Parton Reggeization Approach for gluon-induced processes at Next-to-Leading order

M.A. Nefedov
II. Institut für Theoretische Physik, Universität Hamburg, Luruper Chaussee 149, 22761 Hamburg, Germany;
Samara National Research University, Moskovskoe Shosse, 34, 443086, Samara, Russia

The perturbatively-stable scheme of Next-to-Leading order (NLO) calculations of cross-sections for multi-scale hard-processes in DIS-like kinematics is developed in the framework of Parton Reggeization Approach (PRA). The evolution equation for unintegrated PDF, which resums \( \log \frac{1}{z} \) corrections to the coefficient function in the Leading Logarithmic approximation together with a certain subset of Next-to-Leading Logarithmic and Next-to-Leading Power corrections, necessary for the perturbative stability of the formalism, is formulated and solved in the Double-Logarithmic approximation. An example of DIS-like process, induced by the operator \( \text{tr}[G_{\mu \nu} G_{\mu \nu}] \), which is sensitive to gluon PDF already in the LO, is studied. Moderate \( (O(20\%)) \) NLO corrections to the inclusive structure function are found at small \( x_B < 10^{-4} \), while for the \( p_T \)-spectrum of a leading jet in the considered process, NLO corrections are small \( (<O(20\%)) \) and LO of PRA can be considered as a good approximation. The approach can be straightforwardly extended to the case of multi-scale hard processes in \( pp \)-collisions at high energies.

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I. INTRODUCTION

The High-Energy or \( k_T \)-factorization formalism, first introduced in [1], later had been developed [2,3] as a tool to resum higher-order corrections to coefficient functions of Collinear Parton Model, enhanced by large logarithms \( \log \frac{1}{z} \) of light-cone momentum fraction \( z \), of a parton entering into a hard subprocess, relative to the characteristic light-cone momentum component of a final-state of interest. This kind of corrections become more and more important with increasing collision energy, since more phase-space for additional semi-hard emissions opens up. These emissions generate a transverse momentum recoil, which greatly affects kinematic distributions of the final-state of interest, e.g. di- or multi-jet system [4-6], pair of heavy-flavoured mesons [7,8] or heavy quarkonia [9]. Therefore, \( k_T \)-factorization calculation serves as an interesting alternative to fixed-order calculations of such observables in Collinear Parton Model (CPM) or with conventional Parton Showers (PS) (see Ref. [10] for the review). As one can see, e.g. from references cited above, the \( k_T \)-factorization calculation with a judicious choice of unintegrated-Parton Distribution Function (UPDF) quite often leads to a good description of various correlation observables already in the leading order (LO) in \( \alpha_s \) and significant initial-state PS effects have to be taken into account.

However a multitude of phenomenological approaches to determine the UPDF (see Ref. [11] for a comprehensive list), their seeming inconsistency with each-other and lack of a practical formalism to perform Next-to-Leading order calculations, which goes beyond the results of pioneering papers [12,13], always where major drawbacks of High-Energy factorization program. The perturbative instability of Balitsky-Fadin-Kuraev-Lipatov (BFKL)-formalism [14-16], first observed in a celebrated calculation of NLO BFKL kernel [17-19] is a main reason of a slow development of \( k_T \)-factorization beyond LO. The main source of large NLO corrections to the BFKL kernel was immediately identified [20] – large logarithms of transverse momentum, coming from the collinear region of the NLO correction, which are not reproduced by the iteration of LO kernel. Resummation of this large logarithms requires development of an approach unifying BFKL and Dokshitzer-Gribov-Lipatov-Altarelli-Parizi (DGLAP) [21-23] dynamics, which is a task of formidable complexity. Several approaches to this problem had been proposed [24-27] however their practical implementation in phenomenology had been achieved only recently, see e.g. [28,30]. For that reason, \( k_T \)-factorization phenomenology today is still dominated by various heuristic approaches unifying BFKL and DGLAP evolution, such as different versions of Catani-Chiapoloni-Fiorani-Marchesini equation [31-33], Parton-Branching method [34], Kimber-Martin-Ryskin-Watt (KMRW) prescription [35,37], Collins-Ellis-Blümlein doubly-logarithmic approach [2,38] and many more.

*Electronic address: maxim.nefedov@desy.de
In the present paper we continue development of the technique of NLO calculations in the Parton Reggeization Approach (PRA) \cite{1,7} – the gauge-invariant scheme of $k_T$-factorization, based on Lipatov’s gauge-invariant Effective Field Theory (EFT) for Multi-Regge processes in QCD \cite{39,40} and Modified Multi-Regge Kinematics (MMRK) approximation for QCD amplitudes with multiple real emissions. We write-down the evolution equation for UPDF, based on MMRK-approximation, which besides leading $\log(1/z)$-terms allows one to resum a subset of subleading logarithmic and $O(z)$ power-suppressed corrections to the UPDF. Then we perform an exploratory NLO calculation for the coefficient function of Deep-Inelastic-Scattering-like subprocess, driven by a gauge-invariant operator $\text{tr}[G_{\mu\nu}\tilde{O}^{\mu\nu}]$ (where $G_{\mu\nu}$ is a Non-Abelian field-strength tensor), which couples to gluons already in the LO in $\alpha_s$. In CPM, the coefficients functions of this process are known up to $O(\alpha_s^4)$ \cite{11} and starting from NNLO they contain doubly-logarithmic terms $\propto \alpha_s^4 \log^{2n}(1/z)$ origin of which had been explained in $k_T$-factorization \cite{12}. Besides inclusive DIS cross-section (or “structure-function”) we also study the cross-section of production of a leading jet in this process, which shows, that it solves the major part of the problem of perturbative instability of BFKL formalism at NLO. We trace this results back to the improved treatment of region of initial-state collinear singularity (DGLAP region) in the MMRK approximation. In summary, we have come-up with a practical and manifestly perturbatively-able recipe of NLO calculations in PRA, which allows one to improve accuracy of the predictions and establish the boundaries of applicability of the approach through the smallness of NLO correction.

The paper is organized as follows: in Sec. II we formulate the basic formalism of PRA for the particular process we have chosen to study and derive the evolution equation for UPDF in MRK approximation, then in Sec. III we formulate our MMRK approximation, analyze it’s performance in comparison to an exact QCD amplitude with one additional emission and write-down UPDF-evolution equation in MMRK approximation and corresponding NLO double-counting subtraction terms, in Sec. IV we describe our phase-space slicing strategy, compute corresponding analytic integrals and double-counting subtraction integral in the soft limit, in Sec. V we recall the virtual part of NLO-correction under consideration, computed in Ref. \cite{43}, and derive corresponding virtual subtraction terms, finally in Sec. VI we present and discuss some numerical results and formulate our conclusions. In the Appendix we derive an approximate doubly-logarithmic solution for our UPDF evolution equation, which we use for illustrative numerical calculations throughout this paper.

II. BASIC FORMALISM AND UPDF EVOLUTION IN MRK APPROXIMATION

To simplify our presentation, we will always refer to a particular example of hard process, the DIS-like process (momenta of particles are given in parthenses):

$$O(q) + p(P) \rightarrow X,$$

initiated by the gauge-invariant local QCD operator

$$\lambda O(x) = -\frac{\lambda}{2} \text{tr} [G_{\mu\nu}(x)G^{\mu\nu}(x)],$$

where $\lambda$ is a coupling to an external source and $G_{\mu\nu} = -i [D_\mu, D_\nu]/g_s$ is a field-strength tensor of QCD with covariant derivative expressed as $D_\mu = \partial_\mu + ig_\mu A_\mu$ with gluon field $A_\mu = A_\mu^a T^a$, where $T^a$ are generators of $SU(N_c)$. In the present paper we will concentrate on the case of pure gluodynamics, i.e. the theory with $n_F = 0$. Within Standard Model, the operator (2) can be understood as an effective coupling of gluons to a Higgs boson through a loop of very heavy quark, and therefore process (1) can be visualized as a Higgs-exchange contribution to the usual electron-proton DIS. Of course phenomenologically such contribution is negligible, but since the operator (2) couples to gluons through the two-gluon vertex:

$$G^{(0),\mu_1\mu_2}_{a_1 a_2} = i\lambda \delta_{a_1 a_2} ((k_1 k_2) g^{\mu_1\mu_2} - k_1^{\mu_1} k_2^{\mu_2}),$$

where gluon momenta $k_{1,2}$ are incoming and $q + k_1 + k_2 = 0$, as well as through three and four-gluon vertices, proportional to corresponding vertices of QCD with $q + k_1 + \ldots + k_3 = 0$ and $q + k_1 + \ldots + k_4 = 0$, the process (1) have proven to be a useful tool for the formal studies in QCD, probing various aspects of evolution of gluon PDF, see e.g. Refs. \cite{13,14}.

In the present paper we will consider the dimensionless inclusive “structure-function” of the process (1), which depends on usual DIS kinematic variables $Q^2 = -q^2$ and $x_B = Q^2/2(qP)$ and in the LO of CPM is simply equal to

$$F^{(\text{LO} \ CPM)}_{O}(x_B, Q^2) = \frac{\pi \lambda^2}{4} x_B f_g(x_B, \mu_F = Q),$$

(4)
where $f_g$ is a usual collinear PDF. In our numerical calculations we will put the factor $\pi \lambda^2/4 = 1$. For simplicity, throughout this paper we choose to work in a center-of-momentum frame of $P$ and $q$, where the light-cone components of these momenta can be expressed as:

$$P_\perp = \sqrt{\frac{Q^2}{x_B (1-x_B)}}, \quad P_+ = P_T = 0; \quad q_- = -x_B P_-, \quad q_+ = \frac{Q^2}{x_B P_-}, \quad q_T = 0,$$

i.e. at $x_B \ll 1$ momentum $q$ has large positive (forward) rapidity, while proton flies in the negative direction.

The general expression for inclusive DIS structure-function in CPM is well-known:

$$F_O(x_B, Q^2) = \frac{\pi \lambda^2}{4} \int_{x_B}^1 dz \frac{x_B}{z} f_g \left( \frac{x_B}{z}, \mu_F \right) C(z, Q^2, a_s, \mu_F, \mu^2) + O \left( \frac{\alpha_s^n}{Q^2} \right),$$

where the coefficient function $C$ is computed perturbatively as a power-series in $a_s = \alpha_s(\mu^2)/(2\pi)$ and the first (leading-twist) term of Eq. (6) is valid up to corrections suppressed as $(Q^2)^{-\nu}$ with $\nu > 0$.

The $k_T$-factorization hypothesis states that higher-order corrections to the coefficient function, enhanced by $\log(1/z)$, can be further factorized-out:

$$C(z) = \int \frac{d^2 q_{T1}}{\pi} \int \frac{dx_1}{x_1} \left( \frac{z}{x_1}, q_{T1}, a_s, \mu_F, \mu, \mu_Y \right) H(x_1, q_{T1}, Q^2, a_s, \mu, \mu_Y),$$

where new coefficient function $H$ is free from potentially large corrections and by $\mu_Y$ we have denoted additional scale which arises due to factorization. This statement is proven in QCD in leading-power approximation in $z \ll 1$ for the series of Leading-Logarithmic (LL, $\propto [a_s \log^k(1/z)]^n$ with $k = 1$ or $2$ for some processes) and Next-to-Leading Logarithmic (NLL, $\propto a_s^k [\log(1/z)]^n$) corrections. Evolution factor $\mathcal{C}$ is always single-logarithmic w.r.t. $\log(1/z)$ at leading power in $z$, however additional power of $\log(1/z)$ per $a_s$ can be generated by transverse-momentum integration.

The situation at subleading power is significantly more complicated, see e.g. [45, 46], with doubly-logarithmic corrections arising for some quantities in LLA [47, 49], however this corrections still can be organized into a sum of terms of a form with different coefficient functions $H$ and evolution factors $C$. In the present paper, we work under assumption, that there is a series of subleading-power ($O(z)$) corrections which can be written in a form with the leading-power coefficient function $H$, so that all corrections are absorbed into $C$, and we assume that this series is numerically dominant for most of inclusive quantities. We make such an assumption, instead of systematically going order-by-order in $z$-expansion, because phenomenologically such an expansion can hardly be expected to be quickly-convergent, since higher-order corrections in QCD typically contain functions like $\log(1-1/z)$ or $1/(1-1/z)$ with rather slowly-convergent expansion around $z = 0$.

Substituting Eq. (7) to Eq. (6) and changing the order of integrals in $x_1$ and $z$ one arrives at a standard High-Energy Factorization formula [11, 12]:

$$F_O(x_B, Q^2) = \int \frac{dx_1}{x_1} \int \frac{d^2 q_{T1}}{\pi} \Phi_\delta(x_1, q_{T1}, \mu, \mu_Y) \times \frac{|\mathcal{M}_{\text{PRA}}|^2}{2S x_1} (2\pi)^D \delta(q + q_1 - p_M) d\Pi_M,$$

where $S = P_- q_+, D = 4 - 2\epsilon$, the Unintegrated PDF (UPDF) is

$$\Phi_\delta(x, q_{T}, \mu, \mu_Y) = \int dx \frac{x_B}{z} f_g \left( \frac{x_B}{z}, \mu_F \right) C(z, q_T, a_s, \mu_F, \mu, \mu_Y),$$

and we have rewritten the coefficient-function $H$ in terms of “squared matrix element” (ME) $|\mathcal{M}_{\text{PRA}}|^2$, which at leading power in $z$ is computed as:

$$|\mathcal{M}_{\text{PRA}}|^2 = \frac{1}{(N_c^2 - 1) q_{T1}^2} \left( \frac{q_1^-}{2} \right)^2 |\mathcal{A}_{\text{EFT}}|^2,$$

where $\mathcal{A}_{\text{EFT}}$ is an amputated Green’s function of Lipatov’s EFT for Multi-Regge processes in QCD [39] with one incoming Reggeized gluon $R_-$ with four-momentum $q_{T1}^- = q_1^- n_{T1}^2/2 + q_{F1}^2 = x_1 P_- n_{F1}^2/2 + q_{F1}^2$ and a desired final-state.
with total four-momentum $p_M$. The prescription \cite{10} is introduced (see e.g. Ref. \cite{50}) to ensure, that in the on-shell limit $|q_{T1}| \to 0$ the PRA squared ME reproduces the corresponding squared ME of CPM with Reggeized gluon substituted by an on-shell gluon with momentum $n^\mu q_1^\Lambda / 2$, averaged over its color and polarization states:

$$\int_0^{2\pi} \frac{d\phi_1}{2\pi} \lim_{q_{T1} \to 0} |\mathcal{M}_{\text{PRA}}|^2 = |\mathcal{M}_{\text{CPM}}|^2, \quad (11)$$

where $\phi_1$ is an azimuthal angle of $q_{T1}$. Also in Eq. (8) we have introduced a flux-factor $2S_{x_1}$ which is just a matter of convention, and one have to integrate over the final-state of $\mathcal{M}$ with the Lorentz-invariant phase-space volume element $d\Pi_M$.

The LO PRA subprocess for the process \cite{1} is:

$$\mathcal{O}(q) + R_-(q_1) \to g(q + q_1), \quad (12)$$

with the squared ME, derived from vertex \cite{3} and LO $R_-$ $\to g$-mixing vertex $\Delta_{ab}^{\nu\mu}(q) = (-iq^2)n^\nu_+ \delta_{ab}$ of EFT \cite{39} (see e.g. Eq. (13) in Ref. \cite{43}):

$$|\mathcal{M}_{\text{LO}}|^2 = \left( \frac{\lambda q + q_1}{2} \right)^2. \quad (13)$$

Substituting Eq. (13) into Eq. (8) one obtains the following expression for the structure function in LO of PRA:

$$F_{\mathcal{O}}^{(\text{LO PRA})}(x_B, Q^2) = \frac{\pi \lambda^2}{4} \int_0^{\infty} dq_{T1}^2 \Phi_g(x_1, q_{T1}, \mu, \mu_Y), \quad (14)$$

where

$$x_1 = x_B(Q^2 + q_{T1}^2)/Q^2, \quad (15)$$

due to on-shell condition for final-state gluon.

To set our notation, below we will derive the real-emission term of evolution equation for $C$ in the standard MRK approximation, which up to minor differences in treatment of kinematics coincides with the standard LO BFKL equation. To this end let’s consider the approximation, which up to minor differences in treatment of kinematics coincides with the standard LO BFKL due to on-shell condition for final-state gluon.

$$
\frac{dC_{n+1}}{d\Pi_{M}^{(\text{LO})}} = \int \frac{d^2 q_{T1}}{\pi} \int_{\hat{x}_1}^1 \frac{d\hat{x}_1}{\hat{x}_1} C_n(\hat{x}_1, q_{T1}) \int \frac{dk_{n+1}^- d^{D-2}k_{Tn+1}}{2(k_{n+1}^-)^D-1} \int d^D q_1 \delta(q_1 - k_{n+1} - q_1) \\
\times (2\pi)^D \delta(q_1 + q - p_{M}^{(\text{LO})}) |\mathcal{M}_{(\text{LO}+g)}|^2, \quad (16)
$$

where $\hat{q}_1^\mu = (\hat{x}_1 P_- n^\mu / 2 + \hat{q}_{T1}^\mu$ and we have introduced an intermediate “t-channel” momentum $q_1$. To kinematically factorize-out additional emission one performs the following approximation:

$$
\int \frac{dq_1^+ dq_1^-}{2} 2\delta(k_{n+1}^+ + q_1^+)\delta(q_1^- - k_{n+1}^- - q_1^-) \times 2\delta(q_1^+ + q_+ - (p_{M}^{(\text{LO})} +)^\delta(q_1^- + q_- - (p_{M}^{(\text{LO})})^-), \quad (16)
$$

i.e. neglects the “small” light-cone component in the hard process, thus the light-cone components of $q_1$ are: $q_1^+ = -k_{n+1}^+$, $q_1^- = q_1^- - k_{n+1}^-$, $q_{T1} = q_{T1} + k_{Tn+1}$ and one introduces the variables

$$z_1 = q_1^- / q_1^+, \quad (17)$$

and $x_1 = q_1^- / P_{T1}^-$, in terms of which, the new longitudinal measure of integration is:

$$\frac{d\hat{x}_1 dk_{n+1}^-}{\hat{x}_1 k_{n+1}^-} = \frac{dx_1}{x_1} \frac{dz_1}{1 - z_1}. \quad (18)$$
The approximate t-channel momentum transfer is equal to:

\[ t = q_1^2 = -q_{T1}^2 \left( 1 + \frac{z_1 k_{Tn+1}^2}{(1 - z_1) q_{T1}^2} \right). \]  

(18)

The approximation (15) becomes accurate in the limit \( Q^2 \rightarrow 0 \). In the Regge limit \( z_1 \ll 1 \) one puts \( t \approx -q_{T1}^2 \), but the latter approximation quickly degrades with an increase of \( z_1 \). In the kinematic constraint approach \([51, 52]\) to perturbatively-stabilize BFKL-evolution, one cuts-off the region of real-emission phase-space, where \( t \approx -q_{T1}^2 \) approximation is no longer valid, i.e. one rejects the emissions with:

\[ -q_1^+ q_1^- > q_{T1}^2 \Leftrightarrow z_1 k_{Tn+1}^2 > q_{T1}^2 (1 - z_1). \]

Retaining some realistic approximation for t-channel momentum transfer will provide the smooth cutoff in the same region of phase-space, thus leading to resummation of essentially the same class of collinearly-enhanced higher-order corrections to BFKL-kernel.

The squared matrix element with one gluon emission, in the Regge limit \( z_1 \ll 1 \) factorizes as follows:

\[ |\mathcal{M}(\text{LO}+g)|^2 = \left[ \frac{1}{q_{T1}^2} \left( \frac{q_1^-}{2} \right)^2 \right] \left( \frac{L^2}{2t} \right)^2 \left[ \frac{1}{q_{T1}^2} \left( \frac{q_1^-}{2} \right)^2 \right]^{-1} \times |\mathcal{M}_{\text{LO}}|^2 = \frac{4CA g_s^2}{z_1 k_{Tn+1}^2 \left( \frac{q_{T1}^2}{t} \right)^2} \times |\mathcal{M}_{\text{LO}}|^2, \]  

(19)

where factors in square brackets are introduced due to normalization prescription \([10]\), the factor \( 1/(2t) \) is the Born propagator of Reggeized gluon, in the standard MRK-approximation, the factor in curly brackets is put to one as it was discussed above, and \( L^2 = 16CA g_s^2 (q_{T1}^2 / k_{Tn+1}^2) \) is the square of \( R_-(q_1) \rightarrow R_+(q_1) + g(k_{n+1}) \) amputated Green’s function in the EFT \([39]\), i.e. of Lipatov’s vertex \([14]\):

\[ \tilde{\Delta}^{abc-}\!(\bar{q}_1, q_1) = g_s f^{abc} \left[ -n_+ n_- \right] \left( \frac{q_1 + q_1}{q_1^2} \right) + 2 \left( q_1 n_- + q_1^+ n_+ \right) \].

Collecting all pieces together, one factorizes-out the real-emission term of evolution equation for \( C \):

\[ \frac{dC^{(n+1)}}{d\Pi^{(LO)}_M} = \int \frac{d^2 q_{T1}}{(2\pi)^2} \left[ \alpha_s C_A / \pi \right] \int \frac{dz_1}{z_1 (1 - z_1)} \int d^{D-2} k_{Tn+1} \frac{1}{k_{Tn+1}^2} C_n \left( \frac{z_1}{z}, q_{T1} + k_{Tn+1} \right) \]

\[ \times \left\{ (2\pi)^D \delta(q_1 + g - p_{M}^{(LO)}) \right\} \left[ |\mathcal{M}_{\text{LO}}|^2 \right] \times 2S x_1, \]

(20)

where the lower limit of \( z_1 \)-integration is ought to be \( x_1 \), while integrating up to \( z_1 = 1 \) will lead to un-regularized rapidity divergence. Technically, the divergence arises because approximation (16) violates the conservation of (+)-momentum component and additional emission is allowed to go arbitrarily forward (in the \( q \)-direction) in rapidity. Demanding, that rapidity of this emission is cut-off at some value \( Y_\mu \), one obtains the condition:

\[ y_{n+1} - Y_\mu = \log \left( \frac{|k_{Tn+1}|}{\mu_Y} \right) < 0 \Rightarrow z_1 < \Delta(|k_{Tn+1}|, \mu_Y), \]

\[ \Delta(k_T, \mu) = \frac{\mu}{\mu + k_T}, \]

(21)

where function \( \Delta \) is familiar from the definition of KMRW UPDF \([35, 37]\) and the rapidity-scale \( \mu_Y \) is defined by the relation \( q_1^+ = \mu_Y e^{-Y_\mu} \). In DIS kinematics the good choice for \( Y_\mu \) is the rapidity of a parton emitted in the LO PRA subprocess \([12]\):

\[ Y_\mu \rightarrow Y_H := \frac{1}{2} \log \left( \frac{Q^2 (1 - x_B)}{q_{T1}^2 x_B} \right) \Rightarrow \mu_Y \rightarrow \frac{Q^2 + q_{T1}^2}{|q_{T1}^2|}, \]

(22)

which removes large-logarithmic terms \( \propto Y_H \) form coefficient function \( H \) at NLO, as we will see below.

The rapidity-ordering between \( n \)-th and \( n + 1 \)-th emissions in the evolution cascade leads to a rapidity scale at which the evolution-factor \( C_n \) in Eq. (20) should be evaluated: \( \mu_Y \rightarrow |k_{Tn+1}| / (1 - z_1) \).

Finally, adding the standard virtual part of BFKL-equation \([14, 16]\) with the same rapidity measure as in the real-emission part, we arrive at our evolution equation for \( C \) in MRK-approximation:

\[ C(x, q_T, \mu, \mu_Y) = \pi \delta(x - 1) \delta(q_T) \]

\[ + \int \frac{dz}{z(1 - z)} \left\{ \alpha_s(\mu) C_A / \pi \right\} \int \frac{d^{D-2} k_T}{k_T^2} \left[ \frac{1}{k_T^2} C \left( \frac{x}{z}, q_T + k_T, \mu, \frac{|k_T|}{1 - z} \right) \right] \]

\[ \times \left\{ \theta(\Delta(|k_T|, \mu_Y) - z) + 2\omega_g(q_T^2) \theta(\Delta(|q_T|, \mu_Y) - z) C \left( \frac{x}{z}, q_T, \mu, \mu_Y \right) \right\}, \]

(23)
where \( C_0 = \pi \delta(x-1) \delta(q_{T1}) \) is the perturbative initial condition and

\[
\omega_g(p_T^2) = -\frac{1}{2} \frac{\alpha_s C_A}{2\pi} \int \frac{d^D-2k_T}{k_T^2(p_T - k_T)^2} = -r T \frac{\alpha_s C_A}{2\pi} \frac{(-p_T^2)^{-\epsilon}}{\epsilon},
\]

(24)

is the one-loop Regge trajectory of a gluon with \( r T = \Gamma(1+\epsilon)\Gamma(2)/(\Gamma(1-\epsilon)\Gamma(1-2\epsilon)) \).

It is well-known, that infra-red divergences cancel to all orders in \( \alpha_s \) in Eq. (23), but when convoluted over \( p_T \) with the coefficient-function \( H \), the evolution factor \( C \) generates collinear divergences \( \propto (\alpha_s/\epsilon)^n \), which should be absorbed by usual renormalization of collinear PDF \[3\]. It is most convenient to perform this procedure in Fourier conjugate \( x_T \)-space, because in this space all collinear divergences are contained in the evolution factor. We perform this procedure in Appendix for the simplified version of Eq. (23) which strictly neglects all \( O(\epsilon) \)-corrections in the kernel and thus does not depend on the scale \( \mu_N \). The UPDF obtained from doubly-logarithmic solution of this simplified equation is used for illustrative numerical calculations throughout this paper.

### III. MODIFIED MRK APPROXIMATION: SUBTRACTION TERMS AND UPDF EVOLUTION

The real-emission NLO correction to the cross-section of the process \([1]\) is given by PRA subprocess:

\[
O(q) + R_-(q_1) \rightarrow g(k_1) + g(k_2).
\]

(25)

Let us introduce the convenient variable:

\[
\hat{z} = k_1^-/Q_-, \quad Q_- = q^- + q_1^-,
\]

(26)

which together with \( k_{T1}, k_{T2} \) and \( Q^2 \) completely parametrizes exact \( 2 \rightarrow 2 \) kinematics of this contribution. In terms of this variable, the contribution of subprocess \([25]\) to the SF is given by:

\[
F_O^{(NLO)} = \frac{1}{2!} \frac{\pi \lambda^2}{4} \int \frac{d^D-2k_{T1}d^D-2k_{T2}}{\pi^2(2\pi)^{-2\epsilon}} \int \frac{d\hat{z}}{\hat{z}(1-\hat{z})} w(\hat{z}, Q^2, k_{T1}, k_{T2}, \alpha_s),
\]

(27)

where we have introduced integrand-function \(- w \) and reduced ME \(- f \) which are related with UPDF and squared ME of the subprocess \([25]\) as follows:

\[
w(\hat{z}, Q^2, k_{T1}, k_{T2}) = \frac{\alpha_s C_A}{\pi} \Phi_g(x_1, k_{T1} + k_{T2}, \mu, \mu_Y) f(\hat{z}, Q^2, k_{T1}, k_{T2}),
\]

(28)

\[
f(\hat{z}, Q^2, k_{T1}, k_{T2}) = \frac{[M^{(NLO)}]^2}{4C_A g_s^2 \lambda M^{(NLO)}},
\]

(29)

and

\[
Q_- = \frac{1}{q_+} \left( k_{T1}^2 \hat{z} + k_{T2}^2 \frac{1}{1-\hat{z}} \right), \quad k_1^- = Q_- \hat{z}, \quad k_2^- = Q_-(1-\hat{z}),
\]

(30)

\[
x_1 = \frac{x_B}{Q^2} \left( Q^2 + \frac{k_{T1}^2}{\hat{z}} + \frac{k_{T2}^2}{1-\hat{z}} \right).
\]

(31)

From (30) a simple expression for rapidity difference between gluons follows:

\[
y_2 - y_1 = \log \left[ \frac{k_{T2}}{k_{T1}} \frac{\hat{z}}{1-\hat{z}} \right],
\]

(32)

which tells us, that for fixed transverse momenta, limit \( \hat{z} \to 0 \) corresponds to \( y_1 > y_2 \) (t-channel Regge limit, \(-t/s < 1\)), while in the limit \( \hat{z} \to 1 \) one has \( y_2 > y_1 \) (u-channel Regge limit, \(-u/s << 1\)). In general, the substitution:

\[
\hat{z} \leftrightarrow 1-\hat{z}, \quad k_{T1} \leftrightarrow k_{T2},
\]

(33)

corresponds to permutation of final-state gluons.

The squared ME of the subprocess \([25]\) can be straightforwardly obtained using Feynman rules of EFT \[39\] (see e.g. Refs. \[4\], \[7\], \[43\], \[53\] for the detailed presentation) and is rather long and non-instructive expression, so we refrain from presenting it here. Some relevant limits of it are given below and in Sec. IV.
As it was shown in Sec. [11] the UPDF-evolution is obtained by factorizing-out an additional gluon emission from subprocess [25]. Hence, to remove double-counting of Eq. (27) with the evolution, one has to subtract the corresponding approximation for the squared ME from the exact integrand \( w \). For standard MRK approximation, the \( t \)-channel subtraction term is given by Eq. (28) with the following reduced ME:

\[
f_{\text{sub. } t}^{(\text{MRK})}(\hat{z}, Q^2, k_{T1}, k_{T2}) = \frac{1}{k_{T2}^2} \theta \left( \frac{(1 - \hat{z})^2}{\hat{z}^2} - \frac{k_{T2}^2 (Q^2 + k_{T1}^2)^2}{k_{T1}^2} \right),
\]

and UPDF evaluated at \( x_1^{(\text{MRK})} = x_E (Q^2 + k_{T1}^2/\hat{z}) / Q^2 \) instead of \( \hat{z} \), in accordance with approximation \( \text{(16)} \). The \( \theta \)-function in Eq. (34) enforces the rapidity-ordering condition \( \text{(21)} \) and with \( \mu_Y = (Q^2 + k_{T1}^2) / |k_{T1}| \) it is equivalent to the condition \( y_1 > y_2 \), see Eq. (32). The \( u \)-channel subtraction term is obtained from (34) via substitution (33).

Clearly, the NLO correction will be smaller if the subtraction term provides a better approximation to an exact squared ME. Since evolution equation is constructed by iterating the same approximation, improvement of the subtraction term will also make the evolution to capture more physics. In fact, as we will see in Sec. \( \text{V} \), to obtain meaningful physical results it is crucial to come-up with better approximations to an exact ME, than Eq. (34) can provide. The most important phase-space region, where improvements are necessary, is the DGLAP-region: \( q_{T1}^2 \ll k_{T1}^2 \simeq k_{T2}^2 \ll Q^2 \), integration over which at fixed \( q_{T1}^2 \) generates the contribution enhanced by \( \log(Q^2 / q_{T1}^2) \). The latter large logarithm, when integrated over \( q_{T1}^2 \) with UPDF, leads to sizable numerical effects.

For the squared ME, the DGLAP limit is similar to the on-shell limit \( q_{T1}^2 \rightarrow 0 \) followed by taking \( k_{T2}^2 = k_{T1}^2 \ll Q^2 \) asymptotics. General (initial-state) collinear factorization theorem for squared MEs in QCD is applicable in this case (see e.g. Eq. (4.9) of Ref \[54\]) and reduced ME is given by:

\[
f^{(12)}_{\text{IS-coll.}}(\hat{z}, Q^2, k_{T2}, q_{T1} \rightarrow 0) = \frac{\hat{z} p_{gg}(\hat{z})}{2 q_1 k_2},
\]

with \( 2 q_1 k_2 = k_{T2}^2 / (1 - \hat{z}) \) and \( p_{gg}(z) = z / (1 - z) + (1 - z) / z + z (1 - z) \), Eq. (35) is a non-trivial function of \( \hat{z} \), but in Eq. (34) this function is approximated by a constant. Our goal is to improve this situation but to leave the cancellation of infra-red divergences in Eq. (33) intact. To this end we restore the \( (q_{T1}^2 / t)^2 \) (“propagator-factor”) in Eq. (19) with the following approximation for the \( t \)-channel momentum transfer (compare it with Eq. (18)):

\[
t_{\text{MMRK}} = -k_{T1}^2 = \frac{k_{T2}^2 \hat{z}}{1 - \hat{z}},
\]

so that reduced ME for the subtraction term in the modified MRK-approximation takes the form:

\[
f_{\text{sub. } t}^{(\text{MMRK})}(\hat{z}, Q^2, k_{T1}, k_{T2}) = \frac{1}{k_{T2}^2} \left( 1 + \frac{2 k_{T2}^2}{(1 - \hat{z}) k_{T1}^2} \right)^{-2} \theta \left( \frac{(1 - \hat{z})^2}{\hat{z}^2} - \frac{k_{T2}^2 (Q^2 + k_{T1}^2)^2}{k_{T1}^2} \right).
\]

In the on-shell limit Eq. (37) reproduces Eq. (35) up to \( O(\hat{z}^2) \)-terms, so we have partially achieved our goal. The idea to improve MRK approximation for tree-level MEs by propagator factors on this kind is not new, and such factors where first introduced in the High-Energy Jets (HEJ) approach [55, 56]. Also, the new factor, introduced in Eq. (37) strongly suppresses squared ME in the region which is completely removed in the kinematic constraint approach [51, 52]. In Ref. [52] it was shown that kinematic constraint approach correctly reproduces certain leading large-logarithmic terms of collinear origin which can be found in the NLO and NNLO expressions for BFKL kernel (the latter is not known in QCD but has been conjectured in \( \mathcal{N} = 4 \) Supersymmetric Yang-Mils theory [57]). But e.g. the approach of Ref. [25] resums the same series of collinear corrections by matching DGLAP and BFKL evolutions. So it seems, that kinematic constraint, MMRK-HEJ and direct resummation approaches are solving the same physical problems of BFKL evolution in a compatible way, but further investigations are needed to confirm this hypothesis.

Adding the same factor to real-emission term of evolution equation [23] one ends-up with:

\[
\mathcal{C}(x, q_T, \mu, \mu_Y) = \pi \delta(x - 1) \delta(q_T) + \int \frac{dz}{z(1 - z)} \left\{ \alpha_s(\mu) C_A \left\{ \int \frac{dD-2k_T}{\pi} \frac{1}{k_T^2} \left( 1 + \frac{z k_T^2}{(1 - z) \sqrt{q_T^2 \mu_Y}} \right)^{-2} \times C \left( \frac{x}{z}, q_T + k_T, \mu, |k_T| / (1 - z) \right) \theta(\Delta(|k_T|, \mu_Y) - z) + 2 \omega_g(q_T^2) \theta(\Delta(|q_T|, \mu_Y) - z) \mathcal{C} \left( \frac{x}{z}, q_T, \mu, \mu_Y \right) \right\},
\]
FIG. 1: Plots of integrand function \( w(z) \) as a function of \( z \). Red solid line – exact integrand function, dashed line – standard MRK approximation, blue solid line – MMRK approximation. Plots (a) – (f) correspond to different regions of phase-space with different hierarchies of \( k_{T1,2}, q_{T1} \) and \( Q \). On panels (c) and (f) the function \( w \) averaged over azimuthal angle of \( q_{T1} \) is plotted for the correct on-shell limit.

where the scale-choice \( \delta_s \) was used to express the difference between variables \( \hat{z} \) and \( z_1 \). This equation have to be solved to obtain the UPDF in MMRK-approximation.

In the Fig. 1 we compare our subtraction terms with an exact integrand function numerically. From the plot 1(a) one can see, that in the limit \( |k_{T2}| \ll |k_{T1}| \) (or \( |k_{T1}| \ll |k_{T2}| \)), both MRK and MMRK subtraction terms approximate an exact integrand very well, except from the region of final-state collinear singularity, which is located at \( \Delta \phi_{21}^2 + \Delta y_{21}^2 \ll 1 \) (Fig. 1(b)). When all transverse momenta are of the same order, the MRK subtraction term significantly overshoots an exact integrand outside the Regge limits \( \hat{z} \to 0 \) and \( \hat{z} \to 1 \) (plots (d) and (e) in the Fig. 1), while MMRK expression gives a more reasonable approximation in a whole range of \( \hat{z} \). The same behavior is observed in the DGLAP (Fig. 1(c)) and on-shell (Fig. 1(f)) limits. In general, the MMRK subtraction term is smaller than standard MRK subtraction in whole phase-space, which solves the problem of large negative NLO corrections, typical for BFKL-type calculations, as we will see in Sec. IV. For example, the similar severe over-subtraction problem was observed in the calculation of \( p_T \)-spectrum of leading forward hadron in proton-nucleus collisions within Color-Glass-Condensate formalism at NLO \cite{58–60} and was solved in Ref. \cite{61} by improvement of the kinematics of the subtraction term.

IV. PHASE-SPACE SLICING, SOFT AND FINAL-STATE COLLINEAR INTEGRALS, DOUBLE-COUNTING SUBTRACTION IN THE SOFT REGION

To make our NLO calculations more transparent, we decided to use a simple phase-space slicing method, similar to one proposed in Ref. \cite{62}. Our matrix element has non-integrable singularities in two non-overlapping phase-space regions: soft region, which we define by following cuts on dimensionless energies of gluons:

\[
\frac{k_{T1}^+ + k_{T2}^-}{q_+} < \delta_s \quad \text{or} \quad \frac{k_{T2}^+ + k_{T1}^-}{q_+} < \delta_s,
\]

where phase-space slicing parameter \( 0 < \delta_s \ll 1 \). And (final-state) hard-collinear region, where:

\[
\Delta \phi_{12}^2 + \Delta y_{12}^2 < \delta_c,
\]

with \( 0 < \delta_c \ll \delta_s \) and gluons 1 and 2 not satisfying condition (39).

In terms of variables \( k_{T1,2} \) and \( \hat{z} \), the soft condition for the first gluon has the form:

\[
k_{T1}^2 < k_{T2}^2 \hat{z} \left( \delta_s - \hat{z} e^{-2\gamma_E} \right),
\]
where \( k_{T2} \simeq q_{T1} \), \( 0 < \hat{z} < \hat{z}_{\text{max}} \) with \( \hat{z}_{\text{max}} = \delta_e e^{2Y_H} \), and for the second gluon it can be obtained using the substitution \( \tilde{z} = z \).

To facilitate the integration over hard-collinear region, we parametrize \( k_{T1,2} \) in terms of \( q_{T1} = k_{T1} + k_{T2} \) and new transverse vector \( \Delta \):

\[
k_{T1} = \hat{z}q_{T1} + \Delta, \ k_{T2} = (1 - \hat{z})q_{T1} - \Delta, \tag{42}
\]

which in particular allows one to conveniently express the invariant mass of the pair as:

\[
s = 2k_1k_2 = \frac{\Delta^2}{\hat{z}(1 - \hat{z})}. \tag{43}
\]

In terms of new variable, collinear condition \( \text{(40)} \) has the form:

\[
\Delta^2 < q_{T1}^2\hat{z}^2(1 - \hat{z})^2 \delta_e, \tag{44}
\]

and requirement of both gluons to be non-soft translates into limits on \( \hat{z} \):

\[
\min(\hat{z}, 1 - \hat{z}) > \hat{z}_{\text{min}} = \frac{\delta_s}{1 + e^{-2Y_H}}. \tag{45}
\]

The soft limit of squared PRA amplitude can be computed using usual Eikonal Feynman rule for the emission of a soft gluon with four-momentum \( k \) from the hard gluon leg with momentum \( p \): \( g_s f_{abc} n_\mu^c / (kp) \). To take into account the presence of incoming Reggeized gluon \( R_\perp \), an additional contribution, proportional to \( (-g_s) f_{abc} n_\perp^c / (k_T) \) should be added to Eikonal amplitude. For the case at hands, the soft limit is:

\[
\mathcal{M}^{ab\mu\nu}_{(\text{NLO,soft})} = \mathcal{M}^{\mu}_{(\text{LO})} \times g_s f_{abc} \left( \frac{-n_\mu^c}{k_T^0} + \frac{k_T^0}{(k_1k_2)} \right),
\]

which leads to the following reduced squared amplitude in the \( k_T^0 \to 0 \) soft limit:

\[
f_{\text{soft-k}_1} = \frac{k_T^2 \hat{z}^2}{k_{T1}^2 (k_{T1}^2 - \hat{z}k_{T2})^2}, \tag{46}
\]

Eq. \( \text{(46)} \) has been cross-checked with an exact amplitude \( \text{(29)} \) in the soft limit.

The hard-collinear limit again can be obtained using the standard collinear factorization theorem for squared MEs \( \text{(54)} \), but this time for final-state singularity:

\[
f_{\text{FS-coll.}} = \frac{p_{gg}(\hat{z})}{s}, \tag{47}
\]

with \( s \) given by Eq. \( \text{(43)} \). Eq. \( \text{(47)} \) also has been verified against an exact squared PRA squared amplitude.

Substituting Eq. \( \text{(47)} \) to Eq. \( \text{(27)} \) with the parametrization \( \text{(42)} \) and cuts \( \text{(44)} \) and \( \text{(45)} \) one finds, that up to effects suppressed as \( O(\delta_c, \delta_s) \) the UPDF can be taken out of \( \Delta \) and \( \hat{z} \) integrals:

\[
F_0^{(\text{coll.})} = \frac{\pi\lambda^2}{4} \int_{0}^{\infty} dq_{T1}^2 \Phi_g(x_1, q_{T1}, \mu, \nu Y) F_0^{(\text{NLO})}(q_{T1}, Y_H),
\]

with \( x_1 \) computed by Eq. \( \text{(15)} \) and the following contribution to the NLO coefficient function:

\[
H_{\text{coll.}}^{(\text{NLO})} = \frac{1}{2\pi} \frac{\lambda_s C_A}{\pi} \int_{\hat{z}_{\text{min}}}^{1 - \hat{z}_{\text{min}}} d\hat{z} p_{gg}(\hat{z}) \int \frac{d^{D-2} \Delta}{(2\pi)^{D-2} \Delta^2} \theta (q_{T1}^2 \hat{z}^2 (1 - \hat{z})^2 \delta_c - \Delta^2),
\]

which can be straightforwardly integrated to give:

\[
H_{\text{coll.}}^{(\text{NLO})} = \frac{\lambda_s C_A}{2\pi} \frac{\mu^2}{q_{T1}} \left[ \frac{1}{\epsilon} \left( \frac{11}{6} + 2\log \hat{z}_{\text{min}} \right) - \left( \frac{67}{9} + \frac{2\pi^2}{3} + 2 \log^2 (\hat{z}_{\text{min}}) \right) + O(\epsilon) \right], \tag{48}
\]

where we have introduced \( \lambda_s = \mu^{-2} \alpha_s (4\pi)^\epsilon / \Gamma(1 - \epsilon) \).
Integrating over the soft region [39] we can either add up contributions of $k_0^0 \to 0$ and $k_0^1 \to 0$ limits and divide the cross-section by $2!$, or just integrate Eq. (27) with reduced ME (46) over the region (41) and omit the factor $1/(2!)$:

$$H_{\text{soft}}^{(NLO)} = \frac{\alpha_s C_A}{\pi} \int_0^{|z_{\text{max}}|} \frac{dz}{1 - z} \int \frac{dD_2 k_T}{\pi (2\pi)^{-2z}} \frac{q_{T1}^{2} (\hat{z})}{k_{T1}^{2} - z q_{T2}^{2}})^2 \theta \left(q_{T1}^{2} (\hat{z}) (\hat{d}_s - \hat{z} e^{-2Y_H}) - k_{T1}^{2} \right),$$

(49)

This integral can be calculated using the well-known Mellin-space representation of the $\theta$-function:

$$\theta(x - y) = \lim_{\gamma \to 0^+} \int_{-i \infty}^{+i \infty} \frac{dz}{\pi i} \left(\frac{x}{y}\right)^\gamma$$

(50)

together with the formula for two-dimensional Euclidean “bubble” integral with general indices:

$$J_{d \hat{s}}(p_T) = \int \frac{dD_2 k_T}{(k_T^2)^{a/(p_T - k_T^2)}} = \frac{\pi^{1-\epsilon} (p_T^2)^{-a-b-\epsilon}}{\Gamma(1 - a - \epsilon) \Gamma(1 - b - \epsilon) \Gamma(a + b + \epsilon - 1)}.$$ 

(51)

Also one can notice, that since $\hat{z} < \hat{z}_{\text{max}} \ll 1$, the factor $1/(1 - \hat{z})$ in Eq. (49) can be omitted up to terms $O(\hat{d}_s)$. Hence after expansion on $\hat{d}_s$ and $\epsilon$ we get:

$$H_{\text{soft}}^{(NLO)} = \frac{\alpha_s C_A}{2\pi} \times 2 \left(\frac{\mu^2 q_{T1}^2}{\epsilon} \frac{2 \epsilon}{6 + \log(1 + \epsilon)} + O(\epsilon^2) \right) \left[\frac{\xi}{e^2} + \frac{\log(1 + \epsilon)}{\epsilon} + \log \xi (1 + \xi) - \text{Li}_2(-\xi) + O(\epsilon)\right],$$

(52)

where $\xi = e^{-2Y_H}$.

Finally, we have to take into account the double-counting subtraction with the evolution. It doesn’t influence the collinear limit, since subtraction terms [34] or [37] are not singular in the region (41) and hence lead to $O(\hat{d}_s)$-suppressed contributions. However, subtraction terms [34] or [37] have non-trivial soft limit. The $\hat{d}$-channel subtraction term, which is obtained from Eq. (34) or [37] by the substitution [53], in the region (41) reduces to:

$$I_{\text{sub. } \hat{d}}^\mu = \frac{1}{k_{T1}^2} \theta \left(\hat{z} q_{T1}^2 (\hat{d}_s - \hat{z} \xi) - k_{T1}^2\right),$$

(53)

where $\xi = (\mu^2 q_{T1}^2)/(Q^2 + q_{T1}^2)^2 = e^{-2(Y_H - Y_s)}$, see Eq. (22). One should integrate this expression over region (41) and subtract the result from Eq. (52). We have checked by explicit calculation, that the “propagator-factor” in Eq. (37) makes no difference, up to $O(\hat{d}_s)$-terms, so the double-counting subtraction in the soft limit turns out to be the same for MRK and MMRK approximations.

Due to a rapidity-ordering $\theta$-function in Eq. (53), we have to split the integration over $\hat{z}$ at a point $\hat{z}_m = \hat{d}_s/\xi + \xi$, so that the subtraction form for the coefficient function takes the form:

$$H_{\text{sub.}}^{(NLO)} = \frac{\alpha_s C_A}{\pi} \frac{\Omega_{2-\epsilon}}{(2\pi)^{1-\epsilon}} \left[\frac{\hat{z}_m}{\hat{z}} \int d\hat{z} \frac{q_{T1}^{2 \epsilon} (\hat{z})}{k_{T1}^{2 \epsilon}} \frac{d\epsilon}{d\hat{z}} + \frac{\hat{d}_s}{\epsilon} \int d\epsilon \frac{q_{T1}^{2 \epsilon} (\hat{d}_s - \hat{z})}{k_{T1}^{2 \epsilon}} \frac{d\epsilon}{d\hat{z}}\right],$$

(54)

where $\Omega_{2-\epsilon} = 2\pi^{1-\epsilon}/\Gamma(1 - \epsilon)$. Calculating this integral, one obtains:

$$H_{\text{sub.}}^{(NLO)} = \frac{\alpha_s C_A}{2\pi} \left(\frac{\mu^2}{q_{T1}^2}\right)^\epsilon \left[\frac{\xi_{\epsilon} (\hat{z}_m^{2 \epsilon})}{e^2} + \frac{\delta_s^{2 \epsilon} \epsilon}{3} \left(\frac{2 \log x_0}{\epsilon} - \frac{\pi^2}{3} - \log^2 x_0 + 2 \text{Li}_2(x_0) + O(\epsilon)\right)\right],$$

(55)

with $x_0 = \xi/(\xi + \xi)$. Taking all the results of this section together we get:

$$H_{\text{analyt. real}}^{(NLO)} = H_{\text{coll.}}^{(NLO)} + H_{\text{soft}}^{(NLO)} - H_{\text{sub.}}^{(NLO)} = \frac{\alpha_s C_A}{2\pi} \left(\frac{\mu^2}{q_{T1}^2}\right)^\epsilon \left[\frac{1}{e^2} + \frac{1}{\epsilon} \left(\frac{11}{6} + \log \xi\right) + \frac{67}{9} \frac{2 \pi^2}{3} + 2 \log(1 + \xi) \left(\log \xi - \log(1 + \xi)\right) + \log \delta_s \left(2 \log(1 + \xi) - \frac{11}{6} - 2 \log \delta_s\right) + \frac{1}{2} \log \xi + 4 \log(\xi + \xi) - 4 \log(\xi_0 + \xi) - 2 \log^2(\xi_0 + \xi) - 2 \text{Li}_2(\xi) - 2 \text{Li}_2 \left(\frac{\xi}{\xi_0 + \xi}\right) + O(\epsilon)\right].$$

(56)
This expression has several important features. First, logarithms of $\delta_s$ in the coefficient in front of $1/\epsilon$, which are present in Eqns. (48), (52) and (55) have canceled, giving IR-divergence a chance to cancel against the loop correction. The term $\log \xi_\mu/\epsilon$ will also do so, as we will show in the next section. Second, subtraction (55) removed all terms proportional to $\log \xi = -2Y_\mu$, since this logarithm is resummed in the evolution, and only terms decreasing as $e^{-2Y_\mu}$ are left. And third, if one takes the choice of rapidity scale (22), corresponding to $\xi_\mu = 1$, then all terms $\propto \log \xi_\mu$ will be gone and one will be left with:

$$H^{\text{NLO},\ \text{analyt. real}}_{\text{Y}_\mu=Y_\mu} = \frac{\tilde{\alpha}_s C_A}{2\pi} \left( \frac{\mu^2}{q_T^2} \right)^\epsilon \left[ \frac{1}{\epsilon^2} + \frac{11}{6} \frac{1}{\epsilon} + \frac{67}{9} - \frac{2\pi^2}{3} \right] - \frac{11}{6} \log \delta_c - 2 \log \delta_c \log \delta_a + 2 \log \delta_c \log (1 + \xi) + 2 \log \xi \log (1 + \xi) - 2 \log^2 (1 + \xi) + O(\epsilon) \right].$$

(57)

Let us discuss a bit the physical meaning of singularities, arising in the subtraction term (55). The integration region of Eq. (54) is sketched in the Fig. 2 with the lines of constant rapidity of the first gluon and it’s constant energy overlaid.

Going along the line of constant energy in the direction of decreasing rapidity $y_1$ (i.e. in a direction of a proton), one ends-up in the region of small $k_{T1}^2$. Hence, the $1/k_{T1}^2$-singularity in Eq. (53) is actually a rapidity divergence, corresponding to a fact, that probability of emitting a soft gluon is flat in rapidity. The actual soft divergence is located in a corner ($\hat{z} \to 0$, $k_{T1}^2 \to 0$), where one arrives going along the line of constant rapidity in a direction of decreasing energy. These two divergences overlap in a corner ($\hat{z} \ll \hat{z}_{\text{max}}$, $k_{T1}^2 \ll q_T^2$), producing an $1/\epsilon^2$-pole in Eq. (55).

In the soft integral (52), the $2/\epsilon^2$-term has two sources. First mechanism is the same overlap of rapidity and soft divergences as in subtraction term, and second – the overlap of soft and final-state collinear divergences. The $1/\epsilon^2$-pole contribution from the second source, surviving after double-counting subtraction, will cancel against the loop correction.

V. VIRTUAL CORRECTION AND SUBTRACTIONS

We have computed the one-loop correction to an amplitude of the process (12) in Ref. [43]. Apart from IR and UV divergences, which we regularize dimensionally, it contains rapidity divergence, physically corresponding to rapidity of a gluon in a loop going far negative. This divergence required additional regularization, which we do by tilting the Wilson lines in the definition of EFT [39] from the light-cone, as was first proposed in [63]:

$$n^\mu_\pm \to \tilde{n}^\mu_\pm = n^\mu_\pm + r \cdot n^\mu_\mp,$$

where $0 < r \ll 1$ is the regularization parameter. Such regularization roughly corresponds to a smooth cutoff for gluon rapidity at $(-\log r^{-1})/2$ and rapidity divergence manifests itself as $\log r$-term, arising before expanding loop integrals in $\epsilon$.  

FIG. 2: The sketch of integration region (shaded area above $\hat{z}$ axis) for soft double-counting subtraction term. Red dashed lines correspond to constant value of $k_{T1}^2/q_T^2$, while blue dashed lines correspond to constant $y_1$. 

V. VIRTUAL CORRECTION AND SUBTRACTIONS
The one-loop correction is proportional to the Born vertex (3) (See the last equation in Sec. 4.2 of Ref. [43]), and since we are computing \( O(\alpha_s) \) correction to the cross-section, we need an interference term:

\[
H^{(\text{NLO}), \mathcal{O}}_{\text{virt. unsubtr.}} = 2\text{Re} \left( C \left( G^{(0)} \right) \right) = \frac{\alpha_s}{2\pi} \left\{ -\frac{C_A}{\epsilon^2} + \frac{1}{\epsilon} \left[ \beta_0 - C_A(1 + L_1) \right] - C_A \left( \frac{1}{\epsilon} + \log \frac{\mu^2}{t_1} \right) \log \tilde{r} + C_A \left[ 2\text{Li}_2 \left( 1 - \frac{Q^2}{t_1} \right) + \frac{L_2^2}{2} - L_2 - \frac{1}{2} L_1(L_1 + 2) + \frac{\pi^2}{6} - \frac{2}{3} \right] + \beta_0 \left[ \frac{10}{6} + L_1 + L_2 \right] + O(r, \epsilon) \right\},
\]

where \( \beta_0 = 11C_A/3 - 2n_F/3, L_1 = \log(\mu^2/Q^2), L_2 = \log(Q^2/t_1), t_1 = q^2 / q^2 \) and \( \tilde{r} = rQ^2/q^2 \).

In Ref. [43] we have also shown, that in the full amplitude, which includes one-loop corrections to both scattering vertices and t-channel Reggeized gluon propagator (see the right panel of Fig. 5 in Ref. [43]), the log \( r \)-terms cancel and the (one-Reggeon exchange) EFT result precisely reproduces the (negative-signature part of) the dimensionally-regularized one-loop QCD amplitude of the process \( O(g) + g(P) \to g(k_1) + g(k_2) \) in the Regge limit. The EFT expression for the contribution of the latter process to CPM coefficient-function looks like:

\[
C^{(\text{NLO}), g}_\mathcal{O} = H^{(\text{NLO}), \mathcal{O}}_{\text{virt. unsubtr.}}(q^2_{T1}, y_1, \log r) + H^{(\text{NLO}), g}_{\text{virt. unsubtr.}}(q^2_{T1}, y_2, \log r) - 2\Pi^{(1)}(q^2_{T1}, \log r),
\]

where \( H^{(\text{NLO}), \mathcal{O}}_{\text{virt. unsubtr.}} \) is the one-loop correction to gluon-scattering vertex in EFT, \( \Pi^{(1)} \) is the one-loop correction to Reggeized gluon propagator (with Born propagator 1/(2\( t_1 \)) factorized-away, see Refs. [63,65] or Eq. (53) in Ref. [43]):

\[
\Pi^{(1)}(q^2_{T1}, \log r) = \frac{\alpha_s}{4\pi} \left[ -2C_A(\log r + 1) \left( \frac{1}{\epsilon} + \log \frac{\mu^2}{q^2_{T1}} \right) + \beta_0 \left( \frac{1}{\epsilon} + \frac{5}{3} + \log \frac{\mu^2}{q^2_{T1}} \right) - \frac{8}{3} C_A \right].
\]

On the other hand, one iteration of virtual part of evolution equation (23) or (38) can be written as:

\[
C^{(\text{NLO})}_\mathcal{O} = (Y^{(1)}_\mu - Y^{(2)}_\mu) \times 2\omega_g(q^2_{T1}),
\]

where \( Y^{(1,2)}_\mu \) are rapidity scales for \( \mathcal{O} \) and \( g \) scattering vertices, with an obvious optimal choice \( Y^{(1,2)}_\mu \to y_{1,2} \) and \( \omega_g \) is a gluon Regge trajectory (24). Subtracting the evolution contribution from Eq. (68) and rearranging the terms, one obtains the following expressions for subtracted one-loop corrections to the scattering vertices:

\[
H^{(\text{NLO}), \mathcal{O}}_{\text{virt. unsubtr.}}(q^2_{T1}, y_1, Y^{(1)}_\mu) = H^{(\text{NLO}), \mathcal{O}}_{\text{virt. unsubtr.}}(q^2_{T1}, y_1, \log r) - \Pi^{(1)}(q^2_{T1}, \log r) - 2Y^{(1)}_\mu \omega_g(q^2_{T1}),
\]

\[
H^{(\text{NLO}), g}_{\text{virt. unsubtr.}}(q^2_{T1}, y_2, Y^{(2)}_\mu) = H^{(\text{NLO}), g}_{\text{virt. unsubtr.}}(q^2_{T1}, y_2, \log r) - \Pi^{(1)}(q^2_{T1}, \log r) + 2Y^{(2)}_\mu \omega_g(q^2_{T1}),
\]

in which \( \log r \)-divergence cancels. By similar reasoning, one can obtain the subtracted one-loop correction to the scattering vertex with any regularization for rapidity divergences, including one proposed in Ref. [66], which opens-up a possibility to automatize the NLO calculations in a variety of small-\( x \) physics frameworks. The non rapidity-divergent-part of Eq. (59) depends on a chosen rapidity regulator, but the subtracted results (60) and (61) should be regularization-independent.

Using Eq. (60) and subtracting the known (see Ref. [41] and references therein) counter-term for the UV-renormalization of the operator (2) in the \( \overline{\text{MS}} \)-scheme: \( 2\delta Z^{(1)}_\alpha = (\tilde{\alpha}_s/(2\pi))\beta_0^{(n_F=0)} / \epsilon \) we obtain the following subtracted one-loop coefficient function:

\[
H^{(\text{NLO}), \mathcal{O}}_{\text{virt. unsubtr.}} = \frac{\alpha_s C_A}{2\pi} \left[ \frac{\mu^2}{q^2_{T1}} \left\{ \left\{ -\frac{1}{\epsilon^2} - \frac{1}{\epsilon} \right\} \left( \frac{11}{6} + \log \xi_\mu \right) + \frac{67}{18} + \frac{\pi^2}{6} + \frac{11}{3} \log \left( \frac{\mu^2}{q^2_{T1}} \right) + 2\text{Li}_2 \left( 1 - \frac{Q^2}{q^2_{T1}} \right) + \mathcal{O}(\epsilon) \right\},
\]

which we have rewritten in terms of \( \log \xi_\mu = 2(Y_\mu - Y_H) \). The divergence structure in Eq. (62) precisely matches that of Eq. (60), so the final answer for analytic part of our NLO correction is:

\[
H^{(\text{NLO})}_{\text{analyt.}} = H^{(\text{NLO}), \mathcal{O}}_{\text{virt. unsubtr.}} + H^{(\text{NLO})}_{\text{analyt. real}} = \frac{\alpha_s C_A}{2\pi} \left[ \frac{67}{6} - \frac{\pi^2}{2} + \frac{11}{3} \log \left( \frac{\mu^2}{q^2_{T1}} \right) + 2\text{Li}_2 \left( 1 - \frac{Q^2}{q^2_{T1}} \right) + 2\log \left( 1 + \xi \right) \left( \log \xi + \log \left( \log \xi + \frac{\xi}{\xi_\mu + \xi} \right) \right) \right.
\]

\[
\left. - \frac{1}{2} \log \left( \log \xi + \log \left( \log \xi \xi_\mu + \xi + 4 \log \xi \xi_\mu + \frac{\xi}{\xi_\mu + \xi} \right) \right) \right] + \mathcal{O}(\epsilon).
\]
UPDF used in PRA calculation (see Appendix). This extremely quick growth is an artifact of doubly-logarithmic LO and NLO of CPM, calculated with the same collinear PDF, which was used to generate the doubly-logarithmic $x$ subtractions at various $\delta_s$ which double-precision calculation is sufficiently stable and accuracy NLO cross-section better than 1% is reached.

Rather computationally-costly. In the calculations below, we will use fixed values of $\delta_s$ of $c \times 10^{-3}$ both for MRK and MMRK subtraction terms. To obtain reliable numerical results at two smallest values of $\delta_s$, one have to use quadruple-precision arithmetic in the squared PRA ME subroutine, however such calculations are rather quickly done. In the calculations below, we will use fixed values of $\delta_s = 3 \times 10^{-3}$, red lines - $\delta_c = 10^{-5} \delta_s$. Left panel - MRK subtraction (34) in the numerical part, right panel - MMRK subtraction (37). Relative integration accuracy is $5 \times 10^{-4}$. Quadruple precision arithmetic is used in the evaluation of exact squared ME.

The remaining $\mu$-dependence in the coefficient cancels against the running of a coupling $\lambda$ of $O(x)$ to an external source. The $\xi_s$-dependence should cancel with the UPDF evolution, and with the rapidity-scale choice (22) one obtains:

$$ H^{(\text{NLO})}_{\text{analyt.}}, \quad Y_s=\frac{\bar{\alpha}_s C_A}{2\pi} \left[ \frac{67}{6} - \frac{\pi^2}{2} + \frac{11}{3} \log \left( \frac{\mu^2}{Q^2} \right) + 2 \text{Li}_2 \left( 1 - \frac{Q^2}{Q^2} \right) \right] + \frac{11}{6} \log \delta_c - 2 \log \delta_s \log \delta_s + 2 \log \delta_c \log(1 + \delta_s) + 2 \log \xi \log(1 + \xi) - 2 \log^2(1 + \xi) + O(\epsilon) \right]. \quad (64) $$

VI. NUMERICAL RESULTS

The analytic part of NLO correction, obtained in the previous section, should be added to the numerical integral (27) evaluated in $D = 4$ space-time dimensions over the region of phase-space where neither condition (40) nor condition (41) is satisfied and with subtractions (34) or (37) included at the integrand level. Then, for sufficiently small values of $\delta_s \ll \delta_c \ll 1$, the dependence on this parameters is guaranteed to cancel. In the present section we will show some results of exploratory numerical calculations, performed with UPDF described in the Appendix.

Throughout this section we use the scale-choice $\mu_F = \mu_R = Q$. The numerical calculations have been performed with the help of parallel version of the well-known VEGAS adaptive Monte-Carlo integration algorithm, implemented in the CUBA library [67]. The main purpose of this section is to show, that MMRK subtraction term (37) indeed leads to NLO correction with more physical behavior than MRK subtraction term (34).

The cancellation of $\delta_s$ and $\delta_c$ dependence is demonstrated in the Fig. 3. One can see, that the result for NLO correction is independent on $\delta_s$ within integration accuracy practically for all points and the plateau in $\delta_c$ is reached rather quickly both for MRK and MMRK subtraction terms. To obtain reliable numerical results at two smallest values of $\delta_s$, one have to use quadruple-precision arithmetic in the squared PRA ME subroutine, however such calculations are rather computationally-costly. In the calculations below, we will use fixed values of $\delta_s = 5 \times 10^{-5}$ and $\delta_c = 10^{-5} \delta_s$ with which double-precision calculation is sufficiently stable and accuracy NLO cross-section better than 1% is reached.

Next, let us examine the relative size of NLO correction to inclusive structure function with MRK and MMRK subtractions at various $x_B$ and $Q$ (Fig. 4). The LO PRA curve in Fig. 4 grows sufficiently faster at small $x_B$ than LO and NLO of CPM, calculated with the same collinear PDF, which was used to generate the doubly-logarithmic UPDF used in PRA calculation (see Appendix). This extremely quick growth is an artifact of doubly-logarithmic
FIG. 4: Inclusive structure function of the process (1) as function of $x_B$ for $Q = 10$ GeV (left panel) and 50 GeV (right panel). Dashed line – LO PRA (14), solid lines: yellow – NLO PRA with MRK subtraction term, red – NLO PRA with MMRK subtraction. Dotted lines: blue – LO (4) and orange – NLO [41] of CPM.

approximation, which produces UPDF with nonphysically hard $q_T$ tail. With the solution of the full MMRK evolution equation (38) the small-$x_B$ growth of the structure-function will be significantly slowed-down. However the detailed $q_T$-shape of UPDF does not really affect the relative size of NLO correction. We also refrain from showing the scale-variation bands, since doubly-logarithmic UPDF does not depend on scale $\mu_y$ and corresponding logarithms in the NLO correction will have nothing to cancel against. The more detailed numerical study will be performed once the solution of Eq. (38) will become available.

Both for $Q = 10$ and 50 GeV, the NLO correction with MRK subtraction is negative and for $Q = 50$ GeV it becomes larger than LO term at $x_B > 10^{-4}$, demonstrating a severe perturbative instability. On the contrary, MMRK-subtracted NLO results look reasonable for $Q = 10$ GeV over the whole range of $x_B < 0.1$ and for $Q = 50$ GeV, the NLO correction becomes larger than 50% only at $x > 10^{-2}$.

The reason, why NLO correction appears to be larger at higher scales, has to do with the quality of our subtraction term in the DGLAP region $q_T^2 \ll k_{T,1,2}^2 \ll Q^2$. In CPM this is a region of initial-state collinear divergence, which is subtracted from NLO correction and governs PDF evolution. In PRA, there is no initial-state collinear divergence in the coefficient-function calculation. Collinear divergences are subtracted at the level of UPDF. But the DGLAP region still generates large contribution, enhanced by $\log(Q^2/q_T^2)$ and proportional to the DGLAP splitting function $p_{gg}(\hat{z})$, see Eq. (35). Double-counting subtraction term partially subtracts this contribution, but for MRK-approximation, the quality of this subtraction is very poor, resulting in a negative NLO correction. MMRK subtraction term approximates an exact DGLAP splitting function better, as it was discussed in Sec. III, but the remaining mismatch still generates the “collinear” logarithmic term. However, at smaller values of $x_B$ and $Q$ this logarithmic term does not present such a big problem as for $x_B$ closer to one and at higher $Q \gg \Lambda$, where $\Lambda \sim 1$ GeV is a scale of non-perturbative transverse momentum, which is present in UPDF.

An interesting feature of PRA is, that many observables related with transverse-momentum are available already in the LO. For the process at hands, such an observable is a leading jet $p_T$-spectrum, which at LO is just a structure function (14), differential in transverse momentum $q_{T1}$ and jet rapidity (22). Therefore now we have an opportunity to study the relative size of NLO correction in PRA also for jet-$p_T$ spectrum.

To meaningfully define the jet observable at NLO we need to take into account, that UPDF evolution is not ordered in $p_T$, so it can produce jets with $p_T$ higher than that of a hard process. But evolution is ordered in rapidity, so we can avoid the need of fully-exclusive Monte-Carlo simulation by reasonably defining the “most forward” high-$p_T$ jet. We do this as follows:

1. If both gluons in the NLO subprocess (25) have $\Delta y_{1,2}^2 + \Delta \phi_{1,2}^2 < R^2$ with jet-radius parameter $R = 0.4$ in our numerical calculations below, their four-momenta are added to form a four-momentum of a jet.

2. Otherwise the four-momentum of a gluon leading in $p_T$ and lying within rapidity-acceptance is taken as a jet four-momentum.
3. If rapidity of the gluon subleading in $p_T$ is $y_{\text{subl.}} < y_{\text{jet}}$, then it is unconstrained

4. If $y_{\text{subl.}} > y_{\text{jet}}$, we require it’s $p_{T\text{ (subl.)}} < p_{T\text{ (veto)}} = 10$ GeV.

In all other respects, our NLO calculation for jet-$p_T$ spectrum proceeds the same way as for inclusive structure function, with no need to re-calculate the analytic part, just the phase-space slicing parameters have to be taken sufficiently small to avoid interference with jet definition.
Numerical results for jet-$p_T$ spectrum are shown in the Fig. for two different values of $Q = 10$ and 50 GeV and the same value of “center-of-mass energy” $S = Q^2/x_B$. For jet $p_T$-spectrum we use the same factorization and renormalization scale-choice as for inclusive SF. Here we find larger NLO corrections at smaller scales, which most likely reflects a steeper decrease of PDF with increasing values of $x$ at smaller scales. The MMRK approximation again leads to smaller NLO correction and at $Q = 50$ GeV, the NLO correction to jet-$p_T$ spectrum is negligible in most bins, suggesting that LO PRA is a good approximation for this observable in this kinematic region.

Finally, the Fig. allows us to examine, how NLO correction to the inclusive SF and jet $p_T$-spectrum is distributed w.r.t. transverse momentum of incoming Reggeon $q_{T1}^2$. As it was anticipated above, the bulk of negative NLO contribution is located at small values of $q_{T1}^2 < 1$ GeV$^2$, corresponding to DGLAP region, and the MMRK-term provides smaller subtraction there. For the jet-$p_T$ spectrum, the loop correction and IR-cancellation effects are present only in two bins in the right panel of Fig. where the LO term is non-zero, and one can see, that NLO result in these bins is very close to LO, so NLO correction in those bins is negligible. It is the behavior of subtraction term, which is responsible for the bulk of NLO correction to the jet-$p_T$ spectrum.

As a conclusion we emphasize, that in the present paper we have formulated the technique of NLO calculations in PRA for gluon-induced processes and the MMRK-approximation for squared matrix element in QCD with emission of additional partons. This approximation should be used consistently as the subtraction term in NLO correction and in the UPDF evolution, leading to improved perturbative stability of the calculation both for inclusive structure function and jet cross-section.

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Appendix: UPDF in doubly-logarithmic approximation

To demonstrate how collinear divergences can be subtracted to all orders from the UPDF we will closely follow Ref. [3]. Let us simplify Eq. (23) by omitting all $O(z)$ corrections to the kernel, which in turn leads to disappearance of the $\mu_Y$-scale dependence:

$$\tilde{C}(x, q_T, \mu) = \delta(x - 1)\delta(q_T) + \frac{\alpha_s(\mu)C_A}{\pi} \int_0^1 \frac{dz}{z} \left\{ \int \frac{d^{D-2}k_T}{k_T^2} \tilde{C}\left(\frac{x}{z}, q_T + k_T, \mu\right) + r_T \pi^{-\epsilon} \frac{q_T^{-\epsilon}}{\epsilon} \tilde{C}\left(\frac{x}{z}, q_T, \mu\right) \right\},$$

(65)

where $\tilde{C} = C/\pi$. To facilitate taking the iterations of this kernel and subtraction of collinear divergences, we pass to the Mellin representation (which we define as in Eq. (2.7) of Ref. [3]) for the $x$-dependence of the evolution factor and transverse-position space for it’s $q_T$-dependence:

$$\hat{C}(N, x_T, \mu) = \int_0^1 dx x^N \int d^{D-2}q_T e^{ix_T q_T} \tilde{C}(x, q_T, \mu).$$

(66)

In this representation, Eq. (65) takes the form:

$$\hat{C}(N, x_T, \mu) = 1 + \hat{\alpha}_s \frac{\Gamma(1-\epsilon)(\mu^2)^\epsilon}{N} \left(\frac{4\pi^2}{(4\pi)^{\epsilon}}\right) \int d^{D-2}y_T \left[(x_T^2)^\epsilon \delta(x_T - y_T) + \frac{\epsilon \Gamma(1-\epsilon)}{\pi^{1-\epsilon}} \left((x_T - y_T)^2\right)^{-1+2\epsilon}\right] \hat{C}(N, y_T, \mu),$$

(67)

where $\hat{\alpha}_s = \alpha_s(\mu)(\mu^2)^{-\epsilon}C_A/\pi$. Taking iterations of this equation reduces to repeated use of Eq. (51), so for the $n+1$-th iteration one finds (compare with Eqns. (3.3) – (3.5) of Ref. [3]):

$$\hat{C}_{n+1}(N, x_T, \mu) = \frac{\alpha_s}{N} \frac{\Gamma(1-\epsilon)}{(4\pi)^{\epsilon}} \frac{(\mu^2 x_T^2)^\epsilon}{n+1} J_n(\epsilon) \times \hat{C}_n(N, x_T, \mu),$$

(68)

$$J_n(\epsilon) = n + 1 - \frac{n \Gamma(1-\epsilon) \Gamma(1-\epsilon(n+1)) \Gamma(1+\epsilon(n+1)) \Gamma(1+\epsilon(n-1))}{\Gamma(1-n\epsilon) \Gamma(1-2\epsilon) \Gamma(1+n\epsilon)}.$$
The advantage of working in $x_T$-space is, that inverse Fourier transform of the product of evolution factor and $x_T$-space coefficient function will not produce any additional divergences, so all collinear divergences are contained in the evolution factor in $x_T$-space and can be subtracted from it. The renormalization factor which subtracts collinear divergences from the hard process can be defined in $N$-space as (see Eq. (2.28) in Ref. [8]):

$$Z_{\text{coll}}(\epsilon, N) = \exp \left[ \frac{1}{\epsilon} \sum_{k=1}^{\infty} \frac{(\hat{\alpha}_s S_{\epsilon}(\mu^2/\mu_F^2))^k}{k} \gamma_k(N) \right],$$

where $S_{\epsilon} = \exp [\epsilon (\log 4\pi - \gamma_E)]$ is the usual factor defining the $\overline{MS}$-scheme, $\mu_F$ is the factorization scale and $\gamma_k(N)$ are the coefficients of expansion of DGLAP anomalous dimension $\gamma_{\text{DGLAP}}(N, \alpha_s)$ in powers of $\hat{\alpha}_s$. In agreement with known results [3, 8], we find that the following series of coefficients leads to subtraction of collinear divergences from $\tilde{C}(N, x_T, \mu)$ up to $O(\hat{\alpha}_s^3)$:

$$\gamma_1 = \frac{1}{N}, \ \gamma_2 = \gamma_3 = \gamma_5 = 0, \ \gamma_4 = \frac{2\zeta(3)}{N^4}, \ \gamma_6 = \frac{2\zeta(5)}{N^6}, \ \gamma_7 = \frac{12\zeta(3)}{N^7}, \ \gamma_8 = \frac{2\zeta(7)}{N^8}, \ \gamma_9 = \frac{32\zeta(3)\zeta(5)}{N^9}.$$  

We have checked up to $O(\hat{\alpha}_s^3)$, that the finite part of the coefficient function can be represented as:

$$\tilde{\sigma}_{\text{coll, ren}}(N, x_T, \mu) = \exp \left[ -\frac{\hat{\alpha}_s}{N} \log(\mu_F^2 x_T^2) \right] \left\{ 1 + \left[ \frac{\hat{\alpha}_s^3}{N^3} \frac{2\zeta(3)}{4} - \frac{\hat{\alpha}_s^4}{N^2} \left( 2\zeta(3) \log(\mu_F^2 x_T^2) + \frac{\pi^4}{120} \right) + O(\hat{\alpha}_s^5) \right] \right\},$$

where $x_T = x_T e^{\gamma_E}/(4\pi)$, the non-cancellation of $\gamma_E$ and $4\pi$ is a consequence of working in $x_T$-space. The exponential factor in last equation resums double-logarithms of the form $\hat{\alpha}_s \log(\xi_T^2/\mu_F^2) \log(1/x)$ and corrections to it are at best single-logarithmic and start at $O(\hat{\alpha}_s^3)$. Therefore, the double-logarithmic approximation is the basic approximation for UPDF. Converting the exponential factor back to $x$-space one obtains:

$$\tilde{C}_{\text{DL-pert}}(x, x_T, \mu) = \delta(1-x) - \frac{1}{x} \sqrt{\frac{\hat{\alpha}_s}{N} \log(\mu_F^2 x_T^2)} I_1 \left( 2 \sqrt{\frac{\hat{\alpha}_s}{N} \log(\mu_F^2 x_T^2)} \log(x) \right),$$

(70)

where $I_1(x)$ is the Bessel function of the first kind. Before Fourier-transforming this expression numerically back to $q_T$-space, we multiply it by the non-perturbative shape-function, which we take in a Gaussian form, suppressing large values of $x_T$:

$$F_{\text{NP}}(x_T) = \exp \left[ -\Lambda^2 x_T^2 \right],$$

where parameter $\Lambda$, equal to 1 GeV in our numerical calculations, characterizes the spread of “intrinsic” transverse-momentum of a gluon in a proton. Then, to obtain the UPDF we convolute the evolution factor with the collinear PDF by Eq. (9). In the numerical calculations of the present paper we have used the NLO set of CTEQ-14 PDFs [69] as a collinear input used the NLO running of $\alpha_s$ corresponding to this PDF set with $\alpha_s(M_Z) = 0.106$, as provided by LHAPDF library [70].

In the Fig. 7 we compare Doubly-Logarithmic UPDF with several other phenomenological UPDFs known in the literature, all UPDFs are obtained from HERAPDF-20-NLO-EIG PDF set [70] as a collinear input used the NLO running of $\alpha_s$ corresponding to this PDF set with $\alpha_s(M_Z) = 0.106$, as provided by LHAPDF library [70].

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Also, Collins-Ellis-Bliemlein-approach does not contain any non-perturbative shape-function, which explains different small-$q_T$ behavior.

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FIG. 7: Comparison of $q_T^2$-dependence of Doubly-logarithmic UPDF (solid blue line) proposed in the present paper with several widely-used phenomenological UPDFs: solid orange line – KMRW [35–37] formula with exact normalization [72], dashed green line – Parton-Branching method [11, 34], green-dotted line – Collins-Ellis-Blümlein formula [2, 38], all with the parameters $\mu_F = \mu_R = 50$ GeV and $x = 10^{-2}$ (left panel) and $10^{-4}$ (right panel, the UPDFs are multiplied by $1/6$). All UPDFs are obtained from HERAPDF20-NLO-EIG PDF set [70, 71] as a collinear input.

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For any four-momentum $k$ we define Sudakov decomposition as $k^\mu = (n_+^\mu k_+ + n_-^\mu k_-)/2 + k_T^\mu$ with $k_+ = n_+ k$, $n_-^2 = 0$, $n_+ n_- = 2$ and $n_+ k_T = 0$, so that $k^2 = k_+ k_- - k_T^2$ and we do not distinguish covariant and contravariant light-cone components: $k_\pm = k^{\pm}$.