Universal three-body physics at finite energy near Feshbach resonances

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Abstract. We find that universal three-body physics extends beyond the threshold regime to non-zero energies. For ultracold atomic gases with a negative two-body s-wave scattering length near a Feshbach resonance, we show that the resonant peaks characteristic of Efimov physics persist in three-body recombination to higher collision energies. For this and other inelastic processes, we use the adiabatic hyperspherical representation to derive universal analytical expressions for their dependence on the scattering length, the collision energy and—for narrow resonances—the effective range. These expressions are supported by full numerical solutions of the Schrödinger equation and display log-periodic dependence on energy characteristic of Efimov physics. This dependence is robust and might be used to experimentally observe several Efimov features.
1. Introduction

Ultracold quantum degenerate gases near Feshbach resonances have been studied intensively in recent years [1]. The tunability of the interactions between the atoms, characterized by their s-wave scattering length $a$, makes ultracold quantum gases an ideal testbed for studying a variety of few-body problems. Three-body systems in the strongly interacting regime $|a| \rightarrow \infty$ especially interest people due to their importance in many areas of physics, including condensed matter, atomic, molecular and nuclear physics [2, 3].

Central among all ultracold three-body phenomena is the Efimov effect, which occurs when the scattering length $a$ is much larger than the characteristic length scale $r_0$ of the atomic interaction [4, 5]. First predicted in the early 1970s by the nuclear physicist V Efimov [6], this effect refers to the existence of an infinite number of weakly bound three-body states when $|a| \rightarrow \infty$. It has a profound impact on the three-body scattering observables and leads to universal three-body behavior irrespective of the short-range interactions between the particles [3]. In particular, when $a < 0$, the rate $K_3$ for three-boson recombination, $B + B + B \rightarrow B_2 + B$, features a log-periodic series of resonance peaks on top of an overall $a^4$ scaling near zero energy. Explicitly [3]–[5], in atomic units,

$$K_3 = \frac{4590 \sinh(2\eta)}{\sin^2 s_0 \ln(|a|/r_0) + \Phi + 1.53} + \sinh^2 \eta \frac{a^4}{m},$$

(1)

where $s_0 \approx 1.00624$ is a universal constant, $\Phi$ is the three-body short-range phase [3, 7, 8], $\eta$ is a parameter characterizing short-range inelastic transitions [3, 7] and $m$ is the atomic mass. Physically, the peaks appear in $K_3$ when an Efimov state moves across the three-body break-up threshold and becomes bound [4, 9]. Another three-body inelastic process that is also important in ultracold experiments is the vibrational relaxation of a weakly bound molecule: $B^*_2 + B \rightarrow B_2 + B$ ($a > 0$). The rate for vibrational relaxation $V_{\text{rel}}$ has the same functional form as equation (1), with the overall $a^4$ scaling changed to just $a$; the overall coefficient, to 20.03; and the constant in the phase, to 1.47.

Near a narrow Feshbach resonance, however, the universal three-body physics is modified [10]–[13]. The dependence of the three-body physics on the resonance width in
magnetic field can be parameterized with the two-body effective range \( r_{\text{eff}} \), which is inversely proportional to the resonance width \([10]\). It has recently been predicted that near a narrow resonance three-body inelastic processes dominated by short-range transitions are suppressed for bosons and enhanced for mixed spin fermions \([13]\).

The Efimov and universal three-body physics introduced above are studied through the dependence of the three-body observables on \( a \) in the limit of zero scattering energy. Studies away from zero energy have mainly considered only the scattering length dependence at non-zero collision energies \([14]–[19]\). But, interestingly, it has recently been shown that the energy dependence itself also displays signatures of the Efimov effect for \( a > 0 \) \([20]\). For energies beyond the threshold regime, however, the higher partial wave contributions to the inelastic rates are not suppressed by the Wigner threshold law \([21]\) and tend to mask these energy-dependent Efimov features. These contributions, combined with thermal averaging effects, make observing energy-dependent Efimov features in this case quite challenging. To overcome these difficulties, a scheme to collide Bose–Einstein condensates has therefore been proposed \([20]\).

In this paper, we show that signatures of the Efimov effect also appear at non-zero collision energies in recombination into deeply bound molecules for \( a < 0 \). In contrast to the finite-energy Efimov features identified previously in three-body recombination into weakly bound molecules for \( a > 0 \) \([20]\), the Efimov signatures we show here have negligible higher partial wave contributions and are much less sensitive to thermal averaging. We also study the energy dependence of the vibrational relaxation rate of a weakly bound molecule into a deeply bound one when \( a > 0 \). Finally, we show that Efimov and universal features remain for large negative \( r_{\text{eff}} \), corresponding to a narrow Feshbach resonance. In most cases, we find analytic expressions for the rates and confirm them with numerical solutions of the Schrödinger equation.

2. Method

To study three-body universality at finite energies, we have numerically solved the three-body Schrödinger equation for identical bosons using the adiabatic hyperspherical representation \([22]\). In this representation, the Schrödinger equation reduces to a set of coupled equations for the hyperradial wave functions \( F_\nu (R) \),

\[
\left[ -\frac{1}{2\mu} \frac{d^2}{dR^2} + W_{\nu\nu}(R) \right] F_\nu + \sum_{\nu' \neq \nu} W_{\nu\nu'} F_{\nu'} = EF_\nu. \tag{2}
\]

The hyperradius \( R \) is a measure of the overall size of the system and \( \mu = m/\sqrt{3} \) is the three-body reduced mass. The effective three-body potentials are

\[
W_{\nu\nu}(R) = U_\nu(R) - \frac{1}{2\mu} Q_{\nu\nu}(R), \tag{3}
\]

where \( U_\nu(R) \) are calculated from the adiabatic equation

\[
H_{\text{ad}}(R; \Omega) \Phi_\nu(R; \Omega) = U_\nu(R) \Phi_\nu(R; \Omega). \tag{4}
\]

The adiabatic Hamiltonian,

\[
H_{\text{ad}} = -\frac{\Lambda^2 + 15/4}{2\mu R^2} + v(r_{12}) + v(r_{23}) + v(r_{31}), \tag{5}
\]

includes the hyperangular kinetic energy in the first term via the grand angular momentum operator \( \Lambda^2 \) \([22]\) and the pairwise sum of two-body interactions \( v(r_{ij}) \) between atoms \( i \) and \( j \).
The diagonal couplings $Q_{\nu \nu}(R)$ are determined from
\[ Q_{\nu \nu} = -\left\langle \left( \frac{d\Phi_{\nu}}{dR} \right) \Phi_{\nu} \mid \Phi_{\nu} \right\rangle, \tag{6} \]
where the double brackets indicate integration over the hyperangles $\Omega$. The non-adiabatic couplings $W_{\nu \nu'} (\nu \neq \nu')$ govern the transitions between channels. The effective three-body potentials $W_{\nu \nu}$ are crucial for the study of three-body scattering, especially at low energies, since they determine the universal behavior of the rates like equation (1) [7].

To solve equation (4) numerically, we must specify the interaction potential. Fortunately, when $|a| \gg r_0$, the universality of low-energy three-body physics allows us to model the interactions between the atoms near a Feshbach resonance by single channel interactions with large $|a|$, as the details of the atomic interactions affect three-body observables only through the short-range parameters $\Phi$ and $\eta$. This model is equivalent to a multi-channel treatment as long as there is only one open channel in the two-body system. Therefore, when solving equation (4), we simply use a pairwise sum of two-body potentials
\[ v(r) = -D \sech^2(r/r_0), \tag{7} \]
where $r$ is the interparticle distance and $D$ is a parameter that we use to control the scattering length as well as the number of two-body bound states.

Our numerical approach has been described in detail in [22]–[25]. Briefly, we first solve equation (4) by expanding $\Phi_{\nu}(R; \Omega)$ on Wigner $D$ functions for the Euler angle dependence of the overall rotations [22] and on a two-dimensional (2D), direct product B-spline basis for the hyperangles representing relative motion in the body frame [23]. Inelastic rates are then calculated by solving equation (2) with the eigenchannel $R$-matrix method [24, 25].

We use these numerical solutions to verify our analytical expressions for the universal behavior. These expressions are obtained by solving the hyperradial equation, equation (2), using the universal results for the hyperradial potential $W_{\nu \nu}(R)$ [7, 26]. Moreover, for recombination with $a < 0$ and vibrational relaxation with $a > 0$, we need only know the behavior of the initial channel. In figure 1, we show $W_{\nu \nu}(R)$ for $a = -180$ a.u. numerically calculated from equation (4). The lowest-energy curve is the initial channel for recombination. It has a barrier at $R \approx |a|$ [27]. For smaller $R$, the potential has Efimov character (attractive $1/R^2$), and for larger $R$, it has free particle character (repulsive $1/R^2$). For $a < 0$ and $|a| \gg r_0$, we can idealize the behavior in the incident channel for recombination as
\[ W_{\nu \nu}(R) = \begin{cases} s_0^2 + 1/4 - 2\mu R^2, & r_0 \ll R \ll |a|, \\ \frac{15/4}{2\mu R^2}, & R \gg |a|. \end{cases} \tag{8} \]

For $R \ll r_0$, $W_{\nu \nu}$ is not universal and the solution from that region will simply be parameterized at $R = r_0$ with $\Phi$ and $\eta$. For vibrational relaxation when $a > 0$, the potential for the incident channel is only changed for $R \gg a$ to
\[ W_{\nu \nu} = -E_{2b} + \frac{l(l+1)}{2\mu R^2}, \tag{9} \]
where $E_{2b} = 1/ma^2$ is the binding energy of the weakly bound molecule and $l$ is the angular momentum between the atom and the molecule.
Figure 1. The adiabatic hyperspherical potentials for $a = -180$ a.u. The deeply bound atom–molecule potentials are not shown, so all of the curves represent the three-body continuum, $B + B + B$, asymptotically.

For both processes, the coupling between the initial channel and the final deeply bound atom–molecule channel is only significant for $R \lesssim r_0$, and we can treat the channels as uncoupled for $R > r_0$. The solutions are thus completely determined by the potential equation (8), and by assuming that each potential equation (8) holds over the entire domain, the hyperradial wave functions $F_\nu(R)$ are [15]

$$
F_\nu(R) = \begin{cases} 
C_0 \sin(\Phi + i\eta), 
& R = r_0, \\
C_1 R^{1/2} \{ J_{l_0}(k R) - \tan \delta_1 N_{l_0}(k R) \}, 
& r_0 \leq R \leq \beta |a|, \\
C_2 R^{1/2} \{ J_{l_{eff}+1/2}(k R) - \tan \delta N_{l_{eff}+1/2}(k R) \}, 
& R \geq \beta |a|,
\end{cases}
$$

(10)

where $J_x(x)$ and $N_x(x)$ are Bessel functions of the first and second kind, respectively; and $k = \sqrt{2\mu E}$ with $E$ being the total energy relative to the three-body break-up threshold. Since the boundary between regions is not exactly at $R = |a|$ and cannot be precisely defined, we include the constant $\beta$ as a free parameter to be determined by matching to numerical solutions [13]. The effective angular momentum $l_{eff}$ is 3/2 for recombination [21] and 0 for relaxation. The parameters $\Phi$ and $\eta$ are introduced in the same fashion as in [3, 7] to characterize the non-universal, short-range physics, including the inelastic transition. They will generally be taken to be free parameters and fitted to the rates for a particular system.

By matching the hyperradial wave function $F_\nu(R)$ at each boundary, the asymptotic phase shift $\delta$ can be expressed in terms of $\beta |a|$ and the short-range parameters at any energy. To obtain the probability for inelastic transitions in such a single-channel model, we first calculate the probability for elastic scattering $R$, which is given by the reflection coefficient:

$$
R = \left| 1 - i \tan \delta \right|^2.
$$

(11)

As long as $\eta$ is not zero (which corresponds to coupling to deeply bound states), $\delta$ will be complex and $R$ will be less than unity. The overall probability for an inelastic transition is thus $P = 1 - R$. For three-body recombination of identical bosons, the recombination rate $K_3$
is related to $\mathcal{P}$ by [4]

$$K_3 = \frac{192\sqrt{3}\pi^2}{mk^4}\mathcal{P}.$$  

Equation (1) can then be reproduced by deriving $\mathcal{P}$ for recombination near zero energy ($k \to 0$).

3. Three-body inelastic processes near broad Feshbach resonances

3.1. Three-body recombination at finite energies

In general, each region identified in equation (8) becomes important when the de Broglie wavelength becomes small enough to sample that length scale. Specifically, at energies above $E_s = 1/\mu r_0^2$, the details in the short-range region will start to play a role and the rates will not be universal. Below $E_s$, that is, at $R \gtrsim r_0$—the three-body potentials and couplings are all universal. We therefore expect universal behavior in the three-body scattering for energies below $E_s$.

Because the Efimov region of the lowest potential in figure 1 and equation (8) has a barrier at $R \approx |a|$, it can support a three-body shape resonance when an Efimov state exists behind the barrier. When the energy of the Efimov state is above the three-body break-up threshold $E = 0$, increasing the collision energy across the resonance energy will lead to a resonant peak in $K_3$ [4, 9]. This can be understood from the fact that near the resonant energy, the amplitude of the hyperradial wave function behind the barrier in $W_{vv}$ is greatly enhanced. The three-body system thus has much larger probability in the small-$R$ region where the coupling between the incident channel and the deeply bound atom–molecule channel peaks. However, because the height of the barrier is $\sim E_b = 1/\mu a^2$, the resonant behavior in $K_3$ can only be observed for energies below $E_b$. And, below this energy, the barrier can only confine a resonant Efimov state with size of the order of $|a|$, implying that at most one Efimov resonance can be observed as a function of energy for a fixed $a$.

In figure 2, we show the evolution of the first Efimov resonance peak from finite energy to zero energy as $|a|$ increases (see also [9]). These recombination probabilities were calculated numerically. Interestingly, when a peak moves below zero energy, an oscillatory structure with $|a|$ zero energy as energies below the height of the barrier is $\sim E_b = 1/\mu a^2$, and equation (2) becomes important when the de Broglie wavelength becomes small enough to sample that length scale. Specifically, at energies above $E_s = 1/\mu r_0^2$, the details in the short-range region will start to play a role and the rates will not be universal. Below $E_s$, that is, at $R \gtrsim r_0$—the three-body potentials and couplings are all universal. We therefore expect universal behavior in the three-body scattering for energies below $E_s$.

In figure 2, we show the evolution of the first Efimov resonance peak from finite energy to zero energy as $|a|$ increases (see also [9]). These recombination probabilities were calculated numerically. Interestingly, when a peak moves below zero energy, an oscillatory structure with one full period appears in $\mathcal{P}$ for $E > E_b$, as shown in figure 2(b).

To understand the oscillations revealed in these numerical results, we employ the analytical model introduced in section 2. Matching the hyperradial wave functions at finite energies gives

$$\mathcal{R} = \frac{(J_2'-iN_2') [J_{is_0} - \tan \delta_1, N_{is_0}] - 1}{(J_2 - iN_2) [J_{is_0} - \tan \delta_1, N_{is_0}]} \frac{(J_2' + iN_2') [J_{is_0} - \tan \delta_1, N_{is_0}] - 1}{(J_2 + iN_2) [J_{is_0} - \tan \delta_1, N_{is_0}]} - 1)^2. \tag{13}$$

The primes denote derivatives, and the Bessel functions should be evaluated at $R = \beta|a|$. When the scattering energy satisfies $k \ll 1/r_0$, $\tan \delta_1$ is given by

$$\tan \delta_1 \approx \frac{\sin(\phi + i\eta - i\pi s_0/2)}{\cos(\phi + i\eta + i\pi s_0/2)}, \quad \tan \phi = \tan[\Phi - s_0 \ln(kr_0) + \varphi_0 + \pi/4], \tag{14}$$

with

$$\tan \varphi_0 = \frac{\text{Re}[\Gamma(is_0)] - \text{Im}[\Gamma(is_0)]}{\text{Re}[\Gamma(is_0)] + \text{Im}[\Gamma(is_0)]}. \tag{15}$$

For identical bosons, $\varphi_0 = -0.154 18\pi$. Equations (13) and (14) give a complete universal expression for $K_3$ in terms of the scattering length and energy. This analytic expression is
The three-body recombination probability $P(a < 0)$ for three identical atoms with the mass of Cs when a resonant Efimov trimer state is above the three-body break-up threshold. (a) As $|a|$ increases, the resonant peak moves towards the break-up threshold. (b) With increasing $|a|$ further after the Efimov state becomes bound, the resonant peak disappears, leaving an oscillatory structure behind. Here $r_0 \approx 50$ a.u.

Figure 2. The three-body recombination probability $P(a < 0)$ for three identical atoms with the mass of Cs when a resonant Efimov trimer state is above the three-body break-up threshold. (a) As $|a|$ increases, the resonant peak moves towards the break-up threshold. (b) With increasing $|a|$ further after the Efimov state becomes bound, the resonant peak disappears, leaving an oscillatory structure behind. Here $r_0 \approx 50$ a.u.

evaluated and plotted in figure 3. In panel (a), the evolution of the resonant peaks with energy can be clearly seen, but the oscillatory structure—which should appear at fixed, large $|a|$ as a function of $E$—is essentially invisible due to its relatively small modulation compared to the resonant peaks. Figure 3(b) replots the data to better show the energy-dependent oscillations. Note that the fine structure parallel to the $E_b$ line is an artifact of the discontinuity of our idealized potential in equation (8) at $R = \beta |a|$. The connection between the Efimov resonances and the energy-dependent oscillations is evident in figure 3(b). The peak of an oscillation appears approximately at the energy where the zero-energy position of each Efimov resonance intersects the $E_b$ line. The rate oscillates through a full period when an Efimov resonance moves below the three-body breakup threshold. Therefore, the number of bound Efimov states can be directly read from the number of full oscillations.

Remarkably, when $\eta \ll 1$ (corresponding to a small probability for an inelastic transition at small $R$ as is expected), equations (13) and (14) reduce to a simple form in the energy range $E_b \ll E \ll E_s$ ($kr_0 \ll 1$),

$$P = \frac{2 \sinh(\pi s_0) \sinh(2\eta)}{\cosh(\pi s_0) + \sin[-2s_0 \ln(kr_0) + 2\Phi - 2\phi_0]}.$$  \hspace{1cm} (16)

It should be noted that for $E > E_b$, the parameters $\Phi$ and $\eta$ will depend on energy in general. However, for energies $E \ll E_s$, the change in the shape of the short-range hyperradial wave

Collision energies are reported in Kelvin using the conversion $T = E/k_b$.  

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The universal dependence of $K_3$ on scattering length and energy for identical bosons with the mass of $^7\text{Li}$ with $J = 0$ as evaluated from equations (12) and (13). (a) The rate scaled as $K_3E^2/a^4$ to better show the resonant peaks. (b) The same data scaled as $K_3E^2$ to emphasize the energy-dependent oscillation described by equation (16), which can be seen in the upper right half of the plot. Note that the fine structure parallel to the $E_b$ line is an artifact of the discontinuity of our idealized potential in equation (8) at $R = \beta|a|$. In both plots, $\Phi = \pi/2$, $\eta = 0.02$ and $r_0 = 15$ a.u.

The log periodicity of the energy-dependent oscillations in equation (16) with period $e^{2\pi/s_0}$ provides the clear connection with Efimov physics. Like the energy-dependent oscillations observed for $a > 0$ [20], the overall phase of the oscillations does not depend on $a$, as is clearly shown in figure 3(b). Therefore, as $|a|$ increases, the lower limit of validity $E_b = 1/\mu a^2$ of equation (16) decreases and more oscillations will appear towards the zero-energy limit. At the same time, those that have already appeared at higher energies remain unchanged.

These oscillations might thus provide one experimentally accessible means of measuring multiple Efimov features. In the experiments observing the Efimov effect via zero-energy three-body recombination [28]–[35], the temperature must be kept below $E_b$ as $|a|$ is increased [14]. This restriction makes the observation of multiple Efimov features quite challenging. However, if the Efimov features are measured via the energy-dependent oscillations, the experimental requirements might be less demanding. Furthermore, the $a$-independence of equation (16) allows the energy-dependent oscillations to be measured at different fixed $a$, provided that $E \geq E_b (k|a| \geq 1)$. Because the modulation of the oscillations is small compared to the resonant peaks, the scattering length where the measurements are carried out should be chosen to be far away from resonant peaks.

Our results apply equally well to three-body systems $BBX$ with two identical bosons, assuming that $a \gg r_0$ is the $B + X$ scattering length. All that changes is the value of $s_0$ [36]. The density of Efimov states in such systems grows with the mass ratio $\gamma = m_B/m_X$ since $s_0$ grows with $\gamma$, decreasing the spacing $e^{2\pi/s_0}$ of energy-dependent Efimov features [26, 36]. As can be
Figure 4. (a) The three-body recombination probability $\mathcal{P}(E)$ for three identical Cs atoms, two identical Cs atoms with one Cs atom in a different spin state (Cs$'$) and two identical Cs atoms with one Li atom. The symbols are numerical results, and the black solid lines are the analytical result from equation (16). The analytical curves are only shown over their range of validity, $E_b \leq E \leq E_s$, but $E_s$ is off the scale. For CsCsCs, $a = 2 \times 10^5$ a.u.; for CsCsCs$'$, $a = 6 \times 10^5$ a.u.; and for CsCsLi, $a = 10^5$ a.u. (b) The recombination probability after thermal averaging. To make clear the comparison of the relative modulation in the oscillations from the three different systems, $\mathcal{P}$ for CsCsCs$'$ and CsCsLi has been multiplied by 5 and 1.9, respectively. We have used $r_0 \approx 15$ a.u. in these calculations.

seen from equation (16), however, the relative size of the modulation of $\mathcal{P}$ is

$$\Delta \mathcal{P} = 2 \frac{\cosh(\pi s_0)}{\sinh^2(\pi s_0)},$$

which increases with decreasing $s_0$. The two effects thus conflict: to see more Efimov features requires a large $s_0$ ($\gamma$), but seeing them clearly requires a small $s_0$ ($\gamma$). Figure 4(a) shows $\mathcal{P} \propto E^2 K_3$ (see equation (12)) for model systems chosen to have mass ratios matching representative atomic systems. For three identical Cs atoms, equation (17) predicts $\Delta \mathcal{P}$ to be 0.170; for two identical Cs atoms and one Cs atom in a different spin state, the ratio is 1.37; and for two identical Cs atoms with one Li atom, the ratio is only 0.00789. The predicted spacing between features is 515, 3.9 \times 10^6 and 23.8, respectively. The analytic expression, equation (16), was fitted to the numerical data by adjusting only $\Phi$ and $\eta$ and is also plotted in figure 4. It agrees quite well with the numerical results.

A practical balance of contrast versus spacing can likely be found by first determining the smallest contrast experimentally observable. This choice makes $s_0$ as large as experimentally tolerable and thus gives the smallest spacing between features. For instance, if a relative modulation of $\Delta \mathcal{P} = 10\%$ can be observed, then from equation (17) we find $s_0 = 1.1748$.
The corresponding mass ratio is $\gamma \approx 7.1$, and the spacing between Efimov features is 210—a considerable improvement over three identical bosons.

To determine whether these features are experimentally observable even in the presence of a distribution of collision energies, we must thermally average the energy-dependent rates. Assuming a Boltzmann distribution $^{14,37}$, we find the results shown in figure 4(b). For $E_b \leq E \leq E_s$, the oscillatory structure in $P$ generally becomes less clear after thermal averaging. For $BBX$ three-body systems where $s_0$ is small, such as CsCsCs’, the oscillations are well preserved after thermal averaging, with the trade-off that the oscillation periods are huge.

3.2. Vibrational relaxation at finite energies

Although the relaxation rates $V_{rel}$ for three identical bosons with $a > 0$ resemble the $a < 0$ recombination rates in the zero-energy limit $^{3,7}$, they differ dramatically at higher energies. Note that zero scattering energy now refers to the atom–molecule break-up threshold rather than the three-body break-up threshold as for recombination. The usual two-body Wigner threshold law gives a constant rate for atom–molecule collisions in the threshold regime $E < E_{th}$ where generally $E_{th} \approx E_b$. An exception arises when the zero-energy value of $V_{rel}$ is near a resonant peak, which occurs when the atom–molecule scattering length $a_{am}$ is much larger than $r_0$. In this case, $E_{th} \approx 1/\mu_{am}a_{am}^2$, where $\mu_{am}$ is the atom–molecule reduced mass. For $E_{th} \leq E \leq E_s$, $V_{rel}$ scales like

$$V_{rel} = C k^{-1}.$$   \hspace{1cm} (18)

The constant $C$ depends on the short-range physics and is related to the short-range parameter $\eta$. But different from three-body recombination, the relaxation rates do not show any oscillations in our numerical calculations.

4. Three-body inelastic processes near narrow Feshbach resonances

Although Feshbach resonances are multichannel phenomena in general, a single-channel treatment is sufficient to study universal three-body physics so long as there is only one open channel $^{10,13}$. Following $^{13}$, we model a three-body system near a narrow Feshbach resonance by using pairwise two-body potentials that support an s-wave shape resonance. We accomplish this by adding a barrier to the model two-body potentials used in our numerical calculations, ensuring that the resonance has a large, negative effective range $r_{eff}$ $^{10,11,14,15}$.

$$v(r) = -D \text{sech}^2(3r/r_0) + B e^{-2(3r/r_0-2)^2}.$$ \hspace{1cm} (19)

For such $r_{eff}$, our numerical calculations show that the potential $W_{\nu\nu}(R)$ in the region $r_0 \ll R \ll |r_{eff}|$ is replaced by a weak, non-universal Coulomb potential $^{13}$. For identical bosons, when $|\alpha| \gg |r_{eff}| \gg r_0$ the new length scale produces a $1/|r_{eff}|$ suppression in the zero-energy inelastic rates that lead to a deeply bound two-body state $^{13}$.

This new length scale in $W_{\nu\nu}(R)$ introduces another energy scale for the universal behavior when the wavelength is short enough to sample the three-body potentials and couplings in $r_0 \ll R \ll |r_{eff}|$. The new energy scale is $E_{eff} = 1/|\alpha r_{eff}|^2$ and defines a new energy region: $E_{eff} < E < E_s$. The parameter $\alpha \approx 0.28$ plays a role similar to $\beta$ and was determined in $^{13}$ by fitting the universal formula for $K_3$ near a narrow Feshbach resonance to numerical results near
Figure 5. (a) The three-body recombination probability $P(a < 0)$ for identical bosons near narrow Feshbach resonances with $r_{\text{eff}} = -20, -200$ and $-2000$ a.u. In all cases, $a = -10^5$ a.u., and $r_0 \approx 50$ a.u. The symbols represent the numerical results, and the black solid lines are from equations (16) and (23). The analytical curves are only shown over their range of validity, $E_b \leq E \leq E_{\text{eff}}$. Equation (24) is also shown for $r_{\text{eff}} = -2000$ a.u. with Im $A$ fitted to be $1.1 \times 10^{-4}$ a.u. The parameters Re $A$ are directly calculated from the short-range three-body potentials under the single-channel approximation, giving Re $A = 50, 57$ and $180$ a.u. for $r_{\text{eff}} = -20, -200$ and $-2000$ a.u., respectively. The parameter $\alpha$ is found to be 0.25 by fitting, consistent with [13]. (b) The recombination probability from numerical calculations after thermal averaging. Here $r_0 \approx 50$ a.u.

zero energy. Although it was determined at zero energy, it should apply at non-zero energies as well. Where we treated these processes at zero energy in [13], we will now find the non-zero-energy expressions for their rates.

4.1. Three-body recombination at finite energies

We calculate $K_3$ for identical bosons for $a < 0$ numerically up to the short-range energy $E_s$. As shown in figure 5(a), the recombination probability $P$ at $E \approx E_s$ has only a relatively weak dependence on $r_{\text{eff}}$ and $a$, since in our model the short-range behavior does not have strong dependence on $r_{\text{eff}}$ or $a$. From our numerical calculations, we deduce that the energy dependence changes to $P \propto k$ in the energy range $E_{\text{eff}} < E < E_s$. Therefore, as the energy gets smaller, the recombination probability decreases monotonically until $E = E_{\text{eff}}$, where oscillatory behavior similar to equation (16) takes over. Compared to a broad resonance ($r_{\text{eff}} = -20$ a.u. in the figure), $P$ is suppressed by a factor of $1/|r_{\text{eff}}|$ around $E \approx E_{\text{eff}}$ before it connects to the oscillatory behavior at lower energy. Considering that $K_3$ at a fixed $a$ has the same energy dependence for both broad and narrow Feshbach resonances for $E < E_{\text{eff}}$, the recombination rates at lower energies are all suppressed by $1/|r_{\text{eff}}|$.

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To understand this observed $1/|r_{\text{eff}}|$ suppression, we follow the analytical procedure introduced in [13]. The Coulomb-like potential in the region $|r_{\text{eff}}| \ll R \ll |a|$ takes the form

$$ W_{\nu\nu} \approx \frac{c_0}{2\mu |r_{\text{eff}}| R}, \quad (20) $$

with $c_0$ being a non-universal constant of the order of unity that can be positive or negative [13].

We found in [13] that simply setting $W_{\nu\nu}$ to zero gave analytic results consistent with our numerical results and is justified by the fact that $W_{\nu\nu}$ is not universal, changing from attractive to repulsive, while the numerical rates are universal. To derive analytic expressions for the rates that connect most closely with those for a broad resonance, we write the wave function in the region $r_0 < R < \alpha |r_{\text{eff}}|$ as

$$ F_\nu = C \sin(k R + \delta_{\text{eff}}). \quad (21) $$

For $E \ll E_s$, we can use a low-energy expansion for $\delta_{\text{eff}}$,

$$ \delta_{\text{eff}} \approx -A k, \quad (22) $$

where we have introduced the complex three-body short-range scattering length $A$ to parameterize the physics at $R \lesssim r_0$ [13]. The recombination probability $P$ can thus be derived in the same fashion as the broad Feshbach resonance case. In the energy range $E_b < E < E_{\text{eff}}$, this gives the same results as equation (16), with $r_0$ replaced by $\alpha |r_{\text{eff}}|$ and the parameters $\Phi$ and $\eta$ connected to the real and imaginary parts of $A$,

$$ \tan \Phi = 2s_0 \frac{\alpha - \text{Re} A/|r_{\text{eff}}|}{\alpha + \text{Re} A/|r_{\text{eff}}|}, \quad \sinh \eta = 2 \frac{|\text{Im} A|}{|r_{\text{eff}}|} \sin 2\Phi. \quad (23) $$

These analytical results are shown together with the numerical data in figure 5(a). The discrepancy between the numerical and analytical results for $r_{\text{eff}} = -20$ and $-200$ a.u. is because of the fact that the universal requirement $|r_{\text{eff}}| \gg r_0$ is not well satisfied.

For $E_{\text{eff}} < E < E_s$, the recombination probability $P$ has a simple scaling behavior independent of the scattering length,

$$ P = 4k|\text{Im} A|, \quad (24) $$

in agreement with the numerical results.

To check the robustness of these oscillations for a thermal distribution, we show the thermally averaged $P$ with large $|r_{\text{eff}}|$ in figure 5(b). It can be seen that although the modulation of the oscillations at low temperature is reduced, the $P \propto k$ scaling for $E_{\text{eff}} < E < E_s$ is essentially unchanged.

4.2. Vibrational relaxation at finite energies

The behavior of vibrational relaxation for identical bosons with $a > 0$ changes in the same energy range as for three-body recombination: $E_{\text{eff}} \lesssim E \lesssim E_s$. Figure 6 shows that as $|r_{\text{eff}}|$ increases, instead of the $k^{-1}$ scaling seen in equation (18) for broad Feshbach resonances, a plateau region appears in $V_{\text{rel}}$, extending from $E_s$ down to $E_{\text{eff}}$. This change is also connected with the $1/|r_{\text{eff}}|$ suppression of the rates near zero energy and can be used for observing the effective-range effect near narrow Feshbach resonances. Further, this plateau behavior can be derived in the same manner as equation (24).

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Figure 6. The three-body relaxation rate $V_{\text{rel}}$ for identical bosons near narrow Feshbach resonances with $a = 10^4$ a.u. and $r_{\text{eff}} = -200, -1000$ and $-2000$ a.u. The rates are shown for collision energies above the three-body break-up threshold, where no resonant peaks are present so that the energy dependence is more easily seen. The symbols are the numerical results, and the back solid lines are the thermally averaged results. Here $r_0 \approx 50$ a.u.

Since the value of the plateau has only a weak dependence on $r_{\text{eff}}$ and extends down to $E_{\text{eff}}$, the larger the $|r_{\text{eff}}|$ is, the broader the plateau will be. And because $V_{\text{rel}}$ for different $r_{\text{eff}}$ has similar values for $E \approx E_s$, the fact that the plateau extends down to $E_{\text{eff}}$ will make $V_{\text{rel}}$ smaller than for a broad resonance by a factor of $r_0/|r_{\text{eff}}|$ for all energies $E < E_{\text{eff}}$. In other words, we expect $C$ in equation (18) to be proportional to $1/|r_{\text{eff}}|$. The experimental observation of a plateau region in the energy dependence of $V_{\text{rel}}$ will show the universal physics pertaining to a narrow Feshbach resonance. Moreover, the energy where $V_{\text{rel}}$ changes from plateau to $k^{-1}$ scaling can roughly give the value of $|r_{\text{eff}}|$.

In figure 6, we also show the thermally averaged relaxation rates. It can be seen that the plateaus in $E_{\text{eff}} \leq E \leq E_s$ are not dramatically changed by a thermal distribution.

5. Higher partial wave contributions

To determine whether the energy-dependent features we have described will be observable, the contributions of higher partial waves must be considered as they can easily obscure features away from threshold. Key to this consideration is the fact that three-body recombination ($a < 0$) and relaxation ($a > 0$) are dominated by transitions at $R \leq r_0$. For the three-body systems where the Efimov effect does not occur for $J > 0$—for instance, identical bosons and $BBX$ systems with $\gamma \leq 38.61$ [36]—the effective hyperspherical potentials are all repulsive for $R > r_0$, leading to substantial suppression of the inelastic rates when $E \ll E_s$. Following the same method as that used for deriving the energy dependence above, we find that the recombination probabilities for $J > 0$ scale like $(kr_0)^{2p_0}$, where $p_0$ is a universal constant that increases
with $J$ [36]. For identical bosons, $p_0$ is always larger than 2. For $E \ll E_s (kr_0 \ll 1)$, $\mathcal{P}(J > 0)$ are all negligible. Moreover, we have confirmed this conclusion for a few cases numerically. For vibrational relaxation, there is similar suppression for $J > 0$ contributions.

6. Summary

We have studied inelastic three-body collisions and have found that they behave universally for collision energies well out of the ultracold regime—up to nearly a millikelvin in the examples shown. Moreover, we have identified much of the universal energy dependence as a manifestation of the Efimov effect, opening up new dimensions on Efimov physics. Significantly, we have been able to derive analytic expressions, which were motivated and verified by numerical calculations, for these rates in terms of a few parameters. And we have done so for both broad and narrow resonances. Our study thus brings universal three-body physics out of the ultracold regime, suggesting that even richer universal physics can be discovered both theoretically and experimentally.

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References

[1] Chin C, Grimm R, Julienne P and Tiesinga E 2010 Feshbach resonances in ultracold gases Rev. Mod. Phys. 82 1225–86
[2] Jensen A S, Riisager K, Fedorov D V and Garrido E 2004 Structure and reactions of quantum halos Rev. Mod. Phys. 76 215–61
[3] Braaten E and Hammer H W 2006 Universality in few-body systems with large scattering length Phys. Rep. 428 259–390
[4] Esry B D, Chris Greene H and James Burke P 1999 Recombination of three atoms in the ultracold limit Phys. Rev. Lett. 83 1751–4
[5] Nielsen E and Macek J H 1999 Low-energy recombination of identical bosons by three-body collisions Phys. Rev. Lett. 83 1566–9
[6] Efimov V 1970 Energy levels arising from resonant two-body forces in a three-body system Phys. Lett. B 33 563–4
[7] D’Incao J P and Esry B D 2005 Scattering length scaling laws for ultracold three-body collisions Phys. Rev. Lett. 94 213201
[8] D’Incao J P, Greene C H and Esry B D 2009 The short-range three-body phase and other issues impacting the observation of Efimov physics in ultracold quantum gases J. Phys. B: At. Mol. Opt. Phys. 42 044016
[9] Esry B D and D’Incao J P 2007 Efimov physics in ultracold three-body collisions J. Phys. Conf. Ser. 88 012040
[10] Petrov D S 2004 Three-boson problem near a narrow Feshbach resonance Phys. Rev. Lett. 93 143201
[11] Gogolin A O, Mora C and Egger R 2008 Analytical solution of the bosonic three-body problem Phys. Rev. Lett. 100 140404
[12] Jona-Lasinio M and Pricoupenko L 2010 Three resonant ultracold bosons: off-resonance effects Phys. Rev. Lett. 104 023201

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Wang Y, D'Incao J P and Esry B D 2009 Ultracold three-body collisions near narrow Feshbach resonances arXiv:0906.5019

D'Incao J P, Suno H and Esry B D 2004 Limits on universality in ultracold three-boson recombination Phys. Rev. Lett. 93 123201

Jonsell S 2006 Efimov states for systems with negative scattering lengths Europhys. Lett. 76 8

Braaten E, Kang D and Platter L 2007 Universality constraints on three-body recombination for cold atoms: From $^4$He to $^{133}$Cs Phys. Rev. A 75 052714

Platter L and Shepard J R 2008 Scaling functions applied to three-body recombination of $^{133}$Cs atoms Phys. Rev. A 78 062717

Braaten E, Hammer H-W, Kang D and Platter L 2008 Three-body recombination of identical bosons with a large positive scattering length at nonzero temperature Phys. Rev. A 78 043605

Massignan P and Stoof H T C 2008 Efimov states near a Feshbach resonance Phys. Rev. A 78 030701

Wang Y, D'Incao J P, Nägerl H-C and Esry B D 2010 Colliding Bose–Einstein condensates to observe Efimov physics Phys. Rev. Lett. 104 113201

Esry B D, Greene C H and Suno H 2001 Threshold laws for three-body recombination Phys. Rev. A 65 010705

Suno H, Esry B D, Greene C H and Burke J P 2002 Three-body recombination of cold helium atoms Phys. Rev. A 65 042725

Esry B D 1997 Many-body effects in Bose–Einstein condensates of dilute atomic gases PhD Thesis University of Colorado

Aymar M, Greene C H and Luc-Koenig E 1996 Multichannel Rydberg spectroscopy of complex atoms Rev. Mod. Phys. 68 1015–23

Patrick Burke J Jr 1999 Theoretical investigation of cold alkali atom collisions PhD Thesis University of Colorado

Efimov V 1973 Energy levels of three resonantly interacting particles Nucl. Phys. A 210 157

Esry B D, Greene C H, Zhou Y and Lin C D 1996 Role of the scattering length in three-boson dynamics and Bose–Einstein condensation J. Phys. B: At. Mol. Opt. Phys. 29 L51

Kraemer T, Mark M, Waldburger P, Danzl J G, Chin C, Engeser B, Lange A D, Pilek K, Jaakkola A, Nägerl H C and Grimm R 2006 Evidence for Efimov quantum states in an ultracold gas of caesium atoms Nature 440 315–8

Zaccanti M, Deissler B, D’Errico C, Fattori M, Jona-Lasinio M, Müller S, Roati G, Inguscio M and Modugno G 2009 Observation of an Efimov spectrum in an atomic system Nat. Phys. 5 586

Pollack S E, Dries D and Hulet R G 2009 Universality in three- and four-body bound states of ultracold atoms Science 326 1683

Ottenstein T B, Lompe T, Kohnen M, Wenz A N and Jochim S 2008 Collisional stability of a three-component degenerate Fermi gas Phys. Rev. Lett. 101 203202

Huckans J H, Williams J R, Hazlett E L, Stites R W and O’Hara K M 2009 Three-body recombination in a three-state Fermi gas with widely tunable interactions Phys. Rev. Lett. 102 165302

Barontini G, Weber C, Rabatti F, Catani J, Thalhammer G, Inguscio M and Minardi F 2009 Observation of heteronuclear atomic Efimov resonances Phys. Rev. Lett. 103 043201

Gross N, Shotan Z, Kokkelmans S and Khaykovich L 2009 Observation of universality in ultracold $^7$Li three-body recombination Phys. Rev. Lett. 103 163202

Williams J R, Hazlett E L, Huckans J H, Stites R W, Zhang Y and O’Hara K M 2009 Evidence for an excited-state Efimov trimer in a three-component Fermi gas Phys. Rev. Lett. 103 130404

D’Incao J P and Esry B D 2006 Mass dependence of ultracold three-body collision rates Phys. Rev. A 73 030702

Suno H, Esry B D and Greene C H 2003 Recombination of three ultracold fermionic atoms Phys. Rev. Lett. 90 053202

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