Turbulence-induced collisional velocities and density enhancements: large inertial range results from shell models

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\begin{abstract}
To understand the earliest stages of planet formation, it is crucial to be able to predict the rate and the outcome of dust grains collisions, be it sticking and growth, bouncing or fragmentation. The outcome of such collisions depends on the collision speed, so we need a solid understanding of the rate and velocity distribution of turbulence-induced dust grain collisions. The rate of the collisions depends on both the speed of the collisions and the degree of clustering experienced by the dust grains, which is a known outcome of turbulence. We evolve the motion of dust grains in simulated turbulence, an approach that allows a large turbulent inertial range making it possible to investigate the effect of turbulence on meso-scale grains (millimetre and centimetre). We find three populations of dust grains: one highly clustered, cold and collisionless; one warm; and the third ‘hot’. Our results can be fitted by a simple formula, and predict both significantly slower typical collisional velocities for a given turbulent strength than previously considered, and modest effective clustering of the collisional populations, easing difficulties associated with bouncing and fragmentation barriers to dust grain growth. Nonetheless, the rate of high-velocity collisions falls off merely exponentially with relative velocity so some mid- or high-velocity collisions will still occur, promising some fragmentation.

\textbf{Key words:} turbulence – planets and satellites: formation – protoplanetary discs.
\end{abstract}

\section{Introduction}

Collisions between dust grains in protoplanetary discs are a key process in planetesimal formation, and so have seen a significant amount of study, both theoretical (Völk et al. 1980; Markiewicz, Mizuno \& Völk 1991; Cuzzi \& Hogan 2003; Youdin \& Goodman 2005; Ormel \& Cuzzi 2007) and numerical (Dullemond \& Dominik 2005; Johansen et al. 2007). Moreover, the results of collisions between artificial dust grains as a function of relative velocity can be directly observed in laboratory experiments (Blum \& Wurm 2008; Güttler et al. 2010). These collisions can lead to sticking and growth of dust grains, or to fragmentation, repopulating the smallest sizes of dust grains. On the other hand, laboratory experiments suggest that the null result of a collision, bouncing, poses a significant threat to dust growth beyond the centimetre scale (Zsom et al. 2010).

The gas component of protoplanetary discs is believed to be turbulent, and the effect of this turbulence on the collisions between dust particles entrained in the flow can be reasonably estimated analytically, for example as done in the line of work starting with Völk et al. (1980) and more recently elaborated on by Ormel \& Cuzzi (2007). Such analyses give the unsurprising result that dust collisional velocities are comparable to the turbulent velocities of the gas on scales set by the properties of the dust grains. These estimated velocities are, however, expected to be large. The turbulence in protoplanetary discs is often invoked to explain the disc’s accretion on the mega-year time-scales observed (Shakura \& Sunyaev 1973; Russell et al. 2006), and the turbulent motions required to achieve such a feat are expected to drive dust collisions that destroy the participants or merely result in bouncing (Wurm, Blum \& Colwell 2001; Güttler et al. 2010; Wettlaufer 2010). The existence of the bouncing and fragmentation velocity cut-offs means that it is important to determine not merely the rate of collisions, but also the collisional probability distribution as a function of the relative velocity, because outlying events could cross the bouncing barrier (Zsom et al. 2010).

Another important behaviour of particles in turbulence is ‘preferential concentration’ (Maxey 1987; Fessler, Kulick \& Eaton 1994; Cuzzi et al. 2001; Toschi \& Bodenschatz 2009), where inertial particles are ejected from regions of high vorticity due to centrifugal forces, and accumulate in regions of high strain. This has been hypothesized to increase dust collision rates by creating local dust density fluctuations (Pan et al. 2011), as well as possibly leading to dust drag on the turbulence. The latter can cause a streaming instability that further enhances the local dust density (Youdin \& Goodman 2005; Johansen et al. 2007). Beyond the effect on dust...
grain collisions rates, such density enhancements can provide a rapid route to gravitational instabilities and the resulting collapse into planetesimals (Goldreich & Ward 1973; Johansen, Klahr & Henning 2006). This ejection of inertial particles from turbulent (intermittent) vortices, which will be discussed in this paper, should be distinguished from the dust-trapping property of anti-cyclonic long-lived vortices, which are large enough to feel Coriolis forces (Barge & Sommeria 1995; Johansen, Andersen & Brandenburg 2004).

Moreover, the inertial range of the turbulence in these systems is expected to be quite broad, easily five to six orders of magnitude in $k$-space or four orders of magnitude in turbulent turnover time, as their fluid Reynolds numbers are large. This means that there will be dust grains well embedded in the turbulent cascade, too small to notice that they are trapped within huge eddies, yet too large to be meaningfully affected by the smallest scale turbulence. These well-embedded particles are precisely the most important ones, since they include the millimetre–centimetre sized dust grains that populate the fragmentation and bouncing barriers. A resolution of the order of several thousand cubed would be required to fully simulate a system evolving the full Navier–Stokes equations with an upper double digit inertial range, beyond currently available resources. Further, as we will show, turbulent clustering implies that very small particle separations must be considered to extract useful data. The resolution required for performing a full hydrodynamical simulation is, again, of the order of several thousand cubed (Section 4.1).

Carballido, Cuzzi & Hogan (2010) explicitly run into the above-mentioned problems of limited inertial range and low spatial resolution. We resolve these problems by putting particles into a set of turbulent cascade models known as shell models (Bohr et al. 1998, chapter 3), which allows us to determine the clustering and collisional velocity probability distributions with that can be expected for large inertial ranges (up to 256) while resolving small particle separations. This approach of using synthetic turbulence was also used by Bec et al. (2005), although the details of the projection into real space differ. Our work also differs by our focus on the collisional velocities as a key diagnostic, and we find noticeably lower collisional velocities than those estimated analytically. Furthermore, unlike those works, our results are not well approximated by a single effective collisional velocity, but require consideration of a velocity probability distribution. While our clustering results are similar to those of Pan et al. (2011), simultaneously considering the collisional velocities and the streaming instability, but does allow us to examine dust pairs at a very small relative separation.

2 NUMERICAL SETUP

We use the PENCIL CODE\textsuperscript{1} to track the motion of particles in synthetic turbulent cascades. Our synthetic turbulence derives from shell models: we consider spatial Fourier transforms of the turbulent velocity field (from $x$ to $k$), keeping the modes that fit in a limited range of $|k|$ which defines our considered sub-section of the inertial range. We then coarse-grain the sphere of $k$ vectors into logarithmically spaced shells in $k$-space. As such, we consider only the energy spectrum of the turbulence. We will use four different models. Three follow an exact Kolmogorov power spectrum (Kolmogorov 1941, see Section 2.1) and differ only in the implementation of their time variability, if any. The fourth model, discussed in Section 2.1.1, combines the spatial approach of Section 2.1 with an energy spectrum derived from a Gledzer, Ohkitani and Yamada (GOY) model, which solves a set of equations for the turbulent energy cascade. However, it should be noted that the magnitude of the velocity field in a GOY model is strongly fluctuating in time, making the choice of dimensionless quantities less obvious.

A first study with a time-independent energy spectrum is needed to establish the form of turbulent collisional velocities and is closer to existing analytical work.

Since we do not evolve the Navier–Stokes equations to determine the gas motion, we do not need a grid or a sub-grid model: this allows for greater resolution. We evolve $4 \times 10^6$ particles in a periodic box of size $(2\pi)^3$. Our setup does not allow us for the study of dust back-reaction on turbulence (preventing, for example, the streaming instability), but does allow us to examine dust pairs at a very small relative separation.

2.1 Flow field

Each coarse-grained shell is associated with a fluid velocity and turnover time and indexed with $m$ which runs from $m_{\text{min}}$ to $m_{\text{max}}$ with

$$k_m = 2^m, \quad (1)$$

$$\tau_m = 1/v_m k_m. \quad (2)$$

In a Kolmogorov spectrum, the velocity obeys the power wave

$$v_m = v_0 k_m^{-1/3}, \quad (3)$$

this generates the dashed/red velocity spectrum in Fig. 1 by construction. Our choice of $m_{\text{min}}$ and $m_{\text{max}}$ is limited by available numerical resources and by the constraints of the periodicity of the numerical box. We will vary them for different simulations as the choice determines the effective inertial range of our turbulence simulate. This allows us to find the effective inertial range large enough that the particle response becomes scale free (Appendix A).

While equation (3) defines the amplitude of the velocity on spatial scales $k^{-1}$, we need an actual velocity field to evolve our particle positions. To obtain it, we associate each shell $m$ with three ($k_{\text{min}}, \hat{v}_{\text{min}}$) vector pairs indexed by $n$. These vector pairs characterize contributions to the gas velocity directed along $\hat{v}_{\text{min}}$, and varying along $k_{\text{min}}$ (for shell $m$, $|k| = k_m$). We impose $k_{\text{min}} \perp \hat{v}_{\text{min}}$, so that the flow is incompressible. Further, the $k_{\text{max}}$ and $\hat{v}_{\text{max}}$ vectors for pair $i$ of shell $m$ are approximately perpendicular to their counterparts in the other two pairs of that shell. This formulation (three quasi-perpendicular vectors for both $\hat{v}$ and for each shell in $k$-space) allows us to span both $k$-space and velocity space in each shell separately while simultaneously avoiding strongly preferred directions. The projection of the velocity $v_m$ associated with shell $m$ on to the three vector pairs is done through the introduction of a unit vector $a_m$. The three components of $a_m$ define the relative excitation of the three possible vector pairs.
The GOY model (Ohkitani & Yamada 1989) solves an evolution equation for the energy in differing shells under the assumption that only shells adjacent in $k$-space interact (to generate a cascade rather than non-local effects): shells $n$, $n + 1$ and $n + 2$ can interact. We note that we solve the following system for a larger range of shells than are included in the synthetic turbulence that is applied to the particles: the choice $m = 0$ ($k = 1$) is associated with $n = 4$ in what follows. We follow $N = 22$ shells, using a slaved Adams–Bashforth scheme (Pisarenko et al. 1993; Mitra & Pandit 2004). Under this assumption we can write

$$\frac{dv_n}{dt} = \left( a_n v_{n+1} v_{n+2} + b_n v_{n-1} v_{n+1} + c_n v_{n-1} v_{n+2}\right)^3 + f_n,$$

where $v_n$ is the complex velocity associated with shell $n$, $v$ is a viscosity and $f_n = 0$ for all $n \neq 1$ is the forcing term. The choice of constraints of energy and helicity conservation leads to

$$a_n = k_n, \quad b_n = -k_{n-1}/2, \quad c_n = -k_{n-2}/2,$$

with boundary conditions on the largest and smallest pairs of shells ($n = 0, 1, N - 1, N$) of

$$b_1 = b_N = c_1 = c_N = a_{N-1} = a_0 = 0.$$

The GOY shell model has a 3-cycle with shell index (Kadanoff et al. 1995), which we filter using

$$|v_n|^{3/2} = |\text{Im}(v_{n+2}v_{n+1}v_n - v_{n-1}v_nv_{n+1}/4)|^{1/3}.$$n

This model inherently provides time variation for the phase through the complex nature of $u_n$. We combine this with equation (4) by replacing $v_n$ with $|v_n|$, setting $\phi_{mn} = \text{arg}(v_m)$ and allowing $a_{mn}$ to vary as previously described.

In Fig. 1 we show the velocity spectrum and a sample shell velocity time series. We define a characteristic shell velocity $\pi_n$ through the time average of $|u_n|$ and $\tau_n = 1/|\pi_n|k_n$, but this is not as perfectly defined as in the fully imposed Kolmogorov case. The GOY model velocity spectrum is also slightly steeper than a Kolmogorov one, a consequence of its intermittency.

### 2.2 Particles

The motion of inertial particles in a fluid is determined by the friction between the particles and the fluid. The resulting drag force entrains the particles along the fluid motion. Particles with a finite friction time $\tau_p$ (subscript $p$ referring to particles) are referred to as ‘inertial’ and their motion deviates from that of the gas as long as the particles are not neutrally buoyant (i.e. as long as $\rho_{\text{particle}} \neq \rho_{\text{gas}}$).

In this work we assume that $\rho_{\text{particle}} \gg \rho_{\text{gas}}$ which is extremely well satisfied for protoplanetary discs. This allows us to neglect pressure forces on the particles. We initialize our particles with a frictional stopping time $\tau_p$, so that the equation for a particle’s velocity $u_p$ is determined by

$$\frac{du_p}{dt} = -u_p(t) - V(x_p,t),$$

where $V(x_p,t)$ is the gas velocity at the particle’s position $x_p$. It is this deviation of the particle velocity from that of the gas that allows particles to collide even in incompressible flows.

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**Figure 1.** Velocity spectrum and time series for the GOY model. In the upper panel, we see that the power law of the GOY model is steeper than a Kolmogorov $-1/3$ law, closer to $-0.4$. In the lower panel, we show the velocity time series for the $k = 8$ shell, with time normalized to that shell’s own turnover time.
When we evolve our particles in the velocity field given by equation (4), we find a particle velocity dispersion

\[
\langle \langle \mathbf{u}_p^2 \rangle \rangle^{1/2} \simeq \frac{v_{\text{max}}}{(1 + \tau_p/\tau_h)^{1/2}},
\]

which matches the results expected since Völk et al. (1980). Further, in Fig. 2 we show log-scaled histograms of the total distance travelled in box length units (box side = 2\pi, bin size = 0.5) for a population of 1000 particles in our base setup (see Section 3). The time axis is scaled to the turnover time of the largest scale turbulence (time between samples 0.2\tau_h). We overplot 0.67\kappa^{-1/3}(t/\tau_h)^{1/2}, which has the form and approximate scale of a random walk on the length- and time-scales of the largest scale turbulence. We can see that the particles’ displacements are well fitted by the random walk expected to be generated by the largest scale turbulent motion.

In this work we consider only particles with identical stopping time \( \tau_p \) set to the turnover time of the shell \( m_p \) with \( m_{\text{min}} < m_p < m_{\text{max}} \). We will therefore refer to \( k_{\text{mn}} \) and \( v_{\text{mn}} \) as \( k_p \) and \( v_p \). In the limit of an infinite inertial range, this scale is the only scale, and so most of our values will be non-dimensionalized with respect to it. However, we can investigate the dependence of the collisional velocities on both smaller and larger eddies by turning on and off eddies of differing scales, i.e. by changing \( m_{\text{min}} \) and \( m_{\text{max}} \).

We are primarily interested in dust collisions in protoplanetary discs, which allow some simplifications. The large size of the Kolmogorov scale turbulence in protoplanetary discs (at a minimum kilometre scale), along with the small size of the dust grains (quite sub-metre for grains embedded well within the inertial range), means that when considering the collisions of protoplanetary dust grains we must treat our dust grains as point particles, taking limits as the particle separation tends to zero. This is a distinguishing feature compared to atmospheric turbulence, whose Kolmogorov scale might be 1 mm (Shaw 2003; Xu & Bodenschatz 2008), comparable to rain drops or particulates. Finally, the actual collision rate of dust grains in protoplanetary discs is expected to be modest due to their dust grain number density: grains will experience multiple friction times between collisions. Accordingly, given our focus on protoplanetary discs we do not treat collisional scattering of the grains.

It is traditional to non-dimensionalize the particles’ stopping time to the Stokes number by scaling it to the turbulent turnover time at the dissipation scale (or occasionally to the time at the driving scale). Unfortunately neither definition fits our setup well, but one can define \( St \sim (k_{\text{ss}}/k_0)^{2/3} \) as a conceptual equivalent to scaling \( \tau_p \) to the dissipation scale since \( k_{\text{ss}} \) is the smallest scale included in our simulation. Similarly, \( St \sim (k_{\text{ss}}/k_0)^{2/3} \) is a conceptual equivalent to scaling \( \tau_p \) to the turbulent time at the driving scale. However, in Appendix A we explain why we do not consider either approach to be conceptually important as we explicitly hope to achieve large enough inertial ranges that the results are independent of \( St \) (either definition). We do define our effective Stokes number according to the first definition above, \( St' \equiv (k_{\text{ss}}/k_0)^{2/3} \), for use when we need to scale diagnostics to the smallest included scale. While the equivalence of this effective Stokes number with the Stokes number in a system with a dissipation scale is not perfect, this value should be used when comparing with most other work (e.g. Bec et al. 2010a; Pan et al. 2011; but not Ormel & Cuzzi 2007, where the Stokes number is scaled to the largest scale turbulence).

### 2.3 Run details

In Table 1 we collect the details of the simulations we perform. The total inertial range is given by \( 2^{m_{\text{max}}-m_{\text{min}}} \), the effective Stokes number by \( St' = 2^{3/m_{\text{max}}-m_{\text{min}}} \) and the range between the largest scale turbulence and the particles is given by \( 2^{m_{\text{ss}}-m_{\text{max}}} \). In all cases the time-step used to advance the particles was 0.1\tau_h. For runs using the GOY model, the GOY model is advanced for \( 3 \times 10^7 \) time-steps before the introduction of particles, to allow the turbulence to find its statistical steady state. The GOY equations are run with their own time-step of approximately \( 10^{-3} \) in code units (tweaked to fit an integer number of times within a particle’s time-step); this shorter time-step is for the numerical stability of the algorithm. The particles are initialized with \( u = 0 \) at random positions.

We generate the \( k_{\text{run}} \) and \( v_{\text{run}} \) vectors of equation (4) by randomly selecting \( k_{\text{mn}} \) from the set of vectors with approximately appropriate length (the error decreases as the target \( |k| \) increases) that fit in the periodic box. The vector \( v_{\text{m1}} \) is generated as a random unit vector

| Run | \( m_{\text{min}} \) | \( m_p \) | \( m_{\text{max}} \) | Notes |
|-----|----------------|---------|----------------|------|
| B-A | 0 | 4 | 6 | Smallest \( \tau_p/\tau_h \) |
| B-B | 0 | 3 | 5 | |
| B-LI2 | 0 | 3 | 7 | Largest inertial range \( k_{\text{ss}}/k_0 = 128 \) |
| B-C | 1 | 3 | 5 | |
| B-LI1 | 1 | 3 | 7 | Largest inertial range \( k_{\text{ss}}/k_1 = 64 \) |
| Base | 2 | 3 | 5 | Subject of Section 3 |
| B-D | 2 | 3 | 6 | |
| B-E | 2 | 3 | 7 | |
| B-F | 2 | 3 | 8 | Largest \( St' = 10 \) |
| Q-A | 2 | 3 | 5 | Quenched spatial projection (Section 2.1) |
| Q-LI | 0 | 3 | 7 | Largest inertial range \( k_{\text{ss}}/k_0 = 128 \) |
| P-A | 2 | 3 | 5 | Varying phase (Section 2.1) |
| P-LI | 0 | 3 | 7 | Largest inertial range \( k_{\text{ss}}/k_0 = 128 \) |
| G-A | 1 | 3 | 6 | GOY turbulence (Section 2.1.1) |
| G-B | 2 | 3 | 5 | |
| G-C | 2 | 3 | 6 | |
| G-LI | 0 | 3 | 7 | Largest inertial range \( k_{\text{ss}}/k_0 = 128 \) |
perpendicular to \( k_{\perp} \). The vector \( k_{\perp} \) is then selected from the set of vectors of appropriate length that fit in the box and lie within 30° of \( v_{\perp} \). Next we choose \( v_{\perp} = k_{\perp} \times k_{\parallel} / |k_{\perp} \times k_{\parallel}| \), and repeat the process for \( k_{\parallel} \) and \( v_{\parallel} \). If no vector \( k_{\perp} \) or \( k_{\parallel} \) can be found that satisfies the constraints, the process is restarted.

3 ANALYSIS: BEHAVIOUR

We begin by making a full analysis of the run ‘Base’ (Table 1, smallest inertial range) to extract the behaviour of the system, both with respect to clustering and collision speeds, and to find a fit formula that can be applied in simulations of particle size evolution. In Section 4 we will study how both increasing the inertial range and changing our turbulence model affect the behaviour.

We will obtain our particle clustering and collisional velocity data by analysing during run time snapshots taken every full turbulent turnover \( \tau_0 \) for the largest eddy in the system; these snapshots include every particle’s position and velocity. The particle positions are mapped on to a coarse grid, and every particle pair within a critical separation (at most 0.2 of the grid scale) is found. We bin our particle pairs simultaneously linearly in separation \( R \), using either 50 or 100 bins depending on the run, and linearly in relative velocity \( u \). We consider full spheres, so for every \( R \) we consider every particle pair with \( |x_1 - x_2| < R \). A bin listed with velocity \( u_{\text{bin}} \) contains particle pairs with relative velocity \( u_{\text{bin}} - \Delta u < u < u_{\text{bin}} \), where \( \Delta u = u_{\text{bin}/\text{max}} \), \( u_{\text{max}} \) is the largest considered particle-pair relative velocity and \( n_{\text{vel}} = 2000 \) is the number of velocity bins. For run Base, \( \Delta u = 4 \times 10^{-4} \) in code units. Particle pairs with relative velocities larger than \( u_{\text{bin}} \) are included in that bin, so it is discarded. We will refer to the total number of pairs in bin \( (R, u) \) of snapshot \( t \) as \( N(R, u, t) \), with dropped \( u \) implying summation over all velocity bins and dropped \( t \) implying averaging over snapshots.

We took 529 snapshots from run Base, but as discussed below, for most of the analysis we drop the first 20; accordingly, \( N(R, u) \) and \( N(R) \) are averaged over 509 snapshots.

3.1 Clustering

In the top two panels of Fig. 3 we show \( N(R, t) \) as a function of time for a representative maximal separation \( R \). We further define the clustering \( C(R) \) as the ratio of \( N(R) \) to the expected number of particle pairs

\[
\overline{N}(R) = \frac{1}{2} n_p \pi \left( \frac{4 \pi R^3}{3} n_p \right),
\]

\[
C(R) = \frac{N(R)}{\overline{N}(R)},
\]

assuming a spatially homogeneous particle distribution, where \( n_p \) is the number of particles and \( V \) is the volume of the box. A value of \( C(R) > 1 \) implies that particles have been concentrated on length-scales \( R \), while \( C(R) < 1 \) implies some form of segregation or effective repelling. The clustering \( C(R) \) is the same as the \( g(5t, r) \) of Pan et al. (2011), modulo the differences between \( S' \) and their Stokes number. Their length-scale \( \eta \) might be understood as our \( k_{\perp}^{-1} \), although their turbulence at that scale is explicitly affected by dissipation, and so the energy is no longer following a turbulent cascade.

The particles are initialized with zero velocity and random initial position, and in the top and middle panels of Fig. 3 we see that the number of particle pairs takes more than 10 largest scale turnovers to stabilize (more than 16 particle friction times), and still shows significant fluctuations. This long stabilization time is an important observation which helps explain results later in this paper: there is significant structure in the particle distribution that takes time to develop, and studies of particle collisions need to consider a significant time interval. Unfortunately, including a history of 10 full largest scale turnovers is beyond current or foreseeable analytical analyses. Clearly particle positions are quite correlated with one another. In the analysis that follows, we discard the first 20 snapshots to allow the system to stabilize.

In the bottom panel of Fig. 3 we show \( C(R) \) which shows a clear power-law dependence on the maximal separation \( R \). We will denote by \( \mu \) the power-law \( C(R) \propto R^{-\mu} \), and by \( \mu_t \) the intercept where the power-law fit gives 1.

3.2 Average collision energy

A first calculation of the rms collisional velocity is shown in Fig. 4, where we plot

\[
\overline{\pi}_c(R) = \left( \frac{\sum \pi_c N(R, u) u \left( R_{\text{max}}^2 \right)}{\sum N(R, u) u} \right)^{1/2}.
\]

Figure 3. Particle pairs \( N(R, t) \) (unnormalized count) as a function of time for run Base. Top panel: counts for a collisional sphere \( R_{k_p} = 0.02 \). Middle panel: early time blow-up of the top panel. Bottom panel: clustering \( C(R) \), with a power-law fit in red/dashed.

Figure 4. Average collisional speed between particles as a function of the maximal separation for run Base. Note the small scale of the horizontal axis.
The factor of \( u \) in the denominator is needed to convert from particle pairs to particle collision rates: because we are taking well-separated snapshots we need to weight high-velocity pairs more heavily. A naive expectation would be that the collisional velocity decreases with decreasing \( R \), reaching a finite plateau at small separation, such as seen in \( \text{Bec et al.} \ (2010b) \). This finite plateau is made possible by the compressible nature of the particle ‘fluid’, even if the turbulent gas itself is incompressible. In Fig. 4 we are already in this plateau, but see an increasing collisional velocity with decreasing \( R \) (except for the smallest \( R \), where our pair counts are too low for reliability).

This perhaps surprising result does not flow from poor statistics, but rather is a robust and understandable consequence of the clustering seen in the bottom panel of Fig. 3. The clustering seen there increases strongly with decreasing \( R \), implying a relatively large population of highly correlated particles. As \( R \) decreases, the relative particle–particle velocities in this highly correlated population must decrease for the pairs to stay within the cluster radius: the smaller the \( R \), the lower the relative velocities of the correlated population must be. We might guess that the characteristic relative velocity of this population is linear in \( R \), giving a constant crossing time. Since the power-law fit in Fig. 3 is weaker than \( R^{-1} \), the contribution of this population to the denominator of equation (13) decreases with decreasing \( R \). The velocity plateau at larger separations implies that the highly correlated population has already ceased to contribute meaningfully to the numerator due to the extra factor of \( u^2 \). Even at the largest separations \( R \) we consider in Fig. 4, the relative velocity of the highly correlated population is low enough that this population contributes only negligibly to the total collisional power. In effect, the highly correlated population is diluting the collisional energy when an average is taken, and this dilution is decreased by considering smaller \( R \).

Since we are considering point particles, we must consider the limit \( R \to 0 \), and in that limit, the highly correlated population appears to play a dominant role in the pair counts (because \( \mu > 0 \)), but contributes no collisions (since \( \mu < 1 \)). We accordingly define this collisionless population as ‘cold’ (low relative velocities). Unfortunately, numerical resource constraints prevent us from considering values of \( R \) small enough that the cold population has ceased to contribute meaningfully to straightforward calculations of the collision rate (the denominator of equation (13)). Note that this is different from the expectations of, for example, \( \text{Pan et al.} \ (2011) \), where turbulent clustering was hypothesized to lead to more common collisions by increasing the number of nearby potential collisional partners. As we will see, however, there is a significant chance that some level of clustering will persist, albeit not one that scales inversely with \( R \).

The implications of this interpretation of Fig. 4 are seen in Fig. 5, where

\[
N'(R, u) \equiv \frac{N(R, u)}{\Delta u}
\]  

(14)

is the density of pairs within separation \( R \) in velocity space (to convert between binned data and a smooth distribution). The probability distribution is seen to be strongly affected by a low-velocity peak which moves to smaller velocity linearly with \( R \), to a limit of 0 as \( R \to 0 \) (compare with \( \text{Carballido et al.} \ 2010, \text{fig. 5} \), which is lin-log instead of log-log). We define

\[
\omega \equiv u_{\text{cold}} / R
\]  

(15)

as the measure of the dependence of the velocity of the cold population on separation, where \( u_{\text{cold}} \) is the modal pair velocity.

This collisionless behaviour of the cold population does not mean that we expect no collisions. As can be seen in the top panel of Fig. 5, \( N(R, u) \) has a significant high-velocity tail, which can contribute ‘real’ collisions. A process to excise the cold, collisionless population and fit the remainder is discussed below.

### 3.3 Fitting

We have failed to successfully fit the cold, clustered population with a tractable distribution such as a Weibull distribution; however, the tail of the pair-probability distribution in velocity space is well fitted by a simple exponential, and at more modest velocities the addition of a second exponential improves matters. Accordingly, we fit the particle-pair-probability distribution by

\[
N'(R, u > u_c) / N(R) = \frac{C_a}{u_a} e^{-u/u_a} + \frac{C_b}{u_b} e^{-u/u_b},
\]  

(16)

\[
u_c = 0.1v_p
\]  

(17)

but only for \( u > u_c \), where \( u_c \) is a cut-off designed to exclude the cold population. The fit formula (16) identifies two separate populations \( a \) and \( b \) with \( u_a < u_b \). While we do not have an analytical theory for the form of equation (16), the quality of the high-velocity component \( b \) can be seen from the overlap between the blue/dash-dotted curves (the higher velocity component) and the black/solid curve (data) in Fig. 6. The fit for component \( a \) is less well constrained, but it gives a noticeable difference at lower relative velocities, seen in the difference between the blue/dash–dotted curve (only component \( b \)) and the red/solid curve (both components, overlays the data outside of \( u_c \)). We refer to component \( a \), handling the lower relative velocities, as the ‘warm’ population, and the fit component \( b \), handling the higher relative velocities, as the ‘hot’ tail.

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low-velocity (warm) and high-velocity (hot) regions more heavily, allowing the two populations to be distinguished by the fitting algorithm. The values of $N'(R, u)$ in Fig. 6 have a minimum $\sim 10$. Averaging over 509 snapshots with $\Delta u = 4 \times 10^{-4}$ leads to $\sim 2$ pairs per bin which is a consequence of our large number of velocity bins, 2000. To avoid zeros we have smoothed over 20 bins in velocity space. The effect of sampling noise can be gauged from the middle and bottom panels of Fig. 6 as the limit $R \to 0$ is taken with a corresponding reduction in pair count, especially at high relative velocity. As the fitting procedure is somewhat qualitative, we will be more interested in the scale of the results, and any trends with varying $m_{\text{min}}$ and $m_{\text{max}}$ rather than in precision. As we will discuss in Section 5, our collisional velocities are factors of order 5 slower than those in a previous work, well outside any systematic or statistical errors.

4 ANALYSIS: INERTIAL RANGE, TURBULENCE MODELS

4.1 Clustering

In Table 2 we present the basic clustering data for our turbulence models, across an array of inertial ranges and our turbulence models. The parameters $m_{\text{min}}, m_p,m_{\text{max}}$ and $S'$ define the included inertial range, as well as an effective Stokes number. The diagnostics $\mu$ and $R_c$ are the power law and intercept of the power-law fit to $C(R)$ (Section 3.1) while $\omega$ defines the relative velocities of the cold population (equation 15) and $R_c$ defines the contamination radius outside of which the collisionless cold population becomes indistinguishable from the collisional warm and hot populations.

There appears to be a modest decrease of the clustering parameter $\mu$ with the number of shells with turnover times shorter than the particle drag time $t_p$, as measured by $S'$. However, it also appears to increase with the number of shells included with turnover times longer than $t_p$. Comparing runs B-E, B-F, B-LI1 and B-LI2, we feel that we have reached a large enough inertial range to estimate that particles well embedded in turbulence will experience clustering with $\mu \sim 0.4$. This result matches the behaviour with other turbulence models, although run P-A is a clear outlier.

Another interesting column is $\omega t_p$. This value scales slightly slower than $S'$, and $\omega^{-1}$ is the crossing time of a cold cluster.

| Run  | $m_{\text{min}}$ | $m_p$ | $m_{\text{max}}$ | $S'$ | $\mu$ | $R_c t_p$ | $\omega t_p$ | $R_c k_p$ |
|------|----------------|-------|------------------|-----|-------|------------|-------------|----------|
| B-A  | 0              | 4     | 6                | 2.5 | 0.65  | 1.6        | 0.29        | 0.04     |
| B-B  | 0              | 3     | 5                | 2.5 | 0.57  | 2.7        | 0.27        | 0.04     |
| B-LI2| 0              | 3     | 7                | 6.3 | 0.39  | 1.6        | 0.28        | 0.05     |
| B-C  | 1              | 3     | 5                | 2.5 | 0.55  | 1.3        | 0.23        | 0.04     |
| B-LI1| 1              | 3     | 7                | 6.3 | 0.38  | 1.8        | 0.25        | 0.05     |
| Base | 2              | 3     | 5                | 2.5 | 0.49  | 1.6        | 0.29        | 0.03     |
| B-D  | 2              | 3     | 6                | 4   | 0.45  | 1.5        | 0.39        | 0.02     |
| B-E  | 2              | 3     | 7                | 6.3 | 0.38  | 1.2        | 0.52        | 0.02     |
| B-F  | 2              | 3     | 8                | 10  | 0.38  | 1.2        | 0.76        | 0.02     |
| Q-A  | 2              | 3     | 5                | 2.5 | 0.34  | 2.5        | 0.51        | 0.03     |
| Q-LI | 0              | 3     | 7                | 6.3 | 0.44  | 1.3        | 0.50        | 0.03     |
| P-A  | 2              | 3     | 5                | 2.5 | 0.81  | 0.8        | 0.21        | 0.03     |
| P-LI | 0              | 3     | 7                | 6.3 | 0.40  | 2.2        | 0.39        | 0.03     |
| G-A  | 1              | 3     | 6                | 3.7 | 0.36  | 1.9        | 0.14        | 0.02     |
| G-B  | 2              | 3     | 5                | 2.4 | 0.40  | 2.0        | 0.16        | 0.03     |
| G-C  | 2              | 3     | 6                | 3.7 | 0.33  | 2.7        | 0.14        | 0.03     |
| G-LI | 0              | 3     | 7                | 5.9 | 0.38  | 1.8        | 0.20        | 0.02     |

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Accordingly, the result is that the crossing time of the cold cluster scales slightly slower than the turnover time of the smallest included eddy, supporting the extension of the hypothesis that the cold velocity is linear with separation (equation 15) to an infinite cascade without meaningful smallest scale. This is further born out by the results for the other turbulence models, particularly the GOY model, which see weak or no dependence of $u_c$ on $St'$.

A problematic column is $R_kk_p$. This imposes the minimal spatial resolution required to make a meaningful analysis of dust collisions by discarding the cold population. In a $2^{512}$ box with $k_p = 8$ (assuming an energy carrying scale of $k = 4$), a grid scale with $\Delta x = R_k$ requires a resolution of the order of $1700^2$ even for our smallest inertial range. This smallest range is within reach, albeit barely (Bec et al. 2010a,b).

### 4.2 Collisions

In Table 3 we collect fit data for our simulations, which should only be used for intermediate and higher relative velocities ($u > u_c = 0.11v_p$, see equations 16 and 17). The parameters $m_{min}$, $m_p$, $m_{max}$ and $St'$ define the included inertial range, as well as an effective Stokes number, while the diagnostics $C_a$, $C_b$, $u_a$ and $u_b$ are the coefficients in our fit formula (16). The coefficients $C_a$ and $C_b$ describe the effective target number density seen by the warm and hot populations, respectively, and the coefficients $u_a$ and $u_b$ represent the velocity scale of the two populations.

The $C_a$ and $C_b$ columns indicate that the warm and hot populations, in the velocity regime where the fit applies, have individual effective densities comparable to the density that the total dust population would have were it evenly distributed. This does not strictly imply that there is clustering of the collisional population, because many of the collisions are likely at low, unfitted velocities. The results for $C_a$ and $u_a$ are the least useful, as they are most sensitive to the cold population, and $u_a < u_c$. The results for $u_b$ are nicely only weakly sensitive to the set of included shells, while $C_b$ appears to require a significant $St' \approx 10$ to reach a limiting value.

Most of our diagnostics, such as $u_a$, $u_b$, $C_a$ and $R_k$, appear to be relatively insensitive to the set of included shells, i.e. $(m_{min} - m_{max})$ and $St'$. It should nonetheless be noted that since the diagnostic $u_b$ is crucial to the high-velocity collisions which might result in fragmentation, its weak sensitivity to $m_{min}$ is still important. Achieving converged values for $C_b$, an important diagnostic, certainly requires $St' \geq 6$, while the behaviour of $C_a$ in the limit of both larger and smaller eddies is not yet clear (bottom two lines of Table 3). Accordingly, ranges of $k_p/k_s = 8$ and $k_{min}/k_p = 16$ appear to be reasonable minima. Unfortunately, it appears that extreme resolution is required not merely to include an adequate inertial range, but also to capture the details of particle clustering.

Comparing the results across different turbulence models, a first observation is that they are well fitted by equation (16), as seen in Fig. 7, albeit with differing parameters. The fully quenched model (runs Q-A and Q-LI) produces both significantly more concentrated collisions (the $C$ coefficients) and significantly higher collisional velocities. This is not particularly surprising as there is nothing to destroy long time correlations. Putting the time variation of the spatial projection into the phase (runs P-A and P-LI) however results in a significant drop in the collisional velocities. Finally, the GOY model appears to fit with very similar velocity parameters of our base model although the clustering is weaker. This is somewhat surprising as the turbulence is spiky, which might be expected to result in a stronger high-velocity tail. On the other hand, it inherently includes phase variation. A further unexpected detail of the GOY model is that the smallest range of included shells ($m_{min} = 2$, $m_{max} = 5$) showed the strongest high-velocity tail ($u_b = 0.32v_p$).

This may be due to the fact that the GOY scheme has an intermittent energy cascade, and the generation of sub-vortices from larger vortices (a consequence of localized energy spikes in $k$-space, through the coupling terms of equation 5), if appropriately followed by including an adequate inertial range to track the sub-vortices, will decrease particle correlations.

Finally, we suggest that equation (16) be used with parameters of the order of $C_a \sim 1.5$, $C_b \sim 1$, $u_a \approx 0.07v_p$ and $u_b \approx 0.2v_p$, being applied only when relative velocities exceed the cut-off velocity $u_c$. figure 7. As in Fig. 6, but here we show run G-LI. Averaged over 112 snapshots, with $\Delta u = 4.64 \times 10^{-4}$ in code units.
\[ u_c = 0.1 v_p, \text{ i.e.} \]

\[
\frac{N(R, u > 0.1v_p)}{N(R)} = \frac{21}{v_p} e^{-14u/v_p} + \frac{5}{v_p} e^{-5u/v_p}. \tag{20}
\]

If one prefers the GOY model, the concentration parameters should be changed to \(C_a \sim 0.8\) and \(C_b \sim 0.4\).

5 COMPARISON WITH PREVIOUS WORK

5.1 Clustering

Turbulent clustering has been a topic of interest in the protoplanetary community, and has been invoked, for example, in the creation of chondritic meteorites (Cuzzi et al. 2001). Our clustering diagnostics follow those of Pan et al. (2011), rather than those of Fessler et al. (1994) and Hogan & Cuzzi (2001). The previous work found evidence for strong clustering for particles with modest \(St\), i.e. particles reasonably well coupled to turbulence at the dissipation scale. The clustering of Pan et al. (2011) was observed to decrease as their Stokes number became different from 1. Our results fit in magnitude with those of Pan et al. (2011), which scale up to \(\mu \gtrsim 0.6\) (see their fig. 6). However, our clustering is significantly stronger at high \(St\) or \(St'\). Given the differences in the fluid flows used (synthetic turbulence in our case, forced turbulence in Pan et al. 2011), the similarities in behaviour and scale of the exponent are encouraging. However, it is not clear from the \(\mu\) column in Table 2 that clustering will cease for large \(St'\), as previously seen. It should be noted that previous simulations had limited inertial ranges, so their high \(St\) particles are not fully embedded in the turbulence (see Appendix A).

There is reason to expect that clustering will be more effective for particles with \(St = 1\) (Cuzzi et al. 2001; Hogan & Cuzzi 2001). However, we can imagine evolving a system of two species of particles with \(St_1 = 1\) and \(St_2 \ll 1\). Once a statistical steady state is achieved, with the \(St_1\) particles more strongly coupled than the \(St_2\) particles, we decrease the fluid viscosity significantly so that in the new flow, \(St_1 \gg 1\) and \(St_2 = 1\) because the new dissipative scale is smaller. At that point, even if the \(St_2\) particles are more strongly clustered than the \(St_1\) particles, it is not clear why the \(St_1\) particles would be less clustered than originally. Runs Base through B-F in Table 2 match this thought experiment, and do show a drop of \(\mu\) from 0.49 to 0.38, which appears to be a minimal value. This weak decrease in \(\mu\) may result from particle clusters being disrupted by the additional, smaller scale, eddies.

Pan et al. (2011) hypothesized that clustering, especially if it exists for \(St > 1\), might greatly increase the collision rates between particles by generating regions of enhanced particle number density. Our results allow us to contradict the straightforward version of that hypothesis: the clustering exists, but the particle clusters are non-collisional. However, once particle number densities get high enough that the dust fluid density is comparable to the gas density, the dust drag has a significant back-reaction on the gas. An example is the streaming instability. In protoplanetary discs, gas orbits in a sub-Keplerian fashion because of the outwards pointing pressure force. As a result, particles which would naturally orbit in a Keplerian fashion feel a headwind. Similar to drafting on highways or in bicycle races, clumps of particles can then form which are dense enough to back-react on the gas (Cuzzi et al. 2001; Youdin & Goodman 2005; Johansen et al. 2007; Lewellen, Gong & Lewellen 2008).

5.2 Collisions

Much of astrophysical turbulent dust collision work follows the approach of Völk et al. (1980), Markiewicz et al. (1991) and Cuzzi & Hogan (2003); see Ormel & Cuzzi (2007) for a recent one. This approach defines two classes of eddies, Class I large-scale eddies and Class II small-scale eddies. The former have large velocities and time-scales, and so can transport dust grains significant distances: they dominate the turbulent diffusion of dust throughout a disc or atmosphere. On the other hand, they change slowly enough in both space (compared with dust stopping lengths) and time (compare with \(\tau_p\) that nearby dust grains, which could collide, see nearly identical gas motion. As a result, Class I eddies can affect the collisional behaviour of dust grains only slightly. Class II eddies on the other hand vary rapidly compared to both the frictional stopping time of the dust grains and their stopping length. Accordingly, these eddies can affect even nearby dust grains differently, driving collisions. However, their short time and length-scales mean that they provide only weak large-scale transport.

Dropping the contribution of Class I eddies for collisions between identical dust grains, Ormel & Cuzzi (2007) find an rms collisional velocity of

\[ u_{\text{Ormel}}^2 = 2v_p^2. \tag{21} \]

While this result is frequently quoted in terms of the Stokes number defined as \(St = \tau_p/Omega_1\), the above version is more general (see Appendix A). An important assumption is that the relative motion of particles approaches a finite limiting value as their separation goes to zero, which is possible for inertial particles (\(\tau_p \neq 0\)) whose motion must deviate from that of the gas. Such behaviour is seen in the direct numerical simulations of Bec et al. (2010b) and we believe we understand the deviation from that assumption seen in our Fig. 4. This analytical approach cannot however handle very long time-correlations between dust grains.

The result in equation (21) is different from our equation (16) in two ways. First, it is a single number, which does not suffice to describe our results, with their two velocity scales of the warm and hot populations. Our results also have merely exponentially falling probability tails with \(u\), a much broader distribution than, for example, a Maxwellian distribution. They should therefore be treated as a probability distribution. Secondly, if one nevertheless uses equation (13) to extract a single rms-averaged collisional velocity, we find

\[ u_{\text{out}} = \left[ \frac{6(C_a u_a^2 + C_b u_b^2)}{C_a + C_b} \right]^{1/2} \simeq u_{\text{Ormel}}/4, \tag{22} \]

using the suggested parameters of Section 4.2, a significant decrease in the characteristic collisional velocities and a much larger decrease in the turbulent collisional energies. We attribute this difference in predicted collisional velocities to the high level of correlation we have identified, which not only creates the ‘cold’ population, but appears to slow all the collisions.

One can also compare Fig. 4 and the bottom panel of Fig. 5 with fig. 4 of Carballido et al. (2010), which is further limited by the sub-gridscale gas velocity interpolation. Clearly, very high spatial resolution is required to extract actual collisional velocities from numerical simulations, at least for particles with identical stopping times. Our method provides large inertial ranges and high resolutions, whose necessity can be seen in their fig. 3, where the analytical result based on a full inertial range, the analytical result based on the actual inertial range and the numerical result all disagree. The resolution to begin to see the multiple populations we have identified...
in direct numerical simulations of the full Navier–Stokes equation may be becoming available (for example the 2048³ simulation of Bec et al. 2010a,b which is just adequate to include the inertial range assumed in run Base while resolving \( R_s \)). However, the analysis has not been done in the same terms so it is not clear whether they would see the effects we predict, or, if not, it would be because they contradict our results, or do not have adequate resolution.

6 DISCUSSION AND CONCLUSIONS

We arrive at a picture of turbulence-induced particle concentration and collisions which creates spatially small (with respect to turbulent scales), highly correlated ‘cold’ clusters of particles. While these clusters are themselves non-collisional, they periodically pass through each other, resulting in bursts of collisions. These clusters are channeled through the high-strain boundaries between turbulent eddies, and even separate clusters are strongly correlated. This correlation causes the turbulent collision speeds we find to be significantly slower than those predicted by analytical approaches, which cannot treat the long time-scale correlations. This picture of particle–particle collisions occurring when clusters of particles pass through each other is supported by the long stabilization time seen in Fig. 3, where it takes many turbulent turnover times to reach a statistical steady state, implying non-trivial dust spatial structures.

Our results apply most strongly for particles deeply embedded in a turbulent cascade. Our approach does not have the ability to handle the details of the forcing scale or the dissipation scale of the turbulence, both of which could have different behaviour such as the bottleneck effect at the dissipation scale (Dobler et al. 2003). Nevertheless, the qualitative similarity between our low inertial range runs (Basic, Q-A, P-A, G-A) with the large inertial range runs (B-LI2, Q-LI, P-LI, G-LI) suggests that this will only pose a difficulty if the statistics of the turbulence are indeed quite different at those two scales. A second difficulty is that our synthetic turbulence model cannot handle the dust’s back-reaction on the gas, which can be significant if there is strong clustering. An implicit assumption of our work therefore is that the dust spatial density is less than that of the gas.

We find that equation (16) is a workable fit for the number of turbulence-induced particle pairs as a function of collisional velocity. This form has some interesting implications. A simple one is that it is broad: unlike the case of, for example, a Maxwellian distribution, enough of collisional velocity space is populated that is broad: unlike the case of, for example, a Maxwellian distribution, enough of collisional velocity space is populated that it is self-normalized. Bottom panels: solid self-normalized cumulative totals of the collision rate, i.e. the first velocity moment of equation (20); dashed, \( u_c \) its intercepts by the data. The right axis is self-normalized. Bottom panels: solid self-normalized cumulative totals of the collision rate, i.e. the first velocity moment of equation (20); dashed, \( u_c \) its intercepts.

Figure 8. Ad hoc fits for \( u < u_c \). Left-hand column: extending equation (20) to \( u = 0 \); right-hand column: extending the constant value at \( u = u_c \) to \( u = 0 \). Top panels: pair density, i.e. equation (20). Middle panels: solid, cumulative totals of the top panels; dashed, \( u_c \) its intercepts by the data. The right axis is self-normalized. Bottom panels: solid self-normalized cumulative totals of the collision rate, i.e. the first velocity moment of equation (20); dashed, \( u_c \) its intercepts.

Errors in energy, the square root of the third velocity moment, will contribute few collisions due to their slow speeds. Errors in energy, the square root of the third velocity moment, might be explained by the vertical axes of the middle plots of Fig. 8: the limiting value on the left axes gives the effective concentration, i.e. the number of pairs in the warm and hot populations, divided by \( N(R) \). While the difficulties fitting the cold population make the result uncertain, it appears that clustering occurs in the collisional population: particles see more warm and hot collisional partners than would be expected if all dust grains were homogeneously distributed in space. This clustering is different from that of the cold population in that the clustering is independent of \( R \) for small separations (i.e. the effective \( \mu = 0 \); an equivalent plot to the bottom panel of Fig. 3 would be flat). For any clustering to occur, the colliding particles in the warm and hot populations must still be significantly correlated with one another, an aspect which the work of Ormel & Cuzzi (2007) could not fully capture.

Another intriguing result is the low scale of \( u_s \approx 0.2v_p \). This collisional velocity scale is significantly below the predicted rms collisional velocity \( u_{col} = 2^{1/2}v_p \) of Ormel & Cuzzi (2007). This might be explained by the vertical axes of the middle plots of Fig. 8: the limiting value on the left axes gives the effective concentration, i.e. the number of pairs in the warm and hot populations, divided by \( N(R) \). While the difficulties fitting the cold population make the result uncertain, it appears that clustering occurs in the collisional population: particles see more warm and hot collisional partners than would be expected if all dust grains were homogeneously distributed in space. This clustering is different from that of the cold population in that the clustering is independent of \( R \) for small separations (i.e. the effective \( \mu = 0 \); an equivalent plot to the bottom panel of Fig. 3 would be flat). For any clustering to occur, the colliding particles in the warm and hot populations must still be significantly correlated with one another, an aspect which the work of Ormel & Cuzzi (2007) could not fully capture.

In conclusion, we see turbulence-induced dust collisions at significantly (a factor of 3–5) slower velocities than predicted by existing analytical theory. If we consider an approximate \( \sigma \)-disc minimum mass solar nebula with a sound speed of \( c_s = 10^4 \text{ cm s}^{-1} \) (temperature \( T \approx 300 \text{ K} \)) at 1 au, \( \sigma = 10^{-3} \) and centimetre-scale grains of the cold population (because \( \mu > 0 \), see the height of the peak in the middle panel of Fig. 5, which increases as \( R \) decreases). This does not appear to pose unsurpassable difficulties; in Fig 8 we show two alternate ad hoc extensions for equation (20) into the low-velocity regime. From the bottom panels we can see that it is unlikely that errors in the collision rate based on the extension chosen will be more than 25 per cent of the total: low relative velocity dust grains, even if numerous, contribute few collisions due to their slow speeds. Errors in energy, the square root of the third velocity moment, will be significantly less yet, as long as there is not a hidden, highly clustered but still collisional population that we have not been able to detect so far.

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with a stopping time \( \tau_p \omega_k = 0.01 \) and mass \( m \approx 1 \text{ g} \), the turbulent velocity at the largest scale is \( v_1 = 3 \times 10^3 \text{ cm s}^{-1} \) (Shakura & Sunyaev 1973; Hayashi 1981). Under these conditions, the predicted collisional velocity of Ormel & Cuzzi (2007) is \( u = 5 \times 10^2 \text{ cm s}^{-1} \), well into the destructive collision regime of Gütlinger et al. (2010), fig. 11. A five-fold reduction in the collision speed would however lead to bouncing. While this result implies frequent destructive collisions, we also predict large numbers of lower velocity collisions, which could lead to sticking. This would ease the difficulties in planetesimal formation associated with bouncing and fragmentation (Zsom et al. 2010). The velocities we expect will still lead to significant fragmentation, but the inclusion of a slower collisional velocity probability distribution allows for the consideration of ‘lucky’ particles that are in unusually low-velocity collisions to grow large enough that they can survive. While the slower collisional velocities reduce collision rates, we also see a limited enhancement of the effective dust number density through clustering, which mitigates this collision rate reduction.

Finally, the hot population collisional tail falls off only exponentially with velocity, rather than something closer to a Maxwellian distribution. This emphasizes the importance of using a collisional velocity probability distribution instead of a single characteristic collisional velocity. For example, high-velocity outlying events are expected to occur at non-negligible rates, and could contribute to a fragmentation cascade and dust reprocessing. This would occur even if the reduction in collisional velocities results in few enough destructive collisions that the fragmentation barrier to dust growth is lifted.

Our results are well suited to inclusion in a model of collisional dust grain agglomeration in protoplanetary discs such as that of Zsom et al. (2010). The formula given by equation (16) is simple enough for inclusion, while allowing a velocity probability distribution. This enables the full use of experimental results about the critical velocities at which colliding particles stick, bounce or fragment.

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APPENDIX A: STOKES NUMBERS

Studies of turbulent particle transport generally use the Stokes number \( St \equiv \tau_p/\tau_k \) to non-dimensionalize the stopping time, where \( \tau_k \) is the turbulent time-scale associated with the viscous dissipation scale. Astrophysical studies of protoplanetary discs however often use \( St \equiv \tau_p/\Omega_k \), where \( \Omega_k \) is the Keplerian rotation rate, since it is better constrained than \( \tau_k \) and is believed to be a good estimate for the time-scale associated with the largest scale turbulence. As such, the particle stopping time is often scaled to two completely different time-scales, the largest and the smallest associated with the turbulence. Neither formulation is however clearly appropriate for studies of particle–particle relative motion and clustering.

Stating that the relevant quantity is \( St = \tau_p/\tau_k \) implies that either the details of the dissipation process proper, or the lack of fluid energy at scales \( k > k_k \) plays a crucial role in the particle response to the turbulence. For a particle with \( St \ll 1 \), such effects are expected, as particles couple most strongly with eddies with turnover times \( \tau \sim \tau_p \) and the dissipative cutoff means that many of those eddies are missing. However, it is unclear why a dependence on \( \tau_p/\tau_k \) would exist for particles with \( St \gg 1 \). Instead, for particles with \( \tau_p \gg \tau_k \gg \tau_k \), we expect the behaviour to be scale free: the largest scale of the turbulence (\( \tau_k \)) is too large to affect particle–particle relative motion while the dissipation scale is too small, so the turbulence does not set time- or length-scales for the particles.
Table A1. Variables, parameters and diagnostics.

| Name   | Notes                                                                 | Section     |
|--------|----------------------------------------------------------------------|-------------|
| $m_{\text{min}}$ | Shell numbering for the largest scale shell                           | Section 2.1 |
| $m_{\text{max}}$ | Shell numbering for the smallest scale shell                           |             |
| $m_p$    | Shell numbering for shells with $k = k_p$                              |             |
| $l_s$    | Subscript for the large scale, equivalent to $m_{\text{min}}$        | Section 2.1 |
| $l_s$    | Subscript for the small scale, equivalent to $m_{\text{max}}$        |             |
| $p$      | Subscript for the particle, equivalent to $m_p$                       | Section 2.2 |
| $V(x, t)$ | Gas velocity                                                          | Equation (4) |
| $\tau_{ls}$ | Turbulent turnover time at the largest turbulent scale               |             |
| $\tau_{ss}$ | Turbulent turnover time at the smallest turbulent scales             |             |
| $k_{mn}$ | Synthetic turbulence parameters                                       | Section 2.1 |
| $v_{mn}$ | Synthetic turbulence parameters                                       |             |
| $a_{mn}$ | Particle velocity                                                     | Section 2.2 |
| $\phi_{mn}$ | Particle velocity                                                     |             |
| $\mu$   | Power-law exponent for $C(R)$                                         |             |
| $R_1$   | $y = 1$ intercept for the power-law fit of $C(R)$                      | Equation (18) |
| $N'(R, u)$ | Pair density in velocity space: $N(R, u)/\triangle u$               | Equation (14) |
| $\omega$ | Cold population velocity measure                                      | Equation (15) |
| $R_c$   | Contamination radius                                                  |             |

In this regime, instead of measuring the particle stopping time as a function of some turbulent time, one should instead measure the turbulence by the particle stopping time. This approach is implicitly followed by Völk et al. (1980), where the division of eddies into Classes I and II is done by measuring the turbulent turnover time relative to $\tau_p$. They do quote results as a function of $St = \tau_p/\Omega_1$, but that is only possible because of the assumption of a Kolmogorov cascade, which allows them to note that eddies with turnover times $t = \tau_p$ have velocities $v_p = \sqrt{St}v_{ls}$.

This poses difficulties in numerical simulations of particle relative motion in turbulence because the accessible ratio of $\tau_{ss}/\tau_{ls}$ is modest at best. Accordingly, any apparent dependence on $St = \tau_p/\Omega_1$ is difficult to distinguish from a dependence on $St = \tau_p/\tau_{ss}$ and even then would imply little for the case of a particle deeply embedded in a large cascade. For example, when considering the work of Pan et al. (2011), it should be noted that their ability to track the clustering of particles deeply embedded in a turbulent cascade ($\tau_p \gg \tau_{ss}$ but also $\tau_p \ll \tau_{ls}$) is limited by their modest inertial range. Alternatively, fig. 3 of Carballido et al. (2010) plots the difference between theoretical results for a large inertial range (solid) and the theoretical results applied to the numerically obtained an inertial range (dotted). Any extension of their results of decreased clustering for large $St$ is suspect for a protoplanetary disc with a large inertial range. It does, however, certainly imply that clustering will decrease once $\tau_p/\tau_{ss} \simeq \tau_p/\Omega_1$ becomes large.

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