Maskit combinations of Poincaré-Einstein metrics

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Abstract
We establish a boundary connected sum theorem for asymptotically hyperbolic Einstein metrics; this requires no nondegeneracy hypothesis. We also show that if the two metrics have scalar positive conformal infinities, then the same is true for this boundary join.

1 Introduction
Let $M^{n+1}$ be a compact manifold with boundary. A Riemannian metric $g$ on the interior of $M$ is said to be conformally compact if $g = \rho^{-2}\overline{g}$, where $\overline{g}$ is nondegenerate (with some specified regularity) up to the boundary and $\rho$ is a defining function for $\partial M$ (i.e. $\rho^{-1}(0) = \partial M$ and $d\rho \neq 0$ there). Any such $g$ is complete, and if $|d\rho|_{\overline{g}} = 1$ on $\partial M$, then its sectional curvatures tend to $-1$ as $\rho \to 0$. These metrics generalize the Poincaré model of hyperbolic space $\mathbb{H}^{n+1}$, and accordingly, we call $(M, g)$ Poincaré-Einstein (or simply PE) if $g$ is both conformally compact and Einstein. We always normalize so that $\text{Ric}^{\overline{g}} = -ng$.

Poincaré-Einstein metrics, which are also known as asymptotically hyperbolic Einstein (AHE) or conformally compact Einstein metrics, were introduced originally by Fefferman and Graham [8] as a tool in conformal geometry. More recent interest in them has been generated by their role in the AdS/CFT correspondence in string theory, and this has stimulated much interesting work in this area. A number of explicit examples are known, starting from the most elementary one, the hyperbolic space $\mathbb{H}^{n+1}$ and its convex cocompact quotients $\mathbb{H}^{n+1}/\Gamma$, but also including the hyperbolic analogue of the Schwarzschild metric and, when $\dim M = 4$, the Taub-BOLT metrics on disk bundles over Riemann surfaces. Many of these are catalogued in [3].

More general Poincaré-Einstein metrics can be obtained by perturbing known examples, as established first in the work of Graham and Lee [10], and much later, in more general circumstances, by Biquard [5] and Lee [16]. A substantially more comprehensive theory, especially in four dimensions, has been obtained recently by Anderson [1], [2], [3]. He shows first that when $\dim M$ is

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arbitrary, the moduli space $\mathcal{E} = \mathcal{E}(M)$ of Poincaré-Einstein metrics on $M$ is either empty or else a smooth infinite dimensional Banach manifold (with respect to a suitable Sobolev or Hölder completion). Unlike the situation when $M$ is closed without boundary, this deformation theory is always unobstructed and $\mathcal{E}(M)$ has no singularities. In addition, when $\dim M = 4$ it possible to set up a degree theory to obtain more global existence results. We explain this more carefully in the next section.

The basic goal of all of these papers is to solve an asymptotic boundary problem. More specifically, there is a map $\epsilon$ which associates to any conformally compact metric its conformal infinity $\epsilon(g)$, which is by definition the conformal class of the restriction of $\overline{g} = \rho^2 g$ to $\partial M$. A preliminary conjecture is that the restriction of $\epsilon$ to $\mathcal{E}$ is a bijection, or at least a surjection; in other words, every conformal class on $\partial M$ is the conformal infinity of at least one Poincaré-Einstein metric. Subject to a certain nondegeneracy condition, this is true locally, i.e. if $g$ satisfies this nondegeneracy condition, then all conformal classes near to $\epsilon(g)$ are in the image of $\epsilon$. However, explicit metrics are known which do not satisfy this condition [3]; for example, the conformal class of the product metric $\mathbb{S}^{n-1} \times S^1(L)$ is not in the image of $\epsilon$ for $M = B^n \times S^1(L)$ when the length $L$ of the $S^1$ factor is sufficiently large. Anderson shows [3] that $\epsilon : \mathcal{E}(M) \to \text{Conf}(\partial M)$ is always Fredholm of index zero.

Our goal in this paper is to construct a wider class of Poincaré-Einstein metrics using a method which generalizes the boundary connected sum procedure in hyperbolic geometry; this hyperbolic construction is part of the Maskit combination theorems. More specifically, suppose that $(M_j, g_j)$ are two Poincaré-Einstein metrics. Fix points $p_j \in \partial M_j$ and excise small half-balls $B^+_{n+1}(p_j)$. The boundary connected sum $M_1 \#_b M_2$ is obtained by identifying the hemispherical portions of these boundaries. In the following we let $B_j$, denote any such half-ball in $M_j$ centered at $p_j$.

**Theorem 1** The manifold $M = M_1 \#_b M_2$ carries a family of Poincaré-Einstein metrics $g_\varepsilon$ with the following two properties:

- the restriction of $g_\varepsilon$ to $M_j - B_{j,+}$ converges to $g_j$;
- the restriction of $\rho^2 g_\varepsilon$ to $\partial M_j - (B_{j,+} \cap \partial M_j)$ converges to $\rho^2 g_j$.

The convergence in either case is polynomial in a geometrically natural parameter $\varepsilon$.

In the case where the metric $g_i$ are nondegenerate (see Definition 3), the second statement can be improved since, for the metrics we construct, the restriction of $\rho^2 g_\varepsilon$ to $\partial M_j - (B_{j,+} \cap \partial M_j)$ is identically equal to $\rho^2 g_j$. In particular, this implies that $\epsilon(g_\varepsilon)$ is equal to $\epsilon(g_j)$ on $\partial M_j - (B_{j,+} \cap \partial M_j)$.

An important theme in this theory is that Poincaré-Einstein metrics which have conformal infinities of positive (or at least nonnegative) Yamabe type are geometrically more stable, and the existence theory (in dimension four) is certainly more robust in this case. Recall that a conformal class is said to be
positive (nonnegative) if it contains a metric with positive (nonnegative) scalar curvature. As an example of this stability, the main step in Anderson’s development of a $\mathbb{Z}$-valued degree theory for $c$ in dimension 4 is the properness for the restriction of this map to the preimage of $\text{Conf}^+(\partial M)$, the space of positive conformal classes. Another example, in general dimensions, is the well-known result of Witten and Yau [23], cf. also [6], that if $M$ carries a Poincaré-Einstein metric with positive conformal infinity, then $H_n(M, \partial M) = 0$ (and so, in particular, $\partial M$ is connected). A primary motivation for our Theorem 1 is to construct many more Poincaré-Einstein metrics with positive conformal infinities.

**Corollary 1** Suppose that $(M_j, g_j)$, are Poincaré-Einstein and furthermore, that $c(g_j) \in \text{Conf}^+(\partial M_j)$, $j = 1, 2$. Then $c(g_\varepsilon) \in \text{Conf}^+(M_1 \#_\Sigma M_2)$ when $\varepsilon$ is small. In particular, when $\dim M = 4$, every 3-manifold which is a finite connected sum of quotients of $S^3$ and $S^2 \times S^1$ arises as the boundary of a Poincaré-Einstein metric with positive conformal infinity.

A well-known theorem of Schoen and Yau [21], cf. also [12], states that an arbitrary 3-manifold which carries a metric of positive scalar curvature is a connected sum of quotients of $S^2 \times S^1$ and manifolds with finite fundamental group. Contingent on two (big!!) conjectures (the Poincaré and the spherical space-form conjecture), these latter summands are all lens spaces, and hence are known [11] to have explicit Poincaré-Einstein fillings. Thus, modulo these conjectures, or more simply, if one could show that positive conformal classes on quotients of exotic homotopy 3-spheres either have Poincaré-Einstein fillings, or else do not exist, then our Corollary 1 would imply that every 3-manifold $Y$ which admits a metric of positive scalar curvature has a Poincaré-Einstein filling with positive conformal infinity. In any case, combining the results here with Anderson’s, we obtain a large class of new examples of Poincaré-Einstein manifolds (in arbitrary dimension) with positive conformal infinity.

One can also ask whether other surgery constructions are possible in this category. The most direct generalization would be to join the manifolds $M_j$ along a common submanifold $\Sigma \hookrightarrow \partial M_j$ (with isomorphic normal bundles $N_j(\Sigma) \subset T\Sigma \partial M_j$). Unfortunately, our construction does not go through in any direct way, and may even fail. This is not to say that the manifold $M_1 \#_\Sigma M_2$ does not carry any Poincaré-Einstein metrics, but if these exist, they seem to be distant from the initial metrics $g_j$, even well away from a neighbourhood of $\Sigma$. We discuss this further at the end of this paper.

§2 contains a review of some details of the geometry of Poincaré-Einstein metrics and of the analytic methods used in their perturbation theory. This is followed in three subsequent sections by the construction of approximate solutions, the linear estimates and the proof of the main theorem. In §6 we prove Corollary 1, and in §7 we discuss the plausibility of more general gluing constructions.
2 Preliminaries

We now review in more detail some of the geometric and analysis required in our study of Poincaré-Einstein metrics.

2.1 Geometry of Poincaré-Einstein metrics

Suppose that \( g \) is a conformally compact metric on \( M \), so that it can be written \( g = \rho^{-2} \overline{g} \) for some defining function \( \rho \). We shall always suppose that both \( \rho \) and \( g \) are (at least) \( C^2, \alpha(M) \). The precise regularity is not so important in this paper, but we shall address this issue more carefully below.

2.1.1 Boundary normal coordinates

It will be very convenient to have something like Fermi coordinates around \( \partial M \).

Following Graham and Lee \[10\], we have

**Lemma 1** (\[10\]) Assume that \( g \) is a conformally compact metric so that \( g = \rho^{-2} \overline{g} \) for some defining function \( \rho \), and \( h_0 \) is a representative of \( c(g) \), the conformal infinity of \( g \). Further assume that \( |d\rho|^2_{\overline{g}} = 1 \) at \( \rho = 0 \). Then there exists a new defining function \( x \), in terms of which the metric \( g \) can be written as

\[
g = \frac{dx^2 + h}{x^2} \tag{1}
\]

in some neighborhood of \( \partial M \). Here \( h = h(x) \) is a family of Riemannian metrics on \( \partial M \) depending parametrically on \( x \) with \( h(0) = h_0 \).

Since this result is crucial to our construction, we briefly recall its proof now. If \( \rho \) is an initial defining function, then we look for a new one of the form \( x = e^u \rho \). The metric \( g \) will be in the correct form provided \( |dx|_{x^2 g} = 1 \) near \( \partial M \), and this is equivalent to the equation for the function \( u \)

\[
2 < d\rho, du >_{\overline{g}} + \rho |du|^2_{\overline{g}} = \frac{1 - |d\rho|^2_{\overline{g}}}{\rho}. \tag{2}
\]

Since \( d\rho \neq 0 \) and \( \rho = 0 \) on \( \partial M \), equation \[2\] is noncharacteristic with respect to the boundary, and hence can be solved locally with any boundary condition \( u = u_0 \) when \( \rho = 0 \). We take \( u_0 \) so that \( e^{-2u_0} \overline{g}|_{\partial M} = h_0 \). In terms of \( x = e^u \rho \) and any choice of coordinates on \( \partial M \), the metric \( g \) is given by \[1\].

Recall from \[4\] that if two metrics are conformally related, say \( g = e^{2f} \overline{g} \), then the Ricci tensors of \( g \) and \( \overline{g} \) are related by

\[
\text{Ric}^g = \text{Ric}^\overline{g} - (n - 1) (\nabla^\overline{g} df - df \circ df) - (\Delta^\overline{g} f + (n - 1) |df|^2_{\overline{g}}) \overline{g}. \tag{3}
\]

Applying this with \( f = -\log x \), and the metric \( \overline{g} = dx^2 + h \) given in Lemma \[1\], we obtain the expansion

\[
\text{Ric}^g + ng = \frac{1}{2x} ((n - 1)h_1 + (\text{tr} h_0(h_1)) (dx^2 + h_0)) + O(1). \tag{4}
\]
where we have expanded the metric \( h = h_0 + x h_1 + O(x^2) \). As a consequence we have the :

**Lemma 2** ([10]) Under the assumptions and notations of Lemma 1, if we further assume that \( g \) is a Poincaré-Einstein metric, then the family of Riemannian metrics \( h(x) \) on \( \partial M \) can be expanded as \( h(x) = h_0 + x^2 h(x), \) where \( h(x) \) is a family of symmetric 2-tensors on \( \partial M \) which depend parametrically on \( x \).

In other words, the Einstein condition implies that \( h'(0) = 0 \) (or equivalently, that \( \partial M \) is totally geodesic for the metric \( \overline{g} = x^2 g \)), and hence \( h = h_0 + x^2 h_2 + o(x^2) \). Assuming that \( \rho \) and \( \overline{g} \) are polyhomogeneous, Fefferman and Graham [8] produce a complete formal expansion for \( h \); this is justified in four dimensions by the regularity result in [2], and in general dimensions in the forthcoming paper [11].

### 2.1.2 Gauge choice and the Einstein equation

The equation satisfied by \( g \) is \( \text{Ric}^g + n g = 0 \). It is well-known that this equation is not elliptic because of the underlying diffeomorphism invariance, and so one must choose some gauge condition. The best choice is the one adopted by Biquard [5], and later Anderson, called the Bianchi gauge. If \( \tilde{g} = g + k \) is any metric near to \( g \), then we define

\[
B^g(\tilde{g}) = \delta^g \tilde{g} + \frac{1}{2} d \text{tr}^g \tilde{g} = \delta^g k + \frac{1}{2} d \text{tr}^g k
\]

since \( B^g(g) = 0 \). Thus \( B^g \) is a map from symmetric 2-tensors to 1-forms. The system

\[
\text{Ric}^{\tilde{g}} + n \tilde{g} = 0, \quad B^g(\tilde{g}) = 0,
\]

is elliptic in the sense of Agmon-Douglis-Nirenberg, but it is more convenient to work with the single equation

\[
N_g(k) := \text{Ric}^{g+k} + n(g + k) + (\delta^{g+k})^* B^g(k) = 0,
\]

where the symmetric 2-tensor \( k \) is assumed to be small enough so that \( g + k \) is a metric on \( M \).

**Proposition 1** ([5]) Suppose that \( \text{Ric}^{g+k} < 0 \) and \( |B^g(g + k)| \to 0 \) at \( \partial M \); then any solution of \( N_g(k) = 0 \) corresponds to an Einstein metric \( g + k \) which is in the Bianchi gauge relative to \( g \).

The linearization of \( N_g \) is very simple in this gauge:

\[
L_{g\kappa} := 2D N_g|_0(k) = (\nabla^g)^* \nabla^g \kappa - 2 R^g \kappa + \text{Ric}^g \circ \kappa + \kappa \circ \text{Ric}^g + 2 n \kappa.
\]

Here

\[
(R^g \kappa)_{ij} = R^g_{ipjq} \kappa^{pq}, \quad \text{Ric}^g \circ \kappa = \text{Ric}^g_i \kappa_p, \quad \kappa \circ \text{Ric}^g = \kappa_p \text{Ric}^g_{pj},
\]
and all curvatures are computed relative to $g$. Note that when $g$ is Einstein, $\text{Ric}^g = -ng$, and hence

$$L_g = (\nabla^g)^* \nabla^g - 2 \mathring{R}^g$$

then.

This operator $L_g$ is not uniformly elliptic on $M$; rather it has the structure of a uniformly degenerate operator, as studied in detail in [8], [9]. We shall require some of the main results of the theory of uniformly degenerate operators, and so we now digress briefly to explain this setup. These general results will either be stated for, or immediately specialized to, the operator $L_g$.

### 2.2 Uniformly degenerate operators

Choose coordinates $w = (x, y) := (x, y_1, \ldots, y_n)$, where $x = w_0 \geq 0$ is a boundary defining function, near a point $p \in \partial M$. A second order operator $L$ is said to be uniformly degenerate if it can be expressed in the form

$$L = \sum_{j+|\alpha|\leq 2} a_{j,\alpha}(x, y)(x \partial_x)^j(x \partial_y)^\alpha,$$

where the (scalar or matrix-valued) coefficients are bounded. We usually assume that these coefficients are smooth on $M$, but it is easy to extend most of the main conclusions of this theory when they are only polyhomogeneous, or of some finite regularity. Operators of this type arise naturally in geometry, and in particular all of the natural geometric operators associated to a conformally compact metric are uniformly degenerate.

A typical example of such an operator is given by the Laplace-Beltrami operator on hyperbolic space $(\mathbb{H}^{n+1}, g_0)$. Taking coordinates $(x, y) \in (0, \infty) \times \mathbb{R}^n$ in the upper half space model, we have

$$g_0 = \frac{dx^2 + dy^2}{x^2},$$

and we obtain

$$\Delta_{g_0} = x^2 \partial_x^2 + (1 - n) x \partial_x + x^2 \Delta_y.$$  \hspace{1cm} (9)

### 2.2.1 Ellipticity and model operators

There is a well-defined symbol in this context,

$$\sigma(L)(x, y; \xi, \eta) := \sum_{j+|\alpha|\leq 2} a_{j,\alpha}(x, y) \xi^j \eta^\alpha \neq 0 \quad \text{when} \quad (\xi, \eta) \neq 0,$$

and we say that $L$ is elliptic in the uniformly degenerate calculus provided $\sigma(L)(x, y; \xi, \eta)$ is invertible when $(\xi, \eta) \neq 0$.

Ellipticity alone is not enough to guarantee that $L$ is Fredholm between appropriate function spaces; one must also require that certain simpler operators which model $L$ near the boundary be invertible. There are two of these model operators:
The normal operator of $L$ is defined by
\[ N(L) := \sum_{j+|\alpha| \leq 2} a_{j,\alpha}(0, y)(s \partial_s)^j (s \partial_u)^\alpha, \quad (s, u) \in \mathbb{R}^+ \times \mathbb{R}^n; \]
here $y \in \partial M$ enters only parametrically and the operator acts on functions on a half-space $\mathbb{R}^+ \times \mathbb{R}^n$, which is naturally identified with the inward-pointing half tangent space $T^+_{(0, y)}M$.

The indicial operator of $L$ is defined by
\[ I(L) := \sum_{j \leq 2} a_{j,0}(0, y)(s \partial_s)^j. \]

For example, the normal and indicial operators associated to the (scalar) Laplace-Beltrami operator on hyperbolic space $(\mathbb{H}^{n+1}, g_0)$ are given by
\[ N(\Delta_{g_0}) = s^2 \partial_s^2 + (1 - n) s \partial_s + s^2 \Delta_u, \quad I(\Delta_{g_0}) = s^2 \partial_s^2 + (1 - n) s \partial_s. \]

The normal operator can be regarded as $L$ with its coefficients frozen (in an appropriate sense) at the boundary, so the following result is not surprising.

**Proposition 2** If $g = \rho^{-2} \bar{g}$ is a smooth conformally compact metric such that $|d\rho|^2_\bar{g} = 1$ on $\partial M$, then its Laplace-Beltrami operator $\Delta_g$ and the linearization $L_g$ of the gauged Einstein equation are both elliptic uniformly degenerate operators. Furthermore, their normal operators are given by
\[ N(\Delta_g) = \Delta_{g_0}, \quad N(L_g) = (\nabla^{g_0})^* \nabla^{g_0} - 2 \tilde{R}^0, \]
respectively, where $g_0$ is the standard hyperbolic metric on $\mathbb{H}^{n+1}$.

The indicial operator is a much more primitive model, but it captures some fundamental invariants associated to $L$.

**Definition 1** The number $\zeta \in \mathbb{C}$ is said to be an indicial root of $L$ if
\[ L(x^\zeta v(y)) = O(x^{\zeta+1}) \quad \text{for any} \quad v \in \mathcal{C}^\infty(\partial M). \]

We may replace $L$ by $I(L)$ (and $x$ by $s$) here, since the higher order terms which appear in $L$ but not in the indicial operator are irrelevant for this calculation. Thus
\[ I_L(\zeta)s^\zeta := I(L)(s^\zeta) = \left( \sum_{j \leq 2} a_{j,0}(0, y)\zeta^j \right) s^\zeta, \]
and hence $\zeta$ is an indicial root of $L$ if and only if it is a root of the (matrix-valued) polynomial $\zeta \to I_L(\zeta)$.

The operator $L$ acts naturally on weighted Hölder (and Sobolev) spaces, and the indicial roots of $L$ determine the weights for which these mappings do not have closed range.
The calculation of the indicial roots for the linearized Einstein operator $L_g$ is carried out efficiently in [10], see also [16]. There are several pairs of indicial roots, corresponding to the action of $L_g$ on pure-trace and trace-free symmetric 2-tensors; the latter space in turn decomposes into three irreducible summands, corresponding to the normal and tangential components of the 2-tensors.

Before giving the values of these indicial roots, we must fix a basis of sections for the space of symmetric 2-tensors. There are two natural choices: the standard one, consisting of all symmetric products $dw_i dw_j$, and another, consisting of all symmetric products $\frac{dw_i}{x} \frac{dw_j}{x}$. We use this latter choice since it is geometrically more natural – the 1-forms $dw_i/x$ are of length bounded away from infinity and zero with respect to any conformally compact metric $g$ – and so we write any symmetric 2-tensor $k$ as

$$k = \sum_{i,j=0}^{n} k_{ij} \frac{dw_i}{x} \frac{dw_j}{x}.$$  

This is in accord with the notation and terminology of [18], where the role of this normalization is emphasized and exploited consistently. In fact, in the notation of that paper, the singular symmetric 2-tensors $\frac{dw_i}{x} \frac{dw_j}{x}$ are a basis of smooth sections of a bundle which we denote $S_2^0(M)$. Thus, for example, any conformally compact with smooth conformal compactification is an element of $C^\infty S_2^0(M)$. This normalization differs from the ones in [10] and [16] and shifts by 2 the indicial roots. This explains the discrepancy with the numerology in those papers. We have

**Proposition 3** Assume that $g$ is a conformally compact, so that $g = x^{-2} \mathcal{g}$, and assume also that $|dx|\mathcal{g} = 1$ on $\partial M$. Then the set of indicial roots of $L_g$ consists of the pairs:

$$\zeta_{1,2}^\pm = \frac{1}{2} \left( n \pm \sqrt{n^2 + 8n} \right), \quad \zeta_{3}^\pm = \frac{1}{2} \left( n \pm \sqrt{n^2 + 4n + 4} \right),$$

and $\zeta_{4}^\pm = \frac{1}{2} (n \pm n) = 0, n$.

Setting $\mu_- = \sup \zeta_j^-$ and $\mu_+ = \inf \zeta_j^+$, then we have the important inequality

$$\mu_- := 0 < \mu_+ := n.$$  

As already mentioned, the computation of the indicial roots can be found in Lemma 7.1 and Lemma 7.5 of [16]. However, to be explicit, the action of

$$L_{g_0} := (\nabla^{g_0})^* \nabla^{g_0} - 2 R^{g_0},$$

on trace-free symmetric 2-tensors

$$h := h_{00} \frac{ds^2}{s^2} + h_{0i} \frac{ds}{s} \frac{du_i}{s} + h_{ij} \frac{du_i}{s} \frac{du_j}{s},$$

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is given by
\[
L_{g_0} h = (-\Delta_{g_0} h_{00} + 2n h_{00} - 4s \partial_u h_{0i}) \frac{ds^2}{s^2} + \left(-\Delta_{g_0} h_{0i} + (n + 1) h_{0i} - 2s \partial_{u_j} h_{ij}\right) \frac{ds}{s} \frac{du_j}{s} + \left(-\Delta_{g_0} h_{ij} + 2s (\partial_{u_j} h_{0i} + \partial_{u_i} h_{0j}) - 2 h_{00} \delta_{ij}\right) \frac{du_i du_j}{s},
\]
where $\Delta_{g_0}$ is the Laplace-Beltrami operator on hyperbolic space defined in (9).

It is straightforward to check that the indicial roots corresponding to the normal part (i.e. $h_{00}$) are $\zeta_{1}^\pm$, the indicial roots corresponding to the mixed part (the $h_{0i}$ terms) are $\zeta_{2}^\pm$, and the indicial roots corresponding to the tangential part (the $h_{ij}$ terms) are $\zeta_{3}^\pm$. Finally, the action of $L_{g_0}$ on pure-trace symmetric 2-tensors is given by
\[
L_{g_0}(\tau_{g_0}) = (-\Delta_{g_0} \tau + 2n \tau) g_0,
\]
and so the indicial roots here are also given by $\zeta_{1}^\pm$.

### 2.2.2 Function spaces

Let us now recall the scale of weighted scale invariant Hölder spaces $x^\mu \Lambda^{k,\alpha}_0(M)$. (These do not provide the optimal boundary regularity for this problem, but they are sufficient for our goals here.) For simplicity, definitions will be stated primarily for functions, but they transfer easily to sections of vector bundles, as indicated briefly below. We refer to [18] for further discussion and proofs, cf. also [16] and [3].

We first define $\Lambda^{0,\alpha}_0(M)$ as the natural ‘geometric’ Hölder space associated to any fixed smooth conformally compact metric $g = x^{-2} g$. Thus if $w = (x, y)$ is a smooth coordinate chart near $\partial M$, then this space is the closure of $C^\infty(M)$ with respect to the norm
\[
||u||_{0,\alpha} := \sup_B \sup_{w,w' \in B} \frac{|u(x, y) - u(x', y')|(x + x')^\alpha}{|x - x'|^\alpha + |y - y'|^\alpha} \cong \sup_B \sup_{w,w' \in B} \frac{|u(x, y) - u(x', y')|}{\text{dist}_{g'}(w, w')^\alpha} < \infty;
\]
this supremum is taken first over all points $w = (x, y), w' = (x', y'), w \neq w'$, lying in some coordinate cube $B$ centered at a point $w_0 = (x_0, y_0)$ of sidelength $\frac{1}{2} x_0$, and then over all such cubes. We could equivalently replace these cubes by geodesic balls (with respect to $g$) of radius 1. Continuing on, we define $\Lambda^{k,\alpha}_0(M)$ to consist of all functions $u$ such that $(x \partial_x)^j (x \partial_y)^\lambda u \in \Lambda^{0,\alpha}_0(M)$ for all $j + |\lambda| \leq k$. Noting that if we use the vector fields and 1-forms $x \partial_{w_i}, dw_j/x$ and their tensor products as generators for the sections of all of the tensor bundles, then $\nabla^g$ involves only differentiations with respect to $x \partial_x, x \partial_y$. Hence the definitions of these function spaces extend naturally to sections of any of these bundles.
These norms respect the natural scale invariance of uniformly degenerate operators: in fact, for functions \( u \) supported in a coordinate chart near the boundary, the norms of \( u(w) \) and \( u_\varepsilon(w) = u(w/\varepsilon) \) are the same.

For \( \mu \in \mathbb{R} \), define also
\[
x^\mu \Lambda^{k,\alpha}_0(M) := \left\{ u = x^\mu \tilde{u} : \tilde{u} \in \Lambda^{k,\alpha}_0(M) \right\},
\]
with the corresponding norm denoted \( \| \cdot \|_{k,\alpha,\mu} \).

We shall also have occasion to use weighted \( L^2 \) spaces. Using any standard coordinate chart near the boundary, we define the main ‘reference’ \( L^2 \) space
\[
L^2_b(M) = L^2 \left( M; \frac{dx \, dy}{x} \right)
\]
and its weighted versions
\[
x^\delta L^2_b(M) = \{ v = x^\delta \tilde{v} : \tilde{v} \in L^2_b(M) \}.
\]
Notice that the most natural one of these is
\[
L^2(M; \mu^2) = L^2 \left( M; \frac{dx \, dy}{\mu^{n+1}} \right) = x^{n/2} L^2_b(M).
\]
We use \( L^2_b(M) \) as the reference space because of the simple numerology that
\[
x^\mu \Lambda^{k,\alpha}_0(M) \hookrightarrow x^\delta L^2_b(M)
\]
if and only if \( \mu > \delta \).

**Definition 2** The closure in the norm \( \| \cdot \|_{k,\alpha,\mu} \) of the space of smooth symmetric 2-tensors \( C^\infty S^0_2(M) \) is denoted \( x^\mu \Lambda^{k,\alpha}_0 S^2_0(M) \).

In particular, \( k \in x^\mu \Lambda^{k,\alpha}_0 S^2_0(M) \) can be decomposed as
\[
k = \sum_{i,j=0}^n k_{ij} \frac{dw_i}{x} \frac{dw_j}{x},
\]
where \( k_{ij} \in x^\mu \Lambda^{k,\alpha}_0(M) \). Similarly, we can define the spaces \( L^2 S^2_0(M) \) and \( L^2 S^2_0(M; dV_g) \) of sections of symmetric 2-tensors.

### 2.2.3 Mapping properties

Assume that \( g \) is a conformally compact metric, so that it can be written \( g = x^{-2} \bar{g} \). Further assume that \( |dx|_{\bar{g}} = 1 \). It follows immediately from the definitions that
\[
L_g : x^\mu \Lambda^{k+2,\alpha}_0 S^2_0(M) \rightarrow x^\mu \Lambda^{k,\alpha}_0 S^2_0(M)
\]
is bounded for any $\mu \in \mathbb{R}$ and $\ell \geq 0$. However, this map does not have closed range when $\mu$ is equal to one of the indicial roots $\zeta_i^\pm$ of $L_g$. This stems from the fact that when $\mu$ is an indicial root, the equation $I(L_g)u = s^\mu$ has solution $u = c s^\mu(\log s)$, where $c$ is a constant tensor, and this misses lying in $s^\mu \Lambda^2_{0,\alpha}$ on account of the logarithmic factor, although $s^\mu$ does lie in this space.

**Proposition 4 ([18], cf. also [16])** If $\mu \in (0, n)$, then the mapping (11) is Fredholm of index zero.

When $\mu$ is not in this range, and is not equal to an indicial root, then (11) still has closed range and is semi-Fredholm, but has either kernel or cokernel which is infinite dimensional. The operator $L_g$ has analogous mapping properties when acting between weighted Sobolev spaces, and these are actually more elementary than the one here between weighted Hölder spaces, cf. [18].

The fact that the Fredholm index of $L_g$ is zero when $\mu \in (0, n)$ is proved using the self-adjointness of $L_g$ on $L^2 S^0_2(M; dV_g)$. For the local deformation theory for Poincaré-Einstein metrics, it is crucial to know whether (11) is surjective at a weight $\mu$ in this interval. This proposition shows that in this Fredholm range, surjectivity is equivalent to injectivity at the same weight. This motivates why it is necessary to study the nullspace of $L_g$ more closely. We first state a basic regularity theorem:

**Proposition 5** Assume that $g$ is a Poincaré-Einstein metric. Suppose that $L_gk = 0$ and that $k \in x^\mu \Lambda^2_{0,\alpha} S^0_2(M)$ or $k \in x^\mu L^2_2 S^0_2(M)$ for some $\mu > 0$. Then $k$ is polyhomogeneous, and has an expansion

$$k \sim \sum_{j,\ell,r} k_{j,\ell,r}(y) x^{\zeta_j^+ + \ell} (\log x)^r, \quad k_{j,\ell,r} \in C^\infty S^0_0(\partial M),$$

where the $\zeta_j^+$ are the indicial roots of $L_g$ for which $x^{\zeta_j^+} \in x^\mu \Lambda^2_{0,\alpha}(M)$ or $x^{\zeta_j^+} \in x^\mu L^2_2(M)$. Consequently, for any such $k$ we necessarily have $|k|_g = \mathcal{O}(x^n)$.

Note in particular that this result applies to any $k$ which belongs to $\ker L_g \cap L^2 S^0_2(M; dV_g)$. We remark also that elements of $\ker L_g$ which lie in a weighted space with $\mu \leq 0$ are no longer necessarily polyhomogeneous, and their precise regularity is determined by the regularity of their leading asymptotic coefficients, c.f. [18].

An immediate consequence of this result is that the nullspace of $L_g$ on any one of the $x^\mu L^2_2 S^0_2(M)$ or $x^\mu \Lambda^2_{0,\alpha} S^0_2(M)$ for $\mu \in (0, n)$ are all the same.

**Definition 3** A Poincaré-Einstein metric $g$ is said to be nondegenerate if the nullspace of $L_g$ on $L^2 S^0_0(M; dV_g)$ is trivial.

The local deformation theory for Poincaré-Einstein metrics is simplest when the $L^2$ nullspace for $L_g$ is trivial. The triviality of this nullspace is verified, in increasingly general settings, by Graham and Lee [10], Biquard [3] and Lee [16]. Anderson modifies this approach by including the variation of the conformal infinity as an explicit variable, and shows that the resulting map is always surjective. We explain this more carefully now.
Choose a smooth boundary defining function \(x\) according to Lemma 1. The flow lines for \(\nabla g\) determine a product structure \([0, x_0] \times \partial M\) near the boundary, and we let \(\pi\) be the projection onto the second factor. Fix a smooth cutoff function \(\chi(x) \geq 0\) which equals 1 for \(x \leq x_0/2\) and which vanishes for \(x \geq x_0\), and define the extension map

\[
e: C^{2,\alpha} S^2(\partial M) \longrightarrow x^{-2} C^{2,\alpha} S^2(M) \subset \Lambda_0^{2,\alpha} S_0^2(M)
\]

\[
e(\eta) = \frac{\chi(x(x)) \pi^*(\eta)}{x^2}.
\]

When \(\eta\) is small, \(g_\eta := g + e(\eta)\) is a conformally compact metric. Proposition 1.4.6 in [5] asserts that for any \(\eta \in C^{2,\alpha} S^2(\partial M)\) with small norm, there exists a unique diffeomorphism \(\varphi_\eta\) close to the identity such that the metric

\[
\tilde{g}_\eta := \varphi_\eta^* g_\eta
\]

satisfies the Bianchi gauge condition

\[
B^g(\tilde{g}_\eta) = 0.
\]

In particular, \(\varphi_0 = Id\). In fact, the construction yields that \(\varphi_\eta \in D_2^{3,\alpha}(M)\), the group of \(C^{3,\alpha}\) diffeomorphisms induced by vector fields in \(x^2 \Lambda_0^{3,\alpha}(M)\). Note that \(\varphi_\eta\) restricts to the identity map on \(\partial M\).

The mapping

\[
\Gamma = \Gamma^g : C^{2,\alpha} S^2(\partial M) \longrightarrow \Lambda_0^{2,\alpha} S_0^2(M)
\]

\[
\eta \longrightarrow \tilde{g}_\eta
\]

is smooth and has differential

\[
D\Gamma|_{\eta=0}(\gamma) = e(\gamma) + h(\gamma),
\]

where

\[
h : C^{2,\alpha} S^2(\partial M) \longrightarrow x^2 \Lambda_0^{2,\alpha} S_0^2(M)
\]

is bounded. Differentiating (13) with respect to \(\eta\) yields

\[
B^g(e(\gamma) + h(\gamma)) = 0.
\]

Following Anderson [3], define the nonlinear mapping

\[
\tilde{N}_g(\eta, k) := \text{Ric} \tilde{g}_\eta + n(\tilde{g}_\eta + k) + (\delta \tilde{g}_\eta + k)^* B^g(k).
\]

This maps a neighbourhood of the origin in \(C^{2,\alpha} S^2(\partial M) \oplus x^\mu \Lambda_0^{2,\alpha} S_0^2(M)\) into \(x^\mu \Lambda_0^{0,\alpha} S_0^2(M)\), provided \(0 < \mu \leq 2\). Indeed, it follows from (14) together with the choice of defining function \(x\) as in Lemma 2 that \([\text{Ric} \tilde{g}_\eta + n \tilde{g}_\eta]_g \in x^2 \Lambda_0^{0,\alpha} S^2(M)\).
Now
\[ 2 \hat{D} N_g|_{(0,0)}(\gamma, \kappa) := \hat{L}_g(\gamma, \kappa) \]
\[ = L_g(e(\gamma) + h(\gamma) + \kappa) - 2(\delta g)^* B^0(e(\gamma) + h(\gamma)). \]

Using (14) to simplify this, we obtain finally
\[ \hat{L}_g(\gamma, \kappa) = L_g(e(\gamma) + h(\gamma) + \kappa). \] (16)

**Proposition 6 ([3] Theorem 1.2)** Fix a Poincaré-Einstein metric \( g \) on \( M \) and let \( m = \dim \ker L_g \). Then there exists an \( m \)-dimensional subspace \( S \subset C^2,\alpha(\partial M) \), such that
\[ \hat{L}_g : S \oplus x^\mu A^2,\alpha S^0(M) \rightarrow x^\mu A^0,\alpha S^0(M) \] (17)
is surjective for any \( 0 < \mu \leq 2 \).

The proof boils down to showing that for every \( \kappa \in \ker L_g \), there exists a \( \gamma \in C^2,\alpha(\partial M) \) such that
\[ \int_M \langle \hat{L}_g(\gamma, 0), \kappa \rangle dV_g = \int_M \langle L_g(e(\gamma) + h(\gamma)), \kappa \rangle dV_g \neq 0. \] (18)

Integrating by parts, this is the same as the requirement that
\[ \lim_{a \rightarrow 0} \int_{\partial M_a} (\langle N^g_N \gamma, \kappa \rangle - \langle \gamma, N^g_N \kappa \rangle) dV_g \neq 0; \] (19)

here \( M_a = \{ x \geq a \} \) and \( N = \nabla^g x/|\nabla^g x| \) is the unit normal to \( \partial M_a \). Using that \( \kappa \) is polyhomogeneous with leading term \( x^n \) and expanding the left side of (19) as \( a \searrow 0 \), we deduce that (19) fails to hold precisely when \( \gamma \) lies in a codimension 1 subspace \( V(\kappa) \). The intersection of these subspaces as \( \kappa \) varies over \( \ker L_g \) has codimension \( m \), which finishes the proof.

As a corollary, the implicit function theorem now implies that the moduli space \( \mathcal{E}(M) \) of Poincaré-Einstein metrics on \( M \) is always a smooth Banach manifold. It also follows that the restriction of the projection \( \Pi \) to \( \mathcal{E} \) is Fredholm of index zero; by the Sard-Smale theorem, its image is (at worst) a (Banach) variety of finite codimension (provided \( \mathcal{E}(M) \neq \emptyset \)).

In our gluing construction it will be convenient to work with variations \( \gamma \) and diffeomorphisms \( \varphi \) which are supported away from the points \( p_j \in \partial M_j \). We now argue that Proposition 3 and its proof are stable under small perturbations.

Fix a boundary coordinate chart \((x, y)\) centered at \( p \in \partial M \). Letting \( \psi(r) \) be a smooth cutoff function which vanishes for \( r \leq 1/2 \) and equals 1 for \( r \geq 1 \), and \( \psi_\tau(r) := \psi(r/\tau) \), we define
\[ \hat{\varphi}_\eta := (1 - \psi_\tau(r)) \text{Id} + \psi_\tau(r) \varphi_y, \quad r = |(x, y)|, \]
in that chart and $\hat{\varphi}_\eta = \varphi_\eta$ elsewhere; we also define
\[
\hat{g}_\eta := \hat{\varphi}_\eta^* (g + \psi_\tau e(\eta)).
\]

Clearly $\hat{g}_\eta = g$ on the ball of radius $\tau$ around $p$.

Replacing $\tilde{g}_\eta$ by $\hat{g}_\eta$ in the definition of $\tilde{N}_g$ yields a new nonlinear operator $\hat{N}_g$, with linearization $\hat{L}_g$. We use $\varphi_\gamma \in \mathcal{D}^{2,\alpha}_2(M)$ to show that $(\hat{L}_g - \tilde{L}_g)(\gamma,0) \to 0$ as $\tau \to 0$. Now suppose that (19) holds for some $\kappa$ and $\gamma$. To prove that
\[
\lim_{\tau \to 0} \int_{\partial M} \left( \langle \nabla^{g_\gamma}_N (\psi_\tau \gamma), \kappa \rangle - \langle \psi_\tau \gamma, \nabla^{g_\kappa}_N \kappa \rangle \right) dV_g \neq 0
\]
when $\tau$ is small, we only need observe that the commutator $[\nabla^{g_\gamma}_N, \psi_\tau]$ is bounded and supported on an increasingly small set, hence its contribution to the integral becomes increasingly negligible. This establishes the analogue of Proposition 3.

3 The approximate solution

We now commence with the construction. Suppose that $(M_j, g_j)$, $j = 1, 2$, are Poincaré-Einstein, $\dim M_j = n + 1$. Fix points $p_j \in \partial M_j$ and, as in the introduction, consider the boundary connected sum $M_1 \# b M_2$, obtained by excising half-balls around the points $p_j$ and identifying their hemispherical boundaries.

We fix a defining function $x$ as in Lemma 3 so that
\[
g_1 = \frac{dx^2 + h^{(1)}(x)}{x^2}
\]
where $h^{(1)}(x)$ is a $C^{2,\alpha}$ family of $C^{2,\alpha}$ metrics on $\partial M_1$ and
\[
h^{(1)} = h^{(1)}_0 + \mathcal{O}(x^2),
\]
where $h^{(1)}_0 := h^{(1)}(0)$ is a representative of $c(g)$ the conformal infinity of $g_1$. We further fix Riemann normal coordinates $y$ centered at $p_1 \in \partial M$ for the metric $h^{(1)}_0$. By definition of normal coordinates,
\[
h^{(1)}_0 = dy^2 + \mathcal{O}(|y|^2).
\]
Then, in the boundary normal coordinates $w = (x, y)$ we have
\[
g_1 = \frac{dx^2 + dy^2}{x^2} + \sum_{\alpha, \beta = 1}^n k^{(1)}_{\alpha\beta} \frac{dy_\alpha}{x} \frac{dy_\beta}{x}.
\]
Thus
\[
k^{(1)} := \sum_{\alpha, \beta = 1}^n k^{(1)}_{\alpha\beta}(w) \frac{dy_\alpha}{x} \frac{dy_\beta}{x},
\]
is a 2-tensor which measures the discrepancy of $g_1$ from the standard hyperbolic metric $g_0$ and (20) together with (21) give the estimate for the coefficients of the discrepancy tensor
\[ k^{(1)}_{\alpha\beta}(w) = O(|w|^2). \] (23)

Similarly, in terms of boundary normal coordinates $w' = (x', y')$ near $p_2$ on $M_2$, we can decompose $g_2 = g_0 + k^{(2)}$, and the coefficients of this discrepancy tensor (relative to the coframe $dw'_{/x'}$) are $O(|w'|^2)$.

Let $A_\varepsilon$ and $A'_\varepsilon$ denote the annuli $\{\varepsilon/2 \leq |w| \leq 2\varepsilon\}$ and $\{\varepsilon/2 \leq |w'| \leq 2\varepsilon\}$, respectively. Identifying these by means of the inversion mapping $w$ uniformly for $w', w'' \in B_{2\varepsilon}(p_1)$, where $I_{\varepsilon}(w) := \varepsilon^2 w/|w|^2$, we define the smooth manifold with boundary
\[ M_{\varepsilon} = \left( M_1 - B_{\varepsilon/2}(p_1) \right) \bigcup_{I_{\varepsilon}} \left( M_2 - B_{\varepsilon/2}(p_2) \right). \]

Note that the annulus $A_\varepsilon \sim A'_\varepsilon$ is naturally embedded in $M_{\varepsilon}$.

It will be more convenient to use a rescaling of these coordinate systems, so that we may regard the gluing region as a fixed annulus $A$. Thus define the dilation $R_{\varepsilon}$, which sends $w$ to $\varepsilon w$ (and $w'$ to $\varepsilon w''$). The annuli $A$ and $A'$ of inner and outer radii $1/2$ and $2$ in the $w$, $w'$ coordinates are mapped by $R_{\varepsilon}$ to $A_\varepsilon$ and $A'_\varepsilon$, respectively, and are identified by the fixed inversion $I(w) = w/|w|^2$. Observe that $I = R_{\varepsilon}^{-1} I_{\varepsilon} R_{\varepsilon}$.

The metrics $g_{j,\varepsilon} = R_{\varepsilon}^* g_j$ are defined on the half-ball of radius $C/\varepsilon$ for some $C > 0$; these are just isometric forms of the initial metrics $g_j$. We define a family of conformally compact metrics $g_{\varepsilon}$ on $M_{\varepsilon}$ by pasting these together. Thus let $\chi(r)$ be a nonnegative, smooth cutoff function which equals $1$ for $r = |w| \geq 2$ and vanishes for $r \leq 1/2$. Then set
\[ g_{\varepsilon} = \chi(r) g_{1,\varepsilon} + (1 - \chi(r)) I^* (g_{2,\varepsilon}). \]

This is our approximate solution; it is a conformally compact metric on $M_{\varepsilon}$ and agrees with the original metrics $g_1$ and $g_2$ outside of the half-balls $B_{2\varepsilon}(p_j)$.

We now estimate the discrepancy of $g_{\varepsilon}$ from being Einstein. To this end, observe that
\[ g_{1,\varepsilon} = \frac{dx^2}{x^2} + \frac{dy^2}{x^2} + \sum_{\alpha, \beta = 1}^n \left( R_{\varepsilon}^* k^{(1)}_{\alpha\beta}(w) \right) \frac{dy_\alpha}{x} \frac{dy_\beta}{x}, \]

since $R_{\varepsilon}^* (dw_{/x}) = dw_{j/x}$. Clearly
\[ |R_{\varepsilon}^* k^{(1)}_{\alpha\beta}(w)| \leq C\varepsilon^2, \] (24)

uniformly for $w \in A$, with $C$ independent of $\varepsilon$. The coefficients $R_{\varepsilon}^* k^{(2)}$ satisfy the same estimate for $w' \in A'$. To compute $I^* (g_{2,\varepsilon})$, it suffices to concentrate on the term $I^* R_{\varepsilon}^* k^{(2)}$ since $I$ is an isometry of $g_0$. We have
\[ I^* \left( \frac{dy'_\alpha}{x'} \right) = \frac{dy_\alpha}{x} - \frac{2y_\alpha}{r} \frac{x dx}{r x} + \sum_{\beta = 1}^n \frac{y_\beta}{r} \frac{dy_\beta}{x}, \] (25)
where \( r = (x^2 + |y|^2)^{1/2} \). This then gives for all \( w \in A \)

\[
I^* R_*^{(2)} k^{(2)} = O(\varepsilon^2 x^2) \frac{dx^2}{x^2} + \sum_{\alpha=1}^n O(\varepsilon^2 x) \frac{dx \, dy_\alpha}{x} + \sum_{\alpha,\beta=1}^n (\delta_{\alpha\beta} + O(\varepsilon^2)) \frac{dy_\alpha \, dy_\beta}{x},
\]

where \( I(y) = y/|y|^2 \) is the restriction of the inversion \( I \) to the boundary. The expression for the final term uses that for \( w \in A \) and \( x \to 0, r = |y| + O(x^2) \).

Note that the first two terms and the last term on the right in (26) vanish at \( \partial M_\varepsilon \). Hence the conformal infinity of \( g_\varepsilon \) is represented by the metric \( h_{0,\varepsilon} \) which is obtained by identifying the annuli \( 1/2 \leq |y| \leq 2, 1/2 \leq |y| \leq 2 \) in the rescaled normal coordinates on \( \partial M_1 \) and \( \partial M_2 \) using the inversion \( \tilde{I} \), and pasting together the metrics \( h_0^{(1)} \) and \( h_0^{(2)} \) with the cutoff function \( \chi(|y|) \). This will be important in §6 when we discuss the Yamabe type of \( \varepsilon(g_\varepsilon) \).

The expansion (24) has many other consequences. Observe that, in the annulus \( A \), the metric \( g_\varepsilon \) can be expanded as

\[
g_\varepsilon = (1 + O(\varepsilon^2 x^2)) \frac{dx^2}{x^2} + \sum_{\alpha=1}^n O(\varepsilon^2 x) \frac{dx \, dy_\alpha}{x} + \sum_{\alpha,\beta=1}^n (\delta_{\alpha\beta} + O(\varepsilon^2)) \frac{dy_\alpha \, dy_\beta}{x}. 
\]

(27)

In particular, this implies that

\[
|dx|_{g_\varepsilon}^2 = 1 + O(\varepsilon^2 x^2), 
\]

(28)
in \( A \).

It remains to estimate \( N_{g_\varepsilon}(0) \), which measures the discrepancy of \( g_\varepsilon \) from being Einstein (and in the proper gauge). By definition of (3) we have

\[
N_{g_\varepsilon}(0) = \text{Ric}^{g_\varepsilon} + ng_\varepsilon.
\]

This is supported in \( A \), and since both \( R_*^{(1)} k^{(1)} \) and \( I^* R_*^{(2)} k^{(2)} \) are \( O(\varepsilon^2) \), along with their derivatives, this error term is also \( O(\varepsilon^2) \) in \( A \). There is an improved estimate as \( x \to 0 \). Indeed, if \( \mathcal{F}_\varepsilon := x^2 g_\varepsilon \), then \( \text{Ric}^{\mathcal{F}_\varepsilon} = O(\varepsilon^2) \) in \( A \). Furthermore, if \( f = -\log x \), then

\[
\nabla^{\mathcal{F}_\varepsilon} df - df \circ df = \frac{1}{x} O(\varepsilon^2) \quad \text{and} \quad \Delta_{\mathcal{F}_\varepsilon} f - |df|_{\mathcal{F}_\varepsilon} = \frac{1}{x} O(\varepsilon^2)
\]
in \( A \). Now use (3) with \( \mathcal{F}_\varepsilon := x^2 g_\varepsilon \) and \( f := -\log x \) to conclude that

\[
\text{Ric}^{g_\varepsilon} + ng_\varepsilon = \frac{1}{x} O(\varepsilon^2),
\]
in \( A \). In particular, taking the norm with respect to \( g_\varepsilon \), we obtain finally the

**Proposition 7** For the metric \( g_\varepsilon \) on \( M_\varepsilon \), the tensor \( N_{g_\varepsilon}(0) = \text{Ric}^{g_\varepsilon} + ng_\varepsilon \) vanishes outside the annulus \( A \); in \( A \) its pointwise norm with respect to \( g_\varepsilon \) satisfies

\[
|N_{g_\varepsilon}(0)|_{g_\varepsilon} \leq C \varepsilon^2 x,
\]

where \( C \) is independent of \( \varepsilon \).
4 Linear estimates

Let $L_{g_\varepsilon}$ denote the linearization of the map $k \to N_{g_\varepsilon}(k)$ at $k = 0$. Then

$$L_{g_\varepsilon} = L_0 + B_\varepsilon,$$

where

$$L_\varepsilon := (\nabla^{g_\varepsilon})^* \nabla^{g_\varepsilon} - 2 R^{g_\varepsilon},$$

and

$$B_\varepsilon(\kappa) := \text{Ric}^{g_\varepsilon} \circ \kappa + \kappa \circ \text{Ric}^{g_\varepsilon} + 2 n \kappa.$$

Of course $B_\varepsilon$ would vanish if $g_\varepsilon$ were Einstein, and in any case, $B_\varepsilon$ is supported in $A$ and has coefficients which are $O(\varepsilon^2)$. Our goal in this section is to verify a certain weighted estimate for $L_{g_\varepsilon}$, which we now explain.

In the rescaled coordinates $w$ and $w'$, $M_\varepsilon$ contains an expanding annular region $T_\varepsilon = \{ C \varepsilon \leq |w| \leq C/\varepsilon \}$; the outer portion $1/2 \leq |w| \leq C/\varepsilon$ corresponds to a region in $M_1$ and the inner portion $2 \geq |w| \geq C\varepsilon$ corresponds by inversion to a region in $M_2$. We introduce polar coordinates

$$(r, \phi, \omega) \in \mathbb{R}^+ \times [0, \pi/2] \times S^{n-1}, \quad w := (r \cos \phi, r \sin \phi \omega).$$

Setting $s = \log r$, and dropping an irrelevant additive constant, then

$$T_\varepsilon = \{(s, \phi, \omega) : -s_\varepsilon \leq s \leq s_\varepsilon \}, \quad \text{where} \quad s_\varepsilon = -\log \varepsilon.$$

As $\varepsilon \to 0$, $T_\varepsilon$ expands to fill out $T_0 = \mathbb{R} \times S^n_+$, and the metric $g_\varepsilon$ converges (in $C^\infty$ on compact sets) to

$$\frac{1}{\cos^2 \phi} \left( ds^2 + d\phi^2 + \sin^2 \phi d\omega^2 \right),$$

This is the standard hyperbolic metric $g_0$ on $\mathbb{H}^{n+1}$, written in warped product form; the hemisphere $S^n_+$ with metric $(d\phi^2 + \sin^2 \phi d\omega^2)/\cos^2 \phi$ is isometric to $\mathbb{H}^n$. Notice that $L_{g_\varepsilon}$ and $L_\varepsilon$ both converge in this central region to $L_{g_0}$, the linearized Einstein operator on $\mathbb{H}^{n+1}$. Define a weight function $w_\varepsilon$ on $M_\varepsilon$ which is a smoothed version of the function

$$\bar{w}_\varepsilon(s) = \begin{cases} 
\cosh s / \cosh s_\varepsilon & \text{in } T_\varepsilon, \\
1 & \text{in } M_\varepsilon - T_\varepsilon.
\end{cases}$$

(We require that $w_\varepsilon$ is smooth, and agrees with $\bar{w}_\varepsilon$ except on a small neighborhood of $\partial T_\varepsilon$, where it remains bounded between $1/2$ and $2$.) Now let $\rho$ denote a boundary defining function for $M_\varepsilon$ which is $C^\infty$ and agrees (up to a small smoothing near $\partial B_1(p_j)$) with fixed boundary defining functions $\rho_j$ on $M_j - B_1(p_j)$ (in unrescaled coordinates!) and with $t = \cos \phi$ in $T_\varepsilon$. In terms of these, we define the doubly weighted Hölder spaces

$$\rho^\mu (w_\varepsilon)^\nu \Lambda^k_\alpha (M_\varepsilon),$$

17
which contain all functions of the form \( u = \rho^\mu (w_\varepsilon)^\nu v \), with \( v \in \Lambda_0^{k,\alpha}(M_\varepsilon) \). We define the space of symmetric 2-tensors \( \rho^\mu (w_\varepsilon)^\nu \Lambda_0^{2,\alpha}_0 S_0^2(M_\varepsilon) \) similarly. We denote the corresponding norm \( \| \cdot \|_{k,\alpha,\mu,\nu} \).

**Proposition 8** Suppose that neither of the operators \( L_{g_1} \) nor \( L_{g_2} \) has a non-trivial \( L^2 \) nullspace. Also, let \( 0 < \mu = \nu < n/2 \). Then for \( \varepsilon \) sufficiently small, the operator

\[
L_{g_\varepsilon} : \rho^\mu (w_\varepsilon)^\nu \Lambda_0^{2,\alpha}_0 S_0^2(M_\varepsilon) \rightarrow \rho^\mu (w_\varepsilon)^\nu \Lambda_0^{0,\alpha}_0 S_0^2(M_\varepsilon)
\]

is an isomorphism and its inverse \( G_{g_\varepsilon} \) has norm bounded independently of \( \varepsilon \).

**Proof:** Suppose this proposition were false. Then there would exist a sequence \( \varepsilon_j \rightarrow 0 \) and sequences of 2-tensors \( h_j \) for which

\[
\| h_j \|_{2,\alpha,\mu,\mu} = 1,
\]

while

\[
\| L_{g_\varepsilon} h_j \|_{0,\alpha,\mu,\mu} \rightarrow 0.
\]

Rewriting these norm inequalities gives the estimates

\[
|h_j(z)| \leq (\rho(z) w_{\varepsilon_j}(z))^\mu, \quad \text{and} \quad |L_{g_{\varepsilon_j}} h_j(z)| \leq \eta_j(\rho(z) w_{\varepsilon_j}(z))^\mu, \quad (29)
\]

for all \( z \in M_{\varepsilon_j} \), where \( \eta_j \rightarrow 0 \). We shall use a blow-up analysis to show that this leads to a contradiction.

Suppose that the supremum of the pointwise norm \( \rho^{-\mu}(w_{\varepsilon_j})^{-\mu}|h_j| \) occurs at some point \( q_j \). (If this supremum is not attained anywhere in the interior of \( M_\varepsilon \), then it is enough to assume that the value of this function at \( q_j \) is larger than half its supremum.) Possibly passing to a subsequence, there are several cases which may arise:

(i) \( q_j \) converges to a point \( q \) in the interior of \( M_1 \) or \( M_2 \);

(ii) \( q_j \) lies in \( \mathcal{T}_{\varepsilon_j} \) for every \( j \) and its coordinates \( (s_j, t_j, \omega_j) \) (where \( t_j = \cos \phi_j \)) satisfy \( |s_j| \leq C \), \( t_j \geq c > 0 \);

(iii) \( q_j \) lies in \( \mathcal{T}_{\varepsilon_j} \) for every \( j \), and its coordinates \( (s_j, t_j, \omega_j) \) satisfy \( |s_j| \rightarrow \infty \), \( t_j \geq c > 0 \);

(iv) \( q_j \) converges to a point \( q \) on \( \partial M_1 - \{p_1\} \) or \( M_2 - \{p_2\} \);

(v) \( q_j \) lies in \( \mathcal{T}_{\varepsilon_j} \) for every \( j \) and its coordinates \( (s_j, t_j, \omega_j) \) satisfy \( |s_j| \leq C \), \( t_j \rightarrow 0 \);

(vi) \( q_j \) lies in \( \mathcal{T}_{\varepsilon_j} \) for every \( j \) and its coordinates \( (s_j, t_j, \omega_j) \) satisfy \( |s_j| \rightarrow \infty \), \( t_j \rightarrow 0 \).
These will be ruled out in turn. In each case, we define a new sequence $k_j$, either by dividing by a normalizing constant so that $|k_j(q_j)|$ is bounded away from 0, or else by rescaling the independent variable, or both. Extracting a subsequence if this is necessary, we can assume that this new sequence converges to a symmetric 2-tensor $k$ which is defined, either one of the $M_j$ or on $\mathbb{H}^{n+1}$, and which is a solution of the linearized Einstein equation there. The goal is to show that it satisfies certain $L^\infty$ bounds, and then to show that no such solution exists. In the next several paragraphs, we deduce the existence of this limiting tensor $k$ and deduce the bounds it must satisfy. Only afterwards do we show that these bounds preclude its existence.

In case (i), $\rho(q_j) \geq c > 0$ and $w_{\epsilon_j}(q_j) = 1$ (or at least is bounded away from zero), so we take

$$k_j := h_j.$$  

Suppose, for example, that $q_j \to q \in M_1$. Then $k_j$ converges (in $C^\infty$ on compact sets of $M_1$) to a symmetric 2-tensor $k$ on $M_1$ which satisfies $L_{g_0} k = 0$ and also, from (2.9), the bound $|k(z)| \leq \rho_1(z)^\mu$. Furthermore, $k$ is nontrivial since $k(q) \neq 0$. This case can be ruled out immediately since we are assuming that the $L^2$ nullspace of $L_{g_0}$ is trivial, and hence, by Proposition 5, so is its nullspace in $\rho_1^2 \Lambda_0^{2,\alpha} S_0^\alpha(M_1)$ for $\mu > 0$.

Now consider case (ii). As $\epsilon \to 0$, $T_\epsilon \to T_0 = \mathbb{H}^{n+1}$, and by assumption, $q_j$ remains in a compact set of $T_0$. Thus we may assume that $q_j \to q \in T_0$. We have again that $\rho(q_j) \geq c' > 0$, whereas $w_{\epsilon_j}(z) \to 0$. However, for $z \in T_0$,

$$\frac{w_{\epsilon_j}(z)}{w_{\epsilon_j}(q_j)} = \frac{\cosh s}{\cosh s_j} \to c'' \cosh s, \quad c'' > 0.$$  

(As before, and as in all of the remaining cases, this convergence is $C^\infty$ on compact sets.) Thus if we define

$$k_j := (w_{\epsilon_j}(q_j))^{-\mu} h_j,$$

then $k_j \to k$ where $k(q) \neq 0$, $L_{g_0} k = 0$, and $|k(z)| \leq c (\cos \phi \cosh s)^\mu$.

In case (iii), we first recenter the coordinates on $T_\epsilon$ by replacing the independent variable $s$ by $s - s_j$, where $s_j$ is the $s$ coordinate of $q_j$ (so $|s_j| \to \infty$, by hypothesis), and then define

$$k_j(s, \phi, \omega) := (w_{\epsilon_j}(q_j))^{-\mu} h_j(s - s_j, \phi, \omega).$$

But $|h_j(s - s_j, \phi, \omega)| \leq c (w_{\epsilon_j}(s - s_j))^\mu$ and

$$\frac{w_{\epsilon_j}(s - s_j)}{w_{\epsilon_j}(q_j)} = \frac{\cosh(s - s_j)}{\cosh S_{\epsilon_j}} \frac{\cosh S_{\epsilon_j}}{\cosh s_j} \to c e^s, \quad c > 0,$$

so the limit tensor $k$ is nontrivial, satisfies $L_{g_0} k = 0$ and the pointwise bound $|k(z)| \leq c (\cos \phi e^s)^\mu$.  

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In case (iv), suppose that \( q_j \) remains in some boundary coordinate chart \((x, y)\) and has coordinates \((x_j, y_j)\), where \( x_j \to 0 \) and \( y_j \to y_0 \). We may as well assume that \( y_0 = 0 \), and then define

\[
k_j(x, y) := x_j^{-\mu} (R_{x_j})^* h_j.
\]

Then \( k_j \to k \neq 0 \), where \( k \) is defined on all of \( \mathbb{H}^{n+1} \), satisfies \( L_{g_0} k = 0 \), and the bound \( |k| \leq cx^\mu \). Note that this is the exact same bound as in the previous case, once we have change coordinates \( x = \cos \phi e^s \).

In cases (v) and (vi), we can do nearly identical rescalings and obtain to obtain nontrivial tensors \( k \) defined on all of \( \mathbb{H}^{n+1} \) which satisfy \( L_{g_0} k = 0 \) and the bounds \( |k(z)| \leq c (\cos \phi \cosh s)^\mu \) and \( |k(z)| \leq c (\cos \phi e^s)^\mu \), respectively.

We have already eliminated case (i), so it remains to rule out the existence of nontrivial solutions of \( L_{g_0} k = 0 \) on \( \mathbb{H}^{n+1} \) which are bounded either by \( (\cos \phi \cosh s)^\mu \) or \( (\cos \phi e^s)^\mu \). Observe that the latter case is included in the former. It suffices to check that any such solution lies in \( \rho^\delta L^2 \mathbb{H}^{n+1} \) for some \( \delta > 0 \), where \( \rho \) is a global defining function on \( \mathbb{H}^{n+1} \). This is because by Proposition 5, \( k \) would then be polyhomogeneous, and in particular decay like \( \rho^\delta \) on the entire boundary, and this would then be ruled out by nondegeneracy. This is a simple calculation. It suffices to work in the ball \( |w| = |(x, y)| < 1 \) and, since \( |k(z)| \leq c (x/r)^\mu r^{-\mu} \), where \( r = (x^2 + |y|^2)^{1/2} \), we have to show that

\[
\int_{x^2 + |y|^2 \leq 1} \left( \frac{t}{r} \right)^{2\mu} \frac{dx dy}{x} < +\infty.
\]

Performing the change of variable \( x = r \cos \phi \) and \( y = r \sin \phi \omega \), we must show

\[
\int_0^1 \int_0^1 t^{2\mu - 2\delta - 1} r^{-2\mu - 2\delta + n - 1} dr dt < \infty,
\]

and this holds provided

\[
\delta < \mu < \frac{n - 2\delta}{2}.
\]

Since our only other requirements are that \( \delta > 0 \) and \( 0 < \mu < n \), we see that this is easily satisfied if we choose \( \delta \) sufficiently small, provided \( \mu \in (0, n/2) \). \( \square \)

It is fortuitous that all the possible limiting cases which arise here can be handled solely by Proposition 5. However, even if this were not the case, there is a complete theory for the mapping properties and regularity of solutions of elliptic uniformly degenerate operators on doubly weighted spaces \( x^{\mu} r^{\nu} \Lambda_0^{\alpha} (M) \) and \( x^{\mu} r^{\nu} L^2_0 (M) \), which would have given us the same kind of conclusions (this would only extend the range in which the weight parameter \( \mu \) can be chosen to \((0, n)) \). This more intricate linear theory is useful in many other problems and we shall return to it elsewhere.

It remains to adapt this theorem to the case when at least one of the summands \((M_j, g_j)\) is degenerate. Recall that, given \( \eta^{(j)} \in C^{2,\alpha} S^2_0 (\partial M_j) \), we have defined a metric \( g_{\eta^{(j)}} \) on \( M_j \). Since these metrics are identically equal to \( g_j \).
near \( p_j \), they can be glued together as in \( \S 3 \) to produce a metric \( g_{\varepsilon, \eta} \) on \( M_\varepsilon \), \( \eta = (\eta^{(1)}, \eta^{(2)}) \). This allows one to define a nonlinear mapping

\[
\hat{N}_{g_\varepsilon}(\eta, k) := \text{Ric}_{g_\varepsilon} + k + n(g_{\varepsilon, \eta} + k) + (\mathcal{B}_{g_\varepsilon}(k)^* B_{g_\varepsilon}(k),
\]

with linearization at 0 \( \hat{L}_{g_\varepsilon} \).

Let \( K_j = \ker L_{g_\varepsilon} \) and write \( m_j := \dim K_j \). By our current assumption, at least one of the \( m_j \) is nonzero, and to be definite we suppose that both are. According to Proposition 6, or rather, its modification at the end of \( \S 2 \), for \( j = 1, 2 \), there exist symmetric 2-tensors \( \gamma^{(j)}_i \), \( i = 1, \ldots, m_j \), with span \( S_j \), which are supported away from \( p_j \), and such that the mapping \( (17) \) on \( M_j \) is surjective. We write \( S = S_1 \oplus S_2 \).

Recall that \( L_{g_\varepsilon} \) is the linearization of \( \hat{N}_{g_\varepsilon} \) with respect to the second factor, i.e.,

\[ L_{g_\varepsilon} = \hat{L}_{g_\varepsilon}(0, \cdot). \]

This operator is self-adjoint and Fredholm on \( L^2 \mathcal{S}_0^2(M_\varepsilon; dV_{g_\varepsilon}) \), so for some \( a > 0 \), \( \text{spec}(L_{g_\varepsilon}) \cap (-a, a) \) is discrete. The eigenvalues in this range vary continuously with \( \varepsilon \) (so long as they stay within this interval) but the multiplicity of 0 as an eigenvalue may not be constant. To regain stability we consider the set \( \mathcal{P}_\varepsilon \) of all ‘small eigenvalues’, which are by definition those which tend to 0 as \( \varepsilon \to 0 \).

We may choose \( a > 0 \) so that \( \mathcal{P}_\varepsilon \) coincides precisely with the set of eigenvalues of \( L_{g_\varepsilon} \) lying in \((-a, a)\). The sum of the corresponding eigenspaces is denoted \( V_\varepsilon \) and it is standard, cf. \([13]\), that \( V_\varepsilon \) varies continuously with \( \varepsilon \).

Suppose that \( h_j \in K_j \), \( ||h_j||_{L^2} = 1 \); using cutoff functions \( \psi(r_j/\tau) \) as above, with \( r_j = \text{dist}(w, p_j) \), we define \( \tilde{h}_j = \psi(r_j/\tau)h_j \), extended by 0 to the rest of \( M_\varepsilon \) (i.e. on the other factor \( M_{j'} \), \( j' \neq j \)). Then \( \tilde{h}_j \to h_j \) in \( L^2 \) as \( \tau \to 0 \), and moreover, if

\[ \Pi_\varepsilon : L^2(M_\varepsilon, g_\varepsilon) \to V_\varepsilon \]

is the orthogonal projection, then \( ||(I - \Pi_\varepsilon)\tilde{h}_j||_{L^2} \to 0 \). Consequently, \( \Pi_\varepsilon \tilde{h}_j \to h_j \in K_j \), at least uniformly on compact sets of \( M_j \), and \( \Pi_\varepsilon \tilde{h}_j \to h_j \) on the other component \( M_{j'} \), \( j' \neq j \). In particular, the set of all \( \Pi_\varepsilon \tilde{h}_j \), as \( h_j \) varies over \( K_j \), is a basis of \( V_\varepsilon \).

We define the space \( \rho^{\mu}\Lambda_0^{\alpha\cdot} S_0^2(M_\varepsilon) \perp \) to be the closed subspace of elements of \( \rho^{\mu}\Lambda_0^{\alpha\cdot} S_0^2(M_\varepsilon) \) which are \( L^2 \) orthogonal to elements of \( V_\varepsilon \). We prove

**Proposition 9** For \( \varepsilon \) sufficiently small and \( 0 < \mu < \inf(2, n/2) \), the mapping

\[ \hat{L}_{g_\varepsilon} : \mathcal{S} \oplus \rho^{\mu}\Lambda_0^{\alpha\cdot} S_0^2(M_\varepsilon) \perp \to \rho^{\mu}\Lambda_0^{\alpha\cdot} S_0^2(M_\varepsilon) \]

is an isomorphism and its inverse \( \hat{G}_{g_\varepsilon} \) is bounded independently of \( \varepsilon \).

**Proof:** Following the proof of Proposition 8 we deduce that

\[ L_{g_\varepsilon} : \rho^{\mu}\Lambda_0^{\alpha\cdot} S_0^2(M_\varepsilon) \perp \to \rho^{\mu}\Lambda_0^{\alpha\cdot} S_0^2(M_\varepsilon) \perp \]

is an isomorphism whose inverse is bounded independently of \( \varepsilon \).
Next we show that for \( \varepsilon \) small, and all \( h \in V_\varepsilon \), there exists \( \gamma \in S \) for which

\[
\int_{M_\varepsilon} \langle \hat{L}_{g_\varepsilon}(\gamma, 0), h \rangle \neq 0
\]

This follows from the fact that for any \( \gamma \in S \), \( \hat{L}_{g_\varepsilon}(\gamma, 0) = \hat{L}_{g_1}(\gamma^{(1)}, 0) + \hat{L}_{g_2}(\gamma^{(2)}, 0) \), and the first term on the right is supported in \( M_1 \setminus \{p_1\} \) and the second on \( M_2 \setminus \{p_2\} \). Thus we need to find \( \gamma^{(1)} \) and \( \gamma^{(2)} \) such that

\[
\int_{M_\varepsilon} \langle \hat{L}_{g_1}(\gamma^{(1)}, 0), h_1 \rangle + \langle \hat{L}_{g_2}(\gamma^{(2)}, 0), h_2 \rangle \neq 0.
\]

It is enough to check this for all \( h_\varepsilon = \Pi_\varepsilon \hat{h} \), where \( \hat{h} \) is the extension to all of \( M_\varepsilon \) of an arbitrary element of \( K_1 \) or \( K_2 \). Letting \( \varepsilon \to 0 \) on the left gives

\[
\int_{M_1} \langle \hat{L}_{g_1}(\gamma^{(1)}, 0), h_1 \rangle + \int_{M_2} \langle \hat{L}_{g_2}(\gamma^{(2)}, 0), h_2 \rangle
\]

for some elements \( h_j \in K_j \), not both equal to 0. That we can choose \( \gamma^{(j)} \) so that this can be made nonvanishing is the content of the modification of Proposition 6 at the end of \( \S \) 2. The proof is complete. \( \square \)

5 Proof of Theorem 1

Following the development of the linear analysis in the last section, it is now a simple matter to complete the proof of the main theorem. Recall that our goal is to find a correction term \( k_\varepsilon \) to \( g_\varepsilon \) so that \( g_\varepsilon + k_\varepsilon \) is Poincaré-Einstein. The nonlinear operator \( k \to N_{g_\varepsilon}(k) \) is a second order quasilinear operator with coefficients which are polynomial in the entries of \( (\nabla g_\varepsilon)^j k \), \( j = 0, 1, 2 \), and \( (g_\varepsilon + k)^{-1} \). The same is true for the remainder term

\[
Q_{g_\varepsilon}(k) := N_{g_\varepsilon}(k) - N_{g_\varepsilon}(0) - L_{g_\varepsilon} k,
\]

and in addition, if \( \mu > 0 \), there exists a constant \( C > 0 \), which does not depend on \( \varepsilon \), such that

\[
||Q_{g_\varepsilon}(k_2) - Q_{g_\varepsilon}(k_1)||_{0, \alpha, \mu} \leq C (||k_2||_{2, \alpha, \mu} + ||k_1||_{2, \alpha, \mu}) ||k_2 - k_1||_{2, \alpha, \mu},
\]

for all \( k_1, k_2 \in \rho^\mu A^2_{\alpha, \alpha} S_0^2(M_\varepsilon) \) satisfying \( ||k_1||_{2, \alpha, \mu} + ||k_2||_{2, \alpha, \mu} \leq 1 \).

In the case where the metrics \( g_1 \) and \( g_2 \) are both nondegenerate, we fix \( \mu \in (0, 1) \) and use the result of Proposition 8 to shows that

\[
||N_{g_\varepsilon}(0)||_{0, \alpha, \mu} \leq C' \varepsilon^{2-\mu}
\]

Then, Proposition 8 allows use to rephrase the equation \( N_{g_\varepsilon}(k) = 0 \) as a fixed point problem

\[
k = -G_{g_\varepsilon} (N_{g_\varepsilon}(0) + Q_{g_\varepsilon}(k)).
\]
The fact that $G_{g_\varepsilon}$ is uniformly bounded together with (31) and (32) implies that the mapping $k \to -G_{g_\varepsilon}(N_{g_\varepsilon}(0) + Q_{g_\varepsilon}(k))$ is a contraction mapping from a small ball of $\rho^\mu \Lambda_0^{2,0} S_0^2(M_\varepsilon)$ into itself. This complete the proof of the existence of a solution of $N_{g_\varepsilon}(k) = 0$.

The case where one of the metrics $g_1$ or $g_2$ is degenerate can be treated similarly using Proposition 9 instead of Proposition 8. Observe that an estimate similar to (31) is valid for all $(\gamma_1, k_1), (\gamma_2, k_2) \in S \oplus \rho^\mu \Lambda_0^{2,0} S_0^2(M_\varepsilon)$. We leave the details to the reader.

6 Scalar positive conformal infinities

Given any two Poincaré-Einstein metrics $(M_1, g_1)$ and $(M_2, g_2)$, we have shown how to produce a family of Poincaré-Einstein metrics on the boundary connected sum $M_1 \#_b M_2$ in such a way that the new conformal infinity is unchanged away from a small neighborhood of the gluing points, in case the linear Einstein operators on both factors have trivial $L^2$ kernels, or is altered only very slightly, in the general case. From here it is only a small step to prove Corollary 1 concerning how the Yamabe class of the conformal infinities of these new metrics relate to the initial conformal infinities $\mathfrak{e}(g_1)$ and $\mathfrak{e}(g_2)$.

Let us change notation slightly from the last section and write the Poincaré-Einstein metric as $g_{\varepsilon}$, and decompose it as a sum of an explicit approximate solution $\tilde{g}_{\varepsilon}$ and the deformation term $k_{\varepsilon}$. Let us denote by $h_1, h_2$ and $h_{\varepsilon} = \tilde{h}_{\varepsilon} + q_{\varepsilon}$ the explicit metric representatives of the conformal infinities of each of these metrics, where $\tilde{h}_{\varepsilon}$ is the conformal infinity of $\tilde{g}_{\varepsilon}$ and, in the degenerate case, $q_{\varepsilon} \in S$, but equals 0 otherwise. For simplicity, we also write $Y_j = \partial M_j$.

We first recall the connected sum construction for metrics of constant scalar curvature from [20] and [14]. The idea of the construction and much of the implementation is almost exactly the same as what we have done here; indeed, the main substantial difference is the need for the theory of uniformly degenerate operators for the interior problem. In any case, suppose that $h_1$ and $h_2$ are metrics of constant scalar curvature on $Y_1$ and $Y_2$, respectively; [20] handles the case where the scalar curvatures are the same positive number, while [14] treats the more general situation where the constants may differ and possibly even have different signs. Having chosen points $p_j \in Y_j$, one identifies by inversion the small annuli of inner and outer radii $\varepsilon$ and $2\varepsilon$, for example, around these points to define $Y_1 \# Y_2$, and then uses a partition of unity to patch together the metrics to define a family of metrics $h_{\varepsilon}$. (This construction is phrased differently in [20]: there, fixed annuli around the $p_j$ are transformed conformally to long cylinders, and these are then patched together; the approximate solution metric $\tilde{h}_{\varepsilon}$ is given by a conformal factor which has the shape of a cosh curve on this cylinder, hence is exponentially in the middle small relative to its length. The equivalence of this picture with the other one is immediate.) This step is clearly identical to what is happening on the boundary in our construction of the approximate Poincaré-Einstein metrics $\tilde{g}_{\varepsilon}$.

The constant scalar curvature metric $h_{\varepsilon}$ is obtained by solving the scalar
Yamabe equation, and is a conformal deformation from the background metric $h_\varepsilon$. In particular, if $h_1$ and $h_2$ are both scalar positive metrics, then the conformal class of $h_\varepsilon$ on $Y_1 \# Y_2$ is also scalar positive, provided $\varepsilon$ is small enough.

In the case of nondegenerate Poincaré-Einstein gluing, the conformal infinity of $g_\varepsilon$ is the same as that of $\hat{g}_\varepsilon$, and we have just shown that this is scalar positive if this is true for both summands. In the degenerate case, the proof of Theorem 1 shows that $\epsilon(g_\varepsilon)$ is a $C^2$ small perturbation of $\epsilon(\hat{g}_\varepsilon)$, and since the Yamabe functional is continuous in the $C^2$ topology, the conformal class $\epsilon(g_\varepsilon)$ is again scalar positive. This concludes the proof of Corollary 1.

7 Plumbing and surgery?

We have proved that it is possible to perform a boundary connected sum in the category of Poincaré-Einstein metrics. There are many other interesting ways to join together $M_1$ and $M_2$ along their boundaries as differentiable manifolds, and it is natural to ask whether these operations may also be done in the Poincaré-Einstein category. The two operations we have in mind are:

- **Boundary plumbing**: Suppose that $\Sigma$ is a $k$-dimensional manifold which is smoothly embedded in $\partial M_j$, $j = 1, 2$, in such a way that the normal bundles $\nu_j$ of these embeddings are equivalent. If $S\nu_j$ denote the unit sphere bundles, then the identification $S\nu_1 \cong S\nu_2$ extends to an orientation reversing bundle map $\nu_1 \setminus \{0\} \to \nu_2 \setminus \{0\}$ which is homogeneous of degree $-1$ on the fibres, which we call the inversion $I$. The bundles $\nu_j \oplus \mathbb{R}^+$ are diffeomorphic to inward-pointing tubular neighborhoods $T_j^+$ of $\Sigma$ in $M_j$ (both are half-ball bundles over $\Sigma$). The natural extension of $I$ is a diffeomorphism $T_1^+ - \Sigma \to T_2^+ - \Sigma$, and using this we define the boundary join of $M_1$ and $M_2$ along $\Sigma$, $M_1 \#_{\Sigma} M_2$.

- **Boundary surgery**: Suppose that $S^k \subset \partial M_1$ and $S^{n-k-1} \subset \partial M_2$ are spheres, both with trivial normal bundles, and let $T_j$, $T_j^+$, be the corresponding tubular neighborhoods in $\partial M_j$, $M_j$, respectively. Thus $M_j - T_j^+$ are both manifolds with corners, each with two boundary hypersurfaces

$$ (\partial M_1 - T_1) \cup (S^k \times B^{n-k}) \quad \text{and} \quad (\partial M_2 - T_2) \cup (B^{k+1} \times S^{n-k-1}), $$

respectively. A standard construction in topology joins these two spaces using the plug $B^{k+1} \times B^{n-k}$, which has boundary $(S^k \times B^{n-k}) \cup (B^{k+1} \times S^{n-k-1})$, and hence may be inserted between the two summands $M_j - T_j^+$ to define the surgered manifold $M_1 \#_{b,\sigma} M_2$ (the $b, \sigma$ subscript simply means ‘boundary surgery’, but we suppress the dimension of the surgery from the notation).

Let $(M_j, g_j)$ be two Poincaré-Einstein metrics. We ask the following questions:

- Suppose $\Sigma \subset \partial M_j$ and the two normal bundles $\nu_j$ are equivalent. Does the boundary join $M_1 \#_{b,\Sigma} M_2$ admit a family of Poincaré-Einstein metrics which converges nicely to $g_j$ on compact sets of $\overline{M_j} - \Sigma$, and such that the conformal infinity $\epsilon(g_\varepsilon)$ is close to $\epsilon(g_j)$ away from the neck region?
If $S^k$ and $S^{n-k}$ are spheres with trivial normal bundles in $\partial M_1$ and $\partial M_2$, respectively, then does the surgered manifold $M_1 \#_{\partial \Sigma} M_2$ admit a similar family of Poincaré-Einstein metrics $g_\varepsilon$?

The utility of these constructions is obvious, and in particular if the second were always possible, then it would not be unreasonable to hope that any (compact, nullcobordant, simply connected) scalar-positive manifold might admit a Poincaré-Einstein filling. There would be many other applications too. We are not able to answer either of these questions one way or the other, but these are clearly important directions for future research. However, we suspect that if either $M_1 \#_{\partial \Sigma} M_2$ or $M_1 \#_{\partial \sigma} M_2$ admits a Poincaré-Einstein metric $g$, then $g$ is ‘quite far’ from any family of ‘locally constructed’ approximate solution metrics $\tilde{g}_\varepsilon$, unlike the construction above. In particular, it does not seem likely that there should exist families of Poincaré-Einstein metrics $g_\varepsilon$ in either case which converge to $g_1$ and $g_2$ away from the necks and which have restrictions to the necks which are close to any simple model form. As heuristic evidence for this, we note that were such a family $g_\varepsilon$ to exist, then one would expect that rescalings of its restriction to the neck region should converge to some sort of model Einstein metric. For example, in the boundary connected sum construction, this model metric is just hyperbolic space. However, calculations seem to indicate that there are no warped product candidates for the model Einstein metrics in these more general cases, and it is not clear where else to look. An additional nonrigorous counterargument is that if $g_\varepsilon$ were to have negative sectional curvature in the neck region, then one would be able to join or surger together two copies of hyperbolic space and obtain a manifold with negative sectional curvature everywhere, but which is simply connected and has nontrivial higher homotopy groups, which is impossible.

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