Stability and asymptotic stability in the energy space of the sum of $N$ solitons for subcritical gKdV equations

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Abstract

We prove in this paper the stability and asymptotic stability in $H^1$ of a decoupled sum of $N$ solitons for the subcritical generalized KdV equations

$$u_t + (u_{xx} + u^p)_x = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R},$$

$$u(0, x) = u_0(x), \quad x \in \mathbb{R},$$

for $1 < p < 5$ and $u_0 \in H^1(\mathbb{R})$. This model for $p = 2$ was first introduced in the study of waves on shallow water, see Korteweg and de Vries [9]. It also appears for $p = 2$ and $3$, in other areas of Physics (see e.g. Lamb [10]).

Recall that (1) is well-posed in the energy space $H^1$. For $p = 2, 3, 4$, it was proved by Kenig, Ponce and Vega [8] (see also Kato [7], Ginibre and Tsutsumi [5]), that for $u_0 \in H^1(\mathbb{R})$, there exists a unique solution $u \in C(\mathbb{R}, H^1(\mathbb{R}))$ of (1) satisfying the following two conservation laws, for all $t \in \mathbb{R}$,

$$\int u^2(t) = \int u_0^2,$$  

$$E(u(t)) = \frac{1}{2} \int u_x^2(t) - \frac{1}{p + 1} \int u^{p+1}(t) = \frac{1}{2} \int u_0^{2x} - \frac{1}{p + 1} \int u_0^{p+1}. $$

For $p = 2, 3, 4$, global existence of all solutions in $H^1$, as well as uniform bound in $H^1$, follow directly from the Gagliardo–Nirenberg inequality,

$$\forall v \in H^1(\mathbb{R}), \quad \int |v|^{p+1} \leq C(p) \left( \int v^2 \right)^{\frac{p+3}{p-1}} \left( \int v_x^2 \right)^{\frac{p-1}{p-1}},$$

and relations (2), (3), giving a uniform bound in $H^1$ for any solution.

This is in contrast with the case $p = 5$, for which there exist solutions $u(t)$ of (1) such that $|u(t)|_{H^1} \to +\infty$ as $t \to T$, for $0 < T < +\infty$, see [13] and [17]. For $p > 5$ such behavior...
is also conjectured. Thus, for the question of global existence and bound in $H^1$, the case $1 < p < 5$ is called the subcritical case, $p = 5$ the critical case and $p > 5$ the supercritical case.

Equation (4) has explicit traveling wave solutions, called solitons, which play a fundamental role in the generic behavior of the solutions. Let

$$Q(x) = \left( \frac{p+1}{2 \text{ch}^2 \left( \frac{p-1}{2} x \right)} \right)^{\frac{1}{p-1}}$$

be the only positive solution in $H^1(\mathbb{R})$ (up to translation) of $Q_{xx} + Q^p = Q$, and for $c > 0$, let $Q_c(x) = e^{\sqrt{c}x}Q(\sqrt{c}x)$. The traveling waves solutions of (4) are

$$u(t, x) = Q_c(x - ct) = c^{\frac{1}{p-1}}Q(\sqrt{c}(x - ct)),$$

where $c > 0$ is the speed of the soliton.

For the KdV equation ($p = 2$), there is a much wider class of special explicit solutions for (4), called $N$–solitons. They correspond to the superposition of $N$ traveling waves with different speeds that interact and then remain unchanged after interaction. The $N$–solitons behave asymptotically in large time as the sum of $N$ traveling waves, and as for the single solitons, there is no dispersion. We refer to [20] for explicit expressions and further properties of these solutions. For $p \neq 2$, even the existence of solutions behaving asymptotically as the sum of $N$ solitons was not known.

Important notions for these solutions are the stability and asymptotic stability with respect to initial data.

For $c > 0$, the soliton $Q_c(x - ct)$ is stable in $H^1$ if:

$$\forall \delta_0 > 0, \exists \alpha_0 > 0 / |u_0 - Q_c|_{H^1} \leq \alpha_0 \Rightarrow \forall t \geq 0, \exists x(t) / |u(t) - Q_c(., - x(t))|_{H^1} \leq \delta_0.$$  

The family of solitons $\{Q_c(x - x_0 - ct), c > 0, x_0 \in \mathbb{R}\}$ is asymptotically stable if:

$$\exists \alpha_0 > 0 / |u_0 - Q_c|_{H^1} \leq \alpha_0 \Rightarrow \exists c_{\infty}, x(t) / u(t, . + x(t)) \rightarrow_{t \rightarrow +\infty} Q_{c_{\infty}}$$

in $H^1$.

We recall previously known results concerning the notions of stability of solitons and $N$–solitons:

- In the subcritical case: $p = 2, 3, 4$, it follows from energetic arguments that the solitons are $H^1$ stable (see Benjamin [1] and Weinstein [24]). Moreover, Martel and Merle [13] prove the asymptotic stability of the family of solitons in the energy space. The proof relies on a rigidity theorem close to the family of solitons, which was first given for the critical case ([13]), and which is based on nonlinear argument. (Pego and Weinstein [21] prove this result for $p = 2, 3$ for initial data with exponential decay as $x \rightarrow +\infty$.)

In the case of the KdV equation, Maddocks and Sachs [12] prove the stability in $H^N(\mathbb{R})$ of $N$–solitons (recall that there are explicit solutions of the KdV equation): for any initial data $u_0$ close in $H^N(\mathbb{R})$ to an $N$–soliton, the solution $u(t)$ of the KdV equation remains uniformly close in $H^N(\mathbb{R})$ for all time to an $N$ soliton profile with same speeds. Their proof involves $N$ conserved quantities for the KdV equation, and this is the reason why
they need to impose closeness in high regularity spaces. Note that this result is known only with \( p = 2 \) and with this regularity assumption of the initial data. Asymptotic stability is unknown in this context.

- In the critical case \( p = 5 \), any solution with negative energy initially close to the soliton blows up in finite or infinite time in \( H^1 \) (Merle \[13\]), and actually blows up in finite time if the initial data satisfies in addition a polynomial decay condition on the right in space (Martel and Merle \[17\]). (Note that \( E(Q) = 0 \) for \( p = 5 \).) Of course this implies the instability of the soliton. These results rely on rigidity theorems around the soliton.

- In the supercritical case \( p > 5 \), Bona, Souganidis, and Strauss \[2\] proved, using Grillakis, Shatah, and Strauss \[6\] type arguments, \( H^1 \) instability of solitons. Moreover, numerical experiments, see e.g. Dix and McKinney \[4\], suggest existence of blow up solutions arbitrarily close to the family of solitons.

In this paper, for \( p = 2, 3, 4 \), using techniques developed for the critical and subcritical cases in \[3\] and \[3\] as well as a direct variational argument in \( H^1 \), we prove the stability and asymptotic stability of the sum

\[
\sum_{j=1}^{N} Q_{c_j} (x - x_j), \quad \text{where} \quad 0 < c_1^0 < \ldots < c_N^0, \quad x_1 < \ldots < x_N,
\]

in \( H^1(\mathbb{R}) \), for \( t \geq 0 \).

**Theorem 1 (Asymptotic stability of the sum of \( N \) solitons)** Let \( p = 2, 3 \) or \( 4 \). Let \( 0 < c_1^0 < \ldots < c_N^0 \). There exist \( \gamma_0, A_0, L_0, \alpha_0 > 0 \) such that the following is true. Let \( u_0 \in H^1(\mathbb{R}) \) and assume that there exist \( L > L_0, \alpha < \alpha_0, \) and \( x_1^0 < \ldots < x_N^0 \), such that

\[
\left| u_0 - \sum_{j=1}^{N} Q_{c_j^0} (\cdot - x_j^0) \right|_{H^1} \leq \alpha, \quad \text{and} \quad x_j^0 > x_{j-1}^0 + L, \quad \text{for all} \quad j = 2, \ldots, N.
\]  

(5)

Let \( u(t) \) be the solution of \([1]\). Then, there exist \( x_1(t), \ldots, x_N(t) \) such that

(i) **Stability of the sum of \( N \) decoupled solitons.**

\[
\forall t \geq 0, \quad \left| u(t) - \sum_{j=1}^{N} Q_{c_j^0} (x - x_j(t)) \right|_{H^1} \leq A_0 \left( \alpha + e^{-\gamma_0 L} \right).
\]  

(6)

(ii) **Asymptotic stability of the sum of \( N \) solitons.** Moreover, there exist \( c_1^{+\infty}, \ldots, c_N^{+\infty} \), with \( |c_j^{+\infty} - c_j^0| \leq A_0 \left( \alpha + e^{-\gamma_0 L} \right) \), such that

\[
\left| u(t) - \sum_{j=1}^{N} Q_{c_j^{+\infty}} (x - x_j(t)) \right|_{L^2(\mathbb{R}; \mathbb{R}^d)} \rightarrow 0, \quad \dot{x}_j(t) \rightarrow c_j^{+\infty} \quad \text{as} \quad t \rightarrow +\infty.
\]  

(7)

**Remark 1.** It is well-known that for \( p = 2 \) and \( p = 3, \) \([1]\) is completely integrable. Indeed, for suitable \( u_0 \) (\( u_0 \) and its derivatives with exponential decay at infinity) there exist an infinite number of conservation laws, see e.g. Lax \([1]\) and Miura \([20]\). Moreover, many results on these equations rely on the inverse scattering method, which transform
the problem in a sequence of linear problems (but requires strong decay assumption on the solution). In this paper, we do not use integrability.

**Remark 2.** For Schrödinger type equations, Perelman [22] and Buslaev and Perelman [3], with strong conditions on initial data and nonlinearity, and using a linearization method around the soliton, prove asymptotic stability results by a fixed point argument. Unfortunately, this method breaks down without decay assumption on the initial data.

**Remark 3.** In Theorem 1 (ii), we cannot have convergence to zero in $L^2(x > 0)$. Indeed, assumption (3) on the initial data allows the existence in $u(t)$ of an additional soliton of size less that $\alpha$ (thus traveling at arbitrarily small speed). For $p = 2$, an explicit example can be constructed using the $N$-soliton solutions.

Recall that for $p = 2$ any $N$-soliton solution has the form $v(t, x) = U^{(N)}(x; c_j, x_j - c_j t)$, where $\{U^{(N)}(x; c_j, y_j); c_j > 0, y_j \in \mathbb{R}\}$ is the family of explicit $N$-soliton profiles (see e.g. [12], §3.1). As a direct corollary of Theorem 1, for $p = 2$, we prove stability and asymptotic stability of this family.

**Corollary 1 (Asymptotic stability in $H^1$ of $N$-solitons for $p = 2$)** Let $p = 2$. Let $0 < c_0 < \ldots < c_N^0$ and $x_0, \ldots, x_N^0 \in \mathbb{R}$. For all $\delta_1 > 0$, there exists $\alpha_1 > 0$ such that the following is true. Let $u(t)$ be a solution of (1). If $|u(0) - U^{(N)}(\cdot; c_j^0, -x_j^0)|_{H^1} \leq \alpha_1$, then there exist $x_j(t)$ such that

$$\forall t > 0, \quad |u(t) - U^{(N)}(\cdot; c_j^0, -x_j(t))|_{H^1} \leq \delta_1. \quad (8)$$

Moreover, there exist $c_j^{+\infty} > 0$ such that

$$|u(t) - U^{(N)}(\cdot; c_j^{+\infty}, -x_j(t))|_{L^2(x > c_j^{+\infty} t/10)} \to 0, \quad \dot{x}_j(t) \to c_j^{+\infty} \text{ as } t \to +\infty. \quad (9)$$

Note that this improves the result in [12] in two ways. First, stability is proved in $H^1$ instead of $H^N$. Second, we also prove asymptotic stability as $t \to +\infty$. Corollary 1 is proved at the end of §4.

Let us sketch the proof of these results. For Theorem 1, using modulation theory, $u(t) = \sum_{j=1}^{N} Q_c(t) (x - x_j(t)) + \varepsilon(t, x)$, where $\varepsilon(t)$ is small in $H^1$, and $x_i(t)$, $c_i(t)$ are geometrical parameters (see §2). The stability result is equivalent to control both the variation of $c_j(t)$ and the size of $\varepsilon(t)$ in $H^1$ (§3).

Our main arguments are based on $L^2$ properties of the solution. From [3] and [3], the $L^2$ norm of the solution at the right of each soliton is almost decreasing in time. This property together with energy argument allows us to prove that the variation of $c_j(t)$ is quadratic in $|\varepsilon(t)|_{H^1}$, which is a key of the problem.

Let us explain the argument formally by taking $\varepsilon = 0$ and so $u(t) = \sum Q_c(t) (x - x_j(t))$. The energy conservation becomes

$$\sum c_j^{-1/2}(t) = \sum c_j^{-1/2}(0),$$

where $\beta = \frac{2}{p-1}$. The monotonicity of the $L^2$ norm at the right of each soliton gives us

$$\Delta_j(t) = \sum_{k=\pm j}^{N} c_k^{-1/2}(t) - c_k^{-1/2}(0) \leq 0.$$
We claim that \( c_j(t) = c_j(0) \) by a convexity argument. Indeed,
\[
0 = \sum c_j^{\beta + \frac{1}{2}}(t) - c_j^{\beta + \frac{1}{2}}(0) \sim \frac{2\beta + 1}{2\beta - 1} \sum c_j(0)(c_j^{\beta - \frac{1}{2}}(t) - c_j^{\beta - \frac{1}{2}}(0)) \\
= \frac{2\beta + 1}{2\beta - 1} \sum (c_j(0) - c_{j+1}(0)) \Delta_j(t) \geq \sigma_0 \sum |\Delta_j(t)| \geq \sigma_1 \sum |c_j(t) - c_j(0)|.
\]
Thus \( c_j(t) \) is a constant at the first order. In fact, we prove that the variation in time of \( c_j(t) \) is of order 2 in \( \epsilon(t) \).

Then, we control the variation of \( \epsilon(t) \) in \( H^1 \) by a refined version of this argument, using suitable orthogonality conditions on \( \epsilon \).

The asymptotic stability result follows directly from a rigidity property of the flow of equation (\[4\]) around the solitons (see \[3\]) and monotonicity properties of the mass (\S4).

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2 Decomposition and properties of a solution close to the sum of \( N \) solitons

2.1 Decomposition of the solution and conservation laws

Fix \( 0 < c_1^0 < \ldots < c_N^0 \) and let
\[
\sigma_0 = \frac{1}{2} \min(c_1^0, c_2^0 - c_1^0, \ldots, c_N^0 - c_{N-1}^0).
\]

From modulation theory, we claim.

**Lemma 1 (Decomposition of the solution)** There exists \( L_1, \alpha_1, K_1 > 0 \) such that the following is true. If for \( L > L_1, 0 < \alpha < \alpha_1, t_0 > 0 \), we have
\[
\sup_{0 \leq t \leq t_0} \left( \inf_{y_j > y_{j-1} + L} \left\{|u(t, \cdot) - \sum_{j=1}^N Q_{c_j^0}(\cdot - y_j)|_{H^1}\right\} \right) < \alpha,
\]
then there exist unique \( C^1 \) functions \( c_j : [0, t_0] \rightarrow (0, +\infty), x_j : [0, t_0] \rightarrow \mathbb{R} \), such that
\[
\epsilon(t, x) = u(t, x) - \sum_{j=1}^N R_j(t, x), \quad \text{where} \quad R_j(t, x) = Q_{c_j(t)}(x - x_j(t)),
\]
satisfies the following orthogonality conditions
\[
\forall j, \forall t \in [0, t_0], \quad \int R_j(t) \epsilon(t) = \int (R_j(t))_x \epsilon(t) = 0.
\]

Moreover, there exists \( K_1 > 0 \) such that \( \forall t \in [0, t_0], \)
\[
|\epsilon(t)|_{H^1} + \sum_{j=1}^N |c_j(t) - c_j^0| \leq K_1 \alpha,
\]
\[
\forall j, \quad |c_j(t)| + |\dot{c}_j(t) - c_j(t)| \leq K_1 \left( \int e^{-\sqrt{\sigma_0}|x - x_j(t)|^2/2} \epsilon^2(t) \right)^{1/2} + K_1 e^{-\sqrt{\sigma_0}(L+\sigma_0)t/4}.
\]
Thus, by (14),

$$
\varepsilon_t + \varepsilon_{xx} = -\sum_{j=1}^{N} \frac{\dot{c}_j}{2c_j} \left( \frac{2R_j}{p-1} + (x - x_i)(R_j)_x \right) + \sum_{j=1}^{N} (\dot{x}_j - c_j)R_{jx} - \left( (\varepsilon + \sum_{j=1}^{N} R_j)^p - \sum_{j=1}^{N} R_j^p \right)_x.
$$

By taking (formally) the scalar product of this equation by $R_j$ and $(R_j)_x$, and using calculations in the proof of Lemma 8 we prove

$$
|\dot{c}_j(t)| + |\dot{x}_j(t) - c_j(t)| \leq C \left( \int e^{-\sqrt{\sigma_0}|x-x_j(t)|/2} \varepsilon^2(t) \right)^{1/2} + C \sum_{k\neq j} e^{-\sqrt{\sigma_0}|x_k(t)-x_j(t)|/2}.
$$

For $\alpha > 0$ small enough, and $L$ large enough, we have $|x_k(t) - x_j(t)| \geq \frac{L}{2} + \sigma_0 t$, and this proves (14).

Next, by using the conservation of energy for $u(t)$, i.e.

$$
E(u(t)) := \int \frac{1}{2} u_x^2(t, x) - \frac{1}{p+1} u^{p+1}(t, x) \, dx = E(u_0),
$$

and linearizing the energy around $R = \sum_{j=1}^{N} R_j$, we prove the following result.

**Lemma 2 (Energy bounds)** There exist $K_2 > 0$ and $L_2 > 0$ such that the following is true. Assume that $\forall j$, $c_j(t) \geq \sigma_0$, and $x_j(t) - x_{j-1}(t) \geq L \geq L_2$. Then, $\forall t \in [0, t_0]$,

$$
\left| \sum_{j=1}^{N} [E(R_j(t)) - E(R_j(0))] + \frac{1}{2} \int (\varepsilon_x^2 - pR^{p-1}\varepsilon^2)(t) \right| 
\leq K_2 \left\{ |\varepsilon(0)|^2_{H^1} + |\varepsilon(t)|^3_{H^1} + e^{-\sqrt{\sigma_0}L/2} \right\},
$$

(15)

where $K_2$ is a constant.

**Proof.** Insert (14) into $E(u(t))$ and integrate by parts. We have

$$
E(u(t)) = \int \frac{1}{2} R_x^2 - \frac{1}{p+1} R^{p+1} \, dx - \int (R_{xx} + R^p) \varepsilon \, dx + \int \frac{1}{2} \varepsilon_x^2 - \frac{p}{2} \varepsilon R^{p-1} \, dx
+ \int \frac{1}{p+1} \left( -(R + \varepsilon)^{p+1} + R^p \varepsilon + \frac{p}{2} R^{p-1} \varepsilon^2 \right) \, dx
$$

(16)

(17)

We first observe that $|\varepsilon| \leq C \|\varepsilon\|_{H^1}^3$. Next, remark that $\sigma_0 \leq c_j(t)$, $x_j(t) - x_{j-1}(t) \geq L$, implies $|R_{j}(x, t)| + |(R_j)_x(x, t)| \leq C e^{-\sqrt{\sigma_0}|x-x_j(t)|}$, and so

$$
\int R_j(t) R_k(t) \, dx + \int (R_j)_x(t)(R_k)_x(t) \, dx \leq C e^{-\sqrt{\sigma_0}L/2} \quad \text{if } j \neq k.
$$

(18)

Thus, by $(R_j)_{xx} + R_j^p = c_j R_j$, we have

$$
\left| (16) - \sum_{j=1}^{N} E(R_j(t)) + \sum_{j} c_j R_j \varepsilon(t) - \frac{1}{2} \int (\varepsilon_x^2 - pR^{p-1}\varepsilon^2)(t) \right| \leq C e^{-\sqrt{\sigma_0}L/2}.
$$

(19)
From \( \int R_j(t)\varepsilon(t) = 0 \), we obtain

\[
\left| E(u(t)) - \sum_{j=1}^{N} E(R_j(t)) - \frac{1}{2} \int (\varepsilon_x^2 - pR^{p-1}\varepsilon^2)(t) \right| \leq Ce^{-\sqrt{\sigma_0}L/2} + C \|\varepsilon(t)\|_{H^1}^3.
\]

Since \( E(u(t)) = E(u(0)) \), applying the previous formula at \( t = 0 \) and at \( t \), we prove the lemma.

### 2.2 Almost monotonicity of the mass at the right

We follow the proof of Lemma 20 in [13]. Let

\[
\phi(x) = cQ(\sqrt{\sigma_0}x/2), \quad \psi(x) = \int_{-\infty}^{x} \phi(y)dy, \quad \text{where} \quad c = \left( \frac{2}{\sqrt{\sigma_0}} \int_{-\infty}^{\infty} Q \right)^{-1}.
\]  

(20)

Note that \( \forall x \in \mathbb{R}, \psi' > 0, 0 < \psi(x) < 1 \), and \( \lim_{x \to -\infty} \psi(x) = 0, \lim_{x \to +\infty} \psi(x) = 1 \). Let

\[
j \geq 2, \quad I_j = \int u^2(t, x)\psi(x - m_j(t)) \, dx, \quad m_j(t) = \frac{x_{j-1}(t) + x_j(t)}{2}.
\]

(21)

**Lemma 3** (Almost monotonicity of the mass on the right of each soliton [13])

*There exist \( K_3 = K_3(\sigma_0) > 0, L_3 = L_3(\sigma_0) > 0 \) such that the following is true. Let \( t_1 \in [0, t_0) \). Assume that \( \forall t \in [0, t_1), \forall j, \)

\[
\dot{x}_1(t) \geq \sigma_0, \quad \dot{x}_j(t) - \dot{x}_{j-1}(t) \geq \sigma_0, \quad c_j(t) > \sigma_0, \quad \text{and} \quad |\varepsilon(t)|_{H^1}^{p-1} \leq \frac{\sigma_0}{8 \cdot 2^p - 1}.
\]

(22)

*If for \( L > L_3 \), \( \forall j \in \{2, \ldots, N\}, x_j(0) - x_{j-1}(0) \geq L \), then

\[
I_j(t_1) - I_j(0) \leq K_3 e^{-\sqrt{\sigma_0}L/8}.
\]

**Proof** Let \( j \in \{1, \ldots, N\} \). Using equation (1) and integrating by parts several times, we have (see [13] equation (20)),

\[
\frac{d}{dt}I_j(t) = \int \left( -3u_x^2 - \dot{m}u^2 + \frac{2p}{p+1}u^{p+1} \right) \psi' + u^2 \psi^{(3)}.
\]

By definition of \( \psi, \psi^{(3)} \leq \frac{\sigma_0}{4} \psi' \), so that

\[
\int u^2 \psi^{(3)} \leq \frac{\sigma_0}{4} \int u^2 \psi'.
\]

(23)

To bound \( \int u^{p+1} \psi' \), we divide the real line to two regions: \( I = [a, b] \) and its complement \( I^C \), where \( a = a(t) = x_{j-1}(t) + \frac{L}{4} \) and \( b = b(t) = x_j(t) - \frac{L}{4} \). Inside the interval \( I \) we have

\[
\left| \int_I u^{p+1} \psi' \right| \leq \int u^2 \psi' \cdot \sup_I |u|^{p-1}
\]
Since for $x \in I$, for all $k = 1, 2, \ldots, N$, $|x - x_k(t)| \geq \frac{L}{4}$, we have
\[ |u(t, x)|^{p-1} = \left| \sum_{k=1}^{N} R_k(t, x) + \varepsilon(t, x) \right|^{p-1} \leq Ce^{-\sqrt{\sigma_0}L/4} + 2^{p-1} |\varepsilon(t)|^{p-1} \leq \frac{\sigma_0}{4}, \]
for $L > L_3(\sigma_0)$. Thus,
\[ \left| \int_I u^{p+1} \psi' \right| \leq \frac{\sigma_0}{4} \int u^2 \psi'. \quad (24) \]

Next, in $I^C$, by Gagliardo Nirenberg inequality,
\[
\int_{I^C} u^{p+1} \psi' \, dx \leq \int u^{p+1} \, dx \cdot \sup_{I^C} \psi' \leq C \left\| u \right\|_{H^1}^{p+1} \cdot \exp \left\{ -\frac{\sqrt{\sigma_0}}{4} \left[ x_j(t) - x_{j-1}(t) - \frac{L}{4} \right] \right\} \leq Ce^{-\frac{\sqrt{\sigma_0}}{8}(2\sigma_0 t + L)}, \quad (25)
\]
by $x_j(t) - x_{j-1}(t) \geq x_j(0) - x_{j-1}(0) + \sigma_0 t \geq L + \sigma_0 t$. From $\dot{m} \geq \sigma_0$, (23), (24) and (25), we obtain
\[
\frac{d}{dt} I_j(t) \leq \int \left( -3u^2 x - \frac{\sigma_0}{2} u^2 \right) \psi' \, dx + Ce^{-\frac{\sqrt{\sigma_0}}{8}(2\sigma_0 t + L)} \leq Ce^{-\frac{\sqrt{\sigma_0}}{8}(2\sigma_0 t + L)}.
\]
Thus, by integrating between 0 and $t_1$, we obtain the conclusion. Note that $K_3$ and $L_3$ are chosen independently of $t_1$.

### 2.3 Positivity of the quadratic form

By the choice of orthogonality conditions on $\varepsilon(t)$ and standard arguments, we claim the following lemma.

**Lemma 4 (Positivity of the quadratic form)** There exists $L_4 > 0$ and $\lambda_0 > 0$ such that if $\forall j$, $c_j(t) \geq \sigma_0$, $x_j(t) \geq x_{j-1}(t) + L_4$ then, $\forall t \in [0, t_0]$,
\[
\int \varepsilon^2 x(t) - pR^{p-1}(t)\varepsilon^2(t) + c(t, x)\varepsilon^2(t) \geq \lambda_0 |\varepsilon(t)|^2_{H^1}, \quad (26)
\]
where $c(t, x) = c_1(t) + \sum_{j=2}^{N}(c_j(t) - c_{j-1}(t))\psi(x - m_j(t))$.

**Proof of Lemma 4.** It is well known that there exists $\lambda_1 > 0$ such that if $v \in H^1(\mathbb{R})$ satisfies $\int Qv = \int Qxv = 0$, then
\[
\int v^2 - pQ^{p-1}v^2 + v^2 \geq \lambda_1 |v|^2_{H^1}. \quad (27)
\]
(See proof of Proposition 2.9 in Weinstein [23].) Now we give a local version of (27). Let $\Phi \in C^2(\mathbb{R})$, $\Phi(x) = \Phi(-x)$, $\Phi' \leq 0$ on $\mathbb{R}^+$, with

\[
\Phi(0) = 1 \text{ on } [0, 1]; \quad \Phi(x) = e^{-x} \text{ on } [2, +\infty), \quad e^{-x} \leq \Phi(x) \leq 3e^{-x} \text{ on } \mathbb{R}^+.
\]
Let $\Phi_B(x) = \Phi \left( \frac{x}{B} \right)$. The following claim is similar to a part of the proof of some local Virial relation in §2.2 of [16]; see Appendix A, Steps 1 and 2, in [16] for its proof.
Thus the proof of Lemma 4 is complete.

We finish the proof of Lemma 4. Let $B > B_0$ to be chosen later and $L_4 = 4kB$, where $k > 1$ integer is to be chosen later. We have

$$\int \Phi_B \left( v_x^2 - pQ^{p-1}v^2 + v^2 \right) \geq \frac{\lambda_1}{4} \int \Phi_B(v_x^2 + v^2). \quad (28)$$

Next, we make the following observations:

(i) By (28), we have $\forall j$,

$$\int \Phi_B(x - x_j(t)) \left( \varepsilon_x^2 - pR_j^{p-1} \varepsilon^2 + c_j(t) \varepsilon^2 \right) \geq \frac{\lambda_1}{4} \int \Phi_B(x - x_j(t))(\varepsilon_x^2 + c_j(t) \varepsilon^2).$$

(ii) Since $\Phi_B(x) = 1$ for $|x| < B$, by the decay properties of $Q$, we have

$$0 \leq R^{p-1} - \sum_{j=1}^N R_j^{p-1} \leq |R|^{p-1}_{L^\infty(|x-x_j(t)| > B)} + C \sum_{j \neq k} R_j R_k \leq C e^{-\sqrt{\sigma_0}B}.$$  

(iii) Note that $c(t, x) = \sum_{j=1}^N c_j(t) \varphi_j(t, x)$, where $\varphi_1(t, x) = 1 - \psi(x - m_2(t))$, for $j \in \{2, \ldots, N-1\}$, $\varphi_j(t, x) = \varphi(x - m_j(t)) - \psi(x - m_{j+1}(t))$ and $\varphi_N(t, x) = \psi(x - m_N(t))$. Since $\Phi_B(x) \leq 3e^{-\frac{|x|}{kB}}$, by the properties of $\psi$, and $|m_j(t) - x_j(t)| \geq L_4/2 \geq kB$, we obtain

$$|\Phi_B(x - x_j(t))(c(t, x) - c_j(t))| \leq |c(t, x) - c(t)|_{L^\infty(|x-x_j(t)| \leq kB)} + C e^{-k} \leq C e^{-\sqrt{\sigma_0}kB/2} + C e^{-k}.$$

(iv) $1 - \sum_{j=1}^N \Phi_B(x - x_j(t)) \geq 0$.

Therefore, with $\lambda_0 = \frac{1}{2} \min \left( \frac{\lambda_1}{4}, \frac{\lambda_1}{4} \sigma_0, 1, \sigma_0 \right)$, for $B$ and $k$ large enough,

$$\int \varepsilon_x^2 - pR^{p-1} \varepsilon^2 + c(t, x) \varepsilon^2 \geq 2\lambda_0 \left( \varepsilon_x^2 + \varepsilon^2 \right) \geq \lambda_0 \left( \varepsilon_x^2 + \varepsilon^2 \right).$$

Thus the proof of Lemma 4 is complete.
3 Proof of the stability in the energy space

This section is devoted to the proof of stability result. The proof is by a priori estimate.

Let $0 < c_1^0 < \ldots < c_N^0$, $\sigma_0 = \frac{1}{2} \min(c_1^0, c_2^0 - c_1^0, c_3^0 - c_2^0, \ldots, c_N^0 - c_{N-1}^0)$ and $\gamma_0 = \sqrt{\sigma_0}/16$. For $A_0, L, \alpha > 0$, we define

$$V_{A_0}(L, \alpha) = \{ u \in H^1(\mathbb{R}); \inf_{x_j - x_{j-1} \geq L} \left| u - \sum_{j=1}^{N} Q_{c_j}(-x_j) \right|_{H^1} \leq A_0 \left( \alpha + e^{-\gamma_0 L/2} \right) \}. \quad (29)$$

We want to prove that there exists $A_0 > 0$, $L_0 > 0$, and $\alpha_0 > 0$ such that, $\forall u_0 \in H^1(\mathbb{R})$, if for some $L > L_0$, $0 < \alpha < \alpha_0$, $\left| u_0 - \sum_{j=1}^{N} Q_{c_j}(-x_j) \right|_{H^1} \leq \alpha$, where $x_j^0 > x_{j-1}^0 + L$, then $\forall t \geq 0$, $u(t) \in V_{A_0}(L, \alpha)$ (this proves the stability result in $H^1$). By a standard continuity argument, it is a direct consequence of the following proposition.

**Proposition 1 (A priori estimate)** There exists $A_0 > 0$, $L_0 > 0$, and $\alpha_0 > 0$ such that, for all $u_0 \in H^1(\mathbb{R})$, if

$$\left| u_0 - \sum_{j=1}^{N} Q_{c_j}(-x_j^0) \right|_{H^1} \leq \alpha, \quad (30)$$

where $L > L_0$, $0 < \alpha < \alpha_0$, $x_j^0 > x_{j-1}^0 + L$, and if for $t^* > 0,$

$$\forall t \in [0, t^*], \quad u(t) \in V_{A_0}(L, \alpha), \quad (31)$$

then

$$\forall t \in [0, t^*], \quad u(t) \in V_{A_0/2}(L, \alpha). \quad (32)$$

Note that $A_0$, $L_0$ and $\alpha > 0$ are independent of $t^*$.

**Proof of Proposition 1.** Let $A_0 > 0$ to be fixed later. First, for $0 < \alpha_0 < \alpha_1(A_0)$ and $L_0 > L_1(A_0) > L_1$, we have

$$A_0 \left( \alpha_0 + e^{-\gamma_0 L_0/2} \right) \leq \alpha_1, \quad (33)$$

where $\alpha_1$ and $L_1$ are defined in Lemma 11. Therefore, by (33) and Lemma 11, there exist $c_j : [0, t^*) \rightarrow (0, +\infty)$, $x_j : [0, t^*) \rightarrow \mathbb{R}$, such that

$$\varepsilon(t, x) = u(t, x) - \sum_{j=1}^{N} R_j(t, x), \quad \text{where} \quad R_j(t, x) = Q_{c_j(t)}(x - x_j(t)), \quad (34)$$

satisfies $\forall j, \forall t \in [0, t^*]$,

$$\int R_j(t) \varepsilon(t) = \int (R_j(t)) x \varepsilon(t) = 0, \quad (35)$$

$$|c_j(t) - c_j^0| + |c_j| + |x_j - c_j^0| + |\varepsilon(t)|_{H^1} \leq K_1(A_0 + 1) \left( \alpha_0 + e^{-\gamma_0 L_0} \right). \quad (36)$$

Note that by (33), Lemma 8 (see Appendix) and assumptions of the proposition,

$$|\varepsilon(0)|_{H^1} + \sum_{j=1}^{N} |c_j(0) - c_j^0| \leq K_1 \alpha, \quad x_j(0) - x_{j-1}(0) \geq \frac{L}{2}. \quad (37)$$
From (36) and (37), for \( \alpha_0 < \alpha_{II}(A_0) \) and \( L_0 > L_{II}(A_0) > 2 \max(L_2, L_3, L_4) \) \((L_2, L_3\text{ and } L_4 \text{ are defined in Lemmas } 3 \text{ and } 4)\), we have \forall t \in [0, t^*],

\[
\begin{align*}
c_1(t) & \geq \sigma_0, \quad \dot{x}_1(t) \geq \sigma_0, \quad c_j(t) - c_{j-1}(t) \geq \sigma_0, \quad \dot{x}_j(t) - \dot{x}_{j-1}(t) \geq \sigma_0, \quad (38) \\
\sum_{j=1}^{N} |c_j(t) - c_j(0)| & \leq K_4 \left( |\varepsilon(t)|_{H^1}^2 + |\varepsilon(0)|_{H^1}^2 + e^{-\gamma_0 L} \right). \quad (40)
\end{align*}
\]

**Proof.**

**Step 1. Energetic control.** Let \( \beta = \frac{2}{p-1} \). There exists \( C > 0 \) such that

\[
\begin{align*}
\left| \sum_{j=1}^{N} c_j(0) \left[ c_j^{\beta-1/2}(t) - c_j^{\beta-1/2}(0) \right] \right| & \leq C \left( |\varepsilon(t)|_{H^1}^2 + |\varepsilon(0)|_{H^1}^2 + e^{-\gamma_0 L} \right) \\
& \quad + C \sum_{j=1}^{N} |c_j(t) - c_j(0)|^2. \quad (41)
\end{align*}
\]

Let us prove (41). By (15), we have

\[
\begin{align*}
\left| \sum_{j=1}^{N} \left[E(R_j(t)) - E(R_j(0)) \right] \right| & \leq C \left( |\varepsilon(t)|_{H^1}^2 + |\varepsilon(0)|_{H^1}^2 + e^{-\gamma_0 L} \right). \quad (42)
\end{align*}
\]

Since \( E(Q_c) = -\frac{\kappa}{2} e^{\beta+1/2} \int Q^2 \), where \( \kappa = \frac{5-p}{p+3} \), we have

\[
\begin{align*}
- \sum_{j=1}^{N} \left[E(R_j(t)) - E(R_j(0)) \right] = \frac{\kappa}{2} \left( \int Q^2 \right) \sum_{j=1}^{N} \left[ c_j^{\beta+1/2}(t) - c_j^{\beta+1/2}(0) \right].
\end{align*}
\]

By linearization, we have \( c_j^{\beta+1/2}(t) - c_j^{\beta+1/2}(0) = \frac{2\beta+1}{2\beta-1} c_j(0) \left[ c_j^{\beta-1/2}(t) - c_j^{\beta-1/2}(0) \right] + O\left( |c_j(t) - c_j(0)|^2 \right) \). Note that \( \frac{2\beta+1}{2\beta-1} = \frac{1}{\beta} \). Therefore,

\[
\begin{align*}
\left| \sum_{j=1}^{N} \left[E(R_j(t)) - E(R_j(0)) \right] + \frac{1}{2} \left( \int Q^2 \right) \sum_{j=1}^{N} c_j(0) \left[ c_j^{\beta-1/2}(t) - c_j^{\beta-1/2}(0) \right] \right| & \leq C \sum_{j=1}^{N} |c_j(t) - c_j(0)|^2, \quad (43)
\end{align*}
\]

Therefore, we can apply Lemmas 2, 3 and 4 for all \( t \in [0, t^*] \).

Let \( \alpha_0 = \min(\alpha_{II}(A_0), \alpha_{II}(A_0)) \) and \( L_0 = \max(L_{II}(A_0), L_{II}(A_0)) \). Now, our objective is to give a uniform upper bound on \( |\varepsilon(t)|_{H^1} \) and \( |c_j(t) - c_j(0)| \) on \( [0, t^*] \) improving (36) for \( A_0 \) large enough.

In the next lemma, we first obtain a control of the variation of \( c_j(t) \) which is quadratic in \( |\varepsilon(t)|_{H^1} \). This is the key step of the stability result, based on monotonicity property of the local \( L^2 \) norm and energy constraints. It is essential at this point to have chosen by the modulation \( \int R_j \varepsilon = 0 \).

**Lemma 5 (Quadratic control of the variation of \( c_j(t) \))** There exists \( K_4 > 0 \) independent of \( A_0 \), such that, \( \forall t \in [0, t^*], \)

\[
\sum_{j=1}^{N} |c_j(t) - c_j(0)| \leq K_4 \left( |\varepsilon(t)|_{H^1}^2 + |\varepsilon(0)|_{H^1}^2 + e^{-\gamma_0 L} \right). \quad (40)
\]
Therefore, by step 1, we obtain \((\int Q^2) |d_j(t) - d_j(0)| \leq - (\int Q^2) (d_j(t) - d_j(0)) + C \left( \int \varepsilon^2(0) + e^{-\gamma_0 L} \right). \tag{44}
\
Let us prove \((44)\). Recall that using the notation of section \(\S 2.3\), we have
\[
\mathcal{I}_j(t) \leq \mathcal{I}_j(0) + K_j e^{-\gamma_0 L}, \quad \text{where} \quad \mathcal{I}_j(t) = \int \psi(x - m_j(t)) u^2(t, x) dx.
\]
Since \(\int R_j^2(t) = c_j^{-1/2} \int Q^2\), \(\int R_j(t) \varepsilon(t) = 0\), by similar calculations as in Lemma \(2\), we have
\[
\left| \mathcal{I}_j(t) - \left( \int Q^2 \right) d_j(t) - \int \psi(-m_j(t)) \varepsilon^2(t) \right| \leq C e^{-\gamma_0 L}. \tag{45}
\]
Therefore,
\[
\left( \int Q^2 \right) (d_j(t) - d_j(0)) \leq \int \psi(-m_j(0)) \varepsilon^2(0) - \int \psi(-m_j(t)) \varepsilon^2(t) + C e^{-\gamma_0 L}. \tag{46}
\]
Since the second term on the right hand side is negative, \((44)\) follows easily. Note that by conservation of the \(L^2\) norm \(\int u^2(t) = \int u^2(0)\) and
\[
\int u^2(t) = \int R^2(t) + \int \varepsilon^2(t) + 2 \int R(t) \varepsilon(t) = \int R^2(t) + \int \varepsilon^2(t) = d_1(t) + \int \varepsilon^2(t) + O(e^{-\gamma_0 L}),
\]
we obtain
\[
\left( \int Q^2 \right) (d_1(t) - d_1(0)) \leq \int \varepsilon^2(0) - \int \varepsilon^2(t) + C e^{-\gamma_0 L}. \tag{47}
\]

**Step 3. Resummation argument.** By Abel transform, we have
\[
\sum_{j=1}^{N} c_j(0) \left[ c_j^{-1/2}(t) - c_j^{-1/2}(0) \right] = \sum_{j=1}^{N-1} c_j(0) \left[ d_j(t) - d_{j+1}(t) - (d_j(0) - d_{j+1}(0)) \right] + c_N(0) \left[ d_N(t) - d_N(0) \right]
\]
\[
= c_1(0) \left[ d_1(t) - d_1(0) \right] + \sum_{j=2}^{N} (c_j(0) - c_{j-1}(0))(d_j(t) - d_j(0)). \tag{48}
\]
Therefore, by step 1,
\[
- \left( c_1(0) \left[ d_1(t) - d_1(0) \right] + \sum_{j=2}^{N} (c_j(0) - c_{j-1}(0))(d_j(t) - d_j(0)) \right)
\]
\[
\leq C \left( \| \varepsilon(t) \|_{H^1}^2 + \| \varepsilon(0) \|_{H^1}^2 + e^{-\gamma_0 L} \right) + C \sum_{j=1}^{N} [c_j(t) - c_j(0)]^2. \tag{49}
\]
Since \( c_1(0) \geq \sigma_0 \), \( c_j(0) - c_{j-1}(0) \geq \sigma_0 \), by (44), we have
\[
\sigma_0 \sum_{j=1}^{N} |d_j(t) - d_j(0)| \leq c_1(0)|d_1(t) - d_1(0)| + \sum_{j=2}^{N} (c_j(0) - c_{j-1}(0))|d_j(t) - d_j(0)|
\]
\[
\leq - \left[ c_1(0) [d_1(t) - d_1(0)] + \sum_{j=2}^{N} (c_j(0) - c_{j-1}(0))(d_j(t) - d_j(0)) \right] + C \int \varepsilon^2(0) + Ce^{-\gamma_0 L}.
\]
Thus, by (49), we have
\[
\sum_{j=1}^{N} |d_j(t) - d_j(0)| \leq C \left( |\varepsilon(t)|_{H^1}^2 + |\varepsilon(0)|_{H^1}^2 + e^{-\gamma_0 L} \right) + C \sum_{j=1}^{N} |c_j(t) - c_j(0)|^2.
\]
Since
\[
|c_j(t) - c_j(0)| \leq C|c_j^{\beta-1/2}(t) - c_j^{\beta-1/2}(0)|
\]
\[
\leq C(|d_j(t) - d_j(0)| + |d_{j+1}(t) - d_{j+1}(0)|),
\]
we obtain,
\[
\sum_{j=1}^{N} |c_j(t) - c_j(0)| \leq C \left( |\varepsilon(t)|_{H^1}^2 + |\varepsilon(0)|_{H^1}^2 + e^{-\gamma_0 L} \right) + C \sum_{j=1}^{N} |c_j(t) - c_j(0)|^2.
\]
Choosing a smaller \( \alpha_0(A_0) \) and a larger \( L_0(A_0) \), by (36), we assume \( C|c_j(t) - c_j(0)| \leq 1/2 \) and so
\[
\sum_{j=1}^{N} |c_j(t) - c_j(0)| \leq C \left( |\varepsilon(t)|_{H^1}^2 + |\varepsilon(0)|_{H^1}^2 + e^{-\gamma_0 L} \right).
\]
(50)
Thus, Lemma 5 is proved.

Now, we prove the following lemma, giving uniform control on \( |\varepsilon(t)|_{H^1} \) on \([0, t^*] \).

Lemma 6 (Control of \( |\varepsilon(t)|_{H^1} \)) There exists \( K_5 > 0 \) independent of \( A_0 \), such that, \( \forall t \in [0, t^*] \),
\[
|\varepsilon(t)|_{H^1}^2 \leq K_5 \left( |\varepsilon(0)|_{H^1}^2 + e^{-\gamma_0 L} \right).
\]

Proof. It follows from direct calculation on the energy, and the previous estimates obtained by Abel transform, freezing the \( c_j(t) \) at the first order.

By (15), (43), (44) and (50), we have
\[
\frac{1}{2} \int |\varepsilon^2(t) - pR^q(t)\varepsilon^2(t)| \leq - \sum_{j=1}^{N} [E(R_j(t)) - E(R_j(0))] + K_2 \left( |\varepsilon(0)|_{H^1}^2 + |\varepsilon(t)|_{H^1}^2 + e^{-\gamma_0 L} \right)
\]

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Therefore, from (51), we obtain
\[
\int \varepsilon_x^2(t) - pR^{p-1}(t)\varepsilon^2(t) \leq - (c_1(0) \int \varepsilon^2(t) + \sum_{j=2}^{N} (c_j(0) - c_{j-1}(0)) \int \psi(x - m_j(t))\varepsilon^2(t)) + C \left( |\varepsilon(0)|^2_{H^1} + |\varepsilon(t)|^2_{H^1} + e^{-\gamma_0 L} \right)
\]
where \(c(t, x) = c_1(t) + \sum_{j=2}^{N} (c_j(t) - c_{j-1}(t)) \psi(x - m_j(t))\).
By Lemma 3,
\[
\int \varepsilon_x^2(t) - pR^{p-1}(t)\varepsilon^2(t) + c(t, x)\varepsilon^2(t) \geq \lambda_0 |\varepsilon(t)|^2_{H^1}.
\]
Therefore, from (51), we obtain
\[
|\varepsilon(t)|^2_{H^1} \leq C \left( |\varepsilon(0)|^2_{H^1} + |\varepsilon(t)|^3_{H^1} + e^{-\gamma_0 L} \right),
\]
and so
\[
|\varepsilon(t)|^2_{H^1} \leq K_5 \left( |\varepsilon(0)|^2_{H^1} + e^{-\gamma_0 L} \right),
\]
for some constant \(K_5 > 0\), independent of \(A_0\). Thus Lemma 3 is proved.

We conclude the proof of proposition 1 and of the stability result. By (17) and Lemmas 3 and 8, we have
\[
\left| u(t) - \sum_{j=1}^{N} Q_{\psi}(x - x_j(t)) \right|_{H^1}
\]
\[
\leq \left| u(t) - \sum_{j=1}^{N} R_j(t) \right|_{H^1} + \left| \sum_{j=1}^{N} R_j(t) - \sum_{j=1}^{N} Q_{\psi}(x - x_j(t)) \right|_{H^1}
\]
\[
\leq |\varepsilon(t)|_{H^1} + C \sum_{j=1}^{N} |c_j(t) - c_j^0|
\]
\[
\leq |\varepsilon(t)|_{H^1} + C \sum_{j=1}^{N} |c_j(t) - c_j(0)| + C \sum_{j=1}^{N} |c_j(0) - c_j^0|
\]
\[
\leq |\varepsilon(t)|_{H^1} + C K_4 (|\varepsilon(0)|^2_{H^1} + e^{-\gamma_0 L}) + C K_1 \alpha
\]
\[
\leq K_6 \left( \alpha + e^{-\gamma_0 L/2} \right),
\]

where $K_6 > 0$ is a constant independent of $A_0$.

Choosing $A_0 = 4K_6$, we complete the proof of Proposition 1 and thus the proof of Theorem 1 (i).

## 4 Proof of the asymptotic stability result

This section is devoted to the proof of the asymptotic stability result (Theorem 1 (ii)).

### 4.1 Asymptotic stability around the solitons

In this subsection, we prove the following asymptotic result on $\varepsilon(t)$ as $t \to +\infty$.

**Proposition 2 (Convergence around solitons, $p = 2, 3, 4$)** Under the assumptions of Theorem 1, the following is true:

(i) **Convergence of $\varepsilon(t)$**: $\forall j \in \{1, \ldots, N\}$,

\[
\varepsilon(t, + x_j(t)) \to 0 \quad \text{in} \ H^1(\mathbb{R}) \quad \text{as} \quad t \to +\infty. \tag{52}
\]

(ii) **Convergence of geometric parameters**: there exists $0 < c_1^{+\infty} \leq \cdots \leq c_N^{+\infty}$, such that $c_j(t) \to c_j^{+\infty}$, $\dot{x}_j(t) \to c_j^{+\infty}$ as $t \to +\infty$.

The proof of this result is very similar to the proof of the asymptotic stability of a single soliton in Martel and Merle [13] for the subcritical case (see also the previous paper [13] concerning the critical case $p = 5$). The proof is based on the following rigidity result of solutions of (1) around solitons.

**Theorem (Liouville property close to $R_{c_0}$ for $p = 2, 3, 4$)** Let $p = 2, 3$ or 4, and let $c_0 > 0$. Let $u_0 \in H^1(\mathbb{R})$, and let $u(t)$ be the solution of (1) for all time $t \in \mathbb{R}$. There exists $\alpha_0 > 0$ such that $|u_0 - R_{c_0}|_{H^1} < \alpha_0$, and if there exists $y(t)$ such that

\[
\forall \delta_0 > 0, \exists A_0 > 0 / \forall t \in \mathbb{R}, \quad \int_{|x| > A_0} u^2(t, x + y(t))dx \leq \delta_0, \quad (L^2 \text{ compactness}) \tag{53}
\]

then there exists $c^* > 0$, $x^* \in \mathbb{R}$ such that

\[
\forall t \in \mathbb{R}, \forall x \in \mathbb{R}, \quad u(t, x) = Q_{c^*}(x - x^* - c^*t). \tag{54}
\]

**Proof of Proposition 2 (i).** Consider a solution $u(t)$ satisfying the assumptions of Theorem 1. Then, by §3, we known that $u(t)$ is uniformly close in $H^1(\mathbb{R})$ to the superposition of $N$ solitons for all time $t \geq 0$. With the decomposition introduced in section §2, it is equivalent that $\varepsilon(t)$ is uniformly small in $H^1(\mathbb{R})$ and $\sum_{j=1}^N |c_j(t) - c_j(0)|$ is uniformly small. Therefore, we can assume that, $\forall t \geq 0$,

\[
c_1(t) \geq \sigma_0, \quad c_j(t) - c_{j-1}(t) \geq \sigma_0.
\]

The proof of Proposition 3 is by contradiction. Let $j \in \{1, \ldots, N\}$. Assume that for some sequence $t_n \to +\infty$, we have

\[
\varepsilon(t_n, + x_j(t_n)) \not\to 0 \quad \text{in} \quad H^1(\mathbb{R}) \quad \text{as} \quad t \to +\infty.
\]
Since $0 < \sigma_0 < c_j(t) < \overline{c}$ and $|\varepsilon(t)|_{H^1} \leq C$ for all $t \geq 0$, there exists $\bar{\varepsilon}_0 \in H^1(\mathbb{R})$, $\bar{\varepsilon}_0 \neq 0$, and $\bar{c}_0 > 0$ such that for a subsequence of $(t_n)$, still denoted $(t_n)$, we have

$$\varepsilon(t_n, + x_j(t_n)) \to \bar{\varepsilon}_0 \quad \text{in} \quad H^1(\mathbb{R}), \quad c_j(t_n) \to \bar{c}_0 \quad \text{as} \quad n \to +\infty. \quad (54)$$

Moreover, by weak convergence and stability result, $|\bar{\varepsilon}_0|_{H^1} \leq \sup_{t \geq 0} |\varepsilon(t)|_{H^1} \leq C(\alpha_0 + e^{-\gamma L_0})$, and therefore $|\bar{\varepsilon}_0|_{H^1}$ is as small as we want by taking $\alpha_0$ small and $L_0$ large.

Let now $\tilde{\varepsilon}(0) = Q_{c_0} + \bar{\varepsilon}_0$, and let $\tilde{\varepsilon}(t)$ be the global solution of (1) for $t \in \mathbb{R}$, with $\tilde{\varepsilon}(0)$ as initial data. Let $\tilde{x}(t)$ and $\tilde{c}(t)$ be the geometrical parameters associated to the solution $\tilde{\varepsilon}(t)$ (apply the modulation theory for a solution close to a single soliton).

We claim that the solution $\tilde{\varepsilon}(t)$ is $L^2$ compact in the sense of (53).

**Lemma 7** (L^2 compactness of the asymptotic solution)

$$\forall \delta_0 > 0, \exists A_0 > 0/ \forall t \in \mathbb{R}, \quad \int_{|x| > A_0} \tilde{\varepsilon}^2(t, x + \tilde{x}(t))dx \leq \delta_0. \quad (55)$$

Assuming this lemma, we finish the proof of Proposition 2 (i). Indeed, by choosing $\alpha_0$ small enough and $L_0$ large enough, we can apply the Liouville theorem to $\tilde{\varepsilon}(t)$. Therefore, there exists $c^* > 0$ and $x^* \in \mathbb{R}$ such that $\tilde{\varepsilon}(t) = Q_{c^*}(x - x^*) - c^*t$. In particular, $\tilde{\varepsilon}(0) = Q_{c_0} + \bar{\varepsilon}_0 = Q_{c^*}(x - x^*)$. Since by weak convergence $\int \bar{\varepsilon}_0 Q_{c_0} = 0$, we have easily $x^* = 0$. Next, since $\int \bar{\varepsilon}_0 Q = 0$, we have $c^* = \bar{c}_0$ and so $\bar{\varepsilon}_0 = 0$. This is a contradiction.

Thus Proposition 2 (i) is proved assuming Lemma 7. The proof of Lemma 7 is based only on arguments of monotonicity of the $L^2$ mass in the spirit of [15], [16].

**Proof of Lemma 7.** We use the function $\psi$ introduced in §2.2. For $y_0 > 0$, we introduce two quantities:

$$J_L(t) = \int (1 - \psi(x - (x_j(t) - y_0)))u^2(t, x)dx, \quad J_R(t) = \int \psi(x - (x_j(t) + y_0))u^2(t, x)dx. \quad (56)$$

The strategy of the proof is the following. We prove first that $J_L(t)$ is almost increasing and $J_R(t)$ is almost decreasing in time. Then, assuming by contradiction that $\tilde{u}(t)$ is not $L^2$ compact, using the convergence of $u(t)$ to $\tilde{u}(t)$ for all time, we prove that the $L^2$ norm of $u(t)$ in the compact set $[-y_0, y_0]$, for $y_0$ large enough, oscillates between two different values. This proves that there are infinitely many transfers of mass from the right hand side of the soliton $j$ to the left hand side of the soliton $j$. This is of course impossible since the $L^2$ norm of $u(t)$ is finite.

**Step 1.** Monotonicity on the right and on the left of a soliton. We claim

**Claim.** There exists $C_1, y_1 > 0$ such that $\forall y_0 > y_1, \forall t' \in [0, t]$,

$$J_L(t) \geq J_L(t') - C_1e^{-\gamma y_0}, \quad J_R(t) \leq J_R(t') + C_1e^{-\gamma y_0}. \quad (57)$$

We prove this claim. First note that it is sufficient to prove (57) for $J_L(t)$. Indeed, since $u(-t, -x)$ is also solution of (1), and since $1 - \psi(-x) = \psi(x)$, we can argue backwards

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in time (from \(t\) to \(t'\)) to obtain the result for \(J_R(t)\). By using the same argument as in Lemma \([3]\) we prove easily, for \(y_0\) large enough, for all \(0 < t' < t\),
\[
\int \psi(. - (x_j(t) - y_0 - \frac{4}{7}(t - t'))u^2(t) \leq \int \psi(. - (x_j(t') - y_0))u^2(t') + C_1e^{-\gamma_0y_0}
\]
\[
\leq \int u^2(t') - J_L(t') + C_1e^{-\gamma_0y_0}.
\]
Since \(\int u^2(t) = \int u^2(t')\) and
\[
\int u^2(t) - J_L(t) = \int \psi(. - (x_j(t) - y_0))u^2(t) \leq \int \psi(. - (x_j(t) - y_0 - \frac{4}{7}(t - t'))u^2(t),
\]
we obtain the result.

**Step 2.** Conclusion of the proof. Recall from \([15]\) that we have stability of \((1)\) by weak convergence in \(H^1(\mathbb{R})\) in the following sense
\[
\forall t \in \mathbb{R}, \quad u(t + t_n, + x_j(t + t_n)) \rightarrow \bar{u}(t, + \bar{x}(t)) \quad \text{in } L^2_{\text{loc}}(\mathbb{R}) \quad \text{as } n \rightarrow +\infty. \tag{58}
\]
This was proved in \([15]\) by using the fact that the Cauchy problem for \((1)\) is well posed both in \(H^1(\mathbb{R})\) and in \(H^{s*}(\mathbb{R})\), for some \(0 < s^* < 1\), for any \(p = 2, 3, 4\) (see \([8]\)).

We prove Lemma \([7]\) by contradiction. Let
\[
m_0 = \int \bar{u}^2(0) = \int \bar{u}^2(t).
\]
Assume that there exists \(\delta_0 > 0\) such that for any \(y_0 > 0\), there exists \(t_0(y_0) \in \mathbb{R}\), such that
\[
\int_{|x|<2y_0} \bar{u}^2(t_0(y_0), x + \bar{x}(t_0(y_0)))dx \leq m_0 - \delta_0. \tag{59}
\]
Fix \(y_0 > 0\) large enough so that
\[
\int (\psi(x + y_0) - \psi(x - y_0))u^2(0, x)dx \geq m_0 - \frac{1}{10}\delta_0, \tag{60}
\]
\[
C_1e^{-\gamma_0y_0} + m_0 \sup_{|x| > 2y_0} \{\psi(x + y_0) - \psi(x - y_0)\} \leq \frac{1}{10}\delta_0.
\]
Assume that \(t_0 = t_0(y_0) > 0\) and, by possibly considering a subsequence of \((t_n)\), that \(\forall n, t_{n+1} \geq t_n + t_0\).

Observe that, since \(0 < \psi < 1\) and \(\psi' > 0\), by the choice of \(y_0\) and \([3]\), we have
\[
\int (\psi(x - (\bar{x}(t_0) - y_0)) - \psi(x - (\bar{x}(t_0) + y_0)))\bar{u}^2(t_0, x)dx
\]
\[
\leq \int_{|x|<2y_0} \bar{u}^2(t_0, x + \bar{x}(t_0))dx + m_0 \sup_{|x| > 2y_0} \{\psi(x + y_0) - \psi(x - y_0)\}
\]
\[
\leq \int_{|x|<2y_0} \bar{u}^2(t_0, x + \bar{x}(t_0))dx + \frac{1}{10}\delta_0 \leq m_0 - \frac{9}{10}\delta_0. \tag{61}
\]
Then, by \([60]\), \([71]\) and \([58]\), there exists \(N_0 > 0\) large enough so that \(\forall n \geq N_0,
\[
\int (\psi(x - (x_j(t_n) - y_0)) - \psi(x - (x_j(t_n) + y_0)))u^2(t_n, x)dx \geq m_0 - \frac{1}{5}\delta_0. \tag{62}
\]

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\[
\int (\psi(x - (x_j(t_n + t_0) - y_0)) - \psi(x - (x_j(t_n + t_0) + y_0)))u^2(t_n + t_0, x)dx \leq m_0 - \frac{4}{5}\delta_0. \quad (63)
\]

Recall that from Step 1, and the choice of \(y_0\), we have \(J_R(t_n + t_0) \leq J_R(t_n) + \frac{1}{10}\delta_0\).

Therefore, by conservation of the \(L^2\) norm and (53), (52), we have
\[
J_L(t_n + t_0) \geq J_L(t_n) + \frac{1}{2}\delta_0.
\]

Since \(J_L(t_{n+1}) \geq J_L(t_n + t_0) - \frac{1}{10}\delta_0\) by Step 1, we finally obtain
\[
\forall n \geq N_0, \quad J_L(t_{n+1}) \geq J_L(t_n) + \frac{2}{5}\delta_0.
\]

Of course, this is a contradiction. Thus the proof of Lemma 7 is complete.

**Proof of Proposition 2 (ii).** The proof is similar to the proof of Proposition 3 in [13]. It follows again from monotonicity arguments and the fact that we consider the subcritical case \(1 < p < 5\).

Let \(\delta > 0\) be arbitrary. Since \(\int R_2^2(t) = c_j^{\frac{5-p}{2(p-1)}}(t)\int Q^2\) and \(\varepsilon(t, ., x_j(t)) \to 0\) in \(L^2_{\text{loc}}\) as \(t \to +\infty\), there exists \(T_1(\delta) > 0\) and \(y_1(\delta)\) such that \(\forall t > T_1(\delta), \forall y_0 > y_1(\delta),\)
\[
\left| \int (\psi(x - (x_j(t) - y_0)) - \psi(x - (x_j(t) + y_0)))u^2(t, x)dx - c_j^{\frac{5-p}{2(p-1)}}(t)\int Q^2 \right| \leq \delta.
\]

By Step 1 of the proof of Lemma 7, there exists \(y_2(\delta)\), such that we have, for all \(0 < t' < t, \forall y_0 > y_2(\delta),\)
\[
J_L(t) \geq J_L(t') - \delta, \quad J_R(t) \leq J_R(t') + \delta.
\]

Fix \(y_0 = \max(y_1(\delta), y_2(\delta))\), it follows that there exists \(T_2(\delta), J_L^{+\infty} \geq 0\) and \(J_R^{+\infty} \geq 0\) such that
\[
\forall t \geq T_2(\delta), \quad |J_L(t) - J_L^{+\infty}| \leq 2\delta, \quad |J_R(t) - J_R^{+\infty}| \leq 2\delta.
\]

Therefore, by conservation of \(L^2\) mass, we have, for all \(0 < \min(T_1, T_2) < t' < t,\)
\[
\left| c_j^{\frac{5-p}{2(p-1)}}(t) - c_j^{\frac{5-p}{2(p-1)}}(t') \right| \leq C\delta.
\]

Since \(\delta\) is arbitrary, it follows that \(c_j^{\frac{5-p}{2(p-1)}}(t)\) has a limit as \(t \to +\infty\). Thus there exists \(c_j^{+\infty} > 0\) such that \(c_j(t) \to c_j^{+\infty}\) as \(t \to +\infty\). The fact that \(\dot{x}_j(t) \to c_j^{+\infty}\) is a direct consequence of (13).

### 4.2 Asymptotic behavior on \(x > ct\)

In this subsection, using the same argument of monotonicity of \(L^2\) mass, we prove the following proposition.

**Proposition 3 (Convergence for \(x > c_j^0 t/10\)) Under the assumptions of Theorem 4, the following is true**
\[
|\varepsilon(t)|_{L^2(x > c_j^0 t/10)} \to 0 \quad \text{as} \ t \to +\infty. \quad (64)
\]
Proof. By arguing backwards in time (from $t$ to 0) and using the conservation of $L^2$ norm, we have

$$\int \psi(\cdot, (x_N(t) + y_0))^2(t) \leq \int \psi(\cdot, (x_N(0) + \frac{a_0}{2}t + y_0))^2(0) + C_1 e^{-\gamma y_0}.$$ 

Therefore,

$$\int_{x>x_N(t)+y_0} \varepsilon^2(t) \leq 2 \int \psi(\cdot, (x_N(0) + \frac{a_0}{2}t + y_0))^2(0) + C e^{-\gamma y_0}.$$

Since for fixed $y_0$, $\int_{x_N(t)<x<x_N(t)+y_0} \varepsilon^2(t) \to 0$ as $t \to +\infty$, we conclude $\int_{x>x_N(t)} \varepsilon^2(t) \to 0$ as $t \to +\infty$.

Now, let us prove $\int_{x>x_N(t)} \varepsilon^2(t) \to 0$ as $t \to +\infty$ by backwards induction on $j$. Assume that for $j_0 \in \{2, \ldots, N\}$, we have $\int_{x>x_j(t)} \varepsilon^2(t) \to 0$ as $t \to +\infty$. For $t \geq 0$ large enough, there exists $0 < t' = t'(t) < t$, satisfying

$$x_{j_0}(t') - x_{j_0-1}(t') - \frac{a_0}{2}(t - t') = 2y_0.$$

Indeed, for $t$ large enough, $x_{j_0}(t) - x_{j_0-1}(t) \geq \frac{a_0}{2}t \geq 2y_0$, and $x_{j_0}(0) - x_{j_0-1}(0) - \frac{a_0}{2}t < 0 < 2y_0$. Then,

$$\int \psi(\cdot, (x_{j_0-1}(t) + y_0))^2(t) \leq \int \psi(\cdot, (x_{j_0-1}(t') + \frac{a_0}{2}(t - t') + y_0))^2(t') + C e^{-\gamma y_0}$$

$$\leq \int \psi(\cdot, (x_{j_0}(t') - y_0))^2(t') + C e^{-\gamma y_0}. \quad (65)$$

Let $\delta > 0$ be arbitrary. By $L^2_{loc}$ convergence of $\varepsilon(\cdot, x_{j_0}(t))$ and the induction assumption, we have, for fixed $y_0$,

$$\int_{x>x_{j_0}(t)+2y_0} \varepsilon^2(t) \to 0 \quad \text{as} \quad t \to +\infty.$$

Therefore, by Proposition 4 there exists $T = T(\delta) > 0$, such that $\forall t > T$, $\forall y_0 > y_0(\delta),$

$$|\int \psi(\cdot, (x_{j_0}(t) - y_0))^2(t) - \left( \int Q^2 \sum_{k=j_0}^{N} (c_k^+)^{\frac{2(\gamma - p)}{\gamma (p - 1)}} \right)| \leq \delta. \quad (66)$$

Moreover, since $t'(t) \to +\infty$ as $t \to +\infty$, by possibly taking a larger $T(\delta)$, we also have

$$|\int \psi(\cdot, (x_{j_0}(t') - y_0))^2(t') - \left( \int Q^2 \sum_{k=j_0}^{N} (c_k^+)^{\frac{2(\gamma - p)}{\gamma (p - 1)}} \right)| \leq \delta, \quad (67)$$

and so

$$|\int \psi(\cdot, (x_{j_0}(t) - y_0))^2(t) - \int \psi(\cdot, (x_{j_0}(t') - y_0))^2(t')| \leq 2\delta. \quad (68)$$

Thus, by (68), we have

$$\int \psi(\cdot, (x_{j_0-1}(t) + y_0))^2(t) \leq \int \psi(\cdot, (x_{j_0}(t) - y_0))^2(t) + 2\delta + C e^{-\gamma y_0}. \quad (69)$$
Since $\psi(x) \geq 1/2$ for $x > 0$, by the decay properties of $Q$ and (69), we obtain
\[
\int_{x_{j_0}-1(t) + y_0 < y < x_{j_0} - y_0} \varepsilon^2(t)
\leq 2\left( \int \psi(x, (x_{j_0}-1(t) + y_0)) u^2(t) - \int \psi(x, (x_{j_0} - y_0)) u^2(t) \right) + Ce^{-\gamma_0 y_0}
\leq 4\delta + C'e^{-\gamma_0 y_0}.
\]

Thus, $\int_{x > x_{j_0}-1(t)} \varepsilon^2(t) \to 0$ as $t \to +\infty$.

Finally, we prove $\int_{x > c_{0}t/10} \varepsilon^2(t) \to 0$ as $t \to +\infty$. Indeed, let $0 < t' = t'(t) < t$ such that $x_1(t') - \frac{c_0}{20}(t + t') = y_0$. Then, for $\sup_{t \geq 0} |\varepsilon(t)|_{H^1}$ small enough,
\[
\int \psi(x - \frac{c_0}{10}t) u^2(t) \leq \int \psi(x - \left( \frac{c_0}{10}t' + \frac{c_0}{20}(t - t') \right)) u^2(t') + Ce^{-\gamma_0 y_0}
\leq \int \psi(x - (x_1(t') - y_0)) u^2(t') + Ce^{-\gamma_0 y_0}.
\]

Arguing as before, this is enough to conclude the proof.

**Proof of Corollary 1.** Note first that
\[
\left| U(N) (\cdot; c^0_j, -y_j) - \sum_{j=1}^{N} Q_{c^0_j} (\cdot, -y_j) \right|_{H^1} \to 0 \quad \text{as} \quad \inf(y_{j+1} - y_j) \to +\infty. \quad (70)
\]
For $\gamma_0, A_0, L_0$ and $\alpha_0$ as in the statement of Theorem 1, let $\alpha < \alpha_0, L > L_0$ be such that $A_0 \left( \alpha + e^{-\gamma_0 L} \right) < \delta_1/2$ and
\[
\left| U(N) (\cdot; c^0_j, -y_j) - \sum_{j=1}^{N} Q_{c^0_j} (\cdot, -y_j) \right|_{H^1} \leq \delta_1/2, \quad \text{for} \quad y_{j+1} - y_j > L. \quad (71)
\]
Let $v(t, x) = U(N)(x; c^0_j, - (x^0_j + c^0_j t))$ be an $N$-soliton solution. Let $T > 0$ be such that
\[
\forall t \geq T_1, \quad \left| v(t) - \sum_{j=1}^{N} Q_{c^0_j} (\cdot, - (x^0_j + c^0_j t)) \right|_{H^1} \leq \alpha/2, \quad (72)
\]
and $\forall j, x^0_{j+1} + c^0_{j+1} T \geq x^0_j + c^0_j T + 2L$.

By continuous dependence of the solution of (9) with respect to the initial data (see (8)), there exists $\alpha_1 > 0$ such that if $|u(0) - v(0)|_{H^1} \leq \alpha_1$, then $|u(T) - v(T)|_{H^1} \leq \alpha/2$. Therefore, by (72)
\[
\left| u(T) - \sum_{j=1}^{N} Q_{c^0_j} (\cdot, - (x^0_j + c^0_j T)) \right|_{H^1} \leq \alpha.
\]

Thus, by Theorem 1 (i), there exists $x_j(t)$, for all $t \geq T$ such that
\[
\forall t \geq T, \quad \left| u(t) - \sum_{j=1}^{N} Q_{c^0_j} (\cdot, - x_j(t)) \right|_{H^1} \leq A_0 \left( \alpha + e^{-\gamma_0 L} \right) < \delta_1/2.
\]

Moreover, $x_{j+1}(t) > x_j(t) + L$. Together with (71), this gives the stability result.

Finally, Theorem 1 (ii) and (70) prove the asymptotic stability of the family of $N$-solitons.
Appendix : Modulation of a solution close to the sum of \( N \) solitons

In this appendix, we prove the following lemma.

Let \( 0 < c_1^0 < \ldots < c_N^0, \sigma_0 = \frac{1}{2} \min(c_1^0, c_2^0 - c_1^0, c_3^0 - c_2^0, \ldots, c_N^0 - c_{N-1}^0) \). For \( \alpha, L > 0 \), we consider the neighborhood of size \( \alpha \) of the superposition of \( N \) solitons of speed \( c_j^0 \), located at a distance larger than \( L \)

\[
U(\alpha, L) = \{ u \in H^1(\mathbb{R}) : \inf_{x_j > x_{j-1} + L} \left| u - \sum_{j=1}^{N} Q_{c_j^0}(\cdot - x_j) \right|_{H^1} \leq \alpha \}. \tag{73}
\]

(Note that functions in \( U(\alpha, L) \) have no time dependency.)

**Lemma 8 (Choice of the modulation parameters)** There exists \( \alpha_1 > 0, L_1 > 0 \) and unique \( C^1 \) functions \((c_j, x_j) : U(\alpha_1, L_1) \to (0, +\infty) \times \mathbb{R} \), such that if \( u \in U(\alpha_1, L_1) \), and

\[
\varepsilon(x) = u(x) - \sum_{j=1}^{N} Q_{c_j}(\cdot - x_j), \tag{74}
\]

then

\[
\int Q_{c_j}(x - x_i)\varepsilon(x)dx = \int (Q_{c_j})_x(x - x_i)\varepsilon(x)dx = 0. \tag{75}
\]

Moreover, there exists \( K_1 > 0 \) such that if \( u \in U(\alpha, L) \), with \( 0 < \alpha < \alpha_1, L > L_1 \), then

\[
|\varepsilon|_{H^1} + \sum_{j=1}^{N} |c_j - c_j^0| \leq K_1 \alpha, \quad x_j > x_{j-1} + L - K_1 \alpha. \tag{76}
\]

**Proof.** Let \( u \in U(\alpha, L) \). It is clear that for \( \alpha \) small enough and \( L \) large enough, the infimum

\[
\inf_{x_j \in \mathbb{R}} \left| u - \sum_{j=1}^{N} Q_{c_j}(\cdot - x_j) \right|_{H^1}
\]

is attained for \((x_j)\) satisfying \( x_j > x_{j-1} + L - C\alpha \), for some constant \( C > 0 \) independent of \( L \) and \( \alpha \). By using standard arguments involving the implicit function theorem, there exists \( \alpha_1, L_1 > 0 \) such that there exists unique \( C^1 \) functions \((r_j) : U(\alpha_1, L_1) \to \mathbb{R} \), such that for all \( u \in U(\alpha, L) \), for \( 0 < \alpha < \alpha_1, L > L_1 \), we have

\[
\left| u - \sum_{j=1}^{N} Q_{c_j}(\cdot - r_j(u)) \right|_{H^1} = \inf_{x_j \in \mathbb{R}} \left| u - \sum_{j=1}^{N} Q_{c_j}(\cdot - x_j) \right|_{H^1} \leq \alpha.
\]

Moreover, \( r_j(u) - r_{j-1}(u) > L - C\alpha \).

For some \( c_j, y_j, u \in H^1(\mathbb{R}) \), let

\[
Q_{c_j, y_j}(x) = Q_{c_j}(x - r_j(u) - y_j), \quad \varepsilon(x) = u(x) - \sum_{j=1}^{N} Q_{c_j, y_j}(x).
\]
Define the following functionals:
\[ \rho^{1,j}(c_1, \ldots, c_N, y_1, \ldots, y_N, u) = \int Q_{c_j,y_j}(x) \epsilon(x) dx, \]
\[ \rho^{2,j}(c_1, \ldots, c_N, y_1, \ldots, y_N, u) = \int (Q_{c_j,y_j})_x(x) \epsilon(x) dx, \]
and \(\rho = (\rho^{1,1}, \rho^{2,1}, \ldots, \rho^{1,N}, \rho^{2,N})\). Let \(M_0 = (c^0_1, \ldots, c^0_N, 0, \ldots, 0, \sum_{j=1}^N Q_{c^0_j,0})\). We claim the following.

Claim. (i) \(\forall j\),
\[ \frac{\partial \rho^{1,j}}{\partial c_j}(M_0) = -\frac{5-p}{4(p-1)}(c^0_j)^{\frac{7-p}{2(p-1)}} \int Q^2, \quad \frac{\partial \rho^{1,j}}{\partial y_j}(M_0) = 0, \]
\[ \frac{\partial \rho^{2,j}}{\partial c_j}(M_0) = 0, \quad \frac{\partial \rho^{2,j}}{\partial y_j}(M_0) = (c^0_j)^{\frac{p+3}{2(p-1)}} \int Q^2_x. \]

(ii) \(\forall j \neq k\),
\[ \left| \frac{\partial \rho^{1,j}}{\partial c_k}(M_0) \right| + \left| \frac{\partial \rho^{1,j}}{\partial y_k}(M_0) \right| + \left| \frac{\partial \rho^{2,j}}{\partial c_k}(M_0) \right| + \left| \frac{\partial \rho^{2,j}}{\partial y_k}(M_0) \right| \leq Ce^{-\sqrt{\sigma_0}L/2}. \]

Proof of the claim. Since
\[ \frac{\partial Q_{c_j,y_j}}{\partial c_j}(c^0_j,0) = \frac{1}{2c^0_j} \left( \frac{2}{p-1} Q_{c_j,0} + (x - r_j)(Q_{c_j,0})_x \right), \quad \frac{\partial Q_{c_j,y_j}}{\partial y_j}(c^0_j,0) = -(Q_{c_j,0})_x, \]
we have by direct calculations:
\[ \frac{\partial \rho^{1,j}}{\partial c_j}(M_0) = -\int Q_{c_j,0} \frac{\partial Q_{c_j,y_j}}{\partial c_j}(c^0_j,0) dx = -\frac{1}{2c^0_j} \int Q_{c_j,0} \left( \frac{2}{p-1} Q_{c_j,0} + (x - r_j)(Q_{c_j,0})_x \right) \]
\[ = -\frac{1}{2c^0_j} \int Q \left( \frac{2}{p-1} Q + xQ_x \right) = -\frac{1}{2c^0_j} \int Q \left( \frac{7-p}{2(p-1)} \right) \frac{5-p}{2(p-1)} \int Q^2, \]
by change of variable and integration by parts. For \(j \neq k\),
\[ \left| \frac{\partial \rho^{1,j}}{\partial c_k}(M_0) \right| = \frac{1}{2c^0_k} \int Q_{c_k,0} \left( \frac{2}{p-1} Q_{c_k,0} + (x - r_k)(Q_{c_k,0})_x \right) \]
\[ \leq C \int e^{-\sqrt{\sigma_0}(|x-r_j|+|x-r_k|)} dx \leq e^{-\sqrt{\sigma_0}r_j-r_k}|x|/2 \leq Ce^{-\sqrt{\sigma_0}L/2}. \]
The rest is done in a similar way, using \(\int Q Q_x = 0\), and \(\int (Q_x)_x = e^{\frac{p+3}{2(p-1)}} \int Q^2_x\).

It follows that \(\nabla \rho(M_0) = D + P\), where \(D\) is a diagonal matrix with nonzero coefficients of order one on the diagonal, and \(\|P\| \leq Ce^{-\sqrt{\sigma_0}L/2}\). Therefore, for \(L\) large enough, the absolute value of the Jacobian of \(\rho\) at \(M_0\) is larger than a positive constant depending only on the \(c^0\). Thus, by the implicit function theorem, by possibly taking a smaller \(\alpha_1\), there exists \(C^1\) functions \((c_j, y_j)\) of \(u \in U(\alpha_1, L_1)\) in a neighborhood of \((c^0_1, \ldots, c^0_N, 0, \ldots, 0)\) such that \(\rho(c_1, \ldots, c_N, y_1, \ldots, y_N, u) = 0\). Moreover, for some constant \(K_1 > 0\), if \(u \in U(\alpha, L_1)\), where \(0 < \alpha < \alpha_1\), then
\[ \sum_{j=1}^N |c_j - c^0_j| + \sum_{j=1}^N |y_j| \leq K_1 \alpha. \]
The fact that \(|\epsilon|_{H^1} \leq K_1 \alpha\) then follows from its definition. Finally, we choose \(x_j(u) = r_j(u) + y_j(u)\).
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