Geometry and cosmological perturbations in the bulk inflaton model

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Recently, the idea that our universe is a (mem)brane in a higher dimensional spacetime and all the interactions but gravity are confined on the brane, the so-called “brane world scenario” [1], has attracted much attention as an alternative to the standard 4-dimensional cosmology [3] because of its attractive feature that the standard 4-dimensional gravity is recovered on the brane in the low energy limit [2, 4]. The inflationary cosmology is also widely accepted because of its success in explaining cosmological observations [5]. It is therefore natural to consider the inflationary cosmology from the point of view of the braneworld scenario [6]. Brane inflation in the quantum cosmological context is also of great interest [7, 8].

In this paper, we consider a model of braneworld inflation driven by the dynamics of a scalar field living in the 5-dimensional bulk [9-13] in the context of the RS2 scenario, and investigate the cosmological perturbations on superhorizon scales, i.e., in the long wavelength limit. The existence of such a scalar field is supported from the unified theoretical point of view, as the reduction of a higher dimensional theory to 5 dimensions will give rise to scalar-tensor type effective theories. For a relevant form of the potential in the bulk, the scalar field dynamics can give rise to inflation on the brane [9]. A theoretically interesting aspect of this bulk inflaton model is that the bulk is not inflating at all; inflation and the subsequent hot Friedmann stage of the universe can be realized solely by the geometrical dynamics of the bulk. Furthermore, in the low energy limit $H\ell \ll 1$ where $H$ is the Hubble parameter on the brane and $\ell$ is the curvature radius of the bulk, as far as the spatially homogeneous dynamics is concerned, the dynamics on the brane has been found to be indistinguishable from the conventional slow-roll inflation [12]. It was also shown that the quantum fluctuations projected on the brane [10] as well as reheating after inflation [13] also mimic the 4-dimensional standard inflation, as long as we consider the low energy limit.

Thus the bulk inflaton model may be a viable alternative to the conventional 4-dimensional inflationary scenario. It is then of importance to clarify if the cosmological perturbations produced in the bulk inflaton model also have the same desirable features as those in the 4-dimensional theory in the low energy limit, and if there exists any signature specific to the braneworld scenario that can be observationally tested. Very recently, cosmological perturbations in a bulk inflaton model with an exponential potential and dilatonic coupling to the brane tension have been investigated by Koyama and Takahashi [14]. In this paper, we consider a tachionic bulk potential with the potential maximum at $\phi = 0$, with no coupling to the brane tension, and investigate the bulk geometry and cosmological perturbations.
in this model. We take the geometrical approach to the effective gravitational equations on the brane developed by Shiromizu, Maeda and Sasaki [15] and by Maeda and Wands [16]. In this formalism, the bulk gravitational effects on the brane are described by the projected 5-dimensional Weyl tensor,

$$E_{\mu\nu} = C_{\mu\nu a b} n^a n^b,$$

where $n^a$ is the unit vector normal to the brane (Latin indices run over 5 dimensions, while Greek indices over 4 dimensions, with the choice of Gaussian normal coordinates such that $n^a \partial_a = \partial_5$).

This paper is organized as follows. In Sec. II, we briefly review the bulk inflaton model, based on the geometrical formalism [15, 16]. In Sec. III, we derive the evolution equations for $E_{\mu\nu}$ in the bulk when there exists a non-trivial bulk energy momentum tensor, and solve them in the case of a spatially homogeneous brane, under the assumption that the amplitude of the scalar field $\phi$ is small. In Sec. IV, we consider the cosmological perturbations on superhorizon scales on the brane by solving the perturbation equations for $\phi$ and $E_{\mu\nu}$ in the bulk. Since the tensor perturbations in this model are identical to the vacuum AdS$_5$ bulk model at leading order, we focus on the scalar-type perturbations. We find the standard 4-dimensional result is recovered in the low energy limit $H^2\ell^2 \ll 1$, and possible signatures of the braneworld appear only at $O(H^4\ell^4)$. In Sec. V, we summarize our results and discuss the implications. Some formulas used in the text are summarized in Appendices A and B. Properties of the Green function for $E_{\mu\nu}$ in the bulk are analyzed in Appendix C.

II. REVIEW OF THE BULK INFATON MODEL

First, we review the bulk inflaton model [9]. We choose the Gaussian normal coordinates,

$$ds^2 = (n_a n_b + q_{ab})dx^a dx^b = dr^2 + q_{\mu\nu} dx^\mu dx^\nu.$$  \hspace{1cm} (2.1)

We assume that the brane is located at $r = r_0$. By extremizing the action of the 5-dimensional Einstein-scalar system with a brane, the gravitational equations in the bulk take the form

$$(5) G_{ab} + \Lambda_5 g_{ab} = \kappa_5^2 \left( T_{ab}[\phi] + S_{ab} \delta(r - r_0) \right),$$  \hspace{1cm} (2.2)

where $\kappa_5^2$ and $\Lambda_5$ are the 5-dimensional gravitational and cosmological constants, respectively. $T_{ab}[\phi]$ is the energy-momentum tensor of the bulk scalar field for which we assume the form,

$$T_{ab}[\phi] = \partial_a \phi \partial_b \phi - g_{ab} \left( \frac{1}{2} g^{cd} \partial_c \phi \partial_d \phi + V[\phi] \right),$$  \hspace{1cm} (2.3)

and $S_{ab}$ is the energy-momentum tensor on the brane for which we assume the vacuum form,

$$S_{ab} = -\sigma q_{ab},$$  \hspace{1cm} (2.4)

where $\sigma$ is the brane tension. Following the spirit of the RS2 scenario, we assume the tuning between $\Lambda_5$ and $\sigma$ as

$$\Lambda_5 = -\frac{\kappa_5^2 \sigma^2}{6} = -\frac{6}{\ell_0^2},$$  \hspace{1cm} (2.5)

where $\ell_0$ is the AdS$_5$ curvature radius of the RS2 braneworld, and assume the $Z_2$ symmetry with respect to the brane. Furthermore, we assume that the potential takes a tachionic form

$$V(\phi) = V_0 + \frac{1}{2} m^2 \phi^2; \quad V_0 > 0, \quad m^2 < 0,$$  \hspace{1cm} (2.6)

in the vicinity of $\phi = 0$. We tacitly assume that there exists a minimum of the potential somewhere at $\phi = \phi_{\text{min}} \neq 0$ at which $V(\phi_{\text{min}}) = 0$, where the RS2 flat brane is recovered.

The field equation for $\phi$ in the bulk is

$$\nabla_a \nabla_a \phi - \frac{d}{d\phi} V[\phi] = 0,$$  \hspace{1cm} (2.7)

where $\nabla_a$ is the 5-dimensional covariant derivative. Here, for simplicity, we impose the Neumann boundary condition

$$\partial_r \phi|_{r = r_0} = 0,$$  \hspace{1cm} (2.8)

on the scalar field. This implies that the scalar field does not couple to the brane tension.

From Israel’s junction condition on the brane,

$$[K_{\mu\nu}] = -\kappa_5^2 \left( S_{\mu\nu} - \frac{1}{3} q_{\mu\nu} S \right) = -\frac{1}{3} \kappa_5^2 \sigma q_{\mu\nu},$$  \hspace{1cm} (2.9)
and the $Z_2$ symmetry, we obtain the 4-dimensional effective gravitational equations on the brane \cite{9,15}

\begin{equation}
G_{\mu\nu}^{(4)} = \kappa_5^2 T_{\mu\nu}^{(b)} - E_{\mu\nu},
\end{equation}

where

\begin{equation}
T_{\mu\nu}^{(b)} = \frac{2}{3} \left[ T_{\alpha\beta} q_{\alpha}^a q_{\beta}^d + \left( T_{\alpha\beta} n^\alpha n^\beta - \frac{1}{4} T \right) q_{ab} \right],
\end{equation}

\begin{equation}
E_{\mu\nu} \equiv C_{abcd} q_{\alpha}^a q_{\beta}^b n^\alpha n^\beta.
\end{equation}

(5) $C_{abcd}$ is five-dimensional Weyl tensor.

In general, the projected Weyl tensor term $E_{\mu\nu}$ cannot be determined without solving the bulk dynamics. However, for a spatially homogeneous and isotropic brane, $E_{\mu\nu}$ on the brane may be evaluated without solving the bulk. By using the 4-dimensional contracted Bianchi identities, one finds \cite{9}

\begin{equation}
E_{tt} = \frac{\kappa_5^2}{2a^4(t)} \int^{t} dt' a^4(t') \left( \ddot{\phi} + \frac{\dot{\phi}}{a} \dot{a} \right),
\end{equation}

where $a(t)$ is the cosmic scale factor on the brane and $t$ the cosmic proper time. The other components of $E_{\mu\nu}$ are determined by the isotropy of the brane and the traceless nature of $E_{\mu\nu}$.

For the potential of the form \( (2.6) \), we then focus on the region \(|m^2| \phi^2 \ll V_0\) and consider the perturbation with respect to the amplitude of $\phi$. In the zeroth order, the background metric is determined as \cite{7}

\begin{equation}
d s^2 = b^2(z) \left( dz^2 - dt^2 + a(t)^2 \delta_{ij} dx^i dx^j \right),
\end{equation}

\begin{equation}
b(z) = \frac{H\ell}{\sinh[H(z) + z_0]} , \quad a(t) = \frac{e^{Ht}}{H}.
\end{equation}

Here $dz = dr/b(r)$ and the brane is located at $z = 0$, $H \ell = \sinh[H z_0]$, and

\begin{equation}
\ell = \sqrt{-\frac{6}{\Lambda_5 + \kappa_5^2 V_0}}
\end{equation}

is the effective AdS$_5$ radius. Note that $\ell > \ell_0$. The braneworld inflates with the Hubble rate $H$ given by

\begin{equation}
H^2 = \frac{\kappa_5^2 V_0}{6}.
\end{equation}

In the first order in $\phi$, the evolution of the scalar field on the background metric is determined. Expressing $\phi$ as the sum over all possible modes, $\phi = \sum_n u_n(z) \psi_n(t)$, the late time behavior of $\phi$ on the brane is dominated by the bound-state mode for which the effective 4-dimensional mass-squared for $\phi(t)$ is given by \cite{9}

\begin{equation}
M_{\text{eff}}^2 \approx \frac{m^2}{2},
\end{equation}

for $H^2 \ell^2 \ll 1$ and $|m^2|/H^2 \ll 1$. In this case, one has

\begin{equation}
\partial_z^2 \phi |_b = \frac{m^2}{2} \phi = -\ddot{\phi} - 3H \dot{\phi},
\end{equation}

on the brane.

The effective equations (2.10) at second order is then derived as follows. Using Eq. (2.18), $E_{tt}$ is evaluated as

\begin{equation}
E_{tt} = -\frac{\kappa_5^2}{2a^4} \int^{t} dt a^4 \dot{\phi}^2 + 2H \dot{\phi} = -\frac{\kappa_5^2}{4} \dddot{\phi}^2 + \frac{C}{a^4},
\end{equation}

where $C$ is an integration constant which depends on the initial condition. The term $C/a^4$ is called dark radiation term. In the present case, since we are interested in the late time behavior at the inflationary stage, the dark radiation term can be safely neglected. Then, the effective Friedmann equation becomes

\begin{equation}
3 \left( \frac{\dot{a}}{a} \right)^2 = \kappa_4^2 \rho^{(b)} - E_{tt} \equiv \kappa_4^2 \rho_{\text{eff}},
\end{equation}

where

\begin{equation}
\rho^{(b)} = \ell_0 \left( \frac{1}{4} \dot{\phi}^2 + \frac{1}{2} V(\phi) \right); \quad \kappa_4^2 = \frac{\kappa_5^2 \sigma}{6} = \frac{\kappa_5^2}{\ell_0^2},
\end{equation}

\begin{equation}
(2.19)
\end{equation}
and

$$\rho_{\text{eff}} = \ell_0 \left( \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} V(\phi) \right).$$  

(2.22)

Therefore, by introducing the effective 4-dimensional field $\varphi$ as

$$\varphi = \sqrt{\ell_0} \phi,$$  

(2.23)

the system is equivalent to the 4-dimensional Einstein-scalar system with the potential,

$$U(\varphi) = \frac{\ell_0}{2} V(\varphi/\sqrt{\ell_0}).$$  

(2.24)

Thus the effective field $\varphi$ behaves as a conventional inflaton on the brane, and the conventional slow-roll inflation is realized on the brane in the low energy approximation $H^2 \ell^2 \ll 1$ [9, 12].

III. EVOLUTION EQUATIONS IN THE BULK

When the bulk scalar field is spatially inhomogeneous, it is necessary to solve the bulk geometry to evaluate $E_{\mu\nu}$ on the brane. In this section, we derive the evolution equations for $E_{\mu\nu}$. We then apply the resulting equations to the spatially homogeneous and isotropic case to see how the previous result reviewed in Sec. II is recovered from the bulk point of view.

A. General equations

First, we recapitulate the definitions of the two projected Weyl tensors in the bulk,

$$E_{ab} = C_{cedf}n^c q^d_0 q^f_0,$$  

(3.1)

with

$$B_{abc} = q^d_a q^e_b C_{decf} n^f.$$  

(3.2)

By definition, $E_{ab}$ is symmetric with respect to the indices $(a,b)$ whereas $B_{abc}$ is anti-symmetric with respect to $(a,b)$. Furthermore $E_{ab}$ is traceless $E^a_a = 0$.

We start from the 5-dimensional Bianchi identities

$$\nabla^{(5)}_{[a} R_{bc]de} = 0,$$  

(3.3)

where $R_{abcd}$ is the 5-dimensional Riemann tensor. We can derive the equations for $E_{\mu\nu}$ and $B_{abc}$ from Eq. (3.2) by using the 5-dimensional Einstein equations (2.2). We find

$$\mathcal{L}_n E_{ab} = D^c B_{c(ab)} + K^{cd} C_{acbd} + 4 K_{(a} E_{b)c} - \frac{3}{2} K E_{ab} - \frac{1}{2} q_{ab} K^{cd} E_{cd}$$  

(3.4)

$$+ 2 \tilde{K}_{ac} \tilde{K}_{bd} - \frac{7}{6} \tilde{K}_{ab} \tilde{K}^{cd} \tilde{K}_{cd} + \kappa_5^2 P_{ab} [T_{ab}]$$

(3.5)

with

$$F_{ab} = T_{fg} q^f_a q^g_b - (T_{fg} n^f n^g - \frac{1}{4} T) q_{ab} - \frac{1}{2} T_{fg} q^f a q_{ab},$$

$$\tilde{K}_{ab} = K_{ab} - \frac{1}{4} K q_{ab},$$

and

$$\mathcal{L}_n B_{abc} = -2 D_{[a} E_{b)c} + K^2 B_{ab} - 2 B_{ceg[a} K^g_{b]} + \kappa_5^2 Q_{abc} [T_{ab}],$$

(3.6)

with

$$Q_{abc} [T_{ab}] = -\frac{2}{3} T_{de} n^e (q^d_a K_{b)c} - q^d_{b} K_{a)\tilde{g}_{c]} + \frac{1}{3} \mathcal{L}_n T_{de} n^e q^d_{[a} q_{b)c]}$$

$$- \frac{2}{3} D_{[a} [q_{b)c} (T_{de} n^e - \frac{T}{2})] - \frac{2}{3} D_{[a} [q_{b]e} q_{c]} T_{de}].$$

(3.7)

They are first-order partial differential equations for $E_{ab}$ and $B_{abc}$.
B. Second order equations for $E_{\mu\nu}$

We assume the tachionic potential form (2.6), and take the perturbation approach with respect to the amplitude of $\phi$.

First we expand the energy momentum tensor of the scalar field as

$$T_{ab}[\phi] = T_{ab}^{(0)}[\phi] + T_{ab}^{(2)}[\phi];$$

$$T_{ab}^{(0)}[\phi] = -V_0 g_{ab},$$

$$T_{ab}^{(2)}[\phi] = \partial_\alpha \phi \partial^\alpha \phi - \frac{1}{2} g_{ab} \left( g^{\nu\rho} \partial_\nu \phi \partial_\rho \phi + m^2 \phi^2 \right).$$

(3.8)

So far, we have not specified the coordinate system. Now, for definiteness, we choose the conformal (Gaussian normal) coordinate (2.14). The extrinsic curvature of each $z = \text{constant}$ hypersurface is expanded as

$$K_{\mu\nu} = K_{\mu\nu}^{(0)} + K_{\mu\nu}^{(2)};$$

$$K_{\mu\nu}^{(0)} = \frac{1}{2b(z)} \partial_\nu q_{\mu\nu} = \frac{b'(z)}{b^2(z)} q_{\mu\nu},$$

$$K_{\mu\nu}^{(2)} = O(\phi^2),$$

(3.9)

corresponding to the amplitude of the scalar field.

Substituting the above expansions of $T_{ab}$ and $K_{\mu\nu}$ into Eqs. (3.3) and (3.6), eliminating $B_{\mu\nu\alpha}$ in favor of $E_{\mu\nu}$, and keeping terms up to $O(\phi^2)$, we obtain the desired second order equation for $E_{\mu\nu}$ as

$$\mathcal{L} \dot{E}_{\mu\nu} = \kappa_5^2 S_{\mu\nu} \left[ T_{ab}^{(2)} \right],$$

(3.10)

where

$$\dot{E}_{\mu\nu} = b^2(z)E_{\mu\nu},$$

$$\mathcal{L} \equiv b(z) \frac{\partial}{\partial z} \frac{1}{b(z)} \frac{\partial}{\partial z} + \Box - 4H^2,$$

$$S_{\mu\nu} \left[ T_{ab}^{(2)} \right] = b^4(z) \left[ D^\alpha Q_{\alpha(\mu)} \left[ T_{ab}^{(2)} \right] + \mathcal{L} n \left( P_{\mu\nu} \left[ T_{ab}^{(2)} \right] \right) + 2 \left( \frac{b'(z)}{b^2(z)} \right) P_{\mu\nu} \left[ T_{ab}^{(2)} \right] + D_{\mu} \Sigma_{\nu} \left[ T_{ab}^{(2)} \right] \right],$$

$$\Sigma_{\nu} \left[ T_{ab}^{(2)} \right] \equiv D^{\alpha} T_{\nu}^{(b,2)} + 2 \left( \frac{b'(z)}{b^2(z)} \right) \left( g^{\sigma,\gamma} T_{\nu}^{(2)} n^\gamma \right),$$

(3.11)

where $T_{\nu}^{(b,2)}$ is the second order part of $T_{\nu}^{(b)}$ given by Eq. (2.11). The operator $\mathcal{L}$ is the same as the one derived in [17]. What is new is that the source term due to the presence of the bulk scalar field is taken into account. The boundary condition on the brane is

$$\frac{\partial}{\partial z} \left( \dot{E}_{\mu\nu} \right) \bigg|_b = \kappa_5^2 P_{\mu\nu} \left[ T_{ab}^{(2)} \right] \bigg|_b,$$

(3.12)

which is obtained from Eq. (3.3) and Israel’s junction conditions (2.9). Thus, $E_{\mu\nu}$ has a jump at the brane in general, whereas $\phi$ is smooth as given by Eq. (2.8).

In addition to the above, we have a set of constraint equations that come from the 4-dimensional contracted Bianchi identities on each $z$ = constant hypersurface. They are

$$D^{\mu} E_{\mu\nu} = \kappa_5^2 \Sigma_{\nu} \left[ T_{ab}^{(2)} \right].$$

(3.13)

If we regard $E_{\mu\nu}$ as the ‘electric field’, these constraints imply that $\Sigma_{\nu} \left[ T_{ab}^{(2)} \right]$ plays the role of the ‘charge density’ in the bulk.

Although the bulk equation (3.10) is perfectly legitimate, we find it is more convenient to deal with equations that are reduced from Eq. (3.10) by using the constraint equation (3.13), which we call the reduced equations. The components of Eq. (3.10) are explicitly written down as

$$b(z) \partial_\tau \left( \frac{1}{b(z)} \partial_\tau \right) \dot{E}_{tt} + 4H^2 \dot{E}_{tt} - \partial_t^2 \dot{E}_{tt} - 3H \partial_t \dot{E}_{tt} + \frac{1}{a^2} \nabla^2 \dot{E}_{tt} - \frac{4H}{a^2} \partial^k \dot{E}_{kt} = \kappa_5^2 S_{tt},$$

(3.14)

$$b(z) \partial_\tau \left( \frac{1}{b(z)} \partial_\tau \right) \dot{E}_{ti} - 2H^2 \dot{E}_{ti} - \partial_t^2 \dot{E}_{ti} - H \partial_t \dot{E}_{ti} + 6H^2 \dot{E}_{ti}$$

$$+ \frac{1}{a^2} \nabla^2 \dot{E}_{ti} - 2H \partial_t \dot{E}_{tt} - \frac{2H}{a^2} \partial^k \dot{E}_{kt} = \kappa_5^2 S_{ti},$$

(3.15)

$$b(z) \partial_\tau \left( \frac{1}{b(z)} \partial_\tau \right) \dot{E}_{ij} - \partial_t^2 \dot{E}_{ij} + H \partial_t \dot{E}_{ij}$$

$$+ \frac{1}{a^2} \nabla^2 \dot{E}_{ij} - 2H (\partial_t \dot{E}_{ij} + \partial_j \dot{E}_{ti}) + 2a^2 H^2 \delta_{ij} \dot{E}_{tt} = \kappa_5^2 S_{ij}.$$  

(3.16)
In the above equations, the components of $E_{\mu\nu}$ are coupled each other, which makes it very difficult to solve them. However, this situation can be improved by using the constraint equation (3.13). The components of Eq. (3.13) are written down as

$$-\partial_t \dot{E}_{tt} + \frac{1}{a^2} \partial^k \dot{E}_{kt} - 4H \dot{E}_{tt} = \kappa_5^2 b^4(z) \Sigma_t, \quad (3.17)$$

$$-\partial_t \dot{E}_{ti} + \frac{1}{a^2} \partial^k \dot{E}_{ki} - 3H \dot{E}_{ti} = \kappa_5^2 b^4(z) \Sigma_i. \quad (3.18)$$

Using these, we may eliminate $\partial^k \dot{E}_{kt}$ and $\partial^k \dot{E}_{ki}$ from Eqs. (3.14) and (3.15), respectively, to obtain

$$b(z) \partial_z \left( \frac{1}{b(z)} \partial_z \right) \dot{E}_{tt} - 12H^2 \dot{E}_{tt} - \partial_t^2 \dot{E}_{tt} - 7H \partial_t \dot{E}_{tt} + \frac{1}{a^2} \nabla^2 \dot{E}_{tt} = \kappa_5^2 \dot{S}_{tt};$$

$$\dot{S}_{tt} = S_{tt} + 4H b^4(z) \Sigma_t; \quad (3.19)$$

$$b(z) \partial_z \left( \frac{1}{b(z)} \partial_z \right) \dot{E}_{ti} - 2H^2 \dot{E}_{ti} - \partial_t^2 \dot{E}_{ti} - 3H \partial_t \dot{E}_{ti} + \frac{1}{a^2} \nabla^2 \dot{E}_{ti} = \kappa_5^2 \dot{S}_{ti};$$

$$\dot{S}_{ti} = S_{ti} + 2H b^4(z) \Sigma_i + \frac{2H}{\kappa_5^2} \partial_t \dot{E}_{tt}; \quad (3.20)$$

and Eq. (3.16) is rewritten as

$$b(z) \partial_z \left( \frac{1}{b(z)} \partial_z \right) \dot{E}_{ij} - \partial_t^2 \dot{E}_{ij} + H \partial_t \dot{E}_{ij} + \frac{1}{a^2} \nabla^2 \dot{E}_{ij} = \kappa_5^2 \dot{S}_{ij};$$

$$\dot{S}_{ij} = S_{ij} + \frac{1}{\kappa_5^2} \left( 2H (\partial_i \dot{E}_{ij} + \partial_j \dot{E}_{ii}) - 2a^2 H^2 \delta_{ij} \dot{E}_{tt} \right). \quad (3.21)$$

Thus, one can determine the components of $E_{\mu\nu}$ step by step; solving first Eq. (3.19) for $E_{tt}$, next Eq. (3.20) for $E_{ti}$, and finally Eq. (3.21) for $E_{ij}$.

C. The spatially homogeneous and isotropic background

Here, we solve the evolution equation (3.19) in the spatially homogeneous and isotropic case. In this case, we have already seen that $E_{\mu\nu}$ on the brane can be obtained without solving the bulk, as given by Eq. (2.19). The purpose of this subsection is to determine $E_{\mu\nu}$ in the bulk and to take the limit of it to the brane to examine if $E_{\mu\nu}$ thus obtained gives the correct answer for $E_{\mu\nu}$ on the brane, as a check of our derivation given above.

Equation (3.19) in the spatially homogeneous case is

$$\left[ b(z) \partial_z \left( \frac{1}{b(z)} \partial_z \right) - \partial_t^2 - 7H \partial_t - 12H^2 \right] \dot{E}_{tt} = \kappa_5^2 \dot{S}_{tt}. \quad (3.22)$$

To solve this, we introduce the retarded Green function that satisfies

$$\left[ b(z) \partial_z \left( \frac{1}{b(z)} \partial_z \right) - \partial_t^2 - 7H \partial_t - 12H^2 \right] G(t, z; t', z') = -\delta(t - t')\delta(z - z') b(z). \quad (3.23)$$

The Green function satisfies the reciprocity condition,

$$G(t, z; t', z') = G(-t', z'; -t, z), \quad (3.24)$$

and the Neumann-type boundary condition on the brane,

$$\partial_z G(t, 0+; t', z') = 0. \quad (3.25)$$

An explicit construction of the retarded Green function is given in Appendix C. Using the Green function, the formal solution of $E_{tt}$ is expressed as

$$\dot{E}_{tt}(t, z) = -\int_0^\infty dt' \int_0^{\infty} \frac{dz'}{b(z')} G(t, z; t', z') \partial_{z'} \dot{E}_{tt}(t', z') \bigg|_{z'=0}$$

$$-\int_0^\infty \frac{dz'}{b(z')} \left[ \partial_{t'} G(t, z; t', z') - G(t, z; t', z') \partial_{t'} + 7HG(t, z; t', z') \right] \dot{E}_{tt}(t', z') \bigg|_{t'=0}$$

$$-\kappa_5^2 \int_0^\infty dt' \int_0^\infty \frac{dz'}{b(z')} G(t, z; t', z') \dot{S}_{tt}(t', z'), \quad (3.26)$$
where we have assumed a regular initial data at $t = 0$ so that there is no contribution from $z = \infty$. In the above, the first term is the contribution from the brane-boundary of the extra-dimension, the second term is that from the initial surface, and the third term is that from the source generated from the scalar field. In Appendix C, we show that the Green function $G(t, z; t', z')$ at late times behaves as $\propto a(t)^{-4}$. Therefore, the second term behaves as a radiation, i.e., it describes the dark radiation term. Thus for a de Sitter brane, or for any observer on the $z =$constant hypersurface, we can neglect it after a sufficient lapse of time.

The third term can be evaluated as follows. Using the field equation (2.7), we find the relations among several terms in the source term as

\begin{equation}
D^a Q_{tt} + \frac{\dot{a}}{a} \Sigma_t = 0, \tag{3.27}
\end{equation}

\begin{equation}
\partial_t \left( a^4 b^3 P_{tt} \right) = -\partial_z \left( a^4 b^4 \Sigma_t \right). \tag{3.28}
\end{equation}

In particular, the latter equality implies the existence of an integrability condition. Namely, if we define

\begin{equation}
f(t, z) \equiv a^4(t) b^3(z) P_{tt}, \quad g(t, z) \equiv -a^4(t) b^4(z) \Sigma_t, \tag{3.29}
\end{equation}

Eq. (3.28) implies the existence of a function $M(t, z)$ such that

\begin{equation}
f(t, z) = \partial_z M(t, z), \quad g(t, z) = \partial_t M(t, z). \tag{3.30}
\end{equation}

Introducing a rescaled function of $M(t, z)$ as

\begin{equation}
\tilde{M}(t, z) \equiv a^{-4}(t) M(t, z), \tag{3.31}
\end{equation}

the source function $\tilde{S}_{tt}$ is expressed as

\begin{equation}
\tilde{S}_{tt}(t, z) = \left[ b(z) \partial_z \left( \frac{1}{b(z)} \partial_z \right) - \partial_z^2 - 7H \partial_t - 12H^2 \right] \tilde{M}(t, z). \tag{3.32}
\end{equation}

We see that the operator acting on $\tilde{M}(t, z)$ is the same as that acting on the Green function in Eq. (3.23). Therefore, by repeating integration by part twice, the third term of Eq. (3.26) can be rewritten in the form,

\begin{equation}
\int_0^\infty dt' \frac{1}{b(z')} G(t, z; t', z') \partial_z \tilde{M}(t', z') \bigg|_{z' = 0} + \tilde{M}(t, z). \tag{3.33}
\end{equation}

Here again, assuming the regularity of the scalar field, we have neglected the contribution from $z = \infty$. The boundary condition of $\tilde{M}(t, z)$ on the brane-boundary is obtained from its definition as

\begin{equation}
\partial_z \tilde{M}(t, 0+) = P_{tt}(t, 0+). \tag{3.34}
\end{equation}

This is the same as the boundary condition for $\hat{E}_{tt}$, Eq. (3.12). Thus the boundary contribution in Eq. (3.33) just cancels the first term in Eq. (3.26), and we have

\begin{equation}
\hat{E}_{tt}(t, z) = \kappa_5^2 \tilde{M}(t, z), \tag{3.35}
\end{equation}

that is, the solution for $\hat{E}_{tt}$ is given by $\tilde{M}$, apart from the initial surface contribution that can be neglected at late times.

From Eqs. (3.30) and (3.31), it immediately follows that two different representations of $\tilde{M}$, hence of $\hat{E}_{tt}$, are possible; the $t$-integral representation and the $z$-integral representation.

1. $t$-integral representation

The $t$-integral representation of the solution is

\begin{equation}
E_{tt}(t, z) = -\kappa_5^2 \frac{b^2(z)}{a(t)^4} \int_0^t dt' a^4(t') \left( D^\mu T^{(b, 2)}_{\mu t}(t', z) + 2 \frac{b'(z)}{b(z)} T^{(2)}_{tt}(t', z) \right). \tag{3.36}
\end{equation}

Taking the limit of it to the brane, and eliminating the mass term using the field equation, we obtain

\begin{equation}
E_{tt}(t, 0+) = \frac{1}{a^4(t)} \left[ \kappa_5^2 \int_0^t dt' a^4(t') \left( \partial_z^2 \phi + \frac{\dot{a}}{a} \dot{\phi} \right) + a^4(0) E_{tt}(0, 0+) \right] \tag{3.37}
\end{equation}

where we have recovered the initial surface contribution. This is the same as Eq.(2.19) obtained from the contracted Bianchi identities on the brane.
2. z-integral representation

The z-integral representation of the solution is

$$E_{tt}(t, z) = \frac{1}{b(z)^2} \left[ \kappa_5^2 \int_0^z dz' b^3(z') P_{tt}(t, z') + E_{tt}(t, 0+) \right].$$  (3.38)

This is a new form that has not been obtained before. A nice feature of this representation is that it automatically satisfies the boundary condition on the brane. One immediate consequence is that the regularity at $z = \infty$ implies the equality,

$$E_{tt}(t, 0+) = -\kappa_5^2 \int_0^\infty dz b^3(z) P_{tt}(t, z).$$  (3.39)

Thus, $E_{tt}$ on the brane may be expressed in terms of an integral of over the extra-dimension on the $t =$constant hypersurface with the integrand given by a certain combination of second derivatives of the energy momentum tensor. A point to be emphasized is that Eq. (3.39) has no explicit dependence on the initial condition; $E_{tt}$ on the brane is solely determined by the field configuration on the $t =$ constant hypersurface. This is similar to the local mass conservation law that holds in a spherically symmetric spacetime. Apparently, Eqs. (3.37) and (3.39) must coincide with each other up to the term proportional to $a^{-4}(t)$. This is guaranteed by Eq. (3.28). The representation (3.38) may be useful for analyzing the behavior of $E_{tt}$ in the bulk, though we have not explored it yet.

Given the solution for $E_{tt}$, it is trivial to obtain the other components $E_{ti}$ and $E_{ij}$. For $E_{ti}$ and the traceless part of $E_{ij}$, the source terms for a spatially homogeneous $\phi$ are easily found to be zero. Hence they are zero except for the contributions from the initial data that decay rapidly in time anyway. The trace part of $E_{ij}$ is simply given by $E_{tt}$ because of the traceless nature of $E_{\mu\nu}$.

IV. COSMOLOGICAL PERTURBATIONS

To investigate cosmological perturbations on the brane, we take the following approach. First we derive the perturbation equations for the scalar field $\delta \phi$ and the projected Weyl tensor $\delta E_{\mu\nu}$ in the bulk. Then we solve them with the assumption that the scale of interest is much greater than the Hubble horizon size, and take their projections on the brane to evaluate their effects.

Note that there are two small parameters in our approach. As before, we assume the amplitude of $\phi$ to be small. In addition to it, we assume $\delta \phi$ is much smaller than $\phi$, say $\delta \phi = O(\epsilon \phi)$ with $\epsilon \ll 1$. In what follows, we consider the perturbations accurate to $O(\epsilon^2)$ in the scalar field amplitude and linear in $\epsilon$. An important consequence is that the perturbation of the energy momentum tensor becomes effectively gauge-invariant as will be shown below. Our calculations are considerably simplified by this fact.

We focus on the so-called scalar-type cosmological perturbations, because the tensor-type perturbations (which are spatially transverse-traceless on the brane) are identical to those in the vacuum AdS bulk model discussed in the literature [7, 18] to the accuracy of $O(\epsilon^2)$.

A. The perturbation equations in the bulk

First, we write down the perturbation equations in the bulk. As usual, we expand all the perturbation variables in terms of the spatial scalar harmonics $Y(x^i)$ that satisfy

$$\left( \delta^{ij} \partial_i \partial_j + k^2 \right) Y = 0,$$  (4.1)

and the associated vector and tensor harmonics

$$Y_i = -\frac{1}{k} \partial_i Y, \quad Y_{ij} = \frac{1}{k^2} \partial_i \partial_j Y + \frac{1}{3} \delta_{ij} Y,$$  (4.2)

where we have assumed spatially flat slicing of each $z =$constant hypersurface for simplicity.

The perturbation of the scalar field equation is simply given by

$$\left[ \frac{1}{b^3} \partial_z b^3 \partial_z - \frac{1}{a^3} \partial_i a^3 \partial_i - \frac{k^2}{a^2} - m^2 b^2 \right] \chi(t, z) = 0,$$  (4.3)

with the boundary condition $\partial_z \chi|_{b} = 0$, where we have set $\delta \phi = \chi(t, z) Y(x^i)$. Note that we do not need to take account of the metric perturbation in the bulk because it will be of $O(\epsilon^2)$. 


To write down the perturbation equations for $\delta E_{\mu\nu}$, we expand $\delta E_{\mu\nu}$ in terms of the scalar harmonics as

$$
\delta \hat{E}_{tt} = EY,
\delta \hat{E}_{ti} = aE_i Y_i,
\delta \hat{E}_{ij} = a^2 \left( \frac{1}{3} EY \delta_{ij} + E_i Y_j \right),
$$

(4.4)

and their source terms $S_{\mu\nu}[\delta T_{ab}^{(2)}]$ and $\Sigma_{\nu}[\delta T_{ab}^{(2)}]$ as

$$
S_{tt}[\delta T_{ab}^{(2)}] = SY,
S_{ti}[\delta T_{ab}^{(2)}] = aS_i Y_i,
S_{ij}[\delta T_{ab}^{(2)}] = a^2 \left( \frac{1}{3} SY \delta_{ij} + S_2 Y_{ij} \right),
$$

(4.5)

$$
\Sigma_t[\delta T_{ab}^{(2)}] = \Sigma Y,
\Sigma_i[\delta T_{ab}^{(2)}] = a\Sigma_i Y_i.
$$

(4.6)

The evolution equations of $\delta E_{\mu\nu}$ are written down as

$$
\left[ b\partial_t b^{-1} \partial_z - a^{-7} \partial_t a^7 \partial_t - 12H^2 - \frac{k^2}{a^2} \right] E = \kappa_3^2 S + 4H b^4 \Sigma,
$$

$$
\left[ b\partial_t b^{-1} \partial_z - a^{-5} \partial_t a^5 \partial_t - 6H^2 - \frac{k^2}{a^2} \right] E_1 = \kappa_2^2 (S_1 + 2H b^4 \Sigma_1) - 2\frac{k}{a} HE,
$$

$$
\left[ b\partial_t b^{-1} \partial_z - a^{-3} \partial_t a^3 \partial_t - 2H^2 - \frac{k^2}{a^2} \right] E_2 = \kappa_3^2 S_2 - 4\frac{k}{a} HE_1,
$$

(4.7)

and the constraint equations are written as

$$
- \left( \partial_t + 4H \right) E + \frac{k}{a} E_1 = \kappa_3^2 b^4 \Sigma,
$$

$$
- \left( \partial_t + 4H \right) E_1 - \frac{k}{3a} (E - 2E_2) = \kappa_2^2 b^4 \Sigma_1.
$$

(4.8)

Using the expression of $S_{\mu\nu}[\delta T_{ab}^{(2)}]$ given in Eq. (3.11), we may further deduce the source term as follows. We decompose $P_{\mu\nu}[\delta T_{ab}^{(2)}]$ as

$$
P_{tt}[\delta T_{ab}^{(2)}] = PY,
P_{ti}[\delta T_{ab}^{(2)}] = aP_i Y_i,
P_{ij}[\delta T_{ab}^{(2)}] = a^2 \left( \frac{1}{3} PY \delta_{ij} + P_2 Y_{ij} \right),
$$

(4.9)

and $D^\alpha Q_{\alpha\mu\nu}[\delta T_{ab}^{(2)}]$ as

$$
D^\alpha Q_{\alpha tt}[\delta T_{ab}^{(2)}] = QY,
D^\alpha Q_{\alpha ti}[\delta T_{ab}^{(2)}] = aQ_1 Y_i,
D^\alpha Q_{\alpha ij}[\delta T_{ab}^{(2)}] = a^2 (Q_0 Y \delta_{ij} + Q_2 Y_{ij}),
$$

(4.10)

where, from the traceless condition of the whole source term, $Q_0$ is expressed as

$$
Q_0 = \frac{1}{3} \left[ Q + a^{-3} \partial_t (a^3 \Sigma) - \frac{k}{a} \Sigma_1 \right].
$$

(4.11)

With these expressions, $S$, $S_1$ and $S_2$ are expressed as

$$
S = b^4 \left[ b^{-3} \partial_z (b^2 P) + Q + \partial_t \Sigma \right],
$$

$$
S_1 = b^4 \left[ b^{-3} \partial_z (b^2 P_1) + Q_1 - \frac{1}{2} \left( \frac{k}{a} \Sigma - a\partial_t (a^{-1} \Sigma_1) \right) \right],
$$

$$
S_2 = b^4 \left[ b^{-3} \partial_z (b^2 P_2) + Q_2 - \frac{k}{a} \Sigma_1 \right].
$$

(4.12)
The explicit forms of these source terms in terms of the scalar field perturbation are given in Appendix B.

The boundary conditions of $\delta E_{\mu\nu}$ on the brane are

$$\partial_z E = \kappa_3^2 P, \quad \partial_z E_1 = \kappa_3^2 P_1, \quad \partial_z E_2 = \kappa_3^2 P_2,$$

(4.13)

by using Eq. (3.12).

The bulk effects on the brane can be analyzed by solving first the scalar field perturbation equation (4.3), inserting it to the source term of Eq. (4.7), and solving it under the boundary conditions (4.13). Although to solve them analytically is quite difficult or almost impossible, it is possible to solve them in the long wavelength limit $k/aH \ll 1$.

### B. $\delta \phi$ in the long wavelength limit

Let us analyze the behavior of the scalar field perturbation $\chi$ in the long wavelength limit. We expand it as

$$\chi(t, z) = \chi_b(t, z) + \int_{3H/2}^{\infty} dM \chi_M(t, z),$$

(4.14)

where the first term denotes the contribution from the bound state mode with the effective 4-dimensional mass-squared $M_{\text{eff}}^2 = m^2/2$ (under the assumption $m^2 < H^2$), and the second term is a superposition of the continuous Kaluza-Klein (KK) modes. Since the KK modes have mass-squared $M^2$ larger than $9H^2/4$, once their wavelengths become larger than the Hubble horizon size, they will damp out exponentially as $a^{-3/2}$. Hence we expect that their contributions to be unimportant. At least in the limit $H\ell \ll 1$, there are many pieces of evidence that support this expectation. As in the conventional inflationary scenario, the scalar field perturbations will come from the quantum fluctuations, and this expectation should be carefully examined in the context of quantum theory. Here, however, we simply assume so and focus on the bound state mode. Then we have the relation,

$$\frac{1}{b^4} \partial_z(b^2 \partial_z \chi) = \left( m^2 b^2 - \frac{m^2}{2} \right) \chi = 0.$$  

(4.15)

In particular, on the brane, this gives

$$\partial^2_z \chi_b |_{b} = \frac{m^2}{2} \chi = -\frac{1}{a^3} \partial_t(a^3 \partial_t \chi) - \frac{k^2}{a^2} \chi.$$  

(4.16)

### C. $\delta E_{\mu\nu}$ in the long wavelength limit

By inspecting the explicit forms of the source terms $S$, $S_1$, $S_2$, $\Sigma$ and $\Sigma_1$, we find their behaviors as

$$S, \Sigma = O(1), \quad S_1, \Sigma_1 = O(k), \quad S_2 = O(k^2) \quad \text{for} \quad k \to 0.$$  

(4.17)

Then from Eqs. (4.7) we find

$$E = O(1), \quad E_1 = O(k), \quad E_2 = O(k^2).$$  

(4.18)

Turning to the constraint equations (4.8), we then see that the long wavelength limit of $E$ and $E_1$ can be obtained by simply integrating Eqs. (4.8), i.e., without solving Eqs. (4.7), while we need to solve the 2nd order bulk equation to obtain $E_2$. Therefore, let us first solve for $E$ and $E_1$. Discussion on $E_2$ is deferred to the next subsection.

At leading order in the long wavelength limit, Eqs. (4.8) reduce to

$$-\frac{1}{a^4} \partial_t(a^4 E) = \kappa_3^2 b^4 \Sigma + O(k^2),$$  

(4.19)

$$-\frac{1}{a^4} \partial_t(a^4 E_1) = \kappa_3^2 b^4 \Sigma_1 + \frac{k}{3a} E + O(k^3),$$  

(4.20)

Thus $E$ is given by

$$E(t, z) = -\kappa_3^2 b^4 \left[ \frac{b'}{b} (\dot{\phi} \dot{\chi} + \dot{\phi} \dot{\chi}') + (\phi' \dot{\chi} + \dot{\phi} \chi') - (\phi'' \chi + \dot{\phi} \dot{\chi}') - 2H \ddot{\phi} \dot{\chi} \right],$$  

(4.21)

where

$$\Sigma = \frac{1}{2b^2} \left[ \frac{b'}{b} (\phi' \dot{\chi} + \phi \dot{\chi}') + (\phi' \dot{\chi} + \dot{\phi} \chi') - (\phi'' \chi + \dot{\phi} \dot{\chi}') - 2H \ddot{\phi} \dot{\chi} \right].$$  

(4.22)
With thus given $E$, $E_1$ is given by
\[ E_1(t, z) = -\frac{b^4(z)}{a^4(t)} \int t dt' a^4(t') \left( \kappa_5^2 \Sigma_1 + \frac{k}{3 \kappa_5^3 ab^4} E \right) (t', z) + O(k^2), \] (4.23)
where
\[ \Sigma_1 = -\frac{1}{2b^2} \frac{k}{a} \left[ -m^2 b^2 \phi \chi + \frac{b}{b^2} \phi' \chi - 4H \phi \chi - \frac{4}{3} \phi \chi + \phi' \chi + \frac{1}{3} \phi \chi \right]. \] (4.24)

On the brane, the time integrals in Eqs. (4.30) and (4.31) can be explicitly performed. Using Eq. (4.16) and the boundary condition $\phi' = \chi' = 0$, $\Sigma$ on the brane can be rewritten as
\[ \Sigma(t, 0+) = \frac{1}{2} (\phi \dot{\chi} + \phi \dot{\chi}) + 2H \phi \dot{\chi} + \frac{k^2}{2a^2} \phi \chi \]
\[ = \frac{1}{2a^4} (a^4 \phi \dot{\chi} + \frac{k^2}{2a^2} \phi \chi). \] (4.25)
Hence we obtain
\[ E(t, 0+) = -\frac{\kappa_5^2}{2} \phi \dot{\chi} + \frac{C}{a^4} + O(k^2), \] (4.26)
where $C$ is an integration constant. Since the term $C/a^4$ decays rapidly, we may neglect it at late times. Then, with a similar manipulation, we find
\[ \left( \kappa_5^2 \Sigma_1 + \frac{k}{3a} E \right) (t, 0+) = -\frac{\kappa_5^2}{3} \frac{k}{a} (\phi \dot{\chi} + \phi \dot{\chi} + 3H \phi \dot{\chi}) = -\frac{\kappa_5^2}{3} \frac{k}{a^4} (a^4 \phi \dot{\chi}), \] (4.27)
which gives
\[ E_1(t, 0+) = \frac{\kappa_5^2}{3} \frac{k}{a} \phi \dot{\chi}, \] (4.28)
where we have neglected the term $\propto a^{-4}$ as before. Thus we have obtained $E$ and $E_1$ on the brane on superhorizon scales.

D. Bulk anisotropic effect

Now we turn to solving $E_2$ that describes the spatially anisotropic part of $\delta E_{\mu\nu}$. In the long wavelength limit, the equation for $E_2$ in Eqs. (4.7) reduces to
\[ \left[ b \partial_z \left( \frac{1}{b} \partial_z \right) - \frac{1}{a^2} \partial_t \left( a^3 \partial_t \right) - 2H^2 \right] E_2 = \kappa_5^2 S_2 - 4\frac{k}{a} HE_1 + O(k^4). \] (4.29)
The $E_1$ term on the right-hand side of the above equation is given by the time integral in Eq. (4.23). In general, it is impossible to perform the integral explicitly. However, if we assume that $\phi$ is dominated by the bound-state mode, its time variation is small; i.e., we may adopt the slow-roll approximation. We have already assumed that the perturbation $\chi$ is dominated by the bound-state mode as well. Hence, under the assumption $m^2 \ll H^2$, $E_1$ to the lowest order in $m^2$ is found as
\[ E_1 \approx -\frac{\kappa_5^2}{6} \frac{b^2}{aH} \frac{k}{a} \left( \phi'' \chi + H \phi' \chi - \frac{b'}{b} \phi' \chi \right). \] (4.30)
Note, however, that this approximation will not be valid at large values of $H|z|$, since the regularity of the spacetime at $H|z| = \infty$ implies the breakdown of the bound-state mode dominance. Nevertheless, it seems physically reasonable to assume that the precise behavior of the source term of $E_2$ at large values of $H|z|$ will not significantly affect the value of $E_2$ on the brane. Under the same approximation, $S_2$ is given by
\[ S_2 \approx -\frac{b^2}{3} \frac{k^2}{a^2} \left( 4 \phi'' \chi + 2H \phi' \chi \right). \] (4.31)
From Eqs. (4.30) and (4.31), the source term is evaluated as
\[ \kappa_5^2 S_2 - 4H \frac{k}{a} E_1 \approx -\frac{\kappa_5^2}{3} \frac{2b^2}{a^4} \frac{k^2}{a^2} \left( \phi'' \chi + \frac{b'}{b} \phi' \chi \right). \] (4.32)
Given the source term, the formal solution of $E_2$ is given by

$$E_2(t, z) = -\int_0^\infty dt' \int_0^\infty \frac{dz'}{b(z')} G_2(t, z; t', z') \left( \kappa_5^2 S_2 - 4H_1^2 \right) (t', z'),$$  \hspace{1cm} (4.33)

where $G_2$ is the retarded Green function for $E_2$ that satisfies

$$\left[ b\partial_z b^{-1} \partial_z - a^{-3} \partial_t a^2 \partial_t - 2H^2 \right] G_2(t, z; t', z') = -b(z) \delta(t-t') \delta(z-z'),$$  \hspace{1cm} (4.34)

in the long wavelength limit. In the above, we have neglected the contribution from the initial surface $t = 0$ since $\phi$ is assumed to be very near the top of the potential at $t = 0$. The contribution from the brane-boundary is absent because $P_2 = 0$ on the brane. For $Hl \ll 1$, and at late times, the retarded Green function behaves as

$$G_2(t, z; t', z') = (Hl) \theta(t-t') a^{-2} (t) a^2 (t'),$$  \hspace{1cm} (4.35)

as shown in Appendix C.

Then, at late times, the $z$-integral in Eq. (4.33) can be easily evaluated. From Eq. (4.32) we find

$$\int_0^\infty \frac{dz}{b(z)} \left( \kappa_5^2 S_2 - 4H_1^2 \right) (t, z) \propto \int_0^\infty dz \left[ (b\phi')' - b\phi' \phi' \right] = -\int_0^\infty dz b\phi' \phi'. \hspace{1cm} (4.36)$$

Now, under the assumptions that $\phi$ and $\chi$ are both dominated by the bound-state mode and $m^2 \ell^2 \ll H_1^2 \ell^2 \ll 1$, the derivatives $\phi'$ and $\chi'$ are both of $O(m^2)$. Hence $E_2$ turns out to be of $O(m^4)$. This is a very interesting result. As long as we focus on the effects that persist after a sufficiently long lapse of time, signatures of the braneworld in this bulk inflaton model do not appear at $O(m^2)$ but only at $O(m^4)$. Since we have neglected $O(m^4)$ terms when deriving the source term (4.32), we are not able to evaluate the $O(m^4)$ corrections quantitatively here, but leave it for future work. It may be worth mentioning that analyzes of the decay of the scalar field out to the bulk were carried out previously and this effect was shown to appear at the same order, i.e., at $O(m^4)$ [12, 19, 20].

E. Cosmological perturbations on the brane

We are now ready to discuss cosmological perturbations on the brane. General behavior of superhorizon scale cosmological perturbations on the brane has been analyzed with the geometrical approach in [21]. Here we follow their analysis. Since the scalar field perturbation induces only scalar-type perturbations on the brane, the perturbed metric on the brane $\gamma_{\mu\nu} = g_{\mu\nu}/b^2 (0+)$ may be expressed as

$$\gamma_{tt} = -(1 + 2AY),$$
$$\gamma_{ti} = -a(t)BY_i,$$
$$\gamma_{ij} = a^2(t) \left( \delta_{ij} + 2H_L Y_i \delta_{ij} + 2H_F Y_{ij} \right). \hspace{1cm} (4.37)$$

The effective energy-momentum tensor on the brane is

$$T^{(\text{eff})}_{\mu\nu} = \ell_0 \left( T^{(b)}_{\mu\nu} - \frac{1}{\kappa_5^2} E_{\mu\nu} \right),$$  \hspace{1cm} (4.38)

where $T^{(b)}_{\mu\nu}$ is given by Eq. (2.11). The perturbed effective energy-momentum tensor $\delta T^{(\text{eff})}_{\mu\nu}$ is denoted by

$$\delta T^{(\text{eff})}_{\mu\nu} = -\rho \delta Y,$$
$$\delta T^{(\text{eff})}\ i = q Y_i,$$
$$\delta T^{(\text{eff})}_{ij} = p (\pi_L Y_{ij} + \Pi Y^i_j), \hspace{1cm} (4.39)$$

where $\delta$, $q$, $\pi_L$ and $\Pi$ represent the density contrast, momentum density fluctuation, isotropic pressure perturbation and anisotropic stress perturbation, respectively. By taking the perturbation of $T^{(b)}_{\mu\nu}$ and $E_{\mu\nu}$ explicitly, these matter perturbation variables are expressed in terms of $\delta \phi$ and $\delta E_{\mu\nu}$ as

$$\rho \delta = \ell_0 \left( \frac{1}{2} \dot{\phi} \dot{\chi} + \frac{1}{2} m^2 \dot{\phi} \chi - \frac{1}{\kappa_5^2} E \right),$$
$$q = \ell_0 \left( \frac{2k}{3a} \dot{\phi} \chi + \frac{1}{\kappa_5^2} E_1 \right),$$
\[ p \pi_L = \ell_0 \frac{5}{6} \dot{\phi} \dot{\chi} - \frac{1}{2} m^2 \phi \chi - \frac{1}{3 \kappa_5^2} E, \]
\[ p \Pi = -\frac{\ell_0}{\kappa_5^2} E_2, \]

where it is understood that all the quantities are those evaluated on the brane.

We have just seen in subsection D that \( E_2 \) appears only at \( O(m^4) \). Since \( m^2/H^2 \ll 1 \) is assumed, this implies that anisotropic stress is of \( O(m^4 \ell^4) \) in the low energy approximation \( H^2 \ell^2 \ll 1 \). Hence, to the leading order in the low energy expansion, the anisotropic stress perturbation vanishes; \( \Pi = 0 \). The other matter perturbation variables are expressed in terms of \( \chi \) by substituting to the above the expressions of \( E \) and \( E_1 \), Eqs. (4.26) and (4.28), respectively. We thus find

\[ \rho \delta = \ell_0 \left( \dot{\phi} \dot{\chi} + \frac{1}{2} m^2 \phi \chi \right), \]
\[ q = \ell_0 \frac{k}{a} \dot{\phi} \chi, \]
\[ p \pi_L = \ell_0 \left( \dot{\phi} \dot{\chi} - \frac{1}{2} m^2 \phi \chi \right), \]

in the long wavelength limit. A notable fact is that these expressions are exactly those one would obtain for a 4-dimensional scalar field with mass-squared \( M_{\phi \phi}^2 = m^2/2 \), with a simple rescaling \( \sqrt{\ell_0} \phi \to \varphi \), if we neglect the metric perturbation in them, just as in the case of the homogeneous background. Thus, provided that we can justify the neglect of the metric perturbation in these expressions, the description in terms of the effective 4-dimensional theory with mass-squared \( m^2/2 \) turns out to be valid even for inhomogeneous field configurations, at least for those inhomogeneities whose scales are larger than the Hubble horizon scale and when the low energy approximation is valid.

Now, let us justify our neglect of the metric perturbation in the matter variables. The gauge transformation of the metric perturbation is due to the scalar field perturbation \( \chi \). This implies that the change of \( \delta T^{\mu \nu} \) induced by an infinitesimal coordination transformation \( x^\mu \to \tilde{x}^\mu = x^\mu + \xi^\mu \) is given by

\[ \tilde{h}_{\mu \nu} = h_{\mu \nu} - \xi_{\mu |\nu} - \xi_{\nu |\mu}, \]

where the vertical bar denotes the covariant differentiation with respect to the 4-metric \( \gamma_{\mu \nu} \). Let us assume that a certain ‘geometrical gauge’ is chosen when we solve the effective 4-dimensional Einstein equations. Here a ‘geometrical gauge’ is a gauge for which the gauge condition is given purely in terms of geometrical quantities. An example is the spatially flat slicing \( R = H_L + H_T/3 = 0 \) or the shear-free (Newton) slicing \( k B - H_T = 0 \). In such a gauge, since the metric perturbation is due to the scalar field perturbation \( \chi \), we have \( h_{\mu \nu} = O(\epsilon \phi^2) \), where \( \epsilon \) stands for the amplitude of \( \chi \) relative to \( \phi \). Then, for a gauge transformation between two different geometrical gauges, we have \( \xi^{\mu} = O(\epsilon \phi^3) \). This implies that the change of \( \chi \) by such a gauge transformation is \( \tilde{\chi} - \chi = -\phi_{\mu} \epsilon^{\mu} = O(\epsilon \phi^3) \). Similarly, the change of \( \delta T^{\mu \nu}_{\text{eff}} \) is of \( O(\epsilon \phi^4) \) while \( \delta T_{\mu \nu}^{\text{eff}} \) is of \( O(\epsilon \phi^2) \). Hence, to the accuracy of \( O(\phi^2) \) that we are interested in, \( \chi \) as well as \( \delta T^{\mu \nu}_{\text{eff}} \) are gauge-invariant if we restrict our choice of gauge to a geometrical gauge [22].

The important point to be kept in mind is that the perturbation equations we derived are valid only in geometrical gauges. This is because, if we consider a gauge whose condition involves matter variables, the gauge transformation from a geometrical gauge to such a matter-based gauge will give \( \xi^{\mu} = O(\epsilon) \), and hence \( h_{\mu \nu} = O(\epsilon) \). Our equations can produce \( h_{\mu \nu} \) of \( O(\epsilon \phi^2) \) only.

Nevertheless, we may of course introduce various gauge-invariant quantities that may be defined in matter-based gauges, as in the standard cosmological perturbation theory [23, 24]. To repeat again, what we should keep in mind is that we should evaluate them in a geometrical gauge. As we see below, the most convenient choice is the flat slicing. That is, we simply regard our perturbation equations for \( \chi \) as those on the flat slicing.

Among various gauge-invariant quantities, a convenient one for our discussion is the curvature perturbation on the homogeneous density hypersurface, first introduced by Wands et al. [25],

\[ \zeta = H_L + \frac{H_T}{3} + \frac{\rho \delta}{3(p + \rho)} = R + \frac{\rho \delta}{3(p + \rho)}, \]

The important advantage of using this quantity is that it is known to be conserved in time on superhorizon scales when the entropy perturbation is negligible, irrespective of the gravitational theory. Since our \( T_{\mu \nu}^{\text{eff}} \) is not the real energy-momentum tensor in the 4-dimensional Einstein theory, using \( \zeta \) is preferable. Then, choosing the flat slicing \( R = 0 \), from Eq. (4.41) and the fact that \( \rho + p = \ell_0 \phi^2 \), we find

\[ \zeta = \frac{\dot{\phi} \dot{\chi} + M_{\phi \phi}^2 \phi \chi}{3 \phi^2} \approx \frac{M_{\phi \phi}^2 \phi \chi}{3 \phi^2} \approx -\frac{H \chi}{\dot{\phi}}, \]

where we have used the slow-roll equation, \( 3H \dot{\phi} \approx -M_{\phi \phi}^2 \phi \). This is the same as the one we would obtain for the standard 4-dimensional theory.
Alternatively, we may choose the curvature perturbation on the comoving hypersurface defined by [24, 26]

\[ \mathcal{R}_c = \mathcal{R} - \frac{H \dot{\chi}}{\phi}. \] (4.47)

Provided that \( T_{\mu\nu}^{\text{eff}} \) may be regarded as the actual 4-dimensional energy-momentum tensor, which is indeed appropriate [21], \( \mathcal{R}_c \) is also known to be conserved and equal to \( \zeta \) with good accuracy on superhorizon scales. Again, simply choosing the flat slicing \( \mathcal{R} = 0 \), we see that the result is in agreement with Eq. (4.46) within the accuracy of our interest.

Thus, we conclude that all the predictions we obtain in this bulk inflaton model are indistinguishable from the case of the 4-dimensional inflaton model with mass \( M_2 = m^2/2 \), including the perturbation, in the limit \( H^2t^2 \ll 1 \). In particular, one would obtain the standard, almost scale-invariant spectrum for the large scale cosmological perturbations. In this model, possible signatures of the braneworld appear at most at \( O(m^4t^4) \), or at second order in the slow-roll parameter \( m^2/H^2 \).

V. CONCLUSION

We investigated superhorizon scale cosmological perturbations in the brane inflation model in which slow-roll inflation on the brane is induced by the dynamics of a scalar field \( \phi \) living in the bulk. We took the geometrical approach in which the projected Weyl tensor \( E_{\mu\nu} \) describes the gravitational effect on the brane from the bulk [15].

First we derived the evolution equations for \( E_{\mu\nu} \) in the bulk by assuming a tachionic potential for \( \phi \) and focusing on the dynamics near the top of the potential with negative mass-squared \( m^2 < 0 \). We applied them to the case of spatially homogeneous and isotropic background. We found two different integral forms for \( E_{\mu\nu} \), one with respect to time \( t \) and the other to the extra-dimensional coordinate \( z \). By taking the brane limit of the \( t \)-integral form of \( E_{\mu\nu} \), we recovered the results that had been previously obtained without solving the bulk [9]. Namely, in the low energy limit \( H^2t^2 \), by a rescaling of \( \phi \) to an effective 4-dimensional field, the dynamics on the brane is indistinguishable from the standard 4-dimensional inflaton model with the same tachionic potential but multiplied by \( 1/2 \). In addition, the \( z \)-integral form of \( E_{\mu\nu} \), which may be useful for analyzing the bulk geometry, is found to give a new expression of \( E_{\mu\nu} \) on the brane that depends only on the field configuration on the \( t \)-constant hypersurface.

Then we considered the cosmological perturbations on superhorizon scales. We found that the effective theory on the brane is still the same as the case of the homogeneous and isotropic background. In particular, an anticipated non-trivial contribution from the spatially anisotropic part of \( E_{\mu\nu} \) was found to vanish at first order in the low energy approximation, i.e., at \( O(H^2t^2) \). This implies that possible braneworld signatures may appear only at \( O(H^4t^2) \) or higher, or second order in terms of the slow-roll parameter \( m^2/H^2 \). Very recently, based on their low energy expansion method [27], and by appealing to the AdS/CFT correspondence, Kanno and Soda obtained a low energy effective action for the dilatonic braneworld [28]. Our result that braneworld signatures may appear only at \( O(H^4t^2) \) is completely consistent with their recent result [28].

To predict signatures specific to the braneworld scenario, we thus have to investigate the effects of \( O(H^4t^2) \). Also, it will be interesting to consider the other, high energy limit \( H^2t^2 \gg 1 \) in the bulk inflaton model. Further, it is certainly of interest to explore the geometry in the bulk in more details, particularly the structure near the future Cauchy horizon of AdS\(_{5}\). We hope to come back to these issues in the near future.

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APPENDIX A: PROJECTED ENERGY MOMENTUM TENSOR

In this Appendix, we write down all the projected components of the energy-momentum tensor that appear in the effective 4-dimensional Einstein equations on the background metric given by Eq. (2.14). Specifically, we consider the quadratic part of the energy-momentum tensor,

\[ T^{(2)}_{ab} = \partial_a \phi \partial_b \phi - g_{ab} \left( \frac{1}{2} g^{cd} \partial_c \phi \partial_d \phi + \frac{1}{2} m^2 \phi^2 \right), \] (A1)

and write down the components for the spatially homogeneous and inhomogeneous cases separately.
1.Spatially homogeneous background

When $\phi$ is spatially homogeneous, we have

\[
T^{(2)}_{abq_tq_t^b} = \frac{1}{2} \left( \phi'^2 + \phi'' \right) + \frac{1}{2} m^2 b^2 \phi^2, \\
T^{(2)}_{abq_tq_t^b} = 0, \\
T^{(2)}_{abq_tq_t^b} = -a^2 \left[ \frac{1}{2} \left( \phi' - \phi'' \right)^2 + \frac{1}{2} m^2 b^2 \phi^2 \right] \delta_{ij}, \\
T^{(2)}_{abq_tq_t^b} = -\frac{1}{b^2} \left( 2\phi' - \phi'' \right) - 2m^2 \phi^2, \\
T^{(2)}_{abq_tq_t^b} = \frac{\phi'}{b}, \\
T^{(2)}_{abq_tq_t^b} = 0, \\
T^{(2)}_{abq_tq_t^b} = \frac{1}{2b^2} \left( \phi'^2 + \phi'' \right) - \frac{1}{2} m^2 \phi^2, \\
T^{(2)} = T^{(2)}_{ab} = -\frac{3}{2b^2} \left( \phi'^2 - \phi'' \right) - \frac{5}{2} m^2 \phi^2, \\
(A2)
\]

where the dot (') denotes $\partial_t$ and the prime (') denotes $\partial_z$.

The 4-dimensional projection of the bulk energy-momentum tensor is

\[
T^{(b,2)}_{abq_tq_t^b} = \frac{2}{3} \left[ T^{(2)}_{cdq_tq_t^d} + (T^{(2)}_{nn} - \frac{1}{4} T^{(2)}) q_{ab} \right]. \\
(A3)
\]

The components of this tensor are given by

\[
T^{(b,2)}_{tt} = \frac{1}{4} \left( -\phi'^2 + \phi'' + m^2 b^2 \phi^2 \right), \\
T^{(b,2)}_{ti} = 0, \\
T^{(b,2)}_{ij} = \frac{a^2}{12} \left( 3\phi'^2 + 5\phi'' - 3m^2 b^2 \phi^2 \right) \delta_{ij}, \\
T^{(b,2)} = T^{(b,2)}_{tt} = \frac{1}{b^2} \left( \phi'^2 - \phi'' - m^2 \phi^2 \phi^2 \right). \\
(A4)
\]

2. Perturbation

Here we consider the case when $\phi$ has a spatially inhomogeneous perturbation $\delta\phi$. The linear perturbation of $T^{(2)}_{ab}$ is

\[
\delta T^{(2)}_{ab} = 2\partial_{(a}\phi \partial_{b)} \delta\phi - g_{ab} \left( g^{cd} \partial_c \phi \partial_d \delta\phi + m^2 \phi \delta\phi \right). \\
(A5)
\]

The components are

\[
\delta T^{(2)}_{ab} q_t^aq_t^b = \left( \phi' \delta\phi' + \phi' \delta\phi \right) + m^2 b^2 \phi \delta\phi, \\
\delta T^{(2)}_{ab} q_t^aq_t^b = \phi \partial_t \delta\phi, \\
\delta T^{(2)}_{ab} q_t^aq_t^b = -a^2 \left[ (\phi' \delta\phi' - \phi' \delta\phi) + m^2 b^2 \phi \delta\phi \right] \delta_{ij}, \\
\delta T^{(2)}_{ab} q_t^aq_t^b = -\frac{2}{b^2} \left( 2\phi' \delta\phi' - \phi' \delta\phi \right) - 4m^2 \phi \delta\phi, \\
\delta T^{(2)}_{ab} q_t^aq_t^b = \frac{\phi' \delta\phi + \phi' \delta\phi'}{b}, \\
\delta T^{(2)}_{ab} q_t^aq_t^b = \frac{\phi' \partial_t \delta\phi}{b}, \\
\delta T^{(2)}_{ab} n^n a^n = \frac{1}{b^2} \left( \phi' \delta\phi' + \phi' \delta\phi \right) - m^2 \phi \delta\phi, \\
\delta T^{(2)} = -\frac{3}{b^2} \left( \phi' \delta\phi' - \phi' \delta\phi \right) - 5m^2 \phi \delta\phi. \\
(A6)
\]
The perturbation of the 4-dimensional projection of the bulk energy-momentum tensor is
\[
\delta T_{ab}^{(b,2)} = \frac{2}{3} \left[ \delta T_{cd}^{(2)} q_c g_{d}^\alpha + (\delta T_{\alpha n}^{(2)} - \frac{1}{4} \delta T^{(2)}) q_{ab} \right].
\] (A7)

The components are
\[
\delta T_{tt}^{(b,2)} = \frac{1}{2} \left( -\phi' \delta \phi + \phi' \delta \phi + m^2 b^2 \phi \delta \phi \right),
\]
\[
\delta T_{ti}^{(b,2)} = \frac{2}{3} \frac{\delta \phi}{\phi},
\]
\[
\delta T_{ij}^{(b,2)} = \frac{a^2}{6} \left( 3\phi' \delta \phi + 5\phi \delta \phi - 3m^2 b^2 \phi \delta \phi \right) \delta_{ij},
\]
\[
\delta T_{(b,2)} = \frac{2}{b^3} \left( \phi' \delta \phi + \phi \delta \phi - m^2 b^2 \phi \delta \phi \right).
\] (A8)

**APPENDIX B: SOURCE TERMS FOR E_{\mu\nu}**

Here, we give explicit expressions of the terms that contribute to the source term \( S_{\mu\nu} \) in Eq. (3.10) for \( E_{\mu\nu} \) in the bulk. Again, we treat the spatially homogeneous background and the perturbation separately.

The tensors that appear in the source term are
\[
P_{\mu\nu}[T_{ab}^{(2)}] = \frac{8 b'}{3 b^2} \left( T_{ab}^{(2)} - \frac{1}{4} T^{(2)} \right) q_{\mu\nu} - \frac{2 b'}{3 b^2} T_{\mu\nu}^{(2)} - \frac{1}{3} \left( T_{ab}^{(2)} \right)^\alpha q_{\mu\nu} - \frac{2}{3} \left[ \frac{L_\mu}{(T_{ab})^{2}} - \frac{1}{3} T_{ab}^{(2)} \right] q_{\mu\nu}
\]
\[
Q_{\mu\nu}[T_{ab}^{(2)}] = \frac{2}{3} L_\mu \left( T_{ab}^{(2)} q_{\nu(\alpha)} - \frac{1}{3} \frac{D_{\mu} T_{ab}^{(2)}}{T_{ab}^{(2)}} \right) - D_{\nu(\alpha)} T_{ab}^{(2)},
\]
\[
\Sigma_\mu[T_{ab}^{(2)}] = D_{\nu(\alpha)} Q_{\mu\nu}(x) + \frac{2 b'}{b^2} T_{ab}^{(2)} q_{\mu\nu},
\] (B1)

where \( Q_{\mu\nu} \) contributes to the source term only through the form \( D_{\nu(\alpha)} Q_{\alpha(\mu)} \).

### 1. Spatially homogeneous background

For the spatially homogeneous background discussed in Sec. III C, the non-vanishing components are
\[
P_{tt}[T_{ab}^{(2)}] = -\frac{1}{2} m^2 b \phi' + \frac{3b'}{2b^2} \phi'^2 + \frac{\phi''}{2b^2} \frac{\phi'}{ab} - \frac{\phi'}{2b^2} + \frac{\phi'}{b} - \frac{\phi'}{b},
\]
\[
D_{tt} Q_{tt}[T_{ab}^{(2)}] = \frac{\dot{\alpha}}{\alpha} \left( -\frac{1}{2} m^2 \phi' + \frac{b'}{b^2} \phi' + \frac{1}{2b^2} \phi'^2 - \frac{\dot{\alpha}}{ab} \phi'^2 - \frac{b}{2b^2} \phi' - \frac{1}{2b^2} \phi' \right),
\]
\[
\Sigma_\mu[T_{ab}^{(2)}] = -\frac{1}{2} m^2 b \phi' + \frac{b'}{b^2} \phi' - \frac{\dot{\alpha}}{ab} \phi'^2 + \frac{1}{2b^2} \phi' \phi' - \frac{1}{2b^2} \phi' \phi'.
\] (B2)

All the other components of \( P_{\mu\nu} \), \( D_{\nu(\alpha)} Q_{\alpha(\mu)} \) and \( \Sigma_\mu \) vanish.

### 2. Perturbation

We list the explicit expressions for the source terms under the presence of a perturbation \( \delta \phi = \chi(t, z)Y(x') \). As discussed in Sec. IV, we adopt the expansion in terms of spatial harmonics \( Y(x') \).

The relevant coefficients of the harmonic expansion are defined in Eqs. (4.9), (4.10) and (4.12). They are given by
\[
P = -\frac{1}{2} m^2 b \phi' - \frac{1}{2} m^2 b \phi' \chi + \frac{3b'}{b^2} \phi' \chi' + \frac{1}{2b^2} \phi'^2 \chi' - \frac{\dot{\alpha}}{ab} \phi' \chi' - \frac{b}{2b^2} \phi' \chi' + \frac{1}{2b} \phi' \chi'^2
\]
\[
- \frac{\dot{\alpha}}{ab} \phi' \chi' - \frac{b'}{b^2} \phi' \chi' + \frac{1}{2b^2} \phi' \chi' - \frac{b}{2b^2} \phi' \chi' - \frac{k^2}{3a^2 b} \phi' \chi',
\]
\[
P_1 = -\frac{k}{a} \left( \frac{2}{a} \frac{3b'}{3b^2} \phi' - \frac{2b'}{3b^2} \phi' + \frac{1}{3b} \phi' - \frac{1}{3b} \phi' \chi' - \frac{2}{3b} \phi' \chi' \right),
\]
\[
P_2 = -\frac{2k^2}{3b} \frac{3b'}{a^2} \phi' \chi'.
\] (B3)
Here is the parallel to the one given in [12].

rather trivial, we mainly analyze the Green function for $E_{q}$ Equation (4.29) for the spatially anisotropic part $z$ where we have put

Equation (C2) can be rewritten in the standard Schrödinger form by setting $Q_{1} = -\frac{k^{2}}{a^{2}} \left[ \frac{m^{2}}{6} \phi \chi - \frac{b'}{3b^{3}} \phi' \chi + \frac{1}{3b^{2}} \phi'' \chi + \frac{m^{2}}{6} \phi \chi - \frac{\dot{a}}{a^{2}} b^{2} \phi' \chi + \frac{\dot{a}}{a \dot{b}^{2}} \phi \chi - \frac{\dot{b'}}{b^{3}} \phi' \chi \right]$, $Q_{2} = \frac{k^{2}}{a^{2}} \left[ \frac{1}{2} m^{2} \phi \chi - \frac{b'}{3b^{3}} \phi' \chi + \frac{1}{3b^{2}} \phi'' \chi + \frac{\dot{a}}{a \dot{b}^{2}} \phi \chi - \frac{\dot{b'}}{b^{3}} \phi' \chi \right]$, $\Sigma = \frac{1}{4} \frac{m^{2}}{6} \phi \chi - \frac{b'}{3b^{3}} \phi' \chi + \frac{2b'}{b^{3}} \phi' \chi + \frac{1}{3b^{2}} \phi'' \chi + \frac{2b'}{b^{3}} \phi' \chi - \frac{\dot{a}}{a \dot{b}^{2}} \phi \chi - \frac{\dot{b'}}{b^{3}} \phi' \chi$, $\Sigma_{1} = -\frac{k}{a} \left[ \frac{1}{2} m^{2} \phi \chi - \frac{b'}{3b^{3}} \phi' \chi - \frac{2\dot{a}}{a \dot{b}^{2}} \phi \chi - \frac{\dot{b'}}{b^{3}} \phi' \chi + \frac{1}{3b^{2}} \phi'' \chi + \frac{\dot{b'}}{b^{3}} \phi' \chi \right]$. (B4)

APPENDIX C: GREEN FUNCTIONS FOR $E_{\mu \nu}$

In this Appendix, we analyze the late time behavior of the retarded Green functions for Eq. (3.19) for $E_{tt}$ and Eq. (4.29) for the spatially anisotropic part $E_{2}Y_{ij}$. We consider the long wavelength limit. Hence Eq. (3.19) is the same as that for the spatially homogeneous case, Eq. (3.22). Since the difference between these two Green functions is rather trivial, we mainly analyze the Green function for $E_{tt}$. The Green function for $E_{2}$ is given in the last subsection. Our analysis here is parallel to the one given in [12].

1. Mode functions

We construct the Green function from a superposition of mode functions that satisfy the source-free equation,

$$ b(z)\partial_{z} \left( \frac{1}{b(z)} \partial_{z} \right) \Psi(z, t) = 0 . \quad (C1) $$

Putting $\Psi = u_{q}(z)\psi_{q}(t)$, we have

$$ b(z)\partial_{z} \left( \frac{1}{b(z)} \partial_{z} \right) u_{q}(z) = 0 , \quad (C2) $$

$$ \left[ \partial_{t}^{2} + 7H\partial_{t} + (n^{2} + 10)H^{2} \right] \psi_{q}(t) = 0 , \quad (C3) $$

where we have put $n^{2} = q^{2} + 9/4$ for later convenience.

Equation (C2) can be rewritten in the standard Schrödinger form by setting $v_{q}(z) = b^{-1/2}(z)u_{q}(z)$,

$$ \left[ -\frac{d^{2}}{dz^{2}} + H^{2}V(z) \right] v_{q}(z) = n^{2}H^{2}v_{q}(z) , \quad (C4) $$

where

$$ V(z) = 9 \frac{1}{4} - \frac{1}{4 \sinh^{2}[H(|z| + z_{0})]} + H \coth[H(|z| + z_{0})] \delta(z) . \quad (C5) $$

The potential has a positive delta-function singularity at $z = 0$, i.e., it has an ‘anti-volcano’ form. It then follows that there exists no bound-state mode [17], unlike the case of the bulk scalar field itself. As the potential approaches $9/4$ at $|z| \to \infty$, $n^{2} > 9/4$ and the spectrum is continuous. Thus, we may label the mode functions in terms of $q = \pm \sqrt{n^{2} - 9/4}$ as $\Psi_{q}(t, z) = u_{q}(z)\psi_{q}(t)$ with $-\infty < q < \infty$. 

There are two independent mode functions belonging to each eigenvalue \( q \). For a given \( q \), we set

\[
\psi_q(t) = \frac{1}{\sqrt{2\pi}} e^{-iq\mathcal{H}t} e^{-\frac{z}{H}t},
\]

which satisfies the ortho-normality and completeness,

\[
\int_{-\infty}^{\infty} dt \, H e^{\gamma \mathcal{H}t} \psi_q(t) \psi'_q(t) = \delta(q - q'), \quad \int_{-\infty}^{\infty} dq \, \psi_q(t) \psi'_q(t) = \frac{\delta(t - t')}{H e^{\gamma \mathcal{H}t}}.
\]

With this choice of \( \psi_q(t) \), a convenient choice for the two independent solutions is to require the outgoing-wave and ingoing-wave boundary conditions at \( |z| = \infty \). Specifically,

\[
u_q^{(\text{out})}(z) = \frac{1}{\sqrt{\pi}} \frac{\Gamma(1 - iq)}{\Gamma(1/2 - iq)} Q_{1/2 - iq}(\xi) = 2^{-\frac{1}{2} + iq} \xi^{-\frac{1}{2} + iq} \mathbf{2F1} \left[ \begin{array}{c}
\frac{1}{2} - \frac{iq}{2} - \frac{1}{2} \end{array} ; \frac{3 - iq}{2} ; 1 - iq, \frac{1}{\xi^2} \right],
\]

\[
u_q^{(\text{in})}(z) = \frac{1}{\sqrt{\pi}} \frac{\Gamma(1 + iq)}{\Gamma(1/2 + iq)} Q_{1/2 + iq}(\xi) = u_{-q}^{(\text{out})}(z),
\]

where \( Q_q(\xi) \) is the Legendre function of the second kind and \( \mathbf{2F1}[\alpha, \beta; \gamma, \eta] \) denotes the hypergeometric function [29], and \( \xi \equiv \cosh [H (|z| + z_0)] \). Since the regions \( z > 0 \) and \( z < 0 \) are identical, we focus on the region \( z > 0 \) in what follows.

2. Construction of the Green function

We now construct the retarded Green function introduced in Eq. (3.23) from a superposition over the modes,

\[
G(t, z; t', z') = \int_{-\infty}^{\infty} dq \, G_q(z; z') \psi_q(t) \psi_{-q}(t') a^7(t'),
\]

where \( G_q(z, z') \) satisfies

\[
[b(z) \partial_z \left( \frac{1}{b(z)} \partial_z \right) + (n^2 - 2) H^2] G_q(z; z') = -\frac{H^2 l}{\sinh[H(z + z_0)]} \delta(z - z').
\]

To construct \( G_q(z, z') \), we need to require the \( \mathbb{Z}_2 \)-symmetry, or the Neumann boundary condition on the brane, in addition to the outgoing-wave boundary condition at (conformal) infinity. The \( \mathbb{Z}_2 \)-symmetric mode function is given by

\[
u_q^{(\mathbb{Z}_2)}(z) = \nu_q^{(\text{out})}(z) - \gamma_q \nu_q^{(\text{in})}(z), \quad \gamma_q \equiv \frac{\partial_z u_q^{(\text{out})}(z)}{\partial_z u_q^{(\text{in})}(z)} \bigg|_{z=0}.
\]

Then we have

\[
G_q(z; z') = \frac{1}{W_q} \left[ u_q^{(\text{out})}(z') \nu_q^{(\mathbb{Z}_2)}(z) \theta(z' - z) + u_q^{(\text{out})}(z) \nu_q^{(\mathbb{Z}_2)}(z') \theta(z - z') \right],
\]

where \( W_q \) is the Wronskian defined by

\[
W_q \equiv \frac{\sinh[H(z + z_0)]}{H^0} \left[ \partial_z u_q^{(\mathbb{Z}_2)}(z) u_q^{(\text{out})}(z) - u_q^{(\mathbb{Z}_2)}(z) \partial_z u_q^{(\text{out})}(z) \right] = \frac{iq}{H l} \frac{\gamma_q}{H^0}.
\]

The behavior of the Green function at late times can be studied by analyzing poles of \( G_q \), or equivalently zeros of the Wronskian, in the complex \( q \)-plane [12]. For this purpose, here, we give the \( z \)-derivative of \( u^{(\text{out})} \) explicitly,

\[
\partial_z u_q^{(\text{out})}(z) = (-\frac{1}{2} + iq) 2^{-\frac{1}{2} + iq} \xi^{-\frac{1}{2} + iq} \sinh[H(z + z_0)]
\]

\[
\times \left( \mathbf{2F1} \left[ \begin{array}{c}
\frac{1}{2} - \frac{iq}{2} - \frac{1}{2} \end{array} ; \frac{3 - iq}{2} ; 1 - iq, \frac{1}{\xi^2} \right]
\right.
\]

\[
+ \xi^{-\frac{1}{2} + \frac{3}{2} (\frac{1}{2} - iq)} \mathbf{2F1} \left[ \begin{array}{c}
\frac{5}{2} - iq, \frac{7}{2} - iq \end{array} ; 2 - iq, \frac{1}{\xi^2} \right] \right)\).
From this, expressions for $\partial_z u^{(in)}(z)$ and $\partial_z u^{(2z)}$ follow immediately.

Hereafter, we consider the late time behavior of the Green function in the cases $H\ell \ll 1$ and $H\ell \gg 1$ separately.

1. The case $H\ell \ll 1$

   From Eq. (C15),
   $$
   \partial_z u^{(out)}(0+) \approx 2^{1/2 + i\theta} H \sinh(Hz_0)(-\frac{1}{2} + i\theta)z_0^{-3/2 + i\theta} \frac{\Gamma(1 - i\theta)}{\Gamma(\frac{3}{2} - i\theta)\Gamma(\frac{1}{2} + i\theta)(H\ell)^2},
   $$
   (C16)
   and similarly for $\partial_z u^{(in)}(0+)$. Thus, in the limit $H\ell \ll 1$,
   $$
   \gamma_q = \frac{-1/2 + i\theta}{-1/2 - i\theta} \frac{\Gamma(1 - i\theta)\Gamma(\theta/2 + i\theta)}{\Gamma(\theta + i\theta)\Gamma(3/2 - i\theta)},
   $$
   (C17)
   and the Green function $G_q$ becomes
   $$
   G_q(z; z') = (H\ell) P_{\frac{1}{2} + i\theta}(\xi) P_{\frac{1}{2} - i\theta}(\xi')
   + (P_{\frac{1}{2} + i\theta}(\xi) Q_{\frac{1}{2} + i\theta}(\xi') \theta(z' - z) + Q_{\frac{1}{2} + i\theta}(\xi) P_{\frac{1}{2} - i\theta}(\xi') \theta(z - z')).
   $$
   (C18)
   The total Green function is given by Eq. (C10), i.e., by integral with respect to $q$ along the whole real axis on the complex $q$-plane. The retarded condition is guaranteed if there is no pole in the upper half of complex $q$-plane, which is in fact the case.

   For $t > t'$, the integral (C10) is equivalent to the contour integral over the whole region of the lower half complex $q$-plane, or the sum of the residues of these poles. In the limit $H\ell \ll 1$, we find that $G_q$ has poles only at
   $$
   q = -\frac{2s + 1}{2} \quad (s = 0, 1, 2, \cdots).
   $$
   (C19)
   Hence at $t > t'$, we may express the retarded Green function as
   $$
   G(t, z; t', z') = (H\ell) \theta(t - t') \sum_{n=0}^{\infty} P_n(\xi) P_n(\xi') a^{-(n+4)}(t)a^{n+4}(t'),
   $$
   (C20)
   where $P_n(z)$ is Legendre polynomial of order $n$. In particular, after a sufficient lapse of time, we may approximate the Green function the $n = 0$ term, i.e., by taking account of only the pole with the smallest imaginary part $q = -i/2$. This gives
   $$
   G(t, z; t', z') \approx (H\ell) \theta(t - t') a^{-4}(t)a^4(t').
   $$
   (C21)
   Thus, $G \propto a^{-4}$ at sufficiently late times. This gives rise to the radiation-like behavior of $E_{\ell\ell}$, in accordance with our expectation.

2. The case $H\ell \gg 1$

   In this case, we have
   $$
   \gamma_q = \frac{-1/2 + i\theta}{-1/2 - i\theta} 2^{2i\theta} \xi_0^{2i\theta},
   $$
   (C22)
   and the structure of $G_q(z, z')$ is more complicated than the case $H\ell \ll 1$. Nevertheless, it is possible to locate the poles of $G_q$. We find the poles at
   $$
   q = \frac{i}{2}, \quad -is \quad (s = 1, 2, 3, \cdots).
   $$
   (C23)
   Thus at sufficiently late times, the Green function behaves as
   $$
   G(t, z; t', z') \approx \theta(t - t') a^{-4}(t)a^4(t').
   $$
   (C24)
   Apart from the absence of the factor $H\ell$, this is the same as that in the case $H\ell \ll 1$, Eq. (C21). Hence the radiation-like behavior is realized again. Of course, this is also expected, because the transverse-traceless nature of the source-free $E_{\mu\nu}$ is independent of the energy scale.
3. Green function for $E_2$

The Green function $G_2(t, z; t', z')$ for $E_2$ satisfies

\[ b(z) \partial_z \left( \frac{1}{b(z)} \partial_z \right) - \partial_t^2 - 3H \partial_t - 2H^2 \right] G_2(t, z; t', z') = -\delta(t-t')\delta(z-z')b(z), \quad (C25) \]

in the long wavelength limit. In this case, only the difference from the Green function for $E_{tt}$ appears in the temporal part of the mode functions. For an eigenvalue $q$, with $q^2 = n^2 - 9/4$ as before, the $z$-component of the equation is exactly the same as Eq. (C2), while the $t$-component becomes

\[ \left[ \partial_t^2 + 3H \partial_t + n^2 H^2 \right] \psi_{2,q}(t) = 0. \quad (C26) \]

The solution is related to $\psi_q(t)$ for $E_{tt}$ as $\psi_{2,q}(t) = a^2(t)\psi_q(t)$. Hence, it follows that

\[ G_2(t, z; t', z') = a^2(t)G(t, z; t'z')a^{-2}(t'), \quad (C27) \]

and we can use all the results in the previous subsections. In particular, in the limit $H\ell \ll 1$, the Green function at $t > t'$ is expressed as

\[ G_2(t, z; t', z') = (H\ell)\theta(t-t') \sum_{n=0}^{\infty} P_n(\xi)P_n(\xi')a^{-(n+2)}(t)a^{n+2}(t'). \quad (C28) \]

Also, at sufficiently late times, we have $G_2 \propto a^{-2}$, irrespective to $H\ell$.

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