We study the deformation quantization of scalar and abelian gauge classical free fields. Stratonovich-Weyl quantizer, star-products and Wigner functionals are obtained in field and oscillator variables. Abelian gauge theory is particularly intriguing since Wigner functional is factorized into a physical part and other one containing the constraints only. Some effects of non-trivial topology within deformation quantization formalism are also considered.

Key words: Deformation quantization, Field theory
1. Introduction

Recently a great deal of works have been done in deformation quantization theory. Deformation quantization was originally proposed by Bayen et al.\textsuperscript{1} as an alternative to the standard procedure of quantization avoiding the more difficult problem of constructing the relevant Hilbert space of the system. In these pioneer works it was proposed a way of quantizing a classical system by deforming the corresponding algebraic structures. Different mathematical aspects of deformation quantization were also further explored. Among them, the existence proofs of a star-product for any symplectic manifold\textsuperscript{2–4} and also for a Poisson manifold\textsuperscript{5}. The latter result has been motivated in part by string theory. A full treatment of the interplay between string theory and deformation quantization and, more generally, noncommutative geometry was given very recently by Seiberg and Witten\textsuperscript{6}.

Deformation quantization was mainly applied to quantize classical mechanics. However, it seems to be very interesting to formulate quantum field theory within deformation quantization program. Recently some works on this subject have been done\textsuperscript{7–12}.

Our paper is motivated by those works and we attempt to show systematically how all the deformation quantization formalism (Stratonovich-Weyl quantizer, Grossmann operator, Weyl correspondence, Moyal star-product, Wigner function, etc.) can be carry over to quantum field theory. To this end we use the well known particle interpretation of free fields. This approach in the case of scalar field was first considered by Dito\textsuperscript{7,8}

In this interpretation many formulas seem to be more natural and in a sense the deformation quantization in terms of particle interpretation justifies the respective formulas of the previous works\textsuperscript{9–12}, given in field variable language.

The paper is organized as follows: In Section 2 we consider deformation quantization in scalar field theory. Stratonovich-Weyl quantizer, Grossmann operator, Moyal \(\ast\)-product and Wigner functional are defined in terms of the field variables. In Section 3, the same is given for free scalar fields with the use of normal coordinates. Wigner functional of the ground state is found and also Wigner functional for higher states is considered. Finally, the normal ordering within deformation quantization program is given. Section 4 is devoted to deformation quantization of free electromagnetic field. The constraints (Gauss’s law) are analyzed in some detail. Perhaps the most interesting result is that the Wigner functional
can be factorized so that one part contains only the transverse components of the field and
the second part contains the longitudinal variables and this part vanishes when constraints
are not satisfied. Finally, in Section 5 we consider the Casimir effect in terms of deformation
quantization formalism.

This paper is the first one of a series of papers which we are going to dedicate to
deformation quantization of classical fields. The next one deals with interacting and self-
interacting fields.

2. Deformation Quantization in Scalar Field Theory

Consider a real scalar field on Minkowski space time of signature $(+, +, +, -)$. Canonical-
variables of this field will be denoted by $\phi(x,t)$ and $\varpi(x,t)$ with $(x,t) \in \mathbb{R}^3 \times \mathbb{R}$. We
deal with fields at the instant $t = 0$ and we put $\phi(x,0) \equiv \phi(x)$ and $\varpi(x,0) \equiv \varpi(x)$. It is
worth to mention that some of the functional formulas and their manipulations of Sec.2
and 3, will be formal.

2.1. The Stratonovich-Weyl Quantizer

Let $F[\phi, \varpi]$ be a functional on the phase space $\mathcal{Z}$ and let $\tilde{F}[\lambda, \mu]$ be its Fourier trans-
form

$$\tilde{F}[\lambda, \mu] = \int \mathcal{D}\phi \mathcal{D}\varpi \exp\left\{-i \int dx \left( \lambda(x)\phi(x) + \mu(x)\varpi(x) \right) \right\} F[\phi, \varpi].$$

(2.1)

The functional measures are given by $\mathcal{D}\phi = \prod_x d\phi(x), \mathcal{D}\varpi = \prod_x d\varpi(x)$. By analogy
to quantum mechanics$^{13}$ we define the Weyl quantization rule as follows

$$\hat{F} = W(F[\phi, \varpi]) := \int \mathcal{D}(\frac{\lambda}{2\pi}) \mathcal{D}(\frac{\mu}{2\pi}) \tilde{F}[\lambda, \mu] \hat{U}[\lambda, \mu],$$

(2.2)

where $\{\hat{U}[\lambda, \mu] : (\lambda, \mu) \in \mathcal{Z}^*\}$ is a family of unitary operators.
\[
\hat{U}[\lambda, \mu] := \exp \left\{ i \int dx \left( \lambda(x) \hat{\phi}(x) + \mu(x) \hat{\varpi}(x) \right) \right\}, \]  
(2.3)

with \( \hat{\phi} \) and \( \hat{\varpi} \) being the field operators given by \( \hat{\phi}(x)|\phi\rangle = \phi(x)|\phi\rangle \) and \( \hat{\varpi}(x)|\varpi\rangle = \varpi(x)|\varpi\rangle \).

Using the well known Campbell-Baker-Hausdorff formula and the standard commutation rules for \( \hat{\phi} \) and \( \hat{\varpi} \) one can write \( \hat{U}[\lambda, \mu] \) in the following form

\[
\hat{U}[\lambda, \mu] = \exp \left\{ -\frac{i}{\hbar} \int dx \lambda(x) \mu(x) \right\} \exp \left\{ i \int dx \mu(x) \hat{\varpi}(x) \right\} \exp \left\{ i \int dx \lambda(x) \hat{\phi}(x) \right\},
\]  
(2.4)

Employing Eqs. (2.4) and the relations \( \int \mathcal{D} \phi |\phi\rangle \langle \phi| = 1 \) and \( \int \mathcal{D}(\varpi/\hbar)|\varpi\rangle \langle \varpi| = 1 \) we easily get

\[
\hat{U}[\lambda, \mu] = \int \mathcal{D}(\varpi/\hbar) \exp \left\{ i \int dx \mu(x) \varpi(x) \right\} |\varpi + \frac{\hbar \lambda}{2}\rangle \langle \varpi - \frac{\hbar \lambda}{2}|. \]  
(2.5)

This operator satisfies the following properties

\[
Tr \left\{ \hat{U}[\lambda, \mu] \right\} = \int \mathcal{D} \phi \langle \phi| \hat{U}[\lambda, \mu]|\phi\rangle = \delta \left[ \frac{\hbar \lambda}{2\pi} \right] \delta[\mu]
\]  
(2.6)

and

\[
Tr \left\{ \hat{U}[\lambda, \mu] \hat{U}[\lambda', \mu'] \right\} = \delta \left[ \frac{\hbar}{2\pi} (\lambda - \lambda') \right] \delta[\mu - \mu'] .
\]  
(2.7)

(Compare with Refs. 14 and 15).

Substituting (2.1) into (2.2) one has

\[
\hat{F} = W(F[\phi, \varpi]) = \int \mathcal{D} \phi \mathcal{D}(\varpi/\hbar) F[\phi, \varpi] \hat{\Omega}[\phi, \varpi],
\]  
(2.8)
where $\hat{\Omega}$ is given by

$$
\hat{\Omega}[\phi, \varpi] = \int \mathcal{D}\left(\frac{h\lambda}{2\pi}\right) \mathcal{D}\mu \exp\left\{ -i \int dx \left( \lambda(x)\phi(x) + \mu(x)\varpi(x) \right) \right\} \hat{U}[\lambda, \mu].
$$

(2.9)

It is evident that the operator $\hat{\Omega}$ defined by (2.9) is the quantum field analog of the well known Stratonovich-Weyl quantizer playing an important role in deformation quantization of classical mechanics$^{16-21}$. Therefore we call our $\hat{\Omega}$ the Stratonovich-Weyl quantizer. One can easily check the following properties of $\hat{\Omega}$

$$(\hat{\Omega}[\phi, \varpi])^\dagger = (\hat{\Omega}[\phi, \varpi]),
$$

(2.10)

$$
Tr\{\hat{\Omega}[\phi, \varpi]\} = 1,
$$

(2.11)

$$
Tr\left\{ \hat{\Omega}[\phi, \varpi] \hat{\Omega}[\phi', \varpi'] \right\} = \delta[\phi - \phi']\delta\left[\frac{\varpi - \varpi'}{2\pi\hbar}\right].
$$

(2.12)

Multiplying (2.8) by $\hat{\Omega}[\phi, \varpi]$, taking the trace of both sides and using (2.12) we get

$$
F[\phi, \varpi] = Tr\left\{ \hat{\Omega}[\phi, \varpi] \hat{F} \right\}.
$$

(2.13)

One can also express $\hat{\Omega}[\phi, \varpi]$ in a very useful form by inserting (2.5) into (2.9). Thus one gets

$$
\hat{\Omega}[\phi, \varpi] = \int \mathcal{D}\left(\frac{\eta}{2\pi\hbar}\right) \exp\left\{ -i \int dx \eta(x)\phi(x) \right\} \varpi + \frac{\eta}{2} \varpi - \frac{\eta}{2}
$$

$$
= \int \mathcal{D}\xi \exp\left\{ -i \int dx \xi(x)\varpi(x) \right\} \phi - \frac{\xi}{2} \phi + \frac{\xi}{2}.
$$

(2.14)

Consider now the operator

$$
\hat{I} := \frac{1}{2} \int \mathcal{D}(2\phi)|\phi\rangle\langle -\phi|.
$$

(2.15)

Using (2.13) we can define the field analog of the Grossmann operator$^{22,14,15}$ as follows
\[
\hat{I}[\lambda, \mu] := \hat{U}[\lambda, \mu] \hat{I} = \frac{1}{2} \int \mathcal{D}(2\phi) \exp \left\{ i \int dx \lambda(x) \left( \phi(x) - \frac{\hbar \mu(x)}{2} \right) \right\} |\phi - \hbar \mu \rangle \langle -\phi|
\]

\[
= \frac{1}{2} \int \mathcal{D}(2\xi) \exp \left\{ i \int dx \lambda(x) \xi(x) \right\} |\xi - \frac{\hbar \mu}{2} \rangle \langle -\xi - \frac{\hbar \mu}{2}|. \quad (2.16)
\]

Comparing (2.14) with (2.16) one quickly finds that

\[
\hat{\Omega}[\phi, \varpi] = 2 \hat{I} \left[ \frac{2\phi}{\hbar}, -\frac{2\varpi}{\hbar} \right] = \hat{U} \left[ \frac{2\phi}{\hbar}, -\frac{2\varpi}{\hbar} \right] 2\hat{I}.
\]

Hence \(\hat{\Omega}[0, 0] = 2\hat{I}\) and consequently

\[
\hat{\Omega}[\phi, \varpi] = \hat{U} \left[ \frac{2\varpi}{\hbar}, -\frac{2\phi}{\hbar} \right] \hat{\Omega}[0, 0]. \quad (2.17)
\]

Simple manipulations show that (2.18) can be also written in the following form

\[
\hat{\Omega}[\phi, \varpi] = \hat{U} \left[ \frac{2\varpi}{\hbar}, -\frac{2\phi}{\hbar} \right] \hat{\Omega}[0, 0] \hat{U}^\dagger \left[ \frac{2\varpi}{\hbar}, -\frac{2\phi}{\hbar} \right]. \quad (2.19)
\]

It is interesting to note that similarly as in the quantum mechanics\(^{22}\) one can find the relation between the Stratonovich-Weyl quantizer and the quantum field image of the Dirac \(\delta\). Namely we have

\[
\hat{\delta} := W \left( \delta[\phi] \delta \left[ \frac{\varpi}{2\pi \hbar} \right] \right)
\]

\[
= \int \mathcal{D}\phi \mathcal{D}(\varpi) \delta[\phi] \delta \left[ \frac{\varpi}{2\pi \hbar} \right] \hat{\Omega}[\phi, \varpi] = \hat{\Omega}[0, 0] = 2\hat{I}. \quad (2.20)
\]

2.2. The Star Product

Now we are in a position to define the Moyal \(*\)-product in field theory. Let \(F_1 = F_1[\phi, \varpi]\) and \(F_2 = F_2[\phi, \varpi]\) be some functionals on \(Z\) that correspond to the
field operators ˆF₁ and ˆF₂ respectively, i.e.  

\[ F₁[φ, ω] = W^{-1}( ˆF₁) = Tr( ˆΩ[φ, ω] ˆF₁) \]  

\[ F₂[φ, ω] = W^{-1}( ˆF₂) = Tr( ˆΩ[φ, ω] ˆF₂). \]  

The question is to find the functional which corresponds to the product ˆF₁ ˆF₂. This functional will be denoted by (F₁ * F₂)[φ, ω] and we call it the Moyal *-product\textsuperscript{23,1}. So we have

\[
(F₁ * F₂)[φ, ω] := W^{-1}( ˆF₁ ˆF₂) = Tr( ˆΩ[φ, ω] ˆF₁ ˆF₂). \tag{2.21}
\]

Substituting (2.8) into (2.21) and performing simple calculations one gets

\[
(F₁ * F₂)[φ, ω] = \int D(φ')D(φ'')D(\frac{ω'}{πh})D(\frac{ω''}{πh})F₁[φ', ω'] \exp\left\{\frac{2i}{h} \int dx \left( (φ - φ')(ω - ω'') - (φ - φ'')(ω - ω') \right) \right\} F₂[φ'', ω'']. \tag{2.22}
\]

To proceed further we introduce new variables \( ϕ' = φ' - φ, ϕ'' = φ'' - φ, ϕ' = ω' - ω, ϕ'' = ω'' - ω \). Using the expansion of \( F₁[φ', ω'] = F₁[φ + ϕ', ω + ϕ'] \) and \( F₂[φ'', ω''] = F₂[φ + ϕ'', ω + ϕ''] \) in Taylor series and after some laborious manipulations (in principle very similar to the ones given in Ref. 21) we obtain

\[
(F₁ * F₂)[φ, ω] = F₁[φ, ω] \exp\left\{\frac{iℏ}{2} \overset{→}{\overset{→}{\mathcal{P}}} \right\} F₂[φ, ω], \tag{2.23}
\]

where

\[
\overset{→}{\overset{→}{\mathcal{P}}} := \int dx \left( \frac{↓}{δφ(x)} \frac{↓}{δω(x)} - \frac{↓}{δω(x)} \frac{↓}{δφ(x)} \right). \tag{2.24}
\]

and

\[
\exp\left\{\frac{iℏ}{2} \overset{→}{\overset{→}{\mathcal{P}}} \right\} = \sum_{l=0}^{∞} \frac{1}{l!} \left( \frac{iℏ}{2} \right)^{l} \int dx₁ \ldots dx_l ω^{i₁j₁} \ldots ω^{i_lj_l} \frac{↓}{δZ^{i₁}(x₁)} \ldots \frac{↓}{δZ^{i_l}(x_l)} \frac{↓}{δZ^{j₁}(x₁)} \ldots \frac{↓}{δZ^{j_l}(x_l)}, \tag{2.25}
\]

where \( i₁, \ldots, j₁, \ldots = 1, 2, (Z^1, Z^2) := (φ, ω), (ω^{ij}) := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) and the overarrow indicates direction of action of the corresponding operator.
2.3. The Wigner Functional

Finally we are going to define the Wigner functional. Let $\hat{\rho}$ be the density operator of a quantum state. The functional $\rho[\phi, \varpi]$ corresponding to $\hat{\rho}$ reads (see (2.13))

$$\rho[\phi, \varpi] = \text{Tr} \{ \hat{\Omega}[\phi, \varpi] \hat{\rho} \} = \int D\xi \exp \left\{ -\frac{i}{\hbar} \int dx \xi(x) \varpi(x) \right\} \langle \phi + \frac{\xi}{2} | \hat{\rho} - \frac{\xi}{2} \rangle. \quad (2.26)$$

Then in analogy to quantum mechanics\textsuperscript{24,25,26} the Wigner functional $\rho_w[\phi, \varpi]$ corresponding to the state $\hat{\rho}$ is defined by

$$\rho_w[\phi, \varpi] := \int D\xi \frac{\xi}{2\pi\hbar} \exp \left\{ -\frac{i}{\hbar} \int dx \xi(x) \varpi(x) \right\} \langle \phi + \frac{\xi}{2} | \hat{\rho} - \frac{\xi}{2} \rangle. \quad (2.27)$$

For $\hat{\rho} = |\Psi\rangle\langle\Psi|$ (2.27) gives

$$\rho_w[\phi, \varpi] = \int D\xi \frac{\xi}{2\pi\hbar} \exp \left\{ -\frac{i}{\hbar} \int dx \xi(x) \varpi(x) \right\} \Psi^* \left[ \phi - \frac{\xi}{2}\right] \Psi \left[ \phi + \frac{\xi}{2}\right] \quad (2.28)$$

according to Ref. 12.

[Important Remark: It must be noted that many, perhaps most, of our formulas are defined only formally. This resembles the case of the path integral theory. We expect that further development of the formalism presented in this paper will provide us with better mathematical frame.]

3. Free Scalar Field

In this section we deal with free real scalar field of the action\textsuperscript{27–30}

$$S[\phi] = \int dx dt \mathcal{L}(\phi(x, t), \partial_\mu \phi(x, t)) = -\frac{1}{2} \int dx dt (\partial^\mu \phi \partial_\mu \phi - m^2 \phi^2), \quad (3.1)$$

where $\mu = 1, \ldots, 4$ (we put $c = 1$). The conjugate field momentum is $\varpi(x, t) = \frac{\partial \mathcal{L}}{\partial (\partial_t^\mu \phi)} = \dot{\phi}(x, t)$, where $\dot{\phi} \equiv \frac{\partial \phi}{\partial t}$. Then the Hamiltonian reads
\[ H[\phi, \varpi] = \frac{1}{2} \int dx \left( \varpi(x, t)^2 + (\nabla \phi(x, t))^2 + m^2 \phi(x, t)^2 \right). \quad (3.2) \]

The field \( \phi(x, t) \) satisfies the Klein-Gordon equation \((\partial_\mu \partial^\mu - m^2) \phi(x, t) = 0\). For \( \phi(x, t) \) and \( \varpi(x, t) \) we have the usual Poisson brackets

\[ \{ \phi(x, t), \varpi(y, t) \} = \delta(x - y), \]

\[ \{ \phi(x, t), \phi(y, t) \} = 0 = \{ \varpi(x, t), \varpi(y, t) \}. \quad (3.3) \]

Consider the standard expansion of \( \phi \) and \( \varpi \)

\[ \phi(x, t) = \frac{1}{(2\pi)^{3/2}} \int dk \left( \frac{\hbar}{2\omega(k)} \right)^{1/2} \left( a(k, t) e^{ikx} + a^*(k, t) e^{-ikx} \right), \]

\[ \varpi(x, t) = \dot{\phi}(x, t) = \frac{1}{(2\pi)^{3/2}} \int dk \left( \frac{\omega(k) \hbar}{2} \right)^{1/2} \left( a^*(k, t) e^{-ikx} - a(k, t) e^{ikx} \right), \quad (3.4) \]

where \( \omega(k) = \sqrt{k^2 + m^2} \), \( a(k, t) = a(k) e^{\{-i\omega(k)t\}} \) and \( kx \equiv k_j x_j, j = 1, 2, 3 \).

From (3.4) we get

\[ a(k, t) = \frac{1}{(2\pi)^{3/2}(2\omega(k)\hbar)^{1/2}} \int dx \exp\{-ikx\} \left( \omega(k) \phi(x, t) + i\varpi(x, t) \right). \quad (3.5) \]

One can easily check that (3.3) and (3.5) give

\[ \{ a(k, t), a^*(k', t) \} = -\frac{i}{\hbar} \delta(k - k'), \]

\[ \{ a(k, t), a(k', t) \} = 0 = \{ a^*(k, t), a^*(k', t) \}. \quad (3.6) \]
3.1. Canonical Transformation

Then we introduce new canonical field variables\textsuperscript{31,32}

\[
Q(k, t) := \left(\frac{\hbar}{2\omega(k)}\right)^{1/2} \left(a(k, t) + a^*(k, t)\right), \quad P(k, t) := i\left(\frac{\hbar\omega(k)}{2}\right)^{1/2} \left(a^*(k, t) - a(k, t)\right).
\]

(3.7)

Hence

\[
a(k, t) = \left(\frac{\omega(k)}{2\hbar}\right)^{1/2} \left(Q(k, t) + \frac{i}{\omega(k)} P(k, t)\right).
\]

(3.8)

From (3.7) and (3.8) one has

\[
\{Q(k, t), P(k', t)\} = \delta(k - k'),
\]

\[
\{Q(k, t), Q(k', t)\} = 0 = \{P(k, t), P(k', t)\}.
\]

(3.9)

Inserting (3.8) into (3.7) we obtain

\[
Q(k, t) = \frac{1}{(2\pi)^{3/2}\omega(k)} \int dx \left(\varpi(x, t) \sin(kx) + \omega(k) \phi(x, t) \cos(kx)\right),
\]

\[
P(k, t) = \frac{1}{(2\pi)^{3/2}} \int dx \left(\varpi(x, t) \cos(kx) - \omega(k) \phi(x, t) \sin(kx)\right).
\]

(3.10)

Then the inverse transformation reads

\[
\phi(x, t) = \frac{1}{(2\pi)^{3/2}} \int dk \left(Q(k, t) \cos(kx) - \frac{P(k, t)}{\omega(k)} \sin(kx)\right),
\]

\[
\varpi(x, t) = \frac{1}{(2\pi)^{3/2}} \int dk \left(\omega(k) Q(k, t) \sin(kx) + P(k, t) \cos(kx)\right).
\]

(3.11)

We now consider the field at the instant $t = 0$. From (3.8) it follows that (3.10) defines a linear canonical transformation. Consequently the measure $\mathcal{D}\phi\mathcal{D}\varpi = \mathcal{D}Q\mathcal{D}P$ and all the formalism given before can be easily expressed in terms of new variables $Q$ and $P$. 
3.2. The Stratonovich-Weyl Quantizer

Eqs. (2.8) and (2.9) can be rewritten in the following form

\[ \hat{F} = \int DQDP\left(\frac{P}{2\pi\hbar}\right)F[\phi[Q, P], \varpi[Q, P]]\hat{\Omega}[\phi[Q, P], \varpi[Q, P]] \]  

(3.12)

and

\[ \hat{\Omega}[\phi[Q, P], \varpi[Q, P]] = \int D\left(\frac{\hbar\lambda}{2\pi}\right)D\mu exp\left\{ -i \int dk \left( \lambda(k)Q(k) + \mu(k)P(k) \right) \right\} \]

\[ \exp\left\{ i \int dk \left( \lambda(k)\hat{Q}(k) + \mu(k)\hat{P}(k) \right) \right\} \]

(3.13)

with \( \hat{Q} \) and \( \hat{P} \) the field operators and corresponding states \(|Q\rangle\) and \(|P\rangle\) satisfying the relations: \( \hat{Q}(k)|Q\rangle = Q(k)|Q\rangle, \hat{P}(k)|P\rangle = P(k)|P\rangle, \int DQ|Q\rangle\langle Q| = 1 \) and \( \int DP\langle P|\langle P| = 1 \).

Further on we denote the Stratonovich-Weyl quantizer \( \hat{\Omega}[\phi[Q, P], \varpi[Q, P]] \) simply by \( \hat{\Omega}[Q, P] \). Then

\[ \hat{\Omega}[Q, P] = \int D\left(\frac{\eta}{2\pi\hbar}\right)exp\left\{ -\frac{i}{\hbar} \int dk\eta(k)Q(k) \right\}|P + \frac{\eta}{2}\rangle\langle P - \frac{\eta}{2}| \]

\[ = \int D\xi exp\left\{ -\frac{i}{\hbar} \int dk\xi(k)P(k) \right\}|Q - \frac{\xi}{2}\rangle\langle Q + \frac{\xi}{2}|. \]  

(3.14)

It is evident how to write the Grossmann operator within the \((Q, P)\) formalism, so we do not consider this here.

3.3. Star Product and Wigner Functional

One easily shows that the Moyal \(*\)-product can be now expressed by

\[ (F_1 * F_2)[Q, P] = F_1[Q, P]exp\left\{ \frac{i\hbar}{2} \overleftrightarrow{P} \right\} F_2[Q, P], \]
\[
\hat{P} := \int dk \left( \frac{\delta}{\delta Q(k)} \frac{\delta}{\delta P(k)} - \frac{\delta}{\delta P(k)} \frac{\delta}{\delta Q(k)} \right).
\] (3.15)

Finally, the Wigner functional in the \((Q, P)\) formalism is given by

\[
\rho_w[Q, P] = \int D(\xi) \exp \left\{ -\frac{i}{\hbar} \int dk \xi(k) P(k) \right\} \langle Q + \frac{\xi}{2} | \hat{\rho} | Q - \frac{\xi}{2} \rangle,
\] (3.16)

where \(\rho_w[Q, P]\) means \(\rho_w[\phi(Q, P), \varpi(Q, P)]\). Then for \(\hat{\rho} = |\Psi\rangle \langle \Psi|\) one has

\[
\rho_w[Q, P] = \int D(\xi) \exp \left\{ -\frac{i}{\hbar} \int dk \xi(k) P(k) \right\} \Psi^*[Q - \frac{\xi}{2}] \Psi[Q + \frac{\xi}{2}],
\] (3.17)

Now we are going to find the Wigner functional for the ground state \(|\Psi_0\rangle\). From the very definition of \(|\Psi_0\rangle\)

\[
\hat{a}(k) |\Psi_0\rangle = 0.
\] (3.18)

Substituting (3.8) and using the \(Q\) representation we get the functional equation

\[
\left( Q(k) + \frac{\hbar}{\omega(k)} \frac{\delta}{\delta Q(k)} \right) \Psi_0[Q] = 0.
\] (3.19)

Hence

\[
\Psi_0[Q] \propto \exp \left\{ -\frac{1}{2\hbar} \int dk \omega(k) Q^2(k) \right\}.
\] (3.20)

Finally, inserting (3.20) into (3.17) and performing some straightforward calculations one finds the Wigner functional \(\rho_{w_0}\) of the ground state to be

\[
\rho_{w_0}[Q, P] \propto \exp \left\{ -\frac{1}{\hbar} \int \frac{dk}{\omega(k)} \left( P^2(k) + \omega^2(k) Q^2(k) \right) \right\}.
\] (3.21)

Employing (3.10) we obtain the Wigner functional of the ground state in terms of \((\phi, \varpi)\)
\[ \rho_{W_0}[\phi, \varpi] \propto \exp \left\{ -\frac{1}{\hbar} \int dx \left( \phi(x) (\sqrt{-\nabla_x^2 + m^2}) \phi(x) \right) + \left( \varpi(x) (\sqrt{-\nabla_x^2 + m^2})^{-1} \varpi(x) \right) \right\} \]

(3.22)

according to Ref. 12.

Comparing the Wigner functional (3.21) with the harmonic oscillator Wigner function given in Refs. 26, 33 and 34 we conclude that the former one represents the Wigner function of infinite number of harmonic oscillators. It’s nothing strange as the variables \( Q \) and \( P \) are the field theoretical analogs of normal coordinates and their conjugate momenta.

### 3.4. Oscillator Variables and Ordering

Now one can easily find the Wigner functionals corresponding to higher states. Let \(|\ldots k' \ldots k'' \ldots k^{(n)} \ldots\rangle = \ldots \hat{a}^\dagger(k') \ldots \hat{a}^\dagger(k'') \ldots \hat{a}^\dagger(k^{(n)}) \ldots |\Psi_0\rangle\rangle\) be the higher quantum state of the scalar field. The density operator \( \hat{\rho} \) reads

\[ \hat{\rho}_{(\ldots k' \ldots k'' \ldots k^{(n)} \ldots)} \propto \ldots \hat{a}^\dagger(k') \ldots \hat{a}^\dagger(k'') \ldots \hat{a}^\dagger(k^{(n)}) \ldots |\Psi_0\rangle\langle\Psi_0| \ldots \hat{a}(k^{(n)}) \ldots \hat{a}(k'') \ldots \hat{a}(k') \ldots \]

(3.23)

with \( \hat{a}(k) \) and \( \hat{a}^\dagger(k) \) being the annihilation and creation operators, respectively. Hence the corresponding Wigner functional is

\[ \rho_{W(\ldots k' \ldots k'' \ldots k^{(n)} \ldots)} [a, a^*] \propto \ldots * a^*(k') * \ldots * a^*(k'') * \ldots * a^*(k^{(n)}) * \ldots * \rho_{W_0}[a, a^*] * \ldots * a(k^n) * \ldots * a(k'') * \ldots a(k') * \ldots \]

(3.24)

where by (3.7) and (3.21)

\[ \rho_{W_0}[a, a^*] \propto \exp \left\{ -2 \int dk a^*(k)a(k) \right\}. \]

(3.25)

The Moyal \(*\)-product operator in terms of \( a(k) \) and \( a^*(k) \) can be written as follows:

\[ \hat{a}(k) \hat{a}^\dagger(k) = \hat{a}^\dagger(k) \hat{a}(k) + \hat{a}(k) \hat{a}^\dagger(k) - \frac{\hbar}{2} \delta(k-k') \]

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\[ * = \exp\left\{ \frac{i\hbar}{2} \vec{P} \right\} = \exp\left\{ \frac{1}{2} \int dk \left( \frac{\delta}{\delta a(k)} \frac{\delta}{\delta a^*(k)} - \frac{\delta}{\delta a^*(k)} \frac{\delta}{\delta a(k)} \right) \right\}. \] (3.26)

Consequently, any \( * \)-product of \( a^* \)'s or \( a \)'s can be rewritten as usual product of functions and the Wigner functional (3.24) is given by (compare with Ref. 33)

\[ \rho_{W(\ldots k' \ldots k'' \ldots k(n) \ldots)}[a, a^*] \]

\[ \propto \ldots a^*(k') \ldots a^*(k'') \ldots a^*(k(n)) \ldots \rho_{W_0}[a, a^*] \ldots a(k(n)) \ldots a(k'') \ldots a(k') \ldots . \] (3.27)

An interesting question arises if we are able to define the normal ordering within deformation quantization formalism for field theory. Indeed it can be easily done by a suitable generalization of the results by Agarwal and Wolf\(^\text{35}\) (see also Ref. 26). Let \( F[Q, P] \) be a functional over \( \mathcal{Z} \). Define the functional \( F_N[Q, P] \) as follows

\[ F_N[Q, P] := \hat{N} F[Q, P], \]

where

\[ \hat{N} := \exp\left\{ -\frac{\hbar}{4} \int dk \left( \frac{\omega(k)}{\delta^2} \frac{\delta^2}{P^2(k)} + \frac{1}{\omega(k)} \frac{\delta^2}{\delta Q^2(k)} \right) \right\} = \exp\left\{ -\frac{1}{2} \int dk \frac{\delta^2}{\delta a(k) \delta a^*(k)} \right\}. \] (3.28)

This formula was first obtained by Dito\(^7\).

Then the Weyl image of \( F_N[Q, P] \) gives the normal ordering of the Weyl image of \( F[Q, P] \)

\[ : W[F[Q, P]] : \overset{df}{=} W[F_N[Q, P]] \overset{df}{=} W_N(F[Q, P]) \] (3.29)

It is worthwhile to note some interesting property of normal ordering

\[ W_N^{-1} \left( : (\hat{F}_1 :)(\hat{F}_2 :) : \right) = W^{-1} \left( : \hat{F}_1 :)(\hat{F}_2 :) \right) = \left( \hat{N} W^{-1}(\hat{F}_1) \right) * \left( \hat{N} W^{-1}(\hat{F}_2) \right). \] (3.30)
Example: The Hamiltonian

From (3.2) and (3.4) we have

\[ H[a, a^*] = \int dk \hbar \omega(k) a^*(k) a(k). \]  \hspace{1cm} (3.31)

Then

\[ H_N[a, a^*] = \int dk \hbar \omega(k) a^*(k) a(k) - \frac{1}{2} \int dk \hbar \omega(k) \delta(0), \] \hspace{1cm} (3.32)

and by the quantum version of Eq. (3.6) we get

\[ \hat{H} = W \left( H_N[a, a^*] \right) = \frac{1}{2} \int dk \hbar \omega(k) \left( \hat{a}^\dagger(k) \hat{a}(k) + \hat{a}(k) \hat{a}^\dagger(k) \right) - \frac{1}{2} \int dk \hbar \omega(k) \delta(0) \]

\[ = \int dk \hbar \omega(k) \hat{a}^\dagger(k) \hat{a}(k) + \frac{1}{2} \int dk \hbar \omega(k) \delta(0) - \frac{1}{2} \int dk \hbar \omega(k) \delta(0) \]

\[ = \int dk \hbar \omega(k) \hat{a}^\dagger(k) \hat{a}(k), \] \hspace{1cm} (3.33)

where \( \delta(0) \) means here the Dirac delta in three dimensions.

The eigenvalue Schrödinger equation reads

\[ H_N \ast \rho_w = E \rho_w. \] \hspace{1cm} (3.34)

One immediately finds that the vacuum energy is zero i.e.,

\[ H_N \ast \rho_w = 0 \] \hspace{1cm} (3.35)

as it should be.
3.5. Time Evolution

Finally we consider the time evolution of Wigner functional. The von Neumann-Liouville equation reads

$$
\frac{\partial \rho_w[a, a^*; t]}{\partial t} = \{ \hat{N} H[a, a^*], \rho_w[a, a^*; t] \}_M,
$$

where $\{\cdot, \cdot\}_M$ stands for the Moyal bracket

$$
\{F_1, F_2\}_M = \frac{1}{i\hbar} (F_1 * F_2 - F_2 * F_1) = F_1 \frac{2}{\hbar} \sin \left( \frac{\hbar}{2} \hat{P} \right) F_2.
$$

For the Hamiltonian given by (3.31), the Moyal bracket in (3.36) reduces to the Poisson bracket. So we have

$$
\frac{\partial \rho_w[a, a^*; t]}{\partial t} = \{ \hat{N} H[a, a^*], \rho_w[a, a^*; t] \}.
$$

This result within the ($\phi, \varpi$)-formalism has been found previously by Curtright and Zachos$^{12}$.

4. Free Electromagnetic Field

In this section we consider deformation quantization for free electromagnetic field. This case seems to be very interesting as it is the simplest example of a field theory with constraints. (For details see Refs. 27-31).

We choose the temporal gauge where the fourth (temporal) component of the gauge potential $A_4 = 0$. The canonical components are the potential $A = (A_1, A_2, A_3)$ and its conjugate momentum $\varpi = A = -E = -(E_1, E_2, E_3)$ with $E$ being the electric field$^{27-31}$.

Fields $A$ and $E$ satisfy usual relations

$$
\{A_i(x, t), E_j(y, t)\} = -\delta_{ij}\delta(x - y),
$$
\{A_i(x, t), A_j(y, t)\} = 0 = \{E_i(x, t), E_j(y, t)\}, \ i, j = 1, 2, 3. \hspace{1cm} (4.1)

The standard expansion of the field variables at \( t = 0 \) reads

\[
A_j(x) = \frac{1}{(2\pi)^{3/2}} \int dk \left( \frac{\hbar}{2\omega(k)} \right)^{1/2} \left( a_j(k) \exp(ikx) + a_j^*(k) \exp(-ikx) \right),
\]

\[
E_j(x) = \frac{1}{(2\pi)^{3/2}} \int dk i \left( \frac{\hbar \omega(k)}{2} \right)^{1/2} \left( a_j(k) \exp(ikx) - a_j^*(k) \exp(-ikx) \right), \hspace{1cm} (4.2)
\]

where \( \omega(k) = |k| \).

From Eq. (4.2) one gets

\[
a_j(k) = \frac{1}{(2\pi)^{3/2}(2\hbar \omega(k))^{1/2}} \int dx \exp\{-ikx\} \left( \omega(k) A_j(x) - iE_j(x) \right). \hspace{1cm} (4.3)
\]

Then by Eqs. (4.1) and (4.2) we have

\[
\{a_i(k), a_j^*(k')\} = -\frac{i}{\hbar} \delta_{ij} \delta(k - k'),
\]

\[
\{a_i(k), a_j(k')\} = 0 = \{a_i^*(k), a_j^*(k')\}. \hspace{1cm} (4.4)
\]

Similarly as in the previous section one can introduce new coordinates \( Q \) and their conjugate \( P \) momenta\(^{31,32}\)

\[
Q_j(k) := \left( \frac{\hbar}{2\omega(k)} \right)^{1/2} \left( a_j^*(k) + a_j(k) \right),
\]

\[
P_j(k) := i \left( \frac{\hbar \omega(k)}{2} \right)^{1/2} \left( a_j^*(k) - a_j(k) \right),
\]

\[
\{Q_i(k), P_j(k')\} = \delta_{ij} \delta(k - k'),
\]

\[
\{Q_i(k), Q_j(k')\} = 0 = \{P_i(k), P_j(k')\}. \hspace{1cm} (4.5)
\]
Comparing (4.1) with (4.5) we can observe that the transformation between coordinates \((A, -E)\) and \((Q, P)\) is a canonical one.

Then one gets

\[
Q_j(k) = \frac{1}{(2\pi)^{3/2}\omega(k)} \int dx \left( \omega(k) A_j(x) \cos(kx) - E_j(x) \sin(kx) \right),
\]

\[
P_j(k) = -\frac{1}{(2\pi)^{3/2}} \int dx \left( \omega(k) A_j(x) \sin(kx) + E_j(x) \cos(kx) \right) \tag{4.6}
\]

and

\[
A_j(x) = \frac{1}{(2\pi)^{3/2}} \int dk \left( Q_j(k) \cos(kx) - \frac{P_j(k)}{\omega(k)} \sin(kx) \right),
\]

\[
E_j(x) = -\frac{1}{(2\pi)^{3/2}} \int dk \left( \omega(k) Q_j(k) \sin(kx) + P_j(k) \cos(kx) \right). \tag{4.7}
\]

In the standard way we can now split the field objects into their transverse \((T)\) and longitudinal \((L)\) components. To this end we write \(a_j\) in the following form

\[
a_j(k) = h_{jl}(k) a_l(k) + \frac{k_j k_l}{|k|^2} a_l(k), \tag{4.8}
\]

where \(h_{jl}(k)\) is the projector on the space perpendicular to \(k\) i.e., \(h_{jl} = \delta_{jl} - \frac{k_j k_l}{|k|^2}\). Introducing two polarization vectors \(e^1(k)\) and \(e^2(k)\) such that

\[
e_i^\alpha(k)e_i^{\alpha'}(k) = \delta_{\alpha\alpha'}, \quad k_i e_i^\alpha(k) = 0, \quad \alpha, \alpha' = 1, 2 \tag{4.9}
\]

one can write

\[
e_i^\alpha(k) a_T^{\alpha}(k) := h_{ij}(k) a_j(k) \Rightarrow a_T^{\alpha}(k) = e_i^\alpha(k) a_j(k) \tag{4.10}
\]

Defining also \(a_L(k) := \frac{k_i}{|k|} a_j(k)\) we have

\[
a_i(k) = e_i^\alpha(k) a_T^{\alpha}(k) + \frac{k_i}{|k|} a_L(k). \tag{4.11}
\]

One quickly shows that
\{ a_{T\alpha}(k), a_{T\alpha}^*(k') \} = -\frac{i}{\hbar} \delta_{\alpha\alpha'} \delta(k - k'), \quad \{ a_{L}(k), a_{L}^*(k') \} = -\frac{i}{\hbar} \delta(k - k'), \quad (4.12)

with all the remaining Poisson brackets being zero. Substituting (4.11) and (4.12) into (4.5) we obtain the expressions for the \( T \) and \( L \)-components of \( Q \) and \( P \). Then inserting (4.11) into (4.2) one gets

\[
A_j(x) = \frac{1}{(2\pi)^{3/2}} \int dk \left( \frac{\hbar}{2\omega(k)} \right)^{1/2} \left\{ e^\alpha_j(k) \left( a_{T\alpha}(k) \exp(ikx) + a_{T\alpha}^*(k) \exp(-ikx) \right) \right.
\]
\[
+ \frac{k_j}{|k|} \left( a_{L}(k) - a_{L}^*(-k) \right) \exp(ikx) \right\},
\]

\[
E_j(x) = \frac{1}{(2\pi)^{3/2}} \int dk i \left( \frac{\hbar \omega(k)}{2} \right)^{1/2} \left\{ e^\alpha_j(k) \left( a_{T\alpha}(k) \exp(ikx) - a_{T\alpha}^*(k) \exp(-ikx) \right) \right.
\]
\[
+ \frac{k_j}{|k|} \left( a_{L}(k) + a_{L}^*(-k) \right) \exp(ikx) \right\}. \quad (4.13)
\]

Then the Hamiltonian of the electromagnetic field reads

\[
H = \frac{1}{2} \int dx \left( E^2(x) + (\nabla \times A(x))^2 \right)
\]

\[
= \int dk \hbar \omega(k) a_{T\alpha}^*(k) a_{T\alpha}(k) + \frac{1}{4} \int dk \hbar \omega(k) (a_{L}(k) + a_{L}^*(-k))^*(a_{L}(k) + a_{L}^*(-k))
\]

\[
= \frac{1}{2} \int dk (P^2_T(k) + \omega^2(k)Q^2_T(k)) + \frac{1}{8} \int dk ((P_L(k) - P_L(-k))^2 + \omega^2(k)Q_L(k) + Q_L(-k))^2). \quad (4.14)
\]

Given \( H \) we can solve the Hamiltonian equations for \( Q_j(k, t) \) and \( P_j(k, t) \). Simple calculations show that for the transversal part we obtain
\[ Q_{T\alpha}(k, t) = Q_{T\alpha}(k) \cos(\omega(k)t) + \frac{P_{T\alpha}(k)}{\omega(k)} \sin(\omega(k)t), \]

\[ P_{T\alpha}(k, t) = -\omega(k)Q_{T\alpha}(k) \sin(\omega(k)t) + P_{T\alpha}(k) \cos(\omega(k)t). \quad (4.15) \]

While the solutions for the longitudinal part are

\[ Q_L(k, t) = Q_L(k) + \frac{1}{2} \left( P_L(k) - P_L(-k) \right) t \]

\[ P_L(k, t) = P_L(k) - \frac{1}{2} \omega^2(k) \left( Q_L(k) + Q_L(-k) \right) t. \quad (4.16) \]

Consequently time evolution of \( A_j \) and \( E_j \) is given by

\[
A_j(x, t) = \frac{1}{(2\pi)^{3/2}} \int dk \left\{ \left( Q_{T\alpha}(k, t) \cos(kx) - \frac{P_{T\alpha}(k, t)}{\omega(k)} \sin(kx) \right) e_j^\alpha(k) \\
+ \frac{k_j}{\omega(k)} \left( (Q_L(k) + P_L(k)t) \cos(kx) + (\omega(k)Q_L(k)t - \frac{P_L(k)}{\omega(k)}) \sin(kx) \right) \right\}
\]

\[
E_j(x, t) = -\partial_t A_j(x, t) = -\frac{1}{(2\pi)^{3/2}} \int dk \left\{ \omega(k)Q_{T\alpha}(k, t) \sin(kx) + P_{T\alpha}(k, t) \cos(kx) \right\} e_j^\alpha(k) \\
- \frac{1}{(2\pi)^{3/2}} \int dk \frac{k_j}{\omega(k)} \left( P_L(k) \cos(kx) + \omega(k)Q_L(k) \sin(kx) \right), \quad (4.17)
\]

Now the constraint (the Gauss equation)

\[
\partial_j E_j(x) = 0, \quad (4.18)
\]

is equivalent to the following constraint

\[
a_L(k) + a_L^*(-k) = 0 \iff Q_L(k) + Q_L(-k) + i \left( \frac{P_L(k) - P_L(-k)}{\omega(k)} \right) = 0, \quad (4.19)
\]
which is equivalent to the conditions

\[ Q_L(k) + Q_L(-k) = 0, \quad P_L(k) - P_L(-k) = 0. \]  \hspace{1cm} (4.20)

Note that the gauge transformation \( A_j(x, t) \to A_j(x, t) + \partial_j \Lambda(x) \) produces the additional longitudinal term in \( A_j \) of the form \( \int dk k_j i \left( \lambda(k) Q(k) + \mu(k) P(k) \right) \exp\{i k x\} \) which of course doesn’t change \( E_j(x, t), \nabla \times A(x, t), H, \) etc.

\[ \lambda(k) Q(k) := \lambda_{T\alpha}(k) Q_{T\alpha}(k) + \lambda_L(k) Q_L(k), \] etc. and all measures used in the integrals contain the longitudinal \( (DQ_L) \) and the transverse \( (DQ_T) \) components and \( \hat{Q}(k) \) and \( \hat{P}(k) \) are field operators

\[ \hat{Q}(k) |Q\rangle = Q(k) |Q\rangle, \quad \hat{P}(k) |P\rangle = P(k) |P\rangle, \]

\[ \hat{Q}(k) = (\hat{Q}_T(k), \hat{Q}_L(k)), \quad \hat{P}(k) = (\hat{P}_T(k), \hat{P}_L(k)), \]

\[ |Q\rangle = |Q_T\rangle \otimes |Q_L\rangle, \quad |P\rangle = |P_T\rangle \otimes |P_L\rangle, \]
The commutation relations for $\hat{Q}$ and $\hat{P}$ operators read

$$[\hat{Q}_{T\alpha}(k), \hat{P}_{T\alpha'}(k')] = i\hbar \delta_{\alpha \alpha'} \delta(k - k'), \quad [\hat{Q}_{L}(k), \hat{P}_{L}(k')] = i\hbar \delta(k - k')$$

and all remaining commutators are zero.

Then by (4.5) the relation between $\hat{Q}$ and $\hat{P}$ operators and the annihilation and creation operators $\hat{a}$ and $\hat{a}^\dagger$ is

$$\hat{Q}_{T,L}(k) := \left(\frac{\hbar}{2\omega(k)}\right)^{1/2} \left(\hat{a}_{T,L}(k) + \hat{a}_{T,L}^\dagger(k)\right),$$

$$\hat{P}_{T,L}(k) := i \left(\frac{\hbar \omega(k)}{2}\right)^{1/2} \left(\hat{a}_{T,L}(k) - \hat{a}_{T,L}^\dagger(k)\right).$$

We have the usual commutation relations

$$[\hat{a}_{T\alpha}(k), \hat{a}_{T\alpha'}^\dagger(k')] = \delta_{\alpha \alpha'} \delta(k - k'), \quad [\hat{a}_{L}(k), \hat{a}_{L}^\dagger(k')] = \delta(k - k')$$

with all remaining commutators being zero.

The Stratonovich-Weyl quantizer (4.22) can be rewritten as before in the form of (3.14) and it has the standard properties analogous to (2.10), (2.11) and (2.12).

4.2. The Star Product

From (4.21) we get

$$F[Q, P] = W^{-1}(\hat{F}) = Tr\left\{\hat{\Omega}[Q, P] \hat{F}\right\}.$$  (4.27)

The Moyal $*$-product in the case of electromagnetic field theory can be defined in a similar way as for the scalar field. Let $F_1[Q, P]$ and $F_2[Q, P]$ the functionals on $\mathcal{Z}_E$ and let $\hat{F}_1$ and $\hat{F}_2$ be their corresponding operators. Then the analogous calculations as in Sec. 2 lead to the result

$$\int \mathcal{D}Q|Q\rangle\langle Q| = \mathcal{1} \quad \text{and} \quad \int \mathcal{D}\left(\frac{P}{2\pi \hbar}\right)|P\rangle\langle P| = \mathcal{1}.$$  (4.23)
\( (F_1 * F_2)[Q, P] = F_1[Q, P] \exp \left( \frac{i}{\hbar} \, \mathcal{P} \right) F_2[Q, P], \)

\( \mathcal{P} := \int dk \left( \frac{\delta}{\delta Q(k)} \frac{\delta}{\delta P(k)} - \frac{\delta}{\delta P(k)} \frac{\delta}{\delta Q(k)} \right) \)

\[
= \int dk \left( \frac{\delta}{\delta Q_T(k)} \frac{\delta}{\delta P_T(k)} - \frac{\delta}{\delta P_T(k)} \frac{\delta}{\delta Q_T(k)} \right) + \int dk \left( \frac{\delta}{\delta Q_L(k)} \frac{\delta}{\delta P_L(k)} - \frac{\delta}{\delta P_L(k)} \frac{\delta}{\delta Q_L(k)} \right)
\]

\[
= - \frac{i}{\hbar} \left\{ \int dk \left( \frac{\delta}{\delta a_T(k)} \frac{\delta}{\delta a_T^*(k)} - \frac{\delta}{\delta a_T^*(k)} \frac{\delta}{\delta a_T(k)} \right) + \int dk \left( \frac{\delta}{\delta a_L(k)} \frac{\delta}{\delta a_L^*(k)} - \frac{\delta}{\delta a_L^*(k)} \frac{\delta}{\delta a_L(k)} \right) \right\}.
\]

4.3. The Wigner Functional for the Electromagnetic Field

Now we are in a position to consider the quantum version of the Gauss law (4.18). It is well known (see for example Ref. 29) that the operator equation \( \partial_j \hat{E}_j(x) = 0 \) is inconsistent with commutation relations (4.1) or (4.4). To avoid this inconsistency one imposes the weaker constraint on the “physical states”

\[
\partial_j \hat{E}_j(x) |\Psi_{phys}\rangle = 0.
\]

which is equivalent to

\[
(\hat{a}_L(k) + \hat{a}_L^*(-k)) |\Psi_{phys}\rangle = 0
\]

or in terms of \( \hat{Q}_L \) and \( \hat{P}_L \)

\[
(\hat{Q}_L(k) + \hat{Q}_L(-k)) |\Psi_{phys}\rangle = 0 \quad \text{and} \quad (\hat{P}_L(k) - \hat{P}_L(-k)) |\Psi_{phys}\rangle = 0.
\]

The Wigner functional in the case of electromagnetic field is defined similarly as for the scalar field. Let \( \hat{\rho}_{phys} \) be the density operator of a physical quantum state of the electromagnetic field. Then the Wigner functional corresponding to this state is given by
\[ \rho_w[Q, P] = \int \mathcal{D}(\frac{\xi}{2\pi \hbar}) \exp \left\{ -\frac{i}{\hbar} \int dk \xi(k) P(k) \right\} \langle Q + \frac{\xi}{2} | \hat{\rho}^{\text{phys}} | Q - \frac{\xi}{2} \rangle \propto \text{Tr} \left\{ \hat{\Omega}[Q, P] \hat{\rho}^{\text{phys}} \right\}. \] (4.32)

When \( \hat{\rho}^{\text{phys}} = |\Psi^{\text{phys}}\rangle \langle \Psi^{\text{phys}}| \) then Eq. (4.32) gives

\[ \rho_w[Q, P] = \int \mathcal{D}(\frac{\xi}{2\pi \hbar}) \exp \left\{ -\frac{i}{\hbar} \int dk \xi(k) P(k) \right\} \Psi^{\text{phys}*}[Q - \frac{\xi}{2}] \Psi^{\text{phys}}[Q + \frac{\xi}{2}]. \] (4.33)

In deformation quantization formalism Eqs. (4.30) or (4.31) read

\[ (a_L(k) + a_L^*(-k)) * \rho_w[a, a^*] = 0 \] (4.34)

or

\[ (Q_L(k) + Q_L(-k)) * \rho_w[Q, P] = 0 \quad \text{and} \quad (P_L(k) - P_L(-k)) * \rho_w[Q, P] = 0, \] (4.35)

respectively. Using (4.28) one gets

\[ \left\{ Q_L(k) + Q_L(-k) + \frac{i\hbar}{2} \left( \frac{\delta}{\delta P_L(k)} + \frac{\delta}{\delta P_L(-k)} \right) \right\} \rho_w[Q, P] = 0, \]

\[ \left\{ P_L(k) - P_L(-k) - \frac{i\hbar}{2} \left( \frac{\delta}{\delta Q_L(k)} - \frac{\delta}{\delta Q_L(-k)} \right) \right\} \rho_w[Q, P] = 0. \] (4.36)

Employing the fact that the Wigner functional is real we obtain four equations

\[ (Q_L(k) + Q_L(-k)) \rho_w[Q, P] = 0, \quad (P_L(k) - P_L(-k)) \rho_w[Q, P] = 0 \]

\[ \left( \frac{\delta}{\delta Q_L(k)} - \frac{\delta}{\delta Q_L(-k)} \right) \rho_w[Q, P] = 0, \quad \left( \frac{\delta}{\delta P_L(k)} + \frac{\delta}{\delta P_L(-k)} \right) \rho_w[Q, P] = 0. \] (4.37)

The general solution of these equations is given by

\[ \rho_w[Q, P] = \rho_w^T[Q_T, P_T] \delta[Q_L(k) + Q_L(-k)] \delta[P_L(k) - P_L(-k)]. \] (4.38)

Comparing this result with the formula for classical constraints (4.20) one observes that the Wigner functional \( \rho_w \) vanishes on the points of phase space \( \mathcal{Z}_E \) which don’t satisfy these constraints. (Compare with Ref. 10).
Example: The Ground State

The Wigner functional $\rho_{\text{w0}}$ of the ground state is defined by

$$\left(a_L(k) + \hat{a}_L^*(-k)\right) \ast \rho_{\text{w0}} = 0$$

$$a_T(k) \ast \rho_{\text{w0}} = 0.$$  (4.39)

By Eq. (4.38) we are led to the equation

$$a_T(k) \ast \rho_{\text{w0}}^T = 0.$$  (4.40)

Employing (4.28) we have

$$a_T \rho_{\text{w0}}^T + \frac{1}{2} \frac{\delta \rho_{\text{w0}}^T}{\delta a_T^*(k)} = 0.$$  (4.41)

Thus the ground state is given by

$$\rho_{\text{w0}}^T = C \exp\left(-2 \int dk \ a_T^*(k) a_T(k)\right), \quad C > 0.$$  (4.42)

Hence the Wigner functional (4.38) for the ground state reads

$$\rho_{\text{w0}}[Q, P] = C \exp\left\{-\frac{1}{\hbar} \int \frac{dk}{\omega(k)} \left(P_T^2(k) + \omega^2(k) Q_T^2(k)\right)\right\} \delta[Q_L(k) + Q_L(-k)] \delta[P_L(k) - P_L(-k)].$$  (4.43)

Similarly as in the case of scalar field we can find the Wigner functional for any higher state. To this end one must change in the formula (3.27) $a^*$ and $a$ by $a_{T\alpha}^*$ and $a_{T\alpha}$.

Let $\hat{O}$ be a gauge invariant quantum observable and let $\hat{\rho}^{\text{phys}}$ be the density operator of the physical state. The action of $\hat{O}$ on $\hat{\rho}^{\text{phys}}$ is equivalent to the action of transversal part $\hat{O}_T$ of $\hat{O}$ on $\hat{\rho}^{\text{phys}}$. Let $\mathcal{O}_T[Q_T, P_T]$ be the functional corresponding to $\hat{O}_T$, $\mathcal{O}_T[Q_T, P_T] = W^{-1}(\hat{O}_T)$. Then for the expected value $\langle \hat{O} \rangle$ one gets

$$\langle \hat{O} \rangle = \frac{\text{Tr}\{\hat{O} \hat{\rho}^{\text{phys}}\}}{\text{Tr}\{\hat{\rho}^{\text{phys}}\}}.$$
\[ \frac{\int \mathcal{D}Q_T \mathcal{D}(\frac{P_T}{2\pi \hbar}) \mathcal{O}_T[Q_T, P_T] \rho^T_W[Q_T, P_T]}{\int \mathcal{D}Q_T \mathcal{D}(\frac{P_T}{2\pi \hbar}) \rho^T_W[Q_T, P_T]} \]  

(4.44)

4.4. Ordering

Finally we can also define the normal ordering of field operators within deformation quantization formalism. It can be done, as before, with the use of the operator \( \tilde{N}_T \) acting in the phase space \( Z_E \)

\[ \tilde{N}_T := \exp \left\{ -\frac{\hbar}{4} \int dk \left( \frac{\omega(k)}{\delta P^2(k)} + \frac{1}{\omega(k)} \frac{\delta^2}{\delta Q^2_T(k)} \right) \right\} \]

\[ = \exp \left\{ -\frac{1}{2} \int dk \frac{\delta^2}{\delta a_T(k) \delta a^*_T(k)} \right\} \]  

(4.45)

Let \( \mathcal{O}[Q, P] \) be any gauge invariant functional on \( Z_E \) and \( \hat{\mathcal{O}} \) its Weyl image \( \hat{\mathcal{O}} = W(\mathcal{O}[P, Q]) \). Then, as before

\[ : \hat{\mathcal{O}} : \overset{df}{=} W \left( \tilde{N}_T \mathcal{O}[Q, P] \right) \]  

(4.46)

Having done all that one can easily formulate the deformation quantization of electromagnetic field in the Coulomb gauge: \( A_4 = 0 \) and \( \partial_j A_j = 0 \). Here \( \{ A_i(x, t), E_j(y, t) \} = -\delta_{ij} \delta^T(x-y) \), where \( \delta^T \) stands for the transversal \( \delta \)-function. Consequently, from the very beginning \( A = (A_1, A_2, A_3) \) and \( -E = -(E_1, E_2, E_3) \) are no longer independent canonical variables. However, \( Q_{T\alpha} \) and \( P_{T\alpha} \) are such variables. Moreover, the constraint \( \partial_j E_j(x) = 0 \) is automatically satisfied. Therefore results obtained for the temporal gauge can be quickly carry over to the Coulomb gauge by omitting the longitudinal parts in all formulas of temporal gauge.

5. Topological Effects (Casimir Effect) in Deformation Quantization

In this section we are going to compute the vacuum expectation value of the energy
of a real scalar field on the cylinder and on the Möbius strip (twisted scalar field) within the deformation quantization formalism.

5.1. Scalar Field on the Cylinder

Consider a cylinder \( S^1 \times \mathbb{R} \) representing a two dimensional space time. Here \( S^1 \) is the spatial part and \( \mathbb{R} \) is the temporal part. Local coordinates are as usual \((x, t)\). \( S^1 \) has circumference \( L \), then \( k \) is quantized as \( k = \frac{2 \pi}{L} n \) with \( n \in \mathbb{Z} \) and the frequency is given by \( \omega(k) = \sqrt{k^2 + m^2} = \sqrt{\frac{4 \pi^2}{L^2} n^2 + m^2} \). The Hamiltonian operator can be seen as the zero-zero component of the energy-momentum tensor

\[
\hat{T}_{00} = \frac{1}{2}\left((\partial_t \hat{\phi})^2 + (\partial_x \hat{\phi})^2 + m^2 \hat{\phi}^2\right).
\]  

(5.1)

Now we would like to compute the vacuum expectation value \( \langle \hat{T}_{00} \rangle(L) \). In order to do this computation we will use the point splitting method and we write

\[
\langle \hat{T}_{00} \rangle(L) = \lim_{t \to t'} \lim_{x \to x'} \left\{ \langle 0_L | \frac{1}{2} \left( \partial_t \partial_{t'} + \partial_x \partial_{x'} + m^2 \right) \hat{\phi}(x, t) \hat{\phi}(x', t') | 0_L \rangle - \langle 0_{\infty} | \frac{1}{2} \left( \partial_t \partial_{t'} + \partial_x \partial_{x'} + m^2 \right) \hat{\phi}(x, t) \hat{\phi}(x', t') | 0_{\infty} \rangle \right\},
\]  

(5.2)

where \( |O_L\rangle \) and \( |O_{\infty}\rangle \) are the vacuum states for the two dimensional cylindrical and Minkowski space times, respectively.

From the fact that the second term of the right-hand side of the above equation is independent of \( L \) we can rewrite it as follows

\[
\langle \hat{T}_{00} \rangle(L) := \int dL \lim_{t \to t'} \lim_{x \to x'} \left\{ \frac{1}{2} \left( \partial_t \partial_{t'} + \partial_x \partial_{x'} + m^2 \right) \frac{\partial}{\partial L} \langle 0_L | \hat{\phi}(x, t) \hat{\phi}(x', t') | 0_L \rangle \right\}
\]  

(5.3)

where the integration constant is defined by the condition \( \langle \hat{T}_{00} \rangle(\infty) = 0 \).

Thus to compute of \( \langle \hat{T}_{00} \rangle(L) \) it is necessary first to compute the quantity \( \langle 0_L | \hat{\phi}(x, t) \hat{\phi}(x', t') | 0_L \rangle \). In terms of deformation quantization we have (compare with (4.44))

\[
\langle 0_L | \hat{\phi}(x, t) \hat{\phi}(x', t') | 0_L \rangle = \frac{\int \mathcal{D}Q \mathcal{D}(\frac{P}{2\pi \hbar}) \phi(x, t) \ast \phi(x', t') \rho^L_{\hat{W}_0}(Q, P)}{\int \mathcal{D}Q \mathcal{D}(\frac{P}{2\pi \hbar}) \rho^L_{\hat{W}_0}(Q, P)},
\]  

(5.4)
where $\rho_{W_0}^L[Q, P]$ is the Wigner functional of the ground state (see (3.21))

$$\rho_{W_0}^L \propto \exp \left\{ -\frac{1}{\hbar} \sum_k \frac{1}{\omega(k)} \left( P^2(k) + \omega^2(k)Q^2(k) \right) \right\}$$

(5.5)

and (see (3.11))

$$\phi(x, t) = \frac{1}{\sqrt{L}} \sum_k \left( Q(k) \cos(kx - \omega(k)t) - \frac{1}{\omega(k)} P(k) \sin(kx - \omega(k)t) \right).$$

(5.6)

After straightforward calculations we get

$$\langle 0_L | \hat{T}_{00} | 0_L \rangle = \frac{\hbar}{2L} \sum_k \frac{\exp \left\{ i(k(x - x') - \omega(k)(t - t')) \right\}}{\omega(k)}. \quad (5.7)$$

Substituting (5.7) into (5.8) and using some considerations given by Kay\textsuperscript{37} one finds

$$\langle \hat{T}_{00}(L) \rangle = \int dL \left\{ -\frac{m\hbar}{2L^2} + 2\pi\hbar \lim_{\sigma \to 0} \partial_L \left[ \frac{1}{L^2} \left( S(a) + O(z) - \frac{1}{4} \left( \frac{1}{\sin^2(z/2)} - \frac{a^2}{2} \ln(2\sin(z/2)) \right) \right) \right] \right\}$$

(5.8)

where $z = \frac{2\pi}{L} \sigma$, $a = \frac{ML}{2\pi}$ and

$$S(a) = \sum_{n=1}^{\infty} \left( \sqrt{n^2 + a^2} - n - \frac{a^2}{2n} \right),$$

$$O(z) = \sum_{n=1}^{\infty} \left( \sqrt{n^2 + a^2} - n - \frac{a^2}{2n} \right) \left( \cos(nz) - 1 \right). \quad (5.9)$$

Hence

$$\langle \hat{T}_{00}(L) \rangle = 2\pi\hbar \left[ -\frac{1}{12L^2} + \frac{m}{4\pi L} + \frac{S(a)}{L^2} + \frac{m^2}{8\pi^2} \ln(mL) \right] + D, \quad (5.10)$$

where $D$ is the integration constant which can be computed by the condition $\lim_{L \to \infty} \langle \hat{T}_{00}(L) \rangle = 0$. Thus

$$D = -2\pi\hbar \lim_{L \to \infty} \left( \frac{S(a)}{L^2} + \frac{m^2}{8\pi^2} \ln(mL) \right) = -\frac{m^2\hbar}{2\pi} \lim_{a \to \infty} \left( \frac{1}{a^2} F(a^2) \right) - \frac{m^2\hbar}{4\pi} \ln(2\pi)$$
\[ \mathcal{F}(a^2) = \sum_{n=1}^{\infty} \left( \sqrt{n^2 + a^2} - n - \frac{a^2}{2n} \right) + \frac{a^2}{4} \ln a^2. \]  

(5.11)

Employing the relation involving the $K_0$ Bessel function\(^{39}\)

\[ \sum_{n=1}^{\infty} K_0(nx) = \frac{1}{2} \left( C + \ln \frac{x}{4\pi} \right) + \frac{\pi}{2x} + \pi \sum_{l=1}^{\infty} \left( \frac{1}{\sqrt{x^2 + 4\pi^2l^2}} - \frac{1}{2\pi l} \right), \]  

(5.12)

where $C$ is Euler’s constant and setting $x = 2\pi a$ we find the following relation

\[ \frac{\partial \mathcal{F}(a^2)}{\partial a^2} = \sum_{n=1}^{\infty} K_0(2\pi na) - \frac{1}{2} C + \frac{1}{2} \ln 2 - \frac{1}{4a} + \frac{1}{4}. \]  

(5.13)

Integrating (5.13) with respect to $da^2$ and dividing the result by $a^2$ one finally gets

\[ \lim_{a \to \infty} \frac{1}{a^2} \mathcal{F}(a^2) = \frac{1}{2} \left( C - \ln 2 - \frac{1}{2} \right). \]  

(5.14)

Substituting (5.14) into (5.11) we get the integration constant to be

\[ D = \frac{m^2 \hbar}{4\pi} \left( C - \frac{1}{2} - \ln 4\pi \right). \]  

(5.15)

The formulas (5.10) with (5.15) give the final result which for standard quantum field theory has been obtained by Kay\(^{37}\).

For the massless case ($m = 0$) we get\(^{37,38,40}\)

\[ \lim_{m \to 0} \langle \hat{T}_{00} \rangle(L) = -\frac{\pi \hbar}{6L^2}. \]  

(5.16)

5.2. Scalar Field on The Möbius Strip

Here we deal with the case of scalar field on the Möbius strip\(^{38,40}\). Now the quantization rule gives $k = \frac{2\pi(n + \frac{1}{2})}{L}$. Analogous calculations as in the previous case lead to the result

\[ \langle \hat{T}_{00} \rangle(L) = 2\pi \hbar \int dL \lim_{\sigma \to 0} \partial_L \left[ \frac{1}{L^2} \left( S'(a) + O'(z) - \frac{1}{4} \frac{\cos(z/2)}{\sin^2(z/2)} + \frac{a^2}{2} \ln[\cot(z/4)] \right) \right]. \]  

(5.17)
where \( z = \frac{2\pi}{L} \sigma \), \( a = \frac{mL}{2\pi} \) and

\[
S'(a) = \sum_{n=1}^{\infty} \left( \sqrt{(n - \frac{1}{2})^2 + a^2} - (n - \frac{1}{2}) \frac{a^2}{2(n - \frac{1}{2})} \right),
\]

\[
O'(z) = \sum_{n=1}^{\infty} \left( \sqrt{(n - \frac{1}{2})^2 + a^2} - (n - \frac{1}{2}) \frac{a^2}{2(n - \frac{1}{2})} \right) \left( \cos[(n - \frac{1}{2})z] - 1 \right). \tag{5.18}
\]

Hence

\[
\langle \hat{T}_{00} \rangle \langle L \rangle = 2\pi \hbar \left[ \frac{S'(a)}{L^2} + \frac{1}{24L^2} + \frac{m^2}{8\pi^2} \ln(mL) \right] + D'. \tag{5.19}
\]

The integration constant can be computed from the condition \( \lim_{L \to \infty} \langle \hat{T}_{00} \rangle \langle L \rangle = 0 \) and it yields

\[
D' = -2\pi \hbar \lim_{L \to \infty} \left( \frac{S'(a)}{L^2} + \frac{m^2}{8\pi^2} \ln(mL) \right) = -\frac{m^2\hbar}{2\pi} \lim_{a \to \infty} \left( \frac{1}{a^2} \mathcal{F}'(a^2) \right) - \frac{m^2\hbar}{4\pi} \ln(2\pi),
\]

where

\[
\mathcal{F}'(a^2) = \sum_{n=1}^{\infty} \left( \sqrt{(n - \frac{1}{2})^2 + a^2} - (n - \frac{1}{2}) \frac{a^2}{2(n - \frac{1}{2})} \right) + \frac{a^2}{4} \ln a^2. \tag{5.20}
\]

Now using the following relation

\[
\sum_{n=1}^{\infty} (-1)^n K_0(nx) = \frac{1}{2}(C + \ln \frac{x}{4\pi}) + \sum_{l=1}^{\infty} \left( \frac{1}{\sqrt{(2l - 1)^2 + \left( \frac{1}{\pi} \right)^2}} - \frac{1}{2l} \right) \tag{5.21}
\]

and setting \( x = 2\pi a \) we find

\[
\frac{\partial \mathcal{F}'(a^2)}{\partial a^2} = \sum_{n=1}^{\infty} (-1)^n K_0(2\pi na) - \frac{1}{2} C - \frac{1}{2} \ln 2 + \frac{1}{4}. \tag{5.22}
\]

Integrating (5.22) with respect to \( da^2 \) and dividing by \( a^2 \) one obtains

\[
\lim_{a \to \infty} \frac{1}{a^2} \mathcal{F}'(a^2) = -\frac{1}{2} \left( C + \ln 2 - \frac{1}{2} \right). \tag{5.23}
\]
Substituting (5.23) into (5.20) we have

\[ D' = \frac{m^2 \hbar}{4\pi} \left( C - \frac{1}{2} - \ln \pi \right). \]  

(5.24)

Finally, for the twisted scalar field one gets

\[ \langle \hat{T}_{00} \rangle (L) = 2\pi \hbar \left( \frac{1}{24L^2} + \frac{S'(\frac{mL}{2\pi})}{L^2} + \frac{m^2}{8\pi^2} \ln(mL) + \frac{m^2}{8\pi^2} (C - \frac{1}{2} - \ln \pi) \right). \]  

(5.25)

For \( m = 0 \) we recover the result found by Isham\(^40\) (see also Ref. 38)

\[ \langle \hat{T}_{00} \rangle (L) = \frac{\hbar \pi}{12L^2}. \]  

(5.26)

5.3. New Normal Ordering

We end this section with some comments which will be developed in further work. The results obtained suggest that it seems to be reasonable (and perhaps necessary) to deal with new normal ordering \( \hat{N}' \) when spaces of non-trivial topology are considered. In the present case \( \hat{N}' \) should satisfy the following condition

\[ \langle \hat{H}' \rangle (L) = \frac{\int DQD(\frac{P}{2\pi \hbar})(\hat{N}' \hat{H}) \rho_{\hat{W}_0}(Q, P)}{\int DQD(\frac{P}{2\pi \hbar}) \rho_{\hat{W}_0}(Q, P)} \]  

(5.27)

where \( \langle \hat{H}' \rangle (L) \) is the vacuum energy. The simplest and natural assumption is (see (3.28))

\[ \hat{N}' = \exp \left( \sum_k \left( -\frac{1}{2} + \gamma(k) \right) \frac{\partial^2}{\partial a(k) \partial a^*(k)} \right). \]  

(5.28)

Inserting (5.28) into (5.27), employing the formulas (3.31) and (3.35) one gets the condition on \( \gamma(k) \)

\[ \hbar \sum_k \gamma(k) \omega(k) = \langle \hat{H}' \rangle (L). \]  

(5.29)
Thus, for example, in the case of the massless scalar field on cylindrical space time we have (see (5.16))

$$\sum_{n \neq 0} |n| \gamma \left( \frac{2\pi n}{L} \right) = -\frac{1}{12}. \quad (5.30)$$

Of course, there is infinite number of solutions to (5.29) or (5.30).

Given $\hat{N}'$ one can define new star product $*'$ which is cohomologically equivalent to the Moyal $*$-product

$$F_1 *' F_2 = \hat{N}'^{-1} \left( \hat{N}' F_1 * \hat{N}' F_2 \right). \quad (5.31)$$

The star product $*'$ gives a new quantization of classical field.

Consider now $\lambda \phi^4$-field theory on the cylindrical space time. We have

$$H = \frac{1}{2} \int dx [(\omega)^2 + (\partial_x \phi)^2 + m^2 \phi^2 + \lambda \phi^4]. \quad (5.32)$$

To fulfill condition (5.27) in first order of perturbation theory we take now

$$\hat{N}' = exp \left\{ \sum_k \left( -\frac{1}{2} + \gamma(k) \right) \frac{\partial^2}{\partial a(k) \partial a^*(k)} + \nu(k) \frac{\partial^4}{\partial a^2(k) \partial a^*2(k)} \right\}. \quad (5.33)$$

Straightforward calculations show that the condition (5.27) leads to the following relations

$$\sum_k \frac{\gamma(k)}{\omega(k)} = 0, \quad \sum_k \frac{\gamma(k)}{\omega^2(k)} = 0$$

$$\hbar \sum_k \gamma(k) \omega(k) + \frac{3\hbar^2 \lambda}{4L} \sum_k \frac{\nu(k)}{\omega^2(k)} = \langle \hat{H}' \rangle (L), \quad (5.34)$$

where $k = \frac{2\pi n}{L}$ and $\omega(k) = \sqrt{(\frac{2\pi n}{L})^2 + m^2}$ with $n \in \mathbb{Z}$. ($\langle \hat{H}' \rangle (L)$ for $\lambda \phi^4$-field theory has been found by Kay$^{37}$).
Further developing of non-linear field theory in terms of deformation quantization formalism is a very difficult problem as in that case we must look for other cohomologically equivalent star products to avoid divergences (Dito\textsuperscript{8}). We are going to consider this question in a separate paper.

6. Final Remarks

In this paper we have generalized some aspects of deformation quantization to non-interacting field theory. We were able to show that many well known results of deformation quantization in quantum mechanics could be extended to the case of quantum field theory. This was possible because a free field can be represented as an infinite number of independent harmonic oscillators. One can apply the usual deformation quantization formalism to each oscillator to obtain deformation quantization of the whole theory. Consequently it is expected that phase space interpretation of quantum field theory can be also extended to perturbative field theory. Some work in this direction has been done by Dito\textsuperscript{8}, but many problems remain to be investigated. For example the deformation quantization of $\sigma$-model and Chern-Simons gauge theory, which have non-trivial phase spaces. Interesting is if Fedosov’s approach\textsuperscript{4} can be applied in the later cases. We intend to devote a forthcoming work to these questions.

Acknowledgements

This work was partially supported by CONACyT and CINVESTAV (México). One of us (M.P.) is indebted to the staff of Departamento de Física, CINVESTAV, México D.F., for warm hospitality.

We are grateful to Referee for valuable comments and especially for pointing out the error in Sec. 4 of the previous version of our paper.
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