Exact two-particle Matrix Elements in S-Matrix Preserving
Deformation of Integrable QFTs

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Abstract

In a recent paper it was shown that the response of an integrable QFT under variation of the Unruh temperature can be computed from a S-matrix preserving deformation of the form factor approach. We give explicit expressions for the deformed two-particle formfactors for various integrable models: The Sine-Gordon and $SU(2)$ Thirring model, several perturbed minimal CFTs and the real coupling affine Toda series. A uniform pattern is found to emerge when both the S-matrix and the deformed form factors are expressed in terms Barnes’ multi-periodic functions.

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1. Introduction

The formfactor approach provides a powerful and rigid technique to construct and to solve a large class of 1+1 dimensional quantum field theories (QFTs). Its computational rigidity however also provides an obstacle when one tries to apply it to non-standard situations, even when the problem is essentially of a quantum field theoretical nature. Computing the response of a QFT under a variation of the Unruh temperature is such a problem and in a recent paper [1] a generalized formfactor framework was shown to be able to cope with the problem, while retaining the virtues of the usual formfactor approach [2, 3].

In essence each integrable QFT admits a one-parameter deformation that preserves the bootstrap S-matrix. The deformation parameter $\beta$ plays the role of an inverse Unruh temperature, $\beta = 2\pi$ being the QFT value.

The purpose of this letter is to prepare the ground for a more detailed investigation of these systems by computing the deformed two-particle formfactors, i.e. the non-perturbative two-particle matrix elements of a local operator, for many of the commonly considered models. These two-particle matrix elements $F^m(\theta_2 - \theta_1)$ are of particular interest because typically a factor $\prod_{k>l} F^m(\theta_k - \theta_l)$ appears in a successful Ansatz for the n-particle formfactors [4], a feature that seems to be preserved under deformation [1]. Splitting off this factor results in a recurvise problem for the remainder in an often much simpler function space (ideally one of polynomials, see e.g. [4, 5]).

Explicitly the defining relations for the minimal two-particle deformed form factor are

$$F^m(\theta) = S(\theta) F^m(-\theta),$$
$$F^m(\theta + i\beta) = F^m(-\theta),$$

(1)

where $S(\theta)$ denotes the exact two-particle S-matrix. We will consider a version of (1) with indices when needed.

The term “minimal” refers to the condition that the solution searched for should have no poles and zeros in the strip $0 < \text{Im} \, \theta < \beta/2$ and just a simple zero at $\theta = 0$; the kinematical poles can readily be incorporated by multiplying with a suitable $i\beta$-periodic symmetric function. In the limit $\beta = 2\pi$ the equations (1) are known as Watson’s equations; the novel features of the $\beta \neq 2\pi$ case can already be anticipated by noting that e.g. for a $2\pi i$-periodic S-matrix, the period of the S-matrix and the (cyclicity) period of the form factors no longer coincide. Likewise the kinematical poles appear at relative rapidities $\pm i\pi$ and not at $\pm i\beta/2$.

The purpose of the present letter is to present solutions of (1) for generic $\beta$ for many of the commonly considered integrable QFTs: The Sine-Gordon and $SU(2)$ Thirring model,
perturbed minimal CFTs of the \(A_{2N}^{(2)}\) series as well as the simply-laced real coupling affine Toda theories. It turns out that Barnes’ multi-periodic functions \([6]\) provide the proper tool to find and describe the solutions. Moreover a uniform pattern is found to emerge when both the S-matrix and the deformed form factors are expressed in terms of Barnes’ functions.

2. Barnes’ multi-periodic functions

We are going to recall several facts about multi-periodic functions. The mathematical theory of these functions had been developed by Barnes \([6]\) a long time ago. We use the conventions of \([6]\) and borrow results from \([7]\) where additional properties of the functions in question may be found.

Let \(\omega = (\omega_1, \omega_2, \ldots, \omega_r)\) denote the vector of periods, where \(\text{Re} \omega_i > 0\). Set \(|\omega| = \omega_1 + \ldots + \omega_r\), and let \(\omega(i)\) be the vector of periods with the period \(\omega_i\) omitted. Following \([6, 8]\) we define the multi-periodic zeta-function \(\zeta_r(s, x|\omega)\) via its contour integral representation

\[
\zeta_r(s, x|\omega) = -\frac{\Gamma(1-s)}{2\pi i} \int_{C_H} \frac{\exp(-xt)(-t)^{s-1}}{\prod_{i=1}^r (1 - \exp(-\omega_it))} dt.
\]

The integration curve \(C_H\) is of Hankel type (see e.g. \([8]\), chapter 12), which means that after a deformation of the contour we integrate from infinity back to a small circle (counterclockwise) around the origin and then back to positive infinity. It was shown in \([6]\) that the integral in (2) can be asymptotically written in terms of infinite sums, hence resembling the Hurwitz form of \(\zeta\).

Further we introduce the multiple \(\Gamma\)-function via a derivative of the function \(\zeta_r\) in the integral representation (2), i.e.

\[
\log \Gamma_r(x|\omega) = \frac{\partial}{\partial s} \zeta_r(s, x|\omega)|_{s=0} = \frac{1}{2\pi i} \int_{C_H} \frac{\exp(-xt)(\log(-t) - \gamma)}{\prod_{i=1}^r (1 - \exp(-\omega_it))} \frac{dt}{t},
\]

where \(\gamma\) denotes the Euler constant. The functions \(\Gamma_r\) in (3) are meromorphic with poles at \(x = n_1\omega_1 + n_2\omega_2 + \ldots + n_r\omega_r\), for \(n_i\) being non-positive integers.

In order to compare \(\Gamma_1\) with the standard \(\Gamma\)-function, the latter being 1 periodic by definition, we note that

\[
\Gamma_1(x|\omega_1) = \omega_1^{x/\omega_1 - 1/2} \Gamma(x/\omega_1)/\sqrt{2\pi}.
\]

The multi-periodicity of the \(\Gamma\)-functions defined in (3) is reflected in the following relation.
\[
\frac{\Gamma_r(x + \omega | \omega)}{\Gamma_r(x | \omega)} = \frac{1}{\Gamma_{r-1}(x | \omega(i))}.
\] (5)

Finally we define the multiperiodic sine-function by [6]
\[
S_r(x | \omega) = \Gamma_r(x | \omega)^{-1}\Gamma_r(| \omega | - x | \omega |)^{(-1)^r}.
\] (6)

As a consequence of (4) the function \(S_1\) is related to the standard sine-function by
\[
S_1(x | \omega_1) = 2 \sin\left(\frac{\pi x}{\omega_1}\right).
\] (7)

The periodicity of the multi-sine functions can be derived from (6) and (5) as
\[
\frac{S_r(x + \omega | \omega)}{S_r(x | \omega)} = \frac{1}{\Gamma_{r-1}(x | \omega(i))}.
\] (8)

We mention for later purposes that \(S_2\) is meromorphic with poles at \(x = n_1 \omega_1 + n_2 \omega_2\), with \(m_1, m_2\) positive integers, and zeros in \(x = m_1 \omega_1 + m_2 \omega_2\), for \(m_1, m_2\) non-positive integers. Moreover, the function \(S_3\) has no poles but zeros in \(x = m_1 \omega_1 + m_2 \omega_2 + m_3 \omega_3\), for \(m_i \in \mathbb{Z}\).

These definitions are sufficient for the purposes of this letter. In the appendix we present an integral formula for \(S_r\) and two infinite product expansions for \(S_2\) in order to give the reader the possibility to compare our results with related expressions in other papers.

3. The deformed Sine-Gordon model

For a detailed treatment of the undeformed Sine-Gordon model and its form factors we refer to [3, 9]. The \(S\)-matrix in this model contains a scalar part \(S_0(\theta)\), which is expressible in terms of \(S_2\)-functions. We denote by \(\xi\) the non-perturbative coupling constant. The deformed two-particle problem [4] with this \(S\)-matrix was incidentally already considered in the physically completely distinct context of the \(XXZ\)-model in the gapless regime [7]; our solution is related to that in [7] by a suitable redefinition. Explicitly
\[
S_0(\theta) = \frac{S_2(-i\theta|2\pi, \xi)S_2(\pi + i\theta|2\pi, \xi)}{S_2(i\theta|2\pi, \xi)S_2(\pi - i\theta|2\pi, \xi)} = -\exp\left(-i \int_0^\infty \frac{\sin(\theta t) \sinh\left(\frac{\pi - \xi t}{2}\right)}{t \cosh\left(\frac{\pi t}{2}\right) \sinh\left(\frac{\xi t}{2}\right)} dt\right).
\] (9)
It can be shown using the integral representation of the diperiodic sine-functions (27) that the first part of the equation actually reproduces the integral in (1). The unitarity and crossing relations for all the $S$-matrix elements in the sine-Gordon model are readily verified from (9) when taking into account the periodicity equation (8).

The functional equations to be satisfied by the two-particle deformed formfactor in the present model are

$$F_{SG}^m(\theta) = S_0(\theta)F_{SG}^m(-\theta), \quad F_{SG}^m(\theta + i\beta) = F_{SG}^m(-\theta). \quad (10)$$

Up to functions which are both even and $i\beta$-periodic, and a normalization, the unique solution to (10) with the proper analyticity properties (mentioned in the introduction) is given by

$$F_{SG}^m(\theta) = \frac{S_3(-i\theta|2\pi, \beta, \xi)S_3(\beta + i\theta|2\pi, \beta, \xi)}{S_3(\pi - i\theta|2\pi, \beta, \xi)S_3(\pi + \beta + i\theta|2\pi, \beta, \xi)}. \quad (11)$$

Already at this point we observe a feature which turns out to be generic for the cases to be considered in this paper: The two particle form factors possess the same structure as the $S$-matrix of the model, but the number of periods is increased by one.

In the limit $\beta \rightarrow 2\pi$ it can be verified using the integral representation (27) that the expression (11) turns into the corresponding object in [3].

3. The deformed $SU(2)$ invariant Thirring model

We may again refer to [3] for a discussion of the standard $SU(2)$ invariant Thirring model. The scalar part of the $S$-matrix is given by the ratio of $\Gamma$-function, which we rewrite in terms of $\Gamma_1$, using the relation (4)

$$S_{TM}^0(\theta) = \frac{\Gamma\left(\frac{1}{2} + \frac{\theta}{2\pi i}\right)\Gamma\left(-\frac{\theta}{2\pi i}\right)}{\Gamma\left(\frac{1}{2} - \frac{\theta}{2\pi i}\right)\Gamma\left(\frac{\theta}{2\pi i}\right)} = \frac{\Gamma_1(\pi - i\theta|2\pi)\Gamma_1(i\theta|2\pi)}{\Gamma_1(\pi + i\theta|2\pi)\Gamma_1(-i\theta|2\pi)}. \quad (12)$$

In the light of the previous section we should now expect the two-particle formfactor in the present case, $F_{TM}^m(\theta)$, to be built from $\Gamma_2$ functions. Indeed the functional equations to be satisfied by $F_{TM}^m(\theta)$ are equivalent to (10) and the solution is found to be

$$F_{TM}^m(\theta) = \frac{\Gamma_2(\pi - i\theta|2\pi, \beta)\Gamma_2(\pi + \beta + i\theta|2\pi, \beta)}{\Gamma_2(-i\theta|2\pi, \beta)\Gamma_2(\beta + i\theta|2\pi, \beta)}. \quad (13)$$
This expression has again the required analyticity structure, mentioned in the introduction and turns into the proper undeformed expression \ref{1} for $\beta \to 2\pi$. It is intriguing to compare the structure of \ref{13} with the $S$-matrix \ref{12} and also with the two-particle formfactor in the sine-Gordon model \ref{11}. Note that in the limit $\beta \to 2\pi$ the $\Gamma_2$ functions in \ref{13} have two equal periods. It was shown in \ref{6} that this does not allow (apart from infinite product expansions) to rewrite \ref{13} in simple terms of functions with less periods.

4. Affine Toda theories and perturbed minimal models

In this section we compute the two-particle form factors for the deformed affine Toda models of type ADE and of the deformed perturbed minimal models of the $A_{2N}^{(2)}$ series. As it was shown in \ref{10} the latter models can in the standard (undeformed) context be understood as rational reductions of the breather sectors in the Sine-Gordon model. From the point of view of the $S$-matrices and the two-particle formfactors, this process can be easily reversed. It is then straightforward in the deformed case to be described below to reconstruct the deformed two-particle breather formfactors from our results.

In this section, we first have to review a few things about the $S$-matrices of the affine Toda theories and the perturbed minimal models resp., and to introduce some notations.

It is known \ref{11} that the basic constituent of the $S$-matrices for the models under consideration is the following object.

\begin{equation}
(x)_\theta = (x)_+ - (x)_-, \quad (x)_+ = \sinh \frac{1}{2} \left( \theta + \frac{i\pi}{h} x \right),
\end{equation}

where $h$ denotes the Coxeter number of the underlying Lie algebra. The building block of the $S$-matrices is given by

\begin{equation}
\langle x \rangle_\theta = \frac{\langle x \rangle_+}{\langle -x \rangle_+}, \quad \langle x \rangle_+ = \frac{(x+1)_+ - (x-1)_+}{(x+1-B)_+ - (x-1-B)_+} \text{ for perturbed conformal}
\end{equation}

\begin{equation}
\text{for affine Toda models}
\end{equation}

$B$ is the non-perturbative coupling of the affine Toda models \ref{14}.

The $S$-matrices for both cases to be considered in this section can be written as a product, where $x$ takes values in a certain set $A_{ab}$, which is specified by Lie algebraic data, or using Weyl group techniques \ref{12}, where $m_{ab}(p)$ will denote an exponent obtained via this method.

\begin{equation}
S_{ab}(\theta) = \prod_{x \in A_{ab}} \langle x \rangle_\theta = \prod_{p=1}^{h-1} \left( \langle x \rangle_\theta \right)^{m_{ab}(p)}.
\end{equation}
We have now collected enough material to compute the deformed two-particle form factors in the present case. The functional equations to be solved are

\[
F_{ab}^m(\theta) = S_{ab}(\theta) F_{ba}^m(-\theta), \quad F_{ab}^m(\theta + i\beta) = F_{ba}^m(-\theta). \tag{17}
\]

The indices stand for the particle species.

In what follows we adopt the notation used in [5] for the construction of the two-particle formfactors in the standard context. However, we would like to mention that the objects used in that paper are defined by infinite product expansions of \(\Gamma\) functions which are not convergent.

The structure of the \(S\)-matrices suggests to work with \(S_2\)-functions to solve (17). To that end, we define

\[
g_x(\theta) = S_2(-i\theta + \frac{\pi}{h} x| 2\pi, \beta) \tag{18}
\]

Obviously it holds that \(g_x(\theta + i\pi y) = g_{x+y}(\theta)\). Using the periodicity of the \(S_2\)-function (8) we verify the following identities

\[
g_x(\theta + i\beta) = g_x(\theta), \quad g_x(\theta + 2\pi) = - \frac{2i}{\sinh \frac{\pi}{h}x} g_x(\theta + i\pi). \tag{19}
\]

The definition of the function \(S_2\) in (6) allows the identification

\[
S_2(\omega_1 + \omega_2 - x| \omega_1, \omega_2) = \frac{1}{S_2(x| \omega_1, \omega_2)}, \tag{20}
\]

which we need in order to verify the following

\[
g_x(i\pi - \theta) = \frac{1}{g_{3h-x}(\theta)} = -2i \cdot \sinh \frac{1}{2} \left( \theta + \frac{i\pi}{h} (h - x) \right) \frac{1}{g_{h-x}(\theta)}. \tag{21}
\]

As in the case of the \(S\)-matrices we can now construct a building block for the two-particle formfactors

\[
G_x(\theta) = \begin{cases} 
\frac{g_{x-1}(\theta)g_{x+1}(\theta)}{g_{x-1}(\theta)g_{x+1}(\theta)} & \text{for perturbed conformal} \\
\frac{g_{x-1}(\theta)g_{x+1}(\theta)}{g_{x-1}(\theta)g_{x+1}(\theta)} & \text{for affine Toda models}
\end{cases} \tag{22}
\]

Notice that using (28) we get a convergent expansion of these blocks in terms of \(\Gamma\) functions.
Using the identities (19) and (21) we can easily verify that the following expression satisfies the functional equations (17).

\[ F_{m}^{ab}(\theta) = \prod_{x \in A_{ab}} f_{x}(\theta), \quad f_{x}(\theta) = \frac{G_{x}(\theta)}{G_{2h-x}(\theta)}. \]

The analyticity of this solution is the one we required in the introduction.
Comparing this solution (written in terms of \( S_{2} \)-functions) with the corresponding results in the Sine-Gordon and Thirring models, respectively, one sees that their structure is in principle equivalent.

The expressions (23) also facilitate the derivation of additional functional relations for the two-particle formfactors which in the undeformed case allow one to reduce the construction of \( n \)-particle formfactors to a polynomial problem [4, 5]. For the Sinh-Gordon model a similar identity was shown to have essentially the same consequences in the deformed case [4]. To generalise this identity to other affine Toda models we introduce another bit of notation

\[ [x]_{\beta}^{\pm} = \frac{\sinh \frac{\pi}{i} \left( \theta + i \frac{\pi}{\beta} (x - 1) \right) \sinh \frac{\pi}{i} \left( \theta + i \frac{\pi}{\beta} (x + 1) \right)}{\sinh \frac{\pi}{i} \left( \theta + i \frac{\pi}{\beta} (x - 1 + B) \right) \sinh \frac{\pi}{i} \left( \theta + i \frac{\pi}{\beta} (x + 1 - B) \right)}. \]

This is the expression we need for the affine Toda models. According to (13) the corresponding expression for the case of perturbed minimal models is obtained by setting the denominator in (24) equal to one.

The first identity stems from the crossing symmetry of the \( S \)-matrix and reads

\[ F_{m}^{ab}(\theta + i\pi) F_{ab}^{m}(\theta) = \prod_{x \in A_{ab}} [x]_{\beta}^{+}. \]

The second identity is a consequence of the bootstrap equations of the \( S \)-matrix with \( \theta_{ab}^{\pm} \), \( \hat{\theta}_{ab} = \pi - \theta_{ab}^{\pm} \) denoting the fusion angles [11]

\[ \lambda_{ab,d}^{c}(\theta)^{-1} = \frac{F_{abd}^{\min}(\theta + i\hat{\theta}_{ac}^{b}) F_{bd}^{\min}(\theta - i\hat{\theta}_{bc}^{a})}{F_{cd}^{\min}(\theta)} = \prod_{p=0}^{\hat{u}_{ac}^{b} - 1} \left( \left[ \hat{u}_{ac}^{b} - p \right]_{\beta}^{+} \right)^{m_{ad}(p)} \prod_{p=0}^{\hat{u}_{bc}^{a} - 1} \left( \left[ \hat{u}_{bc}^{a} - p \right]_{\beta}^{+} \right)^{m_{bd}(p)}. \]

Using these identities and following the procedures outlined in [4] it is possible after a suitable parameterization of the pole part in the \( n \)-particle formfactor to write down...
polynomial recursion equations for the $n$-particle deformed formfactors. For the Sinh-Gordon case first solutions have been found in [1]. Obviously, the limit $\beta \to 2\pi$ reproduces the standard results for all objects introduced in this section [4, 5].

Appendix: On the multiple sine-function

The multi-periodic sine-function was defined by (6). Using the integral representation (3) for $\Gamma_r$, we can write down an integral representation for $S_r$.

$$\log S_r(x|\omega) = \frac{1}{2^{r+1}\pi i} \int_{C_H} \frac{\log(-t)}{t} \prod_{i=1}^r \sinh \left( \frac{\omega_i t}{2} \right) \left( (-1)^r e^{xt-|\omega|t/2} - e^{-xt+|\omega|t/2} \right).$$

(27)

It is of course possible using standard methods to rewrite this complex line integral as an integral over the positive real line.

The function $S_r$ can be expanded in various ways. Usually, in the literature an infinite product expansion involving $\Gamma$-functions can be seen as minimal form factors. However, many of them can be easily shown not to be convergent.

The correct expansion of $S_2$ in terms of standard $\Gamma$ functions is

$$S_2(x|\omega_1, \omega_2) = e^{(\omega_1+\omega_2-2x)(\gamma+\log(2\pi)+\log(\omega_1/\omega_2))/(2\omega_1)} \frac{\Gamma((\omega_1+\omega_2-x)/\omega_1)}{\Gamma((x-n\omega_2)/\omega_1)} e^{-(\omega_1+\omega_2-2x)/(2n\omega_1)(n\omega_1/\omega_2)-(\omega_1+\omega_2-2x)/\omega_1}. $$

(28)

It may not be apparent, but this expression is symmetric in the two periods of $S_2$. Another product expansion for $S_2$ was given in [13].

$$S_2(x|\omega_1, \omega_2) = \sqrt{i} e^{\frac{\pi i}{12}(\omega_1^2+\omega_2^2)} \prod_{n=0}^{\infty} \frac{(1-q^n e^{-\omega_1/2})}{(1-q^n e^{-\omega_2/2})} \frac{\sin(x-(\omega_1+\omega_2)x)}{2\omega_1\omega_2}, $$

(29)

where $q = \exp(2\pi i \omega_2/\omega_1)$ and $q' = \exp(-2\pi i \omega_1/\omega_2)$.

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