EFFECT OF INSTITUTIONAL DELEVERAGING ON OPTION VALUATION PROBLEMS

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Abstract. This paper studies the valuation problem of European call options when the presence of distressed selling may lead to further endogenous volatility and correlation between the stock issuer’s asset value and the price of the stock underlying the option, and hence influence the option price. A change of numéraire technique, based on Girsanov Theorem, is applied to derive the analytical pricing formula for the European call option when the price of underlying stock is subject to price pressure triggered by the stock issuer’s own distressed selling. Numerical experiments are also provided to study the impacts of distressed selling on the European call option prices.

1. Introduction. If the sales of goods occur at a heavy discount rate, a fire sale is said to occur. When the seller is facing distressed selling for a big loss or even bankruptcy, the fire sales may occur. Fire sales or the sudden deleveraging of large financial portfolios can cause sharp drop of price and even some further serious consequences as people may perceive it as a negative signal of the company [19, 20].

By contributing to unexpected spikes in volatility and correlations of asset returns (Carlson [4]; Khandani and Lo [13]; Brunnermeier [3]), fire sales have been recognized as a factor of market instability in many economic empirical studies. Coval and Stafford [7] gave empirical evidence for fire sales by open end mutual funds by studying the transactions caused by capital flows. They showed that funds in distress experience outflows of capital by investors which result in fire sales in existing positions, creating a price pressure in the securities held in common by distressed funds. Anton and Polk [1] found empirically that common active mutual fund ownership predicts cross-sectional variation in return realized covariance. Ellul et al. [9] pointed out that bonds subject to a high probability of regulatory-induced selling exhibit price declines and subsequent reversals. These price effects appear

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larger during periods when the insurance industry is relatively distressed and other potential buyers capital is scarce.

Besides, many theoretical studies investigate the effect of fire sales on asset return volatilities and correlations. Kyle and Xiong [14] considered three categories of traders: noise traders, long-term investors and convergence traders and proposed an equilibrium model, which takes into account the supply and demand of three categories of traders. They found that the convergence traders can react to a price shock in one asset by deleveraging their positions in both markets, leading to contagion effects. Cont and Wagalath [6] proposed a tractable framework for modeling and estimating the impact of fire sales in multiple institutions on the systemic risks in a multi-asset setting.

In spite of the important role of fire sales played in security markets, the existing literature is somewhat limited and it rarely deals with option valuation problem when taking into account the price impact triggered by stock issuer’s own fire sales. Yang et al. [22] considered the valuation problems of vulnerable options when the option writer subjects to price pressure triggered by other financial institutions’ fire sales. In this paper we consider the valuation problems of European options by taking into account the effect of the stock issuer’s depressing selling on its own asset value and the stock price. By adopting the change of numéraire techniques, we are able to provide an analytical pricing formula for European options in the presence of fire sales.

The rest of the paper is organized as follows. The modeling framework for the price impact of fire sales on the stock issuer’s asset value, the prices of underlying stock and the bond prices is presented in Section 2. In Section 3, we present the change of numéraire technique based on Girsanov Theorem to obtain the unique risk-neutral measure. Section 4 formulates the valuation model for European call options. In Section 5, we discuss the forward measure and the reciprocal forward measure. We derive an analytical formula for the price of a European call option in Section 6. Numerical results are also presented in this section. Finally, Section 7 concludes the paper.

2. The framework. Due to capital requirements set by regulators or target leverage ratios set by asset managers, distressed selling for large losses may occur in a financial institution. The presence of distressed selling may lead to endogenous volatility and correlation in asset prices of the financial institution. This may further influence the option price written on the stock issued by the financial institution. Here we study the aggregate effect of distressed selling on the valuation of European call options. As presented in Cont and Wagalath [6], a function \( f(\cdot) \) which measures the systematic supply/demand generated by the financial institution as a function of its total assets can be introduced to model this aggregating effect. When the total assets of the financial institution moves from \( X(t_1) \) to \( X(t_2) \) over \([t_1, t_2]\), a portion,

\[
f \left( \frac{X(t_1)}{X(0)} \right) - f \left( \frac{X(t_2)}{X(0)} \right),
\]

of the fund will be liquidated, proportionally in each asset detained by the institution.
As pointed in [6], negative returns for a fund lead to outflows of capital from this fund. This implies that $f$ is an increasing function. Fire sales occur when a fund underperforms significantly and its value goes below a threshold. As a consequence, there exists some values $\beta$, $f$ is constant over $[\beta, \infty)$. For ease of calculation, in our model, we release this requirement by adding a trunked condition

$$\lim_{x \to \infty} x^p f(x) = c_p, \quad p = 1, \text{or} 2$$

(1)

where $c_p, p = 1$ or 2 are finite-valued constants, which implies that

- **a1.** $f'(x) = \mathcal{O}(x^{-1})$, the rate of deleveraging is negatively correlated with the asset value. The greater the total asset, the less the possibility of fire sales; and
- **a2.** $f''(x) = \mathcal{O}(x^{-2})$, the change of deleveraging rate is controlled, simultaneously, by the asset value and the change of the value.

In addition, when a drop in fund value is large enough to generate fire sales, and when the trades involved are sizable with respect to the average trading volume, the supply/demand generated by this deleveraging strategy will affect asset prices. Furthermore, we choose $f$ to be concave, capturing the fact that fire sales accelerate as the fund exhibits large losses. The above figure displays an example of such a deleveraging schedule.

In the following sections, we will construct a quantitative model to describe the endogenous risk incurred by financial institution’s fire sales, based on which the value of an affected stock, and the valuation of a European call option writing on the affected stock will be discussed.

### 2.1. Fundamental models

Let $\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathcal{P}$ be a probability measure space. Let $X$ denote the market value of the reference financial institution and $S$ denote value of the optioned stock. Assume that, in the absence of fire sales, their continuous stochastic movements follow a lognormal diffusion process and are modeled by the
following stochastic differential equations:

\[
\begin{align*}
\frac{dX(t)}{X(t)} &= \mu_X dt + \sigma_X dW_{1t} \\
\frac{dS(t)}{S(t)} &= \mu_S dt + \sigma_S d\left(\rho W_{1t} + \sqrt{1-\rho^2} W_{2t}\right)
\end{align*}
\]

(2)

where \(\mu_X\) and \(\mu_S\) are the respective drift rates, \(\sigma_X\) and \(\sigma_S\) are the respective volatility values, \(\{W_{1t}\}\) and \(\{W_{2t}\}\) are two independent standard Brownian motions under \(\mathcal{P}\), and \(\rho\) is the correlation between the stock price and the reference financial institution’s capital structural, with \(|\rho| < 1\):

\[
d\frac{X(t)}{X(t)} \cdot \frac{dS(t)}{S(t)} = \rho \sigma_X \sigma_S dt.
\]

We introduce here a *market impact* function \(\phi(\cdot)\) to capture the effect of the reference financial institution’s distressed behaviors on the return of the optioned stock. We assume that \(\phi : \mathcal{R} \rightarrow \mathcal{R}\) is a monotone increasing function and \(\phi(0) = 0\). In any small time interval from \(t\) to \(t + \Delta t\), the impact of the reference institution’s distressed behaviors on the return of the optioned stock is then

\[
\phi(\rho \cdot \left[f\left(\frac{X(t)}{X(0)}\right) - f\left(\frac{X(t + \Delta t)}{X(0)}\right)\right]).
\]

(3)

To study the market impact within an infinitesimal time interval, we assume that \(\phi(\cdot) \in \mathcal{C}^3(\mathcal{R})\) and \(f(\cdot) \in \mathcal{C}^4_0(\mathcal{R})\), where \(\mathcal{C}^p_0\) denotes the set of real-valued, \(p\)-times continuously differentiable maps whose derivatives have compact supports. Let

\[
h(\Delta t; t) := \phi(\rho \cdot \left[f\left(\frac{X(t)}{X(0)}\right) - f\left(\frac{X(t + \Delta t)}{X(0)}\right)\right]).
\]

(4)

Then, \(h(0; t) = 0\). Let \(h^{c, \rho}(t) = \lim_{u \to 0^+} \frac{h(u; t)}{u}\) denote the *influence rate* at which the optioned stock price is affected by the reference institution’s distressed behaviors. The following lemma provides some information about it.

**Lemma 1.** Under the above assumptions, the influence rate at time \(t\) is given by:

\[
h^{c, \rho}(t) = -\rho \phi'(0) \frac{X(t)}{X(0)} \left[f'(\frac{X(t)}{X(0)}) \mu_X + \frac{1}{2} f''(\frac{X(t)}{X(0)}) \frac{X(t)}{X(0)} \sigma_X^2 + f'(\frac{X(t)}{X(0)}) \sigma_X \cdot \epsilon(t)\right] + \frac{1}{2} \rho^2 \phi''(0) \sigma_X^2 \left[f'(\frac{X(t)}{X(0)}) \frac{X(t)}{X(0)} \right]^2.
\]

where \(\{\epsilon(t)\}\) is the Gaussian white noise process, a generalized derivative of \(\{W_{1t}\}\) (see, for example Hida et al. [12]).

In the presence of market impact, the value of the optioned stock evolves over time according to

\[
\frac{dS(t)}{S(t)} = [\mu_S - h^{c, \rho}(t)] dt + \sigma_S d\left(\rho W_{1t} + \sqrt{1-\rho^2} W_{2t}\right).
\]

(5)

\(^1\)In spite of trajectories of Brownian motion are not differentiable, a generalized derivative exists.
The affected market value of the reference financial institution is given by
\[
\frac{dX(t)}{X(t)} = [\mu_X - h^{e=1}(t)]dt + \sigma_X dW_{1t}. \tag{6}
\]
We can see from Lemma 1 and Eq. (5), the value of the optioned stock, \( S(t) \),
depends on the market impact of the reference financial institution distressed behaviors only through \( \phi'(0) \) and \( \phi''(0) \). The effect on the volatility, one of the model parameters which will eventually enter the price formula of the European call option, depends only on the slope \( \phi'(0) \) of the market impact function at the point 0. Hence one can easily try different \( \phi(x) \). Following Cont and Wagalath [6], here we adopt the linear price impact function
\[
\phi(x) = \frac{x}{L} \tag{7}
\]
to describe the price impact of the reference institution’s distressed behaviors. Here \( L \) describes the reference institution’s market influence and is interpreted as the portion of portfolio the institution has to liquidate in order to decrease the optioned stock price by 1%. The following corollary shows that the mean rate of return and volatility of the optioned stock price are both affected by the reference financial institution’s distressed behavior.

**Corollary 1. (Linear Price Impact).** Assume \( X(0) = x_0 \), let
\[
\psi_1(x) = \frac{x}{x_0} f' \left( \frac{x}{x_0} \right) \quad \text{and} \quad \psi_2(x) = \left( \frac{x}{x_0} \right)^2 f'' \left( \frac{x}{x_0} \right).
\]
Under the assumption of linear price impact (7), the value of the optioned stock, in the presence of fire sales, evolves over time according to
\[
\frac{dS(t)}{S(t)} = \mu_S(X(t)) dt + \rho \gamma(X(t)) dW_{1t} + \sigma_S \sqrt{1 - \rho^2} dW_{2t}, \tag{8}
\]
where
\[
\begin{align*}
\mu_S(x) &= \mu_S + \frac{\rho}{L} \left[ \mu_X \psi_1(x) + \frac{1}{2} \sigma_X^2 \psi_2(x) \right] \\
\gamma(x) &= \sigma_S + \frac{1}{L} \sigma_X \psi_1(x).
\end{align*}
\]
The affected market value of the reference financial institution is
\[
\frac{dX(t)}{X(t)} = \mu_X(X(t)) dt + \sigma_X(X(t)) dW_{1t}, \tag{9}
\]
where
\[
\begin{align*}
\mu_X(x) &= \mu_X \left[ 1 + \frac{1}{L} \psi_1(x) \right] + \frac{1}{2L} \sigma_X^2 \psi_2(x) \\
\sigma_X(x) &= \sigma_X \left[ 1 + \frac{1}{L} \psi_1(x) \right].
\end{align*}
\]
As to the results in Corollary 1, we remark that (concavity property of \( f \))
\[
\psi_1(x) > 0 \quad \text{and} \quad \psi_2(x) < 0.
\]
When the reference institution has a poor financial performance, i.e. \( \mu_X < 0 \), the incurred fair sales will decrease the optioned stock’s mean rate of return, \( \mu_S < \mu_S \), and simultaneously increase the volatility of the stock price, \( \gamma(X(t)) > \sigma_S \). Before fair sales occurs,
\[
\frac{dX(t)}{X(t)} \cdot \frac{dS(t)}{S(t)} = \rho \sigma_X \sigma_S dt.
\]
After fair sales,
\[
\frac{dX(t)}{X(t)} \frac{dS(t)}{S(t)} = \rho \sigma_X [\sigma_S + \frac{1}{\mathcal{L}} \sigma_X \psi_1(X(t))] \cdot [1 + \frac{1}{\mathcal{L}} \psi_1(X(t))] dt + \rho \sigma_X \sigma_S dt.
\]
That is, fire sales will enhance the positive correlation among securities.

By now, we have analyzed the effect of reference institutions’ fire sales on the financial market, i.e. market impact. In the next sections, we will focus on the effect of fire sales on the valuation of a European call option writing on the affected assets. Before we give further discussions, we first introduce the concept of risk-neutral measure that we will use in the following sections.

3. Risk-neutral measure. In this section, we work in an equivalent risk-neutral measure \( \mathcal{Q} \) that corresponds to the money market account
\[
M_t = e^{\int_0^t r(u) du}
\]
as the numéraire, where \( r(t) \) is the risk-free rate. Assume the short-term risk-free rate evolves over time according to a Vasicek type of mean-reverting process
\[
\frac{dr(t)}{r(t)} = a(b - r(t))dt + \sigma_r \sqrt{1 - \rho^2} dW_4t,
\]
where \( a, b \) and \( \sigma_r \) are constant, and \( W_4t \) is a standard Brownian motion under \( \mathcal{P} \), independent of \( (W_1t, W_2t) \). We may define
\[
W_{4t} = \rho_1 r_{1t} + \rho_2 r_{2t} + \sqrt{1 - \rho^2} W_{3t}.
\]
Then \( \{W_{4t}\} \) is a Brownian motion under \( \mathcal{P} \), and
\[
dW_{1t} dW_{4t} = \rho_{1t} dt, \quad dW_{2t} dW_{4t} = \rho_{2t} dt, \quad dW_{3t} dW_{4t} = \sqrt{1 - \rho^2} dW_{3t} dt.
\]
Under the short-rate process, the default-free zero-coupon bond \( B(t, T) \) maturing at time \( T \) is governed by the following process:
\[
\frac{dB(t, T)}{B(t, T)} = r(t) dt - \sigma_r \frac{1 - e^{-a(T-t)}}{a} dW_{4t}.
\]
Define the discount process
\[
D(t) = e^{\int_0^t r(u) du} = \frac{1}{M(t)}.
\]

**Theorem 1.** There is a unique risk-neutral measure \( \mathcal{Q} \) under which the processes \{\( D(t)B(t, T) \), \{\( D(t)X(t) \), and \{\( D(t)S(t) \) are martingales.

**Proof.** The reference institution’s market value \( X(t) \) in units of the money market account has price \( D(t)X(t) \), and this satisfies the stochastic differential equation (SDE)
\[
d(D(t)X(t)) = D(t)X(t) \left[ (\mu_X(X(t)) - r(t)) dt + \sigma_X^2(X(t)) dW_{1t} \right]. \tag{10}
\]
We would like to construct a process
\[
\tilde{W}_{1t} = \int_0^t \Theta_1(u) du + W_{1t}
\]
that permits us to rewrite (10) as
\[
d(D(t)X(t)) = D(t)X(t) \sigma_X^2(X(t)) d\tilde{W}_{1t}. \tag{11}
\]
Equating the right-hand sides of (10) and (11), we see that $\Theta_1(t)$ must be chosen to satisfy the first market price of risk equation

$$\mu_X^e(X(t)) - r(t) = \sigma_X^e(X(t))\Theta_1(t).$$

(12)

The value of optioned stock in units of money market account is $D(t) S(t)$. The differential of this price is

$$d(D(t) S(t)) = D(t) S(t) \left[ (\mu_X^e(X(t)) - r(t)) dt + \rho \gamma(X(t)) dW_{1t} + \sigma_S \sqrt{1 - \rho^2} dW_{2t} \right].$$

(13)

In addition to the process $\tilde{W}_{1t}$, we would like to construct a process

$$\tilde{W}_{2t} = \int_0^t \Theta_2(u) du + W_{2t}$$

that

$$d(D(t) S(t)) = D(t) S(t) \left[ \rho \gamma(X(t)) d\tilde{W}_{1t} + \sigma_S \sqrt{1 - \rho^2} d\tilde{W}_{2t} \right].$$

(14)

Equating the right-hand sides of (13) and (14), we obtain the second market price of risk equation

$$\mu_X^e(X(t)) - r(t) = \rho \gamma(X(t))\Theta_1(t) + \sigma_S \sqrt{1 - \rho^2} \Theta_2(t).$$

(15)

The default-free zero-coupon bond’s value $B(t, T)$ in units of the money market account has price $D(t) B(t, T)$, and this satisfies the SDE

$$\frac{d(D(t) B(t, T))}{D(t) B(t, T)} = -\sigma_r \frac{1 - e^{-a(T-t)}}{a} dW_{4t}$$

$$= -\sigma_r \frac{1 - e^{-a(T-t)}}{a} d \left( \rho_r \left( \tilde{W}_{1t} - \int_0^t \Theta_1(u) du \right) + \rho_{2r} \left( \tilde{W}_{2t} - \int_0^t \Theta_2(u) du \right) + \sqrt{1 - \rho_{1r}^2 - \rho_{2r}^2} W_{3t} \right).$$

(16)

We would like to construct a process

$$\tilde{W}_{3t} = \int_0^t \Theta_3(u) du + W_{3t}$$

that permits us to rewrite (16) as

$$\frac{d(D(t) B(t, T))}{D(t) B(t, T)} = -\sigma_r \frac{1 - e^{-a(T-t)}}{a} d \left( \rho_{1r} \tilde{W}_{1t} + \rho_{2r} \tilde{W}_{2t} + \sqrt{1 - \rho_{1r}^2 - \rho_{2r}^2} \tilde{W}_{3t} \right).$$

(17)

Equating the right-hand sides of (16) and (17), we see that $\Theta_3(t)$ must be chosen to satisfy the third market price of risk equation

$$\rho_{1r} \Theta_1(t) + \rho_{2r} \Theta_2(t) + \sqrt{1 - \rho_{1r}^2 - \rho_{2r}^2} \Theta_3(t) = 0.$$  

(18)

The market price of risk equations (12), (15) and (18) determine processes $\{\Theta_1(t)\}$, $\{\Theta_2(t)\}$ and $\{\Theta_3(t)\}$, respectively. We can solve explicitly for these processes by first solving Eq.(12) for $\Theta_1(t)$, substituting this into Eq.(15) and (18), and then solving Eq.(15) for $\Theta_2(t)$ and Eq.(18) for $\Theta_3(t)$. The particular formulas for $\Theta_1(t)$, $\Theta_2(t)$ and $\Theta_3(t)$ are irrelevant. What matters is that the market price of risk equations have one and only one solution, and so there is unique risk-neutral measure $\mathcal{Q}$. Under this measure, $\tilde{W}_t = (\tilde{W}_{1t}, \tilde{W}_{2t}, \tilde{W}_{3t})$ is a three-dimensional Brownian.
Option valuation problem.

We may also define

\[ \tilde{W}_{4t} = \rho_{1r}\tilde{W}_{1t} + \rho_{2r}\tilde{W}_{2t} + \sqrt{1 - \rho_{1r}^2 - \rho_{2r}^2}\tilde{W}_{3t}. \]

Then \( \{\tilde{W}_{4t}\} \) is a Brownian motion under \( Q \), and

\[ d\tilde{W}_{1t} d\tilde{W}_{4t} = \rho_{1r} dt, \quad d\tilde{W}_{2t} d\tilde{W}_{4t} = \rho_{2r} dt, \quad d\tilde{W}_{3t} d\tilde{W}_{4t} = \sqrt{1 - \rho_{1r}^2 - \rho_{2r}^2} dt. \]

Eq. (17) can then be rewritten as

\[ \frac{d(D(t)B(t,T))}{D(t)B(t,T)} = -\sigma_r \frac{1 - e^{-a(T-t)}}{a} d\tilde{W}_{4t}. \]

4. **Option valuation problem.** Consider a European call option with strike price \( K \) and maturating at \( T \), the same time as the bond \( B(t,T) \). The usual risk-neutral valuation of the option price at time \( t \) is

\[ C(t) = \frac{1}{D(t)} \mathbb{E}^Q \left[ D(T)(S(T) - K)^+ \mid \mathcal{F}_t \right] \]  

(19)

where \( \mathbb{E}^Q[\cdot] \) denotes the expectation under the risk-neutral measure \( Q \).

The computation of the right-hand side of this formula requires that we know something about the dependence between the discount factor \( D(T) \) and the payoff \( (S(T) - K)^+ \) of the option. This, however, can be difficult to model. The change of numéraire result in Geman et al. [10] is frequently used to decompose the conditional expectation on the right-hand side of (19) into two terms which can be interpreted as the conditional probabilities of the option being in-the-money at maturity (the definition of the probability measures \( \mathbb{P}^{Q,S} \) and \( \mathbb{P}^{Q,T} \) will be discussed later):

\[ C(t) = \frac{1}{D(t)} \mathbb{E}^Q \left[ D(T)(S(T) - K)^+ \mid \mathcal{F}_t \right] \]

\[ = \frac{1}{D(t)} \mathbb{E}^Q \left[ D(T)S(T)I_{\{S(T) > K\}} \mid \mathcal{F}_t \right] - \frac{1}{D(t)} \mathbb{E}^Q \left[ D(T)I_{\{S(T) > K\}} \mid \mathcal{F}_t \right] \]

\[ = S(t)\mathbb{P}^{Q,S}(S(T) > K)\mathcal{F}_t - KB(t,T)\mathbb{P}^{Q,T}(S(T) > K)\mathcal{F}_t. \]  

(20)

If we can find a simple model for the evolution of the underlying asset \( S(t) \) under the probability measures \( \mathbb{P}^{Q,S} \) and \( \mathbb{P}^{Q,T} \), we can use (20), in which we only need to estimate \( S(T) \), instead of using (19), which requires us to estimate \( D(T)(S(T) - K)^+ \).

4.1. **Change of numéraire.** A Numéraire is the unit of account in which other assets are denominated. In principle, we can take any positively priced asset as a numéraire and denominate all other assets in terms of the chosen numéraire. Associated with each numéraire, we shall have a risk-neutral measure. In this section, we will discuss the change of numéraire under \( Q \) within a three-dimensional market model.

Let

\[ L_t = \exp \left( -\int_0^t \vartheta(u) \cdot d\tilde{W}_u - \frac{1}{2} \int_0^t \|\vartheta(u)\|^2 du \right), \]  

(21)

where \( \vartheta(u) = (\vartheta_1(u), \vartheta_2(u), \vartheta_3(u)) \) is a three-dimensional adapted, if stochastic, process. The process \( \{L_t\} \) is a Radon-Nikodým derivative parameterized by \( \vartheta(u) \) if
\[ E^Q[L_t] = 1 \text{ for all } 0 \leq t \leq T. \] The following theorem provides the relevant Girsanov theorem based on which the change of numéraire is discussed.

**Theorem 2.** Consider the probability measure space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, Q)\) over the time interval \([0, T]\). Suppose \(\{L_t\}\) is given by (21) and that
\[ E^Q\left[ \int_0^T ||\vartheta(t)||^2 L^2_u du \right] < \infty. \]

Then \(\{L_t\}\) is a Radon-Nikodým derivative of some equivalent measure \(\tilde{Q}\) with respect to \(Q\), that is
\[ L_t = \frac{dP^{\tilde{Q}}}{dP^Q}|_t, \]
and
\[ \tilde{W}_t = \tilde{W}_t + \int_0^t \vartheta(u)du \]
is a three-dimensional standard Brownian motion under \(\tilde{Q}\).

**Proof.** See Appendix 4.1.1.

4.1.1. The forward measure. According to Theorem 2, we can use the volatility vector of \(B(t, T)\), denoted by
\[ \nu_T(t) = -\sigma_t \frac{1 - e^{-a(T-t)}}{a} (\rho_{1T}, \rho_{2T}, \sqrt{1 - \rho_{1T}^2 - \rho_{2T}^2}), \]
to change the measure. Define
\[ \tilde{W}_T^T = -\int_0^t \nu_T(u)du + \tilde{W}_t \]
and a new probability measure
\[ P^{Q,T}(A) = \frac{1}{B(0, T)} \int_A D(T)dP^Q \text{ for all } A \in \mathcal{F} \]
(the so called T-forward measure). We see from Eq.(14) that \(D(T)_{[0,T]}\) is the random variable appearing in (21) if we replace \(\vartheta(u)\) by \(-\nu_T(u)\).

With this replacement, Theorem 2 implies that, under \(P^{Q,T}\), the process \(\{\tilde{W}_T^T\}\) is a three-dimensional Brownian motion. In particular, under \(P^{Q,T}\), the Brownian motions \(\{\tilde{W}_T^{1T}\}, \{\tilde{W}_T^{2T}\}\), and \(\{\tilde{W}_T^{3T}\}\) are mutually independent. The expected value of an arbitrary random variable \(Y\) under \(P^{Q,T}\) can be computed by the formula
\[ E^{Q,T}[Y] = \frac{1}{B(0, T)} E^Q[Y D(T)]. \]

More generally,
\[ \frac{D(t)B(t, T)}{B(0, T)} = E^Q\left[ \frac{D(T)}{B(0, T)} \bigg| \mathcal{F}_t \right], \quad 0 \leq t \leq T, \]
is a Radon-Nikodým derivative process. For any \(0 \leq u \leq t \leq T\) and \(Y \in \mathcal{F}_t\), we have
\[ E^{Q,T}[Y|\mathcal{F}_u] = \frac{1}{D(u)B(u, T)} E^Q[Y D(t)B(t, T)|\mathcal{F}_u]. \]
In financial economics, the stock yield, expressed in terms of units of the bond, is known as the forward price. The T-forward price \( F_{t,T} \) is given as
\[
F_{t,T} = \frac{S(t)}{B(t,T)}.
\]
Under the forward measure \( \mathbb{P}^{Q,T} \), the bond \( B(t,T) \) is numéraire, and the forward price of the stock yield \( F_{t,T} \) must be a martingale.

**Theorem 3.** Under the probability measure \( \mathbb{P}^{Q,T} \), the process \( \{F_{t,T}\} \) is a martingale. Moreover,
\[
dF_{t,T} = F_{t,T} [\nu_S(t) - \nu_T(t)] \cdot d\tilde{W}^T_t,
\]
where \( \nu_S(t) = (\rho \gamma(X(t)), \sigma_S \sqrt{1 - \rho^2}, 0) \) is the volatility vector of \( S(t) \).

**Proof.** The sketch of proof is given in Appendix 4.1.2.

The reciprocal forward measure. We can also use the volatility vector of \( S(t) \), \( \nu_S(t) = (\rho \gamma(X(t)), \sigma_S \sqrt{1 - \rho^2}, 0) \), to change the measure. Define
\[
\tilde{W}^S_t = -\int_0^t \nu_S(u) du + \tilde{W}_t
\]
and a new probability measure
\[
\mathbb{P}^{Q,S}(A) = \frac{1}{S(0)} \int_A D(T) S(T) d\mathbb{P} \quad \text{for all } A \in \mathcal{F}.
\]
Under \( \mathbb{P}^{Q,S} \), the process \( \{\tilde{W}^S_t\} \) is a three-dimensional Brownian motion. For any \( 0 \leq u \leq t \leq T \) and \( Y \in \mathcal{F}_t \), the conditional expected value of \( Y \) is given as
\[
\mathbb{E}^{Q,S}[Y|\mathcal{F}_u] = \frac{1}{D(u) S(u)} \mathbb{E}[Y D(t) S(t)|\mathcal{F}_u].
\]

Now we turn our attention to using the optioned stock price \( S(t) \) as the numéraire. Since \( \frac{B(t,T)}{S(t)} = \frac{1}{F_{t,T}} \), we term the measure corresponding to the stock price, \( \mathbb{P}^{Q,S} \), the reciprocal forward measure.

**Theorem 4.** Under the probability measure \( \mathbb{P}^{Q,S} \), the process \( \{\frac{1}{F_{t,T}}\} \) is a martingale. Moreover,
\[
d\left( \frac{1}{F_{t,T}} \right) = \frac{1}{F_{t,T}} [\nu_T(t) - \nu_S(t)] \cdot d\tilde{W}^S_t.
\]

**Proof.** The sketch of proof is given in Appendix

5. **Solutions and effects.** In this section, we turn our attention to the calculations of the two conditional probabilities \( \mathbb{P}^{Q,S} \{S(T) > K|\mathcal{F}_t\} \) and \( \mathbb{P}^{Q,T} \{S(T) > K|\mathcal{F}_t\} \) appeared in (20).

We first note that
1. \[
\mathbb{P}^{Q,S} \{S(T) > K|\mathcal{F}_t\} = \mathbb{P}^{Q,S} \{F_{T,T} > K|\mathcal{F}_t\} = \mathbb{P}^{Q,S} \left\{ \frac{1}{F_{T,T}} < \frac{1}{K} \right\}.
\]
Under \( \mathbb{P}^{Q,S} \), \( \{\frac{1}{F_{t,T}}\} \) is a martingale.
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2. \[ P^Q.T \{ S(T) > K | T \} = P^Q.T \{ F_{T,T} > K | T \}. \]

Under \( P^Q.T \), \( \{ F_{t,T} \} \) is a martingale.

The key to calculate these conditional probabilities is to find out the distribution of \( \frac{1}{F_{T,T}} \) and \( F_{T,T} \) under the two different measures \( P^Q.S \) and \( P^Q.T \), respectively.

According to Theorem 3 and 4,

\[
F_{T,T} = F_{t,T} \exp \left\{ -\frac{1}{2} \int_t^T ||\nu_S(u) - \nu_T(u)||^2 du + \int_t^T [\nu_S(u) - \nu_T(u)] \cdot d\tilde{W}_u^T \right\},
\]

and

\[
\frac{1}{F_{T,T}} = \frac{1}{F_{t,T}} \exp \left\{ -\frac{1}{2} \int_t^T ||\nu_T(u) - \nu_S(u)||^2 du + \int_t^T [\nu_T(u) - \nu_S(u)] \cdot d\tilde{W}_u^S \right\}.
\]

The main difficulty of finding the distributions of \( F_{T,T} \) and \( \frac{1}{F_{T,T}} \) lies in the volatility term \( \{ \nu_S(t) \} \), which is an adapted process with respect to the filtration \( F \) and may depend on the reference institution’s asset value \( X(t) \).

To find the distributions of \( F_{T,T} \) and \( \frac{1}{F_{T,T}} \), we may need some information about the deleveraging schedule \( f \), which is the linkage between the reference financial institution’s asset value and the value of the optioned stock. As mentioned before, \( f \) need to satisfy the following conditions to be selected as a deleveraging schedule:

- **(Concave)** \( f \) need to be concave to capture the fact that fire sales accelerate as the reference institution exhibits large losses;

- **(Trunked)** \( f \) need to satisfy one more condition (released trunked condition)

\[
\lim_{x \to \infty} x^p f^{(p)}(x) = c_p, \quad p = 1, or 2,
\]

where \( c_p, p = 1 or 2 \) are finite-valued constants.

5.1. **Log-deleveraging schedule.** The most illuminating case is the log-form approximation:

\[
f(x) = \frac{\log(x)}{\eta}, \quad x > 0
\]

where \( \eta > 0 \) can be described as the maximal leverage ratio of the institution. It is easy to verify that \( f(x) \) satisfied all the conditions listed in the above section.

Under the assumption of log-deleveraging schedule,

\[
\nu_S(t) = (\rho(\sigma_S + \frac{\sigma_X}{\eta}), \sigma_S \sqrt{1 - \rho^2}, 0),
\]

and

\[
\nu_T(t) = -\sigma_T \frac{1 - e^{-a(T-t)}}{a} (\rho_1, \rho_2, \sqrt{1 - \rho_1^2 - \rho_2^2}).
\]

By Eq.(25), we discover that the volatility term \( \nu_S(t) \) increases by \( \rho \frac{\sigma_X}{\eta} \) because of the fire sales effect. So the option price will be higher with the fire sales effect. The volatility term in Eq.(22) and (23) is a nonrandom quantity, and so \( \log(F_{T,T}) \) and \( \log(\frac{1}{F_{T,T}}) \) are normally distributed, under the associated probability measures.
Theorem 5. Let \( \tilde{\sigma}_t = \sqrt{\frac{1}{T-t} \int_t^T ||\nu_S(u) - \nu_T(u)||^2du} \), and assume that the reference institution’s deleveraging schedule satisfies (24) with positive constant \( n \). The value at time \( t \in [0, T] \) of a European call on the affected stock, expiring at time \( T \) with strike price \( K \), is

\[
C(t) = S(t)N(d_+(t)) - KB(t,T)N(d_-(t)),
\]

where the adapted processes \( \{d_+(t)\} \) are given by

\[
d_+(t) = \frac{\log \left( \frac{S(t)}{KB(t,T)} \right) + \frac{1}{2}\tilde{\sigma}_t^2(T-t)}{\tilde{\sigma}_t\sqrt{T-t}}.
\]

Furthermore, a short position in the option can be hedged by holding \( N(d_+(t)) \) shares of the asset and shorting \( KN(d_-(t)) \) T-maturity zero-coupon bonds at each time \( t \).

Proof. Using the fact that \( \int_t^T [\nu_S(u) - \nu_T(u)] \cdot d\tilde{W}_t^T \) is normal with mean zero and variance \( \tilde{\sigma}_t^2(T-t) \) under \( \mathbb{P}^{\mathbb{Q},T} \), we compute

\[
\mathbb{P}^{\mathbb{Q},T}\{F_{t,T} > K|\mathcal{F}_t\} = \mathbb{P}^{\mathbb{Q},T}\left\{ \int_t^T [\nu_S(u) - \nu_T(u)] \cdot d\tilde{W}_u^T \frac{T-t}{\tilde{\sigma}_t} > \log \left( \frac{K}{F_{t,T}} \right) + \frac{1}{2}\tilde{\sigma}_t^2(T-t) \mid \mathcal{F}_t \right\}
\]

\[
= \mathbb{P}^{\mathbb{Q},T}\left\{ \int_t^T [\nu_S(u) - \nu_T(u)] \cdot d\tilde{W}_u^T \frac{T-t}{\tilde{\sigma}_t} < \log \left( \frac{K}{F_{t,T}} \right) - \frac{1}{2}\tilde{\sigma}_t^2(T-t) \mid \mathcal{F}_t \right\}
\]

\[
= \mathcal{N}(d_-(t)).
\]

Using the fact that \( \int_t^T [\nu_T(u) - \nu_S(u)] \cdot d\tilde{W}_t^S \) is normal with mean zero and variance \( \tilde{\sigma}_t^2(T-t) \) under \( \mathbb{P}^{\mathbb{Q},S} \), we compute

\[
\mathbb{P}^{\mathbb{Q},S}\left\{ \frac{1}{F_{t,T}} < \frac{1}{K} \mid \mathcal{F}_t \right\}
\]

\[
= \mathbb{P}^{\mathbb{Q},S}\left\{ \int_t^T [\nu_T(u) - \nu_S(u)] \cdot d\tilde{W}_u^T \frac{T-t}{\tilde{\sigma}_t} < \log \left( \frac{F_{t,T}}{K} \right) + \frac{1}{2}\tilde{\sigma}_t^2(T-t) \mid \mathcal{F}_t \right\}
\]

\[
= \mathcal{N}(d_+(t)).
\]

This completes the proof of (26).

We now consider the hedge suggested by formula (26). It is easier to do this when we take the zero-coupon bond as the numéraire rather than when we use currency. Dividing (26) by \( B(t,T) \), we obtain

\[
\frac{C(t)}{B(t,T)} = F_{t,T}N(d_+(t)) - KN(d_-(t)).
\]

This gives us the option price dominated in zero-coupon bonds. Suppose we hedge a short position in the option by

- holding \( N(d_+(t)) \) shares of the optioned stock; and
- shorting \( KN(d_-(t)) \) zero-coupon bonds at each time \( t \).

The value of this portfolio, denominated in units of zero-coupon bond, agrees with (26). To be sure this short option hedge works, however, we must verify that the portfolio just described is self-financing.
The capital gains differential associated with this portfolio, again denominated in units of zero-coupon bond, is
\[ N(d_+(t))dF_{t,T}. \]
The differential of the portfolio, according to Itô’s formula, is
\[ d\left( \frac{C(t)}{B(t,T)} \right) = N(d_+(t))dF_{t,T} + FtTdN(d_+(t)) + dF_{t,T}dN(d_+(t)) - KdN(d-(t)). \]
In order for the portfolio to be self-financing, we must have
\[ F_{t,T}dN(d_+(t)) + dF_{t,T}dN(d_+(t)) - KdN(d-(t)) = 0, \quad (29) \]
so that the change of value in the portfolio is entirely due to capital gains. The verification of (29) is given in the Appendix.

5.2. Effect on the option price: Greeks analysis. The Greeks of European call option under our model are calculated as follows, where \( N \) is the standard normal cumulative distribution function and \( n \) is the standard normal probability density function.

| Table 1. Greeks |
|------------------|
| Option Price \( (C(t)) \) | \( S(t)N(d_+(t)) - KB(t,T)N(d_-(t)) \) |
| Delta(\( \Delta \)) | \( N(d_+(t)) \) |
| Vega(\( V \)) | \( S(t)n(d_+(t))\sqrt{T-t} \) |
| Gamma(\( \Gamma \)) | \( \frac{n(d_+(t))}{S(t)\overline{\sigma}_t\sqrt{T-t}} \) |

| Table 2. Preference parameters |
|--------------------------|
| Parameters | Values |
| Market depth | \( L = 10 \) |
| Volatility | \( \sigma_S = 0.2 \) |
| Volatility | \( \sigma_r = 0.15 \) |
| Initial price | \( S_0 = 40 \) |
| Initial price | \( X_0 = 100 \) |
| Correlation | \( \rho = 0.7 \) |
| Correlation | \( \rho_1 = 0.5 \) |
| Mean-reverting speed | \( a = 100 \) |
| MLR | \( \eta = 1 \) |
| Volatility | \( \sigma_X = 0.1 \) |
| Time to maturity | \( T - t = 1 \) |
| Strike price | \( K = 40 \) |
| Initial price | \( B(t,T) = 0.05 \) |
| Time steps | \( N = 100 \) |
| Correlation | \( \rho_2 = 0.6 \) |
| Long-term interest rate | \( b = 0.0243 \) |

By applying the parameters given in Table 5.2 and the “log deleveraging schedule” \( f = \frac{\log x}{a} \), Figure 1 shows the European option prices as functions of the ratio of current stock price to the option’s exercise price for the cases of “subject to market impact of distressed selling” and “without market impact”. We can see that the European call option prices subject to market impact of distressed selling are higher than those without market impact. This makes intuitive sense since the occurrence of distressed selling increases the volatility of the stock prices when the reference institution has a poor financial performance and result in higher option prices.
prices, see for instance Eq.(8) and Eq.(25). An implication of this result is that, the distressed selling, corresponding to larger volatilities and correlations, produces a positive effect on the European call option price. In Figure 1, for each case, given the value of the underlying asset, for example, $S = 30$, the European call option price increases as the strike price $K$ decreases. The European call option price is an increasing function with respect to the stock price $S$ if the strike price is fixed.

In Figure 2, we show that the sensitivity of call price to changes in the stock prices (the Delta of the call) is also affected by distressed selling behavior. Thus, when the true model is the one that with distressed selling impact, using the model without distressed selling impact to compute option Deltas would yield biased results. It can be observed from Figure 2 that values of Delta with the distressed selling impact are larger than those without distressed selling impact for out of the money options. This is the opposite of the result for in the money options. An implication of this result is that, when the distressed selling impact is incorporated, the out of the money option price is more sensitive to the stock price, whereas the in the money option price is less sensitive to the stock price.

Figures 3 and 4 show the impact of distressed selling on the Vega and Gamma of the call option. From Figure 3 we know that the distressed selling produces a positive effect on the Vega of the call option which means the price of the call option is more sensitive to the volatility of the stock price when the impact of distressed selling is incorporated. It is also seen from Figure 4 that the European call option prices subject to market impact of distressed selling are lower than those without market impact. An implication of this result is that the option holder faces a lower hedge risk with the impact of the distressed selling. Both the Vega and Gamma attain their maximum when the option is at the money and approach to zero when the option is deeply in the money and deeply out of the money in our results which make intuitive sense.

5.3. General deleveraging schedule. We can relax the log-deleveraging schedule assumption and allow for general options. Thus the volatility term $\nu_S(t)$ in Eq. (22)
and Eq. (23) is a function of $X(t)$: $(\rho \gamma(X(t)), \sigma_S \sqrt{1 - \rho^2}, 0)$. However, an explicit formula for the distribution of $X(t)$ (and hence $F_{T,T}$ and $\frac{1}{F_{T,T}}$, and hence the option price $C(t)$) is usually unavailable. In this case, we could compute $C(t)$ numerically by beginning at $X(t) = x$ and simulating the stochastic differential equation under different measures:

$$
\frac{dX(t)}{X(t)} = \begin{cases} 
[r_t + \rho \gamma(X(t))\sigma_X(X(t))]dt + \sigma_X(X(t))d\tilde{W}^{S}_{1t}, & \text{under } \mathbb{P}^{S,\tilde{W}} \\ 
[r_t - \sigma_r \rho \frac{1 - e^{-a(T-t)}}{a} \sigma_X(X(t))]dt + \sigma_X(X(t))d\tilde{W}^{T}_{1t}, & \text{under } \mathbb{P}^{T,\tilde{W}}.
\end{cases}
$$
One way to do this would be to use the Euler method, a particular type of Monte Carlo method: choose a small positive step size $\delta$, and then set (take $P, Q, S$ as an example)

$$X(t+\delta) = x \left[1 + \left[r_t + \rho \gamma(x) \sigma_X^3(x)\right] \delta + \sigma_X^3(x) \sqrt{\delta} \epsilon_1\right],$$

where $\epsilon_1$ is a standard normal random variable. Then set

$$X(t+2\delta) = X(t+\delta) \left[1 + \left[r_t + \rho \gamma(X(t+\delta)) \sigma_X^3(X(t+\delta))\right] \delta + \sigma_X^3(X(t+\delta)) \sqrt{\delta} \epsilon_2\right],$$

where $\epsilon_2$ is a standard normal random variable independent of $\epsilon_1$. By this device one eventually determines a path for $\{X(u), t \leq u \leq T\}$ (assuming $\delta$ is chosen so that $T-t$ is an integer). This gives one realization of $\mathbb{P}^{Q,S}_{\mathcal{F}_T,T}$.

Now repeat this process $N$ times, for example $N = 10,000$. Compute

$$P_{\text{Prob}_1} = \frac{\# \{\mathbb{P}^{Q,S}_{\mathcal{F}_T,T} \leq \frac{1}{K} \}}{N}$$

over all these simulations to get an approximation value for $\mathbb{P}^{Q,S}_{\mathcal{F}_T,T} \leq \frac{1}{K} |\mathcal{F}_T|$. If one were to begin with different $t$ and initial value $x$, one would get a different answer (i.e., the answer is a function of $t$ and $x$), and hence we can use these answers to analyze the effect of the reference institution’s capital structure and fire sales on the option price.

Figures 5 and 6 show simulation results for the prices of European call options against spot-to-strike ratio, $S/K$, for the cases of log deleveraging schedule ($f = \log_2 x$) and exponential deleveraging schedule ($f = \frac{1-x^2}{2}$), respectively. The “triangles line” and “squares line” correspond to the cases of “subject to market impact of distressed selling” and “without market impact”, respectively. We can see from Figures 5 and 6, the call option prices subject to market impact of distressed selling are greater than those without market impact. For a fix strike price $K$, the European

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3The interest rate $r_t$ should be updated simultaneously.

4The procedure need to simulate the paths of $\{\tilde{W}^{S}_{\Delta t}\}$ and $\{\tilde{W}^{Q}_{\Delta t}\}$, simultaneously.
call option price increases as the stock price $S$ increases which is consistent with the result in Figure 1 for each cases. Monte Carlo simulation is employed to obtain the results in Figure 5. The “triangles line” is more volatile when we compare it with the “triangles line” in Figure 1. If the number of simulation paths is large enough, the “triangles line” in Figure 5 will be much smoother and coincident with the “triangles line” in Figure 1.

![Figure 5](image1.png)

**Figure 5.** Variation of European Call Option Price with Respect to Distressed Selling Impact. $f = \frac{\log x}{\eta}$.

![Figure 6](image2.png)

**Figure 6.** Variation of European Call Option Price with Respect to Distressed Selling Impact. $f = \frac{1-e^x}{\eta}$.

6. **Conclusions.** In this paper, we derive and analyze valuation formulas for European call option where the stock issuer’s asset value and it’s stock price are affected
by its distressed selling for large losses. The effect of the distressed selling on the asset value and the underlying stock price is described quantitatively in our model. Based on the Girsanov Theorem, we can find the risk-neutral measure under which the discounted asset value, discounted stock price and the discounted bond price are martingales. So we can provide an analytical pricing formula for the European call options with the impact of distressed selling which is very similar to the Black-Scholes formula (see Black and Scholes [2]). The numerical analysis is also conducted for the impact of distressed selling on the option price and Greek letters. For future study, one can consider pricing different derivatives with the fire sales effect as well as obtaining the analytical formula for a general deleveraging schedule. For a general deleveraging schedule, the problem can be transformed into obtaining the option pricing formula with stochastic volatility.

Appendix.

Proof of Theorem 1.

Proof. Applying Ito’s formula to \( f(X(t)) \) gives:
\[
\begin{align*}
df(X(t)) &= f'(X(t))dX(t) + \frac{1}{2} f''(X(t))(dX)^2 \\
&= \left( f'(X(t))X(t)\mu_X + \frac{1}{2} f''(X(t))X^2(t)\sigma_X^2 \right) dt + f'(X(t))X(t)\sigma_X dW(t).
\end{align*}
\]

For any fixed \( t \in [0, T] \), denote \( K(u; t) = f \left( \frac{X(t + u)}{X(0)} \right) - f \left( \frac{X(t)}{X(0)} \right) \). Then,
\[
\begin{align*}
dK(u; t) &= \left[ f' \left( \frac{X(t + u)}{X(0)} \right) X(t + u) \frac{\mu_X}{X(0)} + \frac{1}{2} f'' \left( \frac{X(t + u)}{X(0)} \right) \left( \frac{X(t + u)}{X(0)} \right)^2 \sigma_X^2 \right] du \\
&\quad + f' \left( \frac{X(t + u)}{X(0)} \right) X(t + u) \frac{\sigma_X}{X(0)} dW(t + u).
\end{align*}
\]

Here and elsewhere in the following section, \( d \) indicates the differential with respect to the variable \( u \).

Note that \( K(0; t) = 0 \), \( h(u; t) = \phi(-\rho K(u; t)) \) and \( h(0; t) = 0 \). According to Ito’s lemma, we obtain
\[
dh(u; t) = -\rho \phi'(-\rho K(u; t))dK(u; t) + \frac{1}{2} \rho^2 \phi''(-\rho K(u; t))d(K(u; t))^2. \quad (30)
\]

Thus, we have
\[
\begin{align*}
h^\mu_{\rho}(t) &= \frac{dh(u; t)}{du} \\
&= -\rho \phi'(0) \frac{X(t)}{X(0)} \left[ f' \left( \frac{X(t)}{X(0)} \right) \mu_X + \frac{1}{2} f'' \left( \frac{X(t)}{X(0)} \right) \left( \frac{X(t)}{X(0)} \right)^2 \sigma_X^2 \right] \\
&\quad + \frac{1}{2} \rho^2 \phi''(0) \sigma_X^2 \left[ f' \left( \frac{X(t)}{X(0)} \right) \left( \frac{X(t)}{X(0)} \right)^2 \right].
\end{align*}
\]

where \( \{\epsilon(t)\} \) is the Gaussian white noise process. \( \square \)

Proof of Theorem 2.

Proof. We first observe that \( L_t \) is a martingale under \( Q \). With
\[
M(t) = \left( -\int_0^t \phi(u) \cdot d\tilde{W}_u - \frac{1}{2} \int_0^t ||\phi(u)||^2 du \right),
\]
and $g(x) = e^x$, we have
\[
\begin{align*}
\frac{dL_t}{L_t} &= \frac{dM(t)}{M(t)} \\
&= g'(M(t))dM(t) + \frac{1}{2}g''(M(t))dM(t)dM(t) \\
&= e^{M(t)}\left(-\theta(t)\cdot d\hat{W}_t - \frac{1}{2}\|\theta(t)\|^2dt\right) + \frac{1}{2}e^{M(t)}\|\theta(t)\|^2dt \\
&= L_t\left(-\theta(t)\cdot d\hat{W}_t\right).
\end{align*}
\]

So $L_t$ is a martingale. In particular, $E^\mathcal{Q}L_t = L_0 = 1$. So $L_t$ is a Radon-Nikodým derivative process. We next show that $\hat{W}_t$ is a martingale under $\mathcal{Q}$. To see this, we calculate
\[
\frac{d(\hat{W}_t L_t)}{L_t} = \frac{\hat{W}_t d\hat{W}_t + L_td\hat{W}_t + d\hat{W}_tdt}{L_t} \\
= -\hat{W}_t L_t \theta(t) \cdot d\hat{W}_t + L_t d\hat{W}_t + L_t \theta(t) dt \\
+ (d\hat{W}_t + \theta(t) dt)(-L_t \theta(t) \cdot d\hat{W}_t) \\
= -\hat{W}_t L_t \theta(t) \cdot d\hat{W}_t + L_t d\hat{W}_t,
\]
so the process $\hat{W}_t L_t$ is a martingale under $\mathcal{Q}$, then
\[
E^\mathcal{Q}[\hat{W}_t L_t] = 1. \
\]
Furthermore,
\[
\frac{d\hat{W}_t \cdot d\hat{W}_t}{L_t} = \frac{(d\hat{W}_t + \theta(t) dt)^2 - d\hat{W}_t \cdot d\hat{W}_t}{L_t} = dt,
\]
so $\hat{W}_t$ is a martingale starting at zero at time zero, with continuous paths and with quadratic variation equal to $t$ at each time $t$. By Lévy’s Theorem, $\hat{W}_t$ is a Brownian motion under $\mathcal{Q}$. \hfill $\square$

**Proof of Theorem 3.**

*Proof.* We have

\[
\begin{align*}
D(t)S(t) &= S(0) \exp\left\{\int_0^t \nu_S(u) \cdot d\hat{W}_u - \frac{1}{2} \int_0^t \|\nu_S(u)\|^2 du\right\}, \\
D(t)B(t, T) &= B(0, T) \exp\left\{\int_0^t \nu_T(u) \cdot d\hat{W}_u - \frac{1}{2} \int_0^t \|\nu_T(u)\|^2 du\right\},
\end{align*}
\]

and hence

\[
F(t, T) = \frac{S(0)}{B(0, T)} \exp\left\{\int_0^t (\nu_S(u) - \nu_T(u)) \cdot d\hat{W}_u - \frac{1}{2} \int_0^t (\|\nu_S(u)\|^2 - \|\nu_T(u)\|^2) du\right\}.
\]

To apply the Itô-Doeblin formula, we first define
\[
X(t) = \int_0^t (\nu_S(u) - \nu_T(u)) \cdot d\hat{W}_u - \frac{1}{2} \int_0^t (\|\nu_S(u)\|^2 - \|\nu_T(u)\|^2) du,
\]
then
\[
\begin{align*}
\frac{dX(t)}{dt} &= (\nu_S(t) - \nu_T(t)) \cdot d\hat{W}_t - \frac{1}{2}(\|\nu_S(t)\|^2 - \|\nu_T(t)\|^2) dt, \\
\frac{dX(t) dX(t)}{dt} &= \|\nu_S(t)\|^2 - 2\nu_S(t) \cdot \nu_T(t) dt + \|\nu_T(t)\|^2 dt.
\end{align*}
\]
With \( f(x) = \frac{S(0)}{B(0,T)}e^x \), we have \( F(t, T) = f(X(t)) \) and
\[
\begin{align*}
dF(t, T) &= f'(X(t))dX(t) + \frac{1}{2} f''(X(t))dX(t)dX(t) \\
&= F(t, T)\left[ (\nu_S(t) - \nu_T(t)) \cdot d\tilde{W}_t - \frac{1}{2} \|\nu_S(t)\|^2 + \frac{1}{2} \|\nu_T(t)\|^2 dt \right. \\
&\quad \left. - \nu_S(t) \cdot \nu_T(t) dt + \frac{1}{2} \|\nu_T(t)\|^2 dt \right] \\
&= F(t, T)(\nu_S(t) - \nu_T(t)) \cdot d\tilde{W}_t^T.
\end{align*}
\]
\[ (33) \]

**Proof of Theorem 4.**

**Proof.** Similar to Proof of Theorem 3. \[ \square \]

**Verification of Eq. (29).**

**Proof.** Using (27), we obtain that
\[
\begin{align*}
d_-(t) &= d_+(t) - \bar{\sigma}_t \sqrt{T-t} \\
d_+^2(t) - d_-^2(t) &= 2 \log \left( \frac{F_{t,T}}{K} \right). \quad (34)
\end{align*}
\]

Eq.(34) implies that
\[
F_{t,T}e^{-d_+^2(t)/2} - Ke^{-d_-^2(t)/2} = 0. \quad (35)
\]

Using (27) and Itô’s lemma, we compute
\[
\begin{align*}
d_+(t) &= \frac{1}{\bar{\sigma}_t \sqrt{T-t}} \left[ \nu_S(t) - \nu_T(t) \cdot d\tilde{W}_t - ||\nu_S(t) - \nu_T(t)||^2 \left( \frac{3}{4} - \frac{\log \left( \frac{F_{t,T}}{K} \right)}{2 \bar{\sigma}_t^2 (T-t)} \right) dt \right].
\end{align*}
\]

Because of
\[
dd_-(t) = dd_+(t) + \frac{||\nu_S(t) - \nu_T(t)||^2}{2 \bar{\sigma}_t \sqrt{T-t}} dt,
\]
we have
\[
\begin{align*}
\int d_+(t)dd_+(t) = \int d_-(t)dd_-(t) = \frac{||\nu_S(t) - \nu_T(t)||^2}{\bar{\sigma}_t^2 (T-t)} dt.
\end{align*}
\]

Applying Itô’s lemma to \( N(x) \) yields
\[
\int dN(d_+(t)) = \frac{1}{\sqrt{2\pi}} e^{-d_+^2(t)/2} \left[ dd_+(t) - d_+(t) \frac{||\nu_S(t) - \nu_T(t)||^2}{2 \bar{\sigma}_t^2 (T-t)} dt \right]. \quad (36)
\]

Combining the results in Eq.(34), (35) and (36), we have
\[
\begin{align*}
F_{t,T}dN(d_+(t)) &= \frac{1}{\sqrt{2\pi}} F_{t,T} e^{-d_+^2(t)/2} \left[ dd_+(t) - d_+(t) \frac{||\nu_S(t) - \nu_T(t)||^2}{2 \bar{\sigma}_t^2 (T-t)} dt \right] \\
&= \frac{1}{\sqrt{2\pi}} Ke^{-d_-^2(t)/2} \left[ dd_-(t) - d_-(t) \frac{||\nu_S(t) - \nu_T(t)||^2}{2 \bar{\sigma}_t^2 (T-t)} dt - \frac{||\nu_S(t) - \nu_T(t)||^2}{\bar{\sigma}_t \sqrt{T-t}} dt \right] \\
&= K \left[ dN(d_-(t)) - \frac{1}{\sqrt{2\pi}} e^{-d_-^2(t)/2} \frac{||\nu_S(t) - \nu_T(t)||^2}{\bar{\sigma}_t \sqrt{T-t}} dt \right].
\end{align*}
\]

Therefore,
\[
F_{t,T}dN(d_+(t)) - KdN(d_-(t)) = -\frac{K}{\sqrt{2\pi}} e^{-d_-^2(t)/2} \frac{||\nu_S(t) - \nu_T(t)||^2}{\bar{\sigma}_t \sqrt{T-t}} dt. \quad (37)
\]
Using the results in Theorem 3, Eq. (35) and Eq. (36), we compute
\[
dF_{t, T} dN(d_+(t)) = \frac{K}{\sqrt{2\pi}} e^{-d_2(t)/2} \left| \frac{\nu_T(t) - \nu_S(t)}{\sigma_t \sqrt{T-t}} \right|^2.
\] (38)

Combining the results in Eq. (37) and (38), we have
\[
F_{t, T} dN(d_+(t)) + dF_{t, T} dN(d_+(t)) - KdN(d_-(t)) = 0.
\]
This completes the verification of Eq. (29).

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REFERENCES
[1] M. Anton and C. Polk, Connected stocks, The Journal of Finance, 69 (2014), 1099–1127.
[2] F. Black and M. Scholes, The pricing of options and corporate liabilities, Journal of Political Economy, 81 (1973), 637-654.
[3] M. K. Brunnermeier, Deciphering the liquidity and credit crunch 2007-2008, Journal of Economic Perspectives, 23 (2009), 77–100.
[4] M. Carlson, A brief history of the 1987 stock market crash with a discussion of the federal reserve response, in Finance and Economics Discussion Series, Federal Reserve Board, Washington, DC., 2006.
[5] R. Cont and P. Tankov, Financial Modelling with Jump Processes, Chapman & Hall/CRC Financial Mathematics Series, Chapman & Hall/CRC, Boca Raton, FL, 2004.
[6] R. Cont and L. Wagalath, Fire sales forensics: Measuring endogenous risk, Mathematical Finance, 26 (2016), 835–866.
[7] J. Coval and E. Stafford, Asset fire sales (and purchases) in equity markets, Journal of Financial Economics, 86 (2007), 479–512.
[8] D. Duffie and K. Singleton, Modeling term structures of defaultable bonds, The Review of Financial Studies, 12 (1999), 687–720.
[9] A. Ellul, C. Jotikasthira and T. C. Lundblad, Regulatory pressure and fire sales in the corporate bond market, Journal of Financial Economics, 101 (2011), 596–620.
[10] H. Geman, N. El Karoui and J.-C. Rochet, Changes of numéraire, changes of probability measure and option pricing, Journal of Applied Probability, 32 (1995), 443–458.
[11] R. Greenwood and D. Thesmar, Stock price fragility, Journal of Financial Economics, 102 (2011), 471–490.
[12] T. Hida, J. Potthoff and L. Streit, Dirichlet forms and white noise analysis, Communications in Mathematical Physics, 116 (1988), 235–245.
[13] A. Khandani and A. W. Lo, What happened to the quants in August 2007? Evidence from factors and transactions data, Journal of Financial Markets, 14 (2011), 1–46.
[14] A. S. Kyle and W. Xiong, Contagion as a wealth effect, The Journal of Finance, 56 (2001), 1401–1440.
[15] R. C. Merton, Option pricing when underlying stock returns are discontinuous, Journal of Financial Economics, 3 (1976), 125–144.
[16] P. Pedler, Occupation times for two state Markov chains, Journal of Applied Probability, 8 (1971), 381–390.
[17] B. Sericola, Occupation times in Markov processes, Communications in Statistics. Stochastic Models, 16 (2000), 479–510.
[18] S. Shreve, Stochastic Calculus for Finance II: Continuous-Time Models, Springer Finance. Springer-Verlag, New York, 2004.
[19] A. Shleifer and R. W. Vishny, Liquidation values and debt capacity: A market equilibrium approach, The Journal of Finance, 47 (1992), 1343–1366.
[20] A. Shleifer and R. W. Vishny, Fire sales in finance and macroeconomics, Journal of Economic Perspectives, 25 (2011), 29–48.
[21] R. Wiggins, T. Piontek and A. Metrick, The Lehman Brothers Bankruptcy A: Overview. Yale Program on Financial Stability Case Study 2014-3A-V1, SSRN, (2015), 23 pp.
[22] Q.-Q. Yang, W.-K. Ching and T.-K. Siu, Pricing vulnerable options under a Markov-modulated jump-diffusion model with fire sales, *Journal of Industrial and Management Optimization*, 15 (2019), 293–318.

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