Ultradiscrete Lotka-Volterra system computes tropical eigenvalue of symmetric tridiagonal matrices

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Abstract. Some of authors’ recent study shows that the time evolution of the integrable ultradiscrete Toda equation computes eigenvalue of tridiagonal matrices over min-plus algebra, where min-plus algebra is a semiring with two binary operations: \(+\) := min and \(\oplus\) := +. In this paper, we first present a Bäcklund transformation between the ultradiscrete Toda equation and the ultradiscrete Lotka-Volterra system. Using the Bäcklund transformation, we show that the ultradiscrete Lotka-Volterra system can also compute eigenvalue of symmetric tridiagonal matrices over min-plus algebra.

1. Introduction

The qd (quotient difference) algorithm is for computing eigenvalues of symmetric tridiagonal matrices or singular values of upper bidiagonal matrices, introduced by Rutishauser [12]. The recursion formula of the qd algorithm can be obtained by a skillful time discretization of integrable Toda equation [8], i.e., the discrete Toda (dToda) equation:

\[
\begin{align*}
q_k^{(n+1)} &= q_k^{(n)} + e_k^{(n)} - e_{k-1}^{(n+1)}, & k &= 1, 2, \ldots, m, \\
e_k^{(n+1)} &= \frac{q_{k+1}^{(n)}}{q_k^{(n+1)}} e_k^{(n)}, & k &= 1, 2, \ldots, m - 1, \\
e_0^{(n)} &= 0, & n &= 0, 1, \ldots,
\end{align*}
\]

where the subscript \(k\) and the superscript \(n\) denote spacial and discrete time variables, respectively. Toda equation is one of the famous integrable systems. Integrable system is a generic term of nonlinear differential or difference equations which can be solved exactly with finite number of quadrature methods.

For difference equation, Tokihiro, Takahashi, Matsukidaira, and Satsuma [16] found an ultradiscretization technique which is sometimes called the tropicalization. By applying ultradiscretization, difference equation is transformed into piecewise linear equations, which can be written using operator max or min. The resulting equation is called ultradiscrete equation and
is associated with max-plus or min-plus algebras. Min-plus algebra has two binary operations \( \oplus := \min \) and \( \otimes := + \) in the set \( \mathbb{R}_{\min} := \mathbb{R} \cup \{ +\infty \} \). Interestingly, there are close relationship between min-plus algebra and weighted directed graphs, hereinafter we call digraph, constructed with sets of vertices and directed edges with weights. Weighted digraphs appear in various fields of mathematics, such as discrete event systems, railway system and so on [7].

The discrete Lotka-Volterra (dLV) system, which is known as a prey-predator model, is given by

\[
\begin{align*}
\begin{cases}
  u_k^{(n+1)} &= u_k^{(n)} + \frac{1 + \delta^{(n)} u_{k+1}^{(n)}}{1 + \delta^{(n+1)} u_{k-1}^{(n+1)}}, & k = 1, 2, \ldots, 2m - 1, \\
  u_0^{(n)} &= 0,
\end{cases}
\end{align*}
\]

where \( u_k^{(n)} \) denotes the population of species \( k \) at discrete time \( n \) and \( \delta^{(n)} \) denotes the discrete step size at discrete time \( n \). The dLV system (2) is also known to be one of the famous integrable system. Throughout this paper, \( \delta^{(n)} \) is set to be 1 for all \( n \).

Variable transformations called Bäcklund transformations are studied in the field of integrable systems. For example, the Bäcklund transformation, which is sometimes called Miura transformation, between the dToda equation (1) and the dLV system (2) plays a significant role in designing numerical algorithm based on the integrable systems The dLV system can also be applied to eigenvalue algorithm for computing symmetric tridiagonal matrices, and the algorithm is named the dLV algorithm [9]. The mdLVs algorithm is improvement of the dLV algorithm and is known to be accurate and fast owing to incorporating the origin shift [10]. Some of authors’ recent study [17] showed that the ultradiscrete Toda equation computes min-plus tridiagonal matrices. As far as we know, [17] is the first result which relate the ultradiscrete integrable systems and the eigenvalue algorithm over min-plus algebra. In this paper, first we derive the Bäcklund transformation between ultradiscrete Toda equation and ultradiscrete Lotka-Volterra system by applying the ultradiscretization to the discrete one. Moreover, using the ultradiscrete Bäcklund transformation, we relate the ultradiscrete Lotka-Volterra system to the matrix eigenvalue on min-plus algebra.

This paper is organized as follows. In section 2, a brief introduction of matrix eigenvalues over min-plus algebra and relationship to weighted directed graphs corresponding to min-plus matrices are presented. In section 3, the Bäcklund transformation between the ultradiscrete Toda equation and ultradiscrete Lotka-Volterra system is derived. In section 4, it is shown that the ultradiscrete Lotka-Volterra system can compute eigenvalue over min-plus algebra. In section 5, concluding remarks are given.

2. Min-plus arithmetic and Eigenvalue problems

Let \( \mathbb{R} \) be the set of real numbers. We define the min-plus algebra \( \mathbb{R}_{\min} \) by \( \mathbb{R}_{\min} \) with the binary operations \( \oplus \) and \( \otimes \):

\[
\begin{align*}
  a \oplus b &= \min \{ a, b \}, \\
  a \otimes b &= a + b.
\end{align*}
\]

It is easy to check that both \( \oplus \) and \( \otimes \) are associative and commutative, and \( \otimes \) is distributive with respect to \( \oplus \). Moreover, we may regard \( c := +\infty \) and \( e := 0 \) as identities with respect to \( \oplus \) and \( \otimes \), respectively. We also use the notation \( a \otimes b = a - b \) for inverse operation of \( \otimes \) for convenience. Here, we consider matrices with entries in \( \mathbb{R}_{\min} \) to be min-plus matrices. For positive integers \( m \) and \( n \), we denote the set of all \( m \)-by-\( n \) min-plus matrices as \( \mathbb{R}_{\min}^{m \times n} \). For two
min-plus matrices $A = (a_{ij})$, $B = (b_{ij}) \in \mathbb{R}^{m \times m}$, the sum $A \oplus B = ([A \oplus B]_{ij})$ and the product $A \otimes B = ([A \otimes B]_{ij})$ are respectively defined as:

$$[A \oplus B]_{ij} = a_{ij} \oplus b_{ij} = \min\{a_{ij}, b_{ij}\},$$

$$[A \otimes B]_{ij} = \bigoplus_{\ell=1}^{m} (a_{i\ell} \otimes b_{\ell j}) = \min_{\ell=1,2,\ldots,m} \{a_{i\ell} + b_{\ell j}\}.$$  

Moreover, the scalar multiplication $\alpha \otimes A = ([\alpha \otimes A]_{ij})$ are given as:

$$[\alpha \otimes A]_{ij} = \alpha \otimes a_{ij}.$$  

The following definition determines the eigenvalue and the eigenvector of the min-plus matrix.

**Definition 2.1.** Given a min-plus matrix $A \in \mathbb{R}^{m \times m}_{\min}$, we say that $\lambda \in \mathbb{R}_{\min}$ is an eigenvalue of $A$ if there exists $x \neq (\varepsilon, \varepsilon, \ldots, \varepsilon)^{T} \in \mathbb{R}^{m}_{\min}$ satisfying:

$$A \otimes x = \lambda \otimes x.$$  

The vector $x$ is called the eigenvector of $A$.

Since not all elements in $\mathbb{R}_{\min}$ have additive inverses, determinants of min-plus matrices are not directly defined as min-plus analogues of linear algebra. The following definition which is one of many known definitions of the determinant in the min-plus algebra. It is called tropical determinants.

**Definition 2.2.** Let $A = (a_{ij}) \in \mathbb{R}^{m \times m}_{\min}$ be a min-plus matrix. The tropical determinant $\text{tropdet}(A)$ of $A$ is defined as

$$\text{tropdet}(A) := \bigoplus_{\sigma \in S_{m}} a_{1\sigma(1)} \otimes a_{2\sigma(2)} \otimes \cdots \otimes a_{m\sigma(m)},$$

where $S_{m}$ is the symmetric group of permutations of $\{1,2,\ldots,m\}$. Moreover, the characteristic polynomial $g_{A}(x)$ of $A$ is defined as

$$g_{A}(x) := \text{tropdet}(A \oplus x \otimes I),$$

where $I$ denotes the identity matrix of order $m$ whose diagonal entries are $0$ and otherwise are $\varepsilon$.

Here we introduce the min-plus polynomial of order $n$ in a single indeterminate $x$ with coefficients in $\mathbb{R}_{\min}$ by the form

$$p(x) = x^{n} \oplus c_{1} \otimes x^{n-1} \oplus \cdots \oplus c_{n-1} \otimes x \oplus c_{n},$$

where $x^{k} := x \otimes x \otimes \cdots \otimes x$. Since the min-plus polynomial $p(x)$ takes the form

$$p(x) = \min\{nx, c_{1} + (n-1)x, \ldots, c_{n-1} + x, c_{n}\}$$

by using the ordinary expression, we see that $p(x)$ is piecewise linear. For two min-plus polynomials $p(x)$ and $\tilde{p}(x)$, we denote $p(x) = \tilde{p}(x)$ if the corresponding coefficients are equal. On the other hand, two distinct min-plus polynomials $p(x)$ and $\tilde{p}(x)$ sometimes provide the same functional graph [2]. These min-plus polynomials $p(x)$ and $\tilde{p}(x)$ are called equivalent and denoted by $p(x) \equiv \tilde{p}(x)$. The following proposition show the factorization of the characteristic polynomial of the min-plus matrix and its minimum root.

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Proposition 2.3 ([11]). Let \( g_A(t) \) be a characteristic polynomial \( g_A(t) \) of the min-plus matrix \( A \in \mathbb{R}_{\min}^{m \times m} \). Then, \( g_A(t) \) is factorized as

\[
g_A(t) \equiv (t \oplus p_1)^{q_1} \otimes (t \oplus p_2)^{q_2} \otimes \cdots \otimes (t \oplus p_k)^{q_k},
\]

where \( p_1 < p_2 < \cdots < p_k \) and \( q_1 + q_2 + \cdots + q_k = m \). The minimum root \( p_1 \) of \( g_A(t) \) coincides with the minimum eigenvalue of \( A \).

A directed graph (digraph) \( G = (V, E) \) consists of finite sets \( V \) and \( E \subset V \times V \); an element \( v \in V \) is called a vertex and an element \( e = (v_i, v_j) \in E \) is called an edge of the digraph \( G \). An edge of the form \((v, v)\) is called a loop. A path \( P \) is an alternating sequence \( P = (v_0, e_1, v_1, e_2, \ldots, e_s, v_s) \) of distinct vertices and edges such that \( e_k = (v_{k-1}, v_k) \in E \) for \( k = 1, 2, \ldots, s \). Vertices \( v_0 \) and \( v_s \) are respectively called the initial and the terminal vertex of \( P \). A path \( P \) is called a circuit, if the initial vertex of \( P \) coincides with the terminal vertex. A digraph \( G \) is called strongly connected if there exists at least one path from \( v_i \) to \( v_j \) for any vertices \( v_i, v_j \in V \). We assign a number \( w(e) \in \mathbb{R}_{\min} \) to each edge \( e \in E \); \( w(e) \) is called the weight of the edge \( e \). Then a digraph \( G \) is called a weighted digraph if all edges \( e \) in \( G \) have the weight \( w(e) \).

**Definition 2.4.** Let \( G \) be a weighted digraph with \( m \) vertices. The weighted adjacency matrix \( A(G) = (a_{ij}) \in \mathbb{R}_{\min}^{m \times m} \) is defined by

\[
a_{ij} = \begin{cases} w(e), & \text{if } e = (v_i, v_j) \in E, \\ \varepsilon, & \text{otherwise}. \end{cases}
\]

Conversely, for an arbitrary min-plus matrix \( A = (a_{ij}) \in \mathbb{R}_{\min}^{m \times m} \), we can construct the weighted digraph \( G(A) \) by

\[
V = \{v_1, v_2, \ldots, v_m\},
E = \{(v_i, v_j)|a_{ij} \neq \varepsilon\},
w(e) = a_{ij} \text{ for } e = (v_i, v_j) \in E.
\]

Moreover, for a circuit \( C \) in a weighted digraph \( G(A) \), we define the length \( \ell(C) \) by the number of edges in \( C \) and the weight \( w(C) \) by the sum of weights of edges in \( C \). Then we can introduce the average weight of the circuit \( C \) as the following definition.

**Definition 2.5.** Let \( C \) be a circuit in the weighted digraph \( G \). The average weight \( w_{\text{ave}}(C) \) is defined by

\[
w_{\text{ave}}(C) = \frac{w(C)}{\ell(C)}
\]

The following proposition shows relationships between the average weights of circuits on the weighted digraph \( G(A) \) and the minimum eigenvalue of \( A \).

**Proposition 2.6 ([1], [6]).** Let \( G(A) \) be a a weighted digraph. If \( G(A) \) is strongly connected, then the weighted adjacency matrix \( A(G) \) has unique eigenvalue whose value is equal to the minimum value of the average weights of circuits in \( G(A) \).
3. Bäcklund transformation between ultradiscrete Toda equation and ultradiscrete Lotka-Volterra system

In this section, we first review the Bäcklund transformation between dToda equation (1) and dLV system (2) [4, 5]. Next we derive the ultradiscrete Toda equation and the ultradiscrete Lotka-Volterra system. Then, the Bäcklund transformation between ultradiscrete Toda equation and the ultradiscrete Lotka-Volterra system can be obtained by applying the ultradiscretization to the discrete one.

For the dToda equation (1), we introduce the following variable transformations:

$$q^{(n+1)}_k = v^{(n)}_{2k}, \quad q^{(n)}_k = v^{(n)}_{2k-1}, \quad e^{(n)}_k = w^{(n)}_{2k}.$$  \hspace{1cm} (3)

Then, the dToda equation (1) is transformed into

$$\begin{align*}
v^{(n)}_{2k} &= v^{(n)}_{2k-1} + w^{(n+1)}_{2k-2}, \quad k = 1, 2, \ldots, m, \\
w^{(n+1)}_{2k} &= \frac{v^{(n)}_{2k+1}}{v^{(n)}_{2k}} w^{(n)}_{2k}, \quad k = 1, 2, \ldots, m - 1, \\
w^{(n)}_0 &= 0, \quad w^{(n)}_{2m} = 0, \quad n = 0, 1, \ldots.
\end{align*}$$  \hspace{1cm} (4)

In the left hand side of (4), spatial variables are only even numbers. In order that (4) holds for arbitrary spatial numbers, we need to be set

$$w^{(n)}_{2k-1} = q^{(n)}_k.$$  \hspace{1cm} (5)

Then we obtain

$$\begin{align*}
v^{(n)}_k &= v^{(n)}_{k-1} + w^{(n)}_k - w^{(n+1)}_{k-2}, \quad k = 1, 2, \ldots, 2m, \\
w^{(n+1)}_k &= \frac{v^{(n)}_{k+1}}{v^{(n)}_k} w^{(n)}_k, \quad k = 1, 2, \ldots, 2m - 1, \\
w^{(n)}_0 &= 0, \quad w^{(n)}_{2m} = 0, \quad n = 0, 1, \ldots.
\end{align*}$$  \hspace{1cm} (6)

We here introduce the following variable transformations.

$$\begin{align*}
w^{(n)}_k &= u^{(n)}_k (1 + v^{(n)}_{k-1}), \\
v^{(n)}_k &= (1 + u^{(n)}_k)(1 + u^{(n)}_{k-1}).
\end{align*}$$  \hspace{1cm} (7)

By applying (7) to (6), we can easily check that both of the equations in (6) is equivalent to the dLV system (2). Transformations (3), (5) and (7) together is called the Bäcklund transformation between dToda equation (1) and the dLV system (2).

Ultradiscrete technique is based on the following limiting procedure. For $A, B \in \mathbb{R}$, it holds that

$$\lim_{\epsilon \to +0} -\epsilon \log(e^{-A/\epsilon} + e^{-B/\epsilon}) = \min\{A, B\}.\hspace{1cm} (8)$$

Using the limiting procedure (8), we ultradiscretize the dToda equation (1). But since the dToda equation (1) have subtraction in the 1st equation, we can not apply the ultradiscretization as it is. Thus we transform the 1st equation using the 1st and the 2nd equation repeatedly as follows.

$$q^{(n+1)}_k = e^{(n)}_k + \frac{\prod_{j=1}^k q^{(n)}_j}{\prod_{j=1}^{k-1} q^{(n+1)}_j}.$$  \hspace{1cm} (9)
Applying the variable transformation (15) to (14), we obtain

\[
\begin{align*}
Q^{(n+1)}_k &= E^{(n)}_k \oplus \bigotimes_{j=1}^{k-1} Q^{(n+1)}_j, \quad k = 1, 2, \ldots, m, \\
E^{(n+1)}_k &= Q^{(n+1)}_{k+1} \otimes Q^{(n)}_k \otimes E^{(n)}_k, \quad k = 1, 2, \ldots, m-1, \\
E^{(n)}_0 &= \varepsilon, \quad E^{(n)}_m = \varepsilon, \quad n = 0, 1, \ldots.
\end{align*}
\]

For (9) and the 2nd equation in (1), applying the variable transformations \( q_k^{(n)} = e^{-Q^{(n)}/\epsilon} \) and \( \varepsilon_k^{(n)} = e^{-E^{(n)}/\epsilon} \) to the dToda equation (1), taking \(-\epsilon \log \) on both sides, and taking the limit \( \epsilon \to +0 \), we have the ultradiscrete Toda (udToda) equation:

\[
Q^{(n+1)}_k = V^{(n)}_{2k}, \quad Q^{(n)}_k = V^{(n)}_{2k-1}, \quad E^{(n)}_k = W^{(n)}_{2k}.
\]

By applying (11) to (10), we obtain

\[
\begin{align*}
V^{(n)}_{2k} = W^{(n)}_{2k} \oplus \bigotimes_{j=1}^{k-1} V^{(n)}_{2j-1} \otimes V^{(n)}_{2j}, \quad k = 1, 2, \ldots, m, \\
W^{(n+1)}_{2k} &= V^{(n)}_{2k+1} \otimes W^{(n)}_{2k}, \quad k = 1, 2, \ldots, m-1, \\
W^{(n)}_0 &= \varepsilon, \quad W^{(n)}_{2m} = \varepsilon, \quad n = 0, 1, \ldots.
\end{align*}
\]

In the left hand side of (12), odd number of independent variables does not appear. As is the same with the discrete one, we add the condition

\[
W^{(n)}_{2k-1} = Q^{(n)}_k.
\]

Then (12) holds for any independent variables as follows.

\[
\begin{align*}
V^{(n)}_{2k} = W^{(n)}_{2k} \oplus \bigotimes_{j=1}^{k-1} V^{(n)}_{2j-1} \otimes V^{(n)}_{2j}, \quad k = 1, 2, \ldots, m, \\
V^{(n)}_{2k-1} &= W^{(n)}_{2k-1} \oplus \bigotimes_{j=1}^{k-1} V^{(n)}_{2j} \otimes V^{(n)}_{2j}, \quad k = 1, 2, \ldots, m, \\
W^{(n+1)}_k &= V^{(n)}_{k+1} \otimes V^{(n)}_k \otimes W^{(n)}_k, \quad k = 1, 2, \ldots, 2m-1, \\
W^{(n)}_0 &= \varepsilon, \quad W^{(n)}_{2m} = \varepsilon, \quad n = 0, 1, \ldots.
\end{align*}
\]

We also apply the ultradiscretization to (7), we have

\[
\begin{align*}
W^{(n)}_k &= U^{(n)}_k \otimes (0 \oplus U^{(n)}_{k-1}), \\
V^{(n)}_k &= (0 \oplus U^{(n)}_k) \otimes (0 \oplus U^{(n)}_{k-1}).
\end{align*}
\]

Applying the variable transformation (15) to (14), we obtain

\[
\begin{align*}
U^{(n+1)}_k &= U^{(n)}_k \otimes (0 \oplus U^{(n)}_{k+1}) \otimes (0 \oplus U^{(n+1)}_{k-1}), \quad k = 1, 2, \ldots, 2m-1, \\
U^{(n)}_0 &= \varepsilon, \quad U^{(n)}_{2m} = \varepsilon, \quad n = 0, 1, \ldots.
\end{align*}
\]
(16) coincide with the ultradiscretization of the dLV system (2) with $\delta^{(n)} = 1$, i.e., applying the variable transformation $u_k^{(n)} = e^{-U_k^{(n)}}$, taking $-\epsilon \log$ on both hand sides and taking the limit $\epsilon \to +0$. We call (11), (13) and (15) together as the Bäcklund transformation between udToda equation (10) and the udLV system (16).

Note here that (14) can also be obtained through the ultradiscretization of (6) by the following procedure. First we transform the 1st equation into the one without subtraction. In the case where $k = 1$ and $k = 2$, we have
\[
\begin{align*}
v_1^{(n)} &= w_1^{(n)} + 1, \\
v_2^{(n)} &= w_2^{(n)} + v_1^{(n)},
\end{align*}
\]
where $v_0^{(n)} := 1$ and $w_1^{(n)} := 0$ for all $n$. For $k = 3$, we have
\[
\begin{align*}
v_3^{(n)} &= w_3^{(n)} + v_2^{(n)} - w_1^{(n+1)} \\
&= w_3^{(n)} + \frac{v_2^{(n)} - v_1^{(n)}}{v_1^{(n)}},
\end{align*}
\]
where we use (17). For $k = 4$, we also have
\[
\begin{align*}
v_4^{(n)} &= w_4^{(n)} + v_3^{(n)} - w_2^{(n+1)} \\
&= w_4^{(n)} + \frac{v_3^{(n)} - v_1^{(n)}}{v_1^{(n)}},
\end{align*}
\]
where we use (18). For $k = 5, 6, \ldots$, in a similar way, we have
\[
\begin{align*}
v_{2k}^{(n)} &= w_{2k}^{(n)} + \frac{v_{2k-1}^{(n)} - v_{2k-3}^{(n)} - \cdots - v_1^{(n)}}{v_{2k-2}^{(n)} - v_{2k-4}^{(n)} - \cdots - v_2^{(n)}}, & k = 1, 2, \ldots, m, \\
v_{2k-1}^{(n)} &= w_{2k-1}^{(n)} + \frac{v_{2k-2}^{(n)} - v_{2k-4}^{(n)} - \cdots - v_2^{(n)}}{v_{2k-3}^{(n)} - v_{2k-5}^{(n)} - \cdots - v_1^{(n)}}, & k = 1, 2, \ldots, m, \\
w_k^{(n+1)} &= w_k^{(n+1)} + w_k^{(n)}, & k = 1, 2, \ldots, 2m - 1, \\
v_0^{(n)} &= 0, & w_0^{(n)} &= 0, & n = 0, 1, \ldots
\end{align*}
\]
Since (19) has no subtraction, we can apply the ultradiscretization to (19). For (19), applying the variable transformations $v_k^{(n)} = e^{-V_k^{(n)}}$ and $w_k^{(n)} = e^{-W_k^{(n)}}$ and taking $-\epsilon \log$ on both sides, and taking the limit $\epsilon \to +0$, then we have (14).

4. The ultradiscrete Lotka-Volterra system and associated min-plus eigenvalue

In this section, some basic properties of the dLV system and the dLV algorithm for eigenvalues of symmetric tridiagonal matrices are given. Then we show that an ultradiscrete analogue of the dLV system has an application to compute the eigenvalue of min-plus tridiagonal matrices.
Now, we introduce the \(m\)-by-\(m\) upper bidiagonal matrices \(B^{(n)}\) involving variable \(w_k^{(n)}\) in (7) as

\[
B^{(n)} := \begin{bmatrix}
\sqrt{w_1^{(n)}} & \sqrt{w_2^{(n)}} & & \\
& \sqrt{w_2^{(n)}} & \ddots & \\
& & \ddots & \sqrt{w_m^{(n)}} \\
& & & \sqrt{w_{2m-1}^{(n)}}
\end{bmatrix},
\]

where we assume that \(u_k^{(0)} > 0\) therefore \(w_k^{(0)} > 0\) for \(k = 1, 2, \ldots, 2m - 1\). Then, it follows that

\[
A^{(n)} := (B^{(n)})^T B^{(n)} = \begin{bmatrix}
w_1^{(n)} & \sqrt{w_1^{(n)} w_2^{(n)}} & & \\
\sqrt{w_1^{(n)} w_2^{(n)}} & w_2^{(n)} + w_3^{(n)} & \ddots & \\
& \ddots & \ddots & \sqrt{w_{2m-3}^{(n)} w_{2m-2}^{(n)}} \\
& & & \sqrt{w_{2m-3}^{(n)} w_{2m-2}^{(n)} + w_{2m-1}^{(n)}}
\end{bmatrix}.
\]

\(B^{(n)} (B^{(n)})^T = A^{(n+1)} = (B^{(n+1)})^T B^{(n+1)}\) is equivalent to 1 step of the Cholesky LR algorithm [12] for eigenvalue computation. We can see that the dLV system (2) appears on each tridiagonal entry of \(B^{(n)} (B^{(n)})^T = (B^{(n+1)})^T B^{(n+1)}\). Since one step of the Cholesky LR algorithm is an similarity transformation \(A^{(n+1)} = B^{(n)} A^{(n)} (B^{(n)})^{-1}\), so eigenvalues of \(A^{(n)}\) are invariant under the time evolution of the dLV system (2) from \(n\) to \(n + 1\).

Asymptotic behavior of the dLV system (2) is shown in [9] as

\[
\lim_{n \to \infty} u_{2k-1}^{(n)} = \lambda_k, \quad k = 1, 2, \ldots, m,
\]

\[
\lim_{n \to \infty} u_{2k}^{(n)} = 0, \quad k = 1, 2, \ldots, m - 1.
\]

Thus, from (7), asymptotic behavior of variables \(w_k^{(n)}\) yields

\[
\lim_{n \to \infty} w_{2k-1}^{(n)} = \lambda_k, \quad k = 1, 2, \ldots, m,
\]

\[
\lim_{n \to \infty} w_{2k}^{(n)} = 0, \quad k = 1, 2, \ldots, m - 1.
\]

Using the discrete time evolution of the dLV system (2), asymptotic behavior of \(A^{(n)}\) is given as

\[
\lim_{n \to \infty} A^{(n)} = \begin{bmatrix}
\lambda_1 \\
\lambda_2 \\
& \ddots \\
& & \ddots \\
& & & \lambda_m
\end{bmatrix}.
\]

Here we define the given initial entry of the bidiagonal matrix as \(b_k := \sqrt{u_k^{(0)}}\), \(k = 1, 2, \ldots, 2m - 1\). From \(u_k^{(0)} (1 + u_k^{(0)}) = b_k^2\), the initial value of the dLV variables are need to be set as

\[
u_k^{(0)} = \frac{b_k^2}{1 + u_k^{(0)}}, \quad k = 1, 2, \ldots, 2m - 1,
\]
The weighted digraph $G(\mathcal{A}^{(n)})$ corresponding to the adjacency matrix $\mathcal{A}^{(n)}$.

which can be recursively determined. Under the initial value, by the time evolution of the dLV system (2), limiting value of $w_{2k-1}^{(n)}$, $k = 1, 2, \ldots, m$ gives the eigenvalues $\lambda_k$ of $A^{(0)}$, which is equivalent to the singular values of $B^{(0)}$.

From (20), we define the following $m$-by-$m$ min-plus matrices as follows.

$$B^{(n)} = \begin{bmatrix}
\frac{1}{2} W_1^{(n)} & \frac{1}{2} W_2^{(n)} & \varepsilon & \cdots & \varepsilon \\
\varepsilon & \frac{1}{2} W_3^{(n)} & \frac{1}{2} W_4^{(n)} & \cdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \varepsilon \\
\varepsilon & \cdots & \cdots & \frac{1}{2} W_{2m-2}^{(n)} & \varepsilon \\
\varepsilon & \cdots & \cdots & \varepsilon & \frac{1}{2} W_{2m-1}^{(n)}
\end{bmatrix},$$

where we note that $\sqrt{W_k^{(n)}} = \frac{1}{2} W_k^{(n)}$. Then the symmetric tridiagonal matrix $\mathcal{A}^{(n)} := (B^{(n)})^T B^{(n)}$ is given by

$$\mathcal{A}^{(n)} = (B^{(n)})^T B^{(n)} = \begin{bmatrix}
W_1^{(n)} & \frac{1}{2} (W_1^{(n)} \otimes W_2^{(n)}) & \varepsilon & \cdots & \varepsilon \\
\frac{1}{2} W_1^{(n)} \otimes W_2^{(n)} & W_2^{(n)} \oplus W_3^{(n)} & \cdots & \cdots & \vdots \\
\varepsilon & \cdots & \cdots & \cdots & \varepsilon \\
\varepsilon & \cdots & \cdots & \frac{1}{2} W_{2m-3}^{(n)} \otimes W_{2m-2}^{(n)} & \frac{1}{2} (W_{2m-3}^{(n)} \otimes W_{2m-2}^{(n)}) \\
\varepsilon & \cdots & \cdots & \varepsilon & W_{2m-2}^{(n)} \oplus W_{2m-1}^{(n)}
\end{bmatrix}.$$
For the characteristic polynomial of $A^{(n)}$, we have the following theorem.

**Theorem 4.1.** Characteristic polynomial $g_{A^{(n)}}(t)$ of $A^{(n)}$ is factorized as follows.

$$g_{A^{(n)}}(t) = \text{tropdet}(A^{(n)} \oplus t \otimes I)$$

$$= (t \oplus W_1^{(n)}) \otimes (t \oplus W_2^{(n)} \oplus W_3^{(n)}) \otimes (t \oplus W_4^{(n)} \oplus W_5^{(n)}) \otimes \cdots \otimes (t \oplus W_{2m-2}^{(n)} \oplus W_{2m-1}^{(n)}).$$

(22)

**Proof.** This proof is conducted by induction concerning the matrix size $m$. $A_k^{(n)}$ denotes $k$-by-$k$ the principal submatrices of $A^{(n)}$. In case of $m = 1$, from $A_1^{(n)} = W_1^{(n)}$, we have $g_{A_1^{(n)}}(t) = \text{tropdet}(A_1^{(n)} \oplus t \otimes I) = t \oplus W_1^{(n)}$. If $m = 2$, the characteristic polynomials $g_{A_2^{(n)}}(t)$ can be factorized as:

$$g_{A_2^{(n)}}(t) = \text{tropdet}\left( \begin{bmatrix} t \oplus W_1^{(n)} \\ \frac{1}{2}(W_1^{(n)} \circledast W_2^{(n)}) & t \oplus W_2^{(n)} \oplus W_3^{(n)} \end{bmatrix} \right)$$

$$= (t \oplus W_1^{(n)}) \otimes (t \oplus W_2^{(n)} \oplus W_3^{(n)}) \oplus (W_1^{(n)} \circledast W_2^{(n)})$$

$$= (t \oplus W_1^{(n)}) \otimes (t \oplus W_2^{(n)} \oplus W_3^{(n)}).$$

As an min-plus analogue of linear cofactor expansions, we consider the cofactor expansions of tropical determinants, which can be obtained in the similar way of conventional linear algebra. Using the cofactor expansions over min-plus algebra to the $m$th column of tropdet($A^{(n)}_m \oplus t \otimes I$), we have

$$g_{A_m^{(n)}}(t) = (t \oplus W_2^{(n)} \oplus W_3^{(n)} \otimes \text{tropdet}(A^{(n)}_{m-1} \oplus t \otimes I)$$

$$\oplus (W_1^{(n)} \oplus W_2^{(n)} \oplus W_3^{(n)} \otimes \text{tropdet}(A^{(n)}_{m-2} \oplus t \otimes I)$$

$$= (t \oplus W_2^{(n)} \oplus W_3^{(n)} \otimes \text{tropdet}(A^{(n)}_{m-1} \oplus t \otimes I).$$

From the assumption that tropdet($A^{(n)}_{m-1} \oplus t \otimes I) = (t \oplus W_1^{(n)} \otimes (t \oplus W_2^{(n)} \oplus W_3^{(n)} \otimes \cdots \otimes (t \oplus W_{2m-4}^{(n)} \oplus W_{2m-3}^{(n)}), we have (22). □

The roots of the characteristic polynomial in Theorem 4.1 are diagonal entries of $A^{(n)}$, that is, the weight of the loops in the weighted digraph $G(A^{(n)})$. Since, from Proposition 2.3, the minimum value of the roots corresponds to the eigenvalue of $A^{(n)}$, eigenvalue $\lambda^{(n)}$ of $A^{(n)}$ is given as follows.

$$\lambda^{(n)} = \bigoplus_{k=1}^{2m-1} W_k^{(n)}.$$

Here, for the dLV system (2), it is shown in [9] that there exists the conserved quantity as follows.

$$\sum_{k=1}^{2m-1} u_k^{(n)} = \sum_{k=1}^{2m-1} u_k^{(n+1)}.$$  

By applying the ultradiscretization to the conserved quantity, we obtain

$$\bigoplus_{k=1}^{2m-1} W_k^{(n)} = \bigoplus_{k=1}^{2m-1} W_k^{(n+1)}.$$  

(23)
It is to be noted here that this conserved quantity is equivalent to the eigenvalue of $A^{(n)}$. Equation (23) imply that $\lambda^{(n)}$ are constants independent of discrete time $n$. Namely, eigenvalue of $A^{(n)}$ is invariant under the discrete time evolution of the udLV system (16). By combining the discussion, we have the following theorem.

**Theorem 4.2.** For any discrete time $n$, an eigenvalue $\lambda = \lambda^{(0)}$ of the tridiagonal min-plus matrices $A = A^{(0)} \in \mathbb{R}^{m \times m}_{\text{min}}$ associated with the udLV system (16) is expressed as:

$$\lambda = \bigoplus_{k=1}^{2m-1} W_k^{(n)}.$$  

Theorem 4.2 also suggests that the udLV system (16) gives the similarity transformations of $A^{(n)}$ over the min-plus algebra.

The entries in the given initial bidiagonal matrix are defined to be $B_k := \frac{1}{2} W_k^{(n)}$. Then we apply the ultradiscretization to (21), we obtain the initial values of the udLV system (16) as follows.

$$U_k^{(0)} = 2B_k \ominus (0 \oplus U_{k-1}^{(0)}), \quad k = 1, 2, \ldots, 2m - 1.$$  

For the udToda equation (10) asymptotic behavior at sufficiently large discrete time is clarified as follows.

**Proposition 4.3 ([15]).** In the udToda equation (10), there exists a discrete time $N$ such that, for any $n \geq N$:

$$Q_1^{(n)} \leq E_k^{(n)}, \quad k = 1, 2, \ldots, m - 1,$$

$$Q_1^{(n)} \leq Q_2^{(n)} \leq \cdots \leq Q_m^{(n)}.$$  

With the help of the Bäcklund transformation (11), (13) and (15) the asymptotic behavior of the udLV system (16) can also be clarified. For Proposition 4.3, applying ultradiscretization, we have the following theorem.

**Theorem 4.4.** In the udLV system (16), there exists a discrete time $N$ such that, for any $n \geq N$:

$$W_1^{(n)} \leq W_{2k}^{(n)}, \quad k = 1, 2, \ldots, m - 1,$$

$$W_1^{(n)} \leq W_{3}^{(n)} \leq \cdots \leq W_{2m-1}^{(n)}.$$  

Theorem 4.4 implies that $W_1^{(n)}$ with sufficiently large $n$ is equal to the minimum value of all the variables $W_1^{(n)}, W_2^{(n)}, \ldots, W_{2m-1}^{(n)}$. From this fact and Theorem 4.2, we have the main theorem of this paper as follows.

**Theorem 4.5.** For sufficiently large $n$, an eigenvalue $\lambda$ of $A = A^{(0)}$ associated with the udLV system (16) is given by

$$\lambda = W_1^{(n)}.$$  

From Theorems 4.2 and 4.5, it can be concluded that, under the initial settings that $W_1^{(0)}, W_2^{(0)}, \ldots, W_{2m-1}^{(0)}$ are given from the min-plus tridiagonal matrix $A = A^{(0)}$, the udLV system (16) generates min-plus similarity transformations of $A^{(n)}$, then the value of $W_1^{(n)}$ coincide with the eigenvalue at a sufficiently large discrete time. Moreover the value of $W_1^{(n)}$ at the discrete time $n$ gives the minimum value of the average weights of all circuits in $G(A^{(0)})$. 

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5. Conclusion
In this paper, we first derive the Bäcklund transformation between the ultradiscrete Toda (udToda) equation and the ultradiscrete Lotka-Volterra (udLV) system based on the ultradiscretization technique. We determine all the roots of the characteristic polynomial of the min-plus tridiagonal matrix associated with the udLV system. The roots coincide with the weights of the loops of the corresponding weighted digraph. Moreover, through the Bäcklund transformation, the conserved quantities and the asymptotic behavior of the udLV system are clarified and then the udLV system is shown to be used to compute an eigenvalue of a symmetric tridiagonal matrix over min-plus algebra.

It is known that eigenvector over min-plus algebra is related to the shortest path problem of the corresponding weighted digraph [1, 2]. Computing eigenvector of the min-plus matrices associated with the udLV system and the udToda equation, and analyzing the structure of corresponding weighted digraphs are left for future works.

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