Born’s Rule From Second Quantization

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Abstract

Complex phase factors are viewed not only as redundancies in the quantum formalism but instead as remnants of unitary transformations under which the probabilistic properties of observables are invariant. It is postulated that a quantum observable corresponds to a unitary representation of an abelian Lie group, the irreducible subrepresentations of which correspond to the observable’s outcomes. It is shown that this identification agrees with the conventional identification as self-adjoint operators. The upshot of this formalism is that one may ‘second quantize’ the representation to which an observable corresponds, thus obtaining the corresponding Fock space representation. This Fock space representation is then too identifiable as an observable in the same sense, the outcomes of which are naturally interpretable as ensembles of outcomes of the corresponding non-second quantized observable. The frequency interpretation of probability is adopted, i.e. probability as the average occurrence, from which Born’s rule is deduced by enforcing the notion ‘average’ to such that are invariant under contextual global phase factors of the second quantized initial state.

Keywords: Complex phase invariance, Born’s Rule, The Ensemble Interpretation, Quantum Probability, Fock Space.

1 Introduction

In this article quantum theory is viewed as epistemological in the sense that it only talks about the results of measurements, see Bohr [1], [2], see also, [3], [4], [5], [6]. It is not claimed that this is the only sense in which science may talk about nature. It is however claimed that it is through the results of measurements that theories of science are verified. From this it follows that this article’s view poses no restriction on quantum theory in terms of its scientific validity. Further, the stance is taken that a measurement outcome is a certain physical reaction of some physical system which have been deemed as a measuring apparatus,
e.g. the expansion of the liquid in a mercury thermometer, clicks of a Geiger
counter, etc. It is this assignment that creates the ‘quantum to classical’
cut-off. In an experimental situation one wants to see what outcome one gets
making a particular measurement given some fixed initial condition. This ‘fixing’
is defined in terms of some physical attribute that describes the initial state.
Hence an initial state is too identified as an outcome of an observable.

Quantum theory is viewed primarily as a theory of probabilities assigned
to ensembles of outcomes of measurements performed on similarly prepared
systems, the latter being identified as the initial state of the measured system
[6,7,8,9,10,11,12]. A quantum superposition is hence in this probabilistic
view not viewed any differently from, say, the state of coin before a toss, cf. [6,7].
The superposition of the possible outcomes of the coin toss, i.e. heads or tails, is
a statistical description of the initial state in terms of its ensemble of outcomes
upon many measurements. From this view a measurement outcome does not
have to reflect a property of the measured system existing irrespective of the
measurements. But this does not mean that quantum theory promotes non-
realism. It is still claimed that there must be some property of the measured
system yielding this specific reaction of the measuring apparatus [12]. The
result of the measurement does not have to reflect a realistic property of the
measured system, but the measurement result yields a realistic property of the
measured system together with the measuring apparatus. Taking these stands,
the probabilities of quantum theory are in their interpretation no different that
those of Kolmogorov probability [6], i.e. coming from an incomplete description
of the initial conditions pre-measurements, cf. [6,7,12,13,14].

The often ignored complex phases in quantum theory can be given a more
rewarding meaning if identified as all possible unitary transformations under
which the probabilities of outcomes of an observable are invariant. Working in
the standard formalism of quantum theory, non-commuting observables have
different unitary transformations under which they are invariant. This gives a
way of expressing non-commutativity in the language of group representation
theory. This is the subject of Section 2. With the previous section in mind, in
Section 3 it is postulated that an observable is a unitary representation of an
abelian group, the observational outcomes of which correspond to its irreducible
subrepresentations. This is furthermore shown to agree with the conventional
notion of observables as self-adjoint operators. In this section it is furthermore
postulated that the initial condition preceding a measurement too is identifiable
as some physical process creating it. As such an initial condition is identified
as an irreducible subrepresentation of some observable. The enforcement of
the symmetry associated to an initial state is shown to be a crucial part in
the derivation of Born’s rule. This takes place in Section 4. In this section
the ensemble of outcomes of measurement results of an observable given the
same initial condition is formalized by postulating that ensembles of outcomes
are the irreducible subrepresentations of the second quantized[1], i.e. Fock space

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[1] The phrase ‘second quantization’ is rejected by some physicist, e.g. Steven Weinberg [15]. This since quantization has already occurred in turning Poisson brackets to operator
representation, of the considered observable. The second quantized version of an observable may naturally too be identified as an observable. With the frequency interpretation of probability in mind the probability of a certain outcome of an observable is postulated as the average of the number operator associated to that outcome. The notion of ‘average’ is restricted to such that are invariant under the action of the second quantized version of the symmetry associated with the initial condition considered. It is shown that any choice of such an average yields the same result, i.e. Born’s rule. An example of such an average is shown to be induced by coherent vectors generated by the vector state to which the initial condition of the measurement corresponds, hence showing the existence of such averages. Projection operators correspond in this formalism to number operators in Fock space. Section 5 contains a summary of the contents and main conclusions of the article. In the appendix, Section 6, it is speculated that the relative phases have a gauge theory-like origin. However, an actual gauge field model is in no way constructed.

2 Complex Phase Invariance

Let \( A \) and \( B \) be dichotomous observables with respective outcomes \( a_i \) and \( b_i \) for \( i = 1, 2 \). Let \( \mathcal{H} \) be a two-dimensional complex Hilbert space on which \( A \) and \( B \) are represented as self-adjoint operators. Let \( \{ |a_i\rangle \}_{i=1,2} \) and \( \{ |b_i\rangle \}_{i=1,2} \) be orthonormal eigenbases of \( A \) respectively \( B \). The initial state \( \psi \) is represented as a normalized vector in \( \mathcal{H} \),

\[
|\psi\rangle = c_{a_1} e^{i\theta_{a_1}} |a_1\rangle + c_{a_2} e^{i\theta_{a_2}} |a_2\rangle 
= c_{b_1} e^{i\theta_{b_1}} |b_1\rangle + c_{b_2} e^{i\theta_{b_2}} |b_2\rangle ,
\]

where the \( c_{a_i} \)'s and \( c_{b_i} \)'s are real valued. As is well known, multiplying \( |\psi\rangle \) by an arbitrary global phase factor \( e^{i\theta} \) makes no observably detectable difference. But that is not the only symmetry present. Consider the relative phases \( \theta_{a_i} \) in (1). By Born’s rule the probabilities of outcomes of \( A \) are invariant under any change of the \( \theta_{a_i} \)'s. This is similarly true for \( B \) with the \( \theta_{b_i} \)'s. But if the eigenbases of \( A \) and \( B \) are not simultaneous, the probabilities of outcomes of \( B \) are not invariant under changes of the \( \theta_{a_i} \)'s and similarly for \( A \) and the \( \theta_{b_i} \)'s. Such symmetries are called relative phase symmetries. But even though any change in the \( \theta_{a_i} \)'s is undetectable by a measurement of \( A \), if \( A \) and \( B \) are non-commuting, such a change in the \( \theta_{a_i} \)'s induces a detectable difference in measurements of \( B \). This difference is what shows up in double-slit type experiments as interference effects.

Let’s formalize the concept of relative phase symmetries in the language of group representation theory. Consider two representations of \( U(1) \times U(1) \) on commutators. Second quantization is merely representing this obtained operator algebra on a Hilbert space. The term ‘second quantization’ comes from the misconception that in quantum field theory the fields which are quantized are themselves wave-functions. They are not, they are classical fields turned into operator valued fields. The term will here exclusively refer the Fock space representation corresponding to a representation.

3
\[
\mathcal{H}_A := (\mathcal{H}, U(1) \times U(1), R_A) \quad \text{and} \quad \mathcal{H}_B := (\mathcal{H}, U(1) \times U(1), R_B),
\]
where \( R_A \) and \( R_B \) are respectively defined as
\[
R_A(e^{i\theta_1}, e^{i\theta_2})(c_1 |a_1\rangle + c_2 |a_2\rangle) = c_1 e^{i\theta_1} |a_1\rangle + c_2 e^{i\theta_2} |a_2\rangle,
\]
and
\[
R_B(e^{i\theta_1}, e^{i\theta_2})(c_1 |b_1\rangle + c_2 |b_2\rangle) = c_1 e^{i\theta_1} |b_1\rangle + c_2 e^{i\theta_2} |b_2\rangle,
\]
for any \( e^{i\theta_1}, e^{i\theta_2} \in U(1) \) and any \( c_1, c_2 \in \mathbb{C} \). These are indeed representations of the relative phase symmetry. \( A \) and \( B \) are invariant under \( R_A \) and \( R_B \) respectively, i.e.
\[
R_A(e^{i\theta_1}, e^{i\theta_2}) AR_A(e^{i\theta_1}, e^{i\theta_2})^{-1} = A
\]
and correspondingly for \( B \). However, if there is no simultaneous eigenbasis of \( A \) and \( B \), then the only unitary intertwining operators from \( \mathcal{H}_A \) to \( \mathcal{H}_B \) are those of the form
\[
T = e^{i\theta_{a_1b_1}} |b_1\rangle \langle a_1| + e^{i\theta_{a_2b_2}} |b_2\rangle \langle a_2|,
\]
where \( \theta_{a_1b_1}, \theta_{a_2b_2} \in \mathbb{R} \). But even though these are unitary they do not conserve transition probabilities. As seen from the following
\[
| \langle b_j | T | a_i \rangle |^2 = \delta_{ij} \neq | \langle b_j | a_i \rangle |^2.
\]
Note that this is equivalent to saying that the inner product spaces \( (\mathcal{H}_A, \langle \cdot, A(\cdot) \rangle) \) are inequivalent \( (\mathcal{H}_B, \langle \cdot, B(\cdot) \rangle) \). It is also equivalent to the identity operator, i.e. a unitary change of basis, not being intertwining.

**Definition 1.** Two representations are called **canonically isomorphic** if the identity operator is intertwining.

It has hence been shown that \( \mathcal{H}_A \) and \( \mathcal{H}_B \) are canonically isomorphic if and only if \( A \) and \( B \) commute. Hence non-commutativity is expressible through the language of group representation theory. In this sense the canonically different representations brakes the invariance of choice of basis. Although, note that the invariance is preserved if restricted to the subgroup
\[
\{(h, h) : h \in U(1)\} \subset U(1) \times U(1),
\]
in which case it corresponds to the global phase.

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\(^2\)Recall: An operator \( O : \mathcal{H}_A \to \mathcal{H}_B \) is intertwining if
\[
OR_A(e^{i\theta_1}, e^{i\theta_2}) = R_B(e^{i\theta_1}, e^{i\theta_2})O
\]
for all \( \theta_1, \theta_2 \in \mathbb{R} \).
Let's introduce some notation. Let $P_v := |v\rangle \langle v|$ denote the usual one-dimensional projection on $\mathcal{H}$ and $I$ the identity operator. Then
\begin{align*}
R_A(e^{i\theta_1}, e^{i\theta_2}) &= e^{i\theta_1} P_{a_1} + e^{i\theta_2} P_{a_2}, \\
R_B(e^{i\theta_1}, e^{i\theta_2}) &= e^{i\theta_1} P_{b_1} + e^{i\theta_2} P_{b_2},
\end{align*}
and that the corresponding Lie algebra actions are:
\begin{align*}
dR_A(\theta_1, \theta_2) &= i\theta_1 P_{a_1} + i\theta_2 P_{a_2}, \\
dR_B(\theta_1, \theta_2) &= i\theta_1 P_{b_1} + i\theta_2 P_{b_2}.
\end{align*}

The main idea of this article is to take phase factors seriously. It is, as mentioned above, known that the relative phase factors contribute to interference effects and are hence not viewed in the same sense as the global phase factor as just mathematical degeneracies. However, considering an initial state $\psi$ upon which a measurement $A$ is performed, all one really knows from the conventional formulation of quantum physics is that multiplying $\psi$ by an arbitrary factor $e^{i\theta}$ leaves the probabilities of outcomes of $A$ invariant. But this is not only true for the global phase factor, i.e $U(1)$ representation $e^{i\theta} I$, (14) but also true for the $U(1) \times U(1)$ representation
\begin{align*}
R_\Psi (\theta_\psi, \theta_\psi^\perp) &= e^{i\theta_\psi} P_\psi + e^{i\theta_\psi^\perp} P_{\psi^\perp},
\end{align*}
where $\psi^\perp$ is a normalized perpendicular vector to $\psi$.

**Definition 2.** Symmetries of the form (14) are referred to as **proper global phases** and those of the form (15) as **contextual global phases**.

The difference between these two types of symmetries is that the proper global phase is in the formalism of quantum physics inherently physically undetectable while the contextual one is detectable, for instance by adding a $\psi^\perp$ term to the initial state $\psi$. Because of this there is a chance that a physical interpretation of the contextual global phase has a real meaning.

Notice that the each state corresponding an outcome of the observable $A$ can be identified as an irreducible subrepresentation of $R_A$. It is this which will be the starting point for the notion of ‘observable’ in section 3.

### 3 Observables

Let $\mathcal{H}$ be a separable complex Hilbert space, $G$ some abelian group and let
\begin{align*}
U(1)^M := U(1) \times \cdots \times U(1),
\end{align*}
(16)
**Postulate 1.** An Observable $A$ with possible outcomes $\{a_n\}_{n=1}^N$ corresponds to a non-trivial unitary representation

$$\mathcal{H}_A := (\mathcal{H}, G, R_A).$$

where $R_A$ denotes a unitary action of $G$ on $\mathcal{H}$.

$G$ is an abelian group, hence all its irreducible representations are one-dimensional. The decomposition of $\mathcal{H}_A$ into irreducible components is hence of the form

$$\mathcal{H}_A = \bigoplus_{n=1}^N \text{span}_\mathbb{C}\{|a_n\rangle\},$$

where $\{|a_n\rangle\}_{n=1}^N$ is some orthonormal basis. In fact, this decomposition is unique.

**Theorem 1.** The decomposition

$$\mathcal{H}_A = \bigoplus_{n \in \mathbb{N}} \text{span}_\mathbb{C}\{|a_n\rangle\},$$

is unique.

**Proof.** By Schur’s Lemma any intertwining isomorphism $T : \mathcal{H}_A \to \mathcal{H}_A$ is necessarily diagonal in the basis $\{|a_n\rangle\}_{n=1}^N$. If there existed another decomposition of $\mathcal{H}_A$, $T$ would necessarily be diagonal in that as well. But this can only happen if the bases differ only by normalization or if $T$ is the identity operator. \qed

**Postulate 2.** The possible outcome $a_n$ of an observable $A$ is identified with the irreducible representation that $|a_n\rangle$ spans.

Theorem 1 removes all ambiguities of this postulate but the ambiguity to which of the irreducible representation each outcome should be identified. However, this remaining ambiguity causes no problems once a choice is made and is consistently maintained. The notation $|a_n\rangle$ is merely a suggestive notation for the matching of the outcome $a_n$ with an irreducible representation.

Notice that Postulate 2 agrees with the conventional formalism of quantum physics. There as well outcomes are identified as rays in the Hilbert space spanned by eigenvectors of the observable.

The following theorem shows why considering $U(1) \times U(1)$-representation as in section 2 is not a restriction. This justifies us using $U(1)^N$ through out this article without loss of generality.

**Theorem 2.** Any unitary representation of an abelian group $G$ can equivalently be considered as a representation of $U(1)^N$, where $N$ denotes the dimension of the representation.

**Proof.** A unitary representation of $G$ on the $N$-dimensional Hilbert space $\mathcal{H}$ is a group homomorphism $h : G \to U(N)$. Since $G$ is abelian so is $h(G)$. Furthermore, every abelian subgroup of $U(N)$ is isomorphic to a subgroup of $U(1)^N$. This finishes the proof. \qed
For the next result some notation needs to established. $U(1)^N$ is parametrized by $\mathbb{R}^N$. So one may just as well write $R(\theta_1, \ldots, \theta_M)$ as $R(e^{i\theta_1}, \ldots, e^{i\theta_N})$. The notation will even be condensed further. Let $\overline{\theta}$ stand for $(\theta_1, \ldots, \theta_N)$ and let (somewhat abusively) $R(\overline{\theta}) = R(e^{i\theta_1}, \ldots, e^{i\theta_N}).$

**Theorem 3.** For every $n = 1, \ldots, N$ there exists integers $\{f_n^k\}_{k=1}^N$ such that

$$R(\overline{\theta}) |a_n\rangle = e^{i\sum_{k=1}^N f_n^k \theta_k} |a_n\rangle$$

for all $e^{i\theta} \in C(S_A, U(1)).$

**Proof.** Since $|a_n\rangle$ spans an irreducible representation, for every $\theta \in \mathbb{R}^n$ there exists a constant $e^{i \int_{\theta}}$

$$R(\overline{\theta}) |a_n\rangle = e^{i \int_{\theta}} |a_n\rangle.$$  \hspace{1cm} (21)

Furthermore, because of the group structure,

$$f_n(\theta_1 + \theta_2) = f_n(\theta_1) + f_n(\theta_2).$$  \hspace{1cm} (22)

Consider the function defined as

$$f_{\overline{\theta}, n} : t \in \mathbb{R} \mapsto f_n(t \overline{\theta}) \in \mathbb{R}.$$  \hspace{1cm} (23)

This map induces an irreducible representation of $U(1)$ on the span of $|a_n\rangle$ via the action

$$e^{it} \cdot |a_n\rangle := e^{i \int_{\theta, n}(t)} |a_n\rangle.$$  \hspace{1cm} (24)

$f_{\overline{\theta}, n}(t)$ is differentiable since it induces a diffeomorphism of Lie groups. Hence

$$f'_{\overline{\theta}, n}(t) = \lim_{h \to 0} \frac{f_{\overline{\theta}, n}(t+h) - f_{\overline{\theta}, n}(t)}{h}.$$  \hspace{1cm} (25)

$$= \lim_{h \to 0} \frac{f_{\overline{\theta}, n}(h)}{h}.$$  \hspace{1cm} (26)

$$= f'_{\overline{\theta}, n}(0).$$  \hspace{1cm} (27)

So $f_{\overline{\theta}, n}(t) = f'_{\overline{\theta}, n}(0)t$. Hence

$$f_n(\overline{\theta}) = f_{\overline{\theta}, n}(1) = f'_{\overline{\theta}, n}(0),$$  \hspace{1cm} (28)

and thus $f_n(t \overline{\theta}) = tf_n(\overline{\theta})$, i.e. $f_n$ is a linear function $f_n : \mathbb{R}^N \to \mathbb{R}$. Hence it can be written as

$$f_n(\overline{\theta}) = (f_n^1, \ldots, f_n^N) \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_N \end{pmatrix}.$$  \hspace{1cm} (29)

Now, setting all but $\theta_i$ to zero a representation of $U(1)$ is obtained, for which it is known [16] that $f_n^i$ must be an integer. \hfill \Box
Let's consider some relations between observables $A$ and $B$ represented by

$$\mathcal{H}_A := (\mathcal{H}, U(1)^N, R_A)$$

respectively

$$\mathcal{H}_B := (\mathcal{H}, U(1)^N, R_B).$$

The same argument holds here as the corresponding one in section 2. So $A$ and $B$ are canonically isomorphic observables if and only if the identity operator is intertwining.

**Definition 3.** Canonically isomorphic observables are called **coequal** and the negation **complementary**.

Notice, as mentioned before, that these are just other expressions for commuting respectively non-commuting observables. The intuition behind these names is that coequal observables in the quantum formalism gives the same statistical description of the initial state while complementary observables do not.

**Postulate 3.** An initial condition $\psi$ of an observable $A$ correspond to an outcome of an observable $\Psi$ corresponding to a representation

$$\mathcal{H}_\Psi := (\mathcal{H}, U(1)^N, R_\Psi).$$

A normalized vector in the irreducible representation to which $\psi$ corresponds is denoted as $|\psi\rangle$.

Postulate 3 formalizes the fact that an initial condition of a measurement $A$ is some fixed physical preparation procedure. In the next section this interpretation will be shown to agree with the ensemble interpretation in which an initial state is an ensemble of similarly prepared systems.

**Theorem 3** gives us a way of finding the generators of the representations. Applying the short-hand notation from (29) from the proof of Theorem 3 and noting that since

$$-i \frac{d}{dt} \bigg|_{t=0} R_A (i\theta) = \sum_{n=1}^{N} f_n(\theta) P_{a_n},$$

the $f_n P_{a_n}$’s are identifiable as the generators of the symmetry. Notice that $f_n$ corresponds to the representation label an irreducible representation of $U(1)^M$.

Notice also that the irreducible representation to which outcome $a_n$ is set to correspond is unambiguously identifiable with the generator $f_n P_{a_n}$.

## 4 Initial states & Probabilities

Again, in this article probabilities are considered as frequencies of occurrence.

For brevity $A$ is restricted to being a dichotomous observable $A$ for which the representation labels are $f_1 = (1, 0)$ and $f_2 = (0, 1)$, i.e. the case considered
in Section 2. It is straightforward to generalize to any finite number of possible outcomes.

For mathematical details about Fock spaces, second quantization and coherent vectors the reader is referred to [17] and [18], in which the relevant mathematical details are presented.

From running an experiment a large number of times one obtains a sequence of frequencies of occurrences of outcomes. Consider the outcome $a_1$. Then the sequence might look something like

$$(1/1, 1/2, 2/3, \ldots),$$

meaning that outcome $a_1$ occurred in the first run, not in the second but then again in the third. Such a sequence is to be considered as a result of a measurement on its own, and hence should be associated to some observable, denoted $A^\infty$, satisfying Postulate 4. There should be a canonical identification between irreducible representations of $A^\infty$ and sequences of outcomes of $A$. To what representation $A^\infty$ corresponds will be postulated below. It will also be shown that it agrees with the intuition about it and with Postulate 4. But before it is presented some more notation needs to be established.

Let $F(\mathcal{H})$ denote the Fock space generated by $\mathcal{H}$ and $F^\vee(\mathcal{H})$ the corresponding symmetric Fock space. Let $\Gamma(R_A)$ denote the second quantization of $R_A$ onto $F^\vee(\mathcal{H})$. That is, the operator defined by

$$\Gamma(R_A)(\bar{\theta}) f_1 \lor \cdots \lor f_m := R_A(\bar{\theta}) f_1 \lor \cdots \lor R_A(\bar{\theta}) f_m.$$ (35)

The corresponding Lie algebra action of $\Gamma(R_A)$, denoted $d\Gamma(dR_A)$ is consequently given by

$$d\Gamma(dR_A)(\bar{\theta}) f_1 \lor \cdots \lor f_m := \sum_{i=1}^m f_1 \lor \cdots \lor dR_A(\bar{\theta}) f_i \lor \cdots \lor f_m.$$ (36)

**Postulate 4.** The observable $A^\infty$ corresponds to the representation

$$(F^\vee(\mathcal{H})) , U(1)^N , \Gamma(R_A)) .$$ (37)

Also referred to as the second quantization of $A$.

Now let’s give some motivation for this identification. Consider associating a sequence (34) to a vector in $F(\mathcal{H})$,

$$d_1 |a_1\rangle + d_2 |a_1\rangle \otimes |a_2\rangle + d_3 |a_1\rangle \otimes |a_2\rangle \otimes |a_1\rangle + \cdots ,$$ (38)

where each $d_i \in \mathbb{C}$. But it is really only the value of the the frequency after the latest run that interests us, and that is invariant under when in the previous runs the outcomes $a_1$ really occurred. Hence the sequences

$$(1/1, 1/2, 2/3)$$ and $$(0/1, 1/2, 2/3)$$

(39)

The $d_i$’s are left without a specific interpretation for now. The purpose here is merely to provide some intuition on why the Fock space is considered.
are viewed as equivalent to \((34)\). This equivalence should be preserved in our association \((38)\). That is, the terms
\[
|a_1\rangle \otimes |a_2\rangle \otimes |a_1\rangle, \quad |a_1\rangle \otimes |a_1\rangle \otimes |a_1\rangle \quad \text{and} \quad |a_2\rangle \otimes |a_1\rangle \otimes |a_1\rangle
\]
should be equivalent. But this is automatic if the association \((38)\) is modified to
\[
d_1 |a_1\rangle + d_2 |a_1\rangle \lor |a_2\rangle + d_3 |a_1\rangle \lor |a_2\rangle \lor |a_1\rangle + \cdots,
\]
i.e. associated to vectors in the symmetric Fock space \(F^\lor(H)\).

The decomposition of \(F^\lor(H)\) into irreducible components under \(\Gamma(R_A)\) is
\[
F^\lor(H) = \bigoplus_{N=0}^{\infty} \bigoplus_{i=0}^{N} \text{span}_C \{ |a_1\rangle \lor \cdots \lor |a_1\rangle \lor |a_2\rangle \lor \cdots \lor |a_2\rangle \}. \tag{43}
\]

Hence after \(N\) measurements of observables \(A\), there is a unique identification of the obtained sequence with an irreducible representation of
\[
\bigoplus_{i=0}^{N} \text{span}_C \{ |a_1\rangle \lor \cdots \lor |a_1\rangle \lor |a_2\rangle \lor \cdots \lor |a_2\rangle \}. \tag{44}
\]

Postulate \(1\) tells us that \(A^\infty\) is an observable indeed. Hence in this formalism one may identify sequences of outcomes as outcomes in their own right, which they indeed should since they are outcomes of actual measurements.

In this paper the ensemble interpretation of probability is applied. States are considered as similarly prepared initial conditions of single runs of experiments. Hence the natural identification of a single outcome \(a_n\) in the ensemble interpretation is to identify it with preparation procedure always giving outcome \(a_n\) upon measurement. In the symmetric Fock space there is a natural representative of this by coherent vectors
\[
E(|a_n\rangle) := \sum_{k=0}^{\infty} \frac{|a_n\rangle \otimes k}{k!}. \tag{45}
\]

Hence the initial condition \(\psi\) of \(A\) is in the ensemble identified as \(E(|\psi\rangle)\). Notice that
\[
\Gamma(R_A(\theta))E(|a_n\rangle) = E(e^{i\theta_n} |a_n\rangle), \tag{46}
\]
i.e \(E\) viewed as a map from \(A\) to \(A^\infty\) is intertwining. But also note that \(E\) is not linear. This agrees with the intuition that the "ensemble map", \(E\), carries the fundamental symmetry structure of \(A\) over to \(A^\infty\).
Now consider the ordinary creation and annihilation operators on $F_\vee(\mathcal{H})$, i.e. those satisfying
\begin{align}
\alpha(f)\Omega &= 0, \\
\alpha(f_1 \vee \cdots \vee f_n) &= \sum_{i=1}^n (f, f_i) f_1 \vee \cdots \vee f_i^{-1} \vee f_{i+1} \vee \cdots f_n \\
\alpha^*(f)\Omega &= f, \\
\alpha^*(f_1 \vee \cdots \vee f_n) &= f \vee f_1 \vee \cdots \vee f_n, \\
[\alpha(f), \alpha^*(g)] &= \langle f, g \rangle I,
\end{align}
for any $f, g \in \mathcal{H}$ and $n \in \mathbb{N}$, and where $\Omega = 1 \in \mathbb{C} \simeq \otimes_{n=0}^\infty \mathcal{H}$ is the distinguished vacuum vector, i.e. zero measured outcomes. Then
\begin{align}
\Gamma(R_A(\theta)) &= e^{i(\theta_1 N_1 + \theta_2 N_2)}, \\
-i \cdot d\Gamma(dR_A(\theta)) &= \theta_1 N_1 + \theta_2 N_2,
\end{align}
where $a^*_i := \alpha^*([a_i])$ and $N_{a_i} = a^*_i a_i$. The latter is referred to as the Number operator of $a_i$. It represents the second quantization of the generator $-i \cdot dR_A(\theta)$. The number operator’s eigenvectors does indeed correspond to the possible outcomes of $A^\infty$. Notice that the annihilation operator $a_i$ shares the same notation as outcome $a_i$, it will however be clear from context which one is meant. Similarly, $b^*_i := \alpha^*([b_i])$ and $\psi^* := \alpha^*([\psi])$.

Now consider the following theorem.
\begin{theorem}
For a normalized $|\psi\rangle$,
\begin{equation}
\frac{\langle E(|\psi\rangle), N_{a_n} E(|\psi\rangle) \rangle}{\langle E(|\psi\rangle), (N_{a_1} + N_{a_2}) E(|\psi\rangle) \rangle} = |\langle a_n |\psi\rangle|^2.
\end{equation}
\end{theorem}
\begin{proof}
First notice that
\begin{equation}
e^{\psi^* \Omega} = E(|\psi\rangle).
\end{equation}
By direct calculation:
\begin{align}
a_n E(|\psi\rangle) &= a_n e^{\psi^* \Omega} \\
&= [a_n, e^{\psi^* \Omega}] \Omega \\
&= \sum_{k=1}^\infty \frac{[a_n, (\psi^* \Omega)^k]}{k!} \Omega \\
&= \langle a_n |\psi\rangle \sum_{k=1}^\infty \frac{(\psi^* \Omega)^k}{k!} \Omega \\
&= \langle a_n |\psi\rangle e^{\psi^* \Omega} \\
&= \langle a_n |\psi\rangle E(|\psi\rangle).
\end{align}
Hence
\begin{align}
\langle E(|\psi\rangle), N_{a_n} E(|\psi\rangle) \rangle &= \langle a_n E(|\psi\rangle), a_n E(|\psi\rangle) \rangle \\
&= |\langle a_n |\psi\rangle|^2 \| E(|\psi\rangle) \|^2.
\end{align}
This finishes the proof. \qed

11
Theorem 4 is really suggestive towards Born’s rule. In the ensemble setting, identifying an initial state as the coherent vector $E(|\psi\rangle)$; which is an intuitively reasonable one since, as mentioned before, $E$ considered as a map $\mathcal{H} \to \mathcal{F}_\psi(\mathcal{H})$ is intertwining; Born’s rule pops out from taking an average of the operator that counts the outcomes! Now, taking the same notion of average of the number operator $N_\psi$, one gets

$$\langle E(|\psi\rangle), N_\psi E(|\psi\rangle) \rangle = \|E(|\psi\rangle)\|^2. \quad (60)$$

So by the intermediary result in the proof of Theorem 4, one gets

$$\langle E(|\psi\rangle), N_\psi E(|\psi\rangle) \rangle = \langle E(|\psi\rangle), (N_{a_1} + N_{a_2}) E(|\psi\rangle) \rangle. \quad (61)$$

With this in mind let’s make a small detour to consider some fundamentals of probability theory. The frequency interpretation of probability is considered. In that setting probabilities [27] are formally though of as

$$\lim_{N \to \infty} \frac{\#N a_m}{N}, \quad (62)$$

where $\#N a_m$ denotes the number of occurrences of outcome $a_m$ after $N$ runs. So the probability of $a_m$ is identified as the expectation value/average of the occurrence of $a_m$. This suggests an identification of the probability of outcome $a_m$ given initial condition $\psi$ as

$$\frac{\langle N_{a_m} \rangle}{\langle N_\psi \rangle}, \quad (63)$$

where $\langle \cdot \rangle$ denotes some form of expectation value. But there is more. In this article the symmetry $\Gamma(R_\theta)$ is taken seriously, meaning that the probabilities should be invariant under its action. Consider the trivial situation where given initial state $\psi$, $\Psi$ is measured. For that situation,

$$\langle N_{\psi^\perp} \rangle = 0 \quad \text{and} \quad \langle N_\psi \rangle \neq 0, \quad (64)$$

where $\psi^\perp$ denotes the initial condition 'not $\psi$', i.e. that represented by the subspace spanned by an orthonormal vector $|\psi^\perp\rangle$ to $|\psi\rangle$. This is just saying that the same initial condition holds in every run of the experiment. In addition

$$\Gamma(R_\psi(\overline{\theta})) N_{a_m} \Gamma(R_\psi(\overline{\theta}))^{-1} = \langle a_m | \psi \rangle^2 N_\psi + \langle a_m | \psi^\perp \rangle^2 N_{\psi^\perp} \quad (65)$$

$$+ e^{i(\theta - \psi^\perp)} \langle a_m | \psi \rangle \langle \psi^\perp | a_m \rangle \psi^* \psi^\perp \quad (66)$$

$$+ e^{i(\theta^\perp - \psi)} \langle a_m | \psi^\perp \rangle \langle \psi^\perp | a_m \rangle \left( \psi^\perp \right)^* \psi. \quad (67)$$

So by assumption [64] the three latter terms drop out upon taking such an average. Since the average additionally is invariant under $\Gamma(R_\psi(\overline{\theta}))$, one gets that

$$\langle \Gamma(R_\psi(\overline{\theta})) N_{a_m} \Gamma(R_\psi(\overline{\theta}))^{-1} - \Gamma(R_\psi(\overline{\theta})) N_{a_m} \Gamma(R_\psi(\overline{\theta}))^{-1} \rangle \quad (68)$$

$$= \left( e^{i(\theta - \psi^\perp)} - 1 \right) \langle a_m | \psi^\perp \rangle \langle \psi | a_m \rangle \langle \psi^* \psi^\perp \rangle \quad (69)$$

$$+ \left( e^{i(\theta^\perp - \psi)} - 1 \right) \langle a_m | \psi^\perp \rangle \langle \psi^\perp | a_m \rangle \left( \psi^\perp \right)^* \psi \quad (70)$$
must be zero for all $\overline{\theta}$. Hence one must require the average to satisfy
\[\langle \psi^* \psi^\perp \rangle = \langle (\psi^\perp)^* \psi \rangle = 0. \tag{71}\]
Put together this leads to
\[\frac{\langle N_{a_m} \rangle}{\langle N_\psi \rangle} = |\langle a_m | \psi \rangle|^2, \tag{72}\]
for any such average. Notice that in order for conditions (64) and (71) to make sense $\langle \cdot \rangle$ needs only be a linear functional on the space $D(F_\vee(H))$ of densely defined operators on $F_\vee(H)$. Let's state this as a definition:

In the experimental context of having initial condition $\psi$ the initial state $\langle \cdot \rangle_\psi$ associated to the initial condition $\psi$ is a linear functional on $D(F_\vee(H))$ such that the conditions (64) and (71) hold.

Notice that for (72) it does not matter which $\langle \cdot \rangle_\psi$ that is used, they yield the same result. So in this sense $\langle \cdot \rangle_\psi$ can be taken as denoting an equivalence class. Regarding the existence of such $\langle \cdot \rangle_\psi$’s, (1) shows an example of a such.

Let’s state the postulate corresponding to Born’s rule:

**Postulate 5.** In the experimental context of having initial condition $\psi$ and performing a measurement $A$ the probability of outcome $a_m$, denoted $P_\psi(a_m)$, is given by
\[P_\psi(a_m) = \frac{\langle N_{a_m} \rangle_\psi}{\langle N_\psi \rangle_\psi}. \tag{73}\]
From the above calculation (72) this postulate is indeed just a rephrasing of Born’s rule. The difference here is that it comes from an actual interpretation of probability, the frequency interpretations. The upshot of this is the agreement with how empirical verifications of Born’s rule are done in practice. Here the conventional form of Born’s rule, which is postulated, is just a very nice shortcut for doing calculations, so that one do not have to go through the second quantization every time one wants to calculate probabilities.

Even though some time was spent on discussing the intuitive nature of the ensemble map $E$ and coherent states they ended up only serving as showing existence of an invariant average. But let’s come back again to the interpretation of coherent states. $\psi$ was purposely called an ‘initial condition’ as opposed to ‘initial state’. The reason for this is that the initial state is interpreted in accordance with the ensemble interpretation, i.e. as a collection of similarly prepared physical systems. Intuitively the closest mathematical analogue to this is in our ensemble setting, $A^\infty$, is the coherent state $E(|\psi\rangle)$. Hence it is natural to think of $E(|\psi\rangle)$ as the initial state even though it is not explicitly postulated to be the case, nor was there a need. But through $E(|\psi\rangle)$ one obtains a representative of $\langle \cdot \rangle_\psi$ (which was indeed called an ‘initial state’), and $E(|\psi\rangle)$ is unambiguously labelled by the initial condition $\psi$. Hence the conventional
identification of $\psi$ as the initial state causes no conflict in terms of labelling. But, as expected, by (11) it even suffices to consider $\psi$ in order to calculate the probabilities of outcomes $a_m$’s. So referring to $\psi$ as the state is not just unambiguous in terms of labelling but even in terms of statistical properties. But this is here viewed merely as a very convenient result. Furthermore, $E(|\psi\rangle)$ provides a way identifying the coefficients $d_i$ from (11). Namely as products of the coefficients in the $|a_i\rangle$-expansion of $|\psi\rangle$, e.g.

$$d_1 = c_1, \ d_2 = c_1c_2, \ d_3 = c_1c_2c_1$$ (74)

ans so forth.

Lastly, coherent states are common practise in quantum optics [26]. So they are not just theoretical peculiarities. So in addition to the map $E$ being intertwining there might be more physical reasons for their occurrence in this article.

5 Summary

It was noted that the invariance of relative phase factors could be identified as the maximal group of unitary transformation leaving the considered quantum observable invariant. This was then used as the defining property of a quantum observable, i.e. a quantum observable is defined by a maximal set of mutually commuting unitary operators. This is what is stated in Postulate 1.

It was made explicit that initial conditions, and not just observable outcomes, are physical processes. Hence in Postulate 3 initial conditions where identified as outcomes of observables.

It was acknowledged that an ensemble of outcomes of measurements of an observable $A$ is too an outcome of a measurement. This was formalized in Postulate 4 where the second quantized version $A^\infty$ of $A$ qualified as an observable according to Postulate 1 that it also had the desired set of possible outcomes.

Postulate 5 stated that the probability of outcome $a_m$ was equal to the average of the corresponding number operator of $a_m$ in $A^\infty$, with the average being invariant under the second quantized contextual global phase associated with the initial state $\psi$. Born’s rule was derived from this postulate.

6 Appendix: Gauge theory-like origin of the contextual global phase?

Here an analogy with occurrence of the contextual global phase as it appears in the previous sections to that of a local gauge transformations in gauge field theory is presented. Just the bare minimum of gauge field theory to get the points across will be presented. For a more thorough introduction the reader is refer to [19], [20], [21] and [22], where the first is a standard textbook on quantum field theory and the rest dealing only with classical gauge theory.
Let $\mathcal{M}$ be a space time-manifold and $G$ some Lie group. Consider the spaces

$$C^\infty(\mathcal{M}, \mathbb{C}^n)$$

and

$$C^\infty(\mathcal{M}, G).$$

Suppose there is a unitary representation of $G$ onto $\mathbb{C}^n$ so that one can define an action $\cdot$ of $C^\infty(\mathcal{M}, G)$ onto $C^\infty(\mathcal{M}, \mathbb{C}^n)$ as

$$(g(x), \Psi(x)) \in C^\infty(\mathcal{M}, G) \times C^\infty(\mathcal{M}, \mathbb{C}^n) \mapsto g(x) \cdot \Psi(x) \in C^\infty(\mathcal{M}, \mathbb{C}^n).$$

(77)

The elements of $C^\infty(\mathcal{M}, \mathbb{C}^n)$ are not to be mistaken as quantum mechanical wave-functions. As mentioned in [16], the action of $C^\infty(\mathcal{M}, G)$ is not upon a finite dimensional phase space of coordinates and momenta, so it has no interpretation in terms of ordinary quantum mechanical quantization. In gauge theory $C^\infty(\mathcal{M}, \mathbb{C}^n)$ is identified as the phase space which is turned into a quantum field theory by identifying the canonical coordinates and canonically quantizing their Poisson bracket [19]. As such the quantized fields are more appropriately analogous position and momentum operators than wave functions.

The dynamics of the matter fields $\Psi \in C^\infty(\mathcal{M}, \mathbb{C}^n)$ are described by classical field equations derived as Euler-Lagrange equations from a Lagrangian

$$\mathcal{L}[\Psi].$$

(78)

As explained in [20], by requiring $\mathcal{L}$ to be invariant under gauge transformations, i.e.

$$\mathcal{L}[\Psi] = \mathcal{L}[g\Psi]$$

(79)

for $g \in C^\infty(\mathcal{M}, G)$, one ensures that if a $\Psi$ is a solution to the field equations, then so is $g\Psi$. The gauge invariance of the Lagrangian is stronger than just requiring gauge invariance of the field equations. The reason physicists use this stronger requirements is that it gives a gauge invariant action functional

$$S[\Psi] = \int_{\mathcal{M}} \mathcal{L}[\Psi]dx$$

(80)

which is a requirement in path-integral quantization. Details of such deeper structure will not be of main interest here. The focus will instead be on the interactions coming from requiring this invariance. This requirement leads to the introduction of gauge fields $A = (A_\mu),$

$$A_\mu \in C^\infty(\mathcal{M}, g),$$

where $\mu$ is the space-time index, acting on the matter fields via the corresponding Lie algebra action of $C^\infty(\mathcal{M}, G)$. These added gauge fields induces coupled nonlinear field equations with the matter and gauge fields $\Psi$ and $A$. This means that there exists interactions between these. It is because of this the gauge fields

\[\text{Solutions of the field equations correspond to critical points of the action functional.}\]
are referred to as force carriers. All fundamental forces of nature are derivable as gauge theories.

An elementary particle’s property with respect to a gauge symmetry is given by the irreducible representation to which it corresponds. For instance, the identification of quarks solely in terms of the strong nuclear force is as the fundamental representation of $SU(3)$ [19]. With this in mind think of associating an observable $O$ with a representation $(C^\infty(\mathcal{M}, \mathbb{C}^n), C^\infty(\mathcal{M}, G_O))$. Given a field $\Psi \in C^\infty(\mathcal{M}, \mathbb{C}^n)$, the possible observable outcomes of $O$ correspond respectively to the irreducible components of $(\mathbb{C}^n, G_O)$. Hence a similar structure of observables as that suggested in this article has been obtained. So here the method of section 4 could be applied to obtain the probabilistic framework of quantum theory. Note, this would give the conventional formalism on $\mathbb{C}^n$. The fields $\Psi$ would hence be ‘wavefunction-valued’ functions on space-time not wavefunctions themselves.

To emphasize, only classical field theories have been considered. Since the representation theory of $U(1)$ essentially is what induces the quantized energy level of Hamiltonians [23], the reader maybe agrees with the conventional name ‘first quantization’ since the mathematics of representation theory is applied here as well. However, if one considers the probabilistic nature, i.e. Born’s rule, as that which defines ‘quantumness’; then it is in the second quantization where the ‘quantumness’ resides. What ever you decide, the probabilities would arise from considering ensembles of outcomes.

In no way has a proper gauge theory to model quantum measurement been constructed. This sections main focus was to show that the notion of symmetries and irreducible representations have been successfully applied in construction of one of the best accomplishments science, i.e. the standard model of particle physics. There just as here is the true nature of gauge symmetries not known. Are they just mathematical redundancies or is there deeper physics beneath? One could speculate on the existence of a gauge field theory of observables and what it would mean. If such were successfully constructed an observable could be identified with a gauge group whose corresponding gauge field could be interpreted as an interaction between measuring device and the measured system, where the latter would correspond to matter fields. This would all be classical, the quantum notion of probability would not appear until second quantized.

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5 Although not all can be unified in a corresponding quantum theoretic framework, e.g. grand unification and quantum gravity.

6 Perhaps more correctly distributions, even.
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