KOBAYASHI-ROYDEN PSEUDOMETRIC VS. LEMPERT FUNCTION

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Abstract. We give an example showing that the Kobayashi-Royden pseudometric for a pseudoconvex domain is, in general, not the derivative of the Lempert function.

Let $\mathbb{D} \subset \mathbb{C}$ be the open unit disc. Fix a domain $D \subset \mathbb{C}^n$. We recall the definitions of the Lempert function $l_D$ and the Kobayashi-Royden pseudometric $\kappa_D$ of $D$:

$$l_D(z, w) = \inf \{|\alpha| : \exists \varphi \in O(D, D) : \varphi(0) = z, \varphi(\alpha) = w\},$$

$$\kappa_D(z; X) = \inf \{|\alpha| : \exists \varphi \in O(D, D) : \varphi(0) = z, \alpha \varphi'(0) = X\},$$

where $z, w \in D$ and $X \in \mathbb{C}^n$.

By a result of M.-Y. Pang (see [7]), the Kobayashi-Royden metric is the "derivative" of the Lempert function for taut domains in $\mathbb{C}^n$ (such domains are pseudoconvex). More precisely, one can show that if $D \subset \mathbb{C}^n$ is a taut domain (i.e. $O(D, D)$ is a normal family), then

$$\kappa_D(z; X) = \lim_{C^* \ni t \to 0, z' \to z, X' \to X} \frac{l_D(z', z' + tX')}{|t|}$$

($C^* := \mathbb{C} \setminus \{0\}$). For a more general result see [4]. There it is also proved that

$$\kappa_D(z; X) \geq \mathcal{D}l_D(z; X) := \limsup_{C^* \ni t \to 0, z' \to z, X' \to X} \frac{l_D(z', z' + tX')}{|t|}$$

for any domain $D \subset \mathbb{C}^n$. Note that there is a bounded pseudoconvex domain $D \subset \mathbb{C}^2$ containing the origin such that $\lim_{C^* \ni t \to 0} \frac{l_D(0, tX)}{|t|}$ does not exist (cf. [9, Example 4.2.10]), where $X := (1, 1)$. Therefore, taking $\limsup$ in the previous definition is needed.

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The aim of this note is to show that, in general, the inequality
\[ \kappa_{D}(z; X) \geq \tilde{D}l_{D}(z; X) := \limsup_{C^* \ni t \to 0} \frac{l_{D}(z, z + tX)}{|t|}. \]
is a strict one.

Denote by \( \mathcal{M}_3 \) the set of all \( 3 \times 3 \) complex matrices and by \( \Omega_3 \subset \mathbb{C}^9 \) the spectral unit ball, i.e. the set of all matrices from \( \mathcal{M}_3 \) with all their eigenvalues in \( D \).

For a matrix \( C \in \mathcal{M}_3 \) with eigenvalues \( \lambda_1, \lambda_2, \lambda_3 \), we define
\[ \sigma(C) = (\lambda_1 + \lambda_2 + \lambda_3, \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1, \lambda_1\lambda_2\lambda_3) \in \mathbb{C}^3. \]
Recall that \( G_3 := \sigma(\Omega_3) \) is the so-called symmetrized three-disc. We will need that \( G_3 \) is a taut domain (even hyperconvex, see e.g. [1]).

Put
\[ A := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad B_t := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 3t & \omega^2 \end{pmatrix}, \quad t \in \mathbb{C}, \]
where \( \omega := e^{2\pi i/3} \). Set \( B := B_0 \). Now we can formulate our result.

**Proposition 1.** (a) \( \kappa_{\Omega_3}(A; B) > 0 = \tilde{D}l_{\Omega_3}(A; B) \).

(b) Moreover, let \( (t_j)_j \subset \mathbb{C}^*, (C_j)_j \subset \mathcal{M}_3 \) \( (C_j = (c_{k,j}^j)) \) be such that \( t_j \to 0, C_j \to B \), and \( \liminf_{j \to \infty} |c_{3,2}^j|/|t_j - 3| > 0 \). Then
\[ \lim_{j \to \infty} \frac{l_{\Omega_3}(A, A + t_jC_j)}{|t_j|} = 0. \]

Since \( \kappa_D \) and \( l_D \) have the product property, it follows that, in general, the inequality \( \frac{2}{2} \) is strict for pseudoconvex domains in \( \mathbb{C}^n \) for any \( n \geq 9 \). In fact, the proof below shows that \( \tilde{D}l_{\Omega_3}(A; B) = 0 \), where \( \Omega_3 \) is the set of all traceless matrices in \( \Omega_3 \). So the inequality in \( \frac{2}{2} \) is strict for the pseudoconvex domain \( \tilde{\Omega}_3 \subset \mathbb{C}^8 \). This remark is due to Pascal J. Thomas.

**Problem.** It would be interesting to find such examples also in lower dimensions, as well as to see if, in general, the inequality \( \frac{1}{1} \) is strict (as it is conjectured in [4]).

Note that the condition in Proposition\( \frac{1}{1} \)(b) implies that the matrices \( A + t_jC_j \) are cyclic for large \( j \) (which is, in fact, what we need in the proof). We point out that without the \( \liminf \)-condition the claim in Proposition\( \frac{1}{1} \)(b) might not hold. Indeed, we have the following result.
Example 2. \( 1 = \kappa_{\tilde{\Omega}_3}(A;B) = \lim_{t \to 0} l_{\tilde{\Omega}_3}(A, A + tB) / |t| \). In particular,
\[
1 = \kappa_{\tilde{\Omega}_3}(A;B) = \kappa_{\Omega_3}(A;B) = Dl_{\tilde{\Omega}_3}(A;B) = Dl_{\Omega_3}(A;B).
\]

Before we prove Proposition \([1]\) we need the following preparation which is based on \([8, Proposition 4.1]\). Recall that \( M \in \mathcal{M}_3 \) is said to be cyclic if \( M \) has a cyclic vector, i.e. \( \text{span}(v, Mv, M^2v) = \mathbb{C}^3 \) for some \( v \in \mathbb{C}^3 \); for many equivalent properties see e.g. \([2]\).

Lemma 3. Let \( M \in \Omega_3 \) be cyclic and \( \varphi \in \mathcal{O}(\mathbb{D}, \mathbb{D}_3) \) be such that \( \varphi(0) = 0 \) and \( \varphi(\alpha) = \sigma(M) (\alpha \in \mathbb{D}) \). Then there exists a \( \psi \in \mathcal{O}(\mathbb{D}, \Omega_3) \) satisfying \( \psi(0) = A, \psi(\alpha) = M \) and \( \varphi = \sigma \circ \psi \) if and only if \( \varphi'_3(0) = 0 \).

In particular,
\[
l_{\Omega_3}(A, M) = \inf \{ |\alpha| : \exists \varphi \in \mathcal{O}(\mathbb{D}, \mathbb{D}_3) : \varphi(0) = 0, \varphi(\alpha) = \sigma(M), \varphi'_3(0) = 0 \}
\]
and (since \( \mathbb{D}_3 \) is a taut domain) there is an extremal disc for \( l_{\Omega_3}(A, M) \).

For the convenience of the Reader we give the proof.

Proof. If such a \( \psi \) exists, then straightforward calculations show that \( \varphi'_3(0) = (\sigma_3 \circ \psi)'(0) = 0 \).

Conversely, assume that \( \varphi'_3(0) = 0 \). Put
\[
\tilde{\psi}(\zeta) := \begin{pmatrix}
0 & \zeta & 0 \\
0 & 0 & 1 \\
\varphi_3(\zeta) / \zeta & -\varphi_2(\zeta) & \varphi_1(\zeta)
\end{pmatrix}, \quad \zeta \in \mathbb{D}.
\]
Then \( \tilde{\psi}(0) = A \) and \( \varphi = \sigma \circ \tilde{\psi} \). Note also that \((0,0,1)\) is a cyclic vector for \( \tilde{\psi}(\zeta) \) if \( \zeta \neq 0 \). So \( \tilde{\psi}(\alpha) \) is a cyclic matrix with the same spectrum as the cyclic matrix \( M \) and hence they are conjugate (cf. \([2]\)). It remains to write \( M \) in the form \( M = e^{-S} \tilde{\psi}(\alpha) e^S \) for some \( S \in \mathcal{M}_3 \) and to set \( \psi(\zeta) = e^{-\zeta S / \alpha} \tilde{\psi}(\zeta) e^{\zeta S / \alpha} \).

Now we are able to present the proof of Proposition \([1]\).

Proof of Proposition \([1]\). In virtue of Example \([2]\) we have only to verify that
\[
\lim_{j \to \infty} l_{\Omega_3}(A, A + t_j C_j) / |t_j| = 0
\]
under the above condition on the \( c_{3,2}^j \).

**STEP 1.** First we prove that the \( \lim\inf \)-condition implies that \( A + t_j C_j \) are cyclic matrices for sufficiently large \( j \)'s. Assume that all \( (\text{otherwise take an appropriate subsequence}) \ A + t_j C_j \) are non cyclic.
matrices. Therefore, their minimal polynomials are of degree less than 3 (cf. [2]). So their degrees are equal to 2 for sufficiently large \( j \). Then
\[
(\lambda + t_jC_j)^2 + x_j(\lambda + t_jC_j) + y_jE = 0, \quad j \in \mathbb{N},
\]
where \( x_j, y_j \in \mathbb{C} \), and \( E \) denotes the unit matrix in \( \mathcal{M}_3 \). So we get 9 equations; each of them is denoted by \( E_{k,\ell}^j \), where the indices \( k \) and \( \ell \) denote the row and the column, respectively. Looking at equation \( E_{2,3}^j \) we get \( x_j/t_j \rightarrow 1 \). Putting this into equation \( E_{1,1}^j \) leads to \( y_j/t_j^2 \rightarrow -2 \). Finally, equation \( E_{2,2}^j \) implies that \( c_{3,2}^j/t_j \rightarrow 2 - \omega - \omega^2 = 3 \); a contradiction.

**STEP 2.** By step 1 we know that all matrices \( A + t_jC_j \) are cyclic and belong to \( \Omega_3 \) if \( j \geq j_0 \). Calculations show that
\[
\sigma(A + t_jC_j) = \sigma(t_jf_1(C_j), t_jf_2(C_j), t_j^2f_3(C_j)) =: (a_j, b_j, c_j),
\]
with \( f_1(C_j) \rightarrow 0 \), \( f_2(C_j) \rightarrow 0 \), and \( f_3(C_j) \rightarrow 0 \).

Put
\[
\varphi_j(\zeta) := (\zeta a_j/r_j, \zeta b_j/r_j, \zeta^2 c_j/r_j^2), \quad \zeta \in \mathbb{D},
\]
where \( r_j := \max\{3|a_j|, 3|b_j|, \sqrt{3|c_j|}\} \). Then \( \varphi_j \in \mathcal{O}(\mathbb{D}, \mathbb{C}_3) \) with \( \varphi_j(0) = 0 \), \( \varphi_j'(0) = 0 \), and \( \varphi_j(r_j) = \sigma(A + t_jC_j) \). Hence, by Lemma 3,
\[
l_{\Omega_3}(A, A + t_jC_j)/|t_j| \leq r_j/|t_j| \rightarrow 0.
\]
Hence the proof is finished. \( \square \)

Finally we present the proof of the example.

**Proof of Example 2** Since \( A + \zeta B \in \tilde{\Omega}_3 \) for any \( \zeta \in \mathbb{D} \), it follows that \( \kappa_{\tilde{\Omega}_3}(A; B) \leq 1 \).

By (1), it remains to show that \( \liminf_{C^- \ni \zeta \rightarrow 0} l_{\Omega_3}(A, A + tB_t) \geq 1 \).

Note that \( A + tB_t \) is similar to the matrix \( D_t = \text{diag}(t, t, -2t) \) and hence \( l_{\Omega_3}(A, A + tB_t) = l_{\Omega_3}(A, D_t) \) (use the same argument as the one at the end of the proof of Lemma 3).

Let \((t_j)_j \subset \mathbb{C}^* \), \( t_j \rightarrow 0 \), such that \( l_{\Omega_3}(A, D_{t_j})/|t_j| \rightarrow c \).

Now choose \( \psi_j = (\psi_j(kl))_{k,l=1,2,3} \in \mathcal{O}(\mathbb{D}, \Omega_3) \) such that \( \psi_j(0) = A \), \( \psi(\alpha_j) = D_{t_j} \), and \( \alpha_j/t_j = |\alpha_j|/|t_j| \rightarrow c \). Setting
\[
\varphi_j = (\varphi_{j,1}, \varphi_{j,2}, \varphi_{j,3}) := \sigma \circ \psi_j,
\]
we have the following equations:
\[
\varphi_{j,1} = \text{trace} \psi_j, \quad \varphi_{j,3} = \text{det} \psi_j,
\]
\[
\varphi_{j,2} = \psi_{j,(11)} \psi_{j,(22)} + \psi_{j,(11)} \psi_{j,(33)} + \psi_{j,(22)} \psi_{j,(33)} - \psi_{j,(12)} \psi_{j,(21)} - \psi_{j,(13)} \psi_{j,(31)} - \psi_{j,(23)} \psi_{j,(32)}.
\]
Then straightforward calculations show that \( \varphi'_{j,3}(0) = 0 \) and
\[
\varphi'_{j,3}(\alpha_j) - t_j \varphi'_{j,2}(\alpha_j) + t_j^2 \varphi'_{j,1}(\alpha_j) = 0.
\]
Writing
\[
\varphi_j(\zeta) = (\zeta \theta_{j,1}(\zeta), \zeta \theta_{j,2}(\zeta), \zeta^2 \theta_{j,3}(\zeta)),
\]
the last condition becomes
\[
(3) \quad t_j^3 = \alpha_j^2 (\alpha_j \theta'_{j,3}(\alpha_j) - t_j \theta'_{j,2}(\alpha_j) + t_j^2 \theta'_{j,1}(\alpha_j))
\]
(use that \( \theta_{j,1}(\alpha_j) = 0 \), \( \theta_{j,2}(\alpha_j) = -3t_j^2/\alpha_j \) and \( \theta_{j,3}(\alpha_j) = -2t_j^3/\alpha_j^2 \)).

Since \( G_3 \) is a taut domain, passing to a subsequence, we may assume that \( \varphi_j \to \varphi = (\zeta \rho_1, \zeta^2 \rho_2, \zeta^3 \rho_3) \in \mathcal{O}(\mathbb{D}, G_3) \) and \( \rho_1(0) = 0 \). Then the equation (3) implies that
\[
\rho_3(0) = k^3 + k \rho_2(0),
\]
where \( k := 1/c \).

It follows by [3] Proposition 1] (see also [1], Proposition 16] that \( h_{G_3}(z) := \max \{ |\lambda| : \lambda^3 - z_1 \lambda^2 + z_2 \lambda - z_3 = 0 \} \) is a (logarithmically) plurisubharmonic function with \( G_3 = \{ z \in \mathbb{C}^3 : h_{G_3}(z) < 1 \} \). In fact, \( h_{G_3} \) is the Minkowski function of the (1, 2, 3)-balanced domain \( G_3 \). Since
\[
|h_{G_3}(\rho_1(\zeta), \rho_2(\zeta), \rho_3(\zeta)) = h_{G_3}(\varphi(\zeta)) < 1, \quad \zeta \in \mathbb{D},
\]
the maximum principle for plurisubharmonic functions implies that \( h_{G_3}(\rho_1, \rho_2, \rho_3) \leq 1 \) on \( \mathbb{D} \). In particular, \( h_{G_3}(\rho_1(0), \rho_2(0), \rho_3(0)) \leq 1. \) Therefore, all zeros of the polynomial \( P(\lambda) := \lambda^3 - \rho_1(0) \lambda^2 + \rho_2(0) \lambda - \rho_3(0) \), lie in \( \mathbb{D} \). Note that \( P(\lambda) = (\lambda - k)(\lambda^2 + k \lambda + k^2 + \rho_2(0)) \); hence \( c \geq 1. \)

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