Deformation Analysis of Binary Tree Structures by Transfer Matrix Method

by

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A complex structure composed of piping is modeled as a simple frame system with an arbitrary curved beam with branches. The transfer matrix method (TMM) is one of the methods used to analyze the displacement and stress fields in such modeled structure. The transfer matrix method (TMM) is highly popular because it reduces the stiffness matrix and facilitates high-speed calculation. However, the application of the TMM has been limited to structural vibration and static problems for a simple straight beam without branches. In this study, we generalized the theory of the TMM for application to binary tree structure, which is a branching structure without a closed paths and with at most three elements connected to branch points. Additionally, a problem based on a statically indeterminate beam with branch was analyzed to verify the generalized theory in this study.

Key words:
Structure analysis, Transfer matrix method, Finite element method, Fractal beam structure

1 Introduction

Many mechanical structures, such as piping systems 1), can be modeled using a combination of bars and beams. These elements have an arbitrary curved shape and branch to other elements. In addition, external loads and torsional torques act at several points in these models. To analyze the displacements and stresses that occur in these structures (hereinafter referred to as "beam structures"), the finite-element method (FEM), and boundary element method are generally applied. In the FEM 2), the domain of a problem is divided into elements and the matrix equation constructed from the displacement \( \vec{u} \) and force \( \vec{f} \) of nodes, which are components of an element, is

\[
K\vec{u} = \vec{f}
\]

where \( K \) is the stiffness matrix, and consists of the material constants and the shape of the element. Thus, a stiffness equation of the entire system is derived based on the continuity of displacement and equilibrium of forces. Therefore, the size of a stiffness matrix becomes enormous on the basis of the number of nodes. From a technical viewpoint, it is complicated to divide a curved beam structure with several branches; this is the focus of this study.

In contrast, the transfer matrix method (TMM) 3), in which all information at a node is transferred to an adjacent node, is a suitable alternative. This method was actively studied in the 1950s, and is now established as a representative structural vibration analysis technique 4)-6). Unlike other numerical analysis methods, such as the FEM, the size of a transfer matrix equivalent to a stiffness matrix in the FEM is constant if the size of the system increases. For this reason, the TMM is also called the "reduction method". However, the structure shape is limited to only straight bars and simple-shaped beams in the TMM. Therefore, attempts have been made to apply the TMM to curved structures by repeatedly introducing a rotation matrix and to the elastic-plastic problem 7).

In this study, we will focus on structures with branches, especially binary tree structures, as an example of more complex structures. Binary tree structure is a branching structure without a closed paths and with at most three elements connected to branch points. As a boundary condition, all other endpoints of the branched element except one are fixed ends, and the displacement and angle are fixed to zero. The theory of the TMM is generalized for application to this binary tree structure. This generalized theory is validated through an analysis of the statically indeterminate beam problem with \( n \) branches.

2 Fundamental equation

2.1 Overview of the transfer matrix method

First, we consider a curved beam structure in a threedimensional space. Some parts of the structure are fixed to a rigid wall and external loads \( \vec{P} \) and torsional torques \( \vec{T} \) act at arbitrary intermediate points. This structure is divided into \( N \) straight beam elements of arbitrary length, as shown in Fig. 1, and element numbers \( (k) \ (k = 1, ... , N) \) are assigned to each element.

Fig. 2 shows the \((k)\)th beam element. The right end of the element is \( a \) and the left end is \( b \). For the local coordinate system, the \((k)\) axis is set along the axis line, and the \((k)\) and \((k)\) axes are set orthogonally to each other. Additionally, the length is denoted as \( l_k \), Young's modulus is \( E_k \), modulus of rigidity is \( G_k \), cross-sectional area is \( A_k \), moment of inertia of the area around the axis \((k)\) \( x_i \) is \( I_{ik} (i = 1,2) \), and polar moment
of inertia of the area is $I_p$. Thus, the state vectors of right end $a$ and left end $b$ are denoted as $(k)_a$ and $(k)_b$, respectively, and the vectors are defined as follows.

$$Z_a = \begin{bmatrix} \delta_a \\ \vec{\delta}_a \end{bmatrix}, \quad Z_b = \begin{bmatrix} \delta_b \\ \vec{\delta}_b \end{bmatrix}$$

where $\delta_a$ and $\delta_b$ are the vectors that consist of the displacements, $\vec{\delta}_a$ and $\vec{\delta}_b$ comprise the deflection angles or torsional angles, $\vec{p}_a$ and $\vec{p}_b$ consist of the concentrated loads, and $\vec{f}_a$ and $\vec{f}_b$ consist of the bending moments or torques of right end $a$ and left end $b$, respectively. $(k)_a$ or $(k)_b$ means that $(k)$ is notated above each elements of that vector or matrix.

Under the assumption that the deformations are sufficiently small and buckling is not considered, these two state vectors are then related as

$$(k)_b = (k)_a^T F_e Z_a$$

where $F_e$ is a transfer matrix for a beam element.

Next, two beam elements, $(k)$ and $(k+1)$, shown in Fig. 3, are considered. The state vector of right end $a$ of the $(k+1)^{th}$ beam element is transferred using the continuity of the state vector of left end $b$ of the $(k)^{th}$ beam element as

$$(k+1)_a = R(\theta_{k+1}) (k)_b$$

where $R(\theta_{k+1})$ is the rotation matrix. Based on this relation, the continuously connected beam elements from $(1)$ to $(N)$ can be solved. Such a beam structure is hereinafter called a "beam set" $\{s\}$. The equation that expresses the relation between the state vectors of both ends of this set is

$$\begin{bmatrix} s \\ Z_b \end{bmatrix} = [S_e] \begin{bmatrix} s \\ Z_a \end{bmatrix}$$

where

$$[S_e] = \begin{bmatrix} s \\ Z_b \end{bmatrix} = \sum_{k=1}^{N} R(\theta_{k-1,k}) (k-1)_b$$

is a transfer matrix for a beam set and the state vectors of both ends of the beam set are

$$\begin{bmatrix} s \\ Z_b \end{bmatrix} = [S_e] \begin{bmatrix} s \\ Z_a \end{bmatrix}$$

Here, the state vector components at the beam set $\{s\}$ are separated as follows.

$$\begin{bmatrix} s \\ Z_b \end{bmatrix} = \begin{bmatrix} s \\ Z_{\theta b} \end{bmatrix} \begin{bmatrix} s \\ Z_{\theta T} \end{bmatrix}$$

where $(s)$ is the state vector for displacements and deflection angles or torsional angles and $Z_{\theta b}$ is the state vector for concentrated loads, bending moments, and torques. The contents of these state vectors are

$$\begin{bmatrix} s \\ Z_{\theta b} \end{bmatrix} = \begin{bmatrix} s \\ Z_{\theta b} \end{bmatrix} \begin{bmatrix} s \\ \vec{\theta}_b \end{bmatrix} \begin{bmatrix} s \\ \vec{\theta}_b \end{bmatrix}$$

The transfer matrix is also separated as

$$[S_e] = [G_e] \begin{bmatrix} s \\ \vec{\theta}_e \end{bmatrix}$$

Based on this separation, the equations that connect the state vectors can be expressed as

$$\begin{bmatrix} s \\ Z_{\theta b} \end{bmatrix} = [G_e] \begin{bmatrix} s \\ Z_{\theta T} \end{bmatrix}$$

The aforementioned fundamental equation is extended to the case involving an external force at the intermediate point between beam elements $(k)$ and $(k+1)$ as shown in Fig. 4.
When the right end of the \((k + 1)^{th}\) beam element is subjected to concentrated loads \(\bar{F}_{ex}\) and bending moments \(\bar{M}_{ex}\) as external forces, the state vector of the left end of the \((k)^{th}\) beam element is transferred as

\[
\begin{align*}
Z_{ex} &= B(\theta_{k,e+1})Z_{ex} + (k + 1)
\end{align*}
\]  
(12)

where \((k + 1)\) is the external force vector at the right end of the \((k + 1)^{th}\) beam element and is denoted as

\[
\begin{align*}
(k + 1)
\end{align*}
\]  
(13)

Therefore, when the external force vectors act at arbitrary intermediate points of the beam set \([s]\), Eq. (5) is expressed as follows.

\[
\begin{align*}
[s]_b &= [s]_e a + \begin{bmatrix} \bar{P}_{ex} \end{bmatrix} Z_{ex} \\
&= [s]_e a + \begin{bmatrix} \bar{P}_{ex} \end{bmatrix} \left( \begin{bmatrix} 0 \\
\begin{array}{c}
(1) \\
(2) \\
(3) \\
(4) \\
(5) \\
(6) \\
(7) \\
(8) \\
(9) \\
(10) \\
(11) \\
(12) \\
(13) \\
(14) \\
(15) \\
(16) \\
(17) \\
(18) \\
(19) \\
(20) \\
(21) \\
(22) \\
(23) \\
(24) \\
(25)
\end{array}
\end{bmatrix} Z_{ex} \right)
\end{align*}
\]  
(14)

where \([s]_e a + \begin{bmatrix} \bar{P}_{ex} \end{bmatrix} Z_{ex}\) is the external force vector, which can be expressed as

\[
\begin{align*}
[s]_{ex} &= (N) \sum_{i=0}^{n-1} \sum_{m=0}^{n-1} B(\theta_{m,m+1})^{(m)} (k) Z_{ex} + (N) Z_{ex}
\end{align*}
\]  
(15)

Eq. (11) then becomes

\[
\begin{align*}
Z_{gh,b} &= [s]_e a + \begin{bmatrix} \bar{G}_{ex} \bar{Z}_{e} \end{bmatrix} + \begin{bmatrix} \bar{P}_{ex} \end{bmatrix} Z_{ex}
\end{align*}
\]  
(16)

where

\[
\begin{align*}
[s]_b &= \begin{bmatrix} \bar{G}_{ex} \bar{Z}_{e} \end{bmatrix} + \begin{bmatrix} \bar{P}_{ex} \end{bmatrix} Z_{ex}
\end{align*}
\]  
(17)

### 2.2 Generalized fundamental equation for beam structure with several branches

![Fig. 5 Beam structure with one branch subjected to external force at intermediate point.](image)

We attempted to apply the extended TMM to a beam structure with several branches. For this purpose, we first address the problem of a beam structure with one branch subjected to external force at the intermediate point, as shown in Fig. 5. The three parts that consist of continuous beam elements with no branch are beam sets \([k, l, m]\) and \([k, l, m]\) indicate the external forces acting at the branch point in the local coordinate system of the beam set \([k]\). The branch point is assumed to be a rigid node with sufficiently large axial, bending, and torsional stiffnesses. The equilibrium of the forces at the branch point leads to the equation for loads and moments in the local coordinate system of the beam set \([k]\).

\[
\begin{align*}
[k] Z_{pr,b} + [l, m] Z_{pr,e} &= \begin{bmatrix} \bar{L}_{pr} \end{bmatrix} + \begin{bmatrix} \bar{M}_{pr} \end{bmatrix}
\end{align*}
\]  
(18)

where \([l, m]\) and \([m]\) indicate the state vectors for the external forces and moments of the right ends of the beam sets \([l]\) and \([m]\), respectively, in the local coordinate system of the beam set \([k]\). These state vectors in their own local coordinate systems are obtained with rotation matrices as follows.

\[
\begin{align*}
[l]_{pr,a} &= R(\theta_{k,1}) [l]_{pr,a} \\
[m]_{pr,a} &= R(\theta_{k,1}) [m]_{pr,a}
\end{align*}
\]  
(19)

Because the inverse matrix of the rotation matrix clearly coincides with that of the rotation matrix for the reverse direction, Eq. (19) can be rearranged as

\[
\begin{align*}
[l]_{pr,a} &= R(\theta_{k,1})^{-1} [l]_{pr,a} \\
[m]_{pr,a} &= R(\theta_{k,1})^{-1} [m]_{pr,a}
\end{align*}
\]  
(20)

Substitution of Eq. (20) into Eq. (18) yields

\[
\begin{align*}
[k] Z_{pr,b} + [l, m] Z_{pr,e} &= \begin{bmatrix} \bar{L}_{pr} \end{bmatrix} + \begin{bmatrix} \bar{M}_{pr} \end{bmatrix}
\end{align*}
\]  
(21)

Subsequently, Eq. (21) is also rearranged for \([l, m] Z_{pr,e}\) as

\[
\begin{align*}
[l]_{pr,a} &= R(\theta_{k,1}) [l]_{pr,a} \\
[m]_{pr,a} &= R(\theta_{k,1}) [m]_{pr,a}
\end{align*}
\]  
(22)

The state vectors for displacements and deflection and torsional angles are denoted as follows because of fixing the left ends of the beam sets \([l]\) and \([m]\).

\[
\begin{align*}
[l]_{pr,b} &= \begin{bmatrix} \bar{G}_{ex} \bar{Z}_{e} \end{bmatrix} + \begin{bmatrix} \bar{P}_{ex} \end{bmatrix} Z_{ex} \\
[m]_{pr,b} &= \begin{bmatrix} \bar{G}_{ex} \bar{Z}_{e} \end{bmatrix} + \begin{bmatrix} \bar{P}_{ex} \end{bmatrix} Z_{ex}
\end{align*}
\]  
(23)

Equations of the state vectors for displacements and angles of the beam sets \([l]\) and \([m]\) are obtained from Eq. (16) as follows:

\[
\begin{align*}
[l]_{pr,b} &= \begin{bmatrix} \bar{G}_{ex} \bar{Z}_{e} \end{bmatrix} + \begin{bmatrix} \bar{P}_{ex} \end{bmatrix} Z_{ex} \\
[m]_{pr,b} &= \begin{bmatrix} \bar{G}_{ex} \bar{Z}_{e} \end{bmatrix} + \begin{bmatrix} \bar{P}_{ex} \end{bmatrix} Z_{ex}
\end{align*}
\]  
(24)

Substitution of Eq. (23) into Eq. (24) yields

\[
0 = \begin{bmatrix} \bar{G}_{ex} \bar{Z}_{e} \end{bmatrix} + \begin{bmatrix} \bar{P}_{ex} \end{bmatrix} Z_{ex}
\]  
(25)
Thus, Eq. (25) is rearranged as follows:

\[
0 = \begin{bmatrix} m \end{bmatrix}_a + B_{m} Z_{PT_a} + Z_{PT ex}
\]

Thus, Eq. (25) is rearranged as follows:

\[
\begin{bmatrix} I \end{bmatrix}_{\text{stag}} = -B_{m} \begin{bmatrix} m \end{bmatrix}_{\text{stag}} A_{m} \begin{bmatrix} m \end{bmatrix}_{\text{stag}} + \begin{bmatrix} m \end{bmatrix}_{\text{stag}} A_{m} \begin{bmatrix} m \end{bmatrix}_{\text{stag}}
\]

Additionally, because the state vector for the displacements and deflection and torsional angles of the right end of the beam set \( \{ I \} \) must coincide with that of the beam set \( \{ m \} \) in the local coordinate system of the beam set \( \{ k \} \), the following relation is required.

\[
\begin{bmatrix} I \end{bmatrix}_{\text{stag}} = B_{m} \begin{bmatrix} m \end{bmatrix}_{\text{stag}} Z_{PT_a}
\]

\[
\begin{bmatrix} m \end{bmatrix}_{\text{stag}} = \begin{bmatrix} m \end{bmatrix}_{\text{stag}} + B_{m} Z_{PT_a} + Z_{PT ex}
\]

Therefore, by substituting Eq. (22) into Eq. (30), we obtain

\[
\begin{bmatrix} I \end{bmatrix}_{\text{stag}} = -B_{m} \begin{bmatrix} m \end{bmatrix}_{\text{stag}} A_{m} \begin{bmatrix} m \end{bmatrix}_{\text{stag}} + \begin{bmatrix} m \end{bmatrix}_{\text{stag}} A_{m} \begin{bmatrix} m \end{bmatrix}_{\text{stag}}
\]

Thus, Eq. (22) is rearranged as

\[
\begin{bmatrix} I \end{bmatrix}_{\text{stag}} = -B_{m} \begin{bmatrix} m \end{bmatrix}_{\text{stag}} A_{m} \begin{bmatrix} m \end{bmatrix}_{\text{stag}} + \begin{bmatrix} m \end{bmatrix}_{\text{stag}} A_{m} \begin{bmatrix} m \end{bmatrix}_{\text{stag}}
\]

\[
\begin{bmatrix} m \end{bmatrix}_{\text{stag}} = \begin{bmatrix} m \end{bmatrix}_{\text{stag}} A_{m} \begin{bmatrix} m \end{bmatrix}_{\text{stag}} - (\begin{bmatrix} m \end{bmatrix}_{\text{stag}} + B_{m} Z_{PT_a} + Z_{PT ex})
\]

Thus, Eq. (32) is rearranged as

\[
\begin{bmatrix} m \end{bmatrix} = \begin{bmatrix} m \end{bmatrix}_{\text{stag}} A_{m} \begin{bmatrix} m \end{bmatrix}_{\text{stag}} - (\begin{bmatrix} m \end{bmatrix}_{\text{stag}} + B_{m} Z_{PT_a} + Z_{PT ex})
\]

The equations of the right and left ends of the beam set \( \{ m \} \) are denoted as

\[
\begin{bmatrix} m \end{bmatrix} = \begin{bmatrix} m \end{bmatrix}_{\text{stag}} A_{m} \begin{bmatrix} m \end{bmatrix}_{\text{stag}} - (\begin{bmatrix} m \end{bmatrix}_{\text{stag}} + B_{m} Z_{PT_a} + Z_{PT ex})
\]
Thus, Eq. (25) is rearranged as follows:

$$[m] = [m]_{Z_{bb}} + [m]_{Z_{ea}} + [m]_{Z_{ee}}$$

Substitution of Eq. (26) into Eq. (29) also yields

$$[m] = [m]_{Z_{bb}} + [m]_{Z_{ea}} + [m]_{Z_{ee}}$$

Thus, the equations of the beam set \([s]\) are denoted as

$$[s]_{Z_{bb}} = [s]_{Z_{ea}} + [s]_{Z_{ee}}$$

Additionally, considering the equations

$$[s]_{Z_{bb}} = [s]_{Z_{ea}} + [s]_{Z_{ee}}$$

and \([s]_{Z_{bb}} = [s]_{Z_{ea}} + [s]_{Z_{ee}}\) are denoted as

Finally, we consider a beam structure with multiple branches, as shown in Fig. 6. All the parts that consist of continuous beam elements with no branches are expressed as beam sets \([k],[l],[m],[n],[o]\). The beam set \([s]\) represents the combination of the beam sets \([k]\) and \([m]\) and the combined transfer matrix \([s]_{Z_{ee}}\) can be obtained using \([m][n]\) and \([o]\) by following the method described previously. In addition, the combined transfer matrix \([s]_{Z_{ee}}\) can be obtained using \([k][s]\) and \([l][o]\). Similarly, when there are three or more branches in a beam structure, a combined transfer matrix can be obtained via the same procedure.

3. Analysis examples

Some examples of beam structures are presented here to validate the fundamental equation introduced in this study. The program based on the fundamental equation was developed using the C language. The analysis results are compared with the exact solutions derived from material mechanics and FEM analysis results.

3.1 Curved cantilever beam

First, we consider a curved cantilever beam that is subjected to a concentrated load at the free end. Fig. 7 shows the analysis model of the TMM. The left end of the circular arc shaped beam AB of radius \(R\) with angle \(\theta_R\) is supported by the fixed support, and the right end of the straight beam BC with length \(l\) is subjected to the concentrated load \(P\) in the direction vertical to the paper. For the coordinate system, the \(x_3\) axis is arranged along the neutral plane from the left end of the cantilever and the \((x_1,x_2)\) axes are arranged orthogonally to each other. The cross-sectional shape of the beam structure is circular. Table 1 shows the material constants, geometry of the TMM model, and external force. Here, the curved beam is modeled as a combination of straight beams, and the linear superposition of bending moment, torsional torque, and axial force is assumed to be valid.
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The increase of number of divisions in beam AB leads to the shape of the TMM model close to the smooth arc shape. The results of TMM analysis approaches the theoretical value as the number of branch sections increases. It indicates that the neutral plane of the beam from the left end of beam AB when only the concentrated load is applied. Symbols indicate the TMM analysis results and the solid line indicates the theoretical result obtained from classical beam theory. Thus, the verification of TMM indicates the TMM analysis result and the solid line indicates the theoretical result.

Next, we consider a beam structure with several branches. Fig. 10 shows the TMM model of the beam structure with one branch. The straight beam branches off at the right end of the straight beam AB of length \( l \) with angle \( \theta \). Both the left ends of the straight beams are fixed at the rigid walls. For the coordinate system, the \( x_3 \) axis is arranged along the neutral plane of beam AB and the \( (x_1, x_2) \) axes are arranged orthogonally to each other. The concentrated load \( P \) is applied in the downward direction or the torsional torque \( T \) is applied clockwise around the \( x_3 \) axis at point B.

Fig. 11 TMM model of the beam structure with several branches.

Next, we consider a beam structure with several branches. Fig. 10 shows the TMM model of the beam structure with one branch. The straight beam branches off at the right end of the straight beam AB of length \( l \) with angle \( \theta \). Both the left ends of the straight beams are fixed at the rigid walls. For the coordinate system, the \( x_3 \) axis is arranged along the neutral plane of beam AB and the \( (x_1, x_2) \) axes are arranged orthogonally to each other. The concentrated load \( P \) is applied in the downward direction or the torsional torque \( T \) is applied clockwise around the \( x_3 \) axis at point B.

Fig. 11 TMM model of the beam structure with several branches. This is the fractal structure of the beam structure with one branch, as shown in Fig. 10. Here, \( n \) stands for the number of branch sections, and the structure shown in Fig. 10 corresponds to the structure of \( n = 1 \). In this analysis, eight cases are analyzed: two conditions each for concentrated load and torsional torque, and four cases for \( n = 1 \sim 4 \). The cross-

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**Table 1 Material constants, geometry of the model, and external force in this analysis.**

| \( E \) [GPa] | \( v \) | \( d \) [mm] | \( R \) [mm] | \( \theta_R \) [deg] | \( l \) [mm] | \( l \) [kN] |
|---|---|---|---|---|---|---|
| 200 | 0.3 | 50 | 300 | 45 | 150 | 20 |

---

**Fig. 7 TMM model of the curved cantilever beam.**

**Fig. 8 Relationship between number of divisions of AB and the deflection at point C.**

Fig. 8 shows the relationship between the number of divisions of arc shaped beam AB in the TMM model and the deflection at point C. Here, beam BC was set to be 10 elements. In this graph, circular symbol is the TMM analysis result and solid line is the result obtained from classical beam theory. It is indicated that the results of TMM analysis approaches the theoretical value as increasing the number of divisions. The increase of number of divisions in beam AB leads to the shape of the TMM model close to the smooth arc shape.

Fig. 9 shows the TMM analysis results of deflection \( \delta_1 \) along the neutral plane of the beam from the left end \( x \). Both beam AB and BC was assumed to be 10 elements. The circle symbol indicates the TMM analysis result and the solid line indicates the theoretical result. It is clear that the TMM result is completely matched to the theoretical result. Thus, the verification of TMM formulation extended in this study was shown.
sec 1.2 Fractal beam structure with several branches

The TMM analysis results are compared with FEM analysis results. The FEM analysis software Marc was used to conduct FEM analysis. The FE element used in the analysis model is three-dimensional straight beam element with solid cross section (element number 52).

Fig. 12 shows the analysis results when only the torsional torque is applied. Fig. 13(a) shows the distribution of deflection $\delta_2$ in beam AB when only the concentrated load is applied. Symbols indicate the TMM analysis results and the solid line indicates FEM analysis results. The comparison shows good agreement in all the number of branch sections. The deflection results indicate that the deflection is decreased with increasing number of branch sections.

Fig. 13 shows the analysis results when only the torsional torque is applied. Fig. 13(b) shows the distribution of the torsional angle $\theta_3$ in beam AB. Symbols indicate the TMM analysis results and the solid line indicates FEM analysis results. Both comparisons show good agreement in all the number of branch sections. The deflection results indicate that the deflection reaches a maximum value not at point B, but at approximately $x_3 = 300$ mm. The torsional angle results indicate that the torsional angle decreases with increasing number of branch sections, and changes linearly for each branch section.

Table 2 Material constants, geometry of the beam structure, and external force in this analysis.

| $E$ [GPa] | $\nu$ | $d$ [mm] | $l$ [mm] | $\theta$ [deg] | $P$ [kN] | $T$ [kN · m] |
|----------|------|----------|----------|----------------|---------|-------------|
| 191      | 0.3  | 50       | 320      | 30             | 20      | 20          |

Fig. 12 Deflection curve of beam structure with several branches subjected to only concentrated load.

Fig. 13 Analysis results of beam structure with several branches subjected to only torsional torque.

Fig. 14 shows the deformed TMM and FEM analysis model of $n = 4$ when only concentrated load is applied, and displacements are magnified 200 times. Fig. 15 shows the deformed TMM and FEM analysis model of $n = 4$ when only torsional torque is applied, and displacements are magnified 10 times. Both comparisons show good agreement. Fig. 16 shows the comparison of the matrix size of TMM and FEM of the model of $n = 4$. This indicates that the TMM can significantly reduce the size of computational matrix.

Consequently, the TMM extended in this study was verified through some structural problems. In future work, we will apply the TMM for the curved structure with several branches to the optimization of a branched beam structure.
4. Conclusion

In this study, the theory of the TMM was generalized for application to binary tree structures including a curved shape and several branches, and a fundamental equation was derived. A curved cantilever beam and beam structures with several branches were analyzed using the program developed on the basis of the extended TMM theory. The theory was validated through a comparison with exact solutions derived from material mechanics and the analysis results of the fractal beam structures with several branches were provided.

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