A class of multidimensional quadratic BSDEs

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Abstract

In this paper we study a multidimensional quadratic BSDE with a particular class of product generators and give a result of existence of solution in a suitable complete metric space under some constraints on parameters. We also use that result to derive the existence and uniqueness of solution to the one dimensional case with bounded terminal values and show the existence of solution to a lower triangular quadratic BSDE with certain bounded terminal values.

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1 Introduction

A multidimensional quadratic Backward Stochastic Differential Equation (BSDE) on \([0, T]\) with \(T\) being the terminal time, according to the formulation put forward by Pardoux and Peng [11], is a stochastic integral equation with

\[
Y_t = \xi + \int_t^T f(s, Y_s, Z_s) \, ds - \int_t^T Z_s \, dW_s
\]

(1.1)

where \(Y_t\) is \(R^d\)-valued, \(Z_t\) is \(R^{d \times k}\)-valued, terminal value \(\xi\) is \(R^d\)-valued and \(\mathcal{F}_T\) measurable. The generator function \(f : [0, T] \times \Omega \times R^d \times R^{d \times k} \to R^d\) is of quadratic growth and \(W\) is a standard \(k\)-dimensional Brownian motion defined on \((\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})\) where \((\mathcal{F}_t)\) is the Brownian filtration.

BSDE with quadratic growth can be used to solve problems such as utility maximization with exponential utility function. They were studied by Kobylanski [10] and extended by others for example [3][4][7] and etc. More precisely in 2000, by using the monotonicity method adopted from PDE theory, Kobylanski [10] solved a class of one dimensional BSDEs with generator function being of quadratic growth in \(Z\). This particular class of quadratic BSDEs with unbound terminal values were further studied by Briand and Hu [3][4] and Delbaen, Hu and Richou [7]. In 2013, Barrieu and El Karoui [1] adopted a different approach to prove the existence under conditions similar to those of Briand and Hu [3], while Briand and Elie [2] gave a concise study for the case when the terminal value \(\xi\) is bounded. The method used in the present paper to get the main result was partially inspired by the method in Tevzadze [12] for solving existence of solutions to a quadratic BSDE driven by a continuous martingale with bounded terminal values. The case of multidimensional quadratic BSDEs seems significantly more difficult than that of Lipschitz BSDEs, and the methods used in literature are often quite involved, and up until now the results about quadratic BSDEs are mostly only for the one-dimensional case, and they heavily relay on comparison theorems. In 2015, P. Cheridito and K. Nam [5] discussed special systems of BSDEs assuming Markovian and subquadracity because of filtration issue. Recently, based on a result for one-dimensional BSDEs in Briand and Hu [3], Hu and Tang [8] proved the existence and uniqueness of solution to a multidimensional BSDE with diagonal quadratic generator assuming that each component \(f^i\) of the generator \(f\) depends only on the \(i\)th row of the matrix variable \(Z\) in the BSDE (1.1).

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In this paper we study a multidimensional quadratic BSDE with a particular class of product generators and give a result of existence of solution in a suitable complete metric space under some constraints on parameters. The corresponding PDEs however have significance in fluid dynamics and in fact they are simplified version of fluid equations. We also use that result to derive the existence and uniqueness of solution to the one dimensional case of our BSDE with bounded terminal values and then use the result for the one dimensional case to show the existence of solution to a lower triangular quadratic BSDE with certain bounded terminal values. The paper gives a result of existence of solution in a suitable complete metric space under some constraints on parameters. Then we show that a contraction map can be found on a suitable complete metric space on a fixed small time interval under some constraints on parameters, which gives a result of existence of solution to our BSDE on the whole time interval by pasting time together. In Section 4, by using the result obtained in Section 3 for our BSDE with small terminal values and pasting space together, we derive a result of existence and uniqueness of solution to the one dimensional case of our BSDE with bounded terminal values. Finally, in Section 5, by using the result obtained in Section 4 for the one dimensional case, we show the existence of solution to a lower triangular quadratic BSDE with some bounded terminal values satisfying a measurability condition.

2 Definitions and assumptions

Let us begin with a few notations and definitions which are nevertheless standard in BSDE literature as follows:

- $\|y\| = \sqrt{\int_y^T y^2}$ for $y \in \mathbb{R}^d$ and $\|z\| = \sqrt{\int_0^T \langle z, z \rangle}$ for $z \in \mathbb{R}^{d \times k}$ denote the Euclidean norms.
- $E[\cdot] := E[\cdot | \mathcal{F}_t]$.
- $W$ is a standard $k$-dimensional Brownian motion defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ and $(\mathcal{F}_t)_{t \in [0,T]}$ is its Brownian filtration.
- $\mathcal{H}^2(\mathbb{R}^d)$ and $\mathcal{H}^2(\mathbb{R}^{d \times k})$ denote respectively the Banach spaces of progressively measurable processes $Y$ and $Z$ such that $\|Y\|_{\mathcal{H}^2}^2 = E \int_0^T \|Y_t\|^2 dt < \infty$ and $\|Z\|_{\mathcal{H}^2}^2 = E \int_0^T \|Z_t\|^2 dt < \infty$.
- $\mathcal{S}^\infty(\mathbb{R}^d)$ denotes the Banach space of bounded progressively measurable processes $Y$.
- $\mathcal{S}^\infty_1(\mathbb{R}^d)$ denotes the collection of bounded progressively measurable processes $Y$ such that $\|Y\|_{\mathcal{S}^\infty} \leq C_1$ and $C_1$ is a non negative constant.

A continuous square integrable martingale $M$ with $M_0 = 0$ is a BMO martingale if

$$\|M\|_{\text{BMO}}^2 = \sup_{\tau} E \frac{\|M_T - M_\tau\|^2}{\mathbb{P}(\tau < T)} < \infty$$

where $T$ is the terminal time and the supremum is taken over all stopping times $\tau$ bounded by $T$, with the convention that if $\mathbb{P}(\tau = T) = 1$ then $\frac{E[\|M_T - M_\tau\|^2]}{\mathbb{P}(\tau < T)} = 0$.

$\mathcal{B}(\mathbb{R}^{d \times k})$ denotes the space of progressively measurable processes $Z$ such that $\int Z_s dW_s$ is a BMO martingale and define $\|Z\|_{\mathcal{B}} = \|\int Z_s dW_s\|_{\text{BMO}}$. Then $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is a Banach space due to the fact that the space of BMO martingales null at zero is Banach and the definition of stochastic integral.

$\mathcal{B}(\mathbb{R}^{d \times k})_{\sqrt{R}} = \{Z \in \mathcal{B}(\mathbb{R}^{d \times k}) : \|Z\|_{\mathcal{B}} \leq \sqrt{R}\}$ with $R$ being a positive constant, which is a closed subset of $\mathcal{B}(\mathbb{R}^{d \times k})$.

Since the definition of BMO space depends on the underlying probability measure, we denote by $\text{BMO}(\mathbb{P})$ the BMO space under $\mathbb{P}$ and by $\text{BMO}(\mathbb{Q})$ the BMO space under $\mathbb{Q}$ respectively in case of necessity. For the same reason, we also denote by $\mathcal{B}(\mathbb{R}^{d \times k})_{\mathbb{P}}$ the space of progressively measurable processes $Z$ such that $\int Z_s dW_s \in \text{BMO}(\mathbb{P})$ and by $\mathcal{B}(\mathbb{R}^{d \times k})_{\mathbb{Q}}$ the space of progressively measurable processes $Z$ such that $\int Z_s dW_s^\mathbb{Q} \in \text{BMO}(\mathbb{Q})$ where $W^\mathbb{Q}$ is a standard $k$-dimensional Brownian motion under $\mathbb{Q}$.
We consider the following BSDE:

\[
\begin{align*}
    dY_t &= Z_t f(Y_t, Z_t) \, dt + Z_t \, dW_t \\
    Y_T &= \xi
\end{align*}
\]  

(2.1)

where \( Y_t \) is \( R^d \)-valued, \( Z_t \) is \( R^{d \times k} \)-valued, \( f \) is \( R^k \)-valued and \( \xi \) is \( R^d \)-valued and \( \mathcal{F}_T \) measurable, which should be interpreted as a stochastic integral equation (1.1).

We make the following assumptions:

(a) \( \parallel \xi \parallel \leq C_1 \) for some constant \( C_1 > 0 \), i.e. \( \xi \) is bounded.

(b) \( f \) satisfies the Lipschitz condition and has a linear growth:

\[
\begin{align*}
    \parallel f(y_1, z_1) - f(y_2, z_2) \parallel &\leq C_2 \parallel y_1 - y_2 \parallel + C_3 \parallel z_1 - z_2 \parallel, \\
    \parallel f(y_1, z_1) \parallel &\leq C_2 \parallel y_1 \parallel + C_3 \parallel z_1 \parallel + C_4,
\end{align*}
\]

for any \( y_1, y_2 \in R^d \) and \( z_1, z_2 \in R^{d \times k} \), so that \( zf(y, z) \) has quadratic growth in \( z \). \( C_2, C_3, C_4 \) are non negative constants. \( f(Y, Z) \) is progressively measurable when \( (Y, Z) \) is progressively measurable.

By a solution to (2.1), we mean a pair of stochastic processes \( (Y, Z) \) on \( (\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}) \), where \( Y = (Y_t) \in \mathcal{S}^\infty(R^d) \) and \( Z = (Z_t) \in \mathcal{B}(R^{d \times k}) \). Moreover \( Z \in \mathcal{B}(R^{d \times k}) \) implies that \( Z \in \mathcal{M}^2(R^{d \times k}) \) due to the fact that

\[
\parallel Z \parallel_{\mathcal{F}_T}^2 = E\left( \int_0^T \parallel Z_s \parallel^2 \, ds \right) = E\left( \int_0^T \parallel Z_s \, dW_s \parallel_{BMO}^2 \right) \leq \parallel \int Z_s \, dW_s \parallel_{BMO}^2 = \parallel Z \parallel_{\mathcal{F}_T}^2.
\]

(2.2)

The following properties about BMO martingales are well known. If \( \int Z_s \, dW_s \) is a BMO martingale, then

\[
E^{\mathcal{F}_t} \left( \int_t^T \parallel Z_s \parallel^2 \, ds \right) \leq \int_t^T \parallel Z_s \, dW_s \parallel_{BMO}^2 = \parallel Z \parallel_{\mathcal{F}_T}^2, \forall t \in [0, T]
\]

and if \( E^{\mathcal{F}_t} \left( \int_t^T \parallel Z_s \parallel^2 \, ds \right) \leq N, \) for every \( t \in [0, T] \) where \( N \) is a non negative constant, then \( \int Z_s \, dW_s \) is a BMO martingale and

\[
\parallel Z \parallel_{\mathcal{F}_T}^2 = \parallel \int Z_s \, dW_s \parallel_{BMO}^2 \leq N,
\]

see Kazamaki [9] for details.

The following lemma is standard, whose proof can be found in Hu and Tang [8] for example, and it plays an important role in some of the subsequent arguments.

**Lemma 1.** For \( K > 0 \), there are constants \( c_1 > 0 \) and \( c_2 > 0 \) such that for any BMO martingale \( M \), we have for any BMO martingale \( N \) with \( \parallel N \parallel_{BMO(\mathbb{P})} \leq K \) that

\[
c_1 \parallel M \parallel_{BMO(\mathbb{P})} \leq \parallel M \parallel_{BMO(\mathbb{Q})} \leq c_2 \parallel M \parallel_{BMO(\mathbb{P})}
\]

where \( \tilde{M} = M - \langle M, N \rangle \) and \( \parallel \tilde{M} \parallel_{\mathcal{F}_t} = \mathcal{B}(N)_t \).

The following corollary can be obtained immediately by Lemma 1.

**Corollary 2.** Assume that \( N \in BMO(\mathbb{P}) \), then \( M \in BMO(\mathbb{P}) \) if and only if \( \tilde{M} \in BMO(\mathbb{Q}) \), where \( \tilde{M} \) and \( \mathbb{Q} \) are defined as in Lemma 1.

### 3 Existence of solution

We will use the iteration method. Let \( \mathcal{S}^\infty \times \mathcal{B} \) denote, for simplicity, the space \( \mathcal{S}^\infty(R^d) \times \mathcal{B}(R^{d \times k}) \). Suppose that \( (Y, Z) \in \mathcal{S}^\infty \times \mathcal{B} \) and \( f \) and \( \xi \) satisfy the above assumptions (a) and (b), then we have, by the linear growth of \( f \), boundedness of \( Y \) and properties of the BMO martingale \( \int Z_s \, dW_s \), that \( \int f(Y_s, Z_s) \, dW_s \) is also a
BMO martingale, which in turn implies that the stochastic exponential of \(-\int f(Y_s, Z_s)\,dW_s\) is a martingale on 
\([0, T]\). Hence we define a probability measure \(\mathbb{Q}\) by
\[
\frac{d\mathbb{Q}}{d\mathbb{P}} \bigg|_{\mathcal{F}_T} = \mathcal{E} \left( -\int f(Y_s, Z_s)\,dW_s \right) \bigg|_T
\]
and define
\[
dW_i^\mathbb{Q} = dW_i + f(Y_s, Z_s)\,dt.
\]
Then \(W^\mathbb{Q}\) is a standard \(k\)-dimensional Brownian motion under probability measure \(\mathbb{Q}\). The lemma below about continuous martingale representation is well known and its proof may be found in Cohen and Elliott [6].

**Lemma 3.** Suppose \(M\) is a \(d\)-dimensional continuous local martingale under \(\mathbb{Q}\) with \(\mathbb{Q}\) defined as above, then there exists a unique predictable process \(H\) such that \(M - M_0 = \int H_i\,dW_i^\mathbb{Q}\).

**Proof.** Since \(M\) is a continuous semi-martingale under \(\mathbb{P}\), \(M - M_0 = N + A\) where \(N\) is a continuous local martingale null at 0 under \(\mathbb{P}\) and \(A\) is a finite variation process. By the martingale representation theorem applying to \(N\), we have that \(N = \int H_i\,dW_i\) for some predictable process \(H\). Thus \(M - M_0 = \int H_i\,dW_i^\mathbb{Q} = \int H_i\,f(Y_s, Z_s)\,ds + A\) by equation (3.2), which implies that the continuous \(\mathbb{Q}\)-local martingale \(M - M_0 = \int H_i\,dW_i^\mathbb{Q}\) is of finite variation and null at 0. Thus we have that \(M - M_0 = \int H_i\,dW_i^\mathbb{Q}\). Uniqueness can be proved in the usual way.

Let \(\delta \in (0, 1)\), and consider time interval \([T - \delta T, T]\). Let \((Y, Z) \in \mathcal{S}^\infty_{C_1} \times \mathcal{B}\) but with duration \([T - \delta T, T]\).

Since \(\|\xi\| \leq C_1, Y = \mathbb{E}_{\mathbb{Q}}^\mathbb{P}(\xi)\) is a continuous martingale under \(\mathbb{Q}\) on \([T - \delta T, T]\). Thus by Lemma 3 there exists a unique predictable process \(\tilde{Z}\) on \([T - \delta T, T]\) such that
\[
\begin{align*}
\frac{d\tilde{Y}_t}{d\mathbb{P}} &= Z_t f(Y_t, Z_t)\,dt + \tilde{Z}_t\,dW_t \\
\tilde{Y}_T &= \tilde{Z}\eta
\end{align*}
\]
with \(\|\tilde{Y}\|_{\mathcal{S}^\infty_{C_1}} \leq C_1\), which means that \(\tilde{Y} \in \mathcal{S}^\infty_{C_1}(\mathbb{R}^d)\).

**Lemma 4.** \(\int \tilde{Z}_t\,dW_t \in \text{BMO}(\mathbb{P})\) where \(\tilde{Z}\) is defined as in (3.3).

**Proof.** Since \(\tilde{Y}\) defined in (3.3) belongs to \(\text{BMO}(\mathbb{Q})\) as it is bounded under \(\mathbb{Q}\), it can be derived immediately by Corollary 2 that \(\int \tilde{Z}_t\,dW_t \in \text{BMO}(\mathbb{P})\).

We prove the following proposition.

**Proposition 5.** If \(C_1C_3 < e^{-\frac{1}{2}}\) where \(e^{-\frac{1}{2}}\) is just a universal constant, and it does not imply that it is optimal. Then there is a non negative constant \(C_6\) depending on \(C_1, C_2, C_3, C_4\) and \(\delta T\) such that
\[
\|\tilde{Z}\|_{\mathcal{S}^\infty_{C_1}}^2 \leq C_6 + \frac{1}{2}\|Z\|_{\mathcal{S}^\infty_{C_1}}^2
\]
for any pairs \((Y, Z)\) and \((\tilde{Y}, \tilde{Z})\) on \([T - \delta T, T]\) defined by BSDE (3.3).

**Proof.** Consider \(\varphi(x) = e^{Kx}\) where \(K\) is a positive constant to be determined later. Let \(\eta = \varphi\left(\|\tilde{Y}\|^2\right)\). Then by Itô’s formula we have
\[
\frac{d\|\tilde{Y}\|^2}{dt} = 2\sum_i \tilde{Y}_i \frac{d\tilde{Y}_i}{dt} + \|\tilde{Z}\|^2\,dt
\]
\[
= 2\sum_i f(Y_s, Z_s)\tilde{Y}_i\tilde{Z}_i\,dt + \|\tilde{Z}\|^2\,dt + 2\sum_{i,j} \tilde{Y}_i \tilde{Z}_j\,dW_i \,dW_j,
\]
and
\[
\frac{d\eta}{dt} = K\eta \frac{d\|\tilde{Y}\|^2}{dt} + 2K^2\eta \sum_j \left(\sum_i \tilde{Y}_i \tilde{Z}_i\right)^2 dt
\]
\[
= K\eta \left[ 2\sum_{i,j} f(Y_s, Z_s)\tilde{Y}_i \tilde{Z}_j\,dt + \|\tilde{Z}\|^2 dt \right] + 2K^2\eta \sum_j \left(\sum_i \tilde{Y}_i \tilde{Z}_j\right)^2 dt
\]
\[
+ 2K\eta \sum_{i,j} \tilde{Y}_i \tilde{Z}_j\,dW_i \,dW_j.
\]
Set vector $\tilde{U}$ with $\tilde{U} = \sum_i \tilde{y}^i Z^i$, so the previous equation can be written as

$$d\eta = K\eta \left[ 2\tilde{U}^T f(Y,Z) + \|\tilde{Z}\|^2 + 2K \|\tilde{U}\|^2 \right] dt + 2K\eta \tilde{U}^T dW.$$ 

Integrating the equality above from $t$ to $T$, we obtain

$$\eta_t = \eta_T - K \int_t^T \eta \left[ 2\tilde{U}^T f(Y,Z) + \|\tilde{Z}\|^2 + 2K \|\tilde{U}\|^2 \right] ds - 2K \int_t^T \eta \tilde{U}^T dW.$$ 

Since it can be derived immediately by Lemma 4 and the boundedness of $\tilde{Y}$ that $\int \eta \tilde{U}^T dW$ is a martingale, we take the conditional expectation with respect to $\mathcal{F}_t$ to get

$$\eta_t = E^{\mathcal{F}_t}(\eta_T) - KE^{\mathcal{F}_t} \left( \int_t^T \eta \left[ 2\tilde{U}^T f(Y,Z) + \|\tilde{Z}\|^2 + 2K \|\tilde{U}\|^2 \right] ds \right).$$ 

Next applying Cauchy-Schwartz inequality, the linear growth condition of $f$ and the bound of $Y$, we deduce that

$$\eta_t \leq E^{\mathcal{F}_t}(\eta_T) - KE^{\mathcal{F}_t} \left( \int_t^T \eta \left[ -2(C_1C_2 + C_3\|Z\| + C_4) \|\tilde{U}\| + \|\tilde{Z}\|^2 + 2K \|\tilde{U}\|^2 \right] ds \right) - KE^{\mathcal{F}_t} \left( \int_t^T \eta \left[ \|\tilde{Z}\|^2 + 2K \|\tilde{U}\|^2 \right] ds \right).$$ 

Then by applying the inequalities with $\alpha, \beta > 0$

$$2 \|\tilde{U}\| \leq \alpha + \frac{1}{\alpha} \|\tilde{Y}\|^2$$

and

$$2 \|Z\| \leq \beta + \frac{1}{\beta} \|\tilde{Y}\|^2,$$

we get that

$$\eta_t \leq E^{\mathcal{F}_t}(\eta_T) + (C_1C_2 + C_4) KE^{\mathcal{F}_t} \left( \int_t^T \alpha \eta ds \right) - KE^{\mathcal{F}_t} \left( \int_t^T \eta \|\tilde{Z}\|^2 ds \right) + KC_3 E^{\mathcal{F}_t} \left( \int_t^T \eta \|\tilde{U}\|^2 ds \right) - KE^{\mathcal{F}_t} \left( \int_t^T \eta \left[ 2K - \frac{C_3}{\beta} \right] \|\tilde{U}\|^2 ds \right).$$

It follows that

$$E^{\mathcal{F}_t} \left( \int_t^T \eta \|\tilde{Z}\|^2 ds \right) \leq \frac{1}{K} E^{\mathcal{F}_t}(\eta_T - \eta_t) + (C_1C_2 + C_4) E^{\mathcal{F}_t} \left( \int_t^T \alpha \eta ds \right) + C_3 \beta E^{\mathcal{F}_t} \left( \int_t^T \eta \|\tilde{Z}\|^2 ds \right) - KE^{\mathcal{F}_t} \left( \int_t^T \eta \left[ 2K - \frac{C_3}{\beta} \right] \|\tilde{U}\|^2 ds \right).$$

Since $\|\tilde{Y}\|_{\mathcal{F}_\infty} \leq C_1$ we deduce that $1 \leq \eta \leq e^{KC_1^2}$. We may choose constants such that

$$2K - \frac{C_3}{\beta} - \frac{C_1C_2 + C_4}{\alpha} \geq 0$$

and

$$C_3 \beta e^{KC_1^2} = \frac{1}{2}.$$

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In order to do this, it requires that
\[ K - C_3^2 e^{KC_1} > 0 \]
which means that
\[ Ke^{-1} - C_3^2 e^{KC_1-1} > 0 \]
which is possible only if \( C_1C_3 < e^{-\frac{1}{2}} \). When \( C_1C_3 < e^{-\frac{\lambda}{2}} \) by considering
\[ KC_1^2 - C_3^2 e^{2KC_1} > 0 \]
we can choose
\[ K = -\frac{2}{C_1^2} \ln (C_1C_3) \]
which implies that
\[ K - C_3^2 e^{KC_1} > 0. \]
Then we can deduce from the previous inequality (3.4) that
\[
E^{\mathcal{F}_t} \left( \int_t^T \| \tilde{Z}_s \|^2 ds \right) \leq \frac{1}{K} e^{KC_1} + \alpha e^{KC_1} (C_1C_2 + C_4) \delta T \\
+ C_3 \beta e^{KC_1} E^{\mathcal{F}_t} \left( \int_t^T \| Z_s \|^2 ds \right)
\]
for all \( t \in [T - \delta T, T] \). Using the properties of BMO martingales we may deduce that
\[
\| \tilde{Z} \|^2_{\mathcal{F}_t} \leq C_6 + \frac{1}{2} \| Z \|^2_{\mathcal{F}_t}
\]
where
\[
C_6 = e^{KC_1} \left[ \frac{1}{K} + \alpha (C_1C_2 + C_4) \delta T \right] \\
= \frac{1}{C_1^2C_3^2} \left[ -\frac{C_1^2}{2 \ln (C_1C_3)} + \alpha (C_1C_2 + C_4) \delta T \right].
\]
(3.7)

The above proposition implies that we get a pair \((\tilde{Y}, \tilde{Z}) \in \mathcal{S}^\infty_{C_1} \times \mathcal{B}\) on \([T - \delta T, T]\), when \( C_1C_3 < e^{-\frac{1}{2}} \). In this case we define the pair \((\tilde{Y}, \tilde{Z}) = \Phi (Y, Z) \) on \([T - \delta T, T]\) and \( \Phi : \mathcal{S}^\infty_{C_1} \times \mathcal{B} \to \mathcal{S}^\infty_{C_1} \times \mathcal{B} \) is well defined.

In order to get a result about the global existence of solution, we firstly consider the time interval \([T - \delta T, T]\) for some \( \delta \in (0, 1) \) and try to find a contraction map on a closed subspace of \( \mathcal{S}^\infty_{C_1} \times \mathcal{B} \). This approach is inspired by the method used in Tevzadze [12]. Then by working backwards with respect to time intervals of length \( \delta T \), we can get our result by pasting time together.

**Theorem 6.** Under the above assumptions (\( \mathcal{A}1 \)) and (\( \mathcal{A}2 \)) on \( f \) and \( \xi \), if \( C_1C_3 < e^{-144} \), then for any terminal time \( T \), there exists a positive constant \( \bar{K} = \left[ \frac{1}{\pi} \right] R \) with \( R = 2C_6 \) and some fixed constant \( \delta \in (0, 1) \) such that the BSDE
\[
\begin{cases}
    dY_t = Z_t f(Y_t, Z_t) \, dt + Z_t \, dW_t \\
    Y_T = \xi
\end{cases}
\]
has a solution pair \((Y, Z) \in \mathcal{S}^\infty_{C_1} (R^d) \times \mathcal{B} \bigotimes_{\mathcal{P}} (R^{d \times k}) \) on \([0, T]\).

**Proof.** Let \( \lambda \in (0, 1) \) be a positive constant to be determined later. We firstly consider the time interval \([T - \delta T, T]\) as above and assume that
\[ C_1C_3 < e^{-\frac{\lambda}{2}} \]
(3.9)
which implies that \( C_1 C_3 < e^{-\frac{1}{2}} \) and we set constant \( R \) to be

\[
R = 2C_6. \tag{3.10}
\]

We may have the following by choosing \( \delta \) small enough.

\[
2 \left( C_1 C_2 + C_4 \right) \sqrt{\delta T} \sqrt{2C_2 \sqrt{\delta T} \sqrt{R} \sqrt{2C_3 \sqrt{R}} \leq \lambda. \tag{3.11}
\]

We can do this because we have condition (3.9) and in equation (3.7):

\[
C_6 = \frac{1}{C_1^2 C_3^2} \left[ \frac{C_1^2}{-2 \ln (C_1 C_3)} + \alpha (C_1 C_2 + C_4) \delta T \right]
\]

which implies that

\[
C_1^2 C_6 = \frac{1}{C_1^2} \left[ \frac{C_1^2}{-2 \ln (C_1 C_3)} + \alpha (C_1 C_2 + C_4) \delta T \right],
\]

where \( \alpha \) is determined by inequality (3.5):

\[
2K - \frac{C_3}{\beta} - \frac{(C_1 C_2 + C_4)}{\alpha} \geq 0
\]

which can be achieved when \( C_1 C_3 < e^{-\frac{1}{2}} \).

Let \( \mathcal{C}_1^\infty \times \mathcal{B}_R \) denote the space \( \mathcal{C}_1^\infty (R^d) \times \mathcal{B}_R (R^{d \times k}) \). Since \( C_1 C_3 < e^{-\frac{1}{2}} \) which is due to condition (3.9), we have \( \Phi : \mathcal{C}_1^\infty \times \mathcal{B}_R \to \mathcal{C}_1^\infty \times \mathcal{B}_R \) as defined above. Then for any pair \( (Y, Z) \in \mathcal{C}_1^\infty \times \mathcal{B}_R \) we can get \( (\tilde{Y}, \tilde{Z}) = \Phi(Y, Z) \) with \( (\tilde{Y}, \tilde{Z}) \in \mathcal{C}_1^\infty \times \mathcal{B}_R \). By Proposition 5 we have that

\[
\| \tilde{Z} \|^2_{\mathcal{B}_R} \leq C_6 + \frac{1}{2} \| Z \|^2_{\mathcal{B}_R}.
\]

Together with condition (3.10) we get that

\[
\| Z \|^2_{\mathcal{B}_R} \leq \frac{R}{2} + \frac{1}{2} \| Z \|^2_{\mathcal{B}_R} \leq R,
\]

which implies that \( (\tilde{Y}, \tilde{Z}) \in \mathcal{C}_1^\infty \times \mathcal{B}_R \). So that \( \Phi : \mathcal{C}_1^\infty \times \mathcal{B}_R \to \mathcal{C}_1^\infty \times \mathcal{B}_R \) is well defined.

For any \( (Y^1, Z^1), (Y^2, Z^2) \in \mathcal{C}_1^\infty \times \mathcal{B}_R \), we set \( (\tilde{Y}^1, \tilde{Z}^1) = \Phi(Y^1, Z^1) \) and \( (\tilde{Y}^2, \tilde{Z}^2) = \Phi(Y^2, Z^2) \). So we have \( (\tilde{Y}^1, \tilde{Z}^1), (\tilde{Y}^2, \tilde{Z}^2) \in \mathcal{C}_1^\infty \times \mathcal{B}_R \). Then by setting

\[
\Delta = Y^1 - Y^2, \quad \tilde{\Delta} = \tilde{Y}^1 - \tilde{Y}^2, \quad \Lambda = Z^1 - Z^2, \quad \tilde{\Lambda} = \tilde{Z}^1 - \tilde{Z}^2,
\]

we get that \( (\Delta, \Lambda), (\tilde{\Delta}, \tilde{\Lambda}) \in \mathcal{C}^\infty (R^d) \times \mathcal{B} (R^{d \times k}) \) with \( \tilde{\Delta}_T = 0 \) and we also get

\[
d\tilde{\Delta}^i = \sum_j \tilde{\Delta}^i_j dW^j + \sum_j f_j (Y^1, Z^1) \tilde{\Delta}^i_j dt + \sum_j [f_j (Y^1, Z^1) - f_j (Y^2, Z^2)] \tilde{Z}^i_j dt.
\]

Then by Itô’s formula we have

\[
d \| \tilde{\Delta} \|^2 = 2 \theta^T f (Y^1, Z^1) dt + 2 \rho^T [f (Y^1, Z^1) - f (Y^2, Z^2)] dt
\]

\[
+ \| \tilde{\Lambda} \|^2 dt + 2 \theta dW,
\]

where the components of vectors \( \theta \) and \( \rho \) are defined as

\[
\begin{align*}
\theta^i &= \sum_l \tilde{\Delta}^i_l \tilde{\Lambda}^l, \\
\rho^i &= \sum_l \tilde{\Delta}^i_l \tilde{Z}^l.
\end{align*}
\]
and $\int \vartheta^T dW$ is a martingale by the boundedness of $\bar{\Delta}$ and the fact that $\bar{\Lambda} \in \mathcal{B}(R^{d \times k})$. Then by taking conditional expectation we get

$$\|\bar{\Delta}_t\|^2 + E^\mathcal{F}_t \left[ \int_t^T \|\bar{\Lambda}\|^2 ds \right] = -2E^\mathcal{F}_t \left[ \int_t^T \vartheta^T f (Y^1, Z^1) ds \right] + 2E^\mathcal{F}_t \left[ \int_t^T \rho^T f (Y^1, Z^1) - f (Y^2, Z^2) ds \right],$$

which implies that

$$\|\bar{\Delta}_t\|^2 + E^\mathcal{F}_t \left[ \int_t^T \|\bar{\Lambda}\|^2 ds \right] \leq 2E^\mathcal{F}_t \left[ \int_t^T \|\bar{\Delta}\| \|\bar{\Lambda}\| f (Y^1, Z^1) ds \right] + 2E^\mathcal{F}_t \left[ \int_t^T \|\bar{\Delta}\| \|\bar{\Lambda}\| f (Y^1, Z^1) - f (Y^2, Z^2) ds \right].$$

Together with the definition of $\vartheta$ and $\rho$, we obtain from the inequality above that

$$\|\bar{\Delta}_t\|^2 + E^\mathcal{F}_t \left[ \int_t^T \|\bar{\Lambda}\|^2 ds \right] \leq 2E^\mathcal{F}_t \left[ \int_t^T (C_1 C_2 + C_4) \|\Delta_s\| \|\bar{\Lambda}_s\| ds \right] + 2E^\mathcal{F}_t \left[ \int_t^T \|\Delta_s\| \|\bar{\Lambda}_s\| \|\bar{Z}_s\| ds \right] \leq 2(C_1 C_2 + C_4) \sqrt{\delta T} \|\bar{\Delta}\|_{\mathcal{F}_T} \|\bar{\Lambda}\|_{\mathcal{F}_T} + 2C_3 \|\Delta\|_{\mathcal{F}_T} \|\bar{\Lambda}\|_{\mathcal{F}_T} + 2C_2 \|\bar{Z}\|_{\mathcal{F}_T} \|\Delta\|_{\mathcal{F}_T} \|\bar{\Lambda}\|_{\mathcal{F}_T},$$

from which we deduce that

$$\|\bar{\Delta}\|_{\mathcal{F}_T} \leq 2(C_1 C_2 + C_4) \sqrt{\delta T} \|\bar{\Lambda}\|_{\mathcal{F}_T} + 2C_3 \|\Delta\|_{\mathcal{F}_T} \|\bar{\Lambda}\|_{\mathcal{F}_T} + 2C_2 \|\bar{Z}\|_{\mathcal{F}_T} \|\Delta\|_{\mathcal{F}_T} \|\bar{\Lambda}\|_{\mathcal{F}_T}, \tag{3.13}$$

and

$$\|\bar{\Lambda}\|^2_{\mathcal{F}_T} \leq 2(C_1 C_2 + C_4) \sqrt{\delta T} \|\bar{\Delta}\|_{\mathcal{F}_T} \|\bar{\Lambda}\|_{\mathcal{F}_T} + 2C_3 \|\Delta\|_{\mathcal{F}_T} \|\bar{\Lambda}\|_{\mathcal{F}_T} + 2C_2 \|\bar{Z}\|_{\mathcal{F}_T} \|\Delta\|_{\mathcal{F}_T} \|\bar{\Lambda}\|_{\mathcal{F}_T} + 2C_2 \sqrt{\delta T} \|\bar{Z}\|_{\mathcal{F}_T} \|\Delta\|_{\mathcal{F}_T} \|\bar{\Lambda}\|_{\mathcal{F}_T}, \tag{3.14}$$

Thus we have that

$$\|\bar{\Delta}\|_{\mathcal{F}_T} \leq 2(C_1 C_2 + C_4) \sqrt{\delta T} \|\bar{\Lambda}\|_{\mathcal{F}_T} + 2C_3 \sqrt{\delta T} \|\bar{\Lambda}\|_{\mathcal{F}_T} + 2C_2 \sqrt{\delta T} \|\Delta\|_{\mathcal{F}_T} + 2C_3 \sqrt{\delta T} \|\Delta\|_{\mathcal{F}_T} \|\bar{\Lambda}\|_{\mathcal{F}_T}, \tag{3.15}$$

and

$$\|\bar{\Lambda}\|^2_{\mathcal{F}_T} \leq 2(C_1 C_2 + C_4) \sqrt{\delta T} \|\bar{\Delta}\|_{\mathcal{F}_T} \|\bar{\Lambda}\|_{\mathcal{F}_T} + 2C_3 \sqrt{\delta T} \|\bar{\Delta}\|_{\mathcal{F}_T} \|\bar{\Lambda}\|_{\mathcal{F}_T} + 2C_2 \sqrt{\delta T} \|\Delta\|_{\mathcal{F}_T} \|\bar{\Lambda}\|_{\mathcal{F}_T} + 2C_3 \sqrt{\delta T} \|\Delta\|_{\mathcal{F}_T} \|\bar{\Lambda}\|_{\mathcal{F}_T}. \tag{3.16}$$

Then by condition (3.11) we get that

$$\|\bar{\Delta}\|_{\mathcal{F}_T} \leq \lambda (2 \|\bar{\Lambda}\|_{\mathcal{F}_T} + \|\Delta\|_{\mathcal{F}_T} + \|\bar{\Lambda}\|_{\mathcal{F}_T}) \tag{3.17}$$
and
\[ \| \tilde{\Lambda} \|_{\mathcal{B}}^2 \leq \lambda \| \Delta \|_{\mathcal{B}} (2 \| \tilde{\Lambda} \|_{\mathcal{B}} + \| \Delta \|_{\mathcal{B}} + \| \Lambda \|_{\mathcal{B}}). \] (3.18)

So by substituting \( \| \tilde{\Lambda} \|_{\mathcal{B}} \) in (3.18) with (3.17) we have that
\[ \| \tilde{\Lambda} \|_{\mathcal{B}} \leq \lambda (2 \| \tilde{\Lambda} \|_{\mathcal{B}} + \| \Delta \|_{\mathcal{B}} + \| \Lambda \|_{\mathcal{B}}). \]

By combining with (3.17) we deduce that
\[ \| \tilde{\Lambda} \|_{\mathcal{B}} \leq 4 \lambda \| \tilde{\Lambda} \|_{\mathcal{B}} + 2 \lambda (\| \Delta \|_{\mathcal{B}} + \| \Lambda \|_{\mathcal{B}}). \]

If \( \lambda < \frac{1}{4} \), we have that
\[ \| \tilde{\Lambda} \|_{\mathcal{B}} \leq \frac{2 \lambda}{(1 - 4 \lambda)} (\| \Delta \|_{\mathcal{B}} + \| \Lambda \|_{\mathcal{B}}). \] (3.19)

If we can choose \( \lambda < \frac{1}{4} \) so that \( \frac{2 \lambda}{1 - 4 \lambda} < 1 \). Then \( \Phi : \mathcal{C}_1 \times \mathcal{B} \rightarrow \mathcal{C}_1 \times \mathcal{B} \) is a contraction map. By the Banach’s fixed point theorem, we deduce that there exists a unique solution pair \((Y, Z) \in \mathcal{C}_1 \times \mathcal{B} \) on the time interval \([T - \delta T, T]\) restricted to \( \mathcal{C}_1 \times \mathcal{B} \). Therefore what left to be shown is that there exists \( \lambda < \frac{1}{4} \) such that assumption (3.9) holds, and this can be achieved when \( C_1 C_3 < e^{-\frac{4}{1+4}} = e^{-144} \).

We then consider the time interval \([T - 2\delta T, T - \delta T]\) if \( T - 2\delta T > 0 \), and \([0, T - \delta T]\) otherwise, and set terminal value \( \xi \) at time \( T - \delta T \) to be \( Y_{T - \delta T} \) which is the initial value of the solution \( Y \) solved above on the time interval \([T - \delta T, T]\). Then by using the above same method, we get a unique solution pair in \( \mathcal{C}_1 \times \mathcal{B} \) on the time interval \([T - 2\delta T, T - \delta T]\) to the BSDE (3.8). By repeating this procedure backwards and pasting the solutions on all the time intervals together we get a solution pair \((Y, Z) \in \mathcal{C}_1 \times \mathcal{B} \) with \( \delta = \frac{1}{4} R \) on the time interval \([0, T]\) to the BSDE (3.8).

If \( \delta \) and \( \lambda \) satisfy the condition that \( \sqrt{\frac{1}{0}} \lambda < \frac{1}{4} \), which may be achievable when \( T \) and \( C_1 C_3 \) are small enough. Then the solution pair \((Y, Z) \in \mathcal{C}_1 \times \mathcal{B} \) on the time interval \([0, T]\) to the BSDE (3.8) is unique. This uniqueness of the solution can be proved as follows. Suppose there exist two pairs of solutions \((Y^1, Z^1), (Y^2, Z^2) \in \mathcal{C}_1 \times \mathcal{B} \) on the time interval \([0, T]\) to the BSDE (3.8). By setting
\[ \delta = Y^1 - Y^2, \Lambda = Z^1 - Z^2, \]
we get that \((\Delta, \Lambda) \in \mathcal{C}_1 \times \mathcal{B} \) with \( \delta T = 0 \). Then on the time interval \([T - \delta T, T]\) by repeating the procedure starting from equation (3.12), we obtain an inequality which is similar to inequality (3.19) as follows:
\[ \| \Delta \|_{\mathcal{B}} + \| \Lambda \|_{\mathcal{B}} \leq 2 \sqrt{\frac{1}{0}} \lambda (\| \Delta \|_{\mathcal{B}} + \| \Lambda \|_{\mathcal{B}}). \]

Since \( \lambda < \frac{1}{6} \), so that \( \frac{2 \sqrt{\lambda}}{(1 - 4 \sqrt{\lambda})} < 1 \), we deduce that \( \| \Delta \|_{\mathcal{B}} = 0 \) and \( \| \Lambda \|_{\mathcal{B}} = 0 \). Thus \((Y^1, Z^1)\) equals \((Y^2, Z^2)\) on the time interval \([T - \delta T, T]\), in particular \(Y^1_{T - \delta T} = Y^2_{T - \delta T}\). We then consider the time interval \([T - 2\delta T, T - \delta T]\) if \( T - 2\delta T > 0 \), and \([0, T - \delta T]\) otherwise, and terminal values at time \( T - \delta T \) are \( Y^1_{T - \delta T} \) and \( Y^2_{T - \delta T} \) respectively for the two solutions. Again by using the same procedure starting from equation (3.12), we deduce that \((Y^1, Z^1)\) equals \((Y^2, Z^2)\) on the time interval \([T - 2\delta T, T - \delta T]\) as well. Thus by repeating this procedure backwards, we conclude that \((Y^1, Z^1)\) equals \((Y^2, Z^2)\) on the time interval \([0, T]\). \( \square \)

Remark 7. Theorem 6 says that, given parameters \( C_2, C_3, C_4 \), if the bound \( C_1 \) of the terminal value is small enough then there exists a solution pair \((Y, Z) \in \mathcal{C}_1 \times \mathcal{B} \) on the time interval \([0, T]\) to the BSDE (3.8).
Theorem 8. Suppose \( \frac{dQ}{dP} |_{\mathcal{F}_t} = \mathcal{E}(N)_t \), where \( N \in \text{BMO}(\mathbb{P}) \), and \( f \) and \( \xi \) satisfy the same above assumptions \((\mathcal{A}1)\) and \((\mathcal{A}2)\) with \( C_1C_3 < e^{-144} \), then the BSDE
\[
\begin{align*}
    dY_t &= Z_t f(Y_t, Z_t) \, dt + Z_t dW^Q_t \\
    Y_T &= \xi
\end{align*}
\] 
(3.20)
where \( W^Q \) is a standard \( k \)-dimensional Brownian motion under \( Q \) defined as
\[dW^Q_t = dW_t - d(W, N)_t,\]
has a solution pair \((Y, Z) \in \mathcal{F}^{\infty}(\mathbb{R}^d) \times \mathcal{B}(\mathbb{R}^{d \times k})(\mathbb{P})\) on \([0, T]\).

Proof. It can be seen clearly that the above proof also works under probability measure \( Q \) with \( W^Q \) instead of probability measure \( P \) with \( W \). Thus there exists a solution pair \((Y, Z) \in \mathcal{F}^{\infty}(\mathbb{R}^d) \times \mathcal{B}(\mathbb{R}^{d \times k})(Q)\) on the time interval \([0, T]\) to the BSDE (3.20) by Theorem 6. It means that \( \int Z_t dW^Q_t \in \text{BMO}(Q) \), which implies that \( \int Z_t dW_t \in \text{BMO}(P) \) by Corollary 2. Thus we deduce that \( Z \in \mathcal{B}(\mathbb{R}^{d \times k})(\mathbb{P}) \) and \((Y, Z) \in \mathcal{F}^{\infty}(\mathbb{R}^d) \times \mathcal{B}(\mathbb{R}^{d \times k})(\mathbb{P})\). \qed

4 One dimensional case with bounded terminal values

As an application of the results in the previous section, we prove the existence of solution for the one dimensional case of our BSDE with bounded terminal values by pasting space together and this approach is also used in Tevzadze [12].

We consider the one dimensional case i.e. \( d = 1 \).

Lemma 9. Given \( \dot{Z} \in \mathcal{B}(\mathbb{R}^{1 \times k})(\mathbb{P}) \), suppose \( f \) and \( \xi \) satisfy the above assumptions \((\mathcal{A}1)\) and \((\mathcal{A}2)\) with \( C_1C_3 < e^{-144} \), then the BSDE
\[
\begin{align*}
    dY_t &= \left[ f(\dot{Z} + Z_t) - f(\dot{Z}_t) \right] dt + Z_t dW_t \\
    Y_T &= \xi
\end{align*}
\] 
(4.1)
has a solution pair \((Y, Z) \in \mathcal{F}^{\infty}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}^{1 \times k})(\mathbb{P})\) on \([0, T]\).

Proof. We rearrange the terms to get
\[
    dY_t = Z_t f(\dot{Z}_t + Z_t) \, dt + \dot{Z}_t \left[ f(\dot{Z} + Z_t) - f(\dot{Z}_t) \right] dt + Z_t dW_t
\]
\[
= Z_t \left[ f(\dot{Z} + Z_t) - f(\dot{Z}_t) \right] dt + \dot{Z}_t f(\dot{Z}_t) dt + \dot{Z}_t \left[ f(\dot{Z} + Z_t) - f(\dot{Z}_t) \right] dt + Z_t dW_t.
\]
For all \( z \in \mathbb{R}^{1 \times k} \) we define \( g \) as
\[
    g(z) = f(\dot{Z} + z) - f(\dot{Z}),
\]
then it can be verified directly that \( g \) satisfies the above assumption \((\mathcal{A}2)\) with the same parameter \( C_3 \) as that of \( f \) and we have
\[
dY_t = Z_t g(Z_t) \, dt + Z_t f(\dot{Z}_t) \, dt + \dot{Z}_t \left[ f(\dot{Z} + Z_t) - f(\dot{Z}_t) \right] dt + Z_t dW_t.
\]
By a similar argument used in Hu and Tang [8], for \( i = 1, 2, \cdots, k \), we can define a vector process \( \beta(i) \) taking values in \( \mathbb{R}^{k \times 1} \) with \( \| \beta(i) \|^2 \leq kC_3^2 \) such that
\[
f_i(\dot{Z} + Z) - f_i(\dot{Z}_t) = Z\beta(i),
\]
where \( f_i \) is the \( i \)-th component of \( f \). Then we may define a process \( \beta \) taking values in \( \mathbb{R}^{k \times k} \) where the \( i \)-th column of \( \beta \) is \( \beta(i) \) and we deduce that \( \| \beta \|^2 \leq k^2C_3^2 \). It implies that
\[
[ f(\dot{Z} + Z) - f(\dot{Z}_t) ] = (Z\beta)^T.
\] 
(4.2)
Thus we get
\[ dY_t = Z_t g(Z_t) dt + Z_t f(\hat{Z}_t) dt + \hat{Z}_t (Z_t \beta_t)^T dt + Z_t dW_t, \]
which can be written as
\[
\begin{align*}
  dY_t &= Z_t g(Z_t) dt + Z_t f(\hat{Z}_t) dt + Z_t \beta_t (\hat{Z}_t)^T dt + Z_t dW_t \\
  &= Z_t g(Z_t) dt + Z_t \left( f(\hat{Z}_t) + \beta_t (\hat{Z}_t)^T \right) dt + dW_t \\
  &= Z_t g(Z_t) dt + Z_t dW_t^Q
\end{align*}
\]
where the probability measure \( Q \) is defined by
\[
\frac{dQ}{dP} = \mathcal{E} \left( - \int \left[ f(\hat{Z}_s)^T + \hat{Z}_s \beta_s^T \right] dW_s \right) \tag{4.3}
\]
and it can be verified that \(- \int \left[ f(\hat{Z}_s)^T + \hat{Z}_s \beta_s^T \right] dW_s \in \text{BMO}(\mathbb{P}) \) as \( \hat{Z} \in \mathcal{B}(\mathbb{R}^{1 \times k})(\mathbb{P}) \) and \( \beta \) is bounded. \( W^Q \) is defined as
\[
W_t^Q = dW_t + \left[ f(\hat{Z}_t) + \beta_t (\hat{Z}_t)^T \right] dt, \tag{4.4}
\]
which is a standard \( k \)-dimensional Brownian motion under \( Q \). Since \( C_1 C_3 < e^{-144} \) then by Theorem 8 the BSDE (4.1) has a solution pair \((Y, Z) \in \mathcal{S}^\infty(R) \times \mathcal{B}(\mathbb{R}^{1 \times k})(\mathbb{P}) \) on \([0, T]\).

**Theorem 10.** When \( d=1 \), suppose \( f \) and \( \xi \) satisfy the above assumptions (\( \mathcal{A} \) and (\( \mathcal{B} \)), then the BSDE
\[
\begin{align*}
  \begin{cases} 
    dY_t &= Z_t f(Z_t) dt + Z_t dW_t \\
    Y_T &= \xi
  \end{cases} \tag{4.5}
\end{align*}
\]
has a unique solution pair \((Y, Z) \in \mathcal{S}^\infty(R) \times \mathcal{B}(\mathbb{R}^{1 \times k}) \) on \([0, T]\).

**Proof.** Given any \( C_1 > 0 \), we can find \( n \) large enough such that \( C_1 C_3 < e^{-144} \). By Theorem 6 the following BSDE
\[
\begin{align*}
  \begin{cases} 
    dY^1_t &= Z^1_t f(Z^1_t) dt + Z^1_t dW_t \\
    Y^1_T &= \xi/\sqrt{n}
  \end{cases} \tag{4.6}
\end{align*}
\]
has a solution pair \((Y^1, Z^1) \in \mathcal{S}^\infty(R) \times \mathcal{B}(\mathbb{R}^{1 \times k}) \) on \([0, T]\). Then by using induction we can show that for \( m = 2, \cdots, n \) the following BSDE
\[
\begin{align*}
  \begin{cases} 
    dY^m_t &= \left( \sum_{j=1}^{m-1} Z^j_t + Z^m_t \right) f \left( \sum_{j=1}^{m-1} Z^j_t + Z^m_t \right) - \left( \sum_{j=1}^{m-1} Z^j_t \right) f \left( \sum_{j=1}^{m-1} Z^j_t \right) dt + Z^m_t dW_t \\
    Y^m_T &= \xi/n
  \end{cases} \tag{4.7}
\end{align*}
\]
has a solution pair \((Y^m, Z^m) \in \mathcal{S}^\infty(R) \times \mathcal{B}(\mathbb{R}^{1 \times k}) \) on \([0, T]\) by Lemma 9. By adding \( Y^1 \) and \( Z^1 \) together, i.e. letting
\[
Z = \sum_{j=1}^{n} Z^j, Y = \sum_{j=1}^{n} Y^j,
\]
we get that
\[
\begin{align*}
  \begin{cases} 
    dY_t &= Z_t f(Z_t) dt + Z_t dW_t \\
    Y_T &= \xi
  \end{cases} \tag{4.8}
\end{align*}
\]
with \((Y, Z) \in \mathcal{S}^\infty(R) \times \mathcal{B}(\mathbb{R}^{1 \times k}) \) on \([0, T]\).

The uniqueness of the solution can be proved as follows. Suppose there exist two pairs of solutions \((Y, Z), (\tilde{Y}, \tilde{Z}) \in \mathcal{S}^\infty(R) \times \mathcal{B}(\mathbb{R}^{1 \times k}) \) on \([0, T]\) to the BSDE (4.5). By the same argument used in (4.2), we can define a process \( \beta \) taking values in \( \mathbb{R}^{k \times k} \) with \( \|\beta\|^2 \leq k^2 C_3^2 \) such that
\[
\left[ f(Z_t) - f(\hat{Z}_t) \right] = \left[ (Z_t - \hat{Z}_t) \beta \right]^T.
\]

It can be verified that $-\int \left[ Z_t \beta^T_t + f(\hat{\mathbf{Z}}_t)^T \right] dW_t \in \text{BMO}(\mathbb{P})$ as $Z$ and $\hat{Z}$ belong to $\mathcal{B}(R^{1 \times k})$ (\mathbb{P}) and $\beta$ is bounded. We may define probability measure $Q$ by

$$\frac{dQ}{d\mathbb{P}} |_{\mathcal{F}_t} = \mathcal{E} \left( - \int \left[ Z_t \beta^T_t + f(\hat{\mathbf{Z}}_t)^T \right] dW_t \right).$$

(4.9)

$W^Q$ is defined as

$$dW^Q_t = dW_t + \left[ \beta_t (\hat{Z}_t)^T + f(\hat{Z}_t) \right] dt,$$

(4.10)

which is a standard $k$-dimensional Brownian motion under $Q$. Then we have that

$$Y_t - \hat{Y}_t = - \int_t^T \left[ Z_s f(\hat{Z}_s) - Z_s f(\hat{Z}_s) \right] ds - \int_t^T (Z_s - \hat{Z}_s) dW_s$$

$$= - \int_t^T Z_s \left[ f(\hat{Z}_s) - f(Z_s) \right] ds - \int_t^T (Z_s - \hat{Z}_s) f(\hat{Z}_s) ds - \int_t^T (Z_s - \hat{Z}_s) dW_s$$

$$= - \int_t^T Z_s \left[ (Z_s - \hat{Z}_s) \beta_s \right] ds - \int_t^T (Z_s - \hat{Z}_s) f(\hat{Z}_s) ds - \int_t^T (Z_s - \hat{Z}_s) dW_s$$

$$= - \int_t^T (Z_s - \hat{Z}_s) dW^Q_s.$$

Since it can be verified by Corollary 2 that $- \int (Z_s - \hat{Z}_s) dW^Q_s \in \text{BMO}(\mathbb{Q})$, then by taking the conditional expectation with respect to $\mathcal{F}_t$ under $Q$ for $t \in [0, T]$ we get that $Y$ equals $\hat{Y}$. Thus we also have that

$$E^Q_{\mathcal{F}_t} \left( \int_t^T \|Z_s - \hat{Z}_s\|^2 ds \right) = 0,$$

for every $t \in [0, T]$, which implies that $Z$ equals $\hat{Z}$. □

5. A lower triangular quadratic example with bounded terminal values

We consider the case when $d = k$. Let $\mathcal{F}^j$ be the Brownian filtration of $W^j$ which is the $i$th component of a standard $k$-dimensional Brownian motion $W$. Then by considering each $i$ as a one dimensional case and working with respect to $\mathcal{F}^i$ for $i = 1, 2, \cdots, k$, we deduce that the BSDE

$$\begin{aligned}
\begin{cases}
    d\hat{Y}^i_t = Z^i_t f_i(\hat{Z}^i_t) dt + Z^i_t dB^i_t, \\
    \hat{Y}^i_0 = \xi^i - \xi^{i-1}
\end{cases}
\end{aligned}$$

(5.1)

where $f_i$ and $\xi^i$ satisfy the above assumptions (A1) and (A2) with $\xi^0 = 0$ and $\xi^i - \xi^{i-1}$ is $\mathcal{F}^j$ measurable for $i = 1, 2, \cdots, k$, has a solution pair $(\hat{Y}^i, Z^i) \in \mathcal{F}^\infty(R) \times \mathcal{B}(R)$ on $[0, T]$ by Theorem 10. Then $Y^i = \sum_{j=1}^i \hat{Y}^j$ solves the following BSDE:

$$\begin{aligned}
\begin{cases}
    dY^i_t = \sum_{j=1}^i Z^j_t f_j(\hat{Z}^j_t) dt + \sum_{j=1}^i Z^j_t dB^j_t, \\
    Y^i_0 = \xi^i
\end{cases}
\end{aligned}$$

(5.2)

for $i = 1, 2, \cdots, k$. Then let $Y^i$ to be the $i$th component of $Y$, $\hat{Y}^i$ to be the $i$th component of $\hat{Z}$ and $f_i(z_{i,j})$ to be the $i$th component of $f(z)$ for all $z \in R^{k \times k}$, we have by defining the lower triangular $Z$ as follows:

$$Z = \begin{bmatrix} Z^1 & Z^2 & \cdots \\ Z^1 & Z^2 & \cdots \\ \vdots & \vdots & \ddots \\ Z^1 & Z^2 & \cdots & Z^k \end{bmatrix}$$

that $(Y, Z) \in \mathcal{F}^\infty(R^k) \times \mathcal{B}(R^{k \times k})$ on $[0, T]$ is a solution pair to the following quadratic BSDE

$$\begin{aligned}
\begin{cases}
    dY_t = Z_t f(\hat{Z}_t) dt + Z_t dB_t, \\
    Y_T = \xi
\end{cases}
\end{aligned}$$

(5.3)
Remark 11. $\xi$ in (5.3) can be any bounded terminal value satisfying the condition that $\xi^i - \xi^{i-1}$ is $\mathcal{F}_T^i$ measurable where $\xi^i$ is the $i$th component of $\xi$ with $\xi^0 = 0$ for $i = 1, 2, \cdots, k$.

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