FOLIATION OF AN ASYMPTOTICALLY FLAT END BY CRITICAL CAPACITORS

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Abstract. We construct a foliation of an asymptotically flat end of a Riemannian manifold by hypersurfaces which are critical points of a natural functional arising in potential theory. These hypersurfaces are perturbations of large coordinate spheres, and they admit solutions of a certain over-determined boundary value problem involving the Laplace-Beltrami operator. In a key step we must invert the Dirichlet-to-Neumann operator, highlighting the nonlocal nature of our problem.

Keywords: Over-determined problem, foliation.

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1. Introduction

Riemannian manifolds with asymptotically flat ends play an important role in general relativity and cosmology, and so their general properties are of great interest. In particular, it is often useful to foliate an asymptotically flat end with special surfaces. Huisken and Yau [4] famously proved one can foliate a three-dimensional, asymptotically flat end with constant mean curvature spheres. Furthermore they prove these spheres share a common center, which one can take as the physical center of mass of the system. Previously, R. Ye [12] had shown one can foliate an asymptotically flat end in any dimension \( n \geq 3 \) provided the mass at infinity is nonzero.

Subsequently, others have found special foliations by constant expansion surfaces [6], by Willmore surfaces [5], and by isoperimetric surfaces [1]. Here we investigate surfaces which are critical points of the Newton capacity. Recall that, if \( K \subset \mathbb{R}^n \), with \( n \geq 3 \), is a compact set, one can define its Newton capacity as

\[
\text{Cap}(K) = \frac{1}{n(n-2)\omega_n} \inf \left\{ \int_{\mathbb{R}^n} |\nabla u|^2 dx : u \in H^1(\mathbb{R}^n), u|_K \equiv 1 \right\},
\]

where \( \omega_n \) is the Euclidean volume of an \( n \)-dimensional unit ball and \( H^1(\mathbb{R}^n) \) is the Sobolev space of functions with one weak derivative in \( L^2(\mathbb{R}^n) \). Standard results in potential theory imply this infimum is realized by the equilibrium potential function \( U_K \), which solves the boundary value problem

\[
\Delta_0 U_K = 0 \text{ in } \Omega = \mathbb{R}^n \setminus K, \quad U_K|_{\partial K} = 1, \quad \lim_{|y| \to \infty} U_K(y) = 0
\]
where $\Delta_0$ is the usual, flat Laplacian. Moreover, the solution to (1.2) is unique among all functions which satisfy an appropriate decay condition.

It is straightforward to generalize both (1.1) and (1.2) to the setting of a compact set $K$ in a complete, noncompact Riemannian manifold $(M, g)$ with an asymptotically flat end. As discussed in [9], Newton capacity plays a role in the study of scalar curvature and conformal geometry.

The functional $\text{Cap}$ is not scale-invariant in Euclidean space, one should not expect it to have critical points as a domain functional. Thus it is natural to seek critical, and even extremal, domains either subject to a constraint or of a modified functional which is scale invariant. One can normalize $\text{Cap}$ using the volume of $K$ or the surface area of $\partial K$; both choices are natural and have roots in physics and potential theory [3]. Below we will seek critical sets of a volume-normalized functional, which leads us to the over-determined boundary value problem

$$\Delta_0 u = 0 \text{ on } \mathbb{R}^n \setminus K, \quad u|_{\partial K} = 1, \quad \lim_{|x| \to \infty} u(x) = 0, \quad \frac{\partial u}{\partial \eta}|_{\partial K} = \Lambda,$$

where $\Lambda$ is a constant. See Section A for a derivation of (1.3) as the Euler-Lagrange equation of our normalized domain functional. This computation is standard, but we include it in Appendix A for the reader's convenience.

Of course, one can also choose to normalize using the surface area of $\partial K$, which leads one to a slightly different over-determined boundary value problem, namely

$$\Delta_0 u = 0 \text{ on } \mathbb{R}^n \setminus K, \quad u|_{\partial K} = 1, \quad \lim_{|x| \to \infty} u(x) = 0, \quad \frac{\partial u}{\partial \eta}|_{\partial K} = \Lambda H,$$

where $H$ is the mean curvature of $\partial K$. We derive this Euler-Lagrange equation as well, even though we do not require it.

A classical theorem of Serrin [10] implies that the only critical capacitors in Euclidean space are round sphere, but one expects the situation to be more complicated in a general Riemannian manifold.

We introduce some notation so that we can state our main theorem. Our setting is that of a Riemannian manifold $(M, g)$ of dimension $n \geq 3$ with one asymptotically flat end. In other words, there exists a compact set $K \subset M$ and a diffeomorphism

$$\Phi : \mathbb{R}^n \setminus \mathbf{B} \to M \setminus K,$$

such that in these coordinates

$$g_{ij}(y) = (1 + \sigma |y|^{-n}) \delta_{ij} + h_{ij}(y), \quad |h_{ij}(y)| = O(|y|^{-n}), \quad \partial^k h_{ij}(y) = O(|y|^{-k-n}).$$

Here $\mathbf{B}$ is the unit ball in $\mathbb{R}^n$ centered at the origin and $\partial^k$ represents any collection of partial derivatives of order less than or equal to $k$, with $k \in \{1, 2, 3, 4\}$.

**Theorem 1.** Let $(M, g)$ be a Riemannian manifold of dimension $n \geq 3$ with one asymptotically flat end $M \setminus K$, parameterized as in (1.4) and (1.5). Then there exists $\rho_0 > 1$ and compact sets $K_\rho$ indexed by $\rho \in (\rho_0, \infty)$ such that the domains $\Omega_\rho = M \setminus K_\rho$
are critical capacitors. In other words, there exist functions $\bar{u}_\rho$ which solve the overdetermined boundary value problem

$$\begin{align*}
\Delta_g \bar{u}_\rho &= 0 & \text{in } & \Omega_\rho \\
\bar{u}_\rho &= 1 & \text{on } & \partial\Omega_\rho = \partial K_\rho \\
\lim_{|y| \to \infty} \bar{u}_\rho(\varphi(y)) &= 0 \\
\frac{\partial \bar{u}_\rho}{\partial \eta} &= C(\rho, \sigma, n) & \text{on } & \partial\Omega_\rho = \partial K_\rho,
\end{align*}$$

(1.6)

where $\eta$ is the unit interior normal to $K_\rho$ and

$$C(\rho, \sigma, n) = \frac{n - 2}{\rho} + \frac{(n - 2)(n - 3)}{2\rho^n} \sigma.$$ 

The hypersurfaces $\{\partial K_\rho\}_{\rho > \rho_0}$ foliate the $M \setminus K_{\rho_0}$.

Our result builds naturally on earlier work, particularly that of the first and second authors [2]. More precisely, they perturb small geodesic balls to produce a family of domains $\Omega_\rho$, parameterized by $\rho \in (0, \rho_0)$, which admit solutions to the overdetermined boundary value problem

$$\Delta_g u = 1 \text{ on } \Omega_\rho, \quad u|_{\partial\Omega_\rho} = 0, \quad \frac{\partial u}{\partial \eta}|_{\partial\Omega} = \text{constant}.$$ 

In our case, the sets $K_\rho$ will be perturbations of large coordinate spheres, as defined by the parameterization $\Phi$ in (1.4).

We end this introduction with a brief outline of the rest of the paper. We begin by reformulating our problem in Section 2 to take place on a fixed set. We parameterize this reformulated problem by a radius $\rho$, a translation $\tau$, and a function $w \in C^{2, \alpha}(S)$. Section 3 has some preliminary computations, such as expansions of the metric and the Laplace-Beltrami operators for our reformulated problem, as well as a study of the mapping properties of the Laplace-Beltrami operator on certain weighted function spaces in Section 3.3. In Section 4 we construct an approximate solution $v$, given in (4.1), and perturb it by a translation to the eventual solution $\hat{u}_{\rho, \tau, w}$, given in (4.10). The function $\hat{u}_{\rho, \tau, w}$ already satisfies most of our desired properties: it is harmonic, decays appropriately, and has constant Dirichlet data. This it only remains for us to choose parameters $\rho$, $\tau$, and $w$ so that $\hat{u}_{\rho, \tau, w}$ also has constant Neumann data. To correctly choose these parameters we must invert the Dirichlet-to-Neumann operator of the Laplace-Beltrami operator. We do this in two steps, first writing out an expansion of the normal derivative of $\hat{u}_{\rho, \tau, w}$ and performing a linear analysis of this expansion in Section 5, and then completing our nonlinear analysis using the implicit function theorem in Section 6. Finally, in Section 7 we show that we do in fact produce a foliation of the asymptotically flat end.
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2. Reformulation of the problem

In this section we reformulate our problem so that we can solve a family of PDEs on the fixed Euclidean domain $\mathbb{R}^n \setminus B$. Intuitively, we accomplish three things with this reformulation. First, we rescale by $\rho > 0$, which one should take to be large. Second, we translate the center of the ball by a small parameter $\tau \in \mathbb{R}^n$. Third, we deform the unit sphere $S = \partial B$ by a function $w \in C^{1,\alpha}(S)$. Putting all these transformations together we obtain a parameterization

$$\Phi_{\rho,w,\tau} : \mathbb{R}^n \setminus B \rightarrow M, \quad \Phi_{\rho,w,\tau}(x) = \Phi\left(\rho\tau + \rho x + \rho w\left(\frac{x}{|x|}\right)x\right)$$

and let $\hat{\Omega}_{\rho,\tau,w} = \Phi_{\rho,\tau,w}(\mathbb{R} \setminus B)$. Finally we can use $\Phi_{\rho,w,\tau}$ to pull problem (1.6) back to $\mathbb{R}^n \setminus B$. Under this change of coordinates, we have now reformulated our original problem (1.6) as

$$\begin{cases}
\Delta_{\hat{g}} \hat{u}_{\rho,w,\tau} = 0 & \text{in } \mathbb{R}^n \setminus B \\
\hat{u}_{\rho,w,\tau} = 1 & \text{on } \partial B \\
\lim_{|x| \rightarrow \infty} \hat{u}_{\rho,w,\tau}(x) = 0 \\
\frac{\partial \hat{u}_{\rho,w,\tau}}{\partial \hat{n}} = \text{Const.} & \text{on } \partial B,
\end{cases}$$

(2.2)

where $\hat{g} = \Phi^*_{\rho,w,\tau}(g)$, $\hat{n}$ is the inward pointing unit normal to $S = \partial B$ with respect to the metric $\hat{g}$. Both our new metric $\hat{g}$ and the function $\hat{u}_{\rho,w,\tau}$ depend on the three parameters $\rho \in (0, \infty)$, $w \in C^{2,\alpha}(S)$, and $\tau \in \mathbb{R}^n$. We should imagine $\rho$ to be large and both $w$ and $\tau$ to be small. So that our parameters are all of the same scale, we require

$$||w||_{C^{2,\alpha}(S)} = \mathcal{O}(\rho^{-n}), \quad |\tau| = \mathcal{O}(\rho^{-n})$$

(2.3)

for the remainder of the paper.

3. Preliminary computations

In this section we carry out some preliminary computations, in preparation for solving (2.2). We first write out a Taylor expansion of the metric $\hat{g}$.
3.1. **Notation.** All our computations in this section are perturbation expansions, when \( \rho \) is large and \( |\tau| \) and \( \|w\|_{C^{k,\alpha}(S)} \) are small. In the computations below we will sometimes wish to extend a function \( v \) defined on the sphere to a tubular neighborhood, and we do so by making it constant in the radial direction, taking \( w(x) = w\left(\frac{x}{|x|}\right) \).

Similarly, for each \( w \in C^{2,\alpha}(S) \) we let

\[
w_i\left(\frac{x}{|x|}\right) = \partial_i w\left(\frac{x}{|x|}\right), \quad w_{ij}\left(\frac{x}{|x|}\right) = \partial_{ij} w\left(\frac{x}{|x|}\right)
\]

and

\[
\Delta_0 w = \sum_{i=1}^{n} w_{ii},
\]

for all \( x \in \mathbb{R}^n \setminus \{0\} \).

To make the computations below tractable, we adopt the following notation throughout the rest of the paper.

For \( i \in \{0, 1, 2\} \) we let \( L^i \) denote a linear partial differential operator of order \( i \) whose coefficients depend smoothly on \( \rho \) and \( x \) and that satisfies the bound

\[
\|L^i(v)\|_{C^{k,\alpha}(\mathbb{R}^n \setminus B)} \leq c\|v\|_{C^{k+i,\alpha}(S)}
\]

for each \( v \in C^{k,\alpha}(S) \), where the constant \( c > 0 \) is independent of \( \rho \). Similarly we let \( Q^i \) denote a nonlinear operator of order \( i \in \{0, 1, 2\} \) such that \( Q^i(0,0) = 0 \) and that satisfies the bound

\[
\|Q^i(v_1, \tau_1) - Q^i(v_2, \tau_2)\|_{C^{k,\alpha}(\mathbb{R}^n \setminus B)} \leq c \left(\|v_1\|_{C^{k+i,\alpha}(S)} + \|v_2\|_{C^{k+i,\alpha}(S)} + |\tau_1| + |\tau_2|\right)
\]

\[
\times \left(\|v_1 - v_2\|_{C^{k+i,\alpha}(S)} + |\tau_1 - \tau_2|\right)
\]

provided \( \|v_1\|_{C^{k+i,\alpha}(S)} + \|v_2\|_{C^{k+i,\alpha}(S)} + |\tau_1| + |\tau_2| \leq 1 \).

Finally we let \( P_i \) be a function of the form

\[
P_i(\rho, x, v, \tau) = \rho^{1-n}|x|^{1-n}L^i(v) + Q^i(v, \tau) + O(\rho^{-n}|x|^{-n})
\]

such that for every \( k, \ell, m \in \mathbb{N}, x \in \mathbb{R}^n \setminus B, \tau \in \mathbb{R}^n \) and \( \rho > \rho_0 > 0 \)

\[
\|\partial^\ell_\rho \partial^k_\ell \partial_\tau P_i(\rho, \cdot, v, \tau)\| \leq c \left(\rho^{-n-\ell} + \rho^{1-n-\ell}(\|v\|_{C^{i,\alpha}(S)} + \|v\|_{C^{i,\alpha}(S)}^2 + |\tau|^2)\right),
\]

for some positive constant depending only on \( h, n, k, \ell, i, \alpha \) and \( \rho_0 \). For brevity we write

\[
P_i(\rho, x, v) = P_i(\rho, x, v, 0).
\]

It is important to observe that the product of any two terms, each of which has the form of either \( L^i \) or \( Q^i \), has the form of \( P_i \).
3.2. Metric expansions and the Laplacian. We have the following expansions.

**Lemma 3.1.** We have

\[ \rho^{-2} g_{ij}(x) = (1 + 2w + \sigma|z|^{1-n})\delta_{ij} + x_iw_j + x_jw_i + P_1(\rho, x, w, \tau) \]  
and

\[ \rho^{2} g^{ij}(x) = (1 + 2w + \sigma|z|^{1-n})^{-1}\delta_{ij} - (x^i w^j + x^j w^i) + P_1(\rho, x, w, \tau), \]

where

\[ |z|^{1-n} := \rho^{1-n} r^{1-n} \left( 1 - \frac{n-1}{r^2} \langle x, \tau \rangle \right). \]  

**Proof.** Letting \{e_1, \ldots, e_n\} be the standard orthonormal basis for \( \mathbb{R}^n \), we see

\[ D\Phi_{\rho, w, \tau}(e_i) = \sum_k D\Phi(e_k) \left( \rho + \rho w \left( \frac{x}{|x|} \right) \right) \delta_{ik} + \rho x^k w_k \]

\[ = \rho \left( 1 + w \left( \frac{x}{|x|} \right) \right) D\Phi(e_i) + \sum_k x^k w_k D\Phi(e_k), \]

where we evaluate derivatives of \( \Phi \) at \( \rho \tau + \rho x + \rho w(x/|x|)x \). Hence

\[ \rho^{-2} g_{ij} = \left( 1 + w \left( \frac{x}{|x|} \right) \right) D\Phi(e_i) + \sum_k x^k w_k D\Phi(e_k) \]  

\[ \times \left( 1 + w \left( \frac{x}{|x|} \right) \right) D\Phi(e_j) + \sum_l x^l w_l D\Phi(e_l) \]

\[ = (1 + w)g_{ij} + (1 + w)w_j x^l g_{il} + (1 + w)w_i x^k g_{jk} + w_i w_j x^k x^l g_{kl}. \]

Next, we write \( |x| = r \) and

\[ y = \rho \left( 1 + w \left( \frac{x}{|x|} \right) \right) x + \rho \tau, \]

so that

\[ |y|^2 = \rho^2 r^2 \left( 1 + 2w + \frac{2}{r^2} \langle x, \tau \rangle + w^2 + 2r^{-2} w \langle x, \tau \rangle + r^{-2} |\tau|^2 \right) \]

\[ |y|^{1-n} = \rho^{1-n} r^{1-n} \left( 1 - (n-1)w - \frac{n-1}{r^2} \langle x, \tau \rangle + w^2 + r^{-2} w \langle x, \tau \rangle + r^{-2} |\tau|^2 + \cdots \right) \]

\[ = \rho^{1-n} r^{1-n} \left( 1 - \frac{n-1}{r^2} \langle x, \tau \rangle \right) + \rho^{1-n} r^{1-n} L^0(w, \tau) + Q^0(w, \tau). \]

Observe that we absorb the term \((1 - n)\rho^{1-n} r^{1-n} w \) above into \( L^0(w) \), while the corresponding linear term with respect to \( \tau \) is kept. Indeed when solving the nonlinear
equation for small $\rho$ in Section 6, we have to replace $w$ with $\rho^{n-1}w$, which increases the power of $\rho$ in $(1 - n)\rho^{n-1}r^{1-n}w$ by $n - 1$.

Using (1.5) and (3.1), it follows from (3.9) and (3.10) that both (3.6) and (3.7) hold.

In the next sections, we will work with the metric

$$g_\rho := \rho^{-2}\tilde{g}.$$  \hfill (3.11)

**Lemma 3.2.** For $\rho$ sufficiently large,

$$\Delta_{g_\rho} = (1 - 2w - \sigma|z|^{1-n})\Delta_0 - \left[ x_iw_j + x_jw_i + P_1(\rho, x, w, \tau) \right] \partial_{ij}^2$$

$$\Delta_0 = \left[ \left( 3w_j - \frac{n-2}{2}\sigma\partial_j|z|^{1-n} \right) + x_iw_j + [\Delta_0w]x_j + P_2(\rho, x, w, \tau) \right] \partial_j,$$

where $\Delta_0$ is the usual flat Laplacian.

Moreover,

$$\partial_j|z|^{1-n} = -(n-1)\rho^{1-n}r^{-1-n}x_j \left( 1 - \frac{n - 3}{r^2} \langle x, \tau \rangle \right) - (n-1)\rho^{1-n}r^{-1-n}\tau_j.$$

**Proof.** Recall that the Laplace-Beltrami operator has the form

$$\Delta_{g_\rho}u = \frac{1}{\sqrt{|g_\rho|}} \partial_i g^{ij}_{\rho} \sqrt{|g_\rho|} \partial_j u, \quad |g_\rho| = \det(g_\rho).$$

Then using (3.6) and (3.7), we have

$$|g_\rho| = 1 + 2nw + n\sigma|z|^{1-n} + P_1(\rho, x, w, \tau)$$

$$\sqrt{|g_\rho|} = 1 + nw + \frac{n\sigma}{2}|z|^{1-n} + P_1(\rho, x, w, \tau)$$

$$\frac{1}{\sqrt{|g_\rho|}} = 1 - nw - \frac{n\sigma}{2}|z|^{1-n} + P_1(\rho, x, w, \tau)$$

$$g^{ij}_{\rho} \sqrt{|g_\rho|} = \left[ \left( 1 + (n-2)w + \frac{n-2}{2}\sigma|z|^{1-n} \right) \delta_{ij} - (w^ix^j + w^jx^i) + P_1(\rho, x, w, \tau) \right]$$

$$\partial_i(g^{ij}_{\rho} \sqrt{|g_\rho|}) = \left( (n-3)w_i + \frac{n-2}{2}\sigma\partial_i|z|^{1-n} \right) \delta_{ij} - w_{ii}x_j - w_{ij}x_i - w_j + P_2(\rho, x, w, \tau)$$

$$\frac{1}{\sqrt{|g|}} \partial_i(\tilde{g}^{ij} \sqrt{\tilde{g}}) = \left( (n-3)w_i + \frac{n-2}{2}\sigma\partial_i|z|^{1-n} \right) \delta_{ij} - w_{ii}x_j - w_{ij}x_i - w_j + P_2(\rho, x, w, \tau)$$
\[ \Delta g, u = g^{ij}_{\rho} \partial_i \partial_j u + \frac{1}{\sqrt{|g_{\rho}|}} \partial_i (g^{ij}_{\rho} \sqrt{|g_{\rho}|}) \partial_j u \]

\[ = [(1 - 2w - \sigma|z|^{1-n}) \delta_{ij} - (x^i w^j + x^j w^i) + P_1(\rho, x, w, \tau)] \partial_i \partial_j u \]

\[ - \left[ (3w_i - \frac{n-2}{2} \sigma \partial_i |z|^{1-n}) \partial_i u - (x^j \Delta_0 w + x_i w_{ij}) \partial_j u \right] + P_2(\rho, x, w, \tau) \partial_j u. \]

In addition we have from (3.8)

\[ \partial_j |z|^{1-n} = -(n - 1) \rho^{1-n} r^{-1-n} x_j \left( 1 - \frac{n-3}{r^2} \langle x, \tau \rangle \right) - (n - 1) \rho^{1-n} r^{-1-n} \tau_j, \]

which yield the expansions in Lemma 3.2.

3.3. Weighted spaces. The best setting in which to perform our linear analysis is that of weighted Hölder spaces. Following Pacard and Rivière, we use the following definition.

**Definition 1.** Let \( \nu \in \mathbb{R}, k \in \mathbb{N} \) and \( 0 < \alpha < 1 \). Then we say \( u \in C^{k,\alpha}_\nu(\mathbb{R}^n) \) if

\[ \|u\|_{k,\alpha,\nu} = \sup_{s>1} (s^{-\nu} [u]_{k,\alpha,s}) < \infty. \]

Here

\[ [u]_{k,\alpha,s} := \sum_{i=0}^{k} s^i \sup_{A_s} |\nabla^i u| + s^{k+\alpha} \sup_{x,x' \in A_s} \frac{|\nabla^k u(x) - \nabla^k u(x')|}{|x-x'|^\alpha}, \]

where \( A_s = \{ x \in \mathbb{R}^n : s < |x| < 2s \} \). We denote the space of functions vanishing on the boundary by

\[ C^{k,\alpha}_{\nu,D}(\mathbb{R}^n) := \{ u \in C^{k,\alpha}_\nu(\mathbb{R}^n) : u|_{\partial B} = 0 \}. \]

Intuitively, one can think of \( C^{0,\alpha}_\nu(\mathbb{R}^n) \) as those functions which grow at most like \( |x|^\nu \) when \( |x| \) is large.

**Remark 1.** Pacard and Rivière perform their analysis on weighted Hölder spaces on \( B \setminus \{0\} \), whereas we want to examine functions on \( \mathbb{R}^n \setminus B \). It is straight-forward to transfer between the two settings using the Kelvin transform \( \mathbb{K} \), defined by

\[ \mathbb{K} : C_{\nu}^{k,\alpha}(B \setminus \{0\}) \rightarrow C_{2-n-\nu}^{k,\alpha}(\mathbb{R}^n \setminus B), \quad \mathbb{K}(u)(x) = |x|^{2-n} u \left( \frac{x}{|x|^2} \right). \]

It will be convenient to also note the transformation law

\[ \Delta_0 (\mathbb{K}(u))(x) = |x|^{-4} \mathbb{K}(\Delta_0 u)(x). \]

One can show the following theorem (see Section 2.2 of [7]).
Theorem 2. The mapping
\[ \Delta_0 : C^{2,\alpha}_{\nu,D}(\mathbb{R}^n \setminus B) \to C^{0,\alpha}_{\nu-2}(\mathbb{R}^n \setminus B) \]
is injective if \( \nu < 0 \) and surjective if \( \nu > 2 - n \).

The mapping properties of \( \Delta_0 \) change when the weight \( \nu \) crosses over one of the indicial roots \( \gamma^\pm_j \), where
\[ \gamma^\pm_j = \frac{2 - n}{2} \pm \sqrt{\frac{(n - 2)^2}{4} + \lambda_j}, \]
and \( \lambda_j \) is the \( j \)th eigenvalue of the Laplace-Beltrami operator on the sphere. Thus one can recover the indicial roots \( \gamma^\pm_j \) as growth/decay rates of solutions to the ODE
\[ w'' + \frac{n - 1}{r} w' - \frac{\lambda_j}{r^2} w = 0, \quad w = r^{\gamma^\pm_j}. \]

A slightly more refined analysis uncovers the following theorem.

Theorem 3. Let \( \nu < 0 \) with \( \nu \not\in \{\gamma^\pm_j : j \in \mathbb{N}\} \), and let \( j_0 \) be the least non-negative integer such that \( \nu > \gamma^-_{j_0} \). Then the cokernel of the mapping
\[ \Delta_0 : C^{2,\alpha}_{\nu,D}(\mathbb{R}^n \setminus B) \to C^{0,\alpha}_{\nu-2}(\mathbb{R}^n \setminus B) \]
has dimension \( j_0 \). Alternatively let \( \nu > 2 - n \) with \( \nu \not\in \{\gamma^\pm_j : j \in \mathbb{N}\} \), and let \( j_0 \) be the least positive integer such that \( \nu < \gamma^+_j \) then kernel of the mapping
\[ \Delta_0 : C^{2,\alpha}_{\nu,D}(\mathbb{R}^n \setminus B) \to C^{0,\alpha}_{\nu-2}(\mathbb{R}^n \setminus B) \]
has dimension \( j_0 \).

Again, we refer the reader to Section 2.2 of [7] for details.

Replacing \( \rho \) by \( 1/\rho \) in Lemmas 3.1-3.2 and keeping the notation \( g_\rho \) for the metric \( g_{1/\rho} \), we have the following result.

Lemma 3.3. There exist \( \rho_0 > 0 \) and \( c_0 > 0 \) such that the map
\[ \mathcal{L} : (0, \rho_0) \times B_{c_0}(0) \times B_{c_0}(0) \times C^{2,\alpha}_{\nu,D}(\mathbb{R}^n \setminus B) \to C^{0,\alpha}_{\nu}(\mathbb{R}^n \setminus B) \]
defined by
\[ \mathcal{L}(\rho, w, \tau, u) := \Delta_{g_\rho} u \]
is well defined and smooth.

Furthermore, for every \( (\rho, w, \tau) \in (0, \rho_0) \times (B_{c_0}(0))^2 \), the linear map
\[ \mathcal{L}_{\rho, w, \tau} : C^{2,\alpha}_{\nu,D}(\mathbb{R}^n \setminus B) \to C^{0,\alpha}_{\nu}(\mathbb{R}^n \setminus B), \quad u \mapsto \mathcal{L}_{\rho, w, \tau}(u) := \mathcal{L}(\rho, w, \tau, u) \]
is invertible and for all \( u \in C^{2,\alpha}_{\nu,D}(\mathbb{R}^n \setminus B) \) we have the inequalities
\[ C \| u \|_{C^{2,\alpha}_{\nu,D}(\mathbb{R}^n \setminus B)} \leq \| \mathcal{L}_{\rho, w, \tau}(u) \|_{C^{0,\alpha}_{\nu}(\mathbb{R}^n \setminus B)} \leq C^{-1} \| u \|_{C^{2,\alpha}_{\nu,D}(\mathbb{R}^n \setminus B)}, \quad (3.15) \]
where \( C > 0 \) is independent of \( (\rho, w, \tau) \).
Proof. By Lemma 3.2 we can find \( \rho_0 > 0 \) and \( c_0 > 0 \) such that the map

\[
\mathcal{L} : (0, \rho_0) \times B_{c_0}(0) \times B_{c_0}(0) \times C^{2,\alpha}_{\nu,D}(\mathbb{R}^n \setminus B) \rightarrow C^{0,\alpha}_{\nu}(\mathbb{R}^n \setminus B)
\]

defined by

\[
\mathcal{L}(\rho, w, \tau, u) = \Delta g^\rho u
\]
is well defined and smooth. We shall show that the linear map

\[
u \mapsto \mathcal{L}(\rho, w, \tau, u) : C^{2,\alpha}_{\nu,D}(\mathbb{R}^n \setminus B) \rightarrow C^{0,\alpha}_{\nu}(\mathbb{R}^n \setminus B)
\]
is invertible for every \( (\rho, w, \tau) \in (0, \rho_0) \times B_{c_0}(0) \). To see this, we pick \( f \in C^{0,\alpha}(\mathbb{R}^n \setminus B) \) and we define

\[
\mathcal{F} : (-\rho_0, \rho_0) \times B_{c_0}(0) \times B_{c_0}(0) \times C^{2,\alpha}_{\nu,D}(\mathbb{R}^n \setminus B) \rightarrow C^{0,\alpha}_{\nu}(\mathbb{R}^n \setminus B)
\]
by

\[
\mathcal{F}(\varepsilon, w, \tau, u) := \mathcal{L}(\varepsilon, |w|, \tau, u) - f.
\]
It is clear from Lemma 3.2 that \( \mathcal{F} \) is of class \( C^2 \), since \( n \geq 3 \). Let \( u_0 \in C^{2,\alpha}_{\nu,D}(\mathbb{R}^n \setminus B) \) be the unique solution to \( \Delta_0 u_0 = f \). We have \( \mathcal{F}(0, 0, 0, u_0) = 0 \) and by Theorem 2 \( \partial_u \mathcal{F}(0, 0, 0, u_0) \) is invertible. By the implicit function theorem, there exists a unique \( u_{\varepsilon, w, \tau} \) satisfying \( \mathcal{F}(\varepsilon, w, \tau, u_{\varepsilon, w, \tau}) = 0 \), with \( u_{0,0,0} = u_0 \). We then conclude that, provided \( c_0 \) and \( \rho_0 \) small, the linear map

\[
\mathcal{L}(\rho, w, \tau, \cdot) : C^{2,\alpha}_{\nu,D}(\mathbb{R}^n \setminus B) \rightarrow C^{0,\alpha}_{\nu}(\mathbb{R}^n \setminus B)
\]
is invertible for every \( \rho, w, \tau \in (0, \rho_0) \times (B_{c_0}(0))^2 \). The bound (3.15) also follows from the implicit function theorem. \( \square \)

4. Approximate and actual solutions

In this section we construct an approximate solution using the standard Greens function in Euclidean space and compare it to the solution of a corresponding Dirichlet problem.

For \( w \in B_{c_0}(0) \), we define

\[
v(x) = v_{\rho,w}(x) = \left(1 + w \left(\frac{x}{|x|}\right)\right)^{2-n}.
\] (4.1)

We have the following expansion.

**Lemma 4.1.** For \( \rho \) sufficiently small the Laplacian of \( v \) is given by

\[
\Delta g^\rho v = \frac{(n-1)(n-2)}{2} \sigma \rho^{n-1} r^{1-2n} \left(\frac{x}{r^2}, \tau\right) + P_2(1/\rho, x, w, \tau).
\] (4.2)
Proof. By definition
\[ v(x) = |x|^{2-n} \left(1 - (n-2)w + Q^0(w, \tau)\right), \] (4.3)
and so
\[
\frac{\partial v}{\partial x_i} = -(n-2)r^{-n} \left(1 - (n-2)w + Q^0(w, \tau)\right)x_i \] (4.4)
\[
\frac{\partial^2 v}{\partial x_i \partial x_j} = -(n-2)r^{-n} \left(1 - (n-2)w\right)\delta_{ij} + n(n-2)r^{-2-n} \left(1 - (n-2)w\right) x_ix_j + (n-2)^2 r^{-n}w_jx_i \]
\[
\Delta_0(v) = -(n-2)r^{-2-n}\Delta_0 w + Q^2(w, \tau). \] (4.5)

With this, we have
\[
(1 - 2w - \sigma |z|^{r-1-n})\Delta_0(v) = -(n-2)r^{2-n}\Delta_0 w + Q^2(w, \tau) \]
\[
\left[ x_i w_j + x_j w_i + P_1(\rho, x, w, \tau) \right] \frac{\partial^2 v}{\partial x_i \partial x_j} = P_2(\rho, x, w, \tau). \]
\[
- \left[ 3w_j + x_i w_j + [\Delta_0 w] x_j + P_2(\rho, x, w, \tau) \right] \partial_j(v) = (n-2)r^{2-n}\Delta_0 w + (n-2)r^{-n}x_ix_j w_{ij} \]
\[
\frac{n-2}{2} \sigma \partial_j |y|^{1-n} \partial_j(v) = \frac{(n-1)(n-2)^2}{2} \sigma r^{1-n} - \frac{(n-1)(n-4)(n-2)^2}{2} \sigma r^{1-n} \chi(\frac{x}{r^2}, \tau). \] (4.6)

However, \( 0 = \partial_i(x^j w_j) = \delta_{ij} w_j + x^j w_{ij} \Rightarrow x^i x^j w_{ij} = -x^i \delta_{ij} w_j = -x^i w_i = 0 \). The expansion in Lemma 4.1 now follows from Lemma 3.2 after replacing \( \rho \) by \( 1/\rho \). \qed

Using Lemma 3.3, we construct a unique solution \( \Psi_{\rho, w}(x) \) to the equation
\[
\begin{aligned}
\Delta_{g_\rho} \Psi_{\rho, \tau, w} &= -\Delta_{g_\rho} v \quad \text{in} \quad \mathbb{R}^n \backslash B \\
\Psi_{\rho, w} &= 1 - v = (n-2)w + Q^0(w) \quad \text{on} \quad \partial B.
\end{aligned} \] (4.7)

First choose \( \chi \in C^\infty_c(\mathbb{R}^+) \) such that \( \chi(t) = 1 \) for \( 1/2 \leq t \leq 1 \), and define
\[
f(\rho, x, w) = -\Delta_{g_\rho} \left( ((n-2)w + Q^0(w))\chi(|x|) \right) - \Delta_{g_\rho} v \in C^{0,\alpha}_v(\mathbb{R}^n \backslash B) \]
for any \( \nu \in (2-n, 0) \). Next we use Lemma 3.3 to let \( \Psi_{\rho, \tau, w} \) be the unique solution of
\[
\Delta_{g_\rho} \Psi_{\rho, \tau, w}(x) = f(\rho, x, w) \quad \text{for} \quad x \in \mathbb{R}^n \backslash B, \quad \Psi_{\rho, \tau, w} = 0 \quad \text{on} \quad \partial B. \] (4.8)
Observe that \( f \) depends smoothly on \((\rho, w, \tau)\) and that the mapping \( A \mapsto (A)^{-1} \) is smooth in the open set of linear invertible operator between two Banach spaces. We deduce that the mapping 
\[
(\rho, \tau, w) \mapsto \tilde{\Psi}_{\rho,\tau,w}
\]
is smooth. We then have 
\[
\Psi_{\rho,\tau,w} = \tilde{\Psi}_{\rho,\tau,w} + ((n - 2)w + Q^0(w)) \chi(|x|),
\]
which solves uniquely (4.7).

Finally we define 
\[
\hat{u}_{\rho,\tau,w}(x) = v(x) + \Psi_{\rho,\tau,w}(x),
\]
which satisfies 
\[
\begin{cases}
\Delta_{g_{\rho}} \hat{u}_{\rho,\tau,w} = 0 \quad \text{in} \quad \mathbb{R}^n \setminus B \\
\hat{u}_{\rho,\tau,w} = 1 \quad \text{on} \quad \partial B.
\end{cases}
\]

Since \( f(\rho, \cdot, w) \in C^{0,\alpha}(\mathbb{R}^n \setminus B) \), (4.8) and (3.15) imply that \( \tilde{\Psi}_{\rho,w}(x) \to 0 \) as \(|x| \to \infty\). We conclude with (4.1), (4.9) and (4.10) that 
\[
\hat{u}_{\rho,\tau,w}(x) \to 0 \quad \text{as} \quad |x| \to \infty.
\]

5. Expansion of the normal derivative and linear analysis

The aim of this section is to derive an expansion of the normal derivative of the solution \( \hat{u}_{\rho,\tau,w} \). We start by the computation of the interior unit normal vector to \( B \).

**Lemma 5.1.** Let \( \Theta : \mathbb{R}^{n-1} \to \mathcal{S} \) parameterize \( \mathcal{S} \) by the inverse of stereographic projection and let 
\[
\Theta_k := \partial_k \Theta \quad k = 1, \ldots, n - 1.
\]
The interior unit normal vector field to \( B \) with respect to the metric \( g_{\rho} \) is given by 
\[
\hat{\nu}_{g_{\rho}} = \frac{\tilde{\eta}}{\sqrt{\langle \tilde{\eta}, \tilde{\eta} \rangle_{g_{\rho}}}} = \left[ 1 - w - \frac{\sigma}{2} \rho^{1-n} \left( 1 - (n-1) \langle \Theta, \tau \rangle \right) + P_1(\rho, \Theta, w, \tau) \right] (-\Theta + \Upsilon),
\]
where 
\[
\Upsilon = \sum_{m=1}^{n-1} a_k \Theta_k \quad \text{and} \quad a_k = \langle \nabla_S w, \Theta_k \rangle + P_0(\rho, \Theta, w, \tau).
\]

**Proof.** For each \( \Theta \in \mathcal{S} \) the vector fields 
\[
\Theta_\ell := \partial_\ell \Theta(s), \quad \ell = 1, \ldots, n - 1
\]
span the tangent space \( T_\Theta \mathcal{S} \). Since \( \langle \Theta, \Theta \rangle = 1 \), we have that \( \langle \Theta, \Theta_\ell \rangle = 0 \) and \( \langle \nabla_S w, \Theta \rangle_{g_S} = 0 \) on \( \mathcal{S} \). Without loss of generality, we may assume that \( \langle \Theta_i, \Theta_j \rangle = \delta_{ij} \).

We look for a normal vector \( \tilde{\eta} \) of \( \mathcal{S} \) with respect to the metric \( \hat{g} \) in the form 
\[
\tilde{\eta} = -\Theta + \Upsilon, \quad \Upsilon = \sum_{m=1}^{n-1} a_k \Theta_k.
\]

(5.3)
The condition that \( \hat{\eta} \) is normal is thus equivalent to
\[
\langle \hat{\eta}, \Theta_\ell \rangle_{g_\rho} = 0, \quad \ell = 1, \ldots, n - 1 \iff \langle \Upsilon, \Theta_\ell \rangle_{g_\rho} = \langle \Theta, \Theta_\ell \rangle_{g_\rho}, \quad \ell = 1, \ldots, n - 1. \tag{5.4}
\]
By Lemma 3.1
\[
\langle \Theta, \Theta_\ell \rangle_{g_\rho} = (1 + 2w + \sigma|z|^{1-n})\langle \Theta, \Theta_\ell \rangle \\
+ \left[ \Theta^j w_j \Theta^i \Theta_\ell + \Theta^j w_i \Theta^i \Theta_\ell + P_1(\rho, \Theta, w, \tau) \right] \\
= \left[ \langle \nabla_s w, \Theta \rangle + P_1(\rho, \Theta, w, \tau) \right] \tag{5.5}
\]
and
\[
\langle \Upsilon, \Theta_\ell \rangle_{g_\rho} = \sum_{k=1}^{n-1} a_k \langle \Theta_k, \Theta_\ell \rangle_{g_\rho} = \sum_{k=1}^{n-1} a_k \left( (1 + 2w + \sigma|z|^{1-n})\delta_{k\ell} + P_1(\rho, \Theta, w, \tau) \right).
\]
Substituting these last two expressions into (5.4), we obtain
\[
\sum_{k=1}^{n-1} a_k \tilde{g}_{k\ell} = b_\ell, \quad \ell = 1, \ldots, n - 1, \tag{5.6}
\]
where \( \tilde{g}_{k\ell} := (1 + 2w + \sigma|y|^{1-n})\delta_{k\ell} + P_1(\rho, \Theta, w, \tau) \) and \( b_\ell := \langle \nabla_s w, \Theta_\ell \rangle + P_1(\rho, \Theta, w, \tau) \), so that (5.6) then implies
\[
a_k = \sum_{\ell=1}^{n-1} b_\ell \tilde{g}_{k\ell} = \langle \nabla_s w, \Theta_k \rangle + P_1(\rho, \Theta, w, \tau). \tag{5.7}
\]
Next we compute
\[
\langle \hat{\eta}, \hat{\eta} \rangle_{g_\rho} = \langle \Theta, \Theta \rangle_{g_\rho} - 2\langle \Theta, \Upsilon \rangle_{g_\rho} + \langle \Upsilon, \Upsilon \rangle_{g_\rho},
\]
We have
\[
\langle \Theta, \Theta \rangle_{g_\rho} = \left( 1 + 2w + \sigma|z|^{1-n} + P_0(\rho, \Theta, w, \tau) \right)
\]
Also, from (5.7), \( \langle \Upsilon, \Upsilon \rangle_{g_\rho} = P_1(\rho, \Theta, w, \tau) \). Using once more (5.5) and (5.7), we obtain
\[
\langle \Theta, \Upsilon \rangle_{g_\rho} = \sum_{k=1}^{n-1} a_k \langle \Theta_k, \Theta_\ell \rangle_{g_\rho} = P_1(\rho, \Theta, w, \tau)
\]
and hence
\[
\langle \hat{\eta}, \hat{\eta} \rangle_{g_\rho} = \left( 1 + 2w + \sigma|z|^{1-n} + P_1(\rho, \Theta, w, \tau) \right). \tag{5.8}
\]
The normal interior unit vector field to \( B \) for the metric \( g_\rho \) is the given by
\[
\hat{\nu}_{g_\rho} = \frac{\hat{\eta}}{\sqrt{\langle \hat{\eta}, \hat{\eta} \rangle_{g_\rho}}} = \left( 1 - w - \frac{\sigma}{2} |z|^{1-n} + P_1(\rho, \Theta, w, \tau) \right) (-\Theta + \Upsilon), \tag{5.8}
\]
and so (5.1) follows from (3.8).

5.1. Expansion of the normal derivative. The following proposition yields the expansion of the normal derivative of \( \hat{u}_{\rho,\tau,w} \) with respect to the metric \( g_\rho \).

**Proposition 4.** For \( \rho \) sufficiently small the normal derivative of \( \hat{u} = \hat{u}_{\rho,\tau,w} = v + \Psi_{\rho,\tau,w} \) with respect to the metric \( g_\rho \) on \( \partial B \) is given by

\[
\frac{\partial \hat{u}}{\partial n_g_\rho} = (n - 2)(1 - \frac{\sigma}{2} \rho^{n-1}) + H(\rho, \tau, w), \tag{5.9}
\]

where

\[
H(\rho, \tau, w)(x) := (n-1)(n-2) \left( \frac{\sigma}{2} \rho^{n-1} (x, \tau) - w + P_1(1/\rho, x, w, \tau) \right) + g_\rho \frac{\partial \Psi_{\rho,\tau,w}}{\partial x_i} \tag{5.10}
\]

and \( \Psi_{\rho,\tau,w} \) is solution of (4.7).

**Proof.** Recall our solution \( \hat{u}_{\rho,\tau,w}(x) = v(x) + \Psi_{\rho,\tau,w}(x) \), where \( v \) is given by (4.1) and \( \Psi_{\rho,\tau,w} \) satisfies (4.7).

We first compute \( g_\rho(\nabla_{g_\rho} v, \hat{g}_\rho) \). By Lemma 3.1 and (4.4)

\[
\nabla^j_{g_\rho} v = \sum_{i=1}^n g_{ij}^\rho \frac{\partial v}{\partial x_i} = \left[ (1 - 2w - \sigma |z|^{1-n}) \delta_{ij} - (x_i w_j + x_j w_i) + P_0(\rho, x, w, \tau) \right] \times \left[ - (n - 2) \left( 1 - (n - 2) w + Q^0(w) \right) x_i - (n - 2) w_i + Q^1(w) \right]
\]

\[
= - (n - 2) \left( 1 - \sigma |z|^{1-n} \right) x_j - nw x_j + P_1(\rho, x, w, \tau) \quad \text{on} \quad \partial B,
\]

which yields

\[
g_\rho(-\Theta, \nabla_{g_\rho} v) = - \left[ (1+2w+\sigma |z|^{1-n}) \langle \Theta, \nabla_{g_\rho} v \rangle + \Theta^j w_j \Theta^l \nabla^j_{g_\rho} v + \Theta^j w_j \Theta^l \nabla^j_{g_\rho} v + P_0(\rho, \Theta, w, \tau) \right].
\]

On the other hand,

\[
\langle \Theta, \nabla_{g_\rho} v \rangle = -(n - 2) \left( 1 - \sigma |z|^{1-n} - nw + P_1(\rho, \Theta, w, \tau) \right),
\]

so we may rewrite our expression for \( g_\rho(-\Theta, \nabla_{g_\rho} v) \) as

\[
g_\rho(-\Theta, \nabla_{g_\rho} v) = (n - 2) \left( 1 + (2 - n) w + P_1(\rho, \Theta, w, \tau) \right). \tag{5.11}
\]

Additionally,

\[
g_\rho(\Gamma, \nabla_{g_\rho} v) = \sum_{k=1}^n a_k g_\rho(\Theta_k, \nabla_{g_\rho} v) = \sum_{k=1}^n a_k g_\rho(\Theta_k, \Theta) + P_1(\rho, \Theta, w, \tau) = P_1(\rho, \Theta, w, \tau), \tag{5.12}
\]
where we have used (5.5) in the last equality. Finally, making use of (5.8), (5.12) and (5.11), we obtain
\[
g_{\rho}(\nabla_{g_{\rho}} v, \hat{\nu}_{g_{\rho}}) = \left(1 - w - \frac{\sigma}{2}|z|^{1-n} + P_1(\rho, \Theta, w, \tau)\right)g_{\rho}(-\Theta + \nabla_{\hat{g}} v)
\]
\[
= (n - 2) \left(1 - w - \frac{\sigma}{2}|z|^{1-n} + P_1(\rho, \Theta, w, \tau)\right) \left(1 + (2 - n)w + P_1(\rho, \Theta, w)\right)
\]
\[
= (n - 2) \left(1 - \frac{\sigma}{2} \rho^{1-n} + \frac{\sigma(n - 1)}{2} \rho^{1-n} + (n - 1)w + P_1(\rho, \Theta, w, \tau)\right)
\]
(5.13)

and
\[
g_{\rho}(\nabla_{g_{\rho}} \Psi_{\rho,\tau,w}, \hat{\nu}_{g_{\rho}}) = g_{\rho}^{ij}(g_{\rho})_{jk} \hat{\nu}^k_{g_{\rho}} \frac{\partial \Psi_{\rho,\tau,w}}{\partial x_i} = \delta_{ik} \hat{\nu}^k_{g_{\rho}} \frac{\partial \Psi_{\rho,\tau,w}}{\partial x_i} = \hat{\nu}^i_{g_{\rho}} \frac{\partial \Psi_{\rho,\tau,w}}{\partial x_i}.
\]
(5.14)

The expansion (5.9) then follows from (5.13) and (5.14) replacing \(\rho\) by \(1/\rho\). \(\square\)

5.2. **Linear analysis of the normal derivative.** The leading term in (5.9) is
\[
(n - 2)(1 - \sigma \rho^{n-1}/2),
\]
and to complete our proof we must thoroughly understand the remainder term \(H(\rho, \tau, w)\).

To assist in this analysis we linearize the function \(G(\rho, \tau, w) = \rho^{1-n}H(\rho, \tau \rho^{n-1}w)\).

**Proposition 5.** Defining
\[
G(\rho, \tau, w) := \rho^{1-n}H(\rho, \tau, \rho^{n-1}w),
\]
we have
\[
[D_{\tau}]|_{(\rho,\tau,w) = (0,0,0)} G(\rho, \tau, w) \cdot \tau = \frac{(n - 1)(n - 2)}{2} \left(1 + \frac{(n - 2)(n - 4)}{2}\right) \sigma\langle x, \tau \rangle
\]
(5.16)

and
\[
L(w) := [D_{w}]|_{(\rho,\tau,w) = (0,0,0)} G(\rho, \tau, w) \cdot w = \partial_{\nu} \psi_w \cdot x - (n - 1)w,
\]
(5.17)

where \(\nu = -x\) and \(\psi_w\) is the unique solution to
\[
\begin{align*}
\Delta_{\delta} \psi_w &= 0 & \text{in} & \quad \mathbb{R}^n \setminus \mathcal{B} \\
\psi_w &= w & \text{on} & \quad \partial \mathcal{B}.
\end{align*}
\]
(5.18)

**Proof.** From (5.15) and (5.10), we have
\[
G(\rho, \tau, w)(x) = (n - 1)(n - 2)\left(\frac{\sigma}{2}\langle x, \tau \rangle - w + \rho^{1-n}P_1(1/\rho, \Theta, \rho^{n-1}w, \tau)\right)
\]
\[
+ \hat{\nu}^i_{g_{\rho}}(\rho^{n-1}w) \frac{\partial \Psi_{\rho,\tau,w}}{\partial x_i},
\]
(5.19)
where

$$\Psi_{\rho,\tau,w} := \rho^{1-n}\Psi_{\rho,\tau,\rho^{n-1}w},$$

and \(\hat{v}^i_{\rho}(\rho^{n-1}w)\) is the \(i\)th-component of the unit vector \(\hat{v}_{\rho}\) in (4.9) with \(\rho^{n-1}w\) in place of \(w\). We see from (4.7) and Lemma 4.1 that \(\Psi_{\rho,\tau,w}\) is solution the unique solution of

\[
\begin{cases}
\Delta g_{\rho} \Psi_{\rho,\tau,w} = -\frac{(n-1)(n-2)^2}{2}\sigma r^{1-2n} + \frac{(n-1)(n-4)(n-2)^2}{2}\sigma r^{1-2n}(\frac{x}{r}, \tau) \\
+ \rho^{1-n}P_2(1/\rho, x, \rho^{n-1}w, \tau) & \text{in } \mathbb{R}^n \setminus B \\
\Psi_{\rho,\tau,w} = (n-2)w + \rho^{1-n}Q^0(\rho^{n-1}w, \tau) & \text{on } \partial B.
\end{cases}
\tag{5.20}
\]

Differentiating (5.20) with respect to \(\tau\) at \((\rho, \tau, w) = (0, 0, 0)\), we see

\[
F_{\tau} := [D_{\tau}\big|_{(\rho,\tau,w)=(0,0,0)} \Psi_{\rho,\tau,w}] \cdot \tau
\]

satisfies

\[
\begin{cases}
\Delta F_{\tau} = C(n, \sigma) r^{1-2n}(\frac{x}{r}, \tau) & \text{in } \mathbb{R}^n \setminus B \\
F_{\tau} = 0 & \text{on } \partial B,
\end{cases}
\tag{5.21}
\]

where

\[
C(n, \sigma) := \frac{(n-1)(n-4)(n-2)^2}{2}\sigma.
\tag{5.22}
\]

We observe from (5.11) that

\[
[D_w]\big|_{(\rho,\tau,w)=(0,0,0)} \hat{v}^i_{\rho}(\rho^{n-1}w) \cdot w = 0 = [D_{\tau}\big|_{(\rho,\tau,w)=(0,0,0)} \hat{v}^i_{\rho}(\rho^{n-1}w)] \cdot \tau,
\tag{5.23}
\]

which combined with (5.10) and (5.15) allow to get

\[
[D_{\tau}\big|_{(\rho,\tau,w)=(0,0,0)} G(\rho, \tau, w)] \cdot \tau = \frac{(n-1)(n-2)}{2}\sigma\langle x, \tau \rangle - x \cdot \nabla F_{\tau}.
\tag{5.24}
\]

Next, we show that

\[
\partial_{\nu} F_{\tau} = \frac{C(n, \sigma)}{2(n-1)}(\cdot, \tau) & \text{on } S, \text{ with } \nu = -x.
\tag{5.25}
\]

Indeed, let

\[
\Pi : L^2(S) \to L^2(S)
\]

be the orthogonal projection on \(\text{span}\{x_1; \ldots; x_n\}\) and consider the function

\[
X^i(x) := \mathbb{K}(x^i) = |x|^{2-n} \frac{x^i}{|x|^2}.
\]
We know that $\Delta \mathbf{0} = 0$ in $\mathbb{R}^n \setminus \{0\}$. We multiply (5.21) with $X^i$ and integrate by parts to get

$$
\int_S \partial_\nu F_\tau(x) x^i d\mu \mathcal{S} = C(n, \sigma) \int_{\mathbb{R}^n \setminus \mathcal{B}} |x|^{-1-n} |x|^{2-n} \langle \frac{x}{|x|^2}, \tau \rangle X^i(x) d\mu \\
= C(n, \sigma) \tau^i \int_S (x^i)^2 d\mu \int_1^\infty r^{1-3n} r^{n-1} dr.
$$

This implies that

$$
\int_S \partial_\nu F_\tau(x) x^i d\mu \mathcal{S} = \frac{C(n, \sigma)|\partial\mathcal{B}|}{2n-1} \tau^i = \frac{C(n, \sigma)}{2n-1} \int_S \langle x, \tau \rangle x^i d\mu. \quad (5.26)
$$

From this, we deduce that

$$
\Pi \partial_\nu F_\tau(x) = \frac{C(n, \sigma) \langle \tau, x \rangle}{2n-1} \quad \text{for } x \in \mathcal{S}. \quad (5.27)
$$

We claimed that

$$
\Pi^\perp \partial_\nu F_\tau = 0. \quad (5.28)
$$

To see this, we let $Y_i^k$ be the spherical harmonics for which $Y_1^i = x^i$ for $i = 1, \ldots, n$, corresponding to the eigenvalues $k(k + n - 2)$ on sphere. We suppose that $k \neq 1$. We then define

$$
X^k_i(x) = \mathbb{R}(Y_i^k)(x).
$$

Then $X^k_i$ are admissible test functions in (5.21). We observe that the right hand side in (5.21) is in $L^2(\mathbb{R}^n \setminus \mathcal{B})$. Therefore by simple arguments, we have that $F_\tau \in H^1(\mathbb{R}^n \setminus \mathcal{B})$. Using the decomposition in spherical harmonics of $F_\tau$, we can see that $F_\tau(x) = f(|x|) \langle x, \tau \rangle$, for some some function $f$. From this we can multiply (5.21) by $X^k_i$ and use the Gauss-Green formula to deduce that

$$
\int_S \partial_\nu F_\tau(x) Y^k_i(x) d\mu \mathcal{S} = \tau^i \int_{\mathbb{R}^n \setminus \mathcal{B}} |x|^{-n} |x|^{-n} x^i X^k_i(x) d\mu = 0, \quad k \neq 1,
$$

as claimed. Gathering (5.22), (5.24) and (5.25), we obtain (5.16).

To complete the proof of Proposition 5, we differentiate (5.20) with respect to $w$ at $(\rho, \tau, w) = (0,0,0)$ to get a function

$$
\psi_w := \frac{1}{n-2} \{ D_w \}_{(\rho, \tau, w) = (0,0,0)} \overline{\Psi}_{\rho, \tau, w} \cdot w \quad (5.29)
$$

which solve uniquely

$$
\begin{cases}
\Delta \psi_w = 0 & \text{in } \mathbb{R}^n \setminus \mathcal{B} \\
\psi_w = w & \text{on } \partial \mathcal{B}.
\end{cases} \quad (5.30)
$$

The equality in (5.17) then follows from (5.19), (5.29) and (5.23). \qed
5.3. The spectral properties of the operator $\mathbb{L}$. We study the spectral properties of the operator $\mathbb{L}$ in (5.17) defined by

$$\mathbb{L}(w) := \partial_\nu \psi_w - (n - 1)w,$$

where $\psi_w$ satisfies (5.18). We consider the Kelvin transform (see (3.13) and (3.14)) of $\psi_w$

$$\mathbb{K}(\psi_w)(x) = |x|^{2-n} \psi_w \left( \frac{x}{|x|^2} \right),$$

which satisfies

$$\Delta(\mathbb{K}(\psi_w)) = 0 \quad \text{in} \ B \setminus \{0\}.$$

Moreover by direct computations, we have

$$\nabla(\mathbb{K}(\psi_w)) \cdot x = -\nabla \psi_w \cdot x + (2 - n)w = \partial_\nu \psi_w + (2 - n)w \quad \text{in} \ \partial B. \quad (5.31)$$

Since $\psi_w \in C^{2,\alpha}(\mathbb{R}^n \setminus B)$ with $\nu \in (2 - n, 0)$, we see that

$$\lim_{|x| \to \infty} |x|^{2-n} \psi_w(x) = 0.$$ 

Therefore

$$\lim_{|x| \to 0} |x|^{n-2} \mathbb{K}(\psi_w)(x) = 0,$$

so the origin is a removable singularity for $\mathbb{K}(\psi_w)$ and thus $\mathbb{K}(\psi_w)$ solves

$$\begin{cases} 
\Delta \mathbb{K}(\psi_w) = 0 & \text{in} \ B \\
\mathbb{K}(\psi_w) = w & \text{on} \ \partial B.
\end{cases} \quad (5.32)$$

By elliptic regularity theory, $\mathbb{K}(\psi_w) \in C^{2,\alpha}(\overline{B})$. From (5.31) and the definition of $\mathbb{L}$ we get

$$w \mapsto \mathbb{L}(w) = \partial_\nu(\mathbb{K}(\psi_w)) - w,$$

where $\nu = \Theta$. 

Thank to [8], the spectrum of the operator $\mathbb{L}$ is given by

$$\lambda_k = k - 1, \quad k \in \mathbb{N} \quad (5.33)$$

meaning that the kernel of the operator $\mathbb{L}$ is given by the space $V_1$ spanned by linear coordinates on the sphere $S$

$$V_1 := \{ \Theta^i, i = 1, ..., n \}.$$ 

Moreover there exists a constant $C > 0$ such that

$$||w||_{C^{2,\alpha}} \leq C||\mathbb{L}(w)||_{C^{1,\alpha}(S)}, \quad (5.34)$$

provided $w \in \Pi^\perp C^{\alpha}(S)$. 

6. Solving the nonlinear problem

Define the mapping
\[
\tilde{G}(\rho, \tau, w) := G(\rho, \tau, w) + \frac{(n - 1)(n - 2)^2}{2} \sigma \partial_2 \mathcal{K} = G(\rho, \tau, w) - \frac{(n - 1)(n - 2)^2}{2} \sigma x \cdot \nabla \mathcal{K},
\]
where \( \mathcal{K} \) is the unique solution of
\[
\begin{cases}
\Delta \mathcal{K} = r^{1-2n} & \text{in } \mathbb{R}^n \setminus \mathcal{B} \\
\mathcal{K} = 0 & \text{on } \partial \mathcal{B}.
\end{cases}
\]
(6.1)

Then from (6.1) and (5.20),
\[
\tilde{G}(0, 0, 0) = 0.
\]
(6.3)

We want to prove that provided \( \rho \) is small, we can find \( \tau \) and \( w \) such that
\[
\tilde{G}(\rho, \tau, w) = 0.
\]
(6.4)

We denote by \( \Pi \) the orthogonal projection from \( C^1, \alpha(S) \) onto \( V_1 \) and \( T : V_1 \rightarrow \mathbb{R}^n \) the isomorphism sending \( x_i \mid_{\partial \mathcal{B}} \) to \( e_i \). We also define \( \tilde{\Pi} := T \circ \Pi \),
\[
\tilde{K} := \tilde{\Pi} \circ \tilde{G} : (0, \rho_0) \times B_{c_0}(0) \times B_{c_0}(0) \rightarrow \mathbb{R}^n
\]
and consider the equation
\[
\tilde{K}(\rho, \tau, w) = 0.
\]
(6.5)

The mapping \( \tilde{K} \) has the following properties:
- \( \tilde{K}(0, 0, 0) = 0. \) This is from (6.3),
- \( D_\tau \mid_{(\rho, \tau, w) = (0, 0, 0)} \tilde{K} \) is the identity in \( \mathbb{R}^n \) times a constant, which follows from (5.16).

Applying the implicit function theorem, we find a unique smooth mapping
\[
(-r_0, r_0) \times B_{k_1}(0) \rightarrow B_{k_2}(0) \subset \mathbb{R}^n, \quad (\rho, w) \mapsto \tau(\rho, w)
\]
defined for some positive constants \( r_0, k_1 \) and \( k_2 \) such that
\[
\tilde{K}(\rho, \tau(\rho, w), w) = 0 \quad \text{for all } (\rho, w) \in (-r_0, r_0) \times B_{k_0}(0).
\]
(6.6)

Next we provide estimates for the function \( \tau(\rho, w) \) in (6.6). Observe that since \( \tilde{\Pi} \circ \mathbb{L} = 0 \), (6.5) is equivalent to
\[
C'(n, \sigma) \tau + \tilde{\Pi}[\rho^{1-n} P_1(1/\rho, x, \rho^{n-1} w, \tau)] = 0 \quad \text{on } S,
\]
(6.7)

where \( C'(n, \sigma) \) is the constant appearing in (5.10). This can be seen by writing the Taylor expansion of \( G \) using (5.16) and (5.17). We then deduce from (3.3) and (3.4) the estimates
\[
|\tau(\rho, w)| \leq C \rho \quad \text{and} \quad |D_w \tau(\rho, w)| \leq C \rho^{n-1}.
\]
(6.8)

Now replace \( \tau \) by \( \tau(\rho, w) \) in (6.1) and consider the equation
\[
\Pi^\perp(\tilde{G}(\rho, \tau, w)) = 0.
\]
(6.9)
From (6.17) and the estimates in (6.8),
\[
\frac{D_w}{\rho,\tau,w} = \Pi^\perp \circ \tilde{G} = \mathbb{L} : V_1^\perp \to \mathbb{L}(V_1^\perp),
\]
which is an isomorphism from Subsection 5.3. Hence, there exists a unique solution \(w(\rho)\) to (6.9) for small \(\rho \in (0, R_0)\).

Using (5.16) and (5.17), we have
\[
\mathbb{L}(w_{\rho}) + \Pi^\perp [\rho^{1-n} P_1(1/\rho, x, \rho^{n-1} w, \tau)] = 0 \quad \text{on} \quad \mathbb{S},
\]
and from (5.34) and (3.4),
\[
||w_{\rho}||_{C^2,\alpha}(\mathbb{S}) \leq C \rho.
\]

Decreasing \(R_0\) if necessary, the analysis of the previous section establishes the first statement of Theorem 1 with \(\rho_0 = \frac{1}{R_0}\) and
\[
\Omega_{\rho} = \widetilde{\Omega}_{\rho}, \quad \rho \in (\rho_0, +\infty),
\]
where
\[
\widetilde{\Omega}_{\rho} := \Phi_{\frac{1}{\rho^2}, \tau(\rho, w(\rho)), \rho^{n-1} w(\rho)}(\mathbb{R}^n \setminus \mathbb{B}), \quad \rho \in (0, R_0).
\]

In addition, recalling (3.11), we have \(\tilde{\nu}_g = \rho \tilde{\nu}_{g_{\rho}}\) for small \(\rho\) and from (5.9) and (6.1), we see that the constant \(C(\rho, \sigma, n)\) in Theorem 1 is given by
\[
C(\rho, \sigma, n) = \frac{n-2}{\rho} (1 - \frac{\sigma}{2} \rho^{1-n}) - \frac{(n-1)(n-2)^2}{2} \sigma \rho^{-n} x \cdot \nabla K(x) \bigg|_{\mathbb{S}},
\]
where \(K\) is the unique solution of (6.2). It is plain that the function \(K\) is radial. By Gauss-Green formula, we get
\[
C'(1) = -\frac{1}{n-1} \quad \text{and we deduce}
\]
\[
C(\rho, \sigma, n) = \frac{n-2}{\rho} + \frac{(n-2)(n-3)}{2\rho^n} \sigma.
\]

It remains to show that the family \((\partial \Omega_{\rho})_{\rho \in (\rho_0, +\infty)}\) constitutes a smooth foliation.

### 7. Foliation by boundaries of extremal capacitors

**Proposition 6.** There exists a constant \(\rho_0 > 1\) such that the family \((\partial K_{\rho})_{\rho > \rho_0}\) constitutes a smooth foliation.

**Proof.** We are proving that the family \((\partial \Omega_{\rho})_{\rho \in (\rho_0, +\infty)}\) constitutes a foliation of \(M \setminus \Omega_{\rho_0}\). The proof is inspired by the argument in \cite[Section 5]{2} and \cite[Pages 9-10]{12}.

Notice that \(\partial \widetilde{\Omega}_{\rho}\) is given by
\[
\partial \widetilde{\Omega}_{\rho} = \Phi_{\frac{1}{\rho^2}, \tau(\rho, w(\rho)), \rho^{n-1} w(\rho)},
\]
where
\[
\widetilde{S}_{\frac{1}{\rho^2}, \tau, w} = \left\{ y = \frac{1}{\rho} \left( x + \tau + w(x)x \right), \quad x \in \mathbb{S} \right\}.
\]
We define the functions
\[ h(\rho, x) := \frac{1}{\rho} \left( x + \tau(\rho, w(\rho)) + \rho^{n-1} w_\rho(x) x \right) \quad \text{and} \quad v(\rho, x) := \frac{h(\rho, x)}{|h(\rho, x)|_{g_\rho}}, \quad x \in \mathbf{S}. \]

By the estimates in (6.8) and (6.12), the function \( v(\rho, \cdot) \) extends smoothly at \( \rho = 0 \) with \( v(0, \cdot) = I_S \) and for all \( \rho \) small \( v(\rho, \cdot) \) is a diffeomorphism from \( S \) into itself. Thus for all \( y \in S \),
\[ h(\rho, v^{-1}(\rho, y)) = |h(\rho, v^{-1}(\rho, y))|_{g_\rho}. \]

We put
\[ \tilde{S}_{\rho, \tau(\rho, w(\rho)), \rho^{n-1} w_\rho} = \tilde{S}_\rho := \left\{ \tilde{\varphi}(\rho, y) y, \quad y \in S \right\}. \]

Using the estimates in (6.8) and (6.12) once again, we find
\[ D_x h = \frac{1}{\rho} \left( I_S + \rho^{n-1} L^1(w_\rho) x + \rho^{n-1} w_\rho(x) I_S \right) = \frac{1}{\rho} (I_S + O(\rho)) \]
\[ \frac{\partial h}{\partial \rho} = -\frac{1}{\rho^2} \left( x + \tau - \rho \frac{\partial \tau}{\partial \rho} - \rho D_w \tau(\rho, w(\rho)) w'_\rho + \rho^n w'_\rho(x) x - (n-2) \rho^{n-1} w_\rho(x) x \right) \]
\[ = -\frac{1}{\rho^2} (x + O(\rho)). \]

Also, since
\[ |v^{-1}(\rho, y)|_{g_\rho} = 1 \quad \text{for all} \quad \rho \in (0, R_0) \quad \text{and} \quad y \in S, \]
we have
\[ \langle v^{-1}(\rho, y), \partial_{\rho} v^{-1}(\rho, y) \rangle |_{g_\rho} = 0 \quad \text{for all} \quad \rho \in (0, R_0) \quad \text{and} \quad y \in S. \]

Using this, we then obtain
\[ \frac{\partial \tilde{\varphi}}{\partial \rho} = \frac{1}{|h(\rho, v^{-1}(\rho, y))|_{g_\rho}} \langle h(\rho, v^{-1}(\rho, y)), \frac{\partial h}{\partial \rho} (\rho, v^{-1}(\rho, y)) + D_x h(\partial_{\rho} v^{-1}) \rangle |_{g_\rho} \]
\[ = \frac{1}{|v^{-1}(\rho, y) + O(\rho)|_{g_\rho}} \langle v^{-1}(\rho, y) + O(\rho), -\frac{1}{\rho} v^{-1}(\rho, y) + \partial_{\rho} v^{-1} + O(\rho) \rangle |_{g_\rho} \]
\[ = \frac{1}{\rho |v^{-1}(\rho, y) + O(\rho)|_{g_\rho}} \langle v^{-1}(\rho, y) + O(\rho), -v^{-1}(\rho, y) + \rho \partial_{\rho} v^{-1} + O(\rho^2) \rangle |_{g_\rho} \]
\[ = -\frac{1}{\rho} (1 + O(\rho)). \]

We conclude the function \( \tilde{\varphi}(\rho, x) \) is strictly decreasing with respect to \( \rho \) for \( \rho \) small or equivalently \( \tilde{\varphi}(\rho^{-1}, x) \) is strictly increasing for \( \rho \) large. Thank to (7.2), the family \( \langle \tilde{S}_{\rho, \tau(\rho, w(\rho)), \rho^{n-1} w_\rho} \rangle_{\rho > \rho_0} \) constitutes a foliation of \( \mathbb{R}^n \setminus \mathbf{B}_{\rho_0} \). Since \( \Phi \) is a diffeomorphism and
\( \partial \Omega_\rho = \Phi(\tilde{S}_\lambda) \), we deduce that the family \((\partial \Omega_\rho)_{\rho \in (\rho_0, +\infty)}\) foliates \(M \setminus \Omega_{\rho_0}\) and the proof of Theorem \([1]\) is complete.

\[ \square \]

**Appendix A. Variational setting**

In this section we define two scale-invariant energies associated with capacity, and compute their first variations. We perform the computations below in Euclidean space, but again it is an easy exercise to carry them out in a Riemannian manifold with one asymptotically flat end.

We earlier defined the capacity function \(\text{Cap} \) on compact sets \(K \subset \mathbb{R}^n\) in (1.1). A change of variables shows us the scaling law

\[ \text{Cap}(RK) = R^{n-2} \text{Cap}(K) \quad (A.1) \]

for any \(R > 0\), and a quick comparison gives \(\text{Cap}(B_R) = R^{n-2}\), with the equilibrium potential function \(U(x) = R^{n-2}|x|^{2-n}\). As we discussed in the introduction, one can either normalize using volume or surface area, leading to the following two scale-invariant functionals:

\[ E_0(K) = \frac{\text{Cap}(K)}{|K|^\frac{n-2}{n}}, \quad E_1(K) = \frac{\text{Cap}(K)}{|\Sigma|^\frac{n-2}{n-1}}. \quad (A.2) \]

By (A.1) both \(E_0\) and \(E_1\) are scale-invariant.

To compute the first variation of both \(E_0\) and \(E_1\) we let \(X : (\epsilon, \epsilon) \times \mathbb{R}^n \to \mathbb{R}^n\) be a vector field, and let \(\xi\) be its flow, defined by

\[ \xi : (-\epsilon, \epsilon) \times \mathbb{R}^n \to \mathbb{R}^n, \quad \frac{\partial \xi}{\partial t}(t, x) = X(t, x), \quad \xi(0, x) = x. \]

Let \(K_t = \xi(t, \cdot)(K)\) and let \(U_t\) be the solution to (1.2) in \(\Omega_t = \mathbb{R}^n \setminus K_t\). It will also be convenient to denote \(\Sigma = \partial K = \partial \Omega\), and \(\Sigma_t = \partial K_t = \partial \Omega_t\).

**Lemma A.1.** We have

\[ \frac{d}{dt} \text{Cap}(K_t) \bigg|_{t=0} = \frac{1}{n(n-2)\omega_n |K|^\frac{n-2}{n}} \left[ \frac{(n-2)^2 \omega_n \text{Cap}(K)}{|K|} \int_\Sigma \langle X, \eta \rangle d\sigma - \int_\Sigma \langle X, \eta \rangle \left( \frac{\partial U}{\partial \eta} \right)^2 d\sigma \right], \quad (A.3) \]

where \(\eta\) is the unit normal vector pointing into \(K\).

**Proof.** Observe that \(U_t|_{\Sigma_t} = 0\). Differentiating this boundary condition, we see

\[ \frac{\partial U}{\partial t} = -\langle X, \eta \rangle \frac{\partial U}{\partial \eta} \text{ on } \Sigma_t. \]
\[
\frac{d}{dt} \bigg|_{t=0} E_0(K_t) = \frac{1}{n(n-2)\omega_n} \frac{d}{dt} \bigg|_{t=0} |K|^\frac{2}{n} \int_{\Omega_t} |\nabla U_t|^2 dx
\]

\[
= \frac{(n-2)}{n} \frac{1}{n(n-2)\omega_n |K|^\frac{n-2}{n-1}} \int_{\Sigma} \langle X, \eta \rangle \, d\sigma
\]

\[
+ \frac{1}{n(n-2)\omega_n |K|^\frac{n-2}{n-1}} \int_{\Sigma} \langle X, \eta \rangle \, d\sigma
\]

\[
= \frac{(n-2)}{n} \frac{1}{n(n-2)\omega_n |K|^\frac{n-2}{n-1}} \int_{\Sigma} \langle X, \eta \rangle \, d\sigma
\]

\[
+ \frac{1}{n(n-2)\omega_n |K|^\frac{n-2}{n-1}} \left[ \int_{\Sigma} \langle X, \eta \rangle \, d\sigma - 2 \int_{\Omega} \frac{\partial U}{\partial t} \Delta U \, dx + 2 \int_{\Sigma} \frac{\partial U}{\partial t} \frac{\partial U}{\partial \eta} \right]
\]

\[
= \frac{(n-2)}{n} \frac{1}{n(n-2)\omega_n |K|^\frac{n-2}{n-1}} \int_{\Sigma} \langle X, \eta \rangle \, d\sigma
\]

\[
= \frac{1}{n(n-2)\omega_n |K|^\frac{n-2}{n}} \left[ \frac{(n-2)^2 \omega_n \text{Cap}(K)}{|K|} \int_{\Sigma} \langle X, \eta \rangle \, d\sigma - \int_{\Sigma} \langle X, \eta \rangle \left( \frac{\partial U}{\partial \eta} \right)^2 \, d\sigma \right].
\]

Here we have used the fact that \( U \) is constant on \( \Sigma \), so \( |\nabla U| = \frac{\partial U}{\partial \eta} \) there. \( \square \)

**Corollary 7.** A compact set \( K \) with nonempty interior \( \mathcal{O} \) and smooth boundary \( \Sigma \) is a critical point of \( E_0 \) if and only if \( \Omega = \mathbb{R}^n \setminus K \) supports a solution to the over-determined boundary value problem (1.3).

**Proof.** Setting

\[
\Lambda^2 = \frac{(n-2)^2 \omega_n \text{Cap}(K)}{|K|},
\]

we use (A.3) to see that \( K \) is a critical point of \( E_0 \) if and only if

\[
\int_{\Sigma} \langle X, \eta \rangle \left[ \Lambda^2 - \left( \frac{\partial U}{\partial \eta} \right)^2 \right] \, d\sigma = 0
\]

for all possible variation fields \( X \), which in turn implies

\[
\frac{\partial U}{\partial \eta} = \Lambda = (n-2)^2 \sqrt{\frac{\omega_n \text{Cap}(K)}{|K|}}
\]

along \( \Sigma \). We notice that if \( K = B_\rho \), this constant reduces to \( \Lambda = \frac{n-2}{\rho} \).

Conversely, let \( \Omega \) admit a solution to (1.3). By the uniqueness of solutions to (1.2), this function must be the equilibrium potential of \( K \), so by (A.3)

\[
\frac{d}{dt} E_0(K_t) \bigg|_{t=0} = 0
\]
for all possible variation fields $X$.  

Though we will not use it, we include the following derivation to satisfy the reader’s curiosity.

**Lemma A.2.** We have

$$\frac{d}{dt}E_{1}\bigg|_{t=0} = \frac{1}{\left|\Sigma\right|^{\frac{2-n}{n-1}}} \left[ \frac{n-2}{n-1} \left( \frac{\text{Cap}(K)}{\left|\Sigma\right|} \right) \int_{\Sigma} vH_{\Sigma}d\sigma + \frac{1}{n(n-2)\omega_{n}} \int_{\Sigma} \sqrt{v} \left( \frac{\partial U}{\partial \eta} \right)^{2} d\sigma \right].$$

**Proof.** We begin by differentiating the boundary condition $U|_{\Sigma} = 1$ to see

$$\frac{\partial U}{\partial t} = -v \frac{\partial U}{\partial \eta}$$
on $\Sigma$. Thus

$$\frac{d}{dt}E_{1}\bigg|_{t=0} = \frac{d}{dt}\bigg|_{t=0} \left( \frac{1}{\left|\Sigma\right|^{\frac{2-n}{n-1}}} \int_{\Omega} \langle \nabla U_{t}, \nabla U_{t} \rangle dx \right)$$

$$= \frac{1}{\left|\Sigma\right|^{\frac{n-2}{n-1}}} \left( \frac{2-n}{n-1} \right) \left|\Sigma\right|^{-1} \frac{1}{n(n-2)\omega_{n}} \int_{\Omega} \left| \nabla U \right|^{2} dx \int_{\Sigma} vH_{\Sigma}d\sigma$$

$$+ \frac{1}{n(n-2)\omega_{n}} \left|\Sigma\right|^{\frac{n-2}{n-1}} \left[ 2 \int_{\Omega} \left( \nabla U, \nabla \frac{\partial U}{\partial t} \right) dx + \int_{\Sigma} v\left| \nabla U \right|^{2} d\sigma \right]$$

$$= -\left( \frac{n-2}{n-1} \right) \frac{\text{Cap}(K)}{\left|\Sigma\right|^{\frac{n-2}{n-1}}} \int_{\Sigma} vH_{\Sigma}d\sigma$$

$$+ \frac{1}{n(n-2)\omega_{n}} \left|\Sigma\right|^{\frac{n-2}{n-1}} \left[ \int_{\Sigma} v\left| \nabla U \right|^{2} d\sigma - 2 \int_{\Omega} \frac{\partial U}{\partial t} \Delta_{0} U dx + 2 \int_{\Sigma} \frac{\partial U}{\partial t} \frac{\partial U}{\partial \eta} d\sigma \right]$$

$$= -\frac{1}{\left|\Sigma\right|^{\frac{n-2}{n-1}}} \left[ \frac{n-2}{n-1} \frac{\text{Cap}(K)}{\left|\Sigma\right|} \int_{\Sigma} vH_{\Sigma}d\sigma + \frac{1}{n(n-2)\omega_{n}} \int_{\Sigma} \left( \frac{\partial U}{\partial \eta} \right)^{2} d\sigma \right].$$

Here we’ve used the fact that $H_{\Sigma}$ is the first variation of $|\Sigma|$ and that $|\nabla U| = -\frac{\partial U}{\partial \eta}$ on $\Sigma$.

One can mimic the proof of Corollary 7 to show that critical points of $E_{1}$ are precisely those sets $K$ which admit a solution to the over-determined boundary value problem

$$\Delta U = 0, \quad U|_{\partial K} = 1, \quad \lim_{|x| \to \infty} U(x) = 0, \quad \frac{\partial U}{\partial \eta} = \Lambda H,$$

where $H$ is the mean curvature of $\Sigma = \partial K$.

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