ON TRANSFINITE NILPOTENCE OF THE VOGEL-LEVINE LOCALIZATION

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Abstract. We construct a finitely-presented group such that its Vogel-Levine localization is not transfinitely nilpotent. This answers a problem of J. P. Levine.

1. Introduction

In the series of papers [8], [9], [10], [11], J. P. Levine developed the theory of algebraic closure of groups and described possible applications in geometric topology, as well as formulated natural problems related to localizations and completions of groups.

A group homomorphism \( f : G \to H \) is called 2-connected if it induces isomorphism on \( H_1(-, \mathbb{Z}) \) and a surjection on \( H_2(-, \mathbb{Z}) \). Denote by \( \Omega \) the collection of all 2-connected homomorphisms \( f : G \to H \) such that \( G \) and \( H \) are finitely presented groups. This concept plays a fundamental role in geometric topology, since a homology equivalence of connected spaces induces a 2-connected homomorphism of their fundamental groups (see [5], [6] for geometric applications of the theory of 2-connected homomorphisms).

A group \( \Gamma \) is local if given any diagram of homomorphisms as follows, with \( G \to H \) from \( \Omega \),

\[
\begin{array}{ccc}
G & \longrightarrow & H \\
\downarrow & & \downarrow \\
\Gamma & \longleftarrow & \Gamma
\end{array}
\]

there is a unique homomorphism \( H \to \Gamma \) making the diagram commute. The Vogel-Levine localization (also called an algebraic closure) of a group \( G \) is a group \( L(G) \) endowed with a homomorphism \( l : G \to L(G) \), such that \( L(G) \) is local and for any local group \( \Gamma \) and a homomorphism \( f : G \to \Gamma \), there is a unique homomorphism \( p : L(G) \to \Gamma \) such that \( p \circ l = f \). The Vogel-Levine localization is an algebraic analog of the localization of CW-complexes considered by J.-Y. Le Dimet [7].

The Vogel-Levine localization is a functor from all groups to the local groups. The existence, uniqueness and different properties of this functor are given in [10], [8], [9]. Recall some of the properties.

(i) Any homomorphism \( f : G \to H \) from \( \Omega \) induces an isomorphism of localizations \( L(G) \cong L(H) \).

(ii) For any \( G \), the localization \( l : G \to L(G) \) is 2-connected.

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For a finitely generated group $G$, the functor $L(G)$ lives in the corner of the following square which plays a fundamental role in the theory of localizations and completions of groups

\[
\begin{array}{ccc}
L(G) & \longrightarrow & E(G) \\
\downarrow & & \downarrow \\
\overline{G} & \longrightarrow & \hat{G}
\end{array}
\]

Here $E(G)$ is the functor of HZ-localization defined by A. K. Bousfield (see [3]), $\hat{G} := \varprojlim_i G/\gamma_i(G)$ is the functor of pro-nilpotent completion, $\overline{G}$ is the $\omega$-closure (or residually nilpotent algebraic closure, see [8], [10], [11]).

Given a finitely presented group $G$, it is a difficult problem how to describe the algebraic closure $L(G)$ and its group-theoretical properties. One tool to recognize $L(G)$ is given in [6]. Suppose one can construct a sequence of 2-connected homomorphisms $G \to K_1 \to K_2 \to \ldots$ such that $K_i$, $i = 1, 2, \ldots$ are finitely presented and $\varinjlim K_i$ is a local group. Then $L(G) = \varinjlim K_i$. This follows immediately from the definition and uniqueness of the algebraic closure. The seed of the idea to describe the group $L(G)$ as injective limit of 2-connected maps is given in [7]. In [7], Le Dimet shows that the Vogel localization of a finite CW-complex is a colimit of a countable sequence of finite CW-complexes. The above tool to recognize $L(G)$ is an algebraic analog of the construction from [7].

For a group $G$, the lower central series are defined inductively as follows:

\[\gamma_1(G) = G, \quad \gamma_{\alpha+1}(G) = [\gamma_\alpha(G), G]\]

and $\gamma_\tau(G) = \cap_{\alpha < \tau} \gamma_\alpha(G)$ for a limit ordinal $\tau$. A group $G$ is called transfinately nilpotent if $\gamma_\alpha(G) = 1$ for some ordinal $\alpha$. The vertical arrows in (1) are quotients of $L(G)$ and $E(G)$ by the intersections of (finite) lower central series $\gamma_\omega := \cap_1 \gamma_i$.

For any group $G$, its HZ-localization $E(G)$ is transfinately nilpotent [3]. In order to compare group-theoretical properties of algebraic closures and HZ-localizations, J. P. Levine asked the following ([10] (Problem 6 (b)): If $G$ is finitely-generated, is $L(G)$ transfinately nilpotent? The following example answers this problem.

**Theorem 1.** Let $H = \langle a, b \mid a^{k^2} = aa^{3^k}, [a, a^b] = 1 \rangle$. Then $L(H)$ is not transfinately nilpotent.

As usual, if $x, y, a_1, \ldots, a_{k+1}$ are elements of a group $G$ we set $[x, y] = x^{-1}y^{-1}xy, x^y = y^{-1}xy$ and define

\[ [a_1, a_2, \ldots, a_{k+1}] = [[a_1, \ldots, a_k], a_{k+1}] \quad (k > 1). \]

\[1\] In [10], J.P. Levine considered localization with respect to 2-connected maps which are normally surjective. Observe that for the group $H$ in theorem 1, $L(G)$ is the localization in that sense as well, since all maps considered in the construction are normally surjective.
2. Proof of theorem

The construction is based on the 1-relator group from \( [12] \)

\[ G = \langle a, b \mid a^{b^2} = aa^3b \rangle \]

with long lower central series. Our group \( H \) is the quotient \( G/\gamma_\omega(G) \). The lower central series length of \( G \) is \( \omega^2 \). The \( 2\omega \)-lower central quotient

\[ G/\gamma_{2\omega}(G) = \langle a, b \mid a^{b^2} = aa^3b, [a, a^b, a] = [a, a^b, a^b] = 1 \rangle \]

lives in the short exact sequence

\[ 1 \to \langle [a, a^b] \rangle \to G/\gamma_{2\omega}(G) \to H \to 1 \]

where \( \langle [a, a^b] \rangle \) is the infinite cyclic group, \( a \in H \) acting on \( [a, a^b] \) trivially and \( b \) by inverting. The lower central quotients \( \gamma_{\omega+k}(G)/\gamma_{\omega+k+1}(G) \) are cyclic groups of order 2 for all \( k = 0, 1, \ldots \) with generators \( [a, a^b]^{2^k} \cdot \gamma_{\omega+k+1}(G) \).

The proof consists of the following three steps:

1. Description of the \( \omega \)-closure \( H \) and the proof that \( H^2(H) = 0 \). Since \( \overline{H} = L(H)/\gamma_\omega(L(H)) \), the part of the 5-term sequence

\[ H_2(L(H)) \to H_2(\overline{H}) \to \gamma_\omega(L(H))/\gamma_{\omega+1}(L(H)) \to 1 \]

implies that \( \gamma_\omega(L(H)) = \gamma_{\omega+1}(L(H)) \). This implies that \( L(H) \) is residually nilpotent if and only if \( L(H) = \overline{H} \).

2. A construction of the sequence of finitely presented groups \( \Gamma_k' \), \( k = 0, 1, 2, \ldots \) and 2-connected homomorphisms

\[ H = \Gamma_0' \to \Gamma_1' \to \cdots \to \Gamma_k' \to \Gamma_{k+1}' \to \cdots \]

Denote the 2-connected maps from this sequence \( h_k : H \to \Gamma_k', k = 1, 2, \ldots \).

3. Proof that the limit \( \Gamma := \lim \Gamma_k' \) is a local group. This implies that \( L(H) = \Gamma \) and that the algebraic closure \( H \to L(H) \) is the limit map

\[ \lim h_k : H \to \lim \Gamma_k'. \]

It will be shown that there is a natural exact sequence

\[ 1 \to C_{2^\infty} \to L(H) \to \overline{H} \to 1 \]

That is, \( \gamma_\omega(L(H)) \) is non-trivial and isomorphic to the 2-quasi-cyclic group \( C_{2^\infty} := \lim \{ \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/8 \to \cdots \} \).

Step 1. Let \( N = \mathbb{Z} \oplus \mathbb{Z} \) be the \( \mathbb{Z}(b) \)-module generated by \( a \) and \( a^b \). The generator \( b \) of the cyclic quotient of \( H \) acts on \( N \) as the matrix

\[ U := \begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix} \]

Recall that (see prop. 3.2. from [11], and telescope theorem from [1], [2]) the \( \omega \)-closure \( \overline{H} \) of \( H \) has the following natural description

\[ \overline{H} = NS \times (b), \]
where $N_S$ is the $S$-localization of $N$ with $S := 1 + \Delta, \Delta = \ker\{\mathbb{Z}[b] \to \mathbb{Z}\}$. By construction, $N_S$ is the direct limit

$$N_S = \lim_{\to}\{N \xrightarrow{s_1} N \xrightarrow{s_2} \ldots\}$$

where $\{s_1, s_2, \ldots\}$ covers all elements from $S$.

For an element $s \in S$, the $s$-map $N \xrightarrow{s} N$ is injective. This follows from the residual nilpotence of $H$ (see [12] for the proof that $H$ is residually nilpotent). Indeed, if $s(n) = 0$, for some $n \in N, n \geq 0$, then $n \in \gamma(H) = 1$.

The multiplication by $s \in S$, induces a map of exterior squares

$$(\mathbb{Z} \simeq \langle a \wedge a^b \rangle \simeq \Lambda^2(N) \to \Lambda^2(N).$$

Any homomorphism $\mathbb{Z} \to \mathbb{Z}$ is a multiplication with some number. In our case denote this number by $|s|$. Let $|s| = 2^{p(s)}v$ for an odd $v$. Observe that $|s| \neq 0$, since, as observed above, any $s$-map $N \xrightarrow{s} N$ is injective.

Since $H_2(H) = \mathbb{Z}/2$, the map $s$ induces a zero map $H_2(H) \to H_2(H)$ if and only if $p(s) \neq 0$. The simplest example of an element from $S$, with zero induced map $H_2(H) \xrightarrow{s} H_2(H)$ is

$$s = 1 - b + b^2.$$ 

Consider an element from $S$ of the form

$$s = n_0 + n_1b + \cdots + n_lb^l, \ n_0 + \cdots + n_l = 1.$$ 

For $a \in N$, we will use the natural notation

$$a^s := a^{n_0a^{n_1b}}a^{n_2b^l}.$$ 

A technical exercise is to show that

$$|s| \equiv 1 + \sum_{j > i, \ 3|j-i}} n_in_j \mod 2$$

Since there are infinitely many indecomposed elements in $S$, which induce the zero map on $H_2(H)$, we conclude that

$$H_2(H) = \lim_{\to}H_2(H) = 0.$$ 

Using the general relation between Vogel-Levine localization and $\omega$-closure, $\overline{\mathcal{T}} = L(H)/\gamma_\omega(L(H))$, the 5-term sequence implies that

$$\gamma_\omega(L(H)) = \gamma_{\omega+1}(L(H)).$$

**Step 2.** Denote the following groups

$$\Gamma_k := \langle a, b \mid a^{b^2} = aa^{b^2}, [a, a^b, a] = [a, a^b, a^b] = 1, [a, a^b, b, \ldots, b] = 1 \rangle, \ k \geq 1$$
Consider an element $s \in S$. For a given $k \geq 0$, we will construct a homomorphism $\phi_s : \Gamma_k \to \Gamma_{k+p(s)}$ such that there is a commutative diagram

$$
\begin{array}{ccc}
\gamma_\omega(\Gamma_k) \cong \mathbb{Z}/2^k & \xrightarrow{\gamma_\omega(\Gamma_{k+p(s)})} \cong \mathbb{Z}/2^{k+p(s)} \\
\downarrow \quad \phi_s \downarrow & & \downarrow \downarrow \\
\Gamma_k & \xrightarrow{s^*} & \Gamma_{k+p(s)} \\
\downarrow \quad \downarrow & & \\
H & \xrightarrow{\phi} & H \\
\end{array}
$$

and $\phi_s$ induces an isomorphism $H_2(\Gamma_k) \to H_2(\Gamma_{k+p(s)})$.

Since the map $N \to N$ induces a well-defined homomorphism $(N \rtimes \langle b \rangle =) H \to H$, there exists $l \geq 0$, such that

$$
(a^s)^{b^2} = a^s(a^{3s}b[a, a^b])^l
$$
in $\Gamma_k$. We define the homomorphism $\phi_s : \Gamma_k \to \Gamma_{k+p(s)}$ as

$$
a \mapsto a^s[a, a^b]^r \\
b \mapsto b
$$
with $r$ such that $3r \equiv l \mod 2^k$. Let's check that $\phi_s$ is well-defined. Indeed, the relation (3) implies that, in $\Gamma_{k+p(s)}$,

$$(a^s[a, a^b])^{b^2} = (a^s[a, a^b]^r)(a^s[a, a^b]^r)^{3b}$$

The group $\Gamma_k$ has another relation which we have to check. We have

$$[[a, a^b]^s, b, \ldots, b] = ([a, a^b]^s)^{2^k} = [a, a^b]^{s|2^k} = 1$$

in $\Gamma_{k+p(s)}$. The relations $[a, a^b, a] = [a, a^b, a^b] = 1$ are preserved by the considered map. Thus, the homomorphism $\phi_s$ is well-defined. Observe that the homomorphism $\phi_s$ is normally surjective, i.e. the normal closure of the image of $\phi_s$ equals to $\Gamma_{k+p(s)}$.

The homology group $H_2(\Gamma_k)$ is isomorphic to $\mathbb{Z}/2$. Looking at presentation of the second homology $H_2(\Gamma_k)$ via the Hopf formula $H_2(\Gamma_k) = \frac{R[F]}{[R, F]}$, with $F = F(a, b)$, we describe the generator of $H_2(\Gamma_k)$ as the coset

$$[a, a^b, \ldots, b, [R, F] = [a, a^b]^{2^k}[R, F].$$

The image of the generator of $H_2(\Gamma_k)$ under the map induced by $\phi_s$ is non-trivial in $H_2(\Gamma_{k+p(s)})$, hence the induced map

$$H_2(\Gamma_k)(\cong \mathbb{Z}/2) \to H_2(\Gamma_{k+p(s)})(\cong \mathbb{Z}/2)$$
is an isomorphism.

Let's illustrate the above construction for the particular case $s = 1 - b + b^2$. In this case, $|s| = 12$ and the induced map $H_2(H) \to H_2(H)$ is zero. We claim that
there is a commutative diagram

\[
\begin{array}{ccc}
\Gamma_2 & \rightarrow & H \\
\downarrow \phi & & \downarrow \phi \\
H & \rightarrow & H
\end{array}
\]

such that \(\phi\) induces isomorphism \(H_2(H) \rightarrow H_2(\Gamma_2)\) and the vertical map is \(\Gamma_2 \rightarrow \Gamma_2/\gamma_\omega(\Gamma_2) = H\). The map \(\phi\) is defined as

\[
a \mapsto aa^{-b}a^b \quad a \mapsto b
\]

One can easily see that, in \(\Gamma_2\),

\[
(aa^{-b}a^b[a, ab])b^2 = aa^{-b}a^b[(aa^{-b}a^b[a, ab])^3b].
\]

hence the map \(\phi\) is well-defined.

To finalize the Step 2, we conclude that, for a sequence of elements \(\{s_1, s_2, \ldots\}\), there exists an infinite tower

\[
\begin{array}{cccc}
\mathbb{Z}/2^{\lceil p(s_1) \rceil} & \rightarrow & \mathbb{Z}/2^{\lceil p(s_1) + p(s_2) \rceil} & \rightarrow & \cdots \\
\Gamma_0 & \phi_{s_1} & \Gamma_{\lceil p(s_1) \rceil} & \phi_{s_2} & \Gamma_{\lceil p(s_1) + p(s_2) \rceil} & \rightarrow & \cdots \\
H & \overset{s_1}{\rightarrow} & H & \overset{s_2}{\rightarrow} & H & \cdots \end{array}
\]

All the homomorphisms \(\phi_{s_i}\) are 2-connected. The group \(\Gamma := \lim_{\rightarrow \{s_1, s_2, \ldots\}} \Gamma_{s_i}\), lies in the short exact sequence

\[
1 \rightarrow C_{2^\infty} \rightarrow \Gamma \rightarrow \overline{H} \rightarrow 1
\]

and the action of \(\overline{H} = NS \rtimes \langle b \rangle\) on the quasi-cyclic group \(C_{2^\infty}\) is given as follows:

\[
y \circ b = -y, \quad y \circ n = y, \quad n \in NS.
\]

**Step 3.** In order to show that \(\Gamma\) is local, recall the definition of local Cohn modules. For a group \(Q\), let \(M\) be a \(\mathbb{Z}[Q]\)-module. We call \(M\) a *local Cohn module* if, for every map \(t : F_1 \rightarrow F_2\) of finitely generated free \(\mathbb{Z}[Q]\)-modules of the same rank, such that the induced map \(1 \otimes \mathbb{Z}[Q] Z : F_1 \otimes \mathbb{Z}[Q] Z \rightarrow F_2 \otimes \mathbb{Z}[Q] Z\) is an isomorphism, and a morphism of \(\mathbb{Z}[Q]\)-modules \(\alpha : F_1 \rightarrow M\), there is a unique morphism \(\beta : F_2 \rightarrow M\) such that \(\beta \circ t = \alpha\).

Recall the following result (see [5], [6]). Let \(Q\) be a local group and \(M\) a Cohn local \(\mathbb{Z}[Q]\)-module. For any extension

\[
0 \rightarrow M \rightarrow \tilde{Q} \rightarrow Q \rightarrow 1,
\]

the group \(\tilde{Q}\) is local.
Observe that $C_{2^\infty}$ is the direct limit of $\mathbb{Z}[\overline{H}]$-modules

$$\lim \{\mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/8 \to \ldots \}$$

Every submodule $\mathbb{Z}/2^k$, $k \geq 1$ is nilpotent $\mathbb{Z}[\overline{H}]$-module, $(\mathbb{Z}/2^k)\Delta^k(\overline{H}) = 0$. Every nilpotent module is Cohn local and since $C_{2^\infty}$ is a direct limit of nilpotent modules, we conclude that $C_{2^\infty}$ is Cohn local module. Hence $\Gamma$ is a local group and the Step 3 is complete. This completes the proof of theorem.

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