Even the Easiest(?) Graph Coloring Problem is not Easy in Streaming!

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Abstract

We study a graph coloring problem that is otherwise easy but becomes quite non-trivial in the one-pass streaming model. In contrast to previous graph coloring problems in streaming that try to find an assignment of colors to vertices, our main work is on estimating the number of conflicting or monochromatic edges given a coloring function that is streaming along with the graph; we call the problem CONFLICT-Est. The coloring function on a vertex can be read or accessed only when the vertex is revealed in the stream. If we need the color on a vertex that has streamed past, then that color, along with its vertex, has to be stored explicitly. We provide algorithms for a graph that is streaming in different variants of the one-pass vertex arrival streaming model, viz. the VERTEX ARRIVAL (VA), Vertex Arrival With Degree Oracle (VADEG), VERTEX ARRIVAL IN RANDOM ORDER (VARAND) models, with special focus on the random order model. We also provide matching lower bounds for most of the cases. The mainstay of our work is in showing that the properties of a random order stream can be exploited to design streaming algorithms for estimating the number of conflicting edges. We have also obtained a lower bound, though not matching the upper bound, for the random order model. Among all the three models vis-a-vis this problem, we can show a clear separation of power in favor of the VARAND model.

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1 Introduction

The chromatic number $\chi(G)$ of an $n$-vertex graph $G = (V, E)$ is the minimum number of colors needed to color the vertices of $V$ so that no two adjacent vertices get the same color. The chromatic number problem is NP-hard and even hard to approximate within a factor of $n^{1-\varepsilon}$ for any constant $\varepsilon > 0$ [FK98, Zuk07, KP06]. For any connected undirected graph $G$ with maximum degree $\Delta$, $\chi(G)$ is at most $\Delta + 1$ [Viz64]. This existential coloring scheme can be made constructive across different models of computation. A seminal result of recent vintage is that the $\Delta + 1$ coloring can be done in the streaming model [ACK19]. Of late, there has been interest in graph coloring problems in the sub-linear regime across a variety of models [AA20a, ACK19, BDH19, BG18, BCG19]. Keeping with the trend of coloring problems, these works look at assigning colors to vertices. Since the size of the output will be as large as the number of vertices, researchers study the semi-streaming model [McG14] for streaming graphs. In the semi-streaming model, $\tilde{O}(n)$ space is allowed.

In a marked departure from the above works that look at the classical coloring problem, the starting point of our work is (inarguably?) the simplest question one can ask in graph coloring – given a coloring function $f : V \to \{1, \ldots, C\}$ on the vertex set $V$ of a graph $G = (V, E)$, is $f$ a valid coloring, i.e., for any edge $e \in E$, do both the endpoints of $e$ have different colors? This is the problem one encounters while proving that the problem of chromatic number belongs to the class NP [GJ79]. CONFLICT-EST, the problem of estimating the number of monochromatic (or, conflicting) edges for a graph $G$ given a coloring function $f$, remains a simple problem in the RAM model; it even remains simple in the one-pass streaming model if the coloring function $f$ is marked on a public board, readable at all times. We show that the problem throws up interesting consequences if the coloring function $f$ on a vertex is revealed only when the vertex is revealed in the stream. For a streaming graph, if the vertices are assigned colors arbitrarily or randomly on-the-fly while it is exposed, our results can also be used to estimate the number of conflicting edges. These problems also find their use in estimating the number of conflicts in a job schedule and verifying a given job schedule in a streaming setting. This can also be extended to problems in various domains like frequency assignment in wireless mobile networks and register allocation [EHKR09]. As the problem, by its nature, admits an estimate or a yes-no answer, the need of the space to store all vertices as in the semi-streaming model goes away and we can focus on space efficient algorithms in the conventional graph streaming models like VERTEX ARRIVAL [CDK19]. We also note in passing that many of the trend setting problems in streaming, like frequency moments, distinct elements, majority, etc. have been simple problems in the ubiquitous RAM model as the coloring problem we solve here.

2 Preliminaries

2.1 Notations and the streaming models

Notations. We denote the set $\{1, \ldots, n\}$ by $[n]$. $G(V(G), E(G))$ denotes a graph where $V(G)$ and $E(G)$ denote the set of vertices and edges of $G$, respectively; $|V| = n$ and $|E| = m$. We will write only $V$ and $E$ for vertices and edges when the graph is clear from the context. We denote $E_M \subseteq E$ as the set of monochromatic edges. The set of neighbors of a vertex $u \in V(G)$ is denoted by $N_G(u)$ and the degree of a vertex $u \in V(G)$ is denoted by $d_G(u)$. Let $N_G(u) = N_G^-(u) \uplus N_G^+(u)$ where $N_G^-(u)$ and $N_G^+(u)$ denote the set of neighbors of $u$ that have been exposed already and are yet to be exposed, respectively in the stream. Also, $d_G(u) = d_G^-(u) + d_G^+(u)$ where $d_G^-(u) = |N_G^-(u)|$ and $d_G^+(u) = |N_G^+(u)|$. For a monochromatic edge $(u, v) \in E_M$, we refer to $u$ and $v$ as monochromatic.

*$\tilde{O}(\cdot)$ hides a polylogarithmic factor.
neighbors of each other. We define $d_M(u)$ to be the number of monochromatic neighbors of $u$ and hence, the monochromatic degree of $u$.

Let $\mathbb{E}[X]$ denote the expectation of the random variable $X$. For an event $\mathcal{E}$, $\overline{\mathcal{E}}$ denotes the complement of $\mathcal{E}$. $\mathbb{P}(\mathcal{E})$ denotes the probability of an event $\mathcal{E}$. The statement “event $\mathcal{E}$ occurs with high probability” is equivalent to $\mathbb{P}(\mathcal{E}) \geq 1 - \frac{1}{c}$, where $c$ is an absolute constant. The statement “$a$ is a $1 \pm \varepsilon$ multiplicative approximation of $b$” means $|b - a| \leq \varepsilon \cdot b$. For $x \in \mathbb{R}$, $\exp(x)$ denotes the standard exponential function, that is, $e^x$. By polylogarithmic, we mean $\mathcal{O}\left((\log n/\varepsilon)^{O(1)}\right)$. The notation $\tilde{\mathcal{O}}(\cdot)$ hides a polylogarithmic term in $\mathcal{O}(\cdot)$.

**Streaming models for graphs.** As alluded to earlier, the crux of the problem depends on the way the coloring function $f$ is revealed in the stream. The details follow.

(i) **Vertex Arrival (VA):** The vertices of $V$ are exposed in an arbitrary order. After a vertex $v \in V$ is exposed, all the edges between $v$ and pre-exposed neighbors of $v$, are revealed. This set of edges are revealed one by one in an arbitrary order. Along with the vertex $v$, only the color $f(v)$ is exposed, and not the colors of any pre-exposed vertices. So, we can check the monochromaticity of an edge $(v, u)$ only if $u$ and $f(u)$ are explicitly stored.

(ii) **Vertex Arrival with Degree Oracle (VAdeg)** [MVV16, BS20]: This model works same as the VA model in terms of exposure of the vertex $v$ and the coloring on it; but we are allowed to know the degree $d_G(v)$ of the currently exposed vertex $v$ from a degree oracle on $G$.

(iii) **Vertex Arrival in Random Order (VARAND)** [SK12, TGRV14]: This model works same as the VA model but the vertex sequence revealed is equally likely to be any one of the permutations of the vertices.

(iv) **Edge Arrival (EA):** The stream consists of edges of $G$ in an arbitrary order. As the edge $e$ is revealed, so are the colors on its endpoints. Thus the conflicts can be easily checked.

(v) **Adjacency List (AL):** The vertices of $V$ are exposed in an arbitrary order. When a vertex $v$ is exposed, all the edges that are incident to $v$, are revealed one by one in an arbitrary order. Note that in this model each edge is exposed twice, once for each exposure of an incident vertex. As in the VA model, here also only $v$’s color $f(v)$ is exposed.

As the conflicts can be checked easily in the EA model in $O(1)$ space, a logarithmic counter is enough to count the number of monochromatic edges. The AL model works almost the same as the VAdeg model. So, we focus on the three models – VA, VAdeg and VARAND in this work and show that they have a clear separation in their power vis-a-vis the problem we solve. A crucial takeaway from our work is that the random order assumption on exposure of vertices has huge improvements in space complexity.

### 2.2 Problem definitions, results and the ideas

**Problem definition.** Let the vertices of $G$ be colored with a function $f : V(G) \rightarrow [C]$, for $C \in \mathbb{N}$. An edge $(u, v) \in E(G)$ is said to be monochromatic or conflicting with respect to $f$ if $f(u) = f(v)$. A coloring function $f$ is called valid if no edge in $E(G)$ is monochromatic with respect to $f$. For a given parameter $\varepsilon \in (0, 1)$, $f$ is said to be $\varepsilon$-far from being valid if at least $\varepsilon \cdot |E(G)|$ edges are monochromatic with respect to $f$. We study the following problems.

**Problem 2.1 (Conflict Estimation aka Conflict-Est).** A graph $G = (V, E)$ and a coloring function $f : V(G) \rightarrow [C]$ are streaming inputs. Given an input parameter $\varepsilon > 0$, the objective is to estimate the number of monochromatic edges in $G$ within a $(1 \pm \varepsilon)$-factor.

**Problem 2.2 (Conflict Separation aka Conflict-Sep).** A graph $G = (V, E)$ and a coloring function $f : V(G) \rightarrow [C]$ are streaming inputs. Given an input parameter $\varepsilon > 0$, the objective is to distinguish if the coloring function $f$ is valid or is $\varepsilon$-far from being valid.
Remark 1. Problem 2.1 is our main focus, but we will mention a result on Problem 2.2 in Section 4.2. Notice that Conflict-Est is a difficult problem than Conflict-Sep.

The results and the ideas involved. All our upper and lower bounds on space are for one-pass streaming algorithms. Table 1 states our results for the Conflict-Est problem, the main problem we solve in this paper, across different variants of the VA model. The main thrust of our work is on estimating monochromatic edges under random order stream. For random order stream, we present both upper and lower bounds in Sections 4 and 5. There is a gap between the upper and lower bounds in the VARAND model, though we have a strong hunch that our upper bound is tight. Apart from the above, using a structural result on graphs, we show in Section 4.2 that the Conflict-Sep problem admits an easy algorithm in the VARAND model. To give a complete picture across different variants of the VA models, we show matching upper and lower bounds for the VA and VADEG models in Section 3 and Appendix E.

| Model   | VA          | VARAND       | VADEG        |
|---------|-------------|---------------|--------------|
| Upper Bound | $\tilde{O}\left(\min\{|V|, \frac{|V|^2}{T}\}\right)$ (Sec. 3, Thm. 3.1) | $\tilde{O}\left(\frac{|V|}{\sqrt{T}}\right)$ (Sec. 4, Thm. 4.1) | $\tilde{O}\left(\min\{|V|, \frac{|E|}{T}\}\right)$ (Sec. 3, Thm. 3.2) |
| Lower Bound | $\Omega\left(\min\{|V|, \frac{|V|^2}{T}\}\right)$ (Sec. 5, Thm. 5.1) | $\Omega\left(\frac{|V|}{\sqrt{T}}\right)$ (Sec. 5, Thm. 5.1) | $\Omega\left(\min\{|V|, \frac{|E|}{T}\}\right)$ (Sec. E.2, Thm. E.2) |

Table 1: This table shows our results on Conflict-Est on a graph $G(V,E)$ across different Vertex Arrival models. Here, $T > 0$ denotes the promised lower bound on the number of monochromatic edges.

The promise $T$ on the number of monochromatic edges is a very standard assumption for estimating substructures in the world of graph streaming algorithms [KKP18, KMSS12, KMPV19, MVV16, BC17].†

We now briefly mention the salient ideas involved. For the simpler variant of Conflict-Est in VA model, we first check if $|V| \geq T$. If yes, we store all the vertices and their colors in the stream to determine the exact value of the number of monochromatic edges. Otherwise, we sample each pair of vertices $\{u, v\}$ in $\binom{|V|}{2}$ with probability $\tilde{O}(1/T)$ independently§ before the stream starts. When the stream comes, we compute the number of monochromatic edges from this sample. The details are in Section 3. Though the algorithm looks extremely simple, it matches the lower bound result for Conflict-Est in VA model, presented in Appendix E. The VADEG model with its added power of a degree oracle, allows us to know $d_G(u)$ for a vertex $u$ and as edges to pre-exposed vertices are revealed, we also know $d^+_G(u)$ and $d^-_G(u)$. This allows us to use sampling to store vertices and to use a technique which we call sampling into the future where indices of random neighbors, out of $d^+_G(u)$ neighbors, are selected for future checking. The upper bound result, for Conflict-Est in VADEG model, is presented in Section 3 and it is tight as we prove a matching lower bound in Appendix E.

The algorithm for Conflict-Est in VARAND model is the mainstay of our work and is presented in Section 4. We redefine the degree in terms of the number of monochromatic neighbors a vertex has in the randomly sampled set. Here, we estimate the high monochromatic degree and low monochromatic degree vertices separately by sampling a random subset of vertices. While the monochromatic degree for the high degree vertices can be extrapolated from the sample, handling

†Here we have cited a few. However, there are huge amount of relevant literature.
‡$\binom{|V|}{2}$ denotes the set of all size 2 subsets of $V(G)$.
§Note that we might sample some pairs that are not forming edges in the graph.
low monochromatic degree vertices individually in the same way does not work. To get around, we

group such vertices having similar monochromatic degrees and treat them as an entity. We also

provide a lower bound for the VARAND model, in Section 5, using a reduction from multi-party set
disjointness; though there is a gap in terms of the exponent in $T$.

The highlights of our work are as follows:

- We show that possibly the easiest graph coloring problem is worth studying over streams.
- For researchers working in streaming, the gold standard is the EA model as most problems
  are non-trivial in this model. We point out a problem that is harder to solve in the VA model
  as compared to the EA model.
- We show that the three VA related models have a clear separation in their space complexities
  vis-a-vis the problem we solve. We could exploit the random order of the arrival of the vertices
to get substantial improvements in space complexity.
- We could obtain lower bounds for all the three models but the lower bounds are matching for
  the VA and VAdeg models.

2.3 Prior works on graph coloring in semi-streaming model.

Bera and Ghosh [BG18] commenced the study of vertex coloring in the semi-streaming model. They
devise a randomized one pass streaming algorithm that finds a $(1 + \varepsilon)\Delta$ vertex coloring in $\tilde{O}(n)$

space. Assadi et al. [ACK19] find a proper vertex coloring using $\Delta + 1$ colors via various classes of
sublinear algorithms. Their state of the art contributions can be attributed to a key result called
the palette-sparsification theorem which states that for an $n$-vertex graph with maximum degree $\Delta,$
if $O(\log n)$ colors are sampled independently and uniformly at random for each vertex from a list
of $\Delta + 1$ colors, then with a high probability a proper $\Delta + 1$ coloring exists for the graph. They

design a randomized one-pass dynamic streaming algorithm for the $\Delta + 1$ coloring using $\tilde{O}(n)$

space. The algorithm takes post-processing $\tilde{O}(n\sqrt{\Delta})$ time and assumes a prior knowledge of $\Delta$. Alon
and Assadi [AA20b] improve the palette sparsification result of [ACK19]. They consider situations
where the number of colors available is both more than and less than $\Delta + 1$ colors. They show

that sampling $O_\varepsilon(\sqrt{\log n})$ colors per vertex is sufficient and necessary for a $(1 + \varepsilon)\Delta$ coloring. Bera et
al. [BCG19] give a new graph coloring algorithm in the semi-streaming model where the number

of colors used is parameterized by the degeneracy $\kappa$. The key idea is a low degeneracy partition,
also employed in [BG18]. The numbers of colors used to properly color the graph is $\kappa + o(\kappa)$

and post-processing time of the algorithm is improved to $O(n)$, without any prior knowledge about
$\kappa.$ Behnezhad et al. [BDH+19] were the first to give one-pass W-streaming algorithms (streaming
algorithms where outputs are produced in a streaming fashion as opposed to outputs given finally
at the end) for edge coloring both when the edges arrive in a random order or in an adversarial

fashion.

3 CONFLICT-EST in VA and VAdeg models

In this Section, we design algorithms for CONFLICT-EST problem in the VA and VAdeg models. We
show matching lower bounds later in Appendix C. Mainly, we prove the following two theorems

here.

**Theorem 3.1.** Given any graph $G = (V, E)$ and a coloring function $f : V \rightarrow [C]$ as input in the

stream, there exists an algorithm that solves the CONFLICT-EST problem in the VA model with high
probability in $\tilde{O}\left(\min\left(|V|, \frac{|V|^2}{T}\right)\right)$ space, where $T$ is a lower bound on the number of monochromatic edges in the graph.

**Theorem 3.2.** Given any graph $G = (V, E)$ and a coloring function $f : V \to [C]$ as input in the stream, there exists an algorithm that solves the CONFLICT-EST problem in the VAdEG model with high probability in $\tilde{O}\left(\min\{|V|, \frac{|E|}{T}\}\right)$ space, where $T$ is a lower bound on the number of monochromatic edges in the graph.

Before going to the algorithms for CONFLICT-EST problem in the VA and VAdEG model, we discuss as a warm-up, a two-pass algorithm for CONFLICT-EST in the VA model that uses $\tilde{O}\left(\min\{|V|, \frac{|E|}{T}\}\right)$ space, where $T$ is the promised lower bound on the number of monochromatic edges in the graph. Here we assume that $|E|$ is known to the algorithm. However, this assumption can be removed easily in a setting with two passes.

**A two-pass algorithm for CONFLICT-EST in VA model (described informally):**

If $T \leq \frac{|E|}{|V|}$: Our algorithm stores all the vertices and their colors. Thus we can determine the number of monochromatic edges exactly. The algorithm in this case is one pass and uses $\tilde{O}(|V|)$ space.

If $T > \frac{|E|}{|V|}$: In the first pass, store each edge with probability $\tilde{O}\left(\frac{1}{T}\right)$. In the second pass, we check each edge stored in the first pass for conflict. In this way, we determine the number of monochromatic edges in the sample, from which, we can obtain a desired approximation of the number of monochromatic edges in the graph. The space complexity of our algorithm in this case is $\tilde{O}\left(\frac{|E|}{T}\right)$.

If only one pass is allowed, the above algorithm, when $T > \frac{|E|}{|V|}$, can not be simulated in VA model because of the following reason. Consider an edge $(u, v) \in E_M$ such that $u$ is exposed before $v$. Note that we will be able to know about the edge only when $v$ is exposed but we will be able to check whether $(u, v) \in E_M$ only when we have stored $u$ and its color. However, there is no clue about the edge $(u, v)$ when $u$ is exposed. So, to solve it in one-pass, we sample each pair of vertices (without bothering if there is an edge between them) with probability $\tilde{O}\left(\frac{1}{T}\right)$, before the start of the stream, and determine the number of monochromatic edges in the sample to get an estimate of the number of monochromatic edges in $G$. This implies that the space complexity of the algorithm for CONFLICT-EST in VA model is $\tilde{O}\left(\frac{|V|^2}{T}\right)$ as stated in Theorem 3.1. In VAdEG model, when $u$ is exposed we will get $d_G(u)$ and hence $d_G^2(u)$. The degree information, when $u$ is exposed, gives some statistics regarding how the vertex $u$ might be useful in the future. We exploit this advantage of VAdEG model over VA model to get an algorithm for CONFLICT-EST that has better space complexity (See Theorem 3.2).

### 3.1 Proof of Theorem 3.1

Our algorithm for CONFLICT-EST for VA model, first checks if $T \leq |V|$. If yes, we store all the vertices along with their colors to estimate the number of monochromatic edges in the graph exactly. So, the space used by the algorithm is $\tilde{O}(|V|)$ when $T \leq |V|$. We will be done by giving an algorithm for CONFLICT-EST in VA model that uses $\tilde{O}\left(\frac{|V|^2}{T}\right)$ space. This algorithm will only be executed when $T > |V|$.
Let \( V = \{v_1, \ldots, v_n\} \) be the vertices of the graph. Our algorithm starts by generating a sample of vertex pairs where each \( \{v_i, v_j\} \) is added to \( Z \), independently, with probability \( 10^{-2} \). Note that \( Z \) is obtained before the start of the stream. Over the stream, we check the following for each \( \{v_i, v_j\} \in Z \): whether \( (v_i, v_j) \in E \) and is monochromatic. Let \( S \subseteq Z \) be the set of monochromatic edges in \( Z \). Note that the expected value of \( |S| \) is given by \( \mathbb{E}[|S|] = \frac{30 \log n}{\tilde{O}(n)} |E_M| \).

We report \( \hat{m} = \frac{\varepsilon^2 T}{30 \log n} |S| \) as our estimate for \( |E_M| \). Applying Chernoff bound (See Lemma A.1 in Appendix A), we can show that

\[
\mathbb{P}(|\hat{m} - |E_M|| \geq \varepsilon |E_M|) \leq \mathbb{P}(|\mathbb{E}[|S|] - \mathbb{E}[|S|]| \geq \varepsilon \mathbb{E}[|S|]) \leq \exp \left( \frac{-\mathbb{E}[|S|] \varepsilon^2}{3} \right) \leq \frac{1}{n^{10}}.
\]

Note that the last inequality holds as \( \mathbb{E}[|S|] = \frac{30 \log n}{\varepsilon^2 T} |E_M| \) and \( |E_M| \geq T \).

Observe that the space used by our algorithm is \( \tilde{O}(|Z|) \) when \( T > \frac{|E|}{|V|} \). Note that \( \mathbb{E}[|Z|] = \frac{30 \log n}{\varepsilon^2 T} \binom{n}{2} \). Applying Chernoff bound (See Lemma A.1 in Appendix A), we can show that \( |Z| = \tilde{O} \left( \frac{n^2}{T^2} \right) \) with high probability.

Putting together the space complexities of our algorithms for the case \( T \leq |V| \) and \( T > |V| \), we have the desired bound on the space.

### 3.2 Proof of Theorem 3.2

For simplicity of presentation, assume that we know the number of edges \( |E| \) in the graph. We will discuss ways to remove this assumption later.

#### 3.2.1 Algorithm for Conflict-Est in VAdeg model when \( |E| \) is known

Our algorithm for Conflict-Est for VAdeg model, first checks if \( T \leq \frac{|E|}{|V|} \). If \( T \leq \frac{|E|}{|V|} \), we store all the vertices along with their colors to estimate the number of monochromatic edges in the graph exactly. So, the space used by the algorithm is \( \tilde{O}(|V|) \) when \( T \leq \frac{|E|}{|V|} \). We will be done by giving an algorithm for Conflict-Est in VAdeg model that uses \( \tilde{O} \left( \frac{|E|}{|V|} \right) \) space. This algorithm will be executed only when \( T > \frac{|E|}{|V|} \).

Let \( V = \{v_1, \ldots, v_n\} \) and w.o.l.g. the vertices are exposed in the order \( v_1, \ldots, v_n \). However, our algorithm does not know about the ordering of the vertices in the stream. Our algorithm stores the following information.

- A random subset \( Y \subseteq V \times [n] \) that will be generated over the stream;
- a subset \( A \) of vertices formed from the first elements in the pairs present in \( Y \); the colors of the vertices are also stored;
- for each vertex \( v \in A \), a number \( \ell_v \) that denotes the number of neighbors in \( N^+_G(v) \) that have been exposed. So, \( \ell_v \) is initialized to 0 when \( v \) gets exposed in the stream and is at most \( |N^+_G(v)| \) at any instance of the stream;
- a subset \( S \subseteq E_M \) of the set of monochromatic edges in \( G \).

When a vertex \( v_j \) is exposed, our algorithm performs the following steps:

(i) Get \( d_G(v_j) \) from the degree oracle and \( d^+_G(v_j) \) from the exposed edges and compute \( d^+_G(v_j) \);
(ii) Add \((v_j, k), k \in [d_G^+(v_j)]\), with probability \(\frac{30 \log n}{\varepsilon^2 T}\) to \(Y\), independently;

(iii) Add \(v_j\) along with its color to \(A\) if at least one \((v_j, k)\) is added to \(Y\).

(iv) For each \(v_i \in A\) such that \((v_i, v_j) \in E\), increment \(\ell_{v_i}\) by 1.

(v) For each \(v_i \in A\) such that \((v_i, \ell_{v_i}) \in Y\), check whether \((v_i, v_j)\) forms a monochromatic edge. If yes, add \((v_i, v_j)\) to \(S\). (This step ensures independence so that Chernoff bounds can be used. See Remark 2 below.)

The main catch of the algorithm for CONFLICT-Est in VADEG model is in Step-(ii). Due to the added power of degree oracle, we are able to sample edges that have not arrived explicitly in the stream. We referred to this phenomenon as sampling into the future in Section 2.

At the end of the stream, we report \(\hat{m} = \frac{\varepsilon^2}{30 \log n} |S|\) as the estimate of \(|E_M|\). Now, we show that \(\mathbb{P}(|\hat{m} - |E_M|| \geq \varepsilon |E_M|) \leq \frac{1}{n^{10}}\). Consider a monochromatic edge \((v_i, v_j) \in E_M\). W.l.o.g., assume that \(v_j\) is exposed sometime after \(v_i\) is exposed in the stream. Let \(r \in [d_G^+(v_i)]\) be such that \(v_i\) has \(r - 1\) neighbors in \(\{v_{i+1}, \ldots, v_{j-1}\}\). So, \(v_j\) is the \(r\)-th neighbor of \(v_i\) exposed after the exposure of \(v_i\). From the description of the algorithm, \((v_i, v_j)\) is added to \(S\) if and only if \((v_i, r)\) is added to \(Y\). Note that \((v_i, r)\) can be added to \(Y\) only when the vertex \(v_i\) is exposed in the stream. Before calculating \(E[|S|]\) and applying Chernoff bound, we focus on the following remark.

**Remark 2.** At the first look, it might appear that the monochromatic edges are not independently added to \(S\). For example, let us consider the following situation. Let \((v_i, r')\), with \(r' \in [d_G^+(v_i)]\) and \(r' \neq r\), is added to \(Y\), that is, \(v_i\) is present in \(A\) and the color of \(v_i\) is stored. So, when \(v_j\) gets exposed along with its color, we can check whether \((v_i, v_j)\) is monochromatic irrespective of \((v_i, r)\) being added to \(Y\). But the crucial point is that we add \((v_i, v_j)\) to \(S\) only when \((v_i, r)\) is added to \(Y\). However, \((v_k, \ell)\), with \(k \in [n]\) and \(\ell \in [d_G^+(v_i)]\), are added to \(Y\), independently. That is, each monochromatic edge in \(E_M\) is added to \(S\), independently.

The probability that a monochromatic edge is added to \(S\) is \(\frac{30 \log n}{\varepsilon^2 T}\). That is, \(\mathbb{E}[|S|] = \frac{30 \log n}{\varepsilon^2 T} |E_M|\). Applying Chernoff bound (See Lemma A.1 in Appendix A), we can guarantee that

\[
\mathbb{P}(|\hat{m} - |E_M|| \geq \varepsilon |E_M|) \leq \mathbb{P}(||S| - \mathbb{E}[|S|]| \geq \varepsilon \mathbb{E}[|S|]) \leq \exp \left( \frac{-\mathbb{E}[|S|]\varepsilon^2}{3} \right) \leq \frac{1}{n^{10}}.
\]

Note that the last inequality holds as \(|E_M| \geq T\). Observe that the space used by the algorithm is \(\tilde{O}(|Y| + |A| + |S|) = \tilde{O}(|Y|)\). Note that \(\mathbb{E}[|Y|] = \sum_{i=1}^{n} d_G^+(v_i) \cdot \frac{30 \log n}{\varepsilon^2 T} = \frac{30 |E| \log n}{\varepsilon^2 T}\). Applying Chernoff bound (See Lemma A.1 in Appendix A), we can say that \(|Y| = \tilde{O}\left(\frac{|E|}{T}\right)\) with high probability. Putting together the space complexities of our algorithms for the case \(T \leq \frac{|E|}{|V|}\) and \(T > \frac{|E|}{|V|}\), we have the desired bound on the space.

### 3.2.2 Modifying the algorithm in Section 3.2.1 when |E| is unknown

In the modified algorithm, we maintain a counter defined as follows.

\[
\text{CNT} := \sum_{v \text{ has been exposed}} d_G^+(v).
\]

Consider the following observation about CNT that will be used in our analysis. As mentioned earlier, \(|V| = n\).
The formal description of the modified algorithm is presented in Algorithm 1. We describe the algorithm and its analysis by breaking the range of complexity of the modified algorithm is less than $\tau$ and $\ell \in [d^+_{G}(v_j)]$. So, we have the following observation that will be used later in our analysis.

**Observation 3.4.** With high probability, $|Y| = \tilde{O}(\log n)$ for all the instances in the stream while CNT is less than $\tau$.  

**Proof.** Let $v_k$ be the first exposed vertex in the stream when CNT is more than $\tau$. Also, let $U = \bigcup_{j=1}^{k-1} \{(v_j, \ell) : \ell \in [d^+_{G}(v_j)]\}$, where $\bigcup$ denotes disjoint union. Observe that $|U| = \sum_{j=1}^{k-1} d^+_{G}(v_j) < \tau$. We construct $Y$ by selecting independently each element of $U$ with probability $\frac{3000 \log n}{e^{\tau} T}$. Recall that $\tau = 100 |V| T \log n$. So, $E[|Y|] = \frac{3000 |E| \log n}{e^{\tau} T} < \frac{3000 \tau \log n}{3000 \log n} = \frac{300000 \log n}{|V|}$. The observation follows by applying Chernoff bound (see Lemma [A.1] (iii) in Appendix [A]).

However, the modified algorithm behaves differently once CNT is more than $\tau = 100 |V| T \log n$. Let $v_k$ be as defined earlier. We maintain two extra objects, as described below, after CNT crosses $\tau$.

- The set of vertices $B = \{v_k, \ldots, v_n\}$ and their colors;
- A counter $C_{>\tau}$ that denotes the number of monochromatic edges having both the endpoints in $B$.

The formal description of the modified algorithm is presented in Algorithm 1. We describe the algorithm and its analysis by breaking the range of $|E|$ into two cases, that is, $T > \frac{|E|}{100 |V| \log n}$ (or $|E| \leq 100 T |V| \log n = \tau$) and $T < \frac{|E|}{100 |V| \log n}$ (or $|E| > 100 T |V| \log n = \tau$). We show that the space complexity of the modified algorithm is $\tilde{O} \left( \frac{|E|}{T} \right)$ in the first case and is $\tilde{O}(|V|)$ in the latter case with high probability. Observe that this will imply the desired result as claimed in Theorem 3.2.1.

**|E| \leq 100 T |V| \log n:** In this case, by Observation 3.3, CNT never goes beyond $\tau = 100 |V| T \log n$. That is, the algorithm behaves exactly same as that of the algorithm presented in Section 3.2.1 for the case $T > \frac{|E|}{100 |V| \log n}$. Hence, the algorithm reports the desired output using $\tilde{O} \left( \frac{|E|}{T} \right)$ space, with high probability.

**|E| > 100 T |V| \log n:** In this case, by Observation 3.3, there will be an instance (say when vertex $v_k$ is exposed) such that CNT goes beyond $\tau$ for the first time. Then we start storing all the vertices and their colors in $B = \{v_k, \ldots, v_n\}$. We stop updating $Y$ and $A$ after $v_k$ is exposed. However, we update $S$ until end of the stream as we were doing previously in Section 3.2.1. Along with $S$, we maintain the number of monochromatic edges (say $C_{>\tau}$) having both the endpoints in $B = \{v_k, \ldots, v_n\}$. Note that $C_{>\tau}$ is maintained exactly. Finally, we report $\hat{m} = C_{\leq\tau} + C_{>\tau}$ as the output, where 0 or $C_{\leq\tau} = \frac{e^{\tau} T}{3000 \log n} |S|$ depending on whether $|S| \leq \frac{60 \log n}{e^{\tau} T}$ or not, respectively. By Observation 3.4, with high probability, $|Y| = \tilde{O}(|V|)$ for all the instances when CNT is less than $\tau$ (that is before the exposure of $v_k$). Also, after the exposure of $v_k$,
Algorithm 1: CONFLICT-EST-DEG(\(\varepsilon, T\)): CONFLICT-EST in VADEG model

**Input:** \(G = (V, E)\) and a coloring function \(f\) on \(V\) in the VADEG model, parameters \(T\) and \(\varepsilon\) where \(\varepsilon, T \geq 0\).

**Output:** \(\hat{m}\), that is, a \((1 \pm \varepsilon)\) approximation to \(|E_M|\).

for (each exposed vertex \(v_j\)) do

- For each \(v_i \in A\) such that \((v_i, v_j) \in E\), increment \(\ell_{v_i}\) by 1;
- For each \(v_i \in A\) such that \((v_i, \ell_{v_i}) \in Y\), check whether \((v_i, v_j)\) forms a monochromatic edge. If yes, add \((v_i, v_j)\) to \(S\);
- Set \(d_G^+(v_j)\) equals to the number of neighbors that \(v_j\) has in \(\{v_1, \ldots, v_{j-1}\}\).

Get \(d_G(v_j)\) from the degree oracle and compute \(d_G^+(v_j)\). Set \(\text{CNT} = \text{CNT} + d_G^+(v_j)\). Then, depending on whether \(\text{CNT} \leq \tau\), our algorithm performs the following steps.

if \((\text{CNT} \leq \tau)\) then

(i) Add \((v_j, \ell, \ell \in [d_G^+(v_j)]\), with probability \(\frac{3000 \log n}{\varepsilon^3 T}\) to \(Y\), independently;

(ii) Add \(v_j\) to \(A\) (with its color stored) if at least one \((v_j, \ell)\) is added to \(Y\).

else if \((\text{CNT} > \tau)\) then

(i) Add \(v_j\) to \(B\) (along with the color of \(v_j\));

(ii) For each \(v_i \in B\), check whether \((v_i, v_j)\) forms a monochromatic edge. If yes, increment \(C_{>\tau}\) by 1.

\[\text{If } |S| \leq \frac{60 \log n}{\varepsilon^2}, \text{ then set } C_{\leq \tau} = 0. \text{ Otherwise, set } C_{\leq \tau} = \frac{\varepsilon^3 T}{3000 \log n} |S|.\]

Report \(\hat{m} = C_{\leq \tau} + C_{>\tau}\) as the OUTPUT.

we are storing all the vertices along with their colors explicitly. So, the space used by the algorithm is \(\tilde{O}(|V|)\), with high probability. To see the correctness of the algorithm, let \(E_M^B\) be the set of monochromatic edges having both the endpoints in \(B = \{v_k, \ldots, v_n\}\). Note that \(|E_M^B| = C_{>\tau}\). Let \(E_M^{V(G)\setminus B}\) be the set of monochromatic edges having at least one vertex in the set \(V(G)\setminus B = \{v_1, \ldots, v_{k-1}\}\), that is, \(E_M^{V(G)\setminus B} = E_M \setminus E_M^B\). Using Chernoff bound arguments (see Lemma A.1 in Appendix A), we have the following lemma. The proof of the following lemma is presented in Appendix B.

**Lemma 3.5.** (i) If \(|E_M^{V(G)\setminus B}| \geq \frac{\varepsilon}{100} T\), then \(\frac{\varepsilon^3 T}{3000 \log n} |S|\) is a \((1 \pm \frac{\varepsilon}{100})\) approximation to \(|E_M^{V(G)\setminus B}|\) with probability at least \(1 - \frac{1}{n voter}\).

(ii) If \(|E_M^{V(G)\setminus B}| \leq \frac{\varepsilon}{100} T\), \(|S| \leq \frac{60 \log n}{\varepsilon^2}\) with probability at least \(1 - \frac{1}{n voter}\).

Now let us divide the analysis into two cases, that is, \(|S| \geq \frac{60 \log n}{\varepsilon^2}\) and \(|S| < \frac{60 \log n}{\varepsilon^2}\).

\(|S| \leq \frac{60 \log n}{\varepsilon^2}\): In this case, we set \(C_{\leq \tau} = 0\). So, \(\hat{m} = C_{>\tau} = |E_M^B|\) is the output, which is always bounded above by \(|E_M|\). By Lemma 3.5 (i), \(|S| \leq \frac{60 \log n}{\varepsilon^2}\) implies \(|E_M^{V(G)\setminus B}| \leq \frac{\varepsilon}{20} T\).
with probability at least $1 - \frac{1}{n^\tau}$. Note that $|E_M| = |E_M^{V(G)}| + |E_M^R|$ and $|E_M| \geq T$.
Putting everything together, $m = C_{\leq \tau} + C_{\geq \tau}$ lies between $(1 - \frac{\varepsilon}{2^n}) |E_M|$ and $|E_M|$, with probability at least $1 - \frac{1}{n^\tau}$.

$|S| > \frac{60 \log n}{\varepsilon^2}$: In this case, we set $C_{\leq \tau} = \frac{\varepsilon^3 T}{3000 \log n} |S|$. By Lemma 3.5 (ii), $|S| > \frac{60 \log n}{\varepsilon^2}$ implies $|E_M^{V(G)}| > \frac{\varepsilon T}{100} |S|$ with probability at least $1 - \frac{1}{n^\tau}$. Also, by Lemma 3.5 (i), $|E_M^{V(G)}| > \frac{\varepsilon T}{100} |S|$ implies $C_{\leq \tau} = \frac{\varepsilon^3 T}{3000 \log n} |S|$ is a $(1 \pm \frac{\varepsilon}{100})$ approximation to $|E_M^{V(G)}|$ with probability at least $1 - \frac{1}{n^\tau}$. Combining it with the fact that $C_{\geq \tau} = |E_M^R|$, we have $m = C_{\leq \tau} + C_{\geq \tau}$ is an $(1 \pm \varepsilon)$-approximation to $|E_M|$, with probability at least $1 - \frac{2}{n^\tau}$.

This finishes the proof for the case $|E| > 100T|V| \log n$.

We have proved the correctness of Algorithm [1] by considering the cases $|E| \leq 100T|V| \log n$ and $|E| > 100T|V| \log n$ separately. We have also shown that the space complexity of Algorithm [1] is $\tilde{O}(\frac{|E|}{T})$ in the former case and is $\tilde{O}(|V|)$ in the latter case with high probability. Hence, we are done with the proof of Theorem 3.2.

4 CONFLICT-EST and CONFLICT-SEP in VARAND model

In this Section, mainly, we show that the power of randomness can be used to design a better solution for the CONFLICT-EST problem in the VARAND model. The CONFLICT-EST problem is the main highlight of our work. We feel that the crucial use of randomness in the input that is used to estimate a substructure (here, monochromatic edges) in a graph, will be of independent interest.

In this variant, we are given an $\varepsilon \in (0, 1)$ and a promised lower bound $T$ on $|E_M|$, the number of monochromatic edges in $G$, as input and our objective is to determine a $(1 \pm \varepsilon)$-approximation to $|E_M|$.

**Theorem 4.1.** Given any graph $G = (V, E)$ and a coloring function $f : V(G) \to |C|$ as input in the stream, the CONFLICT-EST problem in the VARAND model can be solved with high probability in $\tilde{O}(\frac{|V|}{\sqrt{T}})$ space, where $T$ is a lower bound on the number of monochromatic edges in the graph.

We prove the above theorem in Section 4.1. Note that the above algorithm can be used to solve CONFLICT-SEP in VARAND model. In Section 4.2, we give a simple algorithm for CONFLICT-EST that exploits a structural property of the subgraph having only monochromatic edges. However, the space complexity of the algorithm for CONFLICT-SEP (in Section 4.2) is same that of the algorithm for CONFLICT-EST (in Section 4.1).

4.1 CONFLICT-EST in VARAND model (Proof of Theorem 4.1)

**The proof idea**

A random sample comes for free – pick the first few vertices: Let $v_1, \ldots, v_n$ be the random ordering in which the vertices of $V$ are revealed. Let $R$ be a random subset of $\Gamma = \tilde{\Theta}(\frac{n}{\sqrt{T}})$ many vertices of $G$ sampled without replacement. As we are dealing with a random order stream, consider the first $\Gamma$ vertices in the stream; they can be treated as $R$, the random sample. We start by storing all the vertices in $R$ as well as their colors. Observe that if the monochromatic degree of $\tilde{\Theta}()$ hides a polynomial factor of $\log n$ and $\frac{1}{\varepsilon}$ in the upper bound.
any vertex \( v_i \) is large (say roughly more than \( \sqrt{T} \)), then it can be well approximated by looking at the number of monochromatic neighbors that \( v_i \) has in \( R \). As a vertex \( v_i \) streams past, there is no way we can figure out its monochromatic degree, unless we store its monochromatic neighbors that appear before it in the stream; if we could, we were done. Our only savior is the stored random subset \( R \).

Classifying the vertices of the random sample \( R \) based on its monochromatic degree:
Our algorithm proceeds by figuring out the influence of the color of \( v_i \) on the monochromatic degrees of vertices in \( R \). To estimate this, let \( \kappa_{v_i} \) denote the number of monochromatic neighbors that \( v_i \) has in \( R \). We set a threshold \( \tau = \frac{|R| \sqrt{T}}{n} \), where \( t = \lceil \log_{1 + \frac{1}{mR}} n \rceil \). The significance of \( t \) will be clear from the discussion below. Any vertex \( v_i \) will be classified as a high-\( m_R \) or low-\( m_R \) degree vertex depending on its monochromatic degree within \( R \), i.e., if \( \kappa_{v_i} \geq \tau \), then \( v_i \) is a high-\( m_R \) vertex, else it is a low-\( m_R \) vertex, respectively. (We use the subscripts \( m_R \) to stress the fact that the monochromatic degrees are induced by the set \( R \).) Let \( H \) and \( L \) be the partition of \( V \) into the set of high-\( m_R \) and low-\( m_R \) degree vertices in \( G \). Let \( H_R \) and \( L_R \) denote the set of high-\( m_R \) and low-\( m_R \) degree vertices in \( R \). Notice that, because of the definition of high-\( m_R \) and low-\( m_R \) degree vertices, not only the sets \( H_R, L_R \) are subsets of \( R \), but they are determined by the vertices of \( R \) only.

Let \( m_h \) and \( m_\ell \) denote the sum of the monochromatic degrees of all the high-\( m_R \) degree vertices and low-\( m_R \) degree vertices in \( G \), respectively. So, \( m_h = \sum_{v \in H} d_M(v) \) and \( m_\ell = \sum_{v \in L} d_M(v) \). Note that \( \hat{m} = |E_M| = \frac{1}{2} \sum_{v \in V} d_M(v) = \frac{1}{2} (m_h + m_\ell) \). We will describe how to approximate \( m_h \) and \( m_\ell \) separately. The formal algorithm is described in Algorithm \[2\] as \textsc{Random-Order-Est} \((\varepsilon, T)\) that basically executes steps to approximate \( m_h \) and \( m_\ell \) in parallel.

To approximate \( m_h \), the random sample \( R \) comes to rescue: We can find \( \hat{m}_h \), that is, a \((1 \pm \frac{\varepsilon}{10})\) approximation of \( m_h \) as described below. For each vertex \( v_i \in R \) and each monochromatic edge \((u, v_i)\), \( u \in R \), we see in the stream, we increase the value of \( \kappa_u \) for \( u \) and \( \kappa_{v_i} \) for \( v_i \). After all the vertices in \( R \) are revealed, we can determine \( H_R \) by checking whether \( \kappa_{v_i} \geq \tau \) for each \( v_i \in R \). For each vertex \( v_i \in H_R \), we set its approximate monochromatic degree \( \hat{d}_{v_i} \) to be \( \frac{n}{\sqrt{n}} \kappa_{v_i} \). We initialize the estimated sum of the monochromatic degree of high vertices as \( \hat{m}_h = \sum_{v_i \in H_R} \hat{d}_{v_i} \). For each vertex \( v_i \notin R \) in the stream, we can determine \( \kappa_{v_i} \), as we have stored all the vertices in \( R \) along with their colors, and hence we can also determine whether \( v_i \) is a high-\( m_R \) degree vertex in \( G \). If \( v_i \notin R \) is a high-\( m_R \) degree vertex, we determine \( \hat{d}_{v_i} = \frac{n}{\sqrt{n}} \kappa_{v_i} \) and update \( \hat{m}_h \) by \( \hat{m}_h + \hat{d}_{v_i} \). Observe that, at the end, \( \hat{m}_h \) is \( \sum_{v_i \in H} \hat{d}_{v_i} \). Recall that \( H \) is the set of all high-\( m_R \) degree vertices in \( G \). For each \( v_i \in H \), we will show, as in Claim \[43\] that \( \hat{d}_{v_i} \) is a \((1 \pm \frac{\varepsilon}{10})\)-approximation to \( d_M(v_i) \) with high probability. This implies that

\[
(1 - \frac{\varepsilon}{10}) m_h \leq \hat{m}_h \leq (1 + \frac{\varepsilon}{10}) m_h
\]

(1)

To approximate \( m_\ell \), group the vertices in \( L \) based on similar monochromatic degree:
Recall that \( m_\ell = \sum_{v_i \in L} d_M(v_i) \). Unlike the high-\( m_R \) degree vertices, it is not possible to approximate the monochromatic degree of \( v_i \in L \) from \( \kappa_{v_i} \). To cope up with this problem, we partition the vertices of \( L \) into \( t \) buckets \( B_1, \ldots, B_t \) such that all the vertices present in a bucket have similar monochromatic degrees, where \( t = \lceil \log_{1 + \frac{1}{mR}} n \rceil \). The bucket \( B_j \) is defined as follows: \( B_j = \{ v_i \in L : (1 + \frac{\varepsilon}{10})^{j-1} \leq d_M(v_i) < (1 + \frac{\varepsilon}{10})^j \} \).
Algorithm 2: Random-Order-Est($\varepsilon, T$): Conflict-Est in VARAND model

**Input:** $G = (V, E)$ and a coloring function $f$ on $V$ in the VARAND model, parameters $T$ and $\varepsilon$.

**Output:** $\hat{m}$, that is, a $(1 \pm \varepsilon)$ approximation to $|E_M|$.

- $\Gamma = \Theta\left(\frac{n}{\sqrt{T}}\right)$; $v_1, \ldots, v_n$ be the random ordering in which vertices are revealed and $R = \{v_1, \ldots, v_T\}$;
- $\kappa_{v_i}, i \in [n]$, denotes the number of monochromatic neighbors of $v_i$ in $R$;
- $\hat{d}_{v_i}, i \in [n]$, denotes the (estimated) monochromatic neighbors of vertices in $G$.

- $H$ denotes the set of high degree vertex in $R$, i.e., $H = \{v_i : \kappa_{v_i} \geq \frac{|R|}{n} \sqrt{\frac{T}{8t}}\}$ and $L = V \setminus H$;
- $L_R = L \cap R$ and $H_R = H \cap R$;
- The vertices in $L$ are partitioned into $t$ buckets as follows:
  $$B_j = \{v_i \in L : (1 + \frac{1}{40})^{j-1} \leq \hat{d}_M(v_i) < (1 + \frac{1}{40})^j\},$$
  where $j \in [t]$.

Set $t = \lceil \log_{1 + \frac{1}{10}} n \rceil$. If $T < 63t^2$, then store all the vertices in $G$ along with their colors. At the end, report the exact value of $|E_M|$. Otherwise, we proceed through via three building blocks described below and marked as (1), (2), (3) and (4). Refer to the notations described above this pseudocode.

1. **Processing the vertices in $R$, the first $\Gamma$ vertices, in the stream:**
   
   for (each vertex $v_i \in R$ exposed in the stream) do
   
   Store $v_i$ as well as its color $f(v_i)$.
   
   For each edge $(v_{i'}, v_i)$ that arrives in the stream, increase the values of $\kappa_{v_{i'}}$ and $\kappa_{v_i}$.

2. **Computation of some parameters based on vertices in $R$ and their colors:**
   
   for (each $v_i \in R$ with $\kappa_{v_i} \geq \frac{|R|}{n} \sqrt{\frac{T}{8t}}$) do
   
   Add $v_i$ to $H_R$, and set $\hat{d}_{v_i} = \frac{n}{|R|} \kappa_{v_i}$.
   
   $\hat{m}_h = \sum_{v_i \in H} \hat{d}_{v_i}$.
   
   Let $L_R = R \setminus H_R$.
   
   for (each $v_i \in L_R$) do
   
   Set $\hat{d}_{v_i} = \kappa_{v_i}$.

3. **Processing the vertices in $V(G) \setminus R$ in the stream:**
   
   for (each vertex $v_i \notin R$ exposed in the stream) do
   
   Determine the value of $\kappa_{v_i}$. If $\kappa_{v_i} \geq \frac{|R|}{n} \sqrt{\frac{T}{8t}}$, find $\hat{d}_{v_i} = \frac{n}{|R|} \kappa_{v_i}$ and add $\hat{d}_{v_i}$ to the current $\hat{m}_h$.
   
   Also, for each $v_{i'} \in L_R$, increase the value of $\hat{d}_{v_{i'}}$ if $(v_{i'}, v_i)$ is an edge.

4. **Post processing, after the stream ends, to return the output:**
   
   From the values of $\hat{d}_{v_i}$ for all $v_i \in L_R$, determine the buckets for each vertex in $L_R$. Also, for each $j \in [t]$, find $|A_j| = |L_R \cap B_j|$. Then determine
   
   $$\hat{m}_J = \frac{n}{|R|} \sum_{j \in [t]} |A_j| \left(1 + \frac{\varepsilon}{10}\right)^j.$$  

   Report $\hat{m} = \frac{\hat{m}_h + \hat{m}_J}{2}$ as the final output.
Note that our algorithm will not find the buckets explicitly. It will be used for the analysis only. Observe that $\sum_{j\in[t]} |B_j| \left( 1 + \frac{\varepsilon}{10} \right)^j \leq m_\ell < \sum_{j\in[t]} |B_j| \left( 1 + \frac{\varepsilon}{10} \right)^j$. We can surely approximate $m_\ell$ by approximating $|B_j|$ suitably. We estimate $|B_j|$ as follows. After the stream of the vertices in $R$ has gone past, we have the set of low-$m_R$ degree vertices $L_R$ in $R$ and $\hat{d}_v = \kappa_v$ for each $v \in L_R$. For each $v \notin R$ in the stream, we determine the monochromatic neighbors of $v$ in $L_R$. It is possible as we have stored all the vertices in $R$ and their colors. For each monochromatic neighbor $v' \in L_R$ of $v$, we increase the value of $\hat{d}_{v'}$ of $v'$. Observe that, at the end of the stream, $\hat{d}_{v'} = d_M(v')$ for each $v' \in L_R$, i.e., we can accurately estimate the monochromatic degree of each $v' \in L_R$. So, we can determine the bucket where each vertex in $L_R$ belongs. Let $A_j (= L_R \cap B_j)$ be the bucket $B_j$ projected onto $L_R$ in the random sample; note that as $B_j \subseteq L$ and $L_R = L \cap R$, $A_j = R \cap B_j$ also. We determine $\hat{m}_\ell = \frac{\varepsilon T}{63T} \sum_{j\in[t]} |A_j| \left( 1 + \frac{\varepsilon}{10} \right)^j$. We can show that $\frac{\varepsilon T}{63T} |A_j|$ is a $(1 + \frac{\varepsilon}{10})$-approximation of $|B_j|$, with high probability, if $|B_j| \geq \sqrt{T} \frac{\varepsilon T}{63T}$. Also, we can show that, if $|B_j| < \sqrt{T} \frac{\varepsilon T}{63T}$, then $|A_j| \leq \frac{|R|}{n} \sqrt{\frac{T}{8t}}$ with high probability. Now using the fact that we consider bucketing of only low-$m_R$ degree vertices ($L_R$), we can show that

$$
\left( 1 - \frac{\varepsilon}{10} \right) \left( m_\ell - \frac{\varepsilon T}{63T} \right) \leq \hat{m}_\ell \leq \left( 1 + \frac{\varepsilon}{10} \right)^2 \left( m_\ell + \frac{\varepsilon T}{56T} \right).$$

(2)

Note that $\varepsilon \in (0, 1)$ and $t = \lceil \log_{1 + \frac{\varepsilon}{10}} n \rceil$. Assuming $T \geq 63t^2$, Equations [1] and [2] imply that $\hat{m} = \frac{1}{2}(\hat{m}_h + \hat{m}_\ell)$ is a $(1 \pm \varepsilon)$-approximation to $|E_M|$. If $T < 63t^2$, then note that $n = \tilde{O} \left( \frac{n}{\sqrt{T}} \right)$. So, in that case, we store all the vertices along with their colors and compute the exact value of $|E_M|$.

**Proof of correctness**

The correctness of the algorithm follows trivially if $T \geq 63t^2$. So, let us assume that $T \geq 63t^2$.

In the VARPAND model, we consider the first $\tilde{O} \left( \frac{n}{\sqrt{T}} \right)$ vertices as the random sample $R$ without replacement. Using the Chernoff bound for sampling without replacement (See Lemma [A.2] in Appendix [A]), we can have the following lemma (The proof is in Appendix [C]), which will be useful for the correctness proof of Algorithm [2] (RANDOM-ORDER-EST($\varepsilon, T$)) in case of $T \geq 63t^2$.

**Lemma 4.2.** (i) For each $j \in [t]$ with $|B_j| \geq \sqrt{T} \frac{\varepsilon T}{101}$, $P \left( \left| |B_j \cap R| - \frac{|R||B_j|}{n} \right| \geq \frac{\varepsilon}{10} \frac{|R||B_j|}{n} \right) \leq \frac{1}{n^t}$.

(ii) For each $j \in [t]$ with $|B_j| < \sqrt{T} \frac{\varepsilon T}{101}$, $P \left( |B_j \cap R| \geq \frac{|R|}{n} \sqrt{T} \frac{\varepsilon T}{63T} \right) \leq \frac{1}{n^t}$.

(iii) For each vertex $v_i$ with $d_M(v_i) \geq \sqrt{T} \frac{\varepsilon T}{101}$, $P \left( |\kappa_{v_i} - \frac{|R||d_M(v_i)|}{n}| \geq \frac{\varepsilon}{10} \frac{|R||d_M(v_i)|}{n} \right) \leq \frac{1}{n^t}$.

(iv) For each vertex $v_i$ with $d_M(v_i) < \sqrt{T} \frac{\varepsilon T}{101}$, $P \left( |\kappa_{v_i} - \frac{|R|}{n} \sqrt{T} \frac{\varepsilon T}{8t} \right) \leq \frac{1}{n^t}$.

The correctness proof of the algorithm is divided into the following two claims.

**Claim 4.3.** $(1 - \frac{\varepsilon}{10}) m_h \leq \hat{m}_h \leq (1 + \frac{\varepsilon}{10}) m_h$ with probability at least $1 - \frac{1}{n^t}$.

**Claim 4.4.** $(1 - \frac{\varepsilon}{10}) \left( m_\ell - \frac{\varepsilon T}{63T} \right) \leq \hat{m}_\ell \leq (1 + \frac{\varepsilon}{10}) \left( m_\ell + \frac{\varepsilon T}{56T} \right)$ with probability at least $1 - \frac{1}{n^t}$.

Assuming the above two claims hold and taking $\varepsilon \in (0, 1)$, $t = \lceil \log_{1 + \frac{\varepsilon}{10}} n \rceil$ and $T \geq 63t^2$, observe that $\hat{m} = \frac{1}{2}(\hat{m}_h + \hat{m}_\ell)$ is a $(1 \pm \varepsilon)$ approximation of $|E_M| = m_h + m_\ell$ with high probability. Thus, it remains to prove Claims 4.3 and 4.4.
Proof of Claim 4.3. Note that $$m_h = \sum_{v_i : \kappa_{v_i} \geq \frac{|R|}{n} \frac{\sqrt{T}}{8t}} d_M(v_i)$$ and $$\hat{m}_h = \sum_{v_i : \kappa_{v_i} \geq \frac{|R|}{n} \frac{\sqrt{T}}{8t}} \hat{d}_{v_i}$$.

From Lemma 4.2 (iv) and (iii), $$\kappa_{v_i} \geq \frac{|R|}{n} \frac{\sqrt{T}}{8t}$$ implies that $$\hat{d}_{v_i}$$ is an $$(1 \pm \frac{\epsilon}{10})$$ approximation to $$d_M(v_i)$$ with probability at least $$1 - \frac{2}{n^m}$$. Hence, we have $$(1 - \frac{1}{10}) m_h \leq \hat{m}_h \leq (1 + \frac{\epsilon}{10}) m_h$$ with probability at least $$1 - \frac{1}{n^m}$$.

Proof of Claim 4.4. Note that $$m_\ell = \sum_{v_i \in L} d_M(v_i) = \sum_{v_i \in L : \kappa_{v_i} < \frac{|R|}{n} \frac{\sqrt{T}}{8t}} d_M(v_i)$$ and $$\hat{m}_\ell = \frac{|R|}{n} \sum_{j \in [t]} |A_j| \left(1 + \frac{\epsilon}{10}\right)^j$$.

Recall that the vertices in $$L$$ are partitioned into $$t$$ buckets as follows: $$B_j = \{v_i \in L : (1 + \frac{\epsilon}{10})^{j-1} \leq d_M(v_i) < (1 + \frac{\epsilon}{10})^j\}$$, where $$j \in [t]$$. By Lemma 4.2 (iv), $$\kappa_{v_i} < \frac{|R|}{n} \frac{\sqrt{T}}{8t}$$ implies that $$d_M(v_i) \leq \frac{\sqrt{T}}{8t}$$ with probability $$1 - \frac{1}{n^m}$$. So, we have the following observation.

Observation 4.5. Let $$j \in [t]$$ be such that $$|A_j| \neq 0$$ ($$|B_j| \neq 0$$). Then, with probability at least $$1 - \frac{1}{n^m}$$, the monochromatic degree of each vertex in $$A_j$$ as well as $$B_j$$ is at most $$\frac{\sqrt{T}}{8t}$$, that is, $$\left(1 + \frac{\epsilon}{10}\right)^j \leq \frac{\sqrt{T}}{8t}$$.

To upper and lower bound $$\hat{m}_\ell$$ in terms of $$m_\ell$$, we upper and lower bound $$m_\ell$$ in terms of $$|B_j|$$’s as follows; for the upper bound, we break the sum into two parts corresponding to large and small sized buckets:

$$\sum_{j \in [t]} |B_j| \left(1 + \frac{\epsilon}{10}\right)^j - 1 \leq m_\ell < \sum_{j \in [t]} |B_j| \left(1 + \frac{\epsilon}{10}\right)^j$$

$$\sum_{j \in [t]} |B_j| \left(1 + \frac{\epsilon}{10}\right)^j - 1 \leq m_\ell < \sum_{j \in [t]: |B_j| \geq \frac{\sqrt{T}}{8t}} |B_j| \left(1 + \frac{\epsilon}{10}\right)^j + \sum_{j \in [t]: |B_j| < \frac{\sqrt{T}}{8t}} |B_j| \left(1 + \frac{\epsilon}{10}\right)^j$$

By Observation 4.5 we bound $$m_\ell$$ in terms of $$|B_j|$$’s with probability $$1 - \frac{1}{n^m}$$.

$$\sum_{j \in [t]} |B_j| \left(1 + \frac{\epsilon}{10}\right)^j - 1 \leq m_\ell < \sum_{j \in [t]: |B_j| \geq \frac{\sqrt{T}}{8t}} |B_j| \left(1 + \frac{\epsilon}{10}\right)^j + t \cdot \frac{\sqrt{T}}{8t} \frac{\sqrt{T}}{7t}$$

This implies the following Observation:

Observation 4.6. $$\sum_{j \in [t]} |B_j| \left(1 + \frac{\epsilon}{10}\right)^j - 1 \leq m_\ell < \sum_{j \in [t]: |B_j| \geq \frac{\sqrt{T}}{8t}} |B_j| \left(1 + \frac{\epsilon}{10}\right)^j + \frac{\sqrt{T}}{8t} \frac{\sqrt{T}}{7t}$$ holds with probability at least $$1 - \frac{1}{n^m}$$.

Now, we have all the ingredients to show that $$\hat{m}_\ell$$ is a $$(1 \pm \epsilon)$$ approximation of $$m_\ell$$. To get to $$\hat{m}_\ell$$, we need to focus on low-$$m_R$$ vertices of $$R$$, i.e., $$A_j$$’s. Breaking $$\hat{m}_\ell = \frac{|R|}{n} \sum_{j \in [t]} |A_j| \left(1 + \frac{\epsilon}{10}\right)^j$$ depending on small and large values of $$|A_j|$$’s (recall $$A_j = L_R \cap B_j = R \cap B_j$$), we have

$$\hat{m}_\ell = \frac{n}{|R|} \left[ \sum_{j \in [t]: |A_j| \geq \frac{|R|}{n} \frac{\sqrt{T}}{8t}} |A_j| \left(1 + \frac{\epsilon}{10}\right)^j + \sum_{j \in [t]: |A_j| < \frac{|R|}{n} \frac{\sqrt{T}}{8t}} |A_j| \left(1 + \frac{\epsilon}{10}\right)^j \right]$$

Note that $$A_j = B_j \cap R$$. By Lemma 4.2 (ii), $$|A_j| \geq \frac{|R|}{n} \frac{\sqrt{T}}{8t}$$ implies $$|B_j| \geq \frac{\sqrt{T}}{8t}$$ with probability at least $$1 - \frac{1}{n^m}$$. Also, applying Lemma 4.2 (i), $$|B_j| \geq \frac{\sqrt{T}}{8t}$$ implies $$|A_j|$$ is an $$(1 \pm \frac{\epsilon}{10})$$-approximation to $$\frac{|R||B_j|}{n}$$ with probability at least $$1 - \frac{1}{n^m}$$. So, we have the following observation.
Observation 4.7. Let \( j \in [t] \) be such that \( |A_j| \geq \frac{|R|}{n} \sqrt{\frac{8t}{\varepsilon T}} \). Then \( |A_j| \) is an \( (1 \pm \frac{\varepsilon}{10}) \)-approximation to \( \frac{|R||B_j|}{n} \) with probability at least 1 - \( \frac{2}{n^\omega} \), that is, \( \frac{n}{|R|} |A_j| \) is an \( (1 \pm \frac{\varepsilon}{10}) \)-approximation to \( |B_j| \) with probability at least 1 - \( \frac{2}{n^\omega} \).

By the above observation along with Equation 3, we have the following upper bound on \( \hat{m} \ell \) with probability at least 1 - \( \frac{1}{n^\omega} \).

\[
\hat{m} \ell \leq \sum_{j \in [t]: |A_j| \geq \frac{|R|}{n} \sqrt{\frac{8t}{\varepsilon T}}} (1 + \frac{\varepsilon}{10}) |B_j| \left(1 + \frac{\varepsilon}{10}\right)^j + \sum_{j \in [t]: |A_j| < \frac{|R|}{n} \sqrt{\frac{8t}{\varepsilon T}}} \frac{n}{|R|} |A_j| \left(1 + \frac{\varepsilon}{10}\right)^j \]

\[
\leq \left(1 + \frac{\varepsilon}{10}\right)^2 \left[ \sum_{j \in [t]: |A_j| \geq \frac{|R|}{n} \sqrt{\frac{8t}{\varepsilon T}}} |B_j| \left(1 + \frac{\varepsilon}{10}\right)^{j-1} + \sum_{j \in [t]: |A_j| < \frac{|R|}{n} \sqrt{\frac{8t}{\varepsilon T}}} \frac{\sqrt{\varepsilon T}}{8t} \left(1 + \frac{\varepsilon}{10}\right)^{j-2} \right] \]

Now by Observations 4.9 and 4.13 we have the following with probability at least 1 - \( \frac{1}{n^\omega} \).

\[
\hat{m} \ell \leq \left(1 + \frac{\varepsilon}{10}\right)^2 \left( m \ell + t \cdot \frac{\sqrt{\varepsilon T} \sqrt{\varepsilon T}}{8t - 7t} \right) = \left(1 + \frac{\varepsilon}{10}\right)^2 \left( m \ell + \frac{\varepsilon T}{56t} \right) \]

Now, we will lower bound \( \hat{m} \ell \). From Equation 3, we have

\[
\hat{m} \ell \geq \frac{n}{|R|} \sum_{j \in [t]: |A_j| \geq \frac{|R|}{n} \sqrt{\frac{8t}{\varepsilon T}}} |A_j| \left(1 + \frac{\varepsilon}{10}\right)^j \]

By Observation 4.7, \( |A_j| \geq \frac{|R|}{n} \sqrt{\frac{8t}{\varepsilon T}} \) implies \( \frac{n}{|R|} |A_j| \) is an \( (1 \pm \frac{\varepsilon}{10}) \)-approximation to \( |B_j| \) with probability at least 1 - \( \frac{2}{n^\omega} \). So, the following lower bound on \( \hat{m} \ell \) holds with probability at least 1 - \( \frac{1}{n^\omega} \).

\[
\hat{m} \ell \geq \left(1 - \frac{\varepsilon}{10}\right) \sum_{j \in [t]: |A_j| \geq \frac{|R|}{n} \sqrt{\frac{8t}{\varepsilon T}}} |B_j| \left(1 + \frac{\varepsilon}{10}\right)^j \]

By Lemma 4.12(i), if \( |B_j| \geq \frac{\sqrt{\varepsilon T}}{9t} \), then \( |A_j| \geq \frac{\sqrt{\varepsilon T}}{9t} \) with probability at least 1 - \( \frac{1}{n^\omega} \). Hence, we have the following lower bound on \( m \ell \) with probability at least 1 - \( \frac{1}{n^\omega} \).

\[
\hat{m} \ell \geq \left(1 - \frac{\varepsilon}{10}\right) \sum_{j \in [t]: |B_j| \geq \frac{\sqrt{\varepsilon T}}{9t}} |B_j| \left(1 + \frac{\varepsilon}{10}\right)^j \]

Now by Observation 4.6, we have the following with high probability at least 1 - \( \frac{1}{n^\omega} \).

\[
\hat{m} \ell \geq \left(1 - \frac{\varepsilon}{10}\right) \left( m \ell - \frac{\varepsilon T}{63t} \right) .
\]

\[\square\]
4.2 Conflict-Sep in VArand model

Using a structural property of the graph, we design a simple algorithm to solve the Conflict-Sep problem in the VArand model.

**Theorem 4.8.** Given any graph $G = (V, E)$ and a coloring function $f : V(G) \to [C]$ and a parameter $\varepsilon > 0$ as input, there exists an algorithm that solves the Conflict-Sep problem in the VArand streaming model using space $\tilde{O}\left(\frac{|V|}{\sqrt{\varepsilon |E|}}\right)$ with high probability.

Let $G'$ denote the subgraph of $G$ consisting of only monochromatic edges in $G$. The lemma stated below guarantees that either there exists a large matching of size at least $\sqrt{\varepsilon m}$ in $G'$ or there exists a vertex of degree at least $\sqrt{\varepsilon m}$ in $G'$.

**Lemma 4.9 ([Juk11]).** Let $G = (V, E)$ be a graph and $f : V(G) \to [C]$ be a coloring function such that at least $\varepsilon$ fraction of the edges of $E(G)$ are known to be monochromatic. Then, either there is a matching of size at least $\sqrt{\varepsilon m}$ or there exists a vertex of degree at least $\sqrt{\varepsilon m}$ in the subgraph $G'$ defined on the monochromatic edges of $G$.

**Algorithm 3:** Algorithm: Conflict-Sep in Vertex Arrival in Random Order model

**Input:** $G = (V, E)$ and a coloring function $f$ on $V$ in the VArand model

**Output:** The algorithm verifies if $f$ is $\varepsilon$-far from valid or not

Let $S$ be the set of stored vertices and their colors. Initially, $S$ is empty. For $i \leftarrow 1$ to $|V|$

1. Let $u$ be the $i^{th}$ vertex of the stream
2. Store $u$ and its color $f(u)$ in $S$ with probability $O\left(\frac{\log n}{\sqrt{m}}\right)$
3. For every vertex $v$ in $S$
   1. Check if $(v, u)$ is an edge and $f(v) = f(u)$

End

Output $f$ is valid if none of the edges sampled are conflicting, else output that $f$ is $\varepsilon$-far from being valid.

The algorithm is as simple as it can get. We sample independently and uniformly at random the vertices in stream with probability $p = \min\{1, \frac{\log n}{\sqrt{m}}\}$ and store these vertices along with their colors. Let $S \subseteq V$ be the set of sampled vertices. When a vertex appears in a stream, we check if it forms a monochromatic edge with one of the stored vertices in $S$. At the end of the stream, the algorithm declares the graph to be properly colored (valid) if it cannot find a monochromatic edge, else it declares the instance to be $\varepsilon$-far from being monochromatic.

We show that Theorem 4.8 follows easily using Lemma 4.9.

**Proof.** We consider the following two cases.

- Case 1 – There exists a matching of size at least $\sqrt{\varepsilon m}$: Note that all these matched edges are monochromatic. Let $(u, v)$ denote an arbitrary matched edge where $u$ appears in the

1For simplicity of presentation, we assumed that, the number of edges $m$ in graph $G$ is known before the stream starts. However, this assumption can be removed by a simple tweak of starting with a value of $m$ and increasing it in stages and adjusting the random sample accordingly. This is common in streaming algorithms.
stream before \( v \). Now, the edge \((u, v)\) will be detected as monochromatic if vertex \( u \) has been sampled by the algorithm. The probability that vertex \( u \) is sampled is \( \frac{10 \log n}{\sqrt{m}} \). Since, there are \( \sqrt{em} \) matched monochromatic edges, the algorithm will detect at least one of these matched monochromatic edges with probability at least \( (1 - 1/n^2) \).

- Case 2 – There exists a vertex of degree at least \( \sqrt{em} \): In this case most of the monochromatic edges may be incident on very few high degree vertices. To detect these edges, we want to store either the high degree vertices or one of their neighbours. But, if these high degree vertices appear at the beginning of the stream and we fail to sample them, then we may not detect a monochromatic edge. This is where the random order of vertices arriving in the stream comes into play. Now, assuming random order of vertices in the stream, at least \( \frac{1}{5} \sqrt{em} \) neighbors of \( v \) should appear before \( v \) in the stream with probability at least \( (1 - e^{\frac{m}{n^2}} \sqrt{em}) \). Since we sample every vertex with probability \( \frac{10 \log n}{\sqrt{m}} \), with high probability at least \( (1 - 1/n^2) \) one of its neighbors will be stored.

\[ \Box \]

5 Lower bound for CONFLICT-EST in VARAND model

In this Section, we show a lower bound of \( \Omega \left( \frac{n}{\sqrt{T}} \right) \) for CONFLICT-EST in VERTEX ARRIVAL in RANDOM ORDER via a reduction from a variation of MULTIPARTY SET DISJOINTNESS problem called DISJOINTNESS\(_R\)\( (t, n, p) \), played among \( p \) players: Consider a matrix of order \( t \times n \) having \( t \) (rows) vectors \( M_1, \ldots, M_t \in \{0, 1\}^n \) such that each entry of matrix \( M \) is given to one of the \( p \) players chosen uniformly at random. The objective is to determine whether there exists a column where all the entries are 1s. If \( t \geq 2 \) and \( p = \Omega(t^2) \), Chakrabarti et al. showed that any randomized protocol requires \( \Omega \left( \frac{n}{T} \right) \) bits of communication \( \Box \). They showed that the lower bound holds under a promise called the UNIQUE INTERSECTION PROMISE which states that there exists at most a single column where all the entries are 1s and every other column of the matrix has Hamming weight either 0 or 1. Moreover, the lower bound holds even if all the \( p \) players know the random partition of the entries of matrix \( M \).

Theorem 5.1. Let \( n, T \in \mathbb{N} \) be such that \( 4 \leq T \leq \binom{n}{2} \). Any constant pass streaming algorithm that takes the vertices and edges of a graph \( G(V, E) \) (with \( |V| = \Theta(n) \) and \( |E| = \Theta(m) \)) and a coloring function \( f : V \rightarrow [C] \) in the VARAND model, and determines whether the monochromatic edges in \( G \) is 0 or \( \Omega(T) \) with probability \( 2/3 \), requires \( \Omega \left( \frac{n}{\sqrt{T}} \right) \) bits of space.

Proof. Without loss of generality, assume that \( \sqrt{T} \in \mathbb{N} \). Consider the DISJOINTNESS\(_R\) \( \left( \sqrt{T}, \frac{n}{\sqrt{T}}, p \right) \) problem with UNIQUE INTERSECTION PROMISE when all of the \( p \) players know the random partition of the entries of the relevant matrix \( M \). Note that \( M \) is of order \( \lceil \sqrt{T} \rceil \times \left\lfloor \frac{n}{\sqrt{T}} \right\rfloor \) and \( p = AT \) for some suitable constant \( A \in \mathbb{N} \). Also, consider a graph \( G \), with \( V(G) = \{v_j : i \in \left[ \sqrt{T} \right], j \in \left[ \frac{n}{\sqrt{T}} \right] \} \), having \( \frac{n}{\sqrt{T}} \) many vertex disjoint cliques such that \( \{v_{ij_1}, \ldots, v_{ij_j} \} \) forms a clique for each \( j \in [n] \), i.e., a column of \( M \) forms a clique. Also, notice that each clique has \( \Theta(T) \) edges. Let us assume that there is an \( r \)-pass streaming algorithm \( S \), with space complexity \( s \) bits, that solves CONFLICT-EST for the above graph \( G \) in the VARAND model. Now, we give a protocol \( A \) for DISJOINTNESS\(_R\) \( \left( \sqrt{T}, \frac{n}{\sqrt{T}}, p \right) \) with communication cost \( O(rsp) \). Using the fact that the lower bound of DISJOINTNESS \( \left( \sqrt{T}, \frac{n}{\sqrt{T}}, p \right) \) is \( \Omega \left( \frac{n}{\sqrt{T}} \right) \) along with the fact that \( p = AT \) and \( r \) is a constant, we get \( s = \Omega \left( \frac{n}{\sqrt{T}} \right) \).
Protocol $\mathcal{A}$ for $\text{DISJOINTNESS}_R\left(\sqrt{T}, \frac{n}{\sqrt{p}}\right)$: Let $P_1, \ldots, P_p$ denote the set of $p$ players. For $k \in [p]$, $V_k = \{v_{ij} : M_{ij} \text{ is with } P_k\}$, where $M_{ij}$ denotes the element present in the $i$-th row and $j$-th column of matrix $M$. Note that there is a one-to-one correspondence between the entries of $M$ and the vertices in $V(G)$. Furthermore, there is a one-to-one correspondence between the columns of matrix $M$ and the cliques in graph $G$. We assume that all the $p$ players know the graph structure completely as well as both the one-to-one correspondences. The protocol proceeds as follows: for each $k \in [p]$, player $P_k$ determines a random permutation $\pi_k$ of the vertices in $V_k$. Also, for each $k \in [p]$, player $P_k$ determines the colors of the vertices in $V_k$ by the following rule: if $M_{ij} = 1$, then color vertex $v_{ij}$ with color $C_i$. Otherwise, for $M_{ij} = 0$, color vertex $v_{ij}$ with color $C_j$. Player $P_1$ initiates the streaming algorithm and it goes over $r$-rounds.

Rounds 1 to $r - 1$: For $k \in [p]$, each player resumes the streaming algorithm by exposing the vertices in $V_k$, along with their colors, in the order dictated by $\pi_k$. Also, $P_k$ adds the respective edges to previously exposed vertices when the current vertex is exposed to satisfy the basic requirement of VA model. This is possible because all players know the graph $G$ and the random partition of the entries of matrix $M$ among $p$ players. After exposing all the vertices in $V_k$, as described, $P_k$ sends the current memory state to player $P_{k+1}$. Assume that $P_1 = P_{p+1}$.

Round $r$: All the players behave similarly as in the previous rounds, except that, the player $P_p$ does not send the current memory state to $P_1$. Rather, $P_p$ decides whether there is a column in $M$ with all 1s if the streaming algorithm $S$ decides that there are $\Omega(T)$ many monochromatic edges in $G$. Otherwise, if $S$ decides that there is no monochromatic edge in $G$, then $P_p$ decides that all the columns of $M$ have weight either 0 or 1. Then $P_p$ sends the output to all other players.

The vertices of graph $G$ are indeed exposed randomly to the streaming algorithm. It is because the entries of matrix $M$ are randomly partitioned among the players and each player also generates a random permutation of the vertices corresponding to the entries of matrix $M$ available to them. From the description of the protocol $\mathcal{A}$, the memory state of the streaming algorithm (of space complexity $s$) is communicated $(r - 1)p + (p - 1)$ times and $p - 1$ bits is communicated at the end by player $P_p$ to broadcast the output. Hence, the communication cost of the protocol $\mathcal{A}$ is at most $O(rsp)$.

Now we are left to prove the correctness of the protocol $\mathcal{A}$. If there is a column in $M$ with all 1s, then all the vertices corresponding to entries of that column are colored with color $C_s$. Recall that there is a one-to-one correspondence between the columns in matrix $M$ and cliques in the graph $G$. So, all the vertices of the clique, corresponding to the column having all 1s, are colored with the color $C_s$. As the size of each clique in the graph $G$ is $\sqrt{T}$, there are at most $\Omega(T)$ monochromatic edges. To prove the converse, assume that there is no column in the matrix $M$ having all 1s. By UNIQUE INTERSECTION PROMISE, all the columns have hamming weight at most 1. We will argue that there is no monochromatic edge in $G$. Consider an edge $e$ in $G$. By the structure of $G$, the two vertices of $e$ must be in the same clique, say the $j$-th clique, that is, let $e = \{v_{i_1j}, v_{i_2j}\}$. By the coloring scheme used by the protocols, $v_{i_1j}$ and $v_{i_2j}$ are colored according to the values of $M_{i_1j}$ and $M_{i_2j}$, respectively. Note that both $M_{i_1j}$ and $M_{i_2j}$ belong to $j$-th column. As the hamming weight of every column is at most 1, there are three possibilities:

(i) $M_{i_1j} = M_{i_2j} = 0$, that is, $v_{i_1j}$ and $v_{i_2j}$ are colored with color $C_{i_1}$ and $C_{i_2}$, respectively;

(ii) $M_{i_1j} = 0$ and $M_{i_2j} = 1$, that is, $v_{i_1j}$ and $v_{i_2j}$ are colored with color $C_{i_1}$ and $C_s$, respectively;

(iii) $M_{i_1j} = 1$ and $M_{i_2j} = 0$, that is, $v_{i_1j}$ and $v_{i_2j}$ are colored with color $C_s$ and $C_{i_2}$, respectively.
In any case, the edge $e = \{v_{ij}, v_{ij}\}$ is not monochromatic. This establishes the correctness of protocol $\mathcal{A}$ for $\text{DISJOINTNESS}_R\left(\sqrt{T}, \frac{n}{\sqrt{T}}, p\right)$.

6 Conclusion and Discussion

In this paper, we introduced a graph coloring problem to streaming setting with a different flavor – the coloring function streams along with the graph. We study the problem of $\text{CONFLICT}$-$\text{EST}$ (estimating the number of monochromatic edges) and $\text{CONFLICT}$-$\text{SEP}$ (detecting a separation between the number of valid edges) in $\text{VA}$, $\text{VADEG}$, and $\text{VARAND}$ models. Our algorithms for $\text{VA}$ and $\text{VADEG}$ are tight up to polylogarithmic factors. However, a matching lower bound on the space complexity for $\text{VARAND}$ model is still elusive. There is a gap between our upper and lower bound results for $\text{VARAND}$ model in terms of the exponent in $T$. Our hunch is that the upper bound is tight. Specifically, we obtained an upper bound of $\tilde{O} \left( \frac{n}{\sqrt{T}} \right)$ and the lower bound is $\Omega \left( \frac{n}{T^2} \right)$. Here we would like to note that the lower bound also holds in $\text{AL}$ and $\text{VADEG}$ model when the vertices are exposed in a random order. However, we feel that our algorithm for $\text{CONFLICT}$-$\text{EST}$ in $\text{VARAND}$ model is tight up to polylogarithmic factors. We leave this problem open.

We feel the edge coloring counterpart of the vertex coloring problem proposed in the paper will be worthwhile to study. Let the edges of $G$ be colored with a function $f : E(G) \rightarrow [C]$, for $C \in \mathbb{N}$. A vertex $u \in V(G)$ is said to be a validly colored vertex if no two edges incident on $u$ have the same color. An edge coloring is valid if all vertices are validly colored. Consider the AL model for the edge coloring problem. As all edges incident on an exposed vertex $u$ are revealed in the stream, if we can solve a duplicate element finding problem on the colors of the edges incident on $u$, then we are done! It seems at a first glance that all the three models of $\text{VA}$, $\text{AL}$ and $\text{EA}$ will be difficult to handle for the edge coloring problem on streams of graph and edge colors. It would be interesting to see if the edge coloring variant of the problems we considered in this paper, admit efficient streaming algorithms. We plan to look at this problem next.

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A Some probability results

Lemma A.1 ([DP09] (Chernoff-Hoeffding bound)). Let $X_1, \ldots, X_N$ be independent random variables such that $X_i \in [0, 1]$. For $X = \sum_{i=1}^{N} X_i$ and $\mu = \mathbb{E}[X]$, the following holds for any $0 \leq \delta \leq 1$:

(i) $\mathbb{P}(X \geq (1 + \delta)\mu) \leq \exp\left(-\frac{\mu \delta^2}{3}\right)$;

(ii) $\mathbb{P}(X \leq (1 - \delta)\mu) \leq \exp\left(-\frac{\mu \delta^2}{3}\right)$;

(iii) Furthermore, if $\mu \leq t$, then the following holds.

$$\mathbb{P}(X \geq (1 + \epsilon)t) \leq \exp\left(-\frac{t \delta^2}{3}\right).$$

Lemma A.2 ([Mul18]). Let $I = \{1, \ldots, N\}$, $r \in [N]$ be a given parameter. If we sample a subset $R$ without replacement, then the following holds for any $J \subset I$ and $\delta \in (0, 1)$:

(i) $\mathbb{P}(|J \cap R| \geq (1 + \delta)\frac{|J|}{rN}) \leq \exp\left(-\frac{\delta^2|J|r}{3N}\right)$;

(ii) $\mathbb{P}(|J \cap R| \leq (1 - \delta)\frac{|J|}{rN}) \leq \exp\left(-\frac{\delta^2|J|r}{3N}\right)$;

(iii) Furthermore, we have the following if $|J| \leq k$, then the following holds.

$$\mathbb{P}\left(|J \cap R| \geq (1 + \delta)\frac{k}{N}\right) \leq \exp\left(-\frac{\delta^2kr}{3N}\right).$$

B Proof of Lemma 3.5

Lemma B.1 (Restatement of Lemma 3.5). (i) If $|E_M^n| \geq \frac{\epsilon}{100}|T|$, then $\frac{\epsilon^3T}{3000\log n}|S|$ is an $(1 \pm \epsilon)$ approximation to $|E_M^n|$ with probability at least $1 - \frac{1}{n^{10}}$.

(ii) If $|E_M^n| \leq \frac{\epsilon}{100}|T|$, $|S| \leq \frac{60\log n}{\epsilon^2}$ with probability at least $1 - \frac{1}{n^{10}}$.

Proof. We use the similar argument as that of in Section 3.2.1 to show $\hat{m}$ is an $(1 \pm \epsilon)$-approximation of $|E_M^n|$.

Here, $\mu = \mathbb{E}[|S|] = \frac{3000\log n}{\epsilon^2}|E_M^n|$. We prove (i) and (ii) separately.

(i) As $|E_M^n| \geq \frac{\epsilon}{100}|T|$, $\mathbb{E}[|S|] \geq \frac{30\log n}{\epsilon^2}$. Applying Lemma A.1 (i) and (ii),

$$\mathbb{P}\left(|S| - \mathbb{E}[|S|] \geq \epsilon\mathbb{E}[|S|]\right) \leq 2\exp\left(-\frac{\epsilon^2\mathbb{E}[|S|]}{3}\right)$$

$$\mathbb{P}\left(\frac{\epsilon^3T}{3000\log n}|S| - |E_M^n| \geq \epsilon|E_M^n|\right) \leq \frac{1}{n^{10}}.$$

Observe that we are done with the claim.
(ii) As $|E_M^n| \leq \frac{30\log n}{\varepsilon^2} T$, $\mathbb{E}[|S|] \leq \frac{30\log n}{\varepsilon^2}$. Applying Lemma A.1 (iii), by taking $t = \frac{30\log n}{\varepsilon^2}$ and $\delta = 1$, we have

$$\mathbb{P} \left( |S| \geq (1 + \delta)t \right) \leq \exp \left( -\frac{\delta^2 t}{3} \right)$$

$$\mathbb{P} \left( |S| \geq \frac{60\log n}{\varepsilon^2} \right) \leq \frac{1}{n^{10}}.$$

Observe that, we are done with the claim.

\[\square\]

C Proof of Lemma [4.2]

Lemma C.1 (Restatement of Lemma [4.2]).

(i) For each $j \in [t]$ with $|B_j| \geq \frac{\sqrt{cT}}{10T}$, $\mathbb{P} \left( |B_j \cap R| - \frac{|R||B_j|}{n} \geq \frac{\varepsilon}{10} \frac{|R||B_j|}{n} \right) \leq \frac{1}{n^{10}}$.

(ii) For each $j \in [t]$ with $|B_j| < \frac{\sqrt{cT}}{10T}$, $\mathbb{P} \left( |B_j \cap R| \geq \frac{|R|}{n} \frac{\sqrt{cT}}{8T} \right) \leq \frac{1}{n^{10}}$.

(iii) For each vertex $v_i$ with $d_M(v_i) \geq \frac{\sqrt{cT}}{10T}$, $\mathbb{P} \left( \kappa_{v_i} - \frac{|R|d_M(v_i)}{n} \geq \frac{\varepsilon}{10} \frac{|R|d_M(v_i)}{n} \right) \leq \frac{1}{n^{10}}$.

(iv) For each vertex $v_i$ with $d_M(v_i) < \frac{\sqrt{cT}}{10T}$, $\mathbb{P} \left( \kappa_{v_i} \geq \frac{|R|}{n} \frac{\sqrt{cT}}{8T} \right) \leq \frac{1}{n^{10}}$.

Proof. Let us take $N = n, r = |R| = \Gamma = \tilde{\Theta} \left( \frac{n}{\sqrt{T}} \right), I = \{v_1, \ldots, v_n\}$ in Lemma A.2.

(i) Setting $J = B_j$ and $\delta = \frac{\varepsilon}{10}$ in Lemma A.2 (i) and (ii), we have

$$\mathbb{P} \left( \left| B_j \cap R \right| - \frac{|R||B_j|}{n} \geq \frac{\varepsilon}{10} \frac{|R||B_j|}{n} \right) \leq 2\exp \left( -\frac{\varepsilon(10)^2 |B_j| \Gamma}{3n} \right) \leq \frac{1}{n^{10}}.$$

The last inequality holds as $|B_j| \geq \frac{\sqrt{cT}}{10T}, t = [\log_1 + \frac{\varepsilon}{10} n] = \Theta \left( \frac{\log n}{\varepsilon} \right)$ and $\Gamma = \tilde{\Theta} \left( \frac{n}{\sqrt{T}} \right)$.

(ii) Set $J = B_j, k = \frac{\sqrt{cT}}{10T}, \delta = \frac{1}{4}$ in Lemma A.2 (iii). As $|B_j| \leq \frac{\sqrt{cT}}{10T}, |J| \leq k$. Hence,

$$\mathbb{P} \left( |B_j \cap R| \geq \frac{|R|}{n} \frac{\sqrt{cT}}{8T} \right) \leq \exp \left( -\frac{(1/4)^2(\varepsilon T/10T)\Gamma}{3n} \right) \leq \frac{1}{n^{10}}.$$

(iii) Setting $J$ as the set of monochromatic neighbors of $v_i$ in $R$ and $\delta = \frac{\varepsilon}{10}$ in Lemma A.2 (i) and (ii), we get

$$\mathbb{P} \left( \left| \kappa_{v_i} - \frac{|R|d_M(v_i)}{n} \right| \geq \frac{\varepsilon}{10} \frac{|R|d_M(v_i)}{n} \right) \leq \exp \left( -\frac{(\varepsilon/10)^2 |J| \Gamma}{3n} \right) \leq \frac{1}{n^{10}}.$$

The last inequality holds as $|J| = d_M(v_i) \geq \frac{\sqrt{cT}}{10T}, t = [\log_1 + \frac{\varepsilon}{10} n] = \Theta \left( \frac{\log n}{\varepsilon} \right)$ and $\Gamma = \tilde{\Theta} \left( \frac{n}{\sqrt{T}} \right)$. 

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(iv) Set $J$ as the set of monochromatic neighbors of $v_i$ in $R$, $k = \sqrt{\varepsilon T}$, $\delta = \frac{1}{4}$ in Lemma A.2 (iii). Note that $|J| = d_M(v_i) \leq \sqrt{\varepsilon T \frac{10}{t}} = k$. Hence,

$$P\left(\kappa_{v_i} \geq \frac{|R| \sqrt{\varepsilon T}}{n} \frac{1}{8t}\right) \leq \exp \left( -\frac{(1/4)^2 (\sqrt{\varepsilon T / 10t}) \Gamma}{3n} \right) \leq \frac{1}{n^{10}}.$$

\[\square\]

D Communication Complexity

Communication Complexity [KN97] deals with finding the minimum amount of bits that is needed to communicate in order to compute a function when the input to the function is distributed among multiple parties. For the purpose of our work, we are concerned with two player games with one-way communication protocol. The players are traditionally called Alice and Bob. Both of them have a $n$-bit input string and are unaware of each other’s input. The goal is to minimize the bits Alice needs to communicate to Bob so that he can compute a function on both their inputs. No assumption is made on their computational powers and there is no restriction on the amount of time needed for computing the function. Randomized one way communication complexity of a function, is defined as the number of bits sent by Alice, in the worst case, by the best randomized protocol to compute $fS$.

D.1 INDEX problem in the communication complexity model

Usually, the space lower bound results in the streaming model of computation are proved by a reduction from a problem in communication complexity. We establish our lower bounds by considering a reduction from the INDEX problem in the one-way communication protocol for two players to the specific problem in graphs in the VA model. The INDEX problem is defined as follows: There are two parties, Alice and Bob. Alice has a $N$-bit input string $X \in \{0, 1\}^N$ and Bob has an integer $j \in [N]$. Both are unaware of each other’s input, but have an access to a public randomness, and the goal of Bob is to compute $X_j$, the $j^{th}$ bit of $X$, by receiving a single message from Alice.

Lemma D.1. [KN97] The randomized one-way communication complexity of INDEX is $\Omega(N)$

E Lower bounds for CONFLICT-EST

We show a tight lower bound of $\Omega\left(\min\{|V|, \frac{|V|^2}{T}\}\right)$ for the CONFLICT-EST problem in the vertex arrival model in Section E.1. For the CONFLICT-EST problem in the vertex arrival with degree oracle model, we show a tight lower bound of $\Omega\left(\min\{|V|, \frac{|E|}{T}\}\right)$ in Section E.2. These bounds are proved using reductions from the INDEX problem, discussed in Lemma D.1 in Appendix D, in the one-way communication complexity model to the CONFLICT-EST problem in graphs (in the vertex arrival streaming models).

E.1 Lower bound for CONFLICT-EST in VA model

Theorem E.1. Let $n,m,T \in \mathbb{N}$ be such that $1 \leq T \leq \binom{n}{2}$ and $m \geq T$. Any one pass streaming algorithm; that takes the vertices and edges of a graph $G(V,E)$ (with $|V| = \Theta(n)$ and $|E| = \Theta(m)$)
and coloring function $f : V \rightarrow [C]$ on the vertices, in VA model; and determines whether the number of monochromatic edges in $G$ is 0 or $T$ with probability 2/3; requires $\Omega \left( \min \{ n, \frac{n^2}{T} \} \right)$ bits of space.

Proof. We show that the lower bound is $\Omega(n)$ when $T \leq \frac{n}{2}$ and $\Omega \left( \frac{n^2}{T} \right)$ when $T > \frac{n}{2}$, separately, to get the stated lower bound. We give a reduction from the INDEX problem to the CONFLICT-Est problem in graphs with $\Theta(n)$ vertices, $\Theta(m)$ edges and having at least $T$ conflicting edges, in the vertex arrival model. We show our reduction when $m = T$, but we can modify it for any $m \geq T$.

The reduction works as follows. For $T \leq \frac{n}{2}$, Alice has an $N$-bit input string $X \in \{0,1\}^N$. For each input bit $X_i$, Alice creates a vertex $p_i$. If $X_i$ equals 1, then vertex $p_i$ is colored with color $C_1$, else it is colored with color $C_0$. After processing all bits of her input, Alice sends the current memory state to Bob. Bob constructs a gadget $Q$ which is an independent set of $(n - N)$ vertices and colors all the vertices in the gadget with color $C_1$. He adds all the edges from the vertex $p_j$ to the gadget $Q$. The number of vertices in the graph is $(N) + (n - N) = n$ and the number of edges in the graph is $m = T$. We set $N = n - T$. If $X_j = 0$, then the color of $p_j$ is $C_0$ and there are 0 conflicting edges, where as if $X_j = 1$, then the color of $p_j$ is $C_1$ and there will be $T$ conflicting edges. Therefore, for $N = n - T$, deciding whether the number of monochromatic edges in the graph is 0 or $T$, requires $\Omega(n - T)$ or $\Omega(n)$ space.

For $T \geq \frac{n}{2}$, Alice has an $N$-bit input string $X \in \{0,1\}^N$. For each input bit $X_i$, Alice constructs an independent set $P_i$ of size $\frac{2n}{T}$. If $X_i$ equals 1, then the vertices of $P_i$ are colored with color $C_1$, else the vertices are colored with color $C_0$. After processing all bits of her input, Alice sends the current memory state to Bob. Let $j \in [N]$ be the input of Bob. Bob constructs a gadget $Q$ which is an independent set of $\frac{n}{2}$ vertices and colors all the vertices with the color $C_1$. He adds all the edges from the gadget $P_j$ to the gadget $Q$. We set $N = \frac{n^2}{T}$. The number of vertices in the graph is $\frac{2n}{T} \cdot (N) + \frac{n}{2} = \Theta(n)$ and the number of edges in the graph is $m = T$. If $X_j = 0$, then the color of vertices in $P_j$ is $C_0$ and there are 0 conflicting edges, where as if $X_j = 1$, then the color of vertices in $P_j$ is $C_1$ and there will be $T$ conflicting edges. Therefore, for $N = \frac{n^2}{T}$, deciding whether the number of monochromatic edges in the graph is 0 or $T$, requires $\Omega \left( \frac{n^2}{T} \right)$ space.

Recall that we are doing our reductions for $m = T$. We make the above constructions work for any $m \geq T$ by adding a complete subgraph on $\sqrt{m - T}$ vertices such that none of the edges of the complete subgraph are conflicting. 

E.2 Lower bound for CONFLICT-Est in VAdeg model

Theorem E.2. Let $n, T \in \mathbb{N}$ be such that $1 \leq T \leq \binom{n}{2}$. Then there exists an $m$ with $T \leq m \leq \binom{n}{2}$ such that the following happens. Any one pass streaming algorithm; that takes the vertices and edges of a graph $G(V, E)$ (with $|V| = \Theta(n)$ and $|E| = \Theta(m)$) and a coloring function $f : V \rightarrow [C]$ on the vertices, in VAdeg model; and determines whether the number of monochromatic edges in $G$ is 0 or $T$ with probability 2/3; requires $\Omega \left( \min \{ n, \frac{m^2}{T} \} \right)$ bits of space.

Proof. We show that the lower bound is $\Omega(n)$ when $m > nT$ and $\Omega \left( \frac{m^2}{T} \right)$ when $m \leq nT$, separately, to get the stated lower bound. We give a reduction from the INDEX problem to the CONFLICT-Est problem in graphs with $\Theta(n)$ vertices, $\Theta(m)$ edges and having atleast $T$ conflicting edges, in the vertex arrival model. The existence of $m$ will be evident from the construction.

The reduction works as follows. For $m > nT$, Alice has an $N$-bit input string $X \in \{0,1\}^N$. For each input bit $X_i$, Alice creates a vertex $p_i$. If $X_i$ equals 1, then vertex $p_i$ is colored with color

**Note that $\sqrt{m - T} = O(n)$.**

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C_{j1}, else it is colored with color C_{j0}. After processing all bits of her input, Alice sends the current memory state to Bob. Let \( j \in [N] \) be the input of Bob. Bob constructs a gadget \( Q \) of \( n-N \) vertices such that \( Q \) is an independent set of \( (n-N) \) vertices. Bob colors all the vertices in the gadget \( Q \) with color \( C_{j1} \). He adds all the edges from \( Q \) to all the vertices in \( \{l_i : i \in [N] \} \). The number of vertices in the graph is \( N + (n-N) = \Theta(n) \) and the number of edges in the graph is \( m = NT \).

We set \( N = n-T \). If \( X_j = 0 \), then the color of \( p_j \) is \( C_{j0} \) and there are 0 conflicting edges, whereas if \( X_j = 1 \), then the color of \( p_j \) is \( C_{j1} \) and there will be \( T \) many conflicting edges. Therefore, for \( N = n-T \), deciding whether the number of monochromatic edges in the graph is 0 or \( T \), requires \( \Omega \left( \frac{m}{n} \right) \) space. Observe that, the degree of the vertices are independent of the inputs of Alice and Bob. In particular, the degree of every vertex in \( \{p_i : i \in [N] \} \) is \( |Q| = n-N \) and the degree of every vertex in \( Q \) is \( n \). So, the availability of degree oracle will not help in the above construction.

For \( m \leq nT \), Alice has an \( N \)-bit input string \( X \in \{0,1\}^N \). For each input bit \( X_i \), Alice constructs an independent set \( P_i \) of size \( \frac{2nT}{m} \). If \( X_i \) equals 1, then the vertices of \( P_i \) are colored with color \( C_{i1} \), else the vertices are colored with color \( C_{i0} \). After processing all bits of her input, Alice sends the current memory state to Bob. Let \( j \in [N] \) be the input of Bob. Let \( j \in [N] \) be the input of Bob. Bob constructs a gadget \( Q \) where \( Q \) is an independent set of \( \frac{m}{2n} \) vertices. Bob colors the vertices in \( Q \) with \( C_{j1} \) and he adds all the edges from \( Q \) to \( (P_1 \cup \cdots \cup P_N) \). We set \( N = \frac{m}{2n} \). The number of vertices in the graph is \( \frac{2nT}{m} \cdot N + \frac{m}{2n} = \Theta(n) \) as \( T \geq \frac{n}{2} \) and \( m \geq T \) and the number of edges in the graph is \( m = NT \). If \( X_j = 0 \), then the color of vertices in \( P_j \) is \( C_{j0} \) and there are 0 conflicting edges, whereas if \( X_j = 1 \), then the color of vertices in \( P_j \) is \( C_{j1} \) and there will be \( T \) conflicting edges. Therefore, for \( N = \frac{m}{2n} \), deciding whether the number of monochromatic edges in the graph is 0 or \( T \), requires \( \Omega \left( \frac{n}{m} \right) \) space. Observe that, the degree of the vertices are independent of the inputs of Alice and Bob. In particular, the degree of every vertex in \( P_1 \cup \cdots \cup P_N \) is \( |Q| = \frac{m}{2n} \), and the degree of every vertex in \( Q \) is \( N \cdot \frac{2nT}{m} \). So, the availability of degree oracle will not help in the above construction.