ON WAHL’S PROOF OF $\mu(6) = 65$

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INTRODUCTION

In this note we present a short proof of the following theorem of D. Jaffe and D. Ruberman:

Theorem [Ja-Ru]. A sextic hypersurface in $\mathbb{P}^3$ has at most 65 nodes.

The bound is sharp by Barth’s construction [Ba] of a sextic with 65 nodes.

Following Beauville [Be], to a set of $n$ nodes on a surface is associated a linear subspace of $\mathbb{F}^n$ (where $\mathbb{F}$ is the field with two elements) whose elements correspond to the so-called even subsets of the set of the nodes. Studying this code Beauville proved that the maximal number of nodes of a quintic surface is 31.

The same idea was used by Jaffe and Ruberman, but their proof is not so short as the one of Beauville, partly because at that time a complete understanding of the possible cardinalities of an even set of nodes was missing.

Almost at the same time, J. Wahl [Wa] proposed a much shorter proof of the same result. He proved indeed the following (see the beginning of the next section for the missing definitions)

Theorem [Wa]. Let $V \subset \mathbb{F}^{66}$ be a code, with weights in $\{24, 32, 40\}$. Then $\dim(V) \leq 12$.

He claimed that Jaffe-Ruberman’s theorem follows as a corollary since the code associated to a nodal sextic has dimension at least $n - 53$ (see section 1 of [Ca-To] for this computation). In fact, he used an incorrect result stated by Casnati and Catanese in [Ca-Ca], asserting that the possible cardinalities of an even set of nodes on a sextic are only 24, 32 and 40. Recently Catanese and Tonoli showed indeed

Theorem [Ca-To]. On a sextic nodal surface in $\mathbb{P}^3$, an even set of nodes has cardinality in $\{24, 32, 40, 56\}$.

Note however that [Ca-To] used a result by Jaffe and Ruberman, namely that there is no even set of nodes of cardinality 48.

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By the above theorem the proof of the theorem of Jaffe and Ruberman reduces to the following

**Theorem A.** Let \( V \subset \mathbb{F}^66 \) be a code with weights in \( \{24, 32, 40, 56\} \). Then \( \dim(V) \leq 12 \).

This statement is in fact theorem 8.1 of [Ja-Ru]. Anyway, its proof is much more complicated than Wahl’s one and moreover requires computers computations. In this short note we give an elementary proof, using and integrating Wahl’s ideas.

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### 1. Notation and general results from coding theory

A **code** is (in this note) a vector subspace \( V \subset \mathbb{F}^n \), where \( \mathbb{F} \) is the field with two elements. A **word** is a vector \( v = (v_1, \ldots, v_n) \in \mathbb{F}^n \). Its **support** \( \text{Supp}(v) \) is the set \( \{i \mid v_i \neq 0\} \) of coordinates that do not vanish in \( v \), its **weight** \( |v| \) is the cardinality of its support. The **length** of a code is the cardinality of the union of the supports of all its elements. A code \( V \subset \mathbb{F}^n \) is said to be **spanning** if it has length \( n \).

A code is **even** if all its words have even weight, **doubly even** if all its weights are divisible by 4. The number of words of weight \( i \) in the code \( V \) is denoted by \( a_i(V) \) or simply \( a_i \) when no confusion arises. The **weight enumerator** of the code \( V \) is the homogeneous polynomial

\[
W_V(x, y) = \sum a_i x^{n-i} y^i.
\]

The standard scalar product in \( \mathbb{F}^n \) associates to each code its dual code, i.e., its annihilator \( V^* \subset \mathbb{F}^n \), which has complementary dimension. We set \( a_i^* := a_i(V^*) \).

**Remark 1.1.** 1) \( V \subset \mathbb{F}^n \) is spanning if and only if \( a_1^* = 0 \).

2) If \( v^* \in V^* \) has weight 2, the subset of \( V \) given by all words \( v \) with \( \text{Supp}(v) \cap \text{Supp}(v^*) = \emptyset \) is a subcode of codimension at most 1 (and length at most \( n-2 \)).

3) A doubly even code is automatically isotropic, i.e., \( V \subset V^* \).

The **MacWilliams identity** (cf. [McW-Sl]) states that the weight enumerator \( W_{V^*}(x, y) \) of the dual code \( V^* \) equals \( W_V(x+y, x-y)/2^d \), i.e.,

\[
\sum a_i^* x^{n-i} y^i = \frac{1}{2^d} \left( \sum a_i(x+y)^{n-i}(x-y)^i \right).
\]
As explained in [Wa], comparing the coefficients of $x^{n-i}y^i$ for $i \leq 3$ in both sides of \([1.1]\) gives (since $a_0 = a_0^* = 1$):

**Lemma 1.2.** [Wa, Lemma 2.4] Let $V \subset \mathbb{F}^n$ be a spanning code of dimension $d$. Then:

\[
\begin{align*}
\sum_{i>0} a_i &= 2^d - 1 \\
\sum_1 i a_i &= 2^{d-1} n \\
\sum_1 i^2 a_i &= 2^{d-1} (a_2^* + n(n + 1)/2) \\
\sum_1 i^3 a_i &= 2^{d-2} \left(3(a_2^* n - a_3^*) + n^2(n + 3)/2\right)
\end{align*}
\]

The following proposition gives dimension and weights of a projected linear code.

**Proposition 1.3.** [Wa, Prop. 2.8] Let $V \subset \mathbb{F}^n$ be a code of dimension $d$. Fix a word $w \in V$ and consider the projection $\pi: \mathbb{F}^n \rightarrow \mathbb{F}^n - \mid w \mid$ onto the complement of the support of $w$. Then

1. If $w$ is not a sum of two disjoint words in $V$, then $V' := \pi(V)$ is a code of dimension $d' = d - 1$.
2. $\mid \pi(v) \mid = \frac{1}{2} (\mid v \mid + \mid v + w \mid - \mid w \mid)$.

**Proof.** If $\ker \pi|_V$ contains, besides $w$, another word $v$, one can write a disjoint sum $w = v + (w - v)$. Thus, in the hypothesis of (1), $\dim \ker \pi|_V = 1$ and therefore $d' = d - 1$.

For (2), let $r$ be the cardinality of the intersection of the two supports of $v$ and $w$. Then $\mid v \mid = r + \mid \pi(v) \mid$ and $\mid v \mid + \mid w \mid = \mid v + w \mid + 2r$.

2. The proof

**Lemma 2.1.** [Wa, Lemma 2.6] The dimension of a code with weights in $\{24, 32\}$ is at most 9.

**Proof.** Let $n$ be the length of the code and $d$ its dimension. Solving the linear system given by \((1.2a)\) and \((1.2b)\), $a_{24} = 2^{d-4}(64 - n) - 4$, $a_{32} = 2^{d-4}(n - 48) + 3$. Substituting in \((1.2c)\)

$2^8 \left(2^{d-6} \cdot 9 \cdot (2^6 - n) + 2^{d-2} \cdot (n - 48) + 3\right) = 2^{d-1}(a_2^* + n(n + 1)/2)$

If $d > 9$, then $2^{d-1}$ divides the R.H.S. but not the L.H.S., a contradiction.

**Remark 2.2.** A code $V \subset \mathbb{F}^{67}$ with weights $\geq 24$ has necessarily $a_{56} \leq 1$. 
Proof. Indeed, if there are two different words of weight 56, their sum has weight at least 24 and then the cardinality of the intersection of their supports is at least \( \frac{1}{2}(56 + 56 - 24) = 44. \) Therefore their span has length \( \geq 44 + 2 \cdot (56 - 44) = 68. \) \( \square \)

Lemma 2.3. The dimension of a code \( V \subset F^{67} \) with weights in \( \{24, 32, 56\} \) is at most 10.

Proof. If \( a_{56} = 0 \) the result follows by Lemma 2.1.

Otherwise, by Remark 2.2, \( a_{56} = 1. \) The intersection of \( V \) with any hyperplane not containing its unique word of weight 56 is a code \( V' \) of dimension \( \dim(V) - 1 \) with weights in \( \{24, 32\} \) and the result follows again by Lemma 2.1. \( \square \)

Proof of Theorem A. Suppose that there exists a code \( V \subset F^{66} \) with weights in \( \{24, 32, 40, 56\} \) of dimension 13. Let \( n \) be its length and consider \( V \) as a spanning code in \( F^n. \)

By Lemma 2.3 we have \( a_{40} > 0. \) For each word \( w \in V \) of weight 40 we consider the projection \( \pi_w \) onto the complement of the support of \( w. \) By Proposition 1.3 \( V' := \pi_w(V) \subset F^{n-40} \) is a doubly even code of dimension 12. So \( V' \) is an isotropic subspace, \( n - 40 \geq 24 \) and we obtain \( n \geq 64: \) more precisely \( n \in \{64, 65, 66\}. \)

Suppose \( n = 64. \) For each word \( w \in V \) of weight 40, \( \pi_w(V) \) is isotropic of dimension 12 in \( F^{24}, \) so \( \pi_w(V) = (\pi_w(V))^*. \) Let \( \mathbb{I} \in F^{24} \) be the vector with all coordinates 1: \( \mathbb{I} \in (\pi_w(V))^* \) (since \( \pi_w(V) \) is even) and therefore \( \mathbb{I} \in \pi_w(V). \)

If \( v \in V \) is a word such that both the weights \( |v|, |v+w| \) are \( \leq 40, \) then by Proposition 1.3 \( |\pi_w(v)| \leq 20: \) therefore by remark 2.2 \( a_{56}(V) = 1 \) and \( \mathbb{I} = \pi_w(\mathbb{T}) \) for the unique word \( \mathbb{T} \in V \) with \( |\mathbb{T}| = 56. \)

Fix one coordinate not in the support of \( \mathbb{T} \) and let \( V'' \subset V \) be the subcode defined by the vanishing of the given coordinate. Since \( \mathbb{I} = \pi_w(\mathbb{T}), \) the support of \( w \) contains the complementary of the support of \( \mathbb{T}; \) then \( w \notin V''. \) Since this holds for each \( w \in V \) with \( |w| = 40, \) then \( V'' \) has no word of weight 40: it is a code of dimension 12 with weights in \( \{24, 32, 56\}, \) contradicting lemma 2.3.

Suppose \( n = 65. \) Solving the equations (1.2a)-(1.2d), we obtain \( a_{56} = \frac{1}{2}(a_2^* - a_3^* - 5) \) and thus \( a_2^* > 0. \) Let then \( z \in V^* \) be a word of length 2.

For each word \( w \in V \) of weight 40, \( a_2^*(\pi_w(V)) = 0: \) in fact, for any word \( z' \in (\pi_w(V))^* \) of weight 2, \( \operatorname{Span}(V', z') \) is an isotropic subspace of dimension 13 in \( F^{25}, \) absurd. Therefore every word \( w \) of weight 40 satisfies \( \operatorname{Supp}(w) \supset \operatorname{Supp}(z). \)
By remark 1.1 the subset of $V$ given by all words $v$ with $\text{Supp}(v) \cap \text{Supp}(z) = \emptyset$ is a subcode of dimension at least 12 with weights in \{24, 32, 56\}, contradicting Lemma 2.3.

Then $n = 66$. Solving the equations (1.2a)-(1.2d), we obtain $a_{56} = a_2^* - \frac{1}{2}(a_3^* + 13)$ and thus $a_2^* \geq 7$. We choose two words $z_1 \neq z_2$ in $V^*$ of weight 2.

If we show that for each word $w \in V$ of weight 40, $a_2^*(\pi_w(V)) \leq 1$, then $\text{Supp}(w)$ intersects $Z = \text{Supp}(z_1) \cup \text{Supp}(z_2)$. Therefore, by remark 1.1 the subset of $V$ given by all words $v$ with $\text{Supp}(v) \cap Z = \emptyset$ is a code of dimension at least 11 and weights among \{24, 32, 56\}, contradicting again Lemma 2.3.

So it remains to show only that for each word $w \in V$ of weight 40, $a_2^*(\pi_w(V)) \leq 1$.

If $z' \in (\pi_w(V))^*$ is a word of weight 2, then $V'' := \text{Span}(\pi_w(V), z') \subset \mathbb{P}^{26}$ is an isotropic subspace of dimension 13, and thus $\mathbb{I} \subset V'' = (V'')^*$. Being $\pi_w(V)$ doubly even, $\mathbb{I}, z' \in V'' \setminus \pi_w(V)$, and therefore $\mathbb{I} + z'$ is a word in $\pi_w(V)$ of weight 24. Thus $a_2^*(\pi_w(V)) \leq a_{24}(\pi_w(V))$.

If $v \in V$ is a word such that both the weights $|v|, |v+w|$ are $\leq 40$, then by Proposition 1.3 $|\pi_w(v)| \leq 20$; therefore $a_{24}(\pi_w(V)) \leq a_{56}(V) \leq 1$ (the last inequality by remark 2.2). 

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