NONCLASSICAL TYPE REPRESENTATIONS OF THE \( q \)-DEFORMED ALGEBRA \( U'_q(\mathfrak{so}_n) \)

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Abstract

The nonstandard \( q \)-deformation \( U'_q(\mathfrak{so}_n) \) of the universal enveloping algebra \( U(\mathfrak{so}_n) \) has irreducible finite dimensional representations which are a \( q \)-deformation of the well-known irreducible finite dimensional representations of \( U(\mathfrak{so}_n) \). But \( U'_q(\mathfrak{so}_n) \) also has irreducible finite dimensional representations which have no classical analogue. The aim of this paper is to give these representations which are called nonclassical type representations. They are given by explicit formulas for operators of the representations corresponding to the generators of \( U'_q(\mathfrak{so}_n) \).

1. In [1] it was constructed a \( q \)-deformation \( U'_q(\mathfrak{so}_n) \) of the universal enveloping algebra \( U(\mathfrak{so}_n) \) which differ from the quantum algebra \( U_q(\mathfrak{so}_n) \) introduced by V. Drinfeld [2] and M. Jimbo [3] (see also [4]). The algebra \( U'_q(\mathfrak{so}_n) \) permits to construct the reduction of \( U'_q(\mathfrak{so}_n) \) onto \( U'_q(\mathfrak{so}_{n-1}) \) which can be used for construction of an analogue of Gel’fand–Tsetlin bases for irreducible representations.

In the classical case, the imbedding \( SO(n) \subset SU(n) \) (and its infinitesimal analogue) is of great importance for nuclear physics and in the theory of Riemannian spaces. It is well known that in the framework of Drinfeld–Jimbo quantum groups and algebras one cannot construct the corresponding imbedding. The algebra \( U'_q(\mathfrak{so}_n) \) allows to define such an imbedding [5], that is, it is possible to define the imbedding \( U'_q(\mathfrak{so}_n) \subset U_q(\mathfrak{sl}_n) \), where \( U_q(\mathfrak{sl}_n) \) is the Drinfeld-Jimbo quantum algebra.

As a disadvantage of the algebra \( U'_q(\mathfrak{so}_n) \) we have to mention the difficulties with Hopf algebra structure. Nevertheless, \( U'_q(\mathfrak{so}_n) \) turns out to be a coideal in \( U_q(\mathfrak{sl}_n) \).

Finite dimensional irreducible representations of the algebra \( U'_q(\mathfrak{so}_n) \) were constructed in [1]. The formulas of action of the generators of \( U'_q(\mathfrak{so}_n) \) upon the basis (which is a \( q \)-analogue of the Gel’fand–Tsetlin basis) are given there. A proof of these formulas and some their corrections were given in [6]. However, finite dimensional irreducible representations described in [1] and [6] are representations of the classical type. They are \( q \)-deformations of the corresponding irreducible representations of the Lie algebra \( \mathfrak{so}_n \), that is, at \( q \to 1 \) they turn into representations of \( \mathfrak{so}_n \).

The algebra \( U'_q(\mathfrak{so}_n) \) has other classes of finite dimensional irreducible representations which have no classical analogue. These representations are singular at the limit \( q \to 1 \). The aim of this paper is to describe these representations of \( U'_q(\mathfrak{so}_n) \).

Note that the description of these representations for the algebra \( U'_q(\mathfrak{so}_3) \) is given in [7]. A classification of irreducible \( * \)-representations of real forms of the algebra \( U'_q(\mathfrak{so}_3) \) is given in [8].
The basis element defined by tableau \( \{m\} \) in the classical case, its elements are labelled by Gel'fand–Tsetlin tableaux \( U \). The subalgebras \( n \) for the algebras by a presence of nonzero right hand side and by possibility of the reduction relations in the approach of Drinfeld [2] and Jimbo [3] to quantum orthogonal algebras by a presence of nonzero right hand side and by possibility of the reduction relations defining the universal enveloping algebra \( U(so_n) \).

Note also that relations (1) and (2) principally differ from the \( q \)-deformed Serre relations in the approach of Drinfeld [2] and Jimbo [3] to quantum orthogonal algebras by a presence of nonzero right hand side and by possibility of the reduction \( U_q'(so_n) \supset U_q'(so_{n-1}) \supset \cdots \supset U_q'(so_2) \).

2. In this section we describe irreducible finite dimensional representations of the algebras \( U_q'(so_n) \), \( n \geq 3 \), which are \( q \)-deformations of the finite dimensional irreducible representations of the Lie algebra \( so_n \). They are given by sets \( m_n \) consisting of \( \{n/2\} \) numbers \( m_{1,n}, m_{2,n}, \ldots, m_{n/2,n} \) (here \( \{n/2\} \) denotes integral part of \( n/2 \)) which are all integral or all half-integral and satisfy the dominance conditions

\[
m_{1,2p+1} \geq m_{2,2p+1} \geq \ldots \geq m_{p,2p+1} \geq 0, \quad (4)
\]

\[
m_{1,2p} \geq m_{2,2p} \geq \ldots \geq m_{p-1,2p} \geq |m_{p,2p}| \quad (5)
\]

for \( n = 2p + 1 \) and \( n = 2p \), respectively. These representations are denoted by \( T_{m_n} \). For a basis in a representation space we take the \( q \)-analogue of the Gel'fand–Tsetlin basis which is obtained by successive reduction of the representation \( T_{m_n} \) to the subalgebras \( U_q'(so_{n-1}), U_q'(so_{n-2}), \ldots, U_q'(so_2), U_q'(so_2) := U(so_2) \). As in the classical case, its elements are labelled by Gel'fand–Tsetlin tableaux

\[
\{\xi_n\} \equiv \left\{ \begin{array}{c} m_n \\ m_{n-1} \\ \ldots \\ m_2 \end{array} \right\} \equiv \{m_n, \xi_{n-1}\} \equiv \{m_n, m_{n-1}, \xi_{n-2}\}, \quad (6)
\]

where the components of \( m_n \) and \( m_{n-1} \) satisfy the "betweenness" conditions

\[
m_{1,2p+1} \geq m_{1,2p} \geq m_{2,2p+1} \geq \ldots \geq m_{p,2p+1} \geq m_{p,2p} \geq -m_{p,2p+1}, \quad (7)
\]

\[
m_{1,2p} \geq m_{1,2p-1} \geq m_{2,2p} \geq \ldots \geq m_{p-1,2p} \geq |m_{p,2p}|. \quad (8)
\]

The basis element defined by tableau \( \{\xi_n\} \) is denoted as \( |\{\xi_n\}| \) or simply as \( |\xi_n| \). It is convenient to introduce the so-called \( l \)-coordinates

\[
I_{j,2p+1} = m_{j,2p+1} + p - j + 1, \quad I_{j,2p} = m_{j,2p} + p - j, \quad (9)
\]
for the numbers $m_{i,k}$. In particular, $l_{1,3} = m_{1,3} + 1$ and $l_{1,2} = m_{1,2}$. The operator $T_{m_n}(I_{2p+1,2p})$ of the representation $T_{m_n}$ of $U_q'(so_n)$ acts upon Gelfand–Tsetlin basis elements, labelled by (6), by the formula

$$T_{m_n}(I_{2p+1,2p})|ξ_n⟩ = \sum_{j=1}^{p} \frac{A^j_{2p}(ξ_n)}{q^{l_{i,2p}} + q^{-l_{i,2p}}} |(ξ_n)^{+j}_{2p}⟩ - \sum_{j=1}^{p} \frac{A^j_{2p}(ξ_n)^{-j}_{2p}}{q^{l_{i,2p}} + q^{-l_{i,2p}}} |(ξ_n)^{-j}_{2p}⟩ \quad (10)$$

and the operator $T_{m_n}(I_{2p,2p-1})$ of the representation $T_{m_n}$ acts as

$$T_{m_n}(I_{2p,2p-1})|ξ_n⟩ = \sum_{j=1}^{p-1} \frac{B^j_{2p-1}(ξ_n)}{[2l_{j,2p-1} - 1][l_{j,2p-1}]} |(ξ_n)^{+j}_{2p-1}⟩ \quad (11)$$

In these formulas, $(ξ_n)^{+j}_{k}$ means the tableau (6) in which $j$-th component $m_{j,k}$ in $m_k$ is replaced by $m_{j,k} \pm 1$. The coefficients $A^j_{2p}, B^j_{2p-1}, C_{2p-1}$ in (10) and (11) are given by the expressions

$$A^j_{2p}(ξ_n) = \left( \frac{\prod_{i=1}^{p} [l_{i,2p+1} + l_{i,2p}][l_{i,2p+1} - l_{i,2p} - 1] \prod_{i \neq j}^{p} [l_{i,2p+1} + l_{i,2p}][l_{i,2p+1} - l_{i,2p} - 1]}{\prod_{i=1}^{p} [l_{i,2p} + l_{i,2p}][l_{i,2p} - l_{i,2p}][l_{i,2p} + l_{i,2p}]^2} \right)^{1/2} \quad (12)$$

and

$$B^j_{2p-1}(ξ_n) = \left( \frac{\prod_{i=1}^{p} [l_{i,2p} + l_{i,2p-1}][l_{i,2p} - l_{i,2p} - 1] \prod_{i \neq j}^{p} [l_{i,2p+1} + l_{i,2p}][l_{i,2p+1} - l_{i,2p} - 1]}{\prod_{i=1}^{p} [l_{i,2p-1} + l_{i,2p}][l_{i,2p-1} - l_{i,2p}]^2} \right)^{1/2}, \quad (13)$$

$$C_{2p-1}(ξ_n) = \frac{\prod_{i=1}^{p} [l_{i,2p}][l_{i,2p} - 1]}{\prod_{i=1}^{p} [l_{i,2p} - 1]} \quad (14)$$

where numbers in square brackets mean $q$-numbers defined by

$$[a] := (q^a - q^{-a})/(q - q^{-1}).$$

It is seen from (9) that $C_{2p-1}$ in (14) identically vanishes if $m_{p,2p} \equiv l_{p,2p} = 0$.

A proof of the fact that formulas (10)--(14) indeed determine a representation of $U_q'(so_n)$ is given in [6].

3. The representations of the previous section are called representations of the classical type, since under the limit $q \to 1$ the operators $T_{m_n}(I_{j,j-1})$ turn into the
corresponding operators $T_{m_n}(I_{j,j-1})$ for irreducible finite dimensional representations with highest weights $m_n$ of the Lie algebra $so_n$.

The algebra $U'_q(so_n)$ also has irreducible finite dimensional representations $T$ of nonclassical type, that is, such that the operators $T(I_{j,j-1})$ have no classical limit $q \to 1$. They are given by sets $\epsilon := (\epsilon_2,\epsilon_3,\ldots,\epsilon_n)$, $\epsilon_i = \pm 1$, and by sets $m_n$ consisting of \{n/2\} half-integral numbers $m_{1,n}, m_{2,n}, \ldots, m_{(n/2),n}$ (here \{n/2\} denotes integral part of n/2) that satisfy the dominance conditions

$$m_{1,2p+1} \geq m_{2,2p+1} \geq \ldots \geq m_{p,2p+1} \geq \frac{1}{2}, \quad (15)$$

$$m_{1,2p} \geq m_{2,2p} \geq \ldots \geq m_{p-1,2p} \geq m_{p,2p} \geq \frac{1}{2} \quad (16)$$

for $n = 2p+1$ and $n = 2p$, respectively. These representations are denoted by $T_{\epsilon,m_n}$.

For a basis in the representation space we use the analogue of the basis of the previous section. Its elements are labelled by tableaux

$$\{\xi_n\} \equiv \begin{cases} m_n \\ m_{n-1} \\ \ldots \\ m_2 \end{cases} \equiv \{m_n,0\} \equiv \{m_n,m_{n-1},m_{n-2}\}, \quad (17)$$

where the components of $m_{2p+1}$ and $m_{2p}$ satisfy the "betweenness" conditions

$$m_{1,2p+1} \geq m_{1,2p} \geq m_{2,2p+1} \geq m_{2,2p} \geq \ldots \geq m_{p,2p+1} \geq m_{p,2p} \geq \frac{1}{2}, \quad (18)$$

$$m_{1,2p} \geq m_{1,2p-1} \geq m_{2,2p} \geq m_{2,2p-1} \geq \ldots \geq m_{p-1,2p-1} \geq m_{p,2p}. \quad (19)$$

The basis element defined by tableau $\{\xi_n\}$ is denoted as $|\xi_n\rangle$.

As in the previous section, it is convenient to introduce the $l$-coordinates

$$l_{j,2p+1} = m_{j,2p+1} + p - j + 1, \quad l_{j,2p} = m_{j,2p} + p - j. \quad (20)$$

The operator $T_{\epsilon,m_n}(I_{2p+1,2p})$ of the representation $T_{\epsilon,m_n}$ of $U_q(so_n)$ acts upon our basis elements, labelled by (17), as

$$T_{\epsilon,m_n}(I_{2p+1,2p})|\xi_n\rangle = \delta_{m_{p,2p},1/2} \frac{\epsilon_{2p+1}}{q^{1/2} - q^{-1/2}} D_{2p}(\xi_n)|\xi_n\rangle +$$

$$+ \sum_{j=1}^{p} \frac{A_{2p}^{j}(\xi_n)}{q^{l_{j,2p}} - q^{-l_{j,2p}}}|(\xi_n)^{+j}\rangle - \sum_{j=1}^{p} \frac{A_{2p}^{j}(\xi_n)^{-j}}{q^{l_{j,2p}} - q^{-l_{j,2p}}}|(\xi_n)^{-j}\rangle, \quad (21)$$

where $\delta_{m_{p,2p},1/2}$ is the Kronecker symbol and the summation in the last sum must be from 1 to $p-1$ if $m_{p,2p} = 1/2$. The operator $T_{\epsilon,m_n}(I_{2p,2p-1})$ of the representation $T_{\epsilon,m_n}$ acts as

$$T_{\epsilon,m_n}(I_{2p,2p-1})|\xi_n\rangle = \sum_{j=1}^{p-1} \frac{B^{j}_{2p-1}(\xi_n)}{[2l_{j,2p-1} - 1][l_{j,2p-1} + 1]}|(\xi_n)^{+j}\rangle -$$

$$- \sum_{j=1}^{p-1} \frac{B^{-j}_{2p-1}(\xi_n)}{[2l_{j,2p-1} - 1][l_{j,2p-1} + 1]}|(\xi_n)^{-j}\rangle.$$

4
these operators determine a representation of $U$ by $T$. In the paper [6], we prove that the operators given by formulas (10)–(14) satisfy the defining relations by the same formulas as in (10) and (11) (that is, by the formulas (12) and (13)) and

$$
A \text{ direct verification shows that two subspaces } \mathcal{H}
$$

respectively, where $\mathcal{H}$ means the tableau (17) in which $j$-th component $m_{j,k}$ in $m_k$ is replaced by $m_{j,k} \pm 1$. The coefficients $A_{2p}^j$ and $B_{2p-1}^j$ in (21) and (22) are given by the same formulas as in (10) and (11) (that is, by the formulas (12) and (13)) and

$$
C_{2p-1}(\xi_n) = \frac{\prod_{s=1}^p [l_s,2p] + \prod_{s=1}^{p-1} [l_s,2p-2] + \prod_{s=1}^p [l_s,2p-1] + \prod_{s=1}^{p-1} [l_s,2p-1 - 1]}{\prod_{s=1}^p [l_s,2p] + \prod_{s=1}^{p-1} [l_s,2p-2] + \prod_{s=1}^p [l_s,2p-1] + \prod_{s=1}^{p-1} [l_s,2p-1 - 1]},
$$

(23)

$$
D_{2p}(\xi_n) = \frac{\prod_{s=1}^p [l_i,2p+1 - \frac{j}{2}] \prod_{s=1}^{p-1} [l_i,2p-1 - \frac{j}{2}]}{\prod_{s=1}^p [l_i,2p] + \frac{j}{2} [l_i,2p - \frac{j}{2}]}. \tag{24}
$$

The fact that the above operators $T'_{m_n}(I_{k,k-1})$ satisfy the defining relations (1)–(3) of the algebra $U_q'(\mathfrak{so}_n)$ is proved in the following way. We take the formulas (10)–(14) for the classical type representations $T_m$ of $U_q'(\mathfrak{so}_n)$ with half-integral $m_{i,n}$ and replace there every $m_{j,2p+1}$ by $m_{j,2p+1} - i\pi/2h$, every $m_{j,2p}$, $j \neq p$, by $m_{j,2p} - i\pi/2h$ and $m_{p,2p}$ by $m_{p,2p} - \epsilon_{2s} \epsilon_{2s+1} \epsilon_{2p} - i\pi/2h$, where each $\epsilon_{2s}$ is equal to $+1$ or $-1$ and $h$ is defined by $q = e^{i\pi h}$. Repeating almost word by word the reasoning of the paper [6], we prove that the operators given by formulas (10)–(14) satisfy the defining relations (1)–(3) of the algebra $U_q'(\mathfrak{so}_n)$ after this replacement. Therefore, these operators determine a representation of $U_q'(\mathfrak{so}_n)$. We denote this representation by $T'_{m_n}$. After a simple rescaling, the operators $T'_{m_n}(I_{k,k-1})$ take the form

$$
T'_{m_n}(I_{2p+1,2p})|\xi_n\rangle = \sum_{j=1}^p \frac{A_{2p}^j(\xi_n)}{q^{j,l_{2p}} - q^{l_{2p},j}}(|\xi_n\rangle^{+j} - |\xi_n\rangle^{-j}),
$$

$$
-\sum_{j=1}^{p-1} \frac{B_{2p-1}^j(\xi_n)}{2l_{2p-1} - 1} (l_{2p-1})_+ |(\xi_n\rangle^{+j} - |\xi_n\rangle^{-j}) + \epsilon_{2p} \hat{C}_{2p-1}(\xi_n)|\xi_n\rangle,
$$

where $A_{2p}^j$, $B_{2p-1}^j$ and $\hat{C}_{2p-1}$ are such as in the formulas (21) and (22). The representations $T'_{m_n}$ are reducible. We decompose these representations into subrepresentations in the following way. We fix $p$ $(p = 1, 2, \cdots, \lfloor (n-1)/2 \rfloor)$ and decompose the linear space $\mathcal{H}$ of the representation $T'_{m_n}$ into direct sum of two subspaces $\mathcal{H}_{\epsilon_{2p+1}}$, $\epsilon_{2p+1} = \pm 1$, spanned by the basis vectors

$$
|\xi_n\rangle_{\epsilon_{2p+1}} = |\xi_n\rangle - \epsilon_{2p+1} |\xi_n\rangle, \quad m_{p,2p} \geq 1/2,
$$

respectively, where $|\xi_n\rangle$ is obtained from $|\xi_n\rangle$ by replacement of $m_{p,2p}$ by $-m_{p,2p}$. A direct verification shows that two subspaces $\mathcal{H}_{\epsilon_{2p+1}}$ are invariant with respect
to all the operators $T'_{m_n}(I_{k,k-1})$. Now we take the subspaces $\mathcal{H}_{2p+1}$ and repeat the same procedure for some $s$, $s \neq p$, and decompose each of these subspaces into two invariant subspaces. Continuing this procedure further we decompose the representation space $\mathcal{H}$ into a direct sum of $2^{(n-1)/2}$ invariant subspaces. The operators $T'_{m_n}(I_{k,k-1})$ act upon these subspaces by the formulas (21) and (22). We denote the corresponding subrepresentations on these subspaces by $T_{\epsilon,m_n}$. The above reasoning shows that the operators $T_{\epsilon,m_n}(I_{k,k-1})$ satisfy the defining relations (1)–(3) of the algebra $U'_q(\mathfrak{so}_n)$.

**Theorem 1.** The representations $T_{\epsilon,m_n}$ are irreducible. The representations $T_{\epsilon,m_n}$ and $T'_{\epsilon',m'_n}$ are pairwise nonequivalent for $(\epsilon,m_n) \neq (\epsilon',m'_n)$. For any admissible $(\epsilon,m_n)$ and $m'_n$ the representations $T_{\epsilon,m_n}$ and $T_{m'_n}$ are pairwise nonequivalent.

The algebra $U'_q(\mathfrak{so}_n)$ has non-trivial one-dimensional representations. They are special cases of the representations of the nonclassical type. They are described as follows.

Let $\epsilon := (\epsilon_2,\epsilon_3,\ldots,\epsilon_n)$, $\epsilon_i = \pm 1$, and let $m_n = (m_{1,n},m_{2,n},\ldots,m_{\{n/2\},n}) = (\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2})$. Then the corresponding representations $T_{\epsilon,m_n}$ are one-dimensional and are given by the formulas

$$T_{\epsilon,m_n}(I_{k+1,k})|\xi_n\rangle = \frac{\epsilon_{k+1}}{q^{1/2} - q^{-1/2}}|\xi_n\rangle.$$

Thus, to every $\epsilon := (\epsilon_2,\epsilon_3,\ldots,\epsilon_n)$, $\epsilon_i = \pm 1$, there corresponds a one-dimensional representation of $U'_q(\mathfrak{so}_n)$.

**Conjecture.** If $q$ is not a root of unity, then every irreducible finite dimensional representation of $U'_q(\mathfrak{so}_n)$ is equivalent to one of the representations $T_{m_n}$ of the classical type or to one of the representations $T_{\epsilon,m_n}$ of the nonclassical type.

The research of this publication was made possible in part by Award No. UP1–309 of CRDF and by Award No. 1.4/206 of Ukrainian DFFD.

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