R-SYMMETRY OF HETEROISTIC N=2 SUPERGRAVITIES

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ABSTRACT
The topological twist of N=2, D=4 matter-coupled supergravities requires a suitable R-symmetry. This symmetry is realised in the effective supergravities arising at tree level from certain heterotic compactifications. The set of instanton equations (topological gauge-fixings) is thus obtained. The conditions that R-symmetry should satisfy also when these theories are replaced by their “exact” quantum-corrected counterparts are investigated.

1. Introduction
Remarkable progress has been made recently towards the understanding of some non-perturbative phenomena, both in quantum field theory and in string theory. In particular, powerful tools emerged in N=2 supersymmetric theories.

The exact low energy effective theory of an N=2 super Yang–Mills (SYM) system was obtained by Seiberg and Witten [1].

At the string level, very strong evidences have been accumulated [2]-[7] in favour of the so-called “heterotic-type II duality”. To specific heterotic compactifications, having as effective 4D field theories N=2 supergravities coupled to r+1 vector multiplets and m hypermultiplets, string-string duality associates type IIA compactifications on a Calabi–Yau (CY) manifold with the same spectrum. The Hodge numbers of the CY must be $h^{1,1} = r + 1$, $h^{2,1} = m - 1$.

In the heterotic compactifications the dilaton (whose expectation value is related to the string coupling constant) sits in a vector multiplet; it belongs instead to a hypermultiplet in the type II case. It follows then [8, 9] from an N=2 non-renormalization theorem that the tree level effective theory of the vector multiplets is exact in the type II compactifications, while it undergoes perturbative and non-perturbative corrections in the heterotic case; the opposite happens for the hypermultiplets.

In particular a tree level heterotic effective theory for the vectors is described in terms of a certain special Kähler manifold $ST(n)$. String corrections modify it; the exact theory corresponds to a deformed special manifold $\hat{ST}(n)$. By heterotic-type II duality $\hat{ST}(n)$ is identified as the moduli space of (1,1)-forms on the Calabi–Yau manifold representing the type II compactifying space.

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This pattern can be regarded as the analogue, at the level of locally supersymmetric theories, of the evaluation of exact low energy theory for N=2 SYM in terms of suitable Riemann surfaces [1].

From another point of view, N=2 field theories are particularly relevant because they can be twisted. The topological twist associates them with suitable topological field theories (TFT’s), providing also a set of topological gauge-fixings (“instanton equations”) of the latter.

TFT’s have often a deep interest in mathematics. For instance, topological Yang–Mills [10] (obtained by twisting N=2 SYM) with SU(2) gauge group is related to Donaldson theory. The results of Seiberg–Witten on the exact low-energy SU(2) SYM provided, upon twisting, important new tools in Donaldson theory [11]. The “monopole equation” that plays a basic role in this game is an instance of the instanton equations, obtained by twisting SYM theories coupled to hypermultiplets [12, 13]. It is therefore very interesting to consider the topological twist of a generic N=2, D=4 theory, containing supergravity coupled to hyper- and vector multiplets, and in particular to consider the structure of the corresponding instanton equations.

This contribution considers the twist of N=2 effective supergravities arising from heterotic compactifications, keeping in mind that the tree level theories are modified by string corrections, and in certain cases their exact quantum expressions in terms of CY spaces is conjectured.

In section 2 the basic steps to twist a generic N=2 model are reviewed; between them, the identification of a R-symmetry. Section 3 recalls the special geometry of vector multiplets in tree-level heterotic N=2 models. In sec. 4 the requirements that R-symmetry must satisfy are described. In particular, the R-charges of the dilaton-axion vector multiplet must differ from the others. By twisting, the set of instanton equations is obtained in sec. 5. Finally, a case is considered in sec. 6 in which the exact special geometry is expressed in terms of a specific CY manifold. The existence of a discrete R-symmetry, allowing the topological twist, is shown.

2. Topological twist of N=2, D=4 supergravities

In [10] Witten derived a topological reinterpretation of the N=2 YM theory by means of a redefinition of the euclidean Lorentz group. The steps needed to perform this construction in the case of arbitrary N=2 theories, including also gravity and hypermultiplets, were derived in [12, 14], and are as follows:

1. Systematic use of BRST quantization, prior to the twist

2. Redefinition of the Lorentz group SO(4) = SU(2)_L × SU(2)_R:

   \[ SO(4)' = \text{diag}[SU(2)_Q × SU(2)_L] × \text{diag}[SU(2)_I × SU(2)_R]. \]  \hspace{1cm} (1)

   SU(2)_I is the automorphism group of the N=2 algebra; SU(2)_Q acts on the hypermultiplet sector (so that it is irrelevant in the pure YM case)

3. Redefinition of the BRST charge:

   \[ Q'_{\text{BRST}} = Q_{\text{BRST}} + Q^0_{\text{SUSY}} \]  \hspace{1cm} (2)
where $Q_{\mathrm{SUSY}}^0$ is a combination of the components of the N=2 supersymmetry charges that acquire spin zero after the spin redefinition \cite{[1]}. 

4. Redefinition of the ghost numbers of the fields:

$$g' = g + q_R$$

by means of their charges $q_R$ under a suitable R-symmetry, to match the reinterpretation of the fields as physical ($g' = 0$), ghosts ($g' = 1$), antighosts, . . . .

In the case of N=2 Yang–Mills, the fields are organized in N=2 vector multiplets: $(A^I_\mu, \lambda^I A, \lambda^I_A, Y^I)$, $\lambda$ being the gauginos\cite{[2]} and $Y$ the gauge scalars. Their topological reinterpretation is as follows: (phys., ghost, antigh., gh. for ghost). Accordingly, in the N=2 theory there exists a R-symmetry assigning them R-charges $(0, 1, -1, 2)$.

In the general case, besides the vector multiplets, we have the gravitational multiplet $(V^\mu, \psi_A, \psi^A, A^0_\mu)$ including the vierbein $V^\mu$, the gravitinos $\psi$ and the graviphoton $A^0_\mu$, and the hypermultiplets $(q^u, \zeta^\alpha)$, including the scalars $q^u$, $(u = 1, \ldots, 4m)$ that span a quaternionic manifold, and their fermionic partners $\zeta^\alpha$ belonging to the fundamental representation of Sp(2m).

A specific N=2 model is realised by the following geometrical data:

1. A special Kähler manifold for the vector multiplet scalars, of complex dimension $n$, with $n$ being the number of vector multiplets.

2. A quaternionic manifold for the hypermultiplet scalars, of quaternionic dimension $m$, where $m$ is the number of hypermultiplets.

3. A gauge group $G \subset \mathrm{SO}(n)$, acting via special and quaternionic isometries respectively on the two scalar manifolds.

3. Vector multiplets in “heterotic” N=2 supergravities

The lagrangian and transformation rules of N=2 vector multiplets coupled to supergravity \cite{[3]} are expressed in terms of the “special geometry” of the gauge scalar manifold. The complex gauge scalars $Y^I$ parametrize a Kähler manifold whose Kähler potential

$$K = -\log \left[-i\|\Omega\|^2\right] = -\log \left[-i(X^\Lambda F_\Lambda - \bar{F}_\Lambda X^\Lambda)\right]$$

is expressed in terms of the 2$n$ + 2-dimensional holomorphic symplectic section $\Omega = (X^{\Lambda I}(Y), F_\Lambda(Y)), \Lambda = 0, I$; it is invariant under Sp($2n + 2, \mathbb{R}$) rotations of $\Omega$.

The elements of the structural group Sp($2n + 2, \mathbb{R}$), i.e. matrices with the block structure

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad A^T C - C^T A = 0, \quad B^T D - D^T B = 0, \quad A^T D - C^T B = 1,$$

\footnote{The index $A = 1, 2$ enumerates the supersymmetries; $\gamma_5 \lambda^I A = \lambda^I A$ and $\gamma_5 \lambda^I_A = -\lambda^I_A$ while $\gamma_5 \psi_A = \psi_A$ and $\gamma_5 \psi^A = -\psi^A$ for the gravitinos}
induce coordinate transformations on the scalar manifold while acting, at the same time, as duality rotations on the symplectic vector of magnetic and electric field strengths
\[ (\mathbf{F}_{\mu\nu} - \Lambda_{\mu\nu}, \mathbf{G}_{\mu\nu} - \Lambda_{\mu\nu}) \]
where
\[ G_{\Lambda\mu\nu} = -i \frac{\delta L}{\delta \mathbf{F}_{\mu\nu}}. \]

If the scalar manifold admits a continuous or discrete isometry group, this group must be suitably embedded into \( \text{Sp}(2n + 2, \mathbb{R}) \):
\[ z \to \phi(z) \to M_\phi \in \text{Sp}(2n + 2, \mathbb{R}) \text{ such that } \Omega(\phi(z)) = f(z) M_\phi \cdot \Omega(z), \quad (5) \]
the holomorphic rescaling \( f(z) \) being allowed by equation (4). The corresponding duality transformations on the gauge fields leave the system of Bianchi identities plus eq.s of motion form invariant. They can be of classical \((B = C = 0)\), perturbative \((B = 0, C \neq 0)\) or non-perturbative \((B \neq 0)\) type \cite{10}, according to their projective action on the coupling matrix \( N_{\Lambda\Sigma} \), that appears in the kinetic lagrangian of the gauge fields \( \mathcal{L} \propto \text{Im}(N_{\Lambda\Sigma} \mathbf{F}_{\mu\nu}^+ \mathbf{F}_{\mu\nu}^+) \):
\[ \mathcal{N} \to (C + D\mathcal{N})(A + B\mathcal{N})^{-1} \]

In those N=2 supergravities that arise from heterotic compactifications, the dilaton-axion scalar field \( S = \mathcal{A} + i e^D \), with \( \partial_{\sigma} \mathcal{A} = \frac{e_{\mu\nu\rho}}{\sqrt{|g|}} e^{2D} \partial^\mu B^{\nu\rho} \), belongs to a vector multiplet \((A^S_{\mu}, \lambda^S, \lambda_A^S, S)\), to be adjoined to the “usual” vector multiplets\(^4\).

The existence of a distinguished dilaton-axion direction should have an intrinsic meaning; in particular we assume that in the twist it is the dilaton-axion scalar that remains physical, contrary to the usual v.m.’s where the physical fields are the vectors. Twisting the theory with no other vector multiplets apart from that containing the dilaton \((n = 0)\), gives a “string-inspired” 4D topological gravity. The moduli space \( \mathcal{M}_\tau \) of a typical instanton configuration (an ALE manifold with Hirzebruch signature \( \tau \)) will be \((3 + 1) \times \tau \) dimensional, the +1 being due to the axion deformations. Since the observables have typically even ghost numbers, the selection rule \( \sum g = \dim \mathcal{M}_\tau \) would pose problems if \( \dim \mathcal{M}_\tau = 3\tau \), as it would be the case if only the metric was physical.

The tree-level effective theories that we consider are described by the choices:

\[
\begin{align*}
\text{Special manifold} & \quad \text{quatern. manifold} & \quad \text{gauge group} \\
ST(n) & \equiv \frac{\text{SU}(1,1)}{\text{U}(1)} \times \frac{\text{SO}(2,n)}{\text{SO}(2) \times \text{SO}(n)} & \quad HQ(m) & \equiv \frac{\text{SO}(4,m)}{\text{SO}(4) \times \text{SO}(m)} & \quad \mathcal{G} & \subset \text{SO}(n)
\end{align*}
\]

This structure is obtained by certain N=2 truncations of N=4 matter coupled supergravity, which displays a unique coset structure: \( \frac{\text{SU}(1,1)}{\text{U}(1)} \times \frac{\text{SO}(6,n+m)}{\text{SO}(6) \times \text{SO}(n+m)} \). Different truncations may give different quaternionic manifolds\(^5\). Our considerations depend \footnote{We consider therefore now \( n + 1 \) v.m.s. The previous formulae must be modified in an obvious way, e.g. the structural group is \( \text{Sp}(2n + 4, \mathbb{R}) \), the index A runs on 0, S, I, and so on.}

\footnote{Moreover, for the specific compactifications for which a type-II dual is proposed \cite{2,3}, the explicit form of the quaternionic manifold that must have quaternionic dimension equal to the \( h^{2,1} \) Hodge number of the dual CY space, is not known.}
however mainly on the special manifold $ST(n)$, that is uniquely determined as it is
the only special manifold with explicit factorization of the dilaton $[17]$ (in the
$SU(1,1)/U(1)$ factor).

The special geometry of the $ST(n)$ manifolds is conveniently described $[16]$ in
terms of a symplectic section realizing the following embedding of the isometry group
$SO(2, n) \times SL(2, \mathbb{R})$ into $Sp(2n + 4, \mathbb{R})$:

$$A \in SO(2, n) \leftrightarrow \begin{pmatrix} A & 0 \\ 0 & \eta A \eta^{-1} \end{pmatrix} \in Sp(2n + 4, \mathbb{R})$$

$$\left( \begin{array}{cccc} a & b \\ c & d \end{array} \right) \in SL(2, \mathbb{R}) \leftrightarrow \begin{pmatrix} a \eta^{-1} \\ b \eta \\ c \eta \\ d \eta \end{pmatrix} \in Sp(2n + 4, \mathbb{R})$$

(6)

where $A^T \eta A = \eta$. Notice that, in this embedding, the $SO(2, n)$ transformations are of
classical type. Only $SU(1,1)$ generates perturbative and non-perturbative transfor-
mations. The $S$ field, parametrizing the coset $SU(1,1)/U(1)$, plays a distinguished role. The
explicit form of the symplectic section, corresponding to the embedding of eq. (6), is

$$(X^\Lambda, F_\Lambda) = (X^\Lambda, S \eta_{\Lambda \Sigma} X^\Sigma), \quad X^\Lambda = \left( \frac{1}{2} (1 + Y^2), \frac{i}{2} (1 - Y^2), Y^I \right)$$

(7)

In eq. (7), $Y^I$ are the Calabi–Visentini coordinates, for the coset manifold $SO(2, n)/SO(2) \times SO(n)$. The
pseudoorthogonal metric $\eta_{\Lambda \Sigma}$ is $\text{(+, +, −, ..., −)}$. The Kähler potential following
from eq. (4) is

$$K = - \log i(\bar{S} - S) - \log \bar{X}^T \eta X.$$  

The Kähler metric has therefore a block diagonal structure; we have $g_{S\bar{S}}(S, \bar{S}) = \frac{-1}{(S - \bar{S})^2}$ and $g_{IJ}(Y, \bar{Y}) = \partial_I \partial_{\bar{J}} \mathcal{K}$; at $Y = 0$, $g_{IJ}$ reduces to $2 \delta_{IJ}$. The coupling
matrix $\mathcal{N}$ is such that $\text{Re} \mathcal{N}_{\Lambda \Sigma} = A \eta_{\Lambda \Sigma}$; moreover, at $Y = 0$, $\text{Im} \mathcal{N}_{IJ}$ equals $e^D \delta_{IJ}$. Thus at $Y = 0$ the kinetic term for the ordinary gauge vectors $A^I$ reduces $\mathcal{F}^a_{ab} F^I_{ab}$, and we can reinterpret $g_{\text{eff}} = \frac{g}{\sqrt{\text{Im} S}}$ as the effective gauge coupling.

4. Requirements on $R$-symmetry

Rigid minimally coupled Yang–Mills theory possesses, as already said, an $R$-symmetry;
it acts with charge $\pm 1$ on the susy parameters, with charges $(0, 1, 2)$ on $(A^I, \lambda^I, Y^I)$, commutes with supersymmetry and is an off-shell symmetry of the action. In the i.e.
eff. theory it is broken to a discrete subgroup by quantum effects $[11]$.

We consider now the requirements that an analogue symmetry in N=2 local the-
ories, containing supergravity, must satisfy $[18]$.

We require that the fields of the theory have well defined charges, so that the $R$-
symmetry group is either a $U_R(1)$ group if continuous or a cyclic group $\mathbb{Z}_p$ if discrete.

By definition $R$-symmetry acts diagonally with charge $+1$ ($-1$) on the left-(right)-
handed gravitinos (in the same way as it acts on the supersymmetry parameters in
the rigid case):

$$\psi_A \rightarrow e^{i \theta} \psi_A$$
$$\psi^A \rightarrow e^{-i \theta} \psi^A$$

i.e.

$$q_R(\psi_A) = \frac{1}{2} A^I \psi^A$$

(8)

We take here into account the gauge coupling dependence, via the usual redefinition $A^I \rightarrow \frac{1}{g} A^I$. 

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R-symmetry must generate isometries \( z^i \rightarrow (R_{2\theta}z)^i \) of the scalar metric \( g_{ij} \); as such, it has to be embedded into \( Sp(2n + 4, \mathbb{R}) \) by means of a symplectic matrix:

\[
M_{2\theta} = \begin{pmatrix}
a_{2\theta} & b_{2\theta} \\
c_{2\theta} & d_{2\theta}
\end{pmatrix} \in Sp(2n + 4, \mathbb{R}).
\] (9)

In general it acts as a R-duality; only if \( M_{2\theta} \) is block-diagonal it corresponds to a symmetry of the lagrangian (we will see that this happens in the classical case of the \( ST(n) \) manifold).

There is a symplectic action on the section \( (X^A, F_A) \) induced by \( z^i \rightarrow (R_{2\theta}z)^i \):

\[
(X, F) \rightarrow e^{2i\theta} M_{2\theta} \cdot (X, F).
\]

Comparing with eq. (5), we see that the compensating factor \( f(z)_{2\theta} \) that is in principle allowed, has to be fixed: \( f(z)_{2\theta} = e^{2i\theta} \). Indeed the rescaling factor \( f(z) \) in eq. (5) corresponds to a Kähler transformation \( e^{e^K} \rightarrow e^{e^K+f+\bar{f}} \); the gravitino \( \psi_A \) has Kähler weight one, i.e. \( \psi_A \rightarrow e^{f} \psi_A \). So \( f(z) \) is fixed by eq. (8). This is a very simple but crucial constraint on the form of the R-symmetry.

The above requirements on the R-symmetry ensure that it is a symmetry of the bosonic lagrangian, when \( M_{2\theta} \) is diagonal, or a duality leaving invariant the set of Bianchi identities and equations of motion. By explicit analysis [18] of the supersymmetry transformation laws it emerges that such an R-symmetry does indeed commute with supersymmetry, provided moreover that the Jacobian matrix \( (J_{2\theta})^j_i \) of the R-transformation \( z^i \rightarrow (R_{2\theta}z)^i(z) \) is covariantly constant: \( \nabla (J_{2\theta})^j_i = 0 \). This being the case, we have a symmetry (or a duality) of the full N=2 model.

We add one requirement (which is conceptually independent from the others) and which pertains to the effective supergravities of heterotic compactifications. Under the R-action there must be, in the special manifold, a preferred direction, corresponding to the dilaton–axion multiplet, whose R-charges are reversed with respect to those of all the other multiplets.

In the “classical” case of the \( ST(n) \) models, the action of R-symmetry is extremely simple. It is nothing else but the \( SO(2) \sim U(1) \) subgroup in the denominator of the \( SO(2, n)/SO(2) \times SO(n) \) coset. The embedding of eq. (3) induces the transformation:

\[
\begin{pmatrix}
X \\
F
\end{pmatrix} \rightarrow e^{2i\theta} \begin{pmatrix}
m_{2\theta} & 0 \\
0 & (m_{2\theta}^T)^{-1}
\end{pmatrix} \begin{pmatrix}
X \\
F
\end{pmatrix}, \quad m_{2\theta} = \begin{pmatrix}
cos 2\theta & -\sin 2\theta \\
\sin 2\theta & \cos 2\theta \\
0 & 0
\end{pmatrix},
\]

where \( m_{2\theta} \) belongs to \( SO(2, n) \). Having chosen the rescaling factor \( e^{2i\theta} \) as required, the corresponding transformation on the scalar field is \( S \rightarrow S, Y^I \rightarrow e^{2i\theta} Y^I \). The dilaton-axion is neutral, opposite to the ordinary gauge scalars, as we wanted. It is easy to check that the Jacobian matrix is covariantly constant. We need no more checks; this R-symmetry is a true symmetry of the lagrangian and satisfies all the expected properties.

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7The indices \( i, j, \ldots \) are reserved to generic coordinates \( \{z\} \) on the scalar manifold
Notice that in this classical case \( b_{2\theta} = c_{2\theta} = 0 \), the matrix \( \mathcal{Q} \) is completely diagonal and it has the eigenvalues \( (e^{i\theta}, e^{-i\theta}, 1, \ldots, 1) \).

In the quantum corrected models \( \tilde{ST}(n) \), R-symmetry is a discrete \( \mathbb{Z}_p \) group, for some \( p \in \mathbb{N} \). Consider a coordinate basis \( \{ u^i \} (i = 1, \ldots, n + 1) \) that diagonalizes the action of \( R_{2\theta} \) so that:

\[
R_{2\theta} u^i = \alpha^p u^i \quad q_i = 0, 1, \ldots, p - 1 \mod p ; \quad \alpha^p = 1.
\]

A generic \( Sp(2n + 4, \mathbb{R}) \) matrix has eigenvalues \( \left( \lambda_0, \ldots, \lambda_{n+1}, \frac{1}{\lambda_0}, \ldots, \frac{1}{\lambda_{n+1}} \right) \). The R-symmetry symplectic matrix \( M_{2\theta} \) of eq. (8), being the generator of a cyclic group \( \mathbb{Z}_p \), has eigenvalues \( \lambda_0 = \alpha^{k_0}, \ldots, \lambda_{n+1} = \alpha^{k_{n+1}} \), where \( (k_0, \ldots, k_{n+1}) \) is a new set of \( n + 2 \) integers defined modulo \( p \). These numbers are the R-charges of the electro–magnetic fields strengths \( F_{\mu\nu}^0 + i G_{\mu\nu}^0, \ldots, F_{\mu\nu}^{n+1} + i G_{\mu\nu}^{n+1} \).

Since what really matters in the topological twist are the differences of ghost numbers, the interpretation of the scalars \( u^i \ (i = 1, \ldots, n) \) as ghost for ghosts and of the corresponding vector fields as physical gauge fields requires that \( q_i = k_i + 2 \), for \( i = 1, \ldots, n \). On the other hand, if the vector partner of the axion–dilaton field has to be a ghost for ghosts, the \( S \)–field itself being physical, we must have \( k_{n+1} = q_{n+1} + 2 \). In the last section the existence of such a discrete R-symmetry will be checked in an explicit example (furnished by heterotic-type II duality) of quantum \( \tilde{ST}(n) \) manifold.

5. Instanton equations

Having determined a suitable R-symmetry, it is possible to perform all the steps of section 2 to get the topological theory. In particular, the instanton conditions are obtained setting to zero the topological BRST variations of the antighost, at zero unphysical \( (g'_\tau \neq 0) \) fields. These variations are obtained from the \( N=2 \) susy transformations of these fields in the untwisted theory.

The antighosts in the gravity-v.m.’s sector are \( \psi^A, \lambda^{SA}, \lambda^A \); they have indeed R-charge \(-1\) in the \( N=2 \) theory. We report here the form of the instanton equations in the full theory, containing also the hypermultiplets. To be specific we consider the case in which the h.m. scalars span the \( HQ(m) \) manifold. Being a quaternionic space, three quaternionic two-forms \( \Omega^{-x}, x = 1, 2, 3 \) exist that are the curvatures of the one–forms \( \omega^{-x} \) gauging the SU(2)I action on \( HQ(m) \).\(^8\) The hyperinos appear in the superspace parametrization of the vielbeins: \( U_A^t = u_{aA}^t V^a + \epsilon^{AB} \bar{\psi}_B \xi^B \epsilon_B A + \bar{\psi}^A \xi_A^t \), and the antighost is \( \xi^A \).

Consider also a non-trivial action of the gauge group \( G \). Then the Kähler and SU(2)I connections contain a gauge term, and some modifications occur in the susy transformation rules (see \([22, 13, 18]\)). These modifications are expressed in terms

\(^8\)For convenience, here the index \( n + 1 \) represents the dilaton index previously denoted by \( S \)

\(^9\)Another set of one–forms \( \omega^{-x} \) gauging the SU(2)Q action, needed for the spin redefinition \( \mathcal{Q} \), exists. The \( \omega^{-x} \) forms forms can be explicitly written utilizing a parametrization \( [18] \) of \( HQ(m) \) that is the quaternionic analogue of the Calabi–Visentini parametrization of \( \text{SO}(2,4) \times \text{SO}(m) \). The two SU(2) groups correspond to the \( \text{SO}(4) \sim \text{SU}(2)I \times \text{SU}(2)Q \) decomposition of the \( \text{SO}(4) \) holonomy factor; the vielbeins have the index structure \( U_A^t \); \( A, \bar{A}, \bar{t} \) are in the fundamental of \( \text{SU}(2)I, \text{SU}(2)Q, \text{SO}(m) \).
of the Killing vectors and momentum maps $\mathcal{P}_0^\Lambda$ and $\mathcal{P}_x^\Lambda$ for the action of $\mathcal{G}$ on the special and quaternionic spaces.

At this point, after Wick rotation\(^{10}\), one considers the susy variations of the antighosts $\psi^A, \lambda^{SA}, \lambda^I, \zeta^A$, makes the index identifications implied by eq. (1), sets to zero the unphysical fields and projects on suitable Lorentz components; the details can be found in [18]. The following set of topological gauge-fixing is thus obtained:

\[
R^{-ab} - \sum_{u=1}^{3} J_u^{-ab} q^u \hat{\Omega}^{-u} = 0
\]

\[
\partial_a D - \epsilon_{abcd} e^D H_{bcd} = 0
\]

\[
\mathcal{F}^{-Iab} - \frac{g^2}{2 \exp D} \sum_{u=1}^{3} J_u^{-ab} \mathcal{P}_{I}^{-u} = 0
\]

\[
\mathcal{D}_{\mu} q^P - \sum_{u=1}^{3} (j_u)_{\mu} \nu \mathcal{D}_{\nu} q^Q (J_u)_{Q} = 0.
\]  

(10)

$R^{-ab}$ is the antiselfdual part of the Riemann curvature two–form ($a, b$ are indices in the tangent of the space time manifold $M$). $q^u \hat{\Omega}^{-u}$ denotes the pull–back, via a gauged–triholomorphic map: $q : M \rightarrow HQ(m)$ of the “gauged” 2–forms $\hat{\Omega}^{-u}$. $J_u^{-ab}$ is a basis of anti-selfdual matrices in $\mathbb{R}^4$.

The first equation (arising from a component of the gravitino variation) represents the modification of the usual gravitational instantons ($R^{-ab} = 0$) due to the hypermultiplet sector.

The second of equations (10), that comes from the dilatino variation, describes the H–monopole or axion–dilaton instanton of [20]. The H–monopoles have vanishing stress–energy tensor, so that they do not interfere with the gravitational instanton conditions. The presence of these H-monopoles is the main differences with previous analysis of generic 4D TFT’s [14, 12]; as already emphasized, they should allow the calculation of non–vanishing topological correlators between local observables as intersection numbers in a moduli–space that has now an overall complex structure.

The third equation (from the variations of the ordinary gauginos $\lambda^I_A$) expresses the modification (analogue to that of the gravitational instantons) of Yang–Mills instantons due to the “hyperinstantons” [12].

The latter are the solutions of the last equation, that arises from the hyperino variation and that is the condition of triholomorphicity of the map $q$, rewritten with covariant rather than with ordinary derivatives.

6. R-symmetry in a quantum example

Consider a heterotic N=2, D=4 abelian model with 2 vector multiplets and 87 hypermultiplets. There exists a CY manifold with the correct Hodge numbers $(2,86)$ to represent its type II dual, the hypersurface of degree 8 in $\mathbb{CP}^4_{2,2,2,1,1}$. Its mirror, that can be described as the vanishing locus of

\[
W = X_1^8 + X_2^8 + X_3^4 + X_4^4 + X_5^4 - 8\psi X_1 X_2 X_3 X_4 X_5 - 2\phi X_1^4 X_2^4.
\]  

(11)

A non-trivial point is that together with the Wick rotation, the dilaton-axion field $S = \mathcal{A} + ie^D$ is rotated to $S = \mathcal{A} + e^D$.
has been studied in detail in [21]. The moduli space of (2,1)-forms on this manifold was shown in [4] (where the analysis was limited to the v.m. sector) to be a viable candidate to represent the quantum $\hat{ST}(1)$ manifold. The structure of the chiral ring of $W$ ensures that in the classical limit this moduli space reduces to $ST(1)$. The hypersurface (11) contains a double covering of the torus $Z^2 + X^4 + Y^4 - 2\phi X^2 Y^2 = 0$ describing the SW solution of SU(2) SYM theory; such coverings ensure the embedding of the “rigid” monodromies at this “local” level [4, 6]. This heterotic-type II dual pair has recently been thoroughly examined in [7], where also the heterotic perturbative corrections are reproduced.

The potential (11) admits a $Z_8$ symmetry acting on the moduli as $\{\psi, \phi\} \rightarrow \{\alpha \psi, -\phi\}$, where $\alpha^8 = 1$. This symmetry corresponds to the $Z_4$ discrete R-symmetry of the exact SW solution; the transmutation into $Z_8$ is due to the double covering. The $Sp(6, \mathbb{Z})$ matrix representing the $Z_8$ action on the periods [21] was derived in [21]:

$$Sp(6, \mathbb{Z}) \ni A = \begin{pmatrix} -1 & 0 & 1 & -2 & 2 & 0 \\ -2 & 1 & 0 & -2 & 4 & 4 \\ 0 & 1 & -1 & 0 & 0 & 2 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & -1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & -1 \end{pmatrix}$$

Its second power $A^2$ represents the $Z_4$ R-symmetry of the exact theory. Its eigenvalues are $\{-1, i, -i; -1, -i, i\}$. Comparing with the discussion of pag. 7, we see that our requirements for discrete R-symmetry are satisfied; the $\pm i$ couple of eigenvalues corresponds to the graviphoton and gravidilaton directions; the eigenvalue $-1$ corresponds to the unique “physical” vector. Notice moreover that, in the basis in which it is integer symplectic, the discrete R-symmetry is a non-perturbative duality (the block $B$ is non-zero); it is not at all the $Z_4$ subgroup of the “classical” $U(1)$ R-symmetry of the $ST(1)$ manifold. **Acknowledgement:** I am extremely grateful to all the co-authors of [18] and [4] on which this contribution is entirely based, and particularly P. Fré and A. Van Proeyen for help with this manuscript.

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11Recall that, in the type II formulation, the $Sp(2n + 4, \mathbb{Z})$ group of special geometry acts on the periods of the holomorphic $(3, 0)$ form.
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