ON THE FREDHOLM AND WEYL SPECTRUM OF SEVERAL COMMUTING OPERATORS

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Abstract. In the paper one considers the local structure of the Fredholm joint spectrum of commuting n-tuples of operators. A connection between the spatial characteristics of operators and the algebraic invariant of the corresponding coherent sheaves is investigated. A notion of Weyl joint spectrum of commuting n-tuple is introduced.

0. Introduction.

Let \( T = (T_1, \ldots, T_n) \), where \( T_1, \ldots, T_n \) are mutually commuting linear bounded operators acting in the Banach space \( X \). By \( \sigma(T) \subset \mathbb{C}^n \) we will denote the joint Taylor spectrum of \( T \). Recall that \( \sigma(T) \) consist of all points \( \lambda = (\lambda_1, \ldots, \lambda_n) \) in \( \mathbb{C}^n \) such that the Koszul complex \( K_*(T - \lambda, X) \) of the operators \( (T_1 - \lambda_1, \ldots, T_n - \lambda_n) \) is not exact. Suppose that for given \( \lambda \in \sigma(T) \) all the homology spaces \( H_i(K_*(T - \lambda, X)) \) are finite-dimensional: we will call such a point a Fredholm point for \( T \) and write \( \lambda \in \sigma_F(T) \), the Fredholm spectrum of \( T \). The remaining part is called essential spectrum and denoted by \( \sigma_e(T) := \sigma(T) \setminus \sigma_F(T) \).

It is a simple observation (see \[6\]) that any finite Fredholm complex of Banach spaces with differentials, holomorphically depending on the parameters, is holomorphically quasi-isomorphic to a holomorphic complex of finite-dimensional spaces. Therefore, the sheaves \( H_i(T) \) of homologies of the complex of germs of holomorphic functions with values in \( K_*(T - \lambda, X) \) are coherent on \( \mathbb{C}^n \setminus \sigma_e(T) \). Fixing a point \( \lambda^0 \in \sigma_F(T) \), one can consider the stalk of the homology sheaf \( H_i(T)_{\lambda^0} \) as a module over the Noetherian local ring \( \mathcal{O}_{\lambda^0} \) of germs of holomorphic functions at this point. The Fredholm spectrum \( \sigma_F(T) \) is a complex-analytic subspace of \( \mathbb{C}^n \setminus \sigma_e(T) \), its dimension near \( \lambda^0 \) is well defined, and is an integer not exceeding the dimension of the ambient space \( \mathbb{C}^n \); in the present paper we are interested mostly in the case when this dimension is strictly less than \( n \).

To any coherent sheaf, or finitely generated \( \mathcal{O}_{\lambda^0} \)-module, one can attach an element of the cycle group of \( \mathcal{O}_{\lambda^0} \), i.e. a formal sum of prime ideals of \( \mathcal{O}_{\lambda^0} \). Taking the alternated sum of cycles of the modules \( H_i(T)_{\lambda^0} \) for \( i = 0, \ldots, n \), one obtains the cycle of the Koszul complex of the n-tuple. Roughly speaking, any irreducible component of \( \sigma_F(T) \) through \( \lambda^0 \) "participates" in the homology sheaves multiplied by some integer coefficient, and we obtain a set of integers characterizing the homology sheaves of the Koszul complex of \( T \). (In the case of a single operator \( T \) the
corresponding invariant is the index of $T - \lambda^0$.) One of the purposes of the paper is to establish some connections between the these algebraic characteristics of the homology sheaves, and the action of operators of $T$ in $X$.

In the first section of the paper we recall some useful facts from the commutative algebra and the theory of analytic local rings. Especially, we recall the notions of the isolated prime ideals in the support of a module, the cycle of a module, and the functoriality of the cycle map under finite morphisms. We introduce also the notions of the Hilbert-Samuel polynomial and the multiplicity of a module with respect to a prime ideal.

The second section contains the main results of the paper. We give a definition of the cycle $z_{\lambda^0}(T)$ in a point $\lambda^0$ of the Fredholm spectrum of $T$: this is a formal linear combination with integer coefficients of the irreducible components of $\sigma_F(T)$ passing through $\lambda^0$. In other words, to any irreducible component $P$ of $\sigma_F(T)$ containing $\lambda^0$ with corresponding prime ideal $p_P$ one attaches an integer $l_P(T)$ with $z_{\lambda^0}(T) = \sum l_P(T) p_P$. Theorem 2.3 proves the functoriality of the maximal component of the cycle under the holomorphic functional calculus. Next, theorems 2.5 and 2.8 establish some connections between the coefficients of the cycle $z_{\lambda^0}(T)$ and the spacial characteristics of the $n$-tuple $T$; in some particular cases this can be used in the calculations of these coefficients. In particular, theorem 2.8, considering the structure of the last homology sheaf of the Koszul complex of $T$, can be applied to the theory of multidimensional contractions, developed by Arveson ([1], [2]); in more details the connection between the characteristics of a multidimensional contraction and the geometry of its Fredholm spectrum will be considered elsewhere.

The last part of the second section is devoted to the Weyl spectrum of the commuting $n$-tuple $T$ of operators. In the paper [7], M. Putinar defines the Weyl spectrum $\omega(T)$ of $T$ as the complement in $\sigma(T)$ of the set of points $\lambda \in \sigma_F(T)$ such that $\text{ind}(T - \lambda) = 0$. However, in the case when $\dim \sigma_F(T) < n$, it is easy to see that $\text{ind}(T - \lambda) = 0$ for any $\lambda \notin \sigma_e(T)$, and therefore the Weyl spectrum will coincide with the essential spectrum of the $n$-tuple. We propose here an alternative definition: a Fredholm point $\lambda$ is not in the Weyl spectrum iff the cycle $z_\lambda(T)$ is zero, i.e. all $l_P(T) = 0$. In the case when all the components of $\sigma_F(T)$ are of (maximal) dimension $n$, this definition coincides with the definition adopted in [7]. However, even in the case of a single operator there is a certain discrepancy between our approach and the standard one: in our definition, all the isolated point of the Fredholm spectrum of $T$ belong to its Weyl spectrum. Further we give some basic properties of the so-defined Weyl spectrum: any $n$-tuple with SVEP has the "Weyl property". We prove also the property of spectral inclusion under holomorphic functional calculus for this spectrum. In some particular cases one has a stronger result - the spectral mapping theorem (see propositions 2.18 and 2.19).

The present paper continues the investigations on the structure of the Fredholm spectrum of a commuting $n$-tuple of operators started in [6], and systematically uses some basic result of that paper.

1. Some necessary facts from the commutative algebra.

Consider a coherent analytic sheaf $\mathcal{L}$, defined on open subset of $\mathbb{C}^n$. Recall that its (geometric) support $\text{supp} \mathcal{L}$ consists of all point where $\mathcal{L}$ is non-zero; $\text{supp} \mathcal{L}$ is complex-analytic subset of $\mathbb{C}^n$. If $\lambda^0 \in \text{supp} \mathcal{L}$, then the stalk of $\mathcal{L}$ at $\lambda^0$ is a finitely generated module over the local Noetherian ring $\mathcal{O}_{\lambda^0}$. In this section we recall for
a latter use some basic notions from the commutative local algebra, concerning the
Noetherian rings and modules.

Let \( \mathcal{A} \) be a commutative local Noetherian ring, and \( \mathcal{M} \) be finitely generated
\( \mathcal{A} \)-module with annihilator \( \Ann(\mathcal{M}) \). Let \( \Prime(\mathcal{A}) \) be the set of prime ideals of
\( \mathcal{A} \). One denotes by \( \Supp(\mathcal{M}) \) the (algebraic) support of \( \mathcal{M} \) - the set of all prime ideals in \( \mathcal{A} \), containing \( \Ann(\mathcal{M}) \). The set \( \Supp(\mathcal{M}) \) has an natural ordering by inclusion and the minimal elements with respect of this ordering (called isolated
associated primes) play a special role and correspond to irreducible components of
the geometric support of \( \mathcal{M} \). The set \( Iso(\mathcal{M}) \) of all such prime ideals is finite (see
[5], chap. 3).

For \( p \in \Prime(\mathcal{A}) \), denote by \( \mathcal{M}_p \) the localization of the \( \mathcal{A} \)-module \( \mathcal{M} \) with
respect to \( p \). Then \( p \in \Supp(\mathcal{M}) \) if and only if \( \mathcal{M}_p \neq 0 \). Moreover, \( p \in Iso(\mathcal{M}) \)
iff \( l_p(\mathcal{M}) = \dim_{\mathcal{A}_p} \mathcal{M}_p \) is finite and nonzero. Suppose that \( 0 = \mathcal{M}_0 \subset \ldots \subset \mathcal{M}_k = \mathcal{M} \) is a composition series for \( \mathcal{M} \), i.e. for all \( i \) one has \( \mathcal{M}_i/\mathcal{M}_{i-1} = \mathcal{A}/p_i \) with \( p_i \in \Prime(\mathcal{A}) \). If \( p \in Iso(\mathcal{M}) \), then \( p \) appears exactly \( l_p(\mathcal{M}) \) times in the
sequence \( \{p_1, \ldots, p_k\} \). By \( \Iso_{max}(\mathcal{M}) \) we will denote the set of elements of \( Iso(\mathcal{M}) \)
of maximal dimension (equal to the dimension of \( \mathcal{M} \)).

Let \( \mathcal{M} \) and \( \mathcal{A} \) be as above, and let \( q \) be an ideal in \( \mathcal{A} \) such that \( \mathcal{M}/q\mathcal{M} \) is
of finite dimension. Then there exists a polynomial \( P_{q,\mathcal{M}} \) (called the Hilbert - Samuel polynomial
of the pair \( q,\mathcal{M} \)) of degree \( r = \dim_{\mathcal{A}} \mathcal{M} \) such that for \( n \) sufficiently
big one has \( \dim(\mathcal{M}/q^n\mathcal{M}) = P_{q,\mathcal{M}}(n) \). The leading term of \( P_{q,\mathcal{M}} \) has the form
\( e_q(\mathcal{M})n^r \), where \( e_q(\mathcal{M}) \) is a positive integer, called a multiplicity of \( q \) at \( \mathcal{M} \). If
\( q = \mathcal{m} \), where \( \mathcal{m} \) is the maximal ideal of \( \mathcal{A} \), then the corresponding integer is called
multiplicity of \( \mathcal{M} \) and is denoted by \( e(\mathcal{M}) \). The formula (1) of the page 1 of the
book [8] shows how \( e_q(\mathcal{M}) \) can be calculated in homological terms.

Suppose now that \( \overline{\mathcal{M}} = \{\mathcal{M}_0, \ldots, \mathcal{M}_n\} \) is an ordered finite set of finitely
generated modules over \( \mathcal{A} \). Denote \( \Ann(\overline{\mathcal{M}}) = \bigcap_{i=0}^n \Ann(\mathcal{M}_i) \). Let \( \Supp(\overline{\mathcal{M}}) \) be
the set of all prime ideals of \( \mathcal{A} \), containing \( \Ann(\overline{\mathcal{M}}) \), \( Iso(\overline{\mathcal{M}}) \) – the set of minimal
elements of \( \Supp(\overline{\mathcal{M}}) \), and \( \Iso_{max}(\overline{\mathcal{M}}) \) – the set of elements of \( Iso(\overline{\mathcal{M}}) \)
of maximal dimension. Take \( i \in \{0, \ldots, n\} \) and a prime ideal \( p \in \Supp(\mathcal{M}_i) \). Since
\( \Supp(\mathcal{M}_i) \subset \Supp(\overline{\mathcal{M}}) \), then if \( p \) is not minimal in \( \Supp(\mathcal{M}_i) \), it will be not
minimal in \( \Supp(\overline{\mathcal{M}}) \) as well. So, one obtains

**Proposition 1.1.** For any \( p \in Iso(\overline{\mathcal{M}}) \) and \( i \in \{0, \ldots, n\} \) one has one of the
following two possibilities: 1/ \( p \in Iso(\mathcal{M}_i) \), or 2/ \( p \notin \Supp(\mathcal{M}_i) \).

Therefore for any prime \( p \in Iso(\overline{\mathcal{M}}) \) and \( i \) the integer \( l_p(\mathcal{M}_i) = \dim_{\mathcal{A}_p}(\mathcal{M}_i)_p \)
is well-defined (it is positive in the case 1/ and zero in the case 2/). So for such a
\( p \) one can define:

\[
l_p(\overline{\mathcal{M}}) := \sum_{i=0}^n (-1)^i l_p(\mathcal{M}_i). \]

Let \( q \) be an ideal in \( \mathcal{A} \) such that \( \mathcal{M}_i/q\mathcal{M}_i \) is finite-dimensional for all \( i \). One
can introduce the Hilbert-Samuel polynomial of \( \overline{\mathcal{M}} \):

\[
P_{q,\overline{\mathcal{M}}}(n) := \sum_{i=0}^n (-1)^i P_{q,\mathcal{M}_i}(n). \]
If one denote by \( r = \dim_{A} \overline{M} \) the maximal dimension of the \( A \)-modules \( \mathcal{M}_{i} \), \( i = 0, \ldots, n \), then \( P_{q, \overline{M}}(n) \) is a polynomial of dimension \( r \) and with leading term \( \frac{e_{q}(\overline{M})}{r!} n^{r} \), where

\[
e_{q}(\overline{M}) := \sum_{i=0}^{n} (-1)^{i} e_{q}(\mathcal{M}_{i}).
\]

Denote by \( Z(A) \) the group of the cycles of \( A \), i.e. of all formal linear combinations of elements of \( \text{Prime}(A) \) with integer coefficients. Then to any tuple of \( A \)-modules \( \overline{M} \) one can attach the elements \( z(\overline{M}) \) and \( z^{\max}(\overline{M}) \) of \( Z(A) \), defined by the formulæ

\[
z(\overline{M}) = \sum_{p \in \text{Iso}(\overline{M})} l_{p}(\overline{M}) p, \quad z^{\max}(\overline{M}) = \sum_{p \in \text{Iso}_{\max}(\overline{M})} l_{p}(\overline{M}) p
\]

and one has

\[
e_{q}(\overline{M}) = \sum_{p \in \text{Iso}_{\max}(\overline{M})} l_{p}(\overline{M}) e_{q}(A/p)
\]

(see [\text{S}], p. 125-126).

Finally, we will need a functoriality result. Let \( B \) be a subring of the analytic local ring \( A \) such that the monomorphism \( \varphi : B \rightarrow A \) is finite in the sense of [\text{S}], II.2.2, i.e. \( A \) is finitely generated as a \( B \)-module. For any finitely generated \( A \)-module \( \mathcal{M} \), denote by \( \mathcal{M}_{B} \) the underlying \( B \)-module. Then \( \mathcal{M}_{B} \) is a finitely generated \( B \)-module again. For any \( p \in \text{Prime}(A) \), denote by \( \varphi_{p} \) the element \( z^{\max}(p_{B}) \) of \( Z(B) \). By linearity one can extend this mapping up to a morphism \( \varphi_{*} : Z(A) \rightarrow Z(B) \).

**Proposition 1.2.** For any finitely generated \( A \)-module \( \mathcal{M} \) one has \( z^{\max}(\mathcal{M}_{B}) = \varphi_{*} z^{\max}(\mathcal{M}) \).

**Proof.** Indeed, theorem 2 of [\text{S}], II.5.1 asserts that \( \dim_{A} \mathcal{L} = \dim_{B} \mathcal{L}_{B} \) for any finitely generated \( A \)-module \( \mathcal{L} \). In particular, this is true for the modules of the type \( A/p, p \in \text{Prime}(A) \). Taking the composition series for \( \mathcal{M} \) and using the additivity of the mapping \( z^{\max} \) (see [\text{S}], p. 125), we obtain \( z^{\max}(\mathcal{M}_{B}) = \sum_{p \in \text{Iso}_{\max}(\mathcal{M})} l_{p}(\mathcal{M}) p_{B} = \varphi_{*} z^{\max}(\mathcal{M}) \). \( \Box \)

2. **Local structure of the Fredholm and Weyl spectra of \( n \)-tuple**

2.1. **Main definitions.** Consider a commuting \( n \)-tuple \( T = (T_{1}, \ldots, T_{n}) \) of operators acting in the Banach space \( X \), and denote by \( K_{s}(T-\lambda, X) = \{X_{k}, d_{k}(\lambda)\}_{k=0,1,\ldots,n} \) the Koszul complex of \( T \) in the point \( \lambda \in \mathbb{C}^{n} \). Recall that \( X_{k} \) is a direct sum of \( \binom{n}{k} \) copies of the space \( X \), and \( d_{k}(\lambda) \) depend linearly on \( \lambda \). Denote by \( \mathcal{O}X \) the sheaf of germs of \( X \)-valued holomorphic functions on \( \mathbb{C}^{n} \); then one can consider on \( \mathbb{C}^{n} \) the complex \( \mathcal{O}K_{s}(T-\lambda, X) \) of sheaves of holomorphic sections of the complex \( K_{s}(T-\lambda, X) \). Denote by \( H_{i}(T, \lambda) \) the \( i \)-th homology space of the complex \( K_{s}(T-\lambda, X) \), and by \( \mathcal{H}_{i}(T) \) the \( i \)-th sheaf of homologies of the complex of sheaves \( \mathcal{O}K_{s}(T-\lambda, X) \). Let \( \mathcal{H}_{i}(T)_{\lambda_{0}} \) be the stalk of \( \mathcal{H}_{i}(T) \) at \( \lambda_{0} \), considered as a local \( \mathcal{O}_{\lambda_{0}} \)-module.

Since the operators \( T_{i} \) commute with the differentials \( d_{k}(\lambda) \) of the parameterized Koszul complex, one can define the action of operators \( T_{k}, k = 1, \ldots, n \) on the
spaces $H_i(T, \lambda^0)$ and $\mathcal{H}_i(T)$. It is easy to see that the operator $T_k$ acts on the space $H_i(T, \lambda^0)$ as a multiplication by $\lambda_k^0$, and on the sheaf $\mathcal{H}_i(T)$ as a multiplication by the variable $\lambda_k$.

Fix a point $\lambda^0 \in \sigma_F(T)$; this means that all the spaces $H_i(T, \lambda^0)$ are finite-dimensional. As it was noted in [9], there exists a holomorphic complex of finite-dimensional spaces, defined near $\lambda^0$ and holomorphically quasi-isomorphic to $K_\lambda(T - \lambda, X)$. Therefore, all the sheaves $\mathcal{H}_i(T)$ are coherent in a neighborhood of $\lambda^0$, and the stalks $\mathcal{H}_i(T)_{\lambda^0}$ at $\lambda^0$ are finitely generated modules over the local Noetherian ring $O_{\lambda^0}$. Denote by $\mathcal{H}(T)_{\lambda^0}$ the $n + 1$-tuple $\{\mathcal{H}_i(T)_{\lambda^0}\}_{i=0,\ldots,n}$. Then the prime ideals in $Iso(\mathcal{H}(T)_{\lambda^0})$ correspond to the irreducible components of the complex set $\sigma_F(T)$, containing the point $\lambda^0$. Denote by $r = r(\lambda^0)$ the maximal dimension of these components.

**Definition 2.1.** Define

$$z(\lambda^0)(T) = z(\mathcal{H}(T)_{\lambda^0}), \quad z_{\lambda^0}^{\max}(T) = z_{\lambda^0}^{\max}(\mathcal{H}(T)_{\lambda^0}).$$

The element $z(\lambda^0)(T)$ has the form

$$z(\lambda^0)(T) = \sum_{p \in Iso(\mathcal{H}(T)_{\lambda^0})} l_p(T) p$$

where $l_p(T) = l_p(\mathcal{H}(T)_{\lambda^0})$ are integers. We will call the integer $l_p(T)$ a local index of the $n$-tuple $T$ at the point $\lambda^0$ and the prime ideal $p$.

This definition can be considered from the geometric point of view. Let $\mathcal{P}$ be an irreducible component of the complex set $\sigma_F(T)$; then in any point $\lambda \in \mathcal{P}$ it determines a prime ideal $p_\lambda$ in the ring $O_{\lambda}$.

**Proposition 2.2.** The integer $l_{p_\lambda}(T)$ does not depend on the choice of the point $\lambda \in \mathcal{P}$.

**Proof.** It is sufficient to prove that the integers $l_{p_\lambda}(T)$ are locally constant. Choose $\lambda^0 \in \mathcal{P}$ and a sufficiently small ball $U$, centered at $\lambda^0$, such that $U$ does not intersect any irreducible component of $\sigma_F(T)$, not containing $\lambda^0$. Then for any $i \in \{0,\ldots,n\}$ the $O_U$-module $\mathcal{H}_i(T)|_U$ has the same set of isolated primes, and the same composition series, as $\mathcal{H}_i(T)_{\lambda^0}$. The irreducible complex set $\mathcal{P}$ determines a prime ideal $p_U$ in $O_U$. For any $\lambda \in U$ the $O_{\lambda}$-module $\mathcal{H}_i(T)_{\lambda}$. If $\lambda \in \mathcal{P}$, the localization of $p_U$ at $\lambda$ is a nontrivial prime ideal, and the invariance of the composition series under localization shows that $l_{p_{\lambda^0}}(T) = l_{p_{\lambda}}(T) = l_{p_{\lambda}}(T)$. □

Now, to any irreducible component one can attach its local index $l_P(T)$. The set of all irreducible components and the corresponding local indexes contains certain information about the homology sheaves of the Koszul complex of $T$ and will be called a spectral picture of the commuting $n$-tuple $T$. In the rest of paper we will turn back to the algebraic point of view on the local indexes and will give some facts allowing to compute it in some cases.

### 2.2. Particular cases: dimensions 1 and $n$. Suppose that the dimension of $\sigma_F(T)$ at $\lambda^0$ is the maximal one, i.e. equal to $n$. This means that the Fredholm spectrum $\sigma_F(T)$ contains a neighborhood $U$ of $\lambda^0$. It is easy to see (or to derive from theorem 2.5 below) that in this case

$$z(T)_{\lambda^0} = z_{\lambda^0}^{\max}(T)_{\lambda^0} = (-1)^n \text{ind}(T - \lambda^0) \cdot [0],$$
where \( \text{ind} \,(T - \lambda^0) \) is the Euler characteristic of the complex \( K_* \,(T - \lambda^0, X) \) and \([0]\) is the zero ideal in \( \mathcal{O}_{\lambda^0} \).

Suppose in addition that the \( n \)-tuple \( T \) has SVEP. Then from the theory of coherent sheaves it follows that the sheaf \( \mathcal{H}_n(T) \) is a free \( \mathcal{O} \)-module on the open dense subset \( \tilde{U} \) of \( U \). (In fact, \( \tilde{U} \) is a complement of a complex subset of \( U \) of dimension < \( n \).) Then for any \( \lambda = (\lambda_1, \ldots, \lambda_n) \in \tilde{U} \) one has

\[
\text{ind} \,(T - \lambda^0) = \text{ind} \,(T - \lambda) = (-1)^n \dim \left( X/\sum_{i=1}^n (T_i - \lambda_i) X \right).
\]

Now consider the case when \( \sigma_F(T) \) is one-dimensional near \( \lambda^0 \). Then, as proven in [6], the modules \( \mathcal{H}_i(T)_{\lambda^0} \) are nonzero only for \( i = n, n - 1 \). Moreover, prop. 3.6 of [6] shows that

\[
\mathcal{H}_{n-1}(T) \sim \mathcal{H}_n(T^*),
\]

where \( T^* \) is the \( n \)-tuple \( (T_1^*, \ldots, T_n^*) \) acting in the dual space \( X^* \), and \( \sim \) denotes the equivalence modulo sheaves with zero-dimensional support, i.e. concentrated in the point \( \lambda^0 \). So one obtains the equality

\[
z^\text{max} \,(T)_{\lambda^0} = (-1)^n (z^\text{max} \,(\mathcal{H}_n(T)_{\lambda^0}) - z^\text{max} \,(\mathcal{H}_n(T^*)_{\lambda^0})).
\]

Finally, note that if \( \sigma_F(T) \) is of dimension zero at \( \lambda^0 \) (i.e. \( \lambda^0 \) is an isolated point of the Fredholm spectrum), then \( l_{\{\lambda^0\}}(T) \) coincides with the dimension of the spectral subspace of \( T \) corresponding to the point \( \lambda^0 \).

### 2.3. Functoriality.

Let \( f(\lambda) = (f_1(\lambda), \ldots, f_k(\lambda)) \) be a \( k \)-tuple of holomorphic functions, defined in a neighborhood of the spectrum \( \sigma(T) \subset \mathbb{C}^n \) of the commuting \( n \)-tuple \( T \) of operators. It is well-known that \( \sigma(f(T)) = f(\sigma(T)) \) and \( \sigma_f(T) = f(\sigma(T)) \). Take the point \( \mu^0 \in \mathbb{C}^k \) belonging to the Fredholm spectrum \( \sigma_f(f(T)) \) of the operator \( k \)-tuple \( f(T) \). Then its preimage \( f^{-1}(\mu^0) \cap \sigma(T) \) is a finite subset \( \{\lambda_1, \ldots, \lambda_p\} \) of the Fredholm spectrum \( \sigma_{\mu}(T) \) of \( T \). For any \( \lambda \in f^{-1}(\mu^0) \cap \sigma(T) \) the inverse image \( f^* \) by \( f \) provides an embedding \( \mathcal{O}_{\mu^0}^k \rightarrow \mathcal{O}_{\lambda}^k \) and the induced monomorphism \( \mathcal{O}_{\mu^0}^k \rightarrow \mathcal{O}_{\lambda}/\text{Ann} (\overline{\mathcal{H}}(T)_{\lambda}) \) is finite. Denote by \( f_* : Z \left( \mathcal{O}_{\lambda}/\text{Ann} (\overline{\mathcal{H}}(T)_{\lambda}) \right) \rightarrow Z \left( \mathcal{O}_{\mu^0}^k \right) \) the morphism used in prop. 1.2.

#### Theorem 2.3. One has

\[
z^\text{max} \,(\overline{\mathcal{H}}(f(T))_{\mu^0}) = \sum_{\lambda \in f^{-1}(\mu^0) \cap \sigma(T)} f_* z^\text{max} \,(\overline{\mathcal{H}}(T)_{\lambda}).
\]

**Proof.** In proposition 3.2 of [6], one states the equality

\[
\mathcal{H}_i(f(T))_{\mu^0} = \bigoplus_{\lambda \in f^{-1}(\mu^0) \cap \sigma(T)} \mathcal{H}_{i+n-k}(T)_{\lambda},
\]

both sides considered as \( \mathcal{O}_{\mu^0}^k \)-modules. Applying the functor \( z^\text{max} \) and taking into account the equality \( z^\text{max} \,(\mathcal{M}_{\mathcal{B}}) = \varphi_* z^\text{max} \,(\mathcal{M}) \) proved in the previous section, one obtains the theorem. \( \square \)
2.4. Computations for \( z_{\lambda^0}^{max}(T) \). We will try to extract some information on the cycle \( z_{\lambda^0}^{max}(T) \), i.e. on the integers \( l_p(T) := l_p(\overline{\mathcal{P}(T)}_{\lambda^0}) \) for \( p \in Iso_{\lambda^0}(\overline{\mathcal{P}(T)}_{\lambda^0}) \), from the properties of operators of \( T \). Suppose that the coordinates \( \lambda = (\lambda_1, \ldots, \lambda_r) \) form a coordinate system for \( \sigma_F(T) \) at \( \lambda^0 \). This means that \( \lambda^0 \) is an isolated point of \( \Pi^{-1}\lambda^0 \cap \sigma_F(T) \), where \( \Pi \) is the coordinate projection onto first \( r \) coordinates, and \( \lambda^0 = (\lambda^0_0, \ldots, \lambda^0_r) \), \( \lambda^0_0 = (\lambda^0_{r+1}, \ldots, \lambda^0_n) \). This always can be achieved by a small perturbation of the coordinate system. We will denote \( T' = (T'_1, T'_r) \), where \( T'_1 = (T_{11}, \ldots, T_{r1}), T'_r = (T_{r+1}, \ldots, T_n) \).

Consider the Koszul complex \( K_*(T' - \lambda^0, X) \) of the operators \( T_1 - \lambda_1^0, \ldots, T_r - \lambda_r^0 \) in \( X \). Its homology spaces \( H_i(T', \lambda^0), i = 0, \ldots, r \) are in general not separated, but because of the commutativity of \( T \) the action of the operators from \( T'' \) on it is correctly defined. Moreover, one can form the parameterized Koszul complex of operators of \( T'' - \lambda'' \) in \( H_i(T, \lambda^0) \); this complex is exact for \( \lambda'' \notin \Pi^{-1}\lambda^0 \cap \sigma(T) \). The following proposition is obtained in [6]:

**Proposition 2.4.** In the conditions above there exist a decomposition

\[
H_i(T', \lambda^0) = H'_i(T', \lambda^0) \bigoplus H''_i(T', \lambda^0)
\]

into subspaces, invariant under \( T'' \), such that:

a/ \( H'_i(T', \lambda^0) \) is finite-dimensional (or empty), and the joint spectrum of operators of \( T'' \) in \( H'_i(T', \lambda^0) \) (in the non-empty case) consists on the point \( \lambda''^0 \).

b/ The joint spectrum of \( T'' \) in \( H'_i(T', \lambda^0) \) does not contain the point \( \lambda''^0 \).

c/ There exist a finite-dimensional holomorphic subcomplex \( L_*(\lambda') \) of 
\( K_*(T' - \lambda', X) \) such that \( H_*(L_*(\lambda')) \) coincides with \( H'_i(T', \lambda^0, X) \), and 
\( H_*(K_*(T' - \lambda^0, X) / L_*(\lambda^0)) \) - with \( H''_i(T' - \lambda^0, X) \).

d/ Suppose that \( \Pi^{-1}\lambda^0 \) is contained in \( \sigma_F(T) \) and therefore consists of finitely many points \( \lambda_j = (\lambda^0_0, \lambda^0_r), j = 1, \ldots, k \). Denote by \( H'_j(T', \lambda^0) \) the joint root space of the operators \( T'' - \lambda''^j \) in \( H_i(T', \lambda^0) \). Then \( H_i(T', \lambda^0) = \bigoplus H'_j(T', \lambda^0) \).

In other words, \( H'_j(T', \lambda^0) \) is finite-dimensional and coincides with the joint root space of the operators \( T_{r+1} - \lambda^0_{r+1}, \ldots, T_n - \lambda^0_n \), acting in the linear space \( H_i(T', \lambda^0) \).

Denote

\[
\chi' \left( T', \lambda^0 \right) = \sum_{i=0}^{r} (-1)^i \dim H'_i(T', \lambda^0).
\]

Let \( q \) be the ideal in \( \mathcal{O}_{\lambda^0} \), generated by the functions \( \lambda_1 - \lambda^0_1, \ldots, \lambda_r - \lambda^0_r \). Then

**Theorem 2.5.** One has \( \chi' \left( T', \lambda^0 \right) = e_q(\overline{\mathcal{P}(T)}_{\lambda^0}) \).

**Proof.** Let, as above, \( \mathcal{O}_{\lambda^0}X \) be the stalk at \( \lambda^0 \) of the sheaf of germs of \( X \)-valued holomorphic functions, and let \( \mathcal{O}_{\lambda^0}K_k(T - \lambda, X) \) be the Koszul complex of the operators of \( T - \lambda \) in \( \mathcal{O}_{\lambda^0}X \). This is a complex of \( \mathcal{O}_{\lambda^0} \)-modules. In any stage \( \mathcal{O}_{\lambda^0}X_k \) of this complex, \( k = 0, \ldots, n \), one can consider the action of the operators \( T'' - \lambda''^j = (T_1 - \lambda''^j_1, \ldots, T_r - \lambda''^j_r) \), and this action commutes with the differential of the complex. One can form the Koszul complexes of the operators of \( T'' - \lambda''^j \) in \( \mathcal{O}_{\lambda^0}X_k \), obtaining a bicomplex of sheaves with \( r + 1 \) rows and \( n + 1 \) columns. As it is shown below, the cohomology sheaves of its total complex are in fact supported at \( \lambda^0 \) and finite-dimensional, and one may consider its dimension.

The Euler characteristics of the total complex of the bicomplex can be computed in two ways. One may consider the homologies \( H_i(T)_{\lambda^0}, i = 0, \ldots, n \) of the initial
complex, and take the alternated sum of dimensions of the homologies of the Koszul complex of operators \( T' - \lambda^0 \) in it. The alternated sum of these integers for \( i = 0, \ldots, n \) will be equal to the alternated sum of the dimensions of the homologies of the total complex. Since the action of the operators \( T_j \) on \( \mathcal{H}_i(T)\lambda^0 \) coincide with the multiplication by the variable \( \lambda_j \), one comes to the Koszul complex of \( \lambda_1 - \lambda_1^0, \ldots, \lambda_r - \lambda_r^0 \) in \( \mathcal{H}_i(T)\lambda^0 \), and the formula \( * \) of the p. 1 of the book [8] can be applied. It shows that for any \( i = 0, \ldots, n \) one has

\[
\sum_{j=0}^r (-1)^j \dim H_j \left( \lambda' - \lambda^0, \mathcal{H}_i(T)\lambda^0 \right) = e_q \left( \mathcal{H}_i(T)\lambda^0 \right)
\]

and therefore the Euler characteristic of the total complex is equal to \( e_q \left( \mathcal{H}(T)\lambda^0 \right) \).

On the other hand, take the complex \( K_i \left( T' - \lambda^0, X \right) \) with stages \( X_k \) and differentials \( d_k' = d_k' \left( \lambda^0 \right) : X_k \to X_{k+1}, k = 0, \ldots, r \). The Euler characteristic of the total complex is equal to the alternated sum on \( j, j = 1, \ldots, r \) of the integers

\[
e_j = \sum_{i=1}^n (-1)^i \dim \mathcal{H}_i \left( T - \lambda, \mathcal{O}_{\lambda^0} H_j \left( T' - \lambda', X \right) \right)
\]

where by \( \mathcal{O}_{\lambda^0} H_j \left( T' - \lambda', X \right) \) we denote the factor \( \mathcal{O}_{\lambda^0} \)-module \( \mathcal{O}_{\lambda^0} \ker d_j' \left( \lambda^0 \right) / d_{j-1}' \left( \lambda^0 \right) \left( \mathcal{O}_{\lambda^0} X_{j-1} \right) \). Take the finite-dimensional subcomplex \( L_+ \left( \lambda^0 \right) = \{ L_j, a_j \} \) as in point c/) of prop. 3.2 of [8]: then the action of the operators \( T - \lambda \) on the module \( \mathcal{O}_{\lambda^0} (X_j / L_j) \) is regular for any \( j \), and therefore in the formula above one can replace \( \mathcal{O}_{\lambda^0} H_j \left( T' - \lambda', X \right) \) by \( \mathcal{O}_{\lambda^0} L_j / a_j \left( \mathcal{O}_{\lambda^0} L_{j-1} \right) = \mathcal{O}_{\lambda^0} H_j' \left( T' - \lambda^0, X \right) \). One can easy prove the following:

**Lemma 2.6.** Let \( H \) be a finite-dimensional space, \( \mathcal{O}_{\lambda^0} H \) - the \( \mathcal{O}_{\lambda^0} \)-module of germs of \( H \)-valued holomorphic functions in the point \( \lambda^0 \in \mathbb{C}^n \). Let \( T = (T_1, \ldots, T_n) \) be an \( n \)-tuple of operators acting in \( H \) of the form \( T_i = \lambda_i^0 + K_i, i = 1, \ldots, n \), where \( K_i \) are nilpotent operators in \( H \). Then

\[
\dim \mathcal{H}_i \left( T - \lambda, \mathcal{O}_{\lambda^0} H \right) = \begin{cases} 
0 & \text{if } i < n \\
\dim H & \text{for } i = n.
\end{cases}
\]

Indeed, if all \( K_i \) are zero operators, one obtains the Koszul complex of the germs \( \lambda_i^0 - \lambda_i \) in \( \mathcal{O}_{\lambda^0} H \) exact in all terms except in the last one, where the module of homology is \( \mathcal{O}_{\lambda^0} H / \mathfrak{m}_{\lambda^0} H \), and \( \mathfrak{m}_{\lambda^0} \) is the maximal ideal in \( \mathcal{O}_{\lambda^0} \). In the general case one can find a filtration of the linear space \( H \) such that all \( K_i \) are zero on the corresponding graduate space, and the assertion follows from the case of zero operators.

Applying the assertion of the lemma to the space \( H_i' \left( T', \lambda^0 \right) \) and the operators \( \lambda_0^0, \ldots, \lambda_r^0, T_{r+1}, \ldots, T_n \), one obtains \( c_j = \dim H_i' \left( T', \lambda^0 \right) \) for \( j = 1, \ldots, r \), which proves the theorem.

**Corollary 2.7.** Suppose that \( \text{Iso}_{\text{max}} \left( \mathcal{H}(T)\lambda^0 \right) \) consists of a single prime ideal \( p \), (i.e. there is only one irreducible component \( P \) of \( \sigma_F(T) \) containing \( \lambda^0 \)), and the coordinates \( \lambda' = (\lambda_1, \ldots, \lambda_r) \) form a coordinate system for \( \sigma_F(T) \) at \( \lambda^0 \). Then

\[
\chi' \left( T', \lambda^0 \right) = e_q \left( \mathcal{O}_{\lambda^0} / p \right) l_p(T)
\]

where \( q \) is, as above, the ideal generated by \( \lambda_1 - \lambda_1^0, \ldots, \lambda_r - \lambda_r^0 \).
2.5. **Computations for the last homology sheaf** $\mathcal{H}_n(T)_{\lambda^0}$. In this section we will suppose that $\lambda^0 \in \mathbb{C}^n$ is such that the linear subspace of the element of the type $\sum_{i=1}^n (T_i - \lambda_i^0) x_i$ with $x_1, \ldots, x_n \in X$ is of finite codimension in $X$. Then, using the arguments of [3], 4.3., it is easy to see that the last homology sheaf $\mathcal{H}_n(T)$ of the Koszul complex of $T$ is coherent near $\lambda^0$.

Let $m$ be the maximal ideal in the local ring $O_{\lambda^0}$, and denote by $m(T)$ the operator ideal in $L(X)$ generated by the operators $T_i - \lambda_i^0, \ldots, T_n - \lambda_n^0$. Then the assumption above means that $m(T)X$ is of finite codimension in $X$, and therefore this will be true for the subspace $m^k(T)X$ for any natural $k$.

**Theorem 2.8.** In the conditions above, one has

$$\dim X/m^k(T)X = \dim (\mathcal{H}_n(T)_{\lambda^0}/m^k\mathcal{H}_n(T)_{\lambda^0})$$

and therefore for $k$ sufficiently big is a polynomial of $k$ with leading term
e(\mathcal{H}_n(T)_{\lambda^0}) k^r/r!, where $r = \dim \text{supp} \mathcal{H}_n(T)_{\lambda^0}$.

**Proof.** Denote $Y = m^k(T)X$ and $Z = X/Y$. Then $Z$ is finite-dimensional, and the joint spectrum of the operators of $T$ in $Z$ coincides with the point $\lambda^0$. Therefore $\mathcal{H}_i(T - \lambda, Z)_{\lambda^0} = 0$ for all $i < n$. Denote by $\mathcal{H}_{\lambda^0}X$ the $O_{\lambda^0}$-module $\mathcal{H}_n(T)_{\lambda^0} = O_{\lambda^0}X/\sum (T_i - \lambda_i) O_{\lambda^0}X$, and by $\mathcal{H}_{\lambda^0}Y$, $\mathcal{H}_{\lambda^0}Z$ - the corresponding homology modules for $Y$ and $Z$ respectively. The exact sequence $0 \to Y \to X \to Z \to 0$ determines an exact sequence of the corresponding Koszul complexes, and therefore a long exact sequence of the corresponding homology sheaves, ending with the sequence $0 \to \mathcal{H}_{\lambda^0}Y \to \mathcal{H}_{\lambda^0}X \to \mathcal{H}_{\lambda^0}Z \to 0$. We will prove that the image of $\mathcal{H}_{\lambda^0}Y$ in $\mathcal{H}_{\lambda^0}X$ coincides with $m^k\mathcal{H}_{\lambda^0}X$. Choose polynomials $g_l(\lambda), l = 1, \ldots, L$, generating the ideal $m^k$, and denote by $\sim$ the relation of equivalence in $O_{\lambda^0}X$ modulo $\sum (T_i - \lambda_i) O_{\lambda^0}X$. Any element of $\mathcal{H}_{\lambda^0}Y$ can be represented by a germ of $Y$-valued holomorphic function $y(\lambda) = \sum g_l(T)x_l(\lambda) \in Y$. Then $y$ is equivalent to $\sum g_l(\lambda)x_l(\lambda)$ and therefore its image in $\mathcal{H}_{\lambda^0}X$ belongs to $m^k\mathcal{H}_{\lambda^0}X$. We will prove that the image of $\mathcal{H}_{\lambda^0}Y$ is dense in $m^k\mathcal{H}_{\lambda^0}X$ in the sequential topology (see [3], I.6.2.). Indeed, any element $\xi$ of $m^k\mathcal{H}_{\lambda^0}X$ has the form $\sum g_l(\lambda)\varphi_l(\lambda)$, where $\varphi_l(\lambda)$ are holomorphic $X$-valued functions, defined in a neighborhood of $\lambda^0$. $\varphi_l(\lambda)$ can be represented by power series, converging in some neighborhood of the zero: $\varphi_l(\lambda) = \sum_k a_{l,k} (\lambda - \lambda^0)^k$. Taking the partial sums, one obtain that $\xi$ is a limit in the sequential topology of the elements $\xi_N = \sum_l g_l(\lambda) \sum_{|k| \leq N} a_{l,k} (\lambda - \lambda^0)^k$. But the element $\xi_N$ is equivalent to $\sum_l g_l(T) \sum_{|k| \leq N} (T - \lambda^0)^k a_{l,k}$ and therefore belongs to $\mathcal{H}_{\lambda^0}Y$.

By theorem 10 of [3], II.1.3. all ideals are closed in the sequential topology, and we obtain that $\mathcal{H}_{\lambda^0}Y$ and $m^k\mathcal{H}_{\lambda^0}X$ are isomorphic submodules of $\mathcal{H}_{\lambda^0}X$. Taking the corresponding factor-modules, we obtain an isomorphism between $\mathcal{H}_{\lambda^0}Z$ and $\mathcal{H}_{\lambda^0}X/m^k\mathcal{H}_{\lambda^0}X$. Applying lemma 2.6, we obtain

$$\dim Z = \dim (\mathcal{H}_n(T)_{\lambda^0}/m^k\mathcal{H}_n(T)_{\lambda^0})$$.

□
A similar equality can be proved in a more general situation. Suppose that, as in th. 2.3, \( f(\lambda) = (f_1(\lambda), \ldots, f_p(\lambda)) \) is a \( k \)-tuple of holomorphic functions, defined in the neighborhood of \( \sigma(T), \mu^0 \in \sigma_F(f(T)) \), and therefore the intersection \( f^{-1}(\mu^0) \cap \sigma(T) \) is a finite subset \( \{\lambda^1, \ldots, \lambda^J\} \) of \( \sigma_F(T) \). Let \( q \) be the ideal, generated by \( f_1(\lambda) - \mu^0_1, \ldots, f_p(\lambda) - \mu^0_p \), and \( q(T) \) - the operator ideal in \( L(X) \) generated by \( f_1(T) - \mu^0_1, \ldots, f_k(T) - \mu^0_k \). Let \( r = \max \dim_{\mathcal{O}^{0}_{\lambda_j}}(\mathcal{H}_{n}(T)_{\lambda_j}), \, j = 1, \ldots, J \). Then, applying the theorem above for the \( p \)-tuple \( f(T) - \mu^0 \) and using the functoriality stated in 2.3, one obtains

**Corollary 2.9.** In the conditions above, one has

\[
\dim X/q^{k}(T)X = \sum_{j=1}^{J} \dim \left( \mathcal{H}_{n}(T)_{\lambda_j}/q^{k}\mathcal{H}_{n}(T)_{\lambda_j} \right)
\]

and therefore for \( k \) sufficiently big is a polynomial of degree \( r \).

Recall that the \( n \)-tuple \( T \) has the single value extension property (SVEP) at \( \lambda^0 \) if \( \mathcal{H}_{i}(T)_{\lambda^0} = 0 \) for all \( i \neq n \). Then the theorem implies

**Corollary 2.10.** Suppose that, under the conditions of 2.9, the \( n \)-tuple \( T \) has SVEP at the points \( \{\lambda^1, \ldots, \lambda^J\} \) (or, equivalently, \( f(T) \) has SVEP at \( \mu^0 \)). Then \( \dim (X/q^k(T)X) \) for \( m \) sufficiently big is a polynomial of \( k \) with leading term \( (-1)^{n-\epsilon} e(T)(k)^{r!} \), where \( r \) is the maximum dimension of the modules \( \mathcal{H}_{n}(T)_{\lambda_j} \) for \( j = 1, \ldots, J \).

### 2.6. Application to Arveson’s contraction theory

Suppose that the operators of \( T \) act in the Hilbert space \( H \). Take the polynomials \( g_l(\lambda), \, l = 1, \ldots, L \), generating the ideal \( m^k \), and denote by \( g(T) : H^L \rightarrow H \) the operator matrix with entries \( g_l(T), \, l = 1, \ldots, L \). Then the factor-space \( H/m^k(T)H \) is isomorphic to the kernel of the operator \( g(T)g(T)^* = \sum_{l=1}^{L} g_l(T)g_l(T)^* \).

In particular, consider the case when \( T \) is a commuting \( n \)-contraction with finite rank in the sense of Arveson (see 1), i.e. the operator \( 1 - \sum_{i=1}^{n} T_iT_i^* \) is finite-dimensional and positive. Suppose in addition that the essential spectrum \( \sigma_{e}(T) \) does not contains the origin (this is obviously satisfied if all the operators \( T_i \) are essentially normal; in this case the essential spectrum is contained in the unit sphere in \( \mathbb{C}^n \)). Then the Fredholm spectrum of \( T \) is contained in the open unit ball \( B_n \). Let \( \phi : \mathcal{B}(H) \rightarrow \mathcal{B}(H) \) be the Arveson completely positive map defined by the formula \( \phi(A) = \sum_{i=1}^{n} T_iAT_i^* \) (see 2). In the Arveson’s theory an important role plays the expression

\[
\phi^k(1) = \sum_{k_1 + \ldots + k_n = k} \frac{k!}{k_1! \cdots k_n!} T_1^{k_1} \cdots T_n^{k_n} T_1^{-k_1} \cdots T_n^{-k_n}.
\]

Since the functions \( \lambda_{k_i}^{1j} \ldots \lambda_{k_i}^{nj} \) with \( k_1 + \ldots + k_n = k \) form a system of generators for the \( k \)-th power of the maximal ideal in the local ring \( \mathcal{O}_0 \) of germs of holomorphic functions in the point \( 0 \in \mathbb{C}^n \), then from the arguments above it follows:

**Proposition 2.11.** Let \( T \) be a commuting \( n \)-contraction with finite rank. Then one has

\[
\dim \ker \phi^k(1) = \dim H/m^k(T)H = \dim \left( \mathcal{H}_{n}(T)_0/m^k\mathcal{H}_{n}(T)_0 \right)
\]
and therefore $\dim \ker \phi^k(1)$ is a polynomial on $k$ of degree, equal to the dimension of the $O_k$-module $\mathcal{H}_n(T)_0$.

In [2], prop. 7.2, it is proved that rank $(1 - \phi^k(1))$ is a polynomial of $k$ of degree $\leq n$. The degree of this polynomial is called degree of the module $H$ and denoted by $\deg H$. Since

$$\dim \ker \phi^k(1) \leq \text{rank } (1 - \phi^k(1)),$$

then one obtains

**Corollary 2.12.** Under the assumptions of prop. 2.11 the degree of $H$ is greater or equal to the dimension of $\mathcal{H}_n(T)_0$.

**Remark 2.13.** Suppose that the essential spectrum of $T$ is contained in the unit sphere in $\mathbb{C}^n$. Applying an appropriate M"obius transform to the contraction $T$ (see [4]), one can replace in the statement above the point $0$ with an arbitrary point in the open ball $B_n$. One could make a conjecture that the degree of a pure finite-rank contraction coincides with the maximal degree of the corresponding analytic modules, i.e. with the dimension of the support of $\mathcal{H}_n(T)$; this problem will be considered elsewhere.

In [2] Arveson introduces the notion of a graded $n$-contraction as a contraction endowed with a suitable action of the circle group (for the precise definition see the section 6 of [2]). Then, roughly speaking, the Fredholm spectrum and the corresponding homology sheaves are determined by homogeneous polynomials in $\mathbb{C}^n$. In this case the the inequality of the proposition above becomes an equality.

**Proposition 2.14.** Let $T$ be a pure graded finite-rank contraction in the Hilbert space $H$. Then

$$\deg H = \dim_{O_n} \mathcal{H}_n(T)_0,$$

and the leading terms of the polynomial rank $(1 - \phi^k(1))$ and the Hilbert-Samuel polynomial of $\mathcal{H}_n(T)_0$ coincide.

**Proof.** The assertion follows almost immediately from the proof of theorem B of [2]. Indeed, Arveson constructs a submodule $H_0$ of finite codimension in $H$ such that $1_{H_0} - \phi^k_0(1_{H_0})$ is a (finite-dimensional) projection for any $k$, $\phi^k_0(A)$ being the Arveson’s completely positive map for the submodule $H_0$. Therefore

$$\text{rank } (1_{H_0} - \phi^k_0(1_{H_0})) = \dim \ker \phi^k_0(1_{H_0})$$

and the equality above is satisfied for the submodule $H_0$.

On the other hand, corollary 1 of theorem C of [2] shows that for $k$ sufficiently big the polynomials rank $(1_{H_0} - \phi^k_0(1_{H_0}))$ and rank $(1_H - \phi^k(1_H))$ differ by a polynomial of degree strictly less that $\deg H$. (Formally, this is stated only in the case when $\deg H = n$, but it is easy to see that the proof works in the general case as well.)

The same fact is true for the right hand side; indeed, the arguments used in the proof of 2.8 lead to the exact sequence

$$0 \to \mathcal{H}_n(H_0)_0 \to \mathcal{H}_n(H)_0 \to \mathcal{H}_n(H/H_0)_0 \to 0.$$

Since $H/H_0$ is finite-dimensional, then the Hilbert-Samuel polynomial of $\mathcal{H}_n(H/H_0)_0$ is of degree zero, and the additivity property (see prop. II.10 of [5]) shows that the Hilbert-Samuel polynomials of $\mathcal{H}_n(H_0)$ and $\mathcal{H}_n(H)$ differ by a polynomial of lower degree and therefore have identical leading terms. Now, since the statement of the proposition is valid for $H_0$, then it is valid for $H$ also. □
2.7. Weyl spectrum of \( n \)-tuple.

**Proposition 2.15.** Let \( \lambda^0 \in \sigma_F(T) \). Then the following assertions are equivalent:
1/ \( z_{\lambda^0}(T) = 0 \),
2/ \( z'_{\lambda^0}(T) = 0 \) for \( \lambda \) sufficiently close to \( \lambda^0 \).

**Proof.** Suppose that \( z_{\lambda^0}(T) = 0 \), i.e. \( l_p(T) = 0 \) for any ideal \( p \in \text{Iso}(\mathfrak{H}(T)_{\lambda^0}) \). Then, as in the proof 2.2, one can see that \( l_p(T) = 0 \) for any irreducible component of \( \sigma_F(T) \), containing \( \lambda^0 \).

Conversely, suppose that 2/ is satisfied in a neighborhood \( U \) of \( \lambda^0 \), and \( \mathcal{P} \) is an arbitrary irreducible component of \( \sigma_F(T) \), containing \( \lambda^0 \). One can choose a point \( \lambda \in \mathcal{P} \cap U \) such that no other irreducible component of \( \sigma_F(T) \) contains \( \lambda \). Then the condition 2/ in the point \( \lambda \) implies that \( l_p(T) = 0 \), and 1/ is satisfied. \( \Box \)

**Definition 2.16.** The point \( \lambda^0 \in \sigma_F(T) \) will be called a Weyl point for the \( n \)-tuple of commuting operators \( T \) if the conditions of the proposition above are fulfilled.

We will denote by \( \rho_\omega(T) \) the set of all Weyl points of \( T \), and by \( \omega(T) \) - the Weyl spectrum of \( T \), i.e. the complement of \( \rho_\omega(T) \) in \( \sigma(T) \).

Note that in this definition, unlike in the standard one for a single operator, the isolated points of the Fredholm spectrum are not Weyl points.

If the \( n \)-tuple \( T \) possess SVEP, then \( z_{\lambda^0}(T) = (-1)^n z(\mathcal{H}_n(T)_{\lambda^0}) \) and for any irreducible components \( \mathcal{P} \) of \( \sigma_F(T) \) one has \( (-1)^n l_p(T) \neq 0 \). Therefore, in this case \( \omega(T) = \sigma_\omega(T) \), and \( T \) has "Weyl property".

One can derive a criteria for Weyl points using the spacial characteristics of the tuple \( T \) considered above. Let \( \lambda^0 \in \sigma_F(T) \), and suppose that \( \mathcal{P}_1, \ldots, \mathcal{P}_s \) are the irreducible components of \( \sigma_F(T) \) containing \( \lambda^0 \), with dimensions \( r_1, \ldots, r_s \) corr. Let \( U \) be a neighborhood of \( \lambda^0 \) not intersecting other irreducible components, and take \( \lambda^1, \ldots, \lambda^s \in U \) such that \( \lambda^i \in \mathcal{P}_i \) and \( \lambda^i \notin \mathcal{P}_j \) for \( j \neq i \). One can take local coordinate systems in \( U \) for any of components \( \mathcal{P}_i, i = 1, \ldots, s \), and construct the corresponding local Euler characteristics \( \chi' \left( T', \lambda^i \right) \). Then from 2.75 and 2.15 one obtains:

**Proposition 2.17.** \( \lambda^0 \) is a Weyl point iff all \( \chi' \left( T', \lambda^i \right) = 0, i = 1, \ldots, s \).

The functoriality of \( z(T) \), stated in theorem 2.3, implies the functorial properties of \( \omega(T) \).

**Proposition 2.18.** Let \( f(\lambda) = (f_1(\lambda), \ldots, f_k(\lambda)) \) be a \( k \)-tuple of holomorphic functions, defined in the neighborhood of the spectrum \( \sigma(T) \) of \( T \). Then \( \omega(f(T)) \subset f(\omega(T)) \).

**Proof.** Take a point \( \mu^0 \notin f(\omega(T)) \). Then \( f^{-1}(\mu^0) \), if non-empty, is a finite subset of \( \rho_\omega(T) \), and the same is true for the points \( \mu \) in a sufficiently small neighborhood of \( \mu^0 \). Then from theorem 2.3 and 2.15 2/ it follows that \( \mu^0 \in \rho_\omega(f(T)) \). \( \Box \)

In some particular cases the spectral inclusion in the proposition above can be replaced by equality; indeed, the following proposition follows immediately from 2.3 in the same way as above:

**Proposition 2.19.** Suppose that one of the following two conditions is satisfied:

a/ \( f(z) \) is monomorphic on \( \sigma(T) \), or
b/ the operator \( T \) has SVEP.

Then \( \omega(f(T)) = f(\omega(T)) \).
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