Scale symmetry and quantum corrections

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Abstract. We use the scale invariant regularization method to study loop corrections in a scale
symmetric model of two interacting scalar fields. The method resembles the usual procedure of
dimensional regularization, but the renormalization scale $\mu$ is a function of dynamical fields. The
model is free of the trace anomaly and accommodates spontaneous breaking of the symmetry.
It is also necessarily nonrenormalizable. We show how the nonstandard method results in
beta functions and corrections to the effective scalar potential different than in the usual
regularization, while at the same time it does not spoil the Callan-Symanzik equation.

1. Introduction
Scale symmetry is broken in nature. In models of particle interactions this fact is often accounted
for by means of a quantum anomaly of the scale symmetry. An alternative possibility is to
construct a theory, where this symmetry is not anomalous but broken spontaneously [1]. The
idea that a fundamental particle theory possesses the global dilatation invariance is an exciting one
as it could recast the Higgs naturalness problem in a new light, requiring a dynamical explanation
for the hierarchy of masses. This prospect is enabled by using a modification of dimensional
regularization, the scale invariant (SI) method, based on the idea that the renormalization scale
$\mu$ is a function of dynamical fields rather than a parameter, and that it acquires a nonzero value
through spontaneous breaking of the scale symmetry [2, 3, 4, 5, 6, 7].

Using this method we perform perturbative regularization of a simple model with two scalar
fields. The model is classically scale invariant and we arrange it to allow for the spontaneous
breaking by requiring that the effective potential possesses a flat direction. The usage of scale
invariant regularization constitutes a major qualitative change as it enables the Goldstone boson
of the broken scale symmetry to remain massless to all orders. Our goal is to study this
regularization in some detail. We show why it may be used only in the broken phase of the
symmetry and that it requires us to introduce infinitely many nonrenormalizable interactions to
the Lagrangian. Further, we calculate renormalization group functions and corrections to the
effective potential. The calculation is performed up to the level of two-loop Fynnanman diagrams,
in order to obtain modifications introduced by the new regularization as compared to the old
procedure with $\mu = \text{const}$. 
2. Scale transformations

We will focus in this presentation on toy models that involve only scalar fields. To begin, let us consider the Lagrangian for a single field with a quartic interaction

\[ \mathcal{L} = \frac{1}{2} (\partial \phi)^2 - \lambda \phi^4, \]  

(1)

We wish to extend the algebra of Poincare symmetry by adding the generator of scale transformations (dilatations), \( D \). The only new nonzero commutator in the algebra is

\[ [D, P_\mu] = i P_\mu, \]  

(2)

We will assume that any field, \( \Phi \), transforms (scales) under a nontrivial representation of this group,

\[ U(a) = e^{iaD}, \]  

(3)

\[ \Phi(x) \rightarrow U(a) \Phi(x) U^{-1}(a) = e^{i\alpha x} \Phi(e^{\alpha x}) \]  

(4)

\[ \delta \Phi(x) = i[D, \Phi] = (d_\phi + x^\nu \partial_\nu) \Phi(x). \]  

(5)

Denote the number of space-time dimensions by \( n = 4 - 2\epsilon \). \( d_\phi \) is the engineering dimension of the field, \( (d_\varphi = \frac{n}{2} - 1 = 1 - \epsilon \) for scalars). It is fixed by the requirement that the kinetic part of its action is invariant under dilatations,

\[ \frac{1}{2} (\partial \varphi(x))^2 d^n x \rightarrow \frac{1}{2} e^{2(d_\varphi+1)\epsilon} (\tilde{\partial} \varphi(\tilde{x}))^2 e^{-n\epsilon} d^n \tilde{x}, \]  

(6)

where \( \tilde{x} = e^\epsilon x \), \( \tilde{\partial}_\mu = \frac{\partial}{\partial \tilde{x}^\mu} \).

\[ \delta \frac{1}{2} (\partial \varphi)^2 = [2(d_\varphi + 1) + x^\nu \partial_\nu] \frac{1}{2} (\partial \varphi)^2 = [n + x^\nu \partial_\nu] \frac{1}{2} (\partial \varphi)^2 \]  

(7)

By assumption, dilatation transforms dynamical fields but not dimensional parameters, like masses,

\[ \delta \mu = i[D, \mu] = 0. \]  

(8)

This applies in particular to the renormalization scale \( \mu \), introduced to the Lagrangian as part of the procedure of dimensional regularization,

\[ \mathcal{L} = \frac{1}{2} (\partial \varphi)^2 - \lambda \mu^{2\epsilon} \varphi^4. \]  

(9)

As a consequence, the action is no longer invariant wrt dilatations when \( n \neq 4 \),

\[ \varphi^4(x) d^n x \rightarrow e^{-2\epsilon} \varphi^4(\tilde{x}) d^n \tilde{x} \]  

(10)

\[ \delta \varphi^4 = (4 + x^\nu \partial_\nu) \varphi^4 \neq (n + x^\nu \partial_\nu) \varphi^4. \]  

(11)

One can also observe that the Noether current for dilatations,

\[ D^\mu = \frac{\delta \mathcal{L}}{\delta \partial_\mu \varphi} \delta \varphi - x^\mu \mathcal{L}, \]  

(12)

when continued to \( n \neq 4 \) dimensions, is no longer conserved. Upon using the equation of motion \( \partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu \varphi} = \frac{\delta \mathcal{L}}{\delta \varphi} \), and \( \delta \partial_\mu \varphi = (d_\varphi + 1) \partial_\mu \varphi + x^\nu \partial_\nu \partial_\mu \varphi \) we have

\[ \partial_\mu D^\mu = d_\varphi \frac{\delta \mathcal{L}}{\delta \varphi} \varphi + (d_\varphi + 1) \frac{\delta \mathcal{L}}{\delta \partial_\mu \varphi} \partial_\mu \varphi - n \mathcal{L} = 2\epsilon \lambda \mu^{2\epsilon} \varphi^4 \neq 0. \]  

(13)

On the other hand, \( \partial_\mu D^\mu \) is equal to the trace of improved energy-momentum tensor \( T_{\mu\nu} \), see [8],

\[ \partial_\mu D^\mu = T_{\mu\mu}^\mu, \quad T_{\mu\nu} = \Theta_{\mu\nu} - \frac{1}{6} (\partial_\mu \partial_\nu - g_{\mu\nu} \Box) \varphi^2, \quad \Theta_{\mu\nu} = \frac{\delta \mathcal{L}}{\delta \partial_\mu \varphi} \partial_\nu \varphi - g_{\mu\nu} \mathcal{L}. \]  

(14)

We observe that the scale invariance, present in four dimensions, is no longer preserved when the theory is continued to \( n \neq 4 \) dimensions in course of the usual dimensional regularization.
3. Scale invariant regularization

Introduction of the parameter \( \mu \), that is dimensionful but not charged under the transformation generated by \( D \), results in a Lagrangian that is not a homogeneous function of dimension \( n \) under dilatations:

\[
\delta \mathcal{L} \neq (n + x^\nu \partial_\nu) \mathcal{L} = \partial_\nu (x^\nu \mathcal{L}) .
\]  

(15)

This has been recognized as the source of the trace anomaly, that is the fact that \( T^\mu_\mu \neq 0 \) persists in the limit \( n \to 4 \) after including loop corrections [9, 10]. A solution, known as scale-invariant (SI) regularization, originally proposed in [2, 3], and more recently studied and used in [4, 5, 6, 7], is to reinterpret \( \mu \) as a function of fields, such that it does transform under dilatations with dimension 1,

\[
\mu = \mu(\Phi) , \quad i[D, \mu] = \frac{\partial \mu}{\partial \Phi_i} i[D, \Phi_i] = (1 + x^\nu \partial_\nu) \mu .
\]  

(16)

As a simple example we will use

\[
\mu(\phi^2) = t \phi^2 , \quad t \in \mathbb{R}^+
\]  

(17)

\[
\mathcal{L} = \frac{1}{2} \left( \partial_\nu \phi_1 \right)^2 + \frac{1}{2} \left( \partial_\nu \phi_2 \right)^2 - \mu(\phi^2)^{2\epsilon} V(\phi_1, \phi_2) .
\]  

(18)

The parameter \( t \) is a positive number. Its presence exemplifies the freedom in choosing the function \( \mu(\Phi) \). Regularized quantities should be independent of what \( \mu(\Phi) \) was used. This leads to the hypothesis that coupling constants functionally depend on \( \mu, \lambda = \lambda[\mu(\Phi)] \). That dependence reduces to the familiar *running*, \( \lambda = \lambda(t) \), upon a choice like the one in (17).

To use an interaction term like \( \phi^2 \phi^2 \) in perturbative loop calculations, one has to expand \( \phi^2 \) around a nonzero background value \( \phi^2 = \langle \phi^2 \rangle + \phi^2_\text{new} \). This results in an infinite number of interactions involving the dynamical fluctuation \( \phi^2_\text{new} \),

\[
\phi^2_\text{new} = \langle \phi^2 \rangle_\text{new} = \left[ \langle \phi^1 \rangle + \phi^1_\text{new} \right]^{1/2} \left[ \langle \phi_1 \rangle + \phi_1_\text{new} \right]^{1/2} = \left[ \langle \phi_1 \rangle + \phi_1_\text{new} \right]^{1/2} \left[ \langle \phi_2 \rangle + \phi_2_\text{new} \right]^{1/2} \left[ 1 + \frac{4}{3} \frac{\phi_2'}{\langle \phi^2 \rangle} + \frac{4}{3} \frac{\phi_2''}{\langle \phi^2 \rangle^2} + \frac{4}{3} \frac{\phi_2'''}{\langle \phi^2 \rangle^3} + \ldots \right]_{\text{new}},
\]  

(19)

where the ellipsis stands for terms with higher powers of \( \epsilon \) or \( \phi^2_\text{new} \). As marked in (19), we will refer to all the interactions of \( \phi^2_\text{new} \) stemming from the expansion of \( \mu(\phi^2) \) around \( \langle \phi^2 \rangle \) as *new*. We also recover the familiar term \( \mu_0^2 \phi^4_1 \) with \( \mu_0 = \langle \phi^2 \rangle_\text{new} = \text{const} \) identical to the interaction in (9). One needs only a finite number of *new* interactions, as long as a finite number of external \( \phi^2 \) legs and loops is considered.

After having calculated loop corrections one is interested only in finite terms, \( \mathcal{O}(\epsilon^0) \) and first order poles \( \mathcal{O}(\epsilon^{-1}) \). Since the *new* terms contain positive powers of \( \epsilon \), through one-loop diagrams they can contribute only to finite corrections. For the *new* interactions to produce poles, and consequently modifications in renormalization group functions, one has to include them in two-loop diagrams. Notably, among poles produced this way one finds

\[
\frac{1}{\epsilon} \frac{\langle \phi^4_1 \rangle_{\text{new}}^{4+2m}}{\langle \phi^2 \rangle_{\text{new}}^{2m}} , \quad m = 1, 2, \ldots
\]  

(20)

Hence, corresponding interaction terms need to be included in the Lagrangian from the start, in order to regularize the model,

\[
\mathcal{L} \ni \frac{\lambda^B_{(4+2m)}}{4+2m} \phi^2_2 \frac{\phi^4_{2m}}{\phi^2_2^m} , \quad m = 1, 2, \ldots
\]  

(21)

\[
\lambda^B_{(4+2m)} = t^{2\epsilon} Z_{(4+2m)} \lambda_{(4+2m)}(t) ,
\]  

(22)
where the $Z_{(4+2m)}$ factor contains appropriate counterterms.

The expansion in (19) is justified if $\phi_2$ has a nonzero vacuum expectation value that breaks the scale symmetry spontaneously. Similarly, each term in (21) has a clear interpretation only upon the expansion of $\phi_2$, when it is seen to produce an infinite series of nonrenormalizable interactions suppressed by powers of $\langle \phi_2 \rangle$. Consequently, the model can be analyzed perturbatively only when $\langle \phi_2 \rangle \gg \langle \phi_1 \rangle$.

The assumption that the scale symmetry is broken is of course otherwise required, if one hopes to account for the observed nature of elementary particles. Existence of massive states would be excluded by an unbroken scale symmetry. Nontrivial (other than power-like) scaling of Green functions with momenta still arises as a consequence of the symmetry being spontaneously broken rather than anomalous [1].

4. The model with two scalar fields and a flat direction
A simple model that accommodates the spontaneous scale symmetry breaking and uses the SI regularization is

$$\mathcal{L} = \frac{1}{2} (\partial \phi_1)^2 + \frac{1}{2} (\partial \phi_2)^2 - \mu^2 V^{(0)}(\phi_1, \phi_2) + \mathcal{L}_{ct}$$

$$V^{(0)} = \frac{\lambda_0}{4!} \phi_2^4 + \frac{\lambda_2}{4!} \phi_1^2 \phi_2^2 + \frac{\lambda_4}{4!} \phi_4^2 + \sum_{m=1}^\infty \frac{\lambda_{(4+2m)}}{4+2m} \phi_1^{4+2m} \phi_2^{2m}$$

$$\mathcal{L}_{ct} = \frac{1}{2} (Z_{\phi_1} - 1) (\partial \phi_1)^2 + (Z_{\phi_2} - 1) \frac{1}{2} (\partial \phi_2)^2 - \mu^2 \left[ (Z_{\phi_1} - 1) \frac{\lambda_0}{4!} \phi_4^4 + \text{etc.} \right]$$

The effective potential, $V$, is a homogeneous function of the fields,

$$V = V^{(0)} + V^{(1)} + \ldots$$

$$V(\phi_1, \phi_2) = R^4 W(\theta) , \quad R^2 = \phi_1^2 + \phi_2^2 , \quad \tan \theta = \frac{\phi_1}{\phi_2}.$$
In what follows we will choose values of the couplings in nonpolynomial interactions to be
zero,
\[ \lambda_{(4+2m)} = 0 \quad , \quad m = 1, 2, \ldots \] (32)
Nonetheless, one should note that, as explained earlier, due to the presence of new interactions, setting these values to zero does not place them in a fixed point of the renormalization group evolution,
\[ \beta_{(4+2m)}\big|_{\{\lambda_{(4+2m)}=0\}} \neq 0 . \] (33)
After that choice, we have at the tree level
\[ \lambda_0^{(0)} = \frac{9\lambda_2^2}{\lambda_4} \] (34)
\[ \tan^2 \theta^{(0)} = -\frac{3\lambda_2}{\lambda_4} , \quad \lambda_2 < 0 \] (35)
\[ m_h^2(0) = -\lambda_2 \langle \phi_2 \rangle^2 \left(1 - 3\frac{\lambda_2}{\lambda_4}\right) . \] (36)
This model (or one closely similar) was studied previously in [6, 7, 1, 11, 12]. But quantum corrections were considered there at most up to the level of one loop. Computing corrections at higher loop orders is of interest since only there one obtains modifications of the renormalization group caused by the new interactions in (19), when compared to the case of using the regular dimensional regularisation with \( \mu = \text{const} \). Two loop corrections to renormalization group functions and to the effective scalar potential were obtained in [13]. We will now present the main results of that work.

5. Two loop corrections
To regularize the model described in the previous section we use a minimal subtraction scheme. This means that the counterterms contain only poles in \( \epsilon \), and no finite contributions - in particular those arising from the presence of new interactions - are canceled. (We will denote the loop order of an expression by a superscript in parenthesis.)

\[ Z_N = 1 + \sum_{k=1}^{\infty} \frac{a_{N,k}}{\epsilon^k} , \quad a = a^{(1)} + a^{(2)} + \ldots , \quad a^{(n)} = \mathcal{O}(\lambda^n) , \quad N = \phi_1, \phi_2, 0, 2, 4, \ldots \] (37)
Consider obtaining loop corrections using two different Lagrangians, both of which are given by (23) and (32), but differ in the meaning of \( \mu \). The first Lagrangian realizes the SI regularization method and uses (26), while in the second one we use (in the old way) \( \mu = \langle \phi_2 \rangle = \text{const} \). We will present the differences between renormalization group functions and scalar potentials obtained for these two Lagrangians. We will refer to these differences as
new corrections:

\[
V = V^{old} + V^{new}, \quad \beta = \beta^{old} + \beta^{new}, \quad \gamma_\phi = \gamma^{old}_\phi + \gamma^{new}_\phi
\]

\[
\beta^{(1) new}_4 = 0, \quad \text{for all couplings}
\]

\[
\beta^{(1) new}_2 = \frac{3}{(4\pi)^2}(\lambda_4^2 + \lambda_2^2)
\]

\[
\beta^{(1) new}_2 = \frac{1}{(4\pi)^2}(\lambda_4 + 4\lambda_2 + \lambda_0)\lambda_2
\]

\[
\gamma^{(1) new}_\phi = 0, \quad \gamma^{(2) new}_\phi = 0, \quad \text{for both fields}
\]

\[
\beta^{(2) new}_0 = -\frac{1}{2(4\pi)^4}(48\lambda_2^3 + 4\lambda_2^2\lambda_0 + 21\lambda_0^3)
\]

\[
\beta^{(2) new}_2 = -\frac{\lambda_2}{6(4\pi)^4}(48\lambda_4\lambda_2 + 6\lambda_4\lambda_0 + 123\lambda_2^2 + 86\lambda_2\lambda_0 + 3\lambda_0^2)
\]

\[
\beta^{(4) new}_4 = \frac{1}{(4\pi)^4}\left[\lambda_2^2(24\lambda_2 - 7\lambda_0) + \lambda_4(-14\lambda_2^2 + 16\lambda_2\lambda_0 - 3\lambda_0^2) + \lambda_0^2(-80\lambda_2 + 6\lambda_0)\right]
\]

\[
\beta^{(6) new}_6 = \frac{1}{4(4\pi)^4}\lambda_4\lambda_2(7\lambda_4 - 14\lambda_2 + \lambda_0)
\]

\[
\beta^{(8) new}_8 = \frac{1}{2(4\pi)^4}\lambda_4\lambda_2^2
\]

\[
\beta^{(2) new}_{(8+2m)} = 0, \quad m = 1, 2, \ldots
\]

The potential \( V \) is obtained in the background field method, which means that it is computed as a sum of vacuum 1PI diagrams for the propagating fields \( \phi'_1, \phi'_2 \), using \( \langle \phi_1 \rangle \)- and \( \langle \phi_2 \rangle \)-dependent Feynman rules. Thus, it is formally a function of two variables denoted by \( \langle \phi_1 \rangle \) and \( \langle \phi_2 \rangle \), but we will skip the angle brackets in the following presentation, \( V = V(\phi_1, \phi_2) \).

\[
V^{(1) new} = \frac{1}{48(4\pi)^2}\left[\lambda_4\lambda_4\phi_1^6 - (16\lambda_4\lambda_2 + 18\lambda_2^2 - \lambda_4\lambda_0)\phi_1^4 - (48\lambda_2 + 25\lambda_0)\lambda_2 \phi_1^2\phi_2 - 7\lambda_0^2\phi_2^2\right]
\]

\[
V^{(2) new} = \frac{1}{192(4\pi)^4}\left[\ln V(+) + \ln V(-)\right]\left[-(14\lambda_2^2 - 111\lambda_4\lambda_2^2 - 168\lambda_2^3 + 9\lambda_4^3\lambda_2 - 40\lambda_2^2\lambda_0) \phi_1^4
\]

\[
-(192\lambda_4\lambda_2^2 + 705\lambda_2^3 + 374\lambda_4\lambda_2^2 + 63\lambda_2^2\lambda_0 + 106\lambda_2\lambda_0^2) \phi_1^2\phi_2^2
\]

\[
-(48\lambda_2^3 + 46\lambda_2^2\lambda_0 + 63\lambda_0^3) \phi_2^4 + (18\lambda_4\lambda_2^2 - 4\lambda_4\lambda_2^2 + 3\lambda_2^2 + 3\lambda_4\lambda_2\lambda_0) \frac{\phi_6^2}{\phi_2^2}
\]

\[
+ 3\lambda_4\lambda_2^2 \phi_4^8 + t\text{-independent terms}, \quad \text{where } \ln A \equiv \ln \frac{A}{(t\phi_2)^2}4\pi e^{-\gamma_E} - 1,
\]

\[
V_{(+)} = 1/2\left[V_{11} + V_{22} \pm \left[(V_{11} - V_{22})^2 + 4V_{12}^2\right]^{1/2}\right], \quad V_{ij} = \frac{\partial^2 V^{(0)}}{\partial \phi_i \partial \phi_j}.
\]

Consider now the Callan-Symanzik equation,

\[
\frac{dV}{dt} = \left( t \frac{\partial}{\partial t} + \sum_{m=0} \beta_{2m} \frac{\partial}{\partial \lambda_{2m}} - \phi_1 \gamma_{\phi_1} \frac{\partial}{\partial \phi_1} - \phi_2 \gamma_{\phi_2} \frac{\partial}{\partial \phi_2} \right) V = 0.
\]
It too can be split into the old, established result that we can cross out and the new part (that contains expressions in (38) and (39) marked as new). The latter reads

$$\frac{\partial V^{(2)}_{\text{new}}}{\partial \ln t} + \sum_{2m=0}^{4} \beta_{\lambda_{2m}}^{(2)} \frac{\partial V^{(0)}}{\partial \lambda_{2m}} + \sum_{m=0}^{2} \beta_{\lambda_{2m}}^{(1)} \frac{\partial V^{(1)}_{\text{new}}}{\partial \lambda_{2m}} = 0,$$

(41)

where one should remember that $V^{(0)}$ contains $\lambda \phi^6$ and $\lambda \phi^8$.

Using the explicit expressions in (38) and (39) one readily checks that (41) is satisfied. We observe that at the level of two-loop corrections, the SI regularization method results in nontrivial modifications both in the running of coupling constants and in the corrections to the scalar effective potential. Yet these changes do not invalidate the Callan-Symanzik equation.

6. Summary

We have described the scale invariant regularization method and used it to renormalize a model with two interacting scalar fields. It was shown that the method requires presence of nonrenormalizable interactions. Lagrangian of the model is classically scale invariant and this symmetry is not broken by quantum anomaly, but it is broken spontaneously. By performing a perturbative calculation in the broken phase of the symmetry, we obtained renormalization group functions for the coupling constants and corrections to the effective potential at the order of two loops. This enabled us to explicitly check validity of the Callan-Symanzik equation in the framework of scale invariant regularization.

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