Learning Simon’s quantum algorithm

Kwok Ho Wan,1,2 Feiyang Liu,3 Oscar Dahlsten,3,2,4,5,∗ and M.S. Kim2

1Mathematical Physics, Department of Mathematics, Imperial College London, London, SW7 2AZ, United Kingdom
2QOLS, Blackett Laboratory, Imperial College London, London, SW7 2AZ, United Kingdom
3Institute for Quantum Science and Engineering, Department of Physics, South China University of Science and Technology (SUSTech), Shenzhen, China.
4Clarendon Laboratory, University of Oxford, Parks Road, Oxford, OX1 3PU, United Kingdom
5London Institute for Mathematical Sciences, 35a South Street Mayfair, London, W1K 2XF, United Kingdom

(Dated: June 28, 2018)

We consider whether trainable quantum unitaries can be used to discover quantum speed-ups for classical problems. Using methods recently developed for training quantum neural nets, we consider Simon’s problem, for which there is a known quantum algorithm which performs exponentially faster in the number of bits, relative to the best known classical algorithm. We give the problem to a randomly chosen but trainable unitary circuit, and find that the training recovers Simon’s algorithm as hoped.

INTRODUCTION

The power of quantum computation by [1] Simon provides an exponentially faster quantum algorithm compared to a classical randomised search. Simon illustrated a very simple and scalable quantum circuit to solve a mathematical game now known as Simon’s problem. The aim is to learn a property of a black-box function, a secret bit string s, which determines the function within the families of functions under consideration. Simon’s quantum algorithm is an important precursor to Shor’s Algorithm [2], which also provides an exponential speed up over the best known classical algorithms. With a quantum computer, one could employ Shor’s algorithm to quickly break the widely used RSA cryptographic protocol [3]. Simon’s and Shor’s algorithms are both examples of the Hidden Subgroup Problem over Abelian groups [4].

Quantum machine learning, see e.g. [5–20], contains a research direction known as quantum learning [21–27] which concerns learning and optimising with truly quantum objects. In [24] some of the present authors defined a quantum generalisation of feedforward neural networks which could numerically be trained to perform various quantum generalisations of classical tasks. This motivated us to consider whether these networks can also find quantum speed-ups for classical tasks. This could help deal with the shortage of useful quantum algorithms. To test this, Simon’s algorithm is a natural candidate, having an exponential speed-up over the best known classical algorithm at the same time as being a more minimal, and thus more tractable, algorithm than Shor’s.

We here accordingly aim to determine whether a quantum neural net can discover Simon’s algorithm. We design an explicit training procedure, and demonstrate that it works. This gives significant hope that it is possible to discover new algorithms using this method of quantum learning.

1. TECHNICAL INTRO

The notation: \(|a|^N \equiv \bigotimes_{k=1}^{N} |a\rangle\otimes|a\rangle\otimes...\otimes|a\rangle\), will be used throughout. Note that the words “gates” and “unitaries” will be used synonymously throughout. Also, the words “blackbox function” and “oracle” are interchangeable.

A. Simon’s algorithm

Simon’s problem and solution can be summarised as follows [1–28]. There is a blackbox function, or oracle, that holds a secret string s, within it. One can ask the oracle questions by querying it. The goal is to infer the secret string s with the least number of queries. This blackbox function could be represented classically as \(f(x)\) – a function that takes an n bitstring, \(x = x_1 x_2 x_3 ... x_n\), as an input, where \(x_i\) is either zero or one. The n bitstring, \(x\), lives in the set, \(\{0,1\}^n\), which is the collection of all possible n bitstrings. \(f(x)\) is by design guaranteed to either be a particular type of many-to-one functions or a one-to-one function. We restrict, for simplicity, \(f(x)\) further by excluding the one-to-one case.

In the quantum version, the blackbox function generalises to a unitary transformation of states: \(\hat{U}_f \in \mathbb{U}^{2^n}\). In Simon’s solution, the quantum state: \(|x\rangle = |x_1\rangle \otimes |x_2\rangle \otimes ... \otimes |x_n\rangle \otimes |0\rangle^\otimes n\) encodes the same bit string \(x = x_1 x_2 x_3 ... x_n\). The classical to quantum generalisation of the oracle can be pictured through Fig. [1]. The quantum version of the oracle acts on a quantum bitstring of length \(2n\) instead of \(n\). This is a well known technique in quantum generalisations. Since closed quantum evolution is inherently unitary (reversible), we need the oracle

* Correspondence: Oscar Dahlsten (dahlsten@sustc.edu.cn)
and Fig. 2. The matrix $\hat{U}$ of the fixed secret string oracle unitary, $\hat{a}$ to the initial state $\hat{f}(x) \in \{0,1\}^{n}$.

The game is to find the secret string $s$ to the computational basis. One applies the gate $\hat{U} = \vec{a}$, which are maps that perform: $f : \{0,1\}^{n} \rightarrow \{0,1\}^{n}$ and $\hat{f} = \hat{U}|0\rangle \rightarrow |\hat{f}(x)\rangle \rightarrow |f(x)\rangle$ respectively. Note that the qubits live in the space $\mathbb{H}$. The red box in the lower quantum circuit diagram shows the effective input to output of the quantum string as the generalisation of the oracle to a quantum one is reversible.

FIG. 1. This shows the generalisation of a classical function oracle to a quantum unitary operator oracle, which are maps that perform: $f : \{0,1\}^{n} \rightarrow \{0,1\}^{n}$ and $\hat{U} : |0\rangle^{\otimes n} \rightarrow |\hat{f}(x)\rangle^{\otimes n}$ respectively. Note that the qubit lives in the space $\mathbb{H}$. The red box in the lower quantum circuit diagram shows the effective input to output of the quantum string as the generalisation of the oracle to a quantum one is reversible.

to map the initial state, $|x\rangle \otimes |0\rangle^{\otimes n}$, to the final state $|x\rangle \otimes |f(x)\rangle$ in order to make the map reversible. The extra padding of qubits ($|0\rangle^{\otimes n}$ after the effective output) allows the back-tracking of the input given the output, hence the map $\hat{U}$ is reversible. In summary, Simon’s problem is as follows.

1. $s$ is an $n$ bit string.
2. Blackbox function $f : \{0,1\}^{n} \rightarrow \{0,1\}^{n}$
3. $\exists$ secret string $s \in \{0,1\}^{n} \not\equiv 00...000$ such that for all inputs $x \in \{0,1\}^{n}, f(x) = f(x \oplus s)$
4. For all inputs $x, y \in \{0,1\}^{n}, if \not\equiv y \oplus s \Rightarrow f(x) \neq f(y)$,

where $\oplus$ is the bitwise modulo 2 addition of two $n$ bit strings. The game is to find the secret string $s$. Note that the modulo 2 bitwise addition of $s$ to any input $x$ will not change the outcome of the function.

In Simon’s quantum scheme, one has to have access to a $2n$ bit string long quantum state, $|0\rangle^{\otimes 2n}$ in the computational basis. One applies the gate $\hat{U}_{Simon} = \hat{H}_{g}^{\otimes n} \otimes 1_{2}$ to the initial state $|0\rangle^{\otimes 2n}$ before and after the application of the fixed secret string oracle unitary, $\hat{U}$ as shown in Fig. 2. The matrix $\hat{H}$ is the $2$ by $2$ Hadamard matrix and $1_{2}$ is the $2$ by $2$ identity matrix. In the basis of the Pauli-z eigenstates $\{0,1\}$, $\hat{H} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. Hence the final state from the quantum circuit is: $|final\rangle = \hat{U}_{Simon} \cdot \hat{U} \cdot \hat{f} \cdot \hat{U}_{Simon} |0\rangle^{\otimes 2n}$.

This can be represented using the quantum circuit diagram in Fig. 2 for $n = 2$.

FIG. 2. Simon’s quantum circuit for $n = 2$. The 2 top qubits are associated with the input to the function and the two lower to the output. Hadamard unitaries create a superposition of inputs. $\hat{U}$ enacts the function reversibly, storing the input in the upper two qubits. Hadamards again after $\hat{U}$ create further input superposition branching, allowing for interference between terms which originally had different inputs and the same outputs. The number on the right-hand side labels the output port number.

We shall represent the secret string as:

$$s = s_{1}s_{2}...s_{n} \in \{0,1\}^{n}\tag{2}$$

where $s_{i} \in \{0,1\}$ is the $i^{th}$ bit in the $n$ bit string $s$.

In Simon’s Algorithm, one measures the first register (output port 1 to 2 in Fig. 2). This will produce an $n$ bit string, $y$, such that the dot product between $y$ and $s$ in mod 2 is zero, i.e. $y \cdot s = 0 \pmod{2}$. The algorithm requires repeated inquiries to the oracle, hence obtaining many different $n$ bit strings, $y^{(i)}$, with $i$ indexing the results obtained from each inquiry. The $y$ obtained will be $\{y^{(1)}, y^{(2)}, y^{(3)}, ..., y^{(J)}\}$, for $J$ inquiries. Then a classical processing task of Gaussian Elimination in the Galois field $GF(2)$ is carried out to find $s$, represented as follows:

$$\begin{bmatrix} 0 & 0 & \cdots & 0 \\ y_{1}^{(1)} & y_{2}^{(1)} & \cdots & y_{n}^{(1)} \\ \vdots & \ddots & \ddots & \vdots \\ y_{1}^{(n-1)} & y_{2}^{(n-1)} & \cdots & y_{n}^{(n-1)} \end{bmatrix} \begin{bmatrix} s_{1} \\ s_{2} \\ \vdots \\ s_{n} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \pmod{2}
$$

Solving the linear equations in $GF(2)$ is equivalent to only permitting the $s$ vector’s elements to be in $\{0,1\}$, solving the equations in modulo 2.
Since we require that $s$ is not the zero string, only $n - 1$ linearly independent equations are needed. The measurement process is probabilistic, meaning sometimes one might not get a set of linearly independent equations to perform Gaussian elimination on, hence one has to inquire the oracle more than $n - 1$ times to get a unique solution for $s$, which means $J > n - 1$ on average. The Gaussian elimination would have at worst $O(n^{2.3755})$ complexity in time overhead, because the fastest classical algorithm to solve linear equations by Coppersmith and Winograd [29], scales in that manner.

### B. General Unitary Matrix Parameterisation

We shall train over families of unitaries using techniques from [27]. We shall use a general form of a unitary matrix in terms of Pauli matrices. A general 1 qubit unitary circuit could be written in the forms:

\[
\hat{U}^{(1 \text{ qubit})} = \exp \left( i \sum_{j=0}^{3} \alpha_j \hat{\sigma}_j \right), \quad \text{or}
\]

\[
\hat{U}^{(1 \text{ qubit})} = e^{i \alpha \hat{z}} \left( \cos \Omega \mathbb{1} + i \frac{\sin \Omega}{\Omega} \sum_{j=1}^{3} \alpha_j \hat{\sigma}_j \right),
\]

where $\{\hat{\sigma}_0, \hat{\sigma}_1, \hat{\sigma}_2, \hat{\sigma}_3\}$ are the 2 x 2 identity, Pauli-x, y and z matrices respectively and $\alpha_i \in [0, 2\pi)$ and $\Omega = \sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2}$.

A useful special case with one parameter we will use is $U = \cos(\theta)\hat{\sigma}_3 + \sin(\theta)\hat{\sigma}_1$. A general two qubit unitary could be written in a similar form:

\[
\hat{U}^{(2 \text{ qubits})} = \exp \left( i \sum_{j=0}^{3} \sum_{k=0}^{3} \alpha_{j,k} \hat{\sigma}_j \otimes \hat{\sigma}_k \right).
\]

### 2. METHODS

#### A. Overall approach

The overall approach is described in Fig. 3. The black-box unitary is given, and the quantum circuit together with the classical post processing needs to learn which states to inject and what type of post-processing to do. There are single-qubit unitaries (or 2 qubit unitaries) with free parameters which get tuned in a systematic manner during the training procedure. There is also a classical post-processing part, which could in principle be a quantum circuit but the number of qubits required would be impractical for simulation/experiments, so it is kept classical.

![Diagram](image)

**FIG. 3.** Gradient descent is performed on free parameters $\theta_1$, $\theta_2$ and $\theta_3$, with respect to a single cost function, over all training examples. The quantum part is looped over $J > n - 1$ times in line with the technical introduction.

#### B. Number of possible training examples

**Lemma 2.1.** The number of 2 to 1 functions of $n$ bitstrings is $2^n C_{2n-1}$ for a given $s$.

**Proof.** There are $2^n$ possible $n$ bitstrings. The statement of the lemma is equivalent to saying how many different ways are there to pick $2^n$ objects from a set of $2^n$ objects disregarding ordering.

$\Rightarrow 2^n C_{2n-1}$ different 2 to 1 functions of $n$ bitstrings. □

**Lemma 2.2.** For each $n \in \mathbb{Z}$ in Simon's Problem, there are $(2^n - 1) C_{2n-1}$ possible mapping tables.

**Proof.** The oracle can have $2^n C_{2n-1}$ different functions for a given $s$ (using lemma 2.1), and there are $(2^n - 1)$ different possible $s$ as the problem excludes the zero $n$ bit-string.

$\Rightarrow (2^n - 1) C_{2n-1}$ is the number of mapping tables for a specific $n$ in Simon’s Problem. □

#### C. Cost function

The cost function, which is the mathematical representation of the aim of the task, is

\[
C = \sum_{s \neq 000...000} (p_{\text{desired}}^s - p^s)^2,
\]

where $p_{\text{desired}}^s$ is the probability of the output bit string we want it to be and $p^s$ is the probability of the measured value. The value of $C$ depends on the free parameters being tuned. This is a supervised learning scenario as the correct $s$ is known and used to evaluate the cost function [30].
D. Gradient descent

The quantum circuit is systematically tuned until it reaches a minimum turning point in the cost function. This could be achieved through gradient descent with respect to the $\alpha_{j,k}$ parameters from the unitaries in Eq. 4. Gradient descent is defined as:

$$\alpha_{j,k}^{(l)} \rightarrow \alpha_{j,k}^{(l)} - \eta \frac{\partial C}{\partial \alpha_{j,k}^{(l)}}, \quad \forall j, k,$$

(7)

where $\eta \in \mathbb{R}^+$ is the step size of the gradient descent and $l$ labels the different unitaries.

E. Gradient Descent Assisted Genetic Algorithm Search

Whilst gradient descent works well in many examples, it is a reasonable assumption that optimisation in a high dimensional parameter space may have many local minima. In order to get out of a local minimum [31], a Genetic Algorithm is used in the optimisation. This could be done in parallel, which is worthwhile when the computation is done in a supercomputer or using GPUs with a multitude of cores. The gradient assisted genetic algorithm implemented in my system was as follows heuristically:

1. An agent is a random initial guess for the parameters of the unitaries. Start with many agents. This will involve simultaneously initialising many different sets of $\alpha_{j,k}^{(l)}$, with each set being a different agent. The genetic information of each agent is the $\alpha_{j,k}^{(l)}$ parameters.

2. Carry out gradient descent on each agent individually. In this parallelisable procedure, gradient descent is performed for a small number of steps. We shall call this number of steps in gradient descent a generation.

3. After one generation, compare the cost function of each of the agents and find the $\alpha_{j,k}^{(l)}$ of the few agents with the lowest cost. Then repopulate the entire population with the selected few agents with the lowest cost.

4. Apply a small probabilistic random parameter to the $\alpha_{j,k}^{(l)}$ while repopulating the population. This would be analogous to the mutation process in Biology.

5. Repeat the procedure until a minimum is found.

Despite the high computational cost, this method will not guarantee that the final solution will be a global minimum. However, the benefits of this type of search is that the algorithm is now very parallelisable and the mutations added may aid the agents in getting out of a local minima. See Fig. 4 for a pictorial description of the algorithm.

![FIG. 4. A pictorial representation of how a genetic algorithm works. Many agents are initialised and they are all allowed to propagate to find the minimum.](image)

F. Classical post-processing

The classical post-processing takes a set of outputs (the classical bit string $y$'s) and maps them to the corresponding guess for the secret bit-string: $s$. It thus plays an essential role in the algorithm. In Simon’s algorithm this is done by Gaussian elimination modulo 2, as discussed in the technical introduction.

Whilst this part could in principle be enacted with a quantum circuit, as quantum unitary circuits generalise classical computing, that is very costly experimentally and in terms of classical numerical simulation. Our approach here is thus, as depicted in Fig. 3, to include classical post-processing, which takes classical input(s) and then gives the answer: the secret bit string $s$. As Simon’s algorithm and other similar algorithms require several outputs before the classical post-processing, we loop over the quantum part $J$ times to give $J$ classical outputs which are then fed to the classical post-processing.

The classical post-processing amounts to a classical input-output function. This function can be trained as part of the overall training or to make it simpler it can be set by hand to what we want it to be. In the case where only one $s$ was shown to be needed for $n = 2$ and $n = 3$, the table was set by hand. For $n = 2$ we have also tested that the classical part can be trained together with the quantum part rather than set by hand. The training of this part was done by switching one uniformly randomly chosen pair of output bit strings at a time, and accepting the switch only if it decreased the cost function.

3. RESULTS

The main results are summarised as follows.
A. Recovering Simon’s Circuit

Results. A set of restricted unitaries in the form of

\[ \hat{G}(\theta_i) = \begin{bmatrix} \cos(\theta_i) & \sin(\theta_i) \\ \sin(\theta_i) & -\cos(\theta_i) \end{bmatrix} \]  

(8)
could be put in place of the Hadamard matrices in Simon’s original circuit, as shown in Fig. 3, such that gradient
descent could be performed on \( \theta_1 \), \( \theta_2 \) and \( \theta_3 \) with respect
to the cost function of Eq. 6 to recover Simon’s circuit.

Starting with general single-qubit unitaries yields the
same performance as in that restricted single qubit uni-
tary case.

Results. The same training performed on a set of re-
stricted unitaries in the form of Eq. 4 recovers a cir-
cuit with the same performance (same cost function
minimum) as Simon’s circuit, but not necessarily the
Hadamard gates as in Simon’s circuit.

FIG. 5. This is the cost landscape sampled under a constant
post processing permutation matrix. It can be shown that
Simon’s circuit lies in a local minima via the red line which
represented the gradient descent path with a starting point
close to Simon’s quantum circuit solution parameterised with
\( \theta_1 \) and \( \theta_2 \).

Fig. 5 illustrates how Simon’s solution \((\theta_1, \theta_2) = (\pi/4, \pi/4)\)
is a local minimum of the cost function
(for \( n=2 \)). To check if this is truly optimal we did a
brute force search over other circuits in this family and
confirmed that they never achieve a lower cost value than
Simon’s solution. Fig. 6 shows, for \( n = 3 \), parameters
staring at a random initial point, can also find the
minimum of the cost function, which means can recover
Simon’s algorithm. To speed up the classic simulation
on computer or high performance computer, we use the
parallel computing toolbox in MATLAB, which supports
parallel for-loops for running task-parallel algorithms on
multiple processors and make the full use of multicore
processors on the desktop via workers that run locally.

B. Do not need all secret bit strings to train it

We find that we can recover Simon’s algorithm through
training with just 1 secret string example for \( n=2 \), and 1
secret string for \( n=3 \). This is important as there are \( 2^n-1 \)
such secret strings (the null string is not counted, as it
corresponds to a permutation rather than \( 2^{-1} \) function),
as discussed earlier. An initial approach, wherein the
network would be asked to guess \( s \) after just one call to
the oracle, required all \( s \)'s for the training.

FIG. 6. An example of the cost function value as a function
of time during training. In this case, \( n = 3 \) and there are 3
free parameters, like those free parameters in Eq. 8

4. SUMMARY AND OUTLOOK

We trained a unitary network to find the optimal cir-
cuit for solving the task of finding the hidden bit string
associated with Simon’s oracle. The result is indeed that
the circuit associated with Simon’s algorithm is recov-
ered, such that quantum parallelism is used to probe the
oracle unitary. This demonstrates the potential of these
techniques for finding algorithms.

We chose Simon’s algorithm as a clean example of the
hidden subgroup problem algorithms. It is plausible that
the same approach can be used for other problems in that
class. A key challenge is to find a task that is technolog-
ically useful, such as factoring, but which does not yet
have a known quantum algorithm. The current approach
combines a human and machine to find the algorithm and
making the machine discovery more autonomous may in-
crease the chance of discovering algorithms we have not
yet thought of.

Note added: Whilst we were preparing this
manuscript, a paper with related ideas, for the case of
Grover’s algorithm, appeared on the pre-print server:

arXiv:1805.09387, Variationally learning Grover’s Search
Algorithm, Morales et al.
DATA AVAILABILITY STATEMENT

This is a theoretical paper and there is no experimen-
tal data available beyond the numerical simulation data
described in the paper.

ACKNOWLEDGEMENTS

Part of this work forms KHW’s BSc thesis at Impe-
rial College [32]. We acknowledge discussions with Jon
Alcock, Heliang Huang, Sania Jevtic and Doug Plato.
KHW is funded by the President’s PhD Scholarship of
Imperial College London. OD acknowledges funding from
the 1000 Talents Youth project, China and the Lon-
don Institute for Mathematical Sciences. FL acknowled-
ges use of the SUSTech supercomputer Shuguang 6000.
MK acknowledges funding from the Royal Society, a Sam-
sung GRO Grant, and a programme grant from the UK
EPSRC (EP/K034480/1).

[1] Daniel R. Simon, “On the power of quantum computa-
tion,” SIAM Journal on Computing 26 (5), 1474 – 1483
(1997).
[2] Peter W. Shor, “Polynomial-time algorithms for prime
factorization and discrete logarithms on a quantum com-
puter,” SIAM J. Comput. 26, 1484 – 1509 (1997).
[3] N.D. Mermin, Quantum Computer Science: An Intro-
duction (Cambridge University Press, 2007).
[4] Michelangelo Grigni, Leonard Schulman, Monica Vazi-
rani, and Umesh Vazirani, “Quantum mechanical algo-
rithms for the nonabelian hidden subgroup problem,” in
Proceedings of the Thirty-third Annual ACM Symposium
on Theory of Computing STOC ’01 (ACM, New York,
NY, USA, 2001) pp. 68–74.
[5] Biamonte et.al., “Quantum Machine Learning,” (2016),
arXiv:1607.08535.
[6] M. Schuld, I. Sinayskiy, and F. Petruccione, “The quest
for a quantum neural network,” Quantum Information
Processing 13, 2567–2586 (2014).
[7] S. Lloyd, M. Mohseni, and P. Rebentrost, “Quantum
algorithms for supervised and unsupervised machine learn-
ing,” (2013), arXiv:1307.0411 [quant-ph].
[8] S. Lloyd, M. Mohseni, and P. Rebentrost, “Quantum
principal component analysis,” Nature Physics 10, 631–
633 (2014).
[9] A. Montanaro, “Quantum pattern matching fast on av-
erage,” Algorithmica , 1–24 (2015).
[10] S. Aaronson, “Read the fine print,” Nature Physics 11,
291–293 (2015).
[11] S. Garnerone, P. Zanardi, and D. A. Lidar, “Adiabatic
quantum algorithm for search engine ranking,” Phys.
Rev. Lett. 108, 230506 (2012).
[12] A. W. Harrow, A. Hassidim, and S. Lloyd, “Quantum
algorithm for linear systems of equations,” Phys. Rev.
Lett. 103, 150502 (2009).
[13] S. Lloyd, S. Garnerone, and P. Zanardi, “Quantum
algorithms for topological and geometric analysis of big
data,” Nature Communications 7, 10158 (2016).
[14] P. Rebentrost, M. Mohseni, and S. Lloyd, “Quantum
support vector machine for big data classification,” Phys.
Rev. Lett. 113, 130503 (2014).
[15] N. Wiebe, D. Braun, and S. Lloyd, “Quantum algorithm
for data fitting,” Phys. Rev. Lett. 109, 050505 (2012).
[16] J. Adcock, E. Allen, M. Day, S. Frick, J. Hinchliff,
M. Johnson, S. Morley-Short, S. Pallister, A. Price, and
S. Stanisic, “Advances in quantum machine learning,”
(2015), arXiv:1512.02900 [quant-ph].
[17] B. Heim, T. F. Reunow, S. V. Isakov, and M. Troyer,
“Quantum versus classical annealing of Ising spin
glasses,” Science 348, 215–217 (2015).
[18] D. Gross, Y.K. Liu, S. T. Flammia, S. Becker, and
J. Eisert, “Quantum state tomography via compressed
sensing,” Phys. Rev. Lett. 105, 150401 (2010).
[19] V. Dunjko, J. M. Taylor, and H. J. Briegel, “Quantum-
enhanced machine learning,” Phys. Rev. Lett. 117, 130501
(2016).
[20] P. Wittek, ed., Quantum Machine Learning (Academic
Press, Boston, 2014).
[21] A. Bisio, G. Chiribella, G. M. D’Ariano, S. Fucchini,
and P. Perinotti, “Optimal quantum learning of a uni-
itary transformation,” Phys. Rev. A 81, 032324 (2010).
[22] M. Sasaki and A. Carlini, “Quantum learning and univer-
sal quantum matching machine,” Phys. Rev. A 66, 022303
(2002).
[23] J. Bang, J. Lim, M. S. Kim, and J. Lee, “Quan-
tum Learning Machine,” ArXiv e-prints (2008),
arXiv:0803.2976 [quant-ph].
[24] G. Sentís, M. GuTá, and G. Adesso, “Quantum learn-
ing of coherent states,” EPJ Quantum Technology 2, 17
(2015).
[25] L. Banchi, N. Pancotti, and S. Bose, “Quantum gate
learning in qubit networks: Toffoli gate without time-
dependent control,” Npj Quantum Information 2, 16019
EP – (2016).
[26] P. Palittapongarnpim, P. Wittek, E. Zahedinejad,
S Vedaie, and B. C. Sanders, “Learning in quantum
control: high-dimensional global optimization for noisy
quantum dynamics,” (2016).
[27] K. H. Wan, O. Dahlsten, H. Kristjánsson, R. Gard-
er, and M. S. Kim, “Quantum generalisation of feed-
forward neural networks,” npj Quantum Information 2
9 (3) (2017).
[28] Umesh V. Vazirani, “Simons algorithm,” (2004), unpub-
lished.
[29] Don Coppersmith and Shmuel Winograd, “Matrix mul-
tiplication via arithmetic progressions,” Journal of Sym-
metric Computation 9 (3), 251 (1990).
[30] M. A. Nielsen, Neural Networks and Deep Learning
(Determination Press, online book, 1991).
[31] Olivier de Weck, “A basic introduction to genetic algo-
rithms,” (2010), online notes, unpublished.
[32] Kwok Ho Wan, “The power of quantum computa-
tion—a new perspective, Imperial College London,” (2017),
B.Sc. Thesis.