ON GENERALIZATIONS OF THE SYNGE-KŘÍŽEK MAXIMUM ANGLE CONDITION FOR $d$-SIMPLICES

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Abstract. In this note we present a generalization of the maximum angle condition, proposed by J. L. Synge in 1957 and M. Křížek in 1992 for triangular and tetrahedral elements, respectively, for the case of higher-dimensional simplicial finite elements. Its relations to the other angle-type conditions commonly used in finite element methods are analysed.

1. Introduction

Let $\mathcal{F} = \{T_h\}_{h \to 0}$ be a family of conforming (face-to-face) triangulations $T_h$ of a bounded polygonal domain. In 1957, see [18], Synge proved that linear triangular finite elements yield the optimal interpolation order in the $C$-norm provided the following maximum angle condition is satisfied: there exists a constant $\gamma_0 < \pi$ such that for any triangulation $T_h \in \mathcal{F}$ and any triangle $T \in T_h$ the upper bound

$$\gamma_T \leq \gamma_0,$$

holds, where $\gamma_T$ is the maximum angle of $T$. Later, Babuška and Aziz [1], Barnhill and Gregory [2], and Jamet [13] independently derived the optimal interpolation order in the energy norm of finite element approximations under the condition (1), see also [14] in this respect.

In 1992, the Synge-condition (1) was generalized by Křížek [15] to tetrahedral elements as follows: there exists a constant $\gamma_0 < \pi$ such that for any face-to-face tetrahedralization $T_h \in \mathcal{F}$ and any tetrahedron $T \in T_h$ one has

$$\gamma_D \leq \gamma_0 \quad \text{and} \quad \gamma_F \leq \gamma_0,$$

where $\gamma_D$ is the maximum dihedral angles between faces of $T$ and $\gamma_F$ is the maximum angle in all four triangular faces of $T$. The optimal interpolation estimates were obtained in [15] for various norms under the condition (2), thus allowing the usage of many degenerating (skinny or flat) tetrahedra unavoidably appearing during mesh generation and adaptivity processes in various real-life applications [5, 7].

Recently, some higher-dimensional generalization of conditions (1) (and its relation to condition (2)) was proposed and analysed in [11] (see also [12]). However, this generalization is not of the form of an upper estimate for (all or some) angles of the simplices generated (cf. Definition 2.4).

2. Angle conditions in higher dimensions

Recall that a $d$-simplex $S$ in $\mathbb{R}^d$, $d \in \{1, 2, 3, \ldots\}$, is the convex hull of $d + 1$ vertices $A_0, A_1, \ldots, A_d$ that do not belong to the same $(d-1)$-dimensional hyperplane, i.e.,

$$S = \text{conv}\{A_0, A_1, \ldots, A_d\}.$$
Let
\[ F_i = \text{conv}\{A_0, \ldots, A_{i-1}, A_{i+1}, \ldots, A_d\} \]
be the facet of \( S \) opposite to the vertex \( A_i \) for \( i \in \{0, \ldots, d\} \).

For \( d \geq 2 \) the dihedral angle \( \beta_{ij} \) between two facets \( F_i \) and \( F_j \) of \( S \) is defined by means of the inner product of their outward unit normals \( n_i \) and \( n_j \)
\[ \cos \beta_{ij} = -n_i \cdot n_j. \]

In 1978, Eriksson introduced a generalization of the sine function to an arbitrary \( d \)-dimensional spatial angle, see [8, p. 74].

**Definition 2.1.** Let \( \hat{A}_i \) be the angle at the vertex \( A_i \) of the simplex \( S \). Then \( d \)-sine of the angle \( \hat{A}_i \) for \( d > 1 \) is given by
\begin{align*}
\sin_d(\hat{A}_i|A_0A_1 \ldots A_d) &= \frac{d^{d-1} (\text{meas}_d S)^{d-1}}{(d-1)! \prod_{j=0,j \neq i}^{d-1}\text{meas}_d F_j}. 
\end{align*}

**Remark 2.2.** The \( d \)-sine is really a generalization of the classical sine function. In order to see that consider an arbitrary triangle \( A_0A_1A_2 \). Let \( \hat{A}_0 \) be its angle at the vertex \( A_0 \). Then, obviously,
\[ \text{meas}_2(A_0A_1A_2) = \frac{1}{2} |A_0A_1||A_0A_2| \sin \hat{A}_0. \]
Comparing this relation with (3) for \( d = 2 \), we find that
\[ \sin \hat{A}_0 = \sin_2(\hat{A}_0|A_0A_1A_2). \]

**Definition 2.3.** A family \( \mathcal{F} = \{T_h\}_{h \to 0} \) of face-to-face partitions of a polytope \( \Omega \subset \mathbb{R}^d \) into \( d \)-simplices is said to satisfy the generalized minimum angle condition if there exists a constant \( C > 0 \) such that for any \( T_h \in \mathcal{F} \) and any \( S = \text{conv}\{A_0, \ldots, A_d\} \in T_h \) one has
\begin{align*}
\forall \ i & \in \{0,1,\ldots, d\} \quad \sin_d(\hat{A}_i|A_0A_1 \ldots A_d) \geq C > 0.
\end{align*}

This condition is investigated in the paper [3]. It generalizes the well-known Zlámal minimum angle condition for triangles (see [6, 19, 20]), which is stronger than (1).

**Definition 2.4.** A family \( \mathcal{F} = \{T_h\}_{h \to 0} \) of face-to-face partitions of a polytope \( \Omega \subset \mathbb{R}^d \) into \( d \)-simplices is said to satisfy the generalized maximum angle condition if there exists a constant \( C > 0 \) such that for any \( T_h \in \mathcal{F} \) and any \( S = \text{conv}\{A_0, \ldots, A_d\} \in T_h \) one can always choose \( d \) edges of \( S \), which, when considered as vectors, constitute a (higher-dimensional) angle whose \( d \)-sine is bounded from below by the constant \( C \).

**Remark 2.5.** The generalized maximum angle condition is really weaker than the generalized minimum angle condition as it accepts e.g. degenerating path-simplices [4], which obviously violate Definition 2.3.

The main result on the interpolation estimate is given in the following theorem.

**Theorem 2.6.** Let \( \mathcal{F} \) be a family of face-to-face partitions of a polytope \( \Omega \subset \mathbb{R}^d \) into \( d \)-simplices satisfying the generalized maximum angle condition from Definition
Then there exists a constant $C > 0$ such that for any $T_h \in F$ and any $S \in T_h$ we have
\[
\|v - \pi_S v\|_{1, \infty} \leq C h_S |v|_{2, \infty}, \quad \forall v \in C^2(S),
\]
where $\pi_S$ is the standard Lagrange linear interpolant and $h_S = \text{diam} S$.

For the proof see [11].

Definition 2.7. A family $F = \{T_h\}_{h \to 0}$ of face-to-face partitions of a polytope $\Omega$ into $d$-simplices is said to satisfy the $d$-dimensional maximum angle condition if there exists a constant $\gamma_0 < \pi$ such that for $T_h \in F$ and any simplex $S \in T_h$ and any subsimplex $S' \subseteq S$ with vertex set contained in the vertex set of $S$, the maximum dihedral angle in $S'$ is less than or equal to $\gamma_0$.

Remark 2.8. It is worth to mention that the maximum angle condition is only sufficient to provide the convergence of the finite element approximations as shown in [10].

3. Main results

In this section we present the main results of the work.

Lemma 3.1. For a $d$-simplex we observe that
\[
\sin_d(\hat{A}_i|A_0A_1\ldots A_d) = \sin_{d-1}(\hat{A}_i|A_0A_1\ldots A_{d-1}) \prod_{j=0, j \neq i}^{d-1} \sin(\beta_j),
\]
where $\beta_j$ is the dihedral angle between the facet opposite to $A_j$ and the facet opposite to $A_d$.

For the proof see [8, p. 74–76].

Remark 3.2. The immediate consequence of Lemma 3.1 is that $\sin_d$ is always less than or equal to one.

As usual, we denote by $S^{d-1}$ the unit sphere in $\mathbb{R}^d$, and by $(S^{d-1})^N$ the $N$-fold cartesian product, i.e. the space of $N$ unit vectors in $\mathbb{R}^d$.

Lemma 3.3 (Properties of $\sin_d$). We can define $\sin_d$ as a function on the space of $d$ unit vectors $\vec{t}_1, \ldots, \vec{t}_d$ in $\mathbb{R}^d$, $(S^{d-1})^d$, with the following properties:

a) On the open, dense subset of $d$ vectors spanning $\mathbb{R}^d$ (linearly independent),
\[
\sin_d(\vec{t}_1, \ldots, \vec{t}_d) = \text{equal to the expression from [3] with } \hat{A}_i \text{ equal to the origin and the other points from } \{A_0, \ldots, A_d\} \text{ equal to the endpoints of the vectors}.
\]

b) On the closed subspace of $d$ vectors not spanning $\mathbb{R}^d$ (linearly dependent),
\[
\sin_d(\vec{t}_1, \ldots, \vec{t}_d) = 0.
\]

c) $\sin_d$ is continuous.

Proof. All these properties are implicitly found in Eriksson’s work [8].

Part a): On page 72 of [8], he notes that the definition of $\sin_d$ does not change if one of the vectors is multiplied by a nonzero constant. Thus we can normalize all the vectors and use only unit vectors.

Part b): We take this as the definition of $\sin_d$ for linearly dependent vectors.

Part c): On the two sets in a) and in b) considered separately, $\sin_d$ is obviously continuous. We must check that when a tuple of linearly independent unit vectors $(\vec{t}_1, \ldots, \vec{t}_d)$ approaches a linearly dependent limit, $\sin_d$ approaches zero. This is
clear for $d = 2$ and can be proved by induction for any $d$ using the product formula from Lemma 3.1. If any subset of the vectors becomes close to a linearly dependent set, we can apply the product formula so that the evaluation of $\sin_{d-1}$ involves those vectors, so the product tends to zero by the induction hypothesis. Otherwise, to get a degeneration, a dihedral angle must tend to zero or $\pi$, so that the product tends to zero by the continuity of the ordinary (two-dimensional) sine.

\[\square\]

**Remark 3.4.** It is necessary to restrict the domain of definition for $\sin_d$ in order to have continuity. Otherwise we would have the following problem: If one of the vectors tend to zero, the limit would include a zero vector and thus be a linearly dependent set. By part b), $\sin_d$ should be zero. But since multiplying an edge by any nonzero constant leaves $\sin_d$ unchanged, this would violate continuity.

Let $S_n = \text{conv}\{A_0^n, A_1^n, \ldots, A_d^n\}$, $n = 1, 2, \ldots$, be any infinite sequence of simplices, and consider the vectors

$$\ell_{X_nY_n} = \frac{X_nY_n}{|X_nY_n|}$$

for any pair $\{X_n, Y_n\} \subset \{A_0^n, A_1^n, \ldots, A_d^n\}$, where the symbol $| \cdot |$ denotes the length of the vector.

**Lemma 3.5.** There is a subsequence $\{S'_{n'}\} \subset \{S_n\}$ such that all the sequences $\{\ell_{X_nY_n}\}$ converge.

**Proof.** The sequence of tuples $\{(\ell_{X_nY_n})\}$ for all pairs $\{X_n, Y_n\} \subset \{A_0^n, \ldots, A_d^n\}$ is an infinite subset of the space $(S^{d-1})^N$, where $N = \binom{d+1}{2}$ is the number of pairs. Since this space is compact, the sequence has at least one limit point. Let $\{S'_{n'}\}$ be a subsequence converging to such a limit point. \[\square\]

**Theorem 3.6.** Let $S_n = \text{conv}(A_0^n, A_1^n, \ldots, A_d^n)$, $n = 1, 2, \ldots$, be an infinite sequence of simplices. If the sequence violates the condition from Definition 2.7 then it also violates the generalized maximum angle condition in Definition 2.4.

**Proof.** By Lemma 3.5, we can assume that the limiting vectors $\ell_{XY}$ for all pairs $\{X_n, Y_n\} \subset \{A_0^n, \ldots, A_d^n\}$ exist. Let $S'_{n'} \subset S_n$ be as in Definition 2.4, and assume that there is an infinite sequence of dihedral angles of $S'_{n'}$ tending to $\pi$. We need to show that all the $\sin_{d-1}$ tend to zero. Let $d'$ be the dimension of $S'_{n'}$. By reordering, we can assume that the vertices of $S'_{n'}$ are $A_0^n, A_1^n, \ldots, A_{d'}^n$. Then since one of the dihedral angles tends to $\pi$, the set of limit vectors $\ell_{XY}$ for all pairs $\{X_n, Y_n\} \subset \{A_0^n, \ldots, A_d^n\}$ only span a space of dimension $\leq d' - 1$. Adding the vectors (say) $\ell_{A_0A_{d'+1}}, \ell_{A_0A_{d'+2}}, \ldots, \ell_{A_0A_{d}}$ adds no more than $d - d'$ to the dimension of the span. Any remaining vector $\ell_{XY}$ will be in this span, more precisely in the span of $\ell_{A_0X}$ and $\ell_{A_0Y}$. Therefore any choice of $d$ vectors from the set of $\ell_{XY}$ can only span a space of dimension $\leq d - 1$, so $\sin_d$ is zero by Lemma 3.3 b). By continuity of $\sin_d$ (Lemma 3.3 c)), the generalized maximum angle condition in Definition 2.4 is violated. \[\square\]

**Theorem 3.7.** The conditions in Definition 2.7 and in Definition 2.4 are equivalent.

**Proof.** Use the contradiction argument and Theorem 3.6 we observe that the condition of Definition 2.4 implies the condition in Definition 2.7.

To prove the statement of the theorem in the opposite direction, we generalize the construction proposed by M. Krížek in [15]. (See also Remark 3.8 following this
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proof for the precise relationship of two constructions.) Assume that the condition in Definition 2.7 holds. We show that the generalized maximum angle condition in Definition 2.4 also holds. The proof is by induction on the dimension $d$, where the base case is known ($d = 2, 3$). Let $S \in T_h \in \mathcal{F}$ be a $d$-dimensional simplex, and let the bound on the angles be $\gamma_0$ as in Definition 2.7. By induction, for any subsimplex $S' \subset S$ of dimension $d - 1$ with vertex set contained in the vertex set of $S$, one can choose $d - 1$ unit vectors along the edges so that the $\sin_{d-1}$ applied to these vectors is bounded from below by some constant. When restricted to the set of $d$ unit vectors, both $\sin_d$ and $\text{meas}_d$ of the parallelotope spanned by the vectors are continuous functions which are zero precisely on the compact set of linearly dependent sets of vectors. Therefore the existence of a positive lower bound for the values of one of them on a family implies the existence of a positive lower bound for the other. Let $C > 0$ be a lower bound for the measure of these parallelotopes. Order the vertices of $S$ as $A_0 \ldots A_d$ so that the largest dihedral angle $\alpha_{d-1,d}$ is between the facets opposite to $A_{d-1}$ and $A_d$. By induction, choose $d - 1$ unit vectors $\vec{t}_1, \ldots, \vec{t}_{d-1}$ along the edges of the subsimplex $A_0 \ldots A_{d-1}$ so that $\text{meas}_{d-1}(\vec{t}_1 \ldots \vec{t}_{d-1}) > C$. Now look at the subsimplex $A_0 \ldots A_{d-2}A_d$ opposite to $A_{d-1}$. In this subsimplex, we can also choose $d - 1$ unit vectors $\vec{u}_1 \ldots \vec{u}_{d-1}$ with the same property. At least one of these must be along an edge incident to $A_d$; define this vector (or one of these vectors) to be $\vec{t}_d$.

Let $y$ be the height of $\vec{t}_d$ over the subspace spanned by $\{\vec{u}_1 \ldots \vec{u}_{d-1}\} \setminus \{\vec{t}_d\}$. Then $y$ is the height of a parallelotope with bounded volume, and we have:

$$ C < \text{meas}_{d-1}(\vec{u}_1 \ldots \vec{u}_{d-1}) = y \cdot \text{meas}_{d-2}(\{\vec{u}_1 \ldots \vec{u}_{d-1}\} \setminus \{\vec{t}_d\}) \leq y \cdot 1. $$

The last inequality holds because all the involved vectors are unit vectors. Let $z$ be the height of $\vec{t}_d$ over $\{\vec{t}_1 \ldots \vec{t}_{d-1}\}$. Finally,

$$ \text{meas}_d(\vec{t}_1 \ldots \vec{t}_d) = z \text{meas}_{d-1}(\vec{t}_1 \ldots \vec{t}_{d-1}) = y \sin \alpha_{d-1,d} \text{meas}_{d-1}(\vec{t}_1 \ldots \vec{t}_{d-1}). $$

Now $y \geq C$ and $\text{meas}_{d-1}(\vec{t}_1 \ldots \vec{t}_{d-1}) \geq C$. Since $\alpha_{d-1,d}$ is the largest dihedral angle, it lies in the interval $(\gamma_1, \gamma_0)$, where $\gamma_1$ is the dihedral angle in the regular $d$-simplex (see e.g. [16]). Therefore $\sin \alpha_{d-1,d} \geq \min\{\sin \gamma_1, \sin \gamma_0\}$. Thus the measure of the spanned parallelotope is bounded from below:

$$ \text{meas}_d(\vec{t}_1 \ldots \vec{t}_d) \geq C^2 \min\{\sin \gamma_1, \sin \gamma_0\}. $$

We conclude that $\sin_d$ is also bounded from below by $3$.

**Remark 3.8.** M. Křížek uses a similar construction in [15, pp. 517–518] to find three vectors in a tetrahedron satisfying the same conditions. In his proof, he starts with an arbitrary triangle and a large dihedral angle incident to a chosen edge of this triangle. Two unit vectors are chosen along edges in this triangle. He then chooses a vector based on angles in the other triangle used to compute the dihedral angle. His choice of two vectors in the arbitrary triangle is similar to our choice of $\vec{t}_1, \ldots, \vec{t}_{d-1}$. His procedure using angles to choose the third vector is equivalent to our choice of $\vec{t}_d$. Our bounds are slightly easier because we avoid the choice of an arbitrary subsimplex/triangle, so we will only need bounds for the largest, not the second largest, dihedral angle in a given dimension.
Jamet’s definition. In \[13\], Jamet estimates interpolation error in terms of an angle $\theta$ defined as follows (with notation adapted to the present article). Let $E = \{ \vec{e}_i \}_{i=1}^d$ be a set of unit vectors in $\mathbb{R}^d$. For any other unit vector $\vec{u} \in S^{d-1}$, define $\theta_i(\vec{u})$ to be the angle between $\vec{u}$ and the line through $\vec{e}_i$. Then define $\theta$ by

\[
\theta = \max_{\vec{u} \in S^{d-1}} \min_{i=1, \ldots, d} \theta_i(\vec{u}).
\]

Jamet obtains formulas bounding interpolation errors where a factor $1/\cos \theta$ appears. In particular, in Exemple 1 in \[13\], $E$ is chosen to be a set of unit vectors along the edges of a simplex as in Definition 2.4.

**Definition 3.9.** A family $\mathcal{F} = \{ T_h \}_{h \to 0}$ of face-to-face partitions of a polytope $\Omega \subset \mathbb{R}^d$ into $d$-simplices is said to satisfy Jamet’s condition if there exists a constant $\theta_0 < \pi/2$ such that for all simplices $S \in T_h \in \mathcal{F}$, one can choose $d$ unit vectors along edges of $S$ in such a way that the $\theta$ computed in Equation (6) satisfies $\theta \leq \theta_0$.

**Theorem 3.10.** Jamet’s condition in Definition 3.9 is equivalent to the condition in Definition 2.4, and consequently also to the condition in Definition 2.7.

**Proof.** If the maximum in Equation (6) is $\pi/2$, it is clear that the set $E$ of $d$ vectors is in fact not linearly independent, and that the maximum is attained for any vector which is a normal vector to a $(d - 1)$-dimensional subspace that contains all the vectors. This was observed also by Jamet. The existence of a bound $\theta_0 < \pi/2$ means that the set $E$ is separated from the subset of $(S^{d-1})^d$ consisting of linearly dependent sets of vectors. As seen in the proof of Theorem 3.7, this is equivalent to a lower bound for $\sin d$ applied to the same set of vectors. This concludes the proof. \(\square\)

**Remark 3.11.** A priori, checking a finite number of angles is an easier task than finding the maximum angle $\theta$. To prove the existence of a bound in one of the conditions given the existence of a bound for the other is simpler than giving formulas. For $d = 2$, this is easy: If $\gamma_0 < \pi$ is an upper bound for the angles of the triangles, $\theta_0 = \gamma/2$ is an upper bound for the angles in Jamet’s condition. In \[17\], Rand proves the case $d = 3$ of the above theorem by explicitly computing $\theta_0$ given $\gamma_0$ and vice versa.

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