ON EULER CHARACTERISTIC OF EQUIVARIANT SHEAVES

ALEXANDER BRAVERMAN

Abstract. Let $k$ be an algebraically closed field of characteristic $p > 0$ and let $\ell$ be another prime number. O. Gabber and F. Loeser proved that for any algebraic torus $T$ over $k$ and any perverse $\ell$-adic sheaf $F$ on $T$ the Euler characteristic $\chi(F)$ is non-negative.

We conjecture that the same result holds for any perverse sheaf $F$ on a reductive group $G$ over $k$ which is equivariant with respect to the adjoint action. We prove the conjecture when $F$ is obtained by Goresky-MacPherson extension from the set of regular semi-simple elements in $G$. From this we deduce that the conjecture holds for $G$ of semi-simple rank 1.

1. The main conjecture

1.1. Notations. In what follows $k$ denotes an algebraically closed field of characteristic $p > 0$. Let $\ell$ be a prime number different from $p$. For an algebraic variety $X$ over $k$ we denote by $\mathcal{D}(X)$ the derived category of $\ell$-adic sheaves on $X$. Also we denote by $\text{Perv}(X) \subset \mathcal{D}(X)$ the subcategory of perverse sheaves. For any $F \in \mathcal{D}(X)$ we denote by $\chi(F)$ the Euler characteristic of $F$, i.e.

$$
\chi(F) = \sum (-1)^i \dim H^i_c(X, F) = \sum (-1)^i \dim H^i(X, F)
$$

(cf. [4] for the proof of the equality).

Let $T$ be an algebraic torus over $k$. The following theorem is proved in [3] (cf. also [2] for a partial analogue in characteristic zero):

Theorem 1.2. Let $F \in \text{Perv}(T)$. Then $\chi(F) \geq 0$.

Let $G$ be a connected reductive algebraic group over $k$. We shall denote by $G_{rs} \subset G$ the open subspace of regular semi-simple elements. We denote by $\text{Perv}_G(G)$ the subcategory of $\text{Perv}(G)$ consisting of perverse sheaves which are equivariant with respect to the adjoint action. We propose the following generalization of Theorem 1.2.

Conjecture 1.3. Let $F \in \text{Perv}_G(G)$. Then $\chi(F) \geq 0$.

Theorem 1.4. Assume that $F \in \text{Perv}_G(G)$ is equal to the Goresky-MacPherson extension of its restriction to $G_{rs}$. Then $\chi(F) \geq 0$.

Corollary 1.5. Conjecture 1.3 holds for $G$ of semi-simple rank 1.
Proof. Let $\mathcal{N}$ be the cone of unipotent elements in $G$ and let $Z$ be the center of $G$. Clearly it is enough to prove Conjecture 1.3 only for irreducible sheaves. However if $\mathcal{F} \in \Perv(G)$ is irreducible then either $\mathcal{F}$ is equal to the Goresky-MacPherson extension of its restriction to $G_{rs}$ or it is supported on $ZN$. The former case is covered by Theorem 1.4; hence it is enough to deal with the latter.

First of all there exists a connected component $Z'$ of $Z$ such that $\mathcal{F}$ is supported on $Z'N$. Thus one of the following is true:

1) $\mathcal{F}$ is supported on $Z'$.

2) $\mathcal{F}$ is equal to $\mathcal{F}' \boxtimes (\overline{Q}_l)_N[2]$ where $\mathcal{F}'$ is an irreducible perverse sheaf on $Z'$ and $(\overline{Q}_l)_N$ is the constant sheaf on $N$.

3) $\mathcal{F} = \mathcal{F}' \boxtimes \mathcal{E}$ where $\mathcal{F}'$ is an irreducible perverse sheaf on $Z'$ and $\mathcal{E}$ is an irreducible perverse sheaf on $N$ whose restriction to the open orbit is isomorphic to (the only) non-trivial equivariant irreducible local system on this orbit (this local system is of rank 1 and its square is isomorphic to the constant sheaf).

In cases 1 and 2 our result follows immediately from Theorem 1.2 (note that $Z'$ is a torus and that in case 2 we have $\chi(\mathcal{F}) = \chi(\mathcal{F}')$). In case 3 it is known that $H^*(N, \mathcal{E}) = 0$, hence $H^*(G, \mathcal{F}) = 0$, hence $\chi(\mathcal{F}) = 0$.

Remark. Theorem 1.2 has an "ideological" explanation: namely in \cite{3} O. Gabber and F. Loeser construct a "Mellin transform" functor $\mathcal{M}: D^b(T) \rightarrow D^b_{\text{coh}}(\text{Loc}_T)$ where $\text{Loc}_T$ denotes the moduli space of tame rank one local systems on $T$ and $D^b_{\text{coh}}(\text{Loc}_T)$ is the derived category of quasi-coherent sheaves on $\text{Loc}_T$. Moreover, for every $\mathcal{F} \in \Perv(T)$ the complex $\mathcal{M}(\mathcal{F})$ is actually a sheaf and $\chi(\mathcal{F})$ is equal to the generic rank of $\mathcal{M}(\mathcal{F})$ (hence $\chi(\mathcal{F}) \geq 0$). We don’t know whether a similar explanation of Conjecture 1.3 is possible.

We expect that Conjecture 1.3 holds also when $k$ has characteristic 0 and perverse sheaves are replaced by holonomic D-modules. When $G$ is a torus and the D-modules in question have regular singularities a beautiful geometric proof of Theorem 1.2 was given in \cite{2} where the authors explain how to compute the corresponding Euler characteristics using certain non-compact generalization of Kashiwara’s index theorem. It would be interesting to generalize the proof of \cite{2} to the case of arbitrary reductive group. Some results in this direction have recently been obtained by V.Kiritchenko.

2. Proof of Theorem 1.4

2.1. The space $\tilde{G}$. Let $\mathcal{B}$ denote the flag variety of $G$, i.e. the variety of all Borel subgroups of $G$. Let $\tilde{G}$ denote the variety of pairs $(B \in \mathcal{B}, g \in B)$. We have the natural maps $\pi: \tilde{G} \rightarrow G$, $\alpha: \tilde{G} \rightarrow T$ and $\beta: \tilde{G} \rightarrow \mathcal{B}$. It is easy to see that $\alpha$ is smooth and $\pi$ is proper. Let $d = \dim G - \dim T = 2 \dim \mathcal{B}$. Set $\tilde{G}_{rs} = \pi^{-1}(G_{rs}) = \alpha^{-1}(T_{rs})$.

Lemma 2.2. 1. Let $\mathcal{G} \in \Perv(T)$. Assume that $\mathcal{G}$ is equal to the Goresky-MacPherson extension of its restriction on $T_{rs}$. $\pi_! \alpha^* \mathcal{G}[d](\frac{d}{2})$ is a perverse sheaf which is equal to the Goresky-MacPherson extension of its restriction to $G_{rs}$.
2. Every $W$-equivariant structure on a sheaf $G \in \text{Perv}(T)$ as above gives rise to a $W$-action on $\pi_!G[d](\frac{d}{2})$.

3. Let $F \in \text{Perv}_G(\mathcal{G})$. Assume that $F$ is equal to the Goresky-MacPherson extension of its restriction to $G_{rs}$. Then there exists a $W$-equivariant sheaf $\mathcal{G} \in \text{Perv}(T)$ which is equal to the Goresky-MacPherson extension of its restriction to $T_{rs}$ such that

$$F = (\pi_!G[d](\frac{d}{2}))^W.$$  \hfill (2.1)

Proof. The first statement of Lemma 2.2 is well-known (it follows from the smallness property of $\alpha$ – cf. [1] and references therein)

It follows from 1 that in order to prove 2 it is enough to construct an action of $W$ on the restriction of $\pi_!G[d](\frac{d}{2})$ to $G_{rs}$. Note that $W$ acts freely on $G_{rs}$ in the natural way and

$$G_{rs} = \tilde{G}_{rs}/W.$$  \hfill (2.2)

Moreover, the restriction of $\alpha$ to $\tilde{G}_{rs}$ is $W$-equivariant. Hence $\alpha^*G[d](\frac{d}{2})|_{G_{rs}}$ is $W$-equivariant and thus (2.2) implies that $\pi_!G[d](\frac{d}{2})|_{G_{rs}}$ has a natural action of $W$.

Let us prove 3. Choose an embedding $i : T \to G$. Let $i_{rs} : T_{rs} \to G$ be its restriction to $T_{rs}$. Let $G_{rs} = i_{rs}^*F[-d](\frac{-d}{2})$. It follows from the $G$-equivariance of $F$ that $G_{rs}$ is perverse. We let $\mathcal{G}$ be its Goresky-MacPherson extension to $T$. We leave the verification of (2.1) to the reader. \hfill \Box

Proposition 2.3. Let $F$ be as in Theorem 1.4 and let $G$ be as in Lemma 2.2(3). Then $\chi(F) = \chi(G)$.

Clearly Proposition 2.3 together with Theorem 1.2 imply Theorem 1.4.

2.4. Proof of Proposition Proposition 2.3. Set $\mathcal{H} = \alpha^*G[d](\frac{d}{2}) \in \text{Perv}(\tilde{G})$. Clearly, $H^*_c(G,F) = H^*_c(\tilde{G},\mathcal{H})^W$.

Consider $\alpha_!(\mathcal{H})$. By the projection formula it is isomorphic to $G \otimes \alpha_!(\mathcal{Q}_l)\tilde{G}[d](\frac{d}{2})$. It is easy to see that the complex $\alpha_!(\mathcal{Q}_l)\tilde{G}[d](\frac{d}{2})$ is constant with fibers isomorphic to $H^*(B, \mathcal{Q}_l)$. Indeed, let $\delta : B \times T \to T$ denote the projection to the first multiple. Then $\alpha_!(\mathcal{Q}_l)\tilde{G} = \delta_!(\beta \times \alpha)_!(\mathcal{Q}_l)\tilde{G}$. However, the map $(\beta \times \alpha) : \tilde{G} \to B \times T$ is a locally trivial fibration (in Zariski topology) with fiber isomorphic to $A^d$. Thus $(\beta \times \alpha)_!(\mathcal{Q}_l)\tilde{G} \simeq (\mathcal{Q}_l)_{B \times T}[-d](\frac{-d}{2})$ and hence

$$\alpha_!(\mathcal{Q}_l)\tilde{G} \simeq (\mathcal{Q}_l)_T \otimes H^*(B, \mathcal{Q}_l)[-d](\frac{-d}{2}).$$  \hfill (2.3)

This clearly gives rise to the isomorphism

$$H^*_c(\tilde{G},\mathcal{H}) \simeq H^*_c(T,\mathcal{G}) \otimes H^*(B, \mathcal{Q}_l).$$  \hfill (2.4)
We can also characterize the isomorphism (2.3) in the following way. To determine it uniquely it is enough to construct it on $T_{rs}$. Choose as before an embedding $i: T \to G$. There is a canonical Borel subgroup $B$ containing $i(T)$ (recall that the abstract Cartan group $T$ comes with a root system with a preferred set of positive roots). Then for every $t \in T_{rs}$ we have canonical isomorphism

$$\alpha^{-1}(t) \simeq G/T$$

(2.5)

(which depends however on the above choice). Clearly $H^*_c(G/T, \mathcal{Q}_l)[d](\frac{1}{2}) = H^*(B, \mathcal{Q}_l)$. Hence we have constructed an isomorphism between $\alpha_{\mu}(\mathcal{Q}_l) \mathcal{G}^*[d](\frac{1}{2})|_{T_{rs}}$ and the constant complex with fiber $H^*(B, \mathcal{Q}_l)$. It is easy to see that this isomorphism does not depend on the choice of the embedding $T \to G$ and coincides with (2.3).

Let $W$ act on the left hand side by means of the identification $H^*_c(\tilde{G}, \mathcal{H}) = H^*_c(G, \pi_{\mathcal{H}})$ (note that by Lemma 2.2 the group $W$ acts on $\pi_{\mathcal{H}}$). Let $W$ also act on the right hand side by means of the tensor product of the $W$-action on $H^*_c(T, \mathcal{G})$ coming from the $W$-equivariant structure on $\mathcal{G}$ and the natural $W$-action on $H^*_c(B, \mathcal{Q}_l)$.

**Lemma 2.5.** The isomorphism (2.4) is also an isomorphism of $W$-modules with respect to the above actions.

**Proof.** Let $Z = G \times T/_{T/W}$ be the image of $\tilde{G}$ in $G \times T$ under the map $\gamma = \pi \times \alpha$. This is a closed subvariety of $G \times T$. It is invariant with respect to the $W$ action on the second multiple. Let $Z_{rs}$ denote the set of ”regular semi-simple” elements of $Z$ (i.e. the set of all elements of $Z$ whose projection to $Z$ is regular semi-simple).

Let $\mathcal{K} = \gamma_{!}(\mathcal{H})$. Then $\mathcal{K}$ is equal to the Goresky-MacPherson extension of its restriction to $Z_{rs}$. Since $\gamma$ induces an isomorphism $\tilde{G}_{rs} \simeq Z_{rs}$ and since the restriction of $\mathcal{H}$ to $\tilde{G}_{rs}$ is $W$-equivariant it follows that $\mathcal{K}$ also has a natural $W$-equivariant structure as a perverse sheaf on $G \times T$ (where the $W$-action is as before on the second multiple). Thus $W$ acts naturally on $H^*_c(G \times T, \mathcal{K})$. We have the natural isomorphism $H^*_c(\tilde{G}, \mathcal{H}) \simeq H^*_c(G \times T, \mathcal{K})$ and by the definition the action of $W$ on $H^*_c(\tilde{G}, \mathcal{H})$ introduced before Lemma 2.5 corresponds to the action of $W$ on $H^*_c(G \times T, \mathcal{K})$ introduced above.

Consider, on the other hand, the complex $p_!(\mathcal{K})$ where $p : G \times T \to T$ denotes the natural projection. Clearly, we have $p_!(\mathcal{K}) = \alpha_{\mu}(\mathcal{H}) \simeq \mathcal{G} \otimes H^*(B, \mathcal{Q}_l)$. On the other hand, since $p$ commutes with $W$ it follows that $p_!(\mathcal{K})$ admits a natural $W$-equivariant structure. To prove Lemma 2.4 it is enough to show that under the identification $p_!(\mathcal{K}) \simeq \mathcal{G} \otimes H^*(B, \mathcal{Q}_l)[d](\frac{1}{2})$ this structure is equal to the tensor product of the original $W$-equivariant structure on $\mathcal{G}$ with the natural $W$-action on $H^*(B, \mathcal{Q}_l)$. Since every perverse cohomology of $p_!(\mathcal{K})$ is equal to the Goresky-MacPherson extension of its restriction to $T_{rs}$ it follows that it is enough to check this equality only on $T_{rs}$. This, however, follows immediately from the description of the $W$-equivariant structure on $\alpha_{\mu}(\mathcal{Q}_l)\mathcal{G}[d](\frac{1}{2})|_{\tilde{G}_{rs}}$ given in Section 2.4 and the following observation:
Choose as before an embedding \( T \to G \). Let \( N(T) \) denote the normalizer of \( T \) in \( G \). Then \( W = N(T)/T \) acts on \( G/T \) and hence on \( H^*_c(G/T, \mathbb{Q}_l) \). By identifying \( H^*_c(G/T, \mathbb{Q}_l)[d](\frac{d}{2}) \) with \( H^*(\mathcal{B}, \mathbb{Q}_l) \) we get an action of \( W \) on the latter. This is the standard \( W \)-action on \( H^*(\mathcal{B}, \mathbb{Q}_l) \).

It follows from Lemma 2.5 that \( \chi(F) \) is equal to the Euler characteristic of \((H^*_c(T, \mathcal{G}) \otimes H^*(\mathcal{B}, \mathbb{Q}_l))[d]^W \). However, it is known that \( H^*(\mathcal{B}, \mathbb{Q}_l) \) lives only in even degrees and when we forget the grading it is isomorphic to the regular representation \( \mathbb{Q}_l[W] \) of \( W \). Hence

\[
\chi(F) = \chi((H^*_c(T, \mathcal{G}) \otimes H^*(\mathcal{B}, \mathbb{Q}_l))[d]^W) = \\
\chi((H^*_c(T, \mathcal{G}) \otimes \mathbb{Q}_l[W])^W) = \chi(H^*_c(T, \mathcal{G})) = \chi(G)
\]

which finishes the proof.

2.6. Acknowledgements. The author is grateful to F. Loeser for an illuminating discussion on the subject held in the summer of 2001, to V. Kiritchenko for making her results available to the author before publication and to M. Kapranov for explaining the contents of [2].

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Department of Mathematics, Harvard University, 1 Oxford st. Cambridge MA, 02138

E-mail address: bravalmath.harvard.edu