I. INTRODUCTION

Random motion under resetting represents a process in which a stochastically moving particle returns from time to time to its initial position and starts its motion again from the very beginning. Examples of such processes are found in many fields such as biophysics [1–3], biology [4, 5] and computer science [6]. In computer science, random walks with stochastic restarts represent a useful strategy to optimize search algorithms in computationally hard problems [6]. The organisms can use stochastic resetting or switching between different phenotypic states in order to adapt to fluctuating environments [6]. Also the motion of foraging animals, when they are searching for food and return home from time to time is one of the example of resetting. One can also consider a robotic vacuum cleaner performing stochastic motion while cleaning the appartment and returning to its base in order to recharge.

Early works concentrated on the case when the stochastic displacement process is a simple Brownian motion [7], i.e. a Markovian process with stationary increments. Later on, one has also studied other types of motion between the resetting events, such as Lévy flights [8, 9], continuous-time random walks with or without drift [10–12], and scaled Brownian motion [13, 14]. The main attention was always paid to the probability density function (PDF) of displacements as measured at a given time or in a stationary state, and to the mean first passage time (MFPT) – the average time necessary to hit a specified target [16, 17].

With respect to the first hitting properties, early work concentrated on searchers performing one-dimensional Brownian motion with Poissonian resetting [5, 10–12]. While the MFPT to a target for a diffusing particle in absence of resetting may diverge, in presence of resetting it turns finite, and there exists an optimal rate of resetting which minimizes the MFPT. The discussion has been extended to two and higher dimensions in [20], and to the cases when the waiting time for resetting events follows a power-law distribution [22].

Typically, the return to the initial position (assumed at the origin of the coordinate system in what follows) is considered to be instantaneous. In some studies, a random refractory period after the resetting event has been introduced [23–25], but the return to the origin was still described as an instantaneous jump. However, in many situations when the process considered corresponds to a motion of a material object in space, such an assumption is unphysical. In what follows we discuss what happens when the return to the origin follows a given equation of motion, and takes finite time for completion. We note that this situation was just recently considered by Arnab Pal, Lukasz Kuśmierz and Shlomi Reuveni [26], which work was performed parallel to ours. Apart from other results, Ref. [26] contains ones which overlap with Sec. IV of our work, and the corresponding discussion is given in this section. The exemplary trajectories of such a process are depicted in the right panel of Fig. 1 for the case of Brownian motion with return at a constant speed in one dimension (1d).

In the present work, we derive the stationary PDF and the mean squared displacement (MSD) as well as the MFPT for this process for specific cases when resetting takes place at a fixed time after starting the stochastic motion, and when the resetting time follows an exponential distribution.

II. THE MODEL

The reset-return process consists of subsequent runs. Each run is a sequence of two processes, the stochastic displacement process $x(t)$ which is interrupted by the resetting event, and the deterministic return process, which ends when the particle returns to the origin, as depicted in Fig. 2. The waiting time density for a resetting event is given by a function $\psi(t_{res})$, where $t_{res}$ is the time elapsed from the beginning of the run. The PDF of the displacement during the displacement phase at time $t \leq t_{res}$ is given by $p(x|t)$. The motion during the return phase takes place according to the deterministic equation of motion $x = X(t; x_0)$, where $t$ now is the time elapsed since the beginning of the return phase, and $x_0$ is the particle's
Since after finishing the run the particle returns to the stationary probability density, the coordinate and in the first passage properties of the process restarts anew, with the displacement phase. The diffusion coefficient in the displacement phase is \( D = 1 \).

coordinate at the beginning of this phase (i.e. at the end of the preceding displacement one). The equation of motion \( X(t; x_0) \) is chosen such, that the return from each point \( x_0 \) to the origin takes place in a finite time \( t_{\text{ret}}(x_0) \), given by the solution of the equation \( X(t_{\text{ret}}(x_0); x_0) = 0 \). The run finishes by the return to the origin, after which the process restarts anew, with the displacement phase of the next run. Therefore the whole process is a renewal one. The total duration of the run is \( t_{\text{run}} = t_{\text{res}} + t_{\text{ret}} \). The run, however, may be stopped at the observation time, when the coordinate is measured, or by hitting a target. In the present work we are interested in the existence and shape of the stationary probability density \( P(x) \) of the coordinate and in the first passage properties of the reset-return process.

III. PROBABILITY DENSITY FUNCTION

Let us first turn to the existence and shape of the stationary probability density \( P(x) = \lim_{t \to \infty} P(x|t) \), where \( P(x|t) \) is the PDF of the particle’s position at time \( t \). Since after finishing the run the particle returns to the origin, its position at time \( t \) is defined by its displacement in the last, stopped run. In the present work we will concentrate only on the situation when the first moment \( \langle T \rangle \) of the duration of uninterrupted run does exist: general reset-return processes seem to be rich with respect to possible aging phenomena, whose investigation is left for further work. The conditions under which this moment exists will be discussed below.

Let \( \phi(t) \) denote the probability density of the run’s duration, if not stopped. Let \( t \) be the measurement (stopping) time. Then the starting time \( t_0 \) of the last, stopped, run is given by

\[
p(t_0) = \kappa(t_0) \Phi(t - t_0)
\]

with \( \kappa(t_0) \) being the time-dependent rate of the renewals, and \( \Phi(t) = \int_0^t \phi(t') dt' \) is the survival probability. The equation derives from the fact that the probability \( p(t_0)dt_0 \) to find the last renewal event in time interval between \( t_0 \) and \( t_0 + dt_0 \) is the probability \( \kappa(t_0)dt_0 \) that a renewal took place in the corresponding time interval, and that no further renewals occurred until the measurement was performed.

The expression for \( \kappa(t) \) is given by

\[
\kappa(t) = \sum_{n=0}^{\infty} \phi^{*n}(t),
\]

with \( \phi^{*n}(t) \) being the \( n \)-fold convolution of \( \phi(t) \) with itself. The renewal rate \( \kappa(t) \) has a very simple expression in the Laplace domain:

\[
\hat{\kappa}(s) = \frac{1}{1 - \phi(s)},
\]
where tilde denotes the Laplace transform: \( \tilde{f}(s) = \int_0^\infty f(t) e^{-st} dt \). If the first moment \( \langle T \rangle = \int_0^\infty t \phi(t) dt \) of \( \phi(t) \) exists, the value of \( \kappa(t) \) tends to \( \langle T \rangle^{-1} \) for \( t \to \infty \). Otherwise \( \kappa(t) \) stays explicitly time-dependent. The normalization of \( p(t_0) \) follows by noting that in the Laplace domain \( \tilde{\phi}(s) = s^{-1}(1 - \phi(s)) \). Therefore \( I(t) = \int_0^t p(t_0) dt_0 = \int_0^t \kappa(t_0) \Phi(t-t_0) dt_0 \) has a form of the convolution, and its Laplace transform \( \tilde{I}(s) = \tilde{\kappa}(s) \tilde{\phi}(s) \) is equal to \( s^{-1} \), which corresponds to \( I(t) = 1 \) in the time domain. Hence,

\[
\int_0^t p(t_0) dt_0 = \int_0^t \kappa(t_0) \Phi(t-t_0) dt_0 = 1. \tag{2}
\]

The distribution of \( \tau_{\text{run}} = \tau_{\text{res}} + \tau_{\text{ret}} \) can be easily calculated. The distribution of \( \tau_{\text{res}} \) is given, and the return time is the deterministic function of the coordinate \( x_0 \) at the end of the displacement phase which is given by \( \tau_{\text{ret}}(x_0) \). Therefore

\[
\phi(\tau_{\text{run}}) = \int_0^\infty dt_{\text{res}} \psi(\tau_{\text{res}}) \times \\
\int_{-\infty}^{\infty} dx_0 \delta[\tau_{\text{run}} - \tau_{\text{res}} - \tau_{\text{ret}}(x_0)] p(x_0 | \tau_{\text{res}}).
\]

The survival probability is now given by

\[
\Phi(t) = \int_t^\infty dt' \phi(t') = \int_t^\infty dt' \int_0^\infty dt_{\text{res}} \psi(\tau_{\text{res}}) \times \\
\int_{-\infty}^{\infty} dx_0 \delta[t' - \tau_{\text{res}} - \tau_{\text{ret}}(x_0)] p(x_0 | \tau_{\text{res}})
\]

\[
\tag{3}
= \int_0^\infty dt_{\text{res}} \psi(\tau_{\text{res}}) \int_{-\infty}^{\infty} dx_0 \delta[t_{\text{run}} + \tau_{\text{ret}}(x_0) - t] p(x_0 | \tau_{\text{res}}).
\]

\[
q(x | \Delta t; \tau_{\text{res}}, x_0) = p(x | \Delta t) \Theta(\tau_{\text{res}} - \Delta t) + \delta[x - X(\Delta t - \tau_{\text{res}}; x_0)] \Theta(\Delta t - \tau_{\text{res}}) \Theta[\tau_{\text{res}} + \tau_{\text{ret}}(x_0) - \Delta t], \tag{5}
\]

where the first \( \Theta \)-function represents the condition that the stopping time falls into the displacement phase of the run, and the last \( \Theta \)-function gives the condition that the run is stopped, i.e. not finished. To lift the conditioning on \( x_0 \) and \( \tau_{\text{res}} \) we first average over the distribution of \( x_0 \) for given \( \tau_{\text{res}} \) and then over the distribution of \( \tau_{\text{res}} \). The steps give:

\[
Q(x; \Delta t) = \int_0^\infty p(x | \Delta t) \Theta(\tau_{\text{res}} - \Delta t) \psi(\tau_{\text{res}}) d\tau_{\text{res}}
\]

\[
+ \int_0^\infty dt_{\text{res}} \psi(\tau_{\text{res}}) \int_{-\infty}^{\infty} dx_0 \delta[x - X(\Delta t - \tau_{\text{res}}; x_0)] \Theta(\Delta t - \tau_{\text{res}}) \Theta[\tau_{\text{res}} + \tau_{\text{ret}}(x_0) - \Delta t] p(x_0 | \tau_{\text{res}}).
\]

\[
\tag{6}
Let us now denote the integrals in the first and in the second line of the Eq. (6) by \( Q_1(x; \Delta t) \) and \( Q_2(x; \Delta t) \),
respectively: \(Q(x; \Delta t) = Q_1(x; \Delta t) + Q_2(x; \Delta t)\).

The displacement PDF at time \(t\) is given by

\[
P(x, t) = \int_0^t \kappa(t_0) Q(x; t - t_0) dt_0.
\]

(7)

\[
\int_{-\infty}^{\infty} Q(x; \Delta t) dx = \int_0^\infty \Theta(t_{res} - \Delta t) \psi(t_{res}) dt_{res} + \int_0^\infty dt_{res} \psi(t_{res}) \Theta(\Delta t - t_{res}) \int_{-\infty}^{\infty} dx_0 \Theta[t_{res} + t_{ret}(x_0) - \Delta t] p(x_0|t_{res})
\]

\[
= \int_0^\infty dt_{res} \psi(t_{res}) \left[ \Theta(t_{res} - \Delta t) + \Theta(\Delta t - t_{res}) \int_{-\infty}^{\infty} dx_0 \Theta(t_{res} + t_{ret}(x_0) - \Delta t] p(x_0|t_{res}) \right).
\]

Now we note that \(\Theta(-y) = 1 - \Theta(y)\) for any real \(y\). Making this substitution for the last \(\Theta\)-function in the last expression and using that \(\int_{-\infty}^{\infty} dx_0 p(x_0|t_{res}) = 1\) we get:

\[
\int_{-\infty}^{\infty} Q(x; \Delta t) dx = \int_0^\infty \Theta(t_{res} - \Delta t) \psi(t_{res}) dt_{res} + \int_0^\infty dt_{res} \psi(t_{res}) \Theta(\Delta t - t_{res}) \int_{-\infty}^{\infty} dx_0 \Theta[t_{res} + t_{ret}(x_0) - \Delta t] p(x_0|t_{res})
\]

\[
= \int_0^\infty \Theta(t_{res} - \Delta t) \psi(t_{res}) dt_{res} + \int_0^\infty \Theta(\Delta t - t_{res}) \int_{-\infty}^{\infty} dx_0 \Theta[t_{res} + t_{ret}(x_0) - \Delta t] p(x_0|t_{res})
\]

which coincides with our expression for \(\Phi(\Delta t)\), Eq. [5]. The overall normalization then follows from Eq. [2].

Now we return to calculation of the stationary PDF. If the stationary PDF exists, it has to be attained at \(t \to \infty\). Therefore

\[
P(x) = \lim_{t \to \infty} \int_0^t \kappa(t_0) Q(x; t - t_0) dt_0.
\]

Although one cannot exclude that a stationary PDF of coordinate does exist also for non-stationary process of renewals, the situation is really simple only if the renewal process does possess the mean waiting time and is stationary. In this case \(\kappa(t_0)\) tends to a constant at longer \(t_0\), and the expression for \(P(x)\) reads

\[
P(x) = \lim_{t \to \infty} \frac{1}{\langle T \rangle} \int_0^t Q(x; t - t_0) dt_0 = \frac{1}{\langle T \rangle} \int_0^\infty Q(x; t') dt',
\]

(8)

The fact that \(P(x, t)\) is normalized can be seen by integrating this equation over \(x\). Integrating \(Q(x; \Delta t)\), Eq. [6], over \(x\) we get:

\[
P(x) = \frac{I_1(x) + I_2(x)}{\langle T \rangle}
\]

with \(I_1(x) = \int_0^\infty Q_1(x, t') dt'\) and \(I_2(x) = \int_0^\infty Q_2(x, t') dt'\). The integration in the first integral can be performed explicitly and expressed via the survival probability in a displacement phase \(\Psi(t) = \int_0^\infty \psi(t') dt'\):

\[
I_1(x) = \int_0^\infty p(x|t') \Psi(t') dt'.
\]

(9)

For the second integral we get

\[
I_2(x) = \int_0^\infty dt' \int_0^\infty dt_{res} \psi(t_{res}) \int_{-\infty}^{\infty} dx_0 \delta[x - X(t' - R; x_0)] \Theta(t' - t_{res}) \Theta[t_{res} + t_{ret}(x_0) - t'] p(x_0|t_{res}).
\]

(10)

While \(I_1(x)\) has to be directly evaluated for specific cases, the \(I_2(x)\) can not be simplified in general unless the equa-
tion of motion is explicitly known.

IV. SPECIFIC EXAMPLES

In what follows we will consider situations, when the return takes place at a constant speed, just like in Ref. [20]:

\[
X(t; x_0) = (x_0 - vt)\Theta(x_0) + (x_0 + vt)\Theta(-x_0)
\]

\[
= \begin{cases} 
  x_0 - vt & \text{for } x > 0 \\
  x_0 + vt & \text{for } x < 0 
\end{cases}.
\]

(11)

Now we note that if \( x_0 < 0 \) the value of \( x \) stays negative, and if \( x_0 > 0 \) the value of \( x \) stays positive during the whole return phase. Therefore we may write

\[
\delta[x - X(t - t_{res}; x_0)] = \delta[x - x_0 + v(t - t_{res})]\Theta(x)
+ \delta[x - x_0 - v(t - t_{res})]\Theta(-x).
\]

(12)

Due to the presence of theta functions the integral \( I_{\pm}(x) \) separates into two integrals, which we denote by \( I_{+}(x) \) and \( I_{-}(x) \), which are calculated using the same tricks. Thus,

\[
I_{+}(x) = \Theta(x) \int_{0}^{\infty} dt \int_{0}^{\infty} dt_{res} \psi(t_{res}) \times \\
\int_{-\infty}^{\infty} dx_0 \delta[x - x_0 + v(t - t_{res})]\Theta(t - t_{res}) \times
\]

\[
\Theta[t_{res} + x_0/v - t]p(x_0|t_{res}).
\]

To evaluate the integral we first perform integration in \( x_0 \). Due to the presence of the \( \delta \)-function we get

\[
I_{+}(x) = \Theta(x) \int_{0}^{\infty} dt \int_{0}^{\infty} dt_{res} \psi(t_{res}) \times
\]

\[
\Theta(t - t_{res})p(x + v(t - t_{res})|t_{res}).
\]

(14)

Now we change from integration over \( t \) to integration over \( x_0 = x + v(t - t_{res}) \) and note that the condition given by \( \Theta(t - t_{res}) \) corresponds to \( x_0 > x \). Now we get

\[
I_{+}(x) = \Theta(x) \frac{1}{v} \int_{x}^{\infty} dx_0 \int_{0}^{\infty} dt_{res} p(x_0|t_{res})\psi(t_{res}),
\]

which is the final result for this integral. Analogous steps are then performed for calculating \( I_{-}(x) \). The final result reads

\[
P(x) = \frac{1}{\langle t_{res} \rangle + \langle t_{ret} \rangle} \left( \int_{0}^{\infty} dt_{res} \psi(t_{res}) \int_{0}^{t_{res}} p(x|t) dt \right.
+ \frac{1}{v} \int_{x}^{\infty} dx_0 \int_{0}^{\infty} dt \int_{0}^{\infty} \[x_0]p(x_0|t)\psi(t)\right).
\]

(15)

The value of the duration of the return phase is \( t_{ret}(x_0) = \frac{|x_0|}{v} \), and the mean duration of a run is

\[
\langle T \rangle = \int_{0}^{\infty} t_{res} \psi(t_{res}) dt_{res}
+ v^{-1} \int_{0}^{\infty} dt \int_{-\infty}^{\infty} [x_0]p(x_0|t)\psi(t) dx_0 dt.
\]

(16)

In our examples we consider a standard Brownian motion displacement process, with a PDF

\[
p(x|t) = \frac{1}{\sqrt{4\pi Dt}}e^{-\frac{x^2}{4Dt}}.
\]

(17)

In this case the mean duration of the return phase is \( \langle t_{ret}(x_0) \rangle = v^{-1}(x_0) = v^{-1}\sqrt{4Dt_{res}/\pi} \).

As important specific examples we consider in the next section the exponential waiting times distribution for resetting

\[
\psi(t) = r \exp(-rt),
\]

(18)

and resetting at a given time after the beginning of the displacement phase,

\[
\psi(t) = \delta(t - t_r).
\]

(19)

Other types of waiting time distributions [27, 30] can also be considered, such as a power-law distribution of waiting times [31], which will be a subject of future work.

V. SIMULATIONS

The analytical predictions for MSD and PDF, as well as for the mean hitting time, discussed later, have been compared with numerical simulations directly following from the discretization of Langevin equations in the motion phase and of the equation of motion for the return at a constant speed. The time axis is discretized with the step \( dt = t_{i+1} - t_i \), and the time of the first resetting event is generated according to its probability density \( \psi(t) \). For the deterministic resetting this resetting time is fixed. During the displacement phase the particle performs stochastic motion according to a finite-difference analogue of the Langevin equation

\[
x_{i+1} = x_i + \xi_i \sqrt{2Dt}.
\]

(20)

Here \( x_i = x(t_i) \) is the coordinate of the particle at time \( t_i \), and \( \xi_i \) is the random number distributed according to a standard normal distribution generated using the Box-Muller transform. When the resetting event occurs, the particle starts moving to the origin at a constant speed: \( x_{i+1} = x_i \pm vdt \). When the particle crosses the origin, the time of the next resetting event is generated, and the particle starts performing stochastic motion until this resetting event. The simulations are stopped either at a predefined time, or (for simulations of the first passage times) when the particle crosses for the first time a predefined level. All simulations are performed with \( N = 10^5 \) particles.

The results of simulations are given in Figs. [4, 8] and [5] where they are compared to the corresponding analytical results. Fig. [1] represents typical trajectories of particle’s motion under resetting with instantaneous return, and with the return at a constant speed.
a. **Exponential resetting.** For the exponential distribution of the resetting times, Eq. (18), \( \Psi(t) = e^{-rt} \), the mean resetting time is equal to \( \langle t_{\text{res}} \rangle = 1/r \) and the average location at the resetting event \( \langle |x_0| \rangle = \sqrt{D/r} \). The PDF can be then calculated according to Eq. (15) and attains a simple form:

\[
P(x) = \frac{1}{2\sqrt{D/r}} \exp\left(-\frac{|x|}{\sqrt{D/r}}\right).
\]

(21)

We see that this PDF does not depend on \( v \) and has the same form as for instantaneous resetting. The MSD in stationary state reads

\[
\langle x^2 \rangle = 2 \int_0^\infty x^2 P(x') dx' = 2 \langle |x_0| \rangle^2,
\]

(22)

independently of the velocity of resetting.

The MSD (left panel) and the PDF (right panel) for Brownian motion with exponential resetting are depicted at Fig. 3 together with results of computer simulations.

The right panel confirms the result that for the case of the Brownian motion with exponential resetting, the models with instantaneous return and with the return at a constant speed lead to the same stationary probability distributions of the coordinate.

b. **Deterministic resetting.** Let us now consider resetting at a given time after beginning of the displacement phase. The survival probability for resetting events is now \( \Psi(t) = \Theta(t_r - t) \). The mean duration of the displacement phase is \( \langle t_{\text{res}} \rangle = t_r \). We get:

\[
P(x) = \sqrt{\frac{D}{\pi t_r}} \exp\left(-\frac{x^2}{4Dt_r}\right) + \frac{1}{2} \left(1 - \frac{|x|}{D} \right) \text{erfc}\left(\frac{|x|}{\sqrt{4Dt_r}}\right)
\]

\[
\approx t_r + \frac{1}{\sqrt{\pi}} \frac{D(t_r^2 + 2vt_r)}{v}.
\]

(23)

The MSD can be calculated performing similar integration as in Eq. (22):

\[
\langle x^2 \rangle = D t_r \left( 1 + \frac{2\sqrt{Dt_r}}{t_r v \sqrt{\pi} + 2\sqrt{Dt_r}} \right).
\]

(24)
The corresponding results for the PDF and the following MSD are represented in Fig. 4 by dashed lines. In the case of instantaneous return the oscillations of the MSD are observed: the MSD first linearly grows and then abruptly vanishes. When the return at a constant speed is introduced, the MSD stagnates, and the PDF tends to a stationary form.

VI. MEAN FIRST PASSAGE TIME

When turning to mean hitting (or first passage) properties of the process, we will consider the situation, when the hitting only can take place during the displacement phase. In one dimension this is trivially the case, since the interval of the x-axis from the origin to $x_0$ was already covered by the trajectory of the displacement process. Therefore, if there were any targets on this interval, they should already be found during the displacement phase. The situation in higher dimensions is richer. Depending on the properties of the searcher, one can assume that, parallel to the 1d situation, the target can only be found during the displacement phase, or during both, displacement and return phases. We do not consider these high-dimensional situations in the present work. Our approach to the problem will be similar to the one used in [22].

The hitting of the target may occur during the first run, during the second run (the first one is finished, or idle), in the terminology of Ref. [22], the run is called complete (if it is stopped by hitting event), during the third run (two first runs are idle, i.e. finished), etc., and the probability $h(t)$ to reach a target at time $t$ attains the following form:

$$h(t) = \omega(t) + \int_0^t \phi(t')\omega(t-t')dt' + \int_0^t \int_0^{t'} \phi(t'')\phi(t')\omega(t-t'-t'')dt'dt'' + \ldots .$$

Here $\omega(t)$ is the probability density of hitting a target at time $t$ after starting a (complete) run, and $\phi(t)$ is a probability density of finishing an idle run at time $t$ after its beginning. Since the series has a form of a sum of multiple convolutions of $\omega(t)$ and $\phi(t)$, this can be easily evaluated in the Laplace domain:

$$\tilde{h}(s) = \tilde{\omega}(s) + \tilde{\phi}(s)\tilde{\omega}(s) + \tilde{\phi}^2(s)\tilde{\omega}(s) + \ldots = \frac{\tilde{\omega}(s)}{1 - \tilde{\phi}(s)} .$$

Here $\tilde{\omega}(s) = \int_0^\infty dt e^{-ts} \omega(t)$ is the Laplace transform of $\omega(t)$. The full form of $p(t)$ can be obtained only numerically. The MFPT $\tau$ can however be easily found:

$$\tau = -\frac{d}{ds} \tilde{h}(s) \bigg|_{s=0} = -\frac{\tilde{\omega}'(0)}{1 - \tilde{\phi}(0)} - \frac{\tilde{\omega}(0)\tilde{\phi}'(0)}{(1 - \tilde{\phi}(0))^2} = -\frac{\tilde{\omega}'(0)}{1 - \tilde{\phi}(0)} - \frac{\tilde{\phi}'(0)}{(1 - \tilde{\phi}(0))^2} P_c. \quad (25)$$

Here $P_c = \tilde{\omega}(0)$ is a probability that the run is complete, which means that the target has been found during the run, or the probability that the time necessary to find a target, $t_{hit}$, is smaller than the resetting time $t_{res}$. $P_c = \tilde{\phi}(0)$ is the probability that a run is finished (idle). It is equal to the probability that the resetting time $t_{res}$ is smaller than the hitting time of the target $t_{hit}$. Each run is either complete or idle: $P_c + P_i = 1$, therefore we get from Eq. (25)

$$\tau = -\left(\tilde{\omega}'(0) + \tilde{\phi}'(0)\right) / P_c . \quad (26)$$

This equation is similar to the one obtained in [26], which work appeared when the present one was in preparation, and can be put in the similar form.

According to the definition $\tilde{\omega}'(0) = -\int_0^\infty t\omega(t)dt$. Dividing this by $P_c$ we get the mean duration of a complete run: $\langle t_{hit} \rangle = -\tilde{\omega}'(0)/P_c$. Performing similar calculations, one can show that $-\tilde{\phi}'(0) = \langle t_{run} \rangle P_c$. Taking into account that $t_{run} = t_{res} + t_{ret}$ we get

$$\tau = \frac{\langle \min(t_{hit}, t_{res}) \rangle + \langle t_{run} \rangle P_c}{P_c} . \quad (27)$$

Compared to Eq. (2) of Ref. [26] this equation does not contain the third term, corresponding to staying home. Eq. (27) is however awkward for the further usage. It is easier to use the physical meaning of the terms and immediately write them as integrals (at least in 1d and for Brownian motion, and return at a constant speed).

Let $p(x|t; b)$ be the PDF of the particle’s positions on a semi-infinite interval with an absorbing boundary (target) at $b > 0$. This is a non-proper probability density. Then $\Omega(t) = \int_{-\infty}^b p(x'|t; b)$ is the survival probability of the particle in the interval up to time $t$, and $\Omega(t_{res})$ the probability that no hitting took place before resetting at time $t_{res}$. The hitting time density is

$$\omega(t) = -\frac{d}{dt} \Omega(t) = -\frac{d}{dt} \int_{-\infty}^b p(x|t; b)dx . \quad (28)$$

Let us fix the resetting time $t_{res}$, and calculate first the corresponding means in Eqs. (27) or (26) conditioned on $t_{res}$. To do so we note that for $t_{res}$ fixed, the hitting time density conditioned on the fact, that the target was hit before the resetting is $\omega(t)/(1 - \Omega(t_{res}))$ and therefore

$$\langle t_{hit}|t_{res}\rangle \cdot (1 - \Omega(t_{res})) = -\int_0^{t_{res}} \frac{t}{dt} \Omega(t)dt \quad = \int_0^{t_{res}} \Omega(t)dt - t_{res} \Omega(t_{res}) . \quad (29)$$

Similarly, for the return at a constant speed $v$, the mean return time conditioned on $t_{res}$ (and on the fact that the target was not hit during the displacement phase) is given by

$$\langle t_{ret}|t_{res}\rangle \Omega(t_{res}) = \int_{-\infty}^0 \frac{|x|}{v} p(x, t_{res})dx . \quad (30)$$
The mean run time is

\[ \langle t_{\text{run}} | t_{\text{res}} \rangle \Omega(t_{\text{res}}) = t_{\text{res}} \Omega(t_{\text{res}}) + \langle t_{\text{ret}} | t_{\text{res}} \rangle \Omega(t_{\text{res}}). \]  

Therefore, the sum of these two expressions

\[ S(t_{\text{res}}) = \langle t_{\text{hit}} | t_{\text{res}} \rangle \cdot (1 - \Omega(t_{\text{res}})) + \langle t_{\text{run}} | t_{\text{res}} \rangle \Omega(t_{\text{res}}) \]

\[ = \int_{0}^{t_{\text{res}}} dt \int_{-\infty}^{b} p(x, t) dx + \int_{-\infty}^{b} \frac{|x|}{v} p(x, t_{\text{res}}) dx. \]  

Averaging this expression over the distribution of \( t_{\text{res}} \) gives us the numerator of Eq. (26):

\[ -\tilde{\omega}'(0) - \tilde{\psi}'(0) = \int_{0}^{\infty} \psi(t_{\text{res}}) S(t_{\text{res}}) dt_{\text{res}} \]  

\[ = \int_{0}^{\infty} dt_{\text{res}} \psi(t_{\text{res}}) \int_{0}^{t_{\text{res}}} dt \int_{-\infty}^{b} p(x, t) dx + \int_{-\infty}^{\infty} dt_{\text{res}} \psi(t_{\text{res}}) \int_{-\infty}^{b} \frac{|x|}{v} p(x, t_{\text{res}}) dx. \]  

Now we change the sequence of integrations in \( t_{\text{res}} \) and in \( t \) in Eq. (33): \[ \int_{0}^{\infty} dt_{\text{res}} \psi(t_{\text{res}}) \int_{0}^{t_{\text{res}}} dt \int_{-\infty}^{b} p(x, t) dx = \int_{0}^{\infty} dt \Psi(t) \int_{-\infty}^{b} p(x, t) dx \] and rename the integration variable to \( t \) everywhere. In such a way we obtain

\[ \tau = \frac{\int_{0}^{\infty} dt \Psi(t) \Omega(t) + \int_{0}^{\infty} dt \Psi(t) \int_{-\infty}^{b} \frac{|x|}{v} p(x, t) dx}{\int_{0}^{\infty} dt \Psi(t) \int_{-\infty}^{\infty} p(x, t) dx}. \]  

For the case of the Brownian motion our \( p(x | t ; b) \) reads

\[ p(x | t ; b) = \frac{1}{\sqrt{4\pi Dt}} \left[ \exp \left( -\frac{x^2}{4Dt} \right) - \exp \left( -\frac{(x - 2b)^2}{4Dt} \right) \right] \]

for \( x < b \) and vanishes otherwise. The survival probability is now

\[ \Omega(t) = \text{erf} \left( \frac{b}{2\sqrt{Dt}} \right), \]

and

\[ \int_{-\infty}^{b} \frac{|x|}{v} p(x, t) dx = \frac{1}{v} \left[ \sqrt{\frac{ADt}{\pi}} \left( 1 - e^{-\frac{a^2}{2b}} \right) \right. \]

\[ + b \left( 1 + \text{erf} \left( \frac{b}{\sqrt{ADt}} \right) - 2\text{erf} \left( \frac{b}{\sqrt{Dt}} \right) \right) \].

For the exponentially distributed resetting times, Eq. (18), and in the case of the \( \delta \)-distribution of the resetting times, Eq. (19), the MFPT attains the following forms,

\[ \tau = \frac{1}{r} \left[ e^{\sqrt{\frac{2b}{r}}} - 1 \right] + \frac{1}{v} \left[ \sqrt{\frac{D}{r}} e^{\sqrt{\frac{2b}{r}}} - b + 2be^{-b\sqrt{\pi}} - \sqrt{\frac{D}{r}} \left( 1 + 2b \sqrt{\frac{r}{D}} e^{-b\sqrt{\pi}} \right) \right], \]  

and

\[ \tau = \frac{2b}{\sqrt{\pi}} \text{erf} \left( \frac{b}{\sqrt{4Dt}} \right) + \frac{4b^2}{\sqrt{\pi} r} \left[ \exp \left( -\frac{b^2}{4D} \right) \sqrt{\frac{4Dt}{b}} - \sqrt{\pi} \text{erfc} \left( \frac{b}{\sqrt{4Dt}} \right) \right] \]

\[ + 1 - \text{erf} \left( \frac{b}{\sqrt{4Dt}} \right) \]

\[ + \left[ \sqrt{\frac{ADt}{\pi}} \left( 1 - \exp \left( -\frac{b^2}{4D} \right) \right) + b \left( 1 + \text{erf} \left( \frac{b}{\sqrt{ADt}} \right) - 2\text{erf} \left( \frac{b}{\sqrt{Dt}} \right) \right) \right], \]  

respectively. The first term in both Eq. (35) and Eq. (36) corresponds to the instantaneous resetting, and the second term gives the correction due to the finite return speed. In the case of Eq. (35) the first term represents the well-known expression for the instantaneous resetting.

The MFPT \( \tau \) for both instantaneous return and for return at a constant speed are shown at Fig. 4 demonstrating good agreement between simulation results and theoretical predictions. The MFPT increases with decreasing of the velocity of the ballistic motion towards the origin. The optimal resetting time \( t_r \) for the delta distribution of the resetting events increases with decreasing of \( v \); the longer the duration of the return period the longer is the time period between subsequent resetting events. Similar effect has been observed for the exponential distribution of the resetting events: the larger is the velocity \( v \) the shorter is the average time between resetting events \( 1/r \), which corresponds to larger values of \( r \).
VII. CONCLUSIONS.

We have considered a model of a resetting process, in which a particle, performing a random motion, returns to its origin not immediately after the resetting event, but, instead, moves towards the origin following a given equation of motion. Once the origin is reached, the particle starts performing the stochastic motion again. In the present work we discuss the specific case of a Brownian motion during the displacement phase, and of the return at a constant speed in one dimension. Two specific resetting protocols were considered, for which analytical results can be obtained in a closed form: the exponential waiting time distribution between the beginning of the stochastic motion and the reset event, an a deterministic resetting after a fixed time since the beginning of the stochastic motion. Interestingly, for the exponential resetting, the return at a constant speed does not change the form of the displacement’s PDF; for the case of deterministic resetting the form changes. The mean first passage times of the process to a given target change considerably for both cases. The reason is that the first hitting cannot take place during the return phase which passes through a region which was already visited in stochastic motion. Therefore, the smaller the velocity of return, the longer is the time necessary to find a target. As a consequence, the optimal time between the two resetting events increases with decreasing return speed.

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