Arc and twin arc domination in Cayley digraphs

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Abstract. Let G be a finite group and let S be a generating subset of G. The Cayley digraph Cay (G, S) is the digraph with node set as the elements of G and there is an arc from u to us whenever u ϵ G and s ϵ S. An arc e₁ in a digraph D out arc dominates itself as well as all arcs eᵢ. Such that {e₁, eᵢ} is a directed path of length 2 in D. While eᵢ in arc dominates both itself and all arcs eᵢ such that {eᵢ, e₁} is a directed path of length 2 in D. The arc domination number is the minimum cardinality of an out arc dominating set of D denoted by ϒ′(D). A set of edges of D is twin arc dominating set if every edge of D is out arc dominated by some edge of S and in arc dominated by some edge of S. The minimum Cardinality of a twin arc dominating set is the twin arc domination number denoted by ϒ∗(D) of D. This paper discusses the arc domination and twin arc domination of Cayley digraphs and attempts to find bounds for the arc domination number of the same class of graphs.

Keywords: Twin arc domination, Arc domination, Domination number, Neighborhood, Bounds, Cayley Digraphs.

1. Introduction

The study of domination in graphs has received much attention in recent years. In fact, The theory of Domination in graphs was formally introduced by Berge in 1958 and by Ore[1] in 1962. Strong and weak domination balance in a graph was studied by Sampath kumar et.al[2]. Dominator colouring of various graphs are also studied by Manju et.al[7],[9]. A directed graph D = (V, A) where V is finite set of vertices and a set A is directed arcs, where A ⊆ V X V. The concept of arc domination number in digraphs was introduced by V.R.Kulli[4]. However, domination in cayley digraphs has been studied in [7],[10]. The analogues concept to the edge domination number of a undirected graph is the out arc domination number of a digraph, defined as the minimum cardinality of a subset of the edge set of D such that each edge of D is either in S or an out neighbour of some edge in S. Out arc neighbourhood of arc e = (u,v) is set of all directed arcs begin with v and is denoted by neg⁺((u,v)) or neg⁺(e). In arc neighbourhood of arc e = (u,v) is set of all directed arcs end with u and is denoted by neg⁻((u,v)) or neg⁻(e). Application of domination theory is in maximum flow of facts from source to sink in a network, scheduling end semester examination, find fastest route from national capital to each state capital and etc., plays an eminent role in real life situations. A subset S of arcs in a digraph D is called an arc dominating set of D if every arc (v, w) in A-S, there exists an arc(u,v) ∈ S such that (u,v),(v,w) ∈ A. The minimum cardinality of an arc dominating set of D is called the arc domination number of D and is denoted by ϒ (D). A set of edges of D is twin arc dominating set if every edge of
D is out arc dominated by some edge of S and in arc dominated by some edge of S. The minimum Cardinality of a twin arc dominating set is the twin arc domination number denoted by $\gamma^*(D)$ of D.

2. Main results

**Theorem 2.1.** Let $C_n$, be a directed cyclic graph $D(V, A)$ with $n \geq 3$ arcs, then the arc domination number of $C_n$ is $\left\lceil \frac{n}{2} \right\rceil$, i.e., $\gamma'(C_n) = \left\lceil \frac{n}{2} \right\rceil$, where $\left\lceil \cdot \right\rceil$ is a least integer function.

**Theorem 2.2.** Arc domination of Cayley digraph $D = \text{Cay}(Z_n, X)$ of the group $Z_n$ with generating set $X$ is

$$\gamma'(D) = \begin{cases} \left\lfloor \frac{|V| \times |X|}{|X|+1} \right\rfloor & \text{if } |X| \leq 2 \\ \left\lfloor \frac{|V|}{|X|} \right\rfloor & \text{if } |X| \geq 3 \end{cases}$$

Where $|X|$ is number of elements of $X$

**Proof:** Let G be a group and the Cayley digraph of the group G with generating set $X$ be $\text{Cay}(G, X)$ with $|X| = d$.

**Case i:** If $d = 1$

From the construction of the Cayley digraph of a group, it is a cyclic graph, if $d = 1$, it has $n$ vertices and $n$ arcs. Let $V = \{v_1, v_2, v_3, \ldots, v_n \}$ and $A = \{e_{ij} = (v_j, v_{(j+1) \text{mod} n}) / 1 \leq j \leq n \} \cup \{v_n v_1 \}$.

Let $S = \{e_{1,2}, e_{1,4}, e_{1,6}, \ldots, e_{1,n}\}$. Then out neighbourhood of $e_{1,j+1}$ is $e_{1,j+2}$ for $j = 1, 2, 3, \ldots, n-2$ and out neighbourhood of $e_{1,n}$ is $e_{1,1}$. i.e., Out neighbourhood of each edge has exactly 1 distinct arcs. So one arc dominates exactly 1 arcs. Since there are $\left\lceil \frac{n}{2} \right\rceil$ arcs in $S$, exactly $\left\lfloor \frac{nd}{d+1} \right\rfloor$ arcs dominates the arcs of $A \setminus S$. i.e., $|S| = \left\lceil \frac{nd}{d+1} \right\rceil$.

Therefore the arc domination number for the Cayley digraph $\text{Cay}(G, X)$ with $|X| = 1$ is

$$\gamma'(D) = \left\lfloor \frac{nd}{d+1} \right\rfloor = \left\lfloor \frac{|V| \times |X|}{|X|+1} \right\rfloor$$

**Case ii:** $d = 2$, Let $X = \{a, b\}$

From the construction of the Cayley digraph of a group, it is a cyclic graph. If we have $n$ vertices then it has $2n$ arcs.

Let $V = \{v_1, v_2, v_3, \ldots, v_n \}$ and $A = \{e_{ij} = (v_j, v_{(j+i) \text{mod} n}) / 1 \leq j \leq n \} \cup \{e_{ij} = (v_j, v_{(i+j) \text{mod} n}) / 1 \leq j \leq n \}$.

Clearly every edge $e_{ij}$ has two out neighbourhoods $e_{i,j+i}$ is $e_{i,j+i}$ for $j = 1, 2, 3, \ldots, n$ and $e_{ij}$ has two out neighbourhoods $e_{i,j}$ and $e_{i,j+1}$. Here each arc in $D$ in dominates two arcs and out dominates two arcs. So the set of arcs in $S$ are not independent. So $\gamma'(D) = \left\lfloor \frac{nd}{d+1} \right\rfloor = \left\lfloor \frac{|V| \times |X|}{|X|+1} \right\rfloor$
Illustration 2.3.

Consider the Figure 1. Let $S = \{ e_{11}, e_{13}, e_{21} \}$, $A \setminus S = \{ e_{12}, e_{22}, e_{23}, e_{24} \}$

$\text{Neg}^+(S) = \text{Neg}^+(e_{13}) \cup \text{Neg}^+(e_{11}) \cup \text{Neg}^+(e_{21})$

Clearly $S = \{ e_{13}, e_{11}, e_{21} \}$ out dominates all the arcs of $A \setminus S$. Therefore $\gamma'(D) = 3$.

i.e., $\gamma'(D) = \left\lceil \frac{nd}{d+1} \right\rceil = \left\lceil \frac{4 \times 2}{2+1} \right\rceil = \left\lceil \frac{8}{3} \right\rceil = 3$.

Case iii: $d \geq 3$ Let $X = \{ a, b, c \ldots \}$

From the construction of the Cayley digraph of a group, it is also a cyclic graph. If we have $n$ vertices then it has $\geq 3n$ arcs. Let $V = \{ v_1, v_2, v_3, \ldots, v_n \}$ and $A = \{ e_{ij} = (v_j, v_{(j+i) \text{mod} n}) / 1 \leq j \leq n \} \cup \{ e_{ij} = (v_j, v_{(j+b) \text{mod} n}) \cap \{ e_{ij} = (v_j, v_{(j+c) \text{mod} n}) / 1 \leq j \leq n \} \}$

Clearly every edge $e_{ij}$ has more than two out neighbourhoods $e_{i,j+1}$, $e_{i,j+1}$, $e_{i,j+1}$, for $j=1,2,3\ldots,n-1$ and $e_{in}$ has more than two out neighbourhoods $e_{i1}, e_{i1}, e_{i1}$, Here each arc in D in dominates more than two arcs and out dominates more than two arcs. So the set of arcs in S are not independent.

Initially choose an arc of S which has maximum out degree. By deleting the outgoing arcs of the maximum degree edge successively we will end up with a set of arcs S, which dominates all the arcs of D:S with minimum cardinality. In this case all edges generated by any one generator dominates all other arcs i.e., $|S| = d$. So $\gamma'(D) = \left\lceil \frac{nd}{d} \right\rceil$.

Illustration 2.4.

Consider the Figure. 2 Cayley digraph for the group $\mathbb{Z}_6 = \{ 0, 1, 2, 3, 4, 5 \}$ with the generating set $X = \{ 2, 3, 5 \}$. The arc domination number for the Cayley graph of $\mathbb{Z}_6$ is shown below.
Figure.2 The arc domination of the Cayley digraph for the group \( Z_6 = \{1, 2, 3, 4, 5, 6\} \) with generator set \( X = \{2, 3, 5\} \)

Let \( S = \{e_{31}, e_{32}, e_{33}, e_{34}, e_{35}, e_{36}\} \)

\[
\text{Neg}^+(S) = \text{Neg}^+(e_{31}) \cup \text{Neg}^+(e_{32}) \cup \text{Neg}^+(e_{33}) \cup \text{Neg}^+(e_{34}) \cup \text{Neg}^+(e_{35}) \cup \text{Neg}^+(e_{36})
\]

Clearly \( S \) out dominates all the arcs of \( A \setminus S \)

Then by the definition of arc domination \( \gamma'(Z_6) = |S| = 6. \)

i.e., \( \gamma'(Z_6) = \left| \frac{|V| \times |X|}{|X|} \right| = \left| \frac{6 \times 3}{3} \right| = \left| \frac{18}{3} \right| = 6. \)

**Definition 2.5**

An Out arc dominating set of a digraph \( D(V, A) \) is a set \( S \) of arcs such that every arc \((u, v)\) of \( A-S \) is dominated by some arc \((w, u)\) of \( S \). The minimum cardinality of an out arc dominating set of \( S \) in \( D \) is the out arc domination number of \( S \) denoted by \( \gamma^+(D) \) which is same as ordinary arc domination in digraphs. An in arc dominating set of \( D \) is a set \( S \) of arcs such that every arc \((w, u)\) of \( A-S \) is dominated by some arc \((u, v)\) of \( S \) and is denoted as \( \gamma^-(D) \).

**Definition 2.6**

The twin arc domination number of a digraph \( D(V, A) \) is the minimum cardinality of a subset \( S \) of the arc set of \( D \) such that every edge of \( D \) is out arc dominated by some arc of \( S \) and in arc dominated by some arc of \( S \) and is denoted by \( \gamma^*(D) \). A set of edges of \( D \) is twin arc dominating set if every edge of \( D \) is out arc dominated by some edge of \( S \) and in arc dominated by some edge of \( S \).
Illustration 2.7 Consider the directed star graph given below.

\[ \gamma^+ (D) = 5 \text{ but } \gamma^- (D) = 1. \]

**Theorem 2.7.** Twin arc domination of Cayley digraph of \((\mathbb{Z}_n, X)\) is \(\gamma^* (D) = \frac{|A|}{d}\)

**Proof:** Let G be a group and the Cayley digraph of the group G with generating set X be Cay(G,X) with |X| = d > 1.

The Cayley digraph of a group is a cyclic graph, if G has n elements then Cay(\(\mathbb{Z}_n, X\)) has n vertices and dn arcs. Let \(V = \{v_1, v_2, v_3, \ldots, v_n\}\), \(X = \{a, b, c, \ldots\}\) and \(A = \{e_{aj} = (v_j, v_{(j+a) \mod n})/ 1 \leq j \leq n\} \cup \{e_{bj} = (v_j, v_{(j+b) \mod n})/ 1 \leq j \leq n\} \cup \{e_{cj} = (v_j, v_{(j+c) \mod n})/ 1 \leq j \leq n\} \cup \ldots \).

Clearly every edge \(e_{aj}\) has \(d > 1\) out neighbourhoods \(e_{a(j+1)}, e_{b(j+1)}, e_{c(j+1)}, \ldots\) for \(j=1,2,3,\ldots n-1\) and \(e_{an}\) has \(d > 1\) in neighbourhoods \(e_{a1}, e_{b1}, e_{c1}, \ldots\). Every arc in a cycle out dominates exactly \(d\) arcs and in dominates exactly \(d\) arcs. Therefore \(|S| = d\).

i.e., \(\gamma^*(D) = \frac{|A|}{d}\).

Observation 2.8 If G is a Cayley digraph Cay \((\mathbb{Z}_n, X)\) with \(|X| \geq 2\) then \(\gamma^*(D) = \gamma^* (D)\).

3. Conclusion

In this paper, the arc domination number for Cayley digraph for the group \(\mathbb{Z}_n\) with generating set X is analysed and also found Twin arc domination number for Cayley digraph for the group \(\mathbb{Z}_n\) with some generating set X. In future, arc domination number for various digraphs can be analysed to establish a characterization of arc domination of digraphs.

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