AN IN-PLACE, SUBQUADRATIC ALGORITHM FOR PERMUTATION INVERSION

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ABSTRACT. We assume the permutation \( \pi \) is given by an \( n \)-element array in which the \( i \)-th element denotes the value \( \pi(i) \). Constructing its inverse in-place (i.e. using \( O(\log n) \) bits of additional memory) can be achieved in linear time with a simple algorithm. Limiting the numbers that can be stored in our array to the range \([1...n]\) still allows a straightforward \( O(n^2) \) time solution. The time complexity can be improved using randomization, but this only improves the expected, not the pessimistic running time. We present a deterministic algorithm that runs in \( O(n^{3/2}) \) time.

1. Problem statement and previous work

In the permutation inversion problem, the algorithm is given a positive integer \( n \) and a permutation \( \pi \) of the set \( V = \{1, ..., n\} \) provided in an array \( t \), where \( t[i] = \pi(i) \) for all \( i \in V \). The goal is to perform a sequence of modifications of \( t \), so that \( t[i] = \pi^{-1}(i) \) for all \( i \in V \). In this paper we focus on algorithms that use \( O(\log n) \) bits of additional memory. Such algorithms are called in-place algorithms.

This problem was considered by Knuth [3], who described two solutions, by Huang and by Boothroyd. These algorithms, however, are allowed to store any value in the range \([-n, ..., n]\) in the array \( t \). This seems to bypass the heart of the problem. In fact, the sign of the values in \( t \) can be used as a vector of \( n \) bits. The problem is trivial when such a vector is allowed. Algorithms described in this paper are only allowed to store values in the range \([1, ..., n]\) in \( t \).

First, we describe an \( O(n^2) \) time in-place algorithm. Observe that it is straightforward to reverse a single cycle of \( \pi \):

```
Reverse-Cycle(start)
1  cur = t[start]
2  prev = start
3  while cur \neq start
4      next = t[cur]
5      t[cur] = prev
6      prev = cur
7      cur = next
8  t[start] = prev
```

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For a cycle in the permutation, let the leader of the cycle be the smallest element in the cycle. The following code finds in $O(n)$ time the leader of the cycle that includes element $\text{start}$:

```
Cycle-Leader(start)
1  cur = t[start]
2  smallest = start
3  while cur ≠ start
4    smallest = min(smallest, cur)
5    cur = t[cur]
6  return smallest
```

To obtain the inverse of $\pi$, it suffices to reverse each of the cycles exactly once:

```
for i = 1 to n
    if Cycle-Leader(i) == i
        Reverse-Cycle(i)
```

In 1995, Fich et al. [2] published a paper on a similar topic: the permutation $\pi$ is given by means of an oracle and the goal is to permute the contents of an array according to $\pi$. They provided an algorithm with running time $O(n \log^2 n)$ that uses $O(\log^2 n)$ bits of additional memory. The concept of a cycle leader comes from this paper.

In 2015, the methods of Fich et al. were extended to the permutation inversion problem by Robertson [5], who gave an algorithm with running time $O(n \log n)$ that uses $O(\log^2 n)$ bits of additional memory.

It is interesting to note that the quadratic algorithm can be easily modified to achieve $O(n \log n)$ expected running time. Fich et al. point out, attributing the idea to a personal communication with Impagliazzo, that we can use a randomly chosen hash function $h$ and choose the cycle leader to be the element $i$ for which $h(i)$ is smallest. They proposed it for their problem, but the idea can be applied to cycle inversion as well. When visiting element $i$ in the main loop, we start reversing its cycle and either complete this operation or encounter an element $j$ such that $h(j) < h(i)$, in which case we revert the operation. This way, assuming no hash collisions, every cycle is reversed once and if $h$ is chosen randomly, the average time spent in the main loop at any single element is $O(\log n)$. Thus, we obtain an in-place algorithm with expected running time $O(n \log n)$.

To our knowledge, we present the first deterministic, in-place algorithm with running time $o(n^2)$. The running time of our algorithm is $O(n^{3/2})$.

2. A SUMMARY OF THE $O(n^{3/2})$ ALGORITHM

Observe that the quadratic algorithm runs in time $O(n^{3/2})$ on any instance in which each cycle is of size at most $O(\sqrt{n})$. The complexity
is worse when a large proportion of the elements belong to cycles of greater sizes. We modify the quadratic algorithm to handle large cycles differently, while keeping its behavior for small cycles.

During the course of the algorithm, with the current state of array \( t \) we associate a directed graph \( G_t \) on the vertex set \( V \) and with edge \((i, t[i])\) for each \( i \in V \). The graph \( G_t \) may contain loops and when we consider cycles, loops are counted among them. Initially, \( G_t \) is a disjoint union of cycles. When we modify \( t \), we can get \( G_t \) to be an arbitrary graph with outdegree of each vertex equal to 1.

We will introduce an alternative way of storing a long cycle in the array \( t \). This alternative representation of a cycle is achieved by redirecting a small number of edges. The result is a graph with multiple connected components; each component has \( O(\sqrt{n}) \) elements and is a directed path leading to a cycle. We will use the lengths of these paths and cycles to encode some information. This information will allow us to revert the edge redirections and obtain the original cycle.

When we first encounter a long cycle, we reverse it and convert the reversed cycle to the alternative representation. Next, whenever in the main loop we visit another vertex of the cycle, we traverse only the component of the vertex, which takes \( O(\sqrt{n}) \) time. Previously this step could take \( O(n) \) and this is the improvement that allows us to achieve \( O(n^{3/2}) \) time complexity. When the loop finishes, all short cycles are reversed and all the long cycles are stored using the alternative representation with minor modifications. Next, we perform one pass over all vertices to remove the modifications and another pass to convert long cycles back from the alternative representation.

3. Segments and cycle detection

We call each connected component of the alternative representation to be a segment. A segment is a disjoint union of two directed graphs: a cycle and a path, together with an edge from the last vertex of the path to one of the vertices in the cycle. The first vertex of the path is called the beginning of the segment, the number of vertices in the segment is called its size, and the path is called its tail.

We use the sizes of the segments, as well as the sizes of their cycles, to store some information. Thus, whenever in the main loop variable \( i \) becomes a vertex of a segment, we need to be able to compute in-place the size of the cycle and the distance from \( i \) to the cycle. The problem is called cycle detection and at least two different algorithms are known to solve it in-place and in time proportional to the size of the segment: the Floyd’s cycle-finding algorithm (also called the tortoise and hare algorithm) and the Brent’s algorithm. We will shortly present the first one. If our description is insufficient, we refer to the famous book by Knuth [4, Section 3.1, Exercise 6]. The idea is as follows. First, we initialize two variables, the tortoise and the hare, to
point at element $i$ (called \textit{start} in the following pseudocode). Next, we simultaneously progress both variables: the tortoise moves one step at a time and the hare moves two steps at a time, where by a step we mean setting $v = t[v]$. We stop when both variables point at the same element (which must happen after a number of steps that is linear in the number of vertices reachable from $i$). Next, we bring the hare back to element $i$ and start progressing the pointers again, this time both by one step at a time. They first meet at the beginning of the cycle, i.e. the only vertex with indegree 2 in the segment, which gives us the distance from \textit{start} to the cycle. Finally, we use the tortoise one last time to compute the size of the cycle.

Tortoise-and-Hare($\text{start}$)

1. tortoise = hare = start
2. cycle\_length = dist\_to\_cycle = 0
3. repeat
   4. tortoise = $t[\text{tortoise}]$
   5. hare = $t[t[\text{hare}]]$
6. until tortoise = hare
7. hare = start
8. repeat
   9. tortoise = $t[\text{tortoise}]$
10. hare = $t[\text{hare}]$
11. dist\_to\_cycle = dist\_to\_cycle + 1
12. until tortoise = hare
13. repeat
14. tortoise = $t[\text{tortoise}]$
15. cycle\_length = cycle\_length + 1
16. until tortoise = hare
17. return (cycle\_length, dist\_to\_cycle)

4. The alternative representation of long cycles

In this section, we define the alternative representation of a long cycle and show how to compute it. Let $k = \lceil \sqrt{n} \rceil$. Up to isomorphism, there are $k^2 \geq n$ different segments of size between $k + 1$ and $2k$ and of cycle length between 1 and $k$. We choose different segments of the above form to represent different elements of $V$. This way, a segment can ‘store’ a pointer to a vertex – we say that such a segment, or a pair of a segment size and a cycle length of the above form, \textit{encodes} a vertex. The encoding is the following bijection from $V$ to a subset of $\{k + 1, \ldots, 2k\} \times \{1, \ldots, k\}$: \textsc{Encode}(v) = ([v−1]/k) + (k + 1), ((v − 1) mod k) + 1). Its inverse is the function \textsc{Decode}(s,c) = (s − (k + 1))k + c + 1.

An intuitive understanding of the alternative representation is that the cycle is split into segments such that all except $O(\sqrt{n})$ edges are
preserved (condition (1) in the definition below), the first segment begins at the leader, and every segment encodes the beginning of the next segment, except the first and the last one, which we need to handle differently. This is illustrated in Figure 1.

We set a threshold for a cycle to be handled using the alternative representation as follows: a cycle is called *long* if it has at least $4k + 3$ vertices, otherwise it is called *short*.

For a directed graph $G$ and a subset $X$ of its vertex set, let $G[X]$ denote the subgraph of $G$ induced by $X$. Let $c_1$ be the leader of a long cycle $C = (c_1, \ldots, c_p)$ on vertex set $V' \subseteq V$, let $R$ be a directed graph on $V'$ with the outdegree of each vertex equal to 1 and let $S$ be an integer from $[k + 1, 2k]$. We say that a pair $(R, S)$ is a segment representation of $C$ if there exist $q \geq 2$ integers $i_1, \ldots, i_q$ such that $1 = i_1 < \ldots < i_q < p$ and:

1. for every $i \in \{1, \ldots, p-1\} \setminus \{i_2 - 1, \ldots, i_q - 1\}$, the graph $R$ contains the edge $(c_i, c_{i+1})$;
2. the graph $R[\{c_1, \ldots, c_{i_2-1}\}]$ is a segment with beginning $c_1$, size between $2k + 2$ and $4k + 1$ and cycle length $y$ such that $(S, y) = \text{Encode}(c_{i_2})$;
3. for every $j \in \{2, \ldots, q-1\}$, the graph $R[\{c_{i_j}, c_{i_{j+1}}-1\}]$ is a segment with beginning $c_{i_j}$, size $x$ and cycle length $y$ such that $(x, y) = \text{Encode}(c_{i_{j+1}})$;
4. the graph $R[\{c_{i_q}, \ldots, c_p\}]$ is a segment with beginning $c_{i_q}$, size $2k + 1$ and cycle length at most $k$.

The graph $R$ is called a *segmentation* of $C$.

The intuition behind this definition is as follows. We want the segment representation of $C$ to store enough information to be able to restore $C$. Condition (3) guarantees that every segment except the first and the last one encodes the beginning of the next one. For the first segment, the beginning of its successor can be decoded from the number $S$ and the length of the cycle in the segment — this is condition (2). The last segment has size $2k + 1$, while the length of the cycle can be any integer from $[1, \ldots, k]$ (condition (4)). The size $2k + 1$ is special, as no other segment can have this size, and indicates that the segment is the last one. The fact that we are free to choose the length of the cycle in the last segment is important and will be used later in the paper.

Recall that for a long cycle, we want to obtain a segment representation of its inverse. We present a procedure that achieves this goal in-place and in time linear in the size of the cycle. We require that when $\text{MAKE-SEGMENTS}$ is called, vertex *leader* is the leader of a long cycle $C$ that is a subgraph of $G_t$. We claim that when the procedure ends, the subgraph of $G_t$ induced by the vertices of $C$ together with the
Figure 1. A long cycle $C$ and a possible segmentation of $C$, where $k = 8$ and $q = 5$. The red color marks the cycle leader, a green line from a segment $X$ to a vertex $v$ indicates that $X$ encodes $v$, a green asterisk indicates the segment of size $2k + 1$.

number $S$ returned by \texttt{MAKE-SEGMENTS} is a segment representation of the inverse of $C$.

Before we proceed with the pseudocode, we have a couple of remarks. First, for readability, the code below and others that follow do not necessarily meet the in-place memory requirements if understood literally, but it is straightforward to rewrite them in such a way that they do. Second, it might be unclear at this point why the procedure returns the value $bg\_of\_sg\_created\_first$. It will become useful later. And third, we use the notation $t^i[v]$ defined as follows: $t^0[v] = v$ and $t^i[v] = t^{i-1}[v]$ for $i \in \{1, 2, \ldots\}$.
MAKE-SEGMENTS(leader)
1  to_encode = leader
2  v1 = t[leader]
3  while leader is not among v1, t[v1], t2[v1], ..., t2k[v1]
4      if to_encode == leader
5          s = 2k + 1
6          cycle_length = 1  // we can choose any length between 1 and k
7          bg_of_sg_created_first = t2k[v1]
8      else (s, cycle_length) = ENCODE(to_encode)
9          let v2 = t[v1], ..., vs = t[vs-1]
10         next_v1 = t[vs]
11         set t[v1] = vi-1 for i = 2, ..., s
12         t[v1] = v_cycle_length
13         to_encode = vs
14         v1 = next_v1
15  S = s
16  let p be the smallest i such that t[i][v1] = leader
17  let vi = t[i-1][v1] for i = 2, ..., p
18  t[v1] = to_encode
19  set t[vi] = vi-1 for i = 2, ..., p
20  return (bg_of_sg_created_first, S)

The code simultaneously reverses the cycle and forms new segments.
First, it forms a segment of size 2k+1 that begins at vs. Then, it creates
a sequence of segments, such that each one encodes the beginning of
the one created previously. Finally, when there are not enough vertices
left, the remaining ones are attached to the last created segment. This
means that we do not have control over the number of vertices in this
segment, but we do not have to – we only need to ensure that in falls
within [2k + 2, 4k + 1] and that the number S together with the length
of the cycle in the segment encodes the beginning of the previously
created segment. We leave the proof of correctness of the algorithm to
the reader. Note that because long cycles are defined to have at least
4k + 3 vertices, at least two segments are created.

Now consider the problem of restoring the original cycle from its
segment representation. We start at the leader, decode the beginning
of the next segment using the value S, redirect a single edge and proceed
to the next segment. This time, we use the size of the segment and
the size of its cycle to decode the beginning of the next segment. We
repeat this step several times until we encounter the segment of size
2k + 1. Then we redirect the last edge to the cycle leader (stored in
a separate variable) and stop. This procedure is implemented in the
following pseudocode:
5. The $O(n^{3/2})$ Algorithm

Let us summarize how we change the $O(n^2)$ algorithm to obtain $O(n^{3/2})$ running time. First, when a long cycle is first visited, it is reversed and converted into its segment representation. The value $S$ is stored in another part of the graph $G_t$ – this step is explained in Section 6, for now assume that this is somehow implemented. Short cycles are treated as before and we add a third case to the main loop: if $i$ belongs to a tail of a segment, we do nothing. As a result, after the main loop is complete, all short cycles are reversed and for every long cycle there is a segmentation of its inverse with the cycles in all segments reversed (notice that for every segment, its cycle $C$ is reversed exactly once – when $i$ becomes the leader of $C$).

It remains to reverse the cycles in all segments again and then restore the original cycle for every segment representation of a long cycle. This is achieved with two more passes over all vertices – first we reverse the cycles in segments and then we restore the long cycles. The whole algorithm is below. We claim that it inverts the permutation in-place in $O(n^{3/2})$ time.
Invert-Permutation()

1. for \( i = 1 \) to \( n \)
2. \((\text{cycle}_\text{length}, \text{dist}_\text{to}_\text{cycle}) = \text{Tortoise-and-Hare}(i)\)
3. if \( \text{dist}_\text{to}_\text{cycle} \geq 1 \)
4.     continue
5. if \( \text{cycle}_\text{length} < 4k + 3 \)
6.     if \( \text{Cycle-Leader}(i) == i \)
7.         \( \text{Reverse-Cycle}(i) \)
8.     else
9.         \((\text{bg}_\text{of}_\text{sg}_\text{created}_\text{first}, S) = \text{Make-Segments}(i)\)
10. store \( S \)
11. for \( i = 1 \) to \( n \)
12. \( (\text{cycle}_\text{length}, \text{dist}_\text{to}_\text{cycle}) = \text{Tortoise-and-Hare}(i)\)
13. if \( \text{dist}_\text{to}_\text{cycle} == 1 \)
14.     \( \text{Reverse-Cycle}(t[i]) \)
15. for \( i = 1 \) to \( n \)
16. \( (\text{cycle}_\text{length}, \text{dist}_\text{to}_\text{cycle}) = \text{Tortoise-and-Hare}(i)\)
17. if \( \text{dist}_\text{to}_\text{cycle} \geq 1 \) // \( i \) is the leader of a long cycle in \( \pi \)
18.     retrieve \( S \)
19.     \( \text{Restore-Long-Cycle}(i, S) \)

Let us prove the correctness of the algorithm. Let \( C \) be a cycle in \( \pi \) with vertex set \( V' \). If \( C \) is short, our algorithm reverses it only once – when \( i \) is the leader of \( C \). If \( C \) is long, consider all the moments when our algorithm modifies \( t[v] \) for \( v \in V' \). First, when \( i \) is the leader of \( C \), the cycle is reversed and split into segments. Then, during the loop in lines 4 – 10 all that happens is that every cycle in every segment is reversed exactly once – when \( i \) becomes the leader of the cycle. During the loop in lines 11 – 14 every cycle in every segment is reversed again, also exactly once (because for every such cycle \( \sigma \) there is exactly one vertex with distance to \( \sigma \) equal to 1). Thus, after this loop, the graph \( G_t[V'] \) is again a segmentation of the inverse of \( C \). Later, when \( i \) becomes the leader of \( C \) in the loop in lines 15 – 19 the line 19 sets \( G_t[V'] \) to be the inverse of \( C \). From this moment on, the code does not modify \( t[v] \) for \( v \in V' \). This completes the correctness proof.

As it is obvious that the code runs in \( O(n^{3/2}) \) time and can be implemented in-place, the analysis of the algorithm is complete.

6. Storing the value \( S \)

Recall that the length of the cycle in the segment of size \( 2k + 1 \) of a segmentation can be any integer between 1 and \( k \). We call this cycle the free cycle of that segmentation and we use it to store information. We now make use of the return value of the procedure \( \text{MAKE-SEGMENTS} \). The procedure returns a pair of integers: the beginning of the segment with the free cycle and the value \( S \).
In our algorithm, there are two passes over all vertices. In both passes, long cycles are visited in the same order, say \(C_1, \ldots, C_r\) (here each \(C_i\) denotes a cycle before its reversal). Let \(C'_1, \ldots, C'_r\) be the respective inverses of \(C_1, \ldots, C_r\). After the first pass, in place of every \(C_i\) there is a segment representation \((R_i, S_i)\) of \(C'_i\). The idea is to store \(S_i\) as the length of the free cycle in \(R_{i-1}\). This way, when restoring \(C'_{i-1}\), we can retrieve the value \(S_i\) and use it to restore \(C'_i\). This is done for \(i \geq 2\), while the value \(S_1\) is stored in a separate variable, named \(\text{first}_S\). We present the updated pseudocode of the \(O(n^{3/2})\) time algorithm. The only difference compared to the previous version is the implementation of the operations \text{store} and \text{retrieve}.

**INVERT-PERMUTATION()**

```plaintext
storage, first_S = NIL
for i = 1 to n
    (cycle_length, dist_to_cycle) = TORTOISE-AND-HARE(i)
    if dist_to_cycle \geq 1
        continue
    if cycle_length < 4k + 3
        if CYCLE-LEADER(i) == i
            Reverse-Cycle(i)
    else
        (bg_of_sg_created_first, S) = MAKE-SEGMENTS(i)
        if storage == NIL
            first_S = S
        else set the length of the cycle in the segment
        beginning at storage to S
        storage = bg_of_sg_created_first
    for i = 1 to n
        (cycle_length, dist_to_cycle) = TORTOISE-AND-HARE(i)
        if dist_to_cycle == 1
            Reverse-Cycle(t[i])
    S = first_S
    for i = 1 to n
        (cycle_length, dist_to_cycle) = TORTOISE-AND-HARE(i)
    if dist_to_cycle \geq 1  // i is the leader of a long cycle in \(\pi\)
        S = RESTORE-LONG-CYCLE(i, S)
```

We omit the correctness, memory consumption and time consumption analysis of the code, as it is analogous to the analysis of the previous version of the code.
7. Suggestions for Further Research and Acknowledgements

As we said earlier, the problem can be solved in $O(n \log n)$ expected time using a randomized algorithm. Whether there exists a deterministic solution running in $O(n \log^c n)$ for some constant $c$ seems to be an interesting question.

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