ON ZERO SETS IN FOCK SPACES

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Abstract. We prove that zero sets for distinct Fock spaces are not the same, this is an answer of a question asked by K. Zhu in [6, Page. 209].

1. Introduction and statement of main results

For $\alpha > 0$ and $p > 0$ the Fock space $F^p_\alpha$ consists of those entire functions $f$ satisfying

$$\|f\|_{p,\alpha}^p := \int_{\mathbb{C}} |f(z)|^p dA_{p\alpha/2}(z) < \infty,$$

where

$$dA_\beta(z) := \frac{\beta}{\pi} e^{-\beta|z|^2} dA(z), \quad \beta > 0,$$

and $A$ represents the Lebesgue area measure on the complex plane $\mathbb{C}$. It is known that the space $F^p_\alpha$ endowed with the norm $\| \cdot \|_{p,\alpha}$ is a Banach space when $p \geq 1$, while for $p < 1$ it is a complete metric space, see for instance [6, Chap. 2].

A sequence $\Lambda$ of complex numbers is called a zero set for $F^p_\alpha$ if there exists a function $f \in F^p_\alpha \setminus \{0\}$ such that the zero set $\{ z \in \mathbb{C} : f(z) = 0 \}$ of $f$, counting multiplicities, coincides with $\Lambda$. At the present time there is no complete characterization of zero sets for Fock spaces. In [5] and [6, Chap. 5] K. Zhu has presented many properties enjoyed by zero sets in $F^p_\alpha$, in particular he proved that the spaces $F^p_\alpha$ and $F^q_\beta$ always possess different zero sets in the case where $\alpha \neq \beta$, regardless of $p$ and $q$. He then asked whether this remains true if $\alpha = \beta$, see [6, Page. 209].

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In this paper, we answer positively to this question by considering a
special translation of simple lattices with a uniform density.

The upper and lower Beurling-Landau density of a sequence \( \Lambda \subset \mathbb{C} \)
is known respectively as the following
\[
D^{+}(\Lambda) := \limsup_{\rho \to \infty} \sup_{z \in \mathbb{C}} \frac{N_{\Lambda}(z, \rho)}{\pi \rho^2}
\]
and
\[
D^{-}(\Lambda) := \liminf_{\rho \to \infty} \inf_{z \in \mathbb{C}} \frac{N_{\Lambda}(z, \rho)}{\pi \rho^2},
\]
where \( N_{\Lambda}(z, \rho) \) is the number of the elements in the intersection of \( \Lambda \)
and the Euclidian open disk \( D(z, \rho) \) of center \( z \in \mathbb{C} \) and radius \( \rho > 0 \).

Our main result is the following theorem.

**Theorem 1.1.** Let \( p \) and \( q \) be two positive numbers such that \( p > q \).
There exists a sequence \( \Lambda \) in \( \mathbb{C} \) satisfying
\[
D^{+}(\Lambda) = D^{-}(\Lambda) = \frac{\alpha}{\pi},
\]
and such that \( \Lambda \) is a zero set for \( F^{p}_{\alpha} \) but it is not for \( F^{q}_{\alpha} \).

The condition (1.1) in Theorem 1.1 shows that our result is not based
on the characterization of sampling and interpolating sets for Fock
spaces, given by K. Seip and R. Wallstén [3, 4]. Indeed, a sequence
of the critical density \( \alpha/\pi \) is neither sampling nor interpolating for \( F^{p}_{\alpha} \).

2. Proof of Theorem 1.1

We start this section with some well known preliminaries. We con-
sider the following square lattice
\[
\Lambda := \{z_{m,n} := a(m + in) : m, n \in \mathbb{Z}\},
\]
where \( a \) is a positive number and \( \mathbb{Z} \) denotes the usual set of integers.
The imaginary axis is clearly a line of symmetry for \( \Lambda \). By translating
the positive real points of \( \Lambda \) away from 0 and keeping this symmetry
unchanged, we define the following modified lattice
\[
\Lambda_{R} := \{w_{m,n} : m, n \in \mathbb{Z}\},
\]
where $\mathbb{R}$ is a positive number and

$$w_{m,n} := \begin{cases} z_{m,n}, & \text{if } n \neq 0 \text{ or } m = n = 0, \\ a(m + Rm/|m|), & \text{if } n = 0 \text{ and } m \neq 0. \end{cases} \tag{2.1}$$

We observe that if $R$ is a positive integer, then $\Lambda_R$ is actually obtained from $\Lambda$ by just removing the following finite symmetric set $\{\pm am : m \in \{1, 2, ..., R\}\}$. The well known Weierstrass function associated to $\Lambda$ is defined by

$$\sigma_a(z) := z \prod_{(m,n) \in \mathbb{Z}^2, (m,n) \neq (0,0)} \left(1 - \frac{z}{w_{m,n}}\right) \exp\left(\frac{z}{w_{m,n}} + \frac{z^2}{2z^2_{m,n}}\right), \quad z \in \mathbb{C},$$

one can see the textbooks [1, 2]. The modified Weierstrass function associated to $\Lambda_R$, introduced by K. Seip, is given by

$$\sigma_{a,R}(z) := z \prod_{(m,n) \in \mathbb{Z}^2, (m,n) \neq (0,0)} \left(1 - \frac{z}{w_{m,n}}\right) \exp\left(\frac{z}{w_{m,n}} + \frac{z^2}{2z^2_{m,n}}\right), \quad z \in \mathbb{C},$$

see for instance [6, Chap. 4]. For $h_1$ and $h_2$ being two positive functions, we use the following notation $h_1 \lesssim h_2$ to mean that $h_1 \leq ch_2$ for some positive constant $c$. We also write $h_1 \asymp h_2$ if both $h_1 \lesssim h_2$ and $h_2 \lesssim h_1$. In the next section we give the proof of the following lemma.

**Lemma 2.1.** Let $\alpha$ be a positive number. We have

$$|\sigma_{a,R}(z)| e^{-\pi|z|^2} \lesssim \frac{d(z, \Lambda_R)}{(1 + |z|)^{2R}}, \quad z \in \mathbb{C}, \tag{2.2}$$

where $a = \sqrt{\pi/\alpha}$ and $R$ is a positive constant.

We now let $p$ and $q$ be two positive numbers such that $p > q$. We take a number $R$ satisfying $\frac{1}{p} < R < \frac{1}{q}$. By using (2.2),

$$|\sigma_{a,R}(z)|^p e^{-\frac{p}{2}|z|^2} \lesssim 1/|z|^{2pR-1}, \quad |z| \geq 1. \tag{2.3}$$

Since $2pR - 1 > 1$, we then obtain $\sigma_{a,R} \in \mathcal{F}^p_{\alpha}$, and hence $\Lambda_R$ is a zero set for $\mathcal{F}^p_{\alpha}$. A standard argument by contradiction shows that $\Lambda_R$ cannot be a zero set for $\mathcal{F}^q_{\alpha}$. For the sake of completeness, we sketch here the
proof. We suppose that there exists a function \( f \in \mathcal{F}_q \setminus \{0\} \) with zero set \( \Lambda_R \). By Hadamard’s factorization theorem, we have

\[
f(z) = ze^{Q(z)} \prod_{(m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \left(1 - \frac{z}{w_{m,n}}\right) \exp \left(\frac{z}{w_{m,n}} + \frac{z^2}{2w_{m,n}^2}\right), \quad z \in \mathbb{C},\]

where \( Q \) is a polynomial of degree at most 2. We also obviously have

\[
\sum_{m \geq 1} \frac{R(2m + R)}{m^2(m + R)^2} = M_R < +\infty.
\]

Thus, for \( z \in \mathbb{C} \),

\[
f(z) = e^{Q(z)} \sigma_{a,R}(z) \prod_{m \neq 0} \exp \left(\frac{1}{2w_m^2} - \frac{1}{2z_m^2}\right) z^2 \exp \left(Q(z) - \frac{M_R}{a^2} z^2\right)
\]

\[
= \sigma_{a,R}(z) e^{L(z)},
\]

where \( L \) is a polynomial of degree at most 2. Using (2.2) and (2.4)

\[
\|f\|_q^q = \frac{\alpha q}{2\pi} \int_{\mathbb{C}} |f(z)|^q e^{-\frac{\alpha q}{2}|z|^2} dA(z)
\]

\[
= \frac{\alpha q}{2\pi} \int_{\mathbb{C}} |\sigma_{a,R}^q(z)| e^{-\frac{\alpha q}{2}|z|^2} |e^{qL(z)}| dA(z)
\]

\[
\gtrsim \int_{|z| \geq \frac{\alpha}{8}} d^q(z, \Lambda_R) |z|^{-2Rq} |e^{qL(z)}| dA(z)
\]

\[
\gtrsim \int_{\mathbb{C} \setminus \mathbb{D}(\Lambda_R, a/8)} |z|^{-2Rq} |e^{qL(z)}| dA(z), \quad (2.5)
\]

where

\[
\mathbb{D}(\Lambda_R, a/8) := \bigcup_{\lambda \in \Lambda_R} \mathbb{D}(\lambda, a/8).
\]

For \( \lambda \in \Lambda_R \setminus \{0\} \) and a point \( w \in \mathbb{D}(\lambda, a/8) \), the subharmonicity of the function \( z \mapsto \phi(z) := |z|^{-2Rq} |e^{qL(z)}| \) in \( \mathbb{C} \setminus \{0\} \) gives

\[
\phi(w) \lesssim \int_{a/4 \leq |z-w| \leq 3a/8} \phi(z) dA(z) \leq \int_{\mathbb{D}(\lambda,a/2) \setminus \mathbb{D}(\lambda,a/8)} \phi(z) dA(z).
\]

It follows

\[
\int_{\mathbb{D}(\lambda,a/8)} \phi(w) dA(w) \lesssim \int_{\mathbb{D}(\lambda,a/2) \setminus \mathbb{D}(\lambda,a/8)} \phi(z) dA(z).
\]
Therefore
\[
\int_{\mathbb{D}(\Lambda_\mathcal{R},a/8)\setminus \mathbb{D}(0,a/8)} \phi(w) dA(w) \lesssim \int_{\mathbb{C}\setminus \mathbb{D}(\Lambda_\mathcal{R},a/8)} \phi(z) dA(z), \tag{2.6}
\]
since
\[
\mathbb{D}(\lambda_1, a/2) \cap \mathbb{D}(\lambda_2, a/2) = \emptyset, \quad \lambda_1 \neq \lambda_2.
\]
By combining (2.5) and (2.6)
\[
\int_{\mathbb{C}\setminus \mathbb{D}(0,a/8)} \phi(z) dA(z) \lesssim \|f\|^q. \tag{2.7}
\]
Using again the subharmonicity of \(\phi\) and taking account of (2.7), we deduce that \(\phi\) is bounded at \(\infty\), and hence \(z \mapsto e^{qL(z)}\) possesses a polynomial growth. Thus \(L\) is a constant and by consequence
\[
\int_{a/8}^{\infty} |z|^{-2Rq+1} d|z| \lesssim \|f\|^q. \tag{2.8}
\]
The inequality (2.8) is in contradiction with the fact that \(2Rq-1 < 1\). Hence \(\Lambda_\mathcal{R}\) is not a zero set for \(\mathcal{F}_q^\alpha\), which finishes the proof of Theorem 1.1.

3. Proof of Lemma 2.1

Let \(\alpha\) be a positive number and consider the lattice \(\Lambda\) generated by \(a = \sqrt{\pi/\alpha}\). By using the symmetry with respect to the imaginary axis enjoyed by the lattices \(\Lambda\) and \(\Lambda_\mathcal{R}\), we simply compute
\[
\sigma_{a,R}(z) = \sigma_a(z) \prod_{m \neq 0} \frac{1 - z/w_m}{1 - z/z_m} \frac{\exp(z/w_m)}{\exp(z/z_m)} = \sigma_a(z) \prod_{m \geq 1} \frac{1 - (z/w_m)^2}{1 - (z/z_m)^2}, \quad z \in \mathbb{C}\setminus a\mathbb{Z}, \tag{3.1}
\]
where \(w_m := w_{m,0}\) and \(z_m := z_{m,0}\). For proving Lemma 2.1, we claim that it is sufficient to show
\[
\psi_R(z) := \prod_{m \geq 1} \left| \frac{m + R - z}{m - z} \right| \frac{m}{m + R} \lesssim \frac{d(z, \mathbb{Z}^+_R)}{d(z, \mathbb{Z}^+) (1 + |z|)^R}, \quad z \in \mathbb{C}\setminus \mathbb{Z}^+, \tag{3.2}
\]
where \(\mathbb{Z}^+\) is the set of non-negative integers and \(\mathbb{Z}^+_R := \{m + R, \ m \in \mathbb{Z}^+\}\).
Indeed, assume that (3.2) holds. Then
\[
\psi_R(z/a) = \prod_{m \geq 1} \left| \frac{1 - (z/a(m + R))}{1 - (z/am)} \right| \asymp \frac{d(z, a\mathbb{Z}^+)}{d(z, a\mathbb{Z}^+(1 + |z|)^R}, \quad z \in \mathbb{C} \setminus a\mathbb{Z}^+,
\]

and
\[
\psi_R(-z/a) = \prod_{m \geq 1} \left| \frac{1 + (z/a(m + R))}{1 + (z/am)} \right| \asymp \frac{d(z, a\mathbb{Z}_R^-)}{d(z, a\mathbb{Z}_R^-(1 + |z|)^R}, \quad z \in \mathbb{C} \setminus a\mathbb{Z}^-,
\]

where \( \mathbb{Z}^- := -\mathbb{Z}^+ \) and \( \mathbb{Z}_R^- := -\mathbb{Z}_R^+ \). We clearly have
\[
d(z, a\mathbb{Z}_R^+) \times d(z, a\mathbb{Z}_R^-) \asymp d(z, \Lambda_R), \quad z \in \mathbb{C} \setminus a\mathbb{Z}.
\]

Thus
\[
\prod_{m \geq 1} \left| 1 - \frac{(z/a(m + R))}{1 - (z/am)} \right|^2 \asymp \frac{d(z, \Lambda_R)}{d(z, \Lambda)(1 + |z|)^2R}, \quad z \in \mathbb{C} \setminus a\mathbb{Z}. \tag{3.3}
\]

By using (3.1) and (3.3) we deduce
\[
|\sigma_{a,R}(z)| \asymp \frac{|\sigma_a(z)| d(z, \Lambda_R)}{d(z, \Lambda)(1 + |z|)^2R}, \quad z \in \mathbb{C} \setminus a\mathbb{Z}. \tag{3.4}
\]

On the other hand, it is known that
\[
|\sigma_a(z)| e^{-\frac{\beta}{2}|z|^2} \asymp d(z, \Lambda), \quad z \in \mathbb{C}, \tag{3.5}
\]

see for instance [6, Corollary 1.21]. Hence
\[
|\sigma_{a,R}(z)| e^{-\frac{\beta}{2}|z|^2} \asymp \frac{d(z, \Lambda_R)}{(1 + |z|)^2R}, \quad z \in \mathbb{C}, \tag{3.6}
\]

which proves Lemma 2.1.

Let us now prove (3.2). For this aim, it is sufficient to consider only the situation when \([R]\), the integer part of \(R\), equals zero. Indeed, we fix a number \(R > 1\). We can factorize \(\psi_R\) as follows
\[
\psi_R(z) = \psi_\beta(z - [R]) \prod_{m=1}^{[R]} \frac{m + \beta}{m - z}, \quad z \in \mathbb{C} \setminus \mathbb{Z}^+,
\]

where
\[
\| \psi_\beta(z - [R]) \prod_{m=1}^{[R]} \frac{1}{m - z} | \|_{L^1(\mathbb{C} \setminus \mathbb{Z}^+)} \leq C |R|^{1/2}.
\]

Thus
\[
|\psi_R(z)| \asymp |\psi_\beta(z - [R])| \prod_{m=1}^{[R]} \frac{|1|}{|m - z|}, \quad z \in \mathbb{C} \setminus \mathbb{Z}^+,
\]

which proves Lemma 2.1.
where $\beta := R - [R]$. We have
\[
\prod_{m=1}^{[R]} \frac{1}{|m - z|} \asymp \frac{1}{d(z, \{1, 2, \ldots, [R]\})(1 + |z|)^{[R]-1}}, \quad z \in \mathbb{C} \setminus \{1, 2, \ldots, [R]\}.
\]
If we show
\[
\psi_\beta(z) \asymp \frac{d(z, \mathbb{Z}_\beta^+)}{d(z, \mathbb{Z}^+)(1 + |z|)^\beta}, \quad z \in \mathbb{C} \setminus \mathbb{Z}^+,
\]
then
\[
\psi_\beta(z - [R]) \asymp \frac{d(z, \mathbb{Z}_\beta^+)_{[R]}}{d(z, \mathbb{Z}^+)(1 + |z|)^\beta}, \quad z \in \mathbb{C} \setminus \mathbb{Z}^+_\beta.[R].
\]
Therefore
\[
\psi_R(z) \asymp \frac{d(z, \mathbb{Z}_R^+)}{d(z, \mathbb{Z}^+)(1 + |z|)^R}, \quad z \in \mathbb{C} \setminus \mathbb{Z}^+,
\]
which proves (3.2). So, in the sequel we suppose that $[R] = 0$. We now set
\[
\mathbb{N}_z := \{m_z - 2, m_z - 1, m_z\} \cap \mathbb{Z}^+,
\]
where
\[
m_z := \min\{m \in \mathbb{Z}^+ : m - x \geq 0\},
\]
and $x =: \text{Re}(z)$ is the real part of $z$. Since $\mathbb{N}_z$ contains at most three elements, we obviously get
\[
\prod_{m \in \mathbb{N}_z} \frac{m}{m + R} \asymp 1, \quad z \in \mathbb{C}.
\]
If $x \leq 1$ we then obtain $m_z = 1$, $d(z, \mathbb{Z}^+) = |1 - z|$ and $d(z, \mathbb{Z}_R^+) = |1 + R - z|$, and if $x > 1$ then $m_z \geq 2$, $d(z, \mathbb{Z}^+) = \min\{|m_z - z|, |m_z - 1 - z|\} = d(z, \mathbb{N}_z)$ and $d(z, \mathbb{Z}_R^+) = d(z - R, \mathbb{N}_z)$. Thus
\[
\prod_{m \in \mathbb{N}_z} \left| \frac{m + R - z}{m - z} \right| \frac{m}{m + R} \asymp \frac{d(z, \mathbb{Z}_R^+)_{\mathbb{Z}^+}}{d(z, \mathbb{Z}^+)^R}, \quad z \in \mathbb{C} \setminus \mathbb{Z}^+.
\]
(3.7)
Taking account of (3.7), for proving (3.2) it remains to show
\[
\prod_{m \in \mathbb{Z}^+ \setminus \mathbb{N}_z} \left| \frac{m + R - z}{m - z} \right| \frac{m}{m + R} \asymp (1 + |z|)^{-R}, \quad z \in \mathbb{C},
\]
for which it is necessary and sufficient to show that

\[ \prod_{m \in \mathbb{Z}^+ \setminus N_z} \left| \frac{m + R - z}{m - z} \right| \frac{m}{m + R} \asymp (1 + |z|)^{-R}, \quad |z| \to \infty. \quad (3.8) \]

We set

\[ \varphi_1(z) := \prod_{m > 2|z|} \left| \frac{m + R - z}{m - z} \right| \frac{m}{m + R}, \quad z \in \mathbb{C}. \]

If \( m > 2|z| \) then \( |m + R - z| \geq |m - z| > m/2 \), and by using the following usual inequality

\[ \log(1 + u) \leq u, \quad u \geq 0, \]

we compute

\[
\begin{align*}
|\log \varphi_1(z)| & \leq \sum_{m > 2|z|} \left| \log \left| \frac{1 - z/(m + R)}{1 - z/m} \right| \right| \\
& = \sum_{m > 2|z|} \max \left\{ \log \left| \frac{1 - z/(m + R)}{1 - z/m} \right|, \log \left| \frac{1 - z/m}{1 - z/(m + R)} \right| \right\} \\
& \leq \sum_{m > 2|z|} \max \left\{ \left| \frac{Rz}{(m + R)(m - z)} \right|, \left| \frac{Rz}{m(m + R - z)} \right| \right\} \\
& \leq 2R|z| \sum_{m > 2|z|} \frac{1}{m^2} = O(1).
\end{align*}
\]

We then deduce

\[ \varphi_1(z) \asymp 1, \quad z \in \mathbb{C}. \quad (3.9) \]

With (3.9) in mind, for proving (3.8) it remains now to show

\[ \varphi_2(z) := \prod_{m \in \mathbb{M}_z} \left| \frac{m + R - z}{m - z} \right| \frac{m}{m + R} \asymp (1 + |z|)^{-R}, \quad |z| \to \infty, \]

where

\[ \mathbb{M}_z := \{ m \in \mathbb{Z}^+ \setminus N_z : m \leq 2|z| \}. \]

We recall the following classical equality

\[ \log |1 + u| = \text{Re}(u) + O(|u|^2), \quad \text{Re}(u) \geq -\frac{1}{2} \text{ and } |u| \leq 1. \quad (3.11) \]
By using (3.11),
\[ \sum_{m \in M} \log \frac{m}{m + R} = - \sum_{m \in M} \log \left(1 + \frac{R}{m}\right) = - \sum_{m \in M} \frac{R}{m} + O(1) = -R \log(|z|) + O(1), \]
which gives
\[ \prod_{m \in M} \frac{m}{m + R} \asymp |z|^{-R} \asymp (1 + |z|)^{-R}, \quad |z| \to \infty. \tag{3.12} \]
We have
\[ \sum_{m \in M} \left| \frac{R}{m - z} \right|^2 \leq \sum_{m \in M} \frac{1}{|m - x|^2} \leq 2 \sum_{m \geq 1} \frac{1}{m^2} < \infty, \]
and since
\[ -\frac{1}{2} \leq \text{Re}(\frac{R}{m - z}) \quad \text{and} \quad \frac{R}{|m - z|} \leq 1, \quad m \in M_z, \]
then, by using again (3.11),
\[ \sum_{m \in M_z} \log \left| 1 + \frac{R}{m - z} \right| = R \sum_{m \in M_z} \frac{m - x}{(m - x)^2 + y^2} + O(1), \tag{3.13} \]
where \( y \) is the imaginary part of \( z \). For \(|z| \geq 3/2\),
\[ M^+_z := \{ m \in M : m - x \geq 0 \} = \{ m_z + 1, m_z + 2, \ldots, [2|z|] \} \neq \emptyset, \]
and by a simple calculation
\[ \sum_{m \in M^+_z} \frac{m - x}{(m - x)^2 + y^2} = \frac{1}{2} \log \frac{([2|z|] - x)^2 + y^2}{(m_z + 1 - x)^2 + y^2} + O(1) \]
\[ = \begin{cases} O(1), & \text{if } x \leq 0, \\ \log \frac{|z|}{1 + |y|} + O(1), & \text{if } x > 0. \end{cases} \tag{3.14} \]
We need to distinguish between two different cases. In the case where \( x \leq 3 \), we obtain \( m_z \leq 3 \) and hence \( M^+_z = M_z \). In this case, we either have \( x \leq 0 \) or \( |z| \sim |y| \), for \( |z| \geq 3/2 \). In both situations we deduce the
desired estimate (3.10) by combining the estimates (3.12), (3.13) and (3.14). In the case where \( x > 3 \), we obtain \( m_z \geq 4 \) and by consequence

\[
M_z \setminus M_z^+ = \{1, 2, \ldots, m_z - 3\} \neq \emptyset.
\]

In this case,

\[
\sum_{m \in M_z \setminus M_z^+} \frac{m - x}{(m - x)^2 + y^2} = \frac{1}{2} \log \frac{(m_z - 3 - x)^2 + y^2}{(1 - x)^2 + y^2} + O(1)
\]

\[
= \log \frac{1 + |y|}{|z|} + O(1).
\]  

(3.15)

We again deduce (3.10) by joining together (3.12), (3.13), (3.14) and (3.15). The proof of Lemma 2.1 is completed.

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