light-front holography and the light-front coupled-cluster method

1 introduction

In order to compute hadronic light-front wave functions, we need a method by which the light-front QCD Hamiltonian eigenvalue problem can be solved nonperturbatively. The standard approach is to expand the eigenstate in a truncated Fock basis, with the wave functions as the expansion coefficients, and solve the resulting integral equations for these wave functions. The light-front coupled-cluster (LFCC) method [1] follows this path, except that the Fock basis is not truncated; instead, the wave functions for higher Fock states are restricted to being determined from the wave functions for the lowest states through functions that satisfy nonlinear integral equations. In the lowest Fock sector, designated the valence sector, there is an eigenvalue problem for an effective LFCC Hamiltonian that approximates the effects of higher Fock states. It is this restricted eigenvalue problem that is related to the light-front holographic eigenvalue problem [2], which is usually presented in the form of a transverse light-front Schrödinger equation for massless quarks. We extend light-front holography to include a longitudinal equation and masses for quarks [3].

The truncation of the Fock basis should be avoided, because it causes uncanceled divergences. The analog in Feynman perturbation theory is to separate diagrams into time-ordered diagrams and discard time orderings that include intermediate states with more particles than some finite limit. This destroys covariance, disrupts regularization, and induces spectator dependence for subdiagrams. For example, the Ward identity of gauge theories is destroyed by truncation, as illustrated in Fig. 1. In the nonperturbative case, this happens not just to some finite order in the coupling but to all orders. The LFCC method is designed to avoid this sort of complication.

As its name implies, the LFCC method is for light-front quantized Hamiltonians. We define light-front coordinates [4] as the light-front time \( x^+ = t + z \) and space \( x_- = t - z, x_\perp = (x, y) \). The corresponding light-front energy and momentum are \( p^- = E - p_z \) and \( p = (p^+ = E + p_z, p_\perp = (p_x, p_y)) \). These imply that the mass-shell condition \( p^2 = m^2 \) becomes \( p^- = \frac{m^2 + p_z^2}{p^+} \), and the mass eigenvalue problem is...
Light-front coordinates have several advantages [5,6]. There are no spurious vacuum contributions to eigenstates, because $p^+ > 0$ for all particles, which prevents particle production from the vacuum without violation of momentum conservation. This permits well-defined Fock-state wave functions with no spurious vacuum structure [7].

### 2 Light-Front Coupled-Cluster Method

The LFCC method solves the Hamiltonian eigenvalue problem by first writing the eigenstate as $|\psi\rangle = \sqrt{Z}e^T|\phi\rangle$, where $|\phi\rangle$ is the valence state, with normalization $\langle\phi'|\phi\rangle = \delta(P' - P)$, and $T$ is an operator that increases particle number. The overall normalization is set by $Z$, such that $\langle\psi'|\psi\rangle = \delta(P' - P)$. $T$ conserves all quantum numbers, including $J_z$, light-front momentum $P$, and charge. Because $p^+$ is positive, $T$ must include annihilation, and powers of $T$ include contractions. The LFCC effective Hamiltonian is constructed as $\mathcal{P}^- = e^{-T}\mathcal{P}^-e^T$. Then, with $P_v$, the projection onto the valence Fock sector, we have the coupled system

$$P_v\mathcal{P}^-|\phi\rangle = \frac{M^2 + p_i^2}{p^+}|\phi\rangle, \quad (1 - P_v)\mathcal{P}^-|\phi\rangle = 0. \quad (2)$$

Formulated in this fashion, the eigenvalue problem is still exact but also still infinite in size, because there are, in general, infinitely many terms in $T$. To have a finite problem, but without truncation of the Fock basis, we truncate $T$ and $1 - P_v$. The simplest truncation of $T$ is to include single-particle emission, such as a gluon from a quark; the corresponding truncation of $1 - P_v$ would be to project onto Fock states with one more gluon than the valence state. The truncation of $1 - P$, then provides just enough equations to solve for the emission vertex function contained in $T$. The truncations can be systematically relaxed, by expanding the number of particles created by $T$ and the range of Fock states used for projections.

The mathematics of the LFCC method have their origin in the nonrelativistic many-body coupled-cluster method [8–10], developed in nuclear physics and quantum chemistry [11–14]. It was first applied to the many-electron problem in molecules by Čížek [15]. The Hamiltonian eigenstate is formed as $e^T|\phi\rangle$, where $|\phi\rangle$ is a product of single-particle states and where terms in $T$ annihilate states in $|\phi\rangle$ and create excited states, to build in correlations. The operator $T$ is then truncated at some number of excitations; however, the number of particles does not change. There are also some applications to quantum field theory in equal-time quantization [16–18].

Once the LFCC eigenvalue problem has been solved, the solution can be used to compute observables, such as form factors. For a dressed fermion, the Dirac and Pauli form factors can be computed from a matrix element of the current $J^+ = \bar{\psi}y^+\psi$, which couples to a photon of momentum $q$. The matrix element is generally [19]

$$\langle\psi^0(P,\underline{q})|16\pi^3 J^+(0)|\psi^\pm(P)\rangle = 2\delta_{\alpha\pm}F_1(q^2) \pm \frac{q^1 \pm iq^2}{M}\delta_{\alpha\mp}F_2(q^2), \quad (3)$$

with $F_1$ and $F_2$ the Dirac and Pauli form factors. Thus, we need to be able to compute matrix elements.

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1 This can, of course, be defeated by zero modes, which can be important for analysis of properties usually associated with vacuum structure [7].
As an example of how a matrix element can be computed, we consider the expectation value for an operator \( \hat{O} \), which in the LFCC method would be expressed as

\[
\langle \hat{O} \rangle = \frac{\langle \phi | e^{T} \hat{O} e^{T} | \phi \rangle}{\langle \phi | e^{T} e^{T} | \phi \rangle}.
\] (4)

Direct computation would require an infinite sum over the untruncated Fock basis. Instead, we define \( \overline{O} = e^{-T} \hat{O} e^{T} \) and \( \langle \tilde{\psi} | = \langle \phi | e^{T} \hat{O} e^{T} | \phi \rangle \), so that \( \langle \hat{O} \rangle = \langle \tilde{\psi} | \overline{O} | \phi \rangle \) and \( \langle \tilde{\psi}' | = \langle \phi | e^{T} \hat{O} e^{T} | \phi \rangle \). The effective operator \( \overline{O} \) is computed from the Baker–Hausdorff expansion, \( \overline{O} = \hat{O} + [\hat{O}, T] + \frac{1}{2}[[\hat{O}, T], T] + \cdots \).

The bra \( \langle \tilde{\psi} | \) is a left eigenvector of \( \overline{D} \), because the following holds:

\[
\langle \tilde{\psi} | \overline{D} = \langle \phi | e^{T} \overline{\rho} e^{T} | \phi \rangle = \langle \phi | e^{T+} e^{T} | \phi \rangle = \frac{M^{2} + P^{2}}{P^{+}} (\tilde{\psi}).
\] (5)

With this technique, the Dirac form factor is approximated by the matrix element

\[
F_{1}(q^{2}) = 8\pi^{3} \langle \tilde{\psi}^{+} (P + q) | J^{+}(0) | \phi^{+} (P) \rangle,
\] (6)

with \( J^{+}(0) = J^{+}(0) + [J^{+}(0), T] + \cdots \).

3 Sample LFCC Application

As an example of the use of the LFCC method [1], we consider a soluble model, a light-front analog [20] of the Greenberg–Schweber model [21]. In this model, a heavy fermionic source emits and absorbs bosons without changing its spin, and we solve for the fermionic eigenstate dressed by a cloud of bosons. A graphical representation of the light-front Hamiltonian \( \overline{\rho} \) is given in Fig. 2.

The model is not fully covariant; in particular, states are all limited to having a fixed total transverse momentum. This hides some of the power of the LFCC method, but is sufficient to show how the method can be applied. Details can be found in Ref. [1].

Here we compare the LFCC method with a traditional truncated-Fock-space approach. The Fock-state expansion of the eigenstate is represented in Fig. 3a. For the LFCC method, we take the valence state to be the bare fermion. The resulting LFCC form of the eigenstate is represented in Fig. 3b.

We truncate the \( T \) operator to include only single boson emission, which corresponds to the terms with \( t_{1} \) in Fig. 3b. A comparable truncation of the Fock-space expansion is to limit the number of bosons to no more than two. The resulting integral equations are represented in Fig. 4. In the Fock-space truncation case, the self-energy contribution is spectator dependent, with an energy denominator that includes the second boson in flight as well as the boson in the loop. Also, the self-energy in the one-boson sector is different from the self-energy correction in bare-fermion sector. In the LFCC equations, the self–energy corrections are the same everywhere they appear and are not spectator dependent. The price paid for this gain is the nonlinear nature of the equation for \( t_{1} \).

To derive the LFCC equations, we must first construct the effective Hamiltonian. This is done by computing the commutators of the Baker–Hausdorff expansion for \( \overline{\rho} \). The necessary commutators are represented in Fig. 5. When added to \( \overline{\rho} \), they provide all that is needed to define the LFCC eigenvalue problem and auxiliary equations for the bare-fermion valence state with the chosen truncation of \( T \). Notice that all three of the diagrams analogous to those for the Ward identity in QED are included.

One can show that the truncation for \( T \) provides an exact solution for the model [1], so that this lowest-order LFCC approximation is actually exact. Solution of the left-hand eigenproblem, for \( \langle \tilde{\psi} | \), then permits calculation of the Dirac form factor from Eq. (6). Details are given in [1].

**Fig. 2** Graphical representation of the light-front Hamiltonian \( \overline{\rho} \) for the soluble model discussed in Sect. 3. A cross represents a kinetic energy contribution. Lines on the right represent annihilation operators; lines on left, creation operators. Solid lines are for the fermion, and dashed lines, for the boson.
4 Light-Front Holographic QCD

The LFCC valence-sector eigenvalue problem for mesons and baryons in QCD can be approximated by models based on light-front holography [2]. A factorized meson wave function in the valence \((q\bar{q})\) sector

\[
\psi = e^{iL\phi} X(x)\phi(\zeta)/\sqrt{2\pi\zeta}
\]  

is subject to an effective potential \(\tilde{U}\) that conserves \(L_z\)

\[
\begin{bmatrix}
\mu_1^2 & \frac{\mu_2^2}{1-x} & \frac{\partial^2}{\partial \zeta^2} & -\frac{1}{4\zeta^2} + \tilde{U}
\end{bmatrix} X(x)\phi(\zeta) = M^2 X(x)\phi(\zeta).
\]
For zero-mass quarks, the longitudinal wave function \( X \) decouples, and the transverse wave function satisfies

\[
\left[ -\frac{d^2}{d\xi^2} - \frac{1 - 4L^2}{4\xi^2} + U(\xi) \right] \phi(\xi) = M^2 \phi(\xi),
\]

(9)

with \( U \) determined by an AdS\(_5\) correspondence.

The softwall model for massless quarks \([22]\) yields an oscillator potential \( U(\xi) = k^4 \xi^2 + 2k^2 (J - 1) \) and a simple spectrum. The masses \( M^2 = 4k^2 (n + (J + L)/2) \) have a linear Regge trajectory and are a good fit for light mesons \([23–28]\). The transverse wave functions are 2D oscillator functions. The longitudinal wave function \( X \) is constrained by a form-factor duality \([29]\), between the Fock-space construction

\[
F(q^2) = \int dx \frac{|X(x)|^2}{x(1-x)} \int 2\pi \xi d\xi J_0 \left( \xi q_\perp \sqrt{x/(1-x)} \right) |\phi(\xi)|^2
\]

(10)

and the form computed in AdS\(_5\)

\[
F(q^2) = \int dx \int 2\pi \xi d\xi J_0 \left( \xi q_\perp \sqrt{x/(1-x)} \right) |\phi(\xi)|^2.
\]

(11)

Thus \( X(x) = \sqrt{x(1-x)} \), when the quarks are massless.

For massive quarks, there is the ansatz by Brodsky and De Téramond \([30]\), to replace \( k_1^2 / (x(1-x)) \) with \( k_1^2 / x(1-x) + \mu_1/x + \mu_2^2/(1-x) \) in the transverse harmonic oscillator eigenfunctions, with \( \mu_i \) as current-quark masses. This yields a longitudinal wave function of the form

\[
X_{\text{BDT}}(x) = N_{\text{BDT}} \sqrt{x(1-x)} e^{-\left(\mu_1^2/x + \mu_2^2/(1-x)/2\right)/x^2},
\]

(12)

Instead of this ansatz, we use a longitudinal equation for \( X \), with an effective potential from the ‘t Hooft model \([31]\)

\[
\left[ \frac{m_1^2}{x} + \frac{m_2^2}{1-x} \right] X(x) + \frac{g^2}{\pi} \int dy \frac{X(x) - X(y)}{(x-y)^2} - CX(x) = M_1^2 X(x),
\]

(13)

where the \( m_i \) are constituent masses. The ‘t Hooft model, which is based on large-\( N \) two-dimensional QCD, incorporates longitudinal confinement in a manner consistent with four-dimensional QCD. The solution of this equation, \( X(x) \), is known \([31,32]\) to be well approximated by \( x^{\beta_1}(1-x)^{\beta_2} \), with the \( \beta_i \) subject to the constraints \( m_i^2 \pi / g^2 - 1 + \pi \beta_i \cot \pi \beta_i = 0 \). For consistency, we should have \( \beta_1 = \beta_2 = 1/2 \) in the zero-current-mass limit. This fixes the coupling to be \( g^2 / \pi = m_1^2 = m_2^2 \).

The longitudinal equation is relatively easy to solve numerically \([32–34]\). We expand the solution as \( X(x) = \sum_n c_n f_n(x) \) with respect to basis functions constructed from Jacobi polynomials \([34]\)

\[
f_n(x) = \frac{N_n x^{\beta_1}(1-x)^{\beta_2} P_n^{(2\beta_2,2\beta_1)}(2x - 1)}{2^{\beta_1 + \beta_2} \beta_1! \beta_2!}.
\]

(14)
The \( n = 0 \) term represents 90% or more of the probability. The matrix representation of the longitudinal equation, for \( M_\parallel = 0 \), is then

\[
\left( \frac{m_1^2}{m_2^2} A_1 + \frac{m_2^2}{m_2^2} A_2 + B \right) \mathbf{c} = \xi \mathbf{c},
\]

with \( \xi \equiv C/m_2^2 \) and

\[
(A_1)_{nm} = \int_0^1 \frac{dx}{x} f_n(x) f_m(x), \quad (A_2)_{nm} = \int_0^1 \frac{dx}{1-x} f_n(x) f_m(x),
\]

\[
B_{nm} = \frac{1}{2} \int_0^1 dx \int_0^1 dy \frac{f_n(x) - f_n(y)}{x-y} \frac{f_m(x) - f_m(y)}{x-y}.
\]

The solution of the matrix problem yields the coefficients for the basis-function expansion.

The wave function can then be used to compute the decay constant \([35–37]\)

\[
f_M = 2\sqrt{6} \int_0^1 dx \int_0^{\infty} \frac{d\kappa_{\perp}^2}{16\pi^2} \psi(x, \kappa_{\perp})
\]

and the parton distribution \([24,38]\)

\[
f(x) = P_{q\bar{q}} \frac{X^2(x)}{x(1-x)},
\]

where \( P_{q\bar{q}} \) is the probability of the \( q\bar{q} \) Fock component in the meson. The chosen parameter values and the resulting decay constants are listed in Table 1 for the pion, kaon, and J/Ψ. The wave functions and parton distributions are very similar to those of the ansatz (12), except for the J/Ψ, as can be seen in Figs. 6, 7, and 8.

### Table 1 Parameters and decay constants, compared with the ansatz of Brodsky and Détéramond [30]

| Meson | Model | \( m_1 \) | \( m_2 \) | \( \mu_1 \) | \( \mu_2 \) | \( P_{q\bar{q}} \) | \( \kappa \) | Decay constant |
|-------|-------|----------|----------|----------|----------|----------------|---------|----------------|
| Pion  | 330   | 330      | 4        | 4        | 0.204    | 951            | 131     | 132            | 130          |
| Kaon  | 330   | 500      | 4        | 101      | 1        | 524            | 160     | 162            | 156          |
| J/Ψ   | 1,500 | 1,500    | 1,270    | 1,270    | 1        | 894            | 267     | 238            | 278          |

All dimensionful parameters are in units of MeV. Parameter and experimental values are from Ref. [24] and the Particle Data Group [39].

![Fig. 6](image-url) Longitudinal wave function (a) and parton distribution function (b) for the pion. The solid lines are wave functions from our model; the dashed lines show the ansatz by Brodsky and Détéramond [30].
In order to avoid truncation of Fock space, the LFCC method [1] generates all the higher Fock-state wave functions from the lower wave functions, based on the solution of nonlinear equations for vertex-like functions. This eliminates sector dependence and spectator dependence from the terms in the effective Hamiltonian. The truncation of the nonlinear equations can be relaxed systematically, to provide ever more sophisticated approximations for the higher wave functions.

The LFCC method divides the hadronic eigenproblem into an effective eigenproblem in the valence sector and auxiliary equations that define the effective Hamiltonian. Light-front holography [2] then provides a model for the valence sector. This model can be augmented to include quark masses and a dynamical equation for the longitudinal wave function [3] that is consistent with the Brodsky-de Téramond ansatz [30]. The numerical solution of the longitudinal equation includes a choice of basis functions that could be useful beyond just the holographic approximation to QCD.

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