On the lower bounds for the number of periodic billiard trajectories in manifolds embedded in Euclidean space

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1 Introduction

We shall study the lower bounds for the number of the periodic billiard trajectories in manifolds embedded in Euclidean space. A $p$-periodic billiard trajectory is a closed polygon consisting of $p$ segments all of whose vertices belong to the given manifold and, at every vertex, the two angles formed by the line and the manifold are equal (the exact definition will be given later). The first who considered this problem was George Birkhoff. He proved the following fact in [1]. Suppose $p$ is an odd prime, $M$ is a strictly convex smooth closed curve. Then there exist in $M$ at least two periodic billiard trajectories for each rotation number from $1$ to $\frac{p-1}{2}$. Ivan Babenko studied the billiards in a 2-dimensional sphere, but his paper [2] contains an error. Peter Pushkar in [3] solved the problem for $p=2$. He showed that in generic case in an $m$-dimensional manifold there are at least $B_2 + (m-1)B_2$ diameters, i.e., 2-periodic trajectories, where $B$ is the sum of Betty numbers modulo 2 of the given manifold. Finally, Michael Farber and Serge Tabachnikov proved in [4] that for $m \geq 3$ in an $m$-dimensional sphere there exist at least $\log_2(p-1) + m(p-1)$ in a generic case $p$-periodic billiard trajectories.

In Section 2 we show how one can apply Morse theory to study periodic billiard trajectories. Section 3 contains the generalized Birkhoff theorem. In Section 4 we prove the Farber-Tabachnikov estimate for generic small perturbations of the standard round $m$-sphere for any $m$. In Section 5 we study 3-periodic billiard trajectories in a 2-dimensional sphere. At last in Section 6 we find the rough estimate for the number of 3-periodic billiard trajectories in any manifold.

2 Morse theory of periodic billiard trajectories

Suppose $M$ is a smooth closed $m$-dimensional manifold embedded in Euclidean space $\mathbb{R}^n$, $p$ is an odd prime. An ordered set of points $(x_1, \ldots, x_p) \in M \times \ldots \times M$ is called a $p$-periodic billiard trajectory if

$$\frac{x_i - x_{i+1}}{\|x_i - x_{i+1}\|} + \frac{x_i - x_{i-1}}{\|x_i - x_{i-1}\|} \perp T_{x_i} M$$

for any $x_i, i \in \mathbb{Z}_p$. We consider sets of points $(x_1, \ldots, x_p) \in M \times \ldots \times M$ up to the action of the dihedral group $D_p$ in the $p$th Cartesian power of $M$. This action is generated by the cyclic permutation and the reflection:

$$(x_1, x_2, \ldots, x_p) \rightarrow (x_2, \ldots, x_p, x_1)$$

$$(x_1, x_2, \ldots, x_p) \rightarrow (x_p, x_{p-1}, \ldots, x_1)$$

Suppose $\Delta = \{(x, x, x_3, \ldots, x_p)\} \subset M \times \ldots \times M/D_p$. Let $f$ be the length function of a closed polygon:

$$f(x_1, \ldots, x_p) = \sum_{i \in \mathbb{Z}_p} p(x_i, x_{i+1}).$$

Thus $\Delta$ is the set of all points where $f$ is not smooth. It is clear that $p$-periodic billiard trajectories are critical points of $f$. Indeed,

$$\frac{\partial f}{\partial x_i} = \frac{x_i - x_{i+1}}{\|x_i - x_{i+1}\|} + \frac{x_i - x_{i-1}}{\|x_i - x_{i-1}\|}, \ x_i \in \mathbb{R}^n.$$

The derivative along any tangent vector vanishes if and only if the gradient is orthogonal to the tangent space.

In this paper we consider the general case: $f$ is a Morse function outside of $\Delta$. By $BT_p(M)$ denote the minimal number of $p$-periodic billiard trajectories in $M$. 

1
Lemma 2.1 Let $M$ be a closed Riemannian manifold, $\dim M = m$. Then there exists $\varepsilon > 0$ such that the following conditions hold for any $x \in M$:

1) The solid sphere $B_{\varepsilon}(x) \subset M$ is diffeomorphic to the disk $D_{\varepsilon}(0) \subset \mathbb{R}^m$; this diffeomorphism maps geodesics passing through $x \in M$ to straight lines passing through $0 \in \mathbb{R}^m$; angles between geodesics preserve.

2) Suppose $y_1(t), y_2(t), \ldots, y_k(t)$ are mutually different geodesics such that $y_1(0) = y_2(0) = \ldots = y_k(0) = x$. Let $\alpha_1, \ldots, \alpha_k > 0$ be real numbers. Put

$$h(t) = \rho(x, y_1(\alpha_1 t)) + \sum_{i=1}^{k-1} \rho(y_i(\alpha_i t), y_{i+1}(\alpha_{i+1} t)), \ t > 0$$

If $\alpha_1 t, \ldots, \alpha_k t < \varepsilon$, then $h'(t) \geq \alpha_k$.

Proof. Since $M$ is compact, it is sufficient to find the required $\varepsilon$ only for one point $x \in M$. We introduce coordinates in a neighborhood of the point $x$ along geodesics passing through $x$. In these coordinates

$$\rho(y, z) = \|y - z\| + O(\|y - z\|^2).$$

Thus for $h(t)$ we have

$$h'(t) = \alpha_1 + \sum_{i=1}^{k-1} \|\alpha_i a_i - \alpha_{i+1} a_{i+1}\| + O(t) \geq \alpha_k$$

by the triangle inequality applied to the points $0, \alpha_1 a_1, \ldots, \alpha_k a_k$, where $a_i$ are the directing vectors of the straight lines corresponding to the geodesics $y_i(t)$. Thus we find the required neighborhood of $x$. □

Remark We can choose $\varepsilon$ so that the 2nd condition holds for each $k = 1, \ldots, K$.

Theorem 1 The minimal number of $p$-periodic billiard trajectories satisfies

$$BT_p(M) \geq \sum_{q=1}^{mp} \dim H_q(M \times \ldots \times M/D_p, \Delta; \mathbb{Z}_2).$$

Proof. We construct a function $g$ on $X = M \times \ldots \times M/D_p$ such that the following conditions hold:

1. $g \geq 0$,
2. $g$ is smooth outside of $\Delta$,
3. Critical points of $g$ are the same as those of $f$,
4. $\Delta = \{g = 0\}$.

Then our theorem follows from Morse theory. Indeed, a small neighborhood of $\Delta$ in $X/\Delta$ is contractible, that’s why we can construct a cell space using the function $g$ in the same way as using any Morse function. Further, we apply Morse inequalities and reduce the relative homology to the absolute:

$$\tilde{H}_s(X/\Delta) = H_s(X, \Delta).$$
Suppose $\varphi(t)$ is a smooth function such that $0 \leq \varphi \leq 1$, $\varphi|_{(-\infty,0]} \equiv 0$, $\varphi|_{[\varepsilon,\infty)} \equiv 1$, $\varphi'|_{(0,\varepsilon)} > 0$. Let us show that the function

$$g(x_1, \ldots, x_p) = f(x_1, \ldots, x_p) \left( \prod_{i \in \mathbb{Z}_p} \varphi(p(x_i, x_{i+1})) \right)$$

is required if $\varepsilon$ is small enough.

By definition, put $\Delta^{(p-k)} = \{(x_1, \ldots, x_1, x_{k+1}, \ldots, x_p) \subset \Delta$. Then $\Delta^{(0)} \subset \ldots \subset \Delta^{(p-2)} = \Delta$.

First we find $\varepsilon = K$ tangent vector to the curve $(x_1, \ldots, x_1)$ is required if $p$ is small enough. Suppose we find $\varepsilon$ implies that we have not need to decrease the constructed neighborhood $U$. Suppose we find $\varepsilon$ implies that we have not need to decrease the constructed neighborhood $U$. Then

$$U^0_\varepsilon(\Delta^{(0)}) = \{(x_1, \ldots, x_p) : \rho(x_1, x_2), \ldots, \rho(x_1, x_p) < \varepsilon\}$$

is a small neighborhood of $\Delta^{(0)}$.

We can obtain any point $(x_1, x_2, \ldots, x_p)$ in this neighborhood if we fix $x_1$, emit $p-1$ geodesics from $x_1$, and put points $x_2, \ldots, x_p$ on these geodesics.

Let $x_j(\alpha_j t), j \geq 2$ be geodesics passing through $x_1$, $t$ is a natural parameter, i.e., $\|\frac{dt}{dt} x_j(\alpha_j t)\| = \alpha_j$. Then $g(x_1, \ldots, x_p) = g(t)$. Compute the derivative:

$$g'(t) = \left( \prod_{i \in \mathbb{Z}_p} \varphi(p(x_i, x_{i+1})) \right)' f(t) + \left( \prod_{i \in \mathbb{Z}_p} \varphi(p(x_i, x_{i+1})) \right) f'(t).$$

Inequalities

$$\varphi > 0, f > 0, \rho(x_1, x_{i+1}) > 0,$$

$$\varphi' \geq 0, f'(t) > 0, \frac{dt}{dt} \rho(x_1(t), x_{i+1}(t)) > 0$$

imply that we have $g'(t) > 0$. Thus at any point $A \in U^0(\Delta^{(0)})$ we have found the vector $\vec{V}$ (the tangent vector to the curve $(x_1, x_2(\alpha_2 t), \ldots, x_p(\alpha_p t))$) such that the derivative of the function $g$ along $\vec{V}$ is greater than 0. Consequently $dg(A) \neq 0$. Suppose we decrease $\varepsilon$. Note that we do not need to decrease the constructed neighborhood $U^0(\Delta^{(0)})$ since the critical points of $g$ cannot appear inside it.

Suppose we find $\varepsilon$ for a neighborhood $U^{p-l-1}(\Delta^{(p-l-1)})$. Replace this $\varepsilon$ by $\frac{\varepsilon}{2}$. We need to consider not the entire $\Delta^{(p-l)}$, but only $\Delta^{(p-1)} - U^{p-l-1}(\Delta^{(p-l-1)})$, i.e., we can assume that any point $A \in \Delta^{(p-l)}$ is of the form $(x_1, \ldots, x_l, x_{l+1}, \ldots, x_p)$, where $\rho(x_1, x_{l+1}) > \varepsilon, \ldots, \rho(x_1, x_p) > \varepsilon$. Arguing as above, we see that the whole neighborhood $\Delta^{p-l}$ can be obtained if we fix $x_1, x_{l+1}, \ldots, x_p$ and put $x_2, \ldots, x_l$ on geodesics passing through $x_1$. Again we have $g(x_1, \ldots, x_p) = g(t)$, and

$$g'(t) = \left( \prod_{i=l+1}^{p} \varphi(p(x_i, x_{i+1})) \right)' f(t) + \left( \prod_{i=l+1}^{p} \varphi(p(x_i, x_{i+1})) \right) f'(t).$$

Note that

$$f'(t) = \frac{dt}{dt} \sum_{i=1}^{p-1} \rho(x_i, x_{i+1}) + \frac{dt}{dt} \rho(x_l(\alpha_l t)(t), x_{l+1}) \geq 0.$$

Indeed, the distance from a fixed point to a point moving along a geodesic cannot change with a velocity greater than 1. Thus $g'(t) > 0$ again.

So we can pass from $\Delta^{(p-l-1)}$ to $\Delta^{(p-l)}$. Since $\Delta^{(p-2)} = \Delta$, this completes the proof. □
3 Periodic billiard trajectories in a circle

Lemma 3.1 There exists an embedding of the circle $S^1$ into the plane $\mathbb{R}^2$ such that the function

$$f(x_1, \ldots, x_p) = \sum_{i \in \mathbb{Z}_p} \rho(x_i, x_{i+1})$$

has $p - 1$ critical points: $\frac{p-1}{2}$ maxima and $\frac{p-1}{2}$ points of Morse index $p - 1$.

Two of the four periodic billiard trajectories for $p = 5$

Proof. This embedding in polar coordinates is given by the formula

$$r = 1 - \varepsilon \cos p \varphi$$

for $\varepsilon$ small enough. Let us show this if $p = 3$ (the proof for other values of $p$ is similar). $3$-periodic billiard trajectories of the non-deformed circle $r = 1$ are inscribed regular triangles. The coordinates of the vertices of such triangle are

$$\varphi_1 = \alpha_0, \ \varphi_2 = \alpha_0 + \frac{2\pi}{3}, \ \varphi_3 = \alpha_0 + \frac{4\pi}{3}.$$

Thus $3$-periodic billiard trajectories of the deformed circle $r = 1 - \varepsilon \cos 3 \varphi$ are

$$\varphi_1 = \alpha_0 + \beta_1, \ \varphi_2 = \alpha_0 + \beta_2 + \frac{2\pi}{3}, \ \varphi_3 = \alpha_0 + \beta_3 + \frac{4\pi}{3},$$

where $\beta_1, \beta_2, \beta_3 \to 0$ as $\varepsilon \to 0$. The length function is

$$f(\varphi_1, \varphi_2, \varphi_3) =$$

$$= \sqrt{(1 - \varepsilon \cos 3 \varphi_1)^2 + (1 - \varepsilon \cos 3 \varphi_2)^2 - 2(1 - \varepsilon \cos 3 \varphi_1)(1 - \varepsilon \cos 3 \varphi_2) \cos(\varphi_1 - \varphi_2)} +$$

$$+ \sqrt{(1 - \varepsilon \cos 3 \varphi_1)^2 + (1 - \varepsilon \cos 3 \varphi_3)^2 - 2(1 - \varepsilon \cos 3 \varphi_1)(1 - \varepsilon \cos 3 \varphi_3) \cos(\varphi_1 - \varphi_3)} +$$

$$+ \sqrt{(1 - \varepsilon \cos 3 \varphi_2)^2 + (1 - \varepsilon \cos 3 \varphi_3)^2 - 2(1 - \varepsilon \cos 3 \varphi_2)(1 - \varepsilon \cos 3 \varphi_3) \cos(\varphi_2 - \varphi_3)}.$$

Its derivatives have the following form:

$$\frac{\partial}{\partial \varphi_1} f(\varphi_1, \varphi_2, \varphi_3) =$$

$$= \frac{3 \varepsilon (1 - \varepsilon \cos 3 \varphi_1) \sin 3 \varphi_1 - (1 - \varepsilon \cos 3 \varphi_2)(3 \varepsilon \sin 3 \varphi_1 \cos(\varphi_1 - \varphi_2) - (1 - \varepsilon \cos 3 \varphi_1) \sin(\varphi_1 - \varphi_2))}{\sqrt{(1 - \varepsilon \cos 3 \varphi_1)^2 + (1 - \varepsilon \cos 3 \varphi_2)^2 - 2(1 - \varepsilon \cos 3 \varphi_1)(1 - \varepsilon \cos 3 \varphi_2) \cos(\varphi_1 - \varphi_2)} +}$$
Substituting \( \varphi_1, \varphi_2, \) and \( \varphi_3 \) for their values in this formula and the same formulas for \( \frac{\partial}{\partial \varphi_2} f(\varphi_1, \varphi_2, \varphi_3) \) and \( \frac{\partial}{\partial \varphi_3} f(\varphi_1, \varphi_2, \varphi_3) \), we obtain

\[
\begin{align*}
\varphi_1 &= \alpha_0 + \beta_1, \quad \varphi_2 = \alpha_0 + \beta_2 + \frac{2\pi}{3}, \quad \varphi_3 = \alpha_0 + \beta_3 + \frac{4\pi}{3}, \\
\cos 3\varphi_1 &= \cos 3\alpha_0 - 3\beta_1 \sin 3\alpha_0 + \ldots, \\
\cos 3\varphi_2 &= \cos 3\alpha_0 - 3\beta_2 \sin 3\alpha_0 + \ldots, \\
\cos 3\varphi_3 &= \cos 3\alpha_0 - 3\beta_3 \sin 3\alpha_0 + \ldots, \\
\sin 3\varphi_1 &= \sin 3\alpha_0 + 3\beta_1 \cos 3\alpha_0 + \ldots, \\
\sin 3\varphi_2 &= \sin 3\alpha_0 + 3\beta_2 \cos 3\alpha_0 + \ldots, \\
\sin 3\varphi_3 &= \sin 3\alpha_0 + 3\beta_3 \cos 3\alpha_0 + \ldots, \\
\cos(\varphi_1 - \varphi_2) &= -\frac{1}{2} + \frac{\sqrt{3}}{2}(\beta_1 - \beta_2) + \ldots, \\
\cos(\varphi_2 - \varphi_3) &= -\frac{1}{2} + \frac{\sqrt{3}}{2}(\beta_2 - \beta_3) + \ldots, \\
\cos(\varphi_1 - \varphi_3) &= -\frac{1}{2} - \frac{\sqrt{3}}{2}(\beta_1 - \beta_3) + \ldots, \\
\sin(\varphi_1 - \varphi_2) &= -\frac{\sqrt{3}}{2} - \frac{1}{2}(\beta_1 - \beta_2) + \ldots, \\
\sin(\varphi_2 - \varphi_3) &= -\frac{\sqrt{3}}{2} - \frac{1}{2}(\beta_2 - \beta_3) + \ldots, \\
\sin(\varphi_1 - \varphi_3) &= \frac{\sqrt{3}}{2} - \frac{1}{2}(\beta_1 - \beta_3) + \ldots.
\end{align*}
\]

Now we write that the derivatives of \( f \) vanish:

\[
\begin{align*}
-2\sqrt{3}\sin 3\alpha_0 + \varepsilon \sqrt{3}\sin 3\alpha_0 + \beta_1(-6\sqrt{3}\cos 3\alpha_0 - \frac{\sqrt{3}}{2} - \sin 3\alpha_0) + \\
+ \beta_2\left(\frac{\sqrt{3}}{4} + \frac{1}{2} \sin 3\alpha_0\right) + \beta_3\left(\frac{\sqrt{3}}{4} + \frac{1}{2} \sin 3\alpha_0\right) + \ldots &= 0, \\
-2\sqrt{3}\sin 3\alpha_0 + \varepsilon \sqrt{3}\sin 3\alpha_0 + \beta_1\left(\frac{\sqrt{3}}{4} + \frac{1}{2} \sin 3\alpha_0\right) + \\
+ \beta_2(-6\sqrt{3}\cos 3\alpha_0 - \frac{\sqrt{3}}{2} - \sin 3\alpha_0) + \beta_3\left(\frac{\sqrt{3}}{4} + \frac{1}{2} \sin 3\alpha_0\right) + \ldots &= 0, \\
-2\sqrt{3}\sin 3\alpha_0 + \varepsilon \sqrt{3}\sin 3\alpha_0 + \beta_1\left(\frac{\sqrt{3}}{4} + \frac{1}{2} \sin 3\alpha_0\right) + \\
+ \beta_2\left(\frac{\sqrt{3}}{4} + \frac{1}{2} \sin 3\alpha_0\right) + \beta_3(-6\sqrt{3}\cos 3\alpha_0 - \frac{\sqrt{3}}{2} - \sin 3\alpha_0) + \ldots &= 0.
\end{align*}
\]

Note that the constant term must be equal to 0. Consequently \( \alpha_0 = 0 \) or \( \alpha_0 = \frac{\pi}{2} \). It is clear that \( \beta_1 = \beta_2 = \beta_3 = 0 \) is a periodic billiard trajectory for any \( \varepsilon \), hence all coefficients of \( \varepsilon^k \) are equal to 0. Thus the dominant terms in this system are \( \beta_1, \beta_2, \) and \( \beta_3 \) with their coefficients. The linear system for \( \beta_1, \beta_2, \beta_3 \) has only the trivial solution. Hence we have the two trajectories: \( (\frac{\pi}{3}, \frac{5\pi}{6}, \frac{2\pi}{3}) \) for the maximum and \( (0, \frac{2\pi}{3}, \frac{4\pi}{3}) \) for the point of index 2. \( \square \)
**Corollary** The Euler characteristic of the space \( T^p / D_p - \Delta \) is equal to 0. Moreover, we can estimate the dimensions of homology:

\[
\dim H_p(T^p / D_p, \Delta; \mathbb{Z}_2) = \dim H_{p-1}(T^p / D_p, \Delta; \mathbb{Z}_2) \leq \frac{p-1}{2},
\]

\[
\dim H_q(T^p / D_p, \Delta; \mathbb{Z}_2) = 0, \ q \leq p - 2.
\]

**Lemma 3.2**

\[
\dim H_p(T^p / D_p, \Delta; \mathbb{Z}_2) \geq \frac{p-1}{2}.
\]

**Proof.** We construct a cellular division of the torus \( T^p \). The torus \( T^p \) is a Cartesian product of \( p \) circles. Let \( x_i \in [0, 1] \) be the cyclic coordinate on the \( i \)th circle. To each permutation \( \sigma \in S_p \) we assign a \( p \)-dimensional cell

\[
e^p_\sigma = \{ x_{\sigma(1)} < \ldots < x_{\sigma(p)} \}.
\]

This division is invariant with respect to the action of the permutation group \( S_p \). Then the sums of cells belonging to one connected component of \( T^p / D_p - \Delta \) form a basis in \( H_p(T^p / D_p, \Delta; \mathbb{Z}_2) \).

We construct a function \( I \) of a cell such that \( I \) is constant on the connected components of \( T^p / D_p - \Delta \).

Suppose we have a cell \( e^p_\sigma \subset T^p \). For each pair of indices \((i, i+1), \ i \in \mathbb{Z}_p\), one of the two inequalities \( x_i < x_{i+1} \) or \( x_i > x_{i+1} \) holds. By definition, put

\[
I(e^p_\sigma) = |\#(>) - \#(<)|.
\]

Cells of the same connected component can be obtained from \( e^p_\sigma \) by a sequence of the following transformations:

1. \( e^p_\sigma \mapsto e^p_{t\sigma}, \ t \in D_p \),
2. \( e^p_\sigma \mapsto \{ x_{\sigma(2)} < \ldots < x_{\sigma(p)} < x_{\sigma(1)} \} \),
3. \( e^p_\sigma \mapsto e^p_{r_i \sigma}, \ r_i = \left(1, \ldots, i, \ i+1, \ldots, p \right), \ p-2 \geq |\sigma(i+1) - \sigma(i)| \geq 2, \ i = 1, \ldots, p-1. \)

Evidently, transformations (1) and (3) do not change \( I(e^p_\sigma) \). Transformation (2) does not change it either:

\[
x_{\sigma(1)} < x_{\sigma(1)+1} \rightarrow x_{\sigma(1)} > x_{\sigma(1)+1}
\]

\[
x_{\sigma(1)-1} > x_{\sigma(1)} \rightarrow x_{\sigma(1)-1} < x_{\sigma(1)}
\]

Now we find \( \frac{p-1}{2} \) \( p \)-cells with different values of \( I \):

\[
\{x_1 < \ldots < x_p\}
\]

\[
\{x_1 < x_3 < x_2 < x_4 < \ldots < x_p\}
\]

\[
\{x_1 < x_4 < x_3 < x_2 < x_5 < \ldots < x_p\}
\]

\[
\vdots
\]

\[
\{x_1 < x_{2\cdot \frac{p-1}{2}} < \ldots < x_2 < x_{p+1} < \ldots < x_p\}
\]

The function \( I \) takes values \( p-2, p-4, \ldots, 1 \) at these cells. Thus \( T^p / D_p - \Delta \) has at least \( \frac{p-1}{2} \) connected components. The lemma is proved. \( \square \)

So we obtain the following result:

**Theorem 2**

\[
BT_p(S^1) = p - 1.
\]

**Remark** George Birkhoff found this estimate, but he did not prove that it is exact.
4 Periodic billiard trajectories in almost round spheres

Consider the standard $n$-sphere embedded in $\mathbb{R}^{n+1}$:

$$S^n = \{(y_0, y_1, \ldots, y_n) \in \mathbb{R}^{n+1} : \sum y_i^2 = 1\}$$

The length function $f(x_1, \ldots, x_p)$ has $\frac{p-1}{2}$ diffeomorphic non-degenerate critical manifolds. Each critical manifold is the set of regular $n$-gons (or "stars") inscribed in the unit circle centered at $0 \in \mathbb{R}^{n+1}$. Denote such a component by $V_{n,p}$.

**Lemma 4.1**

$$\sum_{q=0}^{\infty} \dim H_q(V_{n,p}; \mathbb{Z}_2) = 2n$$

**Proof.** It is clear that $V_{n,p}$ is a bundle over the Grassmannian $G_{2,n+1}$ with fiber $S^1$. The fiber has one-dimensional homologies, so this bundle is $\mathbb{Z}_2$-homology simple. Consider the $\mathbb{Z}_2$-cohomology spectral sequence of this bundle. Differentials of $E_2$ are multiplications by the characteristic class $\alpha \in H^2(G_{2,n+1}; \mathbb{Z}_2)$ of this bundle. Grassmannian $G_{2,n+1}$ is subdivided into Schubert cells $\sigma_{2,\ldots,2,1,\ldots,1}$ — see [3]. In this cell complex all boundary operators vanish, therefore each cell corresponds to a class of homology. By the same symbol $\sigma_{2,\ldots,2,1,\ldots,1}$ denote the Poincaré dual cohomology class. Note that

$$\dim H^q(G_{2,n+1}; \mathbb{Z}_2) = \begin{cases} \frac{q}{2} + 1, & \text{if } q \leq n - 1 \\ \frac{(2n-1-q)}{2} + 1, & \text{if } q > n - 1 \end{cases}$$

Indeed, a number $q$ can be presented as a sum of ones and twos in $\frac{q}{2} + 1$ ways. If $q \leq n - 1$, then each sum corresponds to a Schubert cell of the Grassmannian $G_{2,n+1}$. If $q > n - 1$, then our assertion follows from Poincaré duality. Moreover, we have the Pieri formula:

$$\sigma_a \sim \sigma_{b_1,b_2,\ldots} = \sum_{b_i \leq c_i \leq b_{i-1}} \sigma_{c_1,c_2,\ldots}$$

Let us show that the characteristic class of our bundle is equal to $\sigma_2$. First suppose $n = 2$, $p = 3$. Then $G_{2,n+1}$ is the projective plane $\mathbb{RP}^2$. $V_{2,3}$ is the manifold of all big regular triangles inscribed in the unit sphere. Points of the base $\mathbb{RP}^2$ are the lines orthogonal to the planes of these triangles. $\mathbb{RP}^2$ is divided into three cells $e^2$, $e^1$, and $e^0$ of dimensions 2, 1, and 0. The cell $e^1$ consists of all lines that belong to the coordinate plane $Oxy$ (except the axis $Ox$). All other lines belong to the cell $e_2$. Each of them is defined by a point of the upper hemisphere. To compute the characteristic class we must construct a section of the bundle $s : e^1 \to V_{2,3}$. Suppose this section consists of all vertical triangles, i.e., triangles with one vertex at the north pole of the sphere. By $h : \overline{D^2} \to \mathbb{RP}^2$ denote the characteristic map of the cell $e^2$. We assume that the open disk $D^2$ is the upper hemisphere. We have coordinates $(x, y)$ on the disk. The bundle over the disk is trivial. Introduce coordinates of direct product in it. Suppose $\delta$ is a triangle of $V_{2,3}$ not lying in any vertical plane. The plane of this triangle and the plane $Oxy$ intersect by the line $l$. Rotate this triangle around the line $l$ to the horizontal position (see the picture).

Let $\varphi \in \left[0, \frac{2\pi}{3}\right]$ be the smallest polar angle of its vertices. Then the coordinates $(x, y, \varphi)$ trivialize the bundle over the disk $D^2$. Now consider a bundle induced by the characteristic map $h$ over the closed disk $\overline{D^2}$. We can assume that any element of this fibered space is the big triangle with a normal that looks to the upper hemisphere or lies in the plane $Oxy$. This space has the same coordinates $(x, y, \varphi)$. We have a section $s' : S^1_\varphi \to \overline{D^2} \times S^1_\varphi$ over the boundary of the disk,
where $\varphi = \varphi' \mod \frac{2\pi}{3}$ — if we go around the circle–boundary once, then we make three complete turns of the circle–fiber. Thus the $\mathbb{Z}_2$-characteristic class of this bundle assigns the number 1 to the cell $e^2$. Similarly for $p > 3$ this class is also equal to $\sigma_2$. If $n > 2$, then we have an embedding $G_{2,3} \to G_{2,n+1}$ that preserves the cellular division. There is one 2-cell in $G_{2,3}$ and two 2-cells in $G_{2,n+1}$ for $n > 2$: $\sigma_{2,\ldots,2}$ (the image of the 2-cell of $G_{2,3}$) and $\sigma_{2,\ldots,2,1,1}$. It is clear that the cell $\sigma_{2,\ldots,2,1,1}$ does not make any obstruction for the section over 1-cell $\sigma_{2,\ldots,2,1}$. Thus in this case the characteristic class is also equal to $\sigma_2$.

We have in $H^*(G_{2,n+1}; \mathbb{Z}_2)$:

$$\sigma_2 \sim \sigma_{b_1,\ldots,b_k} = \sigma_2, b_1,\ldots,b_k, \text{ if } k < n - 1,$$

$$\sigma_2 \sim \sigma_{b_1,\ldots,b_{n-1}} = 0.$$

by the Pieri formula. Now we see that all differentials $d^{2j+1}_2 : E^{2j+1}_2 \to E^{2j+2,0}_2$ are either monomorphisms or epimorphisms, $d^{2n-2,1}_2$ is an isomorphism. Thus the term $E_3 = E_\infty$ is

$$
\begin{array}{ccccccc}
0 & 0 & \ldots & 0 & \mathbb{Z}_2 & \mathbb{Z}_2 & \ldots & \mathbb{Z}_2 & \mathbb{Z}_2 \\
\mathbb{Z}_2 & \mathbb{Z}_2 & \ldots & \mathbb{Z}_2 & \mathbb{Z}_2 & 0 & \ldots & 0 & 0
\end{array}
$$

All $H^q(V_n,p; \mathbb{Z}_2) = \mathbb{Z}_2$ for $q = 0,\ldots, 2n - 1$. This completes the proof. $\square$

**Remark** Suppose $f$ is a function with a non-degenerate critical manifold $V \subset \{ f = a \}$. Then its small Morse deformation $\tilde{f}$ has at least $\sum \dim H_q(V; \mathbb{Z}_2)$ critical points.

**Proof.** The function $\tilde{f} |_V$ is Morse in the general case. Suppose $\tilde{f} |_V$ has $N$ critical points. Then $N \geq \sum \dim H_q(V; \mathbb{Z}_2)$. A point $A \in V$ is critical for the function $\tilde{f} |_V$ if $\text{grad} \tilde{f} \perp T_A V$. In a neighborhood of a critical point we have

$$f = x_1^2 + \ldots + x_\alpha^2 - x_{\alpha+1}^2 - x_k^2,$$
\[ \tilde{f} = x_1^2 + \ldots + x_α^2 - x_α^{α+1} - x_k^2 + \sum_{i=1}^{m} \varepsilon_i x_i + o(x), \]
\[ V = \{ x_{k+1} = \ldots = x_m = 0 \}. \]

Since \( \text{grad} \tilde{f} \perp V \), we have that \( \varepsilon_{k+1} = \ldots = \varepsilon_m = 0 \). Hence the function \( \tilde{f} \) has only one critical point in this neighborhood: \( x_i \approx \frac{\varepsilon_i}{2}. \) Thus the function \( \tilde{f} \) has at least \( N \) critical points. □

We have proved the following fact.

**Theorem 3** Suppose \( p \) is an odd prime, \( n \) is any integer. Then a generic small perturbation of a standard round \( n \)-sphere has at least \( n(p - 1) \) \( p \)-periodic billiard trajectories.

### 5 3-periodic billiard trajectories in a 2-dimensional sphere

**Theorem 4** The minimal number of 3-periodic billiard trajectories in the sphere satisfies

\[ BT_3(S^2) \geq 4. \]

**Proof.** We shall construct a cellular division of the space \( S^2 \times S^2 \times S^2/D_3 \). We assume that our sphere is a square with the boundary contracted to a point:

\[ S^2 = \{0, 1\}_{\varphi} \times \{0, 1\}_{\psi}/\{\varphi = 0, \varphi = 1, \psi = 0, \psi = 1\} \]

The list of all cells is

\[
\begin{align*}
\omega_1 & = \left\{ \varphi_1 < \varphi_2 < \varphi_3 \right\}, & \omega_2 & = \left\{ \varphi_1 < \varphi_2 < \varphi_3 \right\}, & \omega_3 & = \left\{ \varphi_1 < \varphi_2 < \varphi_3 \right\}, \\
\omega_4 & = \left\{ \varphi_1 < \varphi_2 < \varphi_3 \right\}, & \omega_5 & = \left\{ \varphi_1 < \varphi_2 < \varphi_3 \right\}, & \omega_6 & = \left\{ \varphi_1 < \varphi_2 < \varphi_3 \right\}, \\
\sigma_1 & = \left\{ \varphi_1 < \varphi_2 < \varphi_3 \right\}, & \sigma_2 & = \left\{ \varphi_1 < \varphi_2 < \varphi_3 \right\}, & \sigma_3 & = \left\{ \varphi_1 < \varphi_2 < \varphi_3 \right\}, \\
\sigma_4 & = \left\{ \varphi_1 < \varphi_2 < \varphi_3 \right\}, & \sigma_5 & = \left\{ \varphi_1 < \varphi_2 < \varphi_3 \right\}, & \sigma_6 & = \left\{ \varphi_1 < \varphi_2 < \varphi_3 \right\}, \\
\delta_1 & = \left\{ \varphi_1 < \varphi_2 < \varphi_3 \right\}, & \delta_2 & = \left\{ \varphi_1 < \varphi_2 < \varphi_3 \right\}, & \delta_3 & = \left\{ \varphi_1 < \varphi_2 < \varphi_3 \right\}, \\
\delta_4 & = \left\{ \varphi_1 < \varphi_2 < \varphi_3 \right\}, & \delta_5 & = \left\{ \varphi_1 < \varphi_2 < \varphi_3 \right\}, & \delta_6 & = \left\{ \varphi_1 < \varphi_2 < \varphi_3 \right\}, \\
\alpha_1 & = \left\{ \varphi_1 = \varphi_2 = \varphi_3 \right\}, & \alpha_2 & = \left\{ \varphi_1 < \varphi_2 < \varphi_3 \right\}, & \beta_1 & = \left\{ \varphi_1 < \varphi_2 < \varphi_3 \right\}, \\
\beta_2 & = \left\{ \varphi_1 < \varphi_2 < \varphi_3 \right\}, & \beta_3 & = \left\{ \varphi_1 < \varphi_2 < \varphi_3 \right\}, & \beta_4 & = \left\{ \varphi_1 < \varphi_2 < \varphi_3 \right\}, \\
\gamma_1 & = \left\{ \varphi_1 = 0, \varphi_2 < \varphi_3 \right\}, & \gamma_2 & = \left\{ \varphi_1 = 0, \varphi_2 < \varphi_3 \right\}
\end{align*}
\]

\[
\begin{align*}
\text{dim} & = 6 \\
\text{dim} & = 5 \\
\text{dim} & = 4 \\
\text{dim} & = 3 
\end{align*}
\]
\( \kappa_1 = \left\{ \varphi_1 = 0, \varphi_2 = \varphi_3 \right\} \quad \kappa_2 = \left\{ \varphi_1 = 0, \varphi_2 < \varphi_3 \right\} \)

Now we compute the boundary operators.

\[ \partial_6 : C_6 \to C_5 \]

\[ \omega_1 \mapsto \delta_1 + \delta_6 + \sigma_1 + \sigma_6, \quad \omega_2 \mapsto \delta_3 + \delta_6 + \sigma_3 + \sigma_6, \]

\[ \omega_3 \mapsto \delta_1 + \delta_4 + \sigma_1 + \sigma_4, \quad \omega_4 \mapsto \delta_2 + \delta_3 + \sigma_4 + \sigma_5, \]

\[ \omega_5 \mapsto \delta_4 + \delta_5 + \sigma_2 + \sigma_3, \quad \omega_6 \mapsto \delta_2 + \delta_5 + \sigma_2 + \sigma_5 \]

\[ \partial_5 : C_5 \to C_4 \]

\[ \sigma_1 \mapsto \alpha_2 + \beta_4, \quad \sigma_2 \mapsto \alpha_2 + \beta_2, \quad \sigma_3 \mapsto \alpha_2 + \beta_3 + \beta_4, \]

\[ \sigma_4 \mapsto \alpha_2 + \beta_1 + \beta_2, \quad \sigma_5 \mapsto \alpha_2 + \beta_3, \quad \sigma_6 \mapsto \alpha_2 + \beta_1 \]

\[ \delta_1 \mapsto \alpha_1 + \beta_1, \quad \delta_2 \mapsto \alpha_1 + \beta_2, \quad \delta_3 \mapsto \alpha_1 + \beta_1 + \beta_3, \]

\[ \delta_4 \mapsto \alpha_1 + \beta_2 + \beta_4, \quad \delta_5 \mapsto \alpha_1 + \beta_3, \quad \delta_6 \mapsto \alpha_1 + \beta_4, \]

\[ \partial_4 : C_4 \to C_3 \]

\[ \alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3, \beta_4 \mapsto 0, \quad \gamma_1, \gamma_2 \mapsto \kappa_1 + \kappa_2 \]

\[ \partial_3 = 0 \]

Now we compute the kernels and the images of the boundary operators and the homology groups.

\[ \ker \partial_6 = \langle \omega_1 + \omega_2 + \omega_3 + \omega_4 + \omega_5 + \omega_6 \rangle, \quad \text{im} \partial_7 = 0, \quad H_6 = \mathbb{Z}_2, \]

\[ \ker \partial_5 = \langle \sigma_1 + \sigma_2 + \sigma_3 + \sigma_4 + \sigma_5 + \sigma_6 \rangle \oplus \text{im} \partial_6, \quad H_5 = \mathbb{Z}_2, \]

\[ \text{im} \partial_5 = \langle \alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3, \beta_4 \rangle, \quad \ker \partial_4 = \langle \gamma_1 + \gamma_2 \rangle \oplus \text{im} \partial_5, \quad H_4 = \mathbb{Z}_2, \]

\[ \ker \partial_3 = \langle \gamma_1, \gamma_2 \rangle, \quad \text{im} \partial_4 = \langle \gamma_1 + \gamma_2 \rangle, \quad H_3 = \mathbb{Z}_2. \]

Thus we obtain that

\[ H_*(S^2 \times S^2 \times S^2 / D_3, \Delta; \mathbb{Z}_2) = \{0, 0, 0, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2\}. \]

This completes the proof. \( \Box \)

### 6 A general estimate for 3-periodic billiard trajectories

**Lemma 6.1** Suppose we have an exact sequence of vector spaces

\[ 0 \xrightarrow{i_m} B_m \xrightarrow{j_m} C_m \xrightarrow{\partial_m} A_{m-1} \xrightarrow{i_{m-1}} B_{m-1} \xrightarrow{j_{m-1}} C_{m-1} \xrightarrow{\partial_{m-1}} \cdots \]

By definition, put \( a_q = \dim A_q, \ b_q = \dim B_q, \ c_q = \dim C_q. \) Then \( \sum c_q \geq \sum b_q - \sum a_q. \)
Finally we use the modified Theorem 1:

Therefore

So it is clear that

Now consider the exact sequence of the triple \((X, \Delta, \Delta^{(0)})\):

Therefore

Finally we use the modified Theorem 1:

This completes the proof. \(\Box\)

**Theorem 5** Suppose \(M\) is a closed manifold, \(B = \sum \dim H_q(M; \mathbb{Z}_3)\). Then we have an estimate

\[
BT_3(M) \geq \frac{B^3 - 3B^2 + 2B}{6}.
\]

**Proof.** The group \(\mathbb{Z}_3\) acts on \(M \times M \times M:\)

\[
t : M \times M \times M \to M \times M \times M, \quad t(x_1, x_2, x_3) = (x_2, x_3, x_1), \quad t^3 = \text{id}.
\]

\(\Delta^{(0)} = \{(x, x, x)\} \subset M \times M \times M\) is the set of all fixed points of this action. \(X = M \times M \times M/\mathbb{Z}_3\). The Smith theory gives us an estimate (see [3]):

\[
\dim H_q(X, \Delta^{(0)}; \mathbb{Z}_3) \geq \frac{\dim H_q(M \times M \times M; \mathbb{Z}_3) - \dim H_q(\Delta^{(0)}; \mathbb{Z}_3)}{3}.
\]

As above, \(\Delta = \{(x, x, y)\} \subset X\) is the diagonal of the cyclic cube \(M \times M \times M/\mathbb{Z}_3\). Now we must estimate the dimensions of the other relative homology groups \(H_q(X, \Delta)\). First we have an exact homological sequence of the pair \((\Delta, \Delta^{(0)})\). The pair \((\Delta, \Delta^{(0)})\) coincides with the pair \((M \times M, M = \{(x, x)\} \subset M \times M)\), and we have

\[
\ldots \to H_q(M) \xrightarrow{i} H_q(M \times M) \xrightarrow{j} H_q(M \times M, M) \xrightarrow{\partial} H_{q-1}(M) \to \ldots
\]

Let us note that \(\ker i_* = \{0\}\). Indeed, suppose \(p : M \times M \to M\) is a projection on any of the two factors. Then \(p \circ i = id\), hence \(p_* \circ i_* = id_*\), i.e., \(\ker i_* = \{0\}\). Further, \(H_q(\Delta) \cong H_q(\Delta, \Delta^{(0)}) \oplus H_q(\Delta^{(0)})\).

So it is clear that

\[
\sum_q \dim H_q(\Delta, \Delta^{(0)}) = B^2 - B.
\]

Now consider the exact sequence of the triple \((X, \Delta, \Delta^{(0)})\):

\[
\ldots \to H_q(\Delta, \Delta^{(0)}) \to H_q(X, \Delta^{(0)}) \to H_q(X, \Delta) \to H_{q-1}(\Delta, \Delta^{(0)}) \to \ldots
\]

Therefore

\[
\sum_q \dim H_q(X, \Delta) \geq \sum_q \dim H_q(X, \Delta^{(0)}) - \sum_q \dim H_q(\Delta, \Delta^{(0)}).
\]

Finally we use the modified Theorem 1:

\[
BT_p(M) \geq \frac{1}{2} \sum_{q=1}^{mp} \dim H_q(M \times \ldots \times M/\mathbb{Z}_p, \Delta; \mathbb{Z}_p)
\]

This completes the proof. \(\Box\)
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