RAREFIED GAS DYNAMICS WITH EXTERNAL FIELDS UNDER SPECULAR REFLECTION BOUNDARY CONDITION

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Abstract. We consider the Boltzmann equation with external fields in strictly convex domains with the specular reflection boundary condition. We construct classical $C^1$ solutions away from the grazing set under the assumption that the external field is $C^2$ and the normal derivative of the field is positive and bounded away from 0.

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1. Introduction

Kinetic theory studies the time evolution of a large number of particles modeled by a distribution function in the phase space: $F(t,x,v)$ for $(t,x,v) \in [0,\infty) \times \Omega \times \mathbb{R}^3$, where $\Omega$ is an open bounded subset of $\mathbb{R}^3$. Dynamics and collision processes of dilute charged particles with a field $E$ can be modeled by the Vlasov-Boltzmann equation

$$\partial_t F + v \cdot \nabla_x F + E \cdot \nabla_v F = Q(F, F). \quad (1.1)$$

The collision operator measures “the change rate” in binary collisions and takes the form of

$$Q(F_1, F_2)(v) := Q_{\text{gain}}(F_1, F_2) - Q_{\text{loss}}(F_1, F_2) := \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - u) \cdot \omega |F_1(u')F_2(v') - F_1(u)F_2(v)|d\omega du, \quad (1.2)$$

where $u' = u - [(u - v) \cdot \omega] \omega$ and $v' = v + [(u - v) \cdot \omega] \omega$.

Here, $B(v - u, \omega) = |v - u|^{\kappa} q_0(\frac{v - u}{|v - u|} \cdot \omega)$, $0 \leq \kappa \leq 1$ (hard potential), and $0 \leq q_0(\frac{v - u}{|v - u|} \cdot \omega) \leq C |\frac{v - u}{|v - u|} \cdot \omega|$ (angular cutoff).

The collision operator enjoys collision invariance: for any measurable function $G$,

$$\int_{\mathbb{R}^3} \left[ 1 - \frac{|v|^2 - 3}{2} \right] Q(G,G)dv = [0 \ 0 \ 0]. \quad (1.3)$$

It is well-known that a global Maxwellian $\mu$ satisfies $Q(\mu, \mu) = 0$ where

$$\mu(v) := \frac{1}{(2\pi)^{3/2}} \exp \left(-\frac{|v|^2}{2}\right). \quad (1.4)$$

Throughout this paper we assume that $\Omega$ is a bounded open subset of $\mathbb{R}^3$ and there exists a $C^3$ function $\xi: \mathbb{R}^3 \to \mathbb{R}$ such that $\Omega = \{x \in \mathbb{R}^3 : \xi(x) < 0\}$, and $\partial \Omega = \{x \in \mathbb{R}^3 : \xi(x) = 0\}$. Moreover we assume the domain is strictly convex:

$$\sum_{i,j} \partial_i \xi(x) \partial_j \xi \geq C_\xi |\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^3 \text{ and for all } x \in \bar{\Omega} = \Omega \cup \partial \Omega. \quad (1.5)$$
We assume that
\[ \nabla \xi(x) \neq 0 \text{ when } |\xi(x)| \ll 1, \] (1.6)
and we define the outward normal as \( n(x) = \frac{\nabla \xi(x)}{\|\nabla \xi(x)\|} \) at the boundary. The boundary of the phase space \( \gamma := \{(x, v) \in \partial \Omega \times \mathbb{R}^3\} \) can be decomposed as
\begin{align*}
\gamma_- &= \{(x, v) \in \partial \Omega \times \mathbb{R}^3 : n(x) \cdot v < 0\}, \quad \text{(the incoming set)}, \\
\gamma_+ &= \{(x, v) \in \partial \Omega \times \mathbb{R}^3 : n(x) \cdot v > 0\}, \quad \text{(the outgoing set)}, \\
\gamma_0 &= \{(x, v) \in \partial \Omega \times \mathbb{R}^3 : n(x) \cdot v = 0\}, \quad \text{(the grazing set)}. \quad \text{(1.7)}
\end{align*}

In general the boundary condition is imposed only for the incoming set \( \gamma_- \) for general kinetic PDEs. In this paper we consider a so-called specular reflection boundary condition
\[ F(t, x, v) = F(t, x, R_x v) \text{ on } (x, v) \in \gamma_- \text{, where } R_x v := v - 2n(x)(n(x) \cdot v). \quad \text{(1.8)} \]

Physically this represents when a gas particle hits the boundary, it bounces back with the opposite normal velocity and the same tangential velocity, just like a billiard. Previous studies on the Boltzmann equation with specular reflection boundary conditions can be found in [7, 9, 14, 15, 16]. For other important physical boundary conditions, such as the diffuse boundary condition, we refer [1, 2, 3, 7, 9] and the references therein.

Due to the importance of the Boltzmann equation in the mathematical theory and application, there have been explosive research activities in analytic study of the equation. Notably the nonlinear energy method has led to solutions of many open problems including global strong solution of Boltzmann equation coupled with either the Poisson equation or the Maxwell system for electromagnetism when the initial data are close to the Maxwellian \( \mu \) in periodic box (no boundary). See [6] and the references therein. In many important physical applications, e.g. semiconductor and tokamak, the charged dilute gas is confined within a container, and its interaction with the boundary plays a crucial role both in physics and mathematics.

However, in general, higher regularity may not be expected for solutions of the Boltzmann equation in physical bounded domains. Such a drastic difference of solutions with boundaries had been demonstrated as the formation and propagation of discontinuity in non-convex domains [17, 5], and a non-existence of some second order derivatives at the boundary in convex domains [7]. Evidently the nonlinear energy method is not generally available to the boundary problems. In order to overcome such critical difficulty, Guo developed a \( L^2-L^\infty \) framework in [9] to study global solutions of the Boltzmann equation with various boundary conditions. The core of the method lies in a direct approach (without taking derivatives) to achieve a pointwise bound using trajectory of the transport operator, which leads substantial development in various directions including [3, 5, 7, 8, 13]. In [7], with the aid of some distance function towards the grazing set, the weighted classical \( C^1 \) solutions of Boltzmann equation \( (E \equiv 0 \text{ in } (1.1)) \) was constructed under various boundary conditions.

In this paper, we extend a result of [7] to the Boltzmann equation \((1.1)\) with a given external field \( (E \neq 0) \) satisfying a crucial sign condition on the boundary:
\[ E(t, x) \cdot n(x) > C_E > 0 \quad \text{for all } t \text{ and all } x \in \partial \Omega. \quad \text{(1.9)} \]

One of the major difficulties when dealing with a field \( E \neq 0 \) is that trajectories are curved and behave in a very complicated way when they hit the boundary.

Let’s clarify some notations. For any function \( z(x, v) : \Omega \times \mathbb{R}^3 \to \mathbb{R} \), denote
\[
\|z\|_\infty = \sup_{(x, v) \in \Omega \times \mathbb{R}^3} |z(x, v)|.
\]

And for any function \( g(t, x) : [0, T] \times \Omega \to \mathbb{R} \), denote
\[
\|g\|_{L^\infty_t} = \sup_{(t, x) \in [0, T] \times \Omega} |g(t, x)|, \quad \text{and } \|g\|_{C^0_{t,x}} = \sum_{0 \leq \alpha + \beta \leq n} \sup_{(t, x) \in [0, T] \times \Omega} |\partial_t^\alpha \partial_x^\beta g(t, x)|. \]

Our main result is a weighted \( C^1 \) estimate for the solution of \((1.1)\) with specular boundary condition \((1.8)\) in a short time. To state the main result, we introduce a distance function \( \alpha(t, x, v) \) towards the grazing set \( \gamma_0 \):
\[ \alpha(t, x, v) \sim \left[ |v \cdot \nabla \xi(x)|^2 + \xi(x)^2 - 2(v \cdot \nabla^2 \xi(x) \cdot v)\xi(x) - 2(E(t, \tau) \cdot \nabla \xi(\tau))\xi(x) \right]^{1/2} \quad \text{(1.10)} \]
Furthermore, we define $f$ formula to expand for some polynomial $P$.

Assume $\gamma > 0$, then there exists a unique solution $F$.

Theorem 1 (Weighted $C^1$ Estimate). Suppose $E$ satisfies the sign condition (1.14), and

$$\|E\|_{C^2_{x,t}} < \infty.$$  (1.11)

Assume $F_0 = \sqrt{pf_0}$, $f_0 \in W^{1,\infty}(\Omega \times R^3)$, and for $2 < \beta < 3$, $0 < \theta < \frac{1}{2}$, and $b > 1$,

$$\left\| \frac{\alpha^{\beta-1}}{\langle \psi \rangle^b} \partial_x f_0 \right\|_\infty + \left\| \frac{\alpha^{\beta-2}}{\langle \psi \rangle^{b-1} \partial_t} f_0 \right\|_\infty + \left\| e^{\theta|\psi|^2} f_0 \right\|_\infty < \infty,$$

and the compatibility condition

$$f_0(x,v) = f_0(x,R_x v) \text{ on } (x,v) \in \gamma_-$$.  (1.12)

Then there exists a unique solution $F(t) = \sqrt{pf(t)}$ for $0 \leq t \leq T$, with $T < 1$ to the system [11], that satisfies, for all $0 \leq t \leq T$,

$$\left\| e^{-\psi(t)} \frac{\alpha^{\beta}}{\langle \psi \rangle^{b+1}} \nabla f(t) \right\|_\infty + \left\| e^{-\psi(t)} \frac{\alpha^{\beta-1}}{\langle \psi \rangle^{b+1} \nabla} f(t) \right\|_\infty \leq \left\| \frac{\alpha^{\beta-1}}{\langle \psi \rangle^b} \partial_t f_0 \right\|_\infty + \left\| \frac{\alpha^{\beta-2}}{\langle \psi \rangle^{b+1} \partial_t} f_0 \right\|_\infty + P \left( \left\| e^{\theta|\psi|^2} f_0 \right\|_\infty \right)$$

for some polynomial $P$. Furthermore, if $f_0 \in C^1$, then $f \in C^1$ away from the grazing set $\gamma_0$.

The proof of Theorem 1 devotes a nontrivial extension of the result in [7]. The idea is to use Duhamel’s formula to expand $f$ along the characteristics to the initial data and then take derivatives. To do this, we need to define the generalized characteristics as follows:

**Definition 1.** For any $(t,v) \in [0,T] \times \Omega \times R^3$, let $(X(s;t,x,v), V(s;t,x,v))$ denotes the characteristics

$$\frac{d}{ds} \begin{bmatrix} X(s;t,x,v) \\ V(s;t,x,v) \end{bmatrix} = \begin{bmatrix} V(s;t,x,v) \\ E(s,X(s;t,x,v)) \end{bmatrix} \text{ for } 0 \leq s, t \leq T, \quad (1.13)$$

with $(X(t;t,x,v), V(t;t,x,v)) = (x,v)$.

We define the backward exit time $t_b(t,x,v)$ as

$$t_b(t,x,v) := \sup \{ s \geq 0 : X(\tau;t,x,v) \in \Omega \text{ for all } \tau \in (t-s,t) \}. \quad (1.14)$$

Furthermore, we define $x_b(t,x,v) := X(t-t_b(t,x,v);t,x,v)$, and $v_b(t,x,v) := V(t-t_b(t,x,v);t,x,v)$.

Now let $(t^0, x^0, v^0) = (t, v)$. We define the specular cycles, for $\ell \geq 0$,

$$(t^{\ell+1}, x^{\ell+1}, v^{\ell+1}) = (t^\ell - t_b(t^\ell, x^\ell, v^\ell), x_b(t^\ell, x^\ell, v^\ell), v_b(t^\ell, x^\ell, v^\ell) - 2n(x^{\ell+1})(v_b(t^\ell, x^\ell, v^\ell) \cdot n(x^{\ell+1})))$$

and we define the generalization as

$$X_{\ell}(s;t,x,v) = \sum_{\ell} 1_{(t^{\ell+1}, x^{\ell+1})}(s) X(t^\ell, x^\ell, v^\ell), \quad V_{\ell}(s;t,x,v) = \sum_{\ell} 1_{(t^{\ell+1}, x^{\ell+1})}(s) V(s,t^\ell, x^\ell, v^\ell). \quad (1.15)$$

The key component of the proof is to estimate the derivatives of the backward trajectory

$$\frac{\partial}{\partial (x,v)} \left( X_{\ell}(s;t,x,v), V_{\ell}(s;t,x,v) \right).$$

This is done through the matrix method where we estimate the multiplication of $\ell^\ell(s;t,x,v)$ many Jacobian matrices

$$\prod_{\ell=0}^{\ell^\ell(s;t,x,v)} \frac{\partial (t^{\ell+1}, x^{\ell+1}, v^{\ell+1})}{\partial (t^\ell, x^\ell, v^\ell)}. \quad (1.16)$$

Here $\ell^\ell(s;t,x,v)$ is the number of bounces it takes for the backward trajectory to reach time $s$ from time $t$, which can be shown to have order $\ell^\ell(s;t,x,v) \sim \frac{|t-s|}{\alpha(t,x,v)}$. And for each bounce, we can calculate the Jacobian matrix $\frac{\partial (t^{\ell+1}, x^{\ell+1}, v^{\ell+1})}{\partial (t^\ell, x^\ell, v^\ell)}$ explicitly.

One major difficulty here, comparing to the Boltzmann equation ($E = 0$ in [11]), is the field $E$ is time dependent, thus the characteristics ODE [11, 12] is not autonomous. This results the $\frac{\partial (t^{\ell+1}, x^{\ell+1}, v^{\ell+1})}{\partial (t^\ell, x^\ell, v^\ell)}$ derivatives in the first column of the matrix $\frac{\partial (t^{\ell+1}, x^{\ell+1}, v^{\ell+1})}{\partial (t^\ell, x^\ell, v^\ell)}$ is not trivially equal to 0, and need careful analysis.
We estimate (1.16) by diagonalizing each matrix and multiplying them together. Here another difficulty arises as the derivatives \( \frac{\partial (n(x^{\ell + 1})_t)}{\partial (x_t^{\ell} v^t)} \) can only be bounded as \( \left| \frac{\partial (n(x^{\ell + 1})_t)}{\partial (x_t^{\ell} v^t)} \right| \lesssim |t^\ell - t^{\ell + 1}| \). And this bound will result the multiplication of the \( |t^\ell| \) many eigenvalues of the matrices to behave as
\[
\prod_{\ell=0}^{\ell^*} \text{eig} \left| \frac{\partial (t^{\ell + 1}, x^{\ell + 1}, v^{\ell + 1})}{\partial (t, x, v)} \right| \sim (1 + \sqrt{\alpha})^{t^\ell} \sim (1 + \sqrt{\alpha})^\beta \rightarrow \infty
\]
as \( \alpha \to 0 \). Where \( \alpha = \alpha(t, x, v) \) in (1.10). This blow up will result the bound on (1.16) becomes too singular and makes it impossible for us to close the estimate. In order to overcome such a difficulty we utilize a crucial cancellation property (5.60), and find that as long as the external field \( E \) satisfies the regularity assumption
\[
\| E(t, x) \|_{C_{t,x}^2} < \infty,
\]
we can improve the estimate and achieve the bound \( \left| \frac{\partial (n(x^{\ell + 1})_t)}{\partial (x_t^{\ell} v^t)} \right| \lesssim |t^\ell - t^{\ell + 1}|^2 \). This extra smallness turns out to be just enough to control the accumulation in the many multiplications of eigenvalues:
\[
\prod_{\ell=0}^{\ell^*} \text{eig} \left| \frac{\partial (t^{\ell + 1}, x^{\ell + 1}, v^{\ell + 1})}{\partial (t, x, v)} \right| \sim (1 + \alpha)^{\frac{\beta}{2}} < C.
\]
With this bound and additional cancellations between two adjacent matrices (5.78), we carefully analyze the multiplications of the matrices and eventually achieve the key estimate in Theorem 2.

Let’s also address some other important differences when comparing the equation (1.1) with the Boltzmann equation (\( E = 0 \)). Because of the presence of the field \( E \) and its sign condition (1.9), we can achieve a better bound on the time gap
\[
|t^\ell - t^{\ell + 1}| \lesssim |n(x^\ell) \cdot v^{\ell + 1}|
\]
when \( v \) is small (5.2). This is because when the velocity is small, the field would always “push” the trajectory back to the boundary in a short time. This fact would eventually allow us to get the bound
\[
|\nabla_v X_{el}(s; t, x, v)| \lesssim \frac{1}{\langle v \rangle}
\]
in Theorem 2 which does not blow up when \( |v| \to 0 \), and this turns out to be necessary for us to close the estimate.

When taking derivatives to the Duhamel’s formula of \( f(t, x, v) \) in (6.3), if \( E \neq 0 \), an extra term would come up as (6.4). In order to bound this term we have to additionally estimate the derivatives \( \partial_x t^\ell \) and \( \partial_v t^\ell \), for any \( 1 \leq \ell \leq \ell^* \). Those estimates are consequences of the matrix method and are obtained in (5.97) and (5.98):
\[
|\partial_x t^\ell| \lesssim \frac{1}{\alpha^2}, \quad |\partial_v t^\ell| \lesssim \frac{1}{\alpha}.
\]
It’s also important to note that in (6.4), we have \( \left| \mathcal{R}_t v^\ell - v^t \right| = 2|n(x^\ell) \cdot v^t| \sim \alpha \). Thus the extra regularity we get by multiplying \( \alpha^\beta \) to \( \partial_x f \) and \( \alpha^{\beta - 1} \) to \( \partial_v f \) will bound the term as
\[
\sum_{1 \leq \ell \leq \ell^*} (\alpha^\beta |\partial_x t^\ell| + \alpha^{\beta - 1} |\partial_v t^\ell|) \max_{\ell} |\mathcal{R}_t v^\ell - v^t| \lesssim \frac{1}{\alpha} \left( \alpha^\beta \frac{1}{\alpha^2} + \alpha^{\beta - 1} \frac{1}{\alpha} \right) \alpha \lesssim \alpha^{\beta - 2} < C,
\]
as long as \( \beta > 2 \).

2. Local existence and in-flow problems with external fields

In this section we state some standard results which we will need to prove Theorem 1. Let \( F(t, x, v) = \sqrt{\mu} f(t, x, v) \). Then the corresponding problem to (1.1), (1.8) becomes
\[
\partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f - \frac{v}{2} \cdot E f = \Gamma_{\text{gain}}(f, f) - \nu(\sqrt{\mu}f) f.
\]
Here
\[
\nu(\sqrt{\mu}f)(v) := \frac{1}{\sqrt{\mu(v)}} Q_{\text{loss}}(\sqrt{\mu}f, \sqrt{\mu}f)(v) = \int_{\mathbb{R}^3} \int_{S^2} |v - u|^\omega q_0 \left( \frac{v - u}{|v - u|} \right) \sqrt{\mu(u)} f(u) d\omega du,
\]
and the gain term of the nonlinear Boltzmann operator is given by
\[ \Gamma_{\text{gain}}(f_1, f_2)(v) := \frac{1}{\sqrt{\mu}} Q_{\text{gain}}(\sqrt{\mu} f_1, \sqrt{\mu} f_2)(v) \]
\[ = \int_{\mathbb{R}^3} \int_{S^2} |v-u|^\kappa q_0 \left( \frac{v-u}{|v-u|} \cdot \omega \right) \sqrt{\mu(u)} f_1(u') f_2(v') d\omega du. \]  
(2.3)

And the specular reflection boundary condition in terms of \( f \) is
\[ f(t, x, v) = f(t, x, R_x v), \quad \text{on } (x, v) \in \gamma_. \]  
(2.4)

We first state a local existence result which is standard:

**Lemma 1.** [Local Existence] Suppose \( \|E\|_{L^\infty_{t,x}} < \infty \), and \( \|\varepsilon^0|v|^2 f_0\|_\infty < \infty \), \( 0 < \theta < \frac{1}{4} \). And \( f_0 \) satisfy the compatibility condition \( (1.12) \). Then there exists \( 0 < T \ll 1 \) small enough such that \( f \in L^\infty([0, T) \times \Omega \times \mathbb{R}^3) \) solves the equation \( (2.1) \) with specular boundary condition \( (2.4) \).

**Proof.** Let \( f^0 = \sqrt{\mu} \). We start with the sequence for \( m \geq 0 \)
\[ (\partial_t + v \cdot \nabla_x + E \cdot \nabla_v - \frac{v}{2} \cdot E + \nu(\sqrt{\mu} f^m)) f^{m+1} = \Gamma_{\text{gain}}(f^m, f^m), \]  
(2.5)

with the initial data \( f^m(0, x, v) = f_0(x,v) \), and boundary condition for all \( (x, v) \in \gamma_. \) be
\[ f^1(t, x, v) = f_0(x, R_x v), \]
\[ f^{m+1}(t, x, v) = f^m(t, x, R_x v), \quad m \geq 1. \]  
(2.6)

Then (see Lemma 7 in [7] for example)
\[ \sup_m \sup_{0 \leq t \leq T} \|\varepsilon^0|v|^2 f^m(t)|| \lesssim \|\varepsilon^0|v|^2 f_0\|_\infty < \infty, \]  
(2.7)

where \( \theta' = \theta - T \). From \( (2.4) \) we have up to a subsequence we have the weak-* convergence:
\[ \varepsilon^0|v|^2 f^m(t, x, v) \rightharpoonup \varepsilon^0|v|^2 f(t, x, v) \]  
(2.8)
in \( L^\infty([0, T) \times \Omega \times \mathbb{R}^3) \cap L^\infty([0, T) \times \gamma) \) for some \( f \). And it’s easy to show \( f \) is the solution of \( (2.1) \) with specular boundary condition \( (2.4) \). \( \square \)

We need some bound on the derivatives of the nonlocal term:

**Lemma 2.** Let \( [Y, W] = [Y(x, v), W(x, v)] \in \Omega \times \mathbb{R}^3 \). For \( 0 < \theta < \frac{1}{4} \) and \( \partial_a \in \{\partial_t, \nabla_x, \nabla_v\} \),
\[ |\partial_a \Gamma_{\text{gain}}(g, g)(Y, W)| \lesssim |\partial_a Y|||\varepsilon^0|v|^2 g||_\infty \int_{\mathbb{R}^3} \frac{e^{-C_\theta|u-W|^2}}{|u-W|^{2-\kappa}} |\nabla_x g(Y, u)| du \]
\[ + |\partial_a W|||\varepsilon^0|v|^2 g||_\infty \int_{\mathbb{R}^3} \frac{e^{-C_\theta|u-W|^2}}{|u-W|^{2-\kappa}} |\nabla_v g(Y, u)| du + \langle v \rangle^\kappa e^{-\theta|v|^2} |\partial_a W|||\varepsilon^0|v|^2 g||_\infty^2. \]

**Proof.** See [7]. \( \square \)

We need a result for the corresponding inflow problem to \( (2.1) \). Consider
\[ \partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f + \nu f = H, \]  
(2.9)
where \( H = H(t, x, v) \) and \( \nu = \nu(t, x, v) \) are given. Let \( \tau_1(x) \) and \( \tau_2(x) \) bet unit tangential vector to \( \partial \Omega \)
satisfying
\[ \tau_1(x) \cdot n(x) = 0 = \tau_2(x) \cdot n(x) \]  
and \( \tau_1(x) \times \tau_2(x) = n(x) \).  
(2.10)

And let \( \partial_\tau g \) be the tangential derivative at direction \( \tau_i \) for \( g \) defined on \( \partial \Omega \). Define
\[ \nabla_x g = \sum_{i=1}^2 \tau_i \partial_\tau g - \frac{n}{n \cdot v_b} \left\{ \partial_t g + \sum_{i=1}^2 (v_b \cdot \tau_i) \partial_\tau g + \nu g - H + E \cdot \nabla_v g \right\}, \]  
(2.11)
Proposition 1. Assume the compatibility condition
\[ f_0(x,v) = g(0,x,v) \quad \text{for} \quad (x,v) \in \gamma_. \]
Let \( p \in [1, \infty) \) and \( 0 < \theta < 1/4 \). \( |\nu(t,x,v)| \lesssim \langle v \rangle \cdot \|E\|_{L^1_{t,x}} + \|\nabla_x E\|_{L^2_{t,x}} < \infty. \)
Assume
\[ R_T > 3. \]
Then for any \( \epsilon > 0 \), \( \eta \in C^0([0,T]; L^p(\Omega \times \mathbb{R}^3)) \) and their traces satisfy
\[ \nabla_v f|_{\gamma_{{\epsilon}}} = \nabla_v g, \nabla_x f|_{\gamma_{{\epsilon}}} = \nabla_x g, \quad \text{on} \quad \gamma_{{\epsilon}}, \]
\[ \nabla_x f(0,x,v) = \nabla_x f_0, \nabla_v f(0,x,v) = \nabla_v f_0, \quad \text{in} \quad \Omega \times \mathbb{R}^3, \]
\[ \partial_t f(0,x,v) = \partial_t f_0, \quad \text{in} \quad \Omega \times \mathbb{R}^3. \]
where \( \nabla_x g \) is given by \((3.10)\).

Proof. See \[2\].

3. Velocity lemma and the nonlocal to local estimate

Recall the definition of specular trajectories in \((1.15)\). In this section we prove some properties of the specular trajectories which are crucial in order to establish the main result.

Let’s give the precise definition for the weight function \( \alpha \). We first need a cutoff function: for any \( \epsilon > 0 \), let \( \chi_\epsilon : [0, \infty) \to [0, \infty) \) be a smooth function satisfying:
\[ \chi_\epsilon(x) = x \quad \text{for} \quad 0 \leq x \leq \frac{\epsilon}{4}, \]
\[ \chi_\epsilon(x) = C_\epsilon \quad \text{for} \quad x \geq \frac{\epsilon}{2}, \]
\[ \chi_\epsilon(x) \text{ is increasing for} \quad \frac{\epsilon}{4} < x < \frac{\epsilon}{2}, \]
\[ \chi_\epsilon'(x) \leq 1. \]

Let \( d(x,\partial \Omega) := \inf_{y \in \partial \Omega} \|x-y\| \). And for any \( \delta > 0 \), let
\[ \Omega^\delta := \{ x \in \Omega : d(x,\partial \Omega) < \delta \}. \]
Since \( \partial \Omega \) is \( C^2 \), we claim that if \( \delta \ll 1 \) is small enough we have:
for any \( x \in \Omega^\delta \) there exists a unique \( \bar{x} \in \partial \Omega \) such that \( d(x,\bar{x}) = d(x,\partial \Omega) \), moreover \( \sup_{x \in \Omega^\delta} \|\nabla_x \bar{x}\| < \infty. \)

To prove the claim, we have by \((1.6)\) WLOG locally we can assume \( \eta \) takes the form
\[ \eta(x) = (x_{|1}, x_{|2}, \eta(x_{|1}, x_{|2})), \]
and \( \bar{x} = \eta(\bar{x}) = (\bar{x}_{|1}, \bar{x}_{|2}, \eta(\bar{x}_{|1}, \bar{x}_{|2})). \) Denote \( \partial_i \bar{\eta} = \frac{\partial}{\partial x_{i,|1}} \bar{\eta}(x_{|1}, x_{|2}), \) and \( \partial_i \bar{\eta} = \frac{\partial}{\partial x_{i,|2}} \bar{\eta}(x_{|1}, x_{|2}). \)

Now since $|\eta(\tilde{x})| - x|^2 = \inf_{y \in \partial\Omega} |y - x|^2$, $\tilde{x}$ satisfies

$$
\omega(x_1, x_2, x_3, \tilde{x}_{1,1}, \tilde{x}_{1,2}) = \left( \begin{array}{c}
(\tilde{x}_{1,1} - x_1) + (\tilde{x}_{1,1}, \tilde{x}_{1,2}) - x_3 \partial_t \tilde{x}(\tilde{x}_{1,1}, \tilde{x}_{1,2}) \\
(\tilde{x}_{1,2} - x_2) + (\tilde{x}_{1,1}, \tilde{x}_{1,2}) - x_3 \partial_t \tilde{x}(\tilde{x}_{1,1}, \tilde{x}_{1,2})
\end{array} \right) = 0.
$$

Since

$$
\det \left( \frac{\partial \omega}{\partial x_i} \right) = \det \left[ 1 + (\partial_1 \bar{\eta})^2 + (\tilde{\eta} - x_3) \partial_1 \bar{\eta} - (\tilde{\eta} - x_3) \partial_1 \bar{\eta} + (\tilde{\eta} - x_3) \partial_1 \bar{\eta} \right] = (1 + (\partial_1 \bar{\eta})^2)(1 + (\partial_2 \bar{\eta})^2) - (\partial_1 \bar{\eta})^2 + O(|\bar{\eta} - x_3|) = 1 + (\partial_1 \bar{\eta})^2 + (\partial_2 \bar{\eta})^2 + O(|\bar{\eta} - x_3|) > 0,
$$

if $|\eta(x)| - x_3$ is small enough. By the implicit function theorem $(\tilde{x}_{1,1}, \tilde{x}_{1,2})$ are functions of $x_1, x_2, x_3$ if $x$ is close enough to $\partial \Omega$.

Moreover,

$$
\frac{\partial \tilde{x}_i}{\partial x_j} = -\left( \frac{\partial \omega}{\partial x_i} \right)^{-1} \frac{\partial \omega}{\partial x_j}$$

is bounded as $\frac{\partial \omega}{\partial x_j}$ is bounded and $\det(\frac{\partial \omega}{\partial x_i})$ is bounded from below if $x$ is close enough to the boundary. Therefore $\nabla_x \tilde{x}$ is bounded. This proves (3.2).

Now define

$$
\beta(t, x, v) = \left( v \cdot \nabla \xi(x) + \xi(x)^2 - 2v \cdot \nabla \xi(x) \cdot v \right) \xi(x) = 2(E(t, x, v) \cdot \nabla \xi(x) \xi(x) \right)^{1/2},
$$

for all $(x, v) \in \Omega^\delta \times \mathbb{R}^3$. Let $\delta' := \min\{\xi(x) : x \in \Omega, d(x, \partial \Omega) = \delta\}$, and let $\chi_{\delta'}$ be a smooth cutoff function satisfies (3.1), then define

$$
\alpha(t, x, v) := \begin{cases}
\chi_{\delta'}(\beta(t, x, v)) & x \in \Omega^\delta, \\
C_{\delta'} & x \in \Omega \setminus \Omega^\delta.
\end{cases}
$$

The following lemmas about $\alpha$ is important for our estimate:

**Lemma 3** (Velocity lemma near boundary). Suppose $E(t, x)$ satisfies $\|E\|_{C^1} < \infty$ and the sign condition (1.11). Then $\alpha$ is continuous, and for $\delta \ll 1$ small enough, we have for any $(t, x, v) \in [0, T] \times \Omega \times \mathbb{R}^3$, and $0 \leq s < t$, $\alpha$ satisfies

$$
e^{-C \int_s^t (|V_{cl}(\bar{r})| + 1) dr'} \alpha(s, X_{cl}(s), V_{cl}(s)) \leq \alpha(t, x, v) \leq e^{C \int_s^t (|V_{cl}(\bar{r})| + 1) dr'} \alpha(s, X_{cl}(s), V_{cl}(s)),
$$

for any $C \geq C_\xi L_{L^\infty}G_{\xi} + \|\nabla E\|_{L^\infty} + \|\partial E\|_{L^\infty} + 1$, where $C_\xi$ is a large constant depending only on $\xi$. Here $X_{cl}(s), V_{cl}(s) = (X_{cl}(s; t, x, v), V_{cl}(s; t, x, v))$ as defined in (1.15).

Similar estimates have been used in [10] and then in [12].

**Proof.** See [2].

**Lemma 4.** Suppose $E$ satisfies (1.10), then for any $y \in \overline{\Omega}$, $1 < \beta < 3, 0 < \kappa < 1$, and $\theta > 0$ we have

$$
\int_{\mathbb{R}^3} \frac{e^{-\theta|v - u|^2}}{|v - u|^{2 - \kappa}|\alpha(s, y, u)|^\beta} du \leq C \left( \frac{1}{(|v|^2 \xi(y) + c(y))^{\frac{1}{2}}} + 1 \right),
$$

where $c(y) = \xi(y)^2 - C E \xi(y)$.

**Proof.** See [2].
Lemma 5. (1) Let \((t, x, v) \in [0, T] \times \Omega \times \mathbb{R}^3, 1 < \beta < 3, 0 < \kappa \leq 1\). Suppose \(E\) satisfies (1.9) and (1.11), then for \(\varepsilon \gg 1\) large enough, we have for any \(0 < \delta \ll 1\),

\[
\int_0^t \int_{\Omega} e^{-|\nabla \cdot (V(t, t, x, v))|} \frac{1}{|V(s) - u|^{2-\kappa} (\alpha(s, X(s), u))^2} du \, ds \lesssim e^{2C_E \varepsilon \|\nabla E\|_{L^\infty}^3} \delta \|\nabla E\|_{L^\infty} \delta^\beta \sum_{\theta \in \{0, 1\}} \|\nabla E\|_{L^\infty}^\beta \frac{1}{\delta^\beta} \sup_{0 \leq s \leq t \leq t_b} \{e^{-\frac{1}{\varepsilon}(s-t)} Z(s, x, v)\},
\]

(2) Let \([X_b(s; t, x, v), V_b(s; t, x, v)]\) be the specular backward trajectory as in (1.13). Let \(Z(s, x, v) \geq 0\) be any bounded non-negative function in the phase space.

Proof of (1) Lemma 5 The proof is similar to the proof of Lemma 11 in [2], but with some modifications made in order to achieve (3.7) later. We separate the proof into several cases.

In Step 1 we prove (3.6) for the case when \(x \in \partial \Omega\) and \(t \leq t_b\).
In Step 2 we prove (3.6) for the case when \(x \in \partial \Omega\) and \(t > t_b\).
In Step 3 we prove (3.6) for the case when \(x \in \Omega\) and \(t \leq t_b\).
In Step 4 we prove (3.6) for the case when \(x \in \Omega\) and \(t > t_b\).

Step 1 Let’s first start with the case \(t \geq t_b\) and prove (3.6). Let’s shift the time variable: \(s \rightarrow t - t_b + s\), and let \(\tilde{X}(s) = X(t - t_b + s), \tilde{V}(s) = V(t - t_b + s)\). Then \(s \in [0, t_b]\) and from (3.5) we only need to bound the integral

\[
\int_0^{t_b} e^{-\int_{t_b}^t \|\nabla \cdot (V(t, t, x, v))|} \frac{1}{|\tilde{V}(s)|^2 |\tilde{X}(s) - \xi|^2} \frac{1}{\varepsilon^{\frac{1}{\sigma^2}}ds}.
\]

Let’s assume \(x \in \partial \Omega\) and \(v \cdot \nabla \xi(x) > 0\). Then by the velocity lemma (Lemma 3) we have \(v_b \cdot \nabla \xi(x_b) < 0\).

Claim: for any \(0 < \delta \ll 1\) small enough, if we let

\[
\sigma_1 = \frac{\delta v_b \cdot \nabla \xi(x_b)}{|v|^2 + \|E\|_{L^\infty}^2 + \|E\|_{L^\infty}^2 + 1}, \quad \text{and} \quad \sigma_2 = \frac{\delta v \cdot \nabla \xi(x)}{|v|^2 + \|E\|_{L^\infty}^2 + \|E\|_{L^\infty}^2 + 1},
\]

then \(|\tilde{X}(\sigma_1)| \geq \frac{\delta (v_b \cdot \nabla \xi(x_b))^2}{2(|v|^2 + \|E\|_{L^\infty}^2 + \|E\|_{L^\infty}^2 + 1)}, \quad |\tilde{X}(\sigma_2)| \geq \frac{\delta (v \cdot \nabla \xi(x))^2}{2(|v|^2 + \|E\|_{L^\infty}^2 + \|E\|_{L^\infty}^2 + 1)},
\]

then \(|\tilde{X}(\sigma_1)| \) is monotonically increasing on \([0, \sigma_1]\), and monotonically decreasing on \([\sigma_1 - \sigma_2, \sigma_1]\). Moreover, we have the following bounds:

\[
|\tilde{X}(\sigma_1)| \geq \frac{3\delta (v_b \cdot \nabla \xi(x_b))^2}{2(|v|^2 + \|E\|_{L^\infty}^2 + \|E\|_{L^\infty}^2 + 1)}, \quad \|\tilde{X}(\sigma_2)| \geq \frac{3\delta (v \cdot \nabla \xi(x))^2}{2(|v|^2 + \|E\|_{L^\infty}^2 + \|E\|_{L^\infty}^2 + 1)},
\]

(3.10)
\[ |\hat{V}(s) \cdot \nabla \xi(\hat{X}(s))| \geq \frac{|v_b \cdot \nabla \xi(x_b)|}{2}, \quad s \in [0, \sigma_1], \]

\[ |\hat{V}(s) \cdot \nabla \xi(\hat{X}(s))| \geq \frac{|v \cdot \nabla \xi(x)|}{2}, \quad s \in [t_b - \sigma_2, t_b]. \]

(3.12)

To prove the claim we first note that \( \frac{d}{ds} \xi(\hat{X}(s))|_{s=0} = v_b \cdot \nabla \xi(x_b) < 0 \), and

\[
\frac{d^2}{ds^2} \xi(\hat{X}(s)) = \frac{d}{ds}(\hat{V}(s) \cdot \nabla \xi(\hat{X}(s))) = \hat{V}(s) \cdot \nabla^2 \xi(\hat{X}(s)) \cdot \hat{V}(s) + E(s, \hat{X}(s)) \cdot \nabla \xi(\hat{X}(s))
\]

\[
\leq C(|\hat{V}(s)|^2 + \|E\|_{L^\infty_{t,x}}) \leq C(2|v|^2 + 2(t_b\|E\|_{L^\infty_{t,x}})^2 + \|E\|_{L^\infty_{t,x}}^2) \leq C_1(|v|^2 + \|E\|_{L^\infty_{t,x}}^2 + \|E\|_{L^\infty_{t,x}}^2 + 1),
\]

(3.13)

for some \( C_1 > 0 \). Thus if \( \delta \) small enough, we have \( \frac{d}{ds} \xi(\hat{X}(s)) < 0 \) for all \( s \in [0, \delta \frac{|v \cdot \nabla \xi(x)|}{|v|^2 + \|E\|_{L^\infty_{t,x}}^2 + \|E\|_{L^\infty_{t,x}}^2 + 1}] \).

Therefore \( \xi(\hat{X}(s)) \) is decreasing on \([0, \sigma_1]\).

Similarly \( \frac{d}{ds} \xi(\hat{X}(s))|_{s=t_b} = v \cdot \nabla \xi(x) > 0 \), and since \( |\frac{d}{ds} \xi(\hat{X}(s))| \leq (|v|^2 + \|E\|_{L^\infty_{t,x}}^2 + \|E\|_{L^\infty_{t,x}}^2 + 1) \)

we have that \( \frac{d}{ds} \xi(\hat{X}(s)) > 0 \) for all \( s \in [t_b - \delta \frac{|v \cdot \nabla \xi(x)|}{|v|^2 + \|E\|_{L^\infty_{t,x}}^2 + \|E\|_{L^\infty_{t,x}}^2 + 1}, t_b \) if \( \delta \) small enough. Therefore \( \xi(\hat{X}(s)) \) is increasing on \([t_b - \sigma_2, t_b]\).

Next we establish the bounds (3.10), (3.11), and (3.12). By (3.13), we have

\[
|\xi(\hat{X}(\sigma_1))| = \int_0^{\sigma_1} -\hat{V}(s) \cdot \nabla \xi(\hat{X}(s))ds
\]

\[
= \int_0^{\sigma_1} \left( \int_0^s -\frac{d}{d\tau}(\hat{V}(\tau) \cdot \nabla \xi(\hat{X}(\tau)))d\tau - v_b \cdot \nabla \xi(x_b) \right) ds
\]

\[
\geq \int_0^{\sigma_1} \left( |v_b \cdot \nabla \xi(x_b)| - C_1(|v|^2 + \|E\|_{L^\infty_{t,x}}^2 + \|E\|_{L^\infty_{t,x}}^2 + 1)s \right) ds
\]

\[
= \sigma_1 |v_b \cdot \nabla \xi(x_b)| - \frac{\sigma_1^2}{2} C_1(|v|^2 + \|E\|_{L^\infty_{t,x}}^2 + \|E\|_{L^\infty_{t,x}}^2 + 1)
\]

\[
= \sigma_1 \left( |v_b \cdot \nabla \xi(x_b)| - \frac{\delta \sigma_1}{2} |v_b \cdot \nabla \xi(x_b)| \frac{\delta (v_b \cdot \nabla \xi(x_b))^2}{2(|v|^2 + \|E\|_{L^\infty_{t,x}}^2 + \|E\|_{L^\infty_{t,x}}^2 + 1)} \right)
\]

And by the same argument we have \( |\xi(\hat{X}(\sigma_2))| \geq \frac{\delta (v \cdot \nabla \xi(x))^2}{2(|v|^2 + \|E\|_{L^\infty_{t,x}}^2 + \|E\|_{L^\infty_{t,x}}^2 + 1)} \) for \( \delta \ll 1 \). This proves (3.10).

To prove (3.11), we have from (3.13), for \( s \in [0, \sigma_1] \),

\[
|\xi(\hat{X}(s))| \leq s \left( |v_b \cdot \nabla \xi(x_b)| + \frac{\delta C_1}{2} |v_b \cdot \nabla \xi(x_b)| \right)
\]

\[
\leq \frac{3s}{2} |v_b \cdot \nabla \xi(x_b)| \leq \frac{3\delta (v_b \cdot \nabla \xi(x_b))^2}{2(|v|^2 + \|E\|_{L^\infty_{t,x}}^2 + \|E\|_{L^\infty_{t,x}}^2 + 1)}.
\]

and \( |\xi(\hat{X}(s))| \leq \frac{3\delta (v \cdot \nabla \xi(x))^2}{2(|v|^2 + \|E\|_{L^\infty_{t,x}}^2 + \|E\|_{L^\infty_{t,x}}^2 + 1)} \) for \( s \in [t_b - \sigma_2, t_b] \). This proves (3.11).

Finally for (3.12), again from (3.13),

\[
|\hat{V}(s) \cdot \nabla \xi(\hat{X}(s))| \geq |v_b \cdot \nabla \xi(x_b)| - \int_0^{\sigma_1} C_1(|v|^2 + \|E\|_{L^\infty_{t,x}}^2 + \|E\|_{L^\infty_{t,x}} + 1)ds
\]

\[
\geq |v_b \cdot \nabla \xi(x_b)| - C_1 \delta |v_b \cdot \nabla \xi(x_b)| \geq \frac{|v_b \cdot \nabla \xi(x_b)|}{2}.
\]

And similarly \( |\hat{V}(s) \cdot \nabla \xi(\hat{X}(s))| \geq \frac{|v \cdot \nabla \xi(x)|}{2} \) for \( s \in [t_b - \delta_2, t_b] \). This proves the claim.
Step 2 
Recall the definition of $\sigma_1, \sigma_2$ in (3.9), and $C_E$ in (1.9). In this step we establish the lower bound:

$$|\xi(\tilde{X}(s))| > \frac{C_E}{10}(\sigma_2)^2, \text{ for all } s \in [\sigma_1, t_b - \sigma_2].$$  \hspace{1cm} (3.14)

Suppose towards contradiction that $I := \{s \in [\sigma_1, t_b - \sigma_2]: |\xi(\tilde{X}(s))| \leq \frac{C_E}{10}(\sigma_2)^2\} \neq \emptyset$. Then from (3.4) and (3.10) we have

$$\frac{C_E}{10}(\sigma_2)^2 \leq \delta^2 \frac{C_E}{10} \frac{(v \cdot \nabla \xi(x))^2}{|v|^2 + \|E\|_{L^\infty_{t,x}}^2 + \|E\|_{L^2_{t,x}}^2 + 1}$$

$$...$$

$$...$$

$$...$$

if $\delta \ll 1$. So $\sigma_1 \notin I$. Let $s^* := \min\{s \in I\}$ be the minimum of such $s$. Then clearly

$$\frac{d}{ds} \xi(\tilde{X}(s))|_{s=s^*} = \tilde{V}(s^*) \cdot \nabla \xi(\tilde{X}(s^*)) \geq 0.$$  

Now expanding around $\tilde{X}(s)$, we have

$$E(s, \tilde{X}(s)) \cdot \nabla \xi(\tilde{X}(s)) = E(s, \overline{\tilde{X}(s)}) \cdot \nabla \xi(\overline{\tilde{X}(s)}) + c(\tilde{X}(s)) \cdot \xi(\tilde{X}(s)),$$

with $|c(\overline{\tilde{X}(s)})| < \frac{C_E(\|E\|_{L^\infty_{t,x}} + \|\nabla E\|_{L^\infty_{t,x}})}{C_E}$. Thus

$$\frac{d}{ds}(\tilde{V}(s) \cdot \nabla \xi(\tilde{X}(s))) = \tilde{V}(s) \cdot \nabla \xi(\tilde{X}(s)) \cdot \nabla \tilde{V}(s) + E(s, \tilde{X}(s)) \cdot \nabla \xi(\tilde{X}(s))$$

$$= \tilde{V}(s) \cdot \nabla \xi(\tilde{X}(s)) \cdot \nabla \tilde{V}(s) + E(s, \overline{\tilde{X}(s)}) \cdot \nabla \xi(\overline{\tilde{X}(s)}) + c(\tilde{X}(s)) \cdot \xi(\tilde{X}(s))$$

$$\geq C_E - \frac{C_E(\|E\|_{L^\infty_{t,x}} + \|\nabla E\|_{L^\infty_{t,x}})}{C_E} \cdot |\xi(\tilde{X}(s))|,$$

so

$$\frac{d}{ds}(\tilde{V}(s) \cdot \nabla \xi(\tilde{X}(s)))|_{s=s^*} \geq C_E - \delta^2 \frac{C_E(\|E\|_{L^\infty_{t,x}} + \|\nabla E\|_{L^\infty_{t,x}})}{C_E} \frac{(v \cdot \nabla \xi(x))^2}{|v|^2 + \|E\|_{L^2_{t,x}}^2 + \|E\|_{L^\infty_{t,x}}^2 + 1} \geq \frac{C_E}{2},$$

for $\delta \ll 1$ small enough. Then we have $\frac{d}{ds}(\tilde{V}(s) \cdot \nabla \xi(\tilde{X}(s)))$ is increasing on the interval $[s^*, t_b]$ as $|\xi(\tilde{X}(s))|$ is decreasing. So

$$\frac{d}{ds}(\tilde{V}(s) \cdot \nabla \xi(\tilde{X}(s))) \geq \frac{C_E}{2}, \quad s \in [s^*, t_b].$$

And therefore

$$|\xi(\tilde{X}(s^*))| = \int_{s^*}^{t_b} \tilde{V}(s) \cdot \nabla \xi(\tilde{X}(s))ds$$

$$= \int_{s^*}^{t_b} \left( \int_{s^*}^{t} (\tilde{V}(\tau) \cdot \nabla \xi(\tilde{X}(\tau)))d\tau + \tilde{V}(s^*) \cdot \nabla \xi(\tilde{X}(s^*)) \right)ds$$

$$\geq \int_{s^*}^{t_b} (s - s^*) \frac{C_E}{2} ds = \frac{C_E}{4}(t_b - s^*)^2 \geq \frac{C_E}{4}(\sigma_2)^2,$$

which is a contradiction. Therefore we conclude (3.14).
Step 3: Let's split the time integration [3.8] as
\[ \int_{0}^{t_{b}} e^{-f_{t_{b}}-\alpha s} \frac{1}{2^{\alpha} \varphi(V(\tau,t,x,v))} d\tau \]
\[ = \int_{0}^{\sigma_{1}} + \int_{\sigma_{1}}^{t_{b}-\sigma_{2}} + \int_{t_{b}-\sigma_{2}}^{t_{b}} = (I) + (II) + (III). \]

Let's first estimate (I), (III):
From Step 2 we have that \(|\xi(\bar{x}(s))|\) is monotonically increasing on \([0, \sigma_{1}]\) and \([t_{b} - \sigma_{2}, t_{b}]\), so we have the change of variables:
\[ ds = \frac{d|\xi|}{|V(s) \cdot \nabla \xi(\bar{x}(s))|}. \]
Using this change of variable and the bounds (3.11), (3.12), and the \(|\tilde{V}(s)|^{2} + 1 \gtrsim |v|^{2} + 1\), (I) is bounded by
\[ (I) \lesssim e^{2C_{\xi}} \frac{|v| + \|E\|_{L_{t,x}^{\infty}} + 1}{C_{E}^{\delta}} \frac{\delta^{3-\beta}}{(v)^{2}(C_{E} + 1)^{2\alpha-1}(\alpha(t,x,v))^{\beta-2}(\|E\|_{L_{t,x}^{\infty}} + 1)^{\frac{3-\beta}{2}}}. \]

And by the same computation we get
\[ (III) \lesssim e^{2C_{\xi}} \frac{|v| + \|E\|_{L_{t,x}^{\infty}} + 1}{C_{E}^{\delta}} \frac{\delta^{3-\beta}}{(v)^{2}(C_{E} + 1)^{2\alpha-1}(\alpha(t,x,v))^{\beta-2}(\|E\|_{L_{t,x}^{\infty}} + 1)^{\frac{3-\beta}{2}}}. \]

Finally for (II), using the lower bound for \(|\xi(\bar{x}(s))|\) in (3.14), we have
\[ (II) = \int_{\sigma_{1}}^{t_{b}} e^{-f_{t_{b}}-\alpha s} \varphi(V(\tau,t,x,v)) d\tau \]
\[ \lesssim \frac{1}{C_{E}^{\delta-1}(\|E\|_{L_{t,x}^{\infty}} + 1)^{\beta-1}} \int_{0}^{t_{b}} e^{f_{t_{b}}-\alpha s} \varphi d\tau d\tau \]
\[ \lesssim \frac{(v)^{2}(C_{E} + 1)^{2\alpha-1}(\alpha(t,x,v))^{\beta-2}(\|E\|_{L_{t,x}^{\infty}} + 1)^{\frac{3-\beta}{2}}}{C_{E}^{\delta-1}(v)^{\beta-1}(\alpha(t,x,v))^{\beta-1}}. \]

This proves (3.9) for the case \(x \in \partial \Omega\) and \(t \leq t_{b}\).
\textbf{Step 4}  
Now suppose \( x \in \partial \Omega \) and \( t_b > t \). It suffices to bound the integral:
\[
\int_0^t e^{-\int_0^s \frac{1}{|V(s)|^2 + \|E\|_{L_x}^2 + \|E\|_{L_x}^2 + 1} ds. \tag{3.21}
\]

Denote
\[
X(0; t, x, v) = x_0, V(0; t, x, v) = v_0.
\]

Let
\[
\sigma_2 = \frac{v \cdot \nabla \xi(x)}{|v|^2 + \|E\|_{L_x}^2 + \|E\|_{L_x}^2 + 1}
\]
as defined in (3.9). If
\[
\sigma_2 \geq t,
\]
then from Step 2 \(|\xi(X(s))|\) is decreasing on \([0, t] \), and by (3.11), (3.12), and the bound for (III) (3.19), we get the desired estimate. Now we assume
\[
\sigma_2 < t.
\]

So from (3.10) we have
\[
|\xi(X(\sigma_2))| \geq \frac{\delta (v \cdot \nabla \xi(x))^2}{2(|v|^2 + \|E\|_{L_x}^2 + \|E\|_{L_x}^2 + 1)}. \tag{3.22}
\]

Now if \(|\xi(x_0)| \leq \delta \frac{\alpha^2(t, x, v)}{10(|v|^2 + \|E\|_{L_x}^2 + \|E\|_{L_x}^2 + 1)}, \]
\[
\alpha^2(t, x, v) \leq e^{C_1 \frac{1}{\|E\|_{L_x}^2 + \|E\|_{L_x}^2 + \|E\|_{L_x}^2}} \alpha^2(0, x_0, v_0)
\]
\[
\leq C_2 e^{C_1 \frac{1}{\|E\|_{L_x}^2 + \|E\|_{L_x}^2 + \|E\|_{L_x}^2}} \frac{((v \cdot \nabla \xi(x_0) \cdot v_0)^2 + (|v_0|^2 + |\xi(x_0)| + \|E\|_{L_x}^2))|\xi(x_0)|}{2|v_0|^2 + \delta \alpha^2(t, x, v)} \tag{3.23}
\]
\[
\leq C_2 e^{C_1 \frac{1}{\|E\|_{L_x}^2 + \|E\|_{L_x}^2 + \|E\|_{L_x}^2}} |\nabla \xi(x_0) \cdot v_0| \tag{3.24}
\]

if \( \delta \ll 1 \) is small enough.

Claim:
\[
\nabla \xi(x_0) \cdot v_0 < 0.
\]

Since otherwise by (3.10) we have
\[
\frac{d}{ds} |\xi(X(s))| < 0,
\]

for all \( s \in [0, t] \), so \(|\xi(X(s))|\) is always decreasing, which contradicts (3.22).

Therefore \( \nabla \xi(x_0) \cdot v_0 < 0 \), and we can run the same argument from Step 1, Step 2, Step 3 with \( \nabla \xi(x_b) \cdot v_b \) replaced by \( \nabla \xi(x_0) \cdot v_0 \), and by (3.24) we get the same estimate.

If \(|\xi(x_0)| \geq \delta \frac{\alpha^2(t, x, v)}{10(|v|^2 + \|E\|_{L_x}^2 + \|E\|_{L_x}^2 + 1)}, \)
then we have
\[
\frac{C_E \sigma_2^2}{10} = \delta^2 \frac{C_E}{10} \frac{(v \cdot \nabla \xi(x))^2}{|v|^2 + \|E\|_{L_x}^2 + \|E\|_{L_x}^2 + 1} < C_E \delta |\xi(x_0)| < |\xi(x_0)|, \tag{3.25}
\]

for \( \delta \ll 1 \) small enough. Therefore by (3.22) and the same argument in Step 3 we get the same lower bound
\[
|\xi(s)| \geq \frac{C_E}{10} \sigma_2^2, \text{ for all } s \in [0, t - \sigma_2]. \tag{3.26}
\]

And therefore we get the desired estimate.

\textbf{Step 5}  
We now consider the case when \( x \in \Omega \) and \( t \geq t_b \). We need to bound the integral \( \frac{v_b}{|v|^2 + \|E\|_{L_x}^2 + \|E\|_{L_x}^2 + 1} \).

\[
\sigma_1 = \delta \frac{v_b \cdot \nabla(x_b)}{|v|^2 + \|E\|_{L_x}^2 + \|E\|_{L_x}^2 + 1},
\]
as defined in \((3.10)\). If
\[ \sigma_1 \geq t, \]
then from Step 2 \(|\xi(\bar{X}(s))|\) is increasing on \([0, t_b]\), and by \((3.11)\), \((3.12)\), and the bound for (I) in \((3.15)\), we get the desired estimate.
Now we assume
\[ \sigma_1 < t. \]
So from \((3.10)\) we have
\[ |\xi(\bar{X}(\sigma_1))| \geq \frac{\delta (v_b \cdot \nabla \xi(x_b))^2}{2(|v|^2 + \|E\|^2_{L_{t,x}^\infty} + \|E\|_{L_{t,x}^\infty} + 1)}, \] 
(3.27)

Now if
\[ |\xi(x)| \leq \frac{\alpha^2(t, x, v)}{10(|v|^2 + \|E\|^2_{L_{t,x}^\infty} + \|E\|_{L_{t,x}^\infty} + 1)}, \]
(3.28)
we have
\[ \alpha^2(t, x, v) \leq (\nabla\xi(x) \cdot v)^2 + C(|v|^2 + \|E\|_{L_{t,x}^\infty} + 1)|\xi(x)| \]
\[ \leq (\nabla\xi(x) \cdot v)^2 + \delta \alpha^2(t, x, v) \leq (\nabla\xi(x) \cdot v)^2 + \frac{1}{10} \alpha^2(t, x, v), \]
(3.29)
if \(\delta \ll 1\) is small enough. So
\[ \frac{1}{2} \alpha(t, x, v) \leq |\nabla\xi(x) \cdot v|. \]
(3.30)

Claim:
\[ \nabla\xi(x) \cdot v > 0. \]

Since otherwise by \((3.16)\) we have
\[ \frac{d}{ds} |\xi(\bar{X}(s))| > 0, \]
for all \(s \in [0, t_b]\), so \(|\xi(\bar{X}(s))|\) is always increasing, thus
\[ |\xi(\bar{X}(s))| \leq \frac{\alpha^2(t, x, v)}{10(|v|^2 + \|E\|^2_{L_{t,x}^\infty} + \|E\|_{L_{t,x}^\infty} + 1)}, \]
for all \(s \in [0, t_b]\), which contradicts \((3.27)\).
Therefore \(\nabla\xi(x) \cdot v > 0\), and we can run the same argument from Step 2, Step 3, Step 4, and by \((3.30)\) we get the same estimate.

If
\[ |\xi(x)| \geq \frac{\alpha^2(t, x, v)}{10(|v|^2 + \|E\|^2_{L_{t,x}^\infty} + \|E\|_{L_{t,x}^\infty} + 1)}, \]
(3.31)
we claim:
\[ |\xi(\bar{X}(s))| \geq \frac{\alpha^2(t, x, v)}{|v|^2 + \|E\|^2_{L_{t,x}^\infty} + \|E\|_{L_{t,x}^\infty} + 1}}, \]
(3.32)
for all \(s \in [\sigma_1, t_b]\). Since otherwise let
\[ s^* := \min\{s \in [\sigma_1, t] : |\xi(\bar{X}(s))| < \delta^2 \frac{\alpha^2(t, x, v)}{|v|^2 + \|E\|^2_{L_{t,x}^\infty} + \|E\|_{L_{t,x}^\infty} + 1} \}. \]

From \((3.27)\) we have \(s^* > \sigma_1\), and
\[ \frac{d}{ds} |\xi(\bar{X}(s^*))| < 0. \]

And from \((3.16)\) we have
\[ \frac{d^2}{ds^2} |\xi(\bar{X}(s))| < 0, \]
for all \(s \in [s^*, t]\). So \(|\xi(\bar{X}(s))|\) is always decreasing on \([s^*, t_b]\). Therefore
\[ |\xi(x)| = |\xi(\bar{X}(t_b))| < |\xi(\bar{X}(s^*))| < \delta^2 \frac{\alpha^2(t, x, v)}{|v|^2 + \|E\|^2_{L_{t,x}^\infty} + \|E\|_{L_{t,x}^\infty} + 1}}. \]
which contradicts (3.31). Therefore the lower bound (3.32) and the estimates (3.20), (3.18) gives the desired bound.

**Step 6** Finally we consider the case \( x \in \Omega \) and \( t < t_b \). First suppose
\[
|\xi(x)| \leq \delta \frac{\alpha^2(t,x,v)}{10(|v|^2 + \|E\|_{L^\infty_{t,x}}^2 + \|E\|_{L^\infty_{t,x}} + 1)}.
\]
From (3.30) we have
\[
\frac{\alpha(t,x,v)}{2} \leq |v \cdot \nabla \xi(x)|.
\]
If \( v \cdot \nabla \xi(x) > 0 \), then by (3.16) we have \( \xi(X(t+t')) = 0 \) for some \( t' \leq \frac{\delta}{|v|} < 1 \). Therefore we can extend the trajectory until it hits the boundary and conclude the desired bound from Step 3.

If \( v \cdot \nabla \xi(x) < 0 \), again by (3.16) we have \( |\xi(X(s))| \) is increasing on \([0,t]\) and \( |V(s) \cdot \nabla \xi(X(s))| \) is decreasing on \([0,t]\). Therefore using the change of variable \( s \mapsto |\xi|:
\[
\int_0^t e^{\int_s^t \frac{1}{v \cdot \nabla \xi(X(s))} ds} \frac{1}{|V(s)|^2 |\xi(X(s)) + \xi^2(X(s) - CE\xi(X(s))|} ds 
\]
which is the desired estimate.

Now suppose
\[
|\xi(x)| > \delta \frac{\alpha^2(t,x,v)}{10(|v|^2 + \|E\|_{L^\infty_{t,x}}^2 + \|E\|_{L^\infty_{t,x}} + 1)},
\]
and
\[
|\xi(x_0)| \leq \delta \frac{\alpha^2(t,x,v)}{10(|v|^2 + \|E\|_{L^\infty_{t,x}}^2 + \|E\|_{L^\infty_{t,x}} + 1)}.
\]
Then by (3.24) we have
\[
\frac{\alpha(t,x,v)}{2} \leq c_{C_0} \frac{|v \cdot \nabla \xi(x_0)|}{v \cdot \nabla \xi(x_0)}. \quad (3.35)
\]
Now if \( v_0 \cdot \nabla \xi(x_0) > 0 \), then from (3.10) we have \( |\xi(X(s))| \) is decreasing for all \( s \in [0,t] \). And this contradicts with (3.34). So we must have
\[
v_0 \cdot \nabla \xi(x_0) < 0.
\]
Then we can define \( \sigma_1 = \frac{|v_0 \cdot \nabla \xi(x_0)|}{\|v_0\|^2 + \|E\|_{L^\infty_{t,x}}^2 + \|E\|_{L^\infty_{t,x}} + 1} \) as before. Now if \( \sigma_1 \geq 1 \) then \( |\xi(X(s))| \) is increasing on \([0,t]\), using the change of variable \( x \mapsto |\xi| \) and the estimate (3.18) and (3.35) we get the desired bound.

If \( \sigma_1 < 1 \), then from (3.10) we have
\[
|\xi(X(s_1))| \geq \frac{\delta}{2} |v_0 \cdot \nabla \xi(x_0)|^2.
\]
And then from the argument for (3.32) we get
\[
|\xi(X(s))| \geq \frac{\delta^2}{2} \frac{\alpha^2(t,x,v)}{|v|^2 + \|E\|_{L^\infty_{t,x}}^2 + \|E\|_{L^\infty_{t,x}} + 1}.
\]
for all \( s \in [\sigma_1,t] \). This lower bound combined with the estimate (3.20), (3.18) gives the desired bound.

Finally we left with the case
\[
|\xi(x)| > \delta \frac{\alpha^2(t,x,v)}{10(|v|^2 + \|E\|_{L^\infty_{t,x}}^2 + \|E\|_{L^\infty_{t,x}} + 1)}.
\]
Then again, from the argument for (3.32) we get
\[ |\xi(X(s))| \geq \delta^2 |v|^2 + \|E\|_{L^\infty}^2 + \|E\|_{L^\infty}^2 + 1 \]
for all \( s \in [0,t] \). This lower bound combined with the estimate (3.20) gives the desired bound. \( \square \)

**Proof of (2) Lemma** Since \( \frac{|u_t|^r}{v} \lesssim \frac{|V_{cl}(s) - u|^r}{v} \lesssim \{1 + |V_{cl}(s) - u|^2\}^{r/2} \) and \( \langle V_{cl}(s) - u \rangle^r e^{-\theta |V_{cl}(s) - u|^2} \}
\[ e^{-C_{e,r} |V_{cl}(s) - u|^2} \], it suffices to consider the case \( r = 0 \). It is important to control the number of bounces,
\[ \ell_+(s) = \ell_+(s; t, x, v) \in \mathbb{N} \] such that \( t^{\ell+1} \leq s < t^{\ell+1} \).

An important consequence of Velocity lemma is that for the specular cycles
\[ \alpha(s, X_{cl}(s; t, x, v), V_{cl}(s; t, x, v)) \gtrsim e^{-c(|v|^2)(t-s)} \alpha(t, x, v), \]
and therefore for the specular cycles
\[ \ell_+(s; t, x, v) \leq |t - s| \min_{0 \leq \ell \leq \ell_+(s; t, x, v)} |t - t^{\ell+1}| \leq \frac{|t - s|}{\min_{0 \leq \ell \leq \ell_+(s; t, x, v)} \alpha(t, x, v)} \alpha(t, x, v) \] \[ \lesssim \frac{|t - s|(|v|^2 + 1)}{\alpha(t, x, v)} \] (3.36)

For fixed \((t, x, v)\) we use the following notation \( \alpha(s) := \alpha(s; t, x, v) := \alpha(X_{cl}(s; t, x, v), V_{cl}(s; t, x, v)) \).

Now we consider the estimate (3.7). From (3.18), (3.19), and (3.20) we have
\[ \int_0^t Z(s, x, v) e^{-\theta |V_{cl}(s) - u|^2} \frac{Z(s, x, v)}{|V_{cl}(s) - u|^{2-\kappa}} d \alpha(s, X_{cl}(s; t, x, v), u) ds \]
\[ \lesssim \sum_{0 \leq \ell \leq \ell_+(0; t, x, v)} \int_{t^{\ell+1}}^{t^{\ell}} Z(s, x, v) e^{-\theta |v|^2 - |u|^2} \frac{Z(s, x, v)}{|v|^2 - |u|^{2-\kappa}} d \alpha(s, X_{cl}(s; t, x, v), u) ds \]
\[ \lesssim \sup_{0 \leq \ell \leq \ell_+(0; t, x, v)} \left\{ e^{-\frac{\mu}{2}(|v|^2)/2} Z(s, x, v) \right\} \]
\[ \times \sum_{0 \leq \ell \leq \ell_+(0; t, x, v)} \left( e^{-\frac{\mu}{2}(|v|^2)/2} \frac{\delta^{2-\beta} \langle \alpha(t', x, v) \rangle^{\beta-2}}{2} + e^{-\frac{\mu}{2}(|v|^2)/2} \frac{\delta^{2-\beta} \langle \alpha(t', x, v) \rangle^{\beta-2}}{2} \right) \int_{t^{\ell+1}}^{t^{\ell}} e^{-\frac{\mu}{2}(|v|^2)/2} ds \]
\[ \lesssim \sup_{0 \leq \ell \leq \ell_+(0; t, x, v)} \left\{ e^{-\frac{\mu}{2}(|v|^2)/2} Z(s, x, v) \right\} \]
\[ \times \sum_{0 \leq \ell \leq \ell_+(0; t, x, v)} \left( \frac{\delta^{2-\beta} \langle |v|^2 \alpha(t', x, v) \rangle^{\beta-2}}{2} + \frac{\delta^{2-\beta} \langle |v|^2 \alpha(t', x, v) \rangle^{\beta-2}}{2} \right) \int_{t^{\ell+1}}^{t^{\ell}} e^{-\frac{\mu}{2}(|v|^2)/2} ds \].

Clearly
\[ \sum_{0 \leq \ell \leq \ell_+(0; t, x, v)} \frac{\delta^{2-\beta} \langle |v|^2 \alpha(t', x, v) \rangle^{\beta-2}}{2} \int_{t^{\ell+1}}^{t^{\ell}} e^{-\frac{\mu}{2}(|v|^2)/2} ds \]
\[ \lesssim \frac{1}{\delta^{2-\beta} \langle \alpha(t, x, v) \rangle^{\beta-2}} \int_{t^{\ell+1}}^{t^{\ell}} e^{-\frac{\mu}{2}(|v|^2)/2} ds \]
\[ \lesssim \frac{1}{\delta^{2-\beta} \langle \alpha(t, x, v) \rangle^{\beta-2}} \int_{t^{\ell+1}}^{t^{\ell}} e^{-\frac{\mu}{2}(|v|^2)/2} ds \].

And for \( \int_{t_0}^{t_1} e^{-\frac{\mu}{2}(|v|^2)/2} ds \), we let \( \tilde{\ell} \) be the bounce that \( t^{\tilde{\ell}} \geq t - \frac{1}{\alpha} \) and \( t^{\tilde{\ell}+1} < t - \frac{1}{\alpha} \),
\[ \sum_{0 \leq \ell \leq \ell_+(t_0, t_1, v)} e^{-\frac{\mu}{2}(|v|^2)/2} \]
and decompose \( \sum_{0 \leq \ell \leq \ell_+(t_0, t_1, v)} = \sum_{\ell=0}^{\tilde{\ell}} + \sum_{\ell=\tilde{\ell}+1}^{\ell_+(t_0, t_1, v)} \). Then from (3.36)
\[ \sum_{\ell=0}^{\tilde{\ell}} e^{-\frac{\mu}{2}(|v|^2)/2} \lesssim |\tilde{\ell}| \lesssim \frac{1}{\alpha(t, x, v) / |v|^2} \lesssim \frac{1}{\alpha(t, x, v) / |v|^2} \].
For $\ell \geq \tilde{\ell} + 1$, we have
\[
|t - t^{\ell+1}| \leq |t - t^\ell| + |t^\ell - t^{\ell+1}| \leq |t - t^\ell| + C \frac{1}{(v)} |t - t^\ell| = (1 + C)|t - t^\ell|.
\]
Thus
\[
\sum_{\ell=\tilde{\ell}+1}^{\ell_*}(0,t,x,v) e^{-\frac{t}{(v)} (t-t^\ell)} \leq \sum_{\ell=\tilde{\ell}+1}^{\ell_*}(0,t,x,v) e^{-\frac{t}{(v)} (t-t^\ell)} e^{-\frac{t}{(v)} (t-t^{\ell+1})}
\]
\[
\leq \max_{\ell} \left\{ \sum_{\ell=0}^{\ell_*}|t^\ell - t^{\ell+1}| e^{-\frac{t}{(v)} (t-t^{\ell+1})} \right\} \sum_{\ell=0}^{\ell_*}|t^\ell - t^{\ell+1}| e^{-\frac{t}{(v)} (t-t^{\ell+1})}
\]
\[
\lesssim \frac{2\pi e^{-\frac{t}{(v)} (t-t^\ell)} e^{C(t^\ell)(t^\ell)}}{\alpha(t,x,v)} \int_0^t e^{-\frac{t}{(v)} (t-s)} ds
\]
\[
\lesssim \frac{(v)(1+C)}{\alpha(t,x,v)}.
\]
Therefore
\[
\sum_{\ell=0}^{\ell_*}(0,t,x,v) e^{-\frac{t}{(v)} (t-t^\ell)} \frac{\delta^{3-\beta}}{(v)^2(\alpha(t,x,v))^{-2}} \lesssim \frac{\delta^{3-\beta}}{(v)^2(\alpha(t,x,v))^{-1}}.
\] (3.39)
Combining (3.37), (3.38) and (3.39) we prove (3.47).

4. Moving frame for specular cycles

We use the moving frame for the specular cycles introduced in [7]. We denote the standard spherical coordinate $x| = x| (\omega) = (x_{\perp}, x_{\parallel})$ for $\omega \in S^2$
\[
\omega = (\cos x_{\perp,1}(\omega) \sin x_{\perp,2}(\omega), \sin x_{\perp,1}(\omega) \sin x_{\perp,2}(\omega), \cos x_{\perp,2}(\omega)),
\]
where $x_{\perp,1}(\omega) \in [0, 2\pi)$ is the azimuth and $x_{\perp,2}(\omega) \in [0, \pi)$ is the inclination.

We define an orthonormal basis of $\mathbb{R}^3$, $\{\hat{r}(\omega), \hat{\phi}(\omega), \hat{\theta}(\omega)\}$, with $\hat{r}(\omega) := \omega$ and
\[
\hat{\phi}(\omega) := \left(\cos x_{\perp,1}(\omega) \cos x_{\perp,2}(\omega), \sin x_{\perp,1}(\omega) \cos x_{\perp,2}(\omega), -\sin x_{\perp,2}(\omega)\right),
\]
\[
\hat{\theta}(\omega) := (-\sin x_{\perp,1}(\omega), \cos x_{\perp,2}(\omega), 0).
\]
Moreover, $\hat{r} \times \hat{\phi} = \hat{\theta}$, $\hat{\phi} \times \hat{\theta} = \hat{r}$, $\hat{\theta} \times \hat{r} = \hat{\phi}$, and
\[
\partial_{x_{\perp,1}} \hat{r} = \sin x_{\perp,2} \hat{\theta}, \quad \partial_{x_{\perp,2}} \hat{r} = \hat{\phi},
\] (4.1)
where $\partial_{x_{\perp,1}} \hat{r}$ does not vanish (non-degenerate) away from $x_{\perp,2} = 0$ or $\pi$.

Without loss of generality we assume $0 = (0, 0, 0) \in \Omega$. For
\[
p = (z, w) \in \partial \Omega \times S^2 \text{ with } n(z) \cdot w = 0,
\]
we define the north pole $N_p \in \partial \Omega$ and the south pole $S_p \in \partial \Omega$ as
\[
N_p := |N_p|(n(z) \times w) \in \partial \Omega, \quad S_p := |S_p|(n(z) \times w) \in \partial \Omega,
\]
where $\partial_{x_{\perp,1}} \hat{r}$ is degenerate. We define the straight-line $L_p$ passing both poles
\[
L_p := \{\tau N_p + (1 - \tau)S_p : \tau \in \mathbb{R}\}.
\]

Lemma 6. Assume $\Omega$ is convex [3]. Fix $p = (z, w) \in \partial \Omega \times S^2$ with $n(z) \cdot w = 0$.

(i) There exists a smooth map (spherical-type coordinate)
\[
\eta_p : [0, 2\pi) \times (0, \pi) \rightarrow \partial \Omega \setminus \{N_p, S_p\},
\]
\[
x_{\parallel,p} := (x_{\parallel,p,1}, x_{\parallel,p,2}) \mapsto \eta_p(x_{\parallel,p}),
\] (4.2)
which is one-to-one and onto. Here on $[0, 2\pi) \times (0, \pi)$ we have $\partial_\eta \eta_p := \frac{\partial \eta_p}{\partial x_{\parallel,p,1}} \neq 0$ and
\[
\frac{\partial \eta_p}{\partial x_{\parallel,p,1}}(x_{\parallel,p}) \times \frac{\partial \eta_p}{\partial x_{\parallel,p,2}}(x_{\parallel,p}) \neq 0.
\] (4.3)
We define
\[ \mathbf{n}_p := n \circ \eta_p : [0, 2\pi) \times (0, \pi) \to S^2. \]

(ii) We define the \( p \)-spherical coordinate in the tubular neighborhood of the boundary:
For \( \delta > 0, \delta_1 > 0, C > 0 \), we have a smooth one-to-one and onto map
\[
\Phi_p : [0, C\delta) \times [0, 2\pi) \times (\delta_1, \pi - \delta_1) \times \mathbb{R} \times \mathbb{R}^2 \to \{ x \in \bar{\Omega} : |\xi(x)| < \delta \} \setminus B_{C\delta_1}(L_p) \times \mathbb{R}^3,
\]
\[
(x_{\perp, p}, x_{\parallel, p, 1}, x_{\parallel, p, 2}, v_{\perp, p}, v_{\parallel, p, 1}, v_{\parallel, p, 2}) \mapsto \Phi_p(x_{\perp, p}, x_{\parallel, p, 1}, x_{\parallel, p, 2}, v_{\perp, p}, v_{\parallel, p, 1}, v_{\parallel, p, 2}),
\]
where \( B_{C\delta_1}(L_p) := \{ x \in \mathbb{R}^3 : |x - y| < C\delta_1 \text{ for some } y \in L_p \} \).
Explicitly,
\[
\Phi_p(x_{\perp, p}, x_{\parallel, p}, v_{\perp, p}, v_{\parallel, p}) := \begin{bmatrix}
  v_{\perp, p}[-n_p(x_{\parallel, p})] + \eta_p(x_{\parallel, p}) \\
  v_{\perp, p} \cdot \nabla \eta_p(x_{\parallel, p}) + x_{\perp, p} v_{\parallel, p} \cdot \nabla[-n_p(x_{\parallel, p})]
\end{bmatrix},
\]
where \( \nabla \eta_p = (\partial_1 \eta_p, \partial_2 \eta_p) = \left( \frac{\partial n_p}{\partial x_{\parallel, 1}}, \frac{\partial n_p}{\partial x_{\parallel, 2}} \right) \) and \( \nabla n_p = (\partial_1 n_p, \partial_2 n_p) = \left( \frac{\partial n_p}{\partial x_{\parallel, 1}}, \frac{\partial n_p}{\partial x_{\parallel, 2}} \right) \).

The Jacobian matrix is
\[
\frac{\partial \Phi(x_{\perp, p}, x_{\parallel, p}, v_{\perp, p}, v_{\parallel, p})}{\partial (x_{\perp, p}, x_{\parallel, p}, v_{\perp, p}, v_{\parallel, p})} = \begin{bmatrix}
  n(x_{\perp, p}) & \frac{\partial n_p}{\partial x_{\parallel, 1}}(-x_{\perp, p}) & \frac{\partial n_p}{\partial x_{\parallel, 2}}(-x_{\perp, p}) & \frac{\partial n_p}{\partial x_{\parallel, 1}}(x_{\perp, p}) & \frac{\partial n_p}{\partial x_{\parallel, 2}}(x_{\perp, p}) & 0_{3,3} \\
  -v_{\perp, p} \cdot \nabla n(x_{\perp, p}) & +x_{\perp, p} \frac{\partial n_p}{\partial x_{\parallel, 1}}(x_{\perp, p}) & +x_{\perp, p} \frac{\partial n_p}{\partial x_{\parallel, 2}}(x_{\perp, p}) & -v_{\perp, p} \cdot \nabla n(x_{\perp, p}) & -x_{\perp, p} \frac{\partial n_p}{\partial x_{\parallel, 1}}(x_{\perp, p}) & -x_{\perp, p} \frac{\partial n_p}{\partial x_{\parallel, 2}}(x_{\perp, p})
\end{bmatrix}.
\]

We fix an inverse map
\[
\Phi_p^{-1} : \{ x \in \bar{\Omega} : |\xi(x)| < \delta \} \setminus B_{C\delta'}(L_p) \times \mathbb{R}^3 \to [0, C\delta) \times [0, 2\pi) \times (\delta_1, \pi - \delta_1) \times \mathbb{R} \times \mathbb{R}^2.
\]
In general this choice is not unique but once we fix the range as above then an inverse map is uniquely determined.
We denote, for \( (x, v) \in \{ x \in \bar{\Omega} : |\xi(x)| < \delta \} \setminus B_{C\delta'}(L_p) \times \mathbb{R}^3 \)
\[
(x_{\perp, p}, x_{\parallel, p, 1}, x_{\parallel, p, 2}, v_{\perp, p}, v_{\parallel, p, 1}, v_{\parallel, p, 2}) = \Phi_p^{-1}(x, v),
\]

(iii) Let \( q = (y, u) \in \partial \Omega \times S^2 \) with \( |n(y) \cdot u| = 0 \) and \( |p - q| \ll 1 \) and
\[
\Phi_p(x_{\perp, q}, x_{\parallel, q}, v_{\perp, q}, v_{\parallel, q}) = (x, v) = \Phi_q(x_{\perp, q}, x_{\parallel, q}, v_{\perp, q}, v_{\parallel, q}).
\]
Then
\[
\frac{\partial (x_{\perp, p}, x_{\parallel, p}, v_{\perp, p}, v_{\parallel, p})}{\partial (x_{\perp, q}, x_{\parallel, q}, v_{\perp, q}, v_{\parallel, q})} = \nabla \Phi_q^{-1} \nabla \Phi_p = \mathbf{I}_{6,6} + O(|p - q|) \quad \left( \begin{array}{c|c}
  0 & 0 \\
  0 & 1 \\
  0 & 1 \\
  0 & 0 \\
  0 & 0 \\
  0 & 0 \\
\end{array} \right).
\]

Proof. See [7].

Lemma 7. (i) For \( |\xi(X_{cl}(s, t, x, v))| < \delta \) and \( |X_{cl}(s; t, x, v) - L_p| > C\delta_1 \) we define
\[
(X_p(s; t, x, v), V_p(s; t, x, v)) := \Phi_p^{-1}(X_{cl}(s; t, x, v), V_{cl}(s; t, x, v))
\]
\[
:= (x_{\perp, p}(s; t, x, v), x_{\parallel, p}(s; t, x, v), v_{\perp, p}(s; t, x, v), v_{\parallel, p}(s; t, x, v)).
\]
Then \(|v| \approx |V_p|\) and
\[
\begin{bmatrix}
\dot{x}_p \\
\dot{x}_i_p \\
\dot{v}_p \\
\dot{v}_i_p
\end{bmatrix}
(s; t, x, v) =
\begin{bmatrix}
v_p \\
v_i_p \\
F_p(x_p, v_p) \\
F_i_p(x_p, v_p)
\end{bmatrix}
(s; t, x, v).
\tag{4.7}
\]

Here
\[
F_{\perp} = F_{\perp}(x_{\perp}, x_{\parallel}, v_{\parallel} ) \\
= \sum_{j,k=1}^{2} v_{\parallel, j} v_{\parallel, k} \partial_j \partial_k \eta_p(x_{\parallel}) \cdot n_p(x_{\parallel}) - x_{\perp} \sum_{k=1}^{2} v_{\parallel, k} (v_{\parallel} \cdot \nabla) \partial_k n_p(x_{\parallel}) \cdot n_p(x_{\parallel}) \\
- E(s, -x_{\perp} n(x_{\parallel}) + \eta(x_{\parallel})) \cdot n(x_{\parallel}),
\tag{4.8}
\]

where
\[
\sum_{j,k=1}^{2} v_{\parallel, j} v_{\parallel, k} \partial_j \partial_k \eta_p(x_{\parallel}) \cdot n_p(x_{\parallel}) \lesssim |v|^2,
\]

and
\[
F_{\parallel} = F_{\parallel}(x_{\perp}, x_{\parallel}, v_{\perp}, v_{\parallel}) \\
= \sum_{i=1,2} G_{p, ij}(x_{\perp}, x_{\parallel}) n_p(x_{\parallel}) \cdot (\partial_1 \eta_p(x_{\parallel}) \times \partial_2 \eta_p(x_{\parallel})) \\
\times \{2v_{\perp} v_{\parallel} \cdot \nabla n_p(x_{\parallel}) - v_{\parallel} \cdot \nabla^2 \eta_p(x_{\parallel}) \cdot v_{\parallel} + x_{\perp} v_{\parallel} \cdot \nabla^2 n_p(x_{\parallel}) \cdot v_{\parallel} \\
- E(s, -x_{\perp} n(x_{\parallel}) + \eta(x_{\parallel})) \} \cdot \{n_p(x_{\parallel}) \times \partial_{i+1} \eta_p(x_{\parallel})\},
\tag{4.9}
\]

where a smooth bounded function \(G_{p, ij}(x_{\perp}, x_{\parallel})\) is specified in \([4.10]\).

(ii) For \(\tau \in (t^{l+1}, t^l)\), if the \(\mathbf{p}\)-spherical coordinate is well-defined in \([\tau, t^l]\) then
\[
[X_{\ell}(\tau; t, x, v), V_{\ell}(\tau; t, x, v)] = [X_{\ell}(\tau; t^l, x_{\parallel}, v_{\perp}), V_{\ell}(\tau; t^l, x_{\parallel}, v_{\perp}, v_{\perp})]
\]

and, for \(\partial_{v_{\ell}} = [\partial_{v_{\ell}}, \partial_{v_{\ell}}]\),
\[
\begin{bmatrix}
|\partial_{x_{\ell}}^1 X_{\ell}(\tau)| \\
|\partial_{x_{\ell}}^1 V_{\ell}(\tau)| \\
|\partial_{x_{\ell}} X_{\ell}(\tau)| \\
|\partial_{v_{\ell}}^1 V_{\ell}(\tau)|
\end{bmatrix} \lesssim \begin{bmatrix}
1 \\
1 \\
(\Omega_{\ell} \|V\|_{L^\infty(t^l, t)}^1 + |v|^2) |\tau - t^l| \\
|\tau - t^l|
\end{bmatrix}.
\tag{4.10}
\]

For \(t^{l+1} < \tau < t^l\) then
\[
[X_{\ell}(\tau; t, x, v), V_{\ell}(\tau; t, x, v)] = [X_{\ell}(\tau; s, x_{\ell}(s), t, x, v), V_{\ell}(\tau; s, x_{\ell}(s), t, x, v), V_{\ell}(\tau; s, x_{\ell}(s), t, x, v), V_{\ell}(\tau; s, t, x, v)]
\]

and
\[
\begin{bmatrix}
|\partial_{x_{\ell}} X_{\ell}(\tau)| \\
|\partial_{x_{\ell}} V_{\ell}(\tau)| \\
|\partial_{v_{\ell}} X_{\ell}(\tau)| \\
|\partial_{v_{\ell}} V_{\ell}(\tau)|
\end{bmatrix} \lesssim \begin{bmatrix}
1 \\
1 \\
(\Omega_{\ell} \|V\|_{L^\infty(t^l, t)}^1 + |v|^2) |\tau - s| \\
|\tau - s|
\end{bmatrix}.
\tag{4.11}
\]

Moreover, for either \(\partial_{x_{\ell}} = [\partial_{x_{\ell}}, \partial_{x_{\ell}}, \partial_{x_{\ell}}, \partial_{x_{\ell}}]\) or \(\partial_{x_{\ell}} = [\partial_{x_{\ell}}, \partial_{x_{\ell}}, \partial_{x_{\ell}}, \partial_{x_{\ell}}]\)
\[
\begin{bmatrix}
|\partial_{x_{\ell}} F(\tau)| \\
|\partial_{v_{\ell}} F(\tau)| \\
|\partial_{x_{\ell}} V(\tau)| \\
|\partial_{v_{\ell}} V(\tau)|
\end{bmatrix} \lesssim \begin{bmatrix}
O_{\ell} \|V\|_{L^\infty(t^l, t)}^1 + |v|^2 \\
O_{\ell} \|V\|_{L^\infty(t^l, t)}^1 + |v|^2 \\
O_{\ell} \|V\|_{L^\infty(t^l, t)}^1 + |v|^2 \\
O_{\ell} \|V\|_{L^\infty(t^l, t)}^1 + |v|^2
\end{bmatrix}.
\tag{4.12}
\]

Proof. From \(\dot{v} = 0\) and the second equation of \([4.4]\) equals
\[
E(s, -x_{\perp} n(x_{\parallel}) + \eta(x_{\parallel})) = \dot{v}_{\perp}(s) [-n(x_{\parallel})] - 2v_{\perp}(s) v_{\parallel} \cdot \nabla n(x_{\parallel}) + \dot{v}_{\parallel}(s) \cdot \nabla \eta(x_{\parallel}) \]
\[
\quad + v_{\parallel} \cdot \nabla^2 \eta(x_{\parallel}) \cdot v_{\parallel} - x_{\perp} \dot{v}_{\perp} \cdot \nabla n(x_{\parallel}) - x_{\perp} v_{\parallel} \cdot \nabla^2 n(x_{\parallel}) \cdot v_{\parallel}.
\tag{4.13}
\]

We take the inner product with \(n(x_{\parallel})\) to the above equation to have
\[
\dot{v}_{\perp}(s) = -E(s, -x_{\perp} n(x_{\parallel}) + \eta(x_{\parallel})) \cdot n(x_{\parallel}) + [v_{\parallel} \cdot \nabla^2 \eta(x_{\parallel}) \cdot v_{\parallel} \cdot n(x_{\parallel})]
\]
\[
\quad - x_{\perp} [v_{\parallel} \cdot \nabla^2 n(x_{\parallel}) \cdot v_{\parallel} \cdot n(x_{\parallel})]
\tag{4.14}
\]
where we have used the fact $\nabla n \perp n$ and $\nabla \eta \perp n$.

Since $0 = \xi(\eta(x_i))$ we take $x_{ij}$ and $x_{ji}$ derivatives to have
\[
0 = \partial_{\eta_{ij}} \left[ \sum_k \partial_k \xi \partial x_{ij}, \eta k \right] = \sum_{k,m} \partial_k \partial_m \xi \partial x_{ij} \eta m \partial x_{ij}, \eta k + \sum_k \partial_k \xi \partial x_{ij}, \partial x_{ij}, \eta k,
\]
and from the convexity (1.3) and $n = \nabla \xi / |\nabla \xi|$, \[
\left[ v_\parallel \cdot \nabla^2 \eta \cdot v_\parallel \right] \cdot n = \sum_{i,j,k} \frac{v_{i,k} \partial_k \xi \partial_j \eta k v_{j,k}}{|\nabla \xi|} = -\sum_{i,j,k,m} \frac{v_{i,k} \partial_k \eta m \partial_j \eta \{ \partial_j \eta m v_{j,k} \}}{|\nabla \xi|} \lesssim -|v_\parallel|^2.
\]
Define $a_{ij}(x_\parallel)$ via
\[
\begin{bmatrix}
    a_{11} & a_{12} \\
    a_{21} & a_{22}
\end{bmatrix} = \begin{bmatrix}
    \partial_1 n \cdot \partial_1 n & \partial_1 n \cdot \partial_2 n & \partial_2 n \cdot \partial_2 n
\end{bmatrix}^{-1}
\]
where $\text{det}(\partial_1 \eta \cdot \partial_2 \eta) = |\partial_1 \eta \times \partial_2 \eta|^2 \neq 0$ due to (1.3). Then $\nabla n$ is generated by $\nabla \eta$:
\[
-\partial_i n(x_\parallel) = \sum_k a_{ik}(x_\parallel) \partial_k \eta (x_\parallel).
\]
We take the inner product (4.13) with $(-1)^{i+1} (n(x_\parallel) \times \partial_1 n(x_\parallel))$ to have
\[
\sum_k (\delta_{ki} + x_\perp a_{ki}) v_{i,k} = \frac{(-1)^{i+1}}{-n(x_\parallel) \cdot (\partial_1 \eta (x_\parallel) \times \partial_2 \eta (x_\parallel))} \times \bigg\{ -2 v_\perp v_\parallel \cdot \nabla n(x_\parallel) + v_\parallel \cdot \nabla^2 \eta (x_\parallel) \cdot v_\parallel - x_\perp v_\parallel \cdot \nabla^2 n(x_\parallel) \cdot v_\parallel \\
- E(s, -x_\perp n(x_\parallel) + \eta (x_\parallel)) \bigg\} \cdot (-n(x_\parallel) \times \partial_{i+1} \eta (x_\parallel)),
\]
where we used the notational convention for $\partial_{i+1} \eta$, the index $i + 1 \mod 2$. For $|\xi(x)| \ll 1$ and therefore $|x_\perp | \ll 1$ the matrix $\delta_{ki} + x_\perp a_{ki}$ is invertible; there exists the inverse matrix $G_{ij}$ such that $\sum_i (\delta_{ki} + x_\perp a_{ki}(x_\parallel)) G_{ij}(x_\perp, x_i) = \delta_{kj}$. Therefore we have
\[
\dot{v}_{i,j} = \sum_i G_{ij}(x_\perp, x_i) \frac{(-1)^{i+1}}{-n(x_\parallel) \cdot (\partial_1 \eta (x_\parallel) \times \partial_2 \eta (x_\parallel))} \times \bigg\{ -2 v_\perp v_\parallel \cdot \nabla n(x_\parallel) + v_\parallel \cdot \nabla^2 \eta (x_\parallel) \cdot v_\parallel - x_\perp v_\parallel \cdot \nabla^2 n(x_\parallel) \cdot v_\parallel \\
- E(s, -x_\perp n(x_\parallel) + \eta (x_\parallel)) \bigg\} \cdot (-n(x_\parallel) \times \partial_{i+1} \eta (x_\parallel))
\]
\[
:= F_{i,j}(x_\perp, x_i, v_\perp, v_\parallel).
\]
Here
\[
\begin{bmatrix}
    G_{11} & G_{12} \\
    G_{21} & G_{22}
\end{bmatrix} = \frac{1}{1 + x_\perp (a_{11} + a_{22}) + (x_\parallel)^2 (a_{11} a_{22} - a_{12} a_{21})} \begin{bmatrix}
    1 + x_\perp a_{22} & -x_\perp a_{12} \\
    -x_\perp a_{21} & 1 + x_\perp a_{11}
\end{bmatrix},
\]
\[
\begin{bmatrix}
    a_{11} & a_{12} \\
    a_{21} & a_{22}
\end{bmatrix} = \frac{1}{|\partial_1 \eta|^2 |\partial_2 \eta|^2 - (\partial_1 \eta \cdot \partial_2 \eta)^2} \times \begin{bmatrix}
    |\partial_1 n|^2 |\partial_2 n|^2 - (\partial_1 n \cdot \partial_2 n)(\partial_1 \eta \cdot \partial_2 \eta) & -|\partial_1 n|^2 (\partial_1 \eta \cdot \partial_2 \eta) + (\partial_1 n \cdot \partial_2 n)|\partial_1 \eta|^2 \\
    (\partial_1 n \cdot \partial_2 n)|\partial_2 n|^2 - |\partial_2 n|^2 (\partial_1 \eta \cdot \partial_2 \eta) & -|\partial_2 n|^2 (\partial_1 \eta \cdot \partial_2 \eta) + |\partial_2 n|^2 |\partial_1 \eta|^2
\end{bmatrix}.
\]
To complete the proof of (4.7), from $\dot{x} = v$ and $\dot{v} = E$, we have

$$v = -v_\bot n + v_\|\cdot \nabla \eta + x_\| [-\nabla n(x_\|)] \dot{x}_\|$$
$$= x_\| (-n(x_\|)) + x_\| [-\nabla n(x_\|)] \dot{x}_\| + \nabla \eta \dot{x}_\|$$

$$E(s, -x_\| n + \eta(x_\|)) = \dot{v}_\| (-n(x_\|)) - v_\| \nabla n \dot{x}_\| + \dot{v}_\| \nabla \eta + v_\| \nabla^2 \eta \dot{x}_\|$$
$$+ x_\| v_\| [-\nabla n(x_\|)] + x_\| \dot{v}_\| [-\nabla n(x_\|)] + x_\| v_\| [-\nabla^2 \eta x_\|].$$

We therefore conclude that $\dot{x}_\| = v_\|$, and $\dot{x}_\| = v_\|$ from $\Phi^{-1}$. We then solve $\dot{v}_\|$ and $\dot{v}_\|$ to obtain (4.7).

Now we prove (4.10) and (4.11). From (4.8) and (4.9), $x_\| = v_\|$, $\dot{x}_\| = v_\|$, and $\dot{v}_\| = F_\|$. Denote $\partial_\| = [\frac{\partial}{\partial \xi}, \frac{\partial}{\partial V_\|}, \frac{\partial}{\partial v_\|}]$. From (4.8) and (4.9),

$$\left[ \frac{\partial F_\|}{\partial F_\|} \right] \lesssim \left[ \frac{(O_\| \|V E\|_{L_t^\infty}^\infty (1) + |V(\tau)|^2) \{ |\partial x_\| + |\partial x_\| \} + |V(\tau)| |\partial v_\| |}{(O_\| \|V E\|_{L_t^\infty}^\infty (1) + |V(\tau)|^2) \{ |\partial x_\| + |\partial x_\| \} + |V(\tau)| \{ |\partial v_\| + |\partial v_\| \} } \right].$$

(4.17)

Now we use a single (rough) bound of $|\partial F_\| + |\partial F_\| \lesssim (O_\| \|V E\|_{L_t^\infty}^\infty (1) + |V(\tau)|^2) \{ |\partial x_\| + |\partial x_\| \} + |V(\tau)| \{ |\partial v_\| + |\partial v_\| \}$. Combining with $\frac{d}{dt}[x_\| (\tau), x_\| (\tau)] = [v_\| (\tau), v_\| (\tau)]$ yields

$$\frac{d}{dt} \left[ \frac{\partial x_\| (\tau)}{\partial v_\| (\tau)} + \frac{\partial x_\| (\tau)}{\partial v_\| (\tau)} \right] \lesssim \left[ \frac{0}{O_\| \|V E\|_{L_t^\infty}^\infty (1) + |V(\tau)|^2} \frac{1}{V(\tau)} \right] \left[ \frac{|\partial x_\| (\tau)}{|\partial v_\| (\tau)} + \frac{|\partial x_\| (\tau)}{|\partial v_\| (\tau)} \right].$$

(4.18)

Now for $M \gg 1$, let first prove (4.11) for $|v| < M$. From (4.18) we have

$$|\partial x_\| (\tau)| + |\partial v_\| (\tau)| \lesssim 1 + \int^t_\tau (1 + O_\| \|V E\|_{L_t^\infty}^\infty (1) + |V(\tau)| + |V(\tau)|^2) |\partial x_\| (\tau)| + |\partial v_\| (\tau)| d\tau'$$

$$\lesssim 1 + \int^t_\tau (1 + O_\| \|V E\|_{L_t^\infty}^\infty (1) + M^2) |\partial x_\| (\tau')| + |\partial v_\| (\tau')| d\tau'.$$

From Gronwall we have

$$|\partial x_\| (\tau)| + |\partial v_\| (\tau)| \lesssim \xi_\| \|V E\|_{L_t^\infty}^\infty \|x_\| 1.$$

(4.19)

For $x_\| = [\frac{\partial}{\partial \xi}, \frac{\partial}{\partial v_\|}]$, from (4.19), we have

$$|\partial x_\| (\tau)| \lesssim \int^t_\tau |\partial x_\| (\tau')| d\tau' \lesssim \xi_\| \|V E\|_{L_t^\infty}^\infty \|x_\| |\tau - t'|.$$

(4.20)

And for $x_\| = \frac{\partial}{\partial \xi}$, from (4.18), (4.19), we have

$$|\partial x_\| (\tau)| \lesssim \int^t_\tau \left[ \frac{O_\| \|V E\|_{L_t^\infty}^\infty (1) + |V(\tau)|^2} {O_\| \|V E\|_{L_t^\infty}^\infty (1) + |V(\tau)|^2} \right] |\partial x_\| (\tau')| + |\partial v_\| (\tau')| d\tau'$$

$$\lesssim \left[ \frac{O_\| \|V E\|_{L_t^\infty}^\infty (1) + |v|^2} {O_\| \|V E\|_{L_t^\infty}^\infty (1) + |v|^2} \right] |\tau - t'| + M \int^t_\tau |\partial x_\| (\tau')| d\tau'.$$

From Gronwall we have

$$|\partial x_\| (\tau)| \lesssim \xi_\| \|V E\|_{L_t^\infty}^\infty \|x_\| (O_\| \|V E\|_{L_t^\infty}^\infty (1) + |v|^2) |\tau - t'|.$$

(4.21)

Combining (4.19), (4.20), and (4.21), we prove (4.10) for $|v| < M$. 20
For the case \(|v| \geq M \gg 1\), we have \(|V(\tau)| < 2|v|\), so
\[
\frac{d}{d\tau} \left[ \frac{\partial X_{|v|}(\tau)}{\partial \nu_{|v|}(\tau)} + \frac{\partial X_{|v|}(\tau)}{\partial \nu_{|v|}(\tau)} \right] \lesssim \left( O_{\xi,\|\nu\|}^{\|\nu\|^{\infty}}(1) + |v|^2 \right) \frac{1}{|v|} \left[ \frac{\partial X_{|v|}(\tau)}{\partial \nu_{|v|}(\tau)} + \frac{\partial X_{|v|}(\tau)}{\partial \nu_{|v|}(\tau)} \right].
\]

By Lemma 9 we prove our claim (4.10) for the case \(|v| \geq M\). The proof of (4.11) is exactly same but we use \(\partial = [\partial X_{\xi}(s), \partial V_{\xi}(s)]\) to conclude the proof.

We prove the first row of (4.12) by (4.17). By taking the time derivative to (4.8), (4.9) and applying (4.7) we prove the second row of row of (4.12).

\[\square\]

5. Derivative estimate for the generalized characteristics

The main goal of this section is to prove the following key estimate for the derivatives of the generalized characteristics \((X_{\alpha}(s, t, x, v), V_{\alpha}(s, t, x, v))\) defined in (1.15).

**Theorem 2.** There exists \(C = C(\Omega, E) > 0\) such that for all \((t, x, v) \in [0, T] \times \bar{\Omega} \times \mathbb{R}^3\), \(0 \leq s \leq t\), with \(s \neq t^i\) for \(i = 1, 2, \cdots, t_\ast\)

\[
|\nabla_{\alpha} X_{\alpha}(s, t, x, v)| \lesssim e^{C|v|(t-s)} \frac{|v| + 1}{\alpha(t, x, v)},
\]

\[
|\nabla_{\alpha} V_{\alpha}(s, t, x, v)| \lesssim e^{C|v|(t-s)} \frac{1}{|v| + 1},
\]

\[
|\nabla_{\alpha} D_{\alpha}(s, t, x, v)| \lesssim e^{C|v|(t-s)} \frac{|v|^3 + 1}{\alpha^2(s, t, x, v)},
\]

\[
|\nabla_{\alpha} V_{\alpha}(s, t, x, v)| \lesssim e^{C|v|(t-s)} \frac{|v| + 1}{\alpha(t, x, v)}.
\]

In order to achieve this, we need a crucial bound on the backward exit time:

**Lemma 8.** Suppose \(E(t, x) \cdot n(x) > c_E\) for all \(x \in \partial \Omega\), then there exists \(C = C(\Omega, E) \gg 1\) and \(0 < T \ll 1\) such that for any \((t, x, v) \in [0, T] \times \bar{\Omega} \times \mathbb{R}^3\), \(t^1(t, x, v) > 0\),

\[
\frac{|t - t^1|}{|v|} + \frac{|t - t^1||v|}{|v|} + \frac{|t - t^1||v|^2}{|v|^2} < C. \tag{5.2}
\]

And for \((t, x, v) \in [0, T] \times \gamma_+ \times \mathbb{R}^3\), \(t^1(t, x, v) < 0\),

\[
\frac{|t|}{|v^1|} + \frac{|t||v|}{|v^1|} + \frac{|t||v|^2}{|v|^2} < C. \tag{5.3}
\]

**Proof.** Let \(N > 10(\|E\|_{L^\infty_{\gamma^+}} + 1)\) be fixed. Let’s first consider the case \((t, x, v) \in [0, T] \times \bar{\Omega} \times \mathbb{R}^3\), \(t^1(t, x, v) > 0\), and prove

\[
\frac{|t - t^1|}{|v^1|} \lesssim 1 \text{ for all } |v| < N. \tag{5.4}
\]

From (4.8) we have
\[
F_{\perp}(s) < -c_\xi |v| - c_E + C_\xi |v^1|^2.
\]

By choosing \(T < \frac{c_E}{2Nc_\xi t}\), we have \(x_\perp < 2NT < \frac{c_\xi}{2c_\xi t}\), thus
\[
F_{\perp}(s) < -c_E - c_\xi |v|^2 + \frac{c_\xi}{2}|v|^2 < -c_E, \text{ for all } t^1 < s < t. \tag{5.5}
\]

Therefore
\[
0 < x_\perp(t) = \int_{t^1}^{t} v_\perp(s) ds = \int_{t^1}^{t} \left(-v^1 + \int_{t^1}^{s} F_{\perp}(\tau)d\tau\right) ds = (t - t^1)(-v^1) + \int_{t^1}^{t} \int_{t^1}^{s} F_{\perp}(\tau) d\tau ds.
\]
So from (5.5) and (5.6),
\[
\frac{CE}{2}(t - t^1)^2 < - \int_{t_1}^t \int_{t_1}^t F_\perp(\tau)d\tau ds < |t - t^1|v_1^1.| 
\] (5.7)
Therefore \(\frac{CE}{2}(t - t^1)^2 < |v_1^1.|\), and this proves (5.4).

Next, for \(|v| \geq N\), let \(d = \max_{x,y \in \Gamma} |x - y|\), then \(\xi(X(t + t')) = 0\) for some \(t' < \frac{D}{2N}\) by extending the field as \(E(s,x) = E(T,x)\) for \(s > T\) if necessary. So we can without loss of generality assume \(x \in \partial \Omega\). We claim
\[
|t - t^1||v|^2 \leq 1 \text{ for all } |v| \geq N. 
\] (5.8)
Since \((x,v) \in \gamma_+\) we have
\[
0 = \xi(x_1) - \int_{t_1}^t \nabla \xi(X(s)) \cdot V(s)ds 
= -(t - t_1)v \cdot \nabla \xi(x) + \int_{t_1}^t \int_s^t \left( \nabla \xi(X(\tau)) \cdot V(\tau) + E(\tau, X(\tau)) \cdot \nabla \xi(X(\tau)) \right) d\tau ds. 
\] (5.9)
Note that for \(T < \frac{N}{4|E|_{x,y}}\), \(\frac{|v|}{t} < |V(\tau)| < 2|v|\) for all \(\tau \in [t_1, t]\). Thus from (5.9)
\[
|t - t^1|(v \cdot \nabla \xi(x)) 
\geq \frac{C}{8}|t - t^1||v|^2 + \int_{t_1}^t \int_s^t E(\tau, X(\tau)) \cdot \nabla \xi(X(\tau)) d\tau ds 
\geq \frac{C}{8}|t - t^1||v|^2 + \left| \frac{|t - t^1|^2}{2} E(t,x) \cdot \nabla \xi(x) - \int_{t_1}^t \int_s^t \int_{\tau}^{t} \frac{d}{d\tau'} (E(\tau', X(\tau'))) \cdot \nabla \xi(X(\tau'))) d\tau' d\tau ds \right. 
\] (5.10)
\[
\geq \frac{C}{8}|t - t^1|^2|v|^2 - |t - t^1|^3 C_E \xi(1 + |v|) 
\geq |t - t^1|^2 \left( \frac{C}{8}|v|^2 - |t - t^1(C_E \xi(1 + |v|)) \right). 
\]
Since \(|v| \geq N\), we have \(\frac{C}{8}|v|^2 - |t - t^1|C_E \xi(1 + |v|) > \frac{C}{20}|v|^2\). Therefore (5.11) gives
\[
(v \cdot \nabla \xi(x)) > \frac{C}{20}|t - t^1||v|^2. 
\] (5.11)
Then using the velocity lemma we have \(|t - t^1||v|^2 \leq |v \cdot \nabla \xi(x)| \leq |v_1^1.|\), and we conclude (5.8).

Now combining (5.4) and (5.8) we actually have for all \((x,v) \in \gamma_+\),
\[
\frac{|t - t_1|}{|v_1^1|} + \frac{|t - t^1||v|^2}{|v_1^1|} \leq 1. 
\]
Therefore
\[
\frac{|t - t^1||v|}{|v_1^1|} \leq \max\{\frac{|t - t^1|}{|v_1^1|}, \frac{|t - t^1||v|^2}{|v_1^1|}\} \leq 1, 
\]
and we conclude (5.2).

The proof of (5.3) is similar. If \(|v| < N\), we have
\[
0 < x_\perp(0) = -\int_0^t v_\perp(s)ds = -\int_0^t \left( v_\perp - \int_s^t F_\perp(\tau)d\tau \right) ds = -tv_\perp + \int_0^t \int_s^t F_\perp(\tau)d\tau ds, 
\] (5.12)
So same as (5.7) we have
\[
\frac{CE}{2}t^2 < -\int_0^t \int_s^t F_\perp(\tau)d\tau ds < t|v_\perp|. 
\]
Therefore $\frac{\xi}{\sqrt{t}} < |v_\perp|$. And if $|v| > N$, similarly we get

$$0 > \xi(X(0))$$

$$= \xi(x) - \int_0^t \nabla \xi(X(s)) \cdot V(s) \, ds$$

$$= -|t|(v \cdot \nabla \xi(x)) + \int_0^t \int_s^t (V(\tau) \cdot \nabla^2 \xi(X(\tau)) \cdot V(\tau) + E(\tau, X(\tau)) \cdot \nabla \xi(X(\tau))) \, d\tau ds. $$

Then by the same argument as lines between (5.9) and (5.11) we get $|v \cdot \nabla \xi(x)| > \frac{C}{\sqrt{t}} |v|^2$, and this proves (5.3).

We need a version of Gronwall’s inequality for matrices:

**Lemma 9.** Let $m > 0$, $a(\tau), b(\tau), f(\tau), g(\tau) \geq 0$ for all $0 \leq \tau \leq t$, and satisfy $|v| > M \gg 1$, and

$$A(\tau) B(\tau)$$

then

$$\begin{bmatrix} a(\tau) \\ b(\tau) \end{bmatrix} \leq \begin{bmatrix} 0 & 1 \\ m + |v|^2 & |v| \end{bmatrix} \begin{bmatrix} \int_0^t a(\tau') \, d\tau' \\ \int_0^t b(\tau') \, d\tau' \end{bmatrix} + \begin{bmatrix} g(t - \tau) \\ h(t - \tau) \end{bmatrix}$$

(5.14)

**Proof.** First we consider $A^\varepsilon, B^\varepsilon$ solving, for $\varepsilon > 0$,

$$\begin{bmatrix} A^\varepsilon(\tau) \\ B^\varepsilon(\tau) \end{bmatrix} = C \begin{bmatrix} 0 & 1 \\ m + |v|^2 & |v| \end{bmatrix} \begin{bmatrix} \int_0^t A^\varepsilon(\tau') \, d\tau' \\ \int_0^t B^\varepsilon(\tau') \, d\tau' \end{bmatrix} + \begin{bmatrix} g(t - \tau) \\ h(t - \tau) \end{bmatrix} + \begin{bmatrix} \varepsilon \\ \varepsilon \end{bmatrix},$$

(5.15)

We claim that

$$\begin{bmatrix} A^\varepsilon(\tau) \\ B^\varepsilon(\tau) \end{bmatrix} \leq e^{C(\tau-t)} \begin{bmatrix} 0 & 1 \\ m + |v|^2 & |v| \end{bmatrix} \begin{bmatrix} \int_0^t A^\varepsilon(\tau') \, d\tau' \\ \int_0^t B^\varepsilon(\tau') \, d\tau' \end{bmatrix} + \begin{bmatrix} g(0) + \varepsilon \\ h(0) + \varepsilon \end{bmatrix}$$

(5.16)

We consider the matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ m + |v|^2 & |v| \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ m + |v|^2 & |v| \end{bmatrix}.$$ Denote

$$r_1 := \frac{1 + \sqrt{5 + 4m}}{2}, \quad r_2 := \frac{1 - \sqrt{5 + 4m}}{2}, \quad r_3 := \frac{1}{\sqrt{5 + 4m}}.$$

Then we diagonalize this matrix as

$$\begin{bmatrix} 0 & 1 \\ m + |v|^2 & |v| \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ r_1 |v| & r_2 |v| \end{bmatrix} \begin{bmatrix} r_1 |v| & 0 \\ 0 & r_2 |v| \end{bmatrix} = \begin{bmatrix} -r_2 r_3 & r_3 \frac{1}{|v|} \\ r_1 r_3 & -r_3 \frac{1}{|v|} \end{bmatrix}.$$

Denote

$$\begin{bmatrix} A^\varepsilon(\tau) \\ B^\varepsilon(\tau) \end{bmatrix} := \begin{bmatrix} -r_2 r_3 & r_3 \frac{1}{|v|} \\ r_1 r_3 & -r_3 \frac{1}{|v|} \end{bmatrix} \begin{bmatrix} A^\varepsilon(\tau) \\ B^\varepsilon(\tau) \end{bmatrix}$$

and rewrite the equations as

$$\frac{d}{d\tau} \begin{bmatrix} A^\varepsilon(\tau) \\ B^\varepsilon(\tau) \end{bmatrix} = C \begin{bmatrix} r_1 |v| & 0 \\ 0 & r_2 |v| \end{bmatrix} \begin{bmatrix} A^\varepsilon(\tau) \\ B^\varepsilon(\tau) \end{bmatrix} + \begin{bmatrix} -r_2 r_3 & r_3 \frac{1}{|v|} \\ r_1 r_3 & -r_3 \frac{1}{|v|} \end{bmatrix} \begin{bmatrix} g'(t - \tau) \\ h'(t - \tau) \end{bmatrix}.$$ Directly we compute

$$\begin{bmatrix} A^\varepsilon(\tau) \\ B^\varepsilon(\tau) \end{bmatrix} = \begin{bmatrix} e^{C r_1 |v| (\tau - t)} A^\varepsilon(t) \\ e^{C r_2 |v| (\tau - t)} B^\varepsilon(t) \end{bmatrix} + \int_t^\tau \begin{bmatrix} e^{C r_2 |v| (\tau - \tau')} A^\varepsilon(\tau') \\ 0 \end{bmatrix} + \begin{bmatrix} -r_2 r_3 & r_3 \frac{1}{|v|} \\ r_1 r_3 & -r_3 \frac{1}{|v|} \end{bmatrix} \begin{bmatrix} g'(t - \tau') \\ h'(t - \tau') \end{bmatrix} \, d\tau'.$$
Then
\[
\begin{bmatrix}
A^\varepsilon(\tau) \\
B^\varepsilon(\tau)
\end{bmatrix} =
\begin{bmatrix}
1 & 1 \\
1 & 2
\end{bmatrix}
\begin{bmatrix}
A^\varepsilon(\tau) \\
B^\varepsilon(\tau)
\end{bmatrix}
\]
\[
+ \int_t^\tau \begin{bmatrix} 1 & 1 \\ r_1 |v| & r_2 |v| \end{bmatrix}
\begin{bmatrix}
\varepsilon^{Cr_1|v|}(\tau-t) \\
\varepsilon^{Cr_2|v|}(\tau-t)
\end{bmatrix}
\begin{bmatrix}
r_3 r_3^{-1} \\
r_3 r_3^{-1}
\end{bmatrix}
\begin{bmatrix}
A^\varepsilon(t) \\
B^\varepsilon(t)
\end{bmatrix}
\]

Directly, the RHS equals
\[
\begin{bmatrix}
r_3 (r_1 \varepsilon^{Cr_2|v|}(\tau-t)) - r_2 \varepsilon^{Cr_1|v|}(\tau-t) \\
- r_1 r_2 r_3 |v| (\varepsilon^{Cr_1|v|}(\tau-t) - \varepsilon^{Cr_2|v|}(\tau-t))
\end{bmatrix}
\begin{bmatrix}
A^\varepsilon(t) \\
B^\varepsilon(t)
\end{bmatrix}
\]
\[
\int_t^\tau \begin{bmatrix} 1 & 1 \\ r_1 |v| & r_2 |v| \end{bmatrix}
\begin{bmatrix}
\varepsilon^{Cr_1|v|}(\tau-t) - \varepsilon^{Cr_2|v|}(\tau-t) \\
- r_1 r_2 r_3 |v| (\varepsilon^{Cr_1|v|}(\tau-t) - \varepsilon^{Cr_2|v|}(\tau-t))
\end{bmatrix}
\begin{bmatrix}
r_3 r_3^{-1} \\
r_3 r_3^{-1}
\end{bmatrix}
\begin{bmatrix}
g'(t - t') \\
h'(t - t')
\end{bmatrix}
\int_t^\tau.
\]

Since $|v| > M$, we have $|r_1 - r_2| < 1$, so by expansion we have $|\varepsilon^{Cr_1|v|}(\tau-t) - \varepsilon^{Cr_2|v|}(\tau-t)| \lesssim_{\varepsilon, \delta} |v| |\tau - t| e^{Cr_1|v|}(\tau-t)$. Therefore we conclude (5.10).

Now we claim
\[a(\tau) \leq A(\tau), \quad b(\tau) \leq B(\tau), \quad \text{for all } \tau \leq t.\] (5.17)

First we claim that $a(\tau) \leq A(\tau)$ and $b(\tau) \leq B(\tau)$ for all $\tau$. Otherwise, we should have at least for some time $\tau_0$ such that $a(\tau) \leq A(\tau)$ and $b(\tau) \leq B(\tau)$ for $\tau_0 \leq \tau \leq t$ but either $a(\tau) > A(\tau)$ or $b(\tau) > B(\tau)$ for a small neighborhood of $\tau > \tau_0$. Especially either $a(\tau_0) = A(\tau_0)$ or $b(\tau_0) = B(\tau_0)$. But this is impossible. Since
\[
\begin{bmatrix}
A^\varepsilon(\tau) - a(\tau) \\
B^\varepsilon(\tau) - b(\tau)
\end{bmatrix} \geq C \begin{bmatrix} 0 & 1 \\
m + |v|^2 & |v|
\end{bmatrix} \begin{bmatrix}
\int_t^\tau (A^\varepsilon(\tau') - a(\tau'))d\tau' \\
\int_t^\tau (B^\varepsilon(\tau') - b(\tau'))d\tau'
\end{bmatrix} + \begin{bmatrix}
\varepsilon \\
\varepsilon
\end{bmatrix},
\]
we have $A^\varepsilon(\tau) - a(\tau) \geq C \begin{bmatrix} \varepsilon \\
\varepsilon
\end{bmatrix} > 0$ as $\tau \to \tau_0^+$. Then we prove the inequalities (5.17) by letting $\varepsilon \to 0$.

Finally we prove the claim (5.14) from (5.10) and (5.17) and letting $\varepsilon \to 0$. \qed

Proof of Theorem 2: First we consider the case of $t < t_b(t, x, v)$. Directly
\[
\left| \frac{\partial(X_{\varepsilon l}(s; t, x, v), V_{\varepsilon l}(s; t, x, v))}{\partial(t, x, v)} \right| \lesssim \left[ \frac{|v| + (t-s)}{L^\varepsilon_{x, t}} + 1 \right],
\]
The computation will be the same as we will get for (5.30).

Now we consider the case of $t \geq t_b(t, x, v)$. We split our proof into 10 steps.

Step 1. Moving frames and grouping with respect to the scaling $t|v| = L_\xi$, with fixed $0 < L_\xi \ll 1$.

Fix $(t, x, v) \in [0, \infty) \times \Omega \times \mathbb{R}^3$. Also we fix small constant $\delta$ such that $\delta \ll \|E\|_{L^\varepsilon_{x, t}}$. We define, at the boundary,
\[r^\varepsilon := \frac{|v^\varepsilon|}{|v|}.\] (5.18)

Bounces $\ell$ (and $(t^\varepsilon, x^\varepsilon, v^\varepsilon)$) are categorized as Type I, Type II, or Type III:

all the bounces are Type I if and only if $|v| \leq \delta$,

a bounce $\ell$ is Type II if and only if $|v| > \delta, r^\varepsilon \leq \sqrt{\delta}$,

a bounce $\ell$ is Type III if and only if $|v| > \delta, r^\varepsilon > \sqrt{\delta}$. (5.19)
Now we choose $T < \frac{\sqrt{\delta}}{\|E\|_{L_t^\infty} + 1}$. Then if $|v| \leq \delta$, we have
\[
\max_{t^{\ell+1} \leq s \leq t^\ell} |\xi(X_{cl}(s; t^\ell, x^t, v^t))| \leq |v|T + \|E\|_{L_t^\infty} T^2 \leq 2\delta.
\]
And if $|v| > \delta, r^\ell \leq \sqrt{\delta}$, we have from (5.2)
\[
\max_{t^{\ell+1} \leq s \leq t^\ell} |\xi(X_{cl}(s; t^\ell, x^t, v^t))| \leq |t^\ell - t^{\ell+1}|^2 |v^t|^2 + (\|E\|_{L_t^\infty}^2 + 1)T^2 \lesssim \left( \frac{|v^t|}{|v^\ell|} \right)^2 + \delta \lesssim \delta.
\]
Therefore if a bounce $\ell$ is Type I or Type II then $\max_{t^\ell \leq \tau \leq t^{\ell+1}} |\xi(X_{cl}(\tau; t, x, v))| \leq C\delta$.

Now we assign a coordinate chart for each bounce $\ell$ (moving frames). For Type I bounces $\ell$ in (5.19) we let $p^\ell = (z^\ell, w^\ell)$ with $z^\ell = x^\ell$ and $w^\ell = \tau_1(x^\ell)$. We choose $p^\ell$-spherical coordinate in Lemma 3 and (4.1) with this $p^\ell$.

For Type II bounce $\ell$, we choose $p^\ell := (z^\ell, w^\ell)$ on $\partial \Omega \times S^2$ with $n(z^\ell) \cdot w^\ell = 0$
\[
z^\ell = x^\ell, \quad w^\ell = \frac{v^\ell - (v^\ell \cdot n(z^\ell))n(z^\ell)}{|v^\ell - (v^\ell \cdot n(z^\ell))n(z^\ell)|}.
\]
(5.20)
Note that, by the definition of Type I bounce, $|v^\ell - (v^\ell \cdot n(z^\ell)n(z^\ell))|^2 = |v^\ell|^2 - |v^\ell|^2 \gtrsim |v^\ell|^2(1 - \delta) \gtrsim \delta |v|^2$
and hence $w^\ell$ is well-defined.

Moreover for Type I and Type II bounce
\[
|X_{cl}(s; t, x, v) - L_{p^\ell}| \gtrsim C\delta > 0,
\]
(5.21)
for $|v|^2|t^\ell - s| \lesssim \min_{x \in \partial \Omega} |x|$. This is due to the fact that the projection of $V_{cl}(s)$ on the plane passing $z^\ell$ and perpendicular to $n(z^\ell) \times w^\ell$ is at most $|v|$ magnitude but the distance from $z^\ell$ to the origin(the projection of poles $N_{p^\ell}$ and $S_{p^\ell}$) has lower bound $\min_{x \in \partial \Omega} |x|, |s - t^\ell| \ll 1$.

For Type III bounce $\ell(t^\ell, x^t, v^t)$, we choose $p^\ell = (z^\ell, w^\ell)$ with $|z^\ell - x^\ell| \lesssim \sqrt{\delta}$ and we choose arbitrary $w^\ell \in S^2$ satisfying $n(z^\ell) \cdot w^\ell = 0$. Note that unlike Type I, this $p^\ell$-spherical coordinate might not be defined for $s \in [t^{\ell+1}, \ell^\ell]$ but only defined near the boundary.

Whenever the moving frame is defined (for all $\tau \in [t^{\ell+1}, \ell^\ell]$ when $\ell$ is Type I or Type II, and $|\tau - t^\ell| \ll 1$ when $\ell$ is Type III) we denote, by (4.1),
\[
(X_{cl}(\tau), V_{cl}(\tau)) = (x_{\perp\ell}(\tau), x_{\|\ell}(\tau), v_{\perp\ell}(\tau), v_{\|\ell}(\tau)) := \Phi_{p^\ell}^{-1}(X_{cl}(\tau), V_{cl}(\tau)).
\]
Especially at the boundary we denote
\[
(x_{\perp\ell}^t, x_{\|\ell}^t, v_{\perp\ell}^t, v_{\|\ell}^t) := \lim_{\tau \to t^\ell}(X_{cl}(\tau), V_{cl}(\tau)), \quad \text{with } x_{\perp\ell}^t = 0, \quad v_{\perp\ell}^t \geq 0.
\]
Then we define
\[
(x_{\perp\ell}^{\ell+1}, x_{\|\ell}^{\ell+1}, v_{\perp\ell}^{\ell+1}) = \lim_{\tau \to t^{\ell+1}}(X_{cl}(\tau), x_{\|\ell}(\tau), v_{\perp\ell}(\tau)),
\]
and
\[
v_{\perp\ell}^{\ell+1} := - \lim_{\tau \to t^{\ell+1}} v_{\perp\ell}(\tau).
\]
(5.22)
Now we regroup the indices of the specular cycles, without order changing, as
\[
\{0, 1, 2, \cdots, \ell_\ast - 1, \ell_\ast\} = \{0\} \cup G_1 \cup G_2 \cup \cdots \cup G_{\lfloor \ell - |x|\rfloor + 1} \cup G_{\lfloor \ell - |x|\rfloor + 1},
\]
where $\lfloor a \rfloor \in \mathbb{N}$ is the greatest integer less than or equal to $a$. Each group is
\[
G_1 = \{1, \cdots, \ell_1 - 1, \ell_1\},
\]
\[
G_2 = \{\ell_1, \ell_1 + 1, \cdots, \ell_2 - 1, \ell_2\},
\]
\[
\vdots
\]
\[
G_{\lfloor \ell - |x|\rfloor + 1} = \{\ell_{\lfloor \ell - |x|\rfloor - 1}, \ell_{\lfloor \ell - |x|\rfloor - 1} + 1, \cdots, \ell_{\lfloor \ell - |x|\rfloor} - 1, \ell_{\lfloor \ell - |x|\rfloor}\},
\]
\[
G_{\lfloor \ell - |x|\rfloor + 1} = \{\ell_{\lfloor \ell - |x|\rfloor}, \ell_{\lfloor \ell - |x|\rfloor} + 1, \cdots, \ell_\ast\},
\]
(5.23)
where $\ell_1 = \inf \{ \ell \in \mathbb{N} : |v| \times |t_0^{\ell} - t_1^{\ell}| \geq L_\xi \}$ and inductively
\[
\ell_i = \inf \{ \ell \in \mathbb{N} : |v| \times |t_0^{\ell_i} - t_1^{\ell_i}| \geq L_\xi \},
\]
and we have denoted $\ell_s = \frac{\ell_{|v|} - |v|}{L_\xi} + 1$.

Our analysis is carried out in each group $G_i$. We note that within each $G_i$, $|t_0^{\ell_i} - t_1^{\ell_i}| |v| < L_\xi$ by our design, so from the velocity lemma, $r_\ell$ is comparable to each other, so is $|v|$. We can also cover the entire $G_i$ via a single chart in Section 8. By the chain rule, with the assigned $p^\ell$–spherical coordinate (moving frame), we have for fixed $0 \leq s \leq t$ and $s \in (t_0^{\ell_i}, t_1^{\ell_i})$
\[
\begin{align*}
\frac{\partial (X_{cl}(s; t, x, v), V_{cl}(s; t, x, v))}{\partial (t, x, v)} &= \frac{\partial (X_{cl}(s), V_{cl}(s))}{\partial (t^\ell, X_{||, \ell}^\ell, V_{||, \ell}^\ell)} \\
&= \prod_{i=1}^{\mathsf{dim}(\mathbb{R}^\ell)} \frac{\partial (t_0^{\ell_i}, X_{||, \ell_i}^{t_0^{\ell_i}}, V_{||, \ell_i}^{t_0^{\ell_i}})}{\partial (t_0^{\ell_{i+1}}, X_{||, \ell_{i+1}}^{t_0^{\ell_{i+1}}}, V_{||, \ell_{i+1}}^{t_0^{\ell_{i+1}}})} \times \cdots \times \frac{\partial (t_0^{\ell_i}, X_{||, \ell_i}^{t_0^{\ell_i}}, V_{||, \ell_i}^{t_0^{\ell_i}})}{\partial (t_0^{\ell_i+1}, X_{||, \ell_i+1}^{t_0^{\ell_i+1}}, V_{||, \ell_i+1}^{t_0^{\ell_i+1}})} \\
&= \prod_{i=1}^{\mathsf{dim}(\mathbb{R}^\ell)} \frac{\partial (t, X_{||, \ell_i}^{t}, V_{||, \ell_i}^{t})}{\partial (t, x, v)}.
\end{align*}
\]

whole intermediate groups

from the $t$–plane to the first bounce

Before we start to calculate the matrix for any bounces, we first prove an claim that will be used later: there exists a constant $C = C(\xi)$ such that for any bounce $\ell$ and any $t^{\ell+1}_i < s < t^{\ell}_i$ we have
\[
\left| \frac{\partial F_i(s)}{\partial \tau} + \frac{\partial F_i(s)}{\partial t^\ell} \right| + \left| \frac{\partial F_i(s)}{\partial \tau} + \frac{\partial F_i(s)}{\partial t^\ell} \right| < C \| \partial E \|_{L^\infty_L}
\]

By direct computation we have
\[
\begin{align*}
\frac{\partial x_{||}(s)}{\partial t^\ell} &= -v_{||}(s) + \int_s^{t^\ell} (\partial_{\tau} F_{||}(\tau') + \partial_{t^\ell} F_{||}(\tau')) d\tau' d\tau, \\
\frac{\partial v_{||}(s)}{\partial t^\ell} &= -v_{||}(s) + \int_s^{t^\ell} (\partial_{\tau} F_{||}(\tau') + \partial_{t^\ell} F_{||}(\tau')) d\tau' d\tau, \\
\frac{\partial v_{\perp}(s)}{\partial t^\ell} &= -v_{\perp}(s) - \int_s^{t^\ell} (\partial_{\tau} F_{\perp}(\tau') + \partial_{t^\ell} F_{\perp}(\tau')) d\tau' d\tau, \\
\frac{\partial v_{\perp}(s)}{\partial t^\ell} &= -v_{\perp}(s) - \int_s^{t^\ell} (\partial_{\tau} F_{\perp}(\tau') + \partial_{t^\ell} F_{\perp}(\tau')) d\tau',
\end{align*}
\]
and
\[
\begin{align*}
\frac{\partial F_i(s)}{\partial \tau} + \frac{\partial F_i(s)}{\partial t^\ell} + \frac{\partial F_i(s)}{\partial \tau} + \frac{\partial F_i(s)}{\partial t^\ell} &= \nabla_{x_{||}} F_{||} \cdot (v_{\perp}(s) + \frac{\partial x_{||}(s)}{\partial t^\ell}) + \nabla_{x_{||}} F_{||} \cdot (v_{\perp}(s) + \frac{\partial x_{||}(s)}{\partial t^\ell}) + \nabla_{v_{||}} F_{\perp} \cdot (F_{||}(s) + \frac{\partial v_{||}(s)}{\partial t^\ell}) - \frac{1}{2} \partial_{\tau} E \cdot n(x_{||}) \\
&+ \nabla_{v_{\perp}} F_{\perp} \cdot (F_{||}(s) + \frac{\partial v_{\perp}(s)}{\partial t^\ell}) - \sum_{i=1,2} G_{p,ij}(x_{\parallel p}, x_{\parallel p}) (-1)^i \partial_x E(s, -x_{\parallel p} n(x_{\parallel p}) + \eta(x_{\parallel p})) \cdot \{ n_p(x_{\parallel p}) \times \partial_{\parallel p} \eta_p(x_{\parallel p}) \} \\
&\quad \frac{\partial p(x_{\parallel p})}{\partial \tau} (\partial_{\parallel p} \eta_p(x_{\parallel p}) \times \partial_{\parallel p} \eta_p(x_{\parallel p}))
\end{align*}
\]
Then from $[5.27], \ [5.29]$, and using the fact that $\|\nabla_{x_1,x_2} F\|_\infty + \|\nabla_{x_1} F\|_\infty + \|\nabla_{x_2,v} F\|_\infty \lesssim |v|^2 + 1$, $\|\nabla_{v_1} F\|_\infty + \|\nabla_{v_2,v} F\|_\infty \lesssim |v| + 1$, we have
\[
\left| \frac{\partial F_2(s)}{\partial \tau} + \frac{\partial F_2(s)}{\partial t^*} \right| + \left| \frac{\partial F_1(s)}{\partial \tau} + \frac{\partial F_1(s)}{\partial t^*} \right| \lesssim (|v|^2 + 1) \int_s^t \int_\tau^{t^*} \left| (\partial_\tau F_2(r') + \partial_{t^*} F_2(r')) + \left| \partial_\tau F_2(r') + \partial_{t^*} F_2(r') \right| \right| \, dr' \, d\tau
\]
\[
\lesssim (|v|^2 + 1) \int_s^t \int_\tau^{t^*} \left| (\partial_\tau F_2(r') + \partial_{t^*} F_2(r')) + \left| \partial_\tau F_2(r') + \partial_{t^*} F_2(r') \right| \right| \, dr' \, d\tau
\]
and this proves $[5.29]$.}

**Step 2. From the last bounce $\ell_*$ to the $s$–plane**

We choose $s^{t_*} \in (s^{t_* - \frac{\xi}{2}}, s^{t_*}) \subset (s, t^{s_*})$ such that $|v||t^{s_*} - s^{t_*}| \ll 1$ and the $\ell_*$–spherical coordinate $(X_{t_*}(s^{s_*}), V_{t_*}(s^{s_*}))$ is well-defined regardless of types of $\ell_*$ in $[5.19]$. Notice that $s^{t_*}$ is independent of $t^{s_*}$ and $s$ so that $\frac{\partial s^{t_*}}{\partial s^{t_*}} = 0 = \frac{\partial s^{t_*}}{s^{t_*}}$.

We first follow the flow in $(x,v)$ co-ordinate to near the boundary at $t = s^{t_*}$, change to the chart to $(X,V)$, then follow the flow in $(X,V)$. Regarding $s^{t_*}$ as a free variable, by the chain rule,
\[
\frac{\partial(X_{t_*}(s), V_{t_*}(s))}{\partial(t^{s_*}, x_{t_*}(s^{s_*}), V_{t_*}(s^{s_*}))} = \frac{\partial(X_{t_*}(s), V_{t_*}(s))}{\partial(s^{s_*}, X_{t_*}(s^{s_*}), V_{t_*}(s^{s_*}))} \frac{\partial(s^{s_*}, X_{t_*}(s^{s_*}), V_{t_*}(s^{s_*}))}{\partial(t^{s_*}, x_{t_*}(s^{s_*}), V_{t_*}(s^{s_*}))}
\]
\[
= \frac{\partial(X_{t_*}(s), V_{t_*}(s))}{\partial(s^{s_*}, X_{t_*}(s^{s_*}), V_{t_*}(s^{s_*}))} \frac{\partial(s^{s_*}, X_{t_*}(s^{s_*}), V_{t_*}(s^{s_*}))}{\partial(t^{s_*}, x_{t_*}(s^{s_*}), V_{t_*}(s^{s_*}))}
\]
Firstly, we claim
\[
\frac{\partial(X_{t_*}(s), V_{t_*}(s))}{\partial(s^{s_*}, X_{t_*}(s^{s_*}), V_{t_*}(s^{s_*}))} = \begin{bmatrix}
\partial V_{t_*}(s^{s_*}) + O(1)|s^{s_*} - s| & O_\xi(1)(1 + |v||s^{s_*} - s|) & O_\xi(1)|s^{s_*} - s|
\partial(1 + |v||s^{s_*} - s|) & O_\xi(1)(v) + |s^{s_*} - s| & O_\xi(1)(1 + |s^{s_*} - s|)
\end{bmatrix}
\]
\[
\frac{\partial(X_{t_*}(s), V_{t_*}(s))}{\partial(s^{s_*}, X_{t_*}(s^{s_*}), V_{t_*}(s^{s_*}))} = \begin{bmatrix}
\partial V_{t_*}(s^{s_*}) + O(1)|s^{s_*} - s| & O_\xi(1)(1 + |v||s^{s_*} - s|) & O_\xi(1)|s^{s_*} - s|
\partial(1 + |v||s^{s_*} - s|) & O_\xi(1)(v) + |s^{s_*} - s| & O_\xi(1)(1 + |s^{s_*} - s|)
\end{bmatrix}
\]
Since
\[
\begin{align*}
X_{t_*}(s) &= X_{t_*}(s^{s_*}) - \int_s^{s^{s_*}} V_{t_*}(\tau) \, d\tau = X_{t_*}(s^{s_*}) - (s^{s_*} - s)V_{t_*}(s^{s_*}) + \int_s^{s^{s_*}} \int_\tau^{s^{s_*}} E(\tau', X_{t_*}(\tau')) \, d\tau' \, d\tau,
V_{t_*}(s) &= V_{t_*}(s^{s_*}) - \int_s^{s^{s_*}} E(\tau, X_{t_*}(\tau)) \, d\tau,
\end{align*}
\]
we have
\[
\frac{\partial X_{cl}(s^{*})}{\partial s^{*}} = -V_{cl}(s^{*}) + \int_{s}^{s^{*}} \left[ E(s^{*}, X_{cl}(s^{*}))) + \int_{s}^{s^{*}} \nabla_{s} E(\tau, X_{cl}(\tau)) \frac{\partial X_{cl}(\tau)}{\partial s^{*}} \right] d\tau \] 
\[
= -V_{cl}(s^{*}) + (s^{*} - s)E(s^{*}, X_{cl}(s^{*})) + \int_{s}^{s^{*}} \nabla_{s} E(\tau, X_{cl}(\tau)) \frac{\partial X_{cl}(\tau)}{\partial s^{*}} d\tau d\tau' \] 
\[
= -V_{cl}(s^{*}) + (s^{*} - s)E(s^{*}, X_{cl}(s^{*})) + \int_{s}^{s^{*}} (\tau' - s)\nabla_{s} E(\tau', X_{cl}(\tau')) \frac{\partial X_{cl}(\tau')}{\partial s^{*}} d\tau'. \tag{5.32}
\]

By Gronwall we have
\[
\left| \frac{\partial X_{cl}(s)}{\partial s^{*}} \right| \leq (|V_{cl}(s^{*})| + (s^{*} - s)|E|) e^{\int_{s}^{s^{*}} (s^{*} - s)|\nabla_{s} E| d\tau'} \leq |V_{cl}(s^{*})| + |s^{*} - s|. \tag{5.33}
\]
Plug \((5.33)\) into \((5.32)\) we get
\[
\frac{\partial X_{cl}(s)}{\partial s^{*}} = -V_{cl}(s^{*}) + O_{\xi, \|\nabla E\|_{\infty, s^{*}}} (1)|s^{*} - s|. \tag{5.34}
\]

Similarly we have
\[
\frac{\partial V_{cl}(s)}{\partial s^{*}} = -E(s^{*}, X_{cl}(s^{*})) - \int_{s}^{s^{*}} \nabla_{s} E(\tau, X_{cl}(\tau)) \frac{\partial X_{cl}(\tau)}{\partial s^{*}} d\tau \] 
\[
= -E(s^{*}, X_{cl}(s^{*})) - O_{\xi, \|\nabla E\|_{\infty, s^{*}}} (1)(s^{*} - s)|v|. \tag{5.35}
\]

Also, using the fact that for \(\partial = [\frac{\partial}{X_{cl}(s^{*})}, \frac{\partial}{V_{cl}(s^{*})}], \|\partial X_{cl}(s)\| + \|\partial V_{cl}(s)\| \leq 1\), we can combine \((5.34)\), \((5.33)\), and \((5.35)\) to get
\[
\frac{\partial (X_{cl}(s), V_{cl}(s))}{\partial (s^{*}, X_{cl}(s^{*}), V_{cl}(s^{*}))} = \begin{bmatrix}
-V_{cl}(s^{*}) + O(1)|s^{*} - s| & \text{Id}_{3, 3} + O(1)|s^{*} - s| & -(s^{*} - s)\text{Id}_{3, 3} + O(1)|s^{*} - s| \\
-E - O(1)(s^{*} - s)|v| & 0_{3, 3} + O(1)|s^{*} - s| & \text{Id}_{3, 3} + O(1)|s^{*} - s|
\end{bmatrix}.
\]

Furthermore due to Lemma\([6]\) we conclude
\[
\frac{\partial(s^{*}, X_{cl}(s^{*}), V_{cl}(s^{*}))}{\partial(t^{*}, X^{*}, V^{*})} = 
\begin{bmatrix}
1_{3} & 0_{1, 3} & 0_{1, 3} \\
0_{1, 3} & -n_{1} & 0_{1, 3} \\
0_{1, 3} & -v_{1} \cdot v_{||} n_{1} & 0_{1, 3}
\end{bmatrix},
\]

where all entries are evaluated at \((X_{cl}(s^{*}), V_{cl}(s^{*})).\) The multiplication of above two matrices gives \((5.30)\).

Secondly, we claim that whenever \(p^{*}\)-spherical coordinate is defined for all \(\tau \in [s^{*}, t^{*}]\), we have following \(7 \times 6\) matrix
\[
\frac{\partial(s^{*}, X_{\perp}(s^{*}), X_{\|}(s^{*}), V_{\perp}(s^{*}), V_{\|}(s^{*}))}{\partial(t^{*}, X^{*}, V^{*})} = 
\begin{bmatrix}
0_{1, 2} & 0_{1, 2} \\
-\frac{1}{2} \|v_{\perp}\|^{2} + 1 & O_{1}(|v_{\perp}|^{2} - \|v_{\perp}\|^{2}) & O_{1}(|v_{\perp}|^{2} - \|v_{\perp}\|^{2}) & O_{1}(|v_{\perp}|^{2} - \|v_{\perp}\|^{2}) & O_{1}(|v_{\perp}|^{2} - \|v_{\perp}\|^{2}) & O_{1}(|v_{\perp}|^{2} - \|v_{\perp}\|^{2}) & O_{1}(|v_{\perp}|^{2} - \|v_{\perp}\|^{2}) & 1_{3} & 0_{1, 3}
\end{bmatrix},
\]
\[
\frac{\partial v_{\perp}(s^{*})}{\partial t^{*}} = -F_{\perp}(t^{*}) - \int_{s^{*}}^{t^{*}} \partial_{t^{*}} F_{\perp}(\tau) d\tau = -F_{\perp}(s^{*}) - \int_{s^{*}}^{t^{*}} (\partial_{t^{*}} F_{\perp}(\tau) + \partial_{\tau} F_{\perp}(\tau)) d\tau. \tag{5.37}
\]
and

\[
\frac{\partial \mathbf{x}_\perp(s^t)}{\partial t^t} = -\mathbf{v}_\perp(s^t) + \int_{s^t}^{t^t} \frac{\partial}{\partial \tau} \mathbf{v}_\perp(\tau) d\tau = -\mathbf{v}_\perp(s^t) + \int_{s^t}^{t^t} F_\perp(\tau) d\tau - \int_{s^t}^{t^t} \frac{\partial}{\partial \tau} \mathbf{v}_\perp(\tau)
\]

(5.38)

Then from (5.20), we get the desired estimate for the first column of (5.36).

Now we turn to other entries in (5.36). From the characteristics ODE, (4.7) in the planar spherical coordinate, (4.10), (4.11), and (4.12), we deduce (5.36) for \(|v|s^t - t^t| \lesssim 1

Step 3. From t-plane to the first bounce.

We choose \(s^1 \in (t^1, \frac{t^1}{2}) \subset (t^1, t)\) such that \(|v|t^1 - s^1| \ll 1\) and the polar coordinate \((X_1(s^1), V_1(s^1))\) is well-defined. More precisely we choose \(0 < \Delta\) such that \(|v|t - \Delta - t^1| \ll 1\) and define

\(s^1 := t - \Delta\) (5.39)

We first follow the flow in the cartesian coordinate to near the boundary at \(s^1\), change to the chart to planar spherical coordinate, then follow the flow in that coordinate.

Then, by the chain rule,

\[
\frac{\partial(t^1, x^1_{\parallel}, v^1_{\parallel}, v^1_{\perp})}{\partial(t, x, v)} = \frac{\partial(t^1, x^1_{\parallel}, v^1_{\parallel}, v^1_{\perp})}{\partial(s^1, X_1(s^1), V_1(s^1))} \frac{\partial(s^1, X_1(s^1), V_1(s^1))}{\partial(t, x, v)}
\]

(5.40)

We fix planar spherical coordinate and drop the index of the chart.

Firstly, we claim

\[
\frac{\partial(t^1, x^1_{\parallel}, v^1_{\parallel}, v^1_{\perp})}{\partial(s^1, x_0(s^1), v_0(s^1), v_1(s^1))}
\]

\[
\begin{array}{c|c|c|c|c|c}
|v| & |v| & |(v^1 + v_0(s^1))|1 - t^1 & |v| & |(v^2 + O(1))|1 - t^1 & |v| \\
\hline
|v| & |v| & \frac{|v|^2 + O(1)}{v} & |v| & \frac{|v|^2 + O(1)}{v} & |v| \\
\hline
|v|^2 + O(1) & \frac{|v|^2 + O(1)}{v} & |v|^2 + O(1) & \frac{|v|^2 + O(1)}{v} & |v|^2 + O(1) & \frac{|v|^2 + O(1)}{v} \\
\hline
|v|^2 + O(1) & \frac{|v|^2 + O(1)}{v} & |v|^2 + O(1) & \frac{|v|^2 + O(1)}{v} & |v|^2 + O(1) & \frac{|v|^2 + O(1)}{v} \\
\end{array}
\]

The \(t^1\) is determined via \(x_\perp(t^1) = 0\), i.e.

\[
0 = x_\perp(s^1) - v_\perp(s^1)(s^1 - t^1) + \int_{t^1}^{s^1} \int_s^{s^1} F_\perp(X(\tau), V_1(\tau)) d\tau ds,
\]

(5.41)

where \(X(\tau) = X(\tau; s^1, X(s^1; t, x, v), V(s^1; t, x, v)), V(\tau) = V(\tau; s^1, X(s^1; t, x, v), V(s^1; t, x, v))\). For \(\partial \in \{\partial_{x_\perp(s^1)}, \partial_{x_0(s^1)}, \partial_{v_\perp(s^1)}, \partial_{v_0(s^1)}\}\)

\[
\mathbf{v}_\perp(s^1) \frac{\partial t^1}{\partial \tau} + \int_{t^1}^{s^1} F_\perp(X(\tau), V(\tau)) d\tau + \partial x_\perp(s^1) - \partial v_\perp(s^1)(s^1 - t^1)
\]

(5.42)
But $v_1 = -\lim_{s \to t} v_\perp(s) = -v_\perp(s^1) + \int_{t}^{s^1} F_\perp(X(\tau), V(\tau)) d\tau$, we apply Lemma 7 and $|s^1 - t^1| \lesssim \xi$ min$\{\frac{|v_1|}{|v||x^1|}, t\}$ and (5.42),

$$
\begin{bmatrix}
\frac{\partial v_1}{\partial x_\perp(s^1)} \\
\frac{\partial v_1}{\partial x_\parallel(s^1)} \\
\frac{\partial v_1}{\partial v_\perp(s^1)} \\
\frac{\partial v_1}{\partial v_\parallel(s^1)}
\end{bmatrix} = \begin{bmatrix}
\frac{1}{v_\perp} \left\{ 1 + \int_{t}^{s^1} \frac{\partial}{\partial x_\perp(s^1)} F_\perp(X(\tau), V(\tau)) d\tau \right\} \\
\frac{1}{v_\parallel} \int_{t}^{s^1} \frac{\partial}{\partial x_\parallel(s^1)} F_\perp(X(\tau), V(\tau)) d\tau \\
\frac{1}{v_\perp} \int_{t}^{s^1} \frac{\partial}{\partial v_\perp(s^1)} F_\perp(X(\tau), V(\tau)) d\tau \\
\frac{1}{v_\parallel} \int_{t}^{s^1} \frac{\partial}{\partial v_\parallel(s^1)} F_\perp(X(\tau), V(\tau)) d\tau
\end{bmatrix} \lesssim \xi, t,
$$

Taking $(x(s^1), v(s^1))$ derivatives of the characteristic equations

$$
v_1^1 = -\lim_{s \to t} v_\perp(s) = -v_\perp(s^1) + \int_{t}^{s^1} F_\perp(X_\text{el}(\tau), V_\text{el}(\tau)) d\tau,
$$

$$
x_\parallel^1 = x_\parallel(s^1) - \int_{t}^{s^1} v_\parallel(s) ds,
$$

$$
v_\parallel^1 = v_\parallel(s^1) - \int_{t}^{s^1} F_\parallel(X_\text{el}(\tau), V_\text{el}(\tau)) d\tau.
$$

and using the above estimates and (5.42) and Lemma 7 yields

$$
\begin{bmatrix}
\frac{\partial x_\parallel}{\partial x_\perp(s^1)} \\
\frac{\partial x_\parallel}{\partial x_\parallel(s^1)} \\
\frac{\partial x_\parallel}{\partial v_\perp(s^1)} \\
\frac{\partial x_\parallel}{\partial v_\parallel(s^1)}
\end{bmatrix} \lesssim \xi, t,
$$

and

$$
\begin{bmatrix}
\frac{\partial v_\perp}{\partial x_\perp(s^1)} \\
\frac{\partial v_\perp}{\partial x_\parallel(s^1)} \\
\frac{\partial v_\perp}{\partial v_\perp(s^1)} \\
\frac{\partial v_\perp}{\partial v_\parallel(s^1)}
\end{bmatrix} \lesssim \xi, t,
$$

Secondly, we claim

$$
\begin{bmatrix}
\frac{\partial X_1(s^1), V_1(s^1)}{\partial (t, x, v)} \\
\frac{\partial X_1(s^1), V_1(s^1)}{\partial (X_\text{el}(s^1), V_\text{el}(s^1))} \\
\frac{\partial X_1(s^1), V_1(s^1)}{\partial (X_\text{el}(s^1), V_\text{el}(s^1))}
\end{bmatrix}
$$

$$
= \begin{bmatrix}
\begin{array}{c}
O(1) + O_\xi(|v| |t - s^1|^2) \\
O(1) + O_\xi(|v| |t - s^1|^2) \\
O(1) + O_\xi(|v| |t - s^1|^2)
\end{array}
\begin{array}{c}
O_\xi(|v|) \\
O_\xi(|v|) \\
O_\xi(|v|)
\end{array}
\begin{array}{c}
O(1) + O_\xi(|v| |t - s^1|^2) \\
O(1) + O_\xi(|v| |t - s^1|^2) \\
O(1) + O_\xi(|v| |t - s^1|^2)
\end{array}
\begin{array}{c}
O_\xi(|v|) \\
O_\xi(|v|) \\
O_\xi(|v|)
\end{array}
\end{bmatrix},
$$

where the entries are evaluated at $(X_1(s^1), V_1(s^1))$. Note that $|v||t^1 - s^1| \lesssim \xi 1$.

From (5.43)

$$
\frac{\partial (X_\text{el}(s^1), V_\text{el}(s^1))}{\partial (x(s^1), v(s^1))} = \frac{\partial \Phi(X(s^1), V(s))}{\partial (X(s^1), V(s))} := \begin{bmatrix}
A & 0_{3,3} \\
B & A
\end{bmatrix} + x_\perp \begin{bmatrix}
0_{3,3} & 0_{3,3} \\
0_{3,3} & 0_{3,3}
\end{bmatrix}.
$$
From direct computation and (4.3),

\[
\det(A) = \det \begin{bmatrix}
-n(x) & \frac{\partial n}{\partial x_1}(x) \\
+\mathbf{x}_1[-\frac{\partial n}{\partial x_1}(x)] & +\mathbf{x}_1[-\frac{\partial n}{\partial x_2}(x)]
\end{bmatrix} = -n(x) \cdot \left( \frac{\partial n}{\partial x_1}(x) \times \frac{\partial n}{\partial x_2}(x) \right) + O_\xi(|\mathbf{x}_\perp|) \neq 0,
\]

\[
A^{-1} = \frac{1}{|n| \cdot (\partial n_\perp, \nabla \times \partial n_\perp)} + O(1)|\mathbf{x}_\perp|
\]

\[
\times \left[ (1 - \mathbf{x}_\perp)(\partial n_\perp, \nabla \times \partial n_\perp)^T, (1 - \mathbf{x}_\perp)(\partial n_\perp, \nabla \times [-n] + (1 - \mathbf{x}_\perp)([-n] \times \partial n_\perp)^T, (1 - \mathbf{x}_\perp)([-n] \times \partial n_\perp)^T \right].
\]

From basic linear algebra

\[
det\left( \begin{bmatrix} \partial(X_{cl}(s'), V_{cl}(s')) \end{bmatrix} \right) = det\left( \begin{bmatrix} \frac{A}{B + \mathbf{x}_\perp D} \cdot 0_{3,3} \end{bmatrix} \right) = \{det(A)\}^2 = \{|-n| \cdot (\partial n \times \partial n_\perp) + O(1)|\mathbf{x}_\perp|\}^2,
\]

and (\(\partial(X_{cl}(s'), V_{cl}(s'))\)) is invertible. By the basic linear algebra

\[
\frac{\partial(X_{cl}(s'), V_{cl}(s'))}{\partial(X_{cl}(s'), V_{cl}(s'))} = \left( \begin{bmatrix} \partial(X_{cl}(s'), V_{cl}(s')) \end{bmatrix} \right)^{-1} = \left[ \begin{bmatrix} \frac{A}{B + \mathbf{x}_\perp D} \cdot 0_{3,3} \end{bmatrix} \right]^{-1}
\]

\[
= \begin{bmatrix} A^{-1} & \mathbf{0}_{3,3} \\
-A^{-1}(B + \mathbf{x}_\perp D)A^{-1} & A^{-1} \end{bmatrix} = \begin{bmatrix} \mathbf{A}^{-1}(x) & \mathbf{0}_{3,3} \\
\mathbf{0}_{3,3} & \mathbf{A}^{-1}(x) \end{bmatrix},
\]

and we obtain

\[
\frac{\partial(X_{cl}(s'), V_{cl}(s'))}{\partial(X_{cl}(s'), V_{cl}(s'))} = \begin{bmatrix} (1 - \mathbf{x}_\perp)^2(\partial n \times \partial n_\perp)^T & 0 \\
-n(\partial n \times \partial n_\perp) + O(1)|\mathbf{x}_\perp| & (1 - \mathbf{x}_\perp)^2(\partial n \times \partial n_\perp)^T \\
-(\partial n \times \partial n_\perp) + O(1)|\mathbf{x}_\perp| & (1 - \mathbf{x}_\perp)^2(\partial n \times \partial n_\perp)^T \\
O_x(1)(|v|) & O_x(1)(|v|) \\
O_x(1)(|v|) & O_x(1)(|v|) \\
O_x(1)(|v|) & O_x(1)(|v|) \end{bmatrix}.
\]

From \(X_{cl}(s'; t, x, v) = x - (t - s')v + \int_{s'}^t E(t)\, d\tau \, d\tau ds = x - \Delta x + v + \int_{t - \Delta}^t \int_s^t E(\tau)\, d\tau ds \), and \(V_{cl}(s'; t, x, v) = v - \int_{t - \Delta}^t E(s)\, ds \), we have

\[
\frac{X_{cl}(s')}{\partial t} = -\int_{t - \Delta}^t E(s)\, ds + \int_{t - \Delta}^t E(t)\, ds + \int_{t - \Delta}^t \int_s^t \partial_t E(\tau)\, d\tau \, d\tau ds
\]

\[
= -\int_{t - \Delta}^t \int_s^t \left( \partial E(\tau) + \partial E(\tau) \right) \, d\tau \, d\tau ds
\]

\[
= -\int_{t - \Delta}^t \int_s^t \partial_t E + \nabla E \cdot \nabla x X \, d\tau \, d\tau ds = O(1)|t - s'|^2
\]

\[
\frac{V_{cl}(s')}{\partial t} = -E(t) + E(t - \Delta) - \int_{t - \Delta}^t \partial E(s)\, ds
\]

\[
= -\int_{t - \Delta}^t \left( \partial E(s) + \partial E(s) \right) \, ds
\]

\[
= -\int_{t - \Delta}^t (\partial_t E + \nabla E \cdot \nabla x X) \, ds = O(1)|t - s'|.
\]

And using \(|\nabla x X_{cl}(s')| + |\nabla x X V_{cl}(s')| \leq 1, \)

\[
\frac{\partial(X_{cl}(s'), V_{cl}(s'))}{\partial(t, x, v)} = \begin{bmatrix} O(1)|t - s'|^2 & \text{Id}_{3,3} + O(1)|t - s'|^2 & -(t - s') \text{Id}_{3,3} + O(1)|t - s'|^2 \\
O(1)|t - s'| & O(1)|t - s'| & O(1)|t - s'| \end{bmatrix} \text{Id}_{3,3} + O(1)|t - s'|^2 \]

Finally, we multiply above two matrices and use \(|\mathbf{x}_\perp(s')| \leq |v| |t - s'| \) to conclude the second claim (5.43).
Step 4. Estimate of $\partial(t^\ell+1, x^\ell_{\|}, v^\ell_{\|}, v^\ell_{\perp})/\partial(t^\ell, x^\ell_{\|}, v^\ell_{\|}, v^\ell_{\perp})$.

Recall $r^\ell$ from (5.18). We show that for $0 < T \ll 1$ small enough, there exists $0 < \delta_1 \ll 1$, such that for all $\ell \in \mathbb{N}$ and $0 \leq t^\ell+1 \leq t^\ell \leq t$, if $\ell$ is Type II or Type III,

$$J^\ell+1 := \frac{\partial(t^\ell+1, x^\ell_{\|}, v^\ell_{\|}, v^\ell_{\perp})}{\partial(t^\ell, x^\ell_{\|}, v^\ell_{\|}, v^\ell_{\perp})}$$

(5.45)

And if $\ell$ is Type I, then

$$J^\ell+1 := \frac{\partial(t^\ell+1, x^\ell_{\|}, v^\ell_{\|}, v^\ell_{\perp})}{\partial(t^\ell, x^\ell_{\|}, v^\ell_{\|}, v^\ell_{\perp})}$$

(5.46)

We also denote the Jacobian matrix within a single $p^\ell$ – spherical coordinate:

$$j^\ell+1 := \frac{\partial(t^\ell+1, x^\ell_{\|}, v^\ell_{\perp})}{\partial(t^\ell, x^\ell_{\|}, v^\ell_{\perp})}.$$
Proof of (5.46) (Type I), and (5.47) when ℓ is Type II: Note that \( p^{\ell} \)-spherical coordinate is well-defined of all \( \tau \in [t^{\ell+1}, t^{\ell}] \) for those cases. Due to the chart changing

\[
\frac{\partial (t^{\ell+1}, x_{\ell+1}^{\ell+1}, v_{\ell+1}^{\ell+1}, v_{\ell}^{\ell+1})}{\partial (t^{\ell}, x_{\ell}^{\ell}, v_{\ell}^{\ell}, v_{\ell}^{\ell})} = \begin{bmatrix}
1 & 0_{1,5} \\
0_{5,1} & \frac{\partial (x_{\ell}^{\ell+1}, v_{\ell+1}^{\ell+1}, v_{\ell}^{\ell+1})}{\partial (x_{\ell}^{\ell}, v_{\ell+1}^{\ell}, v_{\ell}^{\ell})}
\end{bmatrix}
\]

where \( \frac{\partial (x_{\ell}^{\ell+1}, v_{\ell+1}^{\ell+1}, v_{\ell}^{\ell+1})}{\partial (x_{\ell}^{\ell}, v_{\ell+1}^{\ell}, v_{\ell}^{\ell})} \) is the 5 \times 5 right lower submatrix of (4.0).

Note that \( |p^{\ell} - p^{\ell+1}| \leq \xi + \delta \) from (5.20). In order to show (5.45) and (5.46) it suffices to show that \( \tilde{J}_{t+1}^{\ell} \) is bounded:

\[
\tilde{J}_{t+1}^{\ell} \leq J(r^{\ell+1}), \text{ if } \ell \text{ is Type II or Type III,} \quad \tilde{J}_{t+1}^{\ell} \leq J(v_{t+1}^{\ell}), \text{ if } \ell \text{ is Type I.} \tag{5.47}
\]

This is due to the following matrix multiplication

\[
\begin{bmatrix}
1 & 0_{1,5} \\
0_{5,1} & \frac{\partial (x_{\ell}^{\ell+1}, v_{\ell+1}^{\ell+1}, v_{\ell}^{\ell+1})}{\partial (x_{\ell}^{\ell}, v_{\ell+1}^{\ell}, v_{\ell}^{\ell})}
\end{bmatrix}
\begin{bmatrix}
\tilde{J}_{t+1}^{\ell} \\
\tilde{J}_{t+1}^{\ell}
\end{bmatrix}
\leq
\begin{bmatrix}
1 & 0_{1,2} & 0_{1,3} \\
0_{2,1} & 1 + Cr_{t+1}^{\ell} & Cr_{t+1}^{\ell} & 0_{3,3} \\
0_{3,1} & Cr_{t+1}^{\ell} & Cr_{t+1}^{\ell} & 0_{3,3}
\end{bmatrix}
\begin{bmatrix}
\tilde{J}_{t+1}^{\ell} \\
\tilde{J}_{t+1}^{\ell} \\
\tilde{J}_{t+1}^{\ell}
\end{bmatrix}
\]

where we used (4.0) with an adjusted constant \( C > 0 \).

Now we prove the claim (5.47). We fix the \( p^{\ell} \)-spherical coordinate and drop the index \( \ell \) for the chart. If \( v_{t+1}^{\ell} = 0 \) then \( t^{t+1} = t^{\ell} \). Otherwise if \( v_{t+1}^{\ell} \neq 0 \) then \( t^{t+1} \) is determined through

\[
0 = v_{t+1}^{\ell}(t^{t+1} - t^{\ell}) + \int_{t^{\ell}}^{t^{t+1}} \int_{s}^{t^{\ell}} F(\tau; \tau^{t+1}, x^{t+1}, v^{t+1}), V_{\ell}(\tau; x^{t}, v^{t}) \, d\tau \, ds. \tag{5.48}
\]

We first consider the \( \frac{\partial}{\partial t} \) derivatives.

Using the trajectory in the standard coordinates we have

\[
0 = \xi(x^{t+1}) = \xi(x^{t} - (t^{t+1} - t^{\ell})v^{t}) + \int_{t^{\ell}}^{t^{t+1}} \int_{s}^{t^{\ell}} E(\tau, X_{\ell}(\tau)) \, d\tau \, ds. \tag{5.49}
\]
Taking the \( \frac{\partial}{\partial t} \) derivative we get

\[
0 = \nabla \xi(x^{t+1}) \cdot \left[ -(1 - \frac{\partial t^{t+1}}{\partial t})v^t - \frac{\partial t^{t+1}}{\partial t} \int_{t^{t+1}}^{t^t} E(\tau, X(\tau))d\tau + \int_{t^{t+1}}^{t^t} E(t^t, x^t)ds + \int_{t^t}^{t^{t+1}} \int_s^t \frac{\partial E(\tau, X(\tau))}{\partial \tau} d\tau ds \right]
\]

\[
= \nabla \xi(x^{t+1}) \cdot \left[ -v^t + \frac{\partial t^{t+1}}{\partial t} v^{t+1} + \int_{t^{t+1}}^{t^t} E(s, X(s))ds + \int_{t^{t+1}}^{t^t} \left(E(v^t, x^t) - E(s, X(s))\right)ds + \int_{t^t}^{t^{t+1}} \int_s^t \frac{\partial E(\tau, X(\tau))}{\partial \tau} d\tau ds \right]
\]

\[
= \nabla \xi(x^{t+1}) \cdot \left[ -v^{t+1} + \frac{\partial t^{t+1}}{\partial t} v^{t+1} + \int_{t^{t+1}}^{t^t} \int_s^t \left(\frac{\partial E(\tau, X(\tau))}{\partial \tau} + \frac{\partial E(\tau, X(\tau))}{\partial t^t}\right) d\tau ds \right].
\]  

(5.50)

Thus

\[
\frac{\partial t^{t+1}}{\partial t^t} = 1 - \frac{\nabla \xi(x^{t+1})}{\nabla \xi(x^{t+1}) \cdot v^{t+1}} \cdot \int_{t^{t+1}}^{t^t} \int_s^t \left(\frac{\partial E(\tau, X(\tau))}{\partial \tau} + \frac{\partial E(\tau, X(\tau))}{\partial t^t}\right) d\tau ds.
\]  

(5.51)

By (5.34) we have

\[
\left| \frac{\partial E(\tau, X(\tau))}{\partial \tau} + \frac{\partial E(\tau, X(\tau))}{\partial t^t} \right| = |\partial_s E(\tau, X(\tau))_\infty + \nabla_x E \cdot (V(\tau) - V(\tau) + O(1)|t^t - t^{t+1}|)|
\]

\[
= |\partial_s E(\tau, X(\tau))_\infty + O(1)\nabla_x E(\tau, X(\tau))(t^t - t^{t+1})| \leq \|
\partial_t E\|_{L^\infty} + \|\nabla_x E\|_{L^\infty} |t^t - t^{t+1}|.
\]  

(5.52)

Thus from (5.51), (5.52), and (5.2) we have

\[
\frac{\partial t^{t+1}}{\partial t^t} = 1 - \xi.E(1) \frac{|t^t - t^{t+1}|^2}{|v^t_\perp|} = 1 - \xi.E(1)|t^t - t^{t+1}|.
\]  

(5.53)

Now by directly computing \( \frac{\partial v^{t+1}}{\partial t} \) we would have

\[
\frac{\partial v^{t+1}}{\partial t^t} = -F_\perp(t^{t+1}) \frac{v^{t+1}_{\perp}}{v^{t+1}} \int_{t^{t+1}}^{t^t} \int_s^t \left(\frac{\partial F_\perp(\tau)}{\partial \tau} + \frac{\partial F_\perp(\tau)}{\partial t^t}\right) d\tau ds + \int_{t^{t+1}}^{t^t} \left(\frac{\partial F_\perp(s)}{\partial s} + \frac{\partial F_\perp(s)}{\partial t^t}\right) ds.
\]  

(5.54)

Recall

\[
F_\perp = F_\perp(x_\perp, x_\parallel, v_\parallel)
\]

\[
= \sum_{j,k=1}^2 v_{\parallel, j} v_{\parallel, k} \partial_j \partial_k \eta(x_\parallel) \cdot n(x_\parallel) - x_\perp \sum_{k=1}^2 v_{\parallel, k} (v_\parallel \cdot \nabla) \partial_k n(x_\parallel) \cdot n(x_\parallel)
\]

\[
- E(s, -x_\perp n(x_\parallel) + \eta(x_\parallel)) \cdot n(x_\parallel),
\]  

(5.55)

So by direct computation

\[
\dot{F}_\perp(\tau) := \frac{\partial F_\perp(\tau)}{\partial \tau} = v_\perp \nabla_x F_\perp + v_\parallel \nabla_x F_\perp + F_\parallel \nabla v_\parallel F_\perp - \partial_s E \cdot n(x_\parallel),
\]  

(5.56)
so \( \| \nabla_{x_{\perp}} \hat{F}_{\perp} \|_\infty + \| \nabla_{x_{\parallel}} \hat{F}_{\perp} \|_\infty \lesssim |v|^3 + 1 \), and \( \| \nabla_{x_{\perp}} \hat{F}_{\perp} \|_\infty + \| \nabla_{x_{\parallel}} \hat{F}_{\perp} \|_\infty \lesssim |v|^2 + 1 \). Thus together with (5.27) we have

\[
\frac{d}{d\tau} \left( \frac{\partial F_{\perp}(\tau)}{\partial \tau} + \frac{\partial F_{\perp}(\tau)}{\partial t^\ell} \right) = \frac{\partial \hat{F}_{\perp}(\tau)}{\partial \tau} + \frac{\partial \hat{F}_{\perp}(\tau)}{\partial t^\ell} \\
= \nabla_{x_{\perp}} \hat{F}_{\perp} \cdot \left( v_{\perp}(s) + \frac{\partial x_{\perp}(s)}{\partial t^\ell} \right) + \nabla_{x_{\parallel}} \hat{F}_{\perp} \cdot \left( v_{\parallel}(s) + \frac{\partial x_{\parallel}(s)}{\partial t^\ell} \right) + \nabla_{v_{\perp}} \hat{F}_{\perp} \cdot \left( F_{\perp}(s) + \frac{\partial v_{\perp}(s)}{\partial t^\ell} \right) + \nabla_{v_{\parallel}} \hat{F}_{\perp} \cdot \left( F_{\parallel}(s) + \frac{\partial v_{\parallel}(s)}{\partial t^\ell} \right) - \partial^2_{\perp} E \cdot n(x_{\parallel}) \tag{5.57}
\]

\[
\lesssim (|v|^3 + 1) \int_0^t \int_0^{\tau} \left( |\partial F_{\perp}(\tau') + \partial F_{\perp}(\tau')| + |\partial F_{\parallel}(\tau') + \partial F_{\parallel}(\tau')| \right) d\tau' d\tau \\
+ (|v|^2 + 1) \int_0^t \left( |\partial F_{\perp}(\tau') + \partial F_{\perp}(\tau')| + |\partial F_{\parallel}(\tau') + \partial F_{\parallel}(\tau')| \right) d\tau' + \| \partial^2_{\perp} E \|_{L^\infty_t} \\
\lesssim \| \partial^2_{\perp} E \|_{L^\infty_t} + (|v|^3 + 1)(t^\ell - t^{\ell+1})^2 + (|v|^2 + 1)(t^\ell - t^{\ell+1}) \\
\lesssim \| \partial^2_{\perp} E \|_{L^\infty_t} + |v| + 1.
\]

Combing (5.26), (5.54), (5.57), and expanding \( \frac{\partial F_{\perp}(\tau)}{\partial \tau} + \frac{\partial F_{\perp}(\tau)}{\partial t^\ell} \) at \( t^\ell \) we get

\[
\frac{\partial v_{\perp}^{\ell+1}}{\partial t^\ell} = \left( \frac{\partial F_{\perp}(t^\ell)}{\partial \tau} + \frac{\partial F_{\perp}(t^\ell)}{\partial t^\ell} \right) \left( F_{\perp}(t^\ell) \frac{|t^\ell - t^{\ell+1}|^2}{2} - |t^\ell - t^{\ell+1}| \right) + O(1)E_{L^2_{t,x}}(1)|t^\ell - t^{\ell+1}|^2(|v| + 1). \tag{5.58}
\]

Now since we have

\[
0 = x_{\perp}^\ell = x_{\perp}^{\ell+1} + \int_{t^{\ell+1}}^{t^\ell} v_{\perp}(s) ds \\
= \int_{t^{\ell+1}}^{t^\ell} \left( -v_{\perp}^{\ell+1} + \int_{t^{\ell+1}}^{s} F_{\perp}(\tau) d\tau \right) ds \\
= (t^\ell - t^{\ell+1})(-v_{\perp}^{\ell+1}) + \int_{t^{\ell+1}}^{t^\ell} \int_{t^{\ell+1}}^{s} F_{\perp}(\tau) d\tau ds \\
= (t^\ell - t^{\ell+1})(-v_{\perp}^{\ell+1}) + \frac{|t^\ell - t^{\ell+1}|^2}{2} F_{\perp}(t^{\ell+1}) + O(1)E_{L^2_{t,x}} + |v|^3|t^\ell - t^{\ell+1}|^3,
\]

we get the following important cancellation identity:

\[
\frac{F_{\perp}(t^{\ell+1}) |t^\ell - t^{\ell+1}|^2}{2} - |t^\ell - t^{\ell+1}| = O(1)(E_{L^2_{t,x}} + |v|^3) \frac{|t^\ell - t^{\ell+1}|^3}{v_{\perp}^{\ell+1}}. \tag{5.60}
\]

By (5.58) and (5.60) we get

\[
\frac{\partial v_{\perp}^{\ell+1}}{\partial t^\ell} \lesssim \left( \| E_{L^2_{t,x}} \| + \| F_{\perp}^2 E \|_{L^\infty_{t,x}} + 1 \right) \left( |v||t^\ell - t^{\ell+1}|^2 + |t^\ell - t^{\ell+1}|^2 \right), \tag{5.61}
\]

\(35\)
Next, taking \( \frac{\partial}{\partial t^\ell} \) derivative to \( v^\ell_{t+1} = v^\ell_t - \int_{t+1}^{t^\ell} F_l(s) ds \), and \( x^\ell_{t+1} = x^\ell_t - (t^\ell - t^\ell_{t+1}) v^\ell_t + \int_{t^\ell}^{t^\ell_{t+1}} \int_s^{t^\ell} F_l(\tau) d\tau ds \) we get

\[
\frac{\partial v^\ell_{t+1}}{\partial t^\ell} = -F_l(t^\ell) + \frac{\partial v^\ell_{t+1}}{\partial t^\ell} F_l(t^\ell) - \int_{t+1}^{t^\ell} \partial_s F_l(s) ds = F_l(t^\ell_{t+1}) - F_l(t^\ell) + O(1) \frac{|t^\ell - t^\ell_{t+1}|^2}{|v^\ell_{t+1}|} - \int_{t+1}^{t^\ell} \partial_s F_l(s) ds = O(1) \frac{|t^\ell - t^\ell_{t+1}|^2}{|v^\ell_{t+1}|} - \int_{t+1}^{t^\ell} (\partial_s F_l(s) + \partial_s F_l(s)) ds \lesssim |t^\ell - t^\ell_{t+1}|,
\]

and

\[
\frac{\partial x^\ell_{t+1}}{\partial t^\ell} = -v^\ell_{t+1} + \frac{\partial t^\ell_{t+1}}{\partial t^\ell} v^\ell_{t+1} + \int_{t+1}^{t^\ell} F_l(t^\ell) ds + \int_{t+1}^{t^\ell} \int_{s}^{t^\ell} \partial_s F_l(\tau) d\tau ds = v^\ell_{t+1} - v^\ell_t - O(1) \frac{|t^\ell - t^\ell_{t+1}|^2}{|v^\ell_{t+1}|} + \int_{t+1}^{t^\ell} F_l(t^\ell) ds + \int_{t+1}^{t^\ell} \int_{s}^{t^\ell} \partial_s F_l(\tau) d\tau ds = \int_{t+1}^{t^\ell} (F_l(t^\ell) - F_l(s)) ds + \int_{t+1}^{t^\ell} \int_{s}^{t^\ell} \partial_s F_l(\tau) d\tau ds + O(1) \frac{|t^\ell - t^\ell_{t+1}|^2}{|v^\ell_{t+1}|} v^\ell_{t+1} = \int_{t+1}^{t^\ell} \int_{s}^{t^\ell} (\partial_s F_l(\tau) + \partial_s F_l(\tau)) d\tau ds + O(1) |t^\ell - t^\ell_{t+1}| \lesssim |t^\ell - t^\ell_{t+1}|.
\]

Where we’ve used (5.20) and (5.33). This proves the first column of (5.45) and (5.46).

Taking derivatives of (5.48) as before and using \( |t^\ell - t^\ell_{t+1}| \lesssim \varepsilon_{t+1} \min\left\{ \frac{|v^\ell_{t+1}|}{|v^\ell_t|}, 1 \right\} \) and Lemma 7

\[
\begin{bmatrix}
\frac{\partial t^\ell_{t+1}}{\partial x^\ell_t} \\
\frac{\partial t^\ell_{t+1}}{\partial v^\ell_t} \\
\frac{\partial t^\ell_{t+1}}{\partial v^\ell_t}
\end{bmatrix} \lesssim \begin{bmatrix}
\frac{1}{v^\ell_t} \int_{t+1}^{t^\ell} \int_{s}^{t^\ell} \frac{\partial}{\partial s} F_l(X_s(\tau), V_s(\tau)) d\tau ds \\
\frac{1}{v^\ell_t} \int_{t+1}^{t^\ell} \int_{s}^{t^\ell} \frac{\partial}{\partial v^\ell} F_l(X_s(\tau), V_s(\tau)) d\tau ds \\
\frac{1}{v^\ell_t} \int_{t+1}^{t^\ell} \int_{s}^{t^\ell} \frac{\partial}{\partial v^\ell} F_l(X_s(\tau), V_s(\tau)) d\tau ds
\end{bmatrix} \lesssim \begin{bmatrix}
\frac{\varepsilon_{t+1}^\ell}{|v^\ell_{t+1}|} \\
\frac{\varepsilon_{t+1}^\ell}{|v^\ell_{t+1}|} \\
\frac{\varepsilon_{t+1}^\ell}{|v^\ell_{t+1}|}
\end{bmatrix} \lesssim \begin{bmatrix}
\frac{|v^\ell_{t+1}|}{|v^\ell_t|} \\
O(1) \\
\frac{|v^\ell_{t+1}|}{|v^\ell_t|}
\end{bmatrix}, \text{ for } |v| > \delta.
\]

(5.62)

Thus from (5.2) we have

\[
\begin{bmatrix}
\frac{\partial t^\ell_{t+1}}{\partial x^\ell_t} \\
\frac{\partial t^\ell_{t+1}}{\partial v^\ell_t} \\
\frac{\partial t^\ell_{t+1}}{\partial v^\ell_t}
\end{bmatrix} \lesssim \begin{bmatrix}
\frac{1}{v^\ell_t} \int_{t+1}^{t^\ell} \int_{s}^{t^\ell} \frac{\partial}{\partial s} F_l(X_s(\tau), V_s(\tau)) d\tau ds \\
\frac{1}{v^\ell_t} \int_{t+1}^{t^\ell} \int_{s}^{t^\ell} \frac{\partial}{\partial v^\ell} F_l(X_s(\tau), V_s(\tau)) d\tau ds \\
\frac{1}{v^\ell_t} \int_{t+1}^{t^\ell} \int_{s}^{t^\ell} \frac{\partial}{\partial v^\ell} F_l(X_s(\tau), V_s(\tau)) d\tau ds
\end{bmatrix} \lesssim \begin{bmatrix}
\frac{|v^\ell_{t+1}|}{|v^\ell_t|} \\
O(1) \\
\frac{|v^\ell_{t+1}|}{|v^\ell_t|}
\end{bmatrix}, \text{ for } |v| \leq \delta.
\]

(5.63)

Taking \( (x^\ell_t, v^\ell_t) \) derivatives of the characteristic equations

\[
x^\ell_{t+1} = x^\ell_t - \int_{t+1}^{t^\ell} v^\ell_t(s; t^\ell x^\ell_t, v^\ell_t) ds,
\]

by Lemma 7 and (5.2), we estimate directly

\[
\begin{bmatrix}
\frac{\partial x^\ell_{t+1}}{\partial x^\ell_t} \\
\frac{\partial x^\ell_{t+1}}{\partial v^\ell_t} \\
\frac{\partial x^\ell_{t+1}}{\partial v^\ell_t}
\end{bmatrix} \lesssim \begin{bmatrix}
\text{Id} + \frac{|t^\ell - t^\ell_{t+1}|^2 |v^\ell_t|^2}{|v^\ell_{t+1}|^2} + O(1) \frac{|t^\ell - t^\ell_{t+1}|^2 |v^\ell_t|}{|v^\ell_{t+1}|^2} \\
O(1) \frac{|t^\ell - t^\ell_{t+1}|^2 |v^\ell_t|}{|v^\ell_{t+1}|^2} + |t^\ell - t^\ell_{t+1}| \\
\frac{|t^\ell - t^\ell_{t+1}|^2}{|v^\ell_{t+1}|^2} \left( |v^\ell_t|^2 + O(1) |v^\ell_t| \right) + |t^\ell - t^\ell_{t+1}|
\end{bmatrix}.
\]
Thus from (5.2) we have

\[
\begin{bmatrix}
\frac{\partial v^{\ell+1}}{\partial x\\x} \\
\frac{\partial v^{\ell+1}}{\partial v^1\\t} \\
\frac{\partial v^{\ell+1}}{\partial v^1\\x}
\end{bmatrix} \lesssim_{\xi,t} \begin{bmatrix}
\text{Id}_{2,2} + \frac{|v^1|}{|v^1|} \\
\frac{1}{|v^1|} \\
\frac{1}{|v^1|}
\end{bmatrix}, \text{ for } |v| > \delta.
\]

\[
 \begin{bmatrix}
\frac{\partial x^{\ell+1}}{\partial x} \\
\frac{\partial x^{\ell+1}}{\partial v^1\\t} \\
\frac{\partial x^{\ell+1}}{\partial v^1\\x}
\end{bmatrix} \lesssim_{\xi,t} \begin{bmatrix}
\text{Id}_{2,2} + \frac{|v^1|}{|v^1|} \\
\frac{1}{|v^1|} \\
\frac{1}{|v^1|}
\end{bmatrix}, \text{ for } |v| \leq \delta.
\]

Also,

\[
\begin{bmatrix}
\frac{\partial v^{\ell+1}}{\partial x\\x} \\
\frac{\partial v^{\ell+1}}{\partial v^1\\t} \\
\frac{\partial v^{\ell+1}}{\partial v^1\\x}
\end{bmatrix} \lesssim_{\xi,t} \begin{bmatrix}
\frac{|v^1|^2 + O(1)}{|v^1|} + |v| + O(1) \\
\frac{|v^1|^2 + O(1)}{|v^1|} + \frac{|v| + O(1)}{|v^1|} \\
\frac{|v^1|^2 + O(1)}{|v^1|} + \frac{|v| + O(1)}{|v^1|} + |v| + O(1)
\end{bmatrix}.
\]

Thus from (5.2) we have

\[
\begin{bmatrix}
\frac{\partial v^{\ell+1}}{\partial x\\x} \\
\frac{\partial v^{\ell+1}}{\partial v^1\\t} \\
\frac{\partial v^{\ell+1}}{\partial v^1\\x}
\end{bmatrix} \lesssim_{\xi,t} \begin{bmatrix}
1 + \frac{|v^1|}{|v^1|} \\
\text{Id}_{2,2} + \frac{|v^1|}{|v^1|} \\
\frac{1}{|v^1|}
\end{bmatrix}, \text{ for } |v| > \delta.
\]

\[
\begin{bmatrix}
\frac{\partial v^{\ell+1}}{\partial x\\x} \\
\frac{\partial v^{\ell+1}}{\partial v^1\\t} \\
\frac{\partial v^{\ell+1}}{\partial v^1\\x}
\end{bmatrix} \lesssim_{\xi,t} \begin{bmatrix}
\frac{|v^1|^2}{|v^1|} + 1 \\
\text{Id}_{2,2} + |v^1| \\
\frac{1}{|v^1|}
\end{bmatrix}, \text{ for } |v| \leq \delta.
\]

Now we move to \(Dv^{\ell+1}\) estimates. Taking derivatives in (5.74), from the extra cancellation in terms of order of \(t^\ell - t^{\ell+1}\) in (5.60), by (5.62), and plugging the expansion

\[
\frac{\partial}{\partial x} F_\perp(X_\ell(\tau), V_\ell(\tau)) = \frac{\partial}{\partial x} F_\perp(x^\ell, v^\ell) - \int_\tau^{t^\ell} \frac{d}{d\tau'} \left( \frac{\partial}{\partial x} F_\perp(X_\ell(\tau'), V_\ell(\tau')) \right) d\tau'
\]

into

\[
\frac{\partial v^{\ell+1}}{\partial x_\perp} = -\frac{F_\perp(x^{\ell+1}, v^{\ell+1})}{v^{\ell+1}_\perp} \int_{t^{\ell+1}}^{t^\ell} \int_s^{t^\ell} \frac{\partial}{\partial x_\perp} F_\perp(X_\ell(\tau), V_\ell(\tau)) d\tau' + \int_{t^{\ell+1}}^{t^\ell} \frac{\partial}{\partial x_\perp} F_\perp(X_\ell(\tau), V_\ell(\tau)) d\tau,
\]

and using the cancellation (5.60) we obtain

\[
\frac{\partial v^{\ell+1}}{\partial x_\perp} = \left\{ \frac{(t^\ell - t^{\ell+1})F_\perp(x^{\ell+1}, v^{\ell+1})}{-2v^{\ell+1}_\perp} + 1 \right\} (t^\ell - t^{\ell+1}) \frac{\partial}{\partial x_\perp} F_\perp(x^\ell, v^\ell)
\]

\[
+ \frac{F_\perp(x^{\ell+1}, v^{\ell+1})}{v^{\ell+1}_\perp} \int_{t^{\ell+1}}^{t^\ell} \int_s^{t^\ell} \frac{d}{d\tau'} \left( \frac{\partial}{\partial x_\perp} F_\perp(X_\ell(\tau'), V_\ell(\tau')) \right) d\tau' d\tau
\]

\[
+ \int_{t^{\ell+1}}^{t^\ell} \frac{d}{d\tau'} \left( \frac{\partial}{\partial x_\perp} F_\perp(X_\ell(\tau'), V_\ell(\tau')) \right) d\tau' d\tau \lesssim \left\{ -1 + O_\xi(1) \frac{t^\ell - t^{\ell+1}}{v^{\ell+1}_\perp} \frac{t^\ell}{2} + 1 \right\} (t^\ell - t^{\ell+1}) + 1
\]

\[
+ \frac{F_\perp(x^{\ell+1}, v^{\ell+1})}{v^{\ell+1}_\perp} \int_{t^{\ell+1}}^{t^\ell} \int_s^{t^\ell} \frac{d}{d\tau'} \left( \frac{\partial}{\partial x_\perp} F_\perp(X_\ell(\tau'), V_\ell(\tau')) \right) d\tau' d\tau.
\]

(5.64)
Now since

\[
\frac{d}{d\tau'} \left( \frac{\partial}{\partial x_i} F_\perp(X_\ell(\tau'), V_\ell(\tau')) \right) \\
\lesssim |v'|^3 + \left| \frac{d}{d\tau'} \left( E(\tau', X_\ell(\tau')) \cdot n(X_\ell(\tau')) \right) \right| \\
\lesssim |v'|^3 + \left| \frac{d}{d\tau'} \left( n(X_\ell(\tau')) \cdot \nabla_x E(\tau', X_\ell(\tau')) \cdot \frac{\partial X_\ell(\tau')}{\partial x_i} + E(\tau', X_\ell(\tau')) \cdot \nabla_x n(X_\ell(\tau')) \cdot \frac{\partial X_\ell(\tau')}{\partial x_i} \right) \right| \\
\lesssim |v'|^3 + \left| n(X_\ell(\tau')) \cdot \nabla_x E(\tau', X_\ell(\tau')) \cdot \left( \frac{d}{d\tau'} \left( \frac{\partial X_\ell(\tau')}{\partial x_i} \right) \right) + \nabla_x n(X_\ell(\tau')) \cdot \nabla_x E(\tau', X_\ell(\tau')) \cdot \frac{\partial X_\ell(\tau')}{\partial x_i} \right| \\
\lesssim |v'|^3 + \left| n(X_\ell(\tau')) \cdot \nabla_x E(\tau', X_\ell(\tau')) \cdot \frac{\partial V_\ell(\tau')}{\partial x_i} + (\nabla_x n(X_\ell(\tau')) \cdot \nabla_x E(\tau', X_\ell(\tau')) \cdot \frac{\partial X_\ell(\tau')}{\partial x_i} \right| \\
\lesssim |v'|^3 + |v'|^2 \| \nabla_x^2 E \|_{L^\infty} + |\partial_i \nabla_x E|_{L^\infty} \\
\tag{5.65}
\]

where we use the bounds from \([111]\). We have

\[
\frac{\partial v_\ell^{t+1}}{\partial x_i} \lesssim \left\{ \frac{|t^\ell - t^{t+1}|^2 (|v'|^2 + 1)}{v_\perp^2} \right\} \left( |t^\ell - t^{t+1}|^2 (|v'|^3 + 1) + |v'|^3 + |v'|^2 \| \nabla_x^2 E \|_{L^\infty} + |\partial_i \nabla_x E|_{L^\infty} \right) \\
\lesssim \xi \min \left\{ \frac{|v'|^2 + 1}{|v'|}, |v'|^2 \right\} \\
\tag{5.66}
\]

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as long as \( \|\nabla^2_x E\|_{L^\infty_{v,t}} + \|\partial_t \nabla_x E\|_{L^\infty_{v,t}} < \infty \). Similarly,

\[
\frac{\partial v^{t+1}}{\partial v^t} = -1 - \sum_{t=1}^{t-1} \frac{\partial v^{t+1}}{\partial v^t} F_\perp(x^{t+1}, v^{t+1}) + \int_{t-1}^{t+1} \frac{\partial v^{t+1}}{\partial v^t} F_\perp(x^{t+1}, v^{t+1}) \, d\tau
\]

\[
= -1 + \sum_{t=1}^{t-1} \frac{\partial v^{t+1}}{\partial v^t} F_\perp(x^{t+1}, v^{t+1}) - \int_{t-1}^{t+1} \frac{\partial v^{t+1}}{\partial v^t} F_\perp(x^{t+1}, v^{t+1}) \, d\tau
\]

\[
+ \sum_{t=1}^{t-1} \frac{\partial v^{t+1}}{\partial v^t} F_\perp(x^{t+1}, v^{t+1}) \, d\tau
\]

\[
= -1 + 2 + O(1) \left( |v^{t+1} - v^t|^2 + 1 \right)
\]

\[
- \frac{F_\perp(x^{t+1}, v^{t+1})}{v^{t+1}} \left( \frac{t - \tau^{t+1}}{2} \right) \left( \frac{\lim_{t\to \tau^{t+1}} \frac{\partial}{\partial v^t} F_\perp(x^{t+1}, v^{t+1}) + O(1) |v^{t+1} - v^t|^2 + 1} {v^{t+1}} \right)
\]

\[
+ (t - \tau^{t+1}) \left( \frac{\lim_{t\to \tau^{t+1}} \frac{\partial}{\partial v^t} F_\perp(x^{t+1}, v^{t+1}) + O(1) |v^{t+1} - v^t|^2 + 1} {v^{t+1}} \right)
\]

\[
= 1 + O(1) \left( \frac{|v^{t+1} - v^t|^2 + 1} {v^{t+1}} \right)
\]

\[
\lesssim 1 + |t - t^{t+1}|^2 \left( \frac{|v^{t+1} - v^t|^2 + 1} {v^{t+1}} \right)
\]

\[
\lesssim_{\xi, t} 1 + \min\left\{ \frac{|v^{t+1} - v^t|^2}{|v^{t+1}|}, \frac{|v^{t+1} - v^t|^2}{|v^{t+1}|} \right\},
\]

These estimates complete the proof of the claims \( (5.40) \) and \( (5.43) \) when \( t \) is Type II.

**Proof of (5.43) when \( \ell \) is Type III:** Recall that we chose a \( p^\ell \)-spherical coordinate as \( p^\ell = (z^\ell, w^\ell) \) with \( |z^\ell - x^\ell| \leq \sqrt{\theta} \) and any \( w^\ell \in \mathbb{S}^n \) with \( n(z^\ell) \cdot w^\ell = 0 \).

Fix \( \ell \). Let us choose fixed numbers \( \Delta_1, \Delta_2 > 0 \) such that \( |v| \Delta_1 \ll 1 \) and \( |v| \|t^{\ell+1} - (t^\ell - \Delta_1 - \Delta_2)| \ll 1 \) so that

\[
s^{t+1} \equiv t^{t+1} - \Delta_1, \quad s^{t+1} \equiv s^\ell - \Delta_2 = t^\ell - \Delta_1 - \Delta_2,
\]

satisfying \( |v||s^{t+1} - s^\ell| = |v||s^\ell - (t^\ell - \Delta_1 - \Delta_2)| \ll 1 \) and \( |v||t^\ell - s^\ell| = |v||\Delta_1| \ll 1 \) so that the spherical coordinates are well-defined for \( s \in [t^{t+1}, s^{t+1}] \) and \( s \in [s^\ell, t^\ell] \).

Notice that

\[
\frac{\partial s^{t+1}}{\partial s^\ell} = \frac{\partial(s^\ell - \Delta_1)}{\partial s^\ell} = 1, \quad \frac{\partial s^\ell}{\partial t^\ell} = \frac{\partial(t^\ell - \Delta_1)}{\partial t^\ell} = 1.
\]

We first follow the flow in \( p^\ell \)-spherical coordinate, then change to the Euclidian coordinate to near the boundary at \( s^\ell \), follow the flow until \( s^{t+1} \), and then change to the chart to \( p^{t+1} \)-spherical coordinate. By
the chain rule,
\[
\frac{\partial (t^{\ell+1}, x_{\parallel, t}^{\ell+1}, v_{\parallel, t}^{\ell+1}, v_{\perp, t}^{\ell+1})}{\partial (t^t, x_{\parallel, t}^t, v_{\parallel, t}^t, v_{\perp, t}^t)} = \frac{\partial (s^{\ell+1}, x_{\parallel, t}^{s^{\ell+1}}, v_{\parallel, t}^{s^{\ell+1}}, v_{\perp, t}^{s^{\ell+1}})}{\partial (s^t, X_{t}\ell(s^t), V_{t}\ell(s^t))} \frac{\partial (s^{\ell+1}, X_{t}\ell(s^{\ell+1}), V_{t}\ell(s^{\ell+1}))}{\partial (s^t, X_{e\ell}(s^t), V_{e\ell}(s^t))} \\
\times \frac{\partial (s^{\ell+1}, X_{e\ell}(s^{\ell+1}), V_{e\ell}(s^{\ell+1}))}{\partial (s^t, x_{\parallel, t}^{s^t}, v_{\parallel, t}^{s^t}, v_{\perp, t}^{s^t})} \frac{\partial (s^t, x_{\parallel, t}^t, v_{\parallel, t}^t, v_{\perp, t}^t)}{\partial (t^t, x_{\parallel, t}^t, v_{\parallel, t}^t, v_{\perp, t}^t)}.
\]

We can express that \( t^{\ell+1} = t^t - t_b(x^t, v^t) = s^{\ell+1} + \Delta_1 + \Delta_2 - t_b(x^t, v^t). \) Let us regard \( t^{\ell+1} \) as \( t^t \) and \( s^{\ell+1} \) as \( s^t \) and \( \Delta_1 + \Delta_2 \) as \( \Delta \) in (5.39). Then we use (5.40) and (5.2) to have
\[
\frac{\partial (t^{\ell+1}, x_{\parallel, t}^{\ell+1}, v_{\parallel, t}^{\ell+1}, v_{\perp, t}^{\ell+1})}{\partial (s^{\ell+1}, x_{\parallel, t}^{s^{\ell+1}}, x_{\parallel, t}^{s^{\ell+1}}, v_{\parallel, t}^{s^{\ell+1}}, v_{\parallel, t}^{s^{\ell+1}})} \leq \begin{bmatrix}
1 + O(1)|t^t - t^{\ell+1}| & O_{\delta, \xi}(1)|t^t - t^{\ell+1}| & O_{\delta, \xi}(1)\frac{1}{|t^t|} \\
O(1)|t^t - t^{\ell+1}| & O_{\delta, \xi}(1) & O_{\delta, \xi}(1)\frac{1}{|t^t|} \\
O(1)|t^t - t^{\ell+1}| & O_{\delta, \xi}(1) & O_{\delta, \xi}(1)\frac{1}{|t^t|} \\
O(1)|t^t - t^{\ell+1}| & O_{\delta, \xi}(1) & O_{\delta, \xi}(1)\frac{1}{|t^t|}
\end{bmatrix}.
\]

Where we have used From (5.41)
\[
\frac{\partial (s^{\ell+1}, X_{t}\ell(s^{\ell+1}), V_{t}\ell(s^{\ell+1}))}{\partial (s^t, X_{e\ell}(s^t), V_{e\ell}(s^t))} \approx \begin{bmatrix}
1 & 0_{1,3} & 0_{1,3} \\
0_{3,1} & O_{\xi}(1) & 0_{3,3} \\
o_{3,1} & O_{\xi}(1)|v| & O_{\xi}(1)
\end{bmatrix},
\]
and from \( s^{\ell+1} = s^t - \Delta_2, \) \( X_{e\ell}(s^{\ell+1}) = X_{e\ell}(s^t) - (s^{\ell+1} - s^t)V_{e\ell}(s^t), \) \( V_{e\ell}(s^{\ell+1}) = V_{e\ell}(s^t), \)
\[
\frac{\partial (s^{\ell+1}, X_{e\ell}(s^{\ell+1}), V_{e\ell}(s^{\ell+1}))}{\partial (s^t, X_{e\ell}(s^t), V_{e\ell}(s^t))} \approx \begin{bmatrix}
1 & 0_{1,3} & 0_{1,3} \\
o_{3,1} & O_{\xi}(1) & 0_{3,3} \\
o_{3,1} & O_{\xi}(1)|v| & O_{\xi}(1)
\end{bmatrix},
\]
and from (5.5)
\[
\frac{\partial (s^t, X_{e\ell}(s^t), V_{e\ell}(s^t))}{\partial (t^t, x_{\parallel, t}^t, v_{\parallel, t}^t, v_{\perp, t}^t)} \approx \begin{bmatrix}
1 & 0_{1,3} & 0_{1,3} \\
o_{3,1} & O_{\xi}(1) & 0_{3,3} \\
o_{3,1} & O_{\xi}(1)|v| & O_{\xi}(1)
\end{bmatrix}.
\]

Recalling (5.36), we have
\[
\frac{\partial (s^t, x_{\parallel, t}^{t}(s^t), x_{\parallel, t}^{s^t}(s^t), v_{\parallel, t}^{s^t}(s^t), v_{\parallel, t}^{s^t}(s^t))}{\partial (t^t, x_{\parallel, t}^t, v_{\parallel, t}^t, v_{\perp, t}^t)} \approx \begin{bmatrix}
1 & \frac{1}{|t^t|} & \frac{1}{|t^t|} \\
o_{2,1} & 1 & \frac{1}{|t^t|} \\
o_{3,1} & \frac{1}{|t^t|} & 1
\end{bmatrix}.
\]

By direct matrix multiplication
\[
\frac{\partial (t^{\ell+1}, x_{\parallel, t}^{\ell+1}, v_{\parallel, t}^{\ell+1}, v_{\perp, t}^{\ell+1})}{\partial (t^t, x_{\parallel, t}^t, v_{\parallel, t}^t, v_{\perp, t}^t)} \approx \begin{bmatrix}
1 & \frac{M}{10} \sqrt{\delta} & \frac{M}{10} \min\{1, \sqrt{\delta}\} \\
o_{2,1} & M\sqrt{\delta} & \frac{M}{10} \min\{1, \sqrt{\delta}\} \\
o_{3,1} & M|v|\min\{\delta, \sqrt{\delta}\} & M\min\{\delta, \sqrt{\delta}\}
\end{bmatrix} \cdot \begin{bmatrix}
0_{2,1} & 0_{1,2} & 0_{1,3} \\
o_{3,1} & O_{\xi}(1) & 0_{3,3} \\
o_{3,1} & O_{\xi}(1)|v| & O_{\xi}(1)
\end{bmatrix}. \]

Note that for Type III we have \( t^{\ell+1} \geq \sqrt{\delta} \) so that from (5.45)
\[
J(t^{\ell+1}) \geq \begin{bmatrix}
1 & \frac{M}{10} \sqrt{\delta} & \frac{M}{10} \min\{1, \sqrt{\delta}\} \\
o_{2,1} & M\sqrt{\delta} & \frac{M}{10} \min\{1, \sqrt{\delta}\} \\
o_{3,1} & M|v|\min\{\delta, \sqrt{\delta}\} & M\min\{\delta, \sqrt{\delta}\}
\end{bmatrix} \cdot \begin{bmatrix}
1 & \frac{1}{|t^t|} & \frac{1}{|t^t|} \\
o_{2,1} & 1 & \frac{1}{|t^t|} \\
o_{3,1} & \frac{1}{|t^t|} & 1
\end{bmatrix} \cdot \begin{bmatrix}
1 & \frac{M}{10} \sqrt{\delta} & \frac{M}{10} \min\{1, \sqrt{\delta}\} \\
o_{2,1} & M\sqrt{\delta} & \frac{M}{10} \min\{1, \sqrt{\delta}\} \\
o_{3,1} & M|v|\min\{\delta, \sqrt{\delta}\} & M\min\{\delta, \sqrt{\delta}\}
\end{bmatrix} \cdot \begin{bmatrix}
0_{2,1} & 0_{1,2} & 0_{1,3} \\
o_{3,1} & O_{\xi}(1) & 0_{3,3} \\
o_{3,1} & O_{\xi}(1)|v| & O_{\xi}(1)
\end{bmatrix}.
\]

This proves our claim (5.44) for Type III.

Step 5. Eigenvalues and diagonalization of (5.43).
We consider the case when \( \ell \) is \textit{Type II} or \textit{Type III}. By a basic linear algebra (row and column operations), the characteristic polynomial of (5.45) equals, with \( r = r^{\ell+1}, \)

\[
\begin{vmatrix}
1 + 5Mr - \lambda & \frac{\lambda}{v}r & \frac{\lambda}{v}r & \frac{\lambda}{v}r & \frac{\lambda}{v}r & \frac{\lambda}{v}r \\
5Mr|v| & 1 + Mr - \lambda & Mr & Mr & Mr & Mr \\
5Mr|v|^2 & Mr & 1 + Mr - \lambda & Mr^2 & Mr^2 & Mr^2 \\
5Mr^2|v|^2 & Mr^2 & Mr^2 & 1 + Mr - \lambda & Mr^2 & Mr^2 \\
5Mr^3|v|^2 & Mr^3 & Mr^3 & Mr^3 & Mr^3 & Mr^3 \\
5Mr^4|v|^2 & Mr^4 & Mr^4 & Mr^4 & Mr^4 & Mr^4 \\
\end{vmatrix} = (\lambda - 1)^5(\lambda - (10Mr + 1)).
\]

Therefore eigenvalues are

\[
\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 1, \lambda_6 = 1 + 10Mr.
\]

(5.68)

Corresponding eigenvectors are

\[
\begin{pmatrix}
-\frac{1}{5|v|} \\
1 \\
0 \\
0 \\
0 \\
0 \\
\end{pmatrix}, \begin{pmatrix}
-\frac{1}{5|v|} \\
0 \\
1 \\
0 \\
0 \\
0 \\
\end{pmatrix}, \begin{pmatrix}
-\frac{1}{5|v|^2r} \\
0 \\
0 \\
1 \\
0 \\
0 \\
\end{pmatrix}, \begin{pmatrix}
-\frac{1}{5|v|^2} \\
0 \\
0 \\
0 \\
1 \\
0 \\
\end{pmatrix}, \begin{pmatrix}
\frac{1}{|v|^2} \\
0 \\
0 \\
0 \\
0 \\
1 \\
\end{pmatrix}, \begin{pmatrix}
\frac{1}{|v|^2} \\
0 \\
0 \\
0 \\
0 \\
1 \\
\end{pmatrix}.
\]

Write \( P = P(r^{\ell}) \) as a block matrix of above column eigenvectors. Then

\[
P = \begin{bmatrix}
-\frac{1}{5|v|} & -\frac{1}{5|v|} & -\frac{1}{5|v|^2r} & -\frac{1}{5|v|^2} & \frac{1}{|v|^2} & \frac{1}{|v|^2} \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 \\
\end{bmatrix}, \quad P^{-1} = \begin{bmatrix}
\frac{1}{|v|^2} & \frac{9}{10} & \frac{1}{10} & \frac{1}{10|v|r} & \frac{1}{10|v|r} & \frac{1}{10|v|r} \\
\frac{1}{|v|^2} & \frac{1}{10} & \frac{9}{10} & \frac{1}{10|v|r} & \frac{1}{10|v|r} & \frac{1}{10|v|r} \\
\frac{1}{|v|^2} & \frac{1}{10} & \frac{9}{10} & \frac{1}{10|v|r} & \frac{1}{10|v|r} & \frac{1}{10|v|r} \\
\frac{1}{|v|^2} & \frac{1}{10} & \frac{9}{10} & \frac{1}{10|v|r} & \frac{1}{10|v|r} & \frac{1}{10|v|r} \\
\frac{1}{|v|^2} & \frac{1}{10} & \frac{9}{10} & \frac{1}{10|v|r} & \frac{1}{10|v|r} & \frac{1}{10|v|r} \\
\frac{1}{|v|^2} & \frac{1}{10} & \frac{9}{10} & \frac{1}{10|v|r} & \frac{1}{10|v|r} & \frac{1}{10|v|r} \\
\end{bmatrix}.
\]

Therefore

\[
J(r) = P(r)A(r)P^{-1}(r),
\]

and

\[
A(r) := \text{diag}\left[1, 1, 1, 1, 1 + 10Mr\right],
\]

where the notation \( \text{diag}[a_1, \cdots, a_m] \) is a \( m \times m \) matrix with \( a_{ii} = a_i \) and \( a_{ij} = 0 \) for all \( i \neq j \).

Similarly for the case when \( \ell \) is \textit{Type I}, the eigenvalues of the matrix (5.46) are (with \( v_\perp = v^{\ell+1}_\perp \))

\[
\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 1, \lambda_6 = 1 + 10Mv_\perp.
\]

(5.70)

Corresponding eigenvectors are

\[
\begin{pmatrix}
-\frac{1}{5} \\
1 \\
0 \\
0 \\
0 \\
0 \\
\end{pmatrix}, \begin{pmatrix}
-\frac{1}{5} \\
0 \\
1 \\
0 \\
0 \\
0 \\
\end{pmatrix}, \begin{pmatrix}
-\frac{1}{5|v|^2} \\
\frac{1}{5|v|^2} \\
0 \\
0 \\
0 \\
0 \\
\end{pmatrix}, \begin{pmatrix}
1 \\
1 \\
0 \\
0 \\
0 \\
0 \\
\end{pmatrix}, \begin{pmatrix}
1 \\
1 \\
0 \\
0 \\
0 \\
0 \\
\end{pmatrix}, \begin{pmatrix}
v_\perp \\
1 \\
0 \\
0 \\
0 \\
0 \\
\end{pmatrix}.
\]
Write \( P = P(v^j) \) as a block matrix of above column eigenvectors. Then
\[
P = \begin{bmatrix}
-\frac{1}{2} & -\frac{1}{2} & -\frac{1}{10v_1} & -\frac{1}{2} & -\frac{1}{2} & 1 \\
1 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & v_1 \\
0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 \\
\end{bmatrix}, \quad P^{-1} = \begin{bmatrix}
-\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

Therefore
\[
J(v_\perp) = P(v_\perp)A(v_\perp)P^{-1}(v_\perp),
\]
and
\[
A(v_\perp) := \text{diag} \left[ 1, 1, 1, 1, 1 + 10Mv_\perp \right],
\]

**Step 6. The \( i \)-th intermediate group**

If \( \ell \) is Type II or Type III, We claim that, for \( i = 1, 2, \cdots, \left\lfloor \frac{t-|s|}{L_\xi} \right\rfloor \),
\[
\frac{1}{C_i} e^{-C_i t^\ell} \leq r^\ell \leq C_i e^{C_i t^\ell},
\]
and define
\[
r_i \equiv C_i e^{C_i t^\ell}. \tag{5.73}
\]

Then we have
\[
\frac{1}{(C_i)^2} e^{-C_i t^\ell} r_i \leq r^\ell \leq r_i \quad \text{for all} \quad \ell_i + 1 \leq j \leq \ell_i.
\]

From (5.43), we have a uniform bound for all \( \ell_i + 1 \leq j \leq \ell_i \)
\[
J^{\ell_i + 1}_j \leq J(r_i) = P(r_i)A(r_i)P^{-1}(r_i).
\]

Therefore
\[
J^{\ell_i + 1}_1 \times \cdots \times J^{\ell_i + 1}_i \leq P(r_i)A(r_i)P^{-1}(r_i).
\]

Now we only left to prove that \( |\ell_i + 1 - \ell_j| \leq \left\lfloor \frac{t^\ell - t^j}{L_\xi} \right\rfloor \). For any \( \ell_i + 1 \leq j \leq \ell_i \), we have \( \xi(x^j) = 0 = \xi(x^{j+1}) = \xi(x^j - (t^j - t^{j+1})v^j) \). We expand \( \xi(x^{j+1} - (t^j - t^{j+1})v^j) \) in time to have
\[
\xi(x^{j+1}) = \xi(x^j) + \int_{t^j}^{t^{j+1}} \frac{d}{ds} \xi(X_{\xi}(s)) ds = \xi(x^j) + (v^j \cdot \nabla \xi(x^j))(t^{j+1} - t^j) + \int_{t^j}^{t^{j+1}} \int_{t}^{s} \partial_{\tau^2} \xi(X_{\xi}(\tau)) d\tau ds,
\]
and
\[
0 = (v^j \cdot \nabla \xi(x^j))(t^{j+1} - t^j) + \frac{(t^j - t^{j+1})^2}{2} (V_{\xi}(\tau_{s}) \cdot \nabla^2 \xi(X_{\xi}(\tau_{s})) \cdot V_{\xi}(\tau_{s}) + E(\tau, X_{\xi}(\tau_{s})) \cdot \nabla \xi(X_{\xi}(\tau_{s})))
\]
for some \( \tau_{s} \in [t^{j+1}, t^j] \). Therefore
\[
\frac{v^j \cdot \nabla \xi(x^j)}{|v|} = (t^j - t^{j+1}) |v| \frac{V_{\xi}(\tau_{s}) \cdot \nabla^2 \xi(X_{\xi}(\tau_{s})) \cdot V_{\xi}(\tau_{s}) + E(\tau, X_{\xi}(\tau_{s})) \cdot \nabla \xi(X_{\xi}(\tau_{s})))}{2|v|^2}.
\]
Thus there exists $C_2(\delta, \zeta, E) \gg 1$

$$\frac{|\nu_j \cdot \nabla \xi(x^j)|}{|v|} \leq C_2|t^j - t^{j+1}||v|.$$  \hfill (5.74)

Therefore we have a lower bound of $|v||t^j - t^{j+1}|$: $|v||t^j - t^{j+1}| \geq \frac{1}{C_2}|\nu_j| \geq \frac{1}{(c_1)^2c_2} e^{-cc_1} r_i$, where we have used (5.73). Finally, using the definition of one group $1 \leq |v||t^j - t^{j+1}| \leq C_1$, we have the following upper bound of the number of bounces in this one group ($i$-th intermediate group)

$$|\ell_i - \ell_{i+1}| \leq \frac{|v||t^j - t^{j+1}|}{\min_i, j \leq \ell_{i+1} |v||t^j - t^{j+1}|} \leq \frac{C_1}{(c_1)^2c_2} e^{-cc_1} r_i \lesssim \frac{1}{r_i},$$

and this completes our claim (5.72).

Let’s consider the whole intermediate groups

$$J_{t^j - 1} \times \cdots \times J_{t^j + 1} \times J_{t^j - 1} \times \cdots \times J_{t^j} \leq J(r^{t_j}) \times \cdots \times J(r^{t_{j+1}}) \times J(r^t) \times \cdots \times J(r^{2}).$$  \hfill (5.75)

We have from (5.74) that

$$J(r^{t_j + 1}) \times J(r^t) = \mathcal{P}(r^{t_j + 1}) \Lambda(r^{t_j + 1}) \mathcal{P}^{-1}(r^{t_j + 1}) \mathcal{P}(r^t) \Lambda(r^t) \mathcal{P}^{-1}(r^t),$$

and by direct computation

$$\mathcal{P}^{-1}(r^{t_j + 1}) \mathcal{P}(r^t) = \begin{bmatrix} -\frac{1}{v^2} & -\frac{9}{10} & \frac{9}{10} & \frac{1}{10} & \frac{-10}{|v||t^{j+1} + 1|} & \frac{10}{|v|} \\ -\frac{9}{10} & -\frac{1}{10} & \frac{9}{10} & \frac{-10}{|v||t^{j+1} + 1|} & \frac{9}{10} & \frac{-10}{|v|} \\ -\frac{9}{10} & -\frac{1}{10} & \frac{9}{10} & \frac{-10}{|v||t^{j+1} + 1|} & \frac{9}{10} & \frac{-10}{|v|} \\ -\frac{9}{10} & -\frac{1}{10} & \frac{9}{10} & \frac{-10}{|v||t^{j+1} + 1|} & \frac{9}{10} & \frac{-10}{|v|} \\ 1 & 0 & 0 & 0 & 0 & \frac{1}{v^2} \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{v^2} \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$  \hfill (5.76)

Since from the definition of $\mathbf{v}_{\perp}^t$, and (5.66) we have

$$\mathbf{v}_{\perp}^{t+1} = - \lim_{s \downarrow t^{j+1}} \mathbf{v}_{\perp}(s) = -\mathbf{v}_{\perp}^t + \int_{t^{j+1}}^{t^j} \mathbf{F}_{\perp}(\mathbf{X}(\tau; t, x, v), \mathbf{V}(\tau; t, x, v)) d\tau$$

$$= -\mathbf{v}_{\perp}^t + (t^j - t^{j+1}) F_{\perp}(t^j) + O(1)|t^j - t^{j+1}|^2(|v|^3 + 1)$$

$$= -\mathbf{v}_{\perp}^t + 2v_{\perp}^{t+1} + O(1)|t^j - t^{j+1}|^2(|v|^3 + 1).$$  \hfill (5.77)

This implies $\mathbf{v}_{\perp}^t - \mathbf{v}_{\perp}^{t+1} = O(1)|t^j - t^{j+1}|^2(|v|^3 + 1)$. Similarly by plugging in

$$(t^j - t^{j+1}) F_{\perp}(t^j) = 2v_{\perp}^t + O(1)|t^j - t^{j+1}|^2(|v|^3 + 1),$$

(5.77) becomes

$$\mathbf{v}_{\perp}^{t+1} = -\mathbf{v}_{\perp}^t + (t^j - t^{j+1}) F_{\perp}(t^j) + O(1)|t^j - t^{j+1}|^2(|v|^3 + 1) = \mathbf{v}_{\perp}^t + O(1)|t^j - t^{j+1}|^2(|v|^3 + 1).$$

Thus $\mathbf{v}_{\perp}^{t+1} - \mathbf{v}_{\perp}^t = O(1)|t^j - t^{j+1}|^2(|v|^3 + 1)$, therefore

$$|\mathbf{v}_{\perp}^{t+1} - \mathbf{v}_{\perp}^t| = O(1)|t^j - t^{j+1}|^2(|v|^3 + 1).$$  \hfill (5.78)
From [5.78] we have
\[
\left| \frac{1}{r^2} - \frac{1}{r^{\ell + 1}} \right| = \frac{1}{|v|} \left| \frac{v^{\ell + 1} - v_{\perp}^\ell}{|v_{\perp}|} \right| \lesssim \frac{|t^\ell - t^{\ell + 1}|(|v|^2 + 1)}{|v_{\perp}|^2} \lesssim 1,
\]
(5.79)
and
\[
\left| 1 - \frac{r^\ell}{r^{\ell + 1}} \right| = \frac{|v^{\ell + 1} - v_{\perp}^\ell|}{|v_{\perp}|} \lesssim \frac{|t^\ell - t^{\ell + 1}|(|v|^2 + 1)}{|v_{\perp}|} \lesssim r^\ell.
\]
(5.80)
Thus
\[
|\mathcal{P}^{-1}(r^{\ell + 1})\mathcal{P}(r^\ell)| \leq \begin{bmatrix} 1 & 0 & \frac{M}{|v|} & 0 & 0 & \frac{M}{|v|} r^\ell \\ 0 & 1 & \frac{M}{|v|} & 0 & 0 & \frac{M}{|v|} r^\ell \\ 0 & 0 & 1 + Mr^\ell & 0 & 0 & Mr^\ell \\ 0 & 0 & M & 1 & Mr^\ell & 0 \\ 0 & 0 & M & 0 & 1 & Mr^\ell \\ 0 & 0 & M & 0 & 0 & 1 + Mr^\ell \end{bmatrix} := Q(r^\ell).
\]

Now we have
\[
J(r^\ell) \times \cdots \times J(r^{\ell + 1}) \times J(r^\ell) \times \cdots \times J(r^2)
\]
\[
\leq \mathcal{P}(r^\ell)A(r^{\ell - 1})A(r^{\ell - 1}) \cdots \mathcal{P}(r^\ell)A(r^{\ell}) \cdots \mathcal{P}(r^2)A(r^2)\mathcal{P}^{-1}(r^2)
\]
\[
\leq \prod_{j=2}^{\ell} (1 + 10Mr^j)\mathcal{P}(r^j)\mathcal{P}(r^j) \cdots \mathcal{P}(r^2)\mathcal{P}^{-1}(r^2)
\]
\[
\leq C^{(t-s)|v|}\mathcal{P}(r^\ell)Q(r^{\ell - 1}) \cdots \mathcal{P}(r^2)\mathcal{P}^{-1}(r^2),
\]
where we have used \(A(r^j) \leq (1 + 10Mr^j)\mathbf{Id}_{6,6}\), and
\[
\prod_{j=2}^{\ell} (1 + 10Mr^j) \leq \prod_{i=1}^{\ell} \prod_{j=\ell-i}^{\ell} (1 + 10Mr^j) \lesssim \prod_{i=1}^{\ell} (1 + 10Mr) \lesssim C^{(t-s)|v|}.
\]

Next we estimate \(Q(r^{\ell - 1}) \cdots Q(r^2)\). First by diagonalization we have
\[
Q(r) = R(r)\mathcal{B}(r)R^{-1}(r)
\]
\[
= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \frac{1}{|v|} \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{|v|} \\ 0 & 0 & 1 & 0 & -r & r \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.
\]
(5.82)

Thus
\[
\prod_{j=2}^{\ell} Q(r^j) \leq \prod_{i=1}^{\ell} \prod_{j=\ell-i}^{\ell} Q(r^j) \lesssim \prod_{i=1}^{\ell} \prod_{j=\ell-i}^{\ell} [Q(r_i)]^{\ell_i - \ell_{i-1}} \leq \prod_{i=1}^{\ell} \mathcal{R}(r_i)[\mathcal{B}(r_i)]^{\ell_i - \ell_{i-1}} \mathcal{R}^{-1}(r_i),
\]

note that for some \(C \gg 1\)
\[
[\mathcal{B}(r_i)]^{\ell_i - \ell_{i-1}} \lesssim [\mathcal{B}(r_i)]^{\frac{C}{|v|}} \lesssim \text{diag}[1, 1, 1, 1, 1, C].
\]
(5.83)
Next we have again by explicit computation and using \(|\frac{r}{r_i+1}| \lesssim C_\xi

\mathcal{R}^{-1}(r_{i+1})\mathcal{R}(r_i) = 
\begin{bmatrix}
1 & 0 & 0 & 0 & \frac{r}{r_i+1} - 1 & -\frac{1}{r_i} (\frac{r}{r_i+1} - 1) \\
0 & 1 & 0 & 0 & \frac{r}{r_i+1} - 1 & -\frac{1}{r_i} (\frac{r}{r_i+1} - 1) \\
0 & 0 & 1 & 0 & \frac{1}{2} (\frac{r}{r_i+1} - 1) & -\frac{1}{2} (\frac{r}{r_i+1} - 1) \\
0 & 0 & 0 & 1 & \frac{1}{2} (\frac{r}{r_i+1} + 1) & -\frac{1}{2} (\frac{r}{r_i+1} - 1) \\
0 & 0 & 0 & 0 & -\frac{1}{2} (\frac{r}{r_i+1} - 1) & -\frac{1}{2} (\frac{r}{r_i+1} + 1) \\
\end{bmatrix}

(5.84)

\begin{bmatrix}
\frac{C_\xi}{2} & C_\xi & 0 & 0 & 0 & 0 \\
0 & \frac{C_\xi}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}

\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 2C_\xi \\
\end{bmatrix}

\begin{bmatrix}
0 & 0 & 0 & 0 & -\frac{1}{r_i} C_\xi & -\frac{1}{r_i} C_\xi \\
1 & 0 & 0 & 0 & -\frac{1}{r_i} C_\xi & -\frac{1}{r_i} C_\xi \\
0 & 1 & 0 & 0 & -\frac{(2C_\xi+C_\xi)}{2} C_\xi & -\frac{(2C_\xi+C_\xi)}{2} C_\xi \\
0 & 1 & 0 & 0 & \frac{1}{2} C_\xi & \frac{1}{2} C_\xi \\
0 & 0 & 0 & 1 & \frac{1}{2} C_\xi & \frac{1}{2} C_\xi \\
0 & 0 & 0 & 0 & \frac{1}{2} C_\xi & \frac{1}{2} C_\xi \\
\end{bmatrix}

(5.85)

Again we diagonalize \(S\) as

\[ S = \mathcal{F} \mathcal{A} \mathcal{F}^{-1} \]

\[ := \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}

\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 2C_\xi \\
\end{bmatrix}

\begin{bmatrix}
0 & 0 & 0 & 0 & -\frac{1}{r_i} C_\xi & -\frac{1}{r_i} C_\xi \\
1 & 0 & 0 & 0 & -\frac{1}{r_i} C_\xi & -\frac{1}{r_i} C_\xi \\
0 & 1 & 0 & 0 & -\frac{(2C_\xi+C_\xi)}{2} C_\xi & -\frac{(2C_\xi+C_\xi)}{2} C_\xi \\
0 & 1 & 0 & 0 & \frac{1}{2} C_\xi & \frac{1}{2} C_\xi \\
0 & 0 & 0 & 1 & \frac{1}{2} C_\xi & \frac{1}{2} C_\xi \\
0 & 0 & 0 & 0 & \frac{1}{2} C_\xi & \frac{1}{2} C_\xi \\
\end{bmatrix} \mathcal{F}^{-1}, \]

and directly

\[ S^{(\frac{r}{r_i+1})} = \mathcal{F} \mathcal{A}^{(\frac{r}{r_i+1})} \mathcal{F}^{-1} \]

\[ = \mathcal{F} \text{diag}[0, 1, 1, 1, 1, (2C_\xi)^{\frac{r}{r_i+1}}] \mathcal{F}^{-1} \]

\[ = \begin{bmatrix}
1 & 0 & 0 & 0 & \frac{1}{r_i} C_\xi & \frac{1}{r_i} C_\xi \\
0 & 1 & 0 & 0 & \frac{1}{r_i} C_\xi & \frac{1}{r_i} C_\xi \\
0 & 0 & 1 & 0 & \frac{1}{r_i} C_\xi & \frac{1}{r_i} C_\xi \\
0 & 0 & 0 & 1 & \frac{1}{r_i} C_\xi & \frac{1}{r_i} C_\xi \\
0 & 0 & 0 & 0 & \frac{1}{r_i} C_\xi & \frac{1}{r_i} C_\xi \\
\end{bmatrix} \]

\[ \mathcal{F} \text{diag}[0, 1, 1, 1, 1, (2C_\xi)^{\frac{r}{r_i+1}}] \mathcal{F}^{-1} \]

\[ = \frac{1}{2} \begin{bmatrix}
1 & 0 & 0 & 0 & \frac{1}{r_i} C_\xi & \frac{1}{r_i} C_\xi \\
0 & 1 & 0 & 0 & \frac{1}{r_i} C_\xi & \frac{1}{r_i} C_\xi \\
0 & 0 & 1 & 0 & \frac{1}{r_i} C_\xi & \frac{1}{r_i} C_\xi \\
0 & 0 & 0 & 1 & \frac{1}{r_i} C_\xi & \frac{1}{r_i} C_\xi \\
0 & 0 & 0 & 0 & \frac{1}{r_i} C_\xi & \frac{1}{r_i} C_\xi \\
\end{bmatrix} \]

\( := \mathcal{D} \).

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Therefore from (5.83) and (5.86) we have for some $C_1 \gg 1$,

$$
\prod_{j=2}^{\ell_*} Q(r^j) \leq C_1 \left( \frac{1}{|v|} C_{\frac{1}{l_e}} R(r_{1,2}) \right) F A \left( \frac{1}{l_e} \right) F^{-1}(r_1) R^{-1}(r_1) \leq C_1 R(r_{1,2}) D R^{-1}(r_1) \quad (5.87)
$$

Finally using (5.81), for $C_2 \gg 1$,

$$
\prod_{j=2}^{\ell_*} Q(r^j) P(r^j) \prod_{j=2}^{\ell_*} Q(r^j) P^{-1}(r^j) \leq C_2 C_2(\ell-s) C \left( \frac{1}{|v|} C_{\frac{1}{l_e}} R(r_{1,2}) \right) D R^{-1}(r_1) P^{-1}(r_1) \quad (5.88)
$$

where we have used (5.74) and the Velocity lemma (Lemma 3) and (5.2) and

$$
r_i = C_1 e^{\frac{3}{4} C_{\frac{1}{l_e}}} r^i \leq e^{C(\ell-s)} \frac{|v|}{|v|} \quad \text{and} \quad \frac{r_i}{r_1} = \frac{r_i}{r_1} \frac{r_1}{r_1} = \frac{r_i}{r_1} \frac{r_1}{r_1} \leq C_1 e^{\frac{3}{4} C_{\frac{1}{l_e}}} |v|.$$
Now we only left to prove
\[ \text{Step 8. Intermediate summary for the matrix method and the final estimate for Type III} \]

We expand \( \xi \)

Thus

Then directly from (5.89) we have for some \( C \gg 1 \),

\[
\begin{bmatrix}
1 & 2 & 5 & 1 & 1 & 1 \\
1 & 1 & 3 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 5 & 1 & 1 & 1 & 1 \\
1 & 5 & 1 & 1 & 1 & 1 \\
1 & 5 & 1 & 1 & 1 & 1 \\
\end{bmatrix} \leq C \]

\[ \leq C \]

Step 8. Intermediate summary for the matrix method and the final estimate for Type III
Recall from \([5.25]\) and \([5.39], [5.88], \) \([5.40]\),
\[
\frac{\partial (s^\ell_\tau, \mathbf{X}_{\ell_\tau}(s^\ell_\tau), \mathbf{V}_{\ell_\tau}(s^\ell_\tau))}{\partial (s^\ell_1, \mathbf{X}_1(s^\ell_1), \mathbf{V}_1(s^\ell_1))} = \frac{\partial (s^\ell_\tau, \mathbf{X}_{\ell_\tau}(s^\ell_\tau), \mathbf{V}_{\ell_\tau}(s^\ell_\tau))}{\partial (s^\ell_1, \mathbf{X}_1(s^\ell_1), \mathbf{V}_1(s^\ell_1))}.
\]

Then directly since \(|v| > \delta\), we bound it by
\[
\leq \frac{5.36}{C^{[t-s]|v|}} \times \frac{C^{[t-s]|v|}}{O(1)}.
\]

where we have used the Velocity lemma (Lemma 3) and \([5.73], [5.72]\), and
\[
|v||t^\ell - s^\ell| \lesssim |\mathbf{v}_1| \langle t-s||v||v| \rangle \lesssim C^{[t-s]|v|} \min\left\{\frac{|\mathbf{v}_1|}{|v|}, 1\right\}.
\]

Again we use the Velocity lemma (Lemma 3, \([5.72]\)), and
\[
|v||t^\ell - s^\ell| \leq \min\{|v||t^\ell - t^\ell+1|, |t-s||v|\} \lesssim \min\left\{\frac{|\mathbf{v}_1|}{|v|}, |t-s||v|\right\} \lesssim C^{[t-s]|v|} \min\left\{\frac{|\mathbf{v}_1|}{|v|}, 1\right\},
\]

and \(|\mathbf{v}_\perp(s^\ell_\tau)| \lesssim C^{[v](s-t)}|\mathbf{v}_\perp|\) to have, from \([5.92]\)
\[
\frac{\partial (s^\ell_\tau, \mathbf{X}_{\ell_\tau}(s^\ell_\tau), \mathbf{V}_{\ell_\tau}(s^\ell_\tau))}{\partial (s^\ell_1, \mathbf{X}_1(s^\ell_1), \mathbf{V}_1(s^\ell_1))} \leq C^{[t-s]|v|}.
\]

We consider the following case:

There exists \(\ell \in \ell_* (s; t, x, v), 0\) such that \(r^\ell \geq \sqrt{\delta}\).

Therefore \(\ell\) is Type III in \([5.19]\). Equivalently \(\tau \in [t^\ell+1, t^\ell]\) for some \(\ell_* \leq \ell \leq 0\) and \(|\xi(X_\alpha(\tau; t, x, v))| \geq C\delta\).

By the Velocity lemma (Lemma 3), for all \(1 \leq i \leq \ell_* (s; t, x, v),\)
\[
|\mathbf{v}_\perp^i| \geq \delta \geq C_\xi \langle v\rangle |t^\ell - t^\ell+1| |\mathbf{v}_\perp^i| \geq \delta \geq C_\xi \langle v\rangle |t^\ell - t^\ell+1| \sqrt{\delta}.
\]

Especially, for all \(1 \leq i \leq \ell_* (s; t, x, v),\)
\[
|\mathbf{v}_\perp^i| \gtrsim \delta \geq C_\xi \langle v\rangle |t^\ell - t^\ell+1| \sqrt{\delta}, \quad \frac{1}{|\mathbf{v}_\perp^i|} = \frac{1}{|\mathbf{v}_\perp^i|} \lesssim \delta \frac{C_\xi \langle v\rangle |t^\ell - t^\ell+1|}{\sqrt{\delta}}.
\]

Note that \(\ell_* (s; t, x, v) \lesssim \max \{\frac{|\tau^\ell - t^\ell+1|}{|\mathbf{v}_\perp^i|} \lesssim \delta C_\xi \langle v\rangle |t^\ell - t^\ell+1|\}.\)
Now for \(|v| < \delta\), we have

\[
\begin{align*}
\frac{\partial}{\partial s} X_t(s, \ell, \mathbf{V}_t(s)) \lesssim & \delta, \xi \\
\end{align*}
\]

Therefore in the case of (5.94), from (5.93),

\[
\begin{align*}
\frac{\partial}{\partial s} X_t(s, \ell, \mathbf{V}_t(s)) \lesssim & \delta, \xi \\
\end{align*}
\]
Now let’s address the derivatives \( \partial_x t^\ell \) and \( \partial_v t^\ell \) for any \( 1 \leq \ell \leq \ell^* \), as we will need it later. For \( |v| > \delta \), we compute [the first row of \( (5.88) \times (5.40) \) : \( \frac{\partial(s^i, x_i, s^i, x_i, v_1, s^1, v_1, s^1)}{\partial(t, x, v)} \)] and use (5.2) to get

\[
\begin{bmatrix}
\partial_x t^\ell \\
\partial_v t^\ell
\end{bmatrix} \lesssim \begin{bmatrix}
|v|^2 \\
|v|^3 \\
\frac{1}{|v| V_1^2}
\end{bmatrix} \begin{bmatrix}
0_{1,3} & 0_{1,3} \\
1 & |t - s^1| \\
|v|
\end{bmatrix} \lesssim \begin{bmatrix}
\frac{|v|}{|v|^2 V_1^2} \\
\frac{1}{|v| V_1^2}
\end{bmatrix}.
\]

(5.97)

And similarly, for \( |v| \leq \delta \), we compute [the first row of \( (5.91) \times (5.40) \) : \( \frac{\partial(s^i, x_i, s^i, x_i, v_1, s^1, v_1, s^1)}{\partial(t, x, v)} \)] and use (5.2) to get

\[
\begin{bmatrix}
\partial_x t^\ell \\
\partial_v t^\ell
\end{bmatrix} \lesssim \begin{bmatrix}
\frac{1}{|v| V_1^2} \\
\frac{1}{|v| V_1^2}
\end{bmatrix} \begin{bmatrix}
0_{1,3} & 0_{1,3} \\
1 & |t - s^1| \\
|v|
\end{bmatrix} \lesssim \begin{bmatrix}
\frac{1}{|v| V_1^2} \\
\frac{1}{|v| V_1^2}
\end{bmatrix}.
\]

(5.98)

We remark \( \partial x_{\perp \ell^*} \) and \( \partial v_{\perp \ell^*} \) have desired bounds but \( \partial x_{\parallel \ell^*} \) and \( \partial v_{\parallel \ell^*} \) still have undesired bounds in (5.93), (5.96).

We only need to consider the remaining cases, i.e. \( \ell \) is Type I or Type II. Note that in either case the moving frame (p^\ell – spherical coordinate) is well-defined for all \( \tau \in [s, t] \). In next two step we use the ODE method to refine the submatrix of (5.93) and (5.96):

\[
\begin{bmatrix}
\frac{\partial(x_{\parallel \ell^*}, s^{\ell^*})}{\partial(x_{\perp \ell^*}, s^{\ell^*})} \\
\frac{\partial(v_{\parallel \ell^*}, s^{\ell^*})}{\partial(x_{\perp \ell^*}, s^{\ell^*})}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial x_{\parallel \ell^*}}{\partial x_{\perp \ell^*}}(s^{\ell^*}) & \frac{\partial x_{\parallel \ell^*}}{\partial v_{\perp \ell^*}}(s^{\ell^*}) & \frac{\partial x_{\parallel \ell^*}}{\partial v_{\perp \ell^*}}(s^{\ell^*}) & \frac{\partial x_{\parallel \ell^*}}{\partial v_{\parallel \ell^*}}(s^{\ell^*}) \\
\frac{\partial v_{\parallel \ell^*}}{\partial x_{\perp \ell^*}}(s^{\ell^*}) & \frac{\partial v_{\parallel \ell^*}}{\partial v_{\perp \ell^*}}(s^{\ell^*}) & \frac{\partial v_{\parallel \ell^*}}{\partial v_{\perp \ell^*}}(s^{\ell^*}) & \frac{\partial v_{\parallel \ell^*}}{\partial v_{\parallel \ell^*}}(s^{\ell^*})
\end{bmatrix}
\]

Step 9. ODE method within the time scale \( |t - s||v| \approx L_\xi \)

Recall the end points (time) of intermediate groups from (5.23):

\[
s < \ell^* < t - \frac{\ell^* |t - s||v|}{|v|} + 1 < \ell^* < t - \frac{\ell^* |t - s||v|}{|v|} - 1 + 1 < \ell^* < t - \ell^* - 1 + 1 < \ell^* < t^\ell_{1+1} < \ell^* < t^\ell_2 < \ell^* < t^\ell_3 < \ell^* < t^\ell_4 < \ell^* < t^\ell_5 < \ell^* < t,
\]

where the underbraced numbering indicates the index of the intermediate group. We further choose points independently on \( (t, x, v) \) for all \( i = 1, 2, \ldots, \left[ \frac{|t - s||v|}{L_\xi} \right] \):

\[
t^\ell_{i+1} < s < t^\ell_i, \\
t^\ell_{2+1} < s^2 < t^\ell_2, \\
\vdots \\
t^\ell_{i+1} < s^i < \ell^* < \ell^* < \ell^* < \ell^* < s^i < \ell^* - 1, \\
\vdots \\
t^\ell_{\left[ \frac{|t - s||v|}{L_\xi} \right] + 1} < s < \ell^* < \ell^* < \ell^* < \ell^* < \ell^* < s^i < \ell^* - 1 < \ell^* - \frac{|t - s||v|}{L_\xi}.
\]

We claim the following estimate at \( s^{i+1} \) via \( s^i \). Within the \( i \)-th intermediate group, we fix \( p^\ell_{\text{spherical coordinate}} \) in Step 9. The goal is to estimate derivatives with respect to initial (\( x_1, v_1 \)) at \( s^{i+1} \) in terms of

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\[ s^i. \text{ This is a different from previous steps.} \]

\[
\begin{bmatrix}
\frac{\partial x_{ij}}{\partial \nu_{lj}}(s^{i+1}) \\
\frac{\partial x_{ij}}{\partial \nu_{lj}}(s^{i+1}) \\
\frac{\partial x_{ij}}{\partial \nu_{lj}}(s^{i+1}) \\
\end{bmatrix}
\begin{bmatrix}
\frac{\partial x_{ij}}{\partial \nu_{lj}}(s^i) \\
\frac{\partial x_{ij}}{\partial \nu_{lj}}(s^i) \\
\frac{\partial x_{ij}}{\partial \nu_{lj}}(s^i) \\
\end{bmatrix}
\]

\[
\lesssim \delta, \xi \begin{bmatrix}
1 \\
|v| \\
1 \\
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
1 \\
\end{bmatrix}
\] + \( e^{C|v||t-s|} \begin{bmatrix}
1 \\
|v| \\
1 \\
\end{bmatrix}
\begin{bmatrix}
0 \\
|v(1 + \frac{|v|}{|v_1|})| \\
0 \\
\end{bmatrix}
\]

\[
\lesssim \delta, \xi \begin{bmatrix}
1 \\
|v| \\
1 \\
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
1 \\
\end{bmatrix}
\] + \( e^{C|v||t-s|} \begin{bmatrix}
1 \\
|v| \\
1 \\
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
1 \\
\end{bmatrix}
\] .

(5.99)

For the sake of simplicity we drop the index \( \ell_i \).

Denote, from (4.9),

\[
F_{\parallel}(x_{\perp}, x_{\parallel}, v_{\perp}, v_{\parallel}) := D(x_{\perp}, x_{\parallel}, v_{\parallel}) + H(x_{\perp}, x_{\parallel}, v_{\parallel})v_{\perp},
\]

(5.100)

where \( D \) is a \( r^3 \)-vector-valued function and \( H \) is a \( 3 \times 3 \) matrix-valued function:

\[
D(x_{\perp}, x_{\parallel}, v_{\parallel}) = \sum_i G_{ij}(x_{\perp}, x_{\parallel}) \frac{(-1)^{i+1}}{-n(x_{\parallel}) \cdot (\partial_1 \eta(x_{\parallel}) \times \partial_2 \eta(x_{\parallel}))}
\]

\[
\times \left\{ v_{\parallel} \cdot \nabla^2 \eta(x_{\parallel}) \cdot v_{\parallel} - x_{\perp} v_{\parallel} \cdot \nabla^2 n(x_{\parallel}) \cdot v_{\parallel} - E(s, -x_{\perp} n(x_{\parallel}) + \eta(x_{\parallel})) \right\} \cdot (-n(x_{\parallel}) \times \partial_{i+1} \eta(x_{\parallel}))
\]

and

\[
H(x_{\perp}, x_{\parallel}, v_{\parallel}) = \sum_i G_{ij}(x_{\perp}, x_{\parallel}) \frac{(-1)^{i+1}}{-n(x_{\parallel}) \cdot (\partial_1 \eta(x_{\parallel}) \times \partial_2 \eta(x_{\parallel}))} 2v_{\parallel} \cdot \nabla n(x_{\parallel}) \cdot (-n(x_{\parallel}) \times \partial_{i+1} \eta(x_{\parallel})).
\]

Note that \( H \) is linear in \( v_{\parallel} \). Here \( G_{ij}(\cdot, \cdot) \) is a smooth bounded function defined in (4.10) and we used the notational convention \( i \equiv i \mod 2 \).

From Lemma [6] we take the time integration of (4.7) along the characteristics to have

\[
x_{\parallel}(s^{i+1}) = x_{\parallel}(s^i) - \int_{s^{i+1}}^{s^i} v_{\parallel}(\tau) d\tau,
\]

\[
v_{\parallel}(s^{i+1}) = v_{\parallel}(s^i) - \int_{s^{i+1}}^{s^i} \left\{ H(x_{\perp}(\tau), x_{\parallel}(\tau), v_{\parallel}(\tau))v_{\perp}(\tau) + D(x_{\perp}(\tau), x_{\parallel}(\tau), v_{\parallel}(\tau)) \right\} d\tau.
\]

Note that \( v_{\perp}(\tau) \) is not continuous with respect to the time \( \tau \). Using (4.7) we rewrite this time integration as

\[
\int_{s^{i+1}}^{s^i} H(x_{\perp}(\tau), x_{\parallel}(\tau), v_{\parallel}(\tau))v_{\perp}(\tau) d\tau = \int_{s^{i+1}}^{s^i} + \sum_{\ell=i-1}^{\ell-1+1} \int_{s^{i+1}}^{s^i} + \int_{s^{i+1}}^{s^{i+1}}.
\]

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then we use \( \mathbf{v}_\perp(\tau) = \dot{\mathbf{x}}_\perp(\tau) \) and the integration by parts to have

\[
\begin{align*}
\int_{t_{\ell-1}^\tau}^{t_{\ell}^\tau} H(\mathbf{x}_\perp(\tau), \mathbf{x}_\parallel(\tau), \mathbf{v}_\parallel(\tau))\dot{\mathbf{x}}_\perp(\tau)d\tau + \sum_{\ell=\ell-1}^{\ell+1} \int_{t_{\ell}^\tau}^{t_{\ell+1}^\tau} H(\mathbf{x}_\perp(\tau), \mathbf{x}_\parallel(\tau), \mathbf{v}_\parallel(\tau))\dot{\mathbf{x}}_\perp(\tau)d\tau \\
+ \int_{s_{\tau-1}}^{s_{\tau}} H(\mathbf{x}_\perp(\tau), \mathbf{x}_\parallel(\tau), \mathbf{v}_\parallel(\tau))\dot{\mathbf{x}}_\perp(\tau)d\tau \\
= H(s^\tau)\mathbf{x}_\perp(s^\tau) - H(t_{\ell-1}^\tau)\mathbf{x}_\perp(t_{\ell-1}^\tau) + \int_{t_{\ell-1}^\tau}^{t_{\ell}^\tau} \left[ \mathbf{v}_\perp(\tau), \mathbf{v}_\parallel(\tau), F_\parallel(\tau) \right] \cdot \nabla H(\mathbf{x}_\perp(\tau))d\tau \\
+ \sum_{\ell=\ell-1}^{\ell+1} H(t_{\ell}^\tau)\mathbf{x}_\perp(t_{\ell}^\tau) - H(t_{\ell+1}^\tau)\mathbf{x}_\perp(t_{\ell+1}^\tau) - \int_{t_{\ell}^\tau}^{t_{\ell+1}^\tau} \left[ \mathbf{v}_\perp(\tau), \mathbf{v}_\parallel(\tau), F_\parallel(\tau) \right] \cdot \nabla H(\mathbf{x}_\perp(\tau))d\tau \\
+ \int_{s_{\tau-1}}^{s_{\tau}} \left[ \mathbf{v}_\perp(\tau), \mathbf{v}_\parallel(\tau), F_\parallel(\tau) \right] \cdot \nabla H(\mathbf{x}_\perp(\tau))d\tau,
\end{align*}
\]

where we have used the fact \( X_{\partial t}(t_{\ell}^\tau) \in \partial \Omega \) (therefore \( \mathbf{x}_\perp(t_{\ell}^\tau) = 0 \)) and the notation \( H(\tau) = H(\mathbf{x}_\perp(\tau), \mathbf{x}_\parallel(\tau), \mathbf{v}_\parallel(\tau)) \), \( D(\tau) = D(\mathbf{x}_\perp(\tau), \mathbf{x}_\parallel(\tau), \mathbf{v}_\parallel(\tau)) \), \( F_\parallel(\tau) = F_\parallel(\mathbf{x}_\perp(\tau), \mathbf{x}_\parallel(\tau), \mathbf{v}_\parallel(\tau)) \).

Overall we have

\[
\begin{align*}
\mathbf{x}_\perp(s_{\tau+1}^\tau) &= \mathbf{x}_\perp(s^\tau) - \int_{s^\tau}^{s_{\tau+1}^\tau} \mathbf{v}_\parallel(\tau)d\tau, \\
\mathbf{v}_\perp(s_{\tau+1}^\tau) &= \mathbf{v}_\perp(s^\tau) - H(s^\tau)\mathbf{x}_\perp(s^\tau) + H(s_{\tau+1}^\tau)\mathbf{x}_\perp(s_{\tau+1}^\tau) \\
&\quad + \int_{s_{\tau+1}^\tau}^{s_{\tau+1}^\tau} \left[ \mathbf{v}_\perp(\tau), \mathbf{v}_\parallel(\tau), F_\parallel(\tau) \right] \cdot \nabla H(\mathbf{x}_\perp(\tau))d\tau - \int_{s_{\tau}^\tau}^{s_{\tau+1}^\tau} D(\tau)d\tau.
\end{align*}
\]

Denote

\[
\partial = \left[ \partial_{x_\perp(s^\tau)}, \partial_{x_\parallel(s^\tau)}, \partial_{v_\perp(s^\tau)}, \partial_{v_\parallel(s^\tau)} \right] = \left[ \frac{\partial}{\partial x_\perp(s^\tau)}, \frac{\partial}{\partial x_\parallel(s^\tau)}, \frac{\partial}{\partial v_\perp(s^\tau)}, \frac{\partial}{\partial v_\parallel(s^\tau)} \right].
\]

We claim that, in a sense of distribution on \( (s^\tau, \mathbf{x}_\perp(s^\tau), \mathbf{x}_\parallel(s^\tau), \mathbf{v}_\perp(s^\tau), \mathbf{v}_\parallel(s^\tau)) \in [0, \infty) \times (0, C_\xi) \times (0, 2\pi) \times (0, \pi - \delta) \times \mathbb{R} \times \mathbb{R}^2 \),

\[
\left[ \partial x_\perp(s^\tau+1; s^\tau, \mathbf{x}(s^\tau), \mathbf{v}(s^\tau)), \partial x_\parallel(s^\tau+1; s^\tau, \mathbf{x}(s^\tau), \mathbf{v}(s^\tau)), \partial v_\perp(s^\tau+1; s^\tau, \mathbf{x}(s^\tau), \mathbf{v}(s^\tau)), \partial v_\parallel(s^\tau+1; s^\tau, \mathbf{x}(s^\tau), \mathbf{v}(s^\tau)) \right] = \sum_{\ell} \mathbf{1}_{[t_{\ell+1}^\tau, t_{\ell}^\tau)}(s^\tau+1; s^\tau) \{ \partial \mathbf{v}_\perp \mathbf{x}_\perp + \partial \mathbf{v}_\parallel \mathbf{x}_\parallel \}
\]

i.e. the distributional derivatives of \( \mathbf{x}_\perp, \mathbf{x}_\parallel, \mathbf{v}_\perp, \mathbf{v}_\parallel \) and \( \mathbf{v}_\perp \mathbf{x}_\perp, \mathbf{v}_\parallel \mathbf{x}_\parallel \) equal the piecewise derivatives.

**Proof of (5.102).** Let \( \phi(\tau', \mathbf{x}_\perp, \mathbf{x}_\parallel, \mathbf{v}_\perp, \mathbf{v}_\parallel) \in C^\infty_c((0, \infty) \times (0, C_\xi) \times 2^2 \times \mathbb{R} \times \mathbb{R}^2) \). Therefore \( \phi \equiv 0 \) when \( \mathbf{x}_\perp < \delta, |v| > \frac{1}{2} \). For \( \mathbf{x}_\perp \geq \delta \) we use the proof of Lemma 7. For \( x = \eta(x_\parallel) + \mathbf{x}_\perp [-\mathbf{n}(x_\parallel)] \),

\[
\mathbf{x}_\parallel \leq \gamma \varepsilon x(x) + \mathbf{x}_\perp [-\mathbf{n}(x_\parallel)] \leq \varepsilon |\mathbf{x}_\perp|,
\]

and therefore \( \xi(x) \geq \varepsilon x(x) + \mathbf{x}_\perp [-\mathbf{n}(x_\parallel)] \) is \( C_\xi, E \). By the Velocity lemma, for \( (x, v) \in \text{supp}(\phi) \)

\[
\alpha(x, v) \geq \varepsilon e^{-C(|v|+1)|t-t'|} \alpha(x, v) \geq \varepsilon e^{-C(|v|+1)|t-t'|} \geq \varepsilon e^\frac{-C(|v|+1)|t-t'|}{\varepsilon},
\]

where we used the fact that \( \phi \) vanishes away from a compact subset \( \text{supp}(\phi) \). Therefore \( \alpha(t', x, v) = \alpha(t', x_\perp, x_\parallel, \mathbf{v}_\perp, \mathbf{v}_\parallel) \) is \( C^1 \) with respect to \( \mathbf{x}_\perp, \mathbf{x}_\parallel, \mathbf{v}_\perp, \mathbf{v}_\parallel \) locally on \( \text{supp}(\phi) \) and therefore \( M = \{ (\tau', x, v) \in \text{supp}(\phi) : t' = t'(t, x, v) \} \) is a \( C^1 \) manifold.
It suffices to consider the case \( |\tau' - t^f(t, x, v)| \ll 1 \). Denote \( \partial_e \in \{ \partial_{x^1}, \partial_{x^1, 1}, \partial_{x^2, 1}, \partial_{v^1}, \partial_{v^1, 1}, \partial_{v^2, 2} \} \) and \( n_M = e_1 \) to have

\[
\int_{\{ (\tau', x, v) \in \text{supp}(\phi) \}} [\partial_e x_{\perp}(\tau'; t, x, v), \partial_e x_{\parallel}(\tau'; t, x, v), \partial_e v_{\parallel}(\tau'; t, x, v)] \phi(\tau', x, v)dx dv d\tau'
\]

\[
= \int_{\tau' < t^f} + \int_{\tau' \geq t^f}
\]

\[
= \int_{\mathcal{M}} \left( \lim_{\tau' \uparrow t^f} [x_{\perp}(\tau'), x_{\parallel}(\tau'), v_{\parallel}(\tau')] - \lim_{\tau' \downarrow t^f} [x_{\perp}(\tau'), x_{\parallel}(\tau'), v_{\parallel}(\tau')] \right) \phi(\tau', x, v) \{ e \cdot n_M \} dx dv
\]

\[
- \int_{\{ \tau' \neq t^f(t, x, v) \}} [x_{\perp}(\tau'), x_{\parallel}(\tau'), v_{\parallel}(\tau')] \partial_e \phi(\tau', x, v) d\tau' dv dx
\]

\[
= - \int_{\{ \tau' \neq t^f(t, x, v) \}} [x_{\perp}(\tau'; t, x, v) v_{\perp}(\tau'; t, x, v)] \partial_e \phi(\tau', x, v) d\tau' dv dx
\]

where we used the continuity of \( [x_{\perp}(\tau'; t, x, v), x_{\parallel}(\tau'; t, x, v), v_{\parallel}(\tau'; t, x, v)] \) in terms of \( \tau' \) near \( t^f(t, x, v) \).

Note that \( v_{\perp}(\tau'; t, x, v) \) is discontinuous around \( |\tau' - t^f| \ll 1 \). However, with crucial \( x_{\perp}(\tau') \) multiplication we have \( x_{\perp}(t^f) v_{\perp}(t^f) = 0 \) and therefore

\[
\int_{\{ (\tau', x, v) \in \text{supp}(\phi) \}} [\partial_e x_{\perp}(\tau'; t, x, v) v_{\perp}(\tau'; t, x, v)] \phi(\tau', x, v) dx dv d\tau'
\]

\[
= \int_{\tau' < t^f} + \int_{\tau' \geq t^f}
\]

\[
= \int_{\mathcal{M}} \left( \lim_{\tau' \uparrow t^f} [x_{\perp}(\tau') v_{\perp}(\tau')] - \lim_{\tau' \downarrow t^f} [x_{\perp}(\tau') v_{\perp}(\tau')] \right) \phi(\tau', x, v) \{ e \cdot n_M \} dx dv
\]

\[
- \int_{\{ \tau' \neq t^f(t, x, v) \}} [x_{\perp}(\tau') v_{\perp}(\tau')] \partial_e \phi(\tau', x, v) d\tau' dv dx
\]

\[
= - \int_{\{ \tau' \neq t^f(t, x, v) \}} [x_{\perp}(\tau'; t, x, v) v_{\perp}(\tau'; t, x, v)] \partial_e \phi(\tau', x, v) d\tau' dv dx.
\]

This completes the proof of (5.102).

Since \( v_{\perp} \) always is multiplied with \( x_{\perp} \) in (5.101), we may apply (5.102) and take derivative inside each \( \int_{s^f} \) of (5.101), separating the main terms with \( \partial_e x_{\parallel} \) and \( \partial_e v_{\parallel} \), and treating the rest (underbraced terms)
as forcing terms to obtain, for \( \partial e \in \{ \partial \kappa_{\perp}, \partial \kappa_{\parallel}, \partial \nu_{\perp}, \partial \nu_{\parallel}, \partial \nu_{\perp}, \partial \nu_{\parallel} \} \),

\[
\partial e x_{\parallel}(s^{i+1}) = \partial e x_{\parallel}(s^i) - \int_{s^{i+1}}^{s^i} \partial e x_{\parallel}(\tau)d\tau,
\]

\[
\partial v_{\parallel}(s^{i+1}) = \partial e H(s^{i+1})x_{\perp}(s^{i+1}) + H(s^{i+1}) \partial e x_{\perp}(s^{i+1}) + \partial e v_{\parallel}(s^i) - \partial e [H(x_{\perp}, x_{\parallel}, v_{\parallel})x_{\perp}](s^{i+1})
\]

\[
+ \int_{s^{i+1}}^{s^i} \partial e v_{\parallel}(\tau) \partial \kappa_{\perp} H(\tau)x_{\perp}(\tau) + \partial e v_{\parallel}(\tau) \cdot \nabla x_{\perp} H(\tau)x_{\perp}(\tau)d\tau
\]

\[
+ \int_{s^{i+1}}^{s^i} \left\{ \left[ \partial e x_{\perp}(\tau) \partial \kappa_{\perp} H(\tau) + \partial e x_{\perp}(\tau) \cdot \nabla x_{\perp} H(\tau) + \partial e v_{\parallel}(\tau) \cdot \nabla v_{\parallel} H(\tau) \right] v_{\perp}(\tau)
\]

\[
+ H(\tau) \partial e v_{\perp}(\tau) + \partial e x_{\perp}(\tau) \partial \kappa_{\perp} D(\tau) + \partial e v_{\parallel}(\tau) \cdot \nabla x_{\perp} D(\tau) + \partial e v_{\parallel}(\tau) \cdot \nabla v_{\parallel} D(\tau) \right\} \cdot \nabla v_{\parallel} H(\tau)x_{\perp}(\tau)d\tau
\]

\[
+ \int_{s^{i+1}}^{s^i} \left\{ v_{\perp}(\tau) [\partial e x_{\perp}(\tau), \partial e v_{\parallel}(\tau)] \cdot \nabla x_{\perp} H(\tau) + v_{\parallel}(\tau) [\partial e x_{\perp}(\tau), \partial e v_{\parallel}(\tau)] \cdot \nabla v_{\parallel} H(\tau)
\]

\[
+ F_{\parallel}(\tau) \cdot [\partial e x_{\perp}(\tau), \partial e x_{\perp}(\tau), \partial e v_{\parallel}(\tau), \partial e v_{\parallel}(\tau)] \cdot \nabla v_{\parallel} H(\tau) \right\} x_{\perp}(\tau)d\tau
\]

\[
- \int_{s^{i+1}}^{s^i} \left[ \partial e x_{\perp}(\tau), \partial e x_{\perp}(\tau), \partial e v_{\parallel}(\tau) \right] \cdot \nabla D(\tau)d\tau
\]

\[
(5.103)
\]

Now we use (5.93) to control the underbraced term of (5.103). Notice that we cannot directly use (5.93) since now we fix the chart for whole \( i \)-th intermediate group but the estimate (5.93) is for the moving frame. (For clarity, we write the index for the chart for this part.) Note the time of bounces within the \( i \)-th intermediate group \( (|t_{\perp}| - t_{\parallel}||v|) \approx L(x) \) are

\[
t_{\perp}^{i+1} < s^{i+1} < t_{\parallel}^{i} < t_{\parallel}^{i-1} < \ldots < t_{\parallel}^{i-1} < s^{i} < t_{\parallel}^{i-1}.
\]

Now we apply (4.10) and (5.93) to bound, for \( \tau \in (s^{i+1}, s^i) \) and \( \ell \in \{ \ell_{\parallel}, \ell_{\perp} - 1, \ldots, \ell_{\perp} - 1 + 2, \ell_{\perp} - 1 + 1, \ell_{\perp} - 1 \} \)

\[
\frac{\partial(x_{\perp}(\tau), x_{\parallel}(\tau), v_{\perp}(\tau), v_{\parallel}(\tau))}{\partial(x_{\perp}(s^i), x_{\parallel}(s^i), v_{\perp}(s^i), v_{\parallel}(s^i))}
\]

\[
\leq e^{C|t-s|v} \left\{ \text{Id}_{0,6} + O_{\xi}(|p^\ell - p_{\ell-1}^\ell|) \right\} \left( \begin{array}{c|c|c|c|c|c} 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 \\
0 & |v| & |v| & 0 & 0 & 0 \\
0 & |v| & |v| & 0 & 0 & 0 \\
0 & |v| & |v| & 0 & 0 & 0 \end{array} \right) \left( \begin{array}{c|c|c|c|c|c} 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 \\
0 & |v| & |v| & 0 & 0 & 0 \\
0 & |v| & |v| & 0 & 0 & 0 \\
0 & |v| & |v| & 0 & 0 & 0 \end{array} \right)
\]

\[
\leq e^{C|t-s|v} \left[ \begin{array}{c|c|c|c|c|c} |v| + 1 & |v| + 1 & |v| + 1 & |v| + 1 & |v| + 1 & |v| + 1 \\
|v| + 1 & |v| + 1 & |v| + 1 & |v| + 1 & |v| + 1 & |v| + 1 \\
|v| + 1 & |v| + 1 & |v| + 1 & |v| + 1 & |v| + 1 & |v| + 1 \\
|v| + 1 & |v| + 1 & |v| + 1 & |v| + 1 & |v| + 1 & |v| + 1 \\
|v| + 1 & |v| + 1 & |v| + 1 & |v| + 1 & |v| + 1 & |v| + 1 \\
|v| + 1 & |v| + 1 & |v| + 1 & |v| + 1 & |v| + 1 & |v| + 1 \end{array} \right]
\]

\[
\leq e^{C|t-s|v} \left[ \begin{array}{c|c|c|c|c|c} 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \end{array} \right] \left( \begin{array}{c|c|c|c|c|c} 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \end{array} \right)
\]

\[
(5.104)
\]
where we have used $|p^t - p^s| \lesssim 1$.

We plug in (5.103) with (5.104) respectively with

$$|\partial_{x_1} v_\perp(\tau)| \lesssim \frac{|v| + 1}{|v_\perp|}, |\partial_{x_1} v_\perp(\tau)| \lesssim \frac{|v|^3 + 1}{|v_\perp|^2}, |\partial_{v_1} v_\perp(\tau)| \lesssim \min\{\frac{1}{|v|}, 1\}, |\partial_{v_1} v_\perp(\tau)| \lesssim 1,$$

and

$$|\nabla v_\perp H(\tau)| \lesssim 1, |\nabla_{x_1, x_2} H(\tau)| \lesssim |v| + 1, |\nabla v_\perp D(\tau)| \lesssim |v| + 1, |\nabla_{x_1, x_2} D(\tau)| \lesssim |v|^2 + 1,$$

and use the fact that $|s^t - s^{t+1}| \lesssim \frac{1}{|v_\perp|}$ by the way we define $s^t$. Collecting terms with tedious but straightforward bounds, we summarize the results as: for $s \in [s^{t+1}, s^t]$

$$\left[\begin{array}{c}
\frac{\partial x_1(s)}{\partial v_\perp} \\
\frac{\partial v_\perp(s)}{\partial v_\perp}
\end{array}\right] \lesssim \xi \left[\begin{array}{c}
\frac{\partial x_1(s^t)}{\partial v_\perp} + |v|\frac{\partial v_\perp(s^t)}{\partial x_1} \\
\frac{\partial v_\perp(s^t)}{\partial v_\perp}
\end{array}\right] + \left[\begin{array}{c}
\frac{f_s^t(|v|^2 + 1)}{\partial x_1} |v| + (|v| + 1)\frac{\partial v_\perp}{\partial v_\perp} \\
\frac{f_s^t(|v|^2 + 1)}{\partial v_\perp} |v| + (|v| + 1)\frac{\partial v_\perp}{\partial v_\perp}
\end{array}\right] + \left[\begin{array}{c}
0 \\
0
\end{array}\right].$$

From (5.105) we have

$$\langle v \rangle \left[\frac{\partial x_1(s)}{\partial x_1} + \frac{\partial v_\perp(s)}{\partial x_1}(s)\right] \lesssim \left[\frac{\partial v_\perp(s^t)}{\partial x_1} + |v|\frac{\partial v_\perp(s^t)}{\partial x_1} + e^{C|v||t-s|} |v|^2 + 1 \right] \left[\frac{\partial x_1}{\partial v_\perp} + |v|\frac{\partial v_\perp}{\partial v_\perp}ight] + \int_s^{s^t} \langle v \rangle \left[\frac{\partial x_1}{\partial x_1} + \frac{\partial v_\perp}{\partial v_\perp}\right],$$

from Gronwall inequality we get

$$\langle v \rangle \left[\frac{\partial x_1(s)}{\partial v_\perp} + \frac{\partial v_\perp(s)}{\partial v_\perp}(s)\right] \leq C\xi \left[\frac{\partial v_\perp(s^t)}{\partial x_1} + |v|\frac{\partial v_\perp(s^t)}{\partial x_1} + e^{C|v||t-s|} |v|^2 + 1 \right] \left[\frac{\partial x_1}{\partial v_\perp} + |v|\frac{\partial v_\perp}{\partial v_\perp}\right] + \int_s^{s^t} \langle v \rangle \left[\frac{\partial x_1}{\partial x_1} + \frac{\partial v_\perp}{\partial v_\perp}\right].$$

Iterating (5.107) we get

$$\langle v \rangle \left[\frac{\partial x_1(s)}{\partial v_\perp} + \frac{\partial v_\perp(s)}{\partial v_\perp}(s)\right] \leq C^2 \left(\frac{\partial v_\perp(s^t-1)}{\partial x_1} + |v|\frac{\partial v_\perp(s^t-1)}{\partial x_1}\right) + (C^2 + C)e^{C|v||t-s|} |v|^2 + 1 \left[\frac{\partial x_1}{\partial v_\perp} + |v|\frac{\partial v_\perp}{\partial v_\perp}\right] + \int_s^{s^t} \langle v \rangle \left[\frac{\partial x_1}{\partial x_1} + \frac{\partial v_\perp}{\partial v_\perp}\right].$$

And by the same argument as (5.106) - (5.108), we get from (5.108) that

$$\langle v \rangle \left[\frac{\partial x_1(s)}{\partial v_\perp} + \frac{\partial v_\perp(s)}{\partial v_\perp}(s)\right] \leq C|v||s||v_\perp|.$$

Therefore, from (5.108) and (5.109) we get

$$\left[\begin{array}{c}
\frac{\partial x_1(s)}{\partial v_\perp} \\
\frac{\partial v_\perp(s)}{\partial v_\perp}
\end{array}\right] \lesssim C|v||s||v_\perp| \left[\begin{array}{c}
\frac{v}{|v_\perp|} \\
\frac{v}{|v_\perp|}
\end{array}\right].$$
With the estimate $\underline{[5.110]}$, we refine $\underline{[5.93]}$ and $\underline{[5.96]}$ to give a final estimate for the case that some $\ell$ is Type I or Type II:

$$\frac{\partial(s^r, \mathbf{x}_\perp(s^r), \mathbf{v}_\parallel(s^r))}{\partial(s^1, \mathbf{x}_\perp(s^1), \mathbf{v}_\parallel(s^1))} \lesssim C^{|v|(t-s)} \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{|v|+1}{|\mathbf{v}|^2} & \frac{|v|+1}{|\mathbf{v}|^2} & \min\{|v|, \frac{|v|}{\mu}\} & \frac{1}{|v|} \\ \frac{|v|^2}{|\mathbf{v}|^2} & \frac{|v|^2}{|\mathbf{v}|^2} & 1 & \frac{1}{|v|} \\ \frac{|v|^2}{|\mathbf{v}|^2} & \frac{|v|^2}{|\mathbf{v}|^2} & |v|+1 & \frac{1}{|v|} O_\xi(1) \end{bmatrix}, \tag{5.111}$$

and from $\underline{[5.30]}$ and $\underline{[5.43]}$

$$\frac{\partial(X_{cl}(s; t, x, v), V_{cl}(s; t, x, v))}{\partial(t, x, v)} \lesssim C^{|v|(t-s)} \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{|v|+1}{|\mathbf{v}|^2} & \frac{|v|+1}{|\mathbf{v}|^2} & \min\{|v|, \frac{|v|}{\mu}\} & \frac{1}{|v|} \frac{1}{|t-s|^2} O_{1,3}(1) \\ \frac{|v|^2}{|\mathbf{v}|^2} & \frac{|v|^2}{|\mathbf{v}|^2} & 1 & \frac{1}{|v|} O_{1,3}(1) \end{bmatrix} \begin{bmatrix} 0 & 0_{1,3} & 0_{1,3} & 0_{1,3} \\ \frac{|v|+1}{|\mathbf{v}|^2} & \frac{|v|+1}{|\mathbf{v}|^2} & \min\{|v|, \frac{|v|}{\mu}\} & \frac{1}{|v|} \frac{1}{|t-s|^2} O_{1,3}(1) \end{bmatrix} \begin{bmatrix} 0 & 0_{1,3} & 0_{1,3} & 0_{1,3} \\ \frac{|v|^2}{|\mathbf{v}|^2} & \frac{|v|^2}{|\mathbf{v}|^2} & \frac{1}{|v|} & \frac{1}{|t-s|^2} O_{1,3}(1) \end{bmatrix}.$$

Finally from $\underline{[5.95]}$ and $\underline{[5.112]}$ we conclude, for all $\tau \in [s, t]$:

$$\begin{bmatrix} \frac{|v|^2}{|\mathbf{v}|^2} & \frac{|v|}{|\mathbf{v}|^2} & \frac{1}{|v|} \end{bmatrix} \lesssim C e^{C^{|v|(t-s)}} \begin{bmatrix} \frac{|v|^2}{|\mathbf{v}|^2} & \frac{|v|}{|\mathbf{v}|^2} & \frac{1}{|v|} \end{bmatrix} 6 \times 7.$$

From the Velocity lemma (Lemma $\underline{3}$),

$$|v| = |v| \cdot [-n(x^1)] = |V_{cl}(t^1; t, x, v) \cdot n(X_{cl}(t^1; t, x, v))| = \sqrt{\alpha(X_{cl}(t^1), V_{cl}(t^1))} \geq e^{C^{|v|t-t^1}} \alpha(t, x, v) \geq \alpha(t, x, v),$$

and this completes the proof. \hfill \Box

6. Weighted $C^1$ estimate

In this section, we put together all the results we get in previous sections and prove our main theorem.

**Proof of Theorem $\underline{7}$**. We use the approximation sequence $\underline{2.25}$ with $\underline{2.6}$. Due to $\underline{2.7}$ we have

$$\sup_m \sup_{0 \leq t \leq T} \|e^{\theta|v|^2} f^m(t)\|_\infty \lesssim_{\xi, T} P(\|e^{\theta|v|^2} f_0\|_\infty).$$

Now we claim that the distributional derivatives coincide with the piecewise derivatives. This is due to Proposition $\underline{1}$ with an invariant property of $\Gamma(f, f) = \Gamma_{\text{gain}}(f, f) - \nu(\sqrt{\mu} f) f :$ Assume $f^m(v) = f^{m-1}(\mathcal{O} v)$ holds for some orthonormal matrix. Then

$$\Gamma(f^m, f^m)(v) = \Gamma(f^{m-1}, f^{m-1})(\mathcal{O} v). \tag{6.1}$$
Denote

$$\nu^{m-\ell}(s) := \nu^{m-\ell}(s, X_{cl}(s), V_{cl}(s)) := \nu(\sqrt{m} f^{m-\ell})(s, X_{cl}(s), V_{cl}(s)) - \frac{V_{cl}(s)}{2} \cdot E(s, X_{cl}(s), V_{cl}(s)). \tag{6.2}$$

Using (6.1), we apply Proposition 1 to have

$$f^m(t, x, v) = e^{-\int_0^t \sum_{\ell=0}^{\ell_0} 1_{[\mu+\ell, \mu+\ell]}(s) \nu^{m-\ell}(s) ds} f_0(X_{cl}(0), V_{cl}(0))$$

$$+ \int_0^t \sum_{\ell=0}^{\ell_0} 1_{[\mu+\ell, \mu+\ell]}(s) e^{-\int_0^s \sum_{j=0}^{j_0} 1_{[\mu+\ell+1, \mu+\ell+1]}(\tau) \nu^{m-j}(\tau) d\tau} \partial_e \left[ \Gamma_{\text{gain}}(f^{m-\ell}, f^{m-\ell})(s, X_{cl}(s), V_{cl}(s)) \right] ds$$

$$\times \Gamma_{\text{gain}}(f^{m-\ell}, f^{m-\ell})(s, X_{cl}(s), V_{cl}(s)) ds$$

$$- e^{-\int_0^t \sum_{\ell=0}^{\ell_0} 1_{[\mu+\ell, \mu+\ell]}(s) \nu^{m-\ell}(s) ds} f_0(X_{cl}(0), V_{cl}(0)) \int_0^t \sum_{\ell=0}^{\ell_0} 1_{[\mu+\ell, \mu+\ell]}(s) \partial_e \left[ \nu^{m-\ell}(s, X_{cl}(s), V_{cl}(s)) \right] ds,$$

and

$$\text{III}_e = \int_0^t \sum_{\ell=0}^{\ell_0} 1_{[\mu+\ell, \mu+\ell]}(s) \left[ - \partial_e \sum_{s_0, t_0} \nu^{m-\ell}(s) + \partial_e \sum_{s_0, t_0^+} \nu^{m-\ell}(s) \right]$$

$$\times e^{-\int_0^s \sum_{j=0}^{j_0} 1_{[\mu+\ell+1, \mu+\ell+1]}(\tau) \nu^{m-j}(\tau) d\tau} \Gamma_{\text{gain}}(f^{m-\ell}, f^{m-\ell})(s, X_{cl}(s), V_{cl}(s))$$

$$+ \int_0^t \sum_{\ell=0}^{\ell_0} 1_{[\mu+\ell, \mu+\ell]}(s) \left[ - \lim_{\tau \to \tau_j} \nu^{m-j}(s, X_{cl}(s), V_{cl}(s)) + \lim_{\tau \to \tau_j} \nu^{m-j}(s, X_{cl}(s), V_{cl}(s)) \right]$$

$$\times e^{-\int_0^s \sum_{j=0}^{j_0} 1_{[\mu+\ell+1, \mu+\ell+1]}(\tau) \nu^{m-j}(\tau) d\tau} \Gamma_{\text{gain}}(f^{m-\ell}, f^{m-\ell})(s, X_{cl}(s), V_{cl}(s)) ds.$$
For $\text{III}_e$ we rearrange the summation and use (5.22), (6.2) and apply (6.1) to get

$$
\text{III}_e = \sum_{\ell=0}^{\ell,0} \left[ -\mu^{m-\ell}(t^\ell, x^\ell, v^\ell) + \mu^{m-\ell+1}(t^\ell, x^\ell, R_{x^\ell}v^\ell) \right] \partial_\ell t^\ell e^{-f_0^\ell \sum_{\ell=0}^{\ell,0} 1_{[\ell+1, \ell)}(s)e^{m-\ell}(s)}
$$

$$
+ \sum_{\ell=0}^{\ell,0} e^{-f_0^\ell \sum_{\ell=0}^{\ell,0} 1_{[\ell+1, \ell)}(s)e^{m-\ell}(s)} \left[ \Gamma_{\text{gain}}(f^{m-\ell}, f^{m-\ell})(t^\ell, x^\ell, v^\ell) - \Gamma_{\text{gain}}(f^{m-\ell+1}, f^{m-\ell+1})(t^\ell, x^\ell, R_{x^\ell}v^\ell) \right]
$$

$$
+ \int_0^t \sum_{\ell} 1_{[\ell+1, \ell)}(s) - f_0^\ell \sum_{\ell=0}^{\ell,0} 1_{[\ell+1, \ell)}(s)e^{m-\ell}(s) \Gamma_{\text{gain}}(f^{m-\ell}, f^{m-\ell})(s, X_{e}(s), v_{e}(s))
$$

$$
\times \sum_{\ell=0}^{\ell,0} \left[ R_{x^\ell}v^\ell - \frac{v^\ell}{2} \cdot E(t^\ell, x^\ell) \right] \partial_\ell t^\ell e^{-f_0^\ell \sum_{\ell=0}^{\ell,0} 1_{[\ell+1, \ell)}(s)e^{m-\ell}(s)}
$$

$$
+ \int_0^t \sum_{\ell} 1_{[\ell+1, \ell)}(s) - f_0^\ell \sum_{\ell=0}^{\ell,0} 1_{[\ell+1, \ell)}(s)e^{m-\ell}(s) \Gamma_{\text{gain}}(f^{m-\ell}, f^{m-\ell})(s, X_{e}(s), v_{e}(s))
$$

$$
\times \sum_{\ell=0}^{\ell,0} \left[ R_{x^\ell}v^\ell - \frac{v^\ell}{2} \cdot E(t^\ell, x^\ell) \right].
$$

(6.4)

**Proof of (6.1).** The proof is due to the change of variables

$$
\tilde{u} = Ou, \quad \tilde{\omega} = Ow, \quad d\tilde{u} = du, \quad d\tilde{\omega} = dw.
$$

Note

$$
\Gamma(f^m, f^m)(v) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^2} |v - \omega|^\nu q_0 \left( \frac{|v - \omega|}{|v - u|} \right) \mu(u) \left( f^m(u - [(u - v) \cdot \omega] \omega) f^m(v + [(u - v) \cdot \omega] \omega) - f^m(u) f^m(v) \right) \omega du d\omega
$$

$$
= \int_{\mathbb{R}^3} \int_{\mathbb{R}^2} |Ov - Ou|^\nu q_0 \left( \frac{|Ov - Ou|}{|Ov - Ou|} \right) \mu(Ou) \left( f^{m-1}(Ou - [(Ou - Ov) \cdot \omega] \omega) f^{m-1}(Ov + [(Ou - Ov) \cdot \omega] \omega) - f^{m-1}(Ou) f^{m-1}(Ov) \right) \omega du d\omega
$$

$$
= \int_{\mathbb{R}^3} \int_{\mathbb{R}^2} |Ov - \tilde{u}|^\nu q_0 \left( \frac{|Ov - \tilde{u}|}{|Ov - \tilde{u}|} \right) \mu(\tilde{u}) \left( f^{m-1}((\tilde{u} - Ov) \cdot \tilde{\omega}) f^{m-1}(Ov + ((\tilde{u} - Ov) \cdot \tilde{\omega}) \tilde{\omega} - f^{m-1}(\tilde{u}) f^{m-1}(Ov) \right) \omega du d\omega
$$

$$
= \Gamma(f^{m-1}, f^{m-1})(Ov).
$$

This proves (6.1). Especially we can apply (6.1) for the specular reflection BC (24) with $Ov = R_{x^\ell}v$

Using Lemma 2 and (2.7), we obtain for $\partial_\ell \in \{\nabla_x, \nabla_v\}$

$$
\text{II}_e \lesssim \mathbb{P}(\|e^{\theta v} f_0\|_{L^\infty}) \int_{\mathbb{R}^3} \sum_{\ell=0}^{\ell,0} 1_{[\ell+1, \ell)}(s) \partial_\ell X_e(s, u) \int_{\mathbb{R}^3} \frac{e^{-C_e |V e(s) - u|}^2}{|V e(s) - u|^{2-\kappa}} \|\nabla_x f^{m-\ell}(s, X_e(s), u)\| \omega du ds
$$

$$
+ \mathbb{P}(\|e^{\theta v} f_0\|_{L^\infty}) \int_{\mathbb{R}^3} \sum_{\ell=0}^{\ell,0} 1_{[\ell+1, \ell)}(s) \partial_\ell V e(s, u) \int_{\mathbb{R}^3} \frac{e^{-C_e |V e(s) - u|}^2}{|V e(s) - u|^{2-\kappa}} \|\nabla_x f^{m-\ell}(s, X_e(s), u)\| \omega du ds
$$

$$
+ t \mathbb{P}(\|e^{\theta v} f_0\|_{L^\infty}) \langle v \rangle^{\kappa} e^{-\theta v} \left( \|E\|_{L^\infty_{v}} + \|\nabla_x E\|_{L^\infty_{v}} \right) \left( \sup_{0 \leq s \leq t} |\partial_\ell V(e(s, t, x, v))| + \sup_{0 \leq s \leq t} |\partial_\ell X(s, t, x, v)| \right).
$$

We shall estimate the following:

$$
e^{-\omega(v)} \frac{|\omega(t, x, v)|^\beta}{\langle v \rangle^{\beta+1}} |\partial_x f(t, x, v)|, \quad e^{-\omega(v) t} \frac{|\omega(t, x, v)|^{\beta-1}}{\langle v \rangle^{\beta-1}} |\partial_x f(t, x, v)|,$$
From (5.1), the Velocity lemma (Lemma 3), Lemma 1 and $F^m \geq 0$ for all $m$, with $\omega \gg 1$

$$e^{-\omega(t)} \frac{1}{(v)^{b+1}} [\alpha(t, x, v)]^\beta I_x$$

$$\lesssim_{\xi, t} e^{-\omega(t)} \frac{1}{(v)^{b+1}} [\alpha(X_{cl}(0), V_{cl}(0))]^\beta e^{2C|v|t}$$

$$\times \left\{ \frac{(v)}{\alpha(t, x, v)} \partial_x f_0(X_{cl}(0), V_{cl}(0)) + \frac{(v)^3}{\alpha^2(t, x, v)} \partial_v f_0(X_{cl}(0), V_{cl}(0)) \right\}$$

$$\lesssim_{\xi, t} \left\| \frac{(v)}{(v)^{b+1}} \alpha^{-1} \partial_x f_0 \right\|_\infty + \left\| \frac{(v)^3}{(v)^{b+1}} \alpha^{-2} \partial_v f_0 \right\|_\infty$$

and

$$e^{-\omega(t)} \frac{1}{(v)^{b+1}} [\alpha(t, x, v)]^\beta I_v$$

$$\lesssim_{\xi, t} e^{-\omega(t)} \frac{1}{(v)^{b-1}} [\alpha(X_{cl}(0), V_{cl}(0))]^\beta e^{2C|v|t}$$

$$\times \left\{ \frac{1}{(v)} \partial_v f_0(X_{cl}(0), V_{cl}(0)) + \frac{(v)}{\alpha(t, x, v)} \partial_x f_0(X_{cl}(0), V_{cl}(0)) \right\}$$

$$\lesssim_{\xi, t} \left\| \frac{\alpha^{-1}}{(v)^b} \partial_x f_0 \right\|_\infty + \left\| \frac{1}{(v)^{b-2}} \alpha^{-2} \partial_v f_0 \right\|_\infty,$$

where we have used $\alpha(t, x, v) \lesssim \|v^2\|$ and the choice of $\omega \gg 1$.

From Lemma 1 and Lemma 2

$$\Pi_v \lesssim_{\xi, t} P(\|e^{\theta|v|^2 t} f_0\|_\infty) \int_0^t ds \sum_{\ell=0}^{\ell(s)} \int_{\mathbb{R}^3} e^{-C_0 |u - V_{cl}(s)|^2} \frac{1}{|u - V_{cl}(s)|^{2-\kappa}}$$

$$\times \left\{ |\partial_v X_{cl}(s)||\partial_x f_m| \approx |\partial_x f_m - j(s, X_{cl}(s), u)| + |\partial_v X_{cl}(s)|(1 + |\partial_x f_m - j(s, X_{cl}(s), u)|) \right\}.$$
to bound the whole exponents of (6.5) by

\[ - (\varpi - C)\langle v \rangle (t - s) + \varpi |v - u| s - C' |v - u|^2 + \varpi s \]
\[ \leq - (\varpi - C)\langle v \rangle (t - s) - (C - \frac{\sigma \varpi^2}{2}) |v - u|^2 + \frac{s^2}{2\sigma} + \varpi s \]
\[ \leq - (\varpi - C)\langle v \rangle (t - s) - C_{\sigma, \varpi} |v - u|^2 + C'_{\sigma, \varpi} \{s^2 + s\}. \]

Hence we prove the claim (6.5) for \( \varpi \gg 1 \).

Now we use (6.5) to bound

\[ e^{-\varpi \langle v \rangle t} \frac{1}{(\langle v \rangle)^{b+1}} \| \alpha(t, x, v) \|^\beta \Pi_{\mathbf{v}} \]

\[ \lesssim_t, \xi P(||\varepsilon||^2 f_0||_\infty) \times \]
\[ \times \left\{ \int_0^t \int_{\mathbb{R}^3} e^{-\frac{\varpi \langle v \rangle}{2} (t - s)} \frac{e^{C'} |V_{\mathbf{cl}}(s) - u|^2}{|V_{\mathbf{cl}}(s) - u|^2 - (\langle v \rangle)^{b+1}} |\alpha(s, X_{\mathbf{cl}}(s), u)|^{\beta - 1} \sup_m \sup_{0 \leq s \leq t} || e^{-\varpi \langle v \rangle s} \frac{\alpha^\beta}{(\langle v \rangle)^{b+1}} \partial_x f^m(s) ||_\infty \right\} \]

\[ + \left\{ \int_0^t \int_{\mathbb{R}^3} e^{-\varpi \langle v \rangle (t - s)} \frac{e^{-C' |V_{\mathbf{cl}}(s) - u|^2}}{|V_{\mathbf{cl}}(s) - u|^2 - (\langle v \rangle)^{b+1}} |\alpha(s, X_{\mathbf{cl}}(s), u)|^{\beta - 2} \sup_m \sup_{0 \leq s \leq t} || e^{-\varpi \langle v \rangle s} \frac{\alpha^{\beta - 1}}{(\langle v \rangle)^{b+1}} \partial_x f^m(s) ||_\infty \right\}. \]

(6.6)

For (A) we use (3.7) with \( Z = \langle v \rangle |\alpha(t, x, v)|^{\beta - 1} \) and \( l = \frac{\varpi}{2} \) and \( r = b + 1 \). For (B) we use (3.7) with \( \beta \mapsto \beta - 1 \) and \( Z = \langle v \rangle |\alpha(t, x, v)|^{\beta - 2} \) and \( l = \frac{\varpi}{2} \) and \( r = b - 1 \). Then

(\text{A}), (\text{B}) \ll 1.

Similarly, but with a different weight \( e^{-\varpi \langle v \rangle t} \frac{1}{(\langle v \rangle)^{b+1}} |\alpha(t, x, v)|^{\beta - 1} \), we use (6.1) to have

\[ e^{-\varpi \langle v \rangle t} \frac{1}{(\langle v \rangle)^{b+1}} |\alpha(t, x, v)|^{\beta - 1} \Pi_{\mathbf{v}} \]

\[ \lesssim_t, \xi P(||\varepsilon||^2 f_0||_\infty) \times \]
\[ \times \left\{ \int_0^t \int_{\mathbb{R}^3} e^{-C |V_{\mathbf{cl}}(s) - u|^2} e^{-\varpi \langle v \rangle s} e^{C |v||t-s|} |\alpha(t, x, v)|^{\beta - 1} \frac{\langle u \rangle^{b+1}}{\langle v \rangle^{b+1}} \sup_m \sup_{0 \leq s \leq t} || e^{-\varpi \langle v \rangle s} \frac{\alpha^\beta}{(\langle v \rangle)^{b+1}} \partial_x f^m(s, X_{\mathbf{cl}}(s), u)||_\infty \right\} \]

\[ + \left\{ \int_0^t \int_{\mathbb{R}^3} e^{-C |V_{\mathbf{cl}}(s) - u|^2} e^{-\varpi \langle v \rangle s} e^{C |v||t-s|} |\alpha(t, x, v)|^{\beta - 2} \frac{\langle u \rangle^{b+1}}{\langle v \rangle^{b+1}} \sup_m \sup_{0 \leq s \leq t} || e^{-\varpi \langle v \rangle s} \frac{\alpha^{\beta - 1}}{(\langle v \rangle)^{b+1}} \partial_x f^m(s, X_{\mathbf{cl}}(s), u)||_\infty \right\}. \]

Again we use (6.5) and (3.7) exactly as (6.6). Therefore for \( 0 < \delta = \delta(||\varepsilon||^2 f_0||_\infty) \ll 1 \)

\[ e^{-\varpi \langle v \rangle t} \frac{1}{(\langle v \rangle)^{b+1}} |\alpha(t, x, v)|^{\beta} \Pi_{\mathbf{v}} + e^{-\varpi \langle v \rangle t} \frac{|v|}{(\langle v \rangle)^{b}} |\alpha(t, x, v)|^{\beta - 1} \Pi_{\mathbf{v}} \]

\[ \lesssim \delta \left\{ \sup_m \sup_{0 \leq s \leq t} || e^{-\varpi \langle v \rangle s} \frac{\alpha^\beta}{(\langle v \rangle)^{b+1}} \partial_x f^m(s)||_\infty + \sup_m \sup_{0 \leq s \leq t} || e^{-\varpi \langle v \rangle s} \frac{\alpha^{\beta - 1}}{(\langle v \rangle)^{b+1}} \partial_x f^m(s)||_\infty \right\}. \]
Finally using $\frac{R_{x,t}^{\ell} - e_{t}^\ell}{2} = v_{t}^\ell$, the bound on $\partial_{\alpha} t^{\ell}$ in (5.9a) and (5.9b), from (5.11), the Velocity lemma (Lemma 3)
\[ e^{-e^{-(v)t}} \frac{1}{(\nu)^{b+1}} [\alpha(t, x, v)]^{1} \Pi_{x} \]
\[ \lesssim e^{-e^{-(v)t}} \frac{1}{(\nu)^{b+1}} [\alpha(t, x, v)]^{1} \sup_{0 \leq t \leq t} ||\partial_{\alpha} t^{\ell}|| + tP(||e^{0v}f_{0}||_{\infty})||E||_{L_{x}^{\infty}} \]
\[ \lesssim ||E||_{L_{x}^{\infty}} [\alpha(t, x, v)]^{1} + tP(||e^{0v}f_{0}||_{\infty})||E||_{L_{x}^{\infty}}, \]
for $\varpi \gg 1$. And similarly
\[ e^{-e^{-(v)t}} \frac{1}{(\nu)^{b+1}} [\alpha(t, x, v)]^{1} \Pi_{x} \]
\[ \lesssim e^{-e^{-(v)t}} \frac{1}{(\nu)^{b+1}} [\alpha(t, x, v)]^{1} \sup_{0 \leq t \leq t} ||\partial_{\alpha} t^{\ell}|| + tP(||e^{0v}f_{0}||_{\infty})||E||_{L_{x}^{\infty}} \]
\[ \lesssim ||E||_{L_{x}^{\infty}} [\alpha(t, x, v)]^{1} + tP(||e^{0v}f_{0}||_{\infty})||E||_{L_{x}^{\infty}}, \]
Collecting all the terms, for $2 < \beta < 3$ and $b = 1$ with $\varpi \gg 1$ and $0 < \delta \ll 1$, we get
\[ \sup_{m} \sup_{|s - t|} ||e^{-e^{-(v)t}} \frac{1}{(\nu)^{b+1}} \partial_{\alpha} f^{m}(s)||_{\infty} + \sup_{m} \sup_{|s - t|} ||e^{-e^{-(v)t}} \frac{1}{(\nu)^{b+1}} \partial_{\alpha} f^{m}(s)||_{\infty} \]
\[ \lesssim \frac{\alpha^{1-\beta}}{(\nu)^{b+1}} \partial_{\alpha} f_{0}||_{\infty} + \frac{\alpha^{1-\beta}}{(\nu)^{b+1}} \partial_{\alpha} f_{0}||_{\infty} + P(||e^{0v}f_{0}||_{\infty}). \]
We remark that this sequence $f^{m}$ is Cauchy in $L_{x}^{\infty}((0, T_{*}^{\ell}) \times \Omega \times \mathbb{R}^{3})$ for $0 < T_{*}^{\ell} < 1$. Therefore the limit function $f$ is a solution of the Boltzmann equation satisfying the specular reflection BC. On the other hand, due to the weak lower semi-continuity of $L_{p}$, $p > 1$, we pass a limit $\partial f^{m} \rightarrow \partial f$ weakly in the weighted $L_{x}^{\infty}$-norm.

Now we consider the continuity of $e^{-e^{-(v)t}} \frac{1}{(\nu)^{b+1}} \partial_{\alpha} f^{m}$. Remark that $e^{-e^{-(v)t}} \frac{1}{(\nu)^{b+1}} \partial_{\alpha} f^{m}$ and $e^{-e^{-(v)t}} \frac{1}{(\nu)^{b+1}} \partial_{\alpha} f^{m}$ satisfy all the conditions of Proposition 1. Therefore we conclude
\[ e^{-e^{-(v)t}} \frac{1}{(\nu)^{b+1}} \partial_{\alpha} f^{m} \in C^{0}([0, T] \times \Omega \times \mathbb{R}^{3}), e^{-e^{-(v)t}} \frac{1}{(\nu)^{b+1}} \partial_{\alpha} f^{m} \in C^{0}([0, T] \times \Omega \times \mathbb{R}^{3}). \]
Now we follow $W_{1, \infty}$-estimate proof for $e^{-e^{-(v)t}} \frac{1}{(\nu)^{b+1}} \partial_{\alpha} f^{m+1} - \partial_{\alpha} f^{m}$ and $e^{-e^{-(v)t}} \frac{1}{(\nu)^{b+1}} \partial_{\alpha} f^{m}$ to show that $e^{-e^{-(v)t}} \frac{1}{(\nu)^{b+1}} \partial_{\alpha} f^{m}$ is Cauchy in $L_{x}^{\infty}$. Then we pass a limit $e^{-e^{-(v)t}} \frac{1}{(\nu)^{b+1}} \partial_{\alpha} f^{m}$ strongly in $L_{x}^{\infty}$ so that $e^{-e^{-(v)t}} \frac{1}{(\nu)^{b+1}} \partial_{\alpha} f \in C^{0}([0, T^{*}] \times \Omega \times \mathbb{R}^{3})$ and $e^{-e^{-(v)t}} \frac{1}{(\nu)^{b+1}} \partial_{\alpha} f \in C^{0}([0, T^{*}] \times \Omega \times \mathbb{R}^{3})$.

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