P-Laplacian Dirac system on time scales

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ABSTRACT
The p-Laplacian type Dirac systems are nonlinear generalizations of the classical Dirac systems. They can be observed as a bridge between nonlinear systems and linear systems. The purpose of this study is to consider p-Laplacian Dirac boundary value problem on an arbitrary time scale to get forceful results by examining some spectral properties of this problem on time scales. Interesting enough, the p-Laplacian type Dirac boundary value problem exhibits the classical Dirac problem on time scales. Moreover, we prove Picone’s identity for p-Laplacian type Dirac system which is an important tool to prove oscillation criteria on time scales. It generalizes a classical and well-known theorem for $p = 2$ to general case $p > 1$.

1. Introduction
The theory of time scales was introduced by Stephan Hilger in his Ph.D. Thesis in 1988 [1]. A time scale is an arbitrary, non-empty, closed subset of real numbers which is denoted by the symbol $\mathbb{T}$ in the literature. Since the study of dynamic systems on time scales not only unifies continuous and discrete processes but also helps in revealing diversities in the corresponding results, it is an active area of research. The time scale calculus theory can be applicable to any field in which dynamic processes are described by discrete or continuous time models.

The significance of studying boundary value problems on time scales was understood and some important results were obtained within a short time. In 2002, Agarwal et al. [2] studied linear dynamic equations and initial value problems on time scales. Guseinov and Kaymakçalan [3] considered second-order linear dynamic equations and gave some sufficient conditions for non-oscillation on time scales in 2002. Erbe et al. [4] gave several comparison theorems for second-order linear dynamic equation on time scales within the same year. In 2003, Guseinov [5] investigated Riemann and Lebesgue integration on time scales and gave the fundamental theorems of calculus. In 2007, Rynne [6] defined space of square-integrable functions and Sobolev-type spaces on time scales and gave practices of the functional analytic results to Sturm–Liouville type boundary value problem. Grace et al. [7] studied second-order half-linear dynamic equations on time scales and investigated some oscillation criteria in 2009.

In recent years, several authors have obtained many important results about p-Laplacian boundary value problems on time scales [8–11]. Anderson et al. [12] proved that there is at least one positive solution for a one-dimensional p-Laplacian delta-nabla dynamic equations on time scales in 2004. He [13] proved that there exists at least double positive solution with three-point boundary conditions for p-Laplacian dynamic equation on time scales in 2005. Sun and Li [14] considered for a one-dimensional p-Laplacian boundary value problem and showed the existence of this problem on time scales in 2007. Binding et al. [15] considered p-Laplacian nonlinear eigenvalue problem on a half line in 2010. Su and Feng [16] studied a second-order p-Laplacian dynamic boundary value problem on a periodic time scale and showed existence of solutions of this problem in 2017.

We find helpful to introduce the following introductory information for a reader not familiar with the time scale theory. Forward and backward jump operators at $t \in \mathbb{T}$, for $t < \sup \mathbb{T}$ are defined as

$$\sigma(t) = \inf (t, \infty) \cap \mathbb{T}, \quad \rho(t) = \sup (-\infty, t) \cap \mathbb{T},$$

respectively (supplemented by $\inf \phi = \sup \mathbb{T}$ and $\sup \phi = \inf \mathbb{T}$, where $\phi$ denotes the empty set). Also $t$ is said to be left dense, left scattered, right dense and right scattered if $\rho(t) = t, \rho(t) < t, \sigma(t) = t$ and $\sigma(t) > t$, respectively. The forward graininess function $\mu : \mathbb{T} \longrightarrow \mathbb{R}^+_{\infty}$ is defined to be

$$\mu(t) = \sigma(t) - t,$$

for all $t \in \mathbb{T}$. A closed interval on $\mathbb{T}$ is denoted by

$$[a, b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \leq t \leq b\},$$

where $a$ and $b$ are fixed points of $\mathbb{T}$ with $a < b$.

We also need to explain the set $\mathbb{T}^x$ which is derived from $\mathbb{T}$ to define $\Delta$-derivative of a function. If $\mathbb{T}$ has left

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scattered maximum \( m \), then \( T^x = T - \{ m \} \). Otherwise, \( T^x = T \) \(^{[17]}\). Assume \( y : T \to \mathbb{R} \) is a function and let \( t \in T^x \). The function \( y : T \to \mathbb{R} \) is said to be Hilger differentiable at the point \( t \in T^x \), if there is a neighbourhood \( U = (t - \delta, t + \delta) \cap T \) for some \( \delta > 0 \) such that

\[
\left| [y(\sigma(t)) - y(s)] - y^\Delta(t) [\sigma(t) - s] \right| \leq \varepsilon |\sigma(t) - s|,
\]

for all \( s \in U \). We call \( y^\Delta(t) \) as \( \Delta \) or Hilger derivative of \( y \) at \( t \). A function \( y \) is called \( rd \)-continuous provided that is continuous at right dense points in \( T \) and has finite limit at left dense points. The set of all \( rd \)-continuous functions on \( T \) is denoted by \( C_{rd}(T, \mathbb{R}) \). The set of functions that are \( \Delta \)-differentiable and whose \( \Delta \)-derivative is \( rd \)-continuous on \( T \) is denoted by \( C_{rd}^\Delta(T, \mathbb{R}) \). However, \( y \) called regulated provided its right-sided limits exist (finite) at all right dense points in \( T \) and its left-sided limits exist (finite) at all left dense points in \( T \). Let \( y \) be a regulated function on \( T \). The indefinite \( \Delta \)-integral of a regulated function \( y \) is denoted to be

\[
\int y(t) \, \Delta t = Y(t) + C,
\]

where \( C \) is an arbitrary constant and \( Y^\Delta = y \) on \( T^x \). Finally, definite \( \Delta \)-integral of \( y \) is defined by

\[
\int_r^s y(t) \, \Delta t = Y(s) - Y(r),
\]

for all \( r, s \in T \). For standard notions and notations connected to time scale calculus, we refer to \([17]\).

In this study, we want to prove some fundamental spectral properties of \( p \)-Laplacian Dirac eigenvalue problem of the form

\[
Ly^{(p-1)} = By^\Delta y^{(p-1)}(t) + (p - 1)Q(t) y^{\sigma(p-1)}(t) = \lambda (p - 1) y^{\sigma(p-1)}(t), t \in [\mu(a), b] \cap T, \quad (1)
\]

with the separated boundary conditions

\[
\alpha y_1^{(p-1)}(\rho(a)) + \beta y_2^{\sigma(p-1)}(\rho(a)) = 0, \quad (2)
\]

\[
y_1^{(p-1)}(b) + \delta y_2^{\sigma(p-1)}(b) = 0, \quad (3)
\]

where

\[
Q(t) = \begin{pmatrix} q(t) & 0 \\ 0 & r(t) \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\]

and \( \lambda \) is a spectral parameter. Throughout this study, we will assume that \( q, r \in L^1_{\mu}(\mu(a), b] \) are continuous functions; \( a, b \in T \) with \( a < b \), \( p > 1 \), \( y^{\sigma(p-1)} = y^{(p-1)}(\sigma) \) and \( (\alpha^2 + \beta^2)(r^2 + \delta^2) \neq 0 \), \( y^{(p-1)}(t) = (y_1^{(p-1)}(t), y_2^{(p-1)}(t))^T \in C_{\mu}(\mu(a), b], \mathbb{R}) \) is known as eigenfunction of the problem \((1)-(3)\) where \( T \) denotes transpose. After some straightforward computations in \((1)\), one can easily get the following \( p \)-Laplacian Dirac system:

\[
\begin{align*}
\left( y_1^{(p-1)} \right)^\Delta &= (p - 1)(\lambda - q(t)) y_1^{\sigma(p-1)}, \\
\left( y_2^{(p-1)} \right)^\Delta &= (p - 1)(\lambda - r(t)) y_2^{\sigma(p-1)}. \quad (4)
\end{align*}
\]

By taking \( T = \mathbb{R} \) in \((4)\), we obtain classical \( p \)-Laplacian Dirac system as

\[
\begin{align*}
\left( y_1^{(p-1)} \right)' &= (p - 1)(\lambda - q(t)) y_1^{(p-1)}, \\
\left( y_2^{(p-1)} \right)' &= (p - 1)(\lambda - r(t)) y_2^{(p-1)}. \quad (5)
\end{align*}
\]

The system \((5)\) is known as the first canonic form of the \( p \)-Laplacian Dirac system in the literature. Now, we need to give some information about the historical development and physical meaning of classical Dirac system when \( p = 2 \).

Dirac operator which is a modern presentation of the relativistic quantum mechanics of electrons aiming to bring a new mathematical result to a wider audience is the relativistic Schrödinger operator in quantum physics \([18-23]\). There are many studies in spectral theory related to different versions of Dirac system \([24-34]\).

In \([35]\), \( p \)-Laplacian Dirac system was first considered and obtained crucial asymptotic formulas of eigenvalues and eigenfunctions by using Prüfer substitution for classical case. In \([36,37]\), Gulsen et al. took into consideration classical Dirac and conformable fractional Dirac eigenvalue problems, respectively, and showed some spectral properties of these problems on time scales. In this study, our purpose is to generalize the obtained results for classical \( p \)-Laplacian Dirac system to an arbitrary time scale.

This study is organized as follows. In Section 2, we prove some basic theorems and Picone’s identity for the first canonic form of the \( p \)-Laplacian Dirac system on \( T \). In Section 3, we give a conclusion to summarize our study.

### 2. Some spectral properties of \( p \)-Laplacian Dirac system on time scales

In this section, we give some valuable results for \( p \)-Laplacian Dirac system on a time scale \( T \) whose all points are right dense. It is well known that the problem \((1)-(3)\) has only real, simple eigenvalues and eigenfunctions are orthogonal to each other in case of \( T = \mathbb{R} \) \([38]\). The following results will generalize some basic results for \( p \)-Laplacian Dirac system.

**Theorem 2.1:** The eigenvalues of the problem \((1)-(3)\) are all simple.

**Proof:** Let \( \lambda, \mu \in \mathbb{R} \) be spectral parameters where \( \lambda \neq \mu \) and \( u^{(p-1)}(t) = (u_1^{(p-1)}(t), u_2^{(p-1)}(t))^T \) be
By multiplying Equations (6)–(9) by some arrangements are made, we have
\[
\begin{align*}
\left( u_{2}^{(p-1)} \right)^{\Delta} (t, \lambda) &= (p - 1)(\lambda - q(t))u_{1}^{\sigma(p-1)}(t, \lambda), \\
\left( u_{1}^{(p-1)} \right)^{\Delta} (t, \lambda) &= (p - 1)(-\lambda + r(t))u_{2}^{\sigma(p-1)}(t, \lambda), \\
\left( u_{2}^{(p-1)} \right)^{\Delta} (t, \mu) &= (p - 1)(\mu - q(t))u_{1}^{\sigma(p-1)}(t, \lambda), \\
\left( u_{1}^{(p-1)} \right)^{\Delta} (t, \mu) &= (p - 1)(-\mu + r(t))u_{2}^{\sigma(p-1)}(t, \lambda).
\end{align*}
\]

If both sides of Equation (16) are divided by \( \lambda - \mu \) and limit are taken as \( \mu \to \lambda \), it yields
\[
- (p - 1) \lim_{\mu \to \lambda} \left\{ u_{1}^{(p-1)}(t, \lambda)u_{1}^{\sigma(p-1)}(t, \mu) + u_{2}^{\sigma(p-1)}(t, \mu) \right\} = - \lim_{\mu \to \lambda} \left\{ u_{1}^{(p-1)}(t, \lambda)u_{1}^{\sigma(p-1)}(t, \mu) - u_{1}^{(p-1)}(t, \lambda)u_{2}^{\sigma(p-1)}(t, \mu) \right\} \Delta
\]
and
\[
- (p - 1) \left\{ \left[ u_{1}^{(p-1)}(t, \lambda) \right]^{2} + \left[ u_{2}^{\sigma(p-1)}(t, \lambda) \right]^{2} \right\} = \left\{ u_{2}^{\sigma(p-1)}(t, \lambda) \frac{\partial}{\partial \lambda} u_{1}^{(p-1)}(t, \lambda) \right\}.
\]

By applying \( \Delta \)-integral to (17) on \( \rho(a, b) \), we get
\[
- (p - 1) \int_{\rho(a)}^{b} \left\{ \left[ u_{1}^{(p-1)}(t, \lambda) \right]^{2} + \left[ u_{2}^{(p-1)}(t, \lambda) \right]^{2} \right\} \Delta t = u_{2}^{\sigma(p-1)}(b, \lambda) \frac{\partial}{\partial \lambda} u_{1}^{(p-1)}(b, \lambda) - u_{1}^{(p-1)}(b, \lambda) \frac{\partial}{\partial \lambda} u_{2}^{\sigma(p-1)}(b, \lambda).
\]

Therefore, \( \Lambda(\lambda) = -[\gamma u_{1}^{(p-1)}(b, \lambda) + \delta u_{2}^{\sigma(p-1)}(b, \lambda)] \) has only simple zeros. Conversely, assume that \( \lambda^{*} \) be double-decker root. Then,
\[
\gamma u_{1}^{(p-1)}(b, \lambda^{*}) + \delta u_{2}^{\sigma(p-1)}(b, \lambda^{*}) = 0
\]
\[
\gamma \frac{\partial}{\partial \lambda} u_{1}^{(p-1)}(b, \lambda^{*}) + \delta \frac{\partial}{\partial \lambda} u_{2}^{\sigma(p-1)}(b, \lambda^{*}) = 0
\]
\[
\Rightarrow u_{2}^{\sigma(p-1)}(b, \lambda^{*}) \frac{\partial}{\partial \lambda} u_{1}^{(p-1)}(b, \lambda^{*}) - u_{1}^{(p-1)}(b, \lambda^{*}) \frac{\partial}{\partial \lambda} u_{2}^{\sigma(p-1)}(b, \lambda^{*}) = 0.
\]

By (18) and (19), we obtain
\[
- (p - 1) \int_{\rho(a)}^{b} \left\{ \left[ u_{1}^{(p-1)}(t, \lambda) \right]^{2} + \left[ u_{2}^{(p-1)}(t, \lambda) \right]^{2} \right\} \Delta t = 0 \Rightarrow u_{1}^{(p-1)}(t, \lambda) = u_{2}^{\sigma(p-1)}(t, \lambda) = 0,
\]
for \( \lambda = \lambda^{*} \). This is a contradiction. So the proof is completed.

**Theorem 2.2:** Let \( u_{1}^{(p-1)} = (u_{1}^{(p-1)}, u_{2}^{(p-1)})^{T}, v_{1}^{(p-1)} = (v_{1}^{(p-1)}, v_{2}^{(p-1)})^{T} \in C_{p}^{2}(\rho(a, b), \mathbb{R}) \) be the eigenfunctions of the problem (1)–(3). Then, the followings are provided
(a) \[
\begin{align*}
( Lu^{(p-1)} )^T u^{(p-1)} &= ( L u^{(p-1)} )^T u^{(p-1)} \\
&= W^\Lambda ( u^{(p-1)}, v^{(p-1)} ),
\end{align*}
\]
on \{\rho(a), b\} \cap \mathbb{T}.

(b) \[
< ( Lu^{(p-1)} )^T, u^{(p-1)} > = < ( L u^{(p-1)} )^T, u^{(p-1)} >
\]
where \( W(u^{(p-1)}, v^{(p-1)}) = u_1^{\sigma(p-1)} v_1^{\sigma(p-1)} - u_2^{\sigma(p-1)} v_2^{\sigma(p-1)}. \)

**Proof:** (a) Definition of \( W \) and product rule for \( \Delta \) derivative give the proof as follows:

\[
W^\Lambda ( u^{(p-1)}, v^{(p-1)} ) = u_2^{\Lambda(p-1)} v_1^{\sigma(p-1)}
\]
\[
+ u_2^{\sigma(p-1)} v_2^{\Lambda(p-1)} - u_1^{\sigma(p-1)} v_2^{\Lambda(p-1)}
\]
\[
= v_1^{\sigma(p-1)} \left( u_2^{\Lambda(p-1)} + (p-1) q(t) u_1^{\sigma(p-1)} \right)
\]
\[
- u_2^{\sigma(p-1)} \left( v_1^{\Delta(p-1)} + (p-1) r(t) u_2^{\sigma(p-1)} \right)
\]
\[
+ v_2^{\sigma(p-1)} \left( u_1^{\Delta(p-1)} + (p-1) r(t) u_2^{\sigma(p-1)} \right)
\]
\[
- u_1^{\sigma(p-1)} \left( v_2^{\Delta(p-1)} + (p-1) q(t) v_1^{\sigma(p-1)} \right)
\]
\[
= ( Lu^{(p-1)} )^T u^{(p-1)} - ( L u^{(p-1)} )^T u^{(p-1)}. \]

(b) By using definition of \( W \) and inner product on \( L^2_\mathbb{T}(\rho(a), b) \) we have

\[
< ( Lu^{(p-1)} )^T, u^{(p-1)} > = \int_{\rho(a)}^{b} \left[ ( Lu^{(p-1)} )^T u^{(p-1)} \right] \Delta t
\]
\[
= \int_{\rho(a)}^{b} \left\{ ( Bu^{(p-1)} + (p-1) Qu^{(p-1)} )^T u^{(p-1)} \right\} \Delta t
\]
\[
= \int_{\rho(a)}^{b} \left\{ u_2^{\Delta(p-1)} v_1^{\sigma(p-1)} + u_2^{\sigma(p-1)} v_2^{\Delta(p-1)} \right\} \Delta t
\]
\[
- \int_{\rho(a)}^{b} \left\{ u_1^{\Delta(p-1)} v_2^{\sigma(p-1)} - u_1^{\sigma(p-1)} v_2^{\Delta(p-1)} \right\} \Delta t
\]
\[
= \int_{\rho(a)}^{b} \left\{ W(u^{(p-1)}, v^{(p-1)}) \right\} \Delta t
\]
\[
= W(u^{(p-1)}, v^{(p-1)})(b) - W(u^{(p-1)}, v^{(p-1)})(\rho(a)).
\]

Hence, this completes the proof. These are known as \( p \)-Lagrange’s identity and \( p \)-Green’s formula for \( p \)-Laplacian Dirac system, respectively.

**Theorem 2.3:** The eigenfunctions

\[
y^{(p-1)}(t, \lambda_1) = \left( y_1^{(p-1)}(t, \lambda_1), y_2^{(p-1)}(t, \lambda_1) \right)^T
\]
and

\[
z^{(p-1)}(t, \lambda_2) = \left( z_1^{(p-1)}(t, \lambda_2), z_2^{(p-1)}(t, \lambda_2) \right)^T
\]
of the problem (1)–(3) corresponding to distinct eigenvalues \( \lambda_1 \) and \( \lambda_2 \) are orthogonal, i.e

\[
(p-1) \int_{\rho(a)}^{b} ( y^{(p-1)}(t, \lambda_1) )^T z^{(p-1)}(t, \lambda_2) \Delta t = 0.
\]

**Proof:** We have

\[
y_2^{(p-1)} \Delta (t, \lambda_1) = (p-1)(\lambda_1 - q(t)) y_1^{(p-1)}(t, \lambda_1),
\]
\[
y_1^{(p-1)} \Delta (t, \lambda_1) = (p-1)(-\lambda_1 + r(t)) y_2^{(p-1)}(t, \lambda_1),
\]
\[
z_2^{(p-1)} \Delta (t, \lambda_2) = (p-1)(\lambda_2 - q(t)) z_1^{(p-1)}(t, \lambda_2),
\]
\[
z_1^{(p-1)} \Delta (t, \lambda_2) = (p-1)(\lambda_2 + r(t)) z_2^{(p-1)}(t, \lambda_2).
\]

Multiplying above equations by \( z_1^{(p-1)}(t, \lambda_2), z_2^{(p-1)}(t, \lambda_2), y_1^{(p-1)}(t, \lambda_1), y_2^{(p-1)}(t, \lambda_1) \), respectively, we get

\[
(y_2^{(p-1)})^\Delta (t, \lambda_1) z_1^{(p-1)}(t, \lambda_2)
\]
\[
= (p-1)(\lambda_1 - q(t)) y_1^{(p-1)}(t, \lambda_1) z_1^{(p-1)}(t, \lambda_2),
\]
(20)

\[
(y_1^{(p-1)})^\Delta (t, \lambda_1) z_2^{(p-1)}(t, \lambda_2)
\]
\[
= (p-1)(\lambda_1 - r(t)) y_2^{(p-1)}(t, \lambda_1) z_2^{(p-1)}(t, \lambda_2),
\]
(21)

\[
(z_2^{(p-1)})^\Delta (t, \lambda_2) y_1^{(p-1)}(t, \lambda_1)
\]
\[
= (p-1)(\lambda_2 - q(t)) z_1^{(p-1)}(t, \lambda_2) y_1^{(p-1)}(t, \lambda_1),
\]
(22)

\[
(z_1^{(p-1)})^\Delta (t, \lambda_2) y_2^{(p-1)}(t, \lambda_1)
\]
\[
= (p-1)(\lambda_2 + r(t)) z_2^{(p-1)}(t, \lambda_2) y_2^{(p-1)}(t, \lambda_1).
\]
(23)

Subtracting (20), (22) and (21), (23) leads to

\[
(y_2^{(p-1)})^\Delta (t, \lambda_1) z_1^{(p-1)}(t, \lambda_1)
\]
\[
= (p-1)(\lambda_2 - \lambda_1) y_1^{(p-1)}(t, \lambda_1) z_1^{(p-1)}(t, \lambda_2)
\]
(24)

\[
(y_1^{(p-1)})^\Delta (t, \lambda_1) z_2^{(p-1)}(t, \lambda_2)
\]
\[
= (p-1)(\lambda_2 - \lambda_1) y_2^{(p-1)}(t, \lambda_1) z_2^{(p-1)}(t, \lambda_2)
\]
The equality
\[ (y_1^{(p-1)})^\Delta(t,\lambda_1)z_2^{(p-1)}(t,\lambda_2) = -(z_1^{(p-1)})^\Delta(t,\lambda_2) y_2^{(p-1)}(t,\lambda_1) \]
\[ = (p-1)(\lambda_2 - \lambda_1) y_2^{(p-1)}(t,\lambda_1) z_2^{(p-1)}(t,\lambda_2). \]  
(25)

If Equations (24) and (25) are added side by side and some computations are made, we have
\[
\begin{align*}
&\left[ y_1^{(p-1)}(t,\lambda_1) z_2^{(p-1)}(t,\lambda_2) \\
&- y_2^{(p-1)}(t,\lambda_1) z_1^{(p-1)}(t,\lambda_2) \right] \Delta t \\
&\begin{cases}
= (p-1)(\lambda_2 - \lambda_1) y_1^{(p-1)}(t,\lambda_1) z_1^{(p-1)}(t,\lambda_2) \\
+ y_2^{(p-1)}(t,\lambda_1) z_2^{(p-1)}(t,\lambda_2).
\end{cases}
\end{align*}
\]  
(26)

Taking $\Delta$-integral of the last equality from $\rho(a)$ to $b$, we get
\[
(p-1)(\lambda_2 - \lambda_1) \int_{\rho(a)}^b \left[ y_1^{(p-1)}(t,\lambda_1) z_1^{(p-1)}(t,\lambda_2) \\
+ y_2^{(p-1)}(t,\lambda_1) z_2^{(p-1)}(t,\lambda_2) \right] \Delta t
= y_1^{(p-1)}(b) z_2^{(p-1)}(b) - y_2^{(p-1)}(b) z_1^{(p-1)}(b) \\
- y_1^{(p-1)}(\rho(a)) z_2^{(p-1)}(\rho(a)) \\
+ y_2^{(p-1)}(\rho(a)) z_1^{(p-1)}(\rho(a))
= 0,
\]
and then,
\[
(p-1) \int_{\rho(a)}^b \left[ y_1^{(p-1)}(t,\lambda_1) z_1^{(p-1)}(t,\lambda_2) \\
+ y_2^{(p-1)}(t,\lambda_1) z_2^{(p-1)}(t,\lambda_2) \right] \Delta t
= 0,
\]
for $\lambda_1 \neq \lambda_2$. This shows that the eigenfunctions $y^{(p-1)}(t,\lambda_1)$ and $z^{(p-1)}(t,\lambda_2)$ corresponding to distinct eigenvalues are always orthogonal.

**Theorem 2.4:** The equality
\[
\begin{align*}
&u_2^{(p-1)}(t,\lambda) \frac{\partial}{\partial \lambda} u_2^{(p-1)}(t,\lambda) - u_1^{(p-1)}(t,\lambda) \frac{\partial}{\partial \lambda} u_1^{(p-1)}(t,\lambda) \\
&= -(p-1) \int_{\rho(a)}^t \left[ u_1^{(p-1)}(\tau,\lambda) \right]^2 + \left[ u_2^{(p-1)}(\tau,\lambda) \right]^2 \Delta \tau,
\end{align*}
\]
holds for the problem (1)--(3), where $t \in [\rho(a), b] \cap \mathbb{T}$ and $\lambda \in \mathbb{R}$.  

**Proof:** Let $v, \lambda \in \mathbb{R}$ with $v \neq \lambda$. Then,
\[
\begin{align*}
&\left[ u_1^{(p-1)}(t,\lambda) u_2^{(p-1)}(t,\lambda) - u_1^{(p-1)}(t,\lambda) u_2^{(p-1)}(t,\lambda) \right] \Delta t \\
&= (p-1) \left( v - \lambda \right) \left[ u_1^{(p-1)}(t,\lambda) u_2^{(p-1)}(t,\lambda) \right] \\
&+ u_1^{(p-1)}(t,\lambda) u_2^{(p-1)}(t,\lambda).
\end{align*}
\]
Dividing both sides of above equality by $\lambda - v$ and taking limit as $v \to \lambda$, we have
\[
\lim_{v \to \lambda} \left[ u_1^{(p-1)}(t,\lambda) u_2^{(p-1)}(t,\lambda) - u_1^{(p-1)}(t,\lambda) u_2^{(p-1)}(t,\lambda) \right] \Delta t \\
= -(p-1) \lim_{v \to \lambda} \left[ u_1^{(p-1)}(t,\lambda) u_2^{(p-1)}(t,\lambda) \right] \\
+ u_2^{(p-1)}(t,\lambda) u_2^{(p-1)}(t,\lambda)
\]
and
\[
\begin{align*}
&\left[ u_2^{(p-1)}(t,\lambda) \frac{\partial}{\partial \lambda} u_1^{(p-1)}(t,\lambda) \\
&- u_1^{(p-1)}(t,\lambda) \frac{\partial}{\partial \lambda} u_2^{(p-1)}(t,\lambda) \right] \Delta t \\
&= -(p-1) \left[ u_1^{(p-1)}(t,\lambda)^2 + u_2^{(p-1)}(t,\lambda)^2 \right].
\end{align*}
\]
By taking $\Delta$-integral of the last equality from $\rho(a)$ to $t$, we get
\[
\begin{align*}
&\int_{\rho(a)}^t \left[ u_2^{(p-1)}(\tau,\lambda) \frac{\partial}{\partial \lambda} u_1^{(p-1)}(\tau,\lambda) \\
&- u_1^{(p-1)}(\tau,\lambda) \frac{\partial}{\partial \lambda} u_2^{(p-1)}(\tau,\lambda) \right] \Delta \tau \\
&= -(p-1) \int_{\rho(a)}^t \left[ u_1^{(p-1)}(\tau,\lambda)^2 + u_2^{(p-1)}(\tau,\lambda)^2 \right] \Delta \tau.
\end{align*}
\]
Since $u_1^{(p-1)}(\rho(a),\lambda) = \beta$ and $u_2^{(p-1)}(\rho(a),\lambda) = -\alpha$, it yields
\[
\frac{\partial}{\partial \lambda} u_1^{(p-1)}(\rho(a),\lambda) = 0 \text{ and } \frac{\partial}{\partial \lambda} u_2^{(p-1)}(\rho(a),\lambda) = 0.
\]
Finally, after some computations, we obtain
\[
\begin{align*}
&u_2^{(p-1)}(t,\lambda) \frac{\partial}{\partial \lambda} u_1^{(p-1)}(t,\lambda) - u_1^{(p-1)}(t,\lambda) \frac{\partial}{\partial \lambda} u_2^{(p-1)}(t,\lambda) \\
&= -(p-1) \int_{\rho(a)}^t \left[ u_1^{(p-1)}(\tau,\lambda)^2 + u_2^{(p-1)}(\tau,\lambda)^2 \right] \Delta \tau.
\end{align*}
\]
So the proof is completed.

\[ \square \]
Theorem 2.5: All eigenvalues of the problem (1)–(3) are real.

Proof: Let $\lambda_0$ be a complex eigenvalue and $\bar{u}^{(p-1)}(t) = (\bar{u}_1^{(p-1)}(t), \bar{u}_2^{(p-1)}(t))^T$ be an eigenfunction corresponding to the eigenvalue $\lambda_0$ of the problem (1)–(3). Then, we obtain

\[
\begin{align*}
\{u_1^{(p-1)}(t)\bar{u}_2^{(p-1)}(t) - u_1^{(p-1)}(t)u_2^{(p-1)}(t)\}^\Delta &= (p-1)(\lambda_0 - \lambda_0) \left(\left|u_1^{\sigma(p-1)}(t)\right|^2 + \left|u_2^{\sigma(p-1)}(t)\right|^2\right),
\end{align*}
\]

If we take $\Delta$-integral of the last equality from $\rho(a)$ to $b$, we get

\[
(p-1)(\lambda_0 - \lambda_0) \int_{\rho(a)}^b \left(\left|u_1^{\sigma(p-1)}(t)\right|^2 + \left|u_2^{\sigma(p-1)}(t)\right|^2\right) dt = u_1^{(p-1)}(b)\bar{u}_2^{(p-1)}(b) - u_1^{(p-1)}(a)\bar{u}_2^{(p-1)}(a) - u_1^{(p-1)}(a)u_2^{(p-1)}(a) + u_2^{(p-1)}(a)u_1^{(p-1)}(a)
\]

by considering the boundary conditions (2), (3). So, we have

\[
(p-1)\int_{\rho(a)}^b \left(\left|u_1^{\sigma(p-1)}(t)\right|^2 + \left|u_2^{\sigma(p-1)}(t)\right|^2\right) dt = 0
\]

\[
\Rightarrow u_1^{\sigma(p-1)}(t) = 0 \text{ and } u_2^{\sigma(p-1)}(t) = 0,
\]

for $\lambda_0 \neq \lambda_0$. This is a contradiction. Hence, all eigenvalues of the problem (1)–(3) are real. \(\blacksquare\)

Now, we give Picone’s identity for the problem (1)–(3) on time scales which is an important formula to prove oscillation criteria. There are many studies about the Picone’s identity in the literature [39,40]. In 1998, Allergetto and Xi [41] obtained a Picone’s identity for the $p$-Laplace operator and Bal [42] showed a generalized Picone’s identity for the $p$-Laplacian operator and then proved Sturmian comparison principle and a Liouville type theorem.

Theorem 2.6 (Picone’s Identity): Let $u^{(p-1)} = (u_1^{(p-1)}, u_2^{(p-1)}), v^{(p-1)} = (v_1^{(p-1)}, v_2^{(p-1)}) \in C_0^1([\rho(a), b], \mathbb{R})$ and $u, v$ be solutions of $L^{(p-1)} + \lambda(1-p)y^{\sigma(p-1)} = 0$. Then

\[
-\frac{u_1^{\sigma(p-1)}}{u_2^{(p-1)}} \left\{ (L^{(p-1)})^T u^{(p-1)} \right\} + \lambda(p-1) \left\{ u^{(p-1)} \right\}^\Delta = \left[ \frac{u_1^{(p-1)}}{u_2^{(p-1)}} W(u^{(p-1)}, v^{(p-1)}) \right]^\Delta
\]

Proof: Suppose that $(u_2^{(p-1)}u_2^{\sigma(p-1)}(t)) \neq 0$. Then from $p$-Lagrange’s identity, we acquire

\[
\begin{align*}
\left[ \frac{u_1^{(p-1)}}{u_2^{(p-1)}} W(u^{(p-1)}, v^{(p-1)}) \right]^\Delta &= \frac{u_1^{\sigma(p-1)}}{u_2^{(p-1)}} \times W^\Delta(u_{2}^{(p-1)}, v^{(p-1)}) \\
&= -\lambda(p-1) \left\{ u^{\sigma(p-1)} \right\}^T v^{\sigma(p-1)} \\
&= -\lambda(p-1) \left\{ u^{(p-1)} \right\}^T v^{(p-1)} \\
&= -\lambda(p-1) \left\{ u^{\sigma(p-1)} \right\}^T v^{\sigma(p-1)} \\
&= -\lambda(p-1) \left\{ u^{(p-1)} \right\}^T v^{(p-1)}
\end{align*}
\]
and the proof is completed. On the other hand, the formula
\[
\left[ u^{(p-1)}_1 \right] \Delta 
\]
\[
\left[ \frac{u^{(p-1)}_1}{u^{(p-1)}_2} W(u^{(p-1)}_1, v^{(p-1)}) \right] \\
= (p - 1) \left\{ (-\lambda + r) - (\lambda - q) \frac{u^{(p-1)}_1}{u^{(p-1)}_2} \right\} \\
\times W(u^{(p-1)}_1, v^{(p-1)})
\]
is also valid. Actually,
\[
\left[ \frac{u^{(p-1)}_1}{u^{(p-1)}_2} W(u^{(p-1)}_1, v^{(p-1)}) \right] \\
= \frac{u^{\sigma(p-1)}_1}{u^{\sigma(p-1)}_2} \Delta (u^{(p-1)}_1, v^{(p-1)}) \\
+ \left( \frac{u^{(p-1)}_1}{u^{(p-1)}_2} \sigma(p-1) \right) \\
\times W(u^{(p-1)}_1, v^{(p-1)})
\]
\[
= \frac{u^{\sigma(p-1)}_1}{u^{\sigma(p-1)}_2} \left[ -v^{\Delta(p-1)}_2 u^{\sigma(p-1)}_1 + v^{\Delta(p-1)}_1 u^{\sigma(p-1)}_2 \right] \\
+ \frac{u^{(p-1)}_1}{u^{(p-1)}_2} \left[ (-\lambda + r) v^{\sigma(p-1)}_1 u^{\sigma(p-1)}_1 - \lambda u^{(p-1)}_1 v^{\sigma(p-1)}_2 \right] \\
+ \left( \frac{u^{(p-1)}_1}{u^{(p-1)}_2} \sigma(p-1) \right) \\
\times W(u^{(p-1)}_1, v^{(p-1)})
\]
\[
= (p - 1) \left\{ \frac{u^{(p-1)}_1}{u^{(p-1)}_2} \right\} \\
\times \frac{u^{\Delta(p-1)}_2}{u^{\Delta(p-1)}_1} \left[ (-\lambda + r) v^{\sigma(p-1)}_1 u^{\sigma(p-1)}_1 - \lambda u^{(p-1)}_1 v^{\sigma(p-1)}_2 \right] \\
+ \left( \frac{u^{(p-1)}_1}{u^{(p-1)}_2} \sigma(p-1) \right) \\
\times W(u^{(p-1)}_1, v^{(p-1)})
\]
\[
= (p - 1) \left\{ \frac{u^{(p-1)}_1}{u^{(p-1)}_2} \right\} \\
\times W(u^{(p-1)}_1, v^{(p-1)})
\]
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