Transparent boundary condition for the momentum conservative scheme of the shallow water equations

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Abstract. In conducting water wave simulations, the correct implementation of boundary conditions is important to obtain accurate wave dynamics in the computational domain. For assessment of coastal structures such as breakwaters, where both transmitted and reflected waves are present in the computational domain, we often need to observe simulation behavior for a somewhat long period of time. In this case, applying a transparent boundary condition is necessary, a condition that allows transmitted wave propagates to the right, whereas reflected waves propagates to the left at all times. In this paper, we propose a transparent boundary condition which derives from the embedded wave generation method of Liam et al. [5]. In this paper, the method is implemented to the momentum conservative scheme of the shallow water equations, and conduct several wave simulations. First, we use a monochromatic wave to demonstrate the implementation of embedded wave generation for constructing transparent boundary condition. Second, we show how this method has an effect on the backward and onward shoreline motion of Carrier-Greenspan [2] simulation. Finally, we consider a simulation of wave reduction due to a submerged breakwater with a certain dimension.

Keywords: transparent boundary condition, embedded influxing, momentum conservative scheme, shallow water equations, reflection and transmission coefficients

1. Introduction

Wave generation is an important part of water wave simulation since most cases need a correct implementation of wave influx or incoming signal. One method to generate influx in water wave simulation is by imposing wave on the left side of the computational domain boundary. For short time observation, this influx boundary condition is quite good and relatively simpler to implement. However, for some cases that involve reflected wave, restricted computational domain will cause a technical problem. When a signal propagates from the left to the right, then hits a structure, a running left reflected wave will be present. For a certain time, the reflected wave will reach the left boundary and interfere with the incoming wave. To solve this problem, a transparent boundary condition is needed in the simulation. The terminology transparent boundary here means that the right running wave and the left running reflected wave can both propagate without interfering one another.

For a linear wave problem, Riemann invariant form can be applied as transparent boundary to absorb the reflected wave ([1],[4]). But, for nonlinear wave problem, the corresponding Riemann form is more complicated. In this paper, we implement a method to generate wave in the...
computational domain instead of on the boundary. The scheme of internal wave generation was studied by Wei et al. [10] and Chawla and Kirby [3] for Boussinesq-type wave model. The mechanism was then generalized for dispersive surface wave models by Liam et al. [5]. Here, we discuss the use of embedded wave generation in order to construct a transparent boundary condition. On a computational domain $[-a,a]$, implementation of embedded influx at $x = 0$, together with applying radiation boundary condition at $x = -a$ will yield a transparent boundary condition at $x = 0$. Via numerical simulations in comparison with analytical formulation, we show that this transparent boundary condition has been correctly implemented.

The embedded wave generation is implemented on shallow water equations (SWE), which are commonly used to model surface wave in shallow area. SWE has been applied to various water wave problems, especially for cases of long wave (compared to water depth). Some applications that adopt the shallow water theory are tsunami wave simulations, dam break problems, river flooding and open channel flows ([9],[8],[7]).

![Figure 1. Sketch of fluid domain in shallow water equations.](image)

SWE consists of mass conservation and momentum balance equations. Consider a layer of fluid on a bottom topography $z = -d(x)$ with free surface denoted by $z = \eta(x,t)$, where $x$ and $z$ consecutively represent spatial axis in horizontal and vertical directions (see Figure 1). In this model, we assume that the horizontal velocity is independent of $z$ and denoted as $u(x,t)$. The simplified nonlinear shallow water equations are formulated as follow.

\[
\eta_t + (hu)_x = 0 \quad (1)
\]
\[
u_t + uu_x = -g\eta_x, \quad (2)
\]

where $h = \eta + d$ is the water thickness and $g$ is denoted as gravity acceleration.

The organization of this paper is set out as follows. In section 2, the mechanism of embedded wave generation is recalled. Then the implementation of embedded influxing as a transparent boundary is discussed through numerical simulation in section 3. Section 4 describes other simulations with uneven bottom problems. Finally, concluding remark is presented.

2. Embedded wave generation

Embedded wave generation here means to produce wave inside the computational domain. The mechanism is adopted from Liam et al. [5]. It is applicable for linear wave equation with dispersion relation $\omega = \Omega(k)$, where $k$ is the wavenumber and $\omega$ is the frequency. Shallow water model has dispersion relation $\omega^2 = k^2gd \equiv \Omega^2(k)$. In this case, the group velocity $V_g = \partial \Omega/\partial k = \sqrt{gd}$ is constant for all wavenumber.

In the mechanism of embedded influxing, a source function $F(x,t)$ is added to the wave model, and defined by

\[
F(x,t) = \gamma(x)f(t), \quad (3)
\]
where $\gamma$ is a function of space and $f$ is a function of time. The spatial and temporal Fourier transformation of $F(x,t)$, denoted by $\bar{F}(k,\omega)$, is related to $F(x,t)$ by

$$F(x,t) = \int \int \bar{F}(k,\omega)e^{i(kx-\omega t)} dk d\omega.$$  

(4)

Thus we can write

$$\bar{F}(k,\omega) = \hat{\gamma}(k)\hat{f}(\omega),$$  

(5)

where $\hat{\gamma}(k)$ is the spatial Fourier transform of $\gamma(x)$ and $\hat{f}(\omega)$ is the temporal Fourier transform of $f(t)$.

Let $s(t)$ be a wave signal to be generated. By applying spatial Fourier transform to the linear wave equation and factorizing the second order wave equation (see [5] for detail), they obtained the relation between the force function and the targeted signal $s(t)$. The relation is expressed in terms of their Fourier transforms, as follows

$$\bar{F}(k,\omega) = \frac{1}{2\pi} V_g(K(\omega))\bar{s}(\omega).$$  

(6)

In equation (6), $K(\omega)$ is the inverse of dispersion relation $\omega = \Omega(K(\omega))$. In the implementation, we restrict to generate a wave signal at one point, say at $x = 0$, by choosing $\gamma(x) = \delta_{Dw}(x)$, a Dirac delta function. In this case, since $\hat{\gamma}(k) = 1/2\pi$, then from (5) we have $\hat{f}(\omega) = V_g(K(\omega))\bar{s}(\omega)$. Thus, by using inverse Fourier transform, $f(t)$ can be calculated to find the source function.

The source function $F(x,t)$ is then added to the continuity equation. The shallow water equations with bi-directional embedded wave generation read

$$\eta_t + (hu)_x = 2F(x,t)$$  

(7)

$$u_t + uu_x = -g\eta_x.$$  

(8)

The source function is multiplied by two because the resulting wave will be symmetrically separated; to the left and right with half amplitude of the signal $s(t)$. Equation (7) and (8) with forcing $F(x,t)$ as given in equation (3) will produce the wave signal $s(t)$.

3. Embedded wave generation as a transparent boundary

In this section, we demonstrate how the embedded wave mechanism can be used to construct a transparent boundary. The computational set up is illustrated in Figure 2, with domain $x \in [-150, 100]$. Wave influx $s(t) = A \sin(\omega t)$ is generated at $x = 0$, with $A = 0.01$ m, $\omega = \pi/4$ s$^{-1}$, and water depth $d = 1$ m. The result of this simulation is performed at three subsequent times as shown in Figure 3.

As the outcome of embedded wave influx at $x = 0$, bi-directional wave enters the spatial domain. One wave is propagating to the right and the other one is propagating to the left, as shown in Figure 3(a). We can calculate that this wave propagates with wavelength $\lambda = 2\pi/k = 25$ m which $k$ fulfill the dispersion relation $\omega/k = \sqrt{gd}$.

Figure 3(b) records the surface elevation at time $t = 61.5$ s. The right running wave hits the hard wall at $x = 100$, generated reflected waves that propagate to the left. The reflected waves interact with the incoming waves, and produce standing wave with amplitude twice that of the incident wave. In the area $[0, 100]$, we have a standing wave that oscillates back and forth for long period of time without damping. This is a typical situation of wave oscillation in a semi-enclosed basin with hard wall with boundary at $x = 100$ and a transparent boundary at $x = 0$. Meanwhile, left running reflected wave passed the embedded wave location $x = 0$ as illustrated in Figure 3(c). In the area $[-100, 0]$, the wave interacts with the existing left running wave. And the waves are absorbed in area $[-150, -100]$. 

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Figure 2. Sketch of embedded wave generation and computational domain on a flat bottom

Figure 3. Surface elevation for three subsequent times (a) \( t = 30 \) s, (b) \( t = 61.5 \) s, (c) \( t = 77.4 \) s.

4. Implementation of transparent boundary on other simulations

Here we implement the transparent boundary condition for two cases with varying bottom. Staggered grid numerical scheme is applied, and the results are compared with the analytical theory.

4.1. Wave run-up over a sloping beach

Transparent boundary condition will be implemented in the simulation of a monochromatic wave over a sloping beach. This is known as Carrier-Greenspan [2] test case. In this problem, the topography is a sloping area with slope 1:25, and the bottom constant depth is 0.5 m. For the embedded wave simulation, the spatial domain is \( x \in [-100, 5] \) m. A monochromatic wave
influx $A \sin(\omega t)$ is embedded at $x = -30.5$. On the left side of the domain, at $x = -100$, the radiation boundary is adopted, whereas on the right area $[-30.5, 5]$ the monochromatic wave propagates over a sloping beach.

Two different numerical simulations were conducted; the first one applied the transparent boundary condition and the second one applied simple influx boundary condition. The spatial set up for influx boundary simulation is $x \in [-30.5, 5]$ m, with

$$\eta(-30.5, t) = A \sin(\omega t).$$ (9)

In this case, the incoming signal has amplitude $A = 0.03$ m and period $T = 10$ s.

Figure 4 display shoreline positions that are obtained from both simulations in comparison with the analytical shoreline. At the beginning, both simulations show comparable results. However, after $t > 56$ s the influx boundary method produce larger waves that is very far from the analytical solution. This phenomenon happen after the reflected waves interact with the incoming waves at the left boundary.

![Figure 4](image-url)

Figure 4. Shoreline during the simulation time for Carrier-Greenspan problem.

4.2. Wave propagation over a submerged bar

The last test case is a simulation of harmonic wave over a submerged bar. Submerged bars are usually constructed as breakwaters at coastal area. This case is considered by Pudjaprasetya and Chendra [6] that has analytically calculated the optimal dimension of the submerged bar. Here we will compute the amplitude of transmitted wave of the simulation and compare to the analytical formula.

The simulation set up for this problem is shown in Figure 5. The signal is generated at $x = 0$. The bar with length ($L \approx 10$ m) is located at $x \in [50, 60]$. The water depth is 4 m above the bar and 10 m at other locations. The generated signal has amplitude 0.1 m and frequency 1 Hz.

The transmitted wave signal from the numerical simulation is shown in Figure 6. The blue dashed line is the result using embedded wave generation. The signal of transmitted wave was recorded at $x = 100$, and we measured the transmitted wave coefficient $A_t = 0.0898$ m. This value is comparable with the analytical coefficient $A_t = 0.0904$ m, according to Pudjaprasetya and Chendra [6]. We also make comparison with numerical simulation using influx boundary method (magenta dashed line). This method produces good results until certain time. After $t \approx 28$ s, the waves are larger than the analytical solution since the reflected wave has been interfering with the incoming signal.
5. Conclusions
A way to construct transparent boundary condition in nonlinear shallow water model has been discussed in this paper. To demonstrate the correct implementation of this transparent boundary, several numerical simulations have been conducted using momentum conservative scheme; wave generation over a flat bottom, a sloping beach, and a submerged bar. Comparable results are obtained with the analytical theory.

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