Algorithmic applications of the corestriction of central simple algebras

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Abstract

Let $L$ be a separable quadratic extension of either $\mathbb{Q}$ or $\mathbb{F}_q(t)$. We propose efficient algorithms for finding isomorphisms between quaternion algebras over $L$. Our techniques are based on computing maximal one-sided ideals of the corestriction of a central simple $L$-algebra. In order to obtain efficient algorithms in the characteristic 2 case, we propose an algorithm for finding nontrivial zeros of a regular quadratic form in four variables over $\mathbb{F}_2[t]$.

Keywords: Central simple algebras, Corestriction of algebras, Computational complexity.

1 Introduction

In this paper we consider a special case of the following algorithmic problem. Let $K$ be a global field and let $A$ and $B$ be central simple algebras over $K$ given by a $K$-basis and a multiplication table of the basis elements. The product of two basis elements can be written as a linear combination of all the basis elements, the corresponding coefficients are called structure constants. The task is to decide whether $A$ and $B$ are isomorphic, and if so, find an explicit isomorphisms between them. A special case of this problem when $B = M_n(K)$ is referred to as the explicit isomorphism problem which has various applications in arithmetic geometry [2], [14], [13], computational algebraic geometry [10] and coding theory [20], [19]. In 2012, Ivanyos, Rónyai and Schicho [26] proposed an algorithm for the explicit isomorphisms problem in the case where $K$ is an algebraic number field. Their algorithm is a polynomial-time $tt$-algorithm (which means one is allowed to call an oracle for factoring integers and polynomials over finite fields) in the case where the dimension of the matrix algebra, the degree of the number field and the discriminant of the number field are all bounded (i.e., the algorithm is exponential in all these parameters). They also show that finding explicit isomorphisms between central simple $K$-algebras of dimension $n^2$ over $K$ can be reduced to finding an explicit isomorphism between an algebra $A$ and $M_{n^2}(K)$. Then in [20] (and independently in [13]) an algorithm was provided when $A$ is isomorphic to $M_2(\mathbb{Q}(\sqrt{d}))$ where the algorithm is polynomial in $\log(d)$ (whereas the algorithm of [20] is exponential in $\log(d)$). The case where $K = \mathbb{F}_q(t)$, the field of rational functions over a finite field was considered in [22] where the authors propose a randomized polynomial-time

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algorithm. The algorithm is somewhat analogous to the algorithm of [26] but it is polynomial in the dimension of the matrix algebra. Similarly to the number field case, this was extended to quadratic extensions (now with a restriction to odd characteristics) in [23]. A major problem with both [26] and [22] is that although they extend to field extensions, the complexity is exponential in the size of the discriminant and the degree of the field over $\mathbb{Q}$ or $\mathbb{F}_q(t)$. This obstacle comes from the fact that both algorithms use lattice reduction methods and one has to search through all lattice vectors in a ball of large radius. In this paper we initiate a new method for dealing with field extensions which is analogous to Galois descent. It is known that finding an explicit isomorphism between $A$ and $M_n(K)$ is polynomial-time equivalent to finding a rank 1 element in $A$. Thus if one could find a subalgebra of $A$ isomorphic to $M_n(\mathbb{Q})$ or $M_n(\mathbb{F}_q(t))$, then one could apply the known algorithms for the subalgebra and that would give an exponential speed-up in both cases. Furthermore, these types of methods should work equally for the function field and number field case which have completely different applications. In [30] and [23] this type of method is studied. In both cases one finds a central simple algebra over the smaller field in $A$ which is not necessarily a matrix algebra but when it is a division algebra, then it is split by the quadratic field (the center of $A$) which can be exploited. The disadvantage of these methods is that they are based on explicit calculations and reductions to finding nontrivial zeros of quadratic forms which do not generalize easily to higher extensions. In this paper we reprove that result and extend it to the isomorphism problem of two quaternion algebras over a quadratic extension. The main technique is to compute a maximal right-ideal of the corestriction of the algebra $A$ (which is an explicit construction corresponding to the usual corestriction on cohomology groups) and apply it to construct an involution of the second kind on $A$. In general this might not be useful, but when $A$ possesses a canonical involution of the first kind, then composing the two kinds of involutions and taking fixed points gives us the central simple subalgebra over a smaller field. Fortunately, tensor products of quaternion algebras carry a canonical involution of the first kind which is exactly what we need.

Our goal is to have efficient algorithms for computing explicit isomorphisms between quaternion algebras over any separable quadratic global field which thus includes function fields of characteristic two. Our other main contribution is to extend the results of [23] to the characteristic two case, thus we show how to find nontrivial zeros of quadratic forms in four variables and use this result to solve the isomorphisms problem of quaternion algebras over separable quadratic extensions in the characteristic 2 case.

The paper is structured as follows. In Section 2 we recall necessary theoretical and algorithmic results. In Section 3 we outline how to compute involutions of the second kind which is the basis of our descent method. In Section 4 we show how find nontrivial zeros of four-variable quadratic forms over $\mathbb{F}_{2^l}(t)$ and apply it to finding zero divisors in quaternion algebras in separable quadratic extensions of $\mathbb{F}_{2^l}(t)$. Finally, in Section 5 we propose an algorithm for computing isomorphisms between quaternion algebras over quadratic global fields (in all characteristics).

2 Preliminaries

In this section we recall theoretical and algorithmic results needed in later sections.

2.1 Central simple algebras

The following results can be found in [33, Chapter 12]. First we define the center of an algebra.
Definition 2.1. Let \( A \) be an algebra over a field \( K \). Then \( Z(\mathcal{A}) \), the center of \( A \), consists of those elements which commute with every element of the algebra.

Note that the identity element \( 1 \) is always contained in \( Z(\mathcal{A}) \). Moreover, \( K \cdot 1 \) is also contained in the center of \( A \). Identifying \( K \) with \( K \cdot 1 \), we may assume that \( K \subseteq Z(\mathcal{A}) \).

Definition 2.2. An algebra \( \mathcal{A} \) over the field \( K \) is simple if it contains no proper two-sided ideals.

The center of a simple algebra is always a field. This motivates the following definition.

Definition 2.3. Let \( K \) be a field and let \( \mathcal{A} \) be a simple algebra over \( K \). Then \( \mathcal{A} \) is a central simple algebra over \( K \) if \( Z(\mathcal{A}) = K \).

Recall that \( K \subseteq Z(\mathcal{A}) \), whence \( \mathcal{A} \) is a central simple \( K \)-algebra if its center is equal to \( K \).

An important property of central simple algebras is the following:

Theorem 2.4 (Wedderburn). A finite-dimensional central simple algebra \( A \) is isomorphic to the full matrix algebra \( M_n(D) \) for some division ring \( D \).

Definition 2.5. A central simple algebra \( \mathcal{A} \) over the field \( K \) that has dimension 4 over \( K \) is called a quaternion algebra.

Theorem 2.4 implies that a quaternion algebra over \( K \) is either a division algebra or is isomorphic to the algebra of \( 2 \times 2 \) matrices over \( K \).

2.2 The Brauer group

In this section we recall some facts about the Brauer group. Our main reference is [18].

Definition 2.7. We call the central simple \( K \)-algebras \( A \) and \( B \) Brauer equivalent if there exist integers \( m, m' > 0 \) such that \( A \otimes_K M_m(K) \cong B \otimes_K M_{m'}(K) \). The Brauer equivalence classes of central simple \( K \)-algebras form a group under tensor product over \( K \). This group is called the Brauer group \( \text{Br}(K) \) of \( K \).

In order to state the cohomological interpretation of the Brauer group we need to introduce some further notation. For a field \( K \) we put \( K_{\text{sep}} \) for a fixed separable closure of \( K \) and \( G_K := \text{Gal}(K_{\text{sep}}/K) \) for the absolute Galois group.

Theorem 2.8. [18, Thm. 4.4.3] Let \( K \) be a field. Then the Brauer group \( \text{Br}(K) \) is naturally isomorphic to the second Galois cohomology group \( H^2(G_K, K_{\text{sep}}^*) \).

For specific fields one can even determine the Brauer group explicitly. The case of local fields is treated by the following famous result of Hasse.

Proposition 2.9 (Hasse). [18, Prop. 6.3.7] Let \( K \) be a complete discretely valued field with finite residue field. Then we have a canonical isomorphism

\[ \text{Br}(K) \cong \mathbb{Q}/\mathbb{Z}. \]
Moreover for a finite separable extension $L/K$ there are commutative diagrams

$$
\begin{array}{ccc}
\Br(L) & \xrightarrow{\text{Cor}} & \Q / \Z \\
\downarrow \mathrm{id} & & \downarrow \mathrm{id} \\
\Br(K) & \xrightarrow{\text{Res}} & \Br(L) \oplus \Q / \Z
\end{array}
$$

where the right vertical map in the second diagram is the multiplication by the degree $[L: K]$.

The map inducing the isomorphism $\Br(K) \cong \Q / \Z$ is classically called the **Hasse invariant map**. Note that in the archimedean case Frobenius’ Theorem on division rings over the real numbers $\mathbb{R}$ is equivalent to the fact $\Br(\mathbb{R}) = \frac{1}{2}\mathbb{Z} / \mathbb{Z} \subset \Q / \Z$. Finally, since $\mathbb{C}$ is algebraically closed, we have $\Br(\mathbb{C}) = 0$.

Now let $K$ be a **global field**, i.e. either a number field (finite extension of $\Q$) or the function field $K = \mathcal{F}(C)$ of a smooth projective curve $C$ over a finite field $\mathcal{F}$. Denote by $\mathcal{P}$ the set of (finite and infinite) places of $K$, i.e. in the function field case $\mathcal{P}$ is the set $\mathcal{C}_0$ of closed points on $C$ and in the number field case $\mathcal{P}$ consists of the prime ideals in the ring of integers of $K$ and the set of equivalence classes of archimedean valuations on $K$. For a place $P \in \mathcal{P}$ we denote by $K_P$ the completion of $K$ at $P$. If $A$ is a central simple algebra over $K$ then $A_P := A \otimes_K K_P$ is a central simple algebra over $K_P$. This induces a natural map $\Br(K) \to \Br(K_P) \cong \Q / \Z$. Note that every central simple algebra $A$ splits at all but finitely many places, i.e. we have $\inv_P([A_P]) = 0$ for all but finitely many $P$. Using the main results of class field theory one obtains the following classical theorem of Hasse.

**Theorem 2.10 (Hasse).** [18 Cor. 6.5.3, Rem. 6.5.5] For any global field $K$ we have an exact sequence

$$0 \to \Br(K) \to \bigoplus_{P \in \mathcal{P}} \Br(K_P) \xrightarrow{\Sigma \inv_P} \Q / \Z \to 0.$$ 

Note that the Hasse-invariant of a nonsplit quaternion algebra over a local field is $\frac{1}{2}$. In particular, any quaternion algebra $A$ over $K$ splits at an even number of places. Further, for any finite subset $S \subset \mathcal{P}$ of even cardinality there exists a unique quaternion algebra (upto isomorphism) over $K$ that splits exactly at the places in $\mathcal{P} \setminus S$. This is usually referred to as Hilbert’s reciprocity law.

### 2.3 The corestriction of a central simple algebra

Due to the fact that the Brauer group admits a cohomological interpretation, one can use standard techniques from Galois cohomology to analyze central simple algebras. Let $L$ be a finite Galois extension of $K$ (contained in the fixed separable closure $K_{\text{sep}}$). Let $G_K$ and $G_L$ be the absolute Galois group of $K$ and $L$ respectively. There are two standard maps to analyze: restriction, which is a map from $H^2(G_K, K_{\text{sep}}^\times)$ to $H^2(G_L, K_{\text{sep}}^\times)$ and corestriction which is a map from $H^2(G_L, K_{\text{sep}}^\times)$ to $H^2(G_K, K_{\text{sep}}^\times)$.

For our purposes we need explicit descriptions of these maps on central simple algebras. The restriction map is easy, one just considers the extensions of scalars by $L$ (i.e., the map $A \mapsto A \otimes_K L$). However the corestriction map is more complicated. We describe the corestriction map when $L$ is a separable quadratic extension of $K$ (which implies that it is a Galois extension). This discussion is taken from [23 Section 3B] (in that book the corestriction is called the norm of an algebra).
Definition 2.11. Let $L$ be a separable quadratic extension of $K$. Let $\sigma$ be a generator of $\text{Gal}(L/K)$. Let $A$ be a central simple algebra over $L$. Then the conjugate algebra $A^{\sigma}$ is defined in the following way. As a set $A^{\sigma} = \{ a^\sigma \mid a \in A \}$ (i.e., the same elements as $A$ just labelled with $\sigma$). Addition, multiplication, and multiplication by a scalar are defined in the following way:

$$a^{\sigma} + b^{\sigma} = (a + b)^{\sigma}, \quad a^{\sigma}b^{\sigma} = (ab)^{\sigma}, \quad (\lambda \cdot a)^{\sigma} = \sigma(\lambda) a^{\sigma}$$

$A^{\sigma}$ is also a central simple $L$-algebra and the induced map $\sigma$ provides a $K$-isomorphism between them. In terms of structure constants, $A^{\sigma}$ can be obtained by conjugating the structure constants of $A$.

Definition 2.12. The switch map $s$ is a map from $A \otimes_{L} A^{\sigma}$ to itself. The map $s$ is defined on elementary tensors as $s(a \otimes b^{\sigma}) = b \otimes a^{\sigma}$ and extended $K$-linearly. It is semilinear and a $K$-algebra automorphism.

Proposition 2.13. [29, Proposition 3.13.] The elements of $A \otimes A^{\sigma}$ invariant under the switch map form a subalgebra which is a central simple algebra over $K$ of dimension $\dim_{K}(A)^{2}$ over $K$.

The algebra in Proposition 2.13 is called the corestriction of $A$ (with respect to the extension $L/K$). It corresponds to the corestriction map of Galois cohomology (and it is also true that $\text{Cor} \circ \text{Res}$ is multiplication by $n$ in the Brauer group of $K$ but we will not use this fact in this paper). Our main application of the corestriction maps concerns involutions of central simple algebras which we introduce in the next subsection.

2.4 Involutions

This subsection is based on [29, Section 2 and 3]. Let $A$ be an algebra over a field $K$. An involution of $A$ is a map $\tau : A \rightarrow A$ with the following properties:

1. $\tau(a + b) = \tau(a) + \tau(b)$ for every $a, b \in A$
2. $\tau(ab) = \tau(b)\tau(a)$ for every $a, b \in A$
3. $\tau(\tau(a)) = a$ for every $a \in A$

If $A$ is a full matrix algebra over a field $K$, then the transpose of a matrix is the most general example of an involution. Furthermore, if $K$ has an automorphism of order two, then the adjoint of a matrix (i.e., composing the transpose of the matrix with the nontrivial automorphism) is also an involution.

Let $A$ be a central simple algebra over a field $L$ and let $\tau$ be an involution on $A$. Then if one restricts $\tau$ to the centre of $A$, then it is an automorphism of $L$ whose order is at most two. This provides a distinction between two types of involutions:

Definition 2.14. Let $A$ be a central simple algebra over a field $L$ and let $\tau$ be an involution on $A$. If $\tau$ fixes the center of $A$, then $\tau$ is an involution of the first kind, otherwise it is called an involution of the second kind (or a unitary involution).

A classical example of an involution of the first kind is the quaternion conjugation of a quaternion algebra. If two central simple algebras admit an involution of the first kind, then so does their tensor product. Indeed, let $A_1, \tau_1$ and $A_2, \tau_2$ be central simple algebras with respective involutions of the first kind. Then $\tau_1 \otimes \tau_2$ defined on elementary tensors as $\tau_1 \otimes \tau_2(a \otimes b) = \tau_1(a_1) \otimes \tau_2(a_2)$ is an involution of the first kind on $A_1 \otimes A_2$. Furthermore, a central simple algebra admits an involution of the first kind if and only if it has order at most two in the
Brauer group [29, Theorem 3.1] (which is a classical theorem of Albert). By the Merkuriev-Suslin theorem [18, Theorem 2.5.7] such an algebra is always isomorphic to the tensor product of quaternion algebras.

The existence theorem for unitary involutions uses the notion of corestriction [29, Theorem 3.1].

**Theorem 2.15.** Let $L/K$ be a quadratic Galois extension and let $A$ be a central simple algebra over $L$. Then $A$ admits an involution of the second kind if and only if the corestriction of $A$ is split.

The proof of this theorem in [29] is constructive which we will exploit in later sections.

2.5 Quadratic forms and quaternion algebras in characteristic 2

In this subsection we recall important facts about quadratic forms and quaternion algebras in characteristic 2. Our main source is [40, Chapter 6]. From here on $F$ will always denote a field with characteristic 2.

**Lemma 2.16.** [40, Chapter 6] For every quaternion algebra $A$ over $F$ there exists an $F$-basis $1, i, j, k$ of $A$ such that $i^2 + i = a, j^2 = b, \text{ and } k = ij = j(i+1)$ where $a, b \in F$.

We denote the quaternion algebra over $F$ with parameters $a, b$ as $[a, b]_F$.

**Definition 2.17.** A quadratic form over $F$ is a homogeneous polynomial $Q$ of degree two in $n$ variables $x_1, \ldots, x_n$ for some $n$. We say that $Q$ is isotropic if there exist $a_1, \ldots, a_n \in F$ not all zero such that $Q(a_1, \ldots, a_n) = 0$. If $Q$ is not isotropic, we say that $Q$ is anisotropic.

We can also view a quadratic form $Q$ with $n$ variables over $F$ as a $Q : F^n \rightarrow F$ function. This motivates the following definition.

**Definition 2.18.** We say that two quadratic forms $Q_1$ and $Q_2$ are isometric if there exists a $\varphi : F^n \rightarrow F^n$ invertible linear map such that $Q_1 \circ \varphi = Q_2$.

**Definition 2.19.** Let $Q_1$ and $Q_2$ be diagonal quadratic forms in $n$ variables. We call $Q_1$ and $Q_2$ similar if there exist a quadratic form $Q'$ that is isometric to $Q_2$ and such that $Q'$ can be obtained from $Q_1$ by multiplication of $Q_1$ by a non-zero $g \in F$.

Even though if char $F = 2$, not all quadratic forms can be diagonalized (we get $ax^2 + axy + by^2$ as the general form), the following can be said about quadratic forms in four variables.

**Lemma 2.20.** [13, Cor. 7.32] Every regular quadratic form in four variables over $F$ is equivalent to a quadratic form in the form of

$$a_1x_1^2 + x_1x_2 + b_1x_2^2 + a_3x_3^2 + x_3x_4 + b_2x_4^2$$

where $a_1, a_3, b_1, b_2 \in F$.

**Corollary 2.21.** Every regular quadratic form in four variables over $F$ is equivalent to a quadratic form in the form of

$$a_1x_1^2 + a_1x_1x_2 + a_1a_2x_2^2 + a_3x_3^2 + a_3x_3x_4 + a_3a_4x_4^2$$

where $a_1, a_2, a_3, a_4 \in F$. 
Proof. We start from the canonical form described in Lemma 2.20. After substituting \( x_2 \leftarrow a_1 x_2 \) and \( x_4 \leftarrow a_3 x_4 \), we get that \( a_1 x_1^2 + a_1 x_1 x_2 + a_3 x_3 x_4 + a_3 x_3 x_4 + a_3^2 b_1 x_2^2 + a_3 x_3 x_4 + a_3 b_2 x_1^2 \). After setting \( a_2 = a_1 b_1 \) and \( a_4 = a_3 b_2 \) we arrive to the form \( a_1 x_1^2 + a_1 x_1 x_2 + a_3 x_3 x_4 + a_3 x_3 x_4 + a_3 a_4 x_1^2 \).

The following lemma [40, Theorem 6.4.11] highlights a connection between the isotropy of quadratic forms and the splitting of quaternion algebras:

**Lemma 2.22 (Hilbert equation).** A quaternion algebra \( \left\langle \frac{a}{F} \right\rangle \) is split if and only if \( bx^2 + bxy + aby^2 = 1 \) has a solution with \( x, y \in F \).

In order to handle quadratic forms, just like in odd characteristics, we will need to introduce a quadratic residue symbol. If \( F \) is a finite field of characteristic 2 and \( \pi \) is an irreducible polynomial in \( F[t] \), then every element in \( F[t]/(\pi) \) will be a square (as the factor ring is a finite field of characteristic 2), so the definition will need to differ slightly. The following definition and claim with proof can be found in [3].

**Definition 2.23.** For a monic irreducible \( \pi \) in \( F[t] \) and any \( f \in F(t) \) that has no pole at \( \pi \), let

\[
[f, \pi] := \begin{cases} 0, & \text{if } f \equiv x^2 + x \pmod{\pi} \text{ for some } x \in F[t] \\ 1, & \text{otherwise} \end{cases}
\]

If \( [f, \pi] = 0 \), we say that \( f \) is a **quadratic residue** modulo \( \pi \). Similarly, for the place at \( \infty \) we define

\[
[f, \infty] := \begin{cases} 0, & \text{if } f \equiv x^2 + x \pmod{t^{-1}} \text{ for some } x \in F[t^{-1}] \\ 1, & \text{otherwise} \end{cases}
\]

whenever \( f \in F(t) \) has no pole at \( \infty \) (i.e. \( \deg(f) \leq 0 \)). If \( f \) has a pole at the (finite or infinite) place \( \pi \) then \( [f, \pi] \) has no meaning.

**Claim 2.24.** The symbol \([f, \pi]\) has the following properties:

1. if \( f_1 \equiv f_2 \pmod{\pi} \), then \([f_1, \pi] = [f_2, \pi]\),
2. \([f, \pi] \equiv f + f^2 + \ldots + f^{d_{\pi}/2} \pmod{\pi} \), where \( q = |F| \),
3. \([f_1 + f_2, \pi] = [f_1, \pi] + [f_2, \pi]\),
4. \([f^2 + f, \pi] = 0\).

### 2.6 Further local and local to global statements

Here we list some classical results (independent of the residue characteristics) that we need in the sequel.

**Lemma 2.25 (Hensel).** Let \( K \) be a complete valued field, \( O \) its valuation ring and \( P \) the unique maximal ideal in \( O \). If a primitive polynomial \( f(x) \in O[x] \) admits modulo \( P \) a factorization

\[
f(x) \equiv \overline{g}(x) \overline{h}(x) \pmod{O}
\]

into relatively prime polynomials \( \overline{g}, \overline{h} \in k[x] \) then \( f(x) \) admits a factorization

\[
f(x) = g(x)h(x)
\]

into polynomials \( g, h \in O[x] \) such that \( \deg(g) = \deg(\overline{g}) \) and

\[
g(x) \equiv \overline{g}(x) \pmod{P}, \quad h(x) \equiv \overline{h}(x) \pmod{P}.
\]
Theorem 2.27. Let \( \mathbb{F} \) be a finite field and \( f \in \mathbb{F}[x_1, x_2, \ldots, x_n] \) such that \( n > \deg f \). Then, if \( x_1 = x_2 = \ldots = x_n = 0 \) is a solution of \( f(x_1, \ldots, x_n) = 0 \), there also exists a nontrivial solution of \( f(x_1, \ldots, x_n) = 0 \).

We state a variant of the Hasse-Minkowski theorem over the field \( \mathbb{F}(t) \) of rational functions over a finite field \( \mathbb{F} \). It was proved by Hasse’s doctoral student Herbert Rauter in 1926.

Theorem 2.27. A non-degenerate quadratic form over \( \mathbb{F}(t) \) is isotropic over \( \mathbb{F}(t) \) if and only if it is isotropic over every completion of \( \mathbb{F}(t) \).

For ternary quadratic forms there exists a slightly stronger version of this theorem which is a consequence of the product formula for quaternion algebras or Hilbert’s reciprocity law [31, Chapter IX, Theorem 4.6] (see also the discussion after Theorem 2.10).

Theorem 2.28. Let \( Q \) be a ternary non-degenerate quadratic form over \( \mathbb{F}(t) \). Then if it is isotropic in every completion except maybe one then it is isotropic over \( \mathbb{F}(t) \).

2.7 Algorithmic preliminaries

In this subsection we give a brief overview of known algorithmic results in this context. Let \( K \) be a field and let \( A \) be an associative algebra given by the following presentation. One is given a \( K \)-basis \( b_1, \ldots, b_m \) of \( A \) and a multiplication table of the basis elements, i.e. \( b_i b_j \) expressed as a linear combination \( \sum_{k=1}^{m} \gamma_{i,j,k} b_k \). These \( \gamma_{i,j,k} \) are called structure constants and we consider our algebra given by structure constants. It is a natural algorithmic problem to compute the structure of \( A \), i.e., compute its Jacobson radical \( R \), compute the Wedderburn decomposition of \( A / R \) and finally compute an explicit isomorphism between the simple components of \( A / R \) and \( M_n(D_i) \) where the \( D_i \) are division algebras over \( K \) and \( M_n(D_i) \) denotes the algebra of \( n \times n \) matrices over \( D_i \). The problem has been studied for various fields \( K \), including finite fields, the field of complex and real numbers, global function fields and algebraic number fields. There exists a polynomial-time algorithm for computing the radical of \( A \) over any computable field \([4]\). There also exist efficient algorithms for every task over finite fields \([17, 36]\) and the field of real and complex numbers \([11]\). Finally, when \( K = \mathbb{F}(t) \), the field of rational functions over a finite field \( \mathbb{F}_q \), then there exist efficient algorithms for computing Wedderburn decompositions \([27]\).

This motivates the algorithmic study of computing isomorphisms between simple algebras. Over finite fields this can be accomplished in polynomial time using the results from \([17, 36]\) and \([53]\). Over number fields there is an immediate obstacle. Rónyai \([53]\) showed that this task is at least as hard as factoring integers. However, in most interesting applications factoring is feasible, thus it is a natural question to ask whether such an isomorphism can be computed if one is allowed to call an oracle for factoring integers. In \([26]\) the authors propose such an algorithm for number fields, however, their algorithm is exponential in the degree of the number field,
the size of the discriminant of the number field and the degree of the matrix algebra. When all these are bounded then the algorithm runs in polynomial time. This essentially provides an algorithm for a finite number of cases. Then in [30] it is shown that the original algorithm can be improved in the case where \( A \) is isomorphic to \( M_2(\mathbb{Q}(\sqrt{d})) \), i.e., there exists a polynomial-time algorithm (modulo factoring integers) for computing this isomorphism. The key here is that the running time is polynomial in \( \log d \) (which was not true for the original algorithm of [26]).

When \( K = \mathbb{F}_q(t) \) where \( q \) is an odd prime power, the situation is slightly different. In [22] the authors propose a polynomial time algorithm for computing an explicit isomorphism between \( A \) and \( M_2(\mathbb{F}_q(t)) \), i.e., there exists a polynomial-time algorithm (modulo factoring integers) for computing this isomorphism. The key here is that the running time is polynomial in \( \log d \) (which was not true for the original algorithm of [26]).

In [23] and [19] one of the main techniques is to use an explicit bound on the number of monic irreducible polynomials in a given residue class [41]:

**Fact 2.29.** Let \( a, m \in \mathbb{F}_q[t] \) be such that \( \deg(m) > 0 \) and the \( \gcd(a, m) = 1 \). Let \( N \) be a positive integer and let

\[
S_N(a, m) = \# \{ f \in \mathbb{F}_q[t] \text{ monic irreducible} \mid f \equiv a \pmod{m}, \deg(f) = N \}.
\]

Let \( M = \deg(m) \) and let \( \Phi(m) \) denote the number of polynomials in \( \mathbb{F}_q[t] \) relative prime to \( m \) whose degree is smaller than \( M \). Then we have the following inequality:

\[
|S_N(a, m) - \frac{q^N}{\Phi(m)N}| \leq \frac{1}{N} (M + 1) q^N.
\]

The above fact allows for an efficient way of finding an irreducible polynomial of a given degree from a certain residue class. Namely, one chooses a uniformly random polynomial from that residue class (of a prescribed degree) and iterates until finding an irreducible polynomial (irreducibility can be checked with Berlekamp’s algorithm [1]). A detailed analysis of this method can be found in both [23] and [19].

**Remark 2.30.** From now we will not make a distinction between deterministic and randomized polynomial-time algorithms we will refer to them as polynomial-time algorithms.

### 3 The descent method

Let \( K \) be a field and let \( L \) be a separable quadratic extension of \( K \). Let \( A \) be a central simple algebra over \( L \) given by structure constants. Our goal in this section is to find a subalgebra of \( A \) which is a central simple algebra over \( K \). In other words, we would like to decompose \( A \) as a tensor product \( B \otimes_K L \) when this is possible.

Our first step is to construct an involution of the second kind on \( A \) if such an involution exists. The following lemma [29, Theorem 3.17.] provides a useful relationship between certain right ideals of the corestriction of \( A \) and involutions of the second kind:
**Lemma 3.1.** Let $A$ be a central simple algebra over $L$ of dimension $n^2$ where $L$ is a separable quadratic extension of the field $K$. Put $B$ for the corestriction of $A$ with respect to $L/K$. Assume that there exists a right ideal $I$ of $B$ such that $B \otimes_K L = A'' \otimes_L A = I_L \oplus (1 \otimes A)$ where $I_L = I \otimes_K L$. Then $A$ admits an involution of the second kind.

**Proof.** We sketch the proof here. For each $a \in A$ there exists a unique element $\tau_I(a) \in A$ such that

\begin{equation*}
\sigma^a \otimes 1 - 1 \otimes \tau_I(a) \in I_L.
\end{equation*}

One can check that the map $a \mapsto \tau_I(a)$ is indeed an involution of the second kind on $A$. \qed

Suppose we have access to an algorithm which can find a maximal one-sided ideal in an algebra $C$ which is isomorphic to $M_{4^2}(K)$ and is given by a structure constant representation. In the next theorem we show that one can either construct an ideal $I$ in the corestriction of $A$ as required by Lemma 3.1 (which implies by using Lemma 3.1 that we can construct an involution of the second kind on $A$) or one can construct a zero divisor in $A$.

**Theorem 3.2.** Let $L$ be a separable quadratic extension of a field $K$. Let $A$ be a central simple algebra over $L$ of dimension $n^2$ which admits an involution of the second kind. Suppose that we can compute maximal right ideals in algebras given by structure constants which are isomorphic to $M_{4^2}(K)$. Then there exists a polynomial-time algorithm which either returns a zero divisor of $A$ or an involution of the second kind on $A$.

**Proof.** Let $B$ be the corestriction of $A$. Our assumptions together with Theorem 2.15 imply that $B$ is split. By Lemma 3.1 it suffices to compute a right ideal $I$ of $B$ with the property that $A'' \otimes_L A = I_L \oplus (1 \otimes A)$. Compute a maximal right ideal $I$ in $B$. Let $I_L = I \otimes L$ be the scalar extension of $I$ in $A'' \otimes A$. Compute the intersection of $I_L$ and $1 \otimes A$. If this intersection is nontrivial, then we have computed a zero divisor in $A$, since every element in $I_L$ is a zero divisor. Thus we may assume that the intersection of $I_L$ and $1 \otimes A$ is trivial. In that case, however, $I$ is a right ideal with the property that $A'' \otimes_L A = I_L \oplus (1 \otimes A)$ by dimension considerations which allows us to construct an involution of the second kind. \qed

The above proof is particularly interesting when one is looking for zero divisors in quaternion algebras. If it does not return a zero divisor, then it returns an involution of the second kind. In that case, one can compose the involution of the second kind with the canonical involution of the first kind (conjugation) and look at the fixed points of this map. This is clearly a $K$-algebra automorphism, thus the fixed points form a $K$-subalgebra which is a quaternion algebra over $K$. This is summarized in the following proposition:

**Proposition 3.3.** Let $L$ be a separable quadratic extension of $K$ and suppose we know an algorithm for finding explicit isomorphisms between degree 4 split central simple algebras given by structure constants and $M_4(K)$. Let $A$ be a quaternion algebra over $K$. Then one can find a quaternion subalgebra of $A$ over $K$ in polynomial time.

In [30] and in [23] this is proven for $K = Q$ or $K = F_q(t)$ (where $q$ is an odd prime power) using explicit calculations and utilizing algorithms for finding nontrivial zeros of quadratic form. Proposition 3.3 shows a more conceptual method for computing subalgebras which avoids tedious calculations. Furthermore, this proposition applies to quaternion algebras in characteristic 2 as well. Since there exist efficient algorithms for finding maximal left ideals in $M_4(K)$ where $K = F_{2^k}(t)$ [22], Proposition 3.3 implies the following:

**Corollary 3.4.** Let $L$ be a separable quadratic extension of $K = F_{2^k}(t)$ and $A$ be a quaternion algebra over $L$. There exists a polynomial-time algorithm which computes a quaternion subalgebra over $K$ of $A$ if such a quaternion algebra exists.
Let $L$ be a quadratic extension $K = \mathbb{F}_{2^k}(t)$ and $A$ be an algebra isomorphic to $M_2(L)$ given by structure constants. Corollary 4.2 shows that we can follow a similar approach to finding zero divisors in $A$ as laid out in [30]: find a subalgebra $B$ of $A$ which is a quaternion algebra over $K$ and then find a subfield isomorphic to $L$ in $B$. In the next section we provide an algorithm for finding nontrivial zeros of quadratic forms over $\mathbb{F}_{2^k}(t)$ and then apply this algorithm to finding the subfield $L$ in $B$.

4 Finding nontrivial zeros of quadratic forms over $\mathbb{F}_{2^k}(t)$

4.1 Local lemmas

We denote by $v_f$ the $f$-adic valuation on $\mathbb{F}_{2^k}(t)$ for a (finite or infinite) prime $f \in \mathbb{F}_{2^k}(t)$, by $\mathbb{F}_{2^k}(t)(f)$ the $f$-adic completion, and by $\mathbb{F}_{2^k}(t)_{(f)} := \{ u \in \mathbb{F}_{2^k}(t)(f) \mid v_f(u) \geq 0 \}$ its valuation ring.

We are interested in the range of the quadratic form $x^2 + xy + ay^2$ for some $a \in \mathbb{F}_{2^k}(t)$.

Definition 4.1. For $0 \neq a \in \mathbb{F}_{2^k}(t)$ we call the quadratic form $x^2 + xy + ay^2$ minimal if all the poles of $a$ (including $\infty$) have odd multiplicity.

Note that for a finite prime $f$ the multiplicity of the the pole of $a$ is by definition the exponent of $f$ in the denominator of $a$. The multiplicity of the pole of $a$ at $\infty$ is the degree of $a$ if it is positive and 0 otherwise.

Lemma 4.2. Any quadratic form $x^2 + xy + ay^2$ with $0 \neq a \in \mathbb{F}_{2^k}(t)$ is equivalent to a minimal form.

Proof. Assume $a = \frac{g}{f^2h_1}$ with $f$ odd, $g$, $h_1$ for some finite prime $f$. Since $\mathbb{F}_{2^k}[t]/(f)$ is a finite field of characteristic 2, the $2$-Frobenius is bijective on $\mathbb{F}_{2^k}[t]/(f)$. In particular, there exists a polynomial $g \in \mathbb{F}_{2^k}[t]$ such that $f \mid g^2h_1 + g_1$. So we may replace the variable $x$ by $x_1 = x + \frac{g_1}{f^2}$ to obtain

$$x^2 + xy + ay^2 = x_1^2 + \frac{g^2y^2}{f^2} + x_1y + \frac{g_1y^2}{f^2} + \frac{g_1}{f^2h_1} = x_1^2 + x_1y + \frac{g^2h_1 + g_1 + f^2h_1g_1}{f^2h_1}y^2$$

and $a' := \frac{g^2h_1 + g_1 + f^2h_1g_1}{f^2h_1}$ has one less $f$ in the denominator. Repeating the process for all finite primes in the denominator of $a$ we are reduced to handle the case of the infinite prime. This is entirely analogous: assume we have $a = \frac{h_1}{h_1}$ with $2r := \deg g_1 - \deg h_1$ even and positive. Since the leading coefficient of $a$ is a square in $\mathbb{F}_{2^k}$, there exists $0 \neq c \in \mathbb{F}_{2^k}$ such that $\deg (g_1 + c^2t^2h_1) < \deg g_1$. Therefore putting $x_1 = x + ct'y$ we obtain the form

$$x^2 + xy + ay^2 = x_1^2 + c^2t^2y^2 + x_1y + ct'y^2 + \frac{g_1}{h_1}y^2 =$$

$$= x_1^2 + x_1y + \frac{h_1ct' + h_1c^2t^2 + g_1}{h_1}y^2$$

such that $a' = a + ct' + c^2t^2 = \frac{h_1ct' + h_1c^2t^2 + g_1}{h_1}$ has smaller degree than $a$. Repeating this step several times we deduce the statement. \qed
Remark 4.3. The above proof also shows that the minimal form of \( x^2 + xy + ay^2 \) is unique up to an additive constant of the form \( a^2 + a \) with \( a \in \mathbb{F}_2 \).

By the local-global principle (Theorem 2.27) we are reduced to identifying the range of a minimal quadratic form \( x^2 + xy + ay^2 \) locally at each place \( f \) of \( \mathbb{F}_2(t) \). Note that \( c \) is in the range of the quadratic form \( x^2 + xy + ay^2 \) if and only if so is \( cd^2 \) for all \( 0 \neq d \in \mathbb{F}_2(t) \) therefore we may rescale \( c \) by a square element as convenient. (Note that \( c = 0 \) is obviously in the range.) We distinguish two cases whether or not \( a \) has a pole at \( f \). At first we treat the case when \( a \) is an \( f \)-adic integer.

**Lemma 4.4.** Assume \( v_f(a) \geq 0 \).

1. If \( v_f(c) \) is even then the equation \( x^2 + xy + ay^2 = c \) has a solution in \( \mathbb{F}_{2^k}(t)_{(f)} \).
2. If \( v_f(c) \) is odd then the equation \( x^2 + xy + ay^2 = c \) has a solution in \( \mathbb{F}_{2^k}(t)_{(f)} \) if and only if \( [a,f] = 0 \).

**Proof.** Note that by rescaling we may assume without loss of generality that \( v_f(c) = 0 \) or 1.

First assume that \( f \nmid c \). If \( f \nmid a \), choosing \( x \equiv 0 \) (mod \( f \)), \( y^2 \equiv c/a \) (mod \( f \)) will be a solution modulo \( f \). If \( f \mid a \), then let \( y \equiv 0 \) (mod \( f \)), \( x^2 \equiv c \) (mod \( f \)), this provides a nontrivial solution.

Since the derivative of this quadratic form is nonzero at either of these mod \( f \) zeros, by Hensel’s lemma (Lemma 2.28) there exists a solution in \( \mathbb{F}_{2^k}(t)_{(f)} \).

In case \( c \equiv 0 \) (mod \( f \)), we get \( x^2 + xy + ay^2 = 0 \). If \( y \equiv 0 \) (mod \( f \)), it also means that \( x \equiv 0 \) (mod \( f \)), which cannot happen as \( v_f(c) = 1 \). Now dividing the equation by \( y^2 \) and substituting \( z = x/y \), we get \( z^2 + z + a \equiv 0 \) (mod \( f \)). By definition it admits a solution if and only if \( [a,f] = 0 \). Again, since the derivative this quadratic form is nonzero, if it has a solution modulo \( f \) then is it also solvable in \( \mathbb{F}_{2^k}(t)_{(f)} \) by Hensel’s lemma.

**Lemma 4.5.** Let \( a_1, a_3 \in \mathbb{F}_2[t] \) be square-free polynomials with no common divisor and \( a_2, a_4 \in \mathbb{F}_2[t] \).

Let \( f \) be a place, i.e. either a monic irreducible polynomial or \( f = \infty \) such that \( v_f(a_1a_3) \) is odd. Assume that neither \( a_2 \) nor \( a_4 \) has a pole at \( f \). Then the equation \( a_1x_1^2 + a_1x_1x_3 + a_1a_2x_2^2 + a_3x_3^2 + a_3x_3x_4 + a_3a_4x_4^2 = 0 \) has a nontrivial solution in \( \mathbb{F}_{2^k}(t)_{(f)} \) if and only if at least one of the two conditions holds:

1. \([a_2,f] = 0\)
2. \([a_4,f] = 0\)

**Proof.** Assume \( f \) is finite. Then our condition that \( v_f(a_1a_3) \) is odd means \( f \) divides either \( a_1 \) or \( a_3 \). First we show that if these conditions hold then our equation admits a nontrivial solution.

Without loss of generality we can assume that \( f \mid a_1 \). Now our equation reduces to \( x_3^2 + x_3x_4 + a_4x_4^2 = 0 \) since \( f \nmid a_3 \). By Lemma 4.3 if \([a_4,f] = 0\), this has a nontrivial solution.

Now suppose that \([a_4,f] = 1 \) and \([a_2,f] = 0 \). This means that \( x_3 \) and \( x_4 \) must be divisible by \( f \), let \( x_3 = x_3'f \), \( x_4 = x_4'f \). Dividing the equation by \( f \) we get that

\[
(a_1/f)x_1^2 + (a_1/f)x_1x_2 + (a_1/f)a_2x_2^2 + fa_3x_3'x_4 + fa_3a_4x_4'^2 = 0
\]

which reduces to

\[
(a_1/f)x_1^2 + (a_1/f)x_1x_2 + (a_1/f)a_2x_2^2 = 0
\]

modulo \( f \). Again, by Lemma 4.3 this is solvable in \( \mathbb{F}_{2^k}(t)_{(f)} \).

Now we will prove that if \([a_2,f] = 1 \) and \([a_4,f] = 1 \) then there is no solution in \( \mathbb{F}_{2^k}(t)_{(f)} \).

If the equation \( a_1x_1^2 + a_1x_1x_2 + a_1a_2x_2^2 + a_3x_3^2 + a_3x_3x_4 + a_3a_4x_4^2 = 0 \) has a solution in \( \mathbb{F}_{2^k}(t)_{(f)} \),
then it also has a solution in the valuation ring of $\mathbb{F}_2(t)^+$. Let $u_1, u_2, u_3, u_4 \in \mathbb{F}_2(t)^+$ be a solution satisfying that not all of them are divisible by $f$. We can assume that $f \mid a_1$ and $f \mid a_3$. Reducing the equation modulo $f$ and dividing by $a_3$ we obtain

$$u_3^2 + u_3u_4 + a_3u_4^2 \equiv 0 \pmod{f}.$$ 

Since $[a_4, f] = 1$, this implies $f \mid u_3$ and $f \mid u_4$ whence $f^2 \mid a_3u_3^2 + a_3u_3u_4 + a_3a_4u_4^2 = a_1u_1^2 + a_1u_1u_2 + a_2u_2^2$. Since $a_1$ is square-free, we have $f^2 \mid a_1$ showing $f \mid u_1^2 + u_1u_2 + u_2^2$. Now $[a_2, f] = 1$ implies $f \mid u_1$ and $f \mid u_2$, contradiction.

Finally, if $f = \infty$ then we change the variable from $t$ to $t^{-1}$ in order to conclude using the previous case. \qed

Now we turn our attention to the case when $a$ has a pole at $f$. By Lemma 4.2 it must be of odd degree $2r + 1$ therefore the following lemma is relevant. In this case it is more convenient to multiply by $f^{2r + 1}$ and put $b = af^{2r + 1}$ which is an $f$-adic unit.

**Lemma 4.6.** Let $b, c$ be in $\mathbb{F}_2(t)^+_{(f)}$ such that $v_f(b) = 0$ (i.e. $b$ is an $f$-adic unit) and $v_f(c) = 0$ or $1$. Then the equation

$$f^{2r+1}x^2 + f^{2r+1}xy + by^2 = cf^{2r}$$

has a solution in $\mathbb{F}_2(t)^+_{(f)}$ if and only if it has a solution modulo $f^{4r+3}$. All such solutions lie in the valuation ring $\mathbb{F}_2(t)^+_{(f)}$.

**Proof.** $\Rightarrow$: Suppose we have a solution $(u, v) \in \mathbb{F}_2(t)^+_{(f)}$. Assume for contradiction that one of $u$ and $v$ is not in $\mathbb{F}_2(t)^+_{(f)}$. Multiplying by the square of the common denominator $f^l$ of $u$ and $v$ we obtain $u_1 = f^lu, v_1 = f^lv \in \mathbb{F}_2(t)^+_{(f)}$ such that $f^{2r+2l} \mid f^{2r+1}u_1^2 + f^{2r+1}u_1v_1 + bv_1^2$ but $f$ does not divide at least one of $u_1$ and $v_1$. Since $f \mid b$ we obtain $f^{2r+1} \mid v_1^2$ whence $f^{r+1} \mid v_1$. So we deduce $f^{2r+2} \mid f^{2r+1}u_1v_1 + bv_1^2$ and $f^{2r+2} \mid f^{2r+1}u_1^2$ contradicting to $f \mid u_1$. Hence we may reduce the equality $f^{2r+1}u^2 + f^{2r+1}uv + bv^2 = cf^{2r}$ modulo $f^{4r+3}$. 

$\Leftarrow$: Assume we have $u_0, v_0 \in \mathbb{F}_2(t)^+_{(f)}$ such that

$$c_0f^{2r} := f^{2r+1}u_0^2 + f^{2r+1}u_0v_0 + bv_0^2 \equiv cf^{2r} \pmod{f^{4r+2}}.$$ 

Then we must have $f^r \mid v_0$ and put $v_0 = f^rv_1$ so dividing by $f^{2r}$ we deduce

$$c_0 = f^{r+1}u_0v_1 + bv_1^2 \equiv c \pmod{f^{2r+2}}$$

Since $f^2 \nmid c$ at least one of $u_0$ and $v_1$ is not divisible by $f$. Putting $c_1 := \frac{c - c_0}{f^{2r+2}}$, we look for the solution of the original equation in the form $x = u_0 + f^{r+1}x_1$, and $y = v_0 + f^{2r+1}y_1$. So we are reduced to solving the equation

$$f^{2r+1}(u_0 + f^{r+1}x_1)^2 + f^{2r+1}(u_0 + f^{r+1}x_1)(f^rv_1 + f^{2r+1}y_1) + b(f^rv_1 + f^{2r+1}y_1)^2 = f^{2r}(c_0 + f^{2r+2}c_1).$$

Using the equation for $c_0$ and dividing by $f^{4r+2}$ we obtain the equivalent equation

$$fx_1^2 + x_1v_1 + u_0y_1 + f^{r+1}x_1y_1 + by_1^2 = c_1.$$ 

(1)
Now note that Hensel’s lemma applies to (1) since the gradient

\[
\left( \frac{\partial}{\partial x_1}(fx_1^2 + v_0x_1 + u_0y_1 + fx_1y_1 + by_1^2 - c_1), \frac{\partial}{\partial y_1}(fx_1^2 + v_0x_1 + u_0y_1 + fx_1y_1 + by_1^2 - c_1) \right) = \left( v_1 + f^{r+1}y_1, u_0 + f^{r+1}x_1 \right) \quad (\text{mod } f)
\]

is nonzero modulo \( f \). Therefore \((u_0, v_0)\) lifts to a solution modulo \( f^{k+3} \iff (1) \) has a solution modulo \( f \). Hence (1) has a solution in \( \mathbb{F}_{2^k}(t)(f) \) \iff \((u_0, v_0)\) lifts to a solution of \( fx^2 + fy + by^2 \) in \( \mathbb{F}_{2^k}(t)(f) \). \( \square \)

### 4.2 Finding nontrivial zeros

Let \( Q(x_1, x_2, x_3, x_4) = a_1x_1^2 + a_1x_1x_2 + a_1x_2x_2^2 + a_3x_3x_4 + a_3a_4x_4^2 \) where \( a_i \in \mathbb{F}_{2^k}(t) \). In this section we provide an algorithm for deciding whether \( Q \) admits a nontrivial zero and if so, returns a nontrivial solution \((x_1, x_2, x_3, x_4)\). The main idea is similar to the main algorithm of [23]. We replace \( Q \) with a similar form \( Q' \) and then decide whether \( Q' \) has a nontrivial zero using the local-global principle. If so, then we look for a common \( c \in \mathbb{F}_{2^k}(t) \) which is represented by both \( 1x_1^2 + a_1x_1x_2 + a_1x_2x_2^2 + a_3x_3x_4 + a_3a_4x_4^2 \) and \( 3x_2^2 + a_3x_3x_4 + a_3a_4x_4^2 \) and then solve the equations \( a_1x_1^2 + a_1x_1x_2 + a_1x_2x_2^2 = c \) and \( a_3x_3x_4 + a_3a_4x_4^2 = c \), separately using the algorithm from [22].

**Theorem 4.7.** Let \( Q(x_1, x_2, x_3, x_4) = a_1x_1^2 + a_1x_1x_2 + a_1x_2x_2^2 + a_3x_3x_4 + a_3a_4x_4^2 \) where \( a_i \in \mathbb{F}_{2^k}(t) \). Then there exists a polynomial-time algorithm which decides whether \( Q \) is isotropic and if so it finds a nontrivial zero of \( Q \).

**Proof.** We look for a common \( c \in \mathbb{F}_{2^k}(t) \) which is represented by both \( a_1x_1^2 + a_1x_1x_2 + a_1x_2x_2^2 \) and \( a_3x_3x_4 + a_3a_4x_4^2 \). Note that \( c \) is represented by both these forms if and only if it is represented by both forms locally at each place \( f \). By Lemma 4.2, that both \( a_1x_1^2 + a_1x_1x_2 + a_1x_2x_2^2 \) and \( a_3x_3x_4 + a_3a_4x_4^2 \) are minimal (in the sense of Definition 4.1). Denote by \( S \) the set of places where at least one of the following holds:

1. \( a_2 \) has a pole at \( f \);
2. \( a_4 \) has a pole at \( f \);
3. \( v_f(a_1a_3) \) is odd.

We look for \( c \) in the form \( c = f_1f_2 \cdots f_nh \) where \( f_1, \ldots, f_m \in S \) are monic irreducible polynomials and \( h \) is irreducible. If \( f \notin S \) and \( v_f(c) = 0 \) then both forms represent \( c \) locally at \( f \) by Lemma 4.3.1. On the other hand, if \( f \in S \) then we distinguish two cases.

First assume that neither \( a_2 \) nor \( a_4 \) has a pole at \( f \) (whence \( v_f(a_1a_3) \) is odd). Then whether or not a square-free polynomial \( c \) is represented by the form \( a_1x_1^2 + a_1x_1x_2 + a_1x_2x_2^2 \) (resp. \( a_3x_3x_4 + a_3a_4x_4^2 \)) depends only on the class of \( c \) modulo \( f^2 \). So we may decide by checking all the residue classes modulo \( f^2 \) whether there is a common value \( c \) of the two forms. If there is no common value then we are done (the 4-variable form is not isotropic). By Lemma 4.3 this happens if and only if \( (a_2, f) = (a_4, f) = 1 \). We put \( f \) among \( f_1, \ldots, f_m \) if all the common square-free values of the two forms are divisible by \( f \). Either way, there possibly appears a condition on \( c \) modulo \( f^2 \) (which we shall encode in the choice of \( h \)).

Now assume that either \( a_2 \) or \( a_4 \) has a pole at \( f \). Then we have a congruence condition on \( c \) modulo \( f^{k+3} \) by Lemma 4.3.1 where \( 2r + 1 = \max(-v_f(a_2), -v_f(a_4)) \) is the bigger order of the pole at \( f \) of \( a_2 \) and \( a_4 \). Again, if \( v_f(c) \) is odd for all the common values then we put \( f \) into the
finite set \( \{f_1, \ldots, f_m\} \). If there are no common values of the forms \( a_1x_1^2 + a_1x_1x_2 + a_1a_2x_2^2 \) and \( a_3x_3^2 + a_3x_3x_4 + a_3a_4x_4^2 \) modulo \( f^{2r+3} \) then we are done (\( Q \) is not isotropic).

Finally, if \( f = \infty \in S \) then the congruence condition on \( e \) involves a condition on the parity of the degree of \( c \), as well as a condition modulo a power of \( t \). Even if \( \infty \notin S \) then the condition \( v_\infty(c) = 0 \) means the degree of \( c \) must be even.

Now if none of the above congruence conditions were contradictory then we deduce that the 4-variable form is isotropic by Theorem 4.7. So we proceed with finding a nontrivial zero looking for \( c = f_1 \cdots f_mh \) where the monic irreducible polynomials \( f_1, \ldots, f_m \in S \) are determined above and we choose \( h \) irreducible satisfying all the above congruence conditions (including a possibly a condition at \( \infty \) if it belongs to \( S \)). This is possible by Lemma 2.27. By construction, \( c \) is a common value of \( a_1x_1^2 + a_1x_1x_2 + a_1a_2x_2^2 \) and \( a_3x_3^2 + a_3x_3x_4 + a_3a_4x_4^2 \) locally at all places in \( S \). Further, if \( g \neq h \) is a (finite or infinite) place not in \( S \) then \( c \) is also a common value locally at \( g \), so the only exception could be at \( h \). However, by Hilbert’s reciprocity law \( c \) is also a common value locally at \( h \).

\[ \Box \]

4.3 Applications

In this subsection we give two applications of our results. One is to finding zero divisors in quaternion algebras of separable quadratic extensions of \( F_{2^4}(t) \) and the other is constructing quaternion algebras over \( F_{2^4}(t) \) with prescribed Hasse invariants.

**Theorem 4.8.** Let \( L \) be a separable quadratic extension of \( F_{2^4}(t) \) and let \( A \) be a quaternion algebra over \( L \) which is split. Then there exists a polynomial-time algorithm which finds a zero divisor in \( A \).

**Proof.** First we apply Corollary 4.8 to find a subalgebra \( B \) which is a quaternion algebra over \( F_{2^4}(t) \). If \( B \) is split, then one can find a zero divisor in \( B \) in polynomial time using the algorithm from [22] (the algorithm also decides whether \( B \) is split or not). Now suppose that \( B \) is a division algebra. In that case \( B \) contains a maximal subfield isomorphic to \( L \) [40, Lemma 6.4.12]. Let \( L = F_{2^4}(t)(s) \) where \( s^2 + s = c \) and \( c \in F_{2^4}(t) \). If we find an element \( u \in B \) such that \( u^2 + u = c \), then \( u + s \) is a zero divisor as \( u \) is not in the center. Suppose that \( B \) has the following quaternion basis:

\[
\begin{align*}
i^2 + i &= a \\
j^2 &= b \\
i j &= j(i + 1)
\end{align*}
\]

Let us look for \( u \) in the form of \( u = \lambda_1 + \lambda_2i + \lambda_3j + \lambda_4ij \), where \( \lambda_i \in F_{2^4}(t) \).

\[ u^2 + u = \lambda_1^2 + \lambda_2^2a + \lambda_3^2b + \lambda_4^2ab + \lambda_3\lambda_4b + \lambda_1 + i(\lambda_2^2 + \lambda_2) + j(\lambda_2\lambda_3 + \lambda_3) + ij(\lambda_2\lambda_4 + \lambda_4) \]

For this to be in \( F_{2^4}[t] \), \( \lambda_2 = 1 \) must hold. Now we will investigate if the following equation has a non-trivial solution:

\[ \lambda_1^2 + \lambda_3^2b + \lambda_4^2ab + \lambda_3\lambda_4b + \lambda_1 + a + c = 0 \]  
(2)

Let \( \mu_2 \) equal to the product of the denominators of all \( \lambda_i \), \( \mu_1 := \lambda_1\mu_2 \), let us introduce new variables \( \mu_3 := \lambda_3\mu_2 \) and \( \mu_4 := \lambda_4\mu_2 \). Then multiplying (2) by \( \mu_2^2 \) gives

\[ \mu_1^2 + \mu_1\mu_2 + (a + c)\mu_2^2 + b\mu_3^2 + b\mu_3\mu_4 + ab\mu_4^2 = 0 \]

where \( \mu_1, \mu_2, \mu_3, \mu_4 \in F_{2^4}[t] \). Now we find a solution to the above equation using the algorithm from Theorem 4.7 ([40, Lemma 6.4.12] guarantees the existence of a solution) which returns \( u \).  
\[ \Box \]

15
The next proposition shows how to construct a quaternion division algebra with given Hasse invariants.

**Proposition 4.9.** Let \( v_1, \ldots, v_l \) be places of \( \mathbb{F}_2[l] \) such that \( l \) is even. Then there exists a polynomial-time algorithm which constructs a quaternion algebra over \( \mathbb{F}_2[l] \) which is ramified exactly at \( v_1, \ldots, v_l \).

**Proof.** Let \( f_1, \ldots, f_m \) be the finite places amongst the \( v_i \). First we find a monic irreducible polynomial in \( b \in \mathbb{F}_2[l] \) such that \( [b, f_i] = 1 \). This can be accomplished in the following way. One finds quadratic non-square \( r \) modulo every \( f_i \) \((\mathbb{F}_2[l]/(f_i) \) is finite field of cardinality \( 2^\deg(f_i) \)) and then obtains a residue class \( r \) modulo \( f_1 \cdots f_m \) such that \( r \equiv r_i \pmod{f_i} \) by Chinese remaindering. Then using Lemma [229] one finds an irreducible polynomial of suitably large degree which is congruent to \( r \mod f_1 \cdots f_m \) by choosing random elements from the residue class until an irreducible is found.

Let \( a = f_1 \cdots f_m \). We show that the quaternion algebra \( A = [a, b] \) ramifies at every \( f_i \). The algebra \( A \) ramifies at \( f_i \) if and only if the quadratic form \( ax^2 + axy + aby^2 + z^2 \) has a nontrivial zero in \( \mathbb{F}_2[l]/(f_i) \). Since the form is homogeneous, it is enough to show that it does not admit an integral zero. The variable \( z \) must be divisible by \( f_i \) since \( a \) is divisible by \( f_i \). Now setting \( z = f_i z' \) and dividing by \( f_i \) we get the following equation:

\[
a/f_i x^2 + a/f_i xy + a/f_i by^2 + f_i z'^2 = 0
\]

Suppose this equation has a nontrivial solution \((x_0, y_0, z_0)\). One may assume that \( f_i \) does not divide \( x_0, y_0 \) and \( z_0 \) simultaneously. Then the following congruence condition holds:

\[
a/f_i x_0^2 + a/f_i x_0 y_0 + a/f_i y_0^2 \equiv 0 \pmod{f_i}
\]

Since \( a/f_i \) is coprime to \( f_i \), one can divide the congruence by \( a/f_i \). If \( y_0 \) is not divisible by \( f_i \), then \( b \) is a quadratic residue mod \( f_i \), which is a contradiction. If \( y_0 \) is divisible by \( f_i \), then so is \( x_0 \). However, if \( x_0 \) and \( y_0 \) are both divisible by \( f_i \), then \( z_0 \) is not divisible by \( f_i \) and then \( a/f_i x_0^2 + a/f_i x_0 y_0 + a/f_i y_0^2 + f_i z_0^2 \) is not divisible by \( f_i \), which is a contradiction.

The algebra \( A \) is split at \( b \) since the equation \( ax^2 + axy + aby^2 + z^2 = 0 \) has a solution modulo \( b \) (setting \( z = 0 \) and \( x = y = 1 \)) which can be lifted by Hensel’s lemma. \( A \) is clearly split at all the other finite places and has the required splitting condition at \( \infty \) by Hilbert reciprocity.

### 5 Isomorphism problem of quaternion algebras over quadratic fields

In this section we give another application to our descent method, namely to the isomorphism problem of quaternion algebras.

We start with a small observation regarding the isomorphism problem of rational quaternion algebras. It is known that there is a polynomial-time algorithm for this task if one is allowed to call an oracle for factoring integers. Furthermore, there is a polynomial-time reduction from the problem of computing explicit isomorphisms of rational quaternion algebras to factoring, which implies that the factoring oracle is indeed necessary.

In [12, 32] the authors study the following problem: if we are given two quaternion algebras over \( \mathbb{Q} \) and we are also given a maximal order in both quaternion algebras, can we compute an isomorphism between them without relying on a factoring oracle? The motivation for this problem comes from the fact that the endomorphism ring of a supersingular elliptic curve is a maximal order in a quaternion algebra. The authors propose a heuristic algorithm which does not rely on factoring. Here we propose an algorithm for this task which does not rely on any heuristics:
**Proposition 5.1.** Let $A, B$ be quaternion algebras over $\mathbb{Q}$ and let $O_1, O_2$ be maximal orders in $A$ and $B$ respectively. Suppose that $A$ and $B$ are isomorphic. Then there exists a polynomial-time algorithm which computes an isomorphism between $A$ and $B$.

**Proof.** In [26] the authors show that finding an isomorphism between $A$ and $B$ can be reduced to finding a primitive idempotent in $C = A \otimes \mathbb{Q} B^{op}$. First observe that $O_1 \otimes O_2$ is a maximal order in $C$. Now we could use the algorithm from [26] but then it might only find a zero divisor which is not enough for our purposes (as it reduces to finding a zero divisor in a quaternion algebra where we do not have a maximal order). Instead we use the algorithm from [24] which finds a primitive idempotent directly.

The main goal of the remainder of the section is to design an efficient algorithm which computes explicit isomorphisms between isomorphic quaternion algebras over quadratic extensions $L$ of $\mathbb{Q}$ or $\mathbb{F}_q(t)$ (where $q$ is a prime power and can be even). In [26, Section 4] the authors show the following reduction:

**Theorem 5.2.** Let $A_1$ and $A_2$ be isomorphic central simple algebras of degree $n$ over an infinite field $K$. Then there is a polynomial-time reduction from computing an explicit isomorphism between $A_1$ and $A_2$ to computing an explicit isomorphism between $A_1 \otimes A_2^{op}$ and $\mathbb{M}_n(K)$.

Thus if one is given $A_1$ and $A_2$ which are quaternion algebras over $L$ which is a separable quadratic extension of either $K = \mathbb{Q}$ or $K = \mathbb{F}_q(t)$, then it is enough to find an explicit isomorphism between $A_1 \otimes A_2^{op}$ and $\mathbb{M}_4(L)$. Note that when $K = \mathbb{Q}$ the paper [26] proposes such an algorithm but it is exponential in the size of the discriminant of $L/\mathbb{Q}$. We will get around this issue by exploiting the fact that in this case $\mathbb{M}_4(L)$ is not given by a usual structure constant representation but as a tensor product of two quaternion algebras.

In order to have a unified algorithm for both the rational and the function field case we identify certain subroutines which our main algorithm needs:

1. Computing maximal right ideals of an algebra isomorphic to $\mathbb{M}_m(K)$ for $m = 4$ and $m = 16$, given by structure constants
2. Computing zero divisors in an algebra given by structure constants which is isomorphic to $\mathbb{M}_2(D)$ where $D$ is a quaternion division algebra over $K$
3. Computing zero divisors in a split quaternion algebra over $L$

We show that if one has access to these subroutines then there exists an efficient algorithm for computing explicit isomorphisms between quaternion algebras over quadratic global fields.

**Theorem 5.3.** Let $A_1$ and $A_2$ be isomorphic quaternion algebras over $L$ where $L$ is a quadratic extension of either $\mathbb{Q}$ or $\mathbb{F}_q(t)$. Suppose there exist polynomial-time algorithms (in the rational case polynomial-time algorithm with an oracle for factoring integers) for all the above subroutines. Then there exists a polynomial-time algorithm for computing an isomorphism between $A_1$ and $A_2$.

**Proof.** We provide an algorithm for computing an explicit isomorphism between $A_1^{op} \otimes A_2$ and $\mathbb{M}_4(L)$. Then [26, Section 4] implies that one can compute an explicit isomorphism between $A_1$ and $A_2$ in polynomial time.

Let $B = A_1^{op} \otimes A_2$. Then one can compute an involution of the first kind on $B$ since it is given as a tensor product of quaternion algebras (i.e., we take the “product” of the canonical involutions).

**Theorem 3.2** and subroutine 1 imply that one can either construct an involution of the second kind or a zero divisor in $B$. Suppose first that the algorithm from Theorem 3.2 finds a zero...
divisor $a$ in $B$. If the zero divisor has rank 1 or 3, then one can find either a rank 1 or a rank 3 idempotent by computing the left unit of the right ideal generated by $a$. Observe that if an idempotent $e$ has rank 3, then $1 - e$ has rank 1, thus one has actually found a primitive idempotent in both cases which implies an explicit isomorphism between $B$ and $M_4(L)$. If $a$ has rank 2, then we construct an idempotent $e$ of rank 2 in a similar fashion. Then $eBe \cong M_2(L)$ and computing an explicit isomorphism between them can be used to construct an explicit isomorphism between $B$ and $M_4(L)$ (as a rank one element in $eBe \cong M_2(L)$ has rank 1 in $B$). For computing an explicit isomorphism between $eBe$ and $M_2(L)$ we use subroutine 3. Note that discussion also implies that it is enough to find a zero divisor in $B$ as it can be used for constructing an explicit isomorphism between $B$ and $M_4(L)$.

Now we can suppose that the algorithm from Theorem 5.2 has computed an involution of the second kind on $B$. Now we have an involution of the second kind and an involution of the first kind on $A$. Composing them and taking fixed points finds a subalgebra $C$ of $B$ which is central simple algebra of degree 4 over $K$ and $C \otimes_K L = B$. There are 3 kinds of central simple algebras of degree 4: full matrix algebras, division algebras, and $2 \times 2$ matrix algebras over a division quaternion algebra. When $C$ is a full matrix algebra over $K$, then we use subroutine 1 to compute a zero divisor. When $C$ is a $2 \times 2$ matrix algebra over a division quaternion algebra, then we use subroutine 2 to compute a zero divisor in $C$. Finally, $C$ is never a division algebra as it is split by a quadratic extension (the smallest splitting field of a degree 4 central simple algebra has degree 4 over the ground field for global fields). 

Now we analyze the complexity of the subroutines for the function field and the rational case separately.

5.1 Subroutines over function fields

We begin with the case when $K = \mathbb{F}_q(t)$ and $q$ is odd:

1. The first subroutine can be accomplished in polynomial time using the algorithm from [22].
2. The second subroutine can be obtained in polynomial time using the algorithm from [19].
3. The third subroutine admits a polynomial-time algorithm derived in [23].

Now we look at the case where $q$ is even:

1. The first subroutine can be accomplished in polynomial time using the algorithm from [22].
2. The second subroutine admits a polynomial-time algorithm by Proposition 4.9.
3. The third subroutine admits a polynomial-time algorithm by Theorem 4.8.

All these imply the following:

**Corollary 5.4.** Let $L$ be a separable quadratic extension of $\mathbb{F}_q(t)$ where $q$ is a prime power (which can be even). Let $A_1$ and $A_2$ be two isomorphic quaternion algebras over $L$. Then there exists a randomized polynomial-time algorithm which computes an isomorphism between $A_1$ and $A_2$. 

18
5.2 Subroutines over \( \mathbb{Q} \)

Now we turn our attention to the \( K = \mathbb{Q} \) case. The first subroutine can again be accomplished in polynomial time (with the help of an oracle for factoring integers) using the algorithm from \([26]\). The third subroutine can also be obtained in polynomial time using an oracle for factoring integers. One has to use the algorithm from \([30]\).

There are no known algorithms for subroutine 2 in the rational case. In the rest of this section we propose a polynomial-time algorithm for this task which is analogous to \([19]\). The key ingredient of the algorithm is a result by Schwinning \([37]\) (which is referred to and generalized in \([2]\)):

**Fact 5.5.** Suppose one is given a list of places \( v_1, \ldots, v_k \) where \( k \) is even. Then there exists a polynomial-time algorithm which constructs a quaternion algebra which ramifies at exactly those places.

**Proposition 5.6.** Let \( A \) be an algebra isomorphic to \( M_2(D) \) where \( D \) is a division quaternion algebra. Then there exists a polynomial-time algorithm which is allowed to call an oracle for factoring integers which computes a zero divisor in \( A \).

**Proof.** First we compute a maximal order in \( A \) using the algorithm from \([25, Corollary 6.5.4]\). An extension of this algorithm \([21]\) computes the places where the algebra \( A \) ramifies. Now we use Schwinning’s algorithm to compute a division algebra \( D_0 \) which ramifies at exactly those places where \( A \) ramifies and that runs through the basis where the matrix has one nonzero entry and that runs through the basis of \( D_0 \). Then as stated previously, one can construct an explicit isomorphism between \( A \) and \( M_2(D_0) \) from an explicit isomorphism between \( A^{op} \otimes M_2(D_0) \) and \( M_{16}(\mathbb{Q}) \). Finally, the preimage of the matrix \( \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \) is a zero divisor. \( \square \)

An immediate corollary is the following:

**Corollary 5.7.** Let \( L \) be a quadratic extension of \( \mathbb{Q} \) and let \( A_1 \) and \( A_2 \) be isomorphic quaternion algebras over \( L \). Then there exists a polynomial-time algorithm which is allowed to call an oracle for factoring integers, that computes an explicit isomorphism between \( A_1 \) and \( A_2 \).

**Remark 5.8.** We would like to remark that even though both the rational and the function field case are polynomial-time algorithms (in the rational case modulo factoring integers), the function field variant is considerably more efficient in practice and the difference does not come from the difficulty of factoring integers. The algorithm from \([22]\) for general \( n \) (the degree of the matrix algebra) is polynomial in \( n \) and the algorithm from \([26]\) is doubly exponential in \( n \). We only need to use this algorithm for small \( n \), thus our algorithms for the rational case are technically polynomial time, but the provable constant is extremely large. Further experiments are needed on how fast the algorithm from \([26]\) is in practice.

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