EXTENSIONS OF FULL SHIFTS WITH GROUP ACTIONS

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Abstract. We give a sufficient condition for a symbolic topological dynamical system with action of a countable amenable group to be an extension of the full shift, a problem analogous to those studied by Ashley, Marcus, Johnson and others for actions of \( \mathbb{Z} \) and \( \mathbb{Z}^d \).

1. Introduction

A well-known result in the study of symbolic dynamical systems states that any subshift of finite type (SFT) with the action of \( \mathbb{Z} \) and entropy greater or equal than \( \log n \) factors onto the full shift over \( n \) symbols – this was proven in [7] and [1] for the cases of equal and unequal entropy respectively. Extending these results for actions of other groups has been difficult, and it is known that a factor map onto a full shift of equal entropy may not exist in this case (see [2]). Johnson and Madden showed in [5] that any SFT with the action of \( \mathbb{Z}^d \), which has entropy greater than \( \log n \) and satisfies an additional mixing condition (known as corner gluing), has an extension which is finite-to-one (hence of equal entropy) and maps onto the full shift over \( n \) symbols. This result was later improved by Desai in [3] to show that such a system factors directly onto the full shift, without the intermediate extension.

In this paper we use similar methods to show that in the case of amenable group actions, any symbolic dynamical system with entropy greater than \( \log n \) which satisfies a mixing condition (the gluing property, see definition [5.1]), has an equal-entropy symbolic extension which factors onto the full shift over \( n \) symbols.

2. Amenable groups and invariance

Definition 2.1. A countable group \( G \) is called amenable if there exists a sequence \( (F_n) \) of finite subsets of \( G \) (known as Følner sets) such that

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for every \( g \in G \) we have
\[
\lim_{n \to \infty} \frac{|gF_n \triangle F_n|}{|F_n|} = 0,
\]
where \(|\cdot|\) denotes the cardinality of a set, and \( \triangle \) denotes the symmetric difference.

**Definition 2.2.** For a pair of finite sets \( T, D \subset G \) and \( \delta > 0 \), we say that \( T \) is \((D, \delta)\)-invariant, if \( \frac{|DT \setminus T|}{|T|} < \delta \).

Note that if \( D \) contains the neutral element of \( G \), then the above condition simplifies to \( \frac{|DT \setminus T|}{|T|} < \delta \).

**Definition 2.3.** If \( T \) and \( D \) are two finite subsets of \( G \), the \( D \)-core of \( T \) is the set
\[
T_D = \{ t \in T : Dt \subset T \}.
\]

It is easy to check that for every \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that if \( T \) is \((D, \delta)\) invariant, then \( |T \setminus T_D| < \varepsilon |T| \), i.e., \( T_D \) is a relatively large subset of \( T \).

### 3. Symbolic dynamical systems and entropy

Let \( \Lambda \) be any finite set. The **full \( G \)-shift over \( \Lambda \)** (often referred to as just the full shift over \( \Lambda \)) is the dynamical system with the space \( \Lambda^G \) endowed with the product topology, and the action of \( G \) defined as \( (gx)(h) = x(hg) \). A **symbolic dynamical system over \( \Lambda \)** is any closed subset of \( \Lambda^G \) invariant under the action of \( G \). If \( T \subset G \) is a finite set, then a **block with domain \( T \)** is a mapping \( B : T \to \Lambda \). If \( x \in \Lambda^G \) and \( T \subset G \) is a finite set, then by \( x(T) \) we understand a block \( B \) with domain \( T \) such that for every \( t \in T \), \( B(t) = x(t) \). In a slight abuse of notation, we will not distinguish between two blocks if their domains differ only by translation, i.e. for any \( g \) we treat \( x(T) \) and \( x(Tg) \) as the same block.

If \( X \) is a symbolic dynamical system over \( \Lambda \), then we say that a block \( B \) with domain \( T \) **occurs in \( X \)**, if \( x(T) = B \) for some \( x \in X \). Finally, if we denote by \( N_T(X) \) the number of blocks with domain \( T \) which occur in \( X \), we can calculate the **topological entropy of \( X \)** as the limit
\[
h(X) = \lim_{n \to \infty} \frac{1}{|F_n|} \log |N_{F_n}(X)|
\]
(where \( \log \) means logarithm with base 2). It is known that this limit always exists and does not depend on the choice of the Følner sequence (see Theorem 6.1 in [6]). In fact, we will make use of the following consequence of the cited theorem:

**Proposition 3.1.** For any \( \varepsilon > 0 \) there exists an \( N \) and \( \delta \) such that if \( T \) is an \((F_n, \delta)\)-invariant set for some \( n > N \), then \( N_T(X) > 2^{h(X) - \varepsilon |T|} \).
We will also assume, that for every finite set \( D \) we have \( D \subset F_n \) for sufficiently large \( n \).

4. Tilings

We briefly recall the notions and most important results concerning tilings of amenable groups; for details we refer the reader to [4].

**Definition 4.1.** A tiling of an amenable group \( G \) is a collection \( \mathcal{T} \) of finite subsets of \( G \), such that:

- \( T_1 \cap T_2 = \emptyset \) whenever \( T_1, T_2 \) are two different elements of \( \mathcal{T} \).
- \( \bigcup_{T \in \mathcal{T}} T = G \)
- There exists a finite family \( \mathcal{S} = (S_1, \ldots, S_k) \) of finite subsets of \( G \), such that every \( T \) can be uniquely represented in the form \( S_j g \) for some \( j \in \{1, 2, \ldots, k\} \) and \( g \in G \).

The elements of \( \mathcal{T} \) are referred to as tiles, and the elements of \( \mathcal{S} \) are referred to as shapes. Also, we can define a mapping \( \sigma : \mathcal{T} \to \mathcal{S} \) such that \( S_j = \sigma(T) \) if and only if \( T = S_j g \) for some \( g \in G \).

Every tiling \( \mathcal{T} \) with shapes \( S_1, \ldots, S_k \) induces an element \( x_\mathcal{T} \) of \( \{0, 1, \ldots, k\}^G \), defined by requesting that if \( S_j g \in \mathcal{T} \) for some \( j \), then \( x_\mathcal{T}(g) = j \) (the properties constituting the definition of a tiling ensure that such a \( j \) is unique), and if no such \( j \) exists, then \( x_\mathcal{T}(g) = 0 \). This in turn allows us to associate with \( \mathcal{T} \) a symbolic dynamical system \( X_\mathcal{T} \) defined as the orbit closure of \( x_\mathcal{T} \) under the shift action. Note that by reversing the procedure which defines \( x_\mathcal{T} \), we can obtain from every element of \( X_\mathcal{T} \) another tiling of \( G \) which uses the same collection of shapes as \( \mathcal{T} \).

In view of this discussion, it is natural to define the action of \( G \) directly on tilings of \( G \) by putting:

\[
g \mathcal{T} = \{ T g^{-1} : T \in \mathcal{T} \}
\]

Clearly, \( g(x_\mathcal{T})(h) = j \) if and only if \( x_\mathcal{T}(hg) = j \), so \( S_j h \) is a tile of \( g \mathcal{T} \) if and only if \( S_j hg \) is a tile of \( \mathcal{T} \) and this definition is consistent with the shift action of \( G \) on \( X_\mathcal{T} \).

We will need the following result which is an immediate consequence of theorem 5.2 of [4]:

**Theorem 4.2.** If \( G \) is a countable amenable group and \( K \subset G \) is a finite set, then for every \( \varepsilon > 0 \) there exists a tiling \( \mathcal{T} \) of \( G \) such that the shapes of \( \mathcal{T} \) are \( (K, \varepsilon) \)-invariant sets, and the system \( X_\mathcal{T} \) has entropy zero.

5. The main result

We begin by introducing a property analogous to the corner/centre gluing conditions used in the \( \mathbb{Z}^d \) context.
Definition 5.1. We say that a symbolic dynamical system \((X, G)\) has the \textit{gluing property} if there exists a finite set \(D\) (containing the neutral element) such that for any finite subsets \(T_1\) and \(T_2\) of \(G\), such that \(T_2 \cap DT_1 = \emptyset\), and any two blocks \(A\) and \(B\), with domains respectively \(T_1\) and \(T_2\), there exists an \(x \in X\) such that \(x(T_1) = A\) and \(x(T_2) = B\). The set \(D\) will be referred to as the \textit{gluing distance}.

Theorem 5.2. If the symbolic dynamical system \((X, G)\) with topological entropy greater than \(\log k\) has the gluing property, then there exists a symbolic extension \((\tilde{X}, \tilde{G})\) of \(X\), having the same topological entropy as \(X\), and such that \((\tilde{X}, \tilde{G})\) factors onto the full shift over \(k\) symbols.

Proof. Let \(l\) be the number of symbols in the alphabet of \(X\), let \(\gamma\) be a number such that \(1 < \gamma < \frac{h(X)}{\log k}\), and let \(D\) be the gluing distance. There exists a tiling \(T_0\) of \(G\), such that \(X_{T_0}\) has topological entropy zero, and for every shape \(S\) of \(T_0\):

1. \(|S \setminus SD| < ((\gamma - 1)\log k) |S|\).
2. The number of blocks with domain \(S\) occurring in \(X\) is greater than \(k^{\gamma|S|}\).

Indeed, using proposition 3.1 for \(\varepsilon < h(X) - \gamma \log k\) we obtain \(N\) and \(\delta\) such that (2) holds for blocks \(S\) which are \((F_n, \delta)\)-invariant, where \(n \geq N\). Then, by theorem 4.2 we get a \((F_n, \varepsilon)\)-invariant tiling, where \(n \geq N\) and \(0 < \varepsilon < \delta\) are appropriately chosen to guarantee (1) (in particular, we demand that \(D \subset F_n\)).

Combining the properties (1) and (2), we can estimate from below the number of blocks with domain \(S\) occurring in \(X\): If we denote by \(N_1\) the number of blocks with domain \(S\), and by \(N_2\) the number of blocks with domain \(SD\), then we have

\[
k^{\gamma|S|} < N_1 < N_2^{|S \setminus SD|} < N_2^{((\gamma - 1)\log k)|S|} = N_2^{k^{(\gamma - 1)|S|}},
\]

hence

\[
N_2 > k^{|S|}.
\]

It follows that for every shape \(S\) of \(T_0\) we can construct a mapping \(\phi_S\) from the collection of all blocks with domain \(SD\) occurring in \(X\) onto \(\{1, \ldots, k\}^S\).

We can now create the symbolic dynamical system \((\tilde{X}, \tilde{G})\) as the product of \((X, G)\) and \((X_{T_0}, G)\). This is obviously an extension of \(X\), and since the entropy of the product is equal to the sum of entropies of both systems, \(\tilde{X}\) has entropy equal to \(X\). Every element \(\tilde{x}\) of \(\tilde{X}\) consists of a pair \((x, T)\) where \(x \in X\) and \(T\) is a tiling of \(G\) using the same shapes as \(T_0\). We can now define a map \(\phi: \tilde{X} \to \{1, \ldots, k\}^G\) as follows: for every \(\tilde{x} = (x, T)\) let \(y = \phi(\tilde{x})\) be defined by requiring that for every \(T \in T\) we have \(y(T) = \phi_{\sigma(T)}(x(T_D))\). Since \(y(T)\) depends only on \(x(T)\), this map is continuous. It is also easy to verify that
it commutes with the shift: If \( \tilde{x} = (x, T) \) is an element of \( \tilde{X} \), then \( g\tilde{x} = (gx, gT) \). For every tile \( Tg^{-1} \) of \( gT \) we have
\[
(\phi(g\tilde{x}))(Tg^{-1}) = \phi_{\sigma(T^{-1})}(gx((Tg^{-1})D)) = \phi_{\sigma(T)}(x(TD)) = \phi(\tilde{x})(T),
\]
therefore, since \( T \) was arbitrary, for every \( h \in G \) we have
\[
(\phi(g\tilde{x}))(h) = \phi(\tilde{x})(hg).
\]

It remains to verify that \( \phi \) is onto. Let \( y \) be any element of \( \{1, \ldots, k\}^G \) and let \( T_1, T_2, \ldots \) be an enumeration of the tiles of \( T_0 \) and let \( B_i = y(T_i) \). There exists some \( x_1 \in X \) such that \( \sigma(T_i)(x_1((T_i)_D)) = B_1 \), and thus for \( \tilde{x}_1 = (x_1, T_0) \) we have \( \phi(\tilde{x}_1)(T_1) = B_1 \). Now, suppose that for some \( j \) we have already found an \( \tilde{x}_j \in \tilde{X} \) such that for every \( i \leq j \) we have \( \phi(\tilde{x}_j)(T_i) = B_i \) (so far we know this is possible for \( j = 1 \)). There exists some \( x_{j+1} \in X \) such that \( \sigma(T_{j+1})(x_{j+1}((T_{j+1})_D)) = B_{j+1} \). Now, the sets \( D(T_{j+1}D) \) and \( T_1 \cup T_2 \cup \ldots \cup T_j \) are disjoint, so the gluing property means there exists some \( x_{j+1} \) such that \( x_{j+1}(T_i) = x_i(T_i) \) for \( i = 1, \ldots, j \), and \( x_{j+1}((T_{j+1})_D) = x_{j+1}((T_{j+1})_D) \). If we now set \( \tilde{x}_{j+1} = (x_{j+1}, T_0) \), we will have \( \phi(\tilde{x}_{j+1})(T_i) = B_i \) for every \( i \leq j+1 \). By the principle of mathematical induction we obtain that for every \( j \) there exists an \( \tilde{x}_j \in \tilde{X} \) such that for every \( i \leq j \) we have \( \phi(\tilde{x}_j)(T_i) = B_i \). Since \( \tilde{X} \) is compact, there exists a convergent subsequence of \( (\tilde{x}_j) \) converging to some \( \tilde{x} \in \tilde{X} \). Hence for every \( i \) there exists some \( j \geq i \) such that \( \tilde{x}(T_i) = \tilde{x}_j(T_i) \), but then \( \phi(\tilde{x})(T_i) = \phi(\tilde{x}_j)(T_i) = B_i \). Since \( i \) was arbitrary, \( \phi(\tilde{x}) = y \), and thus \( \phi \) is onto.

\[ \square \]

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