Generators of KMS Symmetric Markov Semigroups on $\mathcal{B}(\mathfrak{h})$ Symmetry and Quantum Detailed Balance

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Abstract

We find the structure of generators of norm continuous quantum Markov semigroups on $\mathcal{B}(\mathfrak{h})$ that are symmetric with respect to the scalar product $\text{tr}(\rho^{1/2}x^\ast \rho^{1/2}y)$ induced by a faithful normal invariant state invariant state $\rho$ and satisfy two quantum generalisations of the classical detailed balance condition related with this non-commutative notion of symmetry: the so-called standard detailed balance condition and the standard detailed balance condition with an antiunitary time reversal.

1 Introduction

Symmetric Markov semigroups have been extensively studied in classical stochastic analysis (Fukushima et al. [13] and the references therein) because their generators and associated Dirichlet forms are very well tractable by Hilbert space and probabilistic methods.

Their non-commutative counterpart has also been deeply investigated (Albeverio and Goswami [1], Cipriani [6], Davies and Lindsay [8], Goldstein and Lindsay [15], Guido, Isola and Scarlatti [17], Park [23], Sauvageot [26] and the references therein).
The classical notion of symmetry with respect to a measure, however, admits several non-commutative generalisations. Here we shall consider the so-called KMS-symmetry that seems more natural from a mathematical point of view (see e.g. Accardi and Mohari [3], Cipriani [6], [7], Goldstein and Lindsay [14], Petz [25]) and find the structure of generators of norm-continuous quantum Markov semigroups (QMS) on the von Neumann algebra $B(h)$ of all bounded operators on a complex separable Hilbert space $h$ that are symmetric or satisfy quantum detailed balance conditions associated with KMS-symmetry or generalising it.

We consider QMS on $B(h)$, i.e. weak$^*$-continuous semigroups of normal, completely positive, identity preserving maps $T = (T_t)_{t \geq 0}$ on $B(h)$, with a faithful normal invariant state $\rho$. This defines pre-scalar products on $B(h)$ by $(x, y)_s = \text{tr}(\rho_{1-s}x^*\rho_s y)$ for $s \in [0, 1]$ and allows one to define the $s$-dual semigroup $T'_t$ on $B(h)$ satisfying $\text{tr}(\rho_{1-s}x^*\rho_s T_t(y)) = \text{tr}(\rho_{1-s}T'_t(x)^*\rho_s y)$ for all $x, y \in B(h)$. The above scalar products coincide on an Abelian von Neumann algebra, the notion of symmetry $T = T'$, however, clearly depends on the choice of the parameter $s$.

The most studied cases are $s = 0$ and $s = 1/2$. Denoting $T_*$ the predual semigroup, a simple computation yields $T'_t(x) = \rho^{-(1-s)}T_t(\rho^{1-s}x^*\rho^s)\rho^{-s}$, and shows that for $s = 1/2$ the maps $T'_t$ are positive but, for $s \neq 1/2$ this may not be the case. Indeed, it is well-known that, for $s \neq 1/2$, the maps $T'_t$ are positive if and only if the maps $T_t$ commute with the modular group $(\sigma_t)_{t \in \mathbb{R}}$, $\sigma_t(x) = \rho^{it}x\rho^{-it}$ (see e.g. [18] Prop. 2.1 p. 98, [22] Th. 6 p. 7985, for $s = 0$, [11] Th. 3.1 p. 341, Prop. 8.1 p. 362 for $s \neq 1/2$). This quite restrictive condition implies that the generator has a very special form that makes simpler the mathematical study of symmetry but imposes strong structural constraints (see e.g. [18] and [12]).

Here we shall consider the most natural choice $s = 1/2$ whose consequences are not so stringent and say that $T$ is KMS-symmetric if it coincides with its dual $T'$. KMS-symmetric QMS were introduced by Cipriani [6] and Goldstein and Lindsay [14]; we refer to [7] for a discussion of the connection with the KMS condition justifying this terminology.

All quantum versions of the classical principle of detailed balance (Agarwal [4], Alicki [5], Frigerio, Gorini, Kossakowski and Verri [18], Majewski [20], [21]), which is at the basis of equilibrium physics, are formulated prescribing a certain relationship between $T$ and $T'$ or between their generators, therefore they depending of the underlying notion of symmetry. This work clarifies the structure of generators of QMS that are KMS symmetric or satisfy a quantum detailed balance condition involving the above scalar product with $s = 1/2$ and is a key step towards understanding which is the most natural and flexible in view of the study of their generalisations for quantum
systems out of equilibrium as, for instance, the dynamical detailed balance condition introduced by Accardi and Imafuku [2].

The generator $\mathcal{L}$ of a norm-continuous QMS can be written in the standard Gorini-Kossakowski-Sudarshan [16] and Lindblad [19] (GKSL) form

$$\mathcal{L}(x) = i[H, x] - \frac{1}{2} \sum_{\ell \geq 1} (L^*_{\ell} L_{\ell} x - 2 L^*_{\ell} x L_{\ell} + x L^*_{\ell} L_{\ell})$$

(1)

where $H, L_{\ell} \in \mathcal{B}(h)$ with $H = H^*$ and the series $\sum_{\ell \geq 1} L^*_{\ell} L_{\ell}$ is strongly convergent. The operators $L_{\ell}, H$ in (1) are not uniquely determined by $\mathcal{L}$, however, under a natural minimality condition (Theorem 8 below) and a zero-mean condition $\text{tr}(\rho L_{\ell}) = 0$ for all $\ell \geq 1$, $H$ is determined up to a scalar multiple of the identity operator and the $(L_{\ell})_{\ell \geq 1}$ up to a unitary transformation of the multiplicity space of the completely positive part of $\mathcal{L}$. We shall call a GKSL representation of $\mathcal{L}$ by operators $H, L_{\ell}$ satisfying these conditions. As a result, by the remark following Theorem 8, in a special GKSL representation of $\mathcal{L}$, the operator $G = -2^{-1} \sum_{\ell \geq 1} L^*_{\ell} L_{\ell} - i H$, is uniquely determined by $\mathcal{L}$ up to a purely imaginary multiple of the identity operator and allows us to write $\mathcal{L}$ in the form

$$\mathcal{L}(x) = G^* x + \sum_{\ell \geq 1} L^*_{\ell} x L_{\ell} + x G.$$  

(2)

Our characterisation of QMS that are KMS-symmetric or satisfy a quantum detailed balance condition generalising related with KMS-symmetry is given in terms of the operators $G, L_{\ell}$ (or, in an equivalent way $H, L_{\ell}$) of a special GKSL representation.

Theorem 18 shows that a QMS is KMS-symmetric if and only if the operators $G, L_{\ell}$ of a special GKSL representation of its generator satisfy $\rho^{1/2} G^* = G \rho^{1/2} + i c \rho^{1/2}$ for some $c \in \mathbb{R}$ and $\rho^{1/2} L_{k}^* = \sum_{\ell} u_{k\ell} L_{\ell} \rho^{1/2}$ for all $k$ and some unitary $(u_{k\ell})$ on the multiplicity space of the completely positive part of $\mathcal{L}$ coinciding with its transpose, i.e. such that $u_{k\ell} = u_{\ell k}$ for all $k, \ell$.

In order to describe our results on the structure of generators of QMS satisfying a quantum detailed balance condition we first recall some basic definitions. The best known is due to Alicki [5] and Frigerio-Gorini-Kossakowski-Verri [18]: a norm-continuous QMS $\mathcal{T} = (\mathcal{T}_t)_{t \geq 0}$ on $\mathcal{B}(h)$ satisfies the Quantum Detailed Balance (QDB) condition if there exists an operator $\tilde{\mathcal{L}}$ on $\mathcal{B}(h)$ and self-adjoint operator $K$ on $h$ such that $\text{tr}(\rho \tilde{\mathcal{L}}(x)y) = \text{tr}(\rho x \mathcal{L}(y))$ and $\mathcal{L}(x) - \tilde{\mathcal{L}}(x) = 2i[K, x]$ for all $x, y \in \mathcal{B}(h)$. Roughly speaking we can say that $\mathcal{L}$ satisfies the QDB condition if the difference of $\mathcal{L}$ and its adjoint $\tilde{\mathcal{L}}$ with respect to the pre-scalar product on $\mathcal{B}(h)$ given by $\text{tr}(\rho a^* b)$ is a derivation.
This QDB implies that the operator \( \widetilde{\mathcal{L}} = \mathcal{L} - 2i[K, \cdot] \) is conditionally completely positive and then generates a QMS \( \widetilde{T} \). Therefore \( \mathcal{L} \) and the maps \( T_t \) commute with the modular group. This restriction does not follow if the dual QMS is defined with respect to the symmetric pre-scalar product with \( s = 1/2 \).

The QDB can be readily reformulated replacing \( \widetilde{\mathcal{L}} \) with the adjoint \( \mathcal{L}' \) defined via the symmetric scalar product; the resulting condition will be called *Standard Quantum Detailed Balance* condition (SQDB) (see e.g. [9]).

Theorem 15 characterises generators \( \mathcal{L} \) satisfying the SQDB and extends previous partial results by Park [23] and the authors [11]: the SQDB holds if and only if there exists a unitary matrix \( (u_{k\ell}) \), coinciding with its transpose, i.e. \( u_{k\ell} = u_{\ell k} \) for all \( k, \ell \), such that \( \rho^{1/2}L_k^* = \sum_{\ell} u_{k\ell} L_{\ell} \rho^{1/2} \). This shows, in particular, that the SQDB depends only on the \( L_{\ell} \)'s and does not involve directly \( H \) and \( G \). Moreover, we find explicitly the unitary \( (u_{k\ell})_{k\ell} \) providing also a geometrical characterisation of the SQDB (Theorem 16) in terms of the operators \( L_{\ell} \rho^{1/2} \) and their adjoints as Hilbert-Schmidt operators on \( \mathcal{H} \).

We also consider (Definition 5) another notion of quantum detailed balance, inspired by the original Agarwal’s notion (see [4], Majewski [20], [21], Talkner [27]) involving an antiunitary *time reversal* operator \( \theta \) which does not play any role in Alicki et al. definition. Time reversal appears to keep into the account the parity of quantum observables; position and energy, for instance, are even, i.e. invariant under time reversal, momentum are odd, i.e. change sign under time reversal. The original Agarwal’s definition, however, depends on the \( s = 0 \) pre-scalar product and implies then, that a QMS satisfying this quantum detailed balance condition must commute with the modular automorphism. Here we study the modified version (Definition 5) involving the symmetric \( s = 1/2 \) pre-scalar product that we call the SQDB-\( \theta \) condition.

Theorem 21 shows that \( \mathcal{L} \) satisfies the SQDB-\( \theta \) condition if and only if there exists a special GKSL representation of \( \mathcal{L} \) by means of operators \( H, L_{\ell} \) such that \( G\rho^{1/2} = \rho^{1/2}G^* \theta \) and a unitary self-adjoint \( (u_{k\ell})_{k\ell} \) such that \( \rho^{1/2}L_k^* = \sum_{\ell} u_{k\ell} \theta L_{\ell} \theta \rho^{1/2} \) for all \( k \). Here again \( (u_{k\ell})_{k\ell} \) is explicitly determined by the operators \( L_{\ell} \rho^{1/2} \) (Theorem 22).

We think that these results show that the SQDB condition is somewhat weaker than the SQDB-\( \theta \) condition because the first does not involve the directly the operators \( H, G \). Moreover, the unitary operator in the linear relationship between \( L_{\ell} \rho^{1/2} \) and their adjoints is transpose symmetric and any point of the unit disk could be in its spectrum is while, for generators satisfying the SQDB-\( \theta \), it is self-adjoint and its spectrum is contained in \( \{-1, 1\} \). Therefore, by the spectral theorem, it is possible in principle to find
a standard form for the generators of QMSs satisfying the SQDB-θ generalising the standard form of generators satisfying the usual QDB condition (that commute with the modular group) as illustrated in the case of QMSs on $M_2(\mathbb{C})$ studied in the last section. This classification must be much more complex for generators of QMSs satisfying the SQDB.

The above arguments and the fact that the SQDB-θ condition can be formulated in a simple way both on the QMS or on its generator (this is not the case for the QDB-θ when $\mathcal{L}$ and its Hamiltonian part $i[H,\cdot]$ do not commute), lead us to the conclusion that the SQDB-θ is the more natural non-commutative version of the classical detailed balance condition.

The paper is organised as follows. In Section 2 we construct the dual QMS $\mathcal{T}'$ and recall the quantum detailed balance conditions we investigate, then we study the relationship between the generators of a QMS and its adjoint in Section 3. Our main results on the structure of generators are proved in Sections 4 (QDB without time reversal) and 5 (with time reversal).

2 The dual QMS, KMS-symmetry and quantum detailed balance

We start this section by constructing the dual semigroup of a norm-continuous QMS with respect to the $(\cdot, \cdot)_{1/2}$ pre-scalar product on $\mathcal{B}(\mathcal{H})$ defined by an invariant state $\rho$ and prove some properties that will be useful in the sequel. Although this result may be known, the presentation given here leads in a simple and direct way to the dual QMS avoiding non-commutative $L^p$-spaces techniques.

**Proposition 1** Let $\Phi$ be a positive unital normal map on $\mathcal{B}(\mathcal{H})$ with a faithful normal invariant state $\rho$. There exists a unique positive unital normal map $\Phi'$ on $\mathcal{B}(\mathcal{H})$ such that

$$\text{tr} \left( \rho^{1/2} \Phi'(x) \rho^{1/2} y \right) = \text{tr} \left( \rho^{1/2} x \rho^{1/2} \Phi(y) \right)$$

for all $x, y \in \mathcal{B}(\mathcal{H})$. If $\Phi$ is completely positive, then $\Phi'$ is also completely positive.

**Proof.** Let $\Phi^*$ be the predual map on the Banach space of trace class operators on $\mathcal{H}$ and let $\text{Rk}(\rho^{1/2})$ denote the range of the operator $\rho^{1/2}$. This is clearly dense in $\mathcal{H}$ because $\rho$ is faithful and coincides with the domain of the unbounded self-adjoint operator $\rho^{-1/2}$. 
For all self-adjoint \( x \in \mathcal{B}(h) \) consider the sesquilinear form on the domain \( Rk(\rho^{1/2}) \times Rk(\rho^{1/2}) \)

\[
F(v, u) = \langle \rho^{-1/2}v, \Phi_*(\rho^{1/2}x\rho^{1/2})\rho^{-1/2}u \rangle.
\]

By the invariance of \( \rho \) and positivity of \( \Phi_* \) we have

\[
-\|x\|\rho = -\|x\|\Phi_*(\rho) \le \Phi_*(\rho^{1/2}x\rho^{1/2}) \le \|x\|\Phi_*(\rho) = \|x\|\rho.
\]

Therefore \( |F(u, u)| \le \|x\| \cdot \|v\| \cdot \|u\| \). Thus sesquilinear form is bounded and there exists a unique bounded operator \( y \) such that, for all \( u, v \in Rk(\rho^{1/2}) \),

\[
\langle v, yu \rangle = \langle \rho^{-1/2}v, \Phi_*(\rho^{1/2}x\rho^{1/2})\rho^{-1/2}u \rangle.
\]

Note that, \( \Phi \) being a *-map, and \( x \) self/adjoint

\[
\langle v, y^*u \rangle = \overline{\langle y^*u, v \rangle} = \overline{\langle \rho^{-1/2}u, \Phi_*(\rho^{1/2}x\rho^{1/2})\rho^{-1/2}v \rangle} = \langle \Phi_*(\rho^{1/2}x\rho^{1/2})\rho^{-1/2}u, \rho^{-1/2}v \rangle = \langle \rho^{-1/2}v, \Phi_*(\rho^{1/2}x\rho^{1/2})\rho^{-1/2}u \rangle.
\]

This shows that \( y \) is self-adjoint. Defining \( \Phi'(x) := y \), we find a real-linear map on self-adjoint operators on \( \mathcal{B}(h) \) that can be extended to a linear map on \( \mathcal{B}(h) \) decomposing each self-adjoint operator as the sum of its self-adjoint and anti self-adjoint parts.

Clearly \( \Phi' \) is positive because \( \rho^{1/2}\Phi'(x^*x)\rho^{1/2} = \Phi_*(\rho^{1/2}x^*x\rho^{1/2}) \) and \( \Phi_* \) is positive. Moreover, by the above construction \( \Phi'(\mathbf{1}) = \mathbf{1} \), i.e. \( \Phi' \) is unital. Therefore \( \Phi' \) is a norm-one contraction.

If \( \Phi \) is completely positive, then \( \Phi_* \) is also and formula \( \rho^{1/2}\Phi'(x)\rho^{1/2} = \Phi_*(\rho^{1/2}x\rho^{1/2}) \) shows that \( \Phi' \) is completely positive.

Finally we show that \( \Phi' \) is normal. Let \( (x_\alpha)_\alpha \) be a net of positive operators on \( \mathcal{B}(h) \) with least upper bound \( x \in \mathcal{B}(h) \). For all \( u \in h \) we have then

\[
\sup_\alpha \langle \rho^{1/2}u, \Phi'(x_\alpha)\rho^{1/2}u \rangle = \sup_\alpha \langle u, \Phi_*(\rho^{1/2}x_\alpha\rho^{1/2})u \rangle = \langle u, \Phi_*(\rho^{1/2}x\rho^{1/2})u \rangle = \langle \rho^{1/2}u, \Phi'(x)\rho^{1/2}u \rangle.
\]

Now if \( u \in h \), for every \( \varepsilon > 0 \), we can find a \( u_\varepsilon \in Rk(\rho^{1/2}) \) such that \( \|u - u_\varepsilon\| < \varepsilon \) by the density of the range of \( \rho^{1/2} \). We have then

\[
|\langle u, (\Phi'(x_\alpha) - \Phi'(x))u \rangle| \le \varepsilon \| \Phi'(x_\alpha) - \Phi'(x) \| (\|u\| + \|u_\varepsilon\|) + |\langle u_\varepsilon, (\Phi'(x_\alpha) - \Phi'(x))u_\varepsilon \rangle|
\]

for all \( \alpha \). The conclusion follows from the arbitrariness of \( \varepsilon \) and the uniform boundedness of \( \| \Phi'(x_\alpha) - \Phi'(x) \| \) and \( \|u_\varepsilon\| \). \( \square \)
Theorem 2 Let $\mathcal{T}$ be a QMS on $\mathcal{B}(\mathcal{H})$ with a faithful normal invariant state $\rho$. There exists a QMS $\mathcal{T}'$ on $\mathcal{B}(\mathcal{H})$ such that

$$\rho^{1/2} T'_t(x) \rho^{1/2} = T_{st}(\rho^{1/2} x \rho^{1/2})$$

(3)

for all $x \in \mathcal{B}(\mathcal{H})$ and all $t \geq 0$.

Proof. By Proposition [1], for each $t \geq 0$, there exists a unique completely positive normal and unital contraction $\mathcal{T}'_t$ on $\mathcal{B}(\mathcal{H})$ satisfying (3). The semigroup property follows from the algebraic computation

$$\rho^{1/2} T'_{t+s}(x) \rho^{1/2} = T_{st} (\mathcal{T}'_{s}(\rho^{1/2} x \rho^{1/2})) = T_{st} (\rho^{1/2} T'_s(x) \rho^{1/2}) = \rho^{1/2} T'_{s}(T'_t(x)) \rho^{1/2}.$$

Since the map $t \to \langle \rho^{1/2} v, T'_t(x) \rho^{1/2} u \rangle$ is continuous by the identity (3) for all $u, v \in \mathcal{H}$, and $\|T'_t(x)\| \leq \|x\|$ for all $t \geq 0$, a $2\varepsilon$ approximation argument shows that $t \to T'_t(x)$ is continuous for the weak$^*$-operator topology on $\mathcal{B}(\mathcal{H})$. It follows that $\mathcal{T}' = (T'_t)_{t\geq 0}$ is a QMS on $\mathcal{B}(\mathcal{H})$. □

Definition 3 The quantum Markov semigroup $\mathcal{T}'$ is called the dual semigroup of $\mathcal{T}$ with respect to the invariant state $\rho$.

It is easy to see, using (3), that $\rho$ is an invariant state also for $\mathcal{T}'$.

Remark 1 When $\mathcal{T}$ is norm-continuous it is not clear whether also $\mathcal{T}'$ is norm-continuous. Here, however, we are interested in generators of symmetric or detailed balance QMS. We shall see that these additional properties of $\mathcal{T}$ imply that also $\mathcal{T}'$ is norm continuous. Therefore we proceed studying norm-continuous QMSs whose dual is also norm-continuous.

The quantum detailed balance condition of Alicki, Frigerio, Gorini, Kos-sakowski and Verri modified by considering the pre-scalar product $(\cdot, \cdot)_{1/2}$ on $\mathcal{B}(\mathcal{H})$, usually called standard (see e.g. [9]) because of multiplications by $\rho^{1/2}$ as in the standard representation of $\mathcal{B}(\mathcal{H})$, is defined as follows.

Definition 4 The QMS $\mathcal{T}$ generated by $\mathcal{L}$ satisfies the standard quantum detailed balance condition (SQDB) if there exists an operator $\mathcal{L}'$ on $\mathcal{B}(\mathcal{H})$ and a self-adjoint operator $K$ on $\mathcal{H}$ such that

$$\text{tr}(\rho^{1/2} x \rho^{1/2} \mathcal{L}(y)) = \text{tr}(\rho^{1/2} \mathcal{L}'(x) \rho^{1/2} y), \quad \mathcal{L}(x) - \mathcal{L}'(x) = 2i[K, x]$$

(4)

for all $x \in \mathcal{B}(\mathcal{H})$. 7
The operator $L'$ in the above definition must be norm-bounded because it is everywhere defined and norm closed. To see this consider a sequence $(x_n)_{n \geq 1}$ in $B(h)$ converging in norm to $x \in B(h)$ such that $(L(x_n))_{n \geq 1}$ converges in norm to $b \in B(h)$ and note that

$$\text{tr} \left( \rho^{1/2} L'(x) \rho^{1/2} y \right) = \lim_{n \to \infty} \text{tr} \left( \rho^{1/2} x_n \rho^{1/2} L(y) \right) = \lim_{n \to \infty} \text{tr} \left( \rho^{1/2} L'(x_n) \rho^{1/2} y \right) = \text{tr} \left( \rho^{1/2} b \rho^{1/2} y \right)$$

for all $y \in B(h)$. The elements $\rho^{1/2} y \rho^{1/2}$, with $y \in B(h)$, are dense in the Banach space of trace class operators on $h$ because $\rho$ is faithful. Therefore shows that $L'(x) = b$ and $L'$ is closed.

Since both $L$ and $L'$ are bounded, also $K$ is bounded.

We now introduce another definition of quantum detailed balance, due to Agarwal [4] with the $s = 0$ pre-scalar product, that involves a time reversal $\theta$. This is an antiunitary operator on $h$, i.e. $\langle \theta u, \theta v \rangle = \langle v, u \rangle$ for all $u, v \in h$, such that $\theta^2 = \mathbb{1}$ and $\theta^{-1} = \theta^\ast = \theta$.

Recall that, $\theta$ is antilinear, i.e. $\theta z u = \bar{z} u$ for all $u \in h$, $z \in \mathbb{C}$, and its adjoint $\theta^\ast$ satisfies $\langle u, \theta v \rangle = \langle v, \theta^\ast u \rangle$ for all $u, v \in h$. Moreover $\theta x \theta$ belongs to $B(h)$ (linearity is re-established) and $\text{tr}(\theta x \theta) = \text{tr}(x^\ast)$ for every trace-class operator $x$ ([10] Prop. 4), indeed, taking an orthonormal basis of $h$, we have

$$\text{tr}(\theta x \theta) = \sum_j \langle e_j, \theta x \theta e_j \rangle = \sum_j \langle x \theta e_j, \theta^\ast e_j \rangle = \sum_j \langle \theta e_j, x^\ast \theta^\ast e_j \rangle = \text{tr}(x^\ast).$$

It is worth noticing that the cyclic property of the trace does not hold for $\theta$, since $\text{tr}(\theta x \theta) = \text{tr}(x^\ast)$ may not be equal to $\text{tr}(x)$ for non self-adjoint $x$.

**Definition 5** The QMS $T$ generated by $L$ satisfies the standard quantum detailed balance condition with respect to the time reversal $\theta$ (SQDB-$\theta$) if

$$\text{tr}(\rho^{1/2} x \rho^{1/2} L(y)) = \text{tr}(\rho^{1/2} y^\ast \theta \rho^{1/2} L(\theta x^\ast \theta)),$$ \hspace{1cm} (5)

for all $x, y \in B(h)$.

The operator $\theta$ is used to keep into the account parity of the observables under time reversal. Indeed, a self-adjoint operator $x \in B(h)$ is called even (resp. odd) if $\theta x \theta = x$ (resp. $\theta x \theta = -x$). The typical example of antilinear time reversal is a conjugation (with respect to some orthonormal basis).
This condition is usually stated ([20], [21], [27]) for the QMS $T$ as
\[ \text{tr}(\rho^{1/2}x\rho^{1/2}T_t(y)) = \text{tr}(\rho^{1/2}\theta y^*\theta \rho^{1/2}T_t(\theta x^*\theta)), \] (6)
for all $t \geq 0$, $x, y \in B(h)$. In particular, for $t = 0$ we find that this identity holds if and only if $\rho$ and $\theta$ commute, i.e. $\rho$ is an even observable. This is the case, for instance, when $\rho$ is a function of the energy.

**Lemma 6** The following conditions are equivalent:

(i) $\theta$ and $\rho$ commute,

(ii) $\text{tr}(\rho^{1/2}x\rho^{1/2}y) = \text{tr}(\rho^{1/2}\theta y^*\theta \rho^{1/2}x^*\theta)$ for all $x, y \in B(h)$.

**Proof.** If $\rho$ and $\theta$ commute, from $\text{tr}(\theta a\theta) = \text{tr}(a^*)$, we have
\[ \text{tr}(\rho^{1/2}\theta y^*\theta \rho^{1/2}x^*\theta) = \text{tr}(\rho^{1/2}y\rho^{1/2}) \]
and (ii) follows cycling $\rho^{1/2}$. Conversely, if (ii) holds, taking $x = 1$, we have
\[ \text{tr}(\rho y) = \text{tr}(\rho \theta y^*\theta) = \text{tr}(\theta(\theta y^*\theta^*)\rho \theta) = \text{tr}(\theta \rho \theta y), \]
for all $y \in B(h)$, and $\rho = \theta \rho \theta$. □

**Proposition 7** If $\rho$ and $\theta$ commute then (5) and (6) are equivalent.

**Proof.** Clearly (5) follows from (6) differentiating at $t = 0$.

Conversely, putting $\alpha(x) = \theta x\theta$ and denoting $L_*$ the predual of $L$ we can write (5) as
\[ \text{tr}(L_*(\rho^{1/2}x\rho^{1/2}y)) = \text{tr}(\rho^{1/2}\alpha(y^*)\rho^{1/2}L_*(\alpha(x^*))) = \text{tr}(\rho^{1/2}\alpha(L_*(\alpha(x))\rho^{1/2}y), \]
for all $y \in B(h)$, because $\text{tr}(\alpha(a)) = \text{tr}(a^*)$. Therefore we have
\[ L_*(\rho^{1/2}x\rho^{1/2}) = \rho^{1/2}\alpha(L_*(\alpha(x)))\rho^{1/2} \]
and, iterating, $L_*(\rho^{1/2}x\rho^{1/2}) = \rho^{1/2}\alpha(L^n(\alpha(x)))\rho^{1/2}$ for all $n \geq 1$. It follows that (5) holds for all powers $L^n$ with $n \geq 1$. Since $\rho$ and $\theta$ commute, it is true also for $n = 0$ and we find (6) by the exponentiation formula $T_t = \sum_{n \geq 0} t^n L^n/n!$. □

We do not know whether the SQDB condition (4) of Definition 4 has a simple explicit formulation in terms of the maps $T_t$ if $L$ and $L'$ do not commute.
Remark 2 The SQDB condition \([5]\), by \(\text{tr}(\theta a\theta) = \text{tr}(a^*)\), reads
\[
\text{tr}(\rho^{1/2} x \rho^{1/2} \mathcal{L}(y)) = \text{tr}(\rho^{1/2} (\theta \mathcal{L}(\theta x\theta)\theta) \rho^{1/2} x),
\]
for all \(x, y \in B(h)\), i.e. \(\mathcal{L}'(x) = \theta \mathcal{L}(\theta x\theta)\theta\).

Write \(\mathcal{L}\) in a special GKSL form as in \([1]\) and decompose the generator \(\mathcal{L} = \mathcal{L}_0 + i[H, \cdot]\) into the sum of its dissipative part \(\mathcal{L}_0\) and derivation part \(i[H, \cdot]\). If \(H\) commutes with \(\theta\), by the antilinearity of \(\theta\), we find \(\mathcal{L}'(x) = \theta \mathcal{L}_0(\theta x\theta)\theta - i[H, x]\). Therefore, if the dissipative part is time reversal invariant, i.e. \(\mathcal{L}_0(x) = \theta \mathcal{L}_0(\theta x\theta)\theta\), we end up with \(\mathcal{L}' = \mathcal{L} - 2i[H, \cdot]\).

The relationship with Definition 4 of SQDB, in this case, is then clear.

The SQDB conditions of Definition 4 and 5, however, in general are not comparable.

3 The generator of a QMS and its dual

We shall always consider special GKSL representations of the generator of a norm-continuous QMS by means of operators \(L_\ell, H\). These are described by the following theorem (we refer to [24] Theorem 30.16 for the proof).

**Theorem 8** Let \(\mathcal{L}\) be the generator of a norm-continuous QMS on \(B(h)\) and let \(\rho\) be a normal state on \(B(h)\). There exists a bounded self-adjoint operator \(H\) and a finite or infinite sequence \((L_\ell)_{\ell \geq 1}\) of elements of \(B(h)\) such that:

(i) \(\text{tr}(\rho L_\ell) = 0\) for each \(\ell \geq 1\),

(ii) \(\sum_{\ell \geq 1} L_\ell^* L_\ell\) is a strongly convergent sum,

(iii) if \(\sum_{\ell \geq 0} |c_\ell|^2 < \infty\) and \(c_0 + \sum_{\ell \geq 1} c_\ell L_\ell = 0\) for complex scalars \((c_k)_{k \geq 0}\) then \(c_k = 0\) for every \(k \geq 0\),

(iv) the GKSL representation \([7]\) holds.

If \(H', (L'_\ell)_{\ell \geq 1}\) is another family of bounded operators in \(B(h)\) with \(H'\) self-adjoint and the sequence \((L'_\ell)_{\ell \geq 1}\) is finite or infinite then the conditions (i)–(iv) are fulfilled with \(H, (L_\ell)_{\ell \geq 1}\) replaced by \(H', (L'_\ell)_{\ell \geq 1}\) respectively if and only if the lengths of the sequences \((L_\ell)_{\ell \geq 1}\), \((L'_\ell)_{\ell \geq 1}\) are equal and for some scalar \(c \in \mathbb{R}\) and a unitary matrix \((u_{\ell j})_{\ell, j}\) we have

\[
H' = H + c, \quad L'_\ell = \sum_j u_{\ell j} L_j.
\]
As an immediate consequence of the uniqueness (up to a scalar) of the Hamiltonian $H$, the decomposition of $\mathcal{L}$ as the sum of the derivation $i[H,\cdot]$ and a dissipative part $\mathcal{L}_0 = \mathcal{L} - i[H,\cdot]$ determined by special GKSL representations of $\mathcal{L}$ is unique. Moreover, since $(u_{\ell j})$ is unitary, we have

$$
\sum_{\ell \geq 1} (L^*_\ell)^* L^\prime_\ell = \sum_{\ell,k,j \geq 1} \pi_{\ell k} u_{\ell j} L^*_k L_j = \sum_{k,j \geq 1} \left( \sum_{\ell \geq 1} \pi_{\ell k} u_{\ell j} \right) L^*_k L_j = \sum_{k \geq 1} L^*_k L_k.
$$

Therefore, putting $G = -2^{-1} \sum_{\ell \geq 1} L^*_\ell L_\ell - iH$, we can write $\mathcal{L}$ in the form (2) where $G$ is uniquely determined by $\mathcal{L}$ up to a purely imaginary multiple of the identity operator.

Theorem 8 can be restated in the index free form (24 Thm. 30.12).

**Theorem 9** Let $\mathcal{L}$ be the generator of a uniformly continuous QMS on $\mathcal{B}(h)$, then there exist an Hilbert space $\mathfrak{k}$, a bounded linear operator $L : h \rightarrow h \otimes \mathfrak{k}$ and a bounded self-adjoint operator $H$ in $h$ satisfying the following:

1. $\mathcal{L}(x) = i[H,x] - \frac{1}{2} (L^* L - 2L^* (x \otimes \mathbb{1}_k) L + x L^* L)$ for all $x \in \mathcal{B}(h)$;
2. the set $\{(x \otimes \mathbb{1}_k) L u : x \in \mathcal{B}(h), \ u \in h\}$ is total in $h \otimes \mathfrak{k}$.

**Proof.** Let $\mathfrak{k}$ be a Hilbert space with Hilbertian dimension equal to the length of the sequence $(L_k)_k$ and let $(f_k)$ be an orthonormal basis of $\mathfrak{k}$. Defining $Lu = \sum_k L_k u \otimes f_k$, where $(f_k)$ is an orthonormal basis of $\mathfrak{k}$ and the $L_k$ are as in Theorem 8, a simple calculation shows that 1 is fulfilled.

Suppose that there exists a non-zero vector $\xi$ orthogonal to the set of $(x \otimes \mathbb{1}_k) L u$ with $x \in \mathcal{B}(h)$, $u \in h$; then $\xi = \sum_k v_k \otimes f_k$ with $v_k \in h$ and

$$0 = \langle \xi, (x \otimes \mathbb{1}_k) L u \rangle = \sum_k \langle v_k, x L_k u \rangle = \sum_k \langle L^*_k x^* v_k, u \rangle$$

for all $x \in \mathcal{B}(h)$, $u \in h$. Hence, $\sum_k L^*_k x^* v_k = 0$. Since $\xi \neq 0$, we can suppose $\|v_1\| = 1$; then, putting $p = |v_1\rangle \langle v_1|$ and $x = py^*$, $y \in \mathcal{B}(h)$, we get

$$0 = L^*_1 y v_1 + \sum_{k \geq 2} \langle v_1, v_k \rangle L^*_k y v_1 = \left( L^*_1 + \sum_{k \geq 2} \langle v_1, v_k \rangle L^*_k \right) y v_1. \quad (7)$$

Since $y \in \mathcal{B}(h)$ is arbitrary, equation (7) contradicts the linear independence of the $L_k$’s. Therefore the set in (2) must be total. \(\square\)

The Hilbert space $\mathfrak{k}$ is called the **multiplicity space** of the completely positive part of $\mathcal{L}$. A unitary matrix $(u_{\ell j})_{\ell,j \geq 1}$, in the above basis $(f_k)_{k \geq 1}$,
clearly defines a unitary operator on $k$. From now on we shall identify such matrices with operators on $k$.

We end this section by establishing the relationship between the operators $G, L_\ell$ and $G', L'_\ell$ in two special GKSL representations of $\mathcal{L}$ and $\mathcal{L}'$ when these generator are both bounded.

The dual QMS $\mathcal{T}'$ clearly satisfies
\[
\rho^{1/2} T'_t(x) \rho^{1/2} = T_{sl}(\rho^{1/2} x \rho^{1/2})
\]
where $T_{sl}$ denotes the predual semigroup of $T$. Since $\mathcal{L}'$ is bounded, differentiating at $t = 0$, we find the relationship among the generator $\mathcal{L}'$ of $\mathcal{T}$ and $\mathcal{L}_*$ of the predual semigroup $T_*$ of $\mathcal{T}$
\[
\rho^{1/2} \mathcal{L}'(x) \rho^{1/2} = \mathcal{L}_* (\rho^{1/2} x \rho^{1/2}):
\]

**Proposition 10** Let $\mathcal{L}(a) = G^* a + a G + \sum_{\ell} L^*_\ell a L_\ell$ be a special GKSL representation of $\mathcal{L}$ with respect to a $T$-invariant state $\rho = \sum_k \rho_k |e_k \rangle \langle e_k|$. Then
\[
G^* u = \sum_{k \geq 1} \rho_k \mathcal{L}_* (|u \rangle \langle e_k|) e_k - \text{tr}(\rho G) u
\]
\[
G v = \sum_{k \geq 1} \rho_k \mathcal{L}_* (|v \rangle \langle e_k|) e_k - \text{tr}(\rho G^*) v
\]
for every $u, v \in \mathfrak{h}$.

**Proof.** Since $\mathcal{L}(|u \rangle \langle v|) = |G^* u \rangle \langle v| + |u \rangle \langle G v| + \sum_{\ell} |L^*_\ell u \rangle \langle L_\ell v|$, putting $v = e_k$ we have $G^* u = |G^* u \rangle \langle e_k| e_k$ and
\[
G^* u = \mathcal{L}(|u \rangle \langle e_k|) e_k - \sum_{\ell} \langle e_k, L_\ell e_k \rangle L^*_\ell u - \langle e_k, G e_k \rangle u.
\]
Multiplying both sides by $\rho_k$ and summing on $k$, we find then
\[
G^* u = \sum_{k \geq 1} \rho_k \mathcal{L}(|u \rangle \langle e_k|) e_k - \sum_{\ell, k} \rho_k \langle e_k, L_\ell e_k \rangle L^*_\ell u - \sum_{k \geq 1} \rho_k \langle e_k, G e_k \rangle u
\]
\[
= \sum_{k \geq 1} \rho_k \mathcal{L}(|u \rangle \langle e_k|) e_k - \sum_{\ell} \text{tr}(\rho L_\ell) L^*_\ell u - \text{tr}(\rho G) u
\]
and (11) follows since $\text{tr}(\rho L_\ell) = 0$. The identity (11) is now immediate computing the adjoint of $G$. $\square$

**Proposition 11** Let $\mathcal{T}'$ be the dual of a QMS $\mathcal{T}$ generated by $\mathcal{L}$ with normal invariant state $\rho$. If $G$ and $G'$ are the operators (10) in two GKSL representations of $\mathcal{L}$ and $\mathcal{L}'$ then
\[
G' \rho^{1/2} = \rho^{1/2} G^* + (\text{tr}(\rho G) - \text{tr}(\rho G')) \rho^{1/2}.
\]
Moreover, we have $\text{tr}(\rho G) - \text{tr}(\rho G') = ic$ for some $c \in \mathbb{R}$.  

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Proof. The identities (11) and (8) yield
\[ G'\rho^{1/2} = \sum_{k \geq 1} \mathcal{L}'_{\ell}(\rho^{1/2}|v\rangle\langle e_k|)\rho_k^{1/2}e_k - \text{tr}(\rho G^*)\rho^{1/2}v \]
\[ = \sum_{k \geq 1} \mathcal{L}'_{\ell}(\rho^{1/2}|v\rangle\langle e_k|)\rho_k^{1/2}e_k - \text{tr}(\rho G^*)\rho^{1/2}v \]
\[ = \sum_{k \geq 1} \rho^{1/2}\mathcal{L}(\langle v|e_k\rangle)\rho_k^{1/2}e_k - \text{tr}(\rho G^*)\rho^{1/2}v \]
\[ = \rho^{1/2}G^*v + (\text{tr}(\rho G) - \text{tr}(\rho G^*))\rho^{1/2}v. \]
Therefore, we obtain (11). Right multiplying this equation by \(\rho^{1/2}\) we have \(G'\rho = \rho^{1/2}G^*\rho^{1/2} + (\text{tr}(\rho G) - \text{tr}(\rho G^*))\rho\), and, taking the trace,
\[ \text{tr}(\rho G) - \text{tr}(\rho G^*) = \text{tr}(G'\rho) - \text{tr}(\rho^{1/2}G^*\rho^{1/2}) \]
\[ = \text{tr}(G'\rho) - \text{tr}(G^*\rho) = -(\text{tr}(\rho G) - \text{tr}(\rho G^*)); \]
this proves the last claim. \(\square\)

We can now prove as in [11] Th. 7.2 p. 358 the following

**Theorem 12** For all special GKSL representation of \(\mathcal{L}\) by means of operators \(G, L_{\ell}\) as in (7) there exists a special GKSL representation of \(\mathcal{L}'\) by means of operators \(G', L'_{\ell}\) such that:
1. \(G'\rho^{1/2} = \rho^{1/2}G^* + ic\rho^{1/2} \) for some \(c \in \mathbb{R}\),
2. \(L'_{\ell}\rho^{1/2} = \rho^{1/2}L'^*_\ell\).

**Proof.** Since \(\mathcal{L}'\) is bounded, it admits a special GKSL representation \(\mathcal{L}'(a) = G'^*a + \sum_k L'^*_k aL'_k + aG'\). Moreover, by Proposition (11) we have \(G'\rho^{1/2} = \rho^{1/2}G^* + ic, c \in \mathbb{R}\), and so (8) implies
\[ \sum_k \rho^{1/2}L'^*_k xL'_k\rho^{1/2} = \sum_k L_k\rho^{1/2}x\rho^{1/2}L^*_k. \] (12)
Let \(k\) (resp. \(k'\)) be the multiplicity space of the completely positive part of \(\mathcal{L}\) (resp. \(\mathcal{L}'\)) and define an operator \(X : h \otimes k' \rightarrow h \otimes k\)
\[ X(x \otimes \mathbb{1}_{k'})L'\rho^{1/2}u = (x \otimes \mathbb{1}_k)(\rho^{1/2} \otimes \mathbb{1}_k)L'\rho^{1/2}u \]
for all \(x \in \mathcal{B}(h)\) and \(u \in h\), where \(L : h \rightarrow h \otimes k\), \(Lu = \sum_k L_ku \otimes f_k\), \(L' : h \rightarrow h \otimes k'\), \(L'u = \sum_k L'_k u \otimes f'_k\) \((f_k)_k\) and \((f'_k)_k\) are orthonormal bases of \(k\) and \(k'\) respectively. Thus, by (12),
\[ \langle X(x \otimes \mathbb{1}_{k'})L'\rho^{1/2}u, X(y \otimes \mathbb{1}_{k'})L'\rho^{1/2}v \rangle = \sum_k \langle u, \rho^{1/2}L'^*_k x^*yL'_k\rho^{1/2}v \rangle \]
\[ = \langle (x \otimes \mathbb{1}_{k'})L'\rho^{1/2}u, (y \otimes \mathbb{1}_{k'})L'\rho^{1/2}v \rangle \]
for all \( x, y \in B(h) \) and \( u, v \in h \), i.e. \( X \) preserves the scalar product. Therefore, since the set \( \{(x \otimes \mathbb{1}_k')L'\rho^{1/2}u \mid x \in B(h), u \in h\} \) is total in \( h \otimes k' \) (for \( \rho^{1/2}(h) \) is dense in \( h \) and Theorem 9 holds), we can extend \( X \) to an unitary operator from \( h \otimes k' \) to \( h \otimes k \). As a consequence we have \( X^*X = \mathbb{1}_{h \otimes k'} \).

Moreover, since \( X(y \otimes \mathbb{1}_k') = (y \otimes \mathbb{1}_k)X \) for all \( y \in B(h) \), we can conclude that \( X = \mathbb{1}_{h} \otimes Y \) for some unitary map \( Y : k' \to k \).

The definition of \( X \) implies then
\[
(\rho^{1/2} \otimes \mathbb{1}_k)L^* = XL'\rho^{1/2} = (\mathbb{1}_h \otimes Y)L'\rho^{1/2}.
\]

This means that, replacing \( L' \) by \( (\mathbb{1}_h \otimes Y)L' \), or more precisely \( L'_k \) by \( \sum u_{kj}L'_j \) for all \( k \), we have
\[
\rho^{1/2}L'_k = L'_k\rho^{1/2}.
\]

Since \( \text{tr}(\rho L'_k) = \text{tr}(\rho L'^*_k) = 0 \) and, from \( L'(\mathbb{1}) = 0, G'' + G' = -\sum k L'^*_k L'_k \), the properties of a special GKSL representation follow.

**Remark 3** Condition 2 implies that the completely positive parts \( \Phi(x) = \sum \ell L'_\ell x L_\ell \) and \( \Phi' \) of the generators \( L \) and \( L' \), respectively are mutually adjoint i.e.
\[
\text{tr}(\rho^{1/2}\Phi'(x)\rho^{1/2}y) = \text{tr}(\rho^{1/2}x\rho^{1/2}\Phi(y)) \tag{13}
\]
for all \( x, y \in B(h) \). As a consequence, also the maps \( x \to G^*x + xG \) and \( x \to (G')^*x + xG' \) are mutually adjoint.

### 4 Generators of Standard Detailed Balance QMSs

In this section we characterise the generators of norm-continuous QMSs satisfying the SQDB of Definition 4.

We start noting that, since \( \rho \) is invariant for \( T \) and \( T' \), i.e. \( \mathcal{L}_s(\rho) = \mathcal{L}'_s(\rho) = 0 \), the operator \( K \) commutes with \( \rho \). Moreover, by comparing two special GKSL representations of \( L \) and \( L' + 2i[K, \cdot] \), we have immediately the following

**Lemma 13** A QMS \( T \) satisfies the SQDB \( L - L' = 2i[K, \cdot] \) if and only if for all special GKSL representations of the generators \( L \) and \( L' \) by means of operators \( G, L_k \) and \( G', L'_k \), respectively, we have
\[
G = G' + 2iK + ic \quad L'_k = \sum u_{kj}L_j
\]
for some \( c \in \mathbb{R} \) and some unitary \((u_{kj})_{kj}\) on \( k \).
Since we know the relationship between the operators \( G', L'_k \) and \( G, L_k \) thanks to Theorem 12, we can now characterise generators of QMSs satisfying the SQDB. We emphasize the following definition of \( T \)-symmetric matrix (operator) on \( k \) in order to avoid confusion with the usual notion of symmetric operator \( X \) meaning that \( X^* \) is an extension of \( X \).

**Definition 14** Let \( Y = (y_{k\ell})_{k,\ell \geq 1} \) be a matrix with entries indexed by \( k, \ell \) running on the set (finite or infinite) of indices of the sequence \( (L_\ell)_{\ell \geq 1} \). We denote by \( Y^T \) the transpose matrix \( Y^T = (y_{\ell k})_{k,\ell \geq 1} \). The matrix \( Y \) is called \( T \)-symmetric if \( Y = Y^T \).

**Theorem 15** \( T \) satisfies the SQDB if and only if for all special GKSL representation of the generator \( L \) by means of operators \( G, L_k \) there exists a \( T \)-symmetric unitary \( (u_{m\ell})_{m\ell} \) on \( k \) such that

\[
\rho^{1/2} L_k^* = \sum_{\ell} u_{k\ell} L_\ell \rho^{1/2}, \tag{14}
\]

for all \( k \geq 1 \).

**Proof.** Given a special GKSL of \( L \), Theorem 12 allows us to write the dual \( L' \) in a special GKSL representation by means of operators \( G', L'_k \) with

\[
G' \rho^{1/2} = \rho^{1/2} G^*, \quad L'_k \rho^{1/2} = \rho^{1/2} L_k^* \tag{15}
\]

Suppose first that \( T \) satisfies the SQDB. Since \( L'_k = \sum_j u_{kj} L_j \) for some unitary \( (u_{kj})_{kj} \) by Lemma 13 we can find (14) substituting \( L'_k \) with \( \sum_j u_{kj} L_j \) in the second formula (15).

Finally we show that the unitary matrix \( u = (u_{m\ell})_{m\ell} \) is \( T \)-symmetric. Indeed, taking the adjoint of (14) we find \( L_\ell \rho^{1/2} = \sum_m \tilde{u}_{\ell m} \rho^{1/2} L^*_m \). Writing \( \rho^{1/2} L_m^* \) as in (14) we have then

\[
L_\ell \rho^{1/2} = \sum_{m,k} \tilde{u}_{\ell m} u_{mk} L_k \rho^{1/2} = \sum_k (u^*)^T \tilde{u}_{\ell k} L_k \rho^{1/2}.
\]

The operators \( L_\ell \rho^{1/2} \) are linearly independent by property (iii) Theorem 8 of a special GKSL representation, therefore \( (u^*)^T u \) is the identity operator on \( k \). Since \( u \) is also unitary, we have also \( u^* u = (u^*)^T u \), namely \( u^* = (u^*)^T \) and \( u = u^T \).

Conversely, if (14) holds, by (15), we have \( L'_k \rho^{1/2} = \sum_\ell u_{k\ell} L_\ell \rho^{1/2} \), so that \( L'_k = \sum_\ell u_{k\ell} L_\ell \) for all \( k \) and for some unitary \( (u_{kj})_{kj} \). Therefore, thanks to Lemma 13 to conclude it is enough to prove that \( G = G' + i(2K + c) \) namely, that \( G - G' \) is anti self-adjoint.
To this end note that, since \( \rho \) is an invariant state, we have

\[
0 = \rho G^* + \sum_k L_k \rho L_k^* + G \rho,
\]

with

\[
\sum_k L_k \rho L_k^* = \sum_k (L_k \rho^{1/2})(\rho^{1/2} L_k^*) = \sum_k \sum_{i,j} \mathcal{A}_{k\ell} u_{kj} \rho^{1/2} L_\ell^* L_j \rho^{1/2} = \sum \rho^{1/2} L_\ell^* L_\ell \rho^{1/2} = -\rho^{1/2} (G + G^*) \rho^{1/2},
\]

(for condition (14) holds) and so, by substituting in equation (16) we get

\[
0 = \rho G^* - \rho^{1/2} G \rho^{1/2} - \rho^{1/2} G^* \rho^{1/2} + G \rho = \rho^{1/2} (G^* - G \rho^{1/2})
\]

\[
- (\rho^{1/2} G^* - G \rho^{1/2}) \rho^{1/2} = [G \rho^{1/2} - \rho^{1/2} G^*, \rho^{1/2}],
\]

i.e. \( G \rho^{1/2} - \rho^{1/2} G^* \) commutes with \( \rho^{1/2} \).

We can now prove that \( G - G' \) is anti-self-adjoint. Clearly, it suffices to show that \( \rho^{1/2} G \rho^{1/2} - \rho^{1/2} G' \rho^{1/2} \) is anti-self-adjoint. Indeed, by (15), we have

\[
(\rho^{1/2} G \rho^{1/2} - \rho^{1/2} G' \rho^{1/2})^* = (\rho^{1/2} G^* - \rho^{1/2} G')^* = (\rho^{1/2} (G \rho^{1/2} - \rho^{1/2} G^*))^* = ((G \rho^{1/2} - \rho^{1/2} G^*) \rho^{1/2})^* = \rho G^* - \rho^{1/2} G \rho^{1/2} = \rho^{1/2} G' \rho^{1/2} - \rho^{1/2} G \rho^{1/2}
\]

because \( G \rho^{1/2} - \rho^{1/2} G^* \) commutes with \( \rho^{1/2} \). This completes the proof. \( \Box \)

It is worth noticing that, as in Remark 3, \( T \) satisfies the SQDB if and only if the completely positive part \( \Phi \) of the generator \( \mathcal{L} \) is symmetric. This improves our previous result, Thm. 7.3 [11], where we gave \( G \rho^{1/2} = \rho^{1/2} G^* - (2iK + ic) \rho^{1/2} \) for some \( c \in \mathbb{R} \) as an additional condition. Here we showed that it follows from (14) and the invariance of \( \rho \).

**Remark 4** Note that (14) holds for the operators \( L_\ell \) of a special GKSL representation of \( \mathcal{L} \) if and only if it is true for all special GKSL representations because of the second part of Theorem 8. Therefore the conclusion of Theorem 15 holds true also if and only if we can find a single special GKSL representation of \( \mathcal{L} \) satisfying (14).

The \( T \)-symmetric unitary \((u_{m\ell})_{m\ell}\) is determined by the \( L_\ell \)'s because they are linearly independent. We shall now exploit this fact to give a more geometrical characterisation of SQDB.
When the SQDB holds, the matrices \((b_{kj})_{k,j \geq 1}\) and \((c_{kj})_{k,j \geq 1}\) with
\[ b_{kj} = \text{tr} \left( \rho^{1/2} L_{k}^{*} \rho^{1/2} L_{j}^{*} \right), \quad \text{and} \quad c_{kj} = \text{tr} \left( \rho L_{k} L_{j} \right) \] (17)
define two trace class operators \(B\) and \(C\) on \(k\) by Lemma 26 (see the Appendix); \(B\) is \(T\)-symmetric and \(C\) is self-adjoint. Moreover, it admits a self-adjoint inverse \(C^{-1}\) because \(\rho\) is faithful. When \(k\) is infinite dimensional, \(C^{-1}\) is unbounded and its domain coincides with the range of \(C\).

We can now give the following characterisation of QMS satisfying the SQDB condition which is more direct because the unitary \((u_{k\ell})_{k\ell}\) in Theorem 15 is explicitly given by \(C^{-1}B\).

**Theorem 16** \(T\) satisfies the SQDB if and only if the operators \(G, L_k\) of a special GKSL representation of the generator \(L\) satisfy the following conditions:

(i) the closed linear span of \(\{\rho^{1/2} L_{\ell}^{*} | \ell \geq 1\}\) and \(\{L_{\ell} \rho^{1/2} | \ell \geq 1\}\) in the Hilbert space of Hilbert-Schmidt operators on \(h\) coincide,

(ii) the trace-class operators \(B, C\) defined by (17) satisfy \(CB = BC^{T}\) and \(C^{-1}B\) is unitary \(T\)-symmetric.

**Proof.** If \(T\) satisfies the SQDB then, by Theorem 15 the identity (14) holds. The series in the right-hand side of (14) is convergent with respect to the Hilbert-Schmidt norm because
\[
\left\| \sum_{m+1 \leq \ell \leq n} u_{k\ell} L_{\ell} \rho^{1/2} \right\|_{HS}^{2} = \sum_{m+1 \leq \ell, \ell' \leq n} \bar{u}_{k\ell'} u_{k\ell} \text{tr} \left( \rho L_{\ell'}^{*} L_{\ell} \right) \\
\leq \frac{1}{2} \sum_{m+1 \leq \ell, \ell' \leq n} |u_{k\ell'}|^2 |u_{k\ell}|^2 + \frac{1}{2} \sum_{m+1 \leq \ell, \ell' \leq n} |c_{\ell'}|^2 \\
\leq \frac{1}{2} \left( \sum_{m+1 \leq \ell \leq n} |u_{k\ell}|^2 \right)^2 + \frac{1}{2} \sum_{m+1 \leq \ell, \ell' \leq n} |c_{\ell'}|^2
\]
and the right-hand side vanishes as \(n, m\) go to infinity because the operator \(C\) is trace-class by Lemma 26 and the columns of \(U = (u_{k\ell})_{k\ell}\) are unit vectors in \(k\) by unitarity.

Left multiplying both sides of (14) by \(\rho^{1/2} L_{j}^{*}\) and taking the trace we find \(B = CU^{T} = CU\). It follows that the range of the operators \(B, CU\) and \(C\) coincide and \(C^{-1}B = U\) is everywhere defined, unitary and \(T\)-symmetric.
because $U$ is $T$-symmetric. Moreover, since $B$ is $T$-symmetric by the cyclic property of the trace, we have also

$$BC^T = CU^TC^T = C(CU)^T = CB^T = CB.$$  

Conversely, we show that (i) and (ii) imply the SQDB. To this end notice that, by the spectral theorem we can find a unitary linear transformation $V = (v_{nm})_{m,n \geq 1}$ on $k$ such that $V^*CV$ is diagonal. Therefore, choosing a new GKSL of the generator $L$ by means of the operators $L''_k = \sum_{n \geq 1} v_{nk} L_n$, if necessary, we can suppose that both $(L_\ell \rho^1/2)_{\ell \geq 1}$ and $(\rho^{1/2} L^*_k)_{k \geq 1}$ are orthogonal bases of the same closed linear space. Note that

$$\text{tr}(\rho^{1/2}(L'')_k \rho^{1/2}(L'')_j^*) = \sum_{m,n \geq 1} \tilde{v}_{nk}\tilde{v}_{mj} \text{tr}(\rho^{1/2} L_n^* \rho^{1/2} L_m^*)$$

and the operator $B$, after this change of GKSL representation, becomes $V^*B(V^*)^T$ which is also $T$-symmetric.

Writing the expansion of $\rho^{1/2} L^*_k$ with respect to the orthogonal basis $(L_\ell \rho^{1/2})_{\ell \geq 1}$, for all $k \geq 1$ we have

$$\rho^{1/2} L^*_k = \sum_{\ell \geq 1} \frac{\text{tr}(\rho^{1/2} L^*_\ell \rho^{1/2} L^*_k)}{\|L_\ell \rho^{1/2}\|_{HS}^2} L_\ell \rho^{1/2}. \quad (18)$$

In this way we find a matrix $Y$ of complex numbers $y_{k\ell}$ such that $\rho^{1/2} L^*_k = \sum_{\ell} y_{k\ell} L_\ell \rho^{1/2}$ and the series is Hilbert-Schmidt norm convergent. Clearly, since $C$ is diagonal and $B$ is $T$-symmetric, $y_{k\ell} = (BC^{-1})_{k\ell} = ((B(C^{-1})^T)_{k\ell} = ((C^{-1}B)^T)_{k\ell}$. It follows from (ii) that $Y$ coincides with the unitary operator $(C^{-1}B)^T$ and (14) holds. Moreover, $Y$ is symmetric because

$$y_{k\ell} = (BC^{-1})_{k\ell} = ((B(C^{-1})^T)_{k\ell} = (C^{-1}B)_{k\ell} = y_{k\ell}.$$  

This completes the proof. \qed

Corollary 17 Suppose that a QMS $T$ satisfies the SQDB condition. For every special GKSL representation of $L$ with operators $L_\ell \rho^{1/2}$ that are orthogonal in the Hilbert space of Hilbert-Schmidt operators on $\mathcal{H}$ if $\text{tr}(\rho^{1/2} L^*_\ell \rho^{1/2} L^*_k) \neq 0$ for a pair of indices $k, \ell \geq 1$, then $\text{tr}(\rho L^*_\ell L_\ell) = \text{tr}(\rho L^*_k L_k)$.

Proof. It suffices to note that the matrix $(u_{k\ell})$ with entries

$$u_{k\ell} = \frac{\text{tr}(\rho^{1/2} L^*_\ell \rho^{1/2} L^*_k)}{\|L_\ell \rho^{1/2}\|_{HS}^2} = \frac{\text{tr}(\rho^{1/2} L^*_\ell \rho^{1/2} L^*_k)}{\text{tr}(\rho L^*_\ell L_\ell)}$$

must be $T$-symmetric. \qed
Remark 5 The matrix $C$ can be viewed as the covariance matrix of the zero-mean (recall that $\text{tr}(\rho L_\ell) = 0$) “random variables” $\{ L_\ell \mid \ell \geq 1 \}$ and in a similar way, $B$ can be viewed as a sort of mixed covariance matrix between the previous random variable and the adjoint $\{ L_\ell^* \mid \ell \geq 1 \}$. Thus the SQDB condition holds when the random variables $L_\ell$ right multiplied by $\rho^{1/2}$ and the adjoint variables $L_\ell^*$ left multiplied by $\rho^{1/2}$ generate the same subspace of Hilbert-Schmidt operators and the mixed covariance matrix $B$ is a left unitary transformation of the covariance matrix $C$.

If we consider a special GKSL representation of $\mathcal{L}$ with operators $L_\ell \rho^{1/2}$ that are orthogonal, then, by Corollary 17 and the identity $\| L_\ell \rho^{1/2} \|_{HS} = \| L_k \rho^{1/2} \|_{HS}$, the unitary matrix $U$ can be written as $C^{-1/2}BC^{-1/2}$. This, although not positive definite, can be interpreted as a common correlation coefficient matrix of $\{ L_\ell \mid \ell \geq 1 \}$ and $\{ L_\ell^* \mid \ell \geq 1 \}$.

The characterisation of generators of symmetric QMSs with respect to the $s = 1/2$ scalar product follows along the same lines.

Theorem 18 A norm-continuous QMS $\mathcal{T}$ is symmetric if and only if there exists a special GKSL representation of the generator $\mathcal{L}$ by means of operators $G, L_\ell$ such that

1. $G \rho^{1/2} = \rho^{1/2} G^* + ic \rho^{1/2}$ for some $c \in \mathbb{R}$,
2. $\rho^{1/2} L_k^* = \sum_\ell u_{k\ell} L_\ell \rho^{1/2}$, for all $k$, for some unitary $(u_{k\ell})_{k\ell}$ on $\mathcal{k}$ which is also $T$-symmetric.

Proof. Choose a special GKSL representation of $\mathcal{L}$ by means of operators $G, L_k$. Theorem 12 allows us to write the symmetric dual $\mathcal{L}'$ in a special GKSL representation by means of operators $G', L_k'$ as in (15).

Suppose first that $\mathcal{T}$ is KMS-symmetric. Comparing the special GKSL representations of $\mathcal{L}$ and $\mathcal{L}'$, by Theorem $5$ we find

$$G = G' + ic, \quad L_k^* = \sum_j u_{kj} L_j,$$

for some unitary matrix $(u_{kj})$ and some $c \in \mathbb{R}$. This, together with (15) implies that conditions (1) and (2) hold.

Assume now that conditions (1) and (2) hold. Taking the adjoint of (2) we find immediately $L_k \rho^{1/2} = \sum_k u_{k\ell} \rho^{1/2} L_\ell^*$. Then straightforward computation,
by the unitarity of the matrix \((u_{k\ell})\), yields

\[
\mathcal{L}_s(\rho^{1/2} x \rho^{1/2}) = G \rho^{1/2} x \rho^{1/2} + \sum_k L_k \rho^{1/2} x \rho^{1/2} L_k^* + \rho^{1/2} x \rho^{1/2} G^*
\]

\[
= \rho^{1/2} G^* x \rho^{1/2} + \sum_{\ell kj} u_{k\ell} u_{kj} \rho^{1/2} L_k^* x L_j \rho^{1/2} + \rho^{1/2} x G \rho^{1/2}
\]

\[
= \rho^{1/2} \mathcal{L}(x) \rho^{1/2}
\]

for all \(x \in \mathcal{B}(\mathcal{H})\). Iterating we find \(\mathcal{L}_n^a(\rho^{1/2} x \rho^{1/2}) = \rho^{1/2} \mathcal{L}_n^a(x) \rho^{1/2}\) for all \(n \geq 0\), therefore, exponentiating, we find \(\mathcal{T}_t(\rho^{1/2} x \rho^{1/2}) = \rho^{1/2} \mathcal{T}_t(x) \rho^{1/2}\) for all \(t \geq 0\). This, together with (3), implies that \(\mathcal{T}\) is KMS-symmetric. \(\square\)

**Remark 6** Note that condition (2) in Theorem 18 implies that the completely positive part of \(\mathcal{L}\) is KMS-symmetric. This makes a parallel with Theorem 12 where condition (2) implies that the completely positive parts of the generators \(\mathcal{L}\) and \(\mathcal{L}'\) are mutually adjoint.

The above theorem simplifies a previous result by Park ([23] Thm 2.2) where conditions (1) and (2) appear in a much more complicated way.

## 5 Generators of Standard Detailed Balance (with time reversal) QMSs

We shall now study generators of semigroups satisfying the SQDB-\(\theta\) introduced in Definition 5 involving the time reversal operation. In this section, we always assume that the invariant state \(\rho\) and the anti-unitary time reversal \(\theta\) commute.

The relationship between the QMS satisfying the SQDB-\(\theta\), its dual and their generators is clarified by the following

**Proposition 19** A QMS \(\mathcal{T}\) satisfies the SQDB-\(\theta\) if and only if the dual semigroup \(\mathcal{T}'\) is given by

\[
\mathcal{T}'_t(x) = \theta \mathcal{T}_t(\theta x \theta) \theta \quad \text{for all } x \in \mathcal{B}(\mathcal{H}).
\]  

(19)

In particular, if \(\mathcal{T}\) is norm-continuous, then \(\mathcal{T}'\) is also norm-continuous. Moreover, in this case \(\mathcal{T}'\) is generated by

\[
\mathcal{L}'(x) = \theta \mathcal{L}(\theta x \theta) \theta, \quad x \in \mathcal{B}(\mathcal{H}).
\]  

(20)
Proof. Suppose that $T$ satisfies the SQDB-$\theta$ and put $\sigma(x) = \theta x \theta$. Taking $t = 0$ equation (6) reduces to $\text{tr}(\rho^{1/2}x\rho^{1/2}y) = \text{tr}(\rho^{1/2}\sigma(y^*)\rho^{1/2}\sigma(x^*))$ for all $x, y \in \mathcal{B}(\mathcal{H})$, so that

$$
\text{tr}(\rho^{1/2}x\rho^{1/2}T_t(y)) = \text{tr}(\rho^{1/2}\sigma(y^*)\rho^{1/2}T_t(\sigma(x^*)))
= \text{tr}(\rho^{1/2}\sigma(T_t(\sigma(x^*))\rho^{1/2}\sigma(\sigma(y^*)))
= \text{tr}(\rho^{1/2}\sigma(T_t(\sigma(x)))\rho^{1/2}y)
$$

for every $x, y \in \mathcal{B}(\mathcal{H})$ and (19) follows. Therefore, if $T$ is norm continuous, $T'_t = (\sigma \circ T_t \circ \sigma)_t$ is also.

Conversely, if (19) holds, the commutation between $\rho$ and $\theta$ implies

$$
\text{tr}(\rho^{1/2}T_t'(x)\rho^{1/2}y) = \text{tr}(\rho^{1/2}\theta T_t(\theta x \theta)\theta \rho^{1/2}y)
= \text{tr}(\theta (\rho^{1/2}T_t(\theta x \theta)\theta \rho^{1/2}y) \theta)
= \text{tr}(\rho^{1/2}\theta y^* \rho^{1/2}\theta T_t(\theta x \theta))
$$

and (19) is proved. Now (20) follows from (19) differentiating at $t = 0$. \qed

We can now describe the relationship between special GKSL representations of $\mathcal{L}$ and $\mathcal{L}'$.

**Proposition 20** If $T$ satisfies the SQDB-$\theta$ then, for every special GKSL representation of $\mathcal{L}$ by means of operators $H, L_k$, the operators $H' = -\theta H \theta$ and $L'_k = \theta L_k \theta$ yield a special GKSL representation of $\mathcal{L}'$.

**Proof.** Consider a special GKSL representation of $\mathcal{L}$ by means of operators $H, L_k$. Since $\mathcal{L}'(a) = \theta \mathcal{L}(\theta a \theta) \theta$ by Proposition [19] from the antilinearity of $\theta$ and $\theta^2 = 1$ we get

$$
\theta \mathcal{L}'(a) \theta = i[H, \theta a] - \frac{1}{2} \sum_k (L_k^*L_k \theta a \theta - 2L_k^* \theta a \theta L_k + \theta a \theta L_k^*L_k)
= i\theta (\theta H \theta a - a \theta H \theta) \theta + \sum_k \theta ((\theta L_k^* \theta) a (\theta L_k \theta)) \theta
- \frac{1}{2} \sum_k \theta ((\theta L_k^* \theta)(\theta L_k \theta) a + a (\theta L_k^* \theta)(\theta L_k \theta)) \theta
= \theta (-i[\theta H \theta, a]) \theta - \frac{1}{2} \sum_k \theta (L_k^* L_k' a - 2L_k^* a L_k' + a L_k^* L_k') \theta,
$$

where $L_k' := \theta L_k \theta$. Therefore, putting $H' = -\theta H \theta$, we find a GKSL representation of $\mathcal{L}'$ which is also special because $\text{tr}(\rho L_k') = \text{tr}(\theta \rho L_k \theta) = \text{tr}(L_k^* \rho) = \text{tr}(\rho L_k) = 0$. \qed

The structure of generators of QMSs satisfying the SQDB-$\theta$ is described by the following...
Theorem 21 A QMS $T$ satisfies the SQDB-$\theta$ condition if and only if there exists a special GKSL representation of $L$, with operators $G, L$, such that:

1. $\rho^{1/2}G^{*}\theta = G\rho^{1/2}$,
2. $\rho^{1/2}L^{*}\theta = \sum_{j} u_{kj}L_j\rho^{1/2}$ for a self-adjoint unitary $(u_{kj})_{kj}$ on $k$.

Proof. Suppose that $T$ satisfies the SQDB-$\theta$ condition and consider a special GKSL representation of the generator $L$ with operators $G, L$. The operators $-\theta H \theta$ and $\theta L \theta$ give then a special GKSL representation of $L'$ by Proposition 20. Moreover, by Theorem 12, we have another special GKSL representation of $L'$ by means of operators $G', L'$ such that $G'\rho^{1/2} = \rho^{1/2}G^* + ic\rho^{1/2}$ for some $c \in \mathbb{R}$, and $L'\rho^{1/2} = \rho^{1/2}L_k$. Therefore there exists a unitary $(v_{kj})_{kj}$ on $k$ such that $L'_{k} = \sum_{j} v_{kj}L_{j}\theta$, and $\rho^{1/2}L'_{k} = \sum_{j} u_{kj}L_{j}\theta\rho^{1/2}$. Condition 2 follows then with $u_{kj} = \overline{v}_{kj}$ left and right multiplying by the antiunitary $\theta$.

In order to find condition 1, first notice that by the unitarity of $(v_{kj})_{kj}$

$$\sum_{k} L^*_k L_k = \sum_{k} \theta L^*_k L_k \theta. \quad (21)$$

Now, by the uniqueness of $G'$ up to a purely imaginary multiple of the identity in a special GKSL representation, $H' = (G'^* - G')/(2i)$ is equal to $-\theta H \theta + c_1$ for some $c_1 \in \mathbb{R}$. From (21) and $G'\rho^{1/2} = \rho^{1/2}G^* + ic\rho^{1/2}$ we obtain then

$$\rho^{1/2}G^* + ic\rho^{1/2} = G'\rho^{1/2} = -iH'\rho^{1/2} - \frac{1}{2} \sum_{k} L^*_k L_k \rho^{1/2}$$

$$= i\theta H \theta \rho^{1/2} + ic_1\rho^{1/2} - \frac{1}{2} \sum_{k} \theta L^*_k L_k \theta \rho^{1/2}$$

$$= \theta G \theta \rho^{1/2} + ic_1\rho^{1/2}.$$ 

It follows that $\rho^{1/2}G^* \theta = G\rho^{1/2} + ic_2\rho^{1/2}$ for some $c_2 \in \mathbb{R}$. Left multiplying by $\rho^{1/2}$ and tracing we find

$$c_2 = \text{tr}(\theta \rho G^* \theta) - \text{tr}(\rho G) = \text{tr}(G \rho) - \text{tr}(\rho G) = 0$$

and condition 1 holds.

Finally we show that the square of the unitary $(u_{kj})_{kj}$ on $k$ is the identity operator. Indeed, taking the adjoint of the identity $\rho^{1/2}G^* \theta = \sum_{j} u_{kj}L_j\rho^{1/2}$, we have

$$\rho^{1/2}L\rho^{1/2} = \sum_{j} \bar{u}_{kj}\rho^{1/2}L^*_j.$$
Left and right multiplying by the antilinear time reversal $\theta$ (commuting with $\rho$) we find

$$L_k \rho^{1/2} = \sum_j \theta \bar{u}_{kj} \rho^{1/2} L_j^* \theta = \sum_j u_{kj} \rho^{1/2} L_j^* \theta$$

Writing $\rho^{1/2} L_j^* \theta$ as $\sum_m u_{jm} L_m \rho^{1/2}$ by condition 2 we have then

$$L_k \rho^{1/2} = \sum_{j,m} u_{kj} u_{jm} L_m \rho^{1/2} = \sum_m (u^2)_{km} L_m \rho^{1/2}$$

which implies that $u^2 = 1$ by the linear independence of the $L_m \rho^{1/2}$. Therefore, since $u$ is unitary, $u = u^*$. Conversely, if 1 and 2 hold, we can write $\frac{\rho}{\theta} L_k \rho^{1/2}$ as

$$\frac{\rho}{\theta} L_k \rho^{1/2} = \sum_{j,m} u_{kj} u_{jm} L_m \rho^{1/2} = \sum_{j} L_j \rho^{1/2} L_j^* \rho^{1/2} G \rho^{1/2} = \sum_{j} L_j \rho^{1/2} L_j^* \rho^{1/2}$$

This, by Theorem 12, can be written as

$$G \rho^{1/2} x \rho^{1/2} + \sum_{j} L_j \rho^{1/2} L_j^* \rho^{1/2} = \rho^{1/2} L'(x) \rho^{1/2}$$

It follows that $\theta L(\theta x \theta) \theta = L'(x)$ for all $x \in B(h)$ because $\rho$ is faithful. Moreover, it is easy to check by induction that $\theta L^n(\theta x \theta) \theta = (L')^n(x)$ for all $n \geq 0$. Therefore $\theta T_t(\theta x \theta) \theta = T'_t(x)$ for all $t \geq 0$ and $T$ satisfies the SQDB-\theta condition by Proposition 19.

We now provide a geometrical characterisation of the SQDB-\theta condition as in Theorem 16. To this end we introduce the trace class operator $R$ on $k$

$$R_{jk} = \text{tr} \left( \rho^{1/2} L_j^* \rho^{1/2} L_k^* \right)$$

A direct application of Lemma 26 shows that $R$ is trace class. Moreover it is self-adjoint because, by the property $\text{tr}(\theta x \theta) = \text{tr}(x^*)$ of the antilinear time reversal, we have

$$R_{jk} = \text{tr} \left( \rho^{1/2} L_j^* \rho^{1/2} L_k^* \right) = \text{tr} \left( \rho^{1/2} L_j \rho^{1/2} L_k^* \theta \right) = \text{tr} \left( \rho^{1/2} L_j^* \rho^{1/2} L_k \theta \right) = \text{tr} \left( \rho^{1/2} L_j \rho^{1/2} L_k^* \right) = \text{tr} \left( \rho \left( \rho^{1/2} \theta L_j^* \theta \right) \right) = R_{kj}.$$
Theorem 22. $T$ satisfies the SQDB-$\theta$ if and only if the operators $G, L_k$ of a special GKSL of the generator $L$ fulfill the following conditions:

1. $\rho^{1/2}G^*\theta = G\rho^{1/2}$,

2. the closed linear span of $\{\rho^{1/2}\theta L^*_\ell \theta \mid \ell \geq 1\}$ and $\{L_\ell \rho^{1/2} \mid \ell \geq 1\}$ in the Hilbert space of Hilbert-Schmidt operators on $h$ coincide,

3. the self-adjoint trace class operators $R, C$ defined by (17) and (22) commute and $C^{-1}R$ is unitary and self-adjoint.

Proof. It suffices to show that conditions 2 and 3 above are equivalent to condition 2 of Theorem 21.

If $T$ satisfies the SQBD-$\theta$, then it can be shown as in the proof of Theorem 16 that 2 follows from condition 2 of Theorem 21. Moreover, left multiplying by $\rho^{1/2} L^*_\ell$ the identity $\rho^{1/2} \theta L^*_k \theta = \sum_{j} u_{kj} L_j \rho^{1/2}$ and tracing, we find

$$\text{tr} \left( \rho^{1/2} L^*_\ell \rho^{1/2} \theta L^*_k \theta \right) = \sum_{j} u_{kj} \text{tr} \left( \rho L^*_j L_j \right)$$

for all $k, \ell$ i.e. $R = CU^T$. The operator $U^T$ is also self-adjoint and unitary. Therefore $R$ and $C$ have the same range and, since the domain of $C^{-1}$ coincides with the range of $C$, the operator $C^{-1}R$ is everywhere defined, unitary and self-adjoint. It follows that the densely defined operator $RC^{-1}$ is a restriction of $(C^{-1}R)^* = C^{-1}R$ and $CR = RC$.

In order to prove, conversely, that 2 and 3 imply condition 2 of Theorem 21 we first notice that, by the spectral theorem there exists a unitary $V = (v_{mn})_{m,n \geq 1}$ on the multiplicity space $k$ such that $V^* CV$ is diagonal. Choosing a new GKSL representation of the generator $L$ by means of the operators $L''_k = \sum_{n \geq 1} v_{nk} L_n$, if necessary, we can suppose that both $(L_\ell \rho^{1/2})_{\ell \geq 1}$ and $(\rho^{1/2} L^*_k)_{k \geq 1}$ are orthogonal bases of the same closed linear space. Note that

$$\text{tr} \left( \rho^{1/2}(L'')^*_k \rho^{1/2} \theta (L'')^*_j \theta \right) = \sum_{m,n \geq 1} \bar{v}_{nk} v_{mj} \text{tr}(\rho^{1/2} L^*_m \rho^{1/2} \theta L^*_m \theta)$$

and the operator $R$, in the new GKSL representation, transforms into $V^* RV$ which is also self-adjoint.

Expanding $\rho^{1/2} \theta L^*_k \theta$ with respect to the orthogonal basis $(L_\ell \rho^{1/2})_{\ell \geq 1}$, for all $k \geq 1$, we have

$$\rho^{1/2} \theta L^*_k \theta = \sum_{\ell \geq 1} \frac{\text{tr}(\rho^{1/2} L^*_\ell \rho^{1/2} \theta L^*_k \theta)}{\|L_\ell \rho^{1/2}\|_{\text{HS}}^2} L^*_\ell \rho^{1/2} \quad (23)$$

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i.e. $\rho^{1/2} L_k^* = \sum_{\ell} y_{k\ell} L_\ell \rho^{1/2}$ with a matrix $Y$ of complex numbers $y_{k\ell}$.

Clearly, since $C$ is diagonal and commutes with $R$, we have $y_{k\ell} = (RC^{-1})_{k\ell} = (C^{-1}R)_{k\ell}$. It follows then from \( \text{[3]} \) above that $Y$ coincides with the unitary operator $C^{-1}R$ and condition \( \text{[2]} \) of Theorem \( \text{[21]} \) holds. Moreover, $Y$ is self-adjoint because both $R$ and $C$ are. \( \square \)

As an immediate consequence of the commutation of $R$ and $C$ we have the following parallel of Corollary \( \text{[17]} \) for the SQDB condition

\begin{corollary}
Suppose that a QMS $\mathcal{T}$ satisfies the SQDB-$\theta$ condition. For every special GKSL representation of $\mathcal{L}$ with operators $L_\ell \rho^{1/2}$ orthogonal as Hilbert-Schmidt operators on $h$ if $\text{tr}(\rho^{1/2} L^*_k \rho^{1/2} \theta L_k^* \theta) \neq 0$ for a pair of indices $k, \ell \geq 1$, then $\text{tr}(\rho L^*_k L_\ell) = \text{tr}(\rho L_k^* L_\ell)$.
\end{corollary}

When the time reversal $\theta$ is given by the conjugation $\theta u = \bar{u}$ (with respect to some orthonormal basis of $h$), $\theta x^* \theta$ is equal to the transpose $x^T$ of $x$ and we find the following

\begin{corollary}
$\mathcal{T}$ satisfies the SQDB-$\theta$ condition if and only if there exists a special GKSL representation of $\mathcal{L}$, with operators $G, L_k$, such that:

1. $\rho^{1/2} G^T = G \rho^{1/2}$;
2. $\rho^{1/2} L_k^T = \sum_j u_{kj} L_j \rho^{1/2}$ for some unitary self-adjoint $(u_{kj})_{kj}$.
\end{corollary}

\section{SQDB-$\theta$ for QMS on $M_2(\mathbb{C})$}

In this section, as an application, we find a standard form of a special GKSL representation of the generator $\mathcal{L}$ of a QMS on $M_2(\mathbb{C})$ satisfying the SQDB-$\theta$. The faithful invariant state $\rho$, in a suitable basis, can be written in the form

$$
\rho = \begin{pmatrix}
\nu & 0 \\
0 & 1-\nu
\end{pmatrix} = \frac{1}{2} \left( \sigma_0 + (2\nu - 1)\sigma_3 \right), \quad 0 < \nu < 1
$$

where $\sigma_0$ is the identity matrix and $\sigma_1, \sigma_2, \sigma_3$ are the Pauli matrices

$$
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
$$

The time reversal $\theta$ is the usual conjugation in the above basis.

In order to determine the structure of the operators $G$ and $L_k$ satisfying conditions of Corollary \( \text{[24]} \) we find first a convenient basis of $M_2(\mathbb{C})$. We
choose then a basis of eigenvectors of the linear map $X \rightarrow \rho^{1/2}X^T\rho^{-1/2}$ in $M_2(\mathbb{C})$ given by $\sigma_0, \sigma_1^\nu, \sigma_2^\nu, \sigma_3$ where

$$\sigma_1^\nu = \begin{pmatrix} 0 & \sqrt{2\nu} \\ \sqrt{2(1-\nu)} & 0 \end{pmatrix}, \quad \sigma_2^\nu = \begin{pmatrix} 0 & -i\sqrt{2\nu} \\ i\sqrt{2(1-\nu)} & 0 \end{pmatrix}. $$

Indeed, $\sigma_0, \sigma_1^\nu, \sigma_3$ (resp. $\sigma_2^\nu$) are eigenvectors of the eigenvalue 1 (resp. $-1$).

Every special GKSL representation of $L$ is given by (see [11] Lemma 6.1)

$$L_k = -(2\nu - 1)z_k\sigma_0 + z_k\sigma_1^\nu + z_k\sigma_2^\nu + z_k\sigma_3, \quad k \in \mathcal{J} \subseteq \{1, 2, 3\}$$

with vectors $z_k := (z_{k1}, z_{k2}, z_{k3}) \ (k \in \mathcal{J})$ linearly independent in $\mathbb{C}^3$.

The SQDB-\(\theta\) holds if and only if $G, L_k$ satisfy

(i) $G = \rho^{1/2}GT\rho^{-1/2},$

(ii) $L_k = \sum_{j \in \mathcal{J}} u_{kj}\rho^{1/2}L_j^T\rho^{-1/2}$ for some unitary self-adjoint $U = (u_{kj})_{k, j \in \mathcal{J}}$.

Now, if $\mathcal{J} \neq \emptyset$, since every unitary self-adjoint matrix is diagonalizable and its spectrum is contained in $\{-1, 1\}$, it follows that $U = W^*DW$ for some unitary matrix $W = (w_{ij})_{i, j \in \mathcal{J}}$ and some diagonal matrix $D$ of the form

$$\text{diag}(\epsilon_1, \ldots, \epsilon_{|\mathcal{J}|}), \quad \epsilon_i \in \{-1, 1\},$$

where $|\mathcal{J}|$ denotes the cardinality of $\mathcal{J}$. Therefore, replacing the $L_k$’s by operators $L_k' := \sum_{j \in \mathcal{J}} w_{kj}L_j$ if necessary, we can take $U$ of the form (24).

We now analyze the structure of $L_k$’s corresponding to the different (diagonal) forms of $U$. By condition (ii) we have either $L_k = \rho^{1/2}L_k^T\rho^{-1/2}$ or $L_k = -\rho^{1/2}L_k^T\rho^{-1/2}$; an easy calculation shows that

$$L_k = \rho^{1/2}L_k^T\rho^{-1/2} \quad \text{if and only if} \quad z_{k2} = 0$$

and

$$L_k = -\rho^{1/2}L_k^T\rho^{-1/2} \quad \text{if and only if} \quad z_{k1} = z_{k3} = 0.\quad (26)$$

Therefore, the linear independence of $\{z_j : j \in \mathcal{J}\}$ forces $U$ to have at most two eigenvalues equal to 1 and at most one equal to $-1$ and, with a suitable choice of a phase factor for each $L_k$, we can write

$$L_k = (1 - 2\nu)r_k\sigma_0 + r_k\sigma_3 + \zeta_k\sigma_1^\nu \quad \text{for} \quad k = 1, 2 \quad \text{and} \quad r_k \in \mathbb{R}, \zeta_k \in \mathbb{C} \quad (27)$$

$$L_3 = r_3\sigma_2^\nu \quad r_3 \in \mathbb{R}. \quad (28)$$

Clearly $L_1$ and $L_2$ are linearly independent if and only if $r_1\zeta_2 \neq r_2\zeta_1$. This, together with non triviality conditions leaves us, up to a change of indices, with the following possibilities:
multiplying
there exists a special GKS representation of $v$ as in (30) and
Theorem 25
On the other hand, when $J \neq H, L$
Since
is equal to the sum of a term depending only on $\sigma$
Therefore we have the following possible standard forms for $J$ in the case $k \neq 0$
Then statement (i) is equivalent to
$$\sqrt{\nu} g_{21} = \sqrt{1 - \nu} g_{12}. \quad (29)$$
Since $G = -iH - 2^{-1} \sum_k L_k^* L_k$ with $H = \sum_{j=1}^3 v_j \sigma_j, v_j \in \mathbb{R}$, and $\sum_k L_k^* L_k$
is equal to the sum of a term depending only on $\sigma_0$ and $\sigma_3$ plus
$$\sum_{k=1,2}^{2r_k} \left( 0 \; \zeta_k \sqrt{2\nu(1 - \nu) - \zeta_k \nu \sqrt{2(1 - \nu)}} \right)$$
in the case $J \neq \emptyset$ the identity (29) holds if and only if
$$\begin{align*}
v_1 (\sqrt{1 - \nu} - \sqrt{\nu}) &= -2\nu(1 - \nu) (\sqrt{1 - \nu} + \sqrt{\nu})^2 \sum_{k=1}^{2r_k} r_k \Im \zeta_k \\
v_2 (\sqrt{1 - \nu} + \sqrt{\nu}) &= -2\nu(1 - \nu) (\sqrt{1 - \nu} - \sqrt{\nu})^2 \sum_{k=1}^{2r_k} r_k \Re \zeta_k.
\end{align*} \quad (30)$$
On the other hand, when $J = \emptyset$, condition (29) is equivalent to $\sqrt{\nu}(v_1 + iv_2) = \sqrt{1 - \nu}(v_1 - iv_2)$, i.e.
$$v_1 (\sqrt{1 - \nu} - \sqrt{\nu}) = 0, \quad v_2 = 0 \quad (31)$$
Therefore we have the following possible standard forms for $L$.

**Theorem 25** Let $L_1, L_2, L_3$ be as in (27), (28), $H = \sum_{j=1}^3 v_j \sigma_j$ with $v_1, v_2$ as in (30) and $v_3 \in \mathbb{R}$. The QMS $T$ satisfies the SQDB-$\theta$ if and only if there exists a special GKS representation of $L$ given, up to phase factors multiplying $L_1, L_2, L_3$, in one of the following ways:

- $H$ with $v_1 = v_2 = 0$ if $\nu \neq 1/2$, and $v_1 \in \mathbb{R}, v_2 = 0$ if $\nu = 1/2$,

  a) $H, L_1$ with $r_1 \zeta_1 \neq 0$,

  b) $H, L_3$ with $r_3 \neq 0$,

  c) $H, L_1, L_2$ with $r_1 \zeta_1 r_2 \zeta_2 \neq 0$ and $r_1 \zeta_2 \neq r_2 \zeta_1$,

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d) $H, L_1, L_3$ with $r_3 \neq 0$ and $r_1 \zeta_1 \neq 0$,

e) $H, L_1, L_2, L_3$ with $r_1 \zeta_2 \neq r_2 \zeta_1$, $r_1 \zeta_1 r_2 \zeta_2 \neq 0$ and $r_3 \neq 0$.

Roughly speaking, the standard form of $L$ corresponds, up to degeneracies when some of the parameter vanish or when some linear dependence arises, to the case e).

We know that a QMS satisfying the usual (i.e. with pre-scalar product with $s = 0$) QDB-$\theta$ condition must commute with the modular group. Moreover, when this happens, the SQDB-$\theta$ and QDB-$\theta$ conditions are equivalent (see e.g. [6], [11]).

We finally show how the generators of a QMSs on $M_2(\mathbb{C})$ satisfying the usual QDB-$\theta$ condition can be recovered by a special choice of the parameters $r_1, r_2, r_3, \zeta_1, \zeta_2$ in Theorem 25 describing the generator of a QMS satisfying the SQDB-$\theta$ condition.

To this end, we recall that $T$ fulfills the QDB-$\theta$ when $\text{tr}(\rho x T(y)) = \text{tr}(\rho y^* T(\theta x^* \theta))$ for all $x, y \in B(h)$. In [11] we classified generators of QMS on $M_2(\mathbb{C})$ satisfying the QDB condition without time reversal (i.e., formally, replacing $\theta$ by the identity operator, that is, of course, not antiunitary). The same type of arguments show that, disregarding trivialisations that may occur when some of the parameters below vanish, QMSs on $M_2(\mathbb{C})$ satisfying the QDB-$\theta$ condition have the following standard form

$$
\mathcal{L}(x) = i[H, x] - \frac{|\eta|^2}{2} (L^2 x - 2LxL + xL^2) - \frac{|\lambda|^2}{2} (\sigma^- \sigma^+ - 2 \sigma^- \sigma^+ - 2 \sigma^+ \sigma^- + x \sigma^+ \sigma^-),
$$

where $H = h_0 \sigma_0 + h_3 \sigma_3$ ($h_0, h_3 \in \mathbb{R}$), $L = -(2\nu - 1) \sigma_0 + \sigma_3$, $\sigma^\pm = (\sigma_1 \pm i \sigma_2)/2$ and, changing phases if necessary, $\lambda, \mu, \eta$ can be chosen as non-negative real numbers satisfying

$$
\lambda^2 (1 - \nu) = \nu \mu^2. \quad (33)
$$

Choosing $r_1 = \eta, \zeta_1 = 0$ we find immediately that the operator $L$ in (32) coincides with the operator $L_1$ in (27). Moreover, choosing $r_2 = 0$ we find $v_2 = 0$ and also $v_1 = 0$ for $\nu \neq 1/2$. A straightforward computation yields

$$
\begin{pmatrix}
\lambda \sigma_+ \\
\mu \sigma_-
\end{pmatrix} = \begin{pmatrix}
\lambda/(2 \zeta_2 \sqrt{2 \nu}) & i \lambda/(2 r_3 \sqrt{2 \nu}) \\
\mu/(2 \zeta_2 \sqrt{2 (1 - \nu)}) & -i \mu/(2 r_3 \sqrt{2 (1 - \nu)})
\end{pmatrix} \begin{pmatrix}
L_2 \\
L_3
\end{pmatrix}
$$

and the above $2 \times 2$ matrix is unitary if we choose $\zeta_2 = \lambda/(2 \sqrt{\nu})$, $r_3 = i \mu/(2 \sqrt{1 - \nu})$ = $i \zeta_2$ because of (33) and changing the phase of $r_3$ in order to find a unitary that is also self-adjoint.
This shows that we can recover the standard form (32) choosing $H, L_1, L_2, L_3$ as in Theorem 25 e) with $r_1 = \eta, \zeta_1 = 0, r_2 = 0, \zeta_2 = \lambda/(2\sqrt{\nu}), r_3 = i\mu/(2\sqrt{1-\nu}), v_1 = v_2 = 0$.

Appendix

We denote by $\ell^2(J)$ denote the Hilbert space of complex-valued, square summable sequences indexed by a finite or countable set $J$.

Lemma 26 Let $J$ be a complex separable Hilbert space and let $(\xi_j)_{j \in J}, (\eta_j)_{j \in J}$ be two Hilbertian bases of $J$ satisfying $\sum_{j \in J} \|\xi_j\|^2 < \infty$, $\sum_{j \in J} \|\eta_j\|^2 < \infty$. The complex matrices $A = (a_{jk})_{j,k \in J}, B = (b_{jk})_{j,k \in J}, C = (c_{jk})_{j,k \in J}$ given by

$$a_{jk} = \langle \xi_j, \xi_k \rangle, \quad b_{jk} = \langle \xi_j, \eta_k \rangle, \quad c_{jk} = \langle \eta_j, \eta_k \rangle$$

define trace class operators on $\ell^2(J)$ satisfying $B^*A^{-1}B = C$. Moreover $A$ and $C$ are self-adjoint and positive.

Proof. Note that

$$\sum_{j,k \geq 1} |b_{jk}|^2 \leq \sum_{j,k \geq 1} \|\xi_j\|^2 \cdot \|\eta_k\|^2 = \sum_j \|\xi_j\|^2 \cdot \sum_k \|\eta_k\|^2 < \infty$$

Therefore $B$ defines a Hilbert-Schmidt operator on $\ell^2(J)$.

In a similar way $A$ and $C$ define Hilbert-Schmidt operators on $\ell^2(J)$ that are obviously self-adjoint. These are also positive because for any sequence $(z_m)_{m \in J}$ of complex numbers with $z_m \neq 0$ for a finite number of indices $m$ at most we have

$$\sum_{m,n \in J} \bar{z}_m a_{mn} z_n = \sum_{m,n \in J} \bar{z}_m \langle \xi_m, \xi_n \rangle z_n = \left( \sum_{m \in J} z_m \xi_m \right)^2 \geq 0.$$

Moreover, they are trace class because

$$\sum_{j \in J} a_{jj} = \sum_{j \in J} \|\xi_j\|^2 < \infty, \quad \sum_{j \in J} c_{jj} = \sum_{j \in J} \|\eta_j\|^2 < \infty.$$

Finally, we show that $B$ is also trace class. By the spectral theorem, we can find a unitary $V = (v_{kj})_{k,j \in J}$ on $\ell^2(J)$ such that $V^*AV$ is diagonal. The series $\sum_{m \in J} v_{mj} \xi_m$ is norm convergent because

$$\left\| \sum_m v_{mj} \xi_m \right\|^2 = \sum_{m,n \in J} \bar{v}_{nj} a_{nm} v_{mj} = (V^*AV)_{jj}.$$
The series $\sum_{m \in J} v_{mj} \xi_m$ is norm convergent as well for a similar reason. Therefore, putting $\xi'_j = \sum_{m \in J} v_{mj} \xi_m$ and $\eta'_j = \sum_{m \in J} v_{mj} \eta_m$ we find immediately $(V^*AV)_{kj} = \langle \xi'_k, \xi'_j \rangle = 0$ for $j \neq k$, $(V^*AV)_{jj} = \| \xi'_j \|^2$ and

$$(V^*BV)_{kj} = \sum_{m,n} \bar{v}_{kn} v_{mj} \langle \xi_m, \eta_j \rangle = \langle \xi'_k, \eta'_j \rangle,$$

$$(V^*CV)_{kj} = \sum_{m,n} \bar{v}_{kn} v_{mj} \langle \eta_m, \eta_j \rangle = \langle \eta'_k, \eta'_j \rangle.$$

As a consequence, the following identity

$$(V^*B^*A^{-1}BV)_{kj} = ((V^*B^*V)(V^*AV)^{-1}(V^*BV))_{kj} = \sum_{m \in J} (V^*B^*V)_{km} ((V^*AV)_{mm})^{-1} (V^*BV)_{mj}$$

$$= \sum_{m \in J} \left\langle \eta'_k, \frac{\xi'_m}{\| \xi'_m \|} \right\rangle \left\langle \frac{\xi'_m}{\| \xi'_m \|}, \eta'_j \right\rangle$$

$$= \langle \eta'_k, \eta'_j \rangle = (V^*CV)_{kj}$$

holds because $(\xi'_m/\| \xi'_m \|)_{m \in J}$ is an orthonormal basis of $J$.

This proves that $V^*B^*A^{-1}BV = V^*CV$ i.e. $B^*A^{-1}B = C$. It follows that $|A^{-1/2}B| = C^{1/2}$ is Hilbert-Schmidt as well as $A^{-1/2}B$ and $B = A^{1/2}(A^{-1/2}B)$ is trace class being the product of two Hilbert-Schmidt operators. □

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