On supermembrane actions on Calabi-Yau 3-folds

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Abstract

In this note we examine the supermembrane action on Calabi-Yau 3-folds. We write down the Dirac-Born-Infeld part of the action, and show that it is invariant under the rigid spacetime supersymmetry.

1 Introduction

The study of p-branes as solitonic solutions of supergravity has now become indispensable for the understanding of nonperturbative effects in superstring theory. On flat spacetimes p-branes are described by an action which consists of two parts; the Dirac-Born-Infeld (DBI) part, and the Wess-Zumino part. The latter can only be written down for some specific spacetime dimensions. For supermembranes, for instance, it exists only for 4, 5, 7, and 11 spacetime dimensions [1]. Both parts of the action are invariant under the rigid spacetime supersymmetry transformations, however, for a specific choice of normalization of the Wess-Zumino term, the whole action turns out to have an extra local symmetry known as \( \kappa \)-symmetry. The \( \kappa \)-symmetry allows one to remove the redundant fermionic degrees of freedom leaving only the physical ones.

p-branes (and also D-branes) have so far been studied on flat spacetimes, and in some cases on AdS spaces (look at [1, 2] and the references therein, for instance). On Calabi-Yau manifolds, on the other hand, the action is not known, though, in some cases they have been studied through their low energy effective action [3, 4], which for Dp-branes is just the dimensional reduction of 10d super Yang-Mills theory down to \( p + 1 \) dimensions [4]. Along these lines, an effective action of D-branes on Calabi-Yau 3-folds was also introduced in [3].

The study of p-branes on Calabi-Yau 3-folds becomes crucial if we are to learn about the nonperturbative effects of superstring theory or M-theory compactified on such manifolds.

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So it is of importance to find the effective action of p-branes on Calabi-Yau manifolds. This note is an attempt to construct the action of supermembranes on Calabi-Yau manifolds. In section 2, we begin with the preliminaries. The field decompositions on Calabi-Yau 3-folds are used to write down the supersymmetric invariant one-forms \( \Pi \). However, these forms will not be invariant under supersymmetry transformations as soon as we lift them on Calabi-Yau manifolds. We then show that the supersymmetry transformations on a Calabi-Yau 3-fold can be deformed to look like the BRST transformations. This fact is then used to construct the supersymmetric DBI action of supermembranes.

### 2 Preliminaries and the construction of the action

Consider a Calabi-Yau 3-fold \( M \) which has the holonomy group \( SU(3) \). Take \( \theta \) to be the singlet spinor on \( M \) with the negative chirality. Then, as in \([7, 6]\), we can use the following Fierz identities

\[
\theta \theta^\dagger + \frac{1}{2} \gamma^\alpha \theta^* \theta^\dagger \gamma_{\alpha} = \frac{1}{2} (1 - \gamma_7) \quad (1)
\]

\[
\theta^* \theta^\dagger + \frac{1}{2} \gamma^\alpha \theta \theta^\dagger \gamma_{\alpha} = \frac{1}{2} (1 + \gamma_7) \quad (2)
\]

to decompose a complex spinor \( \Psi = \Psi_L + \Psi_R \) on \( M \). Here we have used \( \alpha, \beta, \gamma, \ldots \) to represent the complex tangent indices on \( M \). This results in

\[
\Psi_L = \theta \psi + \frac{1}{2} \gamma^\alpha \theta^* \psi_{\alpha} \Rightarrow \Psi^\dagger_L = \bar{\psi} \theta^\dagger + \frac{1}{2} \bar{\psi}_{\alpha} \theta^\dagger \gamma^\alpha
\]

\[
\Psi_R = \theta^* \chi + \frac{1}{2} \gamma^\alpha \theta \chi_{\alpha} \Rightarrow \Psi^\dagger_R = \bar{\chi} \theta^\dagger + \frac{1}{2} \bar{\chi}_{\alpha} \theta^\dagger \gamma^\alpha, \quad (3)
\]

where we have defined

\[
\psi = \theta^\dagger \Psi_L \quad , \quad \psi_{\alpha} = \theta^\dagger \gamma_{\alpha} \Psi_L
\]

\[
\chi = \theta^\dagger \Psi_R \quad , \quad \chi_{\alpha} = \theta^\dagger \gamma_{\alpha} \Psi_R.
\]

As \( \theta \) is a singlet, we have chosen \( \gamma_{\alpha} \theta = 0 \). The covariantly constant forms that can be constructed from \( \theta \) are the Kähler 2-form \( k_{\alpha \beta} = i \theta^\dagger \gamma_{\alpha \beta} \theta \), and the holomorphic 3-form \( C_{\alpha \beta \gamma} = \theta^\dagger \gamma_{\alpha \beta \gamma} \theta^* \).

As mentioned in the introduction, the supermembrane action can exist in seven dimensions. Therefore we first write the action on flat \( \mathbb{R}^7 \) spacetime \([4]\) using the above decomposition of fields, then we lift it on to \( M \times \mathbb{R}^1 \), where \( \mathbb{R}^1 \) represent the time direction.

Let us first recall the construction of the action on flat spacetime \([4]\). Let \( m, n, \ldots = 0, 1, 2, \ldots, 6 \) represent the tangent indices on the whole manifold, and \( \mu, \nu, \ldots = 1, 2, \ldots, 6 \) indicate the indices on \( M \). \( i, j, k, \ldots = 1, 2, 3 \) will be the tangent indices on the worldvolume of the brane. As for the gamma matrices we take

\[
\gamma_6 = \gamma_0 \gamma_1 \cdots \gamma_5 \quad , \quad \gamma_7 = i \gamma_0.
\]
where $\gamma_\mu$'s are hermitian and $\gamma_0^\dagger = -\gamma_0$. Define
\[ \Pi^m = dX^m - i\bar{\Psi}\gamma^m\Psi + id\bar{\Psi}\gamma^m\Psi, \]
where $\Psi = \Psi^\dagger\gamma_0$. $\Pi^m$ is invariant under the following rigid supersymmetry transformations
\[ \delta X^m = i\varepsilon\gamma^m\Psi - i\bar{\Psi}\gamma^m\varepsilon, \quad \delta \Psi = \varepsilon. \] (4)
Here $\varepsilon = \varepsilon_L + \varepsilon_R$ is a constant complex six-dimensional spinor. On $M$, $\varepsilon$ decomposes just like $\Psi$, however, as we are interested in the scalar part of the transformations, we drop the triplet part of the $\varepsilon$:
\[ \varepsilon_L = \theta \epsilon, \quad \varepsilon_R = \theta^* \eta. \] (5)

Using the field decompositions in (3) and (5), the supersymmetry transformations (4) read (with $\phi \equiv X^0$)
\[ \delta X^\alpha = \bar{\eta} \psi^\alpha + \epsilon \check{\chi}^\alpha \]
\[ \delta X^{\bar{\alpha}} = -\eta \bar{\psi}^{\bar{\alpha}} - \epsilon \check{\bar{\chi}}^{\bar{\alpha}} \]
\[ \delta \phi = i\bar{\epsilon} \psi + i\epsilon \bar{\psi} + i\bar{\eta} \check{\chi} + i\eta \check{\bar{\chi}} \]
\[ \delta \psi = \epsilon, \quad \delta \check{\chi} = \eta \]
\[ \delta \psi^\alpha = 0, \quad \delta \check{\bar{\chi}}^{\bar{\alpha}} = 0. \] (6)

For the sake of simplicity, in the following, we set $\eta = 0$ in the above transformations and work with just one of the supersymmetries survived on $M$.

The components of $\Pi$, on the other hand, are
\[ \Pi^\alpha = dX^\alpha - \check{\chi} D\psi^\alpha - \psi^\alpha d\check{\chi} - \psi D\check{\chi} - \check{\chi}^\alpha d\psi + \frac{1}{4} C^{\alpha\beta\gamma}(\bar{\psi}_\beta D\chi_\gamma - \chi_\beta D\bar{\psi}_\gamma) \]
\[ \Pi^{\bar{\alpha}} = dX^{\bar{\alpha}} + \chi D\bar{\psi}^{\bar{\alpha}} + \bar{\psi}^{\bar{\alpha}} d\chi + \bar{\psi} D\chi + \chi^{\bar{\alpha}} d\bar{\psi} - \frac{1}{4} C^{\bar{\alpha}\bar{\beta}\bar{\gamma}}(\psi_\beta D\bar{\chi}_\gamma - \bar{\chi}_\beta D\psi_\gamma) \]
\[ \Pi^0 = d\phi - i\bar{\psi} d\psi - i\check{\chi} d\bar{\psi} - \frac{i}{2} g_{\alpha\bar{\beta}} \bar{\psi}^\beta D\psi^\alpha - \frac{i}{2} g_{\alpha\bar{\beta}} \check{\chi}^{\bar{\alpha}} D\bar{\chi}^\beta \]
\[ -i\bar{\psi} d\bar{\psi} - i\chi d\bar{\psi} - \frac{i}{2} g_{\alpha\bar{\beta}} \bar{\psi}^\beta D\bar{\psi}^{\bar{\alpha}} - \frac{i}{2} g_{\alpha\bar{\beta}} \check{\chi}^{\bar{\alpha}} D\check{\bar{\chi}}^\beta, \]
where $D\psi^\alpha = d\psi^\alpha + dX^\beta \Gamma^\alpha_{\beta\gamma} \psi^\gamma$. Further define
\[ M_{ij} = \delta_{mn} \Pi^m_i \Pi^n_j. \] (7)

Now as $\Pi$'s are invariant under the SUSY transformations on flat spacetime, the action
\[ S_{DBI} = -T \int d^4\sigma \sqrt{\det M_{ij}}, \] (8)
is also trivially invariant.

On a Calabi-Yau 3-fold, however, neither $\Pi$ nor the metric is invariant under the SUSY transformations. Therefore $M_{ij} = g_{mn} \Pi^m_i \Pi^n_j$ will not be invariant;
\[ \delta(g_{mn} \Pi^m_i \Pi^n_j) = \Delta(g_{mn} \Pi^m_i \Pi^n_j) = g_{mn} (\Delta \Pi^m_i) \Pi^n_j + g_{mn} \Pi^m_i \Delta \Pi^n_j, \] (9)
where the covariant variation is defined by

$$\Delta \Pi^\alpha = \delta \Pi^\alpha + \delta X^\rho \Gamma^\alpha_{\rho\sigma} \psi^\sigma,$$

and the last equality in (9) follows as the metric $g_{mn}$ is covariantly constant.

To get a supersymmetric action on $M \times \mathbb{R}^1$, firstly we note that on flat spacetimes the transformations (6) square to zero (BRST-like) when acting on any field except $\phi$. Secondly if we define the operator $\tilde{Q}$ by

$$\tilde{Q} = Q + \bar{Q}, \quad \delta = i\epsilon Q + i\bar{\epsilon} \bar{Q},$$

we can see that $M_{ij}$ in the action can be written as a BRST-exact term;

$$M_{ij} = \delta_{mn} \Pi_i^m \Pi_j^n = \{ \tilde{Q}, \frac{i}{2}(\psi + \bar{\psi}) \delta_{mn} \Pi_i^m \Pi_j^n \}.$$

(10)

This follows as $\{ \tilde{Q}, \frac{i}{2}(\psi + \bar{\psi}) \}$ = 1. If we could maintain the BRST property of $\tilde{Q}$ on $M$, with a choice of $M_{ij}$ as in (10), it would be straightforward to construct the supersymmetric action on a Calabi-Yau 3-fold. All we need to do is to compute the term

$$\{ \tilde{Q}, \frac{i}{2}(\psi + \bar{\psi}) g_{mn} \Pi_i^m \Pi_j^n \}$$

(11)

to get $M_{ij}$ on $M \times \mathbb{R}^1$. The invariance of the action under $\tilde{Q}$ then follows as $\tilde{Q}^2 = 0$;

$$\{ \tilde{Q}, S_{DBI} \} = -\frac{T}{2} \int d^3 \sigma \sqrt{\det \bar{M}_{ij} \bar{M}^{ij} \{ \tilde{Q}, M_{ij} \} = -\frac{T}{2} \int d^3 \sigma \sqrt{\det \bar{M}_{ij} \bar{M}^{ij} \{ \tilde{Q}, \{ \tilde{Q}, \frac{i}{2}(\psi + \bar{\psi}) g_{mn} \Pi_i^m \Pi_j^n \} \} = 0}. \quad (12)$$

In the following, we will see how this method works.

First of all, on $M \times \mathbb{R}^1$ we need to covariantize the transformations (6). The only transformation which needs to change is that of $\psi^\alpha$. So we write

$$\delta \psi^\alpha = -\delta X^\rho \Gamma^\alpha_{\rho\sigma} \psi^\sigma.$$

With this change, however, $\delta$ does not square to zero anymore when acting on $\psi^\alpha$. To maintain this property of $\delta$, we add another term proportional to the Riemann tensor as follows

$$\delta \psi^\alpha = -\epsilon \bar{X}^\rho \Gamma^\alpha_{\rho\sigma} \psi^\sigma - i\epsilon \bar{\psi} \chi^\beta \psi^\gamma R^\alpha_{\beta\gamma},$$

or

$$\Delta \psi^\alpha = -i\epsilon \bar{\psi} \chi^\beta \psi^\gamma R^\alpha_{\beta\gamma}.$$

Noting that on Kähler manifolds $R^\alpha_{\beta\gamma} = 0$ and $R^\alpha_{\beta\gamma} = R^\alpha_{\gamma\beta}$, it is easy to check that with this change $\delta_{\epsilon_1} \delta_{\epsilon_2} \psi^\alpha = 0$. 

4
Taking $M_{ij}$ as in (11) we need to work out the variation of $\Pi^a$. Since the holomorphic 3-form $C^{\alpha\beta\gamma}$ is covariantly constant, we obtain

$$\Delta \Pi^a_i = -\bar{\chi} \Delta (D_i \psi^a) + \partial_i \bar{\chi} \Delta \psi^a - \psi \Delta (D_i \bar{\chi}) + \frac{1}{4} C^{\alpha}_{\beta\gamma} \left( \Delta \bar{\psi}^\beta D_i \bar{\chi}^\gamma + \bar{\psi}^\beta \Delta (D_i \bar{\chi}^\gamma) - \bar{\chi}^\beta \Delta (D_i \bar{\psi}^\gamma) \right),$$

$$\Delta \Pi^0_i = -\frac{i}{2} g_{\alpha\beta} (\Delta \bar{\psi}^\beta D_i \psi^a + \bar{\psi}^\beta \Delta (D_i \psi^a) + \bar{\chi}^\alpha \Delta (D_i \bar{\chi}^\alpha)) + \text{h.c.},$$

where

$$\Delta (D_i \psi^a) = -(\epsilon \bar{\chi}^\beta \psi^a X_i^\beta + \bar{\psi}^\beta \chi^a X_i^\beta) R^{\alpha}_{\beta\rho\gamma} - i\epsilon D_i (\bar{\psi}^a \chi^\beta \psi^\gamma R^{\alpha}_{\beta\rho\gamma})$$

$$\Delta (D_i \bar{\chi}^\alpha) = -\epsilon X_i^\alpha \chi^\beta \bar{\chi}^\gamma R^{\alpha}_{\beta\rho\gamma}.$$

So finally we find the following supersymmetric action for membranes on $M \times \mathbb{R}^1$

$$S_{DBI} = -T \int d^3 \sigma \sqrt{|M_{ij}|},$$

with

$$M_{ij} = g_{mn} \Pi^m_i \Pi^n_j$$

$$-\frac{i}{2} (\psi + \bar{\psi}) g_{\alpha\beta} \Pi^0_{ij} \left[ (i \bar{\chi}^\beta \chi^a X_i^\beta + i \bar{\chi}^\beta \psi^a X_i^\beta)
+ \partial_j \bar{\psi}^\beta \chi^a \bar{\psi}^a - i \psi^a \bar{\chi}^\alpha X_i^\alpha \right]
+ \frac{1}{4} C^{\alpha}_{\beta\gamma} \left( (\psi^a \bar{X}^\beta \psi^\gamma D_i) \chi^\gamma + i \bar{\chi}^\beta \chi^a X_i^\alpha \right)
- i \chi^a \bar{\psi}^\gamma X_i^\beta - i \bar{\chi}^\gamma \chi^a \bar{\psi}^\gamma X_i^\beta \right] R^{\beta}_{\alpha\rho\gamma} + \chi D_i (\psi^a \bar{X}^\beta \psi^\gamma R^{\beta}_{\alpha\rho\gamma}) \right)$$

$$-\frac{i}{4} (\psi + \bar{\psi}) g_{\alpha\beta} \Pi^0_{ij} \left[ -i \psi^a \bar{X}^\beta \psi^\gamma D_i) \psi^a R^{\beta}_{\alpha\rho\gamma}
- i \bar{\psi}^\beta (\bar{X}^\gamma \psi^\gamma X_i^\beta + \bar{\chi}^\beta \psi^\gamma X_i^\beta) R^{\alpha}_{\beta\rho\gamma} - i \bar{\psi}^\beta D_i (\bar{\psi}^a \chi^b \psi^\gamma R^{\alpha}_{\beta\rho\gamma}) \right] + \text{h.c.}.$$  

As mentioned earlier, since $\delta_{\epsilon_2} \delta_{\epsilon_1}$ acting on any field$^7$ gives zero it follows that $\bar{Q}^2 = 0$, which ensures the invariance of the action under $\bar{Q}$.

Apart from the DBI action which was constructed above, supermembranes action on flat spacetime has another part, the Wess-Zumino term. One way to obtain the WZ term is to vary the DBI action with respect to the $\kappa$-symmetry transformations. One then looks for a supersymmetric (up to a total derivative) 3-form on the worldvolume such that its variation under the $\kappa$-transformations cancels the variation of DBI action. The whole action $S = S_{DBI} + S_{WZ}$ is then invariant under both the supersymmetry and $\kappa$-symmetry transformations. We hope to return to these issues in future works.

$^7$ $\delta^2$ acting on $\phi$ does not give zero, but this does not cause any harm as this field appears in $\Pi^0$ as $d\phi$ and $\delta^2$ acting on $\Pi^0$ still gives zero.
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