COUNTING RATIONAL POINTS AND LOWER BOUNDS FOR GALOIS ORBITS

HARRY SCHMIDT

ABSTRACT. In this article we present a new method to obtain polynomial lower bounds for Galois orbits of torsion points of one dimensional group varieties.

1. Introduction

In this article we introduce a method that allows in certain situations to obtain lower bounds for the degree of number fields associated to a sequence of polynomial equations. By a Galois bound for such a sequence we mean a lower bound for the degree of these number fields that is polynomial in the degree of the polynomials. Maybe the simplest example is

\[ X^n = 1 \]

where \( n \) runs through the positive integers. For a primitive solution \( \zeta_n \) of this equation, that is, one that does not show up for smaller \( n \) this is just the well-known Galois bound for roots of unity \( \mathbb{Q}(\zeta_n) : \mathbb{Q} \gtrsim n^{1-\varepsilon} \). The Galois bound here was perhaps first pointed out by Gauss, by applying the Eisenstein criterion to (a shift of) the cyclotomic polynomials.

Another example are the equations

\[ B_n(X) = 0 \]

where \( B_n \) is the elliptic cyclotomic polynomial \( B_n \). For a fixed elliptic curve \( E \) given in Weiertrass form

\[ E : Y^2 = 4X^3 - g_2X - g_3 \]

this is defined by \( [n](X,Y) = \left( \frac{A_n(X)}{B_n(X)}, \frac{y_n}{y_n} \right) \) where \( A_n \in \mathbb{Q}(g_2, g_3)[X] \) is a monic polynomial of degree \( n^2 \) and \( B_n \in \mathbb{Q}(g_2, g_3)[X] \) is of degree \( n^2 - 1 \) with leading coefficient \( n^2 \). Here Galois bounds are not known as long as for roots of unity. The only known methods to obtain Galois bounds so far seem to be either through Serre’s open image theorem (in the non CM case), class field theory (in the CM case), or (in both cases) transcendence techniques as for instance applied by Masser [Mas89] and further developed by David [Dav97].

For both of these examples we give a new proof of Galois bounds using essentially the same strategy for each (see Corollary 2 and 3).

In joint work with Boxall and Jones we also consider fields obtained by adjoining all solutions \( z \) of an equation \( p^n(z) = p^n(y) \) for a fixed polynomial \( p \) with coefficients in \( \mathbb{Q} \).
Q and certain fixed $y \in Q$ (here $p^n$ is $p$ iterated $n$ times).

The use of counting of rational points on transcendental varieties in Diophantine geometry was first introduced by Pila and Zannier in [PZ08] to find yet another proof of Manin-Mumford and initiated remarkable developments in Diophantine geometry. There are now some excellent accounts of these developments such as [Sea] and [Zan12] and we refrain from saying much more.

We only concentrate on the counting results. Here the basic idea is that for certain transcendental sets a subpolynomial bound for the number of rational points of height bounded by $H$ should hold. The first result in this direction was proven by Bombieri and Pila [BP89]. Pila then later developed his determinant method to show among other things certain counting results for subanalytic surfaces [Pil04]. A conceptual jump was achieved with the introduction of o-minimal structures in the celebrated Pila-Wilkie counting theorem [PW06].

The question of improving the bound from subpolynomial to poly-logarithmic was perhaps around since the first types of such counting results were proven. It was first shown by Surroca [Sur06] that this is not possible in general. However, Wilkie conjectured that a poly-log bound should hold for the structure $\mathbb{R}_{\exp}$, the expansion of the reals by the real exponential function.

**Conjecture.** Let $X^{\text{trans}}$ be the transcendental part of a set $X \subset \mathbb{R}^k$ definable in $\mathbb{R}_{\exp}$. Then the following holds

$$\# \{x \in X^{\text{trans}} \cap \mathbb{Q}^k; H(x) \leq H \} \leq c_1 \log H c_2,$$

where $c_1, c_2$ are real constants depending on $X$.

We recall that $X^{\text{trans}}$ is $X$ deprived of all positive dimensional connected semi-algebraic sets contained in $X$. Now if we replace $\mathbb{Q}$ by a number field, the same type of bound would follow from the Conjecture. And we could ask how the constants depend on the number field. Pila [Pil07], using essentially real analytic methods, proved a poly-log bound for Pfaffian curves and showed that they depend only on the degree of the number field. Jones and Thomas [JT12] then extended this to restricted Pfaffian surfaces. It is reasonable to conjecture that for Pfaffian varieties in any dimension the same should hold. In a landmark work [BN17] Binyamini and Novikov, who combined complex analytic methods with Khovanskii’s zero-estimates, proved a poly-log bound for sets definable by restricted elementary functions. Another method to prove poly-log bounds for analytic functions using Siegel’s lemma was introduced by Masser [Mas11]. And a variant due to Wilkie [Wil15] was used to give an alternative proof of his theorem with Pila. See also Habegger’s work [Hab] where this approach is used in place of the determinant method, in order to count algebraic points near definable sets. Masser’s result on the Riemann $\zeta$ function seems to be the first with a polynomial dependence on the degree of the number field [Mas11, p.2045 (15)]. Related results for analytic functions were obtained by Boxall and Jones [BJ15a, BJ15b], Besson [Bes14] and Jones and Thomas [JT16]. A reasonable question is whether the dependence on the number field can be made polynomial in the structures we mentioned. So for $X$ as above one might
ask whether the following holds

\[ \# \{ x \in X^{\text{trans}} \cap K^k; H(x) \leq H \} \leq c_1 d^{C_2} \log H^{C_3}, \]

where \( d = [K : \mathbb{Q}] \) and \( c_1, c_2, c_3 \) depend on \( X \). In fact we would expect that a bound as above holds for \( X \) definable by an extension of the reals by the complex exponential function restricted to a certain fundamental domain or the restricted \( j \)-function. Even more generally the above should hold for \( X \) definable in the extension of the reals by the restriction of a uniformization map of a mixed Shimura variety on a suitable fundamental domain.

It seems that the methods leading to a poly-log bound always lead to a polynomial dependence on the degree. We will demonstrate this for Pfaffian curves and restricted Pfaffian surfaces. In what follows we write \( X(K, H) \) for the points of \( X \) with coordinates in the number field \( K \) and multiplicative Weil height at most \( H \). We also set \( X^{\text{size}}(K, H) \) to be the set of \( K \)-points on \( X \) with \( H^{\text{size}}(x) \leq H \) where \( H^{\text{size}} \) is defined as in [Pil09, Definition 6.3]. (The height is extended to tuples by taking the maximum).

Pila introduced the notion of mild parametrization [Pil10, Definition 2.1 and 2.4] and showed that the rational points of a set \( X \) that has mild parametrization can be covered by poly-log many hypersurfaces. In his work one can make the dependence on the number field polynomial.

**Theorem 1.** Let \( X \subset (0,1)^n \) have a \((J, A, C)\)-mild parametrization. There exist effectively computable constants \( C_1, C_2, C_3 \) depending only on \( J, A, C, n \) and \( \dim X \) such that \( X(K, H) \) is contained in the union of \( C_1 d^{C_2} \log T^{C_3} \) hypersurfaces of degree bounded by \( (d^2 \log T)^{\dim X/(n-\dim X)} \).

Pila’s approach was then used by Jones and Thomas to show a poly-log bound for any transcendental implicitly defined Pfaffian curve. In their work [JT12] the dependence on the degree of the number field can again be made polynomial. First for curves.

**Theorem 2.** Suppose that \( I \) is an open interval in \( \mathbb{R} \) and that \( f : I \to \mathbb{R} \) is a transcendental implicitly defined Pfaffian function of complexity \((n, r, \alpha, \beta)\). Then for \( T \geq e \), and the graph \( X \) of \( f \)

\[ |X(K, H)| \leq c(n, r, \alpha, \beta)d^{6n+6r+15} \log H^{3n+3r+8}, \]

where \( c(n, r, \alpha, \beta) = 2^{r(r-1)}6^r \frac{\beta^2}{\alpha^2}(n+2)^{n+3r+1} \alpha^2 \beta^{2n+2r+1} \).

They used this result and a stratification result due to Khovanskii and Vorobjov to show a poly-log bound for restricted Pfaffian surfaces. Again one can get a polynomial dependence on the degree from their work. First let \( X \subset (0,1)^n \) be a restricted semi-Pfaffian surface and assume that \( X \) has a mild parametrization with parameters \( J, A, C \) (with \( J, A, C \) as above) bounded by a constant \( M \) and that \( X \) is not contained in an algebraic hyper-surface. From Jones and Thomas work [JT12] one can deduce the following.

**Theorem 3.** There are effectively computable constants \( C_1, C_2, C_3 \) depending only on the format of \( X \) and on \( M \) such that

\[ |X^{\text{trans}}(K, T)| \leq C_1 d^{C_2} \log T^{C_3}. \]
The assumption on $X$ to not be contained in an algebraic hypersurface could be dropped but we will omit the technical details.

Here is how this article is structured. In the next section 2 we show how to deduce Theorem 2 and 3 from the work of Pila and Jones and Thomas. Then in section 3 we apply these results to find a new proof of polynomial lower Galois bounds for torsion points of $\mathbb{G}_m$ and elliptic curves.

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2. Pfaffian curves and surfaces

We start by recalling work of Pila on Pfaffian curves. We will use the notions from Definition 6.4 in \cite{Pil09}.

Lemma 1. Let $K$ be a real number field with $[K : \mathbb{Q}] = d$. Let $\delta \geq 1, T \geq 1, L \geq 1/T^{4d}$. Put $D = (\delta + 1)(\delta + 2)/2$. Let $I$ be an interval of length at most $L$ and $f : I \to \mathbb{R}$ a function. Suppose that $f$ has $D - 1$ continuous derivatives on $I$, with $|f'| \leq 1$, and let $X$ be the graph of $f$ on $I$. Then $X_{\text{size}}(K, T)$ is contained in the union of at most

$$6(D!)^{2d/(D(D-1))} (LT^{4d})^{4/(3(\delta+3))} A_{L,D-1}(f)$$

plane algebraic curves of degree at most $\delta$.

Proof. The proof is the same as the proof of \cite{Pil09} Lemma 6.5] except that instead of estimating $(D!)^{2/(D(D-1))}$ we leave it as it is and estimate $D^{4d/(3(\delta+3))} \leq 5$ there .

Proposition 1. Let $\delta \geq 1, D = (\delta + 1)(\delta + 2)/2, d \geq 1, T \geq e, L \geq 1/T^{4d}$ and $I \subset \mathbb{R}$ an interval of length $\leq L$. Let $K \subset \mathbb{R}$ be a number field of degree $d$. Let $f : I \to \mathbb{R}$ have $D$ continuous derivatives, with $|f'| \leq 1$ and $f^{(j)}$ either non-vanishing in the interior of $I$ or identically vanishing, for $j = 1, \ldots, D$. Let $X$ be the graph of $f$. Then $X_{\text{size}}(K, T)$ is contained in the union of at most

$$54D(D!)^{2d/(D(D-1))} (LT^{4d})^{4/(3(\delta+3))} \log(eLT^{4d})$$

real algebraic curves of degree at most $\delta$.

Proof. We follow the proof of \cite{Pil09} Prop. 6.7] word by word but using the estimate from Lemma 1] instead of the one given in Lemma 6.5 there.

Finally we state a result of Jones and Thomas a little more explicitly. Recall their definition of implicitly defined Pfaffian function on page 640 of \cite{JT12} with its notion of complexity.

Theorem 4. Suppose that $I$ is an open interval in $\mathbb{R}$ and that $f : I \to \mathbb{R}$ is a transcendental implicitly defined Pfaffian function of complexity $(n, r, \alpha, \beta)$. Then for $T \geq e$, and
the graph $X$ of $f$

$$|X^{size}(K, T)| \leq 2^{(r-1)2}\frac{2}{3}n^r + \frac{23}{2}(n + 2)^{n+3r+1}(\alpha + \beta)^{2n+2r+1}d^{3n+3r+8} \log T^{3n+3r+8}.$$  

**Proof.** We follow their proof and for $c_3, c_4$ there it is not hard to estimate

$$c_3c_4d^{n+r+1}D^{n+r+2} \leq 2^{(r-1)2}\frac{2}{3}n^r + \frac{23}{2}(n + 2)^{n+3r+1}(\alpha + \beta)^{2n+2r+1}d^{n+r+1}D^{n+r+2} \leq 2^{(r-1)2}\frac{2}{3}n^r + \frac{23}{2}(n + 2)^{n+3r+1}(\alpha + \beta)^{2n+2r+1}d^{3n+3r+5}.$$  

Now setting $\delta = [d \log T]$ and $L = 2T$ in Proposition 1 we get that the number of hypersurfaces can be bounded by

$$6^{15}d^3 \log T^3$$  

(note the slightly confusing fact that $d$ is now the degree of the number field) and we obtain the Proposition. \qed

**Corollary 1.** It holds that

$$|X(K, T)| \leq 2^{(r-1)2}\frac{2}{3}n^r + \frac{23}{2}(n + 2)^{n+3r+1}(\alpha + \beta)^{2n+2r+1}d^{6n+6r+16} \log T^{3n+3r+8}.$$  

**Proof.** It is clear that $X(K, T) \subset X^{size}(K, Td^n)$ and the corollary follows. \qed

We continue by recording Pila’s work [Pil04]. Recall the definition of $(J, A, C)$ mild [Pil10] Definition 2.1 and 2.4.

Pila proved that for $X$ admitting a mild parametrization the set $X(K, T)$ is contained in a union of hypersurfaces whose number and degree we can control in terms of $J, A, C$.

In order to get a polynomial dependence on the degree of $K$ we we are going to make a slight adjustment at one of the steps in Pila’s proof (which was more concerned with the dependence on $T$) and show that we can extract a polynomial dependence on the degree.

**Theorem 5.** Let $X \subset (0, 1)^n$ have a $(J, A, C)$-mild parametrization. There exist effectively computable constants $C_1, C_2, C_3$ depending only on $J, A, C, n$ and $\dim X$ such that $X(K, T)$ is contained in the union of $C_1d^{C_2}T^{C_3}$ hypersurfaces of degree bounded by $(d^2 \log T)^{\dim X/(n-\dim X)}$.

**Proof.** We follow the proof in [Pil10] right up to the choice of the degree of the hypersurface on p.503. The degree of the number field there is $f$. Instead of choosing $d$ there equal to $[\log T^{k/(n-k)}]$ we choose it to be equal to $[(f \log T)^k/(n-k)]$ there which kills the $f$ in the exponent. Then in the corollary there we get $f^2$. \qed

We start with our investigation of surfaces. For this we use the notion of a semi-Pfaffian set as in [JT12] p.640 with its notion of format. First we record that the following holds.

Let $X$ be a connected semi-pfaffian surface in $(0, 1)^n$ with a mild parametrization with parameters bounded by a constant $M_X$ that is not contained in any algebraic hypersurface.

**Proposition 2.** There exist effectively computable constants $c_1, c_2, c_3, c_4$ depending only on the format of $X$ and $M_X$ such that for any hypersurface $Z$ in $\mathbb{R}^3$ of degree $d_Z$ holds

$$|(X \cap Z)^{trans}(K, T)| \leq c_1d_Z^2d^3 \log T^{c_4}$$
Proof. We first note that we may assume that the dimension of each component of $X \cap Z$ is at most 1. Otherwise $X^{\text{trans}}$ is empty. Furthermore, by Khovanskii’s zero-estimates the number of connected components of the intersection $X \cap Z$ grows polynomially in $d_Z$ with the growth depending only on the format of $X$ as is already pointed in the proof of Proposition 5.3 of [JT12]. Now if the component is a point the counting becomes straightforward. So we may assume it is a curve. If the curve is algebraic it does not belong to $(X \cap V)^{\text{trans}}$. If the curve is transcendental there is some projection to $\mathbb{R}^3$ that is a transcendental curve as well. For each projection we can follow the proof of [JT12, Proposition 5.3] line by line but in the displayed equation just after (7) we use Proposition 4 to get a polynomial dependence on $d$. There are $\binom{n}{3}$ such projection so we need to multiply the final estimate by this number as well. □

Combining Theorem 5 with Proposition 2 we obtain Theorem 3.

3. Torsion points

Corollary 2. For $n \geq 4$ holds

$$[Q(\zeta_n) : \mathbb{Q}] \geq n^{1/40} \log n^{-1/2}/6.$$  

Proof. We consider the function $\cos(2\pi \theta)$ on the interval $(-\frac{1}{2}, \frac{1}{2})$. This is Pfaffian of degree $(2, 1)$ and order 2. For $n \geq 4$ let $\zeta_n$ be a primitive $n$-th root of unity. Then $\Re(\zeta_n) = \cos(2\pi(k/n))$ or some integer $k$ in $(-n/2, n/2)$ and $(k, n) = 1$. Since $\zeta_n = 1/\zeta_n$ this lies in $Q(\zeta_n)$. We also have that $\zeta_n^l \in Q(\zeta_n)$ for an integer $1 \leq l \leq n - 1$ and that $\Re(\zeta_n^l) = \cos(2\pi l'/n)$ for some $l' \in (-n/2, n/2)$ if we exclude $l = n/2$. From elementary height inequalities follows $H(\cos(2\pi k/n)) \leq 4$. We set $d = [Q(\zeta_n) : \mathbb{Q}]$ and conclude from Theorem 4 that

$$n - 2 \leq c(2, 2, 2, 1)d^{40} \log n^{20}$$

where $c(2, 2, 2, 1) \leq 6^{31}$. □

Of course this is far from the real lower bound but the proof uses the same method that we will use to deduce lower bounds for torsion points of elliptic curves.

In order to keep the exposition short we refrain from proving more explicit bounds. We will only show that the dependence on the elliptic curve is only on the height of the elliptic curve.

We fix a lattice $\Lambda \subset \mathbb{C}$ with generators $\omega_1, \omega_2$. Let $\wp_{\Lambda}$ be the Weierstrass function associated to $\Lambda$ and $E$ the associated elliptic curve. In what follows we denote complex conjugation by an upper bar. We define the function $f_{\Lambda}$ by

$$f_{\Lambda}(b_1, b_2) = (\Re \wp_{\Lambda}(b_1 \omega_1 + b_2 \omega_2), \Im \wp_{\Lambda}(b_1 \omega_1 + b_2 \omega_2)); \quad (b_1, b_2) \in [0, 1)^2, b_1^2 + b_2^2 \neq 0.$$  

Let $X_{\Lambda}$ in $\mathbb{R}^4$ be the graph of $f_{\Lambda}$. For any algebraic hypersurface $Z$ in $\mathbb{R}^4$ the following holds.

Lemma 2. Any positive dimensional component of $Z \cap X_{\Lambda}$ is a transcendental curve.

Proof. It is enough to prove that the intersection $Z \cap X_{\Lambda}$ has no 2 dimensional components and that the one dimensional components are transcendental curves. Suppose
first that there exists a 2 dimensional component \( U \). Then there exist complex analytic functions \( r_1, r_2 \) in some poly-disc in \( \mathbb{C}^2 \) such that the determinant of the Jacobian of \( (r_1, r_2) \) does not vanish and such that their restriction to \( \mathbb{R} \) is real. Further \( \text{trdeg}_C \mathbb{C}(r_1, r_2, \varphi_{\Lambda}(r_1 \omega_1 + r_2 \omega_2), \varphi_{\Lambda}(r_1 \overline{\omega}_1 + r_2 \overline{\omega}_2)) \leq 3 \). By Ax’s theorem \(^{[Ax72]}\) this implies that \( y = (\varphi_{\Lambda}(r_1 \omega_1 + r_2 \omega_2), \varphi_{\Lambda}(r_1 \overline{\omega}_1 + r_2 \overline{\omega}_2)) \) is contained in a translate of an algebraic subgroup of \( E \times \overline{E} \). Thus there exists an isogeny \( \beta : E \to E \) that acts on the tangent space by sending \( b_1 \overline{\omega}_1 + b_2 \overline{\omega}_2 \) to \( b_1^0 \omega_1 + b_2^0 \omega_2 \) where \( (b_1^0, b_2^0) = (b_1, b_2)B \) with \( B \) an integer matrix with negative determinant. There is thus a relation of the form

\[
(r_1, r_2)A_1 + (r_1, r_2)BA_2 = 0 \mod \mathbb{R}^2.
\]

where \( A_1, A_2 \) are integer matrices of positive determinant. The matrix \( A_1 + BA_2 \) has rank at least 1 and so \( r_1, r_2 \) are linearly related over \( \mathbb{Q} \) mod \( \mathbb{R} \) which contradicts our assumption that the Jacobian of \( (r_1, r_2) \) is non-singular. Thus there are no 2-dimensional components.

Now let \( U \) be a 1 dimensional component and assume that it is algebraic. Thus there are complex analytic functions \( r_1, r_2 \) not both constant such that \( \text{trdeg}_C \mathbb{C}(r_1, r_2, \varphi_{\Lambda}(r_1 \omega_1 + r_2 \omega_2), \varphi_{\Lambda}(r_1 \overline{\omega}_1 + r_2 \overline{\omega}_2)) \leq 1 \). Now as above this implies that \( r_1, r_2 \) are linearly related over \( \mathbb{Q} \) mod \( \mathbb{R} \). That is we may assume that we can write \( r_2 = c_1 r_1 + c_2 \) for \( c_1, c_2 \in \mathbb{R} \) and that \( r_1 \) is not constant. Setting \( z_\tau = r_1(\omega_1 + c_1 \omega_2) + c_2 \omega_2 \) we have \( \text{trdeg}_C \mathbb{C}(z_\tau, \varphi_{\Lambda}(z_\tau)) \leq 1 \). By Ax’s theorem this implies that \( z_\tau \) is constant and so \( \omega_1 + c_1 \omega_2 = 0 \) which is absurd since \( \Im(\omega_2/\omega_1) \neq 0 \). This proves the claim. \( \square \)

We borrow some estimates from Masser-Wüstholz. First \(^{[MW90]}\) Lemma 3.2.

**Lemma 3.** There exists an effectively computable absolute constant \( C \) such that

\[
|\varphi(z) - \varphi(\omega_2/2)| \leq C d(z, \Lambda)^{-2}.
\]

where \( d(z, \Lambda) \) is the minimal distance of \( z \) to an element of \( \Lambda \).

Pick generators \( \omega_1, \omega_2 \) of \( \Lambda \) such that \( \tau = \omega_2/\omega_1 \) satisfies \( |\Re(\tau)| \leq \frac{1}{2}, |\tau| \geq 1 \).

**Lemma 4.** For any \( \epsilon > 0 \) let \( B_\epsilon \) be the box consisting of \( z = t_1 \omega_1 + t_2 \omega_2 \) with \( |t_1| \leq \frac{1}{2}, |t_2| \leq 1/2 - \epsilon \). There is an effectively computable constant \( C_\epsilon \) depending only on \( \epsilon \) such that

\[
|1/(\varphi(z) - \varphi(\omega_2/2))| \leq C_\epsilon \exp(\pi \Im(\tau)), z \in B_\epsilon.
\]

**Proof.** We consider the expansion \(^{[MW90]}\) (3.3)]. From the proof there follows that \( F(1/2) \prod_{n=1}^\infty \{F(n)/F(n - 1/2)\} \) is bounded from below. For \( F(1/2) \) we note that for \( q, Q \) there

\[
|1 - q^{1/2}Q^{\pm 2}| \geq 1 - \exp(-2(1/2 \pm t_2)\pi \Im(\tau)) \gg_\epsilon \exp(-2\epsilon \pi \Im(\tau)).
\]

The last thing to check is that \( |\sin w| \ll \exp(2\pi(1/2 - \epsilon)\Im(\tau)). \square \)

We pass to the Legendre family \( E_\lambda \) and set \( X_\lambda = \varphi_\lambda + 4(\lambda + 1) \) where \( \varphi_\lambda \) is associated to the lattice \( \Lambda_\lambda \) generated by the differential \( \frac{dX}{2\sqrt{X(X-1)(X-\lambda)}} \). Now assume that \( \lambda \) satisfies

\[
|\lambda| \leq 1, |1 - \lambda| \leq 1, \Re(\lambda) \leq \frac{1}{2}
\]
and set \( \omega_1, \omega_2 \) to be given by hypergeometric series such as in [JS17, p.5]. From [JS17, Lemma 8] follows that \( X_\lambda(z) = \varphi_\lambda(z) - \varphi_\lambda(\omega_2/2) \).

We first define \( B_1 \) be given by
\[
B_1 = \{ z = b_1 \omega_1 + b_2 \omega_2; \quad b_2 \in (1/30, 29/30), b_1 \in (0, 1) \}
\]

**Lemma 5.** We have
\[
d(z; A_\lambda) \geq |\omega_2|/60, \quad z \in B_1.
\]

**Proof.** This follows almost immediately from [JS17, Lemma 8].

Now we define \( B_2 = \{ z = b_1 \omega_1 + b_2 \omega_2; b_1, b_2 \in [-29/60, 29/60] \} \).

Now we note that for any \( \lambda' \in \mathbb{C} \setminus \{ 0, 1 \} \) we can find \( \lambda \) satisfying (1) such that one of the following holds
\[
(2) \quad \Lambda_{\lambda'} = \Lambda_\lambda, \quad \Lambda_{\lambda'} = \lambda^{\frac{1}{2^m}} \Lambda_\lambda.
\]

This follows from the fact that the transformation \( \lambda \to 1 - \lambda \) preserves the lattice while \( \lambda \to 1/\lambda \) scales the lattice by a factor of \( \sqrt{\Lambda} \). For any \( \lambda' \) the generators \( \omega_1, \omega_2 \) respectively \( \sqrt{\Lambda} \omega_2, \sqrt{\Lambda} \omega_1 \) of \( \Lambda_{\lambda'} \) are such that \( \omega_2/\omega_1 = \tau \) is in the standard fundamental domain as above [JS17, Lemma 8]. If we can choose \( \lambda' \) in (1) such that \( \Lambda_{\lambda'} = \Lambda_\lambda \) we set
\[
f_{\lambda'}(b_1, b_2) = X_\lambda(b_1 \omega_1 + b_2 \omega_2), \quad z = b_1 \omega_1 + b_2 \omega_2 \in B_1 \cup (0, 1),
\]

Otherwise we set
\[
f_{\lambda'}(b_1, b_2) = \lambda/X_\lambda(b_1 \omega_1 + b_2 \omega_2), \quad z = b_1 \omega_1 + b_2 \omega_2 \in B_2
\]

We note that \( X_\lambda = X_{\lambda'} \) if \( \Lambda_{\lambda'} = \Lambda_\lambda \) in the first case and else \( X_\lambda = X_{\lambda'}/\lambda \).

**Lemma 6.** There exists an integer \( T \) and effectively computable absolute constants \( A_1, A_2 \) (not depending on \( \lambda' \)) such that the graph \( G_{\lambda'} \) of \( f_{\lambda'}/T \) restricted to \( (b_1, b_2) \) such that \( z \in B_1 \) respectively \( z \in B_2 \) has a \((1, A_1, A_2)\) mild parametrization.

**Proof.** First suppose that \( \Lambda_\lambda = \Lambda_{\lambda'} \). We use Cauchy’s formula for the \( n \)-th derivative
\[
X_\lambda^{(n)}(z)/n! = \frac{1}{2\pi i} \oint \frac{X_\lambda(w)dw}{(z-w)^{n+1}}
\]
and integrate along the circle \( |z-w| = d(z, A_\lambda)/2 \). By Lemma 5 \( d(z, A_\lambda)/2 \geq |\omega_2|/120 \) for \( z \in B_1 \). So by Lemma 3 there exist absolute constants \( \tilde{A}_1, \tilde{A}_2 \) such that
\[
|\omega_2|^n |X_\lambda^{(n)}| \leq \tilde{A}_1(\tilde{A}_2)^n n!, \quad z \in B_1.
\]

As the absolute value of \( \omega_1 \) is bounded by an absolute constant (see for example [JS17, Lemma 12] ), if we pick an integer \( T \) whose absolute value is greater than the absolute value of \( X_\lambda \) we find that a mild parametrization of \( \Gamma_{\lambda'} \) is given by
\[
(t_1, t_2) \to (t_1, 30/29t_2, \Re f_{\lambda'}(t_1, 30/29t_2)/T, \Im f_{\lambda'}(t_1, 30/29t_2)/T)
\]

Now assume that \( \Lambda_{\lambda'} = \sqrt{\Lambda} \lambda \). We first note that from the Fourier expansion of \( \lambda \) follows that \( |\lambda| \ll \exp(-\pi \Im(\tau)) \) so we find that \( \lambda/X_\lambda \) is bounded by an absolute constant on \( B_{1/120} \). We again use Cauchy’s formula but this time we first note that the distance of \( B_2 \) to \( B_{1/120} \) is bounded from below by an absolute constant \( c_1 \) as \( \omega_1 \) is
bounded from below by 1 [JS17, Lemma 8]. We can then pick the circle \( |z - w| = c/2 \) and using Cauchy’s formula find that we can pick \( \tilde{A}_1, \tilde{A}_2 \) such that

\[
|\left( \lambda X_\lambda \right)^{(n)}(z)| \leq \tilde{A}_1 \tilde{A}_2^n n!.
\]

Now it remains to note again that \( \omega_1 \) is bounded absolutely while \( |\omega_2| \ll -\log |\lambda| \) and so \( |\lambda |^{\frac{1}{2}}|\omega_2| \) is also bounded by an absolute constant. We take \( T \) to be also larger than the maximum of \( X_\lambda \) on \( B_2 \). As in the previous case we have established the mild parametrization of \( G_X \).

Now let \( \Gamma_X \) be the graph of \( f_X \).

**Corollary 3.** There exist effectively computable absolute constants \( \gamma_1, \gamma_2, \gamma_3 \) such that

\[
|\Gamma_X(K, H)| \leq \gamma_1 d^{\gamma_2} \log H^{\gamma_3}.
\]

**Proof.** Theorem 1 of [JS17] implies that \( \Gamma_X \) is a finite union of semi-Pfaffian surfaces with the entries of its format and the number of surfaces in the union bounded by an a constant independent of \( \lambda \). For \( \lambda \) such that \( \Lambda_X = \Lambda \lambda \) the corollary follows directly (after rescaling) from Lemma [6, Lemma 2 and Theorem [8. For \( \lambda \) such that \( \Lambda_X = \sqrt{\lambda} \Lambda \lambda \) we need to also infer the use of Theorem [2 for the piece given by \( f_X \) restricted to \((0, 1)\).

We note here that there is a certain uniformity in the counting, since the constants \( \gamma_1, \gamma_2, \gamma_3 \) do not depend on \( \lambda \).

From the corollary follows another corollary.

**Corollary 4.** For algebraic \( \lambda \), let \( P \in E_\lambda(\overline{Q}) \) be a torsion point of order \( n \) and \( d = [Q(P, \lambda) : Q] \). There exist effectively computable positive absolute constants \( \delta_1, \delta_2, \delta_3 \) such that

\[
d \geq \delta_1 (1 + h(\lambda'))^{-\delta_2 n} \delta_3.
\]

**Proof.** The abscissa of \( P \) is \( X_\lambda(z) \) where \( z = k\omega_1/n + l\omega_2/n \) such that \( (k/n, l/n) \) has coordinates in \( Q \) whose minimal denominator is \( n \). The same holds for \( P, 2P, \ldots , (n-1)P \) whose coordinates lie in the field \( Q(\lambda', P) \). For each of these torsion the logarithmic Weil height of its abscissa is bounded by \( c(1 + h(\lambda')) \) [HJM17, p.467] for \( c \) absolute. An elementary computation shows that the set consisting of \( m(k/n\omega_1 + l/n\omega_2) \) mod \( \Lambda_X, m = 1, \ldots , n-1 \) has \( (1/c)n \) representatives in \( B_1 \cup (0, 1) \) respectively \( B_2 \). Thus we find \( (1/c)n \) points in \( \Gamma_X(Q(\lambda', P), H) \) with \( \log H \leq c(1 + h(\lambda')) \log n \). From Corollary [8] follows the present corollary. □

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