Correlation functions of charged free boson and fermion systems

Naihuan Jing$^{1,2}$, Zhijun Li$^{1,4}$ and Tommy Wuxing Cai$^3$

1 School of Mathematics, South China University of Technology, Guangzhou 510640, People’s Republic of China
2 Department of Mathematics, North Carolina State University, Raleigh, NC 27695, United States of America
3 Department of Mathematics, University of Manitoba, Winnipeg, MB R3T 2N2, Canada
E-mail: jing@ncsu.edu, zhijun1010@163.com and cait@myumanitoba.ca

Received 1 March 2020
Accepted for publication 15 June 2020
Published 3 August 2020

Abstract. Using the idea of the quantum inverse scattering method, we introduce the operators $B(x), C(x)$ and $\tilde{B}(x), \tilde{C}(x)$ corresponding to the off-diagonal entries of the monodromy matrix $T$ for the phase model and $i$-boson model in terms of $bc$ fermions and neutral fermions respectively, thus giving alternative treatment of the KP and BKP hierarchies. We also introduce analogous operators $B^*(x)$ and $C^*(x)$ for the charged free boson system and show that they are in complete analogy to those of $bc$ fermionic fields. It is proved that the correlation function $\langle 0|C(x_N)\cdots C(x_1)B(y_1)\cdots B(y_N)|0 \rangle$ in the $bc$ fermionic fields is the inverse of the correlation function $\langle 0|C^*(x_N)\cdots C^*(x_1)B^*(y_1)\cdots B^*(y_N)|0 \rangle$ in the charged free bosons.

Keywords: correlation functions, quantum integrability (Bethe ansatz), algebraic structures of integrable models, bosonisation

$^4$Author to whom any correspondence should be addressed.
1. Introduction

The KP hierarchy is one of the fundamental examples of integrable systems that can be solved by many methods. Besides being most interesting differential equations, the KP hierarchy and the KdV equation are also one of the successful stories in Lie theory and mathematical physics. In a series of papers [3–7], the Kyoto school used infinite dimensional Lie algebras and the boson–fermion correspondence to study the KP and their associated systems. They have shown that, among many things, the KP tau functions are expressed in terms of the Schur functions from algebraic combinatorics (see also [26]). Similarly the tau functions of the BKP hierarchy are determined by Schur’s Q-functions [21, 25, 27, 34].

The quantum inverse scattering method (QISM) [8] is fundamental in studying various exactly solvable physical models such as the XYZ model, six vertex model, eight vertex model, phase model, and lattice model etc. The most important aspect of the QISM is the algebraic Bethe Ansatz, which provides an effective procedure to construct eigenvectors and calculate the eigenvalues for the Hamiltonian.

The main idea of the Bethe Ansatz in solving the phase model relies on two important operators in the off-diagonals of the monodromy matrix. In the simplest situation, the monodromy matrix $T(x)$ is written as

\[ T(x) = \cdots \begin{pmatrix} B(x_{N-1}) & C(x_{N-1}) \\ B(x_{N-2}) & C(x_{N-2}) \\ \vdots & \vdots \\ B(x_1) & C(x_1) \\ B(x_0) & C(x_0) \end{pmatrix} \cdots \]
Correlation functions of charged free boson and fermion systems

\[ T(x) = \begin{pmatrix} A(x) & B(x) \\ C(x) & D(x) \end{pmatrix} \]  

(1.1)

where the operators \( B(x) \) and \( C(x) \) act on the space of states \( \mathcal{V} \) in the physical model. The \( L \)-matrix and therefore the monodromy matrices obey the famous RTT relation on \( \mathcal{V} \otimes \mathcal{V} \):

\[ R(x, y)L_1(x)L_2(y) = L_2(y)L_1(x)R(x, y), \]  

(1.2)

where the \( R \)-matrix satisfies the Yang–Baxter equation:

\[ R(x, y)R(x, z)R(y, z) = R(y, z)R(x, z)R(x, y). \]  

(1.3)

Recently Bogoliubov [1], Foda, Wheeler and Zuparic [10–12, 33], Tsilevich [30] (see also [2]) have established the relationship of the phase model and \( \imath \)-boson model with the KP and BKP hierarchies by QISM and constructed certain operators \( \mathbb{B}(x), C(x) \) using the phase algebras and \( \imath \)-boson algebras. In the KP case with the \( R \)-matrix given in (B.5), they consider the \( L \)-matrix

\[ L_m(x) = \begin{pmatrix} x^{-\frac{1}{2}} & \psi_m^\dagger \\ \psi_m & x^{\frac{1}{2}} \end{pmatrix} \]

where \( \psi_i, \psi_i^\dagger \) obey the Heisenberg relation \([\psi_i, \psi_j^\dagger] = \delta_{i,j}\pi_i\). Then the monodromy matrix

\[ T(x) = L_M(x) \cdots L_1(x)L_0(x) = \begin{pmatrix} A(x) & B(x) \\ C(x) & D(x) \end{pmatrix} \]

also satisfies the RTT equation (1.2) and the operators

\[ \mathbb{B}(x) = x^{\frac{M}{2}}B(x), \quad \text{and} \quad C(x) = x^{\frac{M}{2}}C(\frac{1}{x}) \]  

(1.4)

satisfy four distinguished properties [1, 10, 33] that can be used to characterize the KP system and compute the correlation functions. The exact meaning of the four properties will be presented in the sequel (see (1.6)).

On one hand, the KP and its associate BKP can be formulated in the context of the boson–fermion correspondence (vertex operators realization [17, 18, 20]) as special realizations of the basic representation of the affine Lie algebra [14], where the KP and BKP correspond to the so-called bosonic/fermionic realization of the Lie algebra \( \hat{gl}_\infty \) and \( o_\infty \) [6, 20, 21].

On the other hand, the boson–boson correspondence gives rise to interesting representations of the \( W_{1+\infty} \)-algebra [13, 32] including the Virasoro algebra just as the boson–fermion correspondence. A bosonic KP hierarchy was proposed in [23], and in [19] we have shown there do not exist polynomial tau functions in the bosonic system and computed some nontrivial tau functions in the completion of the Fock space of the charged free bosons, which poses a question how to understand this new phenomenon. The goal of this paper is to formulate the charged boson system using the idea of QISM, specifically focusing on the operators that can be viewed as the off-diagonal entries of...
the monodromy matrix in the QISM and providing different perspective to understand the dynamic procedure.

In order to do this, we propose a QISM-like approach to the charged fermions by generalizing the phase model in recognition that the atomic quadratic operators satisfy similar bosonic relations. We define two new operators (see (2.3) and (2.4))

\[
\mathbf{B}(x) = \prod_{i \in \{-\infty, M\}} \exp \left( x b(-i - 1)c(i - 2) \right), \quad \mathbf{C}(x) = \prod_{i \in \{-\infty, M\}} \exp \left( x b(-i + 2)c(i - 1) \right),
\]

where the fermionic operators obey the commutation relation

\[
\{b(i), c(j)\} = \delta_{i,-j}, \quad \{b(i), b(j)\} = \{c(i), c(j)\} = 0,
\]

in \(bc\) fermionic fields for any positive integer \(M\). We show that these two operators satisfy similar commutation relations for the monodromy matrix using the Baker–Campbell–Hausdorff (BCH) formula. Namely, we shall prove that the properties of the \(bc\) fermionic fields enjoy the following properties:

(a) \([\mathbf{B}(x), \mathbf{B}(y)] = [\mathbf{C}(x), \mathbf{C}(y)] = 0\), and \(\mathbf{B}(x_1) \cdots \mathbf{B}(x_N)|0\rangle\) is a symmetric function in \(x_1, \ldots, x_N\);

(b) The limit of the operators are given by

\[
\lim_{M \to \infty} \mathbf{B}(x) = \exp \left( \sum_{n=1}^{\infty} \frac{x^n}{n} H_{-n} \right), \quad \lim_{M \to \infty} \mathbf{C}(x) = \exp \left( \sum_{n=1}^{\infty} \frac{x^n}{n} H_n \right),
\]

where \(H_{-n}, H_n\) satisfy the Heisenberg relation \([H_m, H_n] = m \delta_{m,-n}\) (see (2.16));

(c) The Bethe eigenvector \(\mathbf{B}(x_1) \cdots \mathbf{B}(x_N)|0\rangle\) is a KP tau function\(^5\);

(d) \(\mathbf{B}(x)|\nu\rangle = \sum_{\nu < \mu \in \left[\nu_1 - 1, M\right]} x^{\nu_1 - |\nu_1|}\mu|\mu\rangle\) (see (2.25)).

Then the correlation function \(\langle 0| \mathbf{C}(x_N) \cdots \mathbf{C}(x_1) \mathbf{B}(y_1) \cdots \mathbf{B}(y_N)|0\rangle\) can be computed (see \([10, 33]\)).

Our second main result is to formulate a QISM-like approach for the charged free bosonic system (or the bosonic \(\beta\gamma\) system). We will introduce two important operators (see (3.3) and (3.4))

\[
\mathbf{B}^*(x) = \prod_{i \in \{-\infty, M\}} \exp \left( x \varphi_{i-\varphi_{i-1}}^* \right), \quad \mathbf{C}^*(x) = \prod_{i \in \{-\infty, M\}} \exp \left( x \varphi_{i+1-\varphi_{i}}^* \right)
\]

in terms of charged free bosons. Based on the BCH formula, we will show that

1. \([\mathbf{B}^*(x), \mathbf{B}^*(y)] = [\mathbf{C}^*(x), \mathbf{C}^*(y)] = 0\), so \(\mathbf{B}^*(x_1) \cdots \mathbf{B}^*(x_N)|0\rangle\) is a symmetric function;

2. For \(M \to \infty\),

\[
\mathbf{B}^*(x) = \exp \left( \sum_{n=1}^{\infty} \frac{(-1)^n+1}{n} x^n h_{-n} \right), \quad \mathbf{C}^*(x) = \exp \left( \sum_{n=1}^{\infty} \frac{(-1)^n+1}{n} x^n h_n \right),
\]

\(^5\)In the paper the tau function is an algebraic tau function without bosonization.
where the $h_n$ satisfies the Heisenberg relation $[h_m, h_n] = -m\delta_{m,-n}$ (see (3.7)).

3. The vector $B^*(x_1) \cdots B^*(x_N)|0\rangle$ is a bosonic KP tau function and expressible in Schur functions as well;

4. For any positive integer $M$, the correlation function $\langle 0|C^*(x_N) \cdots C^*(x_1)B^*(y_1) \cdots B^*(y_N)|0\rangle$ is equal to the inverse of $\langle 0|C(x_N) \cdots C(x_1)B(y_1) \cdots B(y_N)|0\rangle$.

While fulfilling the above two goals, we also obtain and prove several fundamental combinatorial identities such as the Cauchy identity as well as combinatorial formulae for the Schur symmetric polynomials and Schur’s $Q$-functions, which might offer new insight on these symmetric functions.

The paper is organized as follows. In section 2 (with appendix B), we formulate an alternative QISM approach to the charged free fermions by introducing two operators $B(x)$ and $C(x)$ and deriving their important properties. In particular, we give two commutative diagrams with the operators $B(x)$ and $C(x)$ considered in [1, 10, 33], which then gives a new method to compute the correlation functions. In section 3, we introduce the bosonic analog $B^*(x)$ and $C^*(x)$ in terms of charged free bosons, and use them to compute correlation functions $\langle 0|C^*(x_N) \cdots C^*(x_1)B^*(y_1) \cdots B^*(y_N)|0\rangle$ and relate them to their fermionic counterparts. In section 4 (with appendix C), we define two operators $\tilde{B}(x)$ and $\tilde{C}(x)$ from neutral fermions and establish the connection with $B(x)$ and $C(x)$ as well as computing the correlation functions.

2. $B(x)$ and $C(x)$ of $bc$ fermionic fields and correlation functions

In this section we introduce operators $B(x)$ and $C(x)$ of the $bc$ fermionic fields and obtain their relations based on the Baker–Campbell–Hausdorff (BCH) formula. They correspond to the off-diagonal operators of the monodromy matrix in the phase model (see appendix B).

2.1. Operators $B(x)$ and $C(x)$

Let the $bc$ fermionic fields be

$$b(z) = \sum_{i \in \mathbb{Z}} b(i)z^{-i}, \quad c(z) = \sum_{i \in \mathbb{Z}} c(i)z^{-i-1}$$

with the commutation relations

$$\{b(i), c(j)\} = \delta_{i,-j}, \quad \{b(i), b(j)\} = \{c(i), c(j)\} = 0,$$

where $\{A, B\} = AB + BA$. In particular, $b^2(i) = c^2(i) = 0$ for all $i \in \mathbb{Z}$.

For any fixed $M \in \mathbb{N}$ we define two operators $B(x)$ and $C(x)$ in terms of the $bc$ fermionic fields:

$$B(x) = \exp(xb(-M + 1)c(M - 2)) \exp(xb(-M + 2)c(M - 3)) \cdots$$

$$= \prod_{i \in (-\infty, M]} \exp(xb(-i + 1)c(i - 2)),$$

(2.3)
\begin{equation}
\mathbf{C}(x) = \cdots \exp \left( xb(-M+3)c(M-2) \right) \exp \left( xb(-M+2)c(M-1) \right)
\end{equation}

where the product \( \prod_{i \in (-\infty,M]} \exp \left( xb(-i+2)c(i-1) \right) \) runs from left to right (resp. right to left) as \( i \) goes down from \( M \) to \( a \). Here we always use \( a < M \) and if \( a = -\infty \), \( [a, M] = (-\infty, M] \).

**Remark 2.1.** Though the operator \( \mathbf{B}(x) \) (resp. \( \mathbf{C}(x) \)) involves an infinite sum, its action on our concerned space \( \mathcal{F}^{(0)} \) (resp. \( \mathcal{F}^{*,(0)} \)) is finite (see section 2.2).

**Proposition 2.1.** For any fixed \( M \), one has that

\begin{equation}
\mathbf{B}(x) = \exp \left( \sum_{n=1}^{\infty} \frac{1}{n} x^n \wedge_n \right), \quad \mathbf{C}(x) = \exp \left( \sum_{n=1}^{\infty} \frac{1}{n} x^n \wedge^*_n \right),
\end{equation}

where \( \wedge_n = \sum_{i=-\infty}^{M-1} b(-i)c(i-n) \), \( \wedge^*_n = \sum_{i=-\infty}^{M-1} b(-i+n)c(i) \).

**Proof.** We divide the proof into two steps.

**Step 1.** Let \( X \) and \( Y \) be operators on a Hilbert space. The Baker–Campbell–Hausdorff (BCH) formula [15, pp.162–173] [29] says that \( \exp(X)\exp(Y) = \exp(Z) \), where \( Z \) is a formal series in iterated commutators of \( X \) and \( Y \) with rational coefficients. The first few terms are

\[ Z = X + Y + \frac{1}{2} [X, Y] + \frac{1}{12} [X, [X, Y]] + \frac{1}{12} [[X, Y], Y] + \cdots \]

Write \( [X^{(n)}, Y] \) as \( [X, [X^{(n-1)}, Y]] \) inductively, thus \( [X^{(3)}, Y] = [X, [X, [X, Y]]] \). Similarly, we denote \( [X, Y^{(n)}] = [[X, Y^{(n-1)}], Y], [X, Y^{(n)}]X = [[X, Y^{(n)}], X] \). Let \( C_n \) (resp. \( D_n \)) be the coefficients of \( [X^{(n)}, Y] \) (resp. \( [X, Y^{(n)}] \)) in \( Z \), it is known that

\begin{equation}
C_n = D_n = \frac{(-1)^n}{n!} B_n,
\end{equation}

where \( B_n \) are the Bernoulli numbers [22] defined by

\begin{equation}
\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}, \quad |z| < 2\pi.
\end{equation}

Let \( f(x) = \sum_{n \geq 1} \frac{1}{n} x^n = -\ln(1-x) \) and consider the Taylor expansion of \( (f(x))^s(s \geq 1) \)

\begin{equation}
(f(x))^s = \sum_{t \in \mathbb{Z}_+} F^s_t x^t,
\end{equation}

then \( F^s_t = 0 \) for \( t < s \) and \( F^s_s = 1 \). We have the following technical lemma.
Lemma 2.1. The coefficients $C_j$ and $F^j_n$ satisfy the following identities:

$$\sum_{j=1}^{n} C_j F^j_n = \frac{1}{n+1}, \quad \sum_{s=1}^{l} \frac{F^s_s}{s!} = 1. \quad (2.9)$$

Proof. Setting $z = f(x) = -\ln(1-x)$, we have

$$\sum_{j \geq 1} C_j z^j = \frac{z}{1 - e^{-z}} - 1 = -\frac{\ln(1-x)}{x} - 1 = \sum_{n \geq 1} \frac{x^n}{n+1}$$

by combining (2.6) and (2.7). And by (2.8),

$$\sum_{j \geq 1} C_j z^j = \sum_{j \geq 1} C_j \sum_{n \geq j} F^j_n x^n = \sum_{n \geq 1} x^n \sum_{j=1}^{n} C_j F^j_n. \quad (2.11)$$

Comparing (2.10) and (2.11), we get the first identity. The second one can be checked similarly. \(\square\)

Step 2. We claim that for $m \leq M - 1$,

$$\prod_{i \in [m,M-1]} \exp(xb(-i)c(i-1)) = \exp\left(\sum_{n=1}^{M-m} \frac{x^n}{n} \sum_{j=n+m-1}^{M-1} b(-j)c(j-n)\right). \quad (2.12)$$

We use induction on $m$. First, for $m = M - 2$, set $X = xb(-M+1)c(M-2), Y = xb(-M+2)c(M-3)$. As $[X, Y]$ commutes with $X$ and $Y$, so

$$Z = X + Y + \frac{1}{2} [X, Y] = \sum_{n=1}^{2} \frac{x^n}{n} \sum_{j=M-2}^{M-1} b(-j)c(j-n).$$

Assume (2.12) holds for $m = k + 1$, setting $X = \sum_{n=1}^{M-k-1} \frac{x^n}{n} \sum_{j=n+k}^{M-1} b(-j)c(j-n), Y = xb(-k)c(k-1)$, then the only nontrivial iterated commutators of $X$ and $Y$ in $Z$ are

$$[X^{(s)}, Y] = \sum_{j=0}^{M-1-k-s} F^s_{s+j} x^{s+j+1} b(-k-s-j)c(k-1), \quad 1 \leq s \leq M - 1 - k, \quad (2.13)$$

so

$$Z = X + Y + \sum_{s=1}^{M-1-k} C_s \sum_{j=0}^{M-1-k-s} F^s_{s+j} x^{s+j+1} b(-k-s-j)c(k-1)$$

$$= X + Y + \sum_{t=1}^{M-1-k} \sum_{s=1}^{t} C_s F^s_t x^{t+1} b(-k-t)c(k-1)$$

https://doi.org/10.1088/1742-5468/aba0aa
where we have used lemma 2.1 (also recalled X and Y). Therefore (2.12) is true. The proposition follows by taking \( m \to -\infty \) in (2.12).

**Proposition 2.2.** For fixed \( M \), one has that

\[
B(x)B(y) = B(y)B(x), \quad C(x)C(y) = C(y)C(x).
\]  

(2.14)

**Proof.** Since

\[
\left[ \wedge_m, \wedge_n \right] = \sum_{-\infty}^{M-1} b(-i)c(i-m-n) - \sum_{-\infty}^{M-1} b(-i)c(i-m-n) = \wedge_{m+n} - \wedge_{m+n} = 0,
\]

(2.15)

we have \( \sum_{n=1}^{\infty} \frac{1}{n} x^n \wedge_n, \sum_{n=1}^{\infty} \frac{1}{n} y^n \wedge_n \) = 0, then the commutation relations hold. \( \square \)

**Proposition 2.3.** Setting \( \lim_{M \to \infty} \wedge_n = H_n, \quad \lim_{M \to \infty} \wedge_n^* = H^*_n \), then we have that

\[
[H_m, H_n] = m\delta_{m-n}.
\]

(2.16)

Thus we have the following identities.

**Proposition 2.4.** The limit of the operators satisfy that

\[
\lim_{M \to \infty} B(x) = \exp \left( \sum_{n=1}^{\infty} \frac{x^n}{n} H_n \right), \quad \lim_{M \to \infty} C(x) = \exp \left( \sum_{n=1}^{\infty} \frac{x^n}{n} H_n \right),
\]

(2.17)

\[
\lim_{M \to \infty} C(x)B(y) = \frac{1}{1-xy} \lim_{M \to \infty} B(y)C(x).
\]

(2.18)

**2.2. Action of \( B(x) \) on \( F^{(0)} \) and \( C(x) \) on \( F^*(0) \)**

The Fock space \( F \) is spanned by negative (resp. non-positive) modes of \( c(z) \) (resp. \( b(z) \)) acting on the vacuum vector \( |0\rangle \) satisfying the relations

\[
b(m+1)|0\rangle = 0, \quad c(m)|0\rangle = 0, \quad m \geq 0.
\]

(2.19)

Then \( F \) is spanned by \( b(-m_1) \cdots b(-m_s)c(-n_1) \cdots c(-n_t)|0\rangle \), and decomposes as follows.

\[
F = \bigoplus_{i \in \mathbb{Z}} F^{(i)},
\]

where the subspace \( F^{(i)} \) has a basis consisting of monomials \( b(-m_1) \cdots b(-m_s)c(-n_1) \cdots c(-n_t)|0\rangle \), \( m_1 > \cdots > m_s \geq 0 \) and \( n_1 > \cdots > n_t > 0 \) such that \( s-t = i \). Let us focus on the subspace \( F^{(0)} \).

https://doi.org/10.1088/1742-5468/aba0aa

J. Stat. Mech. (2020) 083101
The dual Fock space $\mathcal{F}^*$ is generated by the right action of $b(n), c(n)$ on the dual vacuum vector $\langle 0 \rangle$ satisfying the relations

$$\langle 0 \rangle|b(i + 1) = 0, \quad \langle 0 \rangle|c(i) = 0, \quad i < 0. \quad (2.20)$$

We are only interested in the subspace $\mathcal{F}^{(0)}$ with a basis $\langle 0 \rangle|b(m_1) \cdots b(m_l)$ $c(n_1) \cdots c(n_k)$, $m_1 > \cdots > m_k > 0, n_1 > \cdots > n_k \geq 0$.

A vector $\tau \in \mathcal{F}$ is called a KP tau function [20, 33] if $\tau$ obeys

$$\text{Res}_z b(z) \otimes c(z)(\tau \otimes \tau) = 0, \quad (2.21)$$

where $\text{Res}_z f(z)$ denotes the coefficient of $z^{-1}$ in $f(z)$.

By the vacuum condition, $\text{Res}_z b(z) \otimes c(z) (\langle 0 \rangle \otimes |0\rangle) = 0$. Further, it can be shown that [19]

$$[\text{Res}_z b(z) \otimes c(z), B(x) \otimes B(x)] = 0,$$

thus $B(x_1)B(x_2) \cdots B(x_N)|0\rangle$ is a KP tau function, and can be called a Bethe eigenvector (see [33]).

A partition $\mu = (\mu_1, \ldots, \mu_l)$ is a weakly decreasing non-negative integers. Its weight is $|\mu| = \sum_{j=1}^{l} \mu_j$ and the length is $l(\mu) = l$. Pick a rectangle $[N, M]$ containing the Young diagram of $\mu$, i.e., $\mu_i \leq M, l(\mu) \leq N$. For any partition $\mu = (\mu_1 \ldots, \mu_l)$, we define

$$|\mu| = |\mu_1, \ldots, \mu_l| = b(-m_1)b(-m_2) \cdots b(-m_l)c(-l)c(-l+1) \cdots c(-1)|0\rangle, \quad (2.22)$$

$$\langle \mu| = \langle \mu_1, \ldots, \mu_l| = \langle 0\rangle b(1)b(2) \cdots b(l)c(m_1)c(m_{l-1}) \cdots c(m_1), \quad (2.23)$$

where $m_i = \mu_i - i$ for all $1 \leq i \leq l$, so $m_1 > \cdots > m_l > -l$. We also define $\text{deg}(|\mu\rangle) = \text{deg} (\langle \mu|) = |\mu|).

**Definition 2.1.** The partition $\mu = (\mu_1, \ldots, \mu_{l+1})$ is said to interlace the partition $\nu = (\nu_1, \ldots, \nu_l)$, written as $\mu \succ \nu$, if for all $1 \leq i \leq l$

$$\mu_i \geq \nu_i \geq \mu_{i+1}. \quad (2.24)$$

**Theorem 2.1.** For fixed $M \in \mathbb{N}$ and vector $|\nu\rangle = |\nu_1, \ldots, \nu_l\rangle$ such that $M \geq \nu_1$ we have that

$$B(x)|\nu\rangle = \sum_{\nu < \mu \leq |l+1, M\rangle} x^{|\mu| - |\nu|}|\mu\rangle. \quad (2.25)$$

**Proof.** It follows from (2.2) that

$$\exp(xb(-j)c(j - 1)) = 1 + xb(-j)c(j - 1). \quad (2.26)$$

As $b(-k)c(k - 1)|\nu\rangle = 0$ for $k \leq -l - 1$, we have that
where $a_{M+1} < a_{M+1} < \ldots < a_2 < a_1 \leq M - 1$ and $b(-a_i)c(a_i - 1)$ are bundled together to act.

For simplicity, we denote $|l\rangle = c(-l)\sum c(-l+1)\cdots c(1)|0\rangle$ in the following.

On one hand, for $|\lambda\rangle = b(-s_1)\cdots b(-s_l)|-l\rangle$, $\lambda_i = s_i + i$ and $s_1 > \cdots > s_l \geq -l + 1$, we have

$$b(-s_i - 1)c(s_i)(|\lambda\rangle) = |\lambda_1, \ldots, \lambda_{i-1}, \lambda_i + 1, \lambda_{i+1}, \ldots, \lambda_l\rangle, \lambda_{i-1} - \lambda_i > 0,$$

$$b(l)c(-l - 1)(|\lambda\rangle) = |\lambda_1, \ldots, \lambda_l, 1\rangle,$$

$$\deg(b(-s_i - 1)c(s_i)(|\lambda\rangle)) = \deg(b(l)c(-l - 1)(|\lambda\rangle)) = \deg(|\lambda\rangle) + 1.$$  

due to the fact that $|b(-j)c(j - 1), b(-i)| = \delta_{i+1,j}b(-j)$ and $c(j)|0\rangle = 0, j \geq 0$. Note that if $l_{i-1} = s_{i-1} - s_i > 1$, we have

$$\prod_{k \in [0, l_{i-1} - 2]} b(-s_i - k - 1)c(s_i + k)(|\lambda\rangle) = |\lambda_1, \ldots, \lambda_{l-1}, \lambda_i + l_{i-1} - 1, \lambda_{i+1}, \ldots, \lambda_l\rangle,$$

$$\prod_{k \in [0, l_{i-1} - 1]} b(-s_i - k - 1)c(s_i + k)(|\lambda\rangle) = 0.$$

These relations imply that the new element $|\mu\rangle$ generated by $b(-j)c(j - 1)$ satisfies that

$$\mu_i \geq \nu_i \geq \mu_{i+1}, \quad 1 \leq i \leq l.$$  

In other words, for $|\mu\rangle = b(-a_1)c(a_1 - 1)b(-a_2)c(a_2 - 1)\cdots b(-a_s)c(a_s - 1)|\nu\rangle$, we have $\nu < \mu$ and $|\mu| - |\nu| = s$.

On the other hand, given an element $|\mu\rangle$ with $\nu < \mu$, say

$$|\mu\rangle = b(-m_1)b(-m_2)\cdots b(-m_i)b(-m_{i+1})c(-l - 1)c(-l)c(-l + 1)\cdots c(-1)|0\rangle,$$

$$|\nu\rangle = b(-n_1)b(-n_2)\cdots b(-n_i)b(-n_{i+1})c(-l - 1)c(-l + 1)\cdots c(-1)|0\rangle,$$

where $m_i = \mu_i - i, n_i = \nu_i - i, -l + 1 \leq n_i, m_{i+1} = l + 1 \geq -m_{i+1}$, and $m_i \geq n_i > m_{i+1}$.

For $1 \leq i \leq l$, we set

$$E_i = \begin{cases} 1, & m_i = n_i \\ b(-m_i)c(m_i - 1)b(-m_i + 1)c(m_i - 2)\cdots b(-n_i - 1)c(n_i), & m_i > n_i, \end{cases}$$

then

$$E_1E_2\cdots E_{i+1}|\nu\rangle = |\mu\rangle.$$
Note that $E_1 E_2 \cdots E_{l+1}$ is generated by $b(-j)c(j-1), -l \leq j \leq M - 1$. This completes the proof. □

Using the same method, we have the following result.

**Corollary 2.1.** For arbitrary positive integer $M$ and vector $(\nu | = (\nu_1, \ldots, \nu_l)$ we have

$$\langle \nu | C(x) = \sum_{\nu < \mu \subseteq [l+1,M]} x^{\mu|-\nu|} | \mu \rangle. \quad (2.30)$$

Following [24], the complete symmetric function $h_k(x)$ in the variables $x_1, x_2, \cdots$ is given by

$$\sum_{k=0}^{\infty} h_k(x) z^k = \prod_{i=1}^{\infty} \frac{1}{1 - x_i z}.$$

To each partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ we associate the Schur function $s_{\lambda}(x)$ defined by

$$s_{\lambda}(x) = \det (h_{\lambda_{i,j}}(x))_{1 \leq i, j \leq k}.$$

For the rest of the paper, we usually consider the Schur function $s_{\mu} \{ x \}$ in finitely many variables $\{ x \} = \{ x_1, x_2, \ldots, x_N \}$, which is obtained by letting $x_{N+1} = x_{N+2} = \cdots = 0$ in $s_{\lambda}(x)$. The following identities are well-known [24]:

$$s_{\mu} \{ x \} = \sum_{\nu < \mu} s_{\nu} \{ \bar{x} \} x_N^{\mu|-\nu|}, \quad (2.31)$$

$$s_{\mu} \{ x \} = 0, \quad l(\mu) > N. \quad (2.32)$$

where $\{ \bar{x} \} = \{ x \} \setminus \{ x_N \} = \{ x_1, \ldots, x_{N-1} \}$.

Using proposition 2.1 and [33], we have

**Proposition 2.5.** For a fixed positive integer $M$ and $\{ x \} = \{ x_1, \ldots, x_N \}$,

$$B(x_1) \cdots B(x_N) | 0 \rangle = \exp \left( \sum_{1 \leq m \leq M} \left( -1 \right)^{n-1} s_{(m,1^{n-1})} \{ x \} b(-m+1)c(-n) \right) | 0 \rangle$$

$$= \sum_{\mu \subseteq [N,M]} s_{\mu} \{ x \} | \mu \rangle, \quad (2.33)$$

$$\langle 0 | C(x_N) \cdots C(x_1) = \langle 0 | \exp \left( \sum_{1 \leq m \leq M} \left( -1 \right)^{n-1} s_{(m,1^{n-1})} \{ x \} b(n)c(m-1) \right)$$

$$= \sum_{\mu \subseteq [N,M]} s_{\mu} \{ x \} \langle \mu |. \quad (2.34)$$
2.3. Correlation function $\langle 0|C(x_N)\cdots C(x_1)B(y_1)\cdots B(y_N)|0\rangle$

For $|\lambda| \in J^{(0)}$ and $|\mu| \in F^{(0)}$, we define the bilinear pairing $\langle \rangle$ by

$$\langle \lambda|\mu \rangle = \delta_{\lambda,\mu},$$

(2.35)

where it is assumed that $\langle 0|1|0 \rangle = 1$.

**Theorem 2.2.** For a fixed positive integer $M$,

$$\langle 0|C(x_N)\cdots C(x_1)B(y_1)\cdots B(y_N)|0\rangle = \sum_{\mu \leq [N,M]} s_\mu \{x\} s_\mu \{y\}, \tag{2.36}$$

where $\{x\} = \{x_1, \ldots, x_N\}$, $\{y\} = \{y_1, \ldots, y_N\}$, and $\mu$ runs through all Young diagrams inside the rectangle $[N,M]$ of height $N$ and width $M$. In particular, when $M \to \infty$, we get the Cauchy identity

$$\langle 0|C(x_N)\cdots C(x_1)B(y_1)\cdots B(y_N)|0\rangle = \prod_{i,j=1}^{N} \frac{1}{1 - x_i y_j} = \sum_{\mu \leq N} s_\mu \{x\} s_\mu \{y\}. \tag{2.37}$$

**Remark 2.2.** The correlation function (2.36) is also known as an example of hypergeometric tau function in [28] (see also [16]). In fact, (2.15) implies that

$$\langle 0|C(x_N)\cdots C(x_1)B(y_1)\cdots B(y_N)|0\rangle = \langle 0|\exp \left( \sum_{n=1}^{\infty} \sum_{i=1}^{N} \frac{x_i^n}{n} \right) \exp \left( \sum_{n=1}^{\infty} \sum_{i=1}^{N} \frac{y_i^n}{n} \right) |0\rangle,$$

which agrees with [28, equation (3.1.12)] by setting $r(x)$ to unity in the range of arguments from $-\infty$ to $M$ and zero elsewhere. Then one can get the function in [[28], equation (3.1.4)] (see [16, 25]), which can be used to give another proof of (2.36).

**Remark 2.3.** Using (2.18) we get the following

$$\lim_{M,N \to \infty} \langle 0|C(x_N)\cdots C(x_1)B(y_1)\cdots B(y_N)|0\rangle = \prod_{i,j=1}^{N} \frac{1}{1 - x_i y_j} = \sum_{\mu} s_\mu \{x\} s_\mu \{y\}. \tag{2.38}$$

In particular, for $x_i = y_i = z^{i-\frac{1}{2}}$, we have

$$\lim_{M \to \infty, N \to \infty} \langle 0|C(z^{N-\frac{1}{2}})\cdots C(z^{1-\frac{1}{2}})B(z^{1-\frac{1}{2}})\cdots B(z^{N-\frac{1}{2}})|0\rangle = \prod_{i=1}^{\infty} \frac{1}{(1 - z^i)},$$

which is the generating function of plane partitions [24] (see [33]).

For $1 \leq m \leq M$ and variables $\{x\} = \{x_1, \ldots, x_N\}$, $\{y\} = \{y_1, \ldots, y_N\}$, let

$$a_m = \sum_{1 \leq n \leq N} (-1)^{n-1} s_{(m,1^{n-1})} \{x\} b(n), \quad a_m^* = \sum_{1 \leq n \leq N} (-1)^{n-1} s_{(m,1^{n-1})} \{y\} c(-n),$$

https://doi.org/10.1088/1742-5468/aba0aa
then
\[ T_{kt} = \{a_k, a_t^*\} = \sum_{1 \leq n \leq N} s_{(k,1^{n-1})}\{x\} s_{(1,1^{n-1})}\{y\}. \] (2.39)

In view of proposition 2.5 we have that
\[ B(y_1) \cdots B(y_N)|0\rangle = \prod_{m \in [1,M]} (1 + b(-m + 1)a_m^*|0\rangle, \]
\[ \langle 0|C(x_N) \cdots C(x_1)B(y_1) \cdots B(y_N)|0\rangle = (1 + a_m c(m - 1)) \]
Therefore, we get that

**Proposition 2.6.** For any positive integer $M$, let $T = (T_{ij})_{1 \leq i,j \leq M}$, then
\[ \langle 0|C(x_N) \cdots C(x_1)B(y_1) \cdots B(y_N)|0\rangle = \det(I + T) = \sum_{\mu \subseteq [N,M]} s_{\mu}\{x\} s_{\mu}\{y\}. \] (2.40)

**Proof.** The result follows from the following simple calculation. For strict partition $\mu = (\mu_1 > \cdots > \mu_l)$ inside the rectangle $[N, M]$, let $T_\mu = (T_{\mu_i\mu_j})_{l \times l}$, then
\[ \langle 0|a_{\mu_1} c(\mu_1 - 1) \cdots a_{\mu_l} c(\mu_l - 1) b(-\mu_1 + 1)a_{\mu_1}^* \cdots b(-\mu_l + 1)a_{\mu_l}^*|0\rangle = \langle 0|a_{\mu_1} \cdots a_{\mu_l} a_{\mu_1}^* \cdots a_{\mu_l}^*|0\rangle = \det(T_\mu). \]

\[ \square \]

3. Correlation functions of charged free bosons

In this section we define the charged free bosonic system and introduce two new operators $B^*(x)$ and $C^*(x)$. We will see that their correlation functions enjoy similar but distinct properties in view of the previous section.

3.1. Operators $B^*(x)$ and $C^*(x)$

Recall that the charged free bosons [23, 32] are given by
\[ \varphi(z) = \sum_{i \in \mathbb{Z}} \varphi_i z^{-i-1}, \quad \varphi^*(z) = \sum_{i \in \mathbb{Z}} \varphi_i^* z^{-i} \] (3.1)
with the commutation relations
\[ [\varphi_i, \varphi_j^*] = \delta_{i,-j}, \quad [\varphi_i, \varphi_j] = [\varphi_i^*, \varphi_j^*] = 0. \] (3.2)

For a positive integer $M$, we define
Correlation functions of charged free boson and fermion systems

\[ B^*(x) = \exp\left( x\varphi_{-M}\varphi^*_M \right) \prod_{i \in (-\infty, M]} \exp\left( x\varphi_{-i}\varphi^*_{i-1} \right), \]  
\[ C^*(x) = \cdots \exp\left( x\varphi_{-M+2}\varphi^*_M \right) \prod_{i \in (-\infty, M]} \exp\left( x\varphi_{-i+1}\varphi^*_i \right). \]  

Proposition 3.1. For a fixed М ∈ Н, we have that

\[ B^*(x) = \exp\left( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n \bar{\lambda}_n \right), \quad C^*(x) = \exp\left( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n \bar{\lambda}_n^* \right), \]  

where \( \bar{\lambda}_n = \sum_{i=-\infty}^{M} \varphi_{i-n}^* \) and \( \bar{\lambda}_n^* = \sum_{i=-\infty}^{M} \varphi_{i+n}^* \).

Proposition 3.2. For any fixed М, we also have that

\[ B^*(x)B^*(y) = B^*(y)B^*(x), \quad C^*(x)C^*(y) = C^*(y)C^*(x). \]  

Proposition 3.3. Setting \( \lim_{M \to \infty} \bar{\lambda}_n = h_{-n}, \quad \lim_{M \to \infty} \bar{\lambda}_n^* = h_n \), then

\[ \lim_{M \to \infty} B^*(x) = \exp\left( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n h_{-n} \right), \quad \lim_{M \to \infty} C^*(x) = \exp\left( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n h_n \right), \]  

\[ \lim_{M \to \infty} C^*(x)B^*(y) = (1 - xy) \lim_{M \to \infty} B^*(y)C^*(x). \]  

3.2. Actions of \( B^*(x) \) on \( \tilde{\mathcal{M}}^{(0)} \) and \( C^*(x) \) on \( \tilde{\mathcal{M}}^{(0)} \)

Let \( \mathcal{M} \) (resp. \( \mathcal{M}^* \)) be the Fock space of the charged free bosons generated by monomials in the bosons \( \varphi_i, \varphi_i^* \) with their actions on the vacuum vector \( |0\rangle \) (resp. \( \langle 0| \)) defined by

\[ \varphi_i |0\rangle = 0, \quad \varphi_i^* |0\rangle = 0, \quad i \geq 0 \quad \text{(resp.} \ \langle 0| \varphi_i = \langle 0| \varphi_i^* = 0, \quad i < 0). \]  

As in section 2.2, we define the subspace \( \mathcal{M}^{(0)} \) of \( \mathcal{M} \) and subspace \( \mathcal{M}^{* (0)} \) of \( \mathcal{M}^* \) with bases

\[ \text{Basis}(\mathcal{M}^{(0)}) = \left\{ \varphi_{n_1}^* \cdots \varphi_{-1}^* \varphi^*_k \cdots \varphi^*_0 |0\rangle \left| \sum_{i=0}^{k} n_{-i} = \sum_{j=1}^{l} n_{j}, n_{i} \geq 0 \right. \right\}, \]  

https://doi.org/10.1088/1742-5468/aba0aa
Theorem 3.1. For any \( M \) and \( B \) are KP tau functions in the completion \( \mathcal{M} \).

Proof. A special case of the BCH theorem says that if \( \phi \) is a KP tau function, then

\[
\exp(A) \exp(B) = \exp(B + [A, B]) \exp(A).
\]

Using this formula to successively move \( e^{x_i \phi_i - M} \phi_{M-1}, e^{x_i \phi_i - M} \phi_{M-2}, \ldots \) to the right and noting that \( \phi_i^* |0\rangle = 0 (i > 0) \), we have that

\[
\mathcal{B}^*(x_1)|0\rangle = \exp(x_1 \phi_i - M) \phi_{M-1} \exp(x_1 \phi_i - M) \phi_{M-2} \cdots \prod_{i \in [1, M-2]} \exp(x_1 \phi_i \phi_{i-1}^* ) |0\rangle
\]

which simplifies to

\[
\prod_{i \in [1, M-2]} \exp(x_1 \phi_i \phi_{i-1}^* ) |0\rangle
\]
Then by (2.31) and (2.32), we have

\[
= \exp \left( (x_1 \varphi_{-(M-2)} - x_1^2 \varphi_{-(M-1)} + x_1^3 \varphi_{-M}) \varphi_{M}^* \right) \prod_{i \in [1,M-3]} \exp \left( x_1 \varphi_{-i} \varphi_{i-1}^* \right) |0\rangle = \ldots
\]

\[
= \exp \left( \sum_{1 \leq m \leq M} (-1)^{m-1} s_m \{x_1\} \varphi_{m} \varphi_{m}^* \right) |0\rangle,
\]

Assuming (3.12) holds for \( N - 1 \) and \( \bar{x} = \{x_1, \ldots, x_{N-1}\} \), we have that

\[
B^*(x_1) \cdots B^*(x_N)|0\rangle = B^*(x_N)B^*(x_1) \cdots B^*(x_{N-1})|0\rangle
\]

\[
= \prod_{i \in [2-N,M]} \exp \left( x_N \varphi_{-(i+1)} \varphi_{i}^* \right) \exp \left( \sum_{1 \leq m \leq M, 1 \leq n \leq N-1} (-1)^{m-1} s_{m,1^{n-1}} \{\bar{x}\} \varphi_{m} \varphi_{1-n} \right) |0\rangle
\]

\[
= \prod_{i \in [3-N,M]} \exp \left( x_N \varphi_{-(i+1)} \varphi_{i}^* \right) \exp \left( \sum_{1 \leq m \leq M, 1 \leq n \leq N-2} (-1)^{m-1} s_{m,1^{n-1}} \{\bar{x}\} \varphi_{m} \varphi_{1-n} \right)
\times \exp \left( x_N \varphi_{2-N} \varphi_{1-N}^* \right) \exp \left( \sum_{1 \leq m \leq M, 1 \leq n \leq N-2} (-1)^{m-1} s_{m,1^{n-2}} \{\bar{x}\} \varphi_{m} \varphi_{2-N}^* \right) |0\rangle
\]

\[
= \prod_{i \in [3-N,M]} \exp \left( x_N \varphi_{-(i+1)} \varphi_{i}^* \right) \exp \left( \sum_{1 \leq m \leq M, 1 \leq n \leq N-2} (-1)^{m-1} s_{m,1^{n-2}} \{\bar{x}\} \varphi_{m} \varphi_{1-n} \right)
\times \exp \left( \sum_{1 \leq m \leq M} (-1)^{m-1} s_{m,1^{N-2}} \{\bar{x}\} \varphi_{m} \varphi_{2-N} \right) \exp \left( \sum_{1 \leq m \leq M, 1 \leq n \leq N-1} (-1)^{m-1} s_{m,1^{n-1}} \{\bar{x}\} \varphi_{m} \varphi_{1-n} \right) |0\rangle
\]

\[
= \prod_{i \in [1,M]} \exp \left( x_N \varphi_{-(i+1)} \varphi_{i}^* \right) \exp \left( \sum_{1 \leq m \leq M, 1 \leq n \leq N-1} (-1)^{m-1} s_{m,1^{n-1}} \{\bar{x}\} \varphi_{m} \varphi_{1-n} \right) \exp \left( \sum_{1 \leq m \leq M, 1 \leq n \leq N-1} (-1)^{m-1} s_{m,1^{n-1}} \{\bar{x}\} \varphi_{m} \varphi_{1-n} \right) |0\rangle.
\]

Denote the argument of the second exp by \( P = \sum_{1 \leq m \leq M} P_m \), where

\[
P_m = \sum_{1 \leq n \leq N-1} (-1)^{m-1} s_{m,1^{n-1}} \{\bar{x}\} \varphi_{-n} \varphi_{1-n}^* + x_N \varphi_{-n} \varphi_{1-n}^*.
\]

We consider \( Q_1 = x_N \varphi_{-1} \varphi_{0}^* + P_1 \) and \( Q_j = [x_N \varphi_{-j} \varphi_{j-1}^*, Q_{j-1}] + P_j, 2 \leq j \leq M \). Then by (2.31) and (2.32), we have

\[
Q_1 = (s_{\{1\}} \{\bar{x}\} + x_N) \varphi_{-1} \varphi_{0}^* + \sum_{2 \leq n \leq N-1} (s_{\{n\}} \{\bar{x}\} + s_{\{n-1\}} \{\bar{x}\} x_N) \varphi_{-n} \varphi_{1-n}^*
\]

\[
+ s_{\{N-1\}} \{\bar{x}\} x_N \varphi_{-N} \varphi_{1-N}^*.
\]

https://doi.org/10.1088/1742-5468/aba0aa
Similarly we have

\[ Q_2 = [x_N \varphi_{-2} \varphi_1^*, Q_1] + P_2 = - \sum_{1 \leq n \leq N} s_{(1^n)} \{ \vec{x} \} x_N + s_{(1^{n-1})} \{ \vec{x} \} x_N^2 \varphi_{-2} \varphi_1^* - \sum_{2 \leq n \leq N-1} s_{(2,1^{n-2})} \{ \vec{x} \} x_N \varphi_{-2} \varphi_1^* \]

Continuing in this way, we have that

\[ Q_j = (-1)^{j-1} \sum_{1 \leq n \leq N} s_{(1^{n-1})} \{ x \} \varphi_{-j} \varphi_1^*, \quad 1 \leq j \leq M. \]

Now we compute by using the BCH formula:

\[
\begin{align*}
\mathbf{B}^*(x_1) \cdots \mathbf{B}^*(x_N)|0\rangle &= \prod_{i \in [1, M]} \exp \left( x_N \varphi_{-i} \varphi_1^* \right) \exp \left( \sum_{1 \leq j \leq M} P_j \right) |0\rangle \\
&= \prod_{i \in [2, M]} \exp \left( x_N \varphi_{-i} \varphi_1^* \right) \exp \left( \sum_{1 \leq j \leq M} P_j + x_N \varphi_{-1} \varphi_0^* \right) |0\rangle \\
&= \prod_{i \in [3, M]} \exp \left( x_N \varphi_{-i} \varphi_1^* \right) \exp \left( x_N \varphi_{-2} \varphi_1^* \right) \exp \left( Q_1 + \sum_{2 \leq j \leq M} P_j \right) |0\rangle \\
&= \prod_{i \in [3, M]} \exp \left( x_N \varphi_{-i} \varphi_1^* \right) \exp \left( Q_1 + Q_2 + \sum_{3 \leq j \leq M} P_j \right) |0\rangle = \cdots \\
&= \exp \left( \sum_{1 \leq j \leq M} Q_j \right) |0\rangle = \exp \left( \sum_{1 \leq m < M} (-1)^{m-1} s_{(m,1^{n-1})} \{ x \} \varphi_{-m} \varphi_1^* \right) |0\rangle.
\]

\[ \square \]
Corollary 3.1. For \( \{ x \} = \{ x_1, \ldots, x_N \} \) and fixed \( M \), we also have the expansion

\[
\langle 0 | C'(x_N) \cdots C'(x_1) = \langle 0 | \exp \left( \sum_{1 \leq m \leq M} (-1)^{m-1}s_{(m,1^{m-1})} x\varphi_{m-1}\varphi_m^* \right). \tag{3.13}
\]

3.3. Correlation function \( \langle 0 | C'(x_N) \cdots C'(x_1)B'(y_1) \cdots B'(y_N)|0 \rangle \)

Setting \( \langle 0 | 1 | 0 \rangle = 1 \), for \( | m \rangle = \langle 0 | \varphi_0^{m_0} \varphi_k^{m_k} \varphi_1^{m_1} \cdots \varphi_N^{m_N} \) and \( | n \rangle = \varphi_{-1}^{n_{-1}} \varphi_{-2}^{n_{-2}} \cdots \varphi_0^{n_0}|0 \rangle \), we have

\[
\langle m | n \rangle = (-1)^{\sum_{s=-k}^l m_s} \prod_{s=-k}^l \delta_{m_s,n_s}m_s!. \tag{3.14}
\]

Theorem 3.2. For any positive integer \( M \) and \( \{ x \} = \{ x_1, \ldots, x_N \} \), \( \{ y \} = \{ y_1, \ldots, y_N \} \), one has that

\[
\langle 0 | C'(x_N) \cdots C'(x_1)B'(y_1) \cdots B'(y_N)|0 \rangle = \frac{1}{\sum_{\mu \subseteq [N,M]} s_{\mu}(x)s_{\mu}(y)}. \tag{3.15}
\]

In particular, when \( M \rightarrow \infty \), we get another proof of the Cauchy identity:

\[
\langle 0 | C'(x_N) \cdots C'(x_1)B'(y_1) \cdots B'(y_N)|0 \rangle = \prod_{i,j=1}^N (1 - x_iy_j) = \frac{1}{\sum_{\mu \subseteq [N,M]} s_{\mu}(x)s_{\mu}(y)}. \tag{3.16}
\]

The proof is contained in appendix A.

Comparing the right sides of (2.37) and (3.15), we have

Corollary 3.2. For a positive integer \( M \),

\[
\langle 0 | C'(x_N) \cdots C'(x_1)B'(y_1) \cdots B'(y_N)|0 \rangle = \frac{1}{\langle 0 | C(x_N) \cdots C(x_1)B(y_1) \cdots B(y_N)|0 \rangle}. \tag{3.17}
\]

Corollary 3.3. Taking the limit \( M \rightarrow \infty, N \rightarrow \infty \) we get that

\[
\lim_{M,N \rightarrow \infty} \langle 0 | C'(x_N) \cdots C'(x_1)B'(y_1) \cdots B'(y_N)|0 \rangle = \prod_{i,j=1}^\infty (1 - x_iy_j) = \frac{1}{\sum_{\mu} s_{\mu}(x)s_{\mu}(y)}. \tag{3.18}
\]

where the sum is over all partitions \( \mu \). In particular,

\[
\lim_{M,N \rightarrow \infty} \langle 0 | C'(z^{N-\frac{3}{2}}) \cdots C'(z^{\frac{1}{2}})B'(z^{\frac{1}{2}}) \cdots B'(z^{N-\frac{3}{2}})|0 \rangle = \prod_{i=1}^\infty (1 - z_i^j).
\]
4. Neutral fermions

This section is parallel to section 2. It is known that the usual KP tau functions can be formulated in terms of Schur functions, which are integral combinations of the power sum \( p_r = \sum_i x_i^r \). There is another family of well-known symmetric functions called Schur’s \( Q \)-function \( Q_\nu \), which are orthogonal rational linear combination of the odd degree power sum symmetric function \( p_{2r-1} \). It turns out that the BKP tau functions can be realized by the Schur \( Q \)-functions [34]. Foda and Wheeler [9] used the QISM to show Bethe eigenvectors of the \( \iota \)-boson model are BKP tau functions (see appendix C), we aim to give a new derivation of this beautiful result.

We start with the neutral fermions defined by infinite operators \( \{ \phi_m \}_{m \in \mathbb{Z}} \) with the relations

\[
\{ \phi_m, \phi_n \} = 2(-1)^m \delta_{m+n,0}. \tag{4.1}
\]

The Fock space \( \mathcal{F}_\phi \) and its dual \( \mathcal{F}_\phi^* \) are the vector spaces generated by \( |0\rangle \) and \( \langle 0| \) subject to

\[
\phi_m |0\rangle = \langle 0| \phi_m = 0, \quad m < 0. \tag{4.2}
\]

One can decompose \( \mathcal{F}_\phi \) and \( \mathcal{F}_\phi^* \) into a direct sum

\[
\mathcal{F}_\phi = \bigoplus_{i \in \{0,1\}} \mathcal{F}_\phi^{(i)}, \quad \mathcal{F}_\phi^* = \bigoplus_{i \in \{0,1\}} \mathcal{F}_\phi^{*(i)},
\]

where \( \mathcal{F}_\phi^{(i)} \) and \( \mathcal{F}_\phi^{*(i)} \) are the subspaces generated by the action of all neutral fermion monomials of length \( i \mod 2 \) on \( |0\rangle \) and \( \langle 0| \), respectively. For fixed \( M \), we introduce

\[
\tilde{B}(x) = \exp \left( \frac{1}{2} x (-1)^M \phi_{-M+1} \phi_M \right) \exp \left( \frac{1}{2} x (-1)^{M-1} \phi_{-M+2} \phi_{M-1} \right) \cdots
\]

\[
\times \exp \left( \frac{1}{2} x (-1)^{-M+2} \phi_{M-1} \phi_{-M+2} \right) \exp \left( \frac{1}{2} x (-1)^{M+1} \phi_M \phi_{-M+1} \right)
\]

\[
= \prod_{j \in [-M+1,M]} \exp \left( \frac{1}{2} x (-1)^i \phi_{-j+1} \phi_j \right), \tag{4.3}
\]

\[
\tilde{C}(x) = \exp \left( \frac{1}{2} x (-1)^{-M} \phi_{M-1} \phi_M \right) \exp \left( \frac{1}{2} x (-1)^{M-1} \phi_{M-2} \phi_{-M+1} \right) \cdots
\]

\[
\times \exp \left( \frac{1}{2} x (-1)^{M-2} \phi_{M-1} \phi_{M-2} \right) \exp \left( \frac{1}{2} x (-1)^{M-1} \phi_{-M} \phi_{M-1} \right)
\]

\[
= \prod_{i \in [-M+1,M]} \exp \left( \frac{1}{2} x (-1)^i \phi_{-i} \phi_{i-1} \right). \tag{4.4}
\]
Theorem 4.1. For any fixed $M$ and strict partition where

$$\lambda_n \geq \lambda = \frac{1}{2} \sum_{n \in 2N+1} (-1)^j \phi_{-j+n} \phi_j,$$

It is clear that $\tilde{B}(x_1) \cdots \tilde{B}(x_N)|0\rangle$ is a BKP tau function. Using the similar method of proposition 2.1, we have

**Proposition 4.1.** For any fixed $M$, one has that

$$\tilde{B}(x) = \exp\left( \sum_{n \in 2N+1} \frac{1}{n} x^n \Lambda_n \right), \quad \tilde{C}(x) = \exp\left( \sum_{n \in 2N+1} \frac{1}{n} x^n \Lambda_n^* \right),$$

where

$$\Lambda_n = \frac{1}{2} \sum_{-M+n}^M (-1)^j \phi_{-j+n} \phi_j, \quad \Lambda_n^* = \frac{1}{2} \sum_{-M}^{M-n} (-1)^j \phi_{j-n} \phi_j.$$  

**Remark 4.1.** Under $M \to \infty$, let $\lambda_n = \Lambda_n^*$, $\lambda_{-n} = \Lambda_n$, $n \geq 1$, we have that

$$[\lambda_m, \lambda_n] = 2m \delta_{m,-n}.$$  

The following result is clear.

**Proposition 4.2.** For any fixed $M$, we have that

$$\tilde{B}(x)\tilde{B}(y) = \tilde{B}(y)\tilde{B}(x), \quad \tilde{C}(x)\tilde{C}(y) = \tilde{C}(y)\tilde{C}(x).$$

A partition $\tilde{\nu} = (\tilde{\nu}_1, \ldots, \tilde{\nu}_l)$ is called strict if $\tilde{\nu}_1 \succ \cdots \succ \tilde{\nu}_l$. Define the basis vectors

$$|\tilde{\nu}\rangle = \begin{cases} \phi_{\tilde{\nu}_1} \phi_{\tilde{\nu}_2} \cdots \phi_{\tilde{\nu}_l} |0\rangle & \ell \text{ even} \\ \phi_{\tilde{\nu}_1} \phi_{\tilde{\nu}_2} \cdots \phi_{\tilde{\nu}_l} |0\rangle & \ell \text{ odd} \end{cases}$$

$$|\tilde{\nu}\rangle = \begin{cases} (-1)^{|\tilde{\nu}|} |0\rangle \phi_{-\tilde{\nu}_1} \cdots \phi_{-\tilde{\nu}_l} & \ell \text{ even} \\ (-1)^{|\tilde{\nu}|} |0\rangle \phi_0 \phi_{-\tilde{\nu}_1} \cdots \phi_{-\tilde{\nu}_l} & \ell \text{ odd} \end{cases}$$

Recall that the Schur $Q$-function $Q_{\tilde{\nu}}$ associated to the strict partition $\tilde{\nu}$ [24] is defined by

$$Q_{\tilde{\nu}}(x_1, \ldots, x_n) = 2^l \sum_{w \in S_n/S_{n-l}} x_{w(1)}^{\tilde{\nu}_1} \cdots x_{w(n)}^{\tilde{\nu}_n} \prod_{i<j} x_{w(i)} - x_{w(j)},$$

where $n \geq l = l(\tilde{\nu})$, $S_n$ acts on $x_1, \ldots, x_n$ and $S_{n-l}$ acts on the last $n-l$ variables.

**Theorem 4.1.** For any fixed $M$ and strict partition $\tilde{\nu}$ of length $l(\tilde{\nu})$, one has that

$$\tilde{B}(x)|\tilde{\nu}\rangle = \sum_{\tilde{\mu} \prec \tilde{\nu}} 2^{|\tilde{\nu}|(|\tilde{\mu}| - |\tilde{\nu}|)} x_{\tilde{\nu}|\tilde{\nu}|\tilde{\mu}},$$

$$|\tilde{\nu}|\tilde{C}(x) = \sum_{\tilde{\mu} \prec \tilde{\nu}} 2^{|\tilde{\mu}|(|\tilde{\nu}| - |\tilde{\nu}|)} x_{\tilde{\nu}|\tilde{\nu}|\tilde{\mu}}.$$
where \( \#(\tilde{\mu} | \tilde{\nu}) \) denotes the number of parts in \( \tilde{\mu} \) that are not in \( \tilde{\nu} \).

Our result is summarized as follows.

**Theorem 4.2.** For a fixed \( M \in \mathbb{N} \), \( \{x\} = \{x_1, \ldots, x_N\} \) and \( \{y\} = \{y_1, \ldots, y_N\} \), we have that

\[
\tilde{B}(x_1) \cdots \tilde{B}(x_N)|0\rangle = \sum_{\tilde{\mu} \in [N,M]} 2^{-\ell(\tilde{\mu})} Q_{\tilde{\mu}}\{x\}|\tilde{\mu}\rangle, \tag{4.13}
\]

\[
\langle 0 | \tilde{C}(x_N) \cdots \tilde{C}(x_1) = \sum_{\tilde{\mu} \in [N,M]} 2^{-\ell(\tilde{\mu})} Q_{\tilde{\mu}}\{x\} \langle \tilde{\mu}, \tag{4.14}
\]

\[
\langle 0 | \tilde{C}(x_N) \cdots \tilde{C}(x_1) \tilde{B}(y_1) \cdots \tilde{B}(y_N)|0\rangle = \sum_{\tilde{\mu} \in [N,M]} 2^{-\ell(\tilde{\mu})} Q_{\tilde{\mu}}\{x\} Q_{\tilde{\mu}}\{y\}, \tag{4.15}
\]

where \( \tilde{\mu} \) runs through all strict partitions inside the box \([N,M]\) of height \( N \) and width \( M \).

**Remark 4.2.** The last identity (4.15) can also be realized as hypergeometric tau functions in [27, equation (1.2.9)].

**Remark 4.3.** Using the Heisenberg relation (4.7), it is easy to derive that under \( M \to \infty \)

\[
\langle 0 | \tilde{C}(x_N) \cdots \tilde{C}(x_1) \tilde{B}(y_1) \cdots \tilde{B}(y_N)|0\rangle = \prod_{i,j=1}^{N} \frac{1+x_i y_j}{1-x_i y_j}, \tag{4.16}
\]

which together with (4.15) gives another proof of the well-known Schur’s identity [24]:

\[
\sum_{\ell(\tilde{\mu}) \leq N} 2^{-\ell(\tilde{\mu})} Q_{\tilde{\mu}}\{x\} Q_{\tilde{\mu}}\{y\} = \prod_{i,j=1}^{N} \frac{1+x_i y_j}{1-x_i y_j}, \tag{4.17}
\]

where \( \tilde{\mu} \) runs through all strict partitions with length \( \leq N \), \( \{x\} = \{x_1, \ldots, x_N\} \) and \( \{y\} = \{y_1, \ldots, y_N\} \).

**Acknowledgments**

Research supported by National Natural Science Foundation of China grant no. 11531004, Simons Foundation Grant No. 523868 and Humboldt Foundation.

**Appendix A. Proof of Theorem 3.2**

This appendix gives a detailed proof of theorem 3.2. We start with an easy lemma.

**Lemma A.1.** The constant term of the function \( (1 - \frac{a_1}{z})^{-1} \cdots (1 - \frac{a_n}{z})^{-1}(1 - bz)^{-1} \) is \( (1 - a_1 b)^{-1} \cdots (1 - a_n b)^{-1} \), and the constant term of \( (c_1 + \frac{a_1}{z})^{-1} \cdots (c_n + \frac{a_n}{z})^{-1}(1 + bz)^{-1} \) is \( (c_1 - a_1 b)^{-1} \cdots (c_n - a_n b)^{-1} \).

https://doi.org/10.1088/1742-5468/aba0aa
Correlation functions of charged free boson and fermion systems

**Proposition A.1.** Let \( T = (T_{ij})_{1 \leq i, j \leq n} \) and \( Z_k = \sum_{1 \leq i \leq n} T_{ik} \frac{z_k}{z_i} \) (1 \( \leq k \leq n \)). Then the constant term \( CT(A) \) of the multivariate function \( A \) in the \( z_i \): \( A = \prod_{k=1}^{n} \frac{1}{z_k} \) is equal to \( \det T^{-1} \).

**Proof.** We use induction on \( n \). \( n = 1 \) is clear. For \( 1 \leq k \leq n \), set \( Z_k^* = \sum_{1 \leq i \leq n-1} T_{ik} \frac{1}{z_i} \), then

\[
Z_k = \begin{cases} 
Z_k^* z_k + \frac{T_{nk} z_k}{z_n}, & 1 \leq k \leq n-1 \\
T_{nn} \left(1 + \frac{Z_n^* z_n}{T_{nn}} \right), & k = n
\end{cases}
\]

so \( A \) can be written as

\[
A = T_{nn}^{-1} \left( Z_1^* z_1 + \frac{T_{n1} z_1}{z_n} \right)^{-1} \cdots \left( Z_{n-1}^* z_{n-1} + \frac{T_{nn-1} z_{n-1}}{z_n} \right)^{-1} \left(1 + \frac{Z_n^* z_n}{T_{nn}} \right)^{-1}.
\]

(A.1)

As a function of \( z_n \), it follows from lemma A.1 that \( CT(A) \) is

\[
A' = T_{nn}^{-1} \left( Z_1^* z_1 - \frac{T_{n1} z_1 Z_n^*}{T_{nn}} \right)^{-1} \left( Z_2^* z_2 - \frac{T_{n2} z_2 Z_n^*}{T_{nn}} \right)^{-1} \cdots \left( Z_{n-1}^* z_{n-1} - \frac{T_{nn-1} z_{n-1} Z_n^*}{T_{nn}} \right)^{-1}.
\]

(A.2)

Put \( T = (T_{ij})_{1 \leq i, j \leq n} \) and \( T_{ij}' = T_{ij} - \frac{T_{ij} T_{nn}}{T_{nn}} \) (1 \( \leq i, j \leq n-1 \)) and \( Z_k' = \sum_{1 \leq i \leq n-1} T_{ik} \frac{z_k}{z_i} \), then

\[
A' = T_{nn}^{-1} \prod_{k=1}^{n-1} \frac{1}{Z_k'}.
\]

(A.3)

Using the induction hypothesis, we have that

\[
CT(A) = \frac{1}{T_{nn}} CT(A') = \frac{1}{T_{nn} \det T'},
\]

(A.4)

where \( T' = (T_{ij}')_{1 \leq i, j \leq n-1} \). Clearly \( \det T = T_{nn} \det T' \), we have shown the proposition. \( \square \)

For a positive integer \( M \), let

\[
D_m = \sum_{1 \leq n \leq N} s_{(m,1^{n-1})}\{ x \} \varphi_{n-1}, \ \{ x \} = \{ x_1, \ldots, x_N \},
\]

\[
F_m = \sum_{1 \leq n \leq N} s_{(m,1^{n-1})}\{ y \} \varphi^*_{1^{n-1}}, \ \{ y \} = \{ y_1, \ldots, y_N \}.
\]

\( ^6 \)Professor Stanley informed us that the proposition may be seen as a special case of the MacMahon Master theorem. We provide a direct proof for completeness.

https://doi.org/10.1088/1742-5468/aba0aa
Then
\[ T_{ij} = [D_i, F_j] = \sum_{1 \leq n \leq N} s_{(i,1^{n-1})} \{x\} s_{(j,1^{n-1})} \{y\}, \]

Let \( T = (T_{ij})_{1 \leq i,j \leq M} \), and consider the functions \( Z_j = \sum_{1 \leq i \leq n} (\delta_{ij} + T_{ij}) \bar{z}_i \). By (3.2), (3.14) and (2.39), we have that
\[
\langle 0 | C^* (x_N) \cdots C^* (x_1) B^* (y_1) \cdots B^* (y_N) | 0 \rangle
\]
\[
= \langle 0 | \sum_{m=0}^{M} \frac{(D_m(-1)^{m-1} \varphi_m)}{M_{m!}} \sum_{m=0}^{M} \frac{(F_m(-1)^{m-1} \varphi_m)}{M_{m!}} | 0 \rangle
\]
\[
= \sum_{m=0}^{M} \langle 0 | \prod_{m=1}^{M} \frac{(D_m)^{m_{km}}}{m_{km}!} \prod_{m=1}^{M} \frac{(F_m)^{m_{km}}}{m_{km}!} | 0 \rangle (-1)^{\sum_{m=1}^{M} m_{km}} \prod_{m=1}^{M} k_{m!}
\]
\[
= \prod_{i=1}^{M} \exp \left( -\frac{D_i}{z_i} \right) \prod_{j=1}^{M} \frac{1}{1 - z_j F_j} | 0 \rangle = CT \left( \prod_{j=1}^{M} \frac{1}{Z_j} \right) = \frac{1}{\det(I + T)}
\]
\[
= \sum_{\nu \in [N,M]} s_{\nu} \{x\} s_{\nu} \{y\}.
\]

Appendix B. The phase model

The vector space \( \mathcal{V} \) (resp. dual space \( \mathcal{V}^* \)) is defined as the linear span of all states of the whole lattice with the bases
\[
\text{Basis}(\mathcal{V}) = \{ | n \rangle = | n_0 \rangle_0 \otimes | n_1 \rangle_1 \otimes \cdots \otimes | n_M \rangle_M : n_i \in \mathbb{Z}_+ \}, \quad (B.1)
\]
\[
\text{Basis}(\mathcal{V}^*) = \{ \langle m | = \langle m_0 |_0 \otimes \langle m_1 |_1 \otimes \cdots \otimes \langle m_M |_M : m_i \in \mathbb{Z}_+ \}. \quad (B.2)
\]
The phase algebra [1, 2] is generated by \{\pi_i, \psi_i^\dagger, \mathcal{N}_i, \psi_i\}_{0 \leq i \leq M} subject to the following relations:
\[
[\psi_i, \psi_j^\dagger] = \delta_{i,j} \pi_i, \quad [\mathcal{N}_i, \psi_j] = -\delta_{i,j} \psi_i, \quad [\mathcal{N}_i, \psi_j^\dagger] = \delta_{i,j} \psi_i^\dagger, \quad \psi_i \pi_i = \pi_i \psi_i^\dagger = 0, \quad 0 \leq i,j \leq M.
\]
\[
(B.3)
\]
They act on the Fock space \( \mathcal{V} \) naturally as annihilation (\( \psi_i \)) and creation (\( \psi_i^\dagger \)) operators as follows.
\[
\psi_i | n_0 \rangle_0 \otimes \cdots \otimes | n_M \rangle_M = \delta_{n_i-1} | n_0 \rangle_0 \otimes \cdots \otimes | n_i - 1 \rangle_i \otimes \cdots \otimes | n_M \rangle_M,
\]
\[
\psi_i^\dagger | n_0 \rangle_0 \otimes \cdots \otimes | n_M \rangle_M = | n_0 \rangle_0 \otimes \cdots \otimes | n_i + 1 \rangle_i \otimes \cdots \otimes | n_M \rangle_M,
\]

https://doi.org/10.1088/1742-5468/aba0aa
Correlation functions of charged free boson and fermion systems

\[ \mathcal{N}|n_0\rangle_0 \otimes \cdots \otimes |n_M\rangle_M = n_i|n_0\rangle_0 \otimes \cdots \otimes |n_M\rangle_M, \]
\[ \pi |n_0\rangle_0 \otimes \cdots \otimes |n_M\rangle_M = \delta_{n,0}|n_0\rangle_0 \otimes \cdots \otimes |n_M\rangle_M, \]

where \( \delta_{n,-1} = 1 \) if \( n_r - 1 \geq 0 \) and 0 otherwise. Similarly the action on the dual space \( \mathcal{V}^* \) can be defined accordingly. The \( L \)-matrix for the phase model on the space \( \mathcal{V}_1 \otimes \mathcal{V}_2 \) (\( \mathcal{V}_i \simeq \mathcal{V} \)) has the form

\[ L_{1m}(x) = \begin{pmatrix} x^{-\frac{1}{2}} & \psi^*_n \psi^*_m \alpha^{\frac{1}{2}} \\ \psi_m & x^\frac{1}{2} \end{pmatrix}, \] (B.4)

which satisfies the RLL equation

\[ R(x, y)L_{1m}(x)L_{2m}(y) = L_{2m}(y)L_{1m}(x)R(x, y), \]

where the \( R \)-matrix is given by

\[ R(x, y) = \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & 0 & x^\frac{1}{2}y^\frac{1}{2} & 0 \\ 0 & x^\frac{1}{2}y^\frac{1}{2} & x - y & 0 \\ 0 & 0 & 0 & x \end{pmatrix}. \] (B.5)

The monodromy matrix \( T(x) \) also satisfies the RLL relation and is of the form

\[ T(x) = L_M(x) \cdots L_0(x) = \begin{pmatrix} A(x) & B(x) \\ C(x) & D(x) \end{pmatrix}, \] (B.6)

where \( L_m = L_{im} \) (\( i = 1, 2 \)) and the off-diagonal operators satisfy

\[ [B(x), B(y)] = [C(x), C(y)] = 0. \] (B.7)

Denote \( \mathbb{B}(x) = x^\frac{1}{2}B(x) \) and \( \mathbb{C}(x) = x^\frac{1}{2}C(x) \). Let \( \mathcal{M}_{bc}: \mathcal{V} \rightarrow \mathcal{F}^{(0)} \) and \( \mathcal{M}_{bc}^*: \mathcal{V}^* \rightarrow \mathcal{F}^{*{(0)}} \) be the linear maps defined by \( \mathcal{M}_{bc}(|n\rangle) = |\nu\rangle \), \( \mathcal{M}_{bc}^*(\langle n|) = \langle \nu| \), where \( |\nu\rangle = |\nu_0, \ldots, 1^n\rangle \in \mathcal{F}^{(0)} \) (see (2.22)) and \( \langle \nu| = (\nu_{M_0}^M, \ldots, 1^n) \in \mathcal{F}^{*{(0)}} \) (see (2.23)). Note that these maps are not one-to-one since they are insensitive to the value of \( n_0 \).

**Proposition B.1.** [1, 33] If \( \mathcal{M}_{bc}(|n\rangle) = |\nu\rangle \), \( \mathcal{M}_{bc}^*(\langle n|) = \langle \nu| \), then

\[ \mathcal{M}_{bc}\mathbb{B}(x)|n\rangle = \sum_{\nu \prec \mu \leq |l+1,M|} x^{l+|\nu|-|\mu|} |\mu\rangle, \] (B.8)
\[ \langle n|\mathbb{C}(x)\mathcal{M}_{bc}^* = \sum_{\nu \prec \mu \leq |l+1,M|} x^{l+|\nu|-|\mu|} \langle \mu|. \] (B.9)

The following result gives a correspondence between \( \mathbb{B}(x), \mathbb{C}(x) \) and \( \mathcal{B}(x), \mathcal{C}(x) \). It can be seen from theorem 2.1, corollary 2.1 and proposition B.1.

**Proposition B.2.** The following commutative diagrams hold.

\[ \begin{array}{ccc}
\mathcal{V} & \xrightarrow{\mathcal{M}_{bc}} & \mathcal{F}^{(0)} \\
\mathbb{B}(x) & \xrightarrow{\mathcal{B}(x)} & \mathbb{C}(x) \\
\mathcal{V} & \xrightarrow{\mathcal{M}_{bc}^*} & \mathcal{F}^{*{(0)}} \\
\mathcal{V} & \xrightarrow{\mathcal{M}_{bc}^*} & \mathcal{F}^{*{(0)}}
\end{array} \]

https://doi.org/10.1088/1742-5468/aba0aa
Remark B.1. Using propositions 2.1–2.4 we get another proof of the following result, which is fundamental in previous study of the phase model in the literature.

\[
\lim_{M \to \infty} \mathcal{M}_{bc}(x)|\nu\rangle = \exp \left( \sum_{n=1}^{\infty} \frac{x^n}{n} H_{-n} \right) |\nu\rangle,
\]

where \( \mathcal{M}_{bc}(|\nu\rangle) = |\nu\rangle \).

Appendix C. The \( i \)-boson model

The vector space \( \tilde{V} \) and its dual \( \tilde{V}^* \) are defined respectively as the linear spans of the bases

\[
\text{Basis}(\tilde{V}) = \{ |\tilde{u}\rangle = |n_0\rangle_0 \otimes |n_1\rangle_1 \otimes \cdots \otimes |n_M\rangle_M : n_0 \geq 0, n_i = 0, 1 \}, \quad \text{(C.1)}
\]

\[
\text{Basis}(\tilde{V}^*) = \{ |\tilde{u}^\dag\rangle = \langle m_0|_0 \otimes \langle m_1|_1 \otimes \cdots \otimes \langle m_M|_M : m_0 \geq 0, m_i = 0, 1 \}, \quad \text{(C.2)}
\]

The \( i \)-boson algebra \([33]\) is generated by \( \{ \tilde{\psi}_i, \tilde{\psi}_i^\dag, \tilde{N}_i, \tilde{\psi}_i \}_{0 \leq i \leq M} \) satisfying the relations:

\[
[\tilde{\psi}_i, \tilde{\psi}_j^\dag] = \delta_{i,j} (-1)^{\nu_i}, \quad [\tilde{N}_i, \tilde{\psi}_j] = -\delta_{i,j} \tilde{\psi}_i, \quad [\tilde{N}_i, \tilde{\psi}_j^\dag] = \delta_{i,j} \tilde{\psi}_i^\dag \quad \text{(C.3)}
\]

for all \( 0 \leq i, j \leq M \). The \( i \)-boson algebra naturally acts on \( \tilde{V} \) by

\[
\tilde{\psi}_j |n_0\rangle_0 \otimes \cdots \otimes |n_M\rangle_M = \frac{1}{\sqrt{2}} \delta_{n_j-1,0} |n_0\rangle_0 \otimes \cdots \otimes |n_j-1\rangle_j \otimes \cdots \otimes |n_M\rangle_M, \quad 1 \leq j \leq M,
\]

\[
\tilde{\psi}_0 |n_0\rangle_0 \otimes \cdots \otimes |n_M\rangle_M = \frac{1 - (-1)^{n_0}}{\sqrt{2}} |n_0-1\rangle_0 \otimes |n_1\rangle_1 \otimes \cdots \otimes |n_M\rangle_M,
\]

\[
\tilde{\psi}_j^\dag |n_0\rangle_0 \otimes \cdots \otimes |n_M\rangle_M = \frac{1 + (-1)^{n_j}}{\sqrt{2}} |n_0\rangle_0 \otimes \cdots \otimes |n_j+1\rangle_j \otimes \cdots \otimes |n_M\rangle_M, \quad 1 \leq j \leq M,
\]

\[
\tilde{\psi}_0^\dag |n_0\rangle_0 \otimes \cdots \otimes |n_M\rangle_M = \frac{1}{\sqrt{2}} |n_0+1\rangle_0 \otimes |n_1\rangle_1 \otimes \cdots \otimes |n_M\rangle_M,
\]

\[
\tilde{N}_j |n_0\rangle_0 \otimes \cdots \otimes |n_M\rangle_M = n_j |n_0\rangle_0 \otimes \cdots \otimes |n_M\rangle_M, \quad 0 \leq j \leq M.
\]

The action on \( \tilde{V}^* \) is defined similarly.

The \( R \)-matrix for the \( i \)-boson model is given by

\[
\tilde{R}(x, y) = \begin{pmatrix}
x + y & 0 & 0 & 0 \\
0 & y - x & 2x^{\frac{1}{2}} y^{\frac{1}{2}} & 0 \\
0 & 2x^{\frac{1}{2}} y^{\frac{1}{2}} & x - y & 0 \\
0 & 0 & 0 & x + y
\end{pmatrix}.
\]

The RLL equation \( \tilde{R}(x, y)\tilde{L}_1(x)\tilde{L}_2(y) = \tilde{L}_2(y)\tilde{L}_1(x)\tilde{R}(x, y) \) on \( \tilde{V} \otimes \tilde{V} \) is satisfied by the \( L \)-matrix:

\[\text{https://doi.org/10.1088/1742-5468/aba0aa}\]
Correlation functions of charged free boson and fermion systems

\[ \tilde{L}_{im}(x) = \begin{pmatrix} x^{\frac{i}{2}} & \sqrt{2} \tilde{\psi}_{im}^* \\ \sqrt{2} \tilde{\psi}_{im} & x^{\frac{i}{2}} \end{pmatrix}, \quad i = 1, 2. \] (C.5)

The monodromy matrix \( \tilde{T}(x) \) has the form (also satisfying the RLL relation)

\[ \tilde{T}(x) = \tilde{L}_M(x) \ldots \tilde{L}_0(x) = \begin{pmatrix} \tilde{A}(x) & \tilde{B}(x) \\ \tilde{C}(x) & \tilde{D}(x) \end{pmatrix}, \] (C.6)

where \( \tilde{L}_m = \tilde{L}_{im} \) and the off-diagonal operators satisfy

\[ [\tilde{B}(x), \tilde{B}(y)] = [\tilde{C}(x), \tilde{C}(y)] = 0. \] (C.7)

Denote \( \tilde{\mathbb{B}}(x) = x^{\frac{i}{2}} \tilde{B}(x) \) and \( \tilde{\mathbb{C}}(x) = x^{\frac{i}{2}} \tilde{C}(x) \). Let \( \mathcal{M}_\phi : \tilde{\mathbb{B}} \to \mathcal{F}^{(0)}_\phi \) and \( \mathcal{M}^*_\phi : \tilde{\mathbb{C}} \to \mathcal{F}^{(0)}_* \) be the two linear maps defined by

\[ \mathcal{M}_\phi(\tilde{\mathbb{B}}) = 2^{-(\ell(p))} |\tilde{\nu}\rangle, \quad \mathcal{M}^*_\phi(\tilde{\mathbb{C}}) = 2^{-(\ell(p))} \langle \tilde{\nu}|, \] (C.8)

where \( |\tilde{\nu}\rangle = |M_{nM}, \ldots, 1^{n_1}\rangle \) if \( \sum_i n_i \) is even and \( |\tilde{\nu}\rangle = |M_{nM}, \ldots, 1^{n_1}, 0\rangle \) if \( \sum_i n_i \) is odd. The dual vectors \( \langle \tilde{\nu}| \) is defined similarly.

**Proposition C.1.** \([33]\) If \( \mathcal{M}_\phi(\tilde{\mathbb{B}}) = |\tilde{\nu}\rangle, \mathcal{M}^*_\phi(\tilde{\mathbb{C}}) = \langle \tilde{\nu}|, \) then

\[ \mathcal{M}_\phi \tilde{\mathbb{B}}(x) |\tilde{\nu}\rangle = \sum_{\tilde{\mu} \prec \tilde{\nu}} 2^{\#(\tilde{\mu}|\tilde{\nu}) - \ell(\tilde{\mu}) + \ell(\tilde{\nu})} x^{\#(\tilde{\mu}|\tilde{\nu})} |\tilde{\mu}\rangle, \] (C.9)

\[ \langle \tilde{\nu}| \tilde{\mathbb{C}}(x) \mathcal{M}^*_\phi = \sum_{\tilde{\mu} \prec \tilde{\nu}} 2^{\#(\tilde{\mu}|\tilde{\nu}) - \ell(\tilde{\mu}) + \ell(\tilde{\nu})} x^{\#(\tilde{\mu}|\tilde{\nu})} \langle \tilde{\mu}|. \] (C.10)

The following can be seen from theorem 4.1 and proposition C.1.

**Proposition C.2.** The following commutative diagrams hold.

\[ \begin{array}{ccc}
\tilde{\mathbb{B}}(x) & \xrightarrow{\mathcal{M}_\phi} & \mathcal{F}^{(0)}_\phi \\
\tilde{\mathbb{C}}(x) & \xrightarrow{\mathcal{M}^*_\phi} & \mathcal{F}^{(0)}_* \\
\tilde{\mathbb{B}}(x) & \xrightarrow{\mathcal{M}_\phi} & \mathcal{F}^{(0)}_\phi \\
\tilde{\mathbb{C}}(x) & \xrightarrow{\mathcal{M}^*_\phi} & \mathcal{F}^{(0)}_* \\
\end{array} \]

**References**

[1] Bogoliubov N M 2005 Boxed plane partitions as an exactly solvable boson model J. Phys. A. 38 9415–30
[2] Bogoliubov N M, Izergin A G and Kitanine N A 1998 Correlation functions for a strongly correlated boson system Nucl. Phys. B 516 501–28
[3] Date E, Kashiwara M and Miwa T 1981 Vertex operators and tau functions: transformation groups for soliton equations II Proc. Jpn. Acad. A 57 387–92
[4] Date E, Jimbo M, Kashiwara M and Miwa T 1981 Operator approach to the Kadomtsev–Petviashvili equation: transformation groups for soliton equations III J. Phys. Soc. Japan 50 3806–12
[5] Date E, Jimbo M, Kashiwara M and Miwa T 1982 Quasi-periodic solutions of the orthogonal KP equation, transformation groups for soliton equations V Publ. Res. Inst. Math. Sci. 18 1111–9 Kyoto Univ.
[6] Date E, Jimbo M, Kashiwara M and Miwa T 1981 KP hierarchies of orthogonal and symplectic type: transformation groups for soliton equations VI J. Phys. Soc. Japan 50 3813–8

https://doi.org/10.1088/1742-5468/aba0aa
Correlation functions of charged free boson and fermion systems

[7] Date E, Jimbo M, Kashiwara M and Miwa T 1982 Euclidean Lie algebras and reduction of the KP hierarchy, transformation groups for soliton equations VII *Publ. Res. Inst. Math. Sci.* 18 1077–110

[8] Faddeev L D, Sklyanin E K and Takhtajan L A 1979 Quantum inverse problem method I *Theor. Math. Phys.* 40 688–706

[9] Foda O and Wheeler M 2007 BKP plane partitions *J. High Energy Phys.* JHEP01(2007)075

[10] Foda O, Wheeler M and Zuparic M 2009 On free fermions and plane partitions *J. Stat. Mech.* P03017

[11] Foda O, Wheeler M and Zuparic M 2009 Domain wall partition functions and KP *Nucl. Phys. B* 820 649–63

[12] Frenkel E, Kac V G, Radul A and Wang W 1995 $W_{1+\infty}$ and $W(gls)$ with central charge $N$ *Commun. Math. Phys.* 170 337–57

[13] Frenkel I B 1981 Two constructions of affine Lie algebra representations and boson–fermion correspondence in quantum field theory *J. Funct. Anal.* 44 259–327

[14] Jacobson N 1962 *Lie Algebras* (New York: Wiley)

[15] Harnad J and Orlov AY 2015 Hypergeometric $\tau$-functions, Hurwitz numbers and enumeration of paths *Commun. Math. Phys.* 338 267–84

[16] Jing N 1991 Vertex operators, symmetric functions and the spin group $\Gamma_n$ *J. Algebra* 138 340–98

[17] Jing N 1991 Vertex operators and Hall–Littlewood symmetric functions *Adv. Math.* 87 226–48

[18] Jing N and Li Z 2019 Tau functions of the charged free bosons (arXiv:1912.01794)

[19] Kac V G, Raina A K and Rozhkovskaya N 2013 *Bombay Lectures on Highest Weight Representations of Infinite Dimensional Lie Algebras* (Adv. Ser. Math. Phys.) vol 29 2nd edn (Singapore: World Scientific)

[20] Jing N and Li Z 2019 Tau functions of the charged free bosons (arXiv:1912.01794)

[21] Kac V G and van de Leur J W 1990 A super boson–fermion correspondence of type B, infinite-dimensional Lie algebras and groups *Adv. Math. Phys.* 7 369–406 (Singapore: World Scientific)

[22] Lehmer D H 1935 Lacunary recurrence formulas for the numbers of Bernoulli and Euler *Ann. Math.* 36 637–49

[23] Liszewski K T 2011 The charged free boson integrable hierarchy *Ph.D Thesis* (Univ. Melbourne) (arXiv:1110.6703v1)

[24] Macdonald I G 1995 *Symmetric Functions and Hall Polynomials* 2nd edn (Oxford: Oxford University Press)

[25] Macdonald I G 1995 *Symmetric Functions and Hall Polynomials* 2nd edn (Oxford: Oxford University Press)

[26] Natanzon S M and Orlov A Y 2017 CKP hierarchy, bosonic tau function and bosonization formulae *Symmetry, Integrability Geometry Methods Appl.* 8 28

[27] Orlov A Y 2003 Hypergeometric functions related to Schur $Q$-polynomials and the BKP equation *Theor. Math. Phys.* 137 1574–89

[28] Orlov A Y and Scherbin D 2000 Fermionic representation for basic hypergeometric functions related to Schur polynomials (arXiv:math/0001001)

[29] Thompson R C 1982 Cyclic relations and the Goldberg coefficients in the Campbell–Baker–Hausdorff formula *Proc. Am. Math. Soc.* 86 12–4

[30] Tsilevich N V 2006 Quantum inverse scattering method for the $q$-boson model and symmetric functions *Funct. Anal. Appl.* 40 207–17

[31] van de Leur J W, Orlov A Y and Shiota T 2012 CKP hierarchy, bosonic tau function and bosonization formulæ *Symmetry, Integrability Geometry Methods Appl.* 8 28

[32] Wang W 1998 $W_{1+\infty}$ algebra and $W_0$ algebra, and Friedan–Martinec–Shenker bosonization *Commun. Math. Phys.* 195 95–111

[33] Wheeler M 2011 Free fermions in classical and quantum integrable models *Ph.D Thesis* (Univ. Melbourne) (arXiv:1110.6703v1)

[34] You Y 1990 Polynomial solutions of the BKP hierarchy and projective representations of symmetric groups, infinite-dimensional Lie algebras and groups *Adv. Math. Phys.* 7 449–64 (Singapore: World Scientific)