Local distinction, quadratic base change and automorphic induction for $GL_n$

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Abstract

Behind this sophisticated title hides an elementary exercise on Clifford theory for index two subgroups and self-dual/conjugate-dual representations. When applied to semi-simple representations of the Weil-Deligne group $W'_F$ of a non Archimedean local field $F$, and further translated in terms of representations of $GL_n(F)$ via the local Langlands correspondence when $F$ has characteristic zero, it yields various statements concerning the behaviour of different types of distinction under quadratic base change and automorphic induction. When $F$ has residual characteristic different from 2, combining of one of the simple results that we obtain with the triviality of conjugate-orthogonal root numbers ([GGP12]), we recover without using the LLC a result of Serre on the parity of the Artin conductor of orthogonal representations of $W'_F$ ([Ser71]). On the other hand we discuss its parity for symplectic representations using the LLC and the Prasad and Takloo-Bighash conjecture.

Introduction

Let $E/F$ be a separable quadratic extension of non Archimedean local fields. Then thanks to the known local Langlands correspondence for $GL_n(E)$ and $GL_n(F)$, one has a base change map $BC^E_F$ from the set of isomorphism classes of irreducible representations of $GL_n(F)$ to that of $GL_n(E)$, and an automorphic induction map $AI^E_F$ from the set or isomorphism classes of irreducible representations of $GL_n(E)$ to that of $GL_{2n}(F)$. A typical statement proved in this note (for $F$ of characteristic zero) is that if $\pi$ is a generic unitary representation of $GL_n(F)$ with orthogonal Langlands parameter (orthogonal in short), then $BC^E_F(\pi)$ is orthogonal and $GL_n(F)$-distinguished, and that the converse holds if $\pi$ is a discrete series (see Corollary 3.1 for the general statement). Corollary 3.1 is itself a translation via the LLC of our main result which concerns representations of the Weil-Deligne group of $F$ (Proposition 3.1). Another lucky application of Proposition 3.1 is that the result of [Ser71] on the parity of Artin conductors of representations of the Weil-Deligne group of $F$ is a consequence of that in [Del76] on root numbers of orthogonal representations, when $F$ has odd residual characteristic, as we show in Corollary 4.1. We also discuss its parity for symplectic representations using the LLC and the Prasad and Takloo-Bighash conjecture in Corollary 4.2.

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1 Notation, definitions and basic facts about self-dual and conjugate-dual representations

For $K$ a non Archimedean local field we denote by $W_K$ the Weil group of $K$ (see [Tat79]), and by $W'_K = W_K \times SL_2(\mathbb{C})$ the Weil-Deligne group of $K$. By a representation of $W_K$ we mean a finite dimensional smooth complex representation of $W_K$. By a representation of $W'_K$ we mean a representation which is a direct sum of representations of the form $\phi \otimes S$, where $\phi$ is an irreducible representation of $W_K$ and $S$ is an irreducible algebraic representation of $SL_2(\mathbb{C})$. We sometimes abbreviate "$\phi$ is a representation of $W'_K$" as "$\phi \in \text{Rep}(W'_K)$". We denote by $\phi^\vee \in \text{Rep}(W'_K)$ the dual of $\phi \in \text{Rep}(W'_K)$.

For the following facts on self-dual and conjugate-dual representations of $W'_K$, we refer to [GGP12 Section 3]. We recall that a representation $\phi$ of $W'_K$ is self-dual if and only if there exists a $\phi \times \phi$ $W'_K$-invariant bilinear form $B$ which is non degenerate: we will say that $B$ is $W'_K$-bilinear (which in particular means non degenerate). If moreover $B$ is alternate, we say that $B$ is $(W'_K, -1)$-bilinear in which case we say that $\phi$ is symplectic or $(-1)$-self-dual, whereas if $B$ is symmetric, and we say that $B$ is $(W'_K, 1)$-bilinear in which case we say that $\phi$ is orthogonal or 1-self-dual. If $\phi$ is irreducible and self-dual, then there is up to nonzero scaling a unique $W'_K$-bilinear form on $\phi \times \phi$, which is either $(W'_K, -1)$-bilinear or $(W'_K, 1)$-bilinear, but not both.

Now suppose that $L/K$ is a separable quadratic extension so that $W_L$ has index two in $W_K$, and fix $s \in W_K - W_L$. For $\phi$ a representation of $W'_L$, we denote by $\phi^s$ the representation of $W'_L$ defined as $\phi^s := \phi(s \cdot s^{-1})$. We say that $\phi$ is $L/K$-dual or conjugate-dual if $\phi^s \cong \phi^\vee$. The representation $\phi \in \text{Rep}(W'_L)$ is conjugate-dual if and only if there is on $\phi \times \phi$ a non-degenerate bilinear form $B$ such that

$$B(w,x,sws^{-1},y) = B(x,y)$$

for all $(w,x,y)$ in $W'_L \times \phi \times \phi$. We say that such a bilinear form $B$ is $L/K$-bilinear (this in particular means non degenerate). If moreover there is $\varepsilon \in \{\pm 1\}$ such that $B$ satisfies

$$B(x, s^2y, \varepsilon) = \varepsilon B(y,x)$$

for all $(x,y)$ in $\phi \times \phi$ we say that $B$ is $(L/K, \varepsilon)$-bilinear, in which case we say that $\phi$ is $(L/K, \varepsilon)$-dual or conjugate-symplectic if $\varepsilon = -1$ and conjugate-orthogonal if $\varepsilon = 1$. All the definitions above do not depend on the choice of $s$. When $\phi$ is $L/K$-dual and also irreducible, then there is up to nonzero scaling a unique $L/K$-bilinear form on $\phi \times \phi$, which is either $(L/K, -1)$-bilinear or $(L/K, 1)$-bilinear, but not both.

2 Preliminary results

2.1 Clifford-Mackey theory for index two subgroups

We refer to [OSSST09 Section 3] for the following standard results.

Theorem 2.1. Let $G$ be a finite group, $H$ a finite subgroup of index 2, $s \in G - H$, and let $\eta: G \to \{\pm 1\}$ be the nontrivial character of $G$ trivial on $H$.

- For $\phi$ a (finite dimensional complex) representation of $H$ which is irreducible, the representation $\text{Ind}_H^G(\phi)$ is irreducible if and only if $\phi^s \neq \phi$, which is also equivalent to the fact that $\phi$ does not extend to $G$. If it is reducible then $\phi$ extends to $G$, and if $\tilde{\phi}$ is such an extension, then $\eta \otimes \tilde{\phi}$ is the only other extension different from $\phi$, and $\text{Ind}_H^G(\phi) \cong \tilde{\phi} \otimes (\eta \otimes \tilde{\phi})$. 

2
• An irreducible representation $\phi'$ of $G$ restricts to $H$ either irreducibly, or breaks into two irreducible pieces, and the second case occurs if and only if $\phi' \simeq \eta \otimes \phi'$, which is also equivalent to $\phi' = \text{Ind}^G_H(\phi)$ for $\phi$ an irreducible representation of $H$ such that $\phi^* \simeq \phi$.

For $E/F$ a separable quadratic extension of non Archimedean local fields, we denote by $\eta_{E/F}: W'_F \to \{\pm 1\}$ the nontrivial character of $W'_F$ trivial on $W'_E$. Theorem 2.1 has the following corollary.

Corollary 2.1. Let $E/F$ be a separable quadratic extension of non Archimedean local fields, and fix $s \in W_F - W_E$.

• For $\phi_E \in \text{Rep}(W'_E)$ an irreducible representation, the representation $\text{Ind}^{W'_F}_{W'_E}(\phi_E)$ is irreducible if and only if $\phi'_E \neq \phi_E$, which is also equivalent to the fact that $\phi_E$ does not extend to $W'_F$. If it is reducible then $\phi_E$ extends to $W'_F$, and if $\phi_F$ is such an extension, then $\eta_{E/F} \otimes \phi_F$ is the only other extension different from $\phi_F$, and $\text{Ind}^{W'_F}_{W'_E}(\phi_E) \simeq \phi_F \otimes (\eta_{E/F} \otimes \phi_F)$.

• An irreducible representation $\phi_F$ of $W'_F$ restricts to $W'_E$ either irreducibly, or breaks into two irreducible pieces, and the second case occurs if and only if $\phi_F = \eta_{E/F} \otimes \phi_F$, which is also equivalent to $\phi_F \simeq \text{Ind}^{W'_F}_{W'_E}(\phi_E)$ for $\phi_E$ and irreducible representation of $W'_E$ such that $\phi_E^* \simeq \phi_E$.

Proof. We recall that by [BH06, 28.6], if $\alpha_K$ is an irreducible representation of $W_K$ for $K$ local and non Archimedean, then there exists an unramified character $\chi_K$ of $W_K$ such that $\chi_K \otimes \alpha_K$ has co-finite kernel.

For the first part of the first point, write $\phi_E = \alpha_E \otimes S$, and suppose first that $\text{Ind}^{W'_F}_{W'_E}(\phi_E)$ is irreducible. Twist $\text{Ind}^{W'_F}_{W'_E}(\alpha_E)$ by an unramified character $\chi_F$ so that $\text{Ind}^{W'_F}_{W'_E}(\phi_E) \otimes \chi_F$ has a co-finite kernel (hence $\text{Res}^{W'_F}_{W'_E}(\chi_F) \otimes \alpha_E$ has co-finite kernel as well, as it has to be trivial on $W_E \cap \text{Ker}(\text{Ind}^{W'_F}_{W'_E}((\text{Res}^{W'_F}_{W'_E}(\chi_E) \otimes \alpha_E))))$. Because $\text{Res}^{W'_F}_{W'_E}(\chi_F)^* = \text{Res}^{W'_F}_{W'_E}(\chi_F)$, one deduces from Theorem 2.1 applied to $\text{Res}^{W'_F}_{W'_E}(\chi_E) \otimes \alpha_E = \alpha'_E$ and that $\alpha'_E \neq \alpha_E$ and that $\alpha_E$ does not extend. This implies the same statements for $\phi_E$. Conversely if $\phi'_E \neq \phi_E$, then the same holds for $\alpha_E$. Take $\chi'_E$ unramified such that $\chi'_E \otimes \alpha_E$ has co-finite kernel, and $\chi_F$ any unramified extension of $\chi_E$ to $W_F$. Then $\text{Ind}^{W'_F}_{W'_E}(\alpha_E) = \chi_F^{-1} \otimes \text{Ind}^{W'_F}_{W'_E}(\chi_E \otimes \alpha_E)$ is irreducible by Theorem 2.1 and so is $\text{Ind}^{W'_F}_{W'_E}(\phi_E) = \text{Ind}^{W'_F}_{W'_E}(\alpha_E) \otimes S$. The second part of the first point is similar, using an unramified character $\chi_F$ of $W_F$ such that $\chi_E \otimes \phi_E$ has cofinite kernel (just take such a $\chi_E$ and extend it to an unramified character of $W_F$).

The proof of the second point is similar. □

We will tacitly use the above corollary from now on.

2.2 Distinction and LLC for $GL_n$

Let $F$ be a non Archimedean local field, we denote by LLC the local Langlands correspondence ([LRS93], [HT02], [Hen00]). For any $n \geq 1$, it restricts as a bijection from the set of isomorphism classes of $n$-dimensional representations of $W'_F$, to that of (smooth and complex) irreducible representations of $GL_n(F)$. If $E/F$ is a quadratic extension, and $\pi = \text{LLC}(\phi_F)$ for $\phi_F$ a representation of $W'_F$, we set $B_{E/F}(\pi) = \text{LLC}(\text{Res}_{W'_E}(\phi_F))$ (the quadratic base change of $\pi$), whereas if $\pi = \text{LLC}(\phi_E)$ for $\phi_E$ a representation of $W'_E$, we set $A_{E}(\pi) = \text{LLC}(\text{Ind}_{W'_E}(\phi_E))$ (the quadratic
Theorem 2.2. Suppose that \( \tau \) is irreducible, we call it a discrete series representation if it has a matrix coefficient \( c \) such that \(|c| \neq 0\). A representation \( \phi \) of \( \mathrm{GL}_n(F) \) is irreducible if and only if \( \mathrm{LLC}(\phi) \) is a discrete series.

Let \( N_0(F) \) be the subgroup of \( \mathrm{GL}_n(F) \) of upper triangular unipotent matrices, and let \( \psi \) be a non trivial character of \( F \), which in turn defines a character \( \tilde{\psi} : u \mapsto \psi(u_1,2+\ldots+u_{n-1},n)^{\frac{1}{2}} \) of \( N_0(F) \).

We say that an irreducible representation \( \pi \) of \( \mathrm{GL}_n(F) \) is generic if \( \mathrm{Hom}_{N_0(F)}(\pi, \tilde{\psi}) \neq \{0\} \) and this does not depend on the choice of \( \psi \). Genericity can be read on the Langlands parameter \( [\mathrm{Zel10}] \) (one way to state it is that \( \mathrm{LLC}(\phi) \) is generic if and only if the adjoint L factor of \( \phi \) is holomorphic at \( s = 1 \)). From this one easily deduces the direct implications of the following proposition, the converse implications being special cases of \([\mathrm{MS20}] \) Theorem 9.1.

**Proposition 2.1.**

- Let \( \pi \) be an irreducible representation of \( \mathrm{GL}_n(F) \). If \( \mathrm{BC}_F(\pi) \) is generic, then \( \pi \) is generic, and conversely if \( \pi \) is generic unitary, then \( \mathrm{BC}_F(\pi) \) is generic (unitary).

- Let \( \tau \) be an irreducible representation of \( \mathrm{GL}_n(E) \). If \( \mathrm{AI}_E(\tau) \) is generic, then \( \tau \) is generic, and conversely if \( \tau \) is generic unitary, then \( \mathrm{AI}_E(\tau) \) is generic (unitary).

We denote by \( \mathrm{GL}_n(F) \) the double cover of \( \mathrm{GL}_n(F) \) defined for example in \([\mathrm{Kap17}] \) Section 2.1. Following \([\mathrm{Kap17}] \) we call a map \( \gamma : F^\times \to \mathbb{C}^\times \) a pseudo-character if it satisfies \( \gamma(xy) = \gamma(x)\gamma(y)(x, y)_2 \) for all \( x \) and \( y \) in \( F^\times \), where \( (\ldots)_2 \) is the Hilbert symbol of \( F^\times \). For \( \gamma \) a pseudo-character of \( F^\times \) we denote by \( \theta_{1,\gamma} \) the corresponding Kazhdan-Patterson exceptional representation of \( \mathrm{GL}_n(F) \) as in \([\mathrm{Kap17}] \) Section 2.5. We say that an irreducible representation \( \pi \) of \( \mathrm{GL}_n(F) \) is \( \Theta_F \)-distinguished if there exist pseudo-characters \( \gamma \) and \( \gamma' \) of \( F^\times \) such that \( \mathrm{Hom}_{\mathrm{GL}_n(F)}(\theta_{1,\gamma} \otimes \theta_{1,\gamma'}, \pi^\vee) \neq \{0\} \) (where \( \theta_{1,\gamma} \otimes \theta_{1,\gamma'} \) indeed factors through \( \mathrm{GL}_n(F) \) so that the definition makes sense).

When \( n \) is even, we denote by \( S_n(F) \) the Shalika subgroup of \( \mathrm{GL}_n(F) \) consisting of matrices of the form \( s(g, x) = \text{diag}(g, g) \begin{pmatrix} I_{n/2} & x \\ I_{n/2} & \end{pmatrix} \) for \( g \in \text{GL}_{n/2}(F) \) and \( x \in M_{n/2}(F) \), and for \( \psi \) a non trivial character of \( F \), we denote by \( \Psi \) the character of \( S_n(F) \) defined by \( \Psi(s(g, x)) = \psi(\text{tr}(x)) \).

We say that an irreducible representation \( \pi \) of \( \mathrm{GL}_n(F) \) is \( \Psi_F \)-distinguished if \( n \) is even and \( \mathrm{Hom}_{S_n(F)}(\pi, \Psi) \neq \{0\} \) for some non trivial character \( \psi \) of \( F \). This does not depend on the choice of \( \psi \).

Finally if \( E/F \) is quadratic separable, identifying \( \eta_{E/F} \) to the character of \( F^\times \) trivial on \( N_{E/F}(E^\times) \) via local class field theory, we say that an irreducible representation \( \tau \) of \( \mathrm{GL}_n(E) \) is \( 1_{E/F} \)-distinguished if \( \mathrm{Hom}_{\mathrm{GL}_n(E)}(\tau, 1) \neq \{0\} \) and \( \eta_{E/F} \)-distinguished if \( \mathrm{Hom}_{\mathrm{GL}_n(E)}(\tau, \eta_{E/F} \circ \det) \neq \{0\} \).

The following theorem follows from \([\mathrm{Hen10}] \), \([\mathrm{Kab04}] \), \([\mathrm{AKT04}] \), \([\mathrm{AR05}] \), \([\mathrm{Mat11}] \), \([\mathrm{KR12}] \), \([\mathrm{Jo20}] \), \([\mathrm{Mat17}] \), \([\mathrm{Yam17}] \), \([\mathrm{Kap17}] \). Parts of it are known to hold when \( E \) is of positive characteristic and odd residual characteristic \( [\mathrm{AKM+21}] \) Appendix A).

**Theorem 2.2.** Suppose that \( F \) has characteristic zero.

- Let \( \pi = \mathrm{LLC}(\phi_F) \) be a generic representation of \( \mathrm{GL}_n(F) \), then \( \phi_F \) is symplectic if and only \( \pi \) is \( \Psi_F \)-distinguished, whereas \( \phi_F \) is orthogonal if and only \( \pi \) is \( \Theta_F \)-distinguished.
• Let $\tau = \text{LLC}(\phi_E)$ be a generic representation of $\text{GL}_n(E)$, then $\phi_E$ is conjugate-symplectic if and only $\tau$ is $\eta_{E/F}$-distinguished, whereas $\phi_E$ is conjugate-orthogonal if and only $\tau$ is $1_{E/F}$-distinguished.

2.3 A reminder on epsilon factors

Let $K'/K$ be a finite separable extension of non Archimedean local fields. We denote by $\varpi_K$ a uniformizer of $K$ and by $P_K$ the maximal ideal of the ring of integers $O_K$ of $K$. If $\psi$ is a non trivial character of $K$, we denote by $\psi_K$ the character $\psi \circ \text{Tr}_{K'/K}$. We call the conductor of $\psi$ and write $d(\psi)$ for the smallest integer $d$ such that $\psi$ is trivial on $P_K^d$. When $K'/K$ is unramified, it follows from [Wei74, Chapter 8, Corollary 3] that

$$d(\psi_K) = d(\psi).$$

Similarly if $\chi$ is a character of $W_K'$ identified by local class field theory with a character of $K^*$, we call the Artin conductor of $\chi$ the integer $a(\chi)$ equal to zero if $\chi$ is unramified, or equal to the smallest integer $a$ such that $\chi$ is trivial on $1 + P_K^a$ if $\chi$ is ramified. More generally one can define the Artin conductor $a(\phi)$ (which is an integer) of any representation $\phi$ of $W_K'$, see [Tat79, 3.4.5] when $\phi$ is a representation of $W_K$ and [GR10, Section 2.2] in general. The Artin conductor is additive:

$$a(\phi \otimes \phi') = a(\phi) + a(\phi')$$

for $\phi$ and $\phi'$ in $\text{Rep}(W_K')$. If $\phi$ is a representation of $W_K'$, and $\psi$ is a non trivial character of $K$, we refer to [Tat79, 3.6.4] and [BH06, 31.3] or [GR10, Section 2.2] for the definition of the root number $\epsilon(1/2, \phi, \psi)$. One then defines the Langlands $\lambda$-constant:

$$\lambda(K'/K, \psi) = \frac{\epsilon(1/2, \text{Ind}_{W_K'}^W(1_{W_K'}), \psi)}{\epsilon(1/2, 1_{W_K'}, \psi_{K'})}.$$ 

For $a \in K^*$, we set $\psi_a = \psi(a \cdot)$. These constants enjoy the following list of properties, which we will freely use later in the paper.

1. $\epsilon(1/2, \phi \circ \phi', \psi) = \epsilon(1/2, \phi, \psi)\epsilon(1/2, \phi', \psi)$ where $\phi'$ is another representation of $W_K'$ ([Tat79, 3.4.2]).

2. $\epsilon(1/2, \phi, \psi_a) = \det(\phi(a)) \epsilon(1/2, \phi, \psi)$ ([Tat79, 3.6.6]).

3. $\epsilon(1/2, \phi, \psi)^2 = \det(\phi)(-1)$ when $\phi$ is self-dual ([GR10, Section 2.3, (11)]).

4. If $d(\psi) = 0$ and $\mu$ is an unramified character of $K^*$, it follows from [GR10, Section 2.3, (9)] that:

$$\epsilon(1/2, \mu \circ \phi, \psi) = \mu(\varpi_K^a(\phi)) \epsilon(1/2, \phi, \psi).$$

5. If $K'/K$ is quadratic with $K$ of characteristic not 2, $\delta \in \ker(\text{Tr}_{K'/K}) - \{0\}$, and $\phi$ is a $K'/K$-orthogonal representation of $W_K'$, then by [GGP12, Proposition 5.2] (generalizing [FQ73, Theorem 3]):

$$\epsilon(1/2, \phi, \psi_{K'}) = \det(\phi)(\delta).$$

6. If $\phi_{K'}$ is an $r$-dimensional representation of $W_K'$, then

$$\epsilon(1/2, \text{Ind}_{W_K'}^{W_{K'}}(\phi_{K'}), \psi) = \lambda(K'/K, \psi)^r \epsilon(1/2, \phi_{K'}, \psi_{K'}).$$
Proposition 3.1. When applied to a $K'/K$ quadratic and $\phi_{K'} = \text{Res}_{W_{K'}'}^W(\phi)$ for $\phi$ a representation of $W_{K'}'$, one gets

$$\varepsilon(1/2, \psi)\varepsilon(1/2, \eta_{K'/K} \otimes \phi, \psi) = \lambda(K'/K, \psi)^\vee \varepsilon(1/2, \text{Res}_{W_{K'}'}^W(\phi), \psi_{K'})$$

7. If $K'/K$ is unramified with $[K'/K] = n$:

$$\lambda(K'/K, \psi) = (-1)^{d(\psi)(n-1)}$$

(for example [Moy86] and [2], together with Equation (1)). In particular if $d(\psi) = 0$ then

$$\lambda(K'/K, \psi) = 1.$$ 

### 3 Distinction, base change, and automorphic induction

From now on $E/F$ is a separable quadratic extension of non Archimedean local fields. Our main result is the following proposition, and we notice that half of its first point is [GGPT12] Lemma 3.5, (i).

**Proposition 3.1.** 1. Let $\phi_E$ be a semi-simple representation of $W_E'$ which is either $\varepsilon$-self-dual or $(E/F, \varepsilon)$-dual, then $\text{Ind}_{W_E'}^{W_E}(\phi_E)$ is $\varepsilon$-selfdual.

2. Conversely if $\phi_E$ is irreducible and $\text{Ind}_{W_E'}^{W_E}(\phi_E)$ is $\varepsilon$-self-dual:

(a) if $\text{Ind}_{W_E'}^{W_E}(\phi_E)$ is irreducible, i.e. $\phi_E^s \not= \phi_E$, then either $\phi_E$ is $\varepsilon$-self-dual or $(E/F, \varepsilon)$-dual, but not both together,

(b) if $\text{Ind}_{W_E'}^{W_E}(\phi_E)$ is reducible, i.e. $\phi_E^s = \phi_E$, then $\phi_E$ is both $\varepsilon$-self-dual and $(E/F, \varepsilon)$-dual.

3. Let $\phi_F$ be a semi-simple representation of $W_F'$ which is $\varepsilon$-self-dual, then $\text{Res}_{W_E'}^{W_E}(\phi_F)$ is $\varepsilon$-self-dual and $(E/F, \varepsilon)$-dual.

4. Conversely, if $\phi_F$ is irreducible and $\text{Res}_{W_E'}^{W_E}(\phi_F)$ is $\varepsilon$-self-dual and $(E/F, \varepsilon)$-dual then $\phi_F$ is $\varepsilon$-self-dual.

**Proof.** 1. First suppose that $B_E$ is a $(E/F, \varepsilon)$-bilinear form on $\phi_E$. Write an element $v$ (resp. $v'$) in $\text{Ind}_{W_E'}^{W_E}(\phi_E)$ under the form $v = x + s^{-1}y$ (resp. $v' = x' + s^{-1}.y'$) for $x, x', y, y'$ in $\phi_E$, and set

$$B_F(v, v') = B_E(x, y') + \varepsilon B_E(x', y).$$

Then $B_F$ is $W_E'$-invariant because $B_E$ is $(W_E', \varepsilon)$-conjugate (it is non-degenerate because so is $B_E$). Finally

$$B_F(s.v, s.x') = B_E(y, s^2.x') + \varepsilon B_E(y', s^2.x) = \varepsilon B_E(x', y) + B_E(x, y') = B_F(v, v').$$

Similarly if $B_E$ is $(W_E', \varepsilon)$-bilinear, then one checks that

$$B_F(x + s^{11}y, x' + s^{1}.y') = B_E(x, x') + B_E(y, y')$$

defines a $(W_E', \varepsilon)$-bilinear form on $\phi_F$. 

6
2. Suppose that $\phi_E$ is irreducible and that $\text{Ind}_{W_E}^{W_F}(\phi_E)$ is $\varepsilon$-self-dual with $(W_F, \varepsilon)$-bilinear form $B_F$.

(a) If $\phi_E^s \neq \phi_E$, because $\text{Ind}_{W_E}^{W_F}(\phi_E)$ is self-dual then either $\phi_E$ is self-dual, or $\phi_E^s = \phi_E'$ but not both together. In the first case, say that $\phi_E$ is $\varepsilon'$-self-dual, then so is $\text{Ind}_{W_E}^{W_F}(\phi_E)$ by \[\Box\] but then $\varepsilon' = \varepsilon$ by irreducibility of $\text{Ind}_{W_E}^{W_F}(\phi_E)$. If $\phi_E^s = \phi_E'$ we conclude in a similar manner.

(b) If $\phi_E^s = \phi_E$ then $\text{Ind}_{W_E}^{W_F}(\phi_E) = B \phi \otimes \eta_{E/F} \otimes \phi$ for $\phi$ extending $\phi_E$, and $\phi \neq \eta_{E/F} \otimes \phi$. Because $\phi \neq \eta_{E/F} \otimes \phi$ there are two disjoint cases. The first is when $\phi$ is self-dual, in which case $\phi \leq \eta_{E/F} \otimes \phi$ and $B_F$ restricts non trivially to $\phi \otimes \phi$ (and $\eta_{E/F} \otimes \phi = \phi \otimes \phi$). Then $\phi_E$ is $\varepsilon$-dual and $(\varepsilon, s)$-dual by \[\Box\] Otherwise $\phi' = \eta_{E/F} \otimes \phi$ and $B_F$ is zero on $\phi \otimes \phi$ and $\eta_{E/F} \otimes \phi \otimes \eta_{E/F} \otimes \phi$. In this case there is up to scaling a unique $W_F$-invariant bilinear form on $\text{Ind}_{W_E}^{W_F}(\phi_E)$, namely $B_F$. Because $\phi_E = \phi_E$ (by restricting the relation $\phi' = \eta_{E/F} \otimes \phi$ to $W_F$), $\phi_E$ must be $\varepsilon'$-self-dual, hence $\text{Ind}_{W_E}^{W_F}(\phi_E)$ as well by \[\Box\] but then we have $\varepsilon' = \varepsilon$ by multiplicity one of $W_F$-invariant bilinear form on $\text{Ind}_{W_E}^{W_F}(\phi_E)$. Moreover because $\phi_E = \phi_E$ the parameter $\phi_E$ is also $(\varepsilon'', s)$-self-dual and by \[\Box\] again we deduce that $\varepsilon'' = \varepsilon$.

3. Let $B_F$ be a $(W_F, \varepsilon)$-bilinear form on $\phi_F$, then it remains a $(W_F, \varepsilon)$-bilinear on $\text{Res}_{W_E}^{W_F}(\phi_F)$, and on the other hand

$$B_E(x, y) = B_F(x, s^{-1}y)$$

is an $(E/F, \varepsilon)$-bilinear form on $\text{Res}_{W_E}^{W_F}(\phi_F)$.

4. We suppose that $\phi_E$ is irreducible and that $\text{Res}_{W_E}^{W_F}(\phi_F)$ is $\varepsilon$-self-dual and also $(E/F, \varepsilon)$-dual. There are two cases to consider.

First if $\text{Res}_{W_E}^{W_F}(\phi_F)$ is irreducible, then denote by $B_E$ the $(W_E, \varepsilon)$-bilinear form on $\text{Res}_{W_E}^{W_F}(\phi_F)$. Now set $D_E(x, y) = B_E(x, s^{-1}y)$ for $x, y \in \text{Res}_{W_E}^{W_F}(\phi_F)$. Clearly $D_E$ is $E/F$-bilinear, but by irreducibility $\text{Res}_{W_E}^{W_F}(\phi_F)$ affords at most one such form up to scalar, hence $D_E$ must be $(E/F, \varepsilon)$-bilinear. This implies that for $x$ and $y$ in $\text{Res}_{W_E}^{W_F}(\phi_F)$ one has

$$B_E(sx, sy) = D_E(sx, s^2y) = \varepsilon D_E(y, sx) = \varepsilon B_E(y, x) = B_E(x, y).$$

All in all, when $\text{Res}_{W_E}^{W_F}(\phi_F)$ is irreducible we deduce that $B_E$ is in fact $W_F$-invariant hence that $\phi_F$ is $\varepsilon$-self-dual.

It remains to treat the case where $\text{Res}_{W_E}^{W_F}(\phi_F)$ is reducible. In this case it is of the form $\phi_E \oplus s^{-1} \phi_E$ where $\phi_E$ is an irreducible of $W_E$ such that $\phi_E^s \neq \phi_E$ and $\phi_E = \text{Ind}_{W_E}^{W_F}(\phi_E)$.

First because $\text{Res}_{W_E}^{W_F}(\phi_F)$ is $\varepsilon$-self-dual, then the $(W_E, \varepsilon)$-bilinear form $B_E$ on $\text{Res}_{W_E}^{W_F}(\phi_F)$ either induces an isomorphism $\phi_E^s \cong \phi_E'$ or $\phi_E \perp s^{-1} \phi_E$ for $B_E$. Similarly the $(E/F, \varepsilon)$-bilinear form $C_E$ on $\text{Res}_{W_E}^{W_F}(\phi_F)$ either induces an isomorphism $(\phi_E')' \cong \phi_E'$ or $\phi_E \perp s^{-1} \phi_E$ for $C_E$. Suppose that $B_E$ induces an isomorphism $\phi_E^s \cong \phi_E'$, then one
must have $\phi_E \perp s^{-1}\phi_E$ for $C_E$ because $\phi_E \not\equiv \phi_E' \equiv \phi_E$. This implies that $C_E$ induces an $(E/F, \varepsilon)$-bilinear form on $\phi_E$ and by point 1 we deduce that $\phi_F$ is $\varepsilon$-self-dual. On the other hand if $\phi_E \perp s^{-1}\phi_E$ for $B_E$ then $B_E$ induces an $(W_E', \varepsilon)$-bilinear form on $\phi_E$ and $\phi_E$ is $\varepsilon$-self-dual again by point 2.

\[\square\]

Supposing that $F$ has characteristic zero, we translate Proposition 3.1 via the LLC, in view of the results recalled in Section 2.2. For this we denote by $\sigma$ the Galois conjugation of $E/F$ and its extension to $\text{GL}_n(E)$, and set $\tau^\sigma = \tau \circ \sigma$ for any representation of $\text{GL}_n(E)$.

**Corollary 3.1.**

1. Let $\tau$ be an irreducible representation of $\text{GL}_n(E)$ such that $\text{AI}_E^F(\tau)$ is generic (for example $\tau$ generic unitary). If $\tau$ is either $\theta_E$-distinguished or $1_{E/F}$-distinguished, then $\text{AI}_E^F(\tau)$ is $\theta_F$-distinguished, whereas if $\tau$ is either $\psi_E$-distinguished or $\eta_{E/F}$-distinguished, then $\text{AI}_E^F(\tau)$ is $\psi_F$-distinguished.

2. Conversely if $\tau$ is a discrete series representation of $\text{GL}_n(E)$.
   (a) Suppose that $\text{AI}_E^F(\tau)$ is $\psi_F$-distinguished:
      i. if $\text{AI}_E^F(\tau)$ is a discrete series, i.e. if $\tau^\sigma \not\equiv \tau$, then either $\tau$ is $\psi_E$-distinguished or $\eta_{E/F}$-distinguished, but not both together,
      ii. if $\text{AI}_E^F(\tau)$ is not a discrete series, i.e. $\tau^\sigma \equiv \tau$, then $\tau$ is both $\psi_E$-distinguished and $\eta_{E/F}$-distinguished.

   (b) Suppose that $\text{AI}_E^F(\tau)$ is $\theta_F$-distinguished:
      i. if $\text{AI}_E^F(\tau)$ is a discrete series, i.e. $\tau^\sigma \not\equiv \tau$, then either $\tau$ is $\theta_E$-distinguished or $1_{E/F}$-distinguished, but not both together,
      ii. if $\text{AI}_E^F(\tau)$ is not a discrete series, i.e. $\tau^\sigma \equiv \tau$, then $\tau$ is both $\theta_E$-distinguished and $1_{E/F}$-distinguished.

3. Let $\pi$ be an irreducible representation of $\text{GL}_n(F)$ such that $\text{BC}_E^F(\pi)$ is generic (for example $\pi$ generic unitary). If $\pi$ is $\theta_F$-distinguished, then $\text{BC}_E^F(\pi)$ is $\theta_E$-distinguished and $1_{E/F}$-distinguished, whereas if $\pi$ is $\psi_F$-distinguished, then $\text{BC}_E^F(\pi)$ is $\psi_E$-distinguished and $\eta_{E/F}$-distinguished.

4. Conversely suppose that $\pi$ is a discrete series. If $\text{BC}_E^F(\pi)$ is $\theta_E$-distinguished and $1_{E/F}$-distinguished, then $\pi$ is $\theta_F$-distinguished, whereas if $\text{BC}_E^F(\pi)$ is $\psi_E$-distinguished and $\eta_{E/F}$-distinguished, then $\pi$ is $\psi_F$-distinguished.

4 Parity of the Artin conductor of self-dual representations

In this section $F$ is again a non Archimedean local field. First, using [GGPT12, Proposition 5.2] (which is itself a quick but non trivial consequence of a difficult result of Deligne [Del76] on root numbers of orthogonal representations), we quickly recover in odd residual characteristic from Proposition 3.1 the following result due to Serre [Ser71] (the result in question also holds in even residual characteristic by [Ser71]). In other words we show that the result of [Del76] implies that of [Ser71] for non Archimedean local fields of odd residual characteristic.

**Corollary 4.1** (of Proposition 3.1 [Ser71]). Let $\phi$ be an orthogonal representation of $W'_E$. We have the following congruence of Artin conductors: $a(\phi) = a(\text{det}(\phi))[2]$.
Proof. As we said the result is true for $F$ of any residual characteristic, and we recover it in this proof for $F$ of residual characteristic different from 2. Let $E$ be the unramified quadratic extension of $F$, and take $\psi$ a character of $F$ of conductor zero. We have according to Section 2.3 Points 6 and 7

$$\epsilon(1/2, \text{Res}_{W_E}^F(\phi), \psi_E) = \epsilon(1/2, \phi, \psi)\epsilon(1/2, \eta_{E/F} \otimes \phi, \psi).$$

(2)

Now denoting by $q$ the residual cardinality of $F$, let $u$ be an element of order $q^2 - 1$ in $E^*$, so that $\delta := u^{(q+1)/2}$ does not belong to $F$ but $\Delta := \delta^2$ belongs to $F$. Note that the image of $\Delta$ generates $O_F^*/1 + P_F$. Then $\epsilon(1/2, \text{Res}_{W_E}^F(\phi), \psi_E^{-1}) = 1$ by Proposition \[Xue21\], \[Sé20\], \[SX20\] and Section 2.3 Point 5, hence

$$\epsilon(1/2, \text{Res}_{W_E}^F(\phi), \psi_E) = \det(\text{Res}_{W_E}^F(\phi))(\delta) = \det(\phi)(N_{E/F}(\delta)) = \det(\phi)(-\Delta)$$

thanks to Section 2.3 Point 2. Now observe that $\det(\phi)$ is a quadratic as $\psi$ is self-dual, but because $q$ is odd it is trivial on $1 + P_F$, hence it has conductor 0 or 1, and it is of conductor zero if and only if $\det(\phi)(\Delta) = 1$, hence $\det(\phi)(\Delta) = (-1)^{a(\det(\phi))}$, so

$$\epsilon(1/2, \text{Res}_{W_E}^F(\phi), \psi_E) = (-1)^{a(\det(\phi))}\det(\phi)(-1).$$

Now $\epsilon(1/2, \eta_{E/F} \otimes \phi, \psi) = (-1)^{\epsilon(\phi)}\epsilon(1/2, \phi, \psi)$ thanks to Section 2.3 Point 1 hence Section 2.3 Point 3 implies the following:

$$\epsilon(1/2, \phi, \psi)\epsilon(1/2, \eta_{E/F} \otimes \phi, \psi) = (-1)^{\epsilon(\phi)}\epsilon(1/2, \phi, \psi)^2 = (-1)^{\epsilon(\phi)}\det(\phi)(-1).$$

The result now follows from Equation (2). \[\square\]

One can legitimately ask about the parity of the Artin conductor of symplectic representations of $W_F$. The answer seems much more complicated, and one way to address it is via the LLC, using the so-called Prasad and Takloo-Bighash conjecture, which is now a theorem when $F$ has characteristic zero and residual characteristic different from 2 (\[Xue21\], \[Sé20\], \[Su21\], \[SX20\]). To this end we recall that for $E/F$ a separable quadratic extension, then the matrix algebra $\mathcal{M}_n(E)$ embeds uniquely up to $GL_{2n}(F)$-conjugacy into $\mathcal{M}_{2n}(F)$ as an $F$-subalgebra by the Skolem-Noether theorem. We fix such an embedding, which in turn gives rise to an embedding of $GL_n(E)$ into $GL_{2n}(F)$. We then say that an irreducible representation $\tau$ of $GL_{2n}(F)$ is $1^{E/F}$-distinguished if and only if $\text{Hom}_{\mathcal{M}_n}(\tau, \pi) \neq \{0\}$. We recall the following theorem, which is a consequence of one part of the Prasad and Takloo-Bighash conjecture.

**Theorem 4.1** (\[Xue21\], \[Sé20\], \[SX20\]). Suppose that $F$ has characteristic zero and residual characteristic different from 2. If $\phi$ is an irreducible symplectic representation of $W_F$ of dimension $2n$, then

$$\epsilon(1/2, \phi \otimes \text{Ind}_{W_E}^F(1)) = \eta_{E/F}(-1)^n$$

if LLC($\phi$) is $1^{E/F}$-distinguished and

$$\epsilon(1/2, \phi \otimes \text{Ind}_{W_E}^F(1)) = -\eta_{E/F}(-1)^n$$

otherwise.

**Remark 4.1.** In the statement above, as the determinant of a symplectic representation is equal to 1, we suppressed the dependence of the root number $\epsilon(1/2, \phi \otimes \text{Ind}_{W_E}^F(1), \psi)$ on the non-trivial additive character $\psi$ of $F$. 

9
As an immediate corollary we obtain the following result on the parity of Artin conductors of symplectic representations.

**Corollary 4.2.** Suppose that $F$ has characteristic zero and residual characteristic different from 2, denote by $E$ the unramified quadratic extension of $F$, and let $\phi$ be an irreducible symplectic representation of $W'_F$. Then $a(\phi)$ is even if and only if $\text{LLC}(\phi)$ is $1^E/F$-distinguished.

**Proof.** It easily follows, along the lines of the proof of Corollary 4.1, from Theorem 4.1, noting that $\eta_{E/F}(-1) = 1$.

**Remark 4.2.** A general symplectic representation $\phi$ of $W'_F$ being a direct sum of the form $\bigoplus_{i=1}^r \phi_i \oplus \bigoplus_{j=1}^s (\phi_j' \oplus \phi_j'^\vee)$ for $\phi_i$ irreducible symplectic and $\phi_j'$ irreducible, we deduce the parity of $a(\phi)$ from Corollary 4.2 and such a decomposition. Namely, by Corollary 4.1 $a(\phi'_i \oplus \phi'^i_j) \equiv 0[2]$. Hence setting $\epsilon_i \in \{\pm 1\}$ being equal to 1 if and only if $\text{LLC}(\phi_i)$ is $1^E/F$-distinguished, we deduce by additivity of the Artin conductor that $(-1)^{a(\phi)} = \prod_{i=1}^r \epsilon_i$.

**Remark 4.3.** Looking at it from another angle, one sees that a symplectic discrete series representation of $\text{GL}_{2n}(F)$ is $1^E/F$-distinguished ($E/F$ unramified) if and only if it has even conductor.

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