What makes quantum information science a science? This paper explores the idea that quantum information science may offer a powerful approach to the study of complex quantum systems.

1 Introduction

My subject in this paper is using quantum information science as an approach to the study of complex quantum systems. The work I describe has involved many collaborators at the University of Queensland and MIT, but I would especially like to emphasize the contribution of Tobias Osborne.

Let me begin by asking what it is that makes quantum information science a science? Friends outside the field sometimes comment that it seems to be largely engineering, with little science. A standard response from physicists is that in the course of building devices like quantum computers, we’ll discover lots of interesting physics. This is undoubtedly true, and is an excellent reason for doing quantum information science. But it seems a little like the argument sometimes used to justify going to the moon, namely, that it resulted in valuable spin-off technologies in fields such as computation and aeronautical engineering. This misses a large part of the point, since going to the moon has an intrinsic worth, a point obvious even to a small child.

What is the intrinsic worth of quantum information science? In this paper I argue, that quantum information science is a powerful approach to the study of complex quantum systems. Related ideas have been advocated previously by many people. Aharonov, Nielsen, and Preskill argued that there may be connections between the quantitative theory of entanglement and many-body quantum systems, and there is a burgeoning literature exploring these connections. More explicitly, the concluding paragraph of DiVincenzo’s paper on the physical requirements for quantum computation suggests that quantum information science may offer valuable insights into complex quantum systems. This theme was explored in more detail by Osborne and Nielsen, and the present paper is an outgrowth of this work.

2 Complex quantum systems

What is a complex system? Complexity is an elusive concept: it is difficult to define, but we know it when we see it. In response to this difficulty one
might ask whether it is possible to quantify complexity. In the 1980s Bennett proposed a measure of complexity called the logical depth. The idea is that a system is complex, or logically deep, if a description of the system can be generated by a few simple rules, but those rules require a long time to run. For example, a human body is complex because it is specified by a relatively small amount of information encoded in DNA, but it takes a great deal of processing to get from DNA to the human body. Another example is a regular pattern on a checkerboard, which is not complex because it can be quickly generated by a simple rule. More subtle is the case of a random pattern on a checkerboard. That is not complex either, because there is no simple rule generating the pattern. Indeed, the simplest rule generating the pattern is simply the program which contains (and prints) a complete listing of the states of all the elements of the checkerboard, and this program runs very quickly.

Let me give an example of something complex, that is, with high logical depth. Suppose we take a point, $x$, in the plane, for example, $x = (0, 0)$. We bounce the point around the plane by repeatedly applying one of the following four rules at random.$^{12}$

1. $x \rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0.16 \end{bmatrix} x$ with probability 0.01
2. $x \rightarrow \begin{bmatrix} 0.85 & 0.04 \\ -0.04 & 0.85 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1.6 \end{bmatrix}$ with probability 0.85
3. $x \rightarrow \begin{bmatrix} 0.2 & -0.26 \\ 0.23 & 0.22 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1.6 \end{bmatrix}$ with probability 0.07
4. $x \rightarrow \begin{bmatrix} -0.15 & 0.28 \\ 0.26 & 0.24 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0.44 \end{bmatrix}$ with probability 0.07.

When this procedure is repeated a few thousand times an interesting thing happens: with very high probability a fern shape fills in, as illustrated in Fig. 1. Furthermore, the fern is complex in that there is a simple rule generating the fern, but it takes a long time to run.

Bennett formalized these intuitions by defining the logical depth of a data set to be the running time of a near-optimal computer program generating that data set. There are some technicalities hidden in this definition, like the precise meaning of “near-optimal”, that I will gloss over. Nonetheless, I would like to give the intuitive flavour of the definition. By “near-optimal” we mean that the program is nearly the shortest possible. The motivating idea is an analogy between computer programs generating data sets and scientific hypotheses. Scientists tend to prefer simple explanations over more complex, so if we think of computer programs as explanations for data sets, then we would prefer short computer programs — simple “explanations” — over longer programs. With this definition, simple repeated structures and random patterns, like the checkerboards described earlier, have low logical depth. Systems like
the fern have high logical depth because they have simple explanations that take a long time to run.

There is an interesting quantum twist to logical depth. As we know, factoring integers is a hard problem, hard enough that RSA systems offers lots of money to people able to factor large integers: US $200,000 for a 2048-bit integer, at the time of writing. Let’s optimistically imagine that it’s ten years from now and somebody wants to prove that they have a functioning quantum computer in their lab, but don’t want to reveal the details of how they built it. One good way of proving this would be to publish a paper containing the prime factors of a large group of big integers — perhaps the prime factors of all numbers between $10^{1000}$ and $10^{1000} + 1000000$.

Is this list of factors a complex system? The answer depends on whether the computer in the definition of logical depth is quantum or classical. If it’s quantum then it seems likely that this system is not logically deep, and thus not complex, because we can quickly generate the list using a short quantum program, namely, Shor’s algorithm. If the computer is classical, and there really is no fast classical factoring algorithm, then the list of factors has high logical depth, since there are simple computer programs capable of generating such a list, but they operate very slowly.

Thus, there are two distinct notions of logical depth, classical logical depth, and quantum logical depth. We can summarize the situation by dividing systems into three distinct types. First, there are systems which have both low classical logical depth and low quantum logical depth; these are “simple”. Then there are systems that have high classical logical depth, but low quantum logical depth, like the list of prime factors discussed above; these systems might be called “classically complex”. Finally, there are systems with both high classical logical depth and high quantum logical depth: the truly “quantum complex” systems. I don’t know of any examples with this property, but consider some systems likely candidates, for example, the output of a quantum cellular automata that’s been running a long time. Note that
systems with high quantum logical depth but low classical logical depth seem unlikely, because a simple, fast classical computer generating a data set can be simulated by a simple, fast quantum computer.

3 Quantum dynamics as a physical resource

I’ve talked about quantifying complexity in quantum systems, and the information-theoretic viewpoint has led us to the idea that there are at least two, if not more, qualitatively different types of complex system. I’d like now to talk about what we can learn about specific complex quantum systems from quantum information science.

Over the past few years great effort has been devoted to developing a quantitative theory of quantum entanglement. In my opinion a major area in which this theory will be applied is to obtain insights into the properties of complex quantum systems. However, static quantum entanglement is only a small part of the story: it is also interesting to obtain a better understanding of the quantum dynamics of complex systems. To achieve this this, my group has focused on quantifying the strength of a quantum dynamical operation for information processing.

The motivation for this idea is the observation that quantum dynamics are a fungible physical resource, in the sense that it is possible to interconvert different dynamical operations, just as it is possible to convert one type of entangled state to another. More precisely, suppose a system contains \( n \) qudits, and the Hamiltonian for the system contains only two-body terms, and so can be represented by a graph whose vertices represent qudits, and whose edges represent the presence of an interaction between those qudits. Finally, suppose the graph is connected, so different qudits aren’t cut off from one another. It turns out that by alternating evolution due to such a Hamiltonian with single-qudit gates, we can, in principle, efficiently simulate any quantum computation.

While theoretically interesting, until recently most of these schemes were not practically useful, requiring extremely frequent local control to do simple operations such as the CNOT. Even an optimistic example required \( \approx 10^4 \) operations to do a CNOT.

Recently, the situation has changed. J.-L. and R. Brylinski have shown that, given any entangling two-qudit unitary \( U \), and local unitaries, it is always possible to do universal quantum computation. However, their proof used ideas from algebraic geometry and the theory of Lie algebras, and it is not clear to me whether their proof implies an efficient constructive method for doing the CNOT. Bremner et al. built on this work by giving a simple, constructive algorithm for doing a CNOT, and thus universal quantum computation, using any entangling two-qubit unitary operation, and local unitaries.

\[\text{See }\] and references therein.
The algorithm of $U$ turns out to be *near-optimal*, using nearly the minimal possible number of uses of $U$ to simulate a CNOT, and thus shows that quantum dynamical operations are fungible not only in principle, but may be in practice as well.

Knowing that quantum dynamics are a fungible physical resource, and thus qualitatively equivalent to one another, we can try to quantify the *strength* of a dynamical operation. We will now examine some strength measures, basing our discussion on $U$. Let’s start by defining a strength measure for an $n$-qubit unitary operation, $U$. (This is just one example among many in $U$.) A metric, $D$, on unitary operations induces a natural strength measure, $K_D(U) \equiv \min_{A, B, \ldots} D(U, A \otimes B \otimes \ldots)$. That is, the strength is the minimal distance between $U$ and the set of local unitaries.

What good is a strength measure? Let me answer by explaining a connection between strength and computational complexity. Imagine we have a strength measure with the following three properties. The first property, chaining, says that the strength of a product of two unitary operations, $U$ and $V$, is less than the sum of their combined strengths, $K(UV) \leq K(U) + K(V)$. The intuition is that the ability to do $U$ and $V$ separately should be at least as powerful as the ability to do $UV$. This property is not always true for the metric-based strength measures, but is true for a large subclass $U$.

The second property, stability, says that if we add an extra qubit to our system and do nothing to it, that should not change the strength, $K(U \otimes I) = K(U)$. The metric-based measures do not always satisfy stability, but they do in some instances. The third property, locality, says that a strength measure should be zero for products of local unitary operations, $K(A \otimes B \otimes \ldots) = 0$. This is true for the metric-based measures of strength.

Imagine $K$ is a strength measure satisfying these properties, and we want to perform a unitary, $U$, using CNOT and single-qubit gates. Imagine the circuit contains $M$ CNOTs. Applying the properties, $K(U)$ can be no more than the sum of the strengths of the CNOTs, so $M \geq K(U)/K($CNOT$)$. By stability, the strength of CNOT is constant, so if $K(U)$ scales superpolynomially, then so must the number of gates needed to do $U$. Thus, developing measures of dynamic strength may enable progress on the notoriously difficult problem of proving lower bounds on computational complexity, and thus to gain insight into the enormously complex space of quantum dynamical processes.

Let me conclude by returning to the big picture. I believe that the major scientific task of quantum information science is to develop tools for the study of complex quantum systems. In quantum mechanics we’re like chess players who’ve learnt the rules of the game, but are still trying to figure out the emergent properties those rules imply. We’re doing so by developing overarching theories, like the theory of entanglement and of dynamic strength, which let us understand ever more complex phenomena. I expect that as these theories are developed they will enable us to better understand complex systems, not only in information processing, but also in other areas of many-body physics.
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