**A cosmological model from emergence of space**

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Abstract

Many studies have been carried out since T.Padmanabhan proposed that the cosmic acceleration can be understood from the perspective that spacetime dynamics is an emergent phenomenon. Motivated by such a new paradigm, we firstly study the de Sitter universe from emergence of space. After that we generalize our investigation to a model of cosmology, which possibly describes our universe classically. Furthermore, a constraint on $Ht$ and a estimated value of $\bar{\Omega}_\Lambda$ (caused by $\rho_{\text{vac}}$) can be derived from our model, the comparison with experiments is also presented. The results show the validity of our model.

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I. INTRODUCTION

The discovery of black hole thermodynamics [1, 2] has helped us to know the nature of gravity. With the deep study of the connection between gravitation and thermodynamics, physicists generally believe that the space is emergent which means gravity may not be a fundamental interaction but an emergent phenomenon now.

It was first shown by Jacobson [3] that the Einstein field equations can be derived from the Clausius relation on a local Rindler causal horizon. Verlinde [4], by suggesting that gravity should be explained as an entropic force caused by changes of entropy associated with the information on the holographic screen, put forward a great step towards understanding the nature of gravity. With the holographic principle and the equipartition law of energy, Verlinde derived the Newton’s law and the Einstein fields equations in a relativistic regime. Earlier, Padmanabhan [5] observed that the equipartition law of energy for horizon degrees of freedom (DOF), combined with the thermodynamics relation \( S = \frac{E}{2T} \), leads to Newton’s law of gravity.

In most cases, only the gravitational field is treated as an emergent phenomenon, with the pre-existing background geometric manifold assumed. A more complete way is to treat spacetime itself as an emergent structure as well, and it was finally proposed by Padmanabhan [6, 7]. He argued that the spatial expansion of our universe is due to the difference between the surface DOF and the bulk DOF in the region of emerged space. Then, he proposed a simple equation \( \frac{dV}{dt} = L_p^2 \Delta N \) [6, 7], where \( V \) is the Hubble volume and \( t \) is the cosmic time. \( \Delta N = N_{\text{sur}} - N_{\text{bulk}} \) with \( N_{\text{sur}} \) being the number of DOF on the boundary and \( N_{\text{bulk}} \) being the number in the bulk. Cai [8] generalized the derivation process to the higher (n+1)-dimensional spacetime. He also obtained the Friedmann equations of a flat FRW universe in Gauss-Bonnet and more general Lovelock cosmology by properly modifying the effective volume and the number of DOF on the holographic surface from the entropy formulae of static
spherically symmetric black holes [8]. In Ref [9], on the other hand, the authors
 generalized the holographic equipartition and derived the Friedmann equations by
 assuming that \((dV/dt)\) is proportional to a general function \(f(\Delta N, N_{\text{sur}})\). Note that
 the authors of [8, 9] only derived the Friedmann equations of the spatially flat FRW
 universe. In Ref [10], Sheykhi derived the Friedmann equations of the FRW universe
 with any spatial curvature. The authors of [11] proposed a general equation which can
 be reduced to the different modified ones in different cases. For more investigations
 about the novel idea see Refs. [10–13].

In this paper, we use the equation proposed by Padmanabhan to find some characters
 of the de Sitter universe at first, and then we generalize the important character
 \(\rho + 3p = \text{constant} \) to general case. By solving the equation \(dV/dt = L_p^2 \Delta N\), we
 can get the solution of \(H(t)\), hence \(a(t)\). Therefore we build a cosmological model
 to study the evolution of \(a(t)\) in detail. It is easy to find that our universe would
 be de Sitter universe far into the future and a constraint on \(H\) and \(t\) is obtained in
 our model. Finally, we give a estimated value of \(\Omega_\Lambda\) (caused by \(\rho_{\text{vac}}\)) and compare
 them with the experiments data. This paper is organized as follows: In Section II,
a brief review of Padmanabhan’s work is presented firstly. In section III, we then
discuss a de Sitter universe from emergence of space. In Section IV we present our
cosmological model in details and the comparison of our model with experiments.
Section V is for conclusions and discussions.

II. EMERGENCE OF SPACE

Padmanabhan [6] noticed that in a pure de Sitter universe with Hubble constant
\(H\), the holographic principle can be expressed in terms of

\[
N_{\text{sur}} = N_{\text{bulk}},
\]

(1)
where $N_{\text{sur}}$ denotes the number of DOF on the spherical surface of Hubble radius $H^{-1}$, namely $N_{\text{sur}} = 4\pi H^{-2}/L_p^2$, with $L_p$ being the Planck length, while the bulk DOF $N_{\text{bulk}} = |E|/(1/2) T$. Here $|E| = |ho + 3p| V$, is the Komar energy with the Hubble volume $V = 4\pi/(3H^3)$ and the horizon temperature $T = H/2\pi$. For the pure de Sitter universe, substituting $\rho = -p$ into Eq. (1), the standard result $H^2 = 8\pi L_p^2 \rho / 3$ is obtained.

From Eq. (1), one can get $|E| = (1/2) N_{\text{sur}} T$, which is the standard equipartition law. Padmanabhan called it holographic equipartition, because it relates the effective DOF residing in the bulk to the DOF on the boundary surface. It is known that our real universe is just asymptotically de Sitter. Padmanabhan further suggested that the emergence of space occurs and relates to the difference $\Delta N = N_{\text{sur}} - N_{\text{bulk}}$. A simple equation was proposed [6]

$$\frac{dV}{dt} = L_p^2 \Delta N. \quad (2)$$

Putting the above definition of each term, one obtains

$$\frac{\ddot{a}}{a} = -\frac{4\pi L_p^2}{3} (\rho + 3p). \quad (3)$$

This is the standard dynamical equation for the FRW universe in general relativity. Using continuity equation $\dot{\rho} + 3H(\rho + p) = 0$, one gets the standard Friedmann equation

$$H^2 + \frac{k}{a^2} = \frac{8\pi L_p^2 \rho}{3}, \quad (4)$$

where $k$ is an integration constant, which can be interpreted as the spatial curvature of the FRW universe. Here, Padmanabhan takes $(\rho + 3p) < 0$, which makes sense only in the accelerating phase. It means that in order to have the asymptotic holographic equipartition, the existence of dark energy is necessary.
III. DE SITTER UNIVERSE FROM EMERGENCE OF SPACE

As Padmanabhan said, this new idea provides a new paradigm for cosmology. Hence we would like to push an investigation on cosmology from the emergence of space. First, we will begin with the most simple case, namely the de Sitter universe.

In this case, the Eq. (2) has the form
\[ \frac{dV}{dt} = L_p^2 (N_{\text{sur}} - N_{\text{bulk}}) = 0. \] (5)

One can easily obtain
\[ \frac{dV}{dt} = \frac{d(\frac{4\pi}{3}H^3)}{dt} = 0, \] (6)

which means \( H \) is a constant, and then \( T, V, N_{\text{sur}}, N_{\text{bulk}} \) and \( |E| \) are all constants respectively. Using \( |E| = |\rho + 3p|V \), one can easily have
\[ |\rho + 3p| = \frac{3H^2}{4\pi L_p^2} = \text{constant} \] (7)
in de Sitter universe. In other words, once \( H \) is a constant, what we can have is \( |\rho + 3p| = \text{constant} \) according to emergence of space and \( \rho = -p \) is just one special case. So it is natural to generalize \( p = -\rho \) to the general equation of state (EOS) \( p = \omega \rho \).

If we take the accelerating phase \( \rho + 3p < 0 \), then Eq. (7) would be the term
\[ \rho + 3p = -\frac{3H^2}{4\pi L_p^2} = \text{constant} = -B_1, \] (8)

In de Sitter universe, combining Eq. (8) and continuity equation \( \dot{\rho} + 3H(\rho + p) = 0 \), one can have solutions of \( \rho \) and \( p \)
\[ \rho = \begin{cases} \frac{1}{2}a^{-2} + \frac{B_1}{2}, & -1 < \omega < -1/3 \ (\dot{\rho} < 0) \\ \frac{B_1}{2}, & \omega = -1 \ (\dot{\rho} = 0) \\ \frac{B_1}{2} - \frac{1}{2}a^{-2}, & \omega < -1 \ (\dot{\rho} > 0) \end{cases}, \] (9)
\[ p = \begin{cases} 
-\frac{1}{6}a^{-2} - \frac{B_1}{2}, & -1 < \omega < -1/3 \ (\dot{\rho} < 0) \\
-\frac{B_1}{2}, & \omega = -1 \ (\dot{\rho} = 0) \\
-\frac{B_1}{2} + \frac{1}{6}a^{-2}, & \omega < -1 \ (\dot{\rho} > 0) 
\end{cases} \tag{10} \]

where \( a = Ae^{Ht} \) has absorbed the integral constant. From Eq. (9) and Eq. (10), we can see that there are strong constraints on \( \rho \) and \( \omega \) in this new picture which give important information about cosmology. Such results motivate us to move on to a cosmological model in a more general case.

IV. A COSMOLOGICAL MODEL FROM EMERGENCE OF SPACE

An arbitrary universe whose \( \rho \) and \( p \) have the same form of Eq. (9) and Eq. (10) (the constant \( B_1 \) can be different) can satisfy continuity equation and the equation

\[ \rho + 3p = \text{constant} = -B_2 \tag{11} \]

For example, a universe which has \( \rho = \frac{1}{2}a^{-2} + \frac{B_2}{2} \) and \( p = -\frac{1}{6}a^{-2} - \frac{B_2}{2} \) would satisfy Eq. (11). What would this kind of universe be like in general?

To investigate the general case of this kind of universe, we go back to solve the Eq. (2) with Eq. (11)

\[ \frac{dV}{dt} = \frac{d(\frac{4\pi}{3}V)}{dt} = -\frac{4\pi}{H^4} \dot{H}, \tag{12} \]

\[ L_p^2(N_{sur} - N_{balk}) = L_p^2(\frac{4\pi}{L_p^2H^2} - \frac{16\pi^2B_2}{3H^4}). \tag{13} \]

Combining and arranging the above equations, one can have

\[ \lambda^2 - H^2 = \frac{dH}{dt}, \tag{14} \]

where \( \lambda = \sqrt{\frac{4\pi B_2 L_p^2}{3}} \). The solutions of Eq. (14) are following (to have an expansion, we choose \( H > 0 \), \( t > 0 \):
1): $0 < H < \lambda$, $dH/dt > 0$

$$H = \lambda - \frac{2\lambda}{C_{1}e^{2\lambda} + 1}, \quad (15)$$

where $C_{1}$ is an integral constant satisfying $C_{1} \geq 1$ for the request of $H > 0$.

2): $H = \lambda$, $dH/dt = 0$

$$H = \lambda \quad (16)$$

3): $H > \lambda$, $dH/dt < 0$

$$H = \frac{2\lambda}{D_{1}e^{2\lambda} - 1} + \lambda, \quad (17)$$

where $D_{1}$ is an integral constant satisfying $D_{1} \geq 1$ for the request of $H > \lambda$.

According to $H = \dot{a}/a$, we have $a(t)$:

1): $0 < H < \lambda$

$$a = \frac{C_{1}e^{2\lambda t} + 1}{C_{2}e^{\lambda t}}, \quad (18)$$

where $C_{2}$ is an integral constant satisfying $C_{2} > 0$.

2): $H = \lambda$

$$a = F_{1}e^{Hz}, \quad (19)$$

where $F_{1}$ is an integral constant satisfying $F_{1} > 0$.

3): $H > \lambda$

$$a = \frac{D_{1}e^{2\lambda t} - 1}{D_{2}e^{\lambda t}}, \quad (20)$$

where $D_{2}$ is an integral constant satisfying $D_{2} > 0$.

Next, let us push forward with a more detailed analysis on these three cases respectively.

1): $0 < H < \lambda$, $dH/dt > 0$

$$H = \lambda - \frac{2\lambda}{C_{1}e^{2\lambda t} + 1} \quad a = \frac{C_{1}e^{2\lambda t} + 1}{C_{2}e^{\lambda t}},$$
where $C_1 \geq 1$ and $C_2 > 0$. In the limit of $t \to 0$, one can have $a_0 = (C_1 + 1)/C_2 > 0$. This means a universe has an initial nonzero scale factor $a_0$. With the normal initial condition $a(0) = 0$ imposed, it is not proper to be the evolution of a real universe.

2): $H = \lambda$, $dH/dt = 0$

$$H = \lambda, \ a = F_1 e^{Ht},$$

where $F_1 > 0$. This is exactly the de Sitter universe which we have discussed in Section III.

3): $H > \lambda$, $dH/dt < 0$

$$H = \frac{2\lambda}{D_1 e^{2\lambda t} - 1} + \lambda, \ a = \frac{D_1 e^{2\lambda t} - 1}{D_2 e^{2\lambda t}},$$

where $D_1 \geq 1$ and $D_2 > 0$. In the limit of $t \to 0$, $a_0 = (D_1 - 1)/D_2$. If one set $D_1 = 1$, then $a_0 = 0$. Then it will have the possibility to describe our universe. Therefore we will take it as a cosmological model from emergence of space and give a detailed investigation.

Setting $D_1 = 1$, the above equations become

$$H = \frac{2\lambda}{e^{2\lambda t} - 1} + \lambda, \tag{21}$$

$$a = \frac{e^{2\lambda t} - 1}{D_2 e^{2\lambda t}} = A(e^{\lambda t} - \frac{1}{e^{\lambda t}}), \tag{22}$$

where $A = 1/D_2 > 0$. It is easy to find that Eq. (22) describes a universe which is asymptotically de Sitter and our universe is in this case as we generally believe.

There is a problem if Eq. (22) is used to describe our universe: it can not explain the inflation of early universe. However, just as Padmanabhan [6] said, Eq. (2) needs modifications at early universe.

Since $t > 0$ and $\lambda > 0$, there should be a constraint on $H$ (hence $a$). To derive this constraint, we will reverse the process to solve $\lambda$ by $H$ and $t$. We are going to
get the constraint under which the $\lambda > 0$ exists. By multiplying $t$ on both sides of Eq. (21) and substituting $x = \lambda t$, one can have

$$Ht = \frac{2x}{e^{2x} - 1} + x \ (x > 0). \quad (23)$$

Let

$$y = \frac{2x}{e^{2x} - 1} + x \ (x > 0).$$

After calculating, one can find

$$y' > 0, \ \lim_{x \to 0} y = 1.$$  

So the existence of a solution of $\lambda > 0$ requires a constraint on $H$ and $t$:

$$tH(t) > 1 \text{ or } t > 1/H(t). \quad (24)$$

Eq. (24) is applicable for any $t > 0$ (except for $t \to 0$ which represents the inflation of early universe). Then one can have

$$t_0H_0 > 1, \quad (25)$$

where $t_0$ is the present age of our universe and $H_0$ is the current value of $H$.

Since the constraint is only derived in our model and has never appeared in other theories before as far as we know, we would like to compare it with the experiments [14–18] from the Wilkinson Microwave Anisotropy Probe (WMAP) and the Planck Mission. The analysis is shown in Table I.

All the experimental data show that $H_0t_0 \approx 1$ just as predicted by our model. However, by comparing the data from the WMAP and the Planck, our model tends to support the WMAP rather than the Planck.
Table I. An analysis of the data of $H_0$ and $t_0$ from the WMAP and the Planck Mission. For obtaining $H_0 t_0$, we have changed km/(Mpc·s) into $s^{-1}$ and Ga into $s$. Using the Eq. (23), we have got the maximum $\lambda$ for each case.

| Observer Data published | $H_0$ km/(Mpc·s) | $t_0$ (Ga) | $H_0 t_0$ km/(Mpc·s) | $\lambda$ |
|------------------------|------------------|-----------|----------------------|------------|
| WMAP 2003              | $71_{-3}^{+5}$   | $13.7_{-0.2}^{+0.2}$ | 0.9393−1.0667        | 31.66      |
| WMAP 2006              | $73.2_{-3.2}^{+3.1}$ | $13.73_{-0.15}^{+0.16}$ | 0.9727−1.8044        | 35.71      |
| WMAP 2008              | $70.5_{-1.3}^{+1.3}$ | $13.72_{-0.12}^{+0.12}$ | 0.9629−1.0168        | 16.86      |
| WMAP 2010              | $70.4_{-1.3}^{+1.3}$ | $13.75_{-0.11}^{+0.11}$ | 0.9644−1.0168        | 15.90      |
| Planck 2013            | $67.80_{-0.77}^{+0.77}$ | $13.798_{-0.037}^{+0.037}$ | 0.9438−0.9707        | no value   |

The above results have shown the availability of our model in describing the real universe. What is more, we can actually calculate the vacuum energy through $\lambda$. To see this, let us go back to the solution of $\rho$ in general case

\[
\rho = \begin{cases} 
\frac{1}{2}a^{-2} + \frac{B_2}{2}, & -1 < \omega < -1/3 (\dot{\rho} < 0) \\
\frac{B_2}{2}, & \omega = -1 (\dot{\rho} = 0) \\
\frac{B_2}{2} - \frac{1}{2}a^{-2}, & \omega < -1 (\dot{\rho} > 0) 
\end{cases} \tag{26}
\]

where $B_2 = 3\lambda^2/(4\pi L_p^2)$ and $a$ is given in the form as in Eq. (18)~(20),(22) respectively. Though it can be read from Eq.(26) that the second line in the RHS should correspond to the vacuum energy, we still would like to give a more confident argument. Even though it is impossible to confirm which one ($\dot{\rho} < 0$ or $\dot{\rho} = 0$ or $\dot{\rho} > 0$) belongs to the universe described by Eq.(22), one may find that the $\rho$ of our universe would be $\dot{\rho} < 0$ by noticing that the energy density for matter $\rho_M$: $\rho_M \propto a^{-3}$ and the energy density for radiation $\rho_R$: $\rho_R \propto a^{-4}$. Hence we should choose the first line in the RHS of Eq.(26) to be the possible energy content of our universe.
With the energy content given in the form

\[ \rho = \frac{1}{2} a^{-2} + \frac{B_2}{2}, \quad (27) \]

it is natural to inquire the possible meaning of \( B_2/2 \). Noticing that \( \lim_{a \to +\infty} \rho = B_2/2 = -p \) is the vacuum energy density (\( \rho_{\text{vac}} \)) of the pure de Sitter which our universe would be in the far future, we argue that \( B_2/2 \) is the \( \rho_{\text{vac}} \) of our universe and

\[ \rho_{\text{vac}} = \frac{B_2}{2} = \frac{3\lambda^2}{8\pi L_p^2}, \quad (28) \]

By dimensional analysis, we have

\[ \frac{3\lambda^2}{8\pi L_p^2} = \left[ \frac{T}{L} \right]^2, \quad [\rho] = \left[ \frac{M}{L^3} \right] = \left[ \frac{T}{L} \right]^2 \cdot \left[ T \right]^2 \cdot \left[ M \right]. \]

We now put back the fundamental constants and get

\[ \rho_{\text{vac}} = \frac{3\lambda^2}{8\pi L_p^2} \frac{t_p^2 m_p}{L_p}. \]

Using the \( \lambda \) in the Table I, one can finally obtain the range of \( \rho_{\text{vac}} \):

\[ 4.733 \times 10^{-28} \sim 2.397 \times 10^{-27} \, \text{kg/m}^3. \quad (29) \]

Comparing with the density of dark energy (about \( 6.91 \times 10^{-27} \, \text{kg/m}^3 \)) [14–17], there is a difference between the two though it almost has the same order of our estimated value.

In the standard \( \Lambda \)CDM cosmological model, it is believed that the dark energy is caused by the cosmological constant. Hence it is convenient to compare our theoretical results with the experiment data of \( \Omega_\Lambda \).

The definition of \( \Omega_\Lambda \) is

\[ \Omega_\Lambda = \frac{\Lambda}{(3H_0^2)}, \quad (30) \]
where $\Lambda = 8\pi \rho_{\text{vac}}$. Using the $\rho_{\text{vac}}$ in our model, one can have

$$\tilde{\Omega}_\Lambda = \frac{\lambda^2}{H_0^2}.$$  \hfill (31)

According to Table I we finally get the range of $\tilde{\Omega}_\Lambda$:

$$0.049 \sim 0.260.$$  \hfill (32)

And the experimental data of $\Omega_\Lambda$ from the WMAP and the Planck Mission [14–18] have the range of:

$$0.683 \sim 0.772.$$  

It can be seen that our model predicts the approximate but not exact value of cosmological constant. Still, the cosmological constant derived in our model has the same order of the experimental data. We take the difference as an indication that there are possibly other sources of dark energy such as the quintessence.

V. CONCLUSIONS AND DISCUSSIONS

To summarize, in this paper, we investigated the novel idea proposed by Padmanabhan [6] that the emergence of space and expansion of the universe are due to the difference between the number of DOF on the holographic surface and the one in the emerged bulk. It is shown that the Friedmann equation of a flat FRW universe can be derived with the help of continuity equation. Since the emergence of space may provide a completely different paradigm to study cosmology [6], we studied the de Sitter universe from emergence of space, and found that there is a constraint on $\rho$ and $p$ (Eq. (7) and Eq. (8)) which can derive solutions of $\rho$ and $p$ (Eq. (9) Eq.(10)). By considering an arbitrary universe whose $\rho$ and $p$ have the same form of Eq.(9) and Eq.(10), we generalized Eq.(8) beyond the de Sitter universe and solved Eq.(2).
Among the solutions we obtained, we found a model which has the possibility to describe our universe. After detailed analysis of our model, we got three important conclusions:

(1) The universe would be de Sitter in its later period. \((t >> 1/\lambda)\).

(2) There is a constraint on \(H\) and \(t\): \(H(t) \cdot t > 1\), and it is applicable for any \(t > 0\) (except for \(t \rightarrow 0\) which represents the inflation of early universe).

(3) The value of vacuum energy and \(\tilde{\Omega}_\Lambda\) can be derived in our model.

We made a comparison of our model with experiments. For conclusion (2), the experimental data show that \(H_0 t_0\) ranges from 0.9438 \(\sim\) 1.8044 and our model tends to support the WMAP rather than the Planck. For conclusion (3), our model predicts a positive tiny cosmological constant, which is approximate to the experimental data. The difference indicates that there are probably other sources contributing to the dark energy.

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