Learning Powers of Poisson Binomial Distributions

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Abstract. We introduce the problem of simultaneously learning all powers of a Poisson Binomial Distribution (PBD). A PBD over \(\{1, \ldots, n\}\) is the distribution of a sum \(X = \sum_{i=1}^{n} X_i\), of \(n\) independent Bernoulli 0/1 random variables \(X_i\), where \(E[X_i] = p_i\). The \(k\)th power of this distribution, for \(k\) in a range \(\{1, \ldots, m\}\), is the distribution of \(P_k = \sum_{i=1}^{n} X_i^{(k)}\), where each Bernoulli random variable \(X_i^{(k)} \in \{0,1\}\) has \(E[X_i^{(k)}] = (p_i)^k\). The learning algorithm can query any power \(P_k\) several times and succeeds in simultaneously learning all powers in the range, if with probability at least 1 \(-\delta\), given any \(k \in \{1, \ldots, m\}\), it returns a probability distribution \(Q_k\) with total variation distance from \(P_k\) at most \(\varepsilon\).

We provide almost matching upper and lower bounds on the query complexity for this problem. We first show an information theoretic lower bound on the query complexity on PBD powers instances with many distinct parameters \(p_i\), which are significantly separated. This lower bound shows that essentially a constant number of queries is required per each distinct parameter. We almost match this lower bound by examining the query complexity of simultaneously learning all the powers of a special class of PBD’s resembling the PBD’s of our lower bound.

We extend the classical minimax risk definition from statistics, dating back to 1930s [Wald 1939], to introduce a framework to study sample complexity of estimating functions of sequences of distributions. Within this framework we show how classic lower bounding techniques, such as Le Cam’s and Fano’s, can be applied to provide sharp lower bounds in our learning model.

We study the most fundamental setting of a Binomial distribution, i.e., \(p_i = p\), for all \(i\), and provide an optimal algorithm which uses \(O(1/\varepsilon^2)\) samples, independent of \(n, m\). Thus, we show how to exploit the additional power of sampling from other powers, that leads to a dramatic increase in efficiency. We also prove a matching lower bound of \(\Omega(1/\varepsilon^2)\) samples for the Binomial powers problem, by employing our minimax framework.

Estimating the parameters of a PBD is known to be hard. Diakonikolas, Kane and Stewart [COLT’16] showed an exponential lower bound of \(\Omega(2^{1/\varepsilon})\) samples to learn the \(p_i\)’s within error \(\varepsilon\). Thus, a natural question is whether sampling from powers of PBDs can reduce this sampling complexity. Using our minimax framework we provide a negative answer to this question, showing that the exponential number of samples is inevitable. We then give a nearly optimal algorithm that learns the \(p_i\)’s of a PBD using \(2^{O(n \max(\log(1/\varepsilon), \log(n)))}\) samples from the powers of the PBD, which almost matches our lower bound.

The Newton-Girard formulae give relations between the power sums \(\sum_{i=1}^{n} z_i^k\), \(k = 1, \ldots, n\), of the roots, and the coefficients of a polynomial \(P(x) = \prod_{i=1}^{n} (x - z_i)\). Thus, if we know the power sums exactly, we can first find the coefficients of \(P(x)\) and then compute the roots \(z_1, \ldots, z_n\) with an arbitrarily good accuracy. In our problem we only have access to approximate values for the power sums since they correspond to the means of the PBD powers. An intriguing question is to which extent these “noisy” power sum estimations can be used to recover the actual values of \(p_1, \ldots, p_n\) within sufficient accuracy. We answer this question by providing close lower and upper bounds on the sample complexity of estimating the parameters of a PBD using samples from its powers.
1 Introduction

1.1 Our Model and the PBD Powers Problem

A Poisson Binomial Distribution (PBD) is the discrete probability distribution of a sum of \( n \) independent Bernoulli indicator random variables, and \( n \) is the order of the distribution. So if \( X \) is a PBD of order \( n \) then \( X = \sum_{i=1}^{n} X_i \) where \( X_1, \ldots, X_n \) are independent Bernoulli 0/1 random variables. The expectations \( \mathbb{E}[X_i] = p_i \), called the parameters of the PBD, do not need to be the same and thus PBD’s capture a quite wide class of distributions. It is believed that Poisson was the first to consider PBD’s, hence their name. Let now a random variable \( Y_{i,k} \) be the product of \( k \) Bernoulli independent random variables, each distributed as \( X_i \). The expectation of \( Y_{i,k} \) is \((p_i)^k\). If \( P_k \) is the sum \( \sum_{i=1}^{n} Y_{i,k} \), we call the PBD \( P_k \) the \( k \)th power of the PBD \( X \). The expectation of \( P_k \) is equal to \( \sum_{i=1}^{n}(p_i)^k \). The powers of a PBD clearly relate to the moments of the PBD.

Suppose an unsupervised learning algorithm knows \( n \) but not the \( p_i \)’s, and aims at approximately and simultaneously learning all the powers \( P_k \) of a PBD \( X \) for \( k \in \{1, \ldots, m\} \), where \( m \) is given and can even be greater than \( n \). The algorithm can ask for independent samples from any \( P_k \) for \( k \) in any subset of the range \( \{1, \ldots, m\} \). A query to \( P_k \) returns an independent sample from distribution \( P_k \). Each such sample has \( \log n \) bits since by definition the maximum value of \( P_k \) is \( n \). The algorithm can proceed in an adaptive way, by getting some samples from some powers, then computing, then asking for more samples, depending on the computations and previous samples. The algorithm is said to succeed with probability at least \( 1 - \delta \), for given \( \delta > 0 \), if the following holds with probability at least \( 1 - \delta \): Given any \( k \in \{1, \ldots, m\} \), the algorithm outputs a distribution \( Q_k \) whose Total Variation Distance from \( P_k \) is at most \( \epsilon \). Here, \( \delta, \epsilon \in (0, 1) \) are given as input. Note, \( Q_k \) is not needed to be a PBD itself. The query complexity of the algorithm is the total number of samples obtained and is a function of \( n, m, 1/\delta, 1/\epsilon \). To compare different algorithms that query for independent samples from a subset in the range and manage to learn all powers in the range, we consider the query complexity per learned power to be the total number of queries divided by the number of powers we learn. We study this problem of simultaneously learning all the powers of a PBD in a given range in terms of query and time complexity efficiency. Ideally, our learning algorithm runs in time polynomial in \( n, m, 1/\delta, 1/\epsilon \), but our primary focus is query complexity. The problem can of course be solved by taking samples per power to learn it approximately for each power in the range. The challenging question is if we can do much better than this in terms of query and time complexity, given the fact that the powers of the unknown PBD are related because they are defined over the same unknown parameters \( p_i \)’s.

1.2 Motivation

Random Coverage Valuations The PBD powers problem arises from the problem of learning a natural class of random coverage valuations. Given a ground set \( X = \{e_1, \ldots, e_n\} \), a function \( v : 2^U \rightarrow \mathbb{N} \) is a coverage valuation if there are \( A_1, \ldots, A_m \subseteq X \) such that for all \( S \subseteq [m] \), \( v(S) = \left| \bigcup_{j \in S} A_j \right| \). Coverage valuations are monotone and submodular and have received considerable attention in optimization (maximizing a coverage valuation under a cardinality constraint), learning and algorithmic mechanism design, see e.g., [21, 3, 20] and the references therein.

Let now each element \( e_i \in X \) be associated with a probability \( p_i \in [0, 1] \) and we generate \( m \) random subsets \( A_1, \ldots, A_m \subseteq X \) by including each \( e_i \in X \) in each \( A_j \) independently with probability \( p_i \). The random sets \( A_1, \ldots, A_m \) are selected independently and are identically distributed. Random sets \( A_1, \ldots, A_m \) define a random coverage valuation function \( v : 2^{[m]} \rightarrow \mathbb{N} \) with \( v(S) = \left| \bigcup_{j \in S} A_j \right| \). Suppose we are interested in approximately learning the distribution of the values of such random
coverage valuations $v$ evaluated over subsets $S \subseteq [m]$. Namely, given a ground set $X = \{e_1, \ldots, e_n\}$ and the probabilities $p_1, \ldots, p_n$, we want to find a family of probability distributions $D(S)$ so that $\Pr[D(S) = i] \approx \Pr \left[ \bigcup_{j \in S} A_j = i \right]$, for all $i \in \{0, \ldots, n\}$ and $S \subseteq [m]$ (the probability in the right-hand-side is taken over the random sets $A_j$ with $j \in S$). Each $D(S)$ approximates the distribution of the values $v(S)$ of a coverage valuation function $v$ chosen randomly from the family of coverage valuations described above. Since the random sets $A_1, \ldots, A_n$ are independently identically distributed, only the cardinality of $S$, and not $S$ itself, matters for the union’s cardinality. Hence, given $X$ and the probabilities $p_i$’s, we aim to compute probability distributions $D_k$ so that $\Pr[D_k = i] \approx \Pr \left[ \bigcup_{j=1}^k A_j = i \right]$, for all $i \in \{0, \ldots, n\}$ and $k \in \mathbb{N}$. Each $D_k$ approximates the distribution of the cardinality of the union of $k$ sets selected randomly and independently from $X$.

Learning random coverage valuations can be reduced to the PBD powers problem, by observing that each distribution $D_k$ is the sum of $n$ independent Bernoulli variables with expectations $1 - (1 - p_1)^k, \ldots, 1 - (1 - p_n)^k$, where each such Bernoulli variable $i$ indicates whether element $e_i$ is included in at least one of the $k$ random sets considered in the union. A natural sampling model is that the learning algorithm selects an index $k \in \mathbb{N}$ and receives the cardinality of the union of $k$ random sets, which is exactly the sampling model in the PBD powers problem.

**Newton’s identities** The Newton-Girard formulae give relations between the power sums $\sum_{i=1}^n z_i^k$, $k = 1, \ldots, n$, of the roots, and the coefficients of a polynomial $P(x) = \prod_{i=1}^n (x - z_i)$. Thus, if we know the power sums exactly, we can first find the coefficients of $P(x)$ and then compute the roots $z_1, \ldots, z_n$ with an arbitrarily good accuracy. A similar approach was used in [12] to derive sparse covers for PBDs. In our problem we only have access to approximate values for the power sums since they correspond to the means of the PBD powers. An intriguing question is to which extent these “noisy” power sum estimations can be used to recover the actual values of $p_1, \ldots, p_n$ within sufficient accuracy. We answer this question by providing close lower and upper bounds on the sample complexity of estimating the parameters of a PBD using samples from its powers.

### 1.3 Our Results

We now state our first lower bound. A vector $p = (p_1, \ldots, p_n) \in [0, 1]^n$ is called $(\nu, \kappa, m)$-separated, for some positive integers $m$ and $\kappa > \nu$, with $n/m$ also a positive integer, if there are $m$ positive integers $a_1, \ldots, a_m \in [\nu]$ so that $p$ contains a group of $n/m$ values $p_i = 1 - a_i/\kappa$ for each $i \in [m]$. Thus, a $(\nu, \kappa, m)$-separated vector $p$ has $n/m$ entries of value $p_1 = 1 - a_1/\kappa$, $n/m$ entries of value $p_2 = 1 - a_2/\kappa^2$, $\ldots$, and $n/m$ entries of value $p_m = 1 - a_m/\kappa^m$. A PBD $X$ is $(\nu, \kappa, m)$-separated if the parameters defining $X$ are given by a $(\nu, \kappa, m)$-separated vector $p$.

Our results indicate that when the separation of the $p_i$’s is substantial the problem of estimating the densities of the PBD powers is equivalent to approximate the PBD’s parameters. The following information-theoretic lower bound shows that for any integer $m \leq n/(\ln n)^4$, learning an appropriate collection of $m$ powers of an $(\ln n, (\ln n)^4, m)$-separated PBD $p$ requires $\Omega(m \ln \ln n / \ln n)$ samples in the worst case. Hence, for the special case of separated PBDs, the sampling complexity should increase almost linearly with the number $m$ of different $p_i$ values in $p$, at least as long as $m \leq n/(\ln n)^4$.

**Theorem 1.1.** For any positive integer $m \leq n/(\ln n)^4$ so that $n/m$ is an integer, any algorithm that succeeds in learning all powers with indices $(\ln n)^{4i-2}$, $i = 1, \ldots, m$, of an $(\ln n, (\ln n)^4, m)$-separated PBD, within total variation distance $\varepsilon \in (0, 1/4]$ and with failure probability $\delta \leq 1/2$, requires $\Omega(m \ln \ln n / \ln n)$ samples in the worst case.
To almost match this lower bound, we show how to learn the following class of PBD’s resembling our lower bound PBD. Let \( p_i = 1 - \alpha_i / (c \cdot \ln(n))^{s-1} \), with \( c > 1 \) any constant, and \( s \) a number such that \( (c \cdot \ln(n))^s = n \), \( i = 0, 1, \ldots, s - 1 \). Notice, \( s \approx \ln(n) / \ln(\ln(n)) \), and assume \( \alpha_i \in \{1, 2, \ldots, \lfloor \sqrt{\ln(n)} \rfloor \} \) for each \( i \). The class \( \mathcal{P} \) of PBD’s instances has \( n_i \) probabilities equal to \( p_i \) for \( i = 0, 1, \ldots, s - 1 \), where \( n_i = n/s \) for each \( i = 0, 1, \ldots, s - 1 \). We assume that \( n \) and \( s \) are known. The mean of the first power of a PBD from \( \mathcal{P} \) is \( \sum_{i=0}^{s-1} n_i p_i \). This PBD is defined by a \((\sqrt{\ln(n)}, c \cdot \ln(n), s)\)-separated vector \( p \). We call a block \( i \), all the \( n_i \) probabilities equal to \( p_i \), and note that \( p_0 > p_1 > \cdots > p_{s-1} \).

**Theorem 1.2.** Let \( c \geq 2 \) be a constant, \( \varepsilon \leq 1/(6c) \), and \( n \geq \max\{e^{2c}, 4/(2 - \sqrt{2})^2\} \). Given an unknown PBD \( X \in \mathcal{P} \), the exact values of \( \alpha_i \) for each \( i = 0, 1, \ldots, s - 1 \) can be learned by Algorithm 7 using \( O \left( \log(s/\delta) / \varepsilon^2 \right) \) samples from each power \( X^{\ell_i} \), where \( \ell_i = (c \cdot \ln(n))^{s-i}/c \) for \( i = 0, 1, \ldots, s - 1 \), with success probability at least \( 1 - \delta \). The total number of samples is \( O \left( s \log(s/\delta) / \varepsilon^2 \right) \).

Although our algorithm finds exact values of \( p_i \)’s, thus learning all the powers, it uses at most \( O(\ln(\ln(n))/ (\delta \ln(\ln(n)))/\varepsilon^2) \) samples per sampled power, which is very close to our lower bound. Interestingly, our lower bound proof shows that the claimed number of samples recover the exact values of \( p_i \)’s. Thus, it essentially shows that \( \Omega(1) \) samples are required for each distinct value of \( p_i \), and our upper bound uses roughly \( O(\ln(\ln(n))) \) samples per distinct value of \( p_i \).

The lower bound of Theorem 1.1 implies that the problem is hard in general, which motivates us to consider the PBD powers problem with few distinct parameters. We ask: does the additional power of the algorithm of being able to sample from many powers make the parameter estimation easier? We answer this question in the negative, by proving an exponential lower bound in this case.

The classic minimax risk framework from statistics is used for investigating lower and upper bounds on the sample complexity of testing and learning a single distribution, cf. [28, Chapter 2]. We generalise and extend this framework from testing and learning a single distribution to sequences of distributions. This generalisation is new to the best of our knowledge. The main ingredients of our framework are generic theorems that reduce the problem of learning a sequence of distributions to testing such sequence and provide generic lower bounds on the minimax risk based on classical results from statistics, see, e.g., [29, 32, 4, 30] and [28, Chapter 2]. For precise formulations of these new theorems, see Proposition 5.1, Lemma 5.1 and Lemma 5.2.

Crucial to our framework and our main conceptual contribution here, is the new definition, Definition 5.1 of the minimax risk for sequences of distributions. This definition unveils the structure of any learning algorithm in our model. Namely, such algorithm has two distinct types of operations related to sampling, that of deciding from which distributions in the sequence to sample, and that of using the samples in its computation phase. The two operations’ types might alternate and the algorithm may be adaptive or non-adaptive about further sampling and using the samples.

Our framework is of independent interest since it can be used for proving lower bounds for estimation of functions of distribution sequences such as their densities or their parameters. It can be instantiated with the classic methods for proving sample lower bounds, that is, Le Cam’s, Fano’s or Assouad’s methods [32]. We present two applications of this framework to prove two lower bounds, in Theorems 1.3 and 1.6. To prove an exponential lower bound for parameter estimation in our model, we use our framework with the Le Cam’s method and a PBD instance introduced in Proposition 15 of [15].

**Theorem 1.3.** If \( n \geq 1/\varepsilon \), then any learning algorithm that draws \( N \) samples from any powers of an PBD of order \( n \) and returns estimates of the parameters of this PBD within additive error \( \varepsilon \) with success probability at least \( 2/3 \) must have \( N \geq 2^{\Omega(1/\varepsilon)} \).
We see that parameter estimation remains very difficult even if we enrich the power of the algorithm to allow for sampling from any power of the PBD with $n = \Theta(1/\varepsilon)$. In sharp contrast, observe, that using the density estimation algorithms from [9,16] for each of the $n$ powers of this PBD, we can learn the densities of all these powers with only $O(n/\varepsilon^2) = O(1/\varepsilon^3)$ samples. This gives a provable separation in the sampling complexity between parameter and density estimation in our model, even if the PBD has a constant number of distinct parameters. This also implies that we cannot use parameter estimation in our model as means for density estimation if the underlying PBD has a constant number of distinct parameters.

We almost match the exponential lower bound of Theorem 1.3 for parameter estimation with a close upper bound in the most general version of the PBD powers model. We use the Newton-Girard identities to reduce our problem to the classical polynomial root finding problem. We present an analysis of the error of this reduction from power sums to coefficients of the polynomial and then to its roots, when power sums are known only approximately. The main obstacle in this approach is that to find the roots of a polynomial with inexact coefficients we need extremely good approximations of the coefficients and this leads to an exponential number of samples. Since the algorithms for root finding are almost optimal, this exponential upper bound cannot be improved, unless one uses a different technique. This leads to the following result with details in Appendix C.2.

**Theorem 1.4.** Let $X$ be a PBD with probability vector $p$. There exists an algorithm which draws $2^{O(n \max(\log(1/\varepsilon), \log n))}$ samples from the powers of $X$ and computes a vector $\hat{p}$ such that $\|p - \hat{p}\|_\infty \leq \varepsilon$.

Given the two lower bounds we turn our attention to investigating the model with few distinct parameters, focusing on a single parameter, i.e., the Binomial case. Here, we prove that the parameter and density estimations are essentially equivalent. That is, we get a dramatic increase in efficiency and design an elegant algorithm which learns all powers of a given Binomial using only a constant $O(1/\varepsilon^2)$ number of samples. Crucial for our solution is to generalise the PBD powers problem allowing for any positive real powers. Below $B(n, p)$ is the PBD with all parameters equal to $p$. (Algorithm 2 can be generalised to allow $p \in [\varepsilon^2/n^d, 1-\varepsilon^2/n^d]$, for any constant $d$, see Section C.2.)

**Theorem 1.5.** Let $\varepsilon \in (0, 1/6), n \in \mathbb{N}$. Then, for any $p \in [\varepsilon^2/n, 1-\varepsilon^2/n]$, Algorithm 2 uses $O((\ln(1/\delta))^2/\varepsilon^2)$ samples and outputs $\hat{a} \in \mathbb{R}_+, \hat{q}_1, \hat{q}_2 \in (0, 1)$ such that $d_{tv}(B(n, \hat{q}_1), B(n, p^{\hat{a}})) = O(\varepsilon)$ for $l \in (1, +\infty)$ and $d_{tv}(B(n, \hat{q}_2), B(n, p^{\hat{a}})) = O(\varepsilon)$ for $l \in (0, 1)$ with probability at least $1 - \delta$.

It’s well known that to distinguish two given Binomial distributions, $O(1/\varepsilon^2)$ samples are required, e.g., [9], implying the same lower bound for learning a single Binomial. This lower bound does not apply to our model, because in our setting the input of the algorithm contains samples from many different distributions. To prove a matching lower bound we use our minimax framework with Fano’s method.

**Theorem 1.6.** Let $A$ be an algorithm that returns probability distributions which are within total variation distance $\varepsilon$ from $B(n, p^i)$ for all $i \in \{1, 2, \ldots\}$, using samples from the distributions $B(n, p^i)$ with probability of success at least $2/3$. Then $A$ uses $\Omega(1/\varepsilon^2)$ samples.

### 1.4 Related Work

The problem of approximately learning a PBD, within a given total variation distance $\varepsilon$, in a sample and time efficient way, is a fundamental problem in unsupervised learning and has received
significant attention. Chebyshev’s inequality gives an optimal bound of $O(1/\varepsilon^2)$ on the number of samples for learning a Binomial distribution of known order $n$ with constant failure probability. Birgé [3] gave an efficient algorithm for learning any continuous unimodal distribution over \{0, \ldots, n\} with $O(\log n/\varepsilon^2)$ samples (this result can be extended to PBDs [2]), and proved that this sample complexity is essentially best possible for unimodal distributions. By an elegant combinatorial construction, Daskalakis and Papadimitriou [12] proved that the family of all PBDs admits a sparse cover, i.e., there is a small subset of PBDs, of size $n^2 + n(1/\varepsilon)O(\log^2(1/\varepsilon))$, so that every PBD is within a total variation distance of $\varepsilon$ to some PBD in the subset. They used the sparse cover of PBDs to efficiently compute an approximate Nash equilibrium in anonymous multiplayer games [11].

Daskalakis, Diakonikolas and Servedio [9] exploited the sparse cover of PBDs (and several other ideas and techniques) to show that a PBD can be learned approximately with $O((\ln(1/\delta)/\varepsilon^2)$ samples, where $\delta$ is the probability of failure, and in $O((1/\varepsilon)^{\text{poly}(\log(1/\varepsilon))} \log n)$ time. Other applications of the sparse cover of PBDs include efficient testing of whether a given distribution is $\varepsilon$-close to some PBD [1]. The result and the techniques of [9] were generalised to learning sums of independent integer random variables [8] and to learning Poisson Multinomial Distributions (PMDs) [10]. Very recently, efficient algorithms have been shown for learning PMDs [7,17], PBDs [15], and sums of independent integer random variables [16], by using Fourier Transforms. The paper [16] implies that PBDs can be learned approximately with $O((1/\varepsilon^2)\sqrt{\log(1/\varepsilon)})$ samples and with the same running time.

In this very active research area (see also [14] for a survey and references on efficient approximate learning of structured probability distributions), we consider the problem of approximately learning, in a sample and time efficient way, a family of many closely related PBDs (instead of a single one). Given some samples, we need to extract information not only about the parameters of the PBD from which the samples come, but also about the relation of the different probability distributions that they generate the samples. To the best of our knowledge, this is the first time that a similar question has been studied in the area of unsupervised learning of structured probability distributions.

1.5 Notation

Our model and the basic definitions are introduced at the beginning of Section 1. We introduce here some additional notation used throughout the paper. Additional notation will be introduced per Section. For any positive integer $k$, we let $[k] = \{1, \ldots, k\}$. We let $\log n$ be the base-2 logarithm of $n$ and let $\ln n$ be the natural logarithm of $n$. We let $E[X]$ and $\text{Var}[X]$ denote the mean value and the variance, respectively, of a probability distribution $X$. We let $B(n,p)$ denote a binomial distribution of order $n$ and probability $p$. We usually identify a PBD with the vector $\mathbf{p} = (p_1, \ldots, p_n)$ of expectations of its Bernoulli trials. We denote $\text{err}(n,p,\varepsilon) = \varepsilon \sqrt{p(1-p)/n}$. Let $\mathbb{R}_{++} = \{a \in \mathbb{R} : a > 0\}$. Given two probability distributions $X$ and $Y$ over the set of natural numbers $\mathbb{N}$, the total variation distance (TVD) of $X$ and $Y$, denoted by $d_{TV}(X,Y)$ is $d_{TV}(X,Y) = \sum_{i \in \mathbb{N}} |\Pr[X = i] - \Pr[Y = i]|/2$. For brevity, we often refer to the total variation distance simply the distance of $X$ and $Y$ or TVD of $X$ and $Y$. We often use $X(i)$ to denote $\Pr[X = i]$, i.e., the probability that $X$ takes the value $i$.

2 Lower Bound for Learning PBD Powers: Separated Case

2.1 Preliminaries

We show a simple lower bound on the total variation distance of two PBDs based on the difference of their expected values. Its proof can be found in Section A.1 of the Appendix.
Lemma 2.1. Let $X$ and $Y$ be two PBDs with expected values $\mu_X = E[X]$ and $\mu_Y = E[Y]$ and variances $\sigma_X^2 = \text{Var}[X]$ and $\sigma_Y^2 = \text{Var}[Y]$. Then, for any $\varepsilon > 0$ such that $\sigma_X^2, \sigma_Y^2 \geq \ln(\frac{2}{1-\varepsilon})$, if $|\mu_Y - \mu_X| > 2\sqrt{\ln(\frac{1}{1-\varepsilon})}(\sigma_X + \sigma_Y)$, then $d_{tv}(X, Y) > \varepsilon$.

2.2 The Proof of Theorem 1.1

We show that learning $m$ appropriately selected powers of an $(\ln n, (\ln n)^4, m)$-separated PBD $p$ requires $\Omega(m \ln \ln n / \ln n)$ samples, i.e., almost as many as the number of different $p_i$ values in $p$, provided $m \leq n/(\ln n)^4$. We assume that $n$ is large enough and consider an $(\ln n, (\ln n)^4, m)$-separated PBD defined by $m \leq n/(\ln n)^4$ integers $a_1, \ldots, a_m$, with $1 \leq a_i \leq \ln n$. The corresponding vector $p$ consists of $m$ groups with $n/m$ entries $p_i = 1 - a_i/(\ln n)^4$ in each group $i = 1, \ldots, m$ (note, $p_1 < p_2 < \cdots < p_m$). For simplicity, we assume that $\ln n, n/m \in \mathbb{N}$. The intuition behind the proof is that given a distribution that approximates the $(\ln n)^{4i-2}$-th power of $p$, we can extract the exact value of $a_i$. Lemma 2.2 helps towards formalizing this intuition.

Lemma 2.2. For any $m \in \mathbb{N}$, $m \leq n/(\ln n)^4$, let $p$ and $q$ be two $(\ln n, (\ln n)^4, m)$-separated vectors defined by positive integers $a_1, \ldots, a_m$ and $b_1, \ldots, b_m$, resp. For any $\varepsilon \in (0, 1/2]$ and any $i \in [m]$, if the $(\ln n)^{4i-2}$-th power of $p$ and the $(\ln n)^{4i-2}$-th power of $q$ are within TVD at most $\varepsilon$, then $a_i = b_i$.

Proof. (sketch) For contradiction, assume that there is $i \in [m]$ so that the $(\ln n)^{4i-2}$-th power of $p$ and the $(\ln n)^{4i-2}$-th power of $q$ are within TVD at most $\varepsilon$ and $a_i < b_i$ (case $a_i > b_i$ is symmetric). Let $x_i = b_i - a_i$, with $1 \leq x_i \leq \ln n$, $k = (\ln n)^{4i-2}$, and let $X$ and $Y$ be the $k$-th power of $p$ and $q$, respectively. Let also $\nu = \ln n$ and $s = n/m$, with $(\ln n)^4 \leq s \leq n$; for simplicity, let $\nu, s \in \mathbb{N}$.

We prove that for each power $k = \nu^{4i-2}$ and for both $p$ and $q$, (i) the Bernoulli trials in each group $j < i$ contribute essentially 0 to the mean value and the variance of both $X$ and $Y$ (because for any $j < i$, $(1 - a_j/\nu^{4(i+1)})^{\nu^{4i-2}} \leq n^{-\ln n}$); (ii) the Bernoulli trials in each group $j > i$ increase the variance of $X$ and $Y$ by at most $2s/\nu^5$; and (iii) the difference in the mean values of $X$ and $Y$ due to the Bernoulli trials in each group $j > i$ is roughly $s/\nu^5$. As for the Bernoulli trials in group $i$, they contribute roughly $s/\nu$ to the variance of $X$ and $Y$ and increase the difference in their means by roughly $\nu s_i/\nu^5$. If $s_i \geq 1$, since $s \geq (\ln n)^4 = \nu^4$ and since $n$ is sufficiently large, the difference in the mean values of $X$ and $Y$, which is $\Omega(s/\nu^5)$, is greater than the sum of their standard deviations, which is $O(\sqrt{s/\nu})$. Thus, by Lemma 2.1, the $k$-th powers $X$ and $Y$ of $p$ and $q$ are at distance larger than $\varepsilon$, a contradiction. So, an $\varepsilon$-approximation to the $\nu^{4i-2}$-th power of $p$ by $q$ is possible only if $p_i = q_i$; thus, only if $a_i = b_i$. The details can be found in Section A.2.

To prove Theorem 1.1, we show that given $\varepsilon$-approximations to the powers of $p$ with indices $(\ln n)^{4i-2}$, for all $i \in [m]$, we can determine the exact values of $a_1, \ldots, a_m$, defining $p$. Namely, given distributions $Y_1, \ldots, Y_m$, each $Y_i$ at distance at most $\varepsilon \leq 1/4$ to the $(\ln n)^{4i-2}$-th power of $p$, we can obtain a $(\ln n, (\ln n)^4, m)$-separated vector $q$ defined by $m$ positive integers $(b_1, \ldots, b_m)$ so that for all $i \in [m]$, the $(\ln n)^{4i-2}$-th power of $q$ is within TVD at most $\varepsilon$ to $Y_i$. To find such a vector $q$, we perform exhaustive search, in time $O((\ln n)^m)$. That is, we try all possible tuples $(b_1, \ldots, b_m)$ and find a tuple whose $(\ln n)^{4i-2}$-th power is within TVD at most $\varepsilon$ to the corresponding power $Y_i$, for all $i \in [m]$. At least one such tuple exists, since $(a_1, \ldots, a_m)$ has this property. By the triangle inequality, we have that for all $i \in [m]$, the $(\ln n)^{4i-2}$-th power of $q$ and the $(\ln n)^{4i-2}$-th power of $p$ are at distance at most $2\varepsilon \leq 1/2$. Thus, if the learning algorithm succeeds in computing an $\varepsilon$-approximation $Y_i$ to each $(\ln n)^{4i-2}$ power of $p$, which happens with probability at least $1 - \delta \geq 1/2$, we can find an $(\ln n, (\ln n)^4, m)$-separated vector $q$ whose $(\ln n)^{4i-2}$-th powers are within distance $2\varepsilon \leq 1/2$ to the corresponding powers of $p$. So, by Lemma 2.2 we have $a_i = b_i$ for all $i \in [m]$. 


Since we need \( m \log \log n \) bits to represent \( a_1, \ldots, a_m \) and a sample from a PBD power has \( \log n \) bits, such an \( \varepsilon \)-approximation to the powers of \( p \) with indices \((\ln n)^{4i-2}\), for \( i \in [m] \), requires \( \Omega(m \log \log n / \log n) \) samples in total. Otherwise, we could use samples from the PBD powers, provided to the learning algorithm as input, as an economic representation of \( a_1, \ldots, a_m \). Then, the learning algorithm together with the exhaustive search procedure for finding \( q \) can be used as a “decoding” algorithm to obtain \( a_1, \ldots, a_m \) from their economic representation with the input samples. Since we can use any values for \( a_1, \ldots, a_m \), such a compression scheme is impossible, see, e.g., [23]. Note, such a learning algorithm would have a certain probability of failure, if the input samples were truly random. But here, since we know \( p \) and want to use the learning algorithm as a compression scheme for \( a_1, \ldots, a_m \), we can compute input samples deterministically, so that the learning algorithm satisfies its error guarantees with certainty (such a sample collection exist, since the learning algorithm has a positive probability of success). We have thus shown that the worst-case sample complexity of any learning algorithm for this class of instances is \( \Omega(m \log \log n / \log n) \).

3 Upper Bound for Learning PBD Powers: Separated Case

3.1 Preliminaries

To estimate the mean of a PBD we use the following Proposition from [9] Lemma 6.

Proposition 3.1 (Lemma 6 from [9]). For all \( n, \varepsilon, \delta > 0 \), there exists an algorithm \( A(n, \varepsilon, \delta) \) with the following properties: given access to a PBD \( X \) of order \( n \), it produces estimates \( \hat{\mu}, \hat{\sigma}^2 \) for \( \mu = E[X], \sigma^2 = \text{Var}[X] \) respectively such that with probability at least \( 1 - \delta \) \( |\mu - \hat{\mu}| \leq \varepsilon \sigma \) and \( |\sigma^2 - \hat{\sigma}^2| \leq \varepsilon \sigma^2 \sqrt{4 + \frac{1}{\sigma^2}} \). Moreover, \( A \) uses \( O((\log(1/\delta))/\varepsilon^2) \) samples and runs in time \( O(\log n \log(1/\delta)/\varepsilon^2) \).

Fact 3.1. Let \( m \) and \( y \) be any real numbers such that \( m \geq 1 \) and \( |y| \leq m \). Then we have that \( e^y(1 - y^2/m) \leq (1 + y/m)^m \leq e^y \).

Proof. See e.g. page 435 of [25].
Algorithm 1 Exact Learning Algorithm for Special Class of PBDs

**Input:** Random samples from powers of an unknown PBD $X \in \mathcal{P}$, parameter $n$, any constant $c \geq 2$, error bound $\varepsilon \in (0, 1/(6c)]$, confidence bound $\delta > 0$.

**Output:** Exact values of $(p_0, p_1, \ldots, p_{i-1})$ from $X$ output with success prob. at least $1 - \delta$.

1. for $i = 0, 1, \ldots, s - 1$ do
2.  Call $A(n, \varepsilon, \delta/s)$ (see Proposition 3.1) and draw $O \left( \log(s/\delta)/\varepsilon^2 \right)$ samples from $X^{\ell_i}$ to obtain $\hat{\mu}_{\ell_i}$.
3.  $\ell_i \leftarrow \left( \frac{c \ln(n)\varepsilon}{\delta} \right)^{-1}$, $n_i \leftarrow 2^\ell_i$, $\hat{\tau}_i \leftarrow \hat{\mu}_{\ell_i}/n_i - \sum_{j=0}^{i-1} p_{j}^{\ell_i}$ (* Note: $\sum_{j=0}^{i-1} p_{j}^{\ell_i} = 0$ *)
4.  Let $\beta_i \in \{1, 2, \ldots, \lfloor \sqrt{\ln(n)} \rfloor \}$ be the smallest number s.t. $(1 - \frac{\alpha_i}{c\ell_i})^{\ell_i} \leq \hat{\tau}_i < (1 - \frac{\beta_i}{c\ell_i})^{\ell_i}$
5.  $a_i \leftarrow (1 - \frac{\alpha_i}{c\ell_i})^{\ell_i}$, $b_i \leftarrow (1 - \frac{\beta_i}{c\ell_i})^{\ell_i}$
6.  if $\hat{\tau}_i < \frac{a_i + b_i}{2}$ then $\alpha_i \leftarrow \beta_i$, else $\alpha_i \leftarrow \beta_i - 1$
7.  $p_i \leftarrow 1 - \frac{\alpha_i}{c\ell_i} \left( \frac{\alpha_i}{c\ell_i} \right)^{\ell_i}$.

### 3.2 The Proof of Theorem 1.2

The following technical lemma will be crucial in our proof of Theorem 1.2. Its proof can be found in Appendix B.1.

**Lemma 3.1.** Let $i \in \{0, 1, \ldots, s - 1 \}$. If $\alpha_i, \beta_i \in \{1, 2, \ldots, \lfloor \sqrt{\ln(n)} \rfloor \}$ and $\alpha_i < \beta_i$, and $c \geq 2$ and $\varepsilon \leq 1/(6c)$, then we have that $(1 - \frac{\alpha_i}{c\ell_i})^{\ell_i} - (1 - \frac{\beta_i}{c\ell_i})^{\ell_i} > \frac{4\varepsilon}{n/s}$ for all $n \geq e^{2c}$.

We next present only a sketch of the proof, its full version can be found in Appendix B.2. The proof is by induction on $i$ and we only sketch the induction step. Assume that for some $i \in \{1, 2, \ldots, s - 1 \}$, values of all $p_0, p_1, \ldots, p_{i-1}$ are known exactly, and we will show how to exactly learn $p_i$.

By Fact 3.1, we observe $p_{i}^{\ell_i} \leq \left( \frac{2}{3} \right)^{1/c}$, and similarly, $p_{i+j}^{\ell_i} \leq \left( \frac{1}{n} \right)^{(c\ln(n))^{1/2}}$ for $j = 1, 2, \ldots$. By geometric series properties, this implies $\mu_{\ell_i} = E[X^{\ell_i}] = n_0 \cdot \left( \sum_{j=0}^{i-1} p_{j}^{\ell_i} + \sum_{j=i+1}^{i} p_{j}^{\ell_i} \right) \leq n_0 \cdot \left( \sum_{j=0}^{i-1} p_{j}^{\ell_i} + p_{i}^{\ell_i} + 1/n \right)$, and letting $\mu_{\ell_i} = n_0 \cdot \left( \sum_{j=0}^{i-1} p_{j}^{\ell_i} + p_{i}^{\ell_i} + r_{\ell_i} \right)$, it implies $r_{\ell_i} \leq 2/n$.

By using inequality $(1 - x/m)^m \geq 1 - x$, true for $m \geq 1$, $x \leq m$, and properties of geometric series we obtain $\text{Var}[X^{\beta_i}] = n_0 \sum_{j=0}^{i-1} p_{j}^{\ell_i} (1 - p_{j}^{\ell_i}) < 2n_0$. Thus, by Proposition 3.1, $| \mu_{\ell_i} - \hat{\mu}_{\ell_i} | \leq \varepsilon \sigma_{\ell_i} < \varepsilon \sqrt{2n_0}$, with probability at least $1 - \delta/s$, so $| \sum_{j=0}^{i-1} p_{j}^{\ell_i} + p_{i}^{\ell_i} + r_{\ell_i} - \hat{\mu}_{\ell_i}/n_0 | < \varepsilon \sqrt{2n_0}$. If we let $\alpha_i \in \{1, 2, \ldots, \lfloor \sqrt{\ln(n)} \rfloor \}$ be such that $p_{i}^{\ell_i} = \left(1 - \frac{\alpha_i}{c\ell_i}\right)^{\ell_i}$, and denote $\hat{\tau}_i = \hat{\mu}_{\ell_i}/n_0 - \sum_{j=0}^{i-1} p_{j}^{\ell_i}$ (recall that $p_0, \ldots, p_{i-1}$ are known), the last inequality rewrites to

$$
\left| \left(1 - \frac{\alpha_i}{c\ell_i}\right)^{\ell_i} + r_{\ell_i} - \hat{\tau}_i \right| < \varepsilon \sqrt{2n_0}.
$$

(1)

We can argue that there exists the smallest $\beta_i \in \{1, 2, \ldots, \lfloor \sqrt{\ln(n)} \rfloor \}$ such that $\left(1 - \frac{\beta_i}{c\ell_i}\right)^{\ell_i} \leq \hat{\tau}_i$, so

$$
\left(1 - \frac{\beta_i}{c\ell_i}\right)^{\ell_i} \leq \hat{\tau}_i < \left(1 - \frac{\beta_i - 1}{c\ell_i}\right)^{\ell_i}.
$$

(2)

Suppose now that $\beta_i \geq \alpha_i + 2$; then by (2), Lemma 3.1 and $n \geq e^{2c}$: $\left(1 - \frac{\alpha_i}{c\ell_i}\right)^{\ell_i} + r_{\ell_i} - \hat{\tau}_i > \left(1 - \frac{\alpha_i}{c\ell_i}\right)^{\ell_i} + r_{\ell_i} - \left(1 - \frac{\beta_i - 1}{c\ell_i}\right)^{\ell_i} \geq \left(1 - \frac{\alpha_i}{c\ell_i}\right)^{\ell_i} + r_{\ell_i} - \left(1 - \frac{\alpha_i + 1}{c\ell_i}\right)^{\ell_i} > \frac{4\varepsilon}{\sqrt{n/s}}$. This is in contradiction with (1); thus, $\beta_i \leq \alpha_i + 1$. If $\beta_i \leq \alpha_i - 1$, then by (2) and Lemma 3.1 $| \hat{\tau}_i - p_{i}^{\ell_i} + r_{\ell_i} | =$
\[ |\hat{r}_i - (1 - \alpha_i) \epsilon_i - r_i | \geq \left| (1 - \frac{\alpha_i - 1}{\epsilon_i}) \epsilon_i - (1 - \frac{\alpha_i}{\epsilon_i}) \epsilon_i - r_i \right| > \frac{4\epsilon}{\sqrt{n/s}} - 2/n \] contradicts with (4).

Thus, \( \beta_i \in \{\alpha_i, \alpha_i + 1\} \).

By Lemma 3.1, the length of the interval \( I_i = \left[ (1 - \frac{\alpha_i - 1}{\epsilon_i}) \epsilon_i, (1 - \frac{\beta_i}{\epsilon_i}) \epsilon_i \right) \) in (2) can be lower bounded as \( (1 - \frac{\beta_i - 1}{\epsilon_i}) \epsilon_i - (1 - \frac{\beta_i}{\epsilon_i}) \epsilon_i > \frac{4\epsilon}{\sqrt{n/s}} \). If \( \alpha_i = \beta_i \), then by (11) the distance between \( (1 - \frac{\beta_i - 1}{\epsilon_i}) \epsilon_i \) and \( \hat{r}_i \) is \( r_i + \epsilon \sqrt{2/n_0} \leq 2/n + \epsilon \sqrt{2/n_0} \), i.e., strictly less than half of the length of \( I_i \) by \( n > \frac{4}{(2-\sqrt{2})^2} \). On the other hand, if \( \alpha_i = \beta_i - 1 \), then by (11) the distance between \( (1 - \frac{\beta_i - 1}{\epsilon_i}) \epsilon_i \) and \( \hat{r}_i \) is \( \epsilon \sqrt{2/n_0} \), i.e., strictly less than half of the length of \( I_i \). We can use this test to decide if \( \alpha_i = \beta_i - 1 \) or \( \alpha_i = \beta_i \). Thus the precise value of \( p_i \) can be learned from \( \hat{r}_i \).

To finish the proof, observe that by the union bound all the sampling estimates hold with probability at least \( 1 - \delta \). Moreover, because this sampling for each \( i = 0, 1, \ldots, s - 1 \) takes \( \lceil \frac{s}{(\epsilon^2 \delta)} \rceil \) samples from \( X^d \), the total number of samples is \( s \cdot \lceil \frac{s}{(\epsilon^2 \delta)} \rceil \).

4 Upper Bound for Learning Binomial Powers

4.1 Preliminaries

To bound the Total Variation Distance of a Binomial and a PBD we shall use the following result of Roos [27, Theorem 2].

Lemma 4.1 (Theorem 2 from [27]). Let \( X = \sum_{i=1}^{d} \) be a PBD with probability vector \( p = (p_i)_{i=1}^{n} \) and let \( p \in (0,1) \). Then

\[ d_{tv} (X, B(n, p)) \leq \sqrt{\frac{\sqrt{\tau(p)}}{2 (1 - \sqrt{\tau(p)})^2}} \text{, where } \tau(p) = \frac{\gamma_1(p)^2 + 2\gamma_2(p)}{2np(1-p)}, \quad \gamma_j(p) = \sum_{i=1}^{n} (p - p_i)^j \]

In the special case of bounding the total variation distance of two Binomial distributions we have the following Corollary of Lemma 4.1.

Corollary 1. Let \( \epsilon < 1/2 \), \( n \geq 1 \). Let \( B(n,p), B(n,q) \) be two Binomial distributions such that \( |p - q| \leq \epsilon \sqrt{\frac{p(1-p)}{n}} \) then \( d_{tv} (B(n,q), B(n,p)) \leq 2\sqrt{\epsilon \epsilon} \).

Proof. Following the notation of Lemma 4.1 we have \( \gamma_1(p) \leq \epsilon \sqrt{np(1-p)}, \gamma_2(p) \leq \epsilon^2 p(1-p), \tau(p) \leq \epsilon^2/2 + \epsilon^2/(2n) \leq \epsilon^2 \). Thus \( d_{tv} (B(n,q), B(n,p)) \leq \frac{\epsilon \epsilon}{2(1-\epsilon)} \leq 2\sqrt{\epsilon \epsilon} \) when \( \epsilon < 1/2 \).

The proofs of the following two facts can be found in Section C.11

Fact 4.1. For any \( \epsilon, \delta \in (0,1/2) \), and \( \psi > 0 \), let \( m = \lceil 4 \ln(1/\delta)/(\epsilon^2 \psi^2) \rceil \) and let \( p = (s_1 + \cdots + s_m)/(nm) \), where \( s_1, \ldots, s_m \) are \( m \) independent samples from a Binomial distribution \( B(n,p) \). Then, \( \Pr[p < p + \psi \text{err}(p,q,n,\epsilon)] \geq 1 - \delta \), \( \Pr[p > p + \psi \text{err}(p,q,n,\epsilon)] \geq 1 - \delta \).

Fact 4.2. Let \( p \in [0,1], \epsilon, \delta \in (0,1/2), \psi > 0, k = \lfloor \ln(4/\delta)/\ln(2) \rfloor, m = \lceil 4 \ln(2k/\delta)/(\epsilon^2 \psi^2) \rceil \). For \( i \in [k] \) let \( w_i = \sum_{j=1}^{m} s_j/(nm) \), with \( s_1, \ldots, s_m \) i.i.d. samples from \( B(n,p) \). If \( \hat{q}_1 = \min_{1 \leq i \leq k} w_i, \hat{q}_2 = \max_{1 \leq i \leq k} w_i \), then \( \Pr[p - \psi \text{err}(p,q,n,\epsilon) < \hat{q}_1 < p] \geq 1 - \delta \), \( \Pr[p < \hat{q}_2 < p + \psi \text{err}(p,q,n,\epsilon)] \geq 1 - \delta \).

The overall number of samples to obtain \( \hat{q}_1, \hat{q}_2 \) is \( km = O \left( \ln(1/\delta)^2/(\epsilon^2 \psi^2) \right) \).
4.2 Discussion

We prove here that \( O(1/\varepsilon^2) \) samples are sufficient to learn all the powers of a Binomial distribution \( B(n, p) \) with constant probability of success. From Corollary 4.1 it follows that to properly learn a Binomial distribution \( B(n, p) \) within total variation distance \( O(\varepsilon) \) it’s sufficient to approximate its parameter \( p \) with error \( \text{err}(n, p, \varepsilon) = \varepsilon \sqrt{p(1-p)/n} \). Suppose first that the unknown \( p \approx 1 - 1/n \), then it is not clear at all that sampling from a constant number of powers would suffice to approximate all the powers. We could first sample from \( B(n, p) \) to obtain an approximation \( \hat{p}_1 \approx p \), but in this case it is useless. On the other extreme, if \( p \approx \text{const} \), then roughly only the first \( \log(n) \) powers matter. In fact, it is not too difficult to show that there always exists a constant power, say \( j \), such that \( \hat{p}_1 \) raised to power \( j' \in \{j + 1, j + 2, \ldots, \log(n)\} \) approximates \( p^{j'} \) well enough. Then we can sample from each power \( i = 2, 3, \ldots, j \) separately. But how to solve the large case \( (p \approx 1 - 1/n) \) and bridge it with the small case \( (p \approx \text{const}) \)?

If \( p \) is large, a natural idea is to use the approximation \( \hat{p}_1 \approx p \) to find a power, \( \ell^* \), such that \( \hat{p}_1^{\ell^*} \approx \text{const} \). If we sample from \( B(n, p^{\ell^*}) \) and obtain an approximation \( \hat{q}_1 \approx p^{\ell^*} \), then one can argue that \( \hat{p}_1 := \hat{q}_1^{1/\ell^*} \) approximates \( p^i \) well enough, for \( j = 2, 3, \ldots, \ell^* - 1 \); that’s like using approximation \( \hat{q}_1 \) backwards. Similarly to the case \( p \approx \text{const} \), it is possible to show that there exists a constant power \( k \) such that \( \hat{q}_j \approx p^{j\ell^*} \) well enough for \( j \geq k + 1 \). The remaining powers \( j\ell^* + i \), for \( j = 2, \ldots, k \) and \( i = 1, \ldots, \ell^* - 1 \), can be approximated by sampling from \( B(n, p^{\ell^*}) \) for \( j = 2, \ldots, k \) (obtaining \( \hat{q}_j \approx p^{j\ell^*} \)), and approximating \( p^{j\ell^* + i} \) by \( \hat{q}_j \hat{p}_1 \), where \( \hat{p}_1 \approx p^i \) was found previously, for \( i = 1, 2, \ldots, \ell^* \). That’s like using the approximations \( \hat{q}_j \) forwards, and filling the “gaps” between powers \( j\ell^* \) and \( (j + 1)\ell^* \) by \( \hat{p}_1 \)'s. It is possible to analyze the error of such method but it features behaviour of five dimensional functions depending on \( n, p, \varepsilon, \delta \) and powers \( \ell \), and thus is complex.

We show how to completely avoid these complications by generalising our problem to allow for continuous powers, i.e., we learn \( B(n, p^\ell) \) for all \( \ell \in \mathbb{R}_{++} \). Considering the powers to be in \( \mathbb{R}_{++} \) rather than \( \mathbb{N} \) unveils the symmetric nature of the problem. To see that, notice that \( B(n, p^\ell) \) eventually converges to “deterministic” distributions since \( \lim_{\ell \to \infty} B(n, p^\ell) = B(n, 0) \) and \( \lim_{\ell \to 0} B(n, p^\ell) = B(n, 1) \). We are able to treat uniformly the backwards and forwards cases which now correspond to powers smaller and greater than one, and there is no need to fill the “gaps” now. This leads to an elegant algorithm that interestingly will need to sample from only two different powers. By Corollary 4.1 the problem of approximating the powers \( B(n, p^\ell) \) reduces to approximating \( p^\ell \) for all \( \ell \in (0, +\infty) \). We will explain the main idea of our algorithm. Suppose that \( p = 1 - 1/n + c \), where \( c < 1/n \). We split the decimal representation of \( p \) in two parts. The first part consists of roughly \( \log n \) 9’s determining the \( p \)’s closeness to 1 (or to 0 in the symmetric case \( p \approx 0 \)) and the second part, referred to as a “constant” part, corresponds to \( c \). The decimal representation of such a \( p \) could for example be: \( p = 0.99 \ldots 9458382 \ldots \). It is clear that, for the first powers, the bits of \( p \)’s “constant” part are insignificant but for higher powers \( \ell \) these bits should be learned in order to have an \( \varepsilon \)-approximation of \( B(n, p^\ell) \) in total variation distance. Using Fact 4.1 to approximate \( p = 1 - 1/n + c \) using samples from the first power we see that we can obtain an estimate \( \hat{p} \) with precision roughly \( \sqrt{p(1-p)/n} \approx 1/n \). Thus, we learn the first \( \log n \) 9’s of the representation of \( p \). To learn the \( p \)’s “constant” part we have to sample from a higher power to be able to distinguish the “higher” bits, given the error of Fact 4.1. To learn the “constant” part of \( p \) in our example one should sample roughly from the \( n \)-th power. This idea suggests that to approximate \( p \) sufficiently for all powers \( \ell \in (0, +\infty) \), we have to obtain a good approximation of power \(-1/\log(p)\), corresponding to the number of initial 0’s or 9’s in the decimal representation of \( p \) and an approximation of the “constant” part \( c \). Our Algorithm 2 follows this intuition. The proof of our upper bound is based on Lemmas 4.2 and 4.3. Lemma 4.3 shows that sampling from
Algorithm 2 Binomial Powers

Input : \( O(\ln(1/\delta)^{2}/\varepsilon^{2}) \) samples from the powers of \( B(n,p) \).
Output : \( \hat{a}, \hat{q}_1, \hat{q}_2 \).

1: Draw \( O(\ln(1/\delta)/\varepsilon^{2}) \) samples from \( B(n,p) \) to obtain the approximation \( \hat{p} \) using Fact 4.1.
2: Let \( \hat{a} \leftarrow -1/\ln(\hat{p}) \).
3: Draw \( O(\ln(1/\delta)^{2}/(\varepsilon^{2}\psi(p^{\delta})^{2})) \) samples from \( B(n,p^{\hat{a}}) \) to get estimations \( \hat{q}_1, \hat{q}_2 \) of \( p, \hat{q}_1 \leq p \leq \hat{q}_2 \), using Fact 4.2.
4: return \( \hat{a}, \hat{q}_1, \hat{q}_2 \)

**Lemma 4.2.** Let \( \psi(p) = D \sqrt{\frac{p}{1-p}} \ln(1/p) \), where \( D \approx 1.24263 \). Let \( p, \hat{q}_1, \hat{q}_2 \in (0,1) \) with \( \hat{q}_1 < p < \hat{q}_2 \). Then if \( \hat{q}_1 \geq \psi(p) \varepsilon (n,p,\varepsilon) \), \( \hat{q}_2 - p \leq \psi(p) \varepsilon (n,p,\varepsilon) \) it holds \( p^l - \hat{q}_1 \leq \varepsilon (n,p',\varepsilon) \) for all \( l \in (1, +\infty) \) and \( \hat{q}_2 - p^l \leq \varepsilon (n,p',\varepsilon) \) for all \( l \in (0,1) \).

**Proof.** As a direct consequence of the Mean Value Theorem applied to the mapping \( x \mapsto x^l \) we obtain \( p^l - \hat{q}_1 \leq \varepsilon (n,p,\varepsilon) \) for \( l \in (1, +\infty) \) and \( \hat{q}_2 - p^l \leq \varepsilon (n,p',\varepsilon) \) for all \( l \in (0,1) \). Next we find a function \( u(p) \) such that for all \( l > 0 \)

\[
\begin{align*}
  u(p) p^l - 1 \Psi ((n,p,\varepsilon)) &\leq \varepsilon (n,p',\varepsilon) \\
  u(p) p^l - 1 \Psi (\frac{p(1-p)}{n}) &\leq \sqrt{\frac{p^l (1-p^l)}{n}} \\
  u^2 (p) l^2 p^{2l-2} p(1-p) &\leq p^l (1-p^l) \\
  u^2 (p) &\leq \frac{p}{1-p} \frac{p^l - 1}{l^2}
\end{align*}
\]

Let \( f(l) = \frac{p^{l-1}}{l^2} \), \( g(p) = 6 - 6p^l + 4l \ln p + l^2 (\ln p)^2 \). Then

\[
\begin{align*}
  f'(l) &= \frac{p^{l-1}(-2 + 2p^l - l \ln p)}{l^3} \\
  f''(l) &= \frac{p^{l-1}(6 - 6p^l + 4l \ln p + l^2 (\ln p)^2)}{l^4} \\
  g'(p) &= \frac{2l (2 - 3p^l + l \ln (p))}{p}
\end{align*}
\]

Set \( p^l = y \) and notice that the maximum of the concave function \( y \mapsto 2 - 3y + \ln(y) \) is \( 1 - \ln(3) < 0 \). Thus \( g \) is a continuous, strictly decreasing function of \( p \) and \( \lim_{p \to 1} g(p) = 0 \). Therefore \( g(p) > 0 \) for all \( p \in (0,1) \). Resultantly, \( f \) is a convex function of \( l \) and attains its minimum at \( \hat{l} = -\frac{C}{\ln p} \) (the root of \( f'(l) = 0 \)), where \( C = 2 + W_n(-2/e^2) \approx 1.59362 \). It’s minimum value is \( f(\hat{l}) = \frac{eC}{\varepsilon} \ln(p)^2 \).

Choosing \( u(p) = D \sqrt{\frac{p}{1-p}} \ln(1/p) \), \( D = \frac{eC}{\varepsilon} \) ensures that inequality 3 holds.

**Lemma 4.3.** Let \( \varepsilon \in (0,1/6) \), \( n \geq 1 \), and \( p \in (\tau,\mu) \) where \( \tau = \frac{1}{2} \left( 1 - \sqrt{1 - 36\varepsilon^2/n} \right) \leq \varepsilon^2/n \), \( \mu = \frac{1}{2} \left( 1 + \sqrt{1 - 36\varepsilon^2/n} \right) \geq 1 - \varepsilon^2/n \). Moreover, let \( a, \hat{a} \in \mathbb{R}_{++} \) such that \( p^a = \hat{p}^\hat{a} = 1/e \). If \( |p - \hat{p}| \leq \varepsilon (n,p,\varepsilon) \) then \( \frac{1}{e^{\varepsilon/2}} \leq p^a \leq \frac{1}{e^{\varepsilon/2}} \).

\(^5 \) \( W_n \) denotes the Lambert W function.
Proof. Let $h = \text{err}(n, p, \varepsilon)$. The Taylor approximation of $f(x) = \ln(x)$ for $x \in (p - h, p + h)$ is $\ln(x) = \ln(p) + R_0(x)$. Since $|f'(x)| = 1/x \leq 1/|p - h|$, we obtain

$$\frac{R_0(x)}{\ln p} \leq \frac{1}{|\ln p| |p - h|} \leq \frac{1}{|(1 - p)p/h + p - 1|} = \frac{1}{\sqrt{\varepsilon} \sqrt{p(1 - p) + p - 1}}.$$ 

To upper bound the above quantity by $1/2$ we find the feasible set of the inequality $\frac{1}{\sqrt{\varepsilon} \sqrt{p(1 - p)}} \geq 3$ which assuming that $\varepsilon < 1/6$ gives $\frac{1}{2} \left(1 - \sqrt{1 - 36\varepsilon^2/n}\right) \leq p \leq \frac{1}{2} \left(1 + \sqrt{1 - 36\varepsilon^2/n}\right)$. Therefore, for every $\hat{p} \in (p - h, p + h)$ we have

$$\frac{1}{2} \leq \ln \frac{\hat{p}}{\ln p} \leq \frac{3}{2} \iff -2 \frac{1}{\ln p} \geq -\frac{2}{3} \ln p \iff 2a \geq \hat{a} \geq \frac{2}{3} \iff \frac{1}{e^2} \leq \hat{p} \leq \frac{1}{e^{3/2}}.$$ 

\[\Box\]

4.3 The Proof of Theorem 1.5

Corollary 4.1 implies that to approximate $B(n, p^\ell)$ within total variation distance $\varepsilon$ we need an approximation $\hat{p}_e$ of $p^\ell$ with $|p^\ell - \hat{p}_e| \leq \text{err}(n, p^\ell, \varepsilon)$. We prove that Algorithm 2 outputs approximations $\hat{q}_1$, $\hat{q}_2$ of $p$ satisfying this bound. We use Lemma 4.3 to show that $1/e^2 \leq p^\hat{a} \leq 1/e^{3/2}$, and thus, $\psi(p^\hat{a}) \geq \psi(1/e^2) = 0.983226$. Using Fact 4.2 we draw $O(\ln(1/\delta^2/\varepsilon^2))$ to obtain estimates $\hat{q}_1$, $\hat{q}_2$ such that $\Pr[\hat{p} - \text{err}(n, p, \varepsilon) < \hat{q}_1 < p] \geq 1 - \delta/2$, $\Pr[p < \hat{q}_2 < p + \text{err}(n, p, \varepsilon)] \geq 1 - \delta/2$, and thus the probability of success of obtaining both $\hat{q}_1, \hat{q}_2$ is at least $1 - \delta$. Having obtained the estimates $\hat{q}_1, \hat{q}_2$ the result follows directly from Lemma 4.2.

\[\Box\]

Remark 4.1. Algorithm 2 can be easily modified to the case where the powers we are allowed to sample from are natural numbers, $\ell \geq 1$. In this case Notice that when $p \leq e^{-C} \leq 0.2$, then the function $f$ of Lemma 4.2 is minimized for $\ell = 1$, since $f$ is convex and the position of its global minimum is $\ell \leq 1$. So it suffices to choose $u(p) = \frac{p(1 - 1/p)}{1 - p} = 1$ and we can learn all powers $\ell \geq 1$ using an estimation $\hat{p}$ obtained by sampling from the first power using Fact 4.1. If $p \geq 0.2$ then we can simply run Algorithm 2 with $[\hat{a}]$ instead of $\hat{a}$. Then, $0.2/e^2 \leq p^{[\hat{a}]} \leq 1/e^{3/2}$, thus $\psi(p) \geq \psi(0.2/e^2)$, which means that Algorithm 2 uses $O(\ln(1/\delta)/\varepsilon^2)$ samples to learn all powers $\ell \geq 1$.

Remark 4.2. Algorithm 2 could use the approximation $\hat{p} \approx p$ from Fact 4.1 instead of approximations $\hat{q}_1, \hat{q}_2$ of $p$, which imply a unified analysis by the Mean Value Theorem in Lemma 4.2.

5 Lower Bounds for Learning Functions of Sequences of Distributions

5.1 Preliminaries

The Kullback-Leibler divergence of two probability measures $P, Q$ is

$$D_{kl}(P\|Q) = \int \log \left(\frac{P}{Q}\right) dP.$$ 

Moreover, the Hellinger distance of $P, Q$ with respect to another probability measure $\mu$ is

$$d_{\text{hel}}(P, Q)^2 = \frac{1}{2} \int \left(\sqrt{\frac{dP}{d\mu}} - \sqrt{\frac{dQ}{d\mu}}\right)^2 d\mu.$$ 

We shall use the well known decoupling identities of Hellinger distance and Kullback-Leibler divergence, for proofs see e.g. [19, Chapter 13].
Then we give first notation for a definition of the minimax risk for learning functions of sequences of distributions. Let \( \mathcal{P} \) be a family of sequences of distributions, indexed by the set \( I \). Since we can sample from every distribution \( P_i \) of \( \mathcal{P} \) we have the sample vector \( X^m = (X_{1,1}, \ldots, X_{1,m_1}, \ldots, X_{k,1}, \ldots, X_{k,m_k}) \) where the \( i \)-th group of \( m_i \) samples is drawn from \( P_i \), and define the multi-index \( m = (m_1, \ldots, m_k) \). All samples are independent, so \( X^m \) follows the \( |m| \)-fold product distribution \( P^m = P_1^{m_1} \times P_2^{m_2} \times \ldots \times P_k^{m_k} \). Let \( \theta : \mathcal{P} \rightarrow \Theta \) be a function of sequences of \( \mathcal{P} \) to be estimated. Let \( \hat{\theta} : X^m \rightarrow \Theta \) be an estimator of \( \theta \), and \( \rho : \Theta \times \Theta \rightarrow \mathbb{R}_+ \) be a semimetric on the space \( \Theta \). Let \( d \) denote a metric in the space of distributions. The natural choice for \( d \) on the space of sequences of distributions is to define \( d(P, Q) = \sup_{i \in I} d(P_i, Q_i) \). For example we define the TVD of the two sequences to be \( d_{tv}(P, Q) = \sup_{i \in I} d_{tv}(P_i, Q_i) \).

Definition 5.1. In the above setting we define the minimax risk to be

\[
\mathcal{M}_N(\theta(\mathcal{P}), \rho) := \inf_{\hat{\theta}} \sup_{P \in \mathcal{P}} \inf_{|m|=N} \mathbb{E}_{P^m} \left[ \rho \left( \hat{\theta}(X^m), \theta(P) \right) \right].
\]

(4)

There, the infimum over all multi-indices \( m \) such that \( |m| = N \) corresponds to the optimal selection of samples from each \( P_i \). Definition 5.1 captures the fact that the estimator \( \hat{\theta} \) can be adaptive in the sense that after the adversarial sequence of distributions is picked, the optimal algorithm for the problem will choose the best distributions from the sequence to draw samples from.

Let us give some intuitions about this definition referring to Algorithm 2. This algorithm follows this definition, in that, before seeing the input data, it samples from the first power, which then allows it to decide from which further power to sample. Note, that the extension of this algorithm to very large \( p \) (see Appendix C.2) shows that the first stage of deciding from which further power to sample can be non-trivial and requires binary search. These operations of deciding from which powers to sample correspond to the inner “inf” in the definition.

5.3 Le Cam and Fano Extensions

Let \( \mathcal{V} \) be a finite set of indices and let \( \mathcal{F}_\mathcal{V} \subseteq \mathcal{P} \) be a set of \( |\mathcal{V}| \) sequences indexed by \( \mathcal{V} \). Let \( V \) be the random variable representing a uniform at random choice of a sequence of \( \mathcal{F}_\mathcal{V} \). Conditioned on the choice \( V = v \), the random sample \( X^m \) is drawn from the \( |m| \)-fold product distribution \( P^m \). Let \( \nu^m \) denote the joint distribution of \( V, X^m \). Let \( \Psi : X^m \rightarrow \mathcal{V} \) be a testing function, namely \( \Psi \) takes samples from the unknown sequence \( P_\mathcal{V} \) and outputs an index \( u \in \mathcal{V} \) corresponding to a candidate sequence of distributions. We remark that the following techniques are standard and similar derivations can be found in [32], [28], and the very good lecture notes of John Duchi [19]. We are now ready to prove the standard reduction from estimation to testing using our new definition of minimax risk.
Proposition 5.1. Let $\mathcal{Y} \subseteq \mathcal{P}$ be a family of sequences of distributions indexed by $v \in \mathcal{V}$ such that $\rho(\theta(\mathcal{P}_v, \mathcal{P}_u)) \geq 2\delta$ for all $\mathcal{P}_v, \mathcal{P}_u \in \mathcal{Y}$, where, $v \neq u \in \mathcal{V}$ and $\delta > 0$. The minimax risk defined in Definition 5.1 has lower bound

$$\mathcal{M}_N(\theta(\mathcal{P}), \rho) \geq \delta \inf_{\nu^m} \inf_{\mathcal{V}} \nu^m(\Psi(X^m) \neq V).$$

Proof. Recall the definitions and the notation from Section 5. Fix an estimator $\hat{\theta}$. To simplify notation we shall use $\theta$ for $\theta(\mathcal{P})$ when the sequence $\mathcal{P}$ is clear from the context, and $\theta_v$ for $\theta(\mathcal{P}_v)$. From Markov’s inequality we have

$$\mathbb{E}_{P^m_v}[\rho(\hat{\theta}, \theta)] \geq \delta P^m_v(\rho(\hat{\theta}, \theta) \geq \delta) = \delta \nu^m(\rho(\hat{\theta}, \theta) \geq \delta | V = v)$$

Now we proceed by defining the testing function $\Psi(X^m) := \arg\min_{v \in \mathcal{V}} \{\rho(\hat{\theta}, \theta_v)\}$. Using the fact that $\rho(\theta_v, \theta_u) \geq 2\delta$ for every $v \neq u \in \mathcal{V}$ we have that $\rho(\hat{\theta}, \theta_v) \leq \delta \Leftrightarrow \Psi(\hat{\theta}) = v$. Now to bound the minimax risk

$$\mathcal{M}_N(\theta(\mathcal{P}), \rho) = \inf_{\hat{\theta}} \sup_{\mathcal{P} \in \mathcal{P}} \inf_{\nu^m} \mathbb{E}_{P^m}[\rho(\hat{\theta}(X^m), \theta(\mathcal{P}))]$$

$$\geq \inf_{\hat{\theta}} \sum_{v \in \mathcal{V}} \left( \frac{1}{|\mathcal{V}|} \inf_{\nu^m} \mathbb{E}_{P^m_v}[\rho(\hat{\theta}, \theta_v)] \right)$$

$$\geq \delta \inf_{\hat{\theta}} \sum_{v \in \mathcal{V}} \left( \frac{1}{|\mathcal{V}|} \inf_{\nu^m} \nu^m(\rho(\hat{\theta}, \theta_v) \geq \delta | V = v) \right)$$

$$= \delta \inf_{\nu^m} \inf_{\theta} \sum_{v \in \mathcal{V}} \left( \frac{1}{|\mathcal{V}|} \nu^m(\rho(\hat{\theta}, \theta_v) \geq \delta | V = v) \right)$$

$$= \delta \inf_{\nu^m} \inf_{\mathcal{V}} \nu^m(\Psi(X^m) \neq V),$$

where for the first inequality we use the fact that, the supremum of a set is larger than the average of a subset of the set, for the second inequality we use (5), and for the second equality we use the fact that $\inf(A + B) = \inf(A) + \inf(B)$ for any nonempty sets $A, B$. The last equality follows from Bayes’ Theorem. \hfill \square

Using Proposition 5.1, we prove an extension of Le Cam’s method for sequences of distributions.

Lemma 5.1. Let $\mathcal{P}, \mathcal{Q} \in \mathcal{P}$ and $\delta > 0$ such that $\rho(\theta(\mathcal{P}), \theta(\mathcal{Q})) \geq 2\delta$ then after $N$ observations (samples) the minimax risk has lower bound

$$\mathcal{M}_N(\theta(\mathcal{P}), \rho) \geq \frac{\delta}{2}(1 - \sqrt{2} \sqrt{1 - (1 - d_{tv}(\mathcal{P}, \mathcal{Q}))^N}).$$

Proof. Since we are doing binary hypothesis testing and we want to distinguish the distributions $P$ and $Q$ the random variable $V$ now represents the uniform choice over the measures $P$ and $Q$. We define the probability measure $\mu$ to be the joint distribution of $X^m$ and $V$. The probability that a testing algorithm $\Psi$ outputs a wrong result in the binary testing problem is $\mu(\Psi(X^m) \neq V) = \frac{1}{2} P^m(\Psi(X^m) \neq 1) + \frac{1}{2} Q^m(\Psi(X) \neq 2)$. Le Cam’s inequality states that

$$\inf_{\Psi} \{P^m(\Psi(X^m) \neq 1) + Q^m(\Psi(X^m) \neq 2)\} = 1 - d_{tv}(P^m, Q^m)$$

(6)
Using (6) and Proposition 5.1 we obtain
\[ \mathcal{M}_N (\theta(\mathcal{Y}), \rho) \geq \frac{\delta}{2} \inf_{|\mathcal{M}|=N} (1 - d_{tv} (P^m, Q^m)) = \frac{\delta}{2} \left( 1 - \sup_{|\mathcal{M}|=N} d_{tv} (P^m, Q^m) \right) \]

Notice that
\[ \sup_{|\mathcal{M}|=N} d_{tv} (P^m, Q^m) \leq \sqrt{2} \sup_{|\mathcal{M}|=N} \left( \frac{1 - \prod_{i=1}^N (1 - d_{hel} (P_i, Q_i))^2}{\sqrt{2}} \right) \]
\[ \leq \sqrt{2} \left( 1 - \left( 1 - \sup_{i \in I} d_{hel} (P_i, Q_i) \right)^2 \right)^N \]
\[ \leq \sqrt{2} \left( 1 - \left( 1 - \sup_{i \in I} d_{tv} (P_i, Q_i) \right) \right)^N \]
\[ = \sqrt{2} \left( 1 - (1 - d_{tv} (P, Q))^N \right) \]

where we used the inequality \( d_{hel} (P, Q)^2 \leq d_{tv} (P, Q) \leq \sqrt{2} d_{hel} (P, Q) \) and Fact 5.1.

Lemma 5.1 has an intuitive explanation: to distinguish two sequences of distributions it suffices to find an index \( i \in I \) such that \( d_{tv} (P_i, Q_i) \) is large. Since our Definition 5.1 of the minimax risk allows the algorithm to choose the element of the sequence to draw samples from, clearly, the hypothetical optimal algorithm of Definition 5.1 will choose to sample from the index where the TVD of the two tested sequences is largest. Therefore, to obtain a lower bound for the testing (and thus for the estimation) problem we need to find two sequences of distributions such that all their elements are close in TVD but their parameters are far.

We now state Fano’s Method modified to lower bound the minimax risk of Definition 5.1.

**Lemma 5.2.** Let \( \mathcal{Y} \) be a set of sequences of distributions. Let \( \mathcal{F}_V \subseteq \mathcal{Y} \) be a subset of \( \mathcal{Y} \) indexed by \( v \in V \) such that \( \rho (\theta(P_v), \theta(P_u)) \geq 2\delta \) for all \( P_v, P_u \in \mathcal{F}_V \), where, \( v \neq u \in V \) and \( \delta > 0 \). The minimax risk from Definition 5.1 has lower bound
\[ \mathcal{M}_N (\theta(\mathcal{Y}), \rho) \geq \frac{\delta}{\ln |V|} \left( \frac{1}{\ln |V|} N \inf_{v, u \in V} D_{kl} (P_v || P_u) + \ln 2 \right) . \]

**Proof.** Using Proposition 5.1 and Fano’s inequality (see e.g. [6]) we can lower bound \( \inf_{\Psi} \nu^m (\Psi(X^m) \neq V) \), and therefore
\[ \mathcal{M}_N (\theta(\mathcal{Y}), \rho) \geq \frac{\delta}{\ln |V|} \left( 1 - \frac{1}{\ln |V|} \left( N \sup_{v, u \in V} D_{kl} (P_v || P_u) + \ln 2 \right) \right) . \]

where \( I(V; X^m) \) is the mutual information of \( V, X^m \). To upper bound the mutual information \( I(V; X^m) \) we use the standard inequality
\[ I(V; X^m) \leq \frac{1}{|\mathcal{V}|^2} \sum_{v, u \in \mathcal{V}} D_{kl} (P^m_v || P^m_u) \leq \sup_{v, u \in \mathcal{V}} D_{kl} (P^m_v || P^m_u) , \]
which can be found in [4] or [32] or page 149 of [19]. We have
\[
\sup_{|m|=N} I(V;X^m) \leq \sup_{|m|=N} \sup_{v,u \in V} D_{kl}(P^m \| P^m_u)
\]
\[
= \sup_{|m|=N} \sup_{v,u \in V} \left( P^{m_1}_{v,1} \times \ldots \times P^{m_k}_{v,k} \| P^{m_1}_{u,1} \times \ldots \times P^{m_k}_{u,k} \right)
\]
\[
= \sup_{v,u \in V} m \sup_{v,u \in V, i} \sum_{i=1}^k m_i D_{kl}(P_{v,i} \| P_{u,i})
\]
\[
\leq N \sup_{v,u \in V, i} D_{kl}(P_{v,i} \| P_{u,i})
\]
\[
= N \sup_{v,u \in V} D_{kl}(P_v \| P_u),
\]
where to obtain the second equality we use Fact 5.2.

5.4 Applications

Application 1: Parameter Estimation for PBDs (Theorem 1.3). Since we are estimating the parameters of the PBD, Le Cam’s method is well suited for this problem. To prove Theorem 1.3 using Lemma 5.1, we extend the argument given in Proposition 15 of [15] to prove that \( \Omega(2^{1/\varepsilon}) \) samples are required even in the case where we are allowed to sample from the powers of the PBDs. The key idea is that the instance used in their proof suffices to prove that the TVD of all the powers is \( O(2^{-1/\varepsilon}) \) whereas the separation of the parameter vector is \( \Omega(\varepsilon) \).

Using the notation introduced in the beginning of this section we denote by \( \mathcal{P} \) the sequence of the powers of a PBD and since we want to estimate the parameters \( p_i \), we have \( \theta(\mathcal{P}) = p \). Our metric in the space of vectors, \((0,1)^n\), is \( \rho(p,\tilde{p}) = \|p - \tilde{p}\|_\infty \) since we want to approximate the vector \( p \) in additive error at most \( \varepsilon \).

We will follow the argument given in the proof of Proposition 15 in [15] and therefore we will present it fully for the sake of completeness. We set the length \( n \) of the PBD vector to be \( n = \Theta(\log(N/\varepsilon)) \) where \( N \) represents the number of samples in the minimax risk definition. We take \( p_j := (1 + \cos(2\pi j/\sqrt{n}))/8 \), \( q_j := (1 + \cos(2\pi j + \pi/\sqrt{n}))/8 \), \( j \in [n] \). Then for \( j = n/4 + O(1) \), we have that \( |p_i - p_j| = \Omega(1/\log(N/\varepsilon)) \) since for all \( i \), \( 2\pi i + \pi/n \) is at least \( \Omega(1/\log(N/\varepsilon)) \) from \( 2\pi n \) and \( 2\pi (n-j) \).

Observe that \( p_1, \ldots, p_n \) resp. \( q_1, \ldots, q_n \) are roots of the Chebyshev’s polynomials, \( (T_n(8x - 1) - 1) \), resp. \( (T_n(8x - 1) + 1) \), where \( T_n \) is the \( n \)-th Chebyshev polynomial. Since these polynomials agree in all coefficients except from their constant terms, the Newton-Girard identities imply that \( \sum_{i=1}^n p_i^l = \sum_{i=1}^n q_i^l \) for all \( l \in \{1,2,\ldots, n-1\} \) and moreover, for \( l \geq n \) it is easy to see that
\[
3^l \sum_{i=1}^n (p_i^l - q_i^l) \leq n(3/4)^n = \log(N/\varepsilon)(3/4)^{\log(N/\varepsilon)}.
\]
For small enough \( \varepsilon \) using Lemma 9 of [15] we have that \( d_{tv}(P_1, Q_1) \leq c/N \) for some constant \( c \). We will show that this in fact is true for all powers \( P_s, Q_s \) of these two PBDs. To show this let us fix any power \( s \in \{1,2,\ldots, n\} \). Then, we have that \( \sum_{i=1}^n p_i^{|s|} = \sum_{i=1}^n q_i^{|s|} \), for any \( l = 1,2,\ldots, \lfloor (n-1)/s \rfloor \) assuming that \( s \leq n-1 \). Moreover, when \( l \in \{(n-1)/s \} + 1, \{(n-1)/s \} + 2, \ldots \), we have \( 3^l \sum_{i=1}^n (p_i^{|s|} - q_i^{|s|}) \leq n \frac{3^l}{n} \leq n \frac{3^l}{n} \leq n \frac{3^l}{n} \), where the last inequality holds because \( s \geq n \). It is easy to see that the same hold when \( s = n \), and once more by Lemma 9 in [15], \( d_{tv}(P_s, Q_s) \leq c/N \). Since the separation of the parameters is \( \Omega(1/\log(N/\varepsilon)) \) and the Total Variation distance of the two sequences is less than \( c/N \) we can use Lemma 5.3 to obtain a minimax lower bound rate of \( 1/\log(\varepsilon/N) \). Notice that an upper bound of \( c/N \) on the total variation distance of the two sequences implies a lower bound of \( \Omega(\delta) \) for the minimax risk. Therefore, since we need to approximate the parameters to additive error \( \varepsilon \), \( \mathfrak{M}_N < \varepsilon \) implies that the number of samples \( N \) should be \( \Omega(2^{1/\varepsilon}) \).
Application 2: Learning Powers of Binomials. (Theorem 1.6) We use Lemma 5.2 to prove a matching lower bound of $\Omega(1/\varepsilon^2)$ for the problem of learning the powers of a Binomial distribution. Its quite technical proof (cf. Section D), is only sketched here.

Using the notation of Lemma 5.2 and since we do density estimation, we have $\theta(\mathcal{P}) = (f_i)_{i \in \mathbb{N}}$. Therefore we will use the metric $\rho(\mathcal{P}, \mathcal{Q}) = d_{\text{tv}}(\mathcal{P}, \mathcal{Q}) = \sup_{i \in \mathbb{N}} d_{\text{tv}}(f_i, \hat{f}_i)$. Let $\delta = \Theta(1/\sqrt{nN})$. Let $p_1 = 1/2$, $p_2 = 1/2 + \delta/4$, $p_3 = 1/2 + \delta/2$. Let $\mathcal{P}_1 = (B(n, (1/2)^i))_{i \in \mathbb{N}}$, $\mathcal{P}_2 = B(n, (1/2 + \delta/4)^i)_{i \in \mathbb{N}}$, $\mathcal{P}_3 = B(n, (1/2 + \delta/2)^i)_{i \in \mathbb{N}}$. The TVD of the first powers of these Binomials is $\Omega(1/\sqrt{N})$. To see this notice that since the variance of the Binomials is $O(n)$ we can approximate the Binomials with Normals with insignificant error. When their variances are close, the TVD of two Normals is roughly proportional to the difference of their means divided by their “common” standard deviation, which is $\Omega(1/\sqrt{N})$. Thus we obtain our lower bound for the TVD. We then prove an upper bound for the KL-Divergence between all powers, namely $D_{kl}(\mathcal{P}_1||\mathcal{P}_3) = O(1/N)$. It’s easy to see that this upper bound holds for the first power. To prove that it holds for all the powers notice that $D_{kl}(B(n,p)||B(n,q))$ is an increasing function of $|p - q|$ (for a proof see Proposition D.2). Thus, since the distances of the $p_i$’s of our three Binomials roughly decrease for higher powers the KL-Divergence of the first power is roughly an upper bound for $D_{kl}(\mathcal{P}_1||\mathcal{P}_3)$. Now, applying Lemma 5.2 we have that $\mathcal{M}_N(\theta(\mathcal{P}), \rho) = \Omega(1/\sqrt{N})$, which in turn implies that to have an estimator that approximates all the powers in distance less than $\epsilon$ we need $\mathcal{M}_N(\theta(\mathcal{P}), \rho) < \epsilon$ and therefore the number of samples $N$ should be $\Omega(1/\varepsilon^2)$. □

6 Upper Bound for Parameter Estimation

Newton’s identities, a.k.a the Newton-Girard formulae, give relations between power sums and elementary symmetric polynomials of variables $x_1, \ldots, x_n$. In that setting, the $k$th power sum is $s_k(x_1, \ldots, x_n) = x_1^k + \cdots + x_n^k$. The $k$th elementary symmetric polynomial $e_k(x_1, \ldots, x_n)$ is the sum of all distinct products of $k$ distinct variables. Newton’s identities allow us to compute the elementary symmetric polynomials if we know the power sums exactly. Moreover, the polynomial with roots $x_i$, i.e., $\prod_{i=1}^n (x - x_i)$, may be expanded as $\sum_{k=0}^n (-1)^{n+k} e_{n-k} x^k$. Thus, if we know the power sums $s_1(x_1, \ldots, x_n), \ldots, s_n(x_1, \ldots, x_n)$ exactly, we can first find the coefficients of the elementary symmetric polynomials and then compute the roots $x_1, \ldots, x_n$ with an arbitrarily good accuracy. A similar approach was used in [12] to derive sparse covers for PBDs.

In this section we provide the analysis of the “noisy” version of Newton’s identities. Given query access to PBD powers, we can obtain good estimations of the power sums $s_k(p_1, \ldots, p_n)$ using a reasonable number of samples, since the expectations of PBD powers are the power sums of the unknown probabilities $p_1, \ldots, p_n$. An intriguing question is to which extent these “noisy” power sum estimations can be used to recover the actual values of $p_1, \ldots, p_n$ within sufficiently good accuracy. In this Section we answer this question by providing an upper bound on the sampling complexity of estimating the parameters of a PBD using samples from its powers. This upper bound matches the corresponding lower bound of Theorem 1.3.

6.1 Preliminaries

Let $x \in \mathbb{R}^n$ be a vector, and $A = (A_{ij})_{i,j \in [n]}$ be a $n \times n$ matrix. Then $\|x\|_\infty = \max_{i \in [n]} |x_i|$, $\|A\|_\infty = \max_{i \in [n]} \sum_{j=1}^n |A_{ij}|$, $|x| = (|x_i|)_{i \in [n]}$, $|A| = (|A_{ij}|)_{i,j \in [n]}$. We use “≤” in $A \preceq B$ to denote element-wise inequality of the matrices $A, B$, namely $A \preceq B \iff A_{ij} \leq B_{ij}$ for all $i, j \in [n]$.

To compute the sensitivity of the solution of a linear system $Ax = b$ to perturbations of $A, b$ we shall use Theorem 7.4 from [21], formulated bellow as Lemma 6.1.
Lemma 6.1 (Theorem 7.4 from [21]). Let $A\mathbf{x} = \mathbf{b}$ and $(A+\Delta A)\mathbf{y} = \mathbf{b}+\Delta \mathbf{b}$, where $|\Delta A| \leq u \mathbf{E}$ and $|\Delta \mathbf{b}| \leq u \mathbf{f}$, and assume that $u \parallel A^{-1} \parallel E < 1$, where $\parallel \cdot \parallel$ is an absolute norm. Then

$$\frac{\|x - y\|}{\|x\|} \leq \frac{u}{1 - u \parallel A^{-1} \parallel E} \parallel A^{-1} \parallel (E|x| + f) \parallel$$

and for the $\infty$-norm this bound is attainable to first order in $u$.

To approximate the roots of the univariate polynomial $P(x)$ we use the nearly optimal root finding algorithm of Pan [26](Theorem 2.1.1) formulated below as Lemma 6.2.

Lemma 6.2 (Theorem 2.1.1 from [26]). Let $P(x) = \sum_{i=0}^{n} c_i x^i = c_n \prod_{i=1}^{n} (x - p_i)$, $c_n \neq 0$, be a polynomial of degree $n$ such that all its complex roots satisfy $|p_j| < 1$ for all $j$. Let $b$ be a fixed real number, $b \geq n \log n$. Then complex numbers $\hat{p}_j$ can be computed by using $O((n \log^2 n)(\log^2 n + \log b))$ arithmetic operations performed with the precision of $O(b)$ bits such that $|\hat{p}_j - p_j| < 2^{-b/n}$ for $j = 1, \ldots, n$.

The following simple bound on the coefficient vector of a polynomial with roots in $[-1, 1]$ will be useful.

Fact 6.1. If all roots of a monic polynomial $P = x^n + a_{n-1}x^{n-1} + \ldots + a_0$ of degree $n$ lie in the interval $[-1, 1]$ then $|a_k| \leq \binom{n}{k} \leq 2^n$.

Proof. Using Vieta’s formulae we have

$$a_{n-k} = (-1)^k \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} x_{i_1} x_{i_2} \ldots x_{i_k}$$

Therefore, $|a_{n-k}|$ is maximum when all $x_i$ are 1 and therefore $|a_{n-k}| \leq \binom{n}{k} = \binom{n}{n-k}$.

6.2 The proof of Theorem 1.4

We denote by $P_j$ the $j$-th power of the PBD with probability vector $\mathbf{p}$, namely $P_j$ is the PBD with probability vector $\mathbf{p}^j = (p_{1,j})_{i=1}^{n}$. Let $P(x) = x^n + c_{n-1}x^{n-1} + \ldots + c_0 = \prod_{i=1}^{n} (x - p_i)$ be the monic polynomial of degree $n$. Notice that the mean value of $P_j$ denoted by $\mu_j$ equals the $j$-th Newton sum of the roots of $P(x)$ since $\mu_j = \sum_{i=1}^{n} p_{i,j}$. Given that $P(x)$ is monic, the coefficients of $P(x)$ and the means $\mu_1, \mu_2, \ldots, \mu_n$ satisfy the following linear system of Newton’s identities:

$$\mu_j + \sum_{i=1}^{j-1} c_{n-i} \mu_{j-i} + j c_{n-j} = 0, \quad j = 1, 2, \ldots, n$$

The system has the following matrix form, where we omit zero elements.

$$\begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
\mu_1 & 2 & 0 & \cdots & 0 \\
\mu_2 & \mu_1 & 3 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
\mu_{n-1} \mu_{n-2} \ldots \mu_1 & n & \cdots & \cdots & 0
\end{pmatrix}
\begin{pmatrix}
c_{n-1} \\
c_{n-2} \\
c_{n-3} \\
\vdots \\
c_0
\end{pmatrix}
= \begin{pmatrix}
-\mu_1 \\
-\mu_2 \\
-\mu_3 \\
\vdots \\
-\mu_n
\end{pmatrix} \leftrightarrow \mathbf{A} \mathbf{c} = \mathbf{b} \quad (8)$$

Using the linear system [8] Algorithm [8] retrieves an approximation of the coefficient vector $\mathbf{c}$ and then finds the roots of the corresponding univariate polynomial.
Algorithm 3 Parameter Estimation

**Input:** \(2^{O(n \max\{\log(1/\varepsilon), \log(n)\})}\) samples from the powers \(P_j, j \in [n]\).

**Output:** An additive \(\varepsilon\) approximation of \(p\).

1: Using \(A\) of Lemma 6.1 draw \(2^{O(n \max\{\log(1/\varepsilon), \log(n)\})}\) samples from each power \(P_j\) to obtain the approximations \(\hat{\mu}_j\) of \(\mu_j\).
2: Solve the system \(A\) and obtain \(\hat{c}\).
3: Use Pan’s Algorithm of Lemma 6.2 to compute approximations \(\hat{p}_j\) to all the roots of the polynomial \(P(x) = \sum_{i=1}^n c_i x^i\).
4: return \(\hat{p}\).

We now proceed to the proof of Theorem 1.4.

Starting from the last step of root finding with Pan’s Algorithm of Lemma 6.2 we have that, to obtain \(\varepsilon\)-approximations of the roots of the polynomial \(P(x)\) we need to obtain an approximating vector \(\hat{c}\) of the coefficient vector \(c\) of \(P(x)\) such that

\[
\|c - \hat{c}\|_{\infty} = 2^{O(-n \max\{\log(1/\varepsilon), \log(n)\})},
\]  

(9)

Next, we proceed to computing the precision needed for the means \(\mu_j\) so that the system of Newton Identities \(8\) can be solved to provide a solution satisfying \(9\). Since in our setting the error of approximating the \(j\)-th mean is proportional to the standard deviation of the \(j\)-th powers, the errors \(E, f\) of Lemma 6.1 are

\[
E = \begin{pmatrix}
\sigma_1 \\
\sigma_2 \sigma_1 \\
\vdots \\
\sigma_n \sigma_2 \ldots \sigma_1 \\
\end{pmatrix} \leq \sqrt{n} \begin{pmatrix} 1 \\
1 \\
1 \\
1 \\
\end{pmatrix},
\]

\[
f = \begin{pmatrix}
\sigma_1 \\
\sigma_2 \\
\vdots \\
\sigma_n \\
\end{pmatrix} \leq \sqrt{n} \begin{pmatrix} 1 \\
1 \\
1 \\
1 \\
\end{pmatrix}.
\]

Since it holds \(\mu_j \leq n\) and it follows that \(A_{ij} \leq n\). Since \(A\) is lower triangular, \(\det(A) = n!\), and it holds that \(\det(A) \geq M_{ij}\), where \(M_{ij}\) is the determinant of of \((n-1) \times (n-1)\) submatrix of \(A\) after deleting row \(i\) and column \(j\). It follows that \(|A^{-1}|_{ij} \leq 1\). Moreover, since the solution vector \(x\) corresponds to the coefficients of \(P(x)\) from Fact 6.1 it follows \(|x|_i \leq \binom{n}{n-i}\). Using these inequalities we bound

\[
|A^{-1}||E| \leq \sqrt{n} \begin{pmatrix} 1 \\
1 \\
1 \\
1 \\
\end{pmatrix} \begin{pmatrix} 1 \\
1 \\
1 \\
1 \\
\end{pmatrix} = \sqrt{n} \begin{pmatrix} 1 \\
2 \\
1 \\
1 \\
\end{pmatrix}.
\]

Moreover, \(|A^{-1}|f| \leq (n \ 2n \ldots n^2)^T\). Combining the above inequalities we can estimate the condition of \(A\).

\[
\|A^{-1}|A|c| + |A^{-1}|b\|_{\infty} \leq \sqrt{n} \sum_{i=0}^{n} (n-i) \binom{n}{n-i} + \sqrt{nn} = n^{3/2}(2^{n-1} + 1) = O\left(n^{3/2}2^n\right)
\]

\[
||A^{-1}||E||_{\infty} = O\left(n^{5/2}\right)
\]

Thus, from Lemma 6.1 we obtain the following absolute error bound with respect to the \(\infty\)-norm

\[
\|c - \hat{c}\|_{\infty} \leq u O\left(n^{3/2}2^n\right)
\]
Since we need to run Algorithm $A$ of Proposition 3.1 $n$ times to obtain approximations $\hat{\mu}_j$ such that $|\mu_j - \hat{\mu}_j| \leq u_j \sigma_j$ for all $j \in [n]$ with probability at least $1 - \delta$ it follows from the union bound that we have to draw $O \left( \log(1/n) / n^2 \right)$ from the powers $P_j$, $j \in [n]$. Therefore, since $uO \left( n^{3/2} / 2n \right)$ should satisfy we conclude that overall we need $2O(n \max(\log(1/e), \log(n)))$ samples.

References

1. J. Acharya and C. Daskalakis. Testing Poisson Binomial Distributions. In Proc. of the 26th ACM-SIAM Symposium on Discrete Algorithms (SODA ’15), pages 1829–1840, 2015.
2. M.-F. Balcan and N.J.A. Harvey. Learning submodular functions. In Proc. of the 43rd ACM Symposium on Theory of Computing (STOC ’11), pages 793–802, 2011.
3. L. Birgé. Estimation of unimodal densities without smoothness assumptions. Annals of Statistics, 25(3):970–981, 1997.
4. Lucien Birgé. Approximation dans les espaces mtriques et thorie de l’estimation. Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete, 65(2):181–237, December 1983.
5. J.T. Chu. On bounds for the normal integral. Biometrika, 42:263–265, 1955.
6. Thomas M. Cover and Joy A. Thomas. Elements of Information Theory 2nd Edition. Wiley-Interscience, Hoboken, N.J, 2 edition edition, July 2006.
7. C. Daskalakis, A. De, G. Kamath, and C. Tzamos. A size-free CLT for Poisson multionomials and its applications. In Proc. of the 48th ACM Symposium on Theory of Computing (STOC ’16), pages 1074–1086, 2016.
8. C. Daskalakis, I. Diakonikolas, R. O’Donnell, R.A. Servedio, and L.-Y. Tan. Learning sums of independent integer random variables. In Proc. of the 54th ACM Symposium on Foundations of Computer Science (FOCS ’13), pages 217–226, 2013.
9. C. Daskalakis, I. Diakonikolas, and R.A. Servedio. Learning Poisson Binomial Distributions. Algorithmica, 72(1):316–357, 2015.
10. C. Daskalakis, G. Kamath, and C. Tzamos. On the Structure, Covering, and Learning of Poisson Multinomial Distributions. In Proc. of the 56th IEEE Symposium on Foundations of Computer Science (FOCS ’15), pages 1203–1217, 2015.
11. C. Daskalakis and C.H. Papadimitriou. Approximate Nash equilibria in anonymous games. J. Economic Theory, 156:207–245, 2015.
12. C. Daskalakis and C.H. Papadimitriou. Sparse Covers for Sums of Indicators. Probability Theory and Related Fields, 162(3):679–705, 2015.
13. C. Daskalakis and V. Syrgkanis. Learning in Auctions: Regret is Hard, Envy is Easy. In Proc. of the 57th IEEE Symposium on Foundations of Computer Science (FOCS ’16), 2016.
14. I. Diakonikolas. Learning structured distributions. In P. Bühlmann, P. Drineas, M. Kane, and M.J. van der Laan, editors, Handbook of Big Data, pages 267–283. Chapman and Hall/CRC, 2016.
15. I. Diakonikolas, D.M. Kane, and A. Stewart. Properly learning poisson binomial distributions in almost polynomial time. In Proceedings of the 29th Conference on Learning Theory, (COLT’16), pages 850–878, 2016.
16. I. Diakonikolas, D.M. Kane, and A. Stewart. Optimal learning via the fourier transform for sums of independent integer random variables. In Proceedings of the 29th Conference on Learning Theory, (COLT’16), pages 831–849, 2016.
17. I. Diakonikolas, D.M. Kane, and A. Stewart. The Fourier Transform of Poisson Multinomial Distributions and its Algorithmic Applications. In Proc. of the 48th ACM Symposium on Theory of Computing (STOC ’16), pages 1060–1073, 2016.
18. D.P. Dubhashi and A. Panconesi. Concentration of Measure for the Analysis of Randomized Algorithms. Cambridge University Press, 2009.
19. John Duchi. Stats311, Lecture Notes. URL: https://stanford.edu/class/stats311/Lectures/full_notes.pdf
20. V. Feldman and P. Kothari. Learning coverage functions and private release of marginals. In Proc. of the 27th Conference on Learning Theory (COLT 2014), volume 35 of JMLR Proceedings, pages 679–702, 2014.
21. Nicholas J. Higham. Accuracy and Stability of Numerical Algorithms. SIAM: Society for Industrial and Applied Mathematics, Philadelphia, 2nd edition edition, August 2002.
22. Kiyosi It and Henry P. Jr McKeon. diffusion processes and their sample paths. springer, berlin ; new york, 1996 edition edition, February 1996.
23. M. Li and P.M.B. Vitányi. An Introduction to Kolmogorov Complexity and Its Applications, 3rd Edition. Texts in Computer Science. Springer, 2008.
24. Qi-Man Shao Louis H.Y. Chen, Larry Goldstein. Normal Approximation by Stein’s Method. Springer, 2010.
25. R. Motwani and P. Raghavan. Randomized Algorithms. Cambridge University Press, 1995.
We use the notation introduced in the proof sketch, in the main part. For simplicity, we let\(\lambda\equiv 2\sqrt{\ln(\frac{2}{1-\varepsilon})}\) and assume that\(\mu_Y > \mu_X + \lambda(\sigma_X + \sigma_Y)\) (the other case is symmetric). By the definition of TVD, we obtain that:

\[
2d_{TV}(X,Y) = \sum_{i=0}^{\infty} |\Pr[X = i] - \Pr[Y = i]| \\
\geq \sum_{i=0}^{\infty} (\Pr[X = i] - \Pr[Y = i]) + \sum_{i=\mu_Y-\lambda\sigma_Y}^{\infty} (\Pr[Y = i] - \Pr[X = i]) \\
= (\Pr[X \leq \mu_X + \lambda\sigma_X] - \Pr[Y \leq \mu_X + \lambda\sigma_X]) + \\
(\Pr[Y \geq \mu_Y - \lambda\sigma_Y] - \Pr[X \geq \mu_Y - \lambda\sigma_Y]) \\
\geq (1 - \Pr[X > \mu_X + \lambda\sigma_X] - \Pr[Y < \mu_Y - \lambda\sigma_Y]) + \\
(1 - \Pr[Y < \mu_Y - \lambda\sigma_Y] - \Pr[X > \mu_X + \lambda\sigma_X]) \\
> (1 - (1 - \varepsilon)/2 - (1 - \varepsilon)/2) + (1 - (1 - \varepsilon)/2 - (1 - \varepsilon)/2) = 2\varepsilon
\]

For the second inequality, we use that\(\mu_X + \lambda\sigma_X < \mu_Y - \lambda\sigma_Y\). Therefore,\(\Pr[Y \leq \mu_X + \lambda\sigma_X] \leq \Pr[X > \mu_X + \lambda\sigma_X]\) and\(\Pr[X \geq \mu_Y - \lambda\sigma_Y] \leq \Pr[X > \mu_X + \lambda\sigma_X]\). For the last inequality, we apply Proposition C.1 with \(\lambda = 2\sqrt{\ln(\frac{2}{1-\varepsilon})}\) and obtain that\(\Pr[X > \mu_X + \lambda\sigma_X] < (1 - \varepsilon)/2\) and that\(\Pr[Y < \mu_Y - \lambda\sigma_Y] < (1 - \varepsilon)/2\). \(\square\)

A.2 The Proof of Lemma 2.2: Technical Details

We use the notation introduced in the proof sketch, in the main part. For simplicity, we let \(X(j)\) (resp. \(Y(j)\)) denote of the sum of \(s\) Bernoulli random variables with expectation \(p_j = (1 - a_j/\nu^{k_j})\) (resp. \(q_j = (1 - b_j/\nu^{k_j})\)), i.e., \(X(j)\) and \(Y(j)\) are the \(k\)-th PBD powers of the Bernoulli trials in group \(j\) of \(p\) and \(q\).

We first observe that for all \(j < i\) and for all \(a \in \nu\),

\[
s \left(1 - \frac{a}{\nu^{k_j}}\right) \nu^{4i-2} \leq s e^{-a
u^{4i-2}/\nu^{k_j}} = se^{-a
u^{4(i-j)-2}} \leq s/n^a \ln n,
\]

\[\]
where for the last inequality, we use that \( i - j \geq 1 \) and that \( \nu = \ln n \). Therefore, since \( a_j, b_j \geq 1 \) and since \( m < n \),
\[
\sum_{j < i} \text{Var}[Y(j)] \leq \sum_{j < i} E[Y(j)] \leq s/n - 1 + \ln n \quad \text{and} \quad \sum_{j < i} \text{Var}[X(j)] \leq \sum_{j < i} E[X(j)] \leq s/n - 1 + \ln n.
\]
Moreover, \( \sum_{j < i} |E[Y(j)] - E[X(j)]| \leq s/n - 1 + \ln n \).

We also have that the difference between the mean values of \( X(i) \) and \( Y(i) \) is:
\[
E[X(i)] - E[Y(i)] = s \left( 1 - \frac{a_i}{\nu^4} \right)^{\nu^{i-2}} - s \left( 1 - \frac{a_i}{\nu^4} - \frac{x_i}{\nu^4} \right)^{\nu^{i-2}}
\]
\[
\geq s \left( \nu^{i-2} \frac{x_i}{\nu^4} \left( 1 - \frac{a_i}{\nu^4} \right)^{\nu^{i-3}} - \left( \frac{\nu^{i-2} x_i}{2\nu^4} \right)^2 \left( 1 - \frac{a_i}{\nu^4} \right)^{\nu^{i-4}} \right)
\]
\[
\geq s \left( \frac{x_i}{2\nu^2} - \frac{1}{2} \left( \frac{x_i}{2\nu^2} \right)^2 \right) \geq \frac{s x_i}{4\nu^2}
\]
The first inequality holds because for any \( i, 1 \geq \frac{a_i}{\nu^4} + \frac{x_i}{\nu^4} \). Therefore,
\[
\left( 1 - \frac{a_i}{\nu^4} - \frac{x_i}{\nu^4} \right)^{\nu^{i-2}} \leq \left( 1 - \frac{a_i}{\nu^4} \right)^{\nu^{i-2}} - \nu^{i-2} \frac{x_i}{\nu^4} \left( 1 - \frac{a_i}{\nu^4} \right)^{\nu^{i-3}} + \left( \frac{\nu^{i-2} x_i}{2\nu^4} \right)^2 \left( 1 - \frac{a_i}{\nu^4} \right)^{\nu^{i-4}}
\]
For the second inequality, we use that \((1 - \frac{a_i}{\nu^4})^{\nu^{i-3}} \geq 1 - \frac{a_i}{\nu^4} \geq 1/2 \) and that \((1 - \frac{a_i}{\nu^4})^{\nu^{i-4}} \geq 1 - \frac{a_i}{\nu^4} \geq 1/2 \). For the last inequality, we use that \( \frac{s}{2\nu^2} \leq 1/2 \).

As for the variance of \( X(i) \) and \( Y(i) \), since \( p_i^k > q_i^k \geq 1/2 \) (assuming that \( n \) is sufficiently large),
\[
\text{Var}[X(i)] \leq \text{Var}[Y(i)] = s \left( 1 - \frac{b_i}{\nu^4} \right)^{\nu^{i-2}} \left( 1 - \left( 1 - \frac{b_i}{\nu^4} \right)^{\nu^{i-2}} \right)
\]
\[
\leq s \left( 1 - \left( 1 - \frac{b_i}{\nu^4} \right)^{\nu^{i-2}} \right)
\]
\[
\leq s \nu^{i-2} \frac{b_i}{\nu^4} \leq s/\nu,
\]
where for the last inequality we use that \( b_i \leq \nu \).

Moreover, we let \( x_j = b_j - a_j \), with \( 0 \leq |x_j| < \nu \), and observe that for all \( j > i \),
\[
|E[X(j)] - E[Y(j)]| = \left| s \left( 1 - \frac{a_j}{\nu^4} \right)^{\nu^{j-2}} - s \left( 1 - \frac{a_j}{\nu^4} - \frac{x_j}{\nu^4} \right)^{\nu^{j-2}} \right|
\]
\[
\leq s \nu^{j-2} \frac{|x_j|}{\nu^{j}}
\]
\[
\leq \frac{s}{\nu^{4(j-i)+1}}
\]
Therefore, \( \sum_{j > i} |E[X(j)] - E[Y(j)]| \leq 2s/\nu^5 \).

As for the variance of \( X(j) \) and \( Y(j) \), we have that for all \( a \in \nu \),
\[
s \left( 1 - \frac{a}{\nu^4} \right)^{\nu^{j-2}} \left( 1 - \left( 1 - \frac{a}{\nu^4} \right)^{\nu^{j-2}} \right) \leq s \left( 1 - \left( 1 - \frac{a}{\nu^4} \right)^{\nu^{j-2}} \right)
\]
\[
\leq s \frac{a}{\nu^{4(j-i)+2}}
\]
\[
\leq s/\nu^{4(j-i)+1}
\]
where for the last inequality we use that \( a \in \nu \). Therefore, \( \sum_{j > i} (\text{Var}[X(j)] + \text{Var}[Y(j)]) \leq 4s/\nu^5 \).

Putting everything together and assuming that \( n \) is large enough, we obtain that

\[
|E[X] - E[Y]| \geq \left| \sum_{j \neq i} (E[X(j)] - E[Y(j)]) \right| + |E[X(i)] - E[Y(i)]| \geq s \left( \frac{x_i}{4\nu^2} - \frac{3}{\nu^5} \right) \geq \frac{x_i s}{5\nu^2} \tag{10}
\]

The first inequality holds because for all numbers \( c, d, |c + d| \geq c - |d| \) (here, we use \( c = E[X(i)] - E[Y(i)] \) and \( d = \sum_{j \neq i} (E[X(j)] - E[Y(j)]) \)). The second inequality holds because (i) \( E[X(i)] - E[Y(i)] \geq sx_i/(4\nu^2) \), (ii) \( \sum_{j < i} |E[Y(j)] - E[X(j)]| \leq s/n - 1 + \ln n \), and (iii) \( \sum_{j > i} |E[X(j)] - E[Y(j)]| \leq 2s/\nu^5 \).

As for the variance of \( X \) and \( Y \), we have that

\[
\text{Var}[X] + \text{Var}[Y] \leq \frac{4s}{\nu} \tag{11}
\]

If we assume that \( n \) is sufficiently large, that \( s \geq (\ln n)^4 \) and that \( a_i > b_i \), and thus \( x_i \geq 1 \), we obtain that for any \( \varepsilon \in (0, 1/2] \),

\[
|E[X] - E[Y]| \geq \frac{s}{5\nu^2} > 4\sqrt{\ln\left(\frac{2}{1-\varepsilon}\right)\frac{4s}{\nu}} \\
\geq 4\sqrt{\ln\left(\frac{2}{1-\varepsilon}\right)}(\text{Var}[X] + \text{Var}[Y]) \\
\geq 2\sqrt{\ln\left(\frac{2}{1-\varepsilon}\right)} (\sqrt{\text{Var}[X]} + \sqrt{\text{Var}[Y]})
\]

Thereby, by Lemma 2.1 we obtain that \( X \) and \( Y \) are at distance larger than \( \varepsilon \), for any \( \varepsilon \in (0, 1/2] \), a contradiction. Hence, it must be \( a_i = b_i \), which concludes the proof of the lemma.

\[\square\]

### B Upper Bound for Learning PBD Powers: Separated Case

#### B.1 The proof of Lemma 3.1

To prove this lemma we will first show the following.

**Lemma B.1.** Let us consider the following function \( h(x) = \left( 1 - \frac{\gamma}{cx} \right)^{x-1} \), where \( c \geq 2 \) is a fixed constant, \( \gamma \geq 1 \) and \( x \geq \gamma \). If \( \gamma \geq 2c \), then function \( h \) is increasing in \([\gamma, +\infty)\).

**Proof.** Let us observe that \( h'(x) = \left( 1 - \frac{\gamma}{cx} \right)^{x-1} \left( \ln(1 - \frac{\gamma}{cx}) + \frac{\gamma}{1 - \frac{\gamma}{cx}} \cdot \frac{x-1}{x} \right) \). Because we have that \( h(x) \geq 0 \), to show the claim it suffices to prove that

\[
\ln \left( 1 - \frac{\gamma}{cx} \right) + \frac{\gamma}{1 - \frac{\gamma}{cx}} \cdot \frac{x-1}{x} \geq 0
\]

when \( x \geq \gamma \geq 2c \).

This inequality can be rewritten to

\[
\ln \left( 1 - \frac{\gamma}{cx} \right)^{-1} \leq \frac{\gamma}{1 - \frac{\gamma}{cx}} \cdot \frac{x-1}{x},
\]

\]

\[\square\]
which after variable change $z = \frac{\gamma}{c} \in (0, \frac{1}{c}]$ is equivalent to

$$\ln \frac{1}{1-z} \geq \frac{z}{1-z} \cdot \left(1 - \frac{c}{\gamma} z\right).$$

We now use the Taylor’s series expansion $\ln \frac{1}{1-z} = \sum_{i=1}^{\infty} \frac{z^i}{i}$, which leads to the following inequality

$$\frac{1-z}{z} \left(z + \frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4} + \ldots\right) \leq 1 - \frac{c}{\gamma} z,$$

or equivalently

$$\left(\frac{c}{\gamma} - \frac{1}{2}\right) z + \left(\frac{1}{3} - \frac{1}{2}\right) z^2 + \left(\frac{1}{4} - \frac{1}{3}\right) z^3 + \left(\frac{1}{4} - \frac{1}{5}\right) z^4 + \ldots \leq 0.$$

Note that our operations on these infinite series are legitimate because they converge: for instance taking series $\sum_{i=1}^{\infty} \frac{a_{i+1}}{a_i} = \sum_{i=1}^{\infty} a_i$ and using the d’Alembert’s ratio test we see that $\lim_{i \to \infty} \frac{|a_{i+1}|}{|a_i|} = \lim_{i \to \infty} \frac{1}{i+2} z = z < 1$, so the series converges.

Let us now observe that because $a_i < 0$ for all $i$ and $\frac{\gamma}{c} - \frac{1}{2} \leq 0$, the inequality $(\frac{c}{\gamma} - \frac{1}{2})z + \sum_{i=2}^{\infty} (\frac{1}{i+1} - \frac{1}{i}) z^i \leq 0$ is true for all $z \in (0, \frac{1}{c}]$.

Now we are ready to prove Lemma 3.1

Proof. Let $f(x) = (1 - \frac{x}{c\ell_i})^{\ell_i}$, then $f'(x) = -\frac{1}{c}(1 - \frac{x}{c\ell_i})^{\ell_i-1}$, and so by the Mean Value Theorem there exists $\gamma_i \in (\alpha_i, \beta_i)$ such that $(1 - \frac{x}{c\ell_i})^{\ell_i-1} - (1 - \frac{\beta_i}{c\ell_i})^{\ell_i} = f'(\gamma)(\alpha_i - \beta_i)$. Using this, and that fact that $\beta_i - \alpha_i \geq 1$, the claimed inequality is implied if the following inequality

$$\left(1 - \frac{\gamma_i}{c\ell_i}\right)^{\ell_i-1} > \frac{4\varepsilon}{\sqrt{n/s}}$$

holds for all $n \geq e^{2c}$. Now because $\gamma_i \leq \sqrt{\ln(n)}$, the last inequality is implied by

$$\left(1 - \frac{\sqrt{\ln(n)}}{c\ell_i}\right)^{\ell_i-1} > \frac{4\varepsilon}{\sqrt{n/s}},$$

and by $\ell_i \geq \ln(n)$ and by Lemma 3.1 this inequality is implied by

$$\left(1 - \frac{\sqrt{\ln(n)}}{c\ln(n)}\right)^{\ln(n)-1} > \frac{4\varepsilon}{\sqrt{n/s}}.$$

The last inequality is equivalent to

$$\left(1 - \frac{1}{c\sqrt{\ln(n)}}\right)^{\ln(n)} > \frac{4\varepsilon}{\sqrt{n/s}} \left(1 - \frac{1}{c\sqrt{\ln(n)}}\right),$$

which by using a known inequality $(1 + y/m)^m \geq e^y(1 - y^2/m)$, see Fact 5.1, is implied by

$$\left(\frac{1}{e}\right)^{\sqrt{\ln(n)}} > \frac{4\varepsilon}{\sqrt{n/s}} \left(1 - \frac{1}{c\sqrt{\ln(n)}}\right) / \left(1 - \frac{1}{c^2}\right).$$
We now observe that \((1 - \frac{1}{c\sqrt{\ln(n)}}) / (1 - \frac{1}{2c}) < 3/2\) and so the last inequality is implied by

\[
\left(\frac{1}{e}\right)^{\frac{\sqrt{\ln(n)}}{c}} \geq \frac{6c^2}{\sqrt{n/s}} = \frac{\ln(n)}{2} \geq \frac{\sqrt{\ln(n)}}{c} + \frac{\ln(s)}{2} + \ln(6c^2).
\]

The last inequality can easily be checked to hold when \(n \geq e^{2c}\).

### B.2 Proof of Theorem 1.2

**Proof.** The main idea of the proof follows that of Algorithm 1. That is as it learns \(p_i\)’s starting from the largest and proceeding towards the smallest, the proofs follows the same order.

Recall that we assume that \(n \geq e^{2c}\) and \(n \geq \frac{4}{(2-\sqrt{2})^2}\).

First we show how to exactly learn \(p_0\). Observe that by the inequality from Fact 3.1 we obtain

\[
p_0^{\ell_0} = \left(1 - \frac{\alpha_0/c}{(c\ln(n))^{s/c}}\right)^{(c\ln(n))^{s/c}} \leq \left(\frac{1}{e}\right)^{\frac{\alpha_0/c}{1/c}} \leq \left(\frac{1}{e}\right)^{\frac{\alpha_0}{1/c}} < 1.
\]

Similarly we see that \(p_1^{\ell_0} \leq 1/n\) and \(p_2^{\ell_0} \leq (1/n)^{(2\ln(n))}\), and in general \(p_i^{\ell_0} \leq (1/n)^{(c\ln(n))^{i-1}}\) for \(i = 1, 2, \ldots\).

By this observation we can upper bound the mean of \(X^{\ell_0}\) as follows (note that \(n_i = n_0 = n/s\) for all \(i\)):

\[
\mathbb{E}[X^{\ell_0}] = \sum_{i=0}^{s-1} n_ip_i^{\ell_0} = n_0 \sum_{i=0}^{s-1} p_i^{\ell_0} = n_0 \cdot \left(p_0^{\ell_0} + \sum_{i=1}^{s-1} p_i^{\ell_0}\right) \leq n_0 \cdot \left(p_0^{\ell_0} + \sum_{i=1}^{\infty} \frac{1}{(1/n)^i}\right) \leq n_0 \cdot \left(p_0^{\ell_0} + 2/n\right), \tag{12}
\]

where the last estimate holds if \(n \geq 2\).

Similarly we can bound the variance \(\sigma_{\ell_0}^2 = \text{Var}[X^{\ell_0}]\):

\[
\text{Var}[X^{\ell_0}] = n_0 \sum_{i=0}^{s-1} n_ip_i^{\ell_0}(1 - p_i^{\ell_0}) \leq n_0 \cdot \left(p_0^{\ell_0} + 2/n\right) < n_0 \cdot (1 + 2/n) \leq 2n_0.
\]

We now draw \(O\left(\frac{\log(s/\delta)}{\varepsilon^2}\right)\) samples from \(X^{\ell_0}\) and obtain by Proposition 3.1 the estimate \(\hat{\mu}_{\ell_0}\) of the mean \(\mu_{\ell_0} = \mathbb{E}[X^{\ell_0}]\) such that

\[
|\mu_{\ell_0} - \hat{\mu}_{\ell_0}| \leq \varepsilon \sigma_{\ell_0} < \varepsilon \sqrt{2n_0},
\]

with probability at least \(1 - \delta/s\). This estimate, after letting \(\mu_{\ell_0} = n_0 \cdot \left(p_0^{\ell_0} + r_{\ell_0}\right)\), implies

\[
|p_0^{\ell_0} + r_{\ell_0} - \hat{\mu}_{\ell_0}/n_0| < \varepsilon \sqrt{2/n_0}, \tag{13}
\]

and \(r_{\ell_0} \leq 2/n\) by (12). Let \(\alpha_0 \in \{1, 2, \ldots, \lfloor\sqrt{\ln(n)}\rfloor\}\) be such that \(p_0^{\ell_0} = \left(1 - \frac{\alpha_0}{e\ell_0}\right)^{\ell_0}\), then by (13) we obtain

\[
\left|\left(1 - \frac{\alpha_0}{e\ell_0}\right)^{\ell_0} + r_{\ell_0} - \frac{\hat{\mu}_{\ell_0}}{n_0}\right| < \varepsilon \sqrt{2/n_0}. \tag{14}
\]
By \( n \geq e^{2c} \), Lemma 3.1 implies that \( (1 - \frac{\beta_0 - 1}{c \ell_0})^{\ell_0} - (1 - \frac{\beta_0}{c \ell_0})^{\ell_0} > \frac{4\epsilon}{\sqrt{n/s}} \) for any \( \beta_0 \in \{1, 2, \ldots, \lfloor \sqrt{\ln(n)} \rfloor \} \). This, together with (14), gives us that there exists the smallest \( \beta_0 \in \{1, 2, \ldots, \lfloor \sqrt{\ln(n)} \rfloor \} \) such that \( (1 - \frac{\beta_0}{c \ell_0})^{\ell_0} \leq \hat{\mu}_{\ell_0}/n_0 \); thus, we have that

\[
\left(1 - \frac{\beta_0}{c \ell_0}\right)^{\ell_0} \leq \frac{\hat{\mu}_{\ell_0}}{n_0} < \left(1 - \frac{\beta_0 - 1}{c \ell_0}\right)^{\ell_0},
\]

and let indeed \( \beta_0 \) be such smallest number from \( \{1, 2, \ldots, \lfloor \sqrt{\ln(n)} \rfloor \} \).

We will now prove that \( \alpha_0 \in \{\beta_0 - 1, \beta_0\} \) and we will also show how to decide if in fact \( \alpha_0 = \beta_0 - 1 \) or \( \alpha_0 = \beta_0 \), which means that we can learn the precise value of \( p_0 \) from \( \hat{\mu}_{\ell_0}/n_0 \).

Suppose first that \( \beta_0 \geq \alpha_0 + 2 \); then by (15) we obtain that

\[
\left(1 - \frac{\alpha_0}{c \ell_0}\right)^{\ell_0} + r_{\ell_0} - \frac{\hat{\mu}_{\ell_0}}{n_0} > \left(1 - \frac{\alpha_0}{c \ell_0}\right)^{\ell_0} + r_{\ell_0} - \left(1 - \frac{\beta_0 - 1}{c \ell_0}\right)^{\ell_0} \geq
\]

\[
\left(1 - \frac{\alpha_0}{c \ell_0}\right)^{\ell_0} + r_{\ell_0} - \left(1 - \frac{\alpha_0 + 1}{c \ell_0}\right)^{\ell_0} > \frac{4\epsilon}{\sqrt{n/s}},
\]

where the last inequality follows by Lemma 3.1 because \( n \geq e^{2c} \). But then this is in contradiction with (14); thus, we must have that \( \beta_0 \leq \alpha_0 + 1 \).

Similarly, if \( \beta_0 \leq \alpha_0 - 1 \), then by (15)

\[
\left|\frac{\hat{\mu}_{\ell_0}}{n_0} - r_{\ell_0}\right| = \left|\hat{\mu}_{\ell_0}/n_0 - \left(1 - \frac{\alpha_0}{c \ell_0}\right)^{\ell_0} - r_{\ell_0}\right| 
\geq
\left|\left(1 - \frac{\alpha_0 - 1}{c \ell_0}\right)^{\ell_0} - \left(1 - \frac{\alpha_0}{c \ell_0}\right)^{\ell_0} - r_{\ell_0}\right| > \frac{4\epsilon}{\sqrt{n/s}} - 2/n,
\]

where the last inequality follows by Lemma 3.1. This again is in contradiction with (14); thus, we have that \( \beta_0 \geq \alpha_0 \). We have shown that \( \beta_0 \in \{\alpha_0, \alpha_0 + 1\} \).

What remains to show is how to decide if \( \alpha_0 = \beta_0 - 1 \) or \( \alpha_0 = \beta_0 \). By Lemma 3.1 the length of the interval \( I_0 = \left[\left(1 - \frac{\beta_0}{c \ell_0}\right)^{\ell_0}, \left(1 - \frac{\beta_0 - 1}{c \ell_0}\right)^{\ell_0}\right] \) in (15) containing number \( \hat{\mu}_{\ell_0}/n_0 \) can be lower bounded as follows

\[
\left(1 - \frac{\beta_0 - 1}{c \ell_0}\right)^{\ell_0} - \left(1 - \frac{\beta_0}{c \ell_0}\right)^{\ell_0} > \frac{4\epsilon}{\sqrt{n/s}}.
\]

If we had \( \alpha_0 = \beta_0 \), then (14) implies that the distance between numbers \( \left(1 - \frac{\beta_0}{c \ell_0}\right)^{\ell_0} \) and \( \hat{\mu}_{\ell_0}/n_0 \) is at most \( r_{\ell_0} + \varepsilon \sqrt{2/n_0} \leq 2/n + \varepsilon \sqrt{2/n_0} \), which is strictly less than half of the length of interval \( I_0 \) by our assumption that \( n > \frac{4}{(2 - \sqrt{2})^2 \varepsilon^2} \). On the other hand if we had that \( \alpha_0 = \beta_0 - 1 \), then (14) implies that the distance between numbers \( \left(1 - \frac{\beta_0 - 1}{c \ell_0}\right)^{\ell_0} \) and \( \hat{\mu}_{\ell_0}/n_0 \) is most \( \varepsilon \sqrt{2/n_0} \), which again is strictly less than half of the length of interval \( I_0 \). We can therefore use this test to decide if \( \alpha_0 = \beta_0 - 1 \) or \( \alpha_0 = \beta_0 \).

This finishes the argument of how to exactly learn \( p_0 \). We will now assume that for some \( i \in \{1, 2, \ldots, s - 1\} \), values of all \( p_0, p_1, \ldots, p_{i-1} \) are known exactly, and we will show how to learn exactly the value of the next \( p_i \).
The high-level argument will be quite similar to the case of learning \( p_0 \) because we now can assume that values of \( p_0, p_1, \ldots, p_{i-1} \) are known.

By the inequality from Fact 3.1 we obtain
\[
\ell_i^t = \left(1 - \frac{\alpha_i/c}{(c \ln(n))^{s-1}/c} \right) \left( \frac{c \ln(n)^{s-1}/c}{\alpha_i/c} \right)^{\alpha_i/c} \leq \left(\frac{1}{e}\right)^{\frac{s-1}{c}} \leq \left(\frac{1}{e}\right)^{1/c} < 1.
\]

Similarly we see that \( \ell_{i+1}^t \leq 1/n \) and \( \ell_{i+2}^t \leq (1/n)^{c \ln(n)} \), and in general \( \ell_{i+j}^t \leq (1/n)^{(c \ln(n))^{j-1}} \) for \( j = 1, 2, \ldots \).

We thus can upper bound the mean of \( X^t \) as follows (note that \( n_j = n_0 = n/s \) for all \( j \)):
\[
\mathbb{E}[X^t] = \sum_{j=0}^{s-1} n_j p_j^t = n_0 \sum_{j=0}^{s-1} p_j^t = n_0 \left( \sum_{j=0}^{i-1} p_j^t + p_i^t + \sum_{j=i+1}^{s-1} p_j^t \right) \leq \sum_{j=0}^{i-1} n_0 \left( \sum_{j=0}^{i-1} p_j^t + p_i^t + \sum_{j=1}^{s-1} (1/n)^j \right) \leq n_0 \left( \sum_{j=0}^{i-1} p_j^t + p_i^t + 2/n \right),
\]
where the last estimate holds if \( n \geq 2 \).

We will now bound the variance \( \sigma^2_{t_i} = \text{Var}[X^t] \). By the inequality \( (1 - x/n)^n \geq 1 - x \), which holds for any \( n \geq 1 \) and \( x \leq n \), and by using that \( \alpha_j \leq \ln(n) \), for all \( j \), we obtain
\[
\ell_{i-1}^t = \left(1 - \frac{\alpha_{i-1}/c}{(c \ln(n))^{s-1}/c} \right) \left( \frac{c \ln(n)^{s-1}/c}{\alpha_{i-1}/c} \right)^{\alpha_{i-1}/c} \geq 1 - \frac{\alpha_{i-1}}{c^2 \ln(n)} \geq 1 - \frac{1}{c^2 \sqrt{\ln(n)}}.
\]

Similarly we see that \( \ell_{i-2}^t \geq 1 - \frac{1}{c^2 \sqrt{\ln(n) \cdot \ln(n)}} \) and \( \ell_{i-3}^t \geq 1 - \frac{1}{c^2 \sqrt{\ln(n) \cdot (c \ln(n))^2}} \), and in general \( \ell_{i-j}^t \geq 1 - \frac{1}{c^2 \sqrt{\ln(n) \cdot (c \ln(n))^{j-1}}} \) for \( j = 1, 2, \ldots \). Thus we obtain
\[
\text{Var}[X^t] = n_0 \sum_{j=0}^{s-1} p_j^t (1 - p_j^t) \leq n_0 \left( \sum_{j=0}^{i-1} (1 - p_j^t) + p_i^t + \sum_{j=i+1}^{s-1} p_j^t \right) \leq n_0 \left( \sum_{j=0}^{i-1} \frac{1}{c^2 \sqrt{\ln(n) \cdot (c \ln(n))^j}} + p_i^t + 2/n \right) < n_0 \cdot (1/7 + 1 + 2/n) \leq 2n_0,
\]
where we used that \( n \geq c^{2e} \) and \( c \geq 2 \) to bound
\[
\left( \sum_{j=0}^{s-1} \frac{1}{c^2 \sqrt{\ln(n) \cdot (c \ln(n))^j}} \right) \leq \left( \sum_{j=0}^{\infty} \frac{1}{c^2 \sqrt{\ln(n) \cdot (c \ln(n))^j}} \right) \leq 1/7.
\]

We now draw \( O \left( \log(s/\delta) \right) \times \sqrt{x} \) samples from \( X^t \) and obtain by Proposition 3.1 the estimate \( \hat{\mu}_t \) of the mean \( \mu_t = \mathbb{E}[X^t] \) such that
\[
|\mu_t - \hat{\mu}_t| \leq \varepsilon \sigma_t < \varepsilon \sqrt{2n_0},
\]
with probability at least \( 1 - \delta / s \). This estimate, after letting \( \mu_t = n_0 \left( \sum_{j=0}^{i-1} p_j^t + p_i^t + r_{t_i} \right) \) implies
\[
\left| \sum_{j=0}^{i-1} p_j^t + p_i^t + r_{t_i} - \mu_t / n_0 \right| < \varepsilon \sqrt{2/n_0},
\]
(17)
and \( r_{\ell_i} \leq 2/n \) by (16). Let \( \alpha_i \in \{1, 2, \ldots, \lceil \ln(n) \rceil \} \) be such that \( p_i^{\ell_i} = \left( 1 - \frac{\alpha_i}{c\ell_i} \right)^{\ell_i} \), and let us also denote \( \hat{\tau}_i = \mu_{\ell_i}/n_0 - \sum_{j=0}^{i-1} p_j^{\ell_j} \). Then by (17) we obtain

\[
\left| \left( 1 - \frac{\alpha_i}{c\ell_i} \right)^{\ell_i} + r_{\ell_i} - \hat{\tau}_i \right| < \varepsilon \sqrt{2/n_0}. \tag{18}
\]

Recall that the values \( p_0, \ldots, p_{i-1} \) are known. Suppose next that we find the smallest \( \beta_i \in \{1, 2, \ldots, \lceil \ln(n) \rceil \} \) such that \( \left( 1 - \frac{\beta_i}{c\ell_i} \right)^{\ell_i} \leq \hat{\tau}_i \) (such \( \beta_i \) exists by the same argument as that for \( \beta_0 \)); thus, we have that

\[
\left( 1 - \frac{\beta_i}{c\ell_i} \right)^{\ell_i} \leq \hat{\tau}_i < \left( 1 - \frac{\beta_i - 1}{c\ell_i} \right)^{\ell_i}. \tag{19}
\]

We will now prove that \( \alpha_i \in \{\beta_i - 1, \beta_i\} \) and we will also show how to decide if \( \alpha_i = \beta_i - 1 \) or \( \alpha_i = \beta_i \), which will imply that the precise value of \( p_i \) can be learned from \( \hat{\tau}_i \).

Suppose first that \( \beta_i \geq \alpha_i + 2 \); then by (19) we obtain that

\[
\left| \hat{\tau}_i - p_i^{\ell_i} - r_{\ell_i} \right| = \left| \hat{\tau}_i - \left( 1 - \frac{\alpha_i}{c\ell_i} \right)^{\ell_i} - r_{\ell_i} \right| \geq
\]

\[
\left| \left( 1 - \frac{\alpha_i}{c\ell_i} \right)^{\ell_i} - \left( 1 - \frac{\alpha_i + 1}{c\ell_i} \right)^{\ell_i} - r_{\ell_i} \right| > \frac{4\varepsilon}{\sqrt{n/s}} - 2/n,
\]

where the last inequality follows by Lemma 3.1 because \( n \geq e^{2c} \). But then this is in contradiction with (18); thus, we must have that \( \beta_i \leq \alpha_i + 1 \).

Similarly, if \( \beta_i \leq \alpha_i - 1 \), then by (19)

\[
\left| \hat{\tau}_i - p_i^{\ell_i} - r_{\ell_i} \right| = \left| \hat{\tau}_i - \left( 1 - \frac{\alpha_i - 1}{c\ell_i} \right)^{\ell_i} - r_{\ell_i} \right| \geq
\]

\[
\left| \left( 1 - \frac{\alpha_i}{c\ell_i} \right)^{\ell_i} - \left( 1 - \frac{\alpha_i - 1}{c\ell_i} \right)^{\ell_i} - r_{\ell_i} \right| > \frac{4\varepsilon}{\sqrt{n/s}} - 2/n,
\]

where the last inequality follows by Lemma 3.1. This again is in contradiction with (18); thus, we have that \( \beta_i \leq \alpha_i \). We have shown that \( \beta_i \in \{\alpha_i, \alpha_i + 1\} \).

The next step is to decide if \( \alpha_i = \beta_i - 1 \) or \( \alpha_i = \beta_i \). By Lemma 3.1 the length of the interval \( I_i = \left[ \left( 1 - \frac{\beta_i}{c\ell_i} \right)^{\ell_i}, \left( 1 - \frac{\beta_i - 1}{c\ell_i} \right)^{\ell_i} \right] \) in (19) containing number \( \hat{\tau}_i \) can be lower bounded as follows

\[
\left( 1 - \frac{\beta_i - 1}{c\ell_i} \right)^{\ell_i} - \left( 1 - \frac{\beta_i}{c\ell_i} \right)^{\ell_i} > \frac{4\varepsilon}{\sqrt{n/s}}.
\]

If \( \alpha_i = \beta_i \), then (18) implies that the distance between numbers \( \left( 1 - \frac{\beta_i}{c\ell_i} \right)^{\ell_i} \) and \( \hat{\tau}_i \) is at most \( r_{\ell_i} + \varepsilon \sqrt{2/n_0} \leq 2/n + \varepsilon \sqrt{2/n_0} \), which is strictly less than half of the length of interval \( I_i \) by our assumption that \( n > \frac{4}{(2-\sqrt{1/2})^2} \). On the other hand, if \( \alpha_i = \beta_i - 1 \), then (18) implies that the distance between numbers \( \left( 1 - \frac{\beta_i - 1}{c\ell_i} \right)^{\ell_i} \) and \( \hat{\tau}_i \) is most \( \varepsilon \sqrt{2/n_0} \), which again is strictly less than half of the length of interval \( I_i \). We can therefore use this test to decide if \( \alpha_i = \beta_i - 1 \) or \( \alpha_i = \beta_i \).

To finish the proof, observe that by the union bound all the sampling estimates for the mean values \( \hat{\mu}_{\ell_i} \) hold with probability at least \( 1 - \delta \). Moreover, because this sampling for each \( i = 0, 1, \ldots, s - 1 \) draws \( O\left( \frac{\log(s/\delta)}{\varepsilon^2} \right) \) samples from \( X^{\ell_i} \), the total number of samples is \( O\left( \frac{s \log(s/\delta)}{\varepsilon^2} \right) \).
C Upper Bound for Learning Binomial Powers

C.1 Preliminaries

We start by stating a useful variant of the standard Chernoff Bound.

**Proposition C.1 (Chernoff Bound).** Let \( X = X_1 + \cdots + X_n, X_i \in [0, 1] \). Let \( \mu = \mathbb{E}[X] \) and \( \sigma^2 = \text{Var}(X) \). Then, for all \( \lambda \in (0, 2\sigma) \), \( \Pr[X > \mu + \lambda \sigma] < e^{-\lambda^2/4} \) and \( \Pr[X < \mu - \lambda \sigma] < e^{-\lambda^2/4} \).

**Proof.** See, e.g., page 8 in the book [18].

We now prove Fact 4.1 and Fact 4.2 on estimating the parameter \( p \) of a Binomial \( B(n,p) \) using Chernoff’s Bound.

**Proof of Fact 4.1**

**Proof.** Let \( X = \sum_{i=1}^{m} s_i/n \). Then \( s_i/n \in [0,1] \) and \( \mathbb{E}[X] = mp, \text{Var}[X] = m/n p(1-p) \), since the samples are i.d.d. We show only that \( \Pr[\hat{p} - p > \psi \text{ err}(n,p,\varepsilon)] \leq \delta \) since the other case is similar. From C.1 we obtain with \( t = m\psi \text{ err}(n,p,\varepsilon) \)

\[
\Pr[\hat{p} - p > \psi \text{ err}(n,p,\varepsilon)] = \Pr[\hat{p} > t/m] = \Pr[X - \mathbb{E}[X] > \sqrt{m\psi\varepsilon \text{ Var}[X] / t/m}] \leq \exp\left(-m\varepsilon^2\psi^2/4\right) \leq \delta,
\]

where, for the last inequality, we use that \( m = \lceil 4 \ln(1/\delta) / (\varepsilon^2 \psi^2) \rceil \).

**Proof of Fact 4.2**

**Proof.** We only prove that \( \Pr[\hat{q} < \hat{q}_2 < p + \text{ err}(n,p,\varepsilon)] \geq 1 - \delta \) since the proof for \( \hat{q}_1 = \min_{1 \leq i \leq k} w_i \) is essentially the same.

\[
\Pr\left[\max_i w_i < p\right] \cup \Pr\left[\max_i w_i > p + \psi \text{ err}(n,p,\varepsilon)\right] \\
\leq \Pr\left[\max_i w_i < p\right] + \Pr\left[\max_i w_i > p + \psi \text{ err}(n,p,\varepsilon)\right] \\
= \Pr\left[\bigcup_{i=1}^{k} (w_i < p)\right] + \Pr\left[\bigcup_{i=1}^{k} (w_i > p + \psi \text{ err}(n,p,\varepsilon))\right] \\
\leq \left(\frac{1}{2}\right)^k + ku \leq \delta,
\]

where the last inequality follows from \( k = \lceil \ln(2/\delta) / \ln(2) \rceil \) and by choosing \( u \leq \delta/(2k) \). From Fact 4.1 we have that

\[
m = \lceil 4 \ln(1/u) / (\varepsilon^2 \psi^2) \rceil = \left\lfloor \frac{\ln\left(\frac{2[\ln(2/\delta) / \ln(2)]}{\delta}ight)}{\varepsilon^2 \psi^2} \right\rfloor = O\left(\frac{\ln(1/\delta)}{\varepsilon^2 \psi^2}\right)
\]

is sufficient to ensure that \( \Pr[w_i < p + \psi \text{ err}(n,p,\varepsilon)] \leq \delta/(2k) \).
C.2 The Case where \( p \in [\varepsilon^2/n^d, 1 - \varepsilon^2/n^d] \).

We generalise Algorithm 2 and its analysis to the case where the value of \( p \) is very close to 1 or 0 and lies in \([\varepsilon^2/n^d, 1 - \varepsilon^2/n^d]\), for some fixed constant integer \( d \in \mathbb{N}_+ \). Hence, we cover all values of \( p \) that can be represented by \( O(\log n) \) bits.

This will lead to the following theorem.

**Theorem C.1.** Let \( \varepsilon \in (0,1/6) \) and \( d \in \mathbb{N}_+ \) be fixed constants, and let \( n \in \mathbb{N} \), \( n \geq 5 \). For any \( p \in [\varepsilon^2/n^d, 1 - \varepsilon^2/n^d] \), an extension of Algorithm 2 uses \( O(\log(d) \log(\log(d)/\delta)/\varepsilon^2) \) samples and outputs \( \hat{t}, \hat{q}_1, \hat{q}_2 \in (0,1) \) such that \( d_{tv}(B(n, \hat{q}_1), B(n, \hat{q}_2)) \leq O(\varepsilon) \) for \( l \in (0,1) \) and \( d_{tv}(B(n, \hat{q}_1^0), B(n, p_t)) \) for \( l \in (1, +\infty) \) with probability at least \( 1 - \delta \).

**Proof.** We will first describe the extension of Algorithm 2. Notice that, in this case we only need to find \( t \in (0, +\infty) \) such that \( p_t \in [\varepsilon^2/n, 1 - \varepsilon^2/n] \). Then, we simply call Algorithm 2 using \( B(n, p_t) \) as the "first" power to obtain \( \hat{q}_1, \hat{q}_2, \hat{a} \) such that \( d_{tv}(B(n, \hat{q}_1), B(n, \hat{a})) \leq O(\varepsilon) \) for \( l \in (0,1) \) and \( d_{tv}(B(n, \hat{q}_1^0), B(n, p_t)) \) for \( l \in (1, +\infty) \). To find \( t \) we first sample from \( B(n, p) \) and using Fact 4.2 we obtain \( \hat{q}_{t,1}, \hat{q}_{t,2} \) such that \( \Pr |p - err(n, p, \varepsilon) < \hat{q}_{t,1} < p < \hat{q}_{t,2} < p + err(n, p, \varepsilon) | \geq 1 - \delta \). We have the following cases:

- \( \hat{q}_{t,1} > \varepsilon^2/n \) and \( \hat{q}_{t,2} < 1 - \varepsilon^2/n \). In this case we can use directly Algorithm 2.
- \( \hat{q}_{t,1} < \varepsilon^2/n \). Let \( I_1 = \{1/(i \ln n) : i \in \{2, \ldots, d\}\} \). Using Fact 4.2 draw \( O(d\ln^2(d/\delta)/\varepsilon^2) \) samples from the powers \( B(n, p^t) \), \( l \in I_1 \), and obtain the set of approximations \( Q_1 = \{\hat{q}_{i,1} : i \in I_1\} \) such that with probability \( 1 - \delta/2 \) all \( \hat{q}_{i,1} \in Q_1 \) satisfy the bounds \( p - err(n, p^t, \varepsilon) < \hat{q}_{i,1} < p^t \). We first prove that there exists an element \( t \) of \( I_1 \) such that \( \hat{q}_{t,1} \geq \varepsilon^2/n \). It suffices to show that such a \( t \) exists when \( p = \varepsilon^2/n^d \). Then \( p_{t}^{1/(d\ln n)} = \varepsilon^2/(d\ln n)/e \) and \( \hat{q}_{t,1}^{1/(d\ln n)} = p_{t}^{1/(d\ln n)} - \frac{\varepsilon^2}{2\sqrt{n}} \geq \varepsilon^2/n^2 \) for all \( n \geq 7 \).

Let \( t \) be the largest element of \( I_1 \) such that \( \hat{q}_{t,1} \geq \varepsilon^2/n \). Then \( p_t^t > \varepsilon^2/n \) since \( \hat{q}_{t,1} < p^t \). Moreover, \( p_t^t < 1 - \varepsilon^2/n \). To show that write \( t = 1/(\rho \ln n) \) for some \( \rho \geq 2 \) and \( t' = 1/((\rho - 1) \ln n) \). Then \( p_t^t \leq \hat{q}_{t,1}^t + err(n, p_t^t, \varepsilon) \leq \varepsilon^2/n + \varepsilon/(2\sqrt{n}) \leq \varepsilon/\sqrt{n} \). Thus, \( p_t^t = p_t^{1/(\rho \ln n)} = p_t^{\left(1/\rho \ln n\right)\ln \frac{\varepsilon/\sqrt{n}}{\varepsilon}} \leq \frac{\varepsilon^2}{n^2} \leq 1 - \varepsilon^2/n \), where the last inequality holds for \( n \geq 2, \varepsilon < 1/2 \).

- \( \hat{q}_{t,2} > 1 - \varepsilon^2/n \). Consider now the set \( I_2 = \{n^{i/3} : i \in \{0\} \cup [3d]\} \). Using Fact 4.2 draw \( O(d \ln n^2(d/\delta)/\varepsilon^2) \) samples and obtain the set of approximations \( Q_2 = \{\hat{q}_{i,2} : i \in I_2\} \) such that with probability \( 1 - \delta/2 \) all \( \hat{q}_{i,2} \in Q_2 \) satisfy the bounds \( p^t < \hat{q}_{i,2} < p^t + err(n, p^t, \varepsilon) \). As we did in the previous case we first prove that there exists a \( t \in I_2 \) such that \( \hat{q}_{i,2} \leq 1 - \varepsilon^2/n \). It suffices to prove it for \( p = 1 - \varepsilon^2/n^d \). Take \( t = n^d \). Then \( p_t^t = (1 - \varepsilon^2/n^d)^{n^d} \leq e^{-\varepsilon^2/n} \leq 1 - \varepsilon^2/2 \leq 1 - \varepsilon^2/n \) for \( \varepsilon < 0.85 \), \( n \geq 2 \).

Starting from 1 find the smallest element \( t \) of \( I_2 \) such that \( \hat{q}_{i,2} < 1 - \varepsilon^2/n \). We argue that \( p_{\hat{q}_{i,2}} > 1 - \varepsilon^2/n \). Obviously \( p_t^t < 1 - \varepsilon^2/n \) since \( p_t^t < \hat{q}_{i,2} \). To prove the other inequality, write \( t = n^{\rho/3} \) and \( t' = n^{(\rho - 1)/3} \) for some \( \rho \in [3d] \). We have that \( \hat{q}_{i,2} \geq 1 - \varepsilon^2/n \) and therefore \( p_t^t \geq 1 - \varepsilon^2/n - err(n, p^t, \varepsilon) \geq 1 - \varepsilon^2/n - \varepsilon/(2\sqrt{n}) \geq 1 - \varepsilon/\sqrt{n} \), for \( n \geq 4 \). Thus, \( p_t^t = p_t^{\left(1/\rho \ln n\right)} \leq (p_t^{1/\rho})^{n^{1/3}} \geq (1 - \varepsilon/\sqrt{n})^{n^{1/3}} \geq e^{-2\varepsilon/\sqrt{n}/n^{1/2}} \geq e^{-2\varepsilon/n^{1/6}} \geq \varepsilon^2/n \), where for the second inequality we used \( 1 - x \geq e^{-2x} \) for \( x \in [0, 0.75] \) and the last inequality holds for \( \varepsilon \leq 1/\sqrt{e} \), \( n \geq 1 \).

It is easy to see that we can improve the sampling complexity by doing binary search on \( d \), which means for each \( p \) tested, the algorithm chooses \( O(\log(\log(d)/\delta)/\varepsilon^2) \) independent samples, which leads to \( O(\log(d) \log(\log(d)/\delta)/\varepsilon^2) \) total number of samples. \(\square\)
D Lower Bound for Learning Binomial Powers

D.1 Notation

We denote by \( \mathcal{N}(\mu, \sigma) \) the Normal distribution with mean \( \mu \) and variance \( \sigma^2 \). The density function of \( \mathcal{N}(\mu, \sigma) \) is \( f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \). We denote by \( \text{erf}(x) \) the Gauss error function, namely \( \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \, dt \), by \( \text{erfc}(x) \) the complementary error function, \( \text{erfc}(x) = 1 - \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} \, dt \).

D.2 Preliminaries

We prove the following proposition which provides an exact expression for the KL-Divergence of two Binomial distributions.

**Proposition D.1 (Binomial KL-Divergence).** Let \( X \sim B(n, p) \), \( Y \sim B(n, q) \) be two Binomial distributions. Then

\[
D_{kl}(X\|Y) = -n \left( (1-p) \log \left( \frac{1-q}{1-p} \right) + p \log \left( \frac{q}{p} \right) \right)
\]

**Proof.** We have

\[
D_{kl}(X\|Y) = \sum_{k=0}^{n} X(k) \log \frac{X(k)}{Y(k)}
\]

\[
= \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \log \left( \frac{p^k (1-p)^{n-k}}{q^k (1-q)^{n-k}} \right)
\]

\[
= \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \left( k \ln \left( \frac{p}{q} \right) + (n-k) \ln \left( \frac{1-p}{1-q} \right) \right)
\]

\[
= \ln \left( \frac{p}{q} \right) \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} + \ln \left( \frac{1-p}{1-q} \right) \sum_{k=0}^{n} (n-k) \binom{n}{k} p^k (1-p)^{n-k}
\]

\[
= np \ln \left( \frac{p}{q} \right) + n(1-p) \ln \left( \frac{1-p}{1-q} \right).
\]

\( \square \)

The following simple proposition formalizes an intuition that when the distance of the parameters \( p, q \) of two Binomial distributions is large, then the Kullback-Leibler divergence of the these distributions is large.

**Proposition D.2.** Let \( X \sim B(n, p) \), \( Y \sim B(n, q) \). Then \( D_{kl}(X\|Y) \) and \( D_{kl}(Y\|X) \) are both increasing functions of \( |p-q| \).

**Proof.** Write \( q = p + x \). We start with \( D_{kl}(B(n, p)\|B(n, p + x)) \). The derivative of \( h(x) := D_{kl}(B(n, p)\|B(n, p + x)) \) with respect to \( x \) is

\[
h'(x) = \frac{n(1-p)}{-p + x + 1} - \frac{np}{p + x}.
\]

It’s easy to see that \( h'(x) > 0 \) for \( x \in (0, 1-p) \) and \( h'(x) < 0 \) for \( x \in (-p, 0) \). Therefore \( h(x) \) is minimized at \( x = 0 \). Similarly if we let \( g(x) = D_{kl}(B(n, p + x)\|B(n, p)) \) we have

\[
g'(x) = n \ln \left( \frac{p+x}{p} \right) - n \ln \left( \frac{-p - x + 1}{1-p} \right).
\]

Again we have \( g'(x) < 0 \) for \( x \in (-p, 0) \) and \( g'(x) > 0 \) for \( x \in (0, 1-p) \). Thus, \( g(x) \) is minimized at \( x = 0 \). \( \square \)
D.3 Discretized Normal Approximation

We first start with some basic results about continuous Normal distributions. Chu [5] proved the following inequality for the Normal Integral

Proposition D.3 (Chu’s Inequalities). For any \( x \geq 0 \):

\[
\sqrt{1 - e^{-ax^2}} \leq \text{erf} (x) \leq \sqrt{1 - e^{-bx^2}},
\]

where \( a = 1 \) and \( b = 4/\pi \).

The following Corollary of D.3 provides a slightly weaker lower bound for \( \text{erf} (x) \).

Corollary D.1. If \( 0 \leq x \leq 1 \), then we have that \( \text{erf} (x) \geq \frac{x}{c} \), where \( c \) is any fixed constant such that \( c \geq \sqrt{e/(e - 1)} \).

Proof. Using lower bound of the inequality of Proposition D.3 with \( a = 1 \) we want to prove that

\[
\frac{x}{c} \leq \sqrt{1 - \frac{1}{e^{x^2}}},
\]

for any \( x \in [0, 1] \). This inequality is equivalent to

\[
f(x) := e^{x^2} (c^2 - x^2) - c^2 \geq 0.
\]

We have that \( f'(x) = 2xe^{x^2}(c^2 - 1 - x^2) \), therefore we see that \( f'(x) \leq 0 \) if \( x \geq \sqrt{c^2 - 1} \). This means that function \( f \) has a maximum at \( x_0 = \sqrt{c^2 - 1} \) and thus its smallest value in \([0, 1]\) is \( \min \{ f(0), f(1) \} = \{ 0, (e - 1)c^2 - e \} \). Now we demand that \( (e - 1)c^2 - e \geq 0 \), which leads to \( c \geq \sqrt{e/(e - 1)} \) and under this condition \( f(x) \geq 0 \) for all \( x \in [0, 1] \). \( \Box \)

To bound \( \text{erfc}(z) \) we shall use Komatsu’s inequality stated e.g. as Problem 1, page 17 in [22]. See [31] for more such results.

Proposition D.4 (Komatsu’s Inequalities). For all \( a \geq 0 \) it holds

\[
\frac{e^{-a^2/2}}{2\sqrt{a^2 + 4a + 1}} \leq \int_a^{+\infty} e^{-t^2/2} \text{dt} \leq \frac{e^{-a^2/2}}{2\sqrt{a^2 + 2a + 1}}
\]

When two continuous Normal distributions have the same standard deviation a simple argument gives an exact expression for their total variation distance.

Proposition D.5. Let \( X \sim \mathcal{N} (\mu_1, \sigma) \), \( Y \sim \mathcal{N} (\mu_2, \sigma) \). Then

\[
d_{\text{tv}} (X, Y) = \text{erf} \left( \frac{|\mu_1 - \mu_2|}{2\sqrt{2}\sigma} \right)
\]

Proof. Let \( f_X \) resp. \( f_Y \) be the density functions for \( X \) resp. \( Y \). We can assume that \( \mu_1 < \mu_2 \) since the proof for the other case is essentially the same. Then

\[
f_X(x) \geq f_Y(x) \iff \frac{1}{\sqrt{2\pi}\sigma} e^{-(x - \mu_1)^2/2\sigma^2} \geq \frac{1}{\sqrt{2\pi}\sigma} e^{-(x - \mu_2)^2/2\sigma^2} \iff |x - \mu_1| \leq |x - \mu_2| \iff x \leq \frac{\mu_1 + \mu_2}{2}
\]

Therefore,

\[
d_{\text{tv}} (X, Y) = \int_{-\infty}^{\frac{\mu_1 + \mu_2}{2}} (f_X(x) - f_Y(x)) \text{dx} = \text{erf} \left( \frac{\mu_2 - \mu_1}{2\sqrt{2}\sigma} \right)
\]

\( \Box \)
When the variances of two Normal distributions differ the following Proposition from \[8 \] provides an upper bound for their total variation distance.

**Proposition D.6 (Proposition B.4 from \[8 \]).** Let \( \mu_1, \mu_2 \in \mathbb{R} \) and \( 0 < \sigma_1 \leq \sigma_2 \). Then

\[
d_{tv}(\mathcal{N}(\mu_1, \sigma_1), \mathcal{N}(\mu_2, \sigma_2)) \leq \frac{1}{2} \left( \frac{|\mu_1 - \mu_2|}{\sigma_1} + \frac{\sigma_2^2 - \sigma_1^2}{\sigma_1^2} \right)
\]

Let \( X \sim \mathcal{N}(\mu, \sigma) \). We denote by \( \text{DN}(\mu, \sigma) \) the discretized Normal distribution, namely if \( X_d \sim \text{DN}(\mu, \sigma) \) then \( X_d \) is a discrete random variable with mass function

\[
\Pr[X_d = k] = \Pr\left[k - \frac{1}{2} < X \leq k + \frac{1}{2}\right],
\]

where \( k \) is any integer.

The following recent result of Chen and Leong \[24 \] (Theorem 7.1) shows that a continuity corrected discretized Normal distribution approximates very well a PBD provided that the variance of the PBD is not very small.

**Lemma D.1 (Theorem 7.1 from \[24 \]).** Let \( X \) be a PBD and let \( \mu = \mathbb{E}[X] \), \( \sigma^2 = \text{Var}[X] \). Let \( Y \sim \text{DN}(\mu, \sigma) \). Then

\[
d_{tv}(X, Y) \leq \frac{7.6}{\sigma}.
\]

The next Lemma shows that 2 discretized Normal distributions are close if and only if the corresponding continuous Normals are close. The needed condition for this to hold is that the variance of the 2 Normals is not too small.

**Lemma D.2 (Discrete-Continuous Error).** Let \( X = \mathcal{N}(\mu_1, \sigma_1) \), \( Y = \mathcal{N}(\mu_2, \sigma_2) \) be two Normal distributions such that \( d_{tv}(X, Y) \geq \varepsilon \), where \( \varepsilon > 0 \). Let \( X_d \sim \text{DN}(\mu_1, \sigma_1) \), \( Y_d \sim \text{DN}(\mu_2, \sigma_2) \). Then

\[
\varepsilon - (\ell + m + u) \leq d_{tv}(X_d, Y_d) \leq \varepsilon
\]

where

\[
\ell = \frac{1}{4} \left( \text{erfc}\left(\frac{\mu_1}{\sqrt{2}\sigma_1}\right) + \text{erfc}\left(\frac{\mu_2}{\sqrt{2}\sigma_2}\right) \right) \quad \text{and} \quad u = \frac{1}{4} \left( \text{erfc}\left(\frac{n - \mu_1}{\sqrt{2}\sigma_1}\right) + \text{erfc}\left(\frac{n - \mu_2}{\sqrt{2}\sigma_2}\right) \right) \quad \text{and} \quad m = \frac{1}{2} \left( \text{erf}\left(\frac{1}{\sqrt{2}\sigma_1}\right) + \text{erf}\left(\frac{1}{\sqrt{2}\sigma_2}\right) \right)
\]

Proof. Let \( f_X \), resp. \( f_Y \) be the density function of \( X \), resp. \( Y \). We have that \( \int_{-\infty}^{+\infty} |f_X(x) - f_Y(x)|dx = 2d_{tv}(X, Y) = 2\varepsilon \). Since the density of a Normal distribution with mean \( \mu \) is increasing in \((-\infty, \mu] \) and decreasing in \([\mu, +\infty) \) we have that there exist at most 2 points \( r_1, r_2 \) where the sign of the difference \( d(x) := f_X(x) - f_Y(x) \) changes. Without loss of generality we assume that \( d(x) \) is positive in \((-\infty, r_1) \) and \((r_2, +\infty) \) and negative in \([r_1, r_2] \). Now assume that \( k_1 = [r_1] \) and
We remark that finding a family of sequences satisfying $\rho$ above expression. We have
\[ d \quad \text{since} \quad \kappa \quad \text{any parameter, like} \quad \text{the constants for the lower bound. Note that these will be absolute constants, independent from} \quad \text{the lower bound only by a constant factor. Thus, to simplify our analysis, we shall not compute} \quad D.4 \quad \text{The proof of Theorem 1.6} \]

\[ 2d_{tv}(X_d, Y_d) = \sum_{k=0}^{n} \left| \Pr \left[ k - \frac{1}{2} < X_d \leq k + 1 \right] - \Pr \left[ k - \frac{1}{2} < Y_d \leq k + 1 \right] \right| \]
\[ = \sum_{k=0}^{n} \left| \int_{k-1/2}^{k+1/2} f_X(x) - f_Y(x) \, dx \right| \]
\[ \geq \sum_{k=0}^{k_1} \int_{k-1/2}^{k+1/2} d(x) \, dx + \sum_{k=k_1+1}^{k_2} \int_{k-1/2}^{k+1/2} -d(x) \, dx + \sum_{k=k_2+1}^{n} \int_{k-1/2}^{k+1/2} d(x) \, dx \]
\[ \geq \int_{0}^{k_1} d(x) \, dx + \int_{k_1+1}^{k_2-1} -d(x) \, dx + \int_{k_2}^{n} d(x) \, dx. \]

Since $d_{tv}(X, Y) = \int_{-\infty}^{\infty} |f_X(x) - f_Y(x)| \, dx$ we need to upper bound the “missing” integrals in the above expression. We have
\[ \int_{-\infty}^{0} |d(x)| \, dx \leq \int_{-\infty}^{0} (f_X(x) + f_Y(x)) \, dx = \frac{1}{2} \left( \text{erfc} \left( \frac{\mu_1}{\sqrt{2\sigma_1}} \right) + \text{erfc} \left( \frac{\mu_2}{\sqrt{2\sigma_2}} \right) \right) \]

Similarly,
\[ \int_{n}^{+\infty} |d(x)| \, dx \leq \frac{1}{2} \left( \text{erfc} \left( \frac{n - \mu_1}{\sqrt{2\sigma_1}} \right) + \text{erfc} \left( \frac{n - \mu_2}{\sqrt{2\sigma_2}} \right) \right) \]

Moreover,
\[ \int_{k_1}^{k_1+1} |d(x)| \, dx + \int_{k_2-1}^{k_2} |d(x)| \, dx \leq \int_{\mu_1-1}^{\mu_1+1} f_X(x) \, dx + \int_{\mu_2-1}^{\mu_2+1} f_Y(x) \, dx \]
\[ \leq \text{erf} \left( \frac{1}{\sqrt{2\sigma_1}} \right) + \text{erf} \left( \frac{1}{\sqrt{2\sigma_2}} \right) \]

To prove the upper bound of inequality (20) notice that using (21) we have
\[ 2d_{tv}(X_d, Y_d) = \sum_{k=0}^{n} \left| \int_{k-1/2}^{k+1/2} f_X(x) - f_Y(x) \, dx \right| \]
\[ \leq \sum_{k=0}^{n} \int_{k-1/2}^{k+1/2} |f_X(x) - f_Y(x) | \, dx \]
\[ \leq \int_{-\infty}^{+\infty} |f_X(x) - f_Y(x) | \, dx \]
\[ = 2\varepsilon \]

\[ \square \]

**D.4 The proof of Theorem 1.6**

We remark that finding a family of sequences satisfying $\rho(\theta(P), \theta(Q)) = \Omega(\delta)$ instead of $2\delta$ changes the lower bound only by a constant factor. Thus, to simplify our analysis, we shall not compute the constants for the lower bound. Note that these will be absolute constants, independent from any parameter, like $\varepsilon, \delta$, etc., in our setting.
We restate explicitly our family of Binomial power sequences for the sake of completeness. Let $\delta = \Theta(1/\sqrt{n} N)$. Let $p_1 = 1/2$, $p_2 = 1/2 + \delta/4$, $p_3 = 1/2 + \delta/2$. Let $P_{1,1} = B(n, p_1)$, $P_{2,1} = B(n, p_2)$, $P_{1,3} = B(n, p_3)$ be three Binomial distributions with corresponding power sequences $P_1 = (B(n, p_1^i))_{i \in (1, +\infty)}$, $P_2 = (B(n, p_2^i))_{i \in (1, +\infty)}$, $P_3 = (B(n, p_3^i))_{i \in (1, +\infty)}$.

For the total variation distance of any of the above pairs $i, j \in \{1, 2, 3\}$, $i \neq j$, we have $d_{tv}(P_i, P_j) = \Omega\left(1/\sqrt{N}\right)$. Without loss of generality we prove that $d_{tv}(P_1, P_2) = \Omega(1/\sqrt{N})$.

From the definition of total variation distance for sequences of distributions we see that to lower bound the metric $\rho$ we just need to prove that the total variation distance of $P_{1,1}$, $P_{2,1}$ is $\Omega(1/\sqrt{N})$, namely we need to consider only the first power of the sequences.

Let $\mu_1 = E[P_{1,1}]$, $\mu_2 = E[P_{2,1}]$, $\sigma_1^2 = \text{Var}[P_{1,1}]$, $\sigma_2^2 = \text{Var}[P_{2,1}]$.

We first use Lemma [4.1] to approximate $P_{1,1}$, $P_{2,1}$ with discretized Normal distributions $DN(\mu_1, \sigma_1)$, $DN(\mu_2, \sigma_2)$. Since $\sigma_1$, $\sigma_2$ are both $O(\sqrt{N})$ the error of the two discretized Normal approximations is $O(1/\sqrt{n})$. From Proposition [4.6] we obtain that we can approximate $N(\mu_2, \sigma_2)$ using a Normal with the same mean but with variance $\sigma_1^2$. Applying Proposition [4.6] yields

$$d_{tv}(N(\mu_2, \sigma_2), N(\mu_2, \sigma_1)) \leq \frac{1}{2} \frac{\sigma_1^2 - \sigma_2^2}{\sigma_2^2} = \frac{1}{2} \frac{n/4 - n(1/4 - \delta^2/16)}{n(1/4 - \delta^2/16)} = O(\delta^2) = O(1/n)$$

Consider now the pair of continuous Normals $N(\mu_1, \sigma_1)$, $N(\mu_2, \sigma_1)$. Using Proposition [4.5] we have that

$$d_{tv}(N(\mu_1, \sigma_1), N(\mu_2, \sigma_1)) = \text{erf}\left(\frac{n\delta}{4\sqrt{2}\sqrt{n}}\right) = \text{erf}\left(\frac{\sqrt{n}\delta}{4\sqrt{2}}\right) = \text{erf}\left(\frac{1}{4\sqrt{2}\sqrt{N}}\right) \geq \frac{1}{9\sqrt{N}}.$$

by Corollary [4.1] Therefore, by the triangle inequality, we have

$$d_{tv}(N(\mu_1, \sigma_1), N(\mu_2, \sigma_2)) \geq \frac{1}{9\sqrt{N}} - O\left(\frac{1}{n}\right)$$

Applying Lemma [4.2] when $\sigma_1$, $\sigma_2$ are $O(\sqrt{N})$ yields $\ell + m + u = O\left(\frac{1}{\sqrt{n}}\right)$, since from Komatsu’s inequalities (Proposition [4.4]) we have $\text{erfc}(\sqrt{n}) = \Theta\left(\frac{e^{-n}}{\sqrt{n}}\right)$ and from Proposition [4.3] we have that $\text{erf}(1/\sqrt{n}) = \Theta\left(\frac{1}{\sqrt{n}}\right)$. Therefore

$$d_{tv}(DN(\mu_1, \sigma_1), DN(\mu_2, \sigma_2)) \geq \frac{1}{9\sqrt{N}} - O\left(\frac{1}{\sqrt{n}}\right)$$

Overall, using triangle inequality and the above bounds we have that

$$d_{tv}(P_{1,1}, P_{2,1}) \geq \frac{1}{9\sqrt{N}} - O\left(\frac{1}{\sqrt{n}}\right)$$

We continue with proving an upper bound for the Kullback-Leibler divergence between all powers, namely the sup$_{a \in \mathbb{N}} D_{kl}(P_{1,a}||P_{3,a})$. To apply Theorem 5.2 it suffices to show that the following holds sup$_{i,j \in [3], a \in \mathbb{N}} D_{kl}(P_{1,a}||P_{3,a}) = O(1/N)$. From Proposition [4.2] it is clear that we only need to bound the Kullback-Leibler distance for the most distant $p_i$’s, namely the distances sup$_{a \in \mathbb{N}} D_{kl}(P_{1,a}||P_{3,a})$, sup$_{a \in \mathbb{N}} D_{kl}(P_{3,a}||P_{1,a})$. We remark that is easy to verify that $D_{kl}(P_{1,a}||P_{3,a}) \approx D_{kl}(P_{3,a}||P_{1,a})$ for all $a \in \mathbb{N}$ and therefore we will bound $D_{kl}(P_{1,a}||P_{3,a})$.

Applying Proposition [4.1] for $P_{1,a}$ and $P_{3,a}$ gives

$$D_{kl}(P_{1,a}||P_{3,a}) = 2^{-a} n \ln \left(2^{-a} \left(\frac{1}{2 + \frac{1}{2}}\right)^{a}\right) + (1 - 2^{-a}) n \ln \left(\frac{1 - 2^{-a}}{1 - (\frac{1}{2 + \frac{1}{2}})^{a}}\right)$$
Let $f(\delta) = D_{kl}(P_{1,a}\|P_{3,a})$ defined by the above expression. Taylor Expanding $f(\delta)$ around 0 gives $f(\delta) = 0 + R_1(z)$ for $z \in [-\delta, \delta]$. To bound the error of the Taylor approximation we bound the derivative $f''(\delta)$

$$f''(\delta) = \frac{an((2^a - 1)a(\delta + 1)^a - (2^a - (\delta + 1)^a)((\delta + 1)^a - 1))}{(\delta + 1)^2 (2^a - (\delta + 1)^a)^2}$$

$$\leq \frac{an((2^a - 1)a(\delta + 1)^a)}{(\delta + 1)^2 (2^a - (\delta + 1)^a)^2}$$

$$\leq \frac{na^2(2^a - 1)(3/2)^a}{(2^a - (3/2)^a)^2}$$

$$\leq \frac{na^23^a}{(2^a/4)^2} = 16n a^2(3/4)^a \leq 105n$$

Therefore, $D_{kl}(P_{1,i}\|P_{3,i}) \leq |R_1(z)| \leq 105n \delta^2 \leq 105/N$ for all $i \in \mathbb{N}$. \qed