Article

Power Exchange Option with a Hybrid Credit Risk under Jump-Diffusion Model

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Abstract: In this paper, we study the valuation of power exchange options with a correlated hybrid credit risk when the underlying assets follow the jump-diffusion processes. The hybrid credit risk model is constructed using two credit risk models (the reduced-form model and the structural model), and the jump-diffusion processes are proposed based on the assumptions of Merton. We assume that the dynamics of underlying assets have correlated continuous terms as well as idiosyncratic and common jump terms. Under the proposed model, we derive the explicit pricing formula of the power exchange option using the measure change technique with multidimensional Girsanov’s theorem. Finally, the formula is presented as the normal cumulative functions and the infinite sums.

Keywords: power exchange option; jump-diffusion model; credit risk; option valuation

1. Introduction

The pricing problem of option with credit risk, which is called the ‘vulnerable option’, has been studied by many researchers. For modeling the credit risk, researchers have adopted the structural model or the reduced-form model in general. The structural model was first used by Johnson and Stulz [1] to price the vulnerable European option under the Black–Scholes framework [2]. Later, Klein [3] considered the correlation between the firm value process of the option writer and the underlying asset process for a more realistic assumption. With these results, there have been many studies on the improvement of the pricing model for the vulnerable option under the structural model. Models such as the stochastic interest rate model [4], early counterparty risk model [5,6], stochastic volatility model [7–10] and jump-diffusion model [11–14] have been used to extend the pricing models of vulnerable options under the structural model. In addition, the pricing models of vulnerable options under the reduced-form model have been studied in recent years. Fard [15] studied the vulnerable European option of the reduced-form model under a generalized jump-diffusion model. Koo and Kim [16] provided the explicit pricing formula of a catastrophe put option with exponential jump and credit risk using the multidimensional Girsanov’s theorem. Wang [17] also proposed a reduced-form model based on a Generalized Autoregressive Conditional Heteroscedasticity (GARCH) process for valuing vulnerable options in discrete time.

Recently, hybrid credit risk models that combine the structural model and reduced-form model were proposed to develop the credit risk models for vulnerable options [18–20]. The hybrid model is more realistic because it reflects the characteristics of both models. In other words, exogenous and endogenous risk are modeled via the hybrid model. We also consider a hybrid credit risk model for the valuation of vulnerable options in this paper. Specifically, we study the valuation of power exchange options under the hybrid credit risk model.

The power exchange option was first studied by Blenman and Clark [21]. They derived the pricing formula of power exchange options under the Black–Scholes model.
There have been various extensions to improve the pricing model of power exchange options. Wang [22] extended the framework of Blenman and Clark by incorporating the correlated jump risk. Pasricha and Goel [23] proposed a Hawkes jump diffusion model to price power exchange options. In addition, there have been studies on the valuation of power exchange options with credit risk. Wang, Song and Wang [24] presented a pricing model of power exchange options with credit risk and jump diffusion under the structural model. They considered the idiosyncratic and common jump components in the correlated dynamics for the pricing model of a vulnerable power exchange option. Xu and Shao and Wang [25] considered the credit risk of the counterparty prior to the maturity, and provided an analytical pricing formula of power exchange options with credit risk. Pasricha and Goel [26] studied the valuation of vulnerable power exchange options under a reduced-form model. The dynamics in [26] are assumed to be driven by jump-diffusion processes.

Goel [26] studied the valuation of vulnerable power exchange options under a reduced-form model. The dynamics in [26] are assumed to be driven by jump-diffusion processes. However, to the best of our knowledge, there have been no studies on vulnerable power exchange options with a hybrid credit risk under the jump-diffusion model, and provide an explicit pricing formula for the option. This paper is organized as follows. In Section 2, we introduce the models for the underlying assets and the intensity, and construct a hybrid credit risk model. In Section 3, we derive the explicit pricing formula for the power exchange option with a hybrid credit risk when underlying assets follow the jump-diffusion models. Section 4 provides the concluding remarks.

2. Model

We consider power exchange options with credit risk under the jump-diffusion model of Merton [27]. For the risk-neutral valuation of options, we assume that the uncertainty in the economy satisfies a filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}, Q)\), where \(Q\) is the risk-neutral measure. Under the measure \(Q\), the dynamics of underlying assets follow jump-diffusion processes

\[
S_1(t) = S_1(0) \exp \left\{ \left( r - \frac{1}{2} \sigma_1^2 \right) t + \sigma_1 W_1(t) + \delta_1 \sum_{k=1}^{N_1(t)} Z_k + \sum_{k=1}^{N_1(t)} Z_k^2 \right\},
\]

\[
S_2(t) = S_2(0) \exp \left\{ \left( r - \frac{1}{2} \sigma_2^2 \right) t + \sigma_2 W_2(t) + \delta_2 \sum_{k=1}^{N_2(t)} Z_k + \sum_{k=1}^{N_2(t)} Z_k^2 \right\},
\]

where \(r\) is the risk-free interest rate, and \(\sigma_i (i = 1,2)\) are constant volatilities of the underlying assets, respectively. For hybrid credit risk modeling, we use the reduced-form model of Fard [15] and the the structural model of Klein [3]. For the structural model, the asset of the option writer is assumed as the following process:

\[
V(t) = V(0) \exp \left\{ \left( r - \frac{1}{2} \sigma_3^2 \right) t + \sigma_3 W_3(t) + \delta_3 \sum_{k=1}^{N_3(t)} Z_k + \sum_{k=1}^{N_3(t)} Z_k^2 \right\},
\]

where \(\sigma_3\) is the volatility of the asset of the option writer. For the reduced-form model, as in Fard [15], we assume that the intensity process \(\lambda(t)\) follows in the following form:

\[
d\lambda(t) = a(b - \lambda(t))dt + \sigma_4 dW_4(t),
\]

where \(a, b, \sigma_4\) are positive constants, and \(W_1(t), W_2(t), W_3(t)\) and \(W_4(t)\) are the correlated standard Brownian motions under the measure \(Q\) with \(dW_i(t)dW_j(t) = \rho_{ij}dt \ (i, j = 1, 2, 3, 4), \ |\rho_{ij}| \leq 1\). The term \(N(t)\), which presents the common jump, is a Poisson process with an intensity \(\lambda\), and \(N_1(t), N_2(t)\) and \(N_3(t)\), which present individual jumps, are Poisson processes with intensities \(\lambda_1, \lambda_2\) and \(\lambda_3\), respectively. We also assume that the Poisson processes \(N_1(t), N_2(t), N_3(t)\) and \(N(t)\) are mutually independent, and the jump...
sizes are controlled by \( Z_k^i \) \((i = 1, 2, 3)\) and \( Z_k \) when the jumps occur over a given period. As in Merton [27], \( Z_k \) follows a normal distribution with mean \( \mu \) and variance \( \gamma^2 \), and \( Z_k^i \) \((i = 1, 2, 3)\) follow normal distributions with mean \( \mu_i \) and variance \( \gamma_i^2 \) \((i = 1, 2)\). Then, we can define the compensator \( k_i \) \((i = 1, 2, 3)\) in the processes as

\[
k_i = \lambda (e^{\rho_i \mu + \frac{1}{2} \gamma_i^2} - 1) + \lambda_i (e^{\mu_i + \frac{1}{2} \gamma_i^2} - 1).
\]

Additionally, it is assumed that \( N_i(t) \) \((i = 1, 2, 3)\), \( N(t) \) and \( Z_k^i \) \((i = 1, 2, 3)\), \( Z_k \) are independent of \( W_i(t) \) \((i = 1, 2, 3, 4)\).

In fact, the jump-diffusion models have been used widely to describe the underlying assets in the real financial market. For real examples, see [28–30]. We derive the pricing formula of power exchange options with credit risk when the underlying assets follow jump-diffusion models in the next section.

### 3. Valuation of Power Exchange Option with a Hybrid Credit Risk

In this section, we provide the pricing formula of power exchange options with a hybrid credit risk based on the models introduced in the previous section. Let \( C \) be the initial price of a power exchange option with the hybrid credit risk under the risk-neutral measure \( Q \), and then the price \( C \) is given by

\[
C = e^{-rT}E^Q \left[ (s_1^\beta_1(T) - S_2^\beta_2(T))^+ \left( 1_{\{\tau > T, V(T) > D\}} + \frac{(1 - \alpha)}{D} (1 - 1_{\{\tau > T, V(T) > D\}}) \right) \right] |F_0
\]

\[
= \frac{(1 - \alpha)}{D} e^{-rT}E^Q \left[ V(T)(s_1^\beta_1(T) - S_2^\beta_2(T))^+ |F_0 \right] + e^{-rT}E^Q \left[ (S_1^\beta_1(T) - S_2^\beta_2(T))^+ 1_{\{\tau > T, V(T) > D\}} |F_0 \right]
\]

\[
- \frac{(1 - \alpha)}{D} e^{-rT}E^Q \left[ V(T)(s_1^\beta_1(T) - S_2^\beta_2(T))^+ 1_{\{\tau > T, V(T) > D\}} |F_0 \right],
\]

where \( \beta_i \) \((i = 1, 2)\) are positive constants, \( T \) is the maturity, \( \tau \) is the default time by \( \lambda(t), \alpha \) is the deadweight cost and \( D \) is the value of the option writer’s liability. Since the default time \( \tau \) is defined by \( P(\tau > t) = E^Q \left[ e^{-\int_0^\tau \lambda(s)ds} \right] \), the price \( C \) is represented as

\[
C = \frac{(1 - \alpha)}{D} e^{-rT}E^Q \left[ V(T)(s_1^\beta_1(T) - S_2^\beta_2(T))^+ |F_0 \right] + e^{-rT}E^Q \left[ e^{-\int_0^T \lambda(s)ds} (s_1^\beta_1(T) - S_2^\beta_2(T))^+ 1_{\{V(T) > D\}} |F_0 \right]
\]

\[
- \frac{(1 - \alpha)}{D} e^{-rT}E^Q \left[ e^{-\int_0^T \lambda(s)ds} V(T)(s_1^\beta_1(T) - S_2^\beta_2(T))^+ 1_{\{V(T) > D\}} |F_0 \right]
\]

\[
:= J_1 + J_2 - J_3.
\]

To obtain \( J_1, J_2 \) and \( J_3 \), we need the following lemmas.

**Lemma 1.** If we define a new measure \( \tilde{Q}_i \) \((i = 1, 2)\) equivalent to \( Q \) by the Radon–Nikodym derivative

\[
\frac{d\tilde{Q}_i}{dQ} = \frac{V(T)s_1^\beta_i(T)}{E[V(T)s_1^\beta_i(T)|F_0]} (i = 1, 2),
\]

then the following results are obtained under the measure \( \tilde{Q}_i \) \((i = 1, 2)\).

1. Under \( \tilde{Q}_i \) \((i = 1, 2)\), \( W_1^\tilde{Q}_i(T) = W_1(T) - \beta_i \sigma_{\rho_1}T - \sigma_{\rho_3} \gamma_3T \), \( W_2^\tilde{Q}_i(T) = W_2(T) - \beta_i \sigma_{\rho_2}T - \sigma_{\rho_3} \gamma_3T \), \( W_3^\tilde{Q}_i(T) = W_3(T) - \beta_i \sigma_{\rho_3}T - \sigma_{\rho_3} \gamma_3T \), \( W_4^\tilde{Q}_i(T) = W_4(T) - \beta_i \sigma_{\rho_4}T - \sigma_{\rho_3} \gamma_3T \) are standard Brownian motions (where \( \rho_{\mu} = 1 \)).
2. Under \( \tilde{Q}_1 \), \( N(T), N_1(T) \) and \( N_3(T) \) are independent Poisson processes with respective intensities

\[
\tilde{\lambda} = \lambda e^{(\beta_1 \delta_1 + \delta_2) \mu + \frac{1}{2} (\beta_1 \delta_1 + \delta_2)^2 \gamma^2}, \quad \tilde{\lambda}_1 = \lambda_1 e^{\beta_1 \mu + \frac{1}{2} (\beta_1 \gamma_1)^2}, \quad \tilde{\lambda}_3 = \lambda_3 e^{\beta_3 \mu + \frac{1}{2} (\beta_3 \gamma_3)^2}.
\]
(3) Under $\hat{Q}_2$, $N(T), N_2(T)$ and $N_3(T)$ are independent Poisson processes with respective intensities

$$\hat{\lambda}' = \lambda e^{(\beta_2 \delta_2 + \delta_2)\mu + \frac{1}{2}(\beta_2 \delta_2 + \delta_2)^2 \gamma^2}, \quad \hat{\lambda}_2' = \lambda_2 e^{\beta_2 \delta_2 + \frac{1}{2}(\beta_2 \gamma_2)\gamma^2}, \quad \hat{\lambda}_3' = \lambda_3 e^{\beta_3 \delta_3 + \frac{1}{2}(\beta_3 \gamma_3)\gamma^3}.$$ 

(4) Under $\hat{Q}_1$, $Z_k \sim N(\mu + (\beta_1 \delta_1 + \delta_2)\gamma^2, \gamma^2), \quad Z_k^1 \sim N(\mu_1 + \beta_1 \gamma_1^2, \gamma_1^2), \quad Z_k^3 \sim N(\mu_3 + \gamma_3^2, \gamma_3^2).$

(5) Under $\tilde{Q}_2$, $Z_k \sim N(\mu + (\beta_2 \delta_2 + \delta_3)\gamma^2, \gamma^2), \quad Z_k^2 \sim N(\mu_2 + \beta_2 \gamma_2^2, \gamma_2^2), \quad Z_k^3 \sim N(\mu_3 + \gamma_3^2, \gamma_3^2).$

(6) $N_2(T)$ and $Z_k^2$ have the same distributions under $\tilde{Q}_1$ and $Q$.

(7) $N_1(T)$ and $Z_k^1$ have the same distributions under $\tilde{Q}_2$ and $Q$.

**Proof.** The Radon–Nikodym derivative for the measure $\hat{Q}_i$ ($i = 1, 2$) is calculated as

$$\frac{d\hat{Q}_i}{dQ} = \frac{V(T)S_i^\beta(T)}{E[V(T)S_i^\beta(T)|F_0]} = \exp \left[ \beta_i \sigma_i W_i(T) + \sigma_3 W_3(T) - \frac{1}{2} (\beta_i^2 \sigma_i^2 + \sigma_3^2 + 2 \beta_i \sigma_i \sigma_3 \rho_{i3}) T + (\beta_i \delta_i + \delta_3) \sum_{k=1}^{N(T)} Z_k + \beta_i \sum_{k=1}^{N_i(T)} Z_k^1 + \sum_{k=1}^{N_3(T)} Z_k^3 \right]

- \lambda T \left( e^{(\beta_i \delta_i + \delta_3)\mu + \frac{1}{2}(\beta_i \delta_i + \delta_3)^2 \gamma^2} - 1 \right) - \lambda_1 T \left( e^{\beta_1 \mu_1 + \frac{1}{2}(\beta_1 \gamma_1)\gamma^2} - 1 \right) - \lambda_3 T \left( e^{\mu_3 + \frac{1}{2}(\beta_3 \gamma_3)\gamma^3} - 1 \right) \quad (i = 1, 2).$$

Then, we obtain the results immediately by Girsanov’s theorem (for more details, see [31,32]).

Similarly, we can obtain the following results.

**Lemma 2.** If we define a new measure $\tilde{Q}_i$ ($i = 1, 2$) equivalent to $Q$ by the Radon–Nikodym derivative

$$\frac{d\tilde{Q}_i}{dQ} = \frac{S_i^\beta(T)}{E[S_i^\beta(T)|F_0]} \quad (i = 1, 2),$$

then the following results are obtained under the measure $\tilde{Q}_i$ ($i = 1, 2$).

(1) Under $\tilde{Q}_1$ ($i = 1, 2$), $W_i^\beta(T) = W_1(T) - \beta_i \sigma_i \rho_{i1} T$, $W_2^\beta(T) = W_2(T) - \beta_2 \sigma_2 \rho_{22} T$, $W_4^\beta(T) = W_4(T) - \beta_2 \sigma_3 \rho_{23} T$, $W_4^\beta(T) = W_4(T) - \beta_1 \sigma_2 \rho_{42} T$ are standard Brownian motions (where $\rho_{ii} = 1$).

(2) Under $\tilde{Q}_1$, $N(T)$ and $N_1(T)$ are independent Poisson processes with respective intensities

$$\hat{\lambda} = \lambda e^{\beta_1 \delta_1 \mu + \frac{1}{2}(\beta_1 \delta_1 \gamma)^2}, \quad \hat{\lambda}_1 = \lambda_1 e^{\beta_1 \mu_1 + \frac{1}{2}(\beta_1 \gamma_1)^2}.$$

(3) Under $\tilde{Q}_2$, $N(T)$ and $N_2(T)$ are independent Poisson processes with respective intensities

$$\hat{\lambda}' = \lambda e^{\beta_2 \delta_2 \mu + \frac{1}{2}(\beta_2 \delta_2 \gamma)^2}, \quad \hat{\lambda}_2 = \lambda_2 e^{\beta_2 \mu_2 + \frac{1}{2}(\beta_2 \gamma_2)^2}.$$

(4) Under $\tilde{Q}_1$ ($i = 1, 2$),

$$Z_k \sim N(\mu + \beta_1 \delta_1 \gamma^2, \gamma^2), \quad Z_k^1 \sim N(\mu_1 + \beta_1 \gamma_1^2, \gamma_1^2).$$

(5) $N_2(T), N_3(T), Z_k^2$ and $Z_k^3$ have the same distributions under $\tilde{Q}_1$ and $Q$.

(6) $N_1(T), N_3(T), Z_k^1$ and $Z_k^3$ have the same distributions under $\tilde{Q}_2$ and $Q$. 
Lemma 3. If we define a new measure $\overline{Q}$ equivalent to $Q$ by the Radon–Nikodym derivative

$$
\frac{d\overline{Q}}{dQ} = \frac{e^{-\int_0^T \lambda(s) ds}}{E[e^{-\int_0^T \lambda(s) ds} | F_0]},
$$

then the following results are obtained under the measure $\overline{Q}$.

(1) Under $\overline{Q}$,

$$
\mathbb{W}_1(T) = W_1(T) + \frac{\sigma \rho P_{14}}{\sigma P_{14}} \int_0^T f(s, T, a) ds, \quad \mathbb{W}_2(T) = W_1(T) + \frac{\sigma \rho P_{24}}{\sigma P_{24}} \int_0^T f(s, T, a) ds,
$$

$$
\mathbb{W}_3(T) = W_3(T) + \frac{\sigma \rho P_{34}}{\sigma P_{34}} \int_0^T f(s, T, a) ds, \quad \mathbb{W}_4(T) = W_4(T) + \frac{\sigma}{\sigma} \int_0^T f(s, T, a) ds
$$

are standard Brownian motions, where $f(s, T, a) = 1 - e^{-a(T-s)}$.

(2) Under $\overline{Q}$, $N(T), N_1(T), N_2(T)$ and $N_3(T)$ are independent Poisson processes with intensities $\lambda, \lambda_1, \lambda_2$, and $\lambda_3$, respectively.

(3) $Z^1_k, Z^2_k$ and $Z^3_k$ have the same distributions under $\overline{Q}$ and $Q$.

Applying the measure change technique repeatedly with the above lemmas, we derive the pricing formula of power exchange options with the hybrid credit risk under the jump-diffusion model in the following propositions.

Proposition 1. In the proposed framework, $J_1$ is given by

$$
J_1 = \frac{1 - \alpha}{D} e^{-rT} \left( \sum_{n=0}^{\infty} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \frac{(\lambda T)^n (\lambda_1 T)^n_1 (\lambda_2 T)^n_2}{n! n_1! n_2!} e^{-(\lambda + \lambda_1 + \lambda_2)T} K_1 \right.
$$

$$
- \sum_{n=0}^{\infty} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \frac{(\lambda T)^n (\lambda_1 T)^n_1 (\lambda_2 T)^n_2}{n! n_1! n_2!} e^{-(\lambda + \lambda_1 + \lambda_2)T} K_2), \quad \text{(10)}
$$

where $\lambda, \lambda’, \bar{\lambda}$, and $\lambda^*_1 (i = 1, 2)$ are defined in Lemma 1, and $K_1$ and $K_2$ are as follows.

$$
K_1 = E \left[ S^{B_1}_1(T) V(T) \right] N \left( A_1 \sqrt{B_1} \right) \quad \text{and} \quad K_2 = E \left[ S^{B_2}_2(T) V(T) \right] N \left( A_2 \sqrt{B_2} \right),
$$

where $N(\cdot)$ is the cumulative standard normal distribution function and

$$
E \left[ S^{B_1}_1(T) V(T) \right] = S^{B_1}_1(0) V(0) \exp \left[ \left( \beta_1 r + \frac{1}{2} \beta_1 \sigma_1^2 (\beta_1 - 1) - k_1 \beta_1 - k_3 + \beta_3 \sigma_3 \rho_3 \right) T \right.
$$

$$
+ \lambda \left( e^{(\beta_1 \delta_1 + \delta_3) T} - 1 \right) + \lambda_1 \left( e^{(\beta_1 \delta_1 + \delta_3) T} - 1 \right) + \lambda_2 \left( e^{(\beta_1 \delta_1 + \delta_3) T} - 1 \right) \right] (i = 1, 2),
$$

$$
A_1 = \ln \frac{S^{B_1}_1(0)}{S^{B_2}_1(0)} + \beta_1 \left( r - \frac{1}{2} \sigma_1^2 - k_1 \right) T + \left( \beta_1 \sigma_1^2 + \beta_1 \sigma_3 \rho_3 \right) T + \sum_{i=1}^{n_1} \delta_i \left( \mu + \beta_1 \delta_1 \gamma^2 + \delta_3 \gamma^2 \right) + \sum_{i=1}^{n_1} \beta_1 \left( \mu + \beta_1 \gamma^2 \right)
$$

$$
- \beta_2 \left( r - \frac{1}{2} \sigma_2^2 - k_2 \right) T - (\beta_1 \beta_2 \sigma_1 \rho_{12} + \beta_2 \sigma_2 \rho_{23}) T - n_2 \beta_2 \left( \mu + \beta_2 \delta_2 \gamma^2 + \delta_3 \gamma^2 \right) - n_2 \beta_2 \mu_2,
$$

$$
B_1 = (\beta_1 \rho_1^2 + \beta_2 \rho_2^2 - 2 \beta_1 \beta_2 \rho_1 \rho_{12}) T + \sum_{i=1}^{n_1} \delta_i \left( \mu + \beta_1 \delta_1 \gamma^2 + \delta_3 \gamma^2 \right) + \sum_{i=1}^{n_1} \beta_1 \left( \mu + \beta_1 \gamma^2 \right)
$$

$$
A_2 = \ln \frac{S^{B_1}_2(0)}{S^{B_2}_2(0)} + \beta_1 \left( r - \frac{1}{2} \sigma_1^2 - k_1 \right) T + \left( \beta_1 \beta_2 \sigma_1 \rho_{12} + \beta_1 \sigma_3 \rho_{13} \right) T + \sum_{i=1}^{n_1} \delta_i \left( \mu + \beta_2 \delta_2 \gamma^2 + \delta_3 \gamma^2 \right) + \sum_{i=1}^{n_1} \beta_1 \left( \mu + \beta_1 \gamma^2 \right)
$$

$$
- \beta_2 \left( r - \frac{1}{2} \sigma_2^2 - k_2 \right) T - (\beta_1 \rho_1^2 + \beta_2 \rho_2^2) T - n_2 \beta_2 \left( \mu + \beta_2 \delta_2 \gamma^2 + \delta_3 \gamma^2 \right) - n_2 \beta_2 \left( \mu + \beta_2 \gamma^2 \right).
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Proof. First, we divide $I_1$ into two expectations.

$$I_1 = \frac{(1-\alpha)}{D} e^{-rT} \mathbb{E}^Q \left[ V(T) (S_{n_1}^{\beta_1}(T) - S_{n_2}^{\beta_2}(T)) \mathbb{I}[F_0] \right]$$

$$= \frac{(1-\alpha)}{D} e^{-rT} \mathbb{E}^Q \left[ V(T) S_{n_1}^{\beta_1}(T) \mathbb{I}_{\{S_{n_1}^{\beta_1}(T) > S_{n_2}^{\beta_2}(T)\}} \mathbb{I}[F_0] \right]$$

$$- \frac{(1-\alpha)}{D} e^{-rT} \mathbb{E}^Q \left[ V(T) S_{n_2}^{\beta_2}(T) \mathbb{I}_{\{S_{n_1}^{\beta_1}(T) > S_{n_2}^{\beta_2}(T)\}} \mathbb{I}[F_0] \right]$$

$$:= I_1 - I_2$$

To calculate $I_1$, we define a new measure $\tilde{Q}_1$ such that

$$\frac{d\tilde{Q}_1}{dQ} = \frac{V(T) S_{n_1}^{\beta_1}(T)}{\mathbb{E}[V(T) S_{n_1}^{\beta_1}(T)|F_0]}.$$

Under the measure $\tilde{Q}_1$,

$$I_1 = \frac{(1-\alpha)}{D} e^{-rT} \mathbb{E}^Q \left[ V(T) S_{n_1}^{\beta_1}(T) | F_0 \right] \mathbb{E}_{\tilde{Q}_1} \left[ \mathbb{I}_{\{S_{n_1}^{\beta_1}(T) > S_{n_2}^{\beta_2}(T)\}} | F_0 \right],$$

(12)

where the first expectation in (12) can be calculated directly. To complete the calculation of $I_1$, we have to obtain the second expectation in (12) by conditioning over the exact number of Poisson jumps. Let $\Omega_1 = \{N(t) = n, N_1(t) = n_1, N_2(T) = n_2\}$; then, the probability that Poisson random variables take these values is defined as

$$\tilde{P}_1(\Omega_1) = \frac{(\lambda T)^n (\lambda_1 T)^{n_1} (\lambda_2 T)^{n_2} e^{-(\lambda + \lambda_1 + \lambda_2)T}}{n! n_1! n_2!}.$$

From Lemma 1, the second expectation in (12) becomes

$$\mathbb{E}_{\tilde{Q}_1} \left[ \mathbb{I}_{\{S_{n_1}^{\beta_1}(T) > S_{n_2}^{\beta_2}(T)\}} | F_0 \right] = \sum_{n=0}^{\infty} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \mathbb{P}_{\tilde{Q}_1} \left( S_{n_1}^{\beta_1}(T) > S_{n_2}^{\beta_2}(T) \right) | N(n) = n, N_1(t) = n_1, N_2(T) = n_2$$

$$= \sum_{n=0}^{\infty} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \mathbb{P}_{\tilde{Q}_1} \left( S_{n_1}^{\beta_1}(T) > S_{n_2}^{\beta_2}(T) \right) \mathbb{I}_{\{S_{n_1}^{\beta_1}(T) > S_{n_2}^{\beta_2}(T)\}}$$

By the properties of the distributions under the measure $\tilde{Q}_1$, the second expectation in (12) is calculated as

$$\mathbb{E}_{\tilde{Q}_1} \left[ \mathbb{I}_{\{S_{n_1}^{\beta_1}(T) > S_{n_2}^{\beta_2}(T)\}} | F_0 \right] = \sum_{n=0}^{\infty} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \mathbb{N} \left( \frac{A_1}{\sqrt{B_1}} \right) \tilde{P}_1(\Omega_1).$$

(13)

Then, $I_1$ is completed. To calculate $I_2$, we define a new measure $\tilde{Q}_2$ such that

$$\frac{d\tilde{Q}_2}{dQ} = \frac{V(T) S_{n_2}^{\beta_2}(T)}{\mathbb{E}[V(T) S_{n_2}^{\beta_2}(T)|F_0]}.$$

Then, by using the results in Lemma 1 under the measure $\tilde{Q}_2$, $I_2$ can be derived similarly to $I_1$. \(\Box\)
Proposition 2. In the proposed framework, \( J_2 \) is given by

\[
J_2 = e^{-rT} \left( \sum_{n=0}^{\infty} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} \frac{(\lambda T)^n (\lambda T)^{n_1} (\lambda T)^{n_2} (\lambda T)^{n_3}}{n! n_1! n_2! n_3!} e^{-\left(\lambda + \lambda_1 + \lambda_2 + \lambda_3\right)T} K_3 \right)
\]

\[
- \sum_{n=0}^{\infty} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} \frac{\sqrt{A} (\lambda T)^n (\lambda T)^{n_1} (\lambda T)^{n_2} (\lambda T)^{n_3}}{n! n_1! n_2! n_3!} e^{-\left(\lambda + \lambda_1 + \lambda_2 + \lambda_3\right)T} K_4, \tag{14}
\]

where \( \lambda, \lambda', \lambda_i \) and \( \lambda'_i \) \( (i = 1, 2) \) are defined in Lemma 2, and \( K_3 \) and \( K_4 \) are as follows.

\[
K_3 = E \left[ e^{-\int_0^T \lambda(s) \, ds} \right] E \left[ \mathbb{E}^T \left( S_1^\beta \right) \right] N_2 \left( \frac{A_3}{\sqrt{B_3}}, \frac{A_4}{\sqrt{B_4}} \right),
\]

\[
K_4 = E \left[ e^{-\int_0^T \lambda(s) \, ds} \right] E \left[ \mathbb{E}^T \left( S_2^\beta \right) \right] N_2 \left( \frac{A_5}{\sqrt{B_3}}, \frac{A_6}{\sqrt{B_4}} \right),
\]

where \( N_2(\cdot) \) is the cumulative bivariate normal distribution function and

\[
E \left[ e^{-\int_0^T \lambda(s) \, ds} \right] = \text{exp} \left[ -BT - \frac{t(0) - b}{a} (1 - e^{-aT}) + \frac{e^{aT}}{2a^2} \int_0^T (1 - e^{-a(T-s)})^2 \, ds \right], f(s, T, a) = 1 - e^{-a(T-s)},
\]

\[
E \mathbb{E}^T \left( S_i^\beta \right) = S_i^\beta(0) \text{exp} \left[ \left( -\frac{1}{2} \beta_i \sigma_i^2 (\beta_i - 1) - k_i \beta_i \right) T + \frac{\beta_i \sigma_i^2 \rho_i^4}{a} \int_0^T f(s, T, a) \, ds \right]
\]

\[
+ \lambda \left( s \beta_i \mu + \frac{1}{2} \beta_i \gamma_i^2 - 1 \right) T + \lambda_1 \left( s \beta_i \mu + \frac{1}{2} \beta_i \gamma_i^2 - 1 \right) T \] \( (i = 1, 2),
\]

\[
A_3 = \ln \frac{S_1^\beta(0)}{S_2^\beta(0)} + \beta_1 \left( r - \frac{1}{2} \sigma_1^2 - k_1 \right) T + \beta_1^2 \sigma_1^2 T - \frac{\beta_1 \sigma_1^2 \rho_1^4}{a} \int_0^T f(s, T, a) \, ds - n (\beta_2 \beta_2 - \beta_1 \beta_1) (\mu + \beta_1 \gamma_1^2)
\]

\[
- \beta_2 \left( r - \frac{1}{2} \sigma_2^2 - k_2 \right) T - \beta_2 \beta_2 \sigma_2 \rho_2 T + \frac{\beta_2 \sigma_2 \rho_2}{a} \int_0^T f(s, T, a) \, ds - n \beta_1 (\mu + \beta_1 \gamma_1^2) - n \beta_2 \beta_2 T,
\]

\[
A_4 = \ln \frac{V(0)}{D} + \left( r - \frac{1}{2} \sigma_3^2 - k_3 \right) T + \beta_3 \sigma_3^2 T - \frac{\beta_3 \sigma_3 \rho_3^4}{a} \int_0^T f(s, T, a) \, ds + n \delta_3 (\mu + \beta_1 \gamma_1^2) + n \delta_3 T,
\]

\[
B_3 = (\beta_1^2 \sigma_1^2 + \beta_2^2 \sigma_2^2 - 2 \beta_1 \beta_2 \sigma_1 \rho_1 \rho_2) T + n (\beta_2 \beta_2 - \beta_1 \beta_1) \gamma_2^2 + n \beta_3 \gamma_1^2 T + n \beta_2^2 \gamma_2^2,
\]

\[
B_4 = \sigma_3^2 T + n \delta_3 \gamma_2^2 + n \delta_3 \gamma_3^2,
\]

\[
A_5 = \ln \frac{S_1^\beta(0)}{S_2^\beta(0)} + \beta_1 \left( r - \frac{1}{2} \sigma_1^2 - k_1 \right) T + \beta_1 \beta_2 \sigma_1 \rho_1 T - \frac{\beta_1 \sigma_1 \rho_1^4}{a} \int_0^T f(s, T, a) \, ds - n (\beta_2 \beta_2 - \beta_1 \beta_1) (\mu + \beta_2 \gamma_2^2)
\]

\[
- \beta_2 \left( r - \frac{1}{2} \sigma_2^2 - k_2 \right) T - \beta_2^2 \sigma_2^2 T + \frac{\beta_2 \sigma_2 \rho_2}{a} \int_0^T f(s, T, a) \, ds - n \beta_1 (\mu + \beta_2 \gamma_2^2) + n \beta_1 \mu T,
\]

\[
A_6 = \ln \frac{V(0)}{D} + \left( r - \frac{1}{2} \sigma_3^2 - k_3 \right) T + \beta_3 \sigma_3 T - \frac{\beta_3 \sigma_3 \rho_3^4}{a} \int_0^T f(s, T, a) \, ds + n \delta_3 (\mu + \beta_2 \gamma_2^2) + n \delta_3 T,
\]

\[
\hat{\rho} = \frac{(\beta_1 \sigma_3 \rho_{13} - \beta_2 \sigma_2 \sigma_3 \rho_{23}) T - n \delta_3 (\beta_2 \beta_2 - \beta_1 \beta_1) \gamma_2^2}{\sqrt{B_3} \sqrt{B_4}}.
\]

Proof. \( J_2 \) in (6) is divided into the following expectations.

\[
\begin{align*}
I_2 &= e^{-rT} E \left[ \left. e^{-\int_0^T \lambda(s) \, ds} S_1^\beta(T) \mathbb{I}_{\left( S_1^\beta(T) > S_2^\beta(T), V(T) > D \right)} \right| F_0 \right] \\
&= e^{-rT} E \left[ \left. e^{-\int_0^T \lambda(s) \, ds} S_2^\beta(T) \mathbb{I}_{\left( S_1^\beta(T) > S_2^\beta(T), V(T) > D \right)} \right| F_0 \right] \\
&= I_3 - I_4 \tag{15}
\end{align*}
\]
We now calculate $I_3$ using Lemmas 2 and 3. We first adopt the new measure $\overline{Q}$ equivalent to measure $Q$ defined by
\[
\frac{d\overline{Q}}{dQ} = \frac{e^{-\int_0^T \lambda(s)ds}}{E[e^{-\int_0^T \lambda(s)ds}|F_0]}.
\]

Then, $I_3$ becomes
\[
I_3 = e^{-rT}E[Q]\left[\hat{S}_1^{\beta_1}(T)1_{\{\hat{S}_1^{\beta_1}(T) > \hat{S}_2^{\beta_2}(T), V(T) > D\}|F_0}\right].
\]

(16)

Since the first expectation in (16) is calculated directly, we focus on the second expectation in (16). By Lemma 3, the dynamic of $S_1^{\beta_1}(t)$ under the measure $\overline{Q}$ is represented as
\[
S_1^{\beta_1}(t) = S_1^{\beta_1}(0)\exp\left\{\beta_1(t - \frac{1}{2}T_1^2 - k_1) + \beta_1 \overline{W}_1(t) - \frac{\beta_1 \sigma_1 \lambda_1}{a} \int_0^t f(s, a_s)ds + \delta_1 \sum_{k=1}^{N(t)} Z_k + \sum_{k=1}^{N(t)} Z_k^1\right\},
\]

where $f(s, T, a) = 1 - e^{-a(T-s)}$. With the dynamic of $S_1^{\beta_1}(t)$, we define a new measure $\overline{Q}_1$ such that
\[
\frac{d\overline{Q}_1}{dQ} = \frac{S_1^{\beta_1}(T)}{E[Q]\{S_1^{\beta_1}(T)|F_0\}}.
\]

Then, the second expectation in (16) becomes
\[
E[Q]\{\hat{S}_1^{\beta_1}(T)1_{\{\hat{S}_1^{\beta_1}(T) > \hat{S}_2^{\beta_2}(T), V(T) > D\}|F_0}\} = E[\overline{Q}\{\hat{S}_1^{\beta_1}(T)|F_0\}] E[\overline{Q}_1\{1_{\{\hat{S}_1^{\beta_1}(T) > \hat{S}_2^{\beta_2}(T), V(T) > D\}|F_0}\}].
\]

(17)

The first expectation in (17) is calculated directly under the measure $\overline{Q}$. Thus, we focus on the calculation of the second expectation in (17) with the measure $\overline{Q}_1$. Let $\Omega = \{N(t) = n, N_1(t) = n_1, N_2(T) = n_2, N_3(T) = n_3\}$, and then the probability that Poisson random variables take these values is defined as
\[
\overline{P}_1(\Omega) = \frac{\lambda T^n (\lambda_1 T)^{n_1} (\lambda_2 T)^{n_2} (\lambda_3 T)^{n_3}}{n! n_1! n_2! n_3!} e^{-(\lambda + \lambda_1 + \lambda_2 + \lambda_3)T}.
\]

Applying Lemma 2, under the measure $\overline{Q}_1$, we have
\[
E[\overline{Q}_1\{1_{\{\hat{S}_1^{\beta_1}(T) > \hat{S}_2^{\beta_2}(T), V(T) > D\}|F_0}\}]
\]
\[
= \sum_{n=0}^{\infty} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} \overline{P}_1(\{\hat{S}_1^{\beta_1}(T) > \hat{S}_2^{\beta_2}(T), V(T) > D\}|N(t) = n, N_1(t) = n_1, N_2(T) = n_2, N_3(T) = n_3\})
\]
\[ \sum_{n=0}^{\infty} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} p_{z_1} \left( \begin{array}{c} \beta_2 \sigma_2 W_{1}^{\hat{G}_1}\left(T\right) - \beta_1 \sigma_1 W_{1}^{\hat{G}_1}\left(T\right) + \left( \beta_2 \delta_2 - \beta_1 \delta_1 \right) \sum_{k=1}^{n} Z_k + \beta_2 \sum_{k=1}^{n_2} Z_k^2 - \beta_1 \sum_{k=1}^{n_1} Z_k^1 < \end{array} \right) \]

\[ \ln \frac{S_1^{\delta_1}(0)}{S_2^{\delta_2}(0)} + \beta_1 \left( r - \frac{1}{2} \sigma_1^2 - k_1 \right) T + \beta_2^2 \sigma_2^2 T - \frac{\beta_1 \sigma_1 \sigma_2 \rho_{14}}{a} \int_0^T f(s, T, a) ds \]

\[ -\beta_2 \left( r - \frac{1}{2} \sigma_2^2 - k_2 \right) T - (\beta_1 \beta_2 \sigma_1 \sigma_2 \rho_{12} + \beta_2 \sigma_2^2 \rho_{23}) T + \frac{\beta_2^2 \sigma_2 \rho_{24}}{a} \int_0^T f(s, T, a) ds, \]

\[ -\beta_3 \sigma_3 W_{3}^{\hat{G}_3}\left(T\right) - \delta_3 \sum_{k=1}^{n} Z_k - \sum_{k=1}^{n_2} Z_k^3 < \ln \frac{V(0)}{D} + \left( r - \frac{1}{2} \sigma_3^2 - k_3 + \beta_1 \sigma_1 \sigma_3 \rho_{13} \right) T - \frac{\sigma_3 \rho_{34}}{a} \int_0^T f(s, T, a) ds \]

Since

\[ \mathbb{E} \hat{G}_1[z_1] = n(\beta_2 \delta_2 - \beta_1 \delta_1)(\mu + \beta_1 \delta_1 \gamma^2) - n_1 \beta_1(\mu_1 + \beta_1 \gamma_1^2) + n_2 \beta_2 \mu_2, \]

\[ \mathbb{E} \hat{G}_2[z_2] = -n \delta_3(\mu + \beta_1 \delta_1 \gamma^2) - n_3 \mu_3, \]

\[ \text{Var} \hat{G}_1[z_1] = (\beta_1^2 \sigma_1^2 + \beta_2^2 \sigma_2^2 - 2 \beta_1 \beta_2 \sigma_1 \sigma_2 \rho_{12}) T + n(\beta_2 \delta_2 - \beta_1 \delta_1)^2 \gamma^2 + n_1 \beta_1^2 \gamma_1^2 + n_2 \beta_2^2 \gamma_2^2, \]

\[ \text{Var} \hat{G}_2[z_2] = \sigma_3^2 T + n_3 \delta_3 \gamma^2 + n_3 \gamma_3^2, \]

\[ \text{Cov} \hat{G}_1[z_1, z_2] = (\beta_1 \sigma_1 \sigma_3 \rho_{13} - \beta_2 \sigma_2 \sigma_3 \rho_{23}) T - n \delta_3(\beta_2 \delta_2 - \beta_1 \delta_1) \gamma^2, \]

we have

\[ \mathbb{E} \hat{G}_1 \left[ \mathbb{I}_{\left\{ S_1^{\delta_1}(T) > S_2^{\delta_2}(T), V(T) > D \right\}} \right] = \sum_{n=0}^{\infty} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} N_2 \left( \frac{A_3}{\sqrt{B_3}}, \frac{A_4}{\sqrt{B_4}}, \tilde{\rho} \right) \hat{P}_1(\Omega), \quad (18) \]

where \( \tilde{\rho} = \text{Cov} \hat{G}_1[z_1, z_2] / (\text{Var} \hat{G}_1[z_1] \cdot \text{Var} \hat{G}_2[z_2]), \) and \( I_3 \) is obtained.

In a similar way, \( I_4 \) can be derived. Then, the proof is completed. \( \square \)

For the calculation of \( I_3 \), we follow similar steps as in Proposition 2. Applying the measure change technique with Lemmas 1 and 3, \( I_3 \) is derived.

**Proposition 3.** In the proposed framework, \( I_3 \) is given by

\[ I_3 = \frac{1 - a}{D} e^{-rT} \left( \sum_{n=0}^{\infty} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} \frac{(\hat{\lambda} T)^n (\hat{\lambda}_1 T)^{n_1} (\hat{\lambda}_2 T)^{n_2} (\hat{\lambda}_3 T)^{n_3}}{n! n_1! n_2! n_3!} e^{-\left(\hat{\lambda} + \hat{\lambda}_1 + \hat{\lambda}_2 + \hat{\lambda}_3\right) T} K_5 \right) \]

\[ \sum_{n=0}^{\infty} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} \frac{(\hat{\lambda}_1 T)^n (\hat{\lambda}_2 T)^{n_1} (\hat{\lambda}_3 T)^{n_2}}{n! n_1! n_2! n_3!} e^{-\left(\hat{\lambda}_1 + \hat{\lambda}_2 + \hat{\lambda}_3\right) T} K_6, \quad (19) \]

where \( \hat{\lambda}, \hat{\lambda}', \hat{\lambda}_i \) and \( \hat{\lambda}_i' \) (\( i = 1, 2 \)) are defined in Lemma 1, and \( K_5 \) and \( K_6 \) are as follows.

\[ K_5 = \mathbb{E} \left[ e^{-\int_0^T \hat{\lambda}(s) ds} \right] \mathbb{E}^\mathcal{Q} \left[ S_1^{\delta_1}(T) V(T) \right] N_2 \left( \frac{A_7}{\sqrt{B_3}}, \frac{A_8}{\sqrt{B_4}}, \tilde{\rho} \right), \]

\[ K_6 = \mathbb{E} \left[ e^{-\int_0^T \hat{\lambda}'(s) ds} \right] \mathbb{E}^\mathcal{Q} \left[ S_2^{\delta_2}(T) V(T) \right] N_2 \left( \frac{A_9}{\sqrt{B_3}}, \frac{A_{10}}{\sqrt{B_4}}, \tilde{\rho} \right), \]

where \( N_2(\cdot) \) is the cumulative bivariate normal distribution function,
\[
E\left[ S^\delta_1(T) V(T) \right] = S^\delta_1(0) V(0) \exp \left[ \left( r(\beta_1 + 1) - \frac{1}{2}(\beta_1^2 + \sigma^2_3) - k_1 - k_3 \right) T - \frac{\beta_1 \sigma^4_3 \rho_{14}}{a} \int_0^T f(s, T, a) ds \right. \\
- \frac{\sigma^2_3 \rho_{34}}{a} \int_0^T f(s, T, a) ds + \lambda \left( e^{(\beta_1 + \delta_3) T + \frac{1}{2}(\beta_1 + \delta_3)^2 T^2} - 1 \right) T \\
+ \lambda T \left( e^{\beta_1 T + \frac{1}{2} \beta_1^2 T^2} - 1 \right) \left( i = 1, 2, \right)
\]

\[
A_7 = \ln \frac{S^\delta_1(0)}{S^\rho_2(0)} + \beta_1 \left( r - \frac{1}{2} \sigma^2_1 - k_1 \right) T + (\beta_1^2 \sigma^2_1 + \beta_1 \sigma_1 \rho_{13}) T - \frac{\beta_1 \sigma^4_3 \rho_{14}}{a} \int_0^T f(s, T, a) ds \\
- \beta_2 \left( r - \frac{1}{2} \sigma^2_2 - k_2 \right) T - (\beta_1 \beta_2 \sigma_1 \rho_{12} + \beta_2 \sigma_2 \rho_{23}) T + \frac{\beta_2 \sigma^2_2 \rho_{24}}{a} \int_0^T f(s, T, a) ds \\
- n(\beta_2 \beta_2 - \beta_1 \delta_1)(\mu + (\beta_1 \delta_1 + \delta_3) T) + n_1 \beta_1 (\mu_1 + \beta_1 \gamma_1^2) - n_2 \beta_2 \mu_2,
\]

\[
A_8 = \ln \frac{V(0)}{D} + \left( r - \frac{1}{2} \sigma^2_3 - k_3 \right) T + (\beta_1 \sigma_3 \rho_{13} + \sigma^2_3 \rho_{13}) T - \frac{\sigma^2_3 \rho_{34}}{a} \int_0^T f(s, T, a) ds \\
+ n_3 (\mu + (\beta_1 \delta_1 + \delta_3) T) + n_3 (\mu_3 + \gamma_3^2),
\]

\[
A_9 = \ln \frac{S^\delta_1(0)}{S^\rho_2(0)} + \beta_1 \left( r - \frac{1}{2} \sigma^2_1 - k_1 \right) T + (\beta_1 \beta_2 \sigma_1 \rho_{12} + \beta_1 \sigma_1 \rho_{13} \rho_{13}) T - \frac{\beta_1 \sigma^4_3 \rho_{14}}{a} \int_0^T f(s, T, a) ds \\
- \beta_2 \left( r - \frac{1}{2} \sigma^2_2 - k_2 \right) T - (\beta_2^2 \sigma^2_2 \rho_{23} + \beta_2 \sigma_2 \rho_{23}) T + \frac{\beta_2 \sigma^2_2 \rho_{24}}{a} \int_0^T f(s, T, a) ds \\
- n(\beta_2 \beta_2 - \beta_1 \delta_1)(\mu + (\beta_2 \beta_2 + \delta_3) T) + n_1 \beta_1 (\mu_1 + \beta_2 \mu_2 + \beta_2 \gamma_2^2) + n_2 \beta_2 \mu_2,
\]

\[
A_{10} = \ln \frac{V(0)}{D} + \left( r - \frac{1}{2} \sigma^2_3 - k_3 \right) T + (\beta_2 \sigma_2 \rho_{23} + \sigma^2_3 \rho_{23}) T - \frac{\sigma^2_3 \rho_{34}}{a} \int_0^T f(s, T, a) ds \\
+ n_3 (\mu + (\beta_2 \delta_2 + \delta_3) T) + n_3 (\mu_3 + \gamma_3^2),
\]

and \( E \left[ e^{-\int_0^T \lambda(s) ds} \right] \), \( B_3, B_4 \), \( f(s, T, a) \), \( \hat{\rho} \) are defined in Proposition 2.

From Propositions 1–3, the explicit pricing formula of power exchange options with the hybrid credit risk and jump diffusion is completed.

4. Concluding Remarks

In this paper, we deal with the valuation of power exchange options with a hybrid credit risk. To construct the hybrid credit risk model, we combine the reduced-form model of Fard [15] and the structural model of Klein [3]. This hybrid model was considered to price vulnerable exchange options in Kim [20]. In fact, the main contribution of this paper is an extension of [20]. To extend the work of [20], we assume that underlying assets follow the correlated jump-diffusion models, and consider vulnerable exchange options in which the payoff at expiry is related to the power of the stock dynamic. For the jump-diffusion model, which is a generalization of the Black–Scholes model, which follows the Geometric Brownian Motion (GBM) process, we adopt the model of Merton [27]. To obtain the pricing formula based on the probabilistic approach, the measure change technique is employed. Finally, we provide the explicit pricing formula of power exchange options with the hybrid credit risk using the cumulative normal functions and infinite sums. This result may apply to the pricing of other types of options with a hybrid credit risk under jump-diffusion model.

Although we derive the option pricing formula mathematically well under the proposed model, this paper has several limitations. Thus, we propose the following directions of future research to overcome the limitations. Firstly, it would be interesting to carry out a computational study for vulnerable power exchange options with jump diffusion to verify the accuracy and efficiency of the formula. The second is to find the impacts of the hybrid credit risk with respect to some parameters. In addition, our model
can be extended with the stochastic volatility models. The above will be considered in future research.

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