Optimal Routing for Constant Function Market Makers

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Outline

Overview of constant function market makers

Routing problem

Problem and solution properties

Conclusion
A quick overview of DEXs

- Decentralized exchanges (DEXs) are venues for exchanging assets requiring no intermediary

- The main trading venue on blockchains (Ethereum, Solana, etc.)
A quick overview of DEXs

- *Decentralized exchanges* (DEXs) are venues for exchanging assets requiring no intermediary.

- The main trading venue on blockchains (Ethereum, Solana, etc.)

- Billions of dollars in volume per day!
Constant function market makers

- Usually DEXs are organized as *constant function market makers* (CFMMs)
- Anyone can propose a trade to a CFMM
- The CFMM accepts or rejects this trade based on a simple rule
- If a trade is accepted then the CFMM pays out the trade from its reserves
Constant function market makers (math version)

- CFMMs can trade baskets of $n$ tokens
- Define a trading function $\varphi : \mathbb{R}_+^n \to \mathbb{R}$ and reserves $R \in \mathbb{R}_+^n$
- Anybody can propose a trade, written $\Delta, \Lambda \in \mathbb{R}_+^n$
- CFMM accepts if
  \[ \varphi(R + \gamma \Delta - \Lambda) \geq \varphi(R) \]
  where $0 < \gamma \leq 1$ is the trading fee
- Pays out $\Lambda$ to user, gets $\Delta$ from user
- New reserves: $R + \Delta - \Lambda$
Examples

▶ Many example trading functions

▶ The most common (Uniswap v1/v2, etc.)

\[ \varphi(R_1, R_2) = \sqrt{R_1 R_2} \]

▶ More general (Balancer)

\[ \varphi(R) = \prod_{i=1}^{n} R_i^{w_i} \]

with \( w \geq 0 \) and \( \mathbf{1}^T w = 1 \)

▶ Many others...
Concavity and trading functions

▶ Will mostly use one property: 

Trading function $\phi$ is concave

This is true in all (!) practical examples

(There are other natural generalizations)
Concavity and trading functions

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- Trading function $\varphi$ is concave
- This is true in all(!) practical examples
- (There are other natural generalizations)
Routing problem
Why routing?

- Usually there is not just one CFMM, even on a single chain
- Often there are many (with overlapping tokens!)
- For example, user wants to trade $A \rightarrow B$
- What about $A \rightarrow C \rightarrow B$?
  - $A \rightarrow D \rightarrow B$? Splitting orders?
- Potentially very complicated...
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The routing problem (set up)

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- Let’s write it as an optimization problem!
- Say we have a network of $i = 1, \ldots, m$ CFMMs
- The network also has $n$ tokens, labeled $1, \ldots, n$
- CFMM $i$ trades subset of $n_i \leq n$ tokens
- Has trading function $\varphi_i : \mathbb{R}^{n_i}_+ \rightarrow \mathbb{R}$, fee $\gamma_i$, reserves $R_i \in \mathbb{R}^{n_i}_+$
The routing problem (set up, cont.)

- We write $\Delta_i, \Lambda_i \in \mathbb{R}^{n_i}_+$ for trade with CFMM $i$

- In terms of the global list of tokens, trader receives net amount from $i$:
  \[ A_i(\Lambda_i - \Delta_i) \]

- Here $A_i \in \mathbb{R}^{n \times n_i}$ maps local token indices to global ones

- Net amount received over CFMMs is the network trade vector:
  \[ \Psi = \sum_{i=1}^{m} A_i(\Lambda_i - \Delta_i) \]
The routing problem

This gives the optimal routing problem:

maximize \( U(\Psi) \)

subject to \( \Psi = \sum_{i=1}^{m} A_i(\Lambda_i - \Delta_i) \)

\( \varphi_i(R_i + \gamma \Delta_i - \Lambda_i) \geq \varphi(R_i), \quad i = 1, \ldots, m \)

\( \Delta_i \geq 0, \quad \Lambda_i \geq 0, \quad i = 1, \ldots, m \)

with variables \( \Psi \in \mathbb{R}^n, \Delta_i, \Lambda_i \in \mathbb{R}^{n_i} \) for \( i = 1, \ldots, m \)

\( U : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{-\infty\} \) is the user’s utility over resulting tokens
The routing problem

This gives the *optimal routing problem*:

\[
\begin{align*}
\text{maximize} & \quad U(\Psi) \\
\text{subject to} & \quad \Psi = \sum_{i=1}^{m} A_i(\Lambda_i - \Delta_i) \\
& \quad \varphi_i(R_i + \gamma\Delta_i - \Lambda_i) \geq \varphi(R_i), \quad i = 1, \ldots, m \\
& \quad \Delta_i \geq 0, \quad \Lambda_i \geq 0, \quad i = 1, \ldots, m
\end{align*}
\]

If the utility $U$ is concave, then *problem is convex*
Example utility functions

- Private valuations $c \in \mathbb{R}_+^n$
  \[ U(\Psi) = c^T \Psi \]

- Liquidating basket $\Delta_{\text{in}} \in \mathbb{R}_+^n$ to token $j$
  \[ U(\Psi) = \Psi_j - I(\Psi + \Delta_{\text{in}} \geq 0) \]

- Optimal arbitrage with valuation $c$ (remember this!)
  \[ U(\Psi) = c^T \Psi - I(\Psi \geq 0) \]

- Markowitz mean $\mu \in \mathbb{R}^n$, covariance $\Sigma \in \mathcal{S}_+^n$, risk $\lambda \geq 0$
  \[ U(\Psi) = \mu^T \Psi - \frac{\lambda}{2} \Psi^T \Sigma \Psi \]
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Convexity and tractability

- Problem is convex \( \approx \) easy to solve in practice
- Even for moderately large \( n \) (tokens) and \( m \) (CFMMs)
- Problem lends itself nicely to decomposition methods
- See our open source Julia package: CFMMRouter.jl (Additional work done with Theo Diamandis)
Example solution

- Solutions are rarely intuitive

- Objective: liquidate $t$ of token 1 for token 3

Problem and solution properties
General optimality conditions

➢ When are the zero trades $\Delta_i = \Lambda_i = 0$ optimal?

➢ When there exist multipliers $\lambda_i \geq 0$ with

$$\gamma \nabla \varphi_i(R_i) \leq \lambda_i A_i^T g \leq \nabla \varphi_i(R_i),$$

where $g \in -\partial(-U)(0)$, for $i = 1, \ldots, m$

➢ We can interpret $P_i = \nabla \varphi_i(R_i)$ as the marginal prices of CFMM $i$
No-arbitrage conditions

- When is there no arbitrage? \( U(\Psi) = c^T \Psi + I(\Psi \geq 0) \)

- Apply condition from before: there exists \( \lambda_i \geq 0 \) such that

\[
\gamma P_i \leq \lambda_i A_i^T g \leq P_i,
\]

where \( g \in \mathbb{R}_+^n \) and \( P_i = \nabla \varphi_i(R_i) \), for \( i = 1, \ldots, m \)

- This means there exists a *market clearing price*, \( g \), consistent with all CFMMs’ prices
More general conditions

- Can derive general first-order solution conditions
- (Not just for $\Delta_i = \Lambda_i = 0$)
- See paper for more details and examples!
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Conclusion and future directions

- Problem of optimally routing through DEXs is (usually) easy!
- Is being used to fairly great effect in practice
- Convexity implies good computational properties
- And also some interesting mathematical ones!
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