Parameter-Dependent S-Procedure And Yakubovich Lemma

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Abstract

The paper considers a linear matrix inequality (LMI) that depends on a parameter varying in a compact topological space. It turns out that if a strict LMI continuously depends on a parameter and is feasible for any value of that parameter, then it has a solution which continuously depends on the parameter. The result holds true for LMIs that arise in S-procedure and Yakubovich lemma. It is shown that the LMI which is polynomially dependent on a vector of parameters can be reduced to a parameter-independent LMI of a higher dimension. The result is based on the recent generalization of Yakubovich lemma proposed by Iwasaki and Hara and another generalization formulated in this paper. The problem of positivity verification for a non-SOS polynomial of two variables is considered as an example. To illustrate control applications, a method of parameter-dependent Lyapunov function construction is proposed for nonlinear systems with parametric uncertainty.

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1. Introduction

The S-procedure losslessness theorem and the Kalman-Yakubovich-Popov (KYP) lemma are important mathematical tools of modern control theory. Both statements deal with inequalities of certain quadratic forms. We consider a situation when these quadratic forms depend continuously on a parameter that varies in a compact topological space.

First, we study solutions of a parameter-dependent linear matrix inequality (LMI). It turns out that if a strict LMI continuously depends on a parameter and is feasible, then it has a solution that continuously depends on the parameter. From this general statement it follows that if S-procedure for strict inequalities is lossless for any value of the parameter, then the Lagrange multipliers can be chosen as continuous functions of the parameter.

We also consider parameter-dependent generalizations of one statement of KYP lemma, namely the equivalence of strict frequency-domain inequality and the strict linear matrix inequality. (This result was first proven by V.A. Yakubovich [1] and was called Yakubovich lemma by R. Kalman [2] and S. Lefschetz [3].)

It is proven that if the matrices in the frequency-domain inequality continuously depend on the parameter, then there exists a solution of the LMI, which continuously depends on the parameter. The result also holds true for the generalized KYP lemma proposed by T. Iwasaki and S. Hara [4] as well as for a new version of KYP lemma with matrix frequency-domain inequality. The different result concerning a parameter-dependent version of KYP lemma was obtained by A. L. Likhtarnikov [5]. In [5] solutions of Lur’e equation are considered that are analytic functions of a parameter.

Using parameter-dependent Yakubovich lemma we show that if an LMI polynomially depends on the parameters, then it can be transformed into an LMI of a larger dimension that does not depend on these parameters. As an illustration of this result, we consider minimization of a polynomial of several variables in a bounded domain. The proposed method can be used to verify the positivity of polynomials that cannot be represented as a sum of squares.

The results are also applied to the construction of a parameter-dependent Lyapunov function for stability analysis of nonlinear systems with parametric uncertainty.

2. Parameter-dependent LMI

Consider the parameter-dependent LMI in \( h \in \mathbb{R}^l \)

\[
L(p, h) > 0
\]  

where \( p \in \mathcal{D} \) is a parameter, \( L : \mathcal{D} \times \mathbb{R}^l \to \mathbb{H}^n \) is affine with respect to \( h \). Hereafter \( \mathbb{H}^n \) denotes the space of Hermitian matrices of dimension \( n \).

\textbf{Theorem 1} Suppose that \( \mathcal{D} \) is a compact topological space, the map \( L \) is continuous and is affine with respect to \( h \), and that LMI [1] is feasible for any \( p \in \mathcal{D} \); then there exists a continuous
function $h(.) : \mathcal{D} \rightarrow \mathbb{R}^l$ that satisfies inequality

$$L(p, h(p)) > 0 \ \forall p \in \mathcal{D}.$$  \hfill (2)

The set of all continuous $h(.)$ satisfying (2) is open in $C(\mathcal{D}, \mathbb{R}^l)$.

Hereafter $C(X, Y)$ denotes the normed space of continuous functions from the topological space $X$ to the normed vector space $Y$.

**Corollary 1** If $\mathcal{F}$ is a dense subset of $C(\mathcal{D}, \mathbb{R}^l)$, then there exists $h(.) \in \mathcal{F}$ satisfying (2).

**Corollary 2** If $\mathcal{D} \subset \mathbb{R}^k$, then there exists a polynomial $h(p)$ satisfying (2).

Suppose that $h(p)$ is a polynomial of a vector variable $p \in \mathbb{R}^k$. Denote the vector of coefficients of $h(p)$ by $\hat{h} \in \mathbb{R}^k$. Let $\hat{L} : \mathcal{D} \times \mathbb{R}^k \rightarrow \mathbb{H}_{\mathbb{M}_n}$ be a map that is defined by substitution of the polynomial $h(p)$ into $L$. From corollary 2 it follows that

$$\hat{L}(p, \hat{h}) > 0 \ \forall p \in \mathcal{D}.$$  \hfill (3)

This way the construction of the parameter-dependent solution $h(p)$ of (1) is reduced to a search of a constant vector $\hat{h}$ that satisfies the system of inequalities (3) indexed by $p$. The most interesting case is the one when this system includes infinite number of LMIs. In section 7 we show that under certain assumptions on $\mathcal{D}$ this system can be reduced to a single finite-dimensional LMI.

### 3. Parameter-dependent S-procedure

Consider two maps $F_0 : \mathcal{D} \times \mathcal{X} \rightarrow \mathbb{R}$ and $F : \mathcal{D} \times \mathcal{X} \rightarrow \mathbb{R}^l$, where $\mathcal{D}$ is a compact space, $\mathcal{X}$ is either $\mathbb{C}^n$ or $\mathbb{R}^n$. Let $K$ be closed convex proper cone in $\mathbb{R}^l$. The dual cone is denoted by $K^+ = \{ z \in \mathbb{R}^l | \langle z, y \rangle \geq 0 \ \forall y \in K \}$. Consider two statements:

I. $\forall p \in \mathcal{D} \quad F(p, x) \in K \Rightarrow F_0(p, x) > 0.$

II. $\exists h : \mathcal{D} \rightarrow K^+$ such that

$$F_0(p, x) > \langle h(p), F(p, x) \rangle \ \forall p \in \mathcal{D}, x \in \mathcal{X}.$$  \hfill (4)

It is clear that II implies I. In the case when parameter $p$ is absent, the operation of replacement of the condition I by the condition II is known as S-procedure. In the presence of parameter we call this method a parameter-dependent S-procedure. The S-procedure is said to be lossless if I is equivalent to II.

**Theorem 2** Suppose that $F_0, F$ are continuous and are quadratic forms of $x$. If parameter-dependent S-procedure is lossless then there exists a continuous function $h(.)$ that satisfies (4). The set of all continuous $h(.)$ satisfying (4) is open in $C(\mathcal{D}, \mathbb{R}^l)$. 

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4. Generalized Yakubovich Lemma

Let us begin from the formulation of the classical result.

Theorem 3 (Yakubovich 1962, 1964) Let $A \in \mathbb{C}^{n_x \times n_x}$, $B \in \mathbb{C}^{n_x \times n_u}$, $G \in \mathbb{H}M_{n_x+n_u}$. The following statements are equivalent:

1. The inequality
   \[
   \left( \begin{array}{c}
   x \\
   u
   \end{array} \right)^* G \left( \begin{array}{c}
   x \\
   u
   \end{array} \right) > 0
   \]
   is fulfilled for all $x \in \mathbb{C}^{n_x \times 1}$, $u \in \mathbb{C}^{n_u \times 1}$, $|x|+|u| \neq 0$, $\omega \in [-\infty, +\infty]$ such that $i\omega x = Ax + Bu$.

2. There exists $H \in \mathbb{H}M_{n_x}$ that satisfies the LMI
   \[
   G > \begin{pmatrix}
   HA + A^* H & HB \\
   B^* H & 0
   \end{pmatrix}.
   \]

Hereafter $i$ is the imaginary unit.

This theorem was first proven by V.A. Yakubovich in [1] for the single-input system (i.e. for $n_u = 1$) and in [6] for the multi-input system. Our formulation follows [7]. Now the statement of the Theorem 3 is known as a part of Kalman-Yakubovich-Popov lemma [1][2][6][8]. In our opinion, it is more convenient for reference purposes, and is more correct from the historical point of view, to refer to Theorem 3 as Yakubovich Lemma. We call this statement a lemma to point out its connection with Kalman-Yakubovich-Popov Lemma. Additional information about history of Kalman-Yakubovich-Popov Lemma can be found in [9].

The theorem below is proven in [4]. It is a generalization of Yakubovich Lemma. The paper [4] is based on generalizations of Yakubovich Lemma that were proposed in [10][11].

Let $\Theta = \Theta^* = \begin{pmatrix}
\theta_{11} & \theta_{12} \\
\theta_{21} & \theta_{22}
\end{pmatrix}$, det $\Theta < 0$. Define the curve $\Gamma_\Theta = \{\lambda \in \mathbb{C} \mid \lambda > 0; \Theta^* \lambda \Theta = 0\}$, and the domain $\Omega_\Theta = \{\lambda \in \mathbb{C} \mid \lambda > 0; \Theta^* \lambda \Theta = 0\}$. Hereafter $\mathbb{C}$ is the closed complex plane $\mathbb{C} = \mathbb{R} \cup \{\infty\}$. Choosing different matrices $\Theta$ it is possible to define any circle or straight line on $\mathbb{C}$. To illustrate, let us present some examples: $\Theta = \begin{pmatrix}
0 & 0 \\
1 & 0
\end{pmatrix}$, $\Gamma_\Theta = \mathbb{R} \cup \{\infty\}$, $\Omega_\Theta = \{\Re \lambda \geq 0\} \cup \{\infty\}$; $\Theta = \begin{pmatrix}
0 & -i \\
i & 0
\end{pmatrix}$, $\Gamma_\Theta = \mathbb{Z} \cup \{\infty\}$, $\Omega_\Theta = \{|\lambda| \geq 0\} \cup \{\infty\}$; $\Theta = \begin{pmatrix}
-1 & 0 \\
0 & r^2
\end{pmatrix}$, $\Gamma_\Theta = \{|\lambda| = r\}$, $\Omega_\Theta = \{|\lambda| \leq r\}$.

Let $M, N \in \mathbb{C}^{n_x \times n_x}$, $S \in \mathbb{H}M_{n_x}$, $H \in \mathbb{H}M_n$. Define the generalized Lyapunov operator $\Lambda_{M,N,\Theta}(S) = (M, N)(\Theta \otimes S)(M, N)^*$. The adjoint operator takes the form $\Lambda_{M,N,\Theta}^*(H) = (M^*, N^*)(\Theta^* \otimes H)(M^*, N^*)^*$. Hereafter $M_1 \otimes M_2$ is a Kronecker product of matrices $M_1$ and $M_2$.

Theorem 4 (Iwasaki and Hara, 2005) Let $M, N \in \mathbb{C}^{n_x \times n_x}$, $G \in \mathbb{H}M_{n_x}$. Suppose the intersection $\Gamma_{\Theta_1} \cap \Omega_{\Theta_2}$ includes more than one point and $n_z > n$; then the following statements are equivalent:

1. The inequality
   \[
   z^* G z > 0
   \]
   is fulfilled for all $z \in \mathbb{C}^{n_x \times 1}$, $z \neq 0$, $\lambda \in \Gamma_{\Theta_1} \cap \Omega_{\Theta_2}$ such that $(\lambda N - M)z = 0$.
5. Yakubovich Lemma for matrix frequency-domain inequality

Let $M, N, \Theta_i, i = 1, 2$ be defined as in previous section. Consider matrices $G_{ij} \in \mathbb{C}^{n_z \times n_z}$, $G_{ij} = G_{ji}^*$, $i, j = 1, \ldots, m$.

**Theorem 5** Suppose the intersection $\Gamma_{\Theta_1} \cap \Omega_{\Theta_2}$ includes more than one point, $n_z = n + 1$, and $\text{rank}(\lambda N - M) = n$ for all $\lambda \in \Gamma_{\Theta_1} \cap \Omega_{\Theta_2}$; then the following statements are equivalent:

1. The matrix inequality
   \[
   \left( \begin{array}{ccc}
   z^* G_{11} z & \cdots & z^* G_{1m} z \\
   \vdots & \ddots & \vdots \\
   z^* G_{m1} z & \cdots & z^* G_{mm} z \\
   \end{array} \right) > 0
   \]

   is fulfilled for all $z \in \mathbb{C}^{n_z \times 1}$, $z \neq 0$, $\lambda \in \Gamma_1 \cap \Omega_2$ such that $(\lambda N - M)z = 0$.

2. There exist $H_1, H_2 \in \mathbb{H}\mathbb{M}_{mn}$, $H_2 > 0$ such that
   \[
   \left( \begin{array}{ccc}
   G_{11} & \cdots & G_{1m} \\
   \vdots & \ddots & \vdots \\
   G_{m1} & \cdots & G_{mm} \\
   \end{array} \right) > N_{m, \Theta_1}(H_1) + N_{m, \Theta_2}(H_2).
   \]

6. Parameter-dependent Yakubovich Lemma

Let $\mathcal{D}$ be a topological space. Consider maps $G_{ij}(\cdot) \in \mathcal{C}(\mathcal{D}, \mathbb{C}^{n_z \times n_z})$, $(G_{ij}(p) = G_{ji}^*(p)) \forall p \in \mathcal{D}$, $i, j = 1, \ldots, m$, $M(\cdot), N(\cdot) \in \mathcal{C}(\mathcal{D}, \mathbb{C}^{n_z \times n_z})$, $\Theta_i(\cdot) \in \mathcal{C}(\mathcal{D}, \mathbb{H}\mathbb{M}_2)$, $(\det \Theta_i(p) < 0 \forall p \in \mathcal{D}, i = 1, 2)$. Denote $G(p) = \left( \begin{array}{ccc}
   G_{11}(p) & \cdots & G_{1m}(p) \\
   \vdots & \ddots & \vdots \\
   G_{m1}(p) & \cdots & G_{mm}(p) \\
   \end{array} \right)$.

**Theorem 6** Suppose $\mathcal{D}$ is a compact space, the intersection $\Gamma_{\Theta_1(p)} \cap \Omega_{\Theta_2(p)}$ includes more than one point for all $p \in \mathcal{D}$, and either $m = 1$, $n_z > n$ or $m = 1$, $n_z = n + 1$, $\text{rank}(\lambda N(p) - M(p)) = n$ for all $\lambda \in \Gamma_{\Theta_1(p)} \cap \Omega_{\Theta_2(p)}$ and all $p \in \mathcal{D}$; then the following statements are equivalent:

1. The matrix inequality
   \[
   \left( \begin{array}{ccc}
   z^* G_{11}(p) z & \cdots & z^* G_{1m}(p) z \\
   \vdots & \ddots & \vdots \\
   z^* G_{m1}(p) z & \cdots & z^* G_{mm}(p) z \\
   \end{array} \right) > 0
   \]

   is fulfilled for all $p \in \mathcal{D}$, $z \in \mathbb{C}^{n_z \times 1}$, $z \neq 0$, $\lambda \in \Gamma_{\Theta_1(p)} \cap \Omega_{\Theta_2(p)}$ such that $(\lambda N(p) - M(p))z = 0$.

2. There exist $H_1(\cdot), H_2(\cdot) \in \mathcal{C}(\mathcal{D}, \mathbb{H}\mathbb{M}_{mn})$ such that for all $p \in \mathcal{D}$
   \[
   H_2(p) > 0, G(p) > N_{m, \Theta_1(p)}(H_1(p)) + N_{m, \Theta_2(p)}(H_2(p)).
   \]

The set of continuous pairs $(H_1(\cdot), H_2(\cdot))$ satisfying (12) is open in $\mathcal{C}(\mathcal{D}, \mathbb{H}\mathbb{M}_{mn} \times \mathbb{H}\mathbb{M}_{mn})$. 

2. There exist $H_1, H_2 \in \mathbb{H}\mathbb{M}_n$, $H_2 > 0$ such that
   \[
   G > N'_{m, \Theta_1}(H_1) + N'_{m, \Theta_2}(H_2).
   \]
7. Parameter-independent solutions of parameter-dependent LMI

Consider the parameter-dependent LMI (1). In contrast to section 2 now we are looking for a constant vector $h$ that satisfies (1) for all $p \in \mathcal{D}$. In section 2 it was shown that when $\mathcal{D} \subset \mathbb{R}^k$ the search of parameter-dependent solution of an LMI can be reduced to the search of constant one for a set of parameter-dependent LMIs with growing dimension of solution. Now we would like to show that in some cases the search of the parameter-independent solution of a parameter-dependent LMI can be reduced to the solution of a set of parameter-independent LMIs with growing dimension.

Suppose that $L$ is polynomial of $p$ and the set $\mathcal{D}$ is given by

$$\mathcal{D} = \{ p \in \mathbb{R}^{1 \times k} | a_1 \leq p_1 \leq b_1, a_i(p_{i-1}) \leq p_i \leq b_i(p_{i-1}), i = 2, \ldots, k \},$$

where $p = (p_1, \ldots, p_k), p^i = (p_1, \ldots, p_i) \in \mathbb{R}^{1 \times i}, a_i < b_i, a_i, b_i, i = 2, \ldots, k$, are polynomials and $a_i(p_{i-1}) < b_i(p_{i-1})$ for all $p \in \mathcal{D}$.

Let us introduce matrix polynomials $H_i(p^k, h), i, j = 1, \ldots, m$, which defines the vector $(p^d, \ldots, p_k, 1) \in \mathbb{R}^{1 \times (d+1)}$; then $L(p, h)$ can be represented as

$$L(p, h) = \begin{pmatrix} \zeta G_{11}(p^{k-1}, h) \zeta^* & \cdots & \zeta G_{1m}(p^{k-1}, h) \zeta^* \\ \vdots & \ddots & \vdots \\ \zeta G_{m1}(p^{k-1}, h) \zeta^* & \cdots & \zeta G_{mm}(p^{k-1}, h) \zeta^* \end{pmatrix},$$

where the matrices $G_{ij}(p^{k-1}, h), i, j = 1, \ldots, m$, are polynomials of $p^{k-1}$ and are affine with respect to $h$. For all $i, j = 1, \ldots, m$, $G_{ij}(p^{k-1}, h) = G_{ji}(p^{k-1}, h)^*$.

Define matrices $M = (I_d, 0), N = (0, I_d) \in \mathbb{R}^{d \times (d+1)}$,

$$\Theta_1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \Theta_2(p^{k-1}) = \begin{pmatrix} -1 & \frac{1}{2}(a_k(p^{k-1}) + b_k(p^{k-1})) \\ \frac{1}{2}(a_k(p^{k-1}) + b_k(p^{k-1})) & -a_k(p^{k-1})b_k(p^{k-1}) \end{pmatrix}.$$ 

Then

$$(p_k N - M)z = 0 \Rightarrow \exists c \in \mathbb{C} : z = c(p^d, \ldots, p_k, 1)^T, \Gamma_{\Theta_1} \cap \Omega_{\Theta_2(p^{k-1})} = [a_k(p^{k-1}), b_k(p^{k-1})].$$

Thus (11) is fulfilled for all $p \in \mathcal{D}$ iff

$$\begin{pmatrix} z^* G_{11}(p^{k-1}, h) z & \cdots & z^* G_{1m}(p^{k-1}, h) z \\ \vdots & \ddots & \vdots \\ z^* G_{m1}(p^{k-1}, h) z & \cdots & z^* G_{mm}(p^{k-1}, h) z \end{pmatrix} > 0$$

for all $p^{k-1} \in \mathcal{D}^{k-1}, z \in \mathbb{C}^{(d+1) \times 1}, z \neq 0, p_k \in \Gamma_{\Theta_1} \cap \Omega_{\Theta_2(p^{k-1})}$, satisfying $(p_k N - M)z = 0$. Here $\mathcal{D}^{k-1} = \{ q \in \mathbb{R}^{1 \times (k-1)} | a_1 \leq q_1 \leq b_1, a_i(q_{i-1}) \leq q_i \leq b_i(q_{i-1}), i = 2, \ldots, k - 1 \}$.

Let us introduce matrix polynomials $H_1(p^{k-1}), H_2(p^{k-1}) \in \mathbb{C}(\mathcal{D}^{k-1}, \mathbb{H}_m^{dm})$. Consider inequalities

$$G(p^{k-1}, h) > A'_m \otimes M A_m \otimes N, \Theta_1 (H_1(p^{k-1})) + A'_m \otimes M A_m \otimes N, \Theta_2(q)(H_2(p^{k-1})), H_2(p^{k-1}) > 0. \quad (15)$$
where

\[ G(p^{k-1}, h) = \begin{pmatrix} G_{11}(p^{k-1}, h) & \ldots & G_{1m}(p^{k-1}, h) \\ \vdots & \ddots & \vdots \\ G_{m1}(p^{k-1}, h) & \ldots & G_{mm}(p^{k-1}, h) \end{pmatrix} \in \mathbb{H} \mathbb{M}_{m(d+1)} \].

From Theorem 6 it follows that if the vector \( h \) and the pair of polynomials \( H_1(p^{k-1}), H_2(p^{k-1}) \) satisfies (15) for all \( p^{k-1} \in D^{k-1} \), then \( h \) satisfies (1) for all \( p \in D \).

Let \( h^{(1)}_{aux} \) be the vector of coefficients of polynomials \( H_1(p^{k-1}), H_2(p^{k-1}) \); then inequalities (15) can be rewritten as follows:

\[ R_{1,i}(p^{k-1}, h, h^{(1)}_{aux}) > 0, \quad i = 1, 2, \quad (16) \]

where \( R_{1,1}(p^{k-1}, h, h^{(1)}_{aux}) = G(p^{k-1}, h) - N_{m_1 \otimes M_1, m_2 \otimes N_2, \Theta_1}(H_1(p^{k-1})) - N_{m_1 \otimes M_1, m_2 \otimes N_2, \Theta_2}(H_2(p^{k-1})) \), \( R_{1,2}(p^{k-1}, h, h^{(1)}_{aux}) = H_2(p^{k-1}) \). The maps \( R_{1,i}, i = 1, 2, \) are polynomials of \( p^{k-1} \) and are affine with respect to the joint vector \( (h, h^{(1)}_{aux}) \).

Applying the same procedure to each inequality in (16), we obtain the system of four inequalities

\[ R_{2,i}(p^{k-2}, h, h^{(2)}_{aux}) > 0, \quad i = 1, 2, 3, 4, \quad (17) \]

where \( h^{(2)}_{aux} \) is the vector of parameters of all polynomials, that were introduced on two steps of procedure, the maps \( R_{2,i}, i = 1, 2, 3, 4, \) are polynomials of \( p^{k-2} \) and are affine with respect to the joint vector \( (h, h^{(2)}_{aux}) \).

Repeating the procedure \( k \)-times, we obtain the system of parameter-independent LMIs that has the form

\[ R_{k,i}(h, h_{aux}) > 0, \quad i = 1, \ldots, 2^k, \quad (18) \]

where \( h_{aux} \) is the vector of parameters of all introduced polynomials, the maps \( R_{k,i}, i = 1, \ldots, 2^k, \) are affine with respect to the joint vector \( (h, h_{aux}) \).

For convenience we rewrite (18) as a single LMI

\[ R(h, h_{aux}) > 0, \quad (19) \]

where \( R \) is block-diagonal matrix with blocks \( R_{k,i}, i = 1, \ldots, 2^k, h_{aux} \in \mathbb{R}^{I_{R}} \).

Let \( \mathcal{R} \) be the set of all affine maps \( R \) that can be constructed from the map \( L \), using described recursive procedure.

**Theorem 7** Suppose that \( L \) is polynomial of \( p \), \( D \) is given by (13); then

\[ \{ h \in \mathbb{R}^l \mid L(p, h) > 0 \forall p \in D \} = \bigcup_{R \in \mathcal{R}} \{ h \in \mathbb{R}^l \mid \exists h_{aux} \in \mathbb{R}^{I_{R}} : R(h, h_{aux}) > 0 \}. \]

8. Example. Minimization of a polynomial in a bounded domain

Let \( g(p) \) be the polynomial of \( p \in \mathbb{R}^{1 \times k} \). Consider the problem of evaluation of the polynomial \( g \) minimum in a domain \( D \) given by (13).
Define \( L(p, h) = g(p) - h \), \( h \in \mathbb{R} \). It is clear that any \( h \) satisfying (1) for all \( p \in \mathcal{D} \) is a lower bound for \( g \) in \( \mathcal{D} \). Moreover from Theorem 7 it follows that

\[
\sup_{R \in \mathcal{R}} \sup \{ h \mid \exists h_{\text{aux}} \in \mathbb{R}^{l_R} : R(h, h_{\text{aux}}) > 0 \} = \min \{ g(p) \mid p \in \mathcal{D} \}. \tag{20}
\]

Taking into account (20), we can say that minimization of the polynomial \( g \) in a domain \( \mathcal{D} \) is reduced to a standard LMI optimization problem.

To illustrate the method we apply this approach to a polynomial of two variables \( g(x, y) \). Let \( \mathcal{D} = \{(x, y) \mid a \leq x \leq b, c(x) \leq y \leq d(x)\} \), where \( a < b \), \( c(x) \), \( d(x) \) are polynomials, \( c(x) < d(x) \) for all \( x \in [a, b] \). Consider the parameter-dependent LMI in \( h \in \mathbb{R} \)

\[
g(x, y) - h > 0, \forall x, y \in \mathcal{D} \tag{21}
\]

Let \( \deg_y g \) be the degree of the polynomial \( g \) with respect to \( y \). Denote \( n_y = \left\lfloor \frac{\deg_y (g) + 1}{2} \right\rfloor \), \( Y = (y^{n_y}, \ldots, y, 1) \). Then \( g(x, y) - h = Y G(x, h) Y^\ast \), where

\[
G(x, h) = \begin{pmatrix}
G_{1,1}(x) & \ldots & G_{1,n_y + 1}(x) \\
\vdots & \ddots & \vdots \\
G_{n_y + 1,1}(x) & \ldots & G_{n_y + 1,n_y + 1}(x) - h
\end{pmatrix} \in \mathbb{H}^{n_y + 1}, G_{ij}(x), i, j = 1, \ldots, n_y + 1, \text{ are polynomials of } x.
\]

Define matrices \( M_y = (I_{n_y}, 0), N_y = (0, I_{n_y}) \in \mathbb{R}^{n_y \times (n_y + 1)} \). Then

\[
(y N_y - M_y) Z = 0 \tag{22}
\]

implies \( Z = c Y^\ast \), \( c \in \mathbb{C} \).

Define matrices

\[
\Theta_1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \Theta_2(x) = \begin{pmatrix} -1 & \frac{1}{2}(c(x) + d(x)) \\ \frac{1}{2}(c(x) + d(x)) & -c(x) d(x) \end{pmatrix}.
\]

Then \( \Gamma_{\Theta_1} \cap \Omega_{\Theta_2} = [c(x), d(x)] \). Thus (21) is equivalent to \( Z^\ast G(y) Z > 0 \ \forall x \in [a, b], y \in \Gamma_{\Theta_1} \cap \Omega_{\Theta_2} \) and \( Z \neq 0 \) satisfying (22).

By Theorem 6 it follows that the latter condition is fulfilled iff there exist polynomial matrices \( H_{G_1}(x), H_{G_2}(x) \in \mathcal{C} (\mathbb{R}, \mathbb{H}^{n_y}) \) satisfying the LMIs

\[
H_{G_2}(x) > 0, \ G(x, h) > \Lambda'_{M_y, N_y, \Theta_1}(H_{G_1}(x)) + \Lambda'_{M_y, N_y, \Theta_2}(H_{G_2}(x)) \tag{23}
\]

Let \( n \) be the maximal degree of \( x \) in (23). Put \( n_x = \left\lfloor \frac{n + 1}{2} \right\rfloor \), \( X = (x^{n_x}, \ldots, x, 1) \). Let us denote the vectors of the coefficients of the polynomials \( H_{G_1}(x), H_{G_2}(x) \) by \( \hat{H}_{G_1}, \hat{H}_{G_2} \) respectively and define matrices \( E(\hat{H}_{G_2}, x) = H_{G_2}(x), \ F(\hat{H}_{G_1}, \hat{H}_{G_2}, x, h) = G(x, h) - \Lambda'_{M_y, N_y, \Theta_1}(H_{G_1}(x)) + \Lambda'_{M_y, N_y, \Theta_2}(H_{G_2}(x)) \). Then LMIs (23) can be rewritten as follows

\[
E(\hat{H}_{G_2}, x) = \begin{pmatrix}
XE_{1,1}(\hat{H}_{G_2}) X^\ast & \ldots & X E_{1,n_y}(\hat{H}_{G_2}) X^\ast \\
\vdots & \ddots & \vdots \\
XE_{n_y,1}(\hat{H}_{G_2}) X^\ast & \ldots & X E_{n_y,n_y}(\hat{H}_{G_2}) X^\ast
\end{pmatrix} > 0, \tag{24}
\]
\[
F(\hat{H}_{G1}, \hat{H}_{G2}, x, h) = \begin{pmatrix}
XF_{1,1}(\hat{H}_{G1}, \hat{H}_{G2})X^* & \ldots & XF_{1,n_y+1}(\hat{H}_{G1}, \hat{H}_{G2})X^* \\
\vdots & \ddots & \vdots \\
XF_{n_y,1}(\hat{H}_{G1}, \hat{H}_{G2})X^* & \ldots & XF_{n_y,n_y+1}(\hat{H}_{G1}, \hat{H}_{G2}, h)X^*
\end{pmatrix} > 0. 
\]

(25)

Define matrices \( M_x = (I_{n_x}, 0) \), \( N_x = (0, I_{n_x}) \) \( \in \mathbb{R}^{n_x \times (n_x + 1)} \). Then

\[
(xN_x - M_x)Z = 0
\]

implies \( Z = cX^T, c \in \mathbb{C} \). Let \( \Theta_3 = \begin{pmatrix}
-1 & \frac{1}{2}(a + b) \\
\frac{1}{2}(a + b) & -ab
\end{pmatrix}. \)

According to Theorem 5, \( \Theta_3 \) and \( \Theta_1 \) are fulfilled iff there exist \( H_{E1}, H_{E2} \in \mathbb{H}_{n_x,n_y} \), \( H_{F1}, H_{F2} \in \mathbb{H}_{n_x(n_y + 1)} \) that satisfy the LMIs

\[
\begin{pmatrix}
E_{1,1}(\hat{H}_{G2}) & \ldots & E_{1,n_y}(\hat{H}_{G2}) \\
\vdots & \ddots & \vdots \\
E_{n_y,1}(\hat{H}_{G2}) & \ldots & E_{n_y,n_y}(\hat{H}_{G2})
\end{pmatrix} > \begin{pmatrix}
N'_{I_{n_y} \otimes M_x, I_{n_y} \otimes N_x, \Theta_1}(H_{E1}) + N'_{I_{n_y} \otimes M_x, I_{n_y} \otimes N_x, \Theta_3}(H_{E2}) \\
N'_{I_{n_y+1} \otimes M_x, I_{n_y+1} \otimes N_x, \Theta_1}(H_{F1}) + N'_{I_{n_y+1} \otimes M_x, I_{n_y+1} \otimes N_x, \Theta_3}(H_{F2})
\end{pmatrix}
\]

(27)

In this way \( \Theta_3 \) is fulfilled iff there exists \( n_x \) such that LMIs \( \Theta_3 \) are feasible. Evidently, system \( \Theta_3 \) can be written as a single LMI \( \Theta_1 \). Maximizing \( h \) over solutions of \( \Theta_3 \) and increasing \( n_x \) we can estimate minimum of \( g \) in \( D \) with any desired accuracy.

The proposed approach can be applied to verification of positivity of polynomials. As an illustration let us consider a numerical example.

It is known that \( g(x, y) = x^4 y^2 + x^2 y^4 - 3x^2 y^2 + 1 + \varepsilon > 0 \) for all \( \varepsilon > 0 \) and all \( x, y \). Besides for any \( \varepsilon \) the polynomial \( g(x, y) \) cannot be represented as sum of squares (SOS) of polynomials \( \Theta_1 \). It means that the well known SOS representation technique cannot be used directly to verify the positivity of \( g \).

It is easy to see that \( g(x, y) > 0 \), when \( |x| > 2 \) or \( |y| > 2 \). So, to verify the positivity of \( g \) it is sufficient to find positive lower bound of \( g(x, y) \) in \( D = \{(x, y) \mid |x| \leq 2, |y| \leq 2 \} \). Calculations using Matlab\textsuperscript{®} LMI toolbox showed that for \( n_x = 2 \) the system of inequalities \( \Theta_3 \) has a solution with \( h > 0 \) if \( \varepsilon = 10^{-11} \). This proves that \( x^4 y^2 + x^2 y^4 - 3x^2 y^2 + 1 + 10^{-11} > 0 \) for all \( x, y \).

9. Construction of parameter-dependent Lyapunov function

Consider the parameter-dependent nonlinear system

\[
\dot{x} = A(p)x + B(p)u, \quad x(0) = x_0, 
\]

(28)

\[
u = \varphi(p, t, x),
\]

(29)

where \( t \geq 0, x, x_0 \in \mathbb{R}^{n_x \times 1}, u \in \mathbb{R}^{m_x \times 1}, p \) is a parameter, \( p \in \mathbb{D} \subset \mathbb{R}^{1 \times k}, A(\cdot) : \mathbb{D} \to \mathbb{R}^{n_x \times n}, B(\cdot) : \mathbb{D} \to \mathbb{R}^{m_x \times n}, \varphi : \mathbb{D} \times [0, +\infty) \times \mathbb{R}^{n_x \times 1} \to \mathbb{R}^{m_x \times 1}. \)
The nonlinearity \( \varphi \) satisfies the quadratic constraint

\[
(x^*, u^*)G(p)(x^*, u^*)^* \geq 0
\]

which is fulfilled for all \( x \in \mathbb{R}^{n \times 1}, t \geq 0, p \in D, u = \varphi(p, t, x) \). Here \( G(p) \in \mathbb{H}M_{m+n} \).

Suppose that \( A, B, G \) are polynomials of \( p \), and \( D \) is defined by (13). Consider the parameter-dependent Lyapunov function candidate \( V(p, x) = x^*H(p)x \), where \( H(.) \in \mathcal{C}(D, \mathbb{H}M_n) \). We are looking for \( H(.) \) that satisfies

\[
H(p) > 0 \forall p \in D,
\]

and

\[
\frac{d}{dt}V(p, x(t)) < 0 \forall t \geq 0,
\]

for all solutions of (28) satisfying (30). If fulfilled these conditions guarantee the asymptotic stability of the closed-loop system (28), (29) for any \( \varphi \) such that (30) holds.

Define matrices \( M(p) = (A(p), B(p)), N = (I_n, 0) \in \mathbb{R}^{n \times(n+m)}, \Theta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \). Then

\[
\Lambda'_{M(p),N,\Theta}(H(p)) = \begin{pmatrix} H(p)A(p) + A^*(p)H(p) & H(p)B(p) \\ B^*(p)H(p) & 0 \end{pmatrix}
\]

and condition (32) takes the form:

\[
\forall p \in D, \forall |x| + |u| \neq 0 \text{(30)} \text{ implies } (x^*, u^*)\Lambda'_{M(p),N,\Theta}(H(p))(x^*, u^*) < 0 \text{ (33)}
\]

Using parameter-dependent S-procedure and Theorem 1 we can see that (33) is fulfilled iff there are a polynomial matrix \( H(p) \) and a polynomial \( \eta(p) \) that satisfy

\[
\Lambda'_{M(p),N,\Theta}(H(p)) + \eta(p)G(p) > 0, \quad \eta(p) > 0 \forall p \in D.
\]

Let \( h \in \mathbb{R}^l \) be the vector of coefficients of polynomials \( H(p), \eta(p) \). Then (34) can be written as a single parameter-dependent LMI (1). Using the procedure described in Section 7 we can define the set of affine maps \( \mathcal{R} \). From Theorem 7 it follows that \( H(p) \) and \( \eta(p) \) satisfying (31) and (34) exists iff there is an affine map \( R \in \mathcal{R} \) such that LMI (19) is feasible. Any solution \( h \) of obtained in this way LMI (19) defines the polynomial matrix \( H(p) \) that satisfies (31) and (32).

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