ABEL MAPS OF GORENSTEIN CURVES

Lucia Caporaso\textsuperscript{a}, Juliana Coelho\textsuperscript{b} and Eduardo Esteves\textsuperscript{c} \footnote{Esteves was supported by CNPq, Processos 301117/04-7 and 470761/06-7, by CNPq/FAPERJ, Processo E-26/171.174/2003, and by the Institut Mittag–Leffler (Djursholm, Sweden).}

\textsuperscript{a} Dipartimento di Matematica, Università Roma Tre
Largo San Leonardo Murialdo 1, I-00146 Roma, Italy

\textsuperscript{b} Departamento de Matemática, Universidade Federal Fluminense,
Rua Mário Santos Braga s/n, 24020-005 Niterói, Brazil

\textsuperscript{c} Rua Almeida Godinho 26, apto. 106
22471-140 Rio de Janeiro RJ, Brazil

Abstract. For a Gorenstein curve $X$ and a nonsingular point $P \in X$, we construct Abel maps $A: X \to J^1_X$ and $A_P: X \to J^0_X$, where $J^i_X$ is the moduli scheme for simple, torsion-free, rank-1 sheaves on $X$ of degree $i$. The image curves of $A$ and $A_P$ are shown to have the same arithmetic genus of $X$. Also, $A$ and $A_P$ are shown to be embeddings away from rational subcurves $L \subset X$ meeting $X - L$ in separating nodes. Finally we establish a connection with Seshadri’s moduli scheme $U_X(1)$ for semistable, torsion-free, rank-1 sheaves on $X$, obtaining an embedding of $A(X)$ into $U_X(1)$.

1. Introduction

Fix an algebraically closed field $k$ of any characteristic and let $X$ be a connected, projective curve over $k$. If $X$ is smooth, there is, for each integer $d \geq 1$, a natural map,

$$A^d: X^d \to \text{Pic}^d X,$$

with $\text{Pic}^d X$ denoting the Picard scheme parameterizing line bundles of degree $d$ on $X$; the map sends $(P_1, \ldots, P_d)$ to $[\mathcal{O}_X(P_1 + \cdots + P_d)]$. Of course $A^d$ factors through a map

$$A^{(d)}: X^{(d)} \to \text{Pic}^d X,$$

where $X^{(d)}$ is the $d$th symmetric product of $X$. The map $A^{(d)}$ is called the degree-$d$ Abel map of $X$.

Classical variants are the degree-$d$ Abel maps with base point $P \in X$,

$$A^d_P: X^{(d)} \to \text{Pic}^0 X,$$

which is simply $A^d$ composed with the translation map $\text{Pic}^d X \to \text{Pic}^0 X$, taking $[\mathcal{L}]$ to $[\mathcal{L} \otimes \mathcal{O}_X(-dP)]$, and the compositions of $A^{(d)}$ and $A^{(d)}_P$ with the duality isomorphisms $\lambda^d: \text{Pic}^d X \to \text{Pic}^{-d} X$, taking $[\mathcal{L}]$ to $[\mathcal{L}^*]$. 
Much of the geometry of $X$ is encoded in the Abel maps, since their fibers are the complete linear systems of $X$. For instance, the gonality of $X$ is the smallest integer $d$ such that $A^{(d)}$ is not an embedding. In particular, $X$ is hyperelliptic if and only if $A^{(2)}$ is not an embedding.

The Abel maps behave naturally in families of smooth curves. As smooth curves degenerate to singular ones, we would like to understand how the Abel maps degenerate. So, how to construct Abel maps for singular curves in a natural way?

If $X$ is integral, Altman and Kleiman [AltK80] defined, for each $d \geq 1$, a natural map
\[ \beta^{(d)}: \text{Hilb}^d X \to J^{-d}_X, \]
where $\text{Hilb}^d X$ is the Hilbert scheme of $X$ parameterizing length-$d$ subschemes, and $J^{-d}_X$ is the compactified Jacobian parameterizing torsion-free, rank-1 sheaves of degree $-d$ on $X$; the map sends $[Y]$ to $[\mathcal{I}_Y/X]$. Again, the fibers of $\beta^{(d)}$ are projective spaces. And, if $X$ is smooth, then $\text{Hilb}^d X = X^{(d)}$, $J^{-d}_X = \text{Pic}^{-d} X$ and $\beta^{(d)} = \lambda^d \circ A^{(d)}$.

On the other hand, if a curve is reducible, the situation is more complex. The current knowledge is concentrated on the two extremes: $d = 1$ and $d = g - 1$. For $d = g - 1$, the image of $A^d$ turns out to be the theta divisor. For work extending the construction and the properties of the theta divisor to singular curves we refer the reader to [So94] and [E97] for irreducible curves, and to the more recent [A04] and [C07] for nodal, possibly reducible, curves.

As for $d = 1$, Edixhoven [Ed98] constructed and studied rational Abel maps of nodal curves to Néron models. As Néron models are seldom complete, his maps are not defined everywhere. In [CE06] the compactifications of Néron models and Picard schemes, constructed in [C05], are used; it is shown that, if $X$ is stable, there exists a globally defined map $\alpha^1_X: X \to \overline{P^1_X}$, where $\overline{P^1_X}$ is the compactified Picard scheme parameterizing equivalence classes of degree-1 “semibalanced” line bundles on semistable curves having $X$ as a stable model.

In this paper we extend the construction of $\alpha^1_X$ to any $G$-stable curve $X$, that is, to any reduced curve $X$ with Gorenstein singularities whose dualizing sheaf is ample. Also, we describe the image and the fibers of our Abel map. More precisely, for any reduced curve $X$ with Gorenstein singularities, we consider the fine moduli schemes $J^d_X$, parameterizing simple, torsion-free, rank-1 sheaves of degree $d$ on $X$, for all integers $d$. And we construct a map (cf. 5.2):
\[ A: X \to J^1_X. \]

The schemes $J^d_X$ are quite large, not even Noetherian, but have an open subscheme of finite type, $J^{d,ss}_X$, parameterizing semistable sheaves (cf. 2.4). The map $A$ is constructed in such a way that $A(X) \subseteq J^{1,ss}_X$, if $X$ is $G$-stable (cf. Theorem 5.4).

If $X$ has no separating nodes (nodes whose removal disconnects the curve), then $A$ sends $Q$ to $[\mathcal{I}_Q^*/X]$. In particular, if $X$ is smooth we recover the classical degree-1 Abel map $A^{(1)}$. On the other hand, if $X$ does admit a separating node $N$, then $\mathcal{I}_{\overline{N}/X}$ is not simple, and thus not parameterized by $J^1_X$. So, for each $Q \in X$ we create a new sheaf $\mathcal{I}^1_Q$ out of $\mathcal{I}^*_Q/X$ by tensoring the latter with suitable so-called “twisters”, along the same lines of what was done in [CE06], and let $A$ send $Q$ to $[\mathcal{I}^1_Q]$. 
In Theorem 6.3 we prove that $A$ contracts every smooth rational subcurve $L \subseteq X$ meeting its complementary curve in separating nodes of $X$, and $A$ is an embedding off these subcurves. Also, $A(X)$ has the same arithmetic genus of $X$ and its singularities are those of $X$, together with ordinary singularities with linearly independent tangent lines.

Unfortunately, the schemes $J^{d,ss}_X$ are not, in general, separated. To get a separated scheme, two alternatives are possible: to use either smaller schemes or quotient schemes. We consider both.

For each nonsingular point $P \in X$ the scheme $J^{d,ss}_X$ has an open subscheme $J^{d,P}_X$, parameterizing sheaves that are $P$-quasistable (cf. 2.4), which is projective over $k$. If $P$ is suitably chosen, and $X$ is $G$-stable, then $A(X) \subseteq J^{1,P}_X$ by our Theorem 5.4.

On the other hand, we consider Seshadri's coarse moduli schemes $U_X(d)$ for equivalence classes of semistable, torsion-free, rank-1 sheaves on $X$ of degree $d$ (cf. 2.6). There are natural maps $\Phi^d : J^{d,ss}_X \to U_X(d)$, taking a semistable sheaf to its class. Our Theorem 7.2 says that $\Phi^1$ restricts to an embedding on $A(X)$.

We also construct Abel maps with base points. More precisely, for each nonsingular point $P \in X$ we construct a natural map (cf. 4.5),

$$A_P : X \longrightarrow J^0_X,$$

with image in $J^{0,P}_X$ (cf. Theorem 4.8). If $X$ has no separating nodes, then $A_P$ sends $Q$ to $[\mathcal{I}_{Q/X} \otimes \mathcal{O}_X(P)]$. So, if $X$ is smooth then $A_P = \lambda^0 \circ A^{(1)}_P$. If $X$ has separating nodes, $A_P$ is constructed with the help of twisters, as done for $A$.

The map $A_P$ has the same description as $A$. In fact, for a suitably chosen $P$, we may view $A$ as the composition of $A_P$ with the duality map $J^0_X \to J^1_X$, taking $[\mathcal{I}]$ to $[\mathcal{I}^*]$ and the translation map $J^0_X \to J^1_X$, taking $[\mathcal{I}]$ to $[\mathcal{I} \otimes \mathcal{O}_X(P)]$. So, up to these isomorphisms, $A$ may be viewed as one of the $A_P$. (In fact, everywhere in the paper we prove properties first for $A_P$ and then extend them to $A$.)

The biggest difference between $A$ and the $A_P$ comes when we consider their composition to Seshadri’s moduli schemes: $\Phi^0 \circ A_P$ may actually collapse components of $X$ that were not collapsed by $A_P$, as we point out in Remark 7.3.

We conclude with a few comments about closely related questions and further developments. First, $A$ is not always a natural map. In fact, if $X$ has a “splitting” node (i.e. a separating node splitting the curve in two equal genus subcurves) then $A$ depends on the choice of one of these subcurves (cf. 5.2). The lack of naturality is a major hurdle to extend our construction to families of curves. In this respect, the map to Seshadri’s moduli space, $\Phi^1 \circ A$, looks more natural, as it is independent of the above choice by Theorem 7.2.

On the other hand, the maps $A_P$ are natural, and it seems possible to extend their construction to families of pointed curves, whereas the compositions $\Phi^0 \circ A_P$ do not behave well. We hope to deal with Abel maps for families in the future.

Second, we don’t treat higher degree Abel maps. It seems possible to define them not on $X^d$, $X^{(d)}$ or $\text{Hilb}_X^d$, but on blowups of them. Very little is known, apart from the case of the degree-2 Abel map for a nodal curve with two components meeting at two points, constructed in [Co06].
Here is a layout of the paper: In Section 2, we introduce the moduli schemes \( J_{X}^{d,ss} \) and \( U_{X}(d) \), and the quotient maps \( \Phi^{d}: J_{X}^{d,ss} \to U_{X}(d) \). In Section 3, we construct the Abel maps \( A \) and \( A_{P} \) when \( X \) has no separating nodes. In Section 4, we construct the maps \( A_{P} \) in general, and in Section 5 we do the same for the map \( A \). In Section 6, we prove properties of the maps \( A \) and \( A_{P} \), describing their images and fibers. Finally, in Section 7 we show that \( \Phi^{1} \) restricts to an embedding on \( A(X) \).

2. Compactified Jacobians

All schemes are assumed to be locally of finite type over a fixed algebraically closed field \( k \). A point of a scheme means a closed point. A curve is a reduced, projective scheme of pure dimension 1 over \( k \). If \( Y \) is a curve, we let \( g_{Y} := 1 - \chi(\mathcal{O}_{Y}) \) and call \( g_{Y} \) the (arithmetic) genus of \( Y \).

Throughout the paper, \( X \) denotes a connected curve, \( \omega \) its dualizing sheaf, \( g \) its (arithmetic) genus and \( P \) a point on the nonsingular locus of \( X \).

2.1. (Preliminaries) A reduced union of irreducible components of \( X \), connected or not, is called a subcurve. If \( Y \) is a proper subcurve of \( X \), let \( Y' \) denote the complementary subcurve, that is, the reduced union of all the irreducible components of \( X \) not contained in \( Y \). The intersection \( Y \cap Y' \) is a finite scheme; let \( \delta_{Y} \) denote its length. Since \( X \) is connected, \( \delta_{Y} > 0 \). Also, observe that
\[
g_{Y} \leq g.
\]

Let \( \mathcal{I} \) be a coherent sheaf on \( X \). We say that \( \mathcal{I} \) is torsion-free if its associated points are generic points of \( X \). We say that \( \mathcal{I} \) is of rank 1 if \( \mathcal{I} \) is invertible on a dense open subset of \( X \). And we say that \( \mathcal{I} \) is simple if \( \text{End}(\mathcal{I}) = k \). Each line bundle on \( X \) is torsion-free of rank 1 and simple.

Suppose \( \mathcal{I} \) is torsion-free of rank 1. We call \( \deg(\mathcal{I}) := \chi(\mathcal{I}) - \chi(\mathcal{O}_{X}) \) the degree of \( \mathcal{I} \). For each vector bundle \( F \) on \( X \),
\[
\chi(\mathcal{I} \otimes F) = \text{rk}(F)\chi(\mathcal{I}) + \deg(F) = \text{rk}(F)(\deg(\mathcal{I}) + 1 - g) + \deg(F).
\]

For each subcurve \( Y \) of \( X \), let \( \mathcal{I}_{Y} \) denote the restriction of \( \mathcal{I} \) to \( Y \) modulo torsion, that is, the image of the natural map
\[
\mathcal{I}|_{Y} \longrightarrow \bigoplus_{i=1}^{m}(\mathcal{I}|_{Y})_{\xi_{i}},
\]
where \( \xi_{1}, \ldots, \xi_{m} \) are the generic points of \( Y \). We let \( \deg_{Y}(\mathcal{I}) \) denote the degree of \( \mathcal{I}_{Y} \), that is, \( \deg_{Y}(\mathcal{I}) := \chi(\mathcal{I}_{Y}) - \chi(\mathcal{O}_{Y}) \).

Let \( Y \) be a proper subcurve of \( X \). By the defining property of the dualizing sheaf, the kernel of the restriction map \( \omega \to \omega|_{Y} \) is the dualizing sheaf \( \omega_{Y} \) of \( Y \). Suppose \( X \) is Gorenstein. Then \( \omega \) is a line bundle of degree \( 2g - 2 \), and it follows that
\[
\chi(\omega|_{Y}) = \chi(\omega_{Y}) + \delta_{Y}.
\]
Thus, by duality,
\[
\deg(\omega|_{Y}) = \chi(\omega_{Y}) + \delta_{Y} - \chi(\mathcal{O}_{Y}) = -2\chi(\mathcal{O}_{Y}) + \delta_{Y} = 2g_{Y} - 2 + \delta_{Y}.
\]
Definition 2.2. The curve $X$ is called $G$-stable if $X$ is Gorenstein of genus $g \geq 2$, and does not contain any smooth rational component $L$ with $\delta_L \geq 2$. Equivalently, using (2.1.3), the curve $X$ is G-stable if $X$ is Gorenstein and $\omega$ is ample.

If the singularities of $X$ are (ordinary) nodes, then $X$ is G-stable if and only if it is stable, in the sense of Deligne and Mumford.

2.3. (Semistable sheaves) Let $F$ be a vector bundle and $\mathcal{I}$ a torsion-free, rank-1 sheaf of degree $d$ on the curve $X$. We call $\mathcal{I}$ semistable with respect to $F$ if

1. $\chi(\mathcal{I} \otimes F) = 0$ and
2. $\chi(\mathcal{N} \otimes F) \geq 0$ for each nonzero quotient $\mathcal{N}$ of $\mathcal{I}$ different from $\mathcal{I}$.

We call $\mathcal{I}$ stable with respect to $F$ if the inequality in Condition 2 is always strict.

By (2.1.2), Condition 1 is verified if the slope $\mu(F) := \deg(F)/\text{rk}(F)$ satisfies $\mu(F) = g - 1 - d$. As for Condition 2, observe that all nonzero torsion-free quotients of $\mathcal{I}$ are of the form $\mathcal{I}_Y$ for a subcurve $Y \subseteq X$. So, Condition 2 holds if and only if

$$(2.3.1) \quad \chi(\mathcal{I}_Y \otimes F|_Y) \geq 0$$

for each proper subcurve $Y$ of $X$. For stability, we require strict inequalities.

We say that $\mathcal{I}$ is $P$-quasistable with respect to $F$ if inequality (2.3.1) holds for each proper subcurve $Y \subseteq X$, with equality only if $P \not\in Y$. Clearly, this notion depends only on which component of $X$ the point $P$ lies.

Suppose $X$ is Gorenstein. Define a vector bundle $E_d$ on $X$ as follows: If $g \geq 2$,

$$E_d := \mathcal{O}_X^{g \geq 2g - 3} \oplus \omega^{g - 1 - d}.$$ 

if $g = 0$, set $E_d := \mathcal{O}_X \oplus \omega^{g - 1 + 1}$; and if $g = 1$, define $E_d$ only if $d = 0$, setting $E_0 := \mathcal{O}_X$. We call $E_d$ the canonical $d$-polarization of $X$. Notice that $E_d$ has slope $\mu(E_d) = g - 1 - d$.

We say that $\mathcal{I}$ is (canonically) stable, semistable or $P$-quasistable if $\mathcal{I}$ is so with respect to $E_d$. If $g \geq 2$, then, for each subcurve $Y \subseteq X$,

$$\chi(\mathcal{I}_Y \otimes E_d|_Y) = (2g - 2)\chi(\mathcal{I}_Y) + \deg(E_d|_Y)$$

$$= (2g - 2)\deg_Y(\mathcal{I}) + (2g - 2)\chi(\mathcal{O}_Y) + (g - 1 - d) \deg(\omega|_Y)$$

$$= (2g - 2)\deg_Y(\mathcal{I}) + (1 - g)(\deg(\omega|_Y) - \delta_Y) + (g - 1 - d) \deg(\omega|_Y)$$

$$= (2g - 2)\deg_Y(\mathcal{I}) + (g - 1)\delta_Y - d \deg(\omega|_Y),$$

where we used (2.1.2) and (2.1.3). Thus, $\mathcal{I}$ is semistable if and only if

$$(2.3.3) \quad \deg_Y(\mathcal{I}) \geq d\left(\frac{\deg_Y(\omega)}{2g - 2} - \frac{\delta_Y}{2}\right)$$

for each proper subcurve $Y$ of $X$. If $g = 0$, then an analogous computation can be done, and the condition for semistability is the same. Finally, if $g = 1$ and $d = 0$, then $\mathcal{I}$ is semistable if and only if

$$(2.3.4) \quad \deg_Y(\mathcal{I}) \geq -\frac{\delta_Y}{2}.$$
for each proper subcurve $Y$ of $X$. Notice that (2.3.3) and (2.3.4) are equal conditions if $d = 0$. We leave it to the reader to formulate the analogous conditions for when $\mathcal{I}$ is stable or $P$-quasistable.

2.4. (The fine compactified Jacobians) There exists a scheme $J_X$ parameterizing torsion-free, rank-1, simple sheaves on the curve $X$; see [E01] Thm. B, p. 3048. More precisely, given a scheme $T$, a $T$-flat coherent sheaf $\mathcal{I}$ on $X \times T$ is called torsion-free (resp. rank-1, resp. simple) on $X \times T/T$ if $\mathcal{I}|_{X \times t}$ is torsion-free (resp. rank-1, resp. simple) for every $t \in T$. The scheme $J_X$ represents the functor that associates to each scheme $T$ the set of torsion-free, rank-1 simple sheaves on $X \times T/T$ modulo equivalence $\sim$. Two such sheaves $\mathcal{I}_1$ and $\mathcal{I}_2$ are called equivalent, $\mathcal{I}_1 \sim \mathcal{I}_2$, if there is a line bundle $N$ on $T$ such that $\mathcal{I}_1 \cong \mathcal{I}_2 \otimes p_2^*N$, where $p_2 : X \times T \to T$ is the projection map.

If $T$ is a connected scheme, and $\mathcal{I}$ is a torsion-free, rank-1 sheaf on $X \times T/T$, then $d := \deg(\mathcal{I}|_{X \times t})$ does not depend on the choice of $t \in T$; we say that $\mathcal{I}$ is a degree-$d$ sheaf on $X \times T/T$. Then there is a natural decomposition

$$J_X = \coprod_{d \in \mathbb{Z}} J_X^d,$$

where $J_X^d$ is the subscheme of $J_X$ parameterizing degree-$d$ sheaves. The schemes $J_X^d$ are universally closed over $k$; see [E01] Thm. 32, (2), p. 3068. However, in general, the $J_X^d$ are neither of finite type nor separated over $k$.

Let $F$ be a vector bundle on $X$ with integer slope, and set $d := g - 1 - \mu(F)$. By [E01] Prop. 34, p. 3071, the subschemes $J_X^{s,s}$ (resp. $J_X^{\phi}$, resp. $J_X^{P,P}$) of $J_X^d$ parameterizing simple and semistable (resp. stable, resp. $P$-quasistable) sheaves on $X$ with respect to $F$ are open. If $X$ is Gorenstein and $F = E_d$ (the canonical $d$-polarization defined in [2.3]), we write

$$J_X^{s,s} := J_{E_d}^{s,s}, \quad J_X^{d,s} := J_{E_d}^{d,s} \quad \text{and} \quad J_X^{d,P} := J_{E_d}^{d,P}.$$ 

By [E01] Thm. A, p. 3047, $J_X^{s,s}$ is of finite type and universally closed, $J_X^{\phi}$ is separated and $J_X^{P,P}$ is complete over $k$. Actually, $J_X^{P,P}$ is projective; see [E07] Prop. 2.4.

2.5. (The $S$-equivalence) Let $F$ be a vector bundle on $X$. For each semistable sheaf $\mathcal{I}$ on $X$ with respect to $F$, there is a maximal filtration

$$\emptyset \subsetneq Y_1 \subsetneq Y_2 \subsetneq \cdots \subsetneq Y_{q-1} \subsetneq Y_q \supseteq X$$

of $X$ by subcurves $Y_i$ such that $\chi(\mathcal{I}|_{Y_i} \otimes F|_{Y_i}) = 0$ for each $i = 1, \ldots, q$, which we call a Jordan–Hölder filtration. There may be many Jordan–Hölder filtrations associated to $\mathcal{I}$, but the collection of subcurves

$$\Theta(\mathcal{I}) := \{Y_1, Y_2 - Y_1, \ldots, Y_{q-1} - Y_{q-2}, X - Y_q\}$$

and the isomorphism class of the sheaf

$$\text{Gr}(\mathcal{I}) := \mathcal{I}|_{Y_1} \oplus \ker(\mathcal{I}|_{Y_2} \to \mathcal{I}|_{Y_2}) \oplus \cdots \oplus \ker(\mathcal{I}|_{Y_{q-1}} \to \mathcal{I}|_{Y_{q-1}}) \oplus \ker(\mathcal{I} \to \mathcal{I}|_{Y_q})$$

depend only on $\mathcal{I}$, by the Jordan–Hölder Theorem.
We say that two semistable sheaves \( I \) and \( K \) on \( X \) are \( S \)-equivalent if \( S(I) = S(K) \) and \( \text{Gr}(I) \cong \text{Gr}(K) \).

(For a higher rank semistable sheaf, a Jordan–Hölder filtration is a filtration of the sheaf. However, in rank 1, this filtration is induced by a filtration of \( X \) as above.)

### 2.6. (The coarse compactified Jacobians)

Let \( X_1, \ldots, X_n \) be the irreducible components of the curve \( X \), and \( a = (a_1, \ldots, a_n) \) a \( n \)-tuple of positive rational numbers summing up to 1. For each subcurve \( Y \subseteq X \), set \( a_Y := \sum_{X_i \subseteq Y} a_i \). According to Seshadri [S82] Déf. 9 and Remarques on p. 153, a torsion-free, rank-1 sheaf \( I \) on \( X \) is \( a \)-semistable if

\[
\chi(I_Y) \geq a_Y \chi(I)
\]

for each proper subcurve \( Y \) of \( X \). Also, \( I \) is \( a \)-stable if the inequalities are strict.

Seshadri’s notions of semistability and stability are encompassed by ours. More precisely, for each integer \( d \) there is a vector bundle \( F \) on \( X \) such that \( a \)-semistability (resp. \( a \)-stability) for degree-\( d \), torsion-free, rank-1 sheaves is equivalent to semistability (resp. stability) with respect to \( F \); see [E99] Obs. 13, p. 584. In fact, any vector bundle \( F \) on \( X \) such that

\[
\mu(F|_{X_i}) = a_i(g - 1 - d) \quad \text{for each } i = 1, \ldots n
\]

has this property.

Two \( a \)-semistable sheaves are called \( S \)-equivalent if they are \( S \)-equivalent in the sense of 2.5 for a (and hence any) vector bundle \( F \) on \( X \) satisfying (2.6.1).

In [S82] Thm. 15, p. 155, Seshadri shows that there is a scheme \( U_X(a, d) \) corepresenting the functor that associates to each scheme \( T \) the set of \( T \)-flat coherent sheaves \( I \) on \( X \times T \) such that \( I|_{X \times t} \) is \( a \)-semistable and of degree \( d \) for every \( t \in T \), modulo the same equivalence \( \sim \) of 2.4. Furthermore, \( U_X(a, d) \) is projective and parameterizes \( S \)-equivalence classes of \( a \)-semistable sheaves.

Let \( F \) be any vector bundle on \( X \) satisfying (2.6.1), and set \( J_X(a, d) := J_{ss}^{F} \). (The particular \( F \) is irrelevant.) Since \( J_{ss}^{F} \) is a fine moduli space, there is a naturally induced morphism

\[
\Phi^d: J_X(a, d) \longrightarrow U_X(a, d)
\]

sending \([I]\) to the \( S \)-equivalence class of \( I \). We call \( \Phi^d \) the \( S \)-map.

If \( X \) is \( G \)-stable (cf. 2.2), let \( U_X(d) := U_X(a, d) \), where

\[
a_i := \frac{\text{deg}_{X_i}(\omega)}{2g - 2} \quad \text{for each } i = 1, \ldots, n.
\]

(Notice that \( a_1 + \cdots + a_n = 1 \) because \( X \) is Gorenstein, and the \( a_i \) are positive because \( \omega \) is ample.) Since, for each integer \( i \),

\[
\mu(E_d|_{X_i}) = a_i(g - 1 - d),
\]

where \( E_d \) is the canonical \( d \)-polarization of \( X \) (cf. 2.3), the \( S \)-map (2.6.2) becomes

\[
\Phi^d: J_X^{d,ss} \longrightarrow U_X(d).
\]
3. Abel maps

Assume from now on until the end of the paper that $X$ is Gorenstein.

Definition 3.1. A separating node of the curve $X$ is a point $N$ for which there is a subcurve $Z$ such that $\delta_Z = 1$ and $Z \cap Z' = \{N\}$.

Being $X$ Gorenstein, a separating node is indeed a node, by [Cat82], Prop. 1.10, p. 59.

3.2. (Degree-0 Abel maps) For each point $Q$ on the curve $X$, its sheaf of ideals $m_Q$ is torsion-free of rank 1 and degree $-1$. Also, if $Q$ is not a separating node, $m_Q$ is simple, as it follows from the discussion in [E01], Ex. 38, p. 3073.

Let $I_\Delta$ be the ideal sheaf of the diagonal $\Delta \subset X \times X$, and put

$$I := I_\Delta \otimes p_1^* \mathcal{O}_X(P),$$

where $p_1 : X \times X \to X$ is the first projection. The sheaf $I$ is flat over $X$ and, for each $Q \in X$,

$$I|_{X \times Q} = m_Q \otimes \mathcal{O}_X(P).$$

If $X$ is free from separating nodes, then $I$ defines a morphism

$$A_P : X \longrightarrow J^0_X; \quad Q \mapsto [m_Q \otimes \mathcal{O}_X(P)].$$

We call $A_P$ the degree-0 Abel map of $X$ with base $P$.

Proposition 3.3. Assume that the curve $X$ is free from separating nodes. Then:

1. $A_P(X) \subseteq J^0_X P$.
2. If $X \not\cong \mathbb{P}^1$ then $A_P$ is an embedding.

Proof. For each $Q \in X$, its sheaf of ideals $m_Q$ satisfies

$$\text{deg}_Y(m_Q) = \begin{cases} -1, & \text{if } Q \in Y, \\ 0, & \text{if } Q \not\in Y, \end{cases}$$

for each subcurve $Y \subseteq X$. If $Y$ is proper, then $\delta_Y > 1$ by hypothesis, and hence $\text{deg}_Y(m_Q \otimes \mathcal{O}_X(P)) \geq -\delta_Y/2$, with equality only if $P \not\in Y$. So $m_Q \otimes \mathcal{O}_X(P)$ is $P$-quasistable, showing that $A_P(X) \subseteq J^0_X P$.

Assume $X \not\cong \mathbb{P}^1$. It remains to show that $A_P$ is an embedding. Since $X$ is complete and $J^0_X P$ is separated, the induced map $X \to J^0_X P$ is proper. Thus, we need only show that $A_P$ separates points and tangent vectors. Equivalently, we need only show that every fiber of $A_P$ is either empty or schematically a point.

Let $Q \in X$ and put $L := A_P^{-1}([m_Q \otimes \mathcal{O}_X(P)])$. From [AltK80] Lemma 5.17, p. 88, it follows that $L$ is isomorphic to an open subscheme of the projective space

$$\mathbb{P}^1(\text{Hom}_X(m_Q, \mathcal{O}_X)),$$

the open subscheme parameterizing injective homomorphisms. However, since $A_P$ is proper over $J^0_X P$, the fiber $L$ is complete, and thus $L$ is a projective space.

We need to show that $L$ is a point. Suppose otherwise, by contradiction. Thus, since $L$ has dimension at most 1, we have $L \cong \mathbb{P}^1$. 
Let $Q_1$ and $Q_2$ be distinct points of $L$ on the nonsingular locus of $X$. Since $L$ is a fiber of $A_P$, we have an isomorphism $m_{Q_1} \to m_{Q_2}$. This isomorphism is given by multiplication by a rational function $h$ of $X$, whose only pole is $Q_1$ and whose only zero is $Q_2$, both with order 1. The function $h$ is constant on all components of $X$ other than $L$, because $h$ has no zeros or poles there. Let $Z := L'$. Since $X$ is not isomorphic to $\mathbb{P}^1$, we have $Z \neq \emptyset$. We claim that $L$ intersects $Z$ transversally. In fact, if $L$ intersected $Z$ nontransversally at a point $R$, then $h|_L - h(R)$ would vanish at $R$ with order at least 2. This is not possible because $h|_L$ has degree 1.

Let $Z_1, \ldots , Z_q$ denote the connected components of $Z$. Since $X$ is connected, each $Z_i$ intersects $L$. If $\#(Z_i \cap L) = 1$, then we would have $\delta_{Z_i} = 1$, as we already know that $Z$ intersects $L$ transversally. Thus, each $Z_i$ intersects $L$ in at least two points. But then $h|_L$ takes the same value on these two points of $L$. This is again not possible because $h|_L$ has degree 1. We have reached a contradiction. \[\square\]

3.4. (Degree-1 Abel maps) As in [3.2] let $\mathcal{I}_\Delta$ be the ideal sheaf of the diagonal $\Delta \subset X \times X$. Then $\mathcal{I}_\Delta$ is flat over $X$ and, for each $Q \in X$, the restriction $\mathcal{I}_\Delta|_{X \times Q}$ is isomorphic to the sheaf of ideals $m_Q$. Since $X$ is Gorenstein, the dual sheaf

$$\mathcal{I}_\Delta^* := \text{Hom}_{X \times X}(\mathcal{I}_\Delta, \mathcal{O}_{X \times X})$$

is also flat over $X$, and $\mathcal{I}_\Delta^*|_{X \times Q} \cong m_Q^*$ for each $Q \in X$.

As mentioned in [3.2], the sheaf $m_Q$ is simple if $X$ is free from separating nodes. Since $X$ is Gorenstein,

$$\text{Hom}_X(m_Q^*, m_Q^*) = \text{Hom}_X(m_Q,m_Q).$$

So, if $X$ is free from separating nodes, $\mathcal{I}_\Delta^*|_{X \times Q}$ is simple for every $Q \in X$, and thus $\mathcal{I}_\Delta^*$ defines a morphism

$$(3.4.1) \quad A: X \longrightarrow J_X^1; \quad Q \mapsto [m_Q^*].$$

We call $A$ the degree-1 Abel map of $X$.

**Proposition 3.5.** Assume that the curve $X$ is free from separating nodes. Then:

1. If $X \not\cong \mathbb{P}^1$ then $A$ is an embedding.
2. If $g \geq 2$ then $A(X) \subseteq J_X^{1,ss}$.
3. If $X$ is $G$-stable then $A(X) \subseteq J_X^{1,s}$.

**Proof.** The map $A$ is the composition of $A_P: X \to J_X^0$ followed by the duality map $\lambda: J_X^0 \to J_X^0$, sending $[\mathcal{I}]$ to $[\mathcal{I}^*]$, and the translation $\tau: J_X^0 \to J_X^1$ by $P$, sending $[\mathcal{I}]$ to $[\mathcal{I} \otimes \mathcal{O}_X(P)]$. Assume $X \not\cong \mathbb{P}^1$; then $A_P$ is an embedding by Proposition [3.3], and since $\lambda$ and $\tau$ are isomorphisms, also $A$ is an embedding.

Let us now see that $A(X) \subseteq J_X^{1,ss}$ if $g \geq 2$. We claim that $\deg(\omega|_Y) \geq 0$ for every subcurve $Y \subseteq X$. Indeed, since the degree is additive, we need only check the claim when $Y$ is irreducible. Then $\deg(\omega|_Y) \geq 0$ by (2.1.3), because $\delta_Y \geq 0$ and, by hypothesis, $\delta_Y \geq 2$. Thus, since $\omega$ has degree $2g - 2$, and since $g \geq 1$ and $\delta_Y \geq 2$,

$$(3.5.1) \quad \deg(\omega|_Y) \leq 2(g - 1) \leq \delta_Y(g - 1).$$
Now, for each $Q \in X$, there is a natural inclusion $\mathcal{O}_X \to m_Q^*$, where $m_Q$ is the sheaf of ideals of $Q$. Thus

$$(3.5.2) \quad \deg_Y(m_Q^*) \geq 0$$

for every subcurve $Y \subseteq X$, and hence $(3.5.1)$ implies that $m_Q^*$ is semistable.

Finally, suppose that $X$ is $G$-stable. Let $Y$ be a proper subcurve of $X$. Because of $(3.5.1)$ and $(3.5.2)$, for the inequality

$$\deg_Y(m_Q^*) \geq \deg_Y(\omega_{\mathcal{O}_Y})$$

to be an equality we would need that $\deg_Y(\omega) = \delta_Y(g - 1)$. Since $\delta_Y \geq 2$, we would need that $\delta_Y = 2$ and $\deg_Y(\omega) = 0$. But this is not possible because $\omega$ is ample. Thus $A(X) \subseteq J_X^{1,s}$.

**Remark 3.6.** There are special cases where $J_X^{1,ss} = J_X^{1,s}$, for instance if

$$(3.6.1) \quad \frac{\deg_Y(\omega)}{2g - 2} - \frac{\delta_Y}{2}$$

is not an integer for any proper subcurve $Y \subset X$. This will be the case when $X$ is $G$-stable and $g$ is odd. Indeed, suppose $(3.6.1)$ is an integer. Using $(2.1.3)$, we have

$$\frac{\deg_Y(\omega)}{2g - 2} - \frac{\delta_Y}{2} = \frac{(2 - g)\delta_Y - 2\chi(\mathcal{O}_Y)}{2g - 2}.$$ 

Thus, if $g$ is odd, $\delta_Y$ must be even, and thus $(2g - 2)$ divides $\deg_Y(\omega)$. However, as we saw in the proof of Proposition 3.5, this implies that $\deg_Y(\omega) = 0$ or $\deg_Y(\omega) = 0$, which is not possible if $\omega$ is ample.

If $X$ is a stable curve, in the sense of Deligne and Mumford, it follows from [CE06] Prop. 3.15 that $J_X^{1,ss} = J_X^{1,s}$, unless $X = Y_1 \cup Y_2$, where $Y_1$ and $Y_2$ are connected proper subcurves of the same genus intersecting at an odd number of points.

**Remark 3.7.** Since $X$ is assumed Gorenstein, the dualizing map

$$\lambda: J_X \longrightarrow J_X; \quad [\mathcal{I}] \mapsto [\mathcal{I}^*]$$

is well-defined and takes $J_X^d$ isomorphically onto $J_X^{-d}$, for every integer $d$. Furthermore, given any vector bundle $F$ on $X$, using duality, we have

$$\lambda(J_F^{ss}) = J_{F^\dagger}^{ss}, \quad \lambda(J_F^P) = J_{F^\dagger}^P \quad \text{and} \quad \lambda(J_F^s) = J_{F^\dagger}^s,$$

where $F^\dagger := F^* \otimes \omega$. In particular, since $\mu(E_d^|Y) = \mu(E_{-d}^|Y)$ for each integer $d$ and each subcurve $Y \subseteq X$, it follows that

$$\lambda(J_X^{d,ss}) = J_X^{-d,ss}, \quad \lambda(J_X^{d,P}) = J_X^{-d,P} \quad \text{and} \quad \lambda(J_X^{d,s}) = J_X^{-d,s}.$$ 

Thus, we could have defined $A_P$ as sending $Q$ to $[\mathcal{I}_Q^* \otimes \mathcal{O}_X(-P)]$, or $A$ as sending $Q$ to $[\mathcal{I}_Q^*]$. Apart from the fact that the latter map would have $J_X^{-1}$ as target instead of $J_X^1$, all the conclusions would remain the same.

Essentially, the same observation applies to the twisted Abel maps to be defined in 4.5 and 5.2.
4. Twisted Abel maps of degree 0

4.1. (Spines and tails). A tail of $X$ is a proper subcurve $Z \subset X$ with $\delta_Z = 1$. If $Z$ is a tail, so is $Z'$, and the unique point $N$ of $Z \cap Z'$ is a separating node. In this case, we say that $Z$ and $Z'$ are the tails attached to $N$, and that $N$ generates $Z$ and $Z'$. Notice that a tail is connected (because $X$ is). A tail is called a $P$-tail if it does not contain $P$. We denote by $T(X)$ the set of all tails of $X$ and by $T_P(X)$ the set of all $P$-tails.

A connected subcurve $Y$ of $X$ is called a spine if every point in $Y \cap \overline{X-Y}$ is a separating node. In this case, each connected component $Z$ of $\overline{X-Y}$ is a tail intersecting $Y$ transversally at a single point on the nonsingular loci of $Y$ and $Z$.

Let $Y$ be a subcurve of $X$. If a singular point of $Y$ is a separating node of $X$, then it is also a separating node of $Y$. Conversely, if $Y$ is a spine then a separating node of $Y$ is a separating node of $X$. As a consequence, if a subcurve $Z$ of $Y$ is a spine of $X$, then $Z$ is a spine of $Y$; conversely, if $Z$ is a spine of $Y$ and $Y$ is a spine of $X$, then $Z$ is a spine of $X$.

If $Y$ is a nonempty proper union of spines of $X$, then any connected component of $Y$ or $X-\overline{Y}$ is a spine. Two intersecting spines with no common component intersect transversally at a separating node of $X$.

A q-tuple $\mathfrak{Z} := (Z_1, \ldots, Z_q)$ of spines covering $X$, each two with no component in common, is called a spine decomposition of $X$. If $Y$ is a spine of $X$, then $Y$ and the connected components of $\overline{X-Y}$ form a spine decomposition of $X$.

The following two lemmas will be much used.

**Lemma 4.2.** Let $Z_1$ and $Z_2$ be tails of the curve $X$. Then

either $Z_1 \cup Z_2 = X$ or $Z_1 \cap Z_2 = \emptyset$ or $Z_1 \subseteq Z_2$ or $Z_2 \subseteq Z_1$.

**Proof.** This is [CE06] Lemma 4.3. □

**Lemma 4.3.** Let $\mathfrak{Z} := (Z_1, \ldots, Z_q)$ be a spine decomposition of $X$. Then there is an isomorphism

$$u: J_X \longrightarrow J_{Z_1} \times \cdots \times J_{Z_q}$$

sending $[I]$ to $([I|Z_1], \ldots, [I|Z_q])$. Furthermore, for each integer $d$,

$$u(J_X^d) = \bigcup_{d_1 + \cdots + d_q = d} J_{Z_1}^{d_1} \times \cdots \times J_{Z_q}^{d_q}.$$

**Proof.** This is [E07] Prop. 3.2. □

4.4. (Twisters on tails.) By Lemma 4.3 or [CE06] Lemma 4.4, for each tail $Z$ of the curve $X$ there is a unique, up to isomorphism, line bundle on $X$ whose restrictions to $Z$ and $Z'$ are $O_Z(-N)$ and $O_{Z'}(N)$, where $N$ is the separating node generating $Z$. Denote this line bundle by $O_X(Z)$. We call it a twister.

For simplification, for each formal sum $\sum a_Z Z$ of tails $Z$ with integer coefficients $a_Z$, set

$$O_X(\sum a_Z Z) := \bigotimes O_X(Z)^{\otimes a_Z}.$$
If \( Z \) is a tail of \( X \) attached to the node \( N \), and \( f : \mathcal{X} \to S \) is a one-parameter regular smoothing of \((X, N)\) (a flat, projective morphism of schemes such that \( S \) has dimension one, \( X = f^{-1}(s) \) for a nonsingular \( s \in S \), and \( \mathcal{X} \) is smooth at \( N \)), then \( Z \) is a Cartier divisor of \( \mathcal{X} \), satisfying \( \mathcal{O}_{\mathcal{X}}(Z)|_X \cong \mathcal{O}_X(Z) \) while \( \mathcal{O}_{\mathcal{X}}(Z)|_{f^{-1}(t)} = \mathcal{O}_{f^{-1}(t)} \) for each \( t \in S \setminus s \). So, \( \mathcal{O}_X(Z) \), though nontrivial, is the limit of a family of trivial sheaves.

**4.5. (Degree-0 twisted Abel maps).** Let \( Q \in X \). If \( Q \) is not a separating node, let \( \mathcal{M}_Q \) be the sheaf of ideals \( \mathfrak{m}_Q \) of \( Q \). Notice that \( \mathcal{M}_Q \) is simple. If \( Q \) is a separating node, let \( Z \) be the \( P \)-tail generated by \( Q \) (so that \( P \not\in Z \)) and let \( \mathcal{M}_Q \) be the unique line bundle on \( X \) such that

\[
\mathcal{M}_Q|_Z \cong \mathcal{O}_Z(-Q) \quad \text{and} \quad \mathcal{M}_Q|_{Z'} \cong \mathcal{O}_{Z'}.
\]

(That \( \mathcal{M}_Q \) exists and is unique follows from Lemma 4.3.) Again, \( \mathcal{M}_Q \) is simple.

Define a map

\[
(4.5.1) \quad A_P : X \longrightarrow J^0_X; \quad Q \mapsto [\mathcal{I}_Q]
\]

where

\[
(4.5.2) \quad \mathcal{I}_Q := \mathcal{M}_Q \otimes \mathcal{O}_X(P) \otimes \mathcal{O}_X(-\sum_{Z \in T_P(X); Z \ni Q} Z),
\]

the sum running over all tails \( Z \) of \( X \) containing \( Q \) but not \( P \). Since \( \mathcal{M}_Q \) is simple, so is \( \mathcal{I}_Q \), and hence \( A_P \) is well-defined. We call \( A_P \) the degree-0 (twisted) Abel map of \( X \) with base \( P \). If \( X \) has no separating nodes then \( (4.5.1) \) coincides with \( (3.2.1) \). We will see in Theorem 4.8 that, in any case, \( A_P \) is a morphism of schemes.

**Lemma 4.6.** *Keep the notation of 4.5.* Let \( W \) be a spine of \( X \), and define

\[
B : W \to J^0_W; \quad Q \mapsto [\mathcal{I}_Q|_W].
\]

Then \( B \) is a well-defined map and the following three statements hold:

1. If \( P \in W \), then \( B \) is the degree-0 Abel map of \( W \) with base \( P \).
2. If \( P \in W' \), then \( B \) is the degree-0 Abel map of \( W \) with base \( N \), where \( N \) is the unique point of \( W \cap W' \) on the same connected component of \( W' \) as \( P \).
3. In any case, the isomorphism class of \( \mathcal{I}_Q|_W \) does not depend on \( Q \in W \).

**Proof.** We will use induction on \( \delta_W \). Suppose first that \( \delta_W = 1 \), i.e., that \( W \) is a tail. Let \( N \) be the separating node generating \( W \).

Let \( Q \in W \). Let \( Z_1, \ldots, Z_n \) be the \( \mathfrak{p} \)-tails of \( X \) containing \( Q \). It follows from Lemma 4.2 that either \( Z_i \subset Z_j \) or \( Z_j \subset Z_i \) for each distinct \( i \) and \( j \). Thus, we may assume that

\[
Z_1 \subset Z_2 \subset \cdots \subset Z_{n-1} \subset Z_n.
\]

By definition, \( \mathcal{I}_Q = \mathcal{M}_Q \otimes \mathcal{K}_Q \), where \( \mathcal{K}_Q := \mathcal{O}_X(P) \otimes \mathcal{O}_X(-Z_1 - \cdots - Z_n) \).

Suppose first that \( P \in W' \). Then \( W \in T_P(X) \). As \( W \ni Q \), we have that \( W = Z_i \) for a certain \( i \). The tails \( Z_1, \ldots, Z_{i-1} \) are also tails of \( W \); in fact, they are all the
In any case, it follows that 

\[ \mathcal{K}_Q|_{W'} \cong \mathcal{O}_W(N) \otimes \mathcal{O}_W(- \sum_{Y \in \mathcal{I}_N(W); Y \ni Q} Y) \]

\[ \mathcal{K}_Q|_{W'} \cong \mathcal{O}_W'(P) \otimes \mathcal{O}_X(- \sum_{Y \in \mathcal{I}_P(X); Y \ni Q} Y)|_{W'} \]

Notice that \( \mathcal{K}_Q|_{W'} \) does not depend on \( Q \).

Since \( Q \in W \) and \( P \in W' \), we have that \( \mathcal{M}_Q|_{W'} \cong \mathcal{O}_{W'} \), and hence \( \mathcal{I}_Q|_{W'} \) does not depend on \( Q \). In addition, if \( Q = N \) or \( Q \) is not a separating node of \( X \), then \( Q \) is not a separating node of \( W \), and \( \mathcal{M}_Q|_W \) is the sheaf of ideals of \( Q \) in \( W \). On the other hand, if \( Q \) is a separating node of \( X \) different from \( N \), then \( Q \) is a separating node of \( W \); and if \( Y \) is the \( N \)-tail of \( W \) generated by \( Q \), then \( Y \) is the \( P \)-tail of \( X \) generated by \( Q \), and hence 

\[ \mathcal{M}_Q|_Y \cong \mathcal{O}_Y(-Q) \quad \text{and} \quad \mathcal{M}_{W-Y} \cong \mathcal{O}_{W-Y} \]

In any case, it follows that \( [\mathcal{I}_Q|_W] \) is the image of \( Q \) under the degree-0 Abel map of \( W \) with base \( N \). This finishes the proof of the lemma in the case where \( P \in W' \).

Now, suppose \( P \in W \). We claim that either \( Z_i \cap W' = \emptyset \), or \( Z_i \supseteq W' \), for each integer \( i \). Indeed, \( Z_i \) and \( W' \) are both \( P \)-tails. If \( Z_i \cap W' \neq \emptyset \), then either \( Z_i \supseteq W' \) or \( Z_i \subseteq W' \) by Lemma 4.2. However, if \( Z_i \subseteq W' \), since \( Q \in Z_i \) and \( P \notin W' \), we have that \( W' = Z_i \) for some \( j \geq i \). But, since \( Q \in W \) as well, it follows that \( Q = N \) and \( i = j = 1 \). But then \( Z_i = W' \), proving the claim. Notice from our reasoning above that \( W' = Z_i \) for a certain \( i \) if and only if \( Q = N \).

If \( Z_i \cap W' = \emptyset \), then \( Z_i \) is tail of \( W \). And if \( Z_i \supseteq W' \), then \( Z_i - W' \) is a tail of \( W \). On the other hand, let \( Y \) be a tail of \( W \). If \( N \notin Y \), then \( Y \) is a tail of \( X \) as well, with \( Y \cap W' = \emptyset \). And if \( N \in Y \), then \( Y \cup W' \) is a tail of \( X \). Thus 

\[ \mathcal{K}_Q|_W \cong \mathcal{O}_W(P) \otimes \mathcal{O}_W(- \sum_{Y \in \mathcal{I}_P(W); Y \ni Q} Y) \otimes \mathcal{L}|_W, \]

and \( \mathcal{K}_Q|_{W'} \cong \mathcal{L}|_{W'} \), where \( \mathcal{L} := \mathcal{O}_X(-W') \) if \( Q = N \) and \( \mathcal{L} := \mathcal{O}_X \) otherwise.

Notice that \( \mathcal{M}_Q|_{W'} \cong \mathcal{O}_{W'} \) unless \( Q = N \), in which case \( \mathcal{M}_Q|_{W'} \cong \mathcal{O}_{W'}(-N) \). In any case, \( \mathcal{I}_Q|_{W'} \) is trivial, whence independent from \( Q \). In addition, \( \mathcal{M}_Q|_W \) is the sheaf of ideals of \( Q \) in \( W \), if \( Q \) is not a separating node of \( X \). On the other hand, if \( Q = N \) then \( \mathcal{M}_Q|_W = \mathcal{O}_W \) and \( \mathcal{L}|_W \cong \mathcal{O}_W(-N) \). And if \( Q \) is a separating node of \( X \) different from \( N \), then \( Q \) is a separating node of \( W \); and if \( Y \) is the \( P \)-tail of \( W \) generated by \( Q \), then either \( Y \) or \( Y \cap W' \) is the \( P \)-tail of \( X \) generated by \( Q \), depending on whether \( N \) is on \( Y \) or not, and whence 

\[ \mathcal{M}_Q|_Y \cong \mathcal{O}_Y(-Q) \quad \text{and} \quad \mathcal{M}_{W-Y} \cong \mathcal{O}_{W-Y} \]

In any case, it follows that \( [\mathcal{I}_Q|_W] \) is the image of \( Q \) under the degree-0 Abel map of \( W \) with base \( P \), finishing the proof of the lemma when \( W \) is a tail.

Now, suppose \( \delta_W > 1 \). Let \( Z \) be a connected components of \( W' \), and \( Y := \overline{X - Z} \). Then \( Y \) is a tail. By induction, the map 

\[ C: Y \to J^0_Y; \quad Q \mapsto [\mathcal{I}_Q|_Y] \]
is well-defined, and is the degree-0 Abel map of $Y$ with base $P$ if $P \in Y$, and base $N'$ if $P \notin Y$, where $N'$ is the separating node of $X$ generating $Y$. Also the isomorphism class of $\mathcal{I}_Q|_Z$ does not depend on $Q \in \bar{Y}$.

Note that $W$ is a spine of $Y$ and $\#W \cap \bar{Y} - \bar{W} = \delta_W - 1$. So, by induction, $B$ is well-defined. Furthermore, the isomorphism class of $\mathcal{I}_Q|_{\bar{Y} - \bar{W}}$ does not depend on $Q \in W$. Since neither does the isomorphism class of $\mathcal{I}_Q|_Z$, and $\bar{Y} - \bar{W}$ does not intersect $Z$, we obtain (3).

By induction, if $P \in W$, and hence $P \in Y$, then $B$ is the degree-0 Abel map of $W$ with base $P$. If $P \notin W$, there are two cases to consider: If $P \notin Y$, that is, if $P \in Z$, then $B$ is the degree-0 Abel map of $W$ with base $N'$; and if $P \in Y$, then $P$ belongs to a connected component of $\bar{X} - \bar{W}$ other than $Z$, and thus $B$ is the degree-0 Abel map of $W$ with base the point of intersection of this component and $W$. □

**Definition 4.7.** A smooth rational component $L$ of the curve $X$ is called a separating line if $L$ is a spine.

**Theorem 4.8.** The degree-0 Abel map $A_P$ (defined in (4.5.1)) is a morphism of schemes. Furthermore:

1. $A_P(X) \subseteq J^0_B$.  
2. If $X$ contains no separating lines, then $A_P$ is an embedding.

**Proof.** If $X$ has no separating nodes, we proved the statements in Proposition 3.3. So, let $W$ be a tail of $X$. We will assume, by induction, that the theorem holds for curves with less separating nodes than $X$. Assume, without loss of generality, that $P \in W'$. Let $N$ be the separating node generating $W$. Notice that the separating nodes of $X$ are $N$ and those of $W$ and $W'$. Thus $W$ and $W'$ have fewer separating nodes than $X$.

By Lemma 4.6, under the identification $J_X = J_W \times J_{W'}$ given by Lemma 4.3, (4.8.1)

$$A_P|_W = (A_1, B): W \longrightarrow J^0_W \times J^0_{W'},$$

where $A_1$ is the degree-0 Abel map of $W$ with base $N$ and $B$ is constant, and (4.8.2)

$$A_P|_{W'} = (B', A_2): W' \longrightarrow J^0_W \times J^0_{W'},$$

where $A_2$ is the degree-0 Abel map of $W'$ with base $P$ and $B'$ is constant. By induction, $A_P|_W$ and $A_P|_{W'}$ are morphisms of schemes. Then so is $A_P$, because $W$ and $W'$ intersect transversally at nonsingular points.

Now, let $Q$ be a point of $X$. Let $Z_1, Z_2, \ldots, Z_n$ be the $P$-tails of $X$ containing $Q$. Using Lemma 4.7 as in the proof of Lemma 4.6 we may assume that $Z_1 \subset Z_2 \subset \cdots \subset Z_{n-1} \subset Z_n$.

Let $\mathcal{K} := \mathcal{O}_X(-\sum Z_i)$. Keep the notation of 4.6. We want to show that $\mathcal{I}_Q$ is $P$-quasistable. Let $Y$ be a connected proper subcurve of $X$. It is enough to show that either $\deg_Y(\mathcal{I}_Q) \geq 0$, or (4.8.3)

$$P \notin Y \quad \text{and} \quad \deg_Y(\mathcal{I}_Q) \geq -1 \geq -\delta_Y/2.$$

By [CE06] Lemma 4.8, we have $\deg_Y(\mathcal{K}) \geq -1$. Suppose first that $\deg_Y(\mathcal{K}) = -1$. By the same lemma, $Y \subseteq Z'_1$. 


Suppose \( Q \in Y \). Then \( Q \in Y \cap Y' \) and \( Q \) is the separating node generating \( Z_1 \). So \( \mathcal{I}_Q|_Y \cong \mathcal{O}_Y(P) \otimes \mathcal{K}|_Y \) if \( P \in Y \) and \( \mathcal{I}_Q|_Y \cong \mathcal{K}|_Y \) otherwise. If \( P \in Y \), then \( \deg_Y(\mathcal{I}_Q) = 0 \). On the other hand, if \( P \not\in Y \), then \( \delta_Y \geq 2 \) and \( \deg_Y(\mathcal{I}_Q) = -1 \); so (4.8.3) holds.

Now, suppose \( Q \not\in Y \). Then \( \deg_Y(\mathcal{I}_Q) = 0 \) if \( P \in Y \) and \( \deg_Y(\mathcal{I}_Q) = -1 \) if \( P \not\in Y \). We need only show now that, if \( Y \) is tail, then \( P \in Y \). Indeed, if \( Y \) is a tail, then \( Y \) is generated by the same node that generates a \( Z_j \), for a certain \( j \), again by Lemma 4.8. Since \( Y \subseteq Z'_1 \), we have \( Y = Z'_j \), and hence \( Y \supseteq P \).

Suppose now that \( \deg_Y(K) \geq 0 \). Then \( \deg_Y(\mathcal{I}_Q) \geq -1 \). If \( Q \not\in Y \) or \( P \in Y \), then \( \deg_Y(\mathcal{I}_Q) \geq 0 \). Now, suppose \( Q \in Y \) and \( P \not\in Y \). If \( \delta_Y \geq 2 \), then (4.8.3) holds. On the other hand, if \( Y \) is a tail, then \( Y = Z_j \) for a certain \( j \), and hence \( \deg_Y(K) = 1 \). In this case, \( \deg_Y(\mathcal{I}_Q) = 0 \).

At any rate, \( \mathcal{I}_Q \) is \( P \)-quasistable. Since this holds for each \( Q \in X \), the map \( A_P \) factors through \( J_{X}^{0,P} \).

Finally, assume \( X \) contains no separating lines. First, observe that a separating node of \( W \) is a separating node of \( X \). So, given a smooth, rational component \( L \) of \( W \), since

\[
L \cap L' \subseteq (L \cap \overline{W-L}) \cup \{N\},
\]

not all points of \( L \cap \overline{W-L} \) are separating nodes of \( W \). By induction, \( A_1 \) is an embedding. By the same reasoning, \( A_2 \) is also an embedding. Hence \( A_P|_W \) and \( A_P|_{W'} \) are embeddings.

Now, let \( Q_1 \in W \) and \( Q_2 \in W' \) and assume that \( A_P(Q_1) = A_P(Q_2) \). Then

\[
A_1(Q_1) = B'(Q_2) \quad \text{and} \quad B(Q_1) = A_2(Q_2).
\]

Since \( A_1 \) is injective, and \( B'(W') = \{A_1(N)\} \), we have that \( Q_1 = N \). Also, since \( A_2 \) is injective and \( B(W) = \{A_2(N)\} \), we have \( Q_2 = N \). Hence \( Q_1 = Q_2 \). It follows that \( A_P \) is injective.

Also, since \( A_P|_W \) and \( A_P|_{W'} \) are immersions, so is \( A_P \) everywhere but possibly at \( N \). But \( A_P \) is an immersion also at \( N \), because \( A_P|_W \) and \( A_P|_{W'} \) are immersions at \( N \), and, under the identification \( J_X = J_W \times J_{W'} \) given by Lemma 1.3, they take the tangent spaces of \( W \) and \( W' \) at \( N \) into the linearly independent subspaces \( T_{J_{W},A_1(N)} \oplus 0 \) and \( 0 \oplus T_{J_{W'},A_2(N)} \) of \( T_{J_X,A_P(N)} \), respectively.

Thus \( A_P \) separates points and tangent vectors. Since \( X \) is complete, and \( A_P \) factors through \( J_{X}^{0,P} \), which is separated, \( A_P \) is an embedding. \( \square \)

5. Twisted Abel maps of degree 1

5.1. (Small tails and splitting nodes.) We set now a rule that associates to every separating node \( N \) of the curve \( X \) exactly one of the two tails that \( N \) generates; we shall call the chosen tail the small tail generated by \( N \), and denote it by \( Z_N \). To do this, let \( Z \) and \( Z' \) be the two tails generated by \( N \); then \( g_Z + g_{Z'} = g \). There are two cases:

(1) If \( g_Z \neq g_{Z'} \), let \( Z_N \) be the one between \( Z \) and \( Z' \) having smaller genus. Thus \( g_{Z_N} < g/2 \).
(2) If \(g_Z = g_{Z'} = g/2\), make an arbitrary choice between \(Z\) and \(Z'\), and set it equal to \(Z_N\). In this case we call \(N\) a splitting node of \(X\).

We denote by \(\mathcal{ST}(X) \subset \mathcal{T}(X)\) the set of all small tails of \(X\). By definition, there is a bijection between the set of separating nodes of \(X\) and \(\mathcal{ST}(X)\). Observe that, if \(X\) has a splitting node, \(\mathcal{ST}(X)\) depends upon the choice made in (2) above.

5.2. (Degree-1 twisted Abel maps). For each \(Q \in X\) define a torsion-free, rank-1 sheaf \(\mathcal{N}_Q\) on \(X\) as follows. If \(Q\) is not a separating node, let \(\mathcal{N}_Q\) be the sheaf of ideals of \(Q\). If \(Q\) is a separating node, let \(Z \in \mathcal{ST}(X)\) be the unique small tail attached to it, and let \(\mathcal{N}_Q\) be the unique line bundle on \(X\) such that
\[
\mathcal{N}_Q|_Z \cong \mathcal{O}_Z(-Q) \quad \text{and} \quad \mathcal{N}_Q|_{Z'} \cong \mathcal{O}_{Z'}.
\]
(That \(\mathcal{N}_Q\) exists and is unique follows from Lemma 4.3.) Note that \(\mathcal{N}_Q\) is simple.

Define a map
\[
(5.2.1) \quad A: X \longrightarrow J^1_X,
\]
sending a point \(Q\) of \(X\) to \([T^1_Q]\), where
\[
(5.2.2) \quad T^1_Q := \mathcal{N}_Q^* \otimes \mathcal{O}_X(\sum_{Z \in \mathcal{ST}(X)} Z),
\]
the sum running over all small tails of \(X\) containing \(Q\). Since \(X\) is Gorenstein, and \(\mathcal{N}_Q\) is simple, so is \(T^1_Q\), and hence \(A\) is well-defined. We call \(A\) the degree-1 Abel map of \(X\). If \(X\) has no separating nodes then (5.2.1) coincides with (3.4.1). But if \(X\) admits a splitting node \(N\), then the definition of \(A\) depends on the choice of the small tail \(Z_N\) associated to \(N\) (see 5.1). We will see in Theorem 5.4 that \(A\) is a morphism of schemes.

Lemma 5.3. If the curve \(X\) is G-stable, then the following statements hold:

1. If \(Z_1\) and \(Z_2\) are tails of \(X\) with \(Z_1 \subsetneq Z_2\), then \(g_{Z_1} < g_{Z_2}\).
2. The curve \(X\) has at most one splitting node (defined in 5.1).
3. There is a point \(Q\) on the nonsingular locus of \(X\) such that \(\mathcal{ST}(X) = \mathcal{T}_Q(X)\).

Proof. We prove Statement (1). As \(X\) is G-stable, \(\omega\) is an ample line bundle. So
\[
2g_{Z_1} - 1 = \deg_{Z_1}(\omega) < \deg_{Z_2}(\omega) = 2g_{Z_2} - 1,
\]
and hence \(g_{Z_1} < g_{Z_2}\).

As for Statement (2), suppose \(X\) has a splitting node \(Q\), and let \(Z\) be a tail it generates. Thus \(g_Z = g_{Z'} = g/2\). By contradiction, assume \(X\) has another splitting node, \(N\); we may assume \(N \in Z\). Then \(N\) generates two tails of \(Z\), one of which is a tail of \(X\). So \(Z\) contains properly a tail of \(X\) of genus \(g/2 = g_Z\), contradicting Statement (1).

Finally, let us prove the last statement. We may assume that \(X\) has a tail, hence a small tail, hence a maximal small tail, \(Z\). Let \(Q\) be a point on the nonsingular locus of \(X\) lying on the irreducible component of \(Z'\) containing \(Z \cap Z'\). We claim that \(\mathcal{ST}(X) = \mathcal{T}_Q(X)\). Indeed, let \(W\) be a tail of \(X\). Suppose first that \(Q \in W\). Then \(W \cap Z \neq \emptyset\) and \(W \not\subseteq Z\). If \(W \cup Z = X\) then \(W \supseteq Z'\), and since \(Z'\) is not
small, neither is \( W \). If instead \( W \cup Z \neq X \), then \( Z \subseteq W \), by Lemma \([4.2]\), and thus \( W \) is again not small, by the maximality of \( Z \). Conversely, suppose \( W \) is not small. Then \( W' \) is. But then, as we have just proved, \( Q \notin W' \), and so \( Q \in W \). \( \square \)

**Theorem 5.4.** The degree-1 Abel map \( A \) (defined in \((5.2.1)\)) is a morphism of schemes. Furthermore:

1. If \( X \) has no separating lines, then \( A \) is an embedding.
2. If \( X \) is \( G \)-stable and \( P \) does not lie on any small tail of \( X \), then \( A(X) \subseteq J_{X}^{1,P} \).
3. If \( X \) is \( G \)-stable then \( A(X) \subseteq J_{X}^{1,ss} \).

**Proof.** By Lemma \([5.3]\) there is a point \( Q \) on the nonsingular locus of \( X \) such that \( ST(X) = T_{Q}(X) \), or equivalently, such that \( Q \) does not lie on any small tail of \( X \). So Statement \((3)\) follows from \((2)\). Also, \( A \) is a morphism of schemes because, as in the proof of Proposition \([3.5]\), it is the composition of \( A_{Q} \), the degree-0 Abel map of \( X \) with base \( Q \), with the duality map and the translation-by-Q map. Furthermore, if \( X \) contains no separating lines, since the dualizing and translation maps are isomorphisms, and since \( A_{Q} \) is an embedding by Theorem \([4.8]\) also \( A \) is an embedding.

Let us prove Statement \((2)\). Let \( Q \in X \), and let \( Z_{1}, Z_{2}, \ldots, Z_{n} \) be the small tails of \( X \) containing \( Q \). Using Lemma \([4.2]\) as in the proof of Lemma \([4.6]\) we may assume that

\[
Z_{1} \subseteq Z_{2} \subseteq \cdots \subseteq Z_{n-1} \subseteq Z_{n}.
\]

By hypothesis, \( P \notin Z_{n} \).

Keep the notation of \([5.2]\). Let \( W \) be a connected proper subcurve of \( X \). We need to show that

\[
(5.4.1) \quad \deg_{W}(I_{Q}^{1}) \geq \frac{\deg_{W}(\omega) − \delta_{W}}{2g - 2} + \frac{\delta_{W}}{2},
\]

with equality only if \( P \notin W \).

Let \( \mathcal{K} := O_{X}(\sum Z_{i}) \). By \([CE06]\) Lemma 4.8, we have \( \deg_{W}(\mathcal{K}) \geq -1 \). Suppose first that \( \deg_{W}(\mathcal{K}) = -1 \). Then \( \deg_{W}(I_{Q}^{1}) \geq -1 \). Also, by the same lemma, there is a unique integer \( j \) such that the separating node generating \( Z_{j} \) is contained in \( W \cap W' \). In addition, \( W \subseteq Z_{j} \), and in particular, \( W \subseteq Z_{n} \). So \( P \notin W \). Since \( W \subseteq Z_{n} \), and since \( Z_{n} \) is a small tail, \( \deg(\omega|_{W}) \leq \deg(\omega|_{Z_{n}}) \leq g - 1 \).

So, if \( \delta_{W} \geq 3 \) then \((5.4.1)\) holds. On the other hand, suppose \( \delta_{W} = 1 \). Since \( W \cap W' \) contains the separating node of \( Z_{j} \), and \( W \subseteq Z_{j} \), we must have \( W = Z_{j} \). In this case, \( \deg_{W}(I_{Q}^{1}) = 0 \), and hence \((5.4.1)\) holds as well.

Now, suppose \( \delta_{W} = 2 \). We need to show that \( \deg_{W}(I_{Q}^{1}) \geq 0 \). Since \( W \) contains the separating node generating \( Z_{j} \), and \( W \subseteq Z_{j} \), we have that \( W = Z_{j} - Y \) for a certain tail \( Y \) of \( X \) properly contained in \( Z_{j} \). Since \( Z_{j} \) is a small tail, so is \( Y \). If \( Y = Z_{i} \) for a certain \( i < j \), then \( W \cap W' \) would also contain the separating node generating \( Z_{i} \), which is not possible. So, since \( Y \) is a small tail, \( Q \notin Y \). Thus \( Q \in W \). If \( Q \notin W \cap W' \) then \( \deg_{W}(N_{Q}^{1}) = 1 \), and hence \( \deg_{W}(I_{Q}^{1}) = 0 \). On the
other hand, if $Q \in W \cap W'$, since $Q \not\in Y$, it follows that $Q$ is the separating node generating $Z_j$. Then $j = 1$ and $\deg_W(N_Q^*) = 1$ as well, implying that $\deg_W(I_Q^1) = 0$.

The upshot is that (5.4.1) holds and $P \not\in W$ if $\deg_W(K) = -1$. Now, suppose $\deg_W(K) \geq 0$. Then $\deg_W(I_Q^1) \geq 0$. If $W$ is not a tail, then (5.4.1) holds, and the inequality is strict because $W \not= X$, and hence $\deg_W(\omega) < 2g - 2$. Suppose now that $W$ is a tail. There are two cases to consider: $Q \in W$ and $Q \not\in W$.

Suppose first that $Q \in W$. If $W \not\subseteq Z'_1$, then $\deg_W(N_Q^*) = 1$, and hence $\deg_W(I_Q^1) \geq 1$. In this case, Inequality (5.4.1) holds and is strict. On the other hand, if $W \subseteq Z'_1$, since $Q \in Z_1$, we get that $Q$ is a separating node and $W = Z'_1$. In this case,

$$\deg_W(I_Q^1) = \deg_W(K) = 1,$$

and thus Inequality (5.4.1) holds as well and is strict.

Now, suppose $Q \not\in W$. Then $W' \ni Q$. If $P \in W$, then $W'$ is a $P$-tail, and hence a small tail. Since $W' \ni Q$, we have that $W' = Z_i$ for a certain integer $i$, and hence $\deg_W(K) = 1$. Again, Inequality (5.4.1) holds and is strict. On the other hand, if $P \not\in W$, then $W$ is a small tail, and hence $\deg_W(\omega) \leq g - 1$. Since $\deg_W(I_Q^1) \geq 0$, Inequality (5.4.1) holds.

At any rate, $I_Q^1$ is $P$-quasistable. Since this holds for every $Q \in X$, it follows that $A(X) \subseteq J^p_X$. \hfill $\square$

6. Properties of the Abel maps

**Lemma 6.1.** The curve $X$ has genus $g = 0$ if and only if every irreducible component of $X$ is a separating line.

**Proof.** By induction on the number of irreducible components of $X$. If $X$ is irreducible, $g = 0$ implies that $X$ is smooth; so the lemma holds trivially. Suppose now that $X$ is reducible, and let $L$ be an irreducible component of $X$ and $Z_1, \ldots, Z_n$ the connected components of $L'$. Since $X$ is Gorenstein and $L$ is irreducible, $L$ is a separating line if and only if $g_L = 0$ and $\text{length}(L \cap Z_i) = 1$ for each $i = 1, \ldots, n$ or, equivalently, $g_L = 0$ and

$$\text{length}(L \cap Z_1) + \cdots + \text{length}(L \cap Z_n) = n. \tag{6.1.1}$$

Consider the cohomology sequence associated to the natural exact sequence

$$0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_L \oplus \mathcal{O}_{Z_1} \oplus \cdots \oplus \mathcal{O}_{Z_n} \rightarrow \mathcal{O}_{L \cap Z_1} \oplus \cdots \oplus \mathcal{O}_{L \cap Z_n} \rightarrow 0. \tag{6.1.2}$$

If $h^1(X, \mathcal{O}_X) = g = 0$ then $g_L = h^1(L, \mathcal{O}_L) = 0$ and (6.1.1) holds, whence $L$ is a separating line.

The converse is immediate, as every irreducible component of $X$ is isomorphic to $\mathbb{P}^1$ and every singularity is a separating node (see Example 9.8 in [C05]). \hfill $\square$

**Definition 6.2.** A separating tree of lines of the curve $X$ is a spine of (arithmetic) genus 0. Equivalently, using Lemma 6.1, a separating tree of lines of $X$ is a connected union of separating lines of $X$.

**Theorem 6.3.** Let $A$ and $A_P$ be the Abel maps of the curve $X$. Assume $g > 0$. Set $B := A$ or $B := A_P$, and let $\bar{X} := B(X)$. Then the following statements hold:
(1) Let $S$ be the union of all separating lines of $X$. Let $Y_1, \ldots, Y_n$ be the connected components of $S'$. Then $B|_{Y_i}$ is an embedding for each $i = 1, \ldots, n$.

(2) For any two distinct points $Q_1, Q_2 \in X$, we have that $B(Q_1) = B(Q_2)$ if and only if $Q_1$ and $Q_2$ belong to the same separating tree of lines.

(3) Let $L$ be a maximal separating tree of lines of $X$, and let $R \in J_X$ for which $B(L) = \{R\}$. Let $N_1, \ldots, N_\delta$ be the points of $L \cap L'$. Then $\overline{X}$ has an ordinary $\delta$-fold singularity at $R$, with linearly independent tangent lines equal to the images of the differentials $d_{N_i}B : T_{X,N_i} \to T_{X,R}$.

(4) $\overline{X}$ is a curve of arithmetic genus $g$.

**Proof.** Assume first that $B := A_p$. For each $Q \in X$, let $\mathcal{I}_Q$ be the simple, torsion-free, rank-1 sheaf on $X$ such that $B(Q) = [\mathcal{I}_Q]$.

We prove Statement (1). First, observe that each $Y_i$ is a spine, and contains no separating lines. Since $Y_i$ is a spine, using Lemma 4.3 it is enough to show that the map

$$B_i : Y_i \to J_{Y_i} ; \quad Q \mapsto [\mathcal{I}_Q|_{Y_i}]$$

is an embedding. But, by Lemma 4.6 the map $B_i$ is a degree-0 Abel map. And, since $Y_i$ contains no separating lines, Theorem 3.8 implies that $B_i$ is an embedding.

Consider Statement (2), keeping the notation of Statement (1). Suppose now that $Q_1$ and $Q_2$ belong to the same separating tree of lines. Since the tree is connected, to show that $B(Q_1) = B(Q_2)$ we may assume that $Q_1$ and $Q_2$ lie on the smooth locus of $X$ and on the same separating line, $L$. But then $\mathcal{I}_{Q_1}|_{L} \cong \mathcal{I}_{Q_2}|_{L}$ and $\mathcal{I}_{Q_1}|_{L'} \cong \mathcal{I}_{Q_2}|_{L'}$, as is easily seen. Since $L$ is spine, Lemma 4.3 yields $\mathcal{I}_{Q_1} \cong \mathcal{I}_{Q_2}$.

Suppose now that $Q_1$ and $Q_2$ do not belong to the same tree of separating lines. We must prove that $B(Q_1) \neq B(Q_2)$. As we have just seen that $B$ is constant along separating lines, we may assume that $Q_1, Q_2 \in Y_1 \cup \cdots \cup Y_n$. Let $L_1, \ldots, L_m$ be the connected components of $S$. Since $X$ is connected, there are a positive integer $t$, and integers $i_1, \ldots, i_t \in \{1, \ldots, n\}$ and $j_1, \ldots, j_{t-1} \in \{1, \ldots, m\}$ such that $Q_1 \in Y_{i_1}$ and $Q_2 \in Y_{i_t}$, while $Y_{i_{\ell}} \cap L_{j_\ell} \neq \emptyset$ and $L_{j_\ell} \cap Y_{i_{\ell+1}} \neq \emptyset$ for each $\ell = 1, \ldots, t-1$. Choose $t$ minimum; then $Y_{i_{\ell}} \neq Y_{i_{\ell+1}}$ for every $\ell < t$. We will show that $B(Q_1) \neq B(Q_2)$ by induction on $t$.

If $t = 1$ then $Q_1$ and $Q_2$ belong to the same $Y_i$, and hence $B(Q_1) \neq B(Q_2)$ by Statement (1). Suppose now that $t \geq 2$. And suppose that $B(Q_1) = B(Q_2)$, by contradiction. In particular, using Lemma 4.3 we have $\mathcal{I}_{Q_1}|_{Y_{i_t}} \cong \mathcal{I}_{Q_2}|_{Y_{i_t}}$. But, since $Y_{i_t}$ is a spine, it follows from Lemma 4.6 that $\mathcal{I}_{Q_1}|_{Y_{i_t}} \cong \mathcal{I}_{Q_2}|_{Y_{i_t}}$ for each $Q$ on the same connected component of $Y_{i_t}'$ as $Q_1$. By connectedness, there will be such $Q$ on $L_{i_{t-1}} \cap Y_{i_t}$. Since $\mathcal{I}_{Q_1}|_{Y_{i_t}} \cong \mathcal{I}_{Q_2}|_{Y_{i_t}}$, and since $B_{i_t}$ is an embedding, as we saw in the proof of Statement (1), it follows that $Q_2 \in L_{i_{t-1}}$. Since $L_{i_{t-1}}$ is a separating tree of lines, $B(Q_2) = B(M)$ for any chosen $M \in Y_{i_{t-1}} \cap L_{i_{t-1}}$. Since $B(Q_1) = B(M)$, it follows by induction that $M = Q_1$. But then $Q_1$ and $Q_2$ are on the same separating tree of lines, namely $L_{i_{t-1}}$, reaching a contradiction.

Finally, we prove Statements (3) and (4). We proceed by induction on the number of separating nodes of $X$. If zero, then (3) is vacuous and (4) follows from (1).

Now, let $L$ be a maximal tree of separating lines of $X$, and $R \in J_X$ such that $B(L) = \{R\}$. Notice that $L \neq X$ because $g > 0$. Let $Z_1, \ldots, Z_n$ be the connected
components of $L'$ and $N_1, \ldots, N_δ$ the points in $L \cap L'$. Since $L$ is a spine with genus 0, we have that $n = δ$ and

\[(6.3.1) \quad g = gz_1 + \cdots + gz_n.\]

Also, up to reordering the $Z_i$, we may assume that $Z_i$ is generated by $N_i$ for each $i = 1, \ldots, n$. Notice that, for each $i$, no separating line of $Z_i$ contains $N_i$, because otherwise its union with $L$ would be a tree of separating lines of $X$ larger than $L$ itself. Also, notice that each $Z_i$ has less separating nodes than $X$.

Since $(L, Z_1, \ldots, Z_n)$ is a spine decomposition of $X$, by Lemma 4.3 there is a natural isomorphism

\[u = (u_L, u_1, \ldots, u_n) : J_X \longrightarrow J_L \times J_{Z_1} \times \cdots \times J_{Z_n}.\]

For each $i = 1, \ldots, n$, let $\widetilde{Z}_i := B(Z_i)$. By Lemma 4.6, as $Q$ moves on $Z_i$, the images $u_L(B(Q))$ and $u_j(B(Q))$, for $j \neq i$, remain constant. Thus $\widetilde{Z}_i \cong u_i(B(Z_i))$. However, by the same Lemma 4.6, the composition $u_i \circ B|_{Z_i}$ is a degree-0 Abel map. So, by induction, $\widetilde{Z}_i$ has genus $g_{Z_i}$.

Now, since $N_i$ is not contained in a separating line of $Z_i$, it follows from Statement (1) that $d_{N_i}B$ is injective on $T_{Z_i,N_i}$, and hence $\tilde{Z}_i$ is nonsingular at $R$, with $T_{\tilde{Z}_i,R} = d_{N_i}B(T_{Z_i,N_i})$. Since $B$ contracts $L$, by Statement (2), it follows that $T_{\tilde{Z}_i,R}$ is the whole image of $d_{N_i}B$.

Using $u$ to make the identification

\[T_{J_X,R} = T_{J_L,u_L(R)} \oplus T_{J_{Z_1},u_1(R)} \oplus \cdots \oplus T_{J_{Z_n},u_n(R)},\]

we may view $T_{\tilde{Z}_i,R}$ as a subspace of

\[0 \oplus 0 \oplus \cdots \oplus 0 \oplus T_{J_{Z_i},u_i(R)} \oplus 0 \oplus \cdots \oplus 0\]

for each $i = 1, \ldots, n$. Thus the $T_{\tilde{Z}_i,R}$ are linearly independent subspaces of $T_{J_X,R}$.

Since $R$ lies on the nonsingular loci of all the $\tilde{Z}_i$, and the $\tilde{Z}_i$ cover $\tilde{X}$, it follows that $\tilde{X}$ has a $n$-fold singularity at $R$, with the $T_{\tilde{Z}_i,R}$ for tangent lines, finishing the proof of Statement (3).

As for Statement (4), since the $\tilde{Z}_i$ intersect transversally at $R$, we have exact sequences of the form:

\[0 \rightarrow O_{\tilde{Z}_1}(-R) \rightarrow O_{\tilde{X}} \rightarrow O_{\tilde{Z}_2 \cup \cdots \cup \tilde{Z}_n} \rightarrow 0,\]
\[0 \rightarrow O_{\tilde{Z}_2}(-R) \rightarrow O_{\tilde{Z}_2 \cup \cdots \cup \tilde{Z}_n} \rightarrow O_{\tilde{Z}_3 \cup \cdots \cup \tilde{Z}_n} \rightarrow 0,\]
\[\vdots\]
\[0 \rightarrow O_{\tilde{Z}_{n-1}}(-R) \rightarrow O_{\tilde{Z}_{n-1} \cup \tilde{Z}_n} \rightarrow O_{\tilde{Z}_n} \rightarrow 0.\]

Computing Euler characteristics, and using that $\tilde{Z}_i$ has genus $g_{Z_i}$ for $i = 1, \ldots, n$,

\[\chi(O_{\tilde{X}}) = -gz_1 - gz_2 \cdots - gz_{n-1} + 1 - gz_n = 1 - g,\]

where the last equality is (6.3.1). Thus $\tilde{X}$ has genus $g$.

Finally, choosing a nonsingular point $Q$ of $X$ away from all small tails of $X$, we have that $A : X \rightarrow J_1^X$ is obtained from $A_Q$ by taking duals and then translating.
by \( \mathcal{O}_X(Q) \). Since these operations are isomorphisms, all of the statements proved above for \( A_Q \) hold for \( A \).

\[ \square \]

7. Abel Maps to Seshadri’s Compactified Jacobian

Recall that for each point \( Q \) of the curve \( X \) we defined the simple, torsion-free, rank-1, degree-1 sheaf \( \mathcal{I}^1_Q \), and let \( A(Q) := [\mathcal{I}^1_Q] \); see [5.2]. When \( X \) has a splitting node \( N \), the definition of \( \mathcal{I}^1_Q \) and that of \( A \) depend on the choice of one of the two tails generated by \( N \) (the small tail \( Z_N \)).

If \( X \) is \( G \)-stable, Theorem 5.4 says that \( \mathcal{I}^1_Q \) is semistable with respect to the canonical 1-polarization \( E_1 \) (cf. [2.3.2]). Its Jordan–Hölder filtrations (cf. [2.5]) are easy to describe.

**Lemma 7.1.** Assume the curve \( X \) is \( G \)-stable, and let \( Q \in X \). Then \( \mathcal{I}^1_Q \) is stable if and only if \( X \) admits no splitting node. If \( X \) has a splitting node \( N \), and \( Z_N \) is the small tail generated by \( N \), then

\[ \emptyset \subsetneq Z_N \subsetneq X \]

is the unique Jordan–Hölder filtration of \( \mathcal{I}^1_Q \).

**Proof.** As mentioned before the statement, \( \mathcal{I}^1_Q \) is semistable by Theorem 5.4. So, given a proper subcurve \( Y \subset X \), we have

\[
\deg_Y(\mathcal{I}^1_Q) \geq \frac{\deg_Y(\omega)}{2g-2} - \frac{\delta_Y}{2}.
\]

We must show that equality holds in (7.1.1) if and only if there is a splitting node \( N \) and \( Y = Z_N \). For this, we may assume that \( Y \) is connected.

Since \( X \) is \( G \)-stable,

\[ 0 < \frac{\deg_Y(\omega)}{2g-2} < 1. \]

Also, as seen in the proof of Theorem 5.4, a consequence of [CE06] Lemma 4.8, we have \( \deg_Y(\mathcal{I}^1_Q) \geq -1 \). Hence, since \( \delta_Y \) is a positive integer, equality in (7.1.1) may hold only in two cases:

1. \( \deg_Y(\mathcal{I}^1_Q) = -1 \), while \( \deg_Y(\omega) = g-1 \) and \( \delta_Y = 3 \).
2. \( \deg_Y(\mathcal{I}^1_Q) = 0 \), while \( \deg_Y(\omega) = g-1 \) and \( \delta_Y = 1 \).

We claim that the first case is not possible. Indeed, suppose by contradiction that it occurs. Since \( \delta_Y = 3 \), we have \( \deg_Y(\omega) = 2g+1 \), which implies that \( g_Y = g/2 - 1 \). But since \( \deg_Y(\mathcal{I}^1_Q) = -1 \), it follows that

\[ \deg_Y(\mathcal{O}_X(\sum_{Z \in ST(X), Z \ni Q} Z)) = -1. \]

So, by [CE06] Lemma 4.8, we have that \( Y \subseteq Z \) for a certain small tail \( Z \). Now, \( Y \neq Z \) because \( \delta_Y = 3 \). Then, since \( \omega \) is ample, \( \deg_Z(\omega) > \deg_Y(\omega) \). On the other hand, since \( Z \) is a small tail, \( \deg_Z(\omega) \leq g-1 \), and hence \( \deg_Z(\omega) \leq \deg_Y(\omega) \), reaching a contradiction.

Now, suppose the second case occurs. Then \( Y \) is a tail of genus \( g/2 \). So \( X \) has a splitting node \( N \), and we must show that \( Y = Z_N \). Suppose by contradiction that
Y \neq Z_N \text{ or, in other words, that } Y' \text{ is a small tail. There are two cases to consider: } Q \in Y \text{ and } Q \in Y'. \text{ If } Q \in Y', \text{ then } Y' \text{ is the largest small tail containing } Q, \text{ and hence } \deg_Y(I_Q^1) = 1, \text{ a contradiction. }

Suppose now that } Q \in Y. \text{ Since } \deg_Y(I_Q^1) \neq 1, \text{ there is a small tail } Z \text{ of } X \text{ containing } Q. \text{ For each such } Z, \text{ we have that } Z \not\supseteq Y, \text{ because otherwise the smallness of } Z \text{ would imply that } Y = Z, \text{ and } Y \text{ is not small. Also, } Z \not\supseteq Y', \text{ because otherwise the smallness of } Z \text{ would imply that } Z = Y', \text{ and thus } Q \in Y \cap Y'. \text{ But in this case, } Y' \text{ would be the unique small tail containing } Q, \text{ implying that } \deg_Y(I_Q^1) = 1, \text{ a contradiction. By Lemma 7.1, the only possibility left is that } Z \not\supseteq Y. \text{ But since this must hold for each small tail } Z \text{ containing } Q, \text{ we would get } \deg_Y(I_Q^1) = 1 \text{ again, a contradiction. }

Conversely, let } N \text{ be a splitting node, and suppose } Y = Z_N. \text{ We must show that } \deg_Y(I_Q^1) = 0. \text{ If } Q \in Y, \text{ then } Y \text{ is the largest small tail containing } Q, \text{ and hence } \deg_Y(I_Q^1) = 0. \text{ On the other hand, if } Q \notin Y, \text{ then all small tails containing } Q \text{ are strictly contained in } Y', \text{ and hence } \deg_Y(I_Q^1) = 0 \text{ as well.}

\[ \text{Theorem 7.2. Assume that the curve } X \text{ is } G\text{-stable. Let } A: X \to J_X^1 \text{ be the degree-1 Abel map (cf. 5.2) and } \Phi^1: J_X^{1,ss} \to U_X(1) \text{ the S-map (cf. 2.6.3). Then the following two statements hold:} \]

1. \( \Phi^1 \circ A \) is independent of the choice of a small tail of genus \( g/2 \).
2. \( \Phi^1 \) restricts to a closed embedding on \( A(X) \).

**Proof.** Suppose that \( X \) admits a splitting node \( N \) (unique by Lemma 5.3). Let \( Z \) be a tail generated by \( N \). Then \( A \) depends on whether we choose \( Z \) or \( Z' \) as small tail or, in other words, whether we set \( Z_N := Z \) or \( Z_N = Z' \). Let \( Q \in X \). Let \( \mathcal{I} \) (resp. \( \mathcal{I}' \)) be the torsion-free, rank-1 sheaf on \( X \) such that \( A(Q) = [\mathcal{I}] \) (resp. \( A(Q) = [\mathcal{I}'] \)), when \( Z_N = Z \) (resp. \( Z_N = Z' \)). By Lemma 7.1, we have \( \mathcal{G}(\mathcal{I}) = \mathcal{G}(\mathcal{I}') = \{ Z, Z' \} \).

So \( \mathcal{I} \) and \( \mathcal{I}' \) are S-equivalent, and thus \( \Phi^1(A(Q)) \) is independent of the choice of \( Z_N \).

As for the second statement, we need only show that \( \Phi^1|_{A(X)} \) separates points and tangent vectors. For each \( Q \in X \), let \( I_Q^1 \) be the torsion-free, rank-1 sheaf on \( X \) such that \( A(Q) = [I_Q^1] \). To show that \( \Phi^1|_{A(X)} \) separates points, we need to show that for each \( Q, Q' \in X \),

\[
(7.2.1) \quad \mathcal{G}(I_Q^1) = \mathcal{G}(I_{Q'}^1) \text{ and } \text{Gr}(I_Q^1) \cong \text{Gr}(I_{Q'}^1) \text{ if and only if } I_Q^1 \cong I_{Q'}^1.
\]

Now, if \( X \) contains no splitting node then Lemma 7.1 asserts that \( I_Q^1 \) and \( I_{Q'}^1 \) are stable, and thus \( (7.2.1) \) holds trivially. On the other hand, if \( X \) has a splitting node \( N \), then Lemma 7.1 implies that \( \mathcal{G}(I_Q^1) = \{ Z, Z' \} \) and

\[
\text{Gr}(I_Q^1) = I_Q^1|_Z \oplus (I_Q^1|_{Z'} \otimes \mathcal{O}_{Z'}(-N)),
\]

where \( Z := Z_N \). An analogous description holds for \( I_{Q'}^1 \). So \( \mathcal{G}(I_Q^1) = \mathcal{G}(I_{Q'}^1) \), and \( \text{Gr}(I_Q^1) \cong \text{Gr}(I_{Q'}^1) \) if and only if \( I_Q^1|_Z \cong I_{Q'}^1|_Z \) and \( I_Q^1|_{Z'} \cong I_{Q'}^1|_{Z'} \), which, by Lemma 4.3, occurs if and only if \( I_Q^1 \cong I_{Q'}^1 \). So \( (7.2.1) \) holds.
Finally, for any $Q \in X$, Lemma 7.1 implies that $\mathcal{G}(\mathcal{I}_Q)$ is a spine decomposition, whether $X$ admits a splitting node or not. So, given any nonzero $v \in T_{\mathcal{J}_X, A(Q)}$, it follows from [E07] Lemma 3.11 and Prop. 4.3 that there is a subscheme $\Theta \subseteq U_X(a, d)$ such that $d\Phi_{A(Q)}(v) \notin T_{\Theta, \Phi(A(Q))}$. Thus $\Phi|_{A(X)}$ separates tangent vectors.

**Remark 7.3.** In general, it is not true that $\Phi^0 : J_X^{0,ss} \to U_X(0)$ restricts to a closed embedding on $A_P(X)$. For a simple example, suppose $X$ is a nodal curve with four irreducible components: two of them, $X_1$ and $X_2$, smooth and rational, meeting at two points, $N_1$ and $N_2$; the third, $X_3$, nonrational, meeting only $X_1$ at a single point $N_3$, and the fourth, $X_4$, also nonrational, meeting only $X_2$ at a single point $N_4$. From its description, $X$ is stable. Also, it contains no separating lines, so $A_P$ is an embedding by Theorem 4.8. Suppose further that $g_{X_3} = g_{X_4}$. Then the only small tails of $X$ are $X_3$ and $X_4$.

Suppose $P \in X_1$. For each $Q \in X$, let $\mathcal{I}_Q$ be the torsion-free, rank-1 sheaf on $X$ such that $A_P(Q) = [\mathcal{I}_Q]$ (cf. 4.5). Let $Q \in X_2 - \{N_1, N_2, N_4\}$. Given a proper subcurve $Y$ of $X$, we have that $\deg_Y(\mathcal{I}_Q) = -\delta_Y/2$ if and only if $Y = X_2 \cup X_4$. Thus $\mathcal{G}(\mathcal{I}_Q) = \{X_1 \cup X_3, X_2 \cup X_4\}$ and

$$\text{Gr}(\mathcal{I}_Q) = \mathcal{O}_{X_2 \cup X_4}(-Q) \oplus \mathcal{O}_{X_1 \cup X_3}(P - N_1 - N_2).$$

Now, since $X_2$ is rational, Lemma 4.3 yields that, as $Q$ moves on $X_2 - \{N_1, N_2, N_4\}$, the isomorphism class of $\mathcal{O}_{X_2 \cup X_4}(-Q)$ does not change. Therefore, the composition $\Phi^0 \circ A_P$ contracts $X_2$.

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