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The heredity and bimeromorphic invariance of the $\partial\bar{\partial}$-lemma property

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Abstract. We give a simple proof of a result on the $\partial\bar{\partial}$-lemma property under a blow-up transformation by Deligne–Griffiths–Morgan–Sullivan's criterion. Here, we use an explicit blow-up formula for Dolbeault cohomology given in our previous work, which can be induced by a morphism expressed on the level of spaces of forms and currents. At last, we discuss the heredity and bimeromorphic invariance of the $\partial\bar{\partial}$-lemma property.

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1. Introduction

In non-Kähler geometry, the heredity and bimeromorphic invariance of the $\partial\bar{\partial}$-lemma property are two interesting problems, extensively studied in [2, 3, 6, 7, 12, 15–17] especially in the recent days. The $\partial\bar{\partial}$-lemma on a compact complex manifold $X$ refers to that for every pure-type $d$-closed form on $X$, the properties of $d$-exactness, $\partial$-exactness, $\bar{\partial}$-exactness and $\partial\bar{\partial}$-exactness are equivalent while a compact complex manifold is called a $\partial\bar{\partial}$-manifold if the $\partial\bar{\partial}$-lemma holds on it.

**Question 1 (Heredity).** Does any closed complex submanifold of an $n$-dimensional $\partial\bar{\partial}$-manifold still satisfy the $\partial\bar{\partial}$-lemma?

**Question 2 (Bimeromorphic invariance).** Does any compact complex manifold being bimeromorphic to an $n$-dimensional $\partial\bar{\partial}$-manifold satisfy the $\partial\bar{\partial}$-lemma?

Clearly, the heredity is true for the $\partial\bar{\partial}$-manifolds of dimensions $\leq 2$. Suppose that $\tilde{X}$ is a modification of a compact complex manifold $X$. A. Parshin [11] and P. Deligne, Ph. Griffiths, J. Morgan, D. Sullivan [6] proved that if $\tilde{X}$ is a $\partial\bar{\partial}$-manifold, then so is $X$. L. Alessandrini [2] posed a question in its inverse direction: if $X$ satisfies the $\partial\bar{\partial}$-lemma, so does $\tilde{X}$? We can easily prove that, Question 2 is equivalent to Alessandrini's one. It is true on complex surfaces by the classical results that each compact complex surface with even first Betti number is Kähler (see [5, 8] for
Moreover, Theorem 4.

For any integer \( k \) such that \( 0 \leq k \leq n \), we weaken Question 1 as

**Question 3 (Hereditity for codimension \( \geq k \)).** Does any closed complex submanifold of codimension \( \geq k \) of an \( n \)-dimensional \( \partial \bar{\partial} \)-manifolds still satisfy the \( \partial \bar{\partial} \)-lemma?

For convenience, Questions 1-3 are denoted by \((H_n), (B_n)\) and \((H_n,k)\), respectively. Obviously, \((H_n) = (H_n,0) \Leftrightarrow (H_n,1)\) and if \( k_1 \leq k_2 \), then \((H_n,k_1) \Rightarrow (H_n,k_2)\).

P. Deligne et al. [6, (5.21)] gave an important result, which related the \( \partial \bar{\partial} \)-lemma property with Hodge filtration and the degeneracy of the Frölicher spectral sequence at \( E_1 \)-page. S. Rao, S. Yang and X.-D. Yang [12, Theorem 1.6] investigated the bimeromorphic invariance of the degeneracy of the Frölicher spectral sequence at \( E_1 \) by their Dolbeault blow-up formula and pointed out that these results are applicable to Question 2 in the remarks after [12, Question 1.2]. Subsequently, their [13, Theorem 1.2] gave an explicit expression of the isomorphism between Dolbeault cohomologies in the blow-up formula to implicitly obtain \((B_n) \Leftrightarrow (H_n,2)\) via Proposition 9 indeed. D. Angella, T. Suwa, N. Tardini and A. Tomassini [3, Theorem 13, Questions 22-24] also studied this equivalence by the Čech–Dolbeault cohomology with additional hypotheses and generalized their results to compact complex orbifolds. In his PhD thesis, by Angella–Tomassini’s characterization [4, Theorems A and B], J. Stelzig [15, Corollary F] claimed that the \( \partial \bar{\partial} \)-lemma property is a bimeromorphic invariant of compact complex manifolds if and only if every submanifold of a \( \partial \bar{\partial} \)-manifold is again a \( \partial \bar{\partial} \)-manifold. Inspired by them, we will prove the following theorem.

**Theorem 4.** For any integer \( k \in \{1,2,\ldots,n\} \), there holds the implication hierarchy

\[
(B_{n+k}) \Rightarrow (H_{n+k,k+1}) \Rightarrow (H_n).
\]

Moreover, \((H_{n,2}) \Rightarrow (B_n)\).

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2. Preliminaries

2.1. A criterion on the \( \partial \bar{\partial} \)-lemma

For a compact complex manifold \( X \), a natural filtration on the complex \( A^\star(X)_{\mathbb{C}} \) of \( \mathbb{C} \)-valued smooth forms on \( X \) is defined as

\[
F^p A^k(X)_{\mathbb{C}} = \bigoplus_{r+s=k} A^{r,s}(X),
\]

for all \( k, p \), which give a spectral sequence \((E_{r}^{p,q}, F^p H^k(X, \mathbb{C}))\), namely, the Frölicher spectral sequence of \( X \). Then \( F_{1}^{p,q} = H_{\partial}^{p,q}(X) \) and

\[
F^p H^k(X, \mathbb{C}) = \{ |\alpha| \in H^k(X, \mathbb{C}) | \alpha \in F^p A^k(X) \text{ and } d\alpha = 0 \}.
\]

Clearly, \( F^p H^k(X, \mathbb{C}) = 0 \) for \( p < 0 \) or \( p > k \). For convenience, we call \( F^* H^k(X, \mathbb{C}) \) the Hodge filtration on \( H^k(X, \mathbb{C}) \). Set \( V^{p,q}(X) = F^p H^k(X, \mathbb{C}) \cap \overline{T}^{q} H^k(X, \mathbb{C}) \) for \( p + q = k \), where \( \overline{T}^{q} H^k(X, \mathbb{C}) \) is
the complex conjugation of the complex subspace \( F^q H^k(X, \mathbb{C}) \) in \( H^k(X, \mathbb{C}) \). We say that \textit{the Hodge filtration gives a Hodge structure of weight} \( k \) \textit{on} \( H^k(X, \mathbb{C}) \), if
\[
H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} V^{p,q}(X),
\]
and
\[
\overline{V}^{p,q}(X) = V^{q,p}(X), \quad \text{for any} \ p+q=k.
\]

P. Deligne, Ph. Griffiths, J. Morgan and D. Sullivan established the well-known criterion on the \( \bar{\partial}\bar{\partial} \)-lemma as follows.

**Theorem 5 (cf. [6, (5.21)]).** For a compact complex manifold \( X \), the following statements are equivalent:

1. \( X \) satisfies the \( \bar{\partial}\bar{\partial} \)-lemma.
2. (a) The Frölicher spectral sequence of \( X \) degenerates at \( E_1 \), and
   
   (b) the Hodge filtration gives a Hodge structure of weight \( k \) on \( H^k(X, \mathbb{C}) \), for every \( k \geq 0 \).

**Remark 6.** For a compact complex manifold \( X \), denote by \( b_k(X), h^{p,q}(X) \) the \( k \)-th Betti, \((p, q)\)-th Hodge numbers respectively.

1. In general, \( b_k(X) \leq \sum_{p+q=k} h^{p,q}(X) \) for all \( k \).
2. The statement of Theorem 5(2a) is equivalent to that \( F^p H^k(X, \mathbb{C})/F^{p+1} H^k(X, \mathbb{C}) \cong H^p_{\bar{\partial}^{k-p}}(X) \) for all \( k, p \), and hence is equivalent to that \( b_k(X) = \sum_{p+q=k} h^{p,q}(X) \) for all \( k \).

We refer to [3, Section 1.5] and [14, Section 2.3] for more discussions on the Frölicher spectral sequence and the Hodge structure.

### 2.2. Some notations

Assume that \( X \) is a complex manifold with complex dimension \( n \). Denote by \( \mathcal{D}^{p,q}(X) \) the space of \((p, q)\)-currents on \( X \), which is defined as the dual of the topological vector space \( A^{n-q,n-q}(X) \) equipped with its natural topology. The operators \( \partial \) and \( \bar{\partial} \) on \( A^{\bullet,\bullet}(X) \) naturally induce two differentials \( \partial \) and \( \bar{\partial} \) on \( \mathcal{D}^{\bullet,\bullet}(X) \). Evidently, \( (A^{\bullet,\bullet}(X), \partial, \bar{\partial}) \) and \( (\mathcal{D}^{\bullet,\bullet}(X), \partial, \bar{\partial}) \) are both double complexes. Denote by \( H^q(\mathcal{D}^{p,\bullet}(X)) \) the \( q \)-th cohomology of the complex \( (\mathcal{D}^{p,\bullet}(X), \bar{\partial}) \). The natural inclusion \( A^{p,\bullet}(X) \hookrightarrow \mathcal{D}^{p,\bullet}(X) \) induces an isomorphism \( \rho_X : H^q_{\bar{\partial}}(X) \cong H^q(\mathcal{D}^{p,\bullet}(X)) \).

Let \( f : X \to Y \) be a proper holomorphic map between complex manifolds. Set \( r = \dim_{\mathbb{C}} X - \dim_{\mathbb{C}} Y \). The pushforward \( f_* : \mathcal{D}^{p,\bullet}(X) \to \mathcal{D}^{p-r,\bullet}(Y) \) of the currents defines a morphism \( f_* : H^q(\mathcal{D}^{p,\bullet}(X)) \to H^{q-r}(\mathcal{D}^{p-r,\bullet}(Y)) \) for any \( p, q \). For convenience, we also denote by \( f_* \) the morphism \( \rho_Y \circ f_* \circ \rho_{\bar{X}}^{-1} : H^q_{\bar{\partial}}(X) \to H^{p-r,q-r}(Y) \).

### 3. The Hodge structures on blow-ups and projective bundles

#### 3.1. Blow-up cases

Let \( \pi : \bar{X} \to X \) be the blow-up of a compact complex manifold \( X \) along a complex submanifold \( Y \) and \( E \) the exceptional divisor. Set \( r = \codim_{\mathbb{C}} Y \geq 2 \) and assume that \( i_E : E \to \bar{X} \) is the inclusion. Let \( t \in \mathcal{A}^{1,1}(E) \) be a Chern form of the universal line bundle \( \mathcal{O}_E(-1) \) on \( E = \mathbb{P}(N_{Y/X}) \). Define a double complex
\[
K^{\bullet,\bullet} = A^{\bullet,\bullet}(X) \oplus \bigoplus_{i=1}^{r-1} A^{r-i,\bullet}(Y).
\]
and a morphism of bounded double complexes
\[
\psi : K^{\bullet,\bullet} \to \mathcal{D}^{\bullet,\bullet}(\bar{X})
\]
as

\[(\alpha, \beta^1, ..., \beta^{r-1}) \mapsto \pi^* \alpha + \sum_{i=1}^{r-1} i_E^* \left( t^{i-1} \wedge (\pi|_E)^* \beta^i \right), \]

where \( \alpha \in A^{*,*}(X) \) and \( \beta^i \in A^{*-i,*,*}(Y) \). By [10, Theorem 1.2], \( \psi \) induces an isomorphism

\[H^*_{\alpha}^*(X) \oplus \bigoplus_{i=1}^{r-1} H^*_{\alpha}^{*-i,*,*}(Y) \xrightarrow{\sim} H^*_{\alpha}^*(\bar{X}), \]

i.e., the isomorphism on \( E_1 \)-pages between the spectral sequences associated to \( K^{*,*} \) and \( \mathcal{D}^{*,*}(\bar{X}) \). Hence \( \psi \) induces an isomorphism \( H^k(Y, \mathbb{C}) \oplus \bigoplus_{i=1}^{r-1} H^{k-2i}(Y, \mathbb{C}) \xrightarrow{\sim} H^k(\bar{X}, \mathbb{C}) \) with the isomorphism on the Hodge filtrations

\[F^* H^k(X, \mathbb{C}) \oplus \bigoplus_{i=1}^{r-1} F^* H^{k-2i}(Y, \mathbb{C}) \xrightarrow{\sim} F^* H^k(\bar{X}, \mathbb{C}) \]

for any \( k \). Moreover, \( \psi \) induces an isomorphism

\[V^{p,q}(X) \oplus \bigoplus_{i=1}^{r-1} V^{p-i,q-i}(Y) \xrightarrow{\sim} V^{p,q}(\bar{X}) \]

for any \( p, q \).

**Lemma 7.** For a given \( k \), the Hodge filtration gives a Hodge structure of weight \( k \) on \( H^k(\bar{X}, \mathbb{C}) \), if and only if, the Hodge filtrations give a Hodge structure of weight \( k \) on \( H^k(X, \mathbb{C}) \) and a Hodge structure of weight \( k - 2i \) on \( H^{k-2i}(Y, \mathbb{C}) \) for all \( 1 \leq i \leq r - 1 \).

By (4), (5) and Remark 6, we easily obtain

**Lemma 8 ([12, Theorem 1.6]).** The Frölicher spectral sequence of \( \bar{X} \) degenerates at \( E_1 \), if and only if, so do those of \( X \) and \( Y \).

Combining Lemmas 7, 8 and Theorem 5, we get

**Proposition 9.** Let \( \bar{X} \) be the blow-up of a compact complex manifold \( X \) along a complex submanifold \( Y \) of complex codimension \( \geq 2 \). Then \( \bar{X} \) satisfies the \( \partial \bar{\partial} \)-lemma, if and only if, \( X \) and \( Y \) do.

**Remark 10.** S. Rao, S. Yang, X.-D. Yang [12, Theorem 1.6] [13, Theorem 1.2] first understood Proposition 9 from the viewpoint of Deligne–Griffiths–Morgan–Sullivan’s criterion for the \( \partial \bar{\partial} \)-lemma and S. Yang, X.-D. Yang [17, Theorem 1.3] studied it from the viewpoint of Angella–Tomassini’s characterization for the case of threefolds. Shortly, D. Angella, T. Suwa, N. Tardini, A. Tomassini [3, Theorem 13] also considered it by use of the Čech–Dolbeault cohomology under some additional assumptions. Eventually, J. Stelzig obtained a blow-up formula for Bott–Chern cohomology and wrote this result out explicitly in [15, Corollary 1.40] [4, Theorems A and B].

**Remark 11.** S. Rao, S. Yang, X.-D. Yang [13, Theorem 1.2] gave an isomorphism for blow-up in the inverse direction of \( \psi \) as

\[\phi : H^*_{\alpha}^*(\bar{X}) \xrightarrow{\sim} H^*_{\alpha}^*(X) \oplus \bigoplus_{i=1}^{r-1} H^*_{\alpha}^{*-i,*,*}(Y), \]

\[\alpha \mapsto (\pi_* \alpha, \beta^1, ..., \beta^{r-1}), \]

where \( i_E^* \alpha = \sum_{i=0}^{r-1} h^i \cup (\pi|_E)^* \beta^i \) for unique \( \beta^i \in H^{*-i,*,*}(Y) \), \( 0 \leq i \leq r - 1 \) and \( h = |t|_\beta \in H^{1,1}(E) \). Actually, \( \phi \) can also be lifted to a morphism between complexes of the spaces of forms and currents, see [9, Lemma 6.5]. Using this morphism, we can also give the relationship between \( V^{p,q}(X) \), \( V^{p,q}(Y) \) and \( V^{p,q}(\bar{X}) \) by above progress.
As we know, the exceptional divisor for the blow-up $\tilde{X}$ of $X$ along $Y$ is biholomorphic to the projective bundle of the normal bundle over $Y$ in $X$. By Proposition 9 and the following Proposition 15, we easily get

**Corollary 12.** Let $\tilde{X}$ be a blow-up of a complex manifold $X$ along a smooth center with the exceptional divisor $E$. Then $\tilde{X}$ is a $\partial \bar{\partial}$-manifold, if and only if, $X$ and $E$ are both $\partial \bar{\partial}$-manifolds.

### 3.2. Projective bundle cases

Let $\pi : \mathbb{P}(E) \to X$ be the projective bundle associated to a holomorphic vector bundle $E$ of rank $r$ over a compact complex manifold $X$. Denote by $t \in \omega^{1,1}(\mathbb{P}(E))$ a Chern form of $O_{\mathbb{P}(E)}(-1)$. Define a morphism

$$
\mu = \sum_{i=0}^{r-1} t^i \wedge \pi^*(\bullet) : \bigoplus_{j=0}^{r-1} A^{i,j-i}(X) \to A^{i,i}(\mathbb{P}(E))
$$

of bounded double complexes. Then $\mu$ induces an isomorphism on $E_1$-pages of the spectral sequences, see [12, Proposition 3.3], [3, Proposition 11] or [10, Corollary 3.2]. With the similar arguments as Section 3.1, we can prove following results

**Lemma 13.** For a given $k$, the Hodge filtration gives a Hodge structure of weight $k$ on $H^k(\mathbb{P}(E), \mathbb{C})$, if and only if, the Hodge filtration gives a Hodge structure of weight $k - 2i$ on $H^{k-2}(X, \mathbb{C})$.

**Lemma 14.** The Frölicher spectral sequence of $\mathbb{P}(E)$ degenerates at $E_1$, if and only if, so does that of $X$.

**Proposition 15.** Let $\mathbb{P}(E)$ be the projective bundle associated to a holomorphic vector bundle $E$ on a compact complex manifold $X$. Then $\mathbb{P}(E)$ is a $\partial \bar{\partial}$-manifold, if and only if, $X$ is a $\partial \bar{\partial}$-manifold.

**Remark 16.** The part of “if” in Proposition 15 was also proved by D. Angella et al. [3, Corollary 12] in a different way.

### 4. A proof of Theorem 4

**Proof.** Here we just prove \((H_{n+k,k+1}) \Rightarrow (H_n)\) and the others are the direct corollary of Proposition 9 and the weak factorization theorem [1, Theorem 0.3.1].

Let $X$ be a $\partial \bar{\partial}$-manifold and $Y$ arbitrary closed complex submanifold of codimension $\geq 1$ in $X$. Note that $X \times \mathbb{C}P^k$ is the projective bundle associated to the trivial bundle $X \times \mathbb{C}^{k+1}$ over $X$ and thus satisfies the $\partial \bar{\partial}$-lemma by Proposition 15. Denote by $\{pt\}$ a set consisting of a single point in $\mathbb{C}P^k$. Then $Y \cong Y \times \{pt\}$ has the codimension $\geq k + 1$ in $X \times \mathbb{C}P^k$ and satisfies the $\partial \bar{\partial}$-lemma by \((H_{n+k,k+1})\).

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