Proper $G_a$-actions on $\mathbb{C}^4$ preserving a coordinate

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Dedicated to my teachers

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ABSTRACT. We prove that the actions mentioned in the title are translations. We show also that for certain $G_a$-actions on affine fourfolds the categorical quotient of the action is automatically an affine algebraic variety and describe the geometric structure of such quotients.

INTRODUCTION

An algebraic action $G_a$ of the additive group $\mathbb{C}_+$ of complex numbers on a complex algebraic variety $X$ is free if it has no fixed points. When $X$ is a Euclidean space $\mathbb{C}^n$ with a coordinate system $(x_1, \ldots, x_n)$ the simplest example of such an action is a translation for which the action of an element $t \in \mathbb{C}_+$ is given by $(x_1, x_2, \ldots, x_n) \rightarrow (x_1 + t, x_2, \ldots, x_n)$. It turns out that for $n \leq 3$ these notions are “essentially” the same. More precisely, when $n \leq 3$ every nontrivial free $G_a$-action on $\mathbb{C}^n$ in a suitable polynomial coordinate system is a translation (see [23] and [40] for $n = 2$, and [30] for $n = 3$).

Starting with $n = 4$ the similar statement does not hold and the basic example of Winkelmann [44] gives a triangular free $G_a$-action which is not a translation. In his example the geometric quotient of the action is not Hausdorff while for a translation on $\mathbb{C}^n$ the geometric quotient is isomorphic to $\mathbb{C}^{n-1}$. Recently Dubouloz, Finston, and Jaradat [12] proved that every triangular action on $\mathbb{C}^n$, which is proper (in particular, it is free and has a Hausdorff geometric quotient), is a translation in a suitable coordinate system. Note that every triangular action preserves at least one of coordinates, and

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1In fact, for $n \leq 3$ every connected one-dimensional unipotent algebraic subgroup of Cremona group of $\mathbb{C}^n$ is conjugate to such a translation [39, Corollary 5].

2Recall that a $G_a$-action on $\mathbb{C}^n$ is triangular if in a suitable polynomial coordinate system it is of form $(x_1, \ldots, x_n) \rightarrow (x_1, x_2 + tp_2(x_1), x_3 + tp_3(x_1, x_2), \ldots, x_n + tp_n(x_1, \ldots, x_{n-1}))$ where each $p_i$ is a polynomial. For $n \geq 3$ not every $G_a$-action on $\mathbb{C}^n$ is triangulable (i.e. triangular in a suitable polynomial coordinate system) [2].
one of the aims of this paper is the following generalization of the Finston-Dubouloz-Jaradat result.

**Theorem 0.1.** Every proper $G_a$-action on $\mathbb{C}^4$ that preserves a coordinate is a translation in a suitable polynomial coordinate system.

Its proof involves investigation of categorical quotients of $G_a$-actions on $\mathbb{C}^4$. Namely, let $X$ be an affine algebraic variety equipped with a $G_a$-action $\Phi : G_a \times X \to X$. Denote by $\mathbb{C}[X]^{\Phi}$ the subring of $\Phi$-invariant regular functions in the ring $\mathbb{C}[X]$ of regular functions on $X$ and by $\text{Spec} \mathbb{C}[X]^{\Phi}$ (resp. $\text{Spec} \mathbb{C}[X]$) the spectrum of $\mathbb{C}[X]^{\Phi}$ (resp $\mathbb{C}[X]$). Then the natural embedding $\mathbb{C}[X]^{\Phi} \hookrightarrow \mathbb{C}[X]$ induces a map $\text{Spec} \mathbb{C}[X] \to \text{Spec} \mathbb{C}[X]^{\Phi}$. The fact that $\mathbb{C}[X]^{\Phi}$ is finitely generated is equivalent to the fact that $\text{Spec} \mathbb{C}[X]^{\Phi}$ can be viewed as an affine algebraic variety denoted by $X//\Phi$. It is called the categorical quotient of the action and the map of the spectra yields the quotient morphism $\varphi : X \to X//\Phi$ in the category of affine algebraic varieties. If $\dim X \leq 3$ then $\mathbb{C}[X]^{\Phi}$ is always finitely generated by a theorem of Zariski [45]. In higher dimensions this fact is not necessarily true by Nagata’s counterexample to the fourteenth Hilbert problem. Furthermore, extending Nagata’s counterexample Daigle and Freudenburg [6] showed for a $G_a$-action on $\mathbb{C}^n$ the ring of invariant functions may not be finitely generated starting from dimension $n \geq 5$. In dimension 4 the same authors showed that the ring of regular functions invariant with respect to a triangular $G_a$-action on $\mathbb{C}^4$ is automatically finitely generated [7] and later Bhatwadekar and Daigle [4] proved that it remains finitely generated if one considers instead of triangular $G_a$-actions the wider class of $G_a$-actions preserving a coordinate. In this paper we establish a stronger fact contained in the next theorem together with a generalization of Theorem 0.1.

**Theorem 0.2.** Let $\varphi : X \to B$ be a surjective morphism of a factorial affine algebraic $G_a$-variety $X$ (i.e. $X$ is equipped with some $G_a$-action $\Phi$) of dimension 4 into a smooth affine curve $B$. Suppose also that

- the action preserves each fiber of $\varphi$;
- the generic fiber of $\varphi$ is a three-dimensional variety $Y$ (over the field $K$ of rational functions on $B$) for which the ring of invariants of the $G_a$-action (induced by $\Phi$) on $Y$ is the polynomial ring $K[z, w]$ in two variables;
- for every $b \in B$ the fiber $X_b = \varphi^{-1}(b)$ admits a nonconstant morphism into a curve if and only if this curve is a polynomial one (i.e. the normalization of the curve is the line $\mathbb{C}$).

Then

1. the ring of $\Phi$-invariant functions is finitely generated and, thus, it can be viewed as the ring of regular functions on an affine algebraic variety $Q = X//\Phi$;

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3Neena Gupta informed the author that she and S. M. Bhatwadekar had obtained recently an independent proof of Theorem 0.1.
4The author is grateful to Neena Gupta for drawing his attention to the paper of Bhatwadekar and Daigle.
5This assumption that the ring of invariants of the induced action is isomorphic to $K[z, w]$ can be replaced by the following: for a general $b \in B$ the categorical quotient of the restriction of $\Phi$ to the fiber $X_b = \varphi^{-1}(b)$ is isomorphic to $\mathbb{C}^2$ (see, Theorem 5.7 below).
(2) there is an affine modification $\psi : Q \to B \times \mathbb{C}^2$ such that for some nonempty Zariski dense subset $B^* \subset B$ the restriction of $\psi$ over $B^*$ is an isomorphism and every singular fiber of $\psi$ is of form $C \times \mathbb{C}$ where $C$ is a polynomial curve.

(3) Furthermore, if one requires additionally that
- $\Phi$ is proper and $X$ is Cohen-Macaulay,
- each fiber $X_b$ is normal,
- and the restriction $\Phi_b$ of $\Phi$ to $X_b$ is a translation (in particular, $X_b$ is naturally isomorphic to a direct product $(X_b/\Phi_b) \times \mathbb{C}$),
then the quotient $Q$ is locally trivial $\mathbb{C}^2$-bundle over $B$ (and in particular it is a vector bundle by [3]), $X$ is naturally isomorphic to $Q \times \mathbb{C}$, and $\Phi$ is generated by a translation on the second factor of $X \simeq Q \times \mathbb{C}$.

Let us emphasize that the fact that $X$ is a direct product $Q \times \mathbb{C}$ is a rather rare event for regular $G_a$-actions while in the category of rational actions of a connected linear algebraic group $G$ on an algebraic variety $Y$ (over an algebraically closed field of any characteristic) this variety is automatically birationally isomorphic to the product of $\mathbb{P}^*$ and the rational quotient of $Y$ with respect to a Borel subgroup of $G$ (see [31] and [38]). It is also worth mentioning that the requirement that the restriction of $\Phi$ to any $X_b$ is a translation in (3) can be can be replaced by some topological assumptions. For instance, if each fiber $X_b$ is smooth and factorial with trivial second and third homology groups then $\Phi|_{X_b}$ is automatically a translation by [30, Theorem 5.1]. In particular, this is true when $X_b$ is a smooth contractible threefold since by the Gurjar theorem [22, Theorem 1] (see also [18]) such a threefold is factorial.

This leads to the following application of Theorem 0.2.

**Corollary 0.3.** Let $\varphi : X \to B$ be a surjective morphism of a smooth factorial affine algebraic fourfold $X$ into a smooth curve $B$ such that $X$ is equipped with a proper $G_a$-action $\Phi$ preserving every fiber $X_b = \varphi^{-1}(b)$, $b \in B$ of $\varphi$. Suppose that each $X_b$ is a smooth contractible threefold such that the restriction of $\Phi$ to $X_b$ has the plane as the quotient $^6$.

Then there exits a categorical quotient of the action $Q = X//\Phi$ in the category of affine algebraic varieties. Furthermore, $Q$ is a two-dimensional vector bundle over $B$ and $X$ is naturally isomorphic to $Q \times \mathbb{C}$ while $\Phi$ is generated by a translation on the second factor of $X \simeq Q \times \mathbb{C}$.

Indeed, it is straightforward that the assumptions of Corollary 0.3 imply all assumptions of Theorem 0.2 with a possible exception of the condition on the generic fiber of $\varphi$. But this last condition follows from a combination of the Kambayashi [32] and the Kraft-Russell [33] theorems (see Theorem 5.4 and Example 5.6 below for details).

When $X \simeq \mathbb{C}^4$, $B \simeq \mathbb{C}$, and $\varphi$ is a coordinate function the assumptions of Corollary 0.3 are automatically true and we have Theorem 0.1.

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$^6$For $X_b = \mathbb{C}^3$ the quotient of a nontrivial $G_a$-action is always isomorphic to $\mathbb{C}^2$ due to Miyanishi’s theorem [36] and this is also true for smooth contractible threefolds with negative logarithmic Kodaira dimension [31] like the Russell cubic $\{x + x^2y + z^2 + t^3 = 0\} \subset \mathbb{C}^4_{x,y,z,t}$. 
Some of the results mentioned before (including Theorem 0.1) are extended in the last section of this paper where using the Lefschetz principle we show that similar facts (see, Theorems 9.11 and 9.12) remain valid if we consider varieties \( X \) and \( B \) not over the field of complex numbers \( \mathbb{C} \) but over any (not necessarily algebraically closed) field of characteristic zero.

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1. General facts

**Theorem 1.1.** Let \( \varphi : X \rightarrow Y \) be a birational morphism of irreducible affine algebraic varieties such that \( Y \) is normal and there is a subvariety \( Z \) of \( Y \) with codimension at least 2 for which the restriction of \( \varphi \) to \( X \setminus \varphi^{-1}(Z) \) is a surjective morphism onto \( Y \setminus Z \). Suppose also that for every point \( y \in Y \setminus Z \) the preimage \( \varphi^{-1}(y) \) is finite. Then \( \varphi : X \rightarrow Y \) is an isomorphism.

**Proof.** The Zariski Main theorem (e.g., see [20, Théorème 8.12.6] or [24, Chapter III, Corollary 11.4]) implies that the restriction of \( \varphi \) yields an isomorphism between \( X \setminus \varphi^{-1}(Z) \) and \( Y \setminus Z \). By the Hartogs theorem the composition of the inverse map \( \varphi^{-1} : Y \setminus Z \rightarrow X \setminus \varphi^{-1}(Z) \) with the inclusion \( X \setminus \varphi^{-1}(Z) \hookrightarrow X \) extends to a morphism \( Y \rightarrow X \) and we are done. \( \square \)

**Corollary 1.2.** Let \( \varphi : X \rightarrow Y \) be a morphism of irreducible affine algebraic varieties, and \( E \subset X \) and \( D \subset Y \) be irreducible divisors such that the restriction of \( \varphi \) yields an isomorphism \( X \setminus E \rightarrow Y \setminus D \). Suppose also that \( Y \) is normal and \( \varphi(E) \) is Zariski dense in \( D \). Then \( \varphi : X \rightarrow Y \) is an isomorphism.

**Proof.** The dimension argument implies that for a general point \( y \) in \( D \) the preimage \( \varphi^{-1}(y) \) is finite. Hence it is finite outside a proper closed subvariety \( Z \) of \( D \) and we are done by Theorem 1.1. \( \square \)

The next fact and the modified proof of Theorem 1.4 below were suggested by the referee.

**Proposition 1.3.** Let \( \varrho : X \rightarrow Q \) be a morphism of irreducible affine algebraic varieties such that \( Q \) is normal and \( R = Q \setminus \varrho(X) \) is of codimension at least 2 in \( Q \). Then every regular function \( f \) on \( X \) which is constant on the general fibers of \( \varrho \) descends to a unique regular function \( g \) on \( Q \).

**Proof.** Let us show first that \( f \) is a lift of a rational function \( g \) on \( Q \). That is, one needs to establish the regularity of \( g \) on a Zariski dense open subset \( Q_0 \) of \( Q \setminus R \). We can suppose that the restriction of \( g \) to \( X_0 = g^{-1}(Q_0) \) is flat by [20, Théorème 6.9.1] and, therefore, being surjective it is faithfully flat [1, Ch. 3, Exercise 16].
Consider $Y = X_0 \times_{Q_0} X_0$. Then the two natural projections $Y \to X_0$ generate two homomorphisms $e_1 : \mathbb{C}[X_0] \to \mathbb{C}[Y]$ and $e_2 : \mathbb{C}[X_0] \to \mathbb{C}[Y]$. Since $f$ is constant on general fibers of $g$ we see that $f|_{X_0} \in \text{Ker}(e_1 - e_2)$. In combination with the faithful flatness of the natural morphism $\mathbb{C}[Q_0] \to \mathbb{C}[X_0]$ this implies that $f$ must be a lift of a regular function $g|_{Q_0}$ [15, Lemma 2.61]. Hence $g$ is a rational function on $Q$.

Assume that there exists a divisor $T$ in $Q$ such that a general point $t_0 \in T$ does not belong to the indeterminacy set of $g$ and $g(t_0) = \infty$. We can suppose that also $t_0 \notin R$, i.e. $g^{-1}(t_0) \neq \emptyset$. Hence for a germ of a curve $C$ in $X$ through $x_0 \in g^{-1}(t_0)$ one has $f(c) \to f(x_0)$ as $c \in C$ approaches $x_0$. This implies that $g(g(c)) \to f(x_0) \neq \infty$ as $g(c)$ approaches $t_0$. A contradiction. That is, $g$ is regular on $Q$ outside a subvariety of codimension at least 2. Since $Q$ is normal the Hartogs’ theorem implies that $g$ is regular on $Q$ and we are done.

\[ \square \]

**Theorem 1.4.** Let $g : X \to Q$ be a morphism of irreducible affine algebraic varieties and $\Phi : G_a \times X \to X$ be a non-trivial $G_a$-action on $X$ which preserves each fiber of $g$. Let $Q$ be normal and $P$ be a subvariety of $Q$ with codimension at least 2 such that $Q \setminus P \subset g(X)$. Suppose also that for every point $q \in Q \setminus P$ the preimage $g^{-1}(q)$ is a curve and for a general point $q \in Q \setminus P$ the preimage $g^{-1}(q)$ is an irreducible curve. Then $g : X \to Q$ is the categorical quotient morphism in the category of affine algebraic varieties. In particular, the subring of $G_a$-invariants in the ring of regular functions on $X$ is finitely generated.

**Proof.** Note that by the assumption every general fiber of $g$ is nothing but an orbit of $\Phi$. Thus every $\Phi$-invariant function $f$ is constant on the general fibers of $g$. By Proposition 1.3 $f$ is a lift of a regular function on $Q$. Hence the ring of regular functions on $Q$ coincides with the ring $\mathbb{C}[X]^\Phi$ of invariants and we are done.

\[ \square \]

Now the dimension argument as in the proof of Corollary 1.2 implies the following.

**Corollary 1.5.** Let $g : X \to Q$ be a morphism of irreducible affine algebraic varieties such that $Q$ is normal and let $\Phi : G_a \times X \to X$ be a non-trivial $G_a$-action on $X$ preserving every fiber of $g$. Let $E \subset X$ and $D \subset Q$ be irreducible divisors such that $X \setminus E$ and $Q \setminus D$ are affine algebraic varieties for which $g(X \setminus E) = Q \setminus D$ and $g|_{X \setminus E} : X \setminus E \to Q \setminus D$ is the categorical quotient morphism of the action $\Phi|_{X \setminus E}$. Suppose also that $g(E)$ is Zariski dense in $D$.

Then $g : X \to Q$ is the categorical quotient morphism of $\Phi$.

**Proposition 1.6.** (cf. [30, Lemma 2.1]) Let $g : X \to Q$ be a dominant morphism of normal affine algebraic varieties. Suppose that the general fibers of $g$ are irreducible and there are no non-constant invertible regular function on such fibers. Suppose also that $Q \setminus g(X)$ is of codimension at least 2 in $Q$. Then

1. for every principal irreducible divisor $D$ in $X$, which does not meet general fibers of $g$, the closure of $g(D)$ is the support of a principal irreducible divisor in $Q$;
2. if $X$ is a factorial variety so is $Q$;
(3) if $X$ is factorial the preimage of any irreducible reduced divisor $T$ in $Q$ is an irreducible reduced divisor in $X$.

Proof. In (1) $D$ is the zero locus of a regular function $f$. Since $f$ does not vanish on general fibers and they are irreducible we see that $f$ is constant on each general fiber. By Proposition $\boxed{1.3}$ $f = g \circ \varrho$ where $g \in \mathbb{C}[Q]$. Let us show that the zero locus of $g$ coincides with the closure of $\varrho(D)$. Assume to the contrary that there is a divisor $F \subset \varrho^{-1}(0) \setminus \varrho(D)$ in $Q$. Choose a rational function $h$ on $Q$ whose poles are contained in the closure $T$ of $F$ and a regular function $e$ on $Q$ that vanishes on $T \cap \varrho(D) = T \cap \varrho(X)$ but not at general points of $T$. Then for sufficiently large $k$ the function $(e^k h) \circ \varrho$ is regular on $X$. Since it is constant on every general fiber of $X$ we conclude by Proposition $\boxed{1.3}$ that $e^k h$ is regular on $Q$ contrary to the fact that this function has poles on $T$. Thus we have (1).

Let $T$ be an irreducible reduced Weil divisor in $Q$ and let $D$ be an irreducible component of $\varrho^{-1}(T)$ that is a divisor in $X$ (such a component exists because $Q \setminus \varrho(X)$ is of codimension at least 2 in $Q$). Under the assumption of (2) $D = f^*(0)$ for a regular function $f$ on $X$ since $X$ is factorial. By (1) $T = \varrho(D)$ and it coincides with the zeros of $g \in \mathbb{C}[X]$ where $f = g \circ \varrho$. Furthermore, if $e$ is another regular function on $Q$ that vanishes on $T$ then it is divisible by $g$ since $e \circ \varrho$ is divisible by $f$, and by Proposition $\boxed{1.3}$ $e \circ g \circ \varrho$ is the lift of a regular function on $Q$. That is, $T = g^*(0)$. This implies that every irreducible element in $\mathbb{C}[Q]$ has a zero locus which is reduced irreducible and principal. In particular, this element is prime and, therefore, we have (2).

Assume that $D_1$ and $D_2$ are reduced irreducible components of $\varrho^{-1}(T)$. Since $X$ is factorial $D_i = f_i^*(0)$. By the argument above $f_i = g_i \circ \varrho$ for regular functions $g_i$ on $Q$. Furthermore $g_i$ vanishes on $T$ only and thus $g_1/g_2$ is an invertible regular function. This implies that $D_1 = D_2$ and we have (3).

\[\square\]

**Corollary 1.7.** Let $X$ be a normal affine algebraic variety and $\varrho : X \rightarrow Q$ be a quotient morphism of a nontrivial $G_\alpha$-action on $X$ (in the category of affine algebraic varieties). Then $Q \setminus \varrho(X)$ has codimension at least 2 in $Q$. Furthermore, if $X$ is factorial so is $Q$.

Proof. Note that $Q$ is normal since $X$ is. If $Q \setminus \varrho(X)$ contains a divisor $F$ of $Q$ then as in Statement (1) of Proposition $\boxed{1.6}$ we can construct a rational function $g$ on $Q$ with poles on $F$ whose lift to $X$ is regular. By construction this lift is $G_\alpha$-invariant, i.e. $g$ must be regular. A contradiction. This implies the first statement while the second one follows from Proposition $\boxed{1.6}$.

\[\square\]

2. Basic definitions and properties of affine modifications

**Definition 2.1.** Recall that an affine modification is any birational morphism $\sigma : \hat{X} \rightarrow X$ of affine algebraic varieties [28]. In particular, there exists a divisor $D \subset X$ such that for $\hat{D} = \sigma^{-1}(D)$ the restriction of $\sigma$ yields an isomorphism $\hat{X} \setminus \hat{D} \rightarrow X \setminus D$. There is some freedom in the choice of $D$ and we can always suppose that $D$ is a principal
effective divisor given by zeros of a regular function \( f \in A := \mathbb{C}[X] \). This is the case which we consider in the present paper. When such \( f \) is fixed we call \( D \) the divisor of modification, \( \hat{D} \) is called the exceptional divisor of modification, and the closure \( Z \) of \( \sigma(\hat{D}) \) in \( X \) is called the center of modification. The advantage of a principal divisor \( D \) is that the algebra \( \hat{A} = \mathbb{C}[\hat{X}] \) can be viewed as a subalgebra \( A[\frac{1}{f}] \) in the field \( \text{Frac}(A) \) of fractions of \( A \) where \( I \) is an ideal in \( A \) (see [28]).

It is worth mentioning that \( I \) is not determined uniquely, i.e., one can find another ideal \( J \subset A \) for which \( \hat{A} = A[\frac{1}{f}] \). We suppose further that \( I \) is the largest among such ideals \( J \) and we call \( I \) the ideal of the modification. (Treating \( A \) as a subalgebra of \( \hat{A} \) one can see that \( I \) is the intersection of \( A \) and the principal ideal generated by \( f \) in \( \hat{A} \).)

**Notation 2.2.** The symbols \( X,Z,D,\hat{X},\hat{D},I,f \) in this section have the same meaning as in Definition 2.1

**Example 2.3.** Let \( I \) be generated by regular functions \( f,g_1,\ldots,g_n \in \mathbb{C}[X] \). Consider the closed subspace \( Y = X \times \mathbb{C}^{n}_{v_1,\ldots,v_n} \) given by the system of equations \( f v_i = g_j \) \( (j = 1,\ldots,n) \) and the proper transform \( \hat{X} \) of \( X \) under the natural projection \( Y \to X \) (i.e. \( \hat{X} \) is the only irreducible component of \( Y \) whose image in \( X \) under the projection is dense in \( X \)). Then the restriction \( \sigma : \hat{X} \to X \) of the natural projection is our affine modification. Note that \( D \) (resp. \( \hat{D} \)) coincides with the zero locus of \( f \) (resp. \( f \circ \sigma \)) and the center \( Z \) is given by the equations \( f = g_1 = \ldots = g_n = 0 \).

**Remark 2.4.** (1) The geometrical construction behind the modification in Example 2.3 is the following. Consider blowing-up \( \tau : \tilde{X} \to X \) of \( X \) with respect to the ideal sheaf generated by \( f \) and \( g_1,\ldots,g_n \). Delete from \( \tilde{X} \) divisors on which the zero multiplicity of \( f \circ \tau \) is more than the zero multiplicity of at least one of the functions \( g_j \circ \tau \). The resulting variety is \( \hat{X} \).

(2) Note that the replacement in Example 2.3 of functions \( f,g_1,\ldots,g_n \) by functions \( hf,hg_1,\ldots,hg_n \) respectively (where \( h \in \mathbb{C}[X] \) is nonzero) does not change the modification \( \sigma : \hat{X} \to X \). In order to avoid this ambiguity we have to fix \( f \). We are not going to specify such \( f \)'s for affine modifications considered below since the choice of \( f \) will be clear from the context in each particular case.

**Definition 2.5.** (1) Let the center of modification \( Z \) in Example 2.3 be a set-theoretical complete intersection in \( X \) given by the zeros \( f = g_1 = \ldots = g_n = 0 \) (i.e. \( Z \) coincides with the set of common zeros of these functions and the codimension of \( Z \) in \( X \) is \( n+1 \)). Then we call such \( \sigma : \hat{X} \to X \) a Davis modification. Its main property is that \( \hat{D} \) is naturally isomorphic to \( Y \times \mathbb{C}^n \) and that the support of \( \hat{X} \) coincides with the support of \( Y \). \( ^7 \)

(2) Let \( Z \) be a strict complete intersection in \( X \) given by \( f = g_1 = \ldots = g_n = 0 \); that is, \( Z \) is not only set-theoretical complete intersection but also the defining ideal of \( Z \) in \( Y \). Hence, the preimage of \( Z \) in \( Y \) is naturally isomorphic to \( Z \times \mathbb{C}^n \) and therefore has dimension \( \dim X - 1 \). On the other hand counting the number of equations defining \( Y \) in \( X \times \mathbb{C}^n \) we see that every irreducible component of \( Y \) has dimension at least \( \dim X \). This implies that \( Y \) is irreducible (since all irreducible components of \( Y \) but one are contained in the preimage of \( Z \) and thus \( \hat{D} \simeq Z \times \mathbb{C}^n \).
the (reduced) subvariety $Z$ in $X$ coincides with the ideal $I$ generated by $f, g_1, \ldots, g_n$. Then we call $\sigma$ a simple modification. In this case $\tilde{X}$ coincides with $Y$ as a scheme. Note also that the zero multiplicity of $f \circ \sigma$ is 1 at general points of $\hat{D}$.

Here are some useful properties of simple modifications from [26] which we shall need later.

**Proposition 2.6.** Let $\sigma : \tilde{X} \rightarrow X$ be a simple modification. Then

1. $\tilde{X}$ is smooth over points from $Z_{\text{reg}} \cap D_{\text{reg}} \cap X_{\text{reg}}$;
2. $\tilde{X}$ is Cohen-Macaulay provided $X$ is Cohen-Macaulay;
3. furthermore, if in (2) $X$ is normal and none of irreducible components of the center $Z$ of $\sigma$ is contained in the singularities of $X$ or $D$ then $\tilde{X}$ is normal.

3. **Pseudo-affine modifications**

From a geometrical point of view it is sometimes convenient for us to consider a neighborhood $U \subset X$ of some point $z \in Z$ in the standard topology (we call such $U$ a Euclidean neighborhood) and the restriction $\sigma |_{\hat{U}} : \hat{U} \rightarrow U$ where $\hat{U}$ is any connected component of $\sigma^{-1}(U)$. Since $U$ and $\hat{U}$ are not algebraic varieties but only complex spaces let us consider the analogue of affine modifications in the analytic setting.

**Definition 3.1.** Let $X$ be an irreducible complex Stein space and $\psi : X \rightarrow W$ be a meromorphic map into a projective algebraic variety $W$ with a fixed ample divisor $H$ such that $\psi$ is holomorphic over $W \setminus H$. A minimal resolution $\pi : \tilde{X} \rightarrow X$ of indeterminacy points for $\psi$ leads to a holomorphic map $\tilde{\psi} : \tilde{X} \rightarrow W$. Removing from $\tilde{X}$ the preimage of $H$ we obtain $\hat{X}$ which, together with the natural projection $\sigma : \hat{X} \rightarrow X$, will be called a pseudo-affine modification. Consider the Weil divisor $\tilde{\psi}^*(H)$ and its push-forward (as a cycle) $D$ by $\pi : \tilde{X} \rightarrow X$. Then $D$ is the divisor of the modification, $\hat{D} = \sigma^{-1}(D) \subset \hat{X}$ is its exceptional divisor, and the closure $Z$ of $\sigma(D)$ is its center. The restriction of $\sigma$ induces, of course, a biholomorphism between $\hat{X} \setminus \hat{D}$ and $X \setminus D$. Note also that similarly to the algebraic setting $\sigma |_{\hat{D}} : \hat{D} \rightarrow \sigma(\hat{D})$ is, by construction, the restriction of the proper morphism $\pi |_{T} : T \rightarrow Z$ where $T$ is the closure of $\hat{D}$ in $\hat{X}$. The latter observation is important for the next remark.

**Remark 3.2.** Though we mostly omit explicit formulations it should be emphasized that practically all the facts valid for affine modifications have similar analytic analogues for pseudo-affine modifications (e.g. see Proposition 3.6 below).

**Example 3.3.** Let us switch to the analytic setting in Example 2.3 by assuming that $X$ is a Stein variety, $f, g_1, \ldots, g_n$ are holomorphic functions on $X$. As before $Y$ is given in $X \times \mathbb{C}^{n}$ by equations $fv_j = g_j$ $(j = 1, \ldots, n)$ and $\tilde{X}$ is the proper transform of $X$ under the natural projection $Y \rightarrow X$. Then the restriction $\sigma : \tilde{X} \rightarrow X$ of the natural projection is a pseudo-affine modification with $D$ (resp. $\hat{D}$) being the zero locus of $f$ (resp. $f \circ \sigma$) and the center $Z$ given by the equations $f = g_1 = \ldots = g_n = 0$.\footnote{One can adhere to the algebraic setting by viewing $U$ as an étale neighborhood of $z$ in $X$.}
Definition 3.4. If in Example 3.3 $Z$ is a set-theoretical complete intersection in $X$ given by the zeros $f = g_1 = \ldots = g_n = 0$ then we call $\sigma$ a Davis pseudo-affine modification. If furthermore $Z$ is a strict complete intersection given by these function then $\sigma$ is a simple pseudo-affine modification.

Remark 3.5. (1) For any pair $U, \hat{U}$ as in the beginning of this section the restriction $\hat{U} \to U$ of a simple (resp. Davis) affine modification $\sigma : \hat{X} \to X$ is automatically a simple (resp. Davis) pseudo-affine modification.

(2) Note that when $X, D, Z$ are smooth for a simple pseudo-affine modification then for every point $z \in Z$ the collection $\{f, g_1, \ldots, g_n\}$ can be extended to a local coordinate system in a Euclidean neighborhood $U$ of $z$ in $X$. In particular, if $n = 1$ we can treat $U$ as a germ of $\mathbb{C}^n$ at the origin with coordinates $(u, v, w_1, \ldots, w_{n-2})$ such that $D$ is given in $U$ by $u = 0$, $Z$ by $u = v = 0$, and the preimage of $U$ in $\hat{X}$ is viewed as a subvariety of $U \times \mathbb{C}_w$ given by the equation $uw = v$.

(3) Similarly to the algebraic setting the exceptional divisor of a Davis pseudo-affine modification is the product of its center and a Euclidean space.

The following analytic version of Proposition 2.6 for simple pseudo-affine modifications remains valid with a verbatim proof.

Proposition 3.6. Let $\sigma : \hat{X} \to X$ be a simple pseudo-affine modification of irreducible Stein spaces. Then

1. $\hat{X}$ is smooth over smooth points from $Z$ that are not contained in the singularities of $D$ or $X$;

2. $\hat{X}$ is Cohen-Macauley provided $X$ is Cohen-Macauley;

3. furthermore, if in (2) $X$ is normal and none of irreducible components of the center $Z$ of $\sigma$ is contained in the singularities of $X$ or $D$ then $\hat{X}$ is normal.

Lemma 3.7. Let $\psi : Y \to X$ be a holomorphic map of irreducible normal Stein spaces such that for some principal effective reduced divisor $D \subset X$ and every $x \in X \setminus D$ the preimage $\psi^{-1}(x)$ is a curve. Suppose that the divisor $E = \psi^*(D)$ is reduced and irreducible, $\psi(E)$ is not contained in the singularities of $X$ or $D$, and for a general point $y \in E$ the variety $\psi^{-1}(\psi(y))$ is a surface. Then for such a point $y \in E$ (resp. for $z = \psi(y) \in X$) there exists a local analytic coordinate system $(u', v', v'', \bar{w}')$ on $Y$ at $y$ (resp. $(u, v, \bar{w})$ on $X$ at $z$) where $\bar{w}' = (w'_1, \ldots, w'_{n-2})$ (resp. $\bar{w} = (w_1, \ldots, w_{n-2})$) for which the local coordinate form of $\psi$ is given by

$$(u', v', v'', \bar{w}') \to (u, v, \bar{w}) = (u', (u')^l q(u', v', v'', \bar{w}'), \bar{w}')$$

where $l \geq 1$ is the minimal zero multiplicity of the $(n \times n)$-minors in the Jacobi matrix of $\psi$ at $y$ and the function $q(0, v', v'', \bar{w}')$ depends on $v'$ or $v''$.

Proof. Since $y$ is a general point of $E$ we see that it is a smooth point of $E$ and $X$ and by the assumption $z$ is a smooth point of $Z = \psi(E), D, X$. Thus locally $D$ is given by $u = 0$ and $Z$ by $u = v = 0$ where the holomorphic functions $u$ and $v$ can be included in a local coordinate system $(u, v, \bar{w})$ on $X$ at $z$. If $u' = u \circ \psi$ then by the assumption $E$ is given locally near $y$ by $u' = 0$. Since $\dim E - \dim Z = 2$, by [5]. Appendix, Theorem
2] there exists a local coordinate system \((v', v'', w')\) on \(E\) such that the coordinate form of \(\psi|_E : E \to D\) is given by \(v = 0\) and \(w = w'\). Extending functions \(v', v''\), and \(w'\) holomorphically to a neighborhood of \(y\) in \(Y\) we get a local coordinate system \((u', v', v'', w')\) in this neighborhood. Furthermore, the extension \(w'\) can be chosen as \(w' = w \circ \psi\). Hence \(\psi\) is given locally by \(u = u', w = w'\), and \(v = (u')^l q(u', v', v'', w')\) where \(q(0, v', v'', w')\) is not identically zero. If \(q(0, v', v'', w')\) is independent of \(v'\) or \(v''\) then replacing \(v\) by \(v - u'q(0, w)\) we increase \(l\) (which is at least 1 since otherwise \(\psi(E)\) is not contained in \(Z\)). On the other hand \(l\) cannot be larger than the minimal zero multiplicity of the \((n \times n)\)-minors in the Jacobi matrix of \(\psi\) at \(y\). Hence we can suppose that \(q(0, v', v'', w')\) depends on \(v'\) or \(v''\) in which case \(l\) is such a multiplicity. \(\square\)

**Proposition 3.8.** Let the assumptions of Lemma 3.7 hold. Suppose also that

\[
\xymatrix{ X_m \ar[r]^{\sigma_n} & X_{m-1} \ar[r]^{\sigma_{m-1}} \ar[r] & \cdots \ar[r]^{\sigma_2} & X_1 \ar[r]^{\sigma_1} & X_0 := X }
\]

is a sequence of simple pseudo-affine modifications such that

(i) there exists a holomorphic map \(\psi_m : Y \to X_m\) for which \(\psi = \sigma_1 \circ \cdots \circ \sigma_m \circ \psi_m\);

(ii) for \(\psi_i = \sigma_{i+1} \circ \cdots \circ \sigma_m \circ \psi_m\) the closure of \(\psi_i(E)\) coincides with \(Z_i \subset D_i\) where \(Z_i\) and \(D_i\) are the center and the divisor of \(\sigma_{i+1}\) respectively;

(iii) general points of \(Z_i\) are contained in the smooth parts of \(X_i\) and \(D_i\).

Then such a sequence cannot be extended to the left indefinitely.

**Proof.** Let \(y\) be a general point in \(E\) such that \(\psi_i(y) = z_i \in Z_i\) which implies that near these points \(Z_i, D_i, E, Y, X\) are smooth. By Lemma 3.7 we can consider local coordinate systems \((u', v', v'', w')\) on \(Y\) at \(y\) and \((u_i, v_i, w_i)\) on \(X_i\) at \(z_i\) such that \(\psi_i\) is given locally by \(u_i = u', \bar{w}_i = w'\), and \(v_i = (u')^l q(u', v', v'', w')\). By Remark 3.5 (2) for a local coordinate system \((u_{i+1}, v_{i+1}, w_{i+1})\) on \(X_{i+1}\) the coordinate form of \(\sigma_{i+1}\) is \((u_i, v_i, \bar{w}_i) = (u_{i+1}, u_{i+1} v_{i+1}, \bar{w}_{i+1})\). Hence a local form of \(\psi_{i+1}\) is \(u_{i+1} = u', \bar{w}_{i+1} = w'\), and \(v_{i+1} = (u')^{l_i-1} q(u', v', v'', w')\). That is, in this construction \(l_{i+1} = l_i - 1\). Since such powers cannot be negative we get the desired conclusion. \(\square\)

**Remark 3.9.** In fact, we showed that \(m\) in Proposition 3.8 cannot exceed the minimal zero multiplicity \(l\) of the \((n \times n)\)-minors in the Jacobi matrix of \(\psi\) from Lemma 3.7.

Similarly, we have the following.

**Proposition 3.10.** Let \(\sigma : \hat{X} \to X\) be a pseudo-affine modification with center \(Z\), divisor \(D\), and exceptional divisor \(\hat{D}\). Suppose that none of components of \(Z\) is contained in \(X_{\text{sing}} \cup D_{\text{sing}}\), and the image of every component of \(\hat{D}\) is of codimension 1 in \(D\). Let

\[
\xymatrix{ X_n \ar[r]^{\sigma_n} & X_{n-1} \ar[r]^{\sigma_{n-1}} \ar[r] & \cdots \ar[r]^{\sigma_2} & X_1 \ar[r]^{\sigma_1} & X_0 := X }
\]

be a sequence of simple modifications with similar conditions on centers and such that \(\sigma\) factors through the composition of these simple modifications. Then such a sequence cannot be extended to the left indefinitely without violating the fact that \(\sigma\) factors through the composition.

\(^9\text{Actually, (iii) follows automatically from the Proposition 3.6.}\)
Proof. For local analytic coordinates systems at general points \( y \in \tilde{D} \) and \( z = \sigma(y) \in Z \subset X \) consider the zero multiplicity \( k \) of the Jacobian of \( \sigma \) at \( y \) (clearly this multiplicity is independent of the choice of local coordinates system and the choice of a general point \( y \)) and let \( k_i \) be the similar multiplicity for the modification \( \sigma_i \circ \cdots \circ \sigma_1 : X_i \to X \). Since \( \sigma_i+1 : X_{i+1} \to X_i \) contracts the exceptional divisor in \( X_{i+1} \) we see that \( k_i+1 > k_i \). On the other hand if \( \sigma \) factors through \( \sigma_i \circ \cdots \circ \sigma_1 \) one must have \( k_i \leq k \). This yields the desired conclusion. \( \square \)

**Definition 3.11.** (1) Given two (pseudo-) affine modifications \( \sigma : \tilde{X} \to X \) and \( \delta : \tilde{X} \to X \) with the same center \( Z \) and divisor \( D \subset X \) we say that \( \sigma \) dominates \( \delta \) if it factors through \( \delta \). For instance, consider the normalization \( \nu : \tilde{X}' \to X \) and let \( \delta' = \delta \circ \nu : \tilde{X}' \to X \). Then \( \delta \) is dominated by \( \delta' \).

(2) Let \( f, g, \) and \( h \) be holomorphic functions on a Stein manifold \( X \) and \( \sigma : \tilde{X} \to X \) be a simple pseudo-affine modification with a smooth divisor \( D = f^*(0) \) and a smooth center \( Z \) given by a strict complete intersection \( f = h = 0 \). Suppose that \( Z \) is also a set-theoretical complete intersection given by \( f^k = g = 0 \) and \( \tilde{X} \subset X \times \mathbb{C}w \) is given by \( f^k w = g \). Then we call the Davis modification \( \delta : \tilde{X} \to X \) (induced by the natural projection) homogeneous (of degree \( k \)) if \( fI^{k-1} \) and \( g \) generate the ideal \( I^k \) where \( I \) is the defining ideal of \( Z \) in the ring of holomorphic functions on \( X \).

**Lemma 3.12.** Let \( X \) be a germ of \( \mathbb{C}^n \) at the origin with coordinates \( (u, v, w_1, \ldots, w_n) \) and let \( \sigma : \tilde{X} \to X \) and \( I \) be as in Definition 2.11 \((2)\) with \( f = u \) and \( h = v \). Suppose also that \( g \) and \( \delta \) are as in Definition 2.11 \((2)\) without \( \delta \) being a priori homogeneous. Then \( \delta \) is homogeneous if and only if the function \( g \) is of form \( g = ev^k + a \) where \( a \in fI^{k-1} \) and \( e \) is an invertible holomorphic function.

**Proof.** Since \( f = u \) and \( I \) is generated by \( u \) and \( v \) one can see that \( fI^{k-1} \) is generated by functions \( u^k, u^{k-1}v, \ldots, uv^{k-1} \). That is, the ideal \( I^k \) is generated by \( fI^{k-1} \) and \( v^k \) and also by \( fI^{k-1} \) and \( g \). This is possible iff \( g = ev^k + a \) where \( a \in fI^{k-1} \) and \( e \) is an invertible holomorphic function. \( \square \)

**Proposition 3.13.** Let \( \sigma : \tilde{X} \to X \) and \( \delta : \tilde{X} \to X \) be as in Definition 2.11 \((2)\). Then \( \tilde{X} \) is a normalization of \( X \) and in particular \( \sigma \) dominates \( \delta \).

**Proof.** Since normalization is a local operation, by Remark 3.5 \((2)\) we can view \( X \) as a germ of \( \mathbb{C}^n \) at the origin with coordinates \( (u, v, w_1, \ldots, w_n) \) such that \( D \) is given by \( u = 0, Z \) by \( u = v = 0 \) (i.e. \( \tilde{X} \) can be viewed as the hypersurface in \( X \times \mathbb{C}w \) given by the equation \( uw = v \)). By Lemma 3.12 \( g = ev^k + a \) where \( a \in fI^{k-1} \) and \( e \) is an invertible holomorphic function. Changing \( e \) we can suppose that the Taylor series of \( a \) does not contain monomials divisible by \( v^k \). Hence \( u = g/f^k = g/u^k = ew^k + c \) where \( c \) is a polynomial in \( w \) of degree at most \( k - 1 \) whose coefficients are holomorphic functions on \( X \). Thus \( w \) is integral over the ring of holomorphic functions on \( X \) which concludes the proof. \( \square \)

**Remark 3.14.** (1) Note that for a simple pseudo-affine modification from Definition 3.11 \((2)\) with a given divisor \( D \) the function \( f \) is determined uniquely up to an invertible
factor. This implies that for a given \( k \geq 1 \) the notion of a homogeneous modification of degree \( k \) from Definition 3.11 (2) is determined not as much by \( f \) and \( h \) but by \( D \) and the defining ideal \( I \) (or equivalently the center \( Z \)).

(2) In particular if \( \delta : \hat{X} \to X \) is a pseudo-affine modification with a divisor \( D \) and a center \( Z \) such that \( \dim Z = \dim X - 2 \) then for smooth point \( z \) of \( Z \) that is not in \( X_{\text{sing}} \cup D_{\text{sing}} \) we can say whether \( \delta \) is locally homogeneous at \( z \) or not (where in the former case there exists a Euclidean neighborhood \( U \) of \( z \) in \( X \) such that for every connected component \( \tilde{U} \) of \( \delta^{-1}(U) \) the modification \( \delta|_{\tilde{U}} : \tilde{U} \to U \) is homogeneous).

**Proposition 3.15.** Let \( X \) be a normal irreducible Stein space which is Cohen-Macaulay, \( \delta : \hat{X} \to X \) be a Davis modification with an irreducible divisor \( D \) and a center \( Z \) of codim \( X Z = 2 \) such that none of irreducible component of \( Z \) is contained in \( X_{\text{sing}} \cup D_{\text{sing}} \). Suppose also that \( Z \) is a strict complete intersection of form \( f = h = 0 \) where \( D = f^*(0) \) and that at general points of \( Z \) this modification \( \delta \) is locally homogeneous. Let \( \sigma : \hat{X} \to X \) be a simple pseudo-affine modification associated with divisor \( D \) and center \( Z \). Then \( \hat{X} \) is a normalization of \( \hat{X} \) and thus \( \sigma \) dominates \( \delta \).

**Proof.** First note that \( \hat{X} \) is a normal Stein space by Proposition 3.10. Let \( z \) be a general point of \( Z \), i.e. \( z \) is a smooth point of \( Z \) and it is not contained in some subvariety \( P \subset Z \) of codim \( Z P \geq 1 \) such that \( Z \cap (X_{\text{sing}} \cup D_{\text{sing}}) \subset P \). Then by Proposition 3.13 there exists a Euclidean neighborhood \( U \) of \( z \) in \( X \) such that \( \sigma^{-1}(U) \) is a normalization of \( \delta^{-1}(U) \). That is, we have a biholomorphism \( \psi \) between \( \sigma^{-1}(X \setminus P) \) and a normalization of \( \delta^{-1}(X \setminus P) \). Since \( P \) is of codimension at least 3 in \( X \) and \( \delta \) is a Davis modification the codimension of \( \delta^{-1}(P) \) in \( \hat{X} \) is at least of 2 (because by Remark 3.13 (3) \( \delta^{-1}(P) \simeq P \times \mathbb{C} \)). Hence the Hartogs theorem implies that \( \psi \) extends to a biholomorphism between \( \hat{X} \) and a normalization of \( \hat{X} \) which is the desired conclusion.

\( \square \)

### 4. Semi-finite modifications

**Notation 4.1.** Let \( T \) be a germ of an analytic set at the origin \( o \) of \( \mathbb{C}^n \), \( \text{Hol}(T) \) be the ring of holomorphic functions on \( T \), and \( D = T \times \mathbb{C} \). We consider a hypersurface \( Z \) in \( D \) such that \( \pi|_Z : Z \to T \) is finite where \( \pi : D \to T \) is the natural projection. We also suppose that \( Z \) coincides (as a set) with the zeros of an analytic function \( h \).

**Lemma 4.2.** Let Notation 4.1 hold and \( w \) be a coordinate on the second factor of \( D = T \times \mathbb{C} \). Then \( h \) can be chosen as a monic polynomial in \( w \) with coefficients from \( \text{Hol}(T) \) (i.e. \( h \in \text{Hol}(T)[w] \)).

**Proof.** Let \( \pi^{-1}(o) \cap Z \) consists of points \( z_1, \ldots, z_n \) where \( z_i = (o, w_i) \in T \times \mathbb{C} \). By the Weierstrass preparation theorem for every \( z_i \) there is a Euclidean neighborhood in \( D \) in which \( h \) coincides with \( e_i h_i(w) \) where \( e_i \) is an invertible function and \( h_i \in \text{Hol}(T)[w] \) is a monic polynomial in \( w - w_i \). Note that \( Z \) coincides with the zeros of the product \( h_1 h_2 \cdots h_m \) because of the finiteness of \( \pi|_Z : Z \to T \). Thus replacing \( h \) with the product \( h_1 h_2 \cdots h_m \) we get the desired conclusion.

\( \square \)
Lemma 4.3. Let Notation 4.1 hold and the zero multiplicity of \( h \) at general points of \( Z \) is \( n \). Then \( g = h^{1/n} \) is a holomorphic function on \( D \) and in particular the defining ideal of \( Z \) in the ring \( \text{Hol}(T)[w] \) is the principal ideal generated by \( g \).

Proof. By Lemma 4.2 we can suppose that \( h \) is a monic polynomial in \( w \). Consider the restriction of \( h \) to any fiber. It is a polynomial whose roots have multiplicities divisible by \( n \) (since for general fibers such multiplicities are exactly \( n \)). Thus the restriction of \( g \) to any fiber of \( \pi : D \to T \) can be chosen as a nonzero monic polynomial in \( w \). Therefore, choosing such a restriction on \( \pi^{-1}(o) \) we define by continuity a unique branch of \( h^{1/n} \) on \( D \) as \( g \). Note that \( g \) is holomorphic outside a subset \( K = Z \cap \pi^{-1}(T_{\text{sing}}) \) and because of the assumption on \( h \) we see that \( g = w^k + r_{k-1}w^{k-1} + \ldots + r_1w + r_0 \) where each \( r_i \) is a continuous function on \( T \) holomorphic on \( T \setminus K \). Hence the first statement will follow from the claim.

Claim. Let \( g \) be a (not a priori continuous) function on \( D \) holomorphic outside a proper analytic subset \( K \) that does not contain any fiber of \( \pi \). Suppose also that \( g \) is a polynomial in \( w \). Then \( g \) is holomorphic on \( D \).

Taking a smaller \( T \), if necessary, one can suppose that the image \( K_0 \) of \( K \) under the natural projection \( D \to \mathbb{C}_w \) is relatively compact. In particular, for every \( w_0 \) in \( \mathbb{C} \setminus K_0 \) the function \( r_kw_0^k + r_{k-1}w_0^{k-1} + \ldots + r_1w_0 + r_0 \) is holomorphic on \( T \). Since we have infinite number of such \( w_0 \)'s every coefficient \( r_i \) is also holomorphic on \( T \) which concludes the proof of the Claim.

To see that \( Z \) is a principal divisor in \( D \) consider any function \( f \in \text{Hol}(T)[w] \) vanishing on \( Z \). Then the quotient \( f/g \) is again holomorphic on \( D \) by the Claim which yields the desired conclusion.

\( \Box \)

Lemma 4.4. Let Notation 4.1 hold and \( T \) be singular. Then so is \( Z \).

Proof. Assume that \( Z \) is smooth at some point \( z \in Z \cap \pi^{-1}(o) \). Let \( g \) be a function as in Lemma 4.3. Extend \( g \) to a function \( \tilde{g} \in \text{Hol}(\mathbb{C}^m,o)[w] \) whose zeros define an extension \( \tilde{Z} \) of \( Z \). Note that if the partial derivative of \( \tilde{g} \) with respect to \( w \) is nonzero at \( z \) then by the implicit function theorem \( \tilde{Z} \) is biholomorphic to \( (\mathbb{C}^m,o) \) and therefore \( Z \) is locally biholomorphic to \( T \) which implies that \( Z \) is singular. Hence we assume that this derivative is zero.

Let \( I \) be the defining ideal of \( T \) in \( \text{Hol}(\mathbb{C}^m,o) \) and \( k \) be the dimension of \( T \). Choose any \( m-k \) elements from \( I \) and consider their Jacobi matrix (with respect to coordinates \( z_1, \ldots, z_m \) of \( \mathbb{C}^m \)). Since \( T \) is singular at \( o \) any \((m-k)\)-minor \( M \) of this matrix vanishes at \( o \).

By Lemma 4.3 the defining ideal \( L \) of \( Z \) in \( (\mathbb{C}^m,o) \times \mathbb{C}_w \) is generated by \( I \) and \( \tilde{g} \). Take any \( m+1-k \) elements from \( L \) and consider their Jacobi matrix with respect to coordinate \((w,z_1, \ldots, z_m)\). It follows from the previous argument about the partial derivative of \( \tilde{g} \) with respect to \( w \) and about \((m-k)\)-minor \( M \) that any \((m+1-k)\)-minor of this new Jacobi matrix vanishes at \( o \). This is contrary to the fact that \( Z \) is smooth at \( z \) which concludes the proof.  

\( \Box \)
Example 4.5. Lemma 4.4 is one of the central technical results in this paper. The assumption that $Z$ coincides with zeros of a global analytic function is very important. Indeed, consider the case when $T$ is a semi-cubic parabola $x^2 - y^3 = 0$ in $\mathbb{C}_x^2$. Then there is a closed immersion of $\mathbb{C}$ into $D = T \times \mathbb{C} \subset \mathbb{C}_x^2$, given by $t \to (t^3, t^2, t)$ such that the image is a smooth Weil divisor whose projection to the singular $T$ is finite. Lemma 4.4 is not applicable here because this Weil divisor is not $\mathbb{Q}$-Cartier.

Remark 4.6. If $T$ is not unibranch at $o$ then the argument is much easier and, furthermore, $Z$ is not unibranch at any point $z_0$ above $o$ as well. Indeed, let $T_1$ and $T_2$ be distinct irreducible components of $T$ at $o$. Then $D$ has irreducible branches $T_1 \times \mathbb{C}$ and $T_2 \times \mathbb{C}$ meeting along the line $L = o \times \mathbb{C}$. Since the zeros of $h$ contain the point $z_0 \in L$ but not the line $L$ itself the zeros $Z_i$ of $h$ in $T_i \times \mathbb{C}$ produce two different branches $Z_1$ and $Z_2$ of $Z$ at $z_0$.

Proposition 4.7. Let $T$ be an affine algebraic variety, $D = T \times \mathbb{C}$, and $Z$ be an algebraic hypersurface in $D$ such that $\pi|_Z : Z \to T$ is finite where $\pi : D \to T$ is the natural projection. Suppose that for every point $t_0 \in T$ there is a Euclidian neighborhood $U \subset T$ such that $Z \cap \pi^{-1}(U)$ is an analytic $\mathbb{Q}$-principal divisor in $\pi^{-1}(U)$, i.e. for some natural $k > 0$ and a holomorphic function $g$ on $\pi^{-1}(U)$ the divisor $kZ \cap \pi^{-1}(U)$ coincides with $g^*(0)$.

Then $Z$ is an effective reduced principal divisor in $D$.

Proof. Since the field $\mathbb{C}(Z)$ of rational functions on $Z$ is a finite separable extension of the field $\mathbb{C}(T)$ of rational functions on $T$ there exists $w \in \mathbb{C}(Z)$ such that $\mathbb{C}(Z)$ coincides with the field of rational functions in $w$ with coefficients from $\mathbb{C}(T)$. Let $g(w) = w^n + r_{n-1}(t)w^{n-1} + \ldots + r_1(t)w + r_0(t)$ be the minimal monic polynomial for $w$ over $\mathbb{C}(T)$. In particular $Z \cap \pi^{-1}(U)$ is given by the zeros of this rational function $g$. By Lemma 4.3 $Z \cap \pi^{-1}(U)$ is a principal divisor in $\pi^{-1}(U)$ which implies that the function $g$ is holomorphic. Therefore $g$ is regular (e.g., see [27, Theorem 5]). This yields the desired conclusion.

Definition 4.8. Let $\sigma : \hat{X} \to X$ be an affine modification of normal affine algebraic varieties such that $X$ is Cohen-Macaulay and

(a) its divisor $D = f^*(0)$ (where $f \in \mathbb{C}[X]$) is a reduced principal divisor in $X$ isomorphic to a direct product $D \simeq T \times \mathbb{C}$;

(b) the restriction $\pi|_Z : Z \to T$ of the natural projection $\pi : D \to T$ to the center $Z$ of $\sigma$ is finite;

(c) none of irreducible components of $Z$ is contained in the singularities of $X$ (note that for the singularities of $D$ the similar fact also holds);

(d) for any point $z$ of $Z$ there is a Euclidian neighborhood $U$ in $X$ for which the restriction $\sigma|_U : \hat{U} \to U$ of $\sigma$ to $\hat{U} = \sigma^{-1}(U)$ factors through some Davis pseudo-affine modification $\delta : \hat{U} \to U$ with the following property: the restriction of $\delta$ over a neighborhood $U' \subset U$ of any general point $z' \in Z \cap U$ is homogeneous of degree $k$ (where $k$ does not depend on different irreducible components of $Z \cap U$).

Then we call such a $\sigma$ a semi-finite affine modification.
Proposition 4.9. Let \( \sigma : \hat{X} \to X \) be a semi-finite affine modification with \( D \simeq T \times \mathbb{C} \) and \( Z \) as in Definition 4.8.

Then \( \sigma \) dominates a simple modification (with the same center and divisor) and if \( T \) is singular so is the center \( Z \).

Proof. Let a Davis modification \( \delta : \check{U} \to U \) from Definition 4.8 (d) be defined by the ideal \((f^k, g)\). By Condition (d) \( \delta \) is locally homogeneous at general points of \( Z \cap U \) which in combination with Lemma 3.12 implies that for some \( k \geq 1 \) the divisor \( kZ \cap U \) coincides with \( g^*(0) \cap U \). That is, we have Condition (i) from Proposition 4.7. Hence \( Z = h^*(0) \) where \( h \in \mathbb{C}[D] \). Thus extending \( h \) to a regular function on \( X \) (denoted by the same symbol) we see that \( Z \) is a strict complete intersection given by \( f = h = 0 \). These functions \( f \) and \( h \) induce a simple affine modification \( \tau : X' \to X \) with divisor \( D \) and center \( Z \) such that \( X' \) is a normal affine algebraic variety by Proposition 2.6. Furthermore, \( \hat{X} \) and \( X' \) are also normal as analytic sets by [46, Chapter 13, Theorem 32]. By Proposition 3.15 \( \tau^{-1}(U) \) is an analytic normalization of \( \check{U} \). Since \( \hat{X} \) is normal then the holomorphic map \( \sigma^{-1}(U) \to \check{U} \) (induced by the domination of \( \delta \) by \( \sigma \)) factors through the normalization of \( \check{U} \). This yields a desired holomorphic map from \( \hat{X} \) to \( X' \) which is a morphism since its restriction over \( X \setminus D \) is an algebraic isomorphism. That is, \( \sigma \) dominates \( \tau \). The second statement follows from Lemma 4.4.

\( \square \)

5. Applications of Kambayashi’s theorem

Notation 5.1. In this section \( \varphi : X \to B \) will be a morphism of complex factorial affine algebraic varieties, \( \Phi \) will be a \( G_a \)-action \( \Phi \) on \( X \) which preserves the fibers of \( \varphi \). We do not assume a priori that the \( \mathbb{C}[B] \)-algebra \( \mathbb{C}[X]^{\Phi} \) of \( \Phi \)-invariant regular functions is finitely generated.

However, such an algebra can be viewed (and will be viewed) as a direct limit \( \lim_{\to} A_\alpha \) of its finitely generated \( \mathbb{C}[B] \)-subalgebras \( A_\alpha \) (with respect to the partial order generated by inclusions) where \( \alpha \) belongs to some index set and each \( A_\alpha \) can be treated as the ring \( \mathbb{C}[Q_\alpha] \) of regular functions of some affine algebraic variety \( Q_\alpha \). Replacing \( A_\alpha \) with its integral closure we suppose that each of these \( Q_\alpha \) is normal (the fact that this transition preserves affineness is a standard result, e.g. [13, Theorem 4.14]). Furthermore, by the Rosenlicht Theorem (e.g., see [37, Theorem 2.3]) \( Q_\alpha \) can be chosen so that the morphism \( \varrho_\alpha : X \to Q_\alpha \) induced by the natural embedding \( A_\alpha \subset \mathbb{C}[X] \) separates general orbits of \( \Phi \). As in [16] we introduce the following.

Definition 5.2. A normal affine variety \( Q_\alpha \) as before will be called a partial quotient of \( X \) by \( \Phi \) and the morphism \( \varrho_\alpha : X \to Q_\alpha \) (separating general orbits of \( \Phi \)) will be called a partial quotient morphism.

Remark 5.3. Note that including the coordinate functions of morphism \( \varphi \) into a ring \( A_\alpha \) we can always choose a partial quotient morphism that is constant on the fibers of \( \varphi \).
Theorem 5.4. Let \( \varphi : X \to B \) and \( \Phi \) be as in Notation 5.1 and the generic fiber of \( \varphi \) be a three-dimensional variety \( Y \) (over the field \( K \) of rational functions on \( B \)). Let \( \hat{K} \) be the algebraic closure \( K \) and \( \hat{Y} \) be the variety over \( \hat{K} \) obtained from \( Y \) by the field extension. Suppose that \( \varphi : X \to Q \) is a partial quotient morphism such that \( \varphi \) factors through \( \varphi \). Suppose also that the ring of invariants of the \( G_\varphi \)-action on \( \hat{Y} \) (induced by \( \Phi \)) is a polynomial ring in two variables over \( \hat{K} \). Then there is a morphism \( \psi : Q \to B \times \mathbb{C}^2 \) such that for some nonempty Zariski dense open subset \( B^* \subset B \) the restriction of \( \psi \) over \( B^* \) is an isomorphism.

Proof. By the Kambayashi theorem \([32]\) the ring of invariants of the induced action on \( Y \) is also a polynomial ring in two variables over \( K \). Hence for some \( B^* \) as above and \( X^* = \varphi^{-1}(B^*) \) the induced action on \( X^* \) has the categorical quotient isomorphic to \( B^* \times \mathbb{C}^2 \). Assume that \( B \setminus B^* \) is a principal effective divisor which can be done without loss of generality. Let \( f \) be a regular function on \( B \) with zero locus \( B \setminus B^* \) (we denote the lifts of \( f \) to \( X \) or \( Q \) by the same symbol). For every regular function \( h \in \mathbb{C}[X^*] \) there exists natural \( k \) for which \( f^k h \) extends to a regular function on \( X \). Hence for sufficiently large \( k_1 \) and \( k_2 \) the composition of the quotient morphism \( \varphi_0 : X^* \to B^* \times \mathbb{C}^2 \) with the isomorphism \( \kappa : B^* \times \mathbb{C}^2 \to B^* \times \mathbb{C}^2 \) given by \((b, u_1, u_2) \mapsto (b, f^{k_1} u_1, f^{k_2} u_2)\) extends to a morphism \( \tau : X \to B \times \mathbb{C}^2 \) (note that \( \tau|_{X^*} : X^* \to B^* \times \mathbb{C}^2 \) can be viewed now as the quotient morphism). Similarly, since \( f \in \mathbb{C}[X] \) and each \( u_i \in \mathbb{C}[X^*] \) are \( \Phi \)-invariant we see that \( f^{k_1} u_1 \) and \( f^{k_2} u_2 \) can be viewed as regular functions on the normal variety \( Q \). Thus \( \tau = \psi \circ \theta \) where \( \theta : X \to Q \) and \( \psi : Q \to B \times \mathbb{C} \) are morphisms. By the universal property of quotient morphisms \( \theta|_{X^*} : X^* \to Q \setminus f^{-1}(0) \) factors through \( \tau|_{X^*} \) which implies that \( \psi \) is invertible over \( B^* \). This yields the desired conclusion. \( \square \)

Remark 5.5. By construction \( \psi \) is an affine modification. Another observation that if the general fibers of \( \varphi \) do not admit nonconstant invertible functions then for every irreducible divisor \( T \) in \( B \) the variety \( \varphi^{-1}(T) \) is reduced and irreducible by Proposition 1.6. Actually, in this case there is no need to assume that \( B \) is factorial since it follows automatically from the fact that \( X \) is factorial.

Example 5.6. Let Notation 5.1 hold and every general fiber of \( \varphi \) be isomorphic to the same affine algebraic threefold \( V \). Suppose that \( \Phi \) induces an action on each general fiber of \( \varphi \) with the categorical quotient isomorphic to \( \mathbb{C}^2 \). Then we claim that the above conclusion about the partial quotient \( Q \) as a modification of \( B \times \mathbb{C}^2 \) over \( B \) is valid.

Indeed, by \([33]\) there exists a Zariski dense open subset \( B^* \subset B \) and an unramified covering \( \hat{B}^* \to B^* \) such that \( \hat{X}^* = X \times B^* \). \( \hat{B}^* \) is naturally isomorphic to \( \hat{B}^* \times \mathbb{C}^2 \) over \( B^* \). In particular, the induced action on the generic fiber of \( \hat{X}^* \to \hat{B}^* \) has the ring of invariants isomorphic to the polynomial ring \( \hat{K}[x_1, x_2] \) where \( \hat{K} \) is the field of rational functions on \( \hat{B}^* \). Note that this ring is obtained from the ring of invariants on the generic fiber \( Y \) of \( X^* \to B^* \) via a field extension \([\hat{K} : K]\). Hence by the Kambayashi theorem the latter ring of invariants is \( K[x_1, x_2] \) and we under the assumption of Theorem 5.4.
In fact, as it was suggested by the referee, we can strengthen Theorem 5.4 and Example 5.6 as follows.

**Theorem 5.7.** Let Notation 5.4 hold and let \( \Phi \) induce an action on each general fiber of \( \varphi \) with the categorical quotient isomorphic to \( \mathbb{C}^2 \). Let \( Y \) and \( K \) be as in Theorem 5.4. Then the ring of invariants of the \( G_\alpha \)-action on \( Y \) (induced by \( \Phi \)) is a polynomial ring in two variables over \( K \). In particular, for a partial quotient morphism \( \varrho : X \to Q \) for which \( \varphi \) factors through \( \varrho \) (i.e. for some morphism \( \kappa : Q \to B \) one has \( \varphi = \kappa \circ \varrho \)) there is a morphism \( \psi : Q \to B \times \mathbb{C}^2 \) over \( B \) such that for a nonempty Zariski dense open subset \( B^* \subset B \) the restriction of \( \psi \) over \( B^* \) is an isomorphism.

In preparation for the proof of this theorem we need the next (certainly well-known) fact.

**Proposition 5.8.** Let \( \kappa : Q \to B \) be a dominant morphism of algebraic varieties such that \( Q \) is normal. Then for a general point \( b \in B \) the variety \( \kappa^{-1}(b) \) is normal.

**Proof.** Replacing \( B \) by a Zariski dense open subset (and \( Q \) by its preimage) we can suppose that \( \kappa \) is flat [20, Théorème 6.9.1]. Let \( \omega \) be a generic point of \( B \). Then the fiber of \( \kappa \) over \( \omega \) is normal (e.g., see [11 Proposition 5.13]) which is equivalent to the fact that this fiber is geometrically normal [11] since we work over a field of characteristic zero. On the other hand the set of (not necessarily closed) points in \( B \) for which the fibers over them are geometrically normal is open [21 Théorème 12.1.6]. Since this set is nonempty it contains general (closed) points of \( B \) and we are done. \( \square \)

### 5.1. Proof of Theorem 5.7

Let \( Q_b \) be the fiber \( \kappa^{-1}(b) \) over a general point \( b \in B \) and \( Q^b \simeq \mathbb{C}^2 \) be the categorical quotient of the action \( \Phi|_{\varphi^{-1}(b)} \). By the universal property of quotient morphisms one has a morphism \( \psi_b : Q^b \to Q_b \). Since \( \varrho \) separates general orbits \( \psi_b \) must be birational.

Assume that for a curve \( C_b \subset Q^b \) the image \( \psi_b(C_b) \) is a point \( q_b \in Q_b \). By Corollary 1.7 the preimage of \( C_b \) in \( \varphi^{-1}(b) \) is a surface \( S_b \), i.e. \( \varrho(S_b) = q_b \). By [13 Chapter I, Section 6.3, Corollary] the closure \( T \) of the set \( \{ q \in Q | \dim \varrho^{-1}(q) = 2 \} \) is a subvariety. Let \( L \) be the union of the one-dimensional components of \( T \) (it is non-empty since \( q_b \in T \) for general \( b \)). Note that \( \varrho^{-1}(L) \) is a closed \( \Phi \)-stable threefold \( V \). By the Hironaka flattening theorem [25] there exists of a proper birational morphism \( \pi : \hat{Q} \to Q \) such that for the irreducible component \( \hat{X} \) of \( X \times_\mathbb{C} \hat{Q} \) dominant over \( X \) the natural projection \( \hat{\varrho} : \hat{X} \to \hat{Q} \) is flat, i.e. all of its fibers are one-dimensional or empty. This implies \( \pi^{-1}(L) \) contains a surface \( \hat{P} = \hat{\varrho}(\hat{V}) \) where \( \hat{V} \) is the proper transform of \( V \) in \( \hat{X} \). Choose a curve \( \hat{C} \subset \hat{P} \) whose image in \( B \) is dense and such that \( \hat{\varrho}^{-1}(\hat{c}) \neq \emptyset \) for a general point \( \hat{c} \in \hat{C} \). Let \( \hat{R} \) be a closed surface in \( \hat{Q} \) that meets \( \hat{P} \) along \( \hat{C} \) and let \( R \) be the closure of \( \pi(\hat{R}) \) (note that \( R \) contains \( L \)). Then the closure of \( \varrho^{-1}(R \setminus L) \) contains a \( \Phi \)-stable threefold \( W \) that meets \( V \) over general points of \( B \) (since \( \varrho^{-1}(\hat{c}) \neq \emptyset \)). Because \( X \) is factorial the threefold \( W \) is the zero locus of a regular function \( h \in \mathbb{C}[X] \) which, by

\[\footnote{Recall that a scheme \( Z \) over a field \( k \) is geometrically normal if it is normal and, furthermore, it remains normal under any field extension of \( k \). In the case of a perfect field \( k \) (e.g. a field of characteristic zero) every normal scheme is automatically geometrically normal.}\]
construction, does not vanish on general fibers of $\varphi$. That is, it is constant on general fibers and, therefore, $\Phi$-invariant. By construction $h$ is not constant on the surface $V \cap \varphi^{-1}(b)$ for general $b$ (this follows from the fact that $W$ meets $V \cap \varphi^{-1}(b)$ along a curve). Enlarging a set of generators of the ring $\mathbb{C}[Q]$ by $h$ (and taking the integral closure to preserve the normality) we obtain another partial quotient for which the image of $V$ is a surface. That is, the set $T$ as before becomes at most finite and for a general $b \in B$ this curve $C_b$ does not exists.

Hence the morphism $\psi_b : Q^b \to Q_b$ is quasi-finite now in addition to being birational. Assume that it is not an embedding. Then by the Zariski Main theorem $Q_b$ is not normal for general $b \in B$ contrary to Proposition 5.8. Hence $\psi_b : Q^b \to Q_b$ is an embedding.

Let $D_b$ be the complement to the image of $Q^b$ in $Q_b$. Suppose that $D_b$ is a curve for a general $b \in B$ (since $Q_b$ and $Q^b$ are affine the alternative is an empty $D_b$). The set $Q_b \setminus \varphi(\varphi^{-1}(b))$ consists of $D_b$ and a finite set by Corollary 1.7. Since $\varphi(X)$ is a constructible set by [24, Chap. II, Exercise 3.19] we see that $Q \setminus \varphi(X)$ is an algebraic variety. Consider the irreducible components of this variety which are dominant over $B$ and remove those of them that have dimension $\dim B$. Then we are left with $D := \bigcup_{b \in B} D_b$ because $Q_b \setminus (\varphi(\varphi^{-1}(b)) \cup D_b)$ is finite. That is, $D$ is an algebraic variety and $Q \setminus D$ is a quasi-affine variety. The general fibers $Q_b \setminus D_b$ of the natural morphism $Q \setminus D \to B$ are isomorphic to $\mathbb{C}^2$. By [29] for a Zariski dense open subset $B^* \subset B$ the variety $Q^* = \kappa^{-1}(B^*) \setminus D$ naturally isomorphic to $B^* \times \mathbb{C}^2$. Hence for $X^* = \varphi^{-1}(B^*)$ we get a partial quotient morphism $\varphi|_{X^*} : X^* \to B^* \times \mathbb{C}^2$.

Let $\varphi' : X^* \to Q'$ be another partial quotient morphism for $\Phi|_{X^*}$ into a normal variety $Q'$ (over $B^*$) such that $\varphi$ factors through $\varphi'$, i.e. $\varphi = \theta \circ \varphi'$ for a morphism $\theta : Q' \to B^* \times \mathbb{C}^2$. Suppose that $Q'_b$ is the fiber of the natural morphism $Q' \to B^*$ over a point $b \in B^*$, i.e. the restriction of $\theta$ yields a morphism $\theta_b : Q'_b \to Q_b \simeq Q^b \simeq \mathbb{C}^2$. By the universal property of quotient morphisms we have a natural morphism $Q^b \to Q'_b$, i.e. $\theta_b$ is invertible. Hence $\theta$ is bijective. By the Zariski Main theorem $\theta$ is an isomorphism. This implies that $\varphi|_{X^*} : X^* \to B^* \times \mathbb{C}^2$ is the categorical quotient morphism. In particular, for $Y$ and $K$ as in Theorem 5.4 the ring of invariant of the $G_m$-action on $Y$ (induced by $\Phi$) is a polynomial ring in two variables over $K$. Now the desired conclusion follows from Theorem 5.4. 

$\Box$

6. CRITERION FOR EXISTENCE OF AN AFFINE QUOTIENT

**Notation 6.1.** Let $B$ be a unibranch germ of a smooth complex algebraic curve at point $o$ (i.e. $o$ is the zero locus of some $f \in \mathbb{C}[B]$) and $\varphi : X \to B$ be a morphism from a complex factorial affine algebraic variety $X$ equipped with a $G_m$-action $\Phi$ which preserves each fiber of $\varphi$. We suppose also that $\varphi : X \to Q$ is a partial quotient morphism of the action $\Phi$. Let $\psi : Q \to Q_0$ be a morphism over $B$ into a smooth affine algebraic variety $Q_0$ over $B$ and $\tau = \psi \circ \varphi$. The divisor in $X$ (resp. $Q$, resp. $Q_0$) over $o$ will be denoted by $E$ (resp. $D$, resp. $D_0$). We suppose that the morphism $Q_0 \to B$ is smooth (and in particular $D_0$ is a smooth reduced divisor) and that $\psi$ induces an isomorphism $Q \setminus D \to Q_0 \setminus D_0$. We suppose also that $E = \varphi^*(o)$ is reduced.
and denote by $Z_0$ the closure of $\tau(E)$ in $D_0$. We consider the Stein factorization (e.g., see [24 Chapter III, Corollary 11.5]) of a proper extension $\bar{E} \rightarrow Z_0$ of the morphism $\tau|_E : E \rightarrow Z_0$. Its restriction to $E$ enables us to treat $\tau|_E : E \rightarrow Z_0$ as a composition of a surjective morphism $\lambda : E \rightarrow V$ with connected general fibers and a quasi-finite morphism $\theta : V \rightarrow Z_0$.

Theorem 1.1 implies the following.

**Lemma 6.2.** Let Notation 6.1 hold and $Z_0$ (and therefore $\psi(D)$) be Zariski dense in $D_0$. Then $\psi$ is an isomorphism.

**Convention 6.3.** With an exception of Corollary 6.10 we suppose throughout this section that $Z_0$ is a divisor in $D_0$ and furthermore

(i) the morphism $\theta : V \rightarrow Z_0$ from Notation 6.1 is in fact finite;

(ii) for every point $z_0 \in Z_0$ the preimage $\tau^{-1}(z_0)$ is a surface.

**Remark 6.4.** Note that Convention 6.3 holds automatically when $E$ is isomorphic to $\mathbb{C}^3$ (under the assumption that $Z_0$ is a curve). Indeed, $\tau^{-1}(z_0)$ cannot be three-dimensional (otherwise it coincides with $E$ and $\tau(E)$ is a point, not a curve). Then both $V$ and $Z_0$ must be polynomial curves and a nonconstant morphism of polynomial curve is always finite. In fact, this argument works not only when $E \simeq \mathbb{C}^3$. It is enough to assume that there is no nonconstant morphism from $E$ into any non-polynomial curve.

**Notation 6.5.** Let Notation 6.1 hold. Our aim is to present $\psi : Q \rightarrow Q_0$ over $B$ as a composition

$$Q =: Q_n \xrightarrow{\sigma_n} Q_{n-1} \xrightarrow{\sigma_{n-1}} \cdots \xrightarrow{\sigma_2} Q_1 \xrightarrow{\sigma_1} Q_0$$

of simple affine modifications $\sigma_i : Q_i \rightarrow Q_{i-1}$ over $B$. Let $\psi_i = \sigma_1 \circ \cdots \circ \sigma_i : Q_i \rightarrow Q_0$ and suppose that $\psi$ can be presented as $\psi = \psi_i \circ \delta_i$ for a modification $\delta_i : Q \rightarrow Q_i$. Let $\tau_i = \delta_i \circ \rho$ and thus $\tau = \psi_i \circ \tau_i$. We suppose that the zero locus of $f \circ \psi_i$ is reduced and coincides with $D_i := \psi_i^{-1}(D_0)$ (which is nothing but the divisor of modification $\sigma_{i+1}$) while the center $Z_i$ of $\sigma_{i+1}$ coincides with $\tau_i(E)$. Note that under such assumption every $Q_i$ is normal and Cohen-Macaulay by Proposition 2.6 and induction starting from smooth $Q_0$ and $D_0$.

**Lemma 6.6.** Suppose that for some $i$ one has a sequence of simple affine modifications

$$Q_i \xrightarrow{\sigma_i} Q_{i-1} \xrightarrow{\sigma_{i-1}} \cdots \xrightarrow{\sigma_2} Q_1 \xrightarrow{\sigma_1} Q_0$$

such that $\psi : Q \rightarrow Q_0$ factors through $\psi_i : Q_i \rightarrow Q_0$ and, therefore, $\psi = \psi_i \circ \delta_i$. Let $\delta_i : Q \rightarrow Q_i$ be a semi-finite affine modification. Then the sequence can be extended to a sequence of simple affine modifications

$$Q_{i+1} \xrightarrow{\sigma_{i+1}} Q_i \xrightarrow{\sigma_i} \cdots \xrightarrow{\sigma_2} Q_1 \xrightarrow{\sigma_1} Q_0$$

such that $D_i$ (resp. $Z_i$) is the divisor (resp. center) of $\sigma_{i+1}$ and $\psi : Q \rightarrow Q_0$ factors through $\psi_{i+1} : Q_{i+1} \rightarrow Q_0$, i.e. $\psi = \psi_{i+1} \circ \delta_{i+1}$. 
Proof. The statement follows from Proposition 4.9.

In order to establish when $\delta_i$ is semi-finite we need the following technical fact.

Lemma 6.7. For any point $x_0 \in E$ there is the analytic germ $S$ at $x_0$ of an algebraic subvariety of $X$ of dimension $\dim X - 2$ such that

(i) $S_0 := \tau(S)$ is a germ of an analytic hypersurface in $Q_0$ at $z_0 = \tau(x_0)$;
(ii) the map $\tau|_S : S \to S_0$ is finite;
(iii) for every general point $z \in Z_0$ near $z_0$ there is a local coordinate system $(u, v, \bar{w})$ on $Q_0$ at $z$ for which $D_0$ is given locally by $u = 0$, $Z_0$ by $u = v = 0$, and every irreducible branch of the germ of $S_0$ at $z$ by an equation of form $e := v - u^l d(u, \bar{w}) = 0$ where $d$ is holomorphic and $l$ is the zero multiplicity of the function $e \circ \tau$ at general points of $E$.

Proof. Consider two regular functions $h$ and $e$ on $E$ with the set $P$ of common zeros such that near any general point $P$ is locally a strict complete intersection given by $h = e = 0$ and for $z_0 = \tau(x_0)$ the set $P \cap \tau^{-1}(z_0)$ contains $x_0$ as an isolated component (which is possible since $\tau^{-1}(z_0)$ is a surface). Extend $h$ and $e$ regularly to $X$. Without loss of generality we can suppose that the set of common zeros of these extensions is an $(n - 2)$-dimensional subvariety $T$ of $X$ where $n = \dim X$. Denote by $S$ (resp. $R$) the analytic germ of $T$ (resp. $P$) at $x_0$ and by $T_0$ the closure of $\tau(T)$ in $Q$. Let $S_0$ be the analytic germ of the hypersurface $T_0 \subset Q_0$ at $z_0$. For the restriction $\kappa : S \to T_0$ of $\tau$ the preimage $\kappa^{-1}(z_0) = x_0$ is a singleton. Hence [19] Chapter 1, Sec. 3, Theorem 2] implies that the holomorphic map $\tau|_S : S \to S_0$ is finite. Thus we have (i) and (ii).

By construction $S$ meets $E$ transversely at a general point $x$ of $R$. Let $z = \tau(x)$ and $(u', v', v'', \bar{w})$ (resp. $(u, v, \bar{w})$) be a local analytic coordinate system at $x \in X$ (resp. $z \in Q_0$) as in Lemma 3.7. That is, locally $E$ (resp. $D_0$, resp. $Z_0$) is given by $u' = 0$ (resp. $u = 0$, resp. $u = v = 0$) and $\tau$ is given by $u = u'$, $\bar{w} = \bar{w}'$, and $v = (u')^l q(u', v', v'', \bar{w}')$.

The finiteness of $\tau|_S$ implies that $\tau|_{S \cap E} : S \cap E \to S_0 \cap D_0$ is étale over the general point $z \in Z_0$. Thus $E \cap S$ is given locally by equations of form $v' = h_1(\bar{w}')$ and $v'' = h_2(\bar{w}')$ where $h_1$ and $h_2$ are holomorphic functions. Since $S$ meets $E$ transversely we also see that $S$ must be given locally by equations of form $v' = h_1(\bar{w}') + u' g_1(u', v', v'', \bar{w}')$ and $v'' = h_2(\bar{w}') + u' g_2(u', v', v'', \bar{w}')$ where $g_1$ and $g_2$ are holomorphic functions. Furthermore, by the implicit function theorem these equations can be rewritten as $v' = h_1(\bar{w}') + u' \tilde{g}_1(u', \bar{w}')$ and $v'' = h_2(\bar{w}') + u' \tilde{g}_2(u', \bar{w}')$ for some other holomorphic functions $\tilde{g}_1$ and $\tilde{g}_2$. Plugging these expressions for $v'$ and $v''$ with $u' = u$ and $\bar{w} = \bar{w}'$ into equation $e = (u')^l q(u', v', v'', \bar{w}')$ we get the desired form of equation $e = 0$ in (iii). Note that the function $e \circ \tau$ is given by

$$e \circ \tau(u', v', v'', \bar{w}') = (u')^l [q(u', v', v'', \bar{w}') - h(u', \bar{w}')].$$

Since $q(0, v', v'', \bar{w}')$ depends on $v'$ or $v''$ by Lemma 3.7, we see that the expression on in the brackets is not identically zero on $E$ which shows that $l$ is the zero multiplicity of $e \circ \tau$ at a general point of $E$. 

□
Lemma 6.7. The morphism $\delta_i$ is always semi-finite unless $Z_i = \tau_i(E)$ is dense in $D_i$.

Proof. As we mentioned in Notation 6.5, the variety $Q_i$ is normal and Cohen-Macaulay, and the divisor $D_i = (f \circ \psi_i)^*(0)$ is principal. Furthermore, by construction $D_i = Z_{i-1} \times \mathbb{C}$ for $i \geq 1$. Thus we have condition (a) from Definition 4.8. The finite morphism $V \to Z_0$ (from Notation 6.4 and Convention 6.3) factors through maps $\theta_i : V \to Z_i$, $Z_i \to Z_{i-1}$ and $Z_{i-1} \to Z_0$ each of which must be, therefore, finite, i.e. we have condition (b) from Definition 4.8. Condition (c) follows from construction and Proposition 2.6. It remains to check conditions (d).

Consider the following construction of an analytic germ of an algebraic variety $S$ in $X$ of dim $S = n - 2$ where dim $X = n$. Let $M$ be the set of points in $V$ above $z_0$ (by Convention 6.3 $M$ is finite and nonempty). For every $y \in M$ choose any $x \in E$ above $y$. Consider the analytic germ $S(y)$ of an $(n - 2)$-dimensional algebraic subvariety of $X$ at $x$ such that $S_0(y) := \tau(S(y))$ is an analytic hypersurface in $Q_0$ and the restriction of $\tau$ yields a finite map $S(y) \to S_0(y)$ (the existence of such an $S(y)$ is provided by Lemma 6.7). We use here the following definition: if $r$ is the degree of a finite morphism $V \to U$ of reduced analytic sets and $W_m$ is the $m$-multiple of $W$ then we put the degree of the morphism $W_m \to U$ equal to $mr$. Treating $W_m$ as a union of $m$ disjoint samples of $W$ one can see that this extended notion of degree has all the properties of the standard degree.

11 We use here the following definition: if $r$ is the degree of a finite morphism $V \to U$ of reduced analytic sets and $W_m$ is the $m$-multiple of $W$ then we put the degree of the morphism $W_m \to U$ equal to $mr$. Treating $W_m$ as a union of $m$ disjoint samples of $W$ one can see that this extended notion of degree has all the properties of the standard degree.
function on the normal variety $Q$. This implies that $\delta_i$ factors through the Davis modification $\kappa: Q' \to Q_i$ of $Q_i$ along $D_i$ with ideal generated by $f^{km_i}$ and $h_i$. By Lemma 6.7 for a general point $z'$ of $Z$, there is a Euclidean neighborhood with a local coordinate system $(u', v', w')$ such that $D_i$ is given locally by $u' = 0$, $Z_i$ by $v' = 0$, and up to an invertible factor $h_i$ can be presented locally as a product of $km_i$ factors of form $v' - u'd(u', w')$. Hence by Lemma 3.12 $\kappa$ is locally homogeneous at $z'$. Thus $\delta_i$ is semi-finite and there is a simple modification $\sigma_{i+1}: Q_{i+1} \to Q_i$ as required in Lemma 6.6.

In order to finish induction it remains to prove the existence of a function $h_{i+1}$. Note that over $z'$ the function $h_i \circ \sigma_{i+1}$ has zero multiplicity $km_i$ at general points of $D_{i+1} \subset Q_{i+1}$. Thus $h_{i+1} = h_i \circ \sigma_{i+1}/f^{km_i}$ is a regular function on the normal variety $Q_{i+1}$ that does not vanish at general points of $D_{i+1}$. In particular $h_{i+1}$ vanishes only on the proper transform of $S_i$ which is, by construction, $S_{i+1}$. This is the desired function and we are done. \hfill $\Box$

**Theorem 6.9.** Let Notation 6.1 and Convention 6.3 hold. Then $\mathbb{C}[X]^\Phi$ is finitely generated and $Q = X/\Phi$ coincides with the affine algebraic variety $Q_n$ from Notation 6.3 for some $n$.

**Proof.** By Proposition 3.8 the sequence
\[
Q_n \xrightarrow{\sigma_n} Q_{n-1} \xrightarrow{\sigma_{i-1}} \cdots \xrightarrow{\sigma_2} Q_1 \xrightarrow{\sigma_1} Q_0
\]
of simple modifications from Notation 6.3 cannot be extended indefinitely to the left. Hence by Lemmas 6.6 and 6.8 we can suppose that for some $n$ the image $\tau_n(E)$ is dense in $D_n$. Suppose that $\varrho: X \to Q$ is any partial quotient morphism over $Q_n$. Then $\delta_n(D)$ is dense in $D_n$ since $\tau_n$ factors through $\delta_n$. Then by Lemma 6.2 $Q$ is isomorphic to $Q_n$.

Recall that $\mathbb{C}[X]^\Phi = \lim\nolimits_{\alpha} \mathbb{C}[Q_\alpha]$ where $Q_\alpha$ is a partial quotient. Since every such a quotient over $Q_n$ is $Q_n$ itself we see that $\mathbb{C}[X]^\Phi = \mathbb{C}[Q_n]$ which yields the desired conclusion. \hfill $\Box$

**Corollary 6.10.** Let $X$ be four-dimensional and let Notation 6.1 hold without assuming Convention 6.3. Suppose that $E$ admits a nonconstant morphism into a curve if and only if this curve is a polynomial one.

Then Convention 6.3 holds and thus $\mathbb{C}[X]^\Phi$ is finitely generated. In particular Statement (1) of Theorem 0.2 is true.

**Proof.** By Remark 6.4 Convention 6.3 is true if $Z_0$ is not a point. Assume it is a point $z_0$. Since $Q_0$ and $D_0 = f^*(0)$ are smooth we can suppose that $z_0$ is locally a strict complete intersection given by $f = g_1 = g_2 = 0$. Note that $g_i \circ \tau$ vanishes on $E$. Hence $(g_i/f) \circ \tau$ is a regular $G_n$-invariant function on the normal variety $X$. This implies that $\psi: Q \to Q_0$ factors through $\sigma_1: Q_1 \to Q_0$ where $\sigma_1$ is the simple modifications along $D_0$ with the ideal generated by $f$, $g_1$ and $g_2$. By construction $Q_1$ is smooth and the exceptional divisor $D_1 \simeq \mathbb{C}^2$. Thus we can replace the pair $(Q_0, D_0)$ with $(Q_1, D_1)$. If the center of $\psi$ after this replacement is still a point we observe that it is again a
strict complete intersection $f = h_1 = h_2 = 0$ with $h_i = (g_i/f) \circ \sigma_1$. That is, the zero multiplicity of the lift $h_i$ (to $X$) on $E$ is less than the one of $g_i \circ \tau$. Continue this procedure. As soon as one of these multiplicities becomes zero the variety $Z_0$ becomes at least a curve. Thus we are done.

Another consequence of the equality $Q = Q_0$ in Theorem 6.9 is that the fiber of the morphism $Q \to B$ over $o$ is $D_n \simeq Z_{n-1} \times \mathbb{C}$. By the assumption of Theorem 0.2 $Z_{n-1}$ is a polynomial curve. Hence we have the following.

Corollary 6.11. Statement (2) of Theorem 0.2 is true.

7. Proper actions

Definition 7.1. Recall that an action of an algebraic group $G$ on a variety $X$ is proper if the morphism $G \times X \to X \times X$ that sends $(g, x) \in G \times X$ to $(x, g \cdot x) \in X \times X$ is proper.

It is well-known that every proper $G_a$-action is automatically free. Some geometrical properties of such actions are described below.

Proposition 7.2. Let $X$ be a normal affine algebraic variety equipped with a proper $G_a$-action $\Phi$ and let $C_i$ be a sequence of general orbits of $\Phi$. Suppose that $C \subset X$ is a closed curve such that for every $c \in C$ and every neighborhood $V \subset X$ of $c$ (in the standard topology) there exists $i_0$ for which $C_i$ meets $V$ whenever $i \geq i_0$. Then $C$ is an orbit of $\Phi$, i.e. $C \simeq \mathbb{C}$ (in particular it is smooth and connected).

Proof. Assume that $C$ meets two disjoint orbits $C'$ and $C''$ of $\Phi$ and choose points $c'_i, c''_i \in C_i$ such that $c'_i \to c' \in C' \cap C$ and $c''_i \to c'' \in C'' \cap C$ as $i \to \infty$. Note that $c''_i = t_i, c'_i$. This sequence of numbers $\{t_i\}$ in the group $\mathbb{C}_+$ goes to $\infty$. Indeed, otherwise switching to a subsequence we can suppose that $\lim_{i \to \infty} t_i = t$ and by continuity $c'' = t \cdot c'$ which is impossible since the last two points are in distinct orbits. But this implies that the preimage (in $G \times X$) of a small neighborhood of the point $(c', c'') \in X \times X$ contains points of form $(t_i, c'_i) \in \mathbb{C}_+ \times X$ going to infinity contrary to properness.

Hence $C$ meets only one orbit $O$ of $\Phi$. Hence $C \setminus O = \emptyset$ since otherwise $C$ meets other orbits. Therefore, $C = O \cap C$ which implies that $C = O \simeq \mathbb{C}$ and we are done.

Corollary 7.3. Suppose that for some $q \in Q$ the fiber $q^{-1}(q)$ is a curve. Then $q^{-1}(q) \simeq \mathbb{C}$.

Let us fix notation for the rest of this section.

Notation 7.4. We suppose that $B$ is a unibranch germ of a smooth complex algebraic curve at point $o = f^*(0)$ (where $f \in \mathbb{C}[B]$) and $\varphi : X \to B$ is a morphism from a complex factorial affine algebraic variety $X$ equipped with a $G_a$-action $\Phi$ which preserves each fiber of $\varphi$ and such that the fiber $E = \varphi^*(o)$ is reduced.

\textsuperscript{12}This implies, in particular, that $\Phi$ is free.
We suppose also that Convention 6.3 is true for some partial quotient morphism \( \varphi : X \to Q \) of the action \( \Phi \) and a morphism \( \psi : Q \to Q_0 \) into a smooth affine algebraic variety \( Q_0 \) over \( B \) described in Notation 6.1. Recall that in this case by Theorem 6.9 \( \mathbb{C}[X]^\Phi \) is finitely generated and \( Q = X/\Phi \) coincides with an affine algebraic variety \( Q_n \) from Notation 6.5 for some \( n \), and, particular, in those notations \( \tau_n : X \to Q_n \) is the quotient morphism. Furthermore, let \( D_n \) be the divisor in \( Q_n \) over \( o \in B \). Then it follows from the construction of \( Q_n \) that \( D_n \cong Z_{n-1} \times \mathbb{C} \) and from the proof of Theorem 6.9 that \( \tau_n(E) \) is dense in \( D_n \).

**Proposition 7.5.** Let \( \lambda : E \to R \) be the quotient morphism of the restriction of \( \Phi \) to \( E \). Then there is a natural affine modification \( \beta : R \to D_n \) over \( Z_{n-1} \).

**Proof.** The density of \( \tau_n(E) \) in \( D_n \) implies that for a general point \( q \in D_n \) its preimage \( \tau_n^{-1}(q) \) is a curve. By Corollary 7.3 this curve is isomorphic to \( \mathbb{C} \).

By the universal property of quotient morphisms the map \( \tau_n|_E : E \to D_n \) factors through \( \lambda \), i.e. there exists \( \beta : R \to D_n \) for which \( \tau_n|_E = \beta \circ \lambda \). Since \( \tau_{n-1} \) (which maps \( E \) onto \( Z_{n-1} \)) factors through \( \tau_1 \) we see that \( \beta \) is a morphism over \( Z_{n-1} \). Note that \( \beta^{-1}(q) \) is a point since \( \tau_n^{-1}(q) \) is a connected curve. Thus \( \beta \) is birational. Any birational morphism of affine algebraic varieties is an affine modification [28] and we are done.

**Remark 7.6.** Proposition 7.5 implies that \( \beta \) is an isomorphism over a Zariski dense open subset \( D_n^* \) of \( D_n \). Furthermore removing a proper subvariety from \( D_n^* \) one can suppose that for every point in \( \beta^{-1}(D_n^*) \) the fiber of \( \lambda \) over this point is an orbit of \( \Phi \). Let \( Q' = (Q \setminus D_n) \cup D_n^* \). Note that by construction every fiber of \( \varphi \) over point of \( Q^* \) is an orbit of \( \Phi \) and the codimension of \( Q \setminus Q' = D_n \setminus D_n^* \) in \( Q \) is at least 2.

Recall that when \( X \) is four-dimensional and there is no nonconstant morphism from \( E \) into any non-polynomial curve then by Remark 6.4 Convention 6.3 holds, \( Z_{n-1} \) is a polynomial curve, and thus the normalization of \( D_n \) is \( Z_{n-1} \times \mathbb{C} \) is \( \mathbb{C}^2 \). In particular, if \( E \) is normal then \( R \) is normal (being the quotient space of normal \( E \)) and we can mention the following interesting fact (which won’t be used later).

**Corollary 7.7.** Let the assumption of Corollary 6.10 hold and \( E \) be normal. Then \( R \) is an affine modification of \( \mathbb{C}^2 \). Furthermore, for a factorial \( E \) one has \( R \cong \mathbb{C}^2 \) and normalization of morphism \( \beta \) is a birational morphism \( \beta' : \mathbb{C}^2 \to \mathbb{C}^2 \).

**Proof.** Since \( \beta \) is a morphism over the curve \( Z_{n-1} \) we see that \( \beta' : R \to \mathbb{C}^2 \) is a morphism over the normalization \( Z_{n-1}^{\text{norm}} \cong \mathbb{C} \) of \( Z_{n-1} \). Note that by Corollary 1.7 \( R \) is factorial being the quotient space of the factorial threefold \( E \). This implies that all fibers of the \( \mathbb{C} \)-fibration \( R \to Z_{n-1}^{\text{norm}} \) are reduced and irreducible. Hence \( R \cong \mathbb{C}^2 \) and we are done.

**Proposition 7.8.** Let \( \beta \) be as in Proposition 7.5 and suppose that one of the following conditions is satisfied:
(a) \( \beta \) is finite, \( E \) is smooth outside codimension 2, and every fiber of \( \alpha : E \to R \) is a curve with a possible exception of a finite number of such fibers;

(b) \( \beta \) is quasi-finite and the assumption of Theorem 0.2 (b) holds (in particular, \( X \) is Cohen-Macaulay, \( \Phi \) is proper, each fiber \( X_\mathfrak{a} \) of \( \varphi : X \to B \) (including \( E \)) is normal, and, being the image of \( E \), the curve \( Z_{n-1} \) is a polynomial one).

Then \( Z_{n-1} \) (and therefore \( D_n = Z_{n-1} \times C \)) is smooth and \( \beta \) is an isomorphism.

**Proof.** The assumptions on \( \alpha \) and \( \beta \) imply that there is a finite subset \( M \subset D_n \) such that for every \( q \in D_n \setminus M \) the preimage \( \tau^{-1}_n(q) \) is a curve.

Recall that \( X \) is Cohen-Macaulay which implies that the morphism \( g \) is faithfully flat over \( Q \setminus M \) (e.g., see [33 Chap. 2 (3.J) and Th. 3] and [13 Th. 18.6]). Hence \( \Phi \) is locally a translation over \( Q \) by [9 Theorem 2.8] (see also [10 Theorem 1.2]). This implies that \( Z_{n-1} \) is smooth since otherwise \( D_n \setminus M \) and therefore \( (D_n \setminus M) \times C \subset E \) are not smooth outside codimension 2. It remains to note that, being birational and finite, \( \beta \) is an isomorphism by the Zariski Main theorem.

For the second statement consider the normalization \( \kappa : R_{\text{norm}} \to D_{\text{norm}} \simeq \mathbb{C}^2 \) of \( \beta \). Since \( \beta \) is birational by Proposition 7.5 and quasi-finite by the assumption, the morphism \( \kappa \) is an embedding. Since both \( R_{\text{norm}} \) and \( D_{\text{norm}} \) are affine we see that \( D_{\text{norm}} \setminus \kappa(R_{\text{norm}}) \) is either empty or a Cartier divisor. Since \( D_{\text{norm}} \simeq \mathbb{C}^2 \) such a divisor must be given by zeros of a regular function \( g \). Then \( g \) admits a regular lift to the normal variety \( E \) that does not vanish. This contradicts the assumption that \( E \) does not admit nonconstant morphisms into a non-polynomial curve. Thus \( \kappa(R_{\text{norm}}) = D_{\text{norm}} \) and as before the assumption on \( \alpha \) implies that there is a finite subset \( M \subset D_n \) such that for every \( q \in D_n \setminus M \) the preimage \( \tau^{-1}_n(q) \) is a curve. Hence we can repeat the previous argument and get the desired conclusion.

\[ \square \]

**Remark 7.9.** (1) Let \( Q' \) be from the Remark 7.6. Note that exactly the same argument as in Lemma 7.8 for the set \( Q \setminus M \) implies that the action \( \Phi \) over \( Q' \) is locally trivial\(^{13}\) even without the assumption that \( \beta \) is quasi-finite.

(2) Actually, following [20] one can extract the Sathaye’s theorem [41] from the argument in the proof of Proposition 7.8. Indeed, under the assumption of the Sathaye’s theorem \( D_n = \mathbb{C}^2 \) and thus \( Z_{n-1} \simeq \mathbb{C} \). Since \( Z_{n-1} \) is smooth every polynomial curve \( Z_i \) is smooth by Lemma 4.4 and therefore \( Z_i \simeq \mathbb{C} \). Hence every \( D_i = Z_i \times C \) is the plane and the Abhyankar-Moh-Suzuki theorem \( Z_i \) can be viewed as a coordinate line in \( D_i \). It is a straightforward fact that a modification \( \sigma_{i+1} : Q_{i+1} \to Q_i \) with such center \( Z_i \) and divisor \( D_i \) leads to \( Q_{i+1} \) isomorphic to \( Q_i \) which implies that \( Q_n \simeq B \times \mathbb{C}^2 \).

(3) The assumption on properness of \( \Phi \) is crucial in Proposition 7.8. In the absence of properness Winkelman [44] constructed a free action on \( X = \mathbb{C}^4_{x_1,x_2,x_3,x_4} \) such that the quotient \( Q \) is isomorphic to the hypersurface in \( \mathbb{C}^4_{x_1,x_2,x_3,x_4,w} \) given by

\[ x_1w = v^2 - u^2 - u^3. \]

\(^{13}\)That is, \( \varphi^{-1}(Q') \) can be covered by \( \Phi \)-stable affine open subsets on each of which the action admits an equivariant trivialization.
Both $X$ and $Q$ are considered over the curve $B = \mathbb{C}_{x_1}$ and in particular the zero fiber $D$ of the morphism $Q \to B$ is given by $v^2 - u^2 - u^3 = 0$ in $\mathbb{C}^3_{u,v,w}$. That is, $D$ is not a plane but it is the product of $\mathbb{C}$ and $\mathbb{Z}_{n-1}$ where $\mathbb{Z}_{n-1}$ is a polynomial curve with one node as singularity (in accordance with Theorem 0.2 (2)).

## 8. Some Facts about Fibered Products

In order to establish quasi-finiteness required in Proposition 7.8 we need some auxiliary facts.

**Notation 8.1.** Let $\varrho : X \to Q$ be a dominant morphism of normal quasiprojective algebraic varieties and $S$ be an irreducible closed subvariety of $X$ such that for a Zariski dense open subset $S'$ of $S$ the restriction $\varrho|_{S'} : S' \to Q$ is quasi-finite. Suppose that $k = \dim X - \dim S$ is the dimension of general fibers of $\varrho$.

**Lemma 8.2.** Consider an irreducible germ $\Gamma \subset S$ of a general curve through $s \in S$ (in particular, $\Gamma \setminus s \subset S'$ and the dimension of $\varrho^{-1}(\varrho(\Gamma \setminus s))$ is $k + 1$). Then the dimension of the fiber $F$ in the closure of $\varrho^{-1}(\varrho(\Gamma \setminus s))$ (in $X$) over $s$ is $k$.

**Proof.** By the semi-continuity theorem [43, Chapter I, Section 6.3, Corollary] the dimension of $F$ is at least $k$. On the other hand it cannot be greater than $k$ since otherwise it is not contained in the closure of the $(k + 1)$-dimensional variety $\varrho^{-1}(\varrho(\Gamma \setminus s))$.

\[ \square \]

Note that for a fixed $s$ this fiber $F$ may in general depend on the choice of $\Gamma$, i.e. $\varrho = F(\Gamma)$.

**Definition 8.3.** Suppose that $F_0(\Gamma)$ is the component of $F(\Gamma)$ containing $s$. We say that the data in Notation 8.1 satisfies Condition (A) if for every $s \in S$ the set $\bigcup_{\Gamma} F_0(\Gamma)$ consists of a finite number of $k$-dimensional varieties.

**Example 8.4.** (1) If $X = S$ then Condition (A) holds automatically.

(2) Suppose that $\varrho : X \to Q$ is the quotient morphism of a proper $G_a$-action $\Phi$ and $S$ is a hypersurface of $X$ such that $\varrho(S)$ is Zariski dense in $Q$. Then for a general germ $\Gamma$ as in Definition 8.3 and every point $s' \in \Gamma \setminus s$ the fiber $\varrho^{-1}(\varrho(s'))$ is a union of orbits of $\Phi$. The set $\bigcup_{\Gamma} F_0(\Gamma)$ contains, of course, the orbit of $\Phi$ through $s$ and by Proposition 7.2 it contains nothing else. Thus Condition (A) holds.

**Proposition 8.5.** Let Notation 8.1 hold and Condition (A) be satisfied. Suppose that $\tilde{X}$ is the irreducible component of $X \times_Q S$ such that it contains $\tilde{S} = \{(s,s) | s \in S\}$. Denote by $\lambda : \tilde{X} \to S$ the restriction to $\tilde{X}$ of the natural projection.

Then for every irreducible subvariety $P \subset S \setminus S'$ the dimension of $\lambda^{-1}(P)$ coincides with $m + k$ where $m = \dim P$.

**Proof.** Suppose that for $l \geq 0$ the subvariety $P(l) \subset P$ consists of points $s \in P$ for which $\dim \lambda^{-1}(s) = k + l$. Our aim is to show that $\dim P(l) \leq m - l$ and thus $\dim \lambda^{-1}(P) = m + k$.

Let $s \in S, \Gamma, F$ be as in Definition 8.3 and let the closure of $\varrho^{-1}(\varrho(\Gamma \setminus s)) \cap S$ meets $F$ at points $s_1, \ldots, s_r$ (i.e. each irreducible component of $F$ passes at least through one
of these points). Note that for a fixed \( s \) this set \( \{s_1, \ldots, s_r\} \) may in general depend on the choice of \( \Gamma \), i.e. \( s_i = s_i(\Gamma) \). Suppose that \( M(s) \) is the union \( \bigcup_{l} s_{i_l}(\Gamma) \). By definition of \( \hat{X} \) for every \( s' \in \Gamma \setminus s \) one has \( \lambda^{-1}(s') \sim \varrho^{-1}((g(s'))) \). Hence by Lemma 8.2 and Condition (A), the variety \( \lambda^{-1}(s) = \bigcup_{\Gamma} F(\Gamma) \) has dimension at most \( \dim M(s) + k \).

In particular \( P(l) \) is contained in the set

\[
\{ s \in P \mid \dim M(s) \geq l \}
\]

and we need to estimate the dimension of this last set.

For this we need to use the Hironaka flattening theorem \cite{25} which implies the existence of a proper birational morphism \( \hat{Q} \to Q \) such that for the union \( \hat{S} \) of irreducible components of \( S \times_Q \hat{Q} \) with the dominant projections to \( S \) the induced morphism \( \hat{S} \to \hat{Q} \) is quasi-finite. We have the following commutative diagram

\[
\begin{array}{ccc}
\hat{S} & \overset{\beta}{\to} & S \\
\theta \downarrow & & \downarrow \gamma \\
\hat{Q} & \overset{\alpha}{\to} & Q
\end{array}
\]

where the morphisms \( \alpha \) and \( \beta \) are birational and the morphism \( \theta \) and \( \gamma|_{S'} \) are quasi-finite.

Let \( P = \bigcup_{i} P_i \) where \( P_i \) are disjoint (not necessarily closed) irreducible varieties such that for any irreducible component \( V \) of \( \beta^{-1}(P_i) \) all fibers of the morphism \( \beta|_{V} : V \to P_i \) are of the same dimension. Varying \( i \) and \( V \) consider the collection \( C_1, C_2, \ldots \) of irreducible subvarieties of \( \hat{Q} \) that are of the form \( C_j = \theta(V) \). Observe that stratifying the varieties \( P_i \)'s further one can suppose that for indices \( j \neq l \) the varieties \( C_j \) and \( C_l \) are either equal or disjoint. Denote by \( I \) the set of all pairs \( (i, j) \) for which there exist \( V \subset \beta^{-1}(P_i) \) with \( \theta(V) = C_j \) and put \( C_{ij} = V \) (in particular \( \dim C_j = \dim C_{ij} \) because of quasi-finiteness of \( \theta \)).

Let \( n_{ij} \) be the dimension of a fiber \( F_{ij} \subset \beta^{-1}(s) \) of the morphism \( \beta|_{C_{ij}} : C_{ij} \to P_i \). In particular \( \dim P_i = \dim C_j - n_{ij} \leq m \). Hence for \( n_j = \min_i \{n_{ij} \mid (i, j) \in I \} \) one has \( \dim P_i \leq m + n_j - n_{ij} \) for any \( (i, j) \in I \).

Let \( s \in P_i \), i.e. \( \beta^{-1}(s) = \bigcup_{j} F_{ij} \). For every where \( (t_{ij}) \in I \) we put \( F_{ij}^l = \theta^{-1}(\theta(F_{ij})) \cap C_{t_{ij}} \), i.e. \( \dim F_{ij}^l = n_{ij} \) because of quasi-finiteness. Note that

\[
\dim \beta(F_{ij}^l) = \max(0, n_{ij} - n_{ij}) \leq n_{ij} - n_j \leq \max(p(n_{ij} - n_j).
\]

Thus \( l := \dim \beta(\bigcup_{j} F_{ij}^l) \leq \max_j(n_{ij} - n_j) \) while \( \dim P_i \leq m - \max_j(n_{ij} - n_j) \leq m - l \).

Since we started with a point \( s \in P_i \) for the desired inequality \( \dim P(l) \leq m - l \) it suffices to show that \( M(s) \subset \bigcup_{j} \beta(\bigcup_{l} F_{ij}^l) \) (since the last set is contained in \( P_i \)). But this follows from the fact that the proper transform of \( \Gamma \) (mentioned in the definition of \( M(s) \)) in \( \hat{S} \) meets \( \beta^{-1}(s) \) and therefore meets some \( F_{ij} \). By quasi-finiteness the proper transform (in \( \hat{S} \)) of every component in the closure of \( g^{-1}(\varrho(\Gamma \setminus s)) \) \( \cap S \) meets some \( F_{ij}^l \). This implies that this closure passes through a point in \( \beta(F_{ij}^l) \) which yields \( M(s) \subset \bigcup_{j} \beta(\bigcup_{l} F_{ij}^l) \) and we get \( \dim \lambda^{-1}(P) \leq m + k \), i.e. \( \dim \lambda^{-1}(P) \leq \dim g^{-1}(\varrho(P)) \).
Corollary 8.6. If Condition (A) holds and \( g^{-1}(\varrho(P)) \) is at least of codimension 2 in \( X \) then \( \lambda^{-1}(P) \) has codimension at least 2 in \( \tilde{X} \).

9. Main Theorems

Notation 9.1. Up to Section 9.1 below we adhere to the assumptions of Proposition 7.5 and, in particular, Notation 6.1. That is, \( B \) is the germ of a smooth algebraic curve at point \( o = f^*(0) \), \( \varphi : X \to B \) is a morphism of factorial varieties, \( \Phi \) is a \( G_a \)-action on \( X \), \( Q = Q_n \) is the categorical quotient \( X//\Phi \) in the category of affine algebraic varieties, \( \tau_n = \varrho \), and \( \varrho(E) \) is dense in \( D_n =: D \). Since \( \beta \) is birational by Proposition 7.5, there is a Zariski dense open \( R^* \subset R \) such that the restriction of \( \beta_{|R^*} : R^* \to D \subset Q \) to \( R^* \) is an embedding.

Suppose also (as in Theorem 0.2 (3)) that the restriction of our proper \( G_a \)-action \( \Phi \) to \( E \) is a translation. In particular, the quotient morphism \( \alpha : E \to R \) admits a section whose image in \( E \), by abuse of notation, will be denoted by \( R \). The image of \( R^* \) in this section will keep notation \( R^* \) as well (i.e. \( \varrho|_{R^*} : R^* \to D \) is an embedding).

The section \( R \) coincides with the zero locus of a regular function on \( E \) which has simple zeros at every point of \( R \). Extend this function to a regular function \( h \) on \( X \) and consider the zero locus of \( h \) on \( X \) and the zero locus \( h^{-1}(0) =: S \subset X \) of this extension.

Lemma 9.2. The function \( h \) from Notation 9.1 can be perturbed so that for \( S^* = S \setminus R \) and \( Q^* = Q \setminus D \) the restriction \( \varrho|_{S^*} : S^* \to Q^* \) is finite. In particular, for \( S' = S^* \cup R^* \) the restriction \( \varrho|_{S'} : S' \to Q \) is quasi-finite.

Proof. Recall that \( X \setminus E \simeq Q^* \times \mathbb{C} \). Let \( w \) be a coordinate on the second factor, i.e. \( h|_{X \setminus E} \) can be treated as a polynomial in \( w \) with coefficients in \( \mathbb{C}[Q^*] \). Let \( n \) be the degree of this polynomial and \( f \) be as in Notation 6.1. Then for sufficiently large \( k \) the function \( h + f^k w^{n+1} \) is regular on \( X \) and the restriction of \( h + f^k w^{n+1} \) to \( E \) has only simple zeros on its zero locus \( R \). This \( h + f^k w^{n+1} \) yields the desired perturbation.

Proposition 9.3. Suppose that \( \tilde{X} \) is the irreducible component of \( X \times_Q S \) such that it contains \( \tilde{S} = \{(s,s) | s \in S \} \). Then \( \tilde{X} \) is naturally isomorphic to \( S \times \mathbb{C} \) with \( \tilde{S} \) being the section of the natural projection \( \lambda : \tilde{X} \to S \).
Lemma 9.4. For $\tilde{X}$ from Proposition 9.3, the restriction $\kappa: \tilde{X} \to X$ of the natural projection $X \times_Q S \to X$ is quasi-finite.

Proof. The restriction of $\kappa$ to $\lambda^{-1}(S')$ is quasi-finite because of quasi-finiteness of $\varphi|_{S'}$. Since $\tilde{X} \setminus \lambda^{-1}(S') = (S \setminus S') \times \mathbb{C} = R \times \mathbb{C}$ it suffices to check that the restriction of $\kappa$ yields a quasi-finite morphism $R \times \mathbb{C} \to E$. The last map is an isomorphism since $R$ is a section of $\Phi|_E$ and we are done. 

\[\square\]

Notation 9.5. By the Grothendieck version of the Zariski Main theorem (e.g., see [20, Théorème 8.12.6]) there is an embedding $\tilde{X} \to \hat{X}$ of $\tilde{X}$ into an algebraic variety $\hat{X}$ such that $\kappa$ can be extended to a finite morphism $\chi: \hat{X} \to X$. Note that the finiteness implies that $\hat{X}$ is affine since $X$ is. Furthermore, replacing if necessary $\tilde{X}, \hat{X},$ and $S$ with their normalizations we suppose that these varieties are normal.

Denote the lift of $f \in \mathbb{C}[B]$ from Notation 9.1 to $X$ (resp. $\tilde{X},$ resp. $\hat{X}$) by the same letter $f$ (resp. $\tilde{f},$ resp. $\hat{f}$). Let $\nu$ be the locally nilpotent vector field on $X$ associated with the action $\Phi$. Since $f \in \text{Ker} \nu$ the field $f^k \nu$ is also locally nilpotent for every $k > 0$ and it is associated with a $G_a$-action $\Phi_k$ on $X$ which has the same quotient morphism $\varphi: X \to Q$. The lift of $\nu$ to $\tilde{X}$ (resp. $\hat{X}$) will be denoted by $\tilde{\nu}$ (resp. $\hat{\nu}$). By construction $\hat{\nu}$ (and, hence, $\hat{f}^k \hat{\nu}$) is locally nilpotent on $\hat{X}$ and $\hat{\nu}$ is a rational vector field on $\hat{X}$. Since $\tilde{X} \setminus \hat{\nu}^{-1}(0) \subset \tilde{X}$ we see that for sufficiently large $k$ the field $\hat{f}^k \hat{\nu}$ is locally nilpotent on $\hat{X}$. We denote by $\hat{\nu}$ the $G_a$-action associated with the last field.

Proposition 9.6. The action $\hat{\Phi}_k$ admits a quotient morphism $\hat{\varphi}: \hat{X} \to \hat{Q}$ in the category of affine algebraic varieties such that the following commutative diagram holds

\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\hat{\varphi}} & \hat{Q} \\
\downarrow{\chi} & & \downarrow{\tau} \\
X & \xrightarrow{\varphi} & Q
\end{array}
\]

where $\tau$ is a finite morphism.

Proof. For every function $g \in \mathbb{C}[\hat{X}]$ and sufficiently large $m > 0$ the function $f^m g$ has a regular extension to $\hat{X}$, i.e. it can be viewed as a function from $\mathbb{C}[\hat{X}]^\Phi_k$. Since by construction $\tilde{X} \setminus \hat{\nu}^{-1}(0) = \tilde{X} \setminus \hat{\nu}^{-1}(0) = \lambda^{-1}(S \setminus R)$ where $\lambda: \tilde{X} \to S$ is the quotient morphism of $\tilde{\Phi}$ (by Proposition 9.3) we see now that a finite number of functions from $\mathbb{C}[\hat{X}]^\Phi_k$ separates the orbits of $\hat{\Phi}_k$ contained in $\lambda^{-1}(S \setminus R)$. Consider a partial quotient morphism $\hat{\varphi}: \tilde{X} \to \hat{Q}$ whose coordinates include these functions. Furthermore, since the lift of every function from $\mathbb{C}[Q]$ to $\tilde{X}$ is a function from $\mathbb{C}[\hat{X}]^\Phi_k$ we can include the lifts of the coordinate functions of $\varphi$ into the set of coordinate functions of $\hat{\varphi}$. Then the commutative diagram as before make sense.

Let us show that $\tau$ is finite. Indeed, by finiteness of $\chi$ every function $g$ from $\mathbb{C}[\hat{X}]^\Phi_k$ is integral over $\mathbb{C}[X]$. That is, $g$ is a root of an irreducible monic polynomial $P(g) := g^n + a_{n-1} g^{n-1} + \ldots + a_1 g + a_0$ where each $a_i \in \mathbb{C}[X]$. Since $g$ is $\Phi_k$-invariant it satisfies also an equation $g^n + a_{n-1} g^{n-1} + \ldots + a_1 g + a_0 = 0$ where $a_i$ is the result of the action of
$t \in \mathbb{C}_+$ (induced by $\hat{\Phi}_k$) on $a_i$. However, the minimal polynomial $P$ should be unique which implies that $a_i$ are $\hat{\Phi}_k$-invariant and thus $\mathbb{C}[\hat{X}]^{\hat{\Phi}_k}$ is integral over $\mathbb{C}[Q]$. That is, every function from $\mathbb{C}[\hat{Q}]$ is integral over $\mathbb{C}[Q]$ and hence $\tau$ is finite.

This leads to the commutative diagram

$$
\begin{array}{ccc}
\hat{E} & \to & \hat{D} \\
\downarrow & & \downarrow \\
E & \to & D
\end{array}
$$

where $\hat{D} := \tau^{-1}(D)$, $\hat{E} = \hat{f}^{-1}(0)$, and the vertical morphism are finite. Since for a general point $q \in D$ its preimage $g^{-1}(q)$ is a curve in $E$ the diagram implies that for a general point $\hat{q} \in \hat{D}$ its preimage $\hat{g}^{-1}(\hat{q})$ is a curve in $\hat{E}$. Thus, by Theorem 1.4 $\hat{g}$ is a quotient morphism which is the desired conclusion.

Lemma 9.7. Let $Q' = Q^* \cup D^*$ be as in Remark 7.9. Then

(i) $\hat{X} \setminus (g \circ \chi)^{-1}(Q')$ has codimension at least 2 in $\hat{X}$;

(ii) for every point in $Q'$ there is a Zariski neighborhood $V \subset Q'$ such that $(g \circ \chi)^{-1}(V)$ is naturally isomorphic to $\hat{V} \times \mathbb{C}$ where $\hat{V} = \tau^{-1}(V)$.

Proof. Recall that $g^{-1}(Q \setminus Q') = E \setminus (g|_E)^{-1}(D^*)$ is of codimension at least 2 in $X$. Hence (i) is a consequence of finiteness of $\chi$.

By Remark 7.9 $\Phi$ is a locally trivial over $Q'$. In particular, for every point in $Q'$ there is a Zariski neighborhood $V \subset Q'$ such that $g^{-1}(V)$ is naturally isomorphic to $V \times \mathbb{C}$ where the projection $g^{-1}(V) \to V$ is the restriction of $g$. Note that the restriction of the commutative diagram in Proposition 9.5 yields another commutative diagram

$$
\begin{array}{ccc}
\hat{W} & \to & \hat{V} \\
\downarrow & & \downarrow \\
g^{-1}(V) & \to & V
\end{array}
$$

where $\hat{W} = (g \circ \chi)^{-1}(V)$. Hence we have $\hat{W} \simeq \hat{V} \times \mathbb{C}$ since $g^{-1}(V) \simeq V \times \mathbb{C}$. This yields (ii).

Proposition 9.8. Let $\hat{S}$ be the closure of $\hat{S}$ in $\hat{X}$. Then $\hat{S}$ is a section of the quotient morphism $\hat{\phi} : \hat{X} \to \hat{Q}$.

Proof. If suffices to construct a $\hat{\Phi}_k$-equivariant morphism $\hat{\psi} : \hat{X} \to \hat{S}$ from $\hat{X}$ to $\hat{S}$. Indeed, then by the universal properties of quotient morphisms this morphism factors through $\hat{Q}$ and, therefore, $\hat{\phi}|_{\hat{S}} : \hat{S} \to \hat{Q}$ is invertible.

Choose as the restriction of such a $\hat{\psi}$ to $\hat{X}$ the natural projection $\lambda : \hat{X} = S \times \mathbb{C} \to S$. Note that for $\hat{V} \simeq V \times \mathbb{C}$ from Lemma 9.7(ii) $\lambda$ agrees on $\hat{X} \cap \hat{V}$ with the natural projection $\hat{V} \to V$. Hence we can extend $\hat{\psi}$ to $\hat{X} \cup g^{-1}(Q')$. Now because of Lemma 9.7(i) and the Hartogs’ theorem this restriction extends to $\hat{X}$ and we are done.
Remark 9.9. The fact that the morphism $\alpha : E \to R$ has a section may be made weaker for Proposition 9.8. It suffices, say, to require that there exists a reduced irreducible principal effective divisor $T = h^*(0)$ in $E$ such that the restriction $\alpha|_T : T \to R$ is surjective and quasi-finite. Then one can extend $h$ to a regular function on $X$ and the same proofs as before imply that the result remains valid.

Corollary 9.10. The morphism $\beta : R \to D$ is quasi-finite.

Proof. Let $\tilde{R}$ be the preimage of $R \subset S$ in $\tilde{S}$. Note that $\tilde{R}$ is a subvariety of $\tilde{X} \subset \hat{X}$. On the other hand since $\tilde{Q}$ is isomorphic to $\hat{X} \subset \hat{S}$ by Proposition 9.8 we can treat $\tilde{R}$ as a subvariety of $\hat{Q}$. Furthermore, by construction the quotient morphism $\hat{\rho}$ yields an automorphism of $\tilde{R}$. Hence the restriction of the morphism $\tau \circ \hat{\rho} = \varrho \circ \chi$ to $\tilde{R} \subset \hat{X}$ yields a quasi-finite morphism $\tilde{R} \to D$ (where $\tau, \chi$ are from Proposition 9.6). Since morphism $\varrho \circ \chi|_{\tilde{R}}$ factors through $\beta = \varrho|_R : R \to D$ the latter is also quasi-finite and we are done. □

9.1. Proof of Theorem 0.1. Note that Convention 6.3 is valid by Remark 6.4. Hence Theorem 6.9 and thus Propositions 7.5 are applicable. For the time being consider the case when $B$ is a germ of a smooth curve at a point $o$ and denote by $E$ the fiber in $X$ over $o$. Since $\Phi|_E$ is a translation in Theorem 0.1 (3), for every $r \in R$ the preimage $\alpha^{-1}(r)$ is a curve. Thus, taking into consideration Corollary 9.10, we see that the assumptions of Proposition 7.8 are valid. This implies that the polynomial curve $Z_{n-1}$ is smooth and we have $Z_{n-1} \simeq \mathbb{C}$ and $D_n \simeq \mathbb{C}^2$. Hence the polynomial curve $Z_{n-1}$ is smooth by Proposition 7.8 and we have $Z_{n-1} \simeq \mathbb{C}$ and $D_n \simeq \mathbb{C}^2$. By assumption $D_n$ is also a smooth reduced fiber of the morphism $Q_n \to B$ and $Q_n \setminus D_n \simeq B^* \times \mathbb{C}^2$. The Sathaye’s theorem [41] implies now that $Q_n$ is naturally isomorphic to $B \times \mathbb{C}$. Similarly $X = Q_n \times \mathbb{C}$ when $B$ is a germ of a curve.

Let us return now to the general case when $B$ is a smooth affine curve. The local statement over germ of $B$ at $o$ implies the first global claim of Theorem 0.1 (3) while the second one is the consequence of the fact (see [42]) that every affine manifold with a free $G_a$-action and an affine geometrical quotient is the direct product of that quotient and $\mathbb{C}$. Thus Statement (3) of Theorem 0.1 is valid while Statements (1) and (2) are true by Corollaries 6.10 and 6.11. Hence we are done with the proof of Theorem 0.1. □

9.2. General case of an affine algebraic variety $X$ over a field $k$ of characteristic zero. Recall that there is a one-to-one correspondence between the set of $G_a$-actions on an affine algebraic variety $X$ and the set of locally nilpotent derivations (LNDs) on the algebra $B$ of regular functions on $X$ (e.g., see [17]). Suppose that $\nu$ is such an LND associated with a proper $G_a$-action $\Phi$ on $X$. Then condition that $\Phi$ is free (resp. a translation) is equivalent to the fact that the ideal generated by $\nu(B)$ coincides with $B$ (resp. $\nu(B) = B$). To show the validity of Theorem 0.1 for any $k$ of characteristic zero one needs to repeat the argument of Daigle from [8, Theorem 3.2].
Namely, if a fact is true for varieties over $\mathbb{C}$ it is true in the case when $X$ is considered over a field which is a universal domain$^{[14]}$. Thus consider a field extension $k'/k$ where $k'$ is a universal domain. Then $\nu$ extends to an LND $\nu'$ on $B' = k' \otimes_k B$ associated with a $G_a$-action $\Phi'$. Note that $\nu'$ is free since $\nu$ is. Similarly, $\Phi'$ is proper, since properness survives base extension $^{[24, \text{Corollary 4.8}]}$. Thus under the assumptions of Theorem $0.1$ (with $\mathbb{C}$ replaced by $k'$) $\Phi'$ is a translation by the argument above, i.e. $\nu': B' \to B'$ is surjective. Since $\nu'$ is obtained by applying the functor $k' \otimes_k -$ to $\nu$ and since $k'$ is a faithfully flat $k$-module we see that $\nu: B \to B$ is also surjective. Therefore $\Phi$ is a translation and we have the following.

**Theorem 9.11.** Let $k$ be a field of characteristic zero and $\Phi$ be a proper $G_a$-action on the four-space $k^4_k$ preserving a coordinate. Then $\Phi$ is a translation in a suitable polynomial coordinate system.

Similarly the argument before implies the following.

**Theorem 9.12.** Let the assumptions of Theorem $0.2$ hold with the only change that $X$ and $B$ are varieties over some field $k$ of characteristic zero (and not over $\mathbb{C}$). Suppose also that $\Phi$ is proper and the restriction of $\Phi$ to every fiber is a translation (as in Theorem $0.2$ (3)). Then $\Phi$ is a translation.

9.3. Conclusive Remark. In dimension 5 the analogue of Theorem $0.1$ is not true by the second Winkelmann’s example in $^{[44]}$. This raises the question whether the properness is the right condition for this type of problems. The author suspects that another condition may be more effective at least in dimension 4 which leads to the following question.

Suppose that the quotient of a free $G_a$-action $\Phi$ on $\mathbb{C}^4$ is isomorphic to $\mathbb{C}^3$. Is it true that $\Phi$ is a translation?

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$^{14}$Recall that a universal domain is an algebraically closed field containing $\mathbb{Q}$ such that it has an infinite transcendence degree over $\mathbb{Q}$. 
33

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