NILPOTENT ELEMENTS OF VERTEX ALGEBRAS

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Abstract. Using the method of commutative algebra, we show that the set \( R \) of nilpotent elements of a vertex algebra \( V \) forms an ideal, and \( V/R \) has no nonzero nilpotent elements.

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1. Introduction

In the development of the axiomatic theory of vertex algebra in \[2\], its analogy with the classical theory of commutative rings was greatly emphasized. In this paper we further explore the analogy between vertex algebras and commutative rings, and the main result is the following.

Theorem 1.1. The set \( R \) of nilpotent elements of a vertex algebra \( V \) forms an ideal, and \( V/R \) has no nonzero nilpotent elements.

Remark 1.2. The first assertion was proved in §3.10 of \[2\]. However, their method was a bit complicated, and the subtleties of formal calculus were quite involved.

This paper is organized as follows. In §2, we give some preliminary results about ideals of vertex algebras. We also introduce the notion of prime ideal, which is crucial for our proof. In §3 we generalize a classical result about commutative ring, of which the main theorem comes as a corollary.

We assume that the readers are familiar with the axiomatic theory of vertex algebra as introduced in §3 and §4 of \[2\]. Throughout this paper \((V, Y, 1)\) will be a fixed vertex algebra.
2. Some Preliminaries

In this section, we present some preliminaries about ideals of vertex algebras.

**Definition 2.1.** An ideal of the vertex algebra \((V,Y,1)\) is a subspace \(I\) such that for any \(v \in V\) and \(w \in I\),
\[ Y(v, x)w \in I((x)), \quad Y(w, x)v \in I((x)). \]

If \(I\) is an ideal, we have \(D I \subset I\). We also have a natural quotient vertex algebra \(V/I\) together with the canonical homomorphism \(V \to V/I\).

**Definition 2.2.** For a subset \(S\) of \(V\), we define \((S)\) to be the smallest ideal containing \(S\) and we call \((S)\) the ideal generated by \(S\). Then \((S)\) is the intersection of all ideals of \(V\) containing \(S\). For \(x_1, \ldots, x_n \in V\), we denote by \((x_1, \ldots, x_n)\) the ideal generated by \(\{x_1, \ldots, x_n\}\). For an \(F(x) \in V[[x, x^{-1}]]\), we also denote by \((F(x))\) the ideal generated by the set of coefficients of \(F(x)\); this notion generalizes in an obvious way to the case of several formal variables and the case of several formal series.

We will use two propositions in §4.5 of \([2]\) many times below; we record them for the reader’s convenience.

**Proposition 2.3.** Let \(u, v, w \in V\), \(p, q \in \mathbb{Z}\), then \(u^p v^q w\) can be expressed as a linear combination of elements of the form \((u_s v)_t w\), with \(s, t \in \mathbb{Z}\).

**Proposition 2.4.** Let \(u, v, w \in V\), \(p, q \in \mathbb{Z}\), then \(u^p v^q w\) can be expressed as a linear combination of elements of the form \(v^s u^t w\), with \(s, t \in \mathbb{Z}\).

Combining proposition 2.3 with the \(D\)-bracket formulas for vertex algebras immediately implies:

**Corollary 2.5.** Let \(S\) be a subset of \(V\). Then
\[
(S) = \text{spac}\{v_n D^i(u)|v \in V, n \in \mathbb{Z}, i \geq 0, u \in S\}
= \text{spac}\{D^i(v_n u)|v \in V, n \in \mathbb{Z}, i \geq 0, u \in S\}
= \text{spac}\{D^i(v_n D^j(u))|v \in V, n \in \mathbb{Z}, i, j \geq 0, u \in S\}.
\]

Motivated by this corollary we introduce the following notation. Let \(\mathfrak{p}, \mathfrak{q}\) be two subsets of \(V\), set
\[
\mathfrak{p}\mathfrak{q} = \text{spac}\{v_n D^i(u)|v \in \mathfrak{p}, n \in \mathbb{Z}, i \geq 0, u \in \mathfrak{p}\}
= \text{spac}\{D^i(v_n u)|v \in \mathfrak{p}, n \in \mathbb{Z}, i \geq 0, u \in \mathfrak{p}\}.
\]
\[ \mathrm{spac}\{D^i(v_nD^j(u))|v \in \mathfrak{p}, n \in \mathbb{Z}, i, j \geq 0, u \in \mathfrak{p}\}. \]

The skew symmetry of vertex algebras immediately implies \( \mathfrak{p}\mathfrak{q} = \mathfrak{q}\mathfrak{p} \).

**Theorem 2.6.** For \( u_1, \ldots, u_m, v_1, \ldots, v_n \in V \), we have
\[ (u_1, \ldots, u_m)(v_1, \ldots, v_n) = (\{Y(u_i, x)v_j\}_{i,j}). \]

**Proof.** It suffices to prove that \( (u)(v) = (Y(u, x)v) \) for any \( u, v \in V \). The inclusion \( \supset \) follows directly from the definitions. For the reverse inclusion, we have
\[ (u)(v) = \text{span}\{D^i(w_lu)_{m}w_n'v|l, m, n \in \mathbb{Z}, i \geq 0, w, w' \in V\} \]
\[ \subseteq \text{span}\{D^i(w_lw_mu)_{m}v|l, m, n \in \mathbb{Z}, i \geq 0, w, w' \in V\} \]
\[ \subseteq \text{span}\{D^iw_lv_nw_mu|l, m, n \in \mathbb{Z}, i \geq 0, w, w' \in V\} \]
\[ \subseteq \text{span}\{D^iw_lv_nw_m'v|l, m, n \in \mathbb{Z}, i \geq 0, w, w' \in V\} \]
\[ = (Y(v, x)u) = (Y(u, x)v), \]
where the first and the third \( \subseteq \) follow from Proposition 2.4, the second \( \subset \) and the last \( = \) follow from the property of skew symmetry of vertex algebras, and the second \( = \) follows from Proposition 2.3. \( \square \)

**Definition 2.7.** An ideal \( I \) of \( V \) is prime if \( I \neq V \) and if \( (Y(u, x)v) \subset I \) implies \( u \in I \) or \( v \in I \).

We have, as in the classical case, the following proposition; the proof is routine.

**Proposition 2.8.** Let \( f : V \to V' \) be a surjective homomorphism of vertex algebras, and let \( I \) be an ideal of \( V' \). Then \( f^{-1}(I) \) is a prime ideal of \( V \) if and only if \( I \) is a prime ideal of \( V' \).

3. The Nilpotent Elements

**Definition 3.1.** An element \( v \) of \( V \) is said to be nilpotent if there exists a positive integer \( r \) such that
\[ Y(v, x_1) \cdots Y(v, x_r) = 0. \]

**Theorem 3.2.** The set \( \mathfrak{N} \) of nilpotent elements \( V \) is the intersection of all the prime ideals of \( V \).

This generalizes the corresponding result about commutative rings [?]; the proof is exactly the same.
Proof. Denote by \( \mathcal{R}' \) the intersection of all the prime ideals of \( V \). We want to show that \( \mathcal{R} = \mathcal{R}' \). The inclusion \( \subseteq \) follows immediately from the definition of prime ideal.

In order to prove the reverse inclusion, suppose that \( v \in V \) is not nilpotent. Then using Proposition 2.3 repeatedly implies
\[
Y(v, x_1) \cdots Y(v, x_{n-1})v \neq 0
\]
for each \( n > 0 \). Let \( \Gamma \) be the set of proper ideals \( \mathfrak{m} \) satisfying
\[
(Y(v, x_1) \cdots Y(v, x_{n-1})v) \not\subseteq \mathfrak{m}
\]
for each \( n > 0 \). Order \( \Gamma \) by inclusion, then Zorn’s lemma implies \( \Gamma \) has a maximal element; let \( \mathfrak{w} \) be a maximal element of \( \Gamma \). We shall show that \( \mathfrak{w} \) is a prime ideal. Suppose \( a, b \notin \mathfrak{w} \). Then the ideals \( \mathfrak{w} + (a) \), \( \mathfrak{w} + (b) \) properly contain \( \mathfrak{w} \) and therefore do not lie in \( \Gamma \). Thus there exist some \( m, n > 0 \) such that
\[
(Y(v, x_1) \cdots Y(v, x_{m-1})v) \subseteq \mathfrak{w} + (a),
\]
\[
(Y(v, x_1) \cdots Y(v, x_{n-1})v) \subseteq \mathfrak{w} + (b).
\]
Now repeatedly using Proposition 2.3 yields
\[
(Y(v, x_1) \cdots Y(v, x_{m+n-1})v) \subseteq (\mathfrak{w} + (a))(\mathfrak{w} + (b)) = \mathfrak{w} + (Y(a, x)b),
\]
therefore we have \( (Y(a, x)b) \not\subseteq \mathfrak{w} \). Hence we find a prime ideal \( \mathfrak{w} \) that does not contain \( v \). This finishes the proof. \( \square \)

Theorem 3.2 immediately implies that the set \( \mathcal{R} \) of nilpotent elements \( V \) forms an ideal. Hence we have the quotient homomorphism \( f : V \to V/\mathcal{R} \).
Now combining Proposition 2.8 with Theorem 3.2 implies that \( V/\mathcal{R} \) has no nonzero nilpotent elements. This finishes the proof of Theorem 1.1.

References

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