STRONG UNIQUE CONTINUATION AND LOCAL ASYMPTOTICS AT THE
BOUNDARY FOR FRACTIONAL ELLIPTIC EQUATIONS

ALESSANDRA DE LUCA, VERONICA FELLI AND STEFANO VITA

Abstract. We study local asymptotics of solutions to fractional elliptic equations at boundary points,
under some outer homogeneous Dirichlet boundary condition. Our analysis is based on a blow-up
procedure which involves some Almgren type monotonicity formulæ and provides a classification of all
possible homogeneity degrees of limiting entire profiles. As a consequence, we establish a strong unique
continuation principle from boundary points.

1. Introduction and main results

Let $N \geq 2$ and $s \in (0, 1)$. Dealing with nontrivial solutions to the following fractional equation
\[(1.1)\] 
$$(-\Delta)^s u = hu \quad \text{in } \Omega$$
where $\Omega \subset \mathbb{R}^N$ is a bounded domain, we are interested in a strong unique continuation property and local
asymptotics of solutions at boundary points where the domain is locally $C^{1,1}$ and some outer homogeneous
Dirichlet boundary condition is prescribed.

A family of solutions to some elliptic equation is said to satisfy the strong unique continuation property
if no element of the family has a zero of infinite order, except for the null function.

Asymptotic expansions of solutions to fractional elliptic equations at interior points of the domain
were derived in [13], even in the presence of singular homogeneous potentials, by combining Almgren
type monotonicity formulæ with blow-up arguments; as a relevant byproduct of such sharp asymptotic
analysis, in [13] unique continuation principles were established. The difficulty of defining a suitable
Almgren’s type frequency function in a non-local setting was overcome in [13] by considering the Caffarelli-
Silvestre extension [6], which provides an equivalent formulation of the fractional equation as a local
problem in one dimension more. For local problems such as second order elliptic equations, the classical
approach developed by Garofalo and Lin [23] allows deriving unique continuation directly from doubling
conditions obtained as a consequence of the boundedness of an Almgren type frequency function. In the
fractional case instead, the monotonicity formula and the doubling type conditions obtained in [13] imply
unique continuation properties only for the extended local problem and not for the fractional one; then
in [13] the further step of classification of blow-up limits is performed in order to derive first asymptotic
estimates, and then unique continuation principles, in the spirit of [19, 20], see also [14].

Since [13], the literature devoted to unique continuation for fractional problems has flourished produc-
ing many important results in several directions; we mention, among others, [31] for unique continuation
in presence of rough potentials by Carleman estimates, [41] for fractional operators with variable coeffi-
cients, and [17, 18, 22, 32, 33, 34, 40] for higher order fractional problems.

The aim of the present paper is to extend the results of [13] to boundary points of the domain, i.e.
to establish sharp asymptotics and unique continuation from boundary points for fractional equations of
type (1.1). Possible loss of regularity and unavoidable interference with the geometry of the domain make
the derivation of monotonicity formulæ around boundary points, and consequently the proof of unique
continuation, more difficult and, at the same time, produce new interesting phenomena: in particular, in
[16] it was shown that, under homogeneous Dirichlet boundary conditions, the possible vanishing rates
of solutions at conical boundary points depend of the opening of the vertex. Unique continuation from
the boundary for elliptic equations was also investigated in [2, 3, 26, 39] under homogeneous Dirichlet
conditions and in [10, 38] under Neumann type conditions. Furthermore, we refer to [15] for unique
continuation from Dirichlet-Neumann junctions for planar mixed boundary value problems and to [9] for
unique continuation from the edge of a crack.

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The fractional Laplacian $(-\Delta)^s$ can be defined as the Riesz isomorphism of $\mathcal{D}^{s,2}(\mathbb{R}^N)$ with respect to the scalar product
\begin{equation}
(u,v)_{\mathcal{D}^{s,2}(\mathbb{R}^N)} := \int_{\mathbb{R}^N} |\xi|^{2s} \hat{u}(\xi)\hat{v}(\xi) \, d\xi
\end{equation}
and the associated norm $\|u\|_{\mathcal{D}^{s,2}(\mathbb{R}^N)} = ((u,u)_{\mathcal{D}^{s,2}(\mathbb{R}^N)})^{1/2}$, where $\hat{u}$ denotes the unitary Fourier transform of $u$ in $\mathbb{R}^N$, i.e.
\[ \hat{u}(\xi) = \mathcal{F}u(\xi) := \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{-ix\cdot\xi} u(x) \, dx. \]
The fractional Laplacian $(-\Delta)^s$ can be defined as the Riesz isomorphism of $\mathcal{D}^{s,2}(\mathbb{R}^N)$ with respect to the scalar product \((1.2)\), i.e.
\[ (-\Delta)^s : (u,v)_{\mathcal{D}^{s,2}(\mathbb{R}^N)} = (u,v)_{\mathcal{D}^{s,2}(\mathbb{R}^N)} \]
for all $u,v \in \mathcal{D}^{s,2}(\mathbb{R}^N)$. Then we can define a weak solution to \((1.1)\) as a function $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ satisfying
\begin{equation}
(u,\varphi)_{\mathcal{D}^{s,2}(\mathbb{R}^N)} = \int_{\Omega} h(x) u(x) \varphi(x) \, dx, \quad \text{for all } \varphi \in C_c^\infty(\Omega).
\end{equation}
As far as the potential term is concerned, we assume that
\begin{equation}
\text{there exists } p > \frac{N}{2s} \quad \text{such that } \quad h \in W^{1,p}(\Omega).
\end{equation}
We observe that the right hand side of \((1.3)\) is well defined in view of assumption \((1.4)\), Hölder’s inequality, and the following well-known Sobolev-type inequality
\begin{equation}
S_{N,s} \|u\|_{L^{2^*(s)}(\mathbb{R}^N)} \leq \|u\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)},
\end{equation}
where $S_{N,s}$ is a positive constant depending only on $N$ and $s$ and
\begin{equation}
2^*(s) = \frac{2N}{N - 2s},
\end{equation}
see [8].

Let $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ be a solution to \((1.3)\). Let us assume that there exists a boundary point $x_0 \in \partial \Omega$ such that the boundary $\partial \Omega$ is of class $C^{1,1}$ in a neighbourhood of $x_0$, i.e. there exist $R > 0$ and $g \in C^{1,1}(\mathbb{R}^{N-1})$ such that, choosing a proper coordinate system $(x',x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$,
\begin{align}
B_R(x_0) \cap \Omega &= \{(x',x_N) \in B_R(x_0) : x_N < g(x') \} \quad \text{and} \\
B'_R(x_0) \cap \partial \Omega &= \{(x',x_N) \in B'_R(x_0) : x_N = g(x') \},
\end{align}
where $B'_R(x_0) = \{x \in \mathbb{R}^N : |x - x_0| < R\}$ is the ball in $\mathbb{R}^N$ centered at $x_0$ with radius $R$. We prescribe for the solution $u$ a local outer homogeneous Dirichlet boundary condition, i.e.
\begin{equation}
u = 0 \quad \text{a.e. in } \Omega^c \cap B'_R(x_0).
\end{equation}
By the extension technique introduced in [6], by adding an additional space variable $t \in [0, +\infty)$, we can reformulate the nonlocal problem \((1.1)\) as a local degenerate or singular problem on the half space \[ \mathbb{R}^{N+1}_+ = \mathbb{R}^N \times (0, +\infty). \]
We denote the total variable $z = (x,t) \in \mathbb{R}^{N+1}_+ \times (0, +\infty)$, with $x = (x',x_N) = (x_1, \ldots, x_{N-1}, x_N)$, and define $\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+, t^{1-2s} \, dz)$ as the completion of $C_c^\infty(\mathbb{R}^{N+1}_+)$ with respect to the norm
\[ \|U\|_{\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+, t^{1-2s} \, dz)} = \left( \int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla U(x,t)|^2 \, dx \, dt \right)^{1/2}. \]
It is well-known that there exists a continuous trace map \( \text{Tr} : \mathcal{D}^{1,2}(\mathbb{R}^{N+1}, t^{1-2s} dz) \to \mathcal{D}^{s,2}(\mathbb{R}^N) \) which is onto, see [5]. By [5], for every \( u \in \mathcal{D}^{s,2}(\mathbb{R}^N) \), the minimization problem
\[
\min \left\{ \|W\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^{N+1}, t^{1-2s} dz)} : W \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}, t^{1-2s} dz), \text{Tr} W = u \right\}
\]
admits a unique minimizer \( U = \mathcal{H}(u) \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}, t^{1-2s} dz) \), which can be obtained by convoluting \( u \) with the Poisson kernel of the half-space \( \mathbb{R}^{N+1}_+ \) and weakly solves
\[
\begin{align*}
-\text{div}(t^{1-2s} \nabla U) &= 0 \quad &\text{in } \mathbb{R}^{N+1}_+, \\
\lim_{t \to 0^+} t^{1-2s} \partial_t U &= \kappa_s (-\Delta)^s u \quad &\text{in } \mathbb{R}^N \times \{0\},
\end{align*}
\]
where
\[
\kappa_s = \frac{\Gamma(1-s)}{2^{2s-1} \Gamma(s)} > 0,
\]
i.e.
\[
\int_{\mathbb{R}^{N+1}_+} t^{1-2s} \nabla \mathcal{H}(u)(x, t) \cdot \nabla W(x, t) \, dx \, dt = \kappa_s (u, \text{Tr} W)_{\mathcal{D}^{s,2}(\mathbb{R}^N)} \quad \text{for all } W \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+, t^{1-2s} dz).
\]
As a consequence, \( u \in \mathcal{D}^{s,2}(\mathbb{R}^N) \) is a solution \((1.3)\) if and only if its extension \( U = \mathcal{H}(u) \) weakly solves
\[
\begin{align*}
-\text{div}(t^{1-2s} \nabla U) &= 0 \quad &\text{in } \mathbb{R}^{N+1}_+, \\
\text{Tr} U &= u \quad &\text{in } \mathbb{R}^N \times \{0\},
\end{align*}
\]
in a weak sense, i.e.
\[
\int_{\mathbb{R}^{N+1}_+} t^{1-2s} \nabla U(x, t) \cdot \nabla \phi(x, t) \, dx \, dt = \kappa_s \int_{\Omega} hu \text{Tr} \phi \, dx
\]
for every \( \phi \in C^\infty(\overline{\mathbb{R}^{N+1}_+}) \) with \( \text{Tr} \phi \in C^\infty(\Omega) \).

The asymptotic behaviour at \( x_0 \in \partial \Omega \) of solutions to \((1.9)\), and consequently to \((1.4)\), will turn out to be related to eigenvalues and eigenfunctions of the following weighted spherical eigenvalue problem with mixed Dirichlet-Neumann boundary conditions
\[
\begin{align*}
- \text{div}_{S^N} \left( t^{-2s} \nabla_{S^N} \psi \right) &= \theta_{N+1}^{-2s} \mu \psi \quad &\text{in } S^N, \\
\psi &= 0 \quad &\text{on } S^{N-1} \cap \{ \theta_N \geq 0 \}, \\
\lim_{t \to 0^+} t^{-2s} \nabla_{S^N} \psi \cdot \nu &= 0 \quad &\text{on } S^{N-1} \cap \{ \theta_N < 0 \},
\end{align*}
\]
on the half-sphere
\[
S^N_+ = \{ (\theta_1, \ldots, \theta_N, \theta_{N+1}) \in S^N : \theta_{N+1} > 0 \}
\]
where \( \nu = (0, 0, \ldots, 0, -1) \) and \( \partial S^N_+ = S^{N-1} \times \{0\} \) is identified with \( S^{N-1} \). In order to write the variational formulation of \((1.11)\), we define \( H^1(S^N_+, \theta_{N+1}^{-2s} dS) \) as the completion of \( C^\infty_c(S^N_+) \) with respect to the norm
\[
\|
\psi
\|_{H^1(S^N_+, \theta_{N+1}^{-2s} dS)} = \left( \int_{S^N_+} \theta_{N+1}^{-2s} \left( |\nabla_{S^N} \psi(\theta)|^2 + \psi^2(\theta) \right) dS \right)^{1/2},
\]
where \( dS \) denotes the volume element on \( N \)-dimensional spheres. Let \( \mathcal{H}_0 \) be the closure of \( C^\infty_c(S^N_+) \) in \( H^1(S^N_+, \theta_{N+1}^{-2s} dS) \), where \( S^N_+ = \{ (\theta', \theta_N, 0) \in S^{N-1} : \theta_N \geq 0 \} \). We say that \( \mu \in \mathbb{R} \) is an eigenvalue of \((1.11)\) if there exists \( \psi \in \mathcal{H}_0 \setminus \{0\} \) such that
\[
\int_{S^N_+} \theta_{N+1}^{-2s} \nabla_{S^N} \psi \cdot \nabla_{S^N} \phi dS = \mu \int_{S^N_+} \theta_{N+1}^{-2s} \psi \phi dS \quad \text{for any } \phi \in \mathcal{H}_0.
\]
By classical spectral theory, problem \((1.11)\) admits a diverging sequence of real eigenvalues with finite multiplicity \( \{k_\mu \}_{k \geq 0} \). In Appendix [13] we obtain the following explicit formula for such eigenvalues
\[
\mu_k = (k + s)(k + N - s), \quad k \in \mathbb{N}.
\]
For all \( k \in \mathbb{N} \), let \( M_k \in \mathbb{N} \setminus \{0\} \) be the multiplicity of the eigenvalue \( \mu_k \) and \( \{Y_{k,m}\}_{m=1,2,\ldots,M_k} \) be a \( L^2(S^N_+, \theta_{N+1}^{-2s} dS) \)-orthonormal basis of the eigenspace of problem \((1.11)\) associated to \( \mu_k \). In particular,
\[
\{Y_{k,m} : k \in \mathbb{N}, \ m = 1, \ldots, M_k \}
\]
is an orthonormal basis of $L^2(S^{N-1}, \theta_{N+1}^{-2}ds)$. 

Remark 1.1. It is worth highlighting the fact that eigenfunctions of problem (1.11) cannot vanish identically on $S^{N-1} \cap \{\theta_N < 0\}$, i.e. on the boundary portion where a Neumann homogeneous condition is imposed. Indeed, if an eigenfunction $\psi$ associated to the eigenvalue $\mu_k = (k+s)(k+N-s)$ vanishes on $S^{N-1} \cap \{\theta_N < 0\}$, then the function $\Psi(\rho \theta) = \rho^{s+1}\psi(\theta)$ would be a weak solution to the equation $\text{div}(t^{1-2s} \nabla \Psi) = 0$ in $\mathbb{R}^{N-1} \times (-\infty, 0) \times (0, +\infty)$ satisfying both Dirichlet and weighted Neumann homogeneous boundary conditions on $\mathbb{R}^{N-1} \times (-\infty, 0) \times \{0\}$; then its trivial extension to $\mathbb{R}^{N-1} \times (-\infty, 0) \times \mathbb{R}$ would violate the unique continuation principle for elliptic equations with Muckenhoupt weights proved in [39] (see also [23], [35, Corollary 3.3], and [41] Proposition 2.2).

Our first result is a sharp description of the asymptotic behaviour of solutions to (1.1) at a boundary point.

Theorem 1.2. Let $\Omega$ be a bounded domain in $\mathbb{R}^N$ such that there exist $g \in C^{1,1}(\mathbb{R}^{N-1})$, $x_0 \in \partial \Omega$ and $R > 0$ satisfying (1.7). Let $h$ satisfy (1.3) and $u \in D^{s,2}(\mathbb{R}^N)$, $u \not\equiv 0$, be a weak solution to (1.1) in the sense of (1.3), satisfying (1.8). Then there exists $k_0 \in \mathbb{N}$ and an eigenfunction $Y$ of problem (1.11) associated to the eigenvalue $\mu_{k_0} = (k_0 + s)(k_0 + N - s)$ such that

$$u(x_0 + \lambda x) \not\equiv |x|^{k_0+s}Y \left(\frac{x}{|x|}\right) \quad \text{in } H^s(B_1^*) \quad \text{as } \lambda \to 0^+,$$

where $H^s(B_1^*)$ is the usual fractional Sobolev space on the $N$-dimensional unit ball $B_1^* = B_1^*(0)$.

Theorem 1.2 will be proved as a consequence of the following description of the asymptotic behaviour of nontrivial solutions to (1.1) near $x_0 \in \partial \Omega$.

Theorem 1.3. Let $\Omega$ be a bounded domain in $\mathbb{R}^N$ such that there exist $g \in C^{1,1}(\mathbb{R}^{N-1})$, $x_0 \in \partial \Omega$ and $R > 0$ satisfying (1.7). Let $h$ satisfy (1.3) and $u \in D^{1,2}(\mathbb{R}^{N+1}, t^{1-2s} dz)$ be a weak solution to (1.10) in the sense of (1.10), with $u \not\equiv 0$ and $\text{Tr} u = u$ satisfying (1.8). Then there exists $k_0 \in \mathbb{N}$ and an eigenfunction $Y$ of problem (1.11) associated to the eigenvalue $\mu_{k_0} = (k_0 + s)(k_0 + N - s)$ such that, letting $z_0 = (x_0, 0)$,

$$U(z_0 + \lambda z) \not\equiv |z|^{k_0+s}Y(z/|z|) \quad \text{in } H^1(B_1^*, t^{1-2s} dz) \quad \text{as } \lambda \to 0^+,$$

where $H^1(B_1^*, t^{1-2s} dz)$ is the weighted Sobolev space on the half ball $B_1^*$ defined in Section 2.

Actually the proof of Theorem 1.3 contains a more precise characterization of the angular limit profile $Y$, as a linear combination of the orthonormalized eigenfunctions $\{Y_{k_0, m}\}_{m=1,2,\ldots,M_{k_0}}$ of (1.11) associated to the eigenvalue $\mu_{k_0}$ with coefficients explicitly given by formula (1.62).

The salient consequence of the precise asymptotic expansions described above is the following strong unique continuation principle for problems (1.1) and (1.9), whose proof follows straightforwardly from Theorems 1.2 and 1.3 taking into account Remark 1.1.

Corollary 1.4.

(i) Under the same assumptions as in Theorems 1.2 let $u \in D^{s,2}(\mathbb{R}^N)$ be a weak solution to (1.1) (in the sense of (1.3)) satisfying (1.8) and such that $u(x) = O(|x - x_0|^k)$ as $x \to x_0$, for any $k \in \mathbb{N}$. Then $u \equiv 0$ in $\mathbb{R}^N$.

(ii) Under the same assumptions as in Theorems 1.3 let $U \in D^{1,2}(\mathbb{R}^{N+1}, t^{1-2s} dz)$ be a weak solution to (1.10) (in the sense of (1.10)) with $\text{Tr} U = u$ satisfying (1.8) and such that $U(z) = O(|z - z_0|^k)$ as $z \to z_0$, for any $k \in \mathbb{N}$. Then $U \equiv 0$ in $\mathbb{R}^{N+1}$.

The proof of Theorem 1.3 makes use of the procedure developed in [13, 19, 20] and consisting in a fine blow-up analysis for scaled solutions based on sharp energy estimates, obtained as a consequence of the existence of the limit of an Almgren type frequency function. Such method is applied to an equivalent auxiliary problem obtained by straightening the boundary of the domain $\Omega$ through a diffeomorphic deformation, which is inspired by [2] and built specifically to ensure that the extended equation is conserved by reflection through a straightened vertical boundary, see Section 2.1.

Significant additional difficulties arise with respect to the non-fractional or interior setting. Since the optimal regularity of solutions to (1.1) is $s$-Hölder continuity, see [39], solutions to (1.9) could have singular gradient at $\partial \Omega$, which represents for problem (1.9) the interface between mixed Dirichlet and Neumann boundary conditions. In fact, problems with mixed boundary conditions raise delicate regularity issues,
which turn out to be more difficult in dimension $N \geq 2$ due to the positive dimension of the junction set and some role played by the geometry of the domain.

In particular, we remark that the regularity results known for non-fractional problems (for which solutions are smooth up to the boundary) or for interior points (see e.g. [23]) are not available here, even excluding a neighborhood of the boundary point if $N \geq 2$. Therefore, for problems that, like (1.9), are characterized by mixed boundary conditions in dimension $N \geq 2$, the development of a monotonicity argument around points located at Dirichlet-Neumann junctions presents substantial new difficulties with respect to both the case treated in [13] of boundary points around which a Neumann condition is given and the case treated in [15] of mixed conditions in dimension $N = 1$.

In the present paper, the difficulties related to lack of regularity at Dirichlet-Neumann junctions are overcome by a double approximation procedure: by approximating the potential $h$ with potentials vanishing near the boundary and the Dirichlet $N$-dimensional region with smooth $(N+1)$-sets with straight boundary, we will be able to construct a sequence of approximating solutions (see Section 2.3) which enjoy enough regularity to derive Pohozaev type identities, needed to obtain Almgren type monotonicity formulae and consequently to perform blow-up analysis. We mention that a similar approximation procedure was developed in [9] for a class of elliptic equations in a domain with a crack.

The paper is structured as follows. In Section 2, after introducing a suitable functional setting for the study of the extended problem (1.9), we present an equivalent auxiliary problem obtained by straightening the boundary; then, after providing some Hardy-Sobolev and coercivity type inequalities, in Subsection 2.6 we perform the approximation procedure which allows us to establish a Pohozaev type inequality. The extension context is defined as the completion of $C^\infty(B^+_R)$ with respect to the norm

$$
\|U\|_{H^1(B^+_R, t^{1-2s} dz)} = \sqrt{\int_{B^+_R} t^{1-2s} (|U|^2 + |\nabla U|^2) dz}.
$$

It is well known, see e.g. [25], Proposition 2.1], that there exists a well-defined continuous trace map $\text{Tr} : H^1(B^+_R, t^{1-2s} dz) \to L^{2s(\cdot)}(B^+_R)$; in particular there exists a positive constant $C_{N,s}$ depending only on $N$ and $s$ such that, for all $r > 0$ and $U \in H^1(B^+_R, t^{1-2s} dz)$,

$$
\|\text{Tr}(U)\|_{L^{2s(\cdot)}(B^+_R)}^2 \leq C_{N,s} \int_{B^+_R} t^{1-2s} (r^{-2}|U(z)|^2 + |\nabla U(z)|^2) dz.
$$

We are interested in boundary qualitative properties of solutions to (1.1) close to a fixed point $x_0 \in \partial \Omega$. Without loss of generality, up to translation and rotation, we can assume $x_0 = 0$ and consider the extension $U = \mathcal{H}(u)$ which, under assumptions (1.7) and (1.8), solves

$$
\begin{cases}
-\text{div} (t^{1-2s}\nabla U) = 0 & \text{in } B^+_R, \\
-\lim_{t \to 0^+} t^{1-2s} \partial_t U = \kappa_s h u & \text{in } \Gamma_R^+, \\
U = 0 & \text{in } \Gamma_{g,R}^+ := \{(x', x_N, 0) \in B^+_R : x_N < g(x')\},
\end{cases}
$$

for some $R > 0$, where $g$ is the function which locally parametrizes the boundary $\partial \Omega$ around $x_0 = 0$ according to (1.7), with

$$
g \in C^{1,1}(\mathbb{R}^{N-1}), \quad g(0) = 0 \quad \text{and} \quad \nabla g(0) = 0.
$$

2. FUNCTIONAL SETTING AND CONSTRUCTION OF THE APPROXIMATING DOMAINS

Let us call the total variable $z = (x, t) \in \mathbb{R}^N \times \mathbb{R}$, with $x = (x', x_N) = (x_1, ..., x_{N-1}, x_N)$. We set the following notations for all $r > 0$:

$$
\begin{align*}
B_r &= \{z \in \mathbb{R}^{N+1} : |z| < r\}, \quad B^+_r = B_r \cap \mathbb{R}^{N+1}_+, \\
B'_r &= \{(x, t) \in B_r : t = 0\}, \quad \partial^+ B^+_r = \partial B_r \cap \mathbb{R}^{N+1}_+
\end{align*}
$$

In the sequel $B'_R$ will be sometimes identified with the ball in $\mathbb{R}^N$ centered at 0 with radius $r$. The weighted Sobolev space $H^1(B^+_R, t^{1-2s} dz)$ in the extension context is defined as the completion of $C^\infty(B^+_R)$ with respect to the norm

$$
\|U\|_{H^1(B^+_R, t^{1-2s} dz)} = \sqrt{\int_{B^+_R} t^{1-2s} (|U|^2 + |\nabla U|^2) dz}.
$$

It is well known, see e.g. [25], Proposition 2.1], that there exists a well-defined continuous trace map $\text{Tr} : H^1(B^+_R, t^{1-2s} dz) \to L^{2s(\cdot)}(B^+_R)$; in particular there exists a positive constant $C_{N,s}$ depending only on $N$ and $s$ such that, for all $r > 0$ and $U \in H^1(B^+_R, t^{1-2s} dz)$,

$$
\|\text{Tr}(U)\|_{L^{2s(\cdot)}(B^+_R)}^2 \leq C_{N,s} \int_{B^+_R} t^{1-2s} (r^{-2}|U(z)|^2 + |\nabla U(z)|^2) dz.
$$

We are interested in boundary qualitative properties of solutions to (1.1) close to a fixed point $x_0 \in \partial \Omega$. Without loss of generality, up to translation and rotation, we can assume $x_0 = 0$ and consider the extension $U = \mathcal{H}(u)$ which, under assumptions (1.7) and (1.8), solves

$$
\begin{cases}
-\text{div} (t^{1-2s}\nabla U) = 0 & \text{in } B^+_R, \\
-\lim_{t \to 0^+} t^{1-2s} \partial_t U = \kappa_s h u & \text{in } \Gamma_R^+, \\
U = 0 & \text{in } \Gamma_{g,R}^+ := \{(x', x_N, 0) \in B^+_R : x_N < g(x')\},
\end{cases}
$$

for some $R > 0$, where $g$ is the function which locally parametrizes the boundary $\partial \Omega$ around $x_0 = 0$ according to (1.7), with

$$
g \in C^{1,1}(\mathbb{R}^{N-1}), \quad g(0) = 0 \quad \text{and} \quad \nabla g(0) = 0.
$$
The suitable weighted Sobolev space for energy solutions to (2.2) is $H^1_{\Gamma_{g,R}^+}(B^+_R, t^{1-2s} dz)$, defined as the closure of $C_c^\infty(B^+_R \setminus \Gamma_{g,R}^+) \subset H^1(B^+_R, t^{1-2s} dz)$. By energy solution to (2.2) we mean a function $U \in H^1_{\Gamma_{g,R}^+}(B^+_R, t^{1-2s} dz)$ such that

$$
\int_{B^+_R} t^{1-2s} \nabla U(x,t) \cdot \nabla \phi(x,t) \, dz - \kappa_s \int_{\Gamma_{g,R}^+} h \text{Tr} U \, \phi \, dx = 0 \quad \text{for all } \phi \in C_c^\infty(B^+_R \cup \Gamma_{g,R}^-).
$$

2.1. A diffeomorphism to straighten the boundary. We follow the construction in [2]. Let us consider the following set of variables $(y, t) \in \mathbb{R}^N \times [0, +\infty)$, with $y = (y', y_N) = (y_1, \ldots, y_N-1, y_N)$. Let $\rho \in C_c^\infty(\mathbb{R}^{N-1})$ be such that $\rho \geq 0$, supp$(\rho) \subset B_1'$ and $\int_{\mathbb{R}^{N-1}} \rho(y') \, dy' = 1$. For every $\delta > 0$ we define

$$
\rho_\delta(y') = \delta^{-N+1} \rho \left( \frac{y'}{\delta} \right).
$$

Let us define also, for every $j = 1, \ldots, N-1$,

$$
G_j(y', y_N) = \begin{cases} 
(\rho_{y_N} * \partial_{y_j} g)(y') & \text{if } y' \in \mathbb{R}^{N-1}, \ y_N > 0, \\
\partial_{y_j} g(y') & \text{if } y' \in \mathbb{R}^{N-1}, \ y_N = 0,
\end{cases}
$$

where $*$ denotes the convolution product.

It is easy to verify that, for all $j = 1, \ldots, N-1$, $G_j \in C^\infty(\mathbb{R}_+^N)$, $G_j'$ is Lipschitz continuous in $\overline{B_1^+}$, and $\frac{\partial G_j}{\partial y_i} \in L^\infty(\mathbb{R}_+^N)$ for every $i \in \{1, \ldots, N\}$. Moreover, for all $j = 1, \ldots, N-1$ and $i = 1, \ldots, N$,

$$
y_N \frac{\partial G_j}{\partial y_i} \text{ is Lipschitz continuous in } \overline{B_1^+}.
$$

As a consequence, we have that, letting

$$
\tilde{G}_j : \mathbb{R}^N \to \mathbb{R}, \quad \tilde{G}_j(y', y_N) := G_j(y', |y_N|)
$$

and

$$
\psi_j : \mathbb{R}^N \to \mathbb{R}, \quad \psi_j(y', y_N) = y_j - y_N \tilde{G}_j(y', y_N),
$$

$\tilde{G}_j$ is Lipschitz continuous in $\mathbb{R}^N$ and $\psi_j \in C^{1,1}(\mathbb{R}^N)$ (i.e. $\psi_j$ is continuously differentiable with Lipschitz gradient) for all $j = 1, \ldots, N-1$. Let

$$
\tilde{G}(y', y_N) = \left( \tilde{G}_1(y', y_N), \tilde{G}_2(y', y_N), \ldots, \tilde{G}_{N-1}(y', y_N) \right)
$$

and denote as $J_{\tilde{G}}(y', y_N)$ the Jacobian matrix of $\tilde{G}$ at $(y', y_N)$. Then $J_{\tilde{G}} \in L^\infty(\mathbb{R}^N, \mathbb{R}^{N(N-1)})$ and

$$
|\tilde{G}(y', y_N) - \nabla g(y')| \leq C |y_N| \quad \text{for all } (y', y_N) \in \mathbb{R}^N,
$$

for some constant $C > 0$ independent of $(y', y_N)$.

Let us consider the local diffeomorphism $F : \mathbb{R}^{N+1} \to \mathbb{R}^{N+1}$ defined as

$$
F(y', y_N, t) = (\psi_1(y', y_N), \ldots, \psi_{N-1}(y', y_N), y_N + g(y', t)).
$$

We observe that $F$ is of class $C^{1,1}$ and $F(y', 0, 0) = (y', g(y'), 0)$, namely $F^{-1}$ is straightening the boundary of the set $\{(x', x_N, 0) : x_N < g(x')\}$.
Direct computations and (2.4) yield that

\begin{equation}
J_F(y', y_N, t) = J(y', y_N)
\end{equation}

\[
\begin{pmatrix}
1 - y_N \frac{\partial \tilde{G}_1}{\partial y_1} & -y_N \frac{\partial \tilde{G}_1}{\partial y_2} & \cdots & -y_N \frac{\partial \tilde{G}_1}{\partial y_{N-1}} & -\tilde{G}_1 & -y_N \frac{\partial \tilde{G}_1}{\partial y_N} & 0 \\
-y_N \frac{\partial \tilde{G}_2}{\partial y_1} & 1 - y_N \frac{\partial \tilde{G}_2}{\partial y_2} & \cdots & -y_N \frac{\partial \tilde{G}_2}{\partial y_{N-1}} & -\tilde{G}_2 & -y_N \frac{\partial \tilde{G}_2}{\partial y_N} & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
-y_N \frac{\partial \tilde{G}_{N-1}}{\partial y_1} & -y_N \frac{\partial \tilde{G}_{N-1}}{\partial y_2} & \cdots & 1 - y_N \frac{\partial \tilde{G}_{N-1}}{\partial y_{N-1}} & -\tilde{G}_{N-1} & -y_N \frac{\partial \tilde{G}_{N-1}}{\partial y_N} & 0 \\
\frac{\partial g}{\partial y}(y') & \frac{\partial g}{\partial y_2}(y') & \cdots & \frac{\partial g}{\partial y_{N-1}}(y') & 1 & 0 & 0
\end{pmatrix}
\]

where \(\nabla g(y')\) is meant as a column vector in \(\mathbb{R}^{N-1}\), 0 is the null column vector in \(\mathbb{R}^{N-1}\) and \((\nabla g(y'))^T\), \(0^T\) are their transpose; from now on, the notation \(O(y_N)\) will be used to denote blocks of matrices with all entries being \(O(y_N)\) as \(y_N \to 0\) uniformly with respect to \(y'\) and \(t\).

From (2.3) and the fact that \(g \in C^{1,1}(\mathbb{R}^{N-1})\), it follows that \(\nabla g(y') = O(|y'|)\) as \(|y'| \to 0\), then

\begin{equation}
det J(y', y_N, t) = 1 + |\nabla g(y')|^2 + O(y_N) = 1 + O(|y'|^2) + O(y_N)
\end{equation}

as \(y_N \to 0\) and \(|y'| \to 0\).

In particular we have that \(det J_F(0) = 1 \neq 0\); therefore, by the Inverse Function Theorem, \(F\) is invertible in a neighbourhood of the origin, i.e. there exists \(R_1 > 0\) such that

\begin{equation}
\alpha(y', y_N) := det J(y', y_N, t) > 0 \quad \text{in } B_{R_1}'
\end{equation}

and \(F\) is a diffeomorphism of class \(C^{1,1}\) from \(B_{R_1}\) to \(U = F(B_{R_1})\) for some \(U\) open neighbourhood of 0 such that \(U \subset B_R\). Furthermore

\[F^{-1}(U \cap \Gamma_{g,R}^-) = \Gamma_{R_1}^- \quad \text{and} \quad F^{-1}(U \cap \Gamma_{g,R}^+) = \Gamma_{R_1}^+\]

where, for all \(r > 0\), we denote

\[\Gamma_r^- := \{(y', y_N, 0) \in B_r' : y_N < 0\}, \quad \Gamma_r^+ := \{(y', y_N, 0) \in B_r' : y_N > 0\}\]

Since

\[F^{-1} \in C^{1,1}(U, B_{R_1}), \quad F \in C^{1,1}(B_{R_1}, U), \quad F(0) = F^{-1}(0) = 0, \quad J_F(0) = J_{F^{-1}}(0) = Id_{N+1},\]

we have that

\begin{align}
J_F(x) &= Id_{N+1} + O(|x|) \quad \text{and} \quad F^{-1}(x) = x + O(|x|^2) \quad \text{as } |x| \to 0, \\
J_F(y) &= Id_{N+1} + O(|y|) \quad \text{and} \quad F(y) = y + O(|y|^2) \quad \text{as } |y| \to 0.
\end{align}

If \(U\) is a solution to (2.2), then \(W = U \circ F\) is solution to

\begin{equation}
\begin{cases}
-\text{div} \left( t^{1-2s} A \nabla W \right) = 0 & \text{in } B_{R_1}^+ \\
\lim_{t \to 0^+} \left( t^{1-2s} A \nabla W \cdot \nu \right) = \kappa_s \tilde{h} \operatorname{Tr} W & \text{in } \Gamma_{R_1}^- \\
W = 0 & \text{in } \Gamma_{R_1}^+
\end{cases}
\end{equation}

where \(\nu = (0, 0, \ldots, 0, -1)\) is the vertical downward unit vector, \(A\) is the \((N + 1) \times (N + 1)\) variable coefficient matrix (not depending on \(t\)) given by

\begin{equation}
A(y) = (J(y))^{-1} ((J(y))^{-1})^T |\det J(y)|,
\end{equation}

and

\[\tilde{h}(y) = \alpha(y) h(F(y, 0)), \quad y \in \Gamma_{R_1}^-\].
Equation (2.11) is meant in a weak sense, i.e. $W$ belongs to $H^1_{\Gamma_R} (B_R^+ \setminus \Gamma_R^ \circ)$ in $H^1 (B_R^+ \setminus \Gamma_R^ \circ)$ and satisfies

\begin{equation}
\int_{B_R^+} t^{1-2s} A(y) \nabla W(y,t) \cdot \nabla \phi(y,t) \, dz - \kappa_s \int_{\Gamma_R} \tilde{h} \mathrm{Tr} W \, \mathrm{Tr} \phi \, dy = 0
\end{equation}

for all $\phi \in C_c^\infty (B_R^+ \setminus \Gamma_R^ \circ)$. We observe that $A$ is symmetric and, in view of (2.10), uniformly elliptic if $R_1$ is chosen sufficiently small; furthermore $A$ has $C^{0,1}$ coefficients. We also remark that, under assumption (1.4),

\begin{equation}
\tilde{h} \in W^{1,p} (\Gamma_R^-).
\end{equation}

From (2.6) it follows that

\begin{equation}
J^{-1} = \begin{pmatrix} M^{-1} & 0 \\ 0^T & 1 \end{pmatrix},
\end{equation}

where $0$ is the null column vector in $\mathbb{R}^N$ and

\begin{equation}
M = M(y', y_N) = \begin{pmatrix} \mathrm{Id}_{N-1} - y_N J_{\tilde{g}} \bigg|_{\Gamma} & -\nabla g(y') + O(y_N) \\ (\nabla g(y'))^T & 1 \end{pmatrix}.
\end{equation}

From (2.6) and (2.8) one can deduce that

\begin{equation}
\det M(y', y_N) = \alpha(y', y_N) > 0 \quad \text{in } B_R^+.
\end{equation}

Let us define

\begin{equation}
B(y', y_N) := \det M(y', y_N) (M(y', y_N))^{-1}.
\end{equation}

By (2.15) and a direct calculation we have that

\begin{equation}
B = \begin{pmatrix} 1 + \sum_{j \neq 1} \frac{\partial g}{\partial y_j}^2 + O(y_N) & -\frac{\partial g}{\partial y_1} \frac{\partial g}{\partial y_j} + O(y_N) & \cdots & -\frac{\partial g}{\partial y_1} \frac{\partial g}{\partial y_{N-1}} + O(y_N) \\ -\frac{\partial g}{\partial y_1} \frac{\partial g}{\partial y_j} + O(y_N) & 1 + \sum_{j \neq 2} \frac{\partial g}{\partial y_j}^2 + O(y_N) & \cdots & -\frac{\partial g}{\partial y_2} \frac{\partial g}{\partial y_{N-1}} + O(y_N) \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{\partial g}{\partial y_1} \frac{\partial g}{\partial y_j} + O(y_N) & -\frac{\partial g}{\partial y_{N-1}} \frac{\partial g}{\partial y_2} + O(y_N) & \cdots & 1 + \sum_{j \neq N-1} \frac{\partial g}{\partial y_j}^2 + O(y_N) \\ -\nabla g + O(y_N) & 1 + O(y_N) \end{pmatrix}.
\end{equation}

Then $(J(y))^{-1}$ can be rewritten as follows

\begin{equation}
(J(y))^{-1} = \begin{pmatrix} \frac{1}{\alpha(y)} B(y) & 0 \\ 0^T & 1 \end{pmatrix},
\end{equation}

thus from (2.12) it turns out that

\begin{equation}
A(y) = \begin{pmatrix} D(y) & 0 \\ 0 & \alpha(y) \end{pmatrix},
\end{equation}

where $D = \frac{1}{\alpha(y)} BB^T$. From (2.17), (2.7), and (2.8) it follows that

\begin{equation}
D(y', y_N) = \begin{pmatrix} \mathrm{Id}_{N-1} + O(|y'|^2) + O(y_N) \\ O(y_N) \end{pmatrix},
\end{equation}

where $O(y_N)$, respectively $O(|y'|^2)$, denotes blocks of matrices with all entries being $O(y_N)$ as $y_N \to 0$, respectively $O(|y'|^2)$ as $|y'| \to 0$. In particular we have that

\begin{equation}
A(y) = \mathrm{Id}_{N+1} + O(|y'|) \quad \text{as } |y'| \to 0.
\end{equation}

Let us define, for every $z = (y,t) \in B_R^+$,

\begin{equation}
\beta(z) = \frac{A(y)z}{\mu(z)} = (\beta'(z), \beta_{N+1}(z)) = \begin{pmatrix} D(y)y \alpha(y)t \\ \mu(z) \end{pmatrix},
\end{equation}

where $\mu(z)$ is given in (2.19).
where
\begin{equation}
(2.22) \quad \mu(z) = \frac{A(y)z \cdot z}{|z|^2}, \quad z \neq 0.
\end{equation}

We observe that, possibly choosing \( R_4 \) smaller, \( \beta \) is well-defined, since \( \mu(z) > 0 \) in \( B_{R_1} \), and
\begin{equation}
(2.23) \quad \| A(y) \|_{L^\infty(\mathbb{R}^{N+1}, \mathbb{R}^{N+1})} \leq 2 \quad \text{for all } y \in B'_{R_1}.
\end{equation}

Moreover, for every \( \xi = (\xi_1, \ldots, \xi_N, \xi_{N+1}) \in \mathbb{R}^{N+1} \) and \( y \in B'_{R_1} \), we define \( dA(y) \xi \in \mathbb{R}^{N+1} \) as the vector in \( \mathbb{R}^{N+1} \) with \( i \)-th component, for \( i = 1, \ldots, N + 1 \), given by
\begin{equation}
(2.24) \quad (dA(y) \xi)_i = \sum_{j,k=1}^{N+1} \partial_i a_{jk}(y) \xi_j \xi_k.
\end{equation}

**Lemma 2.1.** Let \( \mu \) be as in (2.22) and \( A \) as in (2.22). Then
\begin{equation}
(2.25) \quad \mu(z) = 1 + O(|z|) \quad \text{as } |z| \to 0^+
\end{equation}
and
\begin{equation}
(2.26) \quad \nabla \mu(z) = O(1) \quad \text{as } |z| \to 0^+.
\end{equation}

**Proof.** Estimate (2.25) follows directly from (2.22) and (2.24). In order to prove (2.26), we differentiate (2.22), obtaining that, for all \( z = (y, t) \in B_{R_1} \),
\[\nabla \mu(z) = -2 \frac{(A(y)z \cdot z)z}{|z|^2} + \frac{dA(y)zz}{|z|^2} + 2 \frac{A(y)z}{|z|^2} = -2 \frac{\mu(z)z}{|z|^2} + \frac{dA(y)zz}{|z|^2} + 2 \frac{A(y)z}{|z|^2}.\]
Noting that \( dA(y)zz = O(|z|^2) \) as \( |z| \to 0 \), since the matrix \( A \) has Lipschitz coefficients, and using (2.25) and (2.20), we deduce that
\[\nabla \mu(z) = -2z |z|^2 [1 + O(|z|)] + O(1) + \frac{2}{|z|^2} [z + O(|z|^2)] = O(1)\]
as \( |z| \to 0^+ \), thus proving (2.26). \( \square \)

**Lemma 2.2.** Let \( \beta \) be as in (2.21) and \( A \) as in (2.22). Then we have that, as \( |z| \to 0^+ \),
\begin{align}
(2.27) \quad \beta(z) &= z + O(|z|^2) = O(|z|), \\
(2.28) \quad J_\beta(z) &= A(y) + O(|z|) = \text{Id}_{N+1} + O(|z|), \\
(2.29) \quad \text{div} \beta(z) &= N + 1 + O(|z|).
\end{align}

**Proof.** The result follows by combining (2.25), (2.20) and (2.20). \( \square \)

### 2.2. Some inequalities.
We recall from [13, Lemma 2.4] the following Hardy type inequality with boundary terms, which will be used throughout the paper.

**Lemma 2.3.** For all \( r > 0 \) and \( w \in H^1(B_r^+, t_1^{-2s} \, dz) \)
\begin{equation}
\left( \frac{N - 2s}{2} \right)^2 \int_{B_r^+} t_1^{-2s} w^2(z) \, dz \leq \int_{B_r^+} t_1^{-2s} \left( \nabla w(z) \cdot \frac{z}{|z|} \right)^2 \, dz + \left( \frac{N - 2s}{2r} \right) \int_{\partial^+ B_r^+} t_1^{-2s} w^2 \, ds.
\end{equation}

In order to prove a coercivity-type inequality, we provide the following Sobolev type inequality with boundary terms (see Lemma 2.6 in [13]).

**Lemma 2.4.** There exists \( \tilde{S}_{N,s} > 0 \) such that, for all \( r > 0 \) and \( V \in H^1(B_r^+, t_1^{-2s} \, dz) \),
\begin{equation}
(2.30) \quad \left( \int_{B_r^+} |\text{Tr} V|^2 (s) \, dy \right)^{\frac{1}{2s}} \leq \tilde{S}_{N,s} \left[ \frac{N - 2s}{2r} \int_{\partial^+ B_r^+} t_1^{-2s} V^2 \, ds + \int_{B_r^+} t_1^{-2s} |\nabla V|^2 \, dz \right].
\end{equation}

Exploiting (2.30) we now prove the following inequality which will be useful in the sequel.
Lemma 2.5. For every $\bar{\alpha} > 0$, there exists $r(\bar{\alpha}) \in (0, R_1)$ such that, for any $0 < r \leq r(\bar{\alpha})$, $\zeta \in L^p(B_{R_1}^r)$ such that $\|\zeta\|_{L^p(B_{R_1}^r)} \leq \bar{\alpha}$ and $V \in H^1(B_{R_1}^r, t^{1-2s} dz)$,

$$\int_{B_{r}^+} t^{1-2s} A \nabla V \cdot \nabla V \, dz - \kappa_s \int_{B_{r}^+} \zeta |\text{Tr} V|^2 \, dy + \frac{N - 2s}{2r} \int_{\partial^+ B_{r}^+} t^{1-2s} \mu V^2 \, dS$$

$$\geq \tilde{C}_{N,s} \left( \int_{B_{r}^+} t^{1-2s} |\nabla V|^2 \, dz + \left( \int_{B_{r}^+} |\text{Tr} V|^{2^*(s)} \, dy \right)^\frac{2}{2^*(s)} \right),$$

for some positive constant $\tilde{C}_{N,s} > 0$ depending only on $N$ and $s$.

Proof. Let us estimate from below each term in the left hand side of (2.31). To this aim, exploiting (2.28), we can choose $r_1 \in (0, R_1)$ such that, for all $0 < r \leq r_1$ and $V \in H^1(B_{r_1}^+, t^{1-2s} dz)$,

$$\int_{B_{r_1}^+} t^{1-2s} A \nabla V \cdot \nabla V \, dz \geq \frac{1}{2} \int_{B_{r_1}^+} t^{1-2s} |\nabla V|^2 \, dz.$$

Furthermore, thanks to (2.25), we can assert that $\mu \geq 1/4$ in $B_r$ if $0 < r \leq r_2$, for some $r_2 \in (0, R_1)$. Hence, using (2.30), we deduce that, for all $0 < r \leq r_2$ and $V \in H^1(B_{r_2}^+, t^{1-2s} dz)$,

$$\int_{B_{r_2}^+} t^{1-2s} A \nabla V \cdot \nabla V \, dz \geq \frac{1}{2} \int_{B_{r_2}^+} t^{1-2s} |\nabla V|^2 \, dz.$$

By Hölder’s inequality, we infer that, for all $r \in (0, R_1)$, $V \in H^1(B_{r_1}^+, t^{1-2s} dz)$, and $\zeta \in L^p(B_{R_1}^r)$ such that $\|\zeta\|_{L^p(B_{R_1}^r)} \leq \bar{\alpha}$,

$$\int_{B_{r}^+} \zeta |\text{Tr} V|^2 \, dy \leq \tilde{c}_{N,s,p} r^{\gamma} \|\zeta\|_{L^p(B_{R_1}^r)} \left( \int_{B_{r}^+} |\text{Tr} V|^{2^*(s)} \, dy \right)^\frac{2}{2^*(s)}$$

$$\leq \tilde{c}_{N,s,p} \bar{\alpha}^{\frac{2}{2^*(s)}} \left( \int_{B_{r}^+} |\text{Tr} V|^{2^*(s)} \, dy \right)^\frac{2}{2^*(s)}$$

for some $\tilde{c}_{N,s,p} > 0$ (depending only on $p, N, s$), where

$$\gamma = \frac{2sp - N}{p} > 0.$$

Selecting $r_3 = r_3(\bar{\alpha}) \in (0, R_1)$ such that

$$\kappa_s \tilde{c}_{N,s,p} \bar{\alpha}^{\frac{2}{2^*(s)}} \leq \frac{1}{8N,s}$$

for all $0 < r \leq r_3$

and combining (2.32), (2.33), and (2.34), we obtain that, for all $0 < r \leq r(\bar{\alpha}) := \min\{r_1, r_2, r_3\}$ and $V \in H^1(B_{r}^+, t^{1-2s} dz)$,

$$\int_{B_{r}^+} t^{1-2s} A \nabla V \cdot \nabla V \, dz \geq \frac{1}{4} \int_{B_{r}^+} t^{1-2s} |\nabla V|^2 \, dz + \left( \frac{1}{4S_{N,s}} - \kappa_s \tilde{c}_{N,s,p} \bar{\alpha}^{\frac{2}{2^*(s)}} \right) \left( \int_{B_{r}^+} |\text{Tr} V|^{2^*(s)} \, dy \right)^\frac{2}{2^*(s)}$$

$$\geq \tilde{C}_{N,s} \left( \int_{B_{r}^+} t^{1-2s} |\nabla V|^2 \, dz + \left( \int_{B_{r}^+} |\text{Tr} V|^{2^*(s)} \, dy \right)^\frac{2}{2^*(s)} \right),$$

where $\tilde{C}_{N,s} := \min\{\frac{1}{4}, \frac{1}{8N,s}\}$, thus proving (2.31). \hfill $\square$

Remark 2.6. For $\bar{\alpha} > 0$, let $r(\bar{\alpha})$ and $\tilde{c}_{N,s,p}$ be as in Lemma 2.5 and let $\zeta \in L^p(B_{R_1}^r)$ be such that $\|\zeta\|_{L^p(B_{R_1}^r)} \leq \bar{\alpha}$. Then, for every $r \in (0, r(\bar{\alpha})]$ and $V \in H^1(B_{r}^+, t^{1-2s} dz)$, we have that

$$\int_{B_{r}^+} \zeta |\text{Tr} V|^2 \, dy \leq \tilde{S}_{N,s} \tilde{c}_{N,s,p} \bar{\alpha}^{\frac{2(2N - 2s)}{r}} \int_{\partial^+ B_{r}^+} t^{1-2s} \mu V^2 \, dS + \frac{1}{8N,s} \int_{B_{r}^+} t^{1-2s} |\nabla V|^2 \, dz.$
Proof. Applying (2.34) and (2.30), we obtain (2.37), taking into account that, for all 0 < r ≤ r(α), (2.30) holds and µ ≥ 1/4.

2.3. Construction of the approximating sequence and convergence. The main difficulty in the proof of a Pohozaev type identity for problem (2.11), which is needed to differentiate the Almgren quotient, relies in a substantial lack of regularity at Dirichlet-Neumann junctions. Here we face this difficulty by a double approximation procedure, involving both the potential h and the N-dimensional region Γ_Δ, where the solution to (2.11) is forced to vanish.

Let η ∈ C^∞([0, +∞)) be such that
(2.38) 1 ≤ η in [0,1/2], η ≡ 0 in [1, +∞), 0 ≤ η ≤ 1 and η′ ≤ 0.

Let
\[ f : [0, +∞) \rightarrow \mathbb{R}, \quad f(t) = η(t) + (1 - η(t))t^{1/4}. \]

We observe that
(2.39) \[ f ∈ C^∞([0, +∞)), \quad f(t) = 1 \text{ for all } t ∈ [0,1/2], \quad \text{and} \quad f(t) - 4t f'(t) ≥ 0 \text{ for all } t ≥ 0. \]

Furthermore
(2.40) \[ f(t) ≥ \frac{1}{2} \text{ and } f(t) ≥ t^{1/4} \text{ for all } t ≥ 0. \]

For every n ∈ N \ {0}, we consider the function
\[ f_n(t) = \frac{f(nt)}{n^{1/8}}. \]

Then, (2.30) implies that
(2.41) \[ f_n ∈ C^∞([0, +∞)), \quad f_n(t) = n^{-1/8} \text{ for all } t ∈ [0,1/2n], \quad f_n(t) - 4tf'_n(t) ≥ 0 \text{ for all } t ≥ 0, \]

whereas (2.40) yields
(2.42) \[ f_n(t) ≥ \frac{1}{2}n^{-1/8} \text{ and } f_n(t) ≥ n^{1/8}t^{1/4} \text{ for all } t ≥ 0. \]

By (2.41) and density of smooth functions in Sobolev spaces, there exists a sequence of potential terms h_n ∈ C^∞(Γ^-_R) such that
(2.43) \[ h_n → h \text{ in } W^{1,p}(Γ^-_R). \]

Let
(2.44) \[ \overline{α}_0 = \sup_n \|h_n\|_{L^p(Γ^-_R)} \]

and set
(2.45) \[ R_0 = r(\overline{α}_0) \]

according to the notation introduced in Lemma (2.3).

Remark 2.7. Because of the above choice of R_0, we have that (2.31) holds with any ζ ∈ L^p(B^+_R_0) such that \|ζ\| ≤ \|h_n\| a.e. (being h_n trivially extended in B^+_R_0 \ Γ^-_R), for any n ∈ N \ {0}, r ≤ R_0, and for all \( V ∈ H^1(B^+_r, t^{1-2s} \, dz) \).

Let us define, for all n ∈ N \ {0},
(2.46) \[ γ_n = \{(y', y_N, t) ∈ B^+_R_0 : y_N = f_n(t)\}, \]

with R_0 as in (2.45). If \((y', y_N, t) ∈ γ_n\), then from (2.42) it follows that
\[ n^{1/8}t^{1/4} ≤ f_n(t) = y_N ≤ R_0, \]

thus proving that
(2.47) \[ t ≤ \frac{R_0^4}{\sqrt{n}} \text{ if } (y', y_N, t) ∈ γ_n. \]

The approximating domains are defined as
(2.48) \[ U_n = \{(y', y_N) ∈ B^+_R_0 : y_N < f_n(t)\} \]
with topological boundary

$$\partial U_n = \sigma_n \cup \gamma_n \cup \tau_n,$$

where $\gamma_n$ has been defined in (2.46) and

$$\sigma_n = \{(y',y_N) \in B_{R_0}^+ : y_N < \frac{1}{n^{1/8}}\}, \quad \tau_n = \{(y',y_N,t) \in \partial B_{R_0} : t \geq 0, \ y_N < f_n(t)\},$$

see Figure 1.

**Figure 1.** Vertical section of the approximating domain $U_n$.

The functions $f_n$ above have been constructed with the aim of making $U_n$ satisfy the following geometric property, which will be needed in the proof of a monotonicity formula.

**Lemma 2.8.** There exists $\bar{n} \in \mathbb{N} \setminus \{0\}$ such that, for all $n \geq \bar{n}$ and $z = (y,t) \in \gamma_n \cap B_{R_0}^+$,

$$A(y) z \cdot \nu \geq 0 \quad \text{on} \quad \gamma_n,$$

where $\gamma_n$ has been defined in (2.46) and $\nu = \nu(z)$ is the exterior unit normal vector at $z \in \partial U_n$.

**Proof.** For all $z = (y,t) \in \gamma_n \cap B_{R_0}^+$ we have that $\nu = \nu(z) = \frac{n}{|n|}$ where $n = (0,1,-f_n'(t))$. Hence, from (2.18) and (2.19) it follows that

$$A(y)(y,t) \cdot n = (D(y)y,\alpha(y)t) \cdot ((0,1),-f_n'(t)) = y_N(1 + O(|y'|) + O(y_N)) - \alpha(y) t f_n'(t).$$

Therefore, possibly choosing $R_1$ (and consequently $R_0$) smaller from the beginning and recalling (2.7)–(2.8), we obtain that

$$A(y)(y,t) \cdot n \geq \begin{cases} \frac{n^2}{2} - 2tf_n'(t) = \frac{1}{2}(f_n(t) - 4tf_n'(t)) & \text{if } f_n'(t) \geq 0 \\ \frac{n^2}{2} & \text{if } f_n'(t) \leq 0 \end{cases}$$

thus concluding that $A(y)(y,t) \cdot n \geq 0$ in view of (2.41). \qed

Now we construct a sequence $U_n$ of solutions to some suitable approximating problems in the domains $U_n$ defined in (2.48), which converges strongly to $W$ in the weighted Sobolev space $H^1(B_{R_0}^+, t^{1-2s} \, dz)$. The functions $U_n$ will be sufficiently regular to satisfy a Rellich-Nečas identity and make it integrable on $U_n$, thus allowing us to obtain a Pohozaev type identity for $U_n$ with some remainder terms produced by the transition of the boundary conditions, whose sign can anyway be understood thanks to the geometric property (2.49); therefore, passing to the limit in the Pohozaev identity satisfied by $U_n$ with some remainder terms produced by the boundary conditions, whose sign can anyway be understood thanks to the geometric property (2.49); therefore, passing to the limit in the Pohozaev identity satisfied by $U_n$, we end up with inequality (2.61) for $W$, which will be used to estimate from below the derivative of the Almgren frequency function (2.54) and then to prove that such frequency has a finite limit at 0 (Proposition 3.6).

Let $W \in H^1(B_{R_1}^+, t^{1-2s} \, dz)$ be a non-trivial energy solution to (2.11), in the sense clarified in (2.13). By density, there exists a sequence of functions $G_n \in C^\infty_c(B_{R_1}^+ \setminus \Gamma_{R_1}^+)$ such that $G_n \to W$ strongly in $H^1(B_{R_1}^+, t^{1-2s} \, dz)$. Thanks to (2.34), without loss of generality we can assume that $G_n = 0$ on $\gamma_n$.

We construct a sequence of cut-off functions in the following way: letting $\eta \in C^\infty([0, +\infty))$ be as in (2.38), we define

$$\eta_n : \mathbb{R}^N \to \mathbb{R}, \quad \eta_n(y',y_N) = \begin{cases} 1 - \eta \left( -\frac{y_N}{2} \right) & \text{if } y_N \leq 0, \\ 0 & \text{if } y_N > 0. \end{cases}$$
For any fixed $n \in \mathbb{N}$, we consider the following boundary value problem
\begin{align}
-\text{div} (t^{1-2s} A \nabla U_n) = 0 & \quad \text{in } U_n,
\lim_{t \to 0^+} (t^{1-2s} A \nabla U_n \cdot \nu) = \kappa_s \eta_n h_n \text{Tr } U_n & \quad \text{in } \sigma_n,
U_n = G_n & \quad \text{in } \tau_n \cup \gamma_n,
\end{align}
(2.51)
in a weak sense, i.e.
\begin{align}
U_n - G_n & \in \mathcal{H}_n,
\int_{U_n} t^{1-2s} A \nabla U_n \cdot \nabla \Phi \, dz - \kappa_s \int_{\sigma_n} \eta_n h_n \text{Tr } U_n \text{Tr } \Phi \, dy = 0 & \quad \text{for all } \Phi \in \mathcal{H}_n,
\end{align}
(2.52)
where $\mathcal{H}_n$ is defined as the closure of $C^\infty_c(U_n \cup \sigma_n)$ in $H^1(U_n, t^{1-2s} \, dz)$. Existence of solutions to (2.52) and their convergence to $W$ are established in the following proposition.

**Proposition 2.9.** For any fixed $n \in \mathbb{N}$, there exists a unique solution $U_n$ to (2.52). Moreover $U_n \to W$ strongly in $H^1(B_{R_0}^+, t^{1-2s} \, dz)$ (where $U_n$ is extended trivially to zero in $B_{R_0}^+ \setminus U_n$).

**Proof.** $U_n$ solves (2.52) if and only if $V_n = U_n - G_n$ satisfies
\begin{align}
V_n & \in \mathcal{H}_n \quad \text{and} \quad b_n(V_n, \Phi) = \langle F_n, \Phi \rangle \quad \text{for all } \Phi \in \mathcal{H}_n,
\end{align}
where
\begin{align}
b_n : \mathcal{H}_n \times \mathcal{H}_n & \to \mathbb{R},
\quad b_n(V, \Phi) = \int_{U_n} t^{1-2s} A \nabla V \cdot \nabla \Phi \, dz - \kappa_s \int_{\sigma_n} \eta_n h_n \text{Tr } V \text{Tr } \Phi \, dy
\end{align}
and
\begin{align}
F_n : \mathcal{H}_n & \to \mathbb{R},
\quad \langle F_n, \Phi \rangle = -\int_{U_n} t^{1-2s} A \nabla G_n \cdot \nabla \Phi \, dz + \kappa_s \int_{\sigma_n} \eta_n h_n \text{Tr } G_n \text{Tr } \Phi \, dy.
\end{align}
From Hölder’s inequality, (2.30), and the boundedness of $\{b_n\}$ and $\{G_n\}$ respectively in $L^p(\Gamma_{R_1})$ and in $H^1(B_{R_0}^+, t^{1-2s} \, dz)$, it follows that
\begin{align}
\|b_n(V, \Phi)\| & \leq c \|\Phi\|_{\mathcal{H}_n} \quad \text{for all } \Phi \in \mathcal{H}_n
\end{align}
for some constant $c > 0$ which does not depend on $n$. In particular $F_n \in \mathcal{H}_n^*$, being $\mathcal{H}_n^*$ the dual space of $\mathcal{H}_n$, and $\|F_n\|_{\mathcal{H}_n^*} \leq c$ uniformly in $n$.

The idea is to apply the Lax-Milgram Theorem. In order to do this, we remark that, using the Hardy inequality in Lemma 2.3, after extending functions $V_n$ trivially to zero in $B_{R_0}^+ \setminus U_n$, the weighted $L^2$-norm of the gradient
\begin{align}
\left( \int_{U_n} t^{1-2s} |\nabla V_n|^2 \, dz \right)^{1/2}
\end{align}
turns out to be an equivalent norm in the space $\mathcal{H}_n$ that will be denoted as $\|\cdot\|_{\mathcal{H}_n}$. It follows that $b_n$ is coercive: indeed, for every $V \in \mathcal{H}_n$, we have that
\begin{align}
b_n(V, V) & = \int_{U_n} t^{1-2s} A \nabla V \cdot \nabla V \, dz - \kappa_s \int_{\sigma_n} \eta_n h_n \text{Tr } V^2 \, dy
\end{align}
\begin{align}
& = \int_{B_{R_0}^+} t^{1-2s} A \nabla V \cdot \nabla V \, dz - \kappa_s \int_{B_{R_0}^+} \eta_n h_n \text{Tr } V^2 \, dy
\end{align}
\begin{align}
& \geq \tilde{C}_{N,s} \int_{B_{R_0}^+} t^{1-2s} |\nabla V|^2 \, dz + \tilde{C}_{N,s} \int_{U_n} t^{1-2s} |\nabla V|^2 \, dz = \tilde{C}_{N,s} \|V\|_{\mathcal{H}_n}^2,
\end{align}
as a consequence of Lemma 2.3 with $\zeta = \eta_n h_n$, see Remark 2.7. Furthermore, from (2.29) and (2.31) it follows that
\begin{align}
|b_n(V, W)| & \leq \left( 2 + \frac{1}{8} \right) \|V\|_{\mathcal{H}_n} \|W\|_{\mathcal{H}_n} \leq 3\|V\|_{\mathcal{H}_n} \|W\|_{\mathcal{H}_n}
\end{align}
for all $V, W \in \mathcal{H}_n$. In particular $b_n$ is continuous.
Hence, from (2.57), (2.58), and the Lax-Milgram Theorem we can conclude that there exists a unique $V_n \in \mathcal{H}_n$ solving (2.53), which implies also the existence and uniqueness of a solution $U_n$ to (2.52). Moreover, combining (2.57) and (2.58) we also obtain that, extending $V_n$ trivially to zero in $B_{R_0}^+ \setminus \mathcal{U}_n$, 

$$
\|V_n\|_{H^1(B_{R_0}^+, t^{1-2s} dz)} \leq \frac{c}{C_{N,s}} \quad \text{for all } n.
$$

From this, it follows that there exist $V \in H^1(B_{R_0}^+, t^{1-2s} dz)$ and a subsequence $\{V_{n_k}\}$ of $\{V_n\}$ such that 

$$
V_{n_k} \rightharpoonup V \quad \text{weakly in } H^1(B_{R_0}^+, t^{1-2s} dz).
$$

From the fact that $V_n \in \mathcal{H}_n$, it follows easily that $V$ has null trace on $\partial^+ B_{R_0}^+$ and on $\Gamma_{R_0}^+$. Hence it can be taken as a test function in (2.19) yielding 

$$
\int_{B_{R_0}^+} t^{1-2s} A \nabla W \cdot \nabla V \, dz - \kappa_s \int_{\Gamma_{R_0}^-} \tilde{h} \text{Tr} W \text{Tr} V \, dy = 0.
$$

Since $G_n \rightharpoonup W$ strongly in $H^1(B_{R_1}^+, t^{1-2s} dz)$, from (2.21) we deduce that $\text{Tr} G_n \rightharpoonup \text{Tr} W$ in $L^2(\Gamma_{R_1}^+)$. (2.21) and (2.59) imply that $\text{Tr} V_{n_k} \rightharpoonup \text{Tr} V$ weakly in $L^2(\Gamma_{R_1}^+)$. Furthermore $\eta_n h_n \rightharpoonup \tilde{h}$ in $L^\infty(\Gamma_{R_1}^+)$. Hence (2.60) yields 

$$
0 = \int_{B_{R_0}^+} t^{1-2s} A \nabla W \cdot \nabla V \, dz - \kappa_s \int_{\Gamma_{R_0}^-} \tilde{h} \text{Tr} W \text{Tr} V \, dy \\
= \lim_{k \to \infty} \int_{B_{R_0}^+} t^{1-2s} A \nabla G_{n_k} \cdot \nabla V_{n_k} \, dz - \kappa_s \int_{\Gamma_{R_0}^-} \eta_{n_k} h_{n_k} \text{Tr} G_{n_k} \text{Tr} V_{n_k} \, dy \\
= \lim_{k \to \infty} \langle F_{n_k}, V_{n_k} \rangle = - \lim_{k \to \infty} b_{n_k}(V_{n_k}, V_{n_k})
$$

thus concluding that $\|V_{n_k}\|_{H^1(B_{R_0}^+, t^{1-2s} dz)} \to 0$ as $k \to +\infty$ in view of (2.57). Hence $V_{n_k} \to V$ strongly in $H^1(B_{R_0}^+, t^{1-2s} dz)$ and $U_{n_k} = V_{n_k} + G_{n_k} \to W$ as $k \to +\infty$ strongly in $H^1(B_{R_0}^+, t^{1-2s} dz)$. By Urysohn’s subsequence principle, we finally conclude that $U_n \to W$ in $H^1(B_{R_0}^+, t^{1-2s} dz)$ as $n \to +\infty$. \hfill $\square$

2.4. Pohozaev-type inequalities. The aim of this section is to prove a Pohozaev-type inequality for the energy solution $W \in H^1(B_{R_1}^+, t^{1-2s} dz)$ to (2.21), in this situation we have to settle for an inequality instead of a classical Pohozaev-type identity because of the mixed boundary conditions, which produce some extra singular terms with a recognizable sign when integrating the Rellich-Neças identity.

The idea is to obtain the inequality as limit of ones for the approximating sequence $U_n$. For every $r \in (0, R_0)$, $n \in \mathbb{N}$ such that $n > r^{-s}$, and $\delta \in (0, \frac{1}{4n})$, let us consider the following domain 

$$
O_{r,n,\delta} = \mathcal{U}_n \cap \{(y, t) \in B_r : t > \delta\}.
$$

We note that, if $\delta \in \left(0, \frac{1}{4n}\right)$, then $f_n(t) = n^{-1/8}$ for $0 \leq t \leq 2\delta$, see (2.41). We can describe its topological boundary as $\partial O_{r,n,\delta} = \sigma_{r,n,\delta} \cup \gamma_{r,n,\delta} \cup \tau_{r,n,\delta}$, with 

$$
\sigma_{r,n,\delta} = \left\{(y', y_N, t) \in B_r : y_N < \frac{1}{n^{1/8}}, \ t = \delta\right\},
$$

$$
\gamma_{r,n,\delta} = \left\{(y', y_N, t) \in \overline{B_r^+} : y_N = f_n(t), \ t \geq \delta\right\},
$$

$$
\tau_{r,n,\delta} = \left\{(y', y_N, t) \in \partial^+ B_r^+ : y_N < f_n(t), \ t \geq \delta\right\},
$$

see Figure 2.

We also define 

$$
S^-_r = \{(y', y_N, t) \in \partial B_r : t = 0 \text{ and } y_N < 0\}.
$$

Having the matrix $A$ Lipschitz coefficients and being the equation satisfied in a smooth domain containing $O_{r,n,\delta}$, by classical elliptic regularity theory (see e.g. [23] Theorem 2.2.2.3) we have that $U_n \in H^2(O_{r,n,\delta})$.
Lipschitz components, we have that

\[ \text{div} (\tilde{A} \nabla U_n) \beta - 2 (\beta \cdot \nabla U_n) \tilde{A} \nabla U_n = (\tilde{A} \nabla U_n \cdot \nabla U_n) \text{div } \beta - 2 (\beta \cdot \nabla U_n) \text{div} (\tilde{A} \nabla U_n) + (d \tilde{A} \nabla U_n \nabla U_n) \cdot \beta - 2 J_\beta (\tilde{A} \nabla U_n) \cdot \nabla U_n, \]

where \( \tilde{A}(z) = t^{1-2s} A(y) \) and \( \beta \) has been defined in (2.61). Since \( U_n \in H^2(O_{r,n,\delta}) \) and \( \tilde{A} \) and \( \beta \) have Lipschitz components, we have that

\[ (\tilde{A} \nabla U_n \cdot \nabla U_n) \beta - 2 (\beta \cdot \nabla U_n) \tilde{A} \nabla U_n \in W^{1,1}(O_{r,n,\delta}) \]

**Proposition 2.10** (Pohozaev-type inequality). Let \( W \in H^1(B^+_{r_0}, t^{1-2s} \, dz) \) weakly solve (2.11). Then, for almost every \( r \in (0, R_0) \),

\[
\begin{align*}
&\frac{r}{2} \int_{\sigma^+ B^+_r} t^{1-2s} A \nabla W \cdot \nabla W \, dS - r \int_{\sigma^+ B^+_r} t^{1-2s} \frac{|A \nabla W \cdot \nu|^2}{\mu} \, dS \\
&+ \frac{\kappa_s}{2} \int_{\gamma} (\nabla \tilde{h} \cdot \beta + \tilde{h} \text{div} \beta') |\text{Tr}W|^2 \, dy - \frac{\kappa_s r}{2} \int_{\gamma} \tilde{h} |\text{Tr}W|^2 \, dS' \\
&\geq \frac{1}{2} \int_{B^+_r} t^{1-2s} A \nabla W \cdot \nabla W \text{div} \beta \, dz - \int_{B^+_r} t^{1-2s} J_{\beta}(A \nabla W) \cdot \nabla W \, dz \\
&+ \frac{1}{2} \int_{B^+_r} t^{1-2s} (dA \nabla W \nabla W) \cdot \beta \, dz + \frac{1-2s}{2} \int_{B^+_r} t^{1-2s} \frac{\alpha}{\mu} A \nabla W \cdot \nabla W \, dz
\end{align*}
\]

and

\[
\int_{B^+_r} t^{1-2s} A \nabla W \cdot \nabla W \, dz = \int_{\sigma^+ B^+_r} t^{1-2s} (A \nabla W \cdot \nu) W \, dS + \kappa_s \int_{\gamma} \tilde{h} |\text{Tr}W|^2 \, dy.
\]

**Remark 2.11.** The term \( \int_{\gamma} \tilde{h} |\text{Tr}W|^2 \, dS' \) is understood for a.e. \( r \in (0, R_0) \) as the \( L^1 \)-function given by the weak derivative of the \( W^{1,1}(0, R_0) \)-function \( r \mapsto \int_{\gamma} \tilde{h} |\text{Tr}W|^2 \, dy \). Likewise, the two terms

\[ \int_{\sigma^+ B^+_r} t^{1-2s} A \nabla W \cdot \nabla W \, dS \quad \text{and} \quad \int_{\sigma^+ B^+_r} t^{1-2s} \frac{|A \nabla W \cdot \nu|^2}{\mu} \, dS \]

are understood for a.e. \( r \in (0, R_0) \) as the \( L^1 \)-functions given by the weak derivative of the \( W^{1,1}(0, R_0) \)-functions \( r \mapsto \int_{B^+_r} A \nabla W \cdot \nabla W \, dz \) and \( r \mapsto \int_{B^+_r} t^{1-2s} \frac{|A \nabla W \cdot \nu|^2}{\mu} \, dz \) respectively.

**Proof.** Since \( U_n \in H^2(O_{r,n,\delta}) \), the following Rellich-Nečas identity holds in a distributional sense in \( O_{r,n,\delta} \):

\[
(\tilde{A} \nabla U_n \cdot \nabla U_n) \beta - 2 (\beta \cdot \nabla U_n) \tilde{A} \nabla U_n = (\tilde{A} \nabla U_n \cdot \nabla U_n) \text{div } \beta - 2 (\beta \cdot \nabla U_n) \text{div} (\tilde{A} \nabla U_n) + (d \tilde{A} \nabla U_n \nabla U_n) \cdot \beta - 2 J_\beta (\tilde{A} \nabla U_n) \cdot \nabla U_n,
\]

where \( \tilde{A}(z) = t^{1-2s} A(y) \) and \( \beta \) has been defined in (2.61). Since \( U_n \in H^2(O_{r,n,\delta}) \) and \( \tilde{A} \) and \( \beta \) have Lipschitz components, we have that

\[ (\tilde{A} \nabla U_n \cdot \nabla U_n) \beta - 2 (\beta \cdot \nabla U_n) \tilde{A} \nabla U_n \in W^{1,1}(O_{r,n,\delta}) \]
so that we can use the integration by parts formula for Sobolev functions on the Lipschitz domain $O_{r,n,\delta}$ and obtain, in view of (2.66) and (2.52),

$$r \int_{\tau_{r,n,\delta}} t^{1-2s} A \nabla U_n \cdot \nabla U_n \, dS - 2r \int_{\tau_{r,n,\delta}} t^{1-2s} \frac{|A \nabla U_n \cdot \nu|^2}{\mu} \, dS$$

$$- \int_{\tau_{r,n,\delta}} t^{1-2s} \frac{1}{\mu} \left| \partial_t U_n \right|^2 (\nabla \cdot \nu)(A \nabla \cdot \nu) \, dS - \int_{\tau_{r,n,\delta}} t^{1-2s} \frac{\Omega}{\mu} A \nabla U_n \cdot \nabla U_n \, dy$$

$$+ 2 \int_{\tau_{r,n,\delta}} t^{1-2s} \frac{1}{\mu} \partial_t U_n (D \nabla_y U_n \cdot y) \, dy + 2 \int_{\tau_{r,n,\delta}} t^{1-2s} \frac{\Omega^2}{\mu} |\partial_t U_n|^2 \, dy$$

$$= \int_{O_{r,n,\delta}} t^{1-2s} A \nabla U_n \cdot \nabla U_n \, \text{div} \beta \, dz - 2 \int_{O_{r,n,\delta}} t^{1-2s} J_{\beta}(A \nabla U_n) \cdot \nabla U_n \, dz$$

$$+ \int_{O_{r,n,\delta}} t^{1-2s} (dA \nabla U_n \nabla U_n) \cdot \beta \, dz + (1 - 2s) \int_{O_{r,n,\delta}} t^{1-2s} \frac{\Omega}{\mu} A \nabla U_n \cdot \nabla U_n \, dz.$$  

In the previous computation we have used the following facts: on $\tau_{r,n,\delta}$ the outward unit normal vector is $\nu = \frac{\nu}{|\nu|}$, on $\gamma_{r,n,\delta}$ we have that $\nabla U_n = \pm \nabla U_n |\nu|$ due to vanishing of $U_n$ on $\gamma_{r,n,\delta}$, and on $\sigma_{r,n,\delta}$ one has $\nu = (0,0,\ldots,0,-1)$, $A \nabla U_n \cdot \nu = -\alpha \partial_\nu U_n$ and

$$A \nabla U_n \cdot z = (D \nabla_y U_n, \alpha \partial_\nu U_n) \cdot z = D \nabla_y U_n \cdot y + \alpha \delta_\nu U_n.$$  

From Lemma 2.8 and uniform ellipticity of $A$ it follows that

$$\int_{\tau_{r,n,\delta}} t^{1-2s} \frac{1}{\mu} |\partial_t U_n|^2 (A \nabla \cdot \nu)(A \nabla \cdot \nu) \geq 0.$$  

Hence, we get the following inequality

$$(2.67) \quad r \int_{\tau_{r,n,\delta}} t^{1-2s} A \nabla U_n \cdot \nabla U_n \, dS - 2r \int_{\tau_{r,n,\delta}} t^{1-2s} \frac{|A \nabla U_n \cdot \nu|^2}{\mu} \, dS - \int_{\tau_{r,n,\delta}} t^{1-2s} \frac{\Omega}{\mu} A \nabla U_n \cdot \nabla U_n \, dy$$

$$+ 2 \int_{\tau_{r,n,\delta}} t^{1-2s} \frac{1}{\mu} \partial_t U_n (D \nabla_y U_n \cdot y) \, dy + 2 \int_{\tau_{r,n,\delta}} t^{1-2s} \frac{\Omega^2}{\mu} |\partial_t U_n|^2 \, dy$$

$$\geq \int_{O_{r,n,\delta}} t^{1-2s} A \nabla U_n \cdot \nabla U_n \, \text{div} \beta \, dz - 2 \int_{O_{r,n,\delta}} t^{1-2s} J_{\beta}(A \nabla U_n) \cdot \nabla U_n \, dz$$

$$+ \int_{O_{r,n,\delta}} t^{1-2s} (dA \nabla U_n \nabla U_n) \cdot \beta \, dz + (1 - 2s) \int_{O_{r,n,\delta}} t^{1-2s} \frac{\Omega}{\mu} A \nabla U_n \cdot \nabla U_n \, dz.$$  

We are going to pass to the limit as $\delta \to 0$. We denote as $O_{r,n}$ the limit domain with its boundary $\partial O_{r,n} = \sigma_{r,n} \cup \gamma_{r,n} \cup \tau_{r,n}$, i.e.

$$O_{r,n} = U_{r,n} \cap B_r, \quad \tau_{r,n} = \{(y',y_N,t) \in \partial B_r : y_N < f_n(t), t \geq 0 \},$$

$$\gamma_{r,n} = \{(y',y_N,t) \in B_r^+ : y_N = f_n(t) \}, \quad \sigma_{r,n} = \{(y',y_N) \in B_r^+ : y_N < n^{-1/8} \}.$$  

We claim that, for every fixed $r \in (0,R_0)$ and $n > r^{-8}$, there exists a sequence $\delta_k \to 0^+$ such that

$$- \int_{\sigma_{r,n,\delta_k}} t^{1-2s} \frac{\Omega}{\mu} A \nabla U_n \cdot \nabla U_n \, dy + 2 \int_{\sigma_{r,n,\delta_k}} t^{1-2s} \frac{\Omega^2}{\mu} |\partial_t U_n|^2 \, dy \to 0 \quad \text{as } k \to \infty.$$  

Since $\alpha$ is bounded, $\mu \geq 1/4$ in $B_{R_0}$, and $A$ has bounded coefficients, it is enough to prove that there exists a sequence $\delta_k \to 0^+$ such that $\lim_{k \to \infty} \int_{\sigma_{r,n,\delta_k}} t^{1-2s} |\nabla U_n|^2 \, dy = 0$. To prove this, we argue by contradiction and assume that there exist a positive constant $c > 0$ and $\delta_0 > 0$ such that, for any $\delta \in (0,\delta_0)$,

$$\frac{c}{\delta} \leq \int_{\sigma_{r,n,\delta}} t^{1-2s} |\nabla U_n(y,\delta)|^2 \, dy,$$

which, after integration in $(0,\delta_0)$, gives the contradiction

$$\int_0^{\delta_0} \frac{c}{\delta} \, d\delta \leq \int_0^{\delta_0} t^{1-2s} \left( \int_{\sigma_{r,n,\delta}} |\nabla U_n(y,\delta)|^2 \, dy \right) \, d\delta \leq \left\| U_n \right\|^2_{H^1(B_{R_0}^+ \cap t^{1-2s} d\delta)},$$

since the first integral diverges.
In order to prove the convergence
\[ 2 \int_{\sigma_{r, n, \delta}} \delta^{1-2s} \frac{\alpha}{\mu} \eta h_n U_n (D\nabla_y U_n \cdot y) dy \to -2 \kappa_s \int_{\sigma_{r, n}} \frac{1}{\mu} \eta h_n \text{Tr} U_n (D\nabla_y \text{Tr} U_n \cdot y) dy, \]
we exploit a continuity result for \( t^{1-2s} \partial_t U_n \) and \( \nabla_y U_n \) over \( \overline{U_n \cap B_r} \), which allows us to pass to the limit by the Dominated Convergence Theorem. More precisely we claim that, for all \( r \in (0, R_0) \) and \( n > r^{-8} \),
\[(2.68) \quad t^{1-2s} \partial_t U_n \in C^0(\overline{U_n \cap B_r}), \quad \nabla_y U_n \in C^0(\overline{U_n \cap B_r}). \]
The continuity of \( t^{1-2s} \partial_t U_n \) and \( \nabla_y U_n \) away from \( \{ t = 0 \} \) easily follows from classical elliptic regularity theory, since \( U_n \) is solution of an uniformly elliptic equation. Nevertheless, Lemma 3.3 in [13] allows us to prove continuity of \( t^{1-2s} \partial_t U_n \) and \( \nabla_y U_n \) up to \( \{ t = 0 \} \) when we stay away from the corner between \( \sigma_{r, n} \) and \( \gamma_{r, n} \), i.e. away from the edge \( \{(y', y_N, t) \in B_r : t = 0 \text{ and } y_N = n^{-1/8} \} \); to this aim it is enough to apply [13] Lemma 3.3 to the function \( U_n \circ F^{-1} \). Eventually, we can deduce continuity of \( t^{1-2s} \partial_t U_n \) and \( \nabla_y U_n \) also in the set \( \{(y', y_N, t) \in B_r : t \in \left[ 0, \frac{r}{n} \right] \text{ and } y_N \in [0, n^{-1/8}] \} \) as a consequence of the regularity result given in Lemma [13, Lemma 3.3].

Using the fact that, for all \( r \in (0, R_0) \) and \( n > r^{-8} \), the terms integrated over \( \sigma_{r, n, \delta} \) belong to \( L^1(\sigma_{r, n}) \) in view of (2.65) and the terms integrated over \( \partial_{r, n, \delta} \) belong to \( L^1(\overline{U_n \cap B_r}) \), then by absolute continuity of the Lebesgue integral, we can pass to the limit in (2.67) along \( \delta = \delta_k \) as \( k \to \infty \), thus ending up with the inequality
\[(2.69) \quad \int_{\tau_{r, n}} t^{1-2s} A U_n \nabla U_n \nabla U_n dS - 2r \int_{\tau_{r, n}} t^{1-2s} A U_n \nabla U_n \nabla U_n dS - 2 \kappa_s \int_{\sigma_{r, n}} \frac{1}{\mu} \eta h_n \text{Tr} U_n (D\nabla_y \text{Tr} U_n \cdot y) dy \]
\[\geq \int_{\partial_{r, n} B_r} t^{1-2s} A U_n \nabla U_n \nabla U_n \nabla U_n \nabla U_n dS - 2 \int_{\partial_{r, n} B_r} t^{1-2s} \beta(A U_n) \cdot \nabla U_n dS \]
\[+ \int_{\partial_{r, n} B_r} t^{1-2s} (d \Delta U_n \nabla U_n) \cdot \beta dS + (1 - 2s) \int_{\partial_{r, n} B_r} t^{1-2s} \frac{\alpha}{\mu} \nabla U_n \cdot \nabla U_n dS, \]
for all \( r \in (0, R_0) \) and \( n > r^{-8} \). For \( r \in (0, R_0) \) fixed, we are going to pass to the limit in (2.69) as \( n \to +\infty \). We extend the functions \( U_n \) to be zero in \( B^+_r \setminus U_n \). By the strong convergence \( U_n \to W \) in \( H^1(B^+_0, t^{1-2s} dS) \) (see Proposition 2.9), it follows that
\[ \int_0^{R_0} \left( \int_{\partial B^+_0} t^{1-2s} \left( |\nabla (U_n - W)|^2 + |U_n - W|^2 \right) dS \right) dr \to 0, \]
which is the sequence of functions \( u_n(r) = \int_{\partial B^+_r} t^{1-2s} \left( |\nabla (U_n - W)|^2 + |U_n - W|^2 \right) dS \) converges to 0 in \( L^1(0, R_0) \) and hence a.e. along a subsequence \( u_{n_k} \). In particular we have that
\[(2.70) \quad U_n \to W \text{ as } k \to \infty \text{ in } H^1(\partial^+ B^+_r, t^{1-2s} dS) \text{ for a.e. } r \in (0, R_0), \]
where \( H^1(\partial^+ B^+_r, t^{1-2s} dS) \) is the completion of \( C^{\infty}(\partial^+ B^+_r) \) with respect to the norm
\[ ||\psi||_{H^1(\partial^+ B^+_r, t^{1-2s} dS)} = \left( \int_{\partial B^+_r} t^{1-2s} \left( |\nabla \psi|^2 + \psi^2 \right) dS \right)^{1/2}. \]
Let us now discuss the behavior of the term \( \int_{\sigma_{r, n}} \frac{1}{\mu} \eta h_n \text{Tr} U_n (D\nabla_y \text{Tr} U_n \cdot y) dy \) as \( n \to \infty \). Since \( \eta_n(y', y_N) = 0 \) if \( y_N > -\frac{1}{n} \), by the Divergence Theorem we have that
\[(2.71) \quad \int_{\tau_{r, n}} \frac{1}{\mu} \eta h_n \text{Tr} U_n (D\nabla_y \text{Tr} U_n \cdot y) dy = \int_{\Gamma^+_r} \frac{1}{\mu} \eta h_n \text{Tr} U_n (D\nabla_y \text{Tr} U_n \cdot y) dy \]
\[= \frac{1}{2} \int_{\Gamma^+_r} \div_y \left( \frac{1}{\mu} \eta h_n |\text{Tr} U_n|^2 D y \right) dy - \frac{1}{2} \int_{\Gamma^+_r} |\text{Tr} U_n|^2 \div_y \left( \frac{1}{\mu} \eta h_n D y \right) dy \]
\[= \frac{1}{2} \int_{S^+_r} \frac{1}{\mu} \eta h_n |\text{Tr} U_n|^2 D y \cdot \nu dS' - \frac{1}{2} \int_{\Gamma^+_r} |\text{Tr} U_n|^2 \div_y (\eta h_n \beta') dy \]
\[= \frac{r}{2} \int_{S^+_r} \frac{1}{\mu} \eta h_n |\text{Tr} U_n|^2 \langle \eta h_n \nabla_y \beta' \rangle dS' - \frac{1}{2} \int_{\Gamma^+_r} |\text{Tr} U_n|^2 (\eta h_n \nabla_y h_n \cdot \beta' + \eta h_n \div_y \beta') dy \]
\[- \frac{1}{2} \int_{\Gamma^+_r} |\text{Tr} U_n|^2 h_n \nabla_y h_n \cdot \beta' dy, \]
where $\beta'$ has been defined in (2.24). From the strong convergence $U_n \to W$ in $H^1(B_{R_0}^+, t^{1-2s}dz)$ proved in Proposition 2.9 and 2.1, it follows that
\[ \int_0^{R_0} \left( \int_{S_{r,nk}} \left( \eta_n h_n |\text{Tr}U_n|^2 - \tilde{h} |\text{Tr}W|^2 \right) \right) ds' \] converges to 0, i.e. the sequence of functions $r \mapsto \int_{S_{r,nk}} \left( \eta_n h_n |\text{Tr}U_n|^2 - \tilde{h} |\text{Tr}W|^2 \right) ds'$ converges to 0 in $L^1(0, R_0)$ and hence a.e. along a further subsequence, which we still index by $n_k$. In particular we have that
\[ \int_{S_{r,nk}} \eta_n h_n |\text{Tr}U_n|^2 ds' \to \int_{S_{r,nk}} \tilde{h} |\text{Tr}W|^2 ds' \quad \text{as } k \to \infty \text{ for a.e. } r \in (0, R_0). \]

The strong convergence of $U_n$ to $W$ in $H^1(B_{R_0}^+, t^{1-2s}dz)$, which implies that $\text{Tr}U_n \to \text{Tr}W$ in $L^2(\Gamma_{R_0}^+, t^{1-2s}dz)$ as $k \to \infty$, the strong convergence (2.43) of $h_n$ to $\tilde{h}$ in $H^1(\Gamma_{R_0}^+, t^{1-2s}dz)$, and the fact that $\eta_n \to 1$ a.e. in $\Gamma_{R_0}^+$ imply that
\[ \lim_{n \to \infty} \int_{\Gamma_{r,nk}} |\text{Tr}U_n|^2 \left( \eta_n \nabla h_n \cdot \beta' + \eta_n h_n \nabla y \beta' \right) dy = \int_{\Gamma_{r,nk}} |\text{Tr}W|^2 \left( \nabla \tilde{h} \cdot \beta' + \tilde{h} \nabla y \beta' \right) dy \]
for all $r \in (0, R_0)$. Finally, we have that, by (2.50) and (2.21),
\[ \nabla \eta \cdot \beta' = \frac{1}{\mu} \frac{n}{2} \eta' \left( -\frac{ny_n}{\mu} \right) (D(y)y)_N. \]

Hence, since (2.40) implies that $(D(y)y)_N = o(y_N)$ as $y_N \to 0$ and (2.48) yields that $\eta' \left( -\frac{ny_n}{\mu} \right) \neq 0$ only for $y_N \in \left( -\frac{\sqrt{2}}{\sqrt{n}}, -\frac{1}{\sqrt{n}} \right)$, we conclude that
\[ \nabla \eta \cdot \beta' \] is bounded in $\Gamma_{r,n}$ uniformly with respect to $n$.

Therefore, by Hölder’s inequality,
\[ \left| \int_{\Gamma_{r,nk}} |\text{Tr}U_n|^2 h_n \nabla \eta \cdot \beta' dy \right| \leq \text{const} \left\| |\text{Tr}U_n|^2 \right\|_{L^2(\Gamma_{r,nk})} \left\| \eta_n \right\|_{L^p(\Gamma_{r,nk})} \left\{ \left. \left( y', y_N \right) \right|_{\Gamma_{r,nk}} : -\frac{\sqrt{2}}{\sqrt{n}} < y_N < -\frac{1}{\sqrt{n}} \right\} \left( y', y_N \right) \]
where $| \cdot |_N$ stands for the $N$-dimensional Lebesgue measure; hence, by boundedness of \{Tr$U_n$\} in $L^2(\Gamma_{r,nk})$, ensured by (2.41), and boundedness of \{h$n$\} in $L^p(\Gamma_{r,nk})$, we conclude that
\[ \lim_{n \to \infty} \int_{\Gamma_{r,nk}} |\text{Tr}U_n|^2 h_n \nabla \eta \cdot \beta' dy = 0 \]
Combining (2.72), (2.73), and (2.24), we can pass to the limit in (2.71) along the subsequence, obtaining that
\[ \lim_{k \to \infty} \int_{\sigma_{r,nk}} \frac{1}{\mu} \eta_n h_n \text{Tr}U_n(D\nabla y \text{Tr}U_n \cdot y) dy = \frac{r}{2} \int_{S_{r,nk}} \tilde{h} |\text{Tr}W|^2 ds' - \frac{1}{2} \int_{\Gamma_{r,nk}} |\text{Tr}W|^2 \left( \nabla \tilde{h} \cdot \beta' + \tilde{h} \nabla y \beta' \right) dy. \]

In view of (2.70), (2.75), and the strong convergence of $U_n$ to $W$ in $H^1(B_{R_0}^+, t^{1-2s}dz)$, we can pass to the limit as $n \to \infty$ in (2.69) obtaining the desired Pohozaev-type inequality (2.64) for the solution $W$.

Finally, to prove (2.65), we first multiply equation (2.51) by $U_n$ and integrate by parts over $O_{r,n,d}$; then we pass to the limit as $\delta \to 0^+$ using (2.65) and as $n \to \infty$, taking into account (2.70). \(\square\)

3. ALMGREN TYPE FREQUENCY FUNCTION

In this section we analyze the properties of the Almgren frequency function $N(r)$ associated to (2.11), see (3.1). To perform a blow-up analysis, the boundedness of the frequency will be crucial; to this aim we are going to prove that $N$ possesses a nonnegative finite limit as $r \to 0^+$.

Let $W \in H^1_{\Gamma_1}(B_{R_1}^+, t^{1-2s}dz)$ be a nontrivial weak solution of (2.11). For all $r \in (0, R_1)$ we define
\[ E(r) = \frac{1}{r^{N-2s}} \left( \int_{B_{r}^+} t^{1-2s} A \nabla W \cdot \nabla W dz - \kappa_s \int_{\Gamma_{r}} \tilde{h} |\text{Tr} W|^2 dy \right) \]
and

\begin{equation}
H(r) = \frac{1}{r^{N+1-2s}} \int_{\partial^+ B^+_r} t^{-2s} \mu(z) W^2(z) \, dS.
\end{equation}

Let us first estimate the derivative of \(H\).

**Lemma 3.1.** Let \(E\) and \(H\) be the functions defined as in (3.1) and (3.2). Then \(H \in W^{1,1}_{\text{loc}}(0, R_1)\) and

\begin{equation}
H'(r) = \frac{2}{r^{N+1-2s}} \int_{\partial^+ B^+_r} t^{-2s} \mu W \frac{\partial W}{\partial \nu} \, dS + H(r)O(1)
\end{equation}

in a distributional sense and for a.e. \(r \in (0, R_1)\), where \(\nu = \nu(z) = \frac{\nu}{|z|}\) denotes the unit outer normal vector to \(\partial^+ B^+_r\). Moreover

\begin{equation}
H'(r) = \frac{2}{r^2} E(r) + H(r)O(1)
\end{equation}

as \(r \to 0^+\).

**Proof.** We observe that \(H \in L^1_{\text{loc}}(0, R_1)\) by definition and it can be rewritten as

\[H(r) = \int_{B^+_r} \theta^{1-2s} \mu(r \theta) W^2(r \theta) \, dS.\]

Thus, for all test functions \(\varphi \in C^\infty_0(0, R_1)\), we have that

\[-\int_0^{R_1} H(r) \varphi'(r) \, dr = -\int_0^{R_1} \left( \int_{B^+_r} \theta^{1-2s} \mu(r \theta) W^2(r \theta) \, dS \right) \varphi'(r) \, dr \]

\[= -\int_{\partial B^+_r} t^{-2s} \mu(z) W^2(z) \nabla \varphi(z) \cdot z \, dz = \int_{\partial B^+_r} \text{div} \left( \frac{t^{-2s} \mu(z) W^2(z)}{|z|^{N+2-2s}} \varphi(z) \right) \, dz \]

\[= \int_{B^+_r} t^{-2s} \left( 2 \mu(z) W(z) \nabla W(z) + W^2(z) \nabla \mu(z) \right) \varphi(z) \, dz \]

\[= \int_0^{R_1} \int_{B^+_r} \theta^{1-2s} \left( 2 \mu(r \theta) W(r \theta) \nabla W(r \theta) \cdot \theta + W^2(r \theta) \nabla \mu(r \theta) \cdot \theta \right) \, dS \varphi(r) \, dr,\]

where \(\varphi(z) := \varphi(|z|)\), so that \(\varphi'(|z|) = \nabla \varphi(z) \cdot \frac{z}{|z|}\). Hence the distributional derivative of \(H\) in \((0, R_1)\) is given by

\begin{equation}
H'(r) = \frac{2}{r^{N+1-2s}} \int_{\partial^+ B^+_r} t^{-2s} \mu W \frac{\partial W}{\partial \nu} \, dS + \frac{1}{r^{N+1-2s}} \int_{\partial^+ B^+_r} t^{-2s} W^2 \nabla \mu \cdot \nu \, dS.
\end{equation}

Since \(W, \nabla W \in L^2(B^+_r, t^{-2s} \, dz)\), from (2.25) and (2.26) we easily infer that \(H \in W^{1,1}_{\text{loc}}(0, R_1)\) and (3.6) also holds for a.e. \(r \in (0, R_1)\). Moreover, combining (2.25), (2.26), (3.2) and (5.6), we obtain (3.3).

In order to prove (3.1), we define \(\gamma(z) = \frac{\mu(z)(\beta(z) - z)}{|z|}\).

Observing that \(\gamma(z) \cdot z = 0\) by definition, we deduce that, for a.e. \(r \in (0, R_1)\),

\begin{equation}
\int_{\partial^+ B^+_r} t^{-2s} (A\nabla W \cdot \nu) W \, dS = \int_{\partial^+ B^+_r} t^{-2s} \mu W \frac{\partial W}{\partial \nu} \, dS + \frac{1}{2} \int_{\partial^+ B^+_r} t^{-2s} \gamma \cdot \nabla W^2 \, dS
\end{equation}

\[= \int_{\partial^+ B^+_r} t^{-2s} \mu W \frac{\partial W}{\partial \nu} \, dS - \frac{1}{2} \int_{\partial^+ B^+_r} \text{div}(t^{1-2s} \gamma) W^2 \, dS \]

\[= \int_{\partial^+ B^+_r} t^{-2s} \mu \frac{\partial W}{\partial \nu} \, dS + H(r)O(r^{N+1-2s}),\]

where we used that

\(\text{div}(t^{1-2s} \gamma) = t^{1-2s} \text{div} \gamma + (1 - 2s) \gamma_{N+1} t^{-2s}\).
and
\[ \gamma_{N+1}(z) = tO(1) \quad \text{as } |z| \to 0^+, \]
\[ \text{div} \gamma = \left( \nabla \mu(z) - \frac{\mu(z)z}{|z|^2} \right) (\beta(z) - z) + \frac{\mu(z)}{|z|^2} (\text{div} \beta - (N+1)) = O(1) \quad \text{as } |z| \to 0^+, \]
as a consequence of (2.21), (2.7), (2.25), (2.26), (2.27), (2.29). Hence, from (3.3) and (3.7) it follows (3.4). From (3.4), (2.65) and (3.4) we infer that
\[ r^{N-2s} E(r) = \int_{\partial^+ B_r^+} t^{1-2s} (A \nabla W \cdot r) W dS = \frac{r^{N+1-2s}}{2} H'(r) + H(r) O(r^{N+1-2s}), \]
as \( r \to 0^+ \), which gives (3.5), thus proving the lemma. \( \square \)

**Lemma 3.2.** The function \( H \) defined as in (3.2) is strictly positive for every \( 0 < r \leq R_0 \), with \( R_0 \) being defined in (2.45).

**Proof.** We prove the statement by contradiction. To this aim, we suppose that there exists \( R \leq R_0 \) such that \( H(R) = 0 \). Then, since for every \( r \leq R_0 \) it holds that \( \mu \geq 1/4 \) in \( B_r \), \( \int_{\partial^+ B_r^+} t^{1-2s} W^2 = 0 \) and hence \( W \equiv 0 \) on \( \partial^+ B_r^+ \). From (3.5) it follows that \( H \) is differentiable in a classical sense in \( R \) and \( H'(R) = 2R^{-1} E(R) \); on the other hand, \( H(r) \geq 0 = H(R) \) implies that \( 0 = H'(R) = 2R^{-1} E(R) \) and hence \( E(R) = 0 \). Then from (2.31) it follows that
\[ 0 = \int_{B_r^+} t^{1-2s} A \nabla W \cdot \nabla W \ dz - \kappa_s \int_{\Gamma_r} h_t |\text{Tr} W|^2 \ dy \geq \tilde{C}_{N,s} \int_{B_r^+} t^{1-2s} |\nabla W|^2 \ dz. \]
By (3.8) and Lemma 2.4 we can conclude that \( W \equiv 0 \) in \( B^+_{R_0} \), which in turn leads to \( W \equiv 0 \) in \( B^+_{R_0} \cap \{ t > \delta \} \) from classical unique continuation principles for second order elliptic equations with Lipschitz coefficients (see [23]). Since \( \delta > 0 \) can be taken arbitrarily small, we end up with \( W \equiv 0 \) in \( B^+_{R_0} \), which is a contradiction. \( \square \)

As a consequence of Lemma 3.2, the Almgren type frequency function
\[ N(r) = \frac{E(r)}{H(r)} \]
is well defined in \((0, R_0)\), with \( R_0 \) as in (2.45).

In the following lemma we provide an estimate for the derivative of the function \( E \).

**Lemma 3.3.** Let \( E \) be the function defined as in (5.1). Then \( E \in W^{1,1}\text{loc}((0, R_0]) \) and
\[ E'(r) \geq \frac{2}{r^{N-2s}} \int_{\partial^+ B_r^+} t^{1-2s} \frac{|A \nabla W \cdot \nu|^2}{\mu} + O(r^{-1+\delta}) \left[ E(r) + \frac{N-2s}{2} H(r) \right] \quad \text{as } r \to 0^+ \]
for a.e. \( r \in (0, R_0) \), where
\[ \delta = \min \{ \varepsilon, 1 \} \in (0, 1] \]
and \( \varepsilon \) is defined as in (2.45).

**Proof.** From (3.11) we deduce that \( E \in L^1_{\text{loc}}(0, R_0) \). From coarea formula \( E \in W^{1,1}_{\text{loc}}((0, R_0]) \) and
\[ E'(r) = \frac{2s - N}{r^{N+1-2s}} \left( \int_{B_r^+} t^{1-2s} A \nabla W \cdot \nabla W \ dz - \kappa_s \int_{\Gamma_r} h_t |\text{Tr} W|^2 \ dy \right) + \frac{1}{r^{N-2s}} \left( \int_{\partial^+ B_r^+} t^{1-2s} A \nabla W \cdot \nabla W \ dS - \kappa_s \int_{\Gamma_r} h_t |\text{Tr} W|^2 \ dS' \right) \]
in a distributional sense and a.e. in \((0, R_0)\). Using (2.64), Lemma 2.1 and Lemma 2.2 we obtain that
\[ E'(r) \geq \frac{2}{r^{N-2s}} \int_{\partial^+ B_r^+} t^{1-2s} \frac{|A \nabla W \cdot \nu|^2}{\mu} ds + O(r) \frac{1}{r^{N+1-2s}} \int_{B_r^+} t^{1-2s} A \nabla W \cdot \nabla W \ dz \]
\[ + \frac{O(1)}{r^{N+1-2s}} \int_{\Gamma_r} (|h| + |\nabla_y h|)|\text{Tr} W|^2 \ dy \]
as $r \to 0^+$, for a.e. $r \in (0, R_0)$. One can estimate the last two terms of the right hand side in (3.13) using (2.31), thus obtaining
\begin{equation}
(3.14) \quad \frac{O(r)}{r^{N+1-2s}} \int_{B^r_+} t^{1-2s} A \nabla W \cdot \nabla W \, dz = O(1) \left[ E(r) + \frac{N - 2s}{2} H(r) \right]
\end{equation}
and
\begin{equation}
(3.15) \quad \frac{O(1)}{r^{N+1-2s}} \int_{\Gamma_r^+} (|\tilde{h}| + |\nabla \cdot \tilde{h}|) W^2 \, dy = \frac{O(r)}{r^{N+1-2s}} \left( \int_{\Gamma_r^+} |\text{Tr} W|^2(s) \, dy \right)^{2/2'(s)}
= O(r^{-1+\tau}) \left[ E(r) + \frac{N - 2s}{2} H(r) \right],
\end{equation}
where in (3.15) we have taken into account (2.34) as well. Estimate (3.10) follows from (3.13), (3.14), and (3.15).

**Lemma 3.4.** Let $\mathcal{N}$ be the function defined in (3.9). Then, for every $0 < r \leq R_0$,
\begin{equation}
(3.16) \quad \mathcal{N}(r) > -\frac{N - 2s}{2}
\end{equation}
and
\begin{equation}
(3.17) \quad \liminf_{r \to 0^+} \mathcal{N}(r) \geq 0.
\end{equation}

**Proof.** We deduce (3.10) from (2.31). By (3.1), (3.2), (2.37) and (2.32), we obtain that for all $0 < r \leq R_0$
\begin{align*}
r^{N-2s} E(r) &= \int_{B^r_+} t^{1-2s} A \nabla W \cdot \nabla W \, dz - \kappa_s \int_{\Gamma_r^+} h |\text{Tr} W|^2 \, dy
\geq \frac{3}{8} \int_{B^r_+} t^{1-2s} |\nabla W|^2 \, dz - \kappa_s \tilde{S}_{N,s} \tilde{c}_{N,s,p} \tilde{\alpha}_0 \frac{2(N - 2s)}{r} \int_{\partial^+ B^r_+} t^{1-2s} \mu W^2 \, dS
\geq -\tilde{C} r^{\tau + N - 2s} H(r),
\end{align*}
with $\tilde{\alpha}_0$ as in (2.44) and $\tilde{C} := 2(N - 2s)\kappa_s \tilde{S}_{N,s} \tilde{c}_{N,s,p} \tilde{\alpha}_0 > 0$. From this and (3.9) it follows that, for every $0 < r \leq R_0$,
\begin{equation}
(3.18) \quad \mathcal{N}(r) \geq -\tilde{C} r^{\tau},
\end{equation}
which leads to (3.17).

**Lemma 3.5.** Let $\mathcal{N}$ be the function defined in (3.9). Then $\mathcal{N} \in W^{1,1}_{\text{loc}}((0, R_0))$ and
\begin{equation}
(3.19) \quad \mathcal{N}'(r) \geq V_1(r) + V_2(r)
\end{equation}
for almost every $r \in (0, R_0)$, where
\begin{equation}
(3.20) \quad V_1(r) = \frac{2r}{2} \left[ \left( \int_{\partial^+ B^r_+} t^{1-2s} |\nabla W|^2 \, dS \right) \left( \int_{\partial^+ B^r_+} t^{1-2s} \mu W^2 \, dS \right) - \left( \int_{\partial^+ B^r_+} t^{1-2s} (A \nabla W \cdot \nu) W \, dS \right)^2 \right]
\end{equation}
and
\begin{equation}
(3.21) \quad V_2(r) = O(r^{-1+\delta}) \left( \left( \mathcal{N}(r) + \frac{N - 2s}{2} \right) \right) \quad \text{as } r \to 0^+,
\end{equation}
with $\delta$ as in (3.11).

**Proof.** The fact that $\mathcal{N} \in W^{1,1}_{\text{loc}}((0, R_0))$ follows from Lemmas 3.1.3.2 and 3.3. Moreover, exploiting (3.14), (3.3) and (3.10), we obtain that
\begin{align*}
(3.19) \quad \mathcal{N}'(r) &= \frac{E'(r) H(r) - H'(r) E(r)}{H^2(r)} = \frac{E'(r) H(r)}{H^2(r)} - \frac{H'(r) \left( \frac{1}{2} H'(r) + H(r) O(r) \right)}{H^2(r)}
\geq \frac{2r}{2} \left[ \left( \int_{\partial^+ B^r_+} t^{1-2s} |\nabla W|^2 \, dS \right) \left( \int_{\partial^+ B^r_+} t^{1-2s} \mu W^2 \, dS \right) - \left( \int_{\partial^+ B^r_+} t^{1-2s} (A \nabla W \cdot \nu) W \, dS \right)^2 \right]
\end{align*}
\begin{align*}
&\quad + O(r^{-1+\delta}) \left( \mathcal{N}(r) + \frac{N - 2s}{2} \right) + O(r) + O(1) \frac{1}{H(r)} \frac{1}{r^{N-2s}} \int_{\partial^+ B^r_+} t^{1-2s} (A \nabla W \cdot \nu) W \, dS.
\end{align*}
In order to estimate the last term in (3.19), we observe that
\[
O(1) \frac{1}{H(r)} \frac{1}{r^{N-2s}} \int_{\partial^+ B_r^+} t^{1-2s}(A \nabla W \cdot \nu) W dS = \frac{H'(r)}{H(r)} O(r) + O(r) = N(r) O(1) + O(r),
\]
where we used (3.4) and (3.5). Combining (3.19) and (3.20) we obtain that
\[
N'(r) \geq 2r \left[ \int_{\partial^+ B_r^+} t^{1-2s} \frac{|A \nabla W|^2}{\mu} dS \left( \int_{\partial^+ B_r^+} t^{1-2s} \mu W^2 dS \right)^2 \right]
\]
\[
+ \left( \int_{\partial^+ B_r^+} t^{1-2s} \mu W^2 dS \right)^2 \left( \int_{\partial^+ B_r^+} t^{1-2s} (A \nabla W \cdot \nu) W dS \right)
\]
\[
+ N(r) O(1) + O(r) + O(r^{-1+\delta}) \left( N(r) + \frac{N-2s}{2} \right)
\]
as \(r \to 0^+\),
which yields (3.21) in view of (3.17).

\[\square\]

**Proposition 3.6.** Let \(\mathcal{N}\) be the function defined in (3.9). Then there exists \(C_1 > 0\) such that, for every \(r \in (0, R_0]\),
\[
\mathcal{N}(r) \leq C_1.
\]

Moreover the limit
\[
\gamma := \lim_{r \to 0^+} \mathcal{N}(r)
\]
exists, is finite and nonnegative.

**Proof.** From Lemma 3.5 we deduce that \(N'(r) \geq V_2(r)\) a.e. in \((0, R_0)\), since \(V_1(r) \geq 0\) as a consequence of Schwarz’s inequality. Hence there exist \(0 < \hat{R} < R_0\) and \(C_2 > 0\) such that
\[
N'(r) \geq -C_2 r^{-1+\delta} \left( N(r) + \frac{N-2s}{2} \right),
\]
for a.e. \(r \in (0, \hat{R})\). Then
\[
\frac{d}{dr} \left[ \log \left( N(r) + \frac{N-2s}{2} \right) \right] \geq -C_2 r^{-1+\delta} \quad \text{a.e. in } (0, \hat{R}),
\]
and, integrating the above inequality between \((r, \hat{R})\) with \(r < \hat{R}\), we obtain the upper bound
\[
\mathcal{N}(r) \leq \left( N(\hat{R}) + \frac{N-2s}{2} \right) e^{C_2 \frac{\hat{R}}{r}} - \frac{N-2s}{2} \quad \text{for all } r \in (0, \hat{R}),
\]
which yields (3.21), in view of the continuity of \(\mathcal{N}\) in \((0, R_0]\). From (3.23), we derive that
\[
\frac{d}{dr} \left[ e^{C_2 \frac{\hat{R}}{r}} \left( \mathcal{N}(r) + \frac{N-2s}{2} \right) \right] \geq 0 \quad \text{a.e. in } (0, \hat{R}),
\]

hence
\[
r \mapsto e^{C_2 \frac{\hat{R}}{r}} \left( \mathcal{N}(r) + \frac{N-2s}{2} \right)
\]
is a monotonically increasing function in \((0, \hat{R})\), thus its limit as \(r \to 0^+\) does exist, and the same holds true for the limit of the function \(\mathcal{N}\). From Lemma 3.4 and (3.21), we can conclude that the limit \(\gamma := \lim_{r \to 0^+} \mathcal{N}(r)\) is finite and nonnegative. \(\square\)

**Lemma 3.7.** Let \(\gamma = \lim_{r \to 0^+} \mathcal{N}(r)\). Then:
\begin{itemize}
    \item[(i)] there exists \(k_1 > 0\) such that, for all \(r \in (0, R_0]\),
    \[
    H(r) \leq k_1 r^{2\gamma};
    \]
    \item[(ii)] for any \(\sigma > 0\), there exists \(k_2(\sigma) > 0\) such that, for all \(r \in (0, R_0]\),
    \[
    H(r) \geq k_2(\sigma) r^{2\gamma+\sigma}.
    \]
\end{itemize}
Proof. (i) By (3.21), (3.23), and (3.22) \( N' \in L^1(0, R_0) \), then using (3.23) and (3.21)
\[
N'(r) - \gamma = \int_0^r N'(s)ds \geq -C_2 \int_0^r s^{-1+\delta} \left( N(s) + \frac{N-2s}{2} \right) ds
\]
\[
\geq -C_2 \left( C_1 + \frac{N-2s}{2} \right) \int_0^r s^{-1+\delta} ds = -C_2 \left( C_1 + \frac{N-2s}{2} \right) \frac{r^\delta}{\delta} \quad \text{for all } r \in (0, R_0).
\]
Using (3.3), one has
\[
\frac{H'(r)}{H(r)} = \frac{2N(r)}{r} + O(1) \geq \frac{2\gamma}{r} - 2C_2 \left( C_1 + \frac{N-2s}{2} \right) \frac{r^{-1+\delta}}{\delta} + O(1) \quad \text{as } r \to 0^+.
\]
Integrating the above estimate and taking into account that \( H \) is continuous on \( (0, R_0) \), we obtain (4.24).

(ii) Since \( \gamma = \lim_{r \to 0^+} N'(r) \), for any \( \sigma > 0 \) there exists \( r_\sigma > 0 \) such that, for any \( r \in (0, r_\sigma) \),
\[
N'(r) < \gamma + \frac{\sigma}{2},
\]
and hence
\[
\frac{H'(r)}{H(r)} = \frac{2N(r)}{r} + O(1) < \frac{2\gamma + \sigma}{r} + c,
\]
for some positive constant \( c \). Integrating over the interval \( (r, r_\sigma) \) and taking into account that \( H \) is continuous and positive in \( (0, R_0) \), we prove the second statement. \( \square \)

4. Blow-up analysis and local asymptotics

4.1. Blow-up analysis. As in Section 3, let \( W \in H^1(\Gamma_{B_R^+}, t^{1-2s} dz) \) be a nontrivial weak solution of (2.11). For every \( \lambda \in (0, R_0) \), with \( R_0 \) being as in (2.35), let us define
\[
(4.1) \quad w^\lambda(z) = \frac{W(\lambda z)}{\sqrt{H(\lambda)}}.
\]
We have that \( w^\lambda \) is a weak solution to
\[
\begin{cases}
-\text{div} \left( t^{1-2s} A(\lambda^2) \nabla w^\lambda \right) = 0 & \text{in } B_{R_1/\lambda}^+,
\lim_{t \to 0^+} \left( t^{1-2s} A(\lambda^2) \nabla w^\lambda \cdot \nu \right) = \kappa_s \lambda^{2s} \bar{h}(\lambda) \text{Tr} w^\lambda & \text{on } \Gamma_{R_1/\lambda}^+,
w^\lambda = 0 & \text{on } \Gamma_{R_1/\lambda}^+.
\end{cases}
\]
Moreover we have that
\[
(4.3) \quad \int_{\mathbb{S}^n_{N+1}} \theta_{N+1} \mu(\lambda \theta) |w^\lambda(\theta)|^2 dS = 1.
\]

Lemma 4.1. The family of functions \( \{w^\lambda\}_{\lambda \in (0, R_0)} \) is bounded in \( H^1(B_{1}^+, t^{1-2s} dz) \).

Proof. By (3.9) and using (2.32), (2.36) and (2.37), we obtain that, for every \( \lambda \in (0, R_0) \),
\[
N(\lambda) = \frac{\lambda^{2s-N}}{H(\lambda)} \left( \int_{B_{1}^+} t^{1-2s} A \nabla W \cdot \nabla W d\gamma - \kappa_s \int_{\Gamma_{B_{1}^+}^+} \bar{h} |\text{Tr} W|^2 dS \right)
\geq \frac{\lambda^{2s-N}}{H(\lambda)} \left[ \frac{3}{8} \int_{B_{1}^+} t^{1-2s} |\nabla W|^2 d\gamma - \kappa_s \tilde{S}_{N,s} \bar{c}_{N,s} \lambda^2 \sigma_0 \frac{2(N-2s)}{\lambda} \int_{\bar{\Omega}^+} t^{1-2s} \mu W^2 dS \right]
\geq \frac{3}{8} \int_{B_{1}^+} t^{1-2s} |\nabla w^\lambda|^2 d\gamma - 2(N-2s) \kappa_s \tilde{S}_{N,s} \bar{c}_{N,s} \lambda^2 \sigma_0
\geq \frac{3}{8} \int_{B_{1}^+} t^{1-2s} |\nabla w^\lambda|^2 d\gamma - \frac{N-2s}{4},
\]
which together with (3.21) implies that \( \left\{ \|\nabla w^\lambda\|_{L^2(B_{1}^+, t^{1-2s} dz)} \right\}_{\lambda \in (0, R_0)} \) is bounded. From this and (4.3), the boundedness of \( \{w^\lambda\}_{\lambda \in (0, R_0)} \) in \( H^1(B_{1}^+, t^{1-2s} dz) \) follows by Lemma 2.3. \( \square \)

We are going to prove strong convergence in \( H^1(B_{1}^+, t^{1-2s} dz) \) of \( \{w^\lambda\} \) along a proper vanishing sequence of \( \lambda \)’s; to this aim, we first need to establish the following doubling properties.

Lemma 4.2. There exists $C_3 > 0$ such that

\[ \frac{1}{C_3} H(\lambda) \leq H(R\lambda) \leq C_3 H(\lambda), \]

\[ \int_{B^+_n} t^{1-2s}|\nabla w^\lambda|^2 \, dz \leq C_3 2^{N-2s} \int_{B^+_1} t^{1-2s}|\nabla w^{R\lambda}|^2 \, dz, \]

and

\[ \int_{B^+_n} t^{1-2s}|w^\lambda|^2 \, dz \leq C_3 2^{N+2-2s} \int_{B^+_1} t^{1-2s}|w^{R\lambda}|^2 \, dz. \]

for any $\lambda < R_0/2$ and $R \in [1,2]$.

Proof. From (3.5) we deduce that, for a.e. $r \in (0,R_0)$,

\[ \frac{H'(r)}{H(r)} = \frac{2N(r)}{r} + O(1) \quad \text{as } r \to 0^+. \]

Hence there exist a positive constant $C > 0$ and $\tilde{R}_0 < R_0$ such that, for all $r \in (0,\tilde{R}_0)$,

\[ -C - \frac{N-2s}{r} \leq \frac{H'(r)}{H(r)} \leq C + \frac{2C_1}{r}, \]

where we used (3.16) and (3.21). Integrating the above inequalities over the interval $(\lambda, R\lambda)$, with $R \in (1,2)$ and $\lambda < \tilde{R}_0/R$, we obtain that

\[ 2^{-(N-2s)} e\frac{C 2^d}{4^d} (R-1) \leq \frac{H(R\lambda)}{H(\lambda)} \leq 4C e\frac{C 2^d}{4^d} (R-1). \]

The above chain of inequalities trivially extends to the case $R = 1$. Estimate (4.4) follows from (4.7) and the fact that $H$ is continuous and strictly positive in $(0,R_0]$ (Lemmas 3.1 and 3.2). By scaling and (4.4), we easily deduce (4.5) and (4.6) (see [9] for details in a similar situation). \hfill \Box

Lemma 4.3. Let $w^\lambda$ be as in (4.3), with $\lambda \in (0,R_0)$. Then there exist $M > 0$ and $\lambda_0 > 0$ such that, for any $\lambda \in (0,\lambda_0)$, there exists $R_\lambda \in [1,2]$ such that

\[ \int_{\partial^+ B^+_n} t^{1-2s} |\nabla w^\lambda|^2 \, dS \leq M \int_{B^+_n} t^{1-2s} |\nabla w^\lambda|^2 \, dz. \]

Proof. We recall that, by Lemma 4.1, the set \{ $w^\lambda$ $\lambda \in (0,R_0)$ \} is bounded in $H^1(B^+_1, t^{1-2s}d\lambda)$ and

\[ w^\lambda \in H^1_{t^2} (B^+_1, t^{1-2s}d\lambda) \quad \text{for all } \lambda \in (0,R_0). \]

Moreover, by Lemma 4.2 we have that \{ $w^\lambda$ $\lambda \in (0,R_0/2)$ \} is bounded in $H^1(B^+_2, t^{1-2s}d\lambda)$, hence

\[ \limsup_{\lambda \to 0^+} \int_{B^+_2} t^{1-2s} |\nabla w^\lambda|^2 \, dz < +\infty. \]

For every $\lambda \in (0,R_0/2)$, let

\[ f^\lambda(r) = \int_{B^+_r} t^{1-2s} |\nabla w^\lambda|^2 \, dz. \]

Then $f^\lambda$ is absolutely continuous in $[0,2]$ with distributional derivative given by

\[ f'_\lambda(r) = \int_{\partial^+ B^+_r} t^{1-2s} |\nabla w^\lambda|^2 \, dS \quad \text{for almost every } r \in (0,2). \]

Let us suppose by contradiction that for any $M > 0$ there exists a sequence $\lambda_n \to 0^+$ such that

\[ \int_{\partial^+ B^+_2} t^{1-2s} |\nabla w^{\lambda_n}|^2 \, dS > M \int_{B^+_2} t^{1-2s} |\nabla w^{\lambda_n}|^2 \, dz \]

for all $r \in [1,2]$ and $n \in \mathbb{N}$, i.e.

\[ f'_\lambda_n(r) > M f_{\lambda_n}(r) \]

for a.e. $r \in (1,2)$ and any $n \in \mathbb{N}$. Integrating (4.10) over $[1,2]$, we obtain that, for any $n \in \mathbb{N}$, $f_{\lambda_n}(2) > e^M f_{\lambda_n}(1)$, and hence

\[ \liminf_{n \to +\infty} f_{\lambda_n}(1) \leq \limsup_{n \to +\infty} f_{\lambda_n}(1) \leq e^{-M} \limsup_{n \to +\infty} f_{\lambda_n}(2), \]
which implies that
\[ \liminf_{\lambda \to 0^+} f_\lambda(1) \leq e^{-M} \limsup_{\lambda \to 0^+} f_\lambda(2), \]
for all $M > 0$. From (4.11) and (4.2), letting $M \to +\infty$ we deduce that $\liminf_{\lambda \to 0^+} f_\lambda(1) = 0$. Then there exist a sequence $\lambda_n \to 0^+$ and some $w \in H^1(B_1^+, t^{-1-2s}dz)$ such that $w_\lambda \to w$ in $H^1(B_1^+, t^{-1-2s}dz)$ with
\[ \lim_{n \to +\infty} \int_{B_1^+} t^{1-2s}|\nabla w_\lambda|^2dz = 0. \]
However, by compactness of trace map $H^1(B_1^+, t^{-1-2s}dz) \hookrightarrow L^2(\partial^+ B_1^+, t^{-1-2s}dz)$, (4.3) and (2.25), and weak lower semicontinuity of norms, we necessarily have that
\[ \int_{\partial^+ B_1^+} t^{1-2s}|\nabla w|^2dS = 0. \]
Hence there exists $c \in \mathbb{R}$ such that $w \equiv c$ in $B_1^+$ and $c \neq 0$. Since $H^1(B_1^+, t^{-1-2s}dz)$ is weakly closed in $H^1(B_1^+, t^{-1-2s}dz)$, from (4.8) we deduce that $w \equiv c \in H^1(B_1^+, t^{-1-2s}dz)$, so that $0 = \text{Tr} w|_{\Gamma_1^+} = c$, a contradiction.

**Lemma 4.4.** Let $w^\lambda$ and $R_\lambda$ be as in the statement of Lemma 4.3. Then there exists $\overline{M} > 0$ such that
\[ \int_{\partial^+ B_1^+} t^{1-2s}|\nabla w^\lambda R_\lambda|^2dS \leq \overline{M} \]
for any $\lambda \in (0, \min\{\lambda_0, R_0/2\})$.

**Proof.** We observe that, by scaling and (4.1),
\[ \int_{\partial^+ B_1^+} t^{1-2s}|\nabla w^\lambda R_\lambda|^2dS = R_1^{1-N+2s}H(\lambda) \int_{\partial^+ B_R^+} t^{1-2s}|\nabla w|^2dS, \]
so that, in view of Lemmas 4.2, 4.3 and 4.1, we have that
\[ \int_{\partial^+ B_1^+} t^{1-2s}|\nabla w^\lambda R_\lambda|^2dS \leq 2C_3 M \int_{B_R^+} t^{1-2s}|\nabla w|^2dz \]
\[ \leq 2^{1+N-2s}MC_3^2 \int_{B_1^+} t^{1-2s}|\nabla w^\lambda|^2dz \leq \overline{M} < +\infty. \]
The proof is thereby complete. \(\square\)

**Proposition 4.5.** Let $W \in H^1_{R_1^+}((B_1^+, t^{-1-2s}dz), W \neq 0$, be a nontrivial weak solution of (2.11). Let $\gamma$ be as in Proposition 3.6. Then
(i) there exists $k_0 \in \mathbb{N}$ such that $\gamma = s + k_0$;
(ii) for any sequence $\lambda_n \to 0^+$, there exist a subsequence $\{\lambda_{n_k}\}$ and an eigenfunction $\psi$ of problem (1.11) associated to the eigenvalue $\mu_{k_0} = (k_0 + s)(k_0 + N - s)$ such that $\|\psi\|_{L^2(B_R^+, g_{s+2s}^0dz)} = 1$ and
\[ w^{\lambda_{n_k}}(z) = \frac{W(\lambda_{n_k}z)}{H(\lambda_{n_k})} \to |z|^\gamma \psi \left( \frac{z}{|z|} \right) \]
strongly in $H^1(B_1^+, t^{-1-2s}dz)$.

**Proof.** Let $w^\lambda \in H^1_{R_1^+}(B_1^+, t^{-1-2s}dz)$ be as in (4.1) and $R_\lambda$ as in Lemma 4.3. From Lemma 4.1 we deduce that the set $\{w^{\lambda R_\lambda}\}_{\lambda \in (0, \min\{\lambda_0, R_0/2\})}$ is bounded in $H^1(B_1^+, t^{-1-2s}dz)$. Let us consider a sequence $\lambda_n \to 0^+$. Then there exist a subsequence $\{\lambda_{n_k}\}$ and $w \in H^1_{R_1^+}(B_1^+, t^{-1-2s}dz)$ such that $w^{\lambda_{n_k} R_{\lambda_{n_k}}} \rightharpoonup w$ weakly in $H^1(B_1^+, t^{-1-2s}dz)$. Moreover we have that
\[ \int_{\partial^+ B_1^+} t^{1-2s}w^2dS = 1 \]
by compactness of trace map $H^1(B_1^+, t^{-1-2s}dz) \hookrightarrow L^2(\partial^+ B_1^+, t^{-1-2s}dz)$, (4.3), and (2.25). This allows us to conclude that $w$ is non-trivial.
We now claim strong convergence

\[ w^{λ_{n_k} R_{λ_{n_k}}} \to w \quad \text{in} \quad H^1(B^+_τ, t^{1-2s}dz), \]

We note that \( w^{λ_{n_k} R_{λ_{n_k}}} \) weakly solves (4.12) with \( λ = λ_{n_k} R_{λ_{n_k}} \). Since \( B^+_τ \subset B^+_{R_{τ}/(λ_{n_k} R_{λ_{n_k}})} \) for sufficiently large \( k \), we then have that

\[
\int_{B^+_τ} t^{1-2s} A(λ_{n_k} R_{λ_{n_k}} y) \nabla w^{λ_{n_k} R_{λ_{n_k}}} (z) \cdot \nabla Φ(z) dz = \kappa_{λ_{n_k} R_{λ_{n_k}}} 2^s \int_{Γ^+_τ} \tilde{h}(λ_{n_k} R_{λ_{n_k}} y) Tr w^{λ_{n_k} R_{λ_{n_k}}} (y) Tr Φ(y) dy + \int_{∂+B^+_τ} (t^{1-2s} A(λ_{n_k} R_{λ_{n_k}} y) \nabla w^{λ_{n_k} R_{λ_{n_k}}} (z) \cdot ν) Φ(z) dS
\]

for sufficiently large \( k \) and for every \( Φ ∈ C^∞_c(B^+_τ \setminus Γ^+_τ) \), hence by density for every \( Φ ∈ H^1(B^+_τ, t^{1-2s}dz) \).

We are going to pass to the limit in (4.14). To this aim, we observe that (2.20) implies that

\[
\left| \int_{B^+_τ} t^{1-2s} (A(λ_{y}) \nabla w^{λ_{y}}(z) - \nabla w(z)) \cdot \nabla Φ(z) dz \right| ≤ C \int_{B^+_τ} t^{1-2s} |\nabla w^{λ_{y}}| |\nabla Φ| dz
\]

\[
≤ C \left( \int_{B^+_τ} t^{1-2s} |\nabla w^{λ_{y}}|^2 dz \right)^{1/2} \left( \int_{B^+_τ} t^{1-2s} |\nabla Φ|^2 dz \right)^{1/2}
\]

for some \( C > 0 \) and for sufficiently small \( λ \), and

\[
λ^{2s} \int_{Γ^+_τ} \tilde{h}(λ_{y}) Tr w^{λ_{y}}(y) Tr Φ(y) dy
\]

\[
≤ \lambda^{2s} \left( \int_{B^+_τ} |Tr w^{λ_{y}}(y)|^{2s} dy \right)^{\frac{1}{2s}} \left( \int_{B^+_τ} |Tr Φ(y)|^{2s} dy \right)^{\frac{1}{2s}} \left( \int_{Γ^+_τ} |\tilde{h}(λ_{y})|^{\frac{1}{2s}} dy \right)^{\frac{2s}{2}} = O(1) \left( \int_{Γ^+_τ} |\tilde{h}(λ_{y})|^{\frac{1}{2s}} dy \right)^{\frac{2s}{2}} = o(1) \quad \text{as} \quad λ → 0^+,
\]

from Hölder’s inequality, Lemmas 2.3, 4.1 and 4.3, since \( μ(λ_{y}) ≥ 1/4 \) for all \( λ ≤ R_0 \). Taking \( λ = λ_{n_k} R_{λ_{n_k}} \) in (4.15) and (4.16), letting \( k → ∞ \), and recalling that \( w^{λ_{n_k} R_{λ_{n_k}}} \to w \) weakly in \( H^1(B^+_τ, t^{1-2s}dz) \), we obtain that

\[
\lim_{k→∞} \int_{B^+_τ} t^{1-2s} A(λ_{n_k} R_{λ_{n_k}} y) \nabla w^{λ_{n_k} R_{λ_{n_k}}} (z) \cdot \nabla Φ(z) dz = \int_{B^+_τ} t^{1-2s} \nabla w \cdot \nabla Φ dz
\]

and

\[
\lim_{k→∞} (λ_{n_k} R_{λ_{n_k}}) 2s \int_{Γ^+_τ} \tilde{h}(λ_{n_k} R_{λ_{n_k}} y) Tr w^{λ_{n_k} R_{λ_{n_k}}} (y) Tr Φ(y) dy = 0.
\]

Thanks to (2.20), we have that

\[
\frac{∂w^{λ_{n_k} R_{λ_{n_k}}}}{∂ν} \to f \quad \text{weakly in} \quad L^2(∂^+B^+_τ, t^{1-2s}dS)
\]

Moreover, from Lemma 4.4 up to a further subsequence, we have that

(4.19)
for some $f \in L^2(\partial^+B^+_1, t^{1-2s}dS)$. Then, taking $\lambda = \lambda_{n_k} R_{\lambda_{n_k}}$ in (4.19), letting $k \to \infty$, and taking into account Lemma 4.4 we obtain that

$$
\lim_{k \to \infty} \int_{\partial^+B^+_1} (t^{1-2s} A(\lambda_{n_k} R_{\lambda_{n_k}} y) \nabla w^{\lambda_{n_k} R_{\lambda_{n_k}}}(z) \cdot \nu) \Phi(z) dS = \int_{\partial^+B^+_1} t^{1-2s} f \Phi dS.
$$

Hence, passing to the limit as $k \to \infty$ in (4.14) and combining (4.17), (4.18), and (4.21), we find that

$$
\int_{B^+_1} t^{1-2s} \nabla w \cdot \nabla \Phi dS = \int_{\partial^+B^+_1} t^{1-2s} f \Phi dS
$$

for any fixed $\Phi \in H^1_{\Gamma_1} (B^+_1, t^{1-2s}dS)$. On the other hand, if we take $\Phi = w^{\lambda_{n_k} R_{\lambda_{n_k}}}$ in (4.14), we have that

$$
\int_{B^+_1} t^{1-2s} A(\lambda_{n_k} R_{\lambda_{n_k}} y) \nabla w^{\lambda_{n_k} R_{\lambda_{n_k}}}(z) \cdot \nabla w^{\lambda_{n_k} R_{\lambda_{n_k}}}(z) dS
$$

$$
= \kappa_s(\lambda_{n_k} R_{\lambda_{n_k}})^{2s} \int_{\Gamma_1} h(\lambda_{n_k} R_{\lambda_{n_k}} y) |\text{Tr} w^{\lambda_{n_k} R_{\lambda_{n_k}}}(y)|^2 dy
$$

$$
+ \int_{\partial^+B^+_1} (t^{1-2s} A(\lambda_{n_k} R_{\lambda_{n_k}} z) \nabla w^{\lambda_{n_k} R_{\lambda_{n_k}}}(z) \cdot \nu) w^{\lambda_{n_k} R_{\lambda_{n_k}}}(z) dS,
$$

hence, by (2.20), arguing as in (4.19) and (4.11) and using Lemma 4.4 and (4.20), we obtain that

$$
\lim_{k \to \infty} \int_{B^+_1} |\nabla w^{\lambda_{n_k} R_{\lambda_{n_k}}}|^2 = \lim_{k \to \infty} \int_{B^+_1} t^{1-2s} A(\lambda_{n_k} R_{\lambda_{n_k}} y) \nabla w^{\lambda_{n_k} R_{\lambda_{n_k}}}(z) \cdot \nabla w^{\lambda_{n_k} R_{\lambda_{n_k}}}(z) dS
$$

$$
= \lim_{k \to \infty} \int_{\partial^+B^+_1} t^{1-2s} \frac{\partial w^{\lambda_{n_k} R_{\lambda_{n_k}}}}{\partial \nu} w^{\lambda_{n_k} R_{\lambda_{n_k}}} dS
$$

$$
= \int_{\partial^+B^+_1} t^{1-2s} f w dS = \int_{B^+_1} t^{1-2s} |\nabla w|^2,
$$

where we used the compactness of the trace operator $H^1(B^+_1, t^{1-2s}dS) \hookrightarrow L^2(\partial^+B^+_1, t^{1-2s}dS)$ and (4.22) with $\Phi = w$. The weak convergence $w^{\lambda_{n_k} R_{\lambda_{n_k}}} \rightharpoonup w$ in $H^1(B^+_1, t^{1-2s}dS)$ together with (4.23) imply (4.13).

For every $k \in \mathbb{N}$ and $r \in (0, 1]$, let us define

$$
E_k(r) = \frac{1}{r^{N-2s}} \int_{B^+_1} t^{1-2s} A(\lambda_{n_k} R_{\lambda_{n_k}} y) \nabla w^{\lambda_{n_k} R_{\lambda_{n_k}}} \cdot \nabla w^{\lambda_{n_k} R_{\lambda_{n_k}}} dS
$$

$$
- \kappa_s(\lambda_{n_k} R_{\lambda_{n_k}})^{2s} \int_{\Gamma_1} h(\lambda_{n_k} R_{\lambda_{n_k}} y) |\text{Tr} w^{\lambda_{n_k} R_{\lambda_{n_k}}}(y)|^2 dy
$$

and

$$
H_k(r) = \frac{1}{r^{N+1-2s}} \int_{\partial^+B^+_1} t^{1-2s} \mu(\lambda_{n_k} R_{\lambda_{n_k}} z) |w^{\lambda_{n_k} R_{\lambda_{n_k}}}(z)|^2 dS.
$$

We also define, for any $r \in (0, 1]$,

$$
E_w(r) = \frac{1}{r^{N-2s}} \int_{B^+_1} t^{1-2s} |\nabla w(z)|^2 dS
$$

and

$$
H_w(r) = \frac{1}{r^{N+1-2s}} \int_{\partial^+B^+_1} t^{1-2s} w^2(z) dS.
$$

By scaling, one can easily verify that

$$
N_k(r) := \frac{E_k(r)}{H_k(r)} = \frac{E(\lambda_{n_k} R_{\lambda_{n_k}} r)}{H(\lambda_{n_k} R_{\lambda_{n_k}} r)} = N(\lambda_{n_k} R_{\lambda_{n_k}} r) \quad \text{for all } r \in (0, 1].
$$

From (4.13), (2.20), and (4.16), it follows that, for any fixed $r \in (0, 1]$,

$$
E_k(r) \to E_w(r).
$$

On the other hand, by compactness of the trace operator and (2.25), we also have, for any fixed $r \in (0, 1]$,

$$
H_k(r) \to H_w(r).
$$
In order to prove that $H_w$ is strictly positive, we argue by contradiction and assume that there exists $r \in (0,1]$ such that $H_w(r) = 0$; then $r$ is a minimum point for $H_w$ and hence, arguing as in Lemma 3.7, we obtain that necessarily $0 = H_w'(r) = 2^{2s-N-1} \int_{B^+_r} t^{1-2s} |\nabla w(z)|^2 dz$ and hence $w$ is constant in $B^+_r$.

From Lemma 2.3 we conclude that $w \equiv 0$ in $B^+_r$, which implies that $w \equiv 0$ in $B^+_1$ from classical unique continuation principles for second order elliptic equations, thus contradicting (4.12).

Hence $H_w(r) > 0$ for all $r \in (0,1]$, thus the function

$$\mathcal{N}_w : (0,1] \to \mathbb{R}, \quad \mathcal{N}_w(r) := \frac{E_w(r)}{H_w(r)}$$

is well defined and one can easily prove that it belongs to $W^{1,1}_{\text{loc}}((0,1])$. From (4.26), (4.27), (4.28), and Proposition 3.6 we deduce that

$$\mathcal{N}_w(r) = \lim_{k \to \infty} \mathcal{N}(\lambda_{n_k} R\lambda_{n_k}^1 r) = \gamma$$

for all $r \in (0,1]$. Therefore $\mathcal{N}_w$ is constant in $(0,1]$ and hence

$$\mathcal{N}_w'(r) = 0 \quad \text{for any } r \in (0,1).$$

Recalling the equation satisfied by $w$, i.e. (4.22), and arguing as in Lemma 3.6 with $A = \text{Id}_N$ and $\tilde{h} \equiv 0$, we can prove that, for a.e. $r \in (0,1)$,

$$\mathcal{N}_w'(r) = \frac{2r \left( \int_{B^+_r} t^{1-2s} |\partial_r w|^2 ds \right) \left( \int_{\partial^+ B^+_r} t^{1-2s} w^2 ds \right) - \left( \int_{\partial^+ B^+_r} t^{1-2s} \partial_r w w^2 ds \right)^2}{\left( \int_{\partial^+ B^+_r} t^{1-2s} w^2 ds \right)^2}.$$ (4.31)

Combining (4.30) and (4.31) with Schwarz’s inequality, we obtain that, for a.e. $r \in (0,1)$,

$$\left( \int_{\partial^+ B^+_r} t^{1-2s} |\partial_r w|^2 ds \right) \left( \int_{\partial^+ B^+_r} t^{1-2s} w^2 ds \right) - \left( \int_{\partial^+ B^+_r} t^{1-2s} \partial_r w w^2 ds \right)^2 = 0.$$ (4.32)

Hence, for a.e. $r \in (0,1)$, $w$ and $\partial_r w$ have the same direction as vectors in $L^2(\partial^+ B^+_r, t^{1-2s} ds)$, so that there exists a function $\eta = \eta(r)$, defined a.e. in $(0,1)$, such that $\partial_r w(r\theta) = \eta(r) w(r\theta)$ for a.e. $r \in (0,1)$ and for all $\theta \in S^N_+$. It is easy to verify that $\eta(r) = \frac{H_w(r)}{2H_w(r)}$ for a.e $r \in (0,1)$, so that $\eta \in L^1_{\text{loc}}((0,1])$. After integration we obtain that

$$w(r\theta) = e^{\int_0^r \eta(s) ds} w(\theta) = g(r) \psi(\theta), \quad r \in (0,1), \quad \theta \in S^N_+.$$ (4.33)

where $g(r) = e^{\int_0^r \eta(s) ds}$ and $\psi = w|_{S^N_+}$. We observe that (4.12) implies that

$$\|\psi\|_{L^2(S^N_+, g^{1/2} \lambda_+ ds)} = 1.$$ (4.34)

From the fact that $w \in H^1_1(B^+_1, t^{1-2s} dz)$ it follows that $\psi \in H_0$; moreover, plugging (4.32) into (4.22) we obtain that $\psi$ satisfies (4.12) for some $\mu \in \mathbb{R}$, so that $\psi$ is an eigenfunction of (4.11). Recalling (4.13) and letting $k_0 \in \mathbb{N}$ be such that $\mu = \mu_{k_0} = (k_0 + s)(k_0 + N - s)$, we can rewrite the equation

$$-\text{div}( t^{1-2s} \nabla w) = 0$$

in polar coordinates exploiting (4.3) Lemma 2.1, thus obtaining, for all $r \in (0,1)$ and $\theta \in S^N_+$,

$$0 = \frac{1}{r^N} \left( r^{N+1-2s} g' \frac{t^{1-2s}}{N+1} \psi(\theta) + r^{-1-2s} g(r) \text{div}_{S^N} \left( \frac{t^{1-2s}}{N+1} \nabla_{S^N} \psi(\theta) \right) \right)$$

$$- \frac{1}{r^N} \left( r^{N+1-2s} g' \frac{t^{1-2s}}{N+1} \psi(\theta) - r^{-1-2s} g(r) \frac{t^{1-2s}}{N+1} \mu_{k_0} \psi(\theta) \right).$$

Then $g(r)$ solves the equation

$$-\frac{1}{r^N} \left( r^{N+1-2s} g' \frac{t^{1-2s}}{N+1} \psi(\theta) + \mu_{k_0} r^{-1-2s} g(r) \right) = 0 \quad \text{in } (0,1)$$

i.e.

$$-g''(r) - \frac{N + 1 - 2s}{r} g'(r) + \frac{\mu_{k_0}}{r^2} g(r) = 0 \quad \text{in } (0,1).$$

Hence $g(r)$ is of the form

$$g(r) = c_1 r^{k_0 + s} + c_2 r^{s-N-k_0}$$
for some $c_1, c_2 \in \mathbb{R}$. Since $w \in H^1(B^+_1, t^{1-2s} \, dz)$ and the function $|z|^{-1}|z|^{s-N-k_0}\psi(\frac{z}{|z|}) \not\in L^2(B^+_1, t^{1-2s} \, dz)$, from Lemma 2.3 we deduce that necessarily $c_2 = 0$ and $g(r) = c_1 r^{k_0+s}$. Moreover, from $g(1) = 1$, we obtain that $c_1 = 1$ and then

$$w(r\theta) = r^{k_0+s}\psi(\theta), \quad \text{for all } r \in (0, 1) \text{ and } \theta \in S^N_+.$$

Let us now consider the sequence $\{w^{\lambda_n}\}$. Up to a further subsequence still denoted by $\{w^{\lambda_n}\}$, we may suppose that $w^{\lambda_n} \rightharpoonup \overline{w}$ weakly in $H^1(B^+_1, t^{1-2s} \, dz)$ for some $\overline{w} \in H^1(B^+_1, t^{1-2s} \, dz)$ and that $R^{\lambda_n} \to \overline{R}$ for some $\overline{R} \in [1, 2]$.

Strong convergence of $w^{\lambda_n}R^{\lambda_n}$ in $H^1(B^+_1, t^{1-2s} \, dz)$ implies that, up to a subsequence, both $w^{\lambda_n}R^{\lambda_n}$ and $|\nabla w^{\lambda_n}R^{\lambda_n}|$ are a.e. dominated by a $L^2(B^+_1, t^{1-2s} \, dz)$-function uniformly with respect to $k$. Moreover, by (4.3), up to a further subsequence, we may assume that the limit

$$\ell := \lim_{k \to +\infty} \frac{H(\lambda_n, R^{\lambda_n})}{H(\lambda_n)}$$

exists and is finite, with $\ell > 0$. Then, by Dominated Convergence Theorem, we have

$$\lim_{k \to +\infty} \int_{B^+_1} t^{1-2s} w^{\lambda_n}(z)v(z) \, dz = \lim_{k \to +\infty} R^{\lambda_n}_{\lambda_n} \int_{B^+_1} t^{1-2s} w^{\lambda_n}(R^{\lambda_n} z)v(R^{\lambda_n} z) \, dz$$

$$= \lim_{k \to +\infty} \sqrt{t} \int_{B^+_1} t^{1-2s} w^{\lambda_n}(z)v(R^{\lambda_n} z) \, dz$$

$$= \lim_{k \to +\infty} \sqrt{t} \int_{B^+_1} t^{1-2s} w(z)v(Rz) \, dz$$

for any $v \in C^\infty(B^+_1)$. By density, the above convergence actually holds for all $v \in L^2(B^+_1, t^{1-2s} \, dz)$ . This proves that $w^{\lambda_n} \rightharpoonup \sqrt{t} w(\cdot/\overline{R})$ weakly in $L^2(B^+_1, t^{1-2s} \, dz)$. Since we know that $w^{\lambda_n} \rightharpoonup \overline{w}$ weakly in $H^1(B^+_1, t^{1-2s} \, dz)$, we conclude that $\overline{w} = \sqrt{t} w(\cdot/\overline{R})$ and then $w^{\lambda_n} \rightharpoonup \sqrt{t} w(\cdot/\overline{R})$. Moreover

$$\lim_{k \to +\infty} \int_{B^+_1} t^{1-2s} |\nabla w^{\lambda_n}(z)|^2 \, dz = \lim_{k \to +\infty} R^{\lambda_n}_{\lambda_n} \int_{B^+_1} t^{1-2s} |\nabla w^{\lambda_n}(R^{\lambda_n} z)|^2 \, dz$$

$$= \lim_{k \to +\infty} \sqrt{t} \int_{B^+_1} t^{1-2s} |\nabla w(z)|^2 \, dz$$

$$= \int_{B^+_1} t^{1-2s} |\nabla w(z)|^2 \, dz.$$

This shows that $w^{\lambda_n} \rightharpoonup \overline{w} = \sqrt{t} w(\cdot/\overline{R})$ strongly in $H^1(B^+_1, t^{1-2s} \, dz)$.

By (4.34) $w$ is homogeneous of degree $k_0 + s$, hence $\overline{w} = \sqrt{t} R^{-k_0-s} w$. Furthermore (4.3) and the strong convergence $w^{\lambda_n} \rightharpoonup \overline{w}$ in $L^2(\partial^+ B^+_1, t^{1-2s} \, dS)$ imply that

$$1 = \int_{\partial^+ B^+_1} t^{1-2s} w^2 \, dS = t \overline{R}^{-2k_0-2s} \int_{\partial^+ B^+_1} t^{1-2s} w^2 \, dS = t \overline{R}^{-2k_0-2s}$$

in view of (4.12), thus implying that $\overline{w} = w$.

It remains to prove part (i). By (4.34), (4.33) and the fact that $\psi$ is an eigenfunction of (1.11) with associated eigenvalue $\mu_{k_0} = (k_0 + s)(k_0 + N - s)$, we have that

$$\int_{B^+_1} t^{1-2s} |\nabla w(z)|^2 \, dz = \frac{r^{N+2k_0}}{N+2k_0} (k_0 + s)^2 + \mu_{k_0}) = (k_0 + s)r^{N+2k_0}$$

and

$$\int_{\partial^+ B^+_1} t^{1-2s} w^2 \, dS = r^{N+1+2s} \int_{S^N_+} \theta^{1-2s} w^2(r\theta) \, dS = r^{N+2k_0+1}.$$
Therefore, by (4.24), (4.25) and (4.29), it follows that
\[
\gamma = N_w(r) = \frac{E_w(r)}{H_w(r)} = \frac{r \int_{B^+ r} t^{1-2s} |\nabla w(z)|^2 dz}{\int_{\partial_B^+ r} t^{1-2s} w^2 dS} = k_0 + s.
\]
This completes the proof. □

To complete the blow-up analysis and detect the sharp asymptotic behaviour of $W$ at 0, it remains to describe the behavior of $H(\lambda)$ as $\lambda \to 0^+$. 

**Lemma 4.6.** Let $\gamma = \lim_{r \to 0} N(r)$ be as in Proposition 3.6 Then the limit $\lim_{r \to 0^+} r^{-2\gamma} H(r)$ exists and is finite.

**Proof.** Thanks to (3.24), it is enough to show that the limit exists. From (3.5) we deduce that, a.e. in $(0, R_0)$,
\[
\frac{d}{dr} H(r) = \frac{H'(r)}{r^{2\gamma}} - 2 \frac{H(r)}{r^{2\gamma+1}} (E(r) + H(r)O(r) - \gamma H(r)) = 2 \frac{H(r)}{r^{2\gamma+1}} (\gamma N(r) - \gamma + O(r)) = 2 \frac{H(r)}{r^{2\gamma+1}} \left( \int_0^r N'(s) \, ds + O(r) \right)
\]
as $r \to 0^+$. Using the notation of Lemma 3.6, we can write $N' = \alpha_1 + \alpha_2$ in $(0, \bar{R})$, with
\[
\alpha_1(r) = N'(r) + C_2 r^{-1-\delta} \left( C_1 + \frac{N-2s}{2} \right) \quad \text{and} \quad \alpha_2(r) = -C_4 r^{-1-\delta},
\]
where $\delta \in (0, 1]$ has been defined in (3.11) and $C_4 = C_2 \left( C_1 + \frac{N-2s}{2} \right)$. Integrating (4.35) between $(r, \bar{R})$, we obtain that
\[
\frac{H(\bar{R})}{R^{2\gamma}} - \frac{H(r)}{r^{2\gamma}} = \int_r^{\bar{R}} \frac{2H(\rho)}{\rho^{2\gamma}} \left( \int_0^\rho \alpha_1(\tau) d\tau \right) d\rho + \int_r^{\bar{R}} \frac{2H(\rho)}{\rho^{2\gamma}} \left( \int_0^\rho \alpha_2(\tau) d\tau \right) d\rho + \int_r^{\bar{R}} \frac{H(\rho)}{\rho^{2\gamma}} O(1) d\rho
\]
\[= \int_r^{\bar{R}} \frac{2H(\rho)}{\rho^{2\gamma}} \left( \int_0^\rho \alpha_1(\tau) d\tau \right) d\rho - \int_r^{\bar{R}} \frac{H(\rho)}{\rho^{2\gamma}} \left( -C_4 r^{-1-\delta} + O(1) \right) d\rho.
\]
Since $\alpha_1 \geq 0$ by (3.24) and (3.25), we have that $\lim_{r \to 0^+} \int_r^{\bar{R}} \frac{2H(\rho)}{\rho^{2\gamma}} \left( \int_0^\rho \alpha_1(\tau) d\tau \right) d\rho$ exists. On the other hand, estimate (3.24) ensures that $\rho \mapsto \frac{H(\rho)}{\rho^{2\gamma}} \left( -C_4 r^{-1-\delta} + O(1) \right) \in L^1(0, \bar{R})$, so that the limit $\lim_{r \to 0^+} \int_r^{\bar{R}} \frac{H(\rho)}{\rho^{2\gamma}} \left( -C_4 r^{-1-\delta} + O(1) \right) d\rho$ exists and is finite. The lemma is thereby proved. □

The next step is the proof that the limit $\lim_{r \to 0^+} r^{-2\gamma} H(r)$ is actually strictly positive. To this aim, we first define the Fourier coefficients associated with $W$, with respect to the orthonormal basis $\{\varphi_{k,m}(\lambda)\}$ of $L^2(\mathbb{S}^N, \theta_{N-1}^{1-2s} dS)$, as
\[
\varphi_{k,m}(\lambda) = \int_{\mathbb{S}^N} \theta_{N-1}^{1-2s} W(\lambda \theta) Y_{k,m}(\theta) dS, \quad \lambda \in (0, R_1], \quad k \in \mathbb{N}, \quad m = 1, \ldots, M_k.
\]

We also define
\[
\Upsilon_{k,m}(\lambda) = -\int_{B^+_{\delta^2}} t^{1-2s} (A - \text{Id}_{N+1}) \nabla W(z) \cdot \nabla_{\mathbb{S}^N} Y_{k,m}(z/|z|) \frac{dz}{|z|}
\]
\[+ \kappa_s \int_{B^+_{\delta^2}} \tilde{h}(y) TrW(y) Y_{k,m}(y/|y|) dy + \int_{\mathbb{S}^N} t^{1-2s} (A - \text{Id}_{N+1}) \nabla W \cdot \frac{z}{|z|} Y_{k,m}(\frac{z}{|z|}) dS,
\]
for a.e. $\lambda \in (0, R_1]$, $k \in \mathbb{N}$ and $m \in \{1, 2, \ldots, M_k\}$.

**Lemma 4.7.** Let $k_0$ be as in Proposition 4.5 Then, for all $m \in \{1, 2, \ldots, M_k\}$ and $R \in (0, R_0]$,
\[
\varphi_{k_0,m}(\lambda) = \chi_{k_0}^{\lambda} \left( R^{-k_0^{-2s}} \varphi_{k_0,m}(R) + \frac{(k_0 + s) R^{-N-2k_0}}{N + 2k_0} \int_0^R \rho^{k_0 + s - 1} \Upsilon_{k_0,m}(\rho) \, d\rho \right)
\]
\[+ \frac{N - s + k_0}{N + 2k_0} \int_{\lambda}^R \rho^{-N-1+s-k_0} \Upsilon_{k_0,m}(\rho) \, d\rho + O(\chi_{k_0}^{\lambda} \chi_{k_0}^{\lambda + \delta}) \quad \text{as} \quad \lambda \to 0^+,
\]
where $\delta$ is defined in (3.11).
Proof. Let $k \in \mathbb{N}$ and $m \in \{1, 2, ..., M_k\}$. Testing (2.11) with $\phi = \omega(|z|)|z|^{-N+1+2s}Y_{k,m}(z/|z|)$ for any test function $\omega \in C^\infty_0(0, R_1)$ and using (1.12), we can easily verify that $\varphi_{k,m}$ solves the following second order differential equation

$$
(4.39) \quad -\varphi''_{k,m}(\lambda) - \frac{N + 1 - 2s}{\lambda} \varphi'_{k,m}(\lambda) + \frac{\mu_k}{\lambda^2} \varphi_{k,m}(\lambda) = \zeta_{k,m}(\lambda) \quad \text{in} \ (0, R_1)
$$

in a distributional sense, with $\mu_k$ as in (1.13), where the distribution $\zeta_{k,m} \in \mathcal{D}'(0, R_1)$ is defined by

$$
\mathcal{D}'(0, R_1)[\zeta_{k,m}, \omega]_{\mathcal{D}(0, R_1)} = \kappa_s \int_0^{R_1} \omega(\lambda) \left( \int_{S^{N-1}} \bar{h}(\lambda') \text{Tr} W(\lambda') Y_{k,m}(\lambda', 0) dS' \right) d\lambda - \int_{B^+_R} t^{1-2s}(A - \text{Id}_{N+1}) \nabla W \cdot \nabla (\omega(|z|) |z|^{-N+1+2s} Y_{k,m}(z/|z|)) dz
$$

for any $\omega \in C^\infty_0((0, R_1))$, where $S^{N-1} = \{ (\theta_1, ..., \theta_N) \in S^{N-1} : \theta_N \leq 0 \}$. Letting $\Upsilon_{k,m}$ be as in (4.37), by direct calculations we have that $\Upsilon_{k,m} \in L^1(0, R_1)$ and

$$
(4.40) \quad \Upsilon'_{k,m}(\lambda) = \lambda^{N+1-2s} \zeta_{k,m}(\lambda) \quad \text{in} \ \mathcal{D}'(0, R_1).
$$

In view of (4.40) and (1.13), we have that (4.39) is equivalent to

$$
(1.41) \quad \left( \lambda^{-k-s} \varphi_{k,m}(\lambda) \right)' = -\lambda^{-N-1+s-k} \Upsilon_{k,m}(\lambda) - (k + s) \lambda^{-N-1-2k} \left( c_{k,m}(R) + \int_{\lambda}^{R} \rho^{k+s-1} \Upsilon_{k,m}(\rho) d\rho \right),
$$

in the sense of distributions in $(0, R_1)$. From (4.41) we infer that $\varphi_{k,m} \in W^{1,1}_{\text{loc}}((0, R_1))$, thus a new integration leads to

$$
(4.42) \quad \varphi_{k,m}(\lambda) = \lambda^{k+s} \left( \frac{\varphi_{k,m}(R)}{R^{k+s}} - \frac{(k + s) c_{k,m}(R)}{(N + 2k) R^{N+2k}} + \frac{N + k - s}{N + 2k} \int_{\lambda}^{R} \rho^{-N-k+s-1} \Upsilon_{k,m}(\rho) d\rho \right) + \frac{(k + s) \lambda^{-N-k+s}}{N + 2k} \left( c_{k,m}(R) + \int_{\lambda}^{R} \rho^{k+s-1} \Upsilon_{k,m}(\rho) d\rho \right)
$$

for all $\lambda \in (0, R_1]$. From now on, we fix $k_0$ as in Proposition 1.3, $R_0$ as in (2.15), and $m \in \{1, 2, ..., M_{k_0}\}$. We prove that

$$
(4.43) \quad \int_0^{R_0} \rho^{-N-k_0+s-1} |\Upsilon_{k_0,m}(\rho)| d\rho < +\infty.
$$

To this purpose, exploiting (2.20) and using Hölder’s inequality, one can estimate the first term in (4.37) for all $\rho \in (0, R_0)$ as follows

$$
(4.44) \quad \left| \int_{B^+_R} t^{1-2s}(A - \text{Id}_{N+1}) \nabla W(z) \cdot \frac{\nabla_{S^N} Y_{k_0,m}(z/|z|)}{|z|} dz \right| 
\leq \text{const} \left( \int_{B^+_R} t^{1-2s} |\nabla W|^2 dz \right)^{1/2} \left( \int_{B^+_R} t^{1-2s} |\nabla_{S^N} Y_{k_0,m}(z/|z|)|^2 dz \right)^{1/2} 
=: \text{const} \ I_1(\rho) \cdot I_2(\rho),
$$

where

$$
(4.45) \quad I_1(\rho) = \sqrt{\rho^{N-2s} \int_{B^+_R} t^{1-2s} |\nabla W(\rho z)|^2 dz} = \rho^{N-2s} \sqrt{H(\rho)} \sqrt{\int_{B^+_R} t^{1-2s} |\nabla w^\rho(z)|^2 dz} 
\leq \text{const} \rho^{N-2s} \sqrt{H(\rho)},
$$

and

$$
I_2(\rho) = \sqrt{\rho^{1-2s} \int_{B^+_R} t^{1-2s} |\nabla_{S^N} Y_{k_0,m}(z/|z|)|^2 dz}.
$$
as a consequence of Lemma 4.1 and

\[
I_2(\rho) = \sqrt{\int_0^\rho \frac{\theta^{N+1-2s}}{S_{22}} \left( \int_{S^2} \theta^{1-2s} |\nabla_{S^2} Y_{k_0,m}(\theta)|^2 dS \right) dt} = \sqrt{\frac{\rho^{N+2-2s}}{N + 2 - 2s} \frac{\rho^{N+2-2s}}{2s}},
\]

due to (1.12). Combining (4.44), (4.45), (4.46), and (3.24) we obtain that, for every \( R \in (0, R_0] \),

\[
\int_0^R \rho^{N-1+s-k_0} \left( \int_{B^+_R} t^{1-2s}(A - I_{N+1}) \nabla W(z) \cdot \frac{\nabla_{S^2} Y_{k_0,m}(z/|z|)}{|z|} dz \right) d\rho \leq \text{const} \int_0^R \rho^{-s-k_0} \sqrt{H(\rho)} d\rho \leq \text{const} R.
\]

Moreover, Hölder’s inequality implies that

\[
\left( \int_{\Gamma^\rho} \tilde{h}(y) Tr W(y) Tr Y_{k_0,m}(\frac{\rho}{|y|}) dy \right)^{\frac{2}{2-2s}} \leq \left( \int_{\Gamma^\rho} |\tilde{h}(y)| Tr W(y)^2 dy \cdot \int_{\Gamma^\rho} |\tilde{h}(y)||Tr Y_{k_0,m}(\frac{\rho}{|y|})|^2 dy \right)^{\frac{1}{2-2s}}.
\]

From (2.31) and homogeneity of the function \( Y_{k_0,m}(y/|y|) \) it follows that, for all \( \rho \in (0, R_0] \),

\[
\int_{\Gamma^\rho} |\tilde{h}(y)||Tr Y_{k_0,m}(\frac{\rho}{|y|})|^2 dy \leq \nu C_{N,s} \rho^{s}\left( \int_{\Gamma^\rho} |\tilde{h}(y)| Tr Y_{k_0,m}(\frac{\rho}{|y|})^2 dy \right)^{\frac{1}{2-2s}}.
\]

Using (2.31), (2.31), and (3.24), we obtain that, for all \( \rho \in (0, R_0] \),

\[
\int_{\Gamma^\rho} |\tilde{h}(y)||Tr W(y)^2 dy \leq \nu C_{N,s} \rho^{s}\left( \int_{\Gamma^\rho} |\tilde{h}(y)| Tr Y_{k_0,m}(\frac{\rho}{|y|})^2 dy \right)^{\frac{1}{2-2s}} \leq \text{const} \rho^{\frac{s}{2-2s}} \sqrt{H(\rho)}.
\]

Putting the above estimates together and recalling (3.24), we conclude that, for every \( R \in (0, R_0] \),

\[
\int_0^R \rho^{N-k_0+s-1} \left( \int_{\Gamma^\rho} \tilde{h}(y) Tr W(y) Tr Y_{k_0,m}(\frac{\rho}{|y|}) dy \right) d\rho \leq \text{const} \int_0^R \rho^{1+s-k_0-s} \sqrt{H(\rho)} d\rho \leq \text{const} R^{\frac{1}{2-2s}}.
\]

In order to estimate the last term, we observe that, since \( \|Y_{k_0,m}\|_{L^2(S^2, g_{N+1}^{N+2-2s} dS)} = 1 \),

\[
\int_{B^+_R} t^{1-2s}|Y_{k_0,m}(\frac{z}{|z|})|^2 dz = \int_0^1 t^{1-2s} \left( \int_{S^2} \theta^{1-2s}|Y_{k_0,m}(\theta)|^2 dS \right) dt = \frac{\lambda^{N+2-2s}}{N + 2 - 2s}.
\]

Hence, integrating by parts, we have that, for every \( R \in (0, R_0] \),

\[
\int_0^R \rho^{N+s-1-k_0} \left( \int_{B^+_R} t^{1-2s}(A - I_{N+1}) \nabla W \cdot \frac{z}{|z|} Y_{k_0,m}(\frac{z}{|z|}) dS \right) d\rho \leq \text{const} \int_0^R \rho^{N+s-k_0} \left( \int_{B^+_R} t^{1-2s} |\nabla W||Y_{k_0,m}(\frac{z}{|z|})| dS \right) d\rho
\]

\[
= \text{const} \left( \int_{B^+_R} t^{1-2s} |\nabla W||Y_{k_0,m}(\frac{z}{|z|})| dz \right) + (N + k_0 - s) \int_0^R \rho^{N+s-1-k_0} \left( \int_{B^+_R} t^{1-2s} |\nabla W||Y_{k_0,m}(\frac{z}{|z|})| dS \right) d\rho
\]

\[
\leq \text{const} \left( R^{1-s-k_0} \sqrt{H(R)} + \int_0^R \rho^{-s-k_0} \sqrt{H(\rho)} d\rho \right) \leq \text{const} R,
\]
as a consequence of Lemma 2.3, giving rise to a contradiction. Hence (4.54) holds true. From (4.54) and (4.55)

\[ k \] as a direct consequence of (4.43), since \( \int_0^R \rho^{-N-k_0+s-1}\|\mathcal{Y}_{k_0,m}(\rho)\|\,d\rho \leq \text{const} R^\delta \)

for some const \( \delta > 0 \) independent of \( R \). From (4.52), we derive immediately (4.43).

From (4.43) it follows that, for every \( R \in (0, R_0) \),

\[ \lambda^{k_0+s} \left( \frac{\varphi_{k_0,m}(R)}{R_{k_0+s}} \right) - \frac{(k_0 + s)c_{k_0,m}(R)}{(N + 2k_0)N^{N+2k_0}} \frac{N + k_0 - s}{N + 2k_0} \int_0^R \rho^{-N-k_0+s-1}\|\mathcal{Y}_{k_0,m}(\rho)\|\,d\rho \]

\[ = O(\lambda^{k_0+s}) = o(\lambda^{-N-k_0+s}) \quad \text{as} \quad \lambda \to 0^+. \]

Now we prove that, for every \( R \in (0, R_0) \),

\[ c_{k_0,m}(R) + \int_0^R \rho^{k_0+s-1}\|\mathcal{Y}_{k_0,m}(\rho)\|\,d\rho = 0. \]

In order to do this, first we observe that

\[ \int_0^{R_0} \rho^{k_0+s-1}\|\mathcal{Y}_{k_0,m}(\rho)\|\,d\rho < +\infty, \]

as a direct consequence of (4.43), since \( k_0 + s - 1 > -N - k_0 + s - 1 \). Suppose by contradiction that \( (4.54) \) does not hold true for some \( R \in (0, R_0) \); then from (4.42), (4.53) and (4.55), we should have that

\[ \varphi_{k_0,m}(\lambda) \sim \frac{(k_0 + s)\lambda^{-N-k_0+s}}{N + 2k_0} c_{k_0,m}(R) \int_0^R \rho^{k_0+s-1}\|\mathcal{Y}_{k_0,m}(\rho)\|\,d\rho \]

as \( \lambda \to 0^+ \), and hence

\[ \int_0^{R_0} \lambda^{N_1-2s}|\varphi_{k_0,m}(\lambda)|^2\,d\lambda = +\infty. \]

On the other hand, by (4.36), we have that

\[ \int_0^{R_0} \lambda^{N_1-2s}|\varphi_{k_0,m}(\lambda)|^2\,d\lambda \leq \int_0^{R_0} \lambda^{N_1-2s} \left( \int_{B^+_{R_0}} \theta_{N+1}^{1-2s}|W(\lambda\theta)|^2\,dS \right)\,d\lambda = \int_{B^+_{R_0}} t^{1-2s} W^2(z)\,dz < \infty, \]

as a consequence of Lemma 2.3, giving rise to a contradiction. Hence (4.54) holds true. From (4.54) and (4.55) we deduce that, for every \( R \in (0, R_0) \),

\[ \left| \lambda^{-N-k_0+s} \left( c_{k_0,m}(R) + \int_0^R \rho^{k_0+s-1}\|\mathcal{Y}_{k_0,m}(\rho)\|\,d\rho \right) \right| = \lambda^{-N+k_0-s} \int_0^\lambda \rho^{k_0+s-1}\|\mathcal{Y}_{k_0,m}(\rho)\|\,d\rho \]

\[ \leq \lambda^{-N+s-k_0} \int_0^\lambda \rho^{N+2k_0}\|\rho^{-N+1+s-k_0}\mathcal{Y}_{k_0,m}(\rho)\|\,d\rho \]

\[ \leq \lambda^{k_0+s} \int_0^\lambda \rho^{-N+1+s-k_0}\|\mathcal{Y}_{k_0,m}(\rho)\|\,d\rho = O(\lambda^{k_0+s}) \quad \text{as} \quad \lambda \to 0^+. \]

Combining this last information with (4.53) and (4.37), we finally obtain (4.38). \( \square \)

Using Lemma 4.7 we now prove that \( \lim_{r \to 0^+} r^{-2\gamma} H(r) = \lim_{r \to 0^+} r^{-2(k_0+s)} H(r) > 0 \).

**Lemma 4.8.** Let \( \gamma = \lim_{r \to 0^+} \mathcal{N}(r) \) be as in Proposition 3.6. Then

\[ \lim_{r \to 0^+} r^{-2\gamma} H(r) > 0. \]

**Proof.** By (2.24) and using the Parseval identity we have that

\[ H(\lambda) = \int_{S^N_+} \theta_{N+1}^{1-2s}|h(\lambda\theta)||W(\lambda\theta)|^2\,dS \]

\[ = (1 + O(\lambda)) \sum_{k=0}^{M_s} \sum_{m=1}^{M_s} |\varphi_{k,m}(\lambda)|^2. \]
Let \( k_0 \in \mathbb{N} \) be as in Proposition 4.5; thus \( \gamma = k_0 + s \). We argue by contradiction, assuming that
\[
\lim_{\lambda \to 0^+} \lambda^{-2\gamma} H(\lambda) = 0.
\]
Hence from (4.56) it follows that \( \lim_{\lambda \to 0^+} \lambda^{-(k_0+s)} \varphi_{k_0,m}(\lambda) = 0 \) for any \( m \in \{1, 2, \ldots, M_{k_0}\} \). This and Lemma 4.7 lead to
\[
(4.58) \quad R^{-k_0-s} \varphi_{k_0,m}(R) + \frac{(k_0 + s) R^{-N-2k_0}}{N + 2k_0} \int_0^R \rho^{k_0+s-1} \Upsilon_{k_0,m}(\rho) \, d\rho \\
+ \frac{N - s + k_0}{N + 2k_0} \int_0^R \rho^{-N-1+s-k_0} \Upsilon_{k_0,m}(\rho) \, d\rho = 0,
\]
for all \( m \in \{1, 2, \ldots, M_{k_0}\} \) and for every \( R \in (0, R_0) \). From (4.58), (4.38), and (4.52) it follows that
\[
\varphi_{k_0,m}(\lambda) = -\lambda^{k_0+s} \frac{N - s + k_0}{N + 2k_0} \int_0^\lambda \rho^{-N-1+s-k_0} \Upsilon_{k_0,m}(\rho) \, d\rho + O(\lambda^{k_0+s+\delta}) = O(\lambda^{k_0+s+\delta})
\]
as \( \lambda \to 0^+ \) for all \( m \in \{1, 2, \ldots, M_{k_0}\} \). Hence
\[
(4.59) \quad \sqrt{H(\lambda)} (w, \psi)_{L^2(\mathbb{S}_N^+, \theta_{1-2s}^{-1} dS)} = O(\lambda^{k_0+s+\delta}) \quad \text{as } \lambda \to 0^+
\]
for every \( \psi \in \text{span}\{Y_{k_0,m} : m = 1, \ldots, M_{k_0}\} \). From Lemma 4.7(ii), \( \sqrt{H(\lambda)} \geq \sqrt{k_2(\theta)} \lambda^{k_0+s+\delta} \) for \( \lambda \) small, so that (4.59) yields
\[
(4.60) \quad (w, \psi)_{L^2(\mathbb{S}_N^+, \theta_{1-2s}^{-1} dS)} = O(\lambda^{s/2}) \quad \text{as } \lambda \to 0^+
\]
for every \( \psi \in \text{span}\{Y_{k_0,m} : m = 1, \ldots, M_{k_0}\} \). On the other hand, by Proposition 4.3 and continuity of the trace map from \( H^1(B_1^+, t_1^{-2s} dz) \) into \( L^2(\partial^+ B_1^+, t_1^{-2s} dS) \), for any sequence \( \lambda_n \to 0^+ \), there exist a subsequence \( \{\lambda_n\} \) and \( \psi_0 \in \text{span}\{Y_{k_0,m} : m = 1, \ldots, M_{k_0}\} \) such that
\[
(4.61) \quad \|\psi_0\|_{L^2(\mathbb{S}_N^+, \theta_{1-2s}^{-1} dS)} = 1 \quad \text{and} \quad w^{\lambda_n} \to \psi_0 \quad \text{in} \quad L^2(\mathbb{S}_N^+, \theta_{1-2s}^{-1} dS)
\]
From (4.60) and (4.61) we deduce that
\[
0 = \lim_{k \to \infty} (w^{\lambda_k}, \psi_0)_{L^2(\mathbb{S}_N^+, \theta_{1-2s}^{-1} dS)} = \|\psi_0\|^2_{L^2(\mathbb{S}_N^+, \theta_{1-2s}^{-1} dS)} = 1,
\]
thus reaching a contradiction. 

**Theorem 4.9.** Let \( k_0 \in \mathbb{N} \) be as in Proposition 4.5; Let \( M_{k_0} \in \mathbb{N} \setminus \{0\} \) be the multiplicity of the eigenvalue \( \mu_{k_0} = (k_0 + s)(k_0 + N - s) \) and let \( \{Y_{k_0,m} : m = 1, \ldots, M_{k_0}\} \) be a \( L^2(\mathbb{S}_N^+, \theta_{1-2s}^{-1} dS) \)-orthonormal basis of the eigenspace of (1.11) associated to \( \mu_{k_0} \). Then, for every \( m \in \{1, 2, \ldots, M_{k_0}\} \), there exists \( \beta_m \in \mathbb{R} \) such that \( (\beta_1, \beta_2, \ldots, \beta_{M_{k_0}}) \neq (0, 0, \ldots, 0) \),
\[
W(\lambda) \lambda^{k_0+s} \sum_{m=1}^{M_{k_0}} \beta_m Y_{k_0,m}(z/|z|) \quad \text{in} \quad H^1(B_1^+, t_1^{-2s} dz) \quad \text{as} \quad \lambda \to 0^+,
\]
and
\[
(4.62) \quad \beta_m = R^{-(k_0+s)} \varphi_{k_0,m}(R) + \frac{(k_0 + s) R^{-N-2k_0}}{N + 2k_0} \int_0^R \rho^{k_0+s-1} \Upsilon_{k_0,m}(\rho) \, d\rho \\
+ \frac{N - s + k_0}{N + 2k_0} \int_0^R \rho^{-N-1+s-k_0} \Upsilon_{k_0,m}(\rho) \, d\rho \quad \text{for all} \quad R \in (0, R_0],
\]
with \( \varphi_{k_0,m} \) and \( \Upsilon_{k_0,m} \) given by (4.38) and (4.52) respectively.

**Proof.** If we consider any sequence of strictly positive real numbers \( \lambda_n \to 0^+ \), then from Proposition 4.5 and Lemmas 4.4 and 4.7 we deduce that there exist a subsequence \( \{\lambda_{n_k}\} \) and real numbers \( \beta_1, \beta_2, \ldots, \beta_{M_{k_0}} \) not all equal to 0 such that
\[
W(\lambda_{n_k} z) \lambda^{k_0+s} \sum_{m=1}^{M_{k_0}} \beta_m Y_{k_0,m}(z/|z|) \quad \text{in} \quad H^1(B_1^+, t_1^{-2s} dz) \quad \text{as} \quad k \to \infty.
\]
We claim that the coefficients $\beta_m$ depend neither on the sequence $\{\lambda_n\}$, nor on its subsequence $\{\lambda_{n_k}\}$.
To this aim, we observe that (4.38), (4.63), and the continuity of the trace map from $H^1(B_1^+, t^{-2s}dz)$ into $L^2(\partial^d B_1^+, t^{-2s}dz)$ imply that, for all $m \in \{1, 2, \ldots, M_{k_0}\}$,
\[
\lim_{k \to \infty} \lambda_n^{-(k_0+s)} \varphi_{k_0,m}(\lambda_n) = \lim_{k \to \infty} \int_{R^d} \theta_{N+1}^{-2s} \lambda_n^{-\gamma} W(\lambda_n \theta) Y_{k_0,m}(\theta) d\sigma
= \sum_{i=1}^{M_{k_0}} \beta_i \int_{R^d} \theta_{N+1}^{-2s} Y_{k_0,m}(\theta) d\sigma = \beta_m,
\]
for all $m \in \{1, 2, \ldots, M_{k_0}\}$. At the same time, after fixing $R < R_0$, by (4.38) we have that
\[
\lim_{k \to \infty} \lambda_n^{-(k_0+s)} \varphi_{k_0,m}(\lambda_n) = R^{-(k_0+s)} \varphi_{k_0,m}(R) + \frac{(k_0 + s) R^{-N-2k_0}}{N + 2k_0} \int_0^R \rho^{k_0+s-1} \Upsilon_{k_0,m}(\rho) d\rho
+ \frac{N - s + k_0}{N + 2k_0} \int_0^R \rho^{-N-1+s-k_0} \Upsilon_{k_0,m}(\rho) d\rho,
\]
hence, by uniqueness of the limit, we can deduce that, for all $m \in \{1, 2, \ldots, M_{k_0}\}$,
\[
\beta_m = R^{-(k_0+s)} \varphi_{k_0,m}(R) + \frac{(k_0 + s) R^{-N-2k_0}}{N + 2k_0} \int_0^R \rho^{k_0+s-1} \Upsilon_{k_0,m}(\rho) d\rho
+ \frac{N - s + k_0}{N + 2k_0} \int_0^R \rho^{-N-1+s-k_0} \Upsilon_{k_0,m}(\rho) d\rho.
\]
This is enough to conclude that the coefficients $\beta_m$ depend neither on the sequence $\{\lambda_n\}$, nor on its subsequence $\{\lambda_{n_k}\}$. Urysohn’s Subsequence Principle allows us to conclude that the convergence in (4.63) holds as $\lambda \to 0^+$, thus completing the proof. □

We are now in position to prove Theorem 1.3

Proof of Theorem 1.3 Up to a translation, we can assume that $x_0 = 0$. If $U$ is as in the assumptions of Theorem 1.3 then, letting $F$ as in Section 2.1, $W = U \circ F \in H^1_{1+} (B_{R_1}^+, t^{-2s}dz)$ is a nontrivial weak solution of (2.11). We notice that the nontriviality of $U$ in any neighbourhood of 0, and consequently of $W$ in $B_{R_1}^+$, can be easily deduced from nontriviality of $U$ in $\mathbb{R}^{N+1}_+$ and classical unique continuation principles for second order elliptic equations with Lipschitz coefficients [23].

Then, by Proposition 4.3 and Theorem 4.9, there exist $k_0 \in \mathbb{N}$ and an eigenfunction $Y$ of problem (1.11) associated to the eigenvalue $\mu_k = (k_0 + s)(k_0 + N - s)$ such that
\[
W(\lambda z) \to |z|^{k_0+s} Y(z/|z|) \quad \text{in } H^1(B_1^+, t^{-2s}dz) \quad \text{as } \lambda \to 0^+.
\]
We observe that
\[
U(\lambda z) = \frac{W(\lambda G_\lambda(z))}{\lambda^{k_0+s}}, \quad \nabla \left( \frac{U(\lambda z)}{\lambda^{k_0+s}} \right)(z) = \nabla \left( \frac{W(\lambda z)}{\lambda^{k_0+s}} \right)(G_\lambda(z)) J_{G_\lambda}(z),
\]
where
\[
G_\lambda(z) = \frac{1}{\lambda} F^{-1}(\lambda z).
\]
From (2.10) we have that
\[
G_\lambda(z) = z + O(\lambda) \quad \text{and} \quad J_{G_\lambda}(z) = \text{Id}_{N+1} + O(\lambda)
\]
as $\lambda \to 0^+$ uniformly with respect to $z \in B_1^+$. From (4.66) one can easily deduce that, if $f_\lambda \to f$ in $L^2(B_1^+, t^{-2s}dz)$, then $f_\lambda \circ G_\lambda \to f$ in $L^2(B_1^+, t^{-2s}dz)$. In view of (4.64) and (4.65), this yields the conclusion. □

As a direct consequence of Theorem 1.3 and of the equivalent formulation of problem (1.1) given in (1.9), we obtain Theorem 1.2

Proof of Theorem 1.2 If $u \in D^{s,2}(\mathbb{R}^N)$, $u \neq 0$, is a nontrivial weak solution to (1.1), then its extension $U = H(u) \in D^{1,2}(\mathbb{R}^{N+1}, t^{-2s}dz)$ weakly solves (1.3) in the weak sense specified in (1.10), see [6] and Section 4. Then the conclusion follows from Theorem 1.3 applied to $U$ and the continuity of the trace map from $H^s(B_1^+, t^{-2s}dz)$ into $H^s(B_1^+)$, see e.g. [23] Proposition 2.1.
Appendix A. Some boundary regularity results at edges of cylinders

Let us consider the following local problem: \( \Omega \subset \mathbb{R}^N \) is a \( C^{1,1} \) domain, \( x_0 \in \partial \Omega \), \( R, T > 0 \) and \( U \) is a weak solution to

\[
\begin{cases}
\text{div} \left( t^{1-2s} \nabla U \right) = 0 & \text{in } C_{R,T}(x_0), \\
U = 0 & \text{in } D_{R,T}(x_0), \\
\lim_{t \to 0} t^{1-2s} \partial_t U = 0 & \text{in } \sigma_{R,T}(x_0),
\end{cases}
\]

where we denoted

\( C_{R,T}(x_0) := (B'_R(x_0) \cap \Omega) \times (0, T), \quad D_{R,T}(x_0) := (B'_R(x_0) \cap \partial \Omega) \times (0, T), \quad \sigma_{R,T}(x_0) = (B'_R(x_0) \cap \Omega) \times \{0\}; \)

i.e. \( U \) belongs to the space \( \mathcal{H} \) defined as the closure of the set

\( \{ v \in C^0(\overline{C_{R,T}(x_0)}): v = 0 \text{ in a neighbourhood of } D_{R,T}(x_0) \} \)

in \( H^1(C_{R,T}(x_0), t^{1-2s} \, dz) \), and

\[
\int_{C_{R,T}(x_0)} t^{1-2s} \nabla U \cdot \nabla \Phi \, dz = 0 \quad \text{for all } \Phi \in C^\infty(C_{R,T}(x_0) \cup \sigma_{R,T}(x_0)).
\]

The following regularity result holds true.

Lemma A.1. Let \( \alpha \in (0, 1), \beta \in (0, 1) \cap (0, 2 - 2s), r < R, \) and \( \tau < T. \) Then there exists a positive constant \( C \) such that, for every weak solution \( U \) to (A.1),

\[
\|U\|_{C^{\alpha}(\overline{C_{r,T}(x_0)})} + \|t^{1-2s} \partial_t U\|_{C^{0,\beta}(\overline{C_{r,T}(x_0)})} \leq C \|U\|_{L^2(C_{R,T}(x_0), t^{1-2s} \, dz)}.
\]

Proof. Denoting the total variable \( z = (x, t) \in \mathbb{R}^N \times (0, +\infty) \), with \( x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} \), let us consider \( g \in C^{1,1}(\mathbb{R}^{N-1}) \) such that \( B'_R(x_0) \cap \Omega = \{ x = (x', x_N) \in B'_R(x_0) : x_N < g(x') \} \). Without loss of generality we can assume that \( x_0 = 0, g(0) = 0 \) and \( \nabla g(0) = 0 \). Starting from this function \( g \), we can argue as in Section 2.1 and construct a function \( F \) as in (2.5), which turns out to be a diffeomorphism in a neighbourhood of 0. Hence there exist positive constants \( \tau_0 < R \) and \( \tau_0 < T \) such that the composition \( W = U \circ F \) weakly solves the following strengthened problem

\[
\begin{cases}
\text{div} \left( t^{1-2s} A \nabla W \right) = 0 & \text{in } \Gamma_{\tau_0} \times (0, \tau_0), \\
W = 0 & \text{in } (B'_{\tau_0} \cap \{ y_N = 0 \}) \times (0, \tau_0), \\
\lim_{t \to 0} t^{1-2s} A \nabla W \cdot \nu = 0 & \text{in } \Gamma_{\tau_0},
\end{cases}
\]

with \( A = A(y) \) being as in (2.12); in particular the matrix \( A(y) \) does not depend on the vertical variable \( t \), is symmetric, uniformly elliptic, and possesses \( C^{0,1} \) coefficients.

Let us consider the odd reflection of \( W \) (which we still denote as \( W \)) through the hyperplane \( \{ y_N = 0 \} \) in \( B'_{\tau_0} \times (0, \tau_0) \), i.e. we set \( W(y', y_N, t) = -W(y', -y_N, t) \) for \( y_N < 0 \); it is easy to verify that \( W \) weakly satisfies

\[
\begin{cases}
\text{div} \left( t^{1-2s} \tilde{A} \nabla W \right) = 0 & \text{in } B'_{\tau_0} \times (0, \tau_0), \\
\lim_{t \to 0} t^{1-2s} \tilde{A} \nabla W \cdot \nu = 0 & \text{in } B'_{\tau_0},
\end{cases}
\]

where

\[
\tilde{A}(y) = \tilde{A}(y', y_N) := \begin{cases} A(y', y_N), & \text{if } y_N \leq 0, \\ SA(y', -y_N)S, & \text{if } y_N > 0,
\end{cases}
\]

with

\[
S := \begin{pmatrix} I_{N-1} & 0 & 0 \\ 0^T & -1 & 0 \\ 0^T & 0 & 1 \end{pmatrix},
\]

We observe that no discontinuities appear in the coefficients of the matrix \( \tilde{A} \) since, denoting as \( (a_{ij}) \) the entries of the matrix \( A, a_{i,N}(y', 0, t) = 0 \) for all \( i < N \) thanks to (2.13) and (2.19). Then the matrix \( \tilde{A} \) has Lipschitz continuous coefficients. Let us then consider the even reflection of \( W \) (which we still denote as \( W \)) through the hyperplane \( \{ t = 0 \} \) in \( B'_{\tau_0} \times (-\tau_0, \tau_0) \), i.e. we set \( W(y', y_N, t) = W(y', y_N, -t) \) for
If, for some $\gamma$, $t < \tau$, account the particular form of the matrix in (2.6).

From [36] Lemma 7.1 it follows that $V = |t| L^{2s}(B'_{t_0} \times (-\tau, \tau))$, $|t|^{2s-1} dz$ is a weak solution to

$$\text{div} \left( |t|^{1-2s} \hat{A} \nabla V \right) = 0 \quad \text{in} \ B'_{t_0} \times (-\tau, \tau),$$

which is odd with respect to $\{t = 0\}$, i.e. $V(y', y_N, -t) = -V(y', y_N, -t)$.

From [36] Theorem 1.2 it follows that, for all $r \in (0, \tau_0)$ and $\tau \in (0, \tau_0)$, $W \in C^{1, \alpha}(B_r \times (-\tau, \tau))$ and $\|W\|_{C^{1, \alpha}(B_r \times (-\tau, \tau))}$ \leq \text{const} \||W|\|_{L^2(B_{2r}(x_0 \times (\tau, \tau)))}$ for some $\text{const} > 0$ (independent of $W$). Furthermore, [12] ensures that $V$ is locally Hölder continuous. More precisely, [37] Proposition 2.10 yields that the function $\Phi(x, t) = \frac{V(x, t)}{t^\frac{1}{2-2s}}$, which is even in the variable $t$, belongs to the weighted Sobolev space $H^1_{L^2}(B'_{t_0} \times (-\tau_0, \tau_0), |t|^{2s-2} dz)$, and weakly solves

$$\text{div} \left( |t|^{3-2s} \hat{A} \nabla \Phi \right) = 0 \quad \text{in} \ B'_{t_0} \times (-\tau_0, \tau_0),$$

thanks to the fact that the matrix $\hat{A}$ is independent of $t$.

From [36] Theorem 1.2 we have that $\Phi \in C^{0, \gamma}(B_r' \times (-\tau, \tau))$ for all $\gamma \in (0, 1), r \in (0, r_0)$ and $\tau \in (0, \tau_0)$, and

$$\|\Phi\|_{C^{0, \gamma}(B_r' \times (-\tau, \tau))} \leq \text{const} \|V\|_{L^2(B_{2r}(x_0 \times (-\tau, \tau)))},$$

for some $\text{const} > 0$ (independent of $V$). Therefore $V \in C^{0, \delta}(B_r' \times (-\tau, \tau))$ with $\delta = \min\{2 - 2s, \gamma\}$ and $\|V\|_{C^{0, \delta}(B_r' \times (-\tau, \tau))}$ \leq \text{const} \||V|\|_{L^2(B_{2r}(x_0 \times (-\tau, \tau)))}$.

The conclusion follows by recalling that $U = W \circ F^{-1}$ with $F^{-1}$ being of class $C^{1, 1}$ and taking into account the particular form of the matrix in [20].

**APPENDIX B. HOMOGENEITY DEGREES AND EIGENVALUES OF THE SPHERICAL PROBLEM**

In this appendix, we derive an explicit formula for the eigenvalues of problem (1.11), which follows from a complete classification of possible homogeneity degrees of homogeneous weak solutions to the problem

$$\begin{align*}
-\text{div} \left( t^{1-2s} \nabla \Psi \right) &= 0 \quad \text{in} \ R^{N+1}_+, \\
\lim_{t \to 0^+} (t^{1-2s} \nabla \Psi \cdot \nu) &= 0 \quad \text{in} \ \Gamma^-, \\
\Psi &= 0 \quad \text{in} \ \Gamma^+,
\end{align*}$$

where $\Gamma^- := \{(y', y_N, 0) \in R^N \times \{0\} : y_N < 0\}$ and $\Gamma^+ := \{(y', y_N, 0) \in R^N \times \{0\} : y_N \geq 0\}$.

**Proposition B.1.** Let $\Psi \in \cap_{\gamma > 0} H^1_{L^2}(B^+_r, t^{1-2s})$ be a weak solution to (B.1), i.e.

$$\int_{R^{N+1}_+} t^{-1+2s} \nabla \Phi \cdot \nabla \Phi \, dz = 0, \quad \text{for all} \ \Phi \in C^\infty_c(\overline{R^{N+1}_+} \setminus \Gamma^+).$$

If, for some $\gamma \geq 0$, $\Psi(z) = |z|^{\gamma} \Psi\left(\frac{z}{|z|}\right)$, then there exists $j \in \N$ such that $\gamma = j + s$.

The proof of Proposition B.1 requires a polynomial Liouville type theorem for even solutions to degenerate equations with a weight which is possibly out of the $A_2$-Muckenhoupt class. To this aim, Lemma [32] below provides a generalization of Lemma 2.7 in [7]. For all $a \in (-1, +\infty)$ and $r > 0$, we define $H^1(B_r, |t|^a \, dz)$ as the completion of $C^\infty_c(B_r)$ with respect to the norm $\sqrt{\int_{B_r} |t|^a (|\Psi|^2 + |\nabla \Psi|^2) \, dz}$ and $H^1_{L^2}(\overline{R^{N+1}})$ as

$$H^1_{L^2}(\overline{R^{N+1}}) = \{ \Psi \in L^2_{L^2}(\overline{R^{N+1}}, |t|^a \, dz) : \Psi \in H^1(B_r, |t|^a \, dz) \quad \text{for all} \ r > 0 \}.$$

We also define

$$H^1_{L^2}(\overline{R^{N+1}}) = \{ \Psi \in L^2_{L^2}(\overline{R^{N+1}}, t^a \, dz) : \Psi \in H^1(B^+_r, t^a \, dz) \quad \text{for all} \ r > 0 \}.$$
Lemma B.2. Let $a \in (-1, +\infty)$ and $v \in H^{1,a}_{\text{loc}}(\mathbb{R}^{N+1})$ be a weak solution to
\begin{equation}
\div(|t|^a \nabla v) = 0 \quad \text{in} \quad \mathbb{R}^{N+1}
\end{equation}
which is even in $t$, i.e.
$$v(x, -t) = v(x, t) \quad \text{a.e. in} \quad \mathbb{R}^{N+1}.$$ 
If there exist $k \in \mathbb{N}$ and $c > 0$ such that
$$|v(z)| \leq c(1 + |z|^k)$$
for all $z \in \mathbb{R}^{N+1}$, then $v$ is a polynomial.

Proof. Let $a > -1$ and $v \in H^{1,a}_{\text{loc}}(\mathbb{R}^{N+1})$ be a weak solution to (B.2) even in $t$. For $\alpha \in (0, 1)$ and $k \in \mathbb{N}$, let $D^\beta v$ be a partial derivative in the variables $x = (x_1, \ldots, x_N)$ of order $k = |\beta|$, with $\beta \in \mathbb{N}^N$ multiindex. Then, there exists a positive constant $C$ depending only on $N, \alpha, a, k$ such that
\begin{equation}
\sup_{B_{1/2}} |D^\beta v| \leq \frac{C}{r^k} \sup_{B_r} |v|\end{equation}
and
\begin{equation}
[D^\beta v]_{C^{0,\alpha}(B_{1/2})} \leq \frac{C}{r^{k+\alpha}} \sup_{B_r} |v|, \end{equation}
where $[v]_{C^{0,\alpha}(A)} := \sup_{x, x' \in A} |x - x'|^{-\alpha} |v(x) - v(x')|$. In order to prove the previous inequalities, we apply some local regularity estimates for even solutions contained in [36]. If $k = 0$, then the inequalities follow by scaling
$$\|v\|_{C^{0,\alpha}(B_{1/2})} \leq C\|v\|_{L^\infty(B_1)}$$
proved in [36 Theorem 1.2 part i)]. If $k \geq 1$, we remark that any partial derivative in variables $x_i$ for $i \in \{1, \ldots, N\}$ commutes with the operator $\div(|t|^a \nabla \cdot)$ and $D^\beta v$ are actually even solutions to the same equation, (see [36 Section 7] for details). Hence, inequalities (B.3) and (B.4) follow by scaling and iterating the estimate
$$\|v\|_{C^{1,\alpha}(B_{1/2})} \leq C\|v\|_{L^\infty(B_1)}$$
proved in [36 Theorem 1.2 part ii)]. Indeed, fixed a multiindex $\beta_k$, we can choose
$$r_k = 1/2 < r_{k-1} < \ldots < r_0 = 1,$$
then
$$\|D^\beta v\|_{C^{0,\alpha}(B_{1/2})} \leq C_{k-1} \sup_{B_{r_{k-1}}} |D^{\beta_{k-1}} v| \leq C_{k-1} C_{k-2} \sup_{B_{r_{k-2}}} |D^{\beta_{k-2}} v| \leq \ldots \leq \left( \prod_{i=0}^{k-1} C_i \right) \sup_{B_1} |v|.$$
Once we have (B.3) and (B.4), we can proceed exactly as in proof of [7 Lemma 2.7]. We have only to remark that for any $a \in (-1, +\infty)$, given an even solution to (B.2), then $\partial^2_{tt} v + \frac{a}{2} \partial_t v = -\Delta_x v$ is also an even solution to (B.3).

Now we are able to prove Proposition B.1.

Proof of Proposition B.1. Let $\Psi \in H^{1,1-2s}_{\text{loc}}(\mathbb{R}^{N+1})$ be a weak solution to (B.1), such that
$$\Psi(z) = |z|^\gamma \Psi \left( \frac{z}{|z|} \right) \quad \text{in} \quad \mathbb{R}^{N+1},$$
for some $\gamma \geq 0$. The homogeneity condition trivially implies a polynomial global bound on the growth of $\Psi$. The same bound is inherited by the trace $\phi = \text{Tr} \Psi$ on $\mathbb{R}^N = \partial \mathbb{R}^{N+1}_+$, which is also $\gamma$-homogeneous. Moreover, $\phi \in C^{\infty}(\mathbb{R}^N)$ by [36 Theorem 1.1] and $\phi \in C^0(\mathbb{R}^N)$ by [27 Proposition 5.3]. With these premises, we can define the extension $V$ of $\phi$ in the sense of [11 Lemma 3.3]. Actually, we introduce a minor change in the definition of the extension given in [11]; that is, for every $R > 0$ we define
\begin{equation}
\phi_R = \phi \chi_{B^R_R}, \end{equation}
(instead of $\phi_R = \phi \chi_{B^R_R}$), where $\eta_R \in C^\infty_c(B^R_{2R})$ is a radially decreasing cut-off function with $|\eta_R| \leq 1$ and $\eta_R \equiv 1$ in $B^R_R$. We remark that the adjusted family of functions $\phi_R$ convoluted with the usual Poisson kernel of the upper half-space converge in a suitable way to the same extension $V$ obtained by
Abatangelo and Ros-Oton in [1]. Moreover, defining the extension starting from (1.5), we can easily ensure that \( V \in H^{1-2s}_{loc}(\mathbb{R}^{N+1}_+) \) and that it is weak solution to (1.1). Nevertheless, also \( V \) inherits from \( \phi \) an at most polynomial growth. Let us consider \( W = V - \Psi \in H^{1-2s}_{loc}(\mathbb{R}^{N+1}_+) \), which weakly solves
\[
\begin{cases}
\text{div}(t^{1-2s}\nabla W) = 0 & \text{in } \mathbb{R}^{N+1}_+,
\text{Tr } W = 0 & \text{on } \mathbb{R}^N = \partial \mathbb{R}^{N+1}_+.
\end{cases}
\]
Then, denoting as \( \bar{W} \) the odd reflection of \( W \) through \( \partial \mathbb{R}^{N+1}_+ \), by [37, Proposition 2.10]
\[
\bar{W} = \frac{W}{\lfloor t \rfloor^{1+2s}} \in H^{1,1+2s}_{loc}(\mathbb{R}^{N+1})
\]
is an even entire weak solution to (B.2) with \( \alpha = 1 + 2s \in (1, 3) \). We have that \( \bar{W} \) satisfies the assumptions of Lemma [32] being a polynomial bound on its growth ensured by the polynomial bounds of \( \Psi \) and \( V \). From Lemma [32] we can promptly conclude that \( \bar{W} \) is a polynomial. We also have that
\[
t^{1-2s}\partial_t V = t^{1-2s}\partial_t \Psi + t^{1-2s}\partial_t(t^{2s}W) = t^{1-2s}\partial_t \Psi + P_k
\]
for some polynomial \( P_k \) of degree \( k \in \mathbb{N} \). Hence, passing to the trace of the weighted derivative above, by [1] Lemma 3.3 it follows that
\[
(-\Delta)^s \phi \stackrel{k+1}{=} 0 \quad \text{in } \Gamma^-
\]
and \( \phi = 0 \) in \( \Gamma^+ \), where the above identity is meant in the sense of the notion of “fractional Laplacian modulus polynomials of degree at most \( k^+ \)” given in [1] Definition 3.1, see also [11]. Hence, by [1] Theorem 3.10, we have that
\[
\phi(x) = p(x)(x_N)_-^s,
\]
for some polynomial \( p \). By homogeneity of \( \phi \), this implies that necessary there exists \( j \in \mathbb{N} \) such that \( \gamma = j + s \).

We are now going to derive from Proposition [31] the explicit formula [13] for the eigenvalues of problem (1.11). We first observe that, if \( \mu \) is an eigenvalue of (1.11) with an associated eigenfunction \( \psi \), then the function \( \Psi(\rho \theta) = \rho^\sigma \psi(\theta) \) with \( \sigma = \frac{-N-2s}{2} + \sqrt{(\frac{N-2s}{2})^2 + \mu} \) belongs to \( H^{1-2s}_{loc}(\mathbb{R}^{N+1}_+) \) and is a weak solution to (1.1). From Proposition [31] we then deduce that there exists \( j \in \mathbb{N} \) such that \( \sigma = j + s \) and hence
\[
\mu = (j + s)(j + N - s).
\]
Viceversa, we prove now that all numbers of the form \( \mu = (j + s)(j + N - s) \) with \( j \in \mathbb{N} \) are eigenvalues of (1.11). For any fixed \( j \in \mathbb{N} \), we consider the function \( \Psi \) defined, in cylindrical coordinates, as
\[
\Psi(x', r \cos \tau, r \sin \tau) = r^{s+j} \left| \sin \left( \frac{\tau}{2} \right) \right|^{2s} 2F_1 \left( -j, j + 1; 1 - s; \frac{1 + \cos \tau}{2} \right), \quad r > 0, \quad \tau \in [0, 2\pi],
\]
where \( 2F_1 \) is the hypergeometric function. From [29] we have that \( \Psi \in H^{1,1-2s}_{loc}(\mathbb{R}^{N+1}_+) \) is a weak solution to (B.1). Furthermore \( \Psi \) is homogeneous of degree \( s+j \) and therefore the function \( \psi := \Psi|_{S^N_+} \) belongs to \( \mathcal{H}_0 \), \( \psi \not\equiv 0 \), and
\[
\Psi(\rho \theta) = \rho^{s+j} \psi(\theta), \quad \rho \geq 0, \quad \theta \in S^N_+.
\]
Plugging the above characterization of \( \Psi \) into (B.1), we obtain that
\[
\rho^j \left( j + s \right) \left( j + N - s \right) \theta^{1-2s} \psi(\theta) + \text{div}_{S^N} \left( \theta^{1-2s} \nabla_{S^N} \psi \right) = 0, \quad \rho > 0, \quad \theta \in S^N_+,
\]
so that \( (j + s)(j + N - s) \) is an eigenvalue of (1.11).

We then conclude that the set of all eigenvalues of problem (1.11) is \( \{(j + s)(j + N - s) : j \in \mathbb{N}\} \).

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(A. De Luca, V. Felli) Dipartimento di Matematica e Applicazioni
Università degli Studi di Milano - Bicocca
Via Cozzi 55, 20125, Milano, Italy
Email address: a.deluca41@campus.unimib.it, veronica.felli@unimib.it

(S. Vita) Dipartimento di Matematica “Giuseppe Peano”
Università degli Studi di Torino
Via Carlo Alberto 10, 10123, Torino, Italy
Email address: stefano.vita@unito.it