RIEMANN-HILBERT TYPE PROBLEMS WITH SINGULARITIES

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Abstract. This paper is devoted to Riemann-Hilbert problems with singularities. As an application, we study the existence and properties of analytic discs which arise as solutions of such nonlinear problems. We then use such families of analytic discs in order to study jet determination for CR diffeomorphisms between pseudoconvex hypersurfaces of finite type.

Introduction

In the endeavor of studying the geometry of smooth real submanifolds $M \subset \mathbb{C}^n$, and in particular the behavior of holomorphic maps between them, it is important to construct suitable invariants: one of such invariants is given by the family of analytic discs attached to $M$. When $M$ is a totally real submanifold of $\mathbb{C}^n$, the study of (deformations of) analytic discs attached to it leads to the consideration of linear Riemann-Hilbert problems of a special form. More specifically, one needs to examine operators of the kind

$$f \rightarrow Gf + G\overline{f}$$

where $f = (f_1, \ldots, f_n)$ is a vector of holomorphic functions defined on the unit disc $\Delta \subset \mathbb{C}$ and continuous up to $\overline{\Delta}$, while $G : b\Delta \rightarrow GL(n, \mathbb{C})$ is a continuous, matrix valued map defined on the unit circle whose values are nowhere singular. Inspired by the work of F. Forstneriˇc [8], J. Globevnik [9, 10] characterized the existence and dimension of a family of deformations of a given analytic disc attached to $M$ in terms of the surjectivity and the dimension of the kernel of the Riemann-Hilbert operator, which in turn depend on the properties of certain indices associated to the matrix $G$ (see also for instance the works of Y.-G. Oh [17] and M. Černe [3]).

If, however, the map $G$ is not valued in $GL(n, \mathbb{C})$ everywhere and instead admits singularities, the indices mentioned above are no longer well-defined and the procedure falls apart: indeed, in such a case the operator always fails to be surjective. This situation does actually arise: turning back to the geometric setting, it corresponds

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to the case when $M$ is not everywhere totally real but instead admits some complex tangencies.

One of the aims of this paper is to study a class of these singular Riemann-Hilbert problems. In the cases that interest us, $G$ admits a singularity at the point $\zeta = 1$ which is of finite type in the sense that the function $\det G(\zeta)$ vanishes at 1 to finite order. Our approach is, roughly speaking, to “factorize the non-singular part” $\tilde{G}$ of $G$ and to study a modified Riemann-Hilbert operator – with $\tilde{G}$ in place of $G$ – which is, however, defined between different spaces. We then show that this sort of operator can be analyzed (in particular, in relation to its surjectivity and its kernel) by a suitable modification of the approach introduced in [9, 10]. The main results in this regard are collected in Theorem 2.2 and Theorem 2.4.

Our second aim is to show how these analytic results can be applied to geometric problems, and specifically to the construction of stationary discs introduced by L. Lempert [16] (see also [12, 19]). These are special analytic discs, attached to hypersurfaces $M$ of $\mathbb{C}^n$, which admit a lift (possibly with a pole of order at most 1 at 0) to the conormal bundle of $M$. The conormal bundle can be seen as a real $2n$-dimensional submanifold of $\mathbb{C}^{2n}$ and, as it turns out, it is totally real if $M$ is Levi non-degenerate while it admits complex tangencies otherwise [21]. Consequently, if $M$ is Levi non-degenerate the study of stationary discs falls into the framework developed in [9, 10], and indeed the first author and L. Blanc-Centi employed this method to construct stationary discs in [1].

We will be instead concerned with a class of hypersurfaces which are of finite type and whose Levi form vanishes at an isolated point: we will show that the resulting Riemann-Hilbert problem can be studied with the tools of Theorem 2.2 and prove that a family of $k_0$-stationary discs (in a slightly generalized sense) always exists for the hypersurfaces of this class for $n = 2, 3$ (see Theorem 3.4 and Theorem 3.5). In particular, for $n = 2$ we recover the results in [2] in a different way. We remark, though, that while the results in [2] are achieved by using a rather “ad hoc” procedure, the method we are using in the present paper appears to be more systematic and more suited to generalization – in fact we can extend it to dimension 3. The only obstacle to generalize the result to any dimension is of a rather technical nature, that is, the estimation of the partial indices of the Riemann-Hilbert operator we obtain.

Finally, in a way similar to [1], [2], we use the invariant provided by the stationary discs to prove finite jet determination results for CR diffeomorphisms defined between finite type hypersurfaces of our class. In order to do this, the procedure is similar to the one in [1] and [2] and thus we give a full proof only where important changes are needed. It is worth remarking that while the finite jet determination problem for CR diffeomorphisms has been heavily studied and has given rise to an extensive
literature for the case of real-analytic hypersurfaces (see for instance [4, 7, 15, 14], and [5, 6, 13] for $C^\infty$ hypersurfaces), the finitely smooth case is quite different and seems to require new methods, especially when the CR diffeomorphism considered do not extend to a holomorphic map defined in a full neighborhood of $M$ in $\mathbb{C}^n$.

The paper is organized as follows. The first two sections are devoted to linear Riemann-Hilbert problems with pointwise constraints. Section 1 contains results dealing with one- and two-dimensional linear Riemann-Hilbert problems with constraints, which are then used in Section 2 to deal with such problems in arbitrary dimension. In Section 3, we study the existence of analytic discs which arise as solutions of non linear Riemann-Hilbert type problems with singularities. Finally, Section 4 is devoted to finite jet determination problems for CR diffeomorphisms.

1. Preliminaries

We denote by $\Delta$ the unit disc in $\mathbb{C}$ and by $b\Delta$ its boundary.

1.1. Function spaces. Let $k$ be an integer and let $0 < \alpha < 1$. We denote by $C^{k,\alpha} = C^{k,\alpha}(b\Delta, \mathbb{R})$ the space of real-valued functions defined on $b\Delta$ of class $C^{k,\alpha}$. The space $C^{k,\alpha}$ is endowed with its usual norm

\[ \|v\|_{C^{k,\alpha}} = \sum_{j=0}^{k} \|v^{(j)}\|_{\infty} + \sup_{\zeta \neq \eta \in b\Delta} \frac{\|v^{(k)}(\zeta) - v^{(k)}(\eta)\|}{|\zeta - \eta|^\alpha}, \]

where $\|v^{(j)}\|_{\infty} = \max_{b\Delta} |v^{(j)}|$.

We define $C^{k,\alpha}_e$ (resp. $C^{k,\alpha}_o$) to be the closed subspace of $C^{k,\alpha}$ given by the even (resp. odd) functions, i.e. the functions $v \in C^{k,\alpha}$ such that $v(-\zeta) = v(\zeta)$ (resp. $v(-\zeta) = -v(\zeta)$) for all $\zeta \in b\Delta$.

We set $C^{k,\alpha}_c = C^{k,\alpha} + iC^{k,\alpha}$. Hence $v \in C^{k,\alpha}_c$ if and only if $\text{Re} \, v, \text{Im} \, v \in C^{k,\alpha}$. The space $C^{k,\alpha}_c$ is equipped with the norm

\[ \|v\|_{C^{k,\alpha}_c} = \|\text{Re} \, v\|_{C^{k,\alpha}} + \|\text{Im} \, v\|_{C^{k,\alpha}}. \]

We denote by $\mathcal{A}^{k,\alpha}$ the subspace of $C^{k,\alpha}_c$ consisting of functions $f : \overline{\Delta} \to \mathbb{C}$, holomorphic on $\Delta$ with trace on $b\Delta$ belonging to $C^{k,\alpha}_c$.

Let $m$ be an integer. We define $(1-\zeta)^m \mathcal{A}^{k,\alpha}$ to be the subspace of $C^{k,\alpha}_c$ consisting of the functions that can be written as $(1-\zeta)^m f$, with $f \in \mathcal{A}^{k,\alpha}$. Note that $(1-\zeta)^m \mathcal{A}^{k,\alpha}$ is not a closed subspace of $C^{k,\alpha}_c$, thus it is not a Banach space with the induced norm. Instead, we equip $(1-\zeta)^m \mathcal{A}^{k,\alpha}$ with the following norm

\[ \|(1-\zeta)^m f\|_{(1-\zeta)^m \mathcal{A}^{k,\alpha}} = \|f\|_{C^{k,\alpha}_c}. \]
which makes it a Banach space, isomorphic to $A^{k,\alpha}$. Notice that the inclusion of $(1 - \zeta)^m A^{k,\alpha}$ into $A^{k,\alpha}$ is a bounded linear operator.

Finally, we denote by $C^{k,\alpha}_{0m}$ the subspace of $C^{k,\alpha}$ consisting of elements that can be written as $(1 - \zeta)^m v$ with $v \in C^{k,\alpha}_C$. We equip $C^{k,\alpha}_{0m}$ with the norm $\| (1 - \zeta)^m f \|_{C^{k,\alpha}_{0m}} = \| f \|_{C^{k,\alpha}_C}$.

Notice that $C^{k,\alpha}_{0m}$ is a Banach space. Denote by $\tau_m$ the map $C^{k,\alpha}_{0m} \to C^{k,\alpha}_C$ given by $\tau_m((1 - \zeta)^m v) = v$.

**Lemma 1.1.** Define the closed subspace $R_m$ of $C^{k,\alpha}_C$ by

$$R_m = \{ v \in C^{k,\alpha}_C | v(\zeta) = (-1)^m \zeta^{-m} v(\zeta) \quad \forall \zeta \in b\Delta \}.$$ 

Then

(i) $\tau_m$ maps $C^{k,\alpha}_{0m}$ isomorphically to $R_m$;
(ii) if $m = 2m'$ is even, the map $v \mapsto \zeta^{m'} v$ induces an isomorphism between $R_m$ and $R_0 = C^{k,\alpha};$
(iii) if $m = 2m' + 1$ is odd, the map $v \mapsto \zeta^{m'} v$ induces an isomorphism between $R_m$ and $R_1$.

Furthermore, if $m$ is odd the map $v(\zeta) \mapsto i\zeta^m v(\zeta^2)$ sends $R_m$ isomorphically to $C^{k,\alpha}_o$.

**Proof.** A function $v \in C^{k,\alpha}_C$ is in the image of $\tau_m$ exactly when $(1 - \zeta)^m v \in C^{k,\alpha}$, i.e. $(1 - \zeta)^m v = (1 - \zeta)^m \overline{v} = (-1)^m \zeta^{-m}(1 - \zeta)^m \overline{v}$, which gives the first point (note that $\tau_m$ is an isometry by definition of the norms).

If $m = 2m'$ and $v \in R_m$, $u = \zeta^{m'} v$, we have

$$u = \zeta^{m'} v = \zeta^{m'} \zeta^{-2m'} \overline{v} = \zeta^{-m'} \overline{v} = \overline{u},$$

hence $u \in C^{k,\alpha}$.

If $m = 2m' + 1$ and $v \in R_m$, $u = \zeta^{m'} v$, we have

$$u = \zeta^{m'} v = -\zeta^{m'} \zeta^{-2m'+1} \overline{v} = -\overline{\zeta} \zeta^{-m'} v = \overline{u},$$

hence $u \in R_1$.

Finally, letting $u(\zeta) = i\zeta^m v(\zeta^2)$ with $v \in R_m$ and $m$ odd we have

$$u(\zeta) = i\zeta^m v(\zeta^2) = -i\zeta^m \zeta^{-2m} v(\zeta^2) = -i\zeta^{-m} v(\zeta) = \overline{u(\zeta)};$$

and furthermore $u(-\zeta) = (-1)^m u(\zeta) = -u(\zeta)$, hence $u \in C^{k,\alpha}_o$. \qed
1.2. One- and two-dimensional Riemann-Hilbert problems with constraints.

Consider the following classical linear Riemann-Hilbert problem which consists, for a given integer \( r \), in finding \( f \in \mathcal{A}^{k,\alpha} \) such that

\[
\text{Re} [\zeta^{-r} f(\zeta)] = 0 \quad \text{on} \quad \partial \Delta.
\]

(1.2)  

Equation (1.2) leads to

\[
f(\zeta) + \zeta^{2r} f(\zeta) = 0 \quad \text{on} \quad \partial \Delta
\]

We consider a more general situation:

Lemma 1.2. Let \( l \) be an integer. Consider the operator \( L^\pm : \mathcal{A}^{k,\alpha} \rightarrow \mathcal{C}^{k,\alpha} \) defined by

\[
L^\pm (f) := (f \pm \zeta^l f)|_{\partial \Delta}.
\]

Then the kernel of \( L^\pm \) has real dimension equal to \( \max\{l + 1, 0\} \).

Indeed it is straightforward to check that if \( f \in \mathcal{A}^{k,\alpha} \) satisfies

\[
f(\zeta) = \sum_{k=0}^l a_k \zeta^k
\]

where \( a_k = \pm a_{l-k} \) for \( k = 0, \ldots, l \).

Now consider, for \( m \geq 0 \), the restriction of \( L^\pm \) to the subspace \((1 - \zeta)^m \mathcal{A}^{k,\alpha}\).

Equation (1.3) can be written as

\[
(1 - \zeta)^m f'(\zeta) \pm (-1)^m (1 - \zeta)^m \zeta^{l-m} f'(\zeta) = 0
\]

and thus

\[
f'(\zeta) \pm \zeta^{l-m} f'(\zeta) = 0
\]

In view of Lemma 1.2 we obtain:

Lemma 1.3. Let \( l, m \) be integers. Consider the operator \( L_1^\pm : (1 - \zeta)^m \mathcal{A}^{k,\alpha} \rightarrow \mathcal{C}^{k,\alpha} \) defined by

\[
L_1^\pm (f) := (f \pm \zeta^l f)|_{\partial \Delta}.
\]

Then the kernel Ker \( L_1^\pm \) has real dimension equal to \( \max\{l + 1 - m, 0\} \).

Now, we study the surjectivity of the operator introduced in Equation (1.2).

Lemma 1.4. Let \( r \) be an integer. Consider the operator \( L : (1 - \zeta)^m \mathcal{A}^{k,\alpha} \rightarrow \mathcal{C}_{0^m}^{k,\alpha} \) defined by

\[
L(f) := \text{Re} [\zeta^{-r} f]|_{\partial \Delta}.
\]

Then \( L \) is onto if and only if \( 2r - m \geq -1 \).
Proof. Let $\varphi \in C_{0}^{k,\alpha}$. Write $\varphi = (1 - \zeta)^{m}v$ where $v \in \mathcal{R}_{m}$ (see Lemma 1.1). We need to study the following equation:

$$\zeta^{-r}f - \zeta^{r}f = \varphi$$

for $f \in (1 - \zeta)^{m}A^{k,\alpha}$. Writing $f = (1 - \zeta)^{m}f'$ with $f' \in A^{k,\alpha}$ reduces to

$$\zeta^{-r}f' + (-1)^{m}\zeta^{r-m}f' = v$$

We distinguish two cases:

First case: $m = 2m'$ is even. In such case, Equation (1.4) is equivalent to

$$\zeta^{-(r-m')}f' + \zeta^{r-m'}f' = \zeta^{m'}v.$$  

Notice that $u \rightarrow \zeta^{m'}u$ maps $\mathcal{R}_{m}$ isomorphically to $C^{k,\alpha}$, see Lemma 1.1. Equation (1.5) is relatively classical and was treated by J. Globevnik in [10] for instance. There exists such an $h'$ if and only if $r - m' \geq 0$.

Second case: $m = 2m' + 1$ is odd. In this case, Equation (1.4) is equivalent to

$$\zeta^{-(r-m')}f' - \zeta^{r-m'}f' = \zeta^{m'}v.$$  

Set $u = \zeta^{m'}v$. By Lemma 1.1 follows that $u \in \mathcal{R}_{1}$. We write $u = u' + u''$, where $u' = P(u) \in A^{k,\alpha}$, $P$ being the Szegö projection. Since $u = -\zeta u$, we have $u'' = -\zeta u'$. If $r - m' \geq 0$, then $f' = \zeta^{r-m'}u' \in A^{k,\alpha}$ and satisfies (1.6). If $r - m' < 0$, then $\int \zeta^{-(r-m')}f\ d\theta = \int \zeta^{r-m'-1}f\ d\theta = 0$ and so, for instance, $1 - \zeta \in \mathcal{R}_{1}$ is not in the image.

Finally, we will also need the following:

**Lemma 1.5.** Let $r_{1}, r_{2}$ be integers. Set

$$P(\zeta) = \begin{pmatrix} 1 + \zeta & -i(1 - \zeta) \\ i(1 - \zeta) & 1 + \zeta \end{pmatrix} \begin{pmatrix} \zeta^{-r_{1}} & 0 \\ 0 & \zeta^{-r_{2}} \end{pmatrix}.$$  

Consider the operator $T : ( (1 - \zeta)^{m}A^{k,\alpha} )^{2} \rightarrow ( C_{0}^{k,\alpha} )^{2}$ defined by

$$T(f) := \text{Re}[Pf]|_{\Delta}.$$  

Then $T$ is onto if and only if $2r_{1} - m \geq 0$ and $2r_{2} - m \geq 0$.

**Proof.** Let $\varphi \in \left(C_{0}^{k,\alpha}\right)^{2}$. Write $\varphi = (1 - \zeta)^{m}v$ where $v \in \left(C_{0}^{k,\alpha}\right)^{2}$ and $f = (1 - \zeta)^{m}f'$ with $f' \in \left(A^{k,\alpha}\right)^{2}$. We need to study the following equation:

$$Pf' + (-1)^{m}\zeta^{-m}Pf' = v.$$  

First case: $m = 2m'$ is even. In that case, we have

$$\zeta^{m'}Pf' + \zeta^{-m}Pf' = \zeta^{m'}v,$$
which was treated by J. Globevnik \[10\]. In particular, (1.7) admits a solution if and
only if \( r_1 - m' \geq 0 \) and \( r_2 - m' \geq 0 \).

**Second case:** \( m = 2m' + 1 \) is odd. We have

\[
\zeta^{m'} P f' - \zeta^{-m'-1} f = \zeta^{m'} v.
\]

Following J. Globevnik, we make the substitution

\[
\zeta = \xi^2
\]

and get

\[
\xi^m P(\xi^2) f'(\xi^2) - \xi^{-m} \xi P(\xi^2) f'(\xi^2) = \xi^m v(\xi^2),
\]

After multiplying by \( i \),

\[
\text{Re} \left[ i \xi^m P(\xi^2) i f'(\xi^2) \right] = i \xi^m v(\xi^2).
\]

which becomes

\[(1.8) \quad \text{Re} \left[ \left( i \xi^{-(2r_1-m-1)} f_1(\xi^2) \right) \right] = 2i \left( \begin{array}{cc} \text{Re} \xi & \text{Im} \xi \\ -\text{Im} \xi & \text{Re} \xi \end{array} \right) \xi^m v(\xi^2). \]

Notice that, according to Lemma [1.4], \( 2 i \xi^m u(\xi^2) \in \mathcal{C}_o^{k,\alpha} \) and that moreover the map

\[
u \mapsto \left( \begin{array}{cc} \text{Re} \xi & \text{Im} \xi \\ -\text{Im} \xi & \text{Re} \xi \end{array} \right) \nu \]

is an isomorphism between \( (\mathcal{C}_o^{k,\alpha})^2 \) and \( (\mathcal{C}_e^{k,\alpha})^2 \). Thus, (1.8) reduces to a pair of one-dimensional problems

\[
\xi^{-(2r_j-m-1)} f_j'(\xi^2) + \xi^{2r_j-m-1} f_j(\xi^2) = u_j(\xi)
\]

with \( u_j \in \mathcal{C}_e^{k,\alpha}, j = 1, 2 \). Writing \( u_j(\xi) = u_j'(\xi^2) \) with \( u_j' \in \mathcal{C}^{k,\alpha} \), this equation is in



## 2. Linear Riemann-Hilbert Problems with Pointwise Constraints

### 2.1. Birkhoff factorization and indices.

Let \( G : b\Delta \to GL_N(\mathbb{C}) \) be a smooth map. One considers a Birkhoff factorization of \( -G(\zeta)^{-1} G(\zeta) \), i.e. some smooth maps \( B^+ : \Delta \to GL_N(\mathbb{C}) \) and \( B^- : (\mathbb{C} \cup \infty) \setminus \Delta \to GL_N(\mathbb{C}) \) such that

\[
\forall \zeta \in b\Delta, \quad -G(\zeta)^{-1} G(\zeta) = B^+(\zeta) \begin{pmatrix} \zeta^{\kappa_1} & & (0) \\ & \ddots & \\ (0) & & \zeta^{\kappa_N} \end{pmatrix} B^-(\zeta)
\]

where \( B^+ \) and \( B^- \) are holomorphic on \( \Delta \) and \( \mathbb{C} \setminus \Delta \) respectively. Note that according to N.P. Vekua [20], one can find \( B^+ \) and \( B^- \) in such a way that \( B^+ = \Theta \) and \( B^- = \Theta^{-1} \), where \( \Theta : \Delta \to GL_N(\mathbb{C}) \) is a smooth map. The integers \( \kappa_1, \ldots, \kappa_N \) are
called the partial indices of $-G^{-1}G$ and the Maslov index of $-G^{-1}G$ is their sum
\[ \kappa := \sum_{j=1}^{N} \kappa_j. \]

**Remark 2.1.** We recall that $G$ being smooth ($C^1$ being enough for the present state-
ment) the Maslov index coincides with the winding number at the origin of the map
\[ \text{det} \left( -G^{-1}G \right). \]

\[ \text{ind det} \left( -G^{-1}G \right) = \frac{1}{2\pi i} \int_{b\Delta} \left[ \frac{\det \left( -G^{-1}G \right)}{\det \left( -G^{-1}G \right)} \right] d\zeta. \]

### 2.2. Linear Riemann-Hilbert problems with homogeneous pointwise constraints.

Let $k$ be an integer and let $0 < \alpha < 1$. Consider the following operator
\[ L : \left( (1 - \zeta) A^{k,\alpha} \right)^N \rightarrow \left( C^{k,\alpha}_{0,m_j} \right)^N \]
defined by
\[ L(f') = 2 \text{Re} \left[ Gf' \right], \]
where $G : b\Delta \rightarrow GL_N(\mathbb{C})$ is smooth. Denote by $\kappa_1, \ldots, \kappa_N$ and by $\kappa$ the partial
indices and the Maslov index of $-G^{-1}G$.

#### 2.2.1. Surjectivity and kernel of $L$.

**Theorem 2.2.** Under the above assumptions:

(i) The map $L$ is onto if and only if $\kappa_j \geq m - 1$ for all $j = 1, \ldots, N$.

(ii) Assume that $\kappa_j \geq m - 1$ for all $j = 1, \ldots, N$. Then the kernel of $L$ has real
dimension $\kappa + N - Nm$.

Before proving Theorem 2.2, we need a few more observations. Following [10],
one can find a smooth map $V : b\Delta \rightarrow GL_N(\mathbb{R})$ such that $V \overline{G} = M(-i\Theta)^{-1}$, where
$M$ has a special block-diagonal form and where $\Theta : \overline{\Delta} \rightarrow GL_N(\mathbb{C})$ is smooth and
holomorphic on $\Delta$. More precisely,

\[ -G(\zeta)^{-1}G(\zeta) = \Theta(\zeta) \begin{pmatrix} \zeta^{\kappa_1} & (0) \\ \zeta^{\kappa_2} & \ddots \\ (0) & \ddots & \zeta^{\kappa_N} \end{pmatrix} \Theta(\zeta)^{-1} \]

(2.1)

\[ = \Theta(\zeta)M(\zeta)^{-1}\overline{M(\zeta)}\Theta(\zeta)^{-1}. \]
Define the operator

\[ \tilde{L} : ((1 - \zeta)^m A^{k, \alpha})^N \to \left( C^{k, \alpha}_{0/m} \right)^N \]

by setting

\[ \tilde{L}(f') := 2\text{Re} \left[ M(i\Theta)^{-1} f' \right]. \]

Since \( \Theta : \Delta \to GL_N(\mathbb{C}) \) is smooth and holomorphic on \( \Delta \), the map \((i\Theta)^{-1}\) is a Banach space isomorphism of \( ((1 - \zeta)^m A^{k, \alpha})^N \) onto itself. Therefore the kernels of \( L \) and \( \tilde{L} \) are of the same dimension and \( L \) is onto if and only if \( \tilde{L} \) is onto. We will prove Theorem 2.2 for \( \tilde{L} \).

**Proof of Theorem 2.2** We first prove (i). Since \( \kappa \) is even, the number of odd partial indices is even. Without loss of generality, suppose that \( \kappa_j \) is odd for \( j = 1, \cdots, 2r \) and that \( \kappa_j \) is even for \( j = 2r + 1, \cdots, N \). According to [10], the matrix \( M \) can be written as

\[
M(\zeta) = \begin{pmatrix}
P_1(\zeta) & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & P_r(\zeta) \\
\end{pmatrix},
\]

where

\[
P_j(\zeta) = \begin{pmatrix}
1 + \zeta & -i(1 - \zeta) \\
i(1 - \zeta) & 1 + \zeta \\
\end{pmatrix}\begin{pmatrix}
\zeta^{-\kappa_j} \\
\zeta^{-\frac{\kappa_j}{2}} \\
\end{pmatrix},
\]

for \( j = 1, \cdots, r \). Lemmas 1.4 and 1.5 proves part (i).

We now prove (ii). Assume that

\[ 2\text{Re} \left[ M f' \right] = 0. \]

The disc \( f' \in ((1 - \zeta)^m A^{k, \alpha})^N \) satisfies

\[
f' = -M^{-1} M f' = -\begin{pmatrix}
\zeta^{\kappa_1} & \cdots & 0 \\
\zeta^{\kappa_2} & \ddots & \vdots \\
0 & \cdots & \zeta^{\kappa_N} \\
\end{pmatrix} f',
\]

which, by means of Lemma 1.3, proves part (ii). \( \square \)
2.2.2. Finite jet determination of the kernel of $L$. Let $\ell_0, m, N \in \mathbb{N}$. We want to consider the linear map $j_{\ell_0} : ((1 - \zeta)^m A^{k,\alpha})^N \rightarrow \mathbb{C}^{N(\ell_0 + 1)}$ sending $f$ to its $\ell_0$-jet at $\zeta = 1$, i.e. to the vector
\[
(f(1), \partial f(1), \ldots, \partial_{\ell_0} f(1)) \in \mathbb{C}^{N(\ell_0 + 1)}.
\]
where $\partial_{\ell_0} f(1) \in \mathbb{C}^N$ denotes the vector $\frac{\partial^m f}{\partial \zeta^m}(1)$.

**Lemma 2.3.** Suppose that $L$ is onto and let $\ell_0 = \max_{1 \leq i \leq N}\{\kappa_i\}$. Then the restriction of $j_{\ell_0}$ to ker $L$ is injective.

**Proof.** Let $\Pi : (A^{k,\alpha})^N \rightarrow ((1 - \zeta)^m A^{k,\alpha})^N$ be the isomorphism defined by
\[
\Pi(f) = (1 - \zeta)^m f.
\]
It is clear that the restriction of $j_\ell$ to ker $L$ is injective if and only if the restriction of $j_{\ell - m}$ to $\Pi^{-1}(\ker L)$ is injective.

Define the subspace $A(\kappa_1, \ldots, \kappa_N) \subset (A^{k,\alpha})^N$ by the elements $f = (f_1, \ldots, f_N)$ satisfying
\[
f_\ell(\zeta) = \sum_{j=0}^{\kappa_i - m} f_{\ell,j} \zeta^j, \quad f_{\ell,j} = \mathcal{I}_{i,j-1} \quad \forall \ 0 \leq j \leq \kappa_i - m, \ 0 \leq \ell \leq N,
\]
where we set $\sum_{j=0}^{\kappa_i - m} f_{\ell,j} \zeta^j = 0$ in the case $\kappa_i = m - 1$. Let $\ell_1 = \max_{1 \leq i \leq N}\{\kappa_i - m\}$: we claim that $j_{\ell_1}|_{A(\kappa_1, \ldots, \kappa_N)}$ is injective.

Indeed, let $f \in A(\kappa_1, \ldots, \kappa_N)$ such that $j_{\ell_1}(f) = 0$, i.e. $\partial f(1)$ for $0 \leq \ell \leq \ell_1$. By the choice of $\ell_1$, in particular $\frac{\partial^{\kappa_i - m} f}{\partial \zeta^{\kappa_i - m}}(1) = 0$ for all $1 \leq i \leq N$; since $\frac{\partial^{\kappa_i - m} f}{\partial \zeta^{\kappa_i - m}}(1) = (\kappa_i - m)! f_{i,\kappa_i - m} = 0$ for all $1 \leq i \leq N$. This in turn implies that
\[
\frac{\partial^{\kappa_i - m - 1} f}{\partial \zeta^{\kappa_i - m - 1}}(1) = (\kappa_i - m - 1)! f_{i,\kappa_i - m - 1} = 0.
\]
A straightforward inductive argument then shows that $f_{\ell} = 0$ for all $1 \leq \ell \leq N$.

Now, the proof of Theorem 2.2 implies that ker $L = \Pi(\Theta_1 A(\kappa_1, \ldots, \kappa_N))$ where $\Theta_1 = (i\Theta)^{-1}$. We claim that the restriction of $j_{\ell_1}$ to the space $\Theta_1 A(\kappa_1, \ldots, \kappa_N)$ is again injective: by the remarks made at the beginning of the proof, it will follow that the restriction of $j_{\ell_0}$ to ker $L$ is injective.

Indeed, for any $\ell \geq 0$ we have
\[
\partial_{\ell}(\Theta_1 f)(1) = \Theta_1(1) \partial f(1) + R_{\ell-1}
\]
where $R$ is a linear function of the $(\ell - 1)$-jet of $f$ at $1$. It follows that the (well-defined) linear map $\Theta_{\ell_1} : \mathbb{C}^{N(\ell_1 + 1)} \rightarrow \mathbb{C}^{N(\ell_1 + 1)}$ which sends the $\ell_1$-jet of $f$ at 1 to the $\ell_1$-jet of $\Theta_1 f$ at 1 has a block-triangular matrix representation whose $N \times N$ blocks in the diagonal are all equal to the non-singular matrix $\Theta_1(1)$. Therefore $\Theta_{\ell_1}.$
is invertible, and the claim follows from the fact that \( j_{\ell_1} \circ \Theta_1 = \Theta_{\ell_1} \circ j_{\ell_1} \) and that \( j_{\ell_1} \) is injective on \( A(\kappa_1, \ldots, \kappa_N) \).

\[ \square \]

2.3. Linear Riemann-Hilbert problems with pointwise constraints. Let \( G : b\Delta \to GL_N(\mathbb{C}) \) be a smooth map such that for all \( \zeta \in b\Delta \):

\[
G(\zeta) = \begin{pmatrix}
G_1(\zeta) & (\ast) \\
G_2(\zeta) & \\
\vdots & \\
(0) & G_r(\zeta)
\end{pmatrix},
\]

where \( G_j(\zeta) \in GL_{N_j}(\mathbb{C}), j = 1, \ldots, r \). Let \( \kappa_1, \ldots, \kappa_N \) and \( \kappa \) be the partial indices and the Maslov index of \( -G^{-1}G \). Suppose that the following operator

\[
L = \prod_{l=1}^{r} \left( (1 - \zeta)^{m_j} A^{k,\alpha} \right) N_j = \prod_{l=1}^{r} \left( C_{\nu^m_j}^{k,\alpha} \right) N_j
\]

is well-defined. Denote by \( \tilde{G}(\zeta) \) the following matrix

\[
\tilde{G}(\zeta) = \begin{pmatrix}
G_1(\zeta) & 0 \\
G_2(\zeta) & \\
\vdots & \\
0 & G_r(\zeta)
\end{pmatrix}
\]

and by \( \tilde{L} \) the corresponding operator. The kernels of \( L \) and \( \tilde{L} \) are of the same dimension and \( L \) is onto if and only if \( \tilde{L} \) is onto. Therefore a direct application of Theorem 2.2 gives:

**Theorem 2.4.** Under the above assumptions:

(i) The map \( L \) is onto if and only if \( \kappa_l \geq m_j - 1 \) for all \( l = \sum_{1}^{j-1} N_{\nu+1}, \ldots, \sum_{1}^{j} N_{\nu} \) and all \( j = 1, \ldots, r \).

(ii) Assume that \( L \) is onto. Then the kernel of \( L \) has real dimension \( \kappa + N - \sum_{j=1}^{r} N_j m_j \).

3. Construction of \( k_0 \)-stationary discs

Let \( M \subset \mathbb{C}^{n+1} \) be a finitely smooth hypersurface of finite type, whose Levi form vanishes at \( 0 \in M \). In this situation, trying to attach stationary discs to \( M \) gives rise to singular Riemann-Hilbert problems. We want to show how these problems can
be treated in terms of the techniques described in Sections 2.2 and 2.3 for certain classes of finite type hypersurfaces of $\mathbb{C}^2$ and $\mathbb{C}^3$.

We denote by $(z, w) \in \mathbb{C}^{n+1}$ with $z = (z_1, \ldots, z_n)$ the standard coordinates in $\mathbb{C}^{n+1}$.

3.1. Definitions. Let $M = \{r = 0\}$ be a finitely smooth hypersurface defined in a neighborhood of the origin in $\mathbb{C}^{n+1}$. Let $k, k_0$ be integers and let $0 < \alpha < 1$. We say that a holomorphic disc $f \in (\mathcal{A}^{k,\alpha})^{n+1}$ is attached to $M$ if $f(\zeta) \in M$ for all $\zeta \in b\Delta$. We recall the following definition from [2]:

**Definition 3.1.** A holomorphic disc $f \in (\mathcal{A}^{k,\alpha})^{n+1}$ attached to $M = \{r = 0\}$ is $k_0$-stationary if there exists a continuous function $c : b\Delta \to \mathbb{R}^+$ such that the map $\zeta \mapsto \zeta^{k_0} c(\zeta) \partial_r (f(\zeta))$, defined on $b\Delta$, extends as a map in $(\mathcal{A}^{k,\alpha})^{n+1}$.

Now we give a more geometric version:

**Definition 3.2.** A holomorphic disc $f \in (\mathcal{A}^{k,\alpha})^{n+1}$ attached to $M = \{r = 0\}$ is $k_0$-stationary if there exists a holomorphic lift $f = (f, \tilde{f})$ of $f$ to the cotangent bundle $T^*\mathbb{C}^{n+1}$, continuous up to the boundary, such that $\forall \zeta \in b\Delta, f(\zeta) \in \mathcal{N}^{k_0} M(\zeta)$ where

$$
\mathcal{N}^{k_0} M(\zeta) := \{(z, w, \tilde{z}, \tilde{w}) \in T^*\mathbb{C}^{n+1} | (z, w) \in M, (\tilde{z}, \tilde{w}) \in \zeta^{k_0} N^*_r M \setminus \{0\}\}.
$$

We are interested in constructing such discs for pseudoconvex hypersurfaces of finite type. Notice that in such a situation, the fibration $\mathcal{N}^{k_0} M(\zeta)$ is not totally real for all $\zeta \in b\Delta$. In fact we are precisely interested in discs passing through the degeneracy locus of $\mathcal{N}^{k_0} M$. For this purpose, we restrict our attention to discs with pointwise constraints. Define the space

$$(3.1) \quad Y := ((1 - \zeta) A^{k,\alpha})^n \times (1 - \zeta) A^{k,\alpha} \times ((1 - \zeta)^{d-1} A^{k,\alpha})^n \times (1 - \zeta) A^{k,\alpha},$$

endowed with the product norm (see (1.1)). We denote by $S_{k_0,r}$ the set lifts of $k_0$-stationary discs $f \in Y$ for the hypersurface $\{r = 0\}$ and by $S_{0,r}$ the lifts $f = (h, g, \tilde{h}, \tilde{g}) \in S_{k_0,r}$ satisfying $g(1) = 1$.

3.2. Polynomial models. We describe our setting for hypersurfaces of $\mathbb{C}^{n+1}, n \in \mathbb{N}$, and we will restrict to $n = 1, 2$ later on. We consider models of the form $S = \{\rho = 0\} \subset \mathbb{C}^{n+1}$, where

$$(3.2) \quad \rho(z, w) = -\text{Re} w + P(z, \overline{z}) = -\text{Re} w + \sum_{d - k_0 \leq |J| \leq |\overline{K}| \leq k_0} \alpha_{J, K} z^J \overline{z}^K.$$

Here $J = (j_1, \ldots, j_n), K = (k_1, \ldots, k_n)$ are multi-indices and $\alpha_{J, K} = \overline{\sigma}_{K, J}$. We choose $\frac{d}{2} \leq k_0 \leq d - 1$ in such a way that there exists $(\overline{J}, \overline{K})$ with $|\overline{K}| = k_0$ such that
αJK ≠ 0. In analogy with the case of hypersurfaces in $\mathbb{C}^2$ [2], we assume that the Levi form

$$P_{z\overline{z}}(z) = \begin{pmatrix} P_{z_1\overline{z}_1}(z, \overline{z}) & \cdots & P_{z_1\overline{z}_n}(z, \overline{z}) \\ \vdots & \ddots & \vdots \\ P_{z_n\overline{z}_1}(z, \overline{z}) & \cdots & P_{z_n\overline{z}_n}(z, \overline{z}) \end{pmatrix}$$

is positive definite outside of $z = 0$. These choices imply that, defining $h^0(\zeta) = (h^0_1(\zeta), \ldots, h^0_n(\zeta)) = ((1 - \zeta), \ldots, (1 - \zeta))$, $\zeta \in b\Delta$, we have

$$\begin{cases} \zeta^{k_0} P_{z_1\overline{z}_j}(h^0(\zeta), \overline{h^0(\zeta)}) = (1 - \zeta)^{d-2} Q_{ij}(\zeta) \\ \zeta^{k_0} P_{z_iz_j}(h^0(\zeta), \overline{h^0(\zeta)}) = (1 - \zeta)^{d-2} S_{ij}(\zeta) \end{cases}$$

where $Q_{ij}$ and $S_{ij}$ are holomorphic polynomials, and where each $Q_{ij}$ has degree at most $2k_0 - d + 1$ and each $S_{ij}$ has degree at most $2k_0 - d$. Furthermore, each $Q_{ij}$ is divisible by $\zeta$. We also have

$$Q(\zeta) = \det \begin{pmatrix} Q_{1\overline{1}}(\zeta) & \cdots & Q_{1\overline{n}}(\zeta) \\ \vdots & \ddots & \vdots \\ Q_{n\overline{1}}(\zeta) & \cdots & Q_{n\overline{n}}(\zeta) \end{pmatrix} \neq 0 \text{ for } \zeta \in b\Delta.$$

For a generic $P$, $Q(\zeta)$ has exactly degree $n(2k_0 - d + 1)$.

**Lemma 3.3.** Suppose that the Levi form $P_{z\overline{z}}$ is positive definite outside of $0$. Then the index of $Q$ is $n(k_0 - \frac{d}{2} + 1)$.

**Proof.** For any homogeneous polynomial $P(z, \overline{z})$ of degree $d$, denote by $Q_P(\zeta)$ the holomorphic polynomial obtained by applying the procedure described above to $P$. For a small $\epsilon \geq 0$ we define $P_\epsilon$ as

$$P_\epsilon(z, \overline{z}) = |z_1|^d + \ldots + |z_n|^d + \epsilon \|z\|^d.$$

Note that the Levi form of $P_\epsilon$ is positive definite outside 0 if $\epsilon > 0$. One can compute directly that $Q_{P_\epsilon}(\zeta) = C\zeta^{n(k_0 - \frac{d}{2} + 1)}$ for a certain constant $C$, hence the index of $Q_{P_\epsilon}(\zeta)$ is equal to $n(k_0 - \frac{d}{2} + 1)$ for $\epsilon > 0$ small enough. On the other hand, the set of the homogeneous polynomials $P$ of degree $d$ such that $P_{z\overline{z}}$ is positive definite outside 0 is a connected (and indeed convex) subset of the space of the polynomials of degree $d$, and since $Q_P(\zeta)$ depends continuously on $P$ it follows that its index is constant on this set. \hfill $\square$

### 3.3. A special family of $k_0$-stationary discs attached to the models.

We show that it is always possible to construct a non-trivial family of $k_0$-stationary discs attached to polynomial models described in Section 3.2. The construction is basically the same as the one outlined in [2].
Let \( S = \{ \rho = 0 \} = \{-\text{Re } w + P(z, w)\} \subset \mathbb{C}^{n+1} \) be a model of the form (3.2). We have
\[
\left\{ \begin{array}{l}
\frac{\partial \rho}{\partial w}(z, w) = -\frac{1}{2} \\
\frac{\partial \rho}{\partial z}(z, w) = P_z(\hat{\zeta}, w) = \sum_{d-k_0 \leq |j| \leq k_0} \hat{j}_\ell \alpha_{JK} \hat{\zeta}^j \hat{\zeta}^K, \ 1 \leq \ell \leq n,
\end{array} \right.
\]
where we put \( \hat{j}_\ell = (j_1, \ldots, j_\ell - 1, \ldots, j_n) \). Referring to the notation of Definition 3.1 a disc \( f = (h, g) : \Delta \to \mathbb{C}^{n+1}, h = (h_1, \ldots, h_n) \), attached to \( S \) is \( k_0 \)-stationary if there exists a continuous, nowhere vanishing \( c \in \mathcal{C}^{k,\alpha} \) such that \(-\frac{1}{2}\zeta^k c \) extends holomorphically to \( \Delta \), and furthermore \( \zeta^k c P_z(h, \overline{h}) \) extend holomorphically to \( \Delta \) for all \( 1 \leq \ell \leq n \).

In order to find non-trivial solutions \((c, f)\), we choose coefficient functions of the form \( c(\zeta) = (b\zeta + 1 + b\zeta)^{k_0} \) where \( b \in \mathbb{C} \) is such that \(|b| < 1/2\). Set \( c'(\zeta) = b\zeta + 1 + b\zeta \). It follows that \( \zeta c' \), and thus \(-\frac{1}{2}\zeta^k c \), extends holomorphically to the unit disc. We set
\[
h(\zeta) = \left( \frac{1 - \zeta}{1 - a(b)\zeta} v_1, \ldots, \frac{1 - \zeta}{1 - a(b)\zeta} v_n \right),
\]
with \( a(b) = \frac{1+\sqrt{1-4|b|^2}}{2b} \) and \( v_1, \ldots, v_n \in \mathbb{C}^* \). A simple computation shows that \( \zeta c' \gamma_j \in \mathcal{A}^{k,\alpha} \) for all \( 1 \leq j \leq n \) (see [1]). Then for any \( 1 \leq \ell \leq n \) we have
\[
\zeta^k c \zeta P_z(h(\zeta), \overline{h(\zeta)}) = \sum_{|j|+|K| = d} |j| \alpha_{JK}(\zeta c'(\zeta))^{k_0-|K|} h(\zeta)^j (\zeta c'(\zeta))^{K} |h(\zeta)|^K.
\]
Note that in the sum above we have \(|K| \leq k_0\), hence \((\zeta c'(\zeta))^{k_0-|K|} \in \mathcal{A}^{k,\alpha}\). Moreover
\[
(\zeta c'(\zeta))^{K} |h(\zeta)|^K = (\zeta c'(\zeta) h_1(\zeta) \zeta) \cdots (\zeta c'(\zeta) h_n(\zeta)) \zeta^k \in \mathcal{A}^{k,\alpha}
\]
so that every term in the sum belongs to \( \mathcal{A}^{k,\alpha} \). Hence \( \zeta^k c P_z(h, \overline{h}) \in \mathcal{A}^{k,\alpha} \) for all \( 1 \leq \ell \leq n \). Imposing the further condition \( g(1) = 0 \), one can find, by standard results about the Hilbert transform, a map \( g \) such that \( f = (h, g) \) is a \( k_0 \)-stationary disc attached to \( S \) and tied to the origin.

In particular, if \( b = 0 \) and \( v_j = 1 \) for \( 1 \leq j \leq n \), it follows that \( c \) is identically equal to \( 1 \) and that \( h_0(\zeta) = (1 - \zeta, \ldots, 1 - \zeta) \). From now on, we will denote by \( f_0 = (h_0, g_0) \) the corresponding stationary disc.

3.4. Space of allowed deformations. Let \( S = \{ \rho = 0 \} \) be a fixed polynomial model of the form (3.2). We want to define a space \( X \) parametrizing a family of deformations of the defining function \( \rho \) for which it is possible to construct a non-trivial family of \( k_0 \)-stationary discs. Choose \( \delta > 0 \) large enough so that \( f_0(\overline{\Delta}) \) is contained in the polydisc \( \delta \Delta^{n+1} \subset \mathbb{C}^{n+1} \).
The following definition is borrowed from [2]: we take \( X \) to be the (affine) Banach space given by the set of functions \( r \in C^{k+3} \left( \bar{\partial} \Delta^{n+1} \right) \) which can be written as

\[
  r(z, w) = \rho(z, w) + \theta(z, \text{Im } w),
\]

with

\[
  \theta(z, \text{Im } w) = \sum |I| + |J| = d + 1 \left( \Re \sum_{l=1} z^I \bar{\partial}^J (\text{Im } w)^l \cdot r_{IJl}(z, \text{Im } w) \right)
\]

where \( r_{IJ0} \in C^{k+3} \left( \bar{\partial} \Delta^n \right) \), \( r_{IJl} \in C^{k+3} \left( \bar{\partial} \Delta^n \times [-\delta, \delta] \right) \). Furthermore, we will consider the norm

\[
  \| r \|_X = \sup \| r_{IJl} \|_{C^{k+3}},
\]

so that \( X \) is isomorphic to a real closed subspace of \( C^{k+3} \left( \bar{\partial} \Delta \times [-\delta, \delta] \right) \) and, hence is a Banach space.

3.5. Defining equations of the fibration \( \mathcal{N}^{k_0} M \) and singular Riemann-Hilbert problems. Let \( S = \{ \rho = 0 \} = \{ -\Re w + P(z, \bar{z}) \} \subset C^{n+1} \) be a model hypersurface of the form \( \{ 3.2 \} \). In such case, we obtain explicit defining equations for \( \mathcal{N}^{k_0} S(\zeta) \). We have

\[
  (z, w, \bar{z}, \bar{w}) \in \mathcal{N}^{k_0} S(\zeta) \iff \left\{ \begin{array}{l}
  \rho(z, w) = 0 \\
  \exists c : b \Delta \to \mathbb{R}^*, (\bar{z}, \bar{w}) = c^{k_0} c(\zeta) \left( P_z(z, \bar{z}), -\frac{1}{2} \right)
\end{array} \right. 
\]

\[
\iff \left\{ \begin{array}{l}
  \rho(z, w) = 0 \\
  \frac{\bar{w}}{c^{k_0}} \in \mathbb{R} \\
  \bar{z}_j + 2\bar{w}_j P_{z_j}(z, \bar{z}) = 0 \text{ for } j = 1, \ldots, n.
\end{array} \right.
\]
Therefore, the defining equations of $\mathcal{N}^{k_0}S(\zeta)$ are given by
\[
\begin{align*}
\tilde{\rho}_1(\zeta)(z, w) &= -\text{Re } w + P(z, \bar{w}) = 0 \\
\tilde{\rho}_2(\zeta)(z, w) &= (\bar{z}_1 + 2\bar{w}_1P_{z_1}(z, \bar{w})) + \left(\bar{z}_1 + 2\bar{w}_1P_{z_1}(z, \bar{w})\right) = 0 \\
\tilde{\rho}_3(\zeta)(z, w) &= i(\bar{z}_1 + 2\bar{w}_1P_{z_1}(z, \bar{w})) - i\left(\bar{z}_1 + 2\bar{w}_1P_{z_1}(z, \bar{w})\right) = 0 \\
\vdots \\
\tilde{\rho}_{2n}(\zeta)(z, w) &= (\bar{z}_n + 2\bar{w}_nP_{z_n}(z, \bar{w})) + \left(\bar{z}_n + 2\bar{w}_nP_{z_n}(z, \bar{w})\right) = 0 \\
\tilde{\rho}_{2n+1}(\zeta)(z, w) &= i(\bar{z}_n + 2\bar{w}_nP_{z_n}(z, \bar{w})) - i\left(\bar{z}_n + 2\bar{w}_nP_{z_n}(z, \bar{w})\right) = 0 \\
\tilde{\rho}_{2n+2}(\zeta)(z, w) &= i\frac{\bar{w}}{\zeta_0} - i\zeta_{k_0}\bar{w} = 0.
\end{align*}
\]
We set $\tilde{\rho} := (\tilde{\rho}_1, \ldots, \tilde{\rho}_{2n+2})$. For a general hypersurface $M = \{ r = 0 \}$ with $r \in X$, we denote by $\tilde{r}$ the defining functions of $\mathcal{N}^{k_0}M(\zeta)$. This allows to consider lifts of stationary as solutions of a nonlinear Riemann-Hilbert type problem with singularities. Indeed, a holomorphic disc $f$ lifts of stationary as solutions of a nonlinear Riemann-Hilbert type problem with singularities. Indeed, a holomorphic disc $f \in (A^{k,\alpha})^{2n+2}$ is the lift of a $k_0$-stationary disc attached to $M$ if and only if
\[
\tilde{r}(f) = 0 \quad \text{on } b\Delta.
\]
The next two sections are devoted to the study of this problem. By applying the implicit function theorem we will show how solving $\tilde{r}(f)$ can be reduced to a linear Riemann-Hilbert problem which can be treated with the techniques described in Sections 2.2 and 2.3.

3.6. Construction of $k_0$-stationary discs in $\mathbb{C}^2$. Let $k \geq 0$ be an integer and let $0 < \alpha < 1$. Let $S = \{ \rho = 0 \} \subset \mathbb{C}^2$ be a model hypersurface of type $d$ of the form $\tilde{g}_0$. Consider the $k_0$-stationary disc attached to $S$ given by
\[
f_0 = (h_0, g_0, \tilde{h}_0, \tilde{g}_0) = (1 - \zeta, g_0, \tilde{h}_0, -\zeta_{k_0}/2),
\]
where $\tilde{h}_0(\zeta) = \zeta_{k_0}P_\zeta(1 - \zeta, 1 - \overline{\zeta})$ (see Section 3.3). Note that the explicit expression of $g_0$ is not needed for our purpose. Then we have:

**Theorem 3.4.** Under the above assumptions, there exist some open neighborhoods $V$ of $\rho$ in $X$ and $U$ of $0$ in $\mathbb{R}^{k_0-d+3}$, $\eta > 0$ and a map $F : V \times U \to Y$ of class $C^1$, such that:
i) $F(\rho, 0) = f_0$, 
ii) for all $r \in V$ the map $F(r, \cdot) : U \to \{ f \in S^{k_0, r} \mid \| f - f_0 \|_Y < \eta \}$ is one-to-one and onto.

Proof. In a neighborhood of $(\rho, f_0)$, we define the following map

$$ T : X \times Y \to C_0^{k, \alpha} \times C_0^{k, \alpha} \times C_0^{k, \alpha} \times C_0^{k, \alpha} $$

by

$$ T(r, f) := \tilde{r}(f), $$

with the notation $\tilde{r}(f)(\zeta) = \tilde{r}(\zeta)(f(\zeta))$. It follows from the definition of the Banach spaces $X$ and $Y$ that the map $T$ is of class $C^1$; the proof of this claim is completely analogous to the one of [2, Lemma 3.3] (see also [11, Lemma 5.1] and [9, Lemma 11.1]). According to (3.3), for any fixed $r \in X$, the zero set of $T(\tilde{r}, \cdot)$ coincides with $S^{k_0, r}$. The result follows then from the implicit function theorem. To this end, we need to consider the partial derivative of $T$ with respect to $Y$ at the point $(\rho, f_0)$, namely

$$ (3.4) \quad T_Y(\rho, f_0)(f') = 2\text{Re} \left[ \overline{G(\zeta)} f' \right], $$

where the matrix $G(\zeta) := (\rho_{\pi}(f_0), \rho_{\pi}(f_0), \rho_{\pi}(f_0), \rho_{\pi}(f_0)) \in M_4(\mathbb{C})$ has the following expression

$$ G(\zeta) = \begin{pmatrix} \gamma(\zeta) & -1/2 & 0 & 0 \\ \alpha(\zeta) & 0 & 1 & 2\gamma(\zeta) \\ \beta(\zeta) & 0 & -i & -2i\gamma(\zeta) \\ 0 & 0 & 0 & -i\zeta^{k_0} \end{pmatrix}, $$

with

$$ \begin{align*}
\gamma(\zeta) & = P_\pi(1 - \zeta, 1 - \zeta) \\
& = \frac{(\zeta - 1)^{d-1}}{\zeta} R(\zeta) \\
\alpha(\zeta) & = -\zeta^{k_0} P_{z\pi}(1 - \zeta, 1 - \zeta) - \zeta^{-k_0} P_{zz}(1 - \zeta, 1 - \zeta) \\
& = -(\zeta - 1)^{d-2} \left( Q(\zeta) + \frac{S(\zeta)}{\zeta} \right) \\
\beta(\zeta) & = -i\zeta^{k_0} P_{z\pi}(1 - \zeta, 1 - \zeta) + i\zeta^{-k_0} P_{zz}(1 - \zeta, 1 - \zeta) \\
& = -i(\zeta - 1)^{d-2} \left( Q(\zeta) - \frac{S(\zeta)}{\zeta} \right),
\end{align*} $$
where
\[ P_{\pi}(1 - \zeta, 1 - \overline{\zeta}) = \sum_{j=(d-k_0)}^{d} (d-j) \alpha_j (1 - \zeta)^j (1 - \overline{\zeta})^{d-j-1} \]
\[ = \frac{(\zeta-1)^{d-1}}{\zeta^{d-1}} \sum_{j=d-k_0}^{d} (-1)^j (d-j) \alpha_j \zeta^j \]
\[ = \frac{(\zeta-1)^{d-1}}{\zeta^{d-1}} R(\zeta), \]
and
\[ \zeta_{k_0} P_{z\bar{z}}(1 - \zeta, 1 - \overline{\zeta}) = \sum_{j=(d-k_0)}^{d} j (j-1) \alpha_j (1 - \zeta)^{j-2} \zeta_{k_0} (1 - \overline{\zeta})^{d-j} \]
\[ = (\zeta - 1)^{d-2} \sum_{j=d-k_0}^{d} (-1)^{j+1} j (j-1) \alpha_j \zeta_{k_0+j-d} \]
\[ = (\zeta - 1)^{d-2} S(\zeta). \]

After permutating the first two columns, we consider the following equivalent linear operator
\[ L_1(g', h', \tilde{h}', \tilde{g}') := 2 \text{Re} \left[ \sum_{j=0}^{d} (1 - \zeta)^j \alpha_j (1 - \overline{\zeta})^{d-j} \right], \]
where \[ L_1 : Y \to C^{k,\alpha}_0 \times C^{k,\alpha}_{d-1} \times C^{k,\alpha}_{0} \times C^{k,\alpha} \]
and where
\[ G_1(\zeta) := \begin{pmatrix} -1 & \gamma(\zeta) & 0 & 0 \\ 0 & \alpha(\zeta) & 1 & 2 \gamma(\zeta) \\ 0 & \beta(\zeta) & -i & -2i \gamma(\zeta) \\ 0 & 0 & 0 & -i \zeta_{k_0} \end{pmatrix}. \]

The kernels \( T_1(\tilde{\rho}, f_0) \) and \( L_1 \) are of the same dimension and \( T_1(\tilde{\rho}, f_0) \) is onto if and only if \( L_1 \) is onto. The following step is crucial in our approach since it is the argument that allows one to reduce a linear singular Riemann-Hilbert problem to a linear regular one with pointwise constraints, and allows then the use of Theorem 2.4. We factorize
\[ G_1(\zeta) = \tilde{G}_1(\zeta) \Delta(\zeta) \]
where
\[ \tilde{G}_1(\zeta) := \begin{pmatrix} -1 & \frac{\zeta^{d-2}}{(1-\zeta)^{d-2}} \gamma(\zeta) & 0 & 0 \\ 0 & \frac{\zeta^{d-2}}{(1-\zeta)^{d-2}} \alpha(\zeta) & 1 & 2 \gamma(\zeta) \\ 0 & \frac{\zeta^{d-2}}{(1-\zeta)^{d-2}} \beta(\zeta) & -i & -2i \gamma(\zeta) \\ 0 & 0 & 0 & -i \zeta_{k_0} \end{pmatrix} \]
and where \( \Delta(\zeta) \) is the Banach space isomorphism
\[ \Delta(\zeta) : Y \to (1 - \zeta) A^{k,\alpha} \times (1 - \zeta)^{d-1} A^{k,\alpha} \times (1 - \zeta)^{d-1} A^{k,\alpha} \times A^{k,\alpha} \]
defined by
\[ \Delta(\zeta)(g', h', \tilde{h}', \tilde{g}') := (g', (1 - \zeta)^{d-2}h', \tilde{h}', \tilde{g}'). \]

Note that
\[
\begin{cases}
\frac{\zeta^{d-2}}{(1-\zeta)^{d-2}} \alpha(\zeta) = -\left(\zeta^{d-2}Q(\zeta) + S(\zeta)\right) \\
\frac{\zeta^{d-2}}{(1-\zeta)^{d-2}} \beta(\zeta) = -i\left(\zeta^{d-2}Q(\zeta) - S(\zeta)\right)
\end{cases}
\]

It follows that the map \( \tilde{G}_1 \) satisfies the assumption of Theorem 2.4, and the problem therefore reduces to estimating the partial indices of \( -\tilde{G}_1^{-1} \tilde{G}_1 \) and computing its Maslov index.

We first prove that \( L_1 \) is surjective. Recall that one can find \( \Theta : \bar{\Delta} \to GL_4(\mathbb{C}) \) such that
\[
-\tilde{G}_1^{-1}(\zeta) \tilde{G}_1(\zeta) = \Theta(\zeta) \begin{pmatrix} 1 & \zeta^{\kappa_2} & 0 \\
\zeta^{\kappa_3} & \zeta^{2k_0} & \Theta(\zeta)^{-1} \end{pmatrix},
\]
Moreover \( \Theta \) has the following form
\[
\Theta(\zeta) = \begin{pmatrix} 1 & \theta(\zeta) \\
(0) & 1 \end{pmatrix},
\]
where \( \theta(\zeta) \) is a \( 2 \times 2 \) matrix. According to point (i) of Theorem 2.4, we need to prove that \( \kappa_2 \geq d - 2 \) and \( \kappa_3 \geq d - 2 \). Without loss of generality, we suppose that \( \kappa_2 \geq \kappa_3 \). Denote by \( l = (l_1, l_2) \) the second row of the matrix \( \theta^{-1} \). It follows that for all \( \zeta \in b\Delta \)
\[
l(\zeta) \frac{\zeta^{d-2}}{Q(\zeta)} \begin{pmatrix} -S(\zeta) \\
|Q(\zeta)|^2 - |S(\zeta)|^2 \\
1 \\
S(\zeta) \end{pmatrix} = \zeta^{\kappa_3}l(\zeta).
\]
The verification of the fact that \( \kappa_3 \geq d - 2 \) reduces to the problem
\[
l_1(\zeta) + S(\zeta)l_2(\zeta) = \zeta^{\kappa_3-d+2}Q(\zeta)l_2(\zeta)
\]
which implies either \( l_1 = l_2 = 0 \) or \( \kappa_3 - d + 2 \geq 0 \).

We finally compute the Maslov index \( -\tilde{G}_1^{-1} \tilde{G}_1 \). A straightforward computation leads to
\[
det \left( -\tilde{G}_1^{-1}(\zeta) \tilde{G}_1(\zeta) \right) = \zeta^{2k_0 + 2d - 4} \frac{Q(\zeta)}{Q(\zeta)},
\]
which implies
\[
\text{ind det} \left( -\tilde{G}_1^{-1} \tilde{G}_1 \right) = 2k_0 + 2d - 4 + 2(l_0 + 1) = 4k_0 + d - 2.
\]
This proves that the Maslov index of $-\tilde{G}_1^{-1} \tilde{G}_1$ is $4k_0 + d - 2$. It follows directly from this fact and point (ii) of Theorem 2.4 that the kernel of $L_1$ has real dimension $4k_0 - d + 3$.

An application of the implicit function theorem completes the proof of Theorem 3.4. □

3.7. Construction of $k_0$-stationary discs in $\mathbb{C}^3$. In this section, we are going to follow the same procedure to construct an invariant family of $k_0$-stationary discs in $\mathbb{C}^3$. We will slightly modify the argument of the previous section, in particular the treatment of the linearized problem, in a way which appears to be more suited to obtain jet determination results (see Lemma 2.3).

Define for $f = (h, g, \tilde{h}, \tilde{g})$ the following disc

$$f^\# := (h, g, \tilde{h}, (1 - \zeta)\tilde{g} + \zeta k_0).$$

Consider the disc in $Y$ given by

$$f_0 = (h_0, g_0, \tilde{h}_0, \tilde{g}_0) = (1 - \zeta, ..., 1 - \zeta, g_0, \tilde{h}_0, 0).$$

Theorem 3.5. Let $n = 2$ and let $S = \{\rho = 0\} \subset \mathbb{C}^{n+1}$ be a model hypersurface of the form (3.2). Then for any integer $k \geq 0$ and $0 < \alpha < 1$, there exist some open neighborhoods $V$ of $\rho$ in $X$ and $U$ of $0$ in $\mathbb{R}^{2(n+1)k_0 - n(d-2)}$, $\eta > 0$, and a map $F : V \times U \rightarrow Y$ of class $C^1$, such that:

i) $F(\rho, 0) = f_0^\#$,

ii) for all $r \in V$, the map $F(r, \cdot) : U \rightarrow \{f^\# \in S^{k_0, r}_0 \mid \|f^\# - f_0^\#\|_Y < \eta\}$ is one-to-one and onto.

Remark 3.6. Although we are only able to prove the previous theorem for $n = 1, 2$ at the moment, we chose to keep $n$ as a variable in its statement. Indeed we expect the result to hold in any dimension. The main challenge is to generalize Lemma 3.7 to a higher $n$.

Proof. Define, on a neighborhood of $(\rho, f_0)$, the map of Banach spaces

$$T : X \times Y \rightarrow C^{k,\alpha}_0 \times (C^{k,\alpha}_{0d+1})^{2n} \times C^{k,\alpha}_0$$

by setting

$$T(r, f) := \tilde{r}(f^\#).$$

This is the same map considered in Section 3.6 with the only difference that we are replacing $\tilde{g}$ by $(1 - \zeta)\tilde{g} + \zeta k_0$; in this way, we impose the normalization $\tilde{g}(1) = 1$. Note that $1/r_w((1 - \zeta)g, (1 - \zeta)h) = -2 + (1 - \zeta)\ell$ for some function $\ell \in C^{k,\alpha}_C$, hence the last line is indeed an element of $C^{k,\alpha}_0$ and the map $T$ is well-defined. For any fixed $r \in X$, $T(\tilde{r}, f) = 0$ if and only if $f^\# \in S^{k_0, r}_0$. The map $T$ is also class $C^1$. 

Once again, we need to consider the Banach space derivative \( L := T_\gamma(\rho, f_0) \) of \( T \) at the point \((\rho, f_0)\). The linear operator \( L \):

\[
L : Y \to \mathcal{C}_0^k \times (\mathcal{C}_{(d-1)}^{k,\alpha})^{2n} \times \mathcal{C}_0^k
\]

is similar to the one in Equation (3.4).

Out of convenience, though, we manipulate the resulting linear system in the following way: we divide the first and last line by \((1 - \zeta)\), while we divide the other lines by \((1 - \zeta)^{d-1}\) and then multiply them by \(\zeta^s\). The resulting map is a linear operator \((\mathcal{A}^{\alpha})^n \times \mathcal{A}^{\alpha} \times (\mathcal{A}^{\alpha})^n \times \mathcal{A}^{\alpha} \to (\mathcal{R}_1)^{n+2}\) (see Lemma 1.1) which is equivalent to the original one with respect to the properties we are interested in (surjectivity and elements of the kernel). The new linear operator is of the form considered in Theorem 2.2 with \( m = 1 \): more precisely, it is of the form \( G_1 f' + G_1 \Gamma f \), where \( f' \in (\mathcal{A}^{\alpha})^{2n+2} \), \( \Gamma(\zeta) \) is the diagonal matrix with entries \(-\zeta^s\) and the matrix \( G_1(\zeta) \) is given by

\[
G_1(\zeta) = \begin{pmatrix}
-1/2 & (\ast) \\
A(\zeta) & -i\zeta^k_0
\end{pmatrix}
\]

with

\[
A(\zeta) = \begin{pmatrix}
\zeta & Q_{11} \zeta^s + \overline{S}_{11} \zeta^s & 0 & Q_{12} \zeta^s + \overline{S}_{12} \zeta^s & \ldots & 0 & Q_{1n} \zeta^s + \overline{S}_{1n} \zeta^s \\
-\overline{\zeta} & iQ_{11} \overline{\zeta}^s - i\overline{S}_{11} \overline{\zeta}^s & 0 & iQ_{12} \overline{\zeta}^s - i\overline{S}_{12} \overline{\zeta}^s & \ldots & 0 & iQ_{1n} \overline{\zeta}^s - i\overline{S}_{1n} \overline{\zeta}^s \\
0 & Q_{21} \zeta^s + \overline{S}_{21} \zeta^s & \zeta & \overline{Q}_{22} \zeta^s + \overline{S}_{22} \zeta^s & \ldots & 0 & \overline{Q}_{2n} \zeta^s + \overline{S}_{2n} \zeta^s \\
0 & iQ_{21} \overline{\zeta}^s - i\overline{S}_{21} \overline{\zeta}^s & -\overline{\zeta} & iQ_{22} \overline{\zeta}^s - i\overline{S}_{22} \overline{\zeta}^s & \ldots & 0 & iQ_{2n} \overline{\zeta}^s - i\overline{S}_{2n} \overline{\zeta}^s \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & Q_{n1} \zeta^s + \overline{S}_{n1} \zeta^s & 0 & Q_{n2} \zeta^s + \overline{S}_{n2} \zeta^s & \ldots & \zeta & Q_{nn} \zeta^s + \overline{S}_{nn} \zeta^s \\
0 & iQ_{n1} \overline{\zeta}^s - i\overline{S}_{n1} \overline{\zeta}^s & 0 & iQ_{n2} \overline{\zeta}^s - i\overline{S}_{n2} \overline{\zeta}^s & \ldots & -\overline{\zeta} & iQ_{nn} \overline{\zeta}^s - i\overline{S}_{nn} \overline{\zeta}^s
\end{pmatrix}
\]

We first want to show that \( L \) is onto. This entails a careful analysis of the matrix \( \overline{A^{-1}} A \), and from now on we restrict to the case \( n = 2 \). We write

\[
\overline{A^{-1}}(\zeta) A(\zeta) = \frac{1}{\det A} A'(\zeta) = \frac{1}{(2i)^n Q(\zeta)} A'(\zeta)
\]

and denote by \( A'_{j,\ell} \) the \((j, \ell)\) entry of the matrix \( A'(\zeta) \), \( 1 \leq j, \ell \leq 2n \). A direct computation of \( A' \) then gives

\[
A'(\zeta) = 4 \begin{pmatrix}
S_{11} \overline{Q}_{2\gamma} - \overline{Q}_{2\gamma} S_{12} & A'_{12} & \overline{Q}_{1\gamma} S_{12} - \overline{Q}_{1\gamma} S_{11} & A'_{14} \\
-\overline{Q}_{2\gamma} S_{21} - \overline{Q}_{2\gamma} S_{22} & \overline{S}_{21} \overline{Q}_{1\gamma} - \overline{S}_{11} \overline{Q}_{2\gamma} & \overline{Q}_{1\gamma} & \overline{Q}_{1\gamma} S_{21} - \overline{Q}_{1\gamma} S_{22} \\
\overline{Q}_{2\gamma} S_{11} - \overline{Q}_{1\gamma} S_{21} & \overline{Q}_{1\gamma} & \overline{Q}_{1\gamma} S_{21} - \overline{Q}_{1\gamma} S_{22} & A'_{34}
\end{pmatrix}
\]
with
\[
A'_{12} = - \det \begin{pmatrix} Q_{1T} & S_{11} & S_{12} \\ S_{11} & Q_{1T} & S_{12} \\ S_{21} & Q_{2T} & S_{22} \end{pmatrix}, \quad A'_{32} = - \det \begin{pmatrix} Q_{1T} & S_{11} & S_{12} \\ S_{21} & Q_{2T} & S_{22} \\ S_{21} & Q_{2T} & S_{22} \end{pmatrix},
\]
\[
A'_{14} = - \det \begin{pmatrix} Q_{1T} & S_{11} & S_{12} \\ S_{12} & Q_{1T} & S_{12} \\ S_{22} & Q_{2T} & S_{22} \end{pmatrix}, \quad A'_{34} = - \det \begin{pmatrix} Q_{1T} & S_{11} & S_{12} \\ S_{21} & Q_{2T} & S_{22} \\ S_{22} & Q_{2T} & S_{22} \end{pmatrix}.
\]

The surjectivity of \( L \) is then a consequence of Theorem 2.2 (i) (with \( m=1 \)), and of the following lemma:

**Lemma 3.7.** The partial indices of \( G^{-1}G_1 \) are non-negative.

**Proof of Lemma 3.7.** We follow the same lines as in [1] Lemma 3.2: let \( \kappa_1 \geq \ldots \geq \kappa_6 \) be the partial indices of \( G^{-1}G_1 \), and let \( \Lambda \) be the diagonal matrix with entries \( \zeta^{\kappa_1}, \ldots, \zeta^{\kappa_6} \). Then there exists a smooth map \( \Theta: \Delta \to GL_6(\mathbb{C}) \), holomorphic on \( \Delta \), such that \( -\Theta G^{-1}G_1 = \Lambda \Theta \) (see Equation (2.11)). Let \( \ell = (l_1, \ldots, l_6) \) be the last row of the matrix \( \Theta \).

First of all, we have \( l_1 = \zeta^\kappa l_1 \), from which it follows that either \( \kappa_6 \geq 0 \) or \( l_1 = 0 \). Assume that \( l_1 = 0 \). Then we get the following system for \( l_2, \ldots, l_5 \):

\[
\begin{aligned}
A'_{12}l_2 + (S_{11}Q_{1T} - Q_{2T}S_{12})l_2 - Q_{2T}l_3 + (Q_{2T}S_{21} - Q_{2T}S_{22})l_4 + Q_{2T}l_5 &= -Q \zeta^\kappa l_2 \\
A'_{12} + (S_{11}Q_{1T} - S_{11}Q_{2T})l_3 + A'_{34}l_4 + (Q_{2T}S_{11} - Q_{1T}S_{21})l_5 &= -Q \zeta^\kappa l_3 \\
(S_{11}Q_{1T} - Q_{1T}S_{11})l_2 + Q_{1T}l_3 + (S_{22}Q_{1T} - Q_{1T}S_{21})l_4 - Q_{1T}l_5 &= -Q \zeta^\kappa l_4 \\
A'_{14}l_2 + (Q_{1T}S_{22} - Q_{2T}S_{12})l_3 + A'_{34}l_4 + (S_{12}Q_{2T} - S_{22}Q_{1T})l_5 &= -Q \zeta^\kappa l_5.
\end{aligned}
\]

Suppose that \( \kappa_6 \leq -1 \). Then, if \( l_2 \neq 0 \), the right hand side of the first line in the system above is an antiholomorphic function on the unit disc which, by Lemma 3.3 has at least \( 2k_0 - d + 3 \) zeros (counted with multiplicity). In particular, the Fourier expansion of \( -Q \zeta^\kappa l_2 \) must contain at least a power of \( \zeta \) which is greater than \( 2k_0 - d + 2 \) (otherwise it would represent a polynomial of degree at most \( 2k_0 - d + 2 \), and it could not admit as many zeros). In the Fourier expansion of the left hand side, though, cannot appear any power of \( \zeta \) greater than \( 2k_0 - d + 1 \) since \( \text{deg}Q_{j\ell} \leq 2k_0 - d + 1 \). It follows that \( l_2 = 0 \), and arguing in a similar way for the third line we get \( l_4 = 0 \). The first and third equation then reduce to \( -Q_2l_3 + Q_2l_5 = 0 \) and \( Q_2l_3 - Q_1l_5 = 0 \) respectively: using these relations in the second and fourth line we see that the left hand sides vanish identically, from which we conclude that \( l_3 = l_5 = 0 \). In summary, the arguments above show that either \( \kappa_6 \geq 0 \) or \( l_1 = \ldots = l_5 = 0 \). In the second
case, we would have \(-\zeta^{2k_0}l_6 = \zeta^{\kappa_6}l_6\), which implies that \(\kappa_6 \geq 2k_0\) or \(l_6 = 0\). Since \(\Theta\) is invertible, the latter would be a contradiction, hence we conclude that \(\kappa_6 \geq 0\).

Next, we compute the dimension of the kernel of \(L\).

**Lemma 3.8.** The kernel of \(L\) is of dimension \(2(n + 1)k_0 - n(d - 2)\).

**Proof of Lemma 3.8.** By the arguments above and Theorem 2.2 (ii) (with \(m=1\)), the dimension of \(\text{ker} L\) is given by the Maslov index of \(G^{-1}G'\). It follows directly that

\[
\text{ind det} \left( G^{-1}G_1 \right) = 2k_0 + \text{ind det} \left( A^{-1}A \right)
\]

From (3.5), after some row manipulation we get

\[
\det A(\zeta) = \begin{pmatrix}
\zeta^s & \overline{S}_{11} \zeta^s & 0 & \overline{S}_{12} \zeta^s & \ldots & 0 & \overline{S}_{1n} \zeta^s \\
0 & 2iQ_{11} \zeta^s & 0 & 2iQ_{12} \zeta^s & \ldots & 0 & 2iQ_{1m} \zeta^s \\
0 & \overline{S}_{21} \zeta^s & \zeta^s & \overline{S}_{22} \zeta^s & \ldots & 0 & \overline{S}_{2n} \zeta^s \\
0 & 2iQ_{21} \zeta^s & 0 & 2iQ_{22} \zeta^s & \ldots & 0 & 2iQ_{2m} \zeta^s \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \overline{S}_{n1} \zeta^s & 0 & \overline{S}_{n2} \zeta^s & \ldots & \zeta^s & \overline{S}_{nn} \zeta^s \\
0 & 2iQ_{n1} \zeta^s & 0 & 2iQ_{n2} \zeta^s & \ldots & 0 & 2iQ_{nm} \zeta^s
\end{pmatrix} = (2i)^n Q(\zeta),
\]

hence \(\det A^{-1}A = -\frac{Q(\zeta)}{Q(\zeta)}\), so that

\[
\text{ind det} \left( G^{-1}G_1 \right) = 2k_0 + 2\text{ind} Q = 2(n + 1)k_0 - n(d - 2)
\]

by Lemma 3.3.

The proof of Theorem 3.5 now follows from Lemma 3.8 and the surjectivity of \(L\).

4. Finite jet determination of CR maps

4.1. **Statement of the result.** Results obtained in the previous section allow to obtain finite jet determination results for CR diffeomorphisms, generalizing what was obtained in [2]. The proof goes along the same lines as in [2]: indeed, the proofs of some of the lemmas in there can be generalized to higher dimension without significant changes, hence we will omit them here.

We will show that the existence of \(k_0\)-stationary discs allows to prove a finite jet determination result in the spirit of [2, Theorem 1.2]. Since the case \(n = 1\) is already contained in [2], we state it in the case \(n = 2\).

In the previous sections, the smoothness \(C^{k,\alpha}\) of the analytic discs has been left undetermined since it was not crucial for their purposes. From now on, we are going
to fix a large enough $\tilde{k} \in \mathbb{N}$, only depending on the type $d$ of the model $P(z, \overline{z})$ – more precisely, it turns out that we can choose $\tilde{k} = 6k_0 - d + 3 \leq 5d$. This is due to the fact that we need to apply Lemma 2.3 with an $\ell_0$ which depends on the order of jet determination we are going to achieve, and thus we need at least $\tilde{k} \geq \ell_0$.

**Theorem 4.1.** Let $M \subset \mathbb{C}^3$ be a real $C^{k+3}$ pseudoconvex hypersurface of finite type whose defining function is locally written as

$$r(z, w) = -\text{Re} w + P(z, \overline{z}) + O(|z|^{d+1}) + \text{Im} w O(|z, \text{Im} w|^{d-1}),$$

with $P(z, \overline{z})$ as in Equation (3.2). Then the germs at $p = 0 \in M$ of CR diffeomorphisms $H$ of class $C^{k+3}$ such that $H(M) = M$ are uniquely determined by their $(\tilde{k} + 1)$-jet at $p$.

**Remark 4.2.** Assume that a jet determination result of order $k'$ holds in the formal setting, in the sense that every $\ell$-jet of a formal biholomorphism which preserves a formal hypersurface (up to the order $\ell$) and is trivial up to order $k'$. Then the conclusion of Theorem 4.1 holds for $k'$-jet determination as long as the smoothness of $M$ is at least $C^{\max\{k', k+3\}}$. Indeed, the $(\tilde{k} + 1)$-order Taylor expansion of $H$ represents a $(\tilde{k} + 1)$-order biholomorphism jet which preserves the polynomial hypersurface induced by the Taylor expansion of $M$ up to order $(\tilde{k} + 1)$, thus if it is trivial up to order $k'$ it must be trivial up to order $\tilde{k} + 1$: from the theorem it follows in turn that $H$ is the identity.

It follows for instance that, for the version of Theorem 4.1 in $\mathbb{C}^2$ (see [2]), we can always achieve 2-jet determination of CR diffeomorphisms as in the real-analytic case (see [7, 14]). In higher dimension we can also use the order of jet determination established in the formal setting, see for instance [15].

The proof of Theorem 4.1 is achieved by putting together several facts:

1. The family of $k_0$-stationary discs is invariant under CR diffeomorphisms (see [2, Proposition 2.5 and Remark 2.6]);
2. a suitable dilation method allows to choose a change of coordinates $\Lambda_t(z, w) = (tz, t^d w)$, for some $t > 0$, in such a way that the defining function $r$ (which is a priori defined on a small neighborhood of 0) has a pull-back $r_t = \frac{1}{t} r \circ \Lambda_t$ which belongs to the neighborhood $V$ of $\rho$ in $X$ identified in Theorem 3.5;
3. similarly, the conjugated $H_t = \Lambda_t^{-1} \circ H \circ \Lambda_t$ of the CR diffeomorphism $H$ can be made arbitrarily close to the identity (in the $C^1$-norm) for an appropriate choice of the parameter $t$;
4. the lifts of $k_0$-stationary discs attached to $r_t$ and tied to 0 are determined by their $(6k_0 - 2d + 4)$-jet at $\zeta = 1$;
5. the union of the points $f(0)$ for $f \in S_0^{k_0, r_t}$ is an open set of $\mathbb{C}^3$. 
The points 2. and 3. can be proved in the same way as [2, Lemmas 5.2 and 5.3] (see also [1, Lemmas 4.1 and 4.2]) and we shall omit the proof.

To prove point 4., note that it is sufficient to show that, putting \( \ell_0 = \tilde{k} \), the restriction of \( j_0 \) to the tangent space \( T_{\mathbf{f}_0} S_{k, \rho}^0 \) of \( S_{k, \rho}^0 \) at the point \( \mathbf{f}_0 = (f_0, \tilde{f}_0) \) is injective: the statement then follows from Theorem 3.5. Since by the implicit function theorem \( T_{\mathbf{f}_0} S_{k, \rho}^0 = \ker T_Y(\mathbf{f}_0) \), the claim is in turn a consequence of Lemma 3.8 and Lemma 2.3. Indeed, by Lemma 3.7 the partial indices of \( G' \) are non-negative, hence each partial index \( \kappa_j \) can be estimated by the Maslov index \( 6k_0 - 2d + 4 \) of \( G'^{-1}G' \). However, because of the rescaling applied to obtain the matrix \( G' \), to get the actual kernel of our original problem we have to multiply some components by \((1-\zeta)^{d-1}\); this implies that the actual order of the jet determination can be estimated by \( \tilde{k} = 6k_0 - d + 3 \leq 5d \).

We will give a proof of point 5. in the next section: the procedure is similar to the one in [2, Proposition 4.4, (i)], but some substantial modifications are required. Finally, the proof of Theorem 4.1 follows from the points above with the same argument as in [2, Section 5.2]: the only difference is that one needs to apply the argument to the lift of \( H_t \) to the conormal bundle rather than to \( H_t \) itself, and this is achieved as in [1, Section 4.2].

4.2. Surjectivity of the evaluation map. We will show that the restriction of the linear map \( Y \ni \mathbf{f} = (f, \tilde{f}) \mapsto f(0) \in \mathbb{C}^3 \) to \( S_{k, \rho}^0 \) is onto by showing that its restriction to \( T_{\mathbf{f}_0} S_{k, \rho}^0 \) is onto. The claim at point 5. in the previous section will then follow for all \( S_{k, r}^0 \) with \( r \in V \) from Theorem 3.5. In fact, it will be enough to consider the (tangent space of the) submanifold of discs constructed in section 3.3.

We start by choosing more convenient coordinates, using the following lemma:

**Lemma 4.3.** Suppose that \( R(z, \bar{z}) \) is a nontrivial homogeneous polynomial of type \( (a, b) \),

\[
R(z, \bar{z}) = \sum_{0 \leq j \leq a} \beta_{jk} z_j z_2^{a-j} \bar{z}_1 z_2^k.
\]

Then there exists a linear change of coordinates \( \phi \) such that, denoting by \( \beta'_{kl} \) the coefficients of \( R' = R \circ \phi \) we have \( \beta'_{ab} \neq 0 \).

**Proof.** We write a general linear change of coordinates in \( \mathbb{C}^2 \) as

\[
z_1' = \delta_1 z_1 + \epsilon_1 z_2, \quad z_2' = \delta_2 z_1 + \epsilon_2 z_2.
\]

Writing down the coefficients of \( R' \) we get
\[ \beta'_{a0} = \sum_{0 \leq j \leq a \atop 0 \leq k \leq b} \delta_j \delta_j^a-j \epsilon_1^k \beta_{jk}. \]

Consider \( \beta'_{a0} \) as a polynomial in \( \delta_j, \delta_j^a, \epsilon_1, \epsilon_j \): the polynomial identity \( \beta'_{a0} = 0 \) holds if and only if \( \beta_{jk} = 0 \) for all \( 0 \leq j \leq a, 0 \leq k \leq b \), contradicting the assumption that \( R \) is non-trivial.

Applying Lemma 4.3 to \( P(z, \bar{z}) \), and more specifically to its \( (k_0, d-k_0) \) part, we can assume that \( \alpha_{jk} \neq 0 \) for \( \tilde{J} = (k_0, 0), \tilde{K} = (0, d-k_0) \). We will show that the evaluation map at 0 is surjective up to a small perturbation of the original disc \((h_0, g_0)\). Let \( \pi : Y \rightarrow ((1 - \zeta)A^{k,\alpha})^3, \pi(f, \bar{f}) = f \). We consider the submanifold (see Section 3.3)

\[ \mathcal{M}' := \left\{ f = (h, g) \in \pi(S_{0^0}) \mid h(\zeta) = \left( \frac{1 - \zeta}{1 - a\zeta^2} v_1, \frac{1 - \zeta}{1 - a\zeta^2} v_2, a \in \Delta, v_1, v_2 \in \mathbb{C}^* \right) \right\}. \]

Notice that for any such disc \( f \), the tangent disc \( f' \in T_y \mathcal{M}' \) can be written \( f' = (h', g') \) with

\[ h'(\zeta) = (h'_1(\zeta), h'_2(\zeta)) = \left( \frac{(1 - \zeta)^2}{(1 - a\zeta^2)^2} v_1 a' + \frac{1 - \zeta}{1 - a\zeta^2} v_1', \frac{(1 - \zeta)^2}{(1 - a\zeta^2)^2} v_2 a' + \frac{1 - \zeta}{1 - a\zeta^2} v_2' \right) \]

where \((a', v'_1, v'_2) \in \mathbb{C}^3\).

We need to consider the map

\[ \psi : \mathcal{M}' \rightarrow \mathbb{C}^3 \]

given by \( \psi : f \mapsto f(0) \). Recall that \( f = (h, g) \) satisfies for \( \zeta \in b\Delta \)

\[ \text{Re } g(\zeta) = P(h(\zeta), \bar{h}(\zeta)). \]

Thus, from \( h(1) = 0 \) and from classical facts on Cauchy transform (see [18, Lemma 3]), we have

\[ g(0) = \frac{1}{i\pi} \int_{b\Delta} \frac{\text{Re } g(\zeta) d\zeta}{1 - \zeta} = \frac{1}{i\pi} \int_{b\Delta} \frac{P(h(\zeta), \bar{g}(\zeta)) d\zeta}{1 - \zeta}. \]

For \( \theta = (\theta_1, \theta_2) \) sufficiently small, consider the disc \( f_\theta = (h_\theta, g_\theta) \in \mathcal{M}' \) given by \( h_\theta(\zeta) = ((1 - \zeta)e^{i\theta_1}, (1 - \zeta)e^{i\theta_2}) \in \mathcal{M}' \). The derivative of \( \psi \) at \( f_\theta \in \mathcal{M}' \) in the \( h' \) direction is given by \( d_{f_\theta} \psi(h')(\theta') = (h'(0), d_{f_\theta} g(0)(h')) \), where

\[ d_{f_\theta} g(0)(h') = \frac{1}{i\pi} \int_{b\Delta} \frac{P_{z_1}(h_\theta(\zeta), \bar{h}_\theta(\zeta))g_1(\zeta) d\zeta}{1 - \zeta} + \frac{1}{i\pi} \int_{b\Delta} \frac{P_{z_2}(h_\theta(\zeta), \bar{h}_\theta(\zeta))g_2(\zeta) d\zeta}{1 - \zeta} + \frac{1}{i\pi} \int_{b\Delta} \frac{P_{z_2}(h_\theta(\zeta), \bar{h}_\theta(\zeta))g_2(\zeta) d\zeta}{1 - \zeta} \]

\[ + \frac{1}{i\pi} \int_{b\Delta} \frac{P_{z_2}(h_\theta(\zeta), \bar{h}_\theta(\zeta))g_2(\zeta) d\zeta}{1 - \zeta} + \frac{1}{i\pi} \int_{b\Delta} \frac{P_{z_2}(h_\theta(\zeta), \bar{h}_\theta(\zeta))g_2(\zeta) d\zeta}{1 - \zeta} \]
and $h' \in T_{h^0}M'$. Now we fix $(Z_1, Z_2, W) \in \mathbb{C}^3$ and we solve $d_{f^0} \psi(g') = (Z_1, Z_2, W)$. The first two components give $v_1' = Z_1$, $v_2' = Z_2$ and the third component leads to

$$i \pi W = \int_{b\Delta} P_{z_1} \left( h_\theta(\zeta), \overline{h_\theta(\zeta)} \right) Z_1 \frac{d\zeta}{\zeta} + \int_{b\Delta} P_{z_1} \left( h_\theta(\zeta), \overline{h_\theta(\zeta)} \right) e^{i\theta_1} a' d\zeta$$

$$- \int_{b\Delta} P_{z_2} \left( h_\theta(\zeta), \overline{h_\theta(\zeta)} \right) Z_2 \frac{d\zeta}{\zeta} - \int_{b\Delta} P_{z_2} \left( h_\theta(\zeta), \overline{h_\theta(\zeta)} \right) e^{i\theta_2} a' d\zeta$$

Set

$$I_1(\theta) = \int_{b\Delta} \left( P_{z_1} \left( h_\theta(\zeta), \overline{h_\theta(\zeta)} \right) e^{i\theta_1} + P_{z_2} \left( h_\theta(\zeta), \overline{h_\theta(\zeta)} \right) e^{i\theta_2} \right) d\zeta$$

and

$$I_2(\theta) = - \int_{b\Delta} \left( P_{z_1} \left( h_\theta(\zeta), \overline{h_\theta(\zeta)} \right) e^{-i\theta_1} + P_{z_2} \left( h_\theta(\zeta), \overline{h_\theta(\zeta)} \right) e^{-i\theta_2} \right) \frac{d\zeta}{\zeta^3}.$$  

It follows that $d_{f^0} \psi(h')$ is surjective if and only if $|I_1(\theta)|^2 \neq |I_2(\theta)|^2$. Computing explicitly, we get

$$I_1(\theta) = - \sum_{j_1+j_2=\ell, k_1+k_2=d-\ell} \ell \begin{pmatrix} d-1 \\ d-1-\ell \end{pmatrix} \alpha_{JK} e^{i(j_1-k_1)\theta_1} e^{i(j_2-k_2)\theta_2},$$

$$I_2(\theta) = \sum_{j_1+j_2=\ell, k_1+k_2=d-\ell} (d-\ell) \begin{pmatrix} d-1 \\ d-3-\ell \end{pmatrix} \alpha_{JK} e^{i(j_1-k_1)\theta_1} e^{i(j_2-k_2)\theta_2}.$$  

Given a trigonometric polynomial $R(\theta) = \sum_{j,k} \gamma_{jk} e^{ij\theta_1} e^{ik\theta_2}$ and $m \in \mathbb{Z}$, we call part of weight $m$ of $R$ the sum $R(\theta) = \sum_{j+k=m} \gamma_{jk} e^{ij\theta_1} e^{ik\theta_2}$. The part of weight $4k_0 - 2d$ (the maximal one) in $|I_1(\theta)|^2$ is then given by

$$k_0(d-k_0) \begin{pmatrix} d-1 \\ k_0-1 \end{pmatrix} \begin{pmatrix} d-1 \\ d-1-k_0 \end{pmatrix} \Sigma(\theta)^2,$$

where

$$\Sigma(\theta) = \sum_{j_1+j_2=k_0} \alpha_{JK} e^{i(j_1-k_1)\theta_1} e^{i(j_2-k_2)\theta_2}.$$
An analogous computation shows that the $4k_0 - 2d$ part of $|I_2(\theta)|^2$ vanishes if $d - 3 < k_0$ and is given by

$$k_0(d - k_0)\left(\frac{d - 1}{k_0 - 3}\right)\left(\frac{d - 1}{d - 3 - k_0}\right)\Sigma(\theta)^2$$

otherwise. Since $\left(\frac{d - 1}{k_0 - 1}\right)\left(\frac{d - 1}{d - 1 - k_0}\right) < \left(\frac{d - 1}{k_0 - 3}\right)\left(\frac{d - 1}{d - 3 - k_0}\right)$, for the identity $|I_1(\theta)|^2 \equiv |I_2(\theta)|^2$ to hold is thus necessary that $\Sigma(\theta) \equiv 0$. On the other hand, in the trigonometric polynomial $\Sigma(\theta)$ the coefficient of $e^{ik_0\theta_1}e^{i(d-k_0)\theta_2}$ is given by $\alpha_{\bar{J}\bar{K}}$ with $\bar{J} = (k_0, 0), \bar{K} = (0, d - k_0)$: since we are assuming $\alpha_{\bar{J}\bar{K}} \neq 0$, it follows that $|I_1(\theta)| \not\equiv |I_2(\theta)|$ for values of $\theta$ arbitrarily close to $(0, 0)$.

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