Uniting cosmological epochs through the twister solution in cosmology with non-minimal coupling

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Abstract. We investigate dynamics of a flat FRW cosmological model with a barotropic matter and a non-minimally coupled scalar field (both canonical and phantom). In our approach we do not assume any specific form of a potential function for the scalar field and we are looking for generic scenarios of evolution. We show that dynamics of universe can be reduced to a 3-dimensional dynamical system. We have found the set of fixed points and established their character. These critical points represent all important epochs in evolution of the universe: (a) a finite scale factor singularity, (b) an inflation (rapid-roll and slow-roll), (c) a radiation domination, (d) a matter domination and (e) a quintessence era. We have shown that the inflation, the radiation and matter domination epochs are transient ones and last for a finite amount of time. The existence of the radiation domination epoch is purely the effect of a non-minimal coupling constant. We show the existence of a twister type solution wandering between all these critical points.

Keywords: modified gravity, dark energy theory, inflation, cosmic singularity

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Introduction

In modern cosmology a scalar field $\phi$ plays a very special role. The discovery of accelerated cosmic expansion [1, 2] gave a motivation to study dynamics of dark energy models (see [3] for review). In the context of the quintessence idea [4, 5] the simplest dynamical models involving the scalar field $\phi$ with a potential function $V(\phi)$ are used to model a time dependent equation of state parameter $w_\phi$. While the simplest candidate for dark energy seems to be a positive cosmological constant, the $\Lambda$CDM model is favoured by observational data [6–8], such a explanation of cosmic acceleration suffers from the fine tuning problem [9] and the coincidence problem [10]. In order to alleviate those problems many alternatives have been proposed like phantom dark energy [11, 12] or extended quintessence [13–16].

If we are going to generalise the scalar field cosmology minimally coupled with gravity, then inclusion of the non-minimal coupling term of type $-\xi R\phi^2$ [17–19] seems to be natural and the simplest generalisation of the Lagrangian for scalar field dynamics in the background of cosmological models with maximal symmetry of space-like slices. Of course the value of this additional parameter should be estimated from observational data [20–22] or given from some theoretical arguments [14]. The nonzero $\xi$ arises from quantum corrections [23] and it is required by the renormalization [18]. While the simplest minimally coupled scalar field with a quadratic potential function has strong motivations in observations [24, 25] its generalisations with a non-minimal coupling term was studied [26] in the context of origin of the canonical inflaton itself.

In this paper we investigate the dynamical evolution of scalar field cosmological models with a non-vanishing coupling constant between a scalar field and gravity. The role of the non-minimal coupling in evolution of the universe in the context of inflation and quintessence was studied previously by many authors [27–48] and in connection with the development of the Standard Model with a non-minimally coupled Higgs field [49–52]. The dynamical systems methods are used in investigating of evolutional paths of cosmological models which dynamics is parameterised by the energetic variables very useful in this context [53].

We are adopting dynamical systems methods in studying the evolution of the cosmological model and its dynamics can be visualised in the phase space which is a geometric framework for its exploration. Moreover one can investigate all evolutional paths of the system under consideration for all admissible initial conditions. Therefore one should ask whether different models with a desired property are typical (generic) in the class of all possible models. In our opinion physically interesting
cosmological models should be generic in the sense that they do not depend on the special choice of initial conditions which should be determined from quantum models.

To keep generality of our considerations we do not assume any specific form of the potential function of a canonical or phantom scalar field. It will be demonstrated that the parameter of non-minimal coupling $\xi$ plays the crucial role during the cosmological evolution. We will show the emergence of a new phase space structure organised through critical points and trajectories. For completeness we also include the model with the barotropic matter with the constant equation of state parameter $w_m$.

We demonstrate that, in principle, the phase space structure at a finite domain is determined by five critical points corresponding to important events during the cosmic evolution, namely, the singularity (of a finite scale factor type), inflation, radiation and matter dominated epochs and finally the accelerated expansion era. In our previous paper we introduced notion of the twister solutions linking the subsequent cosmological epochs \[54\]. In the present paper we generalise this notion without assuming any form of the potential function of the scalar field. The evolutional scenarios investigated in this paper are obvious only if the non-minimal coupling is different from minimal ($\xi = 0$) and conformal ($\xi = 1/6$) coupling value. In this sense we study a unique type of evolution.

2 The model

In the model under consideration we assume the spatially flat Friedmann-Robertson-Walker (FRW) universe filled with the non-minimally coupled scalar field and barotropic fluid with the equation of state $w_m$. The action assumes following form

$$S = \frac{1}{2} \int d^4x \sqrt{-g} \left( \frac{1}{\kappa^2} R - \varepsilon \left( g^{\mu\nu} \partial_{\mu} \phi \partial_{\nu} \phi + \xi R \phi^2 \right) - 2U(\phi) \right) + S_m, \quad (2.1)$$

where $\kappa^2 = 8\pi G$, $\varepsilon = +1, -1$ corresponds to canonical and phantom scalar field, respectively, the metric signature is $(-, +, +, +)$, $R = 6 \left( \frac{\ddot{a}}{a^2} + \frac{\dot{a}^2}{a^2} \right)$ is the Ricci scalar, $a$ is the scale factor and a dot denotes differentiation with respect to the cosmological time and $U(\phi)$ is the scalar field potential function. $S_m$ is the action for the barotropic matter part.

The dynamical equation for the scalar field we can obtain from the variation $\delta S/\delta \phi = 0$

$$\ddot{\phi} + 3H \dot{\phi} + \xi R \phi + \varepsilon U'(\phi) = 0, \quad (2.2)$$

and energy conservation condition from the variation $\delta S/\delta g^{\mu\nu} = 0$

$$\mathcal{E} = \varepsilon \frac{1}{2} \dot{\phi}^2 + \varepsilon 3 \xi H^2 \phi^2 + \varepsilon 3 \xi H (\phi^2)' + U(\phi) + \rho_m - \frac{3}{\kappa^2} H^2. \quad (2.3)$$

Then conservation conditions read

$$\frac{3}{\kappa^2} H^2 = \rho_{\phi} + \rho_m, \quad (2.4)$$

$$\dot{H} = -\frac{\kappa^2}{2} \left[ (\rho_{\phi} + p_{\phi}) + \rho_m (1 + w_m) \right] \quad (2.5)$$

where the energy density and the pressure of the scalar field are

$$\rho_{\phi} = \varepsilon \frac{1}{2} \dot{\phi}^2 + U(\phi) + \varepsilon 3 \xi H^2 \phi^2 + \varepsilon 3 \xi H (\phi^2)', \quad (2.6)$$

$$p_{\phi} = \varepsilon \frac{1}{2} (1 - 4\xi) \dot{\phi}^2 - U(\phi) + \varepsilon \xi H (\phi^2)' - \varepsilon 2 \xi (1 - 6\xi) \dot{H} \phi^2 - \varepsilon 3 \xi (1 - 8\xi) H^2 \phi^2 + 2\xi \phi U'(\phi). \quad (2.7)$$

Note that, when the non-minimal coupling is present, the energy density $\rho_{\phi}$ and the pressure $p_{\phi}$ of the scalar field can be defined in several possible inequivalent ways. This corresponds to different
ways of writing the field equations. In the case adopted here the energy momentum tensor of the scalar field is covariantly conserved, which may not be true for other choices of $\rho_\phi$ and $p_\phi$ [55, 56]. For example, the redefinition of gravitational constant $\kappa_{\text{eff}}^{-2} = \kappa^{-2} - \varepsilon\xi^2\kappa^{-1/2}$ makes it time dependent. The effective gravitational constant can diverge for a critical value of the scalar field $\phi_c = \pm(\varepsilon\kappa^2\xi)^{-1/2}$. Though the FRW model remains regular at this point, the model is unstable with respect to arbitrary small anisotropic and inhomogeneous perturbations which become infinite there. This results in the formation of a strong curvature singularity prohibiting a transition to the region $\kappa_{\text{eff}}^2 < 0$ [57].

In what follows we introduce the energy phase space variables

$$ x \equiv \frac{\kappa\dot{\phi}}{\sqrt{6H}}, \quad y \equiv \frac{\kappa\sqrt{U(\phi)}}{\sqrt{3H}}, \quad z \equiv \frac{\kappa}{\sqrt{6}}\phi, \quad (2.8) $$

which are suggested by the conservation condition

$$ \frac{\kappa^2}{3H^2}\rho_\phi + \frac{\kappa^2}{3H^2}p_m = \Omega_\phi + \Omega_m = 1 \quad (2.9) $$
or in terms of the newly introduced variables

$$ \Omega_\phi = y^2 + \varepsilon\left[(1 - 6\xi)x^2 + 6\xi(x + z)^2\right] = 1 - \Omega_m. \quad (2.10) $$

The acceleration equation can be rewritten to the form

$$ \dot{H} = -\frac{\kappa^2}{2}\left(\rho_{\text{eff}} + p_{\text{eff}}\right) = -\frac{3}{2}\dot{H}^2(1 + w_{\text{eff}}) \quad (2.11) $$

where the effective equation of the state parameter reads

$$ w_{\text{eff}} = \frac{1}{1 - \varepsilon(1 - 6\xi)z^2}\left[-1 + \varepsilon(1 - 6\xi)(1 - w_m)x^2 + 6\xi(1 - 3w_m)(x + z)^2 + (1 + w_m)(1 - y^2) - \varepsilon2\xi(1 - 6\xi)z^2 - 2\xi y^2\lambda z\right] \quad (2.12) $$

where $\lambda = \frac{\sqrt{6}}{\kappa\sqrt{U(\phi)}} dU(\phi)$. The dynamical system describing the investigated models is in the following form [54, 58]

$$ x' = -(x - \frac{1}{2}\lambda y^2)\left[1 - \varepsilon6\xi(1 - 6\xi)z^2\right] + \frac{3}{2}(x + 6\xi z)\left[-\frac{4}{3} - 2\xi\lambda y^2 z + \varepsilon(1 - 6\xi)(1 - w_m)x^2 + \varepsilon2\xi(1 - 3w_m)(x + z)^2 + (1 + w_m)(1 - y^2)\right], \quad (2.13a) $$

$$ y' = y\left(2 - \frac{1}{2}\lambda x\right)\left[1 - \varepsilon6\xi(1 - 6\xi)z^2\right] + \frac{3}{2}y\left[-\frac{4}{3} - 2\xi\lambda y^2 z + \varepsilon(1 - 6\xi)(1 - w_m)x^2 + \varepsilon2\xi(1 - 3w_m)(x + z)^2 + (1 + w_m)(1 - y^2)\right], \quad (2.13b) $$

$$ z' = x\left[1 - \varepsilon6\xi(1 - 6\xi)z^2\right], \quad (2.13c) $$

$$ \lambda' = -\lambda^2(\Gamma - 1)x\left[1 - \varepsilon6\xi(1 - 6\xi)z^2\right]. \quad (2.13d) $$

where a prime denotes differentiation with respect to time $\tau$ defined as

$$ \frac{d}{d\tau} = \left[1 - \varepsilon6\xi(1 - 6\xi)z^2\right] \frac{d}{d\ln a} \quad (2.14) $$

where the expression in brackets is assumed as a positive quantity to assure that during the evolution with $\tau > 0$ the scale factor $a$ is growing, i.e. the universe expands, and

$$ \Gamma = \frac{U''(\phi)U(\phi)}{U'(\phi)^2}. $$
which can be integrated for some given function $\Gamma(\lambda)$, i.e. we need to define the potential function $U(\phi)$. In the special cases of the system with the cosmological constant or exponential potential, $U = U_0 = \text{const.}$ or $U = U_0 \exp(-\lambda \phi)$, the dynamical system (2.13) can be reduced to the 3-dimensional one due to the relation $\lambda = 0$ and $\Gamma = 0$ in the former case, and $\lambda = \text{const.}$ and $\Gamma = 1$ in the latter case. Then dynamical system consists of three equations (2.13a, 2.13b, 2.13c).

There is another possibility of reduction of the system (2.13) from a 4-dimensional dynamical system to a 3-dimensional one. If we assume that $z = z(\lambda)$ and $\Gamma = \Gamma(\lambda)$, then using (2.13c) and (2.13d) we can find the function $z(\lambda)$ from the differential equation

$$\frac{dz(\lambda)}{d\lambda} = z'(\lambda) = -\frac{1}{\lambda^2 (\Gamma(\lambda) - 1)}, \quad (2.15)$$

which can be integrated for some given function $\Gamma(\lambda)$

$$z(\lambda) = -\int \frac{d\lambda}{\lambda^2 (\Gamma(\lambda) - 1)}. \quad (2.16)$$

Then the dynamical system describing the investigated models is in the following form [54, 58]

$$x' = -(x - \frac{1}{2} \lambda y^2) \left[ 1 - \varepsilon 6 \xi (1 - \varepsilon 6 \xi) z(\lambda)^2 \right] + \frac{3}{2} (x + 6 \xi z(\lambda)) \left[ -\frac{4}{3} - 2 \xi \lambda y^2 z(\lambda) + \varepsilon (1 - 6 \xi)(1 - w_m) x^2 + 2 \xi (1 - 3 w_m)(x + z(\lambda))^2 + (1 + w_m)(1 - y^2) \right], \quad (2.17a)$$

$$y' = y \left[ 2 - \frac{1}{2} \lambda x \right] \left[ 1 - \varepsilon 6 \xi (1 - \varepsilon 6 \xi) z(\lambda)^2 \right] + \frac{3}{2} y \left[ -\frac{4}{3} - 2 \xi \lambda y^2 z(\lambda) + \varepsilon (1 - 6 \xi)(1 - w_m) x^2 + 2 \xi (1 - 3 w_m)(x + z(\lambda))^2 + (1 + w_m)(1 - y^2) \right], \quad (2.17b)$$

$$\lambda' = -\lambda^2 (\Gamma(\lambda) - 1) x \left[ 1 - \varepsilon 6 \xi (1 - \varepsilon 6 \xi) z(\lambda)^2 \right]. \quad (2.17c)$$

where a prime denotes now differentiation with respect to time $\tau$ defined as

$$\frac{d}{d\tau} = \left[ 1 - \varepsilon 6 \xi (1 - \varepsilon 6 \xi) z(\lambda)^2 \right] \frac{d}{d\ln a} \quad (2.18)$$

and we assume that the term in bracket is positive in the phase space during the evolution.
Now we are able to express the acceleration equation (2.11) in terms of the energy phase space variables and time \(\tau\)
\[
\frac{d \ln H^2}{d\tau} = -3 \left[ 1 - \varepsilon 6 \xi (1 - 6 \xi) z(\lambda(\tau))^2 \right] (1 + w_{\text{eff}}) \tag{2.19}
\]
which together with (2.12) results in
\[
\ln \left( \frac{H}{H_{\text{ini}}} \right)^2 = -3 \int_0^\tau \left\{ 1 + w_m + \varepsilon (1 - 6 \xi)(1 - w_m) x(\tau)^2 + \varepsilon 2 \xi (1 - 3 w_m) \left( x(\tau) + z(\lambda(\tau)) \right)^2 - y(\tau)^2 \left( 2 \xi \lambda(\tau) z(\lambda(\tau)) + 1 + w_m \right) - \varepsilon 8 \xi (1 - 6 \xi) z(\lambda(\tau))^2 \right\} d\tau, \tag{2.20}
\]
where \(H_{\text{ini}}\) denotes the initial value of Hubble’s function at time \(\tau = 0\). In what follows we will be using this expression together with the linearised solutions in the vicinity of every critical point to investigate the behaviour of Hubble’s function with respect to the scale factor, as well as Hubble’s radius defined as
\[
R_H = \frac{1}{H}.
\]
For example if the function \(\Gamma(\lambda)\) is assumed in the following form
\[
\Gamma(\lambda) = 1 - \frac{1}{\lambda^2} (\alpha + \beta \lambda + \gamma \lambda^2),
\]
then in Table 1 we have gathered forms of the functions \(z(\lambda)\) and corresponding potential functions \(U(\phi)\) for various configurations of values of parameters \(\alpha, \beta\) and \(\gamma\). As we see there are various potential functions which are the most common used in the literature of the subject. Of course this simple ansatz for the function \(\Gamma(\lambda)\) does not manage all possible potential functions. Let us consider the following function
\[
\Gamma(\lambda) = \frac{3}{4} - \frac{\sigma^2 \lambda^2}{4(2 \pm \sqrt{4 + \sigma^2 \lambda^2})^2}
\]
as one can check from (2.16) we receive
\[
z(\lambda) = -\frac{2 \pm \sqrt{4 + \sigma^2 \lambda^2}}{\lambda} + \text{const.}
\]
and this example corresponds to the Higgs potential
\[
U(\phi) = U_0 \left( (\phi - \text{const.})^2 - \sigma^2 \right)^2.
\]
We need to stress that the discussion presented below is not restricted to the specific potential function but is generic in the sense that it is valid for any function \(\Gamma(\lambda)\) for which the integral defined in (2.16) exists.

3 Dynamics of universe with a potential

In our investigations of dynamics of the system given by eqs. (2.17) we will restrict ourselves to the finite region of the phase space, i.e. we will be interested only in the critical points in the finite domain of the phase space. The full investigations of the dynamics requires examination of critical points at infinity, i.e. compactification of the phase space with the Poincaré sphere. The procedure of transforming dynamical variables into the projective variables associated with the compactification requires that the right hand sides of the dynamical system should be polynomial. In our case this is not always true because of the form of function \(z(\lambda)\) (see table 1). In what follows we present detailed discussion of character of critical points of the system (2.17) corresponding to different stages of cosmological evolution (table 2).
In cosmological investigations one encounters usually various types of singularities such as: initial finite scale factor singularity [59, 60], future finite scale factor singularities: the sudden future singularity \([\xi < 1]\), and the Big Boost singularity [63].

Linearised solutions in the vicinity of this critical point are

\[
x_1(\tau) = x^*_1 + (y^*_1 - x^*_1) + 2(1 - 3\xi)(\lambda_1(\lambda_1 - \lambda_1^*) - (\lambda_1^* - \lambda_1^*) \exp (l_1\tau) - 2(1 - 3\xi)z'\lambda_1(\lambda_1^* - \lambda_1^*) \exp (l_3\tau),
\]

\[
y_1(\tau) = y^*_1 \exp (l_2\tau),
\]

\[
\lambda_1(\tau) = \lambda_1^* + (\lambda_1^* - \lambda_1^*) \exp (l_3\tau).
\]

where

\[
l_1 = 6\xi, \quad l_2 = 6\xi, \quad l_3 = 12\xi
\]

are the eigenvalues of the linearization matrix calculated at this critical point, \(x^*_1, y^*_1, \lambda_1^*\) are initial conditions. For positive values of the coupling constant \(\xi > 0\) this critical point represents an unstable node type critical point. For \(\xi < 0\) which is possible only for the phantom scalar field (\(\varepsilon = -1\)) the critical point is of a stable node type.

Using the time reparameterization (2.18)

\[
d \ln a = \left(1 - \varepsilon 6\xi(1 - 6\xi)z(\lambda(\tau))^2 \right) d\tau
\]

and expansion into the Taylor series around the critical point coordinate \(\lambda^*\)

\[
z(\lambda) = z(\lambda^*) + z'(\lambda^*)(\lambda - \lambda^*)
\]

up to linear terms

\[
z(\lambda)^2 = z(\lambda^*)^2 + 2z(\lambda^*)z'(\lambda^*)(\lambda - \lambda^*)
\]
and then together with (3.2c) we have

\[ z(\lambda(\tau))^2 = z(\lambda^*_i)^2 + 2z(\lambda^*_i)z'(\lambda^*_i)(\lambda^{ini}_i - \lambda^*_i) \exp(l_3\tau). \]

Inserting this expansion into the time reparameterization we receive

\[ d\ln a = -\varepsilon 12\xi (1 - 6\xi)z(\lambda^*_i)z'(\lambda^*_i)(\lambda^{ini}_i - \lambda^*_i) \exp(l_3\tau) \]

which can be directly integrated for \( l_3 = 12\xi > 0 \)

\[ \Delta \ln a = -\varepsilon 12\xi (1 - 6\xi)z(\lambda^*_i)z'(\lambda^*_i)(\lambda^{ini}_i - \lambda^*_i) \int_{-\infty}^{0} \exp(l_3\tau) d\tau \]

(we could take also \( l_3 = 12\xi < 0 \) and the integration in this expression should be taken \((0, \infty)\) because for \( \xi < 0 \) this critical point represents a stable node). Where the result is

\[ \Delta \ln a = \ln \left( \frac{a^{ini}_i}{a_s} \right) = -\varepsilon (1 - 6\xi)z(\lambda^*_i)z'(\lambda^*_i)(\lambda^{ini}_i - \lambda^*_i) \]

Finally we receive

\[ a^{ini}_i = a_s \exp \left\{ -\varepsilon (1 - 6\xi)z(\lambda^*_i)z'(\lambda^*_i)(\lambda^{ini}_i - \lambda^*_i) \right\} \]

where the value in the exponent is finite, and \( a^{ini}_i \) is the value of the scale factor at \( \tau = 0 \) and \( a_s \) is the value of the scale factor at singularity. This equation gives us the scale factor growth from singularity to the some initial point where linear approximation is still valid. From simple considerations we have that

\[ z(\lambda^*_i)z'(\lambda^*_i)(\lambda^{ini}_i - \lambda^*_i) < 0 \]

which show that the critical point under consideration represents the finite scale factor singularity. Moreover it is a past singularity for the canonical scalar field with \( 0 < \xi < 1/6 \) and the phantom scalar field with \( \xi > 1/6 \) and future singularity for the phantom scalar field with \( \xi < 0 \).

Now, using linearised solutions (3.2), we can express (2.18) and (2.20) as parametric functions of time \( \tau \)

\[
\begin{align*}
\ln \left( \frac{a^{ini}_i}{a_s} \right)^2 &= \varepsilon (1 - 6\xi)z(\lambda^*_i)z'(\lambda^*_i)(\lambda^{ini}_i - \lambda^*_i)(1 - \exp(l_3\tau)), \\
\ln \left( \frac{H^{ini}_{1}}{H_{1,ini}} \right)^2 &= 3 \left\{ -4\xi \tau - \varepsilon \frac{1}{2} (1 - 6\xi)z(\lambda^*_i)[(x^{ini}_i - x^*_i) + 2(1 - 3\xi)z'(\lambda^*_i)(\lambda^{ini}_i - \lambda^*_i)](1 - \exp(l_1\tau)) \\
&+ \varepsilon \frac{1}{2} (1 - 6\xi)(1 - 3w_m - 12\xi)z(\lambda^*_i)z'(\lambda^*_i)(\lambda^{ini}_i - \lambda^*_i)(1 - \exp(l_3\tau)) \right\}.
\end{align*}
\]

(3.4)

The linearised solutions used to obtain these relations are valid up to the Lyapunov characteristic time which is equal to the inverse of the largest eigenvalue of the linearization matrix. In our case it is \( \tau_{end} = \frac{1}{l_3} = \frac{1}{12\xi} \). Inserting this in to equations (3.4) we obtain maximal values of the scale factor and the Hubble’s function respectively:

\[
\begin{align*}
\ln \left( \frac{a^{end}_i}{a^{ini}_i} \right)^2 &= \varepsilon (1 - 6\xi)z(\lambda^*_i)z'(\lambda^*_i)(\lambda^{ini}_i - \lambda^*_i)(1 - e), \\
\ln \left( \frac{H^{end}_{1}}{H^{ini}_{1}} \right)^2 &= 3 \left\{ -\frac{1}{3} - \varepsilon \frac{1}{2} (1 - 6\xi)z(\lambda^*_i)[(x^{ini}_i - x^*_i) + 2(1 - 3\xi)z'(\lambda^*_i)(\lambda^{ini}_i - \lambda^*_i)](1 - e) \\
&+ \varepsilon \frac{1}{2} (1 - 6\xi)(1 - 3w_m - 12\xi)z(\lambda^*_i)z'(\lambda^*_i)(\lambda^{ini}_i - \lambda^*_i)(1 - e) \right\}.
\end{align*}
\]

(3.5)

The plot representing the evolution of these quantities together with Hubble’s horizon is presented in figure 1. As one can simply conclude

\[ \xi > 0 : \lim_{\tau \to \infty} H^2 \to \infty. \]

The general conclusion is that any phantom scalar field cosmological model with the negative coupling constant \( \xi < 0 \) and the potential function which can be represented by function \( z(\lambda) \) possesses the finite scale factor future singularity with \( w_{\text{eff}} = \pm \infty \).
In a

/Slash1

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Figure 1. Evolution of \( \ln H^2 \) (left panel) and \( R_H \) (right panel) as a function of a natural logarithm of the scale factor \( \ln a \) for a sample trajectory with \( \varepsilon = +1, \xi = \frac{1}{6}, z'(\lambda^*_{2a}) = \frac{1}{12} = 100 \) in the vicinity of the critical point corresponding to the finite scale factor singularity. The solid black line represents the linearised solution (3.4) and the dotted line corresponds to the numerical solution of the system (2.17).

3.2 Inflation with the non-minimal coupling and arbitrary potential

Now we proceed to the very important phase of the evolution of the universe, namely the inflation.

3.2.1 Fast-roll inflation

The critical point located at

\[
x^*_2 = -6\xi z(\lambda^*_{2a}), \quad (y^*_{2a})^2 = \frac{4\xi}{2\xi \lambda^*_{2a} z(\lambda^*_{2a}) + (1 + w_m)}, \quad \lambda^*_{2a} : z(\lambda) = \frac{1}{\varepsilon 6\xi (1 - 6\xi)}
\]

with

\[ w_{\text{eff}} = w_m - 4\xi \]

we identify as a fast-roll inflation (or rapid-roll) \([64–66]\). The first reason is that \( w_{\text{eff}} \) calculated at this critical point can be made close to \(-1\) especially for the phantom scalar field, and the second one is that the first coordinate of this point, using transformations (2.8) can be put in the following form

\[
\dot{\phi} = -6\xi H \phi
\]

which for the conformal coupling \( \xi = 1/6 \), reduces to condition for the rapid-roll inflation given by Kofman and Mukohyama in \([65]\). That is we identify this critical point as a generalisation to the non-minimally coupled case (both for the canonical and phantom scalar fields) with additional presence of the barotropic matter with the equation of state parameter \( w_m \).

The linearization matrix for this critical point is the following

\[
A_{2a} = \begin{pmatrix}
0 & 0 & \frac{\partial y^*_{2a}}{\partial \lambda_{2a}}
\
0 & 0 & \frac{\partial y^*_{2a}}{\partial \lambda_{2a}}
\end{pmatrix},
\]

where

\[
\frac{\partial y^*_{2a}}{\partial \lambda_{2a}} = -3(y^*_{2a})^2 (1 + w_m + 4\xi(1 - 3\xi)\lambda^*_{2a} z(\lambda^*_{2a})),
\]

\[
\frac{\partial y^*_{2a}}{\partial \lambda_{2a}} = -\varepsilon 12\xi (1 - 6\xi) y^*_{2a} z(\lambda^*_{2a}),
\]

\[
\frac{\partial y^*_{2a}}{\partial \lambda_{2a}} = -3\xi y^*_{2a} [(y^*_{2a})^2 z(\lambda^*_{2a}) + (\varepsilon 6(1 - 6\xi)(1 + w_m)z(\lambda^*_{2a}) + (2 + (y^*_{2a})^2)\lambda^*_{2a}) z'(\lambda^*_{2a})]
\]

The eigenvalues of the linearization matrix are obviously \( l_1 = 0, l_2 = 12\xi \) and \( l_3 = -12\xi \). Thus the fixed point is a non-hyperbolic and we cannot make any conclusions concerning its stability based
Then dynamical system in the vicinity of the critical point representing the fast-roll inflation is in the form

\[ \frac{dx}{dt} = f(x, y), \quad \frac{dy}{dt} = g(x, y) \]

for \( \varepsilon(1 - 3w_m) < 0 \) and the right diagram for a stable case for \( \varepsilon(1 - 3w_m) > 0 \). The example is given for \( z(\lambda) = \frac{1}{\lambda} \) and \( \alpha = 1 \), \( w_m = 0 \) left for \( \varepsilon = -1 \) and \( \xi = 1/4 \), right for \( \varepsilon = 1 \) and \( \xi = 1/8 \).

On the left diagram we present an unstable case for \( \varepsilon \) and the bold line represents the center manifold for the problem. The question of stability or instability lies in the center manifold theory (see appendix A).

We apply following procedure: first, we expand the right hand side of the dynamical system \( (2.17) \) into the Taylor series around the critical point \( (3.6) \) up to second order, and second, we make following change of dynamical variables

\[
\begin{pmatrix}
  u \\
  v \\
  w
\end{pmatrix} = P^{-1}_{2a} \begin{pmatrix}
  x - x^*_{2a} \\
  y - y^*_{2a} \\
  \lambda - \lambda^*_{2a}
\end{pmatrix},
\]

where the matrix \( P_{2a} \) is constructed from eigenvectors of the linearization matrix \( (3.7) \) calculated for corresponding eigenvalues and its inverse is

\[
P^{-1}_{2a} = \begin{pmatrix}
  1 & 0 & -\frac{1}{12} \frac{\partial \gamma'}{\partial \xi} |_{2a} \\
  -\frac{1}{12} \frac{\partial \gamma'}{\partial \xi} |_{2a} & 1 & 0 \\
  0 & 0 & 1
\end{pmatrix}.
\]

Then dynamical system in the vicinity of the critical point representing the fast-roll inflation is in the following form

\[
\begin{align*}
  \dot{u} &= -3y^*_2 \left( 2\lambda^*_{2a} z(\lambda^*_{2a}) + 1 + w_m \right) uv + A_u w^2 + B_u vw + C_u uw, \\
  \dot{v} &= -12\xi v + \varepsilon \frac{1}{2} (1 - 3w_m) g^*_2 w^2 - \frac{9}{2} \left( 2\lambda^*_{2a} z(\lambda^*_{2a}) + 1 + w_m \right) g^*_2 v^2 + \varepsilon 12\xi (1 - 6\xi) z(\lambda^*_{2a}) uv \\
  &\quad + A_v w^2 + B_v vw + C_v vw, \\
  \dot{w} &= 12\xi w + A_w w^2 + B_w vw,
\end{align*}
\]

where \( A_i, B_i \) and \( C_i \) are coefficients consisting of second derivatives of right-hand sides of dynamical system \( (2.17) \) calculated at the critical point under considerations. We can note that this dynamical system admits the invariant submanifold \( w = 0 \), and the dynamics can be well approximated on this submanifold. Then

\[
\begin{align*}
  \dot{u} &= -3y^*_2 \left( 2\lambda^*_{2a} z(\lambda^*_{2a}) + 1 + w_m \right) uv, \\
  \dot{v} &= -12\xi v + \varepsilon \frac{1}{2} (1 - 3w_m) g^*_2 w^2 - \frac{9}{2} \left( 2\lambda^*_{2a} z(\lambda^*_{2a}) + 1 + w_m \right) g^*_2 v^2 + \varepsilon 12\xi (1 - 6\xi) z(\lambda^*_{2a}) uv.
\end{align*}
\]
on the invariant submanifold \( w = 0 \).

From the center manifold theorem (appendix A) we have

\[
v = h(u) = \varepsilon \frac{1}{24 \xi}(1 - 3 w_m) g'_{2a} u^2 + \frac{1 - 6 \xi}{24 \xi}(1 - 3 w_m) z(\lambda'_{2a}) g'_{2a} u^3 + O(u^4)
\]

and inserting this approximation into (3.9a) we receive that the vector field restricted to the center manifold is given by

\[
\eta' = -\frac{1}{2}(1 - 3 w_m) \eta^3 + O(\eta^4),
\]

which indicates that for \( \varepsilon (1 - 3 w_m) < 0 \) it is an unstable and for \( \varepsilon (1 - 3 w_m) > 0 \) it is a stable critical point on the invariant submanifold \( w = 0 \) (see figure 2 example for \( z(\lambda) = \frac{1}{\sqrt{\lambda}} \)). This equation can be simply integrated resulting in

\[
\eta(\tau)^2 = \frac{(\eta^{\text{ini}})^2}{(\tau^{\text{ini}})^2 \varepsilon (1 - 3 w_m) \tau + 1}.
\]

Above equation describes behaviour of the system on the center manifold which constitutes the invariant submanifold. This solution can be used in construction of exact solution of the system (3.9) in the vicinity of the critical point representing the fast-roll inflation epoch.

Using the solution from the center manifold theorem and keeping linear term in \( w \) only

\[
u(\tau)v(\tau) \propto u(\tau) \propto u(\tau)^3 \approx 0, \quad v(\tau)w(\tau) \propto u(\tau)^2 w(\tau) \approx 0, \quad v(\tau)^2 \propto u(\tau)^4 \approx 0, \quad w(\tau)^2 \approx 0, \quad u(\tau)w(\tau) \approx 0
\]

from (2.18) and (2.20) we get the parametric equations for the evolution of the scale factor and Hubble’s function

\[
\begin{align*}
\ln \left( \frac{a}{a^\text{ini}} \right) &= -\varepsilon(1 - 6 \xi) z(\lambda'_{2a}) z'(\lambda'_{2a}) u^{\text{ini}} \left( \exp(12 \xi \tau) - 1 \right), \\
\ln \left( \frac{H}{H^\text{ini}} \right)^2 &= -\frac{1}{4} A w^{\text{ini}} \left( \exp(12 \xi \tau) - 1 \right),
\end{align*}
\]

which can be easy combine resulting in

\[
\ln \left( \frac{H}{H^\text{ini}} \right)^2 = -\frac{A}{\varepsilon 4 \xi (1 - 6 \xi) z(\lambda'_{2a}) z'(\lambda'_{2a})} \ln \left( \frac{a}{a^\text{ini}} \right)
\]

where

\[
A = \frac{1}{72 (2 \xi \lambda'_{2a} z(\lambda'_{2a}) + (1 + w_m))} \left( -\varepsilon 144 \xi (1 - 6 \xi)(1 + w_m) (1 + 3 w_m) z(\lambda'_{2a}) + 96 \xi (2 - 9 \xi) \lambda^2 + +288 \xi^2 (\lambda'_{2a})^2 z(\lambda'_{2a}) - 288 \xi^2 z(\lambda'_{2a}) \right).
\]

One can conclude that needs \( |A| \ll 1 \) in order to achieve \( H^2 \approx \text{const.} \) during the evolution. The linearised solution in \( w \) direction is valid up to the Lyapunov time \( \tau_{\text{end}} = \frac{1}{12 \xi} \), using this we obtain maximal values of the scale factor and the Hubble’s function respectively:

\[
\begin{align*}
\ln \left( \frac{a}{a^\text{ini}} \right) &= -\varepsilon(1 - 6 \xi) z(\lambda'_{2a}) z'(\lambda'_{2a}) u^{\text{ini}} \left( e - 1 \right), \\
\ln \left( \frac{H}{H^\text{ini}} \right)^2 &= -\frac{1}{4} A w^{\text{ini}} \left( e - 1 \right),
\end{align*}
\]

In figure 3 we present the evolution of Hubble’s function and Hubble’s horizon in the vicinity of this critical point.

### 3.2.2 Slow-roll inflation

The critical point located at

\[
x^*_{2b} = 0, \quad (g'_{2b})^2 = \frac{2 \xi (1 - 3 w_m)}{(1 - 6 \xi) (2 \xi \lambda^2 z(\lambda^2) + (1 + w_m))}, \quad \lambda^*_{2b} z(\lambda)^2 = \frac{1}{\varepsilon 6 \xi (1 - 6 \xi)}
\]
can be presented in the following form

\[ A \]

where nonzero elements are

\[ \text{corresponding to the fast-roll inflation. The solid black line represents the linearised solution (3.10) and the dotted line represents the numerical solution of the system (2.17).}\]

\[ \varepsilon \]

\[ \phi, \phi' \]

\[ \text{we identify as representing the phase of a slow-roll inflation due to } x \propto \phi \text{ so the dynamics in the vicinity of this point corresponds to the slow-roll condition } \phi \approx 0. \]

\[ \text{The linearization matrix is in the form}\]

\[ A_{2b} = \begin{pmatrix} \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} & \frac{\partial x'}{\partial \lambda} \\ \frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y} & \frac{\partial y'}{\partial \lambda} \\ \frac{\partial \lambda'}{\partial x} & \frac{\partial \lambda'}{\partial y} & \frac{\partial \lambda'}{\partial \lambda} \end{pmatrix}_{2b}, \tag{3.14} \]

\[ \text{where nonzero elements are}\]

\[ \left. \begin{array}{l}
\frac{\partial x'}{\partial x} \bigg|_{2b} = \frac{6\varepsilon}{1-6\varepsilon}(1-3w_m), \\
\frac{\partial x'}{\partial y} \bigg|_{2b} = -18\varepsilon \left( 2\varepsilon \lambda_2^2 z(\lambda_2^2) + (1 + w_m) \right) \lambda_2^2(\lambda_2^2), \\
\frac{\partial x'}{\partial \lambda} \bigg|_{2b} = \frac{3}{1-6\varepsilon}(y^*_{2b})^2 \left( -\varepsilon + (1-6\varepsilon) z^2(\lambda_2^2)(6\varepsilon^2 \lambda_2^2 z(\lambda_2^2) + (1 + w_m)) \right), \\
\frac{\partial y'}{\partial x} \bigg|_{2b} = \varepsilon(1-3w_m) \lambda_2^2, \\
\frac{\partial y'}{\partial y} \bigg|_{2b} = -\frac{6\varepsilon}{1-6\varepsilon}(1 - 3w_m), \\
\frac{\partial y'}{\partial \lambda} \bigg|_{2b} = -3\varepsilon^2 y^*_{2b} \left( (y^*_{2b})^2 \lambda_2^2 z(\lambda_2^2) + z^2(\lambda_2^2) \left( (y^*_{2b})^2 \lambda_2^2 + \varepsilon z(\lambda_2^2)(1 + w_m - 8\varepsilon) \right) \right),
\end{array} \right. \]

\[ \text{and additionally we have the following relation}\]

\[ \left. \frac{\partial x'}{\partial x} \bigg|_{2b} \frac{\partial y'}{\partial y} \bigg|_{2b} = -\left( \frac{6\varepsilon}{1-6\varepsilon} \right)^2 (1 - 3w_m)^2. \]

\[ \text{The characteristic equation for the linearization matrix gives us vanishing eigenvalues } \lambda_1 = \lambda_2 = \lambda_3 = 0, \text{ so the critical point is degenerated. In this case we cannot use standard procedures, following the Hartman-Grobman theorem [67, 68] of determining qualitative behaviour of the investigated system in the vicinity of this critical point. Instead we can notice that the linearization matrix } A_{2b} \text{ calculated at this critical point is nilpotent of order 3, i.e. } (A_{2b})^3 = 0. \text{ Then solution of the linearised problem can be presented in the following form}\]

\[ x(\tau) = \left[ 1 + A_{2b} \tau + \frac{1}{2} (A_{2b})^2 \tau^2 \right] x_0 \]
the effective equation of the state parameter is

\[ \text{be used to calculate the scale factor growth during the slow roll inflation.} \]

\[ \text{The solid black line represents the linearised solution (3.15)} \]

and the dotted line represents the numerical solution of the system (2.17).

Finally solutions in the vicinity of this degenerated critical point up to linear terms are

\[
x(\tau) = x^{\text{ini}}_{2b} + \left( \frac{\partial x'}{\partial \lambda} \right)_{2b} \left( \frac{\partial x'}{\partial y} \right)_{2b} \left( x^{\text{ini}}_{2b} - x^*_{2b} \right) + \frac{1}{2} \left( \frac{\partial x'}{\partial \lambda} \right)_{2b}^2 \left( \lambda^{\text{ini}}_{2b} - \lambda^*_{2b} \right) \tau^2,
\]

\[
y(\tau) = y^{\text{ini}}_{2b} + \left( \frac{\partial y'}{\partial \lambda} \right)_{2b} \left( \frac{\partial y'}{\partial y} \right)_{2b} \left( y^{\text{ini}}_{2b} - y^*_{2b} \right) + \frac{1}{2} \left( \frac{\partial y'}{\partial \lambda} \right)_{2b}^2 \left( \lambda^{\text{ini}}_{2b} - \lambda^*_{2b} \right) \tau^2,
\]

\[
\lambda(\tau) = \lambda^{\text{ini}}_{2b}.
\]

where

\[
\frac{\partial x'}{\partial \lambda} \bigg|_{2b} + \frac{\partial y'}{\partial \lambda} \bigg|_{2b} = 6 \xi (1 + w_m) \left( 2 + 3 \xi \frac{\lambda^*_{2b}}{\lambda^{\text{ini}}_{2b}} z(\lambda^{\text{ini}}_{2b}) \right) \left( y^{\text{ini}}_{2b} - y^*_{2b} \right) \frac{\partial \xi}{\partial \lambda} \bigg|_{2b} \left( \lambda^{\text{ini}}_{2b} - \lambda^*_{2b} \right) \tau^2,
\]

These linearised solutions are valid up to a maximal value of the time parameter \( \tau = \tau_{\text{max}} \) which can be used to calculate the scale factor growth during the slow roll inflation

\[
\ln a^{\text{end}}_{\text{ini}} = -\varepsilon 24 \xi (1 - 6 \xi) z(\lambda^{\text{ini}}_{2b}) z(\lambda_{3a}^*) \left( \lambda^{\text{ini}}_{2b} - \lambda^*_{2b} \right) \tau_{\text{max}}.
\]

The direct application of the linearised solutions (3.15) to (2.18) and (2.20) gives us the approximated evolution of Hubble’s function in the vicinity of the critical point representing the slow-roll inflation (figure 4).

3.3 Radiation domination epoch generated by non-minimal coupling

Following critical point located at

\[
x^*_{3a} : g(x) = 0, \quad y^*_{3a} = 0, \quad \lambda^*_{3a} : z(\lambda)^2 = \frac{1}{\varepsilon 6 \xi (1 - 6 \xi)}
\]

where \( g(x) = \varepsilon (1 - 4 \xi - w_m)x^2 + \varepsilon 4 \xi (1 - 3 w_m)z(\lambda^*_{3a})x + \frac{2 \xi}{1 - 6 \xi}(1 - 3 w_m) \) and at this point value of the effective equation of the state parameter is

\[
w_{\text{eff}} = \frac{1}{3}
\]
represents the radiation dominated universe. With solutions to \( g(x) = 0 \) equation in the form
\[
x_{1,2} = \frac{1}{\varepsilon 2(1 - 4\xi - w_m)} \left\{ -\varepsilon 4\xi (1 - 3w_m)z(\lambda^*_{3a}) \pm \sqrt{-\varepsilon \frac{16}{3}\xi (1 - 3w_m)} \right\}
\]
which is real only if the expression in square root is positive and for the barotropic matter with \( w_m < \frac{1}{3} \) it is possible only if \( \varepsilon \xi < 0 \). We are interested only in evolution with \( \xi > 0 \) because of the discussion of the critical point representing the finite scale factor singularity, and this is the reason we identify this critical point as a representing radiation dominated epoch only for the phantom scalar field.

The linearization matrix calculated at this point is in the following form
\[
A_{3a} = \begin{pmatrix}
\frac{\partial x'}{\partial x}|_{3a} & 0 & \frac{\partial x'}{\partial x}|_{3a} \\
0 & 0 & \frac{\partial \lambda}{\partial x}|_{3a}
\end{pmatrix}, \quad (3.17)
\]
where
\[
\frac{\partial x'}{\partial x}|_{3a} = \varepsilon 12\xi (1 - 6\xi)z(\lambda^*_{3a})x^*_3a, \\
\frac{\partial x'}{\partial x}|_{3a} = 6\xi z' (\lambda^*_{3a}) \left[ (1 - 6\xi) z(\lambda^*_{3a}) x^*_3a + (1 - 3w_m) (x^*_3a + 6\xi z(\lambda^*_{3a})) (x^*_3a + z(\lambda^*_{3a})) \right], \\
\frac{\partial \lambda}{\partial x}|_{3a} = -\varepsilon 12\xi (1 - 6\xi) z(\lambda^*_{3a}) x^*_3a.
\]

Eigenvalues of the linearization matrix are \( l_1 = -\varepsilon 12\xi (1 - 6\xi) z(\lambda^*_{3a}) x^*_3a, l_2 = 0, l_3 = \varepsilon 12\xi (1 - 6\xi) z(\lambda^*_{3a}) x^*_3a \). This indicates that the critical point is non-hyperbolic one and the standard linearization procedure will be inefficient and we need to proceed with the center manifold theorem (see appendix A) and the procedure described during the discussion of the critical point representing fast-roll inflation. We make following change of dynamical variables
\[
\begin{pmatrix}
u \\
w \\
x - x^*_3a \\
y - y^*_3a \\
\lambda - \lambda^*_{3a}
\end{pmatrix} = P^{-1}_{3a} \begin{pmatrix}
u \\
w \\
x - x^*_3a \\
y - y^*_3a \\
\lambda - \lambda^*_{3a}
\end{pmatrix},
\]
where matrix \( P_{3a} \) is constructed from eigenvectors of the linearization matrix (3.17) and its inverse is
\[
P^{-1}_{3a} = \begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & \chi
\end{pmatrix}, \quad \text{and} \quad \chi = \frac{\partial x'|_{3a}}{\partial x|_{3a}}.
\]

Then dynamical system can be presented in the following form
\[
\begin{align*}
\dot{u} &= -\varepsilon 12\xi (1 - 6\xi) z(\lambda^*_{3a}) x^*_3a u + A_u u^2 + B_u u w, \\
\dot{v} &= A_v u v + B_v u w, \\
\dot{w} &= \varepsilon 12\xi (1 - 6\xi) z(\lambda^*_{3a}) x^*_3a w + A_w u^2 + B_w v^2 + C_w w^2 + D_u u w,
\end{align*}
\]
where \( A_i, B_i, C_i \) and \( D_i \) are coefficients consisting of second derivatives of the right hand sides of dynamical system (2.17) calculated at the critical point (3.16).

One can note that above dynamical system admits two invariant submanifolds namely \( v = 0 \) and \( u = 0 \).

On the first invariant submanifold the system can be simply reduced to
\[
\begin{align*}
\dot{u} &= -\varepsilon 12\xi (1 - 6\xi) z(\lambda^*_{3a}) x^*_3a u \\
\dot{w} &= \varepsilon 12\xi (1 - 6\xi) z(\lambda^*_{3a}) x^*_3a w,
\end{align*}
\]
resulting in the solution representing a saddle type critical point in the form
\[
\begin{align*}
u(\tau) &= u_{\text{ini}} \exp \left( -\varepsilon 12\xi(1 - 6\xi) z(\lambda^*_{3a}) x^*_3a \right), \\
w(\tau) &= w_{\text{ini}} \exp \left( \varepsilon 12\xi(1 - 6\xi) z(\lambda^*_{3a}) x^*_3a \right).
\end{align*}
\]
For an example for point. First, we use approximated solutions (on the right diagram for a stable case. The example is given for $z(\lambda) = \frac{1}{2}, \epsilon = -1, w_m = 0, \xi = 1/4$ and $\alpha = 1$ (left) and $\alpha = -4$ (right).

On the other hand we can also restrict our system to the invariant submanifold defined by $u = 0$, then
\[
v' = B_v \nu \nu, \\
w' = \frac{\partial v}{\partial x}|_{3a} w + B_w v^2 + C_w w^2,
\]
and from the center manifold theorem (see appendix A) we have
\[
w = h(v) = -\frac{B_w}{\partial v/\partial x}|_{3a} v^2 + \frac{B_w^2}{\partial^2 v/\partial x^2}|_{3a} (2B_w - C_w) v^4 + O(v^5)
\]
and inserting this approximation into first equation of the system (3.21) we receive that the vector field restricted to the center manifold is given by
\[
\eta' = -\frac{B_v B_w}{\partial^2 v/\partial x^2}|_{3a} \eta^3 + O(\eta^4)
\]
where
\[
\frac{B_v B_w}{\partial^2 v/\partial x^2}|_{3a} = -\frac{\lambda_{30}^*}{2(1 - 6\xi)z(\lambda_{30}^*)} - \frac{3}{2}(1 + w_m)
\]
and this indicates that for $\frac{B_v B_w}{\partial^2 v/\partial x^2}|_{3a} < 0$ it is an unstable and for $\frac{B_v B_w}{\partial^2 v/\partial x^2}|_{3a} > 0$ it is a stable critical point on the invariant submanifold $u = 0$ (see figure 5 for an example for $z(\lambda) = \frac{1}{2}$).

Now we are ready to present the evolution of Hubble’s function in the vicinity of this critical point. First, we use approximated solutions (3.20) on the invariant submanifold $v = 0$. We have
\[
\ln \left( \frac{\nu}{\nu_{ini}} \right) = \frac{z'(\lambda_{30}^*)}{z_{ini}} u_{ini} \left( \exp \left( -\varepsilon 12\xi(1 - 6\xi)z(\lambda_{30}^*)x_{3a}^* \tau \right) - 1 \right), \\
\ln \left( \frac{H}{H_{ini}} \right)^2 = \frac{1 - 3m_{ini} a}{1 - 6\xi} \frac{1 + 6\xi z(\lambda_{30}^*)}{(x_{3a}^*)^2} u_{ini} \left( \exp \left( \varepsilon 12\xi(1 - 6\xi)z(\lambda_{30}^*)x_{3a}^* \tau \right) - 1 \right) - 4 + \frac{1 - w_m}{4\xi} \frac{x_{3a}^*}{z(\lambda_{30}^*)} \frac{z'(\lambda_{30}^*)}{z_{30}^*} u_{ini} \left( \exp \left( -\varepsilon 12\xi(1 - 6\xi)z(\lambda_{30}^*)x_{3a}^* \tau \right) - 1 \right) - \frac{3}{4}(1 + w_m) \tau - \frac{3}{2}(1 + w_m) \tau
\]

On the other hand, using the solution from a center manifold and keeping only linear terms in $u$
\[
w(\tau)^2 \propto v(\tau)^4 \approx 0, \quad u(\tau)w(\tau) \propto u(\tau)v(\tau)^2 \approx 0, \quad u(\tau)^2 \approx 0
\]
we receive

\[ \begin{align*}
\ln \left( \frac{a_{\text{end}}}{a_{\text{ini}}} \right) &= 12\zeta (1 - 6\zeta) z_{\lambda_{\text{ini}}}^3 x_{\lambda_{\text{ini}}}^3 e^{-1}, \\
\left( \frac{H_{\text{end}}}{H_{\text{ini}}} \right)^2 &= -4 + \frac{1 - w_m}{4\xi} \frac{x_{\lambda_{\text{ini}}}^3}{z_{\lambda_{\text{ini}}}^3} \left( z_{\lambda_{\text{ini}}}^3 x_{\lambda_{\text{ini}}}^3 e^{-1} \right)
\end{align*} \]  

which can be easy to combine as

\[ \ln \left( \frac{H}{H_{\text{ini}}} \right)^2 = -4 + \frac{1 - w_m}{4\xi} \frac{x_{\lambda_{\text{ini}}}^3}{z_{\lambda_{\text{ini}}}^3} \ln \left( \frac{a}{a_{\text{ini}}} \right). \]  

One can notice that this expression resembles behaviour of the Hubble’s function during the pure radiation domination epoch, but with contribution coming from non-minimal coupling. The linearised solutions are valid up to the Lyapunov time \( \tau_{\text{end}} = \frac{1}{e^\xi (1 - 6\zeta) z_{\lambda_{\text{ini}}}^3 x_{\lambda_{\text{ini}}}^3} \) or\( > 0 \), then inserting this in to the latter equations we receive maximal values of the scale factor and Hubble’s function valid in the center manifold approximation

\[ \begin{align*}
\ln \left( \frac{a_{\text{end}}}{a_{\text{ini}}} \right)^2 &= \frac{z'_{\lambda_{\text{ini}}}^3}{x_{\lambda_{\text{ini}}}^3} u_{\text{ini}}^3 (e - 1), \\
\left( \frac{H_{\text{end}}}{H_{\text{ini}}} \right)^2 &= -4 + \frac{1 - w_m}{4\xi} \frac{x_{\lambda_{\text{ini}}}^3}{z_{\lambda_{\text{ini}}}^3} \left( z_{\lambda_{\text{ini}}}^3 x_{\lambda_{\text{ini}}}^3 e^{-1} \right)
\end{align*} \]  

In figure 6 we present evolution of \( \ln H^2 \) and \( R_H \) as a functions of \( \ln a \) in the vicinity of the critical point representing radiation domination epoch for the phantom scalar field.

There is another critical point which represents the radiation dominated universe located at

\[ \begin{align*}
x_{\lambda_{\text{3b}}}^3 &= 0, \quad y_{\lambda_{\text{3b}}}^3 = 0, \quad \lambda_{\text{3b}}^3: \ z(\lambda)^2 = \frac{1}{\epsilon 6\xi}
\end{align*} \]  

with the effective equation of the state parameter

\[ w_{\text{eff}} = \frac{1}{3}. \]

We need to stress that this critical point exists only if \( w_m \neq \frac{1}{3} \). Linearised solutions in the vicinity of
this critical point are

\[ x_{3b}(\tau) = \frac{1 - 3w_m}{2 - 3w_m} \left( x^{ini}_{3b} + z'(\lambda^{*}_{3b})(\lambda^{ini}_{3b} - \lambda^{*}_{3b}) \right) \exp(l_1\tau) + \]
\[ \frac{1}{2 - 3w_m} \left( x^{ini}_{3b} - (1 - 3w_m)z'(\lambda^{*}_{3b})(\lambda^{ini}_{3b} - \lambda^{*}_{3b}) \right) \exp(l_3\tau), \]

\[ y_{3b}(\tau) = y^{ini}_{3b} \exp(l_2\tau), \]

\[ \lambda_{3b}(\tau) = \lambda^{*}_{3b} + \frac{1}{2 - 3w_m} \frac{1}{z'(\lambda^{*}_{3b})} \left( x^{ini}_{3b} + z'(\lambda^{*}_{3b})(\lambda^{ini}_{3b} - \lambda^{*}_{3b}) \right) \exp(l_1\tau) - \]
\[ \frac{1}{2 - 3w_m} \frac{1}{z'(\lambda^{*}_{3b})} \left( x^{ini}_{3b} - (1 - 3w_m)z'(\lambda^{*}_{3b})(\lambda^{ini}_{3b} - \lambda^{*}_{3b}) \right) \exp(l_3\tau). \]

where

\[ l_1 = 6\xi(1 - 3w_m), \quad l_2 = 12\xi, \quad l_3 = -6\xi \]

are the eigenvalues of the linearization matrix calculated at this critical point. Simple inspection of this eigenvalues gives us further constraint on the value of the barotropic matter equation of state parameter \( w_m \), namely \( l_1 \) should be positive resulting in \( w_m < \frac{1}{3} \) to assure that in \((x,\lambda)\) plane the dynamics in the vicinity of this critical point would correspond to a saddle type critical point. This will guarantee that the evolution proceeds towards the next critical point representing matter dominated universe.

Using linearised solutions (3.27) we are able to express (2.18) and (2.20) as a parametric functions of time \( \tau \)

\[
\begin{align*}
\ln \left( \frac{a}{a^{ini}_{3b}} \right) &= 6\xi\tau - e^2 \frac{1 - 6\xi}{1 - 3w_m} z(\lambda^{*}_{3b}) \left( x^{ini}_{3b} + z'(\lambda^{*}_{3b})(\lambda^{ini}_{3b} - \lambda^{*}_{3b}) \right) \left( \exp(l_1\tau) - 1 \right) - \\
&- \frac{1 - 2 - 3w_m}{2 - 3w_m} \left( x^{ini}_{3b} - (1 - 3w_m)z'(\lambda^{*}_{3b})(\lambda^{ini}_{3b} - \lambda^{*}_{3b}) \right) \left( \exp(l_3\tau) - 1 \right), \\
\ln \left( \frac{H}{H^{ini}_{3b}} \right)^2 &= -24\xi\tau - e^2 \frac{1 - 6\xi}{1 - 3w_m} z(\lambda^{*}_{3b}) \left( x^{ini}_{3b} + z'(\lambda^{*}_{3b})(\lambda^{ini}_{3b} - \lambda^{*}_{3b}) \right) \left( \exp(l_1\tau) - 1 \right) - \\
&+ \frac{1 - 2 - 3w_m}{2 - 3w_m} \left( x^{ini}_{3b} - (1 - 3w_m)z'(\lambda^{*}_{3b})(\lambda^{ini}_{3b} - \lambda^{*}_{3b}) \right) \left( \exp(l_3\tau) - 1 \right)
\end{align*}
\]

The linearised solutions (3.27) are valid up to the Lyapunov characteristic time \( \tau_{end} = \frac{1}{l_3} = \frac{1}{12\xi} \) and inserting it in the latter equations we can obtain maximal values of the scale factor and the Hubble’s function valid in the linear approximation.

The zero-order approximation \((x^{ini}_{3b} = 0, \lambda^{ini}_{3b} = \lambda^{*}_{3b}, y^{ini}_{3b} \neq 0 \) but \((y^{ini}_{3b})^2 \approx 0 \) is

\[
\begin{align*}
\ln \left( \frac{a}{a^{ini}_{3b}} \right) &= 6\xi\tau, \\
\ln \left( \frac{H}{H^{ini}_{3b}} \right)^2 &= -24\xi\tau
\end{align*}
\]

and it can be combine to

\[ H^2 = (H^{ini}_{3b})^2 \left( \frac{a}{a^{ini}_{3b}} \right)^{-4}, \]

which is the exact behaviour of Hubble’s function during the radiation domination era. This approximation is also valid up to \( \tau_{end} = \frac{1}{l_3} = \frac{1}{12\xi} \) so one can calculate that during radiation domination epoch the scale factor grows at least

\[ a^{end}_{3b} = a^{ini}_{3b} \sqrt{e}. \]

In figure 7 we present evolution of \( \ln H^2 \) and \( R_H \) as a function of \( \ln a \) in the vicinity of the critical point representing the radiation domination era.

### 3.4 Matter domination

The next critical point is located at

\[ x^*_1 = 0, \quad y^*_1 = 0, \quad \lambda^*_1 : z(\lambda) = 0 \] (3.30)
and \( w_{\text{eff}} \) given by (2.12) calculated at this point is
\[
w_{\text{eff}} = w_m.
\]
We identify this critical point as representing the universe which dynamics is dominated by the barotropic matter included in the model with the equation of state parameter \( w_m \).

The linearised solutions in the vicinity of this critical point are in the form
\[
\begin{align*}
x_4(\tau) &= \frac{l_1}{l_1 - l_2} \left( x_4^{\text{ini}} - z'(\lambda_4^*) l_3 (\lambda_4^{\text{ini}} - \lambda_4^*) \right) \exp(l_1 \tau) - \\
&\quad - \frac{l_2}{l_1 - l_3} \left( x_4^{\text{ini}} - z'(\lambda_4^*) l_1 (\lambda_4^{\text{ini}} - \lambda_4^*) \right) \exp(l_3 \tau), \\
y_4(\tau) &= y_4^{\text{ini}} \exp(l_2 \tau), \\
\lambda_4(\tau) &= \lambda_4^* + \frac{1}{z'(\lambda_4^*) (l_1 - l_2)} \left( x_4^{\text{ini}} - z'(\lambda_4^*) l_3 (\lambda_4^{\text{ini}} - \lambda_4^*) \right) \exp(l_1 \tau) - \\
&\quad - \frac{1}{z'(\lambda_4^*) (l_1 - l_3)} \left( x_4^{\text{ini}} - z'(\lambda_4^*) l_1 (\lambda_4^{\text{ini}} - \lambda_4^*) \right) \exp(l_3 \tau).
\end{align*}
\]  
(3.31)
where
\[
\begin{align*}
l_1 &= -\frac{3}{4} \left( 1 - w_m \right) + \sqrt{\left( 1 - w_m \right)^2 - \frac{16}{3} \xi \left( 1 - 3 w_m \right)}, \\
l_2 &= \frac{3}{2} (1 + w_m), \\
l_3 &= -\frac{3}{4} \left( 1 - w_m \right) - \sqrt{\left( 1 - w_m \right)^2 - \frac{16}{3} \xi \left( 1 - 3 w_m \right)}.
\end{align*}
\]
We need to note that this critical point can became degenerate for two specific values of \( w_m \), namely, for \( w_m = -1 \) the second eigenvalue vanishes and for \( w_m = 1 \) the third eigenvalue vanish, for any value of the coupling constant \( \xi \), which makes the system in the vicinity of this critical point degenerated.

From linearised solution (3.31) we have
\[
x_4(\tau)^2 \approx 0, \quad y_4(\tau)^2 \approx 0, \quad z(\lambda_4(\tau))^2 \approx 0
\]
and from (2.18) and (2.20) parametric equations for evolution of the scale factor and Hubble’s function are
\[
\begin{align*}
\ln \left( \frac{\dot{a}}{a} \right) &= \tau, \\
\ln \left( \frac{H}{H_m} \right)^2 &= -3(1 + w_m) \tau.
\end{align*}
\]  
(3.32)
Combining these two expressions we get the Hubble’s function as function of the scale factor during the barotropic matter domination epoch

$$H^2 = (H^\text{ini})^2 \left( \frac{a}{a^\text{ini}} \right)^{-3(1+w_m)}.$$  

The linearised solutions (3.31) are valid up to the Lyapunov time

$$\tau_{\text{end}} = \frac{1}{l_2} = \frac{2}{\alpha(1+w_m)},$$

One can notice that for dust matter $w_m = 0$ during the matter domination epoch the scale factor at least grows

$$\frac{a^\text{end}}{a^\text{ini}} = e^{\frac{2}{3}} \approx 1.948 \text{ times}.$$

In figure 8 we present evolution of the scale factor and Hubble’s function in the vicinity of the critical point representing the barotropic matter domination epoch.

3.5 The present accelerated expansion epoch

Finally we proceed to the last critical points located at

$$x^*_5 = 0, \quad (y^*_5)^2 = 1 - \varepsilon 6 \xi z(\lambda^*_5)^2, \quad \lambda^*_5: \lambda z(\lambda) + 4z(\lambda) - \frac{\lambda}{\varepsilon 6 \xi} = 0$$

with

$$w_{\text{eff}} = -1.$$ 

There can be more than one such critical points because of the third equation in (3.34) which can have more than one solution. In what follows we will show that at least one of them represents a stable critical point.

In this case the characteristic equation for eigenvalues of the linearization matrix calculated at this critical point is in the form

$$l^3 + pl^2 + ql + r = 0$$

where

$$p = 3(2 + w_m)(1 - \varepsilon 6 \xi (1 - 6 \xi) z(\lambda^*_5)^2),$$

$$q = (1 - \varepsilon 6 \xi (1 - 6 \xi) z(\lambda^*_5)^2) \left[ -\frac{\varepsilon}{2} (y^*_5)^2 + 12\xi (1 + \varepsilon 6 \xi z(\lambda^*_5)^2) + 9(1 + w_m)(1 - \varepsilon 6 \xi (1 - 6 \xi) z(\lambda^*_5)^2) \right],$$

$$r = 3(1 + w_m)(1 - \varepsilon 6 \xi (1 - 6 \xi) z(\lambda^*_5)^2) \left[ -\frac{\varepsilon}{2} (y^*_5)^2 + 12\xi (1 + \varepsilon 6 \xi z(\lambda^*_5)^2) \right].$$

Figure 8. Evolution of $\ln H^2$ (left panel) and $R_H$ (right panel) for a sample trajectory with $\varepsilon = -1$, $\xi = \frac{1}{2}$, $z'(\lambda^*_7) = \frac{1}{2} = 100$ in the vicinity of the critical point corresponding to the barotropic matter dominated universe. The solid black line represents the linear approximation (3.32) and the dotted line represents the numerical solution of the system (2.17).
In the most general case without assuming any specific form of the potential function we are unable to solve this equation. In spite of this we are able to formulate general conditions for stability of this critical point. This requires that the real parts of the eigenvalues must be negative. From the Routh-Hurwitz test [68] we have that the following conditions should be fulfilled to assure stability of this critical point

\[ p > 0 \quad \text{and} \quad r > 0 \quad \text{and} \quad q - \frac{r}{p} > 0 \]

Simple inspection of these conditions gives us that, for any matter with \( w_m > -2 \), \( p \) is always positive because of \( (1 - \varepsilon 6\xi(1 - 6\xi)z(\lambda^*_5)^2) > 0 \) due to time transformation and that if \( r \) is a positive quantity it follows that \( q - \frac{r}{p} \) is positive too. We conclude that if the following condition is fulfilled at the critical point

\[ \text{Re}[l_{1,2,3}] < 0 \iff -\frac{1}{2} \left( \frac{y_5^s}{z(\lambda^*_5)} \right)^2 + 12\xi \left( 1 + \varepsilon 6\xi z(\lambda^*_5)^2 \right) > 0 \]  

(3.35)

it represents a stable critical point with the negative real parts of the eigenvalues.

In order to simplify this condition let us introduce the following function

\[ h(\lambda) = \lambda z(\lambda)^2 + 4z(\lambda) - \frac{\lambda}{\varepsilon 6\xi}, \]  

(3.36)

where location of the critical point is the solution to the equation \( h(\lambda) = 0 \), and obviously \( h(\lambda^*_5) = 0 \).

Let us assume that \( \lambda^*_5 \neq 0 \) it follows from (3.36) that also \( z(\lambda^*_5) \neq 0 \) but \( h(\lambda^*_5) = 0 \). Differentiation of Eq. (3.36) gives

\[ h'(\lambda^*_5) = z(\lambda^*_5)^2 - \frac{1}{\varepsilon 6\xi} + 2z'(\lambda^*_5)(\lambda^*_5 z(\lambda^*_5) + 2) \]

which after little algebra can be transformed to the following form

\[ h'(\lambda^*_5) = -\frac{(y_5^s)^2}{z(\lambda^*_5)^2} + \frac{4 z'(\lambda^*_5)}{(y_5^s)^2} \left( 1 + \varepsilon 6\xi z(\lambda^*_5)^2 \right) \]

and finally we arrive to the reformulated stability condition (3.35) in the form

\[ \text{Re}[l_{1,2,3}] < 0 \iff 3\xi \frac{h'(\lambda^*_5)}{z(\lambda^*_5)^2} (y_5^s)^2 > 0 \]  

(3.37)

We have reduced analysis of stability of the critical point representing accelerated expansion to the simple analysis of the sign of the quantity given by relation (3.37). If we assume that the function \( z(\lambda) \) is a monotonic one, i.e. it is a growing or decreasing function in the interesting region of the phase space then it follows that if we have at least two critical points given by (3.34) one of them is definitely a stable critical point.

To present the evolution of Hubble’s function in the vicinity of this critical point, as an example, we choose the simple form of \( z(\lambda) = \frac{1}{\lambda} \) function, and values of the parameters \( \xi \) and \( \alpha \) in range for which there exists only one critical point corresponding the present accelerated expansion of the universe [54]. The linearised solutions in the vicinity of the critical point located at \( x_5^* = 0 \), \( (y_5^s)^2 = 1 \), \( \lambda^*_5 = 0 \) are

\[ x_5(\tau) = \frac{1}{2\sqrt{3} \alpha} \left\{ (3 + \sqrt{3}) \left[ x_5^{ini} + \frac{1}{2\alpha}(3 - \sqrt{3})\lambda_5^{ini} \right] \exp(l_1\tau) - \right. \]
\[ \left. - (3 - \sqrt{3}) \left[ x_5^{ini} + \frac{1}{2\alpha}(3 + \sqrt{3})\lambda_5^{ini} \right] \exp(l_3\tau) \right\}, \]

\[ y_5(\tau) = y_5^s + (y_5^{ini} - y_5^s) \exp(l_2\tau), \]

\[ \lambda_5(\tau) = -\frac{1}{2\sqrt{3} \alpha} \left\{ \left[ x_5^{ini} + \frac{1}{2\alpha}(3 - \sqrt{3})\lambda_5^{ini} \right] \exp(l_1\tau) - \right. \]
\[ \left. - \left[ x_5^{ini} + \frac{1}{2\alpha}(3 + \sqrt{3})\lambda_5^{ini} \right] \exp(l_3\tau) \right\}, \]  

(3.38)
where \( l_{1,3} = -\frac{1}{2} (3 \pm \sqrt{9 + \varepsilon^2 \alpha - 48\xi}) \) and \( l_2 = -3(1 + w_m) \) are eigenvalues of the linearization matrix and \( \Delta_5 = 9 + \varepsilon^2 \alpha - 48\xi \).

Then keeping only linear terms in initial conditions

\[
x_5(\tau)^2 \approx 0, \quad z(\lambda_5(\tau))^2 \approx 0, \quad \lambda_5(\tau)z(\lambda_5(\tau)) \approx 0,
\]

\[
y_5(\tau)^2 \approx (y_5^*)^2 + 2y_5^*(y_5^{ini} - y_5^*)(1 - \exp(-3(1 + w_m)\tau)),
\]

from (2.18) and (2.20) we receive the parametric equations of evolution of the scale factor and Hubble’s function

\[
\begin{align*}
\ln \left( \frac{a}{a_5^*} \right) &= \tau, \\
\ln \left( \frac{H}{H_5^*} \right)^2 &= 2y_5^*(y_5^{ini} - y_5^*)(1 - \exp(-3(1 + w_m)\tau)).
\end{align*}
\]

Combining these two expressions we get Hubble’s function as a function of the scale factor in the vicinity of the critical point corresponding to the present accelerated expansion of the universe

\[
H^2 = (H_5^{ini})^2 \exp \left( 2y_5^*(y_5^{ini} - y_5^*)(1 - \left( \frac{a}{a_5^*} \right)^{-3(1+w_m)}) \right).
\]

One can notice that taking the following limit

\[
H_{mn}^2 = \lim_{\tau \to \infty} \lim_{a \to \infty} H^2 = (H_5^{ini})^2 \exp \left( 2y_5^*(y_5^{ini} - y_5^*) \right) \approx (H_5^{ini})^2 \left( 1 + 2y_5^*(y_5^{ini} - y_5^*) \right)
\]

we get the asymptotic de Sitter expansion.

In figure 9 we present the evolution of \( \ln H^2 \) and \( R_H \) as a function of \( \ln a \) in the vicinity of this critical point.

4 Summary and Conclusions

Modern cosmology becomes very similar to the particle physics. Both theories have parameters and characteristic energetic cut offs. They are the effective description of deeper physics which is currently unknown. The values of these parameters should be obtained from more fundamental theories or from observations. In cosmology (ΛCDM model is called Standard Cosmological Model) the role of such a parameter plays the cosmological constant. Our proposition is to extend this paradigm in which matter content is described in terms of barotropic perfect fluid by introduction additional scalar field
Figure 10. The phase space portrait for the model with the cosmological constant and the canonical scalar field ($\varepsilon = +1$) with $\xi = 1/8$ and the dust matter $w_m = 0$. The critical points are: $S$ – the finite scale factor singularity, $RI$ – the rapid-roll inflation, $SI$ – the slow-roll inflation, $R$ – the radiation dominated era, $M$ – the barotropic matter dominated era and $Q$ – the quintessence era. Note that the critical points representing the finite scale factor singularity, the rapid-roll inflation and the slow-roll inflation have the same value of coordinate $z$.

non-minimally coupled to gravity. As a result we discover new evolitional path which open new perspectives of description of cosmological evolution in unified way. In this scheme the inflation era appears in natural way and it is not put into the $\Lambda$CDM scenario by hand.

In this paper we have shown that the all important epochs in the evolution of the universe can be represented by the critical points of the dynamical system arising from the non-minimally coupled scalar field cosmology in spite of not assuming a form of the potential function. We have shown that for the positive coupling constant there exists a past finite scale factor singularity for both types of the scalar fields. Additionally all the intermediate states are transient one, i.e. they are represented by an unstable critical points in the phase space and last for an finite amount of time. The existence of the radiation dominated era is purely the result of the evolution of the non-minimally coupled scalar field.

For the canonical scalar field $\varepsilon = +1$ and $0 < \xi < 1/6$ we can construct the unique evolitional path represented by the trajectory in the phase space which travels in the vicinity of the following critical points (figure 10)

$$1 \mapsto 2 \mapsto 3 \mapsto 4 \mapsto 5$$

and for the phantom scalar field $\varepsilon = -1$ and $\xi > 1/6$ (figure 11)

$$1 \mapsto 2 \mapsto 3 \mapsto 4 \mapsto 5$$
Figure 11. The phase space portrait for the model with the cosmological constant and the phantom scalar field ($\varepsilon = -1$) with $\xi = 1/4$ and the dust matter $w_m = 0$. The critical points are: $S$ – the finite scale factor singularity, $RI$ – the rapid-roll inflation, $R$ – the radiation dominated era, $M$ – the barotropic matter dominated era and $Q$ – the quintessence era. In the case of the phantom scalar field the critical point representing slow-roll inflation is not present. The critical points denoted as $S$, $RI$ and $R$ have the same value of coordinate $z$.

Within one framework of non-minimally coupled scalar field cosmology we were able to unify all the major epochs in the history of the universe (see figure 12 for twister type behaviour where trajectories interpolate between radiation era, matter domination era and quintessence epoch).

From the analysis presented in this paper one can draw the general conclusion that if the non-minimal coupling constant is present and is different from the conformal coupling $\xi \neq 1/6$ then new evolutional types emerge forming the structure of the phase space nontrivial and richer. Moreover the coupling constant gives us the effect of continuation (glues the evolution) between different cosmological epochs which is very attractive in cosmology, serving as a potential explanation of the global properties of the universe.

A The Center Manifold Theorem for three-dimensional dynamical systems

For the sake of completeness we present here the theorem concerning behaviour of nonlinear dynamical system in the vicinity of degenerated critical point. Expanded discussion can be found, for example, in books by Perko [67] or Wiggins [68].

Suppose we consider the following 3-dimensional nonlinear dynamical system

$$\dot{x} = f(x), \quad x \in \mathbb{R}^3$$

(A.1)
Figure 12. The phase space portrait representing twister type behaviour. Trajectories in this type solution interpolate between three major epochs in the history of universe: $R$ – the radiation dominated universe with $w_{\text{eff}} = \frac{1}{3}$, the matter domination epoch (an unstable focus type critical point) and $Q$ – the quintessence domination epoch with $w_{\text{eff}} = -1$. This type of evolution does not depend on the form of assumed function $z(\lambda)$ (i.e. the form of the scalar field potential) and is generic for the canonical scalar field cosmologies ($\varepsilon = +1$) with $\xi > 0$ and the barotropic matter with equation of state parameter $-1 < w_m < 1/3$.

We are interested in the nature of solution to this dynamical system near fixed point $\bar{x}$ for which $f(\bar{x}) = 0$.

First, we transform the fixed point $x = \bar{x}$ of (A.1) to the origin using the transformation $y = x - \bar{x}$. Then the system (A.1) becomes

$$\dot{y} = f(\bar{x} + y), \quad y \in \mathbb{R}^3 \tag{A.2}$$

then Taylor expansion of $f(\bar{x} + y)$ about $x = \bar{x}$ gives

$$\dot{y} = Ay + R(y), \quad y \in \mathbb{R}^3 \tag{A.3}$$

where $A = Df(\bar{x})$ is a linearization matrix calculated at the fixed point, $R(y) = \mathcal{O}(|y|^n)$ and we have used $f(\bar{x}) = 0$.

From now on we will assume that the linearization matrix has purely real eigenvalues and one is zero $l_1 = 0$, one positive $l_2 > 0$ and one negative $l_3 < 0$. From elementary linear algebra we can find a linear transformation $P$ which transforms the linear part of equation (A.3) into a diagonal form

$$\begin{pmatrix}
\dot{u} \\
\dot{v} \\
\dot{w}
\end{pmatrix} = \begin{pmatrix}
0 & 0 & 0 \\
0 & l_2 & 0 \\
0 & 0 & l_3
\end{pmatrix} \begin{pmatrix}
u \\
v \\
w
\end{pmatrix} \tag{A.4}$$
with a linear transformation of variables

\[ P^{-1} y \equiv \begin{pmatrix} u \\ v \\ w \end{pmatrix} \]

and the matrix \( P \) is constructed from the corresponding eigenvectors of the linearization matrix \( A \). Using this same linear transformation to transform the coordinates of the nonlinear part of the system \((A.3)\) gives the following

\[
\begin{align*}
\dot{u} &= F(u, v, w), \\
\dot{v} &= l_2 v + G(u, v, w), \\
\dot{w} &= l_3 w + H(u, v, w).
\end{align*}
\]  

(A.5)

where \( F(u, v, w) \), \( G(u, v, w) \) and \( H(u, v, w) \) are polynomial in the coordinates. The fixed point \((u, v, w) = (0, 0, 0)\) is unstable due to the existence of a 1-dimensional unstable manifold associated with the negative eigenvalue \( l_3 < 0 \).

**Definition A.1 (Center Manifold)** An invariant manifold will be called a center manifold for \((A.5)\) if it can be locally represented by

\[ W^c(0) = \left\{ (u, v, w) \in \mathbb{R}^3 \mid v = h_1(u), w = h_2(u), |u| < \delta, h_1(0) = 0, h_1'(0) = 0, i = 1, 2 \right\} \]  

(A.6)

for \( \delta \) sufficiently small.

**Theorem A.1 (Existence)** There exists a \( C^r \) center manifold for \((A.5)\). The dynamics of \((A.5)\) restricted to the center manifold is, for \( \eta \) sufficiently small, given by the following 1-dimensional vector field

\[ \dot{\eta} = F(\eta, h_1(\eta), h_2(\eta)). \]  

(A.7)

From the fact that the center manifold is invariant under the dynamics generated by \((A.5)\) we obtain

\[
\begin{align*}
\dot{u} &= F(u, h_1(u), h_2(u)), \\
\dot{v} &= h_1'(u) \dot{u} = l_2 h_1(u) + G(u, h_1(u), h_2(u)), \\
\dot{w} &= h_2'(u) \dot{u} = l_3 h_2(u) + H(u, h_1(u), h_2(u)),
\end{align*}
\]  

(A.8)

which yields the following quasilinear differential equation for \( h_1(u) \) and \( h_2(u) \)

\[
\begin{align*}
\mathcal{N}(h_1(u)) &= h_1'(u)F(u, h_1(u), h_2(u)) - l_2 h_1(u) - G(u, h_1(u), h_2(u)) = 0, \\
\mathcal{N}(h_2(u)) &= h_2'(u)F(u, h_1(u), h_2(u)) - l_3 h_2(u) - H(u, h_1(u), h_2(u)) = 0,
\end{align*}
\]  

(A.9)

and the following theorem (see Theorem 18.1.4 in Wiggins [68, p. 248]) justify solving \((A.9)\) approximately via power series expansion:

**Theorem A.2 (Approximation)** Let \( \phi: \mathbb{R} \to \mathbb{R} \) be a \( C^1 \) mapping with \( \phi(0) = \phi'(0) = 0 \) such that

\[ \mathcal{N}(\phi(u)) = O(|u|^q) \]  

as \( u \to 0 \) for some \( q > 1 \). Then

\[ |h(u) - \phi(u)| = O(|u|^q) \]  

as \( u \to 0 \).

This theorem allows us to compute the center manifold to any desired degree of accuracy by solving \((A.9)\) to the same degree of accuracy.

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