Nuclear matrix elements for exclusive neutrino-nucleus reactions

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Abstract

The formalism describing (anti)neutrino-nucleus reaction cross-sections in neutral- and charged-current processes is improved. Compact formulae for the single-particle transition matrix elements, based on the multipole expansion treatment of the relevant hadronic currents and the use of harmonic oscillator basis, are presented. As an application, the nucleus $^{127}$I, a promising target for detection of solar- and supernova-neutrinos as well as of cold dark matter candidates, is considered. Our matrix elements refer to exclusive processes of the charge-changing $^{127}$I($\nu_l, l^-$)$^{127}$Xe* reaction and especially to the transition $\frac{5}{2}^+ \rightarrow \frac{3}{2}^+$ leading from the ground state of $^{127}$I to the lowest excitation of $^{127}$Xe nucleus.

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1 Introduction

It is well known that, the neutrinos play a very significant role in many phenomena in nature \[\text{[1]-[11]}\]. In spite of their importance, however, numerous questions concerning their properties \[\text{[2]}\], their roles in the solar-neutrino puzzle \[\text{[4]}\], the atmospheric neutrino anomaly and the dark matter problem \[\text{[12]}\], their oscillation-characteristics \[\text{[13, 14]}\], etc., remain still unanswered. During the last decades, several ideas have been proposed to explain the surprising behavior of neutrinos, but only a few of them have been verified. Among others, this is due to the fact that the neutrinos interact with matter only by means of the weak interactions. The goal of the extensive experimental and theoretical investigations related to neutrinos in nuclear physics, particle physics, astrophysics and cosmology, is to shed light on the above open issues.

Among the probes which involve neutrinos, the neutrino-nucleus interaction possess a prominent position \[\text{[5]-[11]}\]. Thus, the study of neutrino scattering with nuclei is a good way to detect or distinguish neutrinos of different flavor and explore the basic structure of the weak interactions. Also, specific neutrino-induced transitions between discrete nuclear states with good quantum numbers of spin, isospin and parity allows us to study the structure of the weak hadronic currents. Furthermore, terrestrial experiments performed to detect astrophysical neutrinos (solar neutrinos, etc.) as well as neutrino-induced nucleosynthesis interpreted through several neutrino-nucleus interaction theories, constitute good sources of explanation for neutrino properties \[\text{[5]-[11]}\].

There are four categories of neutrino–nucleus processes: the two types of charged-current (CC) reactions of neutrinos and antineutrinos and the two types of neutral-current (NC) ones. In the charged-current reactions a neutrino $\nu_l$ (antineutrino $\bar{\nu}_l$) with $l = e, \mu, \tau$, transforms one neutron (proton) of a nucleus to a proton (neutron), and a charged lepton $l^-$ (anti-lepton $l^+$) is emitted as

$$\nu_l + Z^A N^{-} \rightarrow Z^{A+1} + l^-, \quad \bar{\nu}_l + Z^A N^{-} \rightarrow Z^{-1} + l^+.$$  

(1)

These reactions are also called neutrino (anti-neutrino) capture, since they can be considered as the reverse processes of lepton-capture. They are mediated by exchange of heavy $W^\pm$ bosons according to the (lowest order) Feynman diagram shown in Fig. 1(a). In neutral-current reactions (neutrino scattering) the neutrinos (anti-neutrinos) interact via the exchange of neutral $Z^0$ bosons [see Fig. 1(b)] with a nucleus as

$$\nu + Z^A N \rightarrow Z^A N^* + \nu', \quad \bar{\nu} + Z^A N \rightarrow Z^A N^* + \bar{\nu}',$$  

(2)

where $\nu$ ($\bar{\nu}$) denote neutrinos (anti-neutrinos) of any flavor. The neutrino-nucleus reactions leave the final nucleus mostly in an excited state lying below particle-emission thresholds (semi-inclusive processes) \[\text{[3, 4, 6]}\]. The transitions to energy-levels higher than the particle-bound states usually decay by particle emission and, thus, they supply light particles that can cause further nuclear reactions \[\text{[3]}\].

Theoretically, the various neutrino-nucleus transition rates are calculated by following detailed prescriptions such those described in Refs. \[\text{[5]-[11]}\] (for a review on these methods see \[\text{[1, 2]}\]). From the nuclear structure viewpoint the main task is the evaluation of the matrix elements of multipole operators between the initial and final nuclear many-body states \[\text{[13]-[18]}\]. In Ref. \[\text{[19]}\], by exploiting the multipole decomposition of the hadronic currents, we have constructed a general closed formalism which gives all types of the radial matrix elements required for the description of any electromagnetic and weak process in nuclei. This method uses harmonic oscillator single-particle basis and allows the easy calculation of the basic radial matrix elements by means of analytic formulas.
In the present work, we proceed a step further and extend the method of Ref. [19] so as to provide similar expressions for the reduced matrix elements of any basic single-particle tensor operator entering the neutrino-nucleus cross-sections [5]-[11]. As we shall see, the advantage of this method lies in the separation of the geometrical coefficients from the kinematical parameters of the reaction in question. In this way, the transition matrix elements for every value of the momentum transfer $q$ can be evaluated in a direct way. We apply this formalism in order to determine the needed matrix elements for nuclei in the region of $^{127}$I isotope, which is considered a promising target for detection of solar and supernova neutrinos [4] as well as for detection of cold dark matter candidates [12].

In the remainder of the paper, after a brief review of the relevant expressions involved in the formal description of the neutrino-nucleus cross sections and the multipole decomposition treatment of the hadronic currents (Sect. 2), we proceed with the derivation of closed analytic formulas for the basic single-particle reduced matrix elements (Sect. 3). As an application we calculate the transition matrix elements in the case of the charged-current neutrino-nucleus reaction $^{127}\text{I}(\nu_l, l^-)^{127}\text{Xe}^*$ (Sect. 4). Finally, our conclusions are summarized in Sect. 5.

### 2 Weak interaction Hamiltonian of neutrino–nucleus reactions

The effective Hamiltonian, describing the weak interaction of nuclei with neutrinos at low energies compared to the electroweak scale, can be written in current–current form as

$$
\mathcal{H}_{\text{eff}} = \frac{G}{\sqrt{2}} \left( j^{(-)}_\lambda j^{(+)}_\lambda + j^{(0)}_\lambda j^{(0)}_\lambda + \text{h. c.} \right),
$$

(3)

($G$ denotes the weak interaction coupling constant) where $j^{(c)}_\lambda$ and $j^{(a)}_\lambda$ with $c = -, +, 0$, denote the leptonic and hadronic currents, respectively. For the definition of the leptonic current see e.g. Ref. [2]. The expressions for the hadronic currents, since nucleons are extended objects and Lorentz covariance should be fulfilled, reads

$$
J^{(c)}_\lambda = \bar{\Psi}_N \left[ g^V_1 \gamma_\lambda + \frac{i}{2M} g^V_2 \sigma_{\lambda\nu} q^\nu + g^V_3 q_\lambda \right. \\
+ \left. g^A_1 \gamma_\gamma_5 + \frac{i}{2M} g^A_2 \sigma_{\lambda\nu} q^\nu \gamma_5 + g^A_3 q_\lambda \gamma_5 \right] \tau_c \Psi_N,
$$

(4)

($\Psi_N$ represent the nucleon isospin doublet) where the weak nucleon form factors $g^V_i$, $g^A_i$ ($i = 1, 2, 3$) are complex scalar functions of the momentum transfer $q^2$. The $g^V_i$ are fixed by the conserved vector-current theory (CVC) stating that the isovector part of the electromagnetic current and the charge raising and lowering parts of the weak vector current form an isospin triplet of conserved currents [1]. For the axial form factors $g^A_i$, charge symmetry properties and T-invariance of the hadronic current require that $g^A_2 = 0$. Furthermore, the $g^A_3 q_\lambda \gamma_5$ term gives contributions proportional to the mass of the outgoing lepton, and can therefore be neglected in the extreme relativistic limit [1]. Thus, one arrives at the expressions

$$
J^{(+)}_\lambda = \bar{\Psi}_N \left\{ \frac{1}{2}(F^p_1 - F^n_1) \gamma_\lambda + \frac{i}{4M}(F^p_2 - F^n_2) \sigma_{\lambda\nu} q^\nu + G_A \gamma_\lambda \gamma_5 \right\} \tau_c \Psi_N,
$$

(5)

$$
J^{(0)}_\lambda = \bar{\Psi}_N \left\{ F^Z_1 \gamma_\lambda + \frac{F^Z_2 i \sigma_{\lambda\nu} q^\nu}{2M} + G_A \gamma_\lambda \gamma_5 \right\} \psi_N,
$$

(6)
\( J^{(-)}_\lambda \) is the Hermitian conjugate of \( J^{(+)}_\lambda \). Here \( G_A = -\frac{1}{2} G_A^3 \tau_0 \) (\( \tau_0 = +1 \) for protons and \( \tau_0 = -1 \) for neutrons) and \( F^p_{1,2}, F^n_{1,2} \) denote the charge and electromagnetic form factors of proton and neutron (within the nucleus), respectively. These form factors are discussed in Refs. [2, 3].

### 2.1 The formalism for neutrino–nucleus cross sections

By utilizing the neutrino–nucleus weak interaction Hamiltonian discussed before, the neutrino scattering cross sections can reliably be calculated within the first-order Born approximation. If we assume that the initial \( |i\rangle \) and final \( |f\rangle \) nuclear states have well-defined spins and parities, a multipole analysis of the weak (nuclear-level) hadronic current can be performed \([19]\) (see below Sect. 2.2). This has been carried out in close analogy to electron scattering from nuclei \([20, 21]\) within a unified analysis of charge-changing semi-leptonic weak interactions in nuclei.

The differential (with respect to energy and scattering direction) neutrino–nucleus scattering cross-section is written as \([2]\)

\[
\frac{d^2\sigma_{i\rightarrow f}}{d\Omega\,d\omega} = \frac{G^2 \left| k_f \right| \epsilon_f}{\pi (2J_i + 1)} F(Z, \epsilon_f) \left( \sum_{J=0}^{\infty} \sigma^{J}_{\text{C}} \sum_{J=1}^{\infty} \sigma^{J}_{\text{T}} \right),
\]

where \( \omega = \epsilon_i - \epsilon_f \) is the excitation energy of the nucleus, and \( \epsilon_i, \epsilon_f, k_f \) denote the energy of the incoming neutrino and energy and momentum of the outgoing lepton, respectively. The summations in Eq. \([2]\) contain the contributions of the Coulomb (\( \hat{M}_J \)), longitudinal (\( \hat{L}_J \)), transverse electric (\( \hat{T}^e_J \)) and transverse magnetic (\( \hat{T}^m_J \)) operators stemming from the multipole expansion of the weak hadronic current. The operators \( \hat{M}_J, \hat{L}_J, \hat{T}^e_J \) and \( \hat{T}^m_J \) contain both polar-vector and axial-vector parts of which the contributions are written as \([2]\)

\[
\sigma^{J}_{\text{C}} = (1 + a \cos \Phi) \left| \langle J_f | \hat{M}_J(q) | J_i \rangle \right|^2 + (1 + a \cos \Phi - 2b \sin^2 \Phi) \left| \langle J_f | \hat{L}_J(q) | J_i \rangle \right|^2 \\
+ \left[ \frac{\omega}{q} (1 + a \cos \Phi) + d \right] 2\text{Re} \langle J_f | \hat{L}_J(q) | J_i \rangle \langle J_f | \hat{M}_J(q) | J_i \rangle^*,
\]

\[
\sigma^{J}_{\text{T}} = (1 - a \cos \Phi + b \sin^2 \Phi) \left[ \left| \langle J_f | \hat{T}_J^{m,a}(q) | J_i \rangle \right|^2 + \left| \langle J_f | \hat{T}_J^{e}(q) | J_i \rangle \right|^2 \right] \\
\pm \left[ \frac{(\epsilon_i + \epsilon_f)}{q} (1 - a \cos \Phi) - d \right] 2\text{Re} \langle J_f | \hat{T}_J^{m,a}(q) | J_i \rangle \langle J_f | \hat{T}_J^{e}(q) | J_i \rangle^*,
\]

where \( \Phi \) denotes the scattering angle of the outgoing lepton and \( a, b \) and \( d \) are given by

\[
a = \frac{\left| k_f \right|}{\epsilon_f} = 1 - \left( \frac{m_f c^2}{\epsilon_f} \right)^2, \quad b = \frac{\epsilon_i \epsilon_f a^2}{q^2}, \quad d = \frac{(m_f c^2)^2}{q \epsilon_f},
\]

\((m_f \) is the mass of the outgoing lepton). The magnitude of the three-momentum transfer \( q \) is given by

\[
q = |q| = \left[ \omega^2 + 2\epsilon_i \epsilon_f (1 - a \cos \Phi) - (m_f c^2)^2 \right]^{\frac{1}{2}}.
\]

Notice that the interference term between vector and axial vector current in Eq. \([2]\) has a negative (positive) sign for neutrino (antineutrino) scattering due to their different helicities.

The well-known Fermi function \( F(Z, \epsilon_f) \) takes into account the Coulomb–final-state interaction between nucleus and final lepton (case of charged-current reactions only).
2.2 Multipole–decomposition operators

The standard multipole expansion procedure [1], applied on the matrix elements of hadronic current \( \hat{J}_\mu(r) = (\rho, \mathbf{J}) \), where \( \rho(r) \) the density and \( \mathbf{J}(r) \) the three-current operators, leads to spherical tensor operators which are given in terms of the projection functions

\[
M_M^J(qr) = \delta_{LJ} j_L(qr) Y_M^L(\hat{r}), \quad (12)
\]

\[
M_M^{(L)J}(qr) = j_L(qr) Y_M^{(L)J}(\hat{r}). \quad (13)
\]

Here \( j_L(r) \) stands for the spherical Bessel functions, \( Y_M^L(\hat{r}) \) are the spherical Harmonics and \( Y_M^{(L)J}(\hat{r}) \) are the vector spherical Harmonics. In the case of the polar-vector current \( \hat{J}_\lambda \), the multipole decomposition gives the operators \( \hat{M}_M^{Cout}, \hat{L}_{JM}, \hat{T}_{JM}^{el} \) and \( \hat{T}_{JM}^{mag} \). The first three operators have parity \((-)^J\) (normal parity operators), while the parity of \( \hat{T}_{JM}^{mag} \) is \((-)^{J+1}\) (abnormal parity operator). In the case of axial-vector current \( \hat{J}_5^a \) we obtain the operators \( \hat{M}_M^{Cout}, \hat{L}_5^{JM}, \hat{T}_{JM}^{el} \) and \( \hat{T}_{JM}^{mag} \). The first three axial-vector multipoles have parity \((-)^{J+1}\) while \( \hat{T}_{JM}^{mag} \) is a normal parity operator.

For a conserved vector current (CVC) like the electromagnetic, \( \hat{L}_{JM,J}(q) = (q_0/q)\hat{M}_M^{Cout}(q) \), where \( q_0 \) represent the time component of the four-momentum transfer, \( q_\mu = (q_0, \mathbf{q}) \). In this case, the number of independent operators resulting from the decomposition procedure is reduced to seven. In fact, the matrix elements of these seven basic operators involve isospin dependent form factors \( F_X^{(2)}(q^2) \) (see Eqs. (11), (12) and Ref. [13]). For this reason, we define seven new operators as [1]

\[
T_1^{JM} \equiv M_M^J(qr) = \delta_{LJ} j_L(\rho) Y_M^L(\hat{r}), \quad (14)
\]

\[
T_2^{JM} \equiv \Sigma_M^J(qr) = M_M^J \cdot \sigma, \quad (15)
\]

\[
T_3^{JM} \equiv \Sigma_M^J(qr) = -i \left\{ \frac{1}{q} \nabla \times M_M^J(qr) \right\} \cdot \sigma, \quad (16)
\]

\[
T_4^{JM} \equiv \Sigma_M^{aJ}(qr) = \left\{ \frac{1}{q} \nabla M_M^J(qr) \right\} \cdot \sigma, \quad (17)
\]

\[
T_5^{JM} \equiv \Delta_M^J(qr) = M_M^J(qr) \cdot \frac{1}{q} \nabla, \quad (18)
\]

\[
T_6^{JM} \equiv \Delta_M^{aJ}(qr) = -i \left\{ \frac{1}{q} \nabla \times M_M^J(qr) \right\} \cdot \nabla, \quad (19)
\]

\[
T_7^{JM} \equiv \Omega_M^J(qr) = M_M^J(qr) \sigma \cdot \frac{1}{q} \nabla. \quad (20)
\]

Using properties of the nabla operator \( (\nabla) \), Eqs. (16), (17) and (19) can be rewritten as

\[
T_3^{JM} \equiv \Sigma_M^J = [J]^{-1} \left\{ -J^{1/2} M_M^{J+1J} + (J + 1)^{1/2} M_M^{-1J} \right\} \cdot \sigma, \quad (21)
\]
Throughout this work we use the common symbol $[J] = (2J + 1)^{1/2}$.

Many quantities describing the semi-leptonic electroweak processes in nuclei, including, of course, the neutrino-nucleus reactions, are expressed (to a good approximation) in terms of single-particle nuclear matrix elements of the one-body operators $T_i^J$, $i = 1, 2, \ldots, 7$. By applying the Wigner-Eckart theorem, these matrix elements are written in terms of the following four reduced matrix elements: $\langle j_1||M^J||j_2 \rangle$, $\langle j_1||M^{L,J} \cdot \sigma||j_2 \rangle$, $\langle j_1||M^{L,J} \cdot (\nabla/q)||j_2 \rangle$ and $\langle j_1||M^J \sigma \cdot (\nabla/q)||j_2 \rangle$. In the next section, we will use harmonic oscillator basis in order to find simplified expressions for these reduced matrix elements.

## 3 The Single-particle reduced matrix elements

By applying the re-coupling relations and using the formalism of Ref. [13], the reduced matrix elements of the operators $T_i^J$, $i = 1, 2, \ldots, 7$, can be written in the compact forms shown below.

1. For the operators $M_i^J \equiv O_i^J$ and $M_i^{L,J} \cdot \sigma \equiv O_i^{JM}$ the reduced matrix elements $\langle j_1||O_i^J||j_2 \rangle$, have previously been written as [12, 14]

$$\langle j_1||O_i^{(L,S,J)}||j_2 \rangle = (l_1 L \ l_2) \ U_{L,S}^J \langle n_1 l_1 | j_L(\rho) | n_2 l_2 \rangle, \ \ i = 1, 2, \ldots, 7$$

(with $S_1 = 0$ and $S_2 = 1$) where the symbols $l_1 L \ l_2$ and $U_{L,S}^J$ contain the 3-j and 9-j symbols, respectively (see Ref. [13]).

2. The reduced matrix elements of $\langle j_1||M^{L,J}(\nabla/q)||j_2 \rangle$, after some manipulation can be cast in the form

$$\langle j_1||M^{L,J}(\nabla/q)||j_2 \rangle = \frac{1}{q} \sum_{\alpha} A_L^\pm(j_1 j_2; J) \langle n_1 l_1 | \theta_L^\alpha(\rho) | n_2 l_2 \rangle, \ \ \alpha = \pm,$$

where

$$A_L^\pm(j_1 j_2; J) = \pm(-)^{l_1+L+j_2+1/2} \left( \frac{2l_2 + 1 + 1}{2} \right)^{1/2} \langle j_1[j_2][J](l_1 L \ l_2 \mp 1) \times \mathcal{W}_6(l_1, j_1, 1/2, j_2, l_2) \mathcal{W}_6(L, 1, J, l_2, 1, l_2 \mp 1).$$

Here $\mathcal{W}_6$ represent the common 6-j symbol.

3. Similarly, for the reduced matrix element of $\langle j_1||M^{J}(\nabla/q)||j_2 \rangle$ we can write

$$\langle j_1||M^{J}(\nabla/q)||j_2 \rangle = \frac{1}{q} \sum_{\alpha} B_L^\delta(j_1 j_2; J) \langle n_1 l_1 | \theta_L^\alpha(\rho) | n_2 l_2 \rangle, \ \ \alpha = \pm,$$

where

$$B_L^\delta(j_1 j_2; J) = \pm \delta_{j_2,j_2+1/2} [j_1][j_2](l_1 J 2j_2 - l_2) \mathcal{W}_6(l_1, j_1, 1/2, j_2, 2j_2 - l_2, J).$$

From Eqs. (24), (25) and (27) we notice that all basic single-particle reduced matrix elements required for our purposes rely on the following three types of radial integrals:

$$\langle n_1 l_1 | \theta_L^\alpha(\rho) | n_2 l_2 \rangle \equiv \int dr r^2 R_{n_1 l_1}(r) \theta_L^\alpha(\rho) R_{n_2 l_2}(r), \ \ \alpha = 0, \pm,$$
with
\[
\theta_0^i(\rho) = j_i(\rho),
\]
\[
\theta_1^i(\rho) = j_i(\rho) \left( \frac{d}{d\rho} \pm \frac{2l_2 + 1 \pm 1}{2\rho} \right).
\]

The argument \( \rho \) in Eqs. (11) is equal to \( \rho = qr \).

By inserting in Eqs. (24), (25) and (27), the expressions for the radial matrix elements found in Ref. [19] (see Appendix A) and manipulating properly the appearing summations, we can write the four types of reduced matrix elements entering the basic operators (14)-(20) in closed forms, as follows:

1. For the Fermi- (S=0) and Gamow–Teller-type (S=1) operators \( O_i^{(L,S,i)} \), we have
\[
\langle j_1 || O_i^{(L,S,i)} || j_2 \rangle = e^{-y y / 2} n_{max} \sum_{\mu=0}^{n_{max}} E_\mu(L) y^\mu,
\]
with \( S_1 = 0 \) and \( S_2 = 1 \), and where
\[
E_\mu(L) = (l_1 L l_2) U_{LS_i} \varepsilon_\mu(n_1 l_1 n_2 l_2) \quad i = 1, 2.
\]

2. The reduced matrix element \( \langle j_1 || M^{LJ}(qr) \cdot (\nabla/q) || j_2 \rangle \) take the form
\[
\langle j_1 || M^{LJ}(qr) \cdot \frac{1}{q} \nabla || j_2 \rangle = e^{-y y / (L-1)/2} n_{max} \sum_{\mu=0}^{n_{max}} E_3^\mu(L) y^\mu,
\]
with
\[
E_3^\mu(L) = A_L^- (L_\mu^-) + A_L^+ (L_\mu^+).
\]

3. Similarly for \( \langle j_1 || M^L(qr) \sigma \cdot \frac{1}{q} \nabla || j_2 \rangle \) we derive the formula
\[
\langle j_1 || M^L(qr) \sigma \cdot \frac{1}{q} \nabla || j_2 \rangle = e^{-y y / (L-1)/2} n_{max} \sum_{\mu=0}^{n_{max}} E_4^\mu(L) y^\mu,
\]
where
\[
E_4^\mu(L) = B_L^- (L_\mu^-) + B_L^+ (L_\mu^+).
\]

The coefficients \( \varepsilon_\mu(L) \) and \( \zeta_\mu(L) \) are defined in Ref. [19] (see also Appendix A).

Having available the formalism of Eqs. (32), (35) and (37), we can straightforwardly deduce closed analytic formulas for the reduced matrix elements of the seven basic operators Eqs. (14)-(20). All these fundamental matrix elements can be cast in the compact form
\[
\langle j_1 || T^J || j_2 \rangle = e^{-y y / 2} n_{max} \sum_{\mu=0}^{n_{max}} P_\mu^J y^\mu,
\]
where
\[
n_{max} = (N_1 + N_2 - \beta)/2.
\]

The coefficients \( P_\mu^J \) for each specific case are listed in Table 1. The integer \( \beta \) of Eqs. (39) and (40) is also quoted in this Table. As can be seen from Table 1, the coefficients \( P_\mu^J \) are simply related to the numbers \( E_\mu(L) \), \( i = 1, 2, 3, 4 \), given in Eqs. (34), (36) and (38). The coefficients \( P_0^J \) are much simpler compared to those corresponding to \( \mu > 0 \).
We should remark that, upon mutual interchanging \(n_1(l_1/2)j_1\) with \(n_2(l_2/2)j_2\) in Eq. (39), the following relation holds:

\[
\langle j_2||T^J(qr)||j_1 \rangle = (-)^\lambda \langle j_1||T^J(qr)||j_2 \rangle ,
\]

where \(\lambda = j_1 - j_2\) for the operators \(M, \Delta, \Sigma', \Sigma''\), and \(\lambda = j_1 + j_2\) for the operators \(\Delta'\) and \(\Sigma\) (our phase convention for the matrix elements in Eq. (39) is the same with that of Ref. [14]). The operator \(\Omega\) does not have a simple phase symmetry under the above interchange, \(j_1 \leftrightarrow j_2\), and, for this reason, one traditionally defines the operator \(\Omega^J\)

\[
\Omega^J_M = \Omega_M + \frac{1}{2} \Sigma''_M ,
\]

for which, on applying Eq. (11), one must put \(\lambda = j_1 + j_2\). The matrix elements of the operator \(\Omega^J\) are also given by Eq. (39) but now the coefficients \(P^J_\mu\) are (see Table 1)

\[
P^J_\mu = E^J_\mu(J) + \frac{1}{2} \left(J^{1/2}E^2_\mu(J-1) + (J+1)^{1/2}E^2_{\mu-1}(J+1)\right) .
\]

Before presenting some applications on the above formalism, it should be noted that, the explicit and general expressions of Eq. (39) hold for every combination of the single-particle levels \((n_1l_1j_1, n_2l_2j_2)\). For the determination of the involved polynomials it suffices the evaluation of the geometrical momentum independent coefficients \(P^J_\mu\) (see Table 1). These closed formulas have been derived by inverting properly the multiple summations involved in the corresponding matrix elements (see Ref. [19, 22]), so as the final summation is performed over the major harmonic oscillator quantum number \(N = 2n + l\). It is worth remarking that such a compact formalism provides a very useful insight to those authors who wish to adopt a phenomenological approach and fit the nuclear transition strengths for many semi-leptonic processes in nuclei [19, 22].

4 Discussion, analysis and results

Our primary aim in the present paper, is to deal with the improvement of the formalism for the neutrino-nucleus reaction cross sections. We shall focus our discussion on those reactions that pertain to the terrestrial observation of astrophysical neutrinos and especially on exclusive transitions leading to a definite final state in neutrino-induced processes which have attracted considerable interest in the past few years [2, 4, 8]. A special category of such processes is the one occurring in the radiochemical experiments leading to excited states below particle emission thresholds, i.e. to the particle-bound states (semi-inclusive processes). As an example of this category, the reaction \(\nu_l + ^{127}I \rightarrow ^{127}Xe^* + l^-\) is discussed in some detail below.

As it is known, solar or supernova neutrinos, incident on an iodine liquid target, could produce the noble gas \(^{127}Xe\) which can be recovered and counted as in the \(^{37}Cl\) (radiochemical) experiment. This suggests that, the reaction

\[
\nu_l + ^{127}I \rightarrow ^{127}Xe^* + l^- \quad (44)
\]

may be a promising radiochemical-type detector for \(^7Be\) or \(^8B\) neutrinos [3, 4]. In the above reaction, the ground state to ground state (g.s. \(\rightarrow\) g.s.) transition \(5^+ \rightarrow 1^-\), is forbidden, so that, the first allowed one is the \(5^+ \rightarrow 3^+\) leading to the lowest excitation (at 124.6 KeV) of \(^{127}Xe\) isotope.

On the theoretical nuclear physics side, the many-body nuclear wave-functions for heavy odd-A nuclei with relatively high half-integer (ground state) spins, cannot be easily constructed within
the context of shell model or random phase approximation methods. The medium heavy nuclei $^{127}$I and $^{127}$Xe which enter the reaction (14) are isotopes of this sort. For this reason, the wavefunctions involved in the cross sections of process (14) are currently calculated within the framework of the microscopic quasiparticle-phonon model. This model, has recently been used for reliable descriptions of other similar processes [15, 17].

In the quasi-particle phonon (QPM) model [15], the wave functions of odd-mass spherical nuclei are expressed in the form of expansions in a phonon-basis. Thus, the main idea of the model is to construct this phonon-basis, whereby phonons are defined as solutions of the quasi-particle random phase approximation [13]. The goal of the model is to be able to describe the interplay between collective and single-particle degrees of freedom of low-lying excited states. The effective Hamiltonian in the QPM is written as

$$H_{\text{eff}} = H_{\text{sp}} + H_{\text{pair}} + H_{m} + H_{sm},$$  \hspace{1cm} (45)

where $H_{\text{sp}}$ is the single-particle Hamiltonian, $H_{\text{pair}}$ represents the monopole pairing interaction in the particle-hole ($p-h$) channel, $H_{m}$ is the separable multipole interaction in the $p-h$ channel, and $H_{sm}$ is the separable spin-multipole interaction in the $p-h$ channel.

To calculate the spectrum of the needed excited states for the nuclei $^{127}$I and $^{127}$Xe, the effective Hamiltonian Eq. (45) must be transformed in such a way that, instead of the nucleon degrees of freedom, other simple modes of excitation would be present. These simple modes are Bogolubov’s quasiparticles and phonons (that is why the model has been called quasiparticle-phonon model). The required procedure for these transformations is described in detail in Ref. [15, 18].

In the case of the neutral-current neutrino-$^{127}$I scattering, one needs either the $g.s \to g.s.$ transition (coherent process) of $^{127}$I, i.e. $\frac{5}{2}^{+} \to \frac{5}{2}^{+}$, or the transitions going from the ground state of $^{127}$I to its low-lying (particle-bound) states. In general, under the assumptions discussed in the Introduction, any partial cross section is written in terms of the square of the matrix elements of tensor multipole operators (see Sect. 2) of the form

$$\langle f||T^{j}(qr)||i \rangle = \sum_{j_{1},j_{2}} \langle j_{1}\mid T^{j}(qr)\mid j_{2}\rangle D(j_{1},j_{2};J).$$  \hspace{1cm} (46)

where $j_{1},j_{2}$ run over the configurations of the chosen model space which couple to a given $J$. The functions $D(j_{1},j_{2};J)$ are the one-body transition densities which in our case are provided by the quasiparticle-phonon model. The chosen model space in [18] consists of the oscillator major shells with $N = 3 - 5$ plus the intruder $0i_{13/2}$ from the $N = 6$ major shell.

The single-particle reduced matrix elements $\langle j_{1}\mid T^{j}(qr)\mid j_{2}\rangle$ are evaluated by the expression (39). In Tables 2-4 we quote some coefficients $P_{\mu}^{{j}}$ needed to determine the reduced matrix elements for the normal-parity operators $M^{j}, \Sigma^{j}, \Delta^{j}$ (see Sec. 2.2) by using Table 1 and Eq. (38). Due to space limitations, in these Tables we list the coefficients for configuration only of the type $\frac{5}{2}^{+} \to \frac{2}{2}^{+}$ (for others the reader is referred to Refs. [13, 22]). As can be seen from Tables 2-4, the coefficients of the polynomials involved in Eq. (16) are very simple and readily calculable numbers a fact that illustrates the advantage of the formalism presented in Sect. 2-3.

At this point it is worth remarking that, regarding the neutrino energies $E_{\nu}$ one is interested in, two main treatments can be followed: (i) Include in Eq. (16) the momentum dependence of the multipole operators Eqs. (14)-(20), and (ii) Reduce the multipole operators to their $q \to 0$ limit (long wave-length approximation) [1]. It should be mentioned that, according to their energies the neutrinos are classified in low-energy, $E_{\nu} \leq 20 MeV$ (solar neutrinos, low-energy supernova neutrinos), medium-energy, $20MeV \leq E_{\nu} \leq 50MeV$ (high energy supernova neutrinos) and high-energy, $50MeV \leq E_{\nu} \leq 1 - 2 GeV$ (solar flare-, atmospheric-neutrinos). In our formalism developed in the present work, we keep the exact momentum dependence of the operators, since
the momentum transfer and the neutrino-energies involved in the reaction (45) may, sometimes, be rather high.

Before closing, it is worth mentioning that, the $^{127}\text{I}$ nucleus constitutes a promising detector also for cold dark matter candidates [12]. This probe relies on the (elastic) scattering of a cold-dark-matter candidate off the nucleus $^{127}\text{I}$ and requires the calculation of the low-lying excited states of $^{127}\text{I}$. Roughly speaking, in the latter case the required reduced matrix elements are, in general, similar to those of the reaction $^{127}\text{I}(\nu, \nu')^{127}\text{I}^*$, i.e. the neutral-current neutrino-scattering. A thorough discussion of this topic (including calculations for other similar processes) using the above formalism will be done elsewhere [22].

5 Summary and Conclusions

In the present work, we have focused on the theoretical description of exclusive transitions in neutrino-nucleus reactions. Assuming that, the initial and final nuclear states involved have well-defined spin, isospin and parity, and using the decomposition of the hadronic currents into tensor multipole operators, we derived closed analytic formulas for the matrix elements of the principal one-body operators entering the relevant transition strengths. By employing harmonic oscillator basis, these matrix elements have been written as products of an exponential times simple polynomials of even powers in the momentum transfer $q$. The coefficients of the polynomials are, in general, simple and readily calculable numbers as it becomes evident from Tables 2-4.

The advantage of the above formalism has been illustrated, in the case of the lowest transition $\frac{5}{2}^+ \rightarrow \frac{3}{2}^+$ of the currently interesting charged-current reaction $^{127}\text{I}(\nu_l, l^-)^{127}\text{Xe}^*$. The partial rates of this process are currently evaluated in the framework of the microscopic quasi-particle phonon model employing for the single-particle reduced matrix elements our present expressions. This formalism, in the cases when a phenomenological approach is adopted, offers the possibility one to make fits for the nuclear transition strengths in many semi-leptonic processes in nuclei.

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Appendix A

The radial integrals of Eq. (29) in the case when harmonic oscillator basis is used can take the following compact expressions (see Ref. [19]):

(i) For the operator $\theta_0^L(\rho)$ we found [12, 14]

$$\langle n_1 l_1 | j_L(\rho) | n_2 l_2 \rangle = e^{-y} y^{L/2} \sum_{\mu=0}^{n_{\text{max}}} \varepsilon^{L}_\mu y^\mu, \quad y = (gb/2)^2, \quad (47)$$

$$n_{\text{max}} = (N_1 + N_2 - L)/2,$$

where $N_i = 2n_i + l_i$ and the coefficients $\varepsilon^{L}_\mu(n_1 l_1 n_2 l_2)$ are given by

$$\varepsilon^{L}_\mu(n_1 l_1 n_2 l_2) = C \frac{\pi^{1/2}}{2} \sum_{m_1=\phi}^{n_1} \sum_{m_2=\sigma}^{n_2} n! \Lambda_{m_1}(n_1 l_1) \Lambda_{m_2}(n_2 l_2) \Lambda_{\mu}(nL), \quad (48)$$
with
\[ n = m_1 + m_2 + (l_1 + l_2 - L)/2. \]
In Eq. (48) \( C = b^3 N_{n_1 l_1} N_{n_2 l_2}/2 \) while the other symbols are explained in Ref. [12].

(ii) The radial matrix elements for the operators \( \theta^\pm_L \) are given by [19]
\[
\langle n_1 l_1 | \theta^\pm_L | n_2 l_2 \rangle = e^{-y y (L-1)/2} \sum_{\mu=0}^{n_{\text{max}}} \zeta_L^\pm \mu (L) y^\mu, \tag{49}
\]
where
\[
\zeta_L^\mu (L) = -\frac{1}{2} \left\{ \begin{array}{ll}
(n_2 + l_2 + 3/2)^{1/2} \epsilon^L_\mu (n_1 l_1 n_2 l_2 + 1) + n_2^{1/2} \epsilon^L_{\mu,n_{\text{max}}} (n_1 l_1 n_2 l_2 - 1), & 0 \leq \mu < n_{\text{max}} \\
(n_2 + l_2 + 3/2)^{1/2} \epsilon^L_{\mu,n_{\text{max}}} (n_1 l_1 n_2 l_2 + 1), & \mu = n_{\text{max}}
\end{array} \right.
\]
\[
\zeta_L^+ (L) = \frac{1}{2} \left\{ \begin{array}{ll}
(n_2 + l_2 + 1/2)^{1/2} \epsilon^L_\mu (n_1 l_1 n_2 l_2 - 1) + (n_2 + 1)^{1/2} \epsilon^L_{\mu,n_{\text{max}}} (n_1 l_1 n_2 l_2 - 1), & 0 \leq \mu < n_{\text{max}} \\
(n_2 + 1)^{1/2} \epsilon^L_{\mu,n_{\text{max}}} (n_1 l_1 n_2 l_2 - 1), & \mu = n_{\text{max}}
\end{array} \right. \tag{50}
\]
and
\[
n_{\text{max}} = (N_1 + N_2 - L + 1)/2. \tag{51}
\]
The coefficients \( \epsilon^L_\mu \) and \( \zeta_L^\pm \) are calculated using simple codes.

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Table 1: The coefficients $P^J_\mu$, $0 \leq \mu \leq n_{\text{max}}$ for the seven basic single-body operators entering the description of any semi-leptonic process in nuclei [see Eqs. (14)–(20) in the text]. The values for the parameter $\beta$ of Eqs. (39) and (40) are also given.

Table:

| Operator   | $\beta$ | $P^J_\mu$, $0 \leq \mu \leq n_{\text{max}}$ |
|------------|---------|-----------------------------------------------|
| $T^J_1 = M^J$ | J       | $E^1_\mu(J)$ |
| $T^J_2 = \Sigma^J$ | J       | $E^2_\mu(J)$ |
| $T^J_3 = \Sigma'^J$ | J - 1   | $(J + 1)^{1/2}E^2_\mu(J - 1) - J^{1/2}E^2_{\mu-1}(J + 1)$ |
| $T^J_4 = \Sigma''^J$ | J - 1   | $J^{1/2}E^2_\mu(J - 1) + (J + 1)^{1/2}E^2_{\mu-1}(J + 1)$ |
| $T^J_5 = \Delta^J$ | J - 1   | $E^3_\mu(L)$ |
| $T^J_6 = \Delta'^J$ | J - 2   | $(J + 1)^{1/2}E^3_\mu(J - 1) - J^{1/2}E^3_{\mu-1}(J + 1)$ |
| $T^J_7 = \Omega^J$ | J       | $E^4_\mu(J)$ |

$\Omega^J = \mu - 1$, $E^4_\mu(J) + \frac{1}{2} \left\{ J^{1/2}E^2_\mu(J - 1) + (J + 1)^{1/2}E^2_{\mu-1}(J + 1) \right\}$

Figure Caption

Feynman-diagram of lowest order for: (a) the CC neutrino-nucleus reactions $\nu_l + ZA_N \rightarrow Z+1A^*_N - l^-$, and (b) the NC neutrino-nucleus processes $\nu + ZA_N \rightarrow ZA^*_N + \nu'$. The diagrams which correspond to the anti-neutrino reactions are similar.
Table 2: Coefficients which determine the matrix elements $\langle j_1 || M^J || j_2 \rangle$ describing the $\frac{5}{2}^+ \rightarrow \frac{3}{2}^+$ transition of the charge-changing reaction $^{127}\text{I} (\nu_\ell, l^-) ^{127}\text{Xe}$ in a model space including the major oscillator shells N=3-5 (see the text).

| $(n_1l_1)j_1 - (n_2l_2)j_2$ | $J$ | $\mu = 0$ | $\mu = 1$ | $\mu = 2$ | $\mu = 3$ | $\mu = 4$ |
|-----------------------------|-----|------------|------------|------------|------------|------------|
| $0f_{5/2} - 1p_{3/2}$      | 2   | $\frac{8}{5} \sqrt{\frac{2}{3}}$ | $\frac{104}{35} \sqrt{\frac{1}{6}}$ | $\frac{8}{35} \sqrt{\frac{2}{3}}$ |          |            |
|                            | 4   | $-\frac{48}{35}$ | $\frac{16}{35}$ |            |            |            |
| $1d_{5/2} - 1d_{3/2}$      | 2   | $-\frac{44}{5} \sqrt{\frac{1}{21}}$ | $\frac{376}{35} \sqrt{\frac{1}{21}}$ | $-\frac{152}{35} \sqrt{\frac{1}{21}}$ | $\frac{16}{35} \sqrt{\frac{1}{21}}$ |          |
|                            | 4   | $\frac{152}{35} \sqrt{\frac{2}{7}}$ | $-\frac{96}{35} \sqrt{\frac{2}{7}}$ | $\frac{204}{665} \sqrt{\frac{2}{7}}$ |            |            |
| $1f_{5/2} - 1p_{3/2}$      | 2   | $-\frac{4}{5} \sqrt{3}$ | $\frac{40}{35} \sqrt{3}$ | $-\frac{152}{105} \sqrt{\frac{1}{3}}$ | $\frac{16}{105} \sqrt{\frac{1}{3}}$ |          |
|                            | 4   | $\frac{136}{105} \sqrt{2}$ | $-\frac{32}{35} \sqrt{2}$ | $\frac{16}{105} \sqrt{2}$ |            |            |
| $1f_{5/2} - 2p_{3/2}$      | 2   | $\frac{16}{5} \sqrt{\frac{3}{7}}$ | $\frac{40}{7} \sqrt{\frac{3}{7}}$ | $\frac{208}{21} \sqrt{\frac{1}{21}}$ | $-\frac{232}{105} \sqrt{\frac{1}{21}}$ | $\frac{16}{105} \sqrt{\frac{1}{21}}$ |
|                            | 4   | $-\frac{208}{35} \sqrt{\frac{2}{7}}$ | $\frac{592}{105} \sqrt{\frac{2}{7}}$ | $-\frac{176}{105} \sqrt{\frac{2}{7}}$ | $\frac{16}{105} \sqrt{\frac{2}{7}}$ |            |
| \((n_1l_1)j_1 - (n_2l_2)j_2\) | \(J\) | \(\mu = 0\) | \(\mu = 1\) | \(\mu = 2\) | \(\mu = 3\) | \(\mu = 4\) |
|-----------------------------|---|---------|---------|---------|---------|---------|
| \(0f_{5/2} - 1p_{3/2}\)    | 2 | \(-\frac{8}{3}\) | \(\frac{52}{71}\) | \(-\frac{8}{21}\) |
|                            | 4 | \(\frac{24}{7}\sqrt{\frac{2}{7}}\) | \(-\frac{8}{7}\sqrt{\frac{1}{3}}\) |
| \(1d_{5/2} - 1d_{3/2}\)   | 2 | \(\frac{22}{7}\sqrt{\frac{2}{7}}\) | \(\frac{188}{21}\sqrt{\frac{2}{7}}\) | \(-\frac{76}{21}\sqrt{\frac{2}{7}}\) | \(\frac{8}{21}\sqrt{\frac{2}{7}}\) |
|                            | 4 | \(\frac{76}{7}\sqrt{\frac{2}{35}}\) | \(-\frac{48}{7}\sqrt{\frac{2}{35}}\) | \(\frac{8}{7}\sqrt{\frac{2}{35}}\) |
| \(1f_{5/2} - 1p_{3/2}\)   | 2 | \(2\sqrt{2}\) | \(-\frac{20}{7}\sqrt{2}\) | \(\frac{76}{63}\sqrt{2}\) | \(-\frac{8}{63}\sqrt{2}\) |
|                            | 4 | \(-\frac{68}{21}\sqrt{\frac{2}{7}}\) | \(\frac{48}{21}\sqrt{\frac{2}{7}}\) | \(-\frac{8}{21}\sqrt{\frac{2}{7}}\) |
| \(1f_{5/2} - 2p_{3/2}\)   | 2 | \(-8\sqrt{\frac{2}{7}}\) | \(\frac{100}{7}\sqrt{\frac{2}{7}}\) | \(-\frac{520}{63}\sqrt{\frac{2}{7}}\) | \(\frac{116}{63}\sqrt{\frac{2}{7}}\) | \(-\frac{8}{63}\sqrt{\frac{2}{7}}\) |
|                            | 4 | \(\frac{104}{7}\sqrt{\frac{2}{35}}\) | \(-\frac{296}{21}\sqrt{\frac{2}{35}}\) | \(\frac{88}{21}\sqrt{\frac{2}{35}}\) | \(\frac{8}{21}\sqrt{\frac{2}{35}}\) |

Table 3: Coefficients which determine some reduced matrix elements \(\langle j_1|\Sigma^J||j_2\rangle\). See caption of Table 2 and the text.

| \((n_1l_1)j_1 - (n_2l_2)j_2\) | \(J\) | \(\mu = 0\) | \(\mu = 1\) | \(\mu = 2\) | \(\mu = 3\) | \(\mu = 4\) |
|-----------------------------|---|---------|---------|---------|---------|---------|
| \(0f_{5/2} - 1p_{3/2}\)    | 2 | 0       | \(\frac{2}{3}\) | \(-\frac{4}{21}\) |
|                            | 4 | 0       | \(-\frac{4}{7}\sqrt{\frac{1}{5}}\) |
| \(1f_{5/2} - 1p_{3/2}\)   | 2 | \(\frac{3}{5}\sqrt{2}\) | \(-\frac{32}{35}\sqrt{2}\) | \(\frac{122}{315}\sqrt{2}\) | \(-\frac{20}{315}\sqrt{2}\) |
|                            | 4 | \(-\frac{34}{21}\sqrt{\frac{2}{5}}\) | \(\frac{16}{21}\sqrt{\frac{2}{5}}\) | \(-\frac{4}{21}\sqrt{\frac{2}{5}}\) |
| \(1f_{5/2} - 2p_{3/2}\)   | 2 | 0       | \(2\sqrt{\frac{2}{7}}\) | \(-\frac{12}{7}\sqrt{\frac{2}{7}}\) | \(\frac{38}{63}\sqrt{\frac{2}{7}}\) | \(-\frac{4}{63}\sqrt{\frac{2}{7}}\) |
|                            | 4 | 0       | \(-\frac{52}{21}\sqrt{\frac{2}{35}}\) | \(\frac{8}{7}\sqrt{\frac{2}{35}}\) | \(-\frac{4}{21}\sqrt{\frac{2}{35}}\) |

Table 4: Coefficients which determine some reduced matrix elements \(\langle j_1|\Delta^{f^J}||j_2\rangle\). For details see the text.
