BOUNDED AND UNBOUNDED OSCILLATING SOLUTIONS TO A PARABOLIC-ELLIPTIC SYSTEM IN TWO DIMENSIONAL SPACE

YUKI NAITO
Department of Mathematics, Ehime University
Matsuyama-shi, 790-8577, JAPAN

TAKASI SENBA
Faculty of Engineering, Kyushu Institute of Technology
1-1 Sensuicho Tobata Kitakyushu-shi, 804-8550, JAPAN

(Communicated by Juncheng Wei)

Abstract. In this paper, we consider solutions to a Cauchy problem for a parabolic-elliptic system in two dimensional space. This system is a simplified version of a chemotaxis model, and is also a model of self-interacting particles.

The behavior of solutions to the problem closely depends on the $L^1$-norm of the solutions. If the quantity is larger than $8\pi$, the solution blows up in finite time. If the quantity is smaller than the critical mass, the solution exists globally in time. In the critical case, infinite blowup solutions were found.

In the present paper, we direct our attention to radial solutions to the problem whose $L^1$-norm is equal to $8\pi$ and find bounded and unbounded oscillating solutions.

1. Introduction. In this paper, we consider radially symmetric solutions to the Cauchy problem for a parabolic-elliptic system

\[
\begin{align*}
    u_t &= \nabla \cdot (\nabla u - u \nabla v) \quad \text{in } \mathbb{R}^2 \times (0, \infty), \\
    0 &= \Delta v + u \quad \text{in } \mathbb{R}^2 \times (0, \infty), \\
    u(\cdot, 0) &= u_0 \quad \text{in } \mathbb{R}^2,
\end{align*}
\]

where $u_0$ is a non-negative $L^1$-function.

This system is a simplified version of a chemotaxis model (see [4, 9]). The model was introduced to describe the aggregation of cellular slime molds. In the model, the function $u$ denotes the density of cells and the function $v$ denotes the concentration of a chemoattractant secreted by themselves. The system is also a model of self-interacting particles (see [1]). Here, the function $u$ is the density of particles interacting with themselves through the potential $v$.

It is well known that the behavior of solutions to (1), (2) and (3) closely depends on the quantity $\Lambda = \int_{\mathbb{R}^2} u_0(x) dx$. Actually, radial solutions to (1), (2) and (3) blow...
up in finite time if $\Lambda > 8\pi$ and the condition $\Lambda < 8\pi$ is a sufficient condition for the time-global existence and the boundedness of radial solutions (see [2]). This proposition holds also true without symmetry, if
\[
(1 + |x|^2)u_0, \quad u_0 \log u_0 \in L^1(\mathbb{R}^2)
\] (4)
(see [3, 5]). Under a weaker assumption than (4), Ogawa and Nagai [11] showed the global existence and the boundedness. Recently, Nagai [10] showed that solutions exist globally in time and that these solutions converge to 0 without the assumption (4).

Here, we say that a classical solution to (1), (2) and (3) blows up at a time $T \in (0, \infty]$, if the solution $(u, v)$ satisfies that $u(\cdot, t) \in L^\infty(\mathbb{R}^2) \cap C(\mathbb{R}^2)$ for each $t \in [0, T)$ and that
\[
\limsup_{t \to T} \|u(\cdot, t)\|_{L^\infty(\mathbb{R}^2)} = \infty.
\]
If $T < \infty$, we say that the solution blows up in finite time. If $T = \infty$, we say that the solution blows up in infinite time.

In the critical case $\Lambda = 8\pi$, radial solutions exist globally in time (see [2]). Under the assumption (4), solutions also globally in time without symmetry (see [3]). The system has stationary solutions in the critical case. Some solutions converge to a stationary solution (see [2]) and some solutions are unbounded (see [3]). In [2] the existence of oscillating solutions was conjectured.

On the other hand, there are oscillating solutions to the Cauchy problem
\[
u_t = \Delta u + |u|^{p-1}u \text{ in } \mathbb{R}^N \times (0, \infty),
\]
\[
u(\cdot, 0) = u_0 \text{ in } \mathbb{R}^N
\]
(5) in the case where $N \geq 11$ and $p > \{(N - 2)^2 - 4N + 8\sqrt{N - 1}/\{(N - 2)(N - 10)\}$.

Actually, Poláčik and Yanagida [12] showed the existence of oscillating solutions to (5). In this case, the Cauchy problem has a family of stationary solutions $\{\varphi_\alpha\}_{\alpha \in \mathbb{R}}$ satisfying $\varphi_\alpha(0) = \alpha$, the stationary solution $\varphi_\alpha$ is strictly increasing in $\alpha$, and each stationary solution is stable (see [7, 8]). This property plays an important role in [12].

For a radial solution $(u, v)$ to (1), (2) and (3), the mass distribution function
\[
M(r, t) = \int_{|x| < r} u(x, t)dx \quad \text{for } r > 0 \text{ and } t \geq 0
\]
satisfies
\[
M_t = M_{rr} - \frac{1}{r} M_r + \frac{1}{2\pi r} MM_r \quad \text{in } (0, \infty) \times (0, \infty),
\]
\[
M(0, t) = 0, \quad M(\infty, t) = \lim_{r \to \infty} M(r, t) = \Lambda \quad \text{in } (0, \infty),
\]
\[
M(\cdot, 0) = M^0 \quad \text{in } (0, \infty),
\]
where
\[
M^0(r) = \int_{|x| < r} u_0(x)dx \quad \text{for } r > 0.
\]

For $b > 0$, putting
\[
u_b(x) = \frac{8b}{(1 + b|x|^2)^2} \quad \text{for } x \in \mathbb{R}^2,
\]
we see that \((u_b, \log u_b - \log 8b)\) are stationary solutions to (1) and (2) and that the corresponding mass distribution functions

\[ M_b(r) = \int_{|x| < r} u_b(x) \, dx \quad \text{for } r \in [0, \infty) \]

are also stationary solutions to (6) and (7) with \(\Lambda = 8\pi\). In [2] a stability of stationary solutions \(M_b\) was shown in the sense of \(L^1_{loc}\) norm. Since the stability does not suffice to show the existence of oscillating solutions, we show a stronger stability (see Section 3). By using the stability, we construct oscillating solutions to (6), (7) and (8). Furthermore, we must deduce that the corresponding solution \((u, v)\) to (1), (2) and (3) oscillates by using the parabolic regularity argument. That is the difference between our proof and the one in [12].

Let \(u_0\) be a radial function such that \(u_0 \geq 0\) in \(\mathbb{R}^2\) and that \(\int_{\mathbb{R}^2} u_0(x) \, dx \leq 8\pi\).

There exists a unique radial solution \((u, v)\) to (1), (2) and (3) satisfying

\[ u \in C([0, \infty) : L^\infty(\mathbb{R}^2)) \cap C^{2,1}(\mathbb{R}^2 \times (0, \infty)) \]

and

\[ v \in C^{2,1}(\mathbb{R}^2 \times (0, \infty)) \]

(see [2, Theorem 1 and Proposition 2.3]). If the function \(u\) is bounded globally in time, we can deduce from the parabolic regularity argument that the set \(\{u(\cdot, t)\}_{t \geq 0}\) is relatively compact in \(C_0(\mathbb{R}^2)\) and that \(\omega\)-limit set

\[ \omega(u) = \left\{ f \in C_0(\mathbb{R}^2) : \lim_{n \to \infty} \|u(\cdot, t_n) - f\|_{L^\infty(\mathbb{R}^2)} = 0 \right\} \]

for some sequence \(\{t_n\}_{n \geq 1} \subset (0, \infty)\) with \(\lim_{n \to \infty} t_n = \infty\)

is non-empty, compact and connected in \(C_0(\mathbb{R}^2)\). Here, we define \(C_0(\mathbb{R}^2)\) as \(\{f \in C(\mathbb{R}^2) : \lim_{|x| \to \infty} f(x) = 0\}\).

We find a solution whose \(\omega\)-limit set includes a number of stationary solutions.

**Theorem 1.1.** Suppose that \(E\) is a non-empty subset of \((0, \infty)\) satisfying \(0 < \inf E < \sup E < \infty\). Then, there exists a radial solution \((u, v)\) to (1), (2) and (3) satisfying (9), (10), (11), (12), \(\Lambda = 8\pi\),

\[ \limsup_{t \to \infty} \|u(\cdot, t)\|_{L^\infty(\mathbb{R}^2)} < \infty \]

and

\[ \omega(u) \supset \{u_b\}_{b \in E}. \]

As \(b\) tends to \(\infty\), the stationary solution \(u_b\) tends to \(8\pi\delta_0\) in the sense of measures. Here, \(\delta_0\) is the delta function whose support is the origin. By using this fact, we also find an infinite time blowup solution.

**Theorem 1.2.** For a sequence \(\{b_j\}_{j \geq 1} \subset (0, \infty)\) with \(\lim_{j \to \infty} b_j = \infty\) there exists a radial solution \((u, v)\) to (1), (2) and (3) satisfying (9), (10), (11), (12), \(\Lambda = 8\pi\),

\[ \int_{\mathbb{R}^2} u_0(x) |x|^2 \, dx = \infty \]

and

\[ \omega(u) \supset \{u_{b_j}\}_{j \geq 1}. \]
We deduce from (13) and (14) that
\[
\liminf_{t \to \infty} \|u(\cdot, t)\|_{L^\infty(\mathbb{R}^2)} < \limsup_{t \to \infty} \|u(\cdot, t)\|_{L^\infty(\mathbb{R}^2)} = \infty
\]
and that the initial data of the solutions do not satisfy (4). This means that solutions mentioned in Theorem 1.2 blow up in infinite time and that these solutions are different from blowup solutions mentioned in [3].

For solutions mentioned in Theorem 1.1 and any stationary solutions \(u_b\) with \(b \in E\), there exists a sequence \(\{t_n\}_{n \geq 1} \subset (0, \infty)\) such that \(\lim_{n \to \infty} t_n = \infty\) and that
\[
\lim_{n \to \infty} \|u(\cdot, t_n) - u_b\|_{L^\infty(\mathbb{R}^2)} = 0.
\]
In the functional space \(L^\infty(\mathbb{R}^2)\), the orbit \(\{u(\cdot, t)\}_{t \geq 0}\) passes through any neighborhood of the stationary solution \(u_b\) infinitely many times. This means that these solutions are bounded and oscillate among stationary solutions \(\{u_b\}_{b \in E}\), since the set \(\{u_b\}_{b \in E}\) is bounded in \(L^\infty(\mathbb{R}^2)\). Similarly, the solutions mentioned in Theorem 1.2 are unbounded and oscillate among stationary solutions \(\{u_{b_n}\}_{j \geq 1}\), since the set \(\{u_{b_n}\}_{j \geq 1}\) is unbounded in \(L^\infty(\mathbb{R}^2)\).

Oscillation of solutions is caused by the stability of stationary solutions, the continuity of solutions with respect to initial functions and the asymptotic behavior of initial functions at \(|x| = \infty\). We deduce from Proposition 2 and the parabolic regularity that
\[
\lim_{t \to \infty} \|u(\cdot, t) - u_b\|_{L^\infty(\mathbb{R}^2)} = 0,
\]
if
\[
\int_{|x| < r} u_0(x)dx = \int_{|x| < r} u_b(x)dx \quad \text{for any sufficiently large } r \in (0, \infty).
\]
This means that the limit \(\lim_{r \to \infty} u(\cdot, t)\) is decided by the behavior of the initial function \(u_0\) at \(|x| = \infty\). On the other hand, we deduce from the continuity of solutions with respect to initial functions that
\[
\|u_1(\cdot, t + T) - u_2(\cdot, t + T)\|_{L^\infty(\mathbb{R}^2)} \\
\leq C(T, \|u_1(\cdot, t)\|_{L^\infty(\mathbb{R}^2)}, \|u_2(\cdot, t)\|_{L^\infty(\mathbb{R}^2)})\|u_1(\cdot, t) - u_2(\cdot, t)\|_{L^\infty(\mathbb{R}^2)}.
\]
Then, if the initial function \(u_0\) satisfies that
\[
\int_{|x| < r} u_0(x)dx = \begin{cases} 
\int_{|x| < r} u_a(x)dx & \text{for } r \in (0, L), \\
\int_{|x| < r} u_d(x)dx & \text{for } r \geq L
\end{cases}
\]
for a sufficiently large \(L\), the solution \(u\) stays near the stationary solution \(u_a\) for a long time and finally converges to the stationary solution \(u_d\). Therefore, if \(\int_{|x| < r} u_0(x)dx\) suitably oscillates between \(\int_{|x| < r} u_a(x)dx\) and \(\int_{|x| < r} u_d(x)dx\) at \(r = \infty\), the solution \(u(\cdot, t)\) oscillates between two stationary solutions \(u_a\) and \(u_d\). That is the reason why the solution oscillates.

This mechanism appears in the proof of Proposition 5 directly.

This indicates that the dynamics of solutions to (1), (2) and (3) are complicated. The behavior is a complete contrast to the one of positive solutions to the so-called Keller-Segel system in a bounded domain of \(\mathbb{R}^2\) with zero Neumann boundary conditions. Actually, any positive solution to the problem converges to a stationary solution as time tends to infinity, if the solution exits globally in time and remains uniformly bounded (see [6]).

This paper is organized as follows: In Section 2, we describe the well-posedness of solutions, some integral transformations and a stability of stationary solutions.
In Section 3, we show the continuity of solutions with respect to initial data. In Section 4, we show a special version of Theorem 1.1. The proof is simpler than the proofs of Theorems 1.1 and 1.2, and is essentially the same as those of Theorems 1.1 and 1.2. In Section 5, we describe the proofs of Theorems 1.1 and 1.2.

2. Well-posedness, integral transformations and stability of stationary solutions. The following result is an immediate conclusion from [2, Theorem 2.1 and Proposition 2.3]. Thus, we omit the proof.

**Proposition 1.** Let $u_0$ be a radial function satisfying $u_0 \geq 0$ in $\mathbb{R}^2$ and $\Lambda = \int_{\mathbb{R}^2} u_0(x)dx \leq 8\pi$. There exists a unique radial solution $(u, v)$ to (1), (2) and (3) satisfying (9), (10), (11), (12) and having the property

$$\int_{\mathbb{R}^2} u(x, t)dx = \int_{\mathbb{R}^2} u_0(x)dx \equiv \Lambda \quad \text{for } t \in [0, \infty).$$

For a solution $(u, v)$ in Proposition 1, putting

$$M(r, t) = \int_{|x| \leq r} u(x, t)dx \quad \text{for } (r, t) \in (0, \infty) \times [0, \infty),$$

the function $M$ satisfies (6), (7), (8) and

$$0 \leq M < \Lambda \quad \text{and} \quad 0 \leq M_r < \infty \quad \text{in } (0, \infty) \times (0, \infty).$$

On the other hand, if a Lipschitz continuous function $M^0$ satisfies that

$$M^0 \in L^\infty((0, \infty)), \quad 0 \leq M^0 < \Lambda \quad \text{in } [0, \infty)$$

and that $0 \leq M^0$ in a.e. $(0, \infty)$, (16) there exists a unique solution $M$ to (6), (7) and (8) satisfying (15).

Put

$$u(x, t) = \frac{1}{2\pi|x|} M_r(|x|, t) \quad \text{and} \quad v(x, t) = -\int_0^{|x|} \frac{1}{2\pi r} M(r, t)dr.$$

Then, $(u, v)$ is a solution to (1), (2) and (3) with $u_0(x) = M^0(|x|)/(2\pi|x|)$.

For each positive constant $b > 0$ and each function $f$ satisfying $0 \leq f < 8\pi$ in $(0, \infty)$, we define a functional $\mathcal{F}_b(f)$ as

$$\mathcal{F}_b(f) = \int_0^\infty \left\{ f(r) \log \frac{f(r)}{M_b(r)} + (8\pi - f(r)) \log \left( \frac{8\pi - f(r)}{8\pi - M_b(r)} \right) \right\} rdr.$$

This functional $\mathcal{F}_b$ was defined in [2]. The following proposition is the same as Proposition 3.6 in [2].

**Proposition 2.** Let $M^0$ be a continuous function satisfying (16) with $\Lambda = 8\pi$ and let $M$ be the solution to (6), (7) and (8). If $\mathcal{F}_b(M^0) < \infty$ with some $b > 0$, the solution $M$ satisfies

$$\lim_{t \to \infty} \int_0^R |M(r, t) - M_b(r)| rdr = 0 \quad \text{for each } R \geq 0.$$

For $b > 0$ and $(X, t) \in \mathbb{R}^4 \times (0, \infty)$, put $W(X, t) = |X|^{-2}M(|X|, t)$ and $W_b(X) = |X|^{-2}M_b(|X|)$. The function $W$ satisfies

$$W_t = \Delta X W + \frac{1}{2\pi} W \{ X \cdot \nabla X W + 2W \} \quad \text{in } \mathbb{R}^4 \times (0, \infty),$$

$$W(\cdot, 0) = W^0 \quad \text{in } \mathbb{R}^4,$$
where \( W^0(X) = |X|^{-2}M^0(|X|) \). Here, we write \( X = (X_1, X_2, X_3, X_4) \),
\[
\Delta_X = \sum_{i=1}^{4} \frac{\partial^2}{\partial X_i^2} \quad \text{and} \quad \nabla_X = \left( \frac{\partial}{\partial X_1}, \frac{\partial}{\partial X_2}, \frac{\partial}{\partial X_3}, \frac{\partial}{\partial X_4} \right).
\]
For each \( b > 0 \) and each function \( g \) satisfying \( 0 \leq |X|^2g(X) < 8\pi \) in \( \mathbb{R}^4 \) we defined a functional \( \mathcal{H}_b \) as
\[
\mathcal{H}_b(g) = \int_{\mathbb{R}^4} \left\{ g(X) \log \frac{g(X)}{W_b(X)} + \frac{8\pi - |X|^2g(X)}{|X|^2} \log \left( \frac{8\pi - |X|^2g(X)}{8\pi - |X|^2W_b(X)} \right) \right\} dX,
\]
which is essentially the same as the functional \( \mathcal{F}_b \).

For \( b > 0 \), the function \( W_b \) is a stationary solution to (17), and is stable in the following sense.

**Proposition 3.** Suppose that an initial datum \( W^0 \) satisfies \( \mathcal{H}_b(W^0) < \infty \) with some \( b > 0 \) and that \( W_a \leq W^0 \leq W_d \) in \((0, \infty)\) with some \( 0 < a \leq b \leq d \). Then, the corresponding solution \( W \) to (17) and (18) satisfies
\[
\lim_{t \to \infty} \| W(\cdot, t) - W_b \|_{L^\infty(\mathbb{R}^4)} = 0.
\]
Since the following lemma is an immediate conclusion from the comparison theorem, we omit the proof.

**Lemma 2.1.** Let \( a \) and \( d \) be constants satisfying \( 0 < a < d < \infty \). If \( W_a \leq W^0 \leq W_d \) in \( \mathbb{R}^4 \), then \( W_a \leq W \leq W_d \) in \( \mathbb{R}^4 \times (0, \infty) \), where \( W \) is the solution to (17) and (18).

**Proof of Proposition 3.** The solution \( W \) to (17) and (18) can be represented as
\[
W(X, t) = \int_{\mathbb{R}^4} G(X - Y, t)W^0(Y) dY + \int_0^t \int_{\mathbb{R}^4} G(X - Y, t - s) \cdot \frac{1}{2\pi} \left\{ \left[ Y \cdot \nabla_Y \left(W(Y, s) + 2W(Y, s)\right) \right] dY ds \right\} dY ds
\]
with
\[
G(X, t) = \frac{1}{(4\pi t)^{\frac{3}{2}}} \exp \left( -\frac{|X|^2}{4t} \right).
\]
Since
\[
\| W(\cdot, t) - W_b \|_{L^1(B(R))} = |S^3| \int_0^R |M(r, t) - M_b(r)| r dr \quad \text{for} \quad R > 0,
\]
Proposition 2 entails
\[
\lim_{t \to \infty} \| W(\cdot, t) - W_b \|_{L^1(B(R))} = 0 \quad \text{for} \quad R > 0,
\]
where \(|S^3|\) is the area of the unit sphere in \( \mathbb{R}^4 \) and \( B(R) \) is a closed ball with center \( 0 \) and radius \( R \) in \( \mathbb{R}^4 \). We can represent \( W - W_b \) as
\[
W(X, t) - W_b(X) = \int_{\mathbb{R}^4} G(X - Y, t) \left\{ W^0(Y) - W_b(Y) \right\} dY
\]
\[
+ \frac{1}{4\pi} \int_0^t \int_{\mathbb{R}^4} \left( Y \cdot \nabla_Y \left(G(X - Y, t - s)\right) \cdot \left\{ W(Y, s)^2 - W_b(Y)^2\right\} dY ds
\]
\[
= I + \frac{1}{4\pi} II. \tag{19}
\]
Since we infer from the assumption and Lemma 2.1 that
\[ W_a(X) \leq W(X, t) \leq W_d(X) \quad \text{for } (X, t) \in \mathbb{R}^4 \times [0, \infty) \quad (20) \]
and that
\[ |W(X, t) - W_b(X)| \leq \frac{C}{(|X|+1)^4} \quad \text{for } (X, t) \in \mathbb{R}^4 \times [0, \infty) \quad (21) \]
with a positive constant \( C \), we get
\[
|I| \leq \int_{|Y| \leq R} G(X-Y, t)|W^0(Y) - W_b(Y)|dY \\
+ \int_{|Y| \geq R} G(X-Y, t)|W^0(Y) - W_b(Y)|dY \\
\leq \frac{1}{(2\pi t)^{3/2}} \int_{|Y| \leq R} (|W^0(Y)| + |W_b(Y)|)dY + \frac{C}{R^4} \\
\leq \frac{CR^2}{t^2} + \frac{C}{R^4}.
\]

Here and henceforth, \( C \) represents a positive constant independent of \( C \).

Then, each \( C \) may vary from line to line and may depend on constants \( a, b \) and \( d \). For any sufficiently small \( \delta > 0 \) and \( T \in (0, t - \delta) \), put
\[
II_1 = \int_{t-\delta}^{t} \int_{\mathbb{R}^4} \{Y \cdot \nabla_X G(X-Y, t-s)\} \{W(Y, s)^2 - W_b(Y)^2\} dY ds, \\
II_2 = \int_{T}^{t-\delta} \int_{\mathbb{R}^4} \{Y \cdot \nabla_X G(X-Y, t-s)\} \{W(Y, s)^2 - W_b(Y)^2\} dY ds \\
\text{and} \\
II_3 = \int_{0}^{T} \int_{\mathbb{R}^4} \{Y \cdot \nabla_X G(X-Y, t-s)\} \{W(Y, s)^2 - W_b(Y)^2\} dY ds.
\]

We deduce from (20) and (21) that
\[
|II_1| \leq C \int_{t-\delta}^{t} \int_{\mathbb{R}^4} \frac{1}{(t-s)^{3/4}} \left( \frac{|X-Y|}{\sqrt{t-s}} \right)^{9/2} e^{-\frac{|Y|^2}{4(t-s)}} \frac{|Y|}{|X-Y|^{7/2}(|Y|+1)^{3}} dY ds \\
\leq C \left( \int_{t-\delta}^{t} \frac{ds}{(t-s)^{3/4}} \right) \left( \int_{\mathbb{R}^4} \frac{dY}{|X-Y|^{7/2}(|Y|+1)^{3}} \right) \\
\leq C\delta^{1/4} \left\{ \int_{|X-Y| \leq 1} \frac{dY}{|X-Y|^{7/2}} + \int_{|X-Y| \geq 1, |Y| \leq 1} \frac{dY}{(|Y|+1)^{3}} \right. \\
+ \left. \int_{|X-Y| \geq 1, |Y| \geq 1} \frac{dY}{|X-Y|^{7/2}(|Y|+1)^{3}} \right\} \\
\leq C\delta^{1/4}.
\]
\[ |I_2| \leq C \int_{T}^{t-\delta} \frac{1}{(t-s)^{3/2}} \int_{|Y| \leq R} |Y| |W(Y, s)^2 - W_b(Y)^2| \, dY \, ds \\
+ C \left( \int_{T}^{t-\delta} \frac{ds}{(t-s)^{3/4}} \right) \left( \int_{|Y| \geq R} \frac{dY}{|X-Y|^{3/2}(|Y|+1)^3} \right) \\
\leq C \frac{R}{\delta^{3/2}} \sup_{T \leq s \leq t} \|W(\cdot, t) - W_b\|_{L^1(B(R))} + C \frac{C \delta^{3/4}}{\delta^{3/4} R^{1/3}} \left( \frac{1}{R^{3/4}} \int_{|Y| \geq R, |X-Y| \leq 1} \frac{dY}{|X-Y|^{3/2}} \right) \\
+ \int_{|Y| \geq R, |X-Y| \geq 1} \frac{dY}{|X-Y|^{3/2}(|Y|+1)^3} \\
\leq C \frac{R}{\delta^{3/2}} \sup_{T \leq s \leq t} \|W(\cdot, t) - W_b\|_{L^1(B(R))} + C \frac{C \delta^{3/4}}{\delta^{3/4} R^{1/3}} \\
\text{and that} \\
|I_3| \leq C \left( \int_{0}^{T} \frac{ds}{(t-s)^{3/4}} \right) \left( \int_{\mathbb{R}^4} \frac{dY}{|X-Y|^{3/2}(|Y|+1)^3} \right) \leq C \frac{C \delta^{3/4}}{(t-T)^{3/4}}.
\]
These imply
\[ |II| \leq C \frac{\delta^{1/4}}{\delta^{3/4} R^{1/3}} + C \frac{C \delta^{1/4}}{\delta^{3/4} R^{1/3}} \sup_{T \leq s} \|W(\cdot, s) - W_b\|_{L^1(B(R))} + C \frac{C \delta^{1/4}}{(t-T)^{3/4}} \\
\text{for } t > T + \delta, \ R > 1 \text{ and } \frac{\delta}{\delta^{3/4} R^{1/3}}. \]
Take \( R = \delta^{-3}, T = t/2 \) and a sufficiently small \( \delta > 0 \). Thanks to Proposition 2, we see that
\[ \lim_{t \to \infty} \sup_{t \leq s} \|W(\cdot, t) - W_b\|_{L^\infty(\mathbb{R}^4)} \leq C \delta^{1/4}. \]
Since \( \delta \) is an arbitrary positive constant, we get
\[ \lim_{t \to \infty} \|W(\cdot, t) - W_b\|_{L^\infty(\mathbb{R}^4)} = 0. \]
Thus, we finish the proof. \( \square \)

3. Continuity of solutions with respect to initial data. In this section, we consider the continuity of solutions with respect to initial data \( u_0 \).

For \( \ell > 0 \), we define \( W(\cdot, \cdot) : \ell \) as a solution to a Cauchy problem
\[ W_\ell = \Delta_X W + \frac{1}{2\pi} f_\ell(W) \{ X \cdot \nabla_X W + 2W \} \quad \text{in } \mathbb{R}^4 \times (0, \infty), \quad (22) \]
\[ W(\cdot, 0) = W^0 \quad \text{in } \mathbb{R}^4, \]
where \( f_\ell(W) = \min\{\ell, W\} \). Let us put
\[ C_\omega(\mathbb{R}^4) = \left\{ f \in C(\mathbb{R}^4) : \sup_{X \in \mathbb{R}^4} (1 + |X|^2)|f(X)| < \infty \right\} \]
and
\[ ||f||_{C_\omega} = \sup_{X \in \mathbb{R}^4} (1 + |X|^2)|f(X)|. \]

**Proposition 4.** Assume that \( W_1^0 \) and \( W_2^0 \) are functions satisfying
\[ 0 \leq W_i^0(X) < \frac{8\pi}{|X|^2} \quad \text{for } X \in \mathbb{R}^4 \text{ and } i = 1, 2. \]

For \( \ell > 0 \), let \( W_1(\cdot, \cdot) : \ell \) and \( W_2(\cdot, \cdot) : \ell \) be solutions to \((22)\) with \( W_1(\cdot, 0 : \ell) = W_1^0 \) and \( W_2(\cdot, 0 : \ell) = W_2^0 \), respectively. Then, these solutions \( W_1 \) and \( W_2 \) satisfy
\[ ||W_1(\cdot, t : \ell) - W_2(\cdot, t : \ell)||_{C_\omega} \leq C_1 e^{C_1(t+1)^3} ||W_1^0 - W_2^0||_{C_\omega} \quad \text{for } t \geq 0. \]
with some positive constant $C_1$ independent of $\ell$.

We use the following lemma in the proof of Proposition 4.

**Lemma 3.1.** Let a non-negative function $f \in C([0, \infty))$ satisfy

$$f(t) \leq A(t + 1) + B \int_0^t \left(\frac{1}{\sqrt{t-s}} + (t-s)\right) f(s)ds \quad \text{for } t \geq 0$$

with some positive constants $A$ and $B$. Then, the function $f$ satisfies

$$f(t) \leq 2Ae^{10(B+1)^2t} \quad \text{for } t \geq 0.$$  

**Proof.** Putting $F(t) = \max_{0 \leq s \leq t} f(s)$ for $t \geq 0$, we obtain

$$F(t) \leq A(t + 1) + B \int_0^t \left(\frac{1}{\sqrt{t-s}} + (t-s)\right) F(s)ds \quad \text{for } t \geq 0$$

and

$$B \int_0^t \frac{F(s)}{\sqrt{t-s}} ds = B \int_0^{\max(0,t-(4B)^{-2})} \frac{F(s)}{\sqrt{t-s}} ds + B \int_{\max(0,t-(4B)^{-2})}^t \frac{F(s)}{\sqrt{t-s}} ds$$

$$\leq 4B^2 \int_0^{\max(0,t-(4B)^{-2})} F(s) ds + \frac{1}{2} F(t) \quad \text{for } t \geq 0.$$  

We deduce from this that

$$\frac{1}{2} F(t) \leq A(t + 1) + 4B^2 \int_0^t F(s) ds + B \int_0^t (t-s) F(s) ds \quad \text{for } t \geq 0.$$  

Putting

$$G(t) = A(t + 1) + 4(B+1)^2 \int_0^t F(s) ds + (B+1) \int_0^t (t-s) F(s) ds,$$

we see $G(t) \geq F(t)/2$ and $G(t) \geq A$. These imply

$$G'(t) = A + 4(B+1)^2 F(t) + (B+1) \int_0^t F(s) ds$$

$$\leq G(t) + 8(B+1)^2 G(t) + \frac{G(t)}{4(B+1)}$$

$$\leq 10(B+1)^2 G(t).$$  

So the Gronwall inequality gives

$$f(t) \leq F(t) \leq 2G(t) \leq 2A \exp\left(10(B+1)^2 t\right) \quad \text{for } t \geq 0.$$  

Thus, we finish the proof.  

**Proof of Proposition 4.** The comparison theorem yields

$$0 \leq W_i(X, t : \ell) < \frac{8\pi}{|X|^2} \quad \text{for } (X, t) \in \mathbf{R}^4 \times [0, \infty) \quad \text{and } i = 1, 2.$$  

(23)
We can write \( W_1(\cdot, \cdot : \ell) - W_2(\cdot, \cdot : \ell) \) as

\[
W_1(X, t : \ell) - W_2(X, t : \ell) = \int_{\mathbb{R}^4} G(X - Y, t)(W_1^0(Y) - W_2^0(Y))dY \\
+ \frac{1}{\pi} \int_0^t \int_{\mathbb{R}^4} G(X - Y, t - s) \left\{ f_\ell(W_1(Y, s : \ell))W_1(Y, s : \ell) - f_\ell(W_2(Y, s : \ell))W_2(Y, s : \ell) \right\} dY ds \\
- \frac{1}{2\pi} \int_0^t \int_{\mathbb{R}^4} \nabla_Y \cdot \{YG(X - Y, t - s)\} \\
\cdot \{F_\ell(W_1(Y, s : \ell)) - F_\ell(W_2(Y, s : \ell))\} dY ds
\]

\[
= \int_{\mathbb{R}^4} G(X - Y, t)(W_1^0(Y) - W_2^0(Y))dY \\
- \frac{1}{\pi} \int_0^t \int_{\mathbb{R}^4} G(X - Y, t - s) \left\{ g_\ell(W_1(Y, s : \ell)) - g_\ell(W_2(Y, s : \ell)) \right\} dY ds \\
+ \frac{1}{2\pi} \int_0^t \int_{\mathbb{R}^4} Y \cdot \nabla_X G(X - Y, t - s) \\
\cdot \{F_\ell(W_1(Y, s : \ell)) - F_\ell(W_2(Y, s : \ell))\} dY ds
\]

\[= III - \frac{1}{\pi} IV + \frac{1}{2\pi} V,\]

where

\[F_\ell(W) = \int_0^W f_\ell(\tau)d\tau\]

and

\[g_\ell(W) = 2F_\ell(W) - f_\ell(W)W = \max\{0, \ell W - \ell^2\}.\]

We see

\[
(1 + |X|^2)|III| \\
\leq \int_{\mathbb{R}^4} (2|X - Y|^2 + 1 + 2|Y|^2)G(X - Y, t) |W_1^0(Y) - W_2^0(Y)| dY \\
\leq 2 \int_{\mathbb{R}^4} \left( \frac{|X - Y|^2}{1 + |Y|^2} + 1 \right) G(X - Y, t) (1 + |Y|^2)|W_1^0(Y) - W_2^0(Y)|dY \\
\leq 2 \int_{\mathbb{R}^4} (|X - Y|^2 + 1) G(X - Y, t)dY \|W_1^0 - W_2^0\|_{C^\omega} \\
\leq C(t + 1)\|W_1^0 - W_2^0\|_{C^\omega}.
\]

Since we deduce from (23) that

\[
|F_\ell(W_1(Y, s : \ell)) - F_\ell(W_2(Y, s : \ell))| \leq f_\ell(\max \{W_1(Y, s : \ell), W_2(Y, s : \ell)\}) |W_1(Y, s : \ell) - W_2(Y, s : \ell)| \\
\leq \min \left\{ \ell, \frac{8\pi}{|Y|^2} \right\} \cdot |W_1(Y, s ; \ell) - W_2(Y, s ; \ell)|
\]

and that

\[|g_\ell(W_1) - g_\ell(W_2)| \leq \ell |W_1 - W_2|,
\]
Thus, we finish the proof.

4. Special version of Theorem 1.1 and its proof. The following proposition
is a special version of Theorem 1.1 and is shown by a similar argument as that in
main results. Then, we write the proposition and its proof.

Proposition 5. For any pair of constants \(a\) and \(d\) satisfying \(0 < a < d < \infty\), there
exists a radial solution \((u, v)\) to (1), (2) and (3) satisfying (9), (10), (11), (12),
\(\Lambda = 8\pi\) and having the property \(\{u_n, u_d\} \subset \omega(u)\).

Lemma 4.1. For any pair of constants \(a\) and \(d\) satisfying \(0 < a < d < \infty\), there
exists a radial solution \(W\) to (17) and (18) and a sequence \(\{T_n\}_{n \geq 1} \subset [1, \infty)\) such
Since the comparison theorem yields
\[ \|W(\cdot, t) - W_d\|_{C_w} \leq \frac{1}{2n} \quad \text{for } t \in [T_{2n-1} - 1, T_{2n-1} + 1] \text{ and } n \geq 1, \]
\[ \|W(\cdot, t) - W_a\|_{C_w} \leq \frac{1}{2n} \quad \text{for } t \in [T_{2n} - 1, T_{2n} + 1] \text{ and } n \geq 1, \]
and
\[ T_{n+1} \geq T_n + 2 \quad \text{for } n \geq 1. \]
Furthermore, the corresponding solution \((u, v)\) to (1), (2) and (3) satisfies (9), (10), (11), (12) and \(\Lambda = 8\pi\).

Proof. Let \(W^0\) be a radial, continuous function satisfying
\[ W^0(X) = W_d(X) \quad \text{for } X \in B(L_1) \]
and
\[ W^0(X) \in \left[ W_a(X), \frac{8\pi}{|X|^2} \right) \quad \text{for } X \in \mathbb{R}^4 \setminus B(L_1) \]
for some positive constant \(L_1 \geq 1\). This entails
\[ \|W^0 - W_d\|_{C_w} \leq \sup_{|X| \geq a} \left\{ (1 + |X|^2) \left( \frac{8\pi}{|X|^2} - \frac{8\pi a}{1 + a|X|^2} \right) \right\} \leq \frac{8\pi(1 + L_1^2)}{L_1^2(1 + aL_1^2)}. \]
Let us put \(\ell = 8\pi d + 2\). Take \(L_1\) such that
\[ C_1 e^{2C_1(\ell+1)^3} \frac{8\pi(1 + L_1^2)}{L_1^2(1 + aL_1^2)} \leq 1, \]
where \(C_1\) is the constant in Proposition 4. Let \(W(\cdot, \cdot : \ell)\) be a solution to (22) with \(W(\cdot, 0 : \ell) = W^0\) and \(\ell = 8\pi d + 2\) and let \(W(\cdot, \cdot)\) be a solution to (17) with \(W(\cdot, 0) = W^0\). Putting \(T_1 = 1\), it follows from Proposition 4 that
\[ \|W(\cdot, t : \ell) - W_d\|_{C_w} \leq 1 \quad \text{for } t \in [0, T_1 + 1], \]
which means that \(W(\cdot, \cdot) = W(\cdot, \cdot : \ell)\) in \(\mathbb{R}^4 \times [0, T_1 + 1]\) and that
\[ \|W(\cdot, t) - W_d\|_{C_w} \leq 1 \quad \text{for } t \in [T_1 - 1, T_1 + 1]. \]
Let us put \(W_b(|X|) = W_b(X)\) for \(X \in \mathbb{R}^4\) and \(b > 0\). Since \(L_1^2 W_d(L_1) < 8\pi\), there exists a constant \(\tilde{L}_1 \geq L_1\) such that
\[ \frac{L_1^2 W_d(L_1)}{L_1^2} = W_a(\tilde{L}_1). \]
Let \(W^0_1\) be a radial function satisfying
\[ W^0_1(X) = \begin{cases} \frac{W_d(X)}{L_1^2 W_d(L_1)} & \text{for } X \in B(L_1), \\ \frac{W_a(X)}{|X|^2} & \text{for } X \in \mathbb{R}^4 \setminus B(\tilde{L}_1), \end{cases} \]
and let \(W_1(\cdot, \cdot : \ell)\) be the solution to (22) with \(W_1(\cdot, 0 : \ell) = W^0_1\) and \(\ell = 8\pi d + 2\). Since the comparison theorem yields \(W_a \leq W_1(\cdot, \cdot : \ell) \leq W_d\) in \(\mathbb{R}^4 \times [0, \infty)\), the solution \(W_1(\cdot, \cdot : \ell)\) satisfies
\[ |W_1(X, t : \ell) - W_a(X)| \leq |W_d(X) - W_a(X)| \leq \frac{8\pi}{|X|^2(1 + a|X|^2)} \] (24)
for \((X, t) \in \mathbb{R}^4 \times [0, \infty)\) and that \(W_1(\cdot, \cdot) = W_1(\cdot, \cdot : \ell)\) in \(\mathbb{R}^4 \times [0, \infty)\), where \(W_1(\cdot, \cdot)\) is a solution to \((17)\) with \(W_1(\cdot, 0) = W_0^1\). Taking \(L \geq \tilde{L}_1 + 1\) such that

\[
\frac{8\pi(1 + L^2)}{L^2(1 + aL^2)} \leq \frac{1}{4}, \tag{25}
\]

Proposition 3 guarantees the existence of a constant \(T_2 \geq T_1 + 2\) such that

\[
(1 + L^2)\|W_1(\cdot, t) - W_a\|_{L^\infty(B(L))} \leq \frac{1}{4} \quad \text{for } t \in [T_2 - 1, \infty). \tag{26}
\]

It follows from \((24), (25)\) and \((26)\) that

\[
\|W(\cdot, t) - W_a\|_{C_\omega} \leq \frac{1}{4} \quad \text{for } t \in [T_2 - 1, \infty). \tag{27}
\]

Take \(L_2 \geq \tilde{L}_1 + 1\) such that

\[
C_1 e^{C_1(\ell + 1)^3(T_2 + 1)} \frac{8\pi(1 + L_2^2)}{L_2^2(1 + aL_2^2)} \leq \frac{1}{4}
\]

and take \(W^0\) such that

\[
W^0(X) = W_a(X) \quad \text{for } X \in B(L_1),
\]

\[
W^0(X) = \frac{L_1^2}{|X|^2} W_a(L_1) \quad \text{for } X \in B(\tilde{L}_1) \setminus B(L_1),
\]

\[
W^0(X) = W_a(X) \quad \text{for } X \in B(L_2) \setminus B(\tilde{L}_1),
\]

\[
W^0(X) = W_a(L_2) \quad \text{for } X \in B(\tilde{L}_2) \setminus B(L_2)
\]

and

\[
W^0(X) \in C_1 e^{C_1(\ell + 1)^3 t} \frac{8\pi(1 + L_2^2)}{L_2^2(1 + aL_2^2)} \leq \frac{1}{4} \quad \text{for } t \in [0, T_2 + 1].
\]

Hence

\[
W(\cdot, \cdot) = W(\cdot, \cdot : \ell) \quad \text{in } \mathbb{R}^4 \times [0, T_2 + 1]
\]

and

\[
\|W(\cdot, t) - W_a\|_{C_\omega} \leq \frac{1}{2} \quad \text{for } t \in [T_2 - 1, T_2 + 1].
\]

Let \(k \geq 2\). Assume that an initial data \(W^0\) satisfies

\[
W_a(X) \leq W^0(X) \leq W_a(X) \quad \text{for } X \in B(\tilde{L}_{2k})
\]

and

\[
W^0(X) \in \left[ W_a(X), \frac{8\pi}{|X|^2} \right] \quad \text{for } X \in \mathbb{R}^4 \setminus B(\tilde{L}_{2k}) \tag{28}
\]

for some positive constant \(\tilde{L}_{2k}\). Assume, in addition, that there exists a sequence \([T_j]_{j=1}^{2k}\) such that the corresponding solution \(W(\cdot, \cdot)\) to \((17)\) satisfies

\[
\|W(\cdot, t) - W_d\|_{C_\omega} \leq \frac{1}{2j - 1} \quad \text{for } t \in [T_{2j-1} - 1, T_{2j-1} + 1] \tag{29}
\]

and

\[
\|W(\cdot, t) - W_a\|_{C_\omega} \leq \frac{1}{2j} \quad \text{for } t \in [T_{2j} - 1, T_{2j} + 1] \tag{30}
\]

for \(j = 1, 2, 3, \cdots, k\). Let \(W_{2k+1}\) be a solution to \((17)\) with initial condition

\[
W_{2k+1}^0(X) = \begin{cases} 
W^0(X) & \text{for } X \in B(\tilde{L}_{2k}), \\
W_d(X) & \text{for } X \in \mathbb{R}^4 \setminus B(\tilde{L}_{2k}).
\end{cases}
\]
The comparison theorem yields $W_a \leq W_{2k+1} \leq W_d$ in $\mathbb{R}^4 \times [0, \infty)$. We can find a constant $T_{2k+1} \geq T_{2k} + 2$ such that
\[
\|W_{2k+1}(\cdot, t) - W_d\|_{C_\omega} \leq \frac{1}{2(2k+1)} \quad \text{for } t \in [T_{2k+1} - 1, \infty)
\]
by a similar argument as for (27). Putting
\[
W^0(X) = W_d(X) \quad \text{for } X \in B(L_{2k+1}) \setminus B(\tilde{L}_{2k}),
\]
\[
W^0(X) \in \left[ W_a(X), \frac{8\pi}{|X|^2} \right) \quad \text{for } X \in \mathbb{R}^4 \setminus B(L_{2k+1}),
\]
Let $W_{2k+1}(\cdot, \cdot : \ell)$ be a solution to (22) with $W_{2k+1}(\cdot, 0 : \ell) = W^0_{2k+1}$ and $\ell = 8\pi d + 2$. Since $W_{2k+1}(\cdot, \cdot) = W_{2k+1}(\cdot, \cdot : \ell)$, Proposition 4 guarantees the existence of a constant $L_{2k+1} \geq \tilde{L}_{2k+1} + 1$ such that
\[
\|W(\cdot, t : \ell) - W_{2k+1}(\cdot, t)\|_{C_\omega} \leq \frac{1}{2(2k+1)} \quad \text{for } t \in [0, T_{2k+1} + 1],
\]
which implies that $W(\cdot, \cdot : \ell) = W(\cdot, \cdot)$ in $\mathbb{R}^4 \times [0, T_{2k+1} + 1]$ and that
\[
\|W(\cdot, t) - W_d\|_{C_\omega} \leq \frac{1}{2k+1} \quad \text{for } t \in [T_{2k+1} - 1, T_{2k+1} + 1].
\]
Let $W_{2k+2}$ be a solution to (17) with initial condition
\[
W^0_{2k+2}(X) = \begin{cases} 
W^0(X) & \text{for } X \in B(L_{2k+1}), \\
L_{2k+1}^2 W_d(L_{2k+1}) & \text{for } X \in B(\tilde{L}_{2k+1}) \setminus B(L_{2k+1}), \\
W_a(X) & \text{for } X \in \mathbb{R}^4 \setminus B(\tilde{L}_{2k+1}), 
\end{cases}
\]
where $\tilde{L}_{2k+1}$ is a constant such that
\[
W_a(\tilde{L}_{2k+1}) = \frac{L_{2k+1}^2 W_d(L_{2k+1})}{\tilde{L}_{2k+1}^2}.
\]
Similarly, we obtain that $W_a \leq W_{2k+2} \leq W_d$ in $\mathbb{R}^4 \times [0, \infty)$ and that
\[
\|W_{2k+2}(\cdot, t) - W_a\|_{C_\omega} \leq \frac{1}{2(2k+2)} \quad \text{for } t \in [T_{2k+2} - 1, \infty)
\]
with some constant $T_{2k+2} \geq T_{2k+1} + 2$. For $L_{2k+2} \geq \tilde{L}_{2k+1} + 1$, put
\[
W^0(X) = \frac{L_{2k+1}^2 W_d(L_{2k+1})}{|X|^2} \quad \text{for } X \in B(\tilde{L}_{2k+1}) \setminus B(L_{2k+1}),
\]
\[
W^0(X) = W_a(X) \quad \text{for } X \in B(L_{2k+2}) \setminus B(\tilde{L}_{2k+2}),
\]
\[
W^0(X) = \left[ W_a(X), \frac{8\pi}{|X|^2} \right) \quad \text{for } X \in \mathbb{R}^4 \setminus B(\tilde{L}_{2k+2}),
\]
where $\tilde{L}_{2k+2}$ is a constant such that $W_d(\tilde{L}_{2k+2}) = W_a(L_{2k+2})$. Proposition 1 ensures the existence of a constant $L_{2k+2} \geq \tilde{L}_{2k+1} + 1$ such that
\[
\|W(\cdot, t : \ell) - W_{2k+2}(\cdot, t)\|_{C_\omega} \leq \frac{1}{2(2k+2)} \quad \text{in } [0, T_{2k+2} + 1].
\]
These entail that $W(\cdot, \cdot : \ell) = W(\cdot, \cdot)$ in $\mathbb{R}^2 \times [0, T_{2k+2} + 1]$ and that
\[
\|W(\cdot, t) - W_a\|_{C_\omega} \leq \frac{1}{2k+2} \quad \text{for } t \in [T_{2k+2} - 1, T_{2k+2} + 1].
\]
Noting that (29) and (30) hold true if the initial data satisfies (28), for \( j = 1, 2, 3, \ldots, k + 1 \) we obtain that
\[
\|W(\cdot, t) - W_d\|_{C_w} \leq \frac{1}{2j - 1} \quad \text{for} \ t \in [T_{2j-1} - 1, T_{2j-1} + 1]
\]
and that
\[
\|W(\cdot, t) - W_d\|_{C_w} \leq \frac{1}{2j} \quad \text{for} \ t \in [T_{2j} - 1, T_{2j} + 1].
\]
Thus, we find a desired solution \( W \). By the way to construct the initial data \( W^0 \), we see that the corresponding initial data \( u_0 \) is radial and non-negative. Combining this with Proposition 1 implies that the corresponding solution \((u, v)\) to (1), (2) and (3) satisfies (9), (10), (11), (12) and \( \Lambda = 8\pi \). Thus, we finish the proof.

**Remark 1.** In the proof of Proposition 5, the initial data \( W^0 \) are represented as

\[
W^0(X) = \begin{cases} 
W_d(X) & \text{for} \ X \in B(L_1), \\
\max \left\{ \frac{L^2_{2j-1}W_d(L_{2j-1})}{|X|^2}, W_a(X) \right\} & \text{for} \ X \in B(L_{2j}) \setminus B(L_{2j-1}) \text{ and } j \geq 1, \\
\min \{W_a(L_{2j}), W_d(X)\} & \text{for} \ X \in B(L_{2j+1}) \setminus B(L_{2j}) \text{ and } j \geq 1.
\end{cases}
\]

**Proof of Proposition 5.** Fix \( X \in \mathbb{R}^d \) with \(|X| \geq 1 \) and \( t \geq 1 \). Let us put
\[
\Phi(\zeta, \tau) = W \left( \left( \frac{\zeta}{|X|} + 1 \right) X, t + \tau \right) \quad \text{for} \ (\zeta, \tau) \in (-1, 1) \times (-1, 1).
\]
The function \( \Phi \) satisfies
\[
\Phi_{\tau} = \Phi_{\zeta \zeta} + \frac{3}{\zeta + |X|} \Phi_{\zeta} + \frac{\Phi}{2\pi} \left\{ \left( \frac{\zeta}{|X|} + 1 \right) |X| \Phi_{\zeta} + 2\Phi \right\}.
\]
Putting
\[
\Phi_b(\zeta) = W_b \left( \left( \frac{\zeta}{|X|} + 1 \right) X \right) \quad \text{for} \ \zeta \in (-1, 1),
\]
we see
\[
(\Phi - \Phi_b)_{\tau} = (\Phi - \Phi_b)_{\zeta \zeta} + \frac{3}{\zeta + |X|} (\Phi - \Phi_b)_{\zeta} + \frac{\Phi}{2\pi} \left( \frac{\zeta}{|X|} + 1 \right) |X| (\Phi - \Phi_b)_{\zeta}
\]
\[
+ \frac{(\Phi - \Phi_b)}{2\pi} \frac{\zeta}{|X|} + 1 |X| \Phi_b + \frac{\Phi}{\pi} (\Phi + \Phi_b) (\Phi - \Phi_b).
\]
Since \((\zeta + |X|)\Phi\) and \((\zeta + |X|)\Phi_b\) are uniformly bounded for \((\zeta, \tau) \in [-2/3, 2/3] \times [-2/3, 2/3], \ |X| \geq 1 \) and \( t \geq 1 \), the parabolic regularity argument guarantees the existence of a positive constant \( C \) such that
\[
|\Phi(0, 0) - \Phi_b(0)| \leq C \sup_{|\tau| \leq 1/2, |\zeta| \leq 1/2} |\Phi(\zeta, \tau) - \Phi_b(\zeta)|,
\]
which implies
\[
(1 + |x|)|u(x, t) - u_b(x)| \leq C \sup_{|\tau| \leq 1/2} \|W(\cdot, t + \tau) - W_b\|_{C_w} \quad (31)
\]
for \( x \in \mathbb{R}^2 \) with \(|x| \geq 1 \) and \( t \in [1, \infty) \).

For \( X \in \mathbb{R}^4 \) with \( 0 < |X| \leq 1 \) and \( t \in [1, \infty) \), putting
\[
\Psi(\zeta, \tau) = W((\zeta + 1)X, t + |X|^2 \tau) \quad \text{for} \ (\zeta, \tau) \in (-1, 1) \times (-1, 1)
\]
and

$$\Psi_b(\zeta) = W_b((\zeta + 1)X) \quad \text{for} \quad \zeta \in (-1, 1),$$

we have

$$\Psi_x = \Psi_{\zeta \zeta} + \frac{3}{\zeta + 1} \Psi_x + \frac{|X|^2}{2\pi} \left\{ (\zeta + 1)\Psi_x + 2\Psi \right\} \quad \text{in} \quad (-1, 1) \times (-1, 1)$$

and

$$(\Psi - \Psi_b)_x = (\Psi - \Psi_b)_{\zeta \zeta} + \frac{3}{\zeta + 1}(\Psi - \Psi_b)_x + \frac{1}{2\pi} \left\{ |X|^2(\zeta + 1)\Psi_x + 2|X|^2(\Psi + \Psi_b)(\Psi - \Psi_b) \right\}$$

in $(-1, 1) \times (-1, 1)$. Since $|X|^2\Psi$ and $|X|^2\Psi_{bc}$ are uniformly bounded for $(\zeta, \tau) \in [-2/3, 2/3] \times [-2/3, 2/3]$, $0 < |X| \leq 1$ and $t \geq 1$, we obtain

$$(1 + |x|) \left| u(x, t) - u_b(x) \right| \leq C \sup_{|\tau| \leq 1/2} \| W(\cdot, t + \tau) - W_b \|_{C_0}$$

for $x \in \mathbb{R}^2$ with $|x| \leq 1$ and $t \in [1, \infty)$ by using similar arguments as for (31). Combining these with Lemma 4.1 implies this proposition. \qed

5. **Proofs of Theorems 1.1 and 1.2.** The proof of Theorem 1.1 is essentially the same as the one in Section 4. Therefore, we describe a sketch of these proofs.

**Proof of Theorem 1.1.** There exists a countable set $\{b_j\}_{j \geq 1}$ such that

$$E \subset \{b_j\}_{j \geq 1} \subset [\inf E, \sup E]$$

and that $b_i \neq b_j$ if $i \neq j$, where $\overline{\{b_j\}_{j \geq 1}}$ is the closure of the set $\{b_j\}_{j \geq 1}$. For the set $\{b_j\}_{j \geq 1}$ we defined a set $\{\beta_j\}_{j \geq 1}$ as

$$\begin{align*}
\beta_1 &= b_1 & \beta_2 &= b_2 \\
\beta_3 &= b_3 & \beta_4 &= b_2 & \beta_5 &= b_3 \\
\beta_6 &= b_1 & \beta_7 &= b_2 & \beta_8 &= b_3 & \beta_9 &= b_4 \\
\beta_{10} &= b_1 & \beta_{11} &= b_2 & \beta_{12} &= b_3 & \beta_{13} &= b_4 & \beta_{14} &= b_5 \\
&\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{align*}$$

Thus, the sequence $\{\beta_j\}_{j \geq 1}$ satisfies

$$\beta_{j+k+1} = b_j \quad \text{for} \quad j = 1, 2, 3, \cdots, k + 1 \quad \text{and} \quad k \geq 1.$$  

According to the representation in Remark 1, we define $W^0$ as follows:

$$W^0(X) = W_{\beta_1}(X) \quad \text{for} \quad X \in B(L_1)$$

and

$$W^0(X) = \begin{cases} 
\max \left\{ W_{\beta_{j+1}}(X), \frac{L_j^2W_{\beta_j}(L_j)}{|X|^2} \right\} & \text{if} \quad \beta_j > \beta_{j+1}, \\
\min \left\{ W_{\beta_{j+1}}(X), W_{\beta_j}(L_j) \right\} & \text{if} \quad \beta_j < \beta_{j+1}
\end{cases}$$

for $X \in B(L_{j+1}) \setminus B(L_{j})$ and $j \geq 1$.

Taking $\{L_j\}_{j \geq 1}$ such that $L_{j+1} \gg L_j$ for $j \geq 1$, we can find a sequence $\{T_j\}_{j \geq 1}$ such that $T_{j+1} \geq T_j + 2$ for each $j \geq 1$, $W^0$ is continuous in $\mathbb{R}^4$ and that

$$\|W(\cdot, t) - W_{\beta_{j}}\|_{C_0} \leq \frac{1}{j} \quad \text{for} \quad t \in [T_j - 1, T_j + 1] \quad \text{and} \quad j \geq 1.$$
by using a similar argument as that in Lemma 4.1, where \( W(\cdot, \cdot) \) is a solution to (17) with \( W(\cdot, 0) = W^0 \). Furthermore, using a similar argument as that in Proposition 5, we obtain that the corresponding solution \((u, v)\) to (1), (2) and (3) satisfies \( \{u_b\}_{j \geq 1} \subset \omega(u) \) and (10). Combining this with the way to choose the sequence \( \{b_j\}_{j \geq 1} \) implies that \( \{u_b\}_{b \in E} \subset \omega(u) \). By using \( \sup_{t \geq 0} \|W(\cdot, t)\|_{C_w} < \infty \) and a similar argument as that in Proposition 5, we obtain that

\[
\sup_{(x, t) \in \mathbb{R}^2 \times [0, \infty)} |u(x, t)| < \infty.
\]

Thus, we finish the proof. \( \square \)

We can obtain also Theorem 1.2, by using a similar argument as that in Proposition 5. However, we must note the way to choose the constant \( \ell \), when we use Proposition 4. Because, the sequence \( \{W_{b_j}\}_{j \geq 1} \) satisfies \( \lim_{j \to \infty} W_{b_j}(0) = \infty \).

**Proof of Theorem 1.2.** For the sequence \( \{b_j\}_{j \geq 1} \), take the same sequence \( \{\beta_j\}_{j \geq 1} \) as the one in the proof of Theorem 1.1. For \( k \geq 1 \), put \( A(k) = \max_{1 \leq j \leq k} b_j \) and \( B = \min_{j \geq 1} b_j \). We assume that \( b_{j+1} \neq b_j \) for \( j \geq 1 \) without loss of generality. Let \( W^0 \) be a radial continuous function satisfying

\[
W^0(X) = W_{\beta_1}(X) \quad \text{for} \; X \in B(L_1)
\]

and

\[
W^0(X) \in \left[ W_B(X), \frac{8\pi}{|X|^2} \right] \quad \text{for} \; X \in \mathbb{R}^4 \setminus B(L_1)
\]

for some positive constant \( L_1 \geq 1 \). This entails

\[
\|W^0 - W_{\beta_1}\|_{C_w} \leq \sup_{|X| \geq L_1} \left\{ (1 + |X|^2) \left( \frac{8\pi}{|X|^2} - \frac{8\pi B}{1 + B|X|^2} \right) \right\} \leq \frac{8\pi(1 + L_1^2)}{L_1^2(1 + BL_1^2)}.
\]

Noting

\[
\int_{\mathbb{R}^2} |x|^2 u_b(x) dx = \infty \quad \text{for} \; b > 0,
\]

we can take \( L_1 \) such that

\[
C_1e^{2C_1(8\pi A(1)+3)^2} \frac{8\pi(1 + L_1^2)}{L_1^2(1 + BL_1^2)} \leq 1
\]

and that

\[
\int_{|x| < L_1} |x|^2 u_{\beta_1}(x) dx \geq 1,
\]

where \( C_1 \) is the constant in Proposition 4. Let \( W(\cdot, \cdot : \ell) \) be a solution to (22) with \( W(\cdot, 0 : \ell) = W^0 \) and \( \ell = 8\pi A(1) + 2 \), and let \( W(\cdot, \cdot) \) be a solution to (17) with \( W(\cdot, 0) = W^0 \). Putting \( T_1 = 1 \), we deduce from Proposition 4 that

\[
\|W(\cdot, t : \ell) - W_{\beta_1}\|_{C_w} \leq 1 \quad \text{for} \; t \in [0, T_1 + 1],
\]

which implies \( W(\cdot, \cdot) = W(\cdot, \cdot : \ell) \) in \( \mathbb{R}^4 \times [0, T_1 + 1] \). We define \( W^0_1 \) as follows:

\[
W^0_1(X) = W_{\beta_1}(X) \quad \text{for} \; X \in B(L_1)
\]

and

\[
W^0_1(X) = \begin{cases} 
\max\{W_{\beta_2}(X), \frac{L_1^2 W_{\beta_1}(L_1)}{|X|^2}\} & \text{if} \; \beta_1 > \beta_2, \\
\min\{W_{\beta_2}(X), W_{\beta_1}(L_1)\} & \text{if} \; \beta_1 < \beta_2
\end{cases}
\]

for \( X \in \mathbb{R}^4 \setminus B(L_1) \).
Let $W_1(\cdot, \cdot)$ be the solution to (17) with $W(\cdot, 0) = W_1^0$. Since the comparison theorem yields $W_B \leq W_1 \leq W_{B(2)}$ in $\mathbb{R}^4 \times [0, \infty)$, the solution $W_1$ satisfies
\[
|W_1(X, t) - W_{\beta_2}(X)| \leq \frac{8\pi}{|X|^2} - W_B(X) \leq \frac{8\pi}{|X|^2(1 + B|X|^2)}
\]
for $(X, t) \in \mathbb{R}^4 \times [0, \infty)$ and that $W_1(\cdot, \cdot) = W_1(\cdot, \cdot : \ell)$ in $\mathbb{R}^4 \times [0, \infty)$, where $W_1(\cdot, \cdot : \ell)$ is a solution to (22) with $W_1(\cdot, 0 : \ell) = W_1^0$ and $\ell = 8\pi A(2) + 2$.

By a similar argument as for (27), we can find a constant $T_2 \geq T_1 + 2$ such that
\[
\|W_1(\cdot, t) - W_{\beta_2}\|_{C_{\infty}} \leq \frac{1}{4} \quad \text{for } t \in [T_2 - 1, \infty).
\]

Take $L_2 \geq L_1 + 1$ such that
\[
C_1 e^{C_1(8\pi A(2) + 3)^3(T_2 + 1)} \frac{8\pi(1 + L_2^2)}{L_2^2(1 + BL_2^2)} \leq \frac{1}{4},
\]

\[
\begin{cases}
W_{\beta_2}(L_2) \geq \frac{L_1^2W_{\beta_2}(L_1)}{L_2^2} & \text{if } \beta_1 > \beta_2, \\
W_{\beta_2}(L_2) \leq W_{\beta_1}(L_1) & \text{if } \beta_1 < \beta_2
\end{cases}
\]

and that
\[
\int_{L_2 < |x| < L_2} |x|^2u_{\beta_2}(x)dx \geq 1,
\]

where $C_1$ is the constant in Proposition 4 and $\hat{L}_2$ is a constant satisfying
\[
W_{\hat{\beta}_2}(\hat{L}_2) = \begin{cases}
\frac{L_1^2W_{\hat{\beta}_2}(L_1)}{L_2^2} & \text{if } \beta_1 > \beta_2, \\
W_{\beta_1}(L_1) & \text{if } \beta_1 < \beta_2
\end{cases}
\]

Take a continuous function $W^0$ such that
\[
W^0(X) = W_1^0(X) \quad \text{for } X \in B(L_2)
\]

and that
\[
W^0(X) \in \left[ W_B(X), \frac{8\pi}{|X|^2} \right] \quad \text{for } X \in \mathbb{R}^4 \setminus B(L_2).
\]

So the solution $W(\cdot, \cdot : \ell)$ to (22) with $\ell = 8\pi A(2) + 2$ satisfies that
\[
\|W(\cdot, t : \ell) - W_1(\cdot, \cdot : \ell)\|_{C_{\infty}} \leq C_1 e^{C_1(8\pi A(2) + 3)^3t}||W^0 - W_1^0\|_{C_{\infty}}
\leq C_1 e^{C_1(8\pi A(2) + 3)^3(T_2 + 1)} \frac{8\pi(1 + L_2^2)}{L_2^2(1 + BL_2^2)} \leq \frac{1}{4} \quad \text{for } t \in [0, T_2 + 1].
\]

Hence $W(\cdot, \cdot : \ell) = W(\cdot, \cdot)$ in $\mathbb{R}^4 \times [0, T_2 + 1]$ and
\[
\|W(\cdot, t) - W_{\beta_2}\|_{C_{\infty}} \leq \frac{1}{2} \quad \text{for } t \in [T_2 - 1, T_2 + 1].
\]

Let $k \geq 2$. Take $L_k$ such that
\[
\int_{|x| < L_k} |x|^2u_{\beta_j}(x)dx \geq k.
\]
Assume that there exists a sequence \( \{T_j\}_{j=1}^k \) such that an initial data \( W^0 \) and the corresponding solution \( W(\cdot, t) \) to (17) satisfy
\[
W_B(X) \leq W^0(X) \leq W_{A(k)}(X) \quad \text{for } X \in B(L_k),
\]
\[
W^0(X) \in \left[ W_B(X), \frac{8\pi}{|X|^2} \right] \quad \text{for } X \in \mathbb{R}^4 \setminus B(L_k),
\]
\[
\|W(\cdot, t)\|_{C_\omega} \leq 8\pi A(k) + 1 \quad \text{for } t \in [0, T_k + 1]
\]
and
\[
\|W(\cdot, t) - W_{\beta_j}\|_{C_\omega} \leq \frac{1}{j} \quad \text{for } t \in [T_j - 1, T_j + 1]
\]
with \( j = 1, 2, 3, \ldots, k \). Let \( W_k(\cdot, \cdot) \) be a solution whose initial data is determined as
\[
W^0(X) = W^0(X) \quad \text{for } X \in B(L_k)
\]
and
\[
W^0_k(X) = \left\{ \begin{array}{ll}
\max \left\{ \frac{\beta_{k+1}}{L^2_k W_{\beta_k}(L_k)} \right\} & \text{if } \beta_k > \beta_{k+1}, \\
\min \left\{ \frac{\beta_{k+1}}{L^2_k W_{\beta_k}(L_k)} \right\} & \text{if } \beta_k < \beta_{k+1}
\end{array} \right.
\]
for \( X \in \mathbb{R}^4 \setminus B(L_k) \).

We can find a constant \( T_{k+1} \geq T_k + 2 \) such that
\[
\|W_k(\cdot, t) - W_{\beta_{k+1}}\|_{C_\omega} \leq \frac{1}{2(k + 1)} \quad \text{for } t \in [T_{k+1} - 1, \infty)
\]
by a similar argument as for (27). Take \( L_{k+1} \geq L_k + 1 \) such that
\[
\left\{ \begin{array}{ll}
\frac{L^2_k W_{\beta_k}(L_k)}{L^2_{k+1}} \leq W_{\beta_{k+1}}(L_{k+1}) & \text{if } \beta_k > \beta_{k+1}, \\
W_{\beta_k}(L_k) \geq W_{\beta_{k+1}}(L_{k+1}) & \text{if } \beta_k < \beta_{k+1},
\end{array} \right.
\]
\[
C_1 e^{C_1(8\pi A(k+1)+3)^2(T_{k+1}+1)} \frac{8\pi(1 + L^2_{k+1})}{L^2_{k+1}(1 + BL^2_{k+1})} \leq \frac{1}{2(k + 1)}
\]
and that
\[
\int_{L_{k+1} < |x| < \tilde{L}_{k+1}} |x|^2 u_{\beta_{k+1}}(x) dx \geq 1,
\]
where \( \tilde{L}_{k+1} \) is a constant satisfying
\[
W_{\beta_{k+1}}(\tilde{L}_{k+1}) = \left\{ \begin{array}{ll}
\frac{L^2_k}{L^2_{k+1}} W_{\beta_k}(L_k) & \text{if } \beta_k > \beta_{k+1}, \\
W_{\beta_k}(L_k) & \text{if } \beta_k < \beta_{k+1}.
\end{array} \right.
\]
Determining
\[
W^0(X) = W^0_k(X) \quad \text{for } X \in B(L_{k+1}),
\]
\[
W^0(X) \in \left[ W_B(X), \frac{8\pi}{|X|^2} \right] \quad \text{for } X \in \mathbb{R}^4 \setminus B(L_{k+1}),
\]
we deduce from the comparison theorem and Proposition 4 with \( \ell = 8\pi A(k+1) + 2 \) that
\[
W_B \leq W_k \leq W_{A(k+1)} \quad \text{in } \mathbb{R}^4 \times [0, \infty)
\]
and that
\[
\|W(\cdot, t : \ell) - W_k(\cdot, t)\|_{C_\omega} \leq \frac{1}{2(k + 1)} \quad \text{for } t \in [0, T_{k+1} + 1].
\]
Then, we get $W(\cdot, \cdot) = W(\cdot, \cdot : \ell)$ in $\mathbb{R}^4 \times [0, T_{k+1} + 1]$ and

$$\|W(\cdot, t) - W_{\beta_{k+1}}\|_{C^0_\omega} \leq \frac{1}{k+1}$$

for $t \in [T_{k+1} - 1, T_{k+1} + 1]$, where $W(\cdot, \cdot : \ell)$ is a solution to (22) with $W(\cdot, 0 : \ell) = W^0$ and $\ell = 8\pi A(k+1) + 2$.

Thus, we can find a solution $W$ to (17) and a sequence $\{T_j\}_{j \geq 1}$ satisfying the following:

$$T_{j+1} \geq T_j + 2 \quad \text{for} \quad j = 1, 2, 3, \ldots .$$

$$\|W(\cdot, t) - W_{\beta_j}\|_{C^0_\omega} \leq \frac{1}{j} \quad \text{for} \quad t \in [T_j - 1, T_j + 1] \quad \text{and} \quad j = 1, 2, 3, \ldots .$$

$$W(X, t) \in \left[ W_B(X), \frac{8\pi}{|X|^2} \right] \quad \text{for} \quad (X, t) \in \mathbb{R}^4 \times [0, \infty).$$

Noting the way to choose the sequence $\{\beta_j\}_{j \geq 1}$ and the initial function $W^0$, and using a similar argument as that in Proposition 5, we obtain that the corresponding solution $(u, v)$ to (1), (2) and (3) satisfies (9), (10), (11), (12), $\Lambda = 8\pi$, (13) and (14). Thus, we finish the proof.

REFERENCES

[1] P. Biler and N. Nadzieja, Existence and nonexistence of solutions for a model of gravitational interaction particles I, Colloq. Math., 66 (1994), 319–334.
[2] P. Biler, G. Karch, P. Laurençot and T. Nadzieja, The $8\pi$ problem for radially symmetric solutions of a chemotaxis model in the plane, Math. Meth. Appl. Sci., 29 (2006), 1563–1583.
[3] A. Blanchet, J. A. Carrillo and N. Masmoudi, Infinite time aggregation for the critical two-dimensional Patlak-Keller-Segel model, Comm. Pure Appl. Math., 61 (2008), 1449–1481.
[4] S. Childress and J. K. Percus, Nonlinear aspects of chemotaxis, Math. Biosci., 56 (1981), 217–237.
[5] J. Dolbeault and B. Perthame, Optimal critical mass in the two-dimensional Keller-Segel model in $\mathbb{R}^2$, C. R. Math. Acad. Sci. Paris, 339 (2004), 611–616.
[6] E. Feireisl, Ph. Laurençot and H. Petzeltová, On convergence to equilibria for the Keller-Segel chemotaxis model, J. Differential Equations, 236 (2007), 551–569.
[7] C. Gui, W.-M. Ni and X. Wang, On the stability and instability of positive steady states of a semilinear heat equation in $\mathbb{R}^n$, Comm. Pure Appl. Math., 45 (1992), 1153–1181.
[8] C. Gui, W.-M. Ni and X. Wang, Further study on a nonlinear heat equation, J. Diff. Eqs., 169 (2001), 588–613.
[9] E. F. Keller and L. A. Segel, Initiation of slime mold aggregation viewed as an instability, J. Theor. Biol., 26 (1970), 399–415.
[10] T. Nagai, Global existence and decay estimates of solutions to a parabolic-elliptic system of a drift-diffusion type in $\mathbb{R}^2$, Differential Integral Equations, 24 (2011), 29–68.
[11] T. Ogawa and T. Nagai, Global existence of solutions to a parabolic-elliptic system of a drift-diffusion type in $\mathbb{R}^2$, preprint.
[12] P. Poláčik and E. Yanagida, On bounded and unbounded global solutions of a supercritical semilinear heat equation, Math. Ann., 337 (2007), 745–771.

Received March 2011; revised October 2012.

E-mail address: yaito@ehime-u.ac.jp
E-mail address: senba@mns.kyutech.ac.jp