I. INTRODUCTION

In a quantum mechanical system characterized by an N-dimensional Hilbert space, the \( N \times N \) (Hermitian) Hamiltonian matrix \( H(x) \) frequently depends on a set of parameters \( x = (x_1, x_2, \ldots) \). In condensed matter physics, such parameters may take the form of a crystal momentum, the intensity of an external field, an applied strain or any mean-field order parameter. A Hamiltonian of this kind has (real) eigenvalues \( E_\alpha(x) \), and eigenstates \( |\psi_\alpha(x)\rangle \), where \( \alpha \in \{1, \ldots, N\} \). The eigenvalues form a band structure, and one may thus speak of an \( N \)-band system, or equivalently a (parametric) SU(\( N \)) system.

In this paper, we will be concerned with the quantum geometric properties of \( N \)-band systems. More specifically, we will focus on a fundamental quantum geometric object, the (Abelian) quantum geometric tensor (QGT) \[1–3\]

\[
T_{\alpha,ij}(x) = g_{\alpha,ij}(x) - i \Omega_{\alpha,ij}(x). 
\]

Here, the quantum metric (tensor) \[4\] \( g_{\alpha,ij}(x) = \text{Re} T_{\alpha,ij}(x) \) is symmetric in the indices \( i, j \), and the Berry curvature (tensor) \[5, 6\] \( \Omega_{\alpha,ij}(x) = -2 \text{Im} T_{\alpha,ij}(x) \) is antisymmetric.

The concept of a Berry curvature has a long history. It popped up early on, for example quite explicitly in Blount’s work \[7\], and implicitly in studies on geometric phases \[8–11\]. Nowadays, the Berry curvature is recognized as a key quantity for explaining many fundamental physical phenomena \[2, 12–16\]. The influence of the quantum metric on physical effects is more subtle but has started to attract growing attention in recent years. On the theory side, many measurable effects influenced by it were identified \[17–26\], several measurement protocols were developed \[27–31\], and recently the full QGT of a two-level system was measured directly \[32, 33\]. It is now well established that the entire QGT, not only the Berry curvature part, is essential for understanding the quantum geometric contributions to observables in \( N \)-band systems.

The standard perspective on the QGT, following the historical development, is the following (see Fig. 1):

(i) The system is described by the eigenstates \( |\psi_\alpha(x)\rangle \), and the concept of quantum geometry arises from the nonzero overlap \( \langle \psi_\alpha(x) | \psi_\alpha(x + dx) \rangle \) of eigenstates. In this language, the QGT reads \[1–3\]

\[
T_{\alpha,ij}(x) = \langle \partial_i \psi_\alpha | (1_N - |\psi_\alpha\rangle \langle \psi_\alpha |) | \partial_j \psi_\alpha \rangle, 
\]

where \( \partial_i \equiv \partial / \partial x_i \).

Here, our goal is to emphasize the utility of an alternative point of view that can be adopted:

(ii) The system is described by the eigenprojectors \( P_\alpha(x) = |\psi_\alpha(x)\rangle \langle \psi_\alpha(x) | \). Thus, quantum geometry can be viewed as being encoded in rates of changes of the eigenprojectors upon variation of the parameters \( x_i \). In this language, the QGT reads \[1\]

\[
T_{\alpha,ij}(x) = \text{Tr} \left[ (\partial_i P_\alpha) (1 - P_\alpha) (\partial_j P_\alpha) \right].
\]

Note that there exists a one-to-one correspondence between each eigenprojector and the associated (generalized) Bloch vector \( b_\alpha(x) \) – the SU(\( N \)) analog of the familiar SU(2) Bloch vector on the unit
at certain points \( x \) and essentially useless for calculations of the QGT for three main reasons. First, the value, eigenstates prove rather cumbersome in actual geometry. Namely, for two-band systems, it is well known quite complicated already for \( N = 2 \) case demonstrates its usefulness for obtaining analytical insight into the relation between the system’s Hamiltonian and its quantum geometry. Namely, for two-band systems, it is well known that

\[
P_\alpha(x) = \frac{1}{2} \left[ 1_2 + \frac{1}{E_\alpha(x)} H(x) \right],
\]

\[
b_\alpha(x) = \frac{1}{E_\alpha(x)} h(x),
\]

with \( h(x) \) a vector that decomposes the Hamiltonian into Pauli matrices [cf. Eq. (6)]. Inserting Eq. (5) into Eq. (3) or (4) immediately allows to make the following observation [see Eq. (14) below]: The QGT of a two-band system can be written in terms of the Hamiltonian vector \( h(x) \), the energy eigenvalues as well as their parametric derivatives. This approach circumvents the need to explicitly compute energy eigenstates.

The main contribution of this paper is to clarify and quantify the generalization of this convenient property to arbitrary \( N \). The QGT \( T_{\alpha,ij}(x) \) will be written in terms of only the Hamiltonian [matrix \( H(x) \) or vector \( h(x) \)] and the band energy \( E_\alpha(x) \), without having to construct energy eigenstates at all.

Over the last few years, some efforts in this spirit have been made, mostly focusing on the \( N = 3 \) case [35–37]. Very recently, Pozo and de Juan [38] pointed out that, quite generally, any observable (not only the QGT) can be computed without energy eigenstates if the eigenenergies are known. While similar in spirit, our work is less ambitious in scope, since we essentially restrict ourselves to the QGT. The issue of computing more complicated physical quantities without using eigenstates is deferred to future work [39].

The setup of this paper is as follows. In Section II, the familiar two-band case is reviewed. This sets the stage for the \( N \)-band generalization, which is developed in Sections III and IV. First, in Section III, we focus on the eigenprojectors and Bloch vectors, \( i.e. \) we derive formulas \( P_\alpha(x) = P_\alpha(E_\alpha(x), H(x)) \) and \( b_\alpha(x) = b_\alpha(E_\alpha(x), h(x)) \) that generalize Eq. (5) to any \( N \). Second, in Section IV, these results are combined with Eqs. (3) & (4), yielding the QGT of \( N \)-band systems as a function of the Hamiltonian and its eigenvalues. Writing the Berry curvature in terms of the Hamiltonian vector, we recover the familiar SU(2) result as well as the SU(3) formula found in Ref. [35], and write down new explicit formulas for the SU(4) and SU(5) case. Section V serves to illustrate the formalism using explicit multifold fermion models. In particular, we introduce a three- (four-) band low-energy model that has exactly the same spectrum as a simple pseudospin \( S = 1 \ (S = 3/2) \), but completely different geometrical and topological properties. Finally, we sum up and conclude in Section VI.

II. SHORT REVIEW OF TWO-BAND SYSTEMS

A. SU(2) Hamiltonian

Consider a two-band system, described by a \( 2 \times 2 \) Hermitian matrix \( H(x) \) with eigenvalues \{\( E_\alpha(x) | \alpha = \pm \)\} and (orthonormal) eigenstates \{\( \psi_\alpha(x) \)\}. Without loss of generality, the trivial part of \( H \) is set equal to zero, \( i.e. \) \( \text{Tr} \ H = 0 \). The Hamiltonian can be expanded in the Pauli matrices \( \sigma = (\sigma_x, \sigma_y, \sigma_z) \) as

\[
H(x) = h(x) \cdot \sigma,
\]

with the Hamiltonian vector \( h(x) = \text{Tr}\{H\sigma\}/2 \). The eigenvalues of Eq. (6) are

\[
E_\alpha(x) = \alpha \sqrt{\text{Tr}(H^2)/2} = \alpha |h|.
\]

B. Eigenprojectors and Bloch vectors

The eigenprojectors \( P_\alpha(x) \equiv |\psi_\alpha(x)\rangle \langle \psi_\alpha(x)| \) of the Hamiltonian (6) can be expanded in the Pauli matrices as

\[
P_\alpha(x) = \frac{1}{2} \left[ 1_2 + b_\alpha(x) \cdot \sigma \right],
\]

which defines a Bloch vector \( b_\alpha(x) \). It is related to the Hamiltonian vector as

\[
b_\alpha(x) = \alpha \frac{h}{|h|}.
\]

Note the agreement with the important expression (5).

In contrast to the eigenstates, eigenprojectors and Bloch vectors are explicitly gauge-independent quantities and well-behaved in parameter space: a singularity can only appear at band touching points in parameter space.
FIG. 2. Schematic visualization of the different Bloch sphere’s appearing in the treatment of SU(N) Hamiltonians. The familiar SU(2) case (Section II C) is illustrated in the first row, while the second row treats the general case of arbitrary N (Section III D). The Hamiltonian’s Bloch sphere $B_{h}^{(N)}$ is the relevant space for the mapping $u_{h}(x)$, and corresponds to a proper unit sphere for all values of N. The eigenprojector’s Bloch sphere $B_{P_{\alpha}}^{(N)}$ associated to the mapping $b_{\alpha}(x)$ is only a proper sphere for $N = 2$. In the $N > 2$ case, where it is often called generalized Bloch sphere, it is of very complicated shape. The eigenstate’s Bloch sphere $B_{|\psi_{\alpha}(x)}\rangle$ is also a proper unit sphere for all $N$, but of different dimension than the Hamiltonian’s Bloch sphere if $N > 2$.

C. Three different SU(2) Bloch spheres

Here, we briefly discuss three different notions of Bloch spheres encountered in the treatment of two-band systems. A precise distinction of these Bloch spheres is necessary to avoid confusion when introducing Bloch vectors for N-band Hamiltonians.

The unit vector $u_{h}(x) \equiv h / |h|$ associated to the Hamiltonian vector can be parametrized as $u_{h} = (\sin \theta_{h} \cos \phi_{h}, \sin \theta_{h} \sin \phi_{h}, \cos \theta_{h})$. The two parameters $(\theta_{h}(x), \phi_{h}(x))$ may be called Hamiltonian’s angles, since $\cos \theta_{h} = b_{z} / |h|$ and $\tan \phi_{h} = h_{y} / h_{x}$. The unit vector defines a map

$$u_{h} : X \rightarrow B_{h}^{(2)} , \quad x \mapsto u_{h}(x) \quad (10)$$

from the parameter space $X$ to the Hamiltonian’s Bloch sphere $B_{h}^{(2)}$. It is simply a unit two-sphere, $B_{h}^{(2)} = S^{2}$, see Fig. 2(a).

The Bloch vector defines a map

$$b_{\alpha} : X \rightarrow B_{P_{\alpha}}^{(2)} , \quad x \mapsto b_{\alpha}(x) \quad (11)$$

from parameter space to a space that may be called the eigenprojector’s Bloch sphere $B_{P_{\alpha}}^{(2)}$. Importantly, the peculiarity and simplicity of two-band systems, as compared to the more general N-band case, consists in the fact that the eigenprojector’s Bloch sphere is a unit two-sphere, $B_{P_{\alpha}}^{(2)} = S^{2}$ [see Fig. 2(b)], and that the Bloch vector is proportional to the Hamiltonian vector, cf. Eq. (9).

The third type of Bloch sphere arises from the eigenstates. A two-band eigenstate can be parametrized as

$$|\psi_{\alpha}(x)\rangle = e^{i\Gamma_{\alpha}} \begin{pmatrix} \cos \theta_{\alpha} \\ \sin \theta_{\alpha} e^{i\phi_{\alpha}} \end{pmatrix} , \quad (12)$$

with a global phase $\Gamma_{\alpha}(x)$ and two eigenstate’s angles $(\theta_{\alpha}(x), \phi_{\alpha}(x))$. These angles define a map

$$|\psi_{\alpha}(x)\rangle : X \rightarrow B_{|\psi_{\alpha}(x)\rangle}^{(2)} , \quad x \mapsto |\psi_{\alpha}(x)\rangle \quad (13)$$

from parameter space to an eigenstate’s Bloch sphere $B_{|\psi_{\alpha}(x)\rangle}^{(2)}$. Since $(\theta_{\alpha}, \phi_{\alpha})$ can be interpreted as spherical coordinates, $B_{|\psi_{\alpha}(x)\rangle}^{(2)}$ is a unit two-sphere, too, see Fig. 2(c). Note that the eigenstates (12) are explicitly gauge-dependent and potentially suffer from singularities, as discussed in Appendix A.

Although, in the two-band case, all three Bloch spheres happen to correspond to a unit two-sphere, it is important to keep in mind that they are a priori distinct spaces. This distinction is necessary to avoid considerable confusion in the SU(N > 2) case, where the three Bloch spheres are completely different from one another [see Fig. 2(d)–(f) and Section III D].
D. Quantum geometric tensor

In Eqs. (1)–(4), the explicit dimension and form of the matrices $g_\alpha$, $p_\alpha$, and $\Omega_\alpha$ clearly depend on the number and type of parameters $x_i$ contained in the chosen set $x$. Some examples for different interesting choices of $x$ are given in Appendix B. If the set of parameters is kept completely general, the most convenient formulation of the QGT follows from Eq. (3):

$$ g_{\alpha,ij}(x) = \frac{1}{4} b^i_{\alpha} \cdot b^j_{\alpha} = \frac{1}{4|\mathbf{h}|^2} \left[ \mathbf{h}^i \cdot \mathbf{h}^j - \frac{\mathbf{h} \cdot \mathbf{h}^j (\mathbf{h} \cdot \mathbf{h})}{|\mathbf{h}|^2} \right], \quad (14) $$

$$ \Omega_{\alpha,ij}(x) = -\frac{1}{2} b^i_{\alpha} \cdot (b^j_{\alpha} \times b^k_{\alpha}) = -\frac{\alpha}{2|\mathbf{h}|^3} \mathbf{h} \cdot (\mathbf{h}^j \times \mathbf{h}^k), \quad (14) $$

with the shorthand notation $\mathbf{m}^i \equiv \partial_i \mathbf{m}$. For obtaining the expressions in terms of Bloch vectors, we used Eq. (8) and the product identity

$$(\mathbf{m} \cdot \mathbf{\sigma})(\mathbf{n} \cdot \mathbf{\sigma}) = \mathbf{m} \cdot \mathbf{n} I_2 + i(\mathbf{m} \times \mathbf{n}) \cdot \mathbf{\sigma}. \quad (15)$$

The expressions in terms of $\mathbf{h}$ then follow from Eq. (5).

III. GENERALIZATION TO $N$-BAND SYSTEMS: PART 1 – EIGENPROJECTORS AND BLOCH VECTORS

We now systematically generalize the two-band discussion of the previous section to arbitrary $N$. More precisely, after introducing the generic $N$-band Hamiltonian (Section IIIA), we will first generalize Eq. (5), both for the eigenprojectors (Section IIIB) and Bloch vectors (Section IIIC). The properties of the (SU($N$) Bloch spheres are also briefly addressed (Section IId).

Once the eigenprojectors $P_{\alpha}(E_\alpha, H)$ and Bloch vectors $b_{\alpha}(E_\alpha, \mathbf{h})$ are found, one can insert them into Eq. (3) or (4), to obtain the $N$-band generalization of the QGT (14). This will be done in Section IV.

A. SU($N$) Hamiltonian

Consider an $N$-band system, described by an $N \times N$ Hermitian matrix $H(x)$ with eigenvalues $\{E_\alpha(x)\} \alpha = 1, \ldots, N$ and (orthonormal) eigenstates $\{\{\psi_\alpha(x)\}\} \{40\}$.

Again, the trivial part is set equal to zero without loss of generality, $\text{Tr} H = 0$. The Hamiltonian can be expanded as

$$ H(x) = \mathbf{h}(x) \cdot \lambda, \quad (16) $$

with the Hamiltonian vector $\mathbf{h}(x) = \text{Tr}\{H\lambda\}/2$. Here, $\lambda = (\lambda_1, \ldots, \lambda_{N^2-1})$ is a vector composed of the $N^2 - 1$ elementary generator matrices of the SU($N$) Lie group. Together with the identity matrix $I_N$ they constitute a basis for the Lie algebra $\mathfrak{su}(N)$.

Throughout this paper, we will choose the generators $\lambda_\alpha$ as (generalized) Gell-Mann matrices $[41, 42]$. For the reader’s convenience, the Gell-Mann matrices for $N = 3$ and $N = 4$ are listed in Appendix C.

B. Eigenprojectors as a function of the Hamiltonian: $P_{\alpha}(E_\alpha, H)$

According to the Cayley-Hamilton theorem $[43, 44]$, any function of the $N \times N$ Hamiltonian matrix $H$ can be written as a matrix polynomial in $H$, where the highest power is $N - 1$. In particular, as demonstrated in detail in Appendix D, the eigenprojector $P_{\alpha}$ can be written as a matrix polynomial

$$ P_{\alpha} = \sum_{n=0}^{N-1} q_{N-1-n}(E_\alpha) H^n \sum_{n=0}^{N-1} q_{N-1-n}(E_\alpha) E_\alpha^n = \sum_{n=0}^{N-1} q_{N-1-n}(E_\alpha) C_n, \quad (17) $$

where the polynomials $q_\alpha(z) \equiv \sum_{k=0}^{\infty} c_k z^{n-k}$ are closely related to the Hamiltonian’s characteristic polynomial. The coefficients $c_k$ appearing here are listed in Table I for $k \leq 5$, and depend solely on the (classical) Casimir invariants $[45]$

$$ C_n \equiv \text{Tr}(H^n) = \sum_{n=0}^{N} E_\alpha^n, \quad (18) $$

where obviously $C_0 = N$ and $C_1 = 0$.

| $k$ | 0 | 2 | 3 | 4 | 5 |
|-----|---|---|---|---|---|
| $c_k$ | 1 | $\frac{C_2}{2}$ | $\frac{C_3}{3}$ | $\frac{C_4}{4}$ | $\frac{C_5}{5}$ |

TABLE I. Coefficients $c_k$ determining the polynomial $q_\alpha(z)$.

The utility of Eq. (17) becomes immediately apparent if we write down the explicit eigenprojectors $P_{\alpha}(E_\alpha, H)$ for the cases $N = 2$ to $N = 5$:
The eigenstates (21) can then be used further to compute

\[ P_\alpha = \frac{1}{2E_\alpha} (E_\alpha 1_2 + H), \]

\[ P_\alpha = \frac{1}{3E_\alpha^2 - C_3^2} \left[ \left( E_\alpha^2 - \frac{C_2}{2} \right) 1_3 + E_\alpha H + H^2 \right], \]

\[ P_\alpha = \frac{1}{4E_\alpha^3 - C_2 E_\alpha - \frac{C_3}{3}} \left[ \left( E_\alpha^3 - \frac{C_2}{2} E_\alpha - \frac{C_3}{3} \right) 1_4 + \left( E_\alpha^2 - \frac{C_2}{2} \right) H + E_\alpha H^2 + H^3 \right], \]

\[ P_\alpha = \frac{1}{5E_\alpha^4 - 3C_2 E_\alpha^2 - \frac{2C_3}{3} E_\alpha + \frac{C_2^2 - 2C_4}{8}} \left[ \left( E_\alpha^4 - \frac{C_2}{2} E_\alpha^2 - \frac{C_3}{3} E_\alpha + \frac{C_2^2 - 2C_4}{8} \right) 1_5 \right. \]

\[ \left. + \left( E_\alpha^3 - \frac{C_2}{2} E_\alpha - \frac{C_3}{3} \right) H + \left( E_\alpha^2 - \frac{C_2}{2} \right) H^2 + E_\alpha H^3 + H^4 \right]. \]

This represents the \( N \)-band generalization of the two-band projectors (5). Since the Hamiltonian is typically known for any given problem, the only ingredient required for explicitly computing the eigenprojector \( P_\alpha \) is the corresponding eigenenergy \( E_\alpha \).

Beyond the computation of eigenprojectors, Eq. (17) has two interesting applications. First, it can be used to rewrite any function \( f(H) \) as a polynomial of order \( N - 1 \) in the Hamiltonian, which corresponds to an alternative version of Sylvester’s formula [44]:

\[ f(H) = \sum_{\alpha=1}^{N} f(E_\alpha) \frac{q_{N-1-n}(E_\alpha) H^n}{\sum_{n=0}^{N-1} q_{N-1-n}(E_\alpha) E_\alpha^n}. \]

Second, the eigenprojector is in some sense a more fundamental object than the eigenstate, namely Eq. (17) may be employed for constructing energy eigenstates

\[ |\psi_\alpha\rangle = \frac{1}{\sqrt{\langle \psi_g | P_\alpha(E_\alpha, H) | \psi_g \rangle}} P_\alpha(E_\alpha, H) | \psi_g \rangle, \]

by projecting onto a gauge freedom state \( | \psi_g \rangle \) that can be chosen arbitrarily (for more details, see Appendix E). The eigenstates (21) can then be used further to compute

\[ b_\alpha = \frac{1}{E_\alpha} h, \]

\[ b_\alpha = \frac{2}{3E_\alpha^2 - \frac{C_2^2}{4}} (E_\alpha h + h_\alpha), \]

\[ b_\alpha = \frac{2}{4E_\alpha^3 - C_2 E_\alpha - \frac{C_3}{3}} \left[ \left( E_\alpha^2 - \frac{C_2}{4} \right) h + E_\alpha h_\alpha + h_{\alpha \alpha} \right], \]

\[ b_\alpha = \frac{2}{5E_\alpha^4 - 3C_2 E_\alpha^2 - \frac{2C_3}{3} E_\alpha + \frac{C_2^2 - 2C_4}{8}} \left[ \left( E_\alpha^3 - \frac{3C_2}{10} E_\alpha^2 - \frac{2C_3}{15} \right) h + \left( E_\alpha^2 - \frac{3C_2}{10} \right) h_\alpha + E_\alpha h_{\alpha \alpha} + h_{\alpha \alpha \alpha} \right]. \]

\[ \]

matrix elements of observables, or the Berry connection and other quantities of interest.

C. Bloch vectors as a function of the Hamiltonian vector: \( b_\alpha(E_\alpha, h) \)

The above key results (17) & (19) on the eigenprojectors are more convenient for some practical purposes when reformulated in a vectorial language. More precisely, just like a Hamiltonian vector was introduced in Eq. (16), one may define a (generalized) Bloch vector \( b_\alpha \) by expanding the eigenprojector as [46]

\[ P_\alpha(x) = \frac{1}{N} 1_N + \frac{1}{2} b_\alpha(x) \cdot \lambda \]

in the SU(\( N \)) generators, where \( b_\alpha \equiv \text{Tr}\{ P_\alpha \lambda \} \). Again, be reminded that throughout this paper we will always use the generators to be (generalized) Gell-Mann matrices. Importantly, however, the functional form of all SU(\( N \)) vector identities discussed below is independent of the particular choice of generator matrices.

Now, by inserting the expansions (16) & (22) into Eq. (17), one obtains the analog of the function \( P_\alpha(E_\alpha, H) \) in the vectorial language, namely \( b_\alpha(E_\alpha, h) \), as explained in Appendix F. The usefulness of this procedure is most apparent when considering the explicit Bloch vector expressions resulting from it, provided here for \( N = 2 \) to \( N = 5 \):
This is the \( N \)-band generalization of the two-band Bloch vectors (5), and contains exactly the same information as Eq. (19).

Two ingredients of Eq. (23) require some more explanation, namely the Casimir invariants \( C_n \), and the "star product vectors" \( \mathbf{h}_\alpha, \mathbf{h}_\alpha^\ast, \) etc. Importantly, both quantities are completely determined by the vector \( \mathbf{h} \).

**Casimir invariants from the Hamiltonian vector**

Since the Casimir invariants are traces of powers of the Hamiltonian matrix, cf. Eq. (18), they can be directly expressed in terms of the Hamiltonian vector (cf. Appendix F):

\[
\begin{align*}
C_2 &= 2|\mathbf{h}|^2, \\
C_3 &= 2 \mathbf{h} \cdot \mathbf{h}_\alpha, \\
C_4 &= 4|\mathbf{h}|^4/N + 2|\mathbf{h}|^2.
\end{align*}
\]

(24)

It is convenient to combine Eq. (24) with Eq. (23), such that the Bloch vector for a given band \( \alpha \) depends only on the single eigenvalue \( E_\alpha \).

**Star products of the Hamiltonian vector**

For given \( N \), each Bloch vector (23) is a kind of "vector polynomial" of degree \( N - 1 \) in \( \mathbf{h} \), just like each eigenprojector is a matrix polynomial in \( H \). In other words, the vectors \( \mathbf{h}_\alpha, \mathbf{h}_\alpha^\ast, \) etc. – though rather unfamiliar objects of dimension \( \text{Energy}^2 \), \( \text{Energy}^3 \), etc. – are uniquely determined by the Hamiltonian vector \( \mathbf{h} \), as explained in the following.

The properties of the Lie algebra \( \mathfrak{su}(N) \) underlying the \( N \)-band Hamiltonian (16) are determined by the commutation and anticommutation relations [47]

\[
\begin{align*}
[\lambda_a, \lambda_b] &= 2i f_{abc} \lambda_c, \\
\{\lambda_a, \lambda_b\} &= \frac{4}{N} \delta_{ab} 1_N + 2d_{abc} \lambda_c,
\end{align*}
\]

(25)

where repeated lower indices imply summation (Einstein convention). In particular, these relations define totally antisymmetric and totally symmetric *structure constants* of \( \mathfrak{su}(N) \), respectively:

\[
\begin{align*}
f_{abc} &\equiv -\frac{i}{4} \text{Tr}(\{\lambda_a, \lambda_b\} \lambda_c), \\
d_{abc} &\equiv \frac{1}{4} \text{Tr}(\{\lambda_a, \lambda_b\} \lambda_c).
\end{align*}
\]

(26)

These are a known set of real numbers once a matrix representation is chosen for the generators \( \lambda_a \). Note that for \( N = 2 \), where \( \lambda = \sigma \), the \( d_{abc} \) vanish identically and \( f_{abc} = \epsilon_{abc} \), where \( \epsilon_{abc} \) is the Levi-Civita tensor.

From the structure constants, one defines dot, star and cross products of \( \text{SU}(N) \) vectors:

\[
\begin{align*}
\mathbf{m} \cdot \mathbf{n} &\equiv m_c n_c, \\
(\mathbf{m} \ast \mathbf{n})_a &\equiv d_{abc} m_b n_c, \\
(\mathbf{m} \times \mathbf{n})_a &\equiv f_{abc} m_b n_c,
\end{align*}
\]

(27)

where \( \mathbf{m} \) and \( \mathbf{n} \) are real and \( (N^2 - 1) \)-dimensional. The star product is unfamiliar because it does not play any role in \( N = 2 \) situations (where \( d_{abc} = 0 \), but it is crucial for \( N > 2 \) systems. Finally, we may introduce the notation for repeated star products of a vector with itself:

\[
\begin{align*}
\mathbf{m}_0 &= \mathbf{m}, \\
\mathbf{m}_1 &= \mathbf{m} \ast \mathbf{m}, \\
\mathbf{m}_2 &= \mathbf{m} \ast \mathbf{m} \ast \mathbf{m}, \\
\mathbf{m}_{k+1} &= \mathbf{m} \ast \mathbf{m}_{k},
\end{align*}
\]

(28)

The vectors \( \mathbf{h}_\alpha, \mathbf{h}_\alpha^\ast, \) etc. are thus simply star products of the Hamiltonian vector with itself.

**D. Three different SU(\( N \)) Bloch spheres**

The Hamiltonian’s, eigenprojector’s and eigenstate’s Bloch spheres introduced in Section II C for the two-band case are here briefly discussed for higher \( N \).

The Hamiltonian vector \( \mathbf{h} \) has \( N^2 - 1 \) real-valued components, such that the unit vector \( \mathbf{u}_h \equiv \mathbf{h}/|\mathbf{h}| \) can be parametrized by \( N^2 - 2 \) Hamiltonian’s angles. In other words, \( \mathbf{u}_h(x) \) defines a map

\[
u_h: X \rightarrow \mathcal{B}_h^{(N)}, \quad x \mapsto \mathbf{u}_h(x)
\]

(29)

from the parameter space to a *Hamiltonian’s Bloch sphere* \( \mathcal{B}_h^{(N)} = S^{N^2-2} \), where \( S^{N^2-2} \) is the unit \( (N^2 - 2) \)-sphere, as depicted in Fig. 2(d).

Naively, since \( \mathbf{b}_\alpha \) also has \( N^2 - 2 \) real-valued components, the map \( \mathbf{b}_\alpha(x) \) (for given \( \alpha \)) would similarly seem to define an \( (N^2 - 2) \)-sphere, but this is prevented by constraints on the Bloch vectors. In particular, the usual orthogonality relation \( P_\alpha P_\beta = \delta_{\alpha \beta} P_\alpha \) and completeness relation \( \sum_\alpha P_\alpha = 1_N \) translate to the Bloch vector picture as

\[
\begin{align*}
\mathbf{b}_\alpha \cdot \mathbf{b}_\beta &= 2 \left( \delta_{\alpha \beta} - \frac{1}{N} \right), \\
\mathbf{b}_\alpha \ast \mathbf{b}_\beta &= \left( \delta_{\alpha \beta} - \frac{2}{N} \right) (\mathbf{b}_\alpha + \mathbf{b}_\beta), \\
\mathbf{b}_\alpha \times \mathbf{b}_\beta &= 0, \\
\sum_\alpha \mathbf{b}_\alpha &= 0.
\end{align*}
\]

(30)

As a consequence, the vector \( \mathbf{b}_\alpha(x) \) defines a map

\[
\mathbf{b}_\alpha: X \rightarrow \mathcal{B}_P^{(N)}, \quad x \mapsto \mathbf{b}_\alpha(x)
\]

(31)
from parameter space not to an \((N^2 - 2)\)-sphere but to a \(2(N - 1)\)-dimensional subset thereof, which may be called the eigenprojector’s Bloch sphere \(\mathcal{B}^{(N)}_\alpha\), or simply the generalized Bloch sphere, see Fig. 2(e). In contrast to the \(N = 2\) case, the Bloch vector is no longer parallel to the Hamiltonian vector for \(N > 2\) [cf. Eq. (23)], such that \(u_\alpha(x)\) and \(b_\alpha(x)\) are completely distinct maps. An understanding of the true geometrical structure of the generalized Bloch sphere is not at all easy to acquire. Many efforts have been undertaken to figure out its properties for \(N > 2\), which is a surprisingly nontrivial issue, see Refs. [48–55] and references therein. For the interested reader, we outline the main results in Appendix H.

The third type of Bloch sphere arises from the eigenstates. Each eigenstate (21) of an \(N\)-band system can be minimally encoded by a global phase \(\Gamma_\alpha(x)\) and \(N - 1\) pairs of eigenstate’s angles \((\theta_\alpha^i(x), \phi_\alpha^j(x))\) \((i = 1, ..., N - 1)\). These angles define a map
\[
|\psi_\alpha\rangle : X \rightarrow \mathcal{B}^{(N)}_\alpha, \quad x \mapsto |\psi_\alpha(x)\rangle
\]
from the parameter space to a \((2N - 1)\)-dimensional eigenstate’s Bloch sphere \(\mathcal{B}^{(N)}_\alpha\), which is depicted, for comparison with the \(N = 2\) case, in Fig. 2(f).

In summary, the Hamiltonian’s, the eigenprojector’s and the eigenstate’s Bloch sphere are all isomorphic spaces for \(N = 2\), but all different spaces for \(N > 2\).

IV. GENERALIZATION TO \(N\)-BAND SYSTEMS: PART 2 – QUANTUM GEOMETRIC TENSOR

The QGT in \(N\)-band systems is usually defined in terms of \(N\)-component eigenstates \(|\psi_\alpha\rangle\). The standard eigenstate-based QGT formula is given by Eq. (2). (For more details on its origin, see Appendix I.) A more popular eigenstate-based QGT expression, which is easily found by treating \(H(x + dx) - H(x)\) as a formal perturbation to \(|\psi_\alpha(x)\rangle\), reads
\[
T_{\alpha,\beta} = \sum_{\beta \neq \alpha} \frac{\langle \psi_\beta | \partial_\beta H | \psi_\beta \rangle \langle \psi_\beta | \partial_\beta H | \psi_\alpha \rangle}{(E_\alpha - E_\beta)^2}.
\] (33)

The advantage of Eq. (33), as compared to Eq. (2), is that it involves the parametric velocity operators \(\partial_\beta H\), thus avoiding derivatives of eigenstates. Therefore, it is the most common formula used for computing the QGT. However, this in general still requires numerical construction of the eigenstates.

The main goal here is to develop a deeper analytical understanding of the geometric tensors and their direct relation to the Hamiltonian. This can be achieved by avoiding eigenstates and drawing on the eigenprojector-based formula (3). (For the relation between Eq. (2) and Eq. (3), see Appendix I.)

First, we will reformulate Eq. (3) in terms of generalized Bloch vectors (Section IV A). Then, inserting Eq. (23) into the result, we will be able to write the QGT in terms of only the Hamiltonian vector \(h\) and the eigenenergy \(E_\alpha\) (Section IV B).

A. QGT in terms of Bloch vectors

In order to obtain a Bloch vector picture of the geometric tensors, one can simply substitute Eq. (22) into Eq. (3). Using the SU(\(N\)) product identity (cf. Appendix F)
\[
(m \cdot \lambda)(n \cdot \lambda) = \frac{2}{N} m \cdot n 1_N + (m \ast n + i m \times n) \cdot \lambda
\] (34)
and the star product (30) of Bloch vectors, one finds
\[
\Omega_{\alpha,ij} = -\frac{1}{2} b_{\alpha i} \cdot (b_{\alpha j} \times b_{\alpha j}^\prime),
\] (35)
for arbitrary values of \(N\) (again with the shorthand notation \(m^i \equiv \partial_i m\). This is the same as Eq. (4) stated in the introduction.

Symbolically, Eq. (35) is of the exactly same form as in the \(N = 2\) case, cf. Eq. (14). However, be aware that for \(N > 2\) the dimension of the Bloch vectors increases, and the cross product has to be interpreted in the SU(\(N\)) sense, cf. Eq. (27). For the SU(3) case, an equivalent way of writing Berry curvature and quantum metric was already encountered in Refs. [35, 36] and [37], respectively; for the SU(\(N\)) case, Pozo and de Juan recently found a similar formula in terms of what they call \(I\)-generators [38].

While Eq. (35) is conceptually important, a certain disadvantage consists in the presence of explicit parametric derivatives of the Bloch vector; this is potentially complicated, since cumbersome expressions follow from applying the product rule to Eq. (23). An alternative form of writing the QGT that does not involve such explicit derivatives of the Bloch vectors is provided in Appendix J.

B. QGT from the Hamiltonian and its eigenvalues

Having established the general expressions (35) for arbitrary values of \(N\), the final step towards obtaining the QGT in terms of only the Hamiltonian vector \(h\) and the eigenenergy \(E_\alpha\) consists in inserting the explicit formula (23).

For the Berry curvature, we obtain a closed form expression for arbitrary \(N\), which is illustrated here for \(N = 3, 4, 5\) for comparison with the simple \(N = 2\) expression (14); the discussion for arbitrary \(N\) is given in Appendix K.

In the \(N = 3\) case, the Berry curvature tensor is given by
\[
\Omega_{\alpha,ij} = -\frac{4(E_\alpha h^i + h_\alpha^i)}{(3E_\alpha^2 - |h|^2)^{\frac{3}{2}}} \left[\left(E_\alpha h^j + h_\alpha^j\right) \times \left(E_\alpha h^j + h_\alpha^j\right)\right],
\] (36)
with $h_\alpha^i \equiv \partial_i h_\alpha$. This expression is equivalent to the result found by Barnett et al. [35] if one inserts a closed-form energy parametrization in terms of trigonometric functions, see for example Ref. [56]. Note that Eq. (36) can be expressed in a more compact form (cf. Appendix K), which allows to explicitly verify the sum rule $\sum_\alpha \Omega_{\alpha,ij} = 0$.

For arbitrary $N = 4$ systems, we have

$$\Omega_{\alpha,ij} = -\frac{4(Q_\alpha h + E_\alpha h_\alpha + h_{*\alpha})}{(4E_\alpha Q_\alpha - \frac{2}{3} |h|^2)^2} \cdot \left[ (Q_\alpha h^i + E_\alpha h_\alpha^i + h_{*\alpha}^i) \times \left( Q_\alpha h^j + E_\alpha h_\alpha^j + h_{*\alpha}^j \right) \right],$$

with $Q_\alpha = E_\alpha^2 - |h|^2/2$. Similarly, for arbitrary $N = 5$ systems, the Berry curvature is given by

$$\Omega_{\alpha,ij} = -\frac{4(R_\alpha h + \tilde{R}_\alpha h + E_\alpha h_\alpha + h_{*\alpha})}{(5E_\alpha R_\alpha + \frac{3}{10} |h|^4 - \frac{1}{2} |h|^2)^2} \cdot \left[ (R_\alpha h^i + \tilde{R}_\alpha h_\alpha^i + E_\alpha h_\alpha^i + h_{*\alpha}^i) \times \left( R_\alpha h^j + \tilde{R}_\alpha h_\alpha^j + E_\alpha h_\alpha^j + h_{*\alpha}^j \right) \right],$$

where $R_\alpha \equiv E_\alpha R_\alpha - \frac{1}{3} h \cdot h$ and $\tilde{R}_\alpha \equiv E_\alpha^2 - \frac{3}{2} |h|^2$. In the same way, one may obtain expressions for $N > 5$.

For the quantum metric, writing down explicit formulas in terms of $h$ and $E_\alpha$ proves too cumbersome for $N > 2$, due to the absence of appropriate orthogonality relations. The quantum metric can therefore be straightforwardly computed by a two-step procedure: first calculate the Bloch vectors (23), then substitute the result into Eq. (35).

V. EXAMPLES: MULTIFOLD FERMIONS

In this section, the utility of the formalism developed above is demonstrated by applying it to concrete Hamiltonians of interest. In particular, we will consider three- and four-band models in the class of multifold (Dirac) fermions [57], i.e. low-energy models $H(q)$ with a linear spectrum and an $N$-fold degeneracy at $q = 0$. Here, $x = q$ should be understood as some generic three-dimensional quasi-momentum vector, which may take various physical meanings [15, 58–63].

A. Pseudospin fermions

As a first simple class of models that lends itself to an analysis within the above language, consider (pseudo)spin-$S$ multifold fermion models, where $S = (N - 1)/2$. Such models involve matrices $S_{x,y,z}$ that satisfy a spin algebra

$$[S_i, S_j] = i\epsilon_{ijk}S_k,$$

as well as $S^2 = S(S + 1)\mathbb{1}_N$. The corresponding Hamiltonian can be written as

$$H(q) = q \cdot S = q_x S_x + q_y S_y + q_z S_z.$$

This monopole carries a topological charge measured by the first Chern number, $C_m = -2m$. As is well known, $C_m$ is odd (even) for half-integer (integer) spin.

We now restrict ourselves to the three- and four-band case for simplicity, i.e. to pseudospin-1 and pseudospin-3/2 fermions, respectively. The respective band structures are shown in Fig. 3(a)&(b). Our goal is to obtain the eigenprojectors and Bloch vectors, which will allow us to compute the QGT without constructing spin eigenstates. In particular, we will recover the result (42) and additionally compute the quantum metric. We now separate the treatment for spin-1 and spin-3/2.

Pseudospin-1 fermions

The spin Hamiltonian (40) can be easily rewritten in the general form of Eq. (16) by using the relation between spin matrices and Gell-Mann matrices. In the
As expected. Similarly, the Bloch vectors follow from Eq. (24), as obtained from Eq. (19) as

\[ \mathbf{b}(\mathbf{q}) = \frac{1}{2} (\sqrt{2}q_x, \sqrt{2}q_y, q_z, 0, 0, \sqrt{2}q_x, \sqrt{2}q_y, \sqrt{3}q_x). \]  

Upon using the spectrum (41), as well as \( C_2 = 2|q|^2 \) and \( C_3 = 0 \), the eigenprojectors for a spin-3/2 fermion are obtained from Eq. (19) as

\[ P_m(\mathbf{q}) = \frac{1}{3m^2 - 1} \left[ (m^2 - 1)1_3 + \frac{m}{|q|} (\mathbf{h} \cdot \lambda) \right. \]
\[ \left. + \frac{1}{|q|^2} (\mathbf{h} \cdot \lambda)^2 \right]. \]  

(44)

It is readily verified that \( \text{Tr}(P_m) = 1 \) and \( \sum_m P_m = 1_3 \) as expected. Similarly, the Bloch vectors follow from Eq. (23) as

\[ \mathbf{b}_m(\mathbf{q}) = \frac{2}{3m^2 - 1} \left( \frac{m}{|q|} \mathbf{h} + \mathbf{h}_s \right), \]  

(45)

where

\[ \mathbf{h}_s(\mathbf{q}) = \left( \frac{q_xq_z - 3q^2 - |q|^2}{4}, \frac{q_x^2 - q_y^2}{2}, \frac{q_xq_y}{\sqrt{2}}, \frac{q_xq_y}{\sqrt{2}}, \frac{q_yq_z}{\sqrt{2}}, \frac{q_yq_z}{\sqrt{2}}, \frac{|q|^2 - 3q^2}{4\sqrt{3}} \right). \]  

(46)

The Berry curvature may now be obtained from Eq. (35) [or more directly from Eq. (36)], and one recovers Eq. (42) as expected. Similarly, the quantum metric follows from Eq. (35) as

\[ g_{m,ij}(\mathbf{q}) = \frac{15}{2|m^2 - 2} \left( \delta_{ij} - \frac{q_iq_j}{|q|^2} \right), \]  

(47)

in agreement with Ref. [65].

**Pseudospin-3/2 fermions**

In the standard representation, the spin-1 matrices are related to the \( N = 3 \) Gell-Mann matrices by Eq. (C1). Thus, the Hamiltonian vector reads

\[ \mathbf{h}(\mathbf{q}) = \frac{1}{2} (\sqrt{3}q_x, \sqrt{3}q_y, q_z, 0, 0, 2q_x, 2q_y, \sqrt{3}q_z). \]  

(48)

The star product vectors \( \mathbf{h}_s \) and \( \mathbf{h}_{ss} \) can be computed from Eq. (48) using Eq. (28).

Upon using the spectrum (41), as well as \( C_2 = 5|q|^2 \) and \( C_3 = 0 \), the eigenprojectors for a spin-3/2 fermion are obtained as

\[ P_m(\mathbf{q}) = \frac{1}{4m (m^2 - \frac{5}{4})} \left[ m \left( m^2 - \frac{5}{2} \right) 1_4 \right. \]
\[ \left. + m^2 - \frac{5}{4} (\mathbf{h} \cdot \lambda) + \frac{m}{|q|^2} (\mathbf{h} \cdot \lambda)^2 + \frac{1}{|q|^3} (\mathbf{h} \cdot \lambda)^3 \right]. \]  

(49)

Similarly, the Bloch vectors are given by

\[ \mathbf{b}_m(\mathbf{q}) = \frac{2}{4m (m^2 - \frac{5}{4})} \left( \left( m^2 - \frac{5}{2} \right) \frac{\mathbf{h}}{|q|} \right. \]
\[ \left. + m^2 - \frac{5}{4} \frac{\mathbf{h}_s}{|q|^2} + \frac{m}{|q|^2} \left( \mathbf{h} \cdot \lambda \right) \right]. \]  

(50)

Importantly, be aware that the coefficient of the linear term (in \( h \)) of the eigenprojector is in general not the same as the coefficient of the linear term (in \( h \)) of the Bloch vector, and similarly for the higher-order terms. This is due to the fact that \( H_{n+1} \neq \mathbf{h}_d^{(n)} \cdot \lambda \) for \( n \geq 0 \).

The Berry curvature follows from Eq. (35) [or more directly from Eq. (37)], and again takes the expected form (42). Similarly, the quantum metric reads

\[ g_{m,ij}(\mathbf{q}) = \frac{15}{4 - m^2} \left( \delta_{ij} - \frac{q_iq_j}{|q|^2} \right), \]  

(51)

again in agreement with Ref. [65].

**B. Beyond pseudospin fermions**

Let us now discuss two examples that go beyond the simple pseudospin models. More precisely, we present a three-band (four-band) multifield fermion model with exactly the same energy spectrum as a spin-1 (spin-3/2), cf. Fig. 3, but with completely different symmetries and quantum geometric properties. It is important to realize that the analytical expressions presented below are easily obtained using our approach, while establishing them from an energy eigenstate approach would be rather cumbersome.

**Three-band example**

Consider the Hamiltonian

\[ H(\mathbf{q}) = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} (q_x - iq_y) & -iq_z \\ ... & 0 & \frac{1}{\sqrt{2}} (q_x + iq_y) \\ ... & ... & 0 \end{pmatrix}, \]  

(52)
where the lower left matrix elements are obtained by complex conjugating the upper right ones. This model has a global chiral symmetry \( S \dagger H(q) S = -H(q) \) with \( S = \text{antidiag}(1, -1, 1, 1) \). The energy spectrum reads

\[
E_\alpha(q) = \alpha |q|, \quad \alpha = 0, \pm 1, \tag{53}
\]

which is the same as for a pseudospin-1 model.

In the language of Eq. (16), the model (52) is characterized by a Hamiltonian vector

\[
h(q) = \frac{1}{\sqrt{2}}(q_x, q_y, 0, 0, \sqrt{2}q_z, q_x, q_y, 0) \tag{54}
\]

and its star product

\[
h_*(q) = \left( \frac{q_y-q_z}{\sqrt{2}}, \frac{3q^2_2 - |q|^2}{4}, \frac{q^2_2 - q^2_0}{2}, q_x q_y, \frac{q_y q_z}{\sqrt{2}}, \frac{q_x q_z}{\sqrt{2}}, \frac{|q|^2 - 3q^2_0}{4\sqrt{3}} \right). \tag{55}
\]

The Bloch vectors follow from Eq. (23) as

\[
b_\alpha(q) = \frac{2}{3q^2 - 1} \left( \frac{\mathbf{h}}{|q|^2} + \mathbf{h}_* \right). \tag{56}
\]

Notably, they are formally completely equivalent to the spin-1 case, but with slightly different vectors \( \mathbf{h} \) and \( \mathbf{h}_* \).

The QGT for all bands \( \alpha = 0, \pm 1 \) can be summarized in compact form as

\[
\Omega_\alpha(q) = (2 - 3\alpha^2)(\mathbf{q} \cdot \mathbf{e}_z) \frac{\mathbf{q}}{|q|^2},
\]

\[
g_{\alpha,ij}(q) = \frac{2 - \alpha^2}{2|q|^2} \left[ \delta_{ij} - \frac{q_i q_j}{|q|^2} \right] - \frac{2 - \alpha^2}{2|q|^2} (\mathbf{q} \times \mathbf{e}_z)_i (\mathbf{q} \times \mathbf{e}_z)_j, \tag{57}
\]

where we again introduced a Berry curvature pseudovector \( \Omega_\alpha \) and where \( \mathbf{e}_z = (0, 0, 1) \). There are several notable features of Eq. (57). First, the Berry curvature is completely different from its spin-1 analog (42) and yields first Chern numbers \( C_\alpha = 0 \). Second, the quantum metric contains a part that is exactly like for a spin-1 [cf. Eq. (47)], plus an additional term. Third, the QGT is anisotropic despite an isotropic band structure.

### Four-band example

Consider now the Hamiltonian

\[
H(q) = \begin{pmatrix}
0 & q_y + \frac{q_x}{2} - i (q_x + \frac{q_y}{2}) & q_x - q_z \\
0 & q_x + q_z & i \left( q_x - \frac{q_y}{2} \right) \\
0 & \frac{q_x}{2} + q_z & i (q_x - q_z) \\
0 & 0 & q_y - \frac{q_z}{2}
\end{pmatrix}. \tag{58}
\]

This model has a global charge conjugation symmetry with \( C = \text{diag}(1, -1, 1, -1) \), but no chiral symmetry, and a band structure

\[
E_{\alpha_1, \alpha_2}(q) = \alpha_1 \left( 1 + \frac{\alpha_2}{2} \right) |q|, \quad \alpha_1 = \pm 1, \alpha_2 = \pm 1 \tag{59}
\]

which is exactly the same as for a pseudospin-3/2. Note that a similar model has been studied in Ref. [66].

The model (58) is described by a Hamiltonian vector

\[
h(q) = \left( q_y + \frac{q_x}{2}, q_x + \frac{q_y}{2}, \frac{q_x}{2} - q_z, q_x - q_z, 0, q_x, q_y + \frac{q_z}{2}, q_y - \frac{q_z}{2}, 0 \right), \tag{60}
\]

from which the vectors \( \mathbf{h}_* \) and \( \mathbf{h}_{**} \) are readily obtained. The Bloch vectors follow from Eq. (23) as

\[
b_{\alpha_1, \alpha_2}(q) = \frac{\alpha_1}{1 + 2\alpha_2} \left[ \frac{\alpha_2}{|q|^2} \mathbf{h}_* \right] + \alpha_1 \left( 1 + \frac{\alpha_2}{2} \right) \left[ \frac{\alpha_2}{|q|^2} \mathbf{h}_* + \frac{\alpha_2}{|q|^2} \mathbf{h}_{**} \right]. \tag{61}
\]

The QGT for all four bands (calculated in the same way as for the spin-3/2 case above) is then given by

\[
\Omega_{\alpha_1, \alpha_2}(q) = -\alpha_1 (1 + \alpha_2) \frac{\mathbf{q}}{|q|^2},
\]

\[
g_{\alpha_1, \alpha_2, ij}(q) = \frac{1}{2|q|^2} \left( \delta_{ij} - q_i q_j \right). \tag{62}
\]

Though Eq. (62) is formally similar to the QGT of a spin-3/2 fermion, there are crucial differences. Most importantly, the first Chern numbers obtained from Eq. (62) are \{2, 0, 0, -2\} from the lowest to the highest band, while they are \{3, 1, -1, -3\} for a spin-3/2. Moreover, the quantum metric is band-independent.

### VI. CONCLUSIONS

For a physical system described by a (Hermitian) parametric Hamiltonian matrix \( H = H(\mathbf{x}) \), there are situations in which it is useful to recall that any eigenstate is more fundamentally encoded in the \textit{eigenprojector matrix} \( P_\alpha(\mathbf{x}) = \langle \psi_\alpha(\mathbf{x}) \rangle \langle \psi_\alpha(\mathbf{x}) \rangle \) [67].

Consequently, for a given quantity of interest whose standard expression is known in terms of eigenstates, it can prove rewarding to aim for a reformulation in terms of eigenprojectors, which will entirely eliminate the necessity for constructing eigenstates explicitly. Such a reformulation can be done for any physical quantity in principle. The present work demonstrates this by focusing on the case where the quantities of interest are the quantum metric and Berry curvature tensors (forming together what is known as the quantum geometric tensor, QGT). Selecting the QGT is natural for two reasons. First, it is relatively simple (compared to more involved quantities like the orbital magnetic susceptibility), and
therefore serves well for illustrating the main features of
the eigenprojector approach, which are preserved in the
treatment of more complicated observables. Second, the
inconvenient features of eigenstates become particularly
problematic when one is interested in geometrical quan-
tities; therefore, the QGT is one of the observables for
which the projector formalism is the most powerful and
beneficial.

The most striking qualitative conclusion that can be
drawn from the eigenprojector approach is that any
quantity of interest can be written in terms of the Hamil-
tonian vector $\mathbf{h}$ and the eigenvalues $E_{\alpha}$, without using
eigenstates, in agreement with recent results of Pozo and
de Juan [38] (who used a somewhat different mathematical
framework). At the origin of this conclusion is the
fact that each eigenprojector is a matrix polynomial of
degree $N - 1$ in $H$, according to the Cayley-Hamilton
theorem. This result, though implicitly known for a long
time [68], is perhaps most useful to physicists when for-
mulated as an explicit function $P_{\alpha}(E_{\alpha}, H)$ of a single
eigenvalue, see Eqs. (17) & (19). These expressions for
$P_{\alpha}(E_{\alpha}, H)$ represent our first intermediate result.

The realization that the knowledge of $H$ as well as
its eigenenergies immediately yields the eigenprojectors
implies that any physical quantity of interest can be com-
puted in the following way:

(1) Express the quantity in terms of eigenprojectors.

(2) Explicitly insert the function $P_{\alpha}(E_{\alpha}, H)$.

In the course of carrying out this procedure, it proves
extremely enlightening to switch between two "lan-
duages" where appropriate. The first language simply
employs the relevant matrices ($H$, $P_{\alpha}$, etc.), while the
second one expands those matrices in the generators of
SU($N$). In this second (vectorial) language the Hamilto-
nian $H$ is given by a real vector $\mathbf{h}$, and similarly the
eigenprojector $P_{\alpha}$ is represented by a real (generalized) Bloch
vector $\mathbf{b}_{\alpha}$. The analog of the matrix function $P_{\alpha}(E_{\alpha}, H)$
is the vector function $\mathbf{b}_{\alpha}(E_{\alpha}, \mathbf{h})$, see in particular the im-
portant result (23). Both languages contain exactly the
same information, such that the above protocol can be
equivalently formulated as:

(1) Express the quantity in terms of Bloch vectors.

(2) Explicitly insert the function $\mathbf{b}_{\alpha}(E_{\alpha}, \mathbf{h})$.

In the present paper, starting in Section IV, the above
protocol was carried out for the QGT, and the systematic
application to other, more complicated observables will
be published elsewhere [39]. The known formula (3) for
the QGT in terms of eigenprojectors can be written in
the vectorial language as described in Section IV A [cf. in
particular Eq. (35)], which proves quite handy for prac-
tical computations and completes step (1). Step (2) was
carried out in Section IV B, where we obtained the Berry
curvature tensor $\Omega_{\alpha,ij}$ in terms of the vector $\mathbf{h}$ and the
relevant energy eigenvalue $E_{\alpha}$ only. This generalizes the
well-known Berry curvature expression (14) to arbitrary $N$.

As a concrete illustration of the results obtained for
the QGT, Section V served to present models belonging
to the class of multifold (Dirac) fermions, characterized
by an $N$-fold degenerate nodal point accompanied by a
linear band crossing. First, we found interesting Bloch
vector expressions for spin-1 and spin-3/2 fermions that
yield a QGT consistent with the literature. Second, we
introduced two multifold fermion models, which empha-
size that different Hamiltonians can exhibit completely
distinct geometrical and topological properties despite
being indistinguishable according to their band structure.
Such Hamiltonians may serve as interesting platforms for
studying effects such as orbital magnetism or Friedel osc-
cillations, similar to the $a - T_3$ model [69].

To close this paper, we state a few immediate per-
tpectives of our work. First, as already mentioned, it is
worthwhile to apply the eigenprojector approach to ob-
servables that are not purely of geometric origin. In par-
cular, the analytical insight afforded by this approach
will help to unveil and distinguish purely spectral and
geometrical contributions to such observables, similar in
spirit to what has been done for the orbital magnetic
susceptibility of two-band models in Ref. [19], supercon-
ducting stiffness or non-linear responses and more gener-
alized thermodynamic stiffnesses.

Second, it would be interesting to establish a better
understanding of the geometry and topology of multi-
fold fermion models with linear but also quadratic band
crossings. More concretely, how to systematically design
models that all share a certain type of energy spectrum
but belong to different topological classes according to
the tenfold way classification [64]?

Finally, note that the validity of the formalism devel-
oped in this paper goes beyond the case of Hermitian
Hamiltonian matrices. In particular, the key expressions
Eq. (19) for the eigenprojector $P_{\alpha}(E_{\alpha}, H)$ and Eq. (23)
for the Bloch vector $\mathbf{b}_{\alpha}(E_{\alpha}, \mathbf{h})$ stay valid for systems
where the Hermiticity condition is relaxed [70, 71].

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Appendix A: Gauge dependency and singularities of eigenstates

Here, we describe some issues of eigenstates that are absent in an eigenprojector approach. Consider the two-band eigenstates (12). The eigenstate for $\alpha = +$ can be written in either of the forms

$$|\psi_+(x)\rangle = e^{i\Gamma(x)} \begin{pmatrix} \cos \theta_+ \\ \sin \theta_+ e^{i\phi_+} \end{pmatrix} = \frac{1}{\sqrt{2(1 + b_+ E_+)}} \begin{pmatrix} 1 + \frac{b_+}{E_+} \\ h_+ + h_+ \end{pmatrix} = \frac{1}{\sqrt{2(1 + \cos \theta_+)}} \begin{pmatrix} 1 + \cos \theta_+ \\ \sin \theta_+ e^{i\phi_+} \end{pmatrix}, \tag{A1}$$

and similarly for $|\psi_-(x)\rangle$. The first expression corresponds to the formal parametrization of a complex-valued two-component unit vector. It expresses the fact that, at each point $x$ in the parameter space, the eigenstate $|\psi_+(x)\rangle$ is minimally encoded by a global phase $\Gamma_+(x)$ and two eigenstate’s angles ($\theta_+(x), \phi_+(x)$). Thus, the first evident problem of Eq. (A1) consists in the fact that it is gauge-dependent. The second expression in Eq. (A1) illustrates a second problem, namely the possible singular behavior of eigenstates. A singularity is evidently to be expected when $h_+(x_0)/E_+(x_0) = -1$ for some point $x = x_0$ in parameter space. Similarly, if the eigenstate components are written in terms of the angles ($\theta_+, \phi_+$), the explicit relation $\tan \phi_+ = h_+/h_+$ points towards further possible singular behavior in parameter space when $h_+(x_0) = 0$. Finally, the closed-form expression of $|\psi_+(x)\rangle$ (in terms of the corresponding energy eigenvalue and the components of the Hamiltonian vector $h$) is rather cumbersome and not very convenient if one is dealing with classes of Hamiltonians beyond a specific Hamiltonian of interest. This issue in particular becomes much more dramatic as soon as $N > 2$. In this case, similar closed-form expressions are practically useless.

Appendix B: Quantum geometric tensor for different parameter spaces

The explicit dimension and form of the matrices $g_\alpha$ and $\Omega_\alpha$ clearly depends on the parameter space. For example, in a two-band system, taking $x = (\theta_h, \phi_h)$ ($d = \dim(x) = 2$) and inserting the state (A1) into Eq. (2), the quantum metric and Berry curvature tensors are $2 \times 2$ matrices that write as

$$g_+(\theta_h, \phi_h) = \frac{1}{4} \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 \theta_h \end{pmatrix}, \quad \Omega_+(\theta_h, \phi_h) = -\frac{1}{2} \begin{pmatrix} 0 & \sin \theta_h \\ -\sin \theta_h & 0 \end{pmatrix}, \tag{B1}$$

which is a well-known result [3]. Taking instead $x = (h_x, h_y, h_z)$ ($d = \dim(x) = 3$) for the same two-band state, the quantum metric and Berry curvature tensors are now $3 \times 3$ matrices with matrix elements

$$g_{+,ij}(h) = \frac{1}{4|h|^2} \left( \delta_{ij} - \frac{h_i h_j}{|h|^2} \right), \quad \Omega_{+,ij}(h) = -\frac{1}{2|h|^2} \epsilon_{ijk} h_k, \quad \tag{B2}$$

where $\epsilon_{ijk}$ is the Levi-Civita antisymmetric tensor. Note that $g_{-,ij} = g_{+,ij}$ and $\Omega_{-,ij} = -\Omega_{+,ij}$ for the state $|\psi_-\rangle$.

More generally, the QGT of interest is often related to the explicit dependency of the Hamiltonian vector $h(x)$ on some vector of external parameters $x$, with $d = \dim(x) \geq 2$. For example, in condensed matter physics, one will often have $x = k$, where $k$ represents crystal momentum. In that situation, the corresponding QGT $T_{\alpha,ij}(x)$ may be obtained from either $T_{\alpha,kl}(h)$ or $T_{\alpha,kl}(\theta_h, \phi_h)$ by a simple composition rule as

$$T_{\alpha,ij}(x) = \sum_{k,l} (\partial_i y_k)(\partial_j y_l) T_{\alpha,kl}(y), \tag{B3}$$

with $y = h$ or $y = (\theta_h, \phi_h)$.

Appendix C: Gell-Mann and spin matrices

Here we list the $N = 3$ ($N = 4$) Gell-Mann matrices and relate them to spin-1 (spin-3/2) matrices. Note that the generalization to $N \geq 5$ Gell-Mann matrices (not listed here explicitly) is straightforward, see for example Refs. [41, 42]: There are $N(N-1)/2$ symmetric matrices (purely real), $N(N-1)/2$ antisymmetric matrices (purely imaginary), and $N - 1$ diagonal matrices.
The $N = 3$ Gell-Mann matrices [72] are given by

\[
\begin{align*}
\lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \lambda_2 &= \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \lambda_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\
\lambda_5 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, & \lambda_6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, & \lambda_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, & \lambda_8 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.
\end{align*}
\]

They can easily be related to spin-1 matrices, which in the standard representation read

\[
S_x = \frac{1}{\sqrt{2}} (\lambda_1 + \lambda_6), \quad S_y = \frac{1}{\sqrt{2}} (\lambda_2 + \lambda_7), \quad S_z = \frac{1}{2} (\lambda_3 + \sqrt{3} \lambda_8). \tag{C1}
\]

The $N = 4$ Gell-Mann matrices [41, 42] are given by the extended SU(3) Gell-Mann matrices,

\[
\begin{align*}
\lambda_1 &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & \lambda_2 &= \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & \lambda_3 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & \lambda_4 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
\lambda_5 &= \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & \lambda_6 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & \lambda_7 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & \lambda_8 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
\end{align*}
\]

plus an additional seven matrices

\[
\begin{align*}
\lambda_9 &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, & \lambda_{10} &= \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, & \lambda_{11} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, & \lambda_{12} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \\
\lambda_{13} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, & \lambda_{14} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, & \lambda_{15} &= \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}.
\end{align*}
\]

This can easily be used to express the standard representation of spin-3/2 matrices:

\[
\begin{align*}
S_x &= \frac{1}{2} (\sqrt{3} \lambda_1 + 2 \lambda_6 + \sqrt{3} \lambda_{13}), \\
S_y &= \frac{1}{2} (\sqrt{3} \lambda_2 + 2 \lambda_7 + \sqrt{3} \lambda_{14}), \\
S_z &= \frac{1}{2} (\lambda_3 + \sqrt{3} \lambda_8 + \sqrt{6} \lambda_{15}). \tag{C2}
\end{align*}
\]
Appendix D: Derivation of Eq. (17)

To find the eigenprojector $P_\alpha$ as a polynomial in the Hamiltonian, our starting point is the following textbook formula, involving the set $\{E_\beta, \beta = 1, ..., N\}$ of all energy eigenvalues [68]:

$$P_\alpha = \prod_{\beta \neq \alpha} \frac{H - E_\beta 1_N}{E_\alpha - E_\beta}. \quad (D1)$$

Note that in the language of matrix theory, Eq. (D1) defines the Frobenius covariants of $H$. Note also that the denominator of Eq. (D1) corresponds to the derivative of the Hamiltonian’s characteristic polynomial. More generally, before proceeding it proves useful to compile some more details on the characteristic polynomial $p_N(z)$.

1. Characteristic polynomial

The characteristic polynomial of an $N \times N$ matrix $A$ is given by [73]

$$p_N(z) = \det(z1_N - A) = \sum_{k=0}^{N} c_k z^{N-k} = \prod_{\alpha=1}^{N} (z - a\alpha), \quad (D2)$$

where $a\alpha$ denotes an eigenvalue of $A$ and $\hat{c}_0 = 1$. According to the Faddeev-Le Verrier algorithm [73], the coefficients $\hat{c}_k$ may be computed from the traces $s_k \equiv \text{Tr} A^k$ of powers of $A$ as

$$\hat{c}_k = -\frac{1}{k} (s_k + \hat{c}_1 s_{k-1} + ... + \hat{c}_{k-1} s_1)$$

$$= \frac{(-1)^k}{k!} Y_k(s_1, ..., (-1)^{k-1}(k-1)!s_k). \quad (D3)$$

The second equality involves (exponential) complete Bell polynomials $Y_k(z_1, ..., z_k)$ [74, 75], the first few of which read explicitly

$$Y_0 = 1,$$

$$Y_1(z_1) = z_1,$$

$$Y_2(z_1, z_2) = z_1^2 + z_2,$$

$$Y_3(z_1, z_2, z_3) = z_1^3 + 3z_1 z_2 + z_3,$$

$$Y_4(z_1, z_2, z_3, z_4) = z_1^4 + 6z_1^2 z_2 + 4z_1 z_3 + 3z_2^2 + z_4. \quad (D4)$$

Focusing now on the case where $A = H$ represents an $N \times N$ Hamiltonian matrix, we have the Hamiltonian’s characteristic polynomial

$$p_N(z) = \det(z1_N - H) = \sum_{k=0}^{N} c_k z^{N-k} = \prod_{\alpha=1}^{N} (z - E\alpha). \quad (D5)$$

Using the traceless character of the Hamiltonian, cf. Eq. (16), and the Casimir invariants defined in Eq. (18), the coefficients are given by

$$c_k = \frac{(-1)^k}{k!} Y_k(0, -C_2, ..., (-1)^{k-1}(k-1)!C_k), \quad (D6)$$

and the first few of them read explicitly

$$c_0 = 1, \quad c_1 = 0,$$

$$c_2 = -\frac{C_2}{2}, \quad c_3 = -\frac{C_3}{3},$$

$$c_4 = \frac{C_2^2}{8} - \frac{C_4}{4}, \quad c_5 = \frac{C_2 C_3}{6} - \frac{C_5}{5},$$

$$c_6 = -\frac{C_2^3}{48} + \frac{C_2 C_4}{18} + \frac{C_2 C_3}{8} - \frac{C_6}{6}. \quad (D7)$$
2. Rewriting Eq. (D1)

The goal is now to eliminate all $E_{\beta \neq \alpha}$ from Eq. (D1), such that $P_\alpha$ becomes a proper polynomial in $H$ with coefficients that depend only on the single eigenvalue $E_\alpha$, i.e. $P_\alpha = P_\alpha(E_\alpha, H)$. Consider first the numerator of Eq. (D1) and note that by explicit multiplication one may write

\[
\prod_{\beta \neq \alpha} (H - E_\beta 1_N) = \sum_{n=0}^{N-1} (-1)^n e_n(E_1, ..., E_{\alpha-1}, E_{\alpha+1}, ..., E_N)H^{N-1-n},
\]

(D8)

where $e_n = e_n(E_1, ..., E_{\alpha-1}, E_{\alpha+1}, ..., E_N)$ are known as \textit{elementary symmetric polynomials} [76]. One has $e_0 = 1$ and all higher $e_n$ are determined recursively by Newton's identities:

\[
e_n = \frac{1}{n} \sum_{k=1}^{n} (-1)^{k-1} (C_k - E_\alpha^k) e_{n-k},
\]

where the $C_k$ are the Casimir invariants of Eq. (18) and it was exploited that $\sum_{\beta \neq \alpha} E_\beta^k = C_k - E_\alpha^k$. This may further be rewritten as

\[
e_n = (-1)^n \sum_{k=0}^{n} c_k E_\alpha^{n-k},
\]

(D9)

where $c_k$ are the coefficients (D6) of the characteristic polynomial. If we now define polynomials

\[
q_n(z) \equiv \sum_{k=0}^{n} c_k z^{n-k},
\]

(D10)

it is clear that $q_N(z) = p_N(z)$ is the characteristic polynomial (D5), and $q_n(E_\alpha) = (-1)^n e_n$. Moreover, inserting into Eq. (D8), we have

\[
\prod_{\beta \neq \alpha} (H - E_\beta 1_N) = \sum_{n=0}^{N-1} q_{N-1-n}(E_\alpha)H^n.
\]

(D12)

Similarly, for the numerator of Eq. (D1), exactly the same procedure as above (where $H$ gets replaced by $E_\alpha$) leads to

\[
\prod_{\beta \neq \alpha} (E_\alpha - E_\beta) = \sum_{n=0}^{N-1} q_{N-1-n}(E_\alpha)E_\alpha^n.
\]

(D13)

As mentioned in the main text, $\prod_{\beta \neq \alpha} (E_\alpha - E_\beta)$ is equal to the derivative $p_N'(E_\alpha)$ of the characteristic polynomial. From this one may also show that $\prod_{\beta \neq \alpha} (E_\alpha - E_\beta) = \sum_{n=0}^{N-1} q_{N-1-n}(E_\alpha)C_n$. Combining all of these results, one arrives at Eq. (17).

Appendix E: Eigenstates from eigenprojectors

The eigenprojector $P_\alpha$ permits to construct the eigenstate $|\psi_\alpha\rangle$ from an arbitrary gauge freedom state $|\psi_g\rangle$. For example, in the two-band case, using the eigenprojector $P_+ \equiv \frac{1}{\sqrt{\langle \psi_g | P_+ | \psi_g \rangle}} P_+ |\psi_g\rangle$, Eq. (19), a state equivalent to Eq. (A1) is easily constructed as

\[
|\psi_+\rangle = \frac{1}{\sqrt{\langle \psi_g | P_+ | \psi_g \rangle}} P_+ |\psi_g\rangle,
\]

(E1)

where the gauge freedom angles $(\theta_\alpha, \phi_\alpha)$ can be chosen at will at any point in parameter space $x$. For example, if $\cos \theta_\alpha = 1$ and $\sin \theta_\alpha = 0$, one exactly recovers the second expression in Eq. (A1). More generally, the $N$-band eigenstate $|\psi_\alpha\rangle$ in an arbitrary gauge may be obtained as

\[
|\psi_\alpha\rangle = \frac{1}{\sqrt{\langle \psi_g | P_\alpha(E_\alpha, H) | \psi_g \rangle}} P_\alpha(E_\alpha, H) |\psi_g\rangle,
\]

(E2)
where the gauge freedom state $|\psi_g\rangle$ can be chosen arbitrarily. Each eigenstate (E2) is minimally encoded by a global phase $\Gamma_\alpha$ and $N-1$ pairs of eigenstate’s angles $(\theta_\alpha^i, \phi_\alpha^i)$ ($i = 1, ..., N-1$).

Two more remarks can be made. First, although this is not immediately obvious, the eigenstate’s angles stay unchanged upon varying $|\psi_g\rangle$ for fixed $x$. In contrast, the global phase is changing, implying $\Gamma_\alpha \equiv \Gamma_\alpha(x, |\psi_g\rangle)$. As a consequence, the possible singular behaviors of the wavefunction in parameter space are gauge-dependent. Second, the state $|\psi_g\rangle$ must not be orthogonal to the projector $P_\alpha(x)$, i.e. one requires $P_\alpha(x) |\psi_g\rangle \neq 0$. This constraint implies that it might be necessary to change the state $|\psi_g\rangle$ when the parameter $x$ is varying because it is never guaranteed that a single $|\psi_g\rangle$ (meaning a fixed gauge) is sufficient to describe a given eigenstate $|\psi_\alpha\rangle$ over the entire parameter space $x$.

### Appendix F: Derivation of Bloch vector formula $b_\alpha(E_\alpha, H)$

Here we detail the steps that allow to find $b_\alpha(h, E_\alpha)$ from the eigenprojector formula (17). The only missing ingredient is $H^n = (h \cdot \lambda)^n$ in terms of the generators of SU($N$). Defining vectors $\eta_n \equiv \text{Tr}(H^n \lambda)/2$, we can write:

$$H^n = (h \cdot \lambda)^n = \frac{C_n}{N} 1_N + \eta_n \cdot \lambda, \quad (F1)$$

where obviously $\eta_0 = 0$ and $\eta_1 = h$. Inserting into Eq. (17) and using Eq. (22), we obtain the intermediate result:

$$b_\alpha = 2 \sum_{n=0}^{N-1} \eta_{n-1}(E_\alpha) \eta_n \frac{N}{\sum_{n=0}^{N-1} q_{n-1}(E_\alpha) E_n}{\sum_{n=0}^{N-1}} \quad (F2)$$

At this point it remains the task to find the explicit form of the vectors $\eta_n(h)$ for $n > 1$. To accomplish this task, we require a product identity generalizing the familiar SU(2) identity (15) to any $N$. This can be readily achieved by using Eq. (25) to obtain

$$\lambda_\alpha \lambda_b = \frac{2}{N} \delta_{ab} 1_N + (d_{abc} + if_{abc}) \lambda_c. \quad (F3)$$

Since all generators are traceless, $\text{Tr} \lambda_c = 0$, the convenient trace orthogonality $\text{Tr}(\lambda_\alpha \lambda_b) = 2 \delta_{ab}$ holds. From Eqs. (F3) & (27), one directly obtains the desired product identity in vector form:

$$(m \cdot \lambda)(n \cdot \lambda) = \frac{2}{N} m \cdot n 1_N + (m \times n + i m \times n) \cdot \lambda. \quad (F4)$$

When computing $H^n = (h \cdot \lambda)^n$ by applying Eq. (F4) repeatedly, it proves useful to introduce the notation (28) for repeated star products of a vector with itself. The resulting vectors have the following properties (with $n_1, n_2 \in \mathbb{N}_0$):

$$m_\star^{(n_1)} \cdot m_\star^{(n_2)} = \left| m_\star^{(\frac{n_1+n_2}{2})} \right|^2 \quad \text{if } n_1 + n_2 \text{ even},$$

$$m_\star^{(n_1)} \times m_\star^{(n_2)} = 0. \quad (F5)$$

The former identity follows directly from the total symmetry of the structure constants (25), and the latter is a consequence of the second Jacobi identity, as can be proved by mathematical induction. More generally, from the generic properties of the algebra, we have established useful generalized vector and scalar Jacobi identities, listed in Appendix G.

With all these prerequisites we may now calculate $(h \cdot \lambda)^n$ i.e. determine $C_n(h)$ and $\eta(h)$. In particular, we obtain the following simple recursion relations:

$$C_{n+1} = 2 h \cdot \eta_n, \quad \eta_{n+1} = h \star \eta_n + \frac{C_n}{N} h, \quad (F6)$$

with initial conditions $C_0 = N$ and $\eta_0 = 0$. Making use of the fact that $\eta_n$ and the structure constants $f_{abc}$ are real by definition, we can also establish the identity

$$\eta_{n_1} \times \eta_{n_2} = 0, \quad \forall n_1, n_2 \in \mathbb{N}_0. \quad (F7)$$
Applying the recursion (F6), for a traceless Hamiltonian matrix, we obtain successively up to \( n = 4 \) the important identities:

\[
\begin{align*}
C_1 &= 0, & \eta_1 &= h, \\
C_2 &= 2|h|^2, & \eta_2 &= h, \\
C_3 &= 2h \cdot h, & \eta_3 &= \frac{C_2}{N} h + h, \\
C_4 &= \frac{4|h|^4}{N} + 2|h|^2, & \eta_4 &= \frac{C_3}{N} h + \frac{C_2}{N} h + h.
\end{align*}
\] (F8)

More generally, for a generic \( N > 1 \), the form of the vector \( \eta_n(h) \) is compactly written as:

\[
\eta_n = \frac{1}{N} \sum_{p=0}^{n-1} C_p h^{(n-1-p)}.
\] (F9)

The final step required for completing our task of finding the explicit expressions of the Bloch vector \( b_\alpha(h, E_\alpha) \) consists in substituting Eqs. (F8) & (F9) into Eq. (F2). One can then write down the explicit Bloch vectors for any \( N \), as done explicitly in Eq. (23) for \( N = 2 \) to \( N = 5 \).

As a final remark, be aware that, for given \( N \), only \( C_{n \leq N} \) and \( \eta_{n < N} \) are relevant, and there are only \( N - 1 \) independent vectors \( h^{(k)} \), \( k = 0, ..., N - 2 \). For example, for \( N = 3 \), all information we need is encoded in \( C_2, C_3, h \) and \( h_* \). This again follows from the Cayley-Hamilton theorem, which states that \( q_{n=N}(H) = p_N(H) = 0 \). From this property it is easy to establish the following useful identities:

\[
\begin{align*}
N = 3 : & \quad h_* = \frac{C_2}{6} h, \quad h \cdot h_* = \frac{C_2}{3} h - \frac{C_2}{6} h, \\
N = 4 : & \quad h_* = \frac{C_3}{12} h + \frac{C_2}{4} h, \quad h \cdot h_* = \frac{C_3}{3} h, \quad h \cdot h_* = \frac{|h|^2}{2} + \frac{C_3}{12} h.
\end{align*}
\] (F10)

Appendix G: SU(N) Jacobi identities

The first Jacobi identity \([47, 77]\) can be written alternatively for the generator matrices, the antisymmetric structure constants and the SU(N) vectors as

\[
\begin{align*}
[[\lambda_i, \lambda_j], \lambda_k] + [[\lambda_j, \lambda_k], \lambda_i] + [[\lambda_k, \lambda_i], \lambda_j] &= 0, \\
\alpha_{ijk}f_{klm} + \alpha_{ikm}f_{jlm} + \alpha_{ilm}f_{jkm} &= 0, \\
(m \times (n \times o)) + n \times (o \times m) + o \times (m \times n) &= 0, \\
(m \times n) \cdot (o \times p) + (m \times o) \cdot (p \times n) + (m \times p) \cdot (n \times o) &= 0,
\end{align*}
\] (G1)

where the third and fourth lines are obtained from the second line depending on whether or not one keeps a free index. Similarly, the second Jacobi identity is given by

\[
\begin{align*}
\{[\lambda_i, \lambda_j], \lambda_k\} + \{[\lambda_j, \lambda_k], \lambda_i\} + \{[\lambda_k, \lambda_i], \lambda_j\} &= 0, \\
\beta_{ijk}d_{klm} + \beta_{ikm}d_{jlm} + \beta_{ilm}d_{jkm} &= 0, \\
(m \times (n \times o)) + n \times (o \times m) + o \times (m \times n) &= 0, \\
(m \times n) \cdot (o \times p) + (m \times o) \cdot (p \times n) + (m \times p) \cdot (n \times o) &= 0.
\end{align*}
\] (G2)

Furthermore, there is the identity

\[
\begin{align*}
[\lambda_i, [\lambda_j, \lambda_k]] &= \{\lambda_k, \{\lambda_i, \lambda_j\}\} - \{\lambda_j, \{\lambda_i, \lambda_k\}\}, \\
f_{ijk}f_{klm} &= \frac{2}{N} (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}) + d_{ikm}d_{jlm} - d_{ilm}d_{jkm}, \\
m \times (n \times o) &= \frac{2}{N} [(m \cdot o)n - (m \cdot n)o] + (m \cdot o) \times n - (m \times n) \times o, \\
(m \times n) \cdot (o \times p) &= \frac{2}{N} [(m \cdot o)(n \cdot p) - (m \cdot p)(n \cdot o)] + (m \cdot o) \cdot (n \cdot p) - (m \cdot p) \cdot (n \cdot o).
\end{align*}
\] (G3)
contains Σ and it comprises the vectors of maximal N eigenprojector. An expansion in the basis of SU(N) is a proper subset of (i) the boundary of the space of mixed states and (ii) the sphere illustrated in Fig. 4; but for higher N = 2 case, these complications are hidden, since all of the relevant sets coincide.

Appendix H: The generalized Bloch sphere

We give here a short summary of the concept of a generalized Bloch sphere [i.e. the SU(N) eigenprojector’s Bloch sphere B_p^{(N)} introduced in Section III C], drawing largely on Refs. [48–55].

For an N-component (mixed state) density matrix ρ_α representing an N-density (mixed) quantum state |ψ_α⟩, with α = 1, ..., N, are hermiticity ρ_α^† = ρ_α, probability conservation Tr ρ_α = 1 and positive semidefiniteness, ρ_α ≥ 0. Pure states have, in addition, ρ_α^2 = ρ_α, in which case ρ_α = P_α is an eigenprojector. An expansion in the basis of SU(N) generator matrices analogous to Eq. (22) can be made for a general (mixed state) density matrix:

ρ_α = \frac{1}{N} 1_N + \frac{1}{2} b_α \cdot \lambda, \quad (H1)

where now |b_α⟩ can take various values depending on the pureness of the state. In the pure state case, we have |b_α⟩ = (2(N − 1))_N. Considering the vector space ℝ^{N^2−1}, and denoting the (N^2−1)-dimensional subspace accessible to the (mixed state) Bloch vector b_α as Σ_{ρ_α}^{(N)}, one needs to distinguish between two kinds of boundaries, namely its (N^2−2)-dimensional topological boundary ∂Σ_{ρ_α}^{(N)} and its extremal boundary B_p^{(N)}. The latter is of dimension 2(N − 1), and it comprises the vectors of maximal |b_α⟩, i.e. it is the space of pure state Bloch vectors, or in other words the generalized Bloch sphere, sketched in Fig. 4.

Another set of immediate interest in this discussion is the sphere

\mathcal{W}^{N^2−2} \equiv \left\{ r \in ℝ^{N^2−1} \mid |r| = \sqrt{\frac{2(N−1)}{N}} \right\}, \quad (H2)

that contains all pure state Bloch vectors; in other words, it is the surface of the ball ℝ^{N^2−1}, the smallest ball that contains Σ_{ρ_α}^{(N)}. For N = 2, one trivially has ∂Σ_{ρ_α}^{(2)} = B_p^{(2)} = ℝ^2 (and \mathcal{W}^2 = S^2, with the unit two-sphere S^2), as illustrated in Fig. 4; but for higher N the three spaces are different. One has B_p^{(N)} ∩ ∂Σ_{ρ_α}^{(N)} and B_p^{(N)} ∩ \mathcal{W}^{N^2−2}, i.e. the generalized Bloch sphere is a proper subset of (i) the boundary of the space of mixed states and (ii) the sphere \mathcal{W}^{N^2−2}; in fact, it is their intersection: ∂Σ_{ρ_α}^{(N)} ∩ \mathcal{W}^{N^2−2} = B_p^{(N)}. In other words, if b_α corresponds to a pure state, it lies on \mathcal{W}^{N^2−2}, but not necessarily the other way around, i.e. not all points on \mathcal{W}^{N^2−2} represent physically valid pure states.

Analogously, the interior of \mathcal{W}^{N^2−2}, i.e. the ball ℝ^{N^2−1}, is not composed of only physically valid mixed states. In fact, calculations [51] of the volume of Σ_{ρ_α}^{(N)} show that vol(Σ_{ρ_α}^{(3)}/vol(ℝ^8) ≈ 0.26 and vol(Σ_{ρ_α}^{(4)}/vol(ℝ^15)) ≈ 0.12, meaning that most points in ℝ^{N^2−1} do actually not represent physically valid states as soon as N > 2.
At the origin of all of these complications is the constraint of positive definiteness, \( \rho_\alpha \geq 0 \): for \( N = 1 \), this condition is trivially fulfilled. For \( N = 2 \), it is equivalent to \( \text{Tr}(\rho_\alpha^2) \leq 1 \), so for a given density matrix there is one additional constraint as compared to \( N = 1 \). In the same way, whenever \( N \) increases by one, one additional constraint involving traces \( D_{\alpha,n} \equiv \text{Tr}(\rho_\alpha^n) \) needs to be fulfilled in order to have positive definiteness. For example, for \( N = 3 \), there are the three constraints \([50]\) \( D_{\alpha,1} = 1, D_{\alpha,2} \leq 1, \) and \(-2D_{\alpha,3} + 3D_{\alpha,2} \leq 1 \), which can be translated to constraints on the Bloch vectors via Eq. (H1). This restrict the elements of \( \mathbb{R}^8 \) that represent physically valid states. As a consequence, the relevant sets (including the generalized Bloch sphere) acquire a very nontrivial shape already for \( N = 3 \) [55]:

\[
\mathcal{B}^{(3)}_{\rho_\alpha} = \left\{ \mathbf{b}_\alpha \in \mathbb{R}^8 \mid |\mathbf{b}_\alpha|^2 = \frac{4}{3}, |\mathbf{b}_\alpha|^2 - \mathbf{b}_\alpha \cdot (\mathbf{b}_\alpha \times \mathbf{b}_\alpha) = \frac{4}{9} \right\},
\]

\[
\partial \Sigma^{(3)}_{\rho_\alpha} = \left\{ \mathbf{b}_\alpha \in \mathbb{R}^8 \mid |\mathbf{b}_\alpha|^2 \leq \frac{4}{3}, |\mathbf{b}_\alpha|^2 - \mathbf{b}_\alpha \cdot (\mathbf{b}_\alpha \times \mathbf{b}_\alpha) = \frac{4}{9} \right\},
\]

\[
\Sigma^{(3)}_{\rho_\alpha} = \left\{ \mathbf{b}_\alpha \in \mathbb{R}^8 \mid |\mathbf{b}_\alpha|^2 \leq \frac{4}{3}, |\mathbf{b}_\alpha|^2 - \mathbf{b}_\alpha \cdot (\mathbf{b}_\alpha \times \mathbf{b}_\alpha) \leq \frac{4}{9} \right\},
\]

where the star product is defined in Eq. (27), and the agreement of the first line with Eq. (30) is to be noted. Similarly, more complicated constraints can be obtained for higher \( N \), see Ref. [50].

Finally, note that there exists another sphere

\[
\mathcal{V}^{N^2-2} \equiv \left\{ \mathbf{r} \in \mathbb{R}^{N^2-1} \mid |\mathbf{r}| = \sqrt{\frac{2}{N(N-1)}} \right\}
\]

inscribed inside \( \Sigma^{(N)}_{\rho_\alpha} \), i.e., \( \partial \Sigma^{(N)}_{\rho_\alpha} \) lies between \( \mathcal{V}^{N^2-2} \) and \( \mathcal{W}^{N^2-2} \).

The essence of all of these established facts is visualized schematically in Fig. 4. Note that the figure is only a rough sketch supposed to highlight the different sets involved. A visual insight into the true (very complicated) shape of those sets can be gained by considering two- or three-sections, see for instance Refs. [49, 53–55].

**Appendix I: From eigenstate to eigenprojector picture of the QGT**

The QGT is a tensor quantifying the overlap of two states that are "infinitesimally close" in parameter space [4]:

\[
\langle \psi_\alpha(x) | \psi_\alpha(x + dx) \rangle = \mathcal{F}_\alpha(x) e^{i\phi_\alpha(x)},
\]

where the modulus \( \mathcal{F}_\alpha(x) \) has come to be known as fidelity [78]. The fidelity is related to the quantum metric, while the phase \( \phi_\alpha(x) \) gives rise to the Berry curvature. This can be seen by expanding

\[
\langle \psi_\alpha(x) | \psi_\alpha(x + dx) \rangle = 1 + \sum_i \langle \psi_\alpha(x) | \partial_i \psi_\alpha(x) \rangle dx_i + \frac{1}{2} \sum_{ij} \langle \psi_\alpha(x) | \partial_{ij} \psi_\alpha(x) \rangle dx_i dx_j + O(|dx|^3).
\]

Noting that \( \text{Re} \langle \psi_\alpha(x) | \partial_i \psi_\alpha(x) \rangle = -\text{Re} \langle \partial_i \psi_\alpha(x) | \partial_i \psi_\alpha(x) \rangle \) and introducing the Berry connection \( A_{\alpha,i}(x) \equiv i \langle \psi_\alpha(x) | \partial_i \psi_\alpha(x) \rangle \) for \( i \langle \psi_\alpha(x) | \partial_i \psi_\alpha(x) \rangle = -\text{Im} \langle \psi_\alpha(x) | \partial_i \psi_\alpha(x) \rangle \), one immediately finds

\[
\mathcal{F}_\alpha^2(x) = [\text{Re} \langle \psi_\alpha(x) | \psi_\alpha(x + dx) \rangle]^2 + [\text{Im} \langle \psi_\alpha(x) | \psi_\alpha(x + dx) \rangle]^2
\]

\[
= 1 - \sum_{ij} [\text{Re} \langle \partial_i \psi_\alpha(x) | \partial_j \psi_\alpha(x) \rangle] - A_{\alpha,i}(x) A_{\alpha,j}(x) dx_i dx_j + O(|dx|^3).
\]

Now, the quantum metric tensor \( g_{\alpha,ij}(x) \) is defined [4, 28] by \( \sum_{ij} g_{\alpha,ij} dx_i dx_j \equiv 1 - \mathcal{F}_\alpha^2 \), such that

\[
g_{\alpha,ij}(x) = \text{Re} \langle \partial_i \psi_\alpha(x) | \partial_j \psi_\alpha(x) \rangle + \langle \psi_\alpha(x) | \partial_i \psi_\alpha(x) \rangle \langle \psi_\alpha(x) | \partial_j \psi_\alpha(x) \rangle.
\]

Upon calculating the Berry curvature \( \Omega_{\alpha,ij}(x) \equiv \partial_i A_{\alpha,j} - \partial_j A_{\alpha,i} \), one can observe that \( g_{\alpha,ij} \) and \( \Omega_{\alpha,ij} \) are parts of the single complex tensor (2).

An eigenstate-based version of the QGT that is used more frequently than Eq. (2) is given by Eq. (33). From the latter, Eq. (2) is easily recovered upon using the identity \( \langle \psi_\alpha | \partial_i | \psi_\beta \rangle = \langle \partial_i \psi_\alpha | \psi_\beta \rangle (E_\alpha - E_\beta) \), valid for \( \alpha \neq \beta \). To obtain the projector-based formula (3), Eq. (33) may be rewritten in an explicitly gauge-invariant form:

\[
T_{\alpha,ij} = \sum_{\beta \neq \alpha} \frac{\text{Tr} \{ P_\alpha (\partial_i H) P_\beta (\partial_j H) \}}{(E_\alpha - E_\beta)^2},
\]

where we used the identity \( \langle \psi_\alpha | O | \psi_\alpha \rangle = \text{Tr} \{ P_\alpha O P_\alpha \} = \text{Tr} \{ P_\alpha O \} \). From there, by inserting \( H = \sum_\gamma E_\gamma P_\gamma \), it is straightforward to arrive at Eq. (3).
Appendix J: Alternative QGT formula in terms of Bloch vectors

A formula equivalent to Eq. (35) but without derivatives acting on Bloch vectors can be obtained by inserting Eq. (22) into Eq. (14) and exploiting the Jacobi identity (G3):

\[ g_{\alpha,ij} = \sum_{\beta \neq \alpha} \frac{S^{ij}_{\alpha\beta}}{(E_{\alpha} - E_{\beta})^2}, \quad \Omega_{\alpha,ij} = -\sum_{\beta \neq \alpha} \frac{A^{ij}_{\alpha\beta}}{(E_{\alpha} - E_{\beta})^2}. \]  

Here, we have defined

\[ S^{ij}_{\alpha\beta} = \frac{4}{N^2} \mathbf{h}^i \cdot \mathbf{h}^j + \frac{1}{N} \left[ (\mathbf{b}_\alpha \cdot \mathbf{h}^i)(\mathbf{b}_\beta \cdot \mathbf{h}^j) + (\mathbf{b}_\alpha \cdot \mathbf{h}^j)(\mathbf{b}_\beta \cdot \mathbf{h}^i) \right] \]
\[ + \frac{2}{N} (\mathbf{b}_\alpha + \mathbf{b}_\beta) \cdot (\mathbf{h}^i \times \mathbf{h}^j) + \frac{1}{2} \left[ (\mathbf{b}_\alpha \times \mathbf{h}^i) \cdot (\mathbf{b}_\beta \times \mathbf{h}^j) + (\mathbf{b}_\alpha \times \mathbf{h}^j) \cdot (\mathbf{b}_\beta \times \mathbf{h}^i) \right], \]
\[ A^{ij}_{\alpha\beta} = \frac{2}{N} (\mathbf{b}_\alpha - \mathbf{b}_\beta) \cdot (\mathbf{h}^i \times \mathbf{h}^j) + (\mathbf{b}_\alpha \times \mathbf{h}^j) \cdot (\mathbf{b}_\beta \times \mathbf{h}^i) + (\mathbf{b}_\alpha \times \mathbf{h}^i) \cdot (\mathbf{b}_\beta \times \mathbf{h}^j). \]

It is straightforward to check that \( S^{ij}_{\alpha\beta} \) is symmetric under the exchange of indices \( i,j \) (or of \( \alpha, \beta \)), whereas \( A^{ij}_{\alpha\beta} \) is antisymmetric under such exchange, and as a consequence \( \sum_{\alpha} \Omega_{\alpha,ij} = 0 \). While apparently much less compact than Eq. (35), the expressions (J1) appear more convenient for numerical implementation. Conceptually, they mainly distinguish themselves from Eq. (35) in that, on the one hand, they involve solely the parametric derivatives of the Hamiltonian vector, while on the other hand they also illustrate the interband nature of the two geometric tensors, and lastly they also show explicitly the importance of the star product for both the \( N \)-band quantum metric and Berry curvature.

Appendix K: Berry curvature for \( N \)-band systems

The Berry curvature formula for arbitrary SU(\( N \)) systems follows from combining Eqs. (F2), (F9) & (35), together with the total antisymmetry of the triple product \( \mathbf{m} \cdot (\mathbf{n} \times \mathbf{o}) = f_{abc} \alpha_{o} \beta_{m} \) and the orthogonality relations (F5) or (F7):

\[ \Omega_{\alpha,ij} = -\frac{2 \mathbf{b}_{\alpha}}{[g_{N}'(E_{\alpha})]^2} \cdot \left( \sum_{n,m=1}^{N-1} q_{N-1-n}(E_{\alpha})q_{N-1-m}(E_{\alpha}) \tau_{(i)}^{(n)} \times \tau_{(j)}^{(m)} \right). \]  

(K1)

Here we use the polynomials \( q_{n}(z) \) and have defined vectors

\[ \tau_{(n)}^{(i)} = \frac{1}{N} \sum_{p=0}^{n-1} C_{p} \partial_{1} \mathbf{h}_{(n-1-p)}. \]  

(K2)

With the notation \( \mathbf{m}^{i} \equiv \partial_{1} \mathbf{m} \), the first few \( \tau_{(n)}^{(i)} \) read

\[ \tau_{(1)}^{(i)} = \mathbf{h}^{i}, \quad \tau_{(2)}^{(i)} = \mathbf{h}^{i}, \quad \tau_{(3)}^{(i)} = \frac{C_{2}}{N} \mathbf{h}^{i} + \mathbf{h}^{i}. \]  

(K3)

From Eq. (K1), one can obtain the Berry curvature for any given \( N \).

For \( N = 2 \), Eq. (14) is immediately obtained. For \( N = 3 \), analogously, we have

\[ \Omega_{\alpha,ij} = -\frac{4(E_{\alpha} \mathbf{h} + \mathbf{h}_{*})}{(3E_{\alpha}^{2} - C_{2})^{3}} \cdot \left[ E_{\alpha}^{2} \mathbf{h}^{i} \times \mathbf{h}^{j} + E_{\alpha} (\mathbf{h}^{i} \times \mathbf{h}^{j} + \mathbf{h}^{i} \times \mathbf{h}^{j}) + \mathbf{h}^{i} \times \mathbf{h}^{j} \right], \]  

(K4)

or equivalently Eq. (36) in the main text. This result can be simplified due to the following identities:

\[ \mathbf{h} \cdot (\mathbf{h}^{i} \times \mathbf{h}^{j}) = \mathbf{h} \cdot (\mathbf{h}^{i} \times \mathbf{h}^{j}) = \mathbf{h} \cdot (\mathbf{h}^{i} \times \mathbf{h}^{j}), \]
\[ \mathbf{h} \cdot (\mathbf{h}^{i} \times \mathbf{h}^{j}) = \mathbf{h} \cdot (\mathbf{h}^{i} \times \mathbf{h}^{j}) = \mathbf{h} \cdot (\mathbf{h}^{i} \times \mathbf{h}^{j}), \]
\[ \mathbf{h} \cdot (\mathbf{h}^{i} \times \mathbf{h}^{j}) = \left[ \frac{2}{3} (\mathbf{h} \cdot \mathbf{h}_{*}) \mathbf{h} + |\mathbf{h}|^{2} \mathbf{h}_{*} \right] \cdot (\mathbf{h}^{i} \times \mathbf{h}^{j}). \]  

(K5)
The first two lines are valid for general $N$ and can be proved using the second Jacobi identity (G2). The proof of the third line requires the Jacobi identity as well, but additionally the SU(3)-specific identities (F10). Inserting all of these identities into Eq. (K4), and exploiting the characteristic equation

$$E_{\alpha} = \frac{C_{\alpha}}{2} E_\alpha + \frac{C_{\alpha}}{3},$$

one obtains the generic SU(3) Berry curvature formula

$$\Omega_{\alpha,ij} = -\frac{4}{(3E_{\alpha}^2 - |h|^2)^3} \left\{ E_\alpha \left[ |h|^2 \mathbf{h} \cdot (\mathbf{h} \times \mathbf{h}) + 3 \mathbf{h} \cdot (\mathbf{h}_i \times \mathbf{h}_j) \right] + (3E_{\alpha}^2 + |h|^2) \mathbf{h}_\alpha \cdot (\mathbf{h}_i \times \mathbf{h}_j) \right\}. \quad (K6)$$

The advantage of this formula, as compared to Eq. (36), consists in the fact that it contains only three terms, and that the Berry curvature sum rule $\sum_{\alpha} \Omega_{\alpha,ij} = 0$ is more evident, since $\sum_{\alpha} E_{\alpha}(3E_{\alpha}^2 - |h|^2)^{-3} = 0$ and $\sum_{\alpha}(3E_{\alpha}^2 + |h|^2)(3E_{\alpha}^2 - |h|^2)^{-3} = 0$.

In principle, one can continue in this way to obtain the Berry curvature for any $N$. As a shortcut, it is however useful to realize that, when the Bloch vectors (23) are substituted into Eq. (35), the derivatives effectively do not act on the prefactors but only on the vectors $\mathbf{h}$, $\mathbf{h}_\alpha$, etc. This is due to the orthogonality relations (F5) & (F7). For example, for the $N = 2$ case, it suffices to replace $b_\alpha^i \to \frac{1}{E_{\alpha}} \mathbf{h}_i^\alpha$ in Eq. (35), which allows to directly read off the Berry curvature: $\Omega_{\alpha,ij} = -\frac{1}{2} \frac{h}{E_{\alpha}} \cdot (\mathbf{h}_i^\alpha \times \mathbf{h}_j^\alpha)$. Similarly, for $N = 3$, it suffices to replace $b_\alpha^i \to \frac{2}{3E_{\alpha}^2 - C_{2/2}} (E_{\alpha} \mathbf{h}_i^\alpha + \mathbf{h}_i^\alpha)$, and so on for higher $N$. In this way, one directly obtains Eqs. (36)–(38) in the main text.

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