SDC-method of large deviations analysis for nonlinear system

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Abstract. The paper presents a new method of large deviations analysis for nonlinear systems in the form of state-dependent coefficients. The large deviations of the controlled process from some regular state is the basis of forecasting of the critical situation. The forecasting problem is reduced to the Lagrange-Pontryagin optimal control problem. The presented approach to state-dependent coefficients for the solution of Lagrange-Pontryagin problem is different from the approach used before for linear cases in the fact that it uses a control in the form of feedback instead of programmable control. This eliminates the need to calculate the boundary value in the final time point for the conjugate variable, which is the most time-consuming task in nonlinear cases. An example of using the presented approach to the problem of vessel tilting prediction is considered.

1. Introduction
For many years researchers have paid special attention to the problem of stabilization of autonomous control systems under conditions of uncertainty. At that, classic approaches to guaranteeing synthesis often prove to be insufficient to ensure functional stability of the system i.e. the ability to perform the set task irrespectively of the external disturbance action [1,2]. It is evident that in order to reach functional stability, the control system must have a multi-level structure. The lowest (in the hierarchy) control loop must ensure stable operation of the system (system robustness) in normal operation mode, while the purpose of the upper control loop is continuous monitoring and prediction of the development of the controlled process. Herewith, if the controlled process gets out of the established functional framework, the upper control loop must reduce these deviations of the normal mode by reconfiguring the regulator of the lower control loop based on the prediction of development of the controlled process, thus ensuring functional stability. This idea of such online assembly is the key for control under uncertainty and for increasing system functional stability.

In [1] the Wentzel-Freidlin method of large deviations analysis (see [3]) is suggested as a practical tool of synthesis of such two-level system. The proposed technique allows estimating the probability of the controlled process deviations from stable (normal mode) states. Work [2] deals with the development of this approach where relevant assessments were received for linear systems.

In this paper the problem of developing a method for critical states prediction based on the large deviations analysis for nonlinear systems is considered. In this regard, one of the most perspective modern approaches will be used: nonlinear control design technique based on matrix equations with state-dependent coefficients (SDC) [4,5]. The SDC approach is applicable for the class of nonlinear affine systems. The idea of this approach is to present the initial nonlinear system in pseudolinear form, wherein the system and the control matrixes have state-dependent coefficients. An accurate solution of the corresponding nonlinear control problem, in general, is quite complex, but here different techniques
of approximate calculation are possible. The structure of the obtained control law aligns with the linear one. It is necessary at every step to solve numerically corresponding matrix equations with state-dependent coefficients for every new state vector.

The SDC approach has a number of advantages:

- it is applicable to quite a large class of affine nonlinear systems: while the nonlinear model is not simplified or linearized;
- it has a relatively simple operation algorithm.

2. Statement of the problem
The basic equation that will be an object of the large deviation analysis is perturbed i.e. contains a small parameter \( \varepsilon > 0 \):

\[
dx / dt = a(x) + \varepsilon \sigma(x) \dot{w}, \quad x(t_0) = x_0,\]

where \( \dot{w} \) is \( k \)-vector of “white noise”, \( a(x), \sigma(x) \) are smooth matrix functions, \( x \in \mathbb{R}^n \) is system state vector. Let us also consider two generating systems: an unperturbed system:

\[
dx / dt = a(x), \quad x(t_0) = x_0,\]

and the so-called system of paths:

\[
d\varphi / dt = a(\varphi) + \sigma(\varphi) u, \quad \varphi(t_0) = x_0,\]

where \( u = \delta \dot{w} \) is some function, such that on the solutions of (3) an action functional is determined [3]:

\[
S_{0\varepsilon JK}(\varphi, u) = \frac{1}{2} \int_{t_0}^{t_K} u^T u \, dt. \tag{4}
\]

Let us describe global properties of the system (1) with the help of the system of paths, considering the possibility of deviations of the system state from zero towards the border \( \partial D \) of the operational region \( D \subset \mathcal{O}_J \) (\( D \) is an open set).

Let \( \partial D = \bigcup_j \partial_j \) and there be finite aggregate \( A = \{Y_j\} \) of such critical states: \( Y_j(t) = C_j x(t), Y_j \in \partial_j \).

Let for some \( t = t_K \) only one critical state \( Y_{k} \in \partial_k \) and any of the given aggregate \( A = \{Y_j\} \) may arise. Then, it can be considered that \( Y = Y_k \) and it can raise the question of predicting this critical state, supposing:

\[
t_K = \inf \{ t : Y(t) \in \partial_k, \ t > t_0, \ x(t_0) \in D \} . \tag{5}
\]

For set \( D \) and system (3) the following equality holds [3]:

\[
\lim_{\varepsilon \to 0} \varepsilon^2 \ln P_{\varepsilon} \{ x_0 \subseteq \mathbb{R}^n \setminus D \} = - \min_{\varphi \in D} S_{0\varepsilon JK}(\varphi, u), \tag{6}
\]

where functional \( S_{0\varepsilon JK}(\varphi, u) \) is determined in accordance with (4) on the solutions of the system (3), for which let us write a boundary condition of the escape to the critical state:

\[
Y(t_K) = C_k \varphi(t_K) = Y_k \in \partial_k, \tag{7}
\]

where \( C_k \) is a full rank matrix.

The probability in (6) can be valued through solving the following optimal control problem (Lagrange-Pontryagin): minimizing the action functional (noise) (4) for the system of paths (3) under additional condition (7).

**Definition 1.** Let us define a critical state potential as the minimal value of the functional (4) with a negative sign:
and formulate a problem for it to minimize the performance index (4) under terminal conditions (7).

The solution of the problem (8), (4), (7) in the form of a feedback can be written by analogy with the linear case [7]:

\[ u = B^T(\phi)W^T C_K^{-1}(C_K W \phi - Y_K), \]

where

\[ \frac{dW}{dt} = -WA(\phi), \quad W(t_k) = I, \]

\[ \frac{dM}{dt} = -C_K WB(\phi)B^T(\phi)W^T C_K^{T}, \quad M(t_k) = 0. \]

and \( I \) is identity \( n \times n \) matrix.

This numerical solution needs integration of the equations (1), (11) backward from \( t_k \) to \( t \), which is impossible to implement, as the state \( \phi \) of the system (8) is not known ahead of time, thus, the state-dependent coefficients of the matrixes \( A(\phi) \), \( B(\phi) \) are unknown as well. The approximate approach for calculation of the control (9) is based on the evaluation of the state-dependent coefficients (10), (11) under current values of the state vector and their freezing for solving these equations from the current time until the final time at each step [9].

Following this method, let us introduce the notation \( A_S \) and \( B_S \) as the value of the matrixes \( A(\phi) \) and \( B(\phi) \) for the current state vector \( \phi \), i.e.:

\[ A_S = A(\phi)_{\phi(t)}, \quad B_S = B(\phi)_{\phi(t)}. \]

Then for the equations (10), (11) we can write an analytical solution:

\[ W(t) = e^{A_S(t-t_k)}, \quad M(t) = -C_K \left\{ \int_{t_k}^{t} e^{A_S(t-t')} B_S B_S^{T} e^{A_S^{T}(t'-t_k)} dt' \right\} C_K^{T}, \quad M(t_k) = 0. \]

The integral in (12) can be expressed in the form of:
\[
\int_{t_k}^{t} e^{A_S(t-t)}B_S e^{A_S^T(t-t)}dt = e^{A_S(t-t)} P e^{A_S^T(t-t)} - P,
\]

(13)

where \( P \) is the solution of Lyapunov algebraic equation:

\[
A_S P + P A_S^T - B_S B_S^T = 0.
\]

(14)

It can be checked by a simple substitution of (13) for (12), where, as a result, under the integral we get a differential of function \( e^{A_S(t-t)} P e^{A_S^T(t-t)} \), from which by the Newton-Leibniz formula we get the ratio (13).

So that (14) will have an unambiguous solution \( P > 0 \), it is necessary and enough for the pair \( (A_S, B_S) \) to be controllable, i.e. the pair \( A(\phi) \) and \( B(\phi) \) must be controllable pointwise for all \( \phi \) [8].

As a result, we have an analytical solution for the matrix \( M(t) \):

\[
M(t) = -C_K \left( e^{A_S(t-t)} P e^{A_S^T(t-t)} - P \right) C_K^T = C_K \left( P - WPW^T \right) C_K^T.
\]

(15)

The solution of the initial problem (8), corresponding to the control (9), has the form of:

\[
d\phi/dt = A(\phi)\phi - B(\phi)B^T(\phi)W^TC_K^T M^{-1}(C_K W\phi - Y_K).
\]

(16)

By integrating the last equation in the direct time, we get an extremal \( \phi \).

The minimal value of the functional (4) – a normalized action functional:

\[
S = \frac{1}{2} \int_{t_0}^{t_f} \left( C_K W \phi - Y_K \right)^T M^{-1} \left( C_K W \phi - Y_K \right) dt
\]

\[
= \frac{1}{2} \int_{t_0}^{t_f} \left( C_K W \phi - Y_K \right)^T M^{-1} \left( -\frac{d}{dt} M \right) M^{-1} \left( C_K W \phi - Y_K \right) dt.
\]

(17)

Taking into account the rule of differentiation of inverse matrix \( dM^{-1}/dt = M^{-1} \left(-dM/dt\right) M^{-1} \), one can obtain:

\[
S = \frac{1}{2} \int_{t_0}^{t_f} \left( C_K W \phi - Y_K \right)^T \left( \frac{d}{dt} M^{-1} \right) \left( C_K W \phi - Y_K \right) dt = \left. \left( C_K W \phi - Y_K \right)^T M^{-1} \left( C_K W \phi - Y_K \right) \right|_{t=t_0}.
\]

Thus, (17) gives an analytical representation for the normalized action functional.

**Remark 1.** The presented approach for the solution of Lagrange-Pontryagin problem is different from the approach used before for linear cases in the fact that it uses control in the form of a feedback instead of a programmable control. This eliminates the need to calculate the boundary value in the final time point \( t_K \) for the conjugate variable, which is the most time-consuming task in the nonlinear problems.

4. An example for the vessel tilting prediction problem

Rolling motion of the vessel in the conditions of sea and wind disturbance can be presented by a second-order system in the form of (8) with matrices [1]:

\[
A(\phi) = A = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & -2h \end{pmatrix}, \quad B(\phi) = \begin{pmatrix} 0 & 0 & \varepsilon \\ \mu \phi & \gamma & 0 \end{pmatrix},
\]

(18)
where the parameters are equal $a_0^2 = 0.36$, $h = 0.0315$, $\gamma = \mu = 0.1$, $\varepsilon = 10^{-3}$, $\phi_1$ is the first element of the vector $\mathbf{\phi}$. The problem is to predict the critical value of the rolling angle $\gamma = C\phi_1(t_f) = \theta_f$, $\theta_f = 0.5$ rad.

Numerical results of the solution of the problem (8), (4), (7) for the given system (18) are shown in fig. 1, where $\phi_1 = \theta$ is an extremal for the rolling angle; $u_1, u_2$ are correspondingly the disturbance of the sea waves and wind fluctuation. The lower curve $\ln P$ is an action functional, which gives owing to (6) the estimation of the logarithm of the critical roll probability. Corresponding graphs, which were obtained by means of solution of Lagrange-Pontryagin problem for bilinear systems based on the programmable control from [1], are shown in these oscillograms with dashed lines.

![Figure 1. Oscillograms for the vessel tilting prediction problem.](image-url)
5. Conclusion
The paper proposes a method of large deviations analysis based on solving the Lagrange-Pontryagin optimal control problem for nonlinear systems in the form of state-dependent coefficients. The feature of the proposed method is to find the control in the form of a feedback (not in the form of a programmable control), which simplifies the solution of the Lagrange-Pontryagin problem, eliminates the need to calculate the boundary values for the conjugate variables.

It is assumed that this feature can be used in solving problems of a long-term forecast of the critical situations occurrence based on the large deviations analysis. This will be the subject of further research.

6. Acknowledgments
The reported study was funded by the Russian Science Foundation (Project No.17-11-01220).

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