A stabilized separation of variables method for the modified biharmonic equation

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Abstract

The modified biharmonic equation is encountered in a variety of application areas, including streamfunction formulations of the Navier-Stokes equations. We develop a separation of variables representation for this equation in polar coordinates, for either the interior or exterior of a disk, and derive a new class of special functions which makes the approach stable. We discuss how these functions can be used in conjunction with fast algorithms to accelerate the solution of the modified biharmonic equation or the “bi-Helmholtz” equation in more complex geometries.

1 Introduction

Many fourth order elliptic partial differential equations of physical interest can be expressed in terms of the composition of two second order elliptic differential operators. We focus here on the modified biharmonic equation in two dimensions, but much of what follows can be applied to other equations, such as the bi-Helmholtz equation, and has a natural extension to the three dimensional setting. In a domain $\Omega$ with boundary $\Gamma$, the modified biharmonic equation can be written in the form

\[(\Delta^2 - \lambda^2\Delta)u = \Delta(\Delta - \lambda^2)u = 0 \text{ in } \Omega,\]

with $\lambda \in \mathbb{R}$, subject to two application-specific, boundary conditions. This equation arises naturally when solving the the Navier-Stokes equations using an implicit marching scheme \cite{1,2,3,4,5}.

Since the Green’s function for the governing equation is \cite{5}

\[G(r) = -\frac{1}{2\pi\lambda^2} \left(\ln r + K_0(\lambda r)\right),\]

any solution to (1) can be expressed using separation of variables in the interior of a disk by

\[u(r,\theta) = \sum_{n=-\infty}^{\infty} \left[\alpha_n r^{|n|} + \beta_n I_n(\lambda r)\right] e^{in\theta}.\]  

In the exterior of a disk, assuming the solution is bounded, the general solution takes the form

\[u(r,\theta) = C_0 + C_1 K_0(\lambda r) + \sum_{n=-\infty}^{\infty} \left[\alpha_n r^{-|n|} + \beta_n K_n(\lambda r)\right] e^{in\theta}.\]
Unfortunately, using this representation naively for interior or exterior boundary value problems on a disk of radius $R$ leads to numerical instabilities when the value $\lambda R$ is small. While this problem is of interest in its own right, such expansions play a role in more general domains when multipoles and local expansions are used to represent outgoing and incoming fields in more general geometries, as explained below. Here, we propose a stabilized separation of variables approach based on a new class of special functions.

**Remark 1.** In the case of the bi-Helmholtz equation,

$$ (\Delta - \lambda_1^2)(\Delta - \lambda_2^2)u = 0, \quad (4) $$

the separation of variables representation in the interior of a disk is given by

$$ u(r, \theta) = \sum_{n=-\infty}^{\infty} [\alpha_n I_n(\lambda_1 r) + \beta_n I_n(\lambda_2 r)] e^{in\theta}. \quad (5) $$

In this setting, it is perhaps clearer that there are two dimensionless quantities involved: $\lambda_1 R$ and $\lambda_2 R$, with obvious ill-conditioning involved when $\lambda_1 \approx \lambda_2$.

The remainder of this paper is organized as follows. In sections 2 and 3, we provide some mathematical preliminaries, review the classical separation of variables approach to the modified biharmonic problem, and define new functions $Q_n$ and $P_n$ for stably representing solutions in the small $\lambda R$ regime. In section 4, we present numerical examples to illustrate the necessity for stabilization and to demonstrate the efficacy of our new functions. In the discussion of section 5, we outline how these functions can be used for stably solving (1) on more complex geometries with an accelerated integral equation method. Such methods will depend on translation operators for series using the $Q_n$ and $P_n$ functions, which we include in appendix B. Appendix A provides some needed properties of Bessel and Laurent series.

## 2 Mathematical preliminaries

### 2.1 Notation

In the following, we use the “big O” notation to describe the order of the remaining terms in a power series. The expression $f(\epsilon) = \mathcal{O}(g(\epsilon))$ implies that there exist positive constants $C$ and $\epsilon_0$ such that

$$ |f(\epsilon)| \leq C g(\epsilon), \quad (6) $$

when $0 < \epsilon < \epsilon_0$. As we are concerned with the asymptotic behavior with respect to two variables, we often write $f(\epsilon, \delta) = \mathcal{O}(g(\epsilon, \delta))$ which implies that there exist positive constants $C$, $\epsilon_0$, and $\delta_0$ such that

$$ |f(\epsilon, \delta)| \leq C g(\epsilon, \delta), \quad (7) $$

when $0 < \epsilon < \epsilon_0$ and $0 < \delta < \delta_0$.

In section 5, we use the “big O” notation to describe computational cost. For this case, the expression $f(N) = \mathcal{O}(g(N))$ implies that there exist positive constants $C$ and $N_0$ such that

$$ |f(N)| \leq C g(N), \quad (8) $$

when $N > N_0$. 

2
2.2 Condition numbers of 2 × 2 linear systems and diagonal scaling

In this section, we review some basic results from linear algebra. Consider an invertible linear system of the form

\[ Ax = b. \]  

(9)

The condition number \( \kappa(A) \) of the matrix \( A \) describes the sensitivity of the problem of recovering \( x \) from \( b \). Suppose that an approximate solution \( x_0 \) is found such that

\[ \frac{\|b - Ax_0\|_2}{\|b\|_2} = \epsilon, \]  

(10)

where \( \|\cdot\|_2 \) denotes the Euclidean norm. Let \( \kappa(A) = \sigma_{\text{max}}(A)/\sigma_{\text{min}}(A) \), where \( \sigma_{\text{max}}(A) \) and \( \sigma_{\text{min}}(A) \) denote the maximum and minimum singular values of \( A \). Then, \( x_0 \) satisfies

\[ \frac{\|x - x_0\|_2}{\|x\|} \leq \kappa(A) \epsilon. \]  

(11)

Suppose that each column \( a_i \) of \( A \) represents some function from a basis and that \( x \) represents the coefficients which reconstruct \( b \) in that basis. In this case, the notion of the sensitivity of \( x \) to changes in \( b \) should be unaffected by scaling the columns of \( A \). Let \( D \) be an invertible diagonal matrix and \( A, x, b, \) and \( x_0 \) be as above. We note that the residual is unaffected by scaling \( x_0 \) and \( A \), i.e. that \( \|b - Ax_0\|_2 = \|b - (AD)(D^{-1}x_0)\|_2 \) but that the condition numbers of \( A \) and \( AD \) can be significantly different. To avoid this ambiguity, we quantify the sensitivity of recovering \( x \) from \( b \) in terms of the condition number of \( \tilde{A} = AD \), where \( D \) is a diagonal matrix with \( D_{ii} = 1/\|a_i\|_2 \).

For 2 × 2 matrices, this is a natural normalization. It is straightforward to prove

**Proposition 1.** Let \( A \) be a 2 × 2 matrix with columns \( a_i \). If \( D \) is a diagonal matrix with \( D_{ii} = 1/\|a_i\|_2 \), then

\[ \kappa(AD) = \min_{v \in \mathbb{R}^2} \kappa(A \text{diag}(v)), \]  

(12)

where \( \text{diag}(v) \) denotes the diagonal matrix whose main diagonal is given by \( v \).

It is particularly simple to characterize the condition number of matrices with this scaling. We have

**Proposition 2.** Let \( A \) be a 2 × 2 matrix with columns denoted by \( a_i \). Suppose that \( \|a_i\| = 1 \). Then

\[ \kappa = \sqrt{\frac{1 + c}{1 - c}}, \]  

(13)

where \( c = |a_1^T a_2| \) is the cosine of the angle between the two columns of \( A \).

2.3 Separation of variables

Consider the modified biharmonic equation where \( \Omega \) is the disk of radius \( R \) centered at the origin. For given functions \( f \) and \( g \), we prescribe Dirichlet boundary conditions on \( u \), i.e.
where \( \partial_n \) denotes the outward normal derivative. Other types of boundary conditions may be considered.

To take advantage of the simplicity of this geometry, we translate the problem to a polar coordinate system. Let \((r, \theta)\) denote the usual polar coordinates for the point \((x, y) \in \mathbb{R}^2\) as in the following change of variables

\[
\begin{align*}
x &= r \cos(\theta) \\
y &= r \sin(\theta)
\end{align*}
\]

(16)

with \(\theta \in [-\pi, \pi)\). Then, the Laplacian is given by

\[
\Delta = \partial_{rr} + \frac{1}{r} \partial_r + \frac{1}{r^2} \partial_{\theta\theta} .
\]

(17)

Consider the boundary data as functions of the coordinate \(\theta\). For \(f\) and \(g\) sufficiently smooth, the Fourier series

\[
f(R, \theta) = \sum_{n=-\infty}^{\infty} f_n e^{i n \theta} ,
\]

(18)

\[
g(R, \theta) = \sum_{n=-\infty}^{\infty} g_n e^{i n \theta} ,
\]

(19)

converge uniformly in \(\theta\) and the coefficients \(f_n\) and \(g_n\) are given by

\[
f_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(R, \theta) e^{-i n \theta} \, d\theta ,
\]

(20)

\[
g_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(R, \theta) e^{-i n \theta} \, d\theta .
\]

(21)

Following standard practice [7, 8], we seek a solution \(u\) of the form

\[
u(r, \theta) = \sum_{n=-\infty}^{\infty} u_n(r) e^{i n \theta} .
\]

(22)

Plugging this form for \(u\) into (1), we obtain, after some simplification,

\[
\sum_{n=-\infty}^{\infty} \left( \partial_{rr} + \frac{1}{r} \partial_r - \frac{n^2}{r^2} \right) \left( \partial_{rr} + \frac{1}{r} \partial_r - \frac{n^2}{r^2} - \lambda^2 \right) u_n(r) e^{i n \theta} = 0 .
\]

(23)

The above implies that the radial function \(u_n(r)\) satisfies the following ordinary differential equation

\[
\left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{n^2}{r^2} \right) \left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{n^2}{r^2} - \lambda^2 \right) u_n(r) = 0 ,
\]

(24)
subject to certain boundary or regularity conditions. For \( n \neq -1, 0, 1 \), we have

\[
\begin{align*}
    u_n(R) &= f_n, & u'_n(R) &= g_n, \\
    u_n(0) &= 0, & u'_n(0) &= 0.
\end{align*}
\] (25)

For \( n = 0 \), we have

\[
\begin{align*}
    u_0(R) &= f_0, & u'_0(R) &= g_0, \\
    u'_0(0) &= 0, & u''_0(0) &= 0.
\end{align*}
\] (26)

Finally, for \( n = -1, 1 \), we have

\[
\begin{align*}
    u_n(R) &= f_n, & u'_n(R) &= g_n, \\
    u_n(0) &= 0, & u''_n(0) &= 0.
\end{align*}
\] (27)

The conditions at \( r = 0 \) are derived by assuming that \( u \) has four continuous derivatives at the origin.

It is well known \[9, 10\] that equation (24) has the following four linearly independent solutions for \( n \neq 0 \):

\[
\begin{align*}
    r^{|n|}, & r^{-|n|}, & I_n(\lambda r), & K_n(\lambda r),
\end{align*}
\]

where \( I_n \) and \( K_n \) are the modified Bessel functions. For \( n = 0 \), the functions \( 1, \log r, I_0(\lambda r), K_0(\lambda r) \) are linearly independent solutions. The regularity of the solution at zero eliminates \( K_n(\lambda r), r^{-|n|}, \log r \) from the acceptable solution set. Therefore, the allowed functions \( u_n \) are linear combinations of

\[
    u_n(r) = \alpha_n r^{|n|} + \beta_n I_n(\lambda r).
\] (28)

The boundary conditions for \( u_n \) determine \( \alpha_n \) and \( \beta_n \), with the conditions at \( r = 0 \) automatically satisfied. From the conditions at \( r = R \), we obtain the following linear system for \( \alpha_n \) and \( \beta_n \)

\[
\begin{pmatrix}
    R^{|n|} \\
    |n|R^{|n|}-1
\end{pmatrix} \begin{pmatrix}
    \lambda \\
    2
\end{pmatrix} \begin{pmatrix}
    I_n(\lambda R) \\
    (I_{n-1}(\lambda R) + I_{n+1}(\lambda R))
\end{pmatrix} \begin{pmatrix}
    \alpha_n \\
    \beta_n
\end{pmatrix} = \begin{pmatrix}
    f_n \\
    g_n
\end{pmatrix}. 
\] (29)

The determinant of the system in (29) is \( \lambda R^{|n|}I_{n+1}(\lambda R) \), which is nonzero for positive \( R \). Therefore, the coefficients \( \alpha_n \) and \( \beta_n \) are determined by the boundary conditions and the above provides an algorithm for computing \( u \).

In the exterior of a disk, the derivation is analogous to the above. For simplicity, we consider solutions of \([1]\) which are bounded with derivatives that are \( o(1/r) \) as \( r \) goes to infinity. This is sufficient for the solutions of the exterior problem to be unique; see, for example, Proposition 3.5 in \[5\]. The functions appropriate for the exterior of a disk are then \( K_n(\lambda r) \) and \( r^{-|n|} \). As in (29), we obtain a linear system for the expansion coefficients

\[
\begin{pmatrix}
    R^{-|n|} \\
    -|n|R^{-|n|-1}
\end{pmatrix} \begin{pmatrix}
    \lambda \\
    2
\end{pmatrix} \begin{pmatrix}
    K_n(\lambda R) \\
    (K_{n-1}(\lambda R) + K_{n+1}(\lambda R))
\end{pmatrix} \begin{pmatrix}
    \alpha_n \\
    \beta_n
\end{pmatrix} = \begin{pmatrix}
    f_n \\
    g_n
\end{pmatrix}. 
\] (30)

The determinant of this system is \( -\lambda K_{n+1}(\lambda R)/R^{|n|} \) so that it is invertible.

In order to make a numerical method out of the above, one simply truncates the Fourier series expansions at some finite \( N \), i.e.
The formulas \((20)\) and \((21)\) for the coefficients \(f_n\) and \(g_n\) can be approximated using the trapezoidal rule with \(M = 2N + 2\) equispaced points on \(\Gamma\) and computed rapidly via the fast Fourier transform \([11]\). The rate of convergence (in \(N\)) depends on the smoothness of the boundary data \(f\) and \(g\), with spectral convergence for analytic \(f\) and \(g\).

3 Analysis of the separation of variables problem and new basis functions

The difficulty with the above procedure is in solving the linear systems \((29)\) and \((30)\). In particular, for small \(\lambda R\), the columns of these system matrices are nearly linearly dependent, i.e., as \(\lambda\) goes to zero, the angle between the columns goes to zero (there is a similar effect for small \(R\)). As noted in section 2.2, this makes the problem of recovering the coefficients, \(\alpha_n\) and \(\beta_n\), from the data, \(f_n\) and \(g_n\), unstable. In this section, we will investigate the nature of this ill-conditioning and derive new bases which are better conditioned.

We first fix some notation. For a pair of functions \((F(r), G(r))\), define the matrix \(A(F, G, R)\) to be

\[
A(F, G, R) = \begin{pmatrix}
F(R) & G(R) \\
F'(R) & G'(R)
\end{pmatrix}.
\]

This is the form of the matrix that appears in the linear systems \((29)\) and \((30)\) used to solve for the coefficients \(\alpha_n\) and \(\beta_n\). Let \(\hat{B}\) denote the matrix \(B\) with its columns normalized to unit length.

For the interior problem, the ill-conditioning of the basis \((r^{[n]}, I_n)\) results from the fact that the \(I_n(\lambda r)\) and \(r^{[n]}\) are very similar functions for small \(r\); they have the same asymptotic behavior to leading order. The power series for \(I_n(\lambda r)\) is given by

\[
I_n(\lambda r) = \sum_{k=0}^{\infty} \left( \frac{\lambda r}{2} \right)^{2k+|n|} \frac{1}{k!(k+|n|)!} (\lambda r)^{|n|} + \frac{1}{2^{|n|+2}(|n|+1)!} (\lambda r)^{|n|+2} + \cdots,
\]

see \([12\text{ Ch. 10}]\) for reference. By substituting this expression into \(A(r^{[n]}, I_n, R)\), we obtain, for \(n \neq 0\),

\[
A(r^{[n]}, I_n, R) = \begin{pmatrix}
\frac{R^{[n]}}{|n|} & \frac{R^{[n]}a_n(\lambda)(1 + \mathcal{O}(\lambda^2 R^2))}{|n|R^{[n]-1}} \\
|n|R^{[n]-1} & |n|R^{[n]-1}a_n(\lambda)(1 + \mathcal{O}(\lambda^2 R^2))
\end{pmatrix},
\]
where \(a_n(\lambda) = \lambda^n/(2^n|n|n!)\). We see that the columns of \(\tilde{A}(r^n, I_n, R)\) are nearly co-linear in the limit as either \(\lambda\) or \(R\) goes to zero. The dependence on \(\lambda\) and \(R\) is not identical (this is wrapped up in the “big O” expressions); in the next section, we see that the condition number of the normalized matrix generally increases faster as \(R\) goes to zero than it does as \(\lambda\) goes to zero. For \(n = 0\), we have the system

\[
A(1, I_0, R) = \begin{pmatrix}
1 & 1 + O(\lambda^2 R^2) \\
0 & 1/2\lambda R(1 + O(\lambda^2 R^2))
\end{pmatrix},
\]

so that the condition number increases faster as \(\lambda\) goes to zero in this case.

To alleviate this ill-conditioning, we construct basis functions which are less “similar” in the small \(R\) and small \(\lambda\) limits. Let us define the functions \(P_n(r)\) by

\[
P_n(r) = I_n(\lambda r) - \left(\frac{\lambda r}{2}\right)^{|n|} \frac{1}{|n|!}
\]

(38)
deleting the first term in the power series for \(I_n\). Note that, \(P_n\) is a solution of (24) because it is a linear combination of \(I_n\) and \(r^n\). The matrix we obtain for the basis \((r^n, P_n)\) is

\[
A(r^n, P_n, R) = \begin{pmatrix}
R^n |n|R_{n-1} & R^{n+2}b_n(\lambda)(1 + O(\lambda^2 R^2)) \\
0 & (|n| + 2)R^{n+1}b_n(\lambda)(1 + O(\lambda^2 R^2))
\end{pmatrix}
\]

(39)

where \(b_n(\lambda) = \lambda^{n+2}/(2^{n+2}(n+1)!\). As \(\lambda\) goes to zero, the columns of \(\tilde{A}(r^n, P_n, R)\) do not converge to the same vector, as in the above. Further, as \(R\) goes to zero, the normalized columns do converge to the same limit but the effect is not as dramatic as for the pair \((r^n, I_n)\). The problem of recovering the coefficients is therefore more stable for the basis \((r^n, P_n)\), which we verify numerically in the next section.

**Remark 2.** It is simple to evaluate \(P_n(r)\) stably. For small \(\lambda r\), the power series for \(I_n\), with the first term omitted, may be used. For larger \(r\), there is no fear of numerical cancellation and the formula (38) may be used directly, along with existing software for evaluating \(I_n\). We choose the pair \((r^n, P_n)\) as opposed to \((I_n, P_n)\) because \(P_n\) and \(r^n\) have different asymptotic behavior for large \(r\), whereas \(P_n\) and \(I_n\) both grow exponentially.

**Remark 3.** If the functions \(I_n\) and \(r^n\) were used as a basis themselves, their asymptotic similarity in the small \(R\) regime would cause other numerical problems for a solution which behaves like \(P_n\). To see this, note that for small \(r\), the function \(P_n(r)\) is \(O(r^n+2)\), while the functions \(I_n(r)\) and \(r^n\) are \(O(r^n)\). Therefore, there is significant numerical cancellation when evaluating \(P_n(r)\) via the formula (38). We illustrate this effect in the next section.

A similar analysis applies to the exterior problem. The power series for \(K_n(\lambda r)\) is given by

\[
K_n(\lambda r) = \frac{1}{2}(\frac{1}{2}\lambda r)^{-|n|} \sum_{k=0}^{|n|-1} \frac{(|n| - k - 1)!}{k!(|n| + 1)!} (-2^{-1}\lambda r^2)^k + (-1)^{|n|+1} \ln \left(\frac{1}{2}\lambda r\right) I_n(\lambda r)
\]

\[
+ (-1)^{|n|} \frac{1}{2}(\frac{1}{2}\lambda r)^{|n|} \sum_{k=0}^\infty \left(\psi(k+1) + \psi(|n|+k+1)\right) \frac{(\frac{1}{2}\lambda r^2)^k}{k!(|n| + k)!}.
\]

(40)
Substituting this expression into \( A(r^{-|n|}, K_n, R) \), we obtain, for \( n \neq 0 \),

\[
A(r^{-|n|}, K_n, R) = \begin{pmatrix}
R^{-|n|} & R^{-|n|}c_n(\lambda)(1 + \mathcal{O}(\lambda^2 R^2)) \\
-|n| R^{-|n|-1} & -|n| R^{-|n|-1} c_n(\lambda)(1 + \mathcal{O}(\lambda^2 R^2))
\end{pmatrix},
\]

(41)

where \( c_n(\lambda) = (|n| - 1)!2^{n-1}\lambda^{-|n|} \). Again, after normalization, this linear system is ill-conditioned as either \( R \) or \( \lambda \) goes to zero because the normalized columns become nearly colinear. For the case \( n = 0 \), the basis \((1, K_n)\) results in the system

\[
A(1, K_n, R) = \begin{pmatrix}
1 & -\gamma + \log(2) - \log(\lambda R)(1 + \mathcal{O}(\lambda^2 R^2)) \\
0 & -\frac{1}{R}(1 + \mathcal{O}(\lambda^2 R^2|\log(\lambda R)|))
\end{pmatrix},
\]

(42)

which is actually well conditioned, after normalization, for small \( \lambda \) and \( R \). For certain exterior problems, the basis \((\log r, K_n)\) is more appropriate. In this case, we have

\[
A(\log r, K_n, R) = \begin{pmatrix}
\log R & -\gamma + \log(2) - \log(\lambda R)(1 + \mathcal{O}(\lambda^2 R^2)) \\
\frac{1}{R} & -\frac{1}{R}(1 + \mathcal{O}(\lambda^2 R^2|\log(\lambda R)|))
\end{pmatrix}.
\]

(43)

After normalization, this system is generally well-conditioned as \( \lambda \) goes to zero. As \( R \) goes to zero, however, the two columns become nearly colinear and the normalized matrix is ill-conditioned. Regardless of these special cases, the instability for the \( n \neq 0 \) coefficients will negatively affect the separation of variables approach.

To avoid this instability, we can define new functions \( Q_n \) for \( n \neq 0 \) as

\[
Q_n(r) = K_n(\lambda r) - \frac{2^{|n|-1}(|n| - 1)!}{\lambda^{|n|}r^{|n|}}.
\]

(44)

The function \( Q_n \) has a different leading order term from \( K_n \) as \( \lambda \) and \( r \) go to zero but is still a solution of (24) as it is a linear combination of \( K_n \) and \( r^{-|n|} \). As noted above, the system (30) is well conditioned for the functions 1 and \( K_0(\lambda r) \). Therefore, the naïve approach works for the zero mode, if you are indeed interested in solving the exterior problem subject to the conditions we have set at infinity. It is convenient, however, to define \( Q_0 \) as

\[
Q_0(r) = K_0(\lambda r) + \log(r).
\]

(45)

This function is closely related to the Green’s function for the modified biharmonic equation.

As before, the ill-conditioning of the coefficient recovery problem (29) is improved if you use the pair of functions \((Q_n, K_n)\) as your basis. For \(|n| > 2\), we obtain a system of the form

\[
A(Q_n, K_n, R) = \begin{pmatrix}
R^{-|n|+2}d_n(\alpha)(1 + \mathcal{O}(\lambda^2 R^2)) & R^{-|n|}c_n(\lambda)(1 + \mathcal{O}(\lambda^2 R^2)) \\
(-|n| + 2)R^{-|n|+1}d_n(\alpha)(1 + \mathcal{O}(\lambda^2 R^2)) & -|n| R^{-|n|-1} c_n(\lambda)(1 + \mathcal{O}(\lambda^2 R^2))
\end{pmatrix},
\]

(46)

where \( d_n(\lambda) = -2^{|n|-3}(|n| - 2)!\lambda^{-|n|+2} \). As \( \lambda \) goes to zero, the normalized columns of this system matrix do not converge to the same vector, as in the above. Further, as \( R \) goes to zero, the columns do converge but the effect is not as dramatic as for the pair \((r^{-|n|}, K_n)\).

The problem of recovering the coefficients is therefore more stable for the basis \((Q_n, K_n)\) than it is for the basis \((r^{-|n|}, K_n)\), which we verify numerically in the next section.
Remark 4. It is simple to evaluate $Q_n(r)$ stably. For small $r$, the power series for $K_n$, with the first term omitted, may be used. For larger $r$, there is no fear of numerical cancellation and the formula (44) may be used directly, along with existing software for evaluating $K_n$. We choose the pair $(Q_n, K_n)$ as opposed to $(r^{-|n|}, Q_n)$ because $Q_n$ and $K_n$ have different asymptotic behavior for large $r$.

Remark 5. If the functions $K_n$ and $r^{-|n|}$ are used as a basis, their asymptotic similarity in the small $R$ regime will cause other numerical problems for a solution with terms like $Q_n$. Suppose that $K_n$ and $r^{-|n|}$ are used to evaluate $Q_n$. For small $r$, the function $Q_n(r)$ is $O(r^{-|n|+2})$, while the functions $K_n(r)$ and $r^{-|n|}$ are $O(r^{-|n|})$. Therefore, there is significant numerical cancellation when evaluating $Q_n(r)$ via the formula (44). We demonstrate this effect as well in the next section.

Before proceeding to the numerical experiments, we briefly describe the edge cases, i.e. the matrices for $|n| \leq 2$. When $|n| = 2$, the pair $(Q_n, K_n)$ results in a linear system of the form

$$A(Q_2, K_2, R) = \begin{pmatrix} -\frac{1}{2} + O(R^2 \lambda^2 |\log(\lambda R)|) \\ -\frac{1}{4} \lambda^2 R \log(\lambda R) + O(\lambda^2 R) \quad -\frac{2}{\lambda^2 R^2} + O(1) \end{pmatrix}.$$ (47)

The columns of this matrix, after normalization, do not become colinear as either $\lambda$ or $R$ tends to zero. For the case that $|n| = 1$, the pair $(Q_n, K_n)$ results in a linear system of the form

$$A(Q_1, K_1, R) = \begin{pmatrix} \frac{1}{2} \lambda R \log(\lambda R) + O(\alpha R) \\ \frac{1}{2} \lambda \log(\lambda R) + (\lambda + R^2) \quad \frac{1}{\lambda R^2} + O(\lambda |\log(\lambda R)|) \end{pmatrix},$$ (48)

which has nearly colinear columns after normalization as $R$ goes to zero. The normalized columns are not colinear in the limit as $\lambda$ goes to zero. Finally, for the case $n = 0$, the pair $(Q_n, K_n)$ results in a linear system of the form

$$\begin{pmatrix} -\log(\lambda/2) - \gamma + O(\lambda^2 R^2 |\log(\lambda R)|) \\ -\frac{1}{2} \lambda^2 R \log(\lambda R) + O(\lambda^2 R) \end{pmatrix} \begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix} = \begin{pmatrix} f_n \\ g_n \end{pmatrix}.$$ (49)

As $\lambda$ goes to zero, the columns of this matrix slowly become colinear after normalization. As $R$ goes to zero, the normalized columns do not become colinear.

4 Numerical tests

In this section we present some numerical experiments which reinforce the ideas of the previous section. The source code used for these calculations is available online. The Bessel functions were evaluated using routines from FMMLIB2D. Discrete Fourier transforms were computed using the FFTPACK. To compute condition numbers, we used the singular value decomposition routine from EISPACK. All code was written in Fortran using double precision arithmetic and compiled with the gfortran compiler on Linux.
4.1 Condition numbers

For the first test, we verify the analytical observations about the condition numbers of the matrices described in the previous section. For a pair of functions \((F(r), G(r))\), define the matrix \(A(F, G, R)\) as in (34). Let \(\hat{B}\) denote the matrix \(B\) with its columns normalized to unit length. We compute the condition number of \(\hat{A}(F, G, R)\) where \((F, G)\) is taken to be each of the bases \((r^{[n]}, I_n(\lambda r)), (r^{[n]}, P_n(\lambda r)), (r^{-[n]}, K_n(\lambda r)),\) and \((Q_n(\lambda r), K_n(\lambda r))\) for a range of values of \(n, \lambda, \) and \(R\). To observe the effect of changing the radius of the domain, we run experiments with \(\lambda\) fixed (at \(\lambda = 0.5\)) and, for each \(j\) from \(-24\) to \(8\), ten values of \(R\) drawn uniformly at random from the interval \([2^j, 2^{j+1}]\). Likewise, to observe the effect of changing the parameter \(\lambda\), we run experiments with \(R\) fixed (at \(R = 0.5\)) and, for each \(j\) from \(-24\) to \(8\), ten values of \(\lambda\) drawn uniformly at random from the interval \([2^j, 2^{j+1}]\). In order to compare these, we plot the condition number as a function of the product \(\lambda R\).

In fig. 1 we plot the results for the interior problem. The new basis \((r^{[n]}, P_n)\) has smaller condition numbers than the basis \((r^{[n]}, I_n)\), as either \(\lambda\) or \(R\) goes to zero. In the limit as \(\lambda\) goes to zero, we see that the condition number remains roughly constant for \((r^{[n]}, P_n)\). In the limit as \(R\) goes to zero, there is some growth in the condition number for \((r^{[n]}, P_n)\), except for the case \(n = 0\), where it again remains roughly constant. For the basis \((r^{[n]}, I_n)\), the condition number grows as either \(\lambda\) or \(R\) tends to zero and is larger than for the basis \((r^{[n]}, P_n)\). The condition number tends to grow faster as \(R\) goes to zero compared to the growth as \(\lambda\) goes to zero, except for the case \(n = 0\), in which we see the opposite trend. These results agree well with the analysis of section 3.

In fig. 2 we plot the results for the exterior problem. The new basis \((Q_n, K_n)\) generally has smaller condition numbers than the basis \((r^{-[n]}, K_n)\), as either \(\lambda\) or \(R\) goes to zero. In the limit as \(\lambda\) goes to zero, we see that the condition number remains roughly constant for \((Q_n, K_n)\). In the limit as \(R\) goes to zero, there is some growth in the condition number for \((Q_n, K_n)\), except for the cases \(n = 0\) and \(n = 2\), where it again remains roughly constant. For the basis \((r^{-[n]}, K_n)\), the condition number grows faster as \(R\) goes to zero compared to the growth as \(\lambda\) goes to zero. The condition number for the basis \((r^{-[n]}, K_n)\) does grow as \(\lambda\) tends to zero, except in the case that \(n = 0\), for which the condition number remains roughly constant. Note that for \(n = 0\) we have used the basis \((\log(r), K_0)\) as this set of functions is required to represent the synthetic solution used for the error analysis of the next section. Again, these results agree well with the analysis of section 3.

Remark 6. In figs. 1 and 2 we observe two distinct behaviors when \(\lambda\) tends to zero for a fixed \(R\) and vice-versa. This phenomenon is related to the two relevant scales of the bi-Helmholtz kernel noted in remark 7, though more extreme: whereas the bi-Helmholtz operator is the composition of two operators with their own dimensionless parameters, the modified biharmonic operator is the composition of two operators, one of which is scale-invariant and the other is not. In the next set of experiments, we see that, in many applications, there is only one relevant scaling for the modified biharmonic equation.

4.2 Error plots for known solution

In order to test the practical effect of the ill-conditioning seen in the last section, we use the separation of variables procedure to solve the modified biharmonic equation (1) on a disk with boundary conditions corresponding to a known solution. We construct...
this solution using the free-space Green’s function for the modified biharmonic equation, which is defined as

\[
G(x, y) = -\frac{1}{2\pi \lambda^2} \left( K_0(\lambda \rho) + \log(\rho) \right),
\]

where \( \rho = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} \). The solution is then set to be

\[
u(x; \lambda) = \sum_{j=1}^{n_s} \lambda^2 c_j G(x, s_j) + \lambda d_j \partial_{v_j,1} G(x, s_j) + q_j \partial_{v_j,2v_j,3} G(x, s_j),
\]
where the $s_j$ are $n_s = 100$ “source” points located outside of the domain, the $c_j$ are drawn uniformly randomly from $[-1, 1]$, the $d_j$ and $q_j$ are drawn uniformly randomly from $[0, 1]$, and the vectors $v_{j,1}, v_{j,2}, v_{j,3}$ are defined by drawing the entries uniformly at random from $[-1/2, 1/2]$ and normalizing. We note that the $\lambda^2$ and $\lambda$ scales in front of the $c_j$ and $d_j$ are included to ensure that these terms are of roughly the same size as $\lambda$ shrinks.

To implement separation of variables, we discretize the boundary with $M = 100$ points, so that $N = 49$ (the separation of variables expansion runs from $-N$ to $N + 1$ as in section 2.3). We evaluate the function $u(x; \lambda)$ and its normal derivative on the boundary of the disk and compute their Fourier coefficients, i.e. the values $f_n$ and $g_n$ as in (20) and (21), using the FFT. We then solve for the coefficients $\alpha_n$ and $\beta_n$ by inverting...
the linear system

\[
\begin{pmatrix}
F_n(R) & G_n(R) \\
F'_n(R) & G'_n(R)
\end{pmatrix}
\begin{pmatrix}
\alpha_n \\
\beta_n
\end{pmatrix}
= 
\begin{pmatrix}
f_n \\
g_n
\end{pmatrix},
\] (52)

where \((F_n(r), G_n(r))\) is an appropriate pair of basis functions. For the sake of stability, the inversion is performed using Gaussian elimination with complete pivoting. Once the coefficients \(\alpha_n\) and \(\beta_n\) have been computed, the approximate solution \(\hat{u}\) can be evaluated using the formula

\[
\hat{u}(x) = \sum_{n=-N}^{N+1} (\alpha_n F_n(\rho) + \beta_n G_n(\rho)) e^{i n \theta},
\] (53)

where \((\rho, \theta)\) are the polar coordinates of \(x\). The derivatives of \(\hat{u}\) can be obtained by differentiating this expression.

To measure the performance of the separation of variables method, we evaluate \(\hat{u}\) and its first and second derivatives at \(n_t = 100\) “target” points \(t_i\) located inside the domain. We then define three error measures

\[
E_u = \sqrt{\frac{\sum_{i=1}^{n_t} (u(t_i; \lambda) - \hat{u}(t_i))^2}{\sum_{i=1}^{n_t} u(t_i; \lambda)^2}},
\] (54)

\[
E_g = \sqrt{\frac{\sum_{i=1}^{n_t} (u_x(t_i; \lambda) - \hat{u}_x(t_i))^2 + (u_y(t_i; \lambda) - \hat{u}_y(t_i))^2}{\sum_{i=1}^{n_t} u_x(t_i; \lambda)^2 + u_y(t_i; \lambda)^2}},
\] (55)

\[
E_h = \sqrt{\frac{\sum_{i=1}^{n_t} (u_{xx}(t_i; \lambda) - \hat{u}_{xx}(t_i))^2 + (u_{xy}(t_i; \lambda) - \hat{u}_{xy}(t_i))^2 + (u_{yy}(t_i; \lambda) - \hat{u}_{yy}(t_i))^2}{\sum_{i=1}^{n_t} u_{xx}(t_i; \lambda)^2 + u_{xy}(t_i; \lambda)^2 + u_{yy}(t_i; \lambda)^2}},
\] (56)

which represent the relative error in the solution, gradient, and Hessian, respectively.

As in the previous section, we run the tests with a variety of values for the radius of the disk \(R\) and the parameter \(\lambda\). To observe the effect of changing the radius of the domain, we run experiments with \(\lambda\) fixed (at \(\lambda = 0.5\)) and, for each \(j\) from \(-24\) to \(8\), ten values of \(R\) drawn uniformly at random from the interval \([2^j, 2^{j+1}]\). Likewise, to observe the effect of changing the parameter \(\lambda\), we run experiments with \(R\) fixed (at \(R = 0.5\)) and, for each \(j\) from \(-24\) to \(8\), ten values of \(\lambda\) drawn uniformly at random from the interval \([2^j, 2^{j+1}]\). In order to compare these, we plot the error measures \(E_u\), \(E_g\), and \(E_h\) as functions of the product \(\lambda R\).

For the interior problem, the source points are drawn uniformly at random from the box \([-2R, 2R] \times [-2R, 2R]\) outside of the disk of radius \(2R\). With this placement of source points, the length \(N = 49\) expansion for \(\hat{u}\) should be sufficient for approximately machine precision accuracy, based on standard multipole estimates [17, 18]. The target points are drawn uniformly at random from the disk of radius \(R\). We plot a sample arrangement of the source and target points for the interior problem in fig. [3a].

Similarly, for the exterior problem, the source points are drawn uniformly at random from the disk of radius \(R/2\). This placement is again chosen so that the length \(N = 49\) expansion for \(\hat{u}\) is sufficient for approximately machine precision accuracy. The target points are drawn uniformly at random from the box \([-2R, 2R] \times [-2R, 2R]\) outside of
Figure 3: Sample geometries for the interior and exterior problems with $R = 0.5$. The targets are marked by crosses and the sources by circles.

Figure 4: Heatmaps of the exact solution $u$ and select derivatives for the interior problem with $R = 0.5$ and $\lambda = 2^{-24}$.

Figure 5: Heatmaps of the exact solution $u$ and select derivatives for the exterior problem with $R = 0.5$ and $\lambda = 2^{-24}$.

the disk of radius $R$. We plot a sample arrangement of the source and target points for the exterior problem in fig. 3b.

In figs. 4 and 5, we plot a sample exact solution $u$ and some select derivatives for the
interior and exterior problems, respectively. For the following error plots, we consider the error in approximating these solutions using the bases we have discussed above as well as using what we call the “exact difference”. We include the exact difference figure to emphasize that it is not only the ability to recover $\alpha_n$ and $\beta_n$ that causes trouble using the naive bases. Because $u$ is defined in terms of the Green’s function $G$, which is simply a scaled sum of the Green’s functions for the Laplace and modified Helmholtz equations, we could reasonably evaluate $u$ by evaluating these parts separately and combining them in the end. Let $u_L$ and $u_H$ be defined as

$$u_L(x; \lambda) = \frac{1}{2\pi \lambda^2} \sum_{j=1}^{n_s} \lambda^2 c_j \log \|x - s_j\|_2 + \lambda d_j \partial_{v_{j,1}} \log \|x - s_j\|_2 + q_j \partial_{v_{j,2v_{j,3}}} \log \|x - s_j\|_2$$

(57)

and

$$u_H(x; \lambda) = -\frac{1}{2\pi \lambda^2} \sum_{j=1}^{n_s} \lambda^2 c_j K_0(\lambda \|x - s_j\|_2) + \lambda d_j \partial_{v_{j,1}} K_0(\lambda \|x - s_j\|_2) + q_j \partial_{v_{j,2v_{j,3}}} K_0(\lambda \|x - s_j\|_2).$$

(58)

The “exact difference” error below is the error in evaluating $u$ as the difference $u_H - u_L$ in floating point arithmetic.

Figure 6: Interior problem. In the top row, we plot the error measures as functions of $\lambda R$ for $R = 0.5$ as $\lambda$ goes to zero. In the bottom row, we plot the error measures as functions of $\lambda R$ for $\lambda = 0.5$ as $R$ goes to zero.
In fig. 6, we plot the error measures as functions of $\lambda R$ for both the limit as $R$ goes to zero with $\lambda$ fixed and vice-versa, for the interior problem. The behavior of the two limits is similar, in contrast with the condition numbers of the previous section. This is an indication that $\lambda R$ is the relevant figure for applications. We see that the new basis $(r^{[n]}, P_n)$ is able to achieve high accuracy, even as $\lambda R$ tends to zero. We note that $E_u$ is around machine precision, and there is some precision loss for the errors of the derivatives, $E_g$ and $E_h$. When using the naïve basis, $(r^{[n]}, I_n)$, on the other hand, there is significant loss of accuracy as $\lambda R$ goes to zero. The error in evaluating the exact difference between the harmonic and modified Helmholtz parts agrees well with the naïve basis $(r^{[n]}, I_n)$ in the small $\lambda R$ limit. This error for the exact difference shows that there is a fundamental problem in using the basis $(r^{[n]}, I_n)$ to represent such solutions. For large $\lambda R$, the exact difference is capable of near machine precision in even the derivatives, as the procedure does not really involve numerical differentiation. We see similar behavior for the exterior problem in fig. 7.

\section{Discussion}

In the preceding, we have shown that the new bases $(r^{[n]}, P_n)$ and $(Q_n, K_n)$ offer significant advantages over the naïve approach when solving the modified biharmonic equation on a disk. We now show how these functions can be used to efficiently solve the modified
biharmonic equation on more complex geometries.

5.1 Fast sums

Let \( \mathbf{x}_i \) be a set of \( N \) points in space. Suppose that we would like to evaluate the sum

\[
    u(\mathbf{x}_i) = \sum_{j \neq i} \lambda^2 c_j \mathcal{G}(\mathbf{x}_i, \mathbf{x}_j) + \lambda d_j \partial_{\nu_j} \mathcal{G}(\mathbf{x}_i, \mathbf{x}_j) + q_j \partial_{\tau_j} \mathcal{G}(\mathbf{x}_i, \mathbf{x}_j),
\]

for each \( \mathbf{x}_i \) efficiently. Direct evaluation would be an \( \mathcal{O}(N^2) \) calculation. When performing the analogous sum with the Laplace Green’s function, the fast multipole method provides a stable \( \mathcal{O}(N) \) algorithm. We do not seek to review the fast multipole method here, but we note that such a method depends on a few key parts: a formula for representing the sum due to a localized subset of the points (a multipole expansion), a formula for representing the sum due to points outside of a disk (a local expansion), formulas for translating between these representations (translation operators), and a hierarchical organization of the points in space. For details, see [17, 19].

Following the results of the previous section, we see that an expansion in terms of the functions \((Q_n, K_n)\) provides a stable representation for the sum due to points contained inside a disk when evaluated at points sufficiently far from that disk. Similarly, an expansion in terms of the functions \((r^{\text{mid}}, P_n)\) can be used to stably represent the sum due to points located sufficiently far outside of a disk. These are then our multipole and local expansions, respectively. Starting with the formulas for translating multipole and local expansions for the Laplace and modified Helmholtz Green’s functions, which are included in appendix A, it is straightforward to derive translation operators for the \((Q_n, K_n)\) and \((r^{\text{mid}}, P_n)\) expansions. We provide these in appendix B. Therefore, the new basis functions derived in this paper result in a stable fast multipole method for the modified biharmonic equation. Such a method has been implemented by the author and results will be reported at a later date.

5.2 Integral equations

Because \((1)\) is a homogeneous, fourth order equation, it is well suited to solution using an integral equation formulation. We do not attempt a review of the literature here but point to [1, 2, 3, 4, 5] for some representative examples. In an integral equation method, the solution is represented by a layer potential with unknown densities defined on the boundary \( \Gamma \), i.e.

\[
    u(\mathbf{x}) = \int_{\Gamma} K_1(\mathbf{x}, \mathbf{y}) \sigma_1(\mathbf{y}) + K_2(\mathbf{x}, \mathbf{y}) \sigma_2(\mathbf{y}) \, dS(\mathbf{y}),
\]

where the kernels \( K_1 \) and \( K_2 \) are typically defined in terms of directional derivatives of the free-space Green’s function \( \mathcal{G} \). For example, in [5], the kernels are \( K_1 = -\mathcal{G}_{\nu\nu} + \mathcal{G}_{\tau\tau} \) and \( K_2 = -2\mathcal{G}_{\nu\nu\nu} + 3(\Delta - \lambda^2)\mathcal{G}_{\nu} + 2\lambda^2\mathcal{G}_{\nu} \), where \( \nu \) and \( \tau \) represent the normal and tangential directions at \( \mathbf{y} \), respectively.

Continuing the example, the authors of [5] then impose gradient boundary conditions on \( u \), i.e. they set \( \partial_{\nu} u = f \) and \( \partial_{\tau} u = g \) for some functions \( f \) and \( g \) defined on the boundary, where \( \nu \) and \( \tau \) denote the normal and tangential directions at a point \( \mathbf{x} \) on the boundary. These boundary conditions are of physical interest because they correspond
to “no-slip” boundary conditions for a stream function representation of a fluid flow. Plugging the form (60) into the boundary conditions, we obtain the integral equation

\[
\left( \begin{array}{cc} D_{11}(x) & D_{12}(x) \\ 0 & D_{22}(x) \end{array} \right) \left( \begin{array}{c} \sigma_1(x) \\ \sigma_2(x) \end{array} \right) + \int_\Gamma \left( \begin{array}{cc} K_{11}(x,y) & K_{12}(x,y) \\ K_{21}(x,y) & K_{22}(x,y) \end{array} \right) \left( \begin{array}{c} \sigma_1(y) \\ \sigma_2(y) \end{array} \right) dS(y) = \left( \begin{array}{c} f(x) \\ g(x) \end{array} \right),
\]

(61)

where \( K_{ij} = \partial_{x_i} K_{ij} \), \( K_{12} = \partial_{y_i} K_{12} \), \( K_{21} = \partial_{y_j} K_{21} \), and \( K_{22} = \partial_{y_j} K_{22} \). See [3] for details, including a simple preconditioner for turning (61) into a well-conditioned second kind integral equation and explicit formulas for the kernels \( K_{ij} \).

In a Nyström discretization of the integral equation (61), the solution is represented by its values at points on the curve \( \Gamma \) which are used for an integration rule. The basis of a Nyström method is then a numerical quadrature of the integral operator on the curve \( \Gamma \). Typically, different integration rules are required for smooth, weakly singular, and singular integral kernels. Fortunately, a number of quadrature rules are available for handling these singularities with high order accuracy [20, 21, 22, 23].

Let \( x_i \) denote the points on \( \Gamma \) of the Nyström discretization, \( \sigma_{1i} = \sigma_1(x_i) \), \( \sigma_{2i} = \sigma_2(x_i) \), \( f_i = f(x_i) \), and \( g_i = g(x_i) \). In a slight abuse of notation, the quadrature rule provides weights \( w_{ij} \) such that

\[
\left( \begin{array}{c} D_{11}(x_i) \\ 0 \\ D_{22}(x_i) \end{array} \right) \left( \begin{array}{c} \sigma_{1i} \\ \sigma_{2i} \end{array} \right) + \sum_j w_{ij} \left( \begin{array}{cc} K_{11}(x_i,x_j) & K_{12}(x_i,x_j) \\ K_{21}(x_i,x_j) & K_{22}(x_i,x_j) \end{array} \right) \left( \begin{array}{c} \sigma_{1j} \\ \sigma_{2j} \end{array} \right) = \left( \begin{array}{c} f_i \\ g_i \end{array} \right),
\]

(62)

is an accurate approximation of the original integral equation at each point \( x_i \). The above is an abuse of notation because the kernels are not often defined when \( x = y \), so that the formula is in general a function of the kernel and the boundary. A key feature of these integral rules is that the weight \( w_{ij} \) is typically a function of \( j \) alone for points \( x_i \) and \( x_j \) which are sufficiently far apart. Therefore, much of the sum (62) is of the form (59) and the fast multipole method described in the previous section can be used to apply the operator on the left-hand-side of (62) rapidly. Combined with an iterative solver such as GMRES [24], this provides a fast solution method for the densities \( \sigma_1 \) and \( \sigma_2 \). The fast multipole method can also be utilized to efficiently evaluate the formula for \( u, \sigma, \) at points inside the domain [23, 25, 26].

6 Conclusion

We have presented new special functions for representing solutions of the modified biharmonic equation on both the inside and outside of a disk. Numerical experiments demonstrate the superiority of using these functions over the naïve approach, which would employ more familiar functions associated with the Laplace and modified Helmholtz equations. Further, we show how such functions can be used to aid in the solution of the modified biharmonic equation on more complex geometries with an integral equation approach. We also present, in appendix [13] translation operators for multipole and local expansions using our new radial basis functions. These are key components of fast multipole-like methods [19, 17].

The basic approach of the present paper applies to a number of other Green’s functions for high-order PDEs, assuming they can be expressed as the difference of Green’s functions for lower order PDEs. This includes, for example, the Green’s function for the
bi-Helmholtz equation \([4]\). Since the essential analysis concerns radial functions only, the approach extends naturally to three dimensions. We will explore these problems in subsequent work.

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Appendix A. Analytical preliminaries

For the Laplace kernel, the fast multipole method is based on the manipulation of multipole and Taylor expansions. The center of a multipole expansion may be shifted using this formula:

**Lemma.** (Adapted from Lemma 2.3 of [17].) Suppose that

\[
\phi(z) = a_0 \log(z - z_0) + \sum_{l=1}^{\infty} \frac{a_l}{(z - z_0)^l}
\]

(63)

is a multipole expansion of the potential due to a charge density which is contained inside the disk of radius \(R\) about \(z_0\). Then, for \(z\) outside the disk of radius \(R + |z_0|\) about the origin

\[
\phi(z) = b_0 \log(z) + \sum_{l=1}^{\infty} \frac{b_l}{z^l},
\]

(64)

where \(b_0 = a_0\) and

\[
b_l = \left( \sum_{m=1}^{l} a_m z_0^{l-m} \binom{l-1}{m-1} \right) - \frac{a_0 z_0^l}{l},
\]

(65)

using the standard notation for binomial coefficients. We also have the following bound for the truncation error: With \(p \geq 1\),

\[
\left| \phi(z) - b_0 \log(z) - \sum_{l=1}^{p} \frac{b_l}{z^l} \right| \leq \frac{F/2\pi}{1 - (|z_0| + R)/|z|} \left( \frac{|z_0| + R}{|z|} \right)^{p+1},
\]

(66)

where \(F\) is the \(L_1\) norm of the density.

A multipole expansion may be converted into a Taylor expansion using the following formula:

**Lemma.** (Adapted from Lemma 2.4 of [17].) Suppose that a charge density is contained inside the disk of radius \(R\) about \(z_0\) with \(|z_0| > (1+c)R\) for some \(c > 1\). Let the multipole expansion due to this density be given as in Lemma [A]. Then, this multipole expansion converges inside the disk of radius \(R\) about the origin and can be represented by a power series there:
\[ \phi(z) = \sum_{l=1}^{\infty} b_l z^l \]  
where,

\[ b_0 = \sum_{m=1}^{\infty} \frac{a_m}{z_0^m} (-1)^m + a_0 \log(-z_0), \]  

and

\[ b_l = \frac{1}{z_0^l} \left( \sum_{m=1}^{\infty} \frac{a_m}{z_0^m} \left( l + m - 1 \right) (-1)^m \right) - \frac{a_0}{z_0^l}, \]

for \( l \geq 1 \). There is a similar error bound for this lemma, see [17] for details. Suppose that the charge density is supported in a box and the evaluation points \( z \) are taken in another. In the case that these two boxes are well separated from each other, we have

\[ \left| \phi(z) - \sum_{l=0}^{p} b_l z^l \right| \leq CF \left( \frac{1}{2} \right)^p, \]

where \( F \) is as in Lemma A and \( C \) is a constant.

The center of a Taylor expansion can be shifted using the following formula:

**Lemma.** (Adapted from Lemma 2.5 in [17]) Let \( z_0, z, \) and \( a_l \) for \( l = 0, 1, \ldots, p \) be complex. Then

\[ \sum_{l=0}^{p} a_l (z - z_0)^l = \sum_{m=0}^{p} \left( \sum_{l=m}^{p} a_l \binom{l}{m} (-z_0)^{l-m} \right) z^m. \]  

This formula is exact.

The fast multipole method for the modified Helmholtz kernel is based on the manipulation of expansions in the Bessel functions \( I_n \) and \( K_n \). The formula for shifting the center of an expansion in the \( K_n \) functions is:

**Lemma.** (Adapted from [18].) Suppose that

\[ \phi(x) = \sum_{l=-\infty}^{\infty} a_l K_l(\lambda \rho') e^{i l \theta'}, \]  

where \((\rho', \theta')\) are the polar coordinates of \( x \) with respect to the point \( x_0 \), is a multipole expansion of the potential due to a charge density which is contained inside the disk of radius \( R \) about \( x_0 \). Let \((\rho_0, \theta_0)\) be the polar coordinates of \( x_0 \) with respect to the origin. Then, for \( x \) outside the disk of radius \( R + \rho_0 \), we have

\[ \phi(x) = \sum_{l=-\infty}^{\infty} b_l K_l(\lambda \rho) e^{i l \theta}, \]  

where \((\rho, \theta)\) are the coordinates of \( x \) with respect to the origin and the translated coefficients are given by
An expansion in the $K_n$ functions may be converted into an expansion in the $I_n$ functions using the formula:

**Lemma.** (Adapted from [18].) Suppose that

$$
\phi(x) = \sum_{l=-\infty}^{\infty} a_l K_l(\lambda \rho') e^{il\theta'},
$$

where $(\rho', \theta')$ are the polar coordinates of $x$ with respect to the point $x_0$, is a multipole expansion of the potential due to a charge density which is contained inside the disk of radius $R < |x_0|$ about $x_0$. Let $(\rho_0, \theta_0)$ be the polar coordinates of $x_0$ with respect to the origin. Then, for $x$ within the disk of radius $\rho_0 - R$, we have

$$
\phi(x) = \sum_{l=-\infty}^{\infty} b_l I_l(\lambda \rho) e^{il\theta},
$$

where $(\rho, \theta)$ are the coordinates of $x$ with respect to the origin and the translated coefficients are given by

$$
b_l = \sum_{m=-\infty}^{\infty} a_m (-1)^m K_{l-m}(\lambda \rho_0) e^{-i(l-m)\theta_0}.
$$

We phrase the following translation operator in terms of a parent and child box. A child box is any of the four boxes resulting from dividing a parent box into equal square quadrants. The center of an expansion in the $I_n$ functions may be shifted using the following formula:

**Lemma.** (Adapted from [18].) Suppose that

$$
\Psi(x) = \sum_{l=-\infty}^{\infty} a_l I_l(\lambda \rho) e^{il\theta},
$$

where $(\rho, \theta)$ are the polar coordinates of $x$ with respect to the origin, is a local expansion for a parent box centered at the origin. Let $(\rho_0, \theta_0)$ be the polar coordinates of the center $x_0$ of a child box with respect to the origin. Then, for $x$ inside the child box, we have

$$
\Psi(x) = \sum_{l=-\infty}^{\infty} b_l I_l(\lambda \rho') e^{il\theta'},
$$

where $(\rho', \theta')$ are the coordinates of $x$ with respect to the child’s center $x_0$ and the new local expansion coefficients are given by

$$
b_l = \sum_{m=-\infty}^{\infty} a_m I_{l-m}(\lambda \rho_0) e^{-i(l-m)\theta_0}.
$$
Appendix B. Translation properties of $Q_n$ and $P_n$

The stabilized fast multipole method for the modified biharmonic equation that we develop is based on manipulating multipole expansions in the functions $(Q_n, K_n)$ and local expansions in the functions $(r^n, P_n)$. The center of such a multipole expansion can be shifted using the following formula:

**Lemma.** Suppose that

$$\phi(x) = \sum_{l = -\infty}^{\infty} (a_l Q_l(\rho') + b_l K_l(\lambda \rho')) e^{i \phi},$$  \hspace{1cm} (81)

where $(\rho', \theta')$ are the polar coordinates of $x$ with respect to the point $x_0$, is a multipole expansion of the potential due to a charge density which is contained inside the disk of radius $R$ about $x_0$. Let $(\rho_0, \theta_0)$ be the polar coordinates of $x_0$ with respect to the origin. Then, for $x$ outside the disk of radius $R + \rho_0$, we have

$$\phi(x) = \sum_{l = -\infty}^{\infty} (c_l Q_l(\rho) + d_l K_l(\lambda \rho)) e^{i \phi},$$  \hspace{1cm} (82)

where $(\rho, \theta)$ are the coordinates of $x$ with respect to the origin. We have that $c_0 = a_0$ and

$$d_0 = \sum_{m = -\infty}^{-1} (a_m + b_m) I_{-m}(\lambda \rho_0) e^{-i(l-m)\theta_0} + a_0 P_0(\lambda \rho_0) + b_0 I_0(\lambda \rho_0)$$

$$+ \sum_{m=1}^{\infty} (a_m + b_m) I_{-m}(\lambda \rho_0) e^{-i(l-m)\theta_0}.$$  \hspace{1cm} (83)

For $l > 0$ the translated coefficients are given by

$$c_l = \sum_{m=0}^{l} a_m \left( \frac{\lambda \rho_0}{2} \right)^{l-m} e^{-i(l-m)\theta_0} \frac{1}{(l-m)!}$$  \hspace{1cm} (84)

and

$$d_l = \sum_{m=-\infty}^{-1} (a_m + b_m) I_{l-m}(\lambda \rho_0) e^{-i(l-m)\theta_0}$$

$$+ \sum_{m=0}^{l} (a_m P_{l-m}(\rho_0) + b_m I_{l-m}(\lambda \rho_0)) e^{-i(l-m)\theta_0}$$

$$+ \sum_{m=l+1}^{\infty} (a_m + b_m) I_{l-m}(\lambda \rho_0) e^{-i(l-m)\theta_0}. $$  \hspace{1cm} (85)

For $l < 0$ the translated coefficients are given by

$$c_l = \sum_{m=l}^{0} a_m \left( \frac{\lambda \rho_0}{2} \right)^{|l-m|} e^{-i(l-m)\theta_0} \frac{1}{|l-m|!}$$  \hspace{1cm} (86)
and
\[
d_l = \sum_{m=-\infty}^{l-1} (a_m + b_m) I_{l-m}(\lambda \rho_0) e^{-i(l-m)\theta_0} \\
+ \sum_{m=l}^{0} (a_m P_{l-m}(\rho_0) + b_m I_{l-m}(\lambda \rho_0)) e^{-i(l-m)\theta_0} \\
+ \sum_{m=1}^{\infty} (a_m + b_m) I_{l-m}(\lambda \rho_0) e^{-i(l-m)\theta_0} .
\] (87)

A multipole expansion can be converted into a local expansion using the following formula:

**Lemma.** Suppose that
\[
\phi(x) = \sum_{l=-\infty}^{\infty} \left( a_l Q_l(\rho') + b_l K_l(\lambda \rho') \right) e^{i l \phi'},
\] (88)
where \((\rho', \theta')\) are the polar coordinates of \(x\) with respect to the point \(x_0\), is a multipole expansion of the potential due to a charge density which is contained inside the disk of radius \(R < |x_0|\) about \(x_0\). Let \((\rho_0, \theta_0)\) be the polar coordinates of \(x_0\) with respect to the origin. Then, for \(x\) within the disk of radius \(\rho_0 - R\), we have
\[
\phi(x) = \sum_{l=-\infty}^{\infty} \left( c_l P_l(\lambda \rho) + d_l(\lambda \rho)^{|l|} \right) e^{i l \theta},
\] (89)
where \((\rho, \theta)\) are the coordinates of \(x\) with respect to the origin. For all \(l\), we have that
\[
c_l = \sum_{m=-\infty}^{\infty} (-1)^m (a_m + b_m) K_{l-m}(\lambda \rho_0) e^{-i(l-m)\theta_0} .
\] (90)

For \(l > 0\), we have
\[
d_l = \frac{1}{2^{|l|} l!} \sum_{m=-\infty}^{0} (-1)^m (a_m Q_{l-m}(\rho_0) + b_m K_{l-m}(\lambda \rho_0)) e^{-i(l-m)\theta_0} \\
+ \frac{1}{2^{|l|} l!} \sum_{m=1}^{\infty} (-1)^m (a_m + b_m) K_{l-m}(\lambda \rho_0) e^{-i(l-m)\theta_0} .
\] (91)

For \(l = 0\), we have
\[
d_0 = \sum_{m=-\infty}^{\infty} (-1)^m (a_m Q_{-m}(\rho_0) + b_m K_{-m}(\lambda \rho_0)) e^{i m \theta_0}
\] (92)

For \(l < 0\), we have
\[
d_l = \frac{1}{2^{|l|} l!} \sum_{m=-\infty}^{-1} (-1)^m (a_m + b_m) K_{l-m}(\lambda \rho_0) e^{-i(l-m)\theta_0} \\
+ \frac{1}{2^{|l|} l!} \sum_{m=0}^{\infty} (-1)^m (a_m Q_{l-m}(\rho_0) + b_m K_{l-m}(\lambda \rho_0)) e^{-i(l-m)\theta_0} .
\] (93)
Finally, we can shift the center of a local expansion in the \((r^n, P_n)\) using the following formula:

**Lemma.** Suppose that

\[
\Psi(x) = \sum_{l=-\infty}^{\infty} (a_l P_l(\rho) + b_l (\lambda \rho |l|) e^{i l \theta}) ,
\]

(94)

where \((\rho, \theta)\) are the polar coordinates of \(x\) with respect to the origin, is a local expansion for a parent box centered at the origin. Let \((\rho_0, \theta_0)\) be the polar coordinates of the center \(x_0\) of a child box with respect to the origin. Then, for \(x\) inside the child box, we have

\[
\Psi(x) = \sum_{l=-\infty}^{\infty} (c_l P_l(\rho') + d_l (\lambda \rho' |l|) e^{i l \theta'}) ,
\]

(95)

where \((\rho', \theta')\) are the coordinates of \(x\) with respect to the child’s center \(x_0\). For all \(l\), we have

\[
c_l = \sum_{m=-\infty}^{\infty} a_m I_{l-m}(\lambda \rho_0) e^{-i(l-m)\theta_0} .
\]

(96)

For \(l > 0\), we have

\[
d_l = \frac{1}{2^{|l|/|l|}} \sum_{m=-\infty}^{|l|-1} a_m I_{l-m}(\lambda \rho_0) e^{-i(l-m)\theta_0} \\
+ \frac{1}{2^{|l|/|l|}} \sum_{m=|l|}^{\infty} a_m P_{l-m}(\lambda \rho_0) e^{-i(l-m)\theta_0} \\
+ \sum_{m=|l|}^{\infty} \left( \begin{array}{c} m \\ |l| \end{array} \right) b_m \rho_0^{|l-m|} e^{-i(l-m)\theta_0} .
\]

(97)

For \(l = 0\), we have

\[
d_0 = \sum_{m=-\infty}^{\infty} a_m P_{l-m}(\lambda \rho_0) e^{i m \theta_0} + \sum_{m=-\infty}^{\infty} b_m \rho_0^{|l-m|} e^{i m \theta_0} .
\]

(98)

For \(l < 0\), we have

\[
d_l = \frac{1}{2^{|l|/|l|}} \sum_{m=-\infty}^{|l|+1} a_m P_{l-m}(\lambda \rho_0) e^{-i(l-m)\theta_0} \\
+ \frac{1}{2^{|l|/|l|}} \sum_{m=|l|}^{\infty} a_m I_{l-m}(\lambda \rho_0) e^{-i(l-m)\theta_0} \\
+ \sum_{m=-\infty}^{l} \left( \begin{array}{c} |m| \\ |l| \end{array} \right) b_m \rho_0^{|l-m|} e^{-i(l-m)\theta_0} .
\]

(99)
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