Spin glass field theory with replica Fourier transforms

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Received 13 March 2014, revised 11 August 2014
Accepted for publication 27 August 2014
Published 23 October 2014

Abstract

We develop a field theory for spin glasses using replica Fourier transforms (RFT). We present the formalism for the case of replica symmetry and the case of replica symmetry breaking on an ultrametric tree, with the number of replicas $n$ and the number of replica symmetry breaking steps $R$ generic integers. We show how the RFT applied to the two-replica fields allows one to construct a new basis which block-diagonalizes the four-replica mass-matrix, into the replicon, anomalous and longitudinal modes. The eigenvalues are given in terms of the mass RFT and the propagators in the RFT space are obtained by inversion of the block-diagonal matrix. The formalism allows one to express any $i$-replica vertex in the new RFT basis and hence enables one to perform a standard perturbation expansion. We apply the formalism to calculate the contribution of the Gaussian fluctuations around the Parisi solution for the free-energy of an Ising spin glass.

Keywords: spin glasses, field theory, replica Fourier transforms

PACS numbers: 64.60.De, 75.10.Nr

1. Introduction

Spin glasses are disordered magnetic systems with frustration [1–4]. These systems exhibit a freezing transition to a low temperature phase with nontrivial properties. Although spin glasses have been studied for over three decades there is still no consensus on the nature of the glassy phase. Two different pictures have been proposed for the spin glass. One corresponds to the Parisi solution [5] of the infinite-range Sherrington–Kirkpatrick (SK) model [6], which represents the mean field theory for spin glasses and predicts a glassy phase described by an infinite number of pure states, organized in an ultrametric structure. The other one is the ‘droplet’ model [7–9], which claims that in the experimentally relevant short-range spin...
glasses the glassy phase is described by only two pure states, related by a global inversion of the spins. The first picture is set within a replica field theory and results from replica symmetry breaking, while the second picture is based on a scaling theory and corresponds to no replica symmetry breaking. An important step for the understanding of spin glasses, lies in the investigation of how the fluctuations, associated into the finite-range interactions modify the mean-field picture.

Edwards and Anderson [10] introduced a model for short-range spin glasses and used the replica method to perform the average over quenched disorder. A field theory is built for the spin glass with the free energy being written as a functional of replica fields \( Q_{iab} \) (where \( a = 1, \ldots, n \) is a replica index), which represent the spin glass order parameter. A perturbation expansion around the mean-field solution, which corresponds to the infinite-range or infinite-dimensional (i.e., spin coordination number \( z \to \infty \)) model is constructed by separating the field \( Q_{iab} \) into its mean field value \( Q_{ab} \) and fluctuations \( \phi_{iab} \) around it. The mean field value of the order parameter \( Q_{ab} \) is provided by the stationarity condition of the free energy and the stability of the solution is determined by the analysis of the Hessian or mass-matrix \( M_{abcd} \), of the fluctuations, that is by the evaluation of the eigenvalues or, in other terms, the diagonalization of the matrix. In turn, to calculate physical properties one needs the propagators \( G^{abcd}_{ab} \) of the fluctuations, the bare propagators being given by the inverse of the mass-matrix. The replica dependence of \( Q_{ab} \), which reflects the structure of the order parameter, naturally determines the form of the mass-matrix, and also the form of the bare propagators and the interaction vertices of the fluctuations for higher order calculations in the perturbation expansion.

In mean field theory it is found that a phase transition occurs at a critical temperature, from a high-temperature phase with replica symmetry (RS) to a low-temperature phase with replica symmetry breaking (RSB). The stability of the RS solution was studied by de Almeida and Thouless [11], who provided the eigenvalues, their multiplicities, and the eigenvectors of the mass-matrix. They found three different sets of modes, later called replicon, anomalous and longitudinal [12], with the longitudinal and anomalous eigenvalues becoming equal for \( n = 0 \). In the presence of a magnetic field the instability of the RS solution against RSB occurs along a line in the temperature-field plane, the Almeida–Thouless (AT) line. In zero field all the modes become critical at the transition temperature, while in nonzero field only the replicon mode becomes critical at the AT line. The propagators for the RS theory were obtained by Bray and Moore [12] and Pytte and Rudnick [13]. The RSB ansatz proposed by Parisi for the spin glass, which turns out to be the exact solution for the SK model [14], represents many states in a hierarchical organization that is described by an ultrametric tree. The study of the stability of the Parisi RSB solution is a nontrivial task that was carried out by De Dominicis and Kondor [15]. They found that the eigenvalues of the mass-matrix form two continuous bands, for \( n = 0, \) corresponding to replicon and longitudinal-anomalous modes, the lower band, associated to the replicon, being bounded below by zero. Fairly involving computations performed by Kondor and De Dominicis [16] and De Dominicis and Kondor [17] led to results for the multiplicities and propagators of the RSB theory. See De Dominicis et al [18] for a review on the spin glass field theory with RSB. The high complexity of the theory has however inhibited the study of the glassy phase.

A fundamental aspect for the study of spin glasses is the diagonalization and inversion of the ultrametric four-replica mass-matrix, which turns out to be a rather difficult problem. Temesvári et al [19] and De Dominicis et al [20] provided results for the block-diagonalization and inversion of the mass-matrix in direct replica space. The block diagonalized form is a consequence of the ultrametric symmetry of the matrix which reflects the residual symmetry of the problem, after the Parisi breaking of replica symmetry. De Dominicis et al
[21] later used the concept of replica Fourier transform to block-diagonalize and invert the mass-matrix, clearly showing the advantage of this method.

In this article we develop a field theory for spin glasses using replica Fourier transforms (RFT). We consider both the case of a replica symmetric theory where the simple RFT is used and the case of replica symmetry breaking where the RFT is defined on a tree. We show how the RFT applied to the two-replica fields leads to a new basis which block-diagonalizes the four-replica mass-matrix, into three sets of modes, replicon, anomalous and longitudinal. The eigenvalues of the replicon, anomalous and longitudinal modes are then given in terms of the RFT of the mass-matrix. The corresponding multiplicities and eigenvectors are provided. The propagators in the RFT space are then readily obtained by inversion of the block-diagonal mass-matrix. The formalism allows to express any i-replica vertex in the new RFT basis, and hence enables to perform a standard perturbation expansion. We keep the number n of replicas a positive integer, the limit n → 0 of the replica method can be taken at the very end, on the final results. The number of replica symmetry breaking steps R is also considered a generic integer, hence our results apply either in a situation where only a single RSB step is needed, or in the case of full RSB R → ∞ proposed by Parisi. We show that many fundamental results for the study of spin glasses, can be simply derived within the RFT formalism.

The outline of this article is as follows. In section 2 we present the field theory for an Ising spin glass in direct replica space. In section 3 we develop the RFT formalism for the replica symmetric case R = 0. In section 4 we generalize the RFT formalism to the case of replica symmetry breaking R ≠ 0. In section 5 we apply the formalism to calculate the contribution of the Gaussian fluctuations around the Parisi solution for the free-energy of an Ising spin glass, which illustrates the physical relevance of the results presented. Section 6 concludes the article with an overview of the work.

2. Spin glass model

We consider an Ising spin glass in a uniform magnetic field $H$, described by the Edwards–Anderson model

$$H = -\sum_{\langle ij \rangle} J_{ij} S_i S_j - H \sum_i S_i$$

for N spins, $S_i = \pm 1$, located on a regular d-dimensional lattice, where the bonds $J_{ij}$, which couple nearest-neighbor spins only, are independent random variables with a Gaussian distribution, characterized by zero mean and variance $\Delta^2 = J^2 / \tau$, $\tau = 2d$ being the coordination number. The summations are over pairs $\langle ij \rangle$ of distinct sites on the lattice and over the lattice sites $i$.

The free energy averaged over the quenched disorder is given, via the replica method, by

$$\bar{F} = -\frac{1}{\beta} \ln \bar{Z} = -\frac{1}{\beta} \lim_{n \to 0} \frac{\bar{Z}^n - 1}{n},$$

where $Z$ is the partition function and $\beta = 1/k_B T$.

Taking the average of n replicas of the partition function $\bar{Z}^n$, with n integer, followed by a Hubbard–Stratonovich transformation, to decouple a four-spin term, leads to
\[
\mathcal{Z}^n = \int \prod_{(ab)1} \frac{dQ_{ab}^{(1)}}{\sqrt{2\pi}} \exp \left\{ \mathcal{L} \left[ Q_{ab}^{(1)} \right] \right\}
\]

with
\[
\mathcal{L} \left[ Q_{ab}^{(1)} \right] = -\frac{Nn(\beta J)^2}{4} + \frac{z(\beta J)^2}{2} \sum_{i,j} Q_{i}^{ab} (K^{-1})_{ij} Q_{j}^{ab} - \sum_{i} \ln \text{Tr} \exp \left\{ (\beta J)^2 \sum_{(ab)} Q_{i}^{ab} S_{i}^{a} S_{i}^{b} + \beta H \sum_{a} S_{i}^{a} \right\},
\]

where \( K_{ij} = 1 \) for nearest neighbor sites and 0 otherwise, and \( S_{i}^{a} \) are spins with replica index \( a = 1, \ldots, n \). The fields \( Q_{i}^{ab} \) are defined on an \( n(n-1)/2 \)-dimensional replica space of pairs \((ab)\) of distinct replicas, since \( Q_{i}^{ab} = Q_{i}^{ba} \) and \( Q_{i}^{aa} = 0 \).

In order to construct a perturbation expansion around the mean-field solution, one separates the field \( Q_{i}^{ab} \) into
\[
Q_{i}^{ab} = Q_{i}^{ab} + \phi_{i}^{ab},
\]

where \( Q_{i}^{ab} \) represents the mean field order parameter and \( \phi_{i}^{ab} \) are fluctuations around it. The Lagrangian \( \mathcal{L} \) is then given by
\[
\mathcal{L} = \mathcal{L}^{(0)} + \mathcal{L}^{(1)} + \mathcal{L}^{(2)} + \ldots,
\]

where, after Fourier transform into momenta space, one has, for contributions up to quadratic order in the fluctuations,
\[
\mathcal{L}^{(0)} = -\frac{Nn(\beta J)^2}{4} + \frac{N(\beta J)^2}{2} \sum_{(ab)} (Q_{i}^{ab})^2 - N \ln \text{Tr} \exp \left\{ (\beta J)^2 \sum_{(ab)} Q_{i}^{ab} S_{i}^{a} S_{i}^{b} + \beta H \sum_{a} S_{i}^{a} \right\},
\]

\[
\mathcal{L}^{(1)} = \sqrt{N} (\beta J)^2 \sum_{(ab)} [Q_{i}^{ab} - \langle S^{a} S^{b} \rangle] \phi_{i}^{ab},
\]

\[
\mathcal{L}^{(2)} = \frac{1}{2} \sum_{(ab)(cd)} \sum_{p} \phi_{i}^{ab} M_{p}^{ab,cd} \phi_{i}^{cd},
\]

with
\[
M_{p}^{ab,cd} = p^{2} \delta_{s_{ab},s_{cd}} + \zeta \left[ \delta_{s_{ab},s_{cd}} - (\beta J)^2 \langle S^{a} S^{b} S^{c} S^{d} \rangle - \langle S^{a} S^{b} \rangle \langle S^{c} S^{d} \rangle \right],
\]

where \( S_{i}^{a} = S_{a}^{i} \) and the expectation value \( \langle \cdots \rangle \) is calculated with the normalized weight \( \zeta(S)/\text{Tr} \zeta(S) \), where
\[
\zeta(S) = \exp \left\{ (\beta J)^2 \sum_{(ab)} Q_{i}^{ab} S_{i}^{a} S_{i}^{b} + \beta H \sum_{a} S_{i}^{a} \right\}.
\]
In (9), the sum in momenta is confined to the range \( 0 < |\mathbf{p}| \ll \Lambda \), with a cutoff \( \Lambda \approx 1 \), the mass-matrix \( M^{ab,cd}(\mathbf{p}) \) is expanded for small \( \mathbf{p} \), keeping only the terms up to second order, and the fields are rescaled \( [\phi(\beta J/\sqrt{\varepsilon}) \rightarrow \phi] \) to allow one to write the coefficient of the momentum equal to unity.

The mean-field value of the order parameter \( Q^{ab} \) is determined by the stationarity condition \( \mathcal{L}^{(1)} = 0 \), which from (8) gives

\[
Q^{ab} = \langle S^a S^b \rangle. \tag{12}
\]

Hence, \( Q^{ab} \) represents the spin overlap between replicas \( a \) and \( b \). Considering a replica symmetric (RS) solution, \( Q^{ab} = Q \), (12) leads to the following results. In zero magnetic field, \( H = 0 \), there is a phase transition at a critical temperature \( T_c = J/k_B \); \( Q = 0 \) for \( T > T_c \), while \( Q \neq 0 \) for \( T < T_c \). However, the RS solution turns out to be unstable in the low-temperature phase, and RSB is required. In a nonzero magnetic field, \( H \neq 0 \), there is a phase transition along a line in the field-temperature plane, the AT line, which in the region of small fields \( H \), and near the zero-field critical temperature \( T_c \), is given by \( (H/J)^2 = (4/3)(1 - T/T_c)^2 \); above the AT line a RS solution \( Q \neq 0 \) is stable, while below the AT line the RS solution becomes unstable and RSB is required.

The normal modes of the fluctuations of the order parameter are obtained by re-writing \( \mathcal{L}^{(2)} \), (9), in a diagonal form. The eigenvalues of the matrix \( M^{ab,cd} \) are then provided, and the propagators can be easily obtained by inversion of the diagonalized matrix.

### 3. Replica symmetric ansatz

Here we consider that the mean-field order parameter is replica symmetric

\[
Q^{ab} = Q, \quad a \neq b. \tag{13}
\]

In this case, there are three distinct masses

\[
M^{ab,ab} = M_{11},
M^{ab,ac} = M^{ab,bc} = M_{10},
M^{ab,cd} = M_{00}. \tag{14}
\]

The Lagrangian term of the fluctuations \( \mathcal{L}^{(2)} \), (9), then takes the form

\[
\mathcal{L}^{(2)} = \frac{1}{2} \left\{ M_1 \sum_{(ab)} \phi_{ab}^2 + M_{10} \sum_{(abc)} \phi_{ab} \phi_{ac} + M_{00} \sum_{(abcd)} \phi_{ab} \phi_{cd} \right\}, \tag{15}
\]

where the dependence on momentum \( \mathbf{p} \) is implicit and the sums are restricted to distinct replicas.

Writing \( \mathcal{L}^{(2)} \) in terms of sums over unrestricted replicas, one obtains

\[
\mathcal{L}^{(2)} = \frac{1}{4} \left\{ (M_{11} - 2M_{10} + M_{00}) \sum_{a,b} \phi_{ab}^2 + (M_{10} - M_{00}) \sum_{a,b,c} \phi_{ab} \phi_{ac} + \frac{1}{2} M_{00} \sum_{a,b,c,d} \phi_{ab} \phi_{cd} \right\}. \tag{16}
\]
with the field constraints

$$\phi_{aa} = 0, \quad a = 1, \ldots, n.$$  \hspace{1cm} (17)

The RFT for a field with a single replica index, and its inverse transformation, are defined as

$$\phi_{\hat{a}} = \frac{1}{\sqrt{n}} \sum_a e^{-i\phi_{aa}\hat{a}} \phi_a$$

$$\phi_a = \frac{1}{\sqrt{n}} \sum_{\hat{a}} e^{i\phi_{a\hat{a}}\hat{a}} \phi_{\hat{a}}$$  \hspace{1cm} (18)

with $a = 1, \ldots, n$, $\hat{a} = 0, \ldots, n - 1$, and $a, \hat{a}$ considered mod ($n$). One has the relation

$$\sum_a e^{i\phi_{a\hat{a}}\hat{a}} = n\delta_{0,\hat{a}}.$$  \hspace{1cm} (19)

(using $\hat{0}$ when $\hat{a} = 0$). From (18), it follows that $\phi_{\hat{a}}^* = \phi_{a\hat{a}}$. For the two-replica fields we then have

$$\phi_{\hat{a}b} = \frac{1}{n} \sum_{ab} e^{i\phi_{a\hat{b}}\hat{a} + b\hat{b}} \phi_{\hat{a}b}$$

$$\phi_{\hat{a}\hat{b}} = \frac{1}{n} \sum_{ab} e^{i\phi_{a\hat{b}}\hat{a} + b\hat{b}} \phi_{ab}$$  \hspace{1cm} (20)

with the symmetry $\phi_{\hat{a}b} = \phi_{b\hat{a}}$ resulting from $\phi_{ab} = \phi_{ba}$. The fields can be written as $\phi_{\hat{a}b} = \phi_{\hat{a},i-\hat{a}}$, where $i = \hat{a} + \hat{b}$. For $i = 0$ the fields are real, while for $i \neq 0$ they are complex, with $\phi_{\hat{a},i-\hat{a}}^* = \phi_{\hat{a},i+\hat{a}}$.

After RFT the Lagrangian $\mathcal{L}^{(2)}$ becomes,

$$\mathcal{L}^{(2)} = \frac{1}{4} \left\{ (M_{11} - 2M_{00} + M_{00}) \sum_{\hat{a}, \hat{\hat{a}}} |\phi_{\hat{a},i-\hat{\hat{a}}}|^2 \right.$$

$$+ \left. 2n \left( M_{11} - M_{00} \right) \sum_i |\phi_{0,i}|^2 + \frac{n^2}{2} M_{00} |\phi_{0,0}|^2 \right\}$$  \hspace{1cm} (21)

with the field constraints in (17) expressed as

$$\sum_{\hat{a}} \phi_{\hat{a},i-\hat{a}} = 0, \quad i = 0, \ldots, n - 1$$  \hspace{1cm} (22)

which follows from taking the RFT of $\phi_{aa}$ over the index $a$. 
Separating in (21) the fields with indices \( \hat{0} \), one obtains

\[
\mathcal{L}^{(2)} = \frac{1}{4} \left\{ (M_{11} - 2M_{10} + M_{00}) \left( \sum_{a} \left| \phi^\prime_{a', -a''} \right|^2 + \sum_{\hat{r}, \hat{a}'} \left| \phi^\prime_{\hat{r}, \hat{a}' - a''} \right|^2 \right) + 2 \left( M_{11} + (n - 2)M_{10} - (n - 1)M_{00} \right) \sum_{\hat{r}} \left| \phi^\prime_{0, \hat{r}} \right|^2 \right. \\
+ \left. \left( M_{11} + 2(n - 1)M_{10} + \left( 1 - 2n + \frac{n^2}{2} \right)M_{00} \right) \left| \phi^\prime_{0, \hat{0}} \right|^2 \right\} 
\]

(23)

where \( \hat{r} \neq \hat{0}, \hat{a}' \neq \hat{0} \) and \( \hat{a}'' \neq \hat{0}, \hat{r}' \).

Now, we define the new fields

\[
\Phi^\prime_{a', -a''} = \phi^\prime_{a', -a''} + \frac{1}{n - 1} \phi^\prime_{0, \hat{0}}
\]

(24)

and

\[
\Phi^\prime_{\hat{r}, \hat{a}' - a''} = \phi^\prime_{\hat{r}, \hat{a}' - a''} + \frac{1}{n - 2} \left( \phi^\prime_{0, \hat{r}} + \phi^\prime_{1, \hat{0}} \right)
\]

(25)

which, from (22), have the constraints

\[
\sum_{a'} \Phi^\prime_{a', -a''} = 0,
\]

(26)

and

\[
\sum_{a'} \Phi^\prime_{\hat{r}, \hat{a}' - a''} = 0.
\]

(27)

Then, by introducing the new fields in (23), one obtains the Lagrangian \( \mathcal{L}^{(2)} \) in the diagonal form,

\[
\mathcal{L}^{(2)} = \frac{1}{2} \left\{ M_R \sum_{a} \left| \Phi^R_{a', -a''} \right|^2 + M_R \sum_{\hat{r}, \hat{a}'} \left| \Phi^R_{\hat{r}, \hat{a}' - a''} \right|^2 \right. \\
+ \left. M_L \sum_{\hat{r}} \left| \Phi^L_{0, \hat{r}} \right|^2 + M_L \left| \Phi^L_{0, \hat{0}} \right|^2 \right\}
\]

(28)

where

\[
M_R = M_{11} - 2M_{10} + M_{00} \\
M_L = M_{11} + (n - 2)M_{10} - (n - 3)M_{00} \
M_L = M_{11} + 2(n - 2)M_{10} + \frac{1}{2}(n - 2)(n - 3)M_{00}
\]

(29)

and

\[
^R \Phi^\prime_{a', -a''} = \frac{1}{\sqrt{2}} \phi^\prime_{a', -a''}; \quad ^R \Phi^\prime_{\hat{r}, \hat{a}' - a''} = \frac{1}{\sqrt{2}} \phi^\prime_{\hat{r}, \hat{a}' - a''} \\
^A \phi^\prime_{0, \hat{r}} = \frac{n}{\sqrt{(n - 2)}} \phi^\prime_{0, \hat{r}}; \quad ^L \phi^\prime_{0, \hat{0}} = \frac{n}{\sqrt{2(n - 1)}} \phi^\prime_{0, \hat{0}}
\]

(30)
with the constraints,
\[
\sum_{\tilde{\alpha}'} R\Phi_{\tilde{\alpha}',-\tilde{\alpha}'} = 0; \quad \sum_{\tilde{\alpha}''} R\Phi_{\tilde{\alpha}'',-\tilde{\alpha}''} = 0. 
\] (31)

The fields \( L\phi_{0,0}, A\phi_{0,i}, R\Phi_{\tilde{\alpha}',-\tilde{\alpha}'} \) and \( R\Phi_{\tilde{\alpha}',-\tilde{\alpha}'} \) are symmetrized: \( \phi_{0,0} = \frac{1}{2}(\phi_{0,0} + \phi_{i,0}) \) and \( \Phi_{\tilde{\alpha}',-\tilde{\alpha}'} = \frac{1}{2}(\Phi_{\tilde{\alpha}',-\tilde{\alpha}'} + \Phi_{\tilde{\alpha}',-\tilde{\alpha}'}), \) for \( i = 0 \) and \( i \neq 0 \); the fields \( L\phi_{0,0} \) and \( A\phi_{0,i} \) are also normalized, the normalization of \( R\Phi_{\tilde{\alpha}',-\tilde{\alpha}'} \) is \( N_1 = \sqrt{(n-3)/(2(n-1))} \) and that of \( R\Phi_{\tilde{\alpha}',-\tilde{\alpha}'} \) is \( N_2 = \sqrt{(\delta_{\tilde{\alpha}',-\tilde{\alpha}''} + (n-4)/(n-2))}/2 \).

One can see from (28) that the fluctuation space is divided into three sectors, which we identify as the replicon (R) with eigenvalue \( M_R \), the anomalous (A) with eigenvalue \( M_A \), and the longitudinal (L) with eigenvalue \( M_L \). The degeneracies of the eigenvalues are given by the multiplicities of the fields, \( \mu_R = (n-3)/2 \) for \( R\Phi_{\tilde{\alpha}',-\tilde{\alpha}'} \) and \( \mu_A = (n-1)(n-3)/2 \) for \( R\Phi_{\tilde{\alpha}',-\tilde{\alpha}'} \) leads to \( \mu_R = \mu_A = n(n-3)/2 \) for the replicon, \( \mu_A = (n-1) \) for the anomalous and \( \mu_L = 1 \) for the longitudinal, so that the total number of modes is recovered \( \mu_R + \mu_A + \mu_L = n(n-1)/2 \).

We note that the replicon, anomalous and longitudinal masses are given in terms of the RFT of the original masses as
\[
M_R = \hat{M}_L, \\
M_A = \hat{M}_L + \frac{1}{4}(n-2)\hat{M}_0, \\
M_L = \hat{M}_L + \frac{1}{2}(n-1)\hat{M}_0, 
\] (32)

where the RFT of the original masses are defined as [21]
\[
\hat{M}_1 = M_1 - 2M_0 + M_{00}, \\
\hat{M}_0 = 4(M_{00} - M_{00}), \\
\hat{M}_{00} = 4(M_{00} - M_{00}) + nM_{00}. 
\] (33)

The propagators for the longitudinal, anomalous and replicon modes can be easily obtained from (28) and are given by
\[
L G_{0,0} = L \Phi_{0,0} = \frac{1}{M_L}, 
\] (34)
\[
A G_{i,i'} = A \Phi_{0,i} = \frac{1}{M_A}, 
\] (35)
\[
R G_{\tilde{\alpha}',-\tilde{\alpha}'} = R \Phi_{\tilde{\alpha}',-\tilde{\alpha}'} = \frac{1}{2} \left[ \delta_{\tilde{\alpha}',-\tilde{\alpha}'} + \delta_{\tilde{\alpha}',-\tilde{\alpha}'} - \frac{2}{n-1} \right] \frac{1}{M_R}. 
\] (36)
We remark that the propagators for the replicon, (36) and (37), are not completely
diagonalized because of the constraints in (31).

The propagators in the direct replica space can be easily obtained, in terms of their RFT
expression, e.g.,

\[ G_{\tilde{a},\tilde{b}';\tilde{a}',\tilde{b}'}^{R} = \frac{1}{2} \left[ \delta_{\tilde{a}',\tilde{b}'} - \frac{2}{n-2} \right] \frac{1}{M_{R}}. \]  

We remark that the propagators for the replicon, (36) and (37), are not completely
diagonalized because of the constraints in (31).

4. Replica symmetry breaking: Parisi’s ansatz

The replica symmetry breaking ansatz proposed by Parisi for the mean-field order parameter
can be described as follows. Consider \( Q_{ab} \) as a symmetric \( n \times n \) matrix with zeros on its
diagonal. One starts with the replica symmetric form, in which all the off-diagonal elements
have the same value, \( Q_{0} \). We then divide the \( n \times n \) matrix into blocks of size \( p_{1} \times p_{1} \), and in
the diagonal blocks replace \( Q_{0} \) by \( Q_{1} \), leaving \( Q_{0} \) in the off-diagonal blocks. Each of the
\( p_{1} \times p_{1} \) blocks on the diagonal is subdivided further into \( p_{2} \times p_{2} \) sub-blocks, and in the
diagonal sub-blocks \( Q_{1} \) is replaced by \( Q_{2} \). This procedure of subdivision of the diagonal
blocks is repeated, and for \( R \) replica symmetry breaking steps, goes down to \( p_{R} \times p_{R} \) blocks,
with off-diagonal elements \( Q_{R} \). That amounts to take a sequence of \( R \) sizes,

\[ \ldots > p_{2} > p_{1} > p_{0} > p_{R} \times p_{R}, \]

where by definition \( p_{0} = n \) and \( p_{R+1} = 1 \), and values \( Q_{0}, Q_{1}, Q_{2}, \ldots, Q_{R} \), having \( Q_{uv} = Q_{R+1} = 0 \). The matrix element

\[ Q_{ab}^{R} = Q_{r}, \]  

depends on the overlap of the replicas \( a \cap b = r \), such that, \( Q_{r} \) belongs to a block of size \( p_{r} \)
but not to a sub-block of size \( p_{r+1} \).

The ansatz can be described equivalently in terms of a tree whose extremities are the \( n \)
replicas \( a = 1, 2, \ldots, n \), and which foliates at the various levels \( r = 0, 1, 2, \ldots, R \) with

Figure 1. Tree representation for an \( R = 2 \) RSB ansatz.
multiplicity $n_r = p_r / \beta_{r+1}$, as illustrated in figure 1. Each replica is associated to a string of tree coordinates,

$$a: [a_0, a_1, \ldots, a_R]$$

(40)

which tells the path to reach replica $a$. Each component takes $n_r$ values, $a_r = 1, 2, \ldots, n_r$. The overlap of replicas $a$ and $b$ is then defined as

$$a \cap b = r, \quad 0 \leq r \leq R + 1$$

if $a_0 = b_0, \ldots, a_{r-1} = b_{r-1}, \quad \text{but} \quad a_r \neq b_r$

(41)

with $a \cap b = R + 1$ corresponding to $a = b$. The overlap $a \cap b = r$ represents a kind of hierarchical distance between replicas $a$ and $b$. At the $r$th level of hierarchy the order parameter takes the value $Q_{ab}^r$. The tree displays the geometric properties of the order-parameter matrix. In particular, it has ultrametricity, that is, given three replicas $a, b, c$, the overlaps between these replicas, $a \cap b = r, a \cap c = s, b \cap c = t$, either are all equal, or one is larger than the others, but then these are equal (e.g., $r = s < t$).

Now we consider the Lagrangian term of the fluctuations $\mathcal{L}^{(2)}$, (9),

$$\mathcal{L}^{(2)} = \frac{1}{2} \sum_{(ab)(cd)} \phi_{ab} M^{ab,cd} \phi_{cd}$$

(42)

again with the dependence on momentum space $p$ implicit. The fields are characterized by the overlap of the replicas,

$$\phi_{ab} = \phi_r, \quad a \cap b = r$$

(43)

with $\phi_{aa} = \phi_{R+1} = 0$, and depend on the tree coordinates of the replicas

$$\phi_r \equiv \begin{bmatrix} a_0, a_{r-1}, a_r, a_R \\ a_0, a_{r-1}, b_r, b_R \end{bmatrix}, \quad a_r \neq b_r$$

$$\equiv \begin{bmatrix} a_0, a_{r-1}, b_r, b_R \\ a_0, a_{r-1}, a_r, a_R \end{bmatrix}$$

(44)

The mass-matrix depends only on the overlaps of the replicas, and can be parametrized as follows,

$$M^{ab,cd} = M^{r,s}_{uv}$$

(45)

with

$$r = a \cap b, \quad s = c \cap d,$$

$$u = \max (a \cap c, a \cap d),$$

$$v = \max (b \cap c, b \cap d).$$

(46)

Ultrametricity implies that with four replicas there are generically three overlaps, i.e., among the overlaps $r, s, u, v$ at least two are equal; $r, s$ are direct-overlaps and $u, v$ are cross-overlaps.

The Lagrangian $\mathcal{L}^{(2)}$, (42), is then written as

$$\mathcal{L}^{(2)} = \sum_{r,s,u,v} \sum_{(a,b,c,d)} \phi_r M^{r,s}_{uv} \phi_v$$

(47)

with $0 \leq r, s \leq R, 0 \leq u, v \leq R + 1$, and where the sum over the set $\{a, b, c, d\}$ depends on the overlaps $r, s, u, v$. The possible geometries of the tree representation of the mass-matrix, (45), are presented in figures 2 and 3. We distinguish two sets of contributions:
the replicon (R) configurations, in figure 2, are characterized by two identical upper indices, \( r = s \), and two lower indices \( u, v \geq r + 1 \),

\[
M_{u,v}^{r,s} = M_{u,v}^{r,r},
\]

(48)

the longitudinal-anomalous (LA) configurations, in figure 3, are characterized by a single lower index, \( t = \max (u, v) \) (the other lower index is \( r, s, \) or \( t \)) and two upper indices \( r, s \) (where it may happen, incidently, that \( r = s \)),

\[
M_{u,v}^{r,s} = M_{t}^{r,s},
\]

(49)

The upper indices take values 0, 1, ..., \( R \) and the lower indices take values 0, 1, ..., \( R + 1 \). The Lagrangian \( \mathcal{L}^{(2)} \) then contains four contributions

\[
\mathcal{L}^{(2)} = \mathcal{L}_{(R)} + \mathcal{L}_{(LA)}^{I} + \mathcal{L}_{(LA)}^{II} + \mathcal{L}_{(LA)}^{III}
\]

(50)

with \( \mathcal{L}_{(R)} \) and \( \mathcal{L}_{(LA)}^{I}, \mathcal{L}_{(LA)}^{II}, \mathcal{L}_{(LA)}^{III} \) corresponding to the tree structures in figures 2 and 3, respectively.

We now generalize the RFT introduced in (18). To RFT with respect to replica \( a \) on a tree, we RFT each of the \( [a_{r}] \) coordinates of \( a \) on the tree. Focusing on \( [a_{r}] \), one defines

\[
\phi [\hat{a}_{r}] = \frac{1}{\sqrt{N_{r}}} \sum_{a_{r}} e^{-2 \pi i n_{r} \hat{a}_{r}} \phi [a_{r}]
\]

\[
\phi [a_{r}] = \frac{1}{\sqrt{N_{r}}} \sum_{\hat{a}_{r}} e^{2 \pi i n_{r} \hat{a}_{r}} \phi [\hat{a}_{r}],
\]

(51)
where \( \hat{a}_r \) takes \( n_r = p_r / p_{r+1} \) values on the circle, \( \hat{a}_r = 0, 1, 2, \ldots, n_r - 1, \) mod \( (n_r) \), having the relation

\[
\sum_{a_r} e^{i2\pi n_r \hat{a}_r} = n_r \delta_{\hat{a}_r, \hat{b}_r},
\]  

(52)

The RFT of the \( [a_r] \) coordinate represents a sum over the \( n_r \) values that the component takes at the \( r \) level of foliation, among which there is permutation symmetry. From (51), it follows that \( \phi^a \[ \hat{a}_r \] = \phi \[ -\hat{a}_r \].

Let us then carry out the steps needed to accomplish the diagonalization of the Lagrangian \( \mathcal{L}^{(2)} \).

(1) Write the expression for the various contributions to \( \mathcal{L}^{(2)} \), in (50), which are given by:

\[
\mathcal{L}_{(LA)}^I = \frac{1}{8} \left\{ \sum_{s=r+1}^{R} \sum_{r=1}^{s-1} \sum_{t=1}^{s-1} M_{r,s}^I \right. \\
\times \sum_{\{a,b,c,d\}} \sum_{a_r \neq b_r, c_r \neq d_r} \left[ \begin{array}{c}
a_0 \cdot a_{r-1} \cdots a_r \cdot a_{r-1} \cdot a_{r+1} \\
b_y \cdot b_{r-1} \cdots b_r \cdot b_{r+1} \\
c_s \cdot c_{r-1} \cdots c_r \cdot c_{r+1} \\
d_s \cdot d_{r-1} \cdots d_r \cdot d_{r+1}
\end{array} \right] \\
+ \sum_{s=r+1}^{R} \sum_{t=1}^{s-1} \sum_{\{a,b,c,d\}} \sum_{a_r \neq b_r, c_r \neq d_r} \left[ \begin{array}{c}
a_0 \cdot a_{r-1} \cdots a_r \cdot a_{r-1} \cdot a_{r+1} \\
b_y \cdot b_{r-1} \cdots b_r \cdot b_{r+1} \\
c_s \cdot c_{r-1} \cdots c_r \cdot c_{r+1} \\
d_s \cdot d_{r-1} \cdots d_r \cdot d_{r+1}
\end{array} \right] \\
+ \sum_{r=0}^{R} \sum_{\{a,b,c,d\}} \sum_{a_r \neq b_r, c_r \neq d_r} \left[ \begin{array}{c}
a_0 \cdot a_{r-1} \cdots a_r \cdot a_{r-1} \cdot a_{r+1} \\
b_y \cdot b_{r-1} \cdots b_r \cdot b_{r+1} \\
c_s \cdot c_{r-1} \cdots c_r \cdot c_{r+1} \\
d_s \cdot d_{r-1} \cdots d_r \cdot d_{r+1}
\end{array} \right]
\right\}
\]

(53)

\[
\mathcal{L}_{(LA)}^{II} = \frac{1}{8} \left\{ \sum_{s=r+1}^{R} \sum_{r=1}^{s-1} \sum_{t=1}^{s-1} M_{r,s}^{II} \right. \\
\times \sum_{\{a,b,c,d\}} \sum_{a_r \neq b_r, c_r \neq d_r} \left[ \begin{array}{c}
a_0 \cdot a_{r-1} \cdots a_r \cdot a_{r-1} \cdot a_{r+1} \\
b_y \cdot b_{r-1} \cdots b_r \cdot b_{r+1} \\
c_s \cdot c_{r-1} \cdots c_r \cdot c_{r+1} \\
d_s \cdot d_{r-1} \cdots d_r \cdot d_{r+1}
\end{array} \right] \\
+ \text{equivalent terms for } t < s < r \text{ and } t = s < r \right\}
\]
\[ L_{(LA)}^{III} = \frac{1}{8} \sum_{i=r+1}^{R+1} \sum_{j=r+1}^{i-1} \sum_{k=r+1}^{j-1} M_{i,j,k}^{R,s} \]

\[ \sum_{\{a,b,c,d\} \atop a \neq b, c \neq d \atop d \neq c_i} \left[ \begin{array}{c} a_0 \ldots a_{r-1} \\ b_0 \ldots b_{r-1} \\ c_0 \ldots c_{r-1} \\ d_0 \ldots d_{r-1} \\ \end{array} \right] \frac{a_{r} \ldots a_{s} \ldots a_R}{b_{r} \ldots b_{s} \ldots b_R} \]

\[ \sum_{\{a,b,c,d\} \atop a \neq b, c \neq d \atop c_i \neq d_i} \left[ \begin{array}{c} a_0 \ldots a_{r-1} \\ b_0 \ldots b_{r-1} \\ c_0 \ldots c_{r-1} \\ d_0 \ldots d_{r-1} \\ \end{array} \right] \frac{a_{r} \ldots a_{s} \ldots a_R}{b_{r} \ldots b_{s} \ldots b_R} \]

\[ \sum_{\{a,b,c,d\} \atop a \neq b, c \neq d \atop d \neq c_i} \left[ \begin{array}{c} a_0 \ldots a_{r-1} \\ b_0 \ldots b_{r-1} \\ c_0 \ldots c_{r-1} \\ d_0 \ldots d_{r-1} \\ \end{array} \right] \frac{a_{r} \ldots a_{s} \ldots a_R}{b_{r} \ldots b_{s} \ldots b_R} \]

\[ + \text{equivalent terms for } s < t < r \text{ and } s < t = r \], \quad (54)
\[ \mathcal{L}_R = \frac{1}{8} \sum_{r=0}^{R} \sum_{i=r+1}^{R+1} M^{r,i}_{aa} \times \left( \sum_{a_r \neq b} \sum_{a_y \neq c, b \neq a_r, a_y \neq b} \sum_{a_z \neq d, a_y \neq b, a_z \neq d} \sum_{a_t \neq e, a_z \neq d, a_t \neq c} \sum_{a_{u,v} \neq b, a_z \neq d, a_t \neq c} \sum_{a_{u,v} \neq b, a_z \neq d, a_t \neq c} \sum_{a_{u,v} \neq b, a_z \neq d, a_t \neq c} \sum_{a_{u,v} \neq b, a_z \neq d, a_t \neq c} \sum_{a_{u,v} \neq b, a_z \neq d, a_t \neq c} \right) \]
For the cross-overlaps \( t, u, v \), transform the restricted sums over the tree coordinates into unrestricted sums, which, with regrouping of terms among the four contributions (53)–(56), leads to:

\[
\mathcal{L}_{(LA)}^I = \frac{1}{8} \left\{ \sum_{s=0}^{R} \sum_{s=r}^{r} \left( M_{t}^{r,s} - M_{r+1}^{r,s} \right) \right\} \\
\times \sum_{a\neq b, c \neq d} \left[ \begin{array}{c} a_0 \ldots a_{r-1} a_r \ldots a_R \\ b_0 \ldots b_{r-1} b_r \ldots b_R \\ c_0 \ldots c_{r-1} c_r \ldots c_R \\ d_0 \ldots d_{r-1} d_r \ldots d_R \\
\end{array} \right] \\
+ \text{ equivalent term for } t \leq s < r \}
\]

(57)

fixing \( M_{r-1}^{r,s} = 0 \),

\[
\mathcal{L}_{(LA)}^II = \frac{1}{8} \left\{ \sum_{s=0}^{R} \sum_{s=r+1}^{r+1} \left( M_{t}^{r,s} - M_{r+1}^{r,s} \right) \right\} \\
\times \sum_{a\neq b, c \neq d} \left[ \begin{array}{c} a_0 \ldots a_{r-1} a_r \ldots a_R \\ b_0 \ldots b_{r-1} b_r \ldots b_R \\ c_0 \ldots c_{r-1} c_r \ldots c_R \\ d_0 \ldots d_{r-1} d_r \ldots d_R \\
\end{array} \right] \\
+ \text{ equivalent term for } s < t \leq r \}
\]

(58)

\[
\mathcal{L}_{(LA)}^III = \frac{1}{8} \left\{ \sum_{t=0}^{R+1} \sum_{s=r}^{r} \left( M_{t}^{r,s} - M_{r+1}^{r,s} \right) \right\} \\
\times \sum_{a\neq b, c \neq d} \left[ \begin{array}{c} a_0 \ldots a_{r-1} a_r \ldots a_R \\ b_0 \ldots b_{r-1} b_r \ldots b_R \\ c_0 \ldots c_{r-1} c_r \ldots c_R \\ d_0 \ldots d_{r-1} d_r \ldots d_R \\
\end{array} \right] \\
+ \text{ equivalent term for } t \leq s \}
\]

(59)
\[
\sum_{\{a,b,c,d\} \atop a \neq b, a \neq c} \left[ \begin{array}{c}
a_0 \ldots a_{r-1} a_1 \ldots a_{r-1} a_r \\ b_1 \ldots b_{r-1} b_1 \ldots b_{r-1} b_r
\end{array} \right] + \sum_{\{a,b,c,d\} \atop a \neq b, a \neq d} \left[ \begin{array}{c}
a_0 \ldots a_{r-1} a_1 \ldots a_{r-1} a_r \\ b_1 \ldots b_{r-1} b_1 \ldots b_{r-1} b_r
\end{array} \right]
\]

+ equivalent term for \( s < t < r \),

\[\mathcal{L}_{(R)} = \frac{1}{8} \sum_{r=0}^{R} \sum_{t=r+1}^{R+1} \left( M_{a,b}^{r,t} - M_{a,b}^{r,t} - M_{a,b}^{r,t} + M_{a,b}^{r,t} \right) \]

\[\times \sum_{\{a,b,c,d\} \atop a \neq b, a \neq d} \left[ \begin{array}{c}
a_0 \ldots a_{r-1} a_1 \ldots a_{r-1} a_r \\ b_1 \ldots b_{r-1} b_1 \ldots b_{r-1} b_r
\end{array} \right] + \sum_{\{a,b,c,d\} \atop a \neq b, a \neq c} \left[ \begin{array}{c}
a_0 \ldots a_{r-1} a_1 \ldots a_{r-1} a_r \\ b_1 \ldots b_{r-1} b_1 \ldots b_{r-1} b_r
\end{array} \right] + \sum_{\{a,b,c,d\} \atop a \neq b, a \neq d} \left[ \begin{array}{c}
a_0 \ldots a_{r-1} a_1 \ldots a_{r-1} a_r \\ b_1 \ldots b_{r-1} b_1 \ldots b_{r-1} b_r
\end{array} \right] \]

(3) Perform the RFT on all the tree coordinates \( a, b, c, d \) of the replicas, which leads to:

\[\mathcal{L}_{(LA)}^{I} = \frac{1}{8} \sum_{s=0}^{R} \sum_{r=0}^{s} \sum_{t=0}^{r} \sum_{t=0}^{r} \sqrt{\delta_r} \sqrt{\delta_t} p \left( M_{a,b}^{r,s} - M_{a,b}^{r,s} \right) \]

\[\times \left[ \begin{array}{c}
\hat{\delta}_0 \ldots \hat{\delta}_{r-1} \hat{0} \ldots \hat{0}_{r-1} \\ \hat{\delta}_0 \ldots \hat{\delta}_{r-1} \hat{0} \ldots \hat{0}_{r-1}
\end{array} \right] \]

+ equivalent term for \( t \leq s < r \),

\[\mathcal{L}_{(LA)}^{II} = \frac{1}{8} \sum_{s=0}^{R} \sum_{r=0}^{s} \sum_{t=0}^{r} \sum_{t=0}^{r} \sqrt{\delta_r} \sqrt{\delta_t} p \left( M_{a,b}^{r,s} - M_{a,b}^{r,s} \right) \]

\[\times \left[ \begin{array}{c}
\hat{\delta}_0 \ldots \hat{\delta}_{r-1} \hat{0} \ldots \hat{0}_{r-1} \\ \hat{\delta}_0 \ldots \hat{\delta}_{r-1} \hat{0} \ldots \hat{0}_{r-1}
\end{array} \right] \]
\[
\mathcal{L}^{\text{III}}_{(LA)} = \frac{1}{8} \sum_{s,t=1}^{R+1} \sum_{r=0}^{t-1} \sum_{p=0}^{s} \sqrt{\delta_r} \sqrt{\delta_s} \delta_p (M^{r,s}_{p} - M^{r,s}_{p+1})
\times \left[ \hat{\phi}_0 \cdot \hat{\phi}_{r+1} \cdots \hat{\phi}_{t-1} \hat{\phi}_{s} \cdot \hat{\phi}_{s} \right]_{R_N}
\times \left[ \hat{\phi}_0 \cdot \hat{\phi}_{r+1} \cdots \hat{\phi}_{t-1} \hat{\phi}_{s} \cdot \hat{\phi}_{s} \right]_{R_N}
\times \left[ \hat{\phi}_0 \cdot \hat{\phi}_{r+1} \cdots \hat{\phi}_{t-1} \hat{\phi}_{s} \cdot \hat{\phi}_{s} \right]_{R_N}
\times \left[ \hat{\phi}_0 \cdot \hat{\phi}_{r+1} \cdots \hat{\phi}_{t-1} \hat{\phi}_{s} \cdot \hat{\phi}_{s} \right]_{R_N}
\times \text{equivalent term for } s < t < r \},
\]

(62)

\[
\mathcal{L}_{(R)} = \frac{1}{8} \sum_{r=0}^{R} \sum_{s,t=1}^{R+1} \sum_{p=0}^{s} \left( \frac{n_r - 1}{n_s} \right)
\times \left[ \hat{\mu}_0 \cdot \hat{\mu}_{r+1} \cdots \hat{\mu}_{t-1} \hat{\mu}_{s} \cdot \hat{\mu}_{s} \right]_{N_R}
\times \left[ \hat{\mu}_0 \cdot \hat{\mu}_{r+1} \cdots \hat{\mu}_{t-1} \hat{\mu}_{s} \cdot \hat{\mu}_{s} \right]_{N_R}
\times \left[ \hat{\mu}_0 \cdot \hat{\mu}_{r+1} \cdots \hat{\mu}_{t-1} \hat{\mu}_{s} \cdot \hat{\mu}_{s} \right]_{N_R}
\times \left[ \hat{\mu}_0 \cdot \hat{\mu}_{r+1} \cdots \hat{\mu}_{t-1} \hat{\mu}_{s} \cdot \hat{\mu}_{s} \right]_{N_R}
\times \text{equivalent term for } s < t < r \},
\]

(63)

where we define
\( \delta_r = (p_r - p_{r+1}) \) \hspace{1cm} (65)

and use the notation

\[
\left[ \tilde{\gamma}_r \right] = \sum_{\tilde{a}_r} \left[ \frac{\tilde{a}_r}{\tilde{\gamma}_r - \tilde{a}_r} \right].
\]

(66)

normalized, \( \left[ \tilde{\gamma}_r \right] = \frac{1}{\sqrt{n_r}} \left[ \tilde{\gamma}_r \right] \). The restrictions on the tree coordinates associate with the direct-overlaps \( r, s \) are incorporated in (61)–(64), by introducing the marker definition

\[
\left( \begin{array}{c}
\hat{\mu}_r \\
\hat{\gamma}_r - \hat{\mu}_r
\end{array} \right) = \left[ \frac{\hat{\mu}_r}{\hat{\gamma}_r - \hat{\mu}_r} \right] - \frac{1}{n_r} \left[ \tilde{\gamma}_r \right]
\]

(67)

which corresponds to the RFT of \( \left[ \frac{a_r}{b_r} \right] \). The marker has the property

\[
\sum_{\hat{\mu}} \left( \begin{array}{c}
\hat{\mu}_r \\
\hat{\gamma}_r - \hat{\mu}_r
\end{array} \right) = 0,
\]

(68)

and normalization,

\[
\left( \begin{array}{c}
\hat{\mu}_r \\
\hat{\gamma}_r - \hat{\mu}_r
\end{array} \right) = \sqrt{n_r - 1} \left( \begin{array}{c}
\hat{\mu}_r \\
\hat{\gamma}_r - \hat{\mu}_r
\end{array} \right) .
\]

(69)

We have also used the relation

\[
\sum_{\hat{\gamma}, \hat{\mu}} \left( \begin{array}{c}
\hat{\mu}_r \\
\hat{\gamma}_r - \hat{\mu}_r
\end{array} \right) \left( \begin{array}{c}
\hat{\mu}_r \\
\hat{\gamma}_r - \hat{\mu}_r
\end{array} \right)^* = \sum_{\hat{\gamma}, \hat{\mu}} \left[ \frac{\hat{\mu}_r}{\hat{\gamma}_r - \hat{\mu}_r} \right] \left[ \frac{\hat{\mu}_r}{\hat{\gamma}_r - \hat{\mu}_r} \right]^* \nonumber
\]

\[
- \sum_{\hat{\gamma}} \left[ \tilde{\gamma}_N \right] \left[ \tilde{\gamma}_N^\dagger \right].
\]

(70)

(4) Separate in the sums over the tree coordinates of the replicas, the \( \hat{0} \) components from the \( \hat{0}' \neq \hat{0}, \hat{\rho}' \neq \hat{0}, \hat{\nu}' \neq \hat{0} \) components.

For \( \mathcal{L}_{(LA)}^I, \mathcal{L}_{(LA)}^{II}, \mathcal{L}_{(LA)}^{III} \) one obtains:

\[
\mathcal{L}_{(LA)}^I = \frac{1}{16} \left\{ \sum_{(\mu, r, i, i') = 0}^R \sum_{(\tilde{\gamma}, s) \in \mathcal{S}} \sqrt{\delta_{(s-1)}} \mathcal{M}_{r,s} \sqrt{\delta_{(s-1)}} \right\}
\]

\[
\times \left[ \tilde{\gamma}_0: \tilde{\gamma}_{i-1} \hat{0}_i: \hat{0}_{i+1} \ldots \hat{0}_R \right] \hat{0}_r \hat{0}_{i+1} \ldots \hat{0}_R_{SN}
\]

\[
\times \left[ \tilde{\gamma}_0: \tilde{\gamma}_{i-1} \hat{0}_i: \hat{0}_{i+1} \ldots \hat{0}_R \right]^* \hat{0}_r \hat{0}_{i+1} \ldots \hat{0}_R_{SN}
\]

+ equivalent term for \( i \leq s < r \).

(71)
\[ L_{(\text{LA})}^{II} = \frac{1}{16} \sum_{r-t-r+1=0}^{R} \sum_{t=0}^{t-1} \sum_{r=0}^{r-1} \sum_{\gamma} \frac{\delta_{\gamma}^{(t-1)}}{t} \hat{M}_{t}^{\text{rs}} \delta_{\gamma}^{(t-1)} \]
\[
\begin{bmatrix}
\hat{\beta}_{t-1} \hat{\beta}_{t-2} \ldots \hat{\beta}_{0} \\
\hat{\beta}_{t+1} \hat{\beta}_{t+2} \ldots \hat{\beta}_{0}
\end{bmatrix}
\begin{bmatrix}
\hat{0}_{t+1} \ldots \hat{0}_{t} \ldots \hat{0}_{R} \\
\hat{0}_{t+1} \ldots \hat{0}_{t} \ldots \hat{0}_{R}
\end{bmatrix}_{\text{SN}}
\]
\[
\times
\begin{bmatrix}
\hat{\beta}_{t-1} \hat{\beta}_{t-2} \ldots \hat{\beta}_{0} \\
\hat{\beta}_{t+1} \hat{\beta}_{t+2} \ldots \hat{\beta}_{0}
\end{bmatrix}
\begin{bmatrix}
\hat{0}_{t+1} \ldots \hat{0}_{t} \ldots \hat{0}_{R} \\
\hat{0}_{t+1} \ldots \hat{0}_{t} \ldots \hat{0}_{R}
\end{bmatrix}_{\text{SN}}
\]
\[ + \text{ equivalent term for } s < t < r, \quad (72) \]

\[ L_{(\text{LA})}^{III} = \frac{1}{16} \sum_{r-t-r+1=0}^{R} \sum_{t=0}^{t-1} \sum_{r=0}^{r-1} \sum_{\gamma} \frac{\delta_{\gamma}^{(t-1)}}{t} \hat{M}_{t}^{\text{rs}} \delta_{\gamma}^{(t-1)} \]
\[
\begin{bmatrix}
\hat{\beta}_{t-1} \hat{\beta}_{t-2} \ldots \hat{\beta}_{0} \\
\hat{\beta}_{t+1} \hat{\beta}_{t+2} \ldots \hat{\beta}_{0}
\end{bmatrix}
\begin{bmatrix}
\hat{0}_{t+1} \ldots \hat{0}_{t} \ldots \hat{0}_{R} \\
\hat{0}_{t+1} \ldots \hat{0}_{t} \ldots \hat{0}_{R}
\end{bmatrix}_{\text{SN}}
\]
\[
\times
\begin{bmatrix}
\hat{\beta}_{t-1} \hat{\beta}_{t-2} \ldots \hat{\beta}_{0} \\
\hat{\beta}_{t+1} \hat{\beta}_{t+2} \ldots \hat{\beta}_{0}
\end{bmatrix}
\begin{bmatrix}
\hat{0}_{t+1} \ldots \hat{0}_{t} \ldots \hat{0}_{R} \\
\hat{0}_{t+1} \ldots \hat{0}_{t} \ldots \hat{0}_{R}
\end{bmatrix}_{\text{SN}}
\]
\[ + \text{ equivalent term for } s < t < r \]
\[ (73) \]

having, at the end, symmetrized the fields at the marker,
\[ \left( \hat{0}_{r} \right)_{S} = \frac{1}{2} \left( \hat{0}_{r} + \hat{\beta}_{r} \right) \]
\[ (74) \]

and normalized, for \( t = r + 1 \),
\[ \left( \hat{0}_{r} \right)_{S} = \sqrt{\frac{\delta_{\gamma}^{(t-1)}}{2t}} \left( \hat{0}_{r} \right)_{\text{SN}} \]
\[ (75) \]

with
\[ \delta_{\gamma}^{(t)} = p_{t}^{(t)} - p_{t+1}^{(t)} \]
\[ (76) \]

\[ p_{t}^{(t)} = \begin{cases} p_{t}, & r \leq l \\ 2p_{t}, & r > l \end{cases} \]
\[ (77) \]

and, for \( t > r + 1 \),
\[ \left( \hat{0}_{r} \right)_{S} = \frac{1}{\sqrt{2}} \left( \hat{0}_{r} \right)_{\text{SN}} \]
\[ (78) \]
In (71)–(73) \( \hat{M}_t^{r,s} \) is the mass RFT [21],

\[
\hat{M}_t^{r,s} = \sum_{k=0}^{R+1} p_k^{(r,s)} \left( M_k^{r,s} - M_{k-1}^{r,s} \right)
\]  

(79)

with

\[
p_k^{(r,s)} = p_k, \quad k \leq r \leq s
\]

\[
p_k^{(r,s)} = 2p_k, \quad r < k \leq s
\]

\[
p_k^{(r,s)} = 4p_k, \quad r \leq s < k
\]  

(80)

the inverse transform being given by

\[
M_k^{r,s} = \sum_{i=0}^{k} \frac{1}{p_i^{(r,s)}} \left( \hat{M}_t^{r,s} - \hat{M}_{i+1}^{r,s} \right).
\]  

(81)

We now process \( L_{(R)} \), by separating out the \( \hat{0} \) components in the sums. This leads to the different contributions:

(I) \( u, v > r + 1 \):

\[
L_{(R)}^I = \frac{1}{8} \sum_{r=0}^{R} \sum_{u=v=r+2}^{R+1} \sum_{\{\mu, \nu, \ell\}} \left( 1 + \delta_{u,\nu} \delta_{\nu+1,v} \right) \left( \frac{1}{n_r - 1} \right) \tilde{\hat{M}}_{uv}^{r,r} - \hat{M}_{uv}^{r,r}
\]

\[
\times \left[ \tilde{\hat{M}}_{t}^{r+1,r} \left( \tilde{\hat{M}}_{t}^{r+1,r} - \hat{\mu}_r \right) \tilde{\hat{M}}_{t}^{r+1,r} \left( \tilde{\hat{M}}_{t}^{r+1,r} - \hat{\mu}_r \right) \tilde{\hat{M}}_{t}^{r+1,r} \left( \tilde{\hat{M}}_{t}^{r+1,r} - \hat{\mu}_r \right) \right]_{SN}
\]

(82)

with field symmetrization at the marker,

\[
\left( \tilde{\hat{M}}_{t}^{r+1,r} \right)_S = \frac{1}{2} \left[ \left( \tilde{\hat{M}}_{t}^{r+1,r} \right)_N + \left( \tilde{\hat{M}}_{t}^{r+1,r} \right)_N \right]
\]  

(83)

and normalization,

\[
\left( \tilde{\hat{M}}_{t}^{r+1,r} \right)_S = \sqrt{\frac{1}{2} \left( 1 + \delta_{u,\nu} \delta_{\nu+1,v} \right) \left( \tilde{\hat{M}}_{t}^{r+1,r} \right)_N}.
\]  

(84)

In (82), \( \tilde{\hat{M}}_{uv}^{r,r} \) is the mass double RFT [21],

\[
\tilde{\hat{M}}_{uv}^{r,r} = \sum_{k=u}^{R+1} \sum_{l=v}^{R+1} p_k p_l \left( M_{uvj}^{r,r} - M_{uvj+1}^{r,r} + M_{uvj1}^{r,r} - M_{uvj11}^{r,r} \right)
\]  

(85)

the inverse double transform being given by
\[ M_{k,l}^{r,r} - M_{k,l}^{r,r} - M_{k,l}^{r,r} + M_{k,l}^{r,r} = \sum_{u=r+1}^{k} \sum_{v=r+1}^{l} \frac{1}{p_u p_v} \left( \hat{M}_{u+1,v+1}^{r,r} - \hat{M}_{u+1,v+1}^{r,r} + \hat{M}_{u+1,v+1}^{r,r} \right). \] (86)

(II) \( u = r + 1, v > r + 1 \) (or \( v = r + 1, u > r + 1 \):

Here one has to separate the \( \hat{0}_r \) component in the marker. We define the new field, with \( \hat{\mu}_r \neq \hat{0}_r \),

\[ \left\{ \hat{\mu}_r - \hat{\mu}_r' \right\} = \left\{ \hat{\mu}_r' \right\} + \frac{1}{n_r - 1} \left( \hat{0}_r \right) \] (87)

which, from (68), has the property

\[ \sum_{\hat{\rho}_r} \left\{ \hat{\mu}_r - \hat{\mu}_r' \right\} = 0. \] (88)

Introducing this field, with symmetrization

\[ \left\{ \hat{\mu}_r - \hat{\mu}_r' \right\}_S = \frac{1}{2} \left[ \left\{ \hat{\mu}_r' \right\} + \left\{ \hat{\mu}_r - \hat{\mu}_r' \right\} \right] \] (89)

and normalization

\[ \left\{ \hat{\mu}_r - \hat{\mu}_r' \right\}_S = \sqrt{\frac{1}{2} \left( \frac{n_r - 2}{n_r - 1} \right)} \left\{ \hat{\rho}_r - \hat{\rho}_r' \right\}_{SN} \] (90)

and using the relation

\[ \sum_{\hat{\mu}_r' \neq 0} \left\{ \hat{\mu}_r - \hat{\mu}_r' \right\} \left\{ \hat{\mu}_r - \hat{\mu}_r' \right\}^* = \sum_{\hat{\rho}_r} \left( \hat{\rho}_r - \hat{\rho}_r \right) \left( \hat{\rho}_r - \hat{\rho}_r' \right)^* \]

one obtains two contributions:

\[ \mathcal{L}_{(R)}^{II} = \frac{1}{8} \sum_{r=0}^{K+1} \sum_{v=r+2}^{K+1} \sum_{\hat{\rho}_r' \neq 0} \left( \frac{n_r - 2}{n_r - 1} \right)^* M_{r+1,v}^{r,r} \]

\[ \times \left[ \hat{0}_r \cdots \hat{0}_{r-1} \left\{ \hat{\mu}_r - \hat{\mu}_r' \right\} \hat{\rho}_{r+1} \cdots \hat{\rho}_{v-1} \hat{0}_r \cdots \hat{0}_v \hat{0}_R \right]_{SN} \]

\[ \times \left[ \hat{0}_r \cdots \hat{0}_{r-1} \left\{ \hat{\mu}_r - \hat{\mu}_r' \right\} \hat{\rho}_{r+1} \cdots \hat{\rho}_{v-1} \hat{0}_r \cdots \hat{0}_v \hat{0}_R \right]_{SN}^* \] (92)

and
\[ L_{(1)}^{(L,A)} = \frac{1}{8} \sum_{r=0}^{R} \sum_{\nu=r+2}^{R+1} \sum_{\gamma}^\nu \hat{\gamma}_0^{r-1} \hat{\gamma}_r \left( \hat{0}_{r+1} \ldots \hat{0}_{r-1} \hat{0}_r \hat{0}_R \right)_{\text{SN}} \times \left[ \hat{\gamma}_0^{r-1} \hat{\gamma}_r^{r-1} \hat{0}_r \hat{0}_{r-1} \hat{0}_r \hat{0}_R \right]_{\text{SN}} \]  

(93)

(III) \( u = r + 1, \nu = r + 1 \):

Here again one has to separate the \( \hat{0}_r \) components in the marker, having now two situations:

(A) \( \hat{\gamma}_r = \hat{0}_r \): we define the new field, with \( \hat{\mu}' \neq \hat{0}_r \),

\[
\left\{ \hat{\mu}', \hat{\mu} \right\} = \left\{ \hat{\mu}', \hat{\mu} \right\} + \frac{1}{n_r - 1} \left( \hat{0}_r \right)
\]

which, from (68), has the property

\[
\sum_{\hat{\mu}} \left\{ \hat{\mu}', \hat{\mu} \right\} = 0.
\]

Introducing this field, with symmetrization

\[
\left\{ \hat{\mu}', \hat{\mu} \right\} = \frac{1}{2} \left[ \left\{ \hat{\mu}', \hat{\mu} \right\} + \left\{ -\hat{\mu}', -\hat{\mu} \right\} \right]
\]

(96)

and normalization

\[
\left\{ \hat{\mu}', \hat{\mu} \right\} = \sqrt{\frac{1}{2} \left( \frac{n_r - 3}{n_r - 1} \right)} \left\{ \hat{\mu}', \hat{\mu} \right\}_{\text{SN}}
\]

(97)

and using the relation

\[
\sum_{\hat{\mu}, \hat{\mu}' \neq 0} \left[ \hat{\mu}', \hat{\mu} \right] \left[ \hat{\mu}', \hat{\mu} \right]_{\text{SN}} = \sum_{\hat{\mu}} \left( \hat{\mu}, \hat{\mu} \right) \left( \hat{\mu}, \hat{\mu} \right)_{\text{SN}}
\]

(98)

one obtains two contributions:

\[
L_{(R)}^{(L)} = \frac{1}{8} \sum_{r=0}^{R} \sum_{\nu=r+2}^{R+1} \left( \frac{n_r - 3}{n_r - 1} \right) \hat{\gamma}_r \hat{\gamma}_r \hat{0}_r \hat{0}_{r-1} \hat{0}_r \hat{0}_R_{\text{SN}}
\]

(99)
one obtains two contributions:

\[
\left[ \hat{\varphi}_{0} \cdots \hat{\varphi}_{r-1} \begin{array}{c} \hat{\mu}_{r} \\ \hat{\nu}_{r} \end{array}, \begin{array}{c} \hat{0}_{r+1} \cdots \hat{0}_{\gamma_{\text{SN}}} \\ \hat{0}_{r+1} \cdots \hat{0}_{\gamma_{\text{SN}}} \end{array} \right] \times \\
\left[ \hat{\varphi}_{0} \cdots \hat{\varphi}_{r-1} \begin{array}{c} \hat{\mu}_{r} \\ \hat{\nu}_{r} \end{array}, \begin{array}{c} \hat{0}_{r+1} \cdots \hat{0}_{\gamma_{\text{SN}}} \\ \hat{0}_{r+1} \cdots \hat{0}_{\gamma_{\text{SN}}} \end{array} \right]^{*}
\]

(99)

and

\[
L^{(2)}_{\text{(LA)}} = \frac{1}{4} \sum_{r=0}^{R} \sum_{k=0}^{r} \sum_{s} \hat{M}_{r+1,k+1}^{s,r}
\]

\[
\times \left[ \hat{\varphi}_{0} \cdots \hat{\varphi}_{k-1} \hat{\frac{0_{r}}{\otimes}} \cdots \hat{0}_{r+1} \cdots \hat{0}_{\gamma_{\text{SN}}} \right] \times \\
\left[ \hat{\varphi}_{0} \cdots \hat{\varphi}_{k-1} \hat{\frac{0_{r}}{\otimes}} \cdots \hat{0}_{r+1} \cdots \hat{0}_{\gamma_{\text{SN}}} \right]^{*}
\]

(100)

after successively splitting \( \hat{\varphi}_{r} \) into \( \hat{\varphi}_{k} \) and \( \hat{\varphi}_{r}^{k} \).

(B) \( \hat{\varphi}_{r}^{k} \neq 0 \): we define the new field, with \( \hat{\mu}_{r}^{k} \neq \hat{0}_{r}, \hat{\varphi}_{r}^{k} \),

\[
\left\{ \hat{\mu}_{r}^{k} \right\} = \left\{ \hat{\mu}_{r}^{k} \right\} + \frac{1}{n_{r} - 2} \left[ \left( \hat{0}_{r} \right) \left( \hat{\varphi}_{r}^{k} \right) + \left( \hat{\varphi}_{r}^{k} \right) \left( \hat{0}_{r} \right) \right]
\]

(101)

which, from (68), has the property

\[
\sum_{\hat{\mu}_{r}} \left\{ \hat{\varphi}_{r}^{k} - \hat{\varphi}_{r}^{k} \right\} = 0.
\]

(102)

Introducing this field, with symmetrization

\[
\left\{ \hat{\varphi}_{r}^{k} - \hat{\varphi}_{r}^{k} \right\}_{S} = \frac{1}{2} \left[ \left\{ \hat{\mu}_{r}^{k} \right\} + \left\{ \hat{\mu}_{r}^{k} \right\} \right]
\]

(103)

and normalization

\[
\left\{ \hat{\varphi}_{r}^{k} - \hat{\varphi}_{r}^{k} \right\}_{S} = \frac{1}{\sqrt{2}} \left( \delta_{\varphi_{r}^{k}, \varphi_{r}^{k}} + \frac{n_{r} - 4}{n_{r} - 2} \right) \left\{ \hat{\mu}_{r}^{k} \right\}_{S}
\]

(104)

and using the relation

\[
\sum_{\hat{\mu}_{r} \neq \hat{0}_{r}, \hat{\varphi}_{s}^{k}} \left\{ \hat{\mu}_{r}^{k} \right\}_{S} \left\{ \hat{\varphi}_{s}^{k} - \hat{\varphi}_{s}^{k} \right\}_{S}^{*} = \sum_{\hat{\mu}_{r}} \left( \hat{\mu}_{r} \right) \left( \hat{\varphi}_{s}^{k} - \hat{\varphi}_{s}^{k} \right)_{S}^{*}
\]

\[- \left( \hat{0}_{r} \right)_{S} \left( \hat{\varphi}_{s}^{k} - \hat{\varphi}_{s}^{k} \right)_{S}^{*}
\]

(105)

one obtains two contributions:
Thus, we observe that in the replicon geometry there is a longitudinal-anomalous contribution, given by the components in (93), (100), (107), which will be used later to calculate the complete longitudinal-anomalous contribution.

Putting together (82), (92), (99) and (106) one obtains the complete replicon contribution, $L_R^{I(R)} + L_R^{II(R)} + L_R^{III(R)} + L_R^{IV(R)}$, 

$$L_R = \frac{1}{2} \sum_{r=0}^{R} \sum_{\gamma, \bar{\gamma}} \left[ \frac{1}{2} M_{r+1}^{r, r} \left( \hat{\phi}_r - \hat{\mu}_r \right)^2 + \frac{1}{2} M_{r+1, r+1}^{r, r} \left( \hat{\phi}_{r+1, r+1} - \hat{\mu}_{r+1, r+1} \right)^2 \right].$$

where $M_{r+1}^{r, r}$ is the replicon mass, given by (85), and $\phi_{r+1, r+1}^{r, r}$ are the replicon fields, defined as:
\* \( u, v > r + 1: \)

\[
\kappa^{\text{r}} \Phi^{\mu}_{\nu, v} \left( \tilde{\rho}_r - \tilde{\rho}_v \right) = \mathcal{N}_1 \left[ \tilde{\rho}_r \cdots \tilde{\rho}_{r-1} \left( \tilde{\rho}_r - \tilde{\rho}_v \right) \tilde{\rho}_{r+1} \cdots \tilde{\rho}_{u-1} \tilde{\rho}_{u-1} \cdots \tilde{\rho}_v \cdots \tilde{\rho}_r \right]_{\text{SN}}
\]

with \( \mathcal{N}_1 = \left( 1 + \delta_{\mu_1, \nu_1} \delta_{\mu_{r-1}, \nu_{r-1}} \right) \left( \frac{n_{\mu-1}}{n_\nu} \right) \), having the property

\[
\sum_{\tilde{\rho}} \kappa^{\text{r}} \Phi^{\mu}_{\nu, v} \left( \tilde{\rho}_r - \tilde{\rho}_v \right) = 0
\]

and multiplicity

\[
\mu(r; u, v) = \frac{1}{2} p_0 \left( p_r - p_{r+1} \right) \left( 1 - \frac{1}{p_r} \right) \left( 1 - \frac{1}{p_{r+1}} \right)
\]

\* \( u = r + 1, v > r + 1 \) (or \( v = r + 1, u > r + 1 \)):

\[
\kappa^{\text{r}} \Phi^{\mu}_{\nu, v} \left( \tilde{\rho}_r - \tilde{\rho}_v \right) = \mathcal{N}_2 \left[ \tilde{\rho}_r \cdots \tilde{\rho}_{r-1} \left( \tilde{\rho}_r - \tilde{\rho}_v \right) \tilde{\rho}_{r+1} \cdots \tilde{\rho}_{u-1} \tilde{\rho}_{u-1} \cdots \tilde{\rho}_v \cdots \tilde{\rho}_r \right]_{\text{SN}}
\]

with \( \mathcal{N}_2 = \left( \frac{n_{r-2}}{n_r} \right) \), having the property

\[
\sum_{\tilde{\rho}} \kappa^{\text{r}} \Phi^{\mu}_{\nu, v} \left( \tilde{\rho}_r - \tilde{\rho}_v \right) = 0
\]

and multiplicity

\[
\mu(r; r + 1, v) = \frac{1}{2} p_0 \left( p_r - 2 \right) \left( 1 - \frac{1}{p_r} \right) \left( 1 - \frac{1}{p_{r+1}} \right)
\]

\* \( u = r + 1, v = r + 1 \):

\[
\kappa^{\text{r}} \Phi^{\mu}_{\nu, v} \left( \tilde{\rho}_r - \tilde{\rho}_v \right) = \mathcal{N}_3 \left[ \tilde{\rho}_r \cdots \tilde{\rho}_{r-1} \left( \tilde{\rho}_r - \tilde{\rho}_v \right) \tilde{\rho}_{r+1} \cdots \tilde{\rho}_v \cdots \tilde{\rho}_r \right]_{\text{SN}}
\]

with \( \mathcal{N}_3 = \left( \frac{n_{r-2}}{n_r} \right) \), having the property

\[
\sum_{\tilde{\rho}} \kappa^{\text{r}} \Phi^{\mu}_{\nu, v} \left( \tilde{\rho}_r - \tilde{\rho}_v \right) = 0
\]

and multiplicity.
\[ \mu_1(r; r + 1, r + 1) = \frac{1}{2} p_0 \left( \frac{P_r}{P_{r+1}} - 3 \right) \frac{1}{P_r} ; \]  

(117)

and

\[ R \phi^{r}_{t+1, r+1} \left( \hat{\rho}_r, \hat{\eta}_r ; \hat{\rho}_r, \hat{\lambda}_r \right) = \mathcal{N}_4 \left[ \hat{\eta}_0 \cdots \hat{\eta}_{r-1} \left\{ \hat{\rho}_r' - \hat{\rho}_r'' \right\} \hat{\psi}_{r+1} \cdots \hat{\psi}_R \right] \]  

(118)

with \( \mathcal{N}_4 = \frac{1}{\left( \frac{r_{r'-r} - r_{r''-r} + \frac{n_r-4}{n_r-2} }{r_{r'-r} - r_{r''-r} + \frac{n_r-4}{n_r-2} } \right)} \), having the property

\[ \sum_{r'} R \phi^{r'}_{t+1, r+1} \left( \hat{\rho}_r' - \hat{\rho}_r'' \right) = 0 \]  

(119)

and multiplicity,

\[ \mu_2(r; r + 1, r + 1) = \frac{1}{2} p_0 \left( \frac{P_r}{P_{r+1}} - 3 \right) \left( \frac{1}{P_{r+1}} - \frac{1}{P_r} \right) ; \]  

(120)

defining, \( \mu = \mu_1 + \mu_2 \), gives

\[ \mu(r; r + 1, r + 1) = \frac{1}{2} p_0 \left( \frac{P_r}{P_{r+1}} - 3 \right) \frac{1}{P_{r+1}} . \]  

(121)

The propagators for the replicon fields, obtained from (108), are given by:

\[ R G^{r, r}_{t, r} \left( \hat{\rho}_r, \hat{\eta}_r ; \hat{\rho}_r, \hat{\lambda}_r \right) = \left\{ R \phi^{r}_{t, r} \left( \hat{\rho}_r - \hat{\rho}_r \right) R \phi^{r}_{t, r} \left( \hat{\rho}_r - \hat{\rho}_r \right) \right\} \]  

\[ = \delta_{r, \lambda} \left[ \delta_{\mu, \eta} - \frac{1}{n_r} \right] \frac{1}{m_r^\mu} . \]  

(122)

\[ R G^{r+1, r}_{t+1, r} \left( \hat{\rho}_r', \hat{\eta}_r'; \hat{\rho}_r', \hat{\lambda}_r \right) = \left\{ R \phi^{r+1, r}_{t+1, r} \left( \hat{\rho}_r' - \hat{\rho}_r'' \right) R \phi^{r+1, r}_{t+1, r} \left( \hat{\rho}_r' - \hat{\rho}_r'' \right) \right\} \]  

\[ = \delta_{r, \lambda} \left[ \delta_{\mu', \eta'} - \frac{1}{n_r} \right] \frac{1}{m_{r+1}^\mu} . \]  

(123)

\[ R G^{r+1, r+1}_{t+1, r+1} \left( \hat{\rho}_r', \hat{\eta}_r'; \hat{\eta}_r' \right) = \left\{ R \phi^{r+1, r+1}_{t+1, r+1} \left( \hat{\rho}_r' - \hat{\rho}_r'' \right) R \phi^{r+1, r+1}_{t+1, r+1} \left( \hat{\rho}_r' - \hat{\rho}_r'' \right) \right\} \]  

\[ = \left[ \frac{1}{\left( \delta_{\mu', \eta'} + \delta_{\mu', -\eta'} \right)} - \frac{1}{n_r} \right] \frac{1}{m_{r+1}^\mu} . \]  

(124)
Putting together (71), (72), (73), (93), (100) and (107) one obtains the complete longitudinal-anomalous contribution, \( \mathcal{L}_{\mathcal{L}, \mathcal{A}} = \mathcal{L}_{\mathcal{L}, \mathcal{A}}^I + \mathcal{L}_{\mathcal{L}, \mathcal{A}}^H I + \mathcal{L}_{\mathcal{L}, \mathcal{A}}^H II + \mathcal{L}_{\mathcal{L}, \mathcal{A}}^I (3) + \mathcal{L}_{\mathcal{L}, \mathcal{A}}^I (2) + \mathcal{L}_{\mathcal{L}, \mathcal{A}}^I (1) \).
\( \mathcal{L}_{\mathcal{A} \mathcal{P}_r} (\hat{0}_r) = \frac{1}{\sqrt{2}} \begin{bmatrix} \hat{\rho}_0 \cdots \hat{\rho}_{r-1} \hat{0}_r \cdots \hat{0}_R \end{bmatrix} \) \( \hat{0}_{r+1} \cdots \hat{0}_R \)_{SN} (127)

\( \mathcal{L}_{\mathcal{A} \mathcal{P}_r} (\hat{0}_r) = \frac{1}{\sqrt{2}} \begin{bmatrix} \hat{\rho}_0 \cdots \hat{\rho}_{r-1} \hat{0}_r \cdots \hat{0}_R \end{bmatrix} \) \( \hat{0}_{r+1} \cdots \hat{0}_R \)_{SN} (128)

\( \mathcal{L}_{\mathcal{A} \mathcal{P}_r} (\hat{0}_r) = \frac{1}{\sqrt{2}} \begin{bmatrix} \hat{\rho}_0 \cdots \hat{\rho}_{r-1} \hat{0}_r \cdots \hat{0}_R \end{bmatrix} \) \( \hat{0}_{r+1} \cdots \hat{0}_R \)_{SN} (129)

with multiplicity
\[ \mu(t) = p_t \left( \frac{1}{p_t} - \frac{1}{p_{t-1}} \right), \]
\[ \mu(0) = 1. \] (130)

Equation (126) can be written in the generic form,
\[ \mathcal{L}_{\mathcal{A} \mathcal{A}} = \frac{1}{2} \sum_{i=0}^{R+1} \sum_{r=0}^{R} \mathcal{L}_{\mathcal{A} \mathcal{P}_r} \begin{bmatrix} \delta^{\mathcal{K} \mathcal{A}_r} & + & \frac{1}{4} \sqrt{\delta^{(i+1)}} \mathcal{M}_{r, t} \delta^{(i+1)} \end{bmatrix} \mathcal{L}_{\mathcal{A} \mathcal{P}_t} \] (131)

with \( \mathcal{A}_r \) defined as
\[ \mathcal{A}_r = \begin{cases} \mathcal{M}_{r, t} & \text{if } t > r + 1, \\ \mathcal{M}_{r+1, t+1} & \text{if } t \leq r + 1. \end{cases} \] (132)

The propagators:
\[ \mathcal{L}_{\mathcal{G}^t} (\hat{\rho}_t; \hat{\rho}_s) = \begin{bmatrix} \mathcal{L}_{\mathcal{A} \mathcal{P}_r} (\hat{0}_r) \mathcal{L}_{\mathcal{A} \mathcal{P}_s} (\hat{0}_s) \end{bmatrix} \] (133)

are given by the inverse of the matrix
\[ \mathcal{M}_{r, t} = \delta^{\mathcal{K} \mathcal{A}_r} + \frac{1}{4} \sqrt{\delta^{(i+1)}} \mathcal{M}_{r, t} \delta^{(i+1)} \] (134)
that is, \([20, 21]\)

\[
\mathcal{L}_A G^{r,s}_{t,s} = \delta_{r,s} \frac{1}{\Lambda_t} + \frac{1}{4} \sqrt{\delta^{(t-1)}_s} \tilde{F}^{r,s}_t \sqrt{\delta^{(t-1)}_s}
\]

(135)

with

\[
\tilde{F}^{r,s}_t = -\frac{1}{\Lambda_t} M^{r,s}_t \frac{1}{\Lambda_t} - \frac{1}{\Lambda_t} \sum_{k=0}^{R} M^{r,s}_t \delta^{(t-1)}_k \frac{1}{4} \tilde{F}^{k,s}_t.
\]

(136)

A fully explicit form for the solution of \(\tilde{F}^{r,s}_t\) can be found in [18].

From (108) and (126) one sees that the Lagrangian, \(\mathcal{L}^{(2)} = \mathcal{L}_A + \mathcal{L}_R\), breaks up into a string of \((R + 1) \times (R + 1)\) blocks followed by a string of \(1 \times 1\) ‘blocks’ along the diagonal. The \((R + 1) \times (R + 1)\) blocks correspond to the longitudinal–anomalous sector, they contain the matrix elements \(\tilde{M}^{r,s}_t\) with \(r, s = 0, \ldots, R\), and are labelled by the index \(t = 0, 1, \ldots, R + 1\), \((t = 0\) is the longitudinal and \(t \neq 0\) are the anomalous). The \(1 \times 1\) ‘blocks’ correspond to the replicon sector, they are the elements \(\tilde{M}^{r,s}_{t,u,v}\) with \(r = 0, \ldots, R\) and \(u, v = r + 1, \ldots, R + 1\). The total number of longitudinal-anomalous modes is

\[
\mu_{LA} = \sum_{r=0}^{R} \sum_{t=0}^{R+1} \mu(t) = (R + 1)p_0,
\]

(137)

and the total number of replicon modes is

\[
\mu_R = \sum_{u=0}^{R+1} \sum_{r=0}^{R} \sum_{v=r+1}^{R+1} \mu(r; u, v) = \frac{1}{2} p_0 (p_0 - 1) - (R + 1)p_0,
\]

(138)

so that the total number of modes is

\[
\mu = \mu_{LA} + \mu_R = \frac{n(n - 1)}{2}.
\]

(139)

We note that for \(R = 0\), (108) and (126), with (32) and (33), naturally lead to (28).

One can easily obtain the propagators in the direct replica space, for general \(R\), in terms of their RFT expression, e.g., for

\[
G^{ab,ab}_{R+1,R+1} = \{\phi^s a \phi^s b\} = \{\phi^s \phi^s r\}
\]

(140)

with the symmetrized field

\[
\phi^s_t = \left[ a_0 \ldots a_{t-1} b_1 \ldots b_R \right]_S
\]

\[
= \frac{1}{2} \left( \left[ a_0 \ldots a_{t-1} b_1 \ldots b_R \right] + \left[ a_0 \ldots a_{t-1} b_1 \ldots b_R \right]_S \right)
\]

(141)
one finds
\[
G_{R+1,R+1}^{(r,r)} = \frac{1}{n_0 \ldots n_{r-1} n_r (n_r - 1) n_{r+1} \ldots n_{R}} \sum_{a, b_i} \left\{ \phi^{a}_r \phi^{b}_r \right\}
\]  
(\ref{eq:142})

\[
= \frac{1}{P_0 (p_0 - p_{R+1})} \sum_{[\hat{\beta}], \hat{\beta}_r, \hat{\gamma}_r} \left\{ \sum_{u=r+2}^{R+1} R G_{u,v}^{(r,r)} (\hat{\rho}_r, \hat{\beta}_r; \hat{\gamma}_r, \hat{\gamma}_r) + \sum_{v=r+2}^{R+1} R G_{r+1,v}^{(r,r)} (\hat{\rho}_r, \hat{\beta}_r; \hat{\gamma}_r, \hat{\gamma}_r) + 2 R G_{r+1,r+1}^{(r,r)} (\hat{\rho}_r, \hat{\beta}_r; \hat{\gamma}_r, \hat{\gamma}_r) + 8 \sum_{i=r+2}^{R+1} C A^{(r,r)} G_{i}^{(r,r)} (\hat{\rho}_r, \hat{\beta}_r; \hat{\gamma}_r, \hat{\gamma}_r) + 2 C A^{(r,r)} G_{r+1}^{(r,r)} (\hat{\rho}_r, \hat{\beta}_r; \hat{\gamma}_r, \hat{\gamma}_r) + 2 \sum_{i=0}^{r} C A^{(r,r)} G_{i}^{(r,r)} (\hat{\rho}_r, \hat{\beta}_r; \hat{\gamma}_r, \hat{\gamma}_r) \right\}.
\]

5. Spin glass free energy with fluctuations

Here we use the RFT formalism to calculate the contribution of the Gaussian fluctuations around the Parisi solution for the free energy of an Ising spin glass. The spin glass free energy, \(2\), calculated with the partition function in \(3\), can be written as

\[
F = F_{\text{mf}} + F_{\text{fluct}},
\]  
(\ref{eq:143})

where

\[
F_{\text{mf}} = \frac{1}{\beta} \lim_{n \to 0} \frac{\mathcal{L}^{(0)}}{n}
\]  
(\ref{eq:144})

provides the mean field value of the free energy, with \(\mathcal{L}^{(0)}\) given by \(7\),

\[
F_{\text{fluct}} = -\frac{1}{\beta} \lim_{n \to 0} \frac{\ln [\overline{Z}_{\text{fluct}}]}{n}
\]  
(\ref{eq:145})

provides the contribution of the fluctuations. For fluctuations up to the quadratic order,

\[
[\overline{Z}]_{\text{fluct}} = \int D (\mathcal{L}^{(2)} \Phi) D (R \Phi) \exp \{ \mathcal{L}^{(2)} \},
\]  
(\ref{eq:146})

where

\[
\mathcal{L}^{(2)} = \mathcal{L}_{R} + \mathcal{L}_{CA}
\]  
(\ref{eq:147})

with \(\mathcal{L}_{R}\) given by \(108\) and \(\mathcal{L}_{CA}\) given by \(126\), the replicon fields verifying the constraints given by \(110\), \(113\), \(116\) and \(119\). Performing the integration over the longitudinal-anomalous and the replicon fields in \(146\) considering the replicon constraints, leads to
\[
[Z^T]_{\text{huc}} = \exp\left\{ -\frac{1}{2} \sum_{r=0}^{R} \sum_{x=0}^{R+1} \mu(t) \ln \hat{\Lambda}_t^r - \frac{1}{2} \sum_{r=0}^{R+1} \mu(t) \ln \det \hat{\Lambda}_t
\right.
\]
\[
+ \frac{1}{2} \sum_{r=0}^{R} \left[ \sum_{x=0}^{r} \mu(t) \ln \left( 2 \left( 1 - \frac{P_{r+1}}{P_x} \right) \right) + \mu(r+1) \ln \left( 1 - 2 \frac{P_{r+1}}{P_r} \right) \right]
\]
\[
+ \sum_{r=r+2}^{R+1} \mu(t) \ln \left( 1 - \frac{P_{r+1}}{P_r} \right) \right\}
\]
\[
(148)
\]
\[
-\frac{1}{2} \sum_{r=0}^{R} \sum_{x=r+2}^{R+1} \mu(r; u, v) \left\{ \ln \hat{M}_{u,v}^r + \frac{1}{P_x} \ln \left( \frac{1}{P_x} - \frac{1}{P_{r+1}} \right) - \ln 2 \right\}
\]
\[
-\frac{1}{2} \sum_{r=0}^{R} \sum_{x=r+2}^{R+1} \mu(r; r+1, v) \left\{ \ln \hat{M}_{r+1,v}^r + \frac{1}{P_x} \ln \left( \frac{1}{P_x} - \frac{1}{P_{r+1}} \right) - \ln 2 \right\}
\]
\[
-\frac{1}{2} \sum_{r=0}^{R} \sum_{x=r+2}^{R+1} \mu(r; u, r+1) \left\{ \ln \hat{M}_{u,r+1}^r + \frac{1}{P_x} \ln \left( \frac{1}{P_x} - \frac{1}{P_{r+1}} \right) - \ln 2 \right\}
\]
\[
-\frac{1}{2} \sum_{r=0}^{R} \mu_1(r; r+1, r+1) \left\{ \ln \hat{M}_{r+1,r+1}^r + \frac{2}{P_x} \ln \left( \frac{1}{P_x} - \frac{1}{P_{r+1}} \right) \right\}
\]
\[
-\frac{1}{2} \sum_{r=0}^{R} \mu_2(r; r+1, r+1) \left\{ \ln \hat{M}_{r+1,r+1}^r + \frac{2}{P_x} \ln \left( \frac{1}{P_x} - \frac{2}{P_{r+1}} \right) \right\},
\]

where \( \hat{\Lambda}_t^r \) is given by (132) and \( \hat{\Delta}_t \) is
\[
\hat{\Delta}_t^{r,t} = \delta_{r,t}^K + \frac{1}{4} \sqrt{\delta_{r,t}^{(t-1)}} \hat{M}_{r,t}^{r,t} \sqrt{\delta_{r,t}^{(t-1)}}
\]

(149)

the longitudinal-anomalous multiplicity \( \mu(t) \) is given by (130) and the replicon multiplicities
\( \mu(r; u, v) \) for the various cases of \( u, v \geq r+1 \) are given by (111), (114), (117) and (120).

One observes that in (148) there is a cancellation of terms between the longitudinal-
anomalous and the replicon contributions. Hence, one obtains for the free-energy fluctuations,
\[ F_{\text{fluct}} = \frac{1}{\beta} \lim_{n \to 0} \frac{1}{2n} \left( \sum_{r=0}^{R+1} R^{r+1} \mu(t) \ln \det \tilde{A} \right) + \sum_{r=0}^{R} \sum_{u,v \in R+1} \tilde{\mu}(r; u, v) \ln \tilde{M}^{r,r}_{uv} \]

\[ = -\frac{1}{2} \sum_{r=0}^{R} P_{r+1} \left[ \ln p_{r+1} + \frac{1}{P_{r+1}} \left( \frac{P_{r+1}}{P_{r} + 1} + P_{r} - P_{r+1} - 3 \right) \ln 2 \right] \]  

(150)

with

\[ \tilde{\mu}(r; u, v) = \frac{1}{2} P_{0} (p_{r} - P_{r+1}) \left( \frac{1}{P_{a}} - \frac{1}{P_{a-1}} \right) \left( \frac{1}{P_{v}} - \frac{1}{P_{v-1}} \right), \quad u, v > r + 1 \]  

(151)

\[ \tilde{\mu}(r; r+1, v) = \frac{1}{2} P_{0} (p_{r} - P_{r+1}) \left( \frac{1}{P_{v}} - \frac{1}{P_{v-1}} \right), \quad v > r + 1 \]  

(152)

\[ \tilde{\mu}(r; r+1, r+1) = \frac{1}{2} P_{0} (p_{r} - P_{r+1}) \frac{1}{P_{r+1}} \]  

(153)

which are the proper multiplicities as remarked in [19, 20] (\( \tilde{\mu} = \mu_{\text{reg}} \) in their notation).

For \( R = 0 \), (150) reduces to

\[ F_{\text{fluct}} = \frac{1}{\beta} \lim_{n \to 0} \frac{1}{2n} \left\{ \ln M_{L} + (n - 1) \ln M_{A} + \frac{1}{2} n (n - 3) \ln M_{R} \right\}. \]  

(154)

A discussion on the fluctuations of the free energy is provided in [22], where it is concluded that the longitudinal-anomalous contribution vanishes, the full answer being then given by the replicon contribution.

### 6. Conclusion

We developed a field theory for spin glasses using replica Fourier transforms, for the case of replica symmetry and the case of replica symmetry breaking on an ultrametric tree, with the number of replicas \( n \) and the number of replica symmetry breaking steps \( R \) generic integers. We defined a new basis in terms of the RFT of the two-replica fields which block-diagonalizes the four-replica mass matrix into the replicon, anomalous and longitudinal modes. As a result, we have a field theory that is directly defined in terms of the replicon, anomalous and longitudinal fields, in RFT space. The corresponding eigenvalues are given in terms of the mass RFT. The propagators in RFT space are obtained by inversion of the block-diagonal matrix, explicit forms are provided for the propagators, which are particularly simple in the replicon sector. The formalism allows to express any \( i \)-replica vertex in the new basis and hence enables to perform a standard perturbation expansion. Via a clear sequence of steps one can transform the interaction vertices of the fluctuations in direct replica space into the
interaction vertices of the replicon, anomalous and longitudinal modes in RFT space, for higher order calculations in the perturbation expansion.

In the field theory developed for spin glasses with replica symmetry breaking in direct replica space [18], the free propagators are given by a fairly complicated set of coupled integral equations, which were solved in different momenta regimes. Also, the block-diagonalization and inversion of the mass-matrix performed in direct replica space [19], using a particular basis, involves a rather difficult procedure. In [20] a Dyson like equation related the propagators to the mass operators. In [21] the block-diagonalization and inversion of the mass-matrix was achieved by applying directly the RFT on the four-replica mass-matrix. This is to contrast with the field theory in RFT space that we present.

The field theory in RFT space provides a new tool to investigate the behaviour of spin glasses. We applied the formalism to calculate the contribution of the Gaussian fluctuations around the Parisi mean field solution for the free energy of an Ising spin glass. We also showed that the propagators in the direct replica space can be simply related to the propagators in the RFT space, which enables to calculate important physical quantities. The Gaussian propagators, in addition of being building blocks of the interacting theory, also have a direct physical meaning [18]. They are related to correlation functions that reflect the structure of the phase space. Various components of the propagator in direct replica space represent overlaps of spin–spin correlation functions inside a single state and between different states. Physical observables such as the spin glass and the nonlinear susceptibilities are expressed in terms of the propagators, having contributions from both intra- and interstate correlations. It is important to evaluate the contribution of the different fluctuation sectors, replicon, anomalous and longitudinal, to the various quantities. An investigation on spin–spin correlation functions in spin glasses was performed in [23]. We expect that the RFT field theory will allow one to further study the properties of the glassy phase, and hence contribute to the understanding of spin glasses.

Acknowledgement

IRP acknowledges the support from Fundação para a Ciência e a Tecnologia, UI0618, Portugal.

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