Mapping of relativistic Green’s functions under extended point canonical transformations

A. D. Alhaidari
Physics Department, King Fahd University of Petroleum & Minerals, Box 5047, Dhahran 31261, Saudi Arabia
E-mail: haidari@mailaps.org

Given a relativistic two-point Green’s function for a spinor system with spherical symmetry we show how to obtain another in the same class by extended point canonical transformations (XPCT).

I. THEORY

Let \( G(\vec{r},\vec{r}',E) \) be the 4×4 Green’s function for a Dirac spinor with relativistic energy \( E \). Moreover, let the particle be coupled to the 4-component spherically symmetric potential \([\tilde{A},\tilde{A}] = [V(r),iW(r)]\) in a non-minimal way as shown in [1,2]. Then, in the atomic units \((m=\hbar=1)\), \( G(\vec{r},\vec{r}',E) \) satisfies the following Dirac equation:

\[
\begin{pmatrix}
1-E+\alpha^2V & -i\alpha\tilde{\sigma}\cdot\vec{\nabla}+i\alpha\hat{r}\cdot\vec{\sigma}W \\
-i\alpha\hat{\sigma}\cdot\vec{\nabla}-i\alpha\hat{r}\cdot\vec{\sigma}W & -1-E+\alpha^2V
\end{pmatrix} G(\vec{r},\vec{r}',E) = \delta(\vec{r}-\vec{r}')II
\]

where \( \alpha \) is the fine structure parameter, \( \vec{\sigma} \) are the three 2×2 Pauli spin matrices, \( II \) is the 4×4 unit matrix and \( \hat{r} = \vec{r}/r \). Spherical symmetry reduces Eq. (1) to the following 2×2 equation for the radial component \( g_\kappa(r,r',E) \) of \( G(\vec{r},\vec{r}',E) \):

\[
\begin{pmatrix}
1-E+\alpha^2V(r) & \alpha\left(\frac{\kappa}{r}+W(r)-\frac{d}{dr}\right) \\
\alpha\left(\frac{\kappa}{r}+W(r)+\frac{d}{dr}\right) & -1-E+\alpha^2V(r)
\end{pmatrix} g_\kappa(r,r',E) = \delta(r-r')I
\]

where \( \kappa \) is the spin-orbit coupling parameter defined as \( \kappa = \pm (j+\frac{1}{2}) \) for \( l = j \pm \frac{1}{2} \) and \( I \) is the 2×2 unit matrix. Eq. (2) can be transformed into its “canonical” form [1-3] using the unitary transformation \( U = \exp(\rho \sigma_3/2) \), where \( \rho \) is a real angular parameter, together with the following constraint relating the even and odd components of the relativistic potential:

\[
SV(r) + \kappa = \frac{\sin(\rho)}{\alpha}
\]

The resulting canonical equation for the radial Green’s function reads

\[
\begin{pmatrix}
C-E+2\alpha S \left( W + \frac{\kappa}{r} \right) & \alpha \left( -\frac{S}{\alpha} + CW + \frac{C\kappa}{r} \frac{d}{dr} \right) \\
\alpha \left( -\frac{S}{\alpha} + CW + \frac{C\kappa}{r} + \frac{d}{dr} \right) & -C-E
\end{pmatrix} G_\kappa(r,r',E) = \delta(r-r')I
\]

where \( S \equiv \sin(\rho) \), \( C \equiv \cos(\rho) \) and \( G_\kappa = U_g U^\dagger \).
Now, we are interested in answering the following question [4]:

**Assuming that $G_c$ is too "difficult" to compute numerically while $\hat{G}_c$, which describes another system that belongs to the same class, is not. How can we map $\hat{G}_c$ into $G_c$ using the extended point canonical transformations (XPCT) defined in [3] while treating $\hat{G}_c$ as a “black box”?**

Since $G_c$ belongs to the same class as $\hat{G}_c$ then, following the formalism developed in [3], we can write $G_c = R\hat{G}_c \hat{R}^{-1}$. Explicitly, this reads,

$$G_c(r, r', E) = R(x)\hat{G}_c(x, x', \hat{E})\hat{R}^{-1}(x')$$

where $r = q(x), R = \begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix}$ and $\hat{R} = \sigma_2 R \sigma_2 = \begin{pmatrix} h & 0 \\ 0 & g \end{pmatrix}$.

$q(x), g(x)$ and $h(x)$ are the transformation functions defined in [3]. Writing Eq. (4) as $(H - E)G_c(r, r', E) = \delta(r - r')I$ and following the steps in section III of reference [3] we can manipulate this equation as follows:

$$(H - E)G_c = R\left[R^{-1}(H - E)R\right]\hat{G}_c \hat{R}^{-1} = \delta(r - r')I$$

$$= RD^{-1}(\hat{H} - \hat{E})\hat{G}_c \hat{R}^{-1} = \delta(r - r')I$$

where the $2 \times 2$ nonsingular matrix $D$ is defined in [3] as

$$D = \begin{pmatrix} q'g/h & 0 \\ 0 & q'h/g \end{pmatrix}$$

and $q' = dq/dx$. Thus, we can write Eq. (6) as follows

$$(\hat{H} - \hat{E})\hat{G}_c(x, x', \hat{E}) = \delta(q(x) - q'(x'))D(x)R^{-1}(x)\hat{R}(x')$$

Now,

$$D(x)R^{-1}(x)\hat{R}(x') = q\begin{pmatrix} h(x')/h(x) & 0 \\ 0 & g(x')/g(x) \end{pmatrix}$$

and,

$$\delta(q(x) - q'(x')) = \frac{1}{dq/dx} \delta(x - x')$$

Hence, Eq. (7) can finally be written as

$$(\hat{H} - \hat{E})\hat{G}_c(x, x', \hat{E}) = \delta(x - x')I$$

Therefore, the XPCT given by Eq. (5) maps $\hat{G}_c$ into $G_c$, which is the Green’s function for another system in the same class. Stated as an answer to the posed question: “given $\hat{G}_c$ which is not too difficult to compute then we choose an XPCT (i.e., $q(x)$) that maps it into $G_c$ of the target problem, which is in the same class but difficult to compute”. It is to be noted that an XPCT that can map two arbitrary Green’s functions into each other may not always exist. A transformation is an XPCT if it keeps the canonical form of the relativistic wave equation invariant. This is equivalent to the statement that $G_c$ and $\hat{G}_c$ belong to the same class.
Aside from the transformation function \( q(x) \), we need the parameters of the reference problem (i.e., \( \hat{\kappa}, \hat{E}, \hat{S} \) and the parameters of the potential \( \hat{W} \)) to carry out the calculation of \( \mathcal{G}_\kappa(r,r',E) \) in Eq. (5). These parameters are to be evaluated as a function of the target problem parameters (\( \kappa, E, S \) and the parameters of the potential \( W \)). Eq. (3.3) and Eq. (3.4) in reference [3] are used to achieve that as demonstrated in the Example below. The remaining objects needed to complete the calculation in Eq. (5) are as follows:

\[
\begin{align*}
g(x) &= \sqrt{dq/dx} \\
h(x) &= \frac{\xi}{g(x)} \\
\xi &= (\hat{E} + \hat{C})/(E + C) \\
\hat{C} &= \sqrt{1 - \hat{S}^2}
\end{align*}
\]  

(9)

II. EXAMPLE

Let \( \hat{\mathcal{G}}_\kappa \) refer to the Dirac-Oscillator problem [3,5] and \( \mathcal{G}_\kappa \) to the Dirac-Coulomb [3,6]. Then, we have the following reference quantities: \( \hat{S} = 0 \), \( \hat{C} = 1 \), \( \hat{V}(x) = 0 \), \( \hat{W}(x) = \lambda^2 x \), and \( q(x) = x^2 \), where \( \lambda \) is the oscillator strength parameter. Eq. (3.3) in [3] gives:

\[
W(r) = 0, \quad \hat{V}(r) = \frac{S\kappa/\alpha}{r} = \frac{Z}{r}, \quad \text{and} \quad \hat{\kappa} = 2C\kappa + \frac{1}{2}
\]  

(10)

resulting in \( S = aZ/\kappa \) and \( C = \sqrt{1-(aZ/\kappa)^2} \). Putting all of that in equation (3.4) of [3], we finally obtain

\[
\hat{E} = \sqrt{1 + 4\alpha^2 \left( \gamma \lambda^2 - 2ZE \right)} \\
\lambda^2 = \frac{2}{\alpha} \sqrt{1 - E^2}
\]  

(11)

where \( \gamma = C\kappa = \sqrt{\kappa^2 - (aZ)^2} \) is the relativistic angular momentum. Therefore, we have all the tools necessary to perform the calculation in Eq. (5) and obtain \( \mathcal{G}_\kappa(r,r',E) \).

Specifically, one starts by deciding on the set of numbers \( \kappa, E, Z, r \) and \( r' \) for which \( \mathcal{G}_\kappa \) will be computed. Then Eq. (10) and Eq. (11) together with \( x = \sqrt{r} \) will be used to give the corresponding set of numbers (\( \hat{\kappa}, \hat{E}, \lambda, x, x' \)) at which \( \hat{\mathcal{G}}_\kappa \) will be evaluated. Note that an energy dependent expression will substitute for the parameter \( \lambda \) in \( \hat{\mathcal{G}}_\kappa \), thus one needs to know \( \hat{\mathcal{G}}_\kappa \) for all \( \lambda \). Moreover, the values of \( \hat{\kappa} \) obtained by Eq. (10) are not likely to be in the form of integers \( \pm 1, \pm 2, \ldots \) etc. This has also to be observed in the construction of \( \hat{\mathcal{G}}_\kappa \). Finally, the objects in Eq. (9) will be used to complete the evaluation of \( \mathcal{G}_\kappa(r,r',E) \) in Eq. (5).
REFERENCES

[1] A. D. Alhaidari, Phys. Rev. Lett. 87, 210405 (2001); 88, 189901 (2002)
[2] A. D. Alhaidari, J. Phys. A 34, 9827 (2001); 35, 3143 (2002)
[3] A. D. Alhaidari, Phys. Rev. A 65, 042109 (2002); 66, 019901 (2002)
[4] Problem posed by E. O. Le Bigot in a private communication, June 2002
[5] M. Moshinsky and A. Szczepaniak, J. Phys. A 22, L817 (1989)
[6] M. K. F. Wong and H-Y Yeh, Phys. Rev. D 25, 3396 (1982); J-Y Su, Phys. Rev. A 32, 3251 (1985); M. K. F. Wong, Phys. Rev. A 34, 1559 (1986); B. Goodman and S. R. Ignjatovic, Am. J. Phys. 65, 214 (1997)