Scalar-Scalar Ladder Model in the Unequal-Mass Case. II
—Numerical Studies of the BS Amplitudes —

Ichio FUKUI and Noriaki SETÔ†

Information Processing Center, Saga University
Saga 840

† Department of Applied Mathematics, Faculty of Engineering
Hiroshima University, Higashi-Hiroshima 739

Abstract

The Bethe-Salpeter amplitudes of the bound states formed by two scalar particles with unequal masses are analyzed in the massive scalar particle exchange ladder model. The norms of the amplitudes are calculated numerically, and it is confirmed that the norm vanishes for the bound state corresponding to the complex eigenvalue of the coupling constant. The behaviour of the Bethe-Salpeter amplitudes in the momentum space is also investigated.
§1. Introduction

It is expected that the homogeneous Bethe-Salpeter (BS) equation describes the bound state in quantum field theory. The BS kernel appearing in the equation is usually approximated by the contribution from the ladder graph only, and this is referred to as the ladder model. In this model, the BS equation can be regarded as an eigenvalue problem for the coupling constant at a given mass of the bound state.

For the scalar-scalar ladder model with scalar particle exchange, if the mass of the exchanged particle is nonzero, and if the two scalar particles have unequal masses, the corresponding eigenvalue equation cannot be reduced to a real form, when the invariant bound-state mass squared lies in the physical region, that is, between zero and the two-particle threshold. In this case, it becomes a nontrivial problem whether the eigenvalues of the coupling constant are real or complex.

Naito and Nakanishi made an argument that all eigenvalues would be real provided that eigenvalues had a certain analytic property. In the same year, zur Linden published a paper suggesting the reality of eigenvalues by numerical calculation, and Kaufmann published a paper suggesting, on the contrary, that eigenvalues could become complex for some mass configurations of three scalar particles and the bound state. Ida pointed out, in the next year, that the analyticity of eigenvalues assumed in Ref. 2 is never satisfied if the eigenvalue becomes non-real. Based on a rapid progress in computing technique made some twenty years in between, a rather extensive numerical calculation was performed to resolve these entangled situation. It was found that complex eigenvalues appeared in the first and second excited states for the bound state mass squared around the pseudothreshold, and it was also argued that the appearance could not be regarded as an artifact of the numerical scheme employed there.

As a continuation of Ref. 6, which exclusively dealt with the behaviour of eigenvalues, this paper concerns with the analysis of BS amplitudes (eigenvectors) of the eigenvalue equation. In the next section, the general formalism for the BS equation is presented. In §3, the norm of the BS amplitude is calculated numerically. The norm of the BS amplitude corresponding to a complex eigenvalue is found, to within the numerical accuracy, to vanish. In §4, the BS amplitudes of some bound states are depicted in the momentum space. Discussions and a further outlook are made in the final section.

§2. BS equation and BS amplitude

The BS(Bethe-Salpeter or bound state) amplitude \( \phi(p, p_4) \) with mass squared \( s (> 0) \), formed by two scalar particles with masses \( 1 + \Delta \) and \( 1 - \Delta \) by exchanging a scalar particle with mass \( \mu \) obeys, in the ladder model, the following BS equation (after the
Wick rotation):
\[
\phi(p, p_4) = \frac{1}{[(1 + \Delta)^2(1 - \sigma) + p^2 - 2i(1 + \Delta)\sqrt{\sigma}p_4]} \times \frac{1}{[(1 - \Delta)^2(1 - \sigma) + p^2 + 2i(1 - \Delta)\sqrt{\sigma}p_4]} \cdot \frac{\lambda}{\pi^2} \int d^4p' \frac{\phi(p', p_4')}{\mu^2 + (p - p')^2},
\]
(2.1)

with \(\sigma = s/4\) and \(0 \leq \Delta < 1\). We shall denote, hereafter, the model specified above as the \([1 + \Delta \leftrightarrow \mu \Rightarrow 1 - \Delta]\) model. We regard Eq. (2.1) as an eigenvalue equation for the coupling strength \(\lambda\), which has mass dimension +2. In this model, the two-particle threshold is equal to 4, while the pseudothreshold is \(4\Delta^2\).

By introducing the polar coordinates in the four-dimensional momentum space \((p, p_4)\),
\[
p_4 = p \cos \beta, \quad p = p \sin \beta (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta),
\]
(2.2)
we can reduce Eq. (2.1) to an infinite system of one-dimensional integral equations. Since the three-dimensional angular momentum \(l\) is a good quantum number, we can assume that the BS amplitude has the form
\[
\phi(p, p_4) = \sum_{L=l}^{\infty} N_{L,l} \cdot (\sin \beta)^l C^{l+1}_{L-l}(\cos \beta)Y_{lm}(\theta, \varphi) \phi_{L,l}(p)
\]
(2.3)
for suitable \(l (= 0, 1, 2, \ldots)\) and \(m (-l \leq m \leq l)\). The four-dimensional spherical harmonics are \((\sin \beta)^l C^{l+1}_{L-l}(\cos \beta)Y_{lm}(\theta, \varphi)\), with associated normalization constants \(N_{L,l}\).

By changing the variable \(p\) (the magnitude of the four-dimensional momentum) to \(z\) defined by
\[
p = \sqrt{\frac{1 + z}{1 - z}}, \quad z = \frac{p^2 - 1}{p^2 + 1}, \quad (z : -1 \to 1 \text{ as } p : 0 \to \infty)
\]
(2.4)
Eq. (2.1) is transformed into
\[
g_{L,l}(z) = \lambda \sum_{L'=l}^{\infty} \int_{-1}^{1} dz' K_{L,L'}(z, z') g_{L',l}(z').
\]
(2.5)
The BS amplitude \(\phi(p, p_4)\) is expressed in terms of the \(g_{L,l}(z)\) 's as
\[
\phi(p, p_4) = \sum_{L=l}^{\infty} p^L \left(\frac{2}{1 + p^2}\right)^{L+3} i^{L-l} \left(\frac{|p|}{p}\right)^l C^{l+1}_{L-l} \left(\frac{p_4}{p}\right) g_{L,l} \left(\frac{p^2 - 1}{p^2 + 1}\right) Y_{lm}(\theta, \varphi).
\]
(2.6)
The derivation of the above equations and the explicit form of the integral kernel
matrix functions $K_{LL',L}(z, z')$ for $l = 0$ and $l = 1$ are given in Ref. 6. These matrix functions are all real, and if the eigenvalue equation (2.5) admit a real eigenvalue $\lambda$ we can take the eigenvectors $g_{LL}(z)$ ($L = l, l + 1, \ldots$) all real. Even in this case, however, since imaginary unit $i$ appears on the right hand side of Eq. (2.6), the BS amplitude $\phi(p, p_4)$ takes a complex value in general.

§ 3. Norm calculation

In Ref. 6, we have analyzed numerically the eigenvalue problem (2.5) for the $s$-wave ($l = 0$) case. We have approximated the infinite system of integral equations by a finite system of algebraic equations: The summation over $L'$ from 0 to $\infty$ in Eq. (2.5) is truncated at some cutoff value $L_c$, and the integration over $z'$ is replaced by the $N$-point Gauss-Legendre quadrature formula. By this procedure, we can reduce Eq. (2.5) to the $(L_c + 1) \cdot N$-th order matrix eigenvalue problem:

$$
g_L(z_j) = \lambda \sum_{L'=0}^{L_c} \sum_{k=1}^{N} K_{LL',0}(z_j, z_k) w_k g_{L'}(z_k),$$

$$\quad (L = 0, 1, \ldots, L_c, \quad j = 1, 2, \ldots, N) \quad (3.1)$$

where the points $z_1, \ldots, z_N$ are the Gauss-Legendre points on the interval $[-1,1]$, and $w_1, \ldots, w_N$ are the corresponding integration weights. We have put $g_L(z_j) := g_{L,0}(z_j)$.

We have found that for the $[1.6 \leftrightarrow 1.0 \Rightarrow 0.4]$ model (that is, for the case $\Delta = 0.6$ and $\mu = 1.0$ ), which is the same as the case zur Linden and Kaufmann treated, complex eigenvalues appear in the range $0.25 < s < 2.65$, where $s$ is the bound-state mass squared. Main calculations were performed for $N = 45$ and $L_c = 7$. We have, in this paper, recalculate the first three eigenvalues (denoted as $\lambda_0, \lambda_1$ and $\lambda_2$) for the case $s = 0$(the left edge of the threshold), $s = 0.23$(near before the start of complex eigenvalues), $s = 0.27$(near after the start of complex eigenvalues), $s = 1.44$( the pseudothreshold), $s = 2.62$(near before the end of complex eigenvalues), $s = 2.67$(near after the end of complex eigenvalues), $s = 3.90$(near the nonrelativistic limit), taking $N = 47$ and $L_c = 9$. The result is summarized in Table I.

| $s$  | 0.00 | 0.23 | 0.27 | 1.44 | 2.62 | 2.67 | 3.90 |
|------|------|------|------|------|------|------|------|
| $\lambda_0$ | 3.448 | 3.342 | 3.323 | 2.755 | 2.113 | 2.083 | 1.109 |
| $\lambda_1$ | 16.433 | 16.459 | 16.526 - 0.127i | 14.887 - 0.734i | 13.059 - 0.159i | 12.794 | 8.285 |
| $\lambda_2$ | 17.345 | 16.702 | 16.526 + 0.127i | 14.887 + 0.734i | 13.059 + 0.159i | 13.157 | 12.455 |
Before calculating the norm of the BS amplitude, we shall check to what accuracy the eigenvalue equation is satisfied. Denoting Eq. (3·1) symbolically by \( g = \lambda K g \), where \( K \) is \( 470 \times 470 \) matrix, we calculate the relative error \( \| g - \lambda K g \| / \| g \| \). As the vector norm \( \| v \| \), we consider two cases; the \( l_2 \)-norm (\( \| v \| := (\sum_i |v_i|^2)^{1/2} \)) and \( l_\infty \)-norm (\( \| v \| := \max_i |v_i| \)). Numerical values of the relative errors are almost same for the \( l_2 \)-norm and \( l_\infty \)-norm. The relative errors corresponding to the ground states (\( \lambda_0 \)) in Table I are all less than \( 10^{-6} \). Rather large errors appear in the first excited state (corresponding to \( \lambda_1 \)) at \( s = 0 \) and in the first and second excited states (corresponding to \( \lambda_1 \) and \( \lambda_2 \)) at \( s = 0.23 \). The errors are, however, of order \( 10^{-4} \). For other states, including those having complex eigenvalues, the relative errors are less than \( 10^{-5} \).

As was explained in Ref. 6), our numerical method is based on the iteration-reduction scheme.\(^7\) The relative error defined above is not referred to the reduced eigenvalue problem (for the excited state), but to the original eigenvalue problem: We must restore the eigenvector from the reduced eigenvector according to a certain prescription.\(^7\) Since the eigenvalues are obtained to within 0.1% relative error, it can be said that our scheme can reproduce the eigenvector to the full accuracy as we can expect.

Having thus confirmed that the eigenfunction \( g_L(z_j) \) satisfies the eigenvalue equation (3·1) accurately, we will calculate the norm of the BS amplitudes corresponding to the states listed in Table I. By the name ”norm of the BS amplitude”, we mean a quantity defined by

\[
\int d^3p \int p_4 \phi^*(p, -p_4) \left[ (1 + \Delta)^2(1 - \sigma) + p^2 - 2i(1 + \Delta)\sqrt{\sigma}p_4 \right] \\
\times \left[ (1 - \Delta)^2(1 - \sigma) + p^2 + 2i(1 - \Delta)\sqrt{\sigma}p_4 \right] \phi(p, -p_4). \tag{3·2}
\]

This is the same as considered by Naito-Nakanishi (the left hand side of Eq. (2·8) in Ref. 2) and by Ida (Eq. (5·37) in Ref. 3)). This quantity can be shown to take a real value. By making use of the expression for the BS amplitude (\( l = 0 \) case of Eq. (2·6)), and using the recursion relation of the Gegenbauer polynomial \( C^L_1 \), we can perform the (four-dimensional) angular part of the integration in Eq. (3·2), to the result

\[
\int_0^1 dz \sum_{L=0}^{\infty} \left\{ (2 - \delta_{L,0})(1 - \Delta)^2\sigma(1 - z^2) + [1 + z + (1 - \Delta)^2(1 - \sigma)(1 - z)] \\
\times [1 + z + (1 + \Delta)^2(1 - \sigma)(1 - z)] \right\} (-1)^L(1 - z^2)^{L+1} \left[ (\text{Re} g_L(z))^2 + (\text{Im} g_L(z))^2 \right] \\
+ \int_0^1 dz \sum_{L=0}^{\infty} 4\Delta \sqrt{\sigma} [1 + z - (1 - \sigma)(1 - \Delta)^2(1 - z)] (-1)^L(1 - z^2)^{L+2} \\
\times [\text{Re} g_L(z)\text{Re} g_{L+1}(z) + \text{Im} g_L(z)\text{Im} g_{L+1}(z)]
\]
\begin{align*}
+ \int_{-1}^{1} dz \sum_{L=0}^{\infty} 2(1 - \Delta^2)\sigma(-1)^{L+1}(1 - z^2)^{L+3} \\
\times [\text{Re} \ g_L(z)\text{Re} \ g_{L+2}(z) + \text{Im} \ g_L(z)\text{Im} \ g_{L+2}(z)] .
\end{align*}

(3.3)

In the above expression, \( \text{Re} \ g_L(z) \) (resp. \( \text{Im} \ g_L(z) \)) means the real (resp. imaginary) part of \( g_L(z) \). If the corresponding eigenvalue is real, it is natural to take \( g_L(z) \) as a real function. In this case \( \text{Im} \ g_L(z) = 0 \) and it can be shown that the equality \( \phi^*(\mathbf{p}, -p_4) = \phi(\mathbf{p}, p_4) \) holds, that is, the real (imaginary) part of the BS amplitude \( \phi(\mathbf{p}, p_4) \) is an even (odd) function of \( p_4 \). If the eigenvalue \( \lambda \) is complex, \( g_L(z) \)'s acquire imaginary parts, and \( \phi(\mathbf{p}, p_4) \) and \( \phi^*(\mathbf{p}, -p_4) \) become linearly independent, and these two BS amplitudes have mutually complex conjugate eigenvalues \( \lambda \) and \( \lambda^* \).

The approximate value of the norm can be obtained by putting \( g_L(z) = 0 \) for \( L \) larger than the cutoff value \( L_c (= 9) \), and by performing the integration according to the \( N(= 47) \)-point Gauss-Legendre formula. Since the norm in Eq. (3.3) depends on the normalization of \( g_L(z) \)'s, that is, an overall multiplication by a constant, it is desirable to remove this arbitrariness. We divide the norm by a manifestly positive quantity obtained by replacing each summand in the integrands in Eq. (3.3) by its absolute value. The result is

Table II. Norms of the BS amplitude in \([1.6 \Leftarrow 1.0 \Rightarrow 0.4]\) model at several \( s \)'s.

| \( s \) | 0.00 | 0.23 | 0.27 | 1.44 | 2.62 | 2.67 | 3.90 |
|---|---|---|---|---|---|---|---|
| \( n_0 \) | 1.000 | 0.992 | 0.990 | 0.921 | 0.761 | 0.750 | 0.189 |
| \( n_1 \) | 0.999 | 0.216 | \(-0.186 \cdot 10^{-5}\) | 0.283 \cdot 10^{-5} | \(-0.324 \cdot 10^{-5}\) | 0.735 \cdot 10^{-1} | 0.238 |
| \( n_2 \) | \(-1.000\) | \(-0.216\) | \(-0.186 \cdot 10^{-5}\) | 0.283 \cdot 10^{-5} | \(-0.324 \cdot 10^{-5}\) | \(-0.729 \cdot 10^{-1}\) | \(-0.280\) |

In the numerical analysis of an eigenvalue problem, it is a general feature that the accuracy of eigenvectors is lower than that of eigenvalues. Before starting our norm calculation, we expect, therefore, that the numerical value of the norm corresponding to the zero-norm state will be at best of order \( 10^{-3} \). We can thus safely say that the norms of the BS amplitudes corresponding to complex eigenvalues vanish. Theoretically it is known that if the norm of a BS amplitude does not vanish, this BS amplitude has a real eigenvalue. Although the converse statement is not true, we regard that our result amounts almost to an analytical proof that in the \([1 + \Delta \Leftarrow \mu \Rightarrow 1 - \Delta]\) model there can appear complex eigenvalues. We have applied our calculation procedure to the \([1.0 \Leftarrow 1.0 \Rightarrow 1.0]\) model (equal-mass scalar-scalar model) and the \([1.6 \Leftarrow 0 \Rightarrow 0.4]\) model (unequal-mass Wick-Cutkosky model). The norms of the first three eigenstates at the same energies as in Table I are found to be all nonzero. The minimum (in absolute value)
is $\sim 1/30$, which corresponds to the (degenerate) first and second excited state at the pseudothreshold in the $[1.6 \leftarrow 0 \Rightarrow 0.4]$ model. This value can be regarded well away from zero.

§4. BS amplitude in momentum space

We have verified in the preceding section that the approximate $s$-wave eigenvector $g^{(k)}_L(z; s)$, corresponding to the $k$-th eigenvalue $\lambda = \lambda_k$ ($k = 0, 1, 2$) at the bound-state mass squared $s$, satisfies the eigenvalue equation (3·1) with a relative error less than 0.001% except for a few cases. We can thus expect that the corresponding approximate BS amplitude $\phi_k(|p|, p_4; s)$ for $k = 0, 1, 2$,

$$
\phi_k(|p|, p_4; s) = \sum_{L=0}^{9} p^L \left( \frac{2}{1+p^2} \right)^{L+3} i^L C^1_L \left( \frac{p_4}{p} \right) g^{(k)}_L \left( \frac{p^2 - 1}{p^2 + 1}; s \right) \frac{1}{2\sqrt{\pi}} \quad (4·1)
$$

reflects the behaviour of the original BS amplitude in the Wick-rotated four-dimensional momentum space rather accurately.

As a typical example of the real eigenvalue case, Figs. 1 (a) and (b) shows, respectively, the real and imaginary part of the BS amplitude of the first excited state ($k = 1$) at $s = 0.23$. The graphs are plotted on the $p_4$-$|p|$ half plane, where $|p|$ denotes the magnitude of the three-dimensional space part of the four-dimensional momentum. The amplitude is normalized so that the largest value of the magnitude of its real part becomes 1.00.

The expression (4·1) takes a form suitable for plotting in the polar coordinates: From the numerical calculation, we get the data $g^{(k)}_L(z_j; s)$ for $L = 0 \sim 9$ and $j = -23 \sim 23$. The corresponding magnitude of the four-dimensional angular momentum $p = p_j := \sqrt{(1+z_j)/(1-z_j)}$ (cf. Eq. (2·4)) takes the values $p_{-23} = 0.02532$ ($z_{-23} = -0.9998$), $p_{-22} = 0.05817$ ($z_{-22} = -0.9933$), $p_0 = 1.0000$ ($z_0 = 0.0000$), $p_{22} = 17.19$ ($z_{22} = 0.9933$), $p_{23} = 39.50$ ($z_{23} = 0.9987$). In spite of this, we have plotted the graph in the Cartesian coordinates on the $p_4$-$|p|$ plane, relying on the third-order interpolation-extrapolation method, in order to make use of several useful graphic software tools. The graph is cut if $p_4$ or $|p|$ exceeds 1.00, since the behaviour of the amplitude is quite smooth in the region omitted. The curves in the $p_4$-$|p|$ plane are contour lines of the surface, and we can read off from these curves that the real (imaginary) part of the BS amplitude is indeed an even (odd) function of $p_4$, as was explained in §3.

The real and imaginary part of the BS amplitude corresponding the the complex
eigenvalue $\lambda_1 = 14.8876 - 0.734i$ at $s = 1.44$ (pseudothreshold) are depicted in Figs. 2 (a) and (b), respectively.

In this case, the symmetry (antisymmetry) of the real (imaginary) part of the amplitude under the $p_4$ inversion is broken. It seems, however, impossible to judge from Fig. 1 and Fig. 2 that the BS norm of the former is $\sim 0.216$ while that of the latter is $\sim 0.0$.

As the last example, the absolute value of the ground state at $s = 3.9$ is plotted in Fig. 3.

The support of the amplitude is concentrated near the origin, especially in the $p_4$ variable, as it should be in the nonrelativistic limit.

§ 5. Summary and discussion

In this paper, we have analyzed numerically, as a continuation of our previous paper, in which the behaviour of eigenvalues of the $[1.6 \leftarrow 1.0 \Rightarrow 0.4]$ model in the $s$-wave case was clarified, the properties of the BS amplitude (eigenvector). We have verified that the BS amplitude satisfies the truncated eigenvalue equation (3-1) rather accurately, especially well for the amplitude of the ground state and of the state having a complex eigenvalue.

Based on this observation, we have calculated numerically the BS norm of the amplitude (Eq. (3-2)), and found out that the norm vanishes, to within our numerical accuracy, for the eigenvector corresponding to the complex eigenvalue. We can thus say with certainty that the $[1 + \Delta \leftarrow \mu \Rightarrow 1 - \Delta]$ model admits complex eigenvalues for some configuration of $\Delta$, $\mu$ and the bound-state mass. We have also depicted the BS amplitude in the momentum space for some typical cases.

A preliminary calculation suggests that the eigenvalues are all real for the $p$-wave case of the $[1.6 \leftarrow 1.0 \Rightarrow 0.4]$ model. If so, it will be of some interest to see whether this is a universal feature valid for other values of $\Delta$ and $\mu$, or not. The appearance of complex eigenvalues will be closely connected with the phenomenon of anomalous threshold in the triangle Feynman graph. These points will be discussed in forthcoming papers.
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Figure 1: The BS amplitude of the first excited state (corresponding to the real eigenvalue) at $s = 0.23$, (a) real part and (b) imaginary part.
Figure 2: The BS amplitude of the first excited state (corresponding to the complex eigenvalue) at $s = 1.44$, (a) real part and (b) imaginary part.
Figure 3: The absolute value of the BS amplitude of the ground state at $s = 3.9$. 