A PRIORI BOUND ON THE VELOCITY IN AXIALLY SYMMETRIC NAVIER-STOKES EQUATIONS

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Abstract. Let \( v \) be the velocity of Leray-Hopf solutions to the axially symmetric three-dimensional Navier-Stokes equations. Under suitable conditions for initial values, we prove the following a priori bound

\[ |v(x, t)| \leq \frac{C}{r^2}, \]

where \( r \) is the distance from \( x \) to the \( z \) axis, and \( C \) is a constant depending only on the initial value.

This provides a pointwise upper bound (worst case scenario) for possible singularities while the recent papers [5] and [11] gave a lower bound. The gap is polynomial order 1.

1. Introduction

In this paper, we prove, under suitable initial condition, that the flow speed in the axially symmetric incompressible flow has an a priori bound which is proportional to the inverse square of the distance to the rotational axis. In order to present the result precisely, let us first recall the basic set ups. In Cartesian coordinates, the incompressible Navier-Stokes equations are

\[
\Delta v - (v \cdot \nabla)v - \nabla p - \partial_t v = 0, \quad \text{div } v = 0,
\]

where \( v = (v_1(x, t), v_2(x, t), v_3(x, t)) : \mathbb{R}^3 \times [0, T] \rightarrow \mathbb{R}^3 \) is the velocity field and \( p = p(x, t) : \mathbb{R}^3 \times [0, T] \rightarrow \mathbb{R} \) is the pressure. In cylindrical coordinates \( r, \theta, z \) with \( (x_1, x_2, x_3) = (r \cos \theta, r \sin \theta, z) \), axially symmetric solutions are of the form

\[
v(x, t) = v_r(r, z, t)e_r + v_{\theta}(r, z, t)e_\theta + v_z(r, z, t)e_z.
\]

The components \( v_r, v_\theta, v_z \) are all independent of the angle of rotation \( \theta \). Here \( e_r, e_\theta, e_z \) are the basis vectors for \( \mathbb{R}^3 \) given by

\[
e_r = \left( \frac{x_1}{r}, \frac{x_2}{r}, 0 \right), \quad e_\theta = \left( \frac{-x_2}{r}, \frac{x_1}{r}, 0 \right), \quad e_z = (0, 0, 1).
\]

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It is well known (see [4] for example) that \( v_r, v_z \) and \( v\theta \) satisfy the equations

\[
\begin{align*}
(\Delta - \frac{1}{r^2})v_r - (b \cdot \nabla)v_r + \frac{v_r^2}{r} - \frac{\partial p}{\partial r} - \frac{\partial v_r}{\partial t} &= 0, \\
(\Delta - \frac{1}{r^2})v_{\theta} - (b \cdot \nabla)v_{\theta} - \frac{v_r v_{\theta}}{r} - \frac{\partial v_{\theta}}{\partial t} &= 0, \\
\Delta v_z - (b \cdot \nabla)v_z - \frac{\partial p}{\partial z} - \frac{\partial v_z}{\partial t} &= 0, \\
\frac{1}{r} r \frac{\partial r v_r}{\partial r} + \frac{\partial v_z}{\partial z} &= 0,
\end{align*}
\]

where \( b(x, t) = (v_r, 0, v_z) \) and the last equation is the divergence-free condition. Here, \( \Delta \) is the cylindrical scalar Laplacian and \( \nabla \) is the cylindrical gradient field:

\[
\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}, \quad \nabla = \left( \frac{\partial}{\partial r}, \frac{1}{r} \frac{\partial}{\partial \theta}, \frac{\partial}{\partial z} \right).
\]

Observe that the equation for \( v_{\theta} \) does not depend on the pressure. Let \( \Gamma = rv_{\theta} \), one sees that the function \( \Gamma \) satisfies

\[
\Delta \Gamma - (b \cdot \nabla)\Gamma - \frac{2}{r} \frac{\partial \Gamma}{\partial r} - \frac{\partial \Gamma}{\partial t} = 0, \quad \text{div } b = 0.
\]

Recall that the vorticity \( \omega = \text{curl } v \) for axially symmetric solutions

\[
\omega(x, t) = \omega_r \vec{e}_r + \omega_{\theta} \vec{e}_{\theta} + \omega_z \vec{e}_z
\]

is given by

\[
\omega_r = -\frac{\partial v_{\theta}}{\partial z}, \quad \omega_{\theta} = \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}, \quad \omega_z = \frac{\partial v_{\theta}}{\partial r} + \frac{v_{\theta}}{r}.
\]

The equations of vorticity \( \omega = \text{curl } v \) in cylindrical form are (again, see [4] for example):

\[
\begin{align*}
(\Delta - \frac{1}{r^2})\omega_r - (b \cdot \nabla)\omega_r + \omega_r \frac{\partial v_r}{\partial r} + \omega_z \frac{\partial v_z}{\partial z} - \frac{\partial \omega_r}{\partial t} &= 0, \\
(\Delta - \frac{1}{r^2})\omega_{\theta} - (b \cdot \nabla)\omega_{\theta} + 2 \frac{v_r}{r} \frac{\partial v_r}{\partial r} + \omega_{\theta} \frac{\partial v_{\theta}}{\partial r} - \frac{\partial \omega_{\theta}}{\partial t} &= 0, \\
\Delta \omega_z - (b \cdot \nabla)\omega_z + \omega_z \frac{\partial v_z}{\partial z} + \omega_r \frac{\partial v_r}{\partial r} - \frac{\partial \omega_z}{\partial t} &= 0.
\end{align*}
\]

Define \( \Omega = \frac{\omega_{\theta}}{r} \), then we have that \( \Omega \) satisfies

\[
\Delta \Omega - (b \cdot \nabla)\Omega + \frac{2}{r} \frac{\partial \Omega}{\partial r} - \frac{\partial \Omega}{\partial t} + \frac{2v_r}{r^2} \frac{\partial v_r}{\partial z} = 0, \quad \text{div } b = 0.
\]

If the swirl \( v_{\theta} = 0 \), then it is known for long time (see O. A. Ladyzhenskaya [12], M. R. Uchovskii and B. I. Yudovich [18]), that finite energy solutions to (1.1) are smooth for all time. See also the paper by S. Leonardi, J. Malek, J. Necas, and M. Pokornyi [15]).

In the presence of swirl, it is not known in general if finite energy solutions blow up in finite time. However a lower bound for the possible blow up rate is known by the recent results of C.-C. Chen, R. M. Strain, T.-P. Tsai, and H.-T. Yau in [4], [5], G. Koch, N. Nadirashvili, G. Seregin, and V. Sverak in [11]. See also the work by G. Seregin and V. Sverak [16] for a localized version. These authors prove that if

\[
|v(x, t)| \leq \frac{C}{r},
\]

then solutions are smooth for all time. Here \( C \) is any positive constant. Solutions satisfying this bound are often refereed to as type I solutions. One reason for this name is that the
bound scales the same way as the reciprocal of the distance. Their result can be rephrased as: type I solutions are regular. See also the paper by two of us [13, 14] on further results in this direction. Without knowing if blow up happens in general, it is desirable to find an upper bound for the growth of velocity. It is expected that the solutions are smooth away from the axis, with certain growing bound when approaching the axis. Our Theorem 1.1 confirms this intuitive idea. Although it did not give the bound (1.5) which is required for smoothness, it reveals the exact gap between what we have and what we need.

This seems to be the first pointwise bound for the speed (velocity) for the axially symmetric Navier-Stokes equation. We mention that a less accurate a priori upper bound for smoothness, it did not give the bound (1.5) which is required from the axis, with certain growing bound when approaching the axis. Our Theorem 1.1 upper bound for the growth of velocity. It is expected that the solutions are smooth away in this direction. Without knowing if blow up happens in general, it is desirable to find an

As type I solutions are regular. See also the paper by two of us [13], [14] on further results

Theorem 1.1. Suppose $v$ is a smooth, axially symmetric solution of the three-dimensional Navier-Stokes equations in $\mathbb{R}^3 \times (-T, 0)$ with initial data $v_0 = v(\cdot, -T) \in L^2(\mathbb{R}^3)$. Assume further $rv_{0, \theta} \in L^\infty(\mathbb{R}^3)$ and let $R = \min\{1, \sqrt{T/2}\}$.

(i). Then for all $(x, t) \in \mathbb{R}^3 \times (-R^2, 0)$, it holds

$$|v_r(x, t)| + |v_\theta(x, t)| \leq \frac{C}{r^{2+}}.$$  

Here $r$ is the distance from $x$ to the $z$ axis, $2^+$ is any number strictly greater than 2 and $C$ is a constant depending only on the initial value and $2^+$.

(ii). Suppose in addition, one assumes $r^2 \omega_\theta(\cdot, -T) \in L^2(\mathbb{R}^3)$ and $rv_{\theta}(\cdot, -T) \in L^3(\mathbb{R}^3)$. Then for all $(x, t) \in \mathbb{R}^3 \times (-R^2, 0)$, it holds

$$|v_r(x, t)| + |v_\theta(x, t)| \leq \frac{C}{r^{2+}}.$$  

The proof of the theorem is based on the following pointwise bound on the vorticity.

Theorem 1.2. Suppose $v$ is a smooth, axially symmetric solution of the three-dimensional Navier-Stokes equations in $\mathbb{R}^3 \times (-T, 0)$ with initial data $v_0 = v(\cdot, -T) \in L^2(\mathbb{R}^3)$, and $\omega$ is the vorticity. Assume further, $rv_{0, \theta} \in L^\infty(\mathbb{R}^3)$ and let $R = \min\{1, \sqrt{T/2}\}$. Then the following a priori estimates hold.

(i). For any number $Q > 4$, there is an absolute constant $C$, depending on $Q$, such that the following holds for all $(x, t) \in \mathbb{R}^3 \times (-R^2, 0)$ with $r = |x'| \in (0, R)$:

$$|\omega_\theta(x, t)| \leq \frac{C}{r^{(10+Q)/4}} \left[ \sup_{s \in [t-r^2, t]} \left( \int_{B(x, 4r)} (v_r^2 + v_\theta^2)(y, s) dy ds \right)^{1/2} + r^{1/2} \Lambda \right]^{Q/2} \times \left[ \left( \int_{t-r^2}^t \int_{B(x, 4r)} \omega_\theta^2(y, s) dy ds \right)^{1/2} + r^{1/2} \Lambda \right].$$

Here $\Lambda = \|rv_{0, \theta}\|_{L^\infty(\mathbb{R}^3)}$ and $C$ is a generic constant.
(ii). If in addition, one assumes \( r^2 \omega_\theta(\cdot, -T) \in L^2(\mathbb{R}^3) \) and \( rv_\theta(\cdot, -T) \in L^3(\mathbb{R}^3) \), then for all \((x, t) \in \mathbb{R}^3 \times (-R^2, 0)\), it holds

\[
|\omega_\theta(x, t)| \leq \frac{C}{r^{7/2}}
\]

where \( C \) is a positive constant depending only on the initial condition.

Remark 1.1. We assume smoothness of the solution only for technical simplicity. One can use standard approximation methods to treat the weak solution case. In fact, if \( v \) is a (suitable) Leray-Hopf solution, then it is smooth except possibly on the \( z \) axis (c.f. [2]). Also, the bound on \( \omega_\theta \) is scaling invariant. Similar bounds can also be proven for the other two components of the vorticity \( \omega_r \) and \( \omega_z \). But we will not do this here.

Here we mention a number of related papers on axially symmetric Navier-Stokes equations. J. Neustupa and M. Pokorny [8] proved that the regularity of one component (either \( v_r \) or \( v_\theta \)) implies regularity of the other components of the solution. Also proving regularity is the work of Q. Jiu and Z. Xin [9] under an assumption of sufficiently small zero-dimension scaled norms. D. Chae and J. Lee [3] also proved regularity results assuming finiteness of another certain zero-dimensional integral. G. Tian and Z. Xin [17] constructed a family of singular axially symmetric solutions with singular initial data. T. Hou and C. Li [6] found a special class of global smooth solutions. See also a recent extension: T. Hou, Z. Lei and C. Li [7].

Let us outline the proof of the theorem. The starting point is the a priori bound for the rotational component of the velocity: \( r|v_\theta(\cdot, t)| \in L^\infty \). A proof of this fact can be found in [3] Section 3, Proposition 1, for example. The first observation is that the basic energy estimate (3.2) is critical when localized in a dyadic ball which is away from the symmetric axis. This enables us to perform a kind of dimension reduction argument and apply two-dimensional Sobolev imbedding inequalities. The second ingredient is a new weighted \( L^2 \) estimate for \( \omega_\theta \) (see Lemma 3.1) which enables us to enhance the Moser’s iterative scheme. More precisely, we will first apply the new weighted \( L^2 \) estimate of \( \omega_\theta \) to derive a localized \( L^{2+} \) estimate of \( b \) (see Lemma 3.2). Then we apply two-dimensional Sobolev imbedding inequalities and the \( L^{2+} \) estimate of \( b \) to get a localized space-time \( L^4 \) estimate for \( \Omega \) (see Lemma 3.3), which will be used to serve as the first step of Moser’s iteration. Then we apply the standard Moser iteration method and two-dimensional Sobolev imbedding inequalities to get an upper bound for \( \Omega \), based on the evolution equation of \( \Omega \) in (1.4). The third ingredient is a novel use of the localized Biot-Savart law. We use axis-symmetry to show that \( L^2 \) integrals of velocity in small dyadic regions are smaller than usual. This fact and the a priori bound on \( \omega_\theta \) implies the point-wise bound on \( |v^r| + |v^z| \).

Throughout this paper we use \( C \) to denote an absolute positive constant. When \( C \) depends on \( p \), we use the notation \( C_p \). The meanings of \( C \) and \( C_p \) may change from line to line. We will also use \( N^+ \) to denote a number which is bigger but sufficiently close to \( N \).

The remainder of the paper is organized as follows. In Section 2, we prove Theorem 1.2 part (i). In Section 3 we prove integral bounds for \( b \) and \( \omega_\theta \) mentioned earlier. We will prove Theorem 1.2 part (ii) in Section 4 and finally Theorem 1.1 in Section 5.
2. A priori bound for \( \omega \), part (i)

In this section, we will prove part (i) of Theorem 1.1.

Let \((x, t)\) be the point in the statement of the theorem. For simplicity we take \( t = 0 \) and \( x_3 = 0 \). During the proof, it is convenient to replace the three dimensional ball \( B(x, 4r) \) by comparable cylindrical type regions. The reason is that a cylindrical region has a fixed profile in the \( e_r, e_z \) plane. This feature allows us to reduce much computations to 2 dimensional setting.

So let us introduce a few more notations. Let \( R > 0, S > 0, \) and \( 0 < A < B \) be constants.

Denote

\[
C_{AR,BR} = \{(x_1, x_2, x_3) : AR \leq r \leq BR, \ 0 \leq \theta \leq 2\pi, \ |x_3| \leq BR\} \subset \mathbb{R}^3
\]

to be the hollowed out cylinder centered at the origin, with inner radius \( AR \), outer radius \( BR \), and height extending up and down \( BR \) units for a total height of \( 2BR \). If \( R = 1 \), we will write \( C_{A,B} \) in place of \( C_{A1,B1} \).

Denote \( P_{AR,BR,SR} \) to be the parabolic region

\[
P_{AR,BR,SR} = C_{AR,BR} \times (-S^2 R^2, 0).
\]

If \( R = 1 \), we will use \( P_{A,B,S} \) to denote \( P_{A1,B1,S1} \).

Proving the theorem is equivalent to finding priori bound for \( \omega \) in the region \( P_{\frac{1}{2} T, \frac{1}{2} R} \) with \( 0 < k < \min\{1, \sqrt{T/2}\} \). We will use the scaling property of the Navier-Stokes equation to shift the consideration to the cube \( P_{\frac{1}{2} T, \frac{1}{2} R} \). We recall that scaling of the equations now: the pair \((v(x, t), p(x, t))\) is a solution to the system, if and only if for any \( k > 0 \) the re-scaled pair \((\tilde{v}(x, \tilde{t}), \tilde{p}(x, \tilde{t}))\) is also a solution, where \( \tilde{v}(x, \tilde{t}) = kv(x, k^2 t), \ \tilde{p}(x, \tilde{t}) = k^2 p(kx, k^2 t) \). Thus, if \((v, p)\) is a solution to the axially symmetric Navier-Stokes equations for \((x, t) \in P_{k,4k,k}\), then \((\tilde{v}(x, \tilde{t}), \tilde{p}(x, \tilde{t}))\) is a solution to the equation in the variables \( \tilde{x} = \frac{x}{k}, \tilde{t} = \frac{t}{k^2} \) when \((\tilde{x}, \tilde{t}) \in P_{1,4,1} \). We note here how certain quantities scale or change due to the above. Here, \( D \) is any domain in \( \mathbb{R}^3 \) and \( kD = \{x : x = ky, \ y \in D\} \):

\[
\|v(x, t)\|_{L^2(kD \times (-kR^2, 0))} = \left( \int_{-kR}^{0} \int_{D} |\tilde{v}(\tilde{x}, \tilde{t})|^2 d\tilde{x} d\tilde{t} \right)^{\frac{1}{2}}
\]

\[
\|v(x, t)\|_{L^2(D \times (-R^2, 0))} = \left( \frac{1}{k^3} \int_{-kR}^{0} \int_{D} |kv(x, t)|^2 \frac{1}{k^5} dxdt \right)^{\frac{1}{2}} = \frac{1}{k^3} \|v(x, t)\|_{L^2(kD \times (-kR^2, 0))}.
\]

\[
b(x, t) = (v_r, 0, v_z) ;
\]

\[
\tilde{b}(x, t) = (kv_r(kx, k^2 t), 0, kv_z(kx, k^2 t)) = kb(kx, k^2 t), \ \ (x, t) \in P_{k,4k,k}
\]

\[
\Rightarrow \tilde{b}(\tilde{x}, \tilde{t}) = kb(x, t).
\]

\[
\|b(x, t)\|_{L^\infty(-kR^2, 0, L^2(kD))} = \|
\]
\[
\|\tilde{b}(\tilde{x}, \tilde{t})\|_{L^\infty(-R^2,0;L^2(D))} = \sup_{-R^2 \leq \tilde{t} < 0} \left( \int_D |\tilde{b}(\tilde{x}, \tilde{t})|^2 d\tilde{x} \right)^{\frac{1}{2}} \\
= \sup_{-(kR)^2 \leq \tilde{t} < 0} \left( \int_{kD} |b(x,t)|^2 \frac{1}{k^{3}} d \tilde{x} \right)^{\frac{1}{2}} = \frac{1}{k^{\frac{3}{2}}} \|b(x,t)\|_{L^\infty(-(kR)^2,0;L^2(kD))}.
\]

\[\omega(x,t) = k^2 \omega(kx,k^2t), \quad (x,t) \in P_{1,k} \Rightarrow \tilde{\omega}(\tilde{x}, \tilde{t}) = k^2 \omega(x,t)\]

\[\|\omega(x,t)\|_{L^2(kD \times (-kR^2,0))} = \left( \int_{-kR^2}^{0} \int_{kD} \tilde{\omega}(\tilde{x}, \tilde{t})^2 d\tilde{x} d\tilde{t} \right)^{\frac{1}{2}} = \frac{1}{k^{\frac{3}{2}}} \|\omega(x,t)\|_{L^2(kD \times (-kR^2,0))}.
\]

One can also show that \(\tilde{\Gamma}(\tilde{x}, \tilde{t}) = \tilde{r}v_\theta(\tilde{x}, \tilde{t})\) is a solution to (1.2) and \(\tilde{\Omega}(\tilde{x}, \tilde{t}) = \frac{\tilde{\omega}(\tilde{x}, \tilde{t})}{\tilde{r}}\) is a solution to (1.4) in the variables \((\tilde{x}, \tilde{t}) \in P_{1,1,1}\). We will do most of our computations on scaled cylinders.

Since \(rv_\theta\) is scaling invariant, using the following result, we know that \(\tilde{r}v_\theta\) is uniformly bounded for all time.

**Proposition 2.1. (23) and [8]** Suppose \(v\) is a smooth, axially symmetric solution of the three-dimensional Navier-Stokes equations with initial data \(v_0 \in L^2(\mathbb{R}^3)\). If \(rv_\theta \in L^p(\mathbb{R}^3)\), then \(rv_\theta \in L^\infty(0,T;L^p(\mathbb{R}^3))\). In particular, if \(p = \infty\),

\[|v_\theta(x,t)| \leq \frac{\|rv_\theta\|_{L^\infty(\mathbb{R}^3)}}{\sqrt{x_1^2 + x_2^2}}.
\]

**Proof of Theorem 1.2** (i).

During the proof, we are going to drop the “tilde” notation for all relevant quantities over a time when computations take place on the scaled cylinders. By the end, we will scale down to the original solution. Although this scaling seems merely a technical move that simplifies the computation, it actually a key step that allows us to carry out a dimension reduction argument mentioned earlier. In the region \(P_{1,1,1}\) we do our analysis on (1.4):

\[\Delta \Omega - (b \cdot \nabla) \Omega + \frac{2}{\tilde{r}} \frac{\partial \Omega}{\partial \tilde{r}} - \frac{\partial \Omega}{\partial \tilde{t}} + \frac{2v_\theta}{\tilde{r}^2} \frac{\partial v_\theta}{\partial \tilde{z}} = 0, \quad \text{div } b = 0.
\]

A flow chart for the argument to prove part (i) of Theorem 1.2 is as follows:

- **Step 1:** Energy Estimates by a refined cut-off function.
- **Step 2:** Estimate drift term \((b \cdot \nabla) \Omega\) using methods similar to [19]. Use dimension reduction. Note this term is more singular than that allowed by standard theory.
- **Step 3:** Estimate a term involving the cut-off.
- **Step 4:** Estimate the term involving the directional derivative \(\partial_\tilde{r}\) using a method similar to that in [4].
- **Step 5:** Estimate the inhomogeneous term utilizing the bound in Proposition 2.1.
- **Step 6:** \(L^2 - L^\infty\) Estimate on Solutions to (1.4) via Moser’s Iteration. Use dimension reduction.
$L^2 - L^\infty$ Estimate on $\omega_\theta$ via re-scaling.

**Energy Estimates:**

**Step 1:** We use a revised cut-off function and the equation to obtain inequality (2.7) below.

Note that (2.3) $\Lambda \equiv \|v_\theta\|_{L^\infty(P_{1,4,1})} \leq \|rv_{0,\theta}\|_{L^\infty(\mathbb{R}^3)} < \infty,$

where we have used the hypothesis that $rv_{0,\theta} \in L^\infty(\mathbb{R}^3)$, the point-wise bound in Proposition 2.1 and the fact that $1 < \sqrt{x_1^2 + x_2^2} < 4$. Let

(2.4) $\overline{\Omega}_+(x, t) = \begin{cases} \Omega(x, t) + \Lambda & \Omega(x, t) \geq 0, \\ \Lambda & \Omega(x, t) < 0. \end{cases}$

Note that $\overline{\Omega}_+ \geq \Lambda$ and all derivatives of $\overline{\Omega}_+$ on the set where $\Omega(x, t) < 0$ are equal to zero. This function is also Lipschitz and $\Omega$ is smooth by assumption. At interfaces boundary terms upon integration by parts will cancel and so we can make sense of the calculations below. Direct computation yields, for $q > 1$, that

(2.5) $\Delta \overline{\Omega}_+^q - (b \cdot \nabla) \overline{\Omega}_+^q + \frac{2}{r} \partial_r \overline{\Omega}_+^q - \partial_t \overline{\Omega}_+^q = -\frac{q\overline{\Omega}_+^{q-1} \partial v_\theta^2}{r^2} + q(q - 1)\overline{\Omega}_+^{q-2} |\nabla \overline{\Omega}_+|^2.$

Let $\frac{5}{8} \leq \sigma_2 < \sigma_1 \leq 1$. Define

(2.6) $P(\sigma_i) = \{(r, \theta, z) : (5 - 4\sigma_i) < r < 4\sigma_i, \ 0 \leq \theta \leq 2\pi, \ |z| < 4\sigma_i\} \times (-\sigma_i^2, 0)$

for $i = 1, 2$. Here for convenience denote the space portion, which is a hollowed out cylinder, as $C(\sigma_i)$. Choose $\psi = \phi(y)\eta(s)$ to be a refined cut-off function satisfying

\[ \text{supp } \phi \subset C(\sigma_1); \ \phi(y) = 1 \text{ for all } y \in C(\sigma_2); \ 0 \leq \phi \leq 1; \]
\[ \frac{|\nabla \phi|}{\phi^\delta} \leq \frac{c_1}{\sigma_1 - \sigma_2} \text{ for } \delta \in (0, 1); \]
\[ \text{supp } \eta \subset (-\sigma_1^2, 0); \ \eta(s) = 1, \text{ for all } s \in [-\sigma_2^2, 0]; \ 0 \leq \eta \leq 1 \]
\[ |\eta'| \leq \frac{c_2}{(\sigma_1 - \sigma_2)^2}. \]

Let $f = \overline{\Omega}_+^q$ and use $f\psi^2$ as a test function in (2.5) to get

\[ \int_{P(\sigma_1)} (\Delta f - (b \cdot \nabla) f - \partial_s f + \frac{2}{r} \partial_r f) f\psi^2 \, dy \, ds \]
\[ = \int_{P(\sigma_1)} q(q - 1)\overline{\Omega}_+^{q-2} |\nabla \overline{\Omega}_+|^2 f\psi^2 \, dy \, ds - \int_{P(\sigma_1)} \frac{q\overline{\Omega}_+^{q-1} \partial v_\theta^2}{r^2} f\psi^2 \, dy \, ds \]
\[ = q(q - 1) \int_{P(\sigma_1)} \overline{\Omega}_+^{q-2} |\nabla \overline{\Omega}_+|^2 f^2 \psi^2 \, dy \, ds - \int_{P(\sigma_1)} \frac{q\overline{\Omega}_+^{q-2} \partial v_\theta^2}{r^2} \psi^2 \, dy \, ds \]
\[ \geq - \int_{P(\sigma_1)} q \Omega_+^{2q-1} \frac{\partial \nu_\theta^2}{r^2} \psi^2 dyds. \]

Integration by parts on the first term implies that
\[ \int_{P(\sigma_1)} \nabla (f \psi^2) \nabla f dyds \]
\[ \leq \int_{P(\sigma_1)} \left( -b \cdot \nabla f (f \psi^2) - \partial_s f (f \psi^2) + \frac{2}{r} \partial_r f (f \psi^2) + \frac{q \Omega_+^{2q-1}}{r^2} \frac{\partial \nu_\theta^2}{\partial z} \psi^2 \right) dyds. \]

A manipulation using the product rule shows that
\[ \int_{P(\sigma_1)} \nabla (f \psi^2) \nabla f dyds = \int_{P(\sigma_1)} (|\nabla (f \psi)|^2 - |\nabla \psi|^2 f^2) dyds. \]

Thus,
\[ \int_{P(\sigma_1)} |\nabla (f \psi)|^2 dyds \leq \int_{P(\sigma_1)} \left( -b \cdot \nabla f (f \psi^2) - \partial_s f (f \psi^2) + \frac{2}{r} \partial_r f (f \psi^2) + \frac{q \Omega_+^{2q-1}}{r^2} \frac{\partial \nu_\theta^2}{\partial z} \psi^2 + |\nabla \psi|^2 f^2 \right) dyds. \]

Integration by parts on the term involving the time derivative yields
\[ \int_{P(\sigma_1)} - (\partial_s f) f \psi^2 dyds = - \frac{1}{2} \int_{P(\sigma_1)} \partial_s (f^2) \psi^2 dyds \]
\[ = - \frac{1}{2} \left( \int_{C(\sigma_1)} f^2 \psi^2 (y, 0) dy - \int_{C(\sigma_1)} f^2 \psi^2 (y, -\sigma_1^2) dy \right) + \frac{1}{2} \int_{P(\sigma_1)} \partial_s (\psi^2) f^2 dyds. \]

Our cut-off functions provide \( \psi^2 = (\phi \eta)^2, \) \( \eta(0) = 1, \) \( \eta(-\sigma_1^2) = 0, \) and \( 0 \leq \phi \leq 1. \) Thus,
\[ \int_{P(\sigma_1)} - (\partial_s f) f \psi^2 dyds = - \frac{1}{2} \int_{C(\sigma_1)} f^2 (y, 0) \phi^2 (y) dy + \int_{P(\sigma_1)} \phi^2 (\eta \partial_\eta \psi) f^2 dyds \]
\[ \leq - \frac{1}{2} \int_{C(\sigma_1)} f^2 (y, 0) \phi^2 (y) dy + \int_{P(\sigma_1)} (\eta \partial_\eta \psi) f^2 dyds, \]
and so,
\[ \int_{P(\sigma_1)} |\nabla (f \psi)|^2 dyds + \frac{1}{2} \int_{C(\sigma_1)} f^2 (y, 0) \phi^2 (y) dy \]
\[ \leq \int_{P(\sigma_1)} -b \cdot \nabla f (f \psi^2) dyds + \int_{P(\sigma_1)} (\eta \partial_\eta \psi + |\nabla \psi|^2) f^2 dyds \]
\[ + \int_{P(\sigma_1)} 2 \frac{\partial_r f (f \psi^2)}{r^2} dyds + \int_{P(\sigma_1)} \frac{q \Omega_+^{2q-1}}{r^2} \frac{\partial \nu_\theta^2}{\partial z} \psi^2 dyds \]
\[ := T_1 + T_2 + T_3 + T_4. \]

**Step 2:** In this step we find an upper bound for \( T_1, \) following an idea in [19], where a parabolic equation with a similar drift term is explored. The new input is that we can
exploit the fact that in the space time domains of concern, all three dimensional integrals are equivalent to two dimensional ones. Therefore we can apply the 2 dimensional Sobolev inequality, which allows us to make gains.

Since $\text{div } b = 0$,

$$T_1 = \int_{P(\sigma_1)} -b \cdot (\nabla f)(f\psi^2)dyds$$

$$= \frac{1}{2} \int_{P(\sigma_1)} -b\psi^2 \cdot \nabla(f^2)dyds = \frac{1}{2} \int_{P(\sigma_1)} \text{div } (b\psi^2)f^2dyds$$

$$= \frac{1}{2} \int_{P(\sigma_1)} \text{div } b(\psi f)^2dyds + \frac{1}{2} \int_{P(\sigma_1)} b \cdot \nabla(\psi^2)f^2dyds$$

$$= \int_{P(\sigma_1)} b \cdot (\nabla \psi)\psi f^2dyds$$

$$\leq \left| \int_{P(\sigma_1)} \left( b\psi^{1+\delta}(f^{2-a}) \left( \frac{\nabla \psi}{\psi^\delta} |f|^a \right) dyds \right) \right|,$$

for $0 < \delta < 1$, $0 < a < 2$ such that

$$(1 + \delta)p = 2, \quad (2 - a)p = 2.$$

Here $p$ is any number in the interval $(0, 2)$.

Observe that $ap' = ap/(p - 1) = 2$. We can apply Hölder’s inequality to deduce:

$$T_1 \leq \left( \int_{P(\sigma_1)} |b|^p\psi^{(1+\delta)p} |f|^{(2-a)p}dyds \right)^{1/p} \left( \int_{P(\sigma_1)} |f|^{ap'}dyds \right)^{1/p'} \sup_{\psi^\delta} |\nabla \psi|.$$  \hspace{1cm} (2.8)

Consider the domains with 2 spatial dimension

$$\overline{C}(\sigma_1) = \{(r,z) | (r,\theta,z) \in C(\sigma_1)\},$$

$$\overline{P}(\sigma_1) = \{(r,z,s) | (r,\theta,z,s) \in P(\sigma_1)\}.$$  \hspace{1cm} (2.9)

We also use the notations

$$\overline{y} = (r,z), \quad d\overline{y} = drdz, \quad \text{if} \quad dy = rdrdzd\theta.$$

Since $r$ is bounded between two positive constants in $P(\sigma_1)$, we deduce from (2.8) that

$$T_1 \leq C \left( \int_{\overline{P}(\sigma_1)} |b|^p(\psi f)^2d\overline{y}ds \right)^{1/p} \left( \int_{\overline{P}(\sigma_1)} f^2d\overline{y}ds \right)^{1/p'} \sup_{\psi^\delta} |\nabla \psi|.$$  \hspace{1cm} (2.10)

$$\leq \epsilon \int_{\overline{P}(\sigma_1)} |b|^p(\psi f)^2d\overline{y}ds + C \epsilon^{-p'/p} \int_{\overline{P}(\sigma_1)} f^2d\overline{y}ds \left( \sup_{\psi^\delta} |\nabla \psi| \right)^{p'}.$$
Using 2 dimensional Sobolev inequality, we have

\[
\int P|b|^p(\psi f)^2 d\gamma ds \\
\leq \int \left( \int_C |b|^2 d\gamma \right)^{p/2} \left( \int_C (\psi f)^{4/(2-p)} d\gamma \right)^{(2-p)/2} ds \\
\leq C \sup_s \left( \int_C |b|^2 d\gamma \right)^{p/2} \int P|\nabla (\psi f)|^2 d\gamma ds \\
\equiv C\overline{K}(b)^p \int P|\nabla (\psi f)|^2 d\gamma ds,
\]

where \(\nabla = (\partial_r, \partial_z)\) is the 2 dimensional gradient and

\[
\overline{K}(b) \equiv \sup_{-\sigma_1^2 \leq s \leq 0} \left( \int_C |b|^2 d\gamma \right)^{1/2}.
\]

Substituting this into (2.10), we obtain

\[
T_1 \leq C_1 \epsilon \overline{K}(b)^p \int P|\nabla (\psi f)|^2 d\gamma ds + C \epsilon^{-p'/p} \int P|f|^2 d\gamma ds \left( \sup_{\psi \leq 0} |\nabla \psi| \right)^{p'}.
\]

Hence

\[
T_1 \leq C_2 \epsilon \overline{K}(b)^p \int P|\nabla (\psi f)|^2 d\gamma ds + C \epsilon^{-p'/p} \int P|f|^2 d\gamma ds \left( \sup_{\psi \leq 0} |\nabla \psi| \right)^{p'}.
\]

Here \(C_2\) is a positive constant comparable with \(C_1\).

Taking \(\epsilon\) such that \(C_2 \epsilon \overline{K}(b)^p = 1/2\), we arrive at

\[
T_1 \leq \frac{1}{2} \int P|\nabla (\psi f)|^2 d\gamma ds + \overline{K}(b)^p \int P|f|^2 d\gamma ds \left( \sup_{\psi \leq 0} |\nabla \psi| \right)^{p'}.
\]

Using properties of the cutoff function we get

\[
\text{(2.11)} \quad T_1 \leq \frac{1}{2} \int P|\nabla (\psi f)|^2 d\gamma ds + \frac{C \overline{K}(b)^p}{(\sigma_1 - \sigma_2)^{p'}} \int P|f|^2 d\gamma ds.
\]

Here \(p \in (1, 2)\) and \(p' = p/(p - 1)\).

**Step 3:** The term \(T_2\) is treated routinely. We use

\[
T_2 = \int P(\eta \partial_s \eta + |\nabla \psi|^2) f^2 d\gamma ds,
\]

and properties of the cutoff,

\[
|\nabla \psi|^2 = |\eta \nabla \phi|^2 \leq \left( \frac{|\nabla \phi|}{\phi^2} \right)^2 \leq \frac{c_1^2}{(\sigma_1 - \sigma_2)^2},
\]

where
and
\[ |\eta \partial_s \eta| \leq |\partial_s \eta| \leq \frac{c_2}{(\sigma_1 - \sigma_2)^2}, \]
to get
\[ (2.12) \quad |T_2| \leq \frac{C}{(\sigma_1 - \sigma_2)^2} \int_{P(\sigma_1)} f^2 dy ds \leq \frac{C}{(\sigma_1 - \sigma_2)^2} \int_{P(\sigma_1)} f^2 d\bar{y} ds. \]

**Step 4:** As we deal with \( T_3 = \int_{P(\sigma_1)} \frac{2}{r} \partial_r f(\psi^2) dy ds \), we note we are assuming the integration takes place away from the singularity set of the solution to the axially symmetric Navier Stokes equations and away from the z-axis in general. Thus, all functions are bounded and smooth and \( r \) varies between two positive constants. We also utilize the cylindrical coordinates of the axially symmetric case, and integration by parts:
\[
T_3 = \int_{P(\sigma_1)} \frac{2}{r} \partial_r f(\psi^2) dy ds = \int_{P(\sigma_1)} \frac{1}{r} \partial_r (f^2) \psi^2 r dr d\theta dz ds \\
= \int_{P(\sigma_1)} \partial_r (f^2) \psi^2 dr d\theta dz ds = - \int_{P(\sigma_1)} \partial_r (\psi^2) f^2 dr d\theta dz ds \\
= - \int_{P(\sigma_1)} \frac{2}{r} \partial_r (\psi f^2) r dr d\theta dz ds = - \int_{P(\sigma_1)} \frac{2}{r} \partial_r (\psi f^2) dy ds \\
= - \int_{P(\sigma_1)} \frac{2}{r} \nabla \psi (\psi f^2) dy ds.
\]
The Cauchy-Schwarz inequality then implies
\[ |T_3| \leq \int_{P(\sigma_1)} \frac{2}{r} \nabla \psi |\psi| f^2 dy ds. \]
This yields
\[ (2.13) \quad |T_3| \leq \frac{C}{(\sigma_1 - \sigma_2)} \int_{P(\sigma_1)} f^2 dy ds \leq \frac{C}{(\sigma_1 - \sigma_2)} \int_{\bar{P}(\sigma_1)} f^2 d\bar{y} ds. \]

**Step 5:** Lastly, we work on the inhomogeneous term of (1.4), that is, \( 2v_\theta \partial_v \psi \), which produced the term \( T_4 \). Recall
\[ \Lambda = \|v_\theta\|_{L^\infty(P_{r,1})} \leq \|r v_0,\theta\|_{L^\infty(\mathbb{R}^3)} < \infty, \]
and that \( \Omega_+ = \begin{cases} \Omega + \Lambda & \Omega \geq 0 \\ \Lambda & \Omega < 0 \end{cases} \), thus \( \Omega_+ \geq \Lambda \). Also, we have let \( f = \Omega_+^q \). Using integration by parts yields
\[
T_4 = \int_{P(\sigma_1)} \frac{q \Omega_+^{2q-1}}{r^2} \partial_v \psi^2 dy ds \\
= - \int_{P(\sigma_1)} \frac{\partial}{\partial z} \left( \frac{\Omega_+^{2q}}{\Omega_+^q} \right) \frac{q}{r^2} \psi^2 dy ds.
\]
Thus continue by fixing $q > \frac{2}{12}$, utilizing $\Lambda = \overline{\Omega}_+$, and $r = \sqrt{y_1^2 + y_2^2} \geq 1$ for all $y \in P(\sigma_1)$, we continue by fixing $\varepsilon_3 > 0$. Apply Young’s inequality with exponents both being 2 to get

$$|T_4| \leq \int_{P(\sigma_1)} 2q|\nabla f\psi| \left| \frac{\partial (f\psi)}{\partial z} \right| dyds + \frac{c_3}{\sigma_1 - \sigma_2} \int_{P(\sigma_1)} f^2 |v_\theta| dyds$$

$$\leq \int_{P(\sigma_1)} \left[ \frac{2q\Lambda}{(2\varepsilon_3)^2} f\psi \right] \times \left[ (2\varepsilon_3)^\frac{1}{2} \frac{\partial (f\psi)}{\partial z} \right] dyds + \frac{c_3\Lambda}{\sigma_1 - \sigma_2} \int_{P(\sigma_1)} f^2 dyds$$

$$\leq \frac{c_12\Lambda^2q^2}{\varepsilon_3} \int_{P(\sigma_1)} f^2 dyds + c_3 \int_{P(\sigma_1)} |\nabla (f\psi)|^2 dyds + \frac{c_3\Lambda}{\sigma_1 - \sigma_2} \int_{P(\sigma_1)} f^2 dyds.$$

Thus

$$|T_4| \leq \frac{1}{4} \int_{P(\sigma_1)} |\nabla (f\psi)|^2 dyds + C \left[ \Lambda^2q^2 + \frac{\Lambda}{\sigma_1 - \sigma_2} \right] \int_{P(\sigma_1)} f^2 dyds.$$

**Step 6:** $L^2 - L^\infty$ Estimate: An $L^2 - L^\infty$ bound is derived using Moser’s iteration. Recall inequality (2.7) from Step 1 and substitute the estimates for $T_1, T_2, T_3, T_4$ in (2.11), (2.12), (2.13), (2.14) respectively, we obtain

$$\int_{P(\sigma_1)} |\nabla (f\psi)|^2 dyds + \frac{1}{2} \int_{C(\sigma_1)} f^2 (y, 0) \phi^2 (y) dy$$

$$\leq \frac{3}{4} \int_{P(\sigma_1)} |\nabla (f\psi)|^2 dyds + C \left[ \frac{CK(b)\psi'}{(\sigma_1 - \sigma_2)^{\psi'}} + 1 + \Lambda^2q^2 + \frac{1}{(\sigma_1 - \sigma_2)^2} \right] \int_{P(\sigma_1)} f^2 dyds.$$

Consequently,

$$\int_{P(\sigma_1)} |\nabla (f\psi)|^2 dyds + \int_{P(\sigma_1)} f^2 (y, 0) \phi^2 (y) dy$$

$$\leq \frac{Cq^2}{(\sigma_1 - \sigma_2)^{\psi'}} \left[ K(b)\psi' (C_1, \Lambda^2) + \Lambda^2 + 1 \right] \int_{P(\sigma_1)} f^2 dyds.$$

The last inequality follows since $q > 1$ and $0 < \sigma_1 - \sigma_2 < 1$. 

Next we carry out Moser’s iteration process on \((2.15)\). Note that Hölder’s inequality and the Sobolev inequality imply, for any \(\mu > 2\)
\[
\int_{\mathbb{R}^2} (f\phi)^{2(1+\frac{2}{\mu})} d\gamma = \int_{\mathbb{R}^2} (f\phi)^{4/\mu} (f\phi)^{2} d\gamma \\
\leq \left( \int_{\mathbb{R}^2} (f\phi)^{2} d\gamma \right)^{\frac{2}{\mu}} \left( \int_{\mathbb{R}^2} (f\phi)^{2\mu} d\gamma \right)^{\frac{\mu-2}{\mu}} \\
\leq c \left( \int_{\mathbb{R}^2} (f\phi)^{2} d\gamma \right)^{\frac{2}{\mu}} \left( \int_{\mathbb{R}^n} |\nabla (f\phi)|^2 d\gamma \right),
\]

Multiply by the time portion of the cut-off function to the correct power, \(\eta^{2(1+\frac{2}{\mu})}(s)\), on both sides and integrate over time; one can deduce that
\[
\int_{-\sigma_1^2}^{0} \int_{\mathbb{R}^2} (f\psi)^{2(1+\frac{2}{\mu})} d\gamma ds \\
\leq c \sup_{-\sigma_1^2 \leq s \leq 0} \left( \int_{\mathbb{R}^2} (f\psi)^{2} d\gamma \right)^{\frac{2}{\mu}} \int_{-\sigma_1^2}^{0} \int_{\mathbb{R}^2} |\nabla (f\psi)|^2 d\gamma ds.
\]
We use properties of the cut-off to obtain
\[
(2.16) \quad \int_{\mathcal{P}(\sigma_1)} (\psi f)^{2(1+\frac{2}{\mu})} d\gamma ds \\
\leq c \left( \sup_{-\sigma_1^2 \leq s < 0} \int_{\mathcal{C}(\sigma_1)} (f\psi)^{2}(y,s) d\gamma \right)^{\frac{2}{\mu}} \int_{\mathcal{P}(\sigma_1)} |\nabla (f\psi)|^2 d\gamma ds.
\]
The above argument can be run for each time level \(-\sigma_2^2 \leq s < 0\) and in fact \((2.15)\) holds for all \(s\) in this interval as the upper time limit of the time cut-off function. Thus, the second-to-last factor on the right-hand side of inequality \((2.16)\) is still controlled by estimate \((2.15)\). So together with the estimate and the cut-off function again, we get
\[
(2.17) \quad \int_{\mathcal{P}(\sigma_2)} \Omega_+^{2\gamma} d\gamma ds \leq c \left[ \frac{C}{\tau^{p'}} (K^{p'}(b,\mathcal{C}_{1,4}) + \Lambda^2 + 1) \int_{\mathcal{P}(\sigma_1)} \Omega_+^{2\gamma} d\gamma ds \right]^\gamma,
\]
where
\[
\gamma = 1 + \frac{2}{\mu}, \quad \tau = \sigma_1 - \sigma_2, \quad p' > 2, \mu > 2.
\]
Let \(\tau_i = 2^{-i-2}\), \(\sigma_0 = 1\), \(\sigma_i = \sigma_{i-1} - \tau_i = 1 - \sum_{j=1}^{i} \tau_j\), \(q = \gamma^i\). Then \((2.17)\) generalizes to
\[
(2.18) \quad \int_{\mathcal{P}(\sigma_2)} \Omega_+^{2^{i+1}\gamma} d\gamma ds \leq c_1 \left[ c_2^{i+1} \gamma^{2i} (K^{p'}(b,\mathcal{C}_{1,4}) + \Lambda^2 + 1) \int_{\mathcal{P}(\sigma_1)} \Omega_+^{2^{i+1}\gamma} d\gamma ds \right]^\gamma,
\]
which, after taking the \(\frac{1}{\gamma}\)-th power of both sides, implies
\[
\left( \int_{\mathcal{P}(\sigma_2)} \Omega_+^{2^{i+1}\gamma} d\gamma ds \right)^\frac{1}{\gamma} \leq c_1^{\frac{1}{\gamma}} c_2^{i+2} \gamma^{2i} (K^{p'}(b,\mathcal{C}_{1,4}) + \Lambda^2 + 1) \int_{\mathcal{P}(\sigma_1)} \Omega_+^{2^{i+1}\gamma} d\gamma ds.
\]
Using (2.18) on the integral on the left and raising both sides to the $\frac{1}{\gamma}$-th power repeatedly, one obtains

$$\left( \int_{\mathcal{P}(\sigma_{i+1})} \Omega_+^{-\frac{2^j+1}{\gamma^j+1}} d\gamma ds \right)^{\frac{1}{\gamma^j+1}} \leq c_1 \sum_{i=1}^{\frac{1}{\gamma}} c_2 \sum_{j=1}^{\frac{2^j+1}{\gamma^j+1}} \gamma \sum_{j=1}^{\frac{2^j+1}{\gamma^j+1}} \left( K^{p'}(b, C_{1,4}) + \Lambda^2 + 1 \right) \sum_{j=1}^{\frac{2^j+1}{\gamma^j+1}} \int_{\mathcal{P}_{1,4}} \Omega_+^2 d\gamma ds.$$ 

Note the sums in the exponents are all from $j = 1$ to $j = i + 1$. Let $i \to \infty$. All the exponent series converge. We deduce

$$\sup_{\mathcal{P}_{2,3,4}} \Omega_+^2 \leq c \left( K^{p'}(b, C_{1,4}) + \Lambda^2 + 1 \right)^{\gamma/(\gamma-1)} \int_{\mathcal{P}_{1,4}} \Omega_+^2 d\gamma ds.$$ 

Next, repeating the argument on $\Omega_-$, we find that

$$\sup_{\mathcal{P}_{2,3,4}} \Omega_-^2 \leq \left( K^{p'}(b, C_{1,4}) + \Lambda^2 + 1 \right)^{\gamma/(\gamma-1)} \int_{\mathcal{P}(\sigma_1)} \Omega_-^2 d\gamma ds$$

Recall $\Omega_+ = \{ \Omega + \Lambda : \Omega \geq 0 \}$ and $\Omega_- = \{ -\Omega + \Lambda : \Omega \leq 0 \}$. Thus, we obtain

$$\sup_{\mathcal{P}_{2,3,4}} \Omega^2 \leq C \left( K(b, C_{1,4}) + \Lambda + 1 \right)^Q \left( \int_{\mathcal{P}_{1,4}} \Omega^2 d\gamma ds + \Lambda^2 \right).$$

where

$$Q = p' \gamma/(\gamma - 1) = \frac{p \gamma}{(p - 1)(\gamma - 1)}.$$ 

Recall that $p$ can be any number in $(1, 2)$ and $\gamma = 1 + \frac{2}{\mu}$ with $\mu$ being any number greater than 2. Thus the number $Q$ can be any number greater than 4. The constant $C$ may blow up when $p$ or $\mu$ goes to 2.

Re-scaling: We now recall that we omitted the “tilde” in the notation in the above computations. So what has actually been proven thus far is

$$\sup_{(\tilde{x}, \tilde{t}) \in \mathcal{P}_{2,3,4}} \tilde{\Omega}^2(\tilde{x}, \tilde{t}) \leq C \left( K(b, C_{1,4}) + \tilde{\Lambda} + 1 \right)^Q \left( \|\tilde{\Omega}\|_{L^2(P_{1,4}, 1)}^2 + \tilde{\Lambda}^2 \right).$$

Recall $\tilde{x} = \frac{x}{t}$, $\tilde{t} = \frac{t}{r}$, $\tilde{\Omega}(\tilde{x}, \tilde{t}) = \tilde{\omega}_b(\tilde{x}, \tilde{t})$. So with $2 \leq \tilde{r} \leq 3$ on the left and $1 \leq \tilde{r} \leq 4$ on the right we can derive

$$\sup_{(\tilde{x}, \tilde{t}) \in \mathcal{P}_{2,3,4}} \tilde{\omega}_b^2(\tilde{x}, \tilde{t}) \leq C \left( K(b, C_{1,4}) + \tilde{\Lambda} + 1 \right)^Q \left( \int_{P_{1,4}} \tilde{\omega}_b^2(\tilde{x}, \tilde{t}) d\tilde{x} d\tilde{t} + \tilde{\Lambda}^2 \right).$$
We recall from the beginning of the section that
\[ K_5(C_{1,4}) = \|\tilde{b}(\tilde{x}, \tilde{t})\|_{L^\infty(-1,0;L^2(C_{1,4}))} = \frac{1}{k^{\frac{1}{2}}} \|\tilde{b}(x, t)\|_{L^\infty(-k^2,0;L^2(C_{1k,4k}))}, \]
and
\[ \|\tilde{\omega}(\tilde{x}, \tilde{t})\|_{L^2(P_{1,4,1})} = \frac{1}{k^2} \|\omega(x, t)\|_{L^2(P_{k,4k,k})}. \]

Also, we note the control on \( \Lambda \) is a scaling invariant quantity. Since \( \Lambda = \|v_0\|_{L^\infty(P_{1,4,1})} \), we use Proposition 2.1
\[ \tilde{\Lambda} = \left( \sup_{P_{1,4,1}} |\tilde{\omega}(\tilde{x}, \tilde{t})| \right) \leq \left( \|\tilde{\omega}(\tilde{x}, -T)\|_{L^\infty(\mathbb{R}^3)} \right) \text{ applying Proposition 2.1} \]
\[ = \|rv_0, \theta\|_{L^\infty(\mathbb{R}^3)}. \]

We utilize \( 0 < k < 1 \) to obtain
\[ \sup_{(x,t) \in P_{2k,3k,\frac{4k}{3}}} k^4 \omega^2_0(x,t) \leq C \left( \frac{1}{k^{1/2}} \|\tilde{b}\|_{L^\infty(-k^2,0;L^2(C_{1k,4k}))} + \|rv_0, \theta\|_{L^\infty(\mathbb{R}^3)} \right)^Q \]
\[ \left( \int_{P_{k,4k,k}} k^4 \omega^2_0(x,t) \frac{1}{k^3} dxdt + \|rv_0, \theta\|_{L^\infty(\mathbb{R}^3)}^2 \right) \]
\[ \leq \frac{C}{k^{1+(Q/2)}} \left( \|\tilde{b}\|_{L^\infty(-k^2,0;L^2(C_{1k,4k}))} + k^{1/2} \|rv_0, \theta\|_{L^\infty(\mathbb{R}^3)} \right)^Q \]
\[ \left( \|\omega_\theta\|_{L^2(P_{k,4k,k})}^2 + k\|rv_0, \theta\|_{L^\infty(\mathbb{R}^3)}^2 \right). \]

Therefore, for any number \( Q > 4 \),
\[ (2.21) \]
\[ \|\omega_\theta(x,t)\|_{L^\infty(P_{2k,3k,\frac{4k}{3}})} \leq \frac{C}{k^{(10+Q)/4}} \left( \|b\|_{L^\infty(-k^2,0;L^2(C_{1k,4k}))} + k^{1/2} \|rv_0, \theta\|_{L^\infty(\mathbb{R}^3)} \right)^{Q/2} \]
\[ \times \left( \|\omega_\theta\|_{L^2(P_{k,4k,k})} + \sqrt{k}\|rv_0, \theta\|_{L^\infty(\mathbb{R}^3)} \right). \]

This proves part (i) of Theorem 1.2.

3. MORE INTEGRAL A PRIORI BOUNDS

In this section we prove an weighted integral bound for \( \omega_\theta \), which will be needed in proving part (ii) of Theorem 1.2.

First let us prove the following lemma which strengthens a result in [3].

**Lemma 3.1.** There exists an absolute positive constant \( C_0 \) such that the following a priori estimate holds for solutions to the axially symmetric incompressible Navier-Stokes equation:

\[ (3.1) \]
\[ \|r^2 \omega_\theta(\cdot, t)\|_{L^2}^2 + \int_0^t \|\nabla (r^2 \omega_\theta)\|_{L^2}^2 ds \]
\[ \leq C_0 \left( \|r^2 \omega_\theta(\cdot, 0)\|_{L^2}^2 + (1 + \|v_0\|_{L^2}^4 + \|\Gamma_0\|_{L^3}^2) \|v_0\|_{L^2}^2 \right) e^t. \]
Here we assume that the right hand side is finite and also $\Gamma_0 = \Gamma(\cdot, 0)$.

Proof. Multiply the equation for $\Omega$ by $r^3$, one has

$$\partial_t(r^2\omega_\theta) + b \cdot \nabla(r^2\omega_\theta) - 3rv_r\omega_\theta = \frac{\partial_z \Gamma^2}{r} + \Delta(r^2\omega_\theta) - \frac{4}{r}\partial_r(r^2\omega_\theta) + 3\omega_\theta.$$  

The standard energy estimate gives that

$$\frac{1}{2} \frac{d}{dt} \|r^2\omega_\theta\|_{L^2}^2 + \|\nabla(r^2\omega_\theta)\|_{L^2}^2 = 3 \int rv_r\omega_\theta r^2\omega_\theta dx + \int \partial_z \Gamma^2 r^2\omega_\theta dx + 3 \int \omega_\theta r^2\omega_\theta dx.$$  

By Proposition 2.1 and Sobolev imbedding, it is easy to estimate that

$$\int \partial_z \Gamma^2 r^2\omega_\theta dx \leq 2\|\Gamma\|_{L^3} \|\nabla v_\theta\|_{L^2} \|r^2\omega_\theta\|_{L^6} \leq C\|\Gamma_0\|_{L^3}^2 \|\nabla v_\theta\|_{L^2}^2 + \frac{1}{4} \|\nabla(r^2\omega_\theta)\|_{L^2}^2.$$  

Next, one also has

$$\int \omega_\theta r^2\omega_\theta dx \leq \|\omega_\theta\|_{L^2}^2 + \|r^2\omega_\theta\|_{L^2}^2.$$  

Thirdly, we estimate that

$$\int rv_r\omega_\theta r^2\omega_\theta dx \leq C\|v_r\|_{L^2} \|(r^2\omega_\theta)\|_{L^4} \|(\omega_\theta)\|_{L^4} \leq \|v_0\|_{L^2}^2 \|\omega_\theta\|_{L^2}^2 + \frac{1}{4} \|\nabla(r^2\omega_\theta)\|_{L^2}^2.$$  

Thus we arrive at

$$\frac{d}{dt} \|r^2\omega_\theta\|_{L^2}^2 + \frac{1}{2} \|\nabla(r^2\omega_\theta)\|_{L^2}^2 \leq C(\|\Gamma_0\|_{L^3}^2 + \|v_0\|_{L^2}^4 + 1)\|\nabla v\|_{L^2}^2 + \|r^2\omega_\theta\|_{L^2}^2.$$  

Clearly, by virtue of the energy inequality

$$\frac{d}{dt} \|v(\cdot, t)\|_{L^2}^2 \leq \|v(\cdot, 0)\|_{L^2}^2,$$  

the proof of the lemma follows by the previous differential inequality and Gronwall’s inequality.  

We mention that the integration by parts in this lemma can be justified by standard approximation argument. For instance, one can first assume that initial data is smooth and has compact support. Then the solutions will have sufficient fast decay near infinity to carry out the integration by parts to obtains the integral bounds. Since the original solutions, which is smooth by our assumption in the theorems, is the pointwise limit of those solutions with fast decay. The integral bounds follows from Fatou’s lemma.

To make gains by using dimension reduction argument, we need the use of a localized $L^2$ estimate of $b$. This will be the task of the next lemma which requires the following notations. Let $x_0 = (x_{10}, x_{20}, z_0)$ be any point in $\mathbb{R}^3$ with $x_{10}^2 + x_{20}^2 = r_0^2$. Since we are
focused on the axially symmetric case, without loss of generality, we may assume that \( x_0 = (r_0, 0, 0) \). For \( 0 < \alpha < \beta \leq 1 \), we will use \( B_\alpha = B(x_0, \alpha) \) to denote a ball of radius \( \alpha \), whose center is \( x_0 \). Let \( \phi_{\alpha \rho, \beta r_0} \in C_0^\infty(\mathbb{R}^3) \) be a radial cutoff function (with respect to \( x_0 \), the center of \( B_{\alpha \rho} \)) satisfying

\[
\begin{cases}
\phi_{\alpha \rho, \beta r_0} \equiv 1 & \text{on supp } B_{\alpha \rho}, \quad \phi_{\alpha \rho, \beta r_0} \equiv 0 & \text{on supp } B_{\beta r_0}, \\
0 \leq \phi_{\alpha \rho, \beta r_0} \leq 1, & |\nabla \phi_{\alpha \rho, \beta r_0}| \leq C_b r_0^{-1} \phi_{\alpha \rho, \beta r_0}, & |\nabla^2 \phi_{\alpha \rho, \beta r_0}| \leq C_b r_0^{-2} \phi_{\alpha \rho, \beta r_0},
\end{cases}
\]

where \( \delta \in [0, 1) \) will be chosen up to the context.

**Lemma 3.2.** Let \( p = 2^+ \) be any number which is bigger than and close to 2 and \( r_0 > 0 \), there holds

\[
\|b\|_{L^p(B_{r_0}/2)} \leq C_p r_0^{-3(\frac{1}{p} - \frac{1}{2})},
\]

where \( C_p \) is a positive constant depending only on \( p \) and on the initial values of \( \|r^2 \omega_0\|_{L^2} \), \( \|\nu_0\|_{L^2} \) and \( \|\Gamma_0\|_{L^3} \).

**Proof.** Take \( \phi = \phi_{3r_0/4,4r_0/5} \) and \( \tilde{\phi} = \phi_{r_0/2,2r_0/3} \) be the standard cutoff functions with the same center \( x_0 \), and \( \delta = 0 \). Clearly, one has

\[
b = -\Delta^{-1} \nabla \times (\omega \theta e_\theta \phi) + h
\]

so that \( h \) is harmonic on \( x \in B_{3r_0/4} \). By interpolation inequality, one has

\[
\|b\|_{L^p(B_{r_0}/2)} \leq C_p \|\tilde{\phi} b\|_{L^2}^{1-3(\frac{1}{p} - \frac{1}{2})} \|\nabla (\tilde{\phi} b)\|_{L^2}^{3(\frac{1}{2} - \frac{1}{p})}
\]

\[
\leq C_p \|b\|_{L^2(B_{2r_0/3})}^{1-3(\frac{1}{p} - \frac{1}{2})} \|\nabla b\|_{L^2(B_{2r_0/3})}^{3(\frac{1}{2} - \frac{1}{p})} + C_p r_0^{-3(\frac{1}{p} - \frac{1}{2})} \|b\|_{L^2(B_{2r_0/3})}.
\]

Then using (3.3) and \( L^q \) boundedness properties of Riesz operator for \( (1 < q < \infty) \), we have

\[
\|\nabla b\|_{L^2(B_{2r_0/3})} \leq C_p \|\nabla \Delta^{-1} \nabla \times (\omega \theta e_\theta \phi)\|_{L^2} + C_p \|\nabla h\|_{L^2(B_{2r_0/3})}
\]

\[
\leq C_p \|\omega \theta e_\theta \phi\|_{L^2} + C_p r_0^{-1} \|h\|_{L^2(B_{3r_0/4})}.
\]

Here we also used the fact that \( h \) is harmonic on \( B_{3r_0/4} \). Now using (3.3) to replace \( h \), one has, after using Hölder and Sobolev inequality, that

\[
\|\nabla b\|_{L^2(B_{2r_0/3})} \leq C_p \|\omega \theta e_\theta \phi\|_{L^2} + C_p r_0^{-1}\left( \|b\|_{L^2(B_{3r_0/4})} + \|\Delta^{-1} \nabla \times (\omega \theta e_\theta \phi)\|_{L^2(B_{3r_0/4})} \right)
\]

\[
\leq C_p \|\omega \theta\|_{L^2(B_{4r_0/5})} + C_p r_0^{-1}\|b\|_{L^2(B_{3r_0/4})} + C_p \|\Delta^{-1} \nabla \times (\omega \theta e_\theta \phi)\|_{L^6}
\]

\[
\leq C_p \|\omega \theta\|_{L^2(B_{2r_0/5})} + C_p r_0^{-1}\|b\|_{L^2(B_{3r_0/4})}.
\]

Substituting this to (3.4), we have

\[
\|b\|_{L^p(B_{r_0}/2)} \leq C_p \|b\|_{L^2(B_{2r_0/3})}^{1-3(\frac{1}{p} - \frac{1}{2})} \|\omega \theta\|_{L^2(B_{4r_0/5})}^{3(\frac{1}{2} - \frac{1}{p})} + C_p r_0^{-3(\frac{1}{p} - \frac{1}{2})} \|b\|_{L^2(B_{3r_0/4})}.
\]

Clearly, Lemma 3.2 follows by applying Lemma 3.1 and the above estimate. \( \square \)
The next lemma is prepared as the first step of Moser’s iteration scheme which will be presented in next section. The advantage of this lemma is remarked at the end of this section.

**Lemma 3.3.** Let $\Omega = \omega_b/r$ again. Then

$$
\|\Omega\|_{L^4(B_{r_0/2} \times [-r_0^2/4,0])} \leq Cr_0^{-1/4},
$$

where $C$ depends only on the initial values of $\|\omega_b\|_{L^2}$, $\|v_0\|_{L^2}$ and $\|\Gamma_0\|_{L^3}$.

**Proof.** Notice that $B_{r_0/2} \times [-r_0^2/4,0] \subset P_{\frac{r_0}{4},\frac{3r_0}{4},r_0}$ the parabolic cylinder defined in (2.2).

So, we will just prove $\|\Omega\|_{L^4(P_{\frac{r_0}{4},\frac{3r_0}{4},r_0})} \leq Cr_0^{-1/4}$.

Take a cut-off function $\psi = \psi(x,t) = \eta(t) \phi(x)$ with the following properties. The support of $\psi$ is in the larger cylinder $P_{\frac{r_0}{4},2r_0,r_0}$ and $\psi = 1$ in the original cylinder $P_{\frac{r_0}{3},\frac{2r_0}{3},r_0}$.

Also we require that $\phi$ to be a function of $r$ and $z$ only and that $|\nabla \phi| \leq C/r_0$, $|\Delta \phi| \leq C/r_0^2$ and $|\eta'(t)| \leq C/r_0^2$.

Multiplying equation (1.4) by $\phi \Omega$ and then integrating over $\mathbb{R}^3$, one has

$$
\frac{1}{2} \frac{d}{dt} \int \psi \Omega^2 \, dx - \int (\Delta + 2 \frac{r}{r} \partial_r \Omega) \psi \Omega \, dx
$$

$$
= \frac{1}{2} \int \psi_l \Omega^2 \, dx - \int \psi \Omega (b \cdot \nabla \Omega) \, dx + \int \frac{\partial_z (v_0)}{r^2} \psi \Omega \, dx.
$$

First of all, we estimate that

$$
- \int (\Delta + 2 \frac{r}{r} \partial_r \Omega) \psi \Omega \, dx
$$

$$
= 2\pi \int \eta(\Delta \Omega^2/2 - |\nabla \Omega|^2 + \frac{1}{r} \partial_r \Omega^2) \phi r dr dz
$$

$$
\geq -C(\|\Delta \phi\|_{L^\infty} + r_0^{-1} \|\partial_r \phi\|_{L^\infty}) \int_{B_{2r_0/3}} \eta \Omega^2 \, dx
$$

$$
+ \int |\nabla (\sqrt{\psi} \Omega) - \Omega \nabla \sqrt{\psi}|^2 \, dx
$$

$$
\geq -C(\|\Delta \phi\|_{L^\infty} + r_0^{-1} \|\partial_r \phi\|_{L^\infty} + \|\nabla \sqrt{\phi}\|_{L^\infty}^2) \int_{C_{\frac{r_0}{3},3r_0}} \eta \Omega^2 \, dx
$$

$$
+ \int |\nabla (\sqrt{\psi} \Omega)|^2 \, dx.
$$

Here $C_{\frac{r_0}{3},3r_0}$ is defined in (2.1). In view of the properties of $\psi$, one has

$$
\frac{1}{2} \frac{d}{dt} \int \psi \Omega^2 \, dx + \int |\nabla (\sqrt{\psi} \Omega)|^2 \, dx
$$

$$
\leq Cr_0^{-2} \int_{C_{\frac{r_0}{3},3r_0}} \eta \Omega^2 \, dx + \frac{1}{2} \int \Omega^2 b \cdot \nabla \psi \, dx - \int \frac{(v_0)^2}{r^2} \sqrt{\psi} \partial_z (\sqrt{\psi} \Omega) \, dx
$$

$$
\leq Cr_0^{-2} \int_{C_{\frac{r_0}{3},3r_0}} \eta \Omega^2 \, dx + \frac{1}{2} \int \Omega^2 b \cdot \nabla \psi \, dx
$$

$$
- \int \frac{(v_0)^2}{r^2} \sqrt{\psi} \partial_z (\sqrt{\psi} \Omega) \, dx.
$$
\[ \leq Cr_{0}^{-5} + \frac{1}{4} \int |\partial_{z}(\sqrt{\psi}\Omega)|^{2} dx + Cr_{0}^{-2} \int_{C_{\frac{1}{2}r_{0},4r_{0}}} \eta\Omega^{2} dx \]
\[ + \frac{1}{2} \int \Omega^{2} b \cdot \nabla \psi dx. \]

It remains to deal with the last term involving \( b \) in the above inequality. We use the two-dimensional nature of the integral and invoke Lemma 3.2. First of all, it is easy to see that
\[ \int \Omega^{2} b \cdot \nabla \psi dx \leq \|b\|_{L^{p}(C_{\frac{1}{2}r_{0},4r_{0}})} \|\sqrt{\psi}\Omega\|_{L^{2p/(p-2)}} \|\Omega \nabla \sqrt{\psi}\|_{L^{2}}. \]

Denote \( \nabla = (\partial_{r}, \partial_{z})^{T} \) and view \( \sqrt{\psi}\Omega \) as a function of \((r, z)\). We apply the two-dimensional Sobolev-Poincaré inequality to derive that
\[ \|\sqrt{\psi}\Omega\|_{L^{2p/(p-2)}} \leq C_{p}r_{0}^{\frac{p-2}{p}} \left( \int \int |\nabla(\sqrt{\psi}\Omega)|^{2} dr dz \right)^{\frac{1}{2}} \]

Consequently, by Lemma 3.2 we have
\[ \int \Omega^{2} b \cdot \nabla \psi dx \leq r_{0}^{-\frac{2}{p}} \|b\|_{L^{p}(C_{\frac{1}{2}r_{0},4r_{0}})} \|\sqrt{\psi}\Omega\|_{L^{2}}(C_{\frac{1}{2}r_{0},4r_{0}}) + \frac{1}{4} \|\nabla(\sqrt{\psi}\Omega)\|_{L^{2}}^{2}. \]

Inserting the above inequality into (3.5), one has
\[ \frac{d}{dt} \int \psi\Omega^{2} dx + \int |\nabla(\sqrt{\psi}\Omega)|^{2} dx \leq Cr_{0}^{-5} + C(r_{0}^{-2} + r_{0}^{-3}) \int_{C_{\frac{1}{2}r_{0},4r_{0}}} \eta\Omega^{2} dx. \]

Using the basic energy inequality (3.2) and integrating the above differential inequality with respect to \( t \) yields
\[ (3.6) \sup_{-(r_{0}/2)^{2} < t < 0} \int \sqrt{\phi}\Omega^{2} dx + \int_{-\infty}^{0} \int |\nabla(\sqrt{\psi}\Omega)|^{2} dx dt \]
\[ \leq Cr_{0}^{-3} + Cr_{0}^{-4} + C_{p}r_{0}^{-5}. \]

We emphasize that the first term in (3.6) is the contribution of the term involving \( v_{\nu} \) in (1.4) which is best, the second term is the contribution from the linear part of (1.4) due to the localization, while the third term is due to the draft term in (1.4) which is also the worst one. The lemma then follows from (3.6), interpolation inequalities and dimension reduction arguments:
\[ \|\Omega\|_{L^{4}(P_{\frac{1}{2}r_{0},\frac{3}{2}r_{0}})} \leq Cr_{0} \int_{-(r_{0}/2)^{2}}^{0} \int |\sqrt{\phi}\Omega|^{4} dr dz dt \]
\[
\begin{align*}
&\leq Cr_0 \int_{-(r_0/2)^2}^0 \iint |\sqrt{\phi}\Omega|^2drdz \int |\nabla(\sqrt{\phi}\Omega)|^2drdzdt \\
&\leq Cr_0^{-1} \sup_{-(r_0/2)^2} \int_{t<0} |\sqrt{\phi}\Omega|^2dx \int_{-(r_0/2)^2}^0 |\nabla(\sqrt{\phi}\Omega)|^2dxdt \\
&\leq r_0^{-11}.
\end{align*}
\]

\[\square\]

**Remark 3.1.** Using the three-dimensional Sobolev imbedding, one seems only to be able to prove the a priori estimate

\[\|\Omega\|_{L^{10/3}(L^{10/3}(P(1/2)))} \leq Cr_0^{-3},\]

which, heuristically, gives \(\omega_\theta = \mathcal{O}(r^{-3.5})\) on average. The a priori estimate in Lemma 3.3 implies that, at a heuristical level, \(\omega_\theta = \mathcal{O}(r^{-3})\). This gain is obtained by using the two-dimensional nature when the ball is away from the axis and the a priori estimate in Lemma 3.2. See section 4 for more details.

4. A priori bound for \(\omega_\theta\), part (ii)

With the help of integral bound on \(\omega_\theta\) in the last section, now we can prove part (ii) of Theorem 1.2.

The proof is similar to part (i). The changes occur at step 2 and step 6 only.

**Step 1.** This is the same as before.

**Step 2.** Pick a number \(\beta > 2\). By Hölder’s inequality and dimension reduction

\[T_1 = \int_{P(\sigma_1)} b \cdot (\nabla \psi) \psi f^2 dyds \]
\[\leq \frac{C}{\sigma_1 - \sigma_2} \int_{-\sigma_2^2}^0 \|\psi f(\cdot, s)\|_{L^2(\sigma_1)} \|f(\cdot, s)\|_{L^2(\sigma_1)} \|b(\cdot, s)\|_{L^2(\sigma_1)} ds \]
\[\leq \frac{C}{\sigma_1 - \sigma_2} \int_{-\sigma_2^2}^0 \|\nabla(\psi f(\cdot, s))\|_{L^2(\sigma_1)} \|f(\cdot, s)\|_{L^2(\sigma_1)} \|b(\cdot, s)\|_{L^2(\sigma_1)} ds \]
\[\leq \frac{C}{\sigma_1 - \sigma_2} \int_{-\sigma_2^2}^0 \|\nabla(\psi f(\cdot, s))\|_{L^2(\sigma_1)} \|f(\cdot, s)\|_{L^2(\sigma_1)} \|b(\cdot, s)\|_{L^2(\sigma_1)} ds.
\]

Here we just used the 2 dimensional Sobolev inequality. Therefore

\[T_1 \leq \frac{1}{4} \|\nabla(\psi f)\|_{L^2(P(\sigma_1))}^2 + \frac{C}{(\sigma_1 - \sigma_2)^2} \|b\|_{L^\infty([-\sigma_2^2, 0], L^2(\sigma_1))}^2 \|f\|_{L^2(P(\sigma_1))}^2.
\]

**Steps 3-5** are the same as before.
Step 6. Substituting the estimates for $T_1, T_2, T_3, T_4$ in (4.1), (2.12), (2.13), (2.14) respectively to (2.7), we obtain

$$
\int_{P(\sigma_1)} |\nabla (f\psi)|^2 dyds + \frac{1}{2} \int_{C(\sigma_1)} f^2(y,0)\phi^2(y) dy \\
\leq \frac{3}{4} \int_{P(\sigma_1)} |\nabla (f\psi)|^2 dyds + \frac{C}{(\sigma_1 - \sigma_2)^2} \left[ \|b\|_{L^\infty([-\sigma_1^2,0],L^3(\sigma_1))} + \Lambda^2 q^2 + 1 \right] \int_{P(\sigma_1)} f^2 d\gamma ds.
$$

Here $P(\sigma_1)$ is again defined in (2.11) and the $\Omega(\sigma_1)$ is defined in (2.9). This implies, as before, after switching to 2 dimensional integrals, that

(4.2) $$
\int_{\Omega(\sigma_1)} |\nabla (f\psi)|^2 d\gamma ds + \sup_{s \in [-\sigma_1^2,0]} \int_{\Omega(\sigma_1)} f^2(y,s)\phi^2(y) d\gamma \\
\leq \frac{C q^2}{(\sigma_1 - \sigma_2)^2} \left[ \|b\|_{L^\infty([-\sigma_1^2,0],L^3(\sigma_1))} + \Lambda^2 + 1 \right] \int_{\Omega(\sigma_1)} f^2 d\gamma ds.
$$

Next we will iterate the above energy type estimate. Comparing to the proof of part (i) of the theorem, we will apply refined interpolation and embedding inequalities involving BMO functions. Applying Lemma 1 in Section 2 of [10], we know that

$$
\| (f\phi)^2 \|_{L^2(\Omega(\sigma_1))} \leq C \| f\phi \|_{L^2(\Omega(\sigma_1))} \| f\phi \|_{BMO(\Omega(\sigma_1))}.
$$

By the 2 dimensional Poincaré inequality, we also have

$$
\| f\phi \|_{BMO(\Omega(\sigma_1))} \leq C \| \nabla (f\phi) \|_{L^2}.
$$

These two inequalities imply that

$$
\| f\phi \|_{L^4(\Omega(\sigma_1))} \leq C \| f\phi \|_{L^2(\Omega(\sigma_1))}^{1/2} \| \nabla (f\phi) \|_{L^2(\Omega(\sigma_1))}^{1/2}.
$$

Consequently

$$
\int_{-\sigma_1^2}^0 \int_{\mathbb{R}^2} (f\psi)^4 d\gamma ds \\
\leq C \sup_{-\sigma_1^2 \leq s \leq 0} \left( \int_{\mathbb{R}^2} (f\psi)^2 d\gamma \right) \int_{-\sigma_1^2}^0 \int_{\mathbb{R}^2} |\nabla (f\psi)|^2 d\gamma ds.
$$

Substituting (4.2) into the right hand side of the above inequality, we deduce

$$
\int_{\Omega(\sigma_2)} f^4 d\gamma ds \leq C \left[ \frac{q^2}{(\sigma_1 - \sigma_2)^2} \left[ \|b\|_{L^\infty([-\sigma_1^2,0],L^3(\sigma_1))} + \Lambda^2 + 1 \right] \int_{\Omega(\sigma_1)} f^2 d\gamma ds \right]^2.
$$

This shows, since $f = \Omega^\gamma_+$ by definition, that

(4.3) $$
\int_{\Omega(\sigma_2)} \Omega^\gamma_+ d\gamma ds \leq C \left[ \frac{q^2}{(\sigma_1 - \sigma_2)^2} \left[ \|b\|_{L^\infty([-\sigma_1^2,0],L^3(\sigma_1))} + \Lambda^2 + 1 \right] \int_{\Omega(\sigma_1)} \Omega^\gamma_+ d\gamma ds \right]^2.
$$
For $i = 1, 2, \ldots$, in (4.3), we take $q = 2^i; \sigma_1 = 2^{-(i+1)}$ and $\sigma_2 = 2^{-(i+2)}$. After iteration, we arrive at

$$\sup_{P_{2,3,3/4}} \Omega^2_+ \leq C \left[ \|b\|_{L^\infty([-\sigma^2_1,0],L^\beta(\mathbb{C}(\sigma_1)))}^2 + \Lambda^2 + 1 \right] \left( \int_{P_{1,4,1}} \Omega^4_+ d\overline{y} ds \right)^{1/2}.$$ 

Repeating the argument on $\Omega_- = \begin{cases} -\Omega + \Lambda & \Omega \leq 0 \\ \Lambda & \Omega > 0 \end{cases}$, we see that

$$\sup_{P_{2,3,3/4}} \Omega^2_- \leq C \left[ \|b\|_{L^\infty([-\sigma^2_1,0],L^\beta(\mathbb{C}(\sigma_1)))}^2 + \Lambda^2 + 1 \right] \left( \int_{P_{1,4,1}} \Omega^4_- d\overline{y} ds \right)^{1/2}.$$ 

Thus

$$\sup_{P_{2,3,3/4}} \Omega^2 \leq C \left[ \|b\|_{L^\infty([-\sigma^2_1,0],L^\beta(\mathbb{C}(\sigma_1)))}^2 + \Lambda^2 + 1 \right] \left( \int_{P_{1,4,1}} (\Omega^4 + \Lambda^4) d\overline{y} ds \right)^{1/2}.$$ 

In the region $P_{1,4,1}$, the quantities $\Omega = \omega_\theta/r$ and $\omega_\theta$ are equivalent. Hence, the above implies

$$\sup_{P_{2,3,3/4}} \omega^2_\theta \leq C \left[ \|b\|_{L^\infty([-\sigma^2_1,0],L^\beta(\mathbb{C}(\sigma_1)))}^2 + \Lambda^2 + 1 \right] \left( \int_{P_{1,4,1}} (\omega^4_\theta + \Lambda^4) d\overline{y} ds \right)^{1/2}.$$ 

After returning to three dimensional domains and scaling as in the proof of part (i) of the theorem, we obtain, for $k \in (0, 1)$, that

$$k^4 \sup_{P_{2k,3k,3k/4}} \omega^2_\theta \leq C \left[ k^{-3+(6/\beta)} \|b\|_{L^\infty([-k\sigma_1^2,0],L^\beta(\mathbb{C}(\sigma_1,k)))}^2 + \Lambda^2 + 1 \right] \left( \int_{P_{1k,4k,1k}} \omega^4_\theta d\overline{y} ds \right)^{1/2} + \Lambda^2.$$ 

In Lemma 3.1 we take $p = \beta$. Then we know that

$$\|b\|_{L^\infty([-k\sigma_1^2,0],L^\beta(\mathbb{C}(\sigma_1,k)))} \leq Ck^{-3+(6/\beta)}.$$ 

By Lemma 3.3 we also have

$$\int_{P_{1k,4k,1k}} \omega^4_\theta d\overline{y} ds \leq k^{-7}.$$ 

Combining the last three inequalities, we deduce

$$\sup_{P_{2k,3k,3k/4}} |\omega_\theta| \leq \frac{C}{k^{7/2}}$$

where $C$ depends only on the initial value. This completes the proof of Theorem 1.2. □
5. Velocity Bound, proof of Theorem 1.1

Based on the bound on $\omega_\theta$, now we prove the a priori velocity bound. In this section we ignore the time variable. Let $b = v_r e_r + v_z e_z$. Then taking the cylindrical curl of $b$ we get

$$\text{curl } b = \omega_\theta e_\theta$$

Taking the curl on both sides of this equation and using the divergence free condition on $b$ to get $\text{curl (curl } b) = -\Delta b$, we have

$$-\Delta b = \text{curl (} \omega_\theta e_\theta \text{)}$$

Choose a smooth cutoff function $\phi$ with support contained in the ball $B_{2r_0} = B(x, 2r_0)$, $0 \leq \phi \leq 1$ in $B_{2r_0}$, $\phi \equiv 1$ in $B_{r_0}^4$, and the following properties:

$$|\nabla \phi| \leq C r_0, \quad |\Delta \phi| \leq C r_0^2$$

Then $\text{supp}(\nabla \phi) \subset B_{2r_0} \setminus B_{r_0}^4$, and we compute

$$\Delta(\phi b) = \Delta \phi b + 2 \nabla \phi \cdot \nabla b + \phi \Delta b$$

$$\Rightarrow \phi b = \int_{B_{2r_0}} \Gamma(x, y) \left( \Delta \phi b + 2 \nabla \phi \cdot \nabla b - \phi \text{curl (} \omega_\theta e_\theta \text{)} \right) dy$$

$$\phi b = \int_{B_{2r_0}} \Gamma(x, y) \Delta \phi b dy + 2 \int_{B_{2r_0}} \Gamma(x, y) \nabla \phi \cdot \nabla b dy - \int_{B_{2r_0}} \Gamma(x, y) \phi \text{curl (} \omega_\theta e_\theta \text{)} dy$$

where $\Gamma(x, y) = \frac{c_0}{|x - y|}$ is the Green’s function. After doing integration by parts, it is easy to see that, for all $p \geq 1$,

$$(5.1) \sup_{B_{r_0}(x)} |b| \leq C r_0^{-3/p} \|b\|_{L^p(B_{2r_0}(x))} + C r_0 \sup_{B_{2r_0}(x)} |\omega_\theta|.$$
Here \(2^+\) is any positive number greater than 2. This is part (i) of Theorem 1.1.

If the extra condition in part (ii) holds, we then use Theorem 1.2 (ii) to conclude that
\[
|b(x, t)| \leq Cr^{-2}
\]
where \(C\) depends only on the initial value. This completes the proof of Theorem 1.1.

\[\square\]

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