Pushout of quasi-finite and flat group schemes over a Dedekind ring

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Abstract. Let $G$, $G_1$ and $G_2$ be quasi-finite and flat group schemes over a complete discrete valuation ring $R$, $\varphi_1 : G \to G_1$ any morphism of $R$-group schemes and $\varphi_2 : G \to G_2$ a model map. We construct the pushout $P$ of $G_1$ and $G_2$ over $G$ in the category of $R$-affine group schemes. In particular when $\varphi_1$ is a model map too we show that $P$ is still a model of the generic fibre of $G$. We also provide a short proof for the existence of cokernels and quotients of finite and flat group schemes over any Dedekind ring.

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1 Introduction

1.1 Aim and scope

We are interested in the construction of the pushout (whose definition will be recalled in §2) in the category of affine group schemes over a given ring as

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described hereafter. It is known that in the category of abstract groups the pushout of two groups over a third one always exists but it is not finite even when the three groups are all finite (unless one takes very particular cases). However for group schemes over a Dedekind ring $R$ something new happens when we consider some special important cases: so let $G$, $G_1$ and $G_2$ be $R$-affine group schemes and consider the diagram

\[
\begin{array}{ccc}
G & \xrightarrow{\phi_1} & G_1 \\
\downarrow & & \downarrow \\
G & \xrightarrow{\phi_2} & G_2
\end{array}
\]

where $\phi_i : G \to G_i$ ($i = 1, 2$) are $R$-group scheme morphisms. We first prove the following

**Theorem 1.1.** (Cf. Theorem 3.2) Assume $R$ is a complete discrete valuation ring and $G, G_1, G_2$ are finite and flat over $R$. Then if $\phi_1$ is a model map (i.e. generically an isomorphism) the pushout of (1) in the category of affine $R$-group schemes exists. Moreover it is finite and flat and its generic fibre is isomorphic to $G_{2,R}$, the generic fibre of $G_2$.

This immediately implies that when $G, G_1$ and $G_2$ are all models of a same $K$-group scheme $G_K$ ($K$ being the fraction field of $R$) then the pushout of (1) exists and is still a model of $G_K$ thus proving the existence of a lower bound for models of finite group schemes. This was already known in the commutative case (cf. [9], Proposition 2.2.2). The same will be true for the quasi-finite case under the assumption that $G_{2,K}$ admits a finite and flat $R$-model:

**Theorem 1.2.** (Cf. Theorem 3.5) Assume $R$ is a complete discrete valuation ring and $G, G_1, G_2$ are quasi-finite and flat over $R$. If $\phi_1$ is a model map and $G_{2,K}$ admits a finite and flat model then the pushout of (1) in the category of affine $R$-group schemes exists. Moreover it is quasi-finite and flat and its generic fibre is isomorphic to $G_{2,K}$.

Using the fact that $G_{2,K}$ always admits, when it is étale, a finite and flat model up to a finite extension of scalars we finally prove the following

**Corollary 1.3.** (Cf. Corollary 3.9) Assume $R$ is a complete discrete valuation ring and $G, G_1, G_2$ are quasi-finite and flat over $R$. Then if $\phi_1$ is a model map and $G_{2,R}$ is étale then the pushout of (1) in the category of affine $R$-group schemes exists. Again it is quasi-finite and flat and its generic fibre is isomorphic to $G_{2,K}$.

All the proofs rest on the computation of the pushout in the category of $R$-Hopf algebras. With the same techniques we briefly study in section §3.3 the existence of cokernels in the category of affine $R$-group schemes where $R$ is any Dedekind ring. This will lead to a new and short proof of the following:
Corollary 1.4. (Cf. Corollary 3.13) Let $R$ be a Dedekind ring, $G$ and $H$ two finite and flat $R$-group schemes with $H$ a closed and normal $R$-subgroup scheme of $G$. Then the quotient $G/H$ exists in the category of $R$-affine group schemes.

This holds over any base scheme and is in fact a consequence of a much bigger theorem (Cf. [4] Théorème 7.1).

1.2 Notations and conventions

Every ring $A$ will be supposed to be associative and unitary, i.e. provided with a unity element denoted by $1_A$, or simply 1 if no confusion can arise. However, unless stated otherwise, a ring will not be supposed to be commutative. Every Dedekind ring, instead, will always be supposed to be commutative. For an $R$-algebra $A$ the morphisms $u_A : R \to A$ and $m_A : A \otimes_R A \to A$ will always denote the unity and the multiplication morphisms (respectively). If moreover $A$ has an $R$-coalgebra structure then $\Delta_A : A \to A \otimes_R A$, $\varepsilon_A : A \to R$ will denote the comultiplication and the counity respectively. Furthermore if $A$ has a $R$-Hopf algebra structure then $\Delta_A : A \to A$ will denote the coinverse. All the coalgebra structures will be supposed to be coassociative. Morphisms of $R$-algebras (resp. $R$-coalgebras, $R$-Hopf algebras) are $R$-module morphisms preserving $R$-algebra (resp. $R$-coalgebra, $R$-Hopf algebra) structure. We denote by $R$-Hopf the category of associative and coassociative $R$-Hopf algebras while $R$-Hopf$_{ff}$ will denote the category of associative and coassociative $R$-Hopf algebras which are finite and flat as $R$-modules. When $R \to T$ is a morphism of commutative algebras, $M$ is a $R$-module, $X$ is a $R$-scheme, $f : M \to N$ is a $R$-module morphism and $\varphi : X \to Y$ a morphism of $R$-schemes then we denote by $M_T$, $X_T$, $f_T : M_T \to N_T$ and $\varphi_T : X_T \to Y_T$ respectively the $T$-module $M \otimes_R T$, the $T$-scheme $X \times_{\text{Spec}(R)} \text{Spec}(T)$, the $T$-module morphism induced by $f$ and the $T$-morphism of schemes induced by $\varphi$. When $R$ is a Dedekind ring and $K$ its field of fractions then a $R$-morphism of schemes $\varphi : X \to Y$ is called a model map if generically it is an isomorphism, i.e. $\varphi_K : X_K \to Y_K$ is an isomorphism.

2 Pushout of Hopf algebras

In this section we first study the pushout of algebras over a commutative ring $R$ then we discuss the existence of the pushout in the category of $R$-Hopf$_{ff}$ when $R$ is a complete discrete valuation ring. Let us first recall that in a category $\mathcal{C}$ the pushout (see for instance [8], III, §3) of a diagram

\[
\begin{array}{c}
\text{A} \\
\downarrow^f \downarrow^g
\end{array}
\begin{array}{c}
\text{B} \\
\rightarrow
\end{array}
\begin{array}{c}
\text{C}
\end{array}
\]

(where clearly $A, B, C$ are objects of $\mathcal{C}$ and $f, g$ morphisms in the same category) is an object of $\mathcal{C}$ that we denote $B \sqcup_A C$ provided with two morphisms $u : B \to$
$B \sqcup_A C$, $v : C \to B \sqcup_A C$ such that $uf = vg$ and satisfying the following universal property: for any object $P$ of $C$ and any two morphisms $u' : B \to P$, $v' : C \to P$ in $C$ such that $u'f = v'g$ then there exists a unique morphism $p : B \sqcup_A C \to P$ making the following diagram commute:

\[
\begin{array}{ccc}
B & \xrightarrow{f} & B \\
\downarrow^u & & \Downarrow^p \\
A & \xrightarrow{v'} & C \\
\end{array}
\]

When $A$ is an initial object (provided it exists) of $C$ then $B \sqcup_A C$ is the coproduct of $B$ and $C$ in $C$. When $C$ is the category of commutative $R$-algebras then the pushout is given by the tensor product $B \otimes_A C$. This is not true anymore if $C$ is the category of $R$-algebras, (cf. Example [2.9] or create easier examples). However we can always find a pushout even when $C$ is the category of $R$-algebras and it will be denoted by $B \ast_A C$. Before introducing, however, the pushout for non (necessarily) commutative $R$-algebras we recall the behavior of the tensor product over $R$. We put ourselves in the following situation:

**Notation 2.1.** By $R$ we will denote a commutative ring while $A$, $B$ and $C$ will be $R$-algebras and $f : A \to B$, $g : A \to C$ two $R$-algebra morphisms. We also denote by $\rho_B : B \to B \otimes_R C$ and $\rho_C : C \to B \otimes_R C$ the morphisms sending respectively $b \mapsto b \otimes 1_C$ and $c \mapsto 1_B \otimes c$.

**Proposition 2.2.** Let $D$ be any $R$-algebra and $u : B \to D$, $v : C \to D$ two $R$-algebra morphisms such that $u \circ B = v \circ C$ and such that $u(b)v(c) = v(c)u(b)$ for all $b \in B$, $c \in C$. Then there exists a unique $R$-algebra morphism $t : B \otimes_R C \to D$ making the following diagram commute:

\[
\begin{array}{ccc}
R & \xrightarrow{u_B} & B \\
\downarrow^{u_C} & & \Downarrow^t \\
C & \xrightarrow{v} & B \otimes_R C \\
\end{array}
\]

**Proof.** Cf. for instance [3], I, §3 Proposition 3.2.

Unfortunately $B \otimes_A C$ behaves badly in general and one can observe that even $A \otimes_A A \simeq A$, as an $R$-algebra, is not a natural quotient of $A \otimes_R A$. So, instead, let us consider the following construction:

---

1The coproduct can be defined, however, without assuming the existence of an initial object.
Definition 2.3. We denote by $B \ast_A C$, and we call it the star product of $B$ and $C$ over $R$, the quotient of $B \otimes_R C$ by the two-sided ideal generated by $A$, i.e. the ideal of $B \otimes_R C$ generated by the set $\{\rho_B f(a) - \rho_C g(a)\}_{a \in A}$.

It is an easy consequence the following universal property of the star product:

**Proposition 2.4.** Let $D$ be any $R$-algebra and $u : B \to D$, $v : C \to D$ two $R$-algebra morphisms such that $uf = vg$ and such that $u(b)v(c) = v(c)u(b)$ for all $b \in B, c \in C$. Then there exists a unique $R$-algebra morphism $t : B \ast_A C \to D$ making the following diagram commute:

$$\begin{array}{ccc}
B & \xrightarrow{f} & D \\
\downarrow{u} & & \downarrow{t} \\
B \ast_A C & \xrightarrow{g} & C \\
\downarrow{v} & & \downarrow{D} \\
A & \xrightarrow{u} & D
\end{array}$$

**Proof.** It is sufficient to take the $R$-algebra morphism $B \otimes_R C \to D$ and observe that it passes to the quotient. $\square$

The star product will only be used in Example 2.8 and §3.3 so finally let us recall the construction of the pushout of $R$-algebras: we follow essentially 2 1.7 and 5.1 with very few modifications in the exposition in order to obtain an easier to handle description. We describe $A$, $B$ and $C$ giving their presentation as $R$-algebras thus getting $R(X_0; S_0)$, $R(X_1; S_1)$ and $R(X_2; S_2)$ respectively, where $X_i$ is a generating set with relations $S_i$ ($i = 0, 1, 2$). We recall that $R(X; S)$ is to be intended as the $R$-algebra whose elements are all $R$-linear combinations of words on the set $X$ quotiented by the two-sided ideal generated by the relations in $S$. Observe that for $y, z \in X$ we are not assuming $zy = yz$; if it is the case the information will appear in $S$. However for any $x \in X$ and any $r \in R$ we do assume $xr = rx$. For example the commutative $R$-algebra $R[x, y]/f(x, y)$ can be presented as $R(x, y; f(x, y) = 0, xy = yx)$. First we observe that the coproduct of $B$ and $C$ (i.e. the pushout of $B$ and $C$ over the initial object $R$) is given by the $R$-algebra $B \ast_R C := R(X_1 \cup X_2; S_1 \cup S_2)$ where the union is of course disjoint. Let us denote by $u : B \to B \ast_R C$ and $v : C \to B \ast_R C$ the canonical inclusions. Then the pushout of $B$ and $C$ over $A$ is given by the $R$-algebra

$$B \ast_A C := R(X_1 \cup X_2; S_1 \cup S_2 \cup S_3) \quad (3)$$

where $S_3$ consists on the relations given by $uf(x) = vg(x)$ for every $x \in X_0$. Now we relate the pushout just described to the tensor product:

**Lemma 2.5.** Assume that $B = R(X_1; S_1)$ and $C = R(X_2; S_2)$. Then $B \otimes_R C$ can be presented as $R(X_1 \cup X_2; S_1 \cup S_2, \{zy = yz\}_{z \in X_1, y \in X_2})$ thus becoming a quotient of $R(X_1 \cup X_2; S_1 \cup S_2) = B \ast_R C$. 

5
Proof. Let $D$ be an $R$-algebra provided with $R$-algebra morphisms $m : B \to D$ and $n : C \to D$ such that $m \circ u_B = n \circ u_C$, and assume moreover that $m(b)n(c) = n(c)m(b)$ for all $b \in B, c \in C$. Let us denote by $u : B \to B \ast_R C$ and $v : C \to B \ast_R C$ the canonical morphisms and by $\lambda : B \ast_R C \to D$ the universal morphism making the following diagram commute:

![Diagram](image)

By assumption $\lambda u(z) \lambda v(y) = \lambda v(y) \lambda u(z)$ so $u(z)v(y) - v(y)u(z) \in \ker(\lambda)$ hence $\lambda$ factors through $R\langle X_1 \cup X_2; S_1 \cup S_2, \{zy = yz\}_{z \in X_1, y \in X_2} \rangle$ providing it with the universal property stated in Proposition 2.2 and this is enough to conclude. □

Let $q : R \to T$ be a $R$-commutative algebra. When $f = f(x_1, \ldots, x_n) \in R[x_1, \ldots, x_n]$ we denote by $q_*(f)$ the polynomial in $T[x_1, \ldots, x_n]$ whose coefficients are the image in $T$ by $q$ of the coefficients of $f$, i.e. the image of $f$ through the morphism $q_* : R[x_1, \ldots, x_n] \to T[x_1, \ldots, x_n] = R[x_1, \ldots, x_n] \otimes_R T$. Now take $R\langle X, S \rangle$: by an abuse of notation we denote by $q_*(S)$ the set of relations $\{q_*(s_i) = 0\}$ on the set $X$. In Lemma 2.6 we observe that the pushout is stable under base change.

**Lemma 2.6.** Let $q : R \to T$ be a $R$-commutative algebra and $R\langle X; S \rangle$ any $R$-algebra, then

1. $R\langle X; S \rangle \otimes_R T \simeq T\langle X; q_*(S) \rangle$,
2. $(B \ast_A C) \otimes_R T \simeq (B \otimes_R T) \ast_{(A \otimes_R T)} (C \otimes_R T)$.

Proof. As a commutative $R$-algebra, $T$ is isomorphic to $R[\{y_i\}]/(\{f_r\})$ where $\{y_i\}$ is a set of generators and $\{f_r\}$ a set of polynomials in the variables $\{y_i\}$ with coefficients in $R$. So by Lemma 2.3 $R\langle X; S \rangle \otimes_R T$ is isomorphic to $R\langle X \cup \{y_i\}; S \cup \{f_r = 0\} \cup \{y_iy_j = y_jy_i\} \cup \{xy_i = y_ix\}_{x \in X} \rangle$ which is isomorphic to $R\langle X \cup \{y_i\}; q_*(S) \cup \{f_r = 0\} \cup \{y_iy_j = y_jy_i\} \cup \{xy_i = y_ix\}_{x \in X} \rangle$ and the latter is isomorphic to $T\langle X; q_*(S) \rangle$ since $T$ commutes with $X$ and this proves 1. Let us denote $A, B$ and $C$ as $R\langle X_0; S_0 \rangle$, $R\langle X_1; S_1 \rangle$ and $R\langle X_2; S_2 \rangle$ respectively. As a consequence of point 1 we have $A \otimes_R T \simeq T\langle X_0; q_*(S_0) \rangle$, $B \otimes_R T \simeq T\langle X_1; q_*(S_1) \rangle$, $C \otimes_R T \simeq T\langle X_2; q_*(S_2) \rangle$ and $(B \ast_A C) \otimes_R T \simeq T\langle X_1 \cup X_2; q_*(S_1 \cup S_2 \cup S_3) \rangle$ where $S_3$ is as described in 3. But $(B \otimes_R T) \ast_{(A \otimes_R T)} (C \otimes_R T)$ is also isomorphic to the latter which enables us to conclude. □

**Notation 2.7.** When $R$ is a Dedekind ring and $M$ an $R$-module, let us denote by $q : M \to F(M)$ the unique quotient (cf. [5], Lemme (2.8.1.1)) of $M$ which is $R$-flat and such the induced map $q_K : M_K \to F(M)_K$ is an isomorphism.
Let us analyze a few examples whose importance will be clear in the following sections:

**Example 2.8.** Let \( f : A \to B \) and \( g : A \to R \) be morphisms of \( R \)-algebras then the canonical morphism \( \varphi : B \ast A R \to B \ast A R \) is an isomorphism. Indeed we observe that \( B \ast R R = B \) and that the canonical morphisms \( u : B \to B \ast R R \) and \( v : R \to B \ast R R \) are nothing else but \( \text{Id}_B \) and the unit morphism \( u_B \) respectively, then for any \( b \in B \) and any \( r \in R \) we have \( u(b)v(r) = v(r)u(b) \). Hence denoting by \( u' : B \to B \ast A R \) and \( v' : R \to B \ast A R \) the canonical morphisms we also have \( u'(b)v'(r) = v'(r)u'(b) \) as \( u' = \lambda u \) and \( v' = \lambda v \) where \( \lambda : B \ast R R \to B \ast A R \) is the universal morphism. Then \( \varphi \) can be inverted according to Proposition 2.4.

Observe that \( B \ast A R \) is finite as an \( R \)-module if \( B \) is finite (it is indeed a quotient of \( B \)).

**Example 2.9.** Let \( R \) be a discrete valuation ring with uniformising element \( \pi \). Let us fix a positive integer \( p \) and let us set \( A := R[x]/x^p \), \( B := R[y]/y^p \) and \( C := R[z]/z^p \) (thus commutative \( R \)-algebras). Consider the morphisms \( f : A \to B, x \mapsto \pi^n y \) and \( g : A \to C, x \mapsto \pi^m z \) where \( m > n > 0 \) are integers. Then \( B \ast A C = R[y,z]/y^p = z^p = 0, \pi^m z = \pi^n y \). Observe that, as an \( R \)-module, \( B \ast A C \) is not flat as \( \pi^n(\pi^m z - y) = 0 \) thus \( \pi^m z - y \) is a \( R \)-torsion element. However if we add the relation \( \pi^m z = y \) then we eliminate torsion from \( B \ast A C \) and what we obtain is (cf. Notation 2.7) \( F(B \ast A C) = R(y,z)/y^p = z^p = 0, \pi^m z - y = 0 \) thus finitely generated and flat and, in this particular case, it is isomorphic to \( F(B \otimes_A C) \).

The following well known result will be used several times in this paper:

**Theorem 2.10.** Let \( R \) be a complete discrete valuation ring with fraction field \( K \) and residue field \( k \). Let \( M \) be a torsion-free \( R \)-module of finite rank \( r \) (i.e. \( r := \text{dim}_R(M \otimes_R K) < +\infty \)). Then \( M \cong_{R\text{-mod}} K^\oplus s \oplus R^\oplus s \), where \( s = \text{dim}_K(M \otimes_R k) \).

**Proof.** This is [4], Chapter 16, Corollary 2, \( \square \)

**Theorem 2.10** is not true when \( R \) is not complete (cf. [6], Theorem 19) and this is why we will often need to restrict to complete discrete valuation rings. The following lemma is crucial in this paper:

**Lemma 2.11.** Let \( R \) be a complete discrete valuation ring and assume that \( f : A \to B \) and \( g : A \to C \) are \( R \)-algebra morphisms where furthermore \( gK : A_K \to C_K \) is an isomorphism. Then if \( A, B \) and \( C \) are finitely generated and flat as \( R \)-modules then the same holds for \( F(B \ast A C) \). Moreover the canonical \( R \)-algebra morphism \( B \to F(B \ast A C) \) induces an isomorphism \( B_K \to F(B \ast A C)_K \).

**Proof.** Let \( \pi \) be an uniformising element of \( R \) and let \( K \) and \( k \) be the fraction and residue fields respectively. As usual let us present by \( R(X_0; S_0), R(X_1; S_1) \) and \( R(X_2; S_2) \) respectively the \( R \)-algebras \( A, B \) and \( C \) where for \( X_0, X_1, X_2 \) we take respectively bases of \( A, B, C \) as \( R \)-modules minus the identity elements so that the cardinality \( x_0, x_1, x_2 \) of those sets is the rank of \( A, B \) and \( C \) (respectively)
minus one; of course $x_0 = x_2$. Now $B_A C = R(X_1 \cup X_2; S_1 \cup S_2 \cup S_3)$ where $S_3$ is as described in (8). In particular $S_3$ is a set made of $x_0$ $R$-linear relations relating the $x_2$ elements of $X_2$ and the $x_1$ elements of $X_1$. As in Example 2.9 the information of being $R$-torsion (if any) is contained in the set $S_3$, so if we want to cut out $R$-torsion we need to add another set of $x_0$ relations $S_4$ obtained as follows: for each relation $(s = 0) \in S_3$ add the relation $(t = 0)$ to $S_4$ where $s = \pi^\prime t$ and $t$ has at least one coefficient equal to an invertible element of $R$. Thus $F(B_A C) = R(X_1 \cup X_2; S_1 \cup S_2 \cup S_3 \cup S_4)$. But since relations in $S_3$ are automatically satisfied if we add $S_4$ then $F(B_A C) = R(X_1 \cup X_2; S_1 \cup S_2 \cup S_3)$.

Now Lemma 2.6 point 1, implies that $F(B_A C) \otimes_R k = k(X_1 \cup X_2; q_s(S_1 \cup S_2 \cup S_3))$ where $q : R \to k$ is the canonical surjection so $F(B_A C) \otimes_R k$ is the quotient of $k(X_1 \cup X_2; q_s(S_1 \cup S_2))$ by the two-sided ideal generated by the relations $q_s(S_4)$. But in $k(X_1 \cup X_2; q_s(S_1 \cup S_2))$ the elements of the set $X_1 \cup X_2$ are $x_1 + x_2$ $k$-linearly independent vectors then if we add the $x_0 = x_2$ $k$-linear relations $q_s(S_4)$ what remains is a set of at least $x_1 = rk(B) - 1$ $k$-linearly independent elements which become $rk(B)$ if we add $1_B$. Combining this with Theorem 2.10 we obtain that $F(B_A C)$ is a finitely generated $R$-free module, as required, as $dim_k F(B_A C) \otimes_R k = dim_K F(B_A C) \otimes_R K$. The last assertion follows easily from Lemma 2.11 point 2.

\[\square\]

Remark 2.12. The construction in Lemma 2.11 does not depend on $A$. That means that if we take $A'$, $f' : A' \to B$ and $g' : A' \to C$ satisfying similar assumptions then $F(B_A C) \simeq F(B_A' C)$. Indeed, again by [5], Lemme 2.8.1.1, we observe that $F(B_A C)$ is isomorphic to the unique quotient of $B_R C$ which is $R$-flat and whose tensor over $K$ gives $B_K$; but the same property is satisfied by $F(B_A' C)$ hence we conclude by unicity.

Proposition 2.13. Let $R$ be any commutative ring. Then the pushout in the category of $R$-bialgebras exists.

Proof. We follow [7], Chapitre 5, §5.1, Proposition. Consider the diagram

\[
\begin{array}{ccc}
B & \xrightarrow{u_D} & A \\
\downarrow{f} & & \downarrow{g} \\
C & \xrightarrow{u_C} & \end{array}
\]

where we assume that $A$, $B$ and $C$ are $R$-bialgebras and the arrows are $R$-Hopf algebra morphisms. Let $D := B_A C$ be the pushout of the diagram in the category of $R$-algebras and let $m_D$ and $u_D$ be respectively the multiplication and the unit morphism. Then we need to provide $D$ with a comultiplication $\Delta_D$ and a counit $\varepsilon_D$ such that $(D, m_D, u_D, \Delta_D, \varepsilon_D)$ is a $R$-bialgebra. We describe how

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\[^{2}\text{In [7], however, Lemaire uses different notations.}\]
to construct $\Delta_D$, the construction of $\varepsilon_D$ being easier. The rest will be standard verification over complicated diagrams. The existence of $\Delta_D$ is explained in the following diagram, taking into account the universal property of $D$:

\[
\begin{array}{ccc}
A & \Delta_A & B \\
\downarrow f & \Delta_B & \downarrow g \\
C & \Delta_C & D
\end{array}
\]

Remark 2.14. Notation being as in Proposition 2.13 one observes that we can define a $R$-module morphism $S_D : D \to D$, candidate to be a co-inverse, as follows: first construct the opposite algebras $A^\text{op}$, $B^\text{op}$, $C^\text{op}$, $D^\text{op}$, the opposite morphisms $f^\text{op}$, $g^\text{op}$, $u^\text{op}$, $v^\text{op}$ and the morphisms of $R$-algebras $S'^A$, $S'^B$, $S'^C$, induced by the $R$-algebra anti-morphisms $S_A$, $S_B$, $S_C$. Then the existence of $S'_D$ follows from the following diagram:

\[
\begin{array}{ccc}
A & S'_A & B \\
\downarrow g & \Delta_A & \downarrow f \\
C & S'_C & D
\end{array}
\]

exploiting the universal property of $D$ then $S_D$ is the anti-morphism induced by $S'_D$. However $S_D$ may fail to be a co-inverse for $D$ as $m_D \circ (S_D \otimes id_D) \circ m_D$ may not be equal to $u_D \circ \varepsilon_D$ (same for $m_D \circ (id_D \otimes S_D) \circ m_D$).

In order to have an explicit description for $S_D$, constructed in Remark 2.14 set, as usual, $A = R(X_0; S_0)$, $A = R(X_1; S_2)$ and $C = R(X_2; S_2)$ so $D = R(X_1 \cup X_2; S_1 \cup S_2 \cup S_3)$ where $S_3$ is as described in [3]; it is sufficient to set $S_D(x_1) := S_B(x_1)$ for any $x_1 \in X_1$, $S_D(x_2) := S_C(x_2)$ for any $x_2 \in X_2$ for any $x_1 \in X_1$, $S_D(x_1x_2) := S_D(x_2)D_D(x_1)$ and $S_D(x_2x_1) := S_D(x_1)D_D(x_2)$. It is well defined and is by construction an anti-isomorphism for the $R$-algebra $D$. A similar construction gives an explicit description of $\Delta_D$, taking into account that $\Delta_D$ is a morphism of $R$-algebras and not an anti-morphism.

Corollary 2.15. Let $R$ be a complete discrete valuation ring and assume that $f : A \to B$ and $g : A \to C$ are $R$-algebra morphisms where furthermore $g_K : A_K \to C_K$ is an isomorphism. Then $F(B \ast_A C)$ has a natural structure of
If moreover $A$, $B$ and $C$ are finitely generated and flat as $R$-modules then $F(B *_A C)$ is the pushout of $B$ and $C$ over $A$ in $R$-Hopf.\[\text{Proof.}\] By [5], (2.8.3) and of course Proposition 2.13 we obtain that $F(B *_A C)$ has a natural structure of $R$-bialgebra. We need to prove the existence of a coinverse $S_{F(B *_A C)}$ that gives $F(B *_A C)$ a natural structure of $R$-Hopf algebra. So let us take for $D := B *_A C$ the $R$-module morphism $S_D$ defined in Remark 2.14. This morphism induces (by [5], Lemme 2.8.3) an $R$-module morphism $S_{F(D)} : F(D) \to F(D)$ which is the required coinverse: indeed

$$m_{F(D)} \circ (S_{F(D)} \otimes \text{id}_{F(D)}) \circ m_{F(D)} = u_D \circ \Delta_D$$

is the zero map $0_D$ and this is clear since $F(D) \subset B_K$ and [5] tensored over $K$ gives rise to the equality

$$m_{B_K} \circ (S_{B_K} \otimes \text{id}_{B_K}) \circ m_{B_K} = u_{B_K} \circ \Delta_{B_K}$$

which holds as $B_K$ is a $K$-Hopf algebra. The same is true for $m_{F(D)} \circ (\text{id}_{F(D)} \otimes S_{F(D)}) \circ m_{F(D)}$. Finally $F(B *_A C)$ is finitely generated and flat as an $R$-module when $A$, $B$ and $C$ are: this is Lemma 2.11.\[\]

**Remark 2.16.** Notation being as in Proposition 2.13 we observe that $B *_A C$ is cocommutative if $A$, $B$, $C$ are. The same conclusion holds, then, for $F(B *_A C)$ in Corollary 2.14. Moreover observe that if $A$, $B$, $C$ are commutative then $F(B *_A C)$ is commutative too since it is contained in $B_K$. So in particular in this case $F(B *_A C) \simeq F(B \otimes A C)$, as it happend in Example 2.9.

### 3 Pushout of group schemes

In this section $R$ is any complete discrete valuation ring with fraction and residue fields respectively denoted by $K$ and $k$.

**3.1 The finite case**

Let $M = \text{Spec}(B)$ and $N = \text{Spec}(C)$ be finite and flat $R$-group schemes, i.e. $B$ and $C$ are free over $R$ and finitely generated as $R$-modules. Let us assume that there is a $K$-group scheme morphism $\psi : M_K \to N_K$. An upper bound for $M$ and $N$ is a finite and flat $R$-group scheme $U$, provided with a model map $U \to M$ and a $R$-group scheme morphism $\varphi : U \to N$ which generically coincides with $\psi : M_K \to N_K$. A lower bound for $M$ and $N$ is a finite and flat $R$-group scheme $L$, provided with a model map $N \to L$ and a $R$-group scheme morphism $\delta : M \to L$ which generically coincides with $\psi : M_K \to N_K$. The construction of an upper bound is easy: it is sufficient to set $U$ as the schematic closure of $M_K$ in $M \times N$ through the canonical closed immersion

}\[\]
\( M_K \hookrightarrow M_K \times N_K \) (and this holds when the base is any Dedekind scheme). Now consider the following commutative diagram

\[
\begin{array}{c}
M \\
\downarrow m \\
U \\
\downarrow n \\
N
\end{array}
\]

where \( U = \text{Spec}(A) \) is any upper bound. We are now going to study the existence of a pushout \( M \sqcup_U N \) in the category of finite and flat \( R \)-group schemes. We prove the following

**Lemma 3.1.** The pushout of (6) in the category of finite and flat \( R \)-group schemes exists. Moreover \( M \sqcup_U N \) is a lower bound for \( M \) and \( N \).

**Proof.** Notation being as in the beginning of this section, we have the following diagram of commutative \( R \)-Hopf algebras:

\[
\begin{array}{c}
A \\
\downarrow f \\
B \\
\downarrow g \\
C
\end{array}
\]

which, dualizing, gives rise to the following diagram of cocommutative (but possibly non commutative) \( R \)-Hopf algebras:

\[
\begin{array}{c}
A^\vee \\
\downarrow f \\
B^\vee \\
\downarrow g \\
C^\vee
\end{array}
\]

Let us consider the cocommutative \( R \)-Hopf algebra \( F(B^\vee \ast_{A^\vee} C^\vee) \) constructed in Corollary 2.15. Now we take the spectrum of its dual \( P := \text{Spec}(F(B^\vee \ast_{A^\vee} C^\vee)^\vee) \). First of all we observe that \( F(B^\vee \ast_{A^\vee} C^\vee)^\vee \) is commutative as \( F(B^\vee \ast_{A^\vee} C^\vee)^\vee \) is cocommutative so that taking its spectrum does make sense. It remains to prove that the commutative diagram

\[
\begin{array}{c}
M \\
\downarrow m \\
U \\
\downarrow n \\
N \\
\downarrow \rightarrow P
\end{array}
\]
is in fact a pushout in the category of finite and flat $R$-group schemes. But this follows from the fact that $F(B^\vee \ast_{A^\vee} C^\vee)$ is a pushout in $R\text{-Hopf}_{ff}$. That $M \sqcup_U N := P$ is a lower bound for $M$ and $N$ is also clear by construction. □

**Theorem 3.2.** The pushout of (6) in the category of affine $R$-group schemes exists.

**Proof.** Consider the commutative diagram

![Diagram](https://via.placeholder.com/150)

where $P$ is the pushout of (6) in the category of finite and flat $R$-group schemes constructed in Lemma 3.1 and $Q = \text{Spec}(D)$ is any affine $R$-group scheme. We are going to show that $P$ is also the pushout of (6) in the category of affine $R$-group schemes. Let us factor $u$ through $M' := \text{Spec}(B')$ via the morphisms $u' : M \to M'$ and $i : M' \to Q$ where $i$ is a closed immersion and $u'$ is a schematically dominant morphism (i.e. the induced morphism $B' \to B$ is injective) so that $M'$ is a finite and flat $R$-group scheme since $M$ is. Likewise we factor $v$ through the finite and flat $R$-group scheme $N' := \text{Spec}(C')$ via the schematically dominant morphism $v' : N \to N'$ and the closed immersion $j : N' \to Q$. Now consider the finite $R$-group scheme (it needs not be flat a priori) $M' \times_Q N'$ and the natural closed immersions $i' : M' \times_Q N' \to M'$ and $j' : M' \times_Q N' \to N'$. So let us denote by $r : U \to M' \times_Q N'$ the universal morphism, then we have the following commutative diagram

![Diagram](https://via.placeholder.com/150)

and in particular we have the following commutative diagram of $R$-algebras

![Diagram](https://via.placeholder.com/150)
where $B' \to B' \otimes_D C'$ is surjective but also injective since $B' \hookrightarrow A$ is (recall that $m : U \to M$ is a model map). Hence $\iota'$ is an isomorphism so there exists a universal morphism $P \to N'$ by Lemma 2.11 and we conclude. 

\textbf{Remark 3.3.} If $U'$ is any other upper bound for $M$ and $N$ then there is a canonical isomorphism $M \sqcup_U N \cong M \sqcup_{U'} N$: this is a consequence of Remark 2.12. However this does not mean that the lower bound for $M$ and $N$ is unique, which is clearly not true in general.

\section{3.2 The quasi-finite case}

Let $M = \text{Spec}(B)$ and $N = \text{Spec}(C)$ be quasi-finite (by this we will always mean affine and of finite type over $R$, with finite special and generic fibers) and flat $R$-group schemes. It is known (see [1], §7.3 p.179) that any quasi-finite $R$-group scheme $H$ has a finite part $H_f$, that is an open and closed subscheme of $H$ which consists of the special fibre $H_k$ and of all points of the generic fibre which specialize to the special fibre. It is thus flat over $R$ if $H$ is.

\textbf{Remark 3.4.} If $H = \text{Spec}(A)$ is a quasi-finite and flat $R$-group scheme then its finite part coincides with $\text{Spec}(A^\vee)$ where $A^\vee$ is the double dual of $A$: this follows from the fact that $A \cong_{R\text{-mod}} K^{\oplus Ax} \oplus R^{\oplus s}$ (cf. Theorem 2.10) so $A^\vee \cong_{R\text{-mod}} R^{\oplus s}$ is an $R$-Hopf algebra and not just an $R$-algebra. Hence the canonical surjection $A \to A^\vee$ gives the desired closed immersion $H_f \hookrightarrow H$ of group schemes. However this fact will not be necessary in the remainder of this paper.

Let us assume that there is a $K$-group scheme morphism $\psi : M_K \to N_K$. We define upper and lower bounds exactly as in the finite case. One can easily construct an upper bound $U$ for $M$ and $N$ simply proceeding as in §3.1 So $U$ will be in general a quasi-finite and flat $R$-group scheme. For the lower bound it will be a little bit more complicated. So consider again the commutative diagram (6) where $U = \text{Spec}(A)$ is any upper bound. We are going to study the existence of a pushout $M \sqcup_U N$ in the category of affine $R$-group schemes.

\textbf{Theorem 3.5.} Assume that $N_K$ admits a finite and flat model over $R$. Then the pushout of (6) in the category of affine $R$-group schemes exists. Moreover $M \sqcup_U N$ is a lower bound for $M$ and $N$.

\textbf{Proof.} Let $N'$ denote a finite and flat $R$-model for $N_K$, i.e. a finite and flat $R$-group scheme whose generic fibre is isomorphic to $N_K$. Consider the finite part $M_f$ and $N_f$ of, respectively, $M$ and $N$. Compose the closed immersion $M_{f,K} \to M_K$ with $\psi_K : M_K \to N_K$ thus obtaining a morphism $M_{f,K} \to N_K$. By Theorem 3.2 we construct a lower bound $L_1$ for $M_f$ and $N'$, which is finite and flat over $R$, generically isomorphic to $N_K$, then it is already a lower bound for $M$ and $N'$. Considering the closed immersion $N_{f,K} \to N_K$ we also construct a lower bound $L_2$ for $N_f$ and $N'$, which is finite and flat over $R$, generically isomorphic to $N_K$, then it is already a lower bound for $N$ and $N'$. So a lower
bound $L$ for $L_1$ and $L_2$ (which are generically isomorphic) exists by Theorem 3.2 and is also a lower bound for $M$ and $N$. We still need to compute the pushout of $M$ and $N$ over $U$: let us set $U_f := \text{Spec}(A_f)$, $M_f := \text{Spec}(B_f)$, $N_f := \text{Spec}(C_f)$ and $L := \text{Spec}(D)$. Consider the natural $R$-bialgebra morphism (cf. Proposition 2.13) $B_f^\vee *_{A_f^\vee} C_f^\vee \to D^\vee$ and factor it as follows

$$
B_f^\vee *_{A_f^\vee} C_f^\vee \longrightarrow E^\vee \longrightarrow D^\vee
$$

where $E$ is a cocommutative $R$-bialgebra which is flat and finitely generated as an $R$-module because $D^\vee$ is. Consider the morphism $S_{B_f^\vee *_{A_f^\vee} C_f^\vee} : B_f^\vee *_{A_f^\vee} C_f^\vee \to B_f^\vee *_{A_f^\vee} C_f^\vee$ constructed in Remark 2.14 the commutative diagram

$$
\begin{array}{ccc}
B_f^\vee *_{A_f^\vee} C_f^\vee & \longrightarrow & E^\vee \\
\downarrow S_{B_f^\vee *_{A_f^\vee} C_f^\vee} & & \downarrow S_{D^\vee} \\
B_f^\vee *_{A_f^\vee} C_f^\vee & \longrightarrow & D^\vee
\end{array}
$$

induces a anti-morphism of $R$-algebras $S_E : E \to E$ which gives $E$ a natural structure of $R$-Hopf algebra: indeed $m_E \circ (id_E \otimes S_E) \circ m_E = u_E \circ \varepsilon_E$ and $m_E \circ (S_E \otimes id_E) \circ m_E = u_E \circ \varepsilon_E$ since the same equalities hold for $D^\vee$. It is now sufficient to take the union of $\text{Spec}(E^\vee)$ and $N_K \simeq L_K$ in order to construct a quasi-finite and flat $R$-group scheme $P$ which is certainly a pushout in the category of quasi-finite and flat $R$-group schemes. Arguing as in the proof of Theorem 3.2 we can deduce that $P$ is also a pushout in the category of affine $R$-group schemes.

Remark 3.6. As in Remark 3.3 one observes that if $U'$ is any other upper bound for $M$ and $N$ then there is a canonical isomorphism $M \cup_{U'} N \simeq M \cup_U N$: indeed $E$, as constructed in the proof, is the only quotient of $B_f^\vee *_{R} C_f^\vee$, $R$-flat which over $K$ gives $B_{f,K}^\vee *_{K} C_{f,K}^\vee \to D_{f,K}^\vee$ and this does not depend on $A_f$. The same will hold for Corollary 3.7 and will be used in Corollary 3.9.

Corollary 3.7. When $N_K$ is étale then after possibly a finite extension of scalars the pushout of (10) in the category of affine $R$-group schemes exists. Again $M \cup_U N$ is a lower bound for $M$ and $N$.

Proof. Clear since after possibly a finite extension $K'$ of $K$ the $K$-group scheme $N_K$ becomes constant then it certainly admits a finite, constant (so flat) model over $R'$, the integral closure of $R$ in $K'$.

Let $K'$ be a finite extension of $K$ and $R'$ the integral closure of $R$ in $K'$ then $R'$ is a complete discrete valuation ring. Assume that $W$ is a torsion-free $R$ module of finite rank $n$ then we have the following
Lemma 3.8. If $W \otimes_R R'$ is finitely generated as an $R'$-module then $W$ is finitely generated as an $R$-module too (thus free).

Proof. By Theorem 2.10 $W \simeq_{R\text{-}mod} K^\oplus n - s \oplus R^\oplus_s$, where $s = \dim_k(W_k)$, hence $W \otimes_R R' \simeq_{R'\text{-}mod} K^\oplus n - s \oplus R'^\oplus_s$ so if $W \otimes_R R'$ is finitely generated as an $R'$-module then $n - s = 0$ and we conclude. \hfill \Box

This will be used in the following

Corollary 3.9. When $R$ is complete and $N_K$ is étale then the pushout of (6) in the category of affine $R$-group schemes exists. Again $M \sqcup U$ is a lower bound for $M$ and $N$.

Proof. Again $U_f = \text{Spec}(A_f)$, $M_f = \text{Spec}(B_f)$, $N_f = \text{Spec}(C_f)$ will denote the finite part of $U$, $M$ and $N$ respectively. Let us consider the duals $A_f^\vee = A^\vee_{\text{f}}$, $B_f^\vee = B^\vee_{\text{f}}$ and $C_f^\vee = C^\vee_{\text{f}}$ and the commutative diagram

\[ (7) \]

where $E$ comes from the factorisation of the universal morphism $B_f^\vee_{\text{f}} * A_f^\vee_{\text{f}} \rightarrow C_f^\vee_{\text{f}}$. Arguing as in Theorem 3.5 we provide it with a natural structure of $K$-Hopf algebra. Using again [5], Lemme (2.8.1.1) we construct the unique quotient

$$B_f^\vee_{\text{f}} * A_f^\vee_{\text{f}} \rightarrow E'$$

which is $R$-flat and which generically gives

$$B_f^\vee_{\text{f}} * A_f^\vee_{\text{f}} \rightarrow E.$$

Thus $E'$ has naturally a structure of a cocommutative $R$-Hopf algebra: indeed it inherits from $B_f^\vee_{\text{f}} * A_f^\vee_{\text{f}} C_f^\vee_{\text{f}}$ a cocommutative $R$-coalgebra structure and by means of [5], (2.8.3) an anti-morphism of $R$-agebras $S_E : E' \rightarrow E'$ which is a coinverse since tensoring it over $K$ we obtain $S_E : E \rightarrow E$ which is a coinverse for $E$. If we prove that $E'$ is finitely generated as a $R$-module then $\text{Spec}(E'^\vee)$ glued to $N$
is the desired pushout. So now it remains to prove that \( E' \) is finitely generated as a \( R \)-module: let \( K \to K' \) be a finite field extension such that \( N_{K'} \) admits a finite and flat model over \( R' \), the integral closure of \( R \) in \( K' \). Then by Corollary 3.7 and Remark 3.6 \( E' \otimes_R R' \) is \( R' \)-finite and flat. Lemma 3.8 implies that \( E' \) is \( R \)-finite and flat too.

**Remark 3.10.** It is less elegant but still true that Corollary 3.9 holds for all those \( N_K \) that admits a finite and flat \( R \)-model after possibly a finite extension of scalars and étale ones are just a particular case. Observe furthermore that, following the proof, in the situation of both Theorem 3.5 and Corollary 3.7 one can find a finite and flat lower bound for \( M \) and \( N \). This can be false in the situation of Corollary 3.9.

### 3.3 Cokernels and quotients

In a category \( C \) with zero object \( 0_C \) (that is an object which is both initial and final), we can define the cokernel of a morphism \( f : A \to B \) (see for instance [8], III, §3) which turns out to be the pushout \( 0_C \sqcup_A B \) of the obvious diagram.

As explained in the introduction in this section we are going to describe, in Proposition 3.12, a new and easy proof for a well known result. First we need a lemma:

**Lemma 3.11.** Let \( R \) be a Dedekind ring or a field, \( A \), \( B \) and \( C \) \( R \)-Hopf algebras provided with \( R \)-Hopf algebra morphisms \( f : A \to B \) and \( g : A \to C \). Then the star product \( B \star_A C \) defined in Definition 2.3 has a natural structure of \( R \)-Hopf algebra.

**Proof.** First we prove the existence of the comultiplication \( \Delta_{B \star_A C} \): it is sufficient to consider the following diagram

\[
\begin{array}{ccc}
B & \xrightarrow{\Delta_B} & B \otimes_R B \\
\downarrow{f} & & \downarrow{f \otimes f} \\
A \star_A C & \xrightarrow{\Delta_{B \star_A C}} & (B \star_A C) \otimes_R (B \star_A C) \\
\downarrow{g} & & \downarrow{g \otimes g} \\
C & \xrightarrow{\Delta_C} & C \otimes_R C
\end{array}
\]

where the existence of \( \Delta_{B \star_A C} \) is ensured by Proposition 2.4. The existence of \( \varepsilon_{B \star_A C} \) is easier and an argument similar to the one used in Remark 2.14 ensures the existence of an anti-morphism \( S_{B \star_A C} : B \star_A C \to B \star_A C \) which is compatible with \( S_{B \otimes_R C} \), i.e. if \( \lambda : B \otimes_R C \to B \star_A C \) denotes the canonical projection then \( \lambda \circ S_{B \otimes_R C} = S_{B \star_A C} \circ \lambda \). From this we deduce that \( S_{B \star_A C} \) is the desired coinverse for \( B \star_A C \).
**Proposition 3.12.** Let $R$ be a Dedekind ring or a field, $G$ and $H$ two finite and flat $R$-group schemes and $f : H \to G$ a morphism of $R$-group schemes. Then the cokernel of $f$ exists in the category of $R$-affine group schemes.

*Proof.* The zero object in the category of $R$-affine group schemes is $\text{Spec}(R)$. Let us set $H = \text{Spec}(A)$ and $G = \text{Spec}(B)$. Then we first compute the pushout in the category of $R$-Hopf algebras of the diagram

\begin{equation}
\begin{array}{ccc}
A^\vee & \xleftarrow{\phantom{f}} & R \\
\downarrow{f} & & \downarrow{\phantom{f}} \\
B^\vee & \xrightarrow{\phantom{f}} & R
\end{array}
\end{equation}

In Example 2.8 we have observed that $W := B^\vee \star_{A^\vee} R \simeq B^\vee \star_{A^\vee} R$ canonically. That $W$ has a natural structure of $R$-Hopf algebra follows from Lemma 3.11. If $R$ is a Dedekind ring and $W$ is not flat then we consider $F(W)$ (cf. Notation 2.7) which is flat and finitely generated and inherits the $R$-Hopf algebra structure. Since the case of a field is similar let us just consider the case of a Dedekind ring $R$: we are now going to prove that $\text{Spec}(F(W)^\vee)$ is the desired pushout. So let $M$ be any affine $R$-group scheme, $v : G \to M$ an $R$-group scheme morphism and $u : \text{Spec}(R) \to M$ the natural inclusion (the unity map). Let us assume we have a commutative diagram

\begin{equation}
\begin{array}{ccc}
\text{Spec}(R) & & \\
\downarrow{u} & \searrow{v} & \downarrow{w} \\
H & \xrightarrow{\phantom{f}} & M \\
\downarrow{f} & & \downarrow{\phantom{f}} \\
G & \xrightarrow{\phantom{f}} & M
\end{array}
\end{equation}

Observe that we can assume $M$ to be finite and flat, for if it is not we can factor $v$ through a finite and flat (since $G$ is) $R$-group scheme that makes a diagram similar to commute. When $M$ is finite and flat it is easy to construct a universal morphism $\text{Spec}(F(W)^\vee) \to M$ since $F(W)$ is easily seen to be the pushout of diagram in $R$-$\text{Hopf}_{ff}$. \hfill \Box

**Corollary 3.13.** Let $R$ be a Dedekind ring, $G$ and $H$ two finite and flat $R$-group schemes with $H$ a closed and normal $R$-subgroup scheme of $G$. Then the quotient $G/H$ exists in the category of $R$-affine group schemes.

*Proof.* This follows directly from Proposition 3.12 where we take for $f : H \to G$ the given closed immersion. \hfill \Box
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