Many-body localization and new critical phenomena in regular random graphs and constrained Erdős-Rényi networks

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We consider from the localization perspective the new critical phenomena discovered recently for perturbed random regular graphs (RRG) and constrained Erdős-Rényi networks (CERN)\textsuperscript{24}. At some critical value of the chemical potential for 3-cycles, $\mu$, the network decays into the maximally possible number of almost full graphs, and the spectrum of the Laplacian matrix acquires the two-zonal structure with a large gap. We find that the Laplacian eigenvalue statistics corresponds to delocalized states in one zone, and to the localized states in the second one. We interpret these findings in terms of the many-body localization problem where the hierarchical structure of the Fock space of some interacting many-body system is approximated by the RGG and/or by the CERN. We suggest to understand 3-cycles in RRGs and CERNs as resonant triples in the Fock space. We show that the scenario of the "localization without disorder", discussed previously in physical space, gets realized in the Fock space. We argue that it is natural to identify particles in the many-body system above the phase transition with clusters in the RRG. We also discuss the controversial issue of an additional phase transition between ergodic and non-ergodic regimes in the delocalized phase in the Fock space and find the strong "memory dependence" of the states in the delocalized phase, thus advocating existence of non-ergodic states.

I. INTRODUCTION

Anderson one-particle localization\textsuperscript{11} provides the unified framework for the disorder-induced metal-insulator transitions in various physical problems. It was found that above some critical diagonal disorder, the wave function becomes exponentially decaying in space and the system behaves as insulator. There are several criteria of the localization transition. The signature of the localization is encoded in the behavior of the "inverse participation ratio" of the wave function, in the statistics of corresponding eigenmodes (energy levels) and in the behavior of the entanglement entropy (see \textsuperscript{2} for review). In this paper we focus attention at the level spacing distribution criterium. In the delocalized regime the level spacing distribution between nearest energy levels shares the Wigner-Dyson (surmise) law, while in the localized regime it follows the Poisson statistics. In $D > 2$, the localized
and delocalized states can coexist in different parts of the spectrum, and the notion of the mobility edge is introduced to separate these regions.

The idea of Anderson localization has acquired recently a new incarnation in physics of interacting many-body systems [3, 4] using the idea of Fock space localization [5] (see [6, 7] for reviews of many-body localization). According this hypothesis, the particle localization occurs not in real space, but in the Fock (Hilbert) space of a many-body system, approximated by a Bethe lattice, for which some exact results are available [8–11]. Schematically, this mapping is as follows [5]. The nodes of the Bethe lattice constitute the basis states in the Hilbert space, while the links correspond to the resonant pairs of states. If the wave function of an effective one-particle system in the Fock space is close to the state of the initial many-body system, then the localization in the Fock space takes place. Hence, we can identify the localized state in the Fock space with the particle in the initial many-body system. However, if the wave function of the effective one-particle system in the Fock space is expanded over a large number of states of the many-body system, this regime is understood as a delocalized in the Fock space. Similarly to the localization in the real space, the notion of a mobility edge can be introduced in the Fock space as well. Statistical properties of Bethe lattices have a lot in common with properties of random regular graphs, and the issue of the Fock space localization in terms of one-particle excitations on RRG attracts nowadays much attention [12–15].

Recently, a surprising indication of an additional phase transition inside the delocalized regime was reported in [12, 16–18]. It has been found in [12, 16] that the critical value of the diagonal disorder separates the "ergodic" and "non-ergodic" delocalization regimes. In the non-ergodic case the absence of thermalization makes the very notion of equilibrium thermodynamics meaningless. This issue has been discussed in several papers, however no proof of existence or absence of the non-ergodic delocalized phase is yet provided. The non-ergodic delocalized phase has been related recently to the one-step replica symmetry breaking (RSB) in disordered system, and a kind of an analytic approach to this problem has been developed [15]. In [15] it was conjectured that the non-ergodic case corresponds to the RSB phase, while ergodic case means the unbroken replica symmetry. The multifractality exponents exhibit a jump at some critical value of the disorder signaling the ergodic/non-ergodic transition happen inside the delocalized phase.

Another scenario of the localization was suggested long time ago [19] and attracts much attention nowadays [20, 21]. It is usually named as "localization without disorder" or "localization in translational-invariant systems". The key point is the presence in the system of clusters of some nature. These clusters can be placed at any space point. No special on-site disorder is added to the system. It was shown in [19] that clusters can provide the localization in the system and, hence, strongly influence the thermalization. Two possibilities have been discussed. On one hand, there can be a single huge immobile cluster which influences thermalization. On the other hand, there can be several mobile clusters whose motion affects thermalization as well (see [22] for review). Let us emphasize that these clusters exist in the physical space.

In the present work we exploit our recent findings concerning the critical behavior of CERN and RRG to gain new insight on the one-particle Anderson localization in the Fock space. We interpret the phase transition found in [23, 24] for constrained Erdos-Renyi networks (CERN) and random regular graphs (RRG), controlled by the chemical potential for 3-cycles, in terms of the many-body localization in the physical space, and discuss related
ergodic properties. It seems necessary first to summarize the main results obtained in [24].

- Above some critical value of the chemical potential, \( \mu \), controlling the number of 3-cycles in the network, both CERN, and RRG get defragmented into the loosely connected collection of almost full subgraphs ("cliques");

- The spectral densities of CERN and RRG form two-zonal structure. In particular:
  - The spectral density in the main zone above the transition point acquires triangle-like shape both for CERN and RRG. This spectral density differs from the one at \( \mu = 0 \) and emerges due to the strong inter-cluster interactions of spectral modes;
  - The second "non-perturbative" zone in spectral density, emerging for \( \mu > \mu_{cr} \), is filled by eigenvalues corresponding to cliques (one eigenvalue per one clique).

In this paper we analyze numerically few questions:

- What is the spectral statistics in RRGs and CERNs in the main and non-perturbative zones?
- Do we have any signature of an intermediate non-ergodic delocalized phase in our systems?

We show numerically that statistics in the main zone corresponds to delocalized states while in the second (non-perturbative) zone – to localized ones. We have no diagonal disorder in RRG and CERN, however the network gets completely defragmentated into clusters above \( \mu_{cr} \). Just these clusters are responsible for the localization phenomena and we obtain the "localization without disorder" scenario in the Fock space which is the analog of a similar phenomena in the physical space found in [19]. We also demonstrate that the spectral density in the main zone strongly depends on initial conditions, thus indicating the non-ergodicity of these modes.

We conjecture that the 3-cycles in the Fock space correspond to the resonant triples in the Hilbert space of many-body system considered in the transport phenomena [25]. Hence by introducing the chemical potential for 3-cycles, \( \mu \), we can to investigate the influence of resonant triples in the Hilbert space on the transport phenomena. Using the mapping between Fock and physical space developed in [3, 5] and the localization transition found in this paper, we argue that clusters in the Fock space can be regarded as the degrees of freedom in the initial system.

The paper is organized as follows. In Section 2 we remind basic ideas behind the many-body localization and its relation to the one-particle Anderson localization on graphs. The spectral properties of the modes in the main and non-perturbative zones are found numerically in Section 3. We also suggest a simple mean field qualitative explanation of the defragmentation phenomena at critical \( \mu \). In Section 4 we discuss the interpretation of our numerical findings in terms of interacting many-body system in the physical space. In Section 5 we pay attention to the numerical evidence of the non-ergodic nature of the delocalized modes. Open questions are outlined in the Discussion.
II. MANY-BODY LOCALIZATION VERSUS LOCALIZATION IN FOCK SPACE

Let us briefly remind the issue of the many-body localization [3]. Consider some interacting many-body electron system with the Hamiltonian written in the second quantized form

$$H = \sum_i E_i c_i^+ c_i + \sum_{ijkl} V_{ijkl} c_i^+ c_j^+ c_k c_l = H_0 + H_{int} \quad (1)$$

Typically, it is assumed that the two-particle interaction, $U(x, y)$ is local in the space, being for example, $U(u, y) = V \delta(x - y)$, where $V$ measures the strength of interaction. For $V = 0$ all electrons are completely localized in space and the conductivity vanishes at small temperature. The key idea of the many-body localization concerns the localization-delocalization transition in the interacting many-body system at a finite temperature.

The conductivity occurs above some critical temperature due to the inelastic process, in which, roughly speaking, the electron creates an electron-hole pair due to the interactions in real space. One can approximately describe this many-body process by counting multiple excited states (for instance, the number of created electron-hole pairs) in the Fock (Hilbert) space, where the point in the Fock space corresponds to some basis state.

One can introduce the metrics in the Fock space as follows. The ”distance” in the Fock space between two distinct many-body states, $x$ and $y$, in the preselected basis is measured via the difference in numbers of excited electron-hole pairs at points $x$ and $y$ respectively. Upon the choice of the basis, the states get mapped on sites of some topological network and hopping rates between sites are provided by the term $H_{int}$ in (1), responsible for the creation of electron-hole pairs. The term $H_{int}$ yields connections between sites and hence determines the topology of the Fock space network. It was argued that upon some approximation, the Fock space network has the structure of the disordered Bethe lattice, which in turn, can be approximated by the random regular graph (we discuss this point below in more details).

Thus, the problem gets reduced to the investigation of an one-particle localization on a Bethe lattice, assuming that $E_i$ are random variables with flat distribution within some support, $-W < E_i < W$ ($i = 1, ..., N$), and all $E_i$ live on vertices of the Bethe lattice. The vertex degree of the Bethe lattice, $p$, according to [3], is proportional to the temperature, $T$. Thus, the set of parameters, $(V, T)$, in the coordinate space for the many-body problem, is mapped to the set of parameters of the one-particle problem in the Fock space with the topology of disordered Bethe lattice. The effective one-particle Hamiltonian, $H_{Bethe}$, in the Fock space reads:

$$H \sim H_{Bethe} = \text{diag}(E_1, \ldots, E_N) + A(p) \quad (2)$$

where $H$ is given by (1), $A(p)$ is the adjacency matrix of the Bethe lattice with fixed vertex degree $p$. Thus, on the Bethe lattice (the topological graph) we have essentially the Anderson model with the diagonal disorder. The vertex degree, $p$, is related to the parameters of the initial interacting many-body system as follows

$$p \propto \frac{T}{\Delta} \quad (3)$$

where $T$ is the temperature in the initial many-body system, and $\Delta$ is the mean single-particle spacing between nearest eigenstates in the vicinity of the Fermi level.
The localization problem in the Anderson model on the Bethe lattice has been solved in [8–11], where it was shown that the critical value of disorder dividing the localized and delocalized states is

\[ W_{\text{crit}} \propto V p \log p \]  

(4)

where \( V \) is the strength of the interaction yielding \( W \) for on-site disorder on a topological Bethe lattice.

The critical value of the Bethe lattice disorder, \( W_{\text{crit}} \), can be translated back to the initial many-body problem: the localized states of an effective one-body system in the Fock space are close to the states in the many-body system, while the delocalized eigenstates of the one-particle system in the Fock space are the superpositions of large number of many-body states of initial problem [5]. The localization in the Fock space has been transferred from the Bethe lattice to the RRG [13, 16, 17] since the average number of 3-cycles in the uniform RRG ensemble is of order of \( \bar{N}_\Delta = (p-1)^3/6 \) for large \( N \) and, hence, is negligible in the thermodynamic limit. In what follows we work with the standard spectral density of the graphs ensemble defined via averaging of eigenvalues of the adjacency matrix over the disorder:

\[ \rho(\lambda) = \left\langle \sum_{i=1}^{N} \delta(\lambda - \lambda_i) \right\rangle_{\text{ensemble}} \]  

(5)

The spectral density for the ensemble of RRG of degree \( p \), differs from the Wigner semicircle and shares the Kesten-MacKay law

\[ \rho(\lambda) = \frac{p}{2\pi(1 - \lambda^2)} \sqrt{\frac{4(p-1)}{p^2} - \lambda^2} \]  

(6)

As we shall see below, the distribution [6] for RRG is essentially transformed in vicinity of localization-delocalization transition driven by the chemical potential for of 3-cycles.

III. LEVEL SPACING DISTRIBUTION IN PERTURBED CERN AND RRG MODELS

A. Numerical results and their interpretation

To begin with, let us remind the critical behavior in the CERN model [23, 24] which is the generalization of the Strauss model [26] solved in [27, 28] in the mean-field approximation. The CERN differs from the standard Erdős-Renyi (ER) network by two additional requirements: i) the constraint of quenched degree in all graph vertices, and ii) the condition that network is forced to increase the number of 3-cycles, \( N_\Delta \).

The initial state of the constrained Erdos-Renyi network is prepared by connecting any randomly taken pair of vertices with the probability \( p \) (the double connections are excluded). When the initial pattern is prepared, one randomly chooses two arbitrary links, say, between vertices \( i \) and \( j \), \( (ij) \), and between \( k \) and \( m \), \( (km) \), and reconnect them, getting new links \( (ik) \) and \( (jm) \). Such a reconnection conserves the vertex degree [29]. Now, one applies the standard Metropolis algorithm with the following rules: a) if under the reconnection the
number of 3-cycles is increased, a move is accepted, b) if the number of 3-cycles is decreased by $\Delta N_\Delta$, or remains unchanged, a move is accepted with the probability $e^{-\mu \Delta N_\Delta}$. Then the Metropolis algorithm runs repeatedly for large set of randomly chosen pairs of links, until it converges. In [30] it was proven that such Metropolis algorithm converges to the Gibbs measure $e^{\mu N_\Delta}$ in the equilibrium ensemble of random undirected constrained Erdős-Renyi networks with fixed vertex degree.

To explain the numeric results obtained for CERN and RRG, it is instructive to make a step back and consider ensemble of unconstrained (i.e. conventional) Erdős-Renyi graphs with non-conserved vertex degree. The Hamiltonian for such system reads

$$H = -\mu N_\Delta, \quad (\mu > 0)$$

which is the Hamiltonian of the Strauss model [26–28]. It has been shown in [27, 28] that if the system tends to form as many 3-cycles as possible, there is a phase transition, driven by the chemical potential $\mu$, and at some $\mu_{cr}(p)$ the so-called Strauss phase consisting of a single full subgraph (clique) is formed for any initial random ER network with fixed number of vertices, $N$, and fixed average number of links at vertex, $pN$.

If the additional constraint of vertex degree conservation is imposed, such model (CERN) becomes in many respects similar to the RRG. The phase transition pattern both in CERN and RRG models is again $\mu$-controlled, however the network decays into the maximally possible number, $[1/p]$, of almost full subgraphs (cliques) [24], where $p$ is the vertex connection probability in the initial network at the preparation. The corresponding adjacency matrices are shown in the Fig.1, where different phases of the ground states are clearly seen. The ground state of the quenched network involves $[p^{-1}]$ almost complete graphs corresponding to blocks (cliques) of the adjacency matrix $A$ with fluctuating sizes $N_i$ ($\sum_i N_i = N$) and the mean value of the vertices in the clique, $N_{cl} = \langle N_i \rangle = N/[p^{-1}] \approx Np$. To visualize the kinetics, we enumerated vertices at the preparation condition in arbitrary order and run the Metropolis stochastic dynamics. After equilibrating the system, and when the cliques are formed, we re-enumerate vertices sequentially according to their belongings to cliques. Then we restore corresponding dynamic pathways back to the initial configuration.

Since we are interested in detailed description of the localization phenomena, the enveloping shape of the spectral density of adjacency matrix, and the level spacing distribution, become of primary importance. The spectral densities of ensembles of CERN and RRG behave similarly above $\mu_{cr}$ (see [24] for details). Namely, when $\mu$ is increasing from zero, the Wigner semicircle gets gradually deformed, acquiring triangular shape of the main zone,
accompanied by number of isolated eigenvalues, which eventually form the second zone. Isolated eigenvalues correspond to clusters \[31\], hence we could think of the second zone as constituted by "non-perturbative" clusters and inter-cluster interactions, while the spectral density in the main zone reflects the "perturbative" excitations inside each cluster.

The investigation of the level spacing distribution for CERN and RRG provides the identification of the Anderson localization for the one-particle dynamics in the Fock space. In the delocalized regime, the distribution is given by the Wigner surmise, while in the localized phase it becomes Poissonian:

\[
\begin{align*}
P_{\text{deloc}}(s) &= A \, s \, e^{-B s^2} \quad \text{below mobility edge, } \lambda_m(\text{GOE}) \\
P_{\text{loc}}(s) &= e^{-s} \quad \text{above mobility edge, } \lambda_m
\end{align*}
\]

where GOE is the abbreviation of the Gaussian Orthogonal Ensemble, \(s = \frac{\lambda_i - \lambda_{i+1}}{\Delta}\). We have studied how the defragmentation of the CERN and RRG into clusters above the phase transition in \(\mu\), is accompanied by changes in the spectral densities and in the level spacing distributions. The results of our numerical simulations are shown in the Fig.2 for CERN, and in the Fig.3 for RRG. It is seen for both, CERN and RRG, that the level spacing distribution in the central zone follows the Wigner surmise, while it becomes Poissonian for the eigenvalues in the second zone. These distributions are separated by the well defined mobility edge, \(\lambda_m\), which in our case lies between two zones. The second zone corresponds to the localized phase and can be naively interpreted physically as the insulator. Fine structure of delocalized states in the main zone and its meaning will be discussed later.

Since the mobility edge separates the localization-delocalization regimes in the Fock space, we can ask a question what our findings mean for the initial many-body system. Following arguments of \([5]\), it seems naturally to identify the states in localized part of the spectrum (the second zone) in the Fock space with the localized states in the initial many-body interacting model. We know precisely that the localized state in the Fock space is the cluster formed in CERN or RRG above the transition point, \(\mu_{\text{cr}}\). Hence, we come to the straightforward identification of the network cluster with the quasiparticle in the underlying many-body system for which CERN or RRG are the Fock spaces.

We arrive at the following picture. Start with some initial real-space many-body system and fix its intrinsic basis in the Hilbert space above the phase transition point (where clusters-cliques are present) by defining the corresponding Fock space representation. We argue that in the localized regime in the CERN and RRG, the eigenstate in the Fock space with the localized states in the initial many-body interacting model. We know precisely that the localized state in the Fock space is the cluster formed in CERN or RRG above the transition point, \(\mu_{\text{cr}}\). Hence, we come to the straightforward identification of the network cluster with the quasiparticle in the underlying many-body system for which CERN or RRG are the Fock spaces.

Below the phase transition (at \(\mu < \mu_{\text{crit}}\)) such a basis does not seem natural since we have no stable clusters in the Fock space representation in this regime. It is worth noting that such situation is familiar for systems which enjoy different spectra of stable particles at weak and strong coupling regimes. At strong couplings, the natural basis involves the soliton-like states and the perturbative modes have to be considered as a kind of composites of the
solitonic states. Contrary, at weak couplings, the natural basis involves the perturbative modes and solitons are considered as made from large number of perturbative modes.

The existence of the mobility edge has straightforward explanation in this model picture. In the delocalized region of the spectrum we have only perturbative modes which can be considered as excitations of the soliton state. According to [24], the interaction between solitons has dramatic influence on the spectral density: instead of separated peaks which constitute the spectrum of almost full or sparse networks, we have the continuum set of modes which has very peculiar triangular enveloping shape. One could say that perturbative modes get collectivized among the solitons-clusters.
B. Simple mean-field arguments for clusterization

In the frameworks of the naive mean-field description, the disintegration of the unicolor RRG into weakly connected cliques, can be understood using the following qualitative arguments. Suppose that the ground state of RRG splits in $s$ groups of $M$ vertices in each group, such that $sM = N$. We consider $s$ as a variational parameter trying to find the minimum of the mean-field system energy as a function of $s$. The value of $s$ is bounded from above by the maximal value, $s_{\text{max}} = N/k$, at which each cluster of the RRG for given vertex degree is an almost complete graph. Taking into account that the average number of small $\ell$-link cycles in the RRG of degree $k$ is equal to $N_\ell = \frac{(k-1)^\ell}{2\ell}$, we may write the mean-field free energy of the RRG with controlled number of $\ell = 3$-cycles, as follows:

$$F = -\mu N_3 s + M \ln M + (N - M) \ln(N - M)$$  \hfill (9)

where $N_3 \equiv N_\Delta$ is the average number of 3-cycles in the system. In (9) the first term accounts for the energy of collection of triangles ($\mu$ for each 3-cycle) in the system, while
the second term represents the entropic contribution coming from the redistribution of $N$ vertices in $s$ groups of $M$ vertices in each group. Taking into account that $M = \frac{N}{s}$, we get from (9) the following explicit expression:

$$F(s) = -\frac{\mu (k-1)^3}{6} s + \frac{N}{s} \ln \left(\frac{N}{s}\right) + N \left(1 - \frac{1}{s}\right) \ln \left(N \left(1 - \frac{1}{s}\right)\right)$$

(10)

The free energy minimum is determined by the equations $\frac{d}{ds} F(s = s_{cr}) = 0$ and $\frac{d^2}{ds^2} F(s_{cr}) > 0$, where the first one determines the critical point, $s_{cr}$:

$$g s_{cr}^2 = \ln (s_{cr} - 1)$$

(11)

In (11) we have introduced the control parameter $g = \frac{\mu (d-1)^3}{6N}$. The graphic solutions of (11) are depicted in the Fig.4 (upper panel) for $g = 0.01, 0.04, 0.1$ correspondingly. If we take $N = 256$ and $d = 20$, the chosen values of $g = 0.01, 0.04, 0.1$ correspond to the values of $\mu = \frac{6Ng}{(d-1)y}$, specifically, $\mu = 0.0022, 0.0089, 0.022$. The plots of the free energy (10) at these values (consistent with the plots in Fig.4) are shown in Fig.4 (lower panel).

![Graphs showing free energy](image)

FIG. 4: Graphic solution of (11) for $g = 0.01, 0.04, 0.1$ (upper panel); Corresponding plots of the free energy $F(s)$ defined in Eq.(6) at $\mu = 0.0022, 0.0089, 0.022$ (lower panel).

We have mentioned that $s \leq s_{max} = \frac{N}{k}$ for some given $k$ and at $s = s_{max}$ the RRG splits in a collection of weakly connected almost full clusters. As one sees from the Fig.4, the free energy could be even lowered by increasing $s$ beyond $s_{max}$, however this is impossible due to topological restrictions of RRG with the vertex degree $k$. Thus, the transition occurs
when the minimum of the free energy, defined by the equation $\frac{d}{ds} F(s) = 0$ matches the value $F(s_{\text{max}})$. Above the critical $\mu$ the RRG splits into the maximal possible number of clusters, $[N/k]$ as we see in the last diagram in Fig.1.

IV. INTERPRETATION: LOCALIZATION WITHOUT DISORDER IN THE FOCK SPACE

Recall that in our model we have no any on-site (diagonal) disorder in the Fock space, hence the natural to ask a question what type of localization pattern we are dealing with. The proper framework is provided by the scenario of localization without disorder in the translation invariant system first suggested in [19] and reconsidered recently in [20, 21] from the perspective of many-body localization. In the original work [19] the resonant spots located in the physical space, produce an effective structural disorder and play the key role for the localization phenomena. This crucially differs from systems with the diagonal disorder, where the possible resonant spots are placed in positions dictated by the disorder distribution. For the translationally invariant systems positions of rare spots in the physical space are not fixed and they are potential carriers in the transport phenomena. Such ergodic regions could provide the delocalization in the system. In models considered in [20, 21], the resonant spots are induced by the temperature and a bit loosely one could claim that the "randomness" is induced by the initial state.

The localization phenomena we have found has a lot in common with this scenario. The translationally invariant system which we are dealing with, and which enjoys the one-particle localization, lives in the Fock space like in the situation with the on-side disorder. Again, we can treat the one-particle localization in the Fock space as the manifestation of the many-body localized (MBL) states in the physical space. In our situation the resonant spots are induced not by the temperature, but by the chemical potential $\mu$ of 3-cycles. The clusterization occurs at $\mu \geq \mu_{\text{crit}}$ and the states involved into some resonant spot, form the cluster and as in [19, 21] the spectrum inside the resonant spot corresponds to delocalized states. The dependence on the initial state occurs in our situation as well.

Let us emphasize that in our model we have actually two sources of the disorder. On one hand, due to the network randomness at the preparation, we have the non-diagonal disorder for any configuration of 3-cycles and any $\mu$. On the other hand, due to the clusterization happen at $\mu \geq \mu_{\text{crit}}$, we induce a structural disorder. One could question whether the non-diagonal disorder can produce the one-particle localization in the Fock space. The answer is negative: it was shown in [23] there are no clusters at $\mu \leq \mu_{\text{crit}}$ and all states in the system are delocalized. The clusterization occurs for any initial state for $\mu \geq \mu_{\text{crit}}$. Hence, the off-diagonal disorder is not responsible for the clusterization. Let us note that the localization phenomena found in constrained Erdos-Renyi networks essentially differs with the localization forced by the diagonal disorder, in conventional Erdos-Renyi networks [32].

Since the structural disorder in the Fock space is induced by chemical potential of 3-cycles, $\mu$ it is natural to ask a question about its physical meaning. To clarify the situation, let us remind the notion of resonant triples discussed in [25]. The resonant triple is defined as the triple of states in the Hilbert space of some quantum interacting many-body system which obey mutual resonant condition. The resonant condition means that the difference between energies of three states in the absence of interaction is small compared to the effects
of interaction. There is also the semiclassical analog of the resonant triple which corresponds to the peculiar behavior of the separatrix width and the Arnold diffusion [25] (see also [33]). The resonant triples yield the thermal chaotic spots [25] which amount to the transport in the system.

In the network model of the Fock space, a link corresponds to a resonant pair, while a 3-cycle corresponds to a resonant triple. The estimation of the number of the resonant triples in the Hilbert space of the real interacting many body system is a very complicated issue [25, 34] and there is no well defined answer yet. Our study shows that the number of resonant triples is crucial for properties of the many-body system, and especially influences localization and transport phenomena. When the number of resonant triples exceeds some critical threshold, the Fock space of the interacting many-body system gets clusterized. Note the although yet we have investigated the impact of the 3-cycles only, we expect that higher cycles (resonant quarters, etc) play the similar role and affect the structural disorder above the critical value of the corresponding chemical potential.

To complete this Section let us make a short comment concerning the inverse participation ration which is defined as

\[ P_n \propto \int dx |\Psi_n(x)|^4 \]  

(12)

It is related to the return probability of the \( n \)-th eigenmode in the momentum space. This variable can be expressed as the path integral over the loops of all lengths. If we insert into path integral the factor \( e^{-mL} \) for the loop of length \( L \), we enforce the loops to be shorter. Physically this factor comes from the mass of the particle in the gapped phase. Certainly, this factor influences \( P_n \) and, therefore, the localization property. Similar argumentation holds in the Fock space, instead of the real one. Namely, we can introduce the inverse participation ratio defined for the state

\[ P_a \propto \sum_n |\langle a|n \rangle|^4 \]  

(13)

which should be identified with the return probability in the Hilbert space (see for instance the discussion in [35, 36]). This object can be considered as the path integral over all loops in the Fock space. The fugacity \( \mu \) plays the role similar to the mass in the real space since it enforces the number of 3-link loops in the path integral. Hence, \( \mu \) plays the role of the "mass" in the Hilbert space and influences the inverse participation ratio. Therefore it can be considered as a kind of the measure for disorder in the Fock space.

V. TRIANGLE-SHAPED SPECTRAL DENSITY AS A SIGNATURE OF NON-ERGODIC DELOCALIZED PHASE

Here we discuss the possible emergence of delocalized non-ergodic phase. Remind that in the many-body localization paradigm there exists such initial state when all degrees of freedom are localized. What would be the counterpart of such a state in our case? Since we have identified clusters with the basis in the many-body system, it is natural to look at system when all states are localized in clusters.

To this aim we have considered the spectral density of the network consisting of clusters prepared \( \textit{ad hoc} \) with exactly the same connectivity parameters as in the last panel of Fig.1.
Since this network does not have any pre-history, it knows nothing about the chemical potential for 3-cycles and, thus it can be considered as a basis of a "non-interacting" many-body system. It would be a perfect reference system to start with if modes are strictly localized. The spectrum of this model is presented in Fig.5 and we see that indeed all eigenvalues are in the localized phase.

FIG. 5: Spectral density and level spacing distribution for the "memory-independent" CERN prepared ad hoc. There is no mobility edge in the system: all eigenmodes are localized in both zones.

Let us now ask about the ergodicity of states in the central zone of the ground state of CERN. Comparing the spectral densities of this "memory-independent" network, shown in Fig.5, with the one of the "memory-dependent" final state of CERN, depicted in Fig.2, we see that the network with ad hoc prepared clusters does not share the triangle spectral density and has very different level spacing distribution. Thus we can expect that the states in the central delocalized zone in the CERN and RRG are non-ergodic and essentially "memory-dependent" and the triangular shape of the central zone of the spectral density presumably is its signature.
The emergence of two different ("ergodic" and "nonergodic") phases within the delocalized region, has been advocated in [12, 16–18, 37] and is the subject of intensive discussions. There is an argument expressed in [13] that in the large-$N$ limit this two-phase structure disappears, thus the non-ergodicity is a finite-size effect. However, it was argued in [15] that the emergence of the non-ergodic phase is supplemented by the one-step replica symmetry breaking (RSB) and the phase transition becomes even more sharp in the large-$N$ limit, being the real phase transition separating the RSB and the replica symmetric phases. The block structure of the adjacency matrix shown in the last panel of Fig.1 implies that we have a kind of the one-step replica symmetry breaking stage, and we could expect that we are in the non-ergodic "memory-dependent" phase.

The general belief is that the non-ergodic phase corresponds to the situation when the modes are delocalized at some finite number of sites, relatively small at large $N$. This can be recognized via the evaluation of the multifractality exponent. In our case the perturbatively delocalized modes indeed are collectivized among some finite number of clusters, and this does not mean the delocalization of the whole network. The strong memory dependence observed in simulations, allows to conjecture that there are some hidden nonlocal conservation laws which is the common feature of MBL pattern [38]. The key property of the spectral density in the non-ergodic delocalized phase formulated in the studies on this issue is as follows: the spectral density does not obey the Wigner semicircle law, however the level spacing distribution probability nevertheless is approximately of the Wigner-Dyson type form [12, 16].

There are the natural candidates for the hidden integrals of motion in our study. Indeed we impose the condition of the degree conservation at all vertices hence we have $N$ conserved integrals of motion. They are expressed non-locally in terms of the natural degrees of freedom – links of the network. The number of conserved quantities is essentially smaller than the total number of degrees of freedom in the network, hence we do not have a complete integrability. It would be interesting to investigate the situation when the degree conservation constraints are imposed on the part of the vertices only. Certainly these points deserve for the detailed study.

VI. CONCLUSION

In this paper we have analyzed the one-particle localization properties of perturbed RRGs and CERNs. In our model we have no diagonal disorder at all and the off-diagonal intrinsic disorder does not lead to the localized phase as well. The localization occurs due to the structural disorder induced above the critical chemical potential for the 3-cycles. This is the realization of the scenario "localization without disorder" in the translationally invariant systems [19]. Since we interpret our network as the model for the Fock space for the interacting many-body system it is crucial to identify the meaning of the 3-cycles in the Fock space. We suggest that they represent the resonant triples [25], hence our study indicates that the number of such resonant triples strongly influences the properties of the system.

We have shown numerically that above the critical value of the chemical potential, $\mu_{cr}$, the spectrum on the network is separated into the delocalized and localized modes and the mobility edge lies between two zones. Localized modes in one zone correspond to clusters,
while delocalized modes correspond to the perturbative modes around the clusters collectivized via inter-cluster interactions. We present arguments based on direct visualization of adjacency matrix structure, favoring the conjecture that the delocalized modes are in the non-ergodic regime with a kind of one-step replica symmetry breaking. We have focused in our paper mainly on the level distribution statistics. It would be useful to extend this analysis to the participation ratio, the entanglement entropy \cite{39}, or other benchmarks of the localization transition. We could mention also the many-body counterpart of the Thouless energy \cite{10,11}. Since in the localization phase the hidden integrals of motion which destroy the ergodicity are expected, it is natural to discuss this mapping in the integrability framework. We plan to present corresponding arguments elsewhere.

We believe that the mechanism of CERN and RRG defragmentation via of eigenvalue tunnelling, when the system is pushed to some "untypical" topological state with the large number of 3-cycles, is quite general. The eigenvalue tunnelling we are dealing with, has very different incarnations is physics: for instance, formation of the "eigenvalue instanton" in many physical situations signals the creation of the stable soliton (see \cite{12} for review). Interacting soliton-like clusters are the "degrees of freedom" in the initial many-body model. Our numerical study shows that in such model of interacting clusters there can be the mobility edge and the identification of the initial and effective degrees of freedom is very different at two sides of the mobility edge.

In this paper we focused only on "colorless" CERN and RRG where vertices do not carry any additional degrees of freedom. In \cite{23,13} the similar analysis for the colored network has been developed, where it was shown that some additional color-dependent critical phenomena can occur.

In the model considered here, there are two types of natural degrees of freedom: clusters and elementary links. We argued that at strong coupling the fundamental degrees of freedom – network cliques – are identified with the soliton-like quasiparticles in the initial interacting many-body problem. It is natural to conjecture a kind of a weak-strong particle-soliton duality behind the many-body localization. Such a hope is consistent with the genus-one Riemann surface for the two-zonal spectral density. The approach to the modular structure behind the Anderson transition developed in \cite{44} seems to be relevant for the identification of the modular structure in the many-body localization in real space and corresponding Anderson localization in the Fock space. Another incarnation of the modular invariance in localization phenomena for ensemble of sparse matrices, is discussed in \cite{45}.

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