Lessons for Loop Quantum Gravity 
from Parametrised Field Theory

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Abstract

In a series of seminal papers, Laddha and Varadarajan have developed in depth the quantisation of Parametrised Field Theory (PFT) in the kind of discontinuous representations that are employed in Loop Quantum Gravity (LQG). In one spatial dimension (circle) PFT is very similar to the closed bosonic string and the constraint algebra is isomorphic to two mutually commuting Witt algebras. Its quantisation is therefore straightforward in LQG like representations which by design lead to non anomalous, unitary, albeit discontinuous representations of the spatial diffeomorphism group. In particular, the complete set of (distributional) solutions to the quantum constraints, a preferred and complete algebra of Dirac observables and the associated physical inner product has been constructed.

On the other hand, the two copies of Witt algebras are classically isomorphic to the Dirac or hypersurface deformation algebra of General Relativity (although without structure functions). The question we address in this paper, also raised by Laddha and Varadarajan in their most recent paper, is whether we can quantise the Dirac algebra in such a way that its space of distributional solutions coincides with the one just described. This potentially teaches us something about LQG where a classically equivalent formulation of the Dirac algebra in terms of spatial diffeomorphism Lie algebras is not at our disposal.

We find that, in order to achieve this, the Hamiltonian constraint has to be quantised by methods that extend those previously considered. The amount of quantisation ambiguities is somewhat reduced but not eliminated. We also show that the algebra of Hamiltonian constraints closes in a precise sense, with soft anomalies, that is, anomalies that do not cause inconsistencies. We elaborate on the relevance of these findings for full LQG.

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1 Introduction

Undoubtedly the major unresolved challenge in LQG \cite{1,2} is to find a proper implementation of the quantum dynamics. While there is a large degree of control as far as the spatial diffeomorphism constraint is concerned \cite{3}, the appropriate quantisation of the Hamiltonian constraint remains the hardest research problem to be solved. The ideal wish list comprises: 1. quantisation without anomalies, 2. faithful representation of the Dirac or hypersurface deformation algebra, 3. sufficient control on the classical limit, 4. sufficient control on the space of (distributional) solutions and the corresponding physical inner product and 5. lack of quantisation ambiguities. So far only partial fulfillment of this wishlist could be achieved. In \cite{4} an anomaly free quantisation of the Hamiltonian constraint was proposed but the remaining issues could not be addressed. By substituting the infinite number of Hamiltonian constraints by the single Master constraint \cite{5} one cancels items 1. and 2. and makes progress on 4. In particular, if one quantises it as a spatially diffeomorphism invariant operator on the unique \cite{6} (kinematical) Hilbert space of LQG selected by covariance with respect to the spatial diffeomorphism group, then one can also make make progress\cite{7} on 3.\cite{7}.

However, despite of this, one may feel uneasy about the current version of the Master Constraint which is basically the weighted squared integral of all the Hamiltonian constraints, because it is possible to take anomalous Hamiltonian constraints and still end up with a well defined operator with good semiclassical behaviour. The anomalies express themselves in the fact the spectrum of the positive Master Constraint Operator has a gap. Its space of solutions is therefore empty unless one substracts the gap by hand (which is finite and proportional to $\hbar$). While this is consistent with taking the semiclassical limit $\hbar \rightarrow 0$ and actually works in several non trivial examples \cite{10} a better quantisation of the Hamiltonian constraints fulfilling items 1. and 2. is certainly desirable. Furthermore, since many quantisations have the same semiclassical limit, fulfilling 1. and 2. could automatically reduce the amount of quantisation ambiguities and thus might imply progress on 5.

Several extensions of the quantisation proposed in \cite{4} have been discussed. All of these ambiguities arise because the unique Hilbert space representation is discontinuous so that the connection has to be approximated by a holonomy along some loop. The choice of that loop and the representation that one takes the holonomy of label the space of ambiguities\cite{2}, see e.g. in \cite{11,2}. For none of them, it is obvious that property 2. is (dis)satisfied. For in order to check it, one would have to compute the commutator between two Hamiltonian constraints on the kinematical Hilbert space and to decide whether the resulting object is a quantisation of the right hand side of the corresponding classical Poisson bracket. This turns out to be very difficult for three independent reasons: A. The classical right hand side involves an infinitesimal spatial diffeomorphism constraint, whose quantum analog does not exist. B. In order to avoid the anomaly, those operators are chosen as graph changing\cite{3} but to date no semiclassical states have been constructed with respect to which graph changing operators have a good semiclassical limit. C. If one recalls the algebraic manipulations that one has to perform in the classical calculation of the Poisson brackets, then it is clear that one can repeat them quantum mechanically only semiclassically, however, no semiclassical states are available as already mentioned.

It is therefore very difficult to decide whether any of the operators proposed leads us into the right direction. It is at this point where input from non trivial toy models, that allow for a complete solution, may guide us further and one such model is parametrised field theory (PFT) in two spacetime dimensions. In general, PFT is a free field theory involving one or more scalar fields $\phi$ on a flat, fixed background

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1 In \cite{7} an algebraic version of LQG was studied. However, the results obtained there are also valid for standard LQG with minor modifications because a spatially diffeomorphism invariant operator must not be graph changing which essentially leads back to the calculation performed in \cite{7}. The only difference is that in \cite{7} subgraphs of the infinite graph considered may change while in LQG this is not allowed. However, these processes are semiclassically irrelevant if one employs the coherent states defined in \cite{8,9}, see \cite{19} for more details.

2 The associated parameter space is discrete because only the diffeomorphism equivalence class of the loop is important and for three valent vertices there are no $\theta$ moduli, see \cite{1}.

3 That is, the loop in question is never contained in the graph considered.
spacetime \((M, \eta)\) of Minkowski signature and of any dimension which one makes diffeomorphism invariant by pulling back the scalar field and the background metric by an arbitrary spacetime diffeomorphism \(X\). The resulting action now also depends on \(X\) as well and reduces in the gauge \(X = \text{id}\) to the original action in adapted (Cartesian) coordinates. PFT is therefore a diffeomorphism invariant theory much like GR and thus serves as an interesting testing ground for the many technical and conceptual issues of full fledged Quantum Gravity as has been stressed and worked out in a series of seminal papers by Kuchař [12]. In particular, since we know a bona fide quantisation of the model in the gauge \(X = \text{id}\) (Fock representation) it appears to be a trivially solvable theory. Surprisingly, things are not that trivial as pointed out by Torre and Varadarajan in [13]: The Fock representations for flat and curved embeddings of spatial slices in \(M\) for \(\text{dim}(M) > 2\) are in general unitarily inequivalent. In other words, different, classically perfectly equivalent gauges lead to unitarily inequivalent QFT’s! Beautifully, by treating the embedding variables as dynamical fields and applying to them LQG like representations while Fock representations for the scalar fields are kept, this obstruction can be overcome [13]. Moreover, the quantum constraint reduction leads to a theory unitarily equivalent to the usual Fock representation in the gauge \(X = \text{id}\).

A natural question is therefore to ask, what kind of QFT would result if one did not fix a gauge but would rather treat the system à la Dirac as a diffeomorphism invariant theory which in the canonical framework thus leads to spatial diffeomorphism and Hamiltonian constraints. Obviously, given the full arsenal of techniques that have been developed for LQG, it is natural to apply LQG methods to quantise the whole system (i.e embedding variables and scalar field) which in turn is the reason for why it is an interesting model for LQG because we know in principle the full solution. As expected from the purely algebraic (or geometric, i.e. action independent) proof in [16], the Poisson algebra of the constraints is the Dirac or hypersurface deformation algebra. Despite the fact that the gauge fixed theory is free, the unfixed theory is interacting and the Dirac algebra closes with non trivial structure functions only unless we are in two spacetime dimensions. The case \(D = 2\) therefore leads to a further simplification, namely the hypersurface deformation algebra is a true (albeit infinite dimensional) Lie algebra, a fact that is being exploited by a close relative of 2D PFT, namely the bosonic string [17].

In particular, if \(M \cong \mathbb{R} \times S^1\) just as in closed string theory, it is possible to switch from the Dirac algebra to a classically equivalent Lie algebra which is simply the direct sum of two spatial diffeomorphism algebras for \(S^1\). This fact and the fact that the LQG representation by design is well adapted to spatial diffeomorphisms asked for a quantisation of the closed bosonic string by LQG methods [13]. Similarly, in 2D PFT one may exploit this fact and completely solve the theory. This has been done in great detail in impressive works by Laddha and Varadarajan [15]. However, in full LQG in 4D this “trick” is not at our disposal and thus in order to serve as a true testing ground for LQG one should not solve 2D PFT using it but using the original Dirac algebra. Yet, in contrast to 4D LQG, we know in 2D PFT what the answer must be and thus 2D PFT may serve as a guideline for how to faithfully represent the 4D Dirac algebra in the LQG representation.

This presents a real challenge: **If we do not manage to quantise the Dirac algebra for 2D PFT as to yield the known and correct result as given to us by the miracle that happens in 2D, then how can we hope for the correct quantisation of the Dirac algebra of 4D LQG which is much more complicated and involves non trivial structure functions?**

This is the basic question that we analyse in this paper: **Is it possible to find a quantisation of the spatial diffeomorphism and Hamiltonian constraints respectively for closed 2D PFT using LQG techniques such that they (rather their algebraic dual) annihilate the solutions to the two, classically equivalent, spatial diffeomorphism constraints?** This important question has been formulated for the first time in the papers [15] where partial answers were announced. Fortunately, the answer is affirmative. Surprisingly, however, as also has been announced in [15], the quan-

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4 One may be puzzled by the fact that PFT is at the same time diffeomorphism invariant and background dependent. This happens because the metric is here not considered as a dynamical field.
tisation of the Dirac algebra is non-trivial in the sense that it uses techniques so far not considered in \cite{4} and their relatives. This strengthens the suspicion that the techniques of \cite{4} should be generalised.

The architecture of this paper is as follows:

In section 2 we review closed 2D PFT following closely \cite{15}. Our treatment will be much less complete than \cite{15} and we will simplify the discussion where possible. We urge the careful reader to refer to \cite{15} for all the missing details. Notice, however, that we consider a quantisation slightly different from the one employed in \cite{15} which is technically somewhat simpler and does not qualitatively affect the main topic of the present article.

In section 3, which contains the main result of our work, we find suitable quantisations of the spatial diffeomorphism and Hamiltonian constraints which annihilate the space of solutions to the classically equivalent two copies of spatial diffeomorphism constraints. Here we follow to some extent the same route as in 4D LQG \cite{4}, in particular we consider density one valued operator valued distributions, as these are the only ones that have a chance to be quantised in LQG like representations as was shown in \cite{4}. Their classical expression (using as in LQG the volume) was already sketched in \cite{15}. Their constraint algebra closes by inspection and we show that there is a precise correspondence between the classical hypersurface algebra and the quantum version, including a soft anomaly which however does not render the quantisation inconsistent. In particular, the corresponding operators have the same kernel as given in \cite{15}. As announced in \cite{15}, in order to achieve this, new regularisation techniques have to be introduced. We find in addition that also a non-trivial renormalisation has to be performed.

Finally, in section 4 we discuss the possible implications for 4D LQG. One of the most important ones is that in order to match the kernels of the hypersurface algebra and the direct sum of the Witt algebras, it was crucial that one did know about the reformulation in terms of Witt algebras, because this fact motivates to quantise a different holonomy flux like algebra than one would consider natural from the point of view of the hypersurface deformation algebra. As this different kinematical algebra and the usual one are represented discontinuously in the quantum theory, it is not possible to represent the usual kinematical algebra in the Hilbert space adapted to the direct sum of Witt algebras. This observation touches on the very starting point of LQG: If similarly in LQG one should work with a kinematical algebra that is perfectly adapted to the quantum dynamics, then one has to completely reformulate LQG! There is no evidence for the emergence for such a more adapted algebra at the moment and even if there was, the technical tools developed for LQG would presumably easily transferrable to the new situation. This is also the case for 2D PFT as we will see.

We have banned some involved calculations concerning the constraint algebra to an appendix.

Finally, as communicated to the present author, in a completely independent research carried out by Laddha and Varadarajan, the authors have obtained in part similar results. Their work will be published shortly.

2 Review of Parametrised Field Theory

In this section we collect all the formulae that we need for our limited purpose. See \cite{15} for all the details. We separate the briefing into classical and quantum theory. Readers who are familiar with \cite{15} can safely skip this section and move on directly to section 3 except for our slightly different choice of representation in section 2.2.

2.1 Classical Theory

We consider the differentiable manifold $M = \mathbb{R} \times S^1$ together with the flat Minkowski metric $\eta = \text{diag}(-1, 1)$. In order to set up the 1+1 formalism we consider arbitrary foliations $X$ of $M$, i.e. one parameter families of embeddings of the circle into $M$. Let $x^0 := t, x^1 := x$ be standard time and angular variables on $\mathbb{R}$ and
respectively. Here $t$ labels the leaves of the foliation. Then
\[ X : \ M \to M ; \ (t,x) \mapsto (T(t,x),X(t,x)) \] (2.1)
defines a diffeomorphism (reparametrisation). We write $X^0 = T$, $X^1 = X$. By means of $X^A$, $A = 0,1$ we can pull back the flat metric to obtain
\[ g = X^* \eta; \ g_{\alpha\beta}(t,x) := \eta_{AB} X^A_\alpha(t,x) X^B_\beta(t,x) \] (2.2)
Given a scalar field $\phi : \ M \to \mathbb{R}$ we may also pull it back by $X^A$ to obtain
\[ \Phi = X^* \phi; \ \Phi(t,x) := \phi(X(t,x)) \] (2.3)
Consider the free, massless scalar field action on the cylinder
\[ S[\phi] = -\frac{1}{2} \int_M d^2x \eta^{AB} \phi_\alpha \phi_\beta \] (2.4)
and the Parametrised Field Theory (PFT) action on the cylinder
\[ S_{\text{PFT}}[T,X,\Phi] = -\frac{1}{2} \int_M d^2x \sqrt{|\det(g)|} \ g^{\alpha\beta} \Phi_\alpha \Phi_\beta \] (2.5)
It is easy to see that (2.4) and (2.5) coincide. However, (2.5) is reparametrisation invariant and thus is an example for a diffeomorphism invariant field theory although it depends on the background $\eta$. At the level of the Euler Lagrange equations one may check that the field equations for $T, X$ are satisfied once those for $\phi$ are, hence $T, X$ are gauge degrees of freedom by construction.

Notice that by assumption the leaves of the foliation are embedded circles $\Sigma_t = X(t, S^1)$ and as such $T$ is a periodic function of $x$ at fixed $t$ while $X$ is periodic modulo $2\pi R$ where $R$ is the Radius of the cylinder. We also take $x$ to be periodic modulo $2\pi$. That is to say, $X$ and $x$ are just angle variables on the circle.

The passage to the canonical formulation is straightforward and will not repeated here in much detail, see e.g. [15]. One defines the momenta conjugate to $T, X, \Phi$ by the functional derivatives
\[ P_T(t,x) := \frac{\delta S_{\text{PFT}}}{\delta \dot{T}(t,x)}, \ P_X(t,x) := \frac{\delta S_{\text{PFT}}}{\delta \dot{X}(t,x)}, \ \Pi(t,x) := \frac{\delta S_{\text{PFT}}}{\delta \dot{\Phi}(t,x)} \] (2.6)
where a dot denotes a partial derivative with respect to $t = x^0$ and discovers that the resulting phase space is subject to the following constraints
\begin{align*}
D & := P_T T' + P_X X' + \Pi \Phi' \\
C & := P_T X' + P_X T' + \frac{1}{2}(\Pi^2 + [\Phi']^2) \tag{2.7}
\end{align*}
where a prime denotes a partial derivative with respect to $x = x^1$. These constraints are primary, that is, the Legendre transform is singular and only allows to solve for $\dot{\Phi}$ but not for $\dot{T}, \dot{X}$ in terms of the momenta.

From the explicit expressions for $P_T$, $P_X, \Pi$ one immediately sees that they are periodic functions of $x^1$ as they depend only on $X'$. Let us smear the constraints $D, C$ with periodic test functions $f : S^1 \to \mathbb{R}; \ x \mapsto f(x)$. We write for instance
\[ C[f] := \int_{S^1} dx \ f(x) \ C(x) \] (2.8)
etc. and we also define the following bracket
\[ [f,g] := f' g - f \ g' \] (2.9)
Then one readily computes the \textit{Hypersurface Deformation Algebra} \( \mathcal{H} \)

\[
\{D[f], D[g]\} = D[[f, g]] \\
\{D[f], C[g]\} = C[[f, g]] \\
\{C[f], C[g]\} = D[[f, g]]
\]

(2.10)
familiar from the ADM formulation of GR. In performing those computations we used the periodicity
of fields so that boundary terms can be dropped. The interpretation of \( D, C \) respectively is therefore that of
a diffeomorphism and Hamiltonian constraint respectively.

One would expect the right hand side of the last line to depend on the inverse of the 1D metric

\[
q := g_{xx} = -[T']^2 + [X']^2
\]

(2.11)
However, the peculiarity of 1D is that a one form and in particular the derivative of a scalar field such as
\( T, X, \Phi \) is the same thing as a density of weight one and that a vector field is the same thing as a density
of weight -1. Thus both \( C, D \) are densities of weight 2 while the smearing test functions are densities of
weight -1 in order that (2.8) is meaningful. Then (2.9) is nothing else than minus the Lie bracket between
vector fields and (2.10) makes sense as it stands. Put differently, the 1D metric (2.11) is a scalar density of
weight 2 and its inverse would be of weight -2. Since the integrand of the right hand side of the last line
of (2.10) must have overall density weight +1 one should densitise \( q^{-1} \) and multiply by \( \sqrt{\text{det}(q)} \) which
gives unity.

The fact that \( \mathcal{H} \) is a true Lie algebra without structure functions is a major simplification that happens
only in 2D. For instance, in a quantisation of the constrained system \( \alpha \) la Dirac one could consider group
averaging methods in order to solve the constraints and define a physical inner product. Due to the structure
functions, this is not possible in higher dimensions. However, to apply group averaging techniques directly
to the system (2.7) is not entirely straightforward. By means of the following canonical transformation

\[
X_{\pm} := T \pm X, \quad P_{\pm} := \frac{1}{2}(P_T \pm P_X)
\]

(2.12)
and the definition

\[
Y_{\pm} := \Pi \pm \Phi'
\]

(2.13)
one readily computes

\[
D_{\pm} := \frac{1}{2}(D \pm C) = P_{\pm} X'_{\pm} \mp \frac{1}{4}[Y']^2
\]

(2.14)
The equivalent constraints (2.14) obey the much simpler \textit{Diffeomorphism Algebra} \( \mathcal{D} \)

\[
\{D_{\pm}[f], D_{\pm}[g]\} = D_{\pm}[[f, g]] \\
\{D_{\pm}[f], D_{\mp}[g]\} = 0
\]

(2.15)
and thus generate the direct sum of two \( \text{diff}(S^1) \) Lie algebras (Witt algebras). All of this is of course well
known from string theory and is generic to diffeomorphism invariant 2D field theories.

The Hamiltonian flow of the Hamiltonian vector fields of \( D_{\pm}[f] \) generate automorphisms on the phase
space (canonical transformations) which are just the spatial diffeomorphisms \( \varphi^{f_{\pm}} \) generated by the vector
field \( f_{\pm} \) on \( S^1 \). Here \( \varphi^{f_{\pm}} \) acts by pull back on the \( \pm \) sector of the theory and leaves invariant the \( \mp \) sector.
Of course, \( P_{\pm}, Y_{\pm} \) are densities of weight one while \( X_{\pm} \) is a scalar under \( \varphi_{\pm} \). Since \( D_{\pm} \) mutually commute
the flow \( \alpha_{\varphi^{f_+}, \varphi^{f_-}} \) of \( D_+[f^+] + D_-[f^-] \) results by concatenation of the actions just described in either order.

To construct gauge invariant (Dirac) observables, we notice that \( P_{\pm} \) can be eliminated via the constraints
while \( X_{\pm} \) are pure gauge, hence the true degrees of freedom can be identified with \( Y_{\pm} \). Therefore we can
proceed as in [19] and consider the gauge fixing conditions \( X_{\pm} - \sigma_{\pm} = 0 \) corresponding to \( D_{\pm} \) and compute
the gauge invariant extension of the scalar \( Y_{\pm}/X_{\pm} \) off the gauge cut \( X_{\pm} - \sigma_{\pm} \). The result is

\[
O_{Y_{\pm}}(\sigma_{\pm}) = \left[ \frac{Y_{\pm}}{X_{\pm}}(x) \right]_{X_{\pm}(x) = \sigma_{\pm}} = \int dx \ Y_{\pm} \delta(X_{\pm} - \sigma_{\pm})
\]

(2.16)
where the $\delta$ distribution is periodic modulo $2\pi R$. Alternatively we can integrate (2.16) against the Fourier modes $\exp(in\sigma_\pm/R)$ to arrive at the Fourier coefficients

$$O_{Y_\pm,n} = \int_{S^1} dx Y_\pm e^{inX_\pm/R}$$

(2.17)

with $n \in \mathbb{Z}$ also considered in [15].

2.2 Quantum Theory

The point of recalling all of these well known facts is that (2.10) or (2.15) bring us into a situation very close to LQG in 4D. We have a constrained Hamiltonian system part of whose constraint algebra generates spatial diffeomorphisms. Therefore one naturally can apply LQG quantisation techniques and one would first of all consider a kinematical Hilbert space representation of the Weyl algebra determined by the phase space with respect to which the spatial diffeomorphism group is implemented unitarily and without anomalies similar to [6]. Then one can apply group averaging techniques in order to solve the spatial diffeomorphism constraints and construct a Hilbert space of spatially diffeomorphism invariant states. With respect to the system (2.10) one would then still be left with the scalar constraints and one could try to define it as in [4]. However, given the reformulation (2.15) it is much more convenient to consider $P_\pm, X_\pm, Y_\pm$ as the elementary variables and to use the constraints (2.15) because, in a sense, we now have two commuting spatial diffeomorphism groups and we can apply the LQG methods to both of them separately. Then, after solving both diffeomorphism constraints, no scalar constraint is left and one arrives at a complete solution of the theory!

This, and much more has been done in the seminal work [15].

As already mentioned in the outlook part of [15], given this complete solution, it would now be very interesting to go back to the original system (2.10), to quantise it by following the steps of [3, 4] and to compare with the results already obtained. In particular, one would like to see whether there is a quantisation of (2.10) such that the corresponding dual operators annihilate the kernel of (2.15). This is what we will do in the next section. In the present section we just recall the elements from [15] that we need. We will, however, deviate somewhat in the precise technical implementation from [15] as we will indicate explicitly.

The classical phase space consists of the embedding sector described by the variables $(X_\pm, P_\pm)$ and the matter sector described by the variables $Y_\pm$. The embedding sector is gravity like, hence we use an LQG like representation [6] for which the $X_\pm(x)$ (“Ein-Bein”) and the $\exp(ikP_\pm[I])$ (“holonomy”) are well defined operators but not $P_\pm(x)$ itself. Here $I$ is a closed interval, $P_\pm[I] = \int_I dx P_\pm$ and $k \in k_0\mathbb{Z}$ where $\hbar k_0/R, \hbar k_0 \notin 2\pi\mathbb{Q}$ is some positive constant. The matter sector is string like, hence we choose the LQG string representation [18] for which neither $Y_\pm(x)$ exist but only the $\exp(ilY_\pm[I])$ where again $l \in l_0\mathbb{Z}$ and $l_0, l_0^2 \notin 2\pi\mathbb{Q}$ is some positive constant. These functions separate the points of the classical phase space since the intervals $I$ can be arbitrarily “small”. The restrictions on $k_0, l_0$ are motivated by trying the match the $\mathcal{D}$ and $\mathcal{H}$ quantisations. Notice that here we differ somewhat from [15]: The authors there oppositely assume that $\hbar k_0/R = 2\pi/L$ for some large positive integer and that $l_0$ is any real number without any restriction and that $l \in l_0\mathbb{Z} + \lambda$ where $\lambda \in \mathbb{R}$ may vary from charge network to charge network. This leads to certain modifications as far as the structure of the quantum observables is concerned. We will comment on this section.

Following the notation of [15] we consider graphs $\gamma$ which are arbitrary partitions of $S^1$ into disjoint open intervals $I$ (modulo the boundary points). Then we consider the “charge (spin) networks”

$$T_{\gamma,k}^\pm := \exp(i \sum_{I \in \gamma} k_I P_\pm[I])$$

(2.18)

5From $Y_\pm$ we can reconstruct $\Phi$ only up to a constant. This zero mode however decouples from the constraints [15] and will therefore not be considered in this paper.

6For simplicity we take all quantities including $\hbar$ as dimensionless in this article.
and the Weyl elements

\[ W_{\gamma,k}^\pm := \exp(i \sum_{I \in \gamma} l_I Y^I) \]  

(2.19)

The \(T_{\gamma,k}^\pm\) form an Abelian algebra where the product of two charge networks \(T_{\gamma,k}^\pm\), \(T_{\gamma',k'}^\pm\) is the charge network \(T_{\gamma'',k'',\gamma'}^\pm\) where \(\gamma''\) is the coarsest partition of \(S^1\) such that every \(I \in \gamma\), \(I' \in \gamma'\) is a union of intervals in \(\gamma''\) while

\[ k''_{I''} = \sum_{I'' \subset I} k_I + \sum_{I'' \subset I'} k'_{I'} \]  

(2.20)

Likewise, the \(W_{\gamma,k}^\pm\) form a Non–Abelian algebra where the product of two Weyl elements is similarly defined up to a phase which follows from the Poisson brackets

\[ \{Y^I_+, Y^J_+\} = \pm \int_0^{2\pi} dx \ [f, g] =: <f, g>, \ \{Y^I_+, Y^J_-\} = 0 \]  

(2.21)

for the smeared functions \(Y^I_+ = \int dx f \ Y^I_+\). Care is needed since the characteristic functions \(\chi_I\) are not smooth but rather \(\chi_I(x) = \delta(x, f_I) - \delta(x, b_I)\) where \(b_I, f_I\) denote beginning and final point of \(I\) [15]. With the usual regularisation for the integral of the \(\delta\) distribution over half of its support we obtain

\[ <\chi_I, \chi_J> = -[\kappa_I(f_I) - \kappa_J(b_I) - \kappa_I(b_J) + \kappa_J(f_I)] \]  

(2.22)

We thus obtain

\[ W_{\gamma,l}^\pm W_{\gamma',l'}^\pm = W_{\gamma'',l''}^\pm \exp(-i \frac{\hbar}{2} \sum_{I \in \gamma, I' \in \gamma'} l_I l'_I <\chi_I, \chi_I'>) \]  

(2.23)

where we used the canonical quantisation rule to replace commutators by \(i\hbar\) times the classical Poisson brackets as well as the BHC formula. The definition of the abstract *algebra \(\mathfrak{A}\) is completed by defining the commutation relations with the the \(X_\pm(x)\)

\[ [X_\pm(x), T_{\gamma,k}^\pm] = \hbar \sum_{I \in \gamma} k_I \kappa_I(x) T_{\gamma,k}^\pm \]  

(2.24)

and all other commutators are zero.

There is an important subtlety, however. Since \(X\) is an angular field, \(X_\pm\) is subject to the boundary condition \(X_\pm(x + 2\pi) = X_\pm(x) \pm 2\pi R\). Thus, as a function on the circle, it is discontinuous. One way to deal with this is to keep explicitly track of this boundary condition in the choice of the Hilbert space representation [15]. Another possibility is to consider instead the continuous \(S^1\) valued functions

\[ S_{\gamma,n}^\pm = \exp(i \sum_{v \in V(\gamma)} n_v X_\pm(v)/R) \]  

(2.25)

with \(n_v \in \mathbb{Z}\) and \(V(\gamma)\) denotes the vertices of the graph \(\gamma\). These Weyl elements still separate the points of the classical phase space, except for the zero mode of \(X\), because for instance

\[ S_{\{v_0, v_1\}, \{1, -1\}}^\pm = \exp(i[X_\pm(v_1) - X_\pm(v_0)]) = 1 + iX'_\pm(v_0)[v_1 - v_0] + O([v_1 - v_0]^2) \]  

(2.26)

allows to extract \(X'_\pm(x)\) as closely as we wish. The \(S_{\gamma,n}^\pm\) are the precise analog of the point holonomies considered for the first time in [4] as a background independent algebra for scalar fields. Despite the fact that only integer charges are considered, in contrast to [20], almost all information about \(X_\pm\) can be extracted, see also [21] for similar remarks in context of Loop Quantum Cosmology (LQC) [22]. The zero mode cannot be extracted in contrast to [15]. Since, however, the zero modes of both \(\Phi, X\) do not play any
role in the classical action and since anyway we consider the PFT only as a toy model for 4D LQG that merely serves to illustrate certain technical constructions, we feel free to do so. Notice also that the $S_{\gamma,n}$ are the only objects needed in the construction of the observables (2.17).

In this spirit, one could consider a mathematical deformation of the PFT model further and treat $X_\pm$ as periodic functions. Doing this actually is not PFT but it leads to certain technical simplifications which still bring us close to the 4D LQG situation. In what follows, we consider both possibilities A. $X_\pm$ is treated as a periodic function and B. $X_\pm$ is not periodic but angular and we consider instead the $S_{\gamma,n}$. We will see that both treatments lead to qualitatively similar results with respect to the main interest of the present article while concrete formulae will be slightly different.

In terms of the $S_{\gamma,n}$, the Heisenberg relations (2.24) are replaced by the Weyl relations

$$S_{\gamma,n}^\pm T_{\gamma,k}^\pm S_{\gamma,-n}^\pm = \exp(i \frac{\hbar}{\ell} \sum_{v \in V(\gamma), I \in \gamma'} n_v k_I \kappa_I(v)) T_{\gamma,k}^\pm$$  \hspace{1cm} (2.27)

The kinematical Hilbert space is simply\footnote{In\textsuperscript{13\textsuperscript{}} for the embedding sector a representation similar to but slightly different from option A is chosen due to the different strategy to implement the discontinuity of $X$. For the scalar field sector the representations coincide. There one also imposes the zero mode constraint $Y_+([0,2\pi]) - Y_-([0,2\pi]) = 0$ which we ignore here.}

$$\mathcal{H} = \mathcal{H}_+ \otimes \mathcal{H}_-,$$  \hspace{1cm} (2.28)

where $\mathcal{H}_\pm^E$ and $\mathcal{H}_\pm^M$ respectively are the GNS Hilbert spaces \textsuperscript{23} defined by the following states on the respective algebras

$$\omega^E_\pm(T_{\gamma,k}^\pm, X_\pm(x_1), \ldots, X_\pm(x_N)) = \delta_{N,0} \delta_{k,0} \quad \text{Possibility A}$$  \hspace{1cm} (2.29a)

$$\omega^E_\pm(T_{\gamma,k}^\pm, S_{\gamma,n}^\pm) = \delta_{n,0} \delta_{k,0} \quad \text{Possibility B}$$  \hspace{1cm} (2.29b)

That these are states (positive linear functionals) on the respective algebras follows from \textsuperscript{6, 18}. These states are $\mathcal{G} := Diff_+(S^1) \times Diff_-(S^1)$ invariant with respect to the automorphism groups on $\mathfrak{g}$ defined by the relations

$$a_{(\varphi, \varphi_-)}[X_\pm(x)] = X_\pm(\varphi(x)) \quad \text{Possibility A}$$  \hspace{1cm} (2.30a)

$$a_{(\varphi, \varphi_-)}[S_{\gamma,n}^\pm] = S_{\gamma,n}^\pm \quad \text{Possibility B}$$  \hspace{1cm} (2.30b)

$$a_{(\varphi, \varphi_-)}[T_{\gamma,k}^\pm] = T_{\varphi_{\pm}(\gamma), \kappa}^\pm$$  \hspace{1cm} (2.30c)

$$a_{(\varphi, \varphi_-)}[W_{\gamma,l}^\pm] = W_{\varphi_{\pm}(\gamma), l}^\pm$$  \hspace{1cm} (2.30d)

whence by general theorems \textsuperscript{23} there is a unitary representation of $\mathcal{G}$ on $\mathcal{H}$ defined by $U(g)\pi(a)\Omega = \pi(a)(\Omega)$ for any $a \in \mathfrak{g}$. Here

$$\Omega = \Omega_+^E \otimes \Omega_-^M \otimes \Omega_+^E \otimes \Omega_-^M, \quad \pi = \pi_+^E \otimes \pi_-^M \otimes \pi_+^E \otimes \pi_-^M$$  \hspace{1cm} (2.31)

are defined via the GNS data $(\mathcal{H}_\pm^E, \pi^E_\pm, \Omega^E_\pm)$ and $(\mathcal{H}_\pm^M, \pi^M_\pm, \Omega^M_\pm)$ induced by $\omega^E_\pm$ and $\omega^M_\pm$ respectively. One may easily check that the vector states

$$|\gamma; k_+, l_+, K_-, l_- > := \pi^E_{\gamma}(T_{\gamma,k_+}) \otimes \pi^M_{\gamma}(W_{\gamma,+}^+) \otimes \pi^E_{\gamma}(T_{\gamma,k_-}) \otimes \pi^M_{\gamma}(W_{\gamma,-}^-) \quad \Omega$$  \hspace{1cm} (2.32)
that the representation $U$ is not strongly continuous. The solution to the quantum constraints are now linear functionals $l$ defined on the dense subspace $\pi(\mathfrak{A})\Omega$ satisfying the constraint equations

$$l[U(g)\pi(a)\Omega] = l[\pi(a)\Omega] \quad \forall \ a \in \mathfrak{A}, \ g \in \mathfrak{G}$$

(2.33)

that is, functionals invariant under both copies of the diffeomorphism group of $S^1$ (the loop group $S^1 \to S^1$). These solutions and the associated physical inner product as well as the action of the (exponentiated) observables (2.16), (2.17) can be obtained explicitly using the group averaging techniques introduced in \cite{3} and are exhibited in great detail in \cite{15}. In particular one finds, due to the non compactness of $\mathfrak{G}$ (in the discrete topology) and due to the different gauge orbit size of different elements $a \in \mathfrak{A}$ the same phenomenon as in LQG, namely that the physical inner product suffers from averaging ambiguities labelled by diffeomorphism equivalence classes $[\gamma]$ of graphs $\gamma$. The associated subspaces are orthogonal and are superselected by the algebra of Dirac observables.

We will not need these results for what follows. For us the relation (2.33) will be sufficient to check whether quantisations of the algebra $\mathfrak{H}$ exist which annihilate the kernel of the algebra $\mathfrak{D}$ defined by (2.33).

### 3 Quantisation of the Hypersurface Deformation Algebra

This section contains the main result of the present work. The strategy will be as follows:

From the point of view of the algebra $\mathfrak{H}$ there is no motivation to introduce the variables $X_\pm, P_\pm, Y_\pm$ and it would be more natural to consider an LQG like kinematical HS based on 1. “fluxes” $T(x), X(x), \Phi(x)$ and 2. “holonomies” of $P_T, P_X, \Pi$. However, if we did that then we would actually consider discontinuous representations of two different $\ast$-algebras and therefore operators that exist in one representation do not exist in the other already at the kinematical level. Thus, in order to compare the quantisations of $\mathfrak{D}$ and $\mathfrak{H}$ we should keep a common kinematical representation and this should be the one that we reviewed in the previous section because it is well adapted to $\mathfrak{D}$ which in turn allows us to arrive at a complete solution.

Therefore we should also write the spatial diffeomorphism and Hamiltonian constraint in terms of the variables $X_\pm, P_\pm, Y_\pm$ adapted to the chosen representation, which is achieved by inverting (2.14), that is

$$D = D_+ + D_-, \quad C = D_+ - D_-$$

(3.1)

We will discuss the quantisation of $D, C$ separately.

#### 3.1 Spatial Diffeormorphism Constraint

We follow exactly the same strategy as in LQG\cite{4} and exploit the fact that $D$ generates a subalgebra (but not an ideal) of $\mathfrak{H}$. Computing the Hamiltonian vector field of $D[f]$ and considering the associated Hamiltonian flow we obtain a canonical transformation $\beta_{\varphi^I}$ which, unsurprisingly, can be written as

$$\beta_{\varphi^I} = \alpha_{\varphi^I, \varphi^I}$$

(3.2)

because $D[f] = D_+[f] + D_-[f]$. The associated group $\text{Diff}(S^1)$ generated is therefore simply the diagonal subgroup

$$\{(\varphi_+, \varphi_-) \in \mathfrak{G}; \quad \varphi_+ = \varphi_-\} \subset \mathfrak{G}$$

(3.3)

The associated automorphism group $\varphi \mapsto \beta_{\varphi}$ then lifts in the same fashion to $\mathfrak{A}$ and results in a unitary representation $\varphi \mapsto V(\varphi)$ of $\text{Diff}(S^1)$ on $\mathcal{H}$ in complete analogy as described in the previous section. From the point of view of $\mathfrak{H}$, in order that a linear functional $l$ on $\pi(\mathfrak{A})\Omega$ is in the kernel of (the dual of) $\mathfrak{H}$ it must satisfy in particular

$$l[V(\varphi)\pi(a)\Omega] = l[\pi(a)\Omega] \quad \forall \ a \in \mathfrak{A}, \ \varphi \in \text{Diff}(S^1)$$

(3.4)

\footnote{This was also suggested in \cite{15}.}
Since $V(\varphi) = U(\varphi, \varphi)$ it is evident from (2.33) that any $l$ satisfying (2.33) also satisfies (3.4). Hence, as expected, there are no obstacles as far as the spatial diffeomorphism constraint is concerned. This confirms the announcement made in [15].

### 3.2 Hamiltonian Constraint

Things are much more interesting with respect to the Hamiltonian constraint. The Hamiltonian constraints $C[f]$ do not generate a subalgebra of $\mathfrak{h}$ and therefore do not exponentiate to a group. Therefore the strategy adopted in full LQG is to directly define the generator $C[f]$ of the would-be group, especially in view of the fact that in full LQG the complete algebra $\mathfrak{h}$ is not even a Lie algebra due to the structure function involved. Even that does not work straightforwardly because $C$ is a scalar density of weight two and as shown in [4] only scalar densities of weight one have a chance to be well-defined operator valued distributions in spatially diffeomorphism covariant representations. For the PFT considered here this is immediately obvious because if $C[f] = D_+ [f] - D_- [f]$ could be defined as a self adjoint operator then we would obtain it by taking the derivative at $t = 0$ of $U(\varphi^f, \varphi^{-t} f)$. However, we already remarked that $U$ is not strongly continuous, therefore this cannot be the case.

This suggests to follow the same route as in LQG [4] and to quantise instead the function $\tilde{C} = C/\sqrt{\text{det}(q)}$ where

$$\text{det}(q) = q = g_{xx} = - |T^t|^2 + |X^t|^2 = - X'_+ X'_-$$

defines the volume element of $S^1$ which is always positive for spacelike embeddings of the leaves of the foliation into the cylinder. In full LQG the Hamiltonian constraint actually is naturally defined in this way by applying the Dirac procedure to the Einstein–Hilbert action. The factor $1/\sqrt{\text{det}(q)}$ will enable us to absorb certain UV singularities. In 4D LQG the volume operator corresponding to the integral of the volume element over 3D regions plays a pivotal role in the definition of the Hamiltonian constraint. In analogy, we will here as well quantise the operator corresponding to the interval lengths

$$V(I) := \int_I dx \sqrt{\left| - X'_+ X'_- \right|}$$

where we have added, as in full LQG, an absolute value which is classically allowed since classically the argument of the square root is positive anyway.

Once we have done that, there is a chance to obtain a well defined expression for the operator corresponding to

$$\tilde{C}[f] = \int dx f \sqrt{\frac{C}{- X'_+ X'_-}}$$

provided one manages to replace the $P_\pm, Y_\pm$ by holonomies, because $P_\pm, Y_\pm$ do not exist as operators. This is in precise analogy to the steps that are performed in full LQG [4] and leads to an operator that suffers from ambiguities related to the choice of holonomies. The ambiguity is somewhat reduced on the space of solutions to the constraint because diffeomorphism equivalent choices lead to the same kernel. Some care is needed also in the choice of ordering in order to obtain an operator free of anomalies.

However, if one follows these exact same steps as in [4] it is quite obvious that the kernel defined by

$$l[\tilde{C}[f] \pi(a) \Omega] = 0 \forall a \in \mathfrak{h}, f$$

and (3.4) cannot be exactly the same kernel as defined by (3.4). Since certainly we trust the kernel defined by (3.4) much more because there are no quantisation ambiguities, the question, first spelled out in [15], arises whether the steps of [4] can be suitably modified and refined in order to match with (2.33). We will answer this question affirmatively in what follows.

Before we come to the technicalities, let us sketch the procedure that we will follow:

---

9This has been suggested also in [15].
i. **Regularisation by Triangulation**

We consider a regularised operator depending on a triangulation of \([0, 2\pi]\) and the limit of infinitesimally fine triangulation corresponds to removing the regulator.

ii. **Inverse Volume Operator**

Adapting the Poisson bracket Identities developed in \([4]\) to the present situation, an inverse volume operator can be defined free of singularities which reduces the action of \(\tilde{C}[f]\) to those intervals of the triangulation which contain a vertex of the graph \(\gamma\) on which the vector state, on which the operator acts, depends.

iii. **Quantisation of \(D_{\pm}\)**

The resulting expression is a sum over vertices of \(\gamma\) of eigenvalues of the inverse volume operator times a **double integral** of \(D_{\pm}\) over the interval of the triangulation containing the vertex. This was precisely the point of dividing by \(\sqrt{\det(q)}\): The single integral in \(C[f]\) which provides insufficient smearing in order to make it a well defined operator was replaced by a double smearing which now has a chance to result in a well defined operator. The key step is now to quantise these doubly smeared \(D_{\pm}\) in such a way, that they annihilate the solutions of (2.33). We do this by starting from the known action of \(U(\varphi, \varphi^{-1})\) for a choice of diffeomorphism \(\varphi\) with support in the given interval of the triangulation and then try to read off in which sense the operator \(U(\varphi, \varphi^{-1}) - \text{id}\) can be recognised as an approximation to the doubly smeared \(D_{+} - D_{-}\). As we will see, this is indeed possible, however, the recognition involves **new quantisation elements** that have not been considered yet in \([4]\). Not only do we need new **regularisation techniques** but also a non-trivial **renormalisation**.

As we see, steps i. and ii. are very similar to \([4]\), however, step iii. involves new techniques. In other words, the Poisson bracket identities involving the volume operator that have been exploited in \([4]\) seem to be robust, however, the simple and ambiguous choice of holonomy approximation to the continuum object employed in \([4]\) seems to be too naive. That the most straightforward quantisation does not lead to the same kernel was already announced in \([15]\). Of course, the way we obtained the “correct” holonomy approximation cannot be repeated in full 4D LQG because there a complete and clean solution as in 2D PFT is not available. However, at the very least one learns that it is worthwhile considering more sophisticated quantisation techniques. In the conclusion we outline what techniques can presumably be transferred from 2D PFT to full 4D LQG which could be directions to further research.

We now carry out in detail the steps sketched above.

### 3.2.1 Step I. Regularisation by Triangulation

We consider the basic building blocks

\[
\tilde{D}_{\pm}^{E} = P_{\pm} X'_{\pm} \quad \text{and} \quad \tilde{D}_{\pm}^{M} = \frac{1}{4} \frac{Y^{2}_{\pm}}{\sqrt{\det(q)}}
\]

whence

\[
\tilde{D}_{\pm} = \tilde{D}_{\pm}^{E} \pm \tilde{D}_{\pm}^{M}, \quad \tilde{C} = \tilde{D}_{+} - \tilde{D}_{-}, \quad \tilde{D} := \frac{D}{\sqrt{\det(q)}} = \tilde{D}_{+} + \tilde{D}_{-}
\]

It therefore suffices to provide operator expressions for approximations of \(\tilde{D}_{\pm}^{E}\) and \(\tilde{D}_{\pm}^{M}\) and these will be of the same algebraic form for both the “+” and “−” sector. Thus we may drop the label \(\pm\) for the purpose of this subsection.

We consider a triangulation \(\tau\) of \([0, 2\pi]\) into disjoint closed intervals \(I\) (modulo boundary points). The dual partition \(\tau^{*}\) is then the triangulation defined by the barycentres\(^{10}\) of the intervals \(I \in \tau\). If \(I, J\) are

\(^{10}\text{Wrt the Euclidian metric on the interval \([0, 2\pi]\).}\)
neighbour intervals, \( f_I = b_J \), then we set \( I' = [I^*, J^*] \). Hence \( I' \in \tau^* \) contains \( f_I \) as an interior point. We could, for instance, choose all intervals of the same coordinate length \( \epsilon = 2\pi/N \) where \( N \) is the number of intervals in \( \tau \). It follows

\[
D^E[f] = \lim_{\tau \to S^1} \sum_{I \in \tau} f(I^*) \frac{P(I')X(\partial I)}{V(I)}
\]

\[
D^M[f] = \lim_{\tau \to S^1} \sum_{I \in \tau} f(I^*) \frac{Y(I')^2}{V(I)}
\]

(3.11)

where \( X(\partial I) = X(f_I) - X(b_I) \). We will give the motivation to consider \( \tau^* \) next to \( \tau \) in a moment.

### 3.2.2 Step II. Inverse Volume Operator

Given an interval \( I \) we introduce a partition \( \mathcal{P} \) of \( I \) into intervals \( J \) and want to define

\[
V(I) = \lim_{\mathcal{P} \to I} \sum_{J \in \mathcal{P}} \sqrt{|X_+(\partial J) X_-(\partial J)|} = R \lim_{\mathcal{P} \to I} \sum_{J \in \mathcal{P}} \sqrt{\sin(X_+(\partial J)/R) \sin(X_-(\partial J)/R)}
\]

(3.12)

via the spectral theorem. The first and second expression on the right hand side of (3.12) will be geared towards possibilities A and B respectively. Indeed, as follows from (3.12), \( X_\pm(x) \) (possibility A) or \( \sin(X_\pm(x)/R) = [S^+_{\{x\},1} - S^-_{\{x\},-1}]/(2i) \) (possibility B) is a self-adjoint operator and \( \pi_\pm(T_{\gamma,k_\pm}^+)\Omega^\mathcal{P}_\gamma \) are eigenvectors with eigenvalue

\[
\lambda_{\gamma,k_\pm}(x) = \left\{ \begin{array}{ll}
\frac{\hbar}{R} \sum_{\gamma \in \gamma} k_\pm_L \kappa_L(x) & \text{ Possibility A} \\
\sin(\frac{\hbar}{R} \sum_{\gamma \in \gamma} k_\pm_L \kappa_L(x)) & \text{ Possibility B}
\end{array} \right.
\]

(3.13)

The eigenvalue for \( X_\pm(\partial J) \) or \( \sin(X_\pm(\partial J)/R) \) respectively is therefore

\[
\lambda_{\gamma,k_\pm}(f_J) - \lambda_{\gamma,k_\pm}(b_J) = \hbar \sum_{\gamma \in \gamma} k^+_L [\kappa_L(f_J) - \kappa_L(b_J)] \text{ or } \\
\sin(\lambda_{\gamma,k_\pm}(f_J) - \lambda_{\gamma,k_\pm}(b_J)) = \sin(\frac{\hbar}{R} \sum_{\gamma \in \gamma} k^+_L [\kappa_L(f_J) - \kappa_L(b_J)])
\]

(3.14)

respectively. Since we take the limit of infinite refinement in (3.12), we may assume that the intervals \( J \in \mathcal{P} \) are much smaller than the intervals \( L \in \gamma \) of the given \( \gamma \). Therefore there are three possibilities for given \( J, L \):

1. both \( b_J, f_J \) are interior points of \( L \)
2. one of \( b_J, f_J \) is a boundary point of \( L \)
3. either \( b_J < b_L < f_J \) or \( b_J < f_L < f_J \).

Case 1 does not give any contribution in (3.13). Case 2 is of measure zero if we avarage over possible limits of partitions. Case 3 gives a contribution with average measure unity and \( \kappa_L(f_J) - \kappa_L(b_J) = \pm 1 \). It follows that the sum over \( J \) in (3.14) eventually reduces to those that overlap a vertex \( v \in V(\gamma) \) of \( \gamma \) and once that is the case the limit \( \mathcal{P} \to I \) becomes trivial and we obtain

\[
V(I)|_{\gamma; \{k_+, l_+, k_-, l_-\} >}
\]

(3.15)

\[
= \left\{ \begin{array}{ll}
\hbar \sum_{v \in V(\gamma) \cap J} \sqrt{\sum_{\gamma \in \gamma} [k^+_L \delta_{v,b_J} - k^-_L \delta_{v,f_J}][k^-_L \delta_{v,b_J} - k^+_L \delta_{v,f_J}]] |_{\gamma; k_+, l_+, k_-, l_-} > & \text{(A)} \\
R \sum_{v \in V(\gamma) \cap J} \sqrt{\sum_{\gamma \in \gamma} \sin(\frac{\hbar}{R} [k^+_L \delta_{v,b_J} - k^-_L \delta_{v,f_J}]) \sin(\frac{\hbar}{R} [k^-_L \delta_{v,b_J} - k^+_L \delta_{v,f_J}]] |_{\gamma; k_+, l_+, k_-, l_-} >} & \text{(B)}
\end{array} \right.
\]

which in structure is very similar to full LQG just that in PFT the volume operator is diagonal in the charge network basis so that its spectrum is under full analytic control.
By inspection the volume operator has a large kernel so that its inverse is not densely defined. To define it we use the Poisson bracket identity

\[
\frac{1}{V(I)} - \frac{X_+ (\partial I) X_-(\partial I)}{V(I)^3} \approx -4 \left\{ P_+(I'), V(I) \right\} \left\{ P_-(I'), V(I) \right\} = \frac{16}{k_0^2} T_{I', k=-k_0}^+ T_{I', k=-k_0}^- \left\{ T_{I', k=k_0}^+, V(I)^{1/2} \right\} \left\{ T_{I', k=k_0}^-, V(I)^{1/2} \right\}
\]

which removes the Volume from the denominator and replaces the \( P_\pm \) by holonomies which in contrast to \( P_\pm \) are well defined in quantum theory. Thus, upon replacing the Poisson brackets by commutators divided by \( i\hbar \), (3.16) will be a densely defined operator because the holonomy operators are bounded.

The non trivial step in this calculation is the second one: We have for arbitrary intervals \( I, J \)

\[
\left\{ P_\pm (J), V(I) \right\} = -\frac{1}{2} \int_J dx \int_I dy \left\{ P_\pm (x), X_\pm (y) \right\} \frac{X_\pm (y)}{\sqrt{-X_+ ^2 + X_- ^2}} \approx -\frac{1}{2} \frac{X_\pm (\partial I)}{V(I)}\]

Hence for \( J = I' \) we have \( \chi_I (f_{J'}) = 0, \chi_I (b_{J'}) = 1 \) and thus

\[
\left\{ P_\pm (I'), V(I) \right\} = -\frac{1}{2} \left[ \frac{X_\pm (y)}{\sqrt{-X_+ ^2 + X_- ^2}} \right] (b_{I'}) \approx -\frac{1}{2} \frac{X_\pm (\partial I)}{V(I)}\]

These approximations become exact in the limit \( \tau \to S^1 \) whence we may insert (3.16) as an equality into the limit (3.11).

Thus our preliminary operators corresponding to (3.11) are written as

\[
D^E[f] = \lim_{\tau \to S^1} \sum_{I \in \tau} f(I^*) \left[ P(I') X(\partial I) \right] Q(I)
\]

\[
D^M[f] = \lim_{\tau \to S^1} \sum_{I \in \tau} f(I^*) \left[ Y(I') \right]^2 Q(I)
\]

where

\[
Q(I) := -\frac{16}{k_0^2} \sum_{k} T_{I', k=-k_0}^+ T_{I', k=-k_0}^- \left\{ T_{I', k=k_0}^+, V(I)^{1/2} \right\} \left\{ T_{I', k=k_0}^-, V(I)^{1/2} \right\}
\]

The quantum expressions \([.]^\wedge\) in (3.19) remain unspecified for the moment.

We claim in the limit \( \tau \to S^1 \) eventually only those \( I \in \tau \) contribute to the action on \( |\gamma, k_+, l_+, k_-, l_- > \) which contain a vertex of \( \gamma \). To see this, it is enough to remark from (3.15) that \( V(I) \) acts only at the vertices of \( \gamma \) contained in \( I \). In fact, it acts only at the common vertices of \( \gamma_\pm' \) where \( \gamma_\pm' \) is the graph defined by the charges \( k_\pm \). These graphs are to be distinguished from \( \gamma_\pm \) which are defined by the joint charges \( k_\pm, l_\pm \). Now consider the outmost right factor in (3.20) given by

\[
T_{I', k=-k_0}^- \left\{ T_{I', k=k_0}^+, V(I)^{1/2} \right\} = V(I)^{1/2} - T_{I', k=-k_0}^- V(I)^{1/2} T_{I', k=k_0}^-
\]

\( ^{11}\) More precisely, we can define the graphs \( \gamma_\pm^{M}, \gamma_\pm^{E} \) which are determined by \( l_\pm, k_\pm \) respectively. Then \( \gamma_\pm = \gamma_\pm^{M} \cup \gamma_\pm^{E} \) as well as \( \gamma = \gamma_+ \cup \gamma_- \).
The first term vanishes on $|\gamma, k_+, l_+, k_-, l_->$ unless $I$ contains a common vertex of $\gamma'_\pm$. The second term adds new vertices $b_{P_\tau}, f_{P_\tau}$ to $\gamma$ before $\sqrt{V(I)}$ acts and then removes them again. Thus $\sqrt{V(I)}$ may act non trivially at $b_{P_\tau}$ even if $I$ does not contain a vertex of $\gamma'_\pm$. However, both terms in (3.21) act non trivially only if $I$ contains a vertex of $\gamma'_\pm$. Since (3.20) is a product of operators of the form (3.21) one for the “+” and one for the “-” sector, it follows that the inverse volume operator only acts non trivially if in particular $I$ contains a common vertex of $\gamma'_\pm, \gamma'^\prime_\pm$.

Thus indeed only those $I = I_v$ contribute containing a common vertex $v$ of $\gamma'_\pm$, which in particular is also a vertex of $\gamma_{\pm}$ because $\gamma'_\pm$ is a subgraph of $\gamma_{\pm}$ and for those $I_v$ the limit $I_v \to v$ becomes eventually trivial giving rise to an operator $Q(v)$ which acts only at the vertex $v$ and it does so diagonally. The corresponding eigenvalue $\lambda_{\gamma,v,k_+,k_-}$ on $|\gamma, k_+, l_+, k_-, l_->$ can be worked out explicitly using (3.15), (3.20) and (3.21) but will not be needed in what follows.\[12\] We can therefore summarise the discussion so far by

$$D^E[f]|_{\gamma, k_+, l_+, k_-, l_-} = \left[ \sum_{v \in V(\gamma)} f(v) \lim_{\tau \to S^1} \sum_{I \in \tau, v \in I} [P(I')X(\partial I)]^\wedge Q(v) \right]_{\gamma, k_+, l_+, k_-, l_-}$$

$$D^M[f]|_{\gamma, k_+, l_+, k_-, l_-} = \left[ \sum_{v \in V(\gamma)} f(v) \lim_{\tau \to S^1} \sum_{I \in \tau, v \in I} [Y(I')^2]^\wedge Q(v) \right]_{\gamma, k_+, l_+, k_-, l_-}$$

### 3.2.3 Step III. Quantisation of $D_{\pm}$

So far the discussion completely parallels the construction in [4]. We could complete the definition of $\tilde{C}[f]$ which is assemblled from the building blocks (3.22) by, for instance, replacing $P_{\pm}(I'), Y_{\pm}(I')$ by $\sin(k_0 P_{\pm}(I'))/k_0, \sin(l_0 Y_{\pm}(I'))/l_0$ respectively which would result in a well defined operator at finite $\tau$. The limit $\tau \to S^1$ would exist, as in [4] in a weak* operator topology that makes use of spatially diffeomorphism invariant linear functionals (i.e. generalised eigenstates of $V(\varphi)$ with unit eigenvalue) and would correspond to choosing for each graph $\gamma$ and each vertex $v$ in $\gamma$ a neighbourhood $I_{\gamma,v}$ containing $v$ and no other vertex of $v$. The choice of $I_{\gamma,v}$ is otherwise unspecified but different choices are equivalent in the afore mentioned topology, making the limit $\tau \to S^1$ trivial. In this sense there is much less ambiguity than in 4D LQG.

However, there remains the representation or discretisation ambiguity, we could have chosen for instance $\sin(nk_0 P_{\pm}(I_{\gamma,v}))/nk_0$ for any integer $n \neq 0$ and similar for $Y_{\pm}(I_{\gamma,v})$ (and also for the definition of $Q(v)$) The commutator $[\tilde{C}[f], \tilde{C}[f']]$ between two so constructed Hamiltonian constraints is not vanishing\[13\] but its dual action vanishes on spatially diffeomorphism invariant states because while the action of $\tilde{C}[f]$ adds new vertices to a graph, these vertices are bounded by intervals (edges) which are only charged with respect to either the positive sector or the negative sector and these vertices are annihilated by the volume operator. This is because the Hamiltonian constraint does not contain additive terms that are products of operators from the positive and negative sector. In other words, the Hamiltonian constraint does not act on the vertices it creates. Again, this property is completely analogous to the situation in 4D LQG.

However, the dual action of the resulting $\tilde{C}[f]$ would surely not annihilate the exact solutions of (2.33). This may not be bad by itself as long as the two resulting quantum theories have the same classical limit. However, given the luxury of a quantisation without ambiguities and with the correct constraint algebra based on the reformulation of $\mathcal{S}$ as $\mathcal{D}$ it is of interest if there is a quantisation of $\tilde{C}$ different from this naive Wilson – like replacement of “connections” $P_{\pm}, Y_{\pm}$ by holonomies $\sin(k_0 P_{\pm}(I)), \sin(l_0 Y_{\pm}(I))$ proposed in [4] such that 1. (2.33) is annihilated and 2. the resulting operator starts from (3.22) and gives a new expression for $[.]^\wedge$. This is what we will analyse now.

\[12\] They are of the form

$$[\sqrt{\mu_\pm(k_+^0 + k_0, k_-^0) - \mu_\pm(k_-^0 + k_0, k_+^0)}][\sqrt{\mu_\pm(k_+^0 + k_0, k_-^0) - \mu_\pm(k_-^0 + k_0, k_+^0)}]$$

where $\mu_\pm(k_+^0, k_-^0)$ is the eigenvalue (3.15) at $v$ depending on the charges $k_+^0$ of the two intervals $W$ with $v$ as a boundary.

\[13\] This is because the attachment of the intervals $I_{\gamma,v}$ depends a priori on the full graph $\gamma$ and a second action of the Hamiltonian constraint therefore depends not only on $\gamma$ but on $\gamma \cup I_{\gamma,v}$. See [4] for more details.
Obviously, in order that (2.33) is annihilated, we must write \(|I|^\) in the form \(U(\varphi_+, \varphi_-) = \text{id}\) or something similar for some diffeomorphisms \(\varphi_\pm\). To simplify the discussion we label states by

\[|\gamma, k_+, l_+, k_-^l, l_-^l > = |\gamma, k_+, l_+ > \otimes |\gamma_-, k_-, l_- > \] (3.23)

with \(\gamma = \gamma_+ \cup \gamma_-\), where \(\gamma_\pm\) are the coarsest graphs so that no neighbouring intervals have both the same \(k_\pm\) and the same \(l_\pm\) charges. The \(k_\pm^l, l_\pm^l\) then result by splitting edges of \(\gamma_\pm\) into those of \(\gamma\). For the term corresponding to the interval \(I_{\gamma_+, v_+}\) with \(v_+\) a vertex of \(\gamma_+\) we consider diffeomorphisms \(\varphi_+^{\gamma, v_+}\) which have support in \(I_{\gamma_+, v_+}\) (i.e. are equal to the identity outside of it) as is motivated by the explicit expression (3.22) and the discussion above. In other words, the choice of “loop attachment” is translated into a support property of the diffeomorphism. We now have

\[ U(\varphi_+^{\gamma_+\cdot v_+, \varphi_-^{\gamma_-\cdot v_-}}) |\gamma_+, k_+, l_+ > \otimes |\gamma_-, k_-, l_- > = |\varphi_+^{\gamma_+\cdot v_+} (\gamma_+), k_+, l_+ > \otimes |\varphi_-^{\gamma_-\cdot v_-} (\gamma_-), k_-, l_- > \] (3.24)

so that we can discuss the positive and negative sector separately and can drop the label \pm for the rest of the discussion. Notice that we can safely restrict the sum over vertices \(v \in \gamma\) with \(\gamma = \gamma_+ \cup \gamma_-\) to either \(v \in \gamma_+\) or \(\gamma_-\) because due to the operator \(Q(v)\) there is no trivial action only on \(v \in V(\gamma_+) \cap V(\gamma_-)\) anyway.

Thus we should study

\[ |\varphi(\gamma), k, l > = |\gamma, k, l > = T_{\varphi(\gamma), k} \otimes W_{\varphi(\gamma), l} - T_{\gamma, k} \otimes W_{\gamma, l} |(3.25) \]

for a diffeomorphism \(\varphi\) with non-trivial action in some neighbourhood \(I\) around a vertex \(v\) of \(\gamma\). Specifically, consider a graph \(\gamma\) defined by \(N\) vertices \(v_1, v_2, ..., v_N \in \{0, 2\pi\}\) where \(v_k < v_{k+1} = 1, ..., N - 1\) and \(v_N + 1 = v_1\) and suppose \(v = v_1\) w.l.o.g. (otherwise relabel the vertices). Then \(v_k = \varphi(v_k) = v_k\) for \(k \neq 1\) and \(v_1 \neq v_1\).

If we denote the edges of \(\gamma\) by \(I_k = [v_k, v_{k+1}]\) then we see that \(I_k = [v_k, v_{k+1}] = I_k\) for \(k = 2, ..., N - 1\) but \(I_1 = [v_1, v_2] \neq I_1\). If \(v_f > v_1\) then \(I_f = I_f \cup [v_1, v_f]\), \(I_N = I_N \cup [v_1, v_1]\) are disjoint decompositions, if \(v_f > v_1\) then \(I_f = I_f \cup [v_1, v_1]\), \(I_N = I_N \cup [v_1, v_1]\) are disjoint decompositions. We define

\[ < v_1, v_1' > = [v_1, v_1'] \text{ for } v_1 < v_1' \text{ otherwise } < v_1, v_1' > = [v_1, v_1']. \]

We compute, abusing the notation\(^{[2]}\)

\[ T_{\varphi(\gamma), k} T_{\gamma, -k} = \exp(i[k_1 P(I_1) + k_N P(I_N)]) \exp(-i[k_1 P(I_1) + k_N P(I_N)]) \]

\[ = \left\{ \begin{array}{ll}
\exp(i[-k_1 P(< v_1, v_1' >) + k_N P(< v_1, v_1' >)]) & \text{ if } v_1 < v_1'
\\
\exp(i[k_1 P(< v_1, v_1' >) - k_N P(< v_1, v_1' >)]) & \text{ if } v_1 > v_1'
\end{array} \right.
\]

\[ = \left\{ \begin{array}{ll}
\exp(-i[k_1 - k_N] P(< v_1, v_1' >)) & \text{ if } v_1 < v_1'
\\
\exp(i[k_1 - k_N] P(< v_1, v_1' >)) & \text{ if } v_1 > v_1'
\end{array} \right.
\]

\[ = \exp(i[k_N - k_1] P([v_1, v_1'])) \] (3.26)

with the understanding that \(P([v_1, v_1']) = \pm P(< v_1, v_1' >)\) for \(v_1 < v_1'\) \(v_1 > v_1'\) respectively. In what follows we just use orientation preserving diffeomorphisms so that \(v_1' > v_1\).

Suppose we could actually expand the exponential in (3.26) then we would obtain to first order in \(P + i[k_1 - k_N] P([v_1, v_1'])\). If we compare this with the piece \(P(I') X(\partial I)\) in (3.22) then we see that we should choose \(I' = [v_1, v_1']\) and \(I = [v_1', v_1']\) with \(v_1' < v_1 < v_1'\) so that \(X(\partial I) = X(v_1') - X(v_1)\) because then

\[ X_{\pm}(\partial I)|\gamma, k > = h(k_N - k_1)|\gamma, k > \] (3.27)

produces precisely the same factor \(k_N - k_1\) that is needed, at least for Possibility A. Notice also that as anticipated, \(I\) has to be chosen as a neighbourhood of \(v_1\) and \(I'\) is naturally a segment in the dual of the partition containing \(I\). Hence that part of the steps leading to (3.22) was precisely correct and was obtained without having any input from the alternative quantization.

\(^{[2]}\) We display formulae as if the operators \(P(\Gamma), Y(\Gamma)\) existed which is not the case. However, this is just for notational convenience, we could redo the same calculation just using the \(T_{\gamma, k}, W_{\gamma, l}\).
However, neither does $P$ in (3.22) exist nor can we expand the exponential in (3.26). Hence there is a step missing in order to match (3.22) and (3.26). A hint comes from the observation that, for possibility A, if $P(I')$ existed and would commute with $X(\partial I)$ then we would get an exact match between (3.22) and (3.26) if we would interpret $[P(I')X(\partial I)]^\wedge$ as

$$\exp(iP(I')X(\partial I)/\hbar) - \text{id} \quad (3.28)$$

This is reminiscent to what happens in Loop Quantum Cosmology [22]: The label of the “holonomy” is turned into an operator. Unfortunately none of these assumptions hold. The would be operator (3.28) has to be written in terms of the $T_{\gamma,k}$ operators and so we have to somehow take $X(\partial I)$ out of the exponent. However, then it is now no longer difficult to guess the correct operator expression for (3.28). We have for version A

$$\exp(i[k_N - k_1]P(I')) |\gamma, k, l> = \sum_{k' \in k_0Z} \delta_{k', k_N - k_1} \exp(ik'P(I')) |\gamma, k, l>$$

$$= \sum_{k' \in k_0Z} \exp(ik'P(I'))\delta_{k', X(\partial I)/\hbar} |\gamma, k, l>$$

$$=: [\exp(iP(I')X(\partial I))]^\wedge \quad (3.29)$$

Here we define the Kronecker $\delta$ of an operator via the presentation

$$\delta_{k,k'} = \lim_{M \to \infty} \frac{1}{2M - 1} \sum_{n=-(M-1)}^{M-1} e^{in(k-k')} \quad (3.30)$$

and substituting $k$ by $X(\partial I)$ yields a limit of a sum of operators $\exp(in(X(\partial I) - k'))$ which are well defined and unitary since $X(x)$ is self adjoint. Notice that due to $k_0 \not\in 2\pi\mathbb{Q}$ the sum in (3.30) equals unity only for $k = k'$ at any finite value of $M$ so that the geometric sum for $k \neq k'$ is bounded by $[1 + 4/|e^{i(k-k')}-1|](2M - 1)$. This then is the indirect but mathematically only way to define the “intuitively obvious” operator $\exp(-iP(I')X(\partial I))$. Notice that it was to expected that an exponentiated form of $P(I')X(\partial I)$ had to be used, both because only exponentiated diffeomorphism generators and $P(I')$ exist.

For version B we proceed similarly: The only thing that needs to be replaced is the definition of $\delta_{k', X(\partial I)/\hbar}$ in (3.29) which we now define as (since $X(\partial I)$ is not directly available)

$$\delta_{k', X(\partial I)/\hbar} := \lim_{M \to \infty} \frac{1}{2M - 1} \sum_{n=-(M-1)}^{M-1} e^{-ink'/\hbar} S_{\{v_1, v_1\}, \{-n,n\}} \quad (3.31)$$

with the agreement that now $k_0 \hbar/\pi \not\in 2\pi\mathbb{Q}$.

We now turn to the other ingredient of (3.25) and proceed again symbolically, using the same diffeomorphism
as before

\[ W_{\varphi(\gamma), l} W_{\gamma, -l} = \exp(i \sum_{k=1}^{N} l_k Y(I'_k)) \exp(-i \sum_{k=1}^{N} l_k Y(I_k)) \]

\[ \begin{align*}
&= \exp(i \sum_{k=1}^{N} l_k [Y(I'_k) - Y(I_k)]) \exp(i \frac{1}{2} \sum_{k=1}^{N} l_k Y(I'_k) - i \sum_{k=1}^{N} l_k Y(I_k)) \\
&= \exp(i[l_N - l_1]Y([v_1, v'_1])) \exp(i \frac{1}{2} [l_N - l_1] \sum_{k=1}^{N} l_k Y([v_1, v'_1]), Y(I_k)) \\
&= \exp(i[l_N - l_1]Y([v_1, v'_1])) \exp(i \sigma \frac{\hbar}{2} [l_N - l_1] \sum_{k=1}^{N} l_k < \chi_{[v_1, v'_1]}, \chi_I >) \tag{3.32} \end{align*} \]

where \( \sigma = \pm \) takes care of whether we treat the positive or negative sector. Recalling (3.22) we see that only \( I_1, I_N \) contribute to the sum in the exponential and we get (for \( v_1 < v'_1 < v_2 \) since \( v'_1 \) is displaced from \( v_1 \) by an arbitrarily small amount)

\[ \begin{align*}
< \chi_{[v_1, v'_1]}, \chi_{I_1} > &= -[\kappa_{I_1}(v'_1) - \kappa_{I_1}(v_1) - \kappa_{[v_1, v'_1]}(f_{I_1}) + \kappa_{[v_1, v'_1]}(b_{I_1})] = -[1 - 1/2 - 0 + 1/2] = -1 \\
< \chi_{[v_1, v'_1]}, \chi_{I_N} > &= -[\kappa_{I_N}(v'_1) - \kappa_{I_N}(v_1) - \kappa_{[v_1, v'_1]}(f_{I_N}) + \kappa_{[v_1, v'_1]}(b_{I_N})] = -[0 - 1/2 - 1/2 + 0] = 1 \tag{3.33} \end{align*} \]

since \( b_{I_1} = f_{I_N} = v_1, b_{I_N} = v_N < v_1, f_{I_1} = v_2 > v'_1. \) Thus

\[ W_{\varphi(\gamma), l} W_{\gamma, -l} = \exp(i \sigma \frac{\hbar}{2} (l_N - l_1)^2) \exp(i(l_N - l_1)Y([v_1, v'_1])) \tag{3.34} \]

To relate (3.34) to \( \sigma Y(I')^2/4 \) as in (3.22) seems entirely hopeless at first: As we just saw, the best we can hope for is that (3.34) corresponds to something like \( \exp(i \sigma Y(I')^2/4) \). But neither is \( Y(I') \) well defined nor is there any obvious way to write it in terms of the well defined holonomies. A hint comes from the observation that \( \exp(i \sigma Y(I')^2/4) \) is a Gaussian and Gaussians are Fourier transforms of Gaussians, that is,

\[ \int_{\mathbb{R}} dx \ e^{+iyx} e^{ikx^2} = e^{+iy^2/(4k^2)}, \quad c := \int_{\mathbb{R}} dx \ e^{ikx^2} \tag{3.35} \]

where \( c \) is a finite complex number which results from “analytic continuation” from imaginary values \( k = ir, \ r > 0 \) to real values (the rigorous evaluation uses Cauchy integral techniques). Something like formula (3.36) could be applied, with the integral over \( x \) replaced by a suitable sum over \( l \in l_0 \mathbb{Z} \) to formally define \( \exp(i \sigma Y(I')^2) \) in terms of the \( W_{\gamma, l} \) and to compare its action with (3.34). But even if this worked, it would not yet give the phase \( \exp(i \sigma \frac{\hbar}{2} (l'_N - l'_1)^2) \) in (3.34). Yet, this idea turns out to be almost correct as we will see shortly.

We start by trivially rewriting (3.34) as in (3.29)

\[ W_{\varphi(\gamma), l} W_{\gamma, -l} |_{\gamma, k, l} > = \sum_{l' \in l_0 \mathbb{Z}} \delta_{|l'|, N - l_1} \ exp(i \sigma \frac{\hbar}{2} (l'^2) \ exp(i l'^2 Y([v_1, v'_1]))) |_{\gamma, k, l} > \tag{3.36} \]

Now we need, as in (3.29), an operator that acts diagonally on \( |\gamma, k, l > \) with eigenvalue \( l_N - l_1 \). For the embedding sector this was easy because this was precisely the action of \( X(\partial I) \) for suitable \( I' = [v_1, v'_1] \). But for the matter sector we do not have such an operator at our disposal. A hint comes from the following formal calculation: Suppose that \( Y(I') \) was a well defined operator. Then, using (3.33)

\[ [Y([v_1, v'_1]), W_{\gamma, l}] = [Y([v_1, v'_1]) - W_{\gamma, l} Y([v_1, v'_1]) W_{\gamma, l}^{-1}] W_{\gamma, l} = -\hbar \sigma (l_N - l_1) W_{\gamma, l} \tag{3.37} \]
Thus, the operator $\text{ad}_Y([v_1,v'_1])$ would do the right thing, but of course it does not exist. What exists is its exponential, formally given by

$$\text{Ad}_{W_{[v_1,v'_1]}_{\tilde{l}}} = \exp(i\tilde{l}\text{ad}_Y([v_1,v'_1]))$$

(3.38)

where $W_{[v_1,v'_1]}_{\tilde{l}}$ is the charge network with graph consisting of the two edges $[v_1, v'_1]$, $[v'_1, v_1 + 2\pi]$ with charges $\tilde{l}, 0$ respectively. The action of (3.38) is given explicitly by (similar as in Tomita – Takesaki theory [23])

$$\text{Ad}_{W_{[v_1,v'_1]}_{\tilde{l}}} |\gamma, k, l > = W_{[v_1,v'_1]}_{\tilde{l}} T_{\gamma,k} W_{[v_1,v'_1]}_{\tilde{l}}^{-1} T_{\gamma,l} \exp(-i\sigma h(l_N - l_1)\tilde{l})$$

(3.39)

where use was made of the fact that the GNS null space ideal is zero for $\omega^M$ so that $a = [a]$ so that the algebra of charge networks $IS$ (a dense subset of) the HS.

We now combine (3.30) and (3.39) and write (using that $h0 \not\in 2\pi\mathbb{Q}$) (3.36) as

$$W_{\varphi(\gamma),l} W_{\gamma,-l} |\gamma, k, l > = \sum_{l' \in \mathbb{Z}} \exp(i\sigma h(l')^2) \exp(il' Y([v_1, v'_1])) \times$$

$$\left[ \lim_{M \to \infty} \frac{1}{2M - 1} \sum_{n = -(M-1)}^{M-1} \exp(i\sigma nl_0 l') \text{Ad}_{W_{[v_1,v'_1],nl_0}} |\gamma, k, l > \right]$$

$$= \lim_{M \to \infty} \frac{1}{2M - 1} \sum_{n = -(M-1)}^{M-1} \sum_{l' \in \mathbb{Z}} \exp(i\sigma h(l')^2) \exp(i\sigma nl_0 l') \times$$

$$\exp(i[l' + nl_0] Y([v_1, v'_1])) |\gamma, k, l > \exp(-inl_0 Y([v_1, v'_1]))$$

$$= \left[ \sum_{l' \in \mathbb{Z}} \exp(i\sigma h(l')^2) \exp(il' Y([v_1, v'_1])) \right] |\gamma, k, l > \times$$

$$\left[ \lim_{M \to \infty} \frac{1}{2M - 1} \sum_{n = -(M-1)}^{M-1} \exp(-i\sigma h(4nl_0)^2) \exp(-inl_0 Y([v_1, v'_1])) \right]$$

(3.40)

where the phase $\exp(i\sigma nl_0 l')$ in the first line of (3.40) had to be included so that the action of $\text{Ad}_{W_{[v_1,v'_1],nl_0}}$ combines to an effective $\exp(inl_0 \sigma (l' - (l_N - l_1)))$ as desired and this phase could be absorbed into the $l'$ summation by correcting for a phase $\exp(-i\sigma h(4nl_0)^2)$ in the last step. In the second step we formally interchanged the sums which is allowed at finite $M$ so that keeping finite $M$ and take the limit only at the end has to be considered as a regularisation.

Let us define the operator

$$[\exp(-i\sigma Y(I')^2/4)]^\wedge_{M} := \sum_{l' \in \mathbb{Z} ; ||l|| \leq M-1} e^{i\sigma h l^2} W_{l', l}$$

(3.41)

with $I' = [v_1, v'_1]$. Then (3.42) may be written as

$$W_{\varphi(\gamma),l} W_{\gamma,-l} |\gamma, k, l > \lim_{M_1, M_2 \to \infty} \frac{1}{2(M_2 - 1)} [\exp(-i\sigma Y(I')^2/4)]^\wedge_{M_1} |\gamma, k, l > [[\exp(-i\sigma Y(I')^2/4)]^\wedge_{M_2}]^* \times$$

(3.42)

Comparing with our initial idea (3.35) we see that our guess was close to the final result, however, there are three non trivial differences:

1. The integral in (3.35) was replaced by an infinite sum. Using the Poisson resummation formula, the naive Gaussian is thus replaced by its periodification (with period $2\pi/l_0$)

$$\sum_{l} e^{i\sigma l} e^{il_y} \propto \sum_{l} e^{-i\sigma(y+l)^2/4}$$

(3.43)

which is quite unexpected.
2. One of the limits $M_1, M_2 \to \infty$ is accompanied by an infinite renormalisation factor $\propto 1/M_2$.

3. Up to this renormalisation factor, the action of $\exp(-i\sigma Y(I')^2/4)|_{\infty}$ is by “conjugation” of $|\gamma, k, l = T_{\gamma,k} \otimes W_{\gamma,l}$ rather than simple action from the left. This is different from the embedding sector and might seem unusual at first sight. However, it is actually not as we will now explain.

The action of the classical diffeomorphism $\varphi^v \in \text{Diff}_\pm(S^1)$ defined by the vector field $v$ on the phase space is determined by its Hamiltonian flow on the Poisson algebra $\mathcal{P}$ of functions

$$\alpha_{\varphi^v}^\pm(a) = \exp(i\mathcal{D}_\pm[v]) \cdot a = \sum_{n=0}^{\infty} \frac{1}{n!} [D_\pm[v], a]_{(n)}$$

with the iterated Poisson bracket $\{b,a\}_{(n+1)} = \{a,\{b,a\}_{(n)}\}$, $\{b,a\}_{(0)} = a$. Upon canonical quantisation, one replaces the classical Poisson algebra by the associated $^*$-algebra $\mathcal{A}$ which arises from the canonical quantisation rule, that is, from replacing Poisson brackets by commutators divided by $i\hbar$. Applied to (3.44), this means (denoting elements of $\mathcal{P}$ and $\mathcal{A}$ by the same letters)

$$\alpha_{\varphi^v}^\pm(a) = \sum_{n=0}^{\infty} \frac{(i\hbar)^{-n}}{n!} [D_\pm[v], a]_{(n)} = \exp\left(\frac{D_\pm[v]}{i\hbar}\right) a \exp\left(-\frac{D_\pm[v]}{i\hbar}\right)$$

provided $D[v]$ exists as an operator in a given representation. We see that the quantum automorphism $\alpha_{\varphi^v}^\pm$ naturally acts by conjugation on $\mathcal{A}$ by the would be unitary operators $\tilde{U}_\pm(\varphi^v) = \exp(-iD_\pm[v]/\hbar)$. However, even if $D_\pm[v]$ and thus $\tilde{U}_\pm$ does not exist, it is still possible to define $\alpha_{\varphi^v}^\pm$ on $\mathcal{A}$ simply by lifting the geometric action of $\varphi^v$ via pull-back from $\mathcal{P}$ to $\mathcal{A}$.

Now the representation we are considering is the GNS representation defined by a $\text{Diff}_\pm(S^1)$ invariant state $\omega_\pm = \omega_*^E \otimes \omega_\pm^M$ on $\mathcal{A}$ and after dividing out the GNS null ideal it is enough to consider the subalgebra generated by $T_{\gamma,k}^\pm \otimes W_{\gamma,l}^\pm$ which defines a dense subspace in the corresponding Hilbert space $\mathcal{H}_\pm$. We can therefore drop the brackets $[,]$ defining the corresponding GNS equivalence class and identify elements $a$ of that subalgebra with vector states in the corresponding GNS Hilbert space. As is well known [23], in this representation the outer automorphisms $\alpha_{\varphi^v}^\pm$ define rigorously unitary operators $W_\pm(\varphi^v)$ via $\alpha_{\varphi^v}^\pm(a) =: W_\pm(\varphi^v) \cdot a$. If these outer automorphisms are also inner, then we can find unitary operators $U_\pm(\varphi)$ constructed (possibly as limits) from $\mathcal{A}$ and so we see that the heuristic $\tilde{U}_\pm$ should be identified with the rigorous $U_\pm$ whenever both exist.

The point to notice, however, is that if $U_\pm$ exists then it acts on the $\mathcal{H}$ (the completion of the linear span by the above subalgebra of $\mathcal{A}$ in the inner product defined by $\langle b, a \rangle_\pm = \omega_\pm(b^* a)$) by conjugation and not by action from the left. It is the unitary operator

$$W_\pm(\varphi) \cdot a := \alpha_{\varphi}^\pm(a) = U_\pm(\varphi) a U_\pm(\varphi)^{-1}$$

that we are representing on $\mathcal{H}$ by action from the left and not $U_\pm(\varphi)$. This is always the case in GNS representations. If the action of $W_\pm$ were continuous, it would define a generator $D_\pm^\ell[v]$ which however could in general not have much to do with $D_\pm[v]$ as is well known from Tomita – Takesaki theory [23]. In fact, it would be $D_\pm^\ell[v] \propto [D_\pm[v], ]$ if the latter existed. If, however, $U_\pm(\varphi) \cdot 1 = 1$ i.e. if the GNS vacuum vector $\Omega = 1$ is invariant under $U_\pm$ (it is obviously so under $W_\pm$) then $W_\pm = U_\pm$ because

$$W_\pm(\varphi) \cdot a = U_\pm(\varphi) \cdot a \cdot U_\pm(\varphi)^{-1} = U_\pm(\varphi) \cdot a \cdot U_\pm(\varphi)^{-1} \cdot 1 = U_\pm(\varphi) \cdot a$$

There is no contradiction with our experience with ordinary quantum mechanics, for there one is usually given the following situation:

By the Stone – von Neumann theorem, the only irreducible and continuous (with respect to the Weyl algebra $\mathcal{A}$) representation is the usual Schrödinger representation $\pi$ on $\mathcal{H} = L_2(\mathcal{R}, dx)$. Suppose a given
Hamiltonian $H$ has a cyclic and separating\footnote{The equation $\pi(a)\Omega = 0$ has only the trivial solution $a = 0$.} (for the Weyl algebra) ground vector state $\Omega$, $||\Omega|| = 1$, that is, $H\Omega = 0$. We have then inner automorphisms $\pi(\alpha_t(a)) = U_t \pi(a) U_t^{-1}$ and $U_t \Psi = \Psi$ where $U_t = \exp(itH)$. However, $\Psi \neq 1$ and 1 is not normalisable so this is not exactly parallel to the discussion just performed.

To do so, consider the state $\omega(.) := \langle \Omega, \pi(.)\Omega \rangle_{\mathcal{H}}$. It is not difficult to see that it produces the unitarily equivalent GNS data $\mathcal{H}_\omega = L_2(\mathbb{R}, |\Psi|^2 \ dx)$, $\pi_\omega = \Omega^{-1} \pi \Omega$ and $\Omega_\omega = 1$. Here the operator $\Omega$ is defined by $(\Omega|\psi)(x) = \Omega(x)|\psi(x)$ and its pointwise inverse (as a function) exists a.e. because of the separating property.

By definition, the unitary operator $W_t$ in this GNS representation is now defined as

$$W_t \cdot \pi_\omega(a) := \pi_\omega(\alpha_t(a)) = \Omega^{-1} \pi(\alpha_t(a))\Omega = \exp(itH_\omega)\pi_\omega(a)\exp(-itH_\omega) \quad (3.48)$$

hence it corresponds also to an inner automorphism $U_t^\omega$ generated by $H_\omega = \Omega^{-1}H\Omega$. Obviously, $H_\omega$ annihilates $\Omega_\omega = 1$ so that in fact $W_t = U_t^\omega$.

This is also the reason why in the embedding sector the action was not by conjugation but by action from the left: Here the would be unitary operator $\exp(iP_\pm(I')X_\pm(\partial I))$ or better its rigorous replacement \textit{\textbf{[3.29]}} does leave the GNS vacuum $\Omega = 1$ invariant because we know from \textit{\textbf{[6]}} that the GNS representation can be identified with a space of square integrable functions of generalised “connections” $P_\pm(x)$, the unit operator is represented as the vector equal to unity while $X_\pm(x)$ acts by functional derivation by $P_\pm(x)$. Thus, \textit{\textbf{[3.29]}} applied to 1 yields 1 again. For the matter sector we have no such result at our disposal and it is in fact also not necessary.

To summarise this discussion, the fact that our quantisation of $(\exp(\pm iY_\pm(I')^2/4)$ acts by conjugation on the GNS Hilbert space is not at all surprising but rather a priori a generic feature of the action of a Hamiltonian flow in an invariant GNS representation and only in special cases can one replace the conjugation by a simple action from the left. Notice that $D_{\pm}[f] = D_{\pm}^E[f] \pm D_{\pm}^M[f]$ and that in fact $\alpha_+^E = \alpha_+^E \otimes \alpha_+^M$ since matter and embedding variables commute. Therefore a possible quantisation of the Hamiltonian constraint resulting from these considerations and \textit{\textbf{[3.22]}} is given by

$$\tilde{C}[f] |\gamma_+, k_+, I_+ > \otimes |\gamma_-, k_-, I_- > = \left[ \sum_{v \in V(\gamma_+)} f(v) \{[\exp(iP_+(I'(\gamma_+, v))X_+(\partial I(\gamma_+, v)))]^\wedge \otimes [\exp(iY_+(I'(\gamma_+, v))^2/4)]^\wedge - \text{id}_{\mathcal{H}_+} \otimes \text{id}_{\mathcal{H}_-}\} Q(v) \right] \cdot |\gamma_+, k_+, I_+ > \otimes |\gamma_-, k_-, I_- > - \left[ \sum_{v \in V(\gamma_-)} f(v) \text{id}_{\mathcal{H}_+} \otimes \{[\exp(iP_-(I'(\gamma_-, v))X_-(\partial I(\gamma_-, v)))]^\wedge \otimes [\exp(-iY_-(I'(\gamma_-, v))^2/4)]^\wedge - \text{id}_{\mathcal{H}_-}\} Q(v) \right] \cdot |\gamma_+, k_+, I_+ > \otimes |\gamma_-, k_-, I_- > \quad (3.49)$$

Here $(\gamma, v) \mapsto I(\gamma, v)$, $(\gamma, v) \mapsto I'(\gamma, v)$ are choices of intervals, for each graph $\gamma$ and vertex $v \in V(\gamma)$ with the following properties:

1. $I'(\gamma, v) = [v, v']$ where $v'$ lies in between $v$ and the next neighbour of $v$ (counted in the direction of the chosen orientation of $S^1$, i.e. to the right of $v$).
2. $I(\gamma, v) = [v_l, v_r]$ where $v \in [v_l, v_r]$ and $v_l, v_r$ lie in between $v$ and both of its next neighbours.

The operators displayed are given explicitly in \textit{\textbf{[3.29]}}, \textit{\textbf{[3.41]}} and \textit{\textbf{[3.42]}}. Let $\varphi_{\gamma,v}$ be any diffeomorphism of $S^1$ with the property that $\varphi_{\gamma,v}(v) = v'$ and that any other vertex of $\gamma$ is left invariant. Then by construction,
\[ \tilde{C}[f] |_{\gamma_+, k_+, l_+} > \otimes |_{\gamma_-, k_-, l_-} > \]
\[
\begin{align*}
&\quad f(v) \{ \{ T_{\varphi_{\gamma_+, v}(\gamma_+), k_+} T_{\gamma_+, -k_+} \otimes W_{\varphi_{\gamma_+, v}(\gamma_+), l_+} W_{\gamma_+, -l_+} - \text{id}_{H_+} \otimes \text{id}_{H_-} \} \} Q(v) \times \\
&\quad \sum_{v \in V(\gamma_+)} f(v) \{ |_{\gamma_+, k_+, l_+} > \otimes |_{\gamma_-, k_-, l_-} > \\
&\quad - \sum_{v \in V(\gamma_-)} f(v) \lambda(v) \{ |_{\varphi_{\gamma_+, v}(\gamma_+), k_+, l_+} > - |_{\gamma_+, k_+, l_+} > \} \otimes |_{\gamma_-, k_-, l_-} > \\
&\quad [ \sum_{v \in V(\gamma_+)} f(v) \{ U(\varphi_{\gamma_+, v}, \text{id}_{\text{Diff(S)}}) - \text{id}_{H} \} Q(v) \{ |_{\gamma_+, k_+, l_+} > \otimes |_{\gamma_-, k_-, l_-} > \\
&\quad - \sum_{v \in V(\gamma_-)} f(v) \{ U(\text{id}_{\text{Diff(S)}}, \varphi_{\gamma_-, v}) - \text{id}_{H} \} Q(v) \{ |_{\gamma_+, k_+, l_+} > \otimes |_{\gamma_-, k_-, l_-} > \} \}
\end{align*}
\]  

(3.50)

where \( \lambda(v) \) is the eigenvalue of \( Q(v) \) on \( |_{\gamma_+, k_+, l_+} > \otimes |_{\gamma_-, k_-, l_-} > \). Obviously,

\[ l[C[f][a] = 0 \quad \forall \quad f, \quad a \in \mathfrak{A} \]

(3.51)

for any linear functional on \( \mathfrak{A} \) satisfying (3.33). On the other hand, if one would expand the exponentials in (3.49) to linear order (which formally would become more and more exact the closer \( \varphi_{\gamma_+, v} \) is to the identity) we would obtain (3.52) which was derived directly from the classical expression. We therefore managed to find a proper quantisation of \( C[f] \) on the kinematical Hilbert space by methods developed for LQG whose kernel includes the solutions of (3.33).

4 Discussion

The most interesting question is of course what aspects of the new quantisation techniques developed for the PFT model are likely to be extendable to the 4D LQG situation in the absence of the luxury of having an exact solution. Since of course we cannot answer this question, the following list can only be speculative in nature:

I. Algebraic Structure

For the PFT model it would have been impossible to match the solution spaces of \( D_+, D_- \) and \( D, C \) respectively if one had not kept the \( C = D_+ - D_- \) term in \( \tilde{C} = C/\sqrt{\det(q)} \) intact. What we mean by this is that in LQG one considers the combination

\[ e_a^j \propto \epsilon_{abc} e^{jkl} E_k^b E_l^c / \sqrt{\det(q)} \propto \{ A^j_a, V \} \]

(4.1)

where \( V \) is the LQG volume operator and directly quantises this combination upon replacing the connection \( A \) by a holonomy. Here the Euclidian piece of the Lorentzian Hamiltonian constraint (which plays an important role in the quantisation of the Lorentzian constraint) is given by

\[ \tilde{C} = C/\sqrt{\det(q)}, \quad C = B_a^j \epsilon_{abc} e^{jkl} E_k^b E_l^c \]

(4.2)

where \( B \) is the magnetic field of \( A \). The classical kernel is determined entirely by the density two object \( C \). Analogously, in the PFT model we could also have quantised the \( P_\pm X'_\pm / \sqrt{\det(q)} \) term.
differently, e.g. based on the identity
\[ P_\pm(x)X'_\pm(x)/\sqrt{\det(q)}(x) \propto P_\pm(x)\sqrt{-X'_\pm(x)/X'_\pm(x)} = 2P_\pm(x)\{V(I), P_\pm(J)\} \quad (4.3) \]
where \( I \) is an arbitrarily short interval containing \( x \) and \( J \) overlaps only the end point of \( I \). The integral over \( I \) would then become well defined in an triangulation regularisation. However, it would not allow to make contact with the \( D_\pm \) quantisation. Similarly, in 4D LQG it could be desirable to leave \( C \) intact and to quantise \( 1/\sqrt{\det(q)} \) as a factor similar as for PFT. This has already been done as a part of the quantisation of scalar field contribution to the Hamiltonian \cite{4}.

II. Role of Diffeomorphisms

In the PFT model it proved convenient to parametrise the ambiguities in the loop attachment, here the choices of the intervals \( I(\gamma, v), I'(\gamma, v) \), in terms of spatial diffeomorphisms of compact support about \( v \). In 4D LQG one could proceed analogously. As a warm – up, a quantisation of the Husain – Kuchař \cite{24} model along those lines, whose complete solution by LQG techniques is well known, is now being completed \cite{25}. That is to say, as already observed in \cite{4}, while the generator of infinitesimal diffeomorphisms \( D_\alpha, a = 1, 2, 3 \) does not exist in 4D LQG due to the discontinuity of its one parameter subgroups, what can be defined is the operator corresponding to \( C_j = E_\alpha^a C_a/\sqrt{\det(q)} \) because it is scalar density of weight one \cite{4} rather than a covector density of weight one as \( C_a \). This is analogous to quantising in PFT \( \tilde{D} = D/\sqrt{\det(q)} \) (which exists, see below) rather than \( D \) (which does not exist, see above). This object was already quantised using the ordinary LQG techniques in \cite{5, 7} which puts the quantisations of the spatial diffeomorphism constraints and the Hamiltonian constraints on equal footing\footnote{This is sometimes spelled out as a cricism: Why should the infinitesimal Hamiltonian constraint exist while only finite spatial diffeomorphisms exist?}. However, its consistency with the Husain – Kuchař quantisation was not yet verified which will be the subject of \cite{25}.

III. Constraint Algebra

Let us now consider the hypersurface deformation algebra. In section 3 we quantised the spatial diffeomorphism constraint as finite diffeomorphisms as is customary in LQG. However, in order to see whether the quantisation of the Hamiltonian constraint reproduces the algebra \( \mathfrak{g} \) at the quantum level we need the infinitesimal generator \( D \) which however does not exist. As just mentioned, the object that does exist is \( \tilde{D} = D/\sqrt{\det(q)} \). Thus, we consider the classically equivalent algebra of \( \tilde{C}, \tilde{D} \). Since \( C = D_+ - D_- \) and \( D = D_+ + D_- \), in view of \( (3.50) \) the quantisation of \( \tilde{D} \) is immediate and differs only by a sign from \( (3.50) \), that is

\[
\tilde{D}[f]|_{\gamma_+, k_+, l_+} > \otimes|\gamma_-, k_-, l_- > = \left[ \sum_{v \in V(\gamma_+)} f(v) \{ U(\varphi_{\gamma_+, v}, id_{\text{Diff}(S^1)}) - id_{\mathcal{H}} \} \right] Q(v) \left| \gamma_+, k_+, l_+ > \otimes|\gamma_-, k_-, l_- > \right.
\]

\[ + \left[ \sum_{v \in V(\gamma_-)} f(v) \{ U(id_{\text{Diff}(S^1)}, \varphi_{\gamma_-, v}) - id_{\mathcal{H}} \} \right] Q(v) \left| \gamma_+, k_+, l_+ > \otimes|\gamma_-, k_-, l_- > \right. \]

(4.4)

Equations \( (3.39) \) and \( (4.4) \) define \( \tilde{D}_\pm[f] \) via

\[
\tilde{C}[f] =: \tilde{D}_+ [f] - \tilde{D}_- [f], \quad \tilde{D}[f] =: \tilde{D}_+ [f] + \tilde{D}_- [f] \quad (4.5)
\]

It is of course immediate that the dual of the commutator between the \( \tilde{D}_\pm[f] \) still annihilates the kernel of \( D_\pm \) defined in \( (2.33) \). But we want to see whether the algebra of the classical \( D_\pm \) is reproduced and not only the kernel. The notationally somewhat tedious calculation is carried out in appendix A. The result is as follows:
For simplicity and equivalently, we compare the algebra of the operators $\tilde{D}_\pm$ with their classical counterpart. The classical Poisson algebra of the $D_\pm$ closes but only with the structure functions. We show, that there does exist a quantisation of the right hand side of that classical Poisson algebra, following the techniques described in section 3, such that the quantum commutators are precisely reproduced. By this we mean that similar as in [4] the right hand side of the Poisson bracket between Hamiltonian constraints can be quantised by the same techniques as for the Hamiltonian constraint itself, in particular it has to be written in terms of the finite diffeomorphism approximation to the classical infinite generators. One may call this a soft anomaly in the sense that while it is not possible to reproduce the algebra $\mathcal{H}$ at the quantum level in its infinitesimal version, there is a substitute to it which 1. can be seen precisely as the quantisation of $\mathcal{H}$ in terms of finite diffeomorphisms, 2. closes with the correct factor ordering so that solutions of the constraints are not subject to any anomalous extra conditions and 3. yields the correct structure functions in some deformation again caused by the discontinuity of the representation. Such anomalies are not troublesome as they do not lead to a mismatch between classical and quantum physical degrees of freedom.

IV. Other Models

Such soft anomalies were also recently observed in 2+1 gravity with a cosmological constant quantised à la LQG [26]. In more detail, in [26] the authors argue that the anomaly they found presents an obstruction to quantise 2+1 gravity with LQG methods as in [4]. We would like to clarify this statement as follows: The constraints of 2+1 gravity with nonvanishing cosmological constant $\Lambda$ are the Gauss constraint $G$ and the curvature constraint $C = F + \Lambda E$ where $F$ is the curvature of the connection and $E$ is the volume 2-form built from its conjugate momentum. Classically we have (symbolically) $\{G, G\} = G$, $\{G, C\} = C$, $\{C, C\} = G$ with structure constants. There is no quantisation ambiguity as far as $G$ is concerned but $C$ involves the choice of a loop attachment just as in 3+1 GR. The commutator algebra yields (symbolically) $[G, G] = G$, $[G, C] = C$, $[C, C] = \text{Tr}(h) G$ where $h$ is the holonomy of a loop depending on the choice of loop attachment (see formula (40) in [26] and below). Could one shrink the loop to a point then we could replace the trace by a constant, however, this is not possible due to the discontinuity of the representation. The point is now that $\text{Tr}(h)$ is gauge invariant. Therefore one can also write $[C, C] = G \text{Tr}(h)$, hence the Gauss constraint correctly appears to the left of the anomalous structure functions. Accordingly, linear functionals satisfying $l(GT) = l(CT) = 0$ for all spin network functions $T$ and all $G, C$ are not subject to any extra conditions coming from the above soft anomaly. Hence, there is no inconsistency. What does not work is to apply group averaging techniques because of the anomalous structure functions. Thus, one has to rely on alternative techniques such as the master constraint [3] or one has to construct the space of solutions by hand and equip it with an inner product wrt which the $*-$algebra of observables is faithfully represented. As the PFT example reveals where something similar happens, this can in principle be done and the presence of the anomalous structure functions does not at all imply that the LQG quantisation techniques fail.

IV. Quantisation Ambiguities

The quantisation of the 4D LQG Hamiltonian constraint suffers from various quantisation ambiguities. It often argued, that these ambiguities might be drastically reduced once one manages to reproduce the hypersurface deformation algebra at the quantum level. One can only hope to be able to do this in terms of finite diffeomorphisms as in the PFT model due to the discontinuity of the representation, that is, one has to be ready to accept a soft anomaly. Now as far as the PFT model is concerned, some aspects of the ambiguities of the Wilson like quantisation of the Hamiltonian constraint are indeed improved by making use of the $D_\pm$ reformulation. More in detail, no factor ordering or representation choice ambiguities in the quantisation of the would be $\exp(iD_\pm[f])$ arise, that part of the quantisation is clean and unambiguous. The ambiguities arise only in the choice of the support of the finite diffeomorphisms and in the representation choice of the holonomies entering the inverse volume operator. Both ambiguities have absolutely no effect on the space of solutions. This suggests that in 4D LQG
the only worrisome ambiguity lies in the choice of the loop attachment. This kind ambiguity becomes worse the higher the spatial dimension, because it depends on the choice of a vector field of compact support whose integral curves define this diffeomorphism. In one spatial dimension, there is only one direction, but in higher dimensions the choice of vector field becomes much less trivial. We will come back to this point in [25].

V. Kinematical Semiclassical States

Some important tool for answering the question whether a set of operators qualifies as the quantisation of a given set of classical functions on the phase space are semiclassical (minimal uncertainty) states. This is especially important in 4D LQG where the complicated volume operator prevents us from doing any analytic calculation so that one has to rely on semiclassical techniques which allow for suitable approximations [7]. In the PFT model the volume operator is of course under full analytical control. However, the kinematical\(^\text{17}\) semiclassical states proposed so far [8] are not suitable to answer the question whether the Hamiltonian constraint of 4D LQG has been quantised correctly. The reason for this is as follows:

The semiclassical states of usual (free) field theories are coherent superpositions of the corresponding Fock basis states. This is not possible in LQG because, while the spin network states provide a Fock like basis, in contrast to Fock representations the LQG representation is not separable. This implies that any semiclassical vector state is sensitive to the excitations of an at most countable set of edges. However, the Hamiltonian constraint of 4D LQG [4] has the peculiar property of always modifying the graph of the state on which it acts, no matter how large it is. As a result, the expectation value of the Hamiltonian constraint in such vector states is always zero even if they ascribe the correct holonomy and flux expectation values to the edges (and dual faces) of the graph on which the vector state depends. In the PFT model, we can ask a similar question, namely whether kinematical semiclassical vector states exist that probe the correctness of the quantum constraint\(^\text{18}\). For PFT this is a somewhat academic question because the finite \(D_{\pm}\) operators have a geometric action and there is no doubt that they have been quantised correctly. However, as a test for 4D LQG this is an interesting question to ask. Notice that also in PFT the constraints of the form \(U(\varphi^{+}_{\gamma_+},v,1) - 1, U(1,\varphi^{-}_{\gamma_+},v,1) - 1\) always modify the graph on which the operator acts, they are never graph preserving. However, there is a difference to 4D LQG: While the 4D LQG operator changes the number of vertices and edges of a graph, the PFT operators do not do that since the finite diffeomorphisms they generate are confined to the circle. This simplification might enable one to improve the situation for the PFT model whose kinematical Hilbert space is also non separable. One idea [11] is to use vector states that are superpositions of a given semiclassical state and all its images under the repeated action of the Hamiltonain constraint (fractal graph coherent state). However, preliminary investigation indicates that this still does not work. If this looks hopeless in the PFT model, then it is probably also in the much more complicated 4D LQG theory and possibly then the question of the correctness of the quantisation can only be answered at the physical Hilbert space level.

VI. Observables

The observables of the theory have been constructed in all detail in [15]. Their dual acts on the space of solutions to (2.33) from which follows that also the dual of their commutator with \(\hat{C}, \hat{D}\) annihilates physical states. However, given the fact that here we have complementary restrictions on \(k_0, l_0\) as compared to [15], this induces different restrictions on the quantisation of the observables

\[
O_f^\pm := O_{Y_{\pm},f} := \sum_{n \in \mathbb{Z}} \hat{f}_n O_{Y_{\pm},n}^\pm
\]

\(^{17}\)The semiclassical states should not be solutions to the proposed quantum constraints as otherwise one cannot check by means of them whether the constraints have been quantised correctly.

\(^{18}\)Physical coherent states have been constructed explicitly in [15] for each of the superselection sectors.
where the $\hat{f}_n$ are complex numbers without a priori restriction, see (2.17). We will sketch these in the following paragraph.

Following Hardy – Littlewood theory [27], there are several means of sequences of real or complex numbers $a_m \in \mathbb{C}$ which are widely used by physicists and which have the property to 1. produce well defined limits even if the sequence itself does not converge and 2. to reproduce the limit of the sequence if it does converge. The Cesaro mean

\[
C[a] := \lim_{M \to \infty} \frac{1}{2M-1} \sum_{m \in \mathbb{Z}; |m| \leq M-1} a_m
\]

used extensively in section is only one of them. To avoid confusion, the Cesaro mean appeared naturally in section 3 in an exact equality, it was not used in order to “sweep divergencies under the rug”.

The Cesaro mean also naturally appears in the representation of the observables as follows: Since $Y_{\pm}[I']$ does not exist, it is natural, following [15], to try to define instead the exponential

\[
\exp(iO_{\pm}^\gamma) |\gamma^\pm, k^\pm, l^\pm >, O_{\pm}^\gamma := \sum_n \hat{f}_n O_n^{\pm}
\]

By exactly the same calculation displayed in [15] one arrives at

\[
\exp(iO_{\pm}^\gamma) |\gamma^\pm, k^\pm, l^\pm >= \exp(i \sum_{I \in \gamma^\pm} Y_{\pm}(I) l_I) |\gamma^\pm, k^\pm, l^\pm >=
\]

which is well defined provided that

\[
l_I := \sum_n \hat{f}_n e^{\imath \alpha n k_I / R} \in l_0 \mathbb{Z}
\]

for any $k_I \in k_0 \mathbb{Z}$. Let us write $\hat{f}_n =: l_0 \hat{g}_n$ and let $\alpha := \hbar k_0 / R$. Then requirement (4.9) simply reads

\[
z_m := \sum_n \hat{g}_n e^{\imath \alpha n m} \in \mathbb{Z}
\]

for any $m \in \mathbb{Z}$ as a condition of the Fourier coefficients $\hat{g}_n$. Using the Cesaro mean, we can translate this into the requirement that the integer valued sequence $(z_m)$ should be such that

\[
\hat{g}_n := C[a^{(n)}], a^{(n)}_m := e^{-\imath \alpha n m} z_m
\]

converges for any $n \in \mathbb{Z}$. Hence, also here we see a reduction in the number of possible $f$ similar to [15].

VII. Methodology

Anybody who has gone through the details of section 3 will admit that nobody would ever have thought of the constructions there, if the alternative formulation in terms of the $D_{\pm}$ constraints would not have been at our disposal. Therefore one can rightfully ask, why one should go through these indirect constructions if a simpler solution is available and vice versa what use these constructions have, if a simpler solution is not available. Perhaps a possible answer could be to look for classical reformulations of a constraint algebra in terms of simpler algebras or for algebras of elementary observables that are more adapted to the dynamics of the theory. This is already the case in free, massive scalar field theories: There one writes the Hamiltonian in terms of the natural creation and annihilation operators induced by the Hamiltonian and not of some completely different free field Hamiltonian (say with a different mass). It is not even possible to define the Hamiltonian with a different mass in a Fock representation with a given mass, one of the consequences of Haag’s theorem [23]. In LQG the
holonomy flux algebra is well adapted to the spatial diffeomorphism constraint but not obviously as far as the Hamiltonian constraint is concerned. Maybe the choice of the holonomy flux algebra as a classical starting point for the quantisation should be reconsidered. On the other hand, at the very least it is reaffirming, that a non trivial fraction of the techniques introduced in [4] turn out to be useful and correct and also in the completely solvable PFT.

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A Constraint Algebra

We compute first the classical algebra of the \( \tilde{D}_\pm = D_\pm / \sqrt{\det(q)} \). Notice that while \( \det(q) = -X'_\pm X'_\mp \) transforms as a scalar density of weight two under \( D_\pm \), it transforms only as a scalar density of weight one under \( D_\pm \). Let \( f, g \) be some test functions, then from (2.15)

\[
\{ \tilde{D}_\pm[f], \tilde{D}_\pm[g] \} = \int dx \int dy \ f(x) g(y) \{ \tilde{D}_\pm(x), \tilde{D}_\pm(y) \}
\]

\[
= \int dx \int dy \ f(x) g(y) \left[ \frac{1}{\sqrt{\det(q)(x) \det(q)(y)}} \{ D_\pm(x), D_\pm(y) \} + \frac{D_\pm(x)}{\sqrt{\det(q)(y)}} \{ \frac{1}{\sqrt{\det(q)(x)}}, D_\pm(y) \} \right]
\]

\[
- \frac{D_\pm(y)}{\sqrt{\det(q)(x)}} \{ \frac{1}{\sqrt{\det(q)(y)}}, D_\pm(x) \} \}
\]

\[
= \{ D_\pm[f], D_\pm[g] \} - \int dx \int dy \ \frac{f(x)D_\pm(x)}{\sqrt{\pm X_\pm(x)}} \{ D_\pm[g], \frac{1}{\sqrt{\pm X_\pm(x)}} \} + \int dy \ g(y) \frac{D_\pm(y)}{\sqrt{\pm X_\pm(y)}} \{ D_\pm[f], \frac{1}{\sqrt{\pm X_\pm(y)}} \}
\]

\[
= D_\pm[[f], g]] - \frac{1}{2} \int dx \ \frac{fD_\pm}{\sqrt{\pm X_\pm(x)}} \{ g, \frac{1}{\sqrt{\pm X_\pm(x)}} \} - \frac{1}{2} g' \frac{1}{\sqrt{\pm X_\pm(x)}} \}
\]

\[
= D_\pm[[f], g]] - \frac{1}{2} \int dx \ \sqrt{\pm X_\pm(x)D_\pm[f g', \frac{1}{\sqrt{\pm X_\pm(x)}}]}
\]

\[
= \frac{1}{2} D_\pm[[f], g]] = \frac{1}{2} \tilde{D}_\pm[\sqrt{\det(q)}[f, g]]
\]

\[
= \frac{1}{2} \tilde{D}_\pm[\frac{1}{\sqrt{\det(q)}}[f, g]]
\]

(A.1)
with \( \tilde{f} = f / \sqrt{\det(q)} \), \( \tilde{g} = f / \sqrt{\det(q)} \) and one computes the Poisson brackets displayed as if \( \tilde{f}, \tilde{g} \) were independent of the phase space. Similarly,

\[
\{ \tilde{D}_\pm[f], \tilde{D}_\mp[g] \} = \int dx \int dy f(x) g(y) \{ \tilde{D}_\pm(x), \tilde{D}_\mp(y) \}
\]

\[
= \int dx \int dy f(x) g(y) \left[ \frac{1}{\sqrt{\det(q)(x)\sqrt{\det(q)(y)}}} \right] \{ D_\pm(x), D_\mp(y) \} - \frac{D_\pm(y)}{\sqrt{\det(q)(x)}} \{ \frac{1}{\sqrt{\det(q)(y)}}, D_\pm(y) \}
\]

\[
= \{ D_\pm[f], D_\mp[g] \} - \int dx f(x) \tilde{D}_\pm(x) \{ D_\mp[g], \frac{1}{\sqrt{+X_\pm(x)}} \} + \int dy g(y) \tilde{D}_\mp(y) \{ D_\pm[f], \frac{1}{\sqrt{+X_\pm(y)}} \}
\]

\[
= - \int dx \frac{f D_\pm}{\sqrt{\pm X_\pm}} [\tilde{g}(\frac{1}{\sqrt{+X_\pm}})' + \frac{1}{2} \tilde{g}' \frac{1}{\sqrt{+X_\pm}}] + \int dx \frac{g D_\mp}{\sqrt{+X_\pm}} [\tilde{f}(\frac{1}{\sqrt{+X_\pm}})' - \frac{1}{2} \tilde{f}' \frac{1}{\sqrt{+X_\pm}}]
\]

\[
= \frac{1}{2} \int dx f \tilde{D}_\pm \sqrt{\pm X_\pm} [\tilde{g}(\frac{1}{\sqrt{+X_\pm}})' + \frac{1}{2} \tilde{g}' \frac{1}{\sqrt{+X_\pm}}] - \frac{1}{2} \int dx g \tilde{D}_\mp \sqrt{\pm X_\pm} [\tilde{f}(\frac{1}{\sqrt{+X_\pm}})' - \frac{1}{2} \tilde{f}' \frac{1}{\sqrt{+X_\pm}}]
\]

\[
= \frac{1}{2} \int dx f \tilde{D}_\pm \sqrt{+X_\pm} [\tilde{g}(\frac{1}{\sqrt{+X_\pm}})'] - \frac{1}{2} \int dx g \tilde{D}_\mp \sqrt{+X_\pm} [\tilde{f}(\frac{1}{\sqrt{+X_\pm}})']
\]

\[
= - \frac{1}{2} \int dx f D_\pm \frac{X_\pm'}{\sqrt{\det(q)}} [g \frac{X_\pm'}{\sqrt{\det(q)}}]' + \frac{1}{2} \int dx g D_\mp \frac{X_\pm'}{\sqrt{\det(q)}} [f \sqrt{X_\pm} \sqrt{\det(q)}]'
\]

(A.2)

with no further simplification possible. The modifications in the algebra of the \( \tilde{D}_\pm \) of course arise because of the nontrivial transformation behaviour of \( \det(q) \) wrt \( D_\pm \). As expected, the new algebra closes albeit with non trivial structure functions.

We now turn to the quantum computation. For notational simplicity, given graphs \( \gamma^\pm \) and \( v \in V(\varphi^\pm) \) we denote \( \varphi^\pm_v := \varphi_{\gamma^\pm,v} \) and \( \gamma^\pm_v := \varphi_{\gamma^\pm,v}^+ \) where \( \gamma, v \in V(\gamma) \) and specified in section 3. Likewise, for \( \varphi^\pm_v := \varphi_{\gamma^\pm,v} \) we denote \( \varphi^\pm_{\gamma^\pm,v} := \varphi_{\gamma^\pm,v} \). Finally, we define for a given charge network \( |\gamma^+, k_+, l_+ > \otimes |\gamma^-, k_-, l_- > \) by \( \lambda(\gamma^+, \gamma^-, v) \) the eigenvalue of \( Q(v) \) on this eigenvector. As follows from (3.21), (3.21) and (3.22), these eigenvalues depend only on those \( k^I_v, l^I_v \in \gamma^\pm \) such that \( v \in \partial I \). In particular it vanishes if \( v \) is not a common vertex of both \( \gamma^+ \) and \( \gamma^- \). Moreover, these eigenvalues are diffeomorphism invariant in the sense that

\[
Q(v)|\gamma^+, k_+, l_+ > \otimes |\gamma^-, k_-, l_- > = U(\varphi^{-1}, \varphi^{-1}) Q(\varphi(v)) |\varphi(\gamma^+), k_+, l_+ > \otimes |\varphi(\gamma^-), k_-, l_- >
\]

(A.3)

With these preparations we compute first the \( \tilde{D}_+, \tilde{D}_+ \) commutator (the \( \tilde{D}_-, \tilde{D}_- \) commutator is completely analogous). We display only the graph dependence for notational simplicity since the charges are carried with the vertices under diffeomorphisms and are not changed. Also the operator \( Q(v) \) does not
change the charges. We begin with

\[
[\hat{D}_+[f], \hat{D}_+[g]] |\gamma^+, \gamma^- > = \sum_{v \in \hat{V}(\gamma^+)} g(v) \lambda(\gamma^+, \gamma^-, v) D_+[f] [|\gamma^+, \gamma^- > - |\gamma^+, \gamma^- >] \\
- \sum_{v \in \hat{V}(\gamma^+)} f(v) \lambda(\gamma^+, \gamma^-, v) D_+[g] [|\gamma^+_v, \gamma^- > - |\gamma^+, \gamma^- >]
\]

\[
= \sum_{v \in \hat{V}(\gamma^+)} g(v) \lambda(\gamma^+, \gamma^-, v) \{ \sum_{\tilde{v} \in \hat{V}(\gamma^+)} f(\tilde{v}) \lambda(\gamma^+_v, \gamma^-, \tilde{v})[|\gamma^+_v, \gamma^- > - |\gamma^+_v, \gamma^- >] \\
- \sum_{\tilde{v} \in \hat{V}(\gamma^+)} f(\tilde{v}) \lambda(\gamma^+_v, \gamma^-, \tilde{v})[|\gamma^+_v, \gamma^- > - |\gamma^+_v, \gamma^- >] \}
\]

\[
- \sum_{v \in \hat{V}(\gamma^+)} f(v) \lambda(\gamma^+, \gamma^-, v) \{ \sum_{\tilde{v} \in \hat{V}(\gamma^+)} g(\tilde{v}) \lambda(\gamma^+_v, \gamma^-, \tilde{v})[|\gamma^+_v, \gamma^- > - |\gamma^+_v, \gamma^- >] \\
- \sum_{\tilde{v} \in \hat{V}(\gamma^+)} g(\tilde{v}) \lambda(\gamma^+_v, \gamma^-, \tilde{v})[|\gamma^+_v, \gamma^- > - |\gamma^+_v, \gamma^- >] \}
\]

(4.4)

Consider the terms proportional to $|\gamma^+, \gamma^- >$ which is independent of $v, \tilde{v}$. These involve a sum over $v, \tilde{v} \in \hat{V}(\gamma^+)$ with coefficient

\[
g(v)\lambda(\gamma^+, \gamma^-, v)f(\tilde{v})\lambda(\gamma^+, \gamma^-, \tilde{v}) - g(v)\lambda(\gamma^+, \gamma^-, v)f(\tilde{v})\lambda(\gamma^+, \gamma^-, \tilde{v})
\]

(4.5)

which is antisymmetric under $v \leftrightarrow \tilde{v}$ and thus the corresponding sum vanishes.

Next consider the terms proportional to $|\gamma^+_v, \gamma^- >$ and $|\gamma^+_v, \gamma^- >$. The former terms involve a sum over $v \in \hat{V}(\gamma^+), \tilde{v} \in \hat{V}(\gamma^+)$ the latter over $v, \tilde{v} \in \hat{V}(\gamma^+)$ in the latter sum. Then these two contributions can be written as

\[
- \sum_{v \in \hat{V}(\gamma^+), \tilde{v} \in \hat{V}(\gamma^+)} \lambda(\gamma^+, \gamma^-, v) \lambda(\gamma^+_v, \gamma^-, \tilde{v})[g(v)f(\tilde{v}) - f(v)g(\tilde{v})] |\gamma^+_v, \gamma^- >
\]

(4.6)

\[
+ \sum_{v, \tilde{v} \in \hat{V}(\gamma^+)} \lambda(\gamma^+, \gamma^-, v) \lambda(\gamma^+_v, \gamma^-, \tilde{v})[g(v)f(\tilde{v}) - f(v)g(\tilde{v})] |\gamma^+_v, \gamma^- >
\]

Notice that $V(\gamma^+_v) = [V(\gamma^+) - \{v\}] \cup \{\varphi^+_v(v)\}$. We split the sum in the first term in (4.6) into $\tilde{v} \neq v$ and $\tilde{v} = \varphi^+_v(v) =: \check{v}$. For $\tilde{v} \neq v$ we have $\lambda(\gamma^+_v, \gamma^-, v) = \lambda(\gamma^+, \gamma^-, \check{v})$ because the vertices in $\gamma^+$ different from $v$ are not moved by $\varphi^+_v$ and the eigenvalue $\lambda(\gamma^+, \gamma^-, v)$ depends only on the infinitesimal neighbourhood of $v$. It follows that the terms $\tilde{v} \neq v$ in the first sum and those with $\tilde{v} \neq v$ in the second sum in (4.6) cancel each other and what remains is (notice that the contribution to the second sum from $\tilde{v} = v$ vanishes trivially)

\[
- \sum_{v \in \hat{V}(\gamma^+)} \lambda(\gamma^+, \gamma^-, v) \lambda(\gamma^+_v, \gamma^-, \check{v})[g(v)f(\check{v}) - f(v)g(\check{v})] |\gamma^+_v, \gamma^- >
\]

(4.7)

Finally consider the terms proportional to $|\gamma_{v\check{v}}, \gamma^- >$ whose contribution is given by

\[
\sum_{v \in \hat{V}(\gamma^+), \tilde{v} \in \hat{V}(\gamma^+_v)} \lambda(\gamma^+, \gamma^-, v) \lambda(\gamma^+_v, \gamma^-, \tilde{v})[g(v)f(\tilde{v}) - f(v)g(\tilde{v})] |\gamma^+_v, \gamma^- >
\]

(4.8)

Again we split the sum over $\tilde{v}$ into $\tilde{v} \in \hat{V}(\gamma^+) - \{v\}$ and $\tilde{v} = v$. Then (4.8) becomes

\[
\sum_{v, \tilde{v} \in \hat{V}(\gamma^+), v \neq \tilde{v}} \lambda(\gamma^+, \gamma^-, v) \lambda(\gamma^+_v, \gamma^-, \tilde{v})[g(v)f(\tilde{v}) - f(v)g(\tilde{v})] |\gamma^+_v, \gamma^- >
\]

(4.9)
As already shown, \( \lambda(\gamma^+, \gamma^-, \tilde{v}) = \lambda(\gamma^+, \gamma^-, \tilde{v}) \) for \( \tilde{v} \neq v \). Therefore the first term in (A.9) can be written

\[
\frac{1}{2} \sum_{v, \tilde{v} \in V(\gamma^+), \tilde{v} \neq v} \lambda(\gamma^+, \gamma^-, v) \lambda(\gamma^+, \gamma^-, \tilde{v}) [g(v) f(\tilde{v}) - f(v) g(\tilde{v})] [\gamma^+_{v\tilde{v}}, \gamma^-] = -[\gamma^+_{v\tilde{v}}, \gamma^- >]
\]  
(A.10)

where we added the same sum relabelled by \( v \leftrightarrow \tilde{v} \) and divided by 2. Now notice that by definition of \( \varphi_{\gamma^+, v}^+ \) this diffeomorphism only notices the next neighbour structure of \( \gamma^+ \). Therefore, \( \varphi_{\gamma^+, \tilde{v}} = \varphi_{\gamma^+, v} \) unless \( \tilde{v} \in \{(v, r(v)\} \) is a left or right next neighbour of \( v \) in \( \gamma^+ \). Consequently, also \( \gamma^+_{v\tilde{v}} = \gamma^+_{\tilde{v}v} \) then. Thus what remains from (A.10) is

\[
\frac{1}{2} \sum_{v \in V(\gamma^+), \tilde{v} \in \{(v, r(v)\) \} \) \lambda(\gamma^+, \gamma^-, v) \lambda(\gamma^+, \gamma^-, \tilde{v}) [g(v) f(\tilde{v}) - f(v) g(\tilde{v})] [\gamma^+_{v\tilde{v}}, \gamma^-] = -[\gamma^+_{v\tilde{v}}, \gamma^- >]
\]  
(A.11)

Combining (A.7), (A.9) and (A.11) we find

\[
[D_+ [f], \tilde{D}_+ [g]] |\gamma^+, \gamma^- > = \frac{1}{2} \sum_{v \in V(\gamma^+), \tilde{v} \in \{(v, r(v)\) \} \) \lambda(\gamma^+, \gamma^-, v) \lambda(\gamma^+, \gamma^-, \tilde{v}) [g(v) f(\tilde{v}) - f(v) g(\tilde{v})] [\gamma^+_{v\tilde{v}}, \gamma^-] = -[\gamma^+_{v\tilde{v}}, \gamma^- >] 
\]  
(A.12)

Consider the first term in (A.12):

Notice that

\[
\gamma^+_{v\tilde{v}} = \varphi_{\gamma^+, v}^+ \circ \varphi_v(\gamma^+), \quad \gamma^+_{\tilde{v}v} = \varphi_{\gamma^+, \tilde{v}}^+ \circ \varphi_{\tilde{v}}(\gamma^+) 
\]  
(A.13)

Here \( \varphi_{\gamma^+, v}^+ \) only act on \( v \) while \( \varphi_{\gamma^+, \tilde{v}}^+ \) only act on \( \tilde{v} \) by shifting these vertices to the right without touching the next neighbour vertex of the graph they act on. However, since e.g. \( \varphi_{\gamma^+, v}^+ \) acts on \( \gamma^+ \) while \( \varphi_{\gamma^+, \tilde{v}}^+ \) acts on \( \gamma^+_v \), the position \( \varphi_{\gamma^+, \tilde{v}}^+(\tilde{v}) \) maybe different from \( \varphi_{\gamma^+, v}(\tilde{v}) \) because the diffeomorphisms depend on the next neighbour structure of the graph they act and these are different in this case. Consequently \( \gamma^+_{v\tilde{v}} \neq \gamma^+_{\tilde{v}v} \) in general. This term is similar in structure to the commutator of two Hamiltonian constraints in 4D LQG.

Now consider the second term in (A.12):

This term arises from a second action of an infinitesimal diffeomorphism on the shifted vertex \( \tilde{v} = \varphi_{\gamma^+, v}(v) \). This term is absent in 4D LQG because due to the properties of the volume operator there (it does not act on coplanar vertices) it does not act on the vertices it creates. Now it is easy to see, again due to the properties of the volume operator, that the second term in (A.12) also vanishes unless by chance the vertex \( \tilde{v} \) is also a vertex of \( \gamma^- \). It is easy to extend the prescription for the \( \varphi_{\gamma^+, v} \) so that this possibility is avoided by hand, thus becoming a prescription of the form \( \varphi_{\gamma^+, v; \gamma^-}, v \in V(\gamma^+) \) and similar for \( \varphi_{\gamma^-; \gamma^+}, v \in V(\gamma^-) \).

Hence with this extended prescription understood, precisely for the same reason as in 4D LQG only the first term in (A.12) survives because the Hamiltonian constraint does not act on the vertices it creates which is ultimately a property of the volume operator.

Comparing (A.11) and (A.12) we recognise a similar structure of the “structure operators” and of the structure functions. Both terms are propotional to an infinitesimal \( D_+ \) diffeomorphism and the structure functions (two factors of \( 1/\sqrt{\det(g)} \) correctly correspond to the two eigenvalues of the inverse volume. The commutator is manifestly local, only next neighbour vertices or its infinitesimal diffeomorphic image are involved in the sum over vertices of \( \gamma^+ \). Moreover, the terms of the form

\[
g(v) f(\tilde{v}) - f(v) g(\tilde{v}) = g(v) [f(\tilde{v}) - f(v)] - f(v) [g(\tilde{v}) - g(v)]
\]  
(A.14)

Here the infinitesimal diffeomorphisms \( \varphi_{\gamma^+, v}^+ \) could be argued to simultaneously create a new vertex \( \tilde{v} \) and annihilate an old vertex \( v \). Or one could say that the charges on the segment \( [v, \tilde{v}] \) were changed from \( k_{t_{[v, \tilde{v}]} f_{[v, \tilde{v}]} t_{[v, \tilde{v}]} g_{[v, \tilde{v}]} t_{[v, \tilde{v}]} f_{[v, \tilde{v}]} \) to \( k_{t_{[v, \tilde{v}]} f_{[v, \tilde{v}]} t_{[v, \tilde{v}]} g_{[v, \tilde{v}]} t_{[v, \tilde{v}]} f_{[v, \tilde{v}]} \). This is similar to 4D LQG but not quite analogous because charges are just shifted but never changed on segments of the graph in question.

\[\text{[Footnote 18]}\]
qualify as discretisations of the bracket \([f, g](v)\). Even the factors 1/2 in (A.11) and (A.12) come out the same. The question is whether (A.12) qualifies as a possible quantisation of (A.11). To answer this question we write (A.12) in the form (dropping the second term as just discussed)

\[
[D_+[f], D_+g] |\gamma^+, \gamma^- > \\
= \frac{1}{2} \sum_{v \in V(\gamma^+), \delta \in \{l(v), r(v)\}} [g(v) f(\tilde{v}) - f(v) g(\tilde{v})] [U(\varphi_{v\delta}^{v\delta} \circ \varphi_{v\delta}^{v\delta}, 1) - U(\varphi_{v\delta}^{v\delta} \circ \varphi_{v\delta}^{v\delta}, 1)] Q(\tilde{v}) Q(v) |\gamma^+, \gamma^- > \\
= \frac{1}{2} \sum_{v \in V(\gamma^+)} \{[g(\delta_r f - f \delta_l g)](v) [U(\varphi_{v\delta(v)}^{v\delta(v)} \varphi_{v\delta(v)}^{v\delta(v)}, 1) - U(\varphi_{v\delta(v)}^{v\delta(v)} \varphi_{v\delta(v)}^{v\delta(v)}, 1)] Q(r(v)) Q(v) |\gamma^+, \gamma^- > \\
- [g(\delta_l f - f \delta_r g)](v) [U(\varphi_{v\delta(v)}^{v\delta(v)} \varphi_{v\delta(v)}^{v\delta(v)}, 1) - U(\varphi_{v\delta(v)}^{v\delta(v)} \varphi_{v\delta(v)}^{v\delta(v)}, 1)] Q(l(v)) Q(v) |\gamma^+, \gamma^- >
\]

(A.15)

where we have introduced the right and left graph difference

\[
[\delta_r f](v) = f(r(v)) - f(v), \quad [\delta_l f](v) = f(v) - f(l(v))
\]

(A.16)

However, this is precisely a possible quantisation of (A.1): The two inverse volume functions would have been ordered to the right and been replaced by two operators \(Q\). There is a freedom whether to locate both factors of \(Q\) at \(v\) or maybe one at \(r(v), l(v)\). The quantum computation decides for the latter possibility, both are equivalent in the continuum limit of graphs with a large number of vertices. The bracket \([f, g]\) would have been replaced by the discrete difference between neighbouring vertices and again there is a choice between left, right or symmetric derivative. The quantum computation decides for a mixture of the two. Finally, in the limit of large graphs, the two terms in (A.15) at given \(v\) combine to

\[
[f, g](v) \{[U(\varphi_{v\delta(v)}^{v\delta(v)} \circ \varphi_{v\delta(v)}^{v\delta(v)}, 1) - U(\varphi_{v\delta(v)}^{v\delta(v)} \circ \varphi_{v\delta(v)}^{v\delta(v)}, 1)] |r(v) - v| - [U(\varphi_{v\delta(v)}^{v\delta(v)} \circ \varphi_{v\delta(v)}^{v\delta(v)}, 1) - U(\varphi_{v\delta(v)}^{v\delta(v)} \circ \varphi_{v\delta(v)}^{v\delta(v)}, 1)] |l(v) - v| \} Q(v)^2
\]

(A.17)

Now by definition \(U(\varphi_{v\delta(v)}^{v\delta(v)}, 1)\) is supposed to be the quantisation of the would be operator \(\text{exp}(iD_+[I_v])\) where \(I_v\) is an interval containing \(v\) corresponding to the support of \(\varphi_{v\delta(v)}^{v\delta(v)}\). Similarly \(U(\varphi_{v\delta(v)}^{v\delta(v)}, 1)\) is supposed to be the quantisation of \(\text{exp}(iD_+[I_v])\) where \(I_v\) is an interval containing \(v\) corresponding to the support of \(\varphi_{v\delta(v)}^{v\delta(v)}\) which in turn depends on \(v\). Now, if the operators \(D_+[I]\) existed then we could expand the curly bracket in (A.17) to first order in the interval lengths as

\[
i(D_+[I_v] + D_+[I_v'] - D_+[I_v] - D_+[I_v'] - D_+[I_v] - D_+[I_v'] + D_+[I_v'] + D_+[I_v'])
\]

(A.18)

which would classically correspond to \(D_+[J]\) where \(J\) is an interval containing \(v\) and of interval length corresponding to absolute value of the signed sum of the intervals in (A.18). Since in the quantisation of (A.1) such a choice of \(J\) would also have to be made, we simply define it by to be the one chosen by the quantum commutator computation. That is to say, we take the point of view spelled out in [3] and consider the right hand side of the classical Poisson bracket between the \(\tilde{D}_+\), which involves structure functions, as a new operator which must be quantised to some extent by prescriptions and techniques independent of those for \(\tilde{D}_+\).

This is as close as one can hope to see the correspondence between (A.1) and (A.15). The unavoidable ambiguities in the quantisation caused by the discontinuity of the representation can be exploited in order to close the quantum algebra including the structure functions. Also the right hand side of the classical Poisson brackets has to be quantised in terms of finite diffeomorphisms because the generators do not exist. One should maybe call this a “soft anomaly”, i.e. the fact that the classical computation is only reproduced by the finite approximations to the infinitesimal classical counterparts. However, it is not a troublesome anomaly in the sense that the quantum algebra still closes with the correct factor ordering so that the solutions to the constraints do not have to obey any extra properties. It is just that that the structure functions are replaced by somewhat deformed quantum operators due to the unavoidable quantisation ambiguity in the choice of \((\gamma, v) \mapsto \varphi_{\gamma,v}\). A similar “soft” anomaly has been observed in 2+1 gravity with a cosmological
We now turn to the $[\tilde{D}_+, \tilde{D}_-]$ commutator

$$
[\tilde{D}_+[f], \tilde{D}_-[g]] |\gamma^+, \gamma^- > 
= \sum_{v \in V(\gamma^-)} g(v) \lambda(\gamma^+, \gamma^-, v) D_+[f] [|\gamma^+, \gamma^- > -|\gamma^+, \gamma^- >] 
- \sum_{v \in V(\gamma^+)} f(v) \lambda(\gamma^+, \gamma^-, v) D_-[g] [|\gamma^+, \gamma^- > -|\gamma^+, \gamma^- >] 
= \sum_{v \in V(\gamma^-)} g(v) \lambda(\gamma^+, \gamma^-, v) \left\{ \sum_{\tilde{v} \in V(\gamma^+)} f(\tilde{v}) \lambda(\gamma^+, \gamma^-, \tilde{v}) [|\gamma^+, \gamma^- > -|\gamma^+, \gamma^- >] \right\} 
- \sum_{\tilde{v} \in V(\gamma^+)} f(\tilde{v}) \lambda(\gamma^+, \gamma^-, \tilde{v}) [|\gamma^+, \gamma^- > -|\gamma^+, \gamma^- >] 
- \sum_{v \in V(\gamma^+)} f(v) \lambda(\gamma^+, \gamma^-, v) \left\{ \sum_{\tilde{v} \in V(\gamma^-)} g(\tilde{v}) \lambda(\gamma^+, \gamma^-, \tilde{v}) [|\gamma^+, \gamma^- > -|\gamma^+, \gamma^- >] \right\} 
- \sum_{\tilde{v} \in V(\gamma^-)} g(\tilde{v}) \lambda(\gamma^+, \gamma^-, \tilde{v}) [|\gamma^+, \gamma^- > -|\gamma^+, \gamma^- >] \right\} 
$$
(A.19)

Relabelling $v \leftrightarrow \tilde{v}$ in the second term gives

$$
[\tilde{D}_+[f], \tilde{D}_-[g]] |\gamma^+, \gamma^- > 
= \sum_{v \in V(\gamma^-), \tilde{v} \in V(\gamma^+)} g(v) \lambda(\gamma^+, \gamma^-, v) f(\tilde{v}) \times \lambda(\gamma^+, \gamma^-, \tilde{v}) \left\{ \lambda(\gamma^+, \gamma^-, v) [|\gamma^+, \gamma^- > -|\gamma^+, \gamma^- >] - \lambda(\gamma^+, \gamma^-, \tilde{v}) [|\gamma^+, \gamma^- > -|\gamma^+, \gamma^- >] \right\} 
- \sum_{v \in V(\gamma^-), \tilde{v} \in V(\gamma^+)} f(\tilde{v}) \lambda(\gamma^+, \gamma^-, \tilde{v}) g(v) \times \lambda(\gamma^+, \gamma^-, v) [|\gamma^+, \gamma^- > -|\gamma^+, \gamma^- >] - \lambda(\gamma^+, \gamma^-, v) [|\gamma^+, \gamma^- > -|\gamma^+, \gamma^- >] \right\} 
= \sum_{v \in V(\gamma^-), \tilde{v} \in V(\gamma^+)} g(v) f(\tilde{v}) \left\{ \lambda(\gamma^+, \gamma^-, v) \lambda(\gamma^+, \gamma^-, \tilde{v}) - \lambda(\gamma^+, \gamma^-, v) \lambda(\gamma^+, \gamma^-, \tilde{v}) \lambda(\gamma^+, \gamma^-, v) [|\gamma^+, \gamma^- > + \lambda(\gamma^+, \gamma^-, v) \lambda(\gamma^+, \gamma^-, \tilde{v}) - \lambda(\gamma^+, \gamma^-, v) \lambda(\gamma^+, \gamma^-, \tilde{v}) \lambda(\gamma^+, \gamma^-, v) + \lambda(\gamma^+, \gamma^-, v) \lambda(\gamma^+, \gamma^-, \tilde{v}) - \lambda(\gamma^+, \gamma^-, v) \lambda(\gamma^+, \gamma^-, \tilde{v}) \lambda(\gamma^+, \gamma^-, v) \right\} 
$$
(A.20)

Recall that $\lambda(\gamma^+, \gamma^-, v) \neq 0$ only if $v \in V(\gamma^+) \cap V(\gamma^-)$ so that sum over $v, \tilde{v}$ runs over the same effective range. Furthermore, for $\tilde{v} \neq v$ we have

$$
\lambda(\gamma^+, \gamma^-, \tilde{v}) = \lambda(\gamma^+, \gamma^-, v), \quad \lambda(\gamma^+, \gamma^-, v) = \lambda(\gamma^+, \gamma^-, \tilde{v}) \quad (\text{A.21})
$$

Therefore the double sum in the second and third term collapses to $\tilde{v} = v$. In the first term, the contribution for $\tilde{v} \neq v$ again vanishes due to (A.21) while for $v = \tilde{v}$ we have $\lambda(\gamma^+, \gamma^-, v) = \lambda(\gamma^+, \gamma^-, \tilde{v}) = 0$ identically because $v$ is no vertex of $\gamma^+$. Therefore the contribution from the first term vanishes and we are left with

$$
[\tilde{D}_+[f], \tilde{D}_-[g]] |\gamma^+, \gamma^- > 
= \sum_{v \in V(\gamma^+) \cap V(\gamma^-)} f(v) g(v) \lambda(\gamma^+, \gamma^-, v)^2 \{ [|\gamma^+, \gamma^- > -|\gamma^+, \gamma^- >] - [|\gamma^+, \gamma^- > -|\gamma^+, \gamma^- >] \}
= \sum_{v \in V(\gamma^+) \cap V(\gamma^-)} f(v) g(v) \{ [U(\varphi^+_v, 1) - 1] - [U(1, \varphi^-_v) - 1] \} Q(v)^2 |\gamma^+, \gamma^- >
$$
(A.22)
Comparing with \((A.2)\) we see to some extent the correct structure: The result is local in that only a single sum is involved, it is a linear combination of an infinitesimal \(D_+\) and \(D_-\) constraint. However, in contrast to the classical computation no (discrete) derivatives seem to appear and the structure functions do not seem to match the \(Q(v)^2\) operator. Yet, we can match the two expressions as follows:

Recall the end result

\[
\{ \tilde{D}_+[f], \tilde{D}_-[g] \} = -\frac{1}{2} \int dx \; f \; D_+ \frac{X'_+}{\sqrt{\det(q)}} \left[ g \frac{X'}{\sqrt{\det(q)}} \right]' + \frac{1}{2} \int dx \; g \; D_- \frac{X'}{\sqrt{\det(q)}} \left[ f \frac{X'}{\sqrt{\det(q)}} \right]'
\]

(A.23)

To quantise this expression we use the identity

\[
\{ V(I_x), P_{\pm}(I'_x) \} = -\frac{1}{2} \frac{X'_+}{\sqrt{\det(q)}(x)}
\]

(A.24)

where \(I_x\) is any interval containing in \(x\) and \(I'_x = [y, x]\) where \(y \notin I_x\). Thus

\[
\{ \tilde{D}_+[f], \tilde{D}_-[g] \} = -2 \int dx \; f \; D_+ \frac{V(I_x), P_{-}(I'_x)}{\det(q)} \left[ g \{ V(I_x), P_{+}(I'_x) \} \right]' \\
+ 2 \int dx \; g \; D_- \frac{V(I_x), P_{+}(I'_x)}{\det(q)} \left[ f \{ V(I_x), P_{-}(I'_x) \} \right]'
\]

(A.25)

As for the quantisation of \(\tilde{D}_\pm\) we introduce a partition \(\tau\) and replace the derivative by a difference using the intervals \(J\) and their duals \(J'\) of the triangulation

\[
\{ \tilde{D}_+[f], \tilde{D}_-[g] \} = 2 \lim_{\tau \to S^1} \sum_{J \in \tau} \times
\]

\[
\{-f(b_J) \; D_+[J, J'] \frac{\{ V(J), P_{-}(J') \}}{V(J)^2} \left[ g(f_{J_f}) \{ V(J + 1), P_{+}(J' + 1) \} - g(b_J) \{ V(J), P_{+}(J') \} \right] \\
+ g(b_J) \; D_-[J, J'] \frac{\{ V(J), P_{+}(J') \}}{V(J)^2} \left[ f(f_{J_f}) \{ V(J + 1), P_{-}(J' + 1) \} - f(b_J) \{ V(J), P_{-}(J') \} \right] \}
\]

(A.26)

where \(J + 1, J' + 1\) is the (dual) interval next neighbour to \(J, J'\) respectively. Now the inverse volume factors \(1/V(J)\) can be treated as in section 3 giving rise to \(Q(J)^2\) so that

\[
\{ \tilde{D}_+[f], \tilde{D}_-[g] \} = 2 \lim_{\tau \to S^1} \sum_{J \in \tau} \times
\]

\[
\{-f(b_J) \; D_+[J, J'] \{ V(J), P_{-}(J') \} \left[ g(f_{J_f}) \{ V(J + 1), P_{+}(J' + 1) \} - g(b_J) \{ V(J), P_{+}(J') \} \right] \\
+ g(b_J) \; D_-[J, J'] \{ V(J), P_{+}(J') \} \left[ f(f_{J_f}) \{ V(J + 1), P_{-}(J' + 1) \} - f(b_J) \{ V(J), P_{-}(J') \} \right] \}
\]

(A.27)

where \(J + 1, J' + 1\) are (dual) intervals infinitesimally translated from \(I_v, I'_v\) in particular they do not contain any vertex of the graphs. Consequently its contribution vanishes and the operator indeed reduces to \((A.22)\) up to a factor of 1/2 which, however, could be absorbed into the choices of the actual intervals that enter the definition of \(D_\pm[J, J']\) and which cannot be fixed by the arguments of section 3. Hence also the \(\tilde{D}_+, \tilde{D}_+\) commutator closes in the expected way up to a soft anomaly.

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