RANK-ONE CONVEXITY IMPLIES QUASICONVEXITY FOR TWO-COMPONENT MAPS

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Abstract. We prove that, for two-component maps, rank-one convexity is equivalent to quasiconvexity. The essential tool for the proof is a fixed-point argument for a suitable set-valued map going from one component to the other and preserving decomposition directions in the \((H_n)\)-condition formalism; the existence of a fixed point ensures that, in addition to keeping decomposition directions, joint volume fractions are preserved as well. When maps have more than two components, then fixed points exist for every combination of two components, but they do not match in general.

1. Introduction

One of the main ingredients of the direct method of the Calculus of Variations (\cite{10}) to show existence of minimizers for an integral functional of the kind

\[ I(u) = \int_{\Omega} \psi(\nabla u(x)) \, dx \]

is its weak lower semicontinuity. Here \( \Omega \subset \mathbb{R}^N \) is a regular (Lipschitz), bounded domain, and feasible mappings \( u : \Omega \to \mathbb{R}^m \) are smooth or Lipschitz, so that \( \nabla u \) is a \( m \times N \)-matrix at each point \( x \in \Omega \). The weak lower semicontinuity property is in turn equivalent to suitable convexity properties of the continuous integrand \( \psi : M^{m \times N} \to \mathbb{R} \). Morrey (\cite{22}, \cite{23}) proved that this weak lower semicontinuity (in \( W^{1,\infty}(\Omega; \mathbb{R}^m) \)) is equivalent to the quasiconvexity of the integrand \( \psi \), namely,

\[ \psi(F) \leq \frac{1}{|D|} \int_D \psi(F + \nabla v(x)) \, dx \]

for every \( F \in M^{m \times N} \), and every test map \( v \) in \( D \). This concept does not depend on the domain \( D \), and can, equivalently, be formulated in terms of periodic mappings (\cite{36}) so that such a density \( \psi \) is quasiconvex when

\[ \psi(F) \leq \int_Q \psi(F + \nabla v(y)) \, dy \]

for all \( F \in M^{m \times N} \), and every periodic mapping \( v : Q \to \mathbb{R}^m \). Here \( Q \subset \mathbb{R}^N \) is the unit cube.

Unfortunately, the issue is far from settled by simply saying this, since even Morrey realized that it is not at all easy to decide when a given density \( \psi \) enjoys this property. For the scalar case, when either of the two dimensions \( N \) or \( m \) is unity, quasiconvexity reduces to usual convexity. But for genuine vector situations, it is not so. As a matter of

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fact, necessary and sufficient conditions for quasiconvexity in the vector case \((N, m > 1)\) were immediately sought, and important new convexity conditions were introduced:

- **Rank-one convexity.** A continuous integrand \(\psi : M^{m \times N} \to \mathbb{R}\) is said to be rank-one convex if

\[
\psi(t_1 F_1 + t_2 F_2) \leq t_1 \psi(F_1) + t_2 \psi(F_2), \quad t_1 + t_2 = 1, t_1, t_2 \geq 0,
\]

whenever the difference \(F_1 - F_2\) is a rank-one matrix.

- **Polyconvexity.** Such an integrand \(\psi\) is polyconvex if it can be rewritten in the form \(\psi(F) = g(M(F))\) where \(M(F)\) is the vector of all minors of \(F\), and \(g\) is a convex (in the usual sense) function of all its arguments.

It was very soon recognized that quasiconvexity implies rank-one convexity (by using a special class of test fields), and that polyconvexity is a sufficient condition for quasiconvexity. The task suggested itself as trying to prove or disprove the equivalence of these various kinds of convexity. In the scalar case, all three coincide with usual convexity, so that we are facing a purely vector phenomenon. It turns out that these three notions of convexity are different, and counterexamples of various sorts have been found over the years. See [1], [11], [32], [38].

If we focus on the equivalence of rank-one convexity and quasiconvexity, Morrey conjectured that they are not equivalent ([22]), though later he simply stated it as an unsolved problem ([23]). The issue remained undecided until the surprising counterexample by V. Sverak ([36]) after some other additional and very interesting results ([34], [35], [37]). What is quite remarkable is that the original counterexample is only valid when \(m \geq 3\), and later attempts to extend it for \(m = 2\) failed ([4], [27], [29]). Other counterexamples have not been found despite insistent efforts of the author that were definitely discarded in [31]. References [16], and [20] are also relevant here.

The situation for two-component maps has, therefore, stayed unsolved, though some evidence in favor of the equivalence has been gathered throughout the years. See [7], [24], [25], [26]. It is also interesting to point out that for quadratic densities, rank-one convexity and quasiconvexity are equivalent regardless of dimensions. This has been known for a long time ([3], [23]), and it is not difficult to prove it by using Plancherel’s formula. A different point of view is taken in [5]. Another field where the resolution of this equivalence for two components maps would have an important impact is the theory of quasiconformal maps in the plane. There is a large number of references for this topic. See [2] for a rather recent account. In particular, if the equivalence between rank-one convexity and quasiconvexity for two component maps turns out to be true, then the norm of the corresponding Beurling-Ahlfors transform equals \(p^* - 1\) ([19]).

In this note, we prove that indeed for \(m = 2\) rank-one convexity is equivalent to quasiconvexity. The way in which we are going to think about the problem is by using the dual formulation of this equivalence through Jensen’s inequality. What we will actually show is that, when \(m = 2\), every homogeneous gradient Young measure is a laminate. See Chapter 9 in [28].

What is essential or special about \(m = 2\)? This is a question that one has to understand, as it seems quite relevant to a final resolution of the problem. The answer turns out to be quite enlightening: for two component maps, one can define an appropriate map going from one component to the other, and show the existence of a fixed point for
such a map that translates into a rank-one decomposition for any such two-component gradient. For more than two components, more than one map would be involved, and fixed points for every couple of components may not match. This fixed point result (Kakutani’s) is classical and nothing but a natural generalization of the usual Brower fixed point theorem.

More specifically, suppose we are given a periodic gradient \((\nabla u, \nabla v)\) with two components \((u, v)\): 
\[ Q \to \mathbb{R}^2 \]
where \(Q\) is the unit cube in \(\mathbb{R}^N\), which is piecewise-affine with respect to some finite, arbitrary triangulation \(\Gamma\). By a standard density argument about approximation by piece-wise affine mappings, it suffices, to reach our goal, to show that the corresponding discrete underlying gradient Young measure is a laminate. This two-component map establishes a very clear way of moving from operations on the gradient of the first component \(\nabla u\) to the same operations on the gradient \(\nabla v\) of the second component by simply replacing \(u_i\) by the corresponding \(v_i\) in the same element of the triangulation \(\Gamma\), if the finite support of \((\nabla u, \nabla v)\) is the set of pairs \(\{(u_i, v_i)\}_i\). The procedure is incorporated in the definition of a certain mapping. In addition, such map keeps track of decomposition directions as in the definition of laminates and \((H_n)\)-conditions (9). See Appendix I for a reminder of main facts. Given a probability measure supported in the discrete set of vectors \(\{u_i\}_i\) of the first gradient, that is decomposed in the form of a \((H_n)\)-condition along a set of successive directions, we focus on those decompositions, performed in the same way for the second gradient \(\nabla v\), that preserve such family of decomposition directions coming from the first component. Intuitively, a fixed-point for such a map would respect:

1. decomposition directions for both components; and
2. equal volume fractions for the two components jointly, because the passage from one component to the other through the above identification \(u_i \mapsto v_i\) respects such volume fractions for a fixed point.

Therefore fixed points for such a map are identified with joint, i.e. simultaneously in the two components, \((H_n)\)-conditions whose decomposition directions are parallel, i.e. with laminates. Our claim, then, reduces to proving the existence of at least one fixed point for such a map.

Most of the technicalities are related to showing that a suitable framework can be set up so that the appropriate assumptions hold for the fixed-point result to be applied. One crucial issue, though, is to understand what is special about a probability measure associated with a gradient \((\nabla u, \nabla v)\), since we know that not every such probability measure should allow the treatment through such fixed point argument. Indeed, this crucial ingredient is related to the fact that such mapping, together with its domain, is well-defined for the probability measure associated with such a gradient, and the assumptions for the existence of a fixed point are met, while it would not be so for an arbitrary probability measure supported in \(\mathbb{M}^{2 \times N}\).

The proof of our main result is divided in three parts:

1. Section 2 we determine in a suitable way the domain of our underlying map \(T\).
2. Section 3 the set-valued map \(T\) is defined, and Kakutani’s fixed-point theorem is stated.
3. Section 4 the required hypotheses for the fixed-point theorem to be applied are proved.
Since the $(H_n)$-condition formalism will play a fundamental role, we have included a final appendix about it for the convenience of readers.

One of the main applied fields where vector variational problems are relevant is nonlinear elasticity \([3]\). In particular, polyconvexity has played a major role in existence results. See also \([8]\). A main hypothesis to be assumed in this area is the rotationally invariance, as well as the behavior for large deformations. See \([12]\) for a discussion on all these notions of convexity under this invariance. Higher-order theories have also been explored, at least from an abstract point of view \((14, 21)\). More general concepts of quasiconvexity have been introduced in \([17]\). Recent interesting results about approximation by polynomials are worth mentioning \([18]\). Explicit examples of rank-one convex functions can be found in various works; \([5, 11, 34]\), among others. See also \([39, 40]\).

The recent book \([30]\) is to be considered.

2. The domain

We will be working with piecewise affine, two-component maps with respect to a specific family of triangulations of the unit cube \(Q\) of \(\mathbb{R}^N\). This unit cube \(Q\) can be decomposed in a finite number of simplexes and with a finite number \(d(N)\) of normals to the flat faces of those simplexes. By making small copies of this decomposition, we can build a family of triangulations that provide uniform approximations of Lipschitz functions by piecewise affine maps. This is standard and well-known (see, for instance, \([15]\)). For \(N = 2\), three normals suffice, while for dimension \(N = 3\), seven are necessary, and so on. Because of this approximation argument, we can focus on piecewise affine, two-component maps with respect to such families of triangulations.

Let \(\Gamma\) be an arbitrary, regular, finite triangulation of \(Q\), as indicated in the previous paragraph, with elements \(\{T_i\}_i\), \(\lambda_i = |T_i| > 0\), nodes \(\{P_p\}_p\), and planar interfaces \(\gamma_{ij}\) if \(T_i\) and \(T_j\) share a flat boundary. For \(N = 2\), the triangulation \(\Gamma\) can be clearly chosen so that \(|T_i|\) is the same positive number for all the elements of \(\Gamma\). Let \((u, v) : Q \to \mathbb{R}^2\) be a \(Q\)-periodic map, piece-wise affine with respect to \(\Gamma\) so that

\[
(\nabla u(x), \nabla v(x)) = \sum_i \chi_{T_i}(x)(u_i, v_i), \quad x \in Q,
\]

and let

\[
\nu = \sum_i \lambda_i \delta_{(u_i, v_i)}, \quad \nu_u = \sum_i \lambda_i \delta_{u_i}, \quad \nu_v = \sum_i \lambda_i \delta_{v_i},
\]

be the underlying probability measure with vanishing first moment, and its two marginals, respectively. Put

\[
\mathcal{U} = \{u_i\} \subset \mathbb{R}^N, \quad \mathcal{V} = \{v_i\} \subset \mathbb{R}^N.
\]

There is definitely something special about \(\nu\) in \((2.2)\). Indeed, we well know that not every discrete probability measure supported in \(\mathbb{M}^{2 \times N}\) may come from a gradient as in \((2.1)\). Because both components \(u\) and \(v\) are piecewise affine with respect to the same triangulation \(\Gamma\), the set of \(d(N)\) normals across planar interfaces for both components is the same. This is a fundamental fact that will be used in a crucial way below. In addition, we will also make use of nodal values of vector \((u, v)\) on nodes \(\{P_p\}_p\), and again
this would not be possible if the probability measure supported in $\mathbf{M}^{2 \times N}$ is not coming from a gradient.

For each value of $m \in \mathbb{N}$, select two concrete $2^m$-tuples

$$X_m = (x_1, x_2, \ldots, x_{2^m}) \in \mathbb{U}^{2^m} \subset (\mathbb{R}^N)^{2^m},$$
$$Y_m = (y_1, y_2, \ldots, y_{2^m}) \in \mathbb{V}^{2^m} \subset (\mathbb{R}^N)^{2^m},$$

complying with the following fundamental properties:

(1) Compatibility. For every $j$, $1 \leq j \leq 2^m$, there is always $i$ with

$$(x_j, y_j) = (u_i, v_i)$$

corresponding to the same element $T_i$ of $\Gamma$.

(2) Adjacency. For every $k$, $1 \leq k \leq 2^m - 1$, vectors $x_{2k}, x_{2k-1}$ in $X_m$, on the one hand, and $y_{2k}, y_{2k-1}$ in $Y_m$, on the other, are adjacent with respect to the given triangulation $\Gamma$ corresponding to interface $\gamma_{ij}$ if

$$x_{2k} = u_i, \quad x_{2k-1} = u_j,$$
$$y_{2k} = v_i, \quad y_{2k-1} = v_j,$$

in such a way that

$$x_{2k} - x_{2k-1} \parallel y_{2k} - y_{2k-1}$$

and this direction is one of the $d(N)$ normals indicated above.

(3) Nodal organization. Both $X_m$ and $Y_m$ can be organized according to a partition in pairwise-disjoint $2^{m-n}$-tuples $X_{m,p}$ (and $Y_{m,p}$), one for each node $P_p$ of $\Gamma$,

$$X_{m,p} = (x_{j_p+1}, x_{j_p+2}, \ldots, x_{j_p+2^m-n})$$

if there are $2^n$ nodes in $\Gamma$, in such a way that each $X_{m,p}$ only contains values of the gradient of $u$ (or of $v$) corresponding to elements $T_{i,p}$ of $\Gamma$ having $P_p$ as one of its nodes, and all planar interfaces $\gamma_{ij}$ having $P_p$ as one of its nodes are represented in $X_{m,p}$ through the adjacency condition; notice that $\{X_{m,p}\}_p$ will not induce, in general, a partition of the set of elements $\{T_i\}$ of $\Gamma$, i.e. of the set $\mathbb{U}$, or of the full set of flat interfaces $\{\gamma_{ij}\}$, because each triangle $T_i$ of $\Gamma$, and each planar interface $\gamma_{ij}$, has various vertices $P_p$, and each one of these is shared by various simplexes of $\Gamma$, and various different interfaces. In other words, the sets of vectors in each $X_{m,p}$ will not be disjoint, but the sets of indices $\{j_p + 1, j_p + 2, \ldots, j_p + 2^{m-n}\}$, varying with $p$, are. There are always sequences of triangulations of increasing fineness with a number of nodes which is a power of 2. Insisting in that sets $X_{m,p}$ have the same number of elements, and a power of 2, forces us to admit that interfaces having a node in $P_p$ cannot be equally, i.e. occurring the same number of times, represented in $X_{m,p}$. It is as if some interfaces were counted more than once.

(4) Representation. There is one specific collection of weights $\{\nu_j\} \in [0,1]^{2^m}$, such that

$$\nu_u = \sum_j \nu_j \delta_{x_j};$$

(2.4)
as a consequence,

\[ \nu_v = \sum_j \bar{t}_j \delta_{y_j}. \]

It is clear that these tuples can be chosen, in a non-unique way, at least for each value of \( m \) large. In addition, by allowing \( m \) to be larger if necessary, we can assume, without loss of generality and through a standard perturbation argument, that

\[ \bar{t}_{2i} + \bar{t}_{2i-1} = 2^{1-m}, \quad 1 \leq i \leq 2^{m-1}, \]

by choosing a finer representation of both \( \nu_u \) and \( \nu_v \) in (2.4) and (2.5), respectively. This property is not necessary, but it will make things a bit simpler.

Once \( X_m \) and \( Y_m \) have been selected as just indicated, define the set

\[ \Theta_m = \{ t = (t_1, t_2, \ldots, t_{2^m}) \in \mathbb{R}^{2^m} : t_j \geq 0, t_{2i} + t_{2i-1} = 2^{1-m}, 1 \leq i \leq 2^{m-1}, \]

\[ \sum_i t_i \delta_{x_i} = \nu_u \}. \]

Note that, because of the way in which \( X_m \) and \( Y_m \) have been chosen, we also have

\[ \Theta_m = \{ t = (t_1, t_2, \ldots, t_{2^m}) \in \mathbb{R}^{2^m} : t_j \geq 0, t_{2i} + t_{2i-1} = 2^{1-m}, 1 \leq i \leq 2^{m-1}, \]

\[ \sum_i t_i \delta_{y_i} = \nu_v \}. \]

It is interesting to realize that each element \( t \in \Theta_m \) represents a certain partition of the unit cube \( Q \), organized in a suitable way bearing in mind the \((H_n)\)-formalism for laminates. The full set \( \Theta_m \) is a collection of such partitions with a particular structure that is very convenient for our purposes.

**Proposition 2.1.** For \( m \) sufficiently large, the set \( \Theta_m \) so selected, is non-empty, compact, and convex.

**Proof.** The compactness and the convexity of \( \Theta_m \) are straightforward. It is non-empty because, by construction,

\[ \bar{t} = (\bar{t}_1, \ldots, \bar{t}_{2^m}) \]
coming from the representation condition above belongs to \( \Theta_m \). For this, we may have to take \( m \) large enough. \( \square \)

3. **The map and its role**

Each element \( t \in \Theta_m \) gives rise to a whole structure according to the \((H_n)\)-formalism that is defined recursively as follows (check the final Appendix):

1. **Initialization.** Put

\[ t_i^{(m)} = t_i, \quad x_i^{(m)} = x_i, \quad y_i^{(m)} = y_i, \]

for \( 1 \leq i \leq 2^m \).

2. **Recursion.**
(a) Relative weights. For

\[ k = m - 1, m - 2, \ldots, 1, \quad 1 \leq i \leq 2^k, \]

put

\[ t_i^{(k)} = t_{2i-1}^{(k+1)} + t_{2i}^{(k+1)}, \]

and

\[ \lambda_i^{(k)} = \begin{cases} \frac{t_{2i}^{(k+1)}}{t_i^{(k+1)}}, & t_i^{(k)} > 0 \\ 1/2, & t_i^{(k)} = 0 \end{cases}. \]

In this way

\[ t_{2i}^{(k+1)} = t_i^{(k)} \lambda_i^{(k)} \quad t_{2i-1}^{(k+1)} = t_i^{(k)} (1 - \lambda_i^{(k)}), \]

and \( t_i^{(0)} = 1 \). Note however that, because the way in which the set \( \Theta_m \) has been chosen,

\[ t_i^{(m-1)} = 2^{1-m} \quad 1 \leq i \leq 2^{m-1}, \]

and then

\[ \lambda_i^{(k)} = 1/2 \quad 1 \leq k \leq m - 2, 1 \leq i \leq 2^k. \]

(b) Decomposition direction. For

\[ k = m - 1, m - 2, \ldots, 1, \quad 1 \leq i \leq 2^k, \]

define

\[ X_i^{(k)} = x_{2i-1}^{(k+1)} - x_{2i}^{(k+1)}, \quad Y_i^{(k)} = y_{2i-1}^{(k+1)} - y_{2i}^{(k+1)}. \]

(c) New level. For

\[ k = m - 1, m - 2, \ldots, 1, \quad 1 \leq i \leq 2^k, \]

set

\[ X_i^{(k)} = (1 - \lambda_i^{(k)}) x_{2i-1}^{(k+1)} + \lambda_i^{(k)} x_{2i}^{(k+1)}, \]
\[ Y_i^{(k)} = (1 - \lambda_i^{(k)}) y_{2i-1}^{(k+1)} + \lambda_i^{(k)} y_{2i}^{(k+1)}. \]

Decomposition directions \( X_i^{(k)} \) and \( Y_i^{(k)} \), vectors \( x_i^{(k)} \) and \( y_i^{(k)} \), and relative weights \( \lambda_i^{(m-1)} \) as well, depend upon \( t \). To make this dependence explicit we will simply put

\[ X_i^{(k)}(t), \quad Y_i^{(k)}(t), \quad x_i^{(k)}(t), \quad y_i^{(k)}(t), \quad \lambda_i^{(m-1)}(t). \]

Recall that \( \lambda_i^{(k)} = 1/2 \) for all

\[ 0 \leq k \leq m - 2, \quad 1 \leq i \leq 2^k, \]

according to (3.2).

A joint, simultaneous rank-one decomposition of \( \nu \) in (2.2) demands that decomposition directions \( X_i^{(k)} \) and \( Y_i^{(k)} \) are proportional to each other for all

\[ 0 \leq k \leq m - 1, \quad 1 \leq i \leq 2^k. \]
This is guaranteed for \( k = m - 1 \) because of the way sets \( \mathcal{X}_m \) and \( \mathcal{Y}_m \) have been selected (recall the adjacency condition in Section 2). This fundamental property sought motivates the definition of our set-valued mapping

\[ T : \Theta_m \mapsto 2^{\Theta_m}, \]

by putting

\[ T(t) = \{ s \in \Theta_m : Y^{(k)}_i(s) \parallel X^{(k)}_i(t) \text{ for all } 0 \leq k \leq m - 2, 1 \leq i \leq 2^k \}. \]

Note again how \( Y^{(m-1)}_i(s) \) is always parallel to \( X^{(m-1)}_i(s) \) precisely because decomposition directions at the level \( k = m - 1 \) correspond to interfaces between two adjacent elements of the triangulation \( \Gamma \).

The whole point or our concern is the following.

**Proposition 3.1.** The gradient measure \( \nu \) in (2.2) is a laminate if, for some \( m \), there is a fixed point for \( T \), i.e. there is \( t \in \Theta_m \) such that \( t \in T(t) \).

**Proof.** The proof is immediate given the way in which both the set \( \Theta_m \) and the map \( T \) have been defined through sets \( \mathcal{X}_m \) and \( \mathcal{Y}_m \). \( \square \)

We will be using the following classic result to show the existence of a fixed-point of \( T \) for some large \( m \).

**Theorem 3.2.** (Kakutani’s fixed point theorem) Let \( A \subset \mathbb{R}^d \) be a non-empty, compact, convex set, and let \( F : A \mapsto A \) be an upper semicontinuous, set-valued map with non-empty, convex, compact values. Then \( F \) has a fixed point; that is, there is \( \hat{x} \in A \) with \( \hat{x} \in F(\hat{x}) \).

This is a classical theorem on fixed-points for set-valued maps, which is but a generalization of the classic Brower’s fixed point theorem. It is well-known, and can be found in many places, for instance in [33].

The fundamental properties that the application of this result to our framework requires are the non-emptiness, compactness and convexity of \( T(t) \) for each \( t \in \Theta_m \), in addition to the upper semicontinuity.

### 4. Main properties of the map \( T \)

We start with the upper semicontinuity required by Theorem 3.2. This property is, as a matter of fact, elementary, since if

\[ s_j \in T(t_j), \quad s_j \to s, t_j \to t, \]

then, we must necessarily have \( s \in T(t) \). This is straightforward because the dependence of elements in (3.6) on \( t \) is continuous.

On the other hand, the compactness of each subset \( T(t) \) is also clear since all these images are closed subsets of the compact set \([0, 1]^{2m}\).
4.1. Convexity of images. Ensuring this convexity property is responsible for the precise definition of the set $\Theta_m$ we have adopted. It is pretty clear after the following statement.

Proposition 4.1. 

(1) For $k = m - 1, m - 2, \ldots, 1$, $1 \leq i \leq 2^k$,

vectors

$$x^{(k)}_i(t), \quad y^{(k)}_i(t)$$

in (3.4) and (3.5), respectively, depend linearly on $t$ for $t \in \Theta_m$, and consequently, so do decomposition directions

$$X^{(k)}_i(t), \quad Y^{(k)}_i(t)$$

in (3.3).

(2) For each $t \in \Theta_m$, the set $T(t)$ is convex.

Proof. For the first part, note that if we resort to (3.4) and (3.5), we realize that for $k = m - 1$, because

$$i^{(m-1)}_i = 2^{1-m}, \quad 1 \leq i \leq 2^{m-1},$$

and vectors

$$x^{(m)}_j, \quad y^{(m)}_j$$

are given and fixed (taken, respectively, from the sets $X_m$ and $Y_m$, for all $1 \leq j \leq 2^m$ once these have been chosen), those formulas are linear in the components of $t$ because weights $\lambda^{(k-1)}_i$ are. On the other hand, for

$$k = m - 2, m - 3, \ldots, 2, 1,$$

those same formulas indicate that

$$x^{(k)}_i, \quad y^{(k)}_i$$

depend linearly on

$$x^{(k+1)}_j, \quad y^{(k+1)}_j$$

precisely because those relative weights $\lambda^{(k)}_j$, for $t \in \Theta_m$, are exactly $1/2$. By the recursive nature of $(H_n)$-conditions, we have the claimed linear dependence.

The first statement immediately yields the second. If $s_i \in T(t), \quad i = 0, 1,$ and $r \in (0, 1)$, then, for

$$s = rs_1 + (1 - r)s_0,$$

we will have

$$Y^{(k)}_i(s) = rY^{(k)}_i(s_1) + (1 - r)Y^{(k)}_i(s_0)$$

precisely by the previous fact. Hence, if

$$Y^{(k)}_i(s_i) \parallel X^{(k)}_i(t), \quad i = 0, 1,$$

so will $Y^{(k)}_i(s)$ be. This means that $s \in T(t)$. \qed
4.2. Non-emptiness of images. This is the most delicate issue of our proof.

We regard the first component $u$ of our two-component map $(u,v): Q \to \mathbb{R}^2$
as fixed but arbitrary, and allow the second component $v$ to vary. Recall that $\{P_p\}_p$ is an enumeration of the nodes of the triangulation $\Gamma$. The $Q$-periodic function $v$, piecewise affine with respect to $\Gamma$, is uniquely determined by the set of its nodal values $\{v(P_p)\}$, and hence it can be identified in a natural way with $\mathbb{R}^q$ if $q$ is the finite number, depending on dimension $N$ and the fineness of $\Gamma$ (in fact $q$ was chosen as $2^n$ in Section 2). By a natural abuse of language, we will say that $v \in \mathbb{R}^q$.

For an arbitrary $t \in \Theta_m$, determined through the first-component $u$, regarded as given and fixed but otherwise arbitrary, consider the following subset of $\mathbb{R}^q$:

\[(4.1) \quad \Upsilon(t) = \{v \in \mathbb{R}^q : T(t) \neq \emptyset\}.\]

The non-emptiness of every image $T(t)$ amounts to showing the following fact.

**Lemma 4.2.** For every $t \in \Theta_m$, the set $\Upsilon(t)$ in (4.1) is always the full $\mathbb{R}^q$.

**Proof.** The proof proceeds after a typical connectedness argument. We will show that $\Upsilon(t)$ is non-empty, closed, and open, and so it will be the full set $\mathbb{R}^q$.

The non-emptiness of $\Upsilon(t)$ is clear because $u$ itself, through its nodal values $u(P_p)$, belongs to $\Upsilon(t)$. Note that when we take $v = u$, so that our two-component map becomes $(u,u)$, then $t \in T(t)$, and so $\Upsilon(t)$ is non-empty.

It is elementary to realize that $\Upsilon(t)$ is closed, given that $\Theta_m$ is compact, and $T(t)$ is closed. There is no difficulty here.

The crucial step is to show the openness of $\Upsilon(t)$. To this end, if we put

\[V_i^{(k)}(v,s) = X_i^{(k)}(u,t), \quad 0 \leq k \leq m - 1, \quad 1 \leq i \leq 2^k,\]

a fixed collection of decomposition directions, some of which could be null, determined through the first-component $u$ (we have explicitly indicated so in the above notation), Lemma 4.3 below directly shows that $\Upsilon(t)$ is also open. If this lemma is correct, our statement is proved, and so is the non-emptiness of every image $T(t)$. □

As just indicated, the fundamental step necessary to show the non-emptiness of images $T(t)$ for each $t \in \Theta_m$ is the following. Assume decomposition vectors $\{V_i^{(k)}\}_{0 \leq k \leq m - 1, 1 \leq i \leq 2k} \subset \mathbb{R}^N$ are given in such a way that there is a $Q$-periodic function $v: Q \to \mathbb{R}$, piecewise affine with respect to a triangulation $\Gamma$ of $Q$ for which there is $s \in \Theta_m$ with

\[Y_i^{(k)}(v,s) \parallel V_i^{(k)}\]

for all

\[0 \leq k \leq m - 1, \quad 1 \leq i \leq 2^k.\]

We are using here, as already indicated, the more complex notation $Y_i^{(k)}(v,s)$ for decomposition direction to stress that these depend on the underlying function $v$, which is changing. In particular, for the last level $k = m - 1$,

\[V_i^{(m - 1)}, \quad 1 \leq i \leq 2^{m - 1},\]
is one of the finite number \(d(N)\) of normals used in the triangulation \(\Gamma\). Some of those decomposition directions \(V^{(k)}_i\) could vanish, but this is even more advantageous as then we are free to select a parallel direction without any restriction.

**Lemma 4.3.** There is a neighborhood \(V\) of \(v\) in \(\mathbb{R}^q\), through the above identification, such that for every \(\overline{v} \in V\) there is \(\overline{s} \in \Theta_m\) with

\[
Y^{(k)}_i(\overline{v}, \overline{s}) \parallel V^{(k)}_i
\]

for all

\[
0 \leq k \leq m - 1, \quad 1 \leq i \leq 2^k.
\]

**Proof.** If \(\{P_p\}_p\) is an enumeration of the nodes of the triangulation \(\Gamma\), it suffices to focus on perturbations of the function \(v\) produced by changing the nodal value \(v_l\) of \(v\) at a certain fixed, but otherwise arbitrary, node \(P_l\) while retaining the value \(v_p\) of \(v\) at the other nodes \(P_p\), \(p \neq l\). Let \(I(l)\) indicate the set of indices \(i\) of those values \(v_i \in V\) (recall (2.3)) of the gradient \(\nabla v\) in elements of \(\Gamma\) affected by the value \(v_l\) of \(v\) at the node \(P_l\), i.e. \(I(l)\) is the set of those indices of elements of \(\Gamma\) one of whose nodes is \(P_l\). Notice how this is closely related to the nodal organization property of Section 2. Our claim is then that for some small positive \(\epsilon\), the piecewise affine function \(\overline{v}\) that shares the nodal values \(\overline{v}_p = v_p\) with \(v\) for \(p \neq l\), but

\[
|\overline{v}_l - v_l| < \epsilon,
\]

is such that its gradient \(\nabla \overline{v}\) is the result of a \((H_n)\)-condition with the same decomposition directions \(V^{(k)}_i\), preserving relative weights \(1/2\) at all levels except the last one (as required in the definition of \(\Theta_m\)). The value of \(\overline{s}\) is then the result of final weights coming from the top-to-bottom description of \((H_n)\)-conditions (see the final Appendix).

Recall that

\[
\nabla v(x) = \sum_i \chi_{T_i}(x)v_i, \quad x \in Q, \quad \nu_v = \sum_i \lambda_i \delta_{v_i}.
\]

Our hypothesis for \(v\) implies that

\[
\nu_v = \sum_i s_i \delta_{y^{(m)}_i}
\]

and each vector \(y^{(m)}_i\) (which is one of the \(\{v_i\}\) taken on by \(\nabla v\) over the elements of the triangulation \(\Gamma\)) can be written in terms of the decomposition directions \(V^{(k)}_i\), and of the set of scalars

\[
S = \{S^{(k)}_i\}_k_{k=0},..._{k=m-2},i_{i=1},...,2^k \cup \{S^{(m-1)}_{i,\_\_}i_{i=1},...,2m-1 \cup \{S^{(m-1)}_{i,\_\_}i_{i=1},...,2m-1
\]

generated along the process through \((H_n)\)-conditions of the discrete probability measure associated with \(\nabla v\) by using decomposition directions \(V^{(k)}_i\). This exactly means that

\[
y^{(k+1)}_{2i} = y^{(k)}_i + s^{(k)}_i V^{(k)}_i, \quad y^{(k+1)}_{2i-1} = y^{(k)}_i - s^{(k)}_i V^{(k)}_i, \quad 0 \leq k \leq m - 2, 1 \leq i \leq 2^k,
\]

\[
y^{(m)}_{2i} = y^{(m-1)}_i + s^{(m-1)}_{i,\_\_} V^{(m-1)}_i, \quad y^{(m)}_{2i-1} = y^{(m-1)}_i - s^{(m-1)}_{i,\_\_} V^{(m-1)}_i
\]
Note how (4.4) means that relative volume fractions up to level $m - 2$ are exactly $1/2$, so that the corresponding vector $s$ of final weights belongs to $\Theta_m$. This is the reason why we do not have to consider two families of numbers

$$S_{k,+}, \quad S_{k,-}$$

for $k = m - 2, \ldots, 2, 1, 0$.

Nodal values $v_p$ of $v$ at nodes $P_p$ can also be understood as functions of $S$ in (4.3). The important point is to realize that this dependence of $v_p(S)$ on each individual independent variable $S_i^{(k)}$ in (4.3) is affine (eventually constant) when all other components are kept fixed. This is so because of the recursive linear way in which $(H_n)$-conditions are built (check the Appendix at the end of the paper) if decomposition directions $V_i^{(k)}$, regarded as constant vectors, are to be respected: vector values $y_i^{(m)}$ of $\nabla v_i$ depend linearly on each individual $S_i^{(k)}$, as indicated above, and the dependence of the values of $v_j$ on nodal values $v_p$ is also linear. We can conclude that functions $v_p(S)$ are multilinear.

If we now fix our attention on a specific, but arbitrary nodal value $v_l = v(P_l)$, and assume that (4.2) does not hold, then it is elementary to realize that the gradient of $v_l(S)$ with respect to $S$ at the precise value of $S$ furnishing the nodal value $v_l$ must vanish: this value of $S$ with $v(S) = v_l$ ought to be a local extreme for $v_l(S)$, either a local maximum or a local minimum. But because of this multilinear dependence on $S$, it is a fact that the gradient of $v_l(S)$ with respect to $S$ can never vanish even if we restrict these values $S$ by demanding that $v_p(S)$, the nodal values of $v$ at $P_p$, all of which are also multilinear functions of $S$, be given, fixed numbers for all $p \neq l$. As a matter of fact, the nodal organization property in Section 2 has been enforced so that there is a specific subset $S_l$ of independent variables from (4.3) which only affect the nodal value $v_l$, but all other nodal values $v_p$ for $p \neq l$ are independent of those precise variables in $S_l$. In this way, if we keep constant the variables of $S \setminus S_l$, all nodal values $v_p$, $p \neq l$, will stay constant, but the value $v_l$ will depend, in a multilinear fashion, on the variables of $S_l$. Since a non-constant, multilinear function cannot have extreme points, our claim (4.2) is then correct, and the arbitrariness of $l$ implies our statement.

$\square$

Theorem 3.2 can then be applied, and, through Proposition 3.1, we conclude that $(\nabla u, \nabla v)$ is indeed a laminate. The arbitrariness of the triangulation $\Gamma$ implies that every two-component gradient is a laminate, and hence rank-one convexity implies quasiconvexity in the case $2 \times N$.

5. Appendix

We include here, for the convenience of our readers, a short discussion about the notion of $(H_n)$-condition with respect to a given cone $\Lambda$ of admissible directions, as introduced in [9].

We start with a given, discrete probability measure supported in $M^{m \times N}$

$$\nu = \sum_i \lambda_i \delta_{u_i}, \quad \sum_i \lambda_i u_i = 0, \quad \lambda_i > 0, \quad \sum_i \lambda_i = 1, u_i \in M^{m \times N},$$
and put

$$\nu^{(1)} = \delta_0, \quad \text{supp}(\nu^{(1)}) \subset M^{m \times N}.$$ 

Given

$$(5.1) \quad \nu^{(k)} = \sum_i \lambda_i^{(k)} \delta_{\mathbf{u}^{(k)}_i}, \quad \text{supp}(\nu^{(k)}) \subset M^{m \times N}$$

we recursively split the delta measure supported at each $\mathbf{u}^{(k)}_i$ along a certain direction $\mathbf{U}^{(k)}_i$ (which eventually could be the null vector) taken from a selected cone of feasible directions $\Lambda \subset M^{m \times N}$, and with relative weights $t^{(k)}_i$ and $1 - t^{(k)}_i$, so that

$$(5.2) \quad \delta_{\mathbf{u}^{(k)}_i} \mapsto t^{(k)}_i \delta_{\mathbf{u}^{(k)}_i} + (1 - t^{(k)}_i) \mathbf{U}^{(k)}_i + (1 - t^{(k)}_i) \delta_{\mathbf{u}^{(k)}_i - t^{(k)}_i \mathbf{U}^{(k)}_i}.$$ 

Note that weights $t^{(k)}_i$ are given by the various mass points involved, provided decomposition vector $\mathbf{U}^{(k)}_i$ is not zero. Indeed

$$t^{(k)}_i = \frac{|(\mathbf{u}^{(k)}_i - t^{(k)}_i \mathbf{U}^{(k)}_i) - \mathbf{u}^{(k)}_i|}{|(\mathbf{u}^{(k)}_i + (1 - t^{(k)}_i) \mathbf{U}^{(k)}_i) - (\mathbf{u}^{(k)}_i - t^{(k)}_i \mathbf{U}^{(k)}_i)|},$$

$$1 - t^{(k)}_i = \frac{|(\mathbf{u}^{(k)}_i + (1 - t^{(k)}_i) \mathbf{U}^{(k)}_i) - \mathbf{u}^{(k)}_i|}{|(\mathbf{u}^{(k)}_i + (1 - t^{(k)}_i) \mathbf{U}^{(k)}_i) - (\mathbf{u}^{(k)}_i - t^{(k)}_i \mathbf{U}^{(k)}_i)|}.$$ 

The new probability measure is obtained by replacing each such decomposition back into $\nu^{(k)}$ in (5.1)

$$\nu^{(k+1)} = \sum_i \lambda_i^{(k)} \left(t^{(k)}_i \delta_{\mathbf{u}^{(k)}_i} + (1 - t^{(k)}_i) \mathbf{U}^{(k)}_i + (1 - t^{(k)}_i) \delta_{\mathbf{u}^{(k)}_i - t^{(k)}_i \mathbf{U}^{(k)}_i}\right),$$

and reorganizing such representation. One same vector in the support of $\nu^{(k+1)}$ may come from several decompositions in the previous step. Note that if $\mathbf{U}^{(k)}_i = \mathbf{0}$ for all $i$ and some fixed $k$, then $\nu^{(k+1)} = \nu^{(k)}$, and if only some $\mathbf{U}^{(k)}_i = \mathbf{0}$ then the matrix $\mathbf{u}^{(k)}_i$ is passed intact onto the next level. The final measure $\nu^{(m)}$, after a finite number $m$ of steps, should be the one we started with $\nu = \nu^{(m)}$. This is the top-to-bottom procedure.

It is important to stress that the fundamental cone for vector variational problems is the rank-one cone

$$\Lambda = \{ \mathbf{U} \in M^{m \times N} : \text{rank}(\mathbf{U}) \leq 1 \},$$

and that such a cone is the full set of directions if either dimension $m$ or $N$ is unity.

It is enlightening to describe such $(H_n)$-conditions exclusively in terms of vectors and weights. The most direct way of doing this is by keeping record of weights and mass points for the successive probability measures (recall that weights are given and determined by such mass points as indicated above unless denominators vanish), namely

$$(5.3) \quad \{ \{\lambda^{(k)}_i, \mathbf{u}^{(k)}_i\} \}_{1 \leq i \leq 2^k} \{0 \leq k \leq m \}$$
where

\[ u_i^{(k)} = \frac{\lambda_i^{(k+1)}}{\lambda_{2i-1}^{(k+1)} + \lambda_{2i}^{(k+1)}} u_{2i-1}^{(k+1)} + \frac{\lambda_{2i}^{(k+1)}}{\lambda_{2i-1}^{(k+1)} + \lambda_{2i}^{(k+1)}} u_{2i}^{(k+1)}, \]

for all \( 0 \leq k \leq m - 1, 1 \leq i \leq 2^k \). Relative weights are given by

\[ t_i^{(k)} = \frac{\lambda_i^{(k+1)}}{\lambda_{2i-1}^{(k+1)} + \lambda_{2i}^{(k+1)}} = \frac{|u_i^{(k)} - u_{2i-1}^{(k+1)}|}{|u_{2i}^{(k+1)} - u_{2i-1}^{(k+1)}|}, \]
\[ 1 - t_i^{(k)} = \frac{\lambda_{2i}^{(k+1)}}{\lambda_{2i-1}^{(k+1)} + \lambda_{2i}^{(k+1)}} = \frac{|u_i^{(k)} - u_{2i}^{(k+1)}|}{|u_{2i}^{(k+1)} - u_{2i-1}^{(k+1)}|}. \]

When \( u_{2i}^{(k+1)} = u_{2i-1}^{(k+1)} \), the weight \( t_i^{(k)} \) can be chosen in any way in the interval \([0, 1]\). Note that for fixed \( k \), several of the \( u_i^{(k)} \)'s may be the same vector, that \( u_i^{(0)} = 0 \), and that \( u_i^{(k)} \) have to be vectors in the convex hull of the support of \( \nu \). The differences

\[ U_i^{(k)} = u_i^{(k+1)} - u_{2i-1}^{(k+1)} \] or rather \( U_i^{(k)} \| u_{2i}^{(k+1)} - u_{2i-1}^{(k+1)} \)

furnish decomposition directions on each step. The set of vectors

\[ \{ (U_i^{(k)}) \}_{0 \leq k \leq m - 1, 1 \leq i \leq 2^k} \]

is the (complete) set of decomposition directions of the \((H_n)\)-condition. Notice that each vector \( u_i^{(k)} \) goes with a weight \( s_i^{(k)} \) which is the product of \( k \) of the decomposition weights \( t_i^{(j)} \) for \( 0 \leq j \leq k - 1 \), in such a way that \( s_i^{(0)} = 1 \), and

\[ \nu^{(m)} = \nu = \sum_{i=1}^{2m} s_i^{(m)} \delta_{u_i^{(m)}}. \]

There is a whole bunch of intermediate probability measures for fixed \( k \)

\[ \nu^{(k)} = \sum_{i=1}^{2^k} s_i^{(k)} \delta_{u_i^{(k)}}. \]

These are the same as in \((5.1)\). Weights \( \lambda_i^{(k)} \) are obtained by adding together several of the weights \( s_i^{(k)} \) when corresponding vectors \( u_i^{(k)} \) are identical. In addition, each \( u_i^{(k)} \) in the support of \( \nu^{(k)} \) is the first-moment of, at least, one precise (sub)probability measure associated with the \((H_n)\)-condition. Namely,

\[ \nu_i^{(k)} = \sum_j r_{i,j}^{(k)} \delta_{u_j}. \]

These are such that

\[ \nu = \sum_i s_i^{(k)} \left( \sum_j r_{i,j}^{(k)} \delta_{u_j} \right) \]
for all \(k\). Note that these probability measures \(\nu^{(k)}_i\) are associated with a certain sub-\((H_n)\)-condition of the original \((H_n)\)-condition, starting from \(u^{(k)}_i\) as the barycenter.

There is nothing special about the zero vector being the initial vector. The same construction can be made in exactly the same way, had we started out with a different vector \(F \in \mathbb{M}^{m \times N}\), since the basic operation involved in \((H_n)\)-conditions is translation-invariant.

One fundamental observation, after the discussion above, is that the whole \((H_n)\)-condition is completely determined once weights and mass points in the final level have been chosen. This would correspond to the bottom-to-top scheme. Namely, suppose we have

\[
\{ (\lambda_i, u_i) \}_{1 \leq i \leq 2^m}, \quad \nu = \sum_i \lambda_i \delta_{u_i}, \quad \sum_i \lambda_i u_i = 0,
\]

where some of the \(u_i\)’s may be repeated, but they are given. Put

\[
\lambda^{(m)}_i = \lambda_i, \quad u^{(m)}_i = u_i,
\]

and define recursively

\[
\lambda^{(k)}_i = \lambda^{(k+1)}_{2i-1} + \lambda^{(k+1)}_{2i},
\]

and

\[
u^{(k)}_i = \frac{\lambda^{(k+1)}_{2i-1}}{\lambda^{(k)}_i} u^{(k+1)}_{2i-1} + \frac{\lambda^{(k+1)}_{2i}}{\lambda^{(k)}_i} u^{(k+1)}_{2i}
\]

when \(\lambda^{(k)}_i > 0\), but \(u^{(k)}_i\) chosen in any way in the segment \([u^{(k+1)}_{2i-1}, u^{(k+1)}_{2i}]\) if \(\lambda^{(k)}_i = 0\). Despite this ambiguity when \(\lambda^{(k)}_i = 0\), the \((H_n)\)-condition is determined in a unique way because the total mass is carried when there is no such ambiguity. Notice that \(\lambda^{(k)}_i = 0\) implies that both \(\lambda^{(k+1)}_{2i-1}\) and \(\lambda^{(k+1)}_{2i}\) vanish. According to our discussion above, given relative weights \(t^{(k)}_i\) and \(1 - t^{(k)}_i\), for \(0 \leq k \leq m - 1\), \(1 \leq i \leq 2^k\), in all steps, final weights \(\lambda^{(m)}_i\) are given through appropriate products of \(m\) relative weights.

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