Critical amplitudes and mass spectrum of the 2D Ising model in a magnetic field

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Abstract

We compute the spectrum and several critical amplitudes of the two dimensional Ising model in a magnetic field with the transfer matrix method. The three lightest masses and their overlaps with the spin and the energy operators are computed on lattices of a width up to $L_1 = 21$. In extracting the continuum results we also take into account the corrections to scaling due to irrelevant operators. In contrast with previous Monte Carlo simulations our final results are in perfect agreement with the predictions of S-matrix and conformal field theory. We also obtain the amplitudes of some of the subleading corrections, for which no S-matrix prediction has yet been obtained.

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1 Introduction

In these last years there has been much progress in the study of 2d spin models in the neighbourhood of critical points. The scaling limit of such models is described in general by the action $A$ obtained by perturbing the conformal field theory (CFT) which describes the critical point with one (or more) of the relevant operators which appear in the spectrum of the CFT.

$$A = A_0 + \lambda \int d^2x \phi(x)$$

where $A_0$ is the action of the CFT at the critical point [1] and $\phi(x)$ is the perturbing operator. Few years ago A. Zamolodchikov in a seminal paper [2] suggested that in some special cases these perturbed theories are equivalent to relatively simple quantum field theories [3] whose mass spectrum and S-matrix are explicitly known. Later it was realized that these theories had a deep connection with the Dynkin diagrams of suitable Lie algebras and, from the exact knowledge of the S-matrix, several other informations, and in particular some critical amplitudes were obtained (for a review, see for instance [4]). While these results have formally the status of conjectures, they successfully passed in these last years so many tests that they are now universally accepted. The most fascinating example of these S-matrix models is the Ising model perturbed by an external magnetic field, which is also the model which was originally studied by Zamolodchikov in [2]. This model is highly non-trivial. Its spectrum contains 8 stable scalar particles, all with different masses. Both the masses and the entries of the S-matrix are based of the numerology of the $E_8$ exceptional Lie algebra. In particular the ratio between the first two masses is predicted to be the “golden ratio” $m_2/m_1 = 2 \cos(\pi/5)$. The simplest realization of this QFT is the 2d Ising model at $\beta = \beta_c$ in presence of an external magnetic field $h$. However there are several other models which belong to the same universality class. In particular, the first numerical check of the predictions of [2] was performed on the Ising quantum spin chain [5] in which the first few states of the spectrum were precisely observed. Another interesting realization was presented in [6, 7], where the dilute $A_3$ IRF (Interaction Round a Face) model was solved exactly and the predicted spectrum of states was found [7].

Despite these successes, little progress has been achieved in testing Zamolodchikov’s proposal directly in the 2d Ising spin model. Even more, it is exactly
for this model that one faces the only existing discrepancy between Zamolodchikov’s results and Monte Carlo simulations.

The Ising spin model in a magnetic field was studied numerically in [8, 9]. In both papers the authors studied the spin-spin correlator and did not find the spectrum predicted by [2]. Their data were compatible with the presence in the spectrum of only the lowest mass state. The explanation suggested in [8, 9] was that probably the higher masses had a negligible overlap amplitude with the spin operator. However, later, in [10] these overlaps were evaluated explicitly in the S-matrix framework and turned out to be of the same order of magnitude as the overlap with the lowest mass state.

In this paper we shall address this problem. We shall show that Zamolodchikov’s proposal (and the calculations of [10]) is correct also in the case of the 2d Ising spin model and that the apparent disagreement was due to the fact that it is very difficult to extract a complex spectrum from a multi-exponential fit to the spin-spin correlator. We have been prompted to this explanation by another example that we recently studied, in which exactly the same phenomenon happens: the 3d Ising model [11]. In this case also, a multi-exponential fit to the spin-spin correlator seems to indicate the presence of a single state in the spectrum, while using a suitable variational method and diagonalizing a set of improved operators one can clearly see the rich spectrum of the model.

While in previous numerical works [8, 9] the model was studied by using Monte Carlo simulations we tried in the present paper a different approach based on the exact diagonalization of the transfer matrix.

This approach has various advantages: it gives direct access to the mass spectrum of the model and allows to obtain numerical estimates of various quantities with impressively small uncertainties. However it has the serious drawback that only transfer matrices of limited size can be handled and it is difficult to extract from them the continuum limit results in which we are interested. During the last years various strategies have been elaborated to attack this problem, but all of them are affected by systematic errors whose size is in general unknown.

In this paper we propose a new approach based on the fact that, by using the exact solution of the Ising model at the critical point, one can construct very precise expansions for the scaling functions in powers of the perturbing field. More precisely, thanks to the knowledge of the spectrum of the model, it is possible to list all the irrelevant fields which may appear in the effective
Hamiltonian and select them on the basis of the symmetry properties of the observables under study.

Our strategy could be summarized as follows.

- Choose a set of values of $h$ for which the correlation length is much smaller than the maximum lattice size that we can study. Diagonalize the transfer matrix for various values of the transverse size of the lattice and extract all the observables of interest.

- Extrapolate the numbers thus obtained to the thermodynamic limit. Thanks to the very small correlation length, the finite size behaviour is dominated by a rapidly decreasing exponential and the thermodynamic limit can be reached with very small uncertainties (we list in a set of tables at the end of the paper the results that we obtained in this way).

- Construct for each observable the scaling function keeping the first 7 or 8 terms in the expansion in powers of the perturbing field.

- Fit the data with these truncated scaling functions. By varying the number of input data and of subleading terms used in the scaling functions we may then obtain a reliable estimate of the systematic deviations involved in our estimates (see the discussion in sect. 6).

Which are the observables of interest mentioned above?

Usually, when looking at the scaling regime of statistical models one can study only adimensional amplitude ratios which are the only quantities which, thanks to universality, do not depend on the details of the lattice models, but only on the features of the underlying QFT. However the Ising model can be solved exactly at the critical point also on the lattice and explicit expressions for the spin-spin and energy-energy correlators are known. This allows to write the explicit expression in lattice units of the amplitudes evaluated in the framework of the S-matrix theory. Thus one is able to predict not only adimensional amplitude ratios but also the values of the critical amplitudes themselves. This greatly enhances the predictive power of the S-matrix theory and makes much more stringent the numerical test that we perform.

\[1\text{In particular we decided to keep the ratio } \frac{\xi}{L_0} < 0.1. \text{ This means that we only studied values of } h \text{ for which the correlation length was smaller than two lattice spacings.}\]
The final result of our analysis is that all the observables that we can measure perfectly agree with the S-matrix predictions.

In particular we obtain very precise estimates for the first three masses, for several critical amplitudes and, what is more important, for the overlap amplitude of the first two masses with the spin and energy operators, a result which had never been obtained before.

We also measure the amplitude of some of the subleading corrections in the scaling functions, for which no S-matrix prediction exists for the moment. In particular we found that the amplitude of the corrections due to the energy momentum tensor in translationally invariant observables is compatible with zero.

This paper is organized as follows. In sect. 2 we introduce the model in which we are interested, collect some known results from S-matrix theory and finally give the translation in lattice units of the critical amplitudes evaluated in the S-matrix framework. In sect. 3 we construct the scaling functions. This only requires the use of very simple and well known results of Conformal Field Theory. Notwithstanding this it turns out to be a rather non trivial exercise. Since it could be a result of general utility (it could be extended to other models for which the CFT solution is known or to other quantities of the Ising model in a magnetic field that we have not studied in the present paper) instead of simply giving the results, we derived the scaling functions explicitly and tried to give as much details as possible. Sect. 4 is devoted to a description of the transfer matrix method. In sect. 5 we deal with the thermodynamic limit while in sect. 6 we analyze the transfer matrix results and give our best estimates for the critical amplitudes in which we are interested. Sect. 7 is devoted to some concluding remarks.

To help the reader to reproduce our analysis (or to follow some alternative fitting procedure) we list in four tables at the end of the paper the data that we obtained with the transfer matrix approach.

## 2 Ising model in a magnetic field

In this section we shall review the existing theoretical informations on the Ising model in a magnetic field. This will require four steps. First (in sect. 2.1) we shall define the lattice version of the model, discuss its action and define the observables in which we shall be interested in the following.
Then (in sect. 2.2) we shall turn to the continuum version of the theory, described in the present case by the action:

\[ \mathcal{A} = \mathcal{A}_0 + h \int d^2 x \sigma(x) \]  

(2)

where \( \sigma(x) \) is the perturbing operator. In particular we shall discuss, within the framework of the renormalization group, the expected scaling behaviour of the various quantities of interest and define the corresponding critical amplitudes. In sect. 2.3 we shall use the knowledge of the S-matrix of the model to obtain the value of some of the amplitudes of interest by using the Thermodynamic Bethe Ansatz (TBA) and the form factor approach. Finally in sect. 2.4 we shall turn back to the lattice model and show how the continuum results can be translated in lattice units.

2.1 The lattice model

The lattice version of the Ising model in a magnetic field is defined by the partition function

\[ Z = \sum_{\sigma_i = \pm 1} e^{\beta(\sum_{\langle n,m \rangle} \sigma_n \sigma_m + H \sum_n \sigma_n)} \]  

(3)

where the field variable \( \sigma_n \) takes the values \( \{\pm 1\} \); \( n \equiv (n_0, n_1) \) labels the sites of a square lattice of size \( L_0 \) and \( L_1 \) in the two directions and \( \langle n, m \rangle \) denotes nearest neighbour sites on the lattice. In our calculations with the transfer matrix method we shall treat asymmetrically the two directions. We shall denote \( n_0 \) as the “time” coordinate and \( n_1 \) as the space one. The number of sites of the lattice will be denoted by \( N \equiv L_0 L_1 \). In the thermodynamic limit both \( L_0 \) and \( L_1 \) must go to infinity and only in this limit we may recover the results of the continuum theory. In our actual calculations with the transfer matrix method we shall study finite values of \( L_1 \) and then extrapolate the results to infinity. This extrapolation induces systematic errors which are the main source of uncertainty of our results, since the rounding errors in the transfer matrix diagonalization are essentially negligible. In sect. 6 below, we shall discuss these systematic errors and estimate their magnitude.

In order to select only the magnetic perturbation, the coupling \( \beta \) must be fixed to its critical value

\[ \beta = \beta_c = \frac{1}{2} \log (\sqrt{2} + 1) = 0.4406868... \]
by defining $h_l = \beta_c H$ we end up with

$$Z(h_l) = \sum_{\sigma_i = \pm 1} e^{\beta_c \sum_{(m,n)} \sigma_m \sigma_m + h_l \sum_n \sigma_n}.$$ \hfill (4)

$h_l$ denotes the lattice discretization of the magnetic field $h$ which appears in the continuum action eq. (2). It must be, for symmetry reasons, an odd function of $h$.

### 2.1.1 Lattice operators

It is useful to define the lattice analogous of the spin and energy operators of the continuum theory. They will correspond to linear combinations of the relevant and irrelevant operators of the continuum theory with suitable symmetry properties with respect to the $Z_2$ symmetry of the model (odd for the spin operator and even for the energy one). Near the critical point this linear combination will be dominated by the relevant operator and the only remaining freedom will be a conversion constant relating the continuum and lattice versions of the two operators (we shall find this constants in sect. 2.5).

The simplest choices for these lattice analogous are

- **Spin operator**

  $$\sigma_l(x) \equiv \sigma_x$$ \hfill (5)

  i.e. the operator which associates to each site of the lattice the value of the spin at that site.

- **Energy operator**

  $$\epsilon_l(x) \equiv \frac{1}{4} \sigma_x \left( \sum_{y \text{n.n.} x} \sigma_y \right) - \epsilon_b$$ \hfill (6)

  where the sum runs over the four nearest neighbour sites $y$ of $x$. $\epsilon_b$ represents a constant “bulk” term which we shall discuss below.

The index $l$ indicates that these are the lattice discretizations of the continuous operators. We shall denote in the following the normalized sum over all the sites of these operators simply as

$$\sigma_l \equiv \frac{1}{N} \sum_x \sigma_l(x) \quad \quad \epsilon_l \equiv \frac{1}{N} \sum_x \epsilon_l(x) .$$ \hfill (7)
2.1.2 Observables

- **Free Energy**
  The free energy is defined as
  \[
f(h_l) \equiv \frac{1}{N} \log(Z(h_l)) \quad .
\]
  \[\text{(8)}\]
  It is important to stress that \(f(h_l)\) is composed by a “bulk” term \(f_b(h_l)\) which is an analytic even function of \(h_l\) and by a “singular” part \(f_s(h_l)\) which contains the relevant informations on the theory as the critical point is approached. The continuum theory can give informations only on \(f_s\). The value of \(f_b(0)\) can be obtained from the exact solution of the lattice model at \(h_l = 0, \beta = \beta_c\) (see [12])
  \[
f_b = \frac{2G}{\pi} + \frac{1}{2} \log 2 = 0.9296953982... \quad \text{(9)}
\]
  where \(G\) is the Catalan constant.

- **Magnetization**
  The magnetization per site \(M(h_l)\) is defined as
  \[
  M(h_l) \equiv \frac{1}{N} \frac{\partial}{\partial h_l} \log Z(h_l) = \frac{1}{N} \langle \sum_i \sigma_i \rangle \quad .
  \]
  \[\text{(10)}\]
  Hence we have
  \[
  M(h_l) = \langle \sigma_i \rangle \quad .
  \]
  \[\text{(11)}\]

- **Magnetic Susceptibility**
  The magnetic susceptibility \(\chi\) is defined as
  \[
  \chi(h_l) \equiv \frac{\partial M(h_l)}{\partial h_l} \quad .
  \]
  \[\text{(12)}\]

- **Internal Energy**
  We define the internal energy density \(\hat{E}(h_l)\) as
  \[
  \hat{E}(h_l) \equiv \frac{1}{2N} \langle \sum_{(n,m)} \sigma_n \sigma_m \rangle \quad .
  \]
  \[\text{(13)}\]
As for the free energy, also in this case one has a bulk analytic contribution $E_b(h_l)$ which is an even function of $h_l$. Let us define $\epsilon_b \equiv E_b(0)$. The value of $E_b(0)$ can be easily evaluated (for instance by using Kramers-Wannier duality) to be $\epsilon_b = \frac{1}{\sqrt{2}}$. Let us define $E(h_l) \equiv \hat{E}(h_l) - \epsilon_b$, we have

$$E(h_l) = \frac{1}{2N} \langle \sum_{\langle n,m \rangle} \sigma_n \sigma_m \rangle - \frac{1}{\sqrt{2}} . \quad (14)$$

Hence we have

$$E(h_l) = \langle \epsilon_l \rangle . \quad (15)$$

As usual the internal energy can also be obtained by deriving the free energy with respect to $\beta$. However it is important to stress that, due to the magnetic perturbation (see eq.(3)) in performing the derivative we also extract from the Boltzmann factor a term proportional to $H\sigma_l$. Hence we have:

$$\hat{E}(h_l) = \frac{1}{2N} \frac{\partial}{\partial \beta} (\log Z(h_l)) - \frac{h_l}{2\beta_c} \sigma_l . \quad (16)$$

This observation will play an important role in the following.

### 2.1.3 Correlators

We are interested in the spin-spin and in the energy-energy connected correlators defined as

$$G_{\sigma,\sigma}(r) \equiv \langle \sigma_l(0)\sigma_l(r) \rangle - \langle \sigma_l \rangle^2 \equiv \langle \sigma_l(0)\sigma_l(r) \rangle_c \, , \quad (17)$$

$$G_{\epsilon,\epsilon}(r) \equiv \langle \epsilon_l(0)\epsilon_l(r) \rangle - \langle \epsilon_l \rangle^2 \equiv \langle \epsilon_l(0)\epsilon_l(r) \rangle_c \, . \quad (18)$$

For a nonzero magnetic field these correlators are very complicated, unknown, functions of $h$ and $r$, however a good approximation in the large distance regime $r \to \infty$ is\footnote{For a discussion of the limits of this approximation and of the corrections which must be taken into account when the short distance regime is approached see \cite{13}.}

$$\frac{G_{\sigma,\sigma}(r)}{\langle \sigma_l \rangle^2} = \sum_i \frac{|F_i^\sigma(h)|^2}{\pi} K_0(m_i(h)r) \quad (19)$$
where the sum is over the low laying single particle states of the spectrum, $m_i(h)$ denotes their mass the functions $F_i^{\sigma}(h)$ their overlap with the $\sigma$ operator. Similarly we have

$$
\frac{G_{ee}(r)}{(\epsilon_i)^2} = \sum_i \frac{|F_i^{\sigma}(h)|^2}{\pi} K_0(m_i(h)r)
$$

(20)

where the spectrum is the same as for the spin-spin correlator but the overlap constants are different.

A particular role is played by the lowest mass $m_1$ which gives the dominant contribution in the large distance regime. Its inverse corresponds to the (exponential) correlation length $\xi$ of the model and sets the scale for all dimensional quantities in the model. In particular the “large distance regime” mentioned few lines above means “large with respect to the correlation length”.

In the approximation of eqs. (19) and (20) one is neglecting the cut-type contributions which appear above the two-particle threshold i.e. at twice the value of $m_1$. For this reason we shall concentrate in the following only on the three first states of the spectrum which are the only ones which lie below such threshold (see eq. (19) below).

### 2.1.4 Time slice correlators

It is very useful to study the zero momentum projections of the above defined correlators. They are commonly named time slice correlators. The magnetization of a time slice is given by

$$
S_{n_0} \equiv \frac{1}{L_1} \sum_{n_1} \sigma_{(n_0,n_1)}
$$

(21)

The time slice correlation function is then defined as

$$
G_{\sigma\sigma}^0(\tau) \equiv \sum_{n_0} \{ \langle S_{n_0} S_{n_0+\tau} \rangle - \langle S_{n_0} \rangle^2 \}
$$

(22)

where the index 0 indicates that this is the zero momentum projection of the original correlator. Starting from eq. (19) it is easy to show that in the large $\tau$ limit $G_{\sigma\sigma}^0(\tau)$ behaves as

$$
\frac{G_{\sigma\sigma}^0(\tau)}{(\sigma_i)^2} = \sum_i \frac{|F_i^{\sigma}(h)|^2}{m_i(h) L_1} e^{-m_i(h)|\tau|}
$$

(23)
A similar result, with the obvious modifications, holds also for $G^0_{\epsilon,\epsilon}$.

2.2 Critical behaviour

In this section we discuss the critical behaviour of the model by using standard renormalization group methods, keeping in the expansions only the first order in the perturbing field. Both the results and the analysis are well known and can be found in any textbook. We report it here since it will serve us as a starting point for the more refined analysis which we shall perform in sect. 3 below.

2.2.1 Critical indices

The starting point of the renormalization group analysis is the singular part of the free energy $f_s(t, h)$ (where $t$ is the reduced temperature). Standard renormalization group arguments (see for instance [14]) allow to write $f_s$ in terms of a suitable scaling function $\Phi$:

$$
\Phi \left( \frac{u_t}{u_{t_0}} \frac{y_t}{y_{t_0}} \right)
$$

where $u_{t_0}$ and $u_{h_0}$ are reference scales that depend on the model. $u_t$ and $u_h$ denote the scaling variables associated to the magnetic and energy operators respectively and $y_t$, $y_h$ are their RG-exponents. $u_t$ and $u_h$ do not exactly coincide with $t$ and $h$ but are instead analytic functions of them. The only constraint is that they must respect the $\mathbb{Z}_2$ parity of $t$ and $h$. Near the critical point we may suitably rescale $\Phi$ so as to identify $u_t = t$ and $u_h = h$. Thus, setting $t = 0$ we immediately obtain the asymptotic critical behaviour of $f_s$

$$
f_s \propto |h|^{d/y_h}.
$$

Taking the derivative with respect to $h$ (or $t$) and then setting $t = 0$ we can obtain from eq. (24) also the asymptotic critical behaviour of the other observables in which we are interested

$$
M \propto |h|^{d/y_h - 1}
$$

$$
\chi \propto |h|^{d/y_h - 2}
$$
From the exact solution of the Ising model at the critical point we know that \( y_h = \frac{15}{8} \) and \( y_t = 1 \). Inserting these values in the above expressions we find

\[
E \propto |h|^{(d-y_t)/y_h} .
\]  
(28)

The masses \( m_i \) have as scaling exponent, as usual, \( 1/y_h \), hence

\[
m_i \propto |h|^{\frac{8}{15}} .
\]  
(33)

Finally from the definitions of eqs. (19) and (20) we see that the overlap amplitudes behave as adimensional constants.

### 2.2.2 Critical amplitudes

In order to describe the scaling behaviour of the model we also need to know the proportionality constants in the above scaling functions. These constants are usually called critical amplitudes. Using the results collected in eqs. (29)-(32) we have the following definitions:

\[
A_f \equiv \lim_{h \to 0} f h^{-\frac{16}{15}} , \quad A_M \equiv \lim_{h \to 0} M h^{-\frac{1}{15}} , \quad A_{\chi} \equiv \lim_{h \to 0} \chi h^{\frac{14}{15}} ,
\]  
(34)

\[
A_E \equiv \lim_{h \to 0} E h^{-\frac{8}{15}} , \quad A_{m_i} \equiv \lim_{h \to 0} m_i h^{-\frac{8}{15}} ,
\]  
(35)

\[
A_F^\sigma \equiv \lim_{h \to 0} F_i^\sigma , \quad A_F^c \equiv \lim_{h \to 0} F_i^c .
\]  
(36)

Notice for completeness that in the literature (see for instance [15]) the amplitudes \( A_\chi, A_M \) and \( A_{m_1} \) are usually denoted as

\[
A_\chi \equiv \Gamma_c , \quad A_M \equiv D_c^{-\frac{1}{15}} , \quad A_{m_1} \equiv \frac{1}{\xi_c} .
\]  
(37)
We shall show in the next section that all these amplitudes can be exactly evaluated in the framework of the $S$-matrix approach. As a preliminary step let us notice that since $M$ and $\chi$ are obtained as derivatives of $f$ we have

$$A_M = \frac{16}{15} A_f, \quad A_\chi = \frac{1}{15} A_M. \quad (38)$$

### 2.2.3 Universal amplitude ratios

From the above critical amplitudes one can construct universal combinations which do not depend on the particular realization of the model. For this reason they have been widely studied in the literature. In particular there are two “classical” amplitude combinations which involve the critical amplitudes defined above (see for instance [15]). They are:

$$R_\chi \equiv \Gamma D_c B^{14}, \quad Q_2 \equiv (\Gamma/\Gamma_c)(\xi_c/\xi_0)^{\frac{1}{15}} \quad (39)$$

where we used the notations of eq. (37). $\Gamma$ and $\xi_0$ denote the critical amplitudes of the susceptibility and exponential correlation length for $h = 0$ and a small positive reduced temperature, while $B$ denotes the critical amplitude of the magnetization for $h = 0$ and a small negative reduced temperature. Notice however that, since (as we mentioned in the introduction) we are able to give the explicit relation between lattice and continuum expectation values, we are not constrained to study only universal combination but can determine exactly the various critical amplitudes.

### 2.3 S-matrix results

In 1989 A. Zamolodchikov [2] suggested that the scaling limit of the Ising Model in a magnetic field could be described by a a scattering theory which contains eight different species of self-conjugated particles $A_a, a = 1, \ldots, 8$ with masses

$$m_2 = 2m_1 \cos\frac{\pi}{5} = (1.6180339887..) m_1,$$

$$m_3 = 2m_1 \cos\frac{\pi}{30} = (1.9890437907..) m_1,$$

$$m_4 = 2m_2 \cos\frac{7\pi}{30} = (2.4048671724..) m_1.$$
where $m_1(h)$ is the lowest mass of the theory. As mentioned above it coincides with the inverse of the (exponential) correlation length. Few years later, from the knowledge of the S-matrix of the theory V. Fateev [16] obtained explicit predictions for some of the critical amplitudes defined above.

In order to evaluate the amplitudes one must first fix the normalization of the operators involved which can be set, for instance, by fixing the constant in front of the long distance behaviour of the correlators at the critical point. It is important to make explicit this normalization choice, since it will allow us, by comparing with the corresponding correlators in the lattice theory to convert explicitly the continuum results in lattice units. Following the choice of [16] we assume:

$$\langle \sigma(x)\sigma(0) \rangle = \frac{1}{|x|^\frac{1}{4}}, \quad |x| \to \infty$$

$$\langle \epsilon(x)\epsilon(0) \rangle = \frac{1}{|x|^2}, \quad |x| \to \infty. \quad (41)$$

With these conventions one finds [16]:

$$A_{m_1} = C \quad (43)$$

$$A_f = \frac{C^2}{8 (\sin \frac{2\pi}{3} + \sin \frac{2\pi}{5} + \sin \frac{\pi}{15})} \quad (44)$$

where

$$C = \frac{4 \sin \frac{\pi}{5} \Gamma \left( \frac{1}{5} \right)}{\Gamma \left( \frac{2}{3} \right) \Gamma \left( \frac{8}{15} \right)} \left( \frac{4\pi^2 \Gamma \left( \frac{3}{4} \right) \Gamma^2 \left( \frac{13}{16} \right)}{\Gamma \left( \frac{1}{4} \right) \Gamma^2 \left( \frac{3}{16} \right)} \right)^{\frac{1}{4}} = 4.40490858... \quad (45)$$

From $A_f$ one immediately obtains $A_M$ and $A_x$.  

$$m_5 = 2m_2 \cos \frac{2\pi}{15} = (2.9562952015..) m_1,$$

$$m_6 = 2m_2 \cos \frac{\pi}{30} = (3.2183404585..) m_1,$$

$$m_7 = 4m_2 \cos \frac{\pi}{5} \cos \frac{7\pi}{30} = (3.8911568233..) m_1,$$

$$m_8 = 4m_2 \cos \frac{\pi}{5} \cos \frac{2\pi}{15} = (4.7833861168..) m_1.$$
The amplitude $A_E$ requires a more complicated analysis. Its exact expression has been obtained only recently in \[17\]

\[ A_E = 2.00314... \]

Equation 46

We summarize in tab. 1 these S-matrix predictions for the critical amplitudes.

| $A_m$ | 4.40490858... |
|-------|---------------|
| $A_f$ | 1.19773338... |
| $A_M$ | 1.27758227... |
| $A_X$ | 0.08517215... |
| $A_E$ | 2.00314... |

Table 1: Critical amplitudes.

From these critical amplitudes, and using the values of $\Gamma$, $B$ and $\xi_0$ one immediately obtains the classical amplitude ratios defined above (see for instance \[18\]). They are reported in tab. 2.

| $R_x$ | 6.77828502... |
| $Q_2$ | 3.23513834... |

Table 2: Classical amplitude ratios.

Finally, the critical overlap amplitudes $A_{F^\sigma}$ and $A_{F^\epsilon}$ were evaluated in \[10, 19\]. They are reported in tab. 3 and 4.

2.4 Conversion to lattice units

While the values listed in tab. 2, 3 and 4 are universal, the amplitudes listed in tab. 1 depend on the details of the regularization scheme. Thus some further work is needed to obtain their value on the lattice. We shall denote in the following the lattice critical amplitudes with an index $l$. Thus, for instance,

\[ A_{M}^l = \lim_{h_l \to 0} \langle \sigma_l \rangle \ h_l^{-1/4} \]

Equation 47
\[ A_{F_1} = -0.64090211, \]
\[ A_{F_2} = 0.33867436, \]
\[ A_{F_3} = -0.18662854, \]
\[ A_{F_4} = 0.14277176, \]
\[ A_{F_5} = 0.06032607, \]
\[ A_{F_6} = -0.04338937, \]
\[ A_{F_7} = 0.01642569, \]
\[ A_{F_8} = -0.00303607. \]

Table 3: Critical overlap amplitudes for the spin operator.

\[ A_{F^c_1} = -3.70658437, \]
\[ A_{F^c_2} = 3.4228876, \]
\[ A_{F^c_3} = -2.3843446, \]
\[ A_{F^c_4} = 2.26840624, \]
\[ A_{F^c_5} = 1.21338371, \]
\[ A_{F^c_6} = -0.96176431, \]
\[ A_{F^c_7} = 0.45230320, \]
\[ A_{F^c_8} = -0.10584899. \]

Table 4: Critical overlap amplitudes for the energy operator.

to be compared with the continuum critical amplitude defined in eq. (34)

\[ A_M = \lim_{h \to 0} \langle \sigma \rangle h^{-\frac{1}{25}}. \]  

(48)

In order to relate the lattice results with the continuum ones we must study the relationship between the lattice operators and the continuum ones. In general the lattice operators will be given by the most general combination of continuum operators compatible with the symmetries of the lattice operator multiplied by the most general analytic functions of \( t \) and \( h \) (with a parity which is again constrained by the symmetry of the operators involved). Thus, for instance, anticipating the discussion that we shall make in sect. 3, we have

\[ \sigma_I = f_0^\sigma(t, h)\sigma + f_i(t, h)\phi_i \]  

(49)

where \( f_0^\sigma(t, h) \) and \( f_i(t, h) \) are suitable functions of \( t \) and \( h \) and with \( \phi_i \) we
denote all the other fields of the theory (both relevant and irrelevant) which respect the symmetries of the lattice.

A similar relation also holds for the energy operator:

$$\epsilon_l = g^0_\epsilon(t, h) \epsilon + g_\phi(t, h) \phi$$  \hspace{1cm} (50)$$

Finally, also $h_l$ is related to the continuum magnetic field $h$ by a relation of the type

$$h_l = b_0(t, h) h$$  \hspace{1cm} (51)$$

where $b_0(t, h)$ must be an even function of $h$.

At the first order in $t$ and $h$ these combinations greatly simplify and essentially reduce to a different choice of normalization between the continuum operators and their lattice analogous:

$$\sigma_l \equiv R_\sigma \sigma , \quad \epsilon_l \equiv R_\epsilon \epsilon , \quad h_l \equiv R_h h$$  \hspace{1cm} (52)$$

where $R_\sigma$, $R_\epsilon$ and $R_h$ are three constants which correspond to the $h \to 0$, $t \to 0$ limit of the $f^0_\sigma$, $g^0_\epsilon$ and $b_0$ functions.

If we want to compare the S-matrix results discussed in the previous section with our lattice results we must fix these normalizations\footnote{This essentially amounts to measure all the quantities in units of the lattice spacing. For this reason we can fix in the following the lattice spacing to 1 and neglect it.}. The simplest way to do this is to look at the analogous of eqs. (41,42) at the critical point (namely for $h_l = 0$) \footnote{[20]}.

In fact, if $h_l = 0$ it is possible to obtain an explicit expression for the spin-spin and energy-energy correlators (for a comprehensive review see for instance \footnote{[21]}) directly on the lattice, for any value of $\beta$. Choosing in particular $\beta = \beta_c$, and looking at the large distance behaviour of these lattice correlators we may immediately fix the normalization constants. Let us look first at $R_\sigma$.

We know from \footnote{[22]} that:

$$\langle \sigma_i \sigma_j \rangle_{h=0} = \frac{R^2_\sigma}{|r_{ij}|^{1/4}}$$  \hspace{1cm} (53)$$

where $r_{ij}$ denotes the distance on the lattice between the sites $i$ and $j$ and

$$R^2_\sigma = e^{3\xi(-1)25/24} = 0.70338...$$  \hspace{1cm} (54)$$
By comparing this result with eq. 41 we find

$$R_{\sigma} = 0.83868... .$$  \hspace{1cm} (55)

From this we can also obtain the normalization of the lattice magnetic field which must exactly compensate that of the spin operator in the perturbation term $h\sigma$. We find:

$$R_h = (R_{\sigma})^{-1} = 1.1923... .$$ \hspace{1cm} (56)

Combining these two results we obtain the value in lattice units of the constant $A_{\sigma}$

$$A_{M}^l = (R_{\sigma})^{16/15}A_{M} = 1.058... .$$ \hspace{1cm} (57)

From this one can easily obtain also $A_{f}^l$, $A_{\chi}^l$ and $A_{m_1}$.

Let us look now at $R_{\epsilon}$. In the case of the energy operator the connected correlator on the lattice, at $h_l = 0$ and for any value of $\beta$ has the following expression \[23\]:

$$\langle \epsilon_l(0)\epsilon_l(r) \rangle_c = \left( \frac{\delta}{\pi} \right)^2 \left[ K_1^2(\delta r) - K_0^2(\delta r) \right]$$ \hspace{1cm} (58)

where $K_0$ and $K_1$ are modified Bessel functions, $\delta$ is a parameter related to the reduced temperature, defined as

$$\delta = 4|\beta - \beta_c|$$ \hspace{1cm} (59)

and with the index c we denote the connected correlator (notice that thanks to the definition \[14\] no disconnected part must be subtracted at the critical point and the index c becomes redundant). This expression has a finite value in the $\delta \to 0$ limit (namely at the critical point). In fact the Bessel functions difference can be expanded in the small argument limit as

$$\left[ K_1^2(\delta r) - K_0^2(\delta r) \right] = \frac{1}{(\delta r)^2} + ...$$ \hspace{1cm} (60)

thus giving, exactly at the critical point:

$$\langle \epsilon_l(0)\epsilon_l(r) \rangle = \frac{1}{(\pi r)^2} .$$ \hspace{1cm} (61)
By comparing this result with eq. (42) we find
\begin{equation}
R_\epsilon = \frac{1}{\pi}
\end{equation}
and from this we obtain the expression in lattice units of $A_\epsilon$

\begin{equation}
A_\epsilon^l = (R_\sigma)^{8/15}(R_\epsilon)A_\epsilon = 0.58051... \ .
\end{equation}

Our results are summarized in tab. 3.

| $A_{m_1}^l$ | $A_f^l$ | $A_M^l$ | $A_X^l$ | $A_{E}^l$ |
|-------------|--------|--------|--------|--------|
| 4.01039911... | 0.99279949... | 1.05898612... | 0.07059907... | 0.58051... |

Table 5: Critical amplitudes in lattice units.

2.4.1 Alternative derivation of $R_\epsilon$

In this section we discuss, for completeness, an alternative derivation of $R_\epsilon$. It can be used in those cases in which the correlators are not known, but the internal energy is known on a finite size lattice at the critical point. Then $R_\epsilon$ can be obtained by comparing the finite size behaviour of the internal energy on the lattice with that predicted by conformal field theory in the continuum. In the case of the Ising model, thanks to the beautiful work by Ferdinand and Fisher [12], we know that on a square lattice of size $L_0 \times L_1$ with $L_0 > L_1$ with periodic boundary conditions the internal energy must scale as:

\begin{equation}
\langle \epsilon_l \rangle = \frac{\vartheta_2(\tau)\vartheta_3(\tau)\vartheta_4(\tau)}{\vartheta_2(\tau) + \vartheta_3(\tau) + \vartheta_4(\tau)} \frac{1}{L_1}
\end{equation}

where $\vartheta_i(\tau)$ denotes the $i^{th}$ Jacobi theta function and $\tau \equiv i\frac{L_0}{L_1}$.

The same behaviour can be studied in the continuum theory, by using CFT techniques. The result [24] is

\begin{equation}
\langle \epsilon \rangle = \frac{\vartheta_1'(\tau)}{\vartheta_2(\tau) + \vartheta_3(\tau) + \vartheta_4(\tau)} \frac{1}{L_1}
\end{equation}
By using the relation
\[
\vartheta_1(\tau)' = \pi \vartheta_2(\tau)\vartheta_3(\tau)\vartheta_4(\tau)
\] (66)
which allows to express the derivative of the \(\vartheta_1(\tau)\) in terms of ordinary theta functions we see that the two equations (64) and (65) agree only if we choose, as we did in the previous section, \(R_\epsilon = \frac{\pi}{2}\).

3 Scaling functions

In this section we shall construct the scaling functions for the various quantities in which we are interested. Our aim is to give the form (i.e. the value of the scaling exponents) of the first 7-8 terms of the expansion in powers of \(h\) of the scaling functions and at the same time to identify the operators in the lattice Hamiltonian from which they originate. To this end we shall first deal in sect. 3.1 with the theory at the critical point. We shall in particular discuss its spectrum, which can be constructed explicitly by using CFT techniques. Next, in sect. 3.2, we discuss in the framework of the renormalization group approach the origin of the subleading terms in the scaling functions, and show how to obtain their exponents from the knowledge of the renormalization group eigenvalues \(y_i\) of the irrelevant operators. While in general this analysis is only of limited interest since the \(y_i\) of the irrelevant operators are unknown, in the present case, thanks to the CFT solution discussed in sect. 3.1, it becomes highly predictive and will allow us to explicitly construct in sects. 3.3 and 3.4 the scaling functions. In particular in sect. 3.3 we shall list all the irrelevant operators which may appear in the effective Hamiltonian and discuss their symmetry properties, while in sect. 3.4 we shall write the scaling functions and identify the operators involved in the various scaling terms.

3.1 The Ising model at the critical point

The Ising model at the critical point is described by the unitary minimal model with central charge \(c = 1/2\). Its spectrum can be divided into three conformal families characterized by different transformation properties under the dual and \(Z_2\) symmetries of the model. They are the identity, spin
and energy families and are commonly denoted as $[I], [\sigma], [\epsilon]$. Let us discuss their features in detail.

- **Primary fields**
  Each family contains a relevant operator which is called primary field (and gives the name to the entire family). Their conformal weights are $h_I = 0$, $h_\sigma = 1/16$ and $h_\epsilon = 1/2$ respectively. The relationship between conformal weights and renormalization group eigenvalues is: $y = 2 - 2h$. Hence the relevant operators must have $h < 1$.

- **Secondary fields**
  All the remaining operators of the three families (which are called secondary fields) are generated from the primary ones by applying the generators $L_{-i}$ and $\bar{L}_{-i}$ of the Virasoro algebra. In the following we shall denote the most general irrelevant field in the $[\sigma]$ family (which are odd with respect to the $Z_2$ symmetry) with the notation $\sigma_i$ and the most general fields belonging to the energy $[\epsilon]$ or to the identity $[I]$ families (which are $Z_2$ even) with $\epsilon_i$ and $\eta_i$ respectively. It can be shown that by applying a generator of index $k$: $L_{-k}$ or $\bar{L}_{-k}$ to a field $\phi$ (where $\phi = I, \epsilon, \sigma$ depending on the case) of conformal weight $h_\phi$ we obtain a new operator of weight $h = h_\phi + k$. In general any combination of $L_{-i}$ and $\bar{L}_{-i}$ generators is allowed, and the conformal weight of the resulting operator will be shifted by the sum of the indices of the generators used to create it. If we denote by $n$ the sum of the indices of the generators of type $L_{-i}$ and with $\bar{n}$ the sum of those of type $\bar{L}_{-i}$ the conformal weight of the resulting operator will be $h_\phi + n + \bar{n}$. The corresponding RG eigenvalue will be $y = 2 - 2h_\phi - n - \bar{n}$, hence all the secondary fields are irrelevant operators.

- **Nonzero spin operators**
  The secondary fields may have a non zero spin, which is given by the difference $n - \bar{n}$. In general one is interested in scalar quantities and hence in the subset of those irrelevant fields which have $n = \bar{n}$. However on a square lattice the rotational group is broken to the finite subgroup $C_4$ (cyclic group of order four). Accordingly, only spin $0, 1, 2, 3$ are allowed on the lattice. If an operator $\phi$ of the continuum theory has spin $j \in \mathbb{N}$, then its lattice discretization $\phi_l$ behaves as a spin $j \pmod{4}$
operator with respect to the $C_4$ subgroup. As a consequence all the operators which in the continuum limit have spin $j = 4N$ with $N$ non-negative integer can appear in the lattice discretization of a scalar field. This will play a major role in the following.

- **Null vectors**

Some of the secondary fields disappear from the spectrum due to the null vector conditions. This happens in particular for one of the two states at level 2 in the $\sigma$ and $\epsilon$ families and for the unique state at level 1 in the identity family. From each null state one can generate, by applying the Virasoro operators a whole family of null states hence at level 2 in the identity family there is only one surviving secondary field, which can be identified with the stress energy tensor.

- **Secondary fields generated by $L_{-1}$**

Among all the secondary fields a particular role is played by those generated by the $L_{-1}$ Virasoro generator. $L_{-1}$ is the generator of translations on the lattice and as a consequence it has zero eigenvalue on translational invariant observables. Another way to state this results is to notice that $L_{-1}$ can be represented as a total derivative, and as such it gives zero if applied to an operator which can be obtained as the integral over the whole lattice of a suitable density (i.e. a translationally invariant operator).

### 3.2 RG analysis for $h \neq 0$

We shall discuss the higher order corrections to the RG analysis of sect. 2.2 along the lines of [25], to which we refer for a more detailed discussion. The only improvement that we make with respect to [25] is in the part devoted to the contribution due to the irrelevant operators, in which we shall make use of the results discussed in the previous section.

We expect three types of corrections to the asymptotic results reported in sect. 2.2:

a) **Analytic corrections.**

They are due to the fact, already mentioned in sect. 2.2, that the actual scaling variables in the RG approach are not $h_l$ and $t$ but $u_h$ and $u_t$.
which are in principle the most general analytic functions of $h_l$ and $t$ which respect the $Z_2$ parity of $h_l$ and $t$. Let us write the Taylor expansion for $u_h$ and $u_t$, keeping only those first few orders that are needed for our analysis (we use the notations of [25]).

\begin{align}
  u_h &= h_l \left[ 1 + c_h t + d_h t^2 + e_h h_l^2 + O(t^3, t h_l^2) \right] \quad (67) \\
  u_t &= t + b_t h_l^2 + c_t t^2 + d_t t^3 + e_t t h_l^2 + f_t h_l^4 + O(t^4, t^2 h_l^2) \quad (68)
\end{align}

The corrections induced by the higher terms in $u_h$ and $u_t$ are of three types.

- The first one is very simple to understand. It is due to the higher powers of $h_l$ contained in $u_h$ which lead to corrections to the power behaviours listed in sect. 2.2.1 which are shifted by even integer powers of $h_l$. For instance in the free energy, as a consequence of the $e_h h_l^2$ term in $u_h$, we expect a correction of this type:

\[
  f_s(h_l) = A_{f,3}^h |h_l|^{16} \left( 1 + A_{f,3}^h |h_l|^2 + \ldots \right) .
\]  

with $A_{f,3}^h = \frac{22}{15} e_h$. The indices $f, 3$ in $A_{f,3}$ only denote the fact (that we shall discuss in detail in the next section) that this term is the third term in the $h_l$ expansion of the scaling function of the singular part of the free energy.

- The second type of correction is due to the terms that depend on $h_l$ which appear in $u_t$. Their peculiar feature is that, even if they are originated by analytic terms in the scaling variables, they lead in general to non analytic contributions in the scaling functions For instance, as a consequence of the $b_t h_l^2$ term in $u_t$, we find in the free energy a correction of the type:

\[
  f_s(h_l) = A_{f,3}^t |h_l|^{16} \left( 1 + A_{f,3}^t |h_l|^2 - \frac{22}{15} \right). 
\]  

with $A_{f,3}^t = \frac{\Phi''(0)}{\Phi(0)} b_t$ and $2 - \frac{\Phi''(0)}{\Phi(0)} = \frac{22}{15}$.

- The corrections of the third type only appear when studying the internal energy. They are due to the terms linear in $t$ which are present in $u_h$ and $u_t$. The most important of these contributions
is the one due to the $c_h t$ term in $u_h$ which gives a correction proportional to $h_t^{15}$ to the dominant scaling behaviour of the internal energy. We shall discuss these terms in sect. 3.4.3 below.

b) **Corrections due to irrelevant operators in the lattice Hamiltonian.**

These can be treated within the framework of the RG as follows. Let us study as an example the case of an irrelevant operator belonging to the Identity family. Let us call the corresponding scaling variable $u_3$ and its RG eigenvalue $y_3$ (since $u_3$ is irrelevant, $y_3 < 0$). In this case the dependence of $u_3$ on $t$ and $h_t$ is

$$u_3 = u_3^0 + at + bh_t^2 + \cdots . \quad (71)$$

Let us for the moment neglect higher order terms and assume $u_3 = u_3^0$. Then looking again at the singular part of the free energy we find

$$f_s(t, h_t) = |h_t|/h_0^{d/y_n} \Phi\left(\frac{t}{|h_t|^{y_n/y_n}}, u_3^0|h_t|^{y_3/y_n}\right) . \quad (72)$$

Since $u_3^0|h_t|^{y_3/y_n}$ is small as $h_t \to 0$ it is reasonable to assume that we can expand $f_s$ in a Taylor series of $u_3^0|h_t|^{y_3/y_n}$ (notice that in eq. (72) $f_s$ is not singular since it is evaluated at $|h_t| > 0$). Hence we find (setting again $t = 0$)

$$f_s = |h_t|^{d/y_n}(a_1 + a_2u_3^0|h_t|^{y_3/y_n} + \cdots) \quad (73)$$

where $a_1$, $a_2$, $u_3^0$ are non-universal constants.

This analysis can be repeated without changes for any new irrelevant operator: $u_4$, $y_4$ and so on. As a last remark, notice that on top of these non analytic corrections we also expect analytic contributions due to the higher order terms contained in eq. (71).

While in general this analysis is only of limited interest since the $y_i$ of the irrelevant operators are usually unknown, in the present case we may identify the irrelevant operators with the secondary fields discussed in 3.1 and use the corresponding RG-exponents as input of our analysis.

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4In general for the irrelevant operators there is no need to tune $u_3^0$ to 0 to approach the critical point. However we shall see below that, for symmetry reasons, $u_3^0 = 0$ for all the irrelevant operators belonging to the $[\sigma]$ and $[e]$ families.
Logarithmic corrections.

As it is well known, the specific heat of the 2d Ising model at \( h_l = 0 \) approaches the critical point with a logarithmic singularity. This means that in the free energy there must be a term of the type \( \Phi_0 u_t^2 \log(u_t) \). While in general we could expect \( \Phi_0 \) to be a generic function of the ratio \( u_t/u_h^{8/15} \), the absence of leading log corrections in \( M \) and \( \chi \) strongly constrains this function which is usually assumed to be a simple constant. Notwithstanding this, the presence of terms that depend on \( h_l \) in \( u_t \) implies that log type contributions may appear also in the case \( t = 0, h_l \neq 0 \) in which we are interested. These can be easily obtained by inserting eq.(68) into \( \Phi_0 u_t^2 \log(u_t) \) and then making the suitable derivatives and limits \([22]\). In the case of the free energy one obtains a term proportional to \( h_l^4 \log(h_l) \) which is too high to be observed in our fits. However for the internal energy the first contribution is proportional to a smaller power of \( h_l \): \( h_l^2 \log(h_l) \) and must be taken into account in the scaling function.

3.3 The effective lattice Hamiltonian

Let us call \( H_{CFT} \) the Hamiltonian which describes the continuum theory at the critical point. The perturbed Hamiltonian \(^5\) in the continuum is given by:

\[
H = H_{CFT} + h\sigma \ .
\]

The aim of this section is to construct the lattice analogous (which we shall call \( H_{lat} \)) of \( H \).

Notice that \( H_{lat} \) is different from the microscopic Hamiltonian which appears in the exponent of eq. (4). Eq. (4) describes the model at the level of the lattice spacing. We are instead interested in the large distance effective Hamiltonian which one obtains when the short range degrees of freedom are integrated out, i.e. after a large enough number of iterations of the Renormalization Group transformation has been performed. \( H_{lat} \) will contain all the irrelevant operators which are compatible with the symmetries of the lattice model. In this section we shall first discuss the relation between the lattice model.

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\(^5\)Here we follow the convention usually adopted in conformal field theory. In the standard notation of classical statistical mechanics one would denote this quantity “Hamiltonian density” rather than “Hamiltonian”. 

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and the continuum operators, then we shall construct the lattice Hamiltonian in the $h_t = 0$ case and finally we shall extend our results to the $h_t \neq 0$ case.

3.3.1 Relations between lattice and continuum operators.

The lattice operators are given by the most general combination of continuum operators compatible with the symmetries of the lattice operator multiplied by the most general analytic functions of $t$ and $h_t$ (with a parity which is again constrained by the symmetry of the operators involved). In the following, to avoid a too heavy notation, we shall neglect the $t$ dependence\(^6\).

For the spin operator we have

$$\sigma_i = f_0^\sigma(h_t)\sigma + h_t f_0^\epsilon(h_t)\epsilon + f_i^\sigma(h_t)\sigma_i + h_t f_i^\epsilon(h_t)\epsilon_i + h_t f_i^I(h_t)\eta_i, \quad i \in \mathbb{N}$$

(75)

where $f_0^\sigma(h_t)$, $f_0^\epsilon(h_t)$ and $f_i^I(h_t)$ are even functions of $h_t$.

For the energy operator we have

$$\epsilon_i = g_0^\epsilon(h_t)\epsilon + h_t g_0^\sigma(h_t)\sigma + h_t g_i^\sigma(h_t)\sigma_i + g_i^\epsilon(h_t)\epsilon_i + h_t^2 g_i^I(h_t)\eta_i, \quad i \in \mathbb{N}$$

(76)

where again $g_0^\epsilon(h_t)$, $g_i^\epsilon(h_t)$ and $g_i^I(h_t)$ are even functions of $h_t$ and the $h_t^2$ term in front of $g_i^I(h_t)$ is due to the change of sign of the $\epsilon$ operator under duality transformation at $h_t = 0$ (see the discussion at the beginning of sect. 3.3.2).

Among all the possible irrelevant fields only those which respect the lattice symmetries (i.e. those of spin $0 \pmod{4}$) are allowed in the sums. At this stage also irrelevant operators containing $L_{-1}$ or $\bar{L}_{-1}$ appear in the sums. It is only when these operators are applied on translationally invariant states (i.e. on the vacuum) that they disappear. This will happen for instance when we shall study the mean value of the free energy.

3.3.2 Construction of $H_{lat}(h_t = 0)$

In this case all the operators belonging to the $[\sigma]$ family are excluded due to the $Z_2$ symmetry. Also the operators belonging to the $[\epsilon]$ family are excluded for a more subtle reason. The Ising model (both on the lattice and in the continuum) is invariant under duality transformations while the operators

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\(^6\)The $t$ dependence in the scaling variables of the irrelevant operators plays a role only in the construction of the scaling function for the internal energy and we shall resume it in sect. 3.4.3 below.
belonging to the $[\epsilon]$ family change sign under duality, thus they also cannot appear in $H_{\text{lat}}(h_l = 0)$. Thus we expect
\[ H_{\text{lat}} = H_{\text{CFT}} + u_i^0 \eta_i, \quad \eta_i \in [I], \tag{77} \]
where the $u_i^0$ are constants. There are however further restrictions:

- $H_{\text{lat}}$ is a scalar density, hence only operators $\phi$ with angular momentum $j = 4k$, $k = 0, 1, 2 \cdots$ are allowed.

- The operator which acts on the Hilbert space of the theory is the space integral of the Hamiltonian (density) $H_{\text{lat}}$. As such it is translational invariant and only operators $\phi$ which do not contain the generators $L_{-1}$ or $\bar{L}_{-1}$ survive the integration of eq. (77) over the space.

Let us list in order of increasing conformal weight the first few operators which fulfill all the constraints:
\[ \phi_0 = I, \quad \phi_1 = L_{-2}\bar{L}_{-2}I, \quad \phi_2 = (L_{-2})^2 I, \quad \phi_3 = (\bar{L}_{-2})^2 I, \tag{78} \]
\[ \phi_4 = L_{-4}I, \quad \phi_5 = \bar{L}_{-4}I, \quad \phi_6 = L_{-3}\bar{L}_{-3}I \cdots. \tag{79} \]

Some of these fields have a natural interpretation. $\phi_0$ gives rise to the bulk contribution in the free energy (see the discussion of sect. 2.1.2). $\phi_1$, $\phi_2$, $\phi_3$ are related to the energy momentum tensor: $T\bar{T}$, $T^2$, $\bar{T}^2$ respectively. All the fields listed above except the identity and $\phi_6$ have the same conformal weight $h_\phi = 4$. The corresponding RG eigenvalue is $y_\phi = -2$. The field $\phi_6$ has conformal weight $h_{\phi_6} = 6$ and RG eigenvalue $y_{\phi_6} = -4$.

### 3.3.3 Extension to $h_l \neq 0$

Mimicking the continuum case we have, also on the lattice,
\[ H_{\text{lat}}(h_l) = H_{\text{lat}}(h_l = 0) + h_l \sigma_l. \tag{80} \]

Inserting the expression of $\sigma_l$ of eq. (72) we find
\[ H_{\text{lat}}(h_l) = H_{\text{CFT}} + u_i(h_l) \phi_i \tag{81} \]
where this time there is no more restriction coming from the $Z_2$ symmetry and duality, hence $\phi_i$ denotes here the most general operator of the spectrum.
with spin \( j = 4k, k = 0, 1, \ldots \). The \( u_i(h_l) \) are even or odd functions of \( h_l \), depending on the parity of \( \phi_i \) but in the even sector only for the operators belonging to the identity family \( \lim_{h_l \to 0} u_i(h_l) \neq 0 \) (according to eq. (71) we have \( u_i(h_l = 0) = u_i^0 \)). For the operators belonging to the energy family the first nonzero contribution in the \( u_i(h_l) \) functions is of order \( h_l^2 \).

Let us list, starting from those with the lowest conformal weight, the new operators which were not present in \( H_{\text{lat}}(h_l = 0) \). For future convenience let us separate those which do not contain \( L_{-1}, \bar{L}_{-1} \) generators from the remaining ones.

**A] Operators which are not generated by \( L_{-1}, \bar{L}_{-1} \).**

- **Operators belonging to \([\sigma]\)**
  
  In the \([\sigma]\) family the lowest ones are \( L_{-4}\sigma, \bar{L}_{-4}\sigma \) and \( L_{-3}\bar{L}_{-3}\sigma \). In fact \( L_{-1}\sigma \) disappears for translational invariance and due to the null vector equation the \( L_{-2}\sigma \) operator which appears at level 2 can always be rewritten as \( L_{-2}^2\sigma \) with suitable coefficients. The conformal weights of \( L_{-4}\sigma \) and \( L_{-4}\bar{\sigma} \) are \( h_{\sigma,4} = 4 + \frac{1}{8} \). The corresponding RG eigenvalue is \( y_{\sigma,4} = -2 - \frac{1}{8} \). The conformal weight of \( L_{-3}\bar{L}_{-3}\sigma \) is \( h_{\sigma,33} = 6 + \frac{1}{8} \). The corresponding RG eigenvalue is \( y_{\sigma,33} = -4 - \frac{1}{8} \).

- **Operators belonging to \([\epsilon]\)**
  
  The most important contribution from the \([\epsilon]\) family is the one proportional to \( h_l^2\epsilon \) which is responsible for the \( h_l^2 \) term which appears in \( u_t \) as we discussed in the previous section. Besides this one, the lowest operators which appear in the \([\epsilon]\) family must be of the type \( L_{-4}\epsilon \) or \( \bar{L}_{-4}\epsilon \). In fact the same mechanism which allowed us to eliminate the secondary fields of level 2 in the \([\sigma]\) family also works for the \([\epsilon]\) family. On top of this in the \([\epsilon]\) family a new null vector appears at level 3, thus allowing us to eliminate also all the fields at this level. Keeping also into account the fact that the corresponding \( u_i(h_l) \) functions must start from \( h_l^2 \) we immediately see that all these operators have too high powers of \( h_l \) to contribute to the scaling function and can be neglected.

**B] Operators which contain \( L_{-1}, \bar{L}_{-1} \) generators.**

The lowest operators are, in order of increasing weight:
• $L_{-1} \bar{L}_{-1} \sigma$, whose conformal weight is $h_{\sigma,11} = 2 + \frac{1}{8}$. The corresponding RG eigenvalue is $y_{\sigma,11} = -\frac{1}{8}$.
• $L_{-1}^2 \bar{L}_{-1}^2 \sigma$, whose conformal weight is $h_{\sigma,22} = 4 + \frac{1}{8}$. The corresponding RG eigenvalue is $y_{\sigma,22} = -2 - \frac{1}{8}$.
• $L_{-1} \bar{L}_{-1} \epsilon$, whose conformal weight is $h_{\epsilon,11} = 3$. The corresponding RG eigenvalue is $y_{\epsilon,11} = -1$.

3.4 Scaling functions

Using the results of the previous section we are now in the position to write the expression for the scaling functions keeping all the corrections up to the order $h_l^3$.

3.4.1 The free energy

Due to translational invariance, only the secondary fields which are not generated by $L_{-1}, \bar{L}_{-1}$ contribute to the free energy. We find, for the singular part of the lattice free energy:

$$f_s(h_l) = A_l^f |h_l|^{\frac{16}{15}} (1 + A_{f,1}^l |h_l|^{\frac{16}{15}} + A_{f,2}^l |h_l|^{\frac{22}{15}} + A_{f,3}^l |h_l|^{\frac{28}{15}} + A_{f,4}^l |h_l|^{\frac{32}{15}} + A_{f,5}^l |h_l|^{\frac{38}{15}} + A_{f,6}^l |h_l|^{\frac{44}{15}} .......) \quad (82)$$

where $A_{f,n}^l$ denotes the amplitude, normalized to the critical amplitude, of the $n^{th}$ subleading correction.

Let us discuss the origin of the various corrections:
• $A_{f,1}^l |h_l|^{\frac{16}{15}}$
  this term is entirely due to the $T \bar{T}$, $T^2$ and $\bar{T}^2$ irrelevant fields in the Hamiltonian.
• $A_{f,2}^l |h_l|^{\frac{22}{15}}$
  this term is due to the $b_l h_l^2$ term in $u_t$ (or, equivalently, to the appearance of a $h^2 \epsilon$ term in the Hamiltonian).
• $A_{f,3}^l |h_l|^{\frac{30}{15}}$
  this term is due to the $e_h h_l^2$ term in $u_h$. 

...
\begin{itemize}
  \item $A_{f,4}^l |h_l|^{32/15}$
  
  this term keeps into account the second term in the Taylor expansion of the $TT$ like corrections and the contribution of the fields $L_{-3}L_{-3}I$ and $hL_{-3}\sigma$ in the Hamiltonian.
  
  \item $A_{f,5}^l |h_l|^{38/15}$
  
  this is the product of the $A_{f,1}$ and $A_{f,2}$ corrections.
  
  \item $A_{f,6}^l |h_l|^{44/15}$
  
  this is the second term in the Taylor expansion of the $h^7\epsilon$ correction.
\end{itemize}

To these terms we must then add the bulk contributions

$$f_b(h_l) = f_b + f_{b,1}h_l^2 + f_{b,2}h_l^4 + \cdots .$$  \hfill (83)

We have already noticed in sect. 2.1.2 that $f_b$ can be obtained from the exact solution of the Ising model on the lattice at the critical point. Also the next term: $f_{b,1}$ can be evaluated (with a precision of ten digits) by noticing that it corresponds to the constant contribution to the susceptibility at the critical point. This term has been evaluated in \[24\]. We neglect for the moment this information and keep the $f_{b,1}$ amplitude in the scaling function as a free parameter. It is the first subleading term in the scaling function and as such it can be rather precisely estimated with the fitting procedure that we shall discuss below. We shall compare our estimates with the expected value in sect. 6 and use the comparison as a test of the reliability of our results.

Combining eqs. (82) and (83) we find:

$$f(h_l) = f_b + f_{b,1}h_l^2 + f_{b,2}h_l^4 + \cdots .$$  \hfill (84)

where $A_{f,b}$ is $\frac{f_{b,1}}{A_f}$ and $A_{f,2}$ takes also into account now the contribution of $f_{b,2}$.

Deriving this expression with respect to $h_l$ we obtain the scaling functions for the magnetization and the susceptibility\[7\].

\[7\]In this way we obtain directly the lattice definitions of these two quantities, since we are deriving the lattice free energy with respect to the lattice magnetic field. There is no need to go through the continuum definition of the magnetization.
3.4.2 Mass spectrum

The simplest way to deal with the mass spectrum is to fit the square of the masses. The scaling function turns out to be very similar to that which describes the singular part of the free energy eq. (82). The only additional terms are due to the secondary fields which contain $L_{-1}\bar{L}_{-1}$. It turns out that the corresponding scaling dimension exactly match those which already appear in eq. (82). In fact

- $h_{L_{-1}\bar{L}_{-1}}$ gives a contribution which scales with $|h_l|^{16/15}$ and its amplitude can be absorbed in $A_{f,1}$.
- $h_{L_{-1}\bar{L}_{-1}}$ gives a contribution which scales with $|h_l|^{32/15}$ and its amplitude can be absorbed in $A_{f,4}$.
- $h_{L_{-1}\bar{L}_{-1}}$ gives a contribution which scales with $|h_l|^{38/15}$ and its amplitude can be absorbed in $A_{f,5}$.

Thus the functional form of the scaling function for the masses is exactly the same of eq. (82).

$$m_i^2(h_l) = (A_{m_i}^f)^2 |h_l|^{44/15}(1 + A_{m_{i,1}}^f|h_l|^{16/15} + A_{m_{i,2}}^f|h_l|^{32/15} + A_{m_{i,3}}^f|h_l|^{38/15} + A_{m_{i,4}}^f|h_l|^{44/15} + A_{m_{i,5}}^f|h_l|^{44/15} + A_{m_{i,6}}^f|h_l|^{44/15} + ... )$$

(85)

However we shall see below that the presence of these new fields and in particular of $L_{-1}\bar{L}_{-1}$ has very important consequences.

3.4.3 Internal energy

We may obtain the internal energy as a derivative with respect to $t$ of the singular part of the free energy. However in doing this we must resume (as discussed above) the $t$ dependence in the scaling variables. This leads to some new terms in the scaling function with powers $\frac{8}{15}$ (due to the $c_h t$ term in $u_h$), $\frac{24}{15}$ and $\frac{40}{15}$ (due to the $t$ terms in scaling variables of the irrelevant operators).

It is nice to see that the presence of these additional contributions can be understood in another, equivalent, way. Looking at eq. (16) or (76) we see that the internal energy on the lattice contains a term of type $h_l\sigma$. The powers listed above are exactly those that we obtain keeping into account...
the additional $h_i \sigma$ term in the scaling function. Keeping also into account
the bulk contribution we end up with the following scaling function. We
have:

$$E(h_l) = A_{E,b}^l |h_l|^{\frac{8}{15}} + A_{E,2}^l |h_l|^{\frac{15}{15}} + A_{E,3}^l |h_l|^{\frac{22}{15} \log |h_l|} +$$

$$+ A_{E,4}^l |h_l|^{\frac{30}{15}} + A_{E,5}^l |h_l|^{\frac{37}{15}} + A_{E,6}^l |h_l|^{\frac{45}{15}} + A_{E,7}^l |h_l|^{\frac{50}{15}} + \ldots \quad (86)$$

where $A_{E,b}^l$ denotes the amplitude of the $h_l^2$ term in the bulk part of the
internal energy, $A_{E,log}^l$ denotes the amplitude of the $h_l^2 \log |h_l|$ term discussed
in sect.3.1 and the bulk constant term has been already taken into account
in the definition of $E(h_l)$. The first correction which appears in the internal
energy (with amplitude $A_{E,1}^l$) is the one with the lowest power of $h_l$ among
all the subleading terms of the various scaling functions this. Its effect on
the scaling behaviour of the internal energy is very important and it is easily
observable also in standard Monte Carlo simulations [13].

### 3.4.4 Overlaps

Also in this case we fitted the square of the overlap constants. The scaling
functions can be obtained with a straightforward application of the arguments discussed above. Also fields generated by $hL_{-1} \bar{L}_{-1}$ must be taken into account. Moreover, for the overlaps with the internal energy also the $h \sigma$
term must be taken into account. We end up with the following result for
the magnetic overlaps.

$$|F_i^\sigma(h_l)|^2 = |A_{F_i}^\sigma|^2 (1 + A_{F_i,1}^\sigma |h_l|^{\frac{15}{15}} + A_{F_i,2}^\sigma |h_l|^{\frac{22}{15}} +$$

$$+ A_{F_i,3}^\sigma |h_l|^{\frac{30}{15}} + A_{F_i,4}^\sigma |h_l|^{\frac{37}{15}} + A_{F_i,5}^\sigma |h_l|^{\frac{45}{15}} + A_{F_i,6}^\sigma |h_l|^{\frac{50}{15}} + \ldots) \quad (87)$$

While for the energy overlaps we have

$$|F_i^\epsilon(h_l)|^2 = |A_{F_i}^\epsilon|^2 (1 + A_{F_i,1}^\epsilon |h_l|^{\frac{8}{15}} + A_{F_i,2}^\epsilon |h_l|^{\frac{15}{15}} + A_{F_i,3}^\epsilon |h_l|^{\frac{22}{15}} +$$

$$+ A_{F_i,4}^\epsilon |h_l|^{\frac{30}{15}} + A_{F_i,5}^\epsilon |h_l|^{\frac{37}{15}} + \ldots) \quad (88)$$

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4 The transfer matrix method

We computed the mass spectrum and observables by numerical diagonalization of the transfer matrix. The transfer matrix was introduced by Kramers and Wannier [27] in 1941. For a discussion of the transfer matrix see e.g. refs. [28, 29]. The starting point is a simple transformation of the Boltzmann factor

\[
\exp \left( \beta \sum_{<n,m>} \sigma_n \sigma_m + h_l \sum_n \sigma_n \right) = T(u_1, u_2) T(u_2, u_3) \ldots T(u_{L_0}, u_1) \tag{89}
\]

where \( u_{n_0} = (\sigma_{(n_0,1)}, \sigma_{(n_0,2)}, \ldots \sigma_{(n_0,L_1)}) \) is the spin configuration on the time slice \( n_0 \). \( T \) is given by

\[
T(u_{n_0}, u_{n_0+1}) = V(u_{n_0})^{1/2} \quad U(u_{n_0}, u_{n_0+1}) \quad V(u_{n_0+1})^{1/2} \tag{90}
\]

with

\[
U(u_{n_0}, u_{n_0+1}) = \exp \left( \beta \sum_{n_1=1}^{L_1} \sigma_{(n_0,n_1)} \sigma_{(n_0+1,n_1)} \right) \tag{91}
\]

and

\[
V(u_{n_0}) = \exp \left( \beta \sum_{n_1=1}^{L_1} \sigma_{(n_0,n_1)} \sigma_{(n_0,n_1+1)} + h_l \sum_{n_1}^{L_1} \sigma_{(n_0,n_1)} \right). \tag{92}
\]

The partition function becomes

\[
Z = \sum_{\sigma_n \pm 1} \exp \left( \beta \sum_{<n,m>} \sigma_n \sigma_m + h_l \sum_n \sigma_n \right) = \text{tr} \ T^{L_0} = \sum \lambda_i^{L_0} \tag{93}
\]

where \( T \) is interpreted as a matrix. The time-slice configurations are the indices of the matrix. The number of configurations on a time slice is \( 2^{L_1} \). Therefore the transfer matrix is a \( 2^{L_1} \times 2^{L_1} \) matrix. By construction the transfer matrix is positive and symmetric. The \( \lambda_i \) are the eigenvalues of the transfer matrix.

4.1 Computing observables

Observables that are defined on a single time slice can be easily expressed in the transfer matrix formalism. Let us discuss as examples the magnetisation
and the internal energy.

$$< \sigma_{1,1} > = \frac{\sum_{\sigma=\pm 1} \exp \left( \beta \sum_{\langle n,m \rangle} \sigma_n \sigma_m + h_i \sum_n \sigma_n \right) \sigma_{1,1}}{Z}$$

$$= \frac{\text{tr} S T^{L_0}}{\text{tr} T^{L_0}} = \frac{\sum_i \lambda_i^{L_0} < i | \tilde{S} | i >}{\sum_i \lambda_i^{L_0}}$$

(94)

where $S$ is a diagonal matrix. The values on the diagonal are given by $\sigma_{1,1}$ on the configurations. ($S(u, u') = \delta(u, u') u^{(1)}$ where $u^{(1)}$ denotes $\sigma$ on the first site of the time slice). The $|i >$ are normalized eigenvectors of $T$.

In the limit $L_0 \to \infty$ the expression simplifies to

$$< \sigma_{1,1} > = < 0 | S | 0 >$$

(95)

where $|0 >$ is the eigenvector with the largest eigenvalue.

The energy can be computed in a similar way. The diagonal matrix corresponding to the energy is given by

$$E(u, u') = \delta(u, u') u^{(1)} u^{(2)} .$$

(96)

Note that we can only express the product of nearest neighbour spins in this simple form if both spins belong to the same time slice.

In order to understand the relation of the mass spectrum with the eigenvalue spectrum of the transfer matrix we have to compute correlation functions with separation in time direction. The time-slice correlation function eq. (22) becomes in the limit $L_0 \to \infty$

$$< S_0 S_\tau > = \sum_i \exp(-m_i |\tau|) < 0 | \tilde{S} | i > < i | \tilde{S} | 0 >$$

(97)

with

$$m_i = - \log \left( \frac{\lambda_i}{\lambda_0} \right)$$

(98)

and $\tilde{S} = \frac{1}{L_1} \delta(u, u') \sum_n u^{n_1}$. Note that $\tilde{S}$ is translational invariant (in the space direction) and has therefore only overlaps with zero-moment eigenvectors of $T$.

With eq. (23) we get

$$|F_i^\sigma| = \sqrt{m_i L_1} \frac{< 0 | \tilde{S} | i >}{< 0 | \tilde{S} | 0 >} .$$

(99)

An analogous result can be obtained for the energy.
4.2 Computing the eigenvectors and eigenvalues of $T$

The remaining problem is to compute (numerically) eigenvectors and eigenvalues of the transfer matrix. Since we are interested in the thermodynamic limit as well as in the continuum limit we would like to use as large values of $L_1$ as possible. This soon becomes a very difficult task since the dimension of the transfer matrix increases exponentially with $L_1$. The problem slightly simplifies if one is interested in the computation of the the leading eigenvalues and eigenvectors only, and in these last years various methods have been developed to address this task (for a comprehensive discussion of existing approaches see e.g. ref. [29] or the appendix of ref. [30]). In particular there are two approaches which have shown to be the most effective ones.

- The first one reduces the numerical complexity of the problem by writing the transfer matrix as a product of sparse matrices. See refs. [29, 30].
- The second one is to reduce the dimension of the transfer matrix by restricting it to definite channels.

Since we are only interested in the zero-momentum states of the system we decided to follow the second approach and to compute the zero-momentum reduced transfer matrix. The zero-momentum reduced transfer matrix acts on the space of equivalence classes of configurations on slices that transform into each other by translations.

The matrix elements of the reduced transfer-matrix are given by

$$
\tilde{T}(\tilde{u}, \tilde{v}) = \left( n(\tilde{u}) \ n(\tilde{v}) \right)^{-1/2} \sum_{u \in \tilde{u}} \sum_{v \in \tilde{v}} T(u, v) = \left( n(\tilde{u})/n(\tilde{v}) \right)^{1/2} \sum_{v \in \tilde{v}} T(u, v) .
$$

(100)

where $n(\tilde{u})$ is the number of configurations in $\tilde{u}$. For example for $L_0 = 20$ the dimension of the transfer matrix is reduced from 1048576 to 52488.

Still the matrix is too large to save all elements of the matrix in the memory of the computer. Therefore we applied an iterative solver and computed the elements of $\tilde{T}$ whenever they were needed.

As solver we used a generalized power method as discussed in the appendix of ref. [30].
The lattice sizes that we could reach in this way were large enough for our purpose, thus we made no further effort to improve our method and it is well possible that our algorithm might still not be the optimal one.

We propose here, as a suggestion to the interested reader, some directions in which it could be improved.

- One could try to mix the two strategies mentioned above and try to factorize the reduced transfer matrix as a product of sparse matrices. However note that the complexity of the problem increases exponentially with the lattice size. Therefore even a big improvement in the method would allow just to go up in the maximal $L_1$ by a few sites.

- One could study the transfer matrix along the diagonals of the square lattice. Since the distance between two points on the diagonal is $\sqrt{2}$, naively one could increase the accessible lattice size by a factor of $\sqrt{2}$.

5 Thermodynamic limit

In order to take the thermodynamic limit we must know the finite size scaling behaviour of the various observables as a function of $L_1$. This is a very interesting subject in itself and several exact results have been obtained in this context starting from the exact S-matrix solution and using Thermodynamic Bethe Ansatz (TBA) techniques [16, 31].

For instance, it is possible to construct a large $L$ asymptotic expansion for the finite size scaling (FSS) of the energy levels based only on the knowledge of the exact S-matrix of the theory [12, 33]. Let us look to this FSS behaviour in more detail.

Let us define $\Delta m_a(L)$ as the deviation of the mass $m_a$ of the particle $a$ from its asymptotic value:

$$
\Delta m_a(L) \equiv m_a(L) - m_a(\infty) .
$$

(101)

Then in the large $L$ limit, the shift (normalized to the lowest mass $m_1$) is dominated by an exponential decrease of the type

$$
\frac{\Delta m_a(L)}{m_1} \sim -\frac{1}{8m_a^2} \sum_{b,c} \lambda^2_{abc} \mu_{abc} \exp (-\mu_{abc} L)
$$

(102)
where the constants $\mu_{abc}$ and $\lambda_{abc}$ can be obtained from the S-matrix and the prime in the sum of eq. (102) means that the sum must be done only on those combinations of indices that fulfill the condition: $|m_0^2 - m_c^2| < m_a^2$. In particular the $\mu_{abc}$ turn out to be of order one, so that the FSS corrections are dominated (as one could naively expect) by a decreasing exponential of the type $\exp(-L_1/\xi)$ where the correlation length $\xi$ is the inverse of the lowest mass of the theory.

In principle we could use our data to test also the TBA predictions for the FSS. However we preferred to follow a different approach. We chose values of $h$ large enough so as to fulfill the condition $L_1/\xi >> 1$ for the largest values of $L_1$ that we could reach. In this way we could essentially neglect all the details of the FSS functions and approximate them with a single exponential (or, in some cases, with a pair of exponentials). In order to study the FSS functions one should choose smaller values of $h$. We plan to address this issue in a forthcoming paper. With our choice of $h$ we drastically simplify the FSS problem, however nothing is obtained for free. The price we have to pay following this route is that we need to know several terms in the scaling functions to fit such large values of $h$. This explains the major effort that we devoted to this issue in sect. 3.

5.1 Numerical extrapolation

According to the above discussion, for the extrapolation of our data to the thermodynamic limit we made no use of the quantitative theoretical results. We made only use of the qualitative result that the corrections due to the finite $L_1$ vanish exponentially.

We used as ansatz for the extrapolation either

$$A(L_1) = A(\infty) + c_1 \exp(-L_1/z_1)$$

(103)

or

$$A(L_1) = A(\infty) + c_1 \exp(-L_1/z_1) + c_2 \exp(-L_1/z_2)$$

(104)

where $A$ represents any of the quantities that we have studied.

In order to compute the free parameters $A(\infty)$, $c_1$ and $z_1$ or $A(\infty)$, $c_1$, $z_1$, $c_2$ and $z_2$ we solved numerically the system of equations that results from the lattice sizes $L_{1,\text{max}}$, $L_{1,\text{max}} - 1$ and $L_{1,\text{max}} - 2$ or $L_{1,\text{max}}$, $L_{1,\text{max}} - 1$, $L_{1,\text{max}} - 2$, $L_{1,\text{max}} - 3$ and $L_{1,\text{max}} - 4$. 
The error of $A(\infty)$ was estimated by comparing results where $L_{1,max}$ is the largest lattice size that is available and from $L'_{1,max} = L_{1,max} - 1$. Mostly ansatz (104) was used to obtain the final result. In some cases however the numerical accuracy was not sufficient to resolve the second exponential term. Then the final result was taken from the ansatz (103).

| $L_1$ | $f$ | $f$, eq. (103) | $f$, eq. (104) |
|-------|-----|----------------|----------------|
| 4     | 0.993343441146 |                |                |
| 5     | 0.992384038449 |                |                |
| 6     | 0.992160642059 | 0.992092835647 |                |
| 7     | 0.992102865951 | 0.992082710940 |                |
| 8     | 0.992086845804 | 0.992080699493 | 0.992080279141 |
| 9     | 0.992082188258 | 0.992080185903 | 0.992080180502 |
| 10    | 0.992080787928 | 0.992080185903 | 0.992080161487 |
| 11    | 0.992080356320 | 0.992080164020 | 0.992080157709 |
| 12    | 0.992080220728 | 0.992080158619 | 0.992080156931 |
| 13    | 0.992080177480 | 0.992080157225 | 0.992080156758 |
| 14    | 0.992080163514 | 0.992080156853 | 0.992080156721 |
| 15    | 0.992080158958 | 0.992080156752 | 0.992080156716 |
| 16    | 0.992080157458 | 0.992080156722 | 0.992080156709 |
| 17    | 0.992080156961 | 0.992080156715 | 0.992080156713 |
| 18    | 0.992080156795 | 0.992080156712 | 0.992080156710 |
| 19    | 0.992080156739 | 0.992080156710 | 0.992080156710 |
| 20    | 0.992080156721 | 0.992080156712 | 0.992080156713 |
| 21    | 0.992080156714 | 0.992080156710 | -               |

Table 6: Extrapolation of the free energy at $h_t = 0.075$ to the thermodynamic limit. In the first column we give the lattice size $L_1$. In the second column the free energy for this lattice size is given. In the third column we present the extrapolation with a single exponential and in the fourth column the extrapolation with a double exponential ansatz.

In tab. 6 we give, as example, the extrapolation to the thermodynamic limit of the free energy at $h_t = 0.075$. As input for the extrapolation we used the free energy computed up to 12 digits. We consider all these digits save of rounding errors. Within the given precision the free energy has not yet converged at $L_1 = 21$. The single exponential extrapolation (103) converges
(within the given precision) at \( L_1 = 18 \). For larger lattices the result fluctuates in the last digit due to rounding errors of the input data. The double exponential extrapolation \(^{103}\) converges at \( L_1 = 16 \). As final result for the thermodynamic limit we quote \( f(0.075) = 0.99208015671 \).

6 Analysis of the results

The major problem in extracting the continuum limit results from the data listed in tabs. 10-13 is to estimate the systematic errors involved in the truncation of the scaling functions that we use in the fits. We shall devote the first part of this section to a detailed description of the procedure that we followed to estimate this uncertainty. We shall give upper and lower bounds for the critical amplitudes which turn out to be very near to each other and allow for high precision predictions (in some cases we can fix 5 or even 6 significative digits). We then compare our predictions with the results obtained in the framework of the S-matrix approach. In all cases we find a perfect agreement within our bounds. Finally in sect. 6.3, we give, assuming as fixed input the S-matrix predictions for the critical amplitudes, our best estimates for the amplitudes of some of the subleading terms involved in the fits.

6.1 Systematic errors

In order to estimate the systematic errors involved in our estimates of the critical amplitudes we performed for each observable several independent fits starting with a fitting function containing only the dominant scaling dimension and then adding the subleading fields one by one. For each fitting function we tried first to fit all the exiting data (those listed in tabs. 10-13) and then eliminated the data one by one starting from the farthest from the critical point (i.e. from those with the highest values of \( h_l \)). Among the (very large) set of estimates of the critical amplitudes we selected only those fulfilling the following requirements:

1] The reduced \( \chi^2 \) of the fit must be of order unity \(^8\). In order to fix precisely a threshold we required the fit to have a confidence level larger

---

8This is a slightly incorrect use of the \( \chi^2 \) function since the input data are affected by errors which are of systematic more than statistic nature. Notice however that we do not use it to determine best fit values for the observables that we fit (we shall only give upper
than 30%.

2] The number of degrees of freedom of the fit (i.e. the number of data fitted minus the number of free parameters in the fitting function) must be larger than 3.

3] For all the subleading fields included in the fitting function, the amplitude estimated from the fit must be larger than the corresponding errors, otherwise the field is eliminated from the fit.

4] The amplitudes of the subleading fields (in units of the critical amplitude) must be such that when multiplied for the corresponding power of $h_l$ (for the largest value of $h_l$ involved in the fit) must give a contribution much smaller than 1 (in order to fix a threshold we required it to be strictly smaller than 0.3).

In general only a small number of combinations of data and degrees of freedom fulfills simultaneously all these requirements. Among all the corresponding estimates of the critical amplitude we then select the smallest and the largest ones as lower and upper bounds.

6.2 Critical amplitudes

In tab. 7 we report as an example the fits to the magnetization (with the scaling function obtained by deriving eq. (84)) fulfilling the above requirements. For each value of $N_f$ we only report the fits with the minimum and maximum allowed number of d.o.f., since the best fit result for $A_{sM}$ changes monotonically as the data are eliminated from the fit. This is a general pattern for all the observables that we studied and greatly simplifies the analysis of the data. Looking at the table one can see that at least four parameters are needed in the fit to have a reasonable confidence level, due to the very small error of the data that we use. In the last line we report the only fit in which all the 25 data reported in tab. 10 have been used. It required taking into account the first eight terms of the scaling function. For $N_f > 8$, even

and lower bounds for them) but only as a tool to eliminate those situations in which the fitting functions are clearly unable to describe the input data.

\textsuperscript{9}In making this choice we also keep into account the errors in the estimates induced by the systematic errors of the input data.
if we use all the data at our disposal we cannot fulfil requirement 3. It is interesting to notice that the fits which give the best approximations to the exact value of $A_M'$ are those in which we use the largest possible number of terms of the scaling function. This is a general pattern for all the observables that we studied. All the fits were performed using the double precision NAG routine GO2DAF. The bounds that we obtained are listed in tab. together with the S-matrix predictions. From these results we immediately obtain the upper and lower bounds for the universal amplitude ratios of tab. They are reported in the last two lines of tab. 8.

| $A_M'$          | $N_f$ | d.o.f | C.L. |
|-----------------|-------|-------|------|
| 1.05898893(196) | 4     | 4     | 83%  |
| 1.05899447(58)  | 4     | 6     | 50%  |
| 1.05898584(74)  | 5     | 6     | 94%  |
| 1.05898882(22)  | 5     | 7     | 48%  |
| 1.05898178(156) | 6     | 6     | 98%  |
| 1.05898375(8)   | 6     | 10    | 98%  |
| 1.05898433(8)   | 7     | 14    | 80%  |
| 1.05898694(18)  | 8     | 15    | 99%  |
| 1.05898729(13)  | 8     | 17    | 96%  |

Table 7: Fits to the magnetization fulfilling the requirements 1-4 (see text). In the first column the best fit results for the critical amplitude (with in parenthesis the error induced by the systematic errors of the input data), in the second column the number of parameters in the fit, in the third column the number of degrees of freedom and in the last column the confidence level. For each value of $N_f$ we only report the fits with the minimum and maximum allowed number of d.o.f, since the best fit of $A_f'$ changes monotonically as the data are eliminated from the fit.

### 6.3 Subleading operators

In principle we could try to estimate in the fits discussed above also the amplitudes of the first two or three subleading terms in the scaling functions, however it is clear that the results that we would obtain would be strongly cross correlated and we would not be able to give reliable estimates for the
| Observable | Lower bound | Upper bound | Theory               |
|------------|-------------|-------------|----------------------|
| $A_l$      | 0.9927985   | 0.9928005   | 0.9927995...         |
| $A_{m1}$   | 1.058980    | 1.058995    | 1.058986...          |
| $A_{m2}$   | 0.07055     | 0.07072     | 0.070599...          |
| $A_{m3}$   | 0.58050     | 0.58059     | 0.58051...           |
| $A_{m1}^f$ | 4.01031     | 4.01052     | 4.01040...           |
| $A_{m2}^f$ | 6.486       | 6.491       | 6.4890...            |
| $A_{m3}^f$ | 7.91        | 8.02        | 7.9769...            |
| $|A_{F_1}^f|$ | 0.6405      | 0.6411      | 0.6409...            |
| $|A_{F_1}^f|$ | 3.699       | 3.714       | 3.7066...            |
| $|A_{F_2}^f|$ | 0.3         | 0.35        | 0.3387...            |
| $|A_{F_2}^f|$ | 3.32        | $\sim 3.45$| 3.4222...            |
| $R_\chi$   | 6.7774      | 6.7789      | 6.77828...           |
| $Q_2$      | 3.2296      | 3.2374      | 3.23514...           |

Table 8: Lower and upper bounds for various critical amplitudes discussed in the text and, in the last two lines, for the two universal amplitude ratios $R_\chi$ and $Q_2$.

corresponding errors (except, at most, for the first one of them, the next to leading term in the scaling function).

In order to obtain some information on the subleading terms we decided to follow another route. The results of the previous section strongly support the correctness of the S-matrix predictions. We decided then to assume these predictions as an input of our analysis, fixing their values in the scaling functions. Then we used the same procedure discussed in sect. 6.1 to identify the amplitude of the first subleading field. Let us look to the various scaling functions in more detail.

6.3.1 Free energy

This is the case for which we have the most precise data. Moreover we may use the data for the magnetization and the susceptibility as a cross check of our estimates.

Combining all the data at our disposal we end up with a rather precise estimate for $A_{f,b}^l$, which turns out to be bounded by:
As mentioned in sect. 3.4.1 it is possible to evaluate this amplitude in a completely different way, by looking at the constant term in the magnetic susceptibility of the model at the critical point. The comparison between our estimate and the expected value represents a test of the reliability of our fitting procedure. The expected value of this amplitude \( A_{f,b} \) is (in our units)

\[
A_{f,b} = -0.0524442... \tag{106}
\]

which is indeed in perfect agreement with our estimates.

We can then use the value of eq. (106) as a fixed input and try to estimate the amplitude of the following subleading field which has a very important physical meaning being the contribution due to the presence of the \( T \bar{T} \) (and related terms) operator in the lattice Hamiltonian. Remarkably enough, it turns out, by applying the usual analysis, that the corresponding amplitude \( A_{f,1} \) is compatible with zero. More precisely we see that, changing the number of input data and of parameters in the scaling function, the sign of \( A_{f,1} \) changes randomly and its modulus is never larger than \( 10^{-4} \). The same pattern is reproduced in the magnetization and in the susceptibility. We summarize these observations with the following bound

\[
|A_{f,1}| < 0.00005 \tag{107}
\]

This result agrees with the observation concerning the absence of corrections due to irrelevant operators in the case \( t \neq 0 \) and \( h = 0 \). (For a thorough discussion of this point see [34] and refs. therein.)

If we also assume that \( A_{f,1} = 0 \) then we may give a reliable estimate for the amplitude of \( A_{f,2} \) which turns out to be bounded by:

\[
0.020 < A_{f,2} < 0.022 \tag{108}
\]

This is the highest subleading term that we could study with a reasonable degree of confidence in our scaling functions.

### 6.3.2 Internal energy

In the case of the internal energy the first subleading amplitude can be studied with very high confidence since it is associated to a very small exponent:

\[
-0.055 < A_{f,b}^I < -0.050 \tag{105}
\]
$|h_1|^{\frac{n}{2}}$. The result turns out to be

$$-0.646 < A_{E,1}^l < -0.644.$$  \hspace{1cm} (109)

In this case the fits are so constrained that we can study with a rather good degree of confidence also the next subleading correction, $A_{E,2}^l$, which is very interesting, since it again contains the $T\bar{T}$ term discussed above. In agreement with the previous observations also in this case the amplitude turns out to be compatible with zero. More precisely its sign changes randomly as the input data are changed in the fits and its modulus can be bounded by:

$$|A_{E,2}^l| < 0.005.$$  \hspace{1cm} (110)

which is not as strong as the bound of eq. (107) but clearly goes in the same direction.

### 6.3.3 Masses

The most interesting feature of the scaling functions for the masses is that there is no analytic term and the first subleading contribution $A_{m,1}^l$ is the exact analogous of the $A_{f,1}^l$ term for the free energy.

In this case we find a non zero contribution for $A_{m,1}^l$. In particular we find the following bounds for the three masses that we studied:

$$-0.21 < A_{m_{1},1}^l < -0.20$$  \hspace{1cm} (111)

$$-0.48 < A_{m_{2},1}^l < -0.41$$  \hspace{1cm} (112)

$$-0.65 < A_{m_{3},1}^l < -0.50.$$  \hspace{1cm} (113)

In the case of the masses a preferred direction is singled out. Therefore, one has to expect that there is a finite overlap with the irrelevant operator that breaks the rotational symmetry. Notice that a similar contribution has been observed also in the case of the thermal perturbation of the Ising model in [35] where the authors studied the breaking of rotational invariance in the two point correlator (see sect.IV-G of [35] for a discussion of this point).

Our results on the amplitude of the subleading corrections are summarized in tab. 43.
Table 9: Lower and upper bounds for the amplitudes of some of the subleading corrections.

| Amplitude          | Lower Bound | Upper Bound |
|--------------------|-------------|-------------|
| $A_{f,b}$          | $-0.055$    | $-0.050$    |
| $|A_{f,1}|$         | $0.00005$   | $0.022$     |
| $A_{f,2}$          | $-0.646$    | $-0.644$    |
| $|A_{E,1}|$         | $0.005$     | $0.20$      |
| $A_{m1,1}$         | $-0.21$     | $-0.48$     |
| $A_{m2,1}$         | $-0.21$     | $-0.65$     |
| $A_{m3,1}$         | $-0.21$     | $-0.65$     |

7 Conclusions

The major goal of this paper was to test the S-matrix description proposed by Zamolodchikov in [2] for the 2d Ising model perturbed by a magnetic field. To this end we developed some tools and obtained some results which are rather interesting in themselves. In particular

- We improved the standard transfer matrix calculations by implementing a zero momentum projection which allowed us to drastically reduce the dimension of the matrix.

- We discussed in detail the relationship between continuum and lattice observables.

- By using CFT results at the critical point we constructed the first 7-8 terms of the scaling functions for various quantities on the lattice.

We could obtain in this way very precise numerical estimates for several critical amplitudes (in some cases with 5 or even 6 significative digits) and in all cases we found a perfect agreement between S-matrix predictions and lattice results.

By assuming the S-matrix predictions as an input of our analysis we could estimate some of the subleading amplitudes in the scaling functions. In one case the value of the subleading amplitude was already known and again we found a complete agreement between theoretical prediction and numerical estimate. For the remaining ones there is up to our knowledge no theoretical
prediction. They are collected in tab. 8 and represent the most interesting outcome of our analysis. We leave them as a challenge for theorists working in the field.

Among the others, the most surprising result concerns the $T\bar{T}$ term which turns out to have a negligible amplitude in the scaling functions of the translationally invariant observables. It would be nice to understand which is the reason of such behaviour.

Let us conclude by stressing that the techniques that we have developed can be easily extended to the case in which a combinations of both thermal and magnetic perturbations is present. In this case the exact integrability is lost and our numerical methods could help to test new approaches and suggest new ideas.

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Tables of data

Table 10: Data used in the fits

| $h_t$ | $f$          | $M$          | $E$          |
|-------|--------------|--------------|--------------|
| 0.20  | 1.106272538601(1) | 0.934113075978(1) | 0.182495416253(1) |
| 0.19  | 1.096943627061(1) | 0.931644255995(1) | 0.17910587939(1) |
| 0.18  | 1.087640179593(1) | 0.929017517063(1) | 0.17544472125(1) |
| 0.17  | 1.078363862266(1) | 0.926215008782(1) | 0.17180991529(1)  |
| 0.16  | 1.069116534998(1) | 0.923215694344(1) | 0.167881687799(1) |
| 0.15  | 1.059900287285(1) | 0.919994540350(1) | 0.163741028380(1) |
| 0.14  | 1.05071483321(1)  | 0.916521430645(1) | 0.159366050850(1) |
| 0.13  | 1.041570819851(1) | 0.91275968274(1)  | 0.154730958303(1) |
| 0.12  | 1.032463401585(1) | 0.90866397795(1)  | 0.149804982192(1) |
| 0.11  | 1.023398841451(1) | 0.90417470223(1)  | 0.14455091814(1)  |
| 0.10  | 1.014381396853(1) | 0.89922709483(1)  | 0.13892305302(1)  |
| 0.09  | 1.00541615982(1)  | 0.89371763122(1)  | 0.13286414108(1)  |
| 0.08  | 0.99650933082(1)  | 0.88752055778(1)  | 0.1263008320(1)   |
| 0.075 | 0.99208015671(1)  | 0.8841104491(1)   | 0.1228010112(1)   |
| 0.066103019026467 | 0.98424336850(1) | 0.87741739906(1)  | 0.1161548337(1)   |
| 0.055085849188723 | 0.97462849835(1) | 0.86771676938(2)  | 0.10703505648(2)  |
| 0.05  | 0.97022834(1)     | 0.86255168(1)     | 0.10241966(1)     |
| 0.044068679350978 | 0.96513182856(1) | 0.8558157835(1)   | 0.096641767(1)    |
| 0.033051509513233 | 0.95578360408(2) | 0.840485633(1)    | 0.084459355(1)    |
| 0.03  | 0.95322656(1)     | 0.83533709(5)     | 0.0806726(1)      |
| 0.022034339675489 | 0.9466343776(2)  | 0.81901353(2)     | 0.0695436(1)      |
| 0.02  | 0.94497303(2)     | 0.8139196(1)      | 0.0663409(2)      |
| 0.015 | 0.9409395(1)      | 0.7988985(1)      | 0.057595(1)       |
| 0.01  | 0.936994(1)       | 0.77805(5)        | 0.047045(3)       |
| 0.0088137358702   | 0.93607461(2)     | 0.771605(1)       | 0.044149(2)       |
Table 11: Data used in the fits

| $h_t$     | $1/m_1$                     | $1/m_2$                     | $1/m_3$                     |
|-----------|-----------------------------|-----------------------------|-----------------------------|
| 0.20      | 0.59778522553(1)            | 0.37795775263(1)            | 0.310888(1)                 |
| 0.19      | 0.61388448719(1)            | 0.38765653507(1)            | 0.318578(1)                 |
| 0.18      | 0.63134670477(1)            | 0.39818995529(1)            | 0.326940(1)                 |
| 0.17      | 0.65037325706(1)            | 0.40968266918(1)            | 0.336077(2)                 |
| 0.16      | 0.67120940172(1)            | 0.42228634593(5)            | 0.346115(3)                 |
| 0.15      | 0.69415734924(1)            | 0.43618773124(1)            | 0.357209(3)                 |
| 0.14      | 0.71959442645(1)            | 0.45161985381(4)            | 0.369548(4)                 |
| 0.13      | 0.74799884641(1)            | 0.4688779288(2)             | 0.38338(1)                  |
| 0.12      | 0.77998715416(1)            | 0.488342470(1)              | 0.3990(1)                   |
| 0.11      | 0.81637015277(1)            | 0.510513817(1)              | 0.4168(1)                   |
| 0.10      | 0.85823913569(5)            | 0.5360654(1)                | 0.4374(5)                   |
| 0.09      | 0.9071039295(1)             | 0.5659287(6)                | 0.4624(5)                   |
| 0.08      | 0.965123997(1)              | 0.60144(1)                  | 0.492(1)                    |
| 0.075     | 0.998514180(1)              | 0.62189(1)                  | 0.508(1)                    |
| 0.066103019026467 | 1.067300500(2)  | 0.66405(5)                  | 0.543(1)                    |
| 0.055085849188723 | 1.17524158(3)  | 0.7305(1)                   |                             |
| 0.05      | 1.237044(1)                 | 0.768(1)                    |                             |
| 0.044068679350978 | 1.322589(6)    | 0.82(1)                     |                             |
| 0.033051509513233 | 1.54057(2)    |                             |                             |
| 0.03      | 1.6218(2)                   |                             |                             |
| 0.022034339675489 | 1.91(1)        |                             |                             |
Table 12: Data used in the fits

| $h_t$ | $|F_1^2|^2$ | $|F_2^2|^2$ |
|-------|------------|------------|
| 0.20  | 0.29041938711(1) | 0.03800933(1) |
| 0.19  | 0.29570405694(1) | 0.04039999(1) |
| 0.18  | 0.3017729858(1)  | 0.04291078(1) |
| 0.17  | 0.30653975241(1) | 0.04554676(1) |
| 0.16  | 0.31209194307(1) | 0.04831337(1) |
| 0.15  | 0.31773424601(1) | 0.05121641(1) |
| 0.14  | 0.32346684419(1) | 0.05426214(1) |
| 0.13  | 0.3292896717(1)  | 0.05745711(3) |
| 0.12  | 0.3352023388(1)  | 0.0608082(1)  |
| 0.11  | 0.3412040323(4)  | 0.0643227(2)  |
| 0.10  | 0.3472933781(4)  | 0.068008(1)   |
| 0.09  | 0.3534682486(5)  | 0.07187(1)    |
| 0.08  | 0.359725487(1)   | 0.0759(1)     |
| 0.075 | 0.362883627(1)   | 0.0780(2)     |
| 0.066103019026467 | 0.368548928(2) | 0.0818(5)    |
| 0.055085849188723 | 0.3756378(4)   |            |
| 0.05  | 0.378934(5)      |            |
| 0.044068679350978 | 0.38280(2)     |            |
| 0.033051509513233 | 0.3899(1)      |            |
Table 13: Data used in the fits

| $h_l$        | $|F_1^2|^2$                      | $|F_2^2|^2$                      |
|--------------|---------------------------------|---------------------------------|
| 0.20         | 11.7000114647(1)                | 8.0468067(5)                    |
| 0.19         | 11.8448924368(1)                | 8.3246112(5)                    |
| 0.18         | 11.9888157747(1)                | 8.6017424(5)                    |
| 0.17         | 12.1315853880(1)                | 8.8774943(5)                    |
| 0.16         | 12.2729879270(1)                | 9.1511600(5)                    |
| 0.15         | 12.4127893980(1)                | 9.422023(1)                     |
| 0.14         | 12.5507307560(1)                | 9.689348(2)                     |
| 0.13         | 12.6865220830(5)                | 9.952360(5)                     |
| 0.12         | 12.819834783(1)                 | 10.21022(1)                     |
| 0.11         | 12.950290902(2)                 | 10.46202(3)                     |
| 0.10         | 13.077448185(2)                 | 10.7067(5)                      |
| 0.09         | 13.200778543(2)                 | 10.943(3)                       |
| 0.08         | 13.31963596(3)                  | 11.17(1)                        |
| 0.075        | 13.3771415(1)                   | 11.28(1)                        |
| 0.066103019026467 | 13.475815(5)              | 11.46(2)                        |
| 0.055085849188723 | 13.59037(3)              | 11.6(5)                         |
| 0.05         | 13.6398(5)                      |                                |
| 0.044068679350978 | 13.695(1)                     |                                |
| 0.033051509513233 | 13.78(1)                      |                                |