DEGENERATE $r$-ASSOCIATED STIRLING NUMBERS

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ABSTRACT. For any positive integer $r$, the $r$-associated Stirling number of the second kind $S^r_2(n,k)$ enumerates the number of partitions of the set $\{1,2,3,\ldots,n\}$ into $k$ non-empty disjoint subsets such that each subset contains at least $r$ elements. We introduce the degenerate $r$-associated Stirling numbers of the second kind and of the first kind. They are degenerate versions of the $r$-associated Stirling numbers of the second kind and of the first kind, and reduce to the degenerate Stirling numbers of the second kind and of the first kind for $r = 1$. The aim of this paper is to derive recurrence relations for both of those numbers.

1. INTRODUCTION AND PRELIMINARIES

Explorations for the degenerate versions of some special numbers and polynomials have become lively interests for some mathematicians in recent years, which began from the pioneering work of Carlitz (see [1,2]). These have been done by employing various methods, such as generating functions, combinatorial methods, $p$-adic analysis, umbral calculus, operator theory, differential equations, special functions, probability theory and analytic number theory (see [5,9-13,16,17] and the references therein).

The Stirling number of the second kind $S_2(n,k)$ enumerates the number of partitions of the set $[n] = \{1,2,\ldots,n\}$ into $k$ nonempty disjoint sets, while the $r$-associated Stirling number of the second kind $S^r_2(n,k)$ counts the number of partitions of the set $[n]$ into $k$ non-empty disjoint subsets such that each subset contains at least $r$ elements, for any positive integer $r$. The aim of this paper is to introduce the degenerate $r$-associated Stirling numbers of the second and of the first kind, and to derive their recurrence relations. They are degenerate versions of the $r$-associated Stirling numbers of the second kind and of the first kind, and reduce to the degenerate Stirling numbers of the second kind and of the first kind for $r = 1$. Here we mention that the degenerate Stirling numbers of both kinds appear very frequently when one studies various degenerate versions of some special numbers and polynomials.

The outline of this paper is as follows. In Section 1, we recall the degenerate exponentials and the degenerate logarithms. We state the generating functions and recurrence relations for the Stirling numbers of the second kind and of the first kind. We also remind the reader of the generating functions and recurrence relations of the degenerate Stirling numbers of the second kind and of the first kind. Finally, we recall the $r$-associated Stirling numbers of the second kind, its generating function and the recurrence relation. Section 2 is the main result of this paper. We introduce the degenerate $r$-associated Stirling numbers of the second kind and express the degenerate $r$-associated Bell polynomials in terms of the degenerate $r$-associated Stirling numbers of the second kind. In Theorem 2, we derive a recurrence relation for the degenerate Stirling numbers of the second kind. We also introduce the degenerate $r$-associated Stirling numbers of the first kind and deduce a recurrence relation for those numbers in Theorem 4. For the rest of this section, we recall the facts that are needed throughout this paper.

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For any nonzero $\lambda \in \mathbb{R}$, the degenerate exponential functions are defined by

\begin{equation}
(1) \quad e^t_\lambda = (1 + \lambda t)^r = \sum_{n=0}^{\infty} \frac{(x)_n\lambda^n}{n!} t^n, \quad e_1(t) = e^{(1)}(t), \quad \text{(see [1, 2])},
\end{equation}

where

\begin{equation}
(2) \quad (x)_0,1 = 1, \quad (x)_n,\lambda = x(x-\lambda) \cdots (x-(n-1)\lambda), \quad (n \geq 1), \quad \text{(see [10, 11])}.
\end{equation}

Let $\log_\lambda t$ be the compositional inverse of $e_\lambda(t)$, called the degenerate logarithm, such that $\log_\lambda(e_\lambda(t)) = e_\lambda(\log_\lambda(t)) = t$.

Then we have

\begin{equation}
(3) \quad \log_\lambda(1 + t) = \sum_{n=0}^{\infty} \frac{(1)_{n,1/\lambda}}{n!} \lambda^{n-1} t^n, \quad \text{(see [9])}.
\end{equation}

Note that $\lim_{\lambda \to 0} e_\lambda(t) = e^t$, $\lim_{\lambda \to 0} \log_\lambda(1 + t) = \log(1 + t)$.

For $k \geq 0$, the Stirling numbers of the first kind are defined by

\begin{equation}
(4) \quad \frac{1}{k!} \left( \log(1 + t) \right)^k = \sum_{n=k}^{\infty} \frac{S_1(n, k)}{n!} t^n, \quad \text{(see [3, 13, 14, 17])}.
\end{equation}

The Stirling numbers of the second kind are given by

\begin{equation}
(5) \quad \frac{1}{k!} (e^t - 1)^k = \sum_{n=k}^{\infty} \frac{S_2(n, k)}{n!} t^n, \quad \text{(see [5, 6, 18, 19, 20])}.
\end{equation}

From (4) and (5), we have

\begin{equation}
(6) \quad S_1(n+1, k) = S_1(n, k-1) - nS_1(n, k), \quad S_2(n+1, k) = S_2(n, k-1) + kS_2(n, k),
\end{equation}

where $n, k \geq 0$ with $n \geq k$ (see [10, 15, 18]).

Recently, the degenerate Stirling numbers of the first kind and of the second kind are respectively defined by

\begin{equation}
(7) \quad \frac{1}{k!} \left( \log_\lambda(1 + t) \right)^k = \sum_{n=k}^{\infty} \frac{S_1(n, k)}{n!} t^n,
\end{equation}

and

\begin{equation}
(8) \quad \frac{1}{k!} (e_\lambda(t) - 1)^k = \sum_{n=k}^{\infty} \frac{S_2(n, k)}{n!} t^n, \quad \text{(see [9])},
\end{equation}

where $k$ is a non-negative integer.

By (7) and (8), we get

\begin{equation}
(9) \quad S_{1,\lambda}(n+1, k) = S_{1,\lambda}(n, k-1) + (k\lambda - n)S_{1,\lambda}(n, k), \quad S_{2,\lambda}(n+1, k) = S_{2,\lambda}(n, k-1) + (k - n\lambda)S_{2,\lambda}(n, k),
\end{equation}

where $n, k \geq 0$ with $n \geq k$ (see [9]).

Note that $\lim_{\lambda \to 0} S_{1,\lambda}(n, k) = S_1(n, k)$, $\lim_{\lambda \to 0} S_{2,\lambda}(n, k) = S_2(n, k)$.

The generating function of the $r$-associated Stirling numbers of the second kind is given by

\begin{equation}
(10) \quad \frac{1}{k!} \left( e^{r-1} \frac{t^k}{k!} \right)^k = \sum_{n=kr}^{\infty} \frac{S_2^{(r)}(n, k)}{n!} t^n, \quad \text{(see [4, 6, 7, 8])}.
\end{equation}
Thus, by (10), we obtain the recursion formula of $S^{(r)}_2(n, k)$, $(n, k) \geq 0$, which is given by

$$S^{(r)}_2(n, k) = kS^{(r)}_2(n - 1, k) + \binom{n - 1}{r - 1} S^{(r)}_2(n - r, k - 1),$$

with the initial condition $S^{(r)}_2(n, k) = 0$ if $k = 0$ or $n < kr$ and $S^{(r)}_2(n, k) = 1$ if $n = rk$.

2. DEGENERATE $r$-ASSOCIATED STIRLING NUMBERS

As degenerate versions of the $r$-associated Stirling numbers of the second kind, we consider the degenerate $r$-associated Stirling numbers of the second kind given by

$$S^{(r)}_{2, \lambda}(n, k) = \frac{1}{k!} \sum_{l_1 + \cdots + l_k = n} \frac{n!(1)_{l_1, \lambda}(1)_{l_2, \lambda} \cdots (1)_{l_k, \lambda}}{l_1!l_2!\cdots l_k!},$$

where $r \in \mathbb{N}$ and $k$ is a non-negative integer.

From (12), we note that

$$S^{(r)}_{2, \lambda}(n, k) = \frac{1}{k!} \sum_{m=0}^{k} \binom{k}{m} (-1)^m \sum_{l_1, \ldots, l_m \geq 0} \frac{n!\prod_{i=1}^{m}(1)_{l_i, \lambda}}{l_1!l_2!\cdots l_m!(n - l_1 - l_2 - \cdots - l_m)!}.$$

where $n \geq kr \geq 0$, $k, r \geq 0$.

Note that $S^{(r)}_{2, \lambda}(n, k) = 0$ if $n < kr$.

We define the degenerate $r$-associated Bell polynomials given by

$$e^x(e_t(r) - \sum_{j=0}^{r-1} \frac{(1)_{j, \lambda}}{j!} r^j) = \sum_{n=0}^{\infty} \phi^{(r)}_{n, \lambda}(x) \frac{t^n}{n!}, (r \in \mathbb{N}).$$

Thus, by (13) and (14), we have

$$\sum_{n=0}^{\infty} \phi^{(r)}_{n, \lambda}(x) \frac{t^n}{n!} = \sum_{k=0}^{\infty} \frac{x^k}{k!} \left( e_t(r) - \sum_{l=0}^{r-1} \frac{(1)_{l, \lambda}}{l!} r^l \right)^k = \sum_{k=0}^{\infty} \frac{x^k}{k!} \sum_{n=kr}^{\infty} S^{(r)}_{2, \lambda}(n, k) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{k=0}^{\lceil \frac{n}{r} \rceil} x^k S^{(r)}_{2, \lambda}(n, k) \frac{t^n}{n!},$$

where $\lceil x \rceil$ denotes the greatest integer not exceeding $x$. Therefore, we obtain the following theorem.

**Theorem 1.** For any integers $n, r$ with $n \geq 0, r \geq 1$, we have

$$\phi^{(r)}_{n, \lambda}(x) = \sum_{k=0}^{\lceil \frac{n}{r} \rceil} x^k S^{(r)}_{2, \lambda}(n, k).$$

Now, we want to find a recursion formula for the degenerate $r$-associated Stirling numbers of the second kind.
Thus, by (15) and (16), we get

\[
\sum_{n=kr-1}^{\infty} S_{2,k}^{(r)}(n+1,k) \frac{t^n}{n!} = \sum_{n=kr}^{\infty} S_{2,k,k}^{(r)}(n,k) \frac{t^{n-1}}{(n-1)!} = \frac{1}{dt} \left( e_{\lambda}(t) - \sum_{l=0}^{r-1} \frac{(1)_{r-1,k} t^l}{l!} \right)^k.
\]

Here we note that

\[
\frac{d}{dt} \left( e_{\lambda}(t) - \sum_{l=0}^{r-1} \frac{(1)_{r-1,k} t^l}{l!} \right)^k = \frac{k}{k!} \left( e_{\lambda}(t) - \sum_{l=0}^{r-1} \frac{(1)_{r-1,k} t^l}{l!} \right)^{k-1} \left( e_{\lambda}(t) - \sum_{l=0}^{r-1} \frac{(1)_{r-1,k} t^l}{l!} (1+\lambda t) \right).
\]

Thus, by (15) and (16), we get

\[
\sum_{n=kr-1}^{\infty} \left\{ S_{2,k}^{(r)}(n+1,k) + n\lambda S_{2,k}^{(r)}(n,k) \right\} \frac{t^n}{n!} = (1+\lambda t) \sum_{n=kr-1}^{\infty} S_{2,k}^{(r)}(n+1,k) \frac{t^n}{n!}
\]

\[
= \frac{1}{(k-1)!} \left( e_{\lambda}(t) - \sum_{l=0}^{r-1} \frac{(1)_{r-1,k} t^l}{l!} \right)^{k-1} \left( e_{\lambda}(t) - \sum_{l=0}^{r-2} \frac{(1)_{r-1,k} t^l}{l!} (1-\lambda l) - \lambda \sum_{l=0}^{r-1} \frac{(1)_{r-1,k} t^l}{l!} \right)
\]

\[
= \frac{1}{(k-1)!} \left( e_{\lambda}(t) - \sum_{l=0}^{r-1} \frac{(1)_{r-1,k} t^l}{l!} \right)^{k-1} \left( e_{\lambda}(t) - \sum_{l=0}^{r-1} \frac{(1)_{r-1,k} t^l}{l!} \right) + \frac{1}{(k-1)!} \left( e_{\lambda}(t) - \sum_{l=0}^{r-1} \frac{(1)_{r-1,k} t^l}{l!} \right)^{k-1} \left( \frac{1}{(r-1)!} (1)_{r-1,k} t^{r-1} \right)
\]

\[
= k \left( e_{\lambda}(t) - \sum_{l=0}^{r-1} \frac{(1)_{r-1,k} t^l}{l!} \right)^k + \frac{1}{(k-1)!} \left( e_{\lambda}(t) - \sum_{l=0}^{r-1} \frac{(1)_{r-1,k} t^l}{l!} \right)^{k-1} \left( \frac{(1)_{r-1,k} t^{r-1}}{(r-1)!} \right)
\]

\[
= k \sum_{n=kr}^{\infty} S_{2,k}^{(r)}(n,k) \frac{t^n}{n!} + \sum_{n=r(k-l)}^{\infty} S_{2,k}^{(r)}(n,k-1) \frac{t^n}{n!} \frac{(1)_{r-1,k} t^{r-1}}{(r-1)!} - \lambda \sum_{n=r(k-l)}^{\infty} S_{2,k}^{(r)}(n,k-1) \frac{t^n}{n!} \frac{(1)_{r-1,k} t^{r-1}}{(r-1)!}
\]

Taking the derivative with respect to \( t \) on both sides of (12), we obtain
Remark 3. Here we observe that

\[
S_{2, \lambda}^{(r)}(n+1, k) = \left( k-n\lambda \right) S_{2, \lambda}^{(r)}(n, k) + (1)_{r-1, \lambda} \binom{n}{r-1} \left( S_{2, \lambda}^{(r)}(n-r+1, k-1) - \lambda (r-1)(1)_{r-1, \lambda} \binom{n}{r-1} \right) \frac{t^n}{n!},
\]

where \( n \geq 0 \) and \( r \geq 1 \).

Theorem 2. For \( n, k \geq 0 \) with \( n \geq kr - 1 \), we have

\[
S_{2, \lambda}^{(r)}(n+1, k) = \left( k-n\lambda \right) S_{2, \lambda}^{(r)}(n, k) + (1)_{r-1, \lambda} \binom{n}{r-1} \left( S_{2, \lambda}^{(r)}(n-r+1, k-1) - \lambda (r-1)(1)_{r-1, \lambda} \binom{n}{r-1} \right),
\]

where \( n, k \geq 0 \) with \( n \geq k-1 \). So our result agrees with the fact in (9), as \( S_{2, \lambda}^{(1)}(n, k) = S_{2, \lambda}(n, k) \).

Now, we consider the degenerate \( r \)-associated Stirling numbers of the first kind given by

\[
\frac{1}{k!} \left( \log_\lambda (1+t) - \sum_{l=1}^{r-1} \frac{(1)_{l, \lambda}}{l!} \lambda^{l-1} t^l \right) = \sum_{n=kr}^{\infty} S_{1, \lambda}^{(r)}(n, k) \frac{t^n}{n!},
\]

where \( k \) is a nonnegative integer and \( r \geq 2 \).

Taking the derivative with respect to \( t \) on both sides of (19), we get

\[
\sum_{n=kr}^{\infty} S_{1, \lambda}^{(r)}(n+1, k) \frac{t^n}{n!} = \sum_{n=kr}^{\infty} S_{1, \lambda}^{(r)}(n, k) \frac{t^n}{(n-1)!} = \frac{d}{dt} k! \left( \log_\lambda (1+t) - \sum_{l=1}^{r-1} \frac{(1)_{l, \lambda}}{l!} \lambda^{l-1} t^l \right)^k.
\]

Here we observe that

\[
\frac{d}{dt} k! \left( \log_\lambda (1+t) - \sum_{l=1}^{r-1} \frac{(1)_{l, \lambda}}{l!} \lambda^{l-1} t^l \right)^k = \frac{k}{k!} \left( \log_\lambda (1+t) - \sum_{l=1}^{r-1} \frac{(1)_{l, \lambda}}{l!} \lambda^{l-1} t^l \right)^k \left( \frac{(1+t)^\lambda}{1+t} - \sum_{l=1}^{r-1} \frac{(1)_{l, \lambda}}{(l-1)!} \lambda^{l-1} t^{l-1} \right).
\]

Thus, by (20) and (21), we get
\[
\sum_{n=kr-1}^{\infty} S_{1,\lambda}^{(r)}(n+1,k) \frac{r^n}{n!} (1 + t)
\]
\[
= \frac{1}{(k-1)!} \left( \log_\lambda (1 + t) - \sum_{l=1}^{r-1} \frac{(1)}{l!} \lambda^{l-1} t^{l} \right) (1 + t)^{k-1} \left( \log_\lambda (1 + t) - \sum_{l=1}^{r-1} \frac{(1)}{l!} \lambda^{l-1} t^{l} \right)
\]
\[
= \frac{1}{(k-1)!} \left( \log_\lambda (1 + t) - \sum_{l=0}^{r-2} \frac{(1)}{l!} \lambda^{l-1} t^{l} \right) (1 + t)^{k-1} \left( \log_\lambda (1 + t) - \sum_{l=0}^{r-2} \frac{(1)}{l!} \lambda^{l-1} t^{l} \right)
\]
\[
\times \left( (1 + t)^{k-1} - \sum_{l=1}^{r-2} \frac{(1)}{l!} \lambda^{l-1} t^{l} \right) (1 + t)^{k-1} \left( \log_\lambda (1 + t) - \sum_{l=0}^{r-2} \frac{(1)}{l!} \lambda^{l-1} t^{l} \right)
\]
\[
\times \left( (1 + t)^{k-1} - \sum_{l=1}^{r-2} \frac{(1)}{l!} \lambda^{l-1} t^{l} \right) (1 + t)^{k-1} \left( \log_\lambda (1 + t) - \sum_{l=1}^{r-2} \frac{(1)}{l!} \lambda^{l-1} t^{l} \right)
\]
\[
= \frac{\lambda}{(k-1)!} \left( \log_\lambda (1 + t) - \sum_{l=1}^{r-1} \frac{(1)}{l!} \lambda^{l-1} t^{l} \right) (1 + t)^{k-1} \left( \log_\lambda (1 + t) - \sum_{l=1}^{r-1} \frac{(1)}{l!} \lambda^{l-1} t^{l} \right)
\]
\[
+ \lambda^{r-1} \frac{(1)}{r-1} \sum_{n=kr-1}^{\infty} \frac{r^n}{n!} (1 + t)^{k-1} \left( \log_\lambda (1 + t) - \sum_{l=1}^{r-1} \frac{(1)}{l!} \lambda^{l-1} t^{l} \right)
\]
\[
- \frac{(r-1)}{(r-1)!} \lambda^{r-2} \sum_{n=kr-1}^{\infty} \frac{r^n}{n!} (1 + t)^{k-1} \left( \log_\lambda (1 + t) - \sum_{l=1}^{r-1} \frac{(1)}{l!} \lambda^{l-1} t^{l} \right)
\]
\[
= k \lambda \sum_{n=kr-1}^{\infty} S_{1,\lambda}^{(r)}(n,k) \frac{r^n}{n!} + \lambda^{r-1} \sum_{n=kr-1}^{\infty} \left( \begin{array}{c} n \\ r-1 \end{array} \right) S_{1,\lambda}^{(r)}(n-r+1,k-1) \frac{r^n}{n!}
\]
\[
- (r-1) \lambda^{r-2} \sum_{n=kr-1}^{\infty} \left( \begin{array}{c} n \\ r-1 \end{array} \right) S_{1,\lambda}^{(r)}(n-r+1,k-1) \frac{r^n}{n!}
\]
\[
= \sum_{n=kr-1}^{\infty} \left\{ k \lambda S_{1,\lambda}^{(r)}(n,k) + \lambda^{r-1} \sum_{n=kr-1}^{\infty} \left( \begin{array}{c} n \\ r-1 \end{array} \right) S_{1,\lambda}^{(r)}(n-r+1,k-1) \frac{r^n}{n!} \right\}
\]
On the other hand, by simple calculation, we get

$$\begin{align*}
S_{1,\lambda}^{(r)}(n+1,k) \frac{t^n}{n!} (1+t) \\
= \sum_{n=kr-1}^{\infty} S_{1,\lambda}^{(r)}(n+1,k) \frac{t^n}{n!} + \sum_{n=kr}^{\infty} S_{1,\lambda}(n,k) \frac{t^n}{n!} \\
= \sum_{n=kr-1}^{\infty} \left(S_{1,\lambda}^{(r)}(n+1,k) + nS_{1,\lambda}(n,k)\right) \frac{t^n}{n!}.
\end{align*}$$

From (22) and (23), we obtain the following theorem.

**Theorem 4.** Let $r \in \mathbb{N}$ with $r \geq 2$. Then, for $n,k \geq 0$ with $n \geq kr - 1$, we have

$$S_{1,\lambda}^{(r)}(n+1,k) + nS_{1,\lambda}(n,k) = k\lambda S_{1,\lambda}^{(r)}(n,k) + \lambda^{r-1}(1)_{r-1,1/\lambda} \left(\frac{n}{r-1}\right) S_{1,\lambda}^{(r)}(n-r+1,k-1)$$

$$- (r-1)(1)_{r-1,1/\lambda} \lambda^{r-2} \left(\frac{n}{r-1}\right) S_{1,\lambda}(n-r+1,k-1).$$

**Remark 5.** If $r = 1$ in Theorem 4, then we have

$$S_{1,\lambda}^{(1)}(n+1,k) = (k\lambda - n)S_{1,\lambda}^{(1)}(n,k) + S_{1,\lambda}(n,k-1),$$

where $n,k \geq 0$ with $n \geq k - 1$. So our result agrees with the fact in (9), as $S_{1,\lambda}^{(1)}(n,k) = S_{1,\lambda}(n,k)$.

**3. Conclusion**

In recent years, studying degenerate versions of some special numbers and polynomials have drawn the attention of many mathematicians with their regained interests not only in combinatorial and arithmetical properties but also in applications to differential equations, identities of symmetry and probability theory.

In this paper, we introduced the degenerate $r$-associated Stirling numbers of the second kind and of the first kind, and derived their recurrence relations. They are degenerate versions of the $r$-associated Stirling numbers of the second kind and of the first kind, and reduce to the degenerate Stirling numbers of the second kind and of the first kind for $r = 1$.

As one of our future research projects, we would like to continue to explore degenerate versions of some special numbers and polynomials and their applications to physics, science and engineering.

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