Cyclic Statistics In Three Dimensions

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Abstract

While 2-dimensional quantum systems are known to exhibit non-permutation, braid group statistics, it is widely expected that quantum statistics in 3-dimensions is solely determined by representations of the permutation group. This expectation is false for certain 3-dimensional systems, as was shown by the authors of [1, 2, 3]. In this work we demonstrate the existence of “cyclic”, or $\mathbb{Z}_n$, non-permutation group statistics for a system of $n > 2$ identical, unknotted rings embedded in $\mathbb{R}^3$. We make crucial use of a theorem due to Goldsmith in conjunction with the so called Fuchs-Rabinovitch relations for the automorphisms of the free product group on $n$ elements.

1 Introduction

Classical systems with configuration spaces having a non-trivial fundamental group allow inequivalent quantisations, each labeled by the unitary irreducible representations of this group. A simple illustration of this can be found in the sum-over-histories quantisation of a particle on a circle wherein the set of paths with fixed initial and final positions fall into classes labeled by the winding number $m$ [4]. The full partition function is expressed as a sum of partitions over these different classes of paths, each multiplied by an overall phase $e^{im\theta}$, where $\theta \in [0, 2\pi]$ labels the unitary irreducible representations of the fundamental group of the circle. Each choice of $\theta$ thus leads to an inequivalent quantisation of the system. This method of quantisation yields several interesting phenomena ranging from the quantum statistics of point particles, to a Hamiltonian interpretation of the QCD theta angle, to spinorial states in quantum gravity [1, 5, 6, 7].

The particular phenomenon of interest to us in this paper is the emergence of quantum statistics in systems of $n$ identical objects like point particles or topological geons. The fundamental group $\pi_1(Q_n)$ of the configuration space $Q_n$ of such a system contains a subgroup which, in spatial dimension $d > 2$, is isomorphic to the permutation group on $n$ elements $S_n$. On quantisation the unitary irreducible representations of $S_n$ play a role in determining the quantum statistics of the system. For $n = 2$, $d > 2$, for example, the permutation group $S_2$, generated by the exchange operation $\mathcal{E}$, has two inequivalent unitary irreducible representations: the trivial one ($\mathcal{E} \to 1$) corresponding to bose statistics and the non-trivial one ($\mathcal{E} \to -1$) corresponding to fermi statistics. For $n > 2$, $d > 2$, $S_n$ has non-abelian unitary irreducible representations which give rise to
parastatistics. In dimension $d = 2$, however, statistics is dictated by an infinite discrete group, the braid group $B_n$, rather than the finite group $S_n$. The resulting statistics is referred to as “anyonic” and plays a central role in the study of 2 dimensional systems.

Since the permutation group $S_n$ is always a subset of $\pi_1(Q_n)$ for $d > 2$ it is generally believed that the occurrence of non-permutation group quantum statistics is restricted to 2-dimensions. However this is not always the case in 3-dimensions [1, 2, 3]. While $S_n$ plays a determining role in the quantum statistics in $d > 2$, it does not play the only role. As demonstrated in [3], quantum statistics depends on how the subgroup $S_n$ “sits” in the larger group $\pi_1(Q_n)$; typically, $\pi_1(Q_n) = P \ltimes S_n$, where $\ltimes$ denotes a semi-direct product and $P$ is a normal subgroup of $\pi_1(Q_n)$. Quantum statistics is then determined not by unitary irreducible representations of $S_n$, but rather those of the little groups (or stability subgroups) $\mathcal{R} \subseteq S_n$ with respect to the action of $S_n$ on the space of representations of $P$.

For a large class of systems the little groups are themselves permutation subgroups $S_m$ of $S_n$, with $m \leq n$. For example, consider a system of 3 identical extended solitons which are allowed to possess spin, i.e., a $2\pi$ rotation of the soliton is non-trivial (see [8] for an example). Even though they are classically identical, one can construct a representation $\{1/2, 1/2, 0\}$ in which two of the solitons are spin half and the third one is spin zero, thus rendering it quantum mechanically distinguishable from the others. Indeed, as expected, the associated little group of $S_3$ can be shown to be $S_2 \subset S_3$ which corresponds to 2 rather than 3 particle quantum statistics.

However, there exist systems in which the little group need not always be a permutation subgroup. Consider a system of $n$ closed, identical unknotted rings embedded in $\mathbb{R}^3$. Such a system could model a collection of ring-like solitons which make their appearance in certain non-linear sigma models [9]. A crucial analogy between this system and that of $n \mathbb{R}P^3$ geons in $3+1$ canonical quantum gravity was made by the authors of [2]. Drawing on earlier results of [1], they demonstrated the existence of sectors with indeterminate statistics for $n = 2$ [2]. The sectors we describe here are distinct in that they do exhibit a definite, albeit non-permutation group statistics. As in [2], the discovery of these sectors was motivated by the analogy with the system of $n \mathbb{R}P^3$ geons. A rigorous analysis of the quantum sectors for a system of $n$ topological geons in $3+1$ canonical quantum gravity was carried out in [3] and the existence of sectors obeying cyclic, or $\mathbb{Z}_n$ statistics was demonstrated for a system of $n \mathbb{R}P^3$ geons. In this work we use techniques developed in [3] to demonstrate the existence of similar cyclic statistics$^3$ for the set of $n \geq 3$ closed rings embedded in $\mathbb{R}^3$. Namely, we show the existence of quantum sectors in which the little group $\mathcal{R}$ is the non-permutation subgroup $\mathbb{Z}_n \subset S_n$, for $n \geq 3$.

The inequivalent quantisations for this system of rings are determined by the unitary irreducible representations of the so-called motion group $\mathcal{G}$ which we present in Section 2. Using a theorem due to Goldsmith [12], combined with the so called Fuchs-Rabinovitch relations for the automorphisms of the free product group on $n$ elements [13], we show that $\mathcal{G}$ has a nested semi-direct product structure. In Section 3 we examine the structure of the unitary irreducible representations of a nested semi-direct product group using

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1 Quantisation of the system of $n$ rings has also been examined by the authors of [10].

2 A reanalysis of these sectors in the case of 2 $\mathbb{R}P^3$ geons in [3] showed that this ambiguity originates from the lack of a canonical exchange operator.

3 An analogue of cyclic statistics in 5-dimensions has been constructed in [11].
Mackey’s theory of induced representations \cite{14}, and present our main result. We end with some remarks in Section 4.

Since the spin of the rings we consider is trivial, the sectors obeying cyclic statistics clearly violate the spin-statistics connection. In \cite{15} a spin-statistics correlation was shown to hold when the configuration space is expanded to allow the creation and annihilation of rings, thus excluding non-permutation group statistics. However, first quantised systems with ring-like structures could very well occur in condensed matter systems; as suggested in \cite{2}, the rings can be stabilised against creation and annihilation by carrying conserved charges. Whether sectors obeying cyclic statistics are physically realised or not is, of course, ultimately a question for experiment to decide.

2 The Motion Group for a System of \( n \) Rings

We consider the system of \( n \) identical, non-intersecting, infinitely thin, unknotted, unlinked, unoriented rings, \( C = C_1 \cup C_2 \ldots C_n \) in \( \mathbb{R}^3 \), which cannot be destroyed or created. The configuration space \( Q_n \) for this system of rings is the space of embeddings of \( C \) in \( \mathbb{R}^3 \) quotiented by an appropriate group of symmetries called the motion group \( \mathcal{G} \) which we will define below. An obvious example of a symmetry is the exchange of a pair of identical rings. The fundamental group of \( Q_n \) for this system is isomorphic to the motion group \( \mathcal{G} \). This group is non-trivial for all \( n \geq 1 \), and has been extensively studied by Dahm and Goldsmith \cite{12}.

Since the configuration space \( Q_n \) is multiply connected, on quantisation, the Hilbert space splits into inequivalent quantum sectors. A systematic study of such quantum sectors can be found in \cite{6}. The wavefunctions \( \psi : \tilde{Q}_n \to \mathbb{C} \), where \( \tilde{Q}_n \) is the universal cover of \( Q_n \), so that \( \pi_1(\tilde{Q}_n) \) acts non-trivially on \( \psi \). Since physically measurable quantities like inner products should only be functions on the classical configuration space \( Q_n \), the action of \( \pi_1(\tilde{Q}_n) \) on \( \psi \) must be represented as a “phase”, which can be non-abelian for \( n \geq 2 \). Thus, at every point \( \tilde{q} \in \tilde{Q}_n \), \( \psi(\tilde{q}) \) is valued in the carrier spaces of the unitary irreducible representations of \( \pi_1(\tilde{Q}_n) \). The inequivalent unitary irreducible representations of \( \pi_1(\tilde{Q}_n) \) then correspond to inequivalent quantum sectors.

The motion group \( \mathcal{G} \) for this system of rings is defined as follows \cite{12}. Let \( H(\mathbb{R}^3) \) denote the space of continuous maps or homeomorphisms of \( \mathbb{R}^3 \) into itself and \( H(\mathbb{R}^3,C) \) the subspace of homeomorphisms which leave \( C \) invariant. Let \( H_\infty(\mathbb{R}^3) \) and \( H_\infty(\mathbb{R}^3,C) \) be subspaces of \( H(\mathbb{R}^3) \) and \( H(\mathbb{R}^3,C) \), respectively, consisting of homeomorphisms with compact support. A motion is then defined as a path \( h_t \) in \( H_\infty(\mathbb{R}^3) \) such that \( h_0 \) is the identity map from \( \mathbb{R}^3 \) to itself and \( h_1 = H_\infty(\mathbb{R}^3,C) \). The product of two motions can then be defined and the inverse \( g^{-1} \) of the motion \( g \) is the path \( g_{(1-t)} \circ g_1^{-1} \) \cite{12}. Two motions \( h, h' \) are taken to be equivalent if \( h'^{-1}h \) is homotopic to a path which lies entirely in \( H_\infty(\mathbb{R}^3,C) \). The motion group \( \mathcal{G} \) is then the set of equivalence classes of motions of \( C \) in \( \mathbb{R}^3 \) with multiplication induced by \( \circ \).

We will use Hendricks’ definition of a rotation \cite{16} to describe the generators of the motion group. A 3-ball \( \mathbb{B}^3 \subset \mathbb{R}^3 \) will be said to be rotated by an angle \( \alpha \) in the following sense: take a collar neighbourhood \( S^2 \times [0,1) \) of \( \partial \mathbb{B}^3 \approx S^2 \) and let the \( S^2 \)'s be differentially rotated from 0 to \( \alpha \) with \( S^2 \times \{0\} = \partial \mathbb{B}^3 \) rotated by \( \alpha \) and \( S^2 \times \{1\} \) not rotated at all. The rotation by an angle \( \alpha \) of a solid torus \( U = \mathbb{B}^2 \times S^1 \) in the direction

\footnote{For brevity of expression we will henceforth refer to an equivalence class of motions as a motion.}
of its non-contractible circle $S^1$ is similarly defined as a differential rotation of a collar neighbourhood $T^2 \times [0, 1]$ of $\partial U \approx T^2$, with $T^2 \times \{0\} = \partial U$ rotated by $\alpha$ and $T^2 \times \{1\}$ not rotated at all.

$G$ is generated by three types of motions which are quite easily visualised [12]. The first is the flip motion $f_i$ which corresponds to “flipping” the $i$th ring $C_i$. This motion corresponds to a rotation by $\pi$ of an open ball in $\mathbb{R}^3$ containing $C_i$, about an axis lying in the plane of $C_i$. Since the rings are embedded in three dimensions, $f_i^2 = e$, so that each flip generates a $\mathbb{Z}_2$ subgroup. Next is the exchange motion $e_i$ which exchanges the $i$th ring with the $(i+1)$th ring. This can be thought of as a $\pi$ rotation of a solid torus in $\mathbb{R}^3$ containing both $C_i$ and $C_{i+1}$ (but no others). These motions generate the permutation group $S_n$. Finally, one has the slide motion $s_{ij}$ which requires a slightly more detailed description. A point in the configuration space (i.e. $\mathbb{R}^3 - C$ modulo the action of the motion group) is itself a multiply connected space with $\pi_1(\mathbb{R}^3 - C)$ isomorphic to the free product group on $n$ generators $F(x_1, x_2, \ldots, x_n) \approx \mathbb{Z} \ast \mathbb{Z} \ast \ldots \ast \mathbb{Z}$, each factor of $\mathbb{Z}$ isomorphic to the fundamental group of a single ring in $\mathbb{R}^3$. $s_{ij}$ is then the motion of $C_i$ along one of these $\mathbb{Z}$ factors, specifically, the generator of $\mathbb{Z} \subset \pi_1(\mathbb{R}^3 - C)$ passing through $C_j$. Again, one can define the slide using a rotation: consider a solid torus containing $C_i$ and “threading” $C_j$, without intersecting it. A slide is then a $2\pi$ rotation of this solid torus. The existence of slide motions is key to the present analysis, and is what makes the analogy with the system of topological geons explicit.

We denote the three subgroups generated by the flips, the exchanges and the slides as $F$, $S_n$ and $S$, respectively. We will also need to identify the subgroup $G$ generated by only the flips and the exchanges. The structure of $S_n$ is known: it is simply the permutation group on $n$ elements. However, the structures of $F$ and $S$ need to be deduced, as does information on how these groups sit in $G$. While the generators of $G$ have been known for some years, its structure has not been obtained until now. We now show that $G$ has the nested semi-direct product structure

$$G = S \rtimes (F \times S_n).$$

We also show that $S$ is the non-abelian group made up of the free product group on $n(n-1)$ generators

$$\mathbb{Z} \ast \mathbb{Z} \ast \ldots \ast \mathbb{Z}_{n(n-1)}$$

subject to the conditions

$$s_{ij}s_{kl} = s_{kl}s_{ij}, \quad s_{ij}s_{kj} = s_{kj}s_{ij}, \quad s_{ik}s_{jk}s_{ij} = s_{ij}s_{ik}s_{jk}.$$  

$F$, on the other hand, can be shown to be the abelian group isomorphic to the direct product group of the $\mathbb{Z}_2$ flips of each ring

$$F = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2.$$  

**Lemma:** $G$ has the nested semi-direct product structure [11]. Moreover, $S$ is the group [2] subject to the conditions [3], and $F$ is the group [4].

\[5\] In the case of oriented rings, this motion yields a configuration distinct from the first and is not a symmetry.
Proof: The induced action of the motion group on $\pi_1(\mathbb{R}^3 - C)$ has been examined by Goldsmith [12], and provides us with a crucial step in deducing the structure of $\mathcal{G}$. As noted earlier, $\pi_1(\mathbb{R}^3 - C)$ is isomorphic to $F(x_1, \ldots, x_n)$, the free product group on $n$-generators, $x_i$, $i = 1, \ldots, n$. In [12] the “Dahm” homomorphism $D: \mathcal{G} \to \text{Aut}(F(x_1, \ldots, x_n))$ is defined where $\text{Aut}(F(x_1, \ldots, x_n))$ is the group of automorphisms of $F(x_1, \ldots, x_n)$. For each motion $g \in \mathcal{G}$, $D$ induces an automorphism of $F(x_1, \ldots, x_n)$. The following theorem then states:

**Goldsmith’s Theorem [12]:** The group of motions $\mathcal{G}$ of the trivial $n$-component link $C$ in $\mathbb{R}^3$ is generated by the following types of motions:

1. $f_i$ or flips. Turn the $i^{th}$ ring over. This induces the automorphism $\phi_i: x_i \to x_i^{-1}, x_k \to x_k, k \neq i$, of $F(x_1, \ldots, x_n)$.

2. $e_i$ or exchange. Interchange the $i^{th}$ and the $(i+1)^{th}$ rings. The induced automorphism of $F(x_1, \ldots, x_n)$ is $\epsilon_i: x_i \to x_{i+1}, x_{i+1} \to x_i$ and $x_k = x_k$ for $k \neq i, i+1$.

3. $s_{ij}$ or slides. Pull the $i^{th}$ ring through the $j^{th}$ ring. This induces the automorphism $\sigma_{ij}: x_i \to x_ix_j^{-1}, x_k \to x_k, k \neq i$, of $F(x_1, \ldots, x_n)$.

Moreover, the Dahm homomorphism, $D: \mathcal{G} \to \text{Aut}(F(x_1, \ldots, x_n))$ is an isomorphism onto the subgroup $\overline{\mathcal{G}}$ of $\text{Aut}(F(x_1 \ldots x_n))$ generated by $\phi_i, \epsilon_i$ and $\sigma_{ij}$, where $1 \leq i, j \leq n, i \neq j$.

Let us denote the subgroups of $\overline{\mathcal{G}}$ generated by the automorphisms $\sigma_{ij}$, $\phi_i$ and $\epsilon_i$ as $\mathcal{S}$, $\mathcal{F}$ and $\overline{\mathcal{S}}$, respectively. We may now employ the Fuchs-Rabinovitch relations for $\text{Aut}(F(x_1, \ldots, x_n))$ which provides a complete set of relations for the generators of $\overline{\mathcal{G}}$ [13]. For $\pi_1(\mathbb{R}^3 - C) = \mathbb{Z} * \mathbb{Z} * \ldots * \mathbb{Z}$, in particular, these relations are simple and imply that $\overline{\mathcal{G}} \subset \text{Aut}(F(x_1, \ldots, x_n))$ has the nested semi-direct product structure

$$\overline{\mathcal{G}} = \mathcal{S} \ltimes (\mathcal{F} \ltimes \overline{\mathcal{S}}) = \overline{\mathcal{S}} \rtimes \mathcal{G},$$

where $\overline{\mathcal{G}} = \mathcal{F} \ltimes \overline{\mathcal{S}}$. Moreover, these relations imply that $\overline{\mathcal{S}}$ is the free product group on $n(n-1)$ generators $\mathbb{Z} * \mathbb{Z} * \ldots * \mathbb{Z}$ subject to the conditions $\sigma_{ij}\sigma_{kl} = \sigma_{kl}\sigma_{ij}$, $\sigma_{ij}\sigma_{kj} = \sigma_{kj}\sigma_{ij}$, and $\sigma_{ik}\sigma_{jk}\sigma_{ij} = \sigma_{ij}\sigma_{ik}\sigma_{jk}$ while $\mathcal{F}$ is the abelian direct product group made up of $n$ factors of $\mathbb{Z}_2$. $\mathcal{F} = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2$. Since $D$ is an isomorphism with $D(S) \subseteq S$, $D(F) \subseteq F$ and $D(S_n) \subseteq \overline{S_n}$, this means that $S \approx \overline{S}$, $F \approx F$ and $S_n \approx \overline{S_n}$. From [13], it is then obvious that $\mathcal{G}$ itself has the nested semi-direct product structure (4). Moreover, $\mathcal{S}$ is the free product group on $n(n-1)$ generators $\mathbb{Z} * \mathbb{Z} * \ldots * \mathbb{Z}$ subject to the relations (3) and $F$ is given by (1). 

While the structure of the motion group can be completely deduced from the Dahm homomorphism and the Fuchs Rabinovitch relations, it is instructive to examine this group without recourse to $\text{Aut}(F(x_1, \ldots, x_n))$. Using just the definition of the motion group we now illustrate the following properties of $\mathcal{G}$: (a) $\mathcal{S}$ is normal in $\mathcal{G}$ and satisfies the relations (3) and (b) that $\mathcal{F}$ is normal in $\mathcal{G}$.

By definition, an element of the motion group is a homotopy equivalence class of paths in the space of homeomorphisms with compact support. Two homeomorphisms $h_1$ and $h_2$ with compact support on the regions $U_1$ and $U_2$ commute if $U_1 \cap U_2 = \phi$ and hence so do the corresponding motions. It is therefore useful to isolate the “minimal”
neighbourhoods in which homeomorphisms representing the generators of the motion group act so as to determine which two motions commute.

Let $U_i$ denote an open ball neighbourhood of $C_i$ in $\mathbb{R}^3$ which contains no other $C_j$, $j \neq i$, and let $U_{ij}$ denote an open ball neighbourhood of $C_i \cup C_j$ containing no other $C_k$, $k \neq i,j$, etc. We will refer to the $U_i$ as “exclusive” neighbourhoods and the $U_{ij}, U_{ijk}, \ldots$ etc. as “common” neighbourhoods. The flip motion $f_i$ is then defined by a homotopy equivalence class of paths in $H_\infty(\mathbb{R}^3)$ which include a “model” path made up of homeomorphisms with support only on $U_i$, i.e., a path in $H_\infty(\mathbb{R}^3)$ along which $C_i$ is flipped without disturbing any of the other rings. Next, the exchange motion $e_i$ is defined by a homotopy equivalence class of paths including a model path with support only on $U_{ij}$, i.e., a path in $H_\infty(\mathbb{R}^3)$ along which the other rings are not disturbed.

Now, the set of exclusive neighbourhoods $\{U_i\}$ remains invariant when acted upon by the subgroup $\tilde{G}$ generated by the flips and by the exchanges. This is obvious for $\mathcal{F}$, since each flip $f_i$ acts within an exclusive neighbourhood. For $S_n$, while the exchange $e_i$ has compact support on $U_{i(i+1)}$, its action can be considered as a pure exchange of $U_i$ with $U_{i+1}$. Thus, one can consider as a model path for the exchange, a localised $\pi$ rotation in $U_{i(i+1)}$ which exchanges $U_i$ with $U_{i+1}$. This, however, is not the case with the slides $s_{ij}$. While the set $\{U_k\}$ for $k \neq j$ remains invariant under the slide $s_{ij}$ of $C_i$ through $C_j$, the exclusive neighbourhood $U_j$ does not. The non-local action of the slide takes $U_j$ into a set $V_j$ which “encloses” $C_i$ even though it does not contain it, i.e. there exists a $U_i$ such that $U_i \cap V_j = \phi$ (see fig). Thus, $V_j$ is not an exclusive neighbourhood of $C_j$. This feature leads to subtleties in what follows.

![Figure 1: Under the slide $s_{ij}$, $C_i$ “tunnels” through the neighbourhood $U_j$ of $C_j$ and maps it onto the region $V_j$, shown by dashed lines. $V_j$ therefore “encloses” $C_i$ without containing it, i.e. $U_i \cap V_j = \phi$.](image)

Since the exchanges and the flips leave the set $\{U_i\}$ invariant, model paths are sufficient to see that $\mathcal{F}$ is normal in $\tilde{G}$, i.e., for all $\tilde{g} \in \tilde{G}$, $i \leq n$, $\tilde{g} f_i \tilde{g}^{-1} \in \mathcal{F}$. To show this, it is sufficient to take $\tilde{g}$ to be an exchange. For the motion $e_i f_j e_i^{-1}$ with $j \neq i, i+1$, the model paths for $e_i$ and $f_j$ have compact support on $U_{i(i+1)}$ and $U_j$, respectively, where $U_{i(i+1)} \cap U_j = \phi$. Hence the motions commute, so that $e_i f_j e_i^{-1} = f_j$. Now consider the motion $e_i f_i e_i^{-1}$. The model paths for $e_i$ exchange $U_i$ with $U_{i+1}$. One can then use a model path for the motion $f_i$ which acts on some $U_{i+1}' \subset U_{i+1}$ so that the final exchange $e_i$ which exchanges $U_{i+1}$ with $U_i$ does not disturb the action of $f_i$ on $U_{i+1}'$. Thus, $e_i f_i e_i^{-1} = f_{i+1}$. Similarly, $e_i f_{i+1} e_i^{-1} = f_i$. 

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However, model paths are insufficient when one wants to deal with the slides. Let us consider a motion whose model path involves homeomorphisms with support only on the compact region $U$. The homotopy class of paths defining this motion also includes the non-model, or "gregarious" paths, which involve homeomorphisms with non-trivial compact support on $\mathbb{R}^3 - U$. In other words, gregarious paths can disturb the other rings; they can contain homeomorphisms with non-trivial support on neighbourhoods of rings left undisturbed by the model path. Consider the motion $f_i$ for simplicity. A model path for $f_i$ has support only on $U_i$ and corresponds to a $\pi$ rotation about an axis in the plane of $C_i$. A gregarious path on the other hand can be constructed piecewise as follows: (a) rotate $C_i$ by $\pi/3$, about an axis $\hat{x}$ in its plane (b) flip another $C_j$, $j \neq i$, (c) rotate $C_i$ by a further $\pi/3$ about $\hat{x}$ in the same sense as before (d) flip $C_j$ again, (e) and complete with a further $\pi/3$ rotation of $C_i$ about $\hat{x}$ in the same sense as before. Such a path clearly corresponds to the motion $f_i$, but involves homeomorphisms of $\mathbb{R}^3$ in $H_\infty(\mathbb{R}^3)$ which have non-trivial support on the ring $C_j$, $j \neq i$.

Both model and gregarious paths are necessary to demonstrate that $S$ is normal in $G$. $S$ is a normal subgroup of $G$ if $\forall g \in G$ and $\forall i, j \leq n$ $g_1 g_2 g^{-1} \in S$. It is sufficient to take $g$ to be a generator of $S_n$ or $F$.

We begin with the exchanges. Let us examine the motion $e_k s_{ij} e_k^{-1}$ by considering only model paths in the appropriate homotopy class. For $k \neq i, j$ $e_k s_{ij} e_k^{-1} = s_{ij}$, since the homeomorphisms that make up model paths for $e_k$ and $s_{ij}$ have compact supports on $U_k$ and $U_{ij}$ with $U_k \cap U_{ij} = \phi$. Model paths are, however, insufficient to show that $e_k s_{ij} e_k^{-1}$ is also a slide for $k = i$, $i - 1$, $j$ or $j - 1$. Consider the motion $e_j s_{ij} e_j^{-1}$ with $k = j, i \neq j, j + 1 e_j^{-1}$ swaps $U_j$ with $U_{j+1}$ by a $\pi$ rotation of a torus containing both $U_j$ and $U_{j+1}$. Next, $s_{ij}$ rotates by $2\pi$ a solid torus containing $C_i$ and threading $C_{j+1}$, thus mapping $U_{j+1}$ into a non-exclusive neighbourhood $V_{j+1}$. A model path for the final exchange $e_j$ would rotate by $\pi$ a solid torus containing new exclusive neighbourhood $U_{j+1}'$ of $C_{j+1}$ and $U_i$. $U_{i(j+1)}$ in which the slide acts, is not left invariant by this final exchange, making the resultant motion difficult to unravel. Instead, we use the following gregarious path to perform the final exchange: consider a path in $H_\infty(\mathbb{R}^3)$ where $U_{i(j+1)}$ and $U_i$ are swapped by performing an appropriate $\pi$ rotation in the common neighbourhood $U_{ij(j+1)}$ of $C_i, C_j$ and $C_{j+1}$. The final exchange motion is then completed by merely moving $C_i$ back to its original position. $U_{i(j+1)}$ is thus left undisturbed so that the full motion is the slide $s_{ij(j+1)}$. A use of a similar gregarious path for the final exchange shows that $e_{(i-1)} s_{ij} e_{(i-1)}^{-1} = s_{i(j-1)}$, $e_i s_{ij} e_i^{-1} = s_{(i+1)j}$ and $e_{(i-1)} s_{ij} e_{(i-1)}^{-1} = s_{(i-1)j}$.

Next, consider the flips. The motion $f_k s_{ij} f_k^{-1}$ can again be examined using only model paths for $k \neq j$, and we can see that it is $s_{ij}$. This is because the model path for $f_k$ has compact support only on $U_k$ which is undisturbed by the slide even when $k = i$. However, the use of model paths is insufficient to examine the motion $f_j s_{ij} f_j^{-1}$: not only does $U_j$ not remain an exclusive neighbourhood under the slide $s_{ij}$, but the $f_j$ moves the points in $U_j$ relative to each other. Rather than consider just a single gregarious path, following $\mathcal{K}$, we use a particular set of homotopy equivalent paths. Let $\kappa$ be the generator of $\pi_1(\mathbb{R}^3 - C)$ through $C_j$ about which the slide $s_{ij}$ takes $C_i$. We define the paths $\gamma_0$ as follows: (a) perform a "part" inverse flip corresponding to a $(\pi - \alpha)$ rotation of $C_j$ about $\hat{x}$ (b) slide $C_i$ through $C_j$ along $\kappa^{-1}$ (c) finish the inverse flip $f_j^{-1}$ of $C_j$ by a rotation $\alpha$ about $\hat{x}$ and (d) finally, perform the flip $f_j$ of $C_j$ about $\hat{x}$. $\gamma_0$ then corresponds to the model path for the motion $f_j s_{ij} f_j^{-1}$ while the path $\gamma_\pi$ corresponds to the slide
s^{-1}_{ij}. Since \( \alpha \) is a continuous parameter \( \alpha \in [0, \pi] \), the \( \gamma_\alpha \) provide a homotopy map from \( \gamma_0 \) to \( \gamma_\pi \), which implies that \( f_j s_{ij} f_j^{-1} = s_{ij}^{-1} \).

Thus, the slide subgroup \( \mathcal{S} \) is a normal subgroup of \( \mathcal{G} \).

We can also demonstrate that the relations \( \mathcal{K} \) are satisfied by \( \mathcal{S} \), using just the definition of the motion group. The first of these relations is clearly satisfied by the generators of \( \mathcal{S} \), since the model paths corresponding to the slides \( s_{ij} \) and \( s_{kl} \) involve homeomorphisms with compact support only on \( U_{ij} \) and \( U_{kl} \) where \( U_{ij} \cap U_{kl} = \phi \). It takes a little more work to show that the other two relations are also satisfied by the generators of \( \mathcal{S} \).

Consider the motion \( s_{ij} s_{kj} s_{ij}^{-1} \). \( s_{ij} \) and \( s_{kj} \) are slides of the two rings \( C_i \) and \( C_k \) through a third ring \( C_j \). These slides are obtained by \( 2\pi \) rotations of the solid tori \( V_{ij} \approx B^2 \times S^1 \) and \( V_{kj} \approx B^2 \times S^1 \) which thread through \( C_j \), with \( V_{ij} \cap V_{kj} = \phi \). Define the paths \( \gamma_\alpha \) as follows: (a) a rotation by \( -\alpha \) of \( V_{ij} \); (b) a \( 2\pi \) rotation of \( V_{kj} \); (c) a \( -(2\pi - \alpha) \) rotation of \( V_{ij} \) and finally, (d) a \( 2\pi \) rotation of \( V_{ij} \). \( \gamma_0 \) then defines a model path for the motion \( s_{ij} s_{kj} s_{ij}^{-1} \), and \( \gamma_{2\pi} \) corresponds to the slide \( s_{kj} \). Since \( \alpha \) is a continuous parameter, \( \gamma_0 \) is homotopic to \( \gamma_{2\pi} \) and hence also corresponds to \( s_{kj} \). Notice that by keeping \( V_{ij} \cap V_{kj} = \phi \) we prevent a mixing of their rotations and hence the deformations of the neighbourhood \( U_j \) by \( s_{ij} \) and by \( s_{kj} \).

Next, consider the motion \( s_{ij} s_{ik} s_{jk} s_{ij}^{-1} \). Although this looks considerably more complicated than the previous motion, the two elements of \( \mathcal{S} \) involved, \( s_{ik} s_{jk} \) and \( s_{ij} \), have compact supports on non-intersecting neighbourhoods. Namely, the element \( s_{ik} s_{jk} \) corresponds to sliding \( C_j \) through a generator \( \rho \) of \( \pi_1 \) of \( C_k \) and then sliding \( C_i \) through the same generator. Under this action, \( U_j \rightarrow U_j \) and \( U_i \rightarrow U_i \), while \( U_k \) is now mapped to a region \( V_k \) which now “encloses” both \( C_i \) and \( C_j \). Thus, there exists a path in \( H_\infty(\mathbb{R}^3) \) corresponding to the motion \( s_{ik} s_{jk} \) made up of homeomorphisms which leave the common neighbourhood \( U_{ij} \) undisturbed. Since there is a model path corresponding to the slide \( s_{ij} \) which has compact support only on \( U_{ij} \), this means that the two motions \( s_{ik} s_{jk} \) and \( s_{ij} \) indeed commute. Thus, the generators of \( \mathcal{S} \) satisfy all the relations \( \mathcal{K} \).

Remark: In [2] a set of relations for the generators in the \( n = 2 \) case was given:
\[
f_i^2 = \mathcal{E}^2 = (f_i \mathcal{E})^4 = (f_i \mathcal{E} s_j \mathcal{E})^2 = c \quad \text{where} \quad i = 1, 2 \text{ and the slides } s_i \text{ generate } \mathcal{S}, \text{ the flips } f_i \text{ generate } \mathcal{F} \text{ and the exchange } \mathcal{E} \text{ generates } S_2. \text{ These follow in a straightforward manner from the relations presented above.}
\]

3 Cyclic Statistics

The inequivalent quantum sectors for our system of \( n \) identical rings are labeled by the unitary irreducible representations of \( \pi_1(\mathbb{Q}_n) \approx \mathcal{G} \). The group \( \mathcal{G} \) represents a gauge symmetry and the action of the individual motions \( g \in \mathcal{G} \) on \( \mathbb{R}^3 - C \) can be used to interpret the associated quantum phases. For example, for a single ring the motion group is simply \( \mathcal{F} = \mathbb{Z}_2 \), which has two unitary irreducible representations: the trivial one and the non-trivial one. The associated quantum theories thus correspond to an “unoriented” quantum ring in which wavefunctions \( \psi \) transform as \( \psi \rightarrow \psi \) under a flip, and an “oriented” quantum ring in which \( \psi \rightarrow -\psi \) under a flip. Similarly, for \( n \geq 2 \)

\[\text{It is perhaps a useful exercise for the reader to see why a similar argument cannot be used to find a set of homotopic paths between } s_{ij} f_{ij} s_{ij}^{-1} \text{ and an element of } \mathcal{F}.\]
the motions corresponding to permutations of the rings can be non-trivially represented, and lead to different quantum statistics (see [11, 13, 6] for a more detailed discussion of quantum phases and statistics for extended objects).

As mentioned in the introduction, the quantum statistics of a system is not solely determined by \( S_n \), but rather by the unitary irreducible representations of its stability subgroup \( R \subseteq S_N \) associated with its action on the unitary irreducible representations of the normal subgroup \( \mathcal{S} \times \mathcal{F} \) of \( \mathcal{G} \). This follows from Mackey’s theory of induced representations for semidirect product groups \( P \rtimes K \) [14]. In this construction, one begins with the space of unitary irreducible representations \( \hat{P} \) of the normal subgroup \( P \). The subgroup \( K \) has the (not necessarily free) action on \( \hat{P} \)

\[
\Delta(p) \rightarrow \hat{\Delta}(p) = \Delta(kpk^{-1}),
\]

where \( \Delta \in \hat{P} \), \( p \in P \) and \( k \in K \). Starting with a particular \( \Delta_1 \in \hat{P} \) one obtains an orbit \( \mathcal{O} = \{\Delta_1, \Delta_2, \ldots, \Delta_r\} \) of the \( K \) action on \( \hat{P} \), and the little group \( \mathcal{R} \) associated with \( \mathcal{O} \). The full unitary irreducible representation of \( P \rtimes K \) is then built up by taking the direct product of \( \{\Delta_1 \oplus \Delta_2 \oplus \ldots \oplus \Delta_r\} \) with a unitary irreducible representation of \( \mathcal{R} \). For example, if one starts with the trivial representation of \( P \), then the orbit consists of a single point and \( \mathcal{R} = K \). The unitary irreducible representation of \( P \rtimes K \) that can be constructed from this orbit are just the unitary irreducible representations of \( K \). On the other hand, one may find an orbit of \( K \) in \( P \) with \( \mathcal{R} = e \). The full unitary irreducible representation is then simply the sum of the unitary irreducible representations in the orbit, \( \oplus_i \Delta_i \). The action of the subgroup \( K \) is then reduced to a canonical map which permutes the carrier spaces \( \mathbb{H}_i \) of \( \Delta_i \) [3] 7.

We now illustrate the importance of the little group in determining quantum statistics with a simple example. Because of the nested semi-direct product structure of the motion group, we may begin by first representing the slides trivially. We thus need to find only the unitary irreducible representations of \( \mathcal{G} = \mathcal{F} \rtimes S_n \). Since \( \mathcal{F} \equiv \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2 \), it is trivial to list its unitary irreducible representations, \( \Delta \equiv (k_1, k_2 \ldots k_n) \), with \( k_i = \pm 1 \). For example, for \( n = 3 \), let us start with the unitary irreducible representation \( \Delta_1 = (-, -, +) \) of the normal subgroup \( \mathcal{F} \) of \( \mathcal{G} \). This choice corresponds to two of the rings being identical and oriented, while the third is unoriented and hence distinguishable from the others. The action of \( S_3 \) on \( \Delta_1 \) generates the orbit \( \{\Delta_1, \Delta_2, \Delta_3\} \equiv \{(-, -, +), (+, -, -), (-, +, -)\} \) in \( \hat{\mathcal{F}} \) whose associated little group is \( S_2 \). The resulting unitary irreducible representation of \( \mathcal{G} \) is then \( \{\Delta_1 \oplus \Delta_2 \oplus \Delta_3\} \otimes \Gamma \), where \( \Gamma \) is a unitary irreducible representation of \( S_2 \). Under a two particle exchange \( \Gamma \) provides either a bosonic (+1) or a fermionic (-1) phase. Since one of the three rings has been rendered quantum mechanically distinguishable from the other two, one obtains an appropriate two ring statistics. The action of the remaining elements of \( S_3 \), namely the cyclic elements, is canonical: they merely permute the carrier spaces \( \mathbb{H}_i \) of the \( \Delta_i \). This general structure continues to hold for all \( n \), and is illustrative for the case of primary interest here when the slides are non-trivially represented.

Before proceeding to construct a quantum sector exhibiting cyclic statistics for \( n \geq 3 \), let us consider the simplest case with the slides non-trivially represented, namely when \( n = 2 \). For \( n = 2 \), the slide subgroup is generated by the two slides \( s_1, s_2 \), the flip

\footnote{As discussed in [3] for \( n \geq 4 \) the possibility of projective statistics exists when \( \pi_1(Q) \) has a semi-direct product structure.}
group statistics for This quantum lifting of indistinguishability by slides is what leads to non-permutation the non-abelian subgroup $S$ representation $\Omega_1$. Thus, there exist a wavefunction property of each object. However, in this representation, it is the non-local action of mechanically distinguishable even if $S$ however, there are elements of $S$ lies in the stability subgroup of $\tilde{G}$. Under the action of $\mathcal{E}$, $\Omega_1(s_i) \rightarrow \tilde{\Omega}_1(s_i) = \Omega_1(\mathcal{E}s_i\mathcal{E}^{-1}) = \Omega_1(s_j) \neq \Omega_1(s_i)$ where $j \neq i$, so that $S_2 \not\subseteq \mathcal{R}$. Thus, the two rings are quantum mechanically distinguishable even if $F$ is trivially represented. This is very unusual, since indistinguishability of a collection of objects is often thought of as a local, intrinsic property of each object. However, in this representation, it is the non-local action of slides which renders the two rings distinguishable: the rings slide through each other differently. Thus, there exist a wavefunction $\psi$ peaked on a configuration of two well-separated rings such that under the action of $s_1$, $\psi \rightarrow \psi$ and under that of $s_2$, $\psi \rightarrow -\psi$. This quantum lifting of indistinguishability by slides is what leads to non-permutation group statistics for $n > 2$.

Let us begin with the case $n = 3$. $S$ is generated by the six generators $s_{ij}$, $i, j = 1, 2, 3$, $i \neq j$, $F$ is generated by the 3 elements $f_1, f_2, f_3$, and the permutations form the non-abelian subgroup $S_3$. We start with the following abelian unitary irreducible representation $\Omega_1$ of $S$:

$$\Omega_1(s_{12}) = \Omega_1(s_{23}) = \Omega_1(s_{31}) = -1, \quad \Omega_1(s_{21}) = \Omega_1(s_{32}) = \Omega_1(s_{13}) = 1. \quad (7)$$

Consider the action of $\tilde{G}$ on $\Omega_1$. The action of a flip $f_k$ on $\Omega_1$ for $k = i$ or $j$ is: $\Omega_1(s_{ij}) \rightarrow \Omega_1(f_k s_{ij} f_k^{-1}) = \Omega_1^{-1}(s_{ij}) = \Omega_1(s_{ij})$, while the action of $f_k$, $k \neq i, j$ is trivial. Thus, $F$ lies in the stability subgroup of $\tilde{G}$. On the other hand, the exchanges $e_i$ do not leave $\Omega_1$ invariant: $\Omega_1(s_{i(i+1)}) \rightarrow \Omega_1(e_i s_{i(i+1)} e_i^{-1}) = \Omega_1(s_{i(i+1)}) = -\Omega_1(s_{i(i+1)})$. Curiously, however, there are elements of $S_3$ which leave $\Omega_1$ invariant, namely the subgroup of cyclic permutations $\mathbb{Z}_3$ of $S_3$ generated by $z = e_2 e_3$. Under the action of $e_2 e_3$ the slides $\{s_{12}, s_{23}, s_{31}\} \rightarrow \{s_{23}, s_{31}, s_{12}\}$, and $\{s_{21}, s_{32}, s_{13}\} \rightarrow \{s_{32}, s_{13}, s_{21}\}$, which leaves $\Omega_1$ invariant. Thus, the stability subgroup is $F \times \mathbb{Z}_3$. The remaining elements of $F \times S_3$ $e_1, e_2$ and $e_3$ generate the two element orbit $O \equiv \{\Omega_1, \Omega_2\}$ in $\tilde{S}$ the space of unitary irreducible representations of $S$, where

$$\Omega_2(s_{12}) = \Omega_2(s_{23}) = \Omega_2(s_{31}) = 1, \quad \Omega_2(s_{21}) = \Omega_2(s_{32}) = \Omega_2(s_{13}) = -1. \quad (8)$$

The associated unitary irreducible representation of $\mathcal{G}$ is therefore non-abelian, and can be symbolically expressed as

$$(\Omega_1 \oplus \Omega_2) \otimes \mathcal{T}, \quad (9)$$

where $\mathcal{T}$ is a unitary irreducible representation of the stability subgroup $F \times \mathbb{Z}_3$.

Let us for simplicity consider the case when $F$ is trivially represented in $\mathcal{T}$, so that $\mathcal{T}$ is a unitary irreducible representation of $\mathbb{Z}_3$. $\mathbb{Z}_3$ has two non-trivial inequivalent unitary irreducible representations (a) $z \rightarrow e^{\frac{2\pi i}{3}}$ and (b) $z \rightarrow e^{\frac{4\pi i}{3}}$. Thus, there exist wavefunctions $\psi_a, \psi_b$ in the corresponding quantum sectors which are peaked on a configuration of well separated rings and which pick up the phases $\psi_a \rightarrow e^{\frac{2\pi i}{3}} \psi_a$ and $\psi_b \rightarrow e^{\frac{4\pi i}{3}} \psi_b$, respectively, under the action of the cyclic permutations. Thus, these sectors exhibit a cyclic, non-permutation group, statistics: the rings are identical only when permuted by a cyclic combination, and not under pair-wise exchange! This is indeed a very curious
behaviour and is, again, linked to the non-locality of slide motions: even though the flips are all trivially represented the slides render the rings pair-wise distinguishable but cyclically indistinguishable. We will say that the rings obey $\mathbb{Z}_3$ cyclic statistics.

The case for arbitrary $n > 2$ follows in a straightforward manner. Namely, we can always isolate a pair of non-trivial subsets from the set of slide generators $\{s_A\}$ and $\{s_B\}$ which are invariant under $\mathbb{Z}_n$. There is a small difference in the construction in the even $n = 2m$ and odd $n = 2m + 1$ cases. For $n = 2m$, $\mathbb{Z}_{2m}$ contains the subgroup $\mathbb{Z}_2$; if $z$ is the generator of $\mathbb{Z}_{2m}$ with $z^{2m} = e$, then $z^m$ generates a $\mathbb{Z}_2$ subgroup corresponding to $m$ commuting exchanges. One can then see that the two sets of generators $\{s_A\}$ and $\{s_B\}$ which are invariant under $\mathbb{Z}_n$ have cardinality $2m(m - 1)$ and $2m^2$ respectively. For $n = 2m + 1$, $\mathbb{Z}_2$ is not a subgroup of $\mathbb{Z}_{2m+1}$. Hence the two sets of generators $\{s_A\}$ and $\{s_B\}$ each have cardinality $m(2m + 1)$. One can thus obtain $\mathbb{Z}_n$ cyclic statistics for arbitrary $n > 2$.

We end this section by commenting on the possibility that sectors with more complicated non-permutation group statistics may exist. To construct the above cyclic statistics sectors we started with a very simple abelian unitary irreducible representations of the slide subgroup. It is conceivable that if one instead started with a non-abelian unitary irreducible representation of $S$ (with certain symmetries) that the stability subgroup $\mathcal{F} \rtimes K$ associated with it is such that $K$ is non-abelian and a non-permutation subgroup of $S_n$. Such a sector would then exhibit a non-abelian, non-permutation group statistics. Our current work provides a framework in which to probe such questions.

4 Remarks

Anyonic statistics in $2 + 1$ dimensions can be modeled by adding a Chern Simon’s term to the $n$ particle Lagrangian [17]. In [2] a stringy generalisation of this was developed to obtain non-trivial phases from the action of the motion group, namely a $B \wedge F$ topological term made up of an abelian gauge field and an axion field was added to the $n$ string Lagrangian along with an interaction term. Similar systems have subsequently been studied in [10]. In [2] it was shown that even though the statistical phases are trivial (i.e. bosonic) the action of the slide subgroup is non-trivial, giving rise to fractional quantum phases. Since slides involve the motion of one ring through a non-trivial generator of the fundamental group of another ring, these fractional phases correspond to Aharnov-Bohm phases rather than to fractional quantum statistics. Indeed, slides can occur between non-identical particles as well and hence the interpretation of such phases as statistics in [10] seems questionable. Since cyclic statistics occur in non-abelian sectors of the system, it would be interesting to construct appropriate non-abelian generalisations of [2] which exhibit this behaviour. We leave this problem to future investigations.

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