Group representation approach to $1 \rightarrow N$ universal quantum cloning machines

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In this work, we revisit the problem of finding an admissible region of fidelities obtained after an application of an arbitrary $1 \rightarrow N$ universal quantum cloner which has been recently solved in [A. Kay et al., Quant. Inf. Comput 13, 880 (2013)] from the side of cloning machines. Using group-theory formalism, we show that the allowed region for fidelities can be alternatively expressed in terms of overlaps of pure states with recently found irreducible representations of the commutant $U \otimes U \otimes \ldots \otimes U \otimes U^*$, which gives the characterization of the allowed region where states being cloned are figure of merit. Additionally, it is sufficient to take pure states with real coefficients only, which makes calculations simpler. To obtain the allowed region, we make a convex hull of possible ranges of fidelities related to a given irrep. Subsequently, two cases: $1 \rightarrow 2$ and $1 \rightarrow 3$ cloners, are studied for different dimensions of states as illustrative examples.

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I. INTRODUCTION

A basic feature of entanglement is that contrary to classical correlations, it is monogamous. For example, if there is maximal entanglement between two parties, then no other party can be entangled with those parties. More generally, if $A$ is entangled with $B$ and $C$, then the entanglement must be considerably weaker. This phenomenon gives rise to the fact that quantum information cannot be copied, in contrast with information from the ‘classical world’. In other words, one is not able to copy perfectly an arbitrary quantum state. In terms of monogamy, if one wants to prepare some number of copies of the initially unknown quantum state, fidelities of cloning cannot be all equal to 1, there is a trade-off. This basic feature is known as ‘no-cloning theorem’ and was recognized by Wootters and Zurek [1], and Dieks [2].

On the other hand, copying is possible, but the quality of the copy can be very bad sometimes. That’s why, the goal of finding the ultimate bounds for the quality of copying is an important task. A big effort has been made to solve it, starting from the work of Hillery and Bužek [3]. In general, the subject was studied intensively, both for symmetric (all fidelities are equal) Universal Quantum Cloning Machines (UQCM) [4–8], and asymmetric (unequal fidelities) UQCM [8–16]. See also [17, 18] for reviews. Nevertheless, for a long time there was a ‘gap’ in studies of quantum cloning - there was no general results on an admissible region of fidelities for universal asymmetric $1 \rightarrow N$ quantum cloning machines. The problem has been solved just recently in a series of papers [14, 15] from the point of cloning machines. In [19] the problem, for qubits, has been revisited using a group representation approach, namely Schur - Weyl duality, where the authors characterized the problem from the side of a cloned state and obtain that regions for fidelities can be obtained from plain and basic calculations of overlaps of pure quantum states with irreps of a symmetric group $S(n)$.

In this Letter, we shall consider a $1 \rightarrow N$ quantum cloning machine for qudits. Our task it to obtain an admissible region of fidelities after an application of that UQCM. In [19], it has been shown that it is possible to solve the problem for qubits using Schur-Weyl duality. Unfortunately it works only for that dimension of states and there is no way to extend it to higher dimensions by the usage of that dualism. Motivated by this, we turn our attention to, recently developed systematic method - decomposition of partially transposed permutation operators into its irreducible components [20, 21], which allows to omit severe restrictions for the dimensions of states that has appeared previously. However, some modifications are necessary first, so the method suits our problem of cloning machines. We want to stress that to our best knowledge, it is also the first systematic application of that algebra in physics, and particularly - quantum information (see, [22, 23], for the examples of some limited applications).

This work is organized as follows. In Section I, we formulate our main problem: which values of fidelities are allowed after applying a $1 \rightarrow N$ quantum cloning machine for qudits. First, we reformulate the cloning problem in term of entanglement sharing and recall that a cloning fidelity can be connected with a singlet fraction value. Then, we point out that the strategy used in [19] to solve a $1 \rightarrow N$ UQCM for qubits is insufficient when one deals with higher dimensions of states $d$, ($d > 2$), since using Schur-Weyl duality, one is not able to find a maximally entangled state that is invariant under $U \otimes U$ transformations, the only thing that is known is the invariance under $U^* \otimes U$ ones. That’s why, the commutant structure of $U^* \otimes U \otimes \ldots \otimes U$ is needed instead of that known from...
Schur-Weyl duality: $U \otimes U \otimes \ldots \otimes U$. In Section II B, mathematical tools from [20] that are necessary to solve the problem are very briefly mentioned, namely, examples of irreducible representations that are needed in our case study problems: $1 \to 2$ and $1 \to 3 UQCM$. Then, we proceed in Section II C with showing how to connect method of calculations of the admissible region of fidelities from [19] with mathematical tools from the previous section. It allows us to present in Section II D the regions (focusing mainly on our examples $1 \to 2$ and $1 \to 3$ machines) that are allowed in the problem of $1 \to N$ cloning. Up to our best knowledge, it is the first graphical presentation of allowed regions for $d > 2$. At the end, we compare our results in Section II E with those obtained in [6], where results for symmetric cloning has been presented and from [14, 15], where the same problem as ours have been solved, but cloning machines were figures of merit. We obtain matching of results in both cases.

II. FORMULATION AND SOLUTION TO THE PROBLEM

A. Background of the problem

Suppose that one has a universal cloning machine that produces clones with cloning fidelities $f_{ik}$, where $k \in 2, 3, \ldots, n$ and the general, admissible region of fidelities is the figure of merit. The question that one can ask is the following:

Which values of cloning fidelities $(f_{12}, f_{13}, \ldots, f_{1n})$ are allowed for a (qudit) universal cloning machine?

But since quantum cloning can be recast in a picture where one wants to share entanglement between some number of parties (see, for example, [19, 24]). Therefore, we can equivalently state our problems in this formalism, where one evaluates singlet fractions $F_{1i}$ between the initial state and one of the copies. This allows to restate our question as:

Which values of $n$-tuples of singlet fractions $(F_{12}, F_{13}, \ldots, F_{1n})$ are allowed for an arbitrary state of a maximally mixed first subsystem?

Remark: Since these two quantities, cloning fidelities and singlet fractions, are connected [24], in the next section we will adapt the term "fidelities" for the latter.

Let us now consider in more details the relation between cloning fidelities $f$ and the fidelities (singlet fractions) $F$.

Suppose that we are given with the maximally entangled qudit state

$$|\psi^+\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |ii\rangle,$$

and we apply the $1 \to N$ cloning machine $\mathcal{CM}$ to the second subsystem of the $|\psi^+\rangle$, when the first is untouched. As a result we obtain $N + 1$-partite mixed state that possesses all information about the cloning map $\tilde{\Lambda}$. The state is of the form

$$\rho_{1\ldots n} = \left(1 \otimes \tilde{\Lambda}\right)\left(|\psi^+\rangle\langle\psi^+|\right),$$

where $n = N + 1$, so that the index $i = 1$ is related to an initial state, and $i = 2, \ldots, N + 1$ are related to clones. The fidelities of clones are strictly related to fidelities of reduced states $\rho_{1k}$ with maximally entangled state [24]:

$$f_i = \frac{F_id + 1}{d + 1}.$$  

Here $f_i = \langle \psi_{in} | \rho_{1i}^\flat | \psi_{in} \rangle$ is fidelity of $i$-th clone where $\langle \ldots \rangle$ is the uniform average over an input state $\psi_{in}$, and $F_i = \langle \psi_+ | \rho_{1i}^\flat | \psi_+ \rangle$.

An allowed region for quantum cloning, can be calculated then by evaluating singlet fractions $F_{1i}$ between the initial state and one of the copies, denoted by

$$F_{1i} = \langle \psi_{1i}^+ | \text{Tr}_{\tilde{\Pi}}(\rho_{1\ldots n}) | \psi_{1i}^+ \rangle \text{ or } F_{1i} = \langle \psi_{1i}^- | \text{Tr}_{\tilde{\Pi}}(\rho_{1\ldots n}) | \psi_{1i}^- \rangle,$$

where $\tilde{\Pi}$ is described by a completely positive, trace preserving map $\tilde{\Lambda}$.
where 1 < i ≤ n, Tr_{\mathcal{T}} means partial trace over all systems except 1, and |\psi_{11}^+\rangle and |\varphi_{1...n}^-\rangle are defined below.

Let us show here, why we have been able to use Schur-Weyl duality and commutant structure of $U^{\otimes n}$ for qubits cloning machines [19] and explain why it does not work for higher dimensions of states ($d > 2$). For qudits, in principle, the vector $|\psi_{11}^-\rangle = U \otimes 1 |\psi_{11}^+\rangle$, |\psi^-\rangle needs to be obtain after an application of $U$. For qubits, one can use Bell states $|\psi^+\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)$ and $|\psi^-\rangle = \frac{1}{\sqrt{2}} (|01\rangle - |10\rangle)$ and show that the vector $|\psi^-\rangle$ is obtained after the action of the Pauli matrix $-i\sigma_y$ on $|\psi^+\rangle$ Using that we can write

$$|\psi_{11}^-\rangle = U \otimes 1 |\psi_{11}^+\rangle,$$

where $U = -i\sigma_y$. The state $\varphi_{1234}$ from equation (4) is obtained after the following transformation:

$$\varphi_{1...n} = (1 \otimes \Lambda) |\psi_{11}^-\rangle (|\psi_{11}^-\rangle (U \otimes 1)^4.$$

The $n$-partites states $\varphi_{1...n}$, with the constraint $\varphi_1 = 1/2$, are in one-to-one correspondence with cloning machines.

However, now the problem is formulated in terms of singlet fractions with states $|\psi^-\rangle$ rather than $|\psi^+\rangle$. The former states are invariant under $U \otimes U$ transformation for any $U$. Therefore to obtain the region of fidelities with $|\psi^-\rangle$ states it is enough to consider states $\rho_{1...n}$ that are invariant under $U^{\otimes n}$ transformations. There exists well known formalism that allows to deal with states possessing such symmetry, called Schur-Weyl duality that combines representation theory for unitary group with that of group of permutations. We have successfully applied this formalism in [19]. However in dimensions $d > 2$ there is no maximally entangled state, that would be $U \otimes U$-invariant. Therefore, the Schur-Weyl formalism cannot be used.

Instead, it is known, that the state $|\psi^+\rangle$ is $U^* \otimes U$ invariant [25], hence we should consider $U^* \otimes U^{\otimes n-1}$ invariant states. The formalism, related to this kind of symmetry is not so well developed as the previous one, and there are quite basic differences between the two. In particular, while the representation of $U^{\otimes n}$ is dual to representation of another group - the symmetric group, it is not the case for $U^* \otimes U^{\otimes n-1}$ which is dual to representation of an algebra, that does not satisfy group axioms - an instance of so called Brauer algebra. While some general results concerning this type of algebras have been known in literature (see, for example, [22] [23] [26]), it has not been described in depth, in contrast to Schur-Weyl theory. In particular, the explicit form of matrix elements of representations of the algebra, have been provided only recently in [20] [21]. In the following we solve the cloning problem applying these new tools.

### B. Mathematical tools

As it was said before, to solve our problem, the knowledge of irreducible representations of a $U^* \otimes U \otimes \ldots \otimes U$ case is necessary. In a recent papers [20] [21] this problem has been addressed, so we can use the formalism presented there.\footnote{See also Appendix IV A for a short review on this topic.}

In the articles, the authors presented irreducible representations of partially transposed permutation operators $V^{\lambda\sigma}$, where $\sigma \in S(n)$ and $t_n$ denotes partial transposition over the last subsystem. In our approach, we need similar results for irreps when partial transposition is taken over the first subsystem, i.e. we need irreps of $V^{\lambda\sigma}(1)$, where $1 \leq k < n$ for $U^* \otimes U \otimes \ldots \otimes U$ instead of $U \otimes \ldots \otimes U \otimes U^*$. That’s why, first, some work needs to be done to adapt the results, so they suit our problem. One can see that to obtain correct results, we have to take irreps for permutations in the form $(in)$, where $1 \leq i \leq n - 1$, i.e. we have the following mapping

$$|12 \rangle \mapsto |1n \rangle, \, |13 \rangle \mapsto |2n \rangle, \ldots, |1n \rangle \mapsto |n-1 \rangle.$$

In the next sections, for the simplicity, we introduce the notation that $t_n \equiv \prime$. Now we are ready to present all irreps that are essential for our paper (case study examples). Of course our method works efficiently for an arbitrary number of particles $n$ and dimensions of Hilbert space $d$, but here we present them only for $n = 3, 4$, because for these cases we are able to represent our results graphically.
• Case when \( n = 3 \). In this case in algebra \( M \) we have only one irrep labeled by trivial partition \( \alpha = (1) \).

\[
V'_a (13) = \frac{1}{2} \begin{pmatrix} d + 1 & -\sqrt{d^2 - 1} \\ -\sqrt{d^2 - 1} & d - 1 \end{pmatrix}, \quad V'_a (23) = \frac{1}{2} \begin{pmatrix} d + 1 & \sqrt{d^2 - 1} \\ \sqrt{d^2 - 1} & d - 1 \end{pmatrix}
\]

(8)

• Case when \( n = 4 \). In this case in algebra \( M \) we have two irreps labeled by partitions \( \alpha_1 = (2) \) and \( \alpha_2 = (1,1) \). For partition \( \alpha_1 \) we deal with matrices 3x3 for any \( d \geq 1 \):

\[
V'_{\alpha_1} (14) = \frac{1}{3} D^{\alpha_1} \begin{pmatrix} \frac{1}{6} & -\frac{1}{2\sqrt{3}} & \frac{1}{3\sqrt{2}} \\ \frac{1}{2\sqrt{3}} & \frac{2}{3} & \frac{1}{3\sqrt{2}} \\ \frac{1}{3\sqrt{2}} & -\frac{2}{\sqrt{3}} & \frac{1}{3} \end{pmatrix} D^{\alpha_1}, \quad V'_{\alpha_1} (24) = \frac{1}{3} D^{\alpha_1} \begin{pmatrix} \frac{1}{6} & 2\sqrt{3} & \frac{1}{3\sqrt{2}} \\ 2\sqrt{3} & \frac{2}{3} & \frac{1}{3\sqrt{2}} \\ \frac{1}{3\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{3} \end{pmatrix} D^{\alpha_1},
\]

(9)

where

\[
D^{\alpha_1} = \begin{pmatrix} \sqrt{d - 1} & 0 & 0 \\ 0 & \sqrt{d - 1} & 0 \\ 0 & 0 & \sqrt{d + 2} \end{pmatrix}
\]

(10)

and \( \varepsilon = 1 \). For partition \( \alpha_2 \) situation is more complicated. Dimension of irrep \( \alpha_2 \) depends on dimension of local Hilbert space \( d \). Namely for any \( d \geq 3 \) we have

\[
V'_{\alpha_2} (14) = \frac{1}{3} D^{\alpha_2} \begin{pmatrix} \frac{1}{6} & -\frac{1}{2\sqrt{3}} & -\frac{1}{3\sqrt{2}} \\ \frac{1}{2\sqrt{3}} & \frac{2}{3} & -\frac{1}{3\sqrt{2}} \\ \frac{1}{3\sqrt{2}} & 0 & \frac{1}{3} \end{pmatrix} D^{\alpha_2}, \quad V'_{\alpha_2} (24) = \frac{1}{3} D^{\alpha_2} \begin{pmatrix} \frac{1}{6} & 2\sqrt{3} & -\frac{1}{3\sqrt{2}} \\ 2\sqrt{3} & \frac{2}{3} & -\frac{1}{3\sqrt{2}} \\ -\frac{1}{3\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{3} \end{pmatrix} D^{\alpha_2},
\]

(11)

where

\[
D^{\alpha_2} = \begin{pmatrix} \sqrt{d + 1} & 0 & 0 \\ 0 & \sqrt{d + 1} & 0 \\ 0 & 0 & \sqrt{d - 2} \end{pmatrix}
\]

(12)

For every \( d < 3 \) (in our case only \( d = 2 \) is interesting) we deal with matrices 2x2:

\[
V'_{\alpha_2} (14) = 3 \begin{pmatrix} \frac{1}{2} & -\frac{1}{2\sqrt{3}} \\ \frac{1}{2\sqrt{3}} & \frac{6}{2\sqrt{3}} \end{pmatrix}, \quad V'_{\alpha_2} (24) = 3 \begin{pmatrix} \frac{1}{2} & \frac{1}{2\sqrt{3}} \\ \frac{1}{2\sqrt{3}} & \frac{6}{2\sqrt{3}} \end{pmatrix}, \quad V'_{\alpha_2} (34) = 3 \begin{pmatrix} 0 & 0 \\ 0 & \frac{2}{3} \end{pmatrix},
\]

(13)

The full knowledge about irreps of \( V'(\sigma_{ab}) \), where \( \sigma_{ab} \in S(n) \) (see Notation 8 in the Appendix [V A]) allows us to decompose these operators and density operators \( \rho_{1...n} \) which are \( U^* \otimes U \otimes \ldots \otimes U \) invariant into block diagonal form

\[
V'(\sigma_{ab}) = \bigoplus_\alpha I_{r(\alpha)} \otimes \psi'_\alpha(\sigma_{ab}), \quad \rho_{1...n} = \bigoplus_\alpha I_{r(\alpha)} \otimes \tilde{\rho}^\alpha,
\]

(14)

where the direct sum runs over all inequivalent irreps \( \alpha \), \( r(\alpha) \) denotes the dimension of irrep \( \alpha \) and \( \tilde{\rho}^\alpha \) is a representation of operator \( \rho_{1...n} \) on irrep \( \alpha \). In the next paragraph we present how to use the decomposition from formula (14) and explicit matrix form of irreps of \( V'(\sigma_{ab}) \) to calculate fidelities.
C. Method of calculations

Since, in principle, calculations techniques are similar to those from [19], in most cases, proofs are skipped and unless specified otherwise, we refer to the above-mentioned work for them.

In this section we provide a general formula for an allowed region of \( N \)-tuples of fidelities in terms of overlaps of pure states with irreducible representations from the previous section. This is contained in Theorem 3.

Lemma 1. Fidelity \( F_{1k} \) as defined in (4) is of the form

\[
F_{1k} = \sum_{\alpha} F_{1k}^{\alpha},
\]

where

\[
F_{1k}^{\alpha} = \frac{1}{d} \text{Tr} \left( \rho^{\alpha} V'_{\alpha}(k - 1n) \right),
\]

the index \((k - 1n)\) means a permutation that swaps \( k - 1 \) and \( n \), and \( \rho^{\alpha} \)'s are arbitrary normalized states on partition \( \alpha \).

Again, from papers [20, 21] we know that algebra of partially transposed permutation operators \( A'_{n}(d) \) splits into sum of two ideals, i.e. we have \( A'_{n}(d) = M \oplus N \). In Lemma 1 we derived formulas for fidelities for elements in ideal \( M \), now we give similar formulas for elements in ideal \( N \). Physically it means that we looking for fidelities between maximally entangled state and some product state between input state and clones.

Fact 2. Fidelity \( F_{1k}^{N} \) between state \(|\psi_{1k}\rangle\) and a product state \( \rho_{1k} = \frac{1}{d} \text{Tr}_{1k} (I_{1} \otimes \rho_{2 \ldots n}) \) is equal to \( \frac{1}{d} \).

Now we are in position to formulate the main theorem of this section:

Theorem 3. The set \( F \) of admissible vectors of fidelities \( \{ F_{12}, \ldots, F_{1n} \} \) is of the form

\[
F = \text{conv} \left( \bigcup_{\alpha} F^{\alpha} \right),
\]

where \text{conv} stands for a convex hull, the union runs over all irreps and

\[
F^{\alpha} = \left\{ (F_{12}^{\alpha}, \ldots, F_{1n}^{\alpha}) : |\psi\rangle \in \mathbb{C}^{d_{\alpha}} \right\},
\]

where \( F_{1k}^{\alpha} \) are of the form: \( F_{1k}^{\alpha} = \frac{1}{d} \langle \psi | V'_{\alpha}(k - 1n) |\psi\rangle \), and where \( |\psi\rangle \) is a pure state.

Let us note that to determine the allowed region of fidelities, it is enough to consider only vectors of real coefficients.

Lemma 4. To generate a convex hull of the allowed region of fidelities, it is sufficient to consider pure states of real coefficients only.

D. Main result

In this section we present our results for two particular cases \( 1 \to 2 \) and \( 1 \to 3 \) universal quantum cloners.

Let us start with noting that to obtain a general answer to our question from Section II, we need to have a mixture of all fidelities connected with our irreps: \( \sum_{\alpha} p_{\alpha} F_{1N}^{\alpha} \). This implies that a convex hull is needed. On Figures 1 and 2 we show plots for \( N = 2, 3 \) and different dimensions \( d \) before taking the convex hull, so one can see a contribution from each irrep. Then, we take one particular case, namely \( 1 \to N \) UQCM and \( d = 3 \) and present the convex hull for it that reproduces the allowed region for fidelities (Figure 3). All plots are obtained using Mathematica software.

Remark 5 Because of the properties of the cloning map \( \Lambda \) (see Sec. II) all possible convex mixtures of the partitions produce a correct quantum cloner, i.e. a trace preserving completely positive map.

\[3\text{By } \text{Tr}_{1k} \text{ we denote partial trace over all subsystems except 1st and } k^{th}.\]
FIG. 1: The plot of allowed regions of fidelities for $1 \rightarrow 2$ UQCM. Views for various dimensions $d$ of the Hilbert space are presented: thin grey line and black point ($d = 2$); thin, dashed grey line and square ($d = 3$); thick line and diamond ($d = 4$); thick, dashed line and triangle ($d = 5$). One can see that for $d \rightarrow \infty$ the ellipse is squeezed to the line $F_{13} = -F_{12} + 1$ and coordinates of the point obtained from the part $\mathcal{N}$ go to zero.
FIG. 2: (Color online) The plot of allowed regions of fidelities for $1 \rightarrow 3$ UQCM. Views for various dimensions $d$ of the Hilbert space and all allowed irreps are presented. From the top: $d = 2$, $d = 3$ and $d = 10$. One can see that for $d = 2$ we match results from [19] and this is the only case where irreps from $M$ are two dimensional (in this case we have an ellipse). For higher dimensions $d \rightarrow \infty$ all regions obtained from the part $M$ are squeezed and coordinates of points from the part $N$ go to zero.
FIG. 3: (Color online) As an example convex hull for $1 \rightarrow 3$ of UQCM and $d = 3$ is presented.

E. Comparison with other methods

First of all, let us notice that our method gives correct results (according to the Werner’s formula [6]) in the case of symmetric cloning (see, [19], for a possible technique of checking that). What is more, the regions of fidelities obtained for $d = 2$ (qubits) match those obtained using Schur-Weyl duality [19]. Last, but not least, our method seems to correctly reproduce results obtained in [15], where the solution to the $1 \rightarrow N$ universal asymmetric qudit cloning problem for which the exact trade-off in the fidelities of the clones for every $N$ and $d$ has been derived. The authors obtained their result using various tools, like the Choi-Jamiołkowski isomorphism [27, 28] and some variance of the Lieb-Mattis theorem [29, 30]. The crucial part of their proof is the observation that the cloning problem can be mapped to some Heisenberg Hamiltonian on a star. Comparing their technique with ours, one can observe that they solve the problem from the side of the cloning map $\tilde{\Lambda}$, when we attack it from the side of the $n$-parties quantum state (see, Eq. (2) and (6)).

III. CONCLUSIONS

We have shown that using a more general version of Schur-Weyl duality, action of the universal $1 \rightarrow N$ quantum cloning machine can be described, allowing to obtain the admissible general region for fidelities. Contrary to other known methods, in our, quantum states are figures of merit. The method exploits decomposition of (usually big) Hilbert space into blocks of smaller dimensions which, of course, are easier to deal with. Fidelity expressions are then quite easy to obtain, one only needs to know representations of all possible irreps for a given case. Another advantage is that one can consider real pure states in each of the block only when generating convex hulls to obtain an allowed region for fidelities. Let us also notice that it is the first physical application of techniques developed in [20] and, up to our best knowledge, the first graphical presentation of allowed regions for $1 \rightarrow 2$ and $1 \rightarrow 3$ cloners for $d > 2$.

Let us now shortly discuss the results. First of all, suppose that we choose some point that lays outside of the allowed convex hull. Then there does not exist a quantum state that would correspond to that point. On the other hand, whenever we choose points from the convex hull (from inside or from the edge) we are able to derive a family of quantum states for which fidelities are fixed and have values determined by the chosen point. apart from the above-mentioned reconstruction of states from the convex hull, we can try to find, for example, all allowed quantum states which satisfy some required condition for relations between fidelities $F_{1k}$. For example, for $1 \rightarrow 3$
universal cloning machines we can demand the following constraint

\[ F_{12} + F_{13} = 2F_{14}, \]  

(19)

where we take maximization over \( F_{12}. \) Such a reconstruction was presented in our previous paper regarding admissible region of fidelities for the qubit case [19]. Finally, having these states we can reconstruct a cloning machine which returns clones with fidelities \( f_i \) corresponding to fidelities \( F_i \) given by the chosen point.

We have also interesting interpretation of the most bottom part of our plots as optimal anti-clones. First of all one can notice that our convex hulls are invariant with respect to rotations around straight line \( F_{12} = F_{13} = F_{14} \) by the angle \( \beta = 2\pi/3 \) in the case \( 1 \rightarrow 3 \) UQCM and they are symmetric with respect to the straight line \( F_{12} = F_{13} \) in the case \( 1 \rightarrow 2 \) UQCM. The most bottom point is determined by the intersection between symmetry line and convex hull and it corresponds to a minimum value of fidelities which are equal in these cases.

In the future, it would be interesting how to obtain optimal clones starting from our method. Numerically, it is not that hard, one just needs to add a cut to the general region to end with optimal region of fidelities. Analytically the answer does not seem to be so trivial, but we still hope that the employed group theoretical techniques are interesting and may provide some new insight into the inner structure of the optimal universal asymmetric quantum cloners.

Finally, let us note that to solve a \( M \rightarrow N \) \((M < N, M + N = n)\) cloning problem, one needs to posses a knowledge of the commutant structure of a \( U^\otimes N \otimes (U^*)^\otimes M \) transformation, where one has \( M \) conjugate elements \( U^* \) and \( N \) elements \( U \) [20][21].

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## IV. APPENDIX

### A. Algebra of partially transposed permutation operators

Here we present short summary of paper [21] which is crucial for construction of our results. For the reader convenience we keep here original notation. It appears that the structure irreducible representations of the algebra \( A_n'(d) \) is closely related to the structure of the representation \( {\text{ind}}_{S(n-1)}^{S(n-2)}(\phi^a) \) of the group \( S(n-1) \) induced by irreducible representations \( \phi^a \) of the group \( S(n-2) \) and the properties of irreducible representations of \( A_n'(d) \) depends strongly on the relation between \( d \) and \( n \). Before presenting the main ideas of this appendix we have to describe briefly some object appearing in the structure of the algebra \( A_n'(d) \), in particular the properties of the induced representation \( {\text{ind}}_{S(n-2)}^{S(n-1)}(\phi^a) \). The irreducible representations of the group \( S(n-2) \) are characterized by the partitions \( \pi = (a_1, ... , a_k) \) of \( n-2 \), which describe also the corresponding Young diagram \( Y(\pi) \). The representation \( {\text{ind}}_{S(n-2)}^{S(n-1)}(\phi^a) \) is completely and simply reducible i.e. we have [31].

**Proposition 6.**

\[ {\text{ind}}_{S(n-2)}^{S(n-1)}(\phi^a) = \bigoplus_v \psi^v, \]

(20)

where the sum is over all partitions \( v = (v_1, ... , v_k) \) of \( n-1 \), such that their Young diagrams \( Y(v) \) are obtained from \( Y(\pi) \) by adding, in a proper way, one box.

**Definition 7.** [32] Let \( \varphi : H \rightarrow M(n, \mathbb{C}) \) be a matrix representation of a subgroup \( H \) of the group \( G \). Then the matrix form of the induced representation \( \pi = {\text{ind}}_{H}^{G}(\varphi) \) of a group \( G \) induced by an irrep. \( \varphi \) of the subgroup \( H \subset G \) has the following block matrix form

\[ \forall g \in G \quad \pi_{ij}^g = (\varphi_{ij}(g^{-1}gs_b)), \]

where \( g_a, a = 1, ... , [G : H] \) are representatives of the left cosets \( G/H \) and

\[ \varphi_{ij}(g^{-1}gs_b) = \begin{cases} \varphi_{ij}(g^{-1}gs_b) & \text{if } g_a^{-1}gs_b \in H, \\ 0 & \text{if } g_a^{-1}gs_b \notin H. \end{cases} \]
Before main considerations for the appendix let us introduce some notation.

**Notation 8.** Any permutation \( \sigma \in S(n) \) defines, in a natural and unique way, two natural numbers \( a, b \in \{1, 2, \ldots, n\} \)
\[
 n = \sigma(a), \quad b = \sigma(n)
\]

Thus we may characterize any permutation by these two numbers in the following way
\[
 \sigma \equiv \sigma(a,b) \equiv \sigma_{ab}.
\]

Note that in general \( a, b \) may be different except the case, when one of them is equal to \( n \), because in this case we have
\[
 a = n \Leftrightarrow b = n.
\]

When \( a = n = b \) then \( \sigma(n) = n \) and we will use abbreviation \( \sigma = \sigma(n,n) \equiv \sigma_n \in S(n-1) \subset S(n) \).

From Proposition 6 and Definition 7 it follows that the induced representation \( \text{ind}^{S(n-1)}_{S(n-2)}(\varphi^a) \) may be described in two bases. The first one, is the basis of the matrix form of the induced representation of the form
\[
 \{ \varphi^d_{ij}(a) : a = 1, \ldots, n-1, \quad i = 1, \ldots, \dim \varphi^a \},
\]
where the index \( a = 1, \ldots, n-1 \) describes the the cosets \( S(n-1)/S(n-2) \) and the the index \( i = 1, \ldots, \dim \varphi^a \) is the index of a matrix form of \( \varphi^a \). The second one is a basis of the reduced form of \( \text{ind}^{S(n-1)}_{S(n-2)}(\varphi^a) \), which is of the form
\[
 \{ \rho^e_{j_k} : \varphi^e_{j_k} \in \text{ind}^{S(n-1)}_{S(n-2)}(\varphi^a), \quad j_k = 1, \ldots, \dim \varphi^e \}.
\]

The next important objects are the following matrices

**Definition 9.** For any irreducible representation \( \varphi^a \) of the group \( S(n-2) \) we define the block matrix
\[
 Q^d_{n-1}(a) \equiv Q(a) = (d^{ab}\varphi^a_{ij}[(an-1)(ab)(bn-1)]) = (Q^e_{ij}(a)), \quad (23)
\]
where \( a, b = 1, \ldots, n-1, \quad i, j = 1, \ldots, \dim \varphi^a \) and the blocks of the matrix \( Q(a) \) are labeled by indices \((a,b)\) whereas the elements of the blocks are labeled by the indices of the irreducible representation \( \varphi^a = (\varphi^a_{ij}) \) of the group \( S(n-2) \) and \( Q(a) \in M((n-1)w^a, \mathbb{C}) \).

The matrices \( Q(a) \) are hermitian and their structure and properties are described in the [21], where it has been shown, that the eigenvalues \( \lambda_v \) of the matrix \( Q(a) \) are labeled by the irreducible representations \( \varphi^v \in \text{ind}^{S(n-1)}_{S(n-2)}(\varphi^a) \) and the multiplicity of \( \lambda_v \) is equal to \( \dim \varphi^v \). The essential for properties of the irreducible representations of the algebra \( A^i_n(d) \) is the fact, that at most one (up to the multiplicity) eigenvalue \( \lambda_v \) of the matrix \( Q(a) \) may be equal to zero [20][21].

The structure of the algebra \( A^i_n(d) \) is the following

**Theorem 10.** The algebra \( A^i_n(d) \) is a direct sum of two ideals
\[
 A^i_n(d) = M \oplus N
\]
and the ideals \( M \) and \( N \) has different structures.

a) The ideal \( M \) is of the form
\[
 M = \bigoplus_a U(a),
\]
where \( U(a) \) are ideals of the algebra \( A^i_n(d) \) characterized by the irreducible representations \( \varphi^a \) of the group \( S(n-2) \), such that \( \varphi^a \in V_d[S(n-2)] \) and
\[
 U(a) = \text{span}_{\mathbb{C}}\{ a^e_{ij}(\alpha) : a, b = 1, \ldots, n-1, \quad i, j = 1, \ldots, w^a \}
\]

\[
 (26)
\]
with
\[ u_{ij}^{ab}(\alpha) u_{kl}^{pq}(\beta) = \delta_{a\beta} Q_{ik}^{bp} (\alpha) u_{jl}^{aq}(\alpha). \] (27)

The ideals \( U(\alpha) \) are matrix ideals such that
\[ U(\alpha) \simeq M(\text{rank} \, Q(\alpha), \mathbb{C}), \] (28)

in particular when \( \det Q(\alpha) \neq 0 \) we have
\[ U(\alpha) \simeq M((n-1) \dim \varphi^a, \mathbb{C}). \] (29)

b) The ideal \( N \) has the following structure
\[ N \simeq \bigoplus_{\nu} M(\dim \psi^\nu, \mathbb{C}), \] (30)

where the matrix ideals \( M(\dim \psi^\nu, \mathbb{C}) \) are generated by irreducible representations \( \psi^\nu \) of the group \( S(n-1) \) that are included in the representation \( V_d[S(n-1)] \) i.e. \( \psi^\nu \) are such that \( d \geq h(\nu) \).

The matrix ideals contained in the ideals \( M \) and \( N \) contains all minimal left ideals i.e. all irreducible representations of the algebra \( A'_n(d) \). The next theorems describes all these representations.

The structure of the irreducible representations of the algebra \( A'_n(d) \), included in the ideal \( M \), is completely determined by irreducible representations \( \varphi^a \) of the group \( S(n-2) \), therefore we will denote them \( \Phi^a_A \).

**Theorem 11.** The irreducible representations \( \Phi^a_A \) of the algebra \( A'_n(d) \) contained in the ideal \( U(\alpha) \subset M \) (see Theorem \[10\]) are indexed by the irreducible representations \( \varphi^a \) of the group \( S(n-2) \), such that \( \varphi^a \in V_d[S(n-2)] \) and if \( \{ f^a_{\nu} : \varphi^a \in \text{ind} S(n-1)(\varphi^a), \, \nu = 1, ..., \dim \psi^\nu \} \) is the reduced basis of the induced representation \( \text{ind} S(n-1)(\varphi^a) \), then the vectors \( \{ f^a_{\nu} : \lambda_\nu \neq 0 \} \) from the basis of the irreducible representation of the algebra \( A'_n(d) \) and the natural generators of \( A'_n(d) \) act on it in the following way
\[ V'(an) f^a_{\nu} (\alpha) = \sum_{p_{ij}^a} \sum_{k} \sqrt{\lambda_\nu} z_{ij}^{\nu}(\alpha) \chi_{kj}^{\nu}(\alpha) \varphi^a_{kj} \sum \sqrt{\lambda_\nu} f^a_{ij} (\alpha), \] (31)

where the summation is over \( p \) such that \( \lambda_\nu \neq 0 \). Due to the condition \( \varphi^a \in V_d[S(n-2)] \) the eigenvalues \( \lambda_\nu \) of \( Q(\alpha) \) are non-negative. The matrix unitary \( Z(\alpha) = (z_{ij}^{\nu}(\alpha)) \) has the form
\[ z_{ij}^{\nu}(\alpha) = \frac{\dim \psi^\nu}{\sqrt{N_{\nu}^j \,(n-1)!}} \sum_{\gamma \in S(n-1)} \psi^\gamma_{ij}^{\nu} (\sigma^{-1}) \delta_{\sigma \gamma}(q) \varphi^\nu_{ij} \{(an-1)\sigma(qn-1)\}, \] (32)

with
\[ N_{\nu}^j = \frac{\dim \psi^\nu}{(n-1)!} \sum_{\gamma \in S(n-1)} \psi^\gamma_{ij}^{\nu} (\sigma^{-1}) \delta_{\sigma \gamma}(q) \varphi^\nu_{ij} \{(qn-1)\sigma(qn-1)\}, \] (33)

where the indices \( q = 1, ..., n-1, \, r = 1, ..., \dim \varphi^\nu \) are fixed and such that \( N_{\nu}^j > 0 \). For more details see \[21\]. Whenever \( \sigma_n \in S(n-1) \) we have
\[ V(\sigma_n) f^a_{\nu} (\alpha) = \sum_{p_{ij}^a} \psi^\nu_{ij}^{\nu} (\sigma_n) f^a_{ij} (\alpha). \] (34)

In particular when \( \det Q(\alpha) \neq 0 \), i.e. when all \( \lambda_\nu \neq 0 \) then the representation \( \Phi^a_A \) is the induced representation \( \text{ind} S(n-1)(\varphi^a) \) (in the reduced form) for the subalgebra \( V_d[S(n-1)] \subset A'_n(d) \). In this case the dimension of the irreducible representation is equal to
\[ \dim \Phi^a_A = (n-1) \dim \varphi^a = \dim (\text{ind} S(n-1)(\varphi^a)). \] (35)
When \( \det Q(\alpha) = 0 \), \( i.e. \) when one, up to the multiplicity, eigenvalue \( \lambda_\theta \) of \( Q(\alpha) \) is equal to \( 0 \), then the irreducible representation of \( A'_n(d) \) is defined on a subspace \( \{ y_{j''}^\nu : \nu \neq \lambda_\theta \} \) of the representation space \( \text{ind}_{S(n-2)}^{S(n-1)}(\varphi^a) \) and the representation has dimension equal to
\[
\dim \Phi^a_A = \dim_{S(n-2)}^{S(n-1)}(\varphi^a)) - \dim \psi^\theta = \text{rank} Q(\alpha).
\] (36)

This case takes the place when
\[
d = i - \alpha_i - 1
\] (37)
for some \( \alpha_i \) in the partition \( \alpha = (\alpha_1, ..., \alpha_i, ..., \alpha_k) \) characterizing the irreducible representation \( \varphi^a \), under condition that \( \nu = (\alpha_1, ..., \alpha_i + 1, ..., \alpha_k) \) characterizes the representation \( \varphi^\nu \) of \( S(n-1) \).

The ideal \( U(\alpha) \) is a direct sum of \( \dim \Phi^a_A \) of irreducible representations \( \Phi^a_A \).

In particular matrices \( z(\alpha)_{k''}^{ij} \) diagonalize matrix \( Q(\alpha)_{ij}^{ab} \), \i.e. we have following

**Proposition 12.**
\[
\sum_{ak} \sum_{bl} z^a_k Q(\alpha)^{ab}_{ij} z(\alpha)^b_{lj} = \delta^\mu_{ij} \lambda_\mu
\] (38)
and the columns of the matrix \( Z(\alpha) = (z(\alpha)_{k''}^{ij}) \) are eigenvectors of the matrix \( Q(\alpha) \).

The formula for the eigenvalues \( \lambda_\nu \) of matrices \( Q(\alpha) \) is derived in the [21].

**Remark 13** Note that even if \( \dim \varphi^a = 1 \), we have \( \dim \Phi^a = n - 1 \).

The matrix forms of these representations are the following

**Proposition 14.** In the reduced matrix basis \( \{ f^\nu_{j''} : \nu \neq \theta \} \) of the ideal \( U(\alpha) \) the natural generators \( V(\sigma_{ab})_{ij} \) and \( V(\sigma_n) \) of \( A'_n(d) \) are represented by the following matrices
\[
[V^\nu_{n}(\sigma_{ab})]_{ij}^{ab} = \sum_{k=1}^{\text{dim} \varphi^a} \sqrt{\lambda_\rho} z^a_k Q(\alpha)^{ab}_{ij} z(\alpha)^b_{lj} \sqrt{\lambda_\nu} : \rho, \nu \neq \theta,
\] (39)
\[
[V^\nu_{n}(\sigma_n)]_{ij}^{ab} = \delta_\nu^{ij} \psi_{ij}^\nu(\sigma_n).
\] (40)

From the properties of the matrix \( Q(\alpha) \) \([21]\) one gets

**Proposition 15.** If \( d > n - 2 \), then \( \det Q(\alpha) \neq 0 \) and the irreducible representations \( \Phi^a_A \) described in Th. [17] are induced representation \( \text{ind}_{S(n-2)}^{S(n-1)}(\varphi^a) \) for the subalgebra \( V_d[S(n-1)] \subset A'_n(d) \), so their dimension is equal to \( (n - 1) \dim \varphi^a \). When \( d \leq n - 2 \), then for some \( \varphi^a \) it may appear that \( \det Q(\alpha) = 0 \) and consequently the irreducible representation \( \Phi^a \) of \( A'_n(d) \) is define on a subspace of the irreducible representation \( \text{ind}_{S(n-2)}^{S(n-1)}(\varphi^a) \).

The representations of the algebra \( A'_n(d) \) included in the ideal \( N \) are much simpler.

**Theorem 16.** Each irreducible representation \( \psi^\nu \) of the group \( S(n-1) \), which appears in the decomposition of the ideal \( N \) given in the Th. [17], \( i.e. \) \( \psi^\nu \in V_d[S(n-1)] \iff d \geq h(\nu) \) defines irreducible representations \( \Psi^\nu \) of the algebra \( A'_n(d) \) in the following way
\[
\Psi^\nu (\alpha) = \begin{cases} 
0 & \text{if } a \in \mathcal{M}, \\
\psi^\nu(\sigma_n) & \text{if } a = \sigma_n \in S(n-1). 
\end{cases}
\] (41)

So in this representation the non-invertible element of the ideal \( \mathcal{M} \) are represented trivially by zero and therefore we call these representation of the algebra \( A'_n(d) \) semi-trivial. The matrix forms of these representations are simply matrix forms of the irreducible representations of the group algebra \( \mathbb{C}[S(n-1)] \subset A'_n(d) \) and zero matrices for the elements of the ideal \( \mathcal{M} \).

**Corollary 17.** All irreducible representations of the algebra \( A'_n(d) \) of dimension one are included in the ideal \( N \). In particular, because the irreducible identity representation \( \psi^M \) of \( S(n-1) \) is always contained in \( V_d[S(n-1)] \), the algebra \( A'_n(d) \) has a trivial representation, in which the elements of the ideal \( M \) are represented by zero and the elements \( V_d(\sigma) : \sigma \in S(n-1) \) are represented by number 1.
B. Auxiliary lemmas

After short summary of paper [21] given in the previous subsection we prove here the crucial lemma which says that matrices \( z(\kappa)_{\kappa j}^{\nu} \) are unitary (real orthogonal) and then we conclude that representation matrices in the reduced matrix basis are hermitian (symmetric). We start from the following proposition:

**Proposition 18.** Suppose that all representations \( \psi^\nu \) of \( S(n-1) \) and \( \phi^\kappa \) of \( S(n-2) \) are unitary (real orthogonal) then the matrix

\[
z(\kappa)_{\kappa j}^{\nu} = \frac{\dim \psi^\nu}{\sqrt{N_{\kappa j}^\nu (n-1)!}} \sum_{\sigma \in S(n-1)} \psi^\nu_{\kappa j \kappa} (\sigma^{-1}) \delta_{\sigma c(\rho)} \psi^\rho_{\kappa r} [(\sigma n - 1) \sigma (\rho n - 1)], \quad (42)
\]

where

\[
N_{\kappa j}^\nu = \frac{\dim \phi^\kappa}{(n-1)!} \sum_{\sigma \in S(n-1)} \phi^\kappa_{\kappa j \kappa} (\sigma^{-1}) \delta_{\sigma \rho c(\gamma)} \psi^\gamma_{\kappa r} [(\sigma n - 1) \sigma (\rho n - 1)], \quad (43)
\]

is unitary (real orthogonal).

**Proof.** We will prove the orthogonal case, proving that

\[
\sum_{c, k} z(\kappa)_{\kappa j}^{\nu} z(\kappa)_{\kappa j}^{\nu} = \delta^{\nu \mu} \delta_{j k j}. \quad (44)
\]

Using the definition of the matrix \( z(\kappa) \) we get that LHS of the above equation is equal to

\[
\frac{\dim \psi^\nu \dim \phi^\kappa}{\sqrt{N_{\kappa j}^\nu \sqrt{N_{\kappa j}^\mu (n-1)!}}} \sum_{\sigma \rho \in S(n-1)} \sum_{c, k} \psi^\nu_{\kappa j \kappa} (\rho^{-1}) \phi^\kappa_{\kappa j \kappa} (\sigma^{-1}) \delta_{\sigma \rho c(\gamma)} \delta_{\gamma \sigma c(\gamma)} \psi^\gamma_{\kappa r} [(\sigma n - 1) \sigma (\rho n - 1)] \phi^\gamma_{\kappa r} [(\sigma n - 1) \sigma (\rho n - 1)] = \quad (45)
\]

\[
\frac{\dim \psi^\nu \dim \phi^\kappa}{\sqrt{N_{\kappa j}^\nu \sqrt{N_{\kappa j}^\mu (n-1)!}}} \sum_{\rho \in S(n-1) \gamma \in S(n-2)} \sum_{k, \nu} \phi^\kappa_{\kappa j \kappa} (\rho^{-1}) \psi^\nu_{\kappa j \kappa} (\sigma^{-1}) \delta_{\rho^{-1} \sigma c(\gamma)} \psi^\gamma_{\kappa r} [(\gamma n - 1) \gamma (\sigma n - 1)]. \quad (46)
\]

Substituting \( \gamma = \rho^{-1} \sigma \in S(n-2) \subset S(n-1) \) (which follows from \( \delta_{\rho^{-1} \sigma c(\gamma)} \)) we get

\[
\sum_{c, k} z(\kappa)_{\kappa j}^{\nu} z(\kappa)_{\kappa j}^{\nu} = \frac{\dim \psi^\nu \dim \phi^\kappa}{\sqrt{N_{\kappa j}^\nu \sqrt{N_{\kappa j}^\mu (n-1)!}}} \sum_{\rho \in S(n-1) \gamma \in S(n-2)} \sum_{k, \nu} \phi^\kappa_{\kappa j \kappa} (\rho^{-1}) \psi^\nu_{\kappa j \kappa} (\gamma^{-1}) \delta_{\gamma \rho^{-1} \sigma c(\gamma)} \delta_{\gamma \sigma c(\gamma)} \psi^\gamma_{\kappa r} [(\gamma n - 1) \gamma (\sigma n - 1)]. \quad (47)
\]

Now using the orthogonality relations for the irreducible representations \( \psi^\nu \) of \( S(n-1) \) we obtain

\[
\sum_{c, k} z(\kappa)_{\kappa j}^{\nu} z(\kappa)_{\kappa j}^{\nu} = \frac{\dim \psi^\nu}{\sqrt{N_{\kappa j}^\nu (n-1)!}} \sum_{\gamma \in S(n-2)} \delta^{\nu \mu} \delta_{j k j} \phi^\kappa_{\kappa j \kappa} (\gamma^{-1}) \delta_{\gamma \rho \sigma c(\gamma)} \psi^\gamma_{\kappa r} [(\gamma n - 1) \gamma (\sigma n - 1)] = \delta^{\nu \mu} \delta_{j k j}. \quad (48)
\]

The proof for the unitary case is similar.

**Corollary 19.** Suppose that all representations \( \psi^\nu \) of \( S(n-1) \) and \( \phi^\kappa \) of \( S(n-2) \) are unitary (real orthogonal) then the representation matrices (in the reduced matrix basis \( \{ f^\nu_{\kappa j} : \nu \neq \theta \} \) of the ideal \( \text{U}(\kappa) \))

\[
\left[ V^\nu_{\kappa j} \right]_{\kappa j}^{(\rho, \nu)} = \sum_{k = 1, \ldots, \dim \phi^\kappa} \sqrt{\lambda^\kappa_{\kappa j} z(\kappa)_{\kappa j}^{\nu}} \z^{\nu}_{\kappa j} \phi^\kappa_{\kappa j} \sqrt{\lambda^\kappa_j} ; \rho, \nu \neq \theta, \quad (49)
\]

are hermitian (real symmetric). In the orthogonal case we have replace hermitian conjugation \( \dagger \) in the equation \( \text{(49)} \) by normal transposition \( T \).

Indeed unitarity (orthogonality) of matrices \( z(\kappa)_{\kappa j}^{\nu} \) from Proposition 18 allows us to write \( z^+(\kappa)_{\kappa j}^{\nu} = z(\kappa)_{\kappa j}^{\nu} \).

Now writing explicitly matrix elements for \( \left[ V^\nu_{\kappa j} \right]_{\kappa j}^{(\rho, \nu)} \) and \( \left[ V^\nu_{\kappa j} \right]_{\kappa j}^{(\rho, \nu)} \) together with unitarity (orthogonality) properties from Proposition 18 we obtain statement of Corollary 19.
C. Proofs of the theorems from the main text

Proof of Lemma 4. From the definition of a fidelity we can write
\[ F_{1k} = \langle \psi_{1k} | \rho_{1k} | \psi_{1k} \rangle = \text{Tr} (\rho_{1k} | \psi_{1k} \rangle \langle \psi_{1k} | ) = \frac{1}{d} \text{Tr} (\rho_{1k} V'(1k)) , \] (50)
where \( \frac{1}{d} V'(1k) = | \psi_{1k} \rangle \langle \psi_{1k} | \), \( \rho_{1k} = \text{Tr}_{\bar{k}} \rho_{1...n} \) and \( \text{Tr}_{\bar{k}} \) denote partial trace over all systems except 1 and \( k \).

Now we can use decomposition of which we mentioned in Eq. 14 to represent \( V(1k) \) and \( \rho_{1...n} \):
\[ V'(1k) = \bigoplus_a 1_r(a) \otimes \mathcal{V}_a'(1k), \quad \rho_{1...n} = \bigoplus_a 1_r(a) \otimes \bar{\rho}^a , \] (51)
where \( a \) runs over all partitions of \( n - 2 \). Inserting (51) into (50), we have:
\[ F_{1k} = \frac{1}{d} \left[ \left( \bigoplus_{\mu} 1_r(\mu) \otimes \bar{\mathcal{V}}^\mu \right) \left( \bigoplus_{a} 1_r(a) \otimes \mathcal{V}_{a}'(1k) \right) \right] = \frac{1}{d} \text{Tr} \left( \bigoplus_{a} 1_r(a) \otimes \bar{\rho}^{a} \mathcal{V}_{a}'(1k) \right) = \frac{1}{d} \sum_a \text{Tr} (\rho^a \mathcal{V}_{a}'(1k)) = \frac{1}{d} \sum_a \text{Tr} (\rho^a \mathcal{V}_{a}'(k - 1n)) , \] (52)
where the last equality follows from Eq. 7. Now, one can see that Eq. (52) can be written as:
\[ F_{1k} = \sum_a F^a_{1k} , \] (53)
where \( F^a_{1k} = \frac{1}{d} \sum_a \text{Tr} (\rho^a \mathcal{V}_{a}'(k - 1n)) \), \( \rho^a = d_a \bar{\rho}^a \) and \( d_a \) stands for the dimension of irrep labeled by partition \( a \).

Proof of Fact 4. Reader can prove this fact by direct calculations. Namely, one has to compute fidelity between state which is a product in \( 1\mid 2\ldots n \) cut and maximally entangled state \( | \psi_{1k} \rangle \):
\[ F^N_{1k} = \frac{1}{d} \langle \psi_{1k} | \text{Tr}_{\bar{k}} (1_1 \otimes \rho_{2...n}) | \psi_{1k} \rangle = \frac{1}{d} \langle \psi_{1k} | 1_1 \otimes \rho_k | \psi_{1k} \rangle = \frac{1}{d} \text{Tr} \rho_k = \frac{1}{d} . \] (54)

Proof of Theorem 3. The proof is similar to that in 19. Only difference is the fact that now the fidelities look like as in Eq. 16.

Proof of Lemma 4. The proof goes as in 19. The only new thing in the proof is that matrices of irreps for transpositions (in), where \( 1 \leq i \leq n - 1 \) are symmetric (see Appendix IV B, Corollary 19).

D. Fidelity region for each irreducible space and some applications

In this section we provide some technical details regarding construction of admissible region of fidelities for \( 1 \rightarrow N \) UQCM. We focus here for clarity on the case when \( N = 3 \), then we have two non-trivial irreps \( a_1 = (2) \) and \( a_2 = (1,1) \). We also restrict here to dimensions \( d \geq 3 \) to omit discussion about dimension of irrep \( a_2 \), but of course construction in this situation is the same. For any \( d \geq 3 \) non-trivial irreps have the same dimension equal to three, thanks to this and Lemma 4 we can write an arbitrary pure state as \( | \psi^{a_1} \rangle = (a_1,a_2,a_3)^T \) and corresponding density matrix as \( \rho^{a_1} = \begin{pmatrix} a_1^2 & a_1 a_2 & a_1 a_3 \\ a_1 a_2 & a_2^2 & a_2 a_3 \\ a_1 a_3 & a_2 a_3 & a_3^2 \end{pmatrix} \), where \( a_1^2 + a_2^2 + a_3^2 = 1 \) and \( i = 1,2 \). Now putting for example density matrix \( \rho^{(2)} \) into equation 16 from Lemma 4 together with irreps \( \mathcal{V}'_{(2)} (k n - 1) \) from formula (9) we obtain following set of equations:
\[ F_{12}^{(2)} = \frac{1}{18d} \left( a_1^2 (d-1) - 2 \sqrt{3} a_1 a_2 (d-1) + 2 \sqrt{2} a_1 a_3 \sqrt{d-1} \sqrt{d+2} + 3 a_2^2 (d-1) - 2 \sqrt{2} a_2 a_3 \sqrt{d-1} \sqrt{d+2} + 2 a_3^2 (d+2) \right) , \]
\[ F_{13}^{(2)} = \frac{1}{18d} \left( a_1^2 (d-1) + a_1 \left( \sqrt{3} a_2 (d-1) + 2 \sqrt{2} a_3 \sqrt{d-1} \sqrt{d+2} \right) + 3 a_2^2 (d-1) + 2 \sqrt{2} a_2 a_3 \sqrt{d-1} \sqrt{d+2} + 2 a_3^2 (d+2) \right) , \]
\[ F_{14}^{(2)} = \frac{1}{9d} \left( 2 a_1^2 (d-1) - 2 \sqrt{2} a_1 a_3 \sqrt{d-1} \sqrt{d+2} + a_3^2 (d+2) \right) . \] (55)
The similar set of equations we can also obtain for partition \((1, 1)\). Moreover we know that the fidelity from ideal \(N\) is always equal to \(1/d\) (see Fact 2). In next step we use Mathematica software to generate parametric plots of regions given by formulas of the form \((55)\) together with normalization condition \(a_1^2 + a_2^2 + a_3^2 = 1\). Thanks to this we get admissible range of fidelities in every irreducible space labeled by partition \(\alpha_i\). Due to Theorem 3 to obtain admissible region of fidelities we have to generate convex hull of allowed regions obtained for every irreducible representation \(\alpha\). To do this we have used Mathematica package ConvexHull3D. One can see that to generate admissible regions for number of clones larger than 3 we need higher-dimensional space to embed convex hull, so we can not represent our results in the graphical form. There is still some way to omit this problem at least partially. Namely we can construct some projection which maps convex hulls from \(d\)-dimensional space to \(3\)-dimensional space, but then of course we lose some information.

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