Levels of complexity in financial markets

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Abstract

We consider different levels of complexity which are observed in the empirical investigation of financial time series. We discuss recent empirical and theoretical work showing that statistical properties of financial time series are rather complex under several ways. Specifically, they are complex with respect to their (i) temporal and (ii) ensemble properties. Moreover, the ensemble return properties show a behavior which is specific to the nature of the trading day reflecting if it is a normal or an extreme trading day.

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1 Introduction

Financial markets can be regarded as model complex systems [1]. In fact, they are systems composed by many agents which are interacting between them in a highly nonlinear way. Financial markets are continuously monitored. Data exist down to the scale of each single communication of bid and ask of a financial asset (quotes) and at the level of each transaction (trade). The availability of this enormous amount of data allows a detailed statistical description of several aspects of the dynamics of asset price in a financial market. The results of these studies show the existence of several levels of complexity in the price dynamics of a financial asset [2–5]. In this presentation we will focus on some of them that have been investigated by econophysicists and by our research group recently.

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The paper is organized as follows: in the next section we illustrate the first level of complexity which is observed in the statistical properties of a single financial time series, Section 3 presents the results obtained by investigating the synchronous correlations which are observed between all pairs of a selected set of stocks, Section 4 describes some recent works on the statistical properties of ensemble return distribution of equities traded in the New York Stock Exchange and in Section 5 we present a brief discussion of our findings.

2 First level of complexity: time series

In any financial market—either well established and highly active as the New York Stock Exchange, “emerging” as the Budapest stock exchange, or “regional” as the Milan stock exchange—the autocorrelation function of returns is a monotonic decreasing function with a very short correlation time. High frequency data analyses have shown that correlation times can be as short as a few minutes in highly traded stocks or indices [8,9].

This observation is consistent with the so-called efficient market hypothesis [6,7]. In fact, the short-range memory between returns is directly related to the necessity of absence of continuous arbitrage opportunities in efficient financial markets. In other words, the presence of time correlation between returns (and then between price changes) would allow devising trading strategies that would provide a net gain continuously and without risk. The continuous search for and exploitation of arbitrage opportunities from traders focused on this kind of activity drastically diminish the redundancy in the time series of price changes.

The absence of time correlation between returns does not mean that returns are identically distributed over time. In fact different authors have observed that nonlinear functions of return such as the absolute value or the square are correlated over a time scale much longer than a trading day. Moreover the functional form of this correlation seems to be power-law up to at least 20 trading days approximately [9–15].

A final observation concerns the degree of stationary behavior of the stock price dynamics. Empirical analysis shows that returns are not strictly-sense stationary stochastic processes. Indeed the volatility (standard deviation of returns) is itself a stochastic process. Although a general proof is still lacking, empirical analyses performed on financial data of different financial markets suggest that price returns and volatility are locally non-stationary but asymptotically stationary. By asymptotically stationary we mean that the probability density function (pdf) of the stochastic variable measured over a wide time interval exists and it is uniquely defined. A paradigmatic example of sim-
ple stochastic processes which are locally non-stationary but asymptotically stationary is provided by ARCH [16] and GARCH [17] processes.

In summary, the statistical properties of a price time series of a financial asset is rather non-trivial. The stochastic process is simultaneously characterized by both short range and long range memories and it is stationary only asymptotically. These characteristics only would be already enough challenging, however, it will not surprise the reader that this is only one of several levels of complexity of the price dynamics in financial markets.

3 Second level of complexity: cross-correlation

The presence of high degree of cross-correlation between the synchronous time evolution of a set of equities is a well known empirical fact observed in financial market [18–20]. For a time horizon of one trading day correlation coefficient as high as 0.7 can be observed for some pair of equities belonging to the same economic sector.

The study of cross-correlation of a set of economic entities can improve economic forecasting and modeling of composed financial entities such as, for example, stock portfolios. There are different approaches to address this problem. The most common one is the principal component analysis of the correlation matrix of the raw data [21]. This method was also used by physicists by using the perspective and theoretical results of the random matrix theory [22,23]. Another approach is the correlation based clustering procedure which allow to get cluster of stocks homogeneous with respect to the sectors of economic activities. Different algorithm exists to perform cluster analysis in finance [24–26].

Recently [25], it has been proposed to detect economic information present in a correlation coefficient matrix with a filtering procedure based on the estimation of the subdominant ultrametric [27] associated with a metric distance obtained form the correlation coefficient matrix of set of n stocks. This method, already used in other fields, allows to obtain a metric distance and to extract from it a minimum spanning tree (MST) and a hierarchical tree from each correlation coefficient matrix by means of a well defined algorithm known as nearest neighbor single linkage clustering [28]. This allows to reveal geometrical (throughout the MST) and taxonomic (throughout the hierarchical tree) aspects of the correlation present between the stock pairs.

In previous work we have shown that this method gives a meaningful taxonomy for stock time series [25,29] and for market indices of worldwide stock exchanges [30]. Here we discuss the results obtained in [29] for stock price
The correlation coefficient is defined as

$$\rho_{ij}(\Delta t) = \frac{<Y_i Y_j> - <Y_i><Y_j>}{\sqrt{(<Y_i^2> - <Y_i>^2)(<Y_j^2> - <Y_j>^2)}}$$  \hspace{1cm} (1)$$

where $i$ and $j$ are numerical labels of the stocks, $Y_i = \ln P_i(t) - \ln P_i(t - \Delta t)$, $P_i(t)$ is the value of the stock price $i$ at the trading time $t$ and $\Delta t$ is the time horizon which is, in the present discussion, one trading day. The correlation coefficient for logarithm price differences (which almost coincides with stock returns) is computed between all the possible pairs of stocks present in the considered portfolio. The empirical statistical average, indicated in this paper with the symbol $< . >$, is here a temporal average always performed over the investigated time period.

By definition, $\rho_{ij}(\Delta t)$ can vary from -1 (completely anti-correlated pair of stocks) to 1 (completely correlated pair of stocks). When $\rho_{ij}(\Delta t) = 0$ the two stocks are uncorrelated. The matrix of correlation coefficient is a symmetric matrix with $\rho_{ii}(\Delta t) = 1$ in the main diagonal. Hence for each value of $\Delta t$, $n (n - 1)/2$ correlation coefficients characterize each correlation coefficient matrix completely.

A metric distance between pair of stocks can be rigorously determined [31] by defining

$$d_{i,j}(\Delta t) = \sqrt{2(1 - \rho_{ij}(\Delta t))}.$$  \hspace{1cm} (2)$$

With this choice $d_{i,j}(\Delta t)$ fulfills the three axioms of a metric – (i) $d_{i,j}(\Delta t) = 0$ if and only if $i = j$; (ii) $d_{i,j}(\Delta t) = d_{j,i}(\Delta t)$ and (iii) $d_{i,j}(\Delta t) \leq d_{i,k}(\Delta t) + d_{k,j}(\Delta t)$. The distance matrix $D(\Delta t)$ is then used to determine the MST connecting the $n$ stocks.

The MST, a theoretical concept of graph theory [32], is a graph with $n - 1$ links which selects the most relevant connections of each element of the set. The MST allows to obtain, in a direct and essentially unique way, the subdominant ultrametric distance matrix $D^<(\Delta t)$ and the hierarchical organization of the elements (stocks in our case) of the investigated data set.
Fig. 1. Gray scale table of the distance matrix of our portfolio. Equities are ordered in alphabetical order from left to right and from bottom to top. Each gray spot indicates the distance $d_{i,j}$ between stock $i$ and stock $j$. The gray scale used is shown at the right side of the figure. No simple pattern is detected in the distance matrix.

The subdominant ultrametric distance between $i$ and $j$ objects, i.e. the element $d_{i,j}^{\leq}$ of the $D^{\leq}(\Delta t)$ matrix, is the maximum value of the metric distance $d_{k,t}$ detected by moving in single steps from $i$ to $j$ through the path connecting $i$ and $j$ in the MST. The method of constructing a MST linking a set of $n$ objects is direct and it is known in multivariate analysis as the nearest neighbor single linkage cluster analysis [28]. A pedagogical exposition of the determination of the MST in the contest of financial time series is provided in ref. [3].

Subdominant ultrametric space [27] has been fruitfully used in the description of frustrated complex systems. The archetype of this kind of systems is a spin glass [33].

In ref. [29], we investigate a set of 100 highly capitalized stocks traded in the major US equity markets during the period January 1995 - December 1998. At that time, most of them were used to compute the Standard and Poor’s 100 index. The prices are transaction prices stored in the Trade and Quote database of the New York Stock Exchange.

The time horizons investigated in the cited study varies from $\Delta t = d = 6$ h and 30 min (a trading day time interval), to $\Delta t = d/20 = 19$ min and 30 sec. Here, we only discuss the case of the one day time horizon to present the simplest aspect of this kind of complexity detected in the synchronous dynamics of price returns.

In Fig. 1 we show the distance matrix obtained from the correlation matrix for
Fig. 2. Hierarchical tree of the set of 100 stocks traded in the US equities markets obtained starting from the return time series computed with a $\Delta t = 6$ h and 30 min time horizon (1 trading day) during the time period Jan 1995-Dec 1998. Each stock is indicated by a vertical line. The hierarchical tree is highly structured. Two stocks (lines) links when a horizontal line is drawn between two vertical lines. The height of the horizontal line indicates the ultrametric distance at which the two stocks are connected. The tick symbols of the investigated stocks are from left to right: SLB, HAL, BHI, MOB, CHV, XON, ARC, OXY, CGP, JPM, BAC, MER, ONE, WFC, AXP, AIG, KO, GE, PG, CL, USB, MRK, BMY, JNJ, AGC, CPB, PEP, WMT, MAY, S, DIS, CI, UTX, MCD, NT, NSC, BNI, RAL, MSFT, INTC, TXN, CSCO, SUNW, NSM, IBM, HWP, ORCL, AVP, HON, BAX, GM, F, BA, HRS, DOW, DD, MMM, IFF, HNZ, VO, XRX, WY, CHA, IP, BCC, FDX, DAL, LTD, AA, CSC, BEL, AIT, GTE, SO, AEP, UCM, ETR, GD, MTC, MO, ROK, TAN, PNU, WMB, BDK, TOY, MKG, RTNB, CEN, EK, PRD, UIS, TEK, BS, T, COL, FLR, BC, KM, HM. For a description of the investigated stocks see a financial web site as, for example, www.quicken.com.
Fig. 3. Gray scale table of the distance matrix of our portfolio. Equities are ordered with the order obtained from the hierarchical tree of Fig. 2 (from left to right and from bottom to top). Each gray spot indicates the distance $d_{i,j}$ between stock $i$ and stock $j$. The gray scale used is the same as in Fig. 1. Clusters are clearly observable with the present ordering. From the left to the right prominent examples of clusters are the ones of oil (from 1 to 9), financial, conglomerates and consumer/non-cyclical (from 10 to 21), technology (from 39 to 47), basic materials (from 62 to 65) and utility (from 71 to 76).

The hierarchical tree obtained starting from the distance matrix described in Fig. 1 is shown in Fig. 2. In the figure, each vertical line indicates an equity. Each of the investigated stocks is indicated with its tick symbol in the figure caption. Several clusters are clearly identified. From left to right, the most prominent are (i) the cluster of energy stocks (SLB, HAL, BHI, MOB, CHV, XON, ARC, OXY and CGP), (ii) the cluster of financial stocks (JPM, BAC, MER, ONE, WFC, AXP and AIG), (iii) technology cluster (MSFT, INTC, TXN, CSCO, SUNW, NSM, IBM, HWP and ORCL), (iv) basic materials cluster (WY, CHA, IP and BCC) and (v) utility cluster (BEL, AIT, GTE, SO, AEP, UCM and ETR).

The direct interpretation of the economic clusters of Fig. 2 shows that a set of one-day time horizon time series of returns carries information about the economic sector of the stocks considered.

The hierarchical tree provides an order of portfolio equities that can be used to rearrange the distance matrix. By using this order obtained by investigating the subdominant ultrametric of the distance matrix we plot the distance matrix in a form which is much more readable than in the case shown in Fig. 1. In fact in Fig. 3 we observe the presence of groups of stocks which form clusters (to black areas in the matrix) and we also observe regions of longer
distance (to white regions in the matrix).

Several clusters are directly observable in the distance matrix built up with the present ordering. From left to right prominent examples of clusters are the ones of (i) oil which is, to be precise, a cluster composed by two separated sub-clusters to which belongs the companies SLB, HAL, BHI (companies which are providing financial services to the oil industry) and MOB, CHV, XON, ARC, OXY, CGP (company of the oil and gas industry); (ii) financial (JPM, BAC, MER, BAC, ONE, WFC, APX, etc) and consumer/non-cyclical companies (KO, GE, PG, CL, JNJ, etc); (iii) technology companies (MSFT, INTC, TXN, CSCO, NSM, IBM, HWP, ORCL); (d) basic materials (paper industry WY, CHA, IP, BCC) and (iv) utility companies (BEL, AIT, GTE, SO, AEP, UCM, ETR). It may be worth noting that financial and consumers/non-cyclical companies are characterized by short distances also with some companies which are located outside their cluster whereas the distances of oil, technology and paper companies with companies outside their cluster are usually homogeneously much longer than distances inside their clusters. This effect is most evident for the utility cluster.

Equity time series are then carrying economic information which can be detected by using specialized filtering procedures as the one we discuss in the present paragraph. In summary, price time series in a financial market are not only complex in their time statistical properties but they are also rather complex with respect to the intricate synchronous interaction of each time series with all the others.

4 Third level of complexity: Collective behavior during extreme market events

In the present paragraph we discuss a third level of complexity. Specifically, we discuss the different statistical behavior observed in a set of equities simultaneously traded in a financial market during typical and extreme market days.

The investigation of the return distribution of an ensemble of stocks simultaneously traded in a financial market was introduced in [34]. The statistical properties of price return distribution of an ensemble of stocks are discussed in [35] for the typical trading days and in [36] for the extreme market days.

In both studies, the investigated market is the New York Stock Exchange (NYSE) during the 12-year period from January 1987 to December 1998 which corresponds to 3032 trading days. The total number of assets $n$ traded in NYSE is rapidly increasing and it ranges from 1128 in 1987 to 2788 in 1998.
Fig. 4. Daily ensemble return distribution of all the equities traded in the New York Stock Exchange for the trading days: 19th October 1987 (top) and 6th May 1997 (bottom). The 19th October 1987 is the Black Monday, the worst trading day in financial markets since last 50 years whereas the 6th May 1997 is a typical trading day. The shape of the ensemble distribution dramatically changes from typical (bottom) to extreme (top) market days.

The total number of data records exceeds 6 million.

The variable investigated in our analysis is the daily price return, which is defined as $R_i(t) \equiv \frac{(P_i(t+1) - P_i(t))}{P_i(t)}$, where $P_i(t)$ is the closure price of $i$-th asset at day $t$ ($t = 1, 2, \ldots$). In our studies, we consider only the trading days and we remove the weekends and the holidays from the data set. Moreover we do not consider price returns which are in absolute values greater than 50% because some of these returns might be attributed to errors in the database and may affect in a considerable way the statistical analyses. We extract the $n$ returns of the $n$ stocks for each trading day and we consider the normalized pdf of price returns. The distribution of these returns gives an idea about the general activity of the market at the selected trading day. In the absence of extreme events, the central part of the distribution is conserved for long time periods. In these periods the shape of the distribution is systematically non-Gaussian and approximately symmetrical [35]. During extreme trading days the pdf changes abruptly its shape either in the presence of positive or negative mean return.

In Fig. 4 we show the daily ensemble return distribution of all the equities
traded in the New York Stock Exchange for two representative trading days, the famous 19th October 1987 (top) and the anonymous 6th May 1997. The 19th October 1987 is the Black Monday, the worst trading day in financial markets since last 50 years whereas the 6th May 1997 is a typical trading day. The shape of the ensemble distribution dramatically changes from typical (bottom) to extreme (top) market days. The most surprising change in the shape of the ensemble return distribution concerns its symmetry property. In ref. [36], one shows that the shape of the ensemble return distribution is symmetrical in the typical trading day whereas in extreme days (crashes or rallies) the distribution is skewed (negatively or positively respectively). A quantitative estimate of the asymmetry of the pdf is difficult in finite statistical sets because the skewness parameter depends on the third moment of the distribution. Moments higher than the second are essentially affected by rare events rather than by the central part of the distribution. Due to the finite number of stocks in our statistical ensemble, a measure of the asymmetry of the distribution based on its skewness is not statistically robust.

We overcome this problem in ref. [36] by considering an alternative measure of the asymmetry of the distribution. Specifically, we extract the median and the mean of the distribution for all trading days. When a probability distribution is symmetric the median coincides with the mean. Therefore the difference between the mean and the median is a measure of the degree of asymmetry of the distribution. For positively (negatively) skewed distribution the median is smaller (greater) than the mean. The median depends weakly on the rare events of the random variable and therefore is much less affected than the skewness by the finiteness of the number of records of the ensemble.

Figure 5 shows the difference between the mean and the median as a function of the mean for each trading day of the investigated period. In the Figure each circle refers to a different trading day. The circles cluster in a pattern which has a sigmoid shape. In days in which the mean is positive (negative) the difference between mean and median is positive (negative). In extreme days, the corresponding circles are characterized by a great absolute value of the mean and a great absolute value of the difference between mean and median. Another result summarized in Fig. 5 is that symmetry alteration is not exclusive of the days of extreme crash or rally but it is also evident for trading days of intermediate absolute mean return. The change of the shape and of the symmetry properties during the days of large absolute returns suggests that in extreme days the behavior of the market cannot be statistically described in the same way of the 'normal' periods. Moreover Figure 5 indicates that the difference from normal to extreme behavior increases gradually with the absolute value of the average return.

In ref. [35,37] we compare the results of empirical analysis discussed above with the ones predicted by a simple model: the single-index model. The single-
Fig. 5. Degree of symmetry of the ensemble return distribution quantified by the difference between the mean and the median of the ensemble return distribution as a function of the mean for each trading day. Each circle refers to a single trading day of the investigated time period. The change in asymmetry is detected both for crashes (left region of the figure) and for rallies (right region of the figure). Typical and extreme market events are collapsing over a sigmoid function.

Index model \[19,20\] assumes that the returns of all assets are controlled by one factor, usually called the market. For any asset \(i\), we have

\[
R_i(t) = \alpha_i + \beta_i R_M(t) + \epsilon_i(t),
\]

where \(R_i(t)\) and \(R_M(t)\) are the return of the asset \(i\) and of the market at day \(t\), respectively, \(\alpha_i\) and \(\beta_i\) are two real parameters and \(\epsilon_i(t)\) is a zero mean noise term characterized by a variance equal to \(\sigma^2_{\epsilon_i}\). The noise terms of different assets are uncorrelated, \(<\epsilon_i(t)\epsilon_j(t)> = 0\) for \(i \neq j\). Moreover the covariance between \(R_M(t)\) and \(\epsilon_i(t)\) is set to zero for any \(i\).

Each asset is correlated with the market and the presence of such a correlation induces a correlation between any pair of assets. It is customary to adopt a broad-based stock index for the market \(R_M(t)\). Our choice for the market is the Standard and Poor’s 500 index. The best estimation of the model parameters \(\alpha_i, \beta_i\) and \(\sigma^2_{\epsilon_i}\) is usually done with the ordinary least squares method \[20\]. In our comparison \[35,37\], we infer that the correlation which are detected among the stocks can be described by the single-index model only as a first approximation. The degree of approximation of the single-index model progressively becomes worst for market days of increasing absolute average return and fails in properly describing the market behavior of extreme days. Discrepancies between the theoretical predictions of the single-index model
and empirical data have also been documented in ref. [38].

In summary, a third level of complexity is also present in financial markets. Typical and extreme days are different with respect to the statistical properties of the ensemble return distribution. Specifically, in addition to the first moment (mean return) also higher moments governing the shape of the ensemble return distribution change during extreme (crashes or rallies) market events. However, the change of shape of the ensemble return distribution is not arbitrary and statistical regularities can be detected for extreme events occurring after a time interval as long as ten years.

5 Discussion

A complete modeling of the dynamics of financial markets turns out to be extremely challenging. Several levels of complexity arise from the investigation of statistical properties of a set of price of financial assets simultaneously traded in a market. Each single time series has statistical properties which are rather difficult to model (in fact a definitive model is still waiting in spite of all the efforts devoted to this problem since the seminal paper of Bachelier). In addition to this level of complexity, each set of financial assets has associated a specific overall dynamics which connotes economic information driving the global system. The nature and dynamical properties of the correlations between stocks are a key aspect of the complexity of a financial market. The degree of this complexity is enhanced by the observation that the market behave in a different way in typical and in extreme days.

Pointing out all these levels of complexity may sound frustrating for researchers interested in modeling such a system. However, we wish to point out that there may be another interpretation of these results: the results obtained in making formal the existence of these levels of complexity suggest that the system eventually obeys some deep rules that control the statistical properties of the global system both in typical and in extreme days. We believe this is indeed the case that makes so challenging and interesting the study of such complex systems.

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