Application of Abel-Plana formula for collapse and revival of Rabi oscillations in Jaynes-Cummings model

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Abstract

In this paper, we give an analytical treatment to study the behavior of the collapse and the revival of the Rabi oscillations in the Jaynes-Cummings model (JCM). The JCM is an exactly soluble quantum mechanical model, which describes the interaction between a two-level atom and a single cavity mode of the electromagnetic field. If we prepare the atom in the ground state and the cavity mode in a coherent state initially, the JCM causes the collapse and the revival of the Rabi oscillations many times in a complicated pattern in its time-evolution. In this phenomenon, the atomic population inversion is described with an intractable infinite series. (When the electromagnetic field is resonant with the atom, the \( n \)th term of this infinite series is given by a trigonometric function for \( \sqrt{nt} \), where \( t \) is a variable of the time.) According to Klimov and Chumakov’s method, using the Abel-Plana formula, we rewrite this infinite series as a sum of two integrals. We examine the physical meanings of these two integrals and find that the first one represents the initial collapse (the semi-classical limit) and the second one represents the revival (the quantum correction) in the JCM. Furthermore, we evaluate the first and second-order perturbations for the time-evolution of the JCM with an initial thermal coherent state for the cavity mode at low temperature, and write down their correction terms as sums of integrals by making use of the Abel-Plana formula.

1 Introduction

The Jaynes-Cummings model (JCM), which described the interaction between a two-level atom and a single electromagnetic field mode, was originally proposed for examining spontaneous emission in 1960s [1]. This model is derived from applying the rotating wave approximation to an electric dipole coupling. In the interaction term of the Hamiltonian of

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the JCM, the photon creation operator accompanies the atomic de-excitation operator and
the photon annihilation operator accompanies the atomic excitation operator. Because
the JCM is an exactly soluble quantum mechanical model, it is investigated theoretically
by researchers in the field of quantum optics eagerly [2, 3, 4].

If we initially prepare the atom in the ground state and the cavity mode in a coherent
state, the JCM causes the collapse and the revival of the Rabi oscillations many times
in a complicated pattern in its time-evolution and this phenomenon is regarded as the
evidence of the quantum nature of the electromagnetic field [5, 6]. (This phenomenon was
confirmed experimentally in 1980s [7].) Thus, the demonstration of the collapse and the
revival of the Rabi oscillations in the JCM gives the foundation to Planck’s thought [8].
That is, the collapse and the revival of the Rabi oscillations in the JCM tells us that the
photon’s energy is equal to $h\nu$, where $h$ represents the Planck’s constant and $\nu$
represents the frequency of the photon, so that excitation of the photons shows discreteness.

Recently, the JCM has been studied from a new viewpoint by the researchers in the
field of quantum information science. The JCM is often used for investigating the evolu-
tion of entanglement between the atom and the single mode cavity field [9, 10]. The lower
bound of entanglement between the two-level atom and the thermal photons in the JCM
is also discussed [11]. The JCM can be applied to the realization of quantum computa-
tion [12]. So-called sudden death effect (disappearance of entanglement of two isolated
Jaynes-Cummings atoms in a finite time) is predicted [13, 14, 15], and it is experimentally
demonstrated [16]. Thus, some researchers in the field of quantum information science
think that the JCM has to be studied from the new viewpoint.

When we discuss the JCM, we often have to handle an intractable infinite series.
(If the electromagnetic field is resonant with the atom, the $n$th term of this infinite
series is given by a trigonometric function for $\sqrt{nt}$, where $t$ represents a variable of the
time.) For example, the atomic population inversion in the collapse and the revival of
the Rabi oscillations in the JCM is described by this infinite series. In the thermal JCM,
whose initial state of the cavity mode is given by a thermal equilibrium state, the atomic
population inversion is written as a similar intractable infinite series, as well.

In Ref. [17], Klimov and Chumakov discuss the thermal JCM and evaluate the atomic
population inversion, which is described by the intractable infinite series. They change
this intractable infinite series into a sum of two integrals by making use of the Abel-Plana
formula mentioned in Ref. [18]. By carrying out numerical calculations, they find that the
first integral represents a semi-classical limit (the initial collapse) and the second integral
represents a quantum correction (quasi-chaotic behavior).

In this paper, we give an analytical treatment to study the behavior of the collapse
and the revival of the Rabi oscillations in the JCM, according to the method proposed by
Klimov and Chumakov. For applying the Abel-Plana formula to the infinite series that
describes the atomic population inversion, we replace an inverse of a factorial $1/n!$ with
an inverse of the gamma function $1/\Gamma(n + 1)$ and perform the analytical continuation
on the complex plane as $1/\Gamma(z + 1)$. (This prescription is a new key point of this paper
as compared with Ref. [17].) After giving this step, using the Abel-Plana formula, we
write down the atomic population inversion as a sum of two integrals. We examine the
physical meaning of these two integrals and find that the first integral represents the
initial collapse (the semi-classical limit) and the second integral represents the revival
(the quantum correction) in the JCM. In this paper, we clarify that we can separate the quantum correction from the semi-classical limit in the solution of the JCM.

Furthermore, we evaluate the first and second-order perturbations for the time-evolution of the JCM, whose initial state of the cavity mode is given by a thermal coherent state at low temperature [19] [20]. The expansion parameter for the perturbation theory is given by \( \theta(\beta) \equiv \exp(-\beta \epsilon/2) \), where \( \beta = (k_B T)^{-1} \), \( \epsilon = h \nu \), \( \beta \epsilon \gg 1 \) and \( \nu \) is the frequency of the cavity field. A rigorous definition of \( \theta(\beta) \) is given by Refs. [19] and [20]. We obtain the first and second-order correction terms of the atomic population inversion and rewrite them as sums of integrals, using Klimov and Chumakov’s method and the Abel-Plana formula.

After deriving integral forms of the atomic population inversion for both resonant and off-resonant cases and their thermal perturbation corrections, we examine effects of detuning and low temperature against the collapse and the revival of the Rabi oscillations numerically.

This paper is organized as follows. In section 2, we give a review of the JCM and its collapse and revival of the Rabi oscillations. In section 3, we apply Klimov and Chumakov’s method to the atomic population inversion, which is represented by the intractable infinite series. (In this section, we consider the zero-temperature case.) We rewrite it as a sum of two integrals with the Abel-Plana formula. In section 4, we consider the time-evolution of the JCM at low temperature. Preparing the cavity field in a thermal coherent state initially, we evaluate the first and second-order perturbation corrections of the atomic population inversion. Using the Abel-Plana formula, we rewrite them as sums of integrals. In section 5, we examine properties of the integrals obtained in sections 3 and 4 in detail by numerical calculations and analytical methods. In section 6, we give a brief discussion. In Appendix A, we give a derivation of the Abel-Plana formula, which plays an important role in Klimov and Chumakov’s method. In Appendix B, we give some remarks about precise techniques for carrying out the numerical calculations shown in section 5.

2 The JCM and its collapse and revival of the Rabi oscillations

The JCM is a quantum system, which consists of the single two-level atom and the single mode of the electromagnetic field. Its Hamiltonian is given by

\[
H = \frac{\hbar}{2} \omega_0 \sigma_z + \hbar \omega a^\dagger a + \hbar \kappa (\sigma_+ a + \sigma_- a^\dagger),
\]

where \( \hbar = \hbar/2\pi \), \( \sigma_\pm = (1/2)(\sigma_x \pm i\sigma_y) \) and \( [a, a^\dagger] = 1 \). The Pauli matrices \( (\sigma_i, i = x, y, z) \) are operators of the atom, and \( a \) and \( a^\dagger \) are an annihilation and a creation operators of photons, respectively. Moreover, we assume the coupling constant \( \kappa \) to be real.

We can divide \( H \) given by Eq. (1) as follows:

\[
\begin{align*}
H & = \hbar (C_1 + C_2), \\
C_1 & = \omega [(1/2)\sigma_z + a^\dagger a], \\
C_2 & = \kappa (\sigma_+ a + \sigma_- a^\dagger) - (\Delta \omega / 2) \sigma_z,
\end{align*}
\]
where $\Delta \omega = \omega - \omega_0$. Because $[C_1, C_2] = 0$ and we can diagonalize $C_1$ at ease, we take the following interaction picture. We write a state vector of the whole system in the Schrödinger picture as $|\psi_S(t)\rangle$. We define a state vector in the interaction picture as $|\psi_I(t)\rangle = \exp(iC_1t)|\psi_S(t)\rangle$. [We assume $|\psi_I(0)\rangle = |\psi_S(0)\rangle$.] The time-evolution of $|\psi_I(t)\rangle$ is given by $|\psi_I(t)\rangle = U(t)|\psi_I(0)\rangle$, where $U(t) = \exp(-iC_2t)$.

We give the basis vectors for the state of the atom and the photons of the cavity field as follows. (In this section, we consider only the zero-temperature case.) We write the ground and excited states of the atom as two-component vectors,

$$
|g\rangle_A = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad |e\rangle_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix},
$$

(3)

where we assume that $|g\rangle_A$ and $|e\rangle_A$ are eigenvectors of $\sigma_z$ with eigenvalues $-1$ and $1$, respectively. (The index A stands for the atom.) We describe the number states of the photons as $|n\rangle_P$ ($n = 0, 1, 2, ...$), which are eigenstates of $a^\dagger a$. (The index P stands for the photons.)

Describing the atom’s Pauli operators by $2 \times 2$ matrices, we can write down $U(t)$ as follows:

$$
U(t) = \exp[-it \begin{pmatrix} -\Delta \omega/2 & \kappa a \\ \kappa a^\dagger & \Delta \omega/2 \end{pmatrix}] = \begin{pmatrix} u_{00} & u_{01} \\ u_{10} & u_{11} \end{pmatrix},
$$

(4)

where

$$
\begin{align*}
u_{00} &= \cos(\sqrt{D + \kappa^2t}) - \frac{i}{2}\Delta \omega \frac{\sin(\sqrt{D + \kappa^2t})}{\sqrt{D + \kappa^2}}, \\
u_{01} &= \frac{i\kappa}{\sqrt{D}} \sin(\sqrt{D}t)a, \\
u_{10} &= \frac{i\kappa}{\sqrt{D + \kappa^2}} \sin(\sqrt{D + \kappa^2}t)a^\dagger, \\
u_{11} &= \cos(\sqrt{D}t) + \frac{i}{2}\Delta \omega \frac{\sin(\sqrt{D}t)}{\sqrt{D}},
\end{align*}
$$

(5)

and

$$
D = (\Delta \omega/2)^2 + \kappa^2 a^\dagger a.
$$

(6)

Here, we define the initial state of the whole system as follows. We assume that the atom is in the ground state $|g\rangle_A$ at $t = 0$. Moreover, we assume that the cavity mode is in the coherent state $|\alpha\rangle_P$ at $t = 0$, where

$$
|\alpha\rangle_P = \exp(-|\alpha|^2/2) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle_P,
$$

(7)

and $\alpha$ is an arbitrary complex number. (We provide that $0! = 1$.) From now on, to let the discussion be simple, we assume $\alpha$ to be real. Writing the initial state of the whole system as $|\psi_I(0)\rangle = |g\rangle_A|\alpha\rangle_P$, the time-evolution $|\psi_I(t)\rangle$ is given as follows:

$$
|\psi_I(t)\rangle = U(t)|\psi_I(0)\rangle = \begin{pmatrix} u_{00} & u_{01} \\ u_{10} & u_{11} \end{pmatrix} \begin{pmatrix} 0 \\ |\alpha\rangle_P \end{pmatrix} = \begin{pmatrix} u_{01}|\alpha\rangle_P \\ u_{11}|\alpha\rangle_P \end{pmatrix}.
$$

(8)
Figure 1: A graph of $\langle \sigma_z(t) \rangle$ $(t \in [0, 20\pi])$ for $\kappa = 1$, $\Delta \omega = 0$ and $\alpha = 4$. In the numerical calculations of $\langle \sigma_z(t) \rangle$ defined in Eq. (11), the summation of the index $n$ is carried out up to $n = 100$. Looking at this graph, we notice that the initial collapse time is order of unity and the period of the revival is approximately equal to $8\pi$.

Thus, the probability that we observe $|g\rangle_A$ at the time $t$ is given by

$$P_g(t) = |\Lambda \langle g|\psi_I(t)\rangle|^2 = |u_{11}|^2 = \exp(-\alpha^2) \sum_{n=0}^{\infty} \frac{\alpha^{2n}}{n!} \left[ \cos^2\left(\sqrt{(\Delta \omega/2)^2 + n\kappa^2 t}\right) + (\Delta \omega/2)^2 \frac{\sin^2\left(\sqrt{(\Delta \omega/2)^2 + n\kappa^2 t}\right)}{(\Delta \omega/2)^2 + n\kappa^2} \right].$$

The atomic population inversion is given by

$$\langle \sigma_z(t) \rangle = \text{Tr}_P \langle \psi_I(t)|\sigma_z|\psi_I(t)\rangle = 1 - 2P_g(t).$$

Especially, in the case where $\Delta \omega = 0$, that is, the electromagnetic field is resonant with the atom so that the energy gap is equal to the frequency of the photons, we can replace $|\kappa|t$ with $t$ by letting the time $t$ be in units of $|\kappa|^{-1}$ and we obtain

$$\langle \sigma_z(t) \rangle = -\exp(-\alpha^2) \sum_{n=0}^{\infty} \frac{\alpha^{2n}}{n!} \cos(2\sqrt{n}t).$$

We plot $\langle \sigma_z(t) \rangle$ $(t \in [0, 20\pi])$ for $\kappa = 1$, $\Delta \omega = 0$ and $\alpha = 4$ in Fig. 1.

Looking at Fig. 1 we can observe the collapse and the revival of the Rabi oscillations clearly. In general, the larger $|\alpha|$ is, the more distinctly we observe the revival of the Rabi oscillations. From Fig. 1 we can suppose that the time scale of the initial collapse is order of unity and the period of the revival is around $2\pi|\alpha|$.

The time scale of the initial collapse and the period of the revival are explained as follows [21, 22]. Let us evaluate $\langle \sigma_z(t) \rangle$ defined in Eq. (11) with assuming $\alpha^2 \gg 1$. 

\[ \]
Writing the index of the summation as \( n = \alpha^2 + \delta n \), because of the property of the Poisson distribution, the major contribution to the right-hand side of Eq. (11) comes from the terms with \( |\delta n| < \alpha^2 \), so that we can neglect the terms with \( |\delta n| \geq \alpha^2 \). We rewrite \( \sqrt{n} \) as
\[
\sqrt{n} \approx \frac{\alpha^2 + n}{2|\alpha|}.
\] (12)

Then, we can obtain an approximate form \( \langle \sigma_z(t) \rangle \) in Eq. (11) as
\[
\langle \sigma_z(t) \rangle \approx -\exp(-\alpha^2) \sum_{n=0}^{\infty} \frac{\alpha^{2n}}{n!} \left[ \exp\left(2i\frac{\alpha^2 + n}{2|\alpha|}t\right) + \exp\left(-2i\frac{\alpha^2 + n}{2|\alpha|}t\right) \right]
\]
\[
= -\exp[\alpha^2(\cos \frac{t}{|\alpha|} - 1)] \cos(|\alpha|t + \alpha^2 \sin \frac{t}{|\alpha|}).
\] (13)

In the right-hand side of Eq. (13), \( \exp[\alpha^2(\cos(t/|\alpha|) - 1)] \) represents an amplitude envelope of the wave, which causes a kind of the beat, and \( \cos(|\alpha|t + \alpha^2 \sin(t/|\alpha|)) \) represents the Rabi oscillations. If we assume \( 0 \leq t/|\alpha| \ll 1 \), we obtain \( \cos(t/|\alpha|) \approx 1 - (t^2/2\alpha^2) \), and the factor of the amplitude envelope approximates to \( \exp[\alpha^2(\cos(t/|\alpha|) - 1)] \sim \exp(-t^2/2) \), so that we understand that the initial collapse time is approximately equal to unity. Moreover, the factor of the amplitude envelope \( \exp[\alpha^2(\cos(t/|\alpha|) - 1)] \) shows that the period of the revival is given by \( 2\pi|\alpha| \).

Thus, Eq. (13) represents the collapse time and the period of the revival of the genuine \( \langle \sigma_z(t) \rangle \) given by Eq. (11) well. However, to investigate properties of \( \langle \sigma_z(t) \rangle \) precisely, changing the infinite series that appears in Eqs. (9) and (10) into a simple form is favorable. This is the motivation of this paper.

3 Representing the atomic population inversion as a sum of two integrals for the zero-temperature case

To rewrite the atomic population inversion \( \langle \sigma_z(t) \rangle \) given by Eqs. (9) and (10), we use the following formula. If \( n_1 \) and \( n_2 \) are integers and \( \phi(z) \) is a function which is analytical and bounded for all complex values of \( z \) such that \( n_1 \leq \text{Re}(z) \leq n_2 \), then
\[
\frac{1}{2} \phi(n_1) + \phi(n_1 + 1) + \phi(n_1 + 2) + ... + \phi(n_2 - 1) + \frac{1}{2} \phi(n_2)
\]
\[
= \int_{n_1}^{n_2} \phi(x)dx - i \int_{0}^{\infty} \frac{1}{e^{2\pi y} - 1} \times \left[ \phi(n_2 + iy) - \phi(n_1 + iy) - \phi(n_2 - iy) + \phi(n_1 - iy) \right] dy.
\] (14)

(This formula is described in Ref. [18] as an example. The author of Ref. [18] mentions only a suggestion to prove this formula. We give details of the derivation of this formula in Appendix A.) Moreover, if we assume \( \phi(z) \to 0 \) as \( \text{Re}(z) \to +\infty \), we obtain
\[
\frac{1}{2} \phi(0) + \sum_{n=1}^{\infty} \phi(n) = \int_{0}^{\infty} \phi(x)dx + i \int_{0}^{\infty} \frac{\phi(iy) - \phi(-iy)}{e^{2\pi y} - 1} dy.
\] (15)
This equation is called the Abel-Plana formula.

The Abel-Plana formula given by Eq. (15) changes an infinite series into a sum of two integrals. However, we cannot apply this formula to \(\langle \sigma_z(t) \rangle\) given by Eqs. (9) and (10) in a straightforward manner. Thus, we try to extend this formula in the following way.

First, we consider a series,
\[
\sum_{n=0}^{\infty} \frac{1}{n!} f(n + c), \tag{16}
\]
where \(c\) is a real constant and the function \(f(x)\) is infinitely differentiable and bounded for any real value of \(x\). Thus, we can rewrite \(f(n + c)\) as the Taylor series and we obtain
\[
\sum_{n=0}^{\infty} \frac{1}{n!} f(n + c) = \sum_{m=0}^{\infty} \frac{c^m}{m!} f^{(m)}(n) \tag{17}
\]
[In Eq. (17), \(f^{(m)}(n)\) denotes \((d^m/dx^n)f(x)|_{x=n}\).

Second, using the property of the gamma function \(\Gamma(n + 1) = n!\), we rewrite Eq. (17) as
\[
\sum_{n=0}^{\infty} \frac{1}{n!} f(n + c) = \sum_{m=0}^{\infty} \frac{c^m}{m!} \sum_{n=0}^{\infty} \frac{1}{\Gamma(n + 1)} f^{(m)}(n), \tag{18}
\]
where
\[
\Gamma(z) = \int_{0}^{\infty} t^{z-1} e^{-t} dt. \tag{19}
\]
The gamma function \(\Gamma(z)\) given by Eq. (19) converges absolutely only for \(\text{Re}(z) > 0\). However, by analytical continuation, we can let \(\Gamma(z)\) be analytical everywhere in the complex plane except at \(z = 0, -1, -2, \ldots\). Because there are no points at which \(\Gamma(z)\) is equal to zero, \(1/\Gamma(z)\) is analytical at all finite points of the complex plane.

Third, we assume that \(f^{(m)}(z)/\Gamma(z + 1)\) is analytical and bounded for all complex values of \(z\) such that \(0 \leq \text{Re}(z) < +\infty\) and \(f^{(m)}(z)/\Gamma(z + 1) \to 0\) as \(\text{Re}(z) \to +\infty\) for \(m = 0, 1, 2, \ldots\). Then, we can apply the Abel-Plana formula given by Eq. (15) to the right-hand side of Eq. (18) and we obtain
\[
\sum_{n=0}^{\infty} \frac{1}{n!} f(n + c) = \sum_{m=0}^{\infty} \frac{c^m}{m!} \left[ \frac{1}{2} f^{(m)}(0) + \int_{0}^{\infty} \frac{f^{(m)}(x)}{\Gamma(x + 1)} dx ight. \\
+ i \int_{0}^{\infty} \frac{1}{e^{2\pi y} - 1} \left[ \frac{f^{(m)}(iy)}{\Gamma(1 + iy)} - \frac{f^{(m)}(-iy)}{\Gamma(1 - iy)} \right] dy \\
= \frac{1}{2} f(c) + \int_{0}^{\infty} \frac{f(x + c)}{\Gamma(x + 1)} dx \\
\left. - 2 \int_{0}^{\infty} \frac{1}{e^{2\pi y} - 1} \text{Im}\left\{ \frac{f(c + iy)}{\Gamma(1 + iy)} \right\} dy. \right] \tag{20}
\]

Here, applying the formula given by Eq. (20) to Eq. (9), we can change \(P_g(t)\) represented as the infinite series into a sum of integrals. At first, we define
\[
c = (\Delta\omega/2\kappa)^2. \tag{21}
\]
Then, defining \(f(z + c)\) as
\[
f(z + c) = |\alpha|^{2(z+c)}[\cos^2(\sqrt{z + c}\kappa|t|) + \frac{c}{z + c}\sin^2(\sqrt{z + c}\kappa|t|)], \tag{22}
\]
we can confirm that \( f(z + c)/\Gamma(z + 1) \) is analytical and bounded for all complex plane. Moreover, \( f(z + c)/\Gamma(z + 1) \to 0 \) as \( \text{Re}(z) \to +\infty \). [In the limit of \( \text{Re}(z) \to +\infty, |\Gamma(z+1)| \) increases more rapidly than any exponential functions of \( z \).] Thus, we obtain

\[
P_g(t) = \exp(-\alpha^2)\left|\alpha\right|^{-2c} \sum_{n=0}^{\infty} \frac{|\alpha|^{2(n+c)}}{n!} \cos^2(\sqrt{n+c}\kappa|t|) + \frac{c}{n+c} \sin^2(\sqrt{n+c}\kappa|t|)
\]

where

\[
I_1^{(t)} = \int_0^{\infty} \frac{|\alpha|^{2x}}{\Gamma(x+1)} \cos^2(\sqrt{x+c+l}\kappa|t|) + \frac{c}{x+c+l} \sin^2(\sqrt{x+c+l}\kappa|t|) dx.
\]

\[
I_2^{(t)} = \int_0^{\infty} \frac{1}{e^{\pi y} - 1} \text{Im}\left\{ \frac{|\alpha|^{2iy}}{\Gamma(1+iy)} \cos^2(\sqrt{c+l+iy}\kappa|t|) \right\} dy.
\]

In the above equation, \( P_g(t) \) is represented as a sum of two integrals, \( \int_0^{\infty} dx \) and \( \int_0^{\infty} dy \) [namely, \( I_1^{(t)} \) and \( I_2^{(t)} \)]. Thus, using Eq. (10), we can describe \( \langle \sigma_z(t) \rangle \) as a sum of these integrals,

\[
\langle \sigma_z(t) \rangle = 1 - \exp(-\alpha^2)[1 + 2I_1^{(t)}(t) - 4I_2^{(t)}(t)].
\]

In the case where \( \Delta\omega = 0, \kappa = 1 \) and \( c = 0 \), we obtain

\[
P_g(t) = \exp(-\alpha^2)\left|\alpha\right|^{-2c} \sum_{n=0}^{\infty} \frac{|\alpha|^{2(n+c)}}{n!} \cos^2(\sqrt{n+c}\kappa|t|) + \frac{c}{n+c} \sin^2(\sqrt{n+c}\kappa|t|)
\]

where

\[
\frac{1}{2} \int_0^{\infty} \frac{|\alpha|^{2x}}{\Gamma(x+1)} \cos(\sqrt{x}\alpha t) dx - \int_0^{\infty} \frac{1}{e^{\pi y} - 1} \text{Im}\left\{ \frac{|\alpha|^{2iy}}{\Gamma(1+iy)} \cos(\sqrt{c+l+iy}\alpha t) \right\} dy = -\frac{1}{4} + \frac{1}{2} \exp(\alpha^2).
\]

[We can derive Eq. (27) from Eq. (15) at ease.] Thus, we obtain

\[
\langle \sigma_z(t) \rangle = -(1/2) \exp(-\alpha^2) + J_1(t) + J_2(t),
\]

where

\[
J_1(t) = -\exp(-\alpha^2) \int_0^{\infty} \frac{|\alpha|^{2x}}{\Gamma(x+1)} \cos(2\sqrt{x}\alpha t) dx,
\]

\[
J_2(t) = 2 \exp(-\alpha^2) \int_0^{\infty} \frac{1}{e^{2\pi y} - 1} \text{Im}\left\{ \frac{|\alpha|^{2iy}}{\Gamma(1+iy)} \cos(2\sqrt{c+l+iy}\alpha t) \right\} dy.
\]
4 The time-evolution of the JCM with an initial thermal coherent state

4.1 The definition of the thermal coherent state

In Refs. [19] and [20], a thermal coherent state is defined as an extension of the zero-temperature coherent state according to the thermo field dynamics. In this section, preparing the atom in the ground state and the cavity mode in the thermal coherent state initially, and letting the whole system evolve in time with the JCM, we calculate the atomic population inversion up to the second-order perturbation correction. The obtained first and second-order correction terms are represented as the intractable infinite series. In subsection 4.4, we rewrite these correction terms as sums of integrals by making use of the Abel-Plana formula.

In the thermo field dynamics, we have to handle a space that is a direct product of the ordinary zero-temperature Hilbert space $\mathcal{H}$ and a so-called tilde space $\tilde{\mathcal{H}}$. Thus, every number state of the photons $|n\rangle \in \mathcal{H}$ has a corresponding state $|\tilde{n}\rangle \in \tilde{\mathcal{H}}$, so that the orthogonal basis for the whole space in thermo field dynamics is described as $\{ |n\rangle \otimes |\tilde{m}\rangle \in \mathcal{H} \otimes \tilde{\mathcal{H}} : n, m \in \{0, 1, 2, ...\}\}$. In the next paragraph, we give a definition of the thermal coherent state.

First of all, we define creation and annihilation operators acting on $\mathcal{H}$ and $\tilde{\mathcal{H}}$ as follows:

$$[a, a^\dagger] = [\tilde{a}, \tilde{a}^\dagger] = 1, \quad [a, \tilde{a}] = [a, \tilde{a}^\dagger] = 0. \quad (30)$$

We pay attention to the fact that the operators on $\mathcal{H}$ (namely $a$ and $a^\dagger$) and the operators on $\tilde{\mathcal{H}}$ (namely $\tilde{a}$ and $\tilde{a}^\dagger$) commute with each other. Next, we introduce the temperature by the following unitary transformation:

$$\hat{U}_\beta = \exp(-i\hat{G}), \quad (31)$$

$$\hat{G} = -i\theta(\beta)(a^\dagger\tilde{a}^\dagger - a\tilde{a}), \quad (32)$$

$$\cosh \theta(\beta) = [1 - \exp(-\beta \epsilon)]^{-1/2}, \quad \sinh \theta(\beta) = [\exp(\beta \epsilon) - 1]^{-1/2}, \quad (33)$$

where $\beta = (k_B T)^{-1}$ and $\epsilon = \hbar \omega$. We note that $\theta(\beta)$ is real. To emphasize that $\hat{G}$ and $\hat{U}_\beta$ are operators acting on both $\mathcal{H}$ and $\tilde{\mathcal{H}}$, we put an accent (a hat) on them. Then, annihilation operators are transformed as follows:

$$a \rightarrow a(\beta) = \hat{U}_\beta a \hat{U}_\beta^\dagger,$$

$$\tilde{a} \rightarrow \tilde{a}(\beta) = \hat{U}_\beta \tilde{a} \hat{U}_\beta^\dagger. \quad (34)$$

Writing down the zero-temperature vacuum as $|0\rangle \otimes |\tilde{0}\rangle = |0, \tilde{0}\rangle \in \mathcal{H} \otimes \tilde{\mathcal{H}}$, the thermal vacuum is given by

$$|0(\beta)\rangle = \hat{U}_\beta |0, \tilde{0}\rangle. \quad (35)$$

Then, a thermal coherent state is defined as follows:

$$|\alpha, \tilde{\gamma}; \beta\rangle = \exp[\alpha a^\dagger(\beta) + \tilde{\gamma}^* \tilde{a}^\dagger(\beta) - \alpha^* a(\beta) - \tilde{\gamma}\tilde{a}(\beta)] |0(\beta)\rangle, \quad (36)$$

$\tilde{\gamma}$
where \( \alpha \) and \( \tilde{\gamma} \) are arbitrary complex numbers. From now on, for the simplicity, we assume \( \alpha \) and \( \tilde{\gamma} \) to be real.

Writing the initial state of the whole system as \(|\psi_1(0)\rangle = |g\rangle_\Lambda |\alpha, \tilde{\gamma}; \beta\rangle_P\), the time-evolution of \(|\psi_1(t)\rangle\) is described as

\[
|\psi_1(t)\rangle = \left( \begin{array}{c} u_{01}|\alpha, \tilde{\gamma}; \beta\rangle_P \\ u_{11}|\alpha, \tilde{\gamma}; \beta\rangle_P \end{array} \right),
\]

where \( u_{01} \) and \( u_{11} \) are defined in Eqs. (5) and (6). Thus, the probability that we observe \(|g\rangle_\Lambda\) at the time \( t \) is given by

\[
P_g(\beta; t) = |u_{11}|\alpha, \tilde{\gamma}; \beta\rangle|^2.
\]

To calculate \( P_g(\beta; t) \) by the perturbation theory, we rewrite \(|\alpha, \tilde{\gamma}; \beta\rangle\) given by Eqs. (34), (35) and (36) as follows:

\[
|\alpha, \tilde{\gamma}; \beta\rangle = \exp[\hat{U}_\beta(\alpha a^\dagger + \tilde{\gamma} a^\dagger - \alpha a - \tilde{\gamma} \hat{a})]\hat{U}_\beta|0, \tilde{0}\rangle \\
= \hat{U}_\beta \exp[\alpha a^\dagger + \tilde{\gamma} a^\dagger - \alpha a - \tilde{\gamma} \hat{a}]|0, \tilde{0}\rangle \\
= \hat{U}_\beta \exp[\alpha(a^\dagger - a)] \exp[\tilde{\gamma}(\hat{a}^\dagger - \hat{a})]|0, \tilde{0}\rangle \\
= \hat{U}_\beta |\alpha\rangle |\tilde{\gamma}\rangle,
\]

where \(|\alpha\rangle\) and \(|\tilde{\gamma}\rangle\) represent the (zero-temperature) coherent states in \( \mathcal{H} \) and \( \tilde{\mathcal{H}} \), respectively. Thus, we can change Eq. (38) into the following form:

\[
P_g(\beta; t) = \langle \alpha | \langle \tilde{\gamma} | \hat{U}_\beta^\dagger u_{11}^\dagger u_{11} \hat{U}_\beta |\alpha\rangle |\tilde{\gamma}\rangle).
\]

Moreover, using Eqs. (5) and (6), we can rewrite the Hermitian operator \( u_{11}^\dagger u_{11} \) as

\[
u_{11}^\dagger u_{11} = \cos^2(|\kappa| t \sqrt{a^\dagger a + c}) + c \frac{\sin^2(|\kappa| t \sqrt{a^\dagger a + c})}{a^\dagger a + c},
\]

where \( c \) is defined in Eq. (21).

To compute \( P_g(\beta; t) \) approximately, we formulate the perturbation theory as follows. At first, we consider an arbitrary function \( g(x) \), which can be represented as a Taylor series about \( x = 0 \),

\[
g(x) = \sum_{n=0}^{\infty} g^{(n)} x^n,
\]

where \( -\infty < x < +\infty \) and \( g^{(n)} \) is an arbitrary complex number for all \( n \). In Eq. (42), \( g^{(n)} \) is equal to \( (d^n/dx^n)g(x)|_{x=0} \). This notation is slightly different from \( f^{(m)}(n) \) given by Eq. (17). Next, we evaluate \( \hat{U}_\beta^\dagger g(a^\dagger a + c) \hat{U}_\beta \) by the second-order perturbation theory at low temperature. We assume that the temperature \( T \) is quite low, so that \( \beta \epsilon = \epsilon (k_B T)^{-1} \gg 1 \). Then, from \( \beta \epsilon \gg 1 \) and Eq. (33), we understand \( 0 < \theta(\beta) \ll 1 \). Decomposing \( \tanh \theta(\beta) \) into the power series in the small parameter \( \theta(\beta) \),

\[
tanh \theta(\beta) = e^{-\beta \epsilon / 2} = \theta(\beta) - (1/3) \theta(\beta)^3 + O(\theta^5),
\]

where \( \epsilon \) is the energy of the system and \( k_B \) is the Boltzmann constant.
we obtain $\theta(\beta) \simeq e^{-\beta \epsilon/2}$ with neglecting terms of order $O(\theta^3)$. From now on, we consider the second-order perturbation theory with the parameter $\theta(\beta)$.

From the above discussions, we can compute $\hat{U}_\beta^\dagger g(\hat{a}^\dagger \hat{a} + c) \hat{U}_\beta$ approximately as follows:

\[
\hat{U}_\beta^\dagger g(\hat{a}^\dagger \hat{a} + c) \hat{U}_\beta = g(\hat{a}^\dagger \hat{a} + c) + \theta(\beta) \sum_{n=0}^{\infty} \frac{\theta^n}{n!} [\hat{a}^\dagger \hat{a}^\dagger - \hat{a} \hat{a}^\dagger, (\hat{a}^\dagger \hat{a} + c)^n] + O(\theta^3). \tag{44}
\]

(In the above derivation, we use the Baker-Hausdorff theorem [2].) In the following subsections, we calculate the first and second-order perturbation corrections, respectively.

### 4.2 The first-order correction

In this subsection, we calculate the first-order correction given by Eq. (44). To obtain the commutation relation of $\{\hat{a}^\dagger \hat{a}^\dagger - \hat{a} \hat{a}, (\hat{a}^\dagger \hat{a} + c)^n\}$, we use the following notation:

\[
\hat{A} = a^\dagger \hat{a}^\dagger - \hat{a} \hat{a}, \quad \hat{B} = \hat{a}^\dagger \hat{a} + c, \quad \hat{C} = \hat{a}^\dagger \hat{a} + \hat{a} \hat{a}.
\]  \tag{45}

Then, we obtain the following relations:

\[
[\hat{A}, \hat{B}] = -\hat{C}, \quad [\hat{C}, \hat{B}] = -\hat{A}. \tag{46}
\]

Using Eq. (46), we obtain

\[
\begin{align*}
[\hat{A}, \hat{B}] &= -\hat{C}, \\
[\hat{A}, \hat{B}^2] &= \hat{A} - 2\hat{B}\hat{C}, \\
[\hat{A}, \hat{B}^3] &= -\hat{C} + 3\hat{B}\hat{A} - 3\hat{B}^2\hat{C}, \\
[\hat{A}, \hat{B}^4] &= \hat{A} - 4\hat{B}\hat{C} + 6\hat{B}^2\hat{A} - 4\hat{B}^3\hat{C},
\end{align*}
\]  \tag{47}

and so on.

Here, we introduce the following operators:

\[
\hat{\mu} = \hat{a}^\dagger \hat{a}, \quad \hat{\nu} = \hat{a} \hat{a}.
\]  \tag{48}

Because

\[
\hat{A} = \hat{\mu} - \hat{\nu}, \quad \hat{C} = \hat{\mu} + \hat{\nu},
\]  \tag{49}

we can rewrite Eq. (47) as follows:

\[
\begin{align*}
[\hat{A}, \hat{B}] &= (\hat{B} - 1)\hat{\mu} - (1 + \hat{B})\hat{\nu} - \hat{B}\hat{A}, \\
[\hat{A}, \hat{B}^2] &= (\hat{B} - 1)^2\hat{\mu} - (1 + \hat{B})^2\hat{\nu} - \hat{B}^2\hat{A}, \\
[\hat{A}, \hat{B}^3] &= (\hat{B} - 1)^3\hat{\mu} - (1 + \hat{B})^3\hat{\nu} - \hat{B}^3\hat{A}, \\
[\hat{A}, \hat{B}^4] &= (\hat{B} - 1)^4\hat{\mu} - (1 + \hat{B})^4\hat{\nu} - \hat{B}^4\hat{A}.
\end{align*}
\]  \tag{50}
Thus, we obtain

$$[\hat{A}, \hat{B}^n] = (\hat{B} - 1)^n\hat{\mu} - (\hat{B} + 1)^n\hat{\nu} - \hat{B}^n\hat{A} \quad \text{for } n = 1, 2, 3, \ldots$$  \hspace{1cm} (51)

From Eqs. (50) and (51), we obtain the first-order correction as

$$\theta(\beta) \sum_{n=0}^{\infty} g^{(n)} [a^\dagger a^\dagger - a\hat{a}, (a^\dagger a + c)^n]$$

$$= \theta(\beta) [g(\hat{B} - 1)\hat{\mu} - g(\hat{B} + 1)\hat{\nu} - g(\hat{B})\hat{A}].$$  \hspace{1cm} (52)

### 4.3 The second-order correction

In this subsection, we calculate the second-order correction given by Eq. (44). From Eq. (51), we obtain

$$[\hat{A}, [\hat{A}, \hat{B}^n]] = \hat{R}_n + \hat{S}_n \quad \text{for } n = 1, 2, 3, \ldots$$  \hspace{1cm} (53)

where

$$\hat{R}_n = [\hat{A}, (\hat{B} - 1)^n\hat{\mu} - [\hat{A}, (\hat{B} + 1)^n]\hat{\nu} - [\hat{A}, \hat{B}^n]\hat{A},$$

$$\hat{S}_n = (\hat{B} - 1)^n[\hat{A}, \hat{\mu}] + (\hat{B} + 1)^n[\hat{A}, \hat{\nu}].$$  \hspace{1cm} (54)

Thus, we can divide the second-order correction up into the following two parts:

$$\frac{1}{2} \theta(\beta)^2 \sum_{n=0}^{\infty} g^{(n)} \hat{R}_n + \frac{1}{2} \theta(\beta)^2 \sum_{n=0}^{\infty} g^{(n)} \hat{S}_n.$$  \hspace{1cm} (55)

First, we evaluate the part including \{\hat{S}_n\} in Eq. (55). Because of Eqs. (48) and (49), we obtain

$$[\hat{A}, \hat{\mu}] = [\hat{\nu}, \hat{\nu}] = -\hat{D},$$  \hspace{1cm} (56)

where

$$\hat{D} = a^\dagger a + \hat{a}^\dagger\hat{a} + 1.$$  \hspace{1cm} (57)

[We note that the above \( \hat{D} \) is different from \( D \) defined in Eq. (6).] Thus, we can write down \( \hat{S}_n \) as

$$\hat{S}_n = -\left( (\hat{B} - 1)^n + (\hat{B} + 1)^n \right) \hat{D}.$$  \hspace{1cm} (58)

Hence, we can rewrite the term that includes \{\hat{S}_n\} in Eq. (55) as

$$-\frac{1}{2} \theta(\beta)^2 \left( g(\hat{B} - 1) + g(\hat{B} + 1) \right) \hat{D}.$$  \hspace{1cm} (59)

Second, we evaluate the part including \{\hat{R}_n\} in Eq. (55). From Eq. (54), we can confirm \( \hat{R}_0 = \hat{R}_1 = 0 \) at ease. After slightly tough calculations, we obtain

$$\hat{R}_2 = -2[\hat{A}, \hat{B}]\hat{C},$$

$$\hat{R}_3 = -3[\hat{A}, \hat{B}^2]\hat{C} + 3[\hat{A}, \hat{B}]\hat{A},$$

$$\hat{R}_4 = -4[\hat{A}, \hat{B}^3]\hat{C} + 6[\hat{A}, \hat{B}^2]\hat{A} - 4[\hat{A}, \hat{B}]\hat{C},$$  \hspace{1cm} (60)
and so on. Thus, we can write down the part that includes \( \{ \hat{R}_n \} \) in Eq. (51) as

\[
\frac{1}{2} \theta(\beta)^2 \left( -\sum_{n=0}^{\infty} (n+2) g^{(n+2)}[\hat{A}, \hat{B}^{n+1}] \hat{C} \\
+ \frac{1}{2} \sum_{n=0}^{\infty} (n+2)(n+3) g^{(n+3)}[\hat{A}, \hat{B}^{n+1}] \hat{A} \\
- \frac{1}{6} \sum_{n=0}^{\infty} (n+2)(n+3)(n+4) g^{(n+4)}[\hat{A}, \hat{B}^{n+1}] \hat{C} + ... \right)
\]

= \frac{1}{2} \theta(\beta)^2 \left( -\hat{F}_1 \hat{C} + \frac{1}{2} \hat{F}_2 \hat{A} - \frac{1}{3!} \hat{F}_3 \hat{C} + ... \right)

= \frac{1}{2} \theta(\beta)^2 \left( \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \hat{F}_n \hat{\mu} - \sum_{n=0}^{\infty} \frac{1}{n!} \hat{F}_n \hat{\nu} \right)

- \frac{1}{2} \theta(\beta)^2 \left( g(\hat{B} - 1) \hat{\mu} - g(\hat{B} + 1) \hat{\nu} - g(\hat{B}) \hat{A} \right) \hat{A},
\]

where

\[
\hat{F}_n = \left. \frac{d^n}{dx^n} g(x) \right|_{x=B} \hat{\mu} - \left. \frac{d^n}{dx^n} g(x) \right|_{x=B+1} \hat{\nu} - \left. \frac{d^n}{dx^n} g(x) \right|_{x=B} \hat{A}.
\]

[In the above derivation, we use Eq. (51) in an effective manner. The form of \( \hat{F}_n \) in Eq. (62) reflects Eq. (51).]

Here, we introduce the following operators:

\[
e^{\pm d/dx} g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{d^n}{dx^n} g(x).
\]

Using the symbols \( e^{\pm d/dx} \), we can rewrite the first term of Eq. (61) as

\[
\frac{1}{2} \theta(\beta)^2 \left( \left[ e^{-d/dx} g(\hat{B} - 1) \hat{\mu} - e^{-d/dx} g(\hat{B} + 1) \hat{\nu} - e^{-d/dx} g(\hat{B}) \hat{A} \right] \hat{\mu} \right.

\[
\left. - \left[ e^{d/dx} g(\hat{B} - 1) \hat{\mu} - e^{d/dx} g(\hat{B} + 1) \hat{\nu} - e^{d/dx} g(\hat{B}) \hat{A} \right] \hat{\nu} \right).
\]

Then, we apply the following technique to Eq. (64):

\[
e^{\pm d/dx} g(\hat{X}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{d^n}{dx^n} g(x) \bigg|_{x=\hat{X}} = g(\hat{X} \pm 1),
\]

where \( \hat{X} \) is an arbitrary operator. Thus, we can rewrite Eq. (64) as

\[
\frac{1}{2} \theta(\beta)^2 \left[ g(\hat{B} - 2) \hat{\mu} - g(\hat{B}) \hat{\mu} - g(\hat{B} - 1) \hat{A} \right] \hat{\mu}

\[
- \left[ g(\hat{B}) \hat{\mu} - g(\hat{B} + 1) \hat{\nu} - g(\hat{B} + 2) \hat{A} \right] \hat{\nu}.
\]

Finally, from Eqs. (55), (59), (61) and (66), we obtain the second-order correction as

\[
(1/2) \theta(\beta)^2 \sum_{n=0}^{\infty} g^{(n)}[a^\dagger \tilde{a}^\dagger - a\tilde{a}, [a^\dagger \tilde{a}^\dagger - a\tilde{a}, (a^\dagger a + c)^n]]
\]

\[
= (1/2) \theta(\beta)^2 [g(\hat{B} - 2) \hat{\mu}^2 + g(\hat{B} - 1)(-\hat{D} - \hat{\mu} \hat{A} - \hat{A} \hat{\mu})

+ g(\hat{B})(\hat{A}^2 - \hat{\nu} \hat{\mu} - \hat{\mu} \hat{\nu}) + g(\hat{B} + 1)(-\hat{D} + \hat{\nu} \hat{A} + \hat{A} \hat{\nu}) + g(\hat{B} + 2) \hat{\nu}^2].
\]
4.4 The atomic population inversion up to the second-order correction

In this subsection, we compute $P_g(\beta; t)$ defined in Eq. (40) up to the second-order perturbation. According to Eq. (41), we define $g(x)$ as follows:

$$g(x + c) = \cos^2(|\kappa| t \sqrt{x + c}) + \frac{c}{x + c} \sin^2(|\kappa| t \sqrt{x + c}).$$

(68)

From Eqs. (40), (44), (52) and (67), we obtain $P_g(\beta; t)$ as

$$P_g(\beta; t) = P_g(t) + \theta(\beta)P_g^{(1)}(\beta; t) + (1/2)\theta(\beta)^2P_g^{(2)}(\beta; t) + O(\theta^3),$$

where

$$P_g^{(1)}(\beta; t) = \langle \alpha | \tilde{\gamma} | [g(\hat{B} - 1)\hat{\mu} - g(\hat{B} + 1)\hat{\nu} - g(\hat{B})\hat{A}] | \alpha \rangle | \tilde{\gamma} \rangle,$$

$$P_g^{(2)}(\beta; t) = \langle \alpha | \tilde{\gamma} | [g(\hat{B} - 2)\hat{\mu}^2 + g(\hat{B} - 1)(-\hat{D} - \hat{\mu}\hat{A} - \hat{A}\hat{\mu}) + g(\hat{B} + 1)(-\hat{D} + \hat{\nu}\hat{A} + \hat{A}\hat{\nu}) + g(\hat{B} + 2)\hat{\nu}^2] | \alpha \rangle | \tilde{\gamma} \rangle.$$ (70)

First, we compute $P_g^{(1)}(\beta; t)$. Remembering that $|\alpha\rangle$ and $|\tilde{\gamma}\rangle$ are coherent states on $\mathcal{H}$ and $\hat{\mathcal{H}}$ respectively, we can write down $P_g^{(1)}(\beta; t)$ as follows:

$$P_g^{(1)}(\beta; t) = \langle \alpha | \tilde{\gamma} | \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \exp\left(-\frac{\alpha^2 + \tilde{\gamma}^2}{2}\right) \frac{\alpha^n \tilde{\gamma}^m}{\sqrt{n!} \sqrt{m!}}
\times \left([g(a\dagger a + c - 1) - g(a\dagger a + c)]a\dagger a^\dagger
- [g(a\dagger a + c + 1) - g(a\dagger a + c)]a\dagger a\right)|n\rangle |m\rangle
\times \left(\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\alpha^{2n+1} \tilde{\gamma}^{2m+1}}{n! \sqrt{m!}} [g(n + c) - g(n + c + 1)]
- \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\alpha^{2n-1} \tilde{\gamma}^{2m-1}}{(n-1)! (m-1)!} [g(n + c) - g(n + c - 1)]\right)\right)$$

$$= 2\alpha\tilde{\gamma} \exp\left(-\alpha^2 \sum_{n=0}^{\infty} \frac{\alpha^{2n}}{n!} [g(n + c) - g(n + c + 1)]\right)$$

$$= 2\alpha\tilde{\gamma} [P_g(t) - Q_g^{(1)}(t)],$$

(71)

where

$$Q_g^{(1)}(t) = \exp(-\alpha^2) \sum_{n=0}^{\infty} \frac{\alpha^{2n}}{n!} [\cos^2(\sqrt{c + n + l}|\kappa| t) + \frac{c}{c + n + l} \sin^2(\sqrt{c + n + l}|\kappa| t)]$$

$$= \exp(-\alpha^2) |\alpha|^{-2(c+l)} \sum_{n=0}^{\infty} \frac{|\alpha|^{2(n+c+l)}}{n!} [\cos^2(\sqrt{c + n + l}|\kappa| t)
+ \frac{c}{c + n + l} \sin^2(\sqrt{c + n + l}|\kappa| t)]$$

$$= \exp(-\alpha^2) \left[\frac{1}{2} [\cos^2(|\kappa| t \sqrt{c + l}) + \frac{c}{c + l} \sin^2(|\kappa| t \sqrt{c + l})]
+ I_1^{(l)}(t) - 2I_2^{(l)}(t)\right]$$

for $l = 0, 1, 2, ...$, (72)
and \( I_1^{(t)}(t) \) and \( I_2^{(t)}(t) \) are defined in Eq. (24). We note that \( Q_\alpha^{(0)}(t) = P_\alpha(t) \).

Second, we compute \( P_\alpha^{(2)}(\beta; t) \). Before obtaining an explicit form of \( P_\alpha^{(2)}(\beta; t) \), we prepare the following relations:

\[
\begin{align*}
   \hat{\mu}^2|n\rangle|\tilde{m}\rangle &= \sqrt{(n + 1)(n + 2)(m + 1)(m + 2)|n + 2\rangle|m + 2\rangle}, \\
   (-\hat{D} - \hat{\mu}A - \hat{\mu}\hat{v})|n\rangle|\tilde{m}\rangle &= 2nm|n\rangle|\tilde{m}\rangle - 2\sqrt{(n + 1)(n + 2)(m + 1)(m + 2)|n + 2\rangle|m + 2\rangle}, \\
   (\hat{A}^2 - \hat{\mu}A + \hat{\mu}\hat{v})|n\rangle|\tilde{m}\rangle &= 2nm|n\rangle|\tilde{m}\rangle - 2\sqrt{n(n - 1)m(m - 1)|n - 2\rangle|m - 2\rangle}, \\
   \hat{v}^2|n\rangle|\tilde{m}\rangle &= \sqrt{n(n - 1)m(m - 1)|n - 2\rangle|m - 2\rangle} \quad \text{for } n, m \in \{0, 1, 2, \ldots\}. \quad (73)
\end{align*}
\]

Using Eq. (73), we can write down \( P_\alpha^{(2)}(\beta; t) \) as

\[
P_\alpha^{(2)}(\beta; t) = \frac{1}{2}\theta(\beta)^2 \exp(-\frac{\alpha^2 + \tilde{\gamma}^2}{2}) \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\alpha^n \tilde{\gamma}^m}{n! \sqrt{n} \sqrt{m}!} \times \left( [g(n + c) - 2g(n + c + 1) + g(n + c + 2)] \right. \\
\times \sqrt{(n + 1)(n + 2)(m + 1)(m + 2)|n + 2\rangle|m + 2\rangle} + [2nmg(n + c - 1) - 2(2nm + n + m + 1)g(n + c) \\
+ 2nmg(n + c + 1)]|n\rangle|\tilde{m}\rangle + [g(n + c - 2) - 2g(n + c - 1) + g(n + c)] \\
\times \sqrt{n(n - 1)m(m - 1)|n - 2\rangle|m - 2\rangle} \Bigg) \\
= \frac{1}{2}\theta(\beta)^2 \exp(-\alpha^2 + \tilde{\gamma}^2) \sum_{n=0}^{\infty} \frac{\alpha^{2n}}{n!} \times \frac{\tilde{\gamma}^{2m+2}}{m!} [g(n + c) - 2g(n + c + 1) + g(n + c + 2)] \\
\times \frac{\alpha^{2n}}{n!} \tilde{\gamma}^{2m} [g(n + c) - 2g(n + c + 1) + g(n + c + 2)] \\
+ \frac{\alpha^{2n}}{n!} \tilde{\gamma}^{2m} [2nmg(n + c + 1) - (2nm + n + m + 1)g(n + c) \\
+ nmg(n + c - 1)] \\
\times \frac{\alpha^{2n-2}}{(n - 2)!} \tilde{\gamma}^{2m-2} [g(n + c) - 2g(n + c - 1) + g(n + c - 2)] \\
= \frac{1}{2}\theta(\beta)^2 \exp(-\alpha^2) \sum_{n=0}^{\infty} \frac{\alpha^{2n}}{n!} \times (4\tilde{\gamma}^2 \alpha^2 [g(n + c) - 2g(n + c + 1) + g(n + c + 2)])
\]

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Figure 2: The graphs of $\langle \sigma_z(t) \rangle$ given by Eq. (11) and $J_1(t)$ given by Eq. (29) for $t \in [0, 8\pi]$ and $\alpha = 4$. (We consider the resonant case, so that we assume $\Delta \omega = 0$ and $\kappa = 1$.) A thick solid curve represents $J_1(t)$ and a thin dashed curve represents $\langle \sigma_z(t) \rangle$. In the numerical calculation of $\langle \sigma_z(t) \rangle$ given by Eq. (11), the summation of the index $n$ is carried out up to $n = 100$. In the numerical calculation of $J_1(t)$ given by Eq. (29), the integral $\int_0^\infty dx$ is replaced with $\int_0^{100} dx$. The interval of the numerical integration $x \in [0, 100]$ is divided into $10^6$ steps ($\Delta x = 1.0 \times 10^{-4}$) and we apply Simpson’s rule. To obtain the variation of $J_1(t)$ against the time $t$, we divide the interval $t \in [0, 8\pi]$ into 4000 steps ($\Delta t = 2\pi \times 10^{-3}$) and we estimate $J_1(t)$ at each time step. Looking at this figure, we notice the following facts. In the graph of $J_1(t)$, we can observe only the initial collapse and we cannot observe the revival of the Rabi oscillations. $\langle \sigma_z(t) \rangle$ starts to show the revival of the Rabi oscillations around at $t = 5\pi$. By contrast, $J_1(t)$ is nearly equal to zero after $t = \pi$.

\[
-2\alpha^2 g(n + c + 1) - 2\tilde{\gamma}^2 g(n + c) - 2g(n + c)
= \frac{1}{2} \theta(\beta)^2 [2(2\alpha^2 \tilde{\gamma}^2 - \tilde{\gamma}^2 - 1) P_g(t) - 2\alpha^2 (4\tilde{\gamma}^2 + 1) Q_g^{(1)}(t) \\
+ 4\alpha^2 \tilde{\gamma}^2 Q_g^{(2)}(t)].
\]  

(74)

5 Properties of the integrals that form the atomic population inversion

In this section, numerically evaluating the integrals that form the atomic population inversion, we examine their physical meanings.

First, we consider the case where the electromagnetic field is resonant with the atom at zero-temperature, so that we put $\Delta \omega = 0$ and $\kappa = 1$. We show graphs of $J_1(t)$ defined in Eq. (29) and $\langle \sigma_z(t) \rangle$ given by Eq. (11) in Fig. 2. We show graphs of $J_2(t)$ defined in Eq. (29) and $\langle \sigma_z(t) \rangle$ given by Eq. (11) in Fig. 3. In these estimations, we put $\alpha = 4$. Thus, we can neglect the first term of the right-hand side of Eq. (28) $[-(1/2) \exp(-\alpha^2) = -5.63 \times 10^{-8}]$.

Looking at Fig. 2, we can conclude that $J_1(t)$ only represents the initial collapse (the
Figure 3: The graphs of $\langle \sigma_z(t) \rangle$ given by Eq. (11) and $J_2(t)$ given by Eq. (29) for $t \in [0, 8\pi]$ and $\alpha = 4$. (We consider the resonant case, so that we assume $\Delta \omega = 0$ and $\kappa = 1$.) A thick solid curve represents $J_2(t)$ and a thin dashed curve represents $\langle \sigma_z(t) \rangle$. In the numerical calculation of $\langle \sigma_z(t) \rangle$ given by Eq. (11), the summation of the index $n$ is carried out up to $n = 100$. In the numerical calculation of $J_2(t)$ given by Eq. (29), the integral $\int_0^\infty dy$ is replaced with $\int_0^{100} dy$. The interval of the numerical integration $y \in [0, 100]$ is divided into $10^6$ steps ($\Delta y = 1.0 \times 10^{-4}$) and we apply Bode’s rule. To obtain the variation of $J_2(t)$ against the time $t$, we divide the interval $t \in [0, 8\pi]$ into 4000 steps ($\Delta t = 2\pi \times 10^{-3}$) and we estimate $J_2(t)$ at each time step. Looking at this figure, we notice the following facts. In the graph of $J_2(t)$, we can observe only the revival and we cannot observe the initial collapse of the Rabi oscillations. $\langle \sigma_z(t) \rangle$ shows the initial collapse from $t = 0$ until around $t = \pi$. By contrast, $J_2(t)$ is nearly equal to zero from $t = 0$ to $t = 4\pi$.}
and we apply Simpson's rule. To obtain the variation of \( t \) the time revival of the Rabi oscillations. Hence, we can expect that \( J - \) following facts. In the graph of \( \langle \sigma_z(t) \rangle \) given by Eq. (24) for \( t \in [0, 8\pi] \), \( \alpha = 4 \), \( \Delta \omega = 4 \), \( \kappa = 1 \) and \( c = 4 \). (We consider the off-resonant case.) A thick solid curve represents \( [1 - 2 \exp(-\alpha^2)I_1(t)] \) and a thin dashed curve represents \( \langle \sigma_z(t) \rangle \). In the numerical calculation of \( \langle \sigma_z(t) \rangle \) given by Eqs. (9) and (10), the summation of the index \( n \) is carried out up to \( n = 100 \). In the numerical calculation of \( [1 - 2 \exp(-\alpha^2)I_1(t)] \) given by Eq. (24), the integral \( \int_0^\infty dx \) is replaced with \( \int_0^{100} dx \). The interval of the numerical integration \( x \in [0, 100] \) is divided into 100 steps (\( \Delta x = 1.0 \times 10^{-4} \)) and we apply Simpson’s rule. To obtain the variation of \( [1 - 2 \exp(-\alpha^2)I_1(t)] \) against the time \( t \), we divide the interval \( t \in [0, 8\pi] \) into 4000 steps (\( \Delta t = 2\pi \times 10^{-3} \)) and we estimate \( [1 - 2 \exp(-\alpha^2)I_1(t)] \) at each time step. Looking at this figure, we notice the following facts. In the graph of \( [1 - 2 \exp(-\alpha^2)I_1(t)] \), we can observe only the initial collapse and we cannot observe the revival of the Rabi oscillations. \( \langle \sigma_z(t) \rangle \) starts to show the revival of the Rabi oscillations around at \( t = 6\pi \). By contrast, \( [1 - 2 \exp(-\alpha^2)I_1(t)] \) is nearly equal to Const. given by Eq. (25) after \( t = \pi \).

Figure 4: The graphs of \( \langle \sigma_z(t) \rangle \) given by Eqs. (9) and (11) and \( [1 - 2 \exp(-\alpha^2)I_1(t)] \) given by Eq. (24) for \( t \in [0, 8\pi] \), \( \alpha = 4 \), \( \Delta \omega = 4 \), \( \kappa = 1 \) and \( c = 4 \). (We consider the off-resonant case.) A thick solid curve represents \( [1 - 2 \exp(-\alpha^2)I_1(t)] \) and a thin dashed curve represents \( \langle \sigma_z(t) \rangle \). In the numerical calculation of \( \langle \sigma_z(t) \rangle \) given by Eqs. (9) and (10), the summation of the index \( n \) is carried out up to \( n = 100 \). In the numerical calculation of \( [1 - 2 \exp(-\alpha^2)I_1(t)] \) given by Eq. (24), the integral \( \int_0^\infty dx \) is replaced with \( \int_0^{100} dx \). The interval of the numerical integration \( x \in [0, 100] \) is divided into 100 steps (\( \Delta x = 1.0 \times 10^{-4} \)) and we apply Simpson’s rule. To obtain the variation of \( [1 - 2 \exp(-\alpha^2)I_1(t)] \) against the time \( t \), we divide the interval \( t \in [0, 8\pi] \) into 4000 steps (\( \Delta t = 2\pi \times 10^{-3} \)) and we estimate \( [1 - 2 \exp(-\alpha^2)I_1(t)] \) at each time step. Looking at this figure, we notice the following facts. In the graph of \( [1 - 2 \exp(-\alpha^2)I_1(t)] \), we can observe only the initial collapse and we cannot observe the revival of the Rabi oscillations. \( \langle \sigma_z(t) \rangle \) starts to show the revival of the Rabi oscillations around at \( t = 6\pi \). By contrast, \( [1 - 2 \exp(-\alpha^2)I_1(t)] \) is nearly equal to Const. given by Eq. (25) after \( t = \pi \).

semi-classical limit) and it does not represent the revival (the quantum correction) of the Rabi oscillations. By contrast, looking at Fig. 3, we can conclude that \( J_2(t) \) only represents the revival and it does not represent the initial collapse of the Rabi oscillations. In our numerical calculations, we obtain \(|\langle \sigma_z(t) \rangle - J_1(t)\rangle \leq 8.0847 \times 10^{-4} \) for \( t \in [0, 4\pi] \) and \(|\langle \sigma_z(t) \rangle - J_2(t)\rangle \leq 5.6377 \times 10^{-8} \) for \( t \in [4\pi, 8\pi] \).

Here, we pay attention to the representation of \( J_1(t) \) given in Eq. (29). If we let \( t \) be a large value (\( t \gg 1 \)), the trigonometric function \( \cos(2\sqrt{x}t) \) included in the integrand oscillates intensely and rapidly for the small variation of \( x \). Thus, the integral \( \int_0^\infty dx \) of \( J_1(t) \) converges on zero for \( t \gg 1 \). Therefore, we can expect that \( J_1(t) \) never causes the revival of the Rabi oscillations. Hence, we can expect that \( J_2(t) \) lets the revival of the Rabi oscillations happen in the range of \( t \gg 1 \).

Second, we consider the case where the electromagnetic field is off-resonant with the atom at zero-temperature for estimating the effects of detuning. We show graphs of \( [1 - 2 \exp(-\alpha^2)I_1(t)] \) defined in Eq. (24) and \( \langle \sigma_z(t) \rangle \) given by Eqs. (9) and (10) in Fig. 4. We show graphs of \( \text{Const.} + 4 \exp(-\alpha^2)I_2(t) \) defined in Eq. (24) and \( \langle \sigma_z(t) \rangle \) given by
Figure 5: The graphs of $\langle \sigma_z(t) \rangle$ given by Eqs. (9) and (10) and $[\text{Const. } + 4 \exp(-\alpha^2) I_2^{(0)}(t)]$ given by Eqs. (24) and (75) for $t \in [0, 8\pi]$, $\alpha = 4$, $\Delta \omega = 4$, $\kappa = 1$ and $c = 4$. (We consider the off-resonant case.) A thick solid curve represents $[\text{Const. } + 4 \exp(-\alpha^2) I_2^{(0)}(t)]$ and a thin dashed curve represents $\langle \sigma_z(t) \rangle$. In the numerical calculation of $\langle \sigma_z(t) \rangle$ given by Eqs. (9) and (10), the summation of the index $n$ is carried out up to $n = 100$. In the numerical calculation of $[\text{Const. } + 4 \exp(-\alpha^2) I_2^{(0)}(t)]$ given by Eqs. (24) and (75), the integral $\int_0^\infty dy$ is replaced with $\int_0^{100} dy$. The interval of the numerical integration $y \in [0, 100]$ is divided into $10^6$ steps ($\Delta y = 1.0 \times 10^{-4}$) and we apply Bode’s rule. To obtain the variation of $[\text{Const. } + 4 \exp(-\alpha^2) I_2^{(0)}(t)]$ against the time $t$, we divide the interval $t \in [0, 8\pi]$ into 4000 steps ($\Delta t = 2\pi \times 10^{-3}$) and we estimate $[\text{Const. } + 4 \exp(-\alpha^2) I_2^{(0)}(t)]$ at each time step. Looking at this figure, we notice the following facts. In the graph of $[\text{Const. } + 4 \exp(-\alpha^2) I_2^{(0)}(t)]$, we can observe only the revival and we cannot observe the initial collapse of the Rabi oscillations. $\langle \sigma_z(t) \rangle$ shows the initial collapse from $t = 0$ until around $t = \pi$. By contrast, $[\text{Const. } + 4 \exp(-\alpha^2) I_2^{(0)}(t)]$ is nearly equal to Const. given by Eq. (75) from $t = 0$ to $t = 5\pi$.}
Similarly, from Eqs. (23), (72) and (74), we can describe the second-order correction that an amplitude of the atomic population inversion detuning. Shown in Eqs. (23), (71) and (72), we can describe the first-order correction of $I$ given in the previous paragraphs, we can conclude that $\Delta \omega = 0$, $\kappa = 1$, $c = 0$, $\alpha = 4$, $\gamma = 1$ and $\theta(\beta) = 1/40$.

Eqs. (9) and (10) in Fig. 5, where $\Delta \omega = 0$, $\kappa = 1$, $c = 0$, $\alpha = 4$, $\gamma = 1$ and $\theta(\beta) = 1/40$. Thus, replacing $\sin^2(\sqrt{x + c + l|\kappa t|})$ with $\sin^2(\sqrt{x + c + l|\kappa t|})$ included in the integrand of $I_1(\beta; t)$ with their time average $1/2$. Thus, we obtain $\text{Const.} = -0.2086$.

Looking at Figs. 4 and 5, we can conclude that $I_1(\beta; t)$ only represents the initial collapse and $I_2(\beta; t)$ only represents the revival. The reason why $I_1(\beta; t)$ given by Eq. (24) only shows the initial collapse is as follows. If we let $\gamma$ be a large value ($\gamma \gg 1$), we can replace the squares of trigonometric functions $[\cos^2(\sqrt{x + c + l|\kappa t|})$ and $\sin^2(\sqrt{x + c + l|\kappa t|})] included in the integrand of $I_1(\beta; t)$ with their time average $1/2$. Thus, we obtain $\text{Const.} = -0.2086$.

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Figure 7: The graph of $P_g^{(2)}(\beta; t)$ given by Eqs. (72) and (74) for $t \in [0, 8\pi]$ with putting $\Delta \omega = 0$, $\kappa = 1$, $c = 0$, $\alpha = 4$, $\tilde{\gamma} = 1$ and $\theta(\beta) = 1/40$.

effective.

Looking at the graphs of Figs. 6 and 7 and the graphs of Figs. 2 and 3, we understand that the effect of $P_g^{(1)}(\beta; t)$ and $P_g^{(2)}(\beta; t)$ on $P_g(t)$ is quite small. Hence, we can conclude that the collapse and the revival of the Rabi oscillations is robust against the effects of low temperature in the region where the second-order perturbation theory is effective.

6 Discussions

In this paper, we separate the atomic population inversion of the Jaynes-Cummings model into two integrals using the Abel-Plana formula. By numerical calculations, we show that the first integral represents the initial collapse (the semi-classical limit) and the second integral represents the revival (the quantum correction). Moreover, we examine the time-evolution of the JCM with the initial thermal coherent state for the cavity mode at low temperature by the second-order perturbation theory. We describe the first and second-order corrections as sums of integrals, using the Abel-Plana formula.

The Abel-Plana formula and its generalized versions are often made use of for the calculations of the Casimir energies in different configurations [23, 24, 25]. The author thinks that the Abel-Plana formula has a wide application in the field of the quantum optics.

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A The derivation of the Abel-Plana formula

In this section, we give details of the derivation of the summation formula described in Eq. (14). (This formula is introduced in Ref. [18] without precise derivation. The author of Ref. [18] only mentions a suggestion for proving it.)

At first, we assume \( \phi(z) \) to be analytical and bounded for all complex values of \( z \) such that \( n_1 \leq \Re(z) \leq n_2 \), where \( n_1 \) and \( n_2 \) are certain integers. Then, \( \phi(z)/(e^{-2\pi iz} - 1) \) is analytical and bounded \( \forall z \) such that \( n_1 \leq \Re(z) \leq n_2 \) except for \( z = n_1, n_1 + 1, n_1 + 2, ..., n_2 - 1, n_2 \). Hence, if we think about the closed contour \( C_1 \) shown in Fig. 8, \( \phi(z)/(e^{-2\pi iz} - 1) \) is analytical and bounded inside \( C_1 \), so that

\[
\oint_{C_1} \frac{\phi(z)}{e^{-2\pi iz} - 1} \, dz = 0. \tag{76}
\]

(This is the Cauchy integral theorem.)

Similarly, \( \phi(z)/(e^{2\pi iz} - 1) \) is analytical and bounded \( \forall z \) such that \( n_1 \leq \Re(z) \leq n_2 \) except for \( z = n_1, n_1 + 1, n_1 + 2, ..., n_2 - 1, n_2 \). Hence, if we think about the closed contour \( C_2 \) shown in Fig. 9, \( \phi(z)/(e^{2\pi iz} - 1) \) is analytical and bounded inside \( C_2 \), so that

\[
\oint_{C_2} \frac{\phi(z)}{e^{2\pi iz} - 1} \, dz = 0. \tag{77}
\]

Next, we consider the following integrals:

\[
I(n) = \int_{C_+} \frac{\phi(z)}{e^{-2\pi iz} - 1} \, dz + \int_{C_-} \frac{\phi(z)}{e^{2\pi iz} - 1} \, dz, \tag{78}
\]

where \( C_+ \) and \( C_- \) are paths on the complex plane shown in Fig. 10. In Fig. 10, \( n \) represents an integer such that \( n_1 < n < n_2 \) and \( \epsilon \) is a small positive infinitesimal quantity. We can
Figure 9: The closed contour $C_2$ defined on the complex plane.

Figure 10: The paths $C_+$ and $C_-$ defined on the complex plane. Describing $\bar{C}_+$ as an opposite path of $C_+$, $\bar{C}_+$ and $C_-$ form a closed contour.
Figure 11: The paths $C_{R+}$ and $C_{R-}$ defined on the complex plane.

rewrite $I(n)$ defined in Eq. (78) as

$$I(n) = -\int_{C_{R+}} \frac{\phi(z)}{e^{-2\pi i z} - 1} dz + \int_{C_{R-}} \frac{\phi(z)}{e^{2\pi i z} - 1} dz$$

$$= \int_{C_{R+}} \frac{e^{2\pi i z}\phi(z)}{e^{2\pi i z} - 1} dz + \int_{C_{R-}} \frac{\phi(z)}{e^{2\pi i z} - 1} dz$$

$$= \oint_{\overline{C}_{R+} + C_{R-}} \Phi(z) e^{2\pi i z} - 1 dz, \quad (79)$$

where $\overline{C}_{R+}$ is an opposite path of $C_{R+}$ and $\Phi(z)$ is given by

$$\Phi(z) = \begin{cases} e^{2\pi i z}\phi(z) & \text{for } \text{Im}(z) \geq 0 \\ \phi(z) & \text{for } \text{Im}(z) < 0 \end{cases}. \quad (80)$$

Inside of the closed contour $\overline{C}_{R+} + C_{R-}$, $\Phi(z)/(e^{2\pi i z} - 1)$ has only one pole, $z = n$.

If we write $z = n + u$, we can expand $\Phi(z)/(e^{2\pi i z} - 1)$ around $z = n$ as

$$\frac{\Phi(n + u)}{e^{2\pi i (n+u)} - 1} = \frac{\phi(n)}{2\pi i u} + P(u), \quad (81)$$

where $P(u)$ is a power series that includes only terms of nonnegative degrees, so that the residue is given by

$$\text{Res}[\Phi(z)/(e^{2\pi i z} - 1), n] = \phi(n), \quad (82)$$

and we obtain

$$I(n) = \phi(n). \quad (83)$$

Furthermore, we think about the following integrals:

$$I_{R}(n_1) = \int_{C_{R+}} \frac{\phi(z)}{e^{-2\pi i z} - 1} dz + \int_{C_{R-}} \frac{\phi(z)}{e^{2\pi i z} - 1} dz, \quad (84)$$

where $C_{R+}$ and $C_{R-}$ are paths on the complex plane shown in Fig. 11. In Fig. 11, $\epsilon$ is a small positive infinitesimal quantity. [The index R in $I_{R}(n_1)$, $C_{R+}$ and $C_{R-}$ implies that
Figure 12: The paths $C_{L+}$ and $C_{L-}$ defined on the complex plane.

$C_{R+}$ and $C_{R-}$ are right halves of $C_+$ and $C_-$, respectively. In the limit of $\epsilon \to +0$, we obtain the following relation:

$$\lim_{\epsilon \to +0} I_{R}(n_1) = \frac{1}{2} \lim_{\epsilon \to +0} I(n_1) = \frac{1}{2} \phi(n_1). \quad (85)$$

In a similar way, we obtain the following relation:

$$I_L(n_1) = \int_{C_{L+}} \frac{\phi(z)}{e^{-2\pi i z} - 1} dz + \int_{C_{L-}} \frac{\phi(z)}{e^{2\pi i z} - 1} dz, \quad (86)$$

where $C_{L+}$ and $C_{L-}$ are paths on the complex plane shown in Fig. 12 and

$$\lim_{\epsilon \to +0} I_L(n_2) = \frac{1}{2} \lim_{\epsilon \to +0} I(n_2) = \frac{1}{2} \phi(n_2). \quad (87)$$

From Eqs. (76), (77), (83), (85) and (87), we obtain

$$\frac{1}{2} \phi(n_1) + \sum_{n=n_1+1}^{n_2-1} \phi(n) + \frac{1}{2} \phi(n_2) + (\int_{n_1+\epsilon}^{n_1+1-\epsilon} + \int_{n_1+1+\epsilon}^{n_1+2-\epsilon} + \ldots + \int_{n_2-1+\epsilon}^{n_2-\epsilon}) \left[ \frac{\phi(x)}{e^{-2\pi i x} - 1} + \frac{\phi(x)}{e^{2\pi i x} - 1} \right] dx$$

$$+ i \int \phi(n_2 + iy) \frac{dy}{e^{2\pi y} - 1} - \int \phi(n_1 + iy) \frac{dx}{e^{-2\pi i(x+i\infty)} - 1}$$

$$- i \int \phi(n_2 - iy) \frac{dx}{e^{2\pi y} - 1} + i \int \phi(n_1 - iy) \frac{dy}{e^{-2\pi i(x-i\infty)} - 1} = 0. \quad (88)$$

Looking at Eq. (88), we notice the following facts. Because $\phi(z)$ is bounded for all $z$ such that $n_1 \leq \text{Re}(z) \leq n_2$, we obtain

$$\int_{n_1}^{n_2} \frac{\phi(x + i\infty)}{e^{-2\pi i(x+i\infty)} - 1} dx = 0,$$

$$\int_{n_1}^{n_2} \frac{\phi(x - i\infty)}{e^{2\pi i(x-i\infty)} - 1} dx = 0. \quad (89)$$
The integrand of the fourth term in the left-hand side of Eq. (88) can be rewritten as

\[
\frac{\phi(x)}{e^{-2\pi ix} - 1} + \frac{\phi(x)}{e^{2\pi ix} - 1} = \frac{(e^{2\pi ix} - 1)\phi(x) + (e^{-2\pi ix} - 1)\phi(x)}{(e^{-2\pi ix} - 1)(e^{2\pi ix} - 1)} = \frac{(e^{2\pi ix} + e^{-2\pi ix} - 2)\phi(x)}{1 - e^{-2\pi ix} - e^{2\pi ix} + 1} = -\phi(x),
\]

so that the limit of the integral, as \( \epsilon \) approaches zero \( (\epsilon \to +0) \), is equal to

\[
\lim_{\epsilon \to +0} \left( \int_{n_1+1-\epsilon}^{n_1+1+\epsilon} + \int_{n_1+1+\epsilon}^{n_1+2-\epsilon} + \cdots + \int_{n_2-1+\epsilon}^{n_2-1+\epsilon} \right) \left[ \frac{\phi(x)}{e^{-2\pi ix} - 1} + \frac{\phi(x)}{e^{2\pi ix} - 1} \right] dx = - \int_{n_1}^{n_2} \phi(x) dx.
\]

Hence, we can rewrite Eq. (88) as

\[
\frac{1}{2} \phi(n_1) + \sum_{n=n_1+1}^{n_2-1} \phi(n) + \frac{1}{2} \phi(n_2) = \int_{n_1}^{n_2} \phi(x) dx + i \int_0^\infty -\phi(n_2 + iy) + \phi(n_1 + iy) + \phi(n_2 - iy) - \phi(n_1 - iy) \frac{dy}{e^{2\pi y} - 1}.
\]

Thus, we obtain the formula of Eq. (14). If \( \phi(z) \to 0 \) as \( \text{Re}(z) \to +\infty \), we obtain

\[
\frac{1}{2} \phi(0) + \sum_{n=1}^\infty \phi(n) = \int_0^\infty \phi(x) dx + i \int_0^\infty \frac{\phi(iy) - \phi(-iy)}{e^{2\pi y} - 1} dy.
\]

So that, we obtain the formula of Eq. (15). This equation is called the Abel-Plana formula.

\[\text{B} \quad \text{Some remarks about numerical calculations}\]

In section \( \text{B} \) for the numerical calculations of \( J_1(t), J_2(t), I_1(t) \) and \( I_2(t) \), we use the Fortran compiler with quadruple-precision complex (a pair of quadruple-precision real numbers).

We evaluate the Gamma function included in Eqs. (24) and (29) numerically by the Lanczos approximation [26],

\[
\ln \Gamma(z) = (z + \frac{1}{2}) \ln(z + g + \frac{1}{2}) - (z + g + \frac{1}{2}) + \ln \left[ \frac{\sqrt{2\pi}}{z} (c_0 + \sum_{n=1}^6 \frac{c_n}{z + n} + \epsilon) \right] \quad \text{for Re}(z) > 0,
\]

where \( g = 5, \)

\[ c_0 = 1.000 \, 000 \, 000 \, 190 \, 015, \]

26
\[
c_1 = 76.180 \, 091 \, 729 \, 471 \, 46,
\]
\[
c_2 = -86.505 \, 320 \, 329 \, 416 \, 77,
\]
\[
c_3 = 24.014 \, 098 \, 240 \, 830 \, 91,
\]
\[
c_4 = -1.231 \, 739 \, 572 \, 450 \, 155,
\]
\[
c_5 = 0.120 \, 865 \, 097 \, 386 \, 91 \times 10^{-2},
\]
\[
c_6 = -0.539 \, 523 \, 938 \, 495 \, 3 \times 10^{-5}.
\]

The approximation with Eq. (94) gives the error upper bound \( |\epsilon| < 2 \times 10^{-10} \).

For carrying out the numerical integration of \( J_1(t) \) defined in Eq. (29) and \( I_1^{(i)}(t) \) defined in Eq. (24), we use Simpson’s rule [26]. For carrying out the numerical integration of \( J_2(t) \) defined in Eq. (29) and \( I_2^{(i)}(t) \) defined in Eq. (24), we use Bode’s rule [26].

In the numerical integration of \( J_2(t) \), we pay attention to the following fact. In the limit as \( y \to +0 \), the integrand of \( J_2(t) \) converges on a finite value as

\[
\lim_{y \to +0} \frac{1}{e^{2\pi y} - 1} \text{Im}\{ |\alpha|^{2i|y|} \cos(2\sqrt{igt}) \} = \frac{1}{2\pi} (2\ln|\alpha| + \gamma - 2t^2),
\]

(96)

where \( \gamma = 0.577215... \) is the Euler-Mascheroni constant. Similarly, in the limit as \( y \to +0 \), the integrand of \( I_2^{(i)}(t) \) converges on a finite value as

\[
\lim_{y \to +0} \frac{1}{e^{2\pi y} - 1} \text{Im}\{ |\alpha|^{2i|y|} \left[ \cos^2(\sqrt{c + iy}|\kappa|t) + \frac{c}{c + iy} \sin^2(\sqrt{c + iy}|\kappa|t) \right] \} = \frac{1}{4c\pi} [-1 + 2c\gamma + \cos(2\sqrt{c}|\kappa|t) + 4c\ln|\alpha|].
\]

(97)

As shown above, when we calculate \( I_2^{(i)}(t) \) numerically, we have to be careful about taking the limit of the integrand of \( I_2^{(i)}(t) \) as \( y \to +0 \).

The numerical integration of \( J_2(t) \) is more difficult than that of \( J_1(t) \). The numerical evaluation of \( J_2(t) \) for \( t > 8\pi \) does not converge on a reasonable value even if we use Romberg’s method [26]. The exactly same things happen when we calculate \( I_1^{(i)}(t) \) and \( I_2^{(i)}(t) \) numerically.

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