are well-defined, since it is easily seen that there are unique non-trivial classes in the given degrees (cf. Theorem 10.16). The Adams indexing can be determined as follows:

1. $\alpha h_1$ corresponds to a class in $\mathcal{E}xt_{\mathcal{A}'}(F, (\Omega\Sigma^{-1})^{k+1} J)$ of Adams indexing $(5, k+4)$ and the class $b$ has Adams indexing $(8, 4)$;
2. $h_1$ corresponds to a class in $\mathcal{E}xt_{\mathcal{A}'}(F, (\Omega\Sigma^{-1})^{k+1} F)$ of Adams indexing $(1, k+2)$.

The following result constitutes a complete calculation of the $\mathcal{E}xt_{\mathcal{A}'}$-groups for the modules which are classified by Theorem 6.12 (up to shifting by elements of the Picard group $\text{Pic}_{\mathcal{A}'}(1)$).

Theorem 10.16. Let $1 \leq k$ be an integer.

1. There is an isomorphism of $\mathcal{K}$-modules:
   \[ \mathcal{E}xt_{\mathcal{A}'}(F, A_{k,1}) \cong \mathcal{K}(\mu_{k,1}, \lambda_k) / \sim \]
   with relations:
   \[
   a \lambda_k = \kappa \lambda_k = h_0 \lambda_k = 0 \\
   h_1^2 \lambda_k = h_0^{k+1} \mu_{k,1} \\
   h_0^k \kappa = 0;
   \]
   hence $h_0^k a \mu_{k,1} = 0$.
   In particular, $h_0^{k+2} \mathcal{E}xt_{\mathcal{A}'}(F, A_{k,1}) = 0$.
2. There is an isomorphism of $\mathcal{K}$-modules:
   \[ \mathcal{E}xt_{\mathcal{A}'}(F, A_{k,0}) \cong \mathcal{K}(\mu_{k,0}, \nu_k) / \sim \]
   with relations:
   \[
   a \nu_k = \kappa \nu_k = h_0 \nu_k = 0 \\
   h_1^2 \nu_k = h_0^k a \mu_{k,0} \\
   h_0^{k+1} \mu_{k,0} = 0.
   \]
   In particular, $h_0^{k+1} \mathcal{E}xt_{\mathcal{A}'}(F, A_{k,0}) = 0$. 
Proof. In both cases, consider the long exact sequence for $\mathcal{E}xt_{\mathcal{A}}(1)$ associated to the short exact sequence (4); Lemma 10.9 implies that $\alpha$ (hence $b$) is invertible on $\mathcal{E}xt_{\mathcal{A}}(1)(\mathbb{F}, A_{k,\varepsilon})$ hence the long exact sequence can be localized by inverting $\alpha$. The connecting morphism then corresponds to multiplication by $h_0^{k+1-\varepsilon} \kappa$ acting on $\mathcal{K}$, by Lemma 10.13. Since $\kappa^2 = h_0^{k+1}$, in both cases the kernel of multiplication by $h_0^{k+1-\varepsilon} \kappa$ lies in the $h_0$-torsion of $\mathcal{K}$, which coincides with the space annihilated by $h_0$. This torsion gives rise respectively to the classes generated over $\mathcal{K}$ by $\lambda_k$ and $\nu_k$.

From the long exact sequence, the structure of $\mathcal{E}xt_{\mathcal{A}}(1)(\mathbb{F}, A_{k,\varepsilon})$ as a $\text{Pic}_{\mathcal{A}}(1)$-graded vector space follows. To identify the module structure, the only non-trivial points to verify are the relations

1. $h_0^k \lambda_k = h_0^{k+1} \mu_{k,1}$;
2. $h_0^k \nu_k = h_0^{k+1} a \mu_{k,0}$.

These can be established by comparison with other known calculations, for example, via Proposition 7.4 with $\mathcal{E}xt_{\mathcal{A}}(1)(\mathbb{F}, DP_0)$ (see [BG10, Appendix A]). □

Example 10.17. The chart in Figure 11 illustrates the case $A_{(2,1)}$ and Figure 12 the case $A_{2,0}$.

Remark 10.18.

1. Results corresponding to Theorem 10.16 occur in the work of Davis [Dav74, Theorem 3.4 and Lemma 3.10] for truncated projective spaces. The usage of $\mathcal{E}xt_{\mathcal{A}}(1)$ in place of Ext greatly simplifies the presentation of these structures and the separation into the two cases $A_{k,0}$, $A_{k,1}$ makes the behaviour more transparent.

2. By the analysis of the modules $\mathcal{T}(2n)$ in Section 9 and the stable isomorphism $\Sigma^{1+2^r+1} \mathcal{T}(2^{r+1}) \simeq A_{0,1}$, the case $\varepsilon = 1$ of Theorem 10.16 gives a calculation of $\mathcal{E}xt_{\mathcal{A}}(1)(\mathbb{F}, \mathcal{T}(2n))$, for any $n \in \mathbb{N}$. 

---

**Figure 11.** The chart for $\mathcal{E}xt_{\mathcal{A}}(1)(\mathbb{F}, A_{2,1})$

```
\[ \lambda_2 \quad \mu \\
(1,1) \quad \mu \]
```

**Figure 12.** The chart for $\mathcal{E}xt_{\mathcal{A}}(1)(\mathbb{F}, A_{2,0})$

```
\[ \nu_2 \quad \mu \\
(1,1) \quad \mu \]
```
Remark 10.19. There is more structure available when considering $\text{Ext}_{\mathfrak{A}(1)}(\mathcal{F}, \mathcal{T}(2n))$.

1. The definition of $\mathcal{T}(2n)$ via the weight splitting of $\mathcal{T} := \mathbb{F}[\zeta_0^2, \zeta_1, \zeta_2, \ldots]$ provides $\mathfrak{A}(1)$-linear maps
   $$\mathcal{T}(2i) \otimes \mathcal{T}(2j) \rightarrow \mathcal{T}(2i + 2j)$$
   and hence $\text{Ext}_{\mathfrak{A}(1)}(\mathcal{F}, \mathcal{T}(2j))$ has an algebra structure over $\mathcal{K}$.

2. The spaces $\text{Ext}_{\mathfrak{A}(1)}(\mathcal{F}, A_{k,1})$, for $k \in \mathbb{N}$, have analogues of a Dieudonné module structure (cf. [God99]), where multiplication by $h_0$ on $\text{Ext}_{\mathfrak{A}(1)}$ corresponds to multiplication by 2 on stable homotopy groups.

11. RELATING DUAL BROWN-GITLER MODULES AND $P_1$

The existence of a close relationship between the mod 2 reduced cohomology of $\mathbb{RP}^\infty$ and the cohomology of the dual Brown-Gitler spectra $T(k)$ was exhibited algebraically by Miller [Mii84] and Kuhn observed [Kuh01] that this has a topological counterpart, namely $\mathbb{RP}^\infty$ is a stable summand of

$$\text{hocolim}\{T(1) \rightarrow T(2) \rightarrow \ldots \rightarrow T(2^j) \rightarrow \ldots\},$$

for any direct sequence of maps inducing the surjections $H^*(T(2j+1)) \rightarrow H^*(T(2^j))$ of the Mahowald short exact sequences in cohomology (cf. Lemma 9.10). Recall that the reduced cohomology of $\mathbb{RP}^\infty$ is isomorphic to $\Sigma^{-1}\Omega P_0$ (see Section 4), which is denoted $P_1$ in [Bru12].

Theorem 9.14 yields the $\mathfrak{A}(1)$ stable equivalence $H^*(T(2^{\nu+1})) \simeq \Sigma^{-1}A_{\nu,1}$, for $\nu \geq 0$, and it is interesting to consider the analogue of the above in the stable module category.

Lemma 11.1. For $\nu \in \mathbb{N}$, there exists a morphism of $\mathfrak{A}(1)$-modules

$$A_{\nu+1,1} \rightarrow A_{\nu,1}$$

which induces an isomorphism on $H^3(-, Q_1)$.

Proof. (Inductions.) This can be checked by hand using the material of Section 7. (A better method is to calculate $[A_{\nu+1,1}, A_{\nu,1}] \neq 0$; see Proposition 11.5 below.) □

Remark 11.2. Some caution is in order since $\text{Hom}_{\mathfrak{A}(1)}(A_{2,1}, A_{1,1}) = \mathbb{F}^{0,2}$ and, in this case, there are two distinct morphisms satisfying the conclusion of Lemma 11.1. The complication arises from the presence of the Joker at the top of $A_{2,1}$, which allows for a non-trivial composite morphism:

$$A_{2,1} \rightarrow \Sigma^2\mathbb{F} \hookrightarrow A_{1,1}$$

which is clearly trivial on $H^3(-, Q_1)$.

Lemma 11.3. For $1 \leq \nu \in \mathbb{N}$, there exists a morphism of $\mathfrak{A}(1)$-modules

$$\Omega P_0 \rightarrow A_{\nu,1}$$

which induces an isomorphism on $H^3(-, Q_1)$.

Proof. Recall from Section 7 that there is a short exact sequence $0 \rightarrow \Omega P_0 \rightarrow \Sigma R \rightarrow F \rightarrow 0$ of $\mathfrak{A}(1)$-modules and, by construction (see Section 7), there is a short exact sequence $0 \rightarrow A_{\nu,1} \rightarrow (\Omega \Sigma^{-1})^{\nu+1}J \rightarrow F \rightarrow 0$. For connectivity reasons, restricting a morphism $\Sigma R \rightarrow (\Omega \Sigma^{-1})^{\nu+1}J$ to elements of strictly positive degree induces a morphism $\Omega P_0 \rightarrow A_{\nu,1}$.

Tensoring the projection $\Sigma R \rightarrow F$ with $(\Omega \Sigma^{-1})^{\nu+1}J$ induces a stable morphism $\Sigma R \rightarrow (\Omega \Sigma^{-1})^{\nu+1}J \otimes \Sigma R \simeq \Sigma R$, which induces an isomorphism...
on $H^0(-, Q_0)$. The restriction of a representative morphism $\Sigma R \rightarrow (\Omega \Sigma^{-1})_{\nu+1} J$ of $\mathcal{A}(1)$-modules to strictly positive degree provides the required morphism. \hfill \Box

This Lemma can be made more precise by the following:

**Proposition 11.4.** An inverse system of morphisms of $\mathcal{A}(1)$-modules
$$\{f_\nu : A_{\nu+1,1} \rightarrow A_{\nu,1} \mid \nu \geq 1\}$$
such that $H^3(f_\nu, Q_1)$ is an isomorphism, yields an isomorphism
$$\lim_\leftarrow A_{\nu,1} \cong \Omega P_0.$$

**Proof.** (Indications.) In a fixed degree, the inverse system stabilizes to the given isomorphism. \hfill \Box

The indeterminacy evoked in Remark 11.2 is removed upon passage to the stable module category:

**Proposition 11.5.** For $1 \leq k, l \in \mathbb{N}$,
$$[A_{k,1}, A_{l,1}] = \begin{cases} F & k \geq l \\ 0 & k < l \end{cases}$$
and the non-trivial morphism is detected on $H^3(-, Q_1)$.

**Proof.** Consider the long exact sequence associated to
$$0 \rightarrow A_{k,1} \rightarrow (\Omega \Sigma^{-1})_{k+1} J \rightarrow F \rightarrow 0,$$
of which the relevant portion is
$$[(\Omega \Sigma^{-1})_{k+1} J, A_{l,1}] \rightarrow [A_{k,1}, A_{l,1}] \rightarrow [\Omega F, A_{l,1}] \xrightarrow{h^k \mu_k} [\Omega(\Omega \Sigma^{-1})_{k+1} J, A_{l,1}],$$
where the indicated map is given by Lemma 10.13. Theorem 10.16 shows that $[(\Omega \Sigma^{-1})_{k+1} J, A_{l,1}] = 0$ and $[\Omega F, A_{l,1}] = F$, generated by the class $\mu_k$. By Theorem 10.16, $h^k \mu_k \mu_k \neq 0$ if and only if $k < l$; the result follows. \hfill \Box

Hence there is a well defined inverse system
$$\ldots \rightarrow A_{\nu+1,1} \rightarrow A_{\nu,1} \rightarrow \ldots \rightarrow A_{1,1}$$
in the stable module category, where each morphism induces an isomorphism on $H^3(-, Q_1)$. This induces an isomorphism
$$\mathcal{E}xt_{\mathcal{A}(1)}(F, \Omega P_0) \cong \lim_\leftarrow \mathcal{E}xt_{\mathcal{A}(1)}(F, A_{\nu,1}).$$

**Remark 11.6.** Topologically this corresponds to the fact that the stable summand of $\operatorname{hocolim}\{T(1) \rightarrow T(2) \rightarrow \ldots \rightarrow T(2^n) \rightarrow \ldots\}$ complementary to $\Sigma^\infty \mathbb{R}P^\infty$ is $KO^*$-acyclic.

**Remark 11.7.** A similar result holds for a positive integer $n$, when considering
$$T(n) \rightarrow T(2n) \rightarrow \ldots \rightarrow T(2^j n) \rightarrow \ldots,$$
by Theorem 9.14 which provides a stable equivalence (for $n$ odd and $j \geq 1$):
$$\Sigma^{2^n} \mathcal{A}(2^n) \simeq (\Sigma \Omega^{-1})^{1-\alpha(n)}(\Sigma^{-1} A_{j-1,1}).$$
In particular, the term $(\Sigma \Omega^{-1})^{1-\alpha(n)}$ is independent of $j$, hence Proposition 11.5 gives the analogous $\mathcal{E}xt_{\mathcal{A}(1)}$ calculation.
12. Stable decompositions of tensor products

It is interesting to consider the stable isomorphism type of tensor products of modules of the form appearing in Theorem 6.12. This problem has been addressed for truncated projective spaces by Davis in [Dav74, Theorem 3.9].

Over $E(1)$, the calculation is straightforward: the tensor products always split as a direct sum of two, non-trivial indecomposable modules. Over $\mathcal{A}(1)$ the situation is more delicate. For example:

Example 12.1. Recall that $Z$ is the two-dimensional $\mathcal{A}(1)$-module in degrees $-1,0$, with $Sq^1$ acting non-trivially. The tensor product $Z \otimes Z$ is indecomposable, with structure:

Over $E(1)$, the $Sq^2$ is not seen, and the module splits as $\left(\Sigma^{-1}Z \oplus Z\right)|_{E(1)}$. Since $Z$ generates (under $\Sigma^{-3}\Omega$) all stable isomorphism classes of $\mathcal{A}(1)$-modules corresponding to $s = 1$ in Theorem 6.12, a similar statement holds for tensor products of any two such modules.

Recall from Section 7 (in particular Corollary 7.9) that, for $1 \leq k$, the modules $A_{k,0}, A_{k,1}$ give orbit representatives for modules of the form occurring in Theorem 6.12. To consider the stable isomorphism of tensor products of such modules, it suffices to consider $A_{k,\varepsilon} \otimes A_{l,\delta}$ and, without loss of generality, we may suppose that $k \leq l$.

The module $A_{l,\delta}$ is defined by a short exact sequence

$$0 \to A_{l,\delta} \to (\Omega \Sigma^{-1})^{l+1} J^{\otimes \delta} \to F \to 0$$

and the analogous statement holds for $A_{k,\varepsilon}$. Forming the tensor product $A_{k,\varepsilon} \otimes -$ with the above short exact sequence gives

$$0 \to A_{k,\varepsilon} \otimes A_{l,\delta} \to (\Omega \Sigma^{-1})^{l+1} J^{\otimes \delta} \otimes A_{k,\varepsilon} \to A_{k,\varepsilon} \to 0$$

and hence a distinguished triangle in the stable module category:

$$\Omega A_{k,\varepsilon} \to A_{k,\varepsilon} \otimes A_{l,\delta} \to (\Omega \Sigma^{-1})^{l+1} J^{\otimes \delta} \otimes A_{k,\varepsilon} \to$$

with connecting morphism $(\Omega \Sigma^{-1})^{l+1} J^{\otimes \delta} \otimes A_{k,\varepsilon} \to A_{k,\varepsilon}$. This distinguished triangle splits to provide a direct sum decomposition of $A_{k,\varepsilon} \otimes A_{l,\delta}$ if and only if the corresponding element of

$$\omega_{k,l,\delta,\varepsilon} \in [(\Omega \Sigma^{-1})^{l+1} J^{\otimes \delta} \otimes A_{k,\varepsilon}, A_{k,\varepsilon}]$$

is trivial.

The defining short exact sequence for $A_{k,\varepsilon}$ gives an exact sequence:

$$[(\Omega \Sigma^{-1})^{k+l+2} J^{\otimes (\delta+\varepsilon)}, A_{k,\varepsilon}] \to [(\Omega \Sigma^{-1})^{l+1} J^{\otimes \delta} \otimes A_{k,\varepsilon}, A_{k,\varepsilon}] \to [\Omega (\Omega \Sigma^{-1})^{l+1} J^{\otimes \delta}, A_{k,\varepsilon}].$$

Lemma 12.2. For $1 \leq k \leq l \in \mathbb{N}$ and $\delta, \varepsilon \in \{0,1\}$,

$$[(\Omega \Sigma^{-1})^{k+l+2} J^{\otimes (\delta+\varepsilon)}, A_{k,\varepsilon}] = 0$$

Proof. This follows from the stable ext calculations of Theorem 10.16.

If $\delta + \varepsilon \equiv 0 \mod (2)$, then the relevant group is $\mathcal{E}xt_{\mathcal{A}(1)}(F, A_{k,\varepsilon})$ of Adams indexing $l-s = 0$, $s = k+l+2$: for $l-s = 0$, $\mathcal{E}xt_{\mathcal{A}(1)}(F, A_{k,\varepsilon})$ is $F$ concentrated in degree $s = 2$. The hypothesis on $k, l$ implies that $k + l + 2 > 2$, whence the result in this case.
In the remaining case, Proposition \ref{prop4.7} implies that there is a stable isomorphism
\[ J \otimes A_{k,\varepsilon} \simeq \Sigma^4(\Omega^{-1} \Sigma)^2 A_{k,\varepsilon}, \]
and the latter group is trivial. □

It follows that \( \omega_{k,l,\delta,\varepsilon} \in \left[ (\Omega \Sigma^{-1})^{l+1} J \otimes \delta \otimes A_{k,\varepsilon}, A_{k,\varepsilon} \right] \) is zero if and only if its image
\[ \omega_{k,l,\delta,\varepsilon} \in \left[ \Omega(\Omega \Sigma^{-1})^{l+1} J \otimes \delta, A_{k,\varepsilon} \right] \]
is zero. By construction, \( \omega_{k,l,\delta,\varepsilon} \) is the composite of the commutative diagram in the stable module category:
\[
\begin{array}{ccc}
(\Omega \Sigma^{-1})^{l+1} J \otimes \delta \otimes \Omega F & \to & \Omega F \\
\downarrow \omega_{k,l,\delta,\varepsilon} & & \downarrow \mu_{k,\varepsilon} \\
(\Omega \Sigma^{-1})^{l+1} J \otimes \delta \otimes A_{k,\varepsilon} & \to & A_{k,\varepsilon},
\end{array}
\]
where the vertical morphisms are induced by \( \mu_{k,\varepsilon} : \Omega F \to A_{k,\varepsilon} \) and the horizontal morphisms by the non-trivial morphism \( (\Omega \Sigma^{-1})^{l+1} J \otimes \delta \to F \). This provides the identification:
\[ \omega_{k,l,\delta,\varepsilon} = h_{l+1-\delta} \kappa \mu_{k,\varepsilon}. \]

**Theorem 12.3.** For \( 1 \leq k \leq l \in \mathbb{N} \) and \( \delta, \varepsilon \in \{0, 1\} \), the distinguished triangle \( \triangleright \) splits to give a stable isomorphism
\[ A_{k,\varepsilon} \otimes A_{l,\delta} \simeq \Omega A_{k,\varepsilon} \oplus (\Omega \Sigma^{-1})^{l+1} J \otimes \delta \otimes A_{k,\varepsilon} \]
if and only if \( \{k, l, \delta, \varepsilon\} \) does not satisfy both the following conditions
\[ \begin{align*}
k & = l \\
\delta + \varepsilon & \equiv 1 \mod (2).
\end{align*} \]

**Proof.** By the previous discussion, the distinguished triangle splits if and only if
\[ \omega_{k,l,\delta,\varepsilon} = 0, \]
which is equivalent to the condition
\[ h_{l+1-\delta} \kappa \mu_{k,\varepsilon} = 0. \]
Note that \( 1 \leq k \leq l \) by hypothesis.

1. In the case \( \delta = 0 \), by Theorem \ref{thm10.10} \( h_{l+1} \mu_{k,\varepsilon} \neq 0 \) if and only if \( \varepsilon = 1 \) and \( k = l \).

2. If \( \delta = 1 \), since \( \alpha \) is invertible, one can equivalently consider \( \alpha h_{l+1-\delta} \kappa \mu_{k,\varepsilon} = h_{l+1} \mu_{k,\varepsilon} \). By Theorem \ref{thm10.10} this is nonzero if and only if \( k = l \) and \( \varepsilon = 0 \). Thus \( \omega_{k,l,\delta,\varepsilon} \neq 0 \) if and only if \( k = l \) and \( \delta + \varepsilon \equiv 1 \mod (2) \), as required. □

**Example 12.4.** Theorem \ref{thm12.3} implies that \( A_{k,1} \otimes A_{l,1} \) always splits (supposing \( k, l \geq 1 \)). Since truncated projective spaces can lie in either of the components, by Theorem \ref{thm8.3} tensor products of truncated projective spaces need not split, as observed in [Dav74, Theorem 3.9].

13. The Toda complex and the Postnikov tower of \( ko \)

In [Pow14], the cohomology \( ko(n)^*(BV) \) of elementary abelian 2-groups was studied as a functor of \( V \), for \( n \in \mathbb{N} \). The emphasis there was on the functorial structure and the use of the known structure of the periodic theory \( KO^*(BV) \) and the key result was the proof that the morphism of spectra derived from the Postnikov tower for \( ko \)
\[ ko(n) \to KO \vee \Sigma^n H(KO_n) \]
(where \( H \pi \) denotes the Eilenberg-MacLane spectrum for the abelian group \( \pi \)), induces an injection on the cohomology of \( BV \). In [Pow14], this was expressed in terms of the property of detection; from the viewpoint of the current paper, where
functoriality is a less powerful tool, it is natural to reinterpret this as a statement that the Adams spectral sequence for $ko(n)^*(BV)$ collapses at the $E_2$-term. The purpose of this section is to indicate the analogous results which hold for Brown-Gitler spectra.

The main result of [Pea14] studies the $ko$-homology of Brown-Gitler spectra and shows that the Adams spectral sequence collapses at $E_2$ in this case. This fact follows easily from Theorem 10.16 here. A similar statement holds for $ko$-cohomology; moreover, the conclusion extends to the theories $ko(n)$, as in [Pow14].

Recall (cf. [Pow14] and [BG10, Section A.5]) that there is an exact Toda complex

$$
\cdots \to \Sigma^n \mathcal{C}_n \to \Sigma^{n-1} \mathcal{C}_{n-1} \to \cdots
$$

where the $\mathcal{C}_i$ are four-periodic up to suspension, $\mathcal{C}_{i+4} = \Sigma^8 \mathcal{C}_i$, the differentials are of degree zero and

$$
\begin{align*}
\mathcal{C}_0 &= \mathcal{A}(1) \otimes \mathcal{A}(0) F \\
\mathcal{C}_1 &= \Sigma \mathcal{A}(1) \\
\mathcal{C}_2 &= \Sigma^2 \mathcal{A}(1) \\
\mathcal{C}_3 &= \Sigma^4 \mathcal{A}(1) \otimes \mathcal{A}(0) F.
\end{align*}
$$

(See [Pow14, Section 4] and [BG10, Figure A.5.6] for the identification of the differentials; in [Pow14, Section 4], the differentials are taken of degree one, hence the suspension is omitted.)

The exact complex is obtained by splicing short exact sequences:

$$
\begin{array}{ccccccc}
& & & & \Sigma^3 \mathcal{C}_3 & \to & \Sigma^2 \mathcal{C}_2 \\
& & & \Sigma^4 (\Sigma^8 F) & \to & \Sigma^3 K_3 & \to & \Sigma^2 K_2 \\
& & \Sigma^2 \mathcal{C}_1 & \to & \Sigma K_1 & \to & \mathcal{C}_0 \\
& \to & K_0 = F & \to & & \to & & \\
\end{array}
$$

where

$$
\begin{align*}
K_1 &= \Sigma^8 \Omega^{-1} J \\
K_2 &= \Sigma^4 J \\
K_3 &= \Sigma^3 \Omega J.
\end{align*}
$$

In particular, the $K_j$’s represent elements of the Picard group and they are cyclic $\mathcal{A}(1)$-modules. Moreover, for $n \in \mathbb{N}$, there is an isomorphism of $\mathcal{A}$-modules:

$$
HF^*_2(ko(n)) \cong \mathcal{A} \otimes_{\mathcal{A}(1)} K_n.
$$

This makes it transparent that the results of the previous sections can be applied to the Adams spectral sequence calculations for $ko(n)^*$. In particular, on stable ext groups one has:

**Lemma 13.1.** (Cf. [Pow14].) For $M$ a $Q_0$-acyclic $\mathcal{A}(1)$-module and $n \in \mathbb{N}$, the connecting morphisms of the Toda exact sequence induce a natural isomorphism:

$$
\delta xt^*t^*_{\mathcal{A}(1)}(K_n, M) \cong \delta xt^*t^*_{\mathcal{A}(1)}(F, M).
$$

**Proof.** Straightforward: this follows from relative homological algebra for the pair $(\mathcal{A}(1), \mathcal{A}(0))$ (compare Lemma 10.9). □

**Remark 13.2.** In the case of Adams spectral sequence calculations for $ko(n)^*(X)$, the $ko(n)$-cohomology of a spectrum $X$ with $Q_0$-acyclic mod 2-cohomology, this corresponds to the relationship with the periodic cohomology $KO^*(X)$. For example, it applies in the case of the suspension spectra $\Sigma^\infty BV$ and to the 2-torsion Brown-Gitler spectra and their Spanier-Whitehead duals.
In the study of the detection property introduced in [Pow14], the homology of the complex $\text{Hom}(\mathcal{E}_*, M)$ plays a fundamental role. In relation with Lemma 13.1 one has the following identification:

**Lemma 13.3.** For $M$ a $Q_0$-acyclic $\mathcal{A}(1)$-module and $1 \leq n \in \mathbb{Z}$, there is a natural isomorphism:

$$H^n(\text{Hom}_{\mathcal{A}(1)}(\mathcal{E}_*, M)) \cong \text{Ext}^n_{\mathcal{A}(1)}(F, M).$$

**Proof.** Similar to Lemma 13.1. □

The exclusion of the case $n = 0$ above is due to the difference between morphisms in the category of $\mathcal{A}(1)$-modules and in the stable module category.

Putting these results together, one obtains the analogue of the main result of [Pow14] for the cohomology of Brown-Gitler spectra with respect to the theories $\text{ko}(n)$.

Recall from Section 9 that, for $l \in \mathbb{N}$, $B(l)$ denotes the $l$th Brown-Gitler spectrum and $DB(l)$ its Spanier-Whitehead dual.

**Theorem 13.4.** For $0 < l \in \mathbb{N}$ and $n \in \mathbb{N}$, the canonical morphisms of spectra $\text{ko}(n) \to \text{KO}$ and $\text{ko}(n) \to \Sigma^n \text{H}(\text{KO}_n)$ induce an injection

$$\text{ko}(n)^\ast(DB(l)) \hookrightarrow \text{KO}^\ast(DB(l)) \oplus \text{H}(\text{KO}_n)^{\ast+n}(DB(l)).$$

**Proof.** (Indications.) The essential step is the proof that the Adams spectral sequence for $\text{ko}(n)^\ast(DB(l))$ collapses at $E_2$ and hence for $\text{KO}^\ast$, which follows as in [Dav74], for example. The result is then a consequence of the general results on detection established in [Pow14], using Lemmas 13.1 and 13.3 above. □

**Remark 13.5.** This result may be restated as the injectivity of

$$\text{ko}(n)^\ast(B(l)) \hookrightarrow \text{KO}_\ast(B(l)) \oplus \text{H}(\text{KO}_n)^{\ast-n}(B(l)).$$

As explained in [Goe99] Section 11], for $E$ a ring spectrum with representing spaces $\{E_*\}$ of the associated $\Omega$-spectrum, there is a surjective homomorphism

$$E_n(B(l)) \twoheadrightarrow D_1H_\ast(E_{n-1}),$$

that is compatible with the respective Frobenius and Verschiebung morphisms by [Goe99] Proposition 11.3]. Here, for $H$ a graded, bicommutative Hopf algebra, $D_1H$ denotes the associated Dieudonné module; under the appropriate hypotheses, $D_1H$ determines $H$.

It follows that Theorem 13.4 provides information on $H_\ast(\text{ko}(n)_{\ast})$, in particular shedding light on results of Cowen Morton [Mor77].

**References**

[Ada74] J. F. Adams, *Stable homotopy and generalised homology*, University of Chicago Press, Chicago, Ill., 1974, Chicago Lectures in Mathematics. MR 0402720 (53 #6534)

[AM71] J. F. Adams and H. R. Margolis, *Modules over the Steenrod algebra*, Topology 10 (1971), 271–282. MR 0294450 (45 #3520)

[AP76] J. F. Adams and S. B. Priddy, *Uniqueness of $BSO$*, Math. Proc. Cambridge Philos. Soc. 80 (1976), no. 3, 475–509. MR 0431152 (55 #4154)

[Bai10] Scott M. Bailey, *On the spectrum bo ∧ tmf*, J. Pure Appl. Algebra 214 (2010), no. 4, 392–401. MR 2558747 (2010j:55011)

[BG73] Edgar H. Brown, Jr. and Samuel Gitler, *A spectrum whose cohomology is a certain cyclic module over the Steenrod algebra*, Topology 12 (1973), 283–295. MR 0391071 (53 #6534)

[BG10] Robert R. Bruner and J. P. C. Greenlees, *Connective real $K$-theory of finite groups*, Mathematical Surveys and Monographs, vol. 169, American Mathematical Society, Providence, RI, 2010. MR 2723113 (2011k:19007)

[Bru12] Robert R. Bruner, *Idempotents, Localizations and Picard groups of $\mathcal{A}(1)$-modules*, To appear in the proceedings of the fourth Arolla conference, arXiv:1211.0213, 2012.
[Dav74] Donald Davis, Generalized homology and the generalized vector field problem, Quart. J. Math. Oxford Ser. (2) 25 (1974), 169–193. MR 0356053 (50 #8524)

[DGM81] Donald M. Davis, Sam Gitler, and Mark Mahowald, The stable geometric dimension of vector bundles over real projective spaces, Trans. Amer. Math. Soc. 268 (1981), no. 1, 39–61. MR 628445 (83c:55006)

[DGM83] Donald M. Davis, Sam Gitler, and Mark Mahowald, Correction to: “The stable geometric dimension of vector bundles over real projective spaces” [Trans. Amer. Math. Soc. 268 (1981), no. 1, 39–61; MR0628445 (83c:55006)], Trans. Amer. Math. Soc. 280 (1983), no. 2, 841–843. MR 716854 (85g:55006)

[GLM93] P. Goerss, J. Lannes, and F. Morel, Hopf algebras, Witt vectors, and Brown-Gitler spectra, Algebraic topology (Oaxtepec, 1991), Contemp. Math., vol. 146, Amer. Math. Soc., Providence, RI, 1993, pp. 111–128. MR 1224910 (95a:55007)

[Goe99] Paul G. Goerss, Hopf rings, Dieudonn´e modules, and $E_2^2 S^3$, Homotopy invariant algebraic structures (Baltimore, MD, 1998), Contemp. Math., vol. 239, Amer. Math. Soc., Providence, RI, 1999, pp. 115–174. MR 1718079 (2000j:57078)

[Kuh01] Nicholas J. Kuhn, New relationships among loopspaces, symmetric products, and Eilenberg MacLane spaces, Cohomological methods in homotopy theory (Bellaterra, 1998), Progr. Math., vol. 196, Birkhäuser, Basel, 2001, pp. 185–216. MR 1851255 (2002j:55026)

[Mah77] Mark Mahowald, A new infinite family in $2\pi_1 S^3$, Topology 16 (1977), no. 3, 249–256. MR 0445498 (56 #3838)

[Mah81] Paul G. Goerss, Hopf rings, Dieudonn´e modules, and $E_2^2 S^3$, Homotopy invariant algebraic structures (Baltimore, MD, 1998), Contemp. Math., vol. 239, Amer. Math. Soc., Providence, RI, 1999, pp. 115–174. MR 1718079 (2000j:55026)

[Mah84] Paul G. Goerss, Hopf rings, Dieudonn´e modules, and $E_2^2 S^3$, Homotopy invariant algebraic structures (Baltimore, MD, 1998), Contemp. Math., vol. 239, Amer. Math. Soc., Providence, RI, 1999, pp. 115–174. MR 1718079 (2000j:55026)

[Mil75] R. James Milgram, The Steenrod algebra and its dual for connective $K$-theory, Conference on homotopy theory (Evanston, Ill., 1974), Notas Mat. Simpos., vol. 1, Soc. Mat. Mexicana, México, 1975, pp. 127–158. MR 761725

[Mil84] Haynes Miller, The Sullivan conjecture on maps from classifying spaces, Ann. of Math. (2) 120 (1984), no. 1, 39–87. MR 750716 (85j:55012)

[Mor07] Dena S. Cowen Morton, The Hopf ring for $bo$ and its connective covers, J. Pure Appl. Algebra 210 (2007), no. 1, 219–247. MR 2311833 (2009c:55011)

[Pea14] Paul Thomas Pearson, The connective real $K$-theory of Brown-Gitler spectra, Algebr. Geom. Topol. 14 (2014), no. 1, 597–625. MR 3158769

[Pow14] Geoffrey Powell, On connective $KO$-theory of elementary abelian 2-groups, Algebr. Geom. Topol. 14 (2014), no. 5, 2693–2720. MR 3276845

[Rav93] Douglas C. Ravenel, The homology and Morava $K$-theory of $\Omega^2 SU(n)$, Forum Math. 5 (1993), no. 1, 1–21. MR 1190020 (94c:55023)

[Sch94] Lionel Schwartz, Unstable modules over the Steenrod algebra and Sullivan’s fixed point set conjecture, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1994. MR 1282727 (95d:55017)

[Shi84] Don H. Shimamoto, An integral version of the Brown-Gitler spectrum, Trans. Amer. Math. Soc. 283 (1984), no. 2, 383–421. MR 737876 (85m:55012)

[Yu95] Cherng-Yih Yu, The connective real $K$-theory of elementary abelian 2-groups, Ph.D. thesis, University of Notre Dame, 1995.

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