FINITE-TIME CLUSTER SYNCHRONIZATION OF COUPLED DYNAMICAL SYSTEMS WITH IMPULSIVE EFFECTS

TIANHU YU
School of Mathematics, Southeast University
Nanjing 210096, China
Department of Mathematics, Luoyang Normal University
Luoyang 471934, China
JINDE CAO∗
School of Mathematics, Southeast University
Nanjing 210096, China
CHUANGXIA HUANG
Department of Applied Mathematics, Changsha University of Science and Technology
Changsha 410114, China

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Abstract. In our paper, the finite-time cluster synchronization problem is investigated for the coupled dynamical systems in networks. Based on impulsive differential equation theory and differential inequality method, two novel Lyapunov-based finite-time stability results are proposed and be used to obtain the finite-time cluster synchronization criteria for the coupled dynamical systems with synchronization and desynchronization impulsive effects, respectively. The settling time with respect to the average impulsive interval is estimated according to the sufficient synchronization conditions. It is illustrated that the introduced settling time is not only dependent on the initial conditions, but also dependent on the impulsive effects. Compared with the results without stabilizing impulses, the attractive domain of the finite-time stability can be enlarged by adding impulsive control input. Conversely, the smaller attractive domain can be obtained when the original system is subject to the destabilizing impulses. By using our criteria, the continuous feedback control can always be designed to finite-time stabilize the unstable impulsive system. Several existed results are extended and improved in the literature. Finally, typical numerical examples involving the large-scale complex network are outlined to exemplify the availability of the impulsive control and continuous feedback control, respectively.

1. Introduction. The asymptotical or exponential stability problem is an important issue of the dynamical investigations of the coupled dynamical systems [2,3,21,36,54,55]. One typical property of the asymptotical or exponential stability is of that the trajectory of the solution converges to the attractor state as time tends to infinite. In real world applications, it is desired that the trajectory of the
system can tend to a stable state in a finite time rather than merely exponentially or asymptotically, i.e., finite-time stability [6, 15]. It is worth noting that the definition of finite-time stability here is different from the corresponding definition introduced in [1], the finite-time stability definition here is of in the framework of Lyapunov stability, while the finite-time stability definition in [1] is called finite-time boundedness. Because of the capability of finite-time stability, the system may have fast transient and high-precise performances. Thus, the applications of finite-time stability have been studied widely, such as high-quality industrial robot control [13] and finite-time input-to-state stability of the close-loop control system [17]. Until now, several interesting works on finite-time stability have been published from both theoretical and practical perspectives [6, 9, 10, 12, 13, 15, 17, 20, 28, 34, 35, 38, 43, 52, 53, 60]. For example, a basic finite-time stability analysis tool based on Lyapunov-function is introduced in [6], which has been widely used in the literature. Besides the finite-time stability criteria for time-delay-free systems, several works on finite-time stability are reported [12, 28, 34] for the time-delay systems recently. In addition, for the sake of obtaining finite-time stability, several control methods have been introduced, such as backstepping, sliding mode [7, 15, 17, 39].

Impulsive effective can be utilized to describe the abrupt change phenomenon in practical fields. Since 1980s, several interesting and significant works have been published on impulsive differential equation theory and impulsive control theory. According to the above two theories, there are two kinds of basic problems about impulsive effects. On the one hand, the impulsive inputs may ruin the synchronization or stability of the original systems and these impulsive effects are called destabilizing or desynchronization impulses. On the other hand, the unstable or asynchronous system can obtain stability or realize synchronization by designing impulsive input, and this impulse is labeled stabilizing impulse or synchronization impulse. From the point of view of applications, both stabilizing impulse and destabilizing impulse are widely studied in various applications, such as neural networks [19], financial market model [50], mathematical biological systems [41] and secure communication [51], etc. Apart from those works, the impulsive effect can be used to study finite-time stability or finite-time synchronization problems [28, 35, 52]. For instance, the finite-time stability results are derived in the reference [28] for nonlinear impulsive system, and several sufficient conditions are provided for estimating the settling time with respect to the impulsive instants. In the reference [52], an impulsive differential comparison system is given to achieve the finite-time synchronization of complex dynamical networks.

Synchronization is one of the significant dynamical behaviors of complex dynamical systems. Meanwhile, the collective behavior of coupled individual systems receives continuous attention from the engineering researchers [8]. The complete synchronization problem has been widely studied and the investigation methods have been extended applied in variety of fields, such as formation flying [27], energy management in a smart grid [59], time synchronization in wireless sensor networks [16], etc. In recent, more attention has been attracted to the cluster synchronization problem which requires that individual systems belonging to the same cluster to realize the synchronization while different clusters can realize distinct synchronized goals. As be pointed out in [23, 24], there are two basic formation to achieve cluster synchronization, those are self-organization and driving. These formations lead to clusters with dominant intrachannel couplings and dominant intercluster couplings, respectively. Like complete synchronization, there are several applications involving
the cluster synchronization issue, such as segregation of a robotic team [26], IEEE 118-bus test [14] and predicting opinion dynamics in social networks [11]. In order to study cluster synchronization of coupled system, several theoretical method has been introduced in the latest decade [5, 11, 14, 26, 30, 31, 33, 37, 40, 58], including Lyapunov function method [33, 58], constructing vanishing auxiliary control system [31], computational group theory [40], graph-theoretical method [5, 33, 37], external equitable partition [14] and pinning control [30], etc. Recently, the finite-time cluster synchronization problem has been considered and many papers have been published [29, 30, 44, 45, 49]. Particularly, the finite-time stability problem of both delay-free and delay network system has been discussed in [29], and a two-phases-method (2PM) is proposed to study the adaptive finite-time synchronization of network systems. However, to our best knowledge, few study of the finite-time cluster synchronization criteria have been reported for a coupled dynamical systems in impulsive differential equation framework.

Based on the above-mentioned discussion, the finite-time cluster synchronization problem is considered for the coupled dynamical systems in our paper. The object is to develop the novel Lyapunov-based finite-time stability criteria for impulsive systems, to establish the finite-time cluster synchronization criteria for coupled impulsive dynamical systems in networks, and to design the control strategy to realize cluster synchronization. Some numerical examples are presented in order to illustrate that our criteria can be utilized to design the finite-time cluster synchronization controller for the coupled dynamical systems. The contributions here include the following:

(1) Novel Lyapunov-based finite-time stability criteria are proposed for the systems with stabilizing impulses and destabilizing impulses. Compare with the results in [38], our criteria show that the stabilizing impulses can enlarge the attractive domain of the original system, i.e. the impulsive control can switch the local finite-time stability to global finite-time stability. Conversely, the destabilizing impulses can reduce the original attractive domain of finite-time stability.

(2) Compare with the results in [28, 52], not only the impulsive control, but also the impulsive perturbation are considered in the finite-time cluster synchronization controller design. For impulsive perturbation input, the state feedback controller design criteria for global finite-time cluster synchronization are outlined according to Theorem 4.2. For impulsive control input, according to Theorem 4.1, our results point out that the synchronization impulsive effects can be used to replace the state feedback control term \(-ke(t)\) for the sake of designing simple finite-time cluster synchronization controller.

(3) In the synchronization investigation literature [5, 11, 14, 26, 30, 31, 33, 37, 40, 58], several interesting and significant results have been reported about either finite-time synchronization or impulsive cluster synchronization of coupled dynamical systems. Compare with those works, our work can be used to fill the gap in the area of the study of impulsive cluster synchronization in the finite-time stability framework. Meanwhile, the methods developed here can be adopted to investigate the finite-time cluster synchronization issue of general impulsive systems.

The rest of this paper is organized as follows. Several basic model description, definitions and useful lemma are shown in Section 2. The novel finite-time stability criteria for impulsive systems are proposed in Section 3. The finite-time cluster synchronization results for coupled dynamical systems are investigated involving
impulsive effects and feedback control in Section 4. Two numerical examples involving large-scale complex network and the final conclusion are given in Section 5 and Section 6, respectively.

2. Preliminary. In the follows, set \( \mathbb{R} \) and \( \mathbb{R}^n \) denote the set of real numbers and the \( n \)-dimensional real space, respectively. The notations \( K^T \) and \( \lambda_{\text{max}}(K) \) mean the transpose of the matrix \( K \) and the maximal eigenvalue of the symmetrical matrix \( K \). \( I \) means the unit matrix of appropriate dimension, and \( \otimes \) represents the Kronecker product. Let \( Q = \{ Q_1, \ldots, Q_q \} \) be a partition of the index set \( P = \{ 1, 2, \ldots, N \} \) into \( q \) nonempty subsets, i.e. \( Q_i \neq \emptyset \) and \( \bigcup_{i=1}^q Q_i = P \), where \( Q_1 = \{ 1, 2, \ldots, m_1 \}, Q_2 = \{ m_1 + 1, m_1 + 2, \ldots, m_1 + m_2 \}, \ldots, Q_q = \{ m_1 + \ldots + m_{q-1} + 1, \ldots, m_1 + \ldots + m_q \}, 1 < q < N, 1 \leq m_1 < N \) and \( \sum_{i=1}^q m_i = N \). The dynamic of the \( i \)-th node, \( i \in Q_k \), belonging to \( k \)-th cluster can be described as

\[
\dot{u}_i(t) = -B_k u_i(t) + C_k f_k(u_i(t)),
\]

(1)

where \( u_i(t) \in \mathbb{R}^n \) is the state vector. \( B_k u_i(t) \) represents the linear part of the dynamical system (1). \( f_k(u_i(t)) \) are the nonlinear functions. \( C_k = (c_{sj})_{n \times n} \) are the real constant matrices.

Consider the coupled dynamical systems in a network as follows

\[
\dot{u}_i(t) = -B_k u_i(t) + C_k f_k(u_i(t)) + \sum_{j=1}^N h_{ij} l_{ij} \Gamma u_j(t), \quad i \in Q_k,
\]

(2)

where \( i = r_{k-1} + 1, \ldots, r_k \), \( h_{ij} = h_k \) if \( i, j \in Q_k \), \( h_{ij} = 1 \) if \( i \in Q_k, j \in Q_k \), \( k \neq k' \). \( \Gamma = \text{diag} \{ \gamma_1, \ldots, \gamma_n \} \), \( \gamma_1, \ldots, \gamma_n > 0 \) represents the inner coupling matrix. \( L = (l_{sj})_{N \times N} \) is the coupling matrix in which \( l_{sj} > 0 \) if there exists an edge from node \( j \) to node \( s \), otherwise \( l_{sj} = 0 \). For the system (2), it requires that \( l_{ij} = l_{ji} \) and \( l_{ss} = -\sum_{j=1, j \neq s}^N l_{sj}, i, j, s = 1, \ldots, N \). According to the partition \( Q \), rewrite the coupling matrix as follows:

\[
L = \begin{pmatrix}
L_{11} & L_{12} & \cdots & L_{1q} \\
L_{21} & L_{22} & \cdots & L_{2q} \\
\vdots & \vdots & \ddots & \vdots \\
L_{q1} & L_{q2} & \cdots & L_{qq}
\end{pmatrix},
\]

where \( L_{ii}, i = 1, \ldots, q \) is the intra-cluster coupling and \( L_{ij}, i \neq j \) represents the inter-cluster coupling.

The target system for \( k \)-th cluster is given by

\[
\dot{s}_k(t) = -B_k s_k(t) + C_k f_k(s_k(t)),
\]

(3)

where \( s_k(t) \in \mathbb{R}^n \). The initial condition for the system (3) is given as \( s_k(t_0) = s_{k0} \).

Let \( e_i(t) = u_i(t) - s_k(t), i \in Q_k \), then

\[
\sum_{j=1}^n h_{ij} l_{ij} \Gamma e_j(t) = \sum_{j=1}^n h_{ij} l_{ij} \Gamma (e_j(t) + s(t)) = \sum_{j=1}^n h_{ij} l_{ij} \Gamma e_j(t) + \Gamma s(t) \sum_{j=1}^n h_{ij} l_{ij} = \sum_{j=1}^n h_{ij} l_{ij} \Gamma e_j(t).
\]

Therefore, the error system can be written as

\[
\dot{e}_i(t) = -B_k e_i(t) + C_k f_k(e_i(t)) + \sum_{j=1}^N h_{ij} l_{ij} \Gamma e_j(t),
\]

(4)
where $f_k(e_i(t)) = f_k(u_i(t)) - f_k(s_k(t))$, $i \in Q_k$.

**Assumption 1.** Suppose the nonlinear function $f_k(x)$ is continuous and differentiable on $\mathbb{R}^n$. For the function $f_k(x) = [f_k^1(x_1), \ldots, f_k^n(x_n)]$, there exist constants $l^k_j$, $j = 1, \ldots, n$ such that $\forall z_1 = [z_1^1, \ldots, z_1^n]^T$, $z_2 = [z_2^1, \ldots, z_2^n]^T \in \mathbb{R}^n$, there holds $|f_k^j(z_1^j) - f_k^j(z_2^j)| \leq l^k_j |z_1^j - z_2^j|$, $j = 1, \ldots, n$. Denote $\Pi_k = \text{diag}\{l^k_1, \ldots, l^k_n\}$.

Let $e(t) = (e_1^T(t), e_2^T(t), \ldots, e_n^T(t))^T$, the system (4) with impulsive effects

$$
\sum_{k=1}^{+\infty} Ke(t)\delta(t - t_k)
$$

can be rewritten as

$$
\begin{align*}
\dot{e}(t) &= -(B - \tilde{L} \otimes \Gamma)e(t) + Cf(e(t)), \quad t \neq t_k, \\
\Delta e(t) &= e(t) - e(t^-) = Ke(t^-), \quad t = t_k,
\end{align*}
$$

where $K$ is a matrix of appropriate dimension, $\delta(\cdot)$ is the standard Dirac function, $B = \text{diag}\{I_{m_1} \otimes B_1, \ldots, I_{m_q} \otimes B_q\}$, $C = \text{diag}\{I_{m_1} \otimes C_1, \ldots, I_{m_q} \otimes C_q\}$,

$$
f(e(t)) = (f_1(e_1(t))^T, \ldots, f_1(e_m(t))^T, \ldots, f_q(e_{m_1} + \ldots + m_{q-1} + 1)^T, \ldots, f_q(e_N)^T)^T,
$$

$$
\tilde{L} = \begin{pmatrix}
h_1 L_{11} & L_{12} & \cdots & L_{1q} \\
L_{21} & h_2 L_{22} & \cdots & L_{2q} \\
\vdots & \vdots & \ddots & \vdots \\
L_{q1} & L_{q2} & \cdots & h_q L_{qq}
\end{pmatrix}.
$$

**Definition 2.1.** ([32]) The positive scalar $T_a$ is said be the average impulsive interval of impulsive sequence $\zeta = \{t_1, t_2, \ldots\}$ if there exist positive integer $N_0$ such that $T - t - N_0 \leq N_\zeta(T, t) \leq T - t + N_0$, $0 \leq t \leq T$, where $N_\zeta(T, t)$ represents the number of impulsive instants in the time interval $(t, T)$.

**Definition 2.2.** The coupled dynamical system (2) with the partition $Q$ is said to achieve local finite-time cluster synchronization (LFTCS) over the impulsive sequence defined in Definition 2.1 if there exist an open set $U \subset \mathbb{R}^N$ and a function $T = T(x_0, t_0, \{t_k\}) < \infty$ such that from arbitrary initial values in $U$, $i, j \in Q_k$, $k = 1, 2, \ldots, q$,

$$
\lim_{t \to T} \|u_i(t) - u_j(t)\| = 0, \quad u_i(t) = u_j(t), \quad \forall t \geq T
$$

and $u_i(t) \neq u_j(t)$ if $i \in Q_{k_1}, j \in Q_{k_2}, k_1 \neq k_2$. If $U = \mathbb{R}^n$, it is said to be global finite-time cluster synchronization (GFTCS).

**Lemma 2.3.** ([25]) Let $\xi_1, \xi_2, \ldots, \xi_n \geq 0$, $0 < p \leq 1$, $q > 1$. Then

$$
\sum_{i=1}^{n} \xi_i^p \geq \left(\sum_{i=1}^{n} \xi_i\right)^p, \quad \sum_{i=1}^{n} \xi_i^q \geq n^{1-q} \left(\sum_{i=1}^{n} \xi_i\right)^q.
$$

**3. Finite-time stability of impulsive system.**

**Lemma 3.1.** Suppose $V(t)$ is a definite positive function defined on $\mathbb{R}$ and satisfies the following impulsive differential inequality

$$
\begin{align*}
\dot{V}(t) &\leq -l V^h(t) - k V(t), \quad t \neq t_v, \\
V(t_v) &\leq \chi_0 V(t_v^-)
\end{align*}
$$

(6)
with the positive constants $l$, $k$, $\chi_0$ and $h \in (0, 1)$, and $\{t_v\}$ is the impulsive instant sequence satisfying Definition 2.1, then $V(t) = 0$ for all $t \geq T$. In particular:

1. if $0 < \chi_0 \leq 1$, the settling time is

$$T = t_0 + \ln \left( 1 + \frac{V^{1-h}(t_0) \left( k - \frac{\ln \chi_0}{T_a} \right)}{k(1-h) - \frac{(1-h)\ln \chi_0}{T_a}} \right).$$

(7)

2. For $\chi_0 > 1$, there are three cases:

2.1) if $k > \frac{\ln \chi_0}{T_a}$, the settling time is

$$T = t_0 + \frac{\chi_0^{2(1-h)N_0}}{l(1-h)},$$

(9)

2.2) if $k = \frac{\ln \chi_0}{T_a}$, the settling time is

$$T = t_0 + \frac{\chi_0^{2(1-h)N_0}}{l(1-h)}.$$

(10)

2.3) if $k < \frac{\ln \chi_0}{T_a}$ and $V^{1-h}(t_0) < \frac{\chi_0^{2(1-h)N_0}}{l \chi_0 - k}$, the settling time is

$$T = t_0 + \frac{\ln \left( 1 + \frac{V^{1-h}(t_0) \left( \frac{\ln \chi_0}{T_a} - k \right)}{l \chi_0^{2(1-h)N_0}} \right)}{k(1-h) - \frac{(1-h)\ln \chi_0}{T_a}}.$$

(11)

Proof. Set $z(t) = V^{1-h}(t)$, $0 < 1 - h < 1$ and $\chi = \chi_0^{1-h}$. Clearly, $z(t) \geq 0$ for all $t \geq t_0$ and

$$\dot{z}(t) = (1-h)V^{-h}(t)\dot{V}(t) \leq -l(1-h) - k(1-h)V^{1-h}(t),$$

$$z(t_v) \leq \chi_0^{1-h}z(t_v) = \chi z(t_v).$$

(12)

By using the method in [56], one can obtain that

$$z(t) \leq e^{-(1-h)k(t-t_0)} \prod_{t_0 \leq t_v < t} \chi z(t_v) - l(1-h) \int_{t_0}^{t} e^{-(1-h)k(t-s)} \prod_{s \leq t_v < t} \chi z(t_v) ds.$$

(13)

In the sequel, the discussion will be divided to the cases $\chi_0 \geq 1$ and $0 < \chi_0 < 1$.

When $0 < \chi_0 < 1$, it is clear that $0 < \chi < 1$. Combine Definition 2.1 with Eq. (12), then

$$z(t) \leq e^{-(1-h)k(t-t_0)} \chi N,(t_0,t) z(t_0) - l(1-h) \int_{t_0}^{t} e^{-(1-h)k(t-s)} \chi N(s,t) ds.$$
Note that the proof for (13) to be zero because of \( kT_a - \ln \chi_0 > 0 \). As a consequence, there exists \( T > t_0 \) such that \( V(T) = 0 \). Take
\[
\chi_{-N_0}z(t_0) - \frac{l(1 - h)}{k(1 - h) - \frac{\ln \chi_0}{T_a}} \left( e^{(k(1-h) - \frac{\ln \chi_0}{T_a})(T-t_0)} - 1 \right) = 0
\]
which implies that
\[
T = t_0 + \frac{\ln \left( 1 + \frac{kV^{1-h}(t_0)}{l} \right)}{\chi_0^{2(1-h)N_0} - \left( e^{(k(1-h) - \frac{\ln \chi_0}{T_a})(T-t_0)} - 1 \right)}.
\]
Namely, \( 0 \leq V^{1-h}(T) = z(T) \leq 0 \). Therefore, \( V(T) = 0 \). Next, to prove \( V(t) \equiv 0 \) holds for \( t > T \). Otherwise, there must exist \( T_1 > T \) such that \( V(T_1) > 0 \). Take \( T_s = \sup \{ t \in [T, T_1] | V(t) = 0 \} \), then \( T_s < T_1 \), \( V(t) = 0 \) for \( t \in [T, T_s] \) and \( V(T_1) > 0 \) implies that \( \dot{z}(T_1) = (1 - h)V^{1-h}(T_1) \dot{V}(T_1) > 0 \). This contradicts the first inequality of Eq. (11). Consequently, \( V(t) \equiv 0 \) for \( t \geq T \).

When \( \chi_0 = 1 \), it follows from (12) that
\[
T = t_0 + \frac{\ln \left( 1 + \frac{kV^{1-h}(t_0)}{l} \right)}{k(1 - h)}.
\]
When \( \chi_0 > 1 \), it is clear that \( \chi > 1 \). Similarly, Eqs. (12) - (13) implies that
\[
z(t) \leq \chi^{-N_0}e^{-(k(1-h) - \frac{\ln \chi_0}{T_a})(t-t_0)} \left( \chi^{2N_0}z(t_0) - \frac{l(1 - h)}{k(1 - h) - \frac{\ln \chi_0}{T_a}} \left( e^{(k(1-h) - \frac{\ln \chi_0}{T_a})(t-t_0)} - 1 \right) \right).
\]
If \( k > \frac{\ln \chi_0}{T_a} \), then \( k(1 - h) > \frac{\ln \chi_0}{T_a} \). Take \( t = T \) such that the right side of Eq. (16) to be zero, then
\[
T = t_0 + \frac{\ln \left( 1 + \frac{kV^{1-h}(t_0)}{l} \right)}{k(1 - h) - \frac{\ln \chi_0}{T_a}}.
\]
Note that the proof for \( V(t) = 0 \), \( t \geq T \) in the case \( 0 < \chi_0 < 1 \) is just dependent on the first inequality in Eq. (11), the result in case (2.1) thus can be proved. As a consequence, the positive definite function \( V(t) \) satisfying Eq. (11) can reach zero at \( t = T \) and \( V(t) \equiv 0 \) for all \( t \geq T \).

If \( k = \frac{\ln \chi_0}{T_a} \), it is easy to obtain from Eq. (12) that
\[
z(t) \leq \chi^{-N_0}z(t_0) - l\chi^{-N_0}(1 - h)(t - t_0).
\]
Thus, the settling time is \( t_0 + \frac{\chi_0 2^{(1-h)N_0}}{l(1-h)} \).

If \( k < \frac{\ln \chi_0}{T_a} \) and \( V^{1-h}(t_0) < \frac{l \chi_0^{-2(1-h)N_0}}{\ln \chi_0 - k} \), by utilizing the proof of the case (2.1), the result holds.

**Lemma 3.2.** Suppose the function \( V(t) \) is positive definite function on \( \mathbb{R} \) and satisfies the following impulsive differential inequality

\[
\begin{cases}
\dot{V}(t) \leq -l V^{h}(t) + k V(t), & t \neq t_v, \\
V(t_v) \leq \chi_0 V(t_v) &
\end{cases}
\]

with the positive constant \( l, k, \chi_0 \) and \( h \in (0, 1) \), and \( \{t_v\} \) is the impulsive instant sequence satisfying Definition 2.1, then \( V(t) = 0 \) for \( t \geq T \). Particularly:

1. if \( 0 < \chi_0 < 1 \), there are three cases:
   (1.1) if \( k + \frac{\ln \chi_0}{T_a} < 0 \), the settling time is
   \[
   T = t_0 + \ln \left( \frac{1 - \frac{V^{1-h}(t_0)}{l \chi_0^{2(1-h)N_0}} \left( k + \frac{\ln \chi_0}{T_a} \right)}{(h-1)(k + \frac{\ln \chi_0}{T_a})} \right) \quad \text{if } V^{1-h}(t_0) < \frac{l}{k},
   \]
   and
   \[
   T = t_0 + \ln \left( \frac{e^{V^{1-h}(t_0)} \left( 1 - \frac{V^{1-h}(t_0)}{l \chi_0^{2(1-h)N_0}} \left( k + \frac{\ln \chi_0}{T_a} \right) \right)}{(h-1)(k + \frac{\ln \chi_0}{T_a})} \right) \quad \text{if } V^{1-h}(t_0) \geq \frac{l}{k},
   \]
   (1.2) if \( k + \frac{\ln \chi_0}{T_a} = 0 \), the settling time is
   \[
   T = t_0 + \frac{\chi_0^{-2(1-h)N_0} V^{1-h}(t_0)}{l(1-h)},
   \]
   (1.3) if \( k + \frac{\ln \chi_0}{T_a} > 0 \) and \( V^{1-h}(t_0) < \frac{l \chi_0^{-2(1-h)N_0}}{k + \frac{\ln \chi_0}{T_a}} \), the settling time is
   \[
   T = t_0 + \ln \left( \frac{1 - \frac{V^{1-h}(t_0)}{l \chi_0^{2(1-h)N_0}} \left( k + \frac{\ln \chi_0}{T_a} \right)}{k(1-h) + \frac{(1-h) \ln \chi_0}{T_a}} \right).
   \]

2. if \( \chi_0 \geq 1 \) and \( V^{1-h}(t_0) < \frac{l \chi_0^{-2(1-h)N_0}}{k + \frac{\ln \chi_0}{T_a}} \), the settling time is
   \[
   T = t_0 + \ln \left( \frac{1 - \frac{V^{1-h}(t_0)}{l \chi_0^{-2(1-h)N_0}} \left( k + \frac{\ln \chi_0}{T_a} \right)}{k(1-h) + \frac{(1-h) \ln \chi_0}{T_a}} \right). \quad (20)
   \]
Proof. Let \( z(t) = V^{1-h}(t) \), the system (18) is equivalent to the following system
\[
\begin{cases}
  \dot{z}(t) \leq -l(1-h) + k(1-h)z(t) & t \neq t_k, \\
  z(t_k) \leq \chi_0^{1-h}z(t^{-}_k) = \chi z(t^{-}_k).
\end{cases}
\] (21)

Similar to the discussion of Eq. (13), we have
\[
z(t) \leq \chi^{-N_0}e^{(\frac{\ln \chi}{T_a}k(1-h))(t-t_0)}\left(\frac{z(t_0)}{\chi^{2N_0}} + \frac{l}{k + \frac{\ln \chi_0}{T_a}}\left(e^{-((k(1-h) + \frac{\ln \chi}{T_a})(t-t_0)) - 1}\right)\right).
\] (22)

**Case 1.** \( 0 < \chi_0 < 1 \)

If \( k + \frac{\ln \chi_0}{T_a} = 0 \), then
\[
z(t) \leq \chi^{-N_0}z(t_0) - l(1-h)\chi^{N_0}(t - t_0).
\] (23)

Thus, \( T = t_0 + \frac{\chi_0^{-2(1-h)N_0}V^{1-h}(t_0)}{l(1-h)} \).

If \( k + \frac{\ln \chi_0}{T_a} < 0 \). Clearly, \( V(t_k) \leq \chi_0V(t^{-}_k) < V(t^{-}_k), \) \( k \geq 1 \). Note that
\[
\dot{V}(t) \leq -lV^{1-h}(t)\left(1 - \frac{k}{l}V^{1-h}(t)\right) < 0, \quad \forall x \neq 0.
\]

Since \( V(t) \) is positive definite and \( \dot{V} \) is negative if \( 0 < V(t) < \frac{l}{k} \), the set \( \Theta = \{V(t)|0 < V^{1-h}(t) < \frac{l}{k}\} \) is forward invariant. Therefore, there exist two situations:

1). For any solution \( V(t) \) satisfied \( V^{1-h}(t_0) < \frac{l}{k} \), then \( V(t) \in \Theta \) and \( \dot{V}(t) < 0, \forall t \neq t_0 \). Then, the solution of the system (18) can realize finite-time stability, and the settling time \( T \) can be obtained by using the system (21).

By using the proof of the case 1) in Lemma 3.1 and taking the right side of Eq. (22) be zero, the settling time is estimated by
\[
T = t_0 - \frac{\ln \left(1 - \frac{V^{1-h}(t_0)\left(k + \frac{\ln \chi_0}{T_a}\right)}{l\chi_0^{2(1-h)N_0}}\right)}{k(1-h) + \frac{(1-h)\ln \chi_0}{T_a}}.
\] (24)

2). For the solution \( V(t) \) started at \( V^{1-h}(t_0) \geq \frac{l}{k} \), the 2PM method used in [29] is utilized to determine the settling time for the finite-time stability of the system (18): (i) the solution \( V(t) \) approaches to \( \Theta \) in a finite time and (ii) the solution started at the set \( \Theta \) approaches 0 in a finite time. More precisely, if \( V(t) \) initialized at \( V^{1-h}(t_0) \geq \frac{l}{k} \) reaches the set \( \Theta \) in a finite time \( T_2 \) and the solution started at the set \( \Theta \) approaches 0 in a finite time \( T_1 \), then from any initial value \( V(t_0) \) satisfied \( V^{1-h}(t_0) \geq \frac{l}{k} \), \( V(t) \to 0 \) in a finite time \( T = T_1 + T_2 \). Next, we first consider the expression of \( T_2 \) and then give the settling time \( T_1 + T_2 \).

According to Eq. (22),
\[
z(t) \leq \chi^{-N_0}z(t_0)e^{(\frac{\ln \chi}{T_a}+k(1-h))(t-t_0)}
\] (25)
holds if \( kT_a + \ln \chi_0 < 0 \). Set the right side of Eq. (25) to be \( \frac{l}{k} \), then the expression of \( T_1 \) can be written as

\[
T_1 = t_0 + \frac{\ln \frac{kV_{1-h}(t_0)}{l\chi_0}}{(h-1)(k + \frac{\ln \chi_0}{T_a})}
\]  

which implies that \( V^{1-h}(t) < \frac{l}{k} \) for all \( t > T_1 \). Combine Eq. (26) with Eq. (24), the settling time for the solution \( V(t) \) with the initial value \( V_1 - h(t_0) \geq \frac{l}{k} \) can be expressed as

\[
T = t_0 + \frac{\ln \frac{kV_{1-h}(t_0)}{l\chi_0} + \ln \left(1 - \frac{V^{1-h}(t_0)\left(k + \frac{\ln \chi_0}{T_a}\right)}{l\chi_0^{2(1-h)N_0}}\right)}{(h-1)(k + \frac{\ln \chi_0}{T_a})}.
\]  

If \( k + \frac{\ln \chi_0}{T_a} > 0 \) and \( z(t_0) < \frac{\chi_0^{2(1-h)N_0}l}{k + \frac{\ln \chi_0}{T_a}} \), the settling time \( T \) has the same formation in Eq. (24).

**Case 2.** \( \chi_0 = 1 \).

In this case, it is obvious that \( k + \frac{\ln \chi_0}{T_a} > 0 \). According to Eq. (12) with \( -k > 0 \), we have

\[
z(t) \leq e^{k(1-h)(t-t_0)} \left(z(t_0) + \frac{l}{k} \left(e^{-k(1-h)(t-t_0)} - 1\right)\right).
\]  

Clearly, the system (18) with \( \chi_0 = 1 \) can realize finite-time stability if \( V^{1-h}(t_0) < \frac{l}{k} \) and the settling time is

\[
T = t_0 + \frac{\ln \left(1 - \frac{kV^{1-h}(t_0)}{l}\right)}{k(1-h)}.
\]

**Case 3.** \( \chi_0 > 1 \).

Obviously, \( k + \frac{\ln \chi_0}{T_a} > 0, \chi > 1 \) and

\[
z(t) \leq e^{k(1-h)+\frac{\ln \chi_0}{T_a})(t-t_0)} \left(\chi^{N_0}z(t_0) + \frac{l}{\chi_0^{2(1-h)N_0}} \left(e^{-k(1-h)\frac{\ln \chi_0}{T_a}} - 1\right)\right).
\]  

Next, we prove that the system (18) can realize the finite-time stability if \( z(t_0) = \frac{\chi_0^{-2(1-h)N_0}}{k + \frac{\ln \chi_0}{T_a}} \) and the settling time is

\[
T = t_0 + \frac{\ln \left(1 - \frac{V^{1-h}(t_0)\left(k + \frac{\ln \chi_0}{T_a}\right)}{l\chi_0^{-2(1-h)N_0}}\right)}{k(1-h) + \frac{\ln \chi_0}{T_a}}.
\]
Clearly, $V^{1-h}(t_0) < \frac{\lambda_0^{-2(1-h)N_0}}{k + \ln x_0}$ means that $z(t) \leq \frac{\lambda_0^{-N_0}}{k + \ln x_0}$ in Eq. (29). Substituting Eq. (30) into Eq. (29), then $z(T) \leq 0$. Thus, $z(T) = 0$ by means of $z(t) \geq 0, \forall t \geq t_0$. If there exists $T_1 > T$ such that $z(T_1) > 0$, take $T^* = \sup \{t \in [T, T_1]|z(t) = 0\}$. Then $z(T^*) = 0$, $z(t) > 0$ for all $t \in (T^*, T_1]$ and $\dot{z}(T^*) > 0$. However, $z(t) < \frac{\lambda_0^{-2(1-h)N_0}}{k + \ln x_0}$ implies that

$$
\dot{z}(T^*) \leq -l(1-h) + k(1-h)z(T^*)
\leq -l(1-h) + \frac{k(1-h)\lambda_0^{-N_0}}{k + \ln x_0}
< l(1-h)\left(-1 + \frac{k}{k + \ln x_0}\right) < 0.
$$

It is a contraction. Thus, $z(t) \equiv 0, \forall t \geq T$. □

**Corollary 1.** [6, 18, 22] Suppose the positive definite function $V(t)$ is continuous and satisfies

$$
\dot{V}(t) \leq -lV^h(t), \quad t \geq t_0, \quad V(t_0) \geq 0,
$$

with the positive constants $l$ and $h \in (0, 1)$, then

$$
V^{1-h}(t) \leq V^{1-h}(t_0) - l(1-h)(t-t_0), \quad t_0 \leq t \leq t^*,
$$

and $V(t) = 0$ for $t \geq t^* = t_0 + \frac{V^{1-h}(t_0)}{l(1-h)}$.

**Proof.** Consider $V(t_s) = V(t^-)$, $s \geq 1$ and $k = 0$ in Eq. (18), then the system (18) is equivalent to the system (32). According to Eq. (12), the following inequality holds,

$$
z(t) \leq z(t_0) - l(1-h)\int_{t_0}^{t} ds = z(t_0) - l(1-h)(t-t_0), \quad l > 0, \quad 0 < h < 1.
$$

Therefore, $V^{1-h}(t) \leq V^{1-h}(t_0) - l(1-h)(t-t_0)$ for $t_0 < t < t^*$ and $V(t) \equiv 0$ holds for all $t \geq t^* = t_0 + \frac{V^{1-h}(t_0)}{l(1-h)}$. □

**Corollary 2.** ([38]) Assume that a positive definite function $V(x)$ defined on $\mathbb{R}^n$ satisfies

$$
\dot{V}(x) \leq -lV^h(x) + kV(x), \quad l, k > 0, \quad 0 < h < 1.
$$

Then, $V(x) = 0$ for all $t \geq T(x_0)$ and the settling time $T(x_0)$ is estimated by

$$
T(x_0) = \frac{\ln \left(1 - \frac{l}{k}V^{1-h}(x_0)\right)}{l(1-h)}
$$

for all $x_0 \in \{y \in \mathbb{R}^n|V^{1-h}(y) < \frac{1}{k}\}$.

**Proof.** The system (33) can be considered as

$$
\begin{align*}
\dot{V}(\psi(t,x)) &\leq -lV^h(\psi(t,x)) + kV(\psi(t,x)), \quad t \neq t_k, \\
V(\psi(t_k,x)) &= V(\psi(t_k,x)),
\end{align*}
$$

where $l, k > 0, 0 < h < 1$ and $\psi(t,x)$ is the unique solution of Eq. (33). Using the proof of the case 2) in Lemma 3.2, the result holds. □
Corollary 3. (52) Let $V(t)$ be a Lyapunov function satisfied

$$
\begin{align*}
\dot{V}(t) &= \begin{cases} -l, & V(t) > 0, \quad t \neq t_k, \\ 0, & V(t) = 0, \quad t \neq t_k, \\ V(t_k) = \chi_0^2 V(t_k), & t = t_k, \quad k \in \mathbb{N}_+ \end{cases} \quad (34)
\end{align*}
$$

Then the zero equilibrium of the system $(34)$ can realize the finite-time stability. In particular:

1. If $0 < \chi_0 < 1$, the settling time is

$$
T = t_0 + \frac{T_a}{\ln \chi_0} \ln \left( \frac{lT_a}{lT_a - \chi_0^{-2N_0}V(t_0)\ln \chi_0} \right).
$$

2. If $\chi_0 = 1$, the settling time is

$$
T = t_0 + \frac{V(t_0)}{l}.
$$

3. If $\chi_0 > 1$ and $T_a > \frac{V(t_0)\chi_0^{2N_0}\ln \chi_0}{l}$, the settling time is

$$
T = t_0 + \frac{T_a}{\ln \chi_0} \ln \left( \frac{lT_a}{lT_a - \chi_0^{-2N_0}V(t_0)\ln \chi_0} \right).
$$

Proof. When $0 < \chi_0 < 1$, Eq. (12) holds for $h = k = 0$ which implies that

$$
V(t) \leq \prod_{t_0 \leq t_k \leq t} \chi_0 z(t_k) - l \int_{t_0}^{t} \prod_{s \leq t_k \leq t} \chi_0 \, ds 
\leq \chi_0^{N_0} e^{\ln \chi_0 (t-t_0)} \left( \chi_0^{-2N_0} V(t_0) + \frac{lT_a}{\ln \chi_0} \left( e^{\ln \chi_0 (t-t_0)} - 1 \right) \right). \quad (35)
$$

By using the proof in [52], $T = t_0 + \frac{T_a}{\ln \chi_0} \ln \left( \frac{lT_a}{lT_a - \chi_0^{-2N_0}V(t_0)\ln \chi_0} \right)$.

When $\chi_0 = 1$, $h = k = 0$, Eq. (12) degenerates to

$$
V(t) \leq V(t_0) - l \int_{t_0}^{t} \, ds = V(t_0) - l(t-t_0). \quad (36)
$$

Then it is easy to obtain that $T = t_0 + \frac{V(t_0)}{l}$. \hfill \Box

When $\chi_0 > 1$, $h = k = 0$, it is clear from Eq. (16) that

$$
V(t) \leq \chi_0^{\frac{t-t_0}{T_a} - N_0} \left( \chi_0^{2N_0} V(t_0) + \frac{l}{\ln \chi_0} \left( e^{\ln \chi_0 (t-t_0)} - 1 \right) \right) 
= \chi_0^{-N_0} e^{\ln \chi_0 (t-t_0)} \left( \chi_0^{2N_0} V(t_0) + \frac{T_a l}{\ln \chi_0} \left( e^{\ln \chi_0 (t-t_0)} - 1 \right) \right). \quad (37)
$$

The settling time is thus estimated as

$$
T = t_0 + \frac{T_a}{\ln \chi_0} \ln \left( \frac{lT_a}{lT_a - \chi_0^{-2N_0}V(t_0)\ln \chi_0} \right).
$$

Remark 1. The result for $\chi_0 > 1$ in Lemma 3.1 implies that one can use feedback control $u(t)$ to finite-time stabilize the following impulsive perturbation system

$$
\begin{align*}
\dot{V}(x(t)) &= -lV^2(x(t)) + u(t), \quad t \neq t_v, \\
V(x(t_v)) &= \chi_0 V(x(t_v)), \quad \chi_0 > 1,
\end{align*}
$$

(38)
where \( V(t) \geq 0, \forall t \geq 0 \) and \( u(t) = -kV(x(t)) \). Specifically, the system (38) can realize the global and local finite-time stability if the feedback control gain \( k \) satisfies
\[
 k > \frac{\ln \chi_0}{\bar{T}_a} \quad \text{and} \quad k < \frac{\ln \chi_0}{\bar{T}_a},
\]
respectively.

**Remark 2.** In the references [6, 18, 22], the settling time for the finite-time stability is provided as \( t_0 + \frac{V^{1-h}(t_0)}{l(1-h)} \). Obviously, for \( k \gg 1 \),
\[
 T(k) = t_0 + \frac{\ln(1 + \frac{k}{T} V^{1-h}(x_0))}{k(1-h)} \ll t_0 + \frac{\ln(1 + \frac{1}{T} V^{1-h}(x_0))}{(1-h)} < t_0 + \frac{V^{1-h}(x_0)}{l(1-h)}.
\]
Thus, our criterion can be used to obtain the finite-time stability with fast convergence.

**Remark 3.** Compare our result in Lemma 3.2 with the result in [38], the proper impulsive control input can enlarge the original attraction domain of the finite-time stability. Precisely, if the impulsive control satisfies the conditions in the cases (1.1) and (1.2) in Lemma 3.2, the local finite-time stability in [38] can switch to the global finite-time stability. Similarly, according to Lemma 3.1, our results can guarantee the global finite-time stability of the comparison system if the impulsive perturbations satisfy conditions in the cases (2.1) and (2.2). While the criteria derived by the comparison system in [52] can declare the local finite-time stability, rather than the global finite-time stability.

4. Finite-time cluster synchronization. Set \( \chi_0 = \lambda_{\text{max}}((I + K)^T(I + K)) \). According to the value of \( \chi_0 \), the finite-time cluster synchronization problem of the coupled dynamical system (2) are discussed for \( \chi_0 \geq 1 \) and \( 0 < \chi_0 < 1 \), respectively. Next, the impulses satisfying \( \chi_0 \geq 1 \) is called the desynchronization impulsive effects, otherwise the impulses is called synchronization impulsive effects.

4.1. Finite-time synchronization for synchronization impulsive effects. In this subsection, consider the finite-time cluster synchronization controller \( R_1(t) = (R_{i1}^T(t), R_{i2}^T(t), \ldots, R_{iN1}^T(t))^T \) with
\[
 R_{i1}(t) = -\beta_1 \text{sign}(e_i(t))|e_i(t)|^h, \tag{39}
\]
where \( \beta_1 > 0, \ 0 \leq h < 1 \) and
\[
 \text{sign}(e_i(t))|e_i(t)|^h = (\text{sign}(e_{i1}(t))|e_{i1}(t)|^h, \ldots, \text{sign}(e_{in}(t))|e_{in}(t)|^h)^T.
\]

**Theorem 4.1.** Suppose that \( 0 < \chi_0 < 1 \) and the average impulsive interval is \( T_a \).
If the following inequality holds:
\[
 k + \frac{\ln \chi_0}{\bar{T}_a} \leq 0, \tag{40}
\]
where \( k = \lambda_{\text{max}} \left( -(B - \hat{L} \otimes \Gamma) - (B - \hat{L} \otimes \Gamma)^T + C^T C + \Pi^T \Pi \right) \). Then the origin of the error system (5) with controller \( R_1(t) \) is global finite-time stability. Namely, the coupled dynamical systems (2) with synchronization impulsive effects and controller \( R_1(t) \) can achieve GFTCS under the partition \( \{Q_1, Q_2, \ldots, Q_q\} \). In particular:
Consider the Lyapunov function as

\[ V(t) = t_0 + \frac{2\ln \left( 1 - \frac{V \frac{1-h}{T_a} (t_0) \left( k + \frac{\ln \chi_0}{T_a} \right)}{2\chi_0(1-h)N_0\beta_1} \right)}{(h-1)(k + \frac{\ln \chi_0}{T_a})}, \]

for \( V\frac{1-h}{T_a} (t_0) < \frac{2\beta_1}{k} \), the settling time is

\[ T = t_0 + \frac{2\ln \left( 1 - \frac{V \frac{1-h}{T_a} (t_0) \left( k + \frac{\ln \chi_0}{T_a} \right)}{2\beta_1\chi_0(1-h)N_0} \right)}{(h-1)(k + \frac{\ln \chi_0}{T_a})}, \]

for \( V\frac{1-h}{T_a} (t_0) \geq \frac{2\beta_1}{k} \), the settling time is

\[ T = t_0 + \frac{\frac{\chi_0}{\beta_1(1-h)}V \frac{1-h}{T_a} (t_0)}{\ln N_0}, \]

where \( V(t_0) = e^T(t_0)e(t_0) \).

Proof. Consider the Lyapunov function as \( V(t) = e^T(t)e(t) \). For \( t \neq t_k \), it follows from Eq. (5) and Eq. (39) that

\[ \dot{V}(t) = \begin{align*} 2 \dot{e}^T(t) \dot{e}(t) &= 2e^T(t) \left( -(B - \bar{L} \otimes \Gamma)e(t) + Cf(e(t)) + R_1(t) \right) \end{align*} \]

\[ = e^T(t)(-(B - \bar{L} \otimes \Gamma) - (B - \bar{L} \otimes \Gamma)^T)e(t) + 2\dot{e}^T(t)Cf(e(t)) + 2e^T(t)R_1(t). \]

Based on Assumption 1, we obtain

\[ 2e^T(t)C_kf_k(e_i(t)) \leq e^T(t)C_k^T C_k e_i(t) + f_k^T(e_i(t))f_k(e_i(t)) \leq e^T(t)C_k^T C_k e_i(t) + e^T(t)\Pi_k^T \Pi_k e_i(t) \]

which implies that

\[ 2e^T(t)Cf(e(t)) \leq e^T(t)C^T Ce(t) + e^T(t)\Pi^T \Pi e(t). \]

Substituting Eq. (42) and Eq. (43) into Eq. (41), then

\[ \dot{V}(t) \leq e^T(t) \left( -(B - \bar{L} \otimes \Gamma) - (B - \bar{L} \otimes \Gamma)^T + C^T C + \Pi^T \Pi \right) e(t) \]

\[ -2\beta_1 e^T(t)R_1(t) \leq -2\beta_1 e^T(t)R_1(t) + kV(t). \]

As a consequence,

\[ \dot{V}(t) \leq -2\beta_1 e^T(t)\text{sign}(e(t))|e(t)|^h + kV(t). \]
Recall Lemma 2.3 and $0 \leq h < 1$, the following inequality holds:

$$-2\beta_1 e^T(t) \text{sign}(e(t)) e(t) \leq -2\beta_1 |e^T(t)||e(t)| = -2\beta_1 \sum_{i=1}^{n} \sum_{j=1}^{N} (e^2_{ij})^{\frac{1+h}{h}}$$

$$\leq -2\beta_1 \sum_{i=1}^{N} \sum_{j=1}^{n} (e^2_{ij})^{\frac{1+h}{h}} \leq -2\beta_1 \left( \sum_{i=1}^{N} \sum_{j=1}^{n} e^2_{ij} \right)^{\frac{1+h}{h}}.$$  

(46)

Combine Eq. (45)-Eq. (46) with Lemma 2.3, thus

$$\dot{V}(t) \leq -2\beta_1 V^{\frac{1+h}{h}}(t) + kV(t), \quad 0 < \frac{1+h}{2} < 1. \quad (47)$$

For $t = t_v$, we have

$$V(t_k) = e^T(t_v)e(t_v) = e^T(t^-_v)(I + K)^T(I + K)e(t^-_v) \leq \lambda_{\text{max}}((I + K)^T(I + K))e(t^-_v) = \chi_0 V(t^-_v). \quad (48)$$

Based on Eq. (47) - Eq. (48), the following comparison system is considered

$$\begin{cases}
\dot{\eta}(t) = -2\beta_1 \eta^{\frac{1+h}{h}}(t) + k\eta(t), \quad t \neq t_v, \\
\eta(t_k) = \chi_0 \eta(t^-_v), \\
\eta(t_0) = e^T(t_0)e(t_0) = V_0 > 0.
\end{cases} \quad (49)$$

Combine Eq. (47) - Eq. (48) with Eq. (49), we have $0 \leq V(t) \leq \eta(t)$. Thus, the Lyapunov function $V(t)$ can reach origin in a finite time $T > t_0$ if $\eta(t) \equiv 0$ for all $t > T$. Therefore, the finite-time cluster synchronization problem of the coupled dynamical system (2) equals to the finite-time stability problem of the comparison system (49). By using Lemma 3.2 and Eq. (40), $V(t) \to 0$ in a finite time, and the system (2) with synchronization impulsive effects and $R_1(t)$ can realize GFTCS. At the same time, the estimated settling time can be obtained according to Lemma 3.2. \hfill \square

Remark 4. Different from the controller designed in [44, 52], the control term $-y(t)$, $I \geq 0$ is not involved in the controller $R_1(t)$ if Eq. (40) holds for $0 < \chi_0 < 1$. It implies that the impulsive control, rather than the continue state feedback controller $-y(t)$, can be used to obtain the finite-time stability of the error system (5) as long as the parameters $\chi_0$ and $T_a$ are designed properly. Therefore, our controller designed for Theorem 4.1 is simpler than the ones in [44, 52]. In order to show the effectiveness of the controller here, consider the finite-time stabilization of one-order differential system $\dot{y}(t) = y(t) + u(t), \quad y(0) = 1$. Design the controllers as $u_1(t) = -y(t) - y(t) \frac{1}{2}$ and $u_2(t) = -\sum_{k=1}^{N_0} y(t) \delta(t - t_k) - y(t), \quad t_k = 0.2k$ ($T_a = 0.2, N_0 = 1$) is the impulse sequence and $\delta(\cdot)$ is the Dirac impulse function, one can obtain the settling times for the controllers $u_1(t)$ and $u_2(t)$ are 1.5 and 1.579 (using the Lemma 3.2 with $V(y(t)) = \frac{1}{2} y^2(t)$), respectively. Obviously, the continuous feedback control term $-y(t)$ in the $u_1(t)$ is replaced by the equidistant impulsive control $-\sum_{k=1}^{N_0} \frac{1}{2} y(t) \delta(t - t_k), \quad t_k = 0.2k$ in the controller $u_2(t)$ which can be used to realize the finite-time stability effectively.
Remark 5. If the impulsive control satisfies \( k + \frac{\ln \chi_0}{T_a} > 0 \), the coupled dynamical systems can achieve LFTCS according to Lemma 3.2. The domain of attraction and the settling time are

\[
V^{\frac{1-h}{2}}(t_0) < \frac{1}{k} \frac{\chi_0^{(1-h)N_0}}{1 + \frac{\ln \chi_0}{kT_a}} \quad \text{and} \quad T = t_0 + \frac{2 \ln \left(1 - \frac{V^{\frac{1-h}{2}}(t_0) \left(k + \frac{\ln \chi_0}{T_a}\right)}{\chi_0^{(1-h)N_0}}\right)}{(1-h)(k + \frac{\ln \chi_0}{T_a})}
\]

respectively. In addition, one can check that \( \chi_0^{(1-h)N_0} > 1 + \frac{\ln \chi_0}{kT_a} \) holds if \( k + \frac{\ln \chi_0}{T_a} > 0 \) which yields that the impulsive control input can extend the attraction domain of LFTCS obtained in [38].

Remark 6. According to Remark 5 and Lemma 3.2, in order to obtain GFTCS of the coupled dynamical systems (2) with \( k + \frac{\ln \chi_0}{T_a} > 0 \), the state feedback term \(-le(t)\) must be considered in the controller (39) i.e. \( R'_1(t) = -le(t) - \beta_1 \text{sign}(e_i(t))|e_i(t)|^h\).

By using Eq. (44), the system (5) with controller \( R'_1(t) \) can realize GFTCS if \(-2l + k + \frac{\ln \chi_0}{T_a} \leq 0\). Thus, the sign of the term \( k + \frac{\ln \chi_0}{T_a} \) can be viewed as an index of the controller design for GFTCS of the system (2).

Remark 7. In the reference [42], the trade-off issue between control energy and time has been considered in the finite-time stability framework. Similarly, the index [42] of control energy for the controller Eq. (39) introduced in Theorem 4.1 is proposed as

\[
\mathcal{E}_1^C = \int_0^{T_f} ||R_1(t)||_2^2 dt,
\]

where \( ||\cdot||_2 \) denotes 2-norm and \( T_f \) represents the settling time. According to Eqs. (47) - (48) and Lemma S1.2 in [46],

\[
||R_1(t)||_2^2 = \sum_{i=1}^{N} \sum_{j=1}^{n} |e_{ij}(t)|^{2h} = \beta_1^2 ||e(t)||_2^{2h}
\]

\[
\leq (nN)^{1-h} \beta_1^2 ||e(t)||_2^{2h} = (nN)^{1-h} \beta_1^2 V(t)^{2h}.
\]

If \( k + \frac{\ln \chi_0}{T_a} = 0 \), by Eq. (23), then,

\[
\mathcal{E}_1^C \leq (nN)^{1-h} \beta_1^2 \int_0^{T_f} V(t)^{2h} dt
\]

\[
\leq (nN)^{1-h} \beta_1^2 \frac{1}{3h} \int_0^{T_f} (\gamma_1 - \gamma_2 t)^{\frac{4h}{1+3h}} dt
\]

\[
= \frac{(nN)^{1-h} \beta_1^2}{1+3h} (\gamma_1)^{\frac{4h}{1+3h}},
\]

where \( \gamma_1 = \chi_0^{\frac{(1-h)N_0}{2}} V^{\frac{1-h}{2}}(0) > 0 \), \( \gamma_2 = \beta_1 (1-h) \chi_0^{\frac{(1-h)N_0}{2}} > 0 \). Thus, the upper bound of control energy is estimated as \( \mathcal{E}_1^{sup} = \frac{(nN)^{1-h} \beta_1^2}{1+3h} (\gamma_1)^{\frac{4h}{1+3h}} \).
Substituting the estimation of $T$ Combine Eq. (22), Eq. (47) with Eq. (48), one can obtain that

$$V(t) \leq (\gamma_1 - \beta_1(1-h)t)^{\frac{a}{4h}}.$$  

Therefore,

$$\delta_1^{\sup} \leq (nN)^{1-h}\beta_1^2\int_{T_f}^{T_f} (\gamma_1 - \beta_1(1-h)t)^{\frac{4h}{a}} \, dt = \frac{\beta_1(nN)^{1-h}}{1+3h}(\gamma_1^{\frac{4h}{a}} - (\gamma_1 - \beta_1(1-h)T_f)^{\frac{4h}{a}}) \leq \delta_1^{\sup}.$$  

Substituting the estimation of $T_f$ in Theorem 4.1 into Eq. (53), the upper bound of control energy $\delta_1^{\sup}$ can be obtained.

### 4.2. Finite-time synchronization for desynchronization impulsive effects.

In order to overcome the desynchronization impulses, it is desirable to use the state feedback control to stabilize the system (5). Based on the controller designed in the previous subsection, consider the following finite-time cluster synchronization feedback control to stabilize the system (5). Based on the controller designed in the previous subsection, consider the following finite-time cluster synchronization controller $R_2(t) = (R_1^T(t), R_2^T(t), \ldots, R_{qN_2}^T(t))^T$ with

$$R_{i2}(t) = -k_1e_i(t) - \beta_2\text{sign}(e_i(t))|e_i(t)|^h,$$

where $k_1 > 0$ is the state feedback control gain, $\beta_2 > 0$, $0 \leq h < 1$ and

$$\text{sign}(e_i(t))|e_i(t)|^h = (\text{sign}(e_{i1}(t))|e_{i1}(t)|^h, \ldots, \text{sign}(e_{in}(t))|e_{in}(t)|^h)^T.$$

**Theorem 4.2.** Suppose that $\chi_0 \geq 1$ and the average impulsive interval is $T_a$. If there exists nonnegative constant $k_1$ such that

$$\tilde{k} - \frac{\ln \chi_0}{T_a} \geq 0,$$

where $\tilde{k} = 2k_1 - \lambda_{\text{max}} \left( (B - L \otimes \Gamma) - (B - \bar{L} \otimes \Gamma)^T + C^TC + \Pi^T\Pi \right)$. Then the origin of the error system (5) with controller $R_2(t)$ is global finite-time stability. Namely, the coupled dynamical systems (2) with desynchronization impulses and controller $R_2(t)$ can achieve GFTCS under the partition $\{Q_1, Q_2, \ldots, Q_q\}$. In particular:

1. if $\tilde{k} > \frac{\ln \chi_0}{T_a}$, the settling time is

$$T = t_0 + \frac{2\ln \left( 1 + \frac{V_{1-h}(t_0) \left( \tilde{k} - \frac{\ln \chi_0}{T_a} \right)}{2\beta_2\chi_0^{(1-h)N_0}} \right)}{(1-h)(\tilde{k} - \frac{\ln \chi_0}{T_a})},$$

2. if $\tilde{k} = \frac{\ln \chi_0}{T_a}$, the settling time is

$$T = t_0 + \frac{2\chi_0^{(1-h)N_0}V_{1-h}^{-1}(t_0)}{2\beta_2(1-h)},$$

where $V(t_0) = e^T(t_0)e(t_0)$. 

Proof. Set $V(t) = e^T(t)e(t)$. According to Eqs. (41)-(43), it is clear that

$$
\dot{V}(t) \leq e^T(t) \left((B - \tilde{L} \otimes \Gamma) - (B - \tilde{L} \otimes \Gamma)^T + C^T C + \Pi^T \Pi\right) e(t)
- 2k_1 e^T(t)e(t) - 2\beta_2 e^T(t)\text{sign}(e_i(t))|e_i(t)|^h
\leq -2\beta_2 V^{\frac{1+h}{2}}(t) - (2k_1 - k)V(t).
$$

By using Eq. (47), we have

$$
\begin{cases}
\dot{V}(t) \leq -2\beta_2 V^{\frac{1+h}{2}}(t) - (2k_1 - k)V(t), & t \neq t_v, \\
V(t_v) \leq \chi_0 V(t^-), & \chi_0 \geq 1.
\end{cases}
$$

According to Lemma 3.1 and Eq. (55), the system (2) with desynchronization impulsive effects and the controller (54) can realize GFTCS.

Remark 8. In Theorem 4.2, the role of the state feedback control gain $k_1$ is to stabilize the matrices $(B - \tilde{L} \otimes \Gamma) - (B - \tilde{L} \otimes \Gamma)^T + C^T C + \Pi^T \Pi$, namely, to make the maximal eigenvalues of the matrices $(B - \tilde{L} \otimes \Gamma) - (B - \tilde{L} \otimes \Gamma)^T + C^T C + \Pi^T \Pi - 2k_1I$ is less than $-\frac{\ln \chi_0}{T_a}$, where $\chi_0 > 1$ and $T_a$ are determined by the existed impulsive effects.

Remark 9. If the parameter $k$ determined in Theorem 4.2 satisfies $k < \frac{\ln \chi_0}{T_a}$, the coupled dynamical systems can achieve LFTCS according to Lemma 3.1. The domain of attraction and the settling time can be estimated as

$$
V^{\frac{1+h}{2}}(t_0) < \frac{l\chi_0^{-(1-h)N_0}}{\ln \chi_0 - k} \quad \text{and} \quad T = t_0 + \frac{2\ln \left(1 - \frac{V^{\frac{1+h}{2}}(t_0) \left(\frac{\ln \chi_0}{T_a} - k\right)}{l\chi_0^{-(1-h)N_0}}\right)}{(1-h)(k - \frac{\ln \chi_0}{T_a})},
$$

respectively.

Remark 10. The criteria in Theorems 4.1 and 4.2 are independent on the parameter $N_0$. Equivalently, the GFTCS can be realized for arbitrary value of the parameter $N_0$ if Eq. (40) and Eq. (55) hold. Conversely, if Eq. (40) and Eq. (55) are invalid, GFTCS may switch to LFTCS and the attractive domains change over the value of $N_0$. Specifically, the larger $N_0$ is taken, the smaller attractive domains are obtained. Thus, the value of $N_0$ should not be very large in these cases. However, in the references [32, 48], the asymptotical synchronization criteria can hold even if the value of $N_0$ is too large to satisfy Eq. (40) or Eq. (55). That is the fundamental difference between the finite-time synchronization and asymptotical synchronization [32, 48] of impulsive dynamical system.

Remark 11. The control energy for the controller (54) used in Theorem 4.2 is proposed as

$$
E^C_2 = \int_0^T ||R_2(t)||_2^2 dt,
$$
where $T_f$ is the settling time. According to Eq. (57) and Eq. (48),

$$
\|R_2(t)\|_2^2 = \sum_{i=1}^{N} \sum_{j=1}^{n} (k_1e_{ij}(t) + \beta_2\text{sign}(e_{ij}(t))|e_{ij}(t)|^h)^2
\leq 2k_1^2 \sum_{i=1}^{N} \sum_{j=1}^{n} e_{ij}^2(t) + 2\beta_2^2 \sum_{i=1}^{N} \sum_{j=1}^{n} |e_{ij}(t)|^{2h}
\leq 2k_1^2V(t) + 2(nN)^{1-h}\beta_2^2V(t)^{2h}.
$$  

(58)

If $\bar{k} = \frac{\ln \chi_0}{T_a}$, due to Eq. (17), then,

$$
\mathcal{E}_2^C \leq \int_{0}^{T_f} 2k_1^2V(t) + 2(nN)^{1-h}\beta_2^2V(t)^{2h}dt \leq \gamma_5 + \gamma_6,
$$  

(59)

where $\gamma_5 = (1-h)k_1^2V^{\frac{2-h}{2}}(0)$, $\gamma_6 = \frac{\beta_2(1-h)(nN)^{1-h}V^{\frac{1+3h}{4}}(0)}{(1+3h)\chi_0^{-2(4h+N_0)}}$. Thus, the upper bound of control energy is estimated as $\mathcal{E}_2^{Sup} = \gamma_5 + \gamma_6$.

If $\bar{k} > \frac{\ln \chi_0}{T_a}$, according to Eq. (16), one can obtain that

$$
V(t) \leq (\gamma_7 - \gamma_8 t)^{\frac{2-\sigma}{3}}
$$

where $\gamma_7 = \chi_0^{(1-h)N_0}V^{\frac{1-h}{4}}(0)$, $\gamma_8 = 2\chi_0^{-(1-h)N_0}(1-h)\beta_2$. Thus,

$$
\mathcal{E}_2^C \leq \int_{0}^{T_f} 2k_1^2(\gamma_7 - \gamma_8 t)^{\frac{2-\sigma}{3}} + 2(nN)^{1-h}\beta_2^2(\gamma_7 - \gamma_8 t)^{\frac{4h}{3h}}dt
= k_1^2\chi_0^{(1-h)N_0} \frac{(3-h)\beta_2}{(1+3h)} \left(\gamma_7^{\frac{1+3h}{1-h}} - (\gamma_7 - \gamma_8 T_f)^{\frac{1+3h}{1-h}}\right)
+ \frac{(nN)^{1-h}\beta_2\chi_0^{(1-h)N_0}}{(1+3h)} \left(\gamma_7^{\frac{1+3h}{1-h}} - (\gamma_7 - \gamma_8 T_f)^{\frac{1+3h}{1-h}}\right)
\triangleq \mathcal{E}_2^{Sup}.
$$  

(60)

Substituting the expressions of $T_f$ in Theorem 4.2 into Eq. (60), the upper bound of control energy $\mathcal{E}_2^{Sup}$ can be obtained.

5. Numerical examples. Example 1. Consider a network of six agents with two chaotic systems [30, 57] as

$$
\dot{x}(t) = -B_0 x(t) + f_1(x(t)),
$$  

(61)

$$
\dot{y}(t) = -y(t) + C_0 f_2(y(t)),
$$  

(62)

where $x(t), y(t) \in \mathbb{R}^3$ and

$$
B_0 = \begin{pmatrix}
-10 & 10 & 0 \\
28 & -1 & 0 \\
0 & 0 & -\frac{8}{3}
\end{pmatrix},
C_0 = \begin{pmatrix}
1.25 & -3.2 & -3.2 \\
-3.2 & 1.1 & -4.4 \\
-3.2 & 4.4 & 1
\end{pmatrix}.
$$

The phase plots of the systems (61) and (62) are shown in Fig. 1.

Set $Q_1 = \{1, 2\},\ Q_2 = \{3, 4, 5, 6\}$. The nonlinear functions are $f_1(x(t)) = [0, -x_1(t)x_3(t), x_1(t)x_2(t)]^T$ and $f_2(y(t)) = [\phi(y_1(t)), \phi(y_2(t)), \phi(y_3(t))]^T$, where

...
\[ \phi(v) = \frac{1}{2}(|v + 1| - |v - 1|) \]. The coupling matrix for the partition \( \{Q_1, Q_2\} \) is

\[ \tilde{L} = \begin{pmatrix} h_1 L_{11} & L_{12} \\ L_{21} & h_2 L_{22} \end{pmatrix}, \]

where \( h_1 = 1, h_2 = 2, \)

\[ L_{11} = \begin{pmatrix} -2 & 1.5 \\ 1.5 & -2 \end{pmatrix}, \quad L_{12} = \begin{pmatrix} -0.4 & 0.9 & 0.2 & -0.2 \\ 0.4 & -0.7 & 0.3 & 0.5 \end{pmatrix}, \]

\[ L_{21} = L_{12}^T, \quad L_{22} = \begin{pmatrix} -3 & 1 & 1 & 1 \\ 1 & -3.2 & 1 & 1 \\ 1 & 1 & -2.5 & 0 \\ 1 & 1 & 0 & -2.3 \end{pmatrix}. \]

**Figure 1.** Phase plots of (a) the system (61) and (b) the system (62) in Example 1.

**Figure 2.** Time histories of (a) the coupled system without control input, (b-d) the variables \( x_{i1}, x_{i2}, \) and \( x_{i3} \) of the coupled system with synchronization impulsive effect in Example 1.
Obviously, $k = \lambda_{\text{max}} \left( - (B - \bar{L} \otimes \Gamma) - (B - \bar{L} \otimes \Gamma)^T + C^T C + \Pi^T \Pi \right) = 47.1811$. Time histories of the system coupled by Eq. (62) and Eq. (61) without controller are given in Fig. 2 (a). One can observe that the coupled dynamical system cannot achieve FTCS without control input.

**A. Synchronization impulses.** In order to use the synchronization impulses to achieve GFTCS, the parameters $\chi_0$ and $T_a$ need to satisfy $\ln \chi_0 / T_a \leq -k = -47.1881$ due to Theorem 4.1. Set $\chi_0 = 0.1$, then $T_a \leq 0.0488$ according to the criterion (40). Take $\beta_1 = 1$ and $h = 0$ in Eq. (39), then the coupled system can realize GFTCS. Consider the initial value as $x_{11}(0) = x_{21}(0) = 2$, $x_{31}(0) = x_{41}(0) = x_{51}(0) = x_{61}(0) = -2, i = 1, 2, 3$ and take $T_a = 0.04$ and $N_0 = 10$, the settling time can be obtained as $T = 2.5586$. Define an index

$$ E(t) = \sum_{i=1}^{2} \| x_i(t) - s_1(t) \|^2 + \sum_{i=3}^{6} \| x_i(t) - s_2(t) \|^2, \quad (63) $$

where $s_1(t), s_2(t)$ satisfy the systems (61) and (62), respectively. Set the impulses sequence $t_k = 0.4 k$, $k \in \mathbb{N}$, time histories of the state variables $x_i(t)$ and the synchronization error $e_i(t)$ with synchronization impulses are shown in Fig. 2 (b-d) and Fig. 3, respectively.

**B. Desynchronization impulses** Consider the impulsive input matrix $K = 0.5 I$, $T_a = 0.05$ and $N_0 = 5$, then $\chi_0 = 2.25$. In order to reach FTCS of the coupled system with impulsive effects, design the following controller:

$$ R_2(t) = -k_1 e(t) - \text{sign}(e(t)). \quad (64) $$

According to Theorem 4.2, the coupled system can realize GFTCS if $k_1 > \frac{1}{2} (k + \ln \chi_0 / T_a) = 28.6998$. Take $k_1 = 29$ and $x_{11}(0) = -9, x_{12}(0) = -8, \ldots, x_{32}(0) = 7,$
$x_{33}(0) = 8$, then the settling time is $T = 2.1379$. To illustrate the impulsive effects on the finite-time synchronization of the coupled systems, take the equidistant impulses $t_k = 0.05k$ in the numerical simulation. Time histories of $x_i(t)$ and $e_i(t)$, $i = 1, \ldots, 6$ are shown in Fig. 4.

![Figure 4. With desynchronization impulses, time histories of (a-c), the variables $x_{i1}$, $x_{i2}$ and $x_{i3}$ of the coupled system with non-identical nodes (61) and (62), (d-f), the variables $e_{i1}$, $e_{i2}$ and $e_{i3}$ of the synchronization error system in Example 1.](image)

**Example 2.** Now, construct a large-scale network which obeys the distribution of the Barabási-Albert model [4]. The initial graph is complete with $m_0 = 10$ nodes, $m = 6$ edges are randomly added in the complex network if a new node is introduced in, and the final number of all nodes is $N = 200$. Assume that the network is partitioned into three clusters $Q_1 = \{1, \ldots, 20\}$, $Q_2 = \{21, \ldots, 100\}$, $Q_3 = \{101, \ldots, 200\}$. The node dynamical systems for the cluster $Q_1$ and $Q_2$ are described as the systems (61) and (62), respectively. The node dynamic system for the cluster $Q_3$ is given as follows:

$$
\dot{u}_i(t) = -D_k u_i(t) + G_k f_k(u_i(t)), \quad i \in Q_3,
$$

(65)
where \( u_i(t) \in \mathbb{R}^3, \quad f_3(u_i) = [-2.95(\|u_{i1} + 1\| - \|u_{i1} - 1\|), 0, 0]^T \)
and
\[
D_3 = \begin{bmatrix}
3.2 & -10 & 0 \\
-1 & 1 & -1 \\
0 & 14.97 & 0 \\
\end{bmatrix}, \quad G_3 = I_3.
\]

Take the inner coupling matrix \( \Gamma = 10I_3 \) and \( h_1 = h_2 = 1 \). Next, the synchronization impulsive effects and the feedback control \( R_1(t) \) proposed in Theorem 4.1 are used to obtain finite-time cluster synchronization of the complex network with the cluster partition \( \{Q_1, Q_2, Q_3\} \). For the given coupling structure, the value of the parameter \( k \) in Eq. (40) is \( k = -17.1738 \). According to Theorem 4.1, the complex network can achieve GFTCS if \( \frac{\ln \chi_0}{T_a} \leq -k = -17.1738 \). Set \( \chi_0 = 0.01 \), then \( T_a = 0.141 \). Take \( N_0 = 1, \quad T_a = 0.05, \quad \beta_1 = 10 \) and \( x(0) = (x_1(0), \ldots, x_{600}(0)) \), \( x_j(0) = -50 + \frac{100(j-1)}{599}, \quad j = 1, \ldots, 600 \), then the settling time is \( T = 1.3543 \). Set the impulsive sequence \( t_k = 0.1k, \quad k = 1, 2, \ldots \), time histories of \( x_i(t) \) and \( e_i(t) \), \( i = 1, \ldots, 200 \) are shown in Fig. 5. It is clear that the complex network with nonidentical nodes can achieve finite-time cluster synchronization.

**Figure 5.** Time histories of (a-c). the variables \( x_{i1}, x_{i2} \) and \( x_{i3} \) of the complex networks, (d-f). the variables \( e_{i1}, e_{i2} \) and \( e_{i3} \) of the synchronization error system in Example 2.
Remark 12. One can observe that the coupling matrix

\[ L = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} \]

in Example 1 is a symmetrical and zero-row-sum matrix. However, the two blocks \( L_{11} \) and \( L_{22} \) do not satisfy the zero-row-sum condition proposed in [30, 47]. Thus, the requirement for the topology structure of network is removed here.

6. Conclusions. The finite-time cluster synchronization problem of coupled dynamical systems with impulsive effects has been investigated in the present paper. By using impulsive differential equations theory and differential inequality method, the finite-time stability criteria based on Lyapunov function have been proposed for the systems involving the stabilizing impulses and destabilizing impulses. By using the Lyapunov function-based results, the finite-time cluster synchronization criteria have been given based on easy-checked algebraic inequality. The effects of the impulses are explicitly illustrated in the obtained finite-time cluster synchronization criteria. Obviously, the settling times are dependent on both impulsive effects and initial conditions. It is shown that the proper synchronization impulses can enlarge the attractive domain, conversely, the desynchronization impulses can reduce the attractive domain of the original system. Additionally, several classic works have been extended in this paper. Future works focus on the finite-time cluster synchronization problem of other complex systems with impulsive effect, such as impulsive octonion-valued neural networks.

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E-mail address: yuthjianyang@163.com
E-mail address: jdcao@seu.edu.cn
E-mail address: cxiahuang@126.com