MINIMAX RISK OF MATRIX DENOISING
BY SINGULAR VALUE THRESHOLDING

By

David L. Donoho
Matan Gavish

Technical Report No. 2013-03
April 2013

Department of Statistics
STANFORD UNIVERSITY
Stanford, California 94305-4065
MINIMAX RISK OF MATRIX DENOISING
BY SINGULAR VALUE THRESHOLDING

By

David L. Donoho
Matan Gavish
Stanford University

Technical Report No. 2013-03
April 2013

This research was supported in part by
National Science Foundation grant DMS 0906812.

Department of Statistics
STANFORD UNIVERSITY
Stanford, California 94305-4065

http://statistics.stanford.edu
Minimax Risk of Matrix Denoising by Singular Value Thresholding

David L. Donoho ∗ Matan Gavish ∗

April 14, 2013

Abstract

An unknown \( m \) by \( n \) matrix \( X_0 \) is to be estimated from noisy measurements \( Y = X_0 + Z \), where the noise matrix \( Z \) has i.i.d Gaussian entries. A popular matrix denoising scheme solves the nuclear norm penalization problem \( \min_X \|Y - X\|_F^2 / 2 + \lambda \|X\|_* \), where \( \|X\|_* \) denotes the nuclear norm (sum of singular values). This is the analog, for matrices, of \( \ell_1 \) penalization in the vector case. It has been empirically observed that, if \( X_0 \) has low rank, it may be recovered quite accurately from the noisy measurement \( Y \).

In a proportional growth framework where the rank \( r_n \), number of rows \( m_n \) and number of columns \( n \) all tend to \( \infty \) proportionally to each other (\( r_n / m_n \to \rho \), \( m_n / n \to \beta \)), we evaluate the asymptotic minimax MSE

\[
M(\rho, \beta) = \lim_{m_n, n \to \infty} \inf_{\lambda} \sup_{\text{rank}(X) \leq r_n} \text{MSE}(X, \hat{X}_\lambda).
\]

Our formulas involve incomplete moments of the quarter- and semi-circle laws (\( \beta = 1 \), square case) and the Marčenko-Pastur law (\( \beta < 1 \), non square case). We also show that any least-favorable matrix \( X_0 \) has norm “at infinity”.

The nuclear norm penalization problem is solved by applying soft thresholding to the singular values of \( Y \). We also derive the minimax threshold, namely the value \( \lambda^*(\rho) \) which is the optimal place to threshold the singular values.

All these results are obtained for general (non square, non symmetric) real matrices. Comparable results are obtained for square symmetric nonnegative-definite matrices.

Keywords. Matrix Denoising — Nuclear Norm Minimization — Singular Value Thresholding — Soft Thresholding — Least-Favorable Situation — Optimal Threshold — Stein Unbiased Risk Estimate — Anderson Monotonicity — Matrix Completion from Gaussian Measurements — Phase Transition

Acknowledgements. Thanks to Iain Johnstone, Andrea Montanari and Art Owen for advice at several crucial points. This work was partially supported by NSF DMS 0906812 (ARRA). MG was supported by a William R. and Sara Hart Kimball Stanford Graduate Fellowship and a Technion EE Sohns Promising Scientist Award.

∗Department of Statistics, Stanford University
1 Introduction

Suppose we observe a single noisy matrix $Y$, generated by adding noise $Z$ to an unknown matrix $X_0$, so that $Y = X_0 + Z$, where $Z$ is a noise matrix. We wish to recover the matrix $X_0$ with some bound on the mean squared error (MSE). This is hopeless in case $X_0$ is a completely general matrix and the noise $Z$ is arbitrary; but in case $X_0$ happens to be of relatively low rank and the noise matrix is i.i.d standard Gaussian, one can indeed guarantee quantitatively accurate recovery. This paper provides explicit formulas for the best possible guarantees obtainable by a popular, computationally practical procedure.

Specifically, let $Y, X_0$ and $Z$ be $m$-by-$n$ matrices and suppose that $Z$ has i.i.d entries, $Z_{i,j} \sim \mathcal{N}(0, 1)$. Consider the following Nuclear-Norm Penalization (NNP) problem:

$$\text{NNP} \quad \hat{X}_\lambda = \arg\min_{X \in \mathbb{M}_{m \times n}} \frac{1}{2} \|Y - X\|_F^2 + \lambda \|X\|_*,$$

where $\|X\|_*$ denotes the sum of singular values of $X \in \mathbb{M}_{m \times n}$, also known as the nuclear norm, and $\lambda > 0$ is a penalty factor. A solution to (NNP) is efficiently computable by modern convex optimization software \cite{1}; it shrinks away from $Y$ in the direction of smaller nuclear norm.

Measure performance (risk) by mean-squared error (MSE). When the unknown $X_0$ is of known rank $r$ and belongs to a matrix class $X_{m,n} \subset \mathbb{M}_{m \times n}$, the minimax MSE of NNP is

$$\mathcal{M}_{m,n}(r|X) = \inf_{\lambda} \sup_{X_0 \in X_{m,n}} \frac{1}{mn} \mathbb{E}_{X_0} \left\| \hat{X}_\lambda(X_0 + Z) - X_0 \right\|_F^2,$$

namely the worst-case risk of $\hat{X}_\lambda$, where $\lambda$ is the threshold for which this worst-case risk is the smallest possible. For square matrices, $m = n$, we write $\mathcal{M}_n(r|X)$ instead of $\mathcal{M}_{n,n}(r|X)$. In a very clear sense $\mathcal{M}_{m,n}(r|X)$ gives the best possible guarantee for the MSE of NNP, based solely on the rank and problem size, and not on other properties of the matrix $X_0$.

1.1 Minimax MSE Evaluation

In this paper, we calculate the minimax MSE $\mathcal{M}_{m,n}(r|X)$ for two matrix classes $X$:

1. **General Matrices**: $X = \text{Mat}_{m,n}$: The signal $X_0$ is a real matrix $X_0 \in \mathbb{M}_{m \times n}$ ($m \leq n$).

2. **Symmetric Matrices**: $X = \text{Sym}_{n}$: The signal $X_0$ is a real, symmetric positive semidefinite matrix $X_0 \in S^n_+ \subset \mathbb{M}_{n \times n}$.

In both cases, the asymptotic MSE (AMSE) in the “large $n$” asymptotic setting admits considerably simpler and more accessible formulas than the minimax MSE for finite $n$. So in addition to the finite-$n$ minimax MSE, we study the asymptotic setting where a sequence of problem size triples $(r_n, m_n, n)$ is indexed by $n \to \infty$, and where, along this sequence $m/n \to \beta \in (0, 1)$ and $r/m \to \rho \in (0, 1)$. We think of $\beta$ as the matrix shape parameter; $\beta = 1$ corresponds to a square matrix, and $\beta < 1$ to a matrix wider than it is tall. We think of $\rho$ as the fractional rank parameter, with $\rho \approx 0$ implying low rank relative to matrix size. Using these notions we can define the asymptotic minimax MSE (AMSE)

$$\mathcal{M}(\rho, \beta|X) = \lim_{n \to \infty} \mathcal{M}_{m_n,n}(r_n|X).$$

We obtain explicit formulas for the asymptotic minimax MSE in terms of incomplete moments of classical probability distributions: the quarter-circle and semi-circle laws.
(square case $\beta = 1$) and the Marčenko-Pastur distribution (non-square case $\beta < 1$).

Figures 1 and 2 show how the AMSE depends on the matrix class $\mathbf{X}$, the rank fraction $\rho$, and the shape factor $\beta$. We also give explicit formulas for the optimal regularization parameter $\lambda^*$, also as a function of $\rho$; see Figures 3 and 4.

These minimax MSE results constitute best possible guarantees, in the sense that for the procedure in question, the MSE is actually attained at some rank $r$ matrix, so that no better guarantee is possible for the given tuning parameter $\lambda$; but also, no other tuning parameter offers a better such guarantee.

1.2 Motivations

We see four reasons to develop these bounds.

1.2.1 Applications

Several important problems in modern signal and image processing, in network data analysis, and in computational biology can be cast as recovery of low rank matrices from noisy data, and nuclear norm minimization has become a popular strategy in many cases; see for example [2, 3] and references therein. Our results provide sharp limits on what such procedures can hope to achieve, and validate rigorously the idea that low rank alone is enough to provide some level of performance guarantee; in fact, they precisely quantify the best possible guarantee.

1.2.2 Limits on Possible Improvements

One might wonder whether some other procedure offers even better guarantees than NNP. Consider then the minimax risk over all procedures

$$\mathcal{M}^*(\rho, \beta|\mathbf{X}) = \inf_{\hat{X}} \sup_{X_0 \in \mathbf{X}_{m,n}} \mathbb{E}_{X_0} \left\| \hat{X} (X_0 + Z) - X_0 \right\|_F^2,$$

here $\hat{X} = \hat{X}(Y)$ is some measurable function of the observations. Here one wants to find the best possible procedure, without regard to efficient computation. We also prove a lower bound on the minimax MSE over all procedures, and provide an asymptotic evaluation

$$\mathcal{M}^*(\rho, \beta|\mathbf{X}) \geq \mathcal{M}^{-}(\rho, \beta|\mathbf{X}) \equiv \rho + \beta \rho - \beta \rho^2.$$

In the square case ($\beta = 1$), this simplifies to $\mathcal{M}^*(\rho|\mathbf{X}) \geq \mathcal{M}^{-}(\rho|\mathbf{X}) \equiv \rho(2 - \rho)$. The NNP-minimax MSE is by definition larger than the minimax MSE: $\mathcal{M}(\rho, \beta|\mathbf{X}) \geq \mathcal{M}^*(\rho, \beta|\mathbf{X})$.

While there may be procedures outperforming NNP, the performance improvement turns out to be limited. Indeed, our formulas show that

$$\frac{\mathcal{M}(\rho, \beta|\mathbf{X})}{\mathcal{M}^{-}(\rho, \beta|\mathbf{X})} \leq 2 \left( 1 + \frac{\sqrt{\beta}}{1 + \beta} \right),$$

while

$$\lim_{\rho \to 0} \frac{\mathcal{M}(\rho, \beta|\mathbf{X})}{\mathcal{M}^{-}(\rho, \beta|\mathbf{X})} = 2 \left( 1 + \frac{\sqrt{\beta}}{1 + \beta} \right).$$

(4)

For square matrices ($\beta = 1$), this simplifies to

$$\frac{\mathcal{M}(\rho|\mathbf{X})}{\mathcal{M}^{-}(\rho|\mathbf{X})} \leq 3 \quad \lim_{\rho \to 0} \frac{\mathcal{M}(\rho|\mathbf{X})}{\mathcal{M}^{-}(\rho|\mathbf{X})} = 3.$$

(5)
In words, the potential improvement in minimax $\text{AMSE}$ of any other matrix denoising procedure over NNP is at most a factor of 3; and if any such improvement were available, it would only be available in extreme low rank situations. Actually obtaining such an improvement in performance guarantees is an interesting research challenge.

1.2.3 Parallels in Minimax Decision Theory

The low-rank matrix denoising problem stands in a line of now-classical problems in minimax decision theory. Consider the sparse vector denoising problem, where an unknown vector $x$ of interest yields noisy observations $y = x + z$ with noise $z \sim_{\text{i.i.d.}} N(0, 1)$; the vector $x$ is sparsely nonzero - $\# \{ i : x(i) \neq 0 \} \leq \varepsilon \cdot n$ - with $z$ and $x$ independent. In words, a vector with a fraction $\leq \varepsilon$ of non zeros is observed with noise. In this setting, consider the following $\ell_1$-norm penalization problem:

$$ (P_1) \quad \hat{x}_\lambda = \arg\min_{x \in \mathbb{R}^n} \frac{1}{2} \|y - x\|_2^2 + \lambda \|x\|_1. $$

(6)

The sparse vector denoising problem exhibits several striking structural resemblances to low-rank matrix denoising:

- **Thresholding Representation.** For scalar $y$ define the soft thresholding nonlinearity by

$$ \eta_\lambda(y) = \text{sign}(y) \cdot (|y| - \lambda)_+. $$

In words, values larger than $\lambda$ are shifted towards zero by $\lambda$ while those smaller than $\lambda$ are set to zero. The solution vector $\hat{x}_\lambda$ of $(P_1)$ obeys $\hat{x}_\lambda = (\eta_\lambda(y_i))$, namely, it applies $\eta_\lambda$ coordinate-wise. Similarly, the solution of (NNP), *written in the SVD basis for $Y$*, obeys $\hat{X} = \text{diag}(\eta_\lambda(\text{diag}(Y)))$; it applies $\eta_\lambda$ coordinate-wise to the singular values of the noisy matrix $Y$.

**Remark:** by this observation, $(P_1)$ can also be called “soft thresholding” or “soft threshold denoising”, and in fact these other terms are the labels in common use. Similarly, NNP amounts to “soft thresholding of singular values”, this paper will henceforth use the term *Singular Value Soft Thresholding* (SVST).

- **Sparsity/Low Rank Analogy.** The objects to be recovered in the sparse vector denoising problem have sparse entries; those to be recovered in the low rank matrix denoising problem have sparse singular values. Thus the fractional sparsity parameter $\varepsilon$ is analogous to the fractional rank parameter $\rho$. It is natural to ask the same questions about behavior of minimax MSE in one setting (say, asymptotics as $\rho \to 0$) as in the other setting ($\varepsilon \to 0$). In fact, such comparisons turn out to be illuminating.

- **Structure of the Least-Favorable Estimand.** Among sparse vectors $x$ of a given fixed sparsity fraction $\varepsilon$, which of those is the hardest to estimate? This should maximize the mean-squared error of soft thresholding, even under the most clever choice of $\lambda$. This least-favorable configuration is singled out in the minimax AMSE

$$ M_n(\varepsilon) = \inf_{\lambda} \sup_{\# \{ i : x(i) \neq 0 \leq \varepsilon \cdot n \}} \frac{1}{n} \mathbb{E} \|\hat{x}_\lambda - x\|_2^2. $$

(7)

In this min/max, the least favorable situation has all its non zeros in some sense “at infinity”; i.e. all sparse vectors which place large enough values on the nonzeros are nearly least favorable, i.e. essentially make the problem maximally difficult for the estimator, even when it is optimally tuned. In complete analogy, in low-rank matrix denoising we will see that all low rank matrices which are in an appropriate sense “sufficiently large”, are thereby almost least favorable.
• **Structure of the Minimax Smoothing Parameter.** In the sparse vector denoising AMSE (7) the $\lambda = \lambda(\varepsilon)$ achieving the infimum is a type of optimal regularization parameter, or optimal threshold. It decreases as $\varepsilon$ increases, with $\lambda(\varepsilon) \to 0$ as $\varepsilon \to 1$. Paralleling this, we show that the low-rank matrix denoising AMSE (2) has minimax singular value soft threshold $\lambda^*(\rho)$ decreasing as $\rho$ increases, and $\lambda^*(\rho) \to 0$ as $\rho \to 1$.

Despite these similarities, there is one major difference between sparse vector denoising and low-rank matrix denoising. In the sparse vector denoising problem, the soft-thresholding minimax MSE was compared to the minimax MSE over all procedures by Donoho and Johnstone [4]. Let $M_\varepsilon = \lim_{n \to \infty} M_n(\varepsilon)$ denote the soft thresholding AMSE and define the minimax AMSE over all procedures via

$$M^*(\varepsilon) = \lim_{n \to \infty} \inf_{\hat{x}} \sup_{\# \{ i : x(i) \neq 0 \leq \varepsilon \cdot n \}} \frac{1}{n} \mathbb{E} \| \hat{x} - x \|^2_2,$$

where here $\hat{x} = \hat{x}(y)$ denotes any procedure which is measurable in the observations. In the limit of extreme sparsity, Soft Thresholding is asymptotically minimax [4]:

$$\frac{M(\varepsilon)}{M^*(\varepsilon)} \to 1 \text{ as } \varepsilon \to 0.$$

Breaking the chain of similarities, we are not able to show a similar asymptotic minimaxity for SVST in the low rank matrix denoising problem. Although eq. (4) says that soft thresholding of singular values is asymptotically not more than a factor of 3 suboptimal, we doubt that anything better than a factor of 3 can be true; specifically, we conjecture that SVST suffers a minimaxity gap. For example, for $\beta = 1$, we conjecture that

$$\frac{M(\rho | X)}{M^*(\rho | X)} \to 3 \text{ as } \rho \to 0.$$

We believe that interesting new estimators will be found improving upon singular value soft thresholding by essentially this factor of 3. Namely: there may be substantially better guarantees to be had under extreme sparsity, than those which can be offered by SVST. Settling the minimaxity gap for SVST seems a challenging new research question.

### 1.2.4 Indirect Observations.

Evaluating the Minimax MSE of SVST has an intriguing new motivation, [5, 6, 7] arising from the newly evolving fields of compressed sensing and matrix completion.

Consider the problem of recovering an unknown matrix $X_0$ from noiseless, indirect measurements. Let $A : \mathbb{R}^{m \times n} \to \mathbb{R}^p$ be a linear operator, and consider observations

$$y = A(X_0).$$

In words, $y \in \mathbb{R}^p$ contains $p$ linear measurements of the matrix object $X_0$. Can we recover $X_0$? It may seem that $p \geq mn$ measurements are required, and in general this would be true; but if $X_0$ happens to be of low rank, and $A$ has suitable properties, we may need substantially fewer measurements.

Consider reconstruction by **Nuclear Norm Minimization**:

$$\left( P_{nuc} \right) \quad \min \| X \|_* \quad \text{subject to } y = A(X).$$

(8)

Recht and co-authors found that when the matrix representing the operator $A$ has i.i.d $\mathcal{N}(0,1)$ entries, and the matrix is of rank $r$, the matrix $X_0$ is recoverable from
$p < nm$ measurements for certain combinations of $p$ and $r$ \[8\]. The operator $A$ offers so-called Gaussian measurements when the representation of the operator as a matrix has i.i.d. Gaussian entries. Empirical work by by Recht, Xu and Hassibi \[9, 10\], Fazel, Parillo and Recht \[8\], Tanner and Wei \[11\] and Oymak and Hassibi \[12\], documented for Gaussian measurements a phase transition phenomenon, i.e. a fairly sharp transition from success to failure as $r$ increases, for a given $p$. Putting $\rho = r/m$ and $\delta = p/(mn)$ it appears that there is a critical sampling rate $\delta^*(\rho) = \delta^*(\rho; \beta)$, such that, for $\delta > \delta^*(\rho)$, NNM is successful for large $m, n$, while for $\delta < \delta^*(\rho)$, NNM fails. $\delta^*(\rho)$ provides a sharp “sampling limit” for low rank matrices, i.e. a clear statement of how many measurements are needed to recover a low rank matrix, by a popular and computationally tractable algorithm.

In very recent work, \[5, 6, 7\], have shown empirically that the precise location of the phase transition coincides with the minimax MSE:

$$\delta^*(\rho; \beta) = M(\rho, \beta|X), \quad \rho \in (0, 1), \quad \beta \in (0, 1);$$

(9)

A key requirement for discovering and verifying (9) empirically was to obtain an explicit formula for the right-hand side; that explicit formula is derived and proven in this paper. Relationship (9) connects two seemingly unrelated problems: Matrix denoising from direct observations and Matrix recovery from incomplete measurements. Both problems are attracting a large and growing research literature. Equation (9) demonstrates the importance of minimax MSE calculations even in a seemingly unrelated setting where there is no noise and no statistical decision to be made!
2 Results

2.1 Least-Favorable matrix.

For SVST, any matrix of rank \( r \) is least-favorable among \( m \times n \) matrices of rank at most \( r \), in the limit \( \|X_0\| \to \infty \).

**Theorem 1.** The worst-case matrix for SVST has its principal subspace “at \( \infty \)”.

Let \( \lambda > 0 \), \( m \leq n \in \mathbb{N} \) and \( 1 \leq r \leq m \). For the worst-case risk of \( \hat{X}_\lambda \) on \( m \times n \) matrices of rank at most \( r \), we have

\[
\sup_{X_0 \in \mathcal{M}_{m \times n}^r, \text{rank}(X_0) \leq r} R(\hat{X}_\lambda, X_0) = \lim_{\mu \to \infty} R(\hat{X}_\lambda, \mu C), \tag{10}
\]

where \( C \in \mathcal{M}_{m \times n} \) is any fixed matrix of rank exactly \( r \).

2.2 Minimax MSE.

Let \( W_i(m, n) \) denote the marginal distribution of the \( i \)-th largest eigenvalue of a standard central Wishart matrix \( W_m(1, n) \), namely, the \( i \)-th largest eigenvalue of the random matrix \( \frac{1}{n}ZZ' \) where \( Z \in \mathcal{M}_{m \times n} \) has iid \( \mathcal{N}(0, 1) \) entries. Define for \( \Lambda > 0 \) and \( \alpha \in \{1/2, 1\}\)

\[
M_n(\Lambda; r, m, \alpha) = \frac{r}{m} + \frac{r}{n} - \frac{r^2}{mn} + \frac{r(n - r)}{mn} \Lambda^2 + \alpha \frac{(n - r)}{mn} \sum_{i=1}^{m-r} w_i(\Lambda; m - r; n - r), \tag{11}
\]

where

\[
w_i(\Lambda; m, n) = \int_{\Lambda^2} (\sqrt{t} - \Lambda)^2 dW_i(m, n)(t) \tag{12}
\]

is a combination of the complementary incomplete moments of standard central Wishart eigenvalues

\[
\int_{\Lambda^2} t^{k/2} dW_i(m, n)(t)
\]

for \( k = 0, 1, 2 \).

**Theorem 2.** An implicit formula for the finite-\( n \) minimax MSE. The minimax MSE of SVST over \( m \)-by-\( n \) matrices of rank at most \( r \) is given by

\[
\mathcal{M}_n(r, m)_{Mat} = \min_{\Lambda \geq 0} M_n(\Lambda; r, m, 1) \quad \text{and} \quad \mathcal{M}_n(r)_{Sym} = \min_{\Lambda \geq 0} M_n(\Lambda; r, n, 1/2),
\]

where the minimum on the right hand sides is unique.

In fact, we will see that \( M_n(\Lambda; r, m, \alpha) \) is convex in \( \Lambda \). As the densities of the standard central Wishart eigenvalues \( W_i(m, n) \) are known \[13\], this makes it possible, in principle, to tabulate the finite-\( n \) minimax risk.
2.3 Minimax AMSE (Asymptotic MSE).

A more accessible formula is obtained by calculating the large-$n$ asymptotic minimax MSE, where $r = r(n)$ and $m = m(n)$ both grow proportionally to $n$. Let us write AMSE for Asymptotic MSE. For the case $X_{m,n} = \text{Mat}_{m,n}$, we assume a limiting rank fraction $\rho = \lim_{n \to \infty} r/m$ and limiting aspect ratio $\beta = \lim_{n \to \infty} m/n$ and consider

$$M(\rho, \beta | \text{Mat}) = \lim_{n \to \infty} M_n(r, m | \text{Mat}) = \lim_{n \to \infty} \inf \lambda \sup_{\lambda_0 \in M(r, m | \text{Mat})} \frac{1}{mn} \mathbb{E} \left\| \hat{X}_\lambda - X_0 \right\|_F^2. \quad (13)$$

Similarly, for the case $X_{m,n} = \text{Sym}_n$, we assume a limiting rank fraction $\rho = \lim_{n \to \infty} r/n$ and consider

$$M(\rho | \text{Sym}) = \lim_{n \to \infty} M_n(r | \text{Sym}) = \lim_{n \to \infty} \inf \lambda \sup_{\lambda_0 \in \text{Sym}_n} \frac{1}{n^2} \mathbb{E} \left\| \hat{X}_\lambda - X_0 \right\|_F^2. \quad (14)$$

The Marčenko-Pastur distribution \cite{14} gives the asymptotic empirical distribution of Wishart eigenvalues. It has density

$$p_\gamma(t) = \frac{1}{2\pi \gamma t} \sqrt{(\gamma_+ - t)(t - \gamma_-)} \cdot 1_{[\gamma_- \gamma_+]}(t), \quad (15)$$

where $\gamma_\pm = (1 \pm \sqrt{\gamma})^2$. Define the complementary incomplete moments of the Marčenko-Pastur distribution

$$P_\gamma(x; k) = \int_x^{\gamma^+} t^k p_\gamma(t) \, dt. \quad (16)$$

Finally, let

$$M(\Lambda; \rho, \hat{\rho}, \alpha) = \rho + \hat{\rho} - \rho \hat{\rho} + (1 - \hat{\rho}) \left[ \rho \Lambda^2 + \alpha (1 - \rho) \left( P_\gamma(\Lambda^2; 1) - 2\Lambda P_\gamma(\Lambda^2; \frac{1}{2}) + \Lambda^2 P_\gamma(\Lambda^2; 0) \right) \right], \quad (17)$$

with $\gamma = \gamma(\rho, \hat{\rho}) = (\rho - \rho \hat{\rho})/(\rho - \rho \hat{\rho})$.

**Theorem 3.** An explicit formula for the minimax Asymptotic MSE. For the minimax AMSE of SVST we have

$$M(\rho, \beta | \text{Mat}) = \min_{0 \leq \Lambda \leq \gamma_+} M(\Lambda; \rho, \beta \rho, 1) \quad (18)$$

$$M(\rho | \text{Sym}) = \min_{0 \leq \Lambda \leq \gamma_+} M(\Lambda; \rho, 1/2), \quad (19)$$

with $\gamma_+ = \left( 1 + \sqrt{(\beta - \beta \rho)/(1 - \beta \rho)} \right)^2$, where the minimum on the right hand sides is unique. Moreover, for any $0 < \beta \leq 1$, the function $\rho \mapsto M(\rho, \beta | \text{Mat})$ is continuous and increasing on $\rho \in [0, 1]$, with $M(0, \beta | \text{Mat}) = 0$ and $M(1, \beta | \text{Mat}) = 1$. The same is true for $M(\rho | \text{Sym})$.

The curves $\rho \mapsto M(\rho, \beta | \text{Mat})$, for different values of $\beta$, are shown in Figure 1. The curves $\rho \mapsto M(\rho, \beta | \text{Mat})$ and $\rho \mapsto M(\rho, \beta | \text{Mat})$ are shown in Figure 2.
Figure 1: The minimax AMSE curves for case $Mat$, defined in $\text{(18)}$, for a few values of $\beta$.

Figure 2: The minimax AMSE curves for case $Mat$ with $\beta = 1$ and case $Sym$. 
2.4 Computing the minimax AMSE.

To compute $\mathcal{M}(\rho, \beta|\text{Mat})$ and $\mathcal{M}(\rho|\text{Sym})$ we need to minimize (17). Define

$$\Lambda_*(\rho, \beta, \alpha) = \arg\min_\Lambda \mathcal{M}(\Lambda; \rho, \hat{\rho}, \alpha).$$

(20)

**Theorem 4.** A characterization of the minimax AMSE for general $\beta$. For any $\alpha \in \{1/2, 1\}$ and $\beta \in (0, 1]$, the function $\rho \mapsto \Lambda_*(\rho, \beta, \alpha)$ is decreasing on $\rho \in [0, 1]$ with

$$\lim_{\rho \to 0} \Lambda_*(\rho, \beta, \alpha) = \Lambda_*(0, \beta, \alpha) = 1 + \sqrt{\beta} \quad \text{and}$$

$$\lim_{\rho \to 1} \Lambda_*(\rho, \beta, \alpha) = \Lambda_*(1, \beta, \alpha) = 0.$$  

(21)

For $\rho \in (0, 1)$, the minimizer $\Lambda_*(\rho, \beta, \alpha)$ is the unique root of the equation in $\Lambda$

$$P_\gamma(\Lambda^2; \frac{1}{2}) - \Lambda \cdot P_\gamma(\Lambda^2; 0) = \frac{\Lambda \rho}{\alpha(1 - \rho)},$$

(23)

where the left hand side of (23) is a decreasing function of $\Lambda$.

The minimizer $\Lambda_*(\rho, \beta, \alpha)$ can therefore be determined numerically by binary search. (In fact, we will see that $\Lambda_*$ is the unique minimizer of the convex function $\Lambda \mapsto \mathcal{M}(\Lambda; \rho, \hat{\rho}, \alpha)$.) Evaluating $\mathcal{M}(\rho, \beta|\text{Mat})$ and $\mathcal{M}(\rho|\text{Sym})$ to precision $\epsilon$ thus requires $\log(1/\epsilon)$ evaluations of the complementary incomplete Marčenko-Pastur moments (16).

For square matrices ($\beta = 1$), this computation turns out to be even simpler, and only requires evaluation of elementary trigonometric functions.

**Theorem 5.** A characterization of the minimax AMSE for $\beta = 1$. We have

$$\mathcal{M}(\Lambda; \rho, \rho, \alpha) = \rho(2 - \rho) + (1 - \rho) \left[ \rho \Lambda^2 + \alpha(1 - \rho) \left( Q_2(\Lambda) - 2\Lambda Q_1(\Lambda) + \Lambda^2 Q_0(\Lambda) \right) \right],$$

(24)

where

$$Q_0(x) = \frac{1}{\pi} \int_x^2 \sqrt{4 - t^2} \, dt = 1 - \frac{1}{2\pi} \sqrt{4 - x^2} - \frac{2}{\pi} \tan(\frac{x}{\sqrt{4 - x^2}})$$

(25)

$$Q_1(x) = \frac{1}{\pi} \int_x^2 t\sqrt{4 - t^2} \, dt = \frac{1}{3\pi} (4 - x^2)^{3/2}$$

(26)

$$Q_2(x) = \frac{1}{\pi} \int_x^2 t^2\sqrt{4 - t^2} \, dt = 1 - \frac{1}{4\pi} x\sqrt{4 - x^2}(x^2 - 2) - \frac{2}{\pi} \sin(\frac{x}{2})$$

(27)

are the complementary incomplete moments of the Quarter Circle law. Moreover, for $\alpha \in \{1/2, 1\}$

$$\Lambda_*(\rho, \rho, \alpha) = 2 \cdot \sin(\theta_\alpha(\rho)),$$

(28)

where $\theta_\alpha(\rho) \in [0, \pi/2]$ is the unique solution to the transcendental equation

$$\theta + \cot(\theta) \cdot \left( 1 - \frac{1}{3} \cos^2(\theta) \right) = \frac{\pi(1 + \alpha^{-1}\rho - \rho)}{2(1 - \rho)}. $$

(29)

The left hand side of (29) is a decreasing function of $\theta$.

In [15] we make available a Matlab script, and a web-based calculator for evaluating $\mathcal{M}(\rho, \beta|\text{Mat})$ and $\mathcal{M}(\rho|\text{Sym})$. The implementation provided employs binary search to solve (23) (or (29) for $\beta = 1$) and then feeds the minimizer $\Lambda_*$ into (17) (or into (24) for $\beta = 1$).
2.5 Asymptotically optimal tuning for the SVST threshold $\lambda$.

The crucial functional $\Lambda_*$, defined in (20), can now be explained as the optimal (minimax) threshold of SVST in a special system of units. Let $\lambda^*(m, n, r|X)$ denote the minimax tuning threshold, namely

$$\lambda^*(m, n, r|X) = \arg\min_{\lambda} \sup_{X_0 \in \mathbb{X}_{m, n}} \frac{1}{mn} \mathbb{E}_{X_0} \left\| \hat{X}_\lambda (X_0 + Z) - X_0 \right\|_F^2.$$  

**Theorem 6.** Asymptotic minimax tuning of SVST. Consider again a sequence $n \mapsto (m(n), r(n))$ with a limiting rank fraction $\rho = \lim_{n \to \infty} r/m$ and a limiting aspect ratio $\beta = \lim_{n \to \infty} m/n$. For the asymptotic minimax tuning threshold we have

$$\lim_{n \to \infty} \frac{1}{\sqrt{n}} \lambda^*(m, n, r|\text{Mat}) = \sqrt{(1 - \beta \rho)} \cdot \Lambda^*(\rho, \beta, 1) \quad \text{and} \quad \lim_{n \to \infty} \frac{1}{\sqrt{n}} \lambda^*(n, r|\text{Sym}) = \sqrt{(1 - \rho)} \cdot \Lambda^*(\rho, 1, 1/2).$$

The curves $\rho \mapsto \lim_{n \to \infty} \lambda^*(m, n, r|\text{Mat})/\sqrt{n}$, namely the scaled asymptotic minimax tuning threshold for SVST, are shown in Figure 3 for different values of $\beta$. The curves $\rho \mapsto \lim_{n \to \infty} \lambda^*(n, n, r|\text{Mat})/\sqrt{n}$ and $\rho \mapsto \lim_{n \to \infty} \lambda^*(n, r|\text{Sym})/\sqrt{n}$ are shown in Figure 4.

2.6 Parametric representation of the minimax AMSE for square matrices.

For square matrices ($\rho = \tilde{\rho}$, $\beta = 1$) the minimax curves $\mathcal{M}(\rho, 1|\text{Mat})$ and $\mathcal{M}(\rho|\text{Sym})$ admit a parametric representation in the $(\rho, \mathcal{M})$ plane using elementary trigonometric functions.

**Theorem 7.** Parametric representation of the minimax AMSE curve for $\beta = 1$. As $\theta$ ranges over $[0, \pi/2]$,

$$\rho(\theta) = 1 - \frac{\pi/2}{\theta + (\cot(\theta) \cdot (1 - \frac{1}{3} \cos^2(\theta)))}$$

$$\mathcal{M}(\theta) = 2\rho(\theta) - \rho^2(\theta) + 4\rho(\theta)(1 - \rho(\theta))\sin^2(\theta) + \frac{4}{\pi} (1 - \rho)^2 \left[ (\pi - 2\theta)(\frac{5}{4} - \cos(\theta)^2) + \frac{\sin(2\theta)}{12}(\cos(2\theta) - 14) \right]$$

is a parametric representation of $\rho \mapsto \mathcal{M}(\rho, \rho|\text{Mat})$, and similarly

$$\rho(\theta) = 1 - \frac{\pi/2}{\theta + (\cot(\theta) \cdot (1 - \frac{1}{3} \cos^2(\theta)))}$$

$$\mathcal{M}(\theta) = 2\rho(\theta) - \rho^2(\theta) + 4\rho(\theta)(1 - \rho(\theta))\sin^2(\theta) + \frac{2}{\pi} (1 - \rho)^2 \left[ (\pi - 2\theta)(\frac{5}{4} - \cos(\theta)^2) + \frac{\sin(2\theta)}{12}(\cos(2\theta) - 14) \right]$$

is a parametric representation of $\rho \mapsto \mathcal{M}(\rho|\text{Sym})$. 
Figure 3: (Nonsquare Cases.) The scaled asymptotic minimax tuning threshold for SVST, \( \rho \mapsto \lim_{n \to \infty} \lambda\ast(m, n, r | Mat) / \sqrt{n} \) when \( m/n \to \beta \) and \( r/m \to \rho \), for a few values of \( \beta \).

Figure 4: (Square Case.) The scaled asymptotic minimax tuning threshold for SVST, \( \rho \mapsto \lim_{n \to \infty} \lambda\ast(n, n, r | Mat) / \sqrt{n} \) and \( \rho \mapsto \lim_{n \to \infty} \lambda\ast(n, r | Sym) / \sqrt{n} \), when \( r/m \to \rho \).
2.7 Minimax AMSE near $\rho = 0$.

**Theorem 8.** Minimax AMSE to first order in $\rho$ near $\rho = 0$. For the behavior of the minimax curves near $\rho = 0$ we have

\[
M(\rho, \beta | \text{Mat}) = 2 \left( 1 + \sqrt{\beta} + \beta \right) \rho + o(\rho)
\]

and in particular

\[
M(\rho, 1 | \text{Mat}) = 6 \rho + o(\rho).
\]

Moreover,

\[
M(\rho | \text{Sym}) = 6 \rho + o(\rho).
\]

The minimax AMSE curves $\rho \mapsto M(\rho, \beta | \text{Mat})$ for small values of $\rho$, and the corresponding approximation slopes $2(1 + \sqrt{\beta} + \beta)$ are shown in Figure 5 for several values of $\beta$. We find it surprising that asymptotically, symmetric positive definite matrices are no easier to recover than general square matrices. This phenomenon is also seen in the case of sparse vector denoising, where in the limit of extreme sparsity, the non negativity of the non zeros does not allow one to reduce the minimax MSE.\footnote{Compare results in [4] with [16]. To be clear, in both matrix denoising and vector denoising, there is an MSE advantage for each fixed positive rank fraction/sparsity fraction. It’s just that the benefit goes away as either fraction tends to 0.}

2.8 AMSE vs. the asymptotic global minimax MSE

In (3) we have introduced global minimax MSE $M_{m,n}^*(r|X)$, namely the minimax risk over all measurable denoisers $\hat{X} : M_{mxn} \to M_{mxn}$. To define the large-$n$ asymptotic global minimax MSE analogous to (13), consider sequences where $r = r(n)$ and $m = m(n)$ both grow proportionally to $n$, such that both limits $\rho = \lim_{n \to \infty} r/m$ and $\beta = \lim_{n \to \infty} m/n$ exist. Define the asymptotic global minimax MSE

\[
M^*(\rho, \beta | X) = \lim_{n \to \infty} M_{m,n}^*(r, m|X)
\]

**Theorem 9.**

1. For the global minimax MSE we have

\[
M_{m,n}^*(r|X) \geq \frac{r}{m} + \frac{r}{n} - \frac{r^2 + r}{mn}
\]

for case Mat and, if $m = n$, for case Sym.

2. For the asymptotic global minimax MSE we have

\[
M^*(\rho, \beta | X) \geq \rho + \tilde{\rho} - \rho \tilde{\rho}
\]

for case Mat and, if $\beta = 1$, for case Sym. Here $\tilde{\rho} = \beta \rho$.

3. Let

\[
M^-(\rho, \beta) = \rho + \tilde{\rho} - \rho \tilde{\rho}
\]

denote our lower bound on asymptotic global minimax MSE. Then

\[
\frac{M(\rho, \beta | X)}{M^-(\rho, \beta)} \leq 2 \left( 1 + \frac{\sqrt{\beta}}{1 + \beta} \right)
\]

and

\[
\lim_{\rho \to 0} \frac{M(\rho, \beta | X)}{M^-(\rho, \beta)} = 2 \left( 1 + \frac{\sqrt{\beta}}{1 + \beta} \right)
\]
2.9 Outline

The body of the paper proves the above results. Section 3 introduces notation, and proves auxiliary lemmas. In Section 4 we characterize the worst-case MSE of SVST for matrices of fixed size. Section 5 derives formula (11) for the worst-case MSE, and proves Theorem 2. In Section 6 we pass to the large-$n$ limit, deriving formula (17), which provides the worst-case asymptotic MSE in the large-$n$ limit, and prove Theorem 3. In Section 7 we investigate the minimizer of the asymptotic worst-case MSE function, and its minimum, namely the minimax AMSE, and prove Theorems 4 and 5. We then connect the minimizer of the worst-case AMSE function to the minimax tuning threshold for SVST and prove Theorem 6. Finally, we derive a parametric representation of the minimax AMSE curve for square matrices (Theorem 7). In Section 8 we calculate the first order approximation of the minimax AMSE around $\rho = 0$ and prove Theorem 8. In Section 9 we extend the discussion scope from SVST denoisers to all denoisers, investigate the global minimax MSE, and prove Theorem 9. A derivation of the Stein Unbiased Risk Estimate for SVST, which is instrumental in the proof of Theorem 1, is discussed in Appendix A.
3 Preliminaries

3.1 Scaling

Our main object of interest, the worst-case MSE of SVST,

\[
\sup_{X_0 \in \mathbb{M}_{m \times n}} \frac{1}{mn} \mathbb{E} \left\| \hat{X} - X_0 \right\|_F^2 ,
\]

is more conveniently expressed using a specially calibrated risk function. Since the SVST denoisers are scale-invariant -

\[
\mathbb{E} \left\| X - \hat{X} \right\|_F^2 = \sigma^2 \mathbb{E} \left\| \frac{X}{\sigma} - \hat{X} \right\|_F^2 ,
\]

we are free to introduce the scaling \( \sigma = n^{-1/2} \) and define the risk function of a denoiser \( \hat{X} : \mathbb{M}_{m \times n} \to \mathbb{M}_{m \times n} \) at \( X_0 \in \mathbb{M}_{m \times n} \) by

\[
R(\hat{X}, X_0) := \frac{1}{m} \mathbb{E} \left\| \frac{1}{\sqrt{n}} Z \left( X_0 + \frac{1}{\sqrt{n}} Z \right) - X_0 \right\|_F^2 .
\]

Then, the worst-case MSE of \( \hat{X} \) at \( X_0 \) is given by

\[
\sup_{X_0 \in \mathbb{M}_{m \times n}} \frac{1}{mn} \mathbb{E} \left\| \hat{X} - X_0 \right\|_F^2 = \sup_{X_0 \in \mathbb{M}_{m \times n}} R(\hat{X}, X_0) .
\]

To vary the SNR in the problem, it will be convenient to vary the norm of the signal matrix \( X_0 \) instead, namely, to consider \( Y = \mu X_0 + \frac{1}{\sqrt{n}} Z \) with \( \frac{1}{m} \|X_0\|_F^2 = 1 \).

3.2 Notation

- Throughout this text, \( Y \) will denote the data matrix \( Y = \mu X_0 + \frac{1}{\sqrt{n}} Z \).
- \( \mathbb{M}_{m \times n} \) and \( \mathbb{O}_m \) denote the set of real-valued \( m \)-by-\( n \) matrices, and group of \( m \)-by-\( m \) orthogonal matrices, respectively.
- \( \| \cdot \|_F \) denotes the Frobenius matrix norm on \( \mathbb{M}_{m \times n} \), namely the Euclidean norm of a matrix considered as a vector in \( \mathbb{R}^{mn} \).
- We denote matrix multiplication by either \( AB \) or \( A \cdot B \).
- We use the following convenient \( \text{diag} \) notation. For a matrix \( X \in \mathbb{M}_{m \times n} \), we denote by \( X_\Delta \in \mathbb{R}^m \) its main diagonal,

\[
(X_\Delta)_i = X_{i,i} \quad 1 \leq i \leq m .
\]

- Similarly, for a vector \( x \in \mathbb{R}^m \), and \( n \geq m \) that we suppress in our notation, we denote by \( x_\Delta \in \mathbb{M}_{m \times n} \) the “diagonal” matrix

\[
(x_\Delta)_{i,j} = \begin{cases} 
 x_i & 1 \leq i = j \leq m \\
 0 & \text{otherwise} 
\end{cases} .
\]
• We denote a “fat” Singular Value Decomposition (SVD) of $X \in M_{m \times n}$ $X = U_X \cdot \mathbf{x}_\Delta \cdot V_X'$, with $U_X \in M_{m \times m}$ and $V_X \in M_{n \times n}$. Note that the SVD is not uniquely determined, and in particular $\mathbf{x}$ can contain the singular values of $X$ in any order. Unless otherwise noted, we will assume that the entries of $\mathbf{x}$ are non-negative and sorted in non-increasing order, $x_1 \geq \ldots \geq x_m \geq 0$. When $m < n$, the last $n - m$ columns of $V_X$ are not uniquely determined; we will see that our various results do not depend on this choice. Not matter in. Note that with the “fat” SVD, the matrices $Y$ and $U_Y' \cdot Y \cdot V_Y$ have the same dimensionality, which simplifies the notation we will need.

• When appropriate, we let univariate functions act on vectors entry-wise, namely, for $\mathbf{x} \in \mathbb{R}^n$ and $f : \mathbb{R} \rightarrow \mathbb{R}$, we write $f(\mathbf{x}) \in \mathbb{R}^n$ for the vector with entries $f(x_i) = f(x_i)$.

### 3.3 $\hat{X}_\lambda$ acts by soft thresholding of the data singular values

By orthogonal invariance of the Frobenius norm, for a matrix $X \in M_{m \times n}$,

$$||Y - X||_F^2 = ||y - U'_Y \cdot X \cdot V_Y||_F^2 \geq ||y - (U'_Y \cdot X \cdot V_Y)\Delta||_F^2,$$

with equality only if $(U_Y, V_Y)$ diagonalizes $X$. Therefore, (42) is equivalent to

$$\hat{X}_\lambda = \arg\min_{\mathbf{x} \in \mathbb{R}^n} \frac{1}{2} ||\mathbf{y} - \mathbf{x}||_2^2 + \lambda ||\mathbf{x}||_1;$$

through the relation $\hat{X}_\lambda(Y) = U_Y \cdot (\hat{y}_\lambda)\Delta \cdot V_Y'$. It is well known that the solution to (41) is given by $\hat{X}_\lambda = \mathbf{y}_\lambda$, where $\mathbf{y}_\lambda = (y - \lambda)_+$ denotes coordinate-wise soft thresholding of $y$ with threshold $\lambda$. We conclude that the SVST estimator (42) is given by

$$\hat{X}_\lambda : Y \mapsto U_Y \cdot (y_\lambda)\Delta \cdot V_Y'.$$  

Note that (42) is well defined, that is, $\hat{X}_\lambda(Y)$ does not depend on the particular SVD $Y = U_Y \cdot (y)\Delta \cdot V_Y'$ chosen.

In case $Sym$, observe that the solution to (42) is constrained to lie in the linear subspace of symmetric matrices. The solution is the same whether the noise matrix $Z \in M_{n \times n}$ has i.i.d standard normal entries, or whether $Z$ is a symmetric Wigner matrix $\frac{1}{2}(Z_1 + Z_1')$ where $Z \in M_{n \times n}$ has i.i.d standard normal entries. Below, we assume that the data in case $Sym$ is of the form $X_0 + Z$ where $X_0 \in S_n^+$ and $Z$ has this Wigner form, namely, the singular values $y$ are the absolute values of eigenvalues of the symmetric matrix $X_0 + Z$.

### 3.4 The singular values of $Y$ when $||X_0|| \rightarrow \infty$

Our main results depend on the following crucial observations regarding the singular values of $X_0 + Z$ when $X_0$ is a rank $r$ matrix and is rescaled so $||X_0|| \rightarrow \infty$.

**Lemma 1.** Let $Y_\mu = \mu X_0 + Z = U_\mu \cdot (y_\mu)\Delta \cdot V_\mu'$ where $X_0 \in M_{m \times n}$ is of rank $r$ and assume that $y$ is sorted in non-increasing order, $y_1 \geq \ldots \geq y_m$. Then

$$\lim_{\mu \rightarrow \infty} y_{\mu,i} = \infty, \quad i = 1, \ldots, r.$$
Proof. For a symmetric matrix \( S \), let \( \lambda_i(S) \) denote the \( i \)-th eigenvalue in non-increasing order. The Courant-Fischer min-max characterization of eigenvalues states that

\[
\lambda_i(S) = \min_{U: \text{dim}(U) = i-1} \max_{\|v\| = 1} v^T S v.
\] (43)

Of course the squared singular values of \( Y_\mu \) obey \( y_i^2 = \lambda_i(Y_\mu Y_\mu^T) \).

Choose \( 1 \leq i \leq r \). Let \( U \in \mathbb{R}^m \) with \( \text{codim}(U) = i-1 \). Since \( m-r+1 \leq \text{dim}(U) \leq m \) necessarily \( U \cap \text{Im}(X_0) \neq \emptyset \) and we can choose \( v \in U \cap \text{Im}(X_0) \) with \( \|v\| = 1 \). For this \( v \), and with the \( y_i \) again denoting the singular values of \( Y_\mu \),

\[
y_i^2 \geq v^T Y_\mu Y_\mu^T v, \quad 1 \leq i \leq r.
\] (44)

Without loss of generality, we may assume we are working in the basis of the SVD of \( X_0 \), so that \( X_0 \) is vanishing except in the first \( r \) positions of the diagonal. We let \( G_r \) denote the \( r \times r \) sub-matrix of the upper left corner of \( X_0X_0^T \); this is the ‘principal piece’ of \( X_0X_0^T \) where the non zeros live; the rest of \( X_0X_0^T \) is vanishing in this basis. Also let \( Z_r \) denote the upper left \( r \) by \( r \) corner of \( Z_r \), and let \( X_r \) denote the \( r \) by \( r \) diagonal matrix whose diagonal entries are the ordered singular values. Let \( H \) denote the \( m \times m \) matrix which is zero except in the upper left corner, where there is a nonzero \( r \times r \) sub matrix \( H_r \) given by \( H_r = X_r Z_r + Z_r^T X_r^T \). We also need \( W_r \), the upper left \( r \) by \( r \) corner of the Wishart matrix \( ZZ^T \). Finally we note that when represented in this basis, \( v = (v_r, 0_{m-r}) \), i.e. the only non zeros in \( v \) occur in the first \( r \) places, and we let \( v_r \) denote the \( r \times 1 \) column vector of those first \( r \) entries.

For a symmetric matrix \( S \), let \( \lambda_{\min}(S) \) denote the smallest eigenvalue. Note that \( \lambda_{\min}(ZZ^T) \geq 0 \); that \( \lambda_{\min}(G_r) > 0 \) by our hypothesis that the rank of \( X_0 \) is \( r \), and finally note that \( \lambda_{\min}(H_r) \) may be positive or negative, but is in any case well-defined\(^2\).

Now \( Y_\mu Y_\mu^T = \mu^2 (X_0X_0^T) + \mu X_0 Z_r + \mu Z_r X_r^T + ZZ^T \). Taking into account our choice of coordinates, the upper left \( r \) by \( r \) block of \( Y_\mu Y_\mu^T \) is \( \mu^2 G_r + \mu H_r + W_r \). The vector \( v \) ‘sees’ only the first \( r \) coordinates and

\[
v^T Y_\mu Y_\mu^T v = \mu^2 v_r^T G_r v_r + \mu v_r^T H_r v_r + v_r^T W_r v_r \\
\geq \mu^2 \lambda_{\min}(G_r) + \mu \lambda_{\min}(H_r) + \lambda_{\min}(W_r).
\]

This lower bound does not depend on the specific choice of \( U \) or of \( v \). Because \( \lambda_{\min}(G_r) > 0 \), this lower bound tends to \( \infty \) as \( \mu \to \infty \); we finally invoke (44).

\[\square\]

Definition 1. For a pair of matrices \( X_0, Z \in M_{m \times n} \), we denote by \( \zeta(X_0, Z|\text{Mat}) = (\zeta_1, \ldots, \zeta_{m-r}) \) the singular values, in non-increasing order, of

\[
\Pi_m \cdot Z \cdot \Pi_n' \in M_{(m-r) \times (n-r)} \ ,
\] (45)

where \( \Pi_m : \mathbb{R}^m \to \mathbb{R}^{m-r} \) is the projection of \( \mathbb{R}^m \) on \( \text{null}(X_0') = \text{Im}(X_0) \) and \( \Pi_n : \mathbb{R}^n \to \mathbb{R}^{n-r} \) is the projection on \( \text{null}(X_0) \). Similarly, for a pair of matrices \( X_0, Z \in M_{n \times m} \), denote by \( \zeta(X_0, Z|\text{Sym}) = (\zeta_1, \ldots, \zeta_{m-r}) \) the eigenvalues, in non-increasing order, of

\[
\Pi_m \cdot \frac{1}{2} (Z + Z') \cdot \Pi_n' \in M_{(m-r) \times (n-r)} \ .
\] (46)

Lemma 2. Let \( Y_\mu = \mu X_0 + Z = U_\mu \cdot (y_\mu) \Delta \cdot V_\mu^T \), where \( X_0 \in M_{m \times n} \) is of rank \( r \), and write \( y_\mu = (y_{\mu,1}, \ldots, y_{\mu,m}) \) with \( y_{\mu,1} \geq \cdots \geq y_{\mu,m} \). Also, define

\[
\zeta_\mu = (U_\mu \cdot Z \cdot V_\mu) \Delta \\
x_\mu = \mu \cdot (U_\mu \cdot X_0 \cdot V_\mu) \Delta
\]

\(^2\)Moreover, for a fixed choice of \( X_0 \), one could control the distribution of \( \lambda_{\min}(H_r) \).
with \( z_\mu = (z_{\mu,1}, \ldots, z_{\mu,m}) \). Finally, let \((\zeta_1, \ldots, \zeta_{m-r}) = \zeta(X_0, Z|X)\) as in Definition [1] where \( X = Mat \) or \( X = Sym \). Then the \( m - r \) lower singular values of \( Y \) satisfy

\[
\lim_{\mu \to \infty} y_{\mu,r+i} = |\zeta_i| \quad (47)
\]

\[
\lim_{\mu \to \infty} z_{\mu,r+i} = \zeta_i \quad (48)
\]

\[
\lim_{\mu \to \infty} x_{\mu,r+i} = 0. \quad (49)
\]

**Proof.** Assume Case \( Mat \). Write \( \bar{Y}_\sigma = \sigma Y(1/\sigma) = X_0 + \sigma Z \) and let \( \bar{Y}_\sigma = \bar{U}_\sigma \cdot (y_\sigma)_\Delta \cdot \bar{V}_\sigma' \) denote the SVD of \( Y \). Note that \( U_{1/\sigma} = \bar{U}_\sigma \) and \( V_{1/\sigma} = \bar{V}_\sigma \). Write

\[
\bar{U}_\sigma = \begin{pmatrix} \bar{u}_\sigma^1 & \cdots & \bar{u}_\sigma^m \end{pmatrix} \quad \bar{V}_\sigma = \begin{pmatrix} \bar{v}_\sigma^1 & \cdots & \bar{v}_\sigma^m \end{pmatrix},
\]

with \( \bar{u}_\sigma^i \in \mathbb{R}^m \) and \( \bar{v}_\sigma^i \in \mathbb{R}^n \), for the columns of \( \bar{U}_\sigma \) and \( \bar{V}_\sigma \). Similarly write \( u_{\mu}^i \) (resp. \( v_{\mu}^i \)) for the columns of \( U_{\mu} \) (resp. \( V_{\mu} \)).

Let \( \bar{U}_{\sigma,m-r} \) and \( \bar{V}_{\sigma,n-r} \) denote matrices consisting of the last \( m-r \) and \( n-r \) columns of \( \bar{U}_\sigma \) and \( \bar{V}_\sigma \), again in the ordering where the largest singular value has index \( i = 1 \). Similarly, let \( \bar{U}_{0,r} \) and \( \bar{V}_{0,r} \) denote the sub matrices of the first \( r \) columns.

Because the \( r \) nonzero singular values of \( X_0 \) are separated from 0, we can apply the eigenvector perturbation analysis in Theorem 8.1.7 of Golub and Van Loan [17] (originally based on P. Stewart’s work) to obtain the representation

\[
\bar{U}_{\sigma,m-r} = (\bar{U}_{0,m-r} + \bar{U}_{0,r}P_\sigma)Q_\sigma,
\]

\[
\bar{V}_{\sigma,n-r} = (\bar{V}_{0,n-r} + \bar{V}_{0,r}R_\sigma)S_\sigma,
\]

where \( P_\sigma, Q_\sigma, R_\sigma, S_\sigma \) all depend on \( \sigma, X_0 \) and \( Z, P_\sigma \in M_{r \times m-r}, Q_\sigma \in M_{m-r \times m-r}, R_\sigma \in M_{r \times n-r}, S_\sigma \in M_{n-r \times n-r} \). In short, the perturbed singular vectors are representable using the unperturbed ones. In this representation, the cited Theorem provides the following control of the coefficients:

\[
\|P_\sigma\|_2 = O(\sigma), \|R_\sigma\|_2 = O(\sigma), \sigma \to 0; \quad \|Q_\sigma\|_2 \leq 2, \|S_\sigma\|_2 \leq 2, \sigma < \sigma_0,
\]

where \( \|\cdot\|_2 \) is the operator norm on \( \ell_2 \). Noting that

\[
\bar{U}'_{0,m-r}X_0 = 0, \quad \text{and} \quad X_0\bar{V}_{0,n-r} = 0,
\]

we have

\[
\bar{U}'_{\sigma,m-r}X_0\bar{V}_{\sigma,n-r} = P'_\sigma Q'_\sigma \bar{U}'_{0,r}X_0\bar{V}_{0,r}R_\sigma S_\sigma,
\]

and so

\[
|\bar{U}'_{\sigma,m-r}X_0\bar{V}_{\sigma,n-r})_{i,i}| \leq \|P'_\sigma Q'_\sigma \bar{U}'_{0,r}X_0\bar{V}_{0,r}R_\sigma S_\sigma\|_2 \leq \|P'_\sigma\|_2 \cdot \|Q'_\sigma\|_2 \cdot \|X_0\|_2 \cdot \|R_\sigma\|_2 \cdot \|S_\sigma\|_2
\]

\[
= O(\sigma^2), \quad \sigma \to 0.
\]

Recalling \( \sigma = 1/\mu \), and \( u_{\mu}^{r+i} \) is the \( i \)-th column of \( \bar{U}_{\sigma,m-r} \) with \( \sigma = 1/\mu \), analogously for \( v_{\mu}^{r+i} \),

\[
\lim_{\mu \to \infty} x_{\mu,r+i} = \lim_{\mu \to \infty} (u_{\mu}^{r+i})' \cdot (\mu X_0) \cdot v_{\mu}^{r+i}
\]

\[
= \lim_{\mu \to \infty} \mu \cdot |(U'_{1/\mu,m-r}X_0V_{1/\mu,n-r})_{i,i}|
\]

\[
= \lim_{\mu \to \infty} \mu \cdot O(\mu^{-2}) = 0,
\]
for $1 \leq i \leq m - r$.

The projectors $\Pi_m$ and $\Pi_n$ of Definition 1 obey $\bar{u}_i^0 = \Pi_m \bar{u}_i^0$ and $\bar{v}_i^0 = \Pi_n \bar{v}_i^0$ for $i = 1, \ldots, m - r$. As $\lim_{\sigma \to 0} \bar{u}_i = \lim_{\sigma \to 0} \Pi_m \bar{u}_i$ and $\lim_{\sigma \to 0} \bar{v}_i = \lim_{\sigma \to 0} \Pi_n \bar{v}_i$, it follows that for $1 \leq i \leq m - r$ we have $\lim_{\mu \to \infty} y_{\mu, r + i} = ||\Pi_m^\top \Gamma Y_n \Pi_n - A_i||_2$, where $||\cdot||_2$ is the operator norm on $\ell_2$ and $A_i \in M_{(m-r) \times (n-r)}$ is the best rank-$i$ approximation of $\Pi_m^\top Y_n \Pi_n = \Pi_m^\top Z \Pi_n$ in Frobenius norm, namely, $\lim_{\mu \to \infty} y_{\mu, r + i} = \zeta_i$. Finally, this implies $\lim_{\mu \to \infty} y_{\mu, r + i} = \lim_{\mu \to \infty} y_{\mu, r + i} = \lim_{\mu \to \infty} y_{\mu, r + i} = \zeta_i$.

The case $Sym$ involves eigenvalues rather than singular values, making the proof simpler but otherwise identical.

\[\square\]

4 The Least-Favorable Matrix for SVST is at $||X|| = \infty$

We now prove Theorem 1, which characterizes the worst-case MSE of the SVST denoiser $\hat{X}_\lambda$ for a given $\lambda$. The theorem follows from a combination of two classical gems of the statistical literature. The first is Stein’s Unbiased Risk Estimate (SURE) from 1981, which we specialize to the SVST estimator (see also [3]). The second is Anderson’s celebrated monotonicity property for the integral of a symmetric unimodal probability distribution over a symmetric convex set [18], from 1955, and more specifically its implications for monotonicity of the power function of certain tests in multivariate hypothesis testing [19].

To simplify the proof, we introduce the following definitions, which will be used in this section only.

Definition 2. A weak notion of matrix majorization based on singular values. Let $A, B \in M_{m \times n}$ have singular value vectors $a, b \in \mathbb{R}^m$ respectively, which as usual we assume are sorted in non-increasing order: $0 \leq a_m \leq \ldots \leq a_1$ and $0 \leq b_m \leq \ldots \leq b_1$. If $a_i \leq b_i$ for $i = 1, \ldots, m$, we write $A \preceq B$.

Definition 3. Orthogonally invariant function of a matrix argument. We say that $f : M_{m \times n} \rightarrow \mathbb{R}$ is an orthogonally invariant function if $f(U \cdot A \cdot V') = F(A)$ for all $A \in M_{m \times n}$ and all orthogonal $U \in O_m$ and $V \in O_n$.

Definition 4. SV-monotone increasing function of a matrix argument. Let $f : M_{m \times n} \rightarrow \mathbb{R}$ be orthogonally invariant. If, whenever $A \preceq B$ and $\sigma > 0$, $f$ satisfies

$$\mathbb{E} f(A + Z) \leq \mathbb{E} f(B + Z),$$

for $Z \in M_{m \times n}$ and $Z_{i,j} \overset{\text{iid}}{\sim} \mathcal{N}(0, \sigma^2)$, we say that $f$ is singular-value-monotone increasing, or SV-monotone increasing.

We first observe that by rescaling an arbitrary rank-$r$ matrix, it is always possible to majorize any fixed matrix of rank at most $r$ (in the sense of Definition 2).

Lemma 3. Let $C \in M_{m \times n}$ be a matrix of rank $r$ and let $X \in M_{m \times n}$ be a matrix of rank at most $r$. Then there exists $\mu > 0$ for which $X \preceq \mu C$.

Proof. Let $c, x$ be the singular vectors of $C, X$ respectively, sorted in non-increasing order. Then $c_i > 0$. Take $\mu = x_i/c_i$. For $1 \leq i \leq r$ we have $x_i \leq x_1 = \mu c_1 \leq \mu c_i$, and for $r + 1 \leq i \leq m$ we have $\mu c_i = x_i = 0$.

To establish that $f$ is SV-monotone increasing, it is enough to show that $f$ is SV-monotone increasing with respect to each singular value individually:
Lemma 4. Let \( f : M_{m \times n} \to \mathbb{R} \) be an orthogonally invariant function such that

\[
\mathbb{E} f(a_\Delta + Z) \leq \mathbb{E} f((a + \delta e_i)_\Delta + Z)
\]

for any \( a \in \mathbb{R}^m, \delta > 0 \) and \( 1 \leq i \leq m \), where \( e_i \) is the canonical basis vector, \( (e_i)_j = \delta_{i,j} \). Then \( f \) is SV-monotone increasing.

**Proof.** Let \( A, B \in M_{m \times n} \) with singular value decompositions \( A = U_A \cdot a_\Delta \cdot V_A' \) and \( B = U_B \cdot b_\Delta \cdot V_B' \), and assume \( A \preceq B \), namely \( a_i \leq b_i, i = 1, \ldots, m \). Since \( f \) is orthogonally invariant and \( U_A' \cdot Z \cdot V_A \sim Z \), we have \( \mathbb{E} f(A + Z) = \mathbb{E} f(U_A' \cdot A \cdot V_A + U_A' \cdot Z \cdot V_A) = \mathbb{E} f(a_\Delta + Z) \) and similarly \( \mathbb{E} f(B + Z) = \mathbb{E} f(b_\Delta + Z) \). By assumption,

\[
\mathbb{E} f(A + Z) = \mathbb{E} f((a_1, \ldots, a_m)_\Delta + Z) \\
\leq \mathbb{E} f((b_1, a_2, \ldots, a_m)_\Delta + Z) \\
\leq \mathbb{E} f((b_1, b_2, \ldots, b_{m-1}, a_m)_\Delta + Z) \\
\leq \mathbb{E} f((b_1, b_2, \ldots, b_{m-1}, b_m)_\Delta + Z) = \mathbb{E} f(B + Z).
\]

\[\square\]

Das Gupta, Anderson and Mudholkar [19, thm. 1] gave a useful extension of the seminal monotonicity result of Anderson [18]; we present it using our standing notation.

**Theorem 10.** Let \( W \in M_{m \times n} \) be a random matrix whose rows \( w_j \in \mathbb{R}^n (1 \leq j \leq m) \) are mutually independent, each with a vector normal distribution \( w_j \sim N(k_j x_j, \Sigma_i) \), where \( k_j \geq 0 \) and \( w_j \in \mathbb{R}^n \). Assume that \( E \subset M_{m \times n} \) is a convex set, symmetric in each \( w_j \) given the other \( w_\ell (\ell \neq j) \), in the sense that \( W \in E \) iff \( S \cdot W \in E \) for any \( S \in M_{m \times m} \) with \( S_\Delta \in \{ \pm 1 \}^m \). Then \( P(W \in E) \) is monotone non-increasing as long as \( k_j \geq 0 \).

Theorem 10 asserts monotonicity of expectation of indicators. Using the co-area formula for expectation of non-negative random variables, we obtain monotonicity of expectation of quasi-concave (and hence quasi-convex) functions:

**Lemma 5.** Bounded invariant quasi-convex functions are SV-monotone increasing. Let \( f : M_{m \times n} \to \mathbb{R} \) be a bounded orthogonally invariant function. Assume that \( f \) is quasi-convex function, in the sense that for all \( c \in \mathbb{R} \) the set \( f^{-1}((-\infty, c]) \) is convex in \( M_{m \times n} \). Then \( f \) is SV-monotone increasing.

**Proof.** Let \( a \in \mathbb{R}^m, \delta > 0 \) and \( 1 \leq i \leq m \). Define \( b = a + \delta e_i \). By Lemma 4 it is enough to show that

\[
\mathbb{E} f(a_\Delta + Z) \leq \mathbb{E} f(b_\Delta + Z).
\]

Let \( g = -f \) so that \( g \) is quasi-concave, orthogonally invariant and bounded. Without loss of generality we may assume that \( \inf g = 0 \). By the co-area formula we have for the expectation of \( g(a_\Delta + Z) \):

\[
\mathbb{E} g(a_\Delta + Z) = \int_0^\infty \mathbb{P}\{g(a_\Delta + Z) \geq c\} \, dc
\]

and it is therefore enough to show that for all \( c \geq 0 \),

\[
\mathbb{P}\{g(a_\Delta + Z) \geq c\} \geq \mathbb{P}\{g(b_\Delta + Z) \geq c\}.
\]

(52)
We invoke Anderson monotonicity, Theorem [10] with the value of \( i \) chosen above, \( k_i = a_i, w_i \), the standard basis vector \( (w_i)_i = \delta_{i,i} \) and \( \Sigma_i = I_n \), and with \( E = g^{-1}([c, \infty)) \) above. Since \( g \) is quasi-concave, \( E \) is a convex set in \( M_{n \times n} \). Since \( f \) (and hence \( g \)) is orthogonally invariant \( g(a_\Delta + Z) = g(S \cdot (a_\Delta + Z)) \), namely \( g \) is invariant under row sign changes, the symmetry requirement in the theorem is satisfied. The theorem therefore holds, and we obtain

\[
P \{ a_\Delta + Z \in g^{-1}([c, \infty)) \} \geq P \{ b_\Delta + Z \in g^{-1}([c, \infty)) \}
\]
as required.

The sufficient condition for SV-monotonicity we will actually use is:

**Lemma 6.** Assume that \( f : M_{m \times n} \to \mathbb{R} \) can be decomposed as \( f = \sum_{k=1}^{s} f_k \), where for each \( 1 \leq k \leq s \), \( f_k : M_{m \times n} \to \mathbb{R} \) is SV-monotone increasing. Then \( f \) is SV-monotone increasing.

**Proof.** Let \( A, B \in M_{m \times n} \) with \( A \preceq B \). By Lemma [5] we have for each \( k \) that \( \mathbb{E} f_k(A + Z) \leq f_k(B + Z) \). Therefore \( \mathbb{E} f(A + Z) = \sum_{k=1}^{s} \mathbb{E} f_k(A + Z) \leq \sum_{k=1}^{s} \mathbb{E} f_k(B + Z) = \mathbb{E} f(B + Z) \).

The final key ingredient in the proof of Theorem [1] is the Stein Unbiased Risk Estimate for SVST. In Appendix A we prove:

**Lemma 7. The Stein Unbiased Risk Estimate for SVST.** For each \( \lambda > 0 \), there exists an event \( S \subset M_{m \times n} \) and a function, \( SURE_\lambda : S \to \mathbb{R} \), given in Eq. (78) farther below, with the following properties:

1. \( P(S) = 1 \), where \( P \) is the distribution of the matrix \( Z \) with \( Z_{i,j} \overset{iid}{\sim} \mathcal{N}(0,1) \).

2. \( SURE_\lambda \) is a a finite sum of bounded, orthogonally invariant, quasi-convex functions.

3. Denoting as usual \( Y = X_0 + Z/\sqrt{n} \in M_{m \times n} \), where \( X_0, Z \in M_{m \times n} \) and \( Z_{i,j} \overset{iid}{\sim} \mathcal{N}(0,1) \), we have

\[
R(\hat{X}_\lambda, X_0) = \frac{1}{m}\mathbb{E}_{X_0} SURE_\lambda(Y).
\]

Putting together Lemma 6 and Lemma 7, we come to a crucial property of SVST.

**Lemma 8. The risk of SVST is monotone non-decreasing in the signal singular values.** For each \( \lambda > 0 \), the map \( X \mapsto R(\hat{X}_\lambda, X) \) is a bounded, SV-monotone increasing function. In particular, let \( A, B \in M_{m \times n} \) with \( A \preceq B \). Then

\[
R(\hat{X}_\lambda, A) \leq R(\hat{X}_\lambda, B).
\]

**Proof.** By Lemma 7, the function \( SURE_\lambda : M_{m \times n} \to \mathbb{R} \) satisfies the conditions of Lemma 6 and is therefore SV-monotone increasing. It follows that

\[
R(\hat{X}_\lambda, A) = \frac{1}{m} SURE_\lambda(A + Z/\sqrt{n}) \leq \frac{1}{m} SURE_\lambda(B + Z/\sqrt{n}) = R(\hat{X}_\lambda, B).
\]

To see that the risk is bounded, note that for any \( X \in M_{m \times n} \) we have by Lemma 7

\[
\infty < \inf_{Y \in M_{m \times n}} SURE_\lambda(Y) \leq R(\hat{X}_\lambda, X) \leq \sup_{Y \in M_{m \times n}} SURE_\lambda(Y) < \infty.
\]
Proof of Theorem 1. By Lemma 5 the map \( \mu \to R(\hat{X}_\lambda, \mu C) \) is bounded and monotone non-decreasing in \( \mu \), hence \( \lim_{\mu \to \infty} R(\hat{X}_\lambda, \mu C) \) exists and is finite, and

\[
R(\hat{X}_\lambda, \mu_0 C) \leq \lim_{\mu \to \infty} R(\hat{X}_\lambda, \mu C)
\]

for all \( \mu_0 > 0 \). Since \( \text{rank}(C) = r \), obviously

\[
\sup_{\text{rank}(X_0) \leq r} R(\hat{X}_\lambda, X_0) \geq \lim_{\mu \to \infty} R(\hat{X}_\lambda, \mu C),
\]

and we only need to show the reverse inequality. Let \( X_0 \in M_{m \times n} \) be an arbitrary matrix of rank at most \( r \).

By Lemma 3 there exists \( \mu_0 \) such that \( X_0 \leq \mu_0 C \). It now follows from Lemma 8 and (54) that

\[
R(\hat{X}_\lambda, X_0) \leq R(\hat{X}_\lambda, \mu_0 C) \leq \lim_{\mu \to \infty} R(\hat{X}_\lambda, \mu C).
\]

\[\square\]

5 Worst-Case MSE

Let \( \lambda \) and \( r \leq m \leq n \), and consider them fixed for the remainder of this section. Our second main result, Theorem 2, follows immediately from Theorem 1, combined with the following lemma.

Lemma 9. Let \( X_0 \in M_{m \times n} \) be of rank \( r \). Then

\[
\lim_{\mu \to \infty} R(\hat{X}_\lambda, \mu X_0) = M_n \left( \frac{\lambda}{\sqrt{1 - r/n}} ; r, m, \alpha \right),
\]

as defined in (11), with \( \alpha = 1 \) for case \( \text{Mat} \) and \( \alpha = 1/2 \) for case \( \text{Sym} \).

Proof. Assume for simplicity that the data singular vector \( y \) is sorted in non-increasing order: \( y_1 \geq \ldots \geq y_m \). Define

\[
Y_\mu = \mu X_0 + \frac{1}{\sqrt{n}} Z = U_\mu \cdot (y_\mu)_\Delta \cdot V'_\mu
\]

and write \( x_\lambda \) for the entry-wise soft thresholding \( (x - \lambda)_+ \), so that \( \hat{X}_\lambda(Y) = U_Y \cdot (y_\lambda)_\Delta \cdot V'_Y \) is the SVST denoiser. By invariance of the Frobenius norm to orthogonal transformations, we have

\[
\left\| \hat{X}_\lambda(Y_\mu) - X_0 \right\|_F^2 = \left\| X_0 - U_\mu \cdot (y_\mu,\lambda)_\Delta \cdot V'_\mu \right\|_F^2 = \left\| U'_\mu \cdot X_0 \cdot V_\mu - (y_\mu,\lambda)_\Delta \right\|_F^2
\]

\[
= \left\| (U'_\mu \cdot X_0 \cdot V_\mu)_\Delta - (y_\mu,\lambda)_\Delta \right\|_F^2 + \left\| X_0 \right\|_F^2 - \left\| (U'_\mu \cdot X_0 \cdot V_\mu)_\Delta \right\|_F^2.
\]

Introducing the “pinching” notation

\[
x_\mu = (U'_\mu \cdot X_0 \cdot V_\mu)_\Delta
\]

\[
z_\mu = \frac{1}{\sqrt{n}} (U'_\mu \cdot Z \cdot V_\mu)_\Delta
\]

so that \( y_\mu = x_\mu + z_\mu \), and using the fact that

\[
\left\| X_0 \right\|_F^2 - \left\| (U'_\mu \cdot X_0 \cdot V_\mu)_\Delta \right\|_F^2 = \frac{1}{n} \left\| Z \right\|_F^2 - \frac{1}{n} \left\| (U'_\mu \cdot Z \cdot V_\mu)_\Delta \right\|_F^2.
\]
we get
\[
\left\| \hat{X}(Y_\mu) - X_0 \right\|_F^2 = \| y_{\mu,\lambda} - x_\mu \|_2^2 + \frac{1}{n} \| Z \|_F^2 - \| z_\mu \|_2^2.
\]
Therefore,
\[
\lim_{\mu \to \infty} R(\hat{X}_\lambda, \mu X_0) = \lim_{\mu \to \infty} \left( \frac{1}{m} \mathbb{E} \left\| y_{\mu,\lambda} - x_\mu \right\|_2^2 + \frac{1}{m} \mathbb{E} \| z_\mu \|_2^2 \right)
\]
\[
= \frac{1}{m} \sum_{i=1}^r \lim_{\mu \to \infty} \mathbb{E} \left( (y_{\mu,i} - \lambda)_+ - x_{\mu,i} \right)^2
\]
\[
+ \frac{1}{m} \sum_{i=r+1}^m \lim_{\mu \to \infty} \mathbb{E} \left( (y_{\mu,i} - \lambda)_+ - x_{\mu,i} \right)^2
\]
\[
- \frac{1}{m} \sum_{i=1}^r \lim_{\mu \to \infty} \mathbb{E} (z_{\mu,i})^2
\]
\[
- \frac{1}{m} \sum_{i=r+1}^m \lim_{\mu \to \infty} \mathbb{E} (z_{\mu,i})^2 + 1.
\]

We now proceed to evaluate each of the terms (57), (58), and (59) in turn. Starting with (57), observe that \( \mathbb{E} z_{\mu,i} = 0 \) for all \( i \). By Lemma 1, pointwise, \( \lim_{\mu \to \infty} y_{\mu,i} = \infty \), so that \( \lim_{\mu \to \infty} (y_{\mu,i} - \lambda)_+ - x_{\mu,i} = \lim_{\mu \to \infty} (y_{\mu,i} - x_{\mu,i}) \). It follows that for \( 1 \leq i \leq r \),
\[
\lim_{\mu \to \infty} \mathbb{E} \left( (y_{\mu,i} - \lambda)_+ - x_{\mu,i} \right)^2 = \lim_{\mu \to \infty} \mathbb{E} \left( (y_{\mu,i} + z_{\mu,i} - \lambda)_+ - x_{\mu,i} \right)^2
\]
\[
= \lim_{\mu \to \infty} \mathbb{E} (z_{\mu,i})^2
\]
\[
= \lim_{\mu \to \infty} \mathbb{E} z_{\mu,i}^2 + \lambda^2.
\]

Turning to (58). Let \( (\zeta_1, \ldots, \zeta_{m-r}) = \zeta(X_0, Z|X) \) as in Definition 1. By Lemma 2 we have \( \lim_{\mu \to \infty} x_{\mu,i} = 0 \) and \( \lim_{\mu \to \infty} z_{\mu,i} = \zeta_{r-i}/\sqrt{n} \), for \( r+1 \leq i \leq m \), namely,
\[
\lim_{\mu \to \infty} \mathbb{E} ((y_{\mu,i} - \lambda)_+ - x_{\mu,i})^2 = \frac{1}{n} \mathbb{E} (\zeta_{i-r} - \sqrt{n}\lambda)^2.
\]
Finally, as for (59), again by Lemma 2 we have
\[
\lim_{\mu \to \infty} \mathbb{E} (z_{\mu,i})^2 = \frac{1}{n} \mathbb{E} (\zeta_{i-r})^2.
\]
Collecting the terms in (56), we find that the sum does not depend on the particular choice of the signal matrix \( X_0 \), and in fact
\[
R(\hat{X}_\lambda, \mu X_0) = 1 + \frac{r}{m} \lambda^2 + \frac{1}{nm} \sum_{i=1}^{m-r} \mathbb{E} (\zeta_i - \sqrt{n}\lambda)^2 + \frac{1}{nm} \sum_{i=1}^{m-r} \mathbb{E} \zeta_i^2.
\]
Now,
\[
\frac{1}{nm} \sum_{i=1}^{m-r} \mathbb{E} (\zeta_i - \sqrt{n}\lambda)^2 = \frac{n}{nm} \sum_{i=1}^{m-r} \mathbb{E} \left( \frac{\zeta_i}{\sqrt{n-r}} - \frac{\lambda}{\sqrt{1-r/n}} \right)^2,
\]
with \( \alpha = 1 \) in case Mat and \( \alpha = 1/2 \) in case Sym. This factor in case Sym follows from the fact that in case Sym, with probability 1/2 we have \( z_i < 0 \), and conditional
on \( z_i > 0 \), \( z_i \) follows the same distribution as in case \( \text{Mat} \). Observe that in the case \( \text{Mat} \), \( \{ \zeta_i / \sqrt{n} \} \) are the eigenvalues of a standard central Wishart matrix \( \sim W_{m-r}(I, n-r) \), so that

\[
\frac{1}{mn} \sum_{i=1}^{m-r} \mathbb{E}((\zeta_i - \sqrt{n})^2) = \alpha \left( \frac{n-r}{mn} \right) \sum_{i=1}^{m-r} w_i \left( \frac{\lambda}{\sqrt{1-r/n}} ; m-r ; n-r \right),
\]

where \( w_i \) was defined in (12).

Finally, observe that \( \sum_{i=1}^{m-r} \zeta_i^2 \) is the Frobenius norm of \( Z_{(m-r)\times(n-r)} \), hence equals \( (m-r)(n-r) \) in expectation, to the effect that

\[
\frac{1}{mn} \mathbb{E} \sum_{i=1}^{m-r} \zeta_i^2 = \left( \frac{m-r}{mn} \right) = 1 - \frac{r}{m} - \frac{r}{n} + \frac{r^2}{mn}.
\]  \( \text{(63)} \)

Setting \( \Lambda = \lambda / \sqrt{1-r/n} \), we collect the terms and recover (11) as required.

**Lemma 10.** The function \( \Lambda \mapsto \mathcal{M}_n(\Lambda; r, m, \alpha) \), defined in (11) on \( \Lambda \in [0, \infty) \), is convex and obtains a unique minimum.

**Proof.** Differentiating (12) under the integral w.r.t \( \Lambda \), since the upper integration limit does not depend on \( \Lambda \), we get

\[
\frac{d}{d\Lambda} w_i(\Lambda; m, n) = \frac{d}{d\Lambda} \int_{\Lambda^2} (\sqrt{t} - \Lambda)^2 \, dW_i(m, n)(t)
\]

\[
= -\left( \sqrt{\Lambda^2 - \Lambda} \right)^2 \frac{dW_i(m, n)}{dt}(\Lambda^2) \cdot (2\Lambda) + \int_{\Lambda^2} \frac{d}{d\Lambda} (\sqrt{t} - \Lambda)^2 \, dW_i(m, n)
\]

\[
= -2 \int_{\Lambda^2} (\sqrt{t} - \Lambda) \, dW_i(m, n).
\]

Differentiating w.r.t \( \Lambda \) again, the boundary terms vanish again and we get

\[
\frac{d^2}{d\Lambda^2} w_i(\Lambda; m, n) = -2 \int_{\Lambda^2} \frac{d}{d\Lambda} (\sqrt{t} - \Lambda) \, dW_i(m, n) = 2 \int_{\Lambda^2} dW_i(m, n).
\]

Therefore, by (11) we have

\[
\frac{d}{d\Lambda} \mathcal{M}_n(\Lambda; r, m, \alpha) = \frac{d}{d\Lambda} r(\frac{n-r}{mn}) \Lambda^2 + \alpha \left( \frac{n-r}{mn} \right) \sum_{i=1}^{m-r} \frac{d}{d\Lambda} w_i(\Lambda; m-r; n-r)
\]

\[
= 2r(\frac{n-r}{mn}) \Lambda^2 - 2 \int_{\Lambda^2} (\sqrt{t} - \Lambda) \, dW_i(m, n).
\]

and

\[
\frac{d^2}{d\Lambda^2} \mathcal{M}_n(\Lambda; r, m, \alpha) = \frac{d^2}{d\Lambda^2} r(\frac{n-r}{mn}) \Lambda^2 + \alpha \left( \frac{n-r}{mn} \right) \sum_{i=1}^{m-r} \frac{d^2}{d\Lambda^2} w_i(\Lambda; m-r; n-r)
\]

\[
= 2r(\frac{n-r}{mn}) + 2\alpha \left( \frac{n-r}{mn} \right) \sum_{i=1}^{m-r} \int_{\Lambda^2} dW_i(m-r, n-r) > 0.
\]
Therefore, $\Lambda \mapsto M_n(\Lambda; r, m, \alpha)$ is convex on $[0, \infty)$ with
\[
\frac{d}{d\Lambda} M_n(0; r, m, \alpha) < 0 \quad \text{and} \quad \lim_{\Lambda \to \infty} \frac{d}{d\Lambda} M_n(\Lambda; r, m, \alpha) > 0
\]
and the lemma follows.

Finally we can prove our second main result.

**Proof of Theorem 2.** Let $C \in M_{m \times n}$ be an arbitrary fixed matrix of rank $r$. For case \textit{Mat}, by Theorem 1 and Lemma 9,
\[
M_n(r, m \mid \text{Mat}) = \inf_{\lambda} \sup_{X_0 \in M_{m \times n} \atop \text{rank}(X_0) \leq r} R(\hat{X}_\lambda, X_0) = \inf_{\Lambda > 0} \lim_{\mu \to \infty} R(\hat{X}_\lambda, \mu C) = \inf_{\Lambda > 0} M_n \left( \frac{\lambda}{\sqrt{1 - r/n}}; r, m, 1 \right) = \inf_{\Lambda > 0} M_n(\Lambda; r, m, 1) = \min_{\Lambda > 0} M_n(\Lambda; r, m, 1),
\]
where we have used Lemma 10, which also asserts that the minimum is unique.

Now let $C \in S^n_+$ be an arbitrary, fixed symmetric positive semidefinite matrix of rank $r$. For case \textit{Sym}, by the same lemmas,
\[
M_n(r \mid \text{Sym}) = \inf_{\lambda} \sup_{X_0 \in M_{m \times n} \atop \text{rank}(X_0) \leq r} R(\hat{X}_\lambda, X_0) = \inf_{\Lambda > 0} \lim_{\mu \to \infty} R(\hat{X}_\lambda, \mu C) = \inf_{\Lambda > 0} M_n \left( \frac{\lambda}{\sqrt{1 - r/n}}; r, 1/2 \right) = \inf_{\Lambda > 0} M_n(\Lambda; r, 1/2) = \min_{\Lambda > 0} M_n(\Lambda; r, 1/2),
\]
where we have used Lemma 10, which also asserts that the minimum is unique. □

6 Worst-Case AMSE

Toward the proof of our third main result, Theorem 3, let $\lambda$ be fixed. We first show that in the proportional growth framework, where the rank $r(n)$, number of rows $m(n)$ and number of columns $n$ all tend to $\infty$ proportionally to each other, the key quantity in our formulas can be evaluated by complementary incomplete moments of a Marčenko-Pastur distribution, instead of a sum of complementary incomplete moments of Wishart eigenvalues.

**Lemma 11.** Consider sequences $n \mapsto r(n)$ and $n \mapsto m(n)$ and numbers $0 < \beta \leq 1$ and $0 \leq \rho \leq 1$ such that $\lim_{n \to \infty} r(n)/n = \rho$ and $\lim_{n \to \infty} m(n)/n = \beta$. Let $(\zeta_1(n), \ldots, \zeta_{m-r}(n)) =$
\(\zeta(X_0, Z|X)\), as in Definition \[1\] where \(Z \in M_{m \times n}\) has i.i.d \(\mathcal{N}(0, 1)\) entries. Define \(\gamma = (\beta - \rho \beta)/(1 - \rho \beta)\) and \(\gamma_{\pm} = (1 \pm \sqrt{\gamma})^2\), and let \(0 \leq \Lambda \leq \sqrt{\gamma_{+}}\). Then

\[
\lim_{n \to \infty} \frac{1}{m} \sum_{i=1}^{m-r} \mathbb{E} \left( \frac{\zeta_i}{\sqrt{n - r}} - \Lambda \right)^2 = (1 - \rho) \int_{\Lambda^2} (\sqrt{t} - \Lambda)^2 \sqrt{(\gamma_{+} - t)(t - \gamma_{-})} \frac{dt}{2\pi t \gamma}.
\]

**Proof.** Write \(\xi_i = \zeta_i^2/(n - r)\) and recall that by the Marčenko-Pastur law \[14\],

\[
\lim_{n \to \infty} \frac{1}{m - r} \sum_{i=1}^{m-r} \delta_{\xi_i} \equiv P_\gamma,
\]

in the sense of weak convergence of probability measures, where \(P_\gamma\) is the Marčenko-Pastur probability distribution with density \(p_\gamma = dP_\gamma/dt\) given by \[15\]. Now,

\[
\lim_{n \to \infty} \frac{1}{m} \sum_{i=1}^{m-r} \left( \sqrt{\xi_i} - \Lambda \right)_+^2 = \lim_{n \to \infty} \frac{1}{m} \sum_{i=1}^{m-r} \int_0^\infty \left( \sqrt{t} - \Lambda \right)_+^2 \delta_{\xi_i}(t) dt
\]

\[
= \lim_{n \to \infty} \left( 1 - \frac{r}{m} \right) \int_0^\infty \left( \sqrt{t} - \Lambda \right)_+^2 \frac{1}{m - r} \sum_{i=1}^{m-r} \delta_{\xi_i}(t) dt
\]

\[
= (1 - \rho) \int_0^\gamma \left( \sqrt{t} - \Lambda \right)_+^2 p_\gamma(t) dt
\]

as required. \(\square\)

**Lemma 12.** Let \(m(n)\) and \(r(n)\) such that \(\lim_{n \to \infty} m/n = \beta\) and \(\lim_{n \to \infty} r/m = \rho\), and set \(\tilde{\rho} = \beta \rho\). Then

\[
\lim_{n \to \infty} \sup_{x_0 \in M_{m \times n}} \sup_{\text{rank}(X_0) \leq r} R(\hat{X}_\lambda, X_0) = M \left( \frac{\lambda}{\sqrt{1 - \tilde{\rho}}}; \rho, \tilde{\rho}, \alpha \right),
\]

where the right hand side is defined in \[17\], with \(\alpha = 1\) for case \(\text{Mat}\) and \(\alpha = 1/2\) for case \(\text{Sym}\).

**Proof.** For case \(\text{Mat}\), let \(C(n) \in M_{m \times n}\) be an arbitrary fixed matrix of rank \(r\). For case \(\text{Sym}\), \(C(n) \in S^n_+\) an arbitrary, fixed symmetric positive semidefinite matrix of rank \(r\). By Theorem \[1\] and Lemma \[2\]

\[
\lim_{n \to \infty} \sup_{x_0 \in M_{m \times n}} \sup_{\text{rank}(X_0) \leq r} R(\hat{X}_\lambda, X_0) = \lim_{n \to \infty} \lim_{\rho \to \infty} R(\hat{X}_\lambda, \mu C(n))
\]

\[
= \lim_{n \to \infty} M_n \left( \frac{\lambda}{1 - r/n}; r, m, \alpha \right)
\]

\[
= \lim_{n \to \infty} \left[ \frac{r}{m} + \frac{r}{n} \frac{s^2}{mn} + \frac{r}{m} \lambda^2 \right. + \left. \alpha \frac{n - r}{mn} \sum_{i=1}^{m-r} \mathbb{E} \left( \frac{\zeta_i}{\sqrt{n - r}} - \frac{\lambda}{\sqrt{1 - r/n}} \right)_+^2 \right]
\]

\[
= \rho + \tilde{\rho} - \rho \tilde{\rho} + (1 - \tilde{\rho}) \rho \lambda^2
\]

\[
+ \alpha (1 - \rho) (1 - \tilde{\rho}) \int_{\Lambda^2} (\sqrt{t} - \Lambda)^2 M P_\gamma(t) dt
\]

\[
= M \left( \frac{\lambda}{\sqrt{1 - \tilde{\rho}}}; \rho, \tilde{\rho}, \alpha \right),
\]
where we have used Lemma 11 and set $\Lambda = \lambda / \sqrt{1 - \tilde{\rho}}$. □

We now prove a variation of Lemma 10 for the asymptotic setting.

**Lemma 13.** The function $\Lambda \mapsto M(\Lambda; \rho, \tilde{\rho}, \alpha)$, defined in (17) on $\Lambda \in [0, \gamma_+]$, where $\gamma_+ = \left(1 + \sqrt{(\tilde{\rho} - \rho\tilde{\rho}) / (\rho - \rho\tilde{\rho})}\right)^2$, is convex and obtains a unique minimum.

**Proof.** Note that (17) is conveniently expressed as

$$M(\Lambda; \rho, \tilde{\rho}, \alpha) = \rho + \tilde{\rho} + (1 - \tilde{\rho}) \left[ \rho \Lambda^2 + \alpha (1 - \rho) \int_{\Lambda^2} (\sqrt{t} - \Lambda)^2 p_\gamma(t) \, dt \right]. \quad \text{(64)}$$

Differentiating the rightmost term of (64) under the integral, we get

$$\frac{d}{d\Lambda} \int_{\Lambda^2} (\sqrt{t} - \Lambda)^2 p_\gamma(t) \, dt = 2 \left( \sqrt{\gamma+ - \Lambda} \right)^2 p_\gamma(\gamma+) \cdot \frac{\partial \gamma+}{\partial \Lambda} - \left( \sqrt{\Lambda^2 - \Lambda} \right)^2 p_\gamma(\Lambda^2) \cdot (2\Lambda)$$

$$+ \int_{\Lambda^2} \frac{\partial}{\partial \Lambda} (\sqrt{t} - \Lambda)^2 p_\gamma(t) \, dt.$$ 

Since $p_\gamma(\gamma+) = 0$, both boundary terms vanish and therefore

$$\frac{d}{d\Lambda} \int_{\Lambda^2} (\sqrt{t} - \Lambda)^2 p_\gamma(t) \, dt = -2 \int_{\Lambda^2} (\sqrt{\gamma+ - \Lambda}) p_\gamma(t) \, dt = -2P_\gamma(\Lambda^2; 1/2) + 2\Lambda P_\gamma(\Lambda^2; 0),$$

where $P_\gamma$ was defined in (16). Differentiating w.r.t $\Lambda$ again, the boundary terms vanish again and we get

$$\frac{d^2}{d\Lambda^2} \int_{\Lambda^2} (\sqrt{t} - \Lambda)^2 p_\gamma(t) \, dt = 2 \int_{\Lambda^2} p_\gamma(t) \, dt = 2P_\gamma(\Lambda^2; 0).$$

By (64) we obtain

$$\frac{d}{d\Lambda} M(\Lambda; \rho, \tilde{\rho}, \alpha) = 2(1 - \tilde{\rho})\rho \Lambda + 2\alpha (1 - \rho)(1 - \tilde{\rho}) \left( \Lambda P_\gamma(\Lambda^2; 0) - P_\gamma(\Lambda^2; 1/2) \right) \quad \text{(65)}$$

and

$$\frac{d^2}{d\Lambda^2} M(\Lambda; \rho, \tilde{\rho}, \alpha) = 2(1 - \tilde{\rho})\rho + 2\alpha (1 - \rho)(1 - \tilde{\rho}) P_\gamma(\Lambda^2; 0) > 0.$$ 

Therefore, $\Lambda \mapsto M(\Lambda; \rho, \tilde{\rho}, \alpha)$ is convex on $[0, \gamma_+]$ with

$$\frac{d}{d\Lambda} M(0; \rho, \tilde{\rho}, \alpha) < 0 \quad \text{and} \quad \frac{d}{d\Lambda} M(\gamma_+; \rho, \tilde{\rho}, \alpha) > 0$$

and the lemma follows. □

This allows us to prove our third main result.
Proof of Theorem 3. By Lemma 12, 
\[
\mathcal{M}(\rho, \beta|X) = \lim_{n \to \infty} \inf_{\lambda} \sup_{X_0 \in M_{m \times n}^{\text{rank}(X_0) \leq r}} R(\hat{X}_\lambda, X_0)
\]
\[
= \inf_{\lambda} \lim_{n \to \infty} \sup_{X_0 \in M_{m \times n}^{\text{rank}(X_0) \leq r}} R(\hat{X}_\lambda, X_0)
\]
\[
= \inf_{\lambda} M \left( \frac{\lambda}{\sqrt{1 - \rho}} ; \rho, \tilde{\rho}, \alpha \right)
\]
\[
= \inf_{\Lambda} M (\Lambda; \rho, \tilde{\rho}, \alpha)
\]
\[
= \min_{\Lambda} M (\Lambda; \rho, \tilde{\rho}, \alpha)
\]
with \(\alpha = 1\) for case Mat and \(\alpha = 1/2\) for case Sym, where we have used Lemma 13, which also asserts that the minimum is unique.

7 Minimax AMSE

Having established that the asymptotic worst-case MSE (17) satisfies (18) and (19), we turn to its minimizer \(\Lambda^*\). The notation follows (20).

Proof of Theorem 4. By (65) above, the condition

\[
dM(\Lambda; \rho, \tilde{\rho}, \alpha) \bigg|_{\Lambda^*} = 0
\]

is thus equivalent, for any \(\rho \in [0, 1]\), to

\[
f(\Lambda, \rho) := \rho \Lambda - \alpha (1 - \rho) \int_{\Lambda^2} (\sqrt{t} - \Lambda) p_\tau(t) \, dt = 0,
\]

establishing (23) in particular for \(0 < \rho < 1\). By Lemma 13, the minimum exists and is unique, namely this equation has a unique root in \(\Lambda\). One directly verifies that \(f(1 + \sqrt{\beta}, 0) = f(0, 1) = 0\). The limits (21) and (22) follow from the fact that \(\rho \mapsto \Lambda_*(\rho, \cdot)\) is decreasing. To establish this, it is enough to observe that \(\partial f / \partial \rho > 0\) for all \((\Lambda, \rho)\), which can be verified directly.

We proceed to examine the special case \(\beta = 1\).

Proof of Theorem 5. When \(\beta = 1\), \(\gamma_+ = 4\) and \(\gamma_- = 0\) in (15). Changing the integration variable by \(t \mapsto t^2\) in (15) we get

\[
P_1(x; k) = \frac{1}{\pi} \int_{x}^{2} t^2 \sqrt{4 - t^2} \, dt,
\]

namely the \(k\)-th incomplete moment of the Quarter Circle law. Substituting this and \(\rho = \tilde{\rho} = 1\) into (17) we recover (24). The identities (25), (26) and (27) may be directly verified by differentiation.

To show that \(\Lambda_*(\rho, \rho, \alpha)\) satisfies (29), observe that the condition (66), which is equivalent to the general minimizer characterization (23), may in the case \(\beta = 1\) be written
in the equivalent form
\[
\frac{1}{\pi} \int_{-\Lambda}^{\Lambda} (t - \Lambda) \sqrt{4 - t^2} \, dt = \frac{\Lambda \rho}{1 - \rho}.
\]  
(68)

Clearly any solution \( \Lambda_* \) satisfies \( 0 \leq \Lambda_* \leq 2 \) and we define
\[
\theta_\alpha(\rho) = \arcsin(\Lambda_*(\rho, \rho, \alpha)/2).
\]  
(69)

Changing the integration variable \( t \mapsto 2 \sin \psi \) and substituting \( \Lambda = 2 \sin \theta \) in (68), we find that (68) is equivalent to
\[
\frac{4}{\pi} \int_{0}^{\pi/2} (\sin \psi - \sin \theta) \cos^2 \psi \, d\psi = \frac{\rho}{1 - \rho} \sin \theta,
\]  
(70)

which is in turn equivalent to (29) by elementary integration.

Our result regarding parametric representation of the minimax AMSE curve for \( \beta = 1 \) follows immediately:

**Proof of Theorem 7.** The curve is parametrized using the parameter \( \theta \) above. In each of the cases Mat and Sym, the formula is obtained by solving (29) for \( \rho \) to obtain \( \rho(\theta) \), and simplifying \( M(\Lambda(\theta); \rho(\theta), \rho(\theta), \alpha) \), where \( \Lambda(\theta) = 2 \sin \theta \) and \( M \) is defined in (24). We omit the elementary algebra.

Toward the proof of Theorem 6, we recall a trivial fact about convergence of minimizers.

**Lemma 14.** Let \( f_n : [a, b] \to \mathbb{R} \) be a sequence of continuous functions on \([a, b] \subset \mathbb{R}\) and assume that \( \{f_n\} \) converges pointwise to \( f : [a, b] \to \mathbb{R} \). If \( x_n \in [a, b] \) is the unique minimizer of \( f_n \), \( n = 1, 2, \ldots \), and \( x \in [a, b] \) is the unique minimizer of \( f \), then \( \lim_{n \to \infty} x_n = x \).

**Proof.** Let \( \{x_{n_k}\} \) be a convergent subsequence of \( \{x_n\} \), and write \( \lim_{k \to \infty} x_{n_k} = y \). It is enough to show that \( y = x \). Since \( f_n \) is continuous on a compact interval, it is uniformly continuous, hence \( f_{n_k}(x_{n_k}) \to f(y) \). Since \( x_{n_k} \) is a minimizer of \( f_{n_k} \), for all \( k \) we have \( f_{n_k}(x_{n_k}) \leq f_{n_k}(x) \). In the limit \( k \to \infty \) this inequality yields \( f(y) \leq f(x) \). Since \( x \) is a minimizer of \( f \), \( f(x) \leq f(y) \). Therefore \( f(x) = f(y) \). It follows that \( y \) is a minimizer of \( f \), which is unique by assumption, so that \( x = y \).

**Proof of Theorem 6.** Define
\[
\Lambda^n(r, m, \alpha) = \arg\min_{\Lambda} M_n(\Lambda; r, m, \alpha),
\]  
(71)

and recall the definition of \( \Lambda_*(\rho, \beta, \alpha) \) in (20). By Theorem 1 Lemma 9 and Lemma 12 the function sequence \( \Lambda \mapsto M_n(\Lambda; r, m, \alpha) \) converge pointwise to the function \( \Lambda \mapsto M(\Lambda, \rho, \tilde{\rho}, \alpha) \) on \( \Lambda \in [0, \gamma^n] \). We invoke Lemma 14 to obtain that the minimizers of the former converge to the minimizer of the latter, namely
\[
\lim_{n \to \infty} \Lambda^n(r, m, \alpha) = \Lambda_*(\rho, \beta, \alpha).
\]  
(72)
we thus have
\[
\lambda_*(m, n, r|X) = \arg\min_{\lambda} \sup_{X_0 \in \mathbb{X}_{m,n} \atop \text{rank}(X_0) \leq r} \frac{1}{mn} \mathbb{E}_{X_0} \left\| \hat{X}_\lambda (X_0 + Z) - X_0 \right\|^2_F
\]
\[
= \sqrt{n} \cdot \arg\min_{\lambda} \sup_{X_0 \in \mathbb{X}_{m,n} \atop \text{rank}(X_0) \leq r} \frac{1}{mn} \mathbb{E}_{X_0} \left\| \hat{X}_\lambda (X_0 + Z/\sqrt{n}) - X_0 \right\|^2_F
\]
\[
= \sqrt{n} \cdot \arg\min_{\lambda} M_n \left( \frac{\lambda}{\sqrt{1 - r/n}} ; r, m, \alpha \right)
\]
\[
= \sqrt{n} \cdot \sqrt{1 - r/n} \cdot \arg\min_{\lambda} M_n (\Lambda ; r, m, \alpha)
\]
\[
= \sqrt{n} \cdot \sqrt{1 - r/n} \cdot \Lambda_n^*(r, m, \alpha).
\]

Observe that \(\alpha = 1\) for case \(\text{Mat}\) and \(\alpha = 1/2\) for case \(\text{Sym}\). Since \(\lim_{n \to \infty} \sqrt{1 - r/n} = \sqrt{1 - \beta \rho}\), we thus have
\[
\lim_{n \to \infty} \frac{1}{\sqrt{n}} \lambda_*(m, n, r|X) = \lim_{n \to \infty} \sqrt{1 - r/n} \cdot \Lambda_n^*(r, m, \alpha) = \sqrt{1 - \beta \rho} \cdot \Lambda_*(\rho, \beta, \alpha),
\]
where we have used (72).

8 Minimax AMSE in the Low Rank Limit \(\rho \to 0\)

We proceed to evaluate the limit \(\lim_{\rho \to 0} \mathcal{M}(\rho, \beta|X)/\rho\).

Proof of Theorem 8 We first show that
\[
\lim_{\rho \to 0} \frac{1}{\rho} \int_{\Lambda_1^2(\rho)} \gamma_+(\rho) \left( \sqrt{t} - \Lambda_*(\rho) \right) p_{\gamma(\rho)}(t) dt = 0.
\]  

(73)

Observe that by (23), which is equivalent to (66), the minimizer \(\Lambda_*\) satisfies
\[
\int_{\Lambda_1^2(\rho)} \left[ \sqrt{t} - \Lambda_*(\rho) \right] p_{\gamma(\rho)}(t) dt = \frac{\rho \Lambda_*(\rho)}{\alpha(1 - \rho)}
\]
so that
\[
\lim_{\rho \to 0} \int_{\Lambda_1^2(\rho)} \gamma_+(\rho) \left[ \sqrt{t} - \Lambda_*(\rho) \right] p_{\gamma(\rho)}(t) dt = 0.
\]

Differentiating (73) under the integral sign, it remains to show that
\[
\lim_{\rho \to 0} \int_{\Lambda_1^2(\rho)} t^{k/2} \frac{\partial}{\partial \rho} p_{\gamma(\rho)}(t) dt = 0
\]
for \(k = 0, 1, 2\). Note that \(\lim_{\rho \to 0} \Lambda_*^2(\rho) = \lim_{\rho \to 0} \gamma_+(\rho) = (1 + \sqrt{3})^2\). However, it is easy to verify that \(\partial p_{\gamma(\rho)}/\partial \rho \leq C \cdot (\gamma_+(\rho) - t)^{-1/2}\) in a neighborhood of \((1 + \sqrt{3})^2\), for an appropriate constant \(C\), so that
\[
\lim_{\rho \to 0} \int_{\Lambda_1^2(\rho)} t^{k/2} \frac{\partial}{\partial \rho} p_{\gamma(\rho)}(t) dt \leq \lim_{\rho \to 0} C \sqrt{\gamma_+(\rho) - t} \bigg|_{\Lambda_1^2(\rho)} = 0.
\]
We now proceed to calculate the required limit. In the case Mat, by (21) we have
\[
\lim_{\rho \to 0} \Lambda_*(\rho, \beta, \alpha) = 1 + \sqrt{\beta}.
\]
Therefore
\[
\lim_{\rho \to 0} \frac{1}{\rho} \mathcal{M}(\rho, \beta | \text{Mat}) = \lim_{\rho \to 0} \frac{1}{\rho} \mathcal{M}(\Lambda_*(\rho, \beta, 1), \rho, \hat{\rho}, 1) = \lim_{\rho \to 0} \frac{1}{\rho} (1 + \beta + \rho \beta + (1 - \rho \beta)(1 + \sqrt{\beta})^2) = 1 + \beta + (1 + \sqrt{\beta})^2 = 2 \left(1 + \sqrt{\beta} + \beta\right),
\]
where we have used (73) and the relation \(\hat{\rho} = \beta \rho\). The calculation for case Sym is identical. \(\square\)

9 Global Minimax MSE and AMSE

We first give sense to the notion of a general singular-value-based denoiser. This is to be a mapping \(Y \mapsto \hat{X}(Y)\) that acts on \(Y\) only through its singular values, i.e. a mapping of the form \(\hat{X} : M_{m \times n} \to M_{m \times n}\) of the form
\[
\hat{X}(Y) = U_Y \cdot \hat{x}(y) \Delta \cdot V_Y',
\]
where \(Y = U_Y \cdot y_\Delta \cdot V_Y'\) and \(\hat{x} : [0, \infty)^m \to [0, \infty)^m\). The mapping in (74) is not well-defined in general, since the SVD of \(Y\), and in particular the order of the singular values in the vector \(y\), is not uniquely determined. Well-definedness of (74) will obtain when each function \(\hat{x}_i : [0, \infty) \to [0, \infty)\) is invariant under permutations of its coordinates. Since the equality \(Y = U_Y \cdot y_\Delta \cdot V_Y'\) may hold for vectors \(y\) with negative entries, we are led to the following definition.

Definition 5. By Singular Value Denoiser we mean any measurable mapping \(\hat{X} : M_{m \times n} \to M_{m \times n}\) which takes the form (74), where each entry of \(\hat{x}\) is a function \(\hat{x}_i : \mathbb{R}^m \to \mathbb{R}\) that is invariant under permutation and sign changes of its coordinates. We let \(\mathcal{D}\) denote the class of such mappings.

With this definition, \(Y \mapsto \hat{X}(Y)\) in (74) is well defined. For a detailed introduction to real-valued or matrix-valued functions which depend on a matrix variable only through its singular values, see [21, 22].

Proof of Theorem 9. Let \(O_n\) denote the orthogonal group in \(M_{n \times n}\) and let \(O_m \times O_n\) act on \(M_{m \times n}\) by \((U, V) : X \mapsto U' \cdot X \cdot V\). Recall that a decision rule \(\hat{X}\) satisfying \(U \hat{X}(Y) V' = \hat{X}(UYV')\) for all \((U, V) \in G\) is called equivariant with respect this group action (see [23, def. 2.5]). In [2, cor. 7] it is shown that \(\mathcal{D}\) coincides with the family of equivariant decision rules (see also [21, prop. 5.1]). Now, the Hunt-Stein Theorem [23, 24] implies that a lower bound on the minimax MSE over \(\mathcal{D}\) is also a lower bound on the global minimax MSE. Let \(\hat{X}(Y) \in \mathcal{D}\). We will show that
\[
\sup_{X_0 \in \mathcal{X}_{m,n}} \frac{R(\hat{X}, X_0)}{\text{rank}(X_0) \leq r} \geq \frac{r}{m} + \frac{r}{n} - \frac{r^2 + r}{mn}.
\]
Indeed, let $X_0 \in M_{m \times n}$ be a fixed arbitrary matrix of rank $r$. The calculation leading to (55), is valid for any rule in $D$, and implies that $R(\hat{X}(Y), X_0) \geq 1 - \frac{1}{m} \mathbb{E} ||z||_2^2$, where $Y = U_Y \cdot y_\Delta \cdot V'_Y$ and 

$$z = \frac{1}{\sqrt{n}}(U'_Y \cdot Z \cdot V)_\Delta.$$  

(75)

Write $Y_\mu = \mu X_0 + Z/\sqrt{n} = U_\mu \cdot (y_\mu)_\Delta \cdot V'_\mu$ and let $z_\mu = \frac{1}{\sqrt{n}}(U'_\mu \cdot Z \cdot V)_\Delta$. We therefore have 

$$\sup_{\mu, \mu_0 \in M_{m,n}} \frac{\text{rank}(X_0)}{r} \leq r \lim_{\mu \to \infty} R(\hat{X}, \mu X_0) \geq 1 - \frac{1}{m} \lim_{\mu \to \infty} \mathbb{E} ||z_\mu||_2^2.$$ 

Combining (63) and (61), we have seen already seen that 

$$\frac{1}{m} \sum_{i=r+1}^m \lim_{\mu \to \infty} \mathbb{E}(z_{\mu,i})^2 = 1 - \frac{r}{m} - \frac{r}{n} + \frac{r^2}{mn}.$$ 

A similar argument yields 

$$\frac{1}{m} \sum_{i=1}^r \lim_{\mu \to \infty} \mathbb{E}(z_{\mu,i})^2 = \frac{r}{mn},$$

and the first part of the theorem follows. The second part of the theorem follows since, taking the limit $n \to \infty$ as prescribed, we have $r/m \to \rho, r/n \to \tilde{\rho}$ and $r/mn \to 0$. For the third part of the theorem, we have by Theorem 8 

$$\lim_{\rho \to 0} \frac{M(\rho, \beta|X)}{M^-(\rho, \beta)} = \lim_{\rho \to 0} \frac{M(\rho, \beta|X)}{\rho + \beta \rho + \beta \rho^2} = \frac{2(1 + \sqrt{\beta} + \beta)}{1 + \beta} = 2 \left(1 + \frac{\sqrt{\beta}}{1 + \beta}\right).$$

10 Discussion

10.1 Similarities Emerging from our Proofs.

In the introduction, we pointed out several ways that these matrix denoising results for SVST estimation of low-rank matrices parallel results for soft thresholding of sparse vectors. Our derivation of the minimax MSE formulas exposed two more parallels.

- Common Structure of minimax MSE formulas. The minimax MSE formula vector denoising problem involves certain incomplete moments of the standard Gaussian distribution. The matrix denoising problem involves completely analogous incomplete moments, only replacing the Gaussian by the Marčenko-Pastur distribution or (in the square case $\beta = 1$) the quarter-circle law.

- Monotonicity of SURE. In both settings, the least favorable estimand places the signal “at $\infty$”, which yields a convenient formula for Minimax MSE. In each setting, validation of the least-favorable estimation flows from monotonicity, in an appropriate sense, of Stein’s Unbiased Risk Estimate within that specific setting.
10.2 Another Connection: Block Thresholding

The parallels between the problems do not seem accidental. An interesting denoising problem intermediate between sparse vector and low-rank matrices involves block-sparse vectors \[7\]. Expressing that problem in this paper’s notation, the object \( X_0 \) is an \( m \text{-by-} n \) array, and we call the columns “blocks”; so each block has \( m \) entries. We assume at most a fraction \( \epsilon \) of blocks are nonzero: 
\[
\epsilon \cdot n \geq \#\{j : \|X(:,j)\|_2 \neq 0\}.
\]

We observe noisy matrix data \( Y = X_0 + Z \) and we consider the block-shrinkage problem:
\[
(P_{2,1}) \quad \hat{X}_\lambda = \arg\min_{X \in \mathbb{M}_{m \times n}} \frac{1}{2} \|Y - X\|_F^2 + \lambda \sum_j \|X(:,j)\|_2.
\]

By the inequalities
\[
\|X\|_* \leq \sum_j \|X(:,j)\|_2 \leq \sum_{ij} |X(i,j)|,
\]
(NNP) is a relaxation of \((P_{2,1})\), while \((P_{2,1})\) is a relaxation of \((P_1)\) applied to \( \text{vec}(Y) \).

In the case \( m = 1 \) we recover \((P_1)\) and the soft thresholding procedure. In the case \( m > 1 \) we obtain block thresholding; it promotes reconstructions \( \hat{X}_\lambda \) where many blocks are fully zero and a small fraction are nonzero; the nonzero blocks are those where \( \|X(:,j)\|_2 > \lambda \).

The chain of inequalities \((77)\) places \((P_{2,1})\) intermediate between \((P_1)\) and (NNP). All the parallels mentioned so far between soft thresholding and SVST also hold between block soft thresholding and the other two methods.

- All three involve soft thresholding of relevant objects - scalars, column norms, or singular values.
- All three have a least favorable estimand with its nonzero piece “at \( \infty \)”.
- All three have a minimax penalty factor \( \lambda^* (\epsilon) \) monotone decreasing in \( \epsilon \).

Moreover, the two parallels of the last section carry over as well. For block thresholding, the minimax MSE involves incomplete moments, this time of the classical \( \chi_{m}^2 \) distribution. The monotonicity of SURE carries through and implies the structure of the least-favorable estimand.

10.3 Block Thresholding and the Minimaxity Gap

We bring up block soft thresholding because of its relevance to the minimaxity gap of SVST that we conjectured in the introduction. Donoho, Johnstone and Montanari \[7\] considered the following limiting case, where we consider \( n \to \infty \) first, and later \( m \to \infty \). In that setting, the minimax MSE among all measurable procedures under \( \epsilon \)-block sparsity can be evaluated;
\[
M^*_\text{block}(\epsilon) = \lim_{m \to \infty} \lim_{n \to \infty} \inf_X \sup_{\#\{j: X_0(:,j) \neq 0\}} \frac{1}{nm} E \|\hat{X} - X_0\|_F^2;
\]
it obeys \( M(\epsilon) = \epsilon \), and they show it can be attained asymptotically by a particularly lovely method: simply apply the James-Stein shrinkage estimator blockwise! On the other hand, the minimax MSE for soft block thresholding can be evaluated:
\[
M_{\text{block}}(\epsilon) = \lim_{m \to \infty} \lim_{n \to \infty} \inf_{\lambda} \sup_{\#\{j: X_0(:,j) \neq 0\}} \frac{1}{nm} E \|\hat{X}_\lambda - X_0\|_F^2.
\]
They obtain $M_{\text{block}}(\varepsilon) = 2\varepsilon - \varepsilon^2$. Consequently, block soft thresholding is never worse than a factor of 2 from minimaxity, at any level of block sparsity. This bound is achieved as $\varepsilon \to 0$:

$$
\lim_{\varepsilon \to 0} \frac{M_{\text{block}}(\varepsilon)}{M_{\text{block}}(\varepsilon)} = 2;
$$

namely, in the high-dimensional limit, under extreme sparsity, block soft thresholding is a factor 2 worse than minimax. These completed results about the minimaxity gap in high-dimensional block soft thresholding are suggestive from the viewpoint of Singular Value Soft Thresholding. Could there be an estimator improving on SVST, and particularly lovely in form?

**Acknowledgments**

We thank Iain Johnstone, Andrea Montanari and Art Owen for advice at several crucial points. This work was partially supported by NSF DMS 0906812 (ARRA). MG was partially supported by a William R. and Sara Hart Kimball Stanford Graduate Fellowship and a Technion EE Sohnis Promising Scientist Award, and thanks the Technion Department of Electrical Engineering for their hospitality during the preparation of a significant part of this work.

**References**

[1] Michael Grant and Stephen P. Boyd. CVX: Matlab Software for Disciplined Convex Programming, 2010. URL [cxvr.com/cvx](http://cxvr.com/cvx).

[2] Andrey Shabalin and Andrew Nobel. Reconstruction of a Low-rank Matrix in the Presence of Gaussian Noise. *arXiv preprint arXiv:1007.4148*, pages 1–34, 2010. URL [http://arxiv.org/abs/1007.4148](http://arxiv.org/abs/1007.4148).

[3] Emmanuel J. Candès, Carlos A Sing-long, and Joshua D Trzasko. Unbiased Risk Estimates for Singular Value Thresholding and Spectral Estimators. *Stanford Statistical Department technical report no. 2012-12. October 2012*. URL [http://arxiv.org/abs/1210.4139](http://arxiv.org/abs/1210.4139).

[4] David L. Donoho and Iain M. Johnstone. Minimax risk over l p-balls for l q-error. Probability Theory and Related Fields, 303:277–303, 1994. URL [http://www.springerlink.com/index/L77358V34PN470VP.pdf](http://www.springerlink.com/index/L77358V34PN470VP.pdf).

[5] David L. Donoho, Matan Gavish, and Andrea Montanari. The Phase Transition of Matrix Recovery from Gaussian Measurements Matches the Minimax MSE of Matrix Denoising. *Stanford Statistical Department technical report no. 2013-02. February 2013*. URL [http://arxiv.org/abs/1302.2331](http://arxiv.org/abs/1302.2331).

[6] Samet Oymak and Babak Hassibi. On a Relation between the Minimax Denoising and the Phase Transitions of Convex Functions, 2012. URL [http://www.its.caltech.edu/~soymak/Relation.pdf](http://www.its.caltech.edu/~soymak/Relation.pdf).

[7] David L. Donoho, Iain M. Johnstone, and Andrea Montanari. Accurate Prediction of Phase Transitions in Compressed Sensing via a Connection to Minimax Denoising. *Stanford Statistical Department technical report no. 2011-10. November 2011*. URL [http://arxiv.org/abs/1111.1041](http://arxiv.org/abs/1111.1041).
[8] Benjamin Recht, Maryam Fazel, and Pablo A. Parrilo. Guaranteed minimum-rank solutions of linear matrix equations via nuclear norm minimization. *SIAM review*, 52(3):471–501, 2010. URL [http://epubs.siam.org/doi/abs/10.1137/070697835](http://epubs.siam.org/doi/abs/10.1137/070697835).

[9] Benjamin Recht, Weiyu Xu, and Babak Hassibi. Null space conditions and thresholds for rank minimization. *Mathematical Programming*, 127(1):175–202, October 2010. ISSN 0025-5610. doi: 10.1007/s10107-010-0422-2. URL [http://www.springerlink.com/index/10.1007/s10107-010-0422-2](http://www.springerlink.com/index/10.1007/s10107-010-0422-2).

[10] Benjamin Recht, Weiyu Xu, and Babak Hassibi. Necessary and sufficient conditions for success of the nuclear norm heuristic for rank minimization. In *Proceedings of the 47th IEEE conference on Decision and Control Cancun, Mexico*, 2008. URL [http://ieeexplore.ieee.org/xpls/abs_all.jsp?arnumber=4739332](http://ieeexplore.ieee.org/xpls/abs_all.jsp?arnumber=4739332).

[11] Jared Tanner and Ke Wei. Normalized iterative hard thresholding for matrix completion. 2012. URL [http://people.maths.ox.ac.uk/tanner/papers/TaWei_NIHT.pdf](http://people.maths.ox.ac.uk/tanner/papers/TaWei_NIHT.pdf).

[12] Samet Oymak and Babak Hassibi. New Null Space Results and Recovery Thresholds for Matrix Rank Minimization. November 2010. URL [http://arxiv.org/abs/1011.6326](http://arxiv.org/abs/1011.6326).

[13] Alberto Zanella, Marco Chiani, and Moe Z. Win. On the marginal distribution of the eigenvalues of Wishart matrices. *IEEE Transactions on Communications*, 57:1050–1060, 2009. URL [http://ieeexplore.ieee.org/xpls/abs_all.jsp?arnumber=4814372](http://ieeexplore.ieee.org/xpls/abs_all.jsp?arnumber=4814372).

[14] Vladimir Marcenko and Leonid Pastur. Distribution of eigenvalues for some sets of random matrices. *Mathematics USSR Sbornik*, 1(4):457–483, 1967. URL [http://iopscience.iop.org/0025-5734/1/4/A01](http://iopscience.iop.org/0025-5734/1/4/A01).

[15] David L. Donoho and Matan Gavish. Companion website for the article The Phase Transition of Matrix Recovery from Gaussian Measurements Matches the Minimax MSE of Matrix Denoising. [http://www.runmycode.org/CompanionSite/Site265](http://www.runmycode.org/CompanionSite/Site265), 2013. URL [http://www.runmycode.org/CompanionSite/Site265](http://www.runmycode.org/CompanionSite/Site265) accessed 15 February 2013.

[16] David L. Donoho, Iain M. Johnstone, Jeffrey C. Hoch, and Alan S. Stern. Maximum entropy and the nearly black object. *Journal of the Royal Statistical Society. Series B*, 54(1):41–81, 1992. URL [http://www.jstor.org/stable/10.2307/2345948](http://www.jstor.org/stable/10.2307/2345948).

[17] Gene H. Golub and Charles F. Van Loan. *Matrix Computations*. Johns Hopkins University Press, 1996. URL [http://books.google.com/books?hl=en&lr=&id=5U-18U3P-VUC&oi=fnd&pg=PT10&dq=2016&ots=7YHAKo0Ndr&sig=dk1vFHCdOJKjQyZCFk37QP-1cTkB](http://books.google.com/books?hl=en&lr=&id=5U-18U3P-VUC&oi=fnd&pg=PT10&dq=2016&ots=7YHAKo0Ndr&sig=dk1vFHCdOJKjQyZCFk37QP-1cTkB).

[18] Ted W. Anderson. The Integral of a Symmetric Unimodal Function over a Symmetric Convex Set and Some Probability Inequalities. *Proceedings of the American Mathematical Society*, 6(2):170–176, 1955. URL [http://www.jstor.org/stable/10.2307/2032333](http://www.jstor.org/stable/10.2307/2032333).
A Convexity of the Stein Unbiased Risk Estimate for SVST

In this section we prove Lemma 7, whereby the Stein Unbiased Risk Estimate (SURE) for SVST is given as a finite sum of bounded, orthogonally invariant and quasi-convex functions.

This appendix employs the following notation. Following [21, 22], we say that a function $f : \mathbb{R}^m \rightarrow \mathbb{R}$ is absolutely symmetric if it is invariant under permutations and
sign changes of its coordinates. As discussed in Section 9, the function of a matrix argument $F : M_{m \times n} \to \mathbb{R}$ defined by $F(X) = f(x)$, where $X = U_X \cdot x_\Delta \cdot V_X$ for some orthogonal $U_X \in O_m$ and $V_X \in O_n$, is well defined only if $f$ is absolutely symmetric. Denote by $W_m = \{y_1 \geq \cdots \geq y_m \geq 0\} \subset \mathbb{R}^m$ the set of singular value vectors (in non-increasing order) of matrices in $M_{m \times n}$ and by $W_m^+ = \{y_1 > \cdots > y_m > 0\} \subset \mathbb{R}^m$ the space of non-degenerate singular value vectors. We denote by $\Sigma : M_{m \times n} \to W_m$ the map $Y \mapsto y$ that maps a matrix $Y$ to its singular value vector $y$, sorted in non-increasing order. $\Sigma$ is orthogonally invariant, namely invariant under the transformation $X \mapsto U \cdot X \cdot V'$ for any orthogonal $U \in O_m$ and $V \in O_n$. Also note that if $Y = X + \sigma Z \in M_{m \times n}$ with $Z_{i,j} \overset{\text{iid}}{\sim} \mathcal{N}(0, 1)$, then the event $\Sigma(Y) \in W_m^+$ holds almost surely.

Let $\hat{X}$ be a weakly differentiable estimator of $X_0$ from data $Y = X_0 + \sigma Z$, where $Z$ has i.i.d standard normal entries. The Stein Unbiased Risk Estimate [25] is a function of the data, $Y \mapsto SURE(Y)$, for which $\mathbb{E} SURE(Y) = \mathbb{E} \|\hat{X} - X_0\|^2$. In our case, $X_0, Z$ and $Y$ are matrices in $M_{m \times n}$, and Stein’s theorem [25, thm. 1] implies that for

$$SURE(Y) = mn\sigma^2 + \left\|\hat{X}(Y) - Y\right\|^2_F + 2\sigma^2 \sum_{i,j} \frac{\partial(\hat{X}(Y) - Y)_{ij}}{\partial Y_{ij}},$$

we have

$$\left\|\hat{X} - X_0\right\|^2_F = \mathbb{E}_{X_0} SURE(Y).$$

The following lemma specializes the general SURE formula (78) to the case of matrix estimators that act through the data singular values. This calculation has been carried out independently by [3]. While here we use the SURE formula for a theoretical purpose, [3] proposed to use it to select a threshold for matrix denoising by SVST in applications.

**Lemma 15.** Write the singular value decomposition of $Y \in M_{m \times n}$ as $Y = U_Y \cdot y_\Delta \cdot V_Y'$. Let $\hat{X}$ be any weakly differentiable element of $\mathcal{D}$, as in Definition 5. Consider $\hat{X}$ as an estimator of $X_0$ from $Y = X_0 + \sigma Z$. Then for any $Y \in M_{m \times n}$ such that $\Sigma(Y) \in W_m^+$, we have $SURE(Y) = \text{sure}(\Sigma(Y))$, where $\text{sure} : W_m^+ \to \mathbb{R}$ is given by

$$\text{sure}(y) = mn\sigma^2 + \|g(y)\|^2 + 2\sigma^2 \left[ \sum_{1 \leq i \neq j \leq m} \frac{g(y)_j y_j - g(y)_i y_i}{y_j^2 - y_i^2} + \sum_{i=1}^m \frac{\partial g_i}{\partial y_i} \right] + (n - m) \sum_{i=1}^m \frac{g(y)_i}{y_i}.$$  

Here, $g(y) = \hat{x} - y$ and $\partial / \partial y_i$ is a weak derivative.

Note that $y \mapsto \text{sure}_\lambda(\|y\|)$ is absolutely symmetric. Below, we extend the domain of $\text{sure}_\lambda$ by symmetry and consider $\text{sure}_\lambda : \mathbb{R}^m \to \mathbb{R}^m$.

Let us first show that Lemma 15, whereby $SURE_\lambda$ is a bounded, orthogonally invariant and quasi-convex function of a matrix argument, follows from Lemma 15. To establish quasi-convexity if $SURE_\lambda$ we will need to relate the quasi-convexity of $SURE_\lambda$, a function of a matrix argument, to quasi-convexity of $\text{sure}_\lambda$, a function of singular values.

**Lemma 16.** Let $f : \mathbb{R}^m \to \mathbb{R}$ be an absolutely symmetric and lower-semicontinuous function. If $f$ is a quasi-convex then the function $F : M_{m \times n} \to \mathbb{R}$ defined by $F(X) = f(\Sigma(X))$ is quasi-convex on $M_{m \times n}$. 
Corollary 2. Suppose that quasi-convex.

Proof. We use the following characterization of quasi-convexity by subgradients due to Ausell [26, thm. 2.2]: A lower semicontinuous function \(g\) on a Banach space \(\mathcal{X}\) is quasi-convex if and only if the following condition holds for all \(x, y \in \mathcal{X}\):

\[
\exists x^* \in \partial g(x) : \langle x^*, y - x \rangle > 0 \implies \forall y^* \in \partial g(y) : \langle y^*, y - x \rangle \geq 0.
\] (80)

Now, Lewis [22, cor. 2.5] has provided the following characterization of subgradients of \(F\). Let \(X, Y \in M_{m \times n}\). Then \(Y \in \partial F(X)\) if and only if \(\Sigma(Y) \in \partial f(\Sigma(X))\) and there exist orthogonal matrices \(U \in O_m\) and \(V \in O_n\) and vectors \(x, y \in \mathbb{R}^m\) such that

\[
X = U \cdot x \cdot V \quad \text{and} \quad Y = U \cdot y \cdot V.
\]

We also recall von Neumann’s inequality for singular values [22, thm. 2.1], whereby if \(X, Y \in M_{m \times n}\) have singular value vectors \(x, y \in \mathbb{R}^m\), then \(\langle X, Y \rangle = \langle x, y \rangle\). Here and below, \(\langle X, Y \rangle = \sum_{i,j} X_{i,j} Y_{i,j}\) is the Euclidean inner product on \(M_{m \times n}\).

We finally turn to the proof. By [27, thm. 4.2], \(F\) is lower-semicontinuous. Let us show that \(\hat{F}\) is quasi-convex using the characterization (80). Let \(X, Y \in M_{m \times n}\) and assume that \(X^* \in \partial F(X)\) exists such that \(\langle X^*, Y - X \rangle > 0\). We now show that for all \(Y^* \in \partial F(Y)\), we have \(\langle Y^*, Y - X \rangle \geq 0\). Let \(Y^* \in \partial F(Y)\), and let \(x, y, x^*\) and \(y^*\) denote the singular value vectors of \(X, Y, X^*\) and \(Y^*\) respectively. Since \(X^* \in \partial F(X)\) we have \(\langle X^*, X \rangle = \langle x^*, x \rangle\). Also, by von-Neumann’s inequality we have \(\langle X^*, Y \rangle \leq \langle x^*, y \rangle\). We therefore have

\[
\langle x^*, x \rangle = \langle X^*, X \rangle < \langle X^*, Y \rangle \leq \langle x^*, y \rangle.
\]

Since \(f\) is quasi-convex, by assumption, in particular for \(y^*\) we have \(\langle y^*, x \rangle \leq \langle y^*, y \rangle\), since \(y^* \in \partial F(Y)\). Also, \(\langle Y^*, Y \rangle = \langle y^*, y \rangle\). Again by von-Neumann’s inequality we have \(\langle Y^*, X \rangle \leq \langle y^*, x \rangle\). Together, this gives

\[
\langle Y^*, X \rangle \leq \langle y^*, x \rangle \leq \langle y^*, y \rangle = \langle Y^*, Y \rangle,
\]

as required.

\[
\square
\]

Corollary 1. Suppose that \(f : \mathbb{R}^m \to \mathbb{R}\) is representable as \(f(y) = \sum_{i=1}^m f_i(y_i)\) where \(f_i : \mathbb{R} \to \mathbb{R}\) is a bounded, nondecreasing, lower-semicontinuous function such that \(f_i(-y) = f_i(y)\). Then \(F : M_{m \times n} \to \mathbb{R}\) defined by \(F(X) = f(\Sigma(X))\) is bounded and quasi-convex.

Corollary 2. Suppose that \(f : \mathbb{R}^m \to \mathbb{R}\) is representable as \(f(y) = \sum_{1 \leq i \neq j \leq m} f_2(y_i, y_j)\), where \(f_2 : \mathbb{R}^2 \to \mathbb{R}\) is a quasi-convex, lower-semicontinuous function obeying \(f_2(x, y) = f_2(\pm y, \pm x)\). Then \(F : M_{m \times n} \to \mathbb{R}\) defined by \(F(X) = f(\Sigma(X))\) is quasi-convex.

Proof of Lemma 7. Write \(SURE_\lambda\) for the SURE corresponding to the SVST estimator \(\hat{X}_\lambda\). By Lemma 15, let \(suc_{\lambda}\) be the function of the singular values \(SURE_\lambda(Y) = suc_{\lambda}(\Sigma(Y))\). Substituting \(\hat{x}(y)_i = (y_i - \lambda)_+\) and \(\sigma = 1/\sqrt{n}\) in [79], we get \(g(y)_i = -\min \{y_i, \lambda\}\), hence

\[
suc_{\lambda}(y) = m + \sum_{i=1}^m \left( \min \{y_i, \lambda\} \right)^2 - 1_{\{y_i < \lambda\}} - \frac{(n - m) \cdot \min \{y_i, \lambda\}}{y_i},
\] (81)

\[
- \frac{2}{n} \sum_{1 \leq i \neq j \leq m} \frac{\min \{y_j, \lambda\} y_j - \min \{y_i, \lambda\} y_i}{y_j^2 - y_i^2}.
\] (82)
Each of the terms \( \min \{ y_i, \lambda \}, -1_{y_i < \lambda} \) and \(- \min \{ y_i, \lambda \} / y_i \) is non-decreasing, bounded, lower-semicontinuous, and therefore, by Corollary \[ \] the function implicitly defined by the RHS of \( \{ \) is bounded and quasi-convex as required. We now turn to the function implicitly defined by the sum in \( \{ \). Each element in the sum is given by \((2/n) \cdot f_{\lambda}(y_j, y_i)\) for some \(1 \leq i \neq j \leq m\), where \( f_{\lambda} : [0, \infty)^2 \to \mathbb{R} \) is given by

\[
 f_{\lambda}(x, y) = -\frac{\min \{ y, \lambda \} y - \min \{ x, \lambda \} x}{y^2 - x^2} = \begin{cases} 
 -1 & 0 < x, y \leq \lambda \\
 \frac{\lambda x - y^2}{y^2 - x^2} & 0 < y \leq \lambda, x \geq \lambda \\
 -\frac{\lambda}{x+y} & x \geq \lambda, y \geq \lambda 
\end{cases}
\]

with \( f_{\lambda}(x, y) = f_{\lambda}(y, x) \) and \( f_{\lambda}(x, x) \) defined to yield a continuous function. The function \( f_{\lambda} \) is shown in Figure \[ \]. We now show that \( f_{\lambda} \) can be written as a sum of four bounded, lower-semicontinuous, quasi-convex functions. Decompose \( f_{\lambda} \) as

\[
f_{\lambda}(x, y) = -1_{[0, \lambda] \times [0, \lambda]}(x, y) + \frac{\lambda x - y^2}{y^2 - x^2} \cdot 1_{(\lambda, \infty) \times (0, \lambda]}(x, y) + \frac{x^2 - \lambda y}{y^2 - x^2} \cdot 1_{(0, \lambda) \times (\lambda, \infty]}(x, y) - \frac{\lambda}{x+y} \cdot 1_{(\lambda, \infty) \times (\lambda, \infty]}.
\]

The image of each of these terms lies in \([-1, 0]\), hence each is bounded. To see that the second (and hence the third) and the fourth terms are quasi-convex, note that for \(0 \leq c \leq 1\),

\[
\left\{ (x, y) \in [0, \infty)^2 \left| \frac{\lambda x - y^2}{y^2 - x^2} \cdot 1_{(\lambda, \infty) \times (0, \lambda]}(x, y) \leq -c \right. \right\} = \left\{ (x, y) \in (\lambda, \infty) \times (0, \lambda) \left| (1 - c)y^2 \leq -cx^2 + \lambda x \right. \right\}
\]

and

\[
\left\{ (x, y) \in [0, \infty)^2 \left| -\frac{\lambda}{x+y} \cdot 1_{(\lambda, \infty) \times (\lambda, \infty]}(x, y) \leq -c \right. \right\} = \left\{ (x, y) \in (\lambda, \infty) \times (\lambda, \infty) \left| x + y \leq \frac{\lambda}{c} \right. \right\}.
\]

These are easily seen to be convex sets in \( \mathbb{R}^2 \). We conclude that the function \( f_{\lambda} \) can be decomposed \( f_{\lambda} = \sum_{s=1}^4 f_{\lambda}^{(s)} \), where each \( f_{\lambda}^{(s)} \) is quasi-convex. It follows that the sum \( \{ \) can be decomposed into four terms \( T_1 + \cdots + T_4 \), each of which is a sum of quasi-convex functions of pairs of singular values, \( T_s = \sum_{i \neq j} f_{\lambda}^{(s)}(y_i, y_j) \). By Corollary \[ \] each term \( T_s \), for \(1 \leq s \leq 4\), is a bounded, quasi-convex function on matrix space: \( T_s : M_{m \times n} \to \mathbb{R} \). It follows that \( SURE_{\lambda} \) is a sum of five bounded, quasi-convex functions on matrix space, and the proof is complete.

**Proof of Lemma** \[ \]. We first calculate the Jacobian of the SVD, following \[ \]. For the reader’s convenience, for the remainder of this appendix, consider the case \( m \geq n \), in order to conform to their notation. We avoid differential geometry notation and restrict ourselves to the more cumbersome, but more widely used, multivariate calculus notation.
Let \( Y = U_Y \cdot y_\Delta \cdot V_Y' \) denote a full (“fat”) SVD of \( Y \in M_{m \times n} \), where now \( m \ge n \) and \( y_\Delta \in M_{m \times n} \). We can view \( Y \mapsto U_Y \) as a map \( M_{m \times n} \to M_{m \times m} \), where the first \( n \) columns of \( U_Y \) are determined (up to sign of each column) by \( Y \) and the last \( m - n \) columns constitute an arbitrary completion to an orthonormal basis of \( \mathbb{R}^m \). Similarly \( Y \mapsto y \) is a function into the Weyl chamber \( \{ y | y_1 \ge \cdots \ge y_n \} \subset \mathbb{R}^n \) and \( Y \mapsto V_Y \) is a map \( M_{m \times n} \to M_{n \times n} \), determined up to sign of each column. As we will see later, the trace of the Jacobian of the SVD is well defined and does not depend on these arbitrary choices.

[28] have proposed to calculate the Jacobian of each of these multivariate real functions as follows. They show that

\[
\begin{align*}
\frac{\partial U_{k,\ell}}{\partial Y_{i,j}} &= (U \cdot \Omega_U^{i,j})_{k,\ell} \\
\frac{\partial V_{k,\ell}}{\partial Y_{i,j}} &= -(V \cdot \Omega_V^{i,j})_{k,\ell} \\
\frac{\partial y_k}{\partial Y_{i,j}} &= U_{i,k}V_{j,k}
\end{align*}
\]

for \( 1 \le i, k \le m \) and \( 1 \le j, \ell \le n \). Here, \( \Omega_U^{i,j} \in M_{m \times n} \) and \( \Omega_V^{i,j} \in M_{n \times n} \) for any \( 1 \le i \le m \) and \( 1 \le j \le n \). [28] show that both \( \Omega_V^{i,j} \) and the upper \( n \times n \) block of \( \Omega_U^{i,j} \) are antisymmetric, and that each pair \( ((\Omega_U^{i,j})_{k,\ell}, (\Omega_V^{i,j})_{k,\ell}) \) with \( 1 \le k, \ell \le n \) satisfies the \( 2 \times 2 \) linear system

\[
\begin{align*}
y_\ell(\Omega_U^{i,j})_{k,\ell} + y_k(\Omega_V^{i,j})_{k,\ell} &= U_{i,k}V_{j,\ell} \\
y_k(\Omega_U^{i,j})_{k,\ell} + y_\ell(\Omega_V^{i,j})_{k,\ell} &= -U_{i,\ell}V_{j,k}.
\end{align*}
\]

The authors do not explicitly provide the equations that determine the lower \( (m-n) \times n \) block of \( \Omega_U^{i,j} \). Fortunately, their arguments immediately imply that entries in this block satisfy

\[
y_\ell(\Omega_U^{i,j})_{k,\ell} = U_{i,k}V_{j,\ell},
\]

for \( 1 \le i \le m, 1 \le j, \ell \le n \) and \( n + 1 \le k \le m \).
We can now evaluate the divergence term in (78),

\[
\sum_{i=1}^{m} \sum_{j=1}^{n} \frac{\partial(\hat{X}(Y) - Y)_{i,j}}{\partial Y_{i,j}}.
\]

Define \( G(Y) = \hat{X} - Y \) and \( g(y) = \hat{x} - y \). To evaluate \( \frac{\partial G(Y)}{\partial Y_{i,j}} \) we note that \( G(Y)_{i,j} = \sum_k U_{i,k} g(y)_k V_{j,k} \) and hence

\[
\frac{\partial G(Y)}{\partial Y_{i,j}} = \sum_k \frac{\partial}{\partial Y_{i,j}} (U_{i,k} g(y)_k V_{j,k}) = \sum_k \frac{\partial U_{i,k}}{\partial Y_{i,j}} g(y)_k V_{j,k} + \sum_k U_{i,k} g(y)_k \frac{\partial V_{j,k}}{\partial Y_{i,j}} + \sum_k U_{i,k} \frac{\partial g(y)_k}{\partial Y_{i,j}} V_{j,k}.
\]

Since

\[
\frac{\partial g(y)_k}{\partial Y_{i,j}} = \sum_{\ell=1}^{n} \frac{\partial g(y)_k}{\partial y_{\ell}} \frac{\partial y_{\ell}}{\partial Y_{i,j}} = \sum_{\ell=1}^{n} \frac{\partial g(y)_k}{\partial y_{\ell}} U_{i,\ell} V_{j,\ell},
\]

we conclude that

\[
\frac{\partial G(Y)}{\partial Y_{i,j}} = \sum_k \left( U_{i,k} \Omega^{i,j}_U \right) g(y)_k V_{j,k} + \sum_k U_{i,k} g(y)_k \left(V \Omega^{i,j}_V \right)_{j,k} + \sum_k U_{i,k} V_{j,k} \left( \sum_{\ell} \frac{\partial g(y)_k}{\partial y_{\ell}} U_{i,\ell} V_{j,\ell} \right).
\]

Recall that the Jacobian trace \( Tr \left( \frac{\partial G}{\partial Y} \right) = \sum_{i,j,k} \frac{\partial G(Y)}{\partial Y_{i,j}} \) is invariant under change of basis of the underlying linear space \( M_{m \times n} \). Consider the orthonormal basis of \( M_{m \times n} \) given by rank-1 matrices, \( \{ u_i \cdot v_j \}_{i,j} \) where \( u_1, \ldots, u_m \) are the columns of \( U \) and \( v_1, \ldots, v_n \) are the columns of \( V \), respectively. To calculate the trace in this basis, we formally replace \( U \) and \( V \) with identity matrices of their respective dimensions.

The equations that determine \( \Omega^{i,j}_U \) and \( \Omega^{i,j}_V \) for \( 1 \leq i \leq m \) and \( 1 \leq j \leq n \), namely (87) and (88), become

\[
y_{\ell}(\Omega^{i,j}_U)_{k,\ell} + y_k(\Omega^{i,j}_U)_{\ell,i} = \delta_{i,k} \delta_{j,\ell}, \tag{90}
\]

\[
y_{\ell}(\Omega^{i,j}_U)_{k,\ell} + y_k(\Omega^{i,j}_V)_{\ell,i} = -\delta_{i,k} \delta_{j,\ell}, \tag{91}
\]

for \( 1 \leq k, \ell \leq n \), and

\[
y_{\ell}(\Omega^{i,j}_U)_{k,\ell} = \delta_{i,k} \delta_{j,\ell}, \tag{92}
\]

for \( n + 1 \leq k \leq m \) and \( 1 \leq \ell \leq n \). Similarly, in this basis, (89) becomes

\[
Tr \left( \frac{\partial G}{\partial Y} \right) = \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{n} \left[ \left( \Omega^{i,j}_U \right)_{i,k} g(y)_k \delta_{j,k} - \delta_{i,k} g(y)_k \left( \Omega^{i,j}_V \right)_{j,k} + \delta_{i,k} \delta_{j,k} \left( \sum_{\ell} \frac{\partial g(y)_k}{\partial y_{\ell}} \delta_{i,\ell} \delta_{i,\ell} \right) \right] = \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k \neq i} \left[ \left( \Omega^{i,j}_U \right)_{i,j} g(y)_j - g(y)_i \left( \Omega^{i,j}_V \right)_{j,i} \right] + \sum_{j=1}^{n} \frac{\partial g(y)_j}{\partial y_j}.
\]


Solving (90) and (91) for \((\Omega_{i,j}^{U})_{i,j}\) and \((\Omega_{i,j}^{V})_{i,j}\), where \(i \neq j\), we get
\[
(\Omega_{i,j}^{U})_{i,j} = \frac{y_j}{y_j^2 - y_i^2} \quad 1 \leq i, j \leq n
\]
\[
(\Omega_{i,j}^{V})_{i,j} = \frac{1}{y_j} \quad n + 1 \leq i \leq m, 1 \leq j \leq n
\]
\[
(\Omega_{i,j}^{V})_{i,j} = -\frac{y_i}{y_j^2 - y_i^2} \quad 1 \leq i, j \leq n
\]
and \((\Omega_{i,i}^{U})_{i,i} = (\Omega_{i,i}^{V})_{i,i} = 0\), to the effect that
\[
Tr \left( \frac{\partial G}{\partial Y} \right) = \sum_{1 \leq i \neq j \leq n} \frac{y_j g(y)_j - y_i g(y)_i}{y_j^2 - y_i^2} + (m - n) \sum_{j=1}^{n} \frac{g(y)_j}{y_j} + \sum_{i=j}^{n} \frac{\partial g(y)_j}{\partial y_j} .
\] (92)

Changing back to our original notation of \(m \leq n\) by exchanging the symbols \(m\) and \(n\), and using the general SURE formula (78), we have proved Lemma 15.