Abstract. The authors study the geometry of lightlike hypersurfaces on a four-dimensional manifold \((M, c)\) endowed with a pseudoconformal structure \(c = CO(2, 2)\). They prove that a lightlike hypersurface \(V \subset (M, c)\) bears a foliation formed by conformally invariant isotropic geodesics and two isotropic distributions tangent to these geodesics, and that these two distributions are integrable if and only if \(V\) is totally umbilical. The authors also indicate how, using singular points and singular submanifolds of a lightlike hypersurface \(V \subset (M, c)\), to construct an invariant normalization of \(V\) intrinsically connected with \(V\).

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0 INTRODUCTION

A four-dimensional pseudo-Riemannian manifold \((M, g)\) with a metric quadratic form of signature \((3, 1)\) is a geometric model of the classic spacetime in general relativity. Its natural generalization is a pseudo-Riemannian manifold \((M, g)\) of dimension \(n = \text{dim } M\) with a nondegenerate quadratic form of arbitrary signature \((p, q)\), \(p + q = n\). Such manifolds are considered in a construction of multidimensional models of spacetime and in the theory of superstrings.

Let \(x\) be a point of a manifold \((M, g)\), and \(T_x(M)\) the tangent space at the point \(x\). For \(p \cdot q > 0\), the quadratic form \(g\) defines a real isotropic cone \(C_x\) in the space \(T_x\). Its equation is \(g(\xi, \xi) = 0\), \(\xi \in T_x\). This cone is also called the light cone or the null cone.

A hypersurface \(V \subset (M, g)\) is called lightlike if it is tangent to the cone \(C_x\) at each point \(x \in V\). The lightlike hypersurfaces are also called isotropic or null...
hypersurfaces. On the manifold \((M, g)\), such hypersurfaces separate domains with different physical or geometric properties—they are models of physical or geometric horizons (see, for example, [HE 73]).

Many physical and geometric objects on a manifold \((M, g)\) are invariant under conformal transformations of the metric \(g\), that is, under a passage from the metric \(g\) to the metric \(\tilde{g} = \sigma g\), where \(\sigma = \sigma(x)\) is a differentiable function such that \(\sigma(x) \neq 0\), \(x \in M\). Examples of such objects are the light cones and the lightlike hypersurfaces. Hence it is appropriate to study such objects not only on a pseudo-Riemannian manifold \((M, g)\) but also on a manifold \((M, c)\) endowed with a conformal structure \(c = \{\sigma g\}\).

Note that lightlike hypersurfaces arose in the papers of Duggal and Bejancu [DB 91] and [Be 96] (see also their book [DB 96], Section 4.7). They considered them in a pseudo-Riemannian manifold of constant curvature \(c\), and in particular, in pseudo-Euclidean spaces \(\mathbb{R}^4_1\) and \(\mathbb{R}^4_2\). Lightlike hypersurfaces were also studied by Kupeli [Ku 87] (see also his book [Ku 96], Section 4.4). He considered them in a (pseudo-)Riemannian space \((M, g)\) of constant sectional curvature. Lightlike hypersurfaces appeared in the paper Rosca [Ro 71] in which he studied a pair of lightlike hypersurfaces in 1-to-1 correspondence in a Lorentz manifold.

Note also that the totally umbilical lightlike hypersurfaces in Riemannian and pseudo-Riemannian manifolds \((M, g)\) were extensively studied by many authors. They considered their local and global properties. For example, in the papers [Y 75], [Ak 87], [Ra 87], and [Z 96] the authors found necessary and sufficient conditions for a complete spacelike hypersurface to be totally umbilical in \((M, g)\).

The totally umbilical lightlike hypersurfaces in \((M, c)\) endowed with a conformal or pseudoconformal structure were not yet studied extensively. In the papers [AG 99a] and [AG 99b] we have already studied the geometry of lightlike hypersurfaces \(V\) on a manifold \((M, c)\) endowed with a conformal structure \(c\) of Lorentzian signature \((n - 1, 1)\).

In the present paper we consider lightlike hypersurfaces on a four-dimensional manifold \((M, c)\) endowed with a conformal structure of ultrahyperbolic signature \((2, 2)\). We find some properties of the structure of such hypersurfaces and prove that they bear two isotropic two-dimensional distributions in addition to the fibration of isotropic geodesics. We also prove that integrability of these distributions is necessary and sufficient for a lightlike hypersurface to be totally umbilical. We constructed an invariant normalization of \(V \subset (M, c)\) in a fourth differential neighborhood of a point of \(V\). In the case in question, i.e., for \(c = CO(2, 2)\), we were able not only to construct such normalization but also to find a foliation of canonical frames.

For our study of lightlike hypersurfaces on a manifold \((M, c)\), \(\dim M = 4\), \(\text{sign } c = (2, 2)\), we use the apparatus developed in [A 96] (see also [AG 96], Ch. 5). As far as we know, the lightlike hypersurfaces on such manifolds are studied in the present paper for the first time.
1 A MANIFOLD \((M, c)\)

Consider a manifold \((M, c)\) endowed with a conformal structure \(c\) of signature \((p, q)\), \(\dim M = n = p + q\). Let \(x\) be an arbitrary point of \(M\), \(T_x(M)\) be its the tangent space, and \(C_x \subset T_x(M)\) be the isotropic cone in \(T_x(M)\). The space \(T_x(M)\) can be compactified by adding the point at infinity \(y\) and the isotropic cone \(C_y\) with vertex \(y\). After this enlargement, the space \(T_x(M)\) becomes a pseudoconformal space \((C^n_x)\) of the same signature \((p, q)\).

Under the Darboux mapping (see [AG 98] or [AG 96], Ch. 1), the space \((C^n_x)\) will be mapped onto a hyperquadric \((Q^n_x)\) of a projective space \(P^{n+1}_x\) of dimension \(n + 1\). In the space \(P^{n+1}_x\), the hyperquadric \((Q^n_x)\) is defined by the equation \((x, x) = 0\).

We associate a family of projective local frames \(\{A_0, A_i, A_{n+1}\}, i = 1, \ldots, n\), with this hyperquadric in such a way that \(A_0 = x\) and \(A_{n+1} = y\), where \(x\) and \(y\) are points of the hyperquadric \((Q^n_x)\) for which \((x, y) \neq 0\). This implies
\[
(A_0, A_0) = (A_{n+1}, A_{n+1}) = 0, \quad (A_0, A_{n+1}) = -1
\]

The last condition is obtained by taking an appropriate normalization of the points \(A_0\) and \(A_{n+1}\). Here and in what follows the parentheses denote the scalar product with respect to the quadratic form occurring in the left-hand side of the equation of the hyperquadric \((Q^n_x)\).

Denote by \(T_x\) and \(T_y\) the tangent hyperplanes to \((Q^n_x)\) in the points \(x\) and \(y\) and locate the points \(A_i\) at the intersection of these hyperplanes, \(A_i \in T_x \cap T_y, i = 1, \ldots, n\). Then we find that
\[
(A_0, A_i) = (A_{n+1}, A_i) = 0, \quad (A_i, A_j) = g_{ij},
\]
where \(\det(g_{ij}) \neq 0\), \(\text{sign} \ (g_{ij}) = (p, q)\) (see Figure 1).
Now

$$(A_\xi, A_\eta) = (g_{\xi\eta}) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & g_{ij} & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \xi, \eta = 0, 1, \ldots, n + 1, \quad (1)$$

and the equation of the hyperquadric $(Q^n_q)_x \subset P^{n+1}_x$ can be written in the form

$$(x, x) = g_{ij}x^i x^j - 2x^0 x^{n+1} = 0.$$  

The family of projective frames we have constructed is called a first order frame bundle associated with the manifold $(M, c)$.

The isotropic cone $C_x$ is the intersection of the hyperquadric $(Q^n_q)_x$ and the tangent hyperplane $T_x$:

$$C_x = T_x \cap (Q^n_q)_x,$$

and it is defined by the equation

$$g = g_{ij}\xi^i\xi^j = 0, \quad \xi = (\xi^i) \in T_x.$$  

The group of transformations of the tangent space $T_x(M)$ preserving the invariant cone $C_x$ is the group $G = \text{SO}(p, q) \times H$, where $\text{SO}(p, q)$ is a special pseudoorthonormal group of signature $(p, q)$ and $H = \mathbb{R}^+$ is the group of homotheties.

It follows from relations (1) that the equations of infinitesimal displacement in the first-order frame bundle have the form

$$\begin{cases} 
  dA_0 = \omega^0_i A_0 + \omega^0_i A_i, \\
  dA_i = \omega^0_i A_0 + \omega^i_j A_j + \omega^{n+1}_i A_{n+1}, \\
  dA_{n+1} = \omega^{n+1}_i A_i - \omega^0_i A_{n+1},
\end{cases} (2)$$

where

$$\begin{align*}
  \omega^0_i &= \omega^i \\
  \omega^{n+1}_i &= g_{ij} \omega^j, \quad \omega^{n+1}_i = g^{ij} \omega^0_j, \\
  dg_{ij} - g_{ik} \omega^k_j - g_{kj} \omega^k_i &= 0,
\end{align*} (3)$$

and $g^{ij}$ is the inverse tensor of the tensor $g_{ij}$, i.e., $g^{ik} g_{kj} = \delta^i_j$. Note that the tensor $g_{ij}$ and the 1-forms $\omega^i$ are defined in a first-order neighborhood of a point $x \in (M, c)$, the 1-forms $\omega^0_i$ and $\omega^i_j$ are defined in its a second-order neighborhood, and the 1-forms $\omega^{n+1}_i$ are defined in its third-order neighborhood.

The 1-forms $\omega^i$ define a displacement of a point $A_0$ and consequently of a frame $\{A_0, A_i, A_{n+1}\}$ along the manifold $M$. This is the reason that they are called the basis forms. For $\omega^i = 0$, equations (2) take the form

$$\begin{cases} 
  \delta A_0 = \pi^0_0 A_0, \\
  \delta A_i = \pi^0_i A_0 + \pi^i_j A_j, \\
  \delta A_{n+1} = \pi^{n+1}_i A_i - \pi^0_i A_{n+1}.
\end{cases} (4)$$
Here $\delta$ is the symbol of differentiation for $\omega^i = 0$, i.e., with respect to the fiber parameters of the frame bundle, and $\pi_\eta^i = \omega^i_\eta(\delta)$. Formulas (4) define admissible transformations in a fiber of a first-order frame bundle. These transformations form the group $G' = G \ltimes \mathbf{T}(n)$ that is obtained by a differential prolongation of the group $G$ acting in the space $T_x(M)$. Here $\mathbf{T}(n)$ is a subgroup of the group $G'$ which is isomorphic to the group of parallel translations, and the symbol $\ltimes$ is the semidirect product (see [AG 96], Ch. 4). Equations (4) show that the group $G'$ is isomorphic to the subgroup of the group of pseudoconformal transformations of the space $C_n^q$ keeping invariant the point $x = A_0$ of this space.

2 THE TENSOR OF CONFORMAL CURVATURE OF A MANIFOLD $(M, c), \ c = CO(2, 2)$

As was proved in Ch. 4 of the book [AG 96], the structure equations of the manifold $(M, c)$ endowed with a conformal structure of an arbitrary signature $(p, q)$ have the form

$$
\begin{align*}
\text{d}\omega^i &= \omega^0_\eta \wedge \omega^i + \omega^i_\eta \wedge \omega^j, \\
\text{d}\omega^0_\eta &= \omega^i \wedge \omega^0_\eta \\
\text{d}\omega^i_\eta &= \omega^0_\eta \wedge \omega^i + \omega^i_\eta \wedge \omega^j_\eta + \omega^i_{n+1} \wedge \omega^j_{n+1} + C^i_{jkl} \omega^k \wedge \omega^l, \\
\text{d}\omega^j_\eta &= \omega^0_\eta \wedge \omega^j + \omega^j_\eta \wedge \omega^k + C^j_{ikl} \omega^i \wedge \omega^k.
\end{align*}
$$

Here the quantities $C^i_{jkl}$ are defined in a third-order neighborhood of a point $x \in (M, c)$ and form the tensor of conformal curvature, also called the Weyl tensor. Denote it by the letter $C$, where $C = (C^i_{jkl})$.

The quantities $C_{ijk}$ are defined in a fourth-order neighborhood of a point $x \in (M, c)$, and for $n \geq 4$, they do not form a tensor. Denote the object $C_{ijk}$ by $C'$, i.e. $C' = (C_{ijk})$. The reason for this notation is that for $n \geq 4$, this object is expressed in terms of the covariant derivatives of the tensor $C$. For $n \geq 4$, the condition $C = 0$ implies $C' = 0$, and a manifold $(M, c)$ is conformally flat, i.e., it is locally equivalent to a conformal space $C_n^q$.

For $n = 3$, the tensor $C = (C^i_{jkl})$ is identically equal to 0, and the curvature of the space is defined by the object $C' = (C_{ijk})$ which in this case becomes a tensor. In what follows, we will assume that $n \geq 4$.

The components of the tensor $C$ and the object $C'$ satisfy the equations

$$
\begin{align*}
C_{ijkl} &= g_{im} C^m_{jkl}, \\
C_{ijkl} &= -C_{jikl} = -C_{ijlk}, \quad C_{ijkl} = C_{klij}, \\
C_{ijkl} + C_{iklj} + C_{iljk} &= 0, \\
C^i_{jki} &= 0, \quad C_{ijk} = -C_{ikj}.
\end{align*}
$$
Note that we will use the notation $C$ not only for the tensor $C_{ijkl}$ but also for the tensor $C_{ijkl}$.

3 ISOTROPIC FRAMES FOR $(M, c), \ c = CO(2, 2)$

Consider four-dimensional manifold $(M, c)$ endowed with a pseudoconformal structure $c = CO(2, 2)$. The fundamental quadratic form of this signature is reduced to the form

$$g = 2(\xi^2 \xi^3 - \xi^1 \xi^4).$$

(7)

It follows that the matrix of its coefficients is

$$(A_i, A_j) = (g_{ij}) = \begin{pmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix}.$$  

(8)

Note that we changed the signs of components of the tensor $g_{ij}$ in comparison with the book [AG 96] and the paper [A 96].

The isotropic cone $C_x \subset T_x(M)$ is defined by the equation

$$\xi^2 \xi^3 - \xi^1 \xi^4 = 0.$$  

We will clarify a structure of this cone. The last equation can be written in two different ways:

$$\frac{\xi^1}{\xi^4} = \frac{\xi^2}{\xi^4} = -\lambda, \quad \frac{\xi^1}{\xi^2} = \frac{\xi^3}{\xi^4} = -\mu.$$  

It follows that the cone $C_x$ carries two families of two-dimensional plane generators. These are defined by the equations

$$\xi^1 + \lambda \xi^3 = 0, \quad \xi^2 + \lambda \xi^4 = 0$$  

(9)

and

$$\xi^1 + \mu \xi^2 = 0, \quad \xi^3 + \mu \xi^4 = 0.$$  

(10)

The 2-planes defined by equations (9) are called $\alpha$-generators, and the 2-planes
defined by equations (10) are called $\beta$-generators of the cone $C_x$.

Figure 2

Under the projectivization of the tangent space $T_x(M)$ with center at a point $x = A_0$, there corresponds a ruled quadric $PC_x$ for the cone $C_x$ where $PC_x$ belongs to a three-dimensional projective space $P_3^3 = PT_x(M)$. With respect to the frame $\{A_i\}$, where $A_i = PA_i$, the quadric $PC_x$ is defined by the same equation (7) (see Figure 2).

For the conformal structure $CO(2,2)$, the group of transformations of the tangent space $T_x(M)$ preserving the invariant cone is split into three subgroups: $G = SL(2) \times SL(2) \times H$. The first two of these subgroups transfer the families of $\alpha$- and $\beta$-generators of the cones $C_x$ into themselves and are isomorphic to the group of projective transformations on a projective straight line $P^1$. As in the general case, the third subgroup is the group of homotheties.

On the manifold $(M, c)$, the isotropic $\alpha$- and $\beta$-generators of the cone $C_x$ form two fiber bundles $E_\alpha$ and $E_\beta$ with the common base $M$. The fibers of $E_\alpha$ and $E_\beta$ are the families of $\alpha$- and $\beta$-generators of the cones $C_x$. By (9) and (10), these fibers are parameterized by means of nonhomogeneous projective parameters $\lambda$ and $\mu$ and are isomorphic to real projective straight lines $RP_\alpha$ and $RP_\beta$. Thus the fiber bundles $E_\alpha$ and $E_\beta$ can be written as $E_\alpha = (M, RP_\alpha)$ and $E_\beta = (M, RP_\beta)$. These fiber bundles are real twistor fibrations similar to those introduced on four-dimensional manifolds of Lorentzian signature $(3,1)$ by Penrose (see, for example, [PR 86]).

Consider $\alpha$- and $\beta$-generators of the cone $C_x$. For $\omega^i = 0$ (i.e., for fixed principal parameters), they are defined by equations (9) and (10). One can easily prove that these generators intersect one another in an isotropic straight line connecting the point $A_0 = x$ with the point

$$B = \lambda\mu A_1 - \lambda A_2 - \mu A_3 + A_4,$$

and that they belong to a three-dimensional subspace of the space $T_x(M)$ defined by the equation

$$\xi^1 + \mu\xi^2 + \lambda\xi^3 + \lambda\mu\xi^4 = 0.$$
This subspace is tangent to the isotropic cone $C_x$ along its generator $A_0B$ and is also called isotropic.

In the space $T_x(M)$, we specialize our moving frame in such a way that its vertex $A_1$ coincides with the point $B$ and the isotropic straight line $A_0B$ coincides with the straight line $A_0A_1$. Then the nonhomogeneous projective parameters $\lambda$ and $\mu$ occurring in equations (9) and (10) for isotropic 2-planes $A_0A_1A_2$ and $A_0A_1A_3$ become $\infty$, $\lambda = \infty$, $\mu = \infty$, and equations (9) and (10) take the form
\[
\xi^3 = 0, \quad \xi^4 = 0
\]
and
\[
\xi^2 = 0, \quad \xi^4 = 0.
\]
The equation of the isotropic subspace (12) containing these isotropic $\alpha$- and $\beta$-generators becomes
\[
\xi^4 = 0.
\]

4 THE STRUCTURE EQUATIONS OF A MANIFOLD $(M, c)$

For a manifold $(M, c)$ endowed with a conformal structure $c = CO(2,2)$, in the isotropic frame bundle, equations (8) and the last equation of (3) imply that
\[
\begin{align*}
\omega^1_1 &= \omega^0_0 = \omega^3_3 = \omega^4_4 = 0, \\
\omega^2_2 &= \omega^1_1, \\
\omega^3_3 &= \omega^2_2, \\
\omega^4_4 &= \omega^1_1 + \omega^4_4 = 0, \quad \omega^2_2 + \omega^3_3 = 0.
\end{align*}
\]
Thus on the manifold $(M, c)$, among the forms $\omega^j_j$ only the forms $\omega^1_1$, $\omega^2_2$, $\omega^3_3$, $\omega^4_4$, and $\omega^0_0$ are independent. Hence on such a manifold $(M, c)$, the structure equations (5) take the form
\[
\begin{align*}
d\omega^1 &= (\omega^0_0 - \omega^1_1) \wedge \omega^1 + \omega^2 \wedge \omega^3 + \omega^4 \wedge \omega^4, \\
d\omega^2 &= (\omega^0_0 - \omega^2_2) \wedge \omega^2 + \omega^3 \wedge \omega^4 \wedge \omega^4, \\
d\omega^3 &= (\omega^0_0 + \omega^3_3) \wedge \omega^3 + \omega^1 \wedge \omega^4 + \omega^4 \wedge \omega^4, \\
d\omega^4 &= (\omega^0_0 + \omega^4_4) \wedge \omega^4 + \omega^2 \wedge \omega^2 + \omega^3 \wedge \omega^3, \\
d\omega^0_0 &= \omega^1 \wedge \omega^1 + \omega^2 \wedge \omega^2 + \omega^3 \wedge \omega^3 + \omega^4 \wedge \omega^4.
\end{align*}
\]
As a result, the curvature forms of the fiber bundles $A$ and $B$ can be written as

$$\begin{align*}
\Omega_1^2 &= -2 \{b_0 \omega^1 \wedge \omega^3 + b_1 (\omega^1 \wedge \omega^4 + \omega^2 \wedge \omega^3) + b_2 \omega^3 \wedge \omega^4\}, \\
\Omega_1^3 &= +4 \{a_1 \omega^1 \wedge \omega^2 + a_2 (\omega^1 \wedge \omega^4 - \omega^2 \wedge \omega^3) + a_3 \omega^3 \wedge \omega^4\}, \\
\Omega_2^1 &= -2 \{b_0 \omega^1 \wedge \omega^3 + b_1 (\omega^1 \wedge \omega^4 + \omega^2 \wedge \omega^3) + b_2 \omega^3 \wedge \omega^4\}, \\
\Omega_2^2 &= +4 \{b_1 \omega^1 \wedge \omega^2 + b_2 (\omega^1 \wedge \omega^4 + \omega^2 \wedge \omega^3) + b_3 \omega^2 \wedge \omega^4\},
\end{align*}$$

and

$$\begin{align*}
\Omega_3^1 &= +2 \{a_2 \omega^1 \wedge \omega^2 + a_3 (\omega^1 \wedge \omega^4 - \omega^2 \wedge \omega^3) + a_4 \omega^3 \wedge \omega^4\}, \\
\Omega_3^2 &= +2 \{b_2 \omega^1 \wedge \omega^3 + b_3 (\omega^1 \wedge \omega^4 + \omega^2 \wedge \omega^3) + b_4 \omega^2 \wedge \omega^4\}.
\end{align*}$$

From equations (23) and (24) it follows the tensor $C = (C_{ijkl})$ of conformal curvature is split into two subtensors $A$ and $B$, $C = A + B$, where

$$A = \{a_u\}, \quad B = \{b_u\}, \quad u = 0, 1, 2, 3, 4.$$
These are the curvature tensors of the fiber bundles $E_\alpha$ and $E_\beta$.

If one of subtensors $A$ or $B$ vanishes, then a manifold $(M, c)$ is called \textit{conformally semiflat}. In this case the fiber bundle $E_\alpha$ (respectively, $E_\beta$) admits a three-parameter family of two-dimensional integral surfaces $V_\alpha$ (respectively, $V_\beta$).

If both subtensors $A$ and $B$ vanish, then the tensor $C$ also vanishes. In this case a manifold $(M, c)$ becomes \textit{conformally flat} and is locally equivalent to a pseudoconformal space $C^4_2$. Under the Darboux mapping, a hyperquadric $Q^4_2$ of a projective space $P^5$ corresponds for the space $C^4_2$. Under this mapping, two-dimensional plane generators of the hyperquadric $Q^4_2$ correspond for two-dimensional integral surfaces $V_\alpha$ and $V_\beta$ of the fiber bundles $E_\alpha$ and $E_\beta$.

5 PRINCIPAL ISOTROPIC BIVECTORS

Suppose that $\xi$ and $\eta$ are vectors of the space $T_x(M)$, and $p = \xi \wedge \eta$ is a bivector defined by $\xi$ and $\eta$. The coordinates of $p$ are

$$p^{ij} = \xi^i \eta^j = \frac{1}{2}(\xi^i \eta^j - \xi^j \eta^i), \quad p^{ij} = -p^{ji}.$$  

The tensor of conformal curvature $C = (C_{ijkl})$ allows us to define the \textit{relative conformal curvature of the bivector $p$}:

$$C(p) = C_{ijkl} p^{ij} p^{kl}. \quad (25)$$

Let us find relative conformal curvatures of the bivectors $p_\lambda$ and $p_\mu$ defined by $\alpha$- and $\beta$-generators of the isotropic cone $C_x$ of the manifold $(M, c)$. From equations (9) it follows that the vectors $\xi_\lambda$ and $\eta_\lambda$ defining the bivector $p_\lambda$ are defined by the formulas

$$\xi_\lambda = e_3 - \lambda e_1, \quad \eta_\lambda = e_4 - \lambda e_2.$$  

As a result, the coordinates of the bivector $p_\lambda$ are

$$p^{12} = \lambda^2, \quad p^{13} = 0, \quad p^{14} = -\lambda, \quad p^{23} = \lambda, \quad p^{34} = 1, \quad p^{42} = 0.$$  

Substituting these values of coordinates $p^{ij}$ into formula (25) and applying relations (22), we find that

$$\frac{1}{4}C(p_\lambda) = a_0 \lambda^4 - 4a_1 \lambda^3 + 6a_2 \lambda^2 - 4a_3 \lambda + a_4.$$  

Since the right-hand side of the last equation contains only the components of the curvature tensor $A$ of the isotropic fiber bundle $E_\alpha$, this formula can be written as

$$\frac{1}{4}A(p_\lambda) = a_0 \lambda^4 - 4a_1 \lambda^3 + 6a_2 \lambda^2 - 4a_3 \lambda + a_4. \quad (26)$$
Similarly, the bivector $p_\mu$ is defined by the vectors
\[ \xi_\mu = e_2 - \mu e_1, \quad \xi_\mu = e_4 - \mu e_3, \]
and its coordinates are
\[ p^{12} = 0, \quad p^{13} = \mu^2, \quad p^{14} = -\mu, \quad p^{23} = -\mu, \quad p^{34} = 0, \quad p^{42} = -1. \]
This implies that the relative conformal curvatures of the bivector $p_\mu$ is
\[ \frac{1}{4}B(p_\mu) = b_0\mu^4 - 4b_1\mu^3 + 6b_2\mu^2 - 4b_3\mu + b_4. \] (27)

The isotropic bivectors whose relative conformal curvature vanishes are called the principal isotropic bivectors. By (26) and (27), the values of parameters $\lambda$ and $\mu$ defining such bivectors satisfy the algebraic equations
\[ a_0\lambda^4 - 4a_1\lambda^3 + 6a_2\lambda^2 - 4a_3\lambda + a_4 = 0 \] (28)
and
\[ b_0\mu^4 - 4b_1\mu^3 + 6b_2\mu^2 - 4b_3\mu + b_4 = 0. \] (29)
Thus in the general case the isotropic cone $C_x$ bears four principal $\alpha$-generators and the same number of principal $\beta$-generators if we count each of these generators as many times as its multiplicity.

On a manifold $(M, c)$, the principal isotropic bivectors form four principal $\alpha$-distributions and the same number of principal $\beta$-distributions. It was proved in [A 96] that if $\lambda$ is a multiple root of equation (28), then the principal $\alpha$-distribution defined by this root is integrable. In the same way if $\mu$ is a multiple root of equation (29), then the principal $\beta$-distribution defined by this root is integrable.

Suppose that $\lambda$ and $\mu$ are two fixed roots of equations (28) and (29), respectively, and $p_\lambda$ and $p_\mu$ are the principal isotropic distributions defined by these two roots. By means of a frame transformation indicated at the end of Section 3, the values of parameters $\lambda$ and $\mu$ can be made to equal $\infty$, $\lambda = \infty$, $\mu = \infty$. As a result, by (9) and (10), we find that these two distributions are defined by the following two systems of equations:
\[ \omega^3 = 0, \quad \omega^4 = 0 \] (30)
and
\[ \omega^2 = 0, \quad \omega^4 = 0. \] (31)
Moreover, equations (28) and (29) become
\[- 4a_1\lambda^3 + 6a_2\lambda^2 - 4a_3\lambda + a_4 = 0 \] (32)
and
\[- 4b_1\mu^3 + 6b_2\mu^2 - 4b_3\mu + b_4 = 0. \] (33)
If the coefficient $a_1$ in equation (32) vanishes, then the root $\lambda = \infty$ of this equation is a multiple root, and as a result, the principal distribution (30) defined by this root is integrable. Two-dimensional integral surfaces $V_\alpha$ of this distribution form an isotropic fiber bundle on the manifold $(M, c)$. Similarly, if the coefficient $b_1$ in equation (33) vanishes, then the root $\mu = \infty$ of this equation is a multiple root and the principal distribution (31) defined by this root is integrable. In addition, the two-dimensional integral surfaces $V_\beta$ of this distribution form an isotropic fiber bundle on the manifold $(M, c)$.

6 LIGHTLIKE HYPERSURFACES ON $(M, c)$, $c = CO(2, 2)$

As we already said in the introduction, a hypersurface $V$ on a manifold $(M, c)$, $\dim V = 3$, is said to be lightlike if its tangent subspace $T_x(V)$ is tangent to the isotropic cone $C_x$, i.e., this subspace is isotropic. The aim of this paper is to study the geometry of lightlike hypersurfaces on a manifold $(M, c)$, where $c = CO(2, 2)$.

With a point $x$ of a lightlike hypersurface $V$, we associate a moving frame in such a way that its vertex $A_0$ coincide with $x \in V$, $A_0 = x$, the points $A_1, A_2$, and $A_3$ belong to the tangent subspace $T_x(V)$, and the point $A_1$ belongs to the isotropic straight line along which the subspace $T_x(V)$ is tangent to the isotropic cone $C_x$. The subspace $T_x(V)$ contains two isotropic $\alpha$- and $\beta$-planes intersecting one another along the straight line $A_0A_1$. Thus the 2-plane $A_0 \wedge A_1 \wedge A_2$ is an $\alpha$-generator of the cone $C_x$, and the 2-plane $A_0 \wedge A_1 \wedge A_3$ is its $\beta$-generator.

We place points $A_2$ and $A_3$ of our moving frame to these two planes and normalize them by the condition $(A_2, A_3) = 1$. The subspace $A_0 \wedge A_2 \wedge A_3$ is called the screen subspace and is denoted by $S_x$, $S_x = A_0 \wedge A_2 \wedge A_3 \subset T_x(V)$. Further we take a point $A_4$ on the isotropic cone $C_x$ in such a way that the subspace $A_0 \wedge A_1 \wedge A_4$ is conjugate to the subspace $S_x$ with respect to the cone $C_x$. In addition, we normalize the points $A_1$ and $A_4$ by the condition $(A_1, A_4) = -1$.

A straight line $N_x = A_0 \wedge A_4$ does not belong to the tangent subspace $T_x(V)$. This line is called a normalizing straight line. Its location is uniquely determined by the location of the subspace $S_x$.

The matrix of scalar products of the points $A_i$, $i = 1, 2, 3, 4$, now has the form (8).

the family of frames we have constructed is called a family of first-order frames associated with a point $x$ of a lightlike hypersurface $V \subset (M, c)$.

We will now find the equations of a bundle of first-order isotropic frames associated with a lightlike hypersurface $V$. Since its tangent subspace $T_x(V) = A_0 \wedge A_1 \wedge A_2 \wedge A_3$, with respect to this frame bundle the equation of $V$ is

$$\omega^4 = 0,$$

(34)
and as a result, we have
\[ dA_0 = \omega_0^0 A_0 + \omega^1 A_1 + \omega^2 A_2 + \omega^3 A_3. \] (35)

The 1-forms $\omega^1$, $\omega^2$, and $\omega^3$ are independent. They are basis forms of the frame bundle in question and of the hypersurface $V$.

Equations
\[ \omega^2 = \omega^3 = 0 \] (36)
define on $V$ a foliation formed by isotropic lines. As was proved in [AG 99b], these lines are isotropic geodesics for all pseudo-Riemannian metrics $g$ compatible with the conformal structure $c = CO(2, 2)$ on the manifold $(M, c)$.

We will assume that the isotropic geodesics defined by equations (36) can be prolonged indefinitely on a hypersurface $V$. In this case each of these geodesics bears the geometry of a projective straight line $P^1$, and a hypersurface $V$ is the image of the product $M^2 \times P^1$ under its differentiable mapping $f$ into the manifold $(M, c)$: $V = f(M^2 \times P^1)$, $f : M^2 \times P^1 \to (M, c)$.

Equation $\omega^3 = 0$ defines on $V$ a fibration of isotropic $\alpha$-planes $A_0 \wedge A_1 \wedge A_2$, and equation $\omega^2 = 0$ defines on $V$ a fibration of isotropic $\beta$-planes $A_0 \wedge A_1 \wedge A_3$ (cf. these two equations with equations (30) and (31)).

In an isotropic frame bundle, the first fundamental form $I$ of a lightlike hypersurface $V \subset (M, c)$ becomes
\[ I = g|_V = (dA_0, dA_0) = 2\omega^2 \omega^3. \] (37)

This form is of rank 2 and of signature $(1, 1)$, and its coefficients form the matrix
\[ (g_{ab}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad a, b = 2, 3. \] (38)

7 SINGULAR POINTS AND TOTALLY UMBILICAL HYPERSURFACES

By the last equation of (17), exterior differentiation of equation (34) (the basic equation of a lightlike hypersurface $V$) leads to the following exterior quadratic equation:
\[ \omega^2 \wedge \omega^1 + \omega^3 \wedge \omega^1 = 0. \] (39)

Applying Cartan’s lemma to this equation, we that
\[ \begin{cases} 
\omega_3^2 = \lambda_{22} \omega^2 + \lambda_{23} \omega^3, \\
\omega_1^2 = \lambda_{32} \omega^2 + \lambda_{33} \omega^3,
\end{cases} \] (40)

where $\lambda_{23} = \lambda_{32}$.
By means of the Cartan test (see [BCGGG 91] and cf. [AG 99a]), one can prove that lightlike hypersurfaces $V \subset (M, c)$, where $c = CO(2, 2)$, exist and depend on a function of two variables.

Differentiating equation (35), we obtain

$$d^2 A_0 \equiv \left(\omega^2 \omega_2^4 + \omega^3 \omega_3^4\right) A_4 + \left(\omega^2 \omega_2^5 + \omega^3 \omega_3^5\right) A_5 \pmod{T_x(V)}.$$  \hspace{1cm} (41)

But by (3) and (8) we have

$$\omega_2^5 = \omega^1, \quad \omega_3^5 = \omega^2, \quad \omega_2^4 = \omega^3, \quad \omega_3^4 = \omega^4.$$  

Thus by (40) relation (41) takes the form

$$d^2 A_0 \equiv \left(\lambda_{22}(\omega^2)^2 + 2\lambda_{23}\omega^2\omega^3 + \lambda_{33}(\omega^3)^2\right) A_4 + 2\omega^2\omega^3 A_5 \pmod{T_x(V)}. \hspace{1cm} (42)$$

Note that the coefficient in $A_5$ in equation (42) coincides with the first fundamental form (37) of a hypersurface $V \subset (M, c)$.

Denote by $	ilde{I}$ the coefficient in $A_4$ in equation (42):

$$\tilde{I} = \lambda_{22}(\omega^2)^2 + 2\lambda_{23}\omega^2\omega^3 + \lambda_{33}(\omega^3)^2.$$  \hspace{1cm} (43)

Then equation (42) takes the form

$$d^2 A_0 = \tilde{I} A_4 + I A_5 \pmod{T_x(V)}. \hspace{1cm} (44)$$

If we multiply expression (43) by a point $A_1 - xA_0$, then by (1) and (8), we find that

$$(d^2 A_0, A_1 - xA_0) = - (\tilde{I} - xI).$$

The expression in the parentheses of the right-hand side is a pencil of the second fundamental forms of a hypersurface $V \subset (M, c)$:

$$\tilde{I} - xI = \lambda_{22}(\omega^2)^2 + 2(\lambda_{23} - x)\omega^2\omega^3 + \lambda_{33}(\omega^3)^2.$$  \hspace{1cm} (45)

The matrix of their coefficients is

$$\begin{pmatrix} \lambda_{ab} \
\end{pmatrix} = \begin{pmatrix} \lambda_{22} & \lambda_{23} - x \\ \lambda_{23} - x & \lambda_{33} \end{pmatrix}.$$  

From the pencil (43) we will take the form whose matrix is apolar to the matrix $(g_{ab})$, that is, the matrix satisfying the condition

$$\tilde{\lambda}_{ab} g^{ab} = 0.$$  \hspace{1cm} (46)

Since by (38) we have

$$(g_{ab}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
it follows that
\[ \bar{\lambda}_{ab}g^{ab} = 2(\lambda_{23} - x). \]

Thus condition (45) leads to the relation
\[ x = \lambda_{23}. \] (46)

Condition (45) singles out from a pencil (44) of the second fundamental forms of a hypersurface \( V \subset (M, c) \) a conformally invariant fundamental form
\[ II = \bar{I}I - \lambda_{23}I = \lambda_{22}(\omega^2)^2 + \lambda_{33}(\omega^3)^2. \] (47)

Its matrix has the form
\[ (h_{ab}) = (\lambda_{ab}) - \lambda_{23}(g_{ab}) = \begin{pmatrix} \lambda_{22} & 0 \\ 0 & \lambda_{33} \end{pmatrix} \] (48)
and is diagonal.

Consider singular points of the map \( f(M^2 \times P^1) = V^3 \subset (M, c) \). We will look for these points in the form \( X = A_1 - sA_0 \). At these points the dimension of the tangent subspace \( T_X(V) \) must be reduced. By (3), (11), and (34), we have
\[ dA_1 = \omega_0^1A_0 + \omega_1^1A_1 + \omega_2^1A_2 + \omega_3^1A_3. \] (49)

Applying equations (49) and (35), we find that
\[ d(A_1 - sA_0) = (\omega_0^1 - x\omega_0^0 - dx)A_0 + (\omega_1^1 - x\omega_1^0)A_1 + (\omega_2^1 - x\omega_2^0)A_2 + (\omega_3^1 - x\omega_3^0)A_3. \]

Further by (40) we obtain
\[ d(A_1 - sA_0) \equiv (\lambda_{23} - s)A_2 + \lambda_{22}A_3(\omega^2) + (\lambda_{33}A_2 + (\lambda_{23} - s)A_3)(\omega^3) \pmod{A_0, A_1}. \]

The tangent subspace \( T_X(V) \) is determined by the points \( A_0, A_1, (\lambda_{23} - s)A_2 + \lambda_{22}A_3, \) and \( \lambda_{33}A_2 + (\lambda_{23} - s)A_3 \). Thus the dimension of the tangent subspace is reduced only in the points \( X = A_1 - sA_0 \) in which
\[ \det \begin{pmatrix} \lambda_{23} - s & \lambda_{22} \\ \lambda_{33} & \lambda_{23} - s \end{pmatrix} = 0. \]

This equation can be written as
\[ s^2 - 2\lambda_{23}s + (\lambda_{23}^2 - \lambda_{22}\lambda_{33}) = 0. \] (50)

Denote by \( s_1 \) and \( s_2 \) the roots of this equation. They are calculated by the following formula:
\[ s_{1,2} = \lambda_{23} \pm \sqrt{\lambda_{22}\lambda_{33}}. \]

The points \( F_1 = A_1 - s_1A_0 \) and \( F_2 = A_1 - s_2A_0 \) are singular points of an isotropic geodesic \( l = A_0A_1 \) of a hypersurface \( V \).
By Vieta’s theorem, it follows from equation (50) that
\[
s_1 + s_2 = 2\lambda_{23}.
\]
Thus the point \( H = A_1 - \lambda_{23}A_0 \) is the fourth harmonic point \( H \) of the point \( A_0 \) with respect to the points \( F_1 \) and \( F_2 \) on the line \( l = A_0A_1 \). The singular points \( F_1 \) and \( F_2 \) are located symmetrically with respect to the points \( A_0 \) and \( H \).

Now the conformally invariant second fundamental form \( II \) of a hypersurface \( V \subset (M, c) \) can be written as
\[
II = -(d^2A_0, H).
\]

We take a moving frame whose vertex \( A_1 \) coincides with the point \( H \). This implies \( \lambda_{23} = 0 \). As a result, equation (50) becomes
\[
s^2 - h_{22}h_{33} = 0,
\]
and
\[
s_{1,2} = \pm \sqrt{h_{22}h_{33}}. \tag{51}
\]

The following theorem follows from relation (51).

**Theorem 1 (a)** The second fundamental form \( II \) of a hypersurface \( V \subset (M, c) \) at a point \( A_0 \) is positive definite or negative definite if and only if the isotropic geodesic \( l = A_0A_1 \) through the point \( x = A_0 \) bears two real singular points. If at a point \( x = A_0 \) this form is an indeterminate form of rank two, then the singular points on the straight line \( l = A_0A_1 \) are complex conjugate.

(b) The second fundamental form \( II \) of a hypersurface \( V \subset (M, c) \) at a point \( x = A_0 \) has the rank less than two if and only if the singular points on the isotropic geodesic \( l = A_0A_1 \) coincide. In this case the point \( H \) coincides with this multiple singular point.

On a lightlike hypersurface \( V \), 2-planes \( A_0 \wedge A_1 \wedge A_2 \) and \( A_0 \wedge A_1 \wedge A_3 \) of an isotropic frame bundle compose an \( \alpha \)- and \( \beta \)-distribution. Denote them by \( \Delta_{\alpha} \) and \( \Delta_{\beta} \). These distributions are defined on \( V \) by the equations
\[
\omega^3 = 0 \quad (\alpha) \quad \omega^2 = 0 \quad (\beta) \tag{52}
\]

In general, the distributions \( \Delta_{\alpha} \) and \( \Delta_{\beta} \) are not integrable. Let us find the conditions of their integrability.

Exterior differentiation of equation (52α) gives the following exterior quadratic equation
\[
\omega^1 \wedge \omega^3 = 0.
\]
Substituting the value of the form \( \omega^3 \) from (40) into this equation and taking into account (48), we find that the distribution \( \Delta_{\alpha} \) is integrable if and only if
\[
h_{22} = 0. \tag{53}
\]

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Similarly the distribution $\Delta_\beta$ is integrable if and only if
\[ h_{33} = 0. \]  
\[ (54) \]

Comparing the conditions (53) and (54) with relations (51) we arrive at the following result.

**Theorem 2** If at least one of the isotropic distributions $\Delta_\alpha$ and $\Delta_\beta$ on a light-like hypersurface $V \subset (M, c)$ is integrable, then the singular points on each of its isotropic generators coincide.

If both isotropic distributions $\Delta_\alpha$ and $\Delta_\beta$ are integrable on a hypersurface $V$, then conditions (53) and (54) are satisfied simultaneously, and the second fundamental form $II$ of $V$ vanishes. But this means that hypersurface $V$ is totally umbilical. This implies the following result.

**Theorem 3** Both isotropic distributions $\Delta_\alpha$ and $\Delta_\beta$ on a lightlike hypersurface $V \subset (M, c)$ are integrable if and only if the hypersurface $V$ is totally umbilical.

**8 SOME PROPERTIES OF LIGHTLIKE HYPERSURFACES**

We will pass now to the study of properties of a lightlike hypersurface $V \subset (M, c)$ connected with third- and higher-order differential neighborhoods.

We make a reduction in our isotropic second-order frame bundle by taking a specialized frame whose vertex $A_1 \in l$ coincides with the fourth harmonic point $H$ of the point $A_0$ with respect to the singular points $F_1$ and $F_2$ of the straight line $l = A_0A_1$. Then we obtain
\[ \lambda_{23} = 0, \ h_{22} = \lambda_{22}, \ h_{33} = \lambda_{33}, \]
and equations (40) become
\[ \omega_1^3 = h_{22}\omega^2, \ \omega_1^2 = h_{33}\omega^3. \]  
\[ (55) \]

By (12), (14), (15), (18), and (19), exterior differentiation of equations (55) gives
\[ \left\{ \begin{array}{l}
\Delta h_{22} \land \omega^2 + (-\omega_1^3 + h_{22}h_{33}\omega^1 - 2a_1\omega^2 - 2b_1\omega^3) \land \omega^3 = 0, \\
(-\omega_1^3 + h_{22}h_{33}\omega^1 - 2a_1\omega^2 - 2b_1\omega^3) \land \omega^2 + \Delta h_{33} \land \omega^3 = 0,
\end{array} \right. \]  
\[ (56) \]
where
\[ \left\{ \begin{array}{l}
\Delta h_{22} = dh_{22} + h_{22}(\omega_0^0 - 2\omega_2^0 - \omega_1^1) + 2a_0\omega^1, \\
\Delta h_{33} = dh_{33} + h_{33}(\omega_0^0 + 2\omega_2^0 - \omega_1^1) + 2b_0\omega^1.
\end{array} \right. \]  
\[ (57) \]
By Cartan’s lemma, it follows from (56) that
\[
\begin{align*}
\Delta h_{22} &= h_{22}^2 \omega^2 + h_{223}^2 \omega^3, \\
\omega_0^1 &= h_{22} h_{33}^3 \omega^1 - (h_{223} + 2a_1) \omega^2 - (h_{233} + 2b_1) \omega^3, \\
\Delta h_{33} &= h_{233}^2 \omega^2 + h_{333}^2 \omega^3.
\end{align*}
\] (58)

We will apply now equations (58) to totally umbilical hypersurfaces \( V \subset (M, c) \). For such hypersurfaces we have \( h_{22} = h_{33} = 0 \). As a result, equations (55) take the form
\[
\omega_1^3 = 0, \quad \omega_1^2 = 0,
\] (59)
and equations (58) imply that
\[
a_0 = 0, \quad b_0 = 0, \\
h_{222} = h_{223} = h_{233} = 0,
\] (60)
and
\[
\omega_0^1 = -2(a_1 \omega^2 + b_1 \omega^3).
\] (62)

Conditions (60) mean that the isotropic distributions \( \Delta_\alpha \) and \( \Delta_\beta \) are principal. Moreover, it follows now from equations (49) that
\[
dH = \omega_0^0 H - 2(a_1 \omega^2 + b_1 \omega^3) A_0.
\] (63)

This implies the following result.

**Theorem 4** A lightlike totally umbilical hypersurface \( V \subset (M, c) \) possesses the following properties:

(a) The isotropic distributions \( \Delta_\alpha \) and \( \Delta_\beta \) are integrable and principal.

(b) The multiple singular point \( H \) of the isotropic geodesic \( l = A_0 A_1 \) describes an isotropic line tangent to the straight line \( l \) at the point \( H \).

(c) If \( a_1 = b_1 = 0 \), then the point \( H \) is fixed, and a totally umbilical hypersurface is an isotropic cone \( C_H \) with vertex \( H \).

**Proof.** The statement (a) follows from the fact that on a hypersurface \( V \), the isotropic distributions \( \Delta_\alpha \) and \( \Delta_\beta \) are defined by equations (52) and correspond to the values \( \lambda = \infty \) and \( \mu = \infty \) in equations (9) and (10). Hence for \( a_0 = b_0 = 0 \), these values satisfy equations (32) and (33) defining the principal isotropic distributions.

The statement (b) follows immediately from equation (63).

Note that the conditions \( a_1 = b_1 = 0 \) along with conditions (60) imply that the values \( \lambda = \infty \) and \( \mu = \infty \) are multiple roots of equations (32) and (33). This implies that the statement (c) can be also formulated as follows:
A lightlike totally umbilical hypersurface $V \subset (M, c)$ is an isotropic cone if and only if it bears multiple isotropic distributions $\Delta_\alpha$ and $\Delta_\beta$.

Note also that in this case the integral surfaces of the distributions $\Delta_\alpha$ and $\Delta_\beta$ on a hypersurface $V$ are two-dimensional plane generators of the cone $C_H$.

9 CONSTRUCTION OF A CANONICAL DISTRIBUTION OF FRAMES FOR A LIGHTLIKE HYPERSURFACE

We associated a family of the second-order frames with a point $x = A_0$ of a lightlike totally umbilical hypersurface $V \subset (M, c)$ in such a way that the vertex $A_1$ coincides with the harmonic pole $H$ of the isotropic tangent $A_0A_1$. But the points $A_2$ and $A_3$ of these frames can move freely in $\alpha$- and $\beta$-planes containing the straight line $A_0A_4$, and its point $A_4$ can move freely along the isotropic straight line $A_0A_1$ that is conjugate to the screen subspace $A_0 \wedge A_2 \wedge A_3$ with respect to the isotropic cone $C_x$.

For a fixed point $x = A_0$, by equations (16) and (40), we find that

$$\begin{align*}
\delta A_2 &= \pi_0^0 A_0 + \pi_1^1 A_1 + \pi_2^2 A_2, \\
\delta A_3 &= \pi_3^3 A_0 + \pi_4^4 A_1 + \pi_5^5 A_3.
\end{align*}$$

Here $\pi_0^0, \pi_1^1, \pi_3^3,$ and $\pi_4^4$ are fiber forms defining a displacement of the points $A_2$ and $A_3$ in the corresponding isotropic 2-planes.

In order to find the points $A_2$ and $A_3$ uniquely in these 2-planes, one needs to make the above mentioned fiber forms vanish. However, this must be done in such a way that a fixing of the points $A_2$ and $A_3$ would be intrinsically connected with the geometry of a hypersurface $V$. The latter can be achieved by fixing in a certain way the coefficients $h_{abc}$ occurring in equations (58). These coefficients are associated with a third-order neighborhood of a hypersurface $V$.

To this end, we take exterior derivatives of equations (58). As a result, we obtain the following exterior quadratic equations:

$$\begin{align*}
\Delta h_{222} \wedge \omega^2 + \Delta h_{223} \wedge \omega^3 + H_{22} &= 0, \\
\Delta h_{223} \wedge \omega^2 + \Delta h_{233} \wedge \omega^3 + H_{23} &= 0, \\
\Delta h_{233} \wedge \omega^2 + \Delta h_{333} \wedge \omega^3 + H_{33} &= 0,
\end{align*}$$

(64)

where

$$\begin{align*}
\Delta h_{222} &= dh_{222} + h_{222}(2\omega_0^0 - 3\omega_2^2 - \omega_1^1) + 2a_0\omega_1^1 + 3h_{222}\omega_2^2 - 3(h_{222})^2\omega_1^1, \\
\Delta h_{223} &= dh_{223} + h_{223}(2\omega_0^0 - \omega_2^2 - \omega_1^1) + 2a_0\omega_1^1 - h_{223}\omega_2^2 + h_{222}\omega_3^3\omega_1^2, \\
\Delta h_{233} &= dh_{233} + h_{233}(2\omega_0^0 + \omega_2^2 - \omega_1^1) + 2b_0\omega_1^1 - h_{233}\omega_2^2 + h_{223}\omega_3^3\omega_1^3, \\
\Delta h_{333} &= dh_{333} + h_{333}(2\omega_0^0 + 3\omega_2^2 - \omega_1^1) + 2b_0\omega_1^1 + 3h_{333}\omega_3^3 - 3(h_{333})^2\omega_1^2.
\end{align*}$$
Theorem 5

This proves the following result.

Equations (64) imply that the 1-forms $\Delta h_{222}$, $\Delta h_{223}$, $\Delta h_{233}$, and $\Delta h_{333}$ are linear combinations of the basis forms $\omega^1$, $\omega^2$, and $\omega^3$.

For a fixed point $x = A_0$, i.e., for $\omega^1 = \omega^2 = \omega^3 = 0$, these forms vanish, and their expressions become

$$
\begin{align*}
\Delta_3 h_{222} &= \delta h_{222} + h_{222}(2\pi_0^0 - 3\pi_2^2 - \pi_1^1) + 2a_0\pi_2^1 + 3h_{222}\pi_0^0 - 3(h_{222})^2\pi_3^1 = 0, \\
\Delta_3 h_{223} &= \delta h_{223} + h_{223}(2\pi_0^0 - \pi_2^2 - \pi_1^1) + 2a_0\pi_2^3 - h_{222}\pi_3^3 + h_{223}h_{333}\pi_3^1 = 0, \\
\Delta_3 h_{233} &= \delta h_{233} + h_{233}(2\pi_0^0 + \pi_2^2 - \pi_1^1) + 2b_0\pi_2^1 - h_{333}\pi_3^3 + h_{223}h_{333}\pi_3^3 = 0, \\
\Delta_3 h_{333} &= \delta h_{333} + h_{333}(2\pi_0^0 + 3\pi_2^2 - \pi_1^1) + 2b_0\pi_2^3 + 3h_{333}\pi_3^3 - 3(h_{333})^2\pi_3^3 = 0.
\end{align*}
$$

Equations (65) contain the fiber forms $\pi_0^0$, $\pi_1^0$, $\pi_0^1$, and $\pi_3^1$ defining a displacement of the points $A_2$ and $A_3$ in the $\alpha$- and $\beta$-planes $A_0 \cup A_1 \cup A_2$ and $A_0 \cup A_1 \cup A_3$. Consider the determinant $D$ of the matrix of coefficients in these fiber forms in equations (65):

$$
D = \det \begin{pmatrix}
3h_{22} & 2a_0 & 0 & -3(h_{22})^2 \\
0 & h_{223} & -h_{22} & 2a_0 \\
-h_{33} & 2b_0 & 0 & 3h_{223}h_{333} \\
0 & -3(h_{33})^2 & 3h_{333} & 2b_0
\end{pmatrix}.
$$

Calculating this determinant, we find that

$$
D = 4(3h_{22}b_0 + h_{33}a_0)(h_{22}b_0 + 3h_{33}a_0).
$$

If this determinant does not vanish, $D \neq 0$, then equations (65) imply that the quantities $h_{222}$, $h_{223}$, $h_{333}$, and $h_{333}$, occurring in equations (58) can be simultaneously reduced to 0 by means of the fiber forms $\pi_0^0$, $\pi_1^0$, $\pi_0^1$, and $\pi_3^1$ (see [O 62]). As a result, the points $A_2$ and $A_3$ are uniquely determined in the planes $\alpha = A_0 \cup A_1 \cup A_2$ and $\beta = A_0 \cup A_1 \cup A_3$, and we arrive at a family of third-order moving frames associated with a point $x = A_0 \in V \subset (M, c)$.

With respect to a third-order frame we have constructed, equations (58) take the form

$$
\begin{align*}
h_{22} &+ h_{22}(\omega_0^0 - 2\omega_2^2 - \omega_1^1) = -2a_0\omega^1, \\
\omega_0^0 &+ h_{223}\omega^1 - 2b_0\omega^2 - 2b_0\omega^3, \\
h_{33} &+ h_{33}(\omega_0^0 + 2\omega_2^2 - \omega_1^1) = -2b_0\omega^1.
\end{align*}
$$

This proves the following result.

**Theorem 5** If on a lightlike hypersurface $V$ the determinant $D$ does not vanish, then it is possible to construct a third-order frame bundle on $V$ intrinsically connected with the geometry of $V$. In this frame bundle, $h_{abc} = 0$. 

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Note that if the $CO(2, 2)$-structure on a manifold $(M, c)$ is conformally flat, then a third-order frame bundle indicated above cannot be constructed since for a conformally flat structure we have $a_0 = b_0 = 0$, and consequently, $D = 0$. However, for a conformally semiflat $CO(2, 2)$-structure the above construction is possible. A construction of a canonical frame bundle for lightlike totally umbilical hypersurfaces is also impossible since for them $h_{22} = h_{33} = 0$, and consequently, $D = 0$.

In order to complete our construction of a canonical frame bundle, we also have to fix the vertex $A_4$ on the isotropic straight line $A_0 A_4$ which is conjugate to the screen subspace $S_x = A_0 \land A_2 \land A_3$ with respect to the isotropic cone $C_x$. This can be done in the same way as we did for a lightlike hypersurface $V \subset (M, c)$ whose conformal structure $c$ is of Lorentzian signature, $c = CO(n - 1, 1)$. The family of straight lines $A_0 A_4$ associated with a hypersurface $V$ is an isotropic congruence (see [AG 99b]) each ray of which bears two singular points $F'_1$ and $F'_2$. To complete our specialization of moving frames, we choose a frame whose vertex $A_4$ coincides with the harmonic pole $H'$ of the point $A_0$ with respect to singular points $F'_1$ and $F'_2$ (see Figure 3). Since the singular points are defined in a fourth-order differential neighborhood of a point $x \in V$, the point $A_4$ is defined also in this neighborhood.

Thus we arrive at the following result.

**Theorem 6** If $D \neq 0$, a canonical frame bundle on a lightlike hypersurface $V \subset (M, c), c = CO(2, 2)$ is defined by elements of a fourth-order differential neighborhood of a point $x \in V$. 

Figure 3
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Authors’ addresses:

M. A. Akivis
Department of Mathematics
Jerusalem College of Technology
– Mahon Lev, P. O. B. 16031
Jerusalem 91160, Israel

E-mail address: akivis@avoda.jct.ac.il

V. V. Goldberg
Department of Mathematics
New Jersey Institute of Technology
University Heights
Newark, NJ 07102, U.S.A.

E-mail address: vlgold@numerics.njit.edu