Relativistic Motions of Spin-Zero Quantum Oscillator Field in Global Monopole Space-Time with External Potential

Faizuddin Ahmed
(faizuddinahmed15@gmail.com ; faizuddin@ustm.ac.in)
Department of Physics, University of Science & Technology Meghalaya, Ri-Bhoi, Meghalaya-793101, India

Abstract

We analyze spin-zero relativistic quantum oscillator fields in the presence of Aharonov-Bohm (AB) magnetic flux and subsequently with Coulomb-types scalar and vector potentials in the space-time background produced by a point-like global monopole (PGM). We solve the generalized Klein–Gordon oscillator analytically by choosing different types of function $f(r)$ (Coulomb- & Cornell-types) and obtain the bound-states solutions in each cases. We then discuss the influences of topological defect associated with scalar curvature of the PGM space-time, the magnetic flux and Coulomb-types interaction potentials on the energy spectrum of the oscillator field.

Keywords: Relativistic wave-equations, solutions of wave equations: bound-states, Topological defects

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1 Introduction

At present the best theory for describing particles and their interactions is called the Standard Model (SM) of particle physics. It describes all known elementary particles: fermions and bosons. Three of the four fundamental interactions are included in the SM: electromagnetism, strong, and weak forces. Further, the discovery of the Higgs bosons \cite{1,2} stands that symmetries and spontaneous symmetry breaking mechanism play a crucial role on the Universe at the smallest scales \cite{3,4,5}. In spite of its
success, the SM was unable to explain some physical aspects, for instance, strong CP problem, neutrino oscillations \cite{6,7} depending on whether the neutrinos are Dirac or Majorana, matter–antimatter asymmetry, and Dark Matter and Dark Energy. Furthermore, this SM doesn’t incorporate the general theory of relativity. Several researchers proposed theories and/or models that goes beyond the SM by including various extensions. Recently, a model beyond the SM and GUT was proposed in Ref. \cite{8}, where author added new gapped Topological Phase sectors consistent with the nonperturbative global anomaly matching and cobordism constraints via symmetry extension.

Some Grand Unified Theories suggested that global topological defects are formed during the phase transition in the early universe through spontaneous symmetry breaking mechanism \cite{9,10}. Various topological defects includes cosmic strings \cite{11,12}, domain walls \cite{10,13}, and global monopole \cite{14,15,16,17}. In condensed matter physics \cite{18,19,20,21,22,23,24}, these linear defects may be related to dislocations (torsion) and disclinations (curvature) \cite{20,25}. The effects of global monopole have been studied in the context of quantum mechanical systems, for instance, in the non-relativistic limit, the harmonic oscillators \cite{26}, quantum scattering of charged or massive particle \cite{27,28,29}, exact solutions of the Klein-Gordon equation in the presence of a dyon, magnetic flux and scalar potential \cite{30} etc.. In the relativistic limit, studies on hydrogen and pionic atom \cite{31}, exact solutions of scalar bosons in the presence of potential \cite{32}, the Dirac and Klein–Gordon oscillators \cite{33}, and the generalized Klein-Gordon oscillator \cite{34}. In addition, global monopole space-time have been studied in scalar self-energy for a charged particle \cite{35,36}, induced self-energy on a static scalar charged particle \cite{37}, vacuum polarization for a massless scalar field \cite{38}, vacuum polarization for a mass-less spin-1/2 field \cite{39}, vacuum polarization effects in the presence of the Wu–Yang magnetic monopole \cite{40}. Also, the gravitational deflection of light by a rotating global monopole space-time have been studied in Ref. \cite{41}, and charged global monopoles in Ref. \cite{42}.

In this contribution, we will study relativistic scalar fields via the generalized Klein-Gordon oscillator in the presence of the Aharonov-Bohm magnetic flux in the point-like global monopole space-time. In Ref. \cite{43}, oscillator was first studied in the
Dirac equation which is so called now the Dirac oscillator (see also, Refs. [44, 45]). Inspired by the Dirac oscillator, several authors have been studied the Klein-Gordon oscillator in different space-time backgrounds, for instance, in cosmic string space-time Refs. [46, 47, 48], in a point-like global monopole space-time Ref. [33], in five-dimensional Minkowski space-time with external potential Ref. [49], and in the context of Kaluza-Klein theory Refs. [47, 50]. Furthermore, the generalized Klein-Gordon oscillator has also been studied, for example, in five-dimensional magnetic cosmic string space-time with potential in Refs. [50, 51, 52].

We summarize this paper as follows: in section 2, we discuss in details the generalized Klein-Gordon oscillator in the background space-time produced by a point-like global monopole, and then determine its solutions by choosing respectively, Coulomb-type function \( f(r) = \frac{b}{r} \) without external potential (sub-section 2.1), Cornell-type function \( f(r) = \left( a + \frac{b}{r} \right) \) without external potential (sub-section 2.2), Coulomb-type function \( f(r) = \frac{b}{r} \) with Coulomb-types scalar \( S(r,t) (\propto \frac{1}{r}) \) and vector \( A_0 (\propto \frac{1}{r}) \) potentials (sub-section 2.3), Cornell-type function \( f(r) = \left( a + \frac{b}{r} \right) \) with Coulomb-types scalar \( S(r,t) (\propto \frac{1}{r}) \) and vector \( A_0 (\propto \frac{1}{r}) \) potentials (sub-section 2.3); and finally conclusions in section 3. Here we have used the natural units \( c = \hbar = G = 1 \).

## 2 Generalized KG-Oscillator in Global Monopole Space-time Background

We investigate the quantum dynamics of oscillator field in the space-time with a point-like global monopole defect with or without external potential. By solving the generalized Klein-Gordon oscillator analytically, we discuss the influences of the topological defect on the energy profiles of the oscillator field. Thereby, we begin this section by introducing the line element of a point-like global monopole which is static and spherically symmetric metric described by \[ \frac{ds^2}{\alpha^2} = -dt^2 + \frac{dr^2}{\alpha^2} + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2), \]
where $\alpha = \sqrt{1 - 8 \pi \eta_0^2} < 1$ the wedge parameter depends on the energy scale $\eta_0$. The parameter $\eta_0$ represents the dimensionless volumetric mass density of the point-like global monopole (PGM) defect. Here, the different coordinates are in the ranges $-\infty < t < +\infty, r \geq 0, 0 \leq \theta \leq \frac{\pi}{2}$, and $0 \leq \phi < 2 \pi$. This point-like global monopole space-time have some interesting features: it is not flat globally and possesses a naked curvature singularity which is related with the Ricci scalar, $R = R^\mu_{\mu} = \frac{2(1-\alpha^2)}{r^2}$; the area of a sphere of radius $r$ in this manifold is not $4 \pi r^2$ but rather it is $4 \pi \alpha^2 r^2$; the surface $\theta = \frac{\pi}{2}$ presents the geometry of a cone with deficit angle $\delta \Omega = 8 \pi^2 \eta_0^2$; there is no Newtonian-like gravitational potential: $g_{tt} = -1$. Furthermore, in this topological defect space-time geometry the solid angle of a sphere of radius $r$ is $4 \pi \alpha^2 r^2$ which is smaller than $4 \pi^2 r^2$, and hence, there is a solid angle deficit $32 \pi^2 \eta_0^2$. In condensed matter systems, this point-like global monopole space-time describes an effective metric produced in superfluid $^3$He-by a monopole with the angle deficit $\alpha$. In that case, the topological defect has a negative mass $[54, 55]$. As this point-like global monopole geometry has no gravitational fields, some global effects of this geometry has measured, for example, by the scattering cross section for massless bosonic $[56]$, and fermionic $[57]$ particles propagating in it. Note that in the limit $\alpha \rightarrow 1$, we obtain the Minkowski flat space line element in spherical symmetry.

The relativistic quantum dynamics of spin-zero Bosons with an external potential $S(r, t)$ is described by $[30, 47, 50, 51, 52, 58]$

$$
\left\{ \frac{1}{\sqrt{-g}} \left( \partial_\mu - ie A_\mu \right) \left\{ \sqrt{-g} g^{\mu\nu} \left( \partial_\nu + ie A_\nu \right) \right\} - \xi R \right\} \Psi = \left( M + S(r, t) \right)^2 \Psi, \quad (2)
$$

where $e$ is the electric charges, $A_\mu = (-A_0, \vec{A})$ is the electromagnetic four-vector potential, $g$ is determinant of the metric tensor with $g^{\mu\nu}$ its inverse, $\xi$ is an arbitrary coupling constant with the background curvature, $R$ is the Ricci scalar or scalar curvature, and $M$ is the rest mass of the scalar bosons. Here we followed Refs. $[59, 60, 61]$, where it was suggested that a non-electromagnetic potential would be taken into account by modifying the rest mass via $M \rightarrow \left( M + S(r, t) \right)$ in the wave equation. This new formalism has been first used in Ref. $[60]$ to analyse the Dirac equation in the presence of Coulomb and static scalar potentials proportional to the
inverse of the radial distance, i.e., $S(r, t) \propto \frac{1}{r}$ (see, Ref. [61]). Later on, this new formalism has been studied in various space-times background in quantum systems by different researchers (see, Refs. [50, 51, 52, 58] and related references there in).

In this contribution, we examine a spin-zero relativistic quantum oscillator model in the point-like global monopole (PGM) space-time background. For studies of generalized Klein-Gordon oscillator, we perform the replacements in momentum operator $p_\mu \rightarrow (p_\mu - i M \omega X_\mu)$ and $p_\mu^\dagger \rightarrow (p_\mu + i M \omega X_\mu)$ [46, 48, 47, 50, 51, 52], where $\omega$ is the oscillation frequency, and $X_\mu = f(r) \delta_\mu^r$ is an arbitrary four-vector. Note that if one choose $f(r) = r$, then the quantum system is called the Klein-Gordon oscillator.

As we are focusing on the generalized Klein-Gordon oscillator where, $f(r) \neq r$. Thus, one can replace $\vec{p}^2 \rightarrow (p_\mu - i M \omega X_\mu) \cdot (p_\mu + i M \omega X_\mu)$ in the wave equation for the studies of the generalized Klein-Gordon oscillator.

Therefore, the generalized Klein-Gordon oscillator from Eq. (2) becomes

$$\left[ -\frac{1}{\sqrt{-g}} \left( D_\mu + M \omega X_\mu \right) \{ \sqrt{-g} g^{\mu\nu} \left( D_\nu^\dagger - M \omega X_\mu \right) \} + \xi R + \left( M + S(r, t) \right)^2 \right] \Psi = 0, \quad (3)$$

where $D_\mu = (\partial_\mu - i e A_\mu)$.

Equation (3) in the background space-time (1) becomes

$$\left[ -\left( \partial_t + i e A_0 \right)^2 + \frac{\alpha^2}{r^2} \left( \partial_r + M \omega f r \right) \left( r^2 \partial_r - M \omega r^2 f(r) \right) + \frac{1}{r^2 \sin^2 \theta} \partial_\theta \left( \sin \theta \partial_\theta \right) \right] \Psi \ + \left[ \frac{1}{r^2 \sin^2 \theta} \left( \partial_\phi - i e A_\phi \right)^2 - \frac{2 \xi (1 - \alpha^2)}{r^2} - \left( M + S(r, t) \right)^2 \right] \Psi = 0. \quad (4)$$

In quantum mechanics, by the method of separation of variables one can write the total wave function $\Psi(t, r, \theta, \phi)$ in terms of a radial wave function $\psi(r)$ as

$$\Psi(t, r, \theta, \phi) = e^{-i Et} Y_{l,m}(\theta, \phi) \psi(r), \quad (5)$$

where $E$ is the energy of the scalar massive charged particles, $Y_{l,m}(\theta, \phi)$ is the spherical harmonics, and $l, m$ are the angular momentum and magnetic moment quantum numbers, respectively.

Thereby, substituting the ansatz (5) into the Eq. (4), we obtain the following
differential equation:

\[
\frac{1}{\psi(r)} \left[ \alpha^2 \left\{ \left( \frac{\partial}{\partial r} + M \omega f(r) \right) \left( r^2 \frac{\partial \psi(r)}{\partial r} - M \omega^2 f(r) \psi(r) \right) \right\} \right]
+ \frac{1}{\psi(r)} \left\{ -\frac{2 \xi (1 - \alpha^2)}{r^2} + \left( E - e A_0 \right)^2 - \left( M + S(r, t) \right)^2 \right\} \psi(r)
+ \frac{1}{r^2 Y_{l,m}} \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial Y_{l,m}}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \left( \frac{\partial}{\partial \phi} - i \Phi \right)^2 Y_{l,m} \right] = 0. \tag{6}
\]

Here the parameter \( \Phi \) is the amount of the magnetic flux which we will discuss later on. The effective total momentum and the \( z \)-component of the angular momentum operators are

\[
L_{\text{eff}}^z = \left[ -\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) - \frac{1}{\sin^2 \theta} \left( \frac{\partial}{\partial \phi} - i \Phi \right)^2 \right],
\]

\[
L_{\text{eff}}^{z} = -i \left( \frac{\partial}{\partial \phi} - i \Phi \right). \tag{7}
\]

Here the electromagnetic three-vector potential is given by \[30\] 58

\[\bar{A} = (0, A_\phi, 0), \quad A_\phi = \frac{\Phi}{e}, \quad \Phi = (\Phi_B/\Phi_0), \quad \Phi_0 = (2 \pi/e), \tag{8}\]

where \( \Phi_B = \text{const} \) is the Aharonov-Bohm magnetic quantum flux, \( \Phi_0 \) is the quantum of magnetic flux, and \( \Phi \) is called the amount of magnetic flux which is being a positive integer. Note that the Aharonov–Bohm effect \[62\] 63 in a quantum mechanical phenomena and has been investigated in different branches of physics including bound states of massive fermions \[64\], in the context of Kaluza–Klein theory \[49\] 50 51 52 etc.. The eigenvalues of the various operator terms involve in the Eq. \[6\] are as follows \[30\]

\[
L_z Y_{l,m}(\theta, \phi) = m Y_{l,m}(\theta, \phi), \quad L^{\text{eff}}_z Y_{l,m}(\theta, \phi) = (m - \Phi) Y_{l,m}(\theta, \phi),
\]

\[
L^2 Y_{l,m}(\theta, \phi) = \lambda Y_{l,m}(\theta, \phi), \quad L^2_{\text{eff}} Y_{l,m}(\theta, \phi) = \lambda' Y_{l,m}(\theta, \phi)
\]

\[
\lambda = l (l + 1), \quad \lambda' = l' (l' + 1), \quad l' = l - \Phi, \tag{9}
\]

where \( l \) is the angular momentum quantum number, and \( m = 0, \pm 1, \pm 2, \ldots \) is the magnetic moment quantum number.
Thereby, using the above operators in the Eq. (6), we have the radial wave equation for the generalized Klein-Gordon oscillator:

\[
\psi''(r) + \frac{2}{r} \psi'(r) - \left[ 2 M \omega \left( \frac{f'}{r} + \frac{f''}{2} \right) + M^2 \omega^2 f^2 \right] \psi(r)
+ \frac{1}{\alpha^2} \left[ -\frac{2 \xi (1 - \alpha^2) + l' (l' + 1)}{r^2} + \left( E - e A_0 \right)^2 - \left( M + S(r, t) \right)^2 \right] \psi(r) = 0. \tag{10}
\]

To study the modified and/or the generalized Klein-Gordon oscillator in a point-like global monopole space-time, we have chosen two types the function \( f(r) \), namely, a Coulomb-type and a Cornell-type potential form function which are as below.

### 2.1 Coulomb-type function \( f(r) = \frac{b}{r} \) Without Potential \( A_0 = 0 = S \)

We study the relativistic quantum oscillator via the generalized Klein-Gordon oscillator by choosing the function \( f(r) = \frac{b}{r} \) \cite{12, 34, 53}, where \( b > 0 \) in a point-like global monopole space-time with zero scalar and vector potentials, \( A_0 = 0 = S \) subject to the Aharonov-Bohm magnetic flux. We discuss the influences of the topological defects of the space-time on the energy profiles.

Thereby, substituting the function \( f(r) = \frac{b}{r} \) into the Eq. (10), the radial wave equation becomes

\[
\psi''(r) + \frac{2}{r} \psi'(r) + \left( \zeta^2 - \frac{\sigma^2}{r^2} \right) \psi(r) = 0, \tag{11}
\]

where

\[
\zeta = \sqrt{\frac{E^2 - M^2}{\alpha}}, \quad \sigma = \sqrt{\frac{2 \xi (1 - \alpha^2) + l' (l' + 1)}{\alpha^2} + M \omega b + M^2 \omega^2 b^2}. \tag{12}
\]

Now, we perform a transformation \( \psi = \frac{U(r)}{r^{\alpha}} \) into the Eq. (11), we have

\[
r^2 U''(r) + r U'(r) + \left( \zeta^2 r^2 - \tau^2 \right) U(r) = 0, \tag{13}
\]

where \( \tau = \sqrt{\frac{\sigma^2 + 1}{4}} \). Equation (13) is the well-known Bessel’s differential equation \cite{66}. Since \( \tau \) is always positive, the general solution to the Bessel equation (13) is in
the form: \( U(r) = C_1 J_{|\tau|}(\zeta r) + C_2 Y_{|\tau|}(\zeta r) \), where \( J_{|\tau|}(\zeta r) \) and \( Y_{|\tau|}(\zeta r) \) are the Bessel function of first kind and second kind [66], respectively. The Bessel function of second kind \( Y_{|\tau|}(\zeta r) \) diverges at the origin; so, we must take \( C_2 = 0 \) in the general solution, since we are interested in a well-behaved solution. Thus, the regular solution to the Eq. (13) at the origin is given by

\[
U(r) = C_1 J_{|\tau|}(\zeta r).
\] (14)

Let us restrict the motion of the scalar oscillator field to a region where a hard-wall potential is present. This kind of confinement is described by the boundary condition: \( U(r_0) = 0 \), which means that the wave function \( \psi(r) \) vanishes at a fixed radius \( r = r_0 \); that is, this boundary condition corresponds to the scalar oscillator field subject to a hard-wall potential. The hard-wall potential has studied on a scalar particle under noninertial effects in [67, 68], and on harmonic oscillator [69]. Then, let us consider a particular case where \( \eta r_0 >> 1 \). In this particular case, we can write (14) in the form

\[
J_{|\tau|}(\zeta r_0) \propto \cos \left( \zeta r_0 - \frac{\tau \pi}{2} - \frac{\pi}{4} \right).
\] (15)

By substituting Eq. (15) into the Eq. (14), we obtain from the boundary condition \( U(r_0) = 0 \) the relativistic energy levels of the quantum system

\[
E_{n,l} = \pm \sqrt{M^2 + \left\{ n + \frac{1}{2} \left( \frac{1}{4} + \frac{2\xi (1 - \alpha^2) + l' (l' + 1)}{\alpha^2} + M \omega b + M^2 \omega^2 b^2 + \frac{3}{2} \right) \right\}^2 \frac{\alpha^2 \pi^2}{r_0^2}},
\] (16)

where \( n = 0, 1, 2, 3, \ldots \).

We see that the background space-time produced by a point-like global monopole influences the dynamics of the oscillator scalar field subject to a hard-wall potential through the presence of the parameter \( \alpha^2 \) on the relativistic energy level of the system. In addition, the energy eigenvalues \( E_{n,l} \) are influenced by the Aharonov-Bohm magnetic flux \( \Phi_B \). We have plotted few graphs showing the influences of different parameters on the energy spectrum of these oscillators (fig. 1).
Figure 1: Energy eigenvalues $E_{n,l}$ of (16) with different parameters $(\omega, \alpha, \xi)$. 
2.2 Cornell-type function $f(r) = (ar + \frac{b}{r})$ With $A_0 = 0 = S$

In this section, we study the relativistic quantum oscillator in a point-like global monopole space-time by choosing a Cornell-type potential form function, $f(r) = (ar + \frac{b}{r})$ with a zero scalar and vector potential, $V = 0 = S$ subject to an Aharonov-Bohm magnetic flux. This type of potential form function has been used for studies of the generalized Klein-Gordon oscillator (see, Refs. [50, 51, 52, 12, 65, 34] and related references there in), and the generalized Dirac oscillator [70, 71]. We solve the wave equation for generalized oscillator analytically and discuss the influences of the topological defects as well as the function on the energy spectrum.

Thereby, substituting the Cornell-type function $f(r) = (ar + \frac{b}{r})$ into the Eq. (10), we obtain the following radial wave equation:

$$
\psi''(r) + \frac{2}{r} \psi'(r) + \left( \Lambda - \frac{j^2}{r^2} - M^2 \omega^2 a^2 r^2 \right) \psi(r) = 0, \tag{17}
$$

where

$$
\Lambda = \frac{E^2 - M^2 - 3 M \omega a \alpha^2 - 2 a b M^2 \omega^2 \alpha^2}{\alpha^2},
$$

$$
j = \sqrt{\frac{2 \xi (1 - \alpha^2) + \nu (l' + 1)}{\alpha^2}} + M \omega b + M^2 \omega^2 b^2. \tag{18}
$$

Transforming the above equation via $\psi(r) = \frac{U(r)}{r^{3/2}}$, we have

$$
U''(r) - \frac{1}{r} U'(r) + \left[ \Lambda - \frac{(j^2 - \frac{3}{4})}{r^2} - M^2 \omega^2 a^2 r^2 \right] U(r) = 0. \tag{19}
$$

Introducing a new variables via $s = M \omega a r^2$ into the above Eq. (19), we have obtained the following second order differential equation:

$$
U''(s) + \left( \frac{1 - 4 \mu^2}{4 s^2} \right) U(s) + \frac{\nu}{s} U(s) - \frac{1}{4} U(s) = 0, \tag{20}
$$

where different parameters are defined by

$$
2 \mu = \sqrt{j^2 + \frac{1}{4}}, \quad \nu = \frac{\Lambda}{4 M \omega a}. \tag{21}
$$
Equation (20) is the Whittaker differential equation [72] and $U(s)$ is the Whittaker function which can be written in terms of the confluent hypergeometric function $\text{$_1U_1$}(s)$ as

$$U(s) = s^{\frac{1}{2} + \mu} e^{-\frac{s}{2}} \text{$_1F_1$} \left( \mu - \nu + \frac{1}{2}, 2 \mu + 1; s \right). \quad (22)$$

In order to obtain the bound-states solutions, it is necessary that the confluent hypergeometric function $\text{$_1F_1$} \left( \mu - \nu + \frac{1}{2}, 2 \mu + 1; s \right)$ should be a power series of finite degree $n$, and the quantity $(\mu - \nu + \frac{1}{2})$ should be a negative integer, that is, $(\mu - \nu + \frac{1}{2}) = -n$, where $n = 0, 1, 2, ...$. After simplification of this condition, we have the following expression of the energy eigenvalues of the system

$$E_{n,l} = \pm \sqrt{M^2 \left( 1 + 2 a b \omega^2 \alpha^2 \right) + 2 M \omega \alpha^2} \times \left( 2 n + \sqrt{\frac{1}{4} + 2 \xi (1 - \alpha^2) + l' (l' + 1)} + M \omega b + M^2 \omega^2 b^2 + \frac{5}{2} \right). \quad (23)$$

The normalized radial wave functions are given by

$$\psi_{n,l}(s) = D_{n,l} (M \omega)^{3/4} s^{\mu - \frac{1}{4}} e^{-\frac{s}{2}} \text{$_1F_1$} \left( \mu - \nu + \frac{1}{2}, 2 \mu + 1; s \right), \quad (24)$$

where $D_{n,l}$ is a constant which can be determined by the normalization condition for the radial wave function

$$\frac{1}{\alpha} \int_0^{\infty} r^2 dr |\psi(r)|^2 = 1. \quad (25)$$

To solve the integrals of the radial wave function, we can write the confluent hypergeometric function in terms of the associated Laguerre polynomials by the relation [66]

$$\text{$_1F_1$} (-n, 2 \mu + 1; x) = \frac{n! (2 \mu)!}{(n + 2 \mu)!} L_n^{(2 \mu)}(x). \quad (26)$$

Then, taking into account $s = M \omega r^2$, and with the help of [73] to solve the integrals, the normalization constant is given by

$$D_{n,l} = \frac{1}{(2 \mu)!} \sqrt{\frac{2 \alpha (n + 2 \mu)!}{n!}} = \frac{1}{\sqrt{\frac{2 \alpha (n + \sqrt{j^2 + \frac{1}{4}})!}{n!}}} \sqrt{\frac{2 \alpha (n + \sqrt{j^2 + \frac{1}{4}})!}{n!}}. \quad (27)$$

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Figure 2: Energy eigenvalues $E_{n,l}$ of (23) with different parameters $(\alpha, \omega, \xi)$. 
Figure 3: Radial wave function $\psi_{n,l}$ of (24) with $s$.

Equation (23) is the relativistic energy spectrum that stems from the interaction of the scalar field with the generalized Klein-Gordon oscillator in a point-like global monopole space-time background. We see that the topological defects associated with the scalar curvature of the space-time, and the function $f(r) = (a r + \frac{b}{r})$ other than $r$ modified the energy profiles. Furthermore, the energy eigenvalues $E_{n,l}$ are influenced by the Aharonov-Bohm magnetic flux $\Phi_B$. We have plotted few graphs showing the influences of different parameters on the energy spectrum (fig. 2) and on the wave function (fig. 3) of these oscillators.

In the below analysis, we insertion of external potential to study the generalized Klein-Gordon oscillator in a point-like global monopole space-time with the function
function \( f(r) \) considered in the previous two analysis. We have chosen the fourth component of the electromagnetic potential \( A_0 \) and the scalar potential proportional to the inverse of the radial distance \([68] \), i.e.,

\[
A_0 \propto \frac{1}{r} \Rightarrow A_0 = \frac{\kappa}{r} = \pm \frac{|\kappa|}{r}, \quad S \propto \frac{1}{r} \Rightarrow S = \frac{\eta}{r},
\]

where \( \kappa, \eta \) characterize the potential parameters. This type of potential have widely been studied, for instance, 1-dimensional systems \([74, 75] \), topological defects in solids \([76] \), quark-antiquark interaction \([77] \), propagation of gravitational waves \([78] \), quark models \([79] \), and the relativistic quantum systems \([58, 80, 68] \). Here we followed Refs. \([59, 60, 61, 58] \) to introduce the scalar potential in the wave equation by modifying the mass term via transformation \( M \rightarrow M + S(r, t) \), where \( S(r, t) \) is the scalar potential.

### 2.3 Coulomb-type function \( f(r) = \frac{b}{r} \) With \( A_0 = \pm \frac{|\kappa|}{r} \) and \( S = \frac{\eta}{r} \).

In this section, we study the generalized Klein-Gordon oscillator subject to a vector and scalar potentials of Coulomb-types \([28] \) with a Coulomb-type function \( f(r) = \frac{b}{r} \) in the presence of an Aharonov-Bohm magnetic flux in a space-time background produced by a point-like global monopole.

Therefore, the radial wave equation \([10] \) in this case becomes

\[
\psi''(r) + \frac{2}{r} \psi'(r) + \left( -\chi^2 - \frac{\beta^2}{r^2} - \frac{2\gamma}{r} \right) \psi(r) = 0,
\]

where

\[
\beta = \sqrt{\frac{2\xi (1 - \alpha^2) + l'(l' + 1) + \eta^2 - e^2 \kappa^2}{\alpha^2}} + M \omega b + M^2 \omega^2 b^2, \\
\gamma = \frac{(M \eta - e E |\kappa|)}{\alpha}, \quad \chi = \frac{\sqrt{M^2 - E^2}}{\alpha}.
\]

Following the radial wave transformation via \( \psi = \frac{R(r)}{\sqrt{r}} \) into the Eq. \((29)\), we have

\[
R''(r) + \frac{1}{r} R'(r) + \left( -\chi^2 - \frac{\beta^2}{r^2} - \frac{2\gamma}{r} \right) R(r) = 0.
\]
We do another transformation on the radial coordinate via $\rho = 2 \chi r$ into the Eq. (31), we have
\[ R''(\rho) + \frac{1}{\rho} R'(\rho) + \left( -\frac{\beta^2}{\rho^2} - \frac{\gamma}{\chi \rho} - \frac{1}{4} \right) R(\rho) = 0. \] (32)

Now, we impose requirement of the wave function that the radial wave function $R(\rho)$ must be well-behaved at the origin, since it is a singular point of the Eq. (32). In this case, for $\lim_{\rho \to 0} R(\rho) = 0$, the solution is $R(\rho) \sim \rho^\beta$. Furthermore, for $\lim_{\rho \to \infty} R(\rho) = 0$, the solution is $R(\rho) \sim e^{-\rho^2/2}$. Thus, a possible solution to the Eq. (32) is given by
\[ R(\rho) = \rho^\beta e^{-\rho^2/2} F(\rho), \] (33)
where $F(\rho)$ is an unknown function. Substituting this solution (33) into the Eq. (32), we have
\[ \rho F''(\rho) + (2 \beta + 1 - \rho) F'(\rho) + \left( -\beta - \frac{\gamma}{\chi} - \frac{1}{2} \right) F(\rho) = 0. \] (34)

Equation (34) is the confluent hypergeometric second order differential equation and the function $F(\rho) = {}_1F_1(\beta + \frac{\gamma}{\chi} + \frac{1}{2}, 2 \beta + 1; \rho)$ is called the confluent hypergeometric function. In order to find the bound-states solutions of the quantum system, this confluent hypergeometric function must be a finite degree polynomial $n$, and the quantity $\left( \beta + \frac{\gamma}{\chi} + \frac{1}{2} \right)$ should be a negative integer. This condition implies that $\left( \beta + \frac{\gamma}{\chi} + \frac{1}{2} \right) = -n$, where $n = 0, 1, 2, \ldots$. After simplifying this condition, one will find the following expression of the energy spectrum
\[ E_{n,l} = \frac{1}{\alpha^2 \Delta^2 + e^2 \kappa^2} \left[ M e^{|\kappa|} \eta \pm M \Delta \alpha^2 \sqrt{\Delta^2 + \frac{e^2 \kappa^2 - \eta^2}{\alpha^2}} \right], \] (35)
where
\[ \Delta = \left( n + \sqrt{\frac{2 \xi (1 - \alpha^2) + l'(l' + 1) + \eta^2 - e^2 \kappa^2}{\alpha^2} + M \omega b + M^2 \omega^2 b^2 + \frac{1}{2}} \right). \] (36)

The radial wave function is given by
\[ R(\rho) = \rho^{\frac{1}{2} \left( 1 - \alpha^2 \right) + l' \left( l' + 1 \right) + e^2 \kappa^2 + M \omega b + M^2 \omega^2 b^2} e^{-\rho^2/2} {}_1F_1(\beta + \frac{\gamma}{\chi} + \frac{1}{2}, 2 \beta + 1; \rho). \] (37)
Figure 4: Energy $E_{n,l}$ with different parameters.

We can see that the energy spectrum (35) and the radial wave function (37) of oscillator field are influenced by the topological defects parameter $\alpha^2$ which is associated with the curvature of the space-time, the function $f(r) = \frac{b}{r}$ as well as the Coulomb-types scalar and vector potentials present in the system. Furthermore, the energy eigenvalues $E_{n,l}(\Phi_B)$ depends on the Aharonov-Bohm magnetic flux and are influenced by this. We have plotted few graphs showing the influences of different parameters on the energy spectrum (fig. 4) and on the wave function (fig. 5) of these oscillators.
2.4 Cornell-type function $f(r) = (a r + \frac{b}{r})$ With $A_0 = \pm \frac{\kappa}{r}$ and $S = \frac{\eta}{r}$

In this section, we study the generalized Klein-Gordon oscillator subject to a vector and scalar potentials of Coulomb-types [28] with a Cornell-type function, $f(r) = (a r + \frac{b}{r})$ in the presence of an Aharonov-Bohm flux in a point-like global monopole space-time. We discuss the influences on the energy profiles of these oscillators.

Thereby, substituting all into the Eq. [10], we obtain the following equation:

$$\psi''(r) + \frac{2}{r} \psi'(r) + \left( -\delta^2 - \frac{\beta^2}{r^2} - \frac{2\gamma}{r} \right) \psi(r) = 0,$$

(38)
where we have defined
\[
\delta^2 = 3 M \omega a + 2 a b M^2 \omega^2 + \frac{1}{\alpha^2} (M^2 - E^2).
\] (39)

Now, we perform a transformation via \( \psi = \frac{R(r)}{\sqrt{r}} \) into the Eq. (38), we have
\[
R''(r) + \frac{1}{r} R'(r) + \left( -\delta^2 - \frac{\beta}{r^2} \right) R(r) = 0.
\] (40)

Considering a transformation of the radial coordinate via \( \rho = 2 \delta r \) into the Eq. (40), we have
\[
R''(\rho) + \frac{1}{\rho} R'(\rho) + \left( -\rho \frac{\beta}{\rho^2} - \frac{\gamma}{\delta} - \frac{1}{4} \right) R(\rho) = 0.
\] (41)

As state earlier, we impose the radial wave function \( R(\rho) \to 0 \) for \( \rho \to 0 \) and \( \rho \to \infty \).

Suppose, a possible solution to the Eq. (41) given by
\[
R(\rho) = \rho^2 e^{\frac{\gamma}{\delta}} F(\rho),
\] (42)

where \( F(\rho) \) is an unknown function. Substituting this solution into the Eq. (41), we have
\[
\rho F''(\rho) + (2 \beta + 1 - \rho) F'(\rho) + \left( -\beta - \frac{\gamma}{\delta} - \frac{1}{2} \right) F(\rho) = 0.
\] (43)

Equation (43) is the well-known confluent hypergeometric equation which is a second order linear homogeneous differential equation. The solution of this Eq. (43) that is regular for \( \rho \to 0 \) is given in terms of the confluent hypergeometric function as
\[
F(\rho) = {}_1 F_1(\beta + \frac{\gamma}{\delta} + \frac{1}{2}; 2 \beta + 1; \rho).
\] (44)

As state earlier, in order to have a bound-states solutions of this equation, the confluent hypergeometric function \( {}_1 F_1(\beta + \frac{\gamma}{\delta} + \frac{1}{2}; 2 \beta + 1; \rho \to \infty) \) must be a finite degree polynomial of degree \( n \), and the quantity \( (\beta + \frac{\gamma}{\delta} + \frac{1}{2}) \) is a negative integer, that is, \( (\beta + \frac{\gamma}{\delta} + \frac{1}{2}) = -n \), where \( n = 0, 1, 2, \ldots \) After simplifying this condition, we have the following energy eigenvalues of the system
\[
E_{n,l} = \frac{1}{\alpha^2 \Delta^2 + e^2 \kappa^2} \left[ M e |\kappa| \eta \pm M \Delta \right. \left. \alpha^2 \sqrt{\Delta^2 + \frac{e^2 \kappa^2 - \eta^2}{\alpha^2} + \Sigma^2} \right],
\] (45)
where $\Delta$ is given earlier and
\[ \Sigma = \sqrt{\omega a \left( \frac{3}{M} + 2 \omega b \right) (\alpha^2 \Delta^2 + e^2 \kappa^2)} . \] (46)

We can see that the energy spectrum (45)–(46) of oscillator field are influenced by the topological defects parameter $\alpha^2$ of the space-time, the modified function $f(r) = (a r + b/r)$ as well as Coulomb-types scalar and vector potentials. Furthermore, the eigenvalues $E_{n,l}(\Phi_B)$ depends on the geometric quantum phase and therefore are influenced by the Aharonov-Bohm magnetic flux $\Phi_B$.

3 Conclusions

We studied the relativistic quantum oscillator interacting with a gravitational field produced by topological defects via the generalized Klein–Gordon oscillator in a point-like global monopole (PGM) space-time background subject to a scalar potential. We analyze the behaviour of these oscillators in which the topological defects that is associated with the scalar curvature of the space-time influences the energy eigenvalues and the wave function. This deviation in the energy spectrum can be used to investigate the presence of these kind of defects in the Cosmos. The present analysis can be used for simulation of various physical systems, for instance, vibrational spectrum of diatomic molecules [81], binding of heavy quarks [82, 83], quark–antiquark interaction [84] etc.. The presented energy spectrum shows a modification which may be suitable to demonstrate the existence of this kind of topological defects. However, from observational point of view it is clear that to have an observable modification in the energy spectrum, huge amount of particles is needed, otherwise the magnitude of the effect to a real spectrum may not be strong enough to be observed.

In this work, we have investigated the generalized Klein-Gordon oscillator in a point-like global monopole (PGM) space-time in the presence of an Aharonov-Bohm flux subject to an external potentials of Coulomb-types. In sub-section 2.1, we have considered a Coulomb-type function $f(r) = b/r$ for the studies of the generalized
Klein-Gordon oscillator with a zero vector and scalar potentials, $A_0 = 0 = S$. We have derived the radial wave equation and a second-order differential equation of the Bessel form is achieved, and then obtained the eigenvalues given by the Eq. (16) of these oscillators. We see that the energy spectrum are influenced by the topological defects of the manifold that is related with the curvature of the space-time geometry. In sub-section 2.2, we have considered a Cornell-type potential form function, $f(r) = (ar + \frac{b}{r})$ with a zero vector and scalar potentials. The radial wave equation is derived, and achieved the Whittaker differential equation [72] form which is a second-order after a suitable transformation. The solution of this Whittaker equation can be expressed in terms of the confluent hypergeometric function and finally applying the boundary condition, we have obtained the eigenvalues given by the Eq. (23) and the radial wave function given by the Eq. (24) of these oscillators. We see that the topological defects of the space-time geometry as well as the new function $f(r) \neq r$ modified the spectrum of energy and the wave function. In sub-section 2.3, we have considered Coulomb-type function $f(r) = \frac{b}{r}$ subject to a vector and scalar potentials of Coulomb-types (28) for the studies of the generalized Klein-Gordon oscillator in a point-like global monopole space-time in the presence of an Aharonov-Bohm flux. We have derived the radial wave equation, and then achieved the confluent hypergeometric differential equation form. By imposing the boundary condition for the bound-states solutions to be achieved, the energy eigenvalues given by the Eq. (35) of these oscillators is obtained. One can see that the energy eigenvalues are influenced by the topological defects as well as Coulomb-types scalar and vector potentials considered in the quantum system. In sub-section 2.4, we have considered Cornell-type potential form function, $f(r) = (ar + \frac{b}{r})$ subject to a vector and scalar potentials of Coulomb-types (28) for the studies of the generalized Klein-Gordon oscillator in a point-like global monopole space-time in the presence of an Aharonov-Bohm flux. We have derived the radial wave equation and then arrived the confluent hypergeometric differential equation form. By imposing the boundary condition to achieved the bound-states solutions of the confluence hypergeometric equation, we have obtained the energy eigenvalues given by the Eq. (45) of these oscillators. We see here also that the energy spectrum are influenced by the presence of...
the topological defects of the space-time geometry, Coulomb-types scalar and vector potentials and the modified function \( f(r) \neq r \) of these oscillators.

An interesting feature of the presented results is that the energy profiles of these oscillators in addition to the topological defect parameter \( \alpha^2 \) that is associated with the curvature of the space-time as well as other parameters are influenced by the Aharonov-Bohm magnetic flux \( \Phi_B \) produced by the topological defects. This is because the angular momentum quantum number \( l \) is shifted, that is, \( l \rightarrow l_{\text{eff}} = (l - \frac{\Phi_B}{2\pi}) \), an effective angular momentum quantum number. Thus, the relativistic energy eigenvalues of these oscillators is a periodic function of the geometric quantum phase \( \Phi_B \), and hence, we have that \( E_{n,l}(\Phi_B \pm \Phi_0 \tau) = E_{n,l\mp \tau}(\Phi_B) \), where \( \tau = 0, 1, 2, .. \). This dependence of the energy eigenvalues on the geometric quantum phase gives us the gravitational analog of the Aharonov-Bohm effect [62, 63] (see also, Refs. [49, 50, 51, 52]. In condensed matter systems [85, 86, 87], this dependence of the eigenvalues on the quantum phase gives rise to a persistent currents which we will discuss in future work.

**Conflict of Interest**

There is no conflict of interests regarding publication of this paper.

**Data Availability Statement**

All data generated or analysed during this study are included in this published article [and its supplementary information files].

**Contribution**

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