A GLOBAL QUADRATIC SPEED-UP FOR COMPUTING THE PRINCIPAL EIGENVALUE OF PERRON-LIKE OPERATORS

DONG LI AND JIANAN LI

Abstract. We consider a new algorithm in light of the min-max Collatz-Wielandt formalism to compute the principal eigenvalue and the eigenvector (eigen-function) for a class of positive Perron-Frobenius-like operators. Such operators are natural generalizations of the usual nonnegative primitive matrices. These have nontrivial applications in PDE problems such as computing the principal eigenvalue of Dirichlet Laplacian operators on general domains. We rigorously prove that for general initial data the corresponding numerical iterates converge globally to the unique principal eigenvalue with quadratic convergence. We show that the quadratic convergence is sharp with compatible upper and lower bounds. We demonstrate the effectiveness of the scheme via several illustrative numerical examples.

1. Introduction

A fundamental problem in the study of numerical linear algebra and operator theory is the computation of eigenvalue and associated eigenvectors/eigenfunctions. In the past century many powerful algorithms such as the power iteration, the Krylov subspace methods, the Lanczos procedure, the Jacobi method, the LR and QR algorithms (cf. [2] for a review) have been developed to compute efficiently the eigenvalues and eigenvectors for given matrices. The usual Perron-Frobenius theory is concerned with a nonnegative primitive matrix which possesses a simple largest eigenvalue equal to its spectral radius and all other eigenvalues have strictly smaller modulus. The associated eigenvector can be chosen to be completely positive, i.e. all its entries are positive. For generic initial vector (i.e. carrying some nontrivial component along the principal eigenvector direction), the standard power iteration method applied to a given nonnegative primitive matrix leads to the convergence to the Perron eigen-pair with linear convergence speed. To be more concrete, consider $A \in \mathbb{R}^{m \times m}$ with nonnegative entries and assume $A$ is primitive. Denote by $\lambda_\ast = \rho(A) > 0$ where $\rho(A)$ is the spectral radius of $A$. Let $A\phi = \lambda_\ast \phi$ with $\phi > 0$ (i.e. all entries of the vector $\phi \in \mathbb{R}^m$ are positive), and $A^T \psi = \lambda_\ast \psi$ with $\psi > 0$. The standard recipe for the power method is

1. Choose random $v^{(0)} \in \mathbb{R}^m$;
2. While $\|v^{(n+1)} - v^{(n)}\| > \epsilon_{\text{tol}}$, calculate

\begin{align*}
y^{(n+1)} &= Av^{(n)}, \quad v^{(n+1)} = \frac{y^{(n+1)}}{\|y^{(n+1)}\|}, \quad (1.1) \\
\lambda^{(n+1)} &= (v^{(n+1)})^T A v^{(n+1)}. \quad (1.2)
\end{align*}

Here $v^{(0)}$ is usually assumed to have some nontrivial projection along the principal eigenvector $\phi$. This assumption is almost innocuous in the measure sense. On the other hand,
for $A$ being nonnegative and primitive, it suffices to take $v^{(0)} > 0$ since $\phi$ is completely positive. A slight variant of the power method is to take $\lambda > \lambda^*$ and execute the following:

**Algorithm $\lambda$-power ($\lambda$-inverse-power):**

1. Choose $v^{(0)} > 0$;
2. While $\|v^{(n+1)} - v^{(n)}\| > \epsilon_{\text{tol}}$, calculate (below for simplicity we write $\lambda I - A$ as $\lambda - A$)
   \[
   (\lambda - A)y^{(n+1)} = v^{(n)}; \quad v^{(n)} = \frac{y^{(n+1)}}{\|y^{(n+1)}\|};
   \]
   \[
   \lambda^{(n+1)} = (v^{(n+1)})^T A v^{(n+1)}. \tag{1.3}
   \]

To ensure $\lambda > \lambda^*$ without the explicit value of $\lambda^*$, one can appeal to the Collatz-Wielandt formalism, namely for any positive $y \in \mathbb{R}^n$,

\[
\lambda^+_y := \min_i \frac{(Ay)_i}{y_i} \leq \lambda^* \leq \max_i \frac{(Ay)_i}{y_i} =: \lambda^-_y. \tag{1.5}
\]

More precisely, one has (below $y > 0$ means $y_i > 0$ for all $i$)

\[
\lambda^* = \inf_{y > 0} \max_i \frac{(Ay)_i}{y_i} = \sup_{y > 0} \min_i \frac{(Ay)_i}{y_i}. \tag{1.6}
\]

Observe that by definition of $\lambda^+_y$ and $\lambda^-_y$, we have

\[
(A - \lambda^-_y)y \geq 0, \quad (\lambda^+_y - A)y \geq 0. \tag{1.7}
\]

The characterization [1.5] follows easily from taking inner product with $\psi$ on both sides of [1.7]. Thanks to the above simple characterization one can kick-start the algorithm by taking $\lambda > \lambda^+_y(0)$. With a fixed choice of $\lambda$ it is not difficult to check that for Algorithm $\lambda$-power the speed of convergence toward the principal eigenvector is linear.

A natural next step is to consider somewhat more flexible choices of the parameter $\lambda$. In light of Collatz-Wielandt we consider the following iterative algorithm (cf. Chen [16]).

**Algorithm variable-$\lambda$-power:**

1. Choose $v^{(0)} > 0$ and let $\lambda^{(0)} = \max_i \frac{(Av^{(0)})_i}{v^{(0)}_i}$.
2. While $\|v^{(n+1)} - v^{(n)}\| > \epsilon_{\text{tol}}$, calculate
   \[
   (\lambda^{(n)} - A)y^{(n+1)} = v^{(n)}; \quad v^{(n+1)} = \frac{y^{(n+1)}}{\|y^{(n+1)}\|};
   \]
   \[
   \lambda^{(n+1)} = \max_i \frac{(Av^{(n+1)})_i}{v^{(n+1)}_i}. \tag{1.8}
   \]

The purpose of this work, roughly speaking, is to give a rigorous analysis of the new algorithm [1.8]–[1.9] in a general set-up. As it turns out, quite interestingly the numerical iterates of the above algorithm converge globally (i.e. not requiring the initial data to be close to the principal eigen-pair) and super-linearly to the principal eigenvector. In fact the speed of convergence is quadratic with nearly optimal (up to some constant factor) upper and lower bounds. This upgrade of convergence speed also takes place
in the computation of eigen-pair for some infinite-dimensional operators. We develop in detail the corresponding theoretical framework for a class of Perron-Frobenius-type operators which are compact and positive. A prototypical example is $T = (-\Delta_D)^{-1}$ where $-\Delta_D$ is the usual Dirichlet Laplacian on a smooth bounded domain in $\mathbb{R}^d$. We employ the corresponding variable-$\lambda$-power algorithm for $T$ and calculate the principal eigenvalue and eigenfunction of $-\Delta_D$. The convergence speed in this case is shown to be quadratic.

**Remark 1.1.** By now there are a plethora of algorithms in the literature related to power and inverse power methods. To put things into perspective, we mention several standard algorithms below the fold. One should note that a common feature shared by these methods is that their convergence is local, i.e. quadratic or even cubic convergence takes place only when the numerical iterates are close to some eigen-pair with sufficient precision. For simplicity assume $A \in \mathbb{R}^{n \times m}$ and has real eigenvalues $\lambda_1^* > \lambda_2^* \geq \cdots \geq \lambda_m^*$ with corresponding orthonormal eigenvectors $\phi_1, \cdots, \phi_m$.

- **Algorithm inverse power with variable shift:** Choose $0 \neq v^{(0)} \in \mathbb{R}^m$ and $\lambda^{(0)} \in \mathbb{R}$. Fix a given $z \in \mathbb{R}^m$. Recursively execute the following:
  
  If $\lambda^{(n)} - A$ is singular then compute $(\lambda^{(n)} - A)v^{(n)} = 0$ for some nonzero $v^{(n)}$ and halt;

  $$
  (1.10)
  $$

  Otherwise compute
  
  $$
  y^{(n+1)} = (\lambda^{(n)} - A)^{-1}v^{(n)}; \quad k^{(n+1)} = \frac{1}{z \cdot y^{(n+1)}},
  $$

  $$
  v^{(n+1)} = k^{(n+1)}y^{(n+1)}, \quad \lambda^{(n+1)} = \lambda^{(n)} - k^{(n+1)}.
  $$

  (1.11)

  (1.12)

  In the above, a tacit working assumption is that $z \cdot y^{(n+1)} \neq 0$ for all iterates $y^{(n+1)}$. Define a function $F = F(v, \lambda): \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^{m+1}$ as

  $$
  F(v, \lambda) = \left( \frac{\lambda - A}{z^Tv - 1} \right).
  $$

  (1.13)

  The standard Newton iteration implemented on $F$ yields

  $$
  \left( \frac{(\lambda^{(n)} - A)v^{(n)}}{z^Tv^{(n)} - 1} \right) + \left( \frac{\lambda^{(n)} - A}{z^Tv^{(n)}} \right) \left( \frac{v^{(n+1)} - v^{(n)}}{\lambda^{(n+1)} - \lambda^{(n)}} \right) = \left( \begin{array}{c} 0 \\ 0 \end{array} \right).
  $$

  (1.14)

  It is not difficult to verify that (1.14) is equivalent to (1.12). As such the algorithm (1.12) exhibits local quadratic convergence to an eigen-pair.

- **Algorithm inverse power with Rayleigh quotient iteration.**
  
  1. Choose $v^{(0)} \in \mathbb{R}^m$ with $\|v^{(0)}\|_2 = 1$. Define $\lambda^{(0)} = (v^{(0)})^TAv^{(0)}$.
  2. For $n \geq 0$, compute

  $$
  (\lambda^{(n)} - A)y^{(n+1)} = v^{(n)}; \quad v^{(n+1)} = \frac{y^{(n+1)}}{\|y^{(n+1)}\|};
  $$

  (1.15)

  $$
  \lambda^{(n+1)} = (v^{(n+1)})^TAv^{(n+1)}.
  $$

  (1.16)

  It is well-known that (cf. [3]) if $A$ is a Hermitian matrix, then for almost all initial $v^{(0)}$, the iterates converge to an eigen-pair. Furthermore the convergence is cubic. One should note that the key property used in the convergence proof is the minimal residual property, namely $\lambda^{(n)}$ is the minimizer of the residual
∥(λ − A)v^{(n+1)}∥_2 (for fixed v^{(n+1)}). In [11], Parlett extended the analysis to normal and nonnormal matrices.

It is worthwhile mentioning some other approaches for computing eigen-pairs of the aforementioned Perron-Frobenius-like operators. In [1] Lejay and Maire considered a Monte Carlo method for the numerical computation of the principal eigenvalue of the Dirichlet Laplacian in a bounded piecewise smooth domain. The main idea is based on the characterization

\[ \lambda_1 = \lim_{t \to +\infty} \frac{1}{t} \log \mathbb{P}(\tau^x_D > t), \]  

where \( \lambda_1 \) is the principal eigenvalue for \( \Delta_D \), and \( \tau^x_D \) is the exit time from \( D \) of the Brownian motion starting at \( x \). Various numerical schemes such as the Euler scheme, walk on sphere schemes and walk on rectangles schemes are considered in [1]. In [4], Li and Wang considered a family of Perron-like matrices and investigated the principal eigenvalues and the associated generalized eigen-spaces via polynomial approximations of matrix exponentials. For some closely-related developments on the computation of eigen-pairs for matrices, we refer to [5, 6, 7, 8, 9, 10, 12, 13, 14, 15] and the references therein for more extensive discussions.

The rest of this paper is organized as follows. In Section 2 we develop the general theory for the variable \( \lambda \)-power algorithm applied to a class of Perron-Frobenius-like operators. In Section 3 we carry out several numerical experiments to showcase the effectiveness of the new algorithm. The examples range from the usual primitive non-negative matrix to Dirichlet Laplacian operators on polygonal-type domains.

2. COMPUTATION OF THE PRINCIPAL EIGENVALUE: DIRICHLET LAPLACIAN CASE

We first introduce a general set-up. Let \( \Omega \) be a bounded domain in \( \mathbb{R}^d \) with smooth boundary. Denote \( \mathbb{H} = L^2(\Omega) \) which consists real-valued \( L^2 \)-integrable functions on \( \Omega \). We denote by \( \langle \cdot, \cdot \rangle \) the usual \( L^2 \) inner product. Let \( T = (-\Delta_D)^{-1} \) where \( \Delta_D \) is the Dirichlet Laplacian.

We note that \( T \) satisfies the following conditions:

1. \( T\phi = \lambda^* \phi \), where \( \lambda^* > 0 \) is the largest eigen-value of \( T \) and \( \phi \) is the corresponding eigen-vector. By the strong maximum principle, \( \phi(x) \) must take a constant sign in \( \Omega \) and with no loss we may assume \( \phi(x) > 0 \) for all \( x \in \Omega \). We may also assume \( \|\phi\|_2 = 1 \).

2. \( T \) is a completely positive operator, namely if \( v \geq 0 \) and \( v \) is not identically zero, then \( (Tv)(x) > 0 \) for all \( x \in \Omega \).

Remark 2.1. One can consider more generally transfer operators, namely Ruelle–Perron–Frobenius transfer operators. One can also generalize to the usual Krein-Rutman setup but we shall not pursue this more abstract situation here.

Remark 2.2. Note that \( 1/\lambda^* \) is the usual principal eigen-value of \( -\Delta_D \) which is a simple eigenvalue.

\footnote{The corresponding classification of convergence is more complicated. See for example Theorem on pp.685 of [11].}
Lemma 2.1 (Monotonicity of the one-step growth-factor). Let \( B = (\lambda - T)^{-1} \) where \( \lambda > \lambda^* \). Suppose \( v \geq 0 \), \( v \) is not identically zero and let \( w = Bv \). Then \( w(x) > 0 \) for all \( x \in \Omega \), and

\[
\lambda^* \leq \sup_{x \in \Omega} \frac{(Tw)(x)}{w(x)} \leq \sup_{x: v(x) > 0} \frac{(Tv)(x)}{v(x)}. \tag{2.1}
\]

Remark 2.3. Roughly speaking, this lemma asserts that averaging decreases the one-step growth-factor.

Proof. Note that the operator \( B \) commutes with \( T \). Set \( l = \sup_{x: v(x) > 0} \frac{(Tv)(x)}{v(x)} \) and \( l_w = \sup_{x \in \Omega} \frac{(Tw)(x)}{w(x)} \). With no loss we can assume \( l < \infty \). By definition we have \( (l - T)v \geq 0 \). On the other hand

\[
(l - T)w = (l - T)(\lambda - T)^{-1}v = (\lambda - T)^{-1}(l - T)v \geq 0. \tag{2.2}
\]

Thus \( l \geq l_w \). To see \( l_w \geq \lambda^* \) we argue as follows. Clearly by definition \( (l_w - T)w \geq 0 \). Taking the inner produce with \( \phi \), we obtain (below recall \( \langle \cdot, \cdot \rangle \) is the usual \( L^2 \) inner product)

\[
(l_w - \lambda^*)\langle w, \phi \rangle \geq 0. \tag{2.3}
\]

Since \( \langle w, \phi \rangle > 0 \), we obtain \( l_w \geq \lambda^* \).

We consider the following algorithm:

Take nontrivial smooth nontrivial \( v^0 \) such that \( v^0(x) > 0 \) for all \( x \in \Omega \). Assume that

\[
\lambda^{(0)} := \sup_{x \in \Omega} \frac{(Tv^0)(x)}{v^0(x)} < \infty. \tag{2.4}
\]

Iteratively define

\[
v^{n+1} = (\lambda^{(n)} - T)^{-1}(\lambda^{(n)} - \lambda)v^n; \quad \lambda^{(n+1)} = \sup_{x: v^{n+1}(x) > 0} \frac{(Tv^{n+1})(x)}{v^{n+1}(x)}. \tag{2.5}
\]

It should be noted that it is not difficult to check that \( v^{n+1}(x) > 0 \) for all \( n \geq 0 \), and \( x \in \Omega \). Thus in the above definition of \( \lambda^{(n+1)} \) the constraint \( \{x: v^{n+1}(x) > 0\} \) can be replaced by the innocuous condition \( \{x: x \in \Omega\} \). By Lemma 2.1 it is also clear that

\[
\lambda^* \leq \lambda^{(n+1)} \leq \lambda^{(n)} \leq \lambda^{(0)} < \infty \tag{2.6}
\]

for all \( n \geq 0 \). A somewhat delicate technical subtlety is that we need to enforce \( \lambda^{(n)} > \lambda^* \) for all \( n \) so that all the iterates remain well-defined. The next lemma clarifies this issue.

Lemma 2.2. The following hold.

(1) If \( \inf_{x \in \Omega} \frac{(Tv^0)(x)}{v^0(x)} = \sup_{x \in \Omega} \frac{(Tv^0)(x)}{v^0(x)} \), then \( v^0(x) = c_1 \phi(x) \), where \( c_1 > 0 \) is some constant.

(2) If \( \inf_{x \in \Omega} \frac{(Tv^0)(x)}{v^0(x)} < \sup_{x \in \Omega} \frac{(Tv^0)(x)}{v^0(x)} \), then \( \lambda^{(0)} > \lambda^* \).
(3) If \(\inf_{x \in \Omega} \frac{(Tv^0)(x)}{v^0(x)} < \sup_{x \in \Omega} \frac{(Tv^n)(x)}{v^n(x)}\), then for all \(n \geq 0\), we have \(\inf_{x \in \Omega} \frac{(Tv^n)(x)}{v^n(x)} < \sup_{x \in \Omega} \frac{(Tv^n)(x)}{v^n(x)}\) and the strict inequalities:
\[
\lambda^* < \lambda^{(n+1)} < \lambda^{(n)} < \lambda^{(0)}.
\] (2.7)

**Proof.** (1) Clearly in this case we have \(Tv^0 = \lambda^{(0)}v^0\). Taking the inner product with \(\phi\) yields that \(\lambda^{(0)} = \lambda^*\). Since \(\lambda^*\) is a simple eigen-value, we must have \(v^0 = c_1\phi\) for some \(c_1 > 0\).

(2) We employ an orthonormal eigen-basis \((\phi_j)_{j=1}^{\infty}\) for \(T\). Namely \(\phi_1 = \phi\), \(\lambda_1^* = \lambda^*\), and \(\|\phi_j\|_2 = 1\) satisfies \(T\phi_j = \lambda_j^*\phi_j\), \(\lambda_j^* \leq \lambda_{j-1}^*\) for all \(j \geq 2\). Furthermore there is the strict spectral gap \(\lambda_2^* < \lambda_1^* = \lambda^*\). Clearly by assumption we have
\[
v^0(x) = c_1\phi(x) + \sum_{j=2}^{\infty} c_j\phi_j(x),
\] (2.8)
where \(c_1 = \langle v^0, \phi \rangle > 0\), and \(\sum_{j=2}^{\infty} c_j^2 > 0\). We then write
\[
(Tv^0)(x) - \lambda^*v^0(x) = \sum_{j=2}^{\infty} c_j(\lambda_j^* - \lambda^*)\phi_j(x) =: \beta(x).
\] (2.9)
Clearly
\[
\|\beta\|_2^2 = \sum_{j=2}^{\infty} c_j^2(\lambda_j^* - \lambda^*)^2 > 0
\] (2.10)
so that \(\beta\) is not identically zero. On the other hand \(\langle \beta, \phi \rangle = 0\) which implies that \(\beta\) must change sign on \(\Omega\). In particular we must have
\[
\sup_{x \in \Omega} \frac{\beta(x)}{v^0(x)} > 0.
\] (2.11)
This implies that \(\lambda^{(0)} > \lambda^*\).

(3) Since \(v^0\) cannot be a multiple of \(\phi\), we obtain \(v^1\) cannot be a multiple of \(\phi\). By a similar analysis as in Step 2, we obtain \(\lambda^{(1)} > \lambda^*\). A simple induction argument yields \(\lambda^{(n)} > \lambda^*\) for all \(n\). Next we show the strict inequality \(\lambda^{(n)} > \lambda^{(n+1)}\). By definition we have
\[
\lambda^{(n)} - \lambda^{(n+1)} = \lambda^{(n)} - \sup_{x \in \Omega} \frac{Tv^{n+1}}{v^{n+1}}
\] (2.12)
\[
= (\lambda^{(n)} - \lambda^*) \inf_{x \in \Omega} \frac{v^n(x)}{v^n(x)} = \inf_{x \in \Omega} \frac{v^n(x)}{((\lambda^{(n)} - T)^{-1}v^n)(x)}.
\] (2.13)
It suffices for us to show \(\sup_{x \in \Omega} \frac{((\lambda^{(n)} - T)^{-1}v^n)(x)}{v^n(x)} < \infty\). By an argument similar to the proof of Lemma 2.1, we reduce the matter to showing
\[
\sup_{x \in \Omega} \frac{((\lambda^{(n)} - T)^{-1}v^0)(x)}{v^0(x)} < \infty.
\] (2.14)
Proposition 2.1

The computation of the pleasing feature of the new definition is that the inf-part no longer involves any further computation of $\lambda$. Not surprisingly we have the following.

\[
\text{Observe that } \quad ((\lambda^n - T)^{-1}v^0(x) - (\lambda^n)^{-1}v^0(x) = (T \int_0^1 (\lambda^n - \theta T)^{-2}v^0 d\theta)(x); \quad (2.15)
\]

\[
\sup_{x \in \Omega} \frac{(T\hat{u})(x)}{v^0(x)} \leq \sup_{x \in \Omega} \frac{(Tv^0)(x)}{v^0(x)} \cdot \sup_{x \in \Omega} \frac{(T\hat{u})(x)}{(Tv^0)(x)}. \quad (2.16)
\]

By strong maximum principle, it is not difficult to check that

\[
\inf_{x \in \Omega} \frac{(Tv^0)(x)}{\text{dist}(x, \partial \Omega)} \geq \alpha_1 > 0, \quad \sup_{x \in \Omega} \frac{(T\hat{u})(x)}{\text{dist}(x, \partial \Omega)} \leq \alpha_2, \quad (2.17)
\]

\[
\sup_{x \in \Omega} \frac{(T\hat{u})(x)}{(Tv^0)(x)} \leq \frac{\alpha_2}{\alpha_1} < \infty, \quad (2.18)
\]

where in the above $\alpha_1 > 0$, $\alpha_2 > 0$ are constants. Thus \(\inf_{x \in \Omega} \frac{v^n(x)}{((\lambda^n - T)^{-1}v^n)(x)} > 0. \quad \square\)

From the computational point of view, the definition of $\lambda^{(n+1)}$ is somewhat unwieldy since it involves a further computation of $Tv^{n+1}$ which could be costly. A close inspection of \((2.13)\) suggests that the following variant is slightly better:

\[
\mu^{(0)} = \lambda^{(0)}, \quad \tilde{v}^0 = v^0; \quad (2.19)
\]

\[
\tilde{v}^{n+1} = (\mu^{(n)} - T)^{-1}\tilde{v}^n; \quad (2.20)
\]

\[
\mu^{(n+1)} = \mu^{(n)} - \inf_{x \in \Omega} \frac{\tilde{v}^n(x)}{\tilde{v}^{n+1}(x)}. \quad (2.21)
\]

In the practical numerical computation, to ensure numerical stability we often normalize $\tilde{v}^n$ by its $L^2$ norm (more precisely, the discrete $l^2$ norm after full numerical discretization) at each iteration step. Clearly this would not change the definition of $\mu^{(n+1)}$. A very pleasing feature of the new definition is that the inf-part no longer involves any further computation of the $T$ operator. Not surprisingly we have the following.

**Proposition 2.1** (Equivalence of $\lambda^{(n)}$ and $\mu^{(n)}$). Assume $\mu^{(0)} = \lambda^{(0)}$, $\tilde{v}^0 = v^0$ and

\[
\inf_{x \in \Omega} \frac{(Tv^0)(x)}{v^0(x)} < \sup_{x \in \Omega} \frac{(Tv^0)(x)}{v^0(x)} < \infty.
\]

The following hold.

1. $\mu^{(n)} = \lambda^{(n)}$ for all $n \geq 1$. Furthermore $\tilde{v}^n = c_n v^n$ for all $n \geq 1$, where $c_n > 0$ is a constant depending on $n$.

2. For all $n \geq 0$, we have

\[
0 < \inf_{x \in \Omega} \frac{\tilde{v}^n(x)}{\tilde{v}^{n+1}(x)} < \mu^{(n)} - \lambda. \quad (2.22)
\]

**Proof.** The first statement follows from an easy induction argument. The second statement follows from the properties of $\lambda_n$ established earlier. \(\square\)

**Theorem 2.1** (Quadratic convergence of $\lambda^{(n)}$). Suppose

\[
\inf_{x \in \Omega} \frac{(Tv^0)(x)}{v^0(x)} < \sup_{x \in \Omega} \frac{(Tv^0)(x)}{v^0(x)} = \lambda^0 < \infty.
\]
The following hold.

(1) \( \lambda^* < \lambda^{(n+1)} < \lambda^{(n)} < \lambda^0 \) for all \( n \), and \( \lambda^{(n)} \to \lambda^* \) as \( n \to \infty \).

(2) There exist some constants \( C > c > 0 \), and integer \( n_0 \geq 2 \), such that for all \( n \geq n_0 \), we have

\[
0 < c \leq \frac{\lambda^{(n+1)} - \lambda^*}{(\lambda^{(n)} - \lambda^*)^2} \leq C < \infty.
\]

Proof. (1) We show \( \lambda^{(n)} \to \lambda^* \) as \( n \to \infty \). We argue by contradiction. Suppose \( \lambda^{(n)} \to \mu > \lambda^* \). By (2.13), we obtain

\[
\lambda^{(n)} - \lambda^{(n+1)} \geq \inf_{x \in \Omega} \frac{v^n(x)}{((\lambda^{(n)} - T)^{-1}v^n)(x)}.
\]

By an argument similar to the proof of Lemma 2.1, we have

\[
\sup_{x \in \Omega} \frac{((\lambda^{(n)} - T)^{-1}v^n)(x)}{v^n(x)} \leq \sup_{x \in \Omega} \frac{((\lambda^{(n)} - T)^{-1}v^0)(x)}{v^0(x)}.
\]

Since \( \lambda^{(n)} \geq \mu > \lambda^* \) and

\[
((\lambda^{(n)} - T)^{-1} - (\mu - T)^{-1} = -(\lambda^{(n)} - \mu)(\lambda^{(n)} - T)^{-1}(\mu - T)^{-1},
\]

we clearly have

\[
\sup_{x \in \Omega} \frac{((\lambda^{(n)} - T)^{-1}v^0)(x)}{v^0(x)} \leq \sup_{x \in \Omega} \frac{((\mu - T)^{-1}v^0)(x)}{v^0(x)} \leq \beta < \infty,
\]

where \( \beta > 0 \) is independent of \( n \), and the last inequality follows a similar proof of (2.14). It follows that \( \lambda^n - \lambda^{n+1} \geq c_\beta > 0 \) for some fixed constant \( c_\beta > 0 \). This clearly leads to a contradiction.

(2) Since \( v^0 \) is not a multiple of \( \phi \), we may (after a simple normalization) write

\[
v^0 = \phi + \sum_{j=j_0}^\infty c_j^{(0)} \phi_j,
\]

where \( T \phi_j = \lambda_j^* \phi_j \), and \( c_j^{(0)} \neq 0 \) for \( j_0 \geq 2 \) is the first nonzero coefficient. Here if \( \lambda_j = \lambda_{j_0} \) for \( j_0 \leq j \leq j_1 \) (i.e. taking into account the multiplicity) we tacitly assume \( c_j^{(0)} \) is the largest amongst \( |c_j^{(0)}| \) for \( j_0 \leq j \leq j_1 \). With no loss we assume \( c_j^{(0)} > 0 \).

The eigen-functions \( \phi_j \) satisfy \( T \phi_j = \lambda_j \phi_j \) with \( \lambda_{j+1} \leq \lambda_j \) for all \( j \geq 2 \). By a simple induction, we have

\[
v^n = \phi + \sum_{j=j_0}^\infty c_j^{(n)} \phi_j, \quad c_j^{(n)} = c_j^{(0)} \prod_{l=0}^{n-1} \frac{\lambda(l) - \lambda^*}{\lambda(l) - \lambda_j^*}.
\]

By using the definition of \( \lambda^{(n+1)} \), we obtain

\[
\frac{\lambda^{(n+1)} - \lambda^*}{\lambda^{(n)} - \lambda^*} = \sup_{x \in \Omega} \frac{((T - \lambda^*)v^{n+1})(x)}{v^{n+1}(x)} = \sup_{x \in \Omega} \frac{\sum_{j=j_0}^\infty c_j^{(n+1)}(\lambda_j^* - \lambda^*)\phi_j}{\phi + \sum_{j=j_0}^\infty c_j^{(n+1)} \phi_j}.
\]
It is not difficult to check that for $n$ sufficiently large,

$$K_1 \leq \frac{\phi(x) + \sum_{j=j_0}^\infty c_j^{(n+1)} \phi_j(x)}{\phi(x)} \leq K_2, \quad \forall x \in \Omega;$$

$$(2.31)$$

$$\|\sum_{j=j_0}^\infty c_j^{(n+1)} (\lambda_j^* - \lambda^*) \phi_j \|_{L^\infty(\Omega)} \leq c_{j_0}^{(n+1)} K_3,$$

$$(2.32)$$

where $K_i > 0$, $i = 1, 2, 3$ are constants independent of $n$. These imply that

$$\frac{\lambda^{(n+1)} - \lambda^*}{\lambda^{(n)} - \lambda^*} \leq \text{const} \cdot c_{j_0}^{(n+1)}.$$  

$$(2.33)$$

On the other hand, we write

$$\sum_{j=j_0}^\infty c_j^{(n+1)} (\lambda_j^* - \lambda^*) \phi_j$$

$$= (\lambda_{j_0}^* - \lambda^*) (\prod_{l=0}^{n} \frac{\lambda(l) - \lambda^*}{\lambda_{j_0}^* - \lambda_{j_0}^*}) \sum_{j=j_0}^{j_1} c_j^{(0)} \phi_j + \sum_{j> j_1} c_j^{(n+1)} (\lambda_j^* - \lambda^*) \phi_j.$$  

$$(2.34)$$

Here we $j_1 \geq j_0$ accounts for the multiplicity of the eigen-value $\lambda_j^*$, namely we have $\lambda_j^* = \lambda_{j_0}^*$ for $j_0 \leq j \leq j_1$, and $\lambda_j^* < \lambda_{j_0}^*$ for $j > j_1$. Thanks to the spectral gap, the coefficients $c_j^{(n+1)}$ decays faster than $c_{j_0}^{(n+1)}$ for $j > j_1$. Clearly we can fix some $x_0 \in \Omega$ such that

$$d_{x_0} := \sum_{j=j_0}^{j_1} c_j^{(0)} \phi_j(x_0) > 0.$$  

$$(2.35)$$

This is indeed possible since

$$\|\sum_{j=j_0}^{j_1} c_j^{(0)} \phi_j\|_{L^2(\Omega)}^2 \geq (c_{j_0}^{(0)})^2 > 0, \quad \langle \sum_{j=j_0}^{j_1} c_j^{(0)} \phi_j, \phi \rangle = 0.$$  

$$(2.36)$$

It follows that for $n$ sufficiently large

$$\sup_{x \in \Omega} \frac{\sum_{j=j_0}^\infty c_j^{(n+1)} (\lambda_j^* - \lambda^*) \phi_j(x)}{\phi(x)} \geq K_4 c_{j_0}^{(n+1)},$$

$$(2.37)$$

where $K_4 > 0$ is a constant independent of $n$. Thus we have shown for $n$ sufficiently large:

$$\frac{\lambda^{(n+1)} - \lambda^*}{\lambda^{(n)} - \lambda^*} \sim c_{j_0}^{(n+1)} \sim \prod_{l=0}^{n} \frac{\lambda(l) - \lambda^*}{\lambda_{j_0}^* - \lambda_{j_0}^*}.$$  

$$(2.38)$$

The desired result follows easily.
Algorithm 1

Step 1 Choose the initial vector $v^0 > 0$ and compute
\[
\lambda^{(0)} = \max_{1 \leq i \leq N} \frac{[Av^0]_i}{v^0_i} \tag{3.1}
\]

Step 2 For $n = 1, 2, \ldots$, solve the linear equation
\[
(\lambda^{(n-1)} I - A) \omega^n = v^{n-1} \tag{3.2}
\]
and calculate (below $\| \cdot \|$ denotes the usual $l^2$ norm)
\[
v^n = \frac{\omega^n}{\|\omega^n\|}. \tag{3.3}
\]

Update $\lambda^{(n)}$ with
\[
\lambda^{(n)} = \max_{1 \leq i \leq N} \frac{[Av^n]_i}{v^n_i}. \tag{3.4}
\]

Step 3 Enforce the stopping criterion and output $(\lambda^{(n)}, v^n)$.

3. Numerical experiments

3.1. Matrix case. In this subsection, we first apply the aforementioned algorithm to compute the principal eigenvalue of primitive matrices. Let $A$ be a nonnegative primitive matrix. We shall implement the algorithm 1 specified below.

We take $A = H$ or $B$, where $H$ is the well-known Hilbert matrix with entries $H_{ij} = (i + j - 1)^{-1}$, $1 \leq i, j \leq N$, and $B$ is a nonnegative random symmetric tridiagonal matrix
\[
B = \begin{pmatrix}
 a_1 & b_1 & 0 & 0 & \cdots \\
 b_1 & a_2 & b_2 & 0 & \cdots \\
 0 & b_2 & a_3 & b_3 & \cdots \\
 \vdots & \vdots & \vdots & \ddots & \vdots \\
 0 & 0 & 0 & b_{N-1} & a_N
\end{pmatrix}. \tag{3.5}
\]

In the above $a_i \sim U(0, 2)$ are (iid) uniformly randomly distributed on the interval $(0, 2)$, and $b_i \sim U(0, 1)$ are (iid) uniformly randomly distributed on the interval $(0, 1)$ . We take $N = 1000$. We shall compute the principal eigenvalue of the matrix $A$.

Remark 3.1. The Hilbert matrix naturally arises from the least square approximation of general $L^2$ functions on $(0, 1)$ via the polynomial basis \{1, $x$, $x^2$, $x^3$, $\cdots$, $x^{N-1}$\}, namely
\[
E = \min_{a_0, \cdots, a_{N-1}} \int_0^1 (f(x) - q_N(x))^2 \, dx, \tag{3.6}
\]
where $q_N(x) = \sum_{j=0}^{N-1} c_j x^j$. The critical point equations yield
\[
\sum_{j=1}^{N} c_{j-1} \int_0^1 x^{i+j-2} \, dx = \int_0^1 x^{i-1} f(x) \, dx. \tag{3.7}
\]
This in turn leads to the linear system of equations $Hc = f$. A well-known issue is that $H$ is ill-conditioned.

**Stopping criterion.** In light of the Collatz-Wielandt bound, we can take

$$SC1 = \max_i \frac{A v^n}{v^n} - \min_i \frac{A v^n}{v^n} < \varepsilon$$

(3.8)

as a stopping criterion, where $\varepsilon$ is the tolerance threshold. However, for tridiagonal matrices, the principal eigenvector sometimes admit extremely small components and the LHS of (3.8) can remain $O(1)$ when the numerical iterates are in the vicinity of the desired eigen-pair (See Table 1). In lieu of SC1, we enforce the following stopping criterion

$$|\lambda^{(n+1)} - \lambda^{(n)}| < \varepsilon.$$  

(3.9)

**Remark 3.2.** We explain why the LHS of (3.8) can stay $O(1)$ whilst $(\lambda^{(n)}, v^n)$ is close to $(\lambda^*, v)$, where $(\lambda^*, v)$ is the principal eigen-pair. Roughly speaking

$$r_n^i = \frac{(Av^n)_i}{v^n_i} = \sum_j a_{ij} \frac{v^n_j}{v^n_i}$$

(3.10)

which clearly depends on the ratios $\frac{v^n_j}{v^n_i}$. In the scenario where $v = (v_i)$ has vastly different components, the ratio $\frac{v^n_j}{v^n_i}$ can become very large and $\frac{v^n_j}{v^n_i} - \frac{v_j}{v_i} \gtrsim 1$ whilst $v^n$ is close to $v$. As a result the round-off errors accumulated in intermediate computations could spoil the convergence of SC1. In general, it will take more iterations for SC1 to converge than (3.9).

**Number of iterations.** We take the initial vector $v^0 = (1, \cdots, 1)^T$. The exact principal eigenvalue is computed by the function `eig()` of MATLAB. In general it takes 6 to 8 iterations for our algorithm to reach the stopping criterion (See Table 1).

**Quadratic convergence.** As we know the definition of quadratic convergence is

$$\frac{|\lambda^{(n+1)} - \lambda|}{|\lambda^{(n)} - \lambda|^2} \to c > 0$$

(3.11)

as $n \to \infty$. Thus

$$\log(|\lambda^{(n+1)} - \lambda|) - \log(|\lambda^{(n)} - \lambda|) \log(|\lambda^{(n)} - \lambda|) - \log(|\lambda^{(n-1)} - \lambda|) \to 2$$

(3.12)

as $n \to \infty$. As such we shall appeal to the above formula to test the convergence rate (See Table 1). Reassuringly in Table 1 the quadratic convergence takes place when $v^n$ is sufficiently close to the exact eigenvector.

**Remark 3.3.** One should note that in Table 1, the row corresponding to the iteration step 8 belongs to the realm of the machine error. As such the order computed in this row which carries the numerical value 1.193 should be treated as an outlier.

---

2 Denote $a_{n+1} = |\lambda^{n+1} - \lambda|$. Clearly one expects $a_{n+1} = (c+\varepsilon_n) a_n^2 \Rightarrow \log(a_{n+1}) = 2 \log(a_n) + \log(c+\varepsilon_n) \Rightarrow \frac{\log(a_{n+1}) - \log(a_n)}{\log(a_n) - \log(a_{n-1})} \to 2$
**Comparison with Rayleigh quotient.** For comparison we apply the usual Rayleigh quotient iteration to aforementioned two matrices. We take \( v^0 = \text{ones}(N,1) \), \( \text{rand}(N,1) \), with \( N = 100 \) or \( N = 1000 \). In yet other words \( v^0 \) consists of identical ones, or iid random variables on \((0,1)\). Quite interestingly, it is observed that none of them converge to the principal eigenvalue. This corroborates very well with the folklore fact that the Rayleigh quotient iteration usually converges locally.

On the other hand, if we consider a small Hilbert matrix \( H \) with \( N = 50 \) and \( v^0 = 1 \), the cubic convergence to the principal eigen-pair takes place within 6 steps. See the last two columns of Table 1. We shall use (3.9) as the stopping criterion.

**Algorithm 2** Rayleigh quotient iteration

**Step 1** Choose the initial vector \( v^0 > 0 \) and compute

\[
\lambda^{(0)} = \frac{(v^0)^T A v^0}{(v^0)^T v^0}
\]

**Step 2** For \( n = 1, 2, \ldots \), solve the linear equation

\[
(\lambda^{(n-1)} I - A) \omega^n = v^{n-1}
\]

and calculate

\[
v^n = \frac{\omega^n}{\|\omega^n\|}.
\]

Update \( \lambda^{(n)} \) with

\[
\lambda^{(n)} = (v^n)^T A v^n.
\]

**Step 3** Enforce the stopping criterion and output \((\lambda^{(n)}, v^n)\).

| Iterations | Tridiagonal | Hilbert matrix | Hilbert matrix, Rayleigh |
|------------|-------------|---------------|-------------------------|
|            | \( \lambda^{(n)} - \lambda \) | Order | SC1 | \( \lambda^{(n)} - \lambda \) | Order | SC1 | \( |\lambda^{(n)} - \lambda| \) | Order |
| 1          | 4.292e-02   | -         | 2.994 | 0.993 | -         | 4.077 | 0.2915 | -         |
| 2          | 4.173e-03   | -         | 2.771 | 0.441 | -         | 2.517 | 0.1941 | -         |
| 3          | 5.768e-05   | 1.837     | 2.658 | 0.160 | 1.246     | 1.382 | 4.0783e-02 | 3.830 |
| 4          | 1.571e-08   | 1.917     | 2.522 | 3.627e-02 | 1.473 | 0.409 | 1.8191e-04 | 3.482 |
| 5          | 8.449e-16   | 2.039     | 2.322 | 2.611e-03 | 1.766 | 2.638e-02 | 1.3255e-11 | 3.033 |
| 6          | 1.408e-16   | 0.107     | 2.228 | 1.482e-05 | 1.965 | 1.213e-04 | 8.5554e-16 | 0.587 |
| 7          | -           | -         | -     | 4.689e-10 | 2.003 | 3.347e-09 | -         | -         |
| 8          | -           | -         | -     | 1.999e-15 | 1.193 | 1.110e-14 | -         | -         |

**Table 1.** The convergence error, order, stopping criterion v.s. iteration step.

### 3.2. The Dirichlet Laplacian case.

In this subsection, we apply aforementioned algorithm to compute the principal eigenvalue of \( T = (-\Delta_D)^{-1} \), where \(-\Delta_D\) is Dirichlet Laplacian operator. We shall implement the algorithm specified below. To expedite the computation, we shall employ the following identity without explicit mentioning:

\[
(\lambda - T)^{-1} = (-\Delta_D)(\lambda(-\Delta_D) - 1)^{-1}.
\]

(3.17)
In yet other words, we shall employ \((-\Delta_D)(\lambda(-\Delta_D) - 1)^{-1}\) to compute \(w^{n+1}\).

Algorithm 3

**Step 1** Choose the initial vector \(v^0 > 0\) and compute

\[
\lambda^{(0)} = \sup_{x \in \Omega} \frac{T v^0}{v^0} \quad (3.18)
\]

**Step 2** For \(n = 0, 1, 2, \ldots\), solve the linear equation

\[
\omega^{n+1} = \left(\lambda^{(n)} I - T \right)^{-1} v^n \quad (3.19)
\]

and calculate

\[
v^n = \frac{\omega^n}{\|\omega^n\|}. \quad (3.20)
\]

Update \(\lambda^{(n+1)}\) with

\[
\lambda^{(n+1)} = \sup_{x \in \Omega} \frac{T v^{n+1}}{v^{n+1}}, \quad (3.21)
\]

or

\[
\lambda^{(n+1)} = \lambda^{(n)} - \inf_{x \in \Omega} \frac{v^n}{w^{n+1}} \quad (3.22)
\]

**Step 3** Enforce the stopping criterion and output \((\lambda^{(n)}, v^n)\).

**Domain.** We take the two dimensional domain \(\Omega\) as specified in Figure 1. This very nice example is taken from [1] (see Section 4.1 therein).

**Discretization.** We use the finite difference method to compute the Dirichlet Laplacian operator. Denote \(T_h = (-\Delta_h)^{-1}\), where \(\Delta_h\) is the discrete Laplacian operator with \(h = h_x = h_y\). We use the standard 5-stencil scheme:

\[
(-\Delta)_h u = -\frac{u_{i-1,j} + u_{i+1,j} - 4u_{i,j} + u_{i,j-1} + u_{i,j+1}}{h^2}. \quad (3.23)
\]

We shall compute the cases \(h = 1/4, 1/6, 1/10, 1/25, 1/50\). Denote the principal eigenvalue of \(T_h\) as \(\lambda_h\). For simplicity we take the exact principal eigenvalue \(\lambda_*\) of \(T\) on domain \(\Omega\) as the one corresponding to the solution computed by MATLAB/pdetool with 2971838.
nodes, i.e. $\lambda_s \approx 0.676147$. Denote the discrete solution of $v^n$ and $w^n$ as $v^n_h$ and $w^n_h$ respectively.

**Remark 3.4.** By using the pdetool package from MATLAB, we can compute the first eigenvalue of $-\Delta_D$ as a function of the number of nodes in the mesh. The table below collects the values of $\lambda(-\Delta_D)$ and $\lambda_s = \frac{1}{\lambda(-\Delta_D)}$. Here $\lambda(-\Delta_D)$ denotes the eigenvalue of $-\Delta_D$.

| # of nodes | 168  | 698  | 2941 | 12597 | 49660 | 218600 | 764685 | 2971838 |
|------------|------|------|------|-------|-------|--------|--------|---------|
| $\lambda(-\Delta_D)$ | 1.4888 | 1.48278 | 1.47941 | 1.479137 | 1.479022 | 1.478984 | 1.478967 | 1.478967 |
| $\lambda_s$ | 0.67164 | 0.67441 | 0.67560 | 0.67594 | 0.676069 | 0.676122 | 0.676139 | 0.676147 |

**Table 2.** Estimate of $\lambda(-\Delta_D)$ and $\lambda_s$ by Matlab/pdetool v.s. # of nodes.

**Stopping criterion.** We take the stopping criterion to be

$$\max_i T_h v^n_h - \min_i v^n_h < \varepsilon,$$

or equivalently,

$$\max_i v^n_h - \min_i v^n_{h+1} < \varepsilon.$$  \hspace{1cm} (3.25)

**Remark 3.5.** We should mention that in Algorithm 3, the stopping criterion (3.24) & (3.25) are equivalent. Notice that $v^n_h = w^n_h / \|w^n_h\|$ and $v^n_h = (\lambda_h(n) - T_h)w^{n+1}_h$, then

$$\lambda_h = \frac{v^n_h}{w^n_{h+1}} = \frac{(\lambda^{(n)}_h - T_h)w^{n+1}_h}{w^{n+1}_h} = \frac{T_h v^{n+1}_h}{v^n_h}.$$  \hspace{1cm} (3.26)

This means

$$\max_i T_h v^{n+1}_h - \min_i v^{n+1}_h = \max_i v^n_h - \min_i v^n_{h+1},$$  \hspace{1cm} (3.27)

whence the result.

In general the eigenvalues $\lambda_h$ exhibits a subtle dependence on the mesh size parameter $h$. By computational wisdom the order of $|\lambda_s - \lambda_h|$ appears to be no more than 2, i.e. $|\lambda_s - \lambda_h| = O(h^\alpha)$ for some $0 < \alpha \leq 2$. To extract the numerical value of $\alpha$ we employ the formula

$$\alpha \approx \frac{\log(|\lambda_s - \lambda_{h_1}|) - \log(|\lambda_s - \lambda_{h_2}|)}{\log(h_1) - \log(h_2)}.$$  \hspace{1cm} (3.28)

(See Table 3)

The exact $\lambda_h$ is computed by using power method ($v^{n+1}_h = T_h v^n_h$) until $\varepsilon < 1e-14$ in (3.24), i.e.

$$\max_i T_h v^n_h - \min_i v^n_h < 10^{-14}.$$  \hspace{1cm} (3.29)
According to (1.5) we know that
\[
\min_i \frac{T_h v^n_i}{v^n_i} < \lambda_h < \max_i \frac{T_h v^n_i}{v^n_i}.
\] (3.30)

Take \(\lambda_h \approx \max_i \frac{T_h v^n_i}{v^n_i}\), we obtain the value of \(\lambda_h\) with 10\(^{-14}\) precision.

| \(h\)  | \(\lambda_h\)  | Error       | Order |
|-------|----------------|-------------|-------|
| 1/4   | 0.670972       | 5.174e-03   | –     |
| 1/6   | 0.672725       | 3.422e-03   | 1.02  |
| 1/10  | 0.674231       | 1.916e-03   | 1.13  |
| 1/16  | 0.675061       | 1.086e-03   | 1.20  |
| 1/25  | 0.675525       | 0.6213e-03  | 1.25  |
| 1/50  | 0.675893       | 0.2537e-03  | 1.29  |

Table 3. The error \(|\lambda^* - \lambda_h|\) and its order

**Iteration steps.** We take initial vector \(v^0 = T1\), i.e. solving \(-\Delta v^0 = 1\) with the Dirichlet boundary condition. We shall use two different tolerance error thresholds \(\varepsilon = 1e-14\) or \(\varepsilon = h^2/10\). In Table 4, the second and third row correspond to the case \(\varepsilon = 1e-14\), and the fourth and fifth correspond to \(\varepsilon = h^2/10\). The error is computed as \(|\lambda_h - \lambda_h^{(n)}|\). In general it is observed that the convergence takes place within 2 to 4 steps. An interesting observation is that the number of iterations remains almost independent of the mesh size parameter \(h\).

| Criterion (3.25) | h     | 1/4 | 1/6 | 1/10 | 1/16 | 1/25 | 1/50 |
|------------------|-------|-----|-----|------|------|------|------|
| \(\varepsilon = 1e-14\) | Iterations | 4   | 4   | 4    | 4    | 4    | 4    |
| Error            | 2.22e-15 | 9.99e-16 | 1.84e-14 | 9.38e-14 | 7.62e-14 | 7.63e-13 |
| Criterion (3.25) | h     | 1/4 | 1/6 | 1/10 | 1/16 | 1/25 | 1/50 |
| \(\varepsilon = h^2/10\) | Iterations | 2   | 2   | 2    | 2    | 3    | 3    |
| Error            | 6.57e-05 | 6.64e-05 | 6.65e-05 | 6.63e-05 | 8.48e-09 | 8.46e-09 |

Table 4. Number of iterations and errors of the proposed method.

**Quadratic convergence.** We take \(h = \frac{1}{6}, \frac{1}{16}, \frac{1}{50}\) to test the convergence rate. The stopping criterion is (3.25) with tolerance threshold \(\varepsilon = 1e-14\). It is observed that \(\lambda_h^{(n)}\) converges to \(\lambda_h\) quadratically. We employ the empirical formula

\[
\log(|\lambda_h^{(n+1)} - \lambda_h|) - \log(|\lambda_h^{(n)} - \lambda_h|)
\]

\[
\log(|\lambda_h^{(n)} - \lambda_h|) - \log(|\lambda_h^{(n-1)} - \lambda_h|)
\]

(3.31)

to compute the order of convergence.

**Comparison with Rayleigh quotient.** As is well known, the standard Rayleigh quotient iteration has spectacular cubic convergence, if the initial condition is sufficient close to the target eigen-pair.

\(^3\)In view of (3.28), it is natural to take the error tolerance threshold to be \(\frac{1}{10} h^2\).
For comparison we take two different initial conditions \( v^0 = 1 \) or \( v^0 = T1 \) to test the cubic convergence. Since the Rayleigh quotient does not preserve the positivity of \( v^n \) in general, we shall enforce the stopping criterion
\[
|\lambda^{(n)} - \lambda_s| < \varepsilon.
\] (3.32)
When we implement (3.32) for the discretized Dirichlet Laplacian, the corresponding stopping criterion takes the form
\[
|\lambda^{(n)}_h - \lambda_h| < \varepsilon.
\] (3.33)

**Algorithm 4** Rayleigh quotient iteration

**Step 1** Choose the initial vector \( v^0 > 0 \) and compute
\[
\lambda^{(0)} = \frac{(v^0, Tv^0)}{(v^0, v^0)}
\] (3.34)

**Step 2** For \( n = 1, 2, \ldots \), solve the linear equation
\[
(\lambda^{(n-1)} I - T)\omega^n = v^{n-1}
\] (3.35)
and calculate
\[
v^n = \frac{\omega^n}{\|\omega^n\|}.
\] (3.36)

Update \( \lambda^{(n)} \) with
\[
\lambda^{(n)} = (v^n, Tv^n)
\] (3.37)

**Step 3** Enforce the stopping criterion and output \((\lambda^{(n)}, v^n)\).

From Table 5, it is observed that the iterates appear to converge to the principal eigenvalue with cubic convergence.

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Rayleigh Iterations Error
\[ v_0 = 1 \] \[
\begin{array}{cccc}
\text{Rayleigh quotient} & \text{Iterations} & 4 & 4 \\
\text{Error} & 6.66e-16 & 1.33e-15 \\
\text{Our algorithm} & \text{Iterations} & 5 & 5 \\
\text{Error} & 3.11e-14 & 5.96e-13
\end{array}
\]

Rayleigh Iterations Error
\[ v_0 = T1 \] \[
\begin{array}{cccc}
\text{Rayleigh quotient} & \text{Iterations} & 2 & 2 \\
\text{Error} & 5.55e-16 & 1.22e-15 \\
\text{Our algorithm} & \text{Iterations} & 4 & 4 \\
\text{Error} & 2.42e-14 & 3.30e-13
\end{array}
\]

Table 6. The comparison of Rayleigh quotient and our method. The error is \(|\lambda_h - \lambda_h^{(n)}|\). The stopping criterion is \(|\lambda_h^{(n)} - \lambda_h| < 1e-12\).

Rayleigh Iterations Error
\[ h = 1/50 \] \[
\begin{array}{cccc}
\text{Error} & \text{Order} \\
1 & 8.499e-02 & - \\
2 & 2.9410e-03 & - \\
3 & 8.025e-08 & 3.124 \\
4 & 1.332e-15 & 1.704
\end{array}
\]

Table 7. Convergence order of Rayleigh quotient iteration. Here the initial condition is \(v^0 = 1\).

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D. Li, SUSTech International Center for Mathematics, and Department of Mathematics, Southern University of Science and Technology, Shenzhen, P.R. China

Email address: lid@sustech.edu.cn

J. Li, Department of Mathematics, Southern University of Science and Technology, Shenzhen, P.R. China

Email address: 11930493@mail.sustech.edu.cn