K-HARMONIC IMMERSION AND SUBMERSION INTO A SPHERE

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Abstract. J. Eells and L. Lemaire introduced k-harmonic maps, and Wang Shaobo showed the first variational formula. When, k=2, it is called biharmonic maps (2-harmonic maps). There have been extensive studies in the area. In this paper, we study k-harmonic immersion into a sphere, and get the relationship between radius and "k" of k-harmonic. And we also consider k-harmonic submersion. Furthermore, we construct non harmonic k-harmonic by Hopf map.

Introduction

Theory of harmonic maps has been applied into various fields in differential geometry. The harmonic maps between two Riemannian manifolds are critical maps of the energy functional

$$E(\phi) = \int_M \|d\phi\|^2 v_g,$$

for smooth maps $\phi : M \to N$.

On the other hand, in 1981, J. Eells and L. Lemaire [5] proposed the problem to consider the $k$-harmonic maps: they are critical maps of the functional

$$E_k(\phi) = \int_M e_k(\phi)v_g, \quad (k = 1, 2, \cdots),$$

where $e_k(\phi) = \frac{1}{2}\| (d + d^*)^k \phi \|^2$ for smooth maps $\phi : M \to N$. G.Y. Jiang [3] studied the first and second variational formulas of the bi-energy $E_2$, and critical maps of $E_2$ are called biharmonic maps (2-harmonic maps). There have been extensive studies on biharmonic maps.

In 1989, Wang Shaobo [9] studied the first variational formula of the $k$-energy $E_k$, whose critical maps are called $k$-harmonic maps. Harmonic maps are always $k$-harmonic maps by definition. Especially, harmonic maps are always biharmonic maps. Currently, there are a lot of paper on biharmonic maps. But in this paper, we show biharmonic is not always $k$-harmonic ($k \geq 3$). More generally, $s$-harmonic is not always $k$-harmonic ($s < k$). This may imply that we must study $k$-harmonic maps for generalized theory of harmonic maps.

In this paper, we study $k$-harmonic immersion and submersion.

In §1 we introduce notation and fundamental formulas of the tension field.

In §2 we recall $k$-harmonic maps.

In §3 we study $k$-harmonic isometric immersion into a sphere, and obtain the necessary and sufficient condition for $\phi$ to be $k$-harmonic map. Furthermore we get several examples of $k$-harmonic maps.

Finally, in §4 we study $k$-harmonic submersion. And we construct a non harmonic $k$-harmonic map by Hopf map.

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1. Preliminaries

Let \((M, g)\) be an \(m\) dimensional Riemannian manifold, \((N, h)\) an \(n\) dimensional one, and \(\phi : M \to N\), a smooth map. We use the following notation. The second fundamental form \(B(\phi)\) of \(\phi\) is a covariant differentiation \(\nabla d\phi\) of 1-form \(d\phi\), which is a section of \(\otimes^2 T^* M \otimes \phi^{-1} T N\). And we denote \(B(\phi)\) by \(B\). For every \(X, Y \in \Gamma(TM)\), let

\[
B(X, Y) = (\nabla d\phi)(X, Y) = (\nabla_X d\phi)(Y) = \nabla_X d\phi(Y) - d\phi(\nabla X Y).
\]

Here, \(\nabla, \nabla^N, \nabla, \tilde{\nabla}\) are the induced connections on the bundles \(TM, T N, \phi^{-1} TN\) and \(T^* M \otimes \phi^{-1} T N\), respectively.

And we also denote \(B(e_i, e_j)\) by \(B_{ij}\), where \(\{e_i\}_{i=1}^m\) is locally defined orthonormal frame field on \((M, g)\).

If \(M\) is compact, we consider critical maps of the energy functional

\[
E(\phi) = \int_M e(\phi) v_g,
\]

where \(e(\phi) = \frac{1}{2}||d\phi||^2 = \sum_{i=1}^m \frac{1}{2}(d\phi(e_i), d\phi(e_i))\) which is called the energy density of \(\phi\), and the inner product \(\langle \cdot, \cdot \rangle\) is a Riemannian metric \(h\). The tension field \(\tau(\phi)\) of \(\phi\) is defined by

\[
\tau(\phi) = \sum_{i=1}^m (\nabla d\phi)(e_i, e_i) = \sum_{i=1}^m (\nabla e_i d\phi)(e_i).
\]

Then, \(\phi\) is a harmonic map if \(\tau(\phi) = 0\).

The curvature tensor field \(R^N(\cdot, \cdot)\) of the Riemannian metric on the bundle \(TN\) is defined as follows:

\[
R^N(X, Y) = \nabla_X^n \nabla^N_Y - \nabla_X^n \nabla_Y^N - \nabla_{[X, Y]}^N, \quad (X, Y \in \Gamma(TN)).
\]

\(\triangle = \nabla^N \nabla = -\sum_{k=1}^m (\nabla e_k \nabla e_k - \nabla \nabla e_k e_k)\), is the rough Laplacian.

And G.Y.Jiang \[3\] showed that \(\phi : (M, g) \to (N, h)\) is a biharmonic (2-harmonic) if and only if

\[
\triangle \tau(\phi) - R^N(\tau(\phi), d\phi(e_i))d\phi(e_i) = 0.
\]

2. \(k\)-harmonic maps

J. Eells and L. Lemaire \[5\] proposed the notation of \(k\)-harmonic maps. The Euler-Lagrange equation for the \(k\)-harmonic maps was shown by Wang Shaobo \[9\].

In this section, we recall \(k\)-harmonic maps (cf \[6\], \[9\]).

We consider a smooth variation \(\{\phi_t\}_{t \in T, I_\epsilon = (-\epsilon, \epsilon)}\) of \(\phi\) with parameter \(t\), i.e. we consider the smooth map \(F\) given by

\[
F : I_\epsilon \times M \to N, F(t, p) = \phi_t(p),
\]

where \(F(0, p) = \phi_0(p) = \phi(p), \) for all \(p \in M\).

The corresponding variational vector field \(V\) is given by

\[
V(p) = \frac{d}{dt}\bigg|_{t=0} \phi_t(p) \in T_{\phi_t(p)} N,\]

\(V\) are section of \(\phi^{-1} TN\), i.e., \(V \in \Gamma(\phi^{-1} TN)\).
Definition 2.1 ([5]). For \( k = 1, 2, \cdots \) the \( k \)-energy functional is defined by
\[
E_k(\phi) = \frac{1}{k} \int_M \| (d + d^* \phi)^k \|^2 v_g, \quad \phi \in C^\infty(M, N).
\]

Then, \( \phi \) is \( k \)-harmonic if it is a critical point of \( E_k \), i.e., for all smooth variation \( \{ \phi_t \} \) of \( \phi \) with \( \phi_0 = \phi \),
\[
\left. \frac{d}{dt} \right|_{t=0} E_k(\phi_t) = 0.
\]

We say for a \( k \)-harmonic map to be proper if it is not harmonic.

Theorem 2.2 ([9]). Let \( k = 2s \) \( (s = 1, 2, \cdots) \),
\[
\left. \frac{d}{dt} \right|_{t=0} E_{2s}(\phi_t) = - \int_M \langle \tau_{2s}(\phi), V \rangle,
\]
where,
\[
\tau_{2s}(\phi) = \Delta^{2s-1} \tau(\phi) - R^N(\Delta^{2s-2} \tau(\phi), d\phi(e_j))d\phi(e_j)
\]
\[
- \sum_{l=1}^{s-1} \left\{ R^N(\nabla_{e_j} \Delta^{s+l-2} \tau(\phi), \Delta^{s-l-1} \tau(\phi))d\phi(e_j) \right\},
\]
where, \( \Delta^{-1} = 0 \).

Theorem 3.1 ([3]). Let \( \phi : M \to S^{m+1} \) be an isometric immersion having parallel mean curvature vector field with non-zero mean curvature. Then, the necessary and sufficient condition for \( \phi \) to be biharmonic is \( |||B||| = m = \dim M \).
Lemma 3.3. Let \( \phi : (M, g) \rightarrow (N, h) \) be an isometric immersion of which the second fundamental form \( B \) is parallel, i.e., \( \nabla^\perp B = 0 \), where \( \nabla^\perp \) is the induced connection of the normal bundle \( T^\perp M \) by \( \phi \). Then,
\[
\overline{\nabla}B_{st} = h(\overline{\nabla}B_{st}, d\phi(e_i))d\phi(e_i) - h(\nabla_{e_i}B_{st}, d\phi(e_i))B_{ij}.
\]
Proof. Let us recall the definition of \( \nabla^\perp \): For any section \( \xi \in \Gamma(T^\perp M) \), we decompose \( \nabla_X\xi \) according to \( TN|_M = TM \oplus T^\perp M \) as follows.
\[
\nabla_X\xi = \nabla^N_{d\phi(X)}\xi = \nabla^T_{d\phi(X)}\xi + \nabla^\perp_{d\phi(X)}\xi.
\]
By the assumption, \( \nabla^\perp_{d\phi(X)}B_{st} = 0 \) for all \( X \in \Gamma(TM) \), we have
\[
\nabla_XB_{st} = \nabla^T_{d\phi(X)}B_{st} \in \Gamma(\phi_*TM).
\]
Thus, for all \( i = 1, \ldots, m \),
\[
\nabla_{e_i}B_{st} = h(\nabla_{e_i}B_{st}, d\phi(e_j))d\phi(e_j)
\]
because \( \{d\phi(e_j)\}_{j=1}^m \) is an orthonormal basis with respect to \( h \), of \( \phi_*T_xM(x \in M) \).

Now let us calculate
\[
\nabla^2B_{st} = -\{\nabla_{e_i}\nabla_{e_j}B_{st} - \nabla_{\nabla_{e_i}e_j}B_{st}\}.
\]
Indeed, we have
\[
\nabla_{e_i}\nabla_{e_j}B_{st} = h(\nabla_{e_i}\nabla_{e_j}B_{st}, d\phi(e_j))d\phi(e_j)
+ h(\nabla_{e_i}B_{st}, d\phi(e_j))\nabla_{e_j}d\phi(e_j),
\]
and
\[
\nabla_{\nabla_{e_i}e_j}B_{st} = h(\nabla_{\nabla_{e_i}e_j}B_{st}, d\phi(e_j))d\phi(e_j),
\]
so that we have
\[
\nabla^2B_{st} = h(\nabla\nabla B_{st}, d\phi(e_j))d\phi(e_j)
- \{h(\nabla_{e_i}B_{st}, \nabla_{e_j}d\phi(e_j))d\phi(e_j)
+ h(\nabla_{e_i}B_{st}, d\phi(e_j))\nabla_{e_j}d\phi(e_j)\}.
\]
Denoting \( \nabla_{e_i}e_j = \sum_{k=1}^m \Gamma_{ij}^k e_k \), we have \( \Gamma_{ij}^k + \Gamma_{ik}^j = 0 \). Since \( \nabla_{e_i}d\phi(e_j) = \nabla_{e_i}d\phi(e_j) - d\phi(\nabla_{e_i}e_j) \) is a local section of \( T^\perp M \), we have for the second term of the RHS of (7), for each fixed \( i = 1, \ldots, m \),
\[
\begin{align*}
\nabla_{e_i}B_{st} = h(\nabla_{e_i}B_{st}, \nabla_{e_j}d\phi(e_j))d\phi(e_j)
&= h(\nabla_{e_i}B_{st}, d\phi(e_j))d\phi(e_j)
&= h(\nabla_{e_i}B_{st}, d\phi(e_j))d\phi(e_j)
&= h(\nabla_{e_i}B_{st}, d\phi(e_j))d\phi(e_j)
&= h(\nabla_{e_i}B_{st}, d\phi(e_j))d\phi(e_j)
&= h(\nabla_{e_i}B_{st}, d\phi(e_j))d\phi(e_j).
\end{align*}
\]
Substituting (8) into (7), we obtain this lemma.
\[\square\]

Lemma 3.3. Under the same assumption as in Lemma 3.2, we have
\[
\overline{\nabla} = -h(B_{st}, R^N (d\phi(e_j), d\phi(e_k))d\phi(e_j))d\phi(e_j) + h(B_{st}, B_{ij})B_{ij}.
\]
Proof. Since \( h(B_{st}, d\phi(e_j)) = 0 \), differentiating it by \( e_i \), we have
\[
h(\nabla e_i B_{st}, d\phi(e_j)) = -h(B_{st}, \nabla e_i d\phi(e_j)) = -h(B_{st}, (\nabla e_i d\phi)(e_j)).
\]
And we have
\[
h(\nabla e_i, \nabla e_i B_{st}, d\phi(e_j)) + h(\nabla e_i B_{st}, \nabla e_i d\phi(e_j))
= -h(\nabla e_i B_{st}, B_{ij}) - h(B_{st}, \nabla e_i B_{ij})
\]
So we have
\[
(9) \quad 0 = h(\nabla e_i, \nabla e_i B_{st}, d\phi(e_j)) + h(\nabla e_i B_{st}, \nabla e_i d\phi(e_j))
+ h(\nabla e_i B_{st}, B_{ij}) + h(B_{st}, \nabla e_i B_{ij})
= h(\nabla e_i, \nabla e_i B_{st}, d\phi(e_j)) + h(\nabla e_i B_{st}, d\phi(\nabla e_i, e_j))
+ h(B_{st}, \nabla e_i (\nabla e_i d\phi)(e_j)) + h(B_{st}, (\nabla e_i d\phi)(\nabla e_i, e_j)).
\]
Here, using \( h(B_{st}, d\phi(\nabla e_i, e_j)) = 0 \), we get
\[
0 = h(\nabla e_i, B_{st}, d\phi(\nabla e_i, e_j)) + h(B_{st}, \nabla e_i d\phi(\nabla e_i, e_j))
= h(\nabla e_i, B_{st}, d\phi(\nabla e_i, e_j)) + h(B_{st}, (\nabla e_i d\phi)(\nabla e_i, e_j)).
\]
Thus, (9) coincides with
\[
(10) \quad h(\nabla e_i, \nabla e_i B_{st}, d\phi(e_j)) + h(B_{st}, (\nabla e_i \nabla e_i d\phi)(e_j)) = 0.
\]
And using \( h(B_{st}, d\phi(e_j)) = 0 \), we obtain
\[
(11) \quad 0 = h(\nabla e_i, \nabla e_i B_{st}, d\phi(e_j)) + h(B_{st}, \nabla e_i d\phi(e_j))
= h(\nabla e_i, \nabla e_i B_{st}, d\phi(e_j)) + h(B_{st}, (\nabla e_i d\phi)(e_j)).
\]
By (10), and (11), we have
\[
(12) \quad 0 = -h(\nabla e_i, \nabla e_i B_{st}, d\phi(e_j)) - h(B_{st}, (\nabla e_i \nabla e_i d\phi)(e_j))
+ h(\nabla e_i, \nabla e_i B_{st}, d\phi(e_j)) + h(B_{st}, (\nabla e_i d\phi)(e_j))
= h(\Delta B_{st}, d\phi(e_j)) + h(B_{st}, (\nabla e_i d\phi)(e_j)),
\]
where, \( \nabla e_i \nabla e_i = - (\nabla e_i \nabla e_i - \nabla e_i e_i). \)
By Wizenböck formula
\[
\Delta d\phi = \nabla e_i d\phi + Sd\phi,
\]
where,
\[
Sd\phi(X) = -(\tilde{R}(X, e_k) d\phi)(e_k),
\]
\[
\tilde{R}(X, Y) d\phi(Z) = R^{e-1N} (X, Y) d\phi(Z) - d\phi(R^M (X, Y) Z)
= R^N (d\phi(X), d\phi(Y)) d\phi(Z) - d\phi(R^M (X, Y) Z), \quad X, Y, Z \in \Gamma(TM),
\]
and
\[
\Delta d\phi(e_j) = (dd^* + d^* d) \phi(e_j) = dd^* \phi(e_j) = -d\tau(\phi)(e_j) = -\nabla e_i \tau(\phi),
\]
we obtain

\[(13) \quad (\nabla^* \tilde{\nabla} d\phi)(e_j) = \Delta d\phi(e_j) - S d\phi(e_j)
= - \nabla_{e_j} \tau(\phi) + R^N(d\phi(e_j), d\phi(e_k))d\phi(e_k) - d\phi(R^M(e_j, e_k)e_k).
\]

Substituting (13) into (12), we get

\[h(\nabla B_{st}, d\phi(e_j)) = - h(B_{st}, - \nabla_{e_j} \tau(\phi) + R^N(d\phi(e_j), d\phi(e_k))d\phi(e_k)
- d\phi(R^M(e_j, e_k)e_k))
= - h(B_{st}, R^N(d\phi(e_j), d\phi(e_k))d\phi(e_k)).\]

Therefore, we obtain this lemma. □

Lemma 3.4. Let \( \phi : (M, g) \to (N, h) \) be an isometric immersion into a Riemannian manifold with constant sectional curvature \( K \), and second fundamental form of \( \phi \) is parallel. Then,

\[(14) \quad \nabla^k B_{ij} = h(B_{ij}, B_{1i}) \prod_{l=1}^{k-1} h(B_{i(l+1)i}, B_{1(l+1)i} \cdot h_{kl}).\]

Proof. Since the curvature tensor \( R^N \) of \((N, h)\) is given by

\[R^N(U, V)W = K\{h(V, W)U - h(W, U)V\}, \quad U, V, W \in \Gamma(TN),\]

\(R^N(d\phi(e_j), d\phi(e_k))d\phi(e_k)\) is tangent to \( \phi_* TM \). By Lemma 3.3 we have

\[(15) \quad \nabla B_{st} = h(B_{st}, B_{ij})B_{ij}\]

For all \( X \in \Gamma(TM) \), we have

\[\nabla_X \{h(B_{st}, B_{ij})B_{ij}\} = (\nabla_X h(B_{st}, B_{ij}))B_{ij} + h(B_{st}, B_{ij})\nabla_X B_{ij}\]

= \(h(B_{st}, B_{ij})\nabla_X B_{ij}\).

Similarly we have

\[\nabla_X \nabla_X \{h(B_{st}, B_{ij})B_{ij}\} = h(B_{st}, B_{ij})\nabla_X \nabla_X B_{ij}\]

Thus, we obtain

\[\nabla^2 B_{st} = h(B_{st}, B_{ij})\nabla B_{ij} = h(B_{st}, B_{ij})h(B_{kl}B_{kl}).\]

Repeating these steps, we get

\[(16) \quad \nabla^k B_{ij} = h(B_{ij}, B_{1i}) \prod_{l=1}^{k-1} h(B_{i(l+1)i}, B_{1(l+1)i} \cdot h_{kl}).\]

Therefore, we obtain the lemma. □

Using these lemmas, we show the next proposition.

Proposition 3.5. Let \( \phi : M \to S^{m+1} \) be an isometric immersion having parallel second fundamental form. Then, the necessary and sufficient condition for \( \phi \) to be \( k \)-harmonic is

\[||B||^4 - m||B||^2 - (k-2)||\tau(\phi)||^2 = 0.\]
Proof. If we denote by $\xi$ the unit normal vector field to $\phi(M)$, the second fundamental form $B$ is of the form $B(e_i, e_j) = (\nabla_{e_i} \phi)(e_j) = h_{ij}\xi$. Then, we have $\tau(\phi) = B(e_i, e_i) = h_{ii}\xi$ and $\|B\|^2 = h_{ij}h_{ij}$. Using Lemma 3.3, we get $\Delta \tau(\phi) = \|B\|^{2k}\tau(\phi)$. So we have

$$0 = \tau_{2s}(\phi) = \Delta^{2s-1}\tau(\phi) - \{m\Delta^{2s-2}\tau(\phi)\}$$

$$= \sum_{l=1}^{s-1}\{\langle -\{d\phi(e_i), \nabla_{e_i} \Delta_{s-l-1} \tau(\phi)\}l_{s-l-1}\tau(\phi)\rangle + \Delta^{s-l-1}\tau(\phi))\}$$

$$= \|B\|^{2(2s-1)}\tau(\phi) - m\|B\|^{2(2s-2)}\tau(\phi)$$

So $\phi$ is $2s$-harmonic ($s = 1, 2, \cdots$) if and only if

$$\|B\|^4 - m\|B\|^2 - 2(s-1)\|\tau(\phi)\|^2 = 0.$$ 

Similarly, $\phi$ is $2s + 1$-harmonic ($s = 1, 2, \cdots$) if and only if

$$\|B\|^4 - m\|B\|^2 - (2s - 1)\|\tau(\phi)\|^2 = 0.$$ 

\[\square\]

**Corollary 3.6.** Let $\phi : M \to S^{m+1}$ be a $k$-harmonic isometric immersion ($k = 3, 4, \cdots$) having parallel second fundamental form $B \neq 0$. If $\phi$ is biharmonic, then $\phi$ is harmonic.

Proof. By Proposition 3.5

$$0 = m^2 - m \cdot m - (k-2)\|\tau(\phi)\|^2 = -(k-2)\|\tau(\phi)\|^2.$$ 

So we get the corollary. \[\square\]

We define (cf. [9], [3])

$$M^m_p(\lambda) = S^p \left( \frac{1}{\sqrt{1 + \lambda^2}} \right) \times S^{m-p} \left( \frac{\lambda}{\sqrt{1 + \lambda^2}} \right) \quad \lambda > 0, \ 0 \leq p \leq m.$$ 

Indeed, $\phi : M^m_p(\lambda) \to S^{m+1}$ has the parallel second fundamental form, and $\phi$ satisfies that

$$\|\tau(\phi)\| = |p\lambda - (m-p)|\frac{1}{\lambda},$$

and

$$\|B\|^2 = p\lambda^2 + (m-p)|\frac{1}{\lambda^2}.$$ 

Thus, when $\lambda^2 = \frac{m-p}{p}$, $\phi$ is harmonic.

Using Proposition 3.3, $\phi$ is $k$-harmonic if and only if
\[(\lambda^2 - \frac{m - p}{p})((\lambda^6 - (k - 1)\lambda^4 + (k - 1)\frac{(m - p)}{p}\lambda^2 - \frac{m - p}{p}) = 0.\]

Thus \(\phi\) is proper \(k\)-harmonic if and only if
\[\lambda^6 - (k - 1)\lambda^4 + (k - 1)\frac{(m - p)}{p}\lambda^2 - \frac{m - p}{p} = 0, \quad \lambda^2 \neq \frac{m - p}{p}.
\]

So we obtain two theorems.

**Theorem 3.7.** Isometric immersion \(\phi : S^m(\frac{1}{\sqrt{k}}) \rightarrow S^{m+1}\) is a special proper \(k\)-harmonic map.

**Theorem 3.8.** Isometric immersion \(\phi : M^m_p(\lambda) \rightarrow S^{m+1}\) is a proper \(k\)-harmonic map, where \(\lambda\) satisfies that
\[\lambda^6 - (k - 1)\lambda^4 + (k - 1)\frac{(m - p)}{p}\lambda^2 - \frac{m - p}{p} = 0, \quad \lambda^2 \neq \frac{m - p}{p}.
\]

Especially, we have the next two results

**Corollary 3.9.** When, \(m = 2p\),
1) When, \(k = 2, 3, 4\).
2) When, \(k = 5, 6, \ldots\).

Isometric immersion \(\phi : M^m_p(\lambda) \rightarrow S^{m+1}\) is a proper \(k\)-harmonic map if and only if
\[
\lambda = \sqrt{\frac{k - 2 \pm \sqrt{k(k - 4)}}{2}}.
\]

**Corollary 3.10.** When, \(m \neq 2p\), isometric immersion \(\phi : M^m_p(\lambda) \rightarrow S^{m+1}\) is proper \(k\)-harmonic, where,
\[
\lambda = \sqrt{A^\frac{1}{3} + \frac{k^2 - 3a(k - 1) - 2k + 1}{9A^\frac{1}{3}} + \frac{k - 1}{3}},
\]
where,
\[
A = \frac{-a(ak^4 - 4(a^2 + a + 1)k^3 + 12(a + 1)^2k^2 - 4(3a + 10a + 3)k + 4(a - 1)^2)}{234^2} - \frac{2k^3 + 9a(k^2 - 2k - 2) + 2(3k^2 - 3k + 1)}{54},
\]
\[a = \frac{m - p}{p}.
\]

4. \(k\)-HARMONIC SUBMERSION

In this section we generalize Oniciuc’s results \(\Xi\). Let
\[S^n(a) = S^n(a) \times \{b\}\]
\[= \{p = (x^1, \ldots, x^{n+1}, b); \quad (x^1)^2 + \cdots + (x^{n+1})^2 = a^2, a \in (0, 1), \quad a^2 + b^2 = 1\}\]
be a parallel hypersurface of \(S^{n+1}\), and the canonical metric \(\langle \cdot, \cdot \rangle\) on \(S^{n+1}\).

And let
\[\Gamma(TS^n(a)) = \{X = (X^1, \ldots, X^{n+1}, 0) \quad x^1X^1 + \cdots + x^{n+1}X^{n+1} = 0\},\]
be the set of all sections of the tangent bundle of \(S^n(a)\).
Let $\eta = \frac{1}{c}(x^1, \cdots, x^{n+1}, \frac{a^2}{b})$, is a unit section in the tangent bundle of $S^n(a)$ in $S^{n+1}$, where $c > 0$ and $c^2 = a^2 + \frac{4}{b}c$

Thus $\eta$ satisfies

$$\langle \eta, p \rangle = 0, \quad \langle \eta, X \rangle = 0 \quad |\eta| = 1.$$ 

By a direct computation we obtain

$$A = -\frac{1}{c}I, \quad B(X, Y) = -\frac{1}{c}(X, Y)\eta, \quad \nabla^\perp \eta = 0,$$

where $A$ is the shape operator, $B$ the second fundamental form of $S^n(a)$ and $\nabla^\perp$ is the normal connection in the normal bundle of $S^n(a)$ in $S^{n+1}$.

Now, we consider a Riemannian submersion $\varphi : (M, g) \to S^n(a)$, the canonical inclusion $i : S^n(a) \to S^{n+1}$, and $\phi = i \circ \varphi : (M, g) \to S^{n+1}$. The rank of $\phi$ is constant, equal to $n$. The next theorem was proved by C. Oniciuc [1].

**Theorem 4.1** (cf [1]). Assume that $\varphi : (M, g) \to S^n(a)$ is a harmonic Riemannian submersion. Then, $\phi : (M, g) \to S^{n+1}$ is not harmonic, and it is biharmonic if and only if $\alpha = \frac{1}{\sqrt{2}}$ and $b = \pm \frac{1}{\sqrt{2}}$.

We generalize this theorem.

**Proposition 4.2.** Assume that $\varphi : (M, g) \to S^n(a)$ is a harmonic Riemannian submersion. Then, $\phi : (M, g) \to S^{n+1}$ is proper $2s$-harmonic if and only if $a = \frac{1}{\sqrt{2s}}$ and $b = \pm \frac{1}{\sqrt{2s}}$.

Proof. Let $p \in M$, we have $T_pM = T^{\perp}_pM \oplus T^H_pM$, where $T^{\perp}_pM = \ker d\varphi_p$ and $T^H_pM$ is the orthogonal complement of $T^{\perp}_pM$ in $T_pM$ with respect to the metric $g$. Let $W$ be an open subset of $S^n(a)$ such that $\varphi(p) \in W$ and let $\{Y_\alpha\}_{\alpha=1}^m$ be an orthonormal frame field of $W$. Set $U = \varphi^{-1}(W)$, $\{X_s\}_{s=0}^m$, and consider an orthonormal frame field $\{X_s\}_{s=n+1}^m$ on $T^nU$. The tension field of $\varphi$ is given by

$$\tau(\varphi)_p = -\sum_{s=n+1}^m d\varphi_p(\nabla_{X_s}X_s)$$

(cf [1]). Computing the tension field of $\phi$ we obtain

$$\tau(\phi) = di(\tau(\varphi)) + \text{trace} \nabla di(d\varphi, d\varphi) = \sum_{\alpha=1}^n B(Y_\alpha, Y_\alpha) = -\frac{n}{c} \eta,$$

i.e., $\phi$ is not harmonic.

To simplify the notation, we denote the Levi-Civita connection $\nabla^{S^n(a)}$ of $S^n(a)$ by $\nabla^N$. Computing $\Box_\eta$ we get

$$\Box_\eta = -\sum_{s=n+1}^m \langle \nabla_{X_s}\nabla_{X_s} \eta - \nabla_{\nabla_{X_s}X_s} \eta \rangle$$

$$= -\sum_{\alpha=1}^n \langle \nabla_{X_\alpha} \nabla_{X_\alpha} \eta - \nabla_{\nabla_{X_\alpha}X_\alpha} \eta \rangle$$

$$= -\sum_{s=n+1}^m \langle \nabla_{X_s} \nabla_{X_s} \eta - \nabla_{\nabla_{X_s}X_s} \eta \rangle.$$. 
Here we obtain
\[ \nabla X_\alpha \eta = \nabla Y_\alpha^S \eta = \frac{1}{c} Y_\alpha, \]
and using (17) we obtain
\[ \nabla X_\alpha \nabla X_\alpha \eta = \frac{1}{c} \nabla Y_\alpha^N \eta + \frac{1}{c^2} \eta. \]
Further, we have
\[ \nabla \nabla X_\alpha \eta = \nabla Y_\alpha^N \eta, \]
\[ \nabla \nabla X_\alpha \eta = \frac{1}{c} \nabla Y_\alpha^N \eta. \]
So we obtain
\[ \Delta \eta = \frac{n}{c^2} \eta. \]
Repeating these steps, we have
\[ \Delta^l \eta = \left( \frac{n}{c^2} \right)^l \eta, \quad (l = 1, 2, \ldots), \]
and
\[ \Delta^l \tau(\phi) = -\frac{n}{c} \left( \frac{n}{c^2} \right)^l \eta, \quad (l = 1, 2, \ldots). \]
A direct computation shows
\[ \sum_{i=1}^{m} R^{S_{n+1}} (\Delta \tau(\phi), d\phi(X_i)) d\phi(X_i) = \frac{n}{c} \Delta \tau(\phi) = -\frac{n^2}{c} \left( \frac{n}{c^2} \right)^l \eta, \]
\[ \sum_{i=1}^{m} R^{S_{n+1}} (\nabla X_i \Delta \tau(\phi), \nabla X_i \Delta \tau(\phi)) d\phi(X_i) = -\frac{n^2}{c} \left( \frac{n}{c^2} \right)^{l+1} \eta, \]
and
\[ \sum_{i=1}^{m} R^{n+1} (\Delta \tau(\phi), \nabla X_i \Delta \tau(\phi)) d\phi(X_i) = \frac{n^2}{c} \left( \frac{n}{c^2} \right)^{l+1} \eta. \]
Substituting (25), (26), (27) and (28) into (5), we obtain
\[ \tau_{2s}(\phi) = -\frac{n}{c} \left( \frac{n}{c^2} \right)^{2s-1} \eta + \frac{n^2}{c} \left( \frac{n}{c^2} \right)^{2s-2} \eta \]
\[ - \sum_{i=1}^{s-1} \left\{ -\frac{n^2}{c^2} \left( \frac{n}{c^2} \right)^{2s-2} \eta - \frac{n^2}{c} \left( \frac{n}{c^2} \right)^{2s-2} \eta \right\} \]
\[ = -\frac{n}{c} \left( \frac{n}{c^2} \right)^{2s-1} \left( 1 - (2s - 1)c^2 \right) \eta. \]
So \( \phi \) is proper 2s-harmonic if and only if \( c = \frac{1}{\sqrt{2s-1}} \) i.e., \( a = \frac{1}{\sqrt{2s}} \) and \( b = \pm \sqrt{\frac{2s-1}{2s+1}} \).

Similarly we have

**Proposition 4.3.** Assume that \( \varphi : (M, g) \to S^n(a) \) is a harmonic Riemanian submersion. Then, \( \phi : (M, g) \to S^{n+1} \) is proper \( 2s + 1 \)-harmonic if and only if \( a = \frac{1}{\sqrt{2s+1}} \) and \( b = \pm \sqrt{\frac{2s}{2s+1}} \).
Proof. The proof is similar to that of Proposition 4.2.

By Proposition 4.4 and 4.3 we obtain the next theorem.

**Theorem 4.4.** Assume that \( \varphi : (M, g) \rightarrow S^n(a) \) is a harmonic Riemannian submersion. Then, \( \phi : (M, g) \rightarrow S^{n+1} \) is proper k-harmonic if and only if \( a = \frac{1}{\sqrt{k}} \) and \( b = \pm \sqrt{\frac{k-1}{k}} \).

Proof. By Proposition 4.4 and 4.3 we obtain this theorem.

Since the radial projection

\[
S^n \rightarrow S^n(a), \quad x \mapsto ax,
\]

is homothetic, a harmonic Riemannian submersion \( \varphi : (M, g) \rightarrow S^n \) becomes a harmonic Riemannian submersion \( \varphi : (M, a^2 g) \rightarrow S^n(a) \), and using the above theorem, we obtain a proper k-harmonic submersion \( \phi : M \rightarrow S^{n+1} \). Especially we obtain the next corollary.

**Corollary 4.5.** The Hopf map \( \varphi : S^3(\sqrt{k}) = \{(z^1, z^2) \in \mathbb{C}^2; (z^1)^2 + (z^2)^2 = k\} \rightarrow S^2(\sqrt{k}) \), that is

\[
\varphi(z^1, z^2) = \frac{1}{k} (2z^1 z^2, |z^1|^2 - |z^2|^2)
\]

induces a proper k-harmonic map.

Let \( n_1, n_2 \) be two positive integers such that \( n = n_1 + n_2 \) and \( r_1, r_2 \) be two positive real numbers such that \( r_1^2 + r_2^2 = 1 \). Let \( \varphi_j : (M_j, g_j) \rightarrow S^{n_j}(r_j) \) (\( j = 1, 2 \)) be harmonic Riemannian submersions, and \( \phi = i \circ (\varphi_1 \times \varphi_2) \), where \( i : S^{n_1}(r_1) \times S^{n_2}(r_2) \rightarrow S^{n+1} \) is the canonical inclusion.

C. Oniciuc showed the following.

**Theorem 4.6 (\text{[1]}).** The map \( \phi \) is a proper biharmonic submersion if and only if

\[
r_1 = r_2 = \frac{1}{\sqrt{k}} \quad \text{and} \quad n_1 \neq n_2.
\]

We consider general case, and get the next results.

**Theorem 4.7.** The map \( \phi \) is a proper k-harmonic submersion if and only if

\[
\left( \frac{r_2}{r_1} n_1 + \frac{r_1}{r_2} n_2 \right)^2 - n \left( \frac{r_2}{r_1} n_1 + \frac{r_1}{r_2} n_2 \right) - (k - 2) \left( \frac{r_1}{r_2} n_2 - \frac{r_2}{r_1} n_1 \right)^2 = 0,
\]

and

\[
\frac{r_1}{r_2} n_2 - \frac{r_2}{r_1} n_1 \neq 0.
\]

Proof. Let

\[
\xi(p) = \left( \frac{r_2}{r_1} p_1, -\frac{r_1}{r_2} p_2 \right),
\]

where \( p = (p_1, p_2) \) \( \in S^{n_1}(r_1) \times S^{n_2}(r_2) \). Then \( \xi \) is a unit section in the normal bundle of \( S^{n_1}(r_1) \times S^{n_2}(r_2) \) in \( S^{n+1} \). By a straightforward computation we obtain

\[
\tau(\phi) = \frac{r_1}{r_2} n_2 - \frac{r_2}{r_1} n_1 \xi,
\]

\[
\tau_2(\phi) = \frac{r^2 - r_1^2}{r_1 r_2} |\tau(\phi)|^2 \xi.
\]

When, \( k = 3, 4, \cdots \),

\[
\tau_k(\phi) = \left( \frac{r_2}{r_1} n_1 + \frac{r_1}{r_2} n_2 \right)^k \frac{r_1}{r_2} n_2 - \frac{r_2}{r_1} n_1 \}
\]

\[
\left( \left( \frac{r_2}{r_1} n_1 + \frac{r_1}{r_2} n_2 \right)^2 - n \left( \frac{r_2}{r_1} n_1 + \frac{r_1}{r_2} n_2 \right) - (k - 2) \left( \frac{r_1}{r_2} n_2 - \frac{r_2}{r_1} n_1 \right)^2 \right) \xi,
\]
Therefore, we have the theorem.

\[ \therefore \]

**Corollary 4.8.** i) If \( n_2 = n_1 \),

1) When, \( k = 2, 3, 4 \)

There are no proper \( k \)-harmonic submersion.

2) When, \( k = 5, 6, \cdots \)

The map \( \phi \) is a proper \( k \)-harmonic submersion if and only if

\[
\begin{align*}
    r_1 &= \sqrt{\frac{1 + \sqrt{k-4}}{2}}, \\
    r_2 &= \sqrt{\frac{1 - \sqrt{k-4}}{2}},
\end{align*}
\]

or

\[
\begin{align*}
    r_1 &= \sqrt{\frac{1 - \sqrt{k-4}}{2}}, \\
    r_2 &= \sqrt{\frac{1 + \sqrt{k-4}}{2}}.
\end{align*}
\]

**Example 4.9.** i) If \( n_2 = 2n_1 \)

The map \( \phi \) is a proper 3-harmonic submersion if and only if

\[
\begin{align*}
    r_1 &= \sqrt{\frac{3^{1/2}(81\sqrt{19} + 197\sqrt{3})^{3/2} + 113^{1/2}(81\sqrt{19} + 197\sqrt{3})^{3/2} - 14\sqrt{3}}{3^{1/2}(81\sqrt{19} + 197\sqrt{3})^{3/2}}}, \\
    r_2 &= \sqrt{\frac{-3^{1/2}(81\sqrt{19} + 197\sqrt{3})^{3/2} + 163^{1/2}(81\sqrt{19} + 197\sqrt{3})^{3/2} + 14\sqrt{3}}{3^{1/2}(81\sqrt{19} + 197\sqrt{3})^{3/2}}},
\end{align*}
\]

\[ \cdots \text{etc.} \]

ii) If \( n_2 = cn_1 \), (\( c = 2, 3, \cdots \))

The map \( \phi \) is a proper 4-harmonic submersion if and only if

\[
\begin{align*}
    r_1 &= \sqrt{\frac{(c-1)^{3/2} + (c+3)C^{3/2} + (c-1)^{3/2}C^{3/2}}{4C^{3/2}(c+1)}}, \\
    r_2 &= \sqrt{\frac{-(c-1)^{3/2} + (3c+1)C^{3/2} - (c-1)^{3/2}C^{3/2}}{4C^{3/2}(c+1)}},
\end{align*}
\]

where, \( C = c^2 + 4\sqrt{c(c+1)} + 6c + 1 \).

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