ON BOUNDED COHOMOLOGY OF AMALGAMATED PRODUCTS OF GROUPS

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We investigate the structure of the singular part of the second bounded cohomology group of amalgamated products of groups by constructing an analog of the initial segment of the Mayer-Vietoris exact cohomology sequence for the spaces of pseudocharacters.

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1. Introduction. We recall that bounded cohomology $H^k_b(G)$ of a group $G$ (we will be considering only cohomology with coefficients in the additive group of reals $\mathbb{R}$ with trivial action, so, in our notations for cohomology, the coefficient module will be omitted) is defined using the complex

$$
\cdots \rightarrow C_{n+1}^b(G) \xrightarrow{\delta^b_{n+1}} C^n_b(G) \xrightarrow{\delta^b_n} C^{n-1}_b(G) \xrightarrow{\delta^b_{n-1}} \cdots \rightarrow C_2^b(G) \xrightarrow{\delta^b_1} C_1^b(G) \xrightarrow{\delta^b_0} \mathbb{R} \xrightarrow{\delta^b_{-1}} 0
$$

of bounded cochains $f: G \times \cdots \times G \rightarrow \mathbb{R}$, and $\delta_n^b = \delta_n|_{C_n^b(G)}$ is the bounded differential operator. Since $H^0_b(G) = \mathbb{R}$ and $H^1_b(G) = 0$ for any group $G$, investigation of bounded cohomology starts in dimension 2. One observes that $H^2_b(G)$ contains a subspace $H^2_b,2(G)$ (called the singular part of the second bounded cohomology group), which has a simple algebraic description in terms of quasicharacters and pseudocharacters, and the quotient space $H^2_b(G)/H^2_b,2(G)$ is canonically isomorphic to the bounded part of the ordinary cohomology group $H^2(G)$. See [6] for background and available results on bounded cohomology of groups. (For bounded cohomology of topological spaces, see [8].)

We recall that a function $F: G \rightarrow \mathbb{R}$ is called a quasicharacter if there exists a constant $C_F \geq 0$ such that

$$
|F(xy) - F(x) - F(y)| \leq C_F \quad \forall x, y \in G. \quad (1.2)
$$

A function $f: G \rightarrow \mathbb{R}$ is called a pseudocharacter if $f$ is a quasicharacter and, in addition,

$$
f(g^n) = nf(g) \quad \forall g \in G, \ n \in \mathbb{Z}. \quad (1.3)
$$
The notions of a quasicharacter and a pseudocharacter originally arose from the questions of stability of solutions of functional equations [9, 10, 11] and continuous representations of groups [12]. We use the following notation:

(i) \( X(G) \) is the space of additive characters \( G \rightarrow \mathbb{R} \);
(ii) \( QX(G) \) is the space of quasicharacters;
(iii) \( PX(G) \) is the space of pseudocharacters;
(iv) \( B(G) \) is the space of bounded functions.

Then

\[
H^2_{b,2}(G) \cong QX(G)/(X(G) \oplus B(G)) \cong PX(G)/X(G)
\]  

as vector spaces (cf. [6, Proposition 3.2 and Theorem 3.5]). Special interest in \( H^2_{b,2} \) is motivated in part by its connections with other structural properties of groups such as commutator length [1] and bounded generation [6]. (See [3] for a simple proof of triviality of \( H^2_{b,2} \) for Chevalley groups over rings of \( S \)-integers in algebraic number fields using bounded generation.) For example, Grigorchuk [7] (cf. also [5]) proved that the amalgamated product \( A_1 \ast_H A_2 \) does not have bounded generation provided that the number of double cosets of \( A_1 \) modulo \( H \) is at least 3 and \([A_2 : H] \geq 2\) by showing that \( \dim H^2_{b,2}(A_1 \ast_H A_2) = \infty \) in this case. The proof is based on the explicit construction (see Example 4.4) of an infinite family of linearly independent quasicharacters which naturally generalize the construction of quasicharacters for free groups. The quasicharacters for free groups were first constructed by Brooks [2], and Faĭziev showed that they can be used to find a basis for the space of pseudocharacters of a free group [4] (cf. [6, Theorem 5.7] for a shorter and more conceptual proof).

However, no systematic study of bounded cohomology of amalgamated products of groups has been undertaken. The goal of this paper is to provide the first step in an attempt to obtain general information about bounded cohomology of amalgamated products of groups. Since the main technical tool used to compute cohomology of the amalgamated product \( A_1 \ast_H A_2 \) is the Mayer-Vietoris exact sequence (see [14, Theorem 2.3])

\[
\cdots \rightarrow H^n(A_1 \ast_H A_2) \rightarrow H^n(A_1) \oplus H^n(A_2) \rightarrow H^n(H) \rightarrow H^{n+1}(A_1 \ast_H A_2) \rightarrow \cdots,
\]  

(1.5)

it is natural to try to exhibit an analog of this sequence for bounded cohomology. We construct an initial segment of this sequence for bounded cohomology (it starts in dimension 2) and we formulate our results in terms of spaces of pseudocharacters.

We begin by considering the case where the amalgamated subgroup is normal in both factors (Theorem 2.1). In the general case, we restrict our attention to the special class of pseudocharacters which we call \( H \)-spherical (Theorem 4.6); see Section 4 for relevant definitions and discussion.

The sequences (2.11) and (4.14) constructed in Theorems 2.1 and 4.6, respectively, reduce the problem of computation of spaces of pseudocharacters for amalgamated products of groups to that for free products of groups (terms on the left); the structure of the latter spaces is known [6, Proposition 4.3 and Remark 4.4].
We conclude this section with two easy facts which will be used throughout the paper without special reference.

**Lemma 1.1.** Any pseudocharacter is constant on conjugacy classes; a bounded pseudocharacter is trivial.

**Proof.** Let \( f \in PX(G) \) and suppose that \( f(\gamma x \gamma^{-1}) - f(x) = a \neq 0 \) for some \( x, \gamma \in G \). Then the difference \( f(\gamma x^n \gamma^{-1}) - f(x^n) = na \) is unbounded when \( n \to \infty \). On the other hand,

\[
|f(\gamma x^n \gamma^{-1}) - f(x^n)| = |f(\gamma x^n \gamma^{-1}) - f(\gamma) - f(x^n) - f(\gamma^{-1})| \leq 2C_f,
\]

(1.6) a contradiction. The second assertion is obvious. \( \square \)

2. **The case of a normal subgroup.** In this section, we will establish an analog of the initial segment of the Mayer-Vietoris sequence for spaces of pseudocharacters assuming that the amalgamated subgroup \( N \) is normal in both factors \( A_1 \) and \( A_2 \) (in which case it is also normal in the amalgamated product). To describe this sequence, we need to introduce some natural linear maps. First, we define

\[
\beta : PX(A_1 \ast_N A_2) \to PX(A_1) \oplus PX(A_2)
\]

(2.1) as \( \beta = (\beta_1, \beta_2) \), where

\[
\beta_i : PX(A_1 \ast_N A_2) \to PX(A_i), \quad i = 1, 2,
\]

(2.2) is the restriction map associated with the natural embedding \( A_i \hookrightarrow A_1 \ast_N A_2 \). Next, let

\[
y : PX(A_1) \oplus PX(A_2) \to PX(N)
\]

(2.3) be defined by

\[
y(f_1, f_2) = f_1 |_N - f_2 |_N.
\]

(2.4) In contrast to the usual Mayer-Vietoris sequence for the spaces of characters,

\[
0 \to X(A_1 \ast_N A_2) \xrightarrow{\beta} X(A_1) \oplus X(A_2) \xrightarrow{\gamma} X(N),
\]

(2.5) where \( \beta \) and \( \gamma \) are analogous to \( \beta \) and \( \gamma \) introduced above, the sequence for pseudocharacters will contain, at the extreme left, one extra term which is typically an infinite-dimensional vector space. To define it, we consider the embedding

\[
\alpha : PX((A_1/N) \ast (A_2/N)) \to PX(A_1 \ast_N A_2)
\]

(2.6) induced by the natural surjective homomorphism

\[
A_1 \ast_N A_2 \to (A_1/N) \ast (A_2/N)
\]

(2.7)
and let $PX_0((A_1/N) \ast (A_2/N))$ denote the kernel of the linear map

\[
\hat{\beta}: PX((A_1/N) \ast (A_2/N)) \rightarrow PX(A_1/N) \oplus PX(A_2/N),
\]

where

\[
\hat{\beta} = (\hat{\beta}_1, \hat{\beta}_2),
\]

is the restriction map induced by the natural embedding

\[
\begin{align*}
A_i/N &\hookrightarrow (A_1/N) \ast (A_2/N), & i &= 1, 2.
\end{align*}
\]

The above spaces and linear maps align in the following sequence.

**Theorem 2.1.** Let $N$ be a normal subgroup of $A_1$ and $A_2$. Then the sequence of vector spaces

\[
0 \rightarrow PX_0((A_1/N) \ast (A_2/N)) \xrightarrow{\alpha} PX(A_1 \ast_N A_2) \xrightarrow{\beta} PX(A_1/N) \oplus PX(A_2/N) \xrightarrow{\gamma} PX(N)
\]

is exact.

We will prove the theorem in the next section and will now derive two consequences.

**Corollary 2.2.** Given two arbitrary pseudocharacters $f_1$ and $f_2$ on the groups $A_1$ and $A_2$, respectively, there exists a pseudocharacter $f$ on the free product $A_1 \ast A_2$ such that $f|_{A_i} = f_i$, $i = 1, 2$.

**Corollary 2.3.** Let $A$ be an arbitrary group and $N$ its normal subgroup. Then the restriction homomorphism $\rho : PX(A \ast_N A) \rightarrow PX(A)$, induced by embedding $A$ into $A \ast_N A$ as either factor, is surjective. If, moreover, $[A : N] = 2$, then $\rho$ is an isomorphism.

**Proof.** For the first assertion, one needs to observe that for any $f \in PX(A)$, the pair $(f, f)$ belongs to Ker $\gamma$, and is therefore obtained as the restriction of a pseudocharacter on $A \ast_N A$. If $[A : N] = 2$, then $(A/N) \ast (A/N)$ is the infinite dihedral group. Since it is amenable, all its pseudocharacters are in fact characters [6, Theorem 2.1]. On the other hand, since it is generated by elements of order two, it does not have nonzero characters. This, in particular, implies that

\[
PX_0((A_1/N) \ast (A_2/N)) = 0,
\]

hence our second claim. \(\square\)

3. Proof of **Theorem 2.1**

3.1. **Exactness in the term** $PX(A_1) \oplus PX(A_2)$. The inclusion $\text{Im} \hat{\beta} \subset \text{Ker} \gamma$ being obvious, all we need to prove is that, given pseudocharacters $f_i \in PX(A_i)$, $i = 1, 2$, satisfying $f_1|_N = f_2|_N$, there exists a pseudocharacter $f \in PX(A_1 \ast_N A_2)$ such that $f|_{A_i} = f_i$. The following observation saves the (serious) trouble of verifying the condition $f(g^n) = nf(g)$.
**Lemma 3.1.** In the current notation, for the existence of a pseudocharacter $f$, it suffices to construct a quasicharacter $F \in QX(A_1 \ast_N A_2)$ such that the differences $F|_{A_i} - f_i$ are bounded for $i = 1, 2$.

**Proof.** Indeed, it follows from (1.4) that given such an $F$, there exists a pseudocharacter $f \in PX(A_1 \ast_N A_2)$ for which the difference $F - f$ is bounded. Then, for $i = 1, 2$, the difference

$$f_i - f|_{A_i} = (f_i - F|_{A_i}) + (F - f)|_{A_i}$$

is a bounded pseudocharacter of $A_i$, hence zero, proving that $f|_{A_i} = f_i$. □

The construction of such a quasicharacter $F \in QX(A_1 \ast_N A_2)$ rests on a specific choice of systems of representatives $X_i$ of all left cosets $\neq N$ in $A_i/N$ for $i = 1, 2$. Namely, it is possible to choose such systems of representatives $X_i$ having the following property:

(P1) if $x, y \in X_i$ and $xy \in N$, then either $x^2, y^2 \in N$ or $y = x^{-1}$.

Indeed, let $\bar{T}_i$ denote the set of elements of order two in $A_i/N$ and pick an arbitrary system of representatives $\bar{T}_i \subset A_i$ of the cosets from $\bar{T}_i$. Since, for each $x \in \bar{T}_i := (A_i/N) \setminus (\bar{T}_i \cup \{e\})$ (3.2)

(e the identity), we have $x \neq x^{-1}$, there exists a partition $\bar{T}_i = \bar{T}_i' \cup \bar{T}_i''$ such that $\bar{T}_i' \cap \bar{T}_i'' = \emptyset$ and $\bar{T}_i'' = (\bar{T}_i')^{-1}$. Choose an arbitrary system of representatives $T_i' \subset \bar{T}_i'$ of the cosets from $\bar{T}_i'$ and let $T_i = T_i' \cup (T_i')^{-1}$. Finally, let $X_i = S_i \cup T_i$.

Suppose that the systems of representatives $X_i$ with property (P1) have been chosen. We define an involutive transformation $\tau_i : X_i \rightarrow X_i$ by setting

$$\tau_i(x) = \begin{cases} x & \text{if } x \in S_i, \\ x^{-1} & \text{if } x \in T_i. \end{cases}$$

(3.3)

Now, we let $X = X_1 \cup X_2$ (disjoint union) and introduce a function $F$ and an involution $\tau$ on $X$ whose restrictions to $X_i$ are $f_i$ and $\tau_i$, respectively:

$$F(x) = f_i(x), \quad \tau(x) = \tau_i(x) \quad \text{if } x \in X_i.$$ (3.4)

Let $W$ be the set of all words of the form $x_1 \cdots x_n$, where $x_i \in X$ and, for every $i = 1, \ldots, n-1$, the elements $x_i$ and $x_{i+1}$ belong to different parts $X_1$ or $X_2$ of $X$ (by convention, the empty word is included in $W$ and corresponds to $n = 0$). Then each element $g \in G := A_1 \ast_N A_2$ admits a unique canonical presentation of the form

$$g = x_1 \cdots x_nh$$

(3.5)

for some $h \in N$ and some word $x_1 \cdots x_n \in W$ (cf., e.g., [13, Chapter I, Theorem 1]). Using the canonical form (3.5), we can extend $\tau$ to an involutive transformation of $G$ by setting

$$\tau(g) = \tau(x_n) \cdots \tau(x_1)h.$$ (3.6)
Let \( f_0 \in PX(N) \) denote the common restriction of \( f_1 \) and \( f_2 \) to \( N \):

\[
f_0 := f_1|_N = f_2|_N. \tag{3.7}
\]

It follows from (3.6) that for every \( g \in G \), we have \( g\tau(g) \in N \), so the expression \( f_0(g\tau(g)) \) makes sense. Let \( S = S_1 \cup S_2 \) and \( T = T_1 \cup T_2 \). We extend \( F \) to a function on \( A_1 \ast_N A_2 \) by the formula

\[
F(g) = \mu(g) + \eta(g), \tag{3.8}
\]

where

\[
\mu(g) = \frac{1}{2} f_0(g\tau(g)), \quad \eta(g) = \sum_{x_i \in T} F(x_i), \tag{3.9}
\]

and, by convention, \( \eta(g) = 0 \) if \( g \in N \) (in this definition, we use the unique canonical presentation (3.5)).

**Proposition 3.2.** The function \( F \) defined by (3.8) is a quasicharacter of \( A_1 \ast_N A_2 \) such that the differences \( F|_{A_i} - f_i \) are bounded for \( i = 1, 2 \).

First, we observe that for \( h \in N \), we have

\[
\mu(h) = \frac{1}{2} f_0(h^2) = f_0(h). \tag{3.10}
\]

In particular, \( F|_N = \mu|_N \) is a pseudocharacter with constant \( C = C_{f_0} \). Also, writing \( g \in G \) in the canonical form and using the fact that \( f_0 \) is the common restriction of pseudocharacters \( f_1 \) and \( f_2 \) to \( N \), we obtain

\[
\mu(ghg^{-1}) = \mu(h) \quad \forall g \in G, h \in N. \tag{3.11}
\]

Next, we will show that the difference \( F|_{A_i} - f_i \) is a bounded function on \( A_i \) for \( i = 1, 2 \). For \( g \in N \), we have

\[
F(g) = \frac{1}{2} f_0(g^2) = f_0(g). \tag{3.12}
\]

If \( g \in A_i \setminus N \), we write it in the form \( g = xh \) with \( h \in N, x \in X_i \). If \( x \in S_i \), then

\[
|F(g) - f_i(g)| = \left| \frac{1}{2} f_0(xhxh) - f_i(xh) \right| = \left| \frac{1}{2} f_i(x^2([x^{-1}hx]h)) - f_i(xh) \right| \leq \frac{1}{2} \left| f_i(x^2([x^{-1}hx]h)) - f_i(x^2) - 2f_i(h) \right| + \left| f_i(x) + f_i(h) - f_i(xh) \right| \leq 2C_{f_i}. \tag{3.13}
\]
If $x \in T_i$, then
\[
|F(g) - f_i(g)| = \frac{1}{2} |f_0(xhx^{-1}h) + f_i(x) - f_i(xh)| \\
\leq \frac{1}{2} |f_i((xhx^{-1})h) - 2f_i(h)| + |f_i(h) + f_i(x) - f_i(xh)|
\] (3.14)
\[
\leq \frac{3}{2} C f_i.
\]
In particular, we obtain
\[
F|_{A_i} \in QX(A_i).
\] (3.15)
To complete the proof of the proposition, it remains to show that $F$ is a quasicharacter on the entire amalgamated product $A_1 \ast_N A_2$. For a function $f$ on $G$, we define
\[
(\delta f)(g_1, g_2) = f(g_1g_2) - f(g_1) - f(g_2).
\] (3.16)
So, we need to show that $\delta F = \delta \mu + \delta \eta$ is bounded on $G \times G$. For convenience of further reference, we will collect, in the following lemma, some properties of the functions $\mu$ and $\eta$.

**Lemma 3.3.** (i) If $g \in G$, $h \in N$, then $|\mu(gh) - \mu(g) - \mu(h)| \leq C$.
(ii) If $g_1, \ldots, g_k \in G$ and $\tau(g_1 \cdots g_k) = \tau(g_k) \cdots \tau(g_1)$, then
\[
|\mu(g_1 \cdots g_k) - \sum_{i=1}^k \mu(g_i)| \leq \frac{k-1}{2} C.
\] (3.17)
(iii) $\eta(gh) = \eta(g)$ for any $g \in G$ and any $h \in N$.
(iv) If $x_1 \cdots x_n \in W$, then $\eta(x_1 \cdots x_n) = \eta(x_1 \cdots x_i) + \eta(x_{i+1} \cdots x_n)$ for any $i = 1, \ldots, n - 1$.
(v) $\eta(\tau(g)) = -\eta(g)$ for any $g \in G$.

**Proof.** Indeed, if $g \in G$ and $h \in N$, then $\mu(gh) = (1/2)f_0(gh\tau(g)h)$, whence
\[
|\mu(gh) - \mu(g) - \mu(h)| = \frac{1}{2} |f_0(g\tau(g)[\tau(g)^{-1}h\tau(g)]h) - f_0(g\tau(g)) - f_0(h^2)|,
\] (3.18)
and (i) follows. For (ii), one needs to observe that
\[
\mu(g_1 \cdots g_k) = \frac{1}{2} f_0(\left[\left(g_1 \cdots g_{k-1}(g_k \tau(g_k))(g_1 \cdots g_{k-1})^{-1}\right)\right. \\
\left.\times \left[(g_1 \cdots g_{k-2})(g_{k-1} \tau(g_{k-1}))(g_1 \cdots g_{k-2})^{-1}\right)\cdots [g_1 \tau(g_1)]\right].
\] (3.19)
Properties (iii), (iv), and (v) follow immediately from the definition of $\eta$. 

For two given elements $g_1, g_2 \in G$, we pick the canonical presentations
\[
g_1 = x_1 \cdots x_m h_1, \quad g_2 = y_1 \cdots y_n h_2,
\] (3.20)
where \( x_1 \cdots x_m, y_1 \cdots y_n \in W \) and \( h_1, h_2 \in N \). We first consider the easiest case where \( x_m \) and \( y_1 \) belong to different factors \( A_i, i = 1, 2 \). In this case, the canonical presentation of \( g_1 g_2 \) is

\[
 g_1 g_2 = x_1 \cdots x_m y_1 \cdots y_n h,
\]

where

\[
 h = \left[ (y_1 \cdots y_n)^{-1} h_1 (y_1 \cdots y_n) \right] h_2 \in N.
\]

It follows from Lemma 3.3(iii) and (iv) that we conclude from Lemma 3.3(i) and (ii) and (3.11) that

\[
 v
\]

where, by convention,

\[
 w
\]

have the following factorizations:

\[
 w
\]

It follows, in this case, that

\[
 \tau(x_1 \cdots x_m, y_1 \cdots y_n) = \tau(y_1 \cdots y_n) \tau(x_1 \cdots x_m),
\]

we conclude from Lemma 3.3(i) and (ii) and (3.11) that

\[
 |\mu(g_1 g_2) - \mu(h_1) - \mu(h_2) - \mu(x_1 \cdots x_m) - \mu(y_1 \cdots y_n)|
\]

\[
 \leq |\mu(x_1 \cdots x_m y_1 \cdots y_n h) - \mu(x_1 \cdots x_m y_1 \cdots y_n) - \mu(h)|
\]

\[
 + |\mu(h) - \mu(h_1) - \mu(h_2)|
\]

\[
 + |\mu(x_1 \cdots x_m y_1 \cdots y_n) - \mu(x_1 \cdots x_m) - \mu(y_1 \cdots y_n)|
\]

\[
 \leq C + C + \frac{C}{2} = \frac{5}{2} C.
\]

On the other hand,

\[
 |[\mu(g_1) + \mu(g_2)] - [\mu(h_1) + \mu(h_2) + \mu(x_1 \cdots x_m) + \mu(y_1 \cdots y_n)]| \leq 2C.
\]

It follows, in this case, that

\[
 |(\delta F)(g_1, g_2)| \leq \frac{9}{2} C.
\]

To consider the general case, we need to introduce the fragments of \( g_1 \) and \( g_2 \) that cancel out in \( g_1 g_2 \). Let \( k \) be the largest integer less than or equal to \( \min\{m, n\} \) such that \( x_{m-i+1} y_i \in N \) for all \( i = 1, \ldots, k \). We introduce the following elements:

\[
 w_1 = x_1 \cdots x_{m-k-1}, \quad u_1 = x_{m-k}, \quad v_1 = x_{m-k+1} \cdots x_m,
\]

\[
 v_2 = y_1 \cdots y_k, \quad u_2 = y_{k+1} \cdots y_n, \quad w_2 = y_{k+2} \cdots y_n,
\]

where, by convention, \( v_1 = 1 \) if \( k = 0 \), \( w_1 = 1 \) if \( m = k + 1 \), and \( u_1 = 1 \) if \( m = k \), with similar rules for \( v_2, u_2, \) and \( w_2 \). We observe that \( v_2 = \tau(v_1) \), so, letting \( v = v_1 \), we have the following factorizations:

\[
 g_1 = w_1 u_1 v h_1, \quad g_2 = \tau(v) u_2 w_2 h_2.
\]

It follows from our construction that both \( u_1 \) and \( u_2 \) belong to the same factor \( A_i \), so we can write

\[
 u_1 u_2 = zh \quad \text{for some } h \in N, \ z \in X.
\]
We claim that

$$| (δμ)(g_1, g_2) − (δμ)(u_1, u_2) | \leq 10C. \quad (3.30)$$

This estimation is a consequence of the following three inequalities that reflect the three-step transition from $g_1, g_2$ to $u_1, u_2$.

**Lemma 3.4.** (i) Let $s_1 = w_1u_1v, s_2 = τ(v)u_2w_2$, so that $g_i = s_ih_i, i = 1, 2$; then

$$| (δμ)(g_1, g_2) − (δμ)(s_1, s_2) | \leq 4C. \quad (3.31)$$

(ii) Let $t_1 = w_1u_1, t_2 = u_2w_2$, so that $s_1 = t_1v, s_2 = τ(v)t_2$; then

$$| (δμ)(s_1, s_2) − (δμ)(t_1, t_2) | \leq 2C. \quad (3.32)$$

(iii) $| (δμ)(t_1, t_2) − (δμ)(u_1, u_2) | \leq 4C.$

**Proof.** (i) We have

$$g_1g_2 = s_1s_2 ([s_2^{-1}h_1s_2]h_2). \quad (3.33)$$

Using Lemma 3.3(i) and (3.11), we obtain the following two inequalities:

$$| (μ(g_1g_2) − μ(s_1s_2) − μ(h_1) − μ(h_2) | \leq 2C,$n

$$| (μ(g_1) + μ(g_2)) − (μ(s_1) + μ(s_2) + μ(h_1) + μ(h_2)) | \leq 2C, \quad (3.34)$$

from which (i) follows.

(ii) We have

$$s_1s_2 = t_1t_2 (t_2^{-1}[vτ(v)]t_2). \quad (3.35)$$

Using Lemma 3.3(i) and (3.11), we obtain

$$| μ(s_1s_2) − μ(t_1t_2) − μ(vτ(v)) | \leq C. \quad (3.36)$$

Since $τ(s_1) = τ(v)τ(t_1)$ and $τ(s_2) = τ(t_2)v$, Lemma 3.3(ii) combined with the observation that $μ(τ(v)) = μ(v)$ implies that

$$| (μ(s_1) + μ(s_2)) − (μ(t_1) + μ(t_2) + 2μ(v)) | \leq C. \quad (3.37)$$

But $μ(vτ(v)) = f_0(vτ(v)) = 2μ(v)$, and (ii) follows.

(iii) We have

$$t_1t_2 = w_1zw_2 (w_2^{-1}hw_2). \quad (3.38)$$

Since $τ(w_1zw_2) = τ(w_2)τ(z)τ(w_1)$, Lemma 3.3(i) and (ii) and (3.11) imply that

$$| μ(t_1t_2) − (μ(w_1) + μ(z) + μ(w_2) + μ(h)) | \leq 2C, \quad (3.39)$$
and therefore

$$| \mu(t_1 t_2) - (\mu(w_1) + \mu(z h) + \mu(w_2)) | \leq 3C. \quad (3.40)$$

On the other hand, since $\tau(t_1) = \tau(u_1) \tau(w_1)$ and $\tau(t_2) = \tau(w_2) \tau(u_2)$, we have

$$| (\mu(t_1) + \mu(t_2)) - (\mu(w_1) + \mu(u_1) + \mu(w_2) + \mu(u_2)) | \leq C. \quad (3.41)$$

Since $zh = u_1 u_2$, we obtain (iii).

Next, we calculate $(\delta \eta)(g_1, g_2)$ using Lemma 3.3(iii), (iv), and (v):

$$(\delta \eta)(g_1, g_2) = \left[ \eta(w_1) + \eta(z) + \eta(w_2) \right] - \left[ \eta(w_1) + \eta(u_1) + \eta(v) \right] - \left[ \eta(v) + \eta(u_2) + \eta(w_2) \right] = \eta(z) - \eta(u_1) - \eta(u_2) \quad (3.42)$$

From (3.30) and (3.42), we obtain

$$| (\delta F)(g_1, g_2) - (\delta F)(u_1, u_2) | \leq 10C. \quad (3.43)$$

On the other hand, since both $u_1$ and $u_2$ belong to the same factor $A_i$, (3.15) implies that $(\delta F)(u_1, u_2)$ is bounded, and we finally conclude that $(\delta F)(g_1, g_2)$ is bounded, completing the proof of Proposition 3.2.

3.2. Exactness in the term $PX(A_1 *_N A_2)$. The exactness of (2.11) in $PX(A_1 *_N A_2)$ is based on the following fact.

**Lemma 3.5.** Let $G$ be an arbitrary group and $N$ its normal subgroup. If a pseudocharacter $f \in PX(G)$ has zero restriction to $N$, then it satisfies $f(gh) = f(g)$ for all $h \in N$, $g \in G$. In other words, the natural sequence

$$0 \rightarrow PX(G/N) \rightarrow PX(G) \rightarrow PX(N) \quad (3.44)$$

is exact.

**Proof.** Suppose that $f(gh) \neq f(g)$ for some $h \in N$, $g \in G$, and let $a = f(gh) - f(g)$. Then

$$| f((gh)^n) - f(g^n) | = n |a| \rightarrow \infty \quad \text{when} \quad n \rightarrow \infty. \quad (3.45)$$

On the other hand, $(gh)^n$ can be written as $g^n h'$ for $h' \in N$, so

$$| f((gh)^n) - f(g^n) | = | f(g^n h') - f(g^n) - f(h') | \quad (3.46)$$

is bounded independent of $n$. \hfill \Box

If $f \in \text{Ker} \beta$, then $f|_N = 0$. Lemma 3.5 implies that $f$ factors through the group homomorphism

$$A_1 *_N A_2 \rightarrow (A_1 *_N A_2)/N \cong (A_1/N) * (A_2/N), \quad (3.47)$$
immediately implying that $f \in \text{Im} \alpha$ and proving the inclusion $\text{Ker} \beta \subset \text{Im} \alpha$. The opposite inclusion is obvious.

3.3. Remarks. The general construction of a quasicharacter $F \in QX(A_1 \ast_N A_2)$ lifting given pseudocharacters $f_i \in PX(A_i)$ ($i = 1, 2$) with the same restrictions to $N$ essentially simplifies in the following two particular cases:

(1) $S = \emptyset$, that is, when the quotients $A_1/N$ and $A_2/N$ do not have elements of order two;

(2) $T = \emptyset$, that is, when these quotients are groups of exponent two.

In the first case, with the above choice of the coset representative systems, $F$ can be extended “by linearity”:

$$F(x_1 \cdots x_n h) = \sum_{i=1}^n F(x_i) + f_0(h).$$  \hfill (3.48)

4. The case of an arbitrary subgroup. If we do not assume that the amalgamated subgroup $H$ is normal in both factors $A_1$ and $A_2$, then two difficulties arise. First of all, when we switch representatives of cosets modulo $H$ with elements of $H$ in order to write a product of two words in the canonical form, the representative of a coset will change. Secondly, there is no natural candidate for the term on the extreme left. We restrict our attention to special classes of quasicharacters and pseudocharacters which we call strongly $H$-spherical and $H$-spherical, respectively. We would like to point out that the only explicitly known quasicharacters on amalgamated products (see Example 4.4) are strongly $H$-spherical. Below is a brief analysis of what restrictions should be imposed on pseudocharacters.

Let $H$ be a subgroup of a group $G$. The first conjecture, which naturally arises after preliminary considerations, is to look at the following class of pseudocharacters:

$$\{f \in PX(G) \mid f(xb) = f(x) + f(b) \forall x \in G, h \in H\}. \hfill (4.1)$$

However, it has to be discarded as the following observation shows.

**Lemma 4.1.** Let $f \in PX(G)$ and suppose that

$$f(xb) = f(x) + f(b) \hfill (4.2)$$

holds for all $x \in G$ and all $b$ in a certain subset $B \subset G$. Then (4.2) also holds for all $x \in G$ and all $b$ in the normal subgroup $N \subset G$ generated by $B$. Moreover, if $f(xb) = f(x)$ for all $x \in G$ and $b \in B$, then the same is true for all $x \in G$ and all $b \in N$.

**Proof.** It suffices to show that

$$H = \{b \in G \mid f(xb) = f(x) + f(b) \forall x \in G\} \hfill (4.3)$$

is a normal subgroup of $G$. If $b_1, b_2 \in H$, then

$$f(x(b_1b_2)) = f((xb_1)b_2) = f(x) + f(b_1) + f(b_2) = f(x) + f(b_1b_2) \hfill (4.4)$$
for all \(x \in G\), so \(b_1 b_2 \in H\). Similarly, for \(b \in H\), we have

\[
f(x) = f((xb)b^{-1}) = f(xb) - f(b) = f(xb) + f(b^{-1})
\]

which means that

\[
f(yb^{-1}) = f(y) + f(b^{-1})
\]

for all \(y \in G\), that is, \(b^{-1} \in H\). Finally, for fixed \(b \in H\), \(g \in G\), and an arbitrary \(x \in G\), we have

\[
f(x(gbg^{-1})) = f((g^{-1}xg)b) = f(g^{-1}xg) + f(b) = f(x) + f(gbg^{-1}),
\]

proving that \(gbg^{-1} \in H\), hence our first assertion. The argument for the second assertion is similar: one shows that

\[
\{b \in G \mid f(xb) = f(x) \ \forall x \in G\}
\]

is a normal subgroup of \(G\).

**Corollary 4.2.** Suppose that \(G\) has the property that every nontrivial normal subgroup has finite index. If the center of \(G\) is trivial, then, given a nonzero pseudocharacter \(f \in PX(G)\) and an element \(a \in G\), \(a \neq e\), there exists \(x \in G\) such that \(f(xa) \neq f(x) + f(a)\).

**Proof.** Assume the contrary. Then, according to Lemma 4.1, the equality \(f(xb) = f(x) + f(b)\) holds for all \(x \in G\) and all \(b \in N := \text{the normal subgroup of } G \text{ generated by } a\). In particular, the restriction \(f|_N\) is a character of \(N\). Moreover, since \(f\) is constant on conjugacy classes in \(G\), it vanishes on the commutator subgroup \([G,N]\). Since the center of \(G\) is trivial, \([G,N] \neq \{e\}\) and therefore has finite index in \(G\). But then \(f|_{[G,N]} = 0\) implies \(f = 0\) as required.

Further analysis leads to the following definition.

**Definition 4.3.** Let \(H\) be a subgroup of a group \(G\). A quasicharacter \(F \in QX(G)\) is strongly \(H\)-spherical if

(i) \(F(h_1gh_2) = F(h_1) + F(g) + F(h_2)\) for all \(h_1, h_2 \in H\) and \(g \in G\);

(ii) \(F(g^{-1}) = -F(g)\) for all \(g \in G\).

A pseudocharacter \(f \in PX(G)\) is \(H\)-spherical if there exists a strongly \(H\)-spherical quasicharacter \(F\) such that the difference \(f - F\) is bounded.

**Example 4.4.** We briefly recall the construction of quasicharacters used to prove the result in [7] (a similar construction with a more geometric flavor was discovered independently in [5]) as these are essentially the only known quasicharacters on amalgamated products. A product \(u_1 \cdots u_n\) in the amalgamated product \(G = A_1 \ast_H A_2\) is called reduced if the following holds:

(i) every \(u_i\) belongs to either \(A_1\) or \(A_2\);

(ii) \(u_i\) and \(u_{i+1}\) belong to different factors \(A_j, j = 1, 2\);
(iii) if \( n > 1 \), then none of \( u_i \) belongs to \( H \);
(iv) if \( n = 1 \), then \( u_1 \neq 1 \).

Grigorchuk’s construction is based on the fact that if \( u_1 \cdots u_n = v_1 \cdots v_m \) are two reduced products in \( G \), then \( n = m \) and, for every \( i = 1, \ldots, n \), the elements \( u_i \) and \( v_i \) belong to the same double coset modulo \( H \), which is a simple consequence of the structure theorem for reduced words in amalgamated products. Two words \( u \) and \( v \) are called generally equal if there exist reduced products \( u = u_1 \cdots u_n \) and \( v = v_1 \cdots v_m \) such that \( n = m \) and, for every \( i = 1, \ldots, n \), the elements \( u_i \) and \( v_i \) belong to the same double coset modulo \( H \). A reduced word \( w = w_1 \cdots w_k \) is said to generally occur in a reduced word \( u = u_1 \cdots u_n \) if there is a subword \( u_i \cdots u_{i+k-1} \) of \( u \) which is generally equal to \( w_1 \cdots w_k \). We define \( \#_w(u) \) as the number of general occurrences of \( w \) in \( u \), and, for any \( g \in G \), we let

\[
F_w(g) = \#_w(u) - \#_{w^{-1}}(u),
\]

where \( u \) is any reduced word representing \( g \). It turns out that if \( w \) is a reduced word, then the function \( F_w \) is a quasicharacter of \( G \). In case \( |H \backslash A_1/H| \geq 3 \) and \( |A_2:H| \geq 2 \), it is possible to exhibit an infinite sequence of reduced words \( \{w_n\} \) such that the quasicharacters \( \{F_{w_n}\} \) are linearly independent, whence the infinite dimensionality of the second bounded cohomology group. It is immediate from (4.9) that Grigorchuk’s quasicharacters are strongly \( H \)-spherical.

Notice that if \( F \) is a strongly \( H \)-spherical quasicharacter, then the restriction of \( F \) to \( H \) is a character of \( H \); in particular, \( F(1) = 0 \). Also, if \( H_1 \) and \( H_2 \) are subgroups of a group \( G \) and \( F \) is a strongly \( H_i \)-spherical quasicharacter for \( i = 1, 2 \), then \( F \) is a strongly \( H \)-spherical quasicharacter, where \( H \) is the subgroup of \( G \) generated by \( H_1 \) and \( H_2 \). We denote the space of \( H \)-spherical pseudocharacters of \( G \) by \( PX(G)_H \).

In the sequel, we will need the following observation.

**Lemma 4.5.** If \( F \) is a strongly \( H \)-spherical quasicharacter and \( g_1, g_2 \in H \) for some \( g_1, g_2 \in G \), then \( F(g_1 g_2) = F(g_1) + F(g_2) \).

**Proof.** Our claim follows from

\[
F(g_2) = F(g_1^{-1}) + F(g_1 g_2) = -F(g_1) + F(g_1 g_2).
\]

The canonical (surjective) homomorphism

\[
\theta : A_1 \ast A_2 \to A_1 \ast_H A_2
\]

gives rise to the following embedding of the spaces of pseudocharacters:

\[
i : PX(A_1 \ast_H A_2) \hookrightarrow PX(A_1 \ast A_2),
\]

which allows us to identify the former with a subspace of the latter. We denote the kernel of the linear map

\[
\beta : PX(A_1 \ast A_2) \to PX(A_1) \oplus PX(A_2)
\]
by \( PX_0(A_1 \ast A_2) \). The following analog of Theorem 2.1 holds for \( H \)-spherical pseudocharacters in the case when the amalgamated subgroup \( H \) is arbitrary.

**Theorem 4.6.** Let \( H \) be an arbitrary subgroup of \( A_1 \) and \( A_2 \), let \( \theta : A_1 \ast A_2 \to A_1 \ast_H A_2 \) be the canonical homomorphism, let \( \mathcal{H} \) be the subgroup of \( A_1 \ast A_2 \) generated by \( H \ast H \) and \( \text{Ker} \theta \), and let \( PX_{0,\text{Ker} \theta}(A_1 \ast A_2)_\mathcal{H} \) be the subspace of \( PX_0(A_1 \ast A_2)_\mathcal{H} \) consisting of pseudocharacters with trivial restriction to \( \text{Ker} \theta \). Then the sequence of vector spaces

\[
0 \to PX_{0,\text{Ker} \theta}(A_1 \ast A_2)_\mathcal{H} \to PX(A_1 \ast_H A_2)_H \to PX(A_1)_H \oplus PX(A_2)_H \to PX(H)
\]

(4.14)
is exact.

5. Proof of Theorem 4.6

5.1. Exactness in the term \( PX(A_1)_H \oplus PX(A_2)_H \). To prove the exactness of (4.14) in the term \( PX(A_1)_H \oplus PX(A_2)_H \), we need to show that given \( H \)-spherical pseudocharacters \( f_i \in PX(A_i)_H, i = 1,2 \), satisfying \( f_1|_H = f_2|_H \), there exists a \( H \)-spherical pseudocharacter \( f \in PX(A_1 \ast_H A_2)_H \) such that \( f|_{A_i} = f_i \). Let \( F_i \in QX(A_i), i = 1,2 \), be strongly \( H \)-spherical quasicharacters with the property that the differences \( F_i - f_i \) are bounded; also, let \( C = \max\{C_{F_1}, C_{F_2}\} \). An analog of Lemma 3.1 shows that for the existence of \( f \), it suffices to construct a strongly \( H \)-spherical quasicharacter \( F \in QX(A_1 \ast_H A_2) \) with the property that the differences \( F|_{A_i} - F_i \) are bounded for \( i = 1,2 \).

Let \( X_i \) be an arbitrary system of representatives of left cosets \( \neq H \) in \( A_i/H, i = 1,2 \), and let \( X = X_1 \cup X_2 \). Similarly to Section 3, we introduce a function \( F \) on \( X \) whose restriction to \( X_i \) is \( F_i \):

\[
F(x) = F_i(x) \quad \text{if } x \in X_i,
\]

(5.1)

and let \( W \) be the set of all words of the form \( x_1 \cdots x_n \), where \( x_i \in X \) and for every \( i = 1, \ldots, n - 1 \), the elements \( x_i \) and \( x_{i+1} \) belong to different parts \( X_1 \) or \( X_2 \) of \( X \) (by convention, the empty word is included in \( W \) and corresponds to \( n = 0 \)). Then any element \( g \in G := A_1 \ast_H A_2 \) admits a unique canonical presentation of the form

\[
g = x_1 \cdots x_n h
\]

(5.2)

for some \( h \in H \) and some word \( x_1 \cdots x_n \in W \).

Since the restrictions of \( f_i \) to \( H \) coincide, the difference \( F_1|_H - F_2|_H \) is bounded. However, the restrictions \( F_i|_H, i = 1,2 \), are the characters of \( H \), hence

\[
F_1|_H - F_2|_H = 0
\]

(5.3)

and we let \( F_0 \) denote the common restriction of \( F_1 \) and \( F_2 \) to \( H \):

\[
F_0 := F_1|_H - F_2|_H \in X(H).
\]

(5.4)

We now extend \( F \) to a function on \( A_1 \ast_H A_2 \) using the canonical form (5.2):

\[
F(g) = F(x_1) + \cdots + F(x_n) + F_0(h).
\]

(5.5)
To complete the proof of exactness of (4.14) in $PX(A_1)_H \otimes PX(A_2)_H$, it suffices to establish the following.

**Proposition 5.1.** The function $F$ defined by (5.5) is a strongly $H$-spherical quasicharacter of $A_1 \ast_H A_2$ such that the differences $F|_{A_i} - F_i$ are bounded for $i = 1, 2$.

The property that the differences $F|_{A_i} - F_i$ are bounded for $i = 1, 2$ follows immediately from (5.5) (moreover, $F|_{A_i} = F_i$).

Next, we are going to show that $F$ is a quasicharacter of $A_1 \ast_H A_2$. When we switch a representative of a coset modulo $H$ and an element of $H$, both of them will change. Since it is necessary to keep track of all these changes, we introduce the following notation: given elements $x \in X_i$ and $h \in H$, there exist elements $x^{(h)} \in X_i$ and $h^{(x)} \in H$ such that

$$h x = x^{(h)} h^{(x)}.$$  

To simplify notation, we will write $h^{(x_1, x_2)}$ instead of $(h^{(x_1)})^{(x_2)}$ and similarly for $x^{(h_1, h_2)}$. From (5.6), we derive that

$$F(x) + F(h) = F(x^{(h)}) + F(h^{(x)})$$  

which is a crucial equality in our argument. One of the main consequences of this equality is the following fact which follows from (5.7) by induction on $m$.

**Lemma 5.2.** Let $y_1, \ldots, y_m \in X$ and $h \in H$. Then

$$F(y_1^{(h)}) + F(y_2^{(h y_1)}) + F(y_3^{(h y_1 y_2)}) + \cdots + F(y_m^{(h y_1 \cdots y_{m-1})}) + F(h^{y_1 \cdots y_m}) = F(y_1) + \cdots + F(y_m) + F(h).$$  

(5.8)

Given two elements $g_1, g_2 \in G$, we fix their canonical presentations

$$g_1 = x_1 \cdots x_m h_1, \quad g_2 = y_1 \cdots y_n h_2,$$

where $x_1 \cdots x_m, y_1 \cdots y_n \in W$ and $h_1, h_2 \in H$. Suppose first that $x_m$ and $y_1$ belong to different factors $A_i$, $i = 1, 2$. Then the canonical presentation of $g_1 g_2$ is

$$g_1 g_2 = x_1 \cdots x_m y_1^{(h_1)} y_2^{(h y_1)} y_3^{(h y_1 y_2)} \cdots y_n^{(h y_1 \cdots y_{n-1})} (h_1^{y_1 \cdots y_n} h_2)$$  

(5.10)

and

$$|(\delta F)(g_1, g_2)| \leq \left| F(y_1^{(h_1)}) + F(y_2^{(h y_1)}) + \cdots + F(y_n^{(h y_1 \cdots y_{n-1})}) + F(h_1^{y_1 \cdots y_n}) \right| + C.$$  

(5.11)

**Lemma 5.2** implies that, in this case, $|(\delta F)(g_1, g_2)| \leq C$.

In the general case, there might be some cancelation in the middle in the product $g_1 g_2$, and we indicate several steps to write the canonical form of $g_1 g_2$ in a convenient
way. First, we write it in the form (which is not a canonical form in general)

\[ g_1g_2 = x_1 \cdots x_m y_1^{(h_1)} y_2^{(h_2)} y_3^{(h_3)} \cdots y_n^{(h_n)} (h_1^{(y_1 \cdots y_n)}) h_2 \]  \hspace{1cm} (5.12)

and let

\[ z_1 = y_1^{(h_1)}, z_2 = y_2^{(h_2)}, \ldots, z_n = y_n^{(h_n \cdots y_{n-1})} \in X, \]  \hspace{1cm} (5.13)

then (5.12) becomes

\[ g_1g_2 = x_1 \cdots x_m z_1 \cdots z_n h_0. \]  \hspace{1cm} (5.14)

It remains to consider the case when \( x_m \) and \( z_1 \) belong to the same factor \( A_i \); then

\[ x_m z_1 = u_1 a_1, \]  \hspace{1cm} (5.15)

where \( u_1 \in X \) or \( u_1 = e \) and \( a_1 \in H \). If \( u_1 \in X \), then

\[ g_1g_2 = x_1 \cdots x_{m-1} u_1 z_2 \left( a_1^{(x_1)} z_3 \left( a_2^{(x_2)} z_4 \cdots z_n \left( a_1^{(x_2 \cdots x_{n-1})} h_0 \right) \right) \right) \]  \hspace{1cm} (5.16)

is the canonical form of \( g_1g_2 \). If \( u_1 = e \), then

\[ g_1g_2 = x_1 \cdots x_{m-1} a_1 z_2 \cdots z_n h_0 = x_1 \cdots x_{m-1} z_2 \left( a_1^{(x_2)} z_3 \cdots z_n h_0 \right). \]  \hspace{1cm} (5.17)

Notice that we do not transfer \( a_1 \) all the way to the right. Since \( x_{m-1} \) and \( z_2 \) must belong to the same \( X_i \), we next write

\[ x_{m-1} z_2^{(a_1)} a_1^{(z_2)} = u_2 a_2, \]  \hspace{1cm} (5.18)

where \( u_2 \in X \) or \( u_2 = e \) and \( a_2 \in H \). We continue this process until we find a positive integer \( k \) such that

\[ x_{m-j+1} z_j^{(a_{j-1})} a_{j-1} = a_j \in H \quad \text{for} \ 2 \leq j \leq k-1, \]  \hspace{1cm} (5.19)

but

\[ x_{m-k+1} z_k^{(a_{k-1})} a_{k-1} = u_k a_k, \]  \hspace{1cm} (5.20)

where \( u_k \in X \) and \( a_k \in H \). Then the canonical form of \( g_1g_2 \) is

\[ g_1g_2 = x_1 \cdots x_{m-k} u_k z_{k+1}^{(a_k)} z_{k+2}^{(a_k^{(z_{k+1})})} \cdots z_n^{(a_k^{(z_{k+1} \cdots z_{n-1})})} \left( a_k^{(z_{k+1} \cdots z_n)} h_0 \right). \]  \hspace{1cm} (5.21)

Before we can estimate \( |(\delta F)(g_1,g_2)| \), we need the following fact.

**Lemma 5.3.** In the current notation,

\[ |F(u_k) + F(a_k) - [F(x_{m-k+1}) + \cdots + F(x_m) + F(z_1) + \cdots + F(z_k)]| \leq C. \]  \hspace{1cm} (5.22)
PROOF. Since $F_i$, $i = 1, 2$, are strongly $H$-spherical quasicharacters, then, using (5.7), we conclude that

$$F(u_k) + F(a_k) = F_i(u_k) + F_i(a_k) = F_i(u_k a_k)$$

$$= F_i\left(x_{m-k+1}z_k^{(a_{k-1})}a_{k-1}^{(z_k)}\right)$$

$$= F_i\left(x_{m-k+1}z_k^{(a_{k-1})} + F(a_{k-1}^{(z_k)})\right)$$

$$= F_i(a_{k-1}) + F_i(x_{m-k+1}) + F_i(z_k)$$

$$+ \left[F_i\left(x_{m-k+1}z_k^{(a_{k-1})}\right) - F_i(x_{m-k+1}) - F_i(z_k^{(a_{k-1})})\right].$$

For $2 \leq j \leq k-1, x_{m-j+1}z_j^{(a_{j-1})} \in H$ by (5.19), so Lemma 4.5 and (5.7) imply that

$$F_i(a_j) = F_i\left(x_{m-j+1}z_j^{(a_{j-1})}a_{j-1}^{(z_j)}\right)$$

$$= F_i\left(x_{m-j+1}z_j^{(a_{j-1})}\right) + F_i(a_{j-1}^{(z_j)})$$

$$= F_i(x_{m-j+1}) + F_i(z_j^{(a_{j-1})}) + F_i(a_{j-1}^{(z_j)})$$

$$= F_i(x_{m-j+1}) + F_i(z_j) + F_i(a_{j-1}).$$

Finally,

$$F_i(a_1) = F_i(x_m z_1) = F_i(x_m) + F_i(z_1)$$

by Lemma 4.5. \qed

We obtain the following inequalities which show that $F \in QX(A_1 \ast_H A_2)$:

$$\left|\delta F(g_1, g_2)\right| \leq \left|\left[F(x_1) + \cdots + F(x_{m-k}) + F(u_k) + F(a_k)\right.ight.$$  

$$
+ F(z_{k+1}) + \cdots + F(z_n) + F(h_0)\right|$$

$$- \left[F(x_1) + \cdots + F(x_m) + F(h_1)\right]$$

$$- \left[F(y_1) + \cdots + F(y_n) + F(h_2)\right]| + C \quad \text{(Lemma 5.2)}$$

$$\leq \left|\left[F(z_1) + \cdots + F(z_n) + F(h_0)\right.\right.$$  

$$
- \left[F(y_1) + \cdots + F(y_n) + F(h_2) + F(h_1)\right]| + 2C \quad \text{(Lemma 5.3)}$$

$$\leq 3C \quad \text{(Lemma 5.2)}.$$

To finish the proof of Proposition 5.1, it remains to show that $F$ satisfies Definition 4.3(i) and (ii).

To prove (i), we write an arbitrary $g \in G$ in the canonical form $g = x_1 \cdots x_n h$; then, for any $h_1, h_2 \in H$, the canonical form of $h_1 g h_2$ is

$$x_1^{(h_1)} x_2^{(h_1 x_1)} \cdots x_n^{(h_1 x_1 \cdots x_{n-1})} \left(h_1^{(x_1, \ldots, x_n)} h h_2\right).$$

(5.27)

Since the restriction of $F$ to $H$ is a character of $H$, we obtain

$$F\left(h_1^{(x_1, \ldots, x_n)} h h_2\right) = F\left(h_1^{(x_1, \ldots, x_n)}\right) + F(h) + F(h_2)$$

(5.28)
and Lemma 5.2 implies that
\[
F(h_1 g h_2) = F(x_1) + F(x_2) + \cdots + F(x_n) + F(h_1) + F(h) + F(h_2)
\]
\[
= F(h_1) + F(g) + F(h_2)
\]  
(5.29)
as required.

To prove (ii), we first suppose that the canonical form of \( g \in G \) is \( x_1 \cdots x_n \), that is, there is no \( H \)-component. Then the canonical form of \( g^{-1} \) is \( x_n^{-1} \cdots x_1^{-1} \) and
\[
F(g^{-1}) = F_i(x_n^{-1}) + \cdots + F_i(x_1^{-1}) = -F_i(x_n) - \cdots - F_i(x_1) = -F(g).
\]  
(5.30)
In the general case, write \( g = g_0 h \), where the canonical form of \( g_0 \) has no \( H \)-component. Since we have already shown that \( F \) satisfies (i) of strongly \( H \)-spherical quasicharacters, we use (5.30) to obtain
\[
F(g^{-1}) = F(h^{-1} g_0^{-1}) = F(h^{-1}) + F(g_0^{-1}) = -F(h) - F(g_0) = -F(g)
\]  
(5.31)
as required.

5.2. Exactness in the term \( PX(A_1 *) A_2 \). Given \( f \in PX_0(A_1 *) A_2 \) = ker \( \beta \), we let \( F \) denote the corresponding strongly \( H \)-spherical quasicharacter of \( A_1 *) A_2 \). We claim that \( \tilde{F} := F \circ \theta \) is a strongly \( \mathcal{H} \)-spherical quasicharacter of \( A_1 * A_2 \). Indeed, the bounded-ness of \( \delta \tilde{F} \) follows from that of \( \delta F \) and, moreover, \( \tilde{F} \) is both strongly \( (H *) \)-spherical (since \( \theta (H *) = H \)) and strongly ker \( \theta \)-spherical, hence strongly \( \mathcal{H} \)-spherical. There exists a pseudocharacter \( \tilde{f} \in PX(A_1 * A_2) \) such that the difference \( \tilde{F} - \tilde{f} \) is bounded. Thus, \( \tilde{F} \) is an \( \mathcal{H} \)-spherical pseudocharacter of \( A_1 * A_2 \) and, clearly,
\[
\tilde{f}|_{A_1} = \tilde{f}|_{A_2} = \tilde{f}|_{\text{Ker} \theta} = 0.
\]  
(5.32)
This shows the inclusion \( \text{Ker} \beta \subset PX_0, \text{ker} \theta (A_1 * A_2) \).

For the opposite inclusion, we consider \( \tilde{f} \in PX_0, \text{ker} \theta (A_1 * A_2) \) and let \( \tilde{F} \) be the corresponding strongly \( \mathcal{H} \)-spherical quasicharacter. Then
\[
\tilde{F}(g x) = \tilde{F}(g) + \tilde{F}(x) \quad \forall g \in A_1 * A_2, x \in \text{Ker} \theta.
\]  
(5.33)
Since \( \tilde{f}|_{\text{Ker} \theta} = 0 \), the restriction \( \tilde{F}|_{\text{Ker} \theta} \) is bounded. But ker \( \theta \subset \mathcal{H} \) and \( \tilde{F} \) is a character of \( \mathcal{H} \). We conclude that \( \tilde{F}|_{\text{Ker} \theta} = 0 \), and thus
\[
\tilde{F}(g x) = \tilde{F}(g) \quad \forall g \in A_1 * A_2, x \in \text{Ker} \theta.
\]  
(5.34)
Therefore there exists a function \( F \) on \( A_1 * H A_2 \) such that \( \tilde{F} = F \circ \theta \). It is immediate that \( \tilde{F} \) is a strongly \( H \)-spherical quasicharacter of \( A_1 * H A_2 \) with bounded restrictions to \( A_1 \) and \( A_2 \). Hence we can construct an \( H \)-spherical pseudocharacter \( f \) of \( A_1 * H A_2 \) whose restrictions to \( A_1 \) and \( A_2 \) are trivial. Therefore
\[
\text{Ker} \beta \supset PX_0, \text{ker} \theta (A_1 * A_2) \n\]  
(5.35)
and the proof of Theorem 4.6 is complete.
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