ON THE HODGE-BGW CORRESPONDENCE

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ABSTRACT. We establish an explicit relationship between the partition function of certain special cubic Hodge integrals and the generalized Brézin–Gross–Witten (BGW) partition function, which we refer to as the Hodge-BGW correspondence. As an application, we obtain an ELSV-like formula for generalized BGW correlators.

1. Introduction and statements of the results

In this paper, we establish a relationship between two celebrated partition functions: the partition function of certain special cubic Hodge integrals and the generalized BGW partition function, that we will call the Hodge-BGW correspondence.

Let \( \mathcal{M}_{g,n} \) denote the Deligne–Mumford moduli space of stable algebraic curves of genus \( g \) with \( n \) distinct marked points \([9]\). Let \( L_p \) be the \( p \)th tautological line bundle on \( \mathcal{M}_{g,n} \), and \( E_{g,n} \) the Hodge bundle. Denote by \( \psi_p := c_1(L_p) \), \( p = 1, \ldots, n \), the first Chern class of \( L_p \), and by \( \lambda_j := c_j(E_{g,n}) \), \( j = 0, \ldots, g \), the \( j \)th Chern class of \( E_{g,n} \). We also denote by \( \kappa_d := f_*(\psi_{n+1}^{d+1}) \) the \( \kappa \)-class \([4, 31, 46]\), where \( f : \mathcal{M}_{g,n+1} \to \mathcal{M}_{g,n} \) is the forgetful map. Recall that the intersection numbers of mixed \( \psi \)-, \( \lambda \)-, \( \kappa \)-classes are integrals on \( \mathcal{M}_{g,n} \) of the form

\[
\int_{\mathcal{M}_{g,n}} \psi_{i_1}^{i_1} \cdots \psi_{i_n}^{i_n} \lambda_1^{j_1} \cdots \lambda_2^{j_2} \kappa_{d_1} \cdots \kappa_{d_\ell},
\]

where \( i_1, \ldots, i_n, j_1, \ldots, j_g \geq 0, d_1, \ldots, d_\ell \geq 1, \ell \geq 0 \). These integrals vanish unless

\[
(i_1 + \cdots + i_n) + (j_1 + 2j_2 + \cdots + gj_g) + (d_1 + \cdots + d_\ell) = 3g - 3 + n.
\]

When \( \ell = 0 \), they are also called Hodge integrals.

We will be particularly interested in the cubic Hodge integrals of the following form

\[
\int_{\mathcal{M}_{g,n}} \Lambda(-1)^2 \Lambda \left( \frac{1}{2} \right) \psi_{i_1}^{i_1} \cdots \psi_{i_n}^{i_n},
\]

where \( \Lambda(z) := \sum_{j=0}^g \lambda_j z^j \) denotes the Chern polynomial of \( E_{g,n} \). These Hodge integrals belong to the class of cubic Hodge integrals satisfying the Calabi–Yau condition \([26, 36, 44, 48]\). They also appeared in the Gopakumar–Mariño–Vafa conjecture \([26, 44]\) regarding the Chern–Simons/string duality and in the theory of topological vertex \([36, 37, 38]\). Recently, they play important roles in the Hodge-GUE correspondence \([15, 16, 17]\), which implies the ELSV-like (cf. \([22]\)) formula for even GUE and modified GUE correlators \([7, 19, 24]\); see \([2, 3, 54]\) for interesting connections to the KP hierarchy and 2D Toda lattice. Denote by

\[
\mathcal{H}(t; \epsilon) := \sum_{g,n \geq 0} \epsilon^{2g-2} \sum_{i_1, \ldots, i_n \geq 0} \frac{t_{i_1} \cdots t_{i_n}}{n!} \int_{\mathcal{M}_{g,n}} \Lambda(-1)^2 \Lambda \left( \frac{1}{2} \right) \psi_{i_1}^{i_1} \cdots \psi_{i_n}^{i_n}
\]
the generating series of the cubic Hodge integrals \([3]\), called the Hodge free energy associated to \(\Lambda(-1)^2 \Lambda(1/2)\). Denote by

\[
Z_H(t; \epsilon) := e^{\mathcal{H}(t; \epsilon)}
\]

the partition function, and by \(\mathcal{H}_g(t)\) the genus \(g\) part of the Hodge free energy, i.e.,

\[
\mathcal{H}(t; \epsilon) = \sum_{g \geq 0} e^{2g-2} \mathcal{H}_g(t).
\]

We continue and recall terminologies in the BGW side. The BGW partition function was introduced in \([8, 29]\). In \([45]\), a one parameter deformation of this partition function, called the generalized BGW partition function with the parameter \(N\), was given as a particular generalized Kontsevich model \([33]\) (cf. also \([1, 6]\) for the definition). Let us denote this partition function by \(Z_{gBGW}(N, \mathbf{T}; \hbar)\), where \(\mathbf{T} = (T_1, T_3, T_5, \ldots)\) is an infinite vector of indeterminates and \(N\) is an indeterminate. It is often normalized by the following initial condition:

\[
Z_{gBGW}(N, 0; \hbar) = 1.
\]

The logarithm \(\log Z_{gBGW}(N, \mathbf{T}; \hbar) =: \mathcal{F}_{gBGW}(N, \mathbf{T}; \hbar)\), belonging to \(\mathbb{C}[\hbar]][[N, \mathbf{T}]\], is called the generalized BGW free energy with the parameter \(N\). It is known e.g. in \([1]\) that the power series \(\mathcal{F}_{gBGW}(N, \mathbf{T}; \hbar)\) has the form

\[
\mathcal{F}_{gBGW}(N, \mathbf{T}; \hbar) = \sum_{l \geq 1} \sum_{a_1, \ldots, a_l \geq 0} \hbar^{2|l|} \sum_{g=0}^{l} (-2)^{l-g+1} c_g(a_1, \ldots, a_l) N^{2|l|-2g+2} \frac{\prod_{p=1}^{l} T_{2a_p+1}}{l!},
\]

where \(c_g(a_1, \ldots, a_l) (g \geq 0, l \geq 1, a_1, \ldots, a_l \geq 0)\) are numbers, and \(|a| := \sum_{i=1}^{l} a_i\). We call \(c_g(a_1, \ldots, a_l)\) the generalized BGW correlators of genus \(g\). When \(|a| = g - 1\), the numbers \(c_g(a_1, \ldots, a_l)\) are the celebrated BGW correlators of genus \(g\), as they are Taylor coefficients of the BGW free energy, that is, of \(\mathcal{F}_{gBGW}(0, \mathbf{T}; \hbar)\). The numbers \(c_g(a_1, \ldots, a_l)\) vanish when \(|a| < g - 1\).

According to Alexandrov \([1]\) (cf. also \([0, 13, 28, 45, 51]\)), the power series \(Z_{gBGW}(N, \mathbf{T}; \hbar)\) satisfies the following Virasoro constraints:

\[
L_m^{gBGW}(Z_{gBGW}(N, \mathbf{T}; \hbar)) = 0, \quad m \geq 0.
\]

Here \(L_m^{gBGW}\), \(m \geq 0\), are operators given by

\[
L_m^{gBGW} = \frac{1}{2^{m+1}} \sum_{a \geq 0} (2a + 2m + 1)!! T_{2a+1} \frac{\partial}{\partial T_{2a+2m+1}}
\]

\[
+ \frac{\hbar^2}{2^{m+2}} \sum_{a+b=m-1} (2a+1)!! (2b+1)!! \frac{\partial^2}{\partial T_{2a+1} \partial T_{2b+1}} + \left( \frac{1}{16} - \frac{N^2}{4} \right) \delta_{m,0}
\]

with

\[
T_{2a+1} = T_{2a+1} - \delta_{a,0}, \quad a \geq 0.
\]

Here we note that the normalizations of the independent variables \(T_{2a+1}\) differ by simple factors from those of \([1]\). The operators \(L_m^{gBGW}\) satisfy the Virasoro commutation relations:

\[
[L_m^{gBGW}, L_n^{gBGW}] = (m-n) L_{m+n}^{gBGW}.
\]
As we shall see from the uniqueness of solution given in Section 2.2, equations (7), (9) can be used as the defining equations for the generalized BGW partition function with the parameter $N$, i.e. for $Z_{gBGW}(N, T; \hbar)$.

Following Alexandrov [1], introduce

$$x := N \hbar \sqrt{-2}. \quad (13)$$

Let us define

$$Z(x, T; \hbar) := e^{B(x, \hbar)} Z_{gBGW}(x \hbar \sqrt{-2}, T; \hbar), \quad (14)$$

where

$$B(x, \hbar) = \frac{1}{\hbar^2} \left( \frac{x^2}{4} \log \left( -\frac{x}{2} \right) - \frac{3}{8} x^2 \right) + \frac{1}{12} \log \left( -\frac{x}{2} \right) + \sum_{g \geq 2} \frac{\hbar^{2g-2} (\hbar^2)^{2g-1} B_{2g}}{2g (2g-2)} \quad (15)$$

with $B_k$ denoting the $k$th Bernoulli number. We call $Z(x, T; \hbar)$ the generalized BGW partition function, and call its logarithm $\log Z(x, T; \hbar)$ the generalized BGW free energy, denoted by $F(x, T; \hbar)$. Explicitly, by (8) we have

$$F(x, T; \hbar) = \frac{1}{\hbar^2} \left( \frac{x^2}{4} \log \left( -\frac{x}{2} \right) - \frac{3}{8} x^2 \right) + \frac{1}{12} \log \left( -\frac{x}{2} \right) + \sum_{g \geq 2} \frac{\hbar^{2g-2} (\hbar^2)^{2g-1} B_{2g}}{2g (2g-2)}$$

$$+ \sum_{g \geq 0} \hbar^{2g-2} \sum_{i \geq 1} \sum_{a_1 \ldots a_i \geq 0} c_g(a_1, \ldots, a_i) x^{2|a| - 2g + 2} \frac{\prod_{p=1}^i T_{2a_p + 1}}{l!}. \quad (16)$$

Inspired by the Hodge-GUE correspondence [16, 17], we establish in the following theorem an explicit relationship between the cubic Hodge partition function associated to $\Lambda(-1)^2 \Lambda(1/2)$ and the generalized BGW partition function.

**Theorem 1 (Main Theorem).** The following identity

$$e^{\frac{A(x, T)}{\hbar^2}} Z_H(t(x, T); \hbar \sqrt{-4}) = Z(x, T; \hbar) \quad (17)$$

holds true in $\mathbb{C}((\hbar^2))[[x + 2]][[T]]$. Here,

$$t_i(x, T) = \delta_{i,0} x + \delta_{i,1} - \left( -\frac{1}{2} \right)^{i-1} - 2 \sum_{a \geq 0} \left( -\frac{2a + 1}{2} \right)^i \frac{T_{2a+1}}{a!}, \quad i \geq 0, \quad (18)$$

and $A(x, T)$ is a quadratic series given by

$$A(x, T) = \frac{1}{2} \sum_{a,b \geq 0} a! b! (a+b+1) \frac{T_{2a+1} T_{2b+1}}{a! b!} - \sum_{b \geq 0} \frac{x T_{2b+1}}{b! (2b+1)}. \quad (19)$$

The proof of this theorem is in Section 3. We call (17) the Hodge-BGW correspondence.

Let us now present by the following proposition an application of the Hodge-BGW correspondence, expressing an arbitrary generalized BGW correlators of genus $g$ in terms of intersection numbers on the moduli space of curves.
Proposition 1. For $g \geq 0$, $l \geq 1$, and for non-negative integers $a_1, \cdots, a_l$ such that $|a| \geq g - 1$, the generalized BGW correlators of genus $g$ are related to intersection numbers by

$$c_g(a_1, \cdots, a_l) = \frac{(-1)^{g-l} 2^{2g-2+l}}{(2|a| - 2g + 2)! \prod p=1^l a_p!} \int_{\overline{M}_{g,l}+2|a|-2g+2} \Lambda(-1)^2 \Lambda \left( \frac{1}{2} \right) e^{\sum_{d \geq 1} (-1)^{d-1} \kappa_{d} \frac{1}{2^d d}} \prod_{p=1}^l \frac{1 + \frac{2a_p + 1}{2} \psi_p}{1 + \frac{1}{2} \psi_p}.$$  

Moreover, for $g \geq 2$, we have

$$\int_{\overline{M}_{g,0}} \Lambda(-1)^2 \Lambda \left( \frac{1}{2} \right) e^{\sum_{d \geq 1} (-1)^{d-1} \kappa_{d} \frac{1}{2^d d}} = -\frac{B_{2g}}{2g(2g-2)8g-1}. \tag{21}$$

The proof is given in Section 4.

The following corollary follows straightforwardly from Proposition 1.

Corollary 1. For $g \geq 0$, $l \geq 1$, and for non-negative integers $a_1, \cdots, a_l$ such that $|a| = g - 1$, the BGW correlators of genus $g$ admit the ELSV-like formula:

$$c_g(a_1, \cdots, a_l) = \frac{(-1)^{g-l} 2^{2g-2+l}}{\prod p=1^l a_p!} \int_{\overline{M}_{g,l}} \Lambda(-1)^2 \Lambda \left( \frac{1}{2} \right) e^{\sum_{d \geq 1} (-1)^{d-1} \kappa_{d} \frac{1}{2^d d}} \prod_{p=1}^l \frac{1 + \frac{2a_p + 1}{2} \psi_p}{1 + \frac{1}{2} \psi_p}. \tag{22}$$

The rest of the paper is organized as follows. In Section 2 we review in more details about the Hodge partition function and the generalized BGW partition function. In Section 3 we prove Theorem 1. In Section 4 we prove Proposition 1. Three more applications of the Hodge-BGW correspondence are given in Appendix A.

2. Review on Hodge integrals and generalized BGW correlators

2.1. ψ-class intersection numbers and cubic Hodge integrals. When $j_1, \ldots, j_g$ and $\ell$ in (1) all equal 0, the integrals in (1) are also called the ψ-class intersection numbers. It was conjectured by Witten [50], and proved by Kontsevich [35] that the partition function of the ψ-class intersection numbers is a particular tau-function for the Korteweg–de Vries (KdV) hierarchy, now known as the Witten–Kontsevich theorem. To be precise, let $Z_{WK}(t; \epsilon)$ denote this partition function:

$$Z_{WK}(t; \epsilon) := \exp \left( \sum_{g \geq 0} e^{2g-2} \sum_{n \geq 0} \frac{1}{n!} \sum_{i_1, \ldots, i_n \geq 0} \int_{\overline{M}_{g,n}} \psi^{i_1} \cdots \psi^{i_n} t_{i_1} \cdots t_{i_n} \right). \tag{23}$$

Then $u = u_{WK}(t; \epsilon) := e^{2} \partial^{2}_{\epsilon} \left( \log Z_{WK}(t; \epsilon) \right)$ satisfies the KdV hierarchy:

$$\frac{\partial u}{\partial t_i} = \frac{1}{(2i + 1)!!} \left( L^{2i+1}_+ \right)_{+}, \quad i \geq 0, \tag{24}$$
where \( L := \epsilon^2 \partial_{t_0}^2 + 2u \) is the Lax operator of the KdV hierarchy (cf. e.g. [10]). Moreover, \( Z_{WK}(t; \epsilon) \) satisfies the dilaton and string equations

\[
\sum_{i=0}^{\infty} t_i \frac{\partial Z_{WK}(t; \epsilon)}{\partial t_i} + \epsilon \frac{\partial Z_{WK}(t; \epsilon)}{\partial \epsilon} + \frac{1}{24} Z_{WK}(t; \epsilon) = \frac{\partial Z_{WK}(t; \epsilon)}{\partial t_1},
\]

(25)

\[
\sum_{i=0}^{\infty} t_{i+1} \frac{\partial Z_{WK}(t; \epsilon)}{\partial t_i} + \frac{t_0^2}{2\epsilon^2} Z_{WK} = \frac{\partial Z_{WK}(t; \epsilon)}{\partial t_0}.
\]

(26)

(For more about a KdV tau-function cf. [10, 20, 21].)

According to Dijkgraaf, Verlinde, Verlinde [11], the Witten–Kontsevich theorem can be equivalently formulated as follows: the power series \( Z_{WK}(t; \epsilon) \) satisfies the infinite family of linear equations

\[
L_{WK}^k(\epsilon^{-1} \tilde{t}, \epsilon \partial / \partial t) \left( Z_{WK}(t; \epsilon) \right) = 0, \quad k \geq -1,
\]

(27)

where \( \tilde{t}_i := t_i - \delta_{i,1} \), and \( L_{WK}^k = L_{WK}^k(\epsilon^{-1} \tilde{t}, \epsilon \partial / \partial t) \), \( k \geq -1 \), are given by

\[
L_{-1}^{WK} = \sum_{i \geq 1} \tilde{t}_i \frac{\partial}{\partial t_{i-1}} + \frac{t_0^2}{2\epsilon^2},
\]

(28)

\[
L_0^{WK} = \sum_{i \geq 0} \frac{2i + 1}{2} \tilde{t}_i \frac{\partial}{\partial t_i} + \frac{1}{16},
\]

(29)

\[
L_k^{WK} = \frac{\epsilon^2}{2} \sum_{i+j=k-1} \frac{(2i+1)!! (2j+1)!!}{2^{k+1}} \frac{\partial^2}{\partial t_i \partial t_j} + \sum_{i \geq 0} \frac{(2i+2k+1)!!}{2^{k+1}(2i-1)!!} \tilde{t}_i \frac{\partial}{\partial t_{i+k}}, \quad k \geq 1.
\]

(30)

These operators \( L_k^{WK} \) satisfy the following Virasoro commutation relations:

\[
\left[ L_k^{WK}, L_l^{WK} \right] = (k - l) L_{k+l}^{WK}, \quad \forall k, l \geq -1.
\]

(31)

Equations (27) are called the Virasoro constraints for \( Z_{WK}(t; \epsilon) \). Clearly, the \( L_{-1}^{WK} \)-constraint coincides with the string equation (26).

The Hodge partition function defined by (4), (5) can be obtained from the Witten–Kontsevich partition function via the Faber–Pandharipande formula [23]:

\[
Z_H(t; \epsilon) = \exp \left( \sum_{j=1}^{\infty} \frac{(2-2j-1) B_{2j}}{j (2j-1)} D_j(\epsilon^{-1} \tilde{t}, \epsilon \partial / \partial t) \right) \left( Z_{WK}(t; \epsilon) \right),
\]

(32)

where

\[
D_j(\epsilon^{-1} \tilde{t}, \epsilon \partial / \partial t) := - \sum_{i \geq 0} \tilde{t}_i \frac{\partial}{\partial t_{i+2j-1}} + \frac{\epsilon^2}{2} \sum_{a=0}^{2j-2} (-1)^a \frac{\partial^2}{\partial t_a \partial t_{2j-2-a}}, \quad j \geq 1.
\]

(33)

It satisfies the following dilaton equation:

\[
\tilde{L}_\text{dilaton}^H(Z_H(t; \epsilon)) = 0,
\]

(34)
where $\tilde L^H_{\text{dilaton}}$ is the linear operator defined by

$$\tilde L^H_{\text{dilaton}} := \sum_{i \geq 0} \tilde t_i \frac{\partial}{\partial t_i} + \epsilon \frac{\partial}{\partial \epsilon} + \frac{1}{24}.$$  

(35)

Following [55] define the operators $L^H_k(\epsilon^{-1} \tilde t, \epsilon \partial / \partial \epsilon)$ by

$$L^H_k(\epsilon^{-1} \tilde t, \epsilon \partial / \partial \epsilon) = e^G \circ L^W_k(\epsilon^{-1} \tilde t, \epsilon \partial / \partial \epsilon) \circ e^{-G}, \quad k \geq -1,$$

where

$$G := \sum_{j=1}^{\infty} \frac{(2-2j - 1)}{j(2j - 1)} B_{2j} D_j(\epsilon^{-1} \tilde t, \epsilon \partial / \partial \epsilon).$$

(37)

Then we have

$$\left[L^H_k, L^H_l\right] = (k - l) L^H_{k+l}, \quad \forall k, l \geq -1,$$

and moreover,

$$L^H_k(\epsilon^{-1} \tilde t, \epsilon \partial / \partial \epsilon) \left(Z_H(t; \epsilon)\right) = 0, \quad \forall k \geq -1.$$  

(39)

A powerful tool for manipulating Virasoro type operators was introduced by Givental [25]. Let us give a short review as it will be used in Section 3. Convention of the notations will follow from those of [16]. Denote by $V$ the space of Laurent polynomials in $z$ with coefficients in $C$. On $V$ there defines a symplectic bilinear form $\omega$:

$$\omega(f, g) := - \text{Res}_{z=\infty} f(-z) g(z) \frac{dz}{z^2} = - \omega(g, f), \quad \forall f, g \in V.$$  

The pair $(V, \omega)$ is called the Givental symplectic space. For any $f \in V$, write

$$f = \sum_{i \geq 0} q_i z^{-i} + \sum_{i \geq 0} p_i (-z)^{i+1}.$$  

Then $\{q_i, p_i \mid i \geq 0\}$ gives the Darboux coordinates for $(V, \omega)$.

For any infinitesimal symplectic transformation $A$ on $(V, \omega)$, define the Hamiltonian associated to $A$ by

$$H_A(f) = \frac{1}{2} \omega(f, A(f)) = -\frac{1}{2} \text{Res}_{z=\infty} f(-z) A(f(z)) \frac{dz}{z^2}.$$  

This Hamiltonian is a quadratic function on $V$, and its quantization is defined via

$$\hat p_i \hat q_j = \frac{1}{\epsilon^2} \frac{\partial^2}{\partial q_i \partial q_j}; \quad \hat p_i \hat q_j = q_j \frac{\partial}{\partial q_i}; \quad \hat q_i \hat q_j = \frac{1}{\epsilon^2} q_i q_j.$$  

Denote the quantization of $H_A$ by $\hat A$. For two infinitesimal symplectic transformations $A, B$, we have

$$\left[\hat A, \hat B\right] = \left[A, B\right] + C(H_A, H_B),$$

where $C$ is the 2-cocycle term satisfying

$$C(p_i p_j, q_k q_l) = -C(q_k q_l, p_i p_j) = \delta_{i,k} \delta_{j,l} + \delta_{i,l} \delta_{j,k},$$

and $C = 0$ for all other pairs of quadratic monomials of $p, q$.

Following Givental [25], define the operators $\ell_k$ by

$$\ell_k = (-1)^{k+1} z^{3/2} \circ \partial_z^{k+1} \circ z^{-1/2}, \quad k \geq -1.$$  

(40)
It is shown in [25] that $\ell_k$ are infinitesimal symplectic transformations on $\mathcal{V}$, and their quantizations coincide with the Virasoro operators (28)–(30). More precisely,

$$L^W_k(\epsilon^{-1}\tilde{t}, \epsilon \partial / \partial t) = \left. \hat{\ell}_k \right|_{q_i \to i, \theta_{ji} \to \theta_{ji}, i \geq 0} + \frac{\delta_{k,0}}{16}, \quad k \geq -1. \quad (41)$$

Givental [25] also shows that the quantizations of the multiplication operators $z^{1-2j}$, $j \geq 1$, coincide with the operators $D_j$, $j \geq 1$, defined in (33), i.e.,

$$D_j = \left. z^{1-2j} \right|_{q_i \to i, \theta_{ji} \to \theta_{ji}, i \geq 0}. \quad (42)$$

Let

$$\phi(z) := \sum_{k=1}^{\infty} \frac{(2-2k-1)B_{2k}}{k(2k-1)} \frac{1}{z^{2k-1}}. \quad (43)$$

Clearly, $\phi(z)$ gives an infinitesimal symplectic transformation on $\mathcal{V}$. Define $\Phi(z) := e^{\phi(z)}$. The quantization $\hat{\Phi}$ of the symplectomorphism $f(z) \mapsto \Phi(z)f(z)$ is defined by

$$\hat{\Phi} := \left. e^{\phi(z)} \right|_{q_i \to i, \theta_{ji} \to \theta_{ji}, i \geq 0}. \quad (44)$$

We see from (33), (42), (44) that $\hat{\Phi} = e^G$. So the operators $L^H_k(\epsilon^{-1}\tilde{t}, \epsilon \partial / \partial t)$ defined in (36) have the expressions

$$L^H_k(\epsilon^{-1}\tilde{t}, \epsilon \partial / \partial t) = \hat{\Phi} \circ L^W_k(\epsilon^{-1}\tilde{t}, \epsilon \partial / \partial t) \circ \hat{\Phi}^{-1}, \quad k \geq -1. \quad (45)$$

Thus by using (44) we obtain

$$L^H_k(\epsilon^{-1}\tilde{t}, \epsilon \partial / \partial t) = \left. \left[ \hat{\Phi} \circ \left( \hat{\ell}_k + \frac{\delta_{k,0}}{16} \right) \circ \hat{\Phi}^{-1} \right] \right|_{q_i \to i, \theta_{ji} \to \theta_{ji}, i \geq 0}, \quad k \geq -1. \quad (46)$$

Therefore, as in [16], we have

$$L^H_k(\epsilon^{-1}\tilde{t}, \epsilon \partial / \partial t) = \left. \hat{\Phi}_k \right|_{q_i \to i, \theta_{ji} \to \theta_{ji}, i \geq 0} + \frac{\delta_{k,0}}{16} - \frac{\delta_{k,-1}}{16}, \quad k \geq -1, \quad (46)$$

where $\Phi_k(z) = \Phi(z) \circ \ell_k \circ \Phi(z)^{-1}$.

2.2. The generalized BGW partition function. It is proved in [6, 15] that the generalized BGW partition function with the parameter $N$ (i.e. the power series $Z_{gBGW}(N, T; \hbar)$ from the Introduction) is a tau-function of the KdV hierarchy uniquely specified by (7) and the $m = 0$ case of (9); see also [1, 13, 20]. In particular, the power series $u = u_{gBGW}(N, T; \hbar) = \hbar^2 \partial^2_{T_1} \left( \log Z_{gBGW}(N, T; \hbar) \right)$ satisfies the following equations:

$$\frac{\partial u}{\partial T_{2a+1}} = \frac{1}{(2a+1)!!} \left[ \left( L^2_{2a+1} \right)_+, L \right], \quad a \geq 0. \quad (47)$$

Here $L = \hbar^2 \partial^2_{T_1} + 2u$. The initial value of this power series is given as follows [1, 20]:

$$u_{gBGW}(N, T_1, T_3 = T_5 = \cdots = 0; \hbar) = \hbar^2 \frac{1 - \frac{N^2}{2}}{(1 - T_1)^2}. \quad (48)$$

Using these properties, let us give a new proof that $F_{gBGW}(N, T; \hbar)$ has the form given by (5). Indeed, when $l \geq 2$, this statement follows from the tau-structure of the KdV hierarchy [10, 20, 21]. For $l = 1$, the statement then follows from the $m = 0$ case of (9).
As we recall from Introduction, with Alexandrov’s variable \(x\) (see (13)), one recognizes that the expansion (16) for the generalized BGW free energy \(\mathcal{F}(x, T; h)\) is a genus expansion, and it is convenient to write

\[
\mathcal{F}(x, T; h) =: \sum_{g \geq 0} \hbar^{2g-2} \mathcal{F}_g(x, T).
\]  

(49)

We call \(\mathcal{F}_g(x, T)\) the genus \(g\) part of the generalized BGW free energy. The derivatives of \(\mathcal{F}(x, T; h)\) at \(T = 0\)

\[
\left. \frac{\partial \mathcal{F}(x, T; h)}{\partial T_{2a_1+1} \cdots \partial T_{2a_l+1}} \right|_{T=0} =: \langle \sigma_{2a_1+1} \cdots \sigma_{2a_l+1} \rangle(x, \hbar)
\]

are called the generalized BGW correlators, where \(l, a_1, \ldots, a_l \geq 0\). Clearly, for \(l \geq 1\),

\[
\langle \sigma_{2a_1+1} \cdots \sigma_{2a_l+1} \rangle(x, \hbar) = \sum_{g \geq 0} \hbar^{2g-2} c_g(a_1, \ldots, a_l) x^{2|a|-2g+2}.
\]

(51)

It is clear from (13) that for the generalized BGW partition function \(Z(x, T; h)\) (see (14)) the Virasoro constraints (9) translate into

\[
L_m(Z(x, T; h)) = 0, \quad m \geq 0,
\]

(52)

where the operators \(L_m\) are defined by

\[
L_m := \sum_{a \geq 0} \frac{(2a + 2m + 1)!!}{2^{m+1}(2a - 1)!!} \tilde{T}_{2a+1} \frac{\partial}{\partial T_{2a+2m+1}} + \frac{\hbar}{2} \sum_{a+b=m-1} \frac{(2a+1)!!(2b+1)!!}{2^{m+1}} \frac{\partial^2}{\partial T_{2a+1} \partial T_{2b+1}} + \left( \frac{1}{16} + \frac{x^2}{8\hbar^2} \right) \delta_{m,0}.
\]

(53)

We have the following lemma.

**Lemma 1.** The generalized BGW partition function \(Z(x, T; h)\) satisfies the dilaton equation:

\[
L_{\text{dilaton}}(Z(x, T; h)) = 0,
\]

(54)

where \(L_{\text{dilaton}}\) is the linear operator defined by

\[
L_{\text{dilaton}} := \sum_{a \geq 0} \tilde{T}_{2a+1} \frac{\partial}{\partial T_{2a+1}} + x \frac{\partial}{\partial x} + \hbar \frac{\partial}{\partial \hbar} + \frac{1}{24}.
\]

(55)

**Proof.** On one hand, by using (15) and (16) we have

\[
-x \frac{\partial \mathcal{F}_g(x, T)}{\partial x} + \sum_{a \geq 0} 2a \tilde{T}_{2a+1} \frac{\partial \mathcal{F}_g(x, T)}{\partial T_{2a+1}} = (2g - 2) \mathcal{F}_g(x, T) - \frac{x^2}{4} \delta_{g,0} - \frac{1}{12} \delta_{g,1},
\]

(56)

which gives

\[
x \frac{\partial Z}{\partial x} - \sum_{a \geq 0} 2a \tilde{T}_{2a+1} \frac{\partial Z}{\partial T_{2a+1}} + \hbar \frac{\partial Z}{\partial \hbar} - \left( \frac{x^2}{4\hbar^2} + \frac{1}{12} \right) Z = 0.
\]

(57)

On another hand, the \(m = 0\) case of (53) reads

\[
\sum_{a \geq 0} \frac{2a+1}{2} \tilde{T}_{2a+1} \frac{\partial Z(x, T; h)}{\partial T_{2a+1}} + \left( \frac{1}{16} + \frac{x^2}{8\hbar^2} \right) Z(x, T; h) = 0.
\]

(58)
The lemma follows from (57) and (58).

Let us now prove the following lemma.

**Lemma 2 ([1, 6]).** The solution in \( \mathbb{C}((h^2))[x+2][[T]] \) to the Virasoro constraints

\[
L_m(\hat{Z}(x, T; \hbar)) = 0
\]

with the initial value \( \hat{Z}(x, 0; h) = 1 \) is unique.

**Proof.** Following the same arguments as those in [1] we see that if a solution \( \hat{Z} \) in \( \mathbb{C}((\hbar^2))[x+2][[T]] \) exists, it must have the following cut-and-join representation:

\[
\hat{Z}(x, T; \hbar) = e^{\hat{W}(1)},
\]

where

\[
\hat{W} = \sum_{a,b \geq 0} \frac{(2a + 2b + 1)!}{(2a - 1)!(2b - 1)!} T_{2a+1} T_{2b+1} \frac{\partial}{\partial T_{2a+2b+1}}
\]

\[
+ \frac{\hbar^2}{2} \sum_{a,b \geq 0} \frac{(2a + 1)!(2b + 1)!}{(2a + 2b + 1)!} T_{2a+2b+3} \frac{\partial^2}{\partial T_{2a+1} \partial T_{2b+1}} + \left( \frac{1}{8} + \frac{x^2}{4 \hbar^2} \right) T_1.
\]

The lemma is proved.

**Lemma 2** also easily follows from a similar argument used in e.g. [11, 40].

We note that the power series

\[
Z_{gBGW}(\frac{x}{\hbar \sqrt{-2}}, T; \hbar) \in \mathbb{C}((\hbar^2))[x][[T]] \subset \mathbb{C}((\hbar^2))[x+2][[T]]
\]

is a solution to (59) satisfying \( Z_{gBGW}(\frac{x}{\hbar \sqrt{-2}}, 0; \hbar) = 1 \).

### 3. Proof of the Main Theorem

In this section, we prove the Main Theorem. The proof will be similar to that of [16] for the Hodge-GUE correspondence.

Before entering into the details of the proof, we do several preparations. Let us introduce the following linear combinations of \( L_k^H \), \( k \geq -1 \):

\[
\tilde{L}_m^H(\epsilon^{-1} \tilde{t}, \epsilon \partial / \partial t) := - \sum_{k=-1}^{\infty} \frac{(-m)^{k+1}}{(k+1)!} L_k^H(\epsilon^{-1} \tilde{t}, \epsilon \partial / \partial t), \quad m \geq 0.
\]

**Lemma 3.** For a basis \( \{ \alpha_k \mid k \geq -1 \} \) of an infinite dimensional Lie algebra satisfying

\[
[\alpha_k, \alpha_\ell] = (k - \ell) \alpha_{k+\ell}, \quad \forall k, \ell \geq -1,
\]

where \( [\cdot, \cdot] \) denotes the Lie bracket of the Lie algebra, define

\[
\tilde{\alpha}_m := - \sum_{k \geq 1} \frac{(-m)^{k+1}}{(k+1)!} \alpha_k, \quad m \geq 0.
\]

Then

\[
[\tilde{\alpha}_m, \tilde{\alpha}_n] = (m-n) \tilde{\alpha}_{m+n}, \quad \forall m, n \geq 0.
\]
\[ [\tilde{\alpha}_m, \tilde{\alpha}_n] = \sum_{k_1, k_2 = -1}^{\infty} \frac{(-m)^{k_1+1}(-n)^{k_2+1}}{(k_1 + 1)! (k_2 + 1)!} (k_1 - k_2) \alpha_{k_1+k_2} \]

\[ = \sum_{k = -1}^{\infty} \left( \sum_{k_1+k_2 = k \atop k_1 \geq 0, k_2 \geq -1} \frac{(-m)^{k_1+1}(-n)^{k_2+1}}{k_1!(k_2 + 1)!} - \sum_{k_1+k_2 = k \atop k_1 \geq -1, k_2 \geq 0} \frac{(-m)^{k_1+1}(-n)^{k_2+1}}{(k_1 + 1)! k_2!} \right) \alpha_k \]

\[ = \sum_{k = -1}^{\infty} \frac{(m+n)^{k+1} - n (m+n)^{k+1}}{(k+1)!} \alpha_k = (m-n) \tilde{\alpha}_{m+n}. \]

The lemma is proved.

**Proposition 2.** We have

\[ \bar{L}_0^H(\epsilon^{-1} \dot{t}, \epsilon \partial / \partial t) = -\sum_{i \geq 1} \bar{t}_i \frac{\partial}{\partial t_{i-1}} - \frac{t_0^2}{2 \epsilon^2} + \frac{1}{16}, \quad (66) \]

\[ \bar{L}_1^H(\epsilon^{-1} \dot{t}, \epsilon \partial / \partial t) = \sum_{i \geq 0} \sum_{j = 0}^{i+1} \sum_{r = 0}^{i+1-j} \left( \frac{i+1}{j+r} \right) \frac{(-1)^{j-r}}{2^r} \bar{t}_j \frac{\partial}{\partial t_i} \]

\[ - \frac{\epsilon^2}{8} \sum_{i,j \geq 0} \left( -\frac{1}{2} \right)^{i+j} \frac{\partial^2}{\partial t_i \partial t_j} - \frac{t_0^2}{2 \epsilon^2} + \frac{1}{8}, \quad (67) \]

\[ \bar{L}_2^H(\epsilon^{-1} \dot{t}, \epsilon \partial / \partial t) = -\frac{1}{2} \sum_{i_2 \leq i_1 + j + i_3 + j_3 \leq i_2 + 1} (-1)^{i+j+i_2+i_3+1} \left( \frac{i_2+1}{i_3} \right) 3^j 2^{i_3-i-j} \bar{t}_{i_2} \frac{\partial}{\partial t_{i_1}} \]

\[ - \frac{1}{2} \sum_{i_1 \leq i_2 + i_3 + j \leq i_2 + 1} (-1)^{i+j} \left( \frac{-i_2}{i_3} \right) 3^j 2^{i_3-i-j} \bar{t}_{i_2} \frac{\partial}{\partial t_{i_1}} \]

\[ - \frac{\epsilon^2}{2} \sum_{i_1 + i_2 + i_3 + j \leq i_1 + i_2 + 1} (-1)^{i+j+i_2+i_3+1} 3^j 2^{i_3-i-j} \left( \frac{i_2+1}{i_3} \right) \frac{\partial^2}{\partial t_{i_1} \partial t_{i_2}} \]

\[ - \frac{t_0^2}{2 \epsilon^2} + \frac{3}{16}. \quad (68) \]

**Proof.** It follows from (60), (62) that the operators \( \bar{L}_m^H, m \geq 0, \) have the following expressions:

\[ \bar{L}_m^H(\epsilon^{-1} \dot{t}, \epsilon \partial / \partial t) = \tilde{\Psi}_m |_{q_i \rightarrow \bar{t}_i, \partial q_i \rightarrow \partial \bar{t}_i, i \geq 0} + \frac{m+1}{16}, \quad (69) \]

where

\[ \tilde{\Psi}_m(z) := -\sum_{k=-1}^{\infty} \frac{(-m)^{k+1}}{(k+1)!} \Phi_k(z). \quad (70) \]

Let us now prove (66) – (68) one by one. For \( m = 0, \) we have

\[ \bar{L}_0^H(\epsilon^{-1} \dot{t}, \epsilon \partial / \partial t) = -L_{-1}^H(\epsilon^{-1} \dot{t}, \epsilon \partial / \partial t). \quad (71) \]
For $m = 1$,
\[
\Psi_1(z) = -\Phi(z) z^{3/2} e^{\partial_z} \circ z^{-1/2} \circ \Phi^{-1}(z) = -z^{3/2} (z + 1)^{-1/2} \frac{\Phi(z)}{\Phi(z + 1)} e^{\partial_z}.
\]

Note that the following identity
\[
\frac{\Phi(z + 1)}{\Phi(z)} = \frac{z + \frac{1}{2}}{\sqrt{z(z + 1)}}
\]  
(72)
is proved in [5]. Using this identity we get
\[
\Psi_1(z) = -\frac{z^2}{z + \frac{1}{2}} e^{\partial_z}.
\]  
(73)

Then by a straightforward computation similar to the one given in [16] we arrive at formula (67). For $m = 2$, we notice the identity
\[
\frac{\Phi(z + 2)}{\Phi(z)} = \frac{(z + \frac{3}{2}) (z + 1) (z + \frac{1}{2})}{(z + 1) \sqrt{z(z + 2)}}.
\]

Similarly as above we get
\[
\Psi_2(z) = -z^{3/2} (z + 2)^{-1/2} \frac{\Phi(z)}{\Phi(z + 2)} e^{2\partial_z} = -z^2 \frac{z + 1}{(z + \frac{1}{2})(z + \frac{3}{2})} e^{2\partial_z}.
\]  
(74)

Formula (68) then follows. The theorem is proved. \[]

Let us continue the preparations. The following lemma establishes the relationship between the operator $L_{\text{dilaton}}$ and the operator $\tilde{L}_{\text{dilaton}}^H$ defined in the previous section.

**Lemma 4.** We have
\[
e^{-\frac{A(x,T)}{\hbar^2}} \circ L_{\text{dilaton}} \circ e^{\frac{A(x,T)}{\hbar^2}} = \tilde{L}_{\text{dilaton}}^H|_{\epsilon = \hbar \sqrt{-4}}.
\]  
(75)

**Proof.** The identity follows immediately from $[L_{\text{dilaton}}, e^{\frac{A(x,T)}{\hbar^2}}] = 0$ and $[L_{\text{dilaton}}, \tilde{t}_i] = \tilde{t}_i$. \[]

Similarly, we are to establish the relationships between the operators $L_m$ and the operators $\tilde{L}_m^H$. To this end, the following lemma would be helpful.

**Lemma 5.** For $i \geq 0$, let
\[
X_i = \delta_{i,0} x - \sum_{a \geq 0} \left( -\frac{2a + 1}{2} \right)^i \frac{\tilde{T}_{2a+1}}{(a+1)!},
\]  
(66)
\[
Y_i = \frac{2}{3} \delta_{i,0} x - \sum_{a \geq 0} \left( -\frac{2a + 1}{2} \right)^i \frac{\tilde{T}_{2a+1}}{(a+2)!},
\]  
(67)
then
\[ X_i = \frac{1}{2^i} X_0 - \sum_{j=0}^{i-1} \frac{1}{2^{i-j}} \tilde{t}_j, \quad (78) \]
\[ Y_i = \frac{3^i}{2^i} Y_0 - \frac{3^i - 1}{2^i} X_0 + \sum_{j=0}^{i-2} \frac{3^{i-1-j} - 1}{2^{i-j}} \tilde{t}_j. \quad (79) \]

**Proof.** By observing that \( Y_{i+1} = \frac{3}{2} Y_i - X_i \) and \( X_{i+1} = \frac{1}{2} X_i - \frac{1}{2} \tilde{t}_i. \)

**Proposition 3.** The following identities are true:
\[ e^{-\frac{A(x,T)}{\hbar^2}} \circ L_m \circ e^{-\frac{A(x,T)}{\hbar^2}} = \tilde{L}^H_{m|\epsilon=\hbar\sqrt{-4}}, \quad m \geq 0. \quad (80) \]

**Proof.** Because of the Virasoro commutation relations (12) and (65), we just need to prove the \( m = 0, 1, 2 \) cases. For \( m = 0, \)
\[
e^{-\frac{A(x,T)}{\hbar^2}} \circ L_0 \circ e^{-\frac{A(x,T)}{\hbar^2}}
= \sum_{a \geq 0} \frac{2a + 1}{2} \tilde{T}_{2a+1} \frac{\partial}{\partial T_{2a+1}} + \frac{1}{16} + \frac{x^2}{8 \hbar^2} + \frac{1}{\hbar^2} \sum_{a \geq 0} \frac{2a + 1}{2} \tilde{T}_{2a+1} \frac{\partial A}{\partial T_{2a+1}}
=
\sum_{i \geq 0} \sum_{a \geq 0} 2 \left( -\frac{2a + 1}{2} \right) \tilde{T}_{2a+1} \frac{\partial}{\partial t_i} + \frac{1}{16} + \frac{x^2}{8 \hbar^2}
+ \frac{1}{\hbar^2} \sum_{a,b \geq 0} \frac{\tilde{T}_{2a+1} \tilde{T}_{2b+1}}{2a! b!} - \frac{1}{\hbar^2} \sum_{a \geq 0} \frac{x \tilde{T}_{2a+1}}{2a!}
= -\sum_{i \geq 0} \tilde{t}_{i+1} \frac{\partial}{\partial t_i} + \frac{1}{16} + \frac{t_0^2}{8 \hbar^2}
= \tilde{L}^H_0|_{\epsilon=\hbar\sqrt{-4}}.
\]

For \( m = 1, \)
\[
e^{-\frac{A(x,T)}{\hbar^2}} \circ L_1 \circ e^{-\frac{A(x,T)}{\hbar^2}}
= \sum_{a \geq 0} \frac{(2a + 1)(2a + 3)}{4} \tilde{T}_{2a+1} \frac{\partial}{\partial T_{2a+3}} + \frac{\hbar^2}{8} \frac{\partial^2}{\partial T_1^2} + \frac{1}{\hbar^2} \sum_{a \geq 0} \frac{(2a + 1)(2a + 3)}{4} \tilde{T}_{2a+1} \frac{\partial A}{\partial T_{2a+3}}
+ \frac{1}{8 \hbar^2} \frac{\partial A}{\partial T_1} \frac{\partial A}{\partial T_1}
+ \frac{1}{8} \frac{\partial^2 A}{\partial T_1^2}
+ \frac{1}{4} \frac{\partial A}{\partial T_1} \frac{\partial}{\partial T_1}
= (I) + (II) + (III),
\]
Proof of Theorem 1. By using the degree-dimension counting (2) and (18), we find that the validity of (81), (85), (86) we obtain the validity of (80). In a similar way, for (82) we can obtain

\[
(\text{II}) = \sum_{i \geq 0} \left( \sum_{a \geq 0} (2a + 1)(2a + 3) \frac{\tilde{T}_{2a+1}}{(a + 1)!} \left( -\frac{2a + 3}{2} \right)^{i+1} + \frac{1}{2} \left( -\frac{1}{2} \right)^i \left( x - \sum_{a \geq 0} \frac{\tilde{T}_{2a+1}}{(a + 1)!} \right) \right) \frac{\partial}{\partial t_i}
\]

and

\[
(\text{III}) = \sum_{i,j \geq 0} \left( \sum_{r=0}^{i+j} \left( \frac{i+1}{j} \right) (1)^{i+j-r} \frac{1}{2^r} \left( -\frac{1}{2} \right)^j X_0 \right) \frac{\partial}{\partial t_i}.
\]

Here we have used Lemma [5], (81), (85), (86) we obtain the validity of the m = 1 case of (80). In a similar way, for m = 2 we can obtain

\[
e^{-\frac{A(x,T)}{\hbar^2}} \circ L_2 \circ e^{-\frac{A(x,T)}{\hbar^2}} = \sum_{i \geq 0} \left( \sum_{r=0}^{i+j} \left( \frac{i+1}{j} \right) (1)^{i+j-r} \frac{1}{2^r} \right) \frac{\partial}{\partial t_i} + \frac{\hbar^2}{2} \sum_{i,j \geq 0} \left( -\frac{1}{2} \right)^{i+j} 3^{i+1} \frac{\partial^2}{\partial t_i \partial t_j} + \frac{3}{16} + \frac{i_0^2}{8 \hbar^2}.
\]

One can check that this equals \(\tilde{L}_2^H\) given in (88). The proposition is proved. \(\square\)

We are ready to prove Theorem 1.

Proof of Theorem 1. By using the degree-dimension counting (2) and (18), we find that the expression \(H(t(x,T); h \sqrt{-4})\) is a well-defined element in \(\mathbb{C}[[h^2]][[x + 2]][[T]]\), therefore, \(e^{-\frac{A(x,T)}{\hbar^2}} Z_H(t(x,T); h \sqrt{-4})\) is a well-defined element in \(\mathbb{C}((h^2))[[x + 2]][[T]]\). It follows from Proposition 3 that

\[
L_m \left( e^{-\frac{A(x,T)}{\hbar^2}} Z_H(t(x,T); h \sqrt{-4}) \right) = 0, \quad m \geq 0,
\]

where

\[
(\text{I}) := \frac{1}{\hbar^2} \sum_{a \geq 0} \frac{(2a + 1)(2a + 3)}{4} \tilde{T}_{2a+1} \frac{\partial A}{\partial T_{2a+3}} + \frac{1}{8 \hbar^2} \frac{\partial A}{\partial T_1} \frac{\partial A}{\partial T_1} + \frac{1}{8} \frac{\partial^2 A}{\partial T_1^2},
\]

\[
(\text{II}) := \sum_{a \geq 0} \frac{(2a + 1)(2a + 3)}{4} \tilde{T}_{2a+1} \frac{\partial}{\partial T_{2a+3}} + \frac{1}{4} \frac{\partial A}{\partial T_1} \frac{\partial}{\partial T_1},
\]

\[
(\text{III}) := \frac{\hbar^2}{8} \frac{\partial^2}{\partial T_1^2}.
\]
which are the same as the linear equations \([52]\) for the power series \(Z(x, T; h)\). It then follows from Proposition \([2]\) that \(e^{-\frac{A(x, T)}{h^2}} Z_H(t(x, T); h \sqrt{-4})\) and \(Z(x, T; h)\) could only differ by multiplying by a non-zero element in \(\mathbb{C}((h^2))[\![x + 2]\!]\). More precisely, if we denote

\[
R(x, h) := \frac{A(x, 0)}{h^2} + \mathcal{H}(t(x, 0); h \sqrt{-4}) \in h^{-2} \mathbb{C}[[x + 2]][[h^2]],
\]

then it remains to show that \(R(x, h) = B(x, h)\). Write

\[
R(x, h) =: \sum_{g \geq 0} h^{2g-2} R_g(x).
\]

Let us first compute \(R_0(x)\). Let \(v(t) = \frac{\partial^2 \mathcal{H}_0(t)}{\partial t^2}\). It is well known (cf. e.g. \([15]\)) that

\[
v(t) = t_0 + \sum_{k \geq 2} \frac{1}{k} \sum_{i_1 + \cdots + i_k = k-1} \frac{t_{i_1}}{i_1!} \cdots \frac{t_{i_k}}{i_k!},
\]

\[
\mathcal{H}_0(t) = \frac{1}{2} \sum_{i,j \geq 0} \tilde{t}_i \tilde{t}_j v(t)^{i+j+1}.
\]

Since \(t_i(x, 0) = \delta_{i,0} x + \delta_{i,1} - (-\frac{1}{2})^{i-1}\), we find

\[
v(t(x, 0)) = -2 \log \left( -\frac{x}{2} \right),
\]

and therefore,

\[
R_0(x) = A(x, 0) - \frac{1}{8} \sum_{i,j \geq 0} \tilde{t}_i(x, 0) \tilde{t}_j(x, 0) \frac{v(t(x, 0))^{i+j+1}}{i! j! (i+j+1)} = \frac{x^2}{4} \log \left( -\frac{x}{2} \right) - \frac{3}{8} x^2.
\]

Let us now compute \(R_1(x)\). It is known e.g. in \([15, 16, 17]\) that

\[
\mathcal{H}_1(t) = \frac{1}{24} \log(v_0(t)) - \frac{1}{16} v(t).
\]

It then follows from \([93]\) and the fact \(\partial_x = \partial_t\) that

\[
R_1(x) = \frac{1}{12} \log \left( -\frac{x}{2} \right).
\]

Using Lemma \([4]\) we find that for \(g \geq 2\) the element \(R_g(x)\) has the form \(R_g(x) = r_g x^{2g-2}\) for some \(r_g \in \mathbb{C}\). Define

\[
w = w(x, T; h) := \left( 2 - \Lambda^\frac{1}{2} - \Lambda^{-\frac{1}{2}} \right) \left( \mathcal{H}(t(x, T); h \sqrt{-4}) \right).
\]

where \(\Lambda := \exp(2 h \sqrt{-2} \partial_x)\). Noticing that

\[
\frac{\partial}{\partial T_1} = \sum_{i \geq 0} \frac{1}{(-2)^{i-1}} \frac{\partial}{\partial t_i},
\]

we know from the Hodge–FVH correspondence \([11, 12]\) that \(w(x, T; h)\) satisfies the following equation:

\[
- \frac{1}{4} \frac{\partial w}{\partial T_1} = \frac{1}{h \sqrt{-2}} \frac{2 - \Lambda^\frac{1}{2} - \Lambda^{-\frac{1}{2}}}{\Lambda^{-\frac{1}{2}} - \Lambda^\frac{1}{2}} (e^w).
\]
Using the $m = 0$ case of (88) we find that

$$\hbar^2 \frac{\partial^3 \mathcal{H}(t(x,T); \hbar \sqrt{-4})}{\partial x \partial x \partial T_1} \bigg|_{T=0} \equiv \frac{1}{2}. \quad (100)$$

Denote $W(x, \hbar) = w(x, 0; \hbar)$. Using the expressions of $R_0(x)$ and $R_1(x)$ we know that $W(x, \hbar) = \log(-\frac{x}{2}) + O(\hbar^4)$. It follows from (99) and (100) that

$$\partial_x \left( e^{W(x, \hbar)} \right) = -\frac{1}{2}. \quad (101)$$

We therefore conclude that, for $g \geq 2$, $r_g = \frac{(-1)^g 2^{g-1} B_{2g}}{2g (2g-2)}$. The theorem is proved. $\Box$

Since for any fixed parameter $x$, the power series $Z(x, T; \hbar)$ is a tau-function for the KdV hierarchy [15] (cf. [10, 20]) we obtain immediately from the Hodge-BGW correspondence (17) the following corollary.

**Corollary 2.** For any fixed $x$, the power series $e^{\frac{\Delta(x, T)}{\hbar^2}} Z_H(t(x, T); \hbar \sqrt{-4})$ is a tau-function for the KdV hierarchy (47) associated to the solution specified by the initial value

$$\frac{x^2}{4} \left( \frac{1}{1 - T_1} \right)^2 + \frac{\hbar^2}{8} \left( \frac{1}{1 - T_1} \right)^2.$$

4. ELSV-like formula for the generalized BGW correlators

In this section, we prove Proposition 1.

**Proof of Proposition 1.** From the definition (4) of the cubic Hodge free energy and using (18) we obtain that

$$\mathcal{H}(t(x, T); \hbar \sqrt{-4}) = \sum_{g \geq 0} (-4)^{g-1} \hbar^{2g-2} \sum_{n \geq 0} \frac{1}{n!} \int_{\mathcal{M}_{g,n}} \Lambda(-1)^2 \Lambda \left( \frac{1}{2} \right) \prod_{p=1}^n \left( x + 2 + \frac{\psi_p^2}{2 + \psi_p} - 2 \sum_{a \geq 0} \frac{T_{2a+1}/a!}{1 + \frac{2a+1}{2} \psi_p} \right)$$

$$= \sum_{g \geq 0} (-4)^{g-1} \hbar^{2g-2} \sum_{n \geq 0} \frac{1}{n!} \int_{\mathcal{M}_{g,n}} \Lambda(-1)^2 \Lambda \left( \frac{1}{2} \right) \sum_{l=0}^n \binom{n}{l} \prod_{p=l+1}^n \left( x + 2 + \frac{\psi_p^2}{2 + \psi_p} \right)$$

$$\times \prod_{k_1, \ldots, k_l} \frac{T_{2a_p+1}/a_p!}{1 + \frac{2a_p+1}{2} \psi_p}. \quad (102)$$
For $g, l \geq 0$ and for $a_1, \ldots, a_l \geq 0$, by comparing the coefficients of $h^{2g-2} T_{2a_1+1} \cdots T_{2a_l+1}$ of the logarithms of both sides of (17) and using (16), (19), (102) we get

$$c_g(a_1, \ldots, a_l) x^{2|a|-2g+2} \delta_{l,1} = \left( \frac{x^2}{4} \log \left( -\frac{x}{2} \right) - \frac{3}{8} x^2 \right) \delta_{g,0} \delta_{l,0} + \frac{1}{12} \log \left( -\frac{x}{2} \right) \delta_{g,1} \delta_{l,0}$$

$$+ \frac{1}{x^{2g-2}} \frac{(-1)^g 2^{g-1} B_{2g}}{2g (2g-2)} \delta_{g,2} \delta_{l,0}$$

$$= \frac{(-1)^{g+1+l} 2^{2g-2+l}}{(2|a|-2g+2)!} \sum_{k=0}^{l} \frac{1}{k!} \sum_{n'=0}^{\infty} \frac{1}{n'! 2^{n'}}$$

$$\times \int_{\overline{\mathcal{M}}_{g,l+k+n'}} \Lambda(-1)^2 \left( \frac{1}{2} \right)^{l} \prod_{p=1}^{l} \frac{1}{1 + \frac{2a_p+1}{2} \psi_p} \prod_{q=l+k+1}^{l+k+n'} \frac{\psi_q^2}{1 + \frac{1}{2} \psi_q}$$

$$+ \delta_{g,0} \delta_{l,2} \frac{1}{a_1! a_2! (a_1 + a_2 + 1)} - \delta_{g,0} \delta_{l,1} \frac{1}{a_1! (2a_1 + 1)} \left( x + \frac{1}{a_1 + 1} \right). \quad (103)$$

For $l \geq 1$, by comparing the coefficients of $(x+2)^{2|a|-2g+2}$ of both sides of (103) we obtain

$$c_g(a_1, \ldots, a_l) = \frac{(-1)^{g+1+l} 2^{2g-2+l}}{(2|a|-2g+2)!} \sum_{n'=0}^{\infty} \frac{1}{n'! 2^{n'}}$$

$$\times \int_{\overline{\mathcal{M}}_{g,l+2|a|-2g+2+n'}} \Lambda(-1)^2 \left( \frac{1}{2} \right)^{l} \prod_{p=1}^{l} \frac{1}{1 + \frac{2a_p+1}{2} \psi_p} \prod_{q=l+2|a|-2g+2+n'}^{l+2|a|-2g+2+n'} \frac{\psi_q^2}{1 + \frac{1}{2} \psi_q}. \quad (104)$$

Similarly, for $l = 0$ and $g \geq 2$, by comparing the coefficients of $(x+2)^0$ of both sides of (103) we obtain

$$\frac{(-1)^g 2^{g-1} B_{2g}}{2g (2g-2)} = (-1)^{g-1} 4^{2g-2} \sum_{n'=0}^{\infty} \frac{1}{n'! 2^{n'}} \int_{\overline{\mathcal{M}}_{g,n'}} \Lambda(-1)^2 \Lambda \left( \frac{1}{2} \right) \prod_{q=1}^{n'} \frac{\psi_q^2}{1 + \frac{1}{2} \psi_q}. \quad (105)$$

Consider the forgetful map

$$f : \overline{\mathcal{M}}_{g,n+n'} \to \overline{\mathcal{M}}_{g,n} \quad (106)$$

forgetting the last $n'$ marked points. It is known from [41, 32, 39, 43] that

$$\sum_{n' \geq 1} \frac{1}{n'!} \sum_{d_1, \ldots, d_{n'} \geq 1} f_* \left( \Lambda(-1)^2 \Lambda \left( \frac{1}{2} \right) \prod_{p=1}^{l} \frac{1}{1 + \frac{2a_p+1}{2} \psi_p} \prod_{r=1}^{n'} \psi_{d_r+1} \right) \prod_{r=1}^{n'} b_{d_r}(s) = c_{\sum_{d \geq 1} s_d s_d} \Lambda(-1)^2 \Lambda \left( \frac{1}{2} \right) \prod_{p=1}^{l} \frac{1}{1 + \frac{2a_p+1}{2} \psi_p}. \quad (107)$$

where $s = (s_1, s_2, \ldots)$ is an infinite vector of indeterminates and $b_d(s)$ are polynomials of $s$ defined via

$$\exp \left( -\sum_{d \geq 1} s_d z^d \right) = 1 - \sum_{d \geq 1} b_d(s) z^d. \quad (108)$$
(This was also used in [24] to derive the ELSV-like formula for even GUE correlators from the Hodge-GUE correspondence.) By solving the following system of equations

\[- b_d(s) = \left( -\frac{1}{2} \right)^d, \quad d \geq 1,\]

we get \( s_d = -\frac{1}{d} \left( -\frac{1}{2} \right)^d \), thus we obtain identities (20) and (21).

In the above proof we get identities (103) (which hold in \( \mathbb{C}[[x+2]] \)), and obtained Proposition 11 from these identities. We note that one can also get from (103) infinitely many other identities by comparing coefficients of powers of \( x+2 \); for example, for \( l \geq 1 \) and \( a_1, \ldots, a_l \geq 0 \), one can obtain vanishing identities (cf. also [19]) for certain combinations of intersection numbers by taking coefficients of \( (x+2)^k \), for \( k \geq 2|a| - 2g + 3 \).

We hope that the ELSV-like formula (22) for the BGW correlators could be helpful for understanding Norbury’s \( \Theta \)-class integrals [34, 47].

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Appendix A. Three more applications

In this appendix, we give three more applications of the Hodge-BGW correspondence.

Introduce a power series \( Q = Q(x, T) \in \mathbb{C}[[x+2]][[T]] \) by

\[ Q(x, T) = \exp \left( -\frac{v(t(x, T))}{2} \right). \]

Lemma 6. The power series \( Q \) is the unique solution in \( \mathbb{C}[[x+2]][[T]] \) to the initial value problem of the following integrable hierarchy:

\[ \frac{\partial Q}{\partial T_{2a+1}} = -\frac{2}{a!} Q^{2a+1} \frac{\partial Q}{\partial x}, \quad q \geq 0, \]

\[ Q(x, 0) = -\frac{x}{2}. \]

Moreover, \( Q \) satisfies the equation

\[ Q = -\frac{x}{2} + \sum_{a \geq 0} T_{2a+1} \frac{Q^{2a+1}}{a!}. \]

Proof. Equalities (111) follow from (110) and the dispersionless KdV hierarchy

\[ v_{ti} = \frac{v^i}{t^l} v_0; \]

cf. (91). The property (112) follows from the already proved formula (93), and equality (113) is then an elementary exercise. \( \square \)
Proposition 4. The genus zero part of the generalized BGW free energy $F_0(x, T)$ admits the expression (cf. [14])

$$F_0(x, T) = \frac{1}{2} \sum_{a,b \geq 0} \tilde{T}_{2a+1} \tilde{T}_{2b+1} \frac{Q^{2a+2b+2}}{a! b! (a + b + 1)} - x \sum_{b \geq 0} \tilde{T}_{2b+1} \frac{Q^{2b+1}}{b! (2b + 1)} + \frac{x^2}{4} \log Q.$$  

Moreover, the following formula, which is conjectured by Alexandrov [1], is true:

$$F_0(x, T) - \left( \frac{x^2}{4} \log \left( -\frac{x}{2} \right) - \frac{3}{8} x^2 \right) + \frac{x^2}{4} \log(1 - T_1) = \sum_{k \geq 1, j_1, j_2, \cdots \geq 0} \frac{(3j_1 + 5j_2 + \cdots - 1)!}{2^{2k+1} (1 - T_1)^{3j_1 + 5j_2 + \cdots}} \frac{x^{2k+2}}{(2k + 2)!} \frac{T_{3j_1} T_{5j_2} \cdots}{(1!)^{j_1} (2!)^{j_2} j_1! j_2! \cdots}.$$  

Proof. The genus zero part of the dilaton equation (54) reads:

$$2F_0(x, T) = x \frac{\partial F_0(x, T)}{\partial x} + \sum_{a \geq 0} \tilde{T}_{2a+1} \frac{\partial F_0(x, T)}{\partial \tilde{T}_{2a+1}},$$

from which one deduces that

$$F_0(x, T) = \frac{1}{2} \sum_{a,b \geq 0} \tilde{T}_{2a+1} \tilde{T}_{2b+1} \frac{\partial^2 F_0(x, T)}{\partial \tilde{T}_{2a+1} \partial \tilde{T}_{2b+1}} + x \sum_{a \geq 0} \tilde{T}_{2a+1} \frac{\partial^2 F_0(x, T)}{\partial x \partial \tilde{T}_{2a+1}} + \frac{x^2}{4} \frac{\partial^2 F_0(x, T)}{\partial x^2}.$$

By the Hodge-BGW correspondence [17], we know that

$$F_0(x, T) = -\frac{1}{4} \mathcal{H}_0(t(x, T)) + A(x, T).$$

Then by using (18), (110) and the well-known fact that

$$\frac{\partial^2 \mathcal{H}_0(t)}{\partial t_i \partial t_j} = \frac{v(t)^{i+j+1}}{i! j! (i + j + 1)}, \quad i, j \geq 0,$$

with $v(t)$ being defined in (111), we obtain that

$$\frac{\partial^2 F_0(x, T)}{\partial \tilde{T}_{2a+1} \partial \tilde{T}_{2b+1}} = \frac{Q^{2a+2b+2}}{a! b! (a + b + 1)},$$

$$\frac{\partial^2 F_0(x, T)}{\partial x \partial \tilde{T}_{2b+1}} = -\frac{Q^{2b+1}}{b! (2b + 1)},$$

$$\frac{\partial^2 F_0(x, T)}{\partial x \partial x} = \frac{1}{2} \log Q.$$
Here $a, b \geq 0$. Substituting (121)–(123) into (118) we obtain (115).

By using (113) and the Lagrangian inversion formula, we find that

$$Q = - \sum_{k,j_1,j_2,\ldots \geq 0 \atop j_1 + 2j_2 = 2k} \frac{(3j_1 + 5j_2 + \ldots)!}{2^{k+1} (1 - T_1)^{1+3j_1+5j_2+\ldots}} \frac{x^{2k+1}}{(2k+1)!} \frac{T_j}{(1!)^{j_1} (2!)^{j_2} \ldots}.$$ (124)

The proposition is then proved by integrating the $b = 0$ identity of (122) with respect to $x$ and $T_1$, where the integration constants can be fixed by using (121), (122).

We note that formula (116) was also given in [49].

The next application of the Hodge-BGW correspondence is on a way of calculating $F_g(x, T)$ with $g \geq 1$. Recall from e.g. [15, 18] that the genus $g$ part of the Hodge free energy $H_g(t)$, $g \geq 1$, possesses the following jet representation: for $g = 1$, it is given explicitly by (95), and for $g \geq 2$, there exist functions $H_g(v_1, \ldots, v_{3g-2})$ of $(3g-2)$ variables, depending polynomially in $v_2, \ldots, v_{3g-2}$ and rationally in $v_1$, such that

$$H_g(t) = H_g \left( \frac{\partial u(t)}{\partial t_0}, \ldots, \frac{\partial^{3g-2} u(t)}{\partial t_0^{3g-2}} \right),$$ (125)

$$\sum_{k \geq 1} k v_k \frac{\partial H_g}{\partial v_k} = (2g - 2) H_g.$$ (126)

Moreover, $H_1(v_0, v_1) := \frac{1}{24} \log(v_1) - \frac{1}{16} v_0$, and $H_g(v_1, \ldots, v_{3g-2})$, $g \geq 2$, can be computed by using the Dubrovin–Zhang type loop equation [16]. Noticing that according to the Hodge-BGW correspondence [17],

$$F_g(x, T) = (-4)^{g-1} H_g \left( t(x, T) \right), \quad g \geq 1,$$ (127)

and using the simple fact from [18] that $\partial_x = \partial_v$, we can translate the Dubrovin–Zhang type loop equation for $H_g(v_1, \ldots, v_{3g-2})$ to that for higher genus parts of the generalized BGW free energy. More precisely, we arrive at the following proposition.

**Proposition 5.** Let $u(x, T) := -2 \log Q(x, T)$. For any $g \geq 1$, the genus $g$ part of the generalized BGW free energy admits the jet representation: for $g = 1$,

$$F_1(x, T) = \frac{1}{24} \log \left( \frac{\partial u(x, T)}{\partial x} \right) - \frac{1}{16} u(x, T).$$ (128)

and for $g \geq 2$, there exists a function of $3g - 2$ variables $F_g(u_1, \ldots, u_{3g-2})$ such that

$$F_g(x, T) = F_g \left( \frac{\partial u(x, T)}{\partial x}, \ldots, \frac{\partial^{3g-2} u(x, T)}{\partial x^{3g-2}} \right).$$ (129)

Moreover, denote $\partial = \sum_{j \geq 0} u_{j+1} \frac{\partial}{\partial u_j}$, and let

$$B = \sqrt{1 - \frac{4 e^{a_0}}{\lambda}}, \quad \Delta F = \sum_{g \geq 1} h^{2g-2} F_g(u_1, \ldots, u_{3g-2}),$$ (130)
then $\Delta F$ satisfies the following Dubrovin–Zhang type loop equation:

\[
\begin{align*}
\sum_{k \geq 0} \left( \partial^k \left( \frac{1}{B^2} \right) + \sum_{j=1}^{k} \binom{k}{j} \partial^{j-1} \left( \frac{1}{B} \right) \partial^{k-j+1} \left( \frac{1}{B} \right) \right) \frac{\partial \Delta F}{\partial u_k}
+ 2 h^2 \sum_{k,l \geq 0} \partial^{k+1} \left( \frac{1}{B} \right) \partial^{l+1} \left( \frac{1}{B} \right) \left( \frac{\partial^2 F}{\partial u_k \partial u_l} + \frac{\partial \Delta F}{\partial u_k} \frac{\partial \Delta F}{\partial u_l} \right)
+ 2 h^2 \sum_{k \geq 0} \partial^{k+2} \left( \frac{1}{8 B^4} - \frac{1}{4 B^2} \right) \frac{\partial \Delta F}{\partial u_k}
+ \frac{1}{8 B^2} - \frac{1}{16 B^4} = 0. \tag{131}
\end{align*}
\]

We note that there are also other algorithms for computing $H_g$ (cf. [15, 18]), and therefore for computing $F_g = (-4)^{g-1} H_g$, $g \geq 2$.

Now we consider a third application of the Hodge-BGW correspondence. Recall that the Gromov–Witten free energy $F^{\text{WK}}(t; \epsilon)$ of a point is defined as the logarithm of the partition function $Z_{\text{WK}}(t; \epsilon)$ (cf. [23]), and it has the genus expansion:

\[
F^{\text{WK}}(t; \epsilon) = \sum_{g \geq 0} \epsilon^{2g-2} F_g^{\text{WK}}(t). \tag{132}
\]

It is known that for every $g \geq 1$, there exists a function of $(3g-2)$ variables $F_g^{\text{WK}}(z_1, \ldots, z_{3g-2})$ such that

\[
F_g^{\text{WK}}(t) = F_g^{\text{WK}} \left( \frac{\partial v(t)}{\partial t_0}, \ldots, \frac{\partial^{3g-2} v(t)}{\partial t_0^{3g-2}} \right), \tag{133}
\]

where $v(t) = \frac{\partial^2 F^{\text{WK}}(t)}{\partial t_0^2}$ (cf. [21, 30, 52]). For $g = 1$, the explicit expression [50] of $F_1^{\text{WK}}$ is given by

\[
F_1^{\text{WK}}(z_1) = \frac{1}{24} \log z_1. \tag{134}
\]

For every $g \geq 2$, $F_g^{\text{WK}}(z_1, \ldots, z_{3g-2}) \in \mathbb{Q}[z_2, \ldots, z_{3g-2}, z_1^{-1}]$. Dubrovin and Zhang [21] proved that for $g \geq 1$, the functions $F_g^{\text{WK}}(z_1, \ldots, z_{3g-2})$ are uniquely determined by

\[
\sum_{k=1}^{3g-2} k z_k \frac{\partial F_g^{\text{WK}}}{\partial z_k} = (2g - 2) F_g^{\text{WK}} + \frac{\delta_{g,1}}{24}, \quad g \geq 1 \tag{135}
\]

and the following loop equation [21]:

\[
\begin{align*}
- \sum_{k \geq 1} \left( \partial^k \left( \frac{1}{P^2} \right) + \sum_{j=1}^{k} \binom{k}{j} \partial^{j-1} \left( \frac{1}{P} \right) \partial^{k-j+1} \left( \frac{1}{P} \right) \right) \frac{\partial \Delta F^{\text{WK}}}{\partial z_k}
+ \frac{\epsilon^2}{2} \sum_{k,l \geq 0} \partial^{k+1} \left( \frac{1}{P} \right) \partial^{l+1} \left( \frac{1}{P} \right) \left( \frac{\partial^2 F^{\text{WK}}}{\partial z_k \partial z_l} + \frac{\partial \Delta F^{\text{WK}}}{\partial z_k} \frac{\partial \Delta F^{\text{WK}}}{\partial z_l} \right)
+ \frac{\epsilon^2}{16} \partial^{k+2} \left( \frac{1}{P^4} \right) \frac{\partial \Delta F^{\text{WK}}}{\partial z_k} + \frac{1}{16} \frac{1}{P^4} = 0. \tag{136}
\end{align*}
\]
Here \( \partial_1 = \sum_{j \geq 0} z_{j+1} \frac{\partial}{\partial z_j} \), \( P = \sqrt{\lambda - z_0} \), and
\[
\Delta F_{\text{WK}} = \sum_{g \geq 1} \epsilon^{2g-2} F_{\text{WK}}^g(z_1, \ldots, z_{3g-2}).
\]

The following proposition was conjectured by Okuyama and Sakai in [49], and now we give a proof as an application of Theorem 1.

**Proposition 6.** Let \( y = y(x, T) := Q(x, T)^2 = \frac{\partial^2 F_0(x, T)}{\partial T^2} \). For every \( g \geq 1 \), the genus \( g \) part of the generalized BGW free energy satisfy the identity: for \( g = 1 \),
\[
F_1(x, T) = \frac{1}{24} \log \left( \frac{\partial y(x, T)}{\partial T_1} \right) - \frac{\log 2}{24},
\]
and for \( g \geq 2 \),
\[
F_g(x, T) = F_{\text{WK}}^g \left( \frac{\partial y(x, T)}{\partial T_1}, \ldots, \frac{\partial^{3g-2} y(x, T)}{\partial T_1^{3g-2}} \right).
\]

**Proof.** On one hand, noticing (111) and Proposition 6, On the other hand, it follows immediately from (111) that
\[
u(x, T) = -\log(y(x, T)), \quad u_x = -\frac{y x}{y}, \quad y_x = -\frac{y T_1}{2 y^{1/2}}.
\]
For \( g = 1 \), the equality (138) then follows from (128) and (140). For \( g \geq 2 \), using (129), (126), and using (140) iteratively, we know that there exists a function \( \tilde{F}_g(z_0, z_1, \ldots, z_{3g-2}) \) satisfying (135) such that
\[
F_g(x, T) = \tilde{F}_g \left( y(x, T), \frac{\partial y(x, T)}{\partial T_1}, \ldots, \frac{\partial^{3g-2} y(x, T)}{\partial T_1^{3g-2}} \right).
\]

On the other hand, it follows immediately from (111) that
\[
\frac{\partial y}{\partial T_1} = \frac{y^a \partial y}{a! \partial T_1}, \quad a \geq 0.
\]

By using (115), (121), (142), as well as the Virasoro constraints (52), one can derive the loop equation (136) using the method in [21]. By the uniqueness of the solution to (135) and the Dubrovin-Zhang loop equation (136), we see \( \tilde{F}_g(z, z_1, \ldots, z_{3g-2}) = F_{\text{WK}}^g(z_1, \ldots, z_{3g-2}) \). This finishes the proof of (139), and the proposition is proved. \( \square \)

We note that one can also apply the theories of KdV tau-functions [21] (cf. also [10, 20]) to prove Proposition 6. For this purpose, the simple but important observation is the following: for \( g \geq 2 \), the power series
\[
F_{\text{WK}}^g \left( \frac{\partial y(x, T_1, 0)}{\partial T_1^g}, \ldots, \frac{\partial^{3g-2} y(x, T_1, 0)}{\partial T_1^{3g-2}} \right)
\]
does not depend on \( T_1 \), and so
\[
y(x, T_1, 0) + \sum_{g \geq 1} \hbar^{2g} \frac{\partial^2 F_{\text{WK}}^g \left( \frac{\partial y(x, T_1, 0)}{\partial T_1}, \ldots, \frac{\partial^{3g-2} y(x, T_1, 0)}{\partial T_1^{3g-2}} \right)}{\partial T_1 \partial T_1} = \frac{x^2}{4 (1 - T_1)^2} + \frac{\hbar^2}{8} \frac{1}{(1 - T_1)^2},
\]
which coincides with (48). Here, \( y(x, T_1, 0) = \frac{x^2}{4 (1 - T_1)^2} \).
We also remark that using \([138]–[139], [129], [125] \) and \([127]\) one finds that \(H_g, g \geq 1\), are related to \(F^\text{WK}_g\) under the substitution of the invertible map between the jet variables \(z_1, z_2, \ldots\) and \(u_1, u_2, \ldots\) induced by \([140]\); an equivalent relationship with this was obtained in a joint work by Don Zagier and the first-named author of the present paper by a different method.

Following \([30, 56]\) (cf. also \([27, 49, 52]\)), introduce the Itzykson–Zuber variables by

\[
I_1 = 1 - \frac{1}{z_1}, \quad I_{\ell + 1} = \frac{1}{z_1} \partial_1 (I_{\ell}) \quad (\ell \geq 1). \tag{144}
\]

Clearly, \(I_k \in \mathbb{Q}[z_2, \cdots, z_k, z_1^{-1}]\), \(k \geq 1\). For example, \(I_2 = z_2 z_1^{-3}\), \(I_3 = z_3 z_1^{-4} - 3 z_2^2 z_1^{-5}\). It follows from \([113], [138], [139], [144]\) that the following identities hold:

\[
\mathcal{F}_1(x, T) = \frac{1}{24} \log \left( \frac{1}{1 - I_1(x, T)} \right) - \frac{\log 2}{24}, \tag{145}
\]

\[
\mathcal{F}_g(x, T) = \frac{1}{(1 - I_1(x, T))^{2g - 2}} \mathcal{F}^\text{WK}_g \left( 0, 0, \frac{I_2(x, T)}{1 - I_1(x, T)}, \cdots, \frac{I_{3g - 2}(x, T)}{1 - I_1(x, T)} \right), \tag{146}
\]

where \(1/(1 - I_1(x, T)) = \partial_1 (y(x, T)) \in \mathbb{C}[[x + 2]][[T]],\) and

\[
I_k(x, T) = \delta_{k, 1} - x \frac{(-1)^k (2k - 1)!!}{2^{k+1} Q(x, T) 2^{k+1}} + \sum_{a \geq 0} \frac{Q(x, T) 2^{a}}{a!} T_{2a+2k+1} \in \mathbb{C}[[x + 2]][[T]], \quad k \geq 1. \tag{147}
\]

In particular, by taking \(T = 0\) in \((146)\), we obtain that

\[
\frac{x^{4g - 4}}{2^{2g - 2}} \mathcal{F}^\text{WK}_g \left( 0, 0, \frac{2^{13}!!}{x^2}, \frac{2^{25}!!}{x^4}, \cdots, \frac{(-1)^{3g - 2} 2^{3g - 3} (6g - 5)!!}{x^{6g - 6}} \right) = \frac{(-1)^g 2^{g - 1} B_{2g}}{2g (2g - 2) x^{2g - 2}}. \tag{148}
\]

Using this identity and applying the derivations similar to those for \((107)\) (cf. \([32, 39, 43]\), we obtain that

\[
\int_{\mathcal{N}_{g, 0}} e^{\sum_{d \geq 1} \bar{s}_d z^d} = \frac{(-1)^g B_{2g}}{2g (2g - 2)}, \quad g \geq 2, \tag{149}
\]

where \(\bar{s}_d, d \geq 1\), are numbers determined by

\[
\exp \left( - \sum_{d \geq 1} \bar{s}_d z^d \right) = \sum_{d \geq 0} (-1)^d (2d + 1)!! z^d. \tag{150}
\]

We note that the cases with \(g = 2, \ldots, 9\) of identity \((149)\) were proved by Kazarian and Norbury \([34]\); for arbitrary \(g \geq 2\), this identity was also expected in \([34]\) to hold, and is now proved in this paper.

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