An alternative approach of Kirsch-Kress Method for reconstructing the shape of a sound-soft obstacle from several incident fields

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Abstract. The inverse problem we consider is to reconstruct the shape of a plane acoustically sound-soft obstacle from the knowledge of the far-field patterns for several incident fields. An alternatively iterative procedure is proposed based on the numerical method which was developed by Kirsch and Kress. For several incident fields, a computational technique with its corresponding iterative method is presented to reduce the amount of computation by the superposition of the single-layer potential and of the far-field equation. Numerical examples show that the methods give accurate numerical approximations in relatively few iterations for several incoming waves.

1. Introduction

We consider an inverse scattering problem where the shape of a plane acoustically sound-soft obstacle is to be determined from measurements of the far-field patterns for one or several incident fields with wave number $k > 0$. This problem has many important applications, such as remote sensing, ultrasound tomography and seismic imaging. It is difficult to solve since they are nonlinear and also ill-posed, since the solution does not depend continuously on the data. Most of the Newton type methods for approximately solving such inverse scattering problems require the solution of a corresponding direct problem at each iteration step [1, 2, 3].

The aim of this paper is to provide an alternative approach to the Kirsch-Kress method [4, 5, 6] by a boundary operator. Compared with Newton type methods, the method which we describe in this paper belongs to a group of schemes which do not require the forward solver. However, a disadvantage of the Kirsch-Kress method is the increase of computing time when the number $n$ of incoming plane waves is increased. This is due to the fact that the Kirsch-Kress method requires the determination of $n + 1$ unknown functions. On the other hand, we hope to take several incoming plane waves from different directions (so that there is no shadow region) for an accurate reconstruction of the shape. By the superposition of the single-layer potential and of the far-field equation we only need to determine two unknown function for $n$ incoming plane waves, which reduces the amount of computation. Our idea which is very simple and is feasible from the reconstructions of three different shapes.

After precisely formulated the inverse scattering problem in this introduction, the outline of Kirsch-Kress method is presented in section 2. An alternative approach of Kirsch-Kress
method with two iterative procedures is proposed in section 3. For numerical implementation, a parametrization of the boundary curve of the obstacle is found in section 4. Finally, numerical experiments for some two-dimensional examples are given in section 5. Although we can extend our analysis to the Neumann or the impedance boundary condition, for the sake of exposition and simplicity, we confine our presentation to the Dirichlet boundary condition.

Let $D \subset \mathbb{R}^2$ be a simply connected domain with $C^2$-boundary $\partial D$. The incident field $u^i$ is given by $u^i(x) = e^{ikx \cdot d}$, where $k$ is the wave number with $k > 0$ and $d$ is the direction of the incident plane wave. The scattering of time-harmonic acoustic plane wave for the obstacle $D$ with sound-soft boundary condition is modeled by an exterior boundary value problem of the Helmholtz equation. That is, for a given incident plane wave $u^i(x) = e^{ikx \cdot d}$, the direct scattering problem for the sound-soft obstacle is to find the total field $u = u^i + u^s \in C^2(\mathbb{R}^2 \setminus \hat{D}) \cap C(\mathbb{R}^2 \setminus D)$ which satisfies

$$
\begin{cases}
\Delta u + k^2 u = 0, & \text{in } \mathbb{R}^2 \setminus \hat{D}; \\
u = 0, & \text{on } \partial D; \\
\lim_{r \to \infty} \sqrt{r} \left( \frac{\partial u^s}{\partial r} - ik u^s \right) = 0, & r = |x|.
\end{cases}
$$

where $\nu$ is the unit normal vector of $\partial D$ directed into the exterior of $D$. $u^s$ is the scattered field corresponding to the incident field $u^i(x)$. As we know, there exists a unique solution for the direct scattering problem (1) [7]. Furthermore, the forward scattering problem is well posed, i.e., the solution $u$ depend continuously the data $k$, $d$ and the boundary $\partial D$. It is straightforward to show that the scattered field $u^s$ has the following asymptotic behavior

$$
u(x) = \frac{e^{ik|x|}}{\sqrt{|x|}} \left( u_{\infty}(\bar{x}; d) + O \left( \frac{1}{|x|} \right) \right), \quad |x| \to \infty. \tag{2}
$$

Here, $u_{\infty}(\bar{x}; d)$ is the far-field pattern or the scattering amplitude, and $\bar{x} = \frac{x}{|x|}$ belongs to $\Omega$ that is a unit circle.

Now we give the inverse scattering problem: Determine the shape of the scatterer $D$ by the given the far-field pattern $u_{\infty}(\bar{x}; d)$ on the unit circle for one or several incident waves with different directions $d$ and the fixed wave number $k$. This type of inverse scattering problem is well studied. For an overview of this inverse scattering problem, we refer to the monograph by Colton & Kress [6]. And it is both severely ill-posed and nonlinear, thus difficult to be solved. The question of uniqueness for the inverse scattering problem for sound-soft obstacle is not completely solved. According to results of [6] we have the following uniqueness.

**Lemma 1** Let $D_1$ and $D_2$ be two planar scatterers which are contained in a disk of radius $R$ such that $kR < \pi$. If the far patterns coincide for one incident plane wave with wave number $k$, then $D_1 = D_2$.

2. Outline of the Kirsch-Kress method

We now describe a method of approximately solving the inverse scattering problem which was suggested by Kirsch and Kress [4, 5]. The method is divided into two steps. In the first step (S1) the scattered field $u^s$ is determined from the given far-field pattern. Then in the second step (S2) the boundary of the unknown scatterer is found as a zero-level curve of the total field $u^i + u^s$.

**Step(S1).** By a weak *a priori* information about $D$, we can choose an auxiliary closed curve $\Gamma$ contained in the unknown scatterer $D$, so that $k^2$ is not a Dirichlet eigenvalue of the Laplacian in the interior of $\Gamma$. We try to find the scattered field $u^s$ as a single-layer potential

$$
u(x) = \int_\Gamma \Phi(x, y) \varphi(y) ds(y), \tag{3}
$$
with some unknown density \( \varphi \in L^2(\Gamma) \). Here \( \Phi(x, y) = \frac{1}{4} H_0^{(1)}(k|x - y|) \) denotes the fundamental solution to the Helmholtz equation and \( H_0^{(1)}(k|x|) \) is the first kind Hankel function of order zero. From the asymptotic behavior of \( H_0^{(1)}(k|x|) \) [6, 7], the far-field pattern of the single-layer potential is given by

$$ u_\infty(\hat{x}) = \sigma \int_{\Gamma} \varphi(y) e^{-ik\hat{x} \cdot y} ds(y), \quad \hat{x} \in \Omega. \quad (4) $$

where \( \sigma = \frac{\varphi^{\pi/4}}{\sqrt{8\pi k}} \). Now, we define the integral operator \( F : L^2(\Gamma) \to L^2(\Omega) \) by

$$ (F\varphi)(\hat{x}) = \sigma \int_{\Gamma} \varphi(y) e^{-ik\hat{x} \cdot y} ds(y), \quad \hat{x} \in \Omega. \quad (5) $$

Hence, given the far-field pattern \( u_\infty \), we have to solve the integral equation of the first kind

$$ F\varphi = u_\infty. \quad (6) $$

The integral operator \( F \) has an analytic kernel and therefore equation (6) is a severely ill-posed equation of the first kind. Provided \( k^2 \) is not a Dirichlet eigenvalue of the Laplacian in the interior of \( \Gamma \), \( F \) is injective and has dense range in \( L^2(\Omega) \) [6]. Applying the Tikhonov regularization method to (6) [5, 6] leads to the minimization of

$$ \|F\varphi - u_\infty\|_{L^2(\Omega)}^2 + \alpha \|\varphi\|_{L^2(\Gamma)}^2, \quad (7) $$

where \( \alpha \) is the regularization parameter.

Step(S2). Through the solution \( \varphi_\alpha \) (depends on \( \alpha \)) of (7) we obtain a corresponding approximation \( u^\alpha \) for the scattered field by the single-layer potential \( (3) \) with density \( \varphi_\alpha \). Then, given the approximation \( u^\alpha \), we can now seek the boundary of the obstacle \( D \) as the location of the zeros of \( u^i + u^\alpha \) in a minimum norm sense, i.e., we can approximate \( \partial D \) by minimizing the defect

$$ \|u^i + u^\alpha\|_{L^2(\Lambda)} \quad (8) $$

over some suitable class \( U \) of admissible surface \( \Lambda \).

3. An alternative approach to Kirsch-Kress method

From section 2, we know the step(S2) of the Kirsch-Kress method is to seek the unknown boundary \( \partial D \) as the location where the total field \( u = u^i + u^s \) satisfies the boundary condition \( u = 0 \). For a closed \( C^2 \)-surface \( \Lambda \) in the domain of definition of \( u \), we define the operator

$$ G : \Lambda \rightarrow u|_{\Lambda}, \quad (9) $$

which maps \( \Lambda \) onto the trace of the total field \( u \) on \( \Lambda \). Assuming that the boundary \( \partial D \) of the obstacle \( D \) is analytic, the total wave \( u \) can be extended as a solution to the Helmholtz equation into a neighborhood of \( \partial D \). Therefore, the mapping \( G \) is well defined for surfaces \( \Lambda \) in a neighborhood of \( \partial D \). Clearly, finding the boundary \( \partial D \) of the obstacle is equivalent to solving the nonlinear operator equation

$$ G(\Lambda) = 0. \quad (10) $$

Again, we only consider that \( G \) maps \( C^2 \)-surface \( \Lambda \) into \( C(\Lambda) \). From results of [8], the Fréchet derivative of \( G \) is given by

$$ G'(\Lambda)h = \text{grad} u \cdot h, \text{ on } \Lambda. \quad (11) $$

Consequently, the Newton update reads

$$ G(\Lambda) + G'(\Lambda)h = 0. \quad (12) $$
To sum up, the first iterative procedure of this paper is the following:

Step (A1). Given the far-field pattern $u_\infty$ corresponding to the incident wave $u^i$ and the auxiliary closed curve $\Gamma$ contained in the unknown boundary $\partial D$. We approximately solve the ill-posed linear integral equation (6) for the density $\varphi$. 

Step (A2). Given the initial guess $\Lambda$ of $\partial D$ and $\Lambda$ contains auxiliary curve $\Gamma$, we evaluate the scattered field $u^s$ on $\Lambda$ by the single-layer potential (3). 

Step (A3). Solve the nonlinear equation (10) via the linear equation (12) by the Newton iteration method.

Step (A4). If the tolerance is not satisfied or the maximum iteration step is not reached, update $\Lambda$ by $\Lambda := \Lambda + h$, Goto Step (A2).

The convergence [6] of the above algorithm can be obtained by combining the minimization of Tikhonov functional (7) and the defect minimization (8) into one cost functional

$$\|F\varphi - u_\infty\|_{L^2(\Omega)}^2 + \alpha\|\varphi\|_{L^2(\Gamma)}^2 + \|u^i + u^s_\alpha\|_{L^2(\Lambda)}^2.$$  

(13)

If we have the a priori information that the diameter of the unknown obstacle is less than $\frac{2k}{\pi}$, then by Lemma 1 the obstacle can be uniquely determined by the far-field pattern that coincides one incident plane wave with wave number $k$. Hence, due to the lack of a uniqueness result for one wave number and one incoming plane wave, we can not assure that we always have convergence towards the boundary of the unknown scatterer.

We can try to achieve more accurate reconstructions by using more incident waves $u^i_1, u^i_2, \ldots, u^i_n$ with different directions $d_1, d_2, \ldots, d_n$ and corresponding far field patterns $u^i_1, u^i_2, \ldots, u^i_n$. If we solve the ill-posed linear integral equation (6) and calculate the scattered field (3) for each far-field pattern $u^i_\infty$, the computational effort will be large. Due to the linear properties of the far-field operator and of the single-layer potential, we note that

$$\sum_{j=1}^{n} F\varphi_j = \sigma \int_{\Gamma} e^{-ik\hat{x}\cdot y} \sum_{j=1}^{n} \varphi_j(y) ds(y), \ \hat{x} \in \Omega.$$  

(14)

and

$$\sum_{j=1}^{n} S\varphi_j = \int_{\Gamma} \Phi(x, y) \sum_{j=1}^{n} \varphi_j(y) ds(y), \ x \in \Lambda.$$  

(15)

Denoting that $\psi(y) := \sum_{j=1}^{n} \varphi_j(y)$, we obtain the superposition field equation

$$(F\psi)(\hat{x}) = U_\infty := \sum_{j=1}^{n} u^j_\infty, \ \hat{x} \in \Omega.$$  

(16)

and its approximately scattered field

$$U^s = (S\psi)(x) = \int_{\Gamma} \Phi(x, y) \psi(y) ds(y), \ x \in \Lambda.$$  

(17)

Therefore, the superposition total field

$$U = U^i + U^s.$$  

(18)

where $U^i = \sum_{j=1}^{n} e^{ikx\cdot d_j}$. 


We get immediately the second iterative procedure for reconstructing the unknown obstacle $D$ from several incoming wave $u^i$ when we embed (16), (17) and (18) into the first iterative procedure. The second iterative procedure is

**Step(B1).** Given the far-field pattern $u^i_\infty$ corresponding to the incident wave $u^i_j, j = 1, 2, \cdots, n$ and the auxiliary closed curve $\Gamma$ contained in the unknown boundary $\partial D$. We approximately solve the ill-posed linear integral equation (16) for the density $\psi_0$.

**Step(B2).** Given the initial guess $\Lambda$ of $\partial D$ and $\Lambda$ contains the auxiliary curve $\Gamma$, we evaluate the scattered field $U^s$ on $\Lambda$ by the single-layer potential (17).

**Step(B3).** Solve the nonlinear equation (10) via the linear equation (12) with the superposition total field $U$.

**Step(B4).** If the tolerance is not satisfied or the maximum iteration step is not reached, update $\Lambda$ by $\Lambda := \Lambda + h$, Goto Step(B2).

### 4. Numerical implementation

For the practical computation, we represent the curves $\Gamma$ and $\Lambda$ by a parametrization

$$\Lambda = \{x(t) \mid t \in [0, 2\pi]\},$$

where $x(t) : [0, 2\pi] \to \mathbb{R}^2$ is a two times continuously differential $2\pi$-periodic vector. Looking for the location where the boundary condition $u = 0$ is satisfied, we solve the linear equation (12) of (10) with respect to the normal direction at $\Lambda$. To this end, we try to update $\Lambda$ in the following form

$$\Lambda := \Lambda + h = \{x(t) = x(t) + h(t)\nu(t) \mid t \in [0, 2\pi]\},$$

where $h(t) : [0, 2\pi] \to \mathbb{R}$ is a $2\pi$-periodic vector and $\nu(t)$ is the outward normal vector of the former curve $\Lambda$. The normal vector can be evaluated through the parametrization (19) via

$$\nu(t) = \frac{(x'_1(t), -x'_2(t))}{|x'(t)|}, \quad t \in [0, 2\pi],$$

where $x_1(t)$ and $x_2(t)$ are two components of the vector $x(t)$, $|x'(t)|$ is the Euclid norm of tangent vector $x'(t)$. The update $h(t)$ is taken as by trigonometric polynomials of degree less than or equal to $m$, i.e.,

$$h(t) = a_0 + \sum_{j=1}^m (a_j \cos jt + b_j \sin jt).$$

For the above representations of curves $\Gamma$ and $\Lambda$, it is easy to compute the integrals in terms of the parameter $t$. Hence,

$$(F \varphi)(\hat{x}) = \sigma \int_0^{2\pi} \varphi(y(t)) e^{-ikr\gamma(t)|y'(t)|}dt, \quad \hat{x} \in \Omega,$$

and

$$u^s(x(t)) = \int_0^{2\pi} \Phi(x(t), y(\tau)) \varphi(y(\tau)) |y'(\tau)| d\tau, \quad t \in [0, 2\pi],$$

where $x(t) \in \Lambda$ and $y(t) \in \Gamma$. Since the auxiliary closed curve $\Gamma$ is contained in the approximate curve $\Lambda$, there is no singular in the latter integrals. Also, we do not ask the jump relation of the single-layer potential to compute the normal derivative of the scattered field $u^s$.

Next, we show how to get the update $h(t)$ via the linear equation (12). Here, we solve (12) in a least squares sense, that is, the coefficients $a_0, a_j, b_j, j = 1, 2, \cdots, m$ are chosen such that
for a set of collocation points $t_1, t_2, \ldots, t_M$ in $[0, 2\pi)$ the least squares sum

$$\sum_{l=1}^{M} \left| u(x(t_l)) + \frac{\partial u(x(t_l))}{\partial \nu} \left( a_0 + \sum_{j=1}^{m} (a_j \cos j t_l + b_j \sin j t_l) \right) \right|^2.$$  \hspace{1cm} (25)

is minimized. According to results of [8, 9, 10], we can also take a penalized least squares sum for the sake of stability. But we find that we do not need the penalized term in the following example to obtain more accurate reconstructions. Therefore we always take the common form (25) of least squares sum in this paper.

5. Numerical examples

In this section, we present the reconstructions of three different obstacles using the methods of Section 3 and the numerical implementation technique of Section 4. The incident waves are taken with directions $d$ uniformly distributed over the unit circle (if only one wave is used the incoming wave is $e^{ikx \cdot d}$ with $d = (1, 0)$). We also denote by $m$ the degree of the trigonometric polynomial used for the approximation of the boundary, i.e., the parametrization of the boundary

$$x(t) = (x_1(t), x_2(t)), \ t \in [0, 2\pi]$$

where $x_1(t) = b_0^l + \sum_{j=1}^{m} (b_j^l \cos j t + b_j^l \sin j t), l = 1, 2$.

In the following examples, we used $M = 64$ collocation points in (25) and the same number of equidistant quadrature points for the solution of the Tikhonov regularization for (6) and (16). To avoid an inverse crime, we use the far-field patterns obtained by approximately solving the direct scattering problem by the double-layer boundary integral equation method [6, 7]. The double-layer potential approach leads to an integral equation of the second kind which we solved numerically by Nyström’s method [6] with 128 collocation points.

To illustrate the stability for solving the inverse problem, we also give the results for noisy data. For noisy data, random errors were added point-wise to the values of the far-field pattern at the 64 points with the percentage given in terms of the $l^2$-norm. In addition, the wave number $k$ we use is 1.0, and the black arrows in the following figures represent the directions of incoming waves except for the case of twelve incoming waves.

Example 1. we consider the reconstruction of a peanut-shaped boundary curve given by the parametrization

$$\partial D = \{ \sqrt{\cos^2(t) + 0.25 \sin^2(t)} (\cos(t), \sin(t)), \ t \in [0, 2\pi] \}.$$  

For parameters we choose $\alpha = 0.5 \times 10^{-10}$ for the exact far-field and $\alpha = 0.5 \times 10^{-5}$ for noise data with the 3% noise, and $m = 8$. The iterations were started with the auxiliary curve and the initial guess of radius 0.3 and 0.7 centered at origin, respectively. The numerical results after six iterations and with different directions are illustrated in figures 1 and 2.

Example 2. we consider the reconstruction of an acorn-shaped boundary curve given by the parametrization

$$\partial D = \{ (2 + 0.5 \cos(3t))(\cos(t), \sin(t)), \ t \in [0, 2\pi] \}.$$  

For parameters we choose $\alpha = 0.5 \times 10^{-10}$ for the exact far-field and $\alpha = 0.5 \times 10^{-5}$ for noise data with the 3% noise, $m = 6$. The iterations were started with the auxiliary curve and the initial guess of radius 1.0 and 1.8 centered at origin, respectively. The numerical results after eight iterations and with different directions are illustrated in figures 3 and 4.

Example 3. we consider a domain that is contained in the approximation space and given by

$$\partial D = \{ \frac{2 + 1.8 \cos(t) + 0.2 \sin(2t)}{1 + 0.75 \cos(t)} (\cos(t), \sin(t)), \ t \in [0, 2\pi] \}.$$  

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Figure 1: Reconstructions without noise after six iterations.

(a) One incoming wave
(b) Two horizontal incoming wave
(c) Four incoming waves
(d) Twelve incoming waves

Figure 2: Reconstructions with 3% noise after six iterations.

(a) One incoming wave
(b) Two horizontal incoming wave
(c) Four incoming waves
(d) Twelve incoming waves
(a) One incoming wave

(b) Two horizontal incoming wave

(c) Four incoming waves

(d) Twelve incoming waves

Figure 3: Reconstructions without noise after six iterations.

(a) One incoming wave

(b) Two horizontal incoming wave

(c) Four incoming waves

(d) Twelve incoming waves

Figure 4: Reconstructions with 3% noise after six iterations.
For parameters we choose $\alpha = 0.5 \times 10^{-12}$ for the exact far-field and $\alpha = 0.5 \times 10^{-6}$ for noise data with the 3% noise, $m = 16$. The iterations were started with the auxiliary curve and the initial guess of radius 0.5 and 1.6 centered at origin, respectively. The numerical results after ten iterations and with different directions are illustrated in figures 5 and 6.

![Reconstruction curves](image)

Figure 5: Reconstructions without noise after six iterations.

Our numerical results show that the alternative method of Kirsch-Kress method does not need a forward solver and yields reasonable reconstructions of the obstacle, provided the initial guess is close enough to the exact boundary. Moreover, the accuracy of the reconstructions is satisfactory with 4 or more than 4 incoming waves. For a good initial guess, we can combine this method with the probe method [11, 12] or linear sampling method [13, 14]. On the other hand, we note that the regularization parameters were chosen by trial and error; and the number of Newton steps was also chosen by trial and error. Therefore, these are our work in progress. For example, we may choose the regularization parameters by the model function method [15, 16]. Moreover, we can extend this method to handle other boundary conditions such as the sound-hard case (Neumann boundary condition) and the impedance case (Robin boundary condition).

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(a) One incoming wave  
(b) Two horizontal incoming waves  
(c) Four incoming waves  
(d) Twelve incoming waves

Figure 6: Reconstructions with 3% noise after six iterations.

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