Simulations of Multi-Component Relativistic Thermalization

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We report on the development and application of a multi dimensional relativistic Monte-Carlo code to explore the thermalization process in a relativistic multi-component environment. As an illustration we simulate the fully relativistic three dimensional Brownian-motion-like solution to the thermalization of a high mass particle in a bath of relativistic low-mass particles. We follow the thermalization and ultimate equilibrium distribution of the Brownian-like particle as can happen in the cosmic plasma during Big bang nucleosynthesis.

I. INTRODUCTION

Understanding the solution to the Boltzmann equation for fluids is of fundamental importance for its practical implications in chemistry, biophysics, astrophysics, and cosmology. Over the past decades considerable progress has been made toward understanding these solutions for multi-component mixtures or in relativistic environments [1,5]. However, until recently there has been little progress in solving the relativistic Boltzmann equation. Existing solutions are based on the Fokker-Planck equation approximation to the relativistic Boltzmann equation [1,6]. Here we present Monte-Carlo simulations built to replicate the fully relativistic multi-component Boltzmann equation. Such a tool should have widespread applications in the dynamics of mixtures of fluids.

A. Background

Currently, multi-component relativistic simulations have been performed only in 1 dimension where one can vary the number density of each species [3]. However, for 3 dimensions only approximate analytical and numerical solutions for the relativistic Boltzmann equation exist. These are based upon various interpretations of stochastic process for solving the corresponding Fokker-Planck equation [1,6,7].

Cubero et al. [3] have discussed the difficulty in simulating multi-species thermalization in 2 and 3 dimensions. The difficulty being in modelling the complete electromagnetic fields due to all particles in space. This, can be simplified in 1 dimension by treating particles as only undergoing point-like elastic collisions. However, then in 2 and 3 dimensions the collision probability becomes vanishingly small even when including finite cross sections. This increases the computational time for particles to equilibrate. In this paper, however, we present a new Monte-Carlo scheme whereby collisions can be sampled adequately with minimal computation time.

B. Cosmological application

As an illustration, we consider here an application to the thermalization of hadronic species during the epoch of big bang nucleosynthesis (BBN). The relativistic thermalization simulation described here has recently been applied in Ref. [6] to describe the equilibrium in the BBN environment.

BBN occurs during an epoch of the early universe that lasts from about 1 sec to a few minutes and is responsible for the synthesis of nuclei from the pre-existing neutrons and protons to produce light elements such as $^2$H, $^3$He, $^4$He, and $^7$Li. The rates $R_{1\rightarrow3+\ldots}$. For the most part BBN involves two-body nuclear reactions. For this case, each pair of nuclei is directly related to their phase space distribution. These rates are given by

$$R_{1\rightarrow3+\ldots} = n_1 n_2 < \sigma v > = n_1 n_2 \int \sigma v f(v)dv , \quad (1)$$

where $n_1$ and $n_2$ are the number densities of colliding nuclei, $\sigma$ is the cross section, $v$ is the relative velocity between the two nuclei and $f(v)$ is the relative velocity distribution.

At the start of BBN nuclei are immersed in a bath of highly relativistic electrons, positrons and photons. During BBN the universe expands and cools from a temperature of $kT \approx 1 \text{ MeV}$ to $kT \approx 0.01 \text{ MeV}$. During this time frame the electron-positron asymmetry begins to manifest as the temperature falls below the electron or positron’s rest mass $(0.511 \text{ MeV})$. Initially, the electron number density is orders of magnitude higher than the baryon number density (See Table 1 from Sasankan [6]). Even though photons have a high number density w.r.t. baryons $(n_\gamma/n_b \sim 10^{-9})$, they have a low cross section for nuclear scattering. Hence, electron scattering dominates. This implies that nuclei obtain thermal equilibrium, by elastically scattering almost exclusively with mildly relativistic electrons in the cosmic plasma.

A motivation for the present work is that there has been considerable recent interest in the possibility of a modification of the baryon distribution function from Maxwell Boltzmann (MB) statistics. This modification can be in the form of Tsallis statistics [8,11], the influence of inhomogeneous primordial magnetic fields on baryons [12], non-ideal plasma effects at low temperature [13], the injection of nonthermal particles (e.g. [14,20] and Refs. therein), and small relativis-
tic corrections to the MB distribution that arise due to nuclear kinetic drag [21].

Our primary interest here is in the effects of injected non-thermal particles, due for example, to energetic hadronic decays by relic massive super-symmetric particles formed during an earlier epoch. As particles are injected into the primordial plasma one must follow the evolution of the baryon distribution function and the time-dependent effects on the therm-nuclear reaction rates. Thus, it is worthwhile to develop a fully relativistic method to describe the time-dependent evolution toward thermalization within the BBN environment.

To demonstrate the viability of this approach, we here apply our method to several test cases. The cosmological environment poses a good test environment as one component (the baryons) is much heavier than the other (relativistic electrons and photons). Also, as the background temperature changes the lighter particles transition from being relativistic to non-relativistic. This provides as a test case which includes regimes where a heavy (Brownian-like) particle is submerged in either a relativistic or nonrelativistic bath as the injected particles thermalize by relativistic collisions.

C. Monte-Carlo simulation

In this paper we describe a Monte-Carlo simulation that replicates the thermalization of charged nuclei in a background relativistic fluid. As an illustration, we first follow the thermalization of a proton with zero initial momentum in a bath of relativistic electrons. The simulation obeys general physics conservation laws, fully relativistic elastic scattering dynamics, and endeavours to mimic how nuclei would exchange energy with it’s surroundings. In principle, the nuclear distribution obtained during and until the end of thermalization would be the physical distribution contributing to nuclear reaction rates. As a second test case we follow the time evolution toward thermalization of an injected 1 GeV proton in the primordial plasma.

In a sense, this simulation provides an exact solution to the multi-component relativistic Boltzmann equation by a sequence of elastic scattering events in the same way that Nature does. The Boltzmann equation for the one-particle distribution functions \( f_a \) characterizes collisions of constituent \( a \) with constituent particles \( b \). This can be written,

\[
p_a^m \partial_t f_a = \sum_{b=1}^r \int \left( f_a f_b^* - f_a f_b \right) F_{ba} \sigma_{ab} d\Omega \frac{d^4 p_b}{p_{b0}},
\]

where the right-hand side is the one-particle collision term. The quantity \( F_{ba} = \sqrt{(p_a^m p_{b0})^2 - m_a m_b} \) is the invariant flux, while for our purposes \( \sigma_{ba} \) is the invariant differential elastic scattering cross section into an element of solid angle \( d\Omega \) that characterizes the collision of constituent \( a \) with constituents \( b \).

In Section II we discuss the algorithm we have developed for simulating this process. That is followed in Sec. III by numerical results we obtain for the illustrative case of protons with zero initial momentum in a bath of relativistic electrons as would be the case in BBN. In Section IV we describe the evolution of the distribution function of an energetic 1 GeV proton injected by decay into the primordial plasma. We discuss conclusions in Sec. V. In Appendix A we outline transformation of the distribution functions utilized in the Monte-Carlo simulations.

II. METHOD

As an illustration we simulate one test particle undergoing elastic scattering with the background particles. For the case considered here, the test particle is a light nucleus as encountered in BBN. However, the particle mass and scattering cross-section with the background species can be modified to study any other physical environment of interest. The background particles in this illustration are electrons and positrons for BBN, and similarly their mass and cross section with the test particle can be modified to study any other particle bath of interest. In this paper, the terms “test particle” and the “nucleus” are used interchangeably, as are the terms “background particle” and “electron”. We assume that the test particle scatters predominantly with the background species. The physical scenario this setup corresponds to is when the test particle number density is much lower than that of the background particles. This test case corresponds to the environment present during BBN. However, this restriction can easily be lifted to simulate more general fluids and plasmas.

A. Initial conditions for the algorithm

For the illustration considered here we adopt the following initial conditions:

- Total number of scattering events is set to \( 10^7 \).
  The total number of scatterings is set to \( 10^7 \) as we noticed in any simulation with more than \( 3 \times 10^6 \) scattering events, the nucleus assumes its equilibrium distribution and henceforth remains unchanged.

- Temperature of the electron gas is set to values between \( kT = 1 \) MeV to 0.01 MeV.
  The BBN era starts when the temperature of the universe is about 1 MeV and stops when the universe cools down to 0.01 MeV.

- Mass of nucleus is set to the mass of the proton (\( m_p = 939 \) MeV).
  We start with the nucleus at rest. In the simulation the nucleus eventually experiences enough collisions that its initial state is “forgotten”, and hence irrelevant.

- Mass of electron is set to \( m_e = 0.511 \) MeV.

- Initial total relativistic energy \( E = \sqrt{m^2 c^4 + p^2 c^2} \) of the nucleus is set to 939 MeV, i.e. we set the initial momentum of the nucleus to zero.

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B. The Algorithm with details and reasoning.

The algorithm to describe the scattering of an electron from the nucleus involves multiple rotations and Lorentz transformations so that the collision parameters are easier to acquire. These steps are schematically illustrated in Figure 1a-g. Following are the detailed steps of the algorithm:

1. The simulation starts in the background rest frame. This is the frame where the collective background momentum is zero.

2. Rotate the frame to have the velocity of the nucleus be along the +x-axis. We do this to simplify the collision mechanics. This rotation does not affect the background due to the isotropy of the background.

3. Lorentz transform to the co-moving frame of the nucleus. This frame would be the frame at rest w.r.t the background rest frame for only the first collision, prior to which the nucleus is at rest and is located at the origin. After the first iteration onward the nucleus frame will be moving w.r.t the background fluid.

   We Lorentz transform to the co-moving frame of the nucleus to be able to calculate the velocity-dependent flux distribution of the electrons approaching the nucleus. This samples the electron that will interact with the nucleus next.

4. Determine the electron distribution in the co-moving frame using the derivation in the appendix. This is obtained by applying number conservation between the moving and rest frame, finding volume element conversion, followed by converting all variables into their corresponding Lorentz-transformed variable. In 3-D the FD distribution representative of the background fluid in a boosted frame is given by:

\[ f'_{FD,3D}(\mathbf{v'}) = \left( \frac{1}{n} \right) \frac{\gamma'_0}{\gamma_a} \sqrt{s} \left( 1 + \exp \left( \frac{\gamma'_0 (1+\gamma'_a^2)mc^2}{kT} \right) \right) \]  

(3)

Here, \( \frac{1}{n} \) is the normalization constant, \( \gamma'_0 \) and \( \gamma_a \) are the Lorentz factors for the speed of the frames, i.e. they should be \( \gamma'_0 = 1 \) (for cosmic frame, which is at rest with respect to the background cloud) and \( \gamma_a = \frac{1}{\sqrt{1-a^2}} \) (where \( a \) is the speed of the boosted frame). \( \gamma' \) is the background electron velocity and \( \gamma' = \frac{1}{\sqrt{1-a^2}} \). \( f'_{FD,3D} \) is the velocity distribution in 3D in the boosted frame, i.e. rest frame of the nucleus.

5. Select an electron randomly based upon this distribution. Specifically we choose the electron’s velocity vector from the incoming flux rate

\[ R(\theta) \sim \sqrt{s} f'_{FD,3D}(\mathbf{v'}) . \]  

(4)

This electron will be the one that scatters off the nucleus for this iteration and in the process exchanges energy with it.

The electron bath surrounds the nucleus in all directions. The angular part of the distribution of electron velocity depicts the fraction of electrons moving in each direction. We select an electron velocity from the distribution using a Monte-Carlo technique. The direction of the velocity is the direction in which the electron approaches the nucleus starting from an arbitrary distance away.

6. Rotate the frame such that the electron approaches the nucleus from the \((-\) \( x \)-direction and is moving with a positive \( v_x \). Once the electron that collides with a nucleus is chosen, we ignore the rest of the background and this rotation makes it easier to describe the elastic scattering.

7. Lorentz transform to the center of momentum (COM) frame of the nucleus-electron system. Moving to the center of momentum (COM) simplifies the collision. In this frame, the nucleus and electron approach with equal and opposite 3-momenta. When the head-on elastic collision happens, the 3-momenta would simply be exchanged between the two particles.

8. Let the particles collide. The 3-momenta of the two gets exchanged, i.e. the velocities of each particle simply flips sign and retains its magnitude in the COM frame. Once the collision is completed, the electron is no longer considered. The electron moves away from the nucleus and under the assumption of molecular chaos doesn’t interact with the nucleus again, hence is irrelevant and can be ignored.

9. From here we transform the nucleus back to the background rest frame. The transformations that follow are performed to obtain the velocity and energy of the nucleus in the background rest frame.

10. Lorentz transform the velocity of the nucleus back to the pre-collision rest frame of the nucleus.

11. Rotate the velocity of the nucleus to have scattered along the direction the electron was initially approaching.

12. Lorentz transform the velocity of the nucleus to the background rest frame.

13. Repeat from the beginning of the algorithm with this (moving) nucleus as the test particle.

For our test case repetitively performing scattering between the nucleus and electrons is sufficient to produce the distribution the nucleus attains during BBN. However, for more complicated fluids, involving more than one background species
one can very easily expand this simulation technique to include scattering events from the different species onto one test particle. This can be done by adding scattering events from the other species and carefully selecting the incoming particle based on the reaction rate of the test particle with each of the background species.

III. RESULTING DISTRIBUTION FOR A PROTON IN A RELATIVISTIC ELECTRON BATH

We first tested our scattering algorithm to simulate the equilibrium thermalization for a proton initially at rest as the nucleus in an electron bath with various fixed temperatures relevant to the BBN environment. The procedure was as de-
scribed in Section III. We performed simulations for temperatures from the onset of BBN at \( kT = 1 \) MeV to the conclusion of BBN at \( kT = 0.01 \) MeV.

The equilibrium nuclear energy distribution obtained as a result of the simulation is shown in Figures 2 and 3. For reference, the figures also show the FD distribution for electrons and the MB distribution all at the same temperature. Note that the MB distribution very well approximates the FD distribution for nuclei at these temperatures and densities because the nuclei are in the non-degenerate non-relativistic regime \( (kT < \frac{m_e c^2}{\sim} 1 \text{ GeV}). \)

We observe that at all temperatures the equilibrium thermalized proton distribution closely resembles the MB distribution corresponding to the electron temperature. This is independent of whether the background electrons were relativistic or not. This suggests that the two species exchange energy in order to obtain the same analytical distribution, i.e. relativistic Fermi-Dirac distribution, with the same temperature but with their respective masses for each species. These distributions indeed indicate that, even at a common temperature, the energy partition is not the same for species with different masses. The observed distributions corroborate the relativistic Boltzmann equation solution recently solved for a multi-component gas \[3, 6\].

In a previous work \[22\] we reported having observed an anomalous drift to higher energies in the nuclear energy distribution when subjected to a relativistic electron bath. The anomaly arose due to the neglect of the instantaneous viscosity experienced by the nucleus due to it’s motion w.r.t the background. Instantaneous viscosity is the effect that among an electron moving in the opposite direction and another moving in the same direction as the nucleus, the electron moving in the opposite direction is more likely to interact with the nucleus due to its enhanced flux. This was implicitly ignored by assuming an isotropic distribution of electrons in the frame of the nucleus in step. The factor of \( 4 \) of the algorithm by corrects the scattering rate while selecting electrons from the correct electron rest distribution \( f(v) \). In the corrected method the incoming electron is chosen based on it’s flux towards the nucleus given by \( v f(v) \) which correctly reflects the electron flux by it’s velocity and direction of travel \[23, 24\].

We note that this equilibrium simulation can easily be expanded to more than one background specie, by adding another set of instructions on how the test specie interacts with new species. One could then trace and study the specific interactions and dynamics of a test particle undergoing modified Brownian motion in such mixtures.

**IV. EVOLUTION TOWARD THERMALIZATION OF ENERGETIC HADRONS**

As another test application we consider the injection of energetic hadrons (e.g. protons) due, for example, to the decay of a relic massive unstable particle from a previous epoch in the early universe. This could occur by various scenarios described, for example, in \[14–20\]. As one injected particle equilibrates another is injected, so the equilibrium distribution function will then depend upon the abundance and rate of injection of energetic particles by decay.

As an illustration we consider baryons injected with a delta-function kinetic energy of 939 MeV. We then follow the evolution of the distribution in time following multiple scattering. Figure 4 illustrates the spectrum of the injected particles after...
FIG. 3: Monte-Carlo distribution (blue bars) of the kinetic energy distribution of the nucleus scattering in baths of 2 dimensional relativistic $e^+ - e^-$ plasma (black curve) (at $kT = 0.1$ MeV) (upper panel) and 1 dimensional relativistic $e^+ - e^-$ plasma (black curve) (at $kT = 1$ MeV) (lower panel) compared to the kinetic energy distribution of a classical Maxwell-Boltzmann distribution (red curve).

$10^2, 10^3, 10^4, 10^5, 10^6,$ and $10^7$ scatterings. This is compared (red line) with the expected thermalized nucleon MB distribution at $T = 0.1$ keV.

The scattering rate per nucleon $\Gamma$ is approximately given by $\Gamma = n_e \sigma_T v$, where $n_e$ is the electron density, $\sigma_T$ the Thomson scattering cross section and $v$ the relative velocity. So, for an average background electron density of $\sim 10^{19}$ cm$^{-3}$ at $T = 0.1$ MeV, this many scatterings roughly corresponds to times of order $10^{-3}, 10^{-2}, 0.1, 1,$ and 10 seconds during the big bang. Note, that even after many scatterings a remnant tail at high energy remains. This may impact nuclear reaction rates during BBN. This will be explored in a future paper.

V. CONCLUSION

We have presented a Monte-Carlo algorithm for the simulation of multi-dimensional multi-component relativistic thermalization. This method could be used for simulating a bath of multiple different species to replicate environmental conditions any one test particle experiences.

We illustrated two tests of this algorithm for the solution of the distribution function for a heavy particle initially at rest experiencing Brownian-like motion in a bath of relativistic light particles. The test conditions were motivated from big bang nucleosynthesis, as charged nuclei interact with surrounding relativistic constituents, i.e. electrons. The temperature range we choose was between 0.01 MeV to 1 MeV appropriate to BBN. Our simulation of the equilibrium thermalized distribution function corroborates the expected results, i.e. the proton distribution is found to be very close to the MB distribution. To our knowledge this is the first fully relativistic multi-component simulation in three spatial dimensions of such relativistic Brownian motion.

As a second test we have evaluated the thermalization of energetic hadrons injected into a background $e^+ - e^-$ plasma at a temperature of 0.1 MeV. This illustrates how the nuclear spectrum may be distorted due to a continuous injection of non-thermal particles during the big bang.

VI. ACKNOWLEDGMENT

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Appendix A: Derivation of Lorentz transformation of $f(v)$ to $f'(v')$

For our selection of the colliding background particle we needed the background particle distribution in the rest frame of the test particle. Such a distribution would have to be obtained by performing a transformation from the distribution in the cosmic frame. The difficulty in finding a Lorentz-invariant distribution that also satisfies simulation results has been discussed previously[25]. Here we derive the Lorentz transformed distribution of a relativistic gas in a moving frame. We start with relativistic distributions, i.e. relativistic Fermi-Dirac and its non-degenerate approximation Maxwell-Juttner distribution in the rest frame, and find the equivalent distribution in the moving frame.

We begin with the relation regarding the distribution functions[25,26],

$$f'(x', u') = f(x, u)$$

(A1)

where the prime ($'$) denotes quantities in the moving frame and the unprimed quantities are in lab frame, i.e. the frame at rest w.r.t the background fluid. $x$ are the spatial coordinates and $u = \gamma v = \frac{p}{m}$ are spatial parts of the four velocity. Using this we want to find $f'(v')$, and we know $f(v)$ and $f(u)$ are Relativistic Fermi-Dirac distribution and Maxwell-Juttner for the two cases, for electrons as they are in the background fluid’s rest frame.
We know in our case $f(x,u)$ and $f'(x',u')$ are position independent, i.e. we don’t expect the distribution to be different in different coordinates, but only different in different frames. Hence, the integration simply gives the volume in the two frames, albeit contracted by the relevant Lorentz factors. Therefore,

$$\Rightarrow f'(u') = f(u) \times \frac{\gamma_o}{\gamma_a}$$  \hspace{1cm} \text{(A5)}$$

1. 1 Dimension

Now since we want $f'(v')$, multiply Eq (A5) by $du'$ to get:

$$f'(u') du' = f(u) \times \frac{\gamma_o}{\gamma_a} du' \hspace{1cm} \text{(A6)}$$

But we know in 1-D, change of variable from $u'$ to $v'$ is as:

$$f'(u') du' = f'(v') dv' \hspace{1cm} \text{(A7)}$$

Therefore by combining the last two equations,

$$f'(v') dv' = f(u) \times \frac{\gamma_o}{\gamma_a} du' \hspace{1cm} \text{(A8)}$$

$$\Rightarrow f'(v') = f(u) \times \frac{\gamma_o}{\gamma_a} \frac{du'}{dv'} \hspace{1cm} \text{(A9)}$$

FIG. 4: Progression of the distribution function of protons (mass 939 MeV) injected with total energy 1878 MeV as they are thermalized by electron scattering in the background BBN plasma at $T = 0.1 \text{ MeV}$. The red line shows the distribution function expected for an MB distribution at this temperature.
Therefore writing more neatly,
\[ f'(v') = \frac{\gamma_0}{\gamma_0} f(u) v^3 \] (A13)
and we know,
\[ f_{FD,1D}(u) = \left( \frac{1}{n} \right) \left( \frac{1 - \exp\left(\frac{m^2}{kT}\right)}{1 + \exp\left(\frac{m^2}{kT}\right)} \right) \] (A14)
\[ f_{MJ,1D}(u) = \frac{\exp\left(-\frac{m^2}{kT}\right)}{2mcK_1 \left(\frac{m^2}{kT}\right)} \] (A15)
substituting \( \gamma = \gamma_0 (1 + av) \) from [23] gives,
\[ f'_{FD,1D}(v') = \left( \frac{1}{n} \right) \frac{\gamma_0}{\gamma_0} \frac{v^3}{\left(1 + \exp\left(\frac{m^2}{kT}\right)\right)} \] (A16)
\[ f'_{MJ,1D}(v') = \frac{\gamma_0}{\gamma_0} \frac{v^3}{2mcK_1 \left(\frac{m^2}{kT}\right)} \] (A17)
This is the distribution needed. It has been obtained previously by empirical methods as in [41]. In our simulations we will be using \( |v'| f'(v') \) for sampling \( v' \) at which electrons come to hit the nucleus.

2. 2 Dimensions

Now since we want \( f'(v') \), multiply Eq.(A5) by \( d^2u' \) to get:
\[ \implies f'(u') d^2u' = f(u) \times \frac{\gamma_0}{\gamma_0} d^2u' \] (A20)
But we know in 2-D, change of variable from \( u' \) to \( v' \) is as:
\[ f'(u') d^2u' = f'(v') d^2v' \] (A21)
Therefore by combining the last two equations,
\[ f'(v') d^2v' = f(u) \times \frac{\gamma_0}{\gamma_0} d^2u' \] (A22)
\[ \implies f'(v') = f(u) \times \frac{\gamma_0}{\gamma_0} \frac{d^2u'}{d^2v'} \] (A23)

To find \( \frac{d^2u'}{d^2v'} \) we need to find the Jacobian matrix
\[ J = \begin{bmatrix} \frac{\partial u}{\partial v} & \frac{\partial u}{\partial \gamma} \\ \frac{\partial u}{\partial v} & \frac{\partial u}{\partial \gamma} \end{bmatrix} \] (A24)
The change in the volume element in the change of space of integration is given by Jacobian determinant \( |J| \). Therefore,
\[ \frac{d^2u'}{d^2v'} = |J| = \gamma^4 \] (A25)
Therefore, writing more neatly,
\[ f'(v') = \frac{\gamma_0}{\gamma_0} f(u) \gamma^4 \] (A26)
and we know,
\[ f_{FD,2D}(u) = \left( \frac{1}{n} \right) \left( \frac{1}{1 + \exp\left(\frac{m^2}{kT}\right)} \right) \] (A27)
\[ f_{MJ,2D}(u) = \frac{c^2m^2}{2\pi kT(m^2 + kT)} \exp\left(-\frac{(\gamma - 1)m^2}{kT}\right) \] (A28)
where, \( \frac{1}{\gamma} \) is the appropriate normalization constant. The constants independent of \( \gamma \) are irrelevant for our purpose as they are independent of \( v \) and \( u \). Plugging \( f_{FD,2D}(u) \) and \( f_{MJ,2D}(u) \) in eq.(A26) gives,
\[ f'_{FD,2D}(v') = \left( \frac{1}{n} \right) \frac{\gamma_0}{\gamma_0} \frac{\gamma^4}{\gamma_0} \left(1 + \exp\left(\frac{m^2}{kT}\right)\right) \] (A29)
\[ f'_{MJ,2D}(v') = \frac{\gamma_0}{\gamma_0} \frac{\gamma^4}{\gamma_0} \frac{c^2m^2}{2\pi kT(m^2 + kT)} \exp\left(-\frac{(\gamma - 1)m^2}{kT}\right) \] (A30)
Then substituting \( \gamma = \gamma_0 (1 + av) \) from [23] gives,
\[ f'_{FD,2D}(v') = \left( \frac{1}{n} \right) \frac{\gamma_0}{\gamma_0} \frac{\gamma^4}{\gamma_0} \left(1 + \exp\left(\frac{\gamma_0(1 + av)}{kT}\right)\right) \] (A31)
\[ f'_{MJ,2D}(v') = \frac{\gamma_0}{\gamma_0} \frac{\gamma^4}{\gamma_0} \frac{c^2m^2}{2\pi kT(m^2 + kT)} \exp\left(-\frac{(\gamma_0(1 + av) - 1)m^2}{kT}\right) \] (A32)
This distribution is a new result. In our simulations we will be using \( \sigma(v')|v'|f'(v') \) for sampling \( v' \) at which electrons come to hit the nucleus.

3. 3 Dimensions

Now since we want \( f'(v') \), multiply Eq.(A5) by \( d^3u' \) to get:
\[ \implies f'(u') d^3u' = f(u) \times \frac{\gamma_0}{\gamma_0} d^3u' \] (A33)
But we know in 3-D, change of variable from \( u' \) to \( v' \) is as:
\[ f'(u') d^3u' = f'(v') d^3v' \] (A34)
Therefore by combining the last two equations,

\[ f'(v')d^3v' = f(u) \times \frac{\gamma_v}{\gamma_u} d^3u' \]  
(A35)

\[ \implies f'(v') = f(u) \times \frac{\gamma_v}{\gamma_u} d^3u' \]  
(A36)

To find \( \frac{d^3u'}{d^3v'} \) we need to find the Jacobian matrix

\[ J = \begin{vmatrix} \frac{\partial u_1}{\partial v_1} & \frac{\partial u_1}{\partial v_2} & \frac{\partial u_1}{\partial v_3} \\ \frac{\partial u_2}{\partial v_1} & \frac{\partial u_2}{\partial v_2} & \frac{\partial u_2}{\partial v_3} \\ \frac{\partial u_3}{\partial v_1} & \frac{\partial u_3}{\partial v_2} & \frac{\partial u_3}{\partial v_3} \end{vmatrix} \]  
(A37)

The change in the volume element in the change of space of integration is given by Jacobian determinant \(|J|\). Therefore,

\[ \frac{d^3u'}{d^3v'} = |J'| = \gamma^5 \]  
(A38)

Therefore writing more neatly,

\[ f'(v') = \frac{\gamma_v}{\gamma_u} f(u) \gamma^5 \]  
(A39)

and we know,

\[ f_{FD.3D}(u) = \left( \frac{1}{n} \right) \frac{1}{\left( 1 + \exp \left( \frac{mc^2}{kt} \right) \right) } \]  
(A40)

\[ f_{MJ.3D}(u) = \frac{m}{4\pi ckT K_2} \exp \left( -\frac{\gamma mc^2}{kT} \right) \]  
(A41)

where, \( \frac{1}{n} \) is the appropriate normalization constant. The constants independent of \( \gamma \) are irrelevant for our purpose as they are independent of \( v \) and \( u \). Plugging \( f_{FD.3D}(u) \) and \( f_{MJ.3D}(u) \) in eq. (A39) gives,

\[ f_{FD.3D}(u) = \left( \frac{1}{n} \right) \frac{\gamma_v}{\gamma_u} \gamma^5 \frac{1}{\left( 1 + \exp \left( \frac{mc^2}{kt} \right) \right) } \]  
(A42)

\[ f_{MJ.3D}(v') = \frac{\gamma_v}{\gamma_u} \gamma^5 \frac{m}{4\pi ckT K_2} \exp \left( -\frac{\gamma mc^2}{kT} \right) \]  
(A43)

substituting \( \gamma = \gamma' (1 + av_1') \) from (23) gives,

\[ f_{FD.3D}(v') = \left( \frac{1}{n} \right) \frac{\gamma_v}{\gamma_u} \gamma^5 \frac{1}{\left( 1 + \exp \left( \frac{\gamma' (1 + av_1') mc^2}{kT} \right) \right) } \]  
(A44)

\[ f_{MJ.3D}(v') = \frac{\gamma_v}{\gamma_u} \gamma^5 \frac{m}{4\pi ckT K_2} \exp \left( -\frac{\gamma' (1 + av_1') mc^2}{kT} \right) \]  
(A45)

This distribution is a new result we found. In our simulations we will be using \( \sigma(v')|v'|f'(v') \) for sampling \( v' \) at which electrons come to hit the nucleus. The resultant distribution obtained for the nucleus corroborates with analytical solutions (6) and is hence tested via simulation.

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