Some Properties of Balanced Hyperbolic Compact Complex Manifolds

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Abstract. We prove several vanishing theorems for the cohomology of balanced hyperbolic manifolds that we introduced in our previous work and for the $L^2$ harmonic spaces on the universal cover of these manifolds. Other results include a Hard Lefschetz-type theorem for certain compact complex balanced manifolds and the non-existence of certain $L^1$ currents on the universal covering space of a balanced hyperbolic manifold.

1 Introduction

In this paper, we continue the study of compact complex balanced hyperbolic manifolds that we introduced very recently in [MP21] as generalisations in the possibly non-projective and even non-Kähler context of the classical notion of Kähler hyperbolic (in the sense of Gromov) manifolds. Recall that every Kähler hyperbolic manifold is also Kobayashi/Brody hyperbolic.

On the other hand, recall that a Hermitian metric on a complex manifold $X$ identifies with a $C^\infty$ positive definite $(1, 1)$-form $\omega$ on $X$. If we put $\dim_{\mathbb{C}} X = n \geq 2$, a Hermitian metric $\omega$ is said to be balanced (see [Gau77] where these metrics were introduced under the name of semi-Kähler and [Mic83] where they were given this name) if $d\omega^{n-1} = 0$. Moreover, $\omega$ is said to be degenerate balanced (see [Pop15] for the name) if $\omega^{n-1}$ is $d$-exact. Unlike in the Kähler setting, where no $d$-exact Hermitian metric $\omega$ can exist on a compact complex manifold, compact complex manifolds carrying degenerate balanced metrics do exist. These manifolds include:

(i) the connected sums $X = \sharp_k (S^3 \times S^3)$, with $k \geq 2$, endowed with the Friedman-Lu-Tian complex structure constructed via conifold transitions ([Fri89], [LT93], [FLY12]);

(ii) the quotients $X = G/\Gamma$ of any semi-simple complex Lie group $G$ by a lattice $\Gamma \subset G$ ([Yac98]).

In [MP21], we generalised the notion of degenerate balanced compact complex manifolds starting from the observation that this is a kind of hyperbolicity property.

Throughout the text, $\pi_X : \tilde{X} \to X$ will stand for the universal cover of $X$. If $\omega$ is a Hermitian metric on $X$, we let $\tilde{\omega} = \pi_X^* \omega$ be the Hermitian metric on $\tilde{X}$ that is the lift of $\omega$. According to [Gro91], a $C^\infty$ $k$-form $\alpha$ on $X$ is said to be $\tilde{d}$-bounded with respect to $\omega$ if $\pi_X^* \alpha = d\beta$ on $\tilde{X}$ for some $C^\infty$ $(k-1)$-form $\beta$ on $\tilde{X}$ that is bounded w.r.t. $\tilde{\omega}$.

Now, recall that a compact complex manifold $X$ is said to be Kähler hyperbolic in the sense of Gromov (see [Gro91]) if there exists a Kähler metric $\omega$ on $X$ (i.e. a Hermitian metric $\omega$ with $d\omega = 0$) such that $\omega^{n-1}$ is $d$-bounded with respect to itself. In [MP21, Definition 2.1], we introduced the following 1-codimensional analogue of this:

An $n$-dimensional compact complex manifold $X$ is said to be balanced hyperbolic if there exists a balanced metric $\omega$ on $X$ such that $\omega^{n-1}$ is $\tilde{d}$-bounded with respect to $\omega$. 

\[ \text{arXiv:2107.09522v2 [math.CV] 14 Feb 2022} \]
Any such metric $\omega$ is called a balanced hyperbolic metric.

The implications among these notions are summed up in the following diagram. (See [MP21] for relations with other hyperbolicity notions.)

\[
\text{X is Kähler hyperbolic } \implies \text{X is balanced hyperbolic } \implies \text{X is degenerate balanced}
\]

We now outline the specific properties of these classes of manifolds that we prove in this paper.

(I) Case of balanced and degenerate balanced manifolds

In the first part of the paper, we obtain some general results on compact complex manifolds carrying balanced metrics (and, in some cases, results on Gauduchon metrics) and then we use them to infer vanishing results for degenerate balanced manifolds. See §2.1 for a reminder of the terminology used in what follows.

(a) Our first main result, obtained as a consequence of the computation in Lemma 2.1, is a Hard Lefschetz Isomorphism between the De Rham cohomologies of degrees $1$ and $2n-1$ that holds on any compact complex balanced manifold satisfying a mild $\partial \bar{\partial}$-type condition.

For any Hermitian metric $\omega$ on an $n$-dimensional complex manifold, we will use throughout the paper the notation $\omega^p := \omega^{p!}$ for any integer $p$ between 2 and $n$.

Theorem 1.1. Let $X$ be a compact complex manifold with $\text{dim}_\mathbb{C} X = n$.

(i) If $\omega$ is a balanced metric on $X$, the linear map:

\[
\{\omega_{n-1}\}_{DR} \wedge \cdot : H^1_{DR}(X, \mathbb{C}) \rightarrow H^{2n-1}_{DR}(X, \mathbb{C}), \quad \{u\}_{DR} \mapsto \{\omega_{n-1} \wedge u\}_{DR},
\]

is well defined and depends only on the cohomology class $\{\omega_{n-1}\}_{DR} \in H^{2n-2}_{DR}(X, \mathbb{C})$.

(ii) If, moreover, $X$ has the following additional property: for every form $v \in C^\infty_1(X, \mathbb{C})$ such that $dv = 0$, the following implication holds:

\[
v \in \text{Im} \partial \implies v \in \text{Im}(\partial \bar{\partial}),
\]

the map (1) is an isomorphism.

As a consequence of this discussion, we obtain the following vanishing properties for the cohomology of degenerate balanced manifolds.

Proposition 1.2. Let $X$ be a compact degenerate balanced manifold.

(i) The Bott-Chern cohomology groups of types $(1, 0)$ and $(0, 1)$ of $X$ vanish: $H^{1,0}_{BC}(X, \mathbb{C}) = 0$ and $H^{0,1}_{BC}(X, \mathbb{C}) = 0$.

(ii) If, moreover, $X$ satisfies hypothesis (2), its De Rham cohomology group of degree 1 vanishes: $H^1_{DR}(X, \mathbb{C}) = 0$. 

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Note that degenerate balanced manifolds that satisfy hypothesis (2) do exist. Indeed, Friedman showed in [Fri19] that the manifolds $X = \mathbb{P}(S^3 \times S^3)$, with $k \geq 2$, endowed with the Friedman-Lu-Tian complex structure constructed via conifold transitions ([Fri89], [LT93], [FLY12]) are even $\partial\bar{\partial}$-manifolds.

(b) Our study of the cohomology of degree 2 in this setting centres on seeking out positive properties of balanced hyperbolic manifolds. As an alternative to Question 1.4. in [MP21], wondering about possible positivity properties, in the senses defined therein, of the canonical bundle $K_X$ of any balanced hyperbolic manifold $X$, we concentrate this time on whether there are “many” (in a sense to be determined) closed positive currents $T$ of bidegree $(1, 1)$ on such a manifold.

The starting point of this investigation is Proposition 5.4 in [Pop15], reproduced as Proposition 2.10. in [MP21]: a compact complex manifold $X$ is degenerate balanced if and only if there exists no non-zero $d$-closed $(1, 1)$-current $T \geq 0$ on $X$. In other words, the compact degenerate balanced manifolds $X$ are characterised by their pseudo-effective cone $\mathcal{E}_X$ (namely, the set of Bott-Chern cohomology classes of $d$-closed positive $(1, 1)$-currents on $X$) being reduced to the zero class.

This prompts one to ask the following

**Question 1.3.** Let $X$ be a compact complex manifold. Is it true that $X$ is balanced hyperbolic if and only if its pseudo-effective cone $\mathcal{E}_X$ is small (in a sense to be determined)?

In §2.3 and §2.4 we provide some evidence for this by first showing that both the balanced hypothesis on a given Hermitian metric $\omega$ (see Lemma and Definition 2.2) and the Gauduchon hypothesis (see Lemma and Definition 2.12) enable one to define a notion of $\omega$-primitive De Rham cohomology classes of degree 2 (resp. $\omega$-primitive Bott-Chern cohomology classes of bidegree $(1, 1)$). For example, if $\omega$ is balanced, we set

$$H^2_{DR}(X, \mathbb{C})_{\omega-prim} := \ker\left(\{\omega_{n-1}\}_{\text{DR}} \land \cdot\right) \subset H^2_{DR}(X, \mathbb{C}),$$

after we have shown that the linear map:

$$\{\omega_{n-1}\}_{\text{DR}} \land \cdot : H^2_{DR}(X, \mathbb{C}) \rightarrow H^{2n}_{DR}(X, \mathbb{C}) \cong \mathbb{C}, \quad \{\alpha\}_{\text{DR}} \mapsto \{\omega_{n-1} \land \alpha\}_{\text{DR}},$$

is well defined and depends only on the cohomology class $\{\omega_{n-1}\}_{\text{DR}} \in H^{2n}_{DR}(X, \mathbb{C})$. We go on to show that a class $\alpha \in H^2_{DR}(X, \mathbb{C})$ is $\omega$-primitive if and only if it can be represented by an $\omega$-primitive form (cf. Lemma 2.3), a fact that does not seem to hold in the Gauduchon context of §2.4. We then show that, when the balanced metric $\omega$ is not degenerate balanced, the $\omega$-primitive classes form a complex hyperplane $H^2_{DR}(X, \mathbb{C})_{\omega-prim}$ in $H^2_{DR}(X, \mathbb{C})$ that depends only on the balanced class $\{\omega_{n-1}\}_{\text{DR}}$. (See Corollary 2.5.) Finally, we are able to define a positive side $H^2_{DR}(X, \mathbb{R})_{+}^\omega$ and a negative side $H^2_{DR}(X, \mathbb{R})_{-}^\omega$ of the hyperplane $H^2_{DR}(X, \mathbb{C})_{\omega-prim} := H^2_{DR}(X, \mathbb{C})_{\omega-prim} \cap H^2_{DR}(X, \mathbb{C})$ in $H^2_{DR}(X, \mathbb{R})$ and get a partition of $H^2_{DR}(X, \mathbb{R})$:

$$H^2_{DR}(X, \mathbb{R}) = H^2_{DR}(X, \mathbb{R})_{+}^\omega \cup H^2_{DR}(X, \mathbb{R})_{\omega-prim} \cup H^2_{DR}(X, \mathbb{R})_{-}^\omega.$$

A similar study of the case where $\omega$ is only a Gauduchon metric in §2.4 leads to the characterisation of the pseudo-effective cone as the intersection of the non-negative sides of all the hyperplanes $H^{1,1}_{BC}(X, \mathbb{R})_{\omega-prim}$ determined by Aeppli cohomology classes $[\omega_{n-1}]_A$ of Gauduchon metrics $\omega$ on $X$:

$$\mathcal{E}_X = \bigcap_{[\omega_{n-1}]_A \in \mathcal{G}_X} H^{1,1}_{BC}(X, \mathbb{R})_{\omega}^{\geq 0},$$  

(3)
Theorem 1.5. Let \( \tilde{\omega} \) be the Aeppli cohomology of \( \tilde{X} \) when \( X \) is a balanced hyperbolic manifold in the following form. Throughout the paper, \( L^p_{\tilde{\omega}}, L^p_\omega, \) resp. \( L^p_g \) will stand for the space of objects that are \( L^p \) with respect to the metric \( \tilde{\omega}, \omega, \) resp. \( g. \)

Proposition 1.4. Let \((X, \omega)\) be a balanced hyperbolic manifold and let \( \pi : \tilde{X} \to X \) be the universal cover of \( X \). There exists no non-zero \( d \)-closed positive \((1, 1)\)-current \( \tilde{T} \geq 0 \) on \( \tilde{X} \) such that \( \tilde{T} \) is \( L^1_{\tilde{\omega}} \), where \( \tilde{\omega} := \pi^* \omega \) is the lift of \( \omega \) to \( \tilde{X} \).

This result provides the link with the second part of the paper that we now briefly outline.

(II) Case of balanced hyperbolic manifolds

The results in the second part of the paper mirror, to some extent, those in the first part. The main difference is that the stage changes from \( X \) to its universal covering space \( \tilde{X} \). Specifically, when \( X \) is supposed to be balanced hyperbolic, we obtain vanishing theorems for the \( L^2 \) harmonic cohomology of \( \tilde{X} \).

(a) In this setting, our main result in degree 1 and its dual degree \( 2n - 1 \) is the following

**Theorem 1.5.** Let \( X \) be a compact complex balanced hyperbolic manifold with \( \dim_{\mathbb{C}} X = n \). Let \( \pi : \tilde{X} \to X \) be the universal cover of \( X \) and \( \tilde{\omega} := \pi^* \omega \) the lift to \( \tilde{X} \) of a balanced hyperbolic metric \( \omega \) on \( X \).

There are no non-zero \( \Delta_{\tilde{\omega}} \)-harmonic \( L^2_{\tilde{\omega}} \)-forms of pure types and of degrees 1 and \( 2n - 1 \) on \( \tilde{X} \):

\[
H^{1,0}_{\Delta_{\tilde{\omega}}} (\tilde{X}, \mathbb{C}) = H^{0,1}_{\Delta_{\tilde{\omega}}} (\tilde{X}, \mathbb{C}) = 0 \quad \text{and} \quad H^{n,n-1}_{\Delta_{\tilde{\omega}}} (\tilde{X}, \mathbb{C}) = H^{n-1,n}_{\Delta_{\tilde{\omega}}} (\tilde{X}, \mathbb{C}) = 0,
\]

where \( \Delta_{\tilde{\omega}} := dd^*_{\tilde{\omega}} + d_{\tilde{\omega}}^* d \) is the \( d \)-Laplacian induced by the metric \( \tilde{\omega} \).

The differential operators \( d, d^*_{\tilde{\omega}}, \Delta_{\tilde{\omega}} \) and all the similar ones are considered as closed and densely defined unbounded operators on the spaces \( L^2_k (\tilde{X}, \mathbb{C}) \) of \( L^2_{\tilde{\omega}} \)-forms of degree \( k \) on the complete complex manifold \( (\tilde{X}, \tilde{\omega}) \). (See reminder of some basic results on complete Riemannian manifolds and unbounded operators in §3.1.)

(b) To introduce our results in degree 2, we start by reminding the reader of the following facts of [Dem84] (see also [Dem97, VII, §1]). For any Hermitian metric \( \omega \) on a complex manifold \( X \) with \( \dim_{\mathbb{C}} X = n \), one defines the torsion operator \( \tau = \tau_{\omega} := [{\Lambda}_{\omega}, \partial \omega \wedge \cdot] \) of order 0 and of type \((1, 0)\) acting on the differential forms of \( X \), where \( \Lambda_{\omega} \) is the adjoint of the multiplication operator \( \omega \wedge \cdot \) w.r.t. the pointwise inner product \( \langle \cdot, \cdot \rangle_{\omega} \) defined by \( \omega \). The Kähler commutation relations generalise to the arbitrary Hermitian context as

\[
i[\Lambda_{\omega}, \bar{\partial}] = \partial^* + \tau^* \tag{4}
\]

and the three other relations obtained from this one by conjugation and/or adjunction. (See [Dem97, VII, §1, Theorem 1.1.].) Moreover, considering the torsion-twisted Laplacians

\[
\Delta_{\tau} := [d + \tau, d^* + \tau^*] \quad \text{and} \quad \Delta'_{\tau} := [\partial + \tau, \partial^* + \tau^*],
\]
the following formula holds (see [Dem97, VII, §1, Proposition 1.16.]):
\[ \Delta \tau = \Delta' + \Delta''. \]
When the metric \( \omega \) is Kähler, one has \( \tau = 0 \) and one recovers the classical formula \( \Delta = \Delta' + \Delta'' \). However, we will deal with a more general, possibly non-Kähler, case.

In the context of balanced hyperbolic manifolds, our main result in **degree 2** is the following

**Theorem 1.6.** Let \( X \) be a compact complex balanced hyperbolic manifold with \( \dim \mathbb{C} X = n \). Let \( \pi : \tilde{X} \rightarrow X \) be the universal cover of \( X \) and \( \tilde{\omega} := \pi^* \omega \) the lift to \( \tilde{X} \) of a balanced hyperbolic metric \( \omega \) on \( X \).

There are no non-zero semi-positive \( \Delta_{\tilde{\tau}} \)-harmonic \( L^2_{\tilde{\omega}} \)-forms of pure type \((1, 1)\) on \( \tilde{X} \):

\[ \left\{ \alpha^{1,1} \in H^{1,1}_{\Delta_{\tilde{\tau}}}(\tilde{X}, \mathbb{C}) \mid \alpha^{1,1} \geq 0 \right\} = \{0\}, \]

where \( \tilde{\tau} = \tilde{\tau}_{\tilde{\omega}} := [\Lambda_{\tilde{\omega}}, \partial \tilde{\omega} \wedge \cdot] \).

As a piece of notation that will be used throughout the text, whenever \( u \) is a \( k \)-form and \((p, q)\) is a bidegree with \( p + q = k \), \( u^{p,q} \) will stand for the component of \( u \) of bidegree \((p, q)\).

**Acknowledgments.** This work is part of the first-named author’s PhD thesis under the supervision of the second-named author. The former wishes to express his gratitude to the latter for his constant guidance while this work was carried out, as well as to his Tunisian supervisor, Fathi Haggui, for constant support. Both authors are very grateful to the referee for their careful reading of the manuscript and their useful remarks and suggestions.

### 2 Properties of degenerate balanced manifolds

In this section, we investigate the effect of the balanced condition on the cohomology of degrees 1 and 2, while pointing out the peculiarities of the degenerate balanced case.

#### 2.1 Background

Given a complex manifold \( X \) with \( \dim \mathbb{C} X = n \geq 2 \) and a Hermitian metric \( \omega \) (identified with its underlying \( C^\infty \) positive definite \( (1, 1) \)-form \( \omega \)) on \( X \), we will put \( \omega_r := \omega^r/r! \) for \( r = 1, \ldots, n \). Moreover, we denote by \( C^{r,s}(X) = C^{r,s}(X, \mathbb{C}) \) the space of smooth \( \mathbb{C} \)-valued \((r, s)\)-forms on \( X \) for \( r, s = 1, \ldots, n \). If \( X \) is compact, recall the classical definitions of the *Bott-Chern* and *Aeppli* cohomology groups of \( X \) of any bidegree \((p, q)\):

\[
H^{p,q}_{BC}(X, \mathbb{C}) = \frac{\ker(\partial : C^{p,q}(X) \rightarrow C^{p+1,q}(X)) \cap \ker(\bar{\partial} : C^{p,q}(X) \rightarrow C^{p,q+1}(X))}{\text{Im} (\partial \bar{\partial} : C^{p-1,q-1}(X) \rightarrow C^{p,q}(X))},
\]

\[
H^{p,q}_{A}(X, \mathbb{C}) = \frac{\ker(\partial \bar{\partial} : C^{p,q}(X) \rightarrow C^{p+1,q+1}(X))}{\text{Im} (\partial : C^{p-1,q}(X) \rightarrow C^{p,q}(X)) + \text{Im} (\bar{\partial} : C^{p,q-1}(X) \rightarrow C^{p,q}(X))}.
\]
We will use the Serre-type duality (see e.g. [Sch07]):

\[ H^{1,1}_{BC}(X, \mathbb{C}) \times H^{n-1, n-1}_A(X, \mathbb{C}) \rightarrow \mathbb{C}, \quad ([u]_{BC}, [v]_A) \mapsto \{u\}_{BC} \cdot \{v\}_A := \int_X u \wedge v, \]  

(6)
as well as the \textit{pseudo-effective cone} of \( X \) introduced in [Dem92, Definition 1.3] as the set

\[ \mathcal{E}_X := \left\{ [T]_{BC} \in H^{1,1}_{BC}(X, \mathbb{R}) / T \geq 0 \text{ d-closed (1, 1)-current on } X \right\}. \]

Recall that a Hermitian metric \( \omega \) on \( X \) is said to be a \textit{Gauduchon metric} (cf. [Gau77]) if \( \partial \bar{\partial} \omega^{n-1} = 0 \). For any such metric \( \omega \), \( \omega^{n-1} \) defines an Aeppli cohomology class and the set of all these cohomology classes is called the \textit{Gauduchon cone} of \( X \) (cf. [Pop15]):

\[ \mathcal{G}_X := \left\{ \{\omega^{n-1}\}_A \in H^{n-1, n-1}_A(X, \mathbb{R}) \mid \omega \text{ is a Gauduchon metric on } X \right\} \subset H^{n-1, n-1}_A(X, \mathbb{R}). \]

The main link between the cones \( \mathcal{G}_X \) and \( \mathcal{E}_X \) on a compact \( n \)-dimensional \( X \) is provided by the following reformulation observed in [Pop16] of a result of Lamari’s from [Lam99]. The pseudo-effective cone \( \mathcal{E}_X \subset H^{1,1}_{BC}(X, \mathbb{R}) \) and the closure of the Gauduchon cone \( \overline{\mathcal{G}}_X \subset H^{n-1, n-1}_A(X, \mathbb{R}) \) are \textit{dual} to each other under the duality (6). This means that the following two statements hold.

1. Given any class \( \mathcal{c}^{1,1}_{BC} \in H^{1,1}_{BC}(X, \mathbb{R}) \), the following equivalence holds:
   \[ \mathcal{c}^{1,1}_{BC} \in \mathcal{E}_X \iff \mathcal{c}^{1,1}_{BC} \cdot \mathcal{c}^{n-1, n-1}_A \geq 0 \quad \text{ for every class } \mathcal{c}^{n-1, n-1}_A \in \mathcal{G}_X. \]

2. Given any class \( \mathcal{c}^{n-1, n-1}_A \in H^{n-1, n-1}_A(X, \mathbb{R}) \), the following equivalence holds:
   \[ \mathcal{c}^{n-1, n-1}_A \in \overline{\mathcal{G}}_X \iff \mathcal{c}^{1,1}_{BC} \cdot \mathcal{c}^{n-1, n-1}_A \geq 0 \quad \text{ for every class } \mathcal{c}^{1,1}_{BC} \in \mathcal{E}_X. \]

Finally, recall that a compact complex manifold \( X \) is said to be a \textit{\( \partial \bar{\partial} \)-manifold} (see [DGMS75] for the notion, [Pop14] for the name) if for any \( d \)-closed \textit{pure-type} form \( u \) on \( X \), the following exactness properties are equivalent:

\[ u \text{ is } d\text{-exact} \iff u \text{ is } \partial\text{-exact} \iff u \text{ is } \bar{\partial}\text{-exact} \iff u \text{ is } \partial\bar{\partial}\text{-exact}. \]

On a complex manifold \( X \) with \( \dim \mathbb{C} X = n \), we will often use the following standard formula (cf. e.g. [Voi02, Proposition 6.29, p. 150]) for the Hodge star operator \( * = *_\omega \) of any Hermitian metric \( \omega \) applied to \( \omega \)-\textit{primitive} forms \( v \) of arbitrary bidegree \( (p, q) \):

\[ * v = (-1)^{k(k+1)/2} i^{p-q} \omega_{n-p-q} \wedge v, \quad \text{where } k := p + q. \]

(7)
Recall that, for any \( k = 0, 1, \ldots, n \), a \( k \)-form \( v \) is said to be \((\omega)\)-\textit{primitive} if \( \omega_{n-k+1} \wedge v = 0 \) and that this condition is equivalent to \( \Lambda_\omega v = 0 \), where \( \Lambda_\omega \) is the adjoint of the operator \( \omega \wedge \cdot \) (of multiplication by \( \omega \)) w.r.t. the pointwise inner product \( \langle , \rangle_\omega \) defined by \( \omega \).

We will also often deal with \( C^\infty (1, 1) \)-forms \( \alpha \). If \( \alpha = \alpha_{\text{prim}} + f \omega \) is the \textit{Lefschetz decomposition}, where \( \alpha_{\text{prim}} \) is \textit{primitive} and \( f \) is a smooth function on \( X \), we get \( \Lambda_\omega \alpha = nf \), hence

\[ \alpha = \alpha_{\text{prim}} + \frac{1}{n} (\Lambda_\omega \alpha) \omega. \]

(8)
We will often write \((1, 1)\)-forms in this form.

On the other hand, we will often indicate the metric with respect to which certain operators are calculated. For example, \(d^*_\omega\) and \(\Delta = dd^*_\omega + d^*d\) are the adjoint of \(d\), resp. the \(d\)-Laplacian, induced by the metric \(\omega\).

### 2.2 Case of degree 1

The starting point is the following

**Lemma 2.1.** Let \(\omega\) be a Hermitian metric on a complex manifold \(X\) with \(\text{dim}_C X = n\). Fix a form \(u = u^{1,0} + u^{0,1} \in C^\infty_1(X, \mathbb{C})\).

(i) The following formula holds:

\[
\begin{align*}
d^*(\omega_{n-1} \wedge u) &= i(\partial u^{1,0} - \bar{\partial} u^{0,1}) \wedge \omega_{n-2} + i \left( (\partial u^{0,1})_{\text{prim}} - (\bar{\partial} u^{1,0})_{\text{prim}} \right) \wedge \omega_{n-2} \\
&+ \frac{i}{n} \left( \Lambda_\omega(\bar{\partial} u^{1,0}) - \Lambda_\omega(\partial u^{0,1}) \right) \omega_{n-1},
\end{align*}
\]

where \(d^* = d^*_\omega\) is the formal adjoint of \(d\) w.r.t. the \(L^2\) inner product, while the subscript \(\text{prim}\) indicates the \(\omega\)-primitive part in the Lefschetz decomposition of the form to which it is applied.

In particular, if \(du^{1,0} = 0\) and \(du^{0,1} = 0\), we get

\[
d^*(\omega_{n-1} \wedge u) = 0.
\]

(ii) If \(\omega\) is balanced and \(du^{1,0} = 0\) and \(du^{0,1} = 0\), then

\[
\Delta(\omega_{n-1} \wedge u) = 0,
\]

where \(\Delta = \Delta_\omega = dd^* + d^*d\) is the \(d\)-Laplacian induced by \(\omega\).

(iii) If \(X\) is compact, \(\omega\) is degenerate balanced and \(du^{1,0} = du^{0,1} = 0\), then \(u = 0\).

**Proof.** (i) All \(1\)-forms are primitive, so from the standard formula (7) we get: \(\ast u^{1,0} = -i \omega_{n-1} \wedge u^{1,0}\), hence \(\ast (\omega_{n-1} \wedge u^{1,0}) = -iu^{1,0}\). Meanwhile, \(d^* = -\ast d\ast\), so applying \(-\ast d\ast\) to the previous identity and writing the \((1, 1)\)-form \(\bar{\partial} u^{1,0}\) in the form (8), we get the first line below:

\[
\begin{align*}
d^*(\omega_{n-1} \wedge u^{1,0}) &= i \ast \partial u^{1,0} + i \ast (\bar{\partial} u^{1,0})_{\text{prim}} + i \left( \Lambda_\omega(\bar{\partial} u^{1,0}) \right) \ast \omega \\
&= i \partial u^{1,0} \wedge \omega_{n-2} - i (\bar{\partial} u^{1,0})_{\text{prim}} \wedge \omega_{n-2} + \frac{i}{n} \left( \Lambda_\omega(\bar{\partial} u^{1,0}) \right) \omega_{n-1},
\end{align*}
\]

where the second line follows from the standard formula (7).

Running the analogous computations for \(u^{0,1}\) or taking conjugates, we get:

\[
\begin{align*}
d^*(\omega_{n-1} \wedge u^{0,1}) &= -i \bar{\partial} u^{0,1} \wedge \omega_{n-2} + i (\partial u^{0,1})_{\text{prim}} \wedge \omega_{n-2} - \frac{i}{n} \left( \Lambda_\omega \partial u^{0,1} \right) \omega_{n-1}.
\end{align*}
\]

Formula (9) follows by adding up the above expressions for \(d^*(\omega_{n-1} \wedge u^{1,0})\) and \(d^*(\omega_{n-1} \wedge u^{0,1})\).
(ii) If $\omega$ is balanced, we get $d(\omega_{n-1} \wedge u) = \omega_{n-1} \wedge du = 0$ since $du = 0$ under the assumptions. Since we also have $d^*(\omega_{n-1} \wedge u) = 0$ by (i), the contention follows.

(iii) If $\omega$ is degenerate balanced, there exists a smooth $(2n - 3)$-form $\Gamma$ such that $\omega^{n-1} = d\Gamma$. Hence, $\omega^{n-1} \wedge u = d(\Gamma \wedge u) \in \text{Im} \ d$ because we also have $du = 0$ by our assumptions. However, $\omega^{n-1} \wedge u \in \ker \Delta$ by (ii) and $\ker \Delta \perp \text{Im} \ d$ by the compactness assumption on $X$. Thus, the form $\omega^{n-1} \wedge u \in \ker \Delta \cap \text{Im} \ d = \{0\}$ must vanish.

On the other hand, the pointwise map $\omega_{n-1} \wedge \cdot : \Lambda^1 T^* X \longrightarrow \Lambda^{2n-1} T^* X$ is bijective, so we get $u = 0$ from $\omega^{n-1} \wedge u = 0$.

We now use Lemma 2.1 to infer its consequences announced in the introduction.

• Proof of (i) of Proposition 1.2. This follows at once from (iii) of Lemma 2.1.

Another consequence of Lemma 2.1 is that the balanced condition, combined with the mild $\partial \bar{\partial}$-type condition in (ii) of Theorem 1.1, enables one to get a Hard Lefschetz Isomorphism between the De Rham cohomology spaces of degrees 1 and $2n - 1$.

• Proof of Theorem 1.1. (i) Lemma 2.1 is not needed here. Let $u$ be a smooth 1-form. Since $d\omega_{n-1} = 0$, $d(\omega_{n-1} \wedge u) = 0$ whenever $du = 0$, while $\omega_{n-1} \wedge u = df(\omega_{n-1})$ whenever $u = df$ for some smooth function $f$ on $X$. This proves the well-definedness of the map (1).

Similarly, if $\omega_{n-1} = \gamma_{n-1} + d\Gamma$ for some smooth $(2n-2)$-form $\gamma_{n-1}$ and some smooth $(2n-3)$-form $\Gamma$, then $\omega_{n-1} \wedge u = \gamma_{n-1} \wedge u + d(\Gamma \wedge u)$ for every $d$-closed 1-form $u$. Hence, $\{\omega_{n-1} \wedge u\}_{DR} = \{\gamma_{n-1} \wedge u\}_{DR}$ whenever $\{\omega_{n-1}\}_{DR} = \{\gamma_{n-1}\}_{DR}$, so the map (1) depends only on $\{\omega_{n-1}\}_{DR}$.

(ii) Since $H^1_{DR}(X, \mathbb{C})$ and $H^{2n-1}_{DR}(X, \mathbb{C})$ have equal dimensions, by Poincaré duality, it suffices to prove that the map (1) is injective.

Let $u$ be an arbitrary smooth $d$-closed 1-form on $X$. We start by showing that there exists a smooth function $f : X \rightarrow \mathbb{C}$ such that $\partial u^{0,1} = \partial \bar{\partial} f$ on $X$. To see this, notice that the property $du = 0$ translates to the following three relations holding:

\[
\begin{align*}
(a) \quad & \partial u^{1,0} = 0; \\
(b) \quad & \partial u^{0,1} + \bar{\partial} u^{1,0} = 0; \\
(c) \quad & \bar{\partial} u^{0,1} = 0.
\end{align*}
\]

Thus, the $(1, 1)$-form $\partial u^{0,1}$ is $d$-closed (since it is $\bar{\partial}$-closed by (c) of (10)) and $\partial$-exact. Thanks to hypothesis (2), we infer that $\partial u^{0,1}$ is $\partial \bar{\partial}$-exact. Thus, there exists a smooth function $f$ as stated.

Using (b) of (10), we further infer that $\bar{\partial} u^{1,0} = -\partial u^{0,1} = -\partial \bar{\partial} f$, so $\partial(u^{1,0} - \bar{\partial} f) = 0$. From the identities $\partial(u^{0,1} - \bar{\partial} f) = 0$ and $\bar{\partial}(u^{1,0} - \bar{\partial} f) = 0$ and from (a) and (c) of (10), we get:

\[
d(u^{1,0} - \bar{\partial} f) = 0 \quad \text{and} \quad d(u^{0,1} - \bar{\partial} f) = 0.
\]

This means that

\[
(u - df)^{1,0} \in \ker d \quad \text{and} \quad (u - df)^{0,1} \in \ker d.
\]

From this and from (i) of Lemma 2.1, we deduce that

\[
\omega_{n-1} \wedge (u - df) \in \ker d^*.
\]

On the other hand, if $\omega$ is balanced and if $\{\omega_{n-1} \wedge u\}_{DR} = 0 \in H^{2n-1}_{DR}(X, \mathbb{C})$ (i.e. $\omega_{n-1} \wedge u \in \text{Im} \ d$), then

\[
\omega_{n-1} \wedge (u - df) \in \text{Im} \ d.
\]
From (11), (12) and ker $d^* \perp \text{Im} \, d$, we infer that $\omega_{n-1} \wedge (u - df) = 0$. Since $u - df$ is a smooth 1-form on $X$ and the pointwise-defined linear map:

$$\omega_{n-1} \wedge : C^\infty_1(X, \mathbb{C}) \longrightarrow C^\infty_{2n-1}(X, \mathbb{C}), \quad \alpha \mapsto \omega_{n-1} \wedge \alpha,$$

is bijective, we finally get $u - df = 0$, so $\{u\}_{DR} = 0 \in H^1_{DR}(X, \mathbb{C})$.

This proves the injectivity of the map (1) whenever $\omega$ is balanced and $X$ satisfies hypothesis (2).

\[ \square \]

In the degenerate balanced case, we get the vanishing of the first Betti number of the manifold.

**Proof of (ii) of Proposition 1.2.** If $\omega$ is degenerate balanced, the map (1) vanishes identically. Meanwhile, by Theorem 1.1, the map (1) is an isomorphism. We get $H^1_{DR}(X, \mathbb{C}) = 0$. \[ \square \]

### 2.3 Case of degree 2: De Rham cohomology

The balanced property of a metric enables one to define a notion of primitivity for 2-forms.

**Lemma and Definition 2.2.** Let $\omega$ be a balanced metric on a compact complex manifold $X$ with $\dim \mathbb{C} X = n$. The linear map:

$$\{\omega_{n-1}\}_{DR} \wedge : H^2_{DR}(X, \mathbb{C}) \longrightarrow H^2_{DR}(X, \mathbb{C}) \cong \mathbb{C}, \quad \{\alpha\}_{DR} \mapsto \{\omega_{n-1} \wedge \alpha\}_{DR},$$

(13) is well defined and depends only on the cohomology class $\{\omega_{n-1}\}_{DR} \in H^2_{DR}n(X, \mathbb{C})$. We set:

$$H^2_{DR}(X, \mathbb{C})_{\omega-\text{prim}} := \ker \left( \{\omega_{n-1}\}_{DR} \wedge \cdot \right) \subset H^2_{DR}(X, \mathbb{C})$$

and we call its elements ($\omega$-)primitive De Rham 2-classes.

**Proof.** Since $d\omega_{n-1} = 0$, for every $d$-closed (resp. $d$-exact) 2-form $\alpha$, $\omega_{n-1} \wedge \alpha$ is $d$-closed (resp. $d$-exact). This proves the well-definedness of the map.

Meanwhile, if $\Omega \in C^\infty_{n-1,n-1}(X, \mathbb{C})$ is such that $\Omega = \omega_{n-1} + d\Gamma$ for some smooth $(2n-3)$-form $\Gamma$, then, for every $d$-closed 2-form $\alpha$, $\Omega \wedge \alpha = \omega_{n-1} \wedge \alpha + d(\Gamma \wedge \alpha)$. Hence, $\{\Omega \wedge \alpha\}_{DR} = \{\omega_{n-1} \wedge \alpha\}_{DR}$ whenever $\{\Omega\}_{DR} = \{\omega_{n-1}\}_{DR}$. This proves the independence of the map $\{\omega_{n-1}\}_{DR} \wedge \cdot$ of the choice of representative of the class $\{\omega_{n-1}\}_{DR}$. \[ \square \]

We now observe a link between primitive 2-classes and primitive 2-forms.

**Lemma 2.3.** Let $\omega$ be a balanced metric on a compact complex manifold $X$ with $\dim \mathbb{C} X = n$. For any class $\mathfrak{c} \in H^2_{DR}(X, \mathbb{C})$, the following equivalence holds:

$$\mathfrak{c} \text{ is } \omega\text{-primitive} \iff \exists \alpha \in \mathfrak{c} \text{ such that } \alpha \text{ is } \omega\text{-primitive}.$$ 

**Proof.** “$\Rightarrow$” Suppose $\alpha \in C^\infty_2(X, \mathbb{C})$ such that $d\alpha = 0$, $\alpha \in \mathfrak{c}$ and $\alpha$ is $\omega$-primitive. Then, $\omega_{n-1} \wedge \alpha = 0$, hence $\{\omega_{n-1} \wedge \alpha\}_{DR} = 0$. This means that the class $\mathfrak{c} = \{\alpha\}_{DR}$ is $\omega$-primitive.

“$\Rightarrow$” Suppose the class $\mathfrak{c}$ is $\omega$-primitive. Pick an arbitrary representative $\beta \in \mathfrak{c}$. The $\omega$-primitivity of $\mathfrak{c} = \{\beta\}_{DR}$ translates to $\{\omega_{n-1} \wedge \beta\}_{DR} = 0 \in H^2_{DR}(X, \mathbb{C})$. This, in turn, is equivalent to the existence of a form $\Gamma \in C^\infty_{2n-1}(X, \mathbb{C})$ such that $\omega_{n-1} \wedge \beta = d\Gamma$. 

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Meanwhile, we know from the general theory that the map
\[ \omega_{n-1} \wedge \cdot : C^\infty_1(X, \mathbb{C}) \to C^\infty_{2n-1}(X, \mathbb{C}) \]
is an isomorphism. Hence, there exists a unique \( u \in C^\infty_1(X, \mathbb{C}) \) such that \( \Gamma = \omega_{n-1} \wedge u \). We get:
\[ \omega_{n-1} \wedge \beta = d\Gamma = \omega_{n-1} \wedge du, \]
where the last identity follows from the balanced property of \( \omega \). Consequently,
\[ \omega_{n-1} \wedge (\beta - du) = 0, \]
proving that \( \alpha := \beta - du \) is a primitive representative of the class \( \mathfrak{c} = \{ \beta \}_{DR} \).
\[ \square \]

Finally, we can characterise the degenerate balanced property of a given balanced metric in terms of primitivity for 2-classes.

**Lemma 2.4.** Let \( \omega \) be a balanced metric on a compact complex manifold \( X \) with \( \dim \mathbb{C} X = n \). The following equivalence holds:
\[ H^2_{DR}(X, \mathbb{C})_{\omega-prim} = H^2_{DR}(X, \mathbb{C}) \iff \omega \text{ is degenerate balanced.} \]

*Proof.* “\( \Leftarrow \)” Suppose that \( \omega \) is degenerate balanced. Then \( \omega_{n-1} \) is \( d \)-exact, hence \( \omega_{n-1} \wedge \alpha \) is \( d \)-exact (or equivalently \( \{ \omega_{n-1} \wedge \alpha \}_{DR} = 0 \in H^{2n}_{DR}(X, \mathbb{C}) \)) for every \( d \)-closed 2-form \( \alpha \). This means that the map \( \{ \omega_{n-1} \}_{DR} \wedge \cdot : H^2_{DR}(X, \mathbb{C}) \to H^2_{DR}(X, \mathbb{C}) \) vanishes identically, so \( H^2_{DR}(X, \mathbb{C})_{\omega-prim} = H^2_{DR}(X, \mathbb{C}). \)

“\( \Rightarrow \)” Suppose that \( H^2_{DR}(X, \mathbb{C})_{\omega-prim} = H^2_{DR}(X, \mathbb{C}) \). This translates to
\[ \omega_{n-1} \wedge \alpha \in \text{Im} \, d, \quad \forall \alpha \in C^\infty_2(X, \mathbb{C}) \cap \ker d. \tag{14} \]

Since both \( \omega_{n-1} \) and \( \alpha \) are \( d \)-closed, they both have unique \( L^2_\omega \)-orthogonal decompositions:
\[ \omega_{n-1} = (\omega_{n-1})_h + d\Gamma \quad \text{and} \quad \alpha = \alpha_h + du, \]
where \( (\omega_{n-1})_h \) and \( \alpha_h \) are \( \Delta_\omega \)-harmonic, while \( \Gamma \) and \( u \) are smooth forms of respective degrees \( 2n - 3 \) and 1. We get:
\[ \omega_{n-1} \wedge \alpha = (\omega_{n-1})_h \wedge \alpha_h + d\left((\omega_{n-1})_h \wedge u + \Gamma \wedge \alpha_h + \Gamma \wedge du\right), \quad \forall \alpha \in C^\infty_2(X, \mathbb{C}) \cap \ker d. \]

Together with (14), this implies that
\[ (\omega_{n-1})_h \wedge \alpha_h \in \text{Im} \, d, \quad \forall \alpha_h \in \ker \Delta_\omega \cap C^\infty_2(X, \mathbb{C}). \tag{15} \]

Meanwhile, since \( (\omega_{n-1})_h \) is \( \Delta_\omega \)-harmonic (and real), \( *_\omega(\omega_{n-1})_h \) is again \( \Delta_\omega \)-harmonic (and real). Hence,
\[ \text{Im} \, d \ni (\omega_{n-1})_h \wedge *_\omega(\omega_{n-1})_h = |(\omega_{n-1})_h|^2_\omega \, dV_\omega \geq 0, \]
where the first relation follows from (15) by choosing \( \alpha_h = *_\omega(\omega_{n-1})_h \). Consequently, from Stokes’s Theorem we get:
\[ \int_X |(\omega_{n-1})_h|^2_\omega \, dV_\omega = 0, \]
hence \( (\omega_{n-1})_h = 0 \). This implies that \( \omega_{n-1} \) is \( d \)-exact, which means that \( \omega \) is degenerate balanced.
\[ \square \]
Corollary 2.5. Let $\omega$ be a balanced metric on a compact complex manifold $X$ with $\dim_{\mathbb{C}} X = n$. The following dichotomy holds:

(a) if $\omega$ is not degenerate balanced, $H^2_{DR}(X, \mathbb{C})_{\omega-prim}$ is a complex hyperplane in $H^2_{DR}(X, \mathbb{C})$ depending only on the balanced class $\{\omega_{-1}\}_{DR}$;

(b) if $\omega$ is degenerate balanced, $H^2_{DR}(X, \mathbb{C})_{\omega-prim} = H^2_{DR}(X, \mathbb{C})$.

Proof. This follows immediately from Lemma and Definition 2.2, from Lemma 2.4 and from $H^2_{DR}(X, \mathbb{C}) \simeq \mathbb{C}$.

We shall now get a Lefschetz-type decomposition of $H^2_{DR}(X, \mathbb{C})$, induced by an arbitrary balanced metric $\omega$, with $H^2_{DR}(X, \mathbb{C})_{\omega-prim}$ as a direct factor. Recall that the balanced condition $d\omega^{n-1} = 0$ is equivalent to $d^*\omega = 0$.

Thanks to the orthogonal 3-space decompositions:

$$C^\infty_k(X, \mathbb{C}) = \ker \Delta_\omega \oplus \operatorname{Im} d \oplus \operatorname{Im} d^*, \quad k \in \{0, \ldots, 2n\},$$

where $\ker \Delta_\omega \oplus \operatorname{Im} d = \ker d$ and $\ker \Delta_\omega \oplus \operatorname{Im} d^* = \ker d^*$, applied with $k = 2$ and $k = 2n - 2$, we get unique decompositions of $\omega$, resp. $\omega_{-1}$:

$$\ker d^* \ni \omega = \omega_h + d^*\eta_\omega \quad \text{and} \quad \ker d \ni \omega_{-1} = (\omega_{-1})_h + d\Gamma_\omega,$$

where $\omega_h \in \ker \Delta_\omega$ as a 2-form, $(\omega_{-1})_h \in \ker \Delta_\omega$ as a $(2n - 2)$-form, while $\eta_\omega$ and $\Gamma_\omega$ are smooth forms of respective degrees 3 and $2n - 3$. Since $\omega$ and $\omega_{-1}$ are real, so are their harmonic components $\omega_h$ and $(\omega_{-1})_h$.

Moreover, it is well known that $\star_\omega \omega = \omega_{-1}$ and that the Hodge star operator $\star_\omega$ maps $d$-exact forms to $d^*$-exact forms and vice-versa. Hence, we get:

$$\star_\omega \omega_h = (\omega_{-1})_h \quad \text{and} \quad \star_\omega (d^*\eta_\omega) = d\Gamma_\omega.$$  \hfill \text{(17)}

Thus, $\omega_h$ is uniquely determined by $\omega$ and is $d$-closed (because it is even $\Delta_\omega$-harmonic). Therefore, it represents a class in $H^2_{DR}(X, \mathbb{R})$.

Definition 2.6. For any balanced metric $\omega$ on a compact complex manifold $X$, the De Rham cohomology class $\{\omega_h\}_{DR} \in H^2_{DR}(X, \mathbb{R})$ is called the cohomology class of $\omega$.

Of course, if $\omega$ is Kähler, $\omega_h = \omega$, so $\{\omega_h\}_{DR}$ is the usual Kähler class $\{\omega\}_{DR}$.

Lemma 2.7. Suppose there exists a balanced metric $\omega$ on a compact complex manifold $X$. Then, for every $\alpha \in C^\infty_2(X, \mathbb{C})$ such that $d\alpha = 0$ and $\{\alpha\}_{DR} \in H^2_{DR}(X, \mathbb{C})_{\omega-prim}$, we have:

$$\langle \langle \omega_h, \alpha \rangle \rangle = 0,$$

where $\langle \langle \cdot, \cdot \rangle \rangle_\omega$ is the $L^2$ inner product induced by $\omega$.

Proof. Since $\{\alpha\}_{DR} \in H^2_{DR}(X, \mathbb{C})_{\omega-prim}$, there exists a form $\Omega \in C^\infty_{2n-1}$ such that $\omega_{-1} \wedge \alpha = du$. We get:

$$\langle \langle \alpha, \omega_h \rangle \rangle = \int_X \alpha \wedge \star_\omega \omega_h = \int_X \alpha \wedge (\omega_{n-1})_h = \int_X \alpha \wedge (\omega_{n-1} - d\Gamma_\omega) = \int_X \alpha \wedge \omega_{n-1} = \int_X du = 0,$$
where Stokes implies two of the last three equalities (note that $\alpha \wedge d\Gamma_\omega = d(\alpha \wedge \Gamma_\omega)$), while (a) follows from (17) and (b) follows from (16). \qed

**Conclusion 2.8.** Let $X$ be a compact complex manifold with $\text{dim}_{\mathbb{C}} X = n$. Suppose there exists a non-degenerate balanced metric $\omega$ on $X$. Then, the De Rham cohomology space of degree 2 has a Lefschetz-type $L^2_{\omega}$-orthogonal decomposition:

$$H^2_{\text{DR}}(X, \mathbb{C}) = H^2_{\text{DR}}(X, \mathbb{C})_{\omega, \text{prim}} \oplus \mathbb{C} \cdot \{\omega_h\}_{\text{DR}},$$

(18)

where the $\omega$-primitive subspace $H^2_{\text{DR}}(X, \mathbb{C})_{\omega, \text{prim}}$ is a complex hyperplane of $H^2_{\text{DR}}(X, \mathbb{C})$ depending only on the cohomology class $\{\omega_{n-1}\}_{\text{DR}} \in H^{2n-2}_{\text{DR}}(X, \mathbb{C})$, while $\omega_h$ is the $\Delta_\omega$-harmonic component of $\omega$ and the complex line $\mathbb{C} \cdot \{\omega_h\}_{\text{DR}}$ depends on the choice of the balanced metric $\omega$.

If $\omega$ is Kähler, the Lefschetz-type decomposition (18) depends only on the Kähler class $\{\omega\}_{\text{DR}} \in H^2_{\text{DR}}(X, \mathbb{C})$ since $\omega_h = \omega$ in that case.

**Lemma 2.9.** The assumptions are the same as in Conclusion 2.8. For every $\alpha \in C^\infty_2(X, \mathbb{C}) \cap \ker d$, the coefficient of $\{\omega_h\}_{\text{DR}}$ in the Lefschetz-type decomposition of $\{\alpha\}_{\text{DR}} \in H^2_{\text{DR}}(X, \mathbb{C})$ according to (18), namely in

$$\{\alpha\}_{\text{DR}} = \{\alpha\}_{\text{DR, prim}} + \lambda \{\omega_h\}_{\text{DR}},$$

(19)

is given by

$$\lambda = \lambda_\omega(\{\alpha\}_{\text{DR}}) = \frac{\{\omega_{n-1}\}_{\text{DR}} \cdot \{\alpha\}_{\text{DR}}}{||\omega_h||^2_\omega} = \frac{1}{||\omega_h||^2_\omega} \int_X \alpha \wedge \omega_{n-1}. \quad (20)$$

**Proof.** Since $\{\alpha\}_{\text{DR, prim}} \in H^2_{\text{DR}}(X, \mathbb{C})_{\omega, \text{prim}}$, we have $\{\omega_{n-1}\}_{\text{DR}} \cdot \{\alpha\}_{\text{DR, prim}} = 0$, so

$$\{\omega_{n-1}\}_{\text{DR}} \cdot \{\alpha\}_{\text{DR}} = \lambda \int \omega_{n-1} \wedge \omega_h = \lambda \int (\omega_{n-1})_h \wedge \omega_h = \lambda \int (\omega_{n-1})_h \wedge \ast_\omega (\omega_{n-1})_h$$

$$\lambda \int ||(\omega_{n-1})_h||^2_\omega = \lambda ||\omega_h||^2_\omega.$$

This gives (20). \qed

Formula (20) implies that $\lambda_\omega(\{\alpha\}_{\text{DR}})$ is real if the class $\{\alpha\}_{\text{DR}} \in H^2_{\text{DR}}(X, \mathbb{R})$ is real. This enables one to define a positive side and a negative side of the hyperplane $H^2_{\text{DR}}(X, \mathbb{R})_{\omega, \text{prim}} := H^2_{\text{DR}}(X, \mathbb{C})_{\omega, \text{prim}} \cap H^2_{\text{DR}}(X, \mathbb{R})$ in $H^2_{\text{DR}}(X, \mathbb{R})$ by

$$H^2_{\text{DR}}(X, \mathbb{R})^+_{\omega} := \left\{ \{\alpha\}_{\text{DR}} \in H^2_{\text{DR}}(X, \mathbb{R}) \mid \lambda_\omega(\{\alpha\}_{\text{DR}}) > 0 \right\},$$

$$H^2_{\text{DR}}(X, \mathbb{R})^-_{\omega} := \left\{ \{\alpha\}_{\text{DR}} \in H^2_{\text{DR}}(X, \mathbb{R}) \mid \lambda_\omega(\{\alpha\}_{\text{DR}}) < 0 \right\}.$$

These open subsets of $H^2_{\text{DR}}(X, \mathbb{R})$ depend only on the cohomology class $\{\omega_{n-1}\}_{\text{DR}} \in H^{2n-2}_{\text{DR}}(X, \mathbb{R})$.

Since $\{\alpha\}_{\text{DR}}$ is $\omega$-primitive if and only if $\lambda_\omega(\{\alpha\}_{\text{DR}}) = 0$, we get a partition of $H^2_{\text{DR}}(X, \mathbb{R})$:

$$H^2_{\text{DR}}(X, \mathbb{R}) = H^2_{\text{DR}}(X, \mathbb{R})^+_{\omega} \cup H^2_{\text{DR}}(X, \mathbb{R})_{\omega, \text{prim}} \cup H^2_{\text{DR}}(X, \mathbb{R})^-_{\omega}.$$
depending only on the cohomology class \( \{ \omega_{n-1} \}_\text{DR} \subset H^{2n-2}_\text{DR}(X, \mathbb{R}) \).

The next (trivial) observation is that the \( \omega \)-primitive hyperplane \( H^2_\text{DR}(X, \mathbb{C})_{\omega-\text{prim}} \subset H^2_\text{DR}(X, \mathbb{C}) \) depends only on the ray \( \mathbb{R}_{>0} \cdot \{ \omega_{n-1} \}_\text{DR} \) generated by the De Rham cohomology class of \( \omega_{n-1} \) in the De Rham version of the balanced cone \( \mathcal{B}_{X, \text{DR}} \subset H^{2n-2}_\text{DR}(X, \mathbb{R}) \) of \( X \). (We denote by \( \mathcal{B}_{X, \text{DR}} \) the set of De Rham cohomology classes \( \{ \omega_{n-1} \}_\text{DR} \) induced by balanced metrics \( \omega \).

**Lemma 2.10.** Let \( X \) be a compact complex non-degenerate balanced manifold with \( \dim_\mathbb{C} X = n \). Let \( \omega \) and \( \gamma \) be balanced metrics on \( X \). The following equivalence holds:

\[
H^2_\text{DR}(X, \mathbb{C})_{\omega-\text{prim}} = H^2_\text{DR}(X, \mathbb{C})_{\gamma-\text{prim}} \iff \exists c > 0 \text{ such that } \{ \omega_{n-1} \}_\text{DR} = c \{ \gamma_{n-1} \}_\text{DR}.
\]

**Proof.** “\( \Rightarrow \)” This implication follows from proportional linear maps having the same kernel.

“\( \Leftarrow \)” This implication follows from the following elementary fact. Suppose \( T, S : E \to \mathbb{C} \) are \( \mathbb{C} \)-linear maps on a \( \mathbb{C} \)-vector space \( E \) such that \( \ker T = \ker S \subset E \) is of \( \mathbb{C} \)-codimension 1 in \( E \). Then, there exists \( c \in \mathbb{C} \setminus \{ 0 \} \) such that \( T = cS \). To see this, let \( \{ e_j \mid j \in J \} \) be a \( \mathbb{C} \)-basis of \( \ker T = \ker S \) and let \( e \in E \) such that \( \{ e \} \cup \{ e_j \mid j \in J \} \) is a \( \mathbb{C} \)-basis of \( E \). Then, \( T(e) \) and \( S(e) \) are non-zero complex numbers, so there exists a unique \( c \in \mathbb{C} \setminus \{ 0 \} \) such that \( T(e) = cS(e) \). Now, fix an arbitrary \( u \in E \). We will show that \( T(u) = cS(u) \). There is a unique choice of \( \lambda \in \mathbb{C} \) and \( v \in \ker T = \ker S \) such that \( u = \lambda e + v \). Hence, \( T(u) = \lambda T(e) = c(\lambda S(e)) = cS(u) \).

In our case, the assumption \( H^2_\text{DR}(X, \mathbb{C})_{\omega-\text{prim}} = H^2_\text{DR}(X, \mathbb{C})_{\gamma-\text{prim}} \) amounts to \( \ker(\{ \omega_{n-1} \}_\text{DR} \land \cdot) = \ker(\{ \gamma_{n-1} \}_\text{DR} \land \cdot) \). Hence, by the above elementary fact, it amounts to the existence of a constant \( c \in \mathbb{C} \setminus \{ 0 \} \) such that \( \{ \omega_{n-1} \}_\text{DR} \land \cdot = c \{ \gamma_{n-1} \}_\text{DR} \land \cdot \) as \( \mathbb{C} \)-linear maps on \( H^2_\text{DR}(X, \mathbb{C}) \). By the non-degeneracy of the Poincaré duality \( H^2_\text{DR}(X, \mathbb{C}) \times H^{2n-2}_\text{DR}(X, \mathbb{C}) \to \mathbb{C} \), this further amounts to the existence of a constant \( c \in \mathbb{C} \setminus \{ 0 \} \) such that \( \{ \omega_{n-1} \}_\text{DR} = c \{ \gamma_{n-1} \}_\text{DR} \).

Now, since the forms \( \omega_{n-1} \) and \( \gamma_{n-1} \) are real, the constant \( c \) can be chosen real. (Replace \( c \) with \( (c + \bar{c})/2 \) if necessary.) Since the balanced metric \( \omega \) is non-degenerate, \( c \neq 0 \). If \( c < 0 \), then \( \omega_{n-1} - c \gamma_{n-1} > 0 \) would be the \( d \)-exact \((n - 1)\)-st power of a balanced metric. This balanced metric would then be degenerate balanced, contradicting the assumption on \( X \). Thus, \( c \) must be positive. \( \square \)

We will now see that not only do proportional balanced classes \( \{ \omega_{n-1} \}_\text{DR} \) and \( \{ \gamma_{n-1} \}_\text{DR} \) induce the same hyperplane of primitive classes in \( H^2_\text{DR}(X, \mathbb{C}) \), but they can be made to also induce the same Lefschetz-type decomposition (18). This is fortunate since, in general, the complex line \( \mathbb{C} \cdot \{ \omega \}_\text{DR} \) depends on the choice of the balanced metric \( \omega \), unlike \( H^2_\text{DR}(X, \mathbb{C})_{\omega-\text{prim}} \) which depends only on the balanced class \( \{ \omega_{n-1} \}_\text{DR} \subset H^{2n-2}_\text{DR}(X, \mathbb{C}) \).

**Lemma 2.11.** Let \( X \) be a compact complex non-degenerate balanced manifold with \( \dim_\mathbb{C} X = n \).

(i) If \( \omega \) and \( \gamma \) are balanced metrics on \( X \) such that \( \omega_{n-1} = c \gamma_{n-1} \) for some constant \( c > 0 \), there exists a constant \( a > 0 \) such that \( \omega_h = a \gamma_h \).

(ii) For every ray \( \mathbb{R}_{>0} \cdot \{ \omega_{n-1} \}_\text{DR} \) in the De Rham version of the balanced cone \( \mathcal{B}_{X, \text{DR}} \subset H^{2n-2}_\text{DR}(X, \mathbb{R}) \) of \( X \), the balanced metrics representing the classes on this ray can be chosen such that they induce the same Lefschetz-type decomposition (18).

**Proof.** (i) Since \( \omega = c^{1/n-1} \gamma \), we get \( \star_\omega = \text{const} \cdot \star_\gamma \) and \( d^*_\gamma = \text{const} \cdot d^*_\gamma \). The latter identity implies \( \Delta_\omega = \text{const} \cdot \Delta_\gamma \), hence ker \( \Delta_\omega = \text{ker} \Delta_\gamma \). In particular, \( (\omega_{n-1})_h = c(\gamma_{n-1})_h \) and thus

\[
\omega_h = \star_\omega(\omega_{n-1})_h = \text{const} \cdot \star_\gamma(\gamma_{n-1})_h = \text{const} \cdot \gamma_h,
\]
where in all the above identities \textit{const} stands for a positive constant that may change from one occurrence to another.

(ii) Fix a balanced De Rham class \( \{ \gamma_{n-1} \}_{DR} \in \mathcal{B}_{X,DR} \subset H^{2n-2}_{DR}(X, \mathbb{R}) \) and fix a balanced metric \( \gamma \) (whose choice is arbitrary) such that \( \gamma_{n-1} \) represents the class \( \{ \gamma_{n-1} \}_{DR} \). For every constant \( c > 0 \), the balanced class \( c \{ \gamma_{n-1} \}_{DR} \) can be represented by the form \( \omega_{n-1} := c \gamma_{n-1} \) which is induced by the balanced metric \( \omega := c^{n-1} \gamma \). From (i), we get \( \mathbb{C} \{ \omega_h \}_{DR} = \mathbb{C} \{ \gamma_h \}_{DR} \). Since we also have \( H^2_{DR}(X, \mathbb{C})_{\omega_{\text{prim}}} = H^2_{DR}(X, \mathbb{C})_{\gamma_{\text{prim}}} \) by Lemma 2.10, the contention follows. \( \square \)

The proof of (ii) of the above Lemma 2.11 shows that the line \( \mathbb{C} \{ \omega_h \}_{DR} \) in the Lefschetz-type decomposition \( (18) \) induced by a given ray \( \mathbb{R}_{>0} \cdot \{ \omega_{n-1} \}_{DR} \) in the De Rham version of the balanced cone \( \mathcal{B}_{X,DR} \subset H^{2n-2}_{DR}(X, \mathbb{R}) \) of \( X \) still depends on the arbitrary choice of a balanced metric \( \gamma \) such that \( \gamma_{n-1} \) represents a given class \( \{ \gamma_{n-1} \}_{DR} \) on this ray. To tame this dependence, we can fix an arbitrary Hermitian (not necessarily balanced) metric \( \rho \) on \( X \) and make all the choices of harmonic representatives and projections be induced by \( \rho \). Thus, we get \( L^2_{\rho} \)-orthogonal decompositions:

\[
\omega = \omega_{h,\rho} + d^*_{\rho} \eta_{\omega,\rho} \quad \text{and} \quad \omega_{n-1} = \omega_{n-1,h,\rho} + d\Gamma_{\omega,\rho},
\]

where \( \omega_{h,\rho} \in \ker \Delta_{\rho} \) as a 2-form, \( \omega_{(n-1),h,\rho} \in \ker \Delta_{\rho} \) as a \( (2n-2) \)-form, while \( \eta_{\omega,\rho} \) and \( \Gamma_{\omega,\rho} \) are smooth forms of respective degrees 3 and \( 2n-3 \). Since \( \omega \) and \( \omega_{n-1} \) are real, so are their \( \Delta_{\rho} \)-harmonic components \( \omega_{h,\rho} \) and \( \omega_{(n-1),h,\rho} \). In this way, every non-zero balanced class \( \{ \omega_{n-1} \}_{DR} \) induces a Lefschetz-type decomposition analogous to \( (18) \) that depends only on the class \( \{ \omega_{n-1} \}_{DR} \) and on the background metric \( \rho \):

\[
H^2_{DR}(X, \mathbb{C}) = H^2_{DR}(X, \mathbb{C})_{\omega_{\text{prim}}} \oplus \mathbb{C} \cdot \{ \omega_{h,\rho} \}_{DR},
\]

where the hyperplane \( H^2_{DR}(X, \mathbb{C})_{\omega_{\text{prim}}} \) depends only on the class \( \{ \omega_{n-1} \}_{DR} \).

In other words, we remove the dependence of the line \( \mathbb{C} \{ \omega_h \}_{DR} \) in the Lefschetz-type decomposition \( (18) \) on a representative of the class \( \{ \omega_{n-1} \}_{DR} \) and replace it with the dependence on a fixed background metric \( \rho \).

### 2.4 Case of degree 2: Bott-Chern and Aeppli cohomologies

Let us finally point out that the theory developed in §2.3 in the context of the Poincaré duality for the De Rham cohomology spaces of degrees 2 and \( 2n-2 \) can be rerun in the context of the duality (6) between the Bott-Chern and Aeppli cohomology spaces of bidegrees \((1, 1)\), resp. \((n-1, n-1)\).

Since all the results and constructions of §2.3, except for Lemma 2.3, have analogues in the new context with very similar proofs, we will leave most of these proofs to the reader.

In fact, the new cohomological setting allows for the theory of §2.3 to be repeated in the more general context of Gauduchon (not necessarily balanced) metrics and the Aeppli cohomology classes they define in \( H^{n-1,n-1}_A(X, \mathbb{R}) \). We start with the following analogue of Lemma and Definition 2.2.

#### Lemma and Definition 2.12.

Let \( \omega \) be a Gauduchon metric on a compact complex manifold \( X \) with \( \dim_{\mathbb{C}} X = n \). The linear map:

\[
[\omega_{n-1}]_A \wedge : H_{BC}^{1,1}(X, \mathbb{C}) \to H^n_A(X, \mathbb{C}) \simeq \mathbb{C}, \quad [\alpha]_{BC} \mapsto [\omega_{n-1} \wedge \alpha]_A,
\]

\[14\]
is well defined and depends only on the cohomology class $[\omega_{n-1}]_A \in H^{n-1,n-1}_A(X, \mathbb{C})$. We set:

$$H^{1,1}_{BC}(X, \mathbb{C})_{\omega\text{-prim}} := \ker \left( [\omega_{n-1}]_A \cdot \right) \subset H^{1,1}_{BC}(X, \mathbb{C})$$

and we call its elements $(\omega\text{-})$primitive Bott-Chern $(1, 1)$-classes.

**Proof.** The well-definedness follows at once from the identities:

$$\partial \bar{\partial} (\omega_{n-1} \wedge \alpha) = \partial \bar{\partial} \omega_{n-1} \wedge \alpha = 0, \quad \alpha \in C^\infty_{1,1}(X, \mathbb{C}) \cap \ker d,$$

$$\omega_{n-1} \wedge \partial \bar{\partial} \varphi = \partial(\omega_{n-1} \wedge \partial \varphi) + \bar{\partial}(\varphi \partial \omega_{n-1}) \in \text{Im} \partial + \text{Im} \bar{\partial}, \quad \varphi \in C^\infty_{0,0}(X, \mathbb{C}),$$

where the latter takes into account the fact that $\partial \bar{\partial} \omega_{n-1} = 0$.

That the map $[\omega_{n-1}]_A \wedge \cdot$ depends only on the Aeppli cohomology class $[\omega_{n-1}]_A$ follows from:

$$(\omega_{n-1} + \partial \bar{\Gamma} + \bar{\partial} \Gamma) \wedge \alpha - \omega_{n-1} \wedge \alpha = \partial(\bar{\Gamma} \wedge \alpha) + \bar{\partial}(\Gamma \wedge \alpha) \in \text{Im} \partial + \text{Im} \bar{\partial}, \quad \alpha \in C^\infty_{1,1}(X, \mathbb{C}) \cap \ker d.$$

The following result is the analogue of Lemma 2.4.

**Lemma 2.13.** Let $\omega$ be a Gauduchon metric on a compact complex manifold $X$ with $\dim \mathbb{C} X = n$. The following equivalence holds:

$$H^{1,1}_{BC}(X, \mathbb{C})_{\omega\text{-prim}} = H^{1,1}_{BC}(X, \mathbb{C}) \iff \omega_{n-1} \in \text{Im} \partial + \text{Im} \bar{\partial} \quad (\text{i.e. } \omega_{n-1} \text{ is Aeppli-exact}).$$

**Proof.** “$\Leftarrow$” If $\omega_{n-1} \in \text{Im} \partial + \text{Im} \bar{\partial}$, $[\omega_{n-1}]_A = 0$, so the map $[\omega_{n-1}]_A \wedge \cdot$ vanishes identically.

“$\Rightarrow$” Suppose that $H^{1,1}_{BC}(X, \mathbb{C})_{\omega\text{-prim}} = H^{1,1}_{BC}(X, \mathbb{C})$. This translates to

$$\omega_{n-1} \wedge \alpha \in \text{Im} \partial + \text{Im} \bar{\partial}, \quad \forall \alpha \in C^\infty_{1,1}(X, \mathbb{C}) \cap \ker d. \quad (21)$$

Since $\omega_{n-1}$ is $(\partial \bar{\partial})$-closed, it has a unique $L^2_\omega$-orthogonal decomposition:

$$\omega_{n-1} = (\omega_{n-1})_h + (\partial \bar{\Gamma}_\omega + \bar{\partial} \Gamma_\omega),$$

with an $(n-1, n-1)$-form $(\omega_{n-1})_h \in \ker \Delta_{A,\omega}$ and an $(n-1, n-2)$-form $\Gamma_\omega$. (See (23) below.)

On the other hand, $\alpha$ is $d$-closed, so it has a unique $L^2_\omega$-orthogonal decomposition:

$$\alpha = \alpha_h + \partial \bar{\partial} \varphi,$$

where $\alpha_h$ is $\Delta_{BC,\omega}$-harmonic and $\varphi$ is a smooth function on $X$. (See again (23) below.)

Thus, for every $\alpha \in C^\infty_{1,1}(X, \mathbb{C}) \cap \ker d$, we get:

$$\omega_{n-1} \wedge \alpha = (\omega_{n-1})_h \wedge \alpha + \partial(\Gamma_\omega \wedge \alpha) + \bar{\partial}(\Gamma_\omega \wedge \alpha)$$

$$= (\omega_{n-1})_h \wedge \alpha_h + \partial \left( (\omega_{n-1})_h \wedge \partial \varphi \right) + \bar{\partial} \left( \varphi \partial (\omega_{n-1})_h \right) + \partial(\Gamma_\omega \wedge \alpha) + \bar{\partial}(\Gamma_\omega \wedge \alpha),$$

where for the last identity we used the fact that $\bar{\partial} \partial (\omega_{n-1})_h = 0$. 

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Thanks to assumption (21), the last identity implies that

\[(\omega_{n-1})_h \wedge \alpha_h \in \text{Im} \partial + \text{Im} \bar{\partial}, \quad \forall \alpha_h \in C_{1,1}^\infty(X, \mathbb{C}) \cap \ker \Delta_{BC,\omega}.\]  
(22)

Meanwhile, since \((\omega_{n-1})_h\) is \(\Delta_{A,\omega}\)-harmonic (and real), \(\ast_h(\omega_{n-1})_h\) is \(\Delta_{BC,\omega}\)-harmonic (and real). Hence,

\[\text{Im} \partial + \text{Im} \bar{\partial} \ni (\omega_{n-1})_h \wedge \ast_h(\omega_{n-1})_h = |(\omega_{n-1})_h|^2_{\omega} \, dV_\omega \geq 0,\]

where the first relation follows from (22) by choosing \(\alpha_h = \ast_h(\omega_{n-1})_h\). Consequently, from Stokes’s Theorem we get:

\[\int_X |(\omega_{n-1})_h|^2_{\omega} \, dV_\omega = 0,\]

hence \((\omega_{n-1})_h = 0\). This implies that \(\omega_{n-1} \in \text{Im} \partial + \text{Im} \bar{\partial}\) and we are done. \(\square\)

The analogue in this context of Corollary 2.5 is the following

**Corollary 2.14.** Let \(\omega\) be a Gauduchon metric on a compact complex manifold \(X\) with \(\dim_{\mathbb{C}} X = n\). The following dichotomy holds:

1. \(\text{if } \omega_{n-1} \text{ is not Aeppli exact, } H_{BC}^{1,1}(X, \mathbb{C})_{\omega-\text{prim}} \text{ is a complex hyperplane of } H_{BC}^{1,1}(X, \mathbb{C}) \text{ depending only on the Aeppli-Gauduchon class } \{\omega_{n-1}\}_{DR} \in \mathcal{G};\)
2. \(\text{if } \omega_{n-1} \text{ is Aeppli exact, } H_{BC}^{1,1}(X, \mathbb{C})_{\omega-\text{prim}} = H_{BC}^{1,1}(X, \mathbb{C}).\)

To get a Lefschetz-type decomposition of \(H_{BC}^{1,1}(X, \mathbb{C})\) induced by an arbitrary Gauduchon metric \(\omega\), we use the orthogonal 3-space decompositions featuring the Aeppli-, resp. Bott-Chern-Laplacians induced by the metric \(\omega\):

\[
C_{n-1,n-1}^\infty(X, \mathbb{C}) = \ker \Delta_{A,\omega} \oplus (\text{Im} \partial + \text{Im} \bar{\partial}) \oplus (\text{Im} (\partial \bar{\partial})*),
\]
\[
C_{1,1}^\infty(X, \mathbb{C}) = \ker \Delta_{BC,\omega} \oplus (\text{Im} (\partial \bar{\partial}) \oplus (\text{Im} \ast + \text{Im} \bar{\ast})*),
\]
(23)

where \(\ker (\Delta_{A,\omega} \oplus (\text{Im} \partial + \text{Im} \bar{\partial}) = \ker (\partial \bar{\partial})\) and \(\ker (\Delta_{BC,\omega} \oplus (\text{Im} \ast + \text{Im} \bar{\ast})* = \ker (\partial \bar{\partial})*.\) Thus, we get unique decompositions of \(\omega\), resp. \(\omega_{n-1}:

\[
\ker (\partial \bar{\partial})* \ni \omega = \omega_h + (\partial \bar{\ast \omega} + \partial \bar{\omega} \ast u), \quad \text{and} \quad \ker (\partial \bar{\partial}) \ni \omega_{n-1} = (\omega_{n-1})_h + (\partial \bar{\Gamma}_\omega + \partial \bar{\Gamma}_\omega),
\]
(24)

where \(\omega_h \in \ker \Delta_{BC,\omega}\) as a \((1, 1)\)-form, \((\omega_{n-1})_h \in \ker \Delta_{A,\omega}\) as an \((n-1, n-1)\)-form, while \(u\) and \(\Gamma_\omega\) are smooth forms of respective bidegrees \((1, 2)\) and \((n-1, n-2)\). Since \(\omega\) and \(\omega_{n-1}\) are real, so are their harmonic components \(\omega_h\) and \((\omega_{n-1})_h\). Since \(\ast_h \omega = \omega_{n-1}\) and since the Hodge star operator \(\ast_h\) maps Aeppli-harmonic forms to Bott-Chern-harmonic forms and vice-versa, we get:

\[
\ast_h \omega_h = (\omega_{n-1})_h \quad \text{and} \quad \ast_h (\partial \bar{\ast \omega} + \partial \bar{\omega} \ast u) = \partial \bar{\Gamma}_\omega + \partial \bar{\Gamma}_\omega.
\]
(25)

Thus, \(\omega_h\) is uniquely determined by \(\omega\) and is \(d\)-closed (because it is even \(\Delta_{BC,\omega}\)-harmonic). Therefore, it represents a class in \(H_{BC}^{1,1}(X, \mathbb{R})\).

**Definition 2.15.** For any Gauduchon metric \(\omega\) on a compact complex manifold \(X\), the Bott-Chern cohomology class \(\lbrack \omega_h \rbrack_{BC} \in H_{BC}^{1,1}(X, \mathbb{R})\) is called the cohomology class of \(\omega\).
Of course, if \( \omega \) is Kähler, \( \omega_h = \omega \), so \( \{ \omega_h \}_{BC} \) is the usual Bott-Chern Kähler class \( \{ \omega \}_{BC} \).

The analogue of Lemma 2.7 is the following

**Lemma 2.16.** Suppose there exists a Gauduchon metric \( \omega \) on a compact complex manifold \( X \). Then, for every \( \alpha \in C^\infty_{1,1}(X, \mathbb{C}) \) such that \( d\alpha = 0 \) and \( [\alpha]_{BC} \in H^{1,1}_{BC}(X, \mathbb{C})_{\omega-\text{prim}} \), we have:

\[
\langle \langle \omega_h, \alpha \rangle \rangle_\omega = 0,
\]

where \( \langle \langle , \rangle \rangle_\omega \) is the \( L^2 \) inner product induced by \( \omega \).

The analogue in this context of Conclusion 2.8 is the following

**Conclusion 2.17.** Let \( X \) be a compact complex manifold with \( \dim_{\mathbb{C}} X = n \). Let \( \omega \) be a Gauduchon metric on \( X \) such that \( \omega_{n-1} \) is not Aeppli-exact. Then, the Bott-Chern cohomology space of bidegree \((1, 1)\) has a Lefschetz-type \( L^2_\omega \)-orthogonal decomposition:

\[
H^{1,1}_{BC}(X, \mathbb{C}) = H^{1,1}_{BC}(X, \mathbb{C})_{\omega-\text{prim}} \oplus C \cdot [\omega_h]_{BC}, \tag{26}
\]

where the \( \omega \)-primitive subspace \( H^{1,1}_{BC}(X, \mathbb{C})_{\omega-\text{prim}} \) is a complex hyperplane of \( H^{1,1}_{BC}(X, \mathbb{C}) \) depending only on the cohomology class \( [\omega_{n-1}]_A \in H^{n-1,n-1}_A(X, \mathbb{C}) \), while \( \omega_h \) is the \( \Delta_{BC, \omega} \)-harmonic component of \( \omega \) and the complex line \( C \cdot [\omega_h]_{BC} \) depends on the choice of the Gauduchon metric \( \omega \).

We also have the following analogue of Lemma 2.9.

**Lemma 2.18.** The assumptions are the same as in Conclusion 2.17. For every \( \alpha \in C^\infty_{1,1}(X, \mathbb{C}) \cap \ker d \), the coefficient of \( [\omega_h]_{BC} \) in the Lefschetz-type decomposition of \( [\alpha]_{BC} \in H^{1,1}_{BC}(X, \mathbb{C}) \) according to (26), namely in

\[
[\alpha]_{BC} = [\alpha]_{BC,\text{prim}} + \lambda [\omega_h]_{BC}, \tag{27}
\]

is given by

\[
\lambda = \lambda_\omega([\alpha]_{BC}) = \frac{[\omega_{n-1}]_A[\alpha]_{BC}}{||\omega_h||_\omega^2} = \frac{1}{||\omega_h||_\omega^2} \int_X \alpha \wedge \omega_{n-1}. \tag{28}
\]

As in §2.3, formula (28) implies that \( \lambda_\omega([\alpha]_{BC}) \) is real if the class \( [\alpha]_{BC} \in H^{1,1}_{BC}(X, \mathbb{C}) \) is real. Thus, we can define a positive side and a negative side of the hyperplane \( H^{1,1}_{BC}(X, \mathbb{R})_{\omega-\text{prim}} \) := \( H^{1,1}_{BC}(X, \mathbb{C})_{\omega-\text{prim}} \cap H^{1,1}_{BC}(X, \mathbb{R}) \) in \( H^{1,1}_{BC}(X, \mathbb{R}) \) by

\[
H^{1,1}_{BC}(X, \mathbb{R})^+ := \left\{ [\alpha]_{BC} \in H^{1,1}_{BC}(X, \mathbb{R}) \mid \lambda_\omega([\alpha]_{BC}) > 0 \right\},
\]

\[
H^{1,1}_{BC}(X, \mathbb{R})^- := \left\{ [\alpha]_{BC} \in H^{1,1}_{BC}(X, \mathbb{R}) \mid \lambda_\omega([\alpha]_{BC}) < 0 \right\}.
\]

These are open subsets of \( H^{1,1}_{BC}(X, \mathbb{R}) \) that depend only on the cohomology class \( [\omega_{n-1}]_A \in H^{n-1,n-1}_A(X, \mathbb{R}) \).

Since \( [\alpha]_{BC} \) is \( \omega \)-primitive if and only if \( \lambda_\omega([\alpha]_{BC}) = 0 \), we get a partition of \( H^{1,1}_{BC}(X, \mathbb{R}) \):

\[
H^{1,1}_{BC}(X, \mathbb{R}) = H^{1,1}_{BC}(X, \mathbb{R})^+ \cup H^{1,1}_{BC}(X, \mathbb{R})_{\omega-\text{prim}} \cup H^{1,1}_{BC}(X, \mathbb{R})^-\]

depending only on the cohomology class \( [\omega_{n-1}]_A \in H^{n-1,n-1}_A(X, \mathbb{R}) \).

As a consequence of these considerations, we get
Proposition 2.19. Let $X$ be a compact complex manifold with $\dim \mathbb{C} X = n$. The pseudo-effective cone $E_X \subset H^{1,1}_{BC}(X, \mathbb{R})$ of $X$ is the intersection of all the non-negative sides

$$H^{1,1}_{BC}(X, \mathbb{R})_{\omega}^\geq := H^{1,1}_{BC}(X, \mathbb{R})^+ \cup H^{1,1}_{BC}(X, \mathbb{R})_{\omega \text{-prim}}$$

of hyperplanes $H^{1,1}_{BC}(X, \mathbb{R})_{\omega \text{-prim}}$ determined by Aeppli-Gauduchon classes $[\omega_{n-1}]_A \in \mathcal{G}_X$:

$$E_X = \bigcap_{[\omega_{n-1}]_A \in \mathcal{G}_X} H^{1,1}_{BC}(X, \mathbb{R})_{\omega}^\geq, \tag{29}$$

Proof. By the duality between the pseudo-effective cone $E_X$ and the closure $\overline{\mathcal{G}}_X$ of the Gauduchon cone (see §2.1), we know that a given class $[T]_{BC} \in H^{1,1}_{BC}(X, \mathbb{R})$ lies in $E_X$ (i.e. $[T]_{BC}$ can be represented by a closed semi-positive (1, 1)-current) if and only if

$$\int_X T \wedge \omega_{n-1} \geq 0 \quad \text{for all} \ [\omega_{n-1}]_A \in \mathcal{G}_X.$$

The last condition is equivalent to $\lambda_\omega([T]_{BC}) \geq 0$, hence to $[T]_{BC} \in H^{1,1}_{BC}(X, \mathbb{R})_{\omega}^\geq$, for all $[\omega_{n-1}]_A \in \mathcal{G}_X$, so the contention follows. \qed

Based on these considerations, we propose Question 1.3 as a problem for further study.

3 \ Properties of balanced hyperbolic manifolds

The discussion of balanced hyperbolic manifolds featured in this section will mirror that of degenerate balanced manifolds of the previous section.

3.1 \ Background and $L^1$ currents on the universal cover

It is a classical fact due to Gaffney [Gaf54] that certain basic facts in the Hodge Theory of compact Riemannian manifolds remain valid on complete such manifolds. The main ingredient in the proof of this fact is the following cut-off trick of Gaffney’s that played a key role in [Gro91, §1]. It also appears in [Dem97, VIII, Lemma 2.4].

Lemma 3.1. ([Gaf54]) Let $(X, g)$ be a Riemannian manifold. Then, $(X, g)$ is complete if and only if there exists an exhaustive sequence $(K_\nu)_{\nu \in \mathbb{N}}$ of compact subsets of $X$:

$$K_\nu \subset K_{\nu+1} \quad \text{for all} \ \nu \in \mathbb{N} \quad \text{and} \quad X = \bigcup_{\nu \in \mathbb{N}} K_\nu,$$

and a sequence $(\psi_\nu)_{\nu \in \mathbb{N}}$ of $C^\infty$ functions $\psi_\nu : X \rightarrow [0, 1]$ satisfying, for every $\nu \in \mathbb{N}$, the conditions:

$$\psi_\nu = 1 \text{ in a neighbourhood of } K_\nu, \quad \text{Supp} \psi_\nu \subset K_{\nu+1} \quad \text{and} \quad ||d\psi_\nu||_{L^\infty} := \sup_{x \in X} |(d\psi_\nu)(x)|_g \leq \varepsilon_\nu,$$

for some constants $\varepsilon_\nu > 0$ such that $\varepsilon_\nu \downarrow 0$ as $\nu$ tends to $+\infty$. 18
In particular, the cut-off functions $\psi_\nu$ are compactly supported. One can choose $\varepsilon_\nu = 2^{-\nu}$ for each $\nu$ (see e.g. [Dem97, VIII, Lemma 2.4]), but this will play no role here.

An immediate consequence of Gaffney’s cut-off trick is the following classical generalisation of Stokes’s Theorem to possibly non-compact, but complete Riemannian manifolds when the forms involved are $L^1$.

**Lemma 3.2.** ([Gro91, Lemma 1.1.A.]) Let $(X, g)$ be a complete Riemannian manifold of real dimension $m$. Let $\eta$ be an $L^1_g$-form on $X$ of degree $m - 1$ such that $d\eta$ is again $L^1_g$. Then

$$\int_X d\eta = 0.$$

By the form $\eta$ being $L^1$ with respect to the Riemannian metric $g$ ($L^1_g$ for short) we mean that its $L^1$-norm is finite:

$$||\eta||_{L^1_g} := \int_X |\eta(x)|_g dV_g(x) < +\infty,$$

where $dV_g$ is the volume form induced by $g$.

**Proof of Lemma 3.2.** Let $(\psi_\nu)_{\nu \in \mathbb{N}}$ be a sequence of cut-off functions as in Lemma 3.1 whose existence is guaranteed by the completeness of $(X, g)$. The $(m - 1)$-form $\psi_\nu \eta$ is compactly supported for every $\nu \in \mathbb{N}^*$, so the usual Stokes’s Theorem yields:

$$\int_X d(\psi_\nu \eta) = 0, \quad \nu \in \mathbb{N}^*.$$

Meanwhile, $d(\psi_\nu \eta) = d\psi_\nu \wedge \eta + \psi_\nu \ d\eta$, so we get:

$$\left| \int_X \psi_\nu \ d\eta \right| = \left| \int_X d\psi_\nu \wedge \eta \right| \leq ||d\psi_\nu||_{L^\infty} ||\eta||_{L^1_g} \leq \varepsilon_\nu ||\eta||_{L^1_g}, \quad \nu \in \mathbb{N},$$

for some sequence of constants $\varepsilon_\nu \downarrow 0$.

Since $\eta$ is $L^1_g$, $\varepsilon_\nu ||\eta||_{L^1_g} \downarrow 0$ as $\nu \to +\infty$. On the other hand, since $d\eta$ is $L^1_g$, the properties of the functions $\psi_\nu$ imply that

$$\lim_{\nu \to +\infty} \int_X \psi_\nu \ d\eta = \int_X d\eta.$$

Together with (30), these arguments yield $\int_X d\eta = 0$, as desired. \(\square\)

We now apply this standard cut-off function technique to prove Proposition 1.4 stated in the introduction. It is an analogue in our more general context of balanced hyperbolic manifolds of Proposition 5.4 in [Pop15] according to which compact degenerate balanced manifolds are characterised by the absence of non-zero $d$-closed positive $(1, 1)$-currents.

Note that, due to $X$ being compact, any pair of Hermitian metrics $\omega_1$ and $\omega_2$ on $X$ are comparable in the sense that there exists a constant $C > 0$ such that $(1/C) \omega_1 \leq \omega_2 \leq C \omega_1$. Thus, their lifts $\tilde{\omega}_1 := \pi^* \omega_1$ and $\tilde{\omega}_2 := \pi^* \omega_2$ are again comparable on $\tilde{X}$ by means of the same constant: $(1/C) \tilde{\omega}_1 \leq \tilde{\omega}_2 \leq C \tilde{\omega}_1$. Therefore, the $L^1_{\tilde{\omega}}$-assumption on $\tilde{T}$ is independent of the choice of Hermitian metric on $\tilde{X}$ if this metric is obtained by lifting a metric on $X$. However, the $L^1$-condition changes
for metrics on $\tilde{X}$ that are not lifts of metrics on $X$. But we will not deal with the latter type of metrics.

Proof of Proposition 1.4. Let $n = \dim_{\mathbb{C}}X$. The balanced hyperbolic assumption on $X$ means that $\pi^*\omega_{n-1} = d\tilde{\Gamma}$ on $\tilde{X}$ for some smooth $L^\infty_{\omega}$-form $\tilde{\Gamma}$ of degree $(2n - 3)$ on $\tilde{X}$.

If a current $\tilde{T}$ as in the statement existed on $\tilde{X}$, we would have

$$0 < \int_{\tilde{X}} \tilde{T} \wedge \pi^*\omega_{n-1} = \int_{\tilde{X}} d(\tilde{T} \wedge \tilde{\Gamma}) = 0,$$

which is contradictory.

The last identity in (31) follows from Lemma 3.2 applied on the complete manifold $(\tilde{X}, \tilde{\omega})$ to the $L^1_{\omega}$-current $\eta := \tilde{T} \wedge \tilde{\Gamma}$ of degree $2n - 1$ whose differential $d\eta = \tilde{T} \wedge \pi^*\omega_{n-1}$ is again $L^1_{\omega}$. That $\eta$ is $L^1_{\omega}$ follows from $\tilde{T}$ being $L^1_{\omega}$ (by hypothesis) and from $\tilde{\Gamma}$ being $L^\infty_{\omega}$, while $d\eta$ being $L^1_{\omega}$ follows from $\tilde{T}$ being $L^1_{\omega}$ and from $\pi^*\omega_{n-1}$ being $L^\infty_{\omega}$ (as a lift of the smooth, hence bounded, form $\omega_{n-1}$ on the compact manifold $X$).

We now recall the following standard result saying that some further key facts in the Hodge Theory of compact Riemannian manifolds remain valid on complete such manifolds $X$ when the differential operators involved (e.g. $d$, $d^*$, $\Delta$) are considered as closed and densely defined unbounded operators on the spaces $L^k_\omega(X, \mathbb{C})$ of $L^\omega$-forms of degree $k$ on $X$. The only major property that is lost in passing to complete manifolds is the closedness of the images of these operators. As usual, any differential operator $P$ originally defined on $C^\infty_\omega(X, \mathbb{C})$ is extended to an unbounded operator on $L^\omega_\omega(X, \mathbb{C})$ by defining its domain $\text{Dom} P$ as the space of $L^\omega$-forms $u$ such that $Pu$, computed in the sense of distributions, is again $L^\omega$.

**Theorem 3.3.** (see e.g. [Dem97, VIII, Theorem 3.2.]) Let $(X, g)$ be a complete Riemannian manifold of real dimension $m$. Then:

(a) The space $\mathcal{D}_\omega(X, \mathbb{C})$ of compactly supported $C^\infty$ forms of any degree (indicated by a •) on $X$ is dense in the domains $\text{Dom} d$, $\text{Dom} d^*$ and in $\text{Dom} d \cap \text{Dom} d^*$ for the respective graph norms:

$$u \mapsto ||u|| + ||du||, \quad u \mapsto ||u|| + ||d^*u||, \quad u \mapsto ||u|| + ||du|| + ||d^*u||.$$

(b) The extension $d^*$ of the formal adjoint of $d$ to the $L^2$-space coincides with the Hilbert space adjoint of the extension of $d$.

(c) The $d$-Laplacian $\Delta = \Delta_g := dd^* + d^*d$ has the following property:

$$\langle \langle \Delta u, u \rangle \rangle = ||du||^2 + ||d^*u||^2$$

for every form $u \in \text{Dom} \Delta$. In particular, $\text{Dom} \Delta \subset \text{Dom} d \cap \text{Dom} d^*$ and $\ker \Delta = \ker d \cap \ker d^*$.

(d) There are $L^2$-orthogonal decompositions in every degree (indicated by a •):

$$L^2_\omega(X, \mathbb{C}) = \mathcal{H}^\omega_\Delta(X, \mathbb{C}) \oplus \text{Im} d \oplus \text{Im} d^*$$

$$\ker d = \mathcal{H}^\omega_\Delta(X, \mathbb{C}) \oplus \text{Im} d \quad \text{and} \quad \ker d^* = \mathcal{H}^\omega_\Delta(X, \mathbb{C}) \oplus \text{Im} d^*,$$

where $\mathcal{H}^\omega_\Delta(X, \mathbb{C}) := \{u \in L^2_\omega(X, \mathbb{C}) \mid \Delta u = 0\}$ is the space of $\Delta$-harmonic $L^2$-forms, while

$$\text{Im} d := L^2_\omega(X, \mathbb{C}) \cap d(L^2_{\omega-1}(X, \mathbb{C})) \quad \text{and} \quad \text{Im} d^* := L^2_\omega(X, \mathbb{C}) \cap d^*(L^2_{\omega+1}(X, \mathbb{C})).$$
An immediate consequence of (32) applied in degree 0 is that on a connected complete Riemannian manifold \((X, g)\), every \(\Delta\)-harmonic \(L^2\)-function is constant:

\[
\mathcal{H}_\Delta^0(X, \mathbb{C}) \subset \mathbb{C}.
\]  

(34)

3.2 Harmonic \(L^2\)-forms of degree 1 on the universal cover of a balanced hyperbolic manifold

Let \(X\) be a possibly non-compact complex manifold with \(\dim_{\mathbb{C}} X = n\), supposed to carry a complete balanced metric \(\omega\). In subsequent applications, the roles of \(X\) and \(\omega\) will be played by \(\tilde{X}\), the universal cover \(\pi : \tilde{X} \longrightarrow X\) of a compact balanced hyperbolic manifold \((X, \omega)\), resp. \(\tilde{\omega} := \pi^*\omega\).

A well-known consequence of the Kähler commutation relations is the fact that, if \(\omega\) is Kähler, the induced \(d\)-Laplacian \(\Delta = \Delta_\omega\) commutes with the multiplication operator \(\omega^l \wedge \cdot\) acting on differential forms of any degree on \(X\), for every \(l\).

We will see that, when \(\omega\) is merely balanced, the commutation of \(\Delta\) with the multiplication operator \(\omega^{n-1} \wedge \cdot\) acting on differential forms no longer holds. However, we will now compute this commutation defect on 1-forms.

The computation will continue that of (i) in Lemma 2.1. For the sake of generality and for a reason that will become apparent later on, we will work with the more general operators

\[
d_h := h\partial + \bar{\partial}, \quad h \in \mathbb{C}^*,
\]

acting on \(\mathbb{C}\)-valued differential forms on \(X\) and the associated Laplacians \(\Delta_h := d_h d_h^* + d_h^* d_h\).

The first stages of the computation lead to the following result in which no completeness assumption is necessary.

**Lemma 3.4.** Let \(X\) be a complex manifold with \(\dim_{\mathbb{C}} X = n\). Suppose there exists a balanced metric \(\omega\) on \(X\). Then, for any \(h \in \mathbb{C}^*\) and any 1-form \(\varphi\) on \(X\), the following identity holds:

\[
[\Delta_h, L_{\omega_{n-1}}]\varphi = \left(|h|^2 d_{\frac{1}{n}} d_h^* d_{\frac{1}{n}} - d_h^* d_h\right)\varphi \wedge \omega_{n-1} - i\bar{\varphi} d_{\frac{1}{n}} \varphi \wedge (d_h \omega_{n-2} - i(|h|^2 + 1) \partial \bar{\partial}\varphi) \wedge \omega_{n-2}. \tag{35}
\]

**Proof.** • The Jacobi identity yields:

\[
[[d_h, d_h^*], L_{\omega_{n-1}}] - [[d_h^*, L_{\omega_{n-1}}], d_h] + [[L_{\omega_{n-1}}, d_h], d_h^*] = 0.
\]

Since \(\omega\) is balanced, \([L_{\omega_{n-1}}, d_h] = 0\). Writing \(\Delta_h = [d_h, d_h^*]\), the above equality reduces to

\[
[\Delta_h, L_{\omega_{n-1}}] = [d_h^*, L_{\omega_{n-1}}] d_h + d_h [d_h^*, L_{\omega_{n-1}}]. \tag{36}
\]

• Note also the following formula for the **formal adjoint** of \(d_h\) involving the Hodge star operator:

\[
d_h^* = -\bar{h} \star \frac{1}{n} \star. \tag{37}
\]

Indeed, \(d_h^* = (h\partial + \bar{\partial})^* = \bar{h} (-\star \bar{\partial} \star) + (-\star \partial \star) = -\bar{h} \star (\frac{1}{n} \partial + \bar{\partial}) \star = -\bar{h} \star \frac{1}{n} \star.\) No assumption on \(\omega\) is needed here.
• As an application of (37), we observe the following formula for every $(1, 1)$-form $\alpha$:

$$d_h^*(\omega_{n-1} \land \alpha) = -i\hbar d_{-\frac{1}{\hbar}}(\Lambda_\omega \alpha) \land \omega_{n-1}. \quad (38)$$

Again, no assumption on $\omega$ is needed.

To see this, we first multiply the Lefschetz decomposition (8) of $\alpha$ by $\omega_{n-1}$ and we get: $\omega_{n-1} \land \alpha = (\Lambda_\omega \alpha) \omega_n$. Hence, $\ast(\omega_{n-1} \land \alpha) = \Lambda_\omega \alpha$, so we get the first equality below:

$$-\hbar \ast d_{-\frac{1}{\hbar}}(\omega_{n-1} \land \alpha) = -\hbar \ast d_{-\frac{1}{\hbar}}(\Lambda_\omega \alpha) = -\hbar \ast \left(\frac{1}{\hbar} \partial(\Lambda_\omega \alpha)\right) - \hbar \ast \bar{\partial}(\Lambda_\omega \alpha).$$

Applying (37) to the l.h.s. term above and the standard formula (7) to the r.h.s. term, we get:

$$d_h^*(\omega_{n-1} \land \alpha) = i\partial(\Lambda_\omega \alpha) \land \omega_{n-1} - i\hbar \bar{\partial}(\Lambda_\omega \alpha) \land \omega_n.$$ 

Since $i\partial - i\hbar \bar{\partial} = -i\hbar d_{-\frac{1}{\hbar}}$, the above equality is nothing but (38).

- **Computation of the first term on the r.h.s. of (36) on 1-forms $\varphi = \varphi^{0,1} + \varphi^{0,1}$.**

Using formula (38) with $\alpha := h\varphi^{0,1} + \bar{\partial}\varphi^{1,0}$, we get the second equality below:

$$[d_h^*, L_{\omega_{n-1}}] d_h\varphi = d_h^*(\omega_{n-1} \land (h\varphi^{0,1} + \bar{\partial}\varphi^{1,0})) - \omega_{n-1} \land d_h^*d_h\varphi$$

$$= -i\hbar d_{-\frac{1}{\hbar}}\left(h\Lambda_\omega(\varphi^{0,1}) + \Lambda_\omega(\bar{\partial}\varphi^{1,0})\right) \land \omega_{n-1} - d_h^*d_h\varphi \land \omega_{n-1}. \quad (39)$$

On the other hand, the standard formula (7) yields:

$$\ast \varphi = i(\varphi^{0,1} - \varphi^{1,0}) \land \omega_{n-1}.$$ 

Since $\omega$ is balanced, this implies the first equality on each of the two rows below:

$$\partial \ast \varphi = i\partial(\varphi^{0,1} - \varphi^{1,0}) \land \omega_{n-1} = i\varphi^{0,1} \land \omega_{n-1} = i\Lambda_\omega(\varphi^{0,1}) \omega_n$$

$$\bar{\partial} \ast \varphi = i\bar{\partial}(\varphi^{0,1} - \varphi^{1,0}) \land \omega_{n-1} = -i\varphi^{1,0} \land \omega_{n-1} = -i\Lambda_\omega(\varphi^{1,0}) \omega_n.$$ 

Taking $-\ast$ in each of the above two equalities and using the standard identities $-\ast \partial \ast = \bar{\partial} \ast$, $-\ast \bar{\partial} \ast = \partial \ast$, we get:

$$\bar{\partial} \ast \varphi = -i\Lambda_\omega(\varphi^{0,1}) \quad \text{and} \quad \partial \ast \varphi = i\Lambda_\omega(\bar{\partial}\varphi^{1,0}). \quad (40)$$

Putting together (39) and (40), we get:

$$[d_h^*, L_{\omega_{n-1}}] d_h\varphi = -i\hbar d_{-\frac{1}{\hbar}}(ih\bar{\partial} \ast \varphi - i\partial \ast \varphi) \land \omega_{n-1} - d_h^*d_h\varphi \land \omega_{n-1}$$

$$= hh d_{-\frac{1}{h}}d_{-\frac{1}{h}}\varphi \land \omega_{n-1} - d_h^*d_h\varphi \land \omega_{n-1}.$$ 

We have thus obtained the following formula:

$$[d_h^*, L_{\omega_{n-1}}] d_h\varphi = \left(|\hbar|^2d_{-\frac{1}{\hbar}}d_{-\frac{1}{\hbar}} - d_h^*d_h\right)\varphi \land \omega_{n-1} \quad (41)$$

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for every smooth 1-form \( \varphi \) whenever the metric \( \omega \) is balanced.

- Computation of the second term on the r.h.s. of (36) on 1-forms \( \varphi = \varphi^{1,0} + \varphi^{0,1} \).

We start by computing

\[
[d^*_h, L_{\omega_{n-1}}] \varphi = d^*_h(\omega_{n-1} \wedge \varphi) - (d^*_h \varphi) \omega_{n-1}.
\] (42)

Since \( \omega_{n-1} \wedge \varphi^{1,0} = i \ast \varphi^{1,0} \) and \( \omega_{n-1} \wedge \varphi^{0,1} = -i \ast \varphi^{0,1} \), formula (37) yields the first line below:

\[
d^*_h(\omega_{n-1} \wedge \varphi) = i \check{h} \ast d^*_h \varphi = i \ast (\partial \varphi^{1,0} + \check{h} \bar{\varphi}^{1,0} - \partial \varphi^{0,1} - \check{h} \bar{\varphi}^{0,1})
\]

\[
+ i(\partial \varphi^{1,0} - \check{h} \bar{\varphi}^{0,1}) \wedge \omega_{n-2},
\] (43)

where we used the Lefschetz decomposition (8) of the \((1, 1)\)-form \( \check{h} \bar{\varphi}^{1,0} - \partial \varphi^{0,1} \) and then the standard formula (7) to express the value of \( \ast \) on the primitive forms \( \partial \varphi^{1,0} \) (of type \((2, 0)\)), \( \partial \varphi^{0,1} \) (of type \((0, 2)\)) and \( (\check{h} \bar{\varphi}^{1,0} - \partial \varphi^{0,1})_{\text{prim}} \) (of type \((1, 1)\)) and got \( \ast(\partial \varphi^{1,0}) = \partial \varphi^{1,0} \wedge \omega_{n-2} \) and

\[
\ast(\bar{\partial} \varphi^{0,1}) = \bar{\partial} \varphi^{0,1} \wedge \omega_{n-2},
\]

\[
\ast((\check{h} \bar{\varphi}^{1,0} - \partial \varphi^{0,1})_{\text{prim}} = -((\check{h} \bar{\varphi}^{1,0} - \partial \varphi^{0,1})_{\text{prim}} \wedge \omega_{n-2}.
\]

On the other hand, we get

\[
d^*_h \varphi = -\check{h} \ast d^*_h (\varphi^{1,0} + \varphi^{0,1}) = -\check{h} \ast (\frac{1}{\check{h}} \partial + \bar{\partial})(-i \varphi^{1,0} \wedge \omega_{n-1} + i \varphi^{0,1} \wedge \omega_{n-1})
\]

\[
= -\check{h} \ast \left( -\frac{i}{\check{h}} \partial \varphi^{1,0} \wedge \omega_{n-1} + \frac{i}{\check{h}} \partial \varphi^{0,1} \wedge \omega_{n-1} - i \bar{\partial} \varphi^{1,0} \wedge \omega_{n-1} + i \bar{\partial} \varphi^{0,1} \wedge \omega_{n-1} \right)
\]

\[
= -\check{h} \ast \left( i \left( \frac{1}{\check{h}} \partial \varphi^{0,1} - \bar{\partial} \varphi^{1,0} \right) \wedge \omega_{n-1} \right) = -\check{h} \ast (i \Lambda_{\omega} (\frac{1}{\check{h}} \partial \varphi^{0,1} - \bar{\partial} \varphi^{1,0}) \wedge \omega_{n-1})
\]

\[
= i \Lambda_{\omega} (\check{h} \bar{\varphi}^{1,0} - \partial \varphi^{0,1}),
\] (44)

where the balanced assumption on \( \omega \) was used to get (i) and the equalities \( \partial \varphi^{1,0} \wedge \omega_{n-1} = \bar{\partial} \varphi^{0,1} \wedge \omega_{n-1} = 0 \), that hold for bidegree reasons, were used to get (ii).

Noticing that the last term in (44) also features within the first term on the second line in (43), the conclusion of (43) can be re-written as

\[
d^*_h(\omega_{n-1} \wedge \varphi) = \frac{1}{n} (d^*_h \varphi) \omega_{n-1} - i(\check{h} \bar{\varphi}^{1,0} - \partial \varphi^{0,1})_{\text{prim}} \wedge \omega_{n-2} + i(\partial \varphi^{1,0} - \check{h} \bar{\varphi}^{0,1}) \wedge \omega_{n-2}.
\] (45)

From this and from (42), we get:

\[
[d^*_h, L_{\omega_{n-1}}] \varphi = \left( \frac{1}{n} - 1 \right) (d^*_h \varphi) \omega_{n-1} - i(\check{h} \bar{\varphi}^{1,0} - \partial \varphi^{0,1})_{\text{prim}} \wedge \omega_{n-2} + i(\partial \varphi^{1,0} - \check{h} \bar{\varphi}^{0,1}) \wedge \omega_{n-2}.
\]

Hence, using the balanced hypothesis \( d^*_h \omega = 0 \), we get the first two lines below:

\[
d^*_h[d^*_h, L_{\omega_{n-1}}] \varphi = \left( \frac{1}{n} - 1 \right) dd^*_h \varphi \wedge \omega_{n-1} - id^*_h \left( (\check{h} \bar{\varphi}^{1,0} - \partial \varphi^{0,1}) \wedge \omega_{n-2} \right)
\]

\[
+ \frac{n-1}{n} i d^*_h (\Lambda_{\omega} (\check{h} \bar{\varphi}^{1,0} - \partial \varphi^{0,1})) \wedge \omega_{n-1}
\]

\[
+ i(\partial \varphi^{1,0} - \check{h} \bar{\varphi}^{0,1}) \wedge d^*_h \omega_{n-2} - i(\check{h})^2 \partial \bar{\partial} \varphi^{0,1} + \bar{\partial} \partial \varphi^{1,0}) \wedge \omega_{n-2}.
\]
Now, formula (44) shows that the term on the second line above equals minus the first term on the r.h.s. of the first line. Hence, the sum of these two terms vanishes and we get:

\[ d_h[\varphi^*] \big|_{L_{\omega_{n-1}}} \varphi = i \left( \partial \varphi^{1,0} + (\partial \varphi^{0,1} - \bar{h} \bar{\partial} \varphi^{1,0}) - \bar{h} \bar{\partial} \varphi^{0,1} \right) \wedge d_h \omega_{n-2} - i(|h|^2 + 1) \partial \bar{\partial} \varphi \wedge \omega_{n-2} \]

\[ = -i\bar{h} d_{\frac{1}{h}} \varphi \wedge d_h \omega_{n-2} - i(|h|^2 + 1) \partial \bar{\partial} \varphi \wedge \omega_{n-2}. \]

(46)

- **Conclusion.**

Putting together (36), (41) and (46), we get (35). The proof of Lemma 3.4 is complete. □

Recall that for any Hermitian metric $\omega$ on an $n$-dimensional complex manifold $X$, the pointwise Lefschetz map:

\[ L_{\omega_{n-1}} : \Lambda^1 T^* X \rightarrow \Lambda^{2n-1} T^* X, \quad \varphi \mapsto \psi := \omega_{n-1} \wedge \varphi, \]

is bijective and a quasi-isometry (in the sense of Lemma 4.3).

We will now integrate the result of Lemma 3.4 expressing the commutation defect between $\Delta_h$ and $L_{\omega_{n-1}}$ on 1-forms. We need to assume our balanced metric $\omega$ to be complete to ensure that the two meanings of $d_h^*$ coincide and the $L^2_\omega$ inner products can be handled as in the compact case (see (b) and (c) of Theorem 3.3).

**Proposition 3.5.** Let $X$ be a complex manifold with $\text{dim}_\mathbb{C} X = n$. Suppose there exists a complete balanced metric $\omega$ on $X$.

Then, for any $h \in \mathbb{C}^*$ and any 1-form $\varphi \in \text{Dom}(\Delta_{\frac{1}{h}})$ on $X$, the following identity holds:

\[ \langle\langle \Delta_h (\omega_{n-1} \wedge \varphi), \omega_{n-1} \wedge \varphi \rangle\rangle = |h|^2 \langle\langle \Delta_{\frac{1}{h}} \varphi, \varphi \rangle\rangle. \]

(47)

**Proof.** Throughout the proof, $\varphi$ will stand for an arbitrary smooth 1-form on $X$.

- We first notice that $d_h d_{\frac{1}{h}} \varphi = ((|h|^2 + 1)/\bar{h}) \partial \bar{\partial} \varphi$, hence

\[ d_h \left(-i\bar{h} d_{\frac{1}{h}} \varphi \wedge \omega_{n-2}\right) = -i\bar{h} d_{\frac{1}{h}} \varphi \wedge d_h \omega_{n-2} - i(|h|^2 + 1) \partial \bar{\partial} \varphi \wedge \omega_{n-2}. \]

These are the last two terms of formula (35).

Putting $\psi := \omega_{n-1} \wedge \varphi$ and using (35) with its last two terms transformed as above, we get:

\[ \langle\langle \Delta_h \psi, \psi \rangle\rangle = \langle\langle \Delta_h \varphi \wedge \omega_{n-1}, \varphi \wedge \omega_{n-1} \rangle\rangle + \langle\langle |h|^2 d_{\frac{1}{h}} d_{\frac{1}{h}} \varphi \wedge \omega_{n-1}, \varphi \wedge \omega_{n-1} \rangle\rangle 
- i\bar{h} \langle\langle d_{\frac{1}{h}} \varphi \wedge \omega_{n-2}, d_h^* (\omega_{n-1} \wedge \varphi) \rangle\rangle 
= \langle\langle d_h d_{\frac{1}{h}} \varphi \wedge \omega_{n-1}, \varphi \wedge \omega_{n-1} \rangle\rangle + \langle\langle |h|^2 d_{\frac{1}{h}} d_{\frac{1}{h}} \varphi \wedge \omega_{n-1}, \varphi \wedge \omega_{n-1} \rangle\rangle 
- i\bar{h} \langle\langle d_{\frac{1}{h}} \varphi \wedge \omega_{n-2}, d_h^* (\omega_{n-1} \wedge \varphi) \rangle\rangle 
\overset{(\dagger)}{=} \langle\langle d_h d_{\frac{1}{h}} \varphi, \varphi \rangle\rangle + |h|^2 \langle\langle d_{\frac{1}{h}} d_{\frac{1}{h}} \varphi, \varphi \rangle\rangle - i\bar{h} \langle\langle d_{\frac{1}{h}} \varphi \wedge \omega_{n-2}, d_h^* (\omega_{n-1} \wedge \varphi) \rangle\rangle 
= \|d_h^* \varphi\|^2 + |h|^2 \|d_{\frac{1}{h}} \varphi\|^2 - i\bar{h} \langle\langle d_{\frac{1}{h}} \varphi \wedge \omega_{n-2}, d_h^* (\omega_{n-1} \wedge \varphi) \rangle\rangle. \]

(48)
where (i) followed from Lemma 4.1 applied to the (necessarily primitive) 1-forms \( \varphi \), \( d_hd_h^*\varphi \) and \( d_{-\frac{1}{h}}d_{-\frac{1}{h}}^*\varphi \).

- We now transform the last term in (48), namely \( T(\varphi) := -i\hbar \langle \langle d_{-\frac{1}{h}}\varphi \wedge \omega_{n-2}, d_h^*(\omega_{n-1} \wedge \varphi) \rangle \rangle \).

Since the multiplication map \( \omega_{n-2} \wedge \cdot : \Lambda^2 T^*X \rightarrow \Lambda^{2n-2} T^*X \) is bijective, there exists a unique 2-form \( \beta \) such that \( d_h^*(\omega_{n-1} \wedge \varphi) = \omega_{n-2} \wedge \beta \). Thus, using (45) for the second equality below, we get:

\[
\omega_{n-2} \wedge \beta = d_h^*(\omega_{n-1} \wedge \varphi) = \omega_{n-2} \wedge \left( \frac{1}{n(n-1)} (d_h^*\varphi) \omega - i(\hbar \partial \varphi^{1,0} - \partial \varphi^{0,1})_{\text{prim}} + i(\partial \varphi^{1,0} - \hbar \partial \varphi^{0,1}) \right).
\]

The uniqueness of \( \beta \) implies that

\[
\beta = \left( -i(\hbar \partial \varphi^{1,0} - \partial \varphi^{0,1})_{\text{prim}} + i(\partial \varphi^{1,0} - \hbar \partial \varphi^{0,1}) \right) + \frac{1}{n(n-1)} (d_h^*\varphi) \omega, \quad (49)
\]

In particular, the primitive part \( \beta_{\text{prim}} \) of \( \beta \) in the Lefschetz decomposition is the form inside the large parenthesis and \( \Lambda_{\omega, \beta} = \frac{1}{n-1} d_h^*\varphi \).

On the other hand, we have

\[
d_{-\frac{1}{h}}\varphi = -\frac{1}{\hbar} \partial \varphi^{1,0} + (-\frac{1}{\hbar} \partial \varphi^{0,1} + \bar{\partial} \varphi^{1,0})_{\text{prim}} + \bar{\partial} \varphi^{0,1} + \frac{1}{n\hbar} (d_h^*\varphi) \omega, \quad (50)
\]

where the value of the last term follows from formula (44). This implies that \( \beta \) and \( -i\hbar d_{-\frac{1}{h}}\varphi \) have the same primitive part:

\[
\beta_{\text{prim}} = -i\hbar (d_{-\frac{1}{h}}\varphi)_{\text{prim}}. \quad (51)
\]

We get:

\[
\langle d_{-\frac{1}{h}}\varphi \wedge \omega_{n-2}, d_h^*(\omega_{n-1} \wedge \varphi) \rangle = \langle d_{-\frac{1}{h}}\varphi \wedge \omega_{n-2}, \beta \wedge \omega_{n-2} \rangle
\]

\[
= \langle (d_{-\frac{1}{h}}\varphi)_{\text{prim}}, \beta_{\text{prim}} \rangle + (n-1)^2 n \left( \frac{1}{n\hbar} d_h^*\varphi, \frac{1}{n(n-1)} d_h^*\varphi \right),
\]

where the last equality follows from formula (74) in the Appendix.

From this and from (49)-(51), we get:

\[
T(\varphi) = -i\hbar \langle \langle d_{-\frac{1}{h}}\varphi \wedge \omega_{n-2}, d_h^*(\omega_{n-1} \wedge \varphi) \rangle \rangle
\]

\[
= ||\partial \varphi^{1,0}||^2 + ||(\partial \varphi^{0,1} - \hbar \bar{\partial} \varphi^{1,0})_{\text{prim}}||^2 + |\hbar|^2 ||\bar{\partial} \varphi^{0,1}||^2 - (1 - \frac{1}{n}) ||d_h^*\varphi||^2, \quad (52)
\]

- Putting (48) and (52) together and writing \( \frac{1}{n} ||d_h^*\varphi||^2 = |\hbar|^2 \frac{1}{n} ||d_{-\frac{1}{h}}\varphi||^2 \), we get:

\[
\langle \langle \Delta_h \psi, \psi \rangle \rangle = |\hbar|^2 ||d_{-\frac{1}{h}}\varphi||^2 + |\hbar|^2 \frac{1}{n} ||d_{-\frac{1}{h}}^*\varphi||^2
\]

\[
+ |\hbar|^2 \left( ||-\frac{1}{\hbar} \partial \varphi^{1,0}||^2 + ||(-\frac{1}{\hbar} \partial \varphi^{0,1} + \bar{\partial} \varphi^{1,0})_{\text{prim}}||^2 + ||\bar{\partial} \varphi^{0,1}||^2 \right).
\]
Thanks to the expression of $d_{-\frac{1}{h}}\varphi$ obtained in (50), this translates to
\[
\langle\langle \Delta_h \psi, \psi \rangle \rangle = |h|^2 \left( ||d^*_{-\frac{1}{h}}\varphi||^2 + ||d_{-\frac{1}{h}}\varphi||^2 \right) = |h|^2 \langle\langle \Delta_{-\frac{1}{h}}\varphi, \varphi \rangle \rangle,
\]
which is (47).

Proposition 3.5 is proved. □

An immediate consequence of Proposition 3.5 is the following Hard Lefschetz-type result for spaces of harmonic $L^2_\omega$-forms induced by a given complete balanced metric $\omega$ and different operators $\Delta_{-\frac{1}{h}}$ and $\Delta_h$. Note that $h \neq -\frac{1}{h}$ for all $h \in \mathbb{C}^\star$. This is the price we have to pay in the non-Kähler balanced context to get this kind of results.

**Corollary 3.6.** Let $X$ be a complex manifold with $\text{dim}_\mathbb{C} X = n$. Suppose there exists a complete balanced metric $\omega$ on $X$. Then, for any $h \in \mathbb{C}^\star$, the map
\[
\omega_{n-1} \wedge \cdot : \mathcal{H}_{-\frac{1}{h}}^\Delta (X, \mathbb{C}) \longrightarrow \mathcal{H}_{\Delta_h}^{2n-1} (X, \mathbb{C}), \quad \varphi \longmapsto \omega_{n-1} \wedge \varphi,
\]
is well-defined and an isomorphism.

**Proof.** The well-definedness, namely the fact that this map takes $\Delta_{-\frac{1}{h}}$-harmonic $L^2_\omega$-forms to $\Delta_h$-harmonic $L^2_\omega$-forms, follows at once from Proposition 3.5 and from the form $\omega_{n-1}$ being $\omega$-bounded. The fact that this map is an isomorphism follows from the standard fact that the corresponding pointwise map is bijective. □

**Corollary 3.7.** Let $X$ be a complex manifold with $\text{dim}_\mathbb{C} X = n$. Suppose there exists a complete balanced metric $\omega$ on $X$ such that $\omega_{n-1} = d\Gamma$ for an $\omega$-bounded smooth $(2n - 3)$-form $\Gamma$. Then
\[
\langle\langle \Delta \psi, \psi \rangle \rangle \geq \frac{1}{4||\Gamma||^2_{L^\infty_\omega}} ||\psi||^2
\]
fors every pure-type form $\psi \in \text{Dom}(\Delta)$ of degree $2n - 1$.

**Proof.** Taking $h = 1$ in Proposition 3.5, (47) gives:
\[
\langle\langle \Delta \psi, \psi \rangle \rangle = \langle\langle \Delta_{-1} \varphi, \varphi \rangle \rangle = ||(\partial - \bar{\partial})\varphi||^2 + ||(\partial - \bar{\partial})^*\varphi||^2 \geq ||(\partial - \bar{\partial})\varphi||^2,
\]
for every $(2n - 1)$-form $\psi$, where $\varphi$ is the unique 1-form such that $\psi = \omega_{n-1} \wedge \varphi$. (See isomorphism (69) for $r = 1$.) Meanwhile, $\psi$ is of pure type (either $(n, n - 1)$ or $(n - 1, n)$) if and only if $\varphi$ is of pure type (respectively, either $(1, 0)$ or $(0, 1)$). In this case, $\partial \varphi$ and $\bar{\partial} \varphi$ are of different pure types, hence orthogonal to each other, hence $||\varphi|| = ||(\partial + \bar{\partial})\varphi||$. Thus, we get:
\[
\langle\langle \Delta \psi, \psi \rangle \rangle \geq ||d\varphi||^2,
\]
for every pure-type $(2n - 1)$-form $\psi \in \text{Dom}(\Delta)$.

To complete the proof, we adapt the proof of Theorem 1.4.A. in [Gro91] to our context.
Since any 1-form \( \varphi \) is primitive, Lemma 4.1 gives: \( |\psi|^2 = |\omega_{n-1} \wedge \varphi|^2 = |\varphi|^2 \). In particular,
\[
||\psi|| = ||\varphi||.
\] (55)

Meanwhile, we have: \( \psi = \omega_{n-1} \wedge \varphi = d\Gamma \wedge \varphi = d(\Gamma \wedge \varphi) + \Gamma \wedge d\varphi \). In other words,
\[
\psi = d\theta + \psi', \quad \text{where } \theta := \Gamma \wedge \varphi \quad \text{and} \quad \psi' := \Gamma \wedge d\varphi.
\] (56)

To estimate \( \theta \), we write:
\[
||\theta|| \leq ||\Gamma||_{L^\infty} ||\varphi|| = ||\Gamma||_{L^\infty} ||\psi||,
\] (57)
where (55) was used to get the last equality.

To estimate \( \psi' \), we write:
\[
||\psi'|| \leq ||\Gamma||_{L^\infty} ||d\varphi|| \leq ||\Gamma||_{L^\infty} \langle\langle \Delta \psi, \psi \rangle\rangle^{1/2},
\] (58)
where (54) and the fact that \( \varphi \) is of pure type were used to get the last inequality.

To find an upper bound for \( ||\psi|| \), we write:
\[
||\psi||^2 = \langle\langle \psi, d\theta + \psi' \rangle\rangle \leq ||\langle\langle \psi, d\theta \rangle\rangle| + ||\langle\langle \psi, \psi' \rangle\rangle|,
\] (59)
where (56) was used to get the first equality.

For the first term on the r.h.s. of (59), we get:
\[
||\langle\langle \psi, d\theta \rangle\rangle|| = ||\langle\langle d^* \psi, \theta \rangle\rangle|| \leq ||d^* \psi|| ||\theta|| \leq \langle\langle \Delta \psi, \psi \rangle\rangle^{1/2} ||\Gamma||_{L^\infty} ||\psi||,
\] (60)
where (57) was used to get the last inequality.

For the second term on the r.h.s. of (59), we get:
\[
||\langle\langle \psi, \psi' \rangle\rangle|| \leq ||\psi'|| ||\psi|| \leq ||\Gamma||_{L^\infty} \langle\langle \Delta \psi, \psi \rangle\rangle^{1/2} ||\psi||,
\] (61)
where (58) was used to get the last inequality.

Adding up (60) and (61) and using (59), we get
\[
||\psi|| \leq 2 \langle\langle \Delta \psi, \psi \rangle\rangle^{1/2},
\]
which is (53). The proof is complete.

\(\square\)

For the record, if we do not assume \( \psi \) to be of pure type and use the full force of (47) rather than (54), we can run the argument in the proof of Corollary 3.7 with minor modifications starting from the observation that \( \omega_{n-1} = d_{-1} \Gamma_{-1} \), where \( \Gamma_h := h \Gamma^{n,n-3} + \Gamma^{n-1,n-2} + (1/h) \Gamma^{n-2,n-1} + (1/h^2) \Gamma^{n-3,n} \) for every \( h \in \mathbb{C}^* \) and the \( \Gamma^{p,q} \)'s are the pure-type components of \( \Gamma \). Then, we get the following analogue of (53):
\[
||\psi|| \leq C_h ||\Gamma_{-1}|| \left( \langle\langle \Delta_h \psi, \psi \rangle\rangle^{1/2} + \langle\langle \Delta_{-1} \psi, \psi \rangle\rangle^{1/2} \right)
\] (62)
for every form \( \psi \in \text{Dom}(\Delta_h) \cap \text{Dom}(\Delta_{-1}) \) (not necessarily of pure type) of degree \( 2n - 1 \), where \( C_h := \max(1, 1/|h|) \).
The occurrence of two different Laplacians on the r.h.s. of (62) (recall that $h \neq -\frac{1}{\bar{h}}$ for every $h \in \mathbb{C}^*$) is the downside of that estimate that we avoided in Corollary 3.7 by restricting attention to pure-type forms. The advantage of dealing with a single Laplacian is demonstrated by Theorem 1.5 in the introduction that we now prove as a consequence of the above discussion.

**Proof of Theorem 1.5.** The pair $(\tilde{X}, \tilde{\omega})$ satisfies the hypotheses of Corollary 3.7 (playing the role of the pair $(X, \omega)$ therein). When applied to $(n, n-1)$-forms and to $(n-1, n)$-forms $\psi \in \Dom(\Delta_\omega)$, inequality (53) gives the following implication:

$$\Delta_\omega \psi = 0 \implies \psi = 0.$$  

This proves the vanishing of $\mathcal{H}^{n,n-1}_\Delta(\tilde{X}, \mathbb{C})$ and $\mathcal{H}^{n-1,n}_\Delta(\tilde{X}, \mathbb{C})$.

Meanwhile, the Hodge star operator $\star = \star_{\tilde{\omega}}$ commutes with $\Delta_\omega$, so it induces isomorphisms

$$\star_{\tilde{\omega}} : \mathcal{H}^{1,0}_\Delta(\tilde{X}, \mathbb{C}) \rightarrow \mathcal{H}^{n,n-1}_\Delta(\tilde{X}, \mathbb{C}) \quad \text{and} \quad \star_{\tilde{\omega}} : \mathcal{H}^{0,1}_\Delta(\tilde{X}, \mathbb{C}) \rightarrow \mathcal{H}^{n-1,n}_\Delta(\tilde{X}, \mathbb{C}).$$

Therefore, the spaces $\mathcal{H}^{1,0}_\Delta(\tilde{X}, \mathbb{C})$ and $\mathcal{H}^{0,1}_\Delta(\tilde{X}, \mathbb{C})$ must vanish as well. \hfill \Box

### 3.3 Harmonic $L^2$-forms of degree 2 on the universal cover of a balanced hyperbolic manifold

We will discuss 2-forms in a way analogous to the discussion of 1-forms we had in §3.2. The context and the notation are the same. The analogue of Lemma 3.4 is

**Lemma 3.8.** Let $X$ be a complex manifold with $\dim_{\mathbb{C}} X = n$. Suppose there exists a balanced metric $\omega$ on $X$. Then, for any $h \in \mathbb{C}^*$ and any 2-form $\alpha$ on $X$, the following identity holds:

$$[\Delta_h, L_{\omega_{n-1}}] \alpha = -(|h|^2 + 1) i \partial \bar{\partial} (\Lambda_\omega \alpha) \wedge \omega_{n-1} - \omega_{n-1} \wedge \Delta_h \alpha. \quad (63)$$

**Proof.** We compute separately the two terms applied to $\alpha$ on the r.h.s. of the consequence (36) of the Jacobi identity and the balanced hypothesis on $\omega$.

The first term is

$$[d_h^*, L_{\omega_{n-1}}] d_h \alpha = -\omega_{n-1} \wedge d_h^* d_h \alpha, \quad (64)$$

since $d_h^* (\omega_{n-1} \wedge d_h \alpha) = 0$ owing to the vanishing of $\omega_{n-1} \wedge d_h \alpha$ for degree reasons.

To compute $d_h [d_h^*, L_{\omega_{n-1}}] \alpha$, we notice that

$$[d_h^*, L_{\omega_{n-1}}] \alpha = d_h^* (\omega_{n-1} \wedge \alpha) - \omega_{n-1} \wedge d_h^* \alpha = -i \bar{h} d_{-\frac{1}{\bar{h}}} (\Lambda_\omega \alpha) \wedge \omega_{n-1} - \omega_{n-1} \wedge d_h^* \alpha,$$

where the last identity follows from (38). Thus, using the balanced hypothesis on $\omega$, we get:

$$d_h [d_h^*, L_{\omega_{n-1}}] \alpha = -i \bar{h} d_{-\frac{1}{\bar{h}}} (\Lambda_\omega \alpha) \wedge \omega_{n-1} - \omega_{n-1} \wedge d_h d_h^* \alpha. \quad (65)$$

Finally, $d_h d_{-\frac{1}{\bar{h}}} = ((|h|^2 + 1)/\bar{h}) \partial \bar{\partial}$, so (63) follows from (64) and (65). \hfill \Box

We now deduce the following analogue of Proposition 3.5.
Proposition 3.9. Let \((X, \omega)\) be a complete balanced manifold, \(\dim_{\mathbb{C}}X = n \geq 2\).

For any \(h \in \mathbb{C}^*\) and any 2-form \(\varphi \in \text{Dom}(\Delta_{\tau})\) on \(X\), the following identity holds:

\[
\langle\langle \Delta_h(\omega_{n-1} \wedge \alpha), \omega_{n-1} \wedge \alpha \rangle\rangle = (|h|^2 + 1) ||\bar{\partial}(\Lambda_{\omega}\alpha)||^2.
\]

(66)

Proof. An immediate consequence of (63) is the identity

\[
\Delta_h(\omega_{n-1} \wedge \alpha) = -(|h|^2 + 1) i\bar{\partial}(\Lambda_{\omega}\alpha) \wedge \omega_{n-1}.
\]

Taking the pointwise inner product (w.r.t. \(\omega\)) against \(\omega_{n-1} \wedge \alpha\) and using the Lefschetz decomposition \(\alpha^{1,1} = \alpha^{1,1}_{\text{prim}} + (1/n)(\Lambda_{\omega}\alpha^{1,1})\) of the \((1, 1)\)-type component of \(\alpha\), its analogue for the \((1, 1)\)-form \(i\partial\bar{\partial}(\Lambda_{\omega}\alpha)\) and the fact that the product of any primitive 2-form with \(\omega_{n-1}\) vanishes, we get:

\[
\langle\langle \Delta_h(\omega_{n-1} \wedge \alpha), \omega_{n-1} \wedge \alpha \rangle\rangle = -(|h|^2 + 1) \langle\langle \bar{\Delta}_{\omega}(\Lambda_{\omega}\alpha) \omega_n, (\Lambda_{\omega}\alpha) \omega_n \rangle\rangle = -(|h|^2 + 1) \langle\langle \bar{\Delta}_{\omega}(\Lambda_{\omega}\alpha), (\Lambda_{\omega}\alpha) \rangle\rangle,
\]

(67)

where \(\bar{\Delta}_{\omega}f := \Lambda_{\omega}(i\partial\bar{\partial}f)\) for any function \(f\) on \(X\). It is standard that the Laplacian \(\bar{\Delta}_{\omega}\) is a non-negative operator on functions. Identity (71) in Lemma 4.1 with \(k = 0\) and \(r = n\) was used to get the last equality in (67).

Now, we need the following simple observation.

Lemma 3.10. Let \((X, \omega)\) be a complete balanced manifold, \(\dim_{\mathbb{C}}X = n \geq 2\). For any function \(f \in \text{Dom}(\bar{\Delta}_{\omega})\), we have: \(\langle\langle \bar{\Delta}_{\omega}f, f \rangle\rangle = -||\bar{\partial}f||^2\).

Proof of Lemma 3.10. The formula \(\bar{\partial}^* = -\ast \bar{\partial} \ast\) gives the third equality below:

\[
\langle\langle \bar{\Delta}_{\omega}f, f \rangle\rangle = \langle\langle \Lambda_{\omega}(i\partial\bar{\partial}f), f \rangle\rangle = \langle\langle i\partial\bar{\partial}f, \bar{\partial}^*(f\omega) \rangle\rangle = -i \langle\langle \bar{\partial}f, \ast(\bar{\partial}f \wedge \omega_{n-1}) \rangle\rangle,
\]

where we used the balanced hypothesis on \(\omega\) to get the last equality.

Now, \(\bar{\partial}f\) is a \((0, 1)\)-form, hence primitive, so the standard formula (7) yields:

\[
\ast(i\partial\bar{\partial}f) = -\bar{\partial}f \wedge \omega_{n-1}, \quad \text{or equivalently} \quad \ast(\bar{\partial}f \wedge \omega_{n-1}) = i\partial\bar{\partial}f,
\]

since \(\ast = -\text{Id}\) on forms of odd degree.

The contention follows. \(\square\)

End of proof of Proposition 3.9. Integrating (67) and applying Lemma 3.10 with \(f = \Lambda_{\omega}\alpha\), we get (66).

The next consequence of the above discussion can be conveniently worded in terms of Demailly’s torsion operator \(\tau = \tau_{\omega} := [\Lambda_{\omega}, \partial\omega \wedge \cdot]\) and the induced Laplacian \(\Delta_{\tau} := [d + \tau, d^* + \tau^*]\) mentioned in the introduction.

Corollary 3.11. Let \((X, \omega)\) be a connected complete balanced manifold, \(\dim_{\mathbb{C}}X = n \geq 2\). For any \((1, 1)\)-form \(\alpha^{1,1} \in \text{Dom}(\Delta_{\tau})\), the following implication holds:

\[
\Delta_{\tau}\alpha^{1,1} = 0 \implies \Lambda_{\omega}\alpha^{1,1} \text{ is constant}.
\]
Proof. Thanks to (5) and to $\Delta' \geq 0$ and $\Delta'' \geq 0$, the hypothesis $\Delta \alpha^{1,1} = 0$ translates to $\Delta' \alpha^{1,1} = 0$ and $\Delta'' \alpha^{1,1} = 0$. Since $\omega$ is complete, these conditions are further equivalent to

\begin{align*}
(i) \quad (\partial + \tau) \alpha^{1,1} = 0, \\
(ii) \quad (\partial^* + \tau^*) \alpha^{1,1} = 0, \\
(iii) \quad \bar{\partial} \alpha^{1,1} = 0, \\
(iv) \quad \bar{\partial}^* \alpha^{1,1} = 0.
\end{align*}

Thus, we get:

$$
\bar{\partial} (\Lambda_\omega \alpha^{1,1}) = [\bar{\partial}, \Lambda_\omega] \alpha^{1,1} = i (\partial^* + \tau^*) \alpha^{1,1} = 0,
$$

where the first equality follows from (iii) of (68), the second equality follows from Demailly’s Hermitian commutation relation (4) and the third equality follows from (ii) of (68).

We conclude that the hypothesis $\Delta \alpha^{1,1} = 0$ implies $\bar{\partial} (\Lambda_\omega \alpha^{1,1}) = 0$. This implies, thanks to Proposition 3.9 applied with $h = 1$, that $\Delta (\omega_{n-1} \wedge \alpha^{1,1}) = 0$, where $\Delta = \Delta_\omega = dd^* + d^* d$ is the $d$-Laplacian induced by $\omega$. Since $\omega_{n-1} \wedge \alpha^{1,1} = (\Lambda_\omega \alpha^{1,1}) \omega_n = \ast (\Lambda_\omega \alpha^{1,1})$ and since $\Delta$ commutes with $\ast$, we get $\Delta (\Lambda_\omega \alpha^{1,1}) = 0$. By completeness of $\omega$, this means that $d(\Lambda_\omega \alpha^{1,1}) = 0$ on $X$, hence $\Lambda_\omega \alpha^{1,1}$ must be constant since $X$ is connected.

An immediate consequence of Corollary 3.11 is that the following linear map is well defined:

$$
T_{\omega_n} : \mathcal{H}^{1,1}_{\Delta_\omega} (X, \mathbb{C}) \to \mathbb{C}, \quad \alpha^{1,1} \mapsto \Lambda_\omega \alpha^{1,1} = \frac{\alpha^{1,1} \wedge \omega_{n-1}}{\omega_n},
$$

under those assumptions, where $\mathcal{H}^{1,1}_{\Delta_\omega} (X, \mathbb{C})$ is the space of $\Delta_\omega$-harmonic $L^2_{\omega}$-forms of type $(1, 1)$.

Proof of Theorem 1.6. The pair $(\tilde{X}, \tilde{\omega})$ satisfies the hypotheses of Corollary 3.11 (playing the role of the pair $(X, \omega)$ therein). By the balanced hyperbolic hypothesis on $(X, \omega)$, there exists an $\tilde{\omega}$-bounded smooth $(2n - 3)$-form $\tilde{\Gamma}$ on $\tilde{X}$ such that $\tilde{\omega}_{n-1} = d\tilde{\Gamma}$.

Let $\alpha^{1,1} \in \mathcal{H}^{1,1}_{\Delta_{\tilde{\omega}}} (\tilde{X}, \mathbb{C})$ such that $\alpha^{1,1} \geq 0$. Then, $\bar{\partial} \alpha^{1,1} = 0$ (by (iii) of (68)) and real, hence we also have $\partial \alpha^{1,1} = 0$. Thus, $\alpha^{1,1}$ is $d$-closed, so

$$
\tilde{\omega}_{n-1} \wedge \alpha^{1,1} = d (\tilde{\Gamma} \wedge \alpha^{1,1}) \in \text{Im} \ d
$$

because $\tilde{\Gamma} \wedge \alpha^{1,1}$ is $L^2_{\tilde{\omega}}$ and $d (\tilde{\Gamma} \wedge \alpha^{1,1})$ is again $L^2_{\tilde{\omega}}$.

On the other hand,

$$
\tilde{\omega}_{n-1} \wedge \alpha^{1,1} = (\Lambda_{\tilde{\omega}} \alpha^{1,1}) \tilde{\omega}_n \in \mathcal{H}^{2n}_{\Delta_{\tilde{\omega}}} (\tilde{X}, \mathbb{C})
$$

because $\Lambda_{\tilde{\omega}} \alpha^{1,1}$ is constant by Corollary 3.11.

Since the subspaces $\mathcal{H}^{2n}_{\Delta_{\tilde{\omega}}} (\tilde{X}, \mathbb{C})$ and $\text{Im} \ d$ of the space of $L^2_{\tilde{\omega}}$-forms of degree $2n$ on $\tilde{X}$ are orthogonal (see (d) of Theorem 3.3), we deduce that $\tilde{\omega}_{n-1} \wedge \alpha^{1,1} = 0$. Equivalently, $\Lambda_{\tilde{\omega}} \alpha^{1,1} = 0$. This implies that $\alpha^{1,1} = 0$ since $\alpha^{1,1} \geq 0$ by hypothesis. \qed
4 Appendix

A key classical fact used by Gromov in \cite{Gro91} is that some of the Lefschetz maps at the level of differential forms are \textit{quasi-isometries} w.r.t. the $L^2$-inner product. We spell out the equalities involving \textit{pointwise} inner products that lead to more precise statements that were used in earlier parts of our text.

Let $\omega$ be an arbitrary Hermitian metric on an arbitrary complex manifold $X$ with $\dim_{\mathbb{C}} X = n$. As usual, for any $r = 1, \ldots, n$, we put $\omega_r := \omega^{r!}$. Recall the following standard fact.

For every $k \leq n$ and every $r \leq n - k$, the pointwise Lefschetz operator:

$$L^r_\omega : \Lambda^k T^* X \longrightarrow \Lambda^{k+2r} T^* X, \quad L^r_\omega(\varphi) = \omega^r \wedge \varphi,$$

is \textit{injective}. When $r = n - k$, $L^{n-k}_\omega$ is even \textit{bijective}.

We will compare the pointwise inner products $\langle \omega^r \wedge \varphi_1, \omega^r \wedge \varphi_2 \rangle_\omega$ and $\langle \varphi_1, \varphi_2 \rangle_\omega$ for arbitrary $k$-forms $\varphi_1, \varphi_2 \in \Lambda^k T^* X$. We will use the following standard formula (cf. e.g. \cite{Voi02}):

$$[L^r_\omega, \Lambda_\omega] = r(k - n + r - 1) L^{r-1}_\omega \quad \text{on } k\text{-forms},$$

for any integer $r \geq 1$, where $\Lambda = \Lambda_\omega = (\omega \wedge \cdot)^*$ is the adjoint of the Lefschetz operator $L_\omega$ w.r.t. the pointwise inner product $\langle \ , \ \rangle_\omega$ induced by $\omega$.

(1) \textbf{Case of primitive forms}

Recall that for any non-negative integer $k \leq n$, a $k$-form $\varphi$ is said to be \textit{primitive w.r.t.} $\omega$ (or $\omega$-\textit{primitive}, or simply \textit{primitive} when no confusion is likely) if it satisfies any of the following equivalent two conditions:

$$\omega_{n-k+1} \wedge \varphi = 0 \iff \Lambda_\omega \varphi = 0.$$

\textbf{Lemma 4.1.} \textit{For every $k \leq n$, every $r \leq n - k$ and any $k$-forms $\varphi_1, \varphi_2$ one of which is $\omega$-primitive, the following identity holds:}

$$\langle \omega^r \wedge \varphi_1, \omega^r \wedge \varphi_2 \rangle_\omega = (r!)^2 \binom{n-k}{r} \langle \varphi_1, \varphi_2 \rangle_\omega. \quad (71)$$

\textit{In particular, the analogous equality holds for the $L^2_\omega$-inner product $\langle \ , \ \rangle_\omega$.}

\textbf{Proof.} To make a choice, let us suppose that $\varphi_1$ is primitive. We get:

$$\langle \omega^r \wedge \varphi_1, \omega^r \wedge \varphi_2 \rangle_\omega = \langle \Lambda_\omega (\omega^r \wedge \varphi_1), \omega^r \wedge \varphi_2 \rangle_\omega \overset{(i)}{=} \langle [\Lambda_\omega, L^r_\omega] \varphi_1, \omega^r \wedge \varphi_2 \rangle_\omega$$

$$\overset{(ii)}{=} r(n-k-r+1) \langle \omega^{r-1} \wedge \varphi_1, \omega^{r-1} \wedge \varphi_2 \rangle_\omega$$

$$\vdots$$

$$= r(r-1) \cdots 1 (n-k-r+1) (n-k-r+2) \cdots (n-k) \langle \varphi_1, \varphi_2 \rangle_\omega$$

$$= r! \frac{(n-k)!}{(n-k-r)!} \langle \varphi_1, \varphi_2 \rangle_\omega.$$
where (i) follows from \( \varphi_1 \) being primitive, (ii) follows from the standard formula (70), the remaining equalities except for the last one follow from analogues of (i) and (ii), while the last equality proves (71).

Let us also notice that, when the powers of \( \omega \) are distinct, the products involved in the analogue of (71) are actually orthogonal to each other.

**Lemma 4.2.** Let \( r, s, k \in \mathbb{N} \) with \( s > 0 \) and \( k \leq n \). For any \((k - 2s)\)-form \( u \) and any \( \omega \)-primitive \( k \)-form \( v \), the following identity holds:

\[
\langle \omega^{r+s} \wedge u, \omega^r \wedge v \rangle_\omega = 0.
\]  

(72)

In particular, the analogous equality holds for the \( L^2_\omega \)-inner product \( \langle \langle , \rangle \rangle_\omega \).

**Proof.** We have:

\[
\langle \omega^{r+s} \wedge u, \omega^r \wedge v \rangle_\omega = \langle \omega^{r+s-1} \wedge u, \Lambda_\omega(\omega^r \wedge v) \rangle_\omega \overset{(i)}{=} \langle \omega^{r+s-1} \wedge u, [\Lambda_\omega, L^r_\omega] v \rangle_\omega \\
\overset{(ii)}{=} c_1 \langle \omega^{r+s-1} \wedge u, \omega^{r-1} \wedge v \rangle_\omega = \cdots = c_1 \cdots c_r \langle \omega^s \wedge u, v \rangle_\omega \\
= c_1 \cdots c_\ell \langle \omega^{s-1} \wedge u, \Lambda_\omega v \rangle_\omega = 0,
\]

where (i) follows from \( v \) being primitive, (ii) follows from the standard formula (70) with the appropriate constant \( c_1 \) (whose actual value is irrelevant here), the remaining equalities except for the last one follow from analogues of (i) and (ii) with the appropriate constants \( c_2, \ldots, c_r \), while the last equality follows again from \( v \) being primitive and proves (72). \( \square \)

(2) **Case of arbitrary forms**

Let \( \varphi_1, \varphi_2 \) be arbitrary \( k \)-forms and let

\[
\varphi_1 = \varphi_{1, \text{prim}} + \omega \wedge \varphi_{1,1} + \cdots + \omega^l \wedge \varphi_{1,l} \quad \text{and} \quad \varphi_2 = \varphi_{2, \text{prim}} + \omega \wedge \varphi_{2,1} + \cdots + \omega^l \wedge \varphi_{2,l}
\]  

(73)

be their respective Lefschetz decompositions, where \( l \) is the non-negative integer defined by requiring \( 2l = k \) if \( k \) is even and \( 2l = k - 1 \) if \( k \) is odd, while the forms \( \varphi_{j, \text{prim}}, \varphi_{j,1}, \ldots, \varphi_{j,l} \) are primitive of respective degrees \( k, k - 2, \ldots, k - 2l \) for every \( j \in \{1, 2\} \).

The sense in which the Lefschetz operator (69) is a quasi-isometry for the pointwise inner product (hence also the \( L^2 \)-inner product) induced by \( \omega \) is made explicit in the following

**Lemma 4.3.** Fix integers \( 0 \leq k \leq n \), \( 0 \leq r \leq n - k \) and arbitrary \( k \)-forms \( \varphi_1, \varphi_2 \).

(i) The following identity holds:

\[
\langle \omega^r \wedge \varphi_1, \omega^r \wedge \varphi_2 \rangle_\omega = (r!)^2 \binom{n-k}{r}^2 \langle \varphi_{1, \text{prim}}, \varphi_{2, \text{prim}} \rangle_\omega \\
+ \binom{n-k+2}{r+1}(r+1)!^2 \langle \varphi_{1,1}, \varphi_{2,1} \rangle_\omega + \cdots + \binom{n-k+2l}{r+l}(r+1)!^2 \langle \varphi_{1,l}, \varphi_{2,l} \rangle_\omega.
\]  

(74)

(ii) Putting \( C_{n,k,r,s} := ((r+s)!(n-k+s)!)/(s!(n-k-r+s)! \) and

\[
A_{n,k,r} := \min_{s=0,\ldots,l} C_{n,k,r,s}, \quad B_{n,k,r} := \max_{s=0,\ldots,l} C_{n,k,r,s},
\]

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the following inequalities hold:

\[ A_{n,k,r} |\varphi|_\omega^2 \leq |\omega^r \wedge \varphi|^2 \leq B_{n,k,r} |\varphi|_\omega^2. \]

(iii) With the notation of (ii), if \( \langle \varphi_1, s, \varphi_2, s \rangle_\omega \geq 0 \) for every \( s \in \{0,1,\ldots,l\} \), the following inequalities hold:

\[ A_{n,k,r} \langle \varphi_1, \varphi_2 \rangle_\omega \leq \langle \omega^r \wedge \varphi_1, \omega^r \wedge \varphi_2 \rangle_\omega \leq B_{n,k,r} \langle \varphi_1, \varphi_2 \rangle_\omega. \]

Proof. (i) Using the Lefschetz decompositions (73) and Lemma 4.2, we get:

\[ \langle \omega^r \wedge \varphi_1, \omega^r \wedge \varphi_2 \rangle_\omega = \langle \omega^r \wedge \varphi_{1,\text{prim}}, \omega^r \wedge \varphi_{2,\text{prim}} \rangle_\omega + \sum_{s=1}^l \langle \omega^{r+s} \wedge \varphi_{1,s}, \omega^{r+s} \wedge \varphi_{2,s} \rangle_\omega. \]

Identity (74) follows from this and from Lemma 4.1.

(ii) and (iii) follow at once from (i) applied twice, with a given \( 1 \leq r \leq n - k \) and with \( r = 0 \). □

References.

[Dem84] J.-P. Demailly — *Sur l’identité de Bochner-Kodaira-Nakano en géométrie hermitienne* — Séminaire d’analyse P. Lelong, P. Dolbeault, H. Skoda (editors) 1983/1984, Lecture Notes in Math., no. 1198, Springer Verlag (1986), 88-97.

[Dem92] J.-P. Demailly — *Regularization of Closed Positive Currents and Intersection Theory* — J. Alg. Geom., 1 (1992), 361-409.

[Dem97] J.-P. Demailly — *Complex Analytic and Algebraic Geometry* — http://www-fourier.ujf-grenoble.fr/~demailly/books.html

[DGMS75] P. Deligne, Ph. Griffiths, J. Morgan, D. Sullivan — *Real Homotopy Theory of Kähler Manifolds* — Invent. Math. 29 (1975), 245-274.

[FLY12] J. Fu, J. Li, S.-T. Yau – *Balanced Metrics on Non-Kähler Calabi-Yau Threefolds* — J. Differential Geom. 90 (2012), p. 81–129.

[Fri89] R. Friedman — *On Threefolds with Trivial Canonical Bundle* — in Complex Geometry and Lie Theory (Sundance, UT, 1989), Proc. Sympos. Pure Math., vol. 53, Amer. Math. Soc., Providence, RI, 1991, p. 103–134.

[Fri19] R. Friedman — *The \( \partial \bar{\partial} \)-Lemma for General Clemens Manifolds* — Pure and Applied Mathematics Quarterly, vol. 15, no. 4 (2019), 1001–1028.

[Gaf54] M. P. Gaffney — *A Special Stokes’s Theorem for Complete Riemannian Manifolds* — Ann. of Math. 60, No. 1, (1954), 140-145.

[Gau77] P. Gauduchon — *Le théorème de l’excentricité nulle* — C. R. Acad. Sci. Paris, Sér. A, 285 (1977), 387-390.

[Gro91] M. Gromov — *Kähler Hyperbolicity and \( L^2 \) Hodge Theory* — J. Diff. Geom. 33 (1991), 263-292.
[Lam99] A. Lamari — *Courants kähleriens et surfaces compactes* — Ann. Inst. Fourier 49, no. 1 (1999), 263-285.

[LT93] P. Lu, G. Tian – *The Complex Structures on Connected Sums of $S^3 \times S^3$* — in Manifolds and Geometry (Pisa, 1993), Sympos. Math., XXXVI, Cambridge Univ. Press, Cambridge, 1996, p. 284–293.

[MP21] S. Marouani, D. Popovici — *Balanced Hyperbolic and Divisorially Hyperbolic Compact Complex Manifolds* — arXiv e-print math.CV/2107.08972v1.

[Mic83] M. L. Michelsohn — *On the Existence of Special Metrics in Complex Geometry* — Acta Math. 143 (1983) 261-295.

[Pop14] D. Popovici — *Deformation Openness and Closedness of Various Classes of Compact Complex Manifolds; Examples* — Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), Vol. XIII (2014), 255-305.

[Pop15] D. Popovici — *Aeppli Cohomology Classes Associated with Gauduchon Metrics on Compact Complex Manifolds* — Bull. Soc. Math. France 143, no. 4 (2015), p. 763-800.

[Pop16] D. Popovici — *Sufficient Bigness Criterion for Differences of Two Nef Classes* — Math. Ann. 364 (2016), 649-655.

[Voi02] C. Voisin — *Hodge Theory and Complex Algebraic Geometry. I.* — Cambridge Studies in Advanced Mathematics, 76, Cambridge University Press, Cambridge, 2002.

[Yac98] A. Yachou — *Sur les variétés semi-kähleriennes* — PhD Thesis (1998), University of Lille.

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