Distributed optimization on directed graphs based on inexact ADMM with partial participation

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Abstract—We consider the problem of minimizing the sum of cost functions pertaining to agents over a network whose topology is captured by a directed graph (i.e., asymmetric communication). We cast the problem into the ADMM setting, via a consensus constraint, for which both primal subproblems are solved inexactly. In specific, the computationally demanding local minimization step is replaced by a single gradient step, while the averaging step is approximated in a distributed fashion. Furthermore, partial participation is allowed in the implementation of the algorithm. Under standard assumptions on strong convexity and Lipschitz continuous gradients, we establish linear convergence and characterize the rate in terms of the connectivity of the graph and the conditioning of the problem. Our line of analysis provides a sharper convergence rate compared to Push-DIGing. Numerical experiments corroborate the merits of the proposed solution in terms of superior rate as well as computation and communication savings over baselines.

I. INTRODUCTION

Distributed multi-agent optimization has found paramount applications across various fields such as in control [1], signal processing [2], [3], machine learning and data mining [4], [5], and wireless sensor networks [6], [7]. The archetypal problem is to

\[
\begin{align*}
\text{minimize} & \quad f(\hat{x}) = \sum_{i=1}^{n} f_i(\hat{x}), \\
\text{subject to} & \quad \hat{x}\in\mathbb{R}^d
\end{align*}
\]

(1)

where \(f_i(\cdot)\) is a local cost function corresponding to agent \(i\). Distributed optimization amounts to solving (1) over the common decision vector \(\hat{x}\) by a synergy of local computations and communication exchanges. In specific, agent \(i\) holds a local variable \(x_i\) which is updated based on its local cost \(f_i(\cdot)\) alongside information obtained by its neighbors (e.g., their local variables or gradients), and consensus (i.e., \(x_1 = \cdots = x_n\)) is achieved asymptotically.

There has been extensive work on the subject, especially for the case that the communication network topology is symmetric (i.e., an undirected graph) [8], [9], [10], [11]. Nevertheless, it is quite common to have unidirectional communication links in wireless networks due to heterogeneity in transceivers or perceived levels of interference [7]. Algorithms on directed graphs can be roughly classified into: a) primal methods and b) primal-dual methods, most notably based on the Alternating Direction Method of Multipliers (ADMM) [12]. In the former case, a local gradient step is used along with weighted averaging across neighboring agents. A sublinear convergence is established [13], [14] even for strongly convex problems, while linear convergence can be retrieved using gradient tracking [15].

Recent work has developed ADMM-based methods for distributed optimization on directed graphs [16], [17], [18]. In specific, [16] uses dynamically updated weights for local averaging and establishes linear convergence under strong convexity. The authors in [17] adopted \(\epsilon\)-inexact consensus and proposed an asynchronous method that requires a finite number of communication steps per round. [18] allows for both equality and inequality constraints and establishes either sublinear rate to the exact solution or linear rate to a neighborhood of optimality.

Notwithstanding, the prior art based on ADMM requires an exact solution of the local subproblems, which may incur heavy computational burden unsuitable for resource-constrained devices. Besides, it is not amenable to partial agent participation, an imperative requirement in real scenarios where user unavailability is common (due to variable operating conditions such as battery level and network bandwidth) and synchronization is difficult [19]. In order to address these challenges, we propose IPD (Inexact, Partial participation, Directed graph).

Contributions:

1) We propose a primal-dual method for distributed optimization on directed graphs that alleviates the computational load by inexact solution of the local subproblems (using a single gradient step).
2) Under standard assumptions, we establish linear convergence and characterize the rate with respect to the graph connectivity and the conditioning of the problem (Thm. 1 and Cor. 1).
3) The method allows for partial user participation at each iteration of the algorithm. Thm. 2 establishes linear convergence with high probability and reveals its dependency on the activation probability.
4) Our analysis provides a sharper characterization of the rate compared to the state-of-the-art Push-DIGing method with which it shares comparable communication and computation costs.
5) Experiments on two real-life machine learning datasets demonstrate merits in terms of a) faster rate compared to Push-DIGing, b) computation and communication savings over baselines.
II. NOTATION

The network topology is captured by a directed graph $G = \{V, E\}$, where $V$ is the set of agents (with cardinality $n := |V|$) and $E$ is the set of directed communication links: $(i, j) \in E$ if and only if agent $i$ can send a message to agent $j$. We define the set of agent $i$'s in-neighbors as $N_i^\text{in} := \{j : (j, i) \in E\}$, and its out-degree by $d_i := |\{j : (i, j) \in E\}|$. The maximum out-degree is denoted by $d_{\text{max}} := \max_{i \in V} d_i$, while $D := \text{diag}(d_1, d_2, \ldots, d_n)$ is a diagonal matrix with entries the out-degrees of the corresponding agents. The adjacency matrix $A \in \mathbb{R}^{n \times n}$ satisfies $A_{ij} = 1$ if $(j, i) \in E$ and $A_{ij} = 0$ otherwise. We further define $P := (I + AD^{-1})/2$ and for $\lambda_2(P)$ its second largest eigenvalue, while we use $\phi$ for the diameter of the graph. All vectors are meant as column vectors.

The augmented Lagrangian (AL) for (2) is given by:

$$F(x) := \frac{1}{2} \sum_{i=1}^{n} f_i(x_i) + \sum_{i=1}^{n} \lambda_i \left( c_i(x_i) \right) + \frac{\rho}{2} \lVert x - z \rVert^2,$$

where $y \in \mathbb{R}^{nd}$ is the dual variable and $\rho > 0$. The iterations of ADMM are given by sequential alternating minimization of the AL over $x, z$ plus a dual ascent step, as follows:

$$
\begin{align*}
\hat{x}^{k+1} &= \text{argmin}_x \ L_{\rho}(x, z^{k+1}, y^{k+1}), \\
\hat{y}^{k+1} &= \text{argmin}_y \ L_{\rho}(x^{k+1}, z, y^{k+1}), \\
\end{align*}
$$

where $\sum_{j=1}^{n} \phi_{j}^{k+1}$ is the proxy for $\phi_{k+1}$, which is initialized as $\phi_{k+1} = x_{k+1}$ and updated (for $k \geq 0, b \in \{0, \ldots, B - 1\}$) using:

$$z^{k+1} := \phi_{k+1}.$$
so that the update of an active agent will not be affected by the inactivity of its neighbors and an active agent is available to carry all operations in steps 5-9. We emphasize that since only the latest information is stored, the storage requirements do not increase.

IV. CONVERGENCE ANALYSIS

The analysis is carried under the following two assumptions:

Assumption 1. The directed graph \( G \) is strongly connected.

Assumption 2. The objective functions \( f_i \) satisfy:

1) Each \( f_i, i \in [n] \) is strongly-convex, i.e., there exists \( m_f > 0 \), such that \( \forall x, y \in \mathbb{R}^n \),

\[
(\nabla f_i(x) - \nabla f_i(y))^T (x - y) \geq m_f \|x - y\|^2.
\]

2) The gradient of each \( f_i, i \in [n] \) is Lipschitz continuous with constant \( M_f > 0 \), i.e., \( \forall x, y \in \mathbb{R}^n \),

\[
\|\nabla f_i(x) - \nabla f_i(y)\| \leq M_f \|x - y\|.
\]

We denote the primal-dual optimal solution of (2) by \((x^*, z^*, y^*)\); strong convexity guarantees unique primal solutions, while (7c) of the following KKT conditions guarantees uniqueness of the dual optimal solution:

\[
x_1^* = \cdots = x_n^*, \quad (7a)
\]

\[
x^* = z^*, \quad (7b)
\]

\[
\nabla F(x^*) + y^* = 0. \quad (7c)
\]

The following lemma establishes an error bound on the deviation of \( z^k \) from the average \( \bar{z}^k \) that is key in establishing our convergence theorem. The analysis is similar with some steps in establishing [16, Theorem 1], but the obtained upper bound is more general.

Lemma 1. Let \( x^k, z^k \) be the sequence generated by Alg. 1 satisfies

\[
\|z^k\|^2 \leq 2^{\rho \delta B - 1}\|x^k\|^2 + \Delta^k \left( \|x^k - x^*\|^2 + \|x^*\|^2 \right),
\]

\( \forall k \geq \bar{k} \), and \( \Delta^k \to 0 \) geometrically with rate \( \lambda_2(P) \).

Proof: See Appendix.

We proceed to analyze the convergence of Alg. 1 in two steps: a) Thm. 1 establishes linear convergence with full participation; b) Thm. 2 considers the case of partial participation and establishes the convergence with high probability.

Theorem 1 (Full Participation). Under Assumptions 1 and 2, by choosing \( \eta \leq \frac{4}{15(M_f + m_f)}, \rho \leq \frac{4}{97(M_f + m_f)} \),

\[
B \geq \max \left\{ 1, \frac{\ln \left( \frac{\eta}{\ln(\eta)} \right) + 1}{\frac{\ln \left( \frac{8(M_f + m_f)}{M_f + m_f} \right)}{\ln(\eta)} + 1} \right\},
\]

then for any \( \epsilon \in (0, 1) \), with probability at least \( 1 - \epsilon \), the following holds:

\[
\|x^k - x^*\|^2 + \|y^k - y^*\|^2 = O\left( \lambda^k \right),
\]

where \( \lambda_p \) can be arbitrarily chosen from \( \lambda_1, 1 \),

\[
\lambda_1 = \max \left\{ \lambda_2(P), 1 - q_{\text{min}} \frac{1 - \mu_1}{1 + \mu_1} \right\}.
\]

Proof: See Appendix.

Remark 1. Convergence in Thm. 2 is established with high probability, where the dependency on \( \epsilon \) is hidden in \( O(\cdot) \), since the established rate analysis is asymptotic. In specific, for any \( \epsilon \in (0, 1) \), with probability at least \( 1 - \epsilon \), there exists some \( K = K(\epsilon, \lambda_p) \), such that \( \forall k > K \),

\[
\|x^k - x^*\|^2 + \|y^k - y^*\|^2 \leq \lambda^k
\]

V. COMPARISON WITH PUSH-DIGING

Push-DIGing [15] is the most popular gradient-based method for directed graphs that achieves linear convergence through gradient tracking (second line in (11)). The updates
for agent $i$ at iteration $k$ are as follows:

$$u_i(k + 1) = c_{ii}(u_i(k) - \eta y_i(k)) + \sum_{j \in N_{i}^n} c_{ij}(k)(u_j(k) - \eta y_j(k)),$$

(8)

$$v_i(k + 1) = c_{ii}(k)v_i(k) + \sum_{j \in N_{i}^n} c_{ij}(k)v_j(k),$$

(9)

$$x_i(k + 1) = u_i(k + 1)/v_i(k + 1),$$

(10)

$$y_i(k + 1) = c_{ii}(k)y_i(k) + \sum_{j \in N_{i}^n} c_{ij}(k)y_j(k) + \{\nabla f_i(x_i(k + 1)) - \nabla f_i(x_i(k))\}.$$  

(11)

where $\eta$ is the stepsize, $C(k)$ is a column-stochastic matrix, and the initialization uses $v_i(0) = 1$, $x_i(0) = u_i(0)$, and $y_i(0) = \nabla f_i(x_i(0))$.

We first show that our proposed solution has comparable computation/communication cost (in fact lower communication cost for the case $B = 1$):

- for Push-DIGing, in each round the cost for one agent includes one broadcast of $2d + 1$, one gradient evaluation, plus a local averaging cost of $O_B^m(2d + 1)$ where $d_i$ is the in-degree of agent $i$.

- for our method, the cost includes one broadcast of $B(d + 1)$, one gradient evaluation, and averaging cost of $d_i^m(B(d + 1))$.

We conclude that for $B = 1$, IPD has lower computation and communication costs.

We proceed to compare our rate with the one established in [15, Theorem 18]. For Push-DIGing, by denoting $V(k) := \text{diag}(v_1(k), \ldots, v_n(k))$, it holds that $\sup_k ||V(k)^{-1}||_{\text{max}} = \mathcal{O}(n^{-n})$ [15, Equation 49], which results in an upper bound selection of stepsize as $\mathcal{O}(n^{-n})$; this, in turn, leads to a rate of $1 - \mathcal{O}(n^{-n})$. This is reminiscent of the analysis technique which was developed to address a more general problem that also considers time-varying graphs. In contrast, the stepsize in our case depends only on the scaling and not on the population (see Thm. 1). We further study the rate experimentally in Fig. 2, which depicts a substantial acceleration for the same stepsize. This is also translated to substantial computation and communication savings (since $B = 1$ is used in all comparisons). The superior rate achieved by our proposed method can further be explained by the weight balancing process in (5) which converges to a doubly stochastic matrix. In contrast, (11) in Push-DIGing uses a column-stochastic matrix (i.e., it does not apply the push-sum protocol in the gradient estimation step).

VI. EXPERIMENTS

We evaluate IPD on a distributed logistic regression problem:

$$f_i(x) := \frac{1}{m_i} \sum_{j=1}^{m_i} \ln \left(1 + e^{w_j^T x} \right) + (1 - y_j)w_j^T x,$$

where $m_i$ is the number of data points held by each agent and $\{w_j, y_j\}_{j=1}^{m_i} \subset \mathbb{R}^d \times \{0, 1\}$ are labeled samples. We used two datasets from LIBSVM\(^1\) and the UCI Machine Learning Repository\(^2\) for Fig. 1 and Fig. 2 respectively. In each case we take 5,000 data points with dimension $d = 22$, distribute them uniformly at random across $n = 50$ agents. The communication topology is captured by a directed graph obtained by randomly adding edges to the ring graph with probability 0.2. We use the relative cost error as the metric for convergence which is defined as $\frac{\sum_{i=1}^{n} f(x^i) - f(x^*)}{\sum_{i=1}^{n} f(x^i) - f(x^*)}$. Fig. 1.a shows the comparison with the alternative of solving the local optimization problem (3a) exactly as in [16]: it reveals a large computational saving of 87.5% in all cases. Fig. 1.b illustrates a negligible dependency on the number of communication steps $B$. In fact, all other experiments for both datasets were conducted for $B = 1$. Fig. 2.a shows the effect of increasing participation in the speedup of the algorithm (as expected from Thm. 2). Last but not least, we compared against Push-DIGing in Fig. 2.b for two different stepsizes (common for both methods). The superior convergence rate of IPD translates to 90.4% computation and 94.9% communication savings for target accuracy of 0.5. In particular, to reach the target accuracy with stepsize $\eta = 0.01$, our method needs about 100 rounds while Push-DIGing needs 600 rounds and the cost per round is based on the analysis in the previous section.

VII. CONCLUSION

This paper proposed proposed IPD, a primal-dual method for distributed optimization over directed graphs. The

\(^1\)Available at https://www.csie.ntu.edu.tw/~cjlin/libsvm/.
\(^2\)Available at https://archive.ics.uci.edu/ml/index.php.
two primal subproblems are solved inexactly: one step of gradient descent for the local optimization problem \((x-\text{variable})\) and distributed averaging based on weight balancing \((z-\text{variable})\). IPD was shown both theoretically and experimentally to have faster convergence than Push-DIGing (Sec. V) as well as substantial computation and communication savings over baseline methods. The established linear rate gives a decomposition in terms of the problem conditioning and the network connectivity (Thm. 1 and Cor. 1), similar with first order methods on undirected graphs. Furthermore, the lower bound for \(B\) (number of communications per round) was shown to have a logarithmic dependency with conditioning and connectivity (Cor. 1). A distinctive attribute of IPD is the feasibility of partial agent participation, which is crucial in large-scale real systems (the rate was established in Thm. 2).

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APPENDIX

Due to length constraints, we provide proof sketches in the following. The complete proofs are accessible at [20].

Proof of Lemma 1: For each \(k\), we consider the \(l\)-th entry of each \(\xi_k^l\) and combine them to form an \(n\)-dimensional vector denoted by \(\xi^l\), and we let

\[
\xi^l := \left(1 - \frac{1}{n} \mathbf{1}^T / n\right) \xi^l.
\]

We define \(W_k^l := I - (D - A)\text{diag}(w^k)\). Then, according to the first property of [16, Lemma 2], there exists \(\rho^k\), so that \(W_k^l\rho^k = \rho^k\) and \((\rho^k)^T \mathbf{1} = 1\). Following the steps in [16] we can obtain that for all \(k \geq k\)

\[
\|\xi^l_k\| \leq \delta B \left\|\xi^l_k\right\| + (1 + \delta B) \left\|(1/n - \rho^k)\mathbf{1}^T \xi^l_k\right\|.
\]

Letting \(\rho^k = \arg \max_{p \in \mathbb{P}} \left\|\mathbf{p}^T (p^0) - p^k (B)\right\| / (1/n - p)\), we get

\[
\|\xi^l_k\| \leq \delta B \left\|\xi^l_k\right\| + 2 \left\|(1/n - \rho^k)\mathbf{1}^T \xi^l_k\right\|,
\]

which implies (using \(\delta \in (0, 1), (a + b)^2 \leq \frac{1}{4} a^2 + \frac{1}{4} b^2\) for any \(a, b \in \mathbb{R}\), along with Jensen’s inequality, and some algebra):

\[
\|\xi^l_k\| \leq \delta B \left\|\xi^l_k\right\| + 2 \left\|(1/n - \rho^k)\mathbf{1}^T \xi^l_k\right\|^2.
\]

Since \(l\) is arbitrarily chosen from \(\{1, \ldots, d\}\), the inequality above holds for each entry position. Adding the \(d\) inequalities together and defining \(\Delta_k := \frac{2}{n} \left\|(1/n - \rho^k)^2\right\|^2\), which in view of the fact that both \(\left\|\mathbf{1}/n - \rho^k\right\|^2\) and \(\left\|\mathbf{1}/n - \rho^k (B)\right\|^2\) tend to zero geometrically with the rate to be \(\lambda_2 (P)\), completes the proof.

Proof of Theorem 1: For notational simplification, we denote \(\nabla F (x^k)\) by \(\nabla F^x\) and \(\nabla F (x^*)\) by \(\nabla F^*\). From (4) and (7c) we have

\[
\nabla F^k - \nabla F^* = \frac{1}{\eta} (x^{k+1} - x^k) - (y^k - y^*) - \rho (x^k - z^k)
\]
\[ (x^k - x^*)^T (\nabla F^k - \nabla F^*) = (x^{k+1} - x^*)^T (\nabla F^k - \nabla F^*) + (x^k - x^{k+1})^T (\nabla F^k - \nabla F^*). \]

\[ (i) \]

\[ (i) = -\frac{1}{\eta} (x^{k+1} - x^*)^T (x^{k+1} - x^k) \]

\[ - (x^{k+1} - x^*)^T (y^k - y^*) - \rho (x^{k+1} - x^*)^T (x^k - z^k). \]

The following is to bound the three terms. The detailed calculations can be found in [20].

\[ (ii) = \frac{1}{2\eta} \left( \|x^k - x^*\|^2 - \|x^{k+1} - x^*\|^2 - \|x^{k+1} - x^k\|^2 \right), \]

\[ (iii) = - (z_1^{k+1})^T (y^k - y^*) - \frac{1}{2\rho} \left( \|y^{k+1} - y^*\|^2 - \|y^k - y^*\|^2 \right) \]

\[ - \|y^{k+1} - y^*\|^2 - \|y^k - y^*\|^2 \right), \]

\[ (iv) = \frac{\rho}{2} \left( \|x^* - z^k\|^2 + \|x^k - x^{k+1}\|^2 - \|x^* - x^k\|^2 \right) \]

\[ - \|x^k - x^{k+1} - z^k\|^2 \right). \]

Adding the three terms gives:

\[ (i) \leq \frac{1}{2\eta} \left( \|x^k - x^*\|^2 - \|x^{k+1} - x^*\|^2 \right) \]

\[ + \frac{1}{2\rho} \left( \|y^k - y^*\|^2 - \|y^{k+1} - y^*\|^2 \right) \]

\[ + \frac{\rho}{2} \|x^* - z^k\|^2 + \frac{\rho}{2} \left( \|x^k - x^{k+1}\|^2 + \|x^* - x^k\|^2 \right) \]

\[ + \frac{1}{2\rho} \|y^k - y^{k+1}\|^2 - \frac{\rho}{2} \|x^k - x^{k+1} - z^k\|^2 - (z_1^{k+1})^T (y^k - y^*). \]

By bounding (v) and (vi) and invoking the cocoercivity of the gradient, the choices for \( B, \eta, \rho \) in Thm. 1 and moreover \( \gamma = 4, \tau = \frac{3(\tau^1 + \tau^2)}{4}, \zeta = 3(M_f + m_f) \) guarantee that the following inequalities hold:

\[ \frac{1}{4\tau} + \frac{1}{\zeta} - \frac{1}{m_f + M_f} < 0, \]

\[ (12a) \]

\[ \tau + \rho + \frac{1}{\gamma \eta} - \frac{1}{2\eta} < 0, \]

\[ (12b) \]

\[ \frac{1}{2\rho} - \left[ \delta^{2B-1} \left( \frac{5\rho}{4} + \frac{\gamma}{4\eta} + \frac{\zeta}{4} + \rho \right) \right] > 0, \]

\[ (12c) \]

\[ \delta^{2B-1} \left( \frac{17\rho}{4} + \frac{\gamma}{4\eta} + \frac{\zeta}{4} + 3\rho - \frac{m_f M_f}{m_f + M_f} \right) < 0. \]

\[ (12d) \]

Consequently, there exist \( \mu_1, \mu_2, \mu_3, k_1 \), such that \( 0 < \mu_1 < \mu_2 \) and for all \( k > k_1 \), the following holds:

\[ \mu_1 \left( \|x^k - x^*\|^2 + \|y^k - y^*\|^2 \right) + \mu_3 \left( \Delta^k + \Delta^{k+1} \right) \|x^*\|^2 \]

\[ \geq \|x^{k+1} - x^*\|^2 + \|y^{k+1} - y^*\|^2, \]

where \( \mu_1 = \max \left\{ \frac{c_1 + c_2}{2\rho}, \frac{1}{\eta} - c_3 / \left( \frac{1}{2\eta} \right) \right\} \) and \( c_1, c_2, c_3 \) are specified in the statement of the theorem.

**Proof of Theorem 2:** Let \( \Omega^k \in \mathbb{R}^{n 	imes n} \) be a 0-1 diagonal matrix where 1 corresponds to the case agent \( i \) is active and 0 when it is inactive. Let \( E[\Omega^k] = \Omega := \text{diag}(q_1, \ldots, q_n) \). The update of \( w \) can be written compactly as

\[ w^k(b + 1) = w^k(b) + \Omega^k (Pw^k(b) - w^k(b)). \]

Let \( w^\infty = \lim_{t \to \infty} P^t w^0(0) \). By taking conditional expectation on the update of \( w \) (denoted by \( E^k \), where conditioning is on past activations), we have

\[ E^k \left[ \|w^k(b + 1) - w^\infty\|^2_{\Omega^{-1}} \right] \]

\[ = \|w^k(b) - w^\infty\|^2_{\Omega^{-1}} + \|Pw^k(b) - w^k(b)\|^2_{\Omega^{-1}} \]

\[ + 2 (w^k(b) - w^\infty)^T (Pw^k(b) - w^k(b)). \]

From [21, Lemma 1], there exists some positive \( \theta < 1 \) only depending on \( P \), such that

\[ E^k \left[ \|w^k(b + 1) - w^\infty\|^2_{\Omega^{-1}} \right] \leq (1 - q_{mn} \theta) \|w^k(b) - w^\infty\|^2_{\Omega^{-1}}. \]

By induction, we obtain

\[ E^k \left[ \|w^k(b) - w^\infty\|^2_{\Omega^{-1}} \right] \leq c_1 \lambda_2^{B+b}, \]

for some constant \( c_1 \) and \( \lambda_2 := 1 - q_{mn} \theta \), whence invoking Markov’s inequality, we have \( \forall \epsilon > 0, \exists K_1(\epsilon) \), which implies

\[ \Pr \left[ \sum_{k=K_1}^{K_2} \|w^k(b) - w^\infty\|^2_{\Omega^{-1}} < \epsilon_1 \right] \geq 1 - \frac{\epsilon}{2}. \]

We define \( v^k := (x^k)^T, (y^k)^T \), and let \( T_a \) to be the operator corresponding to Alg. 1 for full participation, i.e., \( T_a : (x^k, y^k, z^k) \rightarrow (x^{k+1}, y^{k+1}, z^{k+1}) \) and \( T := \begin{bmatrix} I_d & 0 & 0 \\ 0 & I_d & 0 \end{bmatrix} T_a \).

We define \( \Phi^k := \text{blkdiag} (\Omega_k \otimes I_d, \Omega_k \otimes I_d) \) and \( \Phi^k_a := \text{blkdiag} (\Omega_k \otimes I_d, \Omega_k \otimes I_d) \), then \( E[\Phi^k] = \Phi := \text{blkdiag} (\Omega \otimes I_d, \Omega \otimes I_d) \) so that \( \Phi^k T_a = \Phi^k \begin{bmatrix} I_d & 0 & 0 \\ 0 & I_d & 0 \end{bmatrix} T_a = \begin{bmatrix} I_d & 0 & 0 \\ 0 & I_d & 0 \end{bmatrix} \Phi^k T_a \), which allows the analysis to carry for \( T \) only.

For partial participation, we have

\[ v^{k+1} = v + \Phi^k (Tv^k - v^k), \]

i.e.,

\[ \|v^{k+1} - v^*\|^2_{\Phi^{-1}} = \|v - v^* + \Phi^k (Tv^k - v^k)\|^2_{\Phi^{-1}}. \]

An analogous use of Markov’s inequality in the establishment of (13) concludes there exists some \( K_2 \), such that

\[ \Pr \left[ \sum_{k=K_1}^{K_2} \|v^{k+1} - v^*\|^2_{\Phi^{-1}} \leq \lambda_2^k \right] \geq 1 - \frac{\epsilon}{2}, \]

where \( \lambda_2 \) can be arbitrarily chosen in \( (\lambda_1, 1) \) and \( \lambda_1 \) is as in the statement of Thm. 2.