MINKOWSKI SUPERSPACES AND SUPERSTRINGS AS ALMOST REAL-COMPLEX SUPERMANIFOLDS

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Abstract. In 1996/7, J. Bernstein observed that smooth or analytic supermanifolds that mathematicians study are real or (almost) complex ones, while Minkowski superspaces are completely different objects. They are what we call almost real-complex supermanifolds, i.e., real supermanifolds with a non-integrable distribution, the collection of subspaces of the tangent space, and in every subspace a complex structure is given.

An almost complex structure on a real supermanifold can be given by an even or odd operator; it is complex (without “always”) if the suitable superization of the Nijenhuis tensor vanishes. On almost real-complex supermanifolds, we define the circumcised analog of the Nijenhuis tensor. We compute it for the Minkowski superspaces and superstrings. The space of values of the circumcised Nijenhuis tensor splits into (indecomposable, generally) components whose irreducible constituents are similar to those of Riemann or Penrose tensors. The Nijenhuis tensor vanishes identically only on superstrings of superdimension 1|1 and, besides, the superstring is endowed with a contact structure. We also prove that all real forms of complex Grassmann algebras are isomorphic although singled out by manifestly different anti-involutions.

1. Introduction

1.1. General remark on superizations. In the supermanifold theory, there are several “straightforward” superizations of the classical non-super notions. Definitions of superschemes and supervarieties over any field ([Le0]), of their \(C^\infty\) analogs — supermanifolds ([L1]), and of complex analytic analogs — superspaces ([Va1]) are examples of such “straightforward” superizations. To be just, observe that to figure out which direction of superization should be considered “straightforward” sometimes took a while, some of the above notions are subjects of disputes and “improvements” for more than 40 years by now.

There are also notions of Linear Algebra and algebraic notions of Algebraic and Differential Geometries that have several superizations. Some of these superizations were (and some of them still remain) unexpected and without direct non-super analog. For example, among superdeterminants, there is the well-known Berezinian Ber and several not so well known analogs (e.g., the queer determinant qet, and the “classical limits” of Ber and qet, see [DSB], p. 476).

Studying supersymmetries may sometimes help not only to better understand the classical non-super notion (like integral) but even to distinguish a new notion in the non-super setting. Here we consider one such notion, implicitly introduced together with Minkowski superspaces. Lecturing on the results of this paper we heard from the listeners that a
CR-structure looks similar, and indeed it does to an extent, but a careful comparison immediately reveals that these notions have nothing in common, essentially.

1.2. A new notion: Real-complex supermanifold. During the Special year (1996–97) devoted by IAS, Princeton, to attempts to understand at least some of the mathematics used in physical papers on supersymmetry, J. Bernstein pointed at one more example of an unexpected super structure (see notes of Bernstein’s lectures taken by Deligne and Morgan [Del], p. 94). It dawned upon him that the models of our space-time (Minkowski superspaces) suggested in the physical papers of pioneers, where supersymmetry was discovered, are neither real nor complex supermanifolds, nor real supermanifolds with a(n almost) complex structure; Minkowski superspaces are different from real or complex supermanifolds or real supermanifolds with a(n almost) complex structure studied by mathematicians so far (e.g., see [Va1, MaG]).

The Minkowski superspaces and superstrings introduced by physicists are objects with a structure previously never considered. We give a precise definition of such objects in the next subsection. Meanwhile observe that although the bilinear forms (with Lorentzian signature) given at the tangent space at every point of the Minkowski space are equivalent, there are, nevertheless, several types of Minkowski spaces. These spaces differ, for example, by the Riemannian tensor constructed from the metrics.

Every \( N \)-extended Minkowski superspace and certain of the “super Riemann surfaces” considered in String Theories is a real manifold (Minkowski space or a Riemann surface, respectively) rigged with the sheaf of functions with values in \( \Lambda_C(s) \), the complex Grassmann superalgebra with \( s \) generators. This construction differs from the case of “the sheaf of complex-valued functions on a real manifold” \( (s = 0) \) in two ways:

1) Considering the sheaf of \( \mathbb{C} \)-valued functions on the real (super)manifold we do not get anything new as compared with considering the sheaf of \( \mathbb{R} \)-valued functions on the same (super)manifold. Indeed, there is only one real structure on the target space \( \mathbb{C} \), and hence every \( \mathbb{C} \)-valued function \( f \) can be canonically represented in the form of a sum of a pair of \( \mathbb{R} \)-valued functions: \( f(x) = u(x) + iv(x) \). If \( s > 0 \), there is no canonical, i.e., unique in some way, real structure.

A question arises: how many isomorphism classes of real forms of the Grassmann algebra \( \Lambda_C(s) \), i.e., the space of values of superfunctions at a given point, are there?

Obviously, there are several analogs of the complex conjugation on the Grassmann superalgebra of very distinct shape (mathematicians favor some of them, physicists favor other ones, see [B, MaG, Del]). The answer to the above question was not given anywhere, as far as we know, except [L1], where it was borrowed from the first arXiv version of this paper. We will prove that the answer is as follows: all the real forms of \( \Lambda_C(s) \) are isomorphic (for \( s < \infty \)), but there is no canonical real form if \( s > 0 \).

2) One more peculiarity of \( N \)-extended Minkowski superspace \( \mathcal{M}_N \), whose definition we recall in subsec. 3.3, is the presence of a non-holonomic (i.e., non-integrable) distribution \( \mathcal{M}_N \) is rigged with.

If \( \mathcal{M}_N \) were just Minkowski space \( M \) with functions on it taking values in \( \Lambda_C(2N) \), this would have meant that on the purely odd subspace of the tangent space at each point of \( \mathcal{M}_N \) there is given a complex structure. Having singled out the integral submanifold \( I \) of the distribution of codimension 4|0 on \( \mathcal{M}_N \) we could have offered a pair for characterization of \( \mathcal{M}_N \): the Riemannian tensor on \( M \) and, on \( I \), the Nijenhuis tensor describing the measure of non-flatness of the complex structure on \( I \). There are many distributions

\[1\] The fixed terms “Riemann tensor”, “Nijenhuis tensor” denote, strictly speaking, not tensors but tensor fields. In what follows we have to carefully distinguish tensors from tensor fields.
determined by purely odd subspaces of the tangent spaces at the points of a given supermanifold; some are integrable, some are not. The snag is: each of the distribution that determine the Minkowski superspaces $\mathcal{M}_N$ are non-integrable, so no integral subsupermanifold $I$ exists, and nobody knew how to define the analog of the Nijenhuis tensor in such a situation.

1.3. **Real-complex supermanifold as a supermanifold with a $G$-structure and non-integrable distributions.** There are known various examples when a tensor field of a given type is given and, in the tangent space at a given point, a “flat” shape of the tensor of an equivalence class is selected. The possibility of reducing the tensor field under consideration to the selected “flat” shape in an infinitesimal neighborhood of the given point depends on the obstructions, cocycles representing cohomology we will describe shortly.

**Examples:** (1) on a given vector space $V$ over $\mathbb{C}$, there is just one equivalence class of non-degenerate symmetric bilinear forms $g$; (2) same is true for the non-degenerate anti-symmetric bilinear forms $\omega$; (3) on a given vector space $V$ of dimension $2n$ over $\mathbb{R}$, there is one equivalence class of the automorphism $J$ such that $J^2 = -\text{id}$. (4) There can occur several equivalence classes of tensors of a given type, e.g., non-degenerate symmetric bilinear forms $g$ over $\mathbb{R}$ are distinguished by their signature.

If $V$ is a vector space over $\mathbb{C}$ or $\mathbb{R}$ endowed with a tensor $T$ whose automorphism group is $G$, and $M$ is a manifold such that $\dim M = \dim V$ and endowed with a tensor field whose value at each point is equivalent to $T$, then $M$ is said to be endowed with a $G$-structure.

In what follows we recall and superize the notion of structure functions, i.e., functions on the principal $G$-bundle with values in certain Lie algebra cohomology, see [St]. Superization of this notion is immediate and obvious. Obstructions to “flatness” of the three $G$-structures mentioned in examples above are the Riemann tensor for metrics, $d\omega$ for almost symplectic structures, and the Nijenhuis tensor for almost complex structures.

At every point of $N$-extended Minkowski superspace $\mathcal{M}_N$, we can select any shape of the tensor $J$ that defines the complex structure on the odd subspace of the tangent space for a “flat” one. Question: what are the obstructions to reducing the tensor field $J$ to the flat shape in an infinitesimal neighborhood of the point? Here it is vital (nobody had ever considered such structures) that an almost complex structure (tensor) $J$ is given on the whole tangent spaces but only on the subspaces that constitute a non-integrable distribution.

The example of Minkowski superspaces can be naturally generalized. Here is the most broad definition: an almost real-complex supermanifold of superdimension $(p|q;*r|s)$ is a real supermanifold $\mathcal{E}^{p+2r|q+2s}_\mathbb{R}$ endowed with a non-integrable distribution $D$ whose value at every point — $2r|2s$-dimensional subspace of the tangent space at the point — is endowed with a complex structure $J$.

In [Del], J. Bernstein considered a particular case of real-complex supermanifolds of superdimension $(p|0;*0|s)$ and used a somewhat self-contradictory term cs manifolds, short for complex super manifolds although these supermanifolds are not complex.

Importance of non-integrability of the distribution $D$ is vital: if $D$ were integrable, one could have restricted the problem onto the integral subsupermanifold where the classical Nijenhuis tensor provides with the answer.

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2In a recent preprint [Wi], Witten used asterisk to separate real superdimension of the underlying real supermanifold from that of the complex superdimension of the superspaces of non-holonomic distribution in particular cases $(p|0;*0|s)$. We find this notation very suggestive and adopt it.
In particular, almost real-complex structures exist on manifolds as well, but only the example of Minkowski superspace drew attention to them.

The tensor field $J$ that determines the structure of an almost real-complex supermanifold, being defined on subspaces of the non-holonomic distribution, defines a circumcised (some say reduced) connection, see A.Vershik’s appendix to [Se]. At the time J.Bernstein made his observation not only the definition of the circumcised connection was corrected as compared with [Se], the corresponding curvature tensor was known and even computed for Minkowski superspaces, see [GLs], the details, however, were published much later (see [GL4]) and the invariants of almost real-complex supermanifolds are described for the first time.

1.3.1. The new notion and superstrings. At the seminar on “super Riemann surfaces” and superstrings (for some of its workouts, see [LJ]), our attention was again drawn to a related problem mentioned in [Del]: Whereas all Riemann surfaces are automatically endowed with a complex structure (and vice versa: each complex curve possesses a naturally defined Riemannian metric), it is completely unclear why should the analogs of these statements be true for complex supercurves of superdimension $1|N$, and why real supermanifolds of superdimension $2|2N$ must possess an almost complex structure, to say nothing about complex one, even if the underlying surface is endowed with a Riemannian metric.

As we will show, these great expectations, implicitly assumed in the known to us texts on super analogs of Riemann surfaces, are unjustified: the appropriate analogs of the Nijenhuis tensor are, generally, non-zero, except for superstrings of superdimension $1|1$, and not arbitrary but only those endowed with a contact structure. Since in (super)conformal field theories an important ingredient is the integral over the moduli (super)space of complex structures — (super)Teichmüller space $T$, it is worth to have in mind that the Nijenhuis tensor $N_J$ should vanish for each of the tensors $J$ parameterized by $T$.

1.3.2. Question. What are the obstructions to integrability of the almost complex structure on real supermanifolds and of the almost real-complex structure on real-complex supermanifolds? In this paper we answer this question and give several examples pertaining to currently most popular physical models.

1.4. Integrability of almost complex structures: Analytic and algebraic approaches.

1.4.1. Definitions. The real supermanifold of superdimension $2p|2q$ endowed with an even automorphism $J$ of the tangent space at every point, i.e., a tensor field $J$ of valency $(1, 1)$ such that $J^2 = -\text{id}$ is said to be an almost complex supermanifold.

The supermanifold of superdimension $n|n$ over the ground field $K$ endowed with an odd tensor field $J$ of valency $(1, 1)$ such that $J^2 = -\text{id}$ is said to be an almost $J$-symmetric supermanifold.

The supermanifold of superdimension $n|n$ over the ground field $K$ endowed with an odd tensor field $\Pi$ of valency $(1, 1)$ such that $\Pi^2 = \text{id}$ is said to be an almost $\Pi$-symmetric supermanifold.

Over $\mathbb{C}$, and also if the characteristic of the ground field $K$ is equal to 2, every almost $\Pi$-symmetric structure is isomorphic to an almost $J$-symmetric structure, and the other way round; accordingly, the Lie supergroup $G = GQ_J(n)$ preserving the odd operator $J$ such that $J^2 = -\text{id}$ is isomorphic to the supergroup $G = GQ_{\Pi}(n)$ preserving the odd operator $\Pi$ such that $\Pi^2 = \text{id}$. Contrariwise, the real supergroups $GQ_J(n; \mathbb{R})$ and $G = GQ_{\Pi}(n; \mathbb{R})$ are not isomorphic, see [LT].
A given supermanifold $\mathcal{M}$ is said to be endowed with an almost $G$-structure (the adjective “almost” is often not mentioned causing confusion) if, in the tangent space to every point of $\mathcal{M}$, an action of a supergroup $G$ is given (as in examples above, where $G = GL(p|q;\mathbb{C}) \subset GL(2p|2q;\mathbb{R})$ for $J$ even and $G = GQ_{1|n;\mathbb{K}}$ for $J$ odd). If this $G$-structure is integrable (for details, see [St, GL4]), one can drop the adjective “almost”.

There are two ways to study obstructions to integrability, we expose them in the next two subsections:

1.4.2. **Analytic approach.** The obstruction to integrability of the tensor field $J$ on manifolds were first, as far as we know, computed by Newlander and Nirenberg [NN]. The obstructions constitute what is called the curvature of the almost complex structure given by a tensor $J$ or the Nijenhuis tensor (field) $N_J$ defined, for any vector fields $X, Y$, to be:

$$ \mathcal{N}_J(X, Y) = [J(X), J(Y)] - J([J(X), Y] - J([X, J(Y)]) - [X, Y], $$

where $[\cdot, \cdot]$ is the Lie bracket of vector fields. Nijenhuis showed that the tensor field $N_J$ can be represented in the following form:

$$ N_J = \frac{1}{2} \{ J, J \}_N, $$

where $\{ \cdot, \cdot \}_N$ is the Nijenhuis bracket, see [GT], where there are listed all bilinear differential operators acting in the spaces of tensor fields and invariant with respect to the changes of variables; see also [Nij], where 12 equivalent definitions of the Nijenhuis tensor are given.

A. Vaintrob studied integrability of the even tensor $J$ on supermanifolds and proved that an almost complex supermanifold is complex if the straightforward super analog of the Nijenhuis tensor given by the same expression (1) vanishes, see [Va2]. This fact (rediscovered in [McH]) is used in a rich with results paper [Va1]. The same applies as well to the $J$-symmetry, called in [Va2] odd complex structure. The corresponding Nijenhuis tensor $N_J$ is defined, for any vector fields $X, Y$, to be:

$$ \mathcal{N}_J(X, Y) = (-1)^{p(X)}[J(X), J(Y)] - J([J(X), Y] - J([X, J(Y)]) - [X, Y]. $$

For any $\Pi$-symmetry, the corresponding tensor $N_{\Pi}$ is defined by the same expression (2) with $J$ replaced by $\Pi$.

So far, nobody bothered to investigate if there are other, apart from 2|0, superdimensions $p|q$ over $\mathbb{R}$ in which the Nijenhuis tensor on $\mathcal{M}^{p|q}$ vanishes identically, as in the case of almost complex curves (=Riemannian surfaces). This negligence is understandable because this task is difficult in the analytic approach.

Here we show that the Nijenhuis tensor $N_J$ is irreducible whereas the circumcised Nijenhuis tensors on Minkowski superspaces and on various types of superstrings split into several components, similar to $\alpha$- and $\beta$-components of the Penrose tensor, cf. [Po]. The Nijenhuis tensor on $\mathcal{M}^{p|q}$ vanishes identically if $p|q = 1|1$ and $\mathcal{M}^{1|1}$ is endowed with an almost $J$-symmetry or $\Pi$-symmetry, and also if the $1|1$-dimensional over $\mathbb{C}$ superstring $\mathcal{M}^{1|1}$ is endowed with a contact distribution and a circumcised almost complex structure on the folia of the distribution (as in the Neveu-Schwarz and Ramond cases).

These discoveries were possible since there is, fortunately, an approach more adequate for the task:

1.4.3. **Algebraic approach.** The same problems can be formulated in terms of obstructions to integrability of $G$-structures; compare the definitions in [St] and [GL4]. In these terms superization is immediate and obvious. For example, investigation flatness of the almost complex structure given by a tensor $J$ we consider first the Lie superalgebra $\mathfrak{g}_0 := \mathfrak{gl}(p|q;\mathbb{C})$ preserving the tensor $J$ considered at a given point, identify the
tangent space at this point with the \( \mathfrak{g}_0 \)-module \( \mathfrak{g}_{-1} \), and construct the Cartan prolong \( \mathfrak{g}_* := (\mathfrak{g}_{-1}; \mathfrak{g}_0)_* \) considered as a real Lie superalgebra (for the definition of Cartan prolong and its generalizations we need below, see [Shch] and Appendix). By the usual arguments applicable to any \( G \)-structure ([St]), the obstruction to integrability of the almost \( G \)-structure is a tensor, called \textit{structure function}, whose values at every point lie in the space

\[
H^2(\mathfrak{g}_{-1}; \mathfrak{g}_*).
\]

The curvature tensor itself is embodied by a cocycle representing a non-trivial cohomology class of \( H^2(\mathfrak{g}_{-1}; \mathfrak{g}_*) \).

For example, \( H^2(\mathfrak{g}_{-1}; \mathfrak{g}_*) \) is the space of values (at the point considered) of either the Riemann tensor if \( \mathfrak{g}_0 = \mathfrak{o}(n) \), or of the Nijenhuis tensor if \( \mathfrak{g}_0 = \mathfrak{gl}(2n; \mathbb{R}) \), or of obstructions to integrability of the almost symplectic structure if \( \mathfrak{g}_0 = \mathfrak{sp}(2n) \), etc.

1.4.3a. \textbf{Example}. Consider one example in more details. On a manifold \( M \), let there be given a non-degenerate anti-symmetric bilinear form \( B_m \) at each tangent space \( T_m M \); let \( \omega \) be the exterior 2-form determined by a collection of the forms \( B_m \) for all points \( m \in M \).

\textbf{Definition}. Manifold \( M \) with the above form \( \omega \) is said to be \textit{almost symplectic}. If the form \( \omega \) can be reduced to the chosen flat shape \( \sum dp_i \wedge dq_i \) not only at every point (which is always possible), but also in its (infinitesimal) neighborhood, then the manifold is said to be \textit{symplectic}.

Let \( V \) be a space isomorphic to \( T_m M \), and \( \mathfrak{sp}(V) \) the symplectic Lie algebra preserving the image \( B \) of the form \( B_m \) under this isomorphism. As is known from Linear Algebra, the form \( B \) determines a canonical isomorphism \( V \simeq V^* \), so to every element \( c \in \text{Hom}(V \wedge V, V) \) we can assign an exterior 3-form:

\[
C(u, v, w) = B(c(u, v), w) + B(c(v, w), u) + B(c(u, w), v).
\]

The above described map \( c \mapsto C \) sends the coboundaries, i.e., elements \( c \) of the form

\[
c(u, v) = S(u)v - S(v)u, \quad \text{where} \ S \in \text{Hom}(V, \mathfrak{sp}(V)), \quad \text{and} \ u, v \in V,
\]

to 0, whereas the obstruction to flatness of \( \omega \) at \( m \) is precisely the 3-form \( C \) on \( V \). The collection of all forms \( C \) for all points \( m \in M \) constitute the exterior form \( d\omega \), the analog of the Riemann tensor (field) for the almost symplectic case. If \( d\omega = 0 \), then \( \omega \) can be reduced to the flat form (Darboux’s theorem).

1.4.3b. \textbf{How to express the obstructions to flatness}. It is convenient to represent the space \( H^2(\mathfrak{g}_{-1}; \mathfrak{g}_*) \) as the sum of irreducible \( \mathfrak{g}_0 \)-modules (as this was done long ago for the Riemann tensor represented as the sum of Weyl tensor, traceless Ricci tensor and scalar curvature), and further shorthand this representation by considering only highest weights of the irreducible modules. Such shorthand expression is only possible if the complexification of \( \mathfrak{g}_0 \) is either a semi-simple Lie algebra or a central extension thereof, i.e. is a \textit{reductive complex Lie algebra} if only \( (\mathfrak{g}_0)_0 \) is a reductive complex Lie algebra, then at least one can describe how the irreducible \( \mathfrak{g}_0 \)-modules are glued; for examples of such descriptions, see [Po] and further in this paper.

1.4.4. \textbf{On analogs of Wess-Zumino constraints}. The \( \mathbb{Z} \)-grading of the Lie algebra \( \mathfrak{g}_* \) induces a \( \mathbb{Z} \)-grading on the space \( H^2(\mathfrak{g}_{-1}; \mathfrak{g}_*) \), called the \textit{degree}. As is shown in [St], structure functions of a given degree are only defined provided all the structure functions of smaller degrees vanish; same is true in the non-holonomic situation. In supergravity theory these conditions (vanishing of the structure functions of lesser degrees) are known as \textit{Wess-Zumino constraints}, see [GL4].
1.5. Superstrings and complex structures. Let us begin with supermanifolds of real superdimension $2|2m$, perhaps endowed with an additional structure (such as preserved, infinitesimally, by the centerless Neveu-Schwarz or Ramond superalgebra). In the String Theory, these supermanifolds are usually referred to as super Riemann surfaces. We know that there are not two but four infinite series and several exceptional simple Lie superalgebras analogous to the simple Lie algebra of vector fields on the circle (for the classification, see [GLS]), called stringy superalgebras. Exactly ten of them are distinguished: They are simple, and have non-trivial central extensions, and hence are particularly interesting from the point of view of possible physical applications, since only these centers might act on super versions of certain infinite dimensional Teichmüller spaces (parameterizing deformations of the complex structures), cf. [Kom, BSh]. So we should investigate integrability of the appropriate Nijenhuis tensor on the supersurfaces rigged with one of the structures preserved by the distinguished stringy superalgebra.

As on Minkowski superspaces, some of these structures (e.g., in the centerless Neveu-Schwarz and Ramond cases) are nonholonomic (i.e., non-integrable) distributions, like the contact one.

1.6. Analogs of the curvature tensor in presence of non-integrable distribution. Recall the definition of the analogs of the curvature tensor in presence of non-integrable distribution (for more details, see [GLA]). First of all, in order to distinguish the non-integrability of of the distribution from the (non-)integrability of the curvature field, we will, speaking about distributions, say non-holonomic. The arguments of [GLA] are literally superized, so in this subsection we drop “super”.

Take the $\mathbb{Z}$-graded Lie algebra $g_* := (g_-, g_0)_* = \bigoplus_{i \geq -d} g_i$ associated with the filtered Lie algebra preserving the distribution and the tensor $T$ whose flatness we are studying. Therefore, we identify the tangent space at every point with the space $g_- = \bigoplus_{i < 0} g_i$. If the Lie algebra $g_-$ is generated by the space $g_{-1}$, then this space is precisely the value of the distribution at the point, whereas $g_0$ is the Lie algebra that preserves the tensor $T$.

By analogy with the representation of the Nijenhuis tensor (as well as obstructions to flatness of any other $G$-structure) as a cocycle representing a class from $H^2(g_-, g_*; g_0)$, see [I], we give the following

Definition. The circumsized curvature tensor of the structure given by the tensor $T$ in presence of a non-holonomic distribution $\mathcal{D}$ is a cocycle representing a non-trivial class of

$$H^2(g_{-}; g_*), \text{ where } g_{-} = \bigoplus_{i < 0} g_i.$$

Let $\tilde{g}_* := (g_-, \tilde{g}_0)_*$ be a simple Lie algebra such that $g_0$ has center $\mathfrak{z}$ and the extension of $\tilde{g}_0 = g_0/\mathfrak{z}$ to $g_0$ is trivial, i.e., the center is a direct summand. In [GLA], it is shown that $H^2(g_{-}; \tilde{g}_*)$, the space of values of non-holonomic analog of the Riemannian tensor strictly contains $H^2(g_{-}; g_*)$, the space of values of non-holonomic analog of the Weyl tensor. The case where the Lie algebra of the Lie group $G$ that defines the $G$-structure is $g_0$ (resp. $\tilde{g}_0$) is said to be conformal (resp. reduced).

1.7. Our results. 1) We classified the real structures on the finite dimensional Grassmann algebra $\Lambda_C(s)$, see §2. We prove that, although the anti-automorphisms that single out the real forms of $\Lambda_C(s)$ look completely differently, the real forms they determine are isomorphic, albeit non-canonically.

2) We described the two types of obstructions to integrability:
(2a) of the almost complex structure, as well as almost $J$-symmetric and $\Pi$-symmetric structures, given on the whole tangent space at a point (super versions of the Nijenhuis tensor), and

(2b) of the almost real-complex structure given on the subspaces of a non-holonomic distribution (a circumcised version of the Nijenhuis tensor), see §3.

Obstructions to integrability of an almost complex structure constitute an irreducible $\mathfrak{g}_{0}$-module, while the obstructions to integrability of an almost $J$-structure and $\Pi$-structure form an indecomposable $\mathfrak{g}_{0}$-module, and we described how it is glued of irreducibles.

Obstructions to integrability of an almost real-complex structure are illustrated with examples of Minkowski superspaces and superstrings. In particular, we show that the circumcised Nijenhuis tensor identically vanishes on $1|1$-dimensional complex superstring with contact structure, as predicted by P. Deligne in 1987, see [MaT]. We show further that this is the only superdimension and structure on the superstring (among the four infinite series and four exceptional stringy superalgebras) for which the circumcised Nijenhuis tensor vanishes identically. The Nijenhuis tensors of $\Pi$- and $J$-symmetric ("odd complex") structures vanish identically only in dimension 1|1.

Lorentzian metric on the Minkowski space $M$ underlying the superspace $\mathcal{M}_{N}$ is induced by the non-holonomic distribution of totally even codimension, so the circumcised Nijenhuis tensor replaces, for $N > 0$, the Riemannian tensor on $M$.

The Minkowski space $M$ can be considered as the quotient of the Poincaré group modulo Lorentz group, but can also be considered “conformally”, as the twistor space, see [MaG]. We considered the Minkowski superspaces $\mathcal{M}_{N}$ from super versions of both these approaches.

The statements of §3 are obtained by means of the SuperLie package [Gr] and can be verified by the usual cohomology technique described in detail in [Po].

Minkowski space can possess lots of various metrics that differ by their Riemann tensor — the measure of their “non-flatness”. Similarly, there are lots of different Minkowski superspaces and superstrings that differ by circumcised and usual Nijenhuis tensors, described in theorems 3.1.1, 3.1.3, 3.2.1, 3.3.1.

3) In §4 we showed that there are four ways to superize the notion of Kählerian manifold; there are even more ways to superize the notion of a hyper-Kählerian manifold. We also explained that these superized notions can be endowed with almost real-complex structures. All these superizations seem to be new.

1.8. Related problems. Our methods are applicable to non-standard models of complexified and compactified Minkowski superspaces, such as the ones considered in [MaG, GL].

The Nijenhuis tensor is “inexhaustible as atom”, and recently O. Bogoyavlenskij described, on manifolds, several of its properties previously unnoticed, see [B]. It is interesting to consider the super version of Bogoyavlenskij’s problem.

Mixture of even and odd complex structures. Observe that the odd complex structure such as $J$-symmetry (as well as $\Pi$-symmetry) can be defined over $\mathbb{C}$, so an almost complex supermanifold might also be endowed with an odd almost complex structure ($J$-symmetry or $\Pi$-symmetry). Such “doubly complex” structures exist on the superstrings with Ramond superalgebra (or rather its quotient modulo center) as its Lie superalgebra of infinitesimal symmetries, cf. [LJ], and these even and odd complex structures are integrable or not independently.

Our approach to computing the analogs of the Nijenhuis tensor is applicable to any of these open problems.
2. All real forms of the Grassmann algebra are isomorphic

2.1. Real forms of the complex superalgebra. Given a superalgebra $C$ over $\mathbb{C}$, we say that an even $\mathbb{R}$-linear map $\rho : C \rightarrow C$ is a real structure on $C$ if

$$\rho^2 = \text{id}, \quad \rho(ab) = \rho(a)\rho(b), \quad \text{and} \quad \rho(za) = \bar{z}\rho(a) \quad \text{for any} \quad z \in \mathbb{C} \text{ and} \quad a, b \in C.$$  \hspace{1cm} (6)

We set

$$\text{Re}_\rho C = \{a \in C \mid \rho(a) = a\}, \quad \text{Im}_\rho C = \{a \in C \mid \rho(a) = -a\}. \hspace{1cm} (7)$$

Recall that the realification $C_{\mathbb{R}}$ of $C$ is the same $C$ but considered over $\mathbb{R}$. Clearly, $\text{Re}_\rho C$ is a subalgebra in the realification $C_{\mathbb{R}}$ of $C$, and $\text{Im}_\rho C = i \cdot \text{Re}_\rho C$ whereas $C_{\mathbb{R}} = \text{Re}_\rho C \oplus \text{Im}_\rho C$. The subalgebra $\text{Re}_\rho C$ is said to be a real form of $C$ (corresponding to the real structure $\rho$).

Observe that, on $C_{\mathbb{R}}$, the map $\rho$ is an automorphism, and hence, if $C$ is an algebra with unit, then $\rho(1) = 1$, so, $1 \in \text{Re}_\rho C$.

2.1.1. Examples of real structures on the Grassmann superalgebra $\Lambda_{\mathbb{C}}(n)$. Let $\theta = (\theta_1, \ldots, \theta_n)$ be generators of $\Lambda_{\mathbb{C}}(n)$. For $m = 0$, there is, obviously, just one real structure, the canonical one.

For $n = 1$, there are many real structures; clearly, $\rho(\theta) = \lambda \theta$ for $\lambda \in \mathbb{C}$. Since $\rho$ is involutive, $\lambda \bar{\lambda} = 1$, i.e., $\lambda = \exp(i\varphi)$, where $\varphi \in \mathbb{R}$; and hence

$$\text{Re}_\rho \Lambda_{\mathbb{C}}(1) = \{a + b \exp(i\varphi/2)\theta \mid a, b \in \mathbb{R}\}. \hspace{1cm} (8)$$

For $n = 2k$, set $\theta = (\xi, \eta)$, where $\xi = (\xi_1, \ldots, \xi_k)$, $\eta = (\eta_1, \ldots, \eta_k)$. The following are the main examples of real structures; the first one favored by mathematicians, the second one by physicists; one can (and for $n$ odd, one should) consider a mixture of these structures:

1. $\rho_{\bar{\theta}}(\theta_j) = \theta_j$ for any $j = 1, \ldots, n$ (obviously, one may consider an $n$-parameter generalization $\rho_{\bar{\theta}}(\theta_j) = \exp(i\varphi_j)\theta_j$);

2. $\rho_{\bar{\theta}}(\xi_j) = i \cdot \eta_j$, $\rho_{\bar{\theta}}(\eta_j) = i \cdot \xi_j$ for any $j = 1, \ldots, k$ and $i = \sqrt{-1}$.

In view of the above described diversity of involutions that single out real forms of the complex Grassmann algebra (this diversity is discussed at length in [Ber] and mentioned in [MaG] and [Del]), the following theorem, although obvious to some experts, is worth mentioning. To the best of our knowledge it was not even formulated so far.

2.2. Theorem. All real forms of the Grassmann algebra are isomorphic.

Proof. Let $G := \Lambda_{\mathbb{C}}(n) = \Lambda_{\mathbb{C}}(\theta)$, where $\theta = (\theta_1, \ldots, \theta_n)$ are generators. Set

$$G_k = \bigoplus_{s \geq k} \Lambda^s(\theta), \quad G_0 = \bigoplus_{2s \leq n} \Lambda^{2s}(\theta), \quad G_1 = \bigoplus_{2s - 1 \leq n} \Lambda^{2s-1}(\theta).$$

For $n$ odd, introduce also the space $G^-_1 = \bigoplus_{1 \leq 2k - 1 \leq n} \Lambda^{2k-1}$.

Let $\rho$ be a real structure on $G = \Lambda_{\mathbb{C}}(n)$. Observe that if $U \subset G$ is a $\rho$-invariant complex linear space, i.e., $\rho(U) = U$, then $U$, same as $G$, splits into the direct sum of its real and imaginary parts: $U = \text{Re}_\rho U \oplus \text{Im}_\rho U$, and $(G/U)_{\mathbb{R}} = \text{Re}_\rho G/U \oplus \text{Im}_\rho G/U$ if $U$ is an ideal.

Having selected anticommuting generators $\theta = (\theta_1, \ldots, \theta_n)$ of $G$, we construct the ideals $G_k$. For the natural filtration of the Grassmann superalgebra $G = G_0 \supset G_1 \supset \cdots \supset G_n$ associated with the degree of the elements of $G$ assuming that each generator is of degree 1, set $V = G_1/G_2$, and let $\pi : G_1 \rightarrow V$ be the natural projection.

Observe that

1) $G = \mathbb{C} \cdot 1 \oplus G_1$, as linear space;
2) $G_1 \subset G_1$.

Now observe that $G_1$ has an invariant description: this is the ideal of nilpotent elements of $G$. Therefore $\rho(G_1) = G_1$. But then $\rho(G_k) = G_k$ for all $k = 1, \ldots, n$.

Observe that the center of $G$ is

$$Z = \begin{cases} G_0 & \text{for } n \text{ even} \\ G_0 \oplus G_n & \text{for } n \text{ odd.} \end{cases}$$

For any $n$, we have $\rho(Z) = Z$ and $\rho(Z \cap G_2) = Z \cap G_2$. For $n$ odd, set $G_1^- = \bigoplus_{1 \leq 2k-1 < n} \Lambda^{2k-1}$. 

Now, let $B$ be the real form of $G$ corresponding to the real structure $\rho$, let $B_k$ be the real form of the ideal $G_k$, and $B_Z$ the real form of $Z \cap G_2$. Then $\pi(B_1)$ is the real form of the space $V$, and hence $\dim \pi(B_1) = n$.

Let $x_1, \ldots, x_n \in B_1$ be such that $\pi(x_1), \ldots, \pi(x_n)$ form a basis of $V$. Clearly, $x_1, \ldots, x_n$ generate the algebra $B$ over $\mathbb{R}$ and the algebra $G$ over $\mathbb{C}$. Let us expand each $x_k$ with respect to $Z \cap G_2$ and $G_1$ for $n$ even and with respect to $Z \cap G_2$, $G_1^-$ and $G_n$ for $n$ odd:

$$x_k = y_k + z_k, \text{ where } y_k \in \begin{cases} G_1 & \text{for } n \text{ even,} \\ G_1^- & \text{for } n \text{ odd,} \end{cases}$$

Since $z_k \in G_2$, it follows that $\pi(y_k) = \pi(x_k)$. Therefore the elements $y_1, \ldots, y_n$ also generate $G$ (over $\mathbb{C}$), and anti-commute since they belong to $G_1$. Hence so do their images $\rho(y_1), \ldots, \rho(y_n)$.

Now observe that since $x_k \in B$, it follows that $\rho(x_k) = x_k$, i.e.,

$$\rho(y_k) + \rho(z_k) = y_k + z_k.$$

But $\rho(z_k) \in Z \cap G_2$ implying that $\rho(y_k) = y_k + z'_k$, where $z'_k \in Z \cap G_2$. Since the $\rho(y_1), \ldots, \rho(y_n)$ anticommute, it follows that, for all $k, l$, we have:

$$y_k z'_l + z'_k y_l + z'_k z'_l = 0 \Rightarrow y_k z'_l + z'_k y_l = 0 \text{ and } z'_k z'_l = 0.$$

The second and the third equalities follow from the fact that the first two summands in the first equality lie in $G_1$, and the third summand lies in $G_0$.

Consider now the elements

$$t_k = \frac{1}{2}(y_k + \rho(y_k)) = y_k + \frac{1}{2}z'_k \in B.$$

Since $\pi(t_k) = \pi(y_k)$, the elements $t_k$ generate the algebra $G$ over $\mathbb{C}$ and the algebra $B$ over $\mathbb{R}$. Thanks to $\mathbb{R}$ the elements $t_k$ anticommute.

The theorem now follows from the universality of the Grassmann algebra as algebra with unit and anti-commuting generators. Universality is understood here in the sense that any other algebra with anti-commuting generators is a quotient of the Grassmann algebra (just because there are no other relations in the Grassmann algebra). Since we have found $n$ anti-commuting generators of the algebra $B$ over $\mathbb{R}$, it follows from the dimension considerations that there are no relations which are not corollaries of anti-commutation ones.

3. Obstructions to integrability of almost complex and almost real-complex structures

3.1. Revision of the classical examples. 1) The $2n|2m$-dimensional almost complex supermanifolds with $J$ even. It is well-known that any almost complex structure on the real orientable surface is integrable, see [NN]. Let us investigate if there are other
exceptional cases where a given almost complex structure is always complex, except superdimension 2|0, and investigate if the space of values of the Nijenhuis tensor can be split into the irreducible $\mathfrak{gl}(n|m; \mathbb{C})^\mathbb{R}$-modules. (E.Poletaeva performed similar calculations for the analogs of the Riemann tensor: There is no complete reducibility, and the description of how the irreducible components are glued together is rather intricate, see [Po].)

In terms of subsect. 1.4.3 we consider the Lie superalgebra $\mathfrak{g}_0 = \mathfrak{gl}(n|m; \mathbb{C})^\mathbb{R}$ consisting of supermatrices in the non-standard format of the form

$$
\begin{pmatrix}
A & B \\
-B & A 
\end{pmatrix},
$$

where $A, B \in \mathfrak{gl}(n|m; \mathbb{R})$, and $A + iB \in \mathfrak{gl}(n|m; \mathbb{C})$ for $i = \sqrt{-1}$.

and the tautological $\mathfrak{g}_0$-module $\mathfrak{g}_1 = \mathbb{R}^{2n|2m}$ with the following format of its basis vectors (even | odd | even | odd):

$\text{Span}(\partial_1, \ldots, \partial_n \mid \partial_{n+1}, \ldots, \partial_{n+m} \mid \partial_{n+m+1}, \ldots, \partial_{2n+m} \mid \partial_{2n+m+1}, \ldots, \partial_{2n+2m})$.

The classical results on manifolds state that the tensor field $\mathcal{N}_J$, see (1), is the only obstruction to integrability of the almost complex structure on manifolds; besides, on surfaces, the tensor field $\mathcal{N}_J$ vanishes identically. According to [Va2, McH], on supermanifolds, $\mathcal{N}_J$, see (1), is also the only obstruction to integrability of the almost complex structure.

According to [Va2], the only obstruction to integrability of the almost $J$-symmetry is $\mathcal{N}_J$, see (1).

Let us sharpen these claims. For $n + m > 1$, we consider the natural division of root vectors with respect to which the positive ones are those above the diagonals of $A$ and $B$ in (9).

3.1.1. **Theorem.** The lowest weight (with respect to the $\mathfrak{g}_0$-action) cocycles representing the elements of $H^2(\mathfrak{g}_1; \mathfrak{g}_*)$ for $\mathfrak{g}_0 = \mathfrak{gl}(n|m; \mathbb{C})^\mathbb{R}$ and its tautological module $\mathfrak{g}_1$ are as follows (all of degree 1):

$$
\begin{align*}
n = 1 \mid m = 0 : & \quad 0 \\
n > 1 \mid m = 0 : & \quad \partial_1 \otimes (\partial^*_{n-1} \wedge \partial^*_n \wedge \partial^*_{2n-1}), \quad \partial_{n+1} \otimes (\partial^*_{n-1} \wedge \partial^*_n \wedge \partial^*_{2n-1}) \\
n = 0 \mid m \geq 1 : & \quad \partial_1 \otimes (\partial^*_m \wedge \partial^*_n), \quad \partial_1 \otimes (\partial^*_m)^2 \\
n > 0 \mid m > 0 : & \quad \partial_1 \otimes (\partial^*_{n+m} \wedge \partial^*_n), \quad \partial_1 \otimes (\partial^*_{n+m})^2
\end{align*}
$$

3.1.1a. **Comment.** 1) For $n \mid m = 0 \mid 1$, the cocycles (10) span the whole space $H^2(\mathfrak{g}_1; \mathfrak{g}_*)$, not just that of lowest weight vectors.

2) Since the space $H^2(\mathfrak{g}_1; \mathfrak{g}_*)$ has two $\mathfrak{g}_0$-lowest weight vectors, see (10), the researcher familiar with representations of Lie algebras over $\mathbb{C}$ might think that the Nijenhuis tensor splits into two irreducible components. This is not so since the ground field is $\mathbb{R}$: Each lowest weight vector is obtained from the other one by multiplication by $i$ represented by the matrix $\begin{pmatrix} 0 & 1_{n+m} \\ -1_{n+m} & 0 \end{pmatrix}$, and therefore the $\mathfrak{g}_0$-module $H^2(\mathfrak{g}_1; \mathfrak{g}_*)$ is irreducible; it is the realification of an irreducible module over $\mathbb{C}$.

3.1.2. **Almost $J$-symmetric and almost $\Pi$-symmetric supermanifolds over any ground field $\mathbb{K}$.** Let $A, B \in \mathfrak{gl}(n; \mathbb{K})$, and $p(A) = 0$, $p(B) = 1$; let $\mathfrak{g}_0 = \mathfrak{q}_\Pi(n; \mathbb{K})$ (or $\mathfrak{g}_0 = \mathfrak{q}_J(n; \mathbb{K})$) be the Lie superalgebra consisting of supermatrices of the form

$$
\begin{pmatrix}
A & B \\
-B & A 
\end{pmatrix} \text{ if } \mathfrak{g}_0 = \mathfrak{q}_\Pi(n; \mathbb{K}), \quad \text{or} \quad \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \text{ if } \mathfrak{g}_0 = \mathfrak{q}_J(n; \mathbb{K}),
$$
and \( q_{-1} = \mathbb{K}^{n|n} \) be the tautological \( g_0 \)-module with the following basis vectors in the standard format (even | odd):

\[
\text{Span}(\partial_1, \ldots, \partial_n \mid \partial_{n+1}, \ldots, \partial_{2n}).
\]

### 3.1.3. Theorem

The cocycles representing the elements of \( H^2(g_{-1}; g_+) \) are all of degree 2. The \( g_0 \)-module \( H^2(g_{-1}; g_+) \) is indecomposable with the following \( (g_0)_0 \)-highest weights:

\[
(12)
\]

1. \( n = 1 \): the zero module;
2. \( n = 2 \): (2, 0), and (1, 1) each of multiplicity 2|2 (2 even ones and 2 odd ones);
   contains a submodule with the \( (g_0)_0 \)-highest weights (2, 0), and (1, 1) each of multiplicity 1|1;
3. \( n \geq 3 \): \( \varphi_1 := (2, 1, 0, \ldots, 0, -1), \varphi_2 := (2, 0, \ldots, 0), \text{ and } \varphi_3 := (1, 1, 0, \ldots, 0) \)
   each of multiplicity 2|2;
   contains a submodule with the \( (g_0)_0 \)-highest weights \( \varphi_1 \) of multiplicity 2|2,
   \( \varphi_2 \), and \( \varphi_3 \) each of multiplicity 1|1;
   containing, in turn, a submodule with the \( (g_0)_0 \)-highest weights \( \varphi_2 \), and \( \varphi_3 \)
   each of multiplicity 1|1.

### 3.1.3a. Comment

1) We were able to shorthand the answer in Theorem 3.1.3 by using the fact that the superdimension of the space of vacuum (in our case, lowest weight) vectors of every irreducible finite dimensional \( q_m(n; \mathbb{K}) \)- and \( q_{\tilde{m}}(n; \mathbb{K}) \)-module is of the form \( k|k \) (except for the trivial module \( \mathbb{K} \) and \( \Pi(\mathbb{K}) \), when it is equal to 1|0 and 0|1, respectively). Therefore, it suffices to describe only half of the highest weight vectors, say the even ones.

2) The explicit answer is, however, hardly needed in theoretical constructions neither in this theorem nor in theorem 3.2.1; important is that the Nijenhuis tensor does or does not vanish identically, and if there are several components what is the meaning of vanishing of some of them: compare with the Einstein equations = vanishing of certain components of the Riemann tensor.

### 3.2. The \( N \)-extended Minkowski superspaces \( \mathcal{M}_N \)

Usually, mathematicians represent complexified Minkowski superspaces as homogeneous superspaces \( \mathcal{M}_N^\mathbb{C} = G_N^\mathbb{C}/P_N^\mathbb{C} \), where \( G_N^\mathbb{C} \) and \( P_N^\mathbb{C} \) are certain complex supergroups. In the papers and books written by physicists only the structure of the tangent space \( T_m \mathcal{M}_N \) at a given point \( m \) of Minkowski superspace \( \mathcal{M}_N \), and the Lie superalgebra acting on \( T_m \mathcal{M}_N \) are usually given, actually, cf. [GIOS] and references therein.

If \( \tilde{g} = \text{Lie}(G_N) \) and \( p = \text{Lie}(P_N) \), we have the following: \( g \) consists of supermatrices of the form

\[
(13) \quad g = \text{Span} \begin{pmatrix} A & 0 & 0 \\ Q & B & 0 \\ T & -\bar{Q} & -\bar{A} \end{pmatrix}, \quad \text{where } A \in \mathfrak{sl}(2; \mathbb{C}), \ Q \in \mathfrak{Mat}_\mathbb{C}(2 \times N), \ T = \bar{T}^t, \ B \in \mathfrak{gl}(N; \mathbb{C})
\]

where bar denotes component-wise complex conjugation, the \( \mathbb{Z} \)-grading is block diagonal-wise (\( \deg A = \deg B = 0, \deg Q = -1, \deg T = -2 \)) and the parity is defined as \( \deg \text{ mod } 2 \).

Let the Lie superalgebra \( p \) consist of supermatrices of degree \( \geq 0 \). Then \( T_m \mathcal{M}_N \) is spanned over \( \mathbb{R} \) by matrices \( T \) (constituting \( T_m \mathcal{M} \)) and odd matrices \( Q \).
The presence of a non-holonomic distribution on $\mathcal{M}_N$ is obvious: the subspaces of the tangent spaces that constitute the distribution are spanned by vectors realized by pair of matrices $Q$ and $-\overline{Q}^T$, while the superbracket of such vectors does belong to the linear combination not of them but of the matrices $T$.

We can as well consider the maximal symmetry supergroup $G_N$ of $\mathcal{M}_N$, i.e., assume that $\mathfrak{g} = (\mathfrak{g}_-, \mathfrak{g}_0)_s$, where $\mathfrak{g}_0 = \text{det}_0(\mathfrak{g}_-)$ consists of the grading-preserving derivations of the Lie superalgebra $\mathfrak{g}_-$. Set $\mathfrak{p} \coloneqq \bigoplus_{i \geq 0} \mathfrak{g}_i$. This approach and the natural (especially in the light of successes of twistor models) desire to have a simple Lie superalgebra as the generalized Cartan prolong, or its complexification, imposes restrictions on $B$ and $A$, tying the group of inner symmetries with the Lorentz group.

Namely, this desire is satisfied if in the realization of the Lie superalgebra by supermatrices of the form \[(13)\] we set
\[
B \in \mathfrak{u}(N), \quad \text{tr} B = \text{tr}(A - \overline{A}), \quad \text{where } A \in \mathfrak{gl}(2; \mathbb{C}).
\]

Thus, the desire to have as the generalized Cartan prolong a simple Lie superalgebra with the same negative part as \[(13)\], forces us to consider on the Minkowski space $M$ underlying $\mathcal{M}_N$ not a Lorentzian metric preserved by elements $A \in \mathfrak{sl}(2; \mathbb{C}) \simeq \mathfrak{o}(3, 1)$, but rather a conformal structure that only preserves the conformal class of the metric, allowing to multiply the metric by non-zero constants. Then $\mathfrak{g}_s = (\mathfrak{g}_-, \mathfrak{g}_0)_s$ consists of supermatrices of the form
\[
(14) \quad B \in \mathfrak{u}(N), \quad \text{tr} B = \text{tr}(A - \overline{A}), \quad \text{where } A \in \mathfrak{gl}(2; \mathbb{C}).
\]

3.2.1. Theorem. In this theorem, the supermatrix format is $2|N|2$ and $N = 1$. Let $X_{i,j}$ stand for the $(i, j)$-th matrix unit in the matrix $X = A, B, T$, or $Q$.

1) The “conformal” case. The Lie superalgebra $\mathfrak{g}_s := (\mathfrak{g}_-, \mathfrak{g}_0)_s$ consists of supermatrices of the form \[(15)\].

Let the superscript denote the degree of the cocycle $c$, the subscript its number. The highest weight cocycles representing the basis elements of $H^2(\mathfrak{g}_-; \mathfrak{g}_s)$ are as follows.

\[
c_1^0 = T_{2,2} \otimes (Q_{1,1}^s)^\wedge^2, \quad c_2^0 = T_{2,2} \otimes Q_{1,1}^s \wedge iQ_{1,1}^s
\]

In degrees 1 and 2: None.

In degree 3: (the numbering of cocycles match that of the reduced case)

\[
c_3^1 = 4Q_{1,2} \otimes (T_{1,1}^s \wedge (T_{1,2} + T_{2,1})^s) + i(A_{1,1} - A_{2,2}) \otimes (iQ_{1,1}^s \wedge T_{2,1}^s) + i(A_{1,1} + A_{2,2}) \otimes (iQ_{1,1}^s \wedge T_{1,2}^s) - \\
4A_{1,2} \otimes (Q_{1,1}^s \wedge (T_{1,2} + T_{2,1})^s) + 2A_{1,2} \otimes (iQ_{1,1}^s \wedge (T_{1,2} - T_{2,1})^s) + 4A_{1,2} \otimes (Q_{1,1}^s \wedge (T_{1,2}^s)^s) + \\
2iA_{1,2} \otimes (Q_{1,1}^s \wedge (T_{1,2} + T_{2,1})^s) - 2iR_{1,1} \otimes (Q_{1,1}^s \wedge iQ_{1,1}^s) + 2R_{1,1} \otimes (iQ_{1,1}^s)^\wedge^2
\]

\[
c_3^2 = 4iQ_{1,2} \otimes (T_{1,1}^s \wedge (T_{1,2} + T_{2,1})^s) - i(A_{1,1} - A_{2,2}) \otimes (iQ_{1,1}^s \wedge T_{2,1}^s) - i(A_{1,1} + A_{2,2}) \otimes (iQ_{1,1}^s \wedge T_{1,2}^s) - \\
2A_{1,2} \otimes (Q_{1,1}^s \wedge (T_{1,2} - T_{2,1})^s) - 4A_{1,2} \otimes (iQ_{1,1}^s \wedge (T_{1,2} + T_{2,1})^s) + 4A_{1,2} \otimes (iQ_{1,1}^s \wedge (T_{1,2}^s)^s) - \\
2iA_{1,2} \otimes (Q_{1,1}^s \wedge (T_{1,2} + T_{2,1})^s) + 2iR_{1,1} \otimes (Q_{1,1}^s)^\wedge^2 - 2R_{1,1} \otimes (Q_{1,1}^s \wedge iQ_{1,1}^s)
\]

2) The “reduced” case. Everything is as above but $B = 0$, $A \in \mathfrak{sl}(2; \mathbb{C})$. In this case, $\mathfrak{g}_1 = 0$. The highest weight cocycles representing the basis elements of $H^2(\mathfrak{g}_-; \mathfrak{g}_s)$ are as in the conformal case with the following modifications or additions: Two new cocycles in

\footnote{Recall a criterion of integrability of distributions — Frobenius’s theorem: A distribution is integrable if and only if the sections of the distribution form a Lie sub(super)algebra relative the bracket of vector fields.}
degree 1 appear:
\[ c_3^1 = -Q_{1,1} \otimes (Q_{1,1}^* \wedge iQ_{1,1}) + iQ_{1,1} \otimes (Q_{1,1}^* \wedge iQ_{1,1})^2 - Q_{1,2} \otimes (Q_{1,1}^* \wedge iQ_{1,2}) + iQ_{1,2} \otimes (Q_{1,1}^* \wedge iQ_{1,2}) \]
\[ c_4^1 = -Q_{1,1} \otimes (iQ_{1,1}^*)^2 + iQ_{1,1} \otimes (Q_{1,1}^* \wedge iQ_{1,1}) - Q_{1,2} \otimes (iQ_{1,1}^* \wedge iQ_{1,2}) + iQ_{1,2} \otimes (iQ_{1,1}^* \wedge iQ_{1,2}) \]

Three new cocycles in degree 2 appear:

\[ c_5^2 = -(i(A_{1,1} - A_{2,2}) \otimes (Q_{1,1}^*)^2 - i(A_{1,1} - A_{2,2}) \otimes (iQ_{1,1}^*)^2 + 2A_{1,2} \otimes (Q_{1,1}^* \wedge iQ_{1,2}) - 2iA_{1,2} \otimes (Q_{1,1}^* \wedge iQ_{1,2}) - 2iA_{1,2} \otimes (Q_{1,1}^* \wedge iQ_{1,2}) - Q_{1,1} \otimes (iQ_{1,1}^* \wedge T_{1,1}) + iQ_{1,1} \otimes (Q_{1,1}^* \wedge T_{1,1}) + 2Q_{1,2} \otimes (Q_{1,1}^* \wedge i(T_{1,2} - T_{2,1})^* ) + 2Q_{1,2} \otimes (iQ_{1,1}^* \wedge T_{1,1}) - 2iQ_{1,2} \otimes (Q_{1,1}^* \wedge (T_{1,2} + T_{2,1})^*) + 2iQ_{1,2} \otimes (iQ_{1,1}^* \wedge i(T_{1,2} - T_{2,1})^*) + 3iQ_{1,2} \otimes (Q_{1,2} \wedge T_{1,1}) \]

\[ c_6^2 = (A_{1,1} - A_{2,2}) \otimes (Q_{1,1}^* \wedge Q_{1,2}^*) - (A_{1,1} - A_{2,2}) \otimes (iQ_{1,1}^* \wedge iQ_{1,2}^*) + iA_{1,2} \otimes (Q_{1,1}^* \wedge iQ_{1,2}^*) - iA_{1,2} \otimes (iQ_{1,1}^* \wedge iQ_{1,2}^*) - A_{1,2} \otimes (iQ_{1,2}^*)^2 - A_{1,2} \otimes (iQ_{1,2}^*)^2 + 2iA_{1,2} \otimes (Q_{1,1}^* \wedge iQ_{1,2}^*) - A_{1,2} \otimes (Q_{1,2}^*)^2 + A_{2,1} \otimes (Q_{1,1}^*)^2 + 2A_{2,1} \otimes (Q_{1,1}^* \wedge iQ_{1,2}^*) + Q_{1,1} \otimes (iQ_{1,1}^* \wedge T_{1,2} + T_{2,1})^* ) - Q_{1,1} \otimes (iQ_{1,1}^* \wedge T_{1,2} + T_{2,1})^* ) + Q_{1,2} \otimes (iQ_{1,1}^* \wedge T_{1,2} + T_{2,1})^* ) - Q_{1,2} \otimes (iQ_{1,2}^* \wedge i(T_{1,2} + T_{2,1})^* ) - Q_{1,2} \otimes (iQ_{1,2}^* \wedge i(T_{1,2} + T_{2,1})^* ) - Q_{1,2} \otimes (iQ_{1,2}^* \wedge i(T_{1,2} + T_{2,1})^* ) + iQ_{1,2} \otimes (Q_{1,1}^* \wedge T_{1,2} + T_{2,1})^* ) + Q_{1,2} \otimes (Q_{1,1}^* \wedge T_{1,2} + T_{2,1})^* ) \]

The two cocycles in degree 3 become:

\[ c_3^3 = -A_{1,2} \otimes (Q_{1,1}^* \wedge i(T_{1,2} - T_{2,1})^* ) - A_{1,2} \otimes (iQ_{1,1}^* \wedge T_{1,2} + T_{2,1})^* ) + A_{1,2} \otimes (iQ_{1,2}^* \wedge T_{1,2} + T_{2,1})^* ) - iA_{1,2} \otimes (Q_{1,1}^* \wedge (T_{1,2} + T_{2,1})^* ) + iA_{1,2} \otimes (iQ_{1,1}^* \wedge (T_{1,2} + T_{2,1})^* ) + iA_{1,2} \otimes (Q_{1,1}^* \wedge T_{1,2} + T_{2,1})^* ) + Q_{1,2} \otimes (T_{1,1}^* \wedge i(T_{1,2} - T_{2,1})^* ) + iQ_{1,2} \otimes (T_{1,1}^* \wedge (T_{1,2} + T_{2,1})^* ) \]

\[ c_4^3 = A_{1,2} \otimes (Q_{1,1}^* \wedge (T_{1,2} + T_{2,1})^* ) - A_{1,2} \otimes (iQ_{1,1}^* \wedge i(T_{1,2} - T_{2,1})^* ) - A_{1,2} \otimes (Q_{1,2}^* \wedge T_{1,2} + T_{2,1})^* ) - iA_{1,2} \otimes (Q_{1,1}^* \wedge (T_{1,2} + T_{2,1})^* ) - iA_{1,2} \otimes (iQ_{1,1}^* \wedge (T_{1,2} + T_{2,1})^* ) + iA_{1,2} \otimes (Q_{1,1}^* \wedge T_{1,2} + T_{2,1})^* ) - Q_{1,2} \otimes (T_{1,1}^* \wedge i(T_{1,2} - T_{2,1})^* ) + iQ_{1,2} \otimes (T_{1,1}^* \wedge (T_{1,2} + T_{2,1})^* ) \]

3.2.1a. **Comment.** All cocycles, except \( c_5 \), appear in pairs that differ by multiplication by \( i \), cf. Comment 3.1.1a so the irreducible module generated by any cocycle of the pair splits into two irreducibles after complexification; the cocycle \( c_5 \) corresponds to the real \( g_0 \)-module that remains irreducible after complexification.

3.2.2. **The cases of \( M_N \) for \( N > 1 \) and various supermatrix formats.** Passing to \( N > 1 \), it is convenient to complexify \( g_* = (g_{-1}, g_0)_* \), as well as the \( g_0 \)-module \( H^2(g_{-1}; g_*) \), and consider in this \( g_0 \)-module only highest weight vectors. We have \( g_0^C = \mathfrak{s}(\mathfrak{gl}(2) \oplus \mathfrak{gl}(N) \oplus \mathfrak{gl}(2)) \), where the index identifies a copy of \( \mathfrak{gl}(2; \mathbb{C}) \), whereas the operator \( \mathfrak{s}(\cdot) \) singles out the supertraceless part of the argument. Denote the elements of \( \mathfrak{gl}(2) \) by \( C \) to distinguish from the elements \( A \) of \( \mathfrak{gl}(1) \) and use different names for \( Q \) and \( S := "Q" \) which now are both complex and independent of each other, compare with \( L \):

\[
\begin{pmatrix}
A & V & U \\
Q & B & R \\
T & S & C
\end{pmatrix}, \quad \text{where} \quad S, V \in \text{Mat}_\mathbb{C}(2 \times N), \; Q, R \in \text{Mat}_\mathbb{C}(N \times 2), \; A, C \in \mathfrak{gl}(2; \mathbb{C}), \; B \in \mathfrak{gl}(N), \; \text{tr} B = \text{tr}(A + C).
\]
The elements $T$ are not hermitian now, as in (13), but arbitrary elements of $\mathfrak{gl}(2; \mathbb{C})$. Having found the highest weight vectors of the complex representation of $\mathfrak{g}_0^c$, we have to check which of these irreducible representations are complexifications of already complex representations of $\mathfrak{g}_0$ (these are to be found among those of multiplicity 2) and which are of multiplicity 1 (complexifications of real representations), cf. Comment 3.1.1a.

In [GLs], the calculations of $H^2(\mathfrak{g}_-; \mathfrak{g}_+)$ are performed for $\mathfrak{g}_+ = \mathfrak{sl}(4|N)$ realized in various supermatrix formats (4|N, 2|N|2, and several other ones) for $N = 1, 2, 4$ and 8, for both conformal and reduced cases. Moreover, the calculations are also performed for several types of parabolic subalgebras $\mathfrak{p}$, smaller (if $N > 1$) than the one containing “all above the Q-and-S diagonal”. These smaller subalgebras $\mathfrak{p}$ are chosen so as to have the components of the non-holonomic curvature tensor whose components independent on odd coordinates match those entering the Einstein equations; for details, inessential in this paper, but important, in our opinion, for understanding SUGRA, see [GLs].

3.3. The integrability of almost complex structure of the 1|2n-dimensional over $\mathbb{C}$ supercurves with a distinguished structure. When we study integrability of the almost complex structure (preserved by the Lie (super)algebra $\mathfrak{aut}(J)$) in presence of some other structure (tensor or a distribution) preserved (at the point) by the Lie (super)algebra $\mathfrak{g}_+$, we should replace $\mathfrak{g}_+$ in formulas (4) and (5) by the (generalized) Cartan prolong of $(\mathfrak{g}_-; \mathfrak{h}_0)$, where $\mathfrak{h}_0 := \mathfrak{g}_0 \cap \mathfrak{aut}(J)$. Let $A + Bi \in \mathfrak{g}_0$ be the decomposition into the real and imaginary part. Then the supermatrices of $\mathfrak{h}_0 := \mathfrak{g}_0 \cap \mathfrak{aut}(J)$ are of the form (9) with $A + Bi \in \mathfrak{g}_0$.

We consider superstrings with various additional structures. Let us describe the Lie superalgebras over $\mathbb{C}$ that preserve, infinitesimally, these additional structures. The general algebra $\mathfrak{vect}(m|n)$ does not preserve anything, the divergence-free one $\mathfrak{svect}(m|n)$ preserves a volume element.

To both (centerless) $N$-extended Neveu-Schwarz and Ramond type contact Lie superalgebras with Laurent polynomial coefficients only one vectorial Lie superalgebra with polynomial coefficients corresponds since locally the corresponding superstrings are isomorphic. Similar is the case with the parametric family of stringy superalgebras preserving a volume element. Recall the description of vectorial Lie superalgebras with polynomial coefficients over $\mathbb{C}$:

$$\mathfrak{vect}(m|n) := \mathfrak{dev}\mathbb{C}[x, \theta] = \left\{ \sum f_i \partial x_i + \sum g_j \partial \theta_j \mid f_i, g_j \in \mathbb{C}[x, \theta] \right\},$$

where $x = (x_1, \ldots, x_m)$ are even indeterminates, $\theta = (\theta_1, \ldots, \theta_n)$ are odd ones;

$$\mathfrak{svect}(m|n) := \left\{ \sum f_i \partial x_i + \sum g_j \partial \theta_j \mid \sum \partial x_i (f_i) + \sum (-1)^{p(g_j)} \partial \theta_j (g_j) = 0 \right\};$$

$\mathfrak{v}(2n + 1|m)$ preserves the distribution singled out by the odd form

$$\alpha_1 = dt - \sum_i (p_i dq_i - q_i dp_i) - \sum_j (\xi_j d\eta_j + \eta_j d\xi_j) + \begin{cases} 0 & \text{for } m \text{ even} \\ \theta d\theta & \text{for } m \text{ odd}, \end{cases}$$

where $t, p = (p_1, \ldots, p_n), q = (q_1, \ldots, q_n)$ are even indeterminates and $\xi = (\xi_1, \ldots, \xi_k), \eta = (\eta_1, \ldots, \eta_k)$ for $2k = m$ for $m$ even (and $\theta$ for $m$ odd) are odd ones. Set:

$$K_f = (2 - E)(f) \frac{\partial}{\partial t} - H_f + \frac{\partial f}{\partial t} E \quad \text{for any } f \in \begin{cases} \mathbb{C}[t, p, q, \xi, \eta] & \text{for } m \text{ even} \\ \mathbb{C}[t, p, q, \xi, \eta, \theta] & \text{for } m \text{ odd}, \end{cases}$$
where \( E = \sum y_i \frac{\partial}{\partial y_i} \) (here the \( y_i \) are all the coordinates except \( t \)), and \( H_f \) is the hamiltonian vector field with hamiltonian function \( f \):

\[
H_f = \sum_{i \leq n} \left( \frac{\partial f}{\partial q_i} \frac{\partial}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial}{\partial q_i} \right) - (-1)^{p(f)} \sum_{j \leq k} \left( \frac{\partial f}{\partial q_j} \frac{\partial}{\partial q_j} + \frac{\partial f}{\partial q_j} \frac{\partial}{\partial q_j} \right) + \left\{ \frac{\partial f}{\partial \theta} \frac{\partial}{\partial \theta} \right\} \text{ for } m \text{ even.}
\]

\( \mathfrak{m}(n|n+1) \) preserves the distribution singled out by the even form

\[
\alpha_0 = d\tau + \sum_j (\xi_j dq_j + q_j d\xi_j),
\]

where \( q = (q_1, \ldots, q_n) \) are even indeterminates and \( \xi = (\xi_1, \ldots, \xi_n) \), and \( \tau \) are odd ones. For any \( f \in \mathbb{C}[q, \xi, \tau] \), set:

\[
M_f = (2 - E)(f) \frac{\partial}{\partial \tau} - Le_f - (-1)^{p(f)} \frac{\partial f}{\partial \tau} E,
\]

where \( E = \sum y_i \frac{\partial}{\partial y_i} \) (here the \( y_i \) are all the coordinates except \( \tau \)), and

\[
Le_f = \sum_{i \leq n} \left( \frac{\partial f}{\partial q_i} \frac{\partial}{\partial q_i} + (-1)^{p(f)} \frac{\partial f}{\partial q_i} \frac{\partial}{\partial q_i} \right).
\]

Let \( L_D \) be the Lie derivative along the vector field \( D \). Since

\[
L_{K_f}(\alpha_1) = 2\frac{\partial f}{\partial \tau} \alpha_1 = K_1(f) \alpha_1,
\]

\[
L_{M_f}(\alpha_0) = -(-1)^{p(f)} 2 \frac{\partial f}{\partial \tau} \alpha_0 = -(-1)^{p(f)} M_1(f) \alpha_0,
\]

it follows that \( K_f \in \mathfrak{t}(2n+1|m) \) and \( M_f \in \mathfrak{m}(n) \). It is not difficult to show that

\[
\mathfrak{t}(2n+1|m) = \text{Span} \left\{ K_f \mid f \in \mathbb{C}[t, p, q, \xi, \eta] \text{ for } m \text{ even, } \mathbb{C}[t, p, q, \xi, \eta, \theta] \text{ for } m \text{ odd} \right\},
\]

and \( \mathfrak{m}(n) = \text{Span} \{ M_f \mid f \in \mathbb{C}[q, \xi, \tau] \} \).

These vectorial Lie superalgebras \( \mathfrak{g}_* = \mathfrak{g} \oplus \mathfrak{g}_s \) are considered with their standard \( \mathbb{Z} \)-grading in which the degree of each indeterminate, except \( t \) and \( \tau \), is equal to 1, \( \deg t = \deg \tau = 2 \).

3.3.1. Theorem. The following are the simple vectorial Lie superalgebras \( \mathfrak{g}_* \) with polynomial coefficients in their standard grading corresponding to the distinguished simple stringy Lie superalgebras and the (highest weight with respect to the \( \mathfrak{g}_0 \)-action when appropriate) representatives of the basis elements of \( H^2(\mathfrak{g}_{-1}; \mathfrak{g}_*) \):

- \( \mathfrak{vect}(1|n)^{\mathbb{R}} \subset \mathfrak{vect}(2|2n; \mathbb{R}) \) for \( n = 1, 2 \): these cases are already considered in \([10]\).
- \( \mathfrak{vect}(1|2)^{\mathbb{R}} \subset \mathfrak{vect}(2|4; \mathbb{R}) \): The highest weight cocycles representing the elements of \( H^2(\mathfrak{g}_{-1}; \mathfrak{g}_*) \), where \( \mathfrak{g}_* = \mathfrak{vect}(1|2)^{\mathbb{R}} \), are, in addition to those for \( \mathfrak{vect}(1|2)^{\mathbb{R}} \), as follows:

\[
\mathfrak{t}(1|1)^{\mathbb{R}} \subset \mathfrak{vect}(2|2; \mathbb{R}): \text{We have } \mathfrak{g}_* = \text{Span}(K_1, iK_1, K_0, iK_0), \mathfrak{g}_0 = \text{Span}(K_t, iK_t) \text{ and } H^2(\mathfrak{g}_{-1}; \mathfrak{g}_*) = 0.
\]

- For \( \mathfrak{t}(1|n)^{\mathbb{R}} \subset \mathfrak{vect}(2|2n; \mathbb{R}) \), where \( n \geq 2 \), all cocycles are of degree 0:
\* \(\mathfrak{t}(1|2)^R\): We have \(\mathfrak{g}_- = \text{Span}(K_1, iK_1, K_ζ, iK_ζ, K_η, iK_η)\), \(\mathfrak{g}_0 = \text{Span}(K_t, iK_t, K_ζθ, iK_ζθ, K_ηθ, iK_ηθ)\). cocycles of \(H^2(\mathfrak{g}_-; \mathfrak{g}_*)\) are only the following ones (hereafter, in cocycles, we write just \(f \) instead of \(K_f\)):

\[
\begin{align*}
c_0^0 &= 1 \otimes (\xi^*)^2, & c_0^3 &= 1 \otimes (\xi^* \eta)^2, \\
c_2^0 &= 1 \otimes (\eta^*)^2, & c_2^3 &= 1 \otimes (\eta^* \xi)^2.
\end{align*}
\]

\* \(\mathfrak{t}(1|3)^R\): We have

\[
\begin{align*}
\mathfrak{g}_- &= \text{Span}(K_1, iK_1, K_ζ, iK_ζ, K_η, iK_η, K_θ, iK_θ), \\
\mathfrak{g}_0 &= \text{Span}(K_t, iK_t, K_ζθ, iK_ζθ, K_ηθ, iK_ηθ).
\end{align*}
\]

The highest weight cocycles of \(H^2(\mathfrak{g}_-; \mathfrak{g}_*)\) with respect to the \(\mathfrak{g}_0\)-action are

\[
\begin{align*}
c_1^0 &= 1 \otimes (\eta^*)^2, & c_2^0 &= 1 \otimes (\eta^* \xi)^2.
\end{align*}
\]

\* For \(\mathfrak{t}(1|2n)^R\), where \(n \geq 2\), the highest weight cocycles corresponding to the three irreducible components of the \(\mathfrak{g}_0\)-module \(H^2(\mathfrak{g}_-; \mathfrak{g}_*)\) are as follows:

\[
\begin{align*}
c_0^0 &= 1 \otimes (\xi^*)^2, & c_0^3 &= 1 \otimes (\xi^* \eta)^2, \\
c_1^0 &= 1 \otimes (\eta^*)^2, & c_1^3 &= 1 \otimes (\eta^* \xi)^2, \\
c_2^0 &= 1 \otimes (\eta^* \xi)^2, & c_2^3 &= 1 \otimes (\eta^* \xi)^2.
\end{align*}
\]

\* For \(\mathfrak{t}(1|2n+1)^R\), where \(n \geq 2\), the highest weight cocycles corresponding to the three irreducible components of the \(\mathfrak{g}_0\)-module \(H^2(\mathfrak{g}_-; \mathfrak{g}_*)\) are as follows:

\[
\begin{align*}
c_0^0 &= 1 \otimes (\eta^* \xi)^2, & c_0^3 &= 1 \otimes (\eta^* \xi)^2, \\
c_1^0 &= 1 \otimes (\eta^* \xi)^2, & c_1^3 &= 1 \otimes (\eta^* \xi)^2, \\
c_2^0 &= 1 \otimes (\eta^* \xi)^2, & c_2^3 &= 1 \otimes (\eta^* \xi)^2.
\end{align*}
\]

\* \(\mathfrak{m}(1)^R \subset \text{vect}(2|4; \mathbb{R})\): We have

\[
\begin{align*}
\mathfrak{g}_- &= \text{Span}(M_1, iM_1, M_4, iM_4, M_θ, iM_θ), & \mathfrak{g}_0 &= \text{Span}(M_τ, iM_τ, M_θτ, iM_θτ, M_φτ, iM_φτ).
\end{align*}
\]

The cocycles representing the elements of \(H^2(\mathfrak{g}_-; \mathfrak{g}_*)\) are as follows:

\[
\begin{align*}
c_0^0 &= 1 \otimes (q_1^* \xi)\xi, & c_1^0 &= 1 \otimes (q_1^* \theta_1^* + 1^* + q_1 \otimes i(q_1^* \theta_1^*)^*, \\
c_2^0 &= 1 \otimes (q_2^* \eta)\eta, & c_2^3 &= 1 \otimes (q_2^* \eta)\eta.
\end{align*}
\]

the operators \(M_φτ\) and \(iM_φτ\) trivially act on the cohomology classes represented by the above cocycles.

### 3.3.1a. Remark

Although \(\text{vect}(1|1) \cong \mathfrak{t}(1|2) \cong \mathfrak{m}(1)\) as abstract Lie superalgebras, see [CLS], they are non-isomorphic as filtered or \(\mathbb{Z}\)-graded ones, and preserve completely different structures. So no wonder that the obstructions to integrability to the almost complex (resp. almost real-complex) structure (in the first case, resp. the other two cases) look (and are) completely different.

### 4. On Kähler and Hyper-Kähler Supermanifolds

#### 4.1. Definition on manifolds

Let a real manifold \(M\) possess an almost complex structure \(J\) and a non-degenerate symmetric bilinear form \(h\) such that

\[
h(X, Y) = h(JX, JY)\]

for any vector fields \(X, Y \in \text{vect}(M)\) (such \(h\) is said to be pseudo-Hermitian). The manifold \(M\) is said to be Kähler if \(J\) is covariantly constant with regard to the Levi-Civita connection \(\nabla\) corresponding to the metric \(h\), i.e.,

\[
\nabla J = 0.
\]
Each Kähler manifold is symplectic in a natural way with the non-degenerate 2-form defined by
\[ \omega(X, Y) = h(JX, Y) \]
for any \( X, Y \in \text{vect}(M) \), whereas requirement
\[ d\omega = 0 \]
is one of definitions of Kähler manifolds instead of (24).

Any two of the constituents of the triple \((\omega, h, J)\) determine the third one by means of eq. (25) and, since on supermanifolds these two entities can be even or odd, the notion of Kähler manifold has (at least) four types of superizations.

4.1.1. Remark. For the formula (25) to define any third ingredient of the triple \((\omega, h, J)\) given the other two, we only need non-degeneracy of \(\omega\) and \(h\). The flatness of the \(G\)-structures associated with \(J\) in the conventional definition of the Kähler manifold (i.e., requirements that \(J\) is complex, not almost complex) does not seem to be justified: We do not require flatness of the metric, so why discriminate \(\omega\) and \(J\)? Besides, why should the metric be sign-definite?

M. Verbitsky informed us that, indeed, the sign-definiteness of \(h\) in the traditional definition of the Kähler manifold is unnecessary (published classification results are only known, however, for sign-definite forms \(h\)), whereas the symplectic structure is needed because
\[ d\omega = 0 \implies \nabla J = 0. \]
The condition \(d\omega = 0\) is, actually, a system of two equations (vanishing of both components of \(d\omega \in \Omega^3 = P^3 \oplus \omega \wedge \Omega^1\), where \(P^3\) is the space of primitive (aka harmonic) forms, the ones “not divisible” by \(\omega\), and both are needed to ensure (24).

Moreover, the condition \(d\omega = 0\) is often taken for the definition of the Kähler manifold.

In view of this Remark, we suggest the following definition of Kähler supermanifold suitable also in the non-super setting.

4.2. Definitions on supermanifolds. A non-degenerate supersymmetric bilinear form \(h\) on the superspace \(V\) will be called pseudo-hermitian metric relative the operator \(J \in \text{End}(V)\) such that \(J^2 = \pm \text{id}\) if
\[ h(X, Y) = (-1)^{p(X)p(J)}h(JX, JY) \]
for any vectors \(X, Y \in V\).

Let \(\mathcal{M}\) be a real supermanifold with a complex structure (or a \(J\)-symmetric or \(\Pi\)-symmetric structure over any ground field), \(h\) a non-degenerate pseudo-hermitian metric relative to \(J\) or \(\Pi\), and \(\omega\) a non-degenerate differential 2-form (for details of definition of these notions, see [L1]). The supermanifold \(\mathcal{M}\) is said to be Kähler (an almost one if \(J\)-symmetric or \(\Pi\)-symmetric or the complex structure is an “almost” one) if
\[ \omega(X, Y) = h(JX, Y) \]
for any \(X, Y \in \text{vect}(\mathcal{M})\) provided \(p(h) + p(J) = p(\omega)\) and \(\nabla J = 0\) for the Levi-Civita connection \(\nabla\) corresponding to the metric \(h\), see [Po].

This definition implies the following restrictions on the possible superdimensions of \(\mathcal{M}\):

\[
\begin{array}{c|c|c}
 p(J) = 0 & p(J) \text{ (or } p(\Pi) \text{)} = 1 \\
p(h) = 0 & 2n|2m & 2n|2n \\
p(h) = 1 & 2n|2n & n|n
\end{array}
\]
The integrability conditions are not, however, automatically satisfied and must be verified. (Recall that the odd non-degenerate and closed form $\omega$ obtained in the off-diagonal cases of table (30) is called, as A. Weil suggested, \textit{periplectic}, it is the analog of the symplectic form that leads to the “antibracket” and the corresponding mechanics, discovered in [Lm] and rediscovered by I. Batalin and G. Vilkovissky who applied it to physics [BV]; for a review, see [GPS].)

The Lie superalgebra preserving an odd non-degenerate anti-symmetric bilinear form is called \textit{periplectic} and denoted $\mathfrak{pe}^a(n)$; an odd non-degenerate symmetric bilinear form is preserved by a Lie superalgebra $\mathfrak{pe}(n)$, isomorphic to $\mathfrak{pe}^a(n)$, but having a different matrix realization, see [LL].

Given three (almost) complex structures $J_i$, satisfying the relations of quaternionic units, and one metric $h$ pseudo-hermitian relative each $J_i$, together with three symplectic (or periplectic) forms $\omega_i$ tied together by three relations of the form (29), we arrive at the notion of an (almost) \textit{hyper-Kähler supermanifold}.

Most of the notions of this section had never been distinguished before. It would be interesting to generalize with their help the ideas exposed, e.g., in lectures [Ku] and later works.

5. Appendix: Cartan prolongations and its generalizations (Shch)

5.1. Cartan prolongations. Let $g$ be a Lie algebra, $V$ a $g$-module, and $S^i$ the operator of the $i$th symmetric power. Set $g_{-1} = V$, $g_0 = g$, and for $k > 0$, define the $k$th \textit{Cartan prolongation} of the pair $(g_{-1}, g_0)$ by setting

$$g_k = \{ X \in \text{Hom}(g_{-1}, g_{k-1}) \mid X(v_0)(v_1, v_2, \ldots v_k) = X(v_1)(v_0, v_2, \ldots, v_k) \text{ for any } v_0, v_1, \ldots v_k \in g_{-1} \}.$$

Let

$$i : S^{k+1}(g_{-1})^* \otimes g_{-1} \longrightarrow S^k(g_{-1})^* \otimes g_{-1}^* \otimes g_{-1},$$
$$j : S^k(g_{-1})^* \otimes g_0 \longrightarrow S^k(g_{-1})^* \otimes g_{-1}^* \otimes g_{-1}$$

be the natural embeddings. Then $g_k = i(S^{k+1}(g_{-1})^* \otimes g_{-1}) \cap j(S^k(g_{-1})^* \otimes g_0)$.

The complete Cartan prolong of the pair $(V, g)$ is the space $(g_{-1}, g_0)_* = \bigoplus_{k \geq -1} g_k$. This space is naturally endowed with a Lie algebra structure which is rather bothersome to define in abstract terms. If, however, the $g_0$-module $g_{-1}$ is \textit{faithful}, there is an embedding

$$(g_{-1}, g_0)_* \subset \text{vect}(n) = \text{der}\mathbb{C}[x_1, \ldots, x_n], \text{ where } n = \dim g_{-1} \text{ and }$$
$$g_i = \{ D \in \text{vect}(n) \mid \deg D = i, [D, X] \in g_{i-1} \text{ for any } X \in g_{-1} \},$$

and the Lie algebra structure on $\text{vect}(n)$ induces same on $(g_{-1}, g_0)_*$, the latter structure coincides with the one we were lazy to define in abstract terms.

Of four series of simple vectorial Lie algebras with polynomial coefficients, three are the complete Cartan prolongs:

$$\text{vect}(n) = (\text{id}, \mathfrak{gl}(n))_*, \quad \text{svect}(n) = (\text{id}, \mathfrak{sl}(n))_*, \quad \mathfrak{h}(2n) = (\text{id}, \mathfrak{sp}(n))_*,$$

The fourth series — $\mathfrak{k}(2n + 1)$ — is the result of a bit more general construction to be described in subsec. 5.2.
5.1.1. **Vectorial Lie superalgebras as Cartan prolongs.** Superization of constructions of subsec. 5.1 is direct one: via the Sign Rule. We obtain in this way the following infinite dimensional Lie superalgebras:

\[
\begin{align*}
\mathfrak{vect}(m|n) &= (\text{id}, \mathfrak{gl}(m|n))_*, & \mathfrak{svect}(m|n) &= (\text{id}, \mathfrak{sl}(m|n))_*; \\
\mathfrak{h}(2m|n) &= (\text{id}, \mathfrak{osp}^a(m|2n))_*, & \mathfrak{le}(n) &= (\text{id}, \mathfrak{pe}^a(n))_*,
\end{align*}
\]

where \(\mathfrak{osp}^a(m|2n)\) is assumed preserving a non-degenerate even anti-symmetric bilinear form, while an isomorphic to it Lie superalgebra \(\mathfrak{osp}(m|2n)\) is assumed preserving a non-degenerate even symmetric bilinear form.

**Remarks.**

1) The Cartan prolong \((\text{id}, \mathfrak{osp}(m|2n))_* = (\Pi(\text{id}), \mathfrak{osp}^a(m|2n))_*\) is of finite dimension.

2) Superization of the contact Lie algebras leads to the two series, \(\mathfrak{k}\) and \(\mathfrak{m}\), described below.

5.2. **Generalizations of Cartan prolongation.** Consider a nilpotent \(\mathbb{Z}\)-graded Lie algebra \(\mathfrak{g}_- = \bigoplus_{-d \leq i \leq -1} \mathfrak{g}_i\), and a subalgebra \(\mathfrak{g}_0 \subset \mathfrak{derv}_0 \mathfrak{g}\) of the Lie algebra of its \(\mathbb{Z}\)-grading preserving derivations. Let

\[
\begin{align*}
i: S^{k+1}(\mathfrak{g}_-)^* \otimes \mathfrak{g}_- &\rightarrow S^k(\mathfrak{g}_-)^* \otimes \mathfrak{g}_-^* \otimes \mathfrak{g}_-, \\
j: S^k(\mathfrak{g}_-)^* \otimes \mathfrak{g}_0 &\rightarrow S^k(\mathfrak{g}_-)^* \otimes \mathfrak{g}_-^* \otimes \mathfrak{g}_-
\end{align*}
\]

be natural embeddings analogous to (31). For \(k > 0\), define the \(k\)th prolongation of the pair \((\mathfrak{g}_-, \mathfrak{g}_0)\) by setting

\[
\mathfrak{g}_k = (j(S^*(\mathfrak{g}_-)^* \otimes \mathfrak{g}_0) \cap i(S^*(\mathfrak{g}_-)^* \otimes \mathfrak{g}_-))_k,
\]

where the index \(k\) in the right hand side singles out a component of degree \(k\).

Set \((\mathfrak{g}_-, \mathfrak{g}_0)_* = \bigoplus_{i \geq -d} \mathfrak{g}_i\). If \(\mathfrak{g}_0\)-modules \(\mathfrak{g}_i\) are irreducible for \(i < 0\), then, as is easy to check, \((\mathfrak{g}_-, \mathfrak{g}_0)_*\) is a Lie subalgebra in \(\mathfrak{vect}(\text{dim } \mathfrak{g}_-)\).

Superization of the construction is obvious.

What is the Lie algebra of contact vector fields in these terms? Let \(\mathfrak{hei}(2n)\) be the Heisenberg Lie algebra: its space is \(W \oplus \mathbb{C} \cdot z\), where \(W\) is a \(2n\)-dimensional space endowed with a non-degenerate anti-symmetric bilinear form \(B\), and the bracket is given by the following relations:

\[
(32) \quad z \text{ lies in the center and } [v, w] = B(v, w) \cdot z \text{ for any } v, w \in W.
\]

Clearly, \(\mathfrak{k}(2n + 1) \cong (\mathfrak{hei}(2n), \mathfrak{osp}(2n))_*\), where \(\mathfrak{cg}\) denotes the trivial central extension of the Lie algebra \(\mathfrak{g}\) by means of a 1-dimensional center.

- Define a structure of a Lie superalgebra \(\mathfrak{hei}(2n|m)\) on the direct sum of a \((2n|m)\)-dimensional superspace \(W\) endowed with a non-degenerate even anti-symmetric bilinear form \(B\) and a \((1, 0)\)-dimensional space spanned by a vector \(z\) by the expression (32). Obviously,

\[
\mathfrak{k}(2n + 1|m) = (\mathfrak{hei}(2n|m), \mathfrak{osp}^a(m|2n))_*.
\]

- Pericontact (i.e., “odd” contact) analog of the series \(\mathfrak{k}\) is associated with the following “odd” analog of the Lie superalgebra \(\mathfrak{hei}(2n|m)\). Let \(\mathfrak{ab}(n)\) be the antibracket superalgebra: its space is \(W \oplus \mathbb{C} \cdot z\), where \(W\) is an \(n|m\)-dimensional superspace endowed with a non-degenerate odd anti-symmetric bilinear form \(B\), and the bracket in \(\mathfrak{ab}(n)\) is given by the following relations:

\[
z \text{ is odd and lies in the center, and } [v, w] = B(v, w) \cdot z \text{ for any } v, w \in W.
\]
Clearly, 

$$m(n) = (ab(n), cpe^a(n))^*.$$ 

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