PATH INTEGRALS ON RIEMANNIAN MANIFOLDS WITH SYMMETRY AND STRATIFIED GAUGE STRUCTURE

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Abstract. We study a quantum system in a Riemannian manifold \( M \) on which a Lie group \( G \) acts isometrically. The path integral on \( M \) is decomposed into a family of path integrals on quotient space \( Q = M/G \) and the reduced path integrals are completely classified by irreducible unitary representations of \( G \). It is not necessary to assume that the action of \( G \) on \( M \) is either free or transitive. Hence the quotient space \( M/G \) may have orbifold singularities. Stratification geometry, which is a generalization of the concept of principal fiber bundle, is necessarily introduced to describe the path integral on \( M/G \). Using it we show that the reduced path integral is expressed as a product of three factors: the rotational energy amplitude, the vibrational energy amplitude, and the holonomy factor.

1. Basic Observations and the Questions

Let us consider the usual quantum mechanics of a free particle in the one-dimensional space \( \mathbb{R} \). A solution for the initial-value problem of the Schrödinger equation

\[
\frac{i}{\hbar} \frac{\partial}{\partial t} \phi(x, t) = -\frac{1}{2} \frac{\partial^2}{\partial x^2} \phi(x, t) = \frac{1}{2} \Delta \phi(x, t)
\]

(1.1)

is given by

\[
\phi(x, t) = \int_{-\infty}^{\infty} dy \ K(x, y; t) \phi(y, 0)
\]

(1.2)
with the propagator
\[
K(x, y; t) = \langle x | e^{-\frac{i}{2} \Delta t} | y \rangle = \frac{1}{\sqrt{2\pi i t}} \exp \left[ \frac{i}{2t} (x - y)^2 \right].
\] (1.3)

Their physical meanings are clear; the wave function \( \psi(x, t) \) represents probability amplitude to find the particle at the location \( x \) at the time \( t \). The propagator \( K(x, y; t) \) represents transition probability amplitude of the particle to move from \( y \) to \( x \) in the time interval \( t \).

If the particle is confined in the half line \( \mathbb{R}_{\geq 0} = \{ x \geq 0 \} \), we need to impose a boundary condition on the wave function \( \psi(x, t) \) at \( x = 0 \) to make the initial-value problem (1.1) have a unique solution. As one of possibilities we may chose the Neumann boundary condition
\[
\frac{\partial \psi}{\partial x}(0, t) = 0.
\] (1.4)

Then the solution of (1.1) is given by
\[
\psi(x, t) = \int_{-\infty}^{\infty} dy \, K_N(x, y; t) \psi(y, 0)
\] (1.5)
with the corresponding propagator
\[
K_N(x, y; t) = K(x, y; t) + K(-x, y; t).
\] (1.6)

The physical meaning of the propagator \( K_N(x, y; t) \) is obvious; the first term \( K(x, y; t) \) represents propagation of a wave from \( y \) to \( x \) while the second term \( K(-x, y; t) \) represents propagation of a wave from \( y \) to \( -x \), which is the mirror image of \( x \). Thus the Neumann propagator \( K_N(x, y; t) \) is a superposition of the direct wave with the reflected wave.

As an alternative choice we may impose the Dirichlet boundary condition
\[
\psi(0, t) = 0.
\] (1.7)

Then the solution of (1.1) is given by
\[
\psi(x, t) = \int_{-\infty}^{\infty} dy \, K_D(x, y; t) \psi(y, 0)
\] (1.8)
with the corresponding propagator
\[
K_D(x, y; t) = K(x, y; t) - K(-x, y; t).
\] (1.9)

Thus the Dirichlet propagator \( K_D(x, y; t) \) is also a superposition of the direct wave with the reflected wave but reflection changes the sign of the wave.
The half line \( \mathbb{R}_{\geq 0} \) can be regarded as an orbifold \( \mathbb{R}/\mathbb{Z}_2 \). In the above discussion we have assumed the existence of the propagator \( K(x, y; t) \) in \( \mathbb{R} \) and constructed the propagators in \( \mathbb{R}/\mathbb{Z}_2 \) from \( K(x, y; t) \). There are two inequivalent propagators; the Neumann propagator \( K_N(x, y; t) \) obeys the trivial representation of \( \mathbb{Z}_2 \) whereas the Dirichlet propagator \( K_D(x, y; t) \) obeys the defining representation of \( \mathbb{Z}_2 = \{+1, -1\} \).

Now a question arises; how is a propagator in a general orbifold \( M/G \) constructed? Here \( M \) is a Riemannian manifold and \( G \) is a compact Lie group that acts on \( M \) by isometries. Such an example is easily found; we may take \( M = \mathbb{S}^2 \) and \( G = \text{U}(1) \). Then the quotient space is \( M/G = [-1, 1] \), which has two boundary points.

Let us turn to another aspect of the propagator, namely, the path-integral expression of the propagator. For the general Schrödinger equation

\[
\frac{i}{\hbar} \frac{\partial}{\partial t} \phi(x, t) = H\phi(x, t) = -\frac{1}{2} \frac{\partial^2}{\partial x^2} \phi(x, t) + V(x)\phi(x, t), \quad x \in \mathbb{R},
\]

its solution is formally given by

\[
\phi(x, t) = \int_{-\infty}^{\infty} dy \, K(x, y; t)\phi(y, 0).
\]

The propagator satisfies the composition property

\[
K(x'', x; t + t') = \int_{-\infty}^{\infty} dx' \, K(x'', x'; t')K(x', x; t).
\]

By dividing the time interval \([0, t]\) into short intervals we get

\[
K(x_N, x_0; t) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dx_{N-1} \cdots dx_1 K(x_N, x_{N-1}; \epsilon) \cdots K(x_1, x_0; \epsilon)
\]

with \( t = N\epsilon \). For a short distance and a short time-interval the propagator asymptotically behaves as

\[
K(x + \Delta x, x; \Delta t) \sim \frac{1}{\sqrt{2\pi i\Delta t}} \exp \left[ -\frac{i}{2} \left( \frac{\Delta x}{\Delta t} \right)^2 \Delta t - iV(x)\Delta t \right].
\]

Then “the limit \( N \to \infty \)” gives an infinite-multiplied integration, which is called the path integral,

\[
K(x', x; t) = \int_{x}^{x'} D\delta \, e^{i \int ds} = \int_{x}^{x'} D\delta \exp \left[ i \int_{0}^{t} ds \left( \frac{1}{2} \dot{x}(s)^2 - V(x(s)) \right) \right].
\]
In a rigorous sense, the limit $N \to \infty$ does not exists but physicists use this expression for convenience. The philosophy of the path integral can be symbolically written as

$$ \text{propagation of the wave} = \sum_{\text{trajectories}} \text{motion of the particle}. \quad (1.16) $$

We can construct the path integral on the half line $\mathbb{R}_{\geq 0} = \mathbb{R}/\mathbb{Z}_2$ as well:

$$ K_N(x', x; t) = \sum_{n=0}^{\infty} \int_{x}^{x'} Dx \, e^{i \int L \, ds}, \quad (1.17) $$

$$ K_D(x', x; t) = \sum_{n=0}^{\infty} (-1)^n \int_{x}^{x'} Dx \, e^{i \int L \, ds}, \quad (1.18) $$

where the summations are taken with respect to the number of reflections of the trajectory at the boundary $x = 0$.

Now another question arises; what is the definition of path integrals on a general orbifold $M/G$? Our main concerns are propagators and path integrals in $M/G$.

### 2. Reduction of Quantum System

When a quantum system has a symmetry, it is decomposed into a family of quantum systems that are defined in the subspaces of the original. Here we review the reduction method [5] of quantum system.

A quantum system $(\mathcal{H}, H)$ is defined by a pair of a Hilbert space $\mathcal{H}$ and a Hamiltonian $H$, which is a self-adjoint operator on $\mathcal{H}$. The symmetry of the quantum system is specified by $(G, T)$, where $G$ is a compact Lie group and $T$ is a unitary representation of $G$ over $\mathcal{H}$. The symmetry implies that $T(g)H = HT(g)$ for all $g \in G$. The compact group $G$ is equipped with the normalized invariant measure $dg$.

To decompose $(\mathcal{H}, H)$ into a family of reduced quantum systems, we introduce $(\mathcal{H}^x, \rho^x)$, where $\mathcal{H}^x$ is a finite dimensional Hilbert space of the dimensions $d^x = \text{dim} \mathcal{H}^x$. Besides, $\rho^x$ is an irreducible unitary representation of $G$ over $\mathcal{H}^x$. The set $\{\chi\}$ labels all the inequivalent representations. For each $g \in G$, $\rho^x(g) \otimes T(g)$ acts on $\mathcal{H}^x \otimes \mathcal{H}$ and defines the tensor product representation. The **reduced Hilbert space** is defined as the subspace of the invariant vectors of $\mathcal{H}^x \otimes \mathcal{H}$,

$$ (\mathcal{H}^x \otimes \mathcal{H})^G := \{ \psi \in \mathcal{H}^x \otimes \mathcal{H}; \forall h \in G, (\rho^x(h) \otimes T(h))\psi = \psi \}, \quad (2.1) $$
Let the set \( \{ e_1^X, \ldots, e_d^X \} \) be an orthonormal basis of \( \mathcal{H}^X \). Then the reduction operator \( S^X_i : \mathcal{H} \rightarrow (\mathcal{H}^X \otimes \mathcal{H})^G \) is defined by
\[
f \in \mathcal{H} \mapsto S^X_i f := \sqrt{d^X} \int_G dg (\rho^X(g)e_i^X) \otimes (T(g)f) .\tag{2.2}
\]

**Theorem 2.1.** \( S^X_i \) is a partial isometry. Namely, \( (S^X_i)^*S^X_i \) is an orthogonal projection operator acting on \( \mathcal{H} \) while \( S^X_i(S^X_i)^* \) is the identity operator on \( (\mathcal{H}^X \otimes \mathcal{H})^G \).

**Theorem 2.2.** The family of the projections \( \{(S^X_i)^*S^X_i\} \) forms a resolution of the identity as
\[
\sum_{i,d} (S^X_i)^*S^X_i = I_{\mathcal{H}} .\tag{2.3}
\]
Hence, the Hilbert space is decomposed as
\[
\mathcal{H} = \bigoplus_{X,i} \text{Im}(S^X_i)^*S^X_i \cong \bigoplus_{X,i} (\mathcal{H}^X \otimes \mathcal{H})^G \tag{2.4}
\]
and this decomposition is compatible with the Hamiltonian action. Namely, we have the commutative diagram
\[
\begin{array}{ccc}
\mathcal{H} & \xrightarrow{S^X_i} & (\mathcal{H}^X \otimes \mathcal{H})^G \\
\downarrow & & \downarrow \text{Id} \otimes \text{H} \\
\mathcal{H} & \xrightarrow{S^X_i} & (\mathcal{H}^X \otimes \mathcal{H})^G
\end{array}
\tag{2.5}
\]
Then \( ((\mathcal{H}^X \otimes \mathcal{H})^G, \text{Id} \otimes \text{H}) \) defines a **reduced quantum system**.

The projection \( P^X : \mathcal{H}^X \otimes \mathcal{H} \rightarrow (\mathcal{H}^X \otimes \mathcal{H})^G \) onto the reduced space is defined by
\[
P^X := \int_G dg \rho^X(g) \otimes T(g) .\tag{2.6}
\]

The **reduced time-evolution operator** of the reduced system is
\[
U^X := P^X(\text{Id} \otimes e^{-iHt}) .\tag{2.7}
\]

Theorems 2.1 and 2.2 are easily proved by an application of the Peter–Weyl theorem, which states that the set of the matrix elements of irreducible unitary representations \( \{ \sqrt{d^X} \rho^X_{ij}(g) \}_{X,i,j} \) forms a complete orthonormal set of \( L_2(G) \). Our main purpose is to give a path-integral expression to the time-evolution operator \( U^X \). To describe it we need to introduce some related notions.
Assume that the base space $M$ is equipped with the measure $dx$. Then the space of the square-integrable functions $L_2(M)$ becomes a Hilbert space $\mathcal{H}$. Moreover, assume that the compact Lie group $G$ acts on $M$ preserving the measure $dx$. Then $g \in G$ is represented by the unitary operator $T(g)$ on $f \in L_2(M)$ by

$$(T(g)f)(x) := f(g^{-1}x). \quad (2.8)$$

Let $p: M \to Q = M/G$ be the canonical projection map. Then a measure $dq$ of $Q = M/G$ is induced by the following way. Let $\phi(q)$ be a function on $Q$ such that $\phi(p(x))$ is a measurable function on $M$. The induced measure $dq$ of $Q$ is then defined by

$$\int_Q dq \phi(q) := \int_M dx \phi(p(x)). \quad (2.9)$$

On the other hand, suppose that the time-evolution operator $U(t) := e^{-iHt}$ is expressed in terms of an integral kernel $K: M \times M \times \mathbb{R}_{>0} \to \mathbb{C}$ as

$$(U(t)f)(x) = \int_M dy K(x, y; t)f(y) \quad (2.10)$$

for any $f(x) \in L_2(M)$.

Let us turn to the reduced Hilbert space (2.1) and characterize it for the case $\mathcal{H} = L_2(M)$. A vector $\psi \in \mathcal{H}^x \otimes L_2(M)$ can be identified with a measurable map $\psi: M \to \mathcal{H}^x$. The tensor product $\rho^x(g) \otimes T(g)$ acts on $\psi$ as

$$((\rho^x(g) \otimes T(g))\psi)(x) = \rho^x(g)\psi(g^{-1}x), \quad g \in G \quad (2.11)$$

via the definition (2.8). The definition (2.1) of the invariant vector $\psi \in (\mathcal{H}^x \otimes L_2(M))^G$ implies

$$((\rho^x(g) \otimes T(g))\psi)(x) = \rho^x(g)\psi(g^{-1}x) = \psi(x), \quad (2.12)$$

which is equivalent to

$$\psi(gx) = \rho^x(g)\psi(x). \quad (2.13)$$

A function $\psi: M \to \mathcal{H}^x$ satisfying the above property is called an equivariant function. Hence the reduced Hilbert space is identified with the space of the equivariant functions $L_2(M, \mathcal{H}^x)^G$.

The projection operator $P^x: L_2(M; \mathcal{H}^x) \to L_2(M, \mathcal{H}^x)^G$, is now given by

$$(P^x\psi)(x) = \int_G dg \rho^x(g)\psi(g^{-1}x). \quad (2.14)$$
From (2.7–2.10) and (2.14) the reduced time-evolution operator is given by

\[ (U^x(t)\psi)(x) = \int_G \int_M \, dg \, dy \, \rho^x(g) K(g^{-1}x, y; t)\psi(y) \]  \hspace{1cm} (2.15)

and thus the corresponding reduced propagator is \( K^x : M \times M \times \mathbb{R}_{>0} \to \text{End} \mathcal{H}^x \) is defined by

\[ K^x(x, y; t) := \int_G \rho^x(g) K(g^{-1}x, y; t). \]  \hspace{1cm} (2.16)

Our aim is to express the reduced propagator in terms of path integrals.

3. Stratification Geometry

To write down a concrete form of the path integral we need to equip the base space \( M \) with a Riemannian structure. Namely, now we assume that \( M \) is a differential manifold equipped with a Riemannian metric \( g_M \) and that the Lie group \( G \) acts on \( M \) preserving the metric \( g_M \). Then the volume form induced from the metric defines an invariant measure \( dx \) of \( M \). We do not assume that the action of \( G \) on \( M \) is free. Therefore \( p: M \to M/G \) is not necessarily a principal bundle.

For each point \( x \in M \), \( G_x := \{ g \in G \mid gx = x \} \) is called the isotropy group of \( x \) and \( O_x := \{ gx \mid g \in G \} \) is the orbit through \( x \). It is easy to see that \( O_x \cong G/G_x \). Note that the dimensions of the orbit \( O_x \) can change suddenly when the point \( x \in M \) is moved. The subspace of the tangent space \( T_xM \), \( V_x := T_xO_x \), is called the vertical subspace and its orthogonal complement \( H_x := (V_x)^\perp \) is called the horizontal subspace. \( P_V : T_xM \to V_x \) is the vertical projection while \( P_H : T_xM \to H_x \) is the horizontal projection. A curve in \( M \) whose tangent vector always lies in the horizontal subspace is called a horizontal curve. Although these terms have been introduced in the theory of principal fiber bundle, we use them for a more general manifold that admits group action.

Let \( g \) denote the Lie algebra of the group \( G \). For each \( x \in M \), \( g_x \) is the Lie subalgebra of the isotropy group \( G_x \). The group action \( G \times M \to M \) induces infinitesimal transformations \( g \times M \to TM \) by differentiation. The induced linear map \( \theta_x : g \to T_xM \) has \( \ker \theta_x = g_x \) and \( \Im \theta_x = V_x \). Then it defines an isomorphism \( \rho_x : g/g_x \to V_x \). Now we define the stratified connection form \( \omega \) by

\[ \omega_x := (\rho_x)^{-1} \circ P_V : T_xM \to g/g_x. \]  \hspace{1cm} (3.1)

Actually \( \omega \) is not smooth over the whole \( M \) but it is smooth on each stratum of \( M \).
4. Reduction of Path Integral

The Riemannian structure \((M, g_M)\) defines the Laplacian \(\Delta_M\). Suppose that \(V: M \to \mathbb{R}\) is a potential function such that \(V(gx) = V(x)\) for all \(x \in M\), \(g \in G\). Then the Hamiltonian \(H = \frac{1}{2} \Delta_M + V(x)\), which acts on \(L_2(M)\), commutes with the action of \(G\), which is defined in (2.8). Let us assume that the path integral in \(M\) is formally given by

\[
K(x', x; t) = \int_x^{x'} \mathcal{D}x \exp \left[ i \int_0^t ds \left( \frac{1}{2} \| \dot{x}(s) \|^2 - V(x(s)) \right) \right]. \tag{4.1}
\]

Now we repeat our question; what is the path-integral expression for the reduced propagator (2.16) on \(Q = M/G\)? The answer is our main result which is given below.

**Theorem 4.1.** The reduced path integral on \(Q = M/G\) is

\[
K^x(x', x; t) = \int_q \mathcal{D}q \rho^x(\gamma) \rho^x_\gamma \left( \mathcal{P} \exp \left[ -i \int_0^t ds \Lambda(\tilde{q}(s)) \right] \right) \times \exp \left[ i \int_0^t ds \left( \frac{1}{2} \| \dot{q}(s) \|^2 - V(q(s)) \right) \right]. \tag{4.2}
\]

To read the above equation we need explanation of the symbols. The canonical projection map \(p: M \to Q = M/G\) induces the metric \(g_Q\) of \(Q\) by asserting that the map \(p\) is a stratified Riemannian submersion. For \(x, x' \in M\) we put \(q = p(x)\) and \(q' = p(x')\). The map \(q: [0, t] \to Q\) is a curve connecting \(q = q(0)\) and \(q' = q(t)\). The map \(\tilde{q}: [0, t] \to M\) is a horizontal curve such that \(\tilde{q}(0) = x\) and \(p(\tilde{q}(s)) = q(s)\) for \(s \in [0, t]\). The element \(\gamma \in G\) is a holonomy defined by \(x' = \gamma \cdot \tilde{q}(t)\).

To describe the symbol \(\Lambda\), which is called the **rotational energy operator**, we need more explanation. The metric \(g_M: TM \otimes TM \to \mathbb{R}\) defines an isomorphism \(\hat{g}_M: TM \to T^*M\). Then its inverse map \(\hat{g}_M^{-1}: T^*M \to TM\) defines a symmetric tensor field \(g_M^{-1}: M \to TM \otimes TM\). Thus combining it with the stratified connection \(\omega_x: T_x M \to g/g_x\) we define the rotational energy operator by

\[
\Lambda(x) := -(\omega_x \otimes \omega_x) \circ g_M^{-1}(x) \in (g/g_x) \otimes (g/g_x). \tag{4.3}
\]

The unitary representation \(\rho^x_\gamma\) of the group \(G\) in \(\mathcal{H}^x\) induces the representation \(\rho^x_\gamma\) of the universal enveloping algebra \(U(g)\). Then we have \(\rho^x_\gamma(\Lambda(x)) \in \text{End} \mathcal{H}^x\). Moreover,
\[ \lambda(\tau) = \rho^X_\tau \left( \mathcal{P} \exp \left[ - \frac{i}{2} \int_0^\tau ds \Lambda(\tilde{q}(s)) \right] \right) \in \text{End } \mathcal{H}^X \] (4.4)

is defined as a solution of the differential equation
\[ \frac{d}{d\tau} \lambda(\tau) = -\frac{i}{2} \rho^X_\tau (\Lambda(\tilde{q}(\tau))) \lambda(\tau), \quad \lambda(0) = I \in \text{End } \mathcal{H}^X. \] (4.5)

Now we can read off the physical meaning of the reduced path integral (4.2). The path integral is expressed as a product of three factors:

i) the rotational energy amplitude \( \exp\left[ -\frac{i}{2} \int_0^t ds \Lambda(\tilde{q}(s)) \right] \), which represents motion of the particle along the vertical directions of \( p: M \rightarrow M/G \);

ii) the vibrational energy amplitude \( \exp\left[ i \int_0^t ds \left( \frac{1}{2} \| \dot{q}(s) \|^2 - V(q(s)) \right) \right] \), which represents motion of the particle along the horizontal directions;

iii) the holonomy factor \( \gamma \), which is caused by non-integrability of the horizontal distributions.

Here we give the outline of the proof of the main Theorem 4.1. For the detail see the reference [6]. Essentially, it is only a matter of calculation; from the path integral on \( M \) (4.1)
\[ K(x', x; t) = \int x' D x e^{iI[x]}, \quad I[x] = \int_0^t ds \left( \frac{1}{2} \| \dot{x}(s) \|^2 - V(x(s)) \right) \] (4.6)
with the reduction procedure (2.16) we get
\[ K^X(x', x; t) = \int_G dh \rho^X(h) K(h^{-1}x', x; t) = \int_G dh \rho^X(h) \int_x^{h^{-1}x'} D x e^{iI[x]} \]
\[ = \int_G dh \rho^X(h) \int_q^{q'} \mathcal{D} q \int_q^{h^{-1} \gamma} e^{iI[q]} = \int_q^{q'} \mathcal{D} q \int_G dh \rho^X(h) \int_q^{h^{-1} \gamma} e^{iI[q]} \]
\[ = \int_q^{q'} \mathcal{D} q \int_G dh \rho^X(h) \int_q^{h^{-1} \gamma} e^{iI[q]} \] (4.7)
\[ = \int_q^{q'} \mathcal{D} q \rho^X(\gamma) \int_G dh \rho^X(h) \int_q^{h^{-1} \gamma} D g e^{i\int ds \frac{1}{2} \| \dot{q} \|^2} e^{i \int ds \left( \frac{1}{2} \| \dot{q} \|^2 - V(q) \right)} \]
\[ = \int_q^{q'} \mathcal{D} q \rho^X(\gamma) \rho^X_\gamma \left( \mathcal{P} \exp \left[ -\frac{i}{2} \int_0^t ds \Lambda(\tilde{q}(s)) \right] \right) e^{i \int ds \left( \frac{1}{2} \| \dot{q} \|^2 - V(q) \right)} . \]
5. Example

Finally, we show an example of application of our formulation. Let us begin with the plane $M = \mathbb{R}^2$, which has the standard metric $g_M = dx^2 + dy^2 = dr^2 + r^2 d\theta^2$. It admits the symmetry action of $G = \mathbb{S}O(2)$. The quotient space is a half line $Q = \mathbb{R}^2/\mathbb{S}O(2) = \mathbb{R}_{\geq 0}$. The invariant potential is a function $V(r)$ only of $r$.

The group action

$$\mathbb{S}O(2) \times \mathbb{R}^2 \rightarrow \mathbb{R}^2; \quad \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

induces the action of the Lie algebra

$$\mathfrak{so}(2) \times \mathbb{R}^2 \rightarrow T\mathbb{R}^2; \quad \begin{pmatrix} 0 & -\phi \\ \phi & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

which defines the vertical distribution

$$\theta: \mathfrak{so}(2) \times \mathbb{R}^2 \rightarrow T\mathbb{R}^2; \quad \begin{pmatrix} 0 & -\phi \\ \phi & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \phi \frac{\partial}{\partial \theta}.$$

Then the stratified connection becomes

$$\omega = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} d\theta.$$  

In the cotangent space the metric is given as

$$(g_M)^{-1} = \frac{\partial}{\partial r} \otimes \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial}{\partial \theta} \otimes \frac{\partial}{\partial \theta}.$$  

The rotational energy operator is

$$\Lambda = -(\omega \otimes \omega) \circ (g_M)^{-1} = -\frac{1}{r^2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$  

The irreducible unitary representations of $\mathbb{S}O(2)$ are labeled by the integers $n \in \mathbb{Z}$ and defined by

$$\rho_n: \mathbb{S}O(2) \rightarrow U(1); \quad \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} \mapsto e^{in\phi}.$$  

The differential representation of the Lie algebra of $\mathbb{S}O(2)$ is

$$(\rho_n)_*: \mathfrak{so}(2) \rightarrow \mathfrak{u}(1); \quad \begin{pmatrix} 0 & -\phi \\ \phi & 0 \end{pmatrix} \mapsto in\phi.$$
The rotational energy operator is then represented as

\[ (\rho_n)_\ast(\Lambda) = -\frac{(in)^2}{r^2} = \frac{n^2}{r^2}. \tag{5.9} \]

Finally the reduced path integral is given by

\[
K_n(r', \theta', r, \theta; t) = \int_r^{r'} Dr \, e^{in(\theta'-\theta)}
\times \exp \left[ i \int_0^t ds \left\{ -\frac{n^2}{2r^2} + \frac{1}{2} \dot{r}^2 - V(r) \right\} \right]. \tag{5.10}
\]

So the effective potential for the radius coordinate \( r \) is given by

\[ V_{\text{eff}}(r) = V(r) + \frac{n^2}{2r^2}, \tag{5.11} \]

where the second term represents the centrifugal force.

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