Noncommutative Quantum Mechanics: The Two-Dimensional Central Field

J. Gamboa\textsuperscript{1*}, M. Loewe\textsuperscript{2†}, F. Méndez\textsuperscript{1‡} and J. C. Rojas\textsuperscript{3§}

\textsuperscript{1}Departamento de Física, Universidad de Santiago de Chile, Casilla 307, Santiago 2, Chile
Facultad de Física, Pontificia Universidad Católica de Chile, Casilla 306, Santiago 22, Chile
\textsuperscript{2}Departament ECM, Facultat de Física, Universitat de Barcelona and Institut D’Altes Energies, Diagonal 647, E-08028, Barcelona, Spain

Abstract

Quantum mechanics in a noncommutative plane is considered. For a general two dimensional central field, we find that the theory can be perturbatively solved for large values of the noncommutative parameter ($\theta$) and explicit expressions for the eigenstates and eigenvalues are given. The Green function is explicitly obtained and we show that it can be expressed as an infinite series. For polynomial type potentials, we found a smooth limit for small values of $\theta$ and for non-polynomial ones this limit is necessarily abrupt. The Landau problem, as a limit case of a noncommutative system, is also considered.

E-mail: jgamboa@lauca.usach.cl
E-mail: mloewe@fis.puc.cl
E-mail: fnendez@lauca.usach.cl
E-mail: rojas@sonia.ecm.ub.es
I. INTRODUCTION

Recent results in string theory suggest that spacetime could be noncommutative [1]. If this claim is correct, then important new implications in our conception of space and time could take place [2]. For instance, spatial noncommutativity implies new Heisenberg-type relations, namely

$$\Delta x \Delta y \sim \theta,$$

where $\theta$ is a measure of the noncommutative effects and plays an analogous role of $\hbar$ in standard quantum mechanics [3]. The space-time noncommutativity, however, violates causality at the quantum field theory level although space-time noncommutativity could be consistent in string theory [4].

The goal of this paper is to discuss two dimensional quantum mechanics in a noncommutative space. More precisely, we will consider the central field case which, remarkably, can be solved almost completely and the solution shows quite explicitly the difficulties and virtues found in noncommutative quantum field theory. Thus, in this sense, our model could be considered as a framework where one could explicitly check some properties that appear in quantum field theory.

The paper is organized as follows: in section 2, we review the model proposed in [5] and we discuss their properties and physical interpretation. In section 3 we write a general expression for the Green function. Section 4 is devoted to the calculation of the partition function and the statistical mechanics of simple systems, sketching a perturbative procedure for more general cases. Finally, in section 5 the conclusion and an outlook is presented.

II. QUANTUM MECHANICS IN A NONCOMMUTATIVE SPACE

Noncommutative quantum mechanics is a theory defined on a manifold where the product of functions is the Moyal one. If $A(x)$ and $B(x)$ are two functions, then the Moyal product is defined as
\[ \mathbf{A} \star \mathbf{B}(x) = e^{i\theta_{ij}\delta^{(1)}\delta^{(2)}} \mathbf{A}(x_1)\mathbf{B}(x_2)|_{x_1=x_2=x}. \] (2)

This formula implies that the Schrödinger equation
\[ i\frac{\partial \Psi(x, t)}{\partial t} = \left( \frac{p^2}{2m} + V(x) \right) \Psi(x, t), \] (3)
in the noncommutative space is the same one but with the potential shifted as \( V(x - \frac{\hat{\mathbb{H}}}{\theta}) \), where \( \hat{\mathbb{H}} = \theta \mathbb{L} \), \( \theta_{ij} = \theta \delta_{ij} \) and \( \delta_{ij} \) is the antisymmetric tensor in two dimensions. Although this result appeared in connection with string theory [6] there is also an older version known as Bopp’s shift [7]. This last fact implies that quantum mechanics in a noncommutative plane is highly nontrivial because, as the shifted potential involves in principle arbitrary powers of the momenta, we will have an arbitrary large number of derivatives in the Schrödinger equation.

Thus, the question is how to handle noncommutative systems in a simple way. In order to give an answer to this question, let us consider a central field in two dimensions \( V(x) = V(r^2) \). In the noncommutative space this potential is equivalent to the replacement
\[ V(x) \rightarrow V(\frac{\theta^2}{4}p_x^2 + x^2 + \frac{\theta^2}{4}p_y^2 + y^2 - \theta L_z) \]
\[ = V(\hat{\mathbb{H}}), \] (4)

where the aleph operator (\( \hat{\mathbb{H}} \)) is defined as
\[ \hat{\mathbb{H}} = \hat{H}_{HO} - \theta \hat{L}_z, \] (5)

where \( H_{HO} \) is the hamiltonian for a two dimensional harmonic oscillator with mass \( \frac{2}{\theta^2} \), frequency \( \omega = \theta \) and \( L_z \) is the z-component of the angular momentum defined as \( L_z = xp_y - yp_x \).

The eigenstates and the eigenvalues of the \( \hat{\mathbb{H}} \) operator can be calculated noticing that in two dimensions the harmonic oscillator is associated to the \( SU(2) \) group whose generators are given by
\[ L_x = \frac{1}{2}(a_x^\dagger a_x - a_y^\dagger a_y), \]
\[ L_y = \frac{1}{2}(a_x^\dagger a_y + a_y^\dagger a_x), \]
\[ L_z = \frac{1}{2i}(a_x^\dagger a_y - a_y^\dagger a_x), \]  

and, furthermore, \((\mathcal{H}, \hat{L}^2, J_z = \frac{1}{2}L_z)\) is a complete set of commuting observables. For the calculation of the eigenvalues is more convenient to use

\[ a_\pm = \frac{1}{\sqrt{2}}(a_y \pm ia_x), \]
\[ a_\pm^\dagger = \frac{1}{\sqrt{2}}(a_y^\dagger \pm ia_x^\dagger). \]  

(7)

In the basis \(|n_+, n_-\rangle\), the operators \((\mathcal{H}, \hat{L}^2, J_z)\) are diagonal and the quantum numbers take the values \(n_\pm = 0, 1, 2, 3, \ldots\). By using this, one get that the eigenvalues of \(\mathcal{H}\) are

\[ \lambda_{n_+, n_-} = \theta [2n_- + 1]. \]  

(8)

Note that the spectrum of \(\mathcal{H}\) does not depend on the label \(n_+\) and their eigenvalues are infinitely degenerated.

Remarkably, one can compute exactly the eigenvalues associated to \(V(\mathcal{H})\). Indeed, let \(a_n\) be the eigenvalues associated to the operator \(\hat{A}\) and \(\psi_n\) the corresponding eigenfunction. Then for a small shift of the argument of an arbitrary function \(f(\hat{A})\) one find that

\[ f(\hat{A} + \epsilon)\psi_n = f(a_n + \epsilon)\psi_n. \]  

(9)

By using (9), the following identity is found

\[ V(\mathcal{H})|n_+, n_-\rangle = V(\theta [2n_- + 1])|n_+, n_-\rangle. \]  

(10)

Once equation (10) is obtained, we can compute the eigenvalues and eigenfunctions for a general system described by the hamiltonian

\[ \hat{H} = \frac{1}{2M}p^2 + V(x^2), \]  

(11)

which can be rewritten as follows:
\[ \hat{H} = \frac{p^2}{2M} + V(\hat{\mathbf{r}}), \]
\[ = \frac{2}{M\theta^2} \left( \frac{\theta^2}{4} p^2 + r^2 + V(\hat{\mathbf{r}}) \right) - \frac{2}{M\theta^2} r^2 \]
\[ \equiv H_0 - \frac{2}{M\theta^2} r^2. \] (12)

Then, after using (10), the eigenvalues \( \Lambda_{n_+,n_-} \) of \( H_0 \) are
\[ \Lambda_{n_+,n_-} = \frac{2}{M\theta} [n_+ + n_- + 1] + V[\theta(2n_- + 1)], \] (13)

while the eigenvalues of \( \hat{H} \) are
\[ E_{n_+n_-} = \langle n_+n_-|\hat{H}_0|n_+n_-\rangle - \frac{2}{M\theta^2} \langle n_+n_-|r^2|n_+n_-\rangle \]
\[ = \frac{2}{M\theta} [n_+ + n_- + 1] + V[\theta(2n_- + 1)] - \frac{2}{M\theta^2} \langle n_+n_- | r^2 | n_+n_- \rangle. \] (14)

In this expression, the last term of the R.H.S. can be computed by using perturbation theory for large values of \( \theta \) and, as a consequence, the eigenfunctions that appear in the matrix element \( \langle n_+n_- | r^2 | n_+n_- \rangle \), are those of the two dimensional harmonic oscillator, i.e.
\[ |n_+n_-\rangle = \frac{a_+a_-^{n_+n_-}}{\sqrt{(n_+)!/(n_-)!}} |0,0\rangle. \] (15)

Thus, in this approximation the eigenvalues that correspond to the diagonal part of the hamiltonian are
\[ \tilde{E}_{n_+n_-} = \frac{1}{M\theta}(n_+ + n_- - 1) + V(\theta[2n_- + 1]). \] (16)

Although the higher order corrections to the eigenfunctions and eigenvalues are straightforward calculable the perturbative series cannot be expressed in a closed way.

One should note that quantum mechanics in a noncommutative space has new consequences. In particular, the usual Schrödinger equation in the commutative case
\[ (\triangle + k^2)\psi(x) = V(x)\psi(x), \] (17)

has an integral version
\[ \psi(x) = \varphi(x) + \int dx' G[x, x'] V(x') \psi(x'), \]  

(18)

which cannot be realized in the noncommutative space, since the shift \( x \rightarrow x + \frac{1}{2} \theta \nabla \) in the potential involves unknown powers of the derivatives. These higher powers could spoil the unitarity (conservation of probability) of quantum mechanics in a noncommutative space. However, this is circumvented by the properties satisfied by the Moyal product. These imply that the equation

\[ \nabla \cdot J = -\frac{\partial \rho}{\partial t}, \]  

(19)

with \( \rho = \psi^* \psi \) and \( \psi^* \nabla \psi - \psi \nabla \psi^* \) remains valid and, therefore, the physical meaning of the wave function in a noncommutative space does not change.

III. GREEN FUNCTIONS IN NONCOMMUTATIVE SPACES

In this section we will discuss the calculation of the Green function \( G[x, x'] \) for the previous model in a \( 1/\theta \) power expansion. For \( \theta >> 1 \), we can compute \( G[x, x'] \) to any order in \( 1/\theta \) since we need the matrix element \( < n_+ n_- | r^2 | n_+ n_- > \).

For \( \theta >> 1 \), as we discussed above, the term \( r^2 \) is a non-diagonal perturbation and the 0-order wave function are those of the two dimensional harmonic oscillator, i.e. the basis \( | n_+ n_- > \). Thus, assuming this condition, the Green function is given by

\[ G[x, x'] = < x | e^{-i \hat{H}(t-t')} | x' > \]

\[ = \sum_{n_+ n_- n_+ n_-} < x | n_+ n_- > < n_+ n_- | e^{-i \hat{H}(t-t')} | n_+ n_- > < n_+ n_- | x' >, \]

\[ = \sum_{n_+ n_- n_+ n_-} \psi^*_{n_+ n_-} (x) \psi_{n_+ n_-} (x') < n_+ n_- | e^{-i \hat{H}(t-t')} | n_+ n_- >, \]  

(20)

where the wave function \( \psi_{n_+ n_-} (x) \) are the usual Hermite functions.

As \( r^2 \) does not commute with \( \hat{R} \) and \( L_z \), the matrix element \( < n_+ n_- | e^{-i \hat{H}(t-t')} | n_+ n_- > \) must be calculated by expanding the evolution operator \( e^{-i \hat{H}(t-t')} \), i.e.

\[ < x | e^{-i \hat{H}(t-t')} | x' > = \sum_{k=0}^{\infty} \frac{(-i)^k}{k!} < x | [H(t-t')]^k | x' >. \]  

(21)
If we use recursively the completeness relation for $|n_+ n_->$, we obtain

$$<x|e^{-i\hat{H}(t-t')}|x'>=\sum_{k=0}^{\infty}\sum_{n_+^{(1)}n_-^{(1)}\ldots n_+^{(k)}n_-^{(k)}}\frac{(-i)^k}{k!}\times$$

$$\times<n_+n_-|\hat{H}|n_+^{(1)}n_-^{(1)}><n_+^{(1)}n_-^{(1)}|\hat{H}|n_+^{(2)}n_-^{(2)}>\ldots<n_+^{(n-1)}n_-^{(k-1)}|\hat{H}|n_+^{(k)}n_-^{(k)}>(t-t')^k.$$\text{(22)}

Using the hamiltonian (12) and the properties of the operators $\mathcal{R}$ and $L_z$, then each bracket in (22) becomes

$$<n_+n_-|\hat{H}|n_+^{'}n_-^{'}> = \tilde{E}_{n_+n_-}\delta_{n_+^{'}n_-^{'}}\delta_{n_+n_-} - \frac{1}{M\theta}[\sqrt{(n_+^{'}+1)(n_-^{'}+1)}\delta_{n_+n_-^{'}+1}\delta_{n_-^{'}n_+^{'}+1}$$

$$+\sqrt{n_+n_-}\delta_{n_-n_-^{'}-1}\delta_{n_+n_-^{'}-1}],$$\text{(23)}

where $\tilde{E}_{n_+n_-}$ is defined in (16).

In order to simplify the calculations, it is better to rewrite (23) in terms of the variables $j, m$ defined as $2j = n_+ + n_-$ and $m = n_+ - n_-$, respectively, which take the values $j = 0, \frac{1}{2}, 1, \frac{3}{2}\ldots$ and $m = -j, -j + 1, \ldots, j - 1, j$.

The result is\footnote{This notation must be taken with care since here we are not working with the usual basis in polar coordinates.}

$$<n_+n_-|\hat{H}|n_+^{'}n_-^{'}> = \tilde{E}_{jn_+n_-}\delta_{jn_+^{'}j^{'}n_-^{'}m} - \frac{1}{M\theta}[\sqrt{j^2 - m^2}\delta_{j-1j^{'}'} + \sqrt{(j+1)^2 - m^2}\delta_{j+1j^{'}'}],$$\text{(24)}

and, as a consequence, the Green function becomes

$$G[x,x'] = \sum_{j,m}\left[\psi_{jm}^*(x)\psi_{jm}(x')S_{jm} + \left(-\frac{1}{M\theta}\right)\left(\psi_{j-1,m}^*(x')\psi_{jm}(x)S_{j-1,m} + \psi_{j+1,m}^*(x')\psi_{jm}(x)S_{j+1,m}\right) + \left(-\frac{1}{M\theta}\right)^2\left(\psi_{j-2,m}^*(x')\psi_{jm}(x)S_{j-2,m} + \psi_{j+2,m}^*(x')\psi_{jm}(x)S_{j+2,m}\right) + \ldots\right],$$\text{(25)}

where the $S_{jm}$ are given by (with $\tau = -it$)
\[S_{jm} = e^{-\tau \tilde{E}_{jm}} + \left(\frac{\tau}{M\theta}\right)^2 (j^2 - m^2) + \left(\frac{\tau}{M\theta}\right)^3 [....], \]

\[S_{j-1,m} = \sqrt{j^2 - m^2} \left[\tau + \tau^2 \left(\tilde{E}_{j-1} + \tilde{E}_j\right) + \tau^3 \left(\tilde{E}_{j-1}^2 + \tilde{E}_{j-1} \tilde{E}_j + \tilde{E}_j^2\right) + ....\right], \]

\[S_{j+1,m} = \sqrt{(j+1)^2 - m^2} \left[\tau + \tau^2 \left(\tilde{E}_{j+1} + \tilde{E}_j\right) + \tau^3 \left(\tilde{E}_{j+1}^2 + \tilde{E}_{j+1} \tilde{E}_j + \tilde{E}_j^2\right) + ....\right], \]

\[S_{j-2,m} = \sqrt{j^2 - m^2} \sqrt{(j-1)^2 - m^2} \left[\tau + \tau^2 \left(\tilde{E}_{j-2} + \tilde{E}_{j-1} + \tilde{E}_j\right) + ....\right], \]

\[S_{j+2,m} = \sqrt{(j+2)^2 - m^2} \sqrt{(j+1)^2 - m^2} \left[\tau + \tau^3 \left(\tilde{E}_{j+2} + \tilde{E}_{j+1} + \tilde{E}_j\right) + ....\right]. \]

This is the full Green function for the motion of a particle in a general central field if \(\theta \gg 1\). Thus, in a perturbative sense, our model is exactly solvable although we cannot find a closed form for the perturbative series.

Notice that the potential contains dynamical information induced by the presence of the momentum due to the shift \(x \rightarrow x - \tilde{p}/2\), allowing then to consider the kinetic term \(p^2/2M\) as a perturbation.

In order to do that, one must evaluate the matrix element \(<n_+n_-|p^2|n_+n_->\) as was done previously for \(r^2\). The result is

\[<n_+n_-|p^2|n_+n_-> = \frac{2}{\theta}(n_- + n_+ + 1)\delta_{n_+n'_+}\delta_{n_-n'_-} - \frac{2}{\theta}[\sqrt{(n'_+ + 1)(n'_- + 1)}\delta_{n_+n'_+1}\delta_{n_-n'_-1} + \sqrt{n'_+n'_-}\delta_{n_+n'_+1}\delta_{n_-n'_-1}], \]

which gives the same expression for the Green function as equation (23).

It is interesting to notice, at this point, that the mass \(M\) and the noncommutative parameter \(\theta\) appear always in the combination \(1/(M\theta)\), and as a consequence, our calculation is insensitive to the replacement \(M \leftrightarrow \theta\); this sort of duality explains why the limits \(M \rightarrow \infty\) and \(\theta \rightarrow \infty\) give the same results at perturbative level.

**IV. THE LANDAU PROBLEM AS A NONCOMMUTATIVE SYSTEM**

In this section we will explain how the Landau problem can be understood as a noncommutative one [9]. Although this idea has been discussed recently [10], there is an older
version on this problem \[1\]. Following the results given in section II, if we choose the potential \(V(\aleph)\) in \((12)\) as

\[ V(\aleph) = \Omega \aleph. \]  

(32)

where \(\Omega\) is an appropriate constant, then the hamiltonian becomes

\[ H = \left( \frac{1}{2M} + \frac{\Omega \theta^2}{4} \right) p^2 + \Omega x^2 - \Omega \theta L_z. \]  

(33)

This hamiltonian is diagonal in the basis \(|n_+ n_-\rangle\) and \(|\psi\rangle\)

\[ H|n_+ n_-\rangle = \left[ \sqrt{\frac{2\Omega}{M}} (n_+ + n_- + 1) - \Omega \theta (n_+ - n_-) \right]|n_+ n_-\rangle, \]  

(34)

where \(M\) is an effective mass defined as \(\frac{1}{M} = \frac{1}{M} + \frac{\Omega \theta^2}{2}\).

The propagator can be computed as in the previous section, but now for any value of \(\theta\). Indeed, the Green function becomes the matrix element \(<x|e^{-i[\frac{\theta^2}{2M} + \Omega \aleph]T}|x'\rangle\) and the explicit calculation gives

\[ G[x, x'; T] = <x|e^{-i[\frac{\theta^2}{2M} + \Omega \aleph]T}|x'\rangle, \]  

\[ = \sum_{n_+ n_-} \psi_{n_+ n_-}^*(x) \psi_{n_+ n_-}(x') e^{-i\sqrt{\frac{2\Omega}{M}} (n_+ + n_- + 1) - \Omega \theta (n_+ - n_-)} e^{eta E_{n_+ n_-}}, \]  

(35)

where \(T = t - t'\) and \(\psi_{n_+ n_-}(x)\) are the eigenfunctions of the harmonic oscillator in cartesian coordinates.

From \((35)\) one can compute the associated partition function

\[ Z = \text{Tr} G(x, x') \]  

(36)

once the euclidean rotation \(iT \rightarrow \beta\) has been performed. Using this, one can see that the usual formula in the commutative space

\[ Z = \sum_{n_+, n_- = 0} e^{-\beta E_{n_+ n_-}}, \]  

(37)

2This basis is the same one where \(\aleph\) is diagonal. However, in the coordinates representation, they look due to the definition of the operators \(a_\pm, a_\pm^\dagger\). For details see the Appendix.
hold in the noncommutative case. It is interesting to note that for a general \( V(N) \), equation (25) yields a partition function like (37) only in the large \( \theta \) limit; otherwise, new non exponential contributions should be added.

Finally, the partition function is

\[
Z = \sum_{n_+, n_- = 0} \exp \left( -\beta \left( \sqrt{\frac{2\Omega}{M}} - \Omega \theta \right) n_+ - \left( \sqrt{\frac{2\Omega}{M}} + \Omega \theta \right) n_- + \sqrt{\frac{2\Omega}{M}} \right),
\]

(38)

\[
= \frac{1}{4 \sinh \left( \beta \left( \sqrt{\frac{2\Omega}{M}} - \Omega \theta \right) \right) \sinh \left( \beta \left( \sqrt{\frac{2\Omega}{M}} + \Omega \theta \right) \right)}
\]

(39)

which is the usual partition function of the two dimensional harmonic oscillator.

Now we can map (33) into the Landau problem. If the magnetic field \( \mathbf{H} = H_0 \hat{e}_3 \), then the hamiltonian in the symmetric gauge for a particle with mass \( \mu \)

\[
\hat{H}_{\text{Landau}} = \frac{1}{2\mu} \mathbf{p}^2 + \frac{e^2 H_0^2}{8\mu^2} \mathbf{x}^2 - \frac{eH_0}{2\mu} \mathbf{L}_z,
\]

(40)

just coincides with (33) if we identify

\[
\frac{1}{2\mu} = \frac{1}{2M} + \frac{\Omega \theta^2}{4}, \quad \frac{e^2 H_0^2}{8\mu} = \Omega, \quad \Omega \theta = \frac{eH_0}{2\mu}.
\]

(41)

(42)

These equations are consistent if and only if \( M = \infty \), \textit{i.e.} the equivalence between noncommutative quantum mechanics and the Landau problem is exact for the lowest Landau level.

Thus

\[
\Omega = \frac{e^2 H_0}{8\mu}, \quad \theta = \frac{4}{eH_0}.
\]

(43)

A simple dimensional analysis shows that \( \Omega \) has \([\text{energy/length}^2]\) dimensions and, therefore is enough to change \( \theta \) by \( \tilde{\theta} = \hbar \theta \) in order to have a noncommutative parameter with the appropriated dimensions \([\text{length}]^2\).
Considering magnetic fields in the region of quantum Hall effect ($\sim 12 \, \text{T}$) one can find a bound for the noncommutative parameter \( i.e. \)

\[
\tilde{\theta} = 0.22 \times 10^{-11} \text{cm}^2.
\]

(44)

For this value of \( \tilde{\theta} \) one cannot distinguish between noncommutative quantum mechanics and the Landau problem.

The partition function in the limit \( M \to \infty \) can be computed by noticing that the argument of the first \( \sinh \) in the denominator vanishes and therefore the partition function diverges. This remind us that in (38) we had an infinite sum over \( n_+ \) which must be regularized.

The partition function for the Landau problem becomes

\[
Z = \frac{\mathcal{N}}{2 \sinh\left(\frac{\beta e H_0}{2\mu}\right)},
\]

(45)

where \( \mathcal{N} \) is an infinite constant that is irrelevant for the thermodynamics analysis, although can computed by using \( \zeta \)-function regularization giving \( \mathcal{N} = 1/2 \).

The magnetization and the magnetic susceptibility yield to the usual expressions \[12\], \( i.e. \)

\[
M = -\frac{4e}{\mu} \coth\left(\frac{\beta e H_0}{2\mu}\right)
\]

\[
\chi = -\frac{16e^2}{\mu^2} \text{cosech}\left[\frac{4H_0 e \beta}{\mu}\right]
\]

(46)

and, therefore, the system is diamagnetic.

V. CONCLUSIONS

In this paper, noncommutative quantum mechanics for a two dimensional central field was considered. The main point is that we have established a mapping between any noncommutative quantum mechanical system in two dimensions with a commutative version. In the commutative space the potential is shifted by \( V(\mathbf{x} - \frac{1}{2}\tilde{\mathbf{p}}) \), where \( V(\mathbf{x}) \) is the potential
defined on the noncommutative space. This last fact is a consequence of the Moyal product or, in other words, the Moyal product provides an explicit realization of the Seiberg-Witten map [1].

In the central field case, one can compute the spectrum of the hamiltonian from (16). Indeed, if $V(x)$ has the form $x^m$, then if $m > 0$

$$\tilde{E}_{n_+n_-} = \frac{2}{M\theta}(n_+ + n_- + 1) - \frac{2}{M\theta^2} < n_+n_- \mid r^2 \mid n_+n_- > + [\theta(2n_- + 1)]^{m/2}. \quad (47)$$

If $\theta >> 1$ the second term is computed by using perturbation theory and, the diagonal part of (47) becomes

$$\tilde{E}_{n_+n_-} = \frac{1}{M\theta}(n_+ + n_- + 1) + [\theta(2n_- + 1)]^{m/2}, \quad (48)$$

and, therefore, the dominant part of the spectrum becomes $[\theta(2n_- + 1)]^{m/2}$, i.e. is infinitely degenerated in $n_+$.

If $\theta << 1$, only the first two terms in (47) are the dominant contributions.

If $m < 0$ but $|m| > 2$, then the main contributions to $\tilde{E}_{n_+n_-}$ comes from the terms proportional to $1/M$. When $|m| < 2$, the dominant contributions are due to $[\theta(2n_- + 1)]^{m/2}$.

If the potential is non-polynomial, then the limit $\theta << 1$ could not exist as, e.g. the Coulomb potential.

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APPENDIX A:

In this section we set the definition of the operators $a_{\pm}, a_{\pm}^\dagger$, in order to construct the basis $| n_+n_- >$. 

The operator $\aleph$ can be diagonalized as follows; defining
\[ \hat{\alpha}_x = \frac{1}{\sqrt{\theta}}(\hat{x} + \frac{i}{2\theta} \hat{p}_x), \]  \tag{A1}
\[ \hat{\alpha}_x^\dagger = \frac{1}{\sqrt{\theta}}(\hat{x} - \frac{i}{2\theta} \hat{p}_x). \]  \tag{A2}

This operators satisfy the usual commutation relations $[\hat{\alpha}_i, \hat{\alpha}_j^\dagger] = \delta_{ij}$ and $[\hat{\alpha}_i, \hat{\alpha}_j] = 0$.

Although the aleph operator can be written in terms of these operators, it assumes the diagonal form only in the basis $|n_+, n_−\rangle$ generated by
\[ \hat{\alpha}_+ = \frac{1}{\sqrt{2}}(\hat{a}_y + i\hat{a}_x), \]  \tag{A3}
\[ \hat{\alpha}_- = \frac{1}{\sqrt{2}}(\hat{a}_y - i\hat{a}_x), \]  \tag{A4}

due to the presence of $L_z$. The aleph can be written as
\[ \aleph = \theta(2\hat{\alpha}_+^\dagger \hat{\alpha}_- + 1). \]  \tag{A5}

For the hamiltonian $\hat{H}$ given in the equation (33) the previous definitions (A2) must be modified in order to have a diagonal form.

If we define
\[ \hat{\alpha}_x = \frac{(2\Omega \mu)^{1/4}}{\sqrt{2}}(\hat{x} + \frac{i}{\sqrt{2}(2\Omega \mu)^{1/4}} \hat{p}_x), \]  \tag{A6}
\[ \hat{\alpha}_x^\dagger = \frac{(2\Omega \mu)^{1/4}}{\sqrt{2}}(\hat{x} - \frac{i}{\sqrt{2}(2\Omega \mu)^{1/4}} \hat{p}_x), \]  \tag{A7}
and the corresponding $\alpha_\pm$ defined in (A4), the hamiltonian (33) turn out to be
\[ \hat{H} = \sqrt{\frac{2\Omega}{\mu}}[\hat{\alpha}_+^\dagger \hat{\alpha}_+ + \hat{\alpha}_-^\dagger \hat{\alpha}_- + 1] - \Omega \theta [\hat{\alpha}_+^\dagger \hat{\alpha}_- - \hat{\alpha}_-^\dagger \hat{\alpha}_+]. \]  \tag{A8}
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