Vacuum Expectation Value of the Higgs field and Dyon Charge Quantisation from Spacetime Dependent Lagrangians

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Abstract

The spacetime dependent lagrangian formalism of references [1-2] is used to obtain a classical solution of Yang-Mills theory. This is then used to obtain an estimate of the vacuum expectation value of the Higgs field, viz. $\phi_a = A/e$, where $A$ is a constant and $e$ is the Yang-Mills coupling (related to the usual electric charge). The solution can also accommodate non-commuting coordinates on the boundary of the theory which may be used to construct $D$–brane actions. The formalism is also used to obtain the Deser-Gomberoff-Henneaux-Teitelboim results [10] for dyon charge quantisation in abelian $p$–form theories in dimensions $D = 2(p + 1)$ for both even and odd $p$. PACS: 11.15.-q, 11.27.+d, 11.10.Ef
The spacetime dependent lagrangian formalism [1-2] gives an alternative way to deal with electromagnetic duality [3], weak-strong duality [4] and electro-gravity duality [5]. Here this method will be used (a) to obtain an estimate of the vacuum expectation value (VEV) of the Higgs field in terms of the electric charge $e$ and a constant (b) to show that the ’t Hooft ansatz for obtaining the ’t Hooft-Polyakov monopole solution is sufficiently general to lead to other solutions containing coordinates near the boundary that do not commute (c) to show that the ’t Hooft ansatz for the gauge field is sufficient to yield a solution for the Higgs field for $r \to \infty$ without the necessity of any further ansatz for $\phi$ and (d) to obtain the results of Deser, Gomberoff, Henneaux and Teitelboim [10] regarding dyon charge quantisation in abelian, $p$-form theories. Results (a) to (c) will be obtained in Section 1 while Section 2 deals with (d).

1. VEV of Higgs field

In [1] it was shown that if the lagrangian $L'$ be a function of fields $\eta_\rho$, their derivatives $\eta_\rho,\nu$ and the spacetime coordinates $x_\nu$, and $L'$ be written as $L'(\eta_\sigma, \eta_\sigma, \nu, \ldots x_\nu) = L(\eta_\sigma, \eta_\sigma, \nu) \Lambda(x_\nu)$, then the variational principle [12] gives equations of motion as

$$\int dV \left( \partial_\eta (L \Lambda) - \partial_\mu \partial_\nu \eta (L \Lambda) \right) = 0 \quad (1)$$

$\Lambda(x_\nu)$ is the $x_\nu$ dependent part in $L'$ and is a finite non-dynamical and non-vanishing function. It is like an external field and equations of motion for $\Lambda$ meaningless. Duality invariance is related to finiteness of $\Lambda$ on the boundary. When equations of motion are duality invariant, finiteness of $\Lambda$ on the spatial boundary at infinity leads to new solutions for the fields. The finite behaviour of $\Lambda$ on the boundary encodes the exotic solutions of the theory within the boundary thus reminding one of the holographic principle[9].
Consider the Georgi-Glashow model with [3]

\[ L = \left[-\left(\frac{1}{4}\right)G_{\mu\nu}^a G_{a\mu\nu} + \frac{1}{2}(D_\mu^a\phi)(D_\mu^a\phi) - V(\phi)\right] \]  

(2a)

where usually one takes \( V(\phi) = (\lambda/4)(\phi^a\phi^a - a^2)^2 \). The gauge group is \( SO(3) \), \( a, b, c \) are \( SO(3) \) indices, with the generators \( \tau^a \) satisfying \( [\tau^a, \tau^b] = i\epsilon^{abc}\tau^c \). Gauge fields \( W_\mu = W_\mu^a\tau^a \). The field strength is \( G_{\mu\nu}^a = \partial_\mu W_\nu^a - \partial_\nu W_\mu^a - \epsilon_{abc} W_\mu^b W_\nu^c \); \( \tilde{G}_{\mu\nu}^a = (1/2)e^{a\mu\nu\sigma}G_{\rho\sigma} \); and the matter fields \( \phi \) are in the adjoint representation of \( SO(3) \). The energy density \( \Theta_{00} = (1/2)[(E_i^a)^2 + (B_i^a)^2 + (D_0^a\phi)^2 + (D_i^a\phi)^2 + V(\phi)] \geq 0 \). The non-abelian electric and magnetic fields are defined respectively as: \( E_i^a = -G_{0i}^a \) and \( B_i^a = -(1/2)\epsilon_{ijk}G_{jk}^a \). The vacuum configuration is \( G_{\mu\nu}^a = 0 \); \( D_\mu^a\phi = 0 \); \( V(\phi) = 0 \). There are also the Bianchi identities \( D_\mu\tilde{G}_{\mu\nu}^a = 0 \). Duality invariance means that \( D_\mu^aG_{\mu\nu}^a = 0 \).

In [1] it was shown that for this theory

\[ L' = L\Lambda = \left[-\left(\frac{1}{4}\right)G_{\mu\nu}^a G_{a\mu\nu} + \frac{1}{2}(D_\mu^a\phi)(D_\mu^a\phi) - V(\phi)\right]\Lambda (x_\nu) \]  

(2b)

The equations of motion using (1) were:

\[ \Lambda(D_\mu G_{a\mu\nu}) + (\partial_\mu \Lambda)G_{a\mu\nu} + \Lambda \epsilon_{abc}(\partial_\mu \phi)_b(\phi)_c - \Lambda e^2 \epsilon_{abc}\epsilon_{bc'd'} W_{\nu' c'}(\phi)_c(\phi)_{d'} = 0 \]  

(3a)

\[ (D_\mu D_\nu^a\phi)_a\Lambda + (D_\mu^a\phi)\partial_\mu \Lambda = -(\partial_\mu V)\Lambda \]  

(3b)

and the Bianchi identities were: \( D^\mu \tilde{G}_{\mu\nu}^a = 0 \) Requiring duality invariance (i.e. \( D^\mu G_{a\mu\nu} = 0 \)) gave the solution to \( \Lambda \equiv \Lambda(r) \) as

\[ \Lambda_\infty = \Lambda_0 e^{\exp[-e \int_0^\infty dr \left( (\epsilon_{abc}(D_\nu\phi)_{b}\phi)e(\partial^k r G_{a\nu})^{-1}\right)]] \]  

(4)

where \( \Lambda_0 \) is the value of \( \Lambda \) at \( r = p; a, \nu \) are fixed; and there is a sum over indices \( i, b \) and \( c \). \( \Lambda_\infty \) must be finite. Choose this to be the constant.
This may be realised in various ways, the simplest being \( (D_\nu (\phi)_b \Rightarrow 0, (\phi)_c \Rightarrow \text{finite}, \) and the product \( (D_\nu \phi)_b (\phi)_c \) falls off faster than \( G_{a\ nu} \) for large \( r \). Then a constant value for \( \Lambda \) was perfectly consistent with (3b) and the conditions became analogous to the Higgs’ vacuum condition for the t’Hooft-Polyakov monopole solutions where the duality invariance of the equations of motion and Bianchi identities are attained at large \( r \) by demanding \( (D_\mu \phi)_a \Rightarrow 0 \) and \( \phi_a \Rightarrow a\delta_{a3} \) at large \( r \). The results were perfectly consistent with the usual choice for the Higgs’ potential \( V(\phi) \) even though nothing had been assumed regarding this. So the t’Hooft-Polyakov monopole solutions followed naturally in our formalism. We now discuss two other interesting possibilities.

**Case I**

\[
(\epsilon_{abc}(D_\nu \phi)_b \phi_c) \Rightarrow 0
\]

(i.e. the duality condition \( (D^\mu G_{\nu\mu})_a = 0 \)) and falls off faster than \( G_{a\ nu} \) for large \( r \) \((a\ and\ \nu \) are fixed). A solution is when

\[
D_\nu \phi = \alpha_\nu \phi
\]

where \( \alpha_\nu \) can be any Lorentz four vector field that is consistent with all the relevant equations of motion and the minimum energy requirements. The minimum energy requirements are satisfied because it is straightforward to verify that the gauge fields \( W^\mu_a \) do not change. This is seen by taking the cross product of \( \phi \) (\( \phi \) is a \( SO(3) \) vector) with equation (6). We again arrive at the well known results of Corrigan *et al* [7], viz. \( W^\mu = (1/a^2c)\phi \wedge \partial^\mu \phi + (1/a)\phi A^\mu \), where \( A^\mu \) is arbitrary.

As the gauge fields \( W^\mu_a \) do not change, so even with this solution we obtain
the same gauge field solutions as before and so minimum energy requirem
t is automatically satisfied. However, this new solution allows us to obtain an
estimate of the vacuum expectation value of the Higgs field and to this we now
proceed. Let \( \alpha_\nu = (0, \alpha_i) \equiv \alpha(r) \hat{r} \), where \( \hat{r} \) is the unit radial vector. So the
Bogomolny condition is \( B^i_a = D^i \phi_a = \alpha^i \phi_a \) and the Higgs vacuum condition
obtained from equation (3b) (for \( r \to \infty, \Lambda \) is a constant, say unity) is
\[
[D^i(\alpha_i \phi)]_a = -\partial_{\phi^a} V = 0 \quad (7a)
\]
i.e. we are at a minima of the potential \( V \). If \( \phi_a \neq 0 \), (7a) implies
\[
div \, \alpha + \alpha^2 = 0 \quad (7b)
\]
and the solution is
\[
\alpha(r) = 1/(cr^2 - r) \ , \ \alpha^i = r^i/(cr^3 - r^2) \quad (8)
\]
where we take the constant \( c \) to be negative. Let us now take the ’t Hooft
ansatz for the gauge field, \( viz. \)
\[
W^0_a = 0 \ ; \ W^i_a = -\epsilon_{aik} r^k [1 - K(ae)]/(er^2) \quad (9)
\]
where the function \( K(ae) \) has been well studied \[3\] and goes to zero at
\( r \to \infty \). Then the electric field vanishes while \( G_{a \ jk}, B^i_a \) are
\[
G_{a \ jk} = (1/er^2)[2\epsilon_{ajk}(1 - K) + \epsilon_{aik}r^l \partial_j K - \epsilon_{ajl}r^k \partial_l K]
+ (1/er^4)[2(1 - K)\epsilon_{aik}r^l r_j - \epsilon_{ajl}r^i r_k] + (1 - K)^2(\delta_{aj} \epsilon_{ckl} r^c r^l - \epsilon_{jkl} r^a r^i) \quad (10a)
\]
\[
B^i_a = (1/er^4)[(1 - K)^2 r_a r^i - 2(1 - K)r^i r_a] - (1/er^2)[r^i \partial_a K - \delta^i_a r^m \partial_m K] \quad (10b)
\]
Now \( B^i_a = D^i \phi_a = \alpha^i \phi_a \). Therefore, taking \( c = -A \) so that \( A \) is positive we have
\[
\phi_a = \frac{(1 + Ar)}{(er^2)} [2(1 - K)r_a - (1 - K)^2 r_a + r^2 \partial_a K - r_a r^m \partial_m K] \quad (10c)
\]
It is easily seen that (10c) reduces to the 't Hooft ansatz for \( \phi_a \) for \( A = 0 \) and \( r \to \infty \). Thus we have obtained an expression for \( \phi \) without assuming any ansatz. This had never been possible before. There is another interesting outcome. For \( r \to \infty \) we have \( K \to 0 \) and so

\[
\phi_a \to \frac{A r^a}{e r} + \frac{r^a}{e r^2} = \frac{A}{e} \hat{r}^a + \frac{\hat{r}}{e r} \to \frac{A}{e} \hat{r}^a
\]

for \( r \to \infty \). But \( \phi_a = a \delta_{a3} \) for \( r \to \infty \). Therefore \( a = A/e \). Thus we have obtained the VEV of the Higgs field in terms of \( e \).

Case II

\( \alpha_\nu \) is any Lorentz four vector field as in I but which may also carry internal symmetry indices other than \( SO(3) \) with the generators of the symmetry satisfying some Lie algebra \([T_P, T_Q] = i f_{PQR} T_R\). Let us take the group to be \( SU(2) \). i.e. say, \( \alpha_\nu = \alpha_\nu P T_P \); \( P, Q, R = 1, 2, 3 \); \( T_P \) being the generators of \( SU(2) \). Again choosing \( \alpha_\nu = (0, \alpha_\nu P T_P) \) with \( \vec{\alpha}_P = \alpha_P (r) \hat{r} \) and using the well known properties of the Pauli matrices it is easily seen that the analogue of equation (7b) is

\[
div \vec{\alpha}_P = 0
\]

which has the solution

\[
\alpha^i_P = A_P \frac{r^i}{r^3}
\]

where \( A_P \) are constants. Writing \( r^i_P = A_P r^i = r^3 \alpha^i_P \), we can then define new coordinates

\[
R^i = r^i_P T_P \ ; [R^i, R^j] \neq 0
\]

and these are non-commuting. Moreover, they carry both Lorentz and internal indices and hence are like gauge fields in some different theory. Note that transverse coordinates (i.e. transverse to the brane and lying in the
bulk volume) in D brane theories are often identified with gauge fields [8] and so we can construct such actions with our solutions (14). Under these circumstances, equation (6) should be written as

$$\partial^P_\mu \phi_a - e \epsilon_{abc} (W^b_\mu)^P \phi_c = \alpha^P_\mu \phi_a$$

where the coordinates and their differentials are now matrices and capital alphabets denote the indices of the new symmetry group. For fixed $P$, $(W^b_\mu)^P$ may be identified with the old gauge fields $W^b_\mu$.

A point to note is that we have taken the symmetry group for $\alpha_\nu$ to be some group other than $SO(3)$. This is to ensure in the simplest possible way that the fields $(W^b_\mu)^P$ for fixed $P$ may be identified with the old (i.e.unchanged) gauge fields $W^b_\mu$ ($P$ is now a fixed index) and so the minimum energy requirements are satisfied in each sector of $P = 1, 2, 3$. (This is seen by taking the cross product of $\phi$ with the analogue of equation (6) and proceeding as before). So each sector now contains a monopole. Then we have a configuration that is quite similar to "string" of monopole solutions connecting two D-branes. Such configurations are known in the literature [8]. The other point is that the vacuum expectation value of the Higgs field is proportional to the inverse of the coupling $e$; and this result has been obtained from the classical solutions. This result is similar to that obtained in Ref.[2b] if we are ready to identify the inverse of Newton’s gravitational constant (which is definitely the coupling constant in theories of gravity) as the vacuum expectation value of some field hitherto unknown.

The solutions in equation (6) were hidden in 't Hooft-Polyakov’s work. This had been overlooked before for the simple reason because at that point of time one was more concerned in obtaining solutions from the minimum
(finite) energy principles. This was perfectly justified. We have obtained the solutions from the requirement of duality invariance which is quite relevant at this point of time. However, we have also shown that the duality requirements automatically contain the minimum energy condition (Λ is also finite for $D_\nu \phi = 0$). All the results have been obtained at $r \to \infty$. That is, we are at the boundary of the theory. So the finiteness of Λ at the boundary encodes the duality invariance of the theory within the boundary and thus an analogue of the holographic principle [9] seems to be at work. On the boundary there seems to exist a different gauge field theory together with non-commuting coordinates.

2. Dyon charge quantisation in abelian $p$-form theories

In [1] the formalism of spacetime dependent lagrangians was used to obtain the Dirac quantisation condition. Here we shall follow the same method to obtain the results of Deser,Gomberoff,Henneaux and Teitelboim [10] regarding dyon charge quantisation in abelian, $p$–form theories. Our results will be obtained from a simple generalisation of the lagrangian constructed in [1] and a generalisation of the interaction terms using the completely antisymmetric symmetric tensor $\epsilon_{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and the symmetric tensor $\rho_{ab} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

In [1] we considered a $U(1) \otimes U(1)$ gauge invariant theory. $A_\mu$ and $B_\mu$ were four-vector potentials corresponding to electric ($e$) and magnetic ($g$) charges; $F_{\mu\nu}$, $G_{\mu\nu}$ were the respective field strengths; $j_\mu$, $k_\mu$ were the electric and magnetic (current) sources with interactions between respective currents.
and potentials introduced in the usual way

\[ L_1 = -(1/4)F^{\mu\nu}F_{\mu\nu} - (1/4)G^{\mu\nu}G_{\mu\nu} - j^\mu A_\mu - k^\mu B_\mu \]  \hspace{1cm} (15a)

with \( F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \); \( G^{\mu\nu} = \partial^\mu B^\nu - \partial^\nu B^\mu \); \( \tilde{G}^{\mu\nu} = (1/2)e^{\mu\nu\rho\sigma}G_{\rho\sigma} \); \( \partial^\mu j_\mu = \partial^\mu k_\mu = 0 \) (current conservation); \( \partial^\mu A_\mu = \partial^\mu B_\mu = 0 \) (transversality); \( \partial^\mu F_{\mu\nu} = j_\nu \); \( \partial^\mu \tilde{F}_{\mu\nu} = 0 \); \( \partial^\mu G_{\mu\nu} = k_\nu \); \( \partial^\mu \tilde{G}_{\mu\nu} = 0 \). Defining (note that \( \tilde{\tilde{F}} = -F \) and \( \tilde{\tilde{G}} = -G \))

\[ \xi^{\mu\nu} = F^{\mu\nu} + \tilde{G}^{\mu\nu} \; \tilde{\xi}^{\mu\nu} = F^{\mu\nu} - \tilde{G}^{\mu\nu} \]  \hspace{1cm} (15b)

means \( \partial^\mu \xi_{\mu\nu} = j_\nu \); \( \partial^\mu \tilde{\xi}_{\mu\nu} = -k_\nu \). A complex interaction term \( if(\Lambda)\alpha A^\mu B_\mu j^\nu k^\nu \) was introduced where \( f(\Lambda) \) was a dimensionless function of \( \Lambda \), and the space-time dependent lagrangian was written as

\[ L = \left[ -(1/4)F^{\mu\nu}F_{\mu\nu} - (1/4)G^{\mu\nu}G_{\mu\nu} - j^\mu A_\mu - k^\mu B_\mu + if(\Lambda)\alpha A^\mu B_\mu j^\nu k^\nu \right] \Lambda(x) \]  \hspace{1cm} (15c)

Equations of motion using (1) were set up, duality invariance imposed and the solution for \( \Lambda \) obtained for appropriate sources \( j_\mu, k_\mu \). Finiteness of \( \Lambda \) at \( r \to \infty \) led to the Dirac quantisation condition. The \( U(1) \otimes U(1) \) invariance of the original theory was broken.

We now use the above procedure to obtain the dyon charge quantisation condition for abelian \( p \)-form theories. First consider dimension \( D = 4 \). Then \( p = 1 \). There are now two objects, each of which carries both electric (e) and magnetic (g) charges. Accordingly, there will be two \( F \) ’s, two \( G \) ’s two \( A \) ’s, two \( B \) ’s and two \( j \) ’s and two \( k \) ’s. Let the index \( a = 1, 2 \) denote this. We next choose the interaction term as \( if(\Lambda)e^{bc}\alpha A_a^\mu B_\mu j^\nu k_c^\nu \). Then the generalisation of the lagrangian (15a) becomes

\[ L = \left[ -(1/4)F^{\mu\nu}_a F_{a \mu\nu} - (1/4)G^{\mu\nu}_a G_{a \mu\nu} - j^\mu A_a^\mu - k^\mu B_a^\mu \right] \]
\[ +i f(\Lambda) \alpha \epsilon^{bc} A^\mu_a B_{a \mu} j^\nu_b k_{c \nu} \Lambda(x) \] (16)

As before \( \xi^{\mu \nu}_a = F^{\mu \nu}_a + \tilde{G}^{\mu \nu}_a \); \( \tilde{\xi}^{\mu \nu}_a = F^{\mu \nu}_a - \tilde{G}^{\mu \nu}_a \). Equations of motion that follow from (1) are (for each \( a = 1, 2 \)):

\[ \Lambda(\partial^\mu \xi^{\mu \nu}_a) + [(\partial^\mu \Lambda) F_{a \mu \nu} - \Lambda(j_{a \nu} + ic_{a \nu})] = 0 \] (17a)

\[ \Lambda(\partial^\mu \tilde{\xi}^{\mu \nu}_a) - [(\partial^\mu \Lambda) G_{a \mu \nu} - \Lambda(k_{a \nu} + id_{a \nu})] = 0 \] (17b)

where \( c_{a \nu} = f(\Lambda) \alpha \epsilon^{bc} j^\mu_b k_{c \mu} B_{a \nu} \); \( d_{a \nu} = f(\Lambda) \alpha \epsilon^{bc} j^\mu_b k_{c \mu} A_{a \nu} \). Duality invariance means \( \partial^\mu \xi^{\mu \nu}_a = 0 \) and \( \partial^\mu \tilde{\xi}^{\mu \nu}_a = 0 \). This therefore implies

\[ (\partial^\mu \Lambda) F_{a \mu \nu} - \Lambda(j_{a \nu} + ic_{a \nu}) = 0 \] (18a)

\[ (\partial^\mu \Lambda) G_{a \mu \nu} - \Lambda(k_{a \nu} + id_{a \nu}) = 0 \] (18b)

To solve the above for specific sources we take \( j^\nu_a = e_a \int dx^\nu \delta^4(x) \); \( k^\nu_a = g_a \int dx^\nu \delta(x_3 - b) \delta^3(x) \). Now assume \( \Lambda = \Lambda(x_3) \) and that only the \( \nu = 0 \) component of the sources are present so that \( j^0_1 = e_1 \delta(x_1) \delta(x_2) \delta(x_3) \); \( k^0_1 = g_a \delta(x_1) \delta(x_2) \delta(x_3) \) and \( j^0_2 = e_2 \delta(x_1) \delta(x_2) \delta(x_3 - b) \); \( k^0_2 = g_2 \delta(x_1) \delta(x_2) \delta(x_3 - b) \). Then we get for \( \nu = 0, 1, 2 \)

\[ (\partial^3 \Lambda) F_{a \ 3\nu} = \Lambda(j_{a \nu} + ic_{a \nu}) \] (18a)

\[ (\partial^3 \Lambda) G_{a \ 3\nu} = \Lambda(k_{a \nu} + id_{a \nu}) \] (18b)

For \( \nu = 3 \), \( F_{a \ 33} = G_{a \ 33} = 0 \) for all \( a \), and the solutions to (18a) and (18b) for \( \nu = 0 \) are:

\[ \Lambda_{\infty} = \Lambda_{-\infty} \exp[e_a \delta(x_1) \delta(x_2)/F_{a \ 30}(x_0, x_1, x_2, 0)] \]

\[ \exp[i f(\Lambda) \alpha \epsilon^{bc} e_b g_e P_{a \ 0}(x_0, x_1, x_2, b)] \]

\[ = \Lambda_{-\infty} \exp[e_a \delta(x_1) \delta(x_2)/F_{a \ 30}(x_0, x_1, x_2, 0)] \]
\[
\exp[i f(\Lambda) \alpha (e_1 g_2 - e_2 g_1) P_{a_0}(x_0, x_1, x_2, b)] \quad (19a)
\]
\[
\Lambda_\infty = \Lambda_{-\infty} \exp[g_a \delta(x_1) \delta(x_2)/G_{a_{30}}(x_0, x_1, x_2, 0)]
\]
\[
\exp[i f(\Lambda) \alpha e_b e_c g_c Q_{a_0}(x_0, x_1, x_2, b)] = \Lambda_{-\infty} \exp[e_a \delta(x_1) \delta(x_2)/G_{a_{30}}(x_0, x_1, x_2, 0)]
\]
\[
P_{a_0}(x_0, x_1, x_2, b) = (\delta(x_1))^2(\delta(x_2))^2 \delta(b) B_{a_0}(x_0, x_1, x_2, b)/F_{a_{30}}(x_0, x_1, x_2, b)
\quad (20a)
\]
\[
Q_{a_0}(x_0, x_1, x_2, b) = (\delta(x_1))^2(\delta(x_2))^2 \delta(b) A_{a_0}(x_0, x_1, x_2, b)/G_{a_{30}}(x_0, x_1, x_2, b)
\quad (20b)
\]

Proceeding as in ref. [1], choose \(\Lambda_\infty = \Lambda_{-\infty} = 1\) and consider the set of equations (8a) and (9a). The two exponentials must reduce to unity. For the first exponential this implies the Dirac string configuration where \(F_{a_{30}} \to \infty\), and so the exponential becomes unity. For the second exponential, the numerator in (9a) has singular \(\delta\)-functions and together with \(B_{a_{30}} \to \infty\) since \(F_{a_{30}} \to \infty\). So second exponential is unity if \(\exp[i f(\Lambda) \alpha (e_1 g_2 - e_2 g_1) P_{a_0}] = 1\), i.e. \(\exp[i f(\Lambda) \alpha (e_1 g_2 - e_2 g_1)]P_{a_0}^o = 1\) (as \(P_{a_0}\) is finite). Therefore

\[
f(\Lambda) \alpha (e_1 g_2 - e_2 g_1) = 2\pi n \quad (10)
\]

All the above results are true in each sector, viz., \(a = 1, 2\). As in ref. [1], there are two possibilities: (a) \(f(\Lambda) = 0\). Then the \(U(1) \otimes U(1)\) invariance in each sector of \(L\) is unbroken and we have the Dirac string configuration from the first exponential \(F_{a_{30}} \to \infty\). (b) \(f(\Lambda) = \text{a finite constant}\). Then the \(U(1) \otimes U(1)\) invariance in each sector of \(L\) is broken and putting \(\alpha = (\hbar)^{-1}\), we get the Dirac quantisation condition for dyons. For \(\nu = 1, 2\) a similar analysis will again lead to (10) and similarly for the set of equations 8(b)
and (9b). Note that we have taken the same function $\Lambda$ in each sector of the theory. This is justifiable from the fact that finally we proceed to the case of $\Lambda$ becoming unity in each sector.

Now consider dimension $D = 6$. This means $p = 2$. So we have 2-form potentials $A^{\mu\nu}_a$; $B^{\mu\nu}_a$. Then each of the antisymmetric field strengths $F, G$ will be a 3-form in the Lorentz indices and we have the following constructions:

$$3\text{- form field strengths: } F^{\mu\nu\sigma}_a = \partial^{\mu} A^{\nu\sigma}_a - \partial^\nu A^{\mu\sigma}_a + \partial^\sigma A^{\mu\nu}_a$$

$$G^{\mu\nu\sigma}_a = \partial^{\mu} B^{\nu\sigma}_a - \partial^\nu B^{\mu\sigma}_a + \partial^\sigma B^{\mu\nu}_a$$

$$\xi^{\mu\nu\sigma}_a = F^{\mu\nu\sigma}_a + G^{\mu\nu\sigma}_a; \ \tilde{\xi}^{\mu\nu\sigma}_a = F^{\mu\nu\sigma}_a - G^{\mu\nu\sigma}_a$$

2-form antisymmetric potentials $A^{\mu\nu}_a$; $B^{\mu\nu}_a$ (antisymmetric w.r.t. $\mu, \nu$)

Dual: $\tilde{F}^{\mu\nu\sigma}_a = (1/3!)\epsilon^{\mu\nu\sigma\alpha\beta\gamma} F^a_{\alpha\beta\gamma}; \ \tilde{G}^{\mu\nu\sigma}_a = (1/3!)\epsilon^{\mu\nu\sigma\alpha\beta\gamma} G^a_{\alpha\beta\gamma}$

2-form currents: $j^{\mu\nu}_a = e_a \int dx^\mu \wedge dx^\nu \delta^6(x); \ k^{\mu\nu}_a = g_a \int dx^\mu \wedge dx^\nu \delta^6(x)$

Now assume that the only non-zero currents are $j^{0\nu}_a$ and $k^{0\nu}_a; \ \Lambda = \Lambda(x_5)$ and take the lagrangian as

$$L = [-(1/12)F^{\mu\nu}_a F_a^{\mu\nu} - (1/12)G^{\mu\nu}_a G_a^{\mu\nu} - (1/4)j^\mu_a A^\mu_a - (1/4)k^\mu_a B^\mu_a ] \Lambda(x) + i f(\Lambda) \rho^{bc} A^\mu_a B^\mu_a j^\nu_b k_c \Lambda(x)$$

where the matrix $\rho_{ab} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. It is then straightforward to obtain the dyon quantisation condition by proceeding exactly as before and the result is

$$e_1 g_2 + e_2 g_1 = 2\pi n \hbar$$

Thus the quantisation condition depends on whether $p$ is odd or even.
In fact, the above procedure can be generalised to arbitrary $p$-form fields by constructing appropriate field strengths $F,G$ and choosing the lagrangian as

$$L = [-\frac{1}{2}(1/(p+1))!F_{a \mu_1..\mu_{p+1}}F_{a \mu_1..\mu_{p+1}} - (1/2)(1/(p+1))!]G_{a \mu \nu}G_{a \mu \nu}$$

$$-\frac{1}{2}(1/p!)j_a^{\mu_{1..p}}A_{a \mu_1..\mu_p} - (1/2)(1/p!)k_a^{\mu_{1..p}}B_{a \mu_1..\mu_p}$$

$$+ i f(\Lambda) \alpha \Omega^{bc} A_{a \mu_1..\mu_p} B_{b \mu_1..\mu_p} j_{c \mu_1..\mu_p} A_{b \mu_1..\mu_p} k_{a \mu_1..\mu_p} \Lambda(x)$$

(13)

where the matrix

$$\Omega_{ab} = (1/2)[(1 + (-1)^{p+1})\epsilon_{ab} + (1 + (-1)^p)\rho_{ab}]$$

(14)

Currents will be defined as

$$j_a^{\mu_1..\mu_p} = e_a \int dx^{\mu_1} \wedge dx^{\mu_2} ... \wedge dx^{\mu_p}$$

(15a)

$$k_a^{\mu_1..\mu_p} = g_a \int dx^{\mu_1} \wedge dx^{\mu_2} ... \wedge dx^{\mu_p}$$

(15b)

Assuming as before that the only non vanishing currents are $j_a^{0\mu_1..\mu_{p-1}}$ and $k_a^{0\mu_1..\mu_{p-1}}$ and $\Lambda = \Lambda(x_i)$, where $i$ is some spatial coordinate, one can solve the relevant equations to get the dyon quantisation condition again. Depending on whether $p$ is odd or even we will have

$$e_1g_2 + (-1)^p e_2g_1 = 2\pi n\hbar$$

(16)

We mention that for odd $p$ we will have anti-selfdual field strengths, while for even $p$ we will have selfdual field strengths.

In conclusion, the spacetime dependent lagrangian formalism in conjunction with the 't Hooft-Polyakov results have yielded an expression for the vacuum expectation value of the Higgs field as $A/e$. This result is definitely
susceptible to experiments. We have also shown that the 't Hooft ansatz for
the gauge field is sufficient to obtain an expression for the Higgs field if one
uses our formalism. No additional ansatz for $\phi$ is necessary. The expres-
sion obtained reduces to the 't Hooft ansatz for the Higgs field at $r \to \infty$.
Finally, we have shown that classical solutions of Yang-Mills theory also con-
tain the germ of non-commuting coordinates residing on the boundary. The
structure of these coordinates are like gauge fields and hence are relevant in
constructing $D$-brane actions.

In conclusion, we have shown that the spacetime dependent lagrangian
formulation of electromagnetic duality can also accommodate the results of
[3]. The dependence of the quantisation condition on $p$ [3,4] is also accom-
modated. In our scheme this has to do with the fact that coupling in the
interaction lagrangian depends on $p$ through the matrix $\Omega_{ab}$. The impor-
tance of the dyon charge quantisation in the theory of $D$-branes have been
exhaustibly studied in [3]. So we do not elaborate on this. However, our
formalism provides an alternate interaction lagrangian picture of the same.
The holographic principle [5] is again illustrated—the finite behaviour of $\Lambda$
on the boundary gives rise to the exotic solutions within the bulk volume.

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References

[1] R.Bhattacharyya and D.Gangopadhyay, Mod.Phys.Lett. A15, 901
   (2000).
[2] R.Bhattacharyya and D.Gangopadhyay, Mod.Phys.Lett. A17, 729 (2002); D.Gangopadhyay,R.Bhattacharyya,L.P.Singh, Spacetime Dependent Lagrangians and the Barriola-Vilenkin Monopole Mass, hep-th/0208097.

[3] P.Goddard and D.Olive, Rep.Prog.Phys. 41, 1357 (1978).

[4] S.Coleman, Phys.Rev. D11 (1975) 2088; S.Mandelstam, Phys.Rev. D11 (1975) 3026.

[5] N.Dadhich, Mod.Phys.Lett. A14, 337 (1999);

[6] M.Barriola and A.Vilenkin, Phys.Rev.Lett. 63, 341 (1989).

[7] E.Corrigan and D.Olive, Nucl.Phys. B110,237 (1976); E.Corrigan, D.Olive and J.Nuyts, Nucl.Phys. B106, 475 (1976).

[8] J.Polchinski, String Theory, Vol 2, Cambridge Univ. Press, 1998.

[9] G.t’Hooft, Dimensional Reduction in Quantum Gravity, gr-qc/9310006; L.Susskind, Phys.Rev. D49 (1994) 6606; J.D.Bekenstein, Phys.Rev. D49 (1994) 1912; J.Maldacena, Adv.Theor.Math.Phys. 2 (1998) 231; E.Witten, Adv.Theor.Math.Phys. 2 (1998) 253; L.Susskind and E.Witten, The Holographic bound in Anti-de Sitter Space, hep-th/9805112; O.Aharony, S.S.Gubser,J.Maldacena,H.Ooguri and Y.Oz, Large N Field Theories, String Theory and Gravity, hep-th/9905111; N.Seiberg and E.Witten, String theory and noncommutative geometry, hep-th/9908143.

[10] S.Deser, A.Gomberoff,M.Henneaux, Nucl.Phys. B520, 179 (1998).

[11] S.Deser, M.Henneaux, A.Schwimmer, Phys. Lett. B428, 284 (1998).