The Normalized Laplacian Spectrum Analysis of Fractal Möbius Octagonal Networks and its Applications

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Abstract. The study and calculation of spectrum of networks can be used to describe networks structure and quantify analysis of networks performance. The fractal Möbius octagonal networks, denoted by $Q_n$, is derived from the inverse identification of the opposite lateral edges of fractal linear octagonal networks. In this paper, the normalized Laplacian spectrum of $Q_n$ is determined by two matrices $L_A$ and $L_S$. As an important application of our results, some topological indices (multiplicative degree-Kirchhoff index, the number of spanning trees) formulas of $Q_n$ are obtained.

Keywords: Fractal Möbius Octagonal Networks, Multiplicative Degree-Kirchhoff index, Normalized Laplacian, Spanning trees.

1. Introduction

In recent years, as a tool for studying mathematics, complex network plays a great role in scientific and social research, which has attracted the attention of scholars and achieved some results [1–3]. In this paper, the network is considered as simple, finite and undirected. If there is no special explanation, the terminology and notation we mainly use come from [4].

A network can be regarded as a graph, $G = (V_G, E_G)$, and $V_G$, $E_G$ are its vertex set and edge set, respectively, $V_G = \{u_1, u_2, \ldots, u_n\}$, $E_G = \{e_1, e_2, \ldots, e_m\}$. The adjacent matrix $A_G$ of simple networks can be expressed as

$$(A_G)_{ij} = \begin{cases} 1, & \text{if } u_i \text{ and } u_j \text{ are adjacent;} \\ 0, & \text{otherwise.} \end{cases}$$

The degree of $u_j$ is represented by $d_j$, for $1 \leq j \leq n$. The Laplacian matrix $L_G$ of $G$ is expressed as $L_G = D_G - A_G$, where $D_G = diag_G(d_1, d_2, \ldots, d_n)$. Let $0 = \mu_1 < \mu_2 \leq \cdots \leq \mu_n$ be the eigenvalues of $L_G$. Then the Laplacian spectrum of $L_G$ is expressed as $Sp(L_G) = \{\mu_1, \mu_2, \ldots, \mu_n\}$.

In the past few years, the normalized Laplacian matrix has come into the eyes of academic researchers. Some of its results are not only useful for regular networks, but also suitable for general networks. For any vertices $u_i$ and $u_j$, the normalized Laplacian matrix can be written as

$$(L(G))_{ij} = \begin{cases} 1, & i = j; \\ -\frac{1}{\sqrt{d_i d_j}}, & i \neq j, \ u_i \text{ and } u_j \text{ are adjacent;} \\ 0, & \text{otherwise.} \end{cases}$$

When each edge of a connected network $G$ takes the place of a unit resistor [5], the network can be regarded as a electric circuit. $r_{ij}$ means the effective resistance distance between any vertices $u_i$ and $u_j$. It has been found that this new parameter is also an invariant of network $G$, and it has also played an important role in chemistry [6, 7]. At this time, based on the electrical network, Klein and Randić [5] proposed a new distance function called Kirchhoff index, $Kf(G)$. Similar to Wiener

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index, Kirchhoff index, the sum of the effective resistance distances of any two vertices, is represented as $K_f(G) = \sum_{i<j} r_{ij}$. Later, Gutman, Mohar [8] and Zhu et al. [9] introduced

$$K_f(G) = \sum_{i<j} r_{ij} = n \sum_{j=2}^{n} \frac{1}{\mu_j},$$

where $0 = \mu_1 < \mu_2 < \cdots < \mu_n$ are the eigenvalues of $L_G$.

Chen and Zhang [10] proposed a new index, denoted by $DK(G)$, called multiplicative degree-Kirchhoff index. The mathematical expression is $DK(G) = \sum_{i<j} d_i d_j r_{ij}$. One can see [11, 12]. At the same time, the index is closely related to the normalized Laplacian matrix of network $G$. For the studies of normalized Laplacian of different networks, one can see [13–17].

Furthermore, Kemeny’s constant [18] is the expected number of time steps required for a Markov chain to transition from a starting state $i$ to a random destination state sampled from the Markov chain’s stationary distribution, denoted as $K_c(G)$. The $K_c(G)$ is closely related to the effective resistance of the graph, and the Kemeny’s constant can be connected with the degree-Kirchhoff index by means of the normalized Laplacian matrix.

Recently, a host of scholars have paid attention to study spectrum of networks. In 2016, Huang et al. [19] characterized the normalized Laplacian spectrum, multiplicative degree-Kirchhoff index and the number of spanning trees of linear hexagonal chain networks. In 2018, Li et al. [20] studied the normalized Laplacian spectrum of the penta-graphene and Möbius networks, and obtained the corresponding formulas of degree-Kirchhoff index and the number of spanning trees. In 2018, Li et al. [21] also studied the normalized Laplacian spectrum of linear phenylene and their dicyclobutadieno derivatives. By using the same methods, in 2019, X. Ma and H. Bian, [22,23] not only got the expressions for the Möbius networks, but also calculated some indices of cylinder phenylene networks. More information about normalized spectrum. In 2019, Liu et al. [24] derived the multiplication degree-Kirchhoff index $DK(G)$ and the number of spanning trees $\tau(G)$ of fractal Möbius octagonal networks.

Let $L_n$ be a fractal linear octagonal networks of $n$ octagons, it is shown in Figure 1. Then, the fractal Möbius octagonal networks $Q_n$ is obtained from $L_n$ by identifying the opposite lateral edges in reversed way.

The rest of the work is arranged as follows. In Section 2, in order to get the results of this paper, we introduce some theorems and terms. In Section 3, closed-form formulae of degree-Kirchhoff index and the number of spanning trees are determined for $Q_n$. In Section 4, we have done a full text summary.

![Figure 1: The fractal linear octagonal networks $L_n$.](image)

2. Preliminary

In order to facilitate the calculation in the next section, firstly, we introduce some theorems and lemmas. In this paper, we use $\Phi(G) = det(I - L(G))$ as its characteristic polynomial of $L(G)$. It is well
known that the roots of \( \Phi(G) \) are composed of the normalized Laplacian eigenvalues of \( L(G) \).

Suppose \( \pi = (1, \bar{1}, \ldots, m)(1,1')(2,2') \cdots (n, n') \) is an automorphism of a network \( G = (V_G, E_G) \). From Figure 2, obviously, we find that \( |V_G| = m + 2n \). Provided \( V_0 = \{1, \bar{1}, \ldots, m\}, V_1 = \{1, 2, \ldots, n\}, V_2 = \{1', 2', \ldots, n'\} \). Therefore, \( L(G) \) can be written as

\[
L(G) = \begin{pmatrix}
L_{V_0 V_0} & L_{V_0 V_1} & L_{V_0 V_2} \\
L_{V_1 V_0} & L_{V_1 V_1} & L_{V_1 V_2} \\
L_{V_2 V_0} & L_{V_2 V_1} & L_{V_2 V_2}
\end{pmatrix},
\]

where \( L_{V_i V_j} \) are the matrix formed by the rows and columns of \( L(G) \) corresponding to the vertices in \( V_i \cup V_j, 0 \leq i, j \leq 2 \).

Suppose that

\[
U = \begin{pmatrix}
I_m & 0 & 0 \\
0 & -\frac{1}{\sqrt{2}}I_n & \frac{1}{\sqrt{2}}I_n \\
0 & \frac{1}{\sqrt{2}}I_n & \frac{1}{\sqrt{2}}I_n
\end{pmatrix}.
\]

Consequently,

\[
UL(O_n)U' = \begin{pmatrix}
L_A & 0 \\
0 & L_S
\end{pmatrix},
\]

where \( U' \) is the transposition of \( U \), \( L_A = \begin{pmatrix} L_{V_0 V_0} & \sqrt{2}L_{V_0 V_1} \\
\sqrt{2}L_{V_0 V_1} & L_{V_1 V_1} + L_{V_1 V_2} \end{pmatrix} \), and \( L_S = L_{V_1 V_1} - L_{V_1 V_2} \).

Huang et al. \[19\] proposed the decomposition theorem of normalized Laplacian characteristic polynomial.

**Theorem 2.1.** \[19\] Let \( L(G) \), \( L_A \), and \( L_S \) be defined as above. Then we have

\[
\phi_{L(G)}(z) = \phi_{L_A}(z)\phi_{L_S}(z).
\]

**Lemma 2.2.** \[29\] Let \( G \) be connected graph with \( |V_G| = n \) and \( |E_G| = m \). Then

\[DK(G) = 2m \sum_{j=2}^{n} \frac{1}{\lambda_j},\]

where \( 0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n \) are eigenvalues of \( L(G) \).

**Lemma 2.3.** \[30\] Let \( G \) be \( |V_G| = n \) and \( |E_G| = m \). Then

\[Kc(G) = \sum_{j=2}^{n} \frac{1}{\lambda_j},\]
Lemma 2.4. \[31\] Let \(\tau(G)\) be the number of spanning trees of a connected graph \(G\) of order \(n\) and \(m\) edges. Then
\[
\prod_{j=1}^{n} d_j \prod_{j=2}^{n} \lambda_j = 2m\tau(G).
\]

3. Main results

In this section, we mainly give some results and proofs. Using theorem 2.1 and Vieta’s theorem, \(DK(Q_n)\) and \(\tau(Q_n)\) are determine.

Obviously, \(|V(Q_n)| = 6n\), and \(\pi = (1, 1, 2, 2, \ldots, 3n, 3n)\) is an automorphism of \(Q_n\). Let \(V_0 = \{\emptyset\}, V_1 = \{1, 2, \ldots, 3n\}, \) and \(V_2 = \{1, 2, \ldots, (3n)\}\). Thus \(L_A = L_{V_1 V_1} + L_{V_1 V_2}\) and \(L_S = L_{V_1 V_1} - L_{V_1 V_2}\).

By elementary calculations, we have
\[
L_{V_1 V_1} = \begin{pmatrix}
1 & -\frac{1}{\sqrt{6}} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{\sqrt{6}} & 1 & -\frac{1}{2} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{2} & 1 & -\frac{1}{\sqrt{6}} & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{\sqrt{6}} & 1 & -\frac{1}{\sqrt{6}} & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{\sqrt{6}} & 1 & \cdots & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 1 & -\frac{1}{\sqrt{6}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & -\frac{1}{\sqrt{6}} & 1 & -\frac{1}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -\frac{1}{2} & 1 & 0 \\
\end{pmatrix}_{(3n) \times (3n)}
\]

and
\[
L_{V_1 V_2} = \begin{pmatrix}
-\frac{1}{3} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -\frac{1}{\sqrt{6}} \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{3} & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & -\frac{1}{3} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
-\frac{1}{\sqrt{6}} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\end{pmatrix}_{(3n) \times (3n)}
\]

Hence
\[
L_A = \begin{pmatrix}
\frac{2}{3} & -\frac{1}{\sqrt{6}} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -\frac{1}{\sqrt{6}} \\
-\frac{1}{\sqrt{6}} & 1 & -\frac{1}{2} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{2} & 1 & -\frac{1}{\sqrt{6}} & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{\sqrt{6}} & 1 & -\frac{1}{\sqrt{6}} & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{\sqrt{6}} & 1 & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 1 & -\frac{1}{\sqrt{6}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & -\frac{1}{\sqrt{6}} & 1 & -\frac{1}{2} \\
-\frac{1}{\sqrt{6}} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -\frac{1}{2} & 1 \\
\end{pmatrix}_{(3n) \times (3n)}
\]
and

\[
\mathcal{L}_S = \begin{pmatrix}
\frac{4}{3} & -\frac{1}{\sqrt{6}} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \frac{1}{\sqrt{6}} \\
-\frac{1}{\sqrt{6}} & 1 & \frac{1}{2} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & \frac{1}{2} & 1 & -\frac{1}{\sqrt{6}} & \frac{1}{3} & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{\sqrt{6}} & \frac{4}{3} & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 1 & -\frac{1}{\sqrt{6}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & -\frac{1}{\sqrt{6}} & 1 & -\frac{1}{2} \\
\frac{1}{\sqrt{6}} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -\frac{1}{2} & 1 \\
\end{pmatrix}_{(3n) \times (3n)}
\]

Let \(0 = \alpha_1 < \alpha_2 \leq \cdots \leq \alpha_{3n}\) and \(0 < \rho_1 \leq \rho_2 \leq \cdots \leq \rho_{3n}\) be the eigenvalues of \(\mathcal{L}_A\) and \(\mathcal{L}_S\), respectively. Hence \(Sp(\mathcal{L}(Q_n)) = \{\alpha_1, \alpha_2, \ldots, \alpha_{3n}, \rho_1, \rho_2, \ldots, \rho_{3n}\}\).

**Theorem 3.1.** Let \(0 = \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_{3n}\) be eigenvalues of \(\mathcal{L}_A\). Then

\[
\sum_{j=2}^{3n} \frac{1}{\alpha_j} = \frac{147n^2 - 19}{84}.
\]

**Proof.** Let \(\phi_{\mathcal{L}_A}(z) = z^{3n} + d_1 z^{3n-1} + \cdots + d_{3n-1} z + d_{3n}\) be the characteristic polynomial of \(\mathcal{L}_A\). According to Vieta’s theorem, we have

\[
\sum_{j=2}^{3n} \frac{1}{\alpha_j} = \frac{(-1)^{3n-2} d_{3n-2}}{(-1)^{3n-1} d_{3n-1}}.
\]

To determine \((-1)^{3n-2} d_{3n-2}\) and \((-1)^{3n-2} d_{3n-1}\), we need more preparations. Let

\[
\mathcal{L}_A^0 = \begin{pmatrix}
\frac{2}{3} & -\frac{1}{\sqrt{6}} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
-\frac{1}{\sqrt{6}} & 1 & -\frac{1}{2} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{2} & 1 & -\frac{1}{\sqrt{6}} & \frac{2}{3} & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{\sqrt{6}} & \frac{4}{3} & 1 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 1 & -\frac{1}{\sqrt{6}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & -\frac{1}{\sqrt{6}} & \frac{2}{3} & -\frac{1}{\sqrt{6}} \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -\frac{1}{2} & 1 \\
\end{pmatrix}_{(3n) \times (3n)}
\]

\[
\mathcal{L}_A^1 = \begin{pmatrix}
1 & -\frac{1}{2} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
-\frac{1}{2} & 1 & -\frac{1}{\sqrt{6}} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{\sqrt{6}} & \frac{2}{3} & -\frac{1}{\sqrt{6}} & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{\sqrt{6}} & \frac{4}{3} & 1 & -\frac{1}{2} & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{2} & 1 & \cdots & 0 & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots & -\frac{1}{\sqrt{6}} & 1 & -\frac{1}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & -\frac{1}{2} & 1 & -\frac{1}{\sqrt{6}} \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -\frac{1}{\sqrt{6}} & \frac{2}{3} \\
\end{pmatrix}_{(3n) \times (3n)}
\]
and

$$L_A^2 = \begin{pmatrix}
1 & -\frac{1}{\sqrt{6}} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
-\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{\sqrt{6}} & 1 & \frac{1}{\sqrt{2}} & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{2} & 1 & \frac{1}{\sqrt{6}} & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{\sqrt{6}} & \frac{1}{3} & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & \cdots & -\frac{1}{2} & 1 & \frac{1}{\sqrt{6}} & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & -\frac{1}{\sqrt{6}} & \frac{2}{3} & \frac{1}{\sqrt{6}} \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -\frac{1}{2} & 1
\end{pmatrix}_{(3n)\times(3n)}.$$  

Let $W_j^0$ ($W_j^1/W_j^2$) be the sequential principal minor of order $i$ of $L_A^0$ (resp. $L_A^1/L_A^2$). Take $w_j^0 := \det W_j^0, w_j^1 := \det W_j^1, w_j^2 := \det W_j^2, w_0 = w_0^0 = w_0^1 = 1$. Then we can get some results as below.

**Fact 1.** For $1 \leq j \leq 3n$,

$$w_j^0 = \begin{cases}
(1 + j) \cdot \left(\frac{1}{12}\right)^{\frac{j}{2}}, & \text{if } j \equiv 0 \pmod{3}; \\
(1 + \frac{j}{3}) \cdot \left(\frac{1}{12}\right)^{\frac{j-1}{2}}, & \text{if } j \equiv 1 \pmod{3}; \\
(1 + j) \cdot \left(\frac{1}{12}\right)^{\frac{j-2}{2}}, & \text{if } j \equiv 2 \pmod{3}.
\end{cases}$$

**Proof.** It’s easy to get

$$w_0^0 = \frac{2}{3}, w_0^1 = \frac{1}{2}, w_0^2 = \frac{1}{3}, w_0^3 = \frac{5}{36}, w_0^4 = \frac{1}{12}, w_0^5 = \frac{7}{144},$$

and for $3 \leq j \leq 3n - 1$,

$$w_j^0 = \begin{cases}
w_{j-1}^0 - \frac{1}{4}w_{j-2}^0, & \text{if } j \equiv 0 \pmod{3}; \\
\frac{2}{3}w_{j-1}^0 - \frac{1}{6}w_{j-2}^0, & \text{if } j \equiv 1 \pmod{3}; \\
w_{j-1}^0 - \frac{1}{6}w_{j-2}^0, & \text{if } j \equiv 2 \pmod{3}.
\end{cases}$$

For $1 \leq j \leq n - 1$, let $A_j = w_{3j}^0, B_j = w_{3j+1}^0, C_j = w_{3j+2}^0$. Then $A_1 = \frac{1}{3}, B_0 = \frac{2}{7}, B_1 = \frac{5}{36}, C_0 = \frac{1}{2}, C_1 = \frac{1}{12}$. For $j \geq 2$

$$\begin{cases}
A_j = C_{j-1} - \frac{1}{4}B_{j-1}; \\
B_j = \frac{3}{2}A_j - \frac{1}{6}C_{j-1}; \\
C_j = B_j - \frac{1}{6}A_j.
\end{cases} \quad (3.1)$$

By substituting elimination method into 3.1, we have $A_j = 2B_j + \frac{1}{12}B_{j-1}$, and $C_j = \frac{2}{3}B_j - \frac{1}{12}B_{j-1}$. Finally, we put the results into the second equation of 3.1, and one has

$$144B_j - 24B_{j-1} + B_{j-2} = 0.$$ 

Thus

$$B_j = (c_1 + c_2j)\left(\frac{1}{12}\right)^j,$$

$B_0$ and $B_1$ are introduced into the above formula.

$$\begin{cases}
c_1 \cdot \left(\frac{1}{12}\right)^0 = \frac{2}{7}; \\
(c_1 + c_2) \cdot \left(\frac{1}{12}\right) = \frac{5}{36}.
\end{cases}$$
So, $c_1 = \frac{2}{3}$, $c_2 = 1$.
Thus,

$$
\begin{align*}
A_j &= (1 + 3j) \cdot \left(\frac{1}{12}\right)^j; \\
B_j &= \left(\frac{4}{3} + j\right) \cdot \left(\frac{1}{12}\right)^j; \\
C_j &= \left(\frac{1}{4} + \frac{j}{2}\right) \cdot \left(\frac{1}{12}\right)^j.
\end{align*}
$$

The Fact 1 proved over.
In a similar way, we acquire Fact 2 and Fact 3.

**Fact 2.** For $1 \leq j \leq 3n$,

$$
|w_j| = \begin{cases} 
(1 + j) \cdot \left(\frac{1}{12}\right)^j, & \text{if } j \equiv 0 \pmod{3}; \\
\left(\frac{4}{3} + j\right) \cdot \left(\frac{1}{12}\right)^j, & \text{if } j \equiv 1 \pmod{3}; \\
\left(\frac{1}{4} + \frac{j}{2}\right) \cdot \left(\frac{1}{12}\right)^j, & \text{if } j \equiv 2 \pmod{3}.
\end{cases}
$$

**Fact 3.** For $1 \leq j \leq 3n - 1$,

$$
|w_j|^2 = w_{j-1}^0 - \frac{1}{6} w_{j-2}^1.
$$

**Fact 4.** $(-1)^{3n-1}d_{3n-1} = 21n^2 \left(\frac{1}{12}\right)^n$.

**Proof.**

$$
(-1)^{3n-1}d_{3n-1} = \sum_{x=1}^{3n} \det\mathcal{L}_A[x]
$$

$$
= \sum_{x=3, x \equiv 0 \pmod{3}}^{3n} \det\mathcal{L}_A[x] + \sum_{x=1, x \equiv 1 \pmod{3}}^{3n-2} \det\mathcal{L}_A[x] + \sum_{x=2, x \equiv 2 \pmod{3}}^{3n-1} \det\mathcal{L}_A[x],
$$

where

$$
\det\mathcal{L}_A[x] = \begin{cases} 
\frac{w_{x-1}^0 \cdot w_{3n-x}^0 - \frac{1}{6} w_{x-2}^1 \cdot w_{3n-x-1}^0}{w_{x-1}^0} & \text{if } x \equiv 0 \pmod{3}; \\
\frac{w_{x-1}^0 \cdot w_{3n-x}^0 - \frac{1}{6} w_{x-2}^1 \cdot w_{3n-x-1}^0}{w_{x-1}^0} & \text{if } x \equiv 1 \pmod{3}; \\
\frac{w_{x-1}^0 \cdot w_{3n-x}^0 - \frac{1}{6} w_{x-2}^1 \cdot w_{3n-x-1}^0}{w_{x-1}^0} & \text{if } x \equiv 2 \pmod{3}.
\end{cases}
$$

By the Fact 1 and Fact 2, we obtain

$$
\sum_{x=3, x \equiv 0 \pmod{3}}^{3n} \det\mathcal{L}_A[x] = w_{j-1}^0 \cdot w_{3n-x}^0 - \frac{1}{6} w_{x-2}^1 \cdot w_{3n-x-1}^0
$$

$$
= \sum_{j=3, x \equiv 0 \pmod{3}}^{3n} \left(\frac{x}{6} \left(\frac{1}{12}\right)^{x-3} \cdot (1 + 3n - x) \left(\frac{1}{12}\right)^{3n-3} - \frac{1}{6} \cdot \frac{1}{2} (x-1) \cdot \left(\frac{1}{12}\right)^{x-3} \cdot \frac{1}{6} (3n - x) \cdot \left(\frac{1}{12}\right)^{3n-3-3}\right).
$$

$$
= 6n^2 \left(\frac{1}{12}\right)^n.
$$

In the same way, according to Fact 1 - 3, we get the following results.

$$
\sum_{x=1, x \equiv 1 \pmod{3}}^{3n-2} \det\mathcal{L}_A[x] = 9n^2 \left(\frac{1}{12}\right)^n.
$$

$$
\sum_{x=2, x \equiv 2 \pmod{3}}^{3n-1} \det\mathcal{L}_A[x] = 6n^2 \left(\frac{1}{12}\right)^n.
$$
The desired result holds.

**Fact 5.** $(-1)^{3n-2} d_{3n-2} = \frac{147n^4-19n^2}{4}$.  

**Proof.** $(-1)^{3n-2} d_{3n-2}$ is the sum of all principal minors obtained by deleting two rows and two columns of $L_A$. So

$(-1)^{3n-2} d_{3n-2} = \sum_{1 \leq x < y}^{3n} \det L_A[x, y] = M_0 + M_1 + M_2.$

Note that

$M_0 = \sum_{1 \leq x < y \leq n} \det L_A[3x, 3y] + \sum_{1 \leq x < y \leq n-1} \det L_A[3x, 3y + 1] + \sum_{1 \leq x < y \leq n-1} \det L_A[3x, 3y + 2];$

$M_1 = \sum_{1 \leq x < y \leq n} \det L_A[3x + 1, 3y] + \sum_{1 \leq x < y \leq n-1} \det L_A[3x + 1, 3y + 1] + \sum_{1 \leq x < y \leq n-1} \det L_A[3x + 1, 3y + 2];$

$M_2 = \sum_{1 \leq x < y \leq n} \det L_A[3x + 2, 3y] + \sum_{1 \leq x < y \leq n-1} \det L_A[3x + 2, 3y + 1] + \sum_{1 \leq x < y \leq n-1} \det L_A[3x + 2, 3y + 2].$

**Case 1.** If $3x \ (\text{mod} \ 3) \equiv 0, 3y \ (\text{mod} \ 3) \equiv 0, 1 \leq x < y \leq n$. That is,

$\sum_{1 \leq x < y \leq n} \det L_A[3x, 3y] = \sum_{1 \leq x < y \leq n} w^0_{x-1} \cdot w^0_{y-x-1} \cdot w^0_{3n-y} - \frac{1}{6} w^1_{x-2} \cdot w^1_{y-x-1} \cdot w^1_{3n-y-1}$

$= \sum_{1 \leq x < y \leq n} 4(y - x)(3n - y + x) \cdot \left(\frac{1}{12}\right)^n$

$= 3(n^4 - n^2)\left(\frac{1}{12}\right)^n.$

**Case 2.** For $1 \leq x < y \leq n - 1$,

$\sum_{1 \leq x < y \leq n-1} \det L_A[3x, 3y + 1] = \sum_{1 \leq x < y \leq n-1} w^0_{x-1} \cdot w^0_{y-x-1} \cdot w^1_{3n-y} - \frac{1}{6} w^1_{x-2} \cdot w^0_{y-x-1} \cdot w^1_{3n-y-1}$

$= \sum_{1 \leq x < y \leq n-1} 6(y - x)(3n - y + x) \cdot \left(\frac{1}{12}\right)^n$

$= \left(\frac{9}{2} n^4 - 6n^3 + \frac{3}{2} n^2\right)\left(\frac{1}{12}\right)^n.$

**Case 3.** For $1 \leq x < y \leq n - 1$,

$\sum_{1 \leq x < y \leq n-1} \det L_A[3x, 3y + 2] = \sum_{1 \leq x < y \leq n-1} w^0_{x-1} \cdot w^0_{y-x-1} \cdot w^2_{3n-y} - \frac{1}{6} w^1_{x-2} \cdot w^0_{y-x-1} \cdot w^2_{3n-y-1}$

$= \sum_{1 \leq x < y \leq n-1} 4(y - x)(3n - k + j) \cdot \left(\frac{1}{12}\right)^n$

$= (3n^4 - 2n^3 + n^2 - 2n)\left(\frac{1}{12}\right)^n.$
Case 4. If \((3x + 1) \pmod{3} \equiv 1, 3y \pmod{3} \equiv 0, 1 \leq x < y \leq n\). So,
\[
\sum_{1 \leq x < y \leq n} \det \mathcal{L}_A[3x + 1, 3y] = \sum_{1 \leq x < y \leq n} w^0_{x-1} \cdot w^1_y \cdot w^0_{3n-y} - \frac{1}{6} w^1_{x-2} \cdot w^1_y \cdot w^0_{3n-y-1} = 6(y-x)(3n-y+x) \cdot \left(\frac{1}{12}\right)^n = \left(\frac{9}{2} n^4 + 6n^3 + 3n^2\right) \left(\frac{1}{12}\right)^n.
\]

Case 5. For \(1 \leq x < y \leq n-1\),
\[
\sum_{1 \leq x < y \leq n-1} \det \mathcal{L}_A[3x + 1, 3y+1] = \sum_{1 \leq x < y \leq n-1} w^0_{x-1} \cdot w^1_y \cdot w^1_{3n-y} - \frac{1}{6} w^1_{x-2} \cdot w^1_y \cdot w^1_{3n-y-1} = 9(y-x)(3n-y+x) \cdot \left(\frac{1}{12}\right)^n = \frac{27}{4} (n^4 - n^2) \left(\frac{1}{12}\right)^n.
\]

Case 6. For \(1 \leq x < y \leq n-1\),
\[
\sum_{1 \leq x < y \leq n-1} \det \mathcal{L}_A[3x + 1, 3y+2] = \sum_{1 \leq x < y \leq n-1} w^0_{x-1} \cdot w^2_y \cdot w^1_{3n-y} - \frac{1}{6} w^1_{x-2} \cdot w^2_y \cdot w^0_{3n-y-1} = 6(y-x)(3n-y+x) \cdot \left(\frac{1}{12}\right)^n = \frac{9}{2} (n^4 + 3n^3 + \frac{3}{2} n^2 + 3n) \left(\frac{1}{12}\right)^n.
\]

Case 7. If \((3x + 2) \pmod{3} \equiv 2, 3y \pmod{3} \equiv 0\), for \(1 \leq x < y \leq n\),
\[
\sum_{1 \leq x < y \leq n} \det \mathcal{L}_A[3x + 2, 3y] = \sum_{1 \leq x < y \leq n} w^0_{x-1} \cdot w^2_y \cdot w^1_{3n-y} - \frac{1}{6} w^1_{x-2} \cdot w^2_y \cdot w^1_{3n-y-1} = 4(y-x)(3n-y+x) \cdot \left(\frac{1}{12}\right)^n = (3n^4 + 2n^3 + n^2 + 2n) \left(\frac{1}{12}\right)^n.
\]

Case 8. For \(1 \leq x < y \leq n-1\),
\[
\sum_{1 \leq x < y \leq n-1} \det \mathcal{L}_A[3x + 2, 3y+1] = \sum_{1 \leq x < y \leq n-1} w^0_{x-1} \cdot w^2_y \cdot w^1_{3n-y} - \frac{1}{6} w^1_{x-2} \cdot w^2_y \cdot w^1_{3n-y-1} = 6(y-x)(3n-y+x) \cdot \left(\frac{1}{12}\right)^n = \frac{9}{2} (n^4 - 3n^3 + \frac{3}{2} n^2 - 3n) \left(\frac{1}{12}\right)^n.
\]

Case 9. For \(1 \leq x < y \leq n-1\),
\[
\sum_{1 \leq x < y \leq n-1} \det \mathcal{L}_A[3x + 2, 3y+2] = \sum_{1 \leq x < y \leq n-1} w^0_{x-1} \cdot w^2_y \cdot w^0_{3n-y} - \frac{1}{6} w^1_{x-2} \cdot w^2_y \cdot w^0_{3n-y-1} = 4(y-x)(3n-y+x) \cdot \left(\frac{1}{12}\right)^n = 3(n^4 - n^2) \left(\frac{1}{12}\right)^n.
\]
The proof of Fact 5 completed.

**Theorem 3.2.** Assume that \( \rho_1 < \rho_2 \leq \cdots \leq \rho_{3n} \) are the eigenvalues of \( L_S \). One has

\[
\sum_{i=1}^{3n} \frac{1}{\rho_j} = \frac{37 \sqrt{15} n}{30} \left[ (4 + \sqrt{15})^n - (4 - \sqrt{15})^n \right]
\]

(3.2)

**Proof.** Let \( \phi_{L_S}(z) = z^{3n} + t_1 z^{3n-1} + \cdots + t_{3n-1} z + t_{3n} \) be the characteristic polynomial of \( L_S \). So,

\[
\sum_{j=1}^{3n} \frac{1}{\rho_j} = \frac{(-1)^{3n-1} t_{3n-1}}{(-1)^{3n-1} t_{3n}} = \frac{(-1)^{3n-1} t_{3n-1}}{det L_S}.
\]

To determine \((-1)^{3n-1} t_{3n-1}\), we need more preparations. Let

\[
L_0^0 = \begin{pmatrix}
\frac{1}{3} & -\frac{1}{\sqrt{6}} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\frac{1}{\sqrt{6}} & 1 & \frac{1}{2} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 1 & \frac{1}{2} & \frac{1}{\sqrt{6}} & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{\sqrt{6}} & \frac{1}{3} & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{\sqrt{6}} & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \frac{1}{\sqrt{6}} & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \frac{1}{\sqrt{6}} \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1
\end{pmatrix}
\]

and

\[
L_1^0 = \begin{pmatrix}
1 & -\frac{1}{2} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
-\frac{1}{2} & 1 & -\frac{1}{\sqrt{6}} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & \frac{1}{\sqrt{6}} & \frac{1}{2} & -\frac{1}{\sqrt{6}} & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{\sqrt{6}} & \frac{1}{3} & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{\sqrt{6}} & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \frac{1}{\sqrt{6}} & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \frac{1}{\sqrt{6}} \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1
\end{pmatrix}
\]

Let \( Q_i^0 \) (\( Q_i^1 \)) be the sequential principal minor of order \( i \) of \( L_0^0 \) (resp. \( L_1^0 \)). Take \( q_i^0 = det Q_i^0 \), \( q_i^1 = det Q_i^1 \). Then we get the following results.

**Claim 1.** For \( 1 \leq j \leq 3n \),

\[
q_j^0 = \begin{cases}
\frac{1}{2} + \frac{\sqrt{15}}{6} & \text{if } j \equiv 0 \pmod{3} \\
\frac{2}{3} + \frac{\sqrt{15}}{90} & \text{if } j \equiv 1 \pmod{3} \\
\frac{7}{12} + \frac{\sqrt{15}}{45} & \text{if } j \equiv 2 \pmod{3}
\end{cases}
\]

Proof. According to the specific calculation, one has \( q_1^0 = \frac{4}{5} \), \( q_2^0 = \frac{7}{6} \), \( q_3^0 = \frac{5}{6} \), \( q_4^0 = \frac{11}{12} \), \( q_5^0 = \frac{7}{5} \). For \( 1 \leq j \leq 3n \),

\[
q_j^0 = \begin{cases}
\frac{1}{2} q_{j-1}^0 - \frac{1}{6} q_{j-2}^0 & \text{if } j \equiv 0 \pmod{3} \\
\frac{1}{2} q_{j-1}^0 - \frac{1}{6} q_{j-2}^0 & \text{if } j \equiv 1 \pmod{3} \\
\frac{1}{2} q_{j-1}^0 - \frac{1}{6} q_{j-2}^0 & \text{if } j \equiv 2 \pmod{3}
\end{cases}
\]
For 1 ≤ j ≤ n, Suppose that e_j = q_{3j}. When 0 ≤ j ≤ n − 1, Suppose that f_j = q_{3j+1}, and g_j = q_{3j+2}. Then e_1 = \frac{5}{6}, f_0 = \frac{1}{3}, f_1 = \frac{11}{12}, g_1 = \frac{2}{5}. For j ≥ 2,

\begin{align*}
e_j &= g_j - \frac{1}{6} f_{j-1}; \\
f_j &= \frac{4}{3} e_j - \frac{1}{5} g_{j-1}; \\
g_j &= f_j - \frac{1}{6} e_j.
\end{align*}

From the first and second expressions of 3.3, we can get e_j = \frac{5}{6} f_j + \frac{1}{25} f_{j-1}. Then put the result into the third equation, we have \( g_j = \frac{6}{7} f_j - \frac{1}{168} f_{j-1}, \) so \( g_{j-1} = \frac{6}{7} f_{j-1} - \frac{1}{168} f_{j-2}. \) Finally, substituting e_j and g_{j-1} into the second formula,

\[ 144 f_j - 96 f_{j-1} + f_{j-2} = 0. \]

So, \( f_j = a_1\left(\frac{1}{12} + \frac{\sqrt{15}}{\sqrt{15}}\right)^j + a_2\left(\frac{4\sqrt{15}}{12}\right)^j, \) substituting the initial conditions \( f_1 \) and \( f_2 \) into the above formula, \( a_1 = \frac{2}{3} + \frac{17\sqrt{15}}{90}, \) and \( a_2 = \frac{2}{3} - \frac{17\sqrt{15}}{90} \) are obtained. And then,

\begin{align*}
e_j &= (\frac{1}{2} + \frac{\sqrt{15}}{5})(\frac{4\sqrt{15}}{12})^j + (\frac{1}{2} - \frac{\sqrt{15}}{5})(\frac{4\sqrt{15}}{12})^j; \\
f_j &= (\frac{2}{3} + \frac{17\sqrt{15}}{90})(\frac{4\sqrt{15}}{12})^j + (\frac{2}{3} - \frac{17\sqrt{15}}{90})(\frac{4\sqrt{15}}{12})^j; \\
g_j &= (\frac{7}{12} + \frac{7\sqrt{15}}{45})(\frac{4\sqrt{15}}{12})^j + (\frac{7}{12} - \frac{7\sqrt{15}}{45})(\frac{4\sqrt{15}}{12})^j,
\end{align*}

as desired.

In the same way, we can get Claim 2. Here we omit the proof.

**Claim 2.** For 1 ≤ j ≤ 3n,

\[
g_j^1 = \begin{cases} 
\left(\frac{1}{2} + \frac{\sqrt{15}}{3}\right)(\frac{4\sqrt{15}}{12})^j + \left(\frac{1}{2} - \frac{\sqrt{15}}{3}\right)(\frac{4\sqrt{15}}{12})^j, & \text{if } j \equiv 0 \pmod{3}; \\
\left(\frac{1}{2} + \frac{\sqrt{15}}{3}\right)(\frac{4\sqrt{15}}{12})^j + \left(\frac{1}{2} - \frac{\sqrt{15}}{3}\right)(\frac{4\sqrt{15}}{12})^j, & \text{if } j \equiv 1 \pmod{3}; \\
\left(\frac{1}{3} + \frac{\sqrt{15}}{10}\right)(\frac{4\sqrt{15}}{12})^j + \left(\frac{1}{3} - \frac{\sqrt{15}}{10}\right)(\frac{4\sqrt{15}}{12})^j, & \text{if } j \equiv 2 \pmod{3}.
\end{cases}
\]

Using the properties of determinants, we have

\[
\text{det } \mathcal{L}_S = \begin{vmatrix} 
\frac{4}{3} & -\frac{1}{\sqrt{6}} & 0 & 0 & \cdots & 0 & \frac{1}{\sqrt{6}} \\
-\frac{1}{\sqrt{6}} & 1 & -\frac{1}{\sqrt{6}} & 0 & \cdots & 0 & 0 \\
0 & -1 & 2 & -1 & \cdots & 0 & 0 \\
0 & 0 & -1 & 4 & \cdots & 0 & 0 \\
& \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1 & -\frac{1}{\sqrt{6}} \\
\frac{1}{\sqrt{6}} & 0 & 0 & 0 & \cdots & -\frac{1}{\sqrt{6}} & 1
\end{vmatrix}_{3n \times 3n}
\]

= \begin{vmatrix} 
\frac{4}{3} & -\frac{1}{\sqrt{6}} & 0 & 0 & \cdots & 0 & 0 \\
-\frac{1}{\sqrt{6}} & 1 & -\frac{1}{\sqrt{6}} & 0 & \cdots & 0 & 0 \\
0 & -1 & 2 & -1 & \cdots & 0 & 0 \\
0 & 0 & -1 & 4 & \cdots & 0 & 0 \\
& \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1 & -\frac{1}{\sqrt{6}} \\
\frac{1}{\sqrt{6}} & 0 & 0 & 0 & \cdots & -\frac{1}{\sqrt{6}} & 1
\end{vmatrix}_{3n \times 3n}
Claim 5. In the determinant of \( L \), we have the following Claim.

Next, we focus on calculating \((-1)^{3n-1}t_{3n-1}\).

Claim 4. \((-1)^{3n-1}t_{3n-1} = \frac{37\sqrt[3]{15}n}{30} \left[ (\frac{4+\sqrt{15}}{12})^n - (\frac{4-\sqrt{15}}{12})^n \right] \). \( \] .

Proof. \((-1)^{3n-1}a_{3n-1} \) is the sum of all \( 3n-1 \) order principal subexpression by deleting the row by column of \( L \). One has

\[-\sum_{j=1,j \equiv 1 \pmod{3}}^{3n-2} \] det\( L \) \( + \) \( \sum_{x=2,x \equiv 2 \pmod{3}}^{3n-1} \) det\( L \) \( = \) \( \frac{1}{6} q_{3n-2} + 2 \cdot \left( \frac{1}{12} \right)^n \)

In the determinant of \( \text{det} L \), we obtain

\[
(-1)^{3n-1}t_{3n-1} = \sum_{x=3,x \equiv 0 \pmod{3}}^{3n} \text{det} L[x] + \sum_{x=1,x \equiv 1 \pmod{3}}^{3n-2} \text{det} L[j] + \sum_{x=2,x \equiv 2 \pmod{3}}^{3n-1} \text{det} L[x]. \tag{3.4}
\]

Proof. Let \( \text{det} L[x] = \left( \begin{array}{cc} L & M \\ N & O \end{array} \right) \), where \( L \) is a square matrix of order \( x - 1 \), and \( O \) is \( 3n-x \times (3n-x) \). Obviously

\[
\left( \begin{array}{c} 0 \\ I_{3n-x} \end{array} \right)^T \text{det} L[x] \left( \begin{array}{c} 0 \\ I_{3n-x} \end{array} \right) = \left( \begin{array}{c} O \\ -M \\ -L \end{array} \right) \tag{3.5}
\]

Supposing that \( \left( \begin{array}{c} 0 \\ I_{3n-x} \end{array} \right)^T = S \). Then the above formula can be written as

\[
S^T \text{det} L[x] S = \left( \begin{array}{c} O \\ -M \\ -L \end{array} \right) \]

Subcase 1. If \( x \equiv 0 \pmod{3} \), \( 3 \leq x \leq 3n-3 \),

\[
S^T \text{det} L[x] S = Q_{3n-1}^0 = L_S[3n]
\]
\[ \sum_{x=3, x \equiv 0 \pmod{3}}^{3n} \det \mathcal{L}_S[x] = n q^0_{3n-1} \]
\[ = n \left( \frac{7}{12} + \frac{7\sqrt{15}}{45} \right) \left( \frac{4 + \sqrt{15}}{12} \right)^n + \left( \frac{7}{12} + \frac{7\sqrt{15}}{45} \right) \left( \frac{4 - \sqrt{15}}{12} \right)^{n-1} \]
\[ = \frac{7\sqrt{15} n}{15} \left[ \left( \frac{4 + \sqrt{15}}{12} \right)^n - \left( \frac{4 - \sqrt{15}}{12} \right)^n \right]. \]

Similarly,

**Subcase 2.** If \( x \equiv 1 \pmod{3}, \) \( 1 \leq x \leq 3n - 2, \)

\[ ST \det \mathcal{L}_S[x] S = Q^1_{3n-1} = \mathcal{L}_S[1] \]

and

\[ \sum_{x=1, x \equiv 1 \pmod{3}}^{3n-2} \det \mathcal{L}_S[x] = n q^1_{3n-1} \]
\[ = \frac{3\sqrt{15} n}{10} \left[ \left( \frac{4 + \sqrt{15}}{12} \right)^n - \left( \frac{4 - \sqrt{15}}{12} \right)^n \right]. \]

**Subcase 3.** If \( x \equiv 2 \pmod{3}, \) \( 2 \leq x \leq 3n - 1, \)

\[ ST \det \mathcal{L}_S[x] S = \left( \begin{array}{cccccccccccc}
1 & -\frac{1}{\sqrt{6}} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{\sqrt{6}} & \frac{2}{3} & -\frac{1}{5} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{5} & 1 & -\frac{1}{7} & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{\sqrt{6}} & \frac{3}{4} & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{1}{\sqrt{6}} & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 4 & \frac{1}{3} & \frac{1}{\sqrt{6}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{3} & 1 & -\frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \frac{1}{\sqrt{6}} & 1 & -\frac{1}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \frac{1}{3} & -\frac{1}{6} & -\frac{1}{6} \\
\end{array} \right)_{(3n-1) \times (3n-1)} \]

and

\[ \sum_{x=2, x \equiv 2 \pmod{3}}^{3n-1} \det \mathcal{L}_S[x] = n q^0_{3n-1} \]
\[ = \frac{7\sqrt{15} n}{10} \left[ \left( \frac{4 + \sqrt{15}}{12} \right)^n - \left( \frac{4 - \sqrt{15}}{12} \right)^n \right], \]
as desired.

**Theorem 3.3.** Suppose \( Q_n \) is a fractal M"obius octagonal networks of \( n \) octagons. Then

\[ D_k(Q_n) = 14n \left( \sum_{j=2}^{3n-1} \frac{1}{\alpha_j} + \sum_{j=1}^{3n} \frac{1}{\rho_j} \right) \]
\[ = 14n \left( \frac{147n^2 - 19}{84} \right) + \frac{37\sqrt{15} n}{30} \left[ \left( \frac{4 + \sqrt{15}}{12} \right)^n - \left( \frac{4 - \sqrt{15}}{12} \right)^n \right] \]
\[ = \frac{147n^3}{6} - \frac{19n}{6} + 14n \xi_n, \]
where
\[
\xi_n = \frac{37 \sqrt{15} n}{30} \left[ \left( \frac{4+\sqrt{15}}{12} \right)^n - \left( \frac{4-\sqrt{15}}{12} \right)^n \right]
\]

\[
\frac{\left( \frac{4+\sqrt{15}}{12} \right)^n + \left( \frac{4-\sqrt{15}}{12} \right)^n + 2 \left( \frac{1}{12} \right)^n}
\]

Table 1: The Degree-Kirchhoff indices of \( Q_n \) from \( Q_1 \) to \( Q_{30} \).

| \( G \) | \( DK(G) \) | \( G \) | \( DK(G) \) | \( G \) | \( DK(G) \) |
|-------|-------------|-------|-------------|-------|-------------|
| \( Q_1 \) | 73.13 | \( Q_{11} \) | 33310.28 | \( Q_{21} \) | 228232.34 |
| \( Q_2 \) | 319.17 | \( Q_{12} \) | 43100.48 | \( Q_{22} \) | 262277.55 |
| \( Q_3 \) | 851.80 | \( Q_{13} \) | 54654.69 | \( Q_{23} \) | 299556.76 |
| \( Q_4 \) | 1822.69 | \( Q_{14} \) | 68119.90 | \( Q_{24} \) | 340216.96 |
| \( Q_5 \) | 3381.01 | \( Q_{15} \) | 83643.10 | \( Q_{25} \) | 384405.17 |
| \( Q_6 \) | 5674.24 | \( Q_{16} \) | 101371.31 | \( Q_{26} \) | 432268.38 |
| \( Q_7 \) | 8849.45 | \( Q_{17} \) | 121451.52 | \( Q_{27} \) | 483953.58 |
| \( Q_8 \) | 13053.65 | \( Q_{18} \) | 144030.72 | \( Q_{28} \) | 539607.79 |
| \( Q_9 \) | 18433.86 | \( Q_{19} \) | 169255.93 | \( Q_{29} \) | 599378.00 |
| \( Q_{10} \) | 25137.07 | \( Q_{20} \) | 197274.14 | \( Q_{30} \) | 663411.21 |

**Theorem 3.4.** Suppose \( Q_n \) is a fractal Möbius octagonal networks of length \( n \geq 2 \). Therefore,
\[
Kc(Q_n) = \sum_{j=2}^{3n} \frac{1}{\alpha_j} + \sum_{j=1}^{3n} \frac{1}{\rho_j}
\]
\[
= \frac{147n^2 - 19}{84} + \frac{37 \sqrt{15} n}{30} \left[ \left( \frac{4+\sqrt{15}}{12} \right)^n - \left( \frac{4-\sqrt{15}}{12} \right)^n \right]
\]
\[
\frac{\left( \frac{4+\sqrt{15}}{12} \right)^n + \left( \frac{4-\sqrt{15}}{12} \right)^n + 2 \left( \frac{1}{12} \right)^n}
\]

**Theorem 3.5.** Suppose \( Q_n \) is a fractal Möbius octagonal networks of length \( n \geq 2 \). Then
\[
\tau(Q_n) = \frac{3n}{2} (4 + \sqrt{15})^n + (4 - \sqrt{15})^n + 2.
\]

**Proof.** According to Fact 4, one has
\[
\prod_{j=2}^{3n} \alpha_j = (-1)^{3n-1} d_{3n-1} = 21n^2 \left( \frac{1}{12} \right)^n.
\]

Similarly, by Claim 4, one finds
\[
\prod_{j=1}^{3n} \rho_j = det L_S = \left( \frac{4 + \sqrt{15}}{12} \right)^n + \left( \frac{4 - \sqrt{15}}{12} \right)^n + 2 \left( \frac{1}{12} \right)^n.
\]

It needs to be pointed out \( \prod_{j=1}^{6n} d_j(Q_n) = 2^{4n} 3^{2n} \), and \( |E(Q_n)| = 7n \). Then Theorem 3.5 is acquired by the formula of \( \tau(Q_n) \).

Finally, according to the formula, we calculate the number of spanning trees from \( Q_1 \) to \( Q_{12} \).
Table 2: The complexity of $Q_n$ from $Q_1$ to $Q_{12}$.

| $G$ | $\tau(G)$ | $G$ | $\tau(G)$ | $G$ | $\tau(G)$ | $G$ | $\tau(G)$ |
|-----|------------|-----|------------|-----|------------|-----|------------|
| $Q_1$ | 15         | $Q_4$ | 230,64     | $Q_7$ | 196,863,45 | $Q_{10}$ | 137,241,225,60 |
| $Q_2$ | 192        | $Q_5$ | 226,875    | $Q_8$ | 177,131,568| $Q_{11}$ | 118,854,766,965 |
| $Q_3$ | 2205       | $Q_6$ | 214,329,6  | $Q_9$ | 156,887,293,5| $Q_{12}$ | 102,080,901,875,2 |

4. Conclusion

Under the research of some scholars, we have carried on some expansion, studied a new graph, fractal Möbius Octagonal networks ($Q_n$). In this paper, we first restate the normalized Laplacian decomposition theorem. Then, the product of the sum of reciprocal eigenvalues of $L_A$ and $L_S$ are required, by using the Vieta’s theorem for the characteristic polynomials of $L_A$ and $L_S$. Finally, $DK(Q_n)$, $Kc(Q_n)$ and $\tau(Q_n)$ of fractal Möbius octagonal networks ($Q_n$) are obtained.

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