Simple relations among $E2$ matrix elements of low-lying collective states

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Abstract

A method is developed to derive simple relations among the reduced matrix elements of the quadrupole operator between low-lying collective states. As an example, the fourth order scalars of $Q$ are considered. The accuracy and validity of the proposed relations is checked for the ECQF Hamiltonian of the IBM–1 in the whole parameter space of the Casten triangle. Furthermore these relations are successfully tested for low-lying collective states in nuclei for which all relevant data is available.

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Microscopic shell model wave functions of collective nuclear states need a huge configurational space. However, experimental data indicates that there are comparably simple relations between the wave functions of different collective states, including the ground state. The wave functions of some excited states can be described by actions of one-body operators on the ground state wave function with good accuracy. In even-even nuclei, where the ground state is a $0^+$ state, the first $2^+$ state is given by the quadrupole operator $Q$ acting on the ground state. The generalization of this concept has been named the $Q$-phonon approach [1-7]. In this approach one describes the low-lying collective positive parity states of even-even nuclei in the basis of multiple $Q$-phonon excitations of the ground state, $|0^+_1\rangle$, 

$$|L^+, n\rangle = N^{(L,n)}(Q\ldots Q)^{(L)}|0^+_1\rangle . \tag{1}$$

In the framework of the IBM-1 [8,9] it has been shown over the whole parameter space of the ECQF Hamiltonian [10,11] that each of the wave vectors of the yrast states can be described by only one multiple $Q$-phonon configuration with good accuracy [1-4, 5], which has recently been confirmed by microscopic calculations in [3].

The $Q$-phonon approximation implies the existence of selection rules for the matrix elements of the quadrupole operator. Thus, one finds that $E2$ transitions between $Q$-phonon configurations, that differ by more than one $Q$-phonon, are weak compared to transitions between those configurations that differ by only one $Q$-phonon. During the last years much data on $\gamma$-soft nuclei has been collected, especially in the $A=130$ mass region, which support these selection rules, e.g. [12, 13].

The $Q$-phonon structure of the low-lying collective states allows one to obtain quadrupole shape invariants [14-17] from rather few data. As an example we consider fourth order scalars obtained by coupling the four quadrupole operators in different ways. One obtains several different expressions for the fourth order quadrupole shape invariants in terms of only a few $E2$ matrix elements. These expressions can be used to derive approximate values of various observables, e.g., for the quadrupole moment of the first $2^+$ state or the lifetime of the first excited $0^+$ state, from more easily accessible nuclear data. Such information is desirable for nuclei where complete experimental information about low-lying states is not – or not yet – available, as e.g. nuclei which are produced using rare isotope beams.

There are three possibilities to couple the quadrupole operators to obtain fourth order scalars,

$$q_4^{(0)} = \langle 0^+_1|(Q \cdot Q)(Q \cdot Q)|0^+_1\rangle , \tag{2}$$

$$q_4^{(2)} = \langle 0^+_1|[\{QQ\}^{(2)}[QQ]^{(2)}]^{(0)}|0^+_1\rangle , \tag{3}$$

$$q_4^{(4)} = \langle 0^+_1|[\{QQ\}^{(4)}[QQ]^{(4)}]^{(0)}|0^+_1\rangle , \tag{4}$$

The notation $[\ldots]^{(L)}$ abbreviates the tensor coupling of two operators to angular momentum $L$. These three scalars are proportional to each other according to Dobaczewski, Rohoziński and Srebrny in [18], if the $Q$-operators commute. Then one obtains the relations

$$q_4^{(0)} = \frac{7\sqrt{5}}{2}q_4^{(2)} = \frac{35}{6}q_4^{(4)}. \tag{5}$$
In the IBM-1 the \(Q\)-operators do not commute. The effect of the noncommutativity of the components of the quadrupole operators scales however with \(1/N\) and is therefore neglected in first order. Below, we will check the accuracy of Eq. (5) in the framework of IBM-1.

In order to do this we decompose the scalars into sums over reduced matrix elements,

\[
q_4^{(0)} = \sum_{i,j,k} \langle 0^+_{1}\|Q\|2^+_i\rangle \langle 2^+_i\|Q\|0^+_j \rangle \\
\times \langle 0^+_{1}\|Q\|2^+_k\rangle \langle 2^+_k\|Q\|0^+_j \rangle ,
\]

(6)

\[
q_4^{(2)} = \frac{1}{5\sqrt{5}} \sum_{i,j,k} \langle 0^+_{1}\|Q\|2^+_i\rangle \langle 2^+_i\|Q\|2^+_j \rangle \\
\times \langle 2^+_i\|Q\|2^+_k\rangle \langle 2^+_k\|Q\|0^+_j \rangle ,
\]

(7)

\[
q_4^{(4)} = \frac{1}{15} \sum_{i,j,k} \langle 0^+_{1}\|Q\|2^+_i\rangle \langle 2^+_i\|Q\|4^+_j \rangle \\
\times \langle 4^+_i\|Q\|2^+_k\rangle \langle 2^+_k\|Q\|0^+_j \rangle .
\]

(8)

Using Eqs. (6)-(8), the quantities (2)-(4) have been calculated gridwise – using the code Phint \[19\] – for \(N=10\) bosons over the whole IBM-1 symmetry space spanned by the ECQF-Hamiltonian \[10,11\]

\[
H_{ECQF} = a \left[ (1 - \zeta) n_d - \frac{\zeta}{4N} Q \cdot Q \right].
\]

(9)

The ECQF Hamiltonian interpolates between the symmetry limits of the IBM-1 using two structural parameters, \(\zeta\) and \(\chi\). Here, \(n_d\) is the \(d\)-boson number operator and \(N\) is the total boson number. The parameter \(a\) has no structural meaning as it sets an absolute energy scale, and \(Q\) is the CQF quadrupole operator, both in the Hamiltonian and the \(E2\) transition operator,

\[
1/e_B \ T(E2) = Q = s^+ d + d^+ s + \chi [d^+ d]^{(2)},
\]

(10)

depending on the structural parameter \(\chi\), with \(-\sqrt{7}/2 \leq \chi \leq 0\); \(e_B\) is the effective boson charge. The result of this calculation is a near proportionality of the \(q_4^{(0)}\), \(q_4^{(2)}\) and \(q_4^{(4)}\), in accordance with Eq. (5).

In view of the selection rules of the \(Q\)-phonon scheme, the sums (6)-(8) reduce drastically. In the first approximation the set of \(E2\) matrix elements necessary for the calculation of \(q_4^{(n)}\) \((n = 0, 2, 4)\) reduces to the following matrix elements,

\[
\langle 2^+_i\|Q\|4^+_i \rangle \longrightarrow i = 1 ,
\]

(11)

\[
\langle 2^+_i\|Q\|2^+_i \rangle \longrightarrow i = 1, 2 ,
\]

(12)

\[
\langle 2^+_i\|Q\|0^+_i \rangle \longrightarrow i = 1, 2, 3 .
\]

(13)

The first three \(0^+\) states are taken into account, because the \(0^+_{2,3}\)-eigenstates of the ECQF Hamiltonian are mixtures of two- and three-\(Q\)-phonon \(0^+\) configurations. Of course, if there are low-lying non-collective \(0^+\) states, the ECQF \(0^+_{2,3}\) eigenstates may refer to higher lying physical states. We have introduced the short notation \(0^+_QQ\) by means of
\[\langle 0^+_{Q0} | Q | J \rangle^2 = \langle 0^+_{2} | Q | J \rangle^2 + \langle 0^+_{4} | Q | J \rangle^2 .\]  

(14)

By using only the matrix elements (11)-(13) in (6)-(8) we see that in each sum a factor \(\langle 0^+_{2} | Q | 2^+_1 \rangle^2\) appears, which may be dropped, since we are interested in the ratios. Eqs. (6)-(8) now become:

\[t^{(4)}_4 = \frac{1}{5\sqrt{5}} \left( \langle 2^+_1 | Q | 2^+_1 \rangle^2 + \langle 2^+_1 | Q | 2^+_1 \rangle^2 \right) ,\]

(15)

\[t^{(2)}_4 = \frac{2}{7\sqrt{5}} (N - 1)(N + 5) \]  

(20)

\[t^{(4)}_4 = \frac{6}{35}(N - 1)(N + 5) \]  

(21)

where we use \(t^{(n)}_4\) to distinguish these quantities from the exact \(q^{(n)}_4\) values. These quantities are also approximately proportional to each other, like the quantities \(q^{(n)}_4\). For arbitrary values of the boson number \(N\) the \(t^{(n)}_4\) values are related by factors \(c^{N}_{0i}\) defined by

\[t^{(0)}_4 = \frac{1}{c^{N}_{02}} t^{(2)}_4 = \frac{1}{c^{N}_{04}} t^{(4)}_4 ,\]

(18)

which depend on the boson number and the dynamical symmetry character. To obtain the values of the \(c^{N}_{0i}\), we consider the \(U(5), SU(3)\) and \(O(6)\) dynamical symmetry limits of the IBM-1 at first. The ECQF quadrupole operator (10) is used, and one obtains

\[t^{(0)}_4 = \left\{ \begin{array}{l}
7N - 2 \quad U(5) \\
N(2N + 3) \quad SU(3) \\
N(N + 4) \quad O(6) 
\end{array} \right. ,\]

(19)

\[t^{(2)}_4 = \left\{ \begin{array}{l}
\frac{1}{\sqrt{5}} \left( 2N + \chi^2 - 2 \right) \quad U(5) \\
\frac{2}{7\sqrt{5}}(N - 1)(N + 5) \quad O(6) 
\end{array} \right. ,\]

(20)

\[t^{(4)}_4 = \left\{ \begin{array}{l}
\frac{6}{5}(N - 1) \quad U(5) \\
\frac{6}{35}(N - 1)(N + 5) \quad O(6) 
\end{array} \right. .\]

(21)

Comparing Eq. (8) and Eqs. (19)-(21) one obtains proportionality factors for \(N \to \infty\)

\[c^{\infty}_{02} = \frac{2}{7\sqrt{5}} ,\quad c^{\infty}_{04} = \frac{6}{35} ;\]

(22)

in agreement with the factors of Eq. (4). These values hold also for finite \(N\) in the \(O(6)\) and the \(SU(3)\) dynamical symmetry limits when one neglects \(1/N^2\) terms. Only in the \(U(5)\) limit a \(1/N\) dependence is left, causing a small deviation from the limiting values. The values of the parameters \(c^{N}_{0i}\) with finite \(N\) differ slightly from those with \(N \to \infty\). Using Eqs. (15)-(21) we obtain improved relations in the dynamical symmetry limits including the values of \(c^{N}_{0i}\) for finite \(N\). For nuclei far from symmetries one can calculate the exact \(c^{N}_{0i}\) using Eq. (18) and interpolating in the IBM-1. We have done such calculation for the IBM-1 using the ECQF-Hamiltonian (3). Fig. (4) shows the values of the parameter
for \(N=10\) bosons. Using the limiting values for \(N \to \infty\) results in a systematical error below 10%. We note that some deviations arise from our use of only the \(0^+_2\) state for the \(0^+_QQ\) configuration in this calculation.

From Eqs. (15)-(18) one obtains two relations for the quadrupole moment of the \(2^+_1\) state:

\[
\langle 2^+_1 ||Q||2^+_1 \rangle^2 + \langle 2^+_1 ||Q||2^+_2 \rangle^2
\]

\[
= \left(\frac{c_0^N}{c_0^\infty} \frac{c_4^N}{c_4^\infty} \right) \cdot \frac{5}{9} \left(2^+_1 ||Q||4^+_1 \right)^2 ,
\]

\[
Q_{2^+_i}^2 = \frac{32\pi}{35} \left[ \left(\frac{c_0^N}{c_0^\infty} \frac{c_4^N}{c_4^\infty} \right) \cdot B(E2; 4^+_1 \to 2^+_1)
\]

\[
\qquad \qquad - B(E2; 2^+_2 \to 2^+_1) \right] \right)
\]

and

\[
\langle 2^+_1 ||Q||2^+_1 \rangle^2 + \langle 2^+_1 ||Q||2^+_2 \rangle^2
\]

\[
= \frac{10}{7} \cdot \frac{c_0^N}{c_0^\infty} \cdot (\langle 2^+_1 ||Q||0^+_1 \rangle^2 + \langle 2^+_1 ||Q||0^+_2 \rangle^2) ,
\]

\[
Q_{2^+_i}^2 = \frac{32\pi}{35} \left[ \frac{2}{7} \cdot \frac{c_0^N}{c_0^\infty} \left[ 5B(E2; 2^+_1 \to 0^+_1)
\right.
\]

\[
\qquad \qquad + B(E2; 0^+_QQ \to 2^+_1) \right] - B(E2; 2^+_2 \to 2^+_1) \left. \right] ,
\]

where Eqs. (24), (25) and (26), (27), respectively, differ only in notation. In a first approximation with \(c_{0i}/c_{0i}^\infty=1\), and if we define \(B(E2; 2^+_1 \to 2^+_1) \equiv 1/5(2^+_1 ||Q||2^+_1)^2\), we can write expression (24) in an intuitively interesting way:

\[
B(E2; 2^+_1 \to 2^+_1) + B(E2; 2^+_2 \to 2^+_1) = B(E2; 4^+_1 \to 2^+_1) .
\]

A relation similar to (25) for \(N \to \infty\) has been obtained in [20], but was derived in a much less transparent way and was expressed using a rather difficult notation. Rewriting Eq. (27) we get a relation for \(B(E2; 0^+_QQ \to 2^+_1)\), and a second relation by inserting (27) in (25).

Extending our previous definitions [17] of quadrupole shape invariants, we define now not only \(K_4\), but \(K_4^{(0)}\), \(K_4^{(2)}\) and \(K_4^{(4)}\), depending on the coupling. We want to obtain values, which characterize the nucleus and do not depend strongly on the coupling scheme. Thus, with \(q_2=(0^+_1 ||Q \cdot Q||0^+_1)\), we introduce

\[
K_4^{(0)} = \frac{q_4^{(0)}}{q_2} ,
\]

\[
K_4^{(2)} = \frac{7\sqrt{5}}{2} \frac{q_4^{(2)}}{q_2} ,
\]

\[
K_4^{(4)} = \frac{35}{6} \frac{q_4^{(4)}}{q_2} .
\]
These quantities are all equal if the quadrupole operators commute. We note that in the large $N$ limit of the IBM-1 the theoretical values for the $K_4^{(n)}$ are 1 for the $SU(3)$ and the $O(6)$, and 1.4 for the $U(5)$ dynamical symmetry limit, distinguishing between $\beta$-rigid and vibrational nuclei, respectively. Applying the above results to the $K_4^{(n)}$ leads to an approximation formula for $K_4^{(0)}$ that has already been obtained for $N \to \infty$ in [16],

$$K_4^{(0)} \approx \frac{7}{10} \frac{B(E2; 4_1^+ \to 2_1^+)}{B(E2; 2_1^+ \to 0_1^+)} \equiv K_4^{\text{appr.}}.$$  

(32)

A second approximation is

$$K_4^{(0)} \approx \frac{7}{10} \left[ \frac{35}{32} Q_{2_1^+}^2 + B(E2; 2_2^+ \to 2_1^+) \right] \frac{B(E2; 2_1^+ \to 0_1^+)}{B(E2; 2_1^+ \to 0_1^+)}.$$  

(33)

Due to $K_4^{(0)} \in [1, 1.4]$ it emerges from Eq. (33) that, e.g., in the transition from $O(6)$ to $SU(3)$, where $K_4^{(0)} = 1$, the value of $Q_{2_1^+}^2 / B(E2; 2_1^+ \to 0_1^+)$ rises from zero to $10/7$, while the value of $B(E2; 2_2^+ \to 2_1^+)/B(E2; 2_1^+ \to 0_1^+)$ drops from $10/7$ to zero. Thus, these ratios characterize nicely the change of structure.

In order to compare the relations with experimental data we considered nuclei near dynamical symmetries, for which all needed data is available. This data comes mostly from Coulomb excitation experiments by D. Cline and co-workers [15,21,22,26,29]. In Tables I, II the results are given. The Os and Pt nuclei are considered to be $\gamma$-soft [22–24] or transitional between $\gamma$-soft and axially deformed nuclei, which is indicated by the large values of the quadrupole moments of the $2_1^+$ states. As examples for vibrational nuclei Cd and Pd nuclei are shown, and Gd and Dy nuclei for the axially deformed case. We used $c_0^N$ values from the appropriate dynamical symmetry.

In Table I the relations (18) are tested with satisfactory overall agreement. Additionally, the values of $K_4^{\text{appr.}}$ are given in Table II.

Table I shows the experimental values of $Q_{2_1^+}^2$ and $B(E2; 0_{QQ}^+ \to 2_1^+)$ for the chosen nuclei, compared to the values obtained by the relations. The values of $Q_{2_1^+}^2$, obtained from the relations (24) and (27), agree with the experimental values within the errors in most cases. A high accuracy of data is necessary for significant results, especially for the vibrator-like and the $\gamma$-soft nuclei, for which the quadrupole moments become very small.

As an example for discrepancies, we consider $^{188}$Os for which the $B(E2; 0_2^+ \to 2_1^+)$ value is very small, as expected for an $O(6)$ nucleus. The two-$Q$-phonon content of this state should therefore be very small. However, the values from Eqs. (23),(27) in Table I may refer to a higher lying $0^+$ state with larger two-$Q$-phonon contribution, for which the lifetime is not known. Thus, with the missing $E2$ strength in $^{188}$Os, the value of $Q_{2_1^+}$ is underestimated by relation (27), while Eq. (27) describes the quadrupole moment well.

One finds significant deviations for other nuclei, too. For example, in $^{194}$Pt the large $B(E2; 0_1^+ \to 2_1^+)$ value indicates a two-$Q$-phonon structure for the $0_1^+$ state, in contradiction with the $O(6)$ prediction. Thus, this transition has been included in the calculation of the $B(E2; 0_{QQ}^+ \to 2_1^+)$ value. The value of $K_4^{\text{appr.}} = 0.8$ in $^{192}$Os is considerably smaller than its minimally allowed value: 1. This may be due to the small experimental value of $B(E2; 4_1^+ \to 2_1^+)$. Also $K_4^{\text{appr.}}$ for $^{108}$Pd is unexpectedly small, which does not support the
vibrational character of this nucleus. In the Cd isotopes considered the measured $Q_{2+}$ are smaller than expected from the relations.

To summarize, we propose a simple method to derive sets of relations between the experimentally observable reduced matrix elements of the quadrupole operator. This approach is based on the use of the quadrupole shape invariants, the selection rules of the $Q$–phonon scheme and the fact that corrections from noncommutativity of the components of the quadrupole moment operator in the IBM-1 are small. As an example of the general scheme, fourth order $Q$-invariants of the ground state are given. One can apply the scheme also to higher order invariants, e.g. $q_5$ or $q_6$, or to invariants built on excited states. The accuracy of the derived relations is checked for finite boson number $N$ over the whole parameter space of the ECQF-IBM-1 Hamiltonian and is shown to be rather good. A satisfactory agreement between data and theoretical relations has been obtained in many cases, but some exceptions clearly need further study.

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TABLES

TABLE I. Values of $t_{4}^{(0)}$, $1/c_{0}2^{N}t_{4}^{(2)}$ and $1/c_{0}4^{N}t_{4}^{(4)}$ for various nuclei. The three values should agree according to Eq. [18]

| Data taken from | $t_{4}^{(0)}$ [e$^2$b$^2$] | $1/c_{0}2^{N}t_{4}^{(2)}$ [e$^2$b$^2$] | $1/c_{0}4^{N}t_{4}^{(4)}$ [e$^2$b$^2$] | $K_{4}^{\text{appr.}}$ |
|-----------------|--------------------------|---------------------------------|---------------------------------|-----------------|
| $^{186}$Os      | 22                       | 2.84(7)                         | 2.79(56)                        | 3.06(17)        | 1.06(3)         |
| $^{188}$Os      | 22                       | 2.52(3)                         | 2.72(48)                        | 2.82(8)         | 1.08(1)         |
| $^{190}$Os      | 22                       | 2.36(6)                         | 1.97(40)                        | 2.28(16)        | 0.93(3)         |
| $^{192}$Os      | 22                       | 2.12(3)                         | 2.20(30)                        | 1.84(6)         | 0.82(1)         |
| $^{194}$Pt      | 22                       | 1.56(12)                        | 1.96(12)                        | 1.54(5)         | 1.00(4)         |
| $^{196}$Pt      | 25                       | 1.34(6)                         | 1.53(25)                        | 1.56(9)         | 1.08(7)         |
| $^{106}$Pd      | 20                       | 0.76(7)                         | 0.86(10)                        | 0.83(9)         | 1.19(9)         |
| $^{108}$Pd      | 20                       | 0.92(11)                        | 1.10(13)                        | 0.86(9)         | 1.04(9)         |
| $^{112}$Cd      | 27,28                    | 0.65(5)                         | 0.37(6)                         | 0.76(7)         | 1.41(14)        |
| $^{114}$Cd      | 24                       | 0.60(3)                         | 0.53(8)                         | 0.77(5)         | 1.39(12)        |
| $^{156}$Gd      | 30,31                    | 4.67(13)                        | 4.55(23)                        | 4.66(13)        | 0.98(3)         |
| $^{158}$Gd      | 30,32,31                 | 5.03(15)                        | 5.01(25)                        | 5.20(14)        | 1.02(4)         |
| $^{160}$Gd      | 33,34                    | 5.25(5)                         | 5.36(26)                        | 5.20(13)        | 0.98(2)         |
| $^{164}$Dy      | 31                       | 5.57(8)                         | 5.16(100)                       | 5.12(27)        | 0.91(5)         |
TABLE II. Comparison of the quadrupole moments of the $2^+_1$ state and the reduced transition strengths of the $0^{+}_{QQ} \rightarrow 2^+_1$ transition for various nuclei.

|       | $Q^2_{2^+_1}$ [e$^2$b$^2$] | $B(E2; 0^{+}_{QQ} \rightarrow 2^+_1)$ [e$^2$b$^2$] |       |
|-------|-----------------------------|---------------------------------|-------|
|       | Eq. (25) | exp. | Eq. (27)                  |       |
| $^{186}$Os | 1.97$^{+1.14}_{-1.25}$ | 1.76$^{+0.26}_{-0.44}$ | 1.80$^{+0.10}_{-0.12}$ | 0.25$^{+0.16}_{-0.17}$ | 0.040$^{+0.24}_{-0.16}$ | < 0.35 |
| $^{188}$Os | 1.80$^{+0.7}_{-0.21}$ | 1.72$^{+0.10}_{-0.38}$ | 1.56$^{+0.8}_{-0.8}$ | 0.30$^{+0.8}_{-0.2}$ | 0.0061$^{+0.3}_{-0.2}$ | 0.20$^{+0.15}_{-0.20}$ |
| $^{190}$Os | 1.14$^{+0.15}_{-0.30}$ | 0.90$^{+0.19}_{-0.32}$ | 1.20$^{+0.9}_{-0.9}$ | < 0.10 | 0.014$^{+0.2}_{-0.2}$ | 0 |
| $^{192}$Os | 0.56$^{+0.6}_{-0.19}$ | 0.84$^{+0.24}_{-0.8}$ | 0.78$^{+0.7}_{-0.7}$ | 0 | 0.004$^{+1.1}_{-0.1}$ | 0.08$^{+3.2}_{-0.8}$ |
| $^{194}$Pt | 0 | 0.20$^{+0.7}_{-0.7}$ | < 0.01 | 0.08$^{+0.6}_{-0.8}$ | 0.100$^{+0.6}_{-0.6}$ | 0.50$^{+0.9}_{-0.9}$ |
| $^{196}$Pt | 0.26(9) | 0.24(18) | 0.10(8) | 0.24(11) | 0.02(1) | 0.21(26) |
| $^{106}$Pd | 0.28$^{+0.22}_{-0.22}$ | 0.30$^{+0.19}_{-0.2}$ | 0.23$^{+0.7}_{-0.7}$ | 0.20$^{+0.14}_{-0.11}$ | 0.14$^{+0.2}_{-0.2}$ | 0.23$^{+0.11}_{-0.11}$ |
| $^{108}$Pd | 0.20$^{+0.20}_{-0.20}$ | 0.38$^{+0.10}_{-0.4}$ | 0.24$^{+0.10}_{-0.8}$ | 0.11$^{+0.11}_{-0.11}$ | 0.16$^{+0.2}_{-0.2}$ | 0.35$^{+0.11}_{-0.17}$ |
| $^{112}$Cd | 0.43(6) | 0.14(3) | 0.35(5) | 0.27(8) | 0.16(5) | 0 |
| $^{114}$Cd | 0.31(4) | 0.13(6) | 0.18(3) | 0.26(6) | 0.090(5) | 0.02(9) |
| $^{156}$Gd | 3.79(11) | 3.72(15) | 3.79(15) | < 0.18 | n.a. | < 0.18 |
| $^{158}$Gd | 4.19(11) | 4.04(16) | 4.05(18) | 0.17(20) | n.a. | < 0.28 |
| $^{160}$Gd | 4.20(10) | 4.33(17) | 4.23(14) | < 0.09 | n.a. | 0.11(26) |
| $^{164}$Dy | 4.09(22) | 4.12(81) | 4.46(15) | 0 | n.a. | < 0.60 |
FIG. 1. The deviations of the factors $c^{N}_{02}$ and $c^{N}_{04}$ from the limiting values, calculated gridwise over the whole ECQF symmetry space for $N = 10$ bosons.