Smoothness, Semistability, and Toroidal Geometry

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0. INTRODUCTION

0.1. Statement. We provide a new proof of the following result:

Theorem 0.1 (Hironaka). Let $X$ be a variety of finite type over an algebraically closed field $k$ of characteristic 0, let $Z \subset X$ be a proper closed subset. There exists a modification $f : X_1 \rightarrow X$, such that $X_1$ is a quasi-projective nonsingular variety and $Z_1 = f^{-1}(Z)_{\text{red}}$ is a strict divisor of normal crossings.

Remark 0.2. Needless to say, this theorem is a weak version of Hironaka’s well known theorem on resolution of singularities. Our proof has the feature that it builds on two standard techniques of algebraic geometry: semistable reduction for curves, and toric geometry.

Remark 0.3. Another proof of the same result was discovered independently by F. Bogomolov and T. Pantev [B-P]. The two proofs are similar in spirit but quite different in detail.

0.2. Structure of the proof.

1. As in [dJ], we choose a projection $X \rightarrow P$ of relative dimension 1, and apply semistable reduction to obtain a model $X' \rightarrow B$ over a suitable Galois base change $B \rightarrow P$, with Galois group $G$.

2. We apply induction on the dimension to $B$, therefore we may assume that $B$ is smooth, and that the discriminant locus of $X' \rightarrow B$ is a $G$-strict divisor of normal crossings.

3. A few auxiliary blowups make the quotient $X'/G$ toroidal.

4. Theorem 11* of [KKMS] about toroidal resolutions finishes the argument.

0.3. What we do not show.

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0.3.1. **Canonicity.** Our proof has the drawback that the resolution is noncanonical. Some of the steps are not easily carried out in practice, and in fact we almost always blow up in the smooth locus. However, it follows from the proof, that if \( C \subset X_{ns} \) is a curve, then we can guarantee that \( X' \to X \) is an isomorphism in a neighborhood of \( C \). We do expect a slight modification of our argument to give equivariant resolution of singularities in the case where a finite group acts.

0.3.2. **Positive characteristic.** There is one crucial point where the proof fails if \( \text{char } k = p > 0 \). This is at the point where we claim that the quotient \( X'/G \) is toroidal. Given \( x \in X' \) and \( g \in \text{Stab } x \) of order \( p \), the action of \( g \) on \( \mathcal{O}_{X',x} \) is unipotent. Thus even if \( X' \) is toroidal, we cannot guarantee that the quotient is toroidal as well. It might happen that the quotient is toroidal by accident, and it would be interesting to see to what extent such accidents can be encouraged to happen.

In any case, the quotient step goes through if \( p \not| \# G \). See remark 1.10 for a discussion of a bound on \( \# G \). As a result, there exists a function

\[
M : \{ \text{varieties with subvarieties} \} \to \mathbb{Z}
\]

which is bounded on any bounded family, and which is “describable” in a geometrically meaningful way (\( M \) for “multi-genus”), such that whenever \( p > M([X \supset Z]) \), our proof goes through for the pair \( X \supset Z \). We were informed by T. Scanlon and E. Hrushovski that the existence of a function \( M \) which is bounded on bounded families is known, by an application of the compactness theorem in model theory, for any resolution process, in particular Hironaka’s. A proof of this was given in [2].

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0.5. **Terminology.** We recall some definitions; we restrict ourselves to the case of varieties over \( k \).

A **modification** is a proper birational morphism of irreducible varieties.

An **alteration** \( a : B_1 \to B \) is a proper, surjective, generically finite morphism of irreducible varieties, see [13, 2.20]. The alteration \( a \) is a **Galois alteration** if there is a finite group \( G \subset \text{Aut}_B(B_1) \) such that the associated morphism \( B_1/G \to B \) is birational, compare [12, 5.3].

Let a finite group \( G \) act on a (possibly reducible) variety \( Z \). Let \( Z = \bigcup Z_i \) be the decomposition of \( Z \) into irreducible components. We say that \( Z \) is **\( G \)-strict** if the union of translates \( \bigcup_{g \in G} g(Z_i) \) of each component \( Z_i \) is a normal variety.
We simply say that $Z$ is strict if it is $G$-strict for the trivial group, namely every $Z_i$ is normal.

A divisor $D \subset X$ is called a divisor of normal crossings if étale locally at every point it is the zero set of $u_1 \cdots u_k$ where $u_1, \ldots, u_k$ is part of a regular system of parameters. Thus a strict divisor of normal crossings is what is usually called a divisor of strict normal crossings, i.e., all components of $D$ are nonsingular.

An open embedding $U \hookrightarrow X$ is called a toroidal embedding if locally in the étale topology (or classical topology in case $k = \mathbb{C}$, or formally) it is isomorphic to a torus embedding $T \hookrightarrow V$, (see [KKMS], II§1). If $D = X \setminus U$, we will sometimes denote this toroidal embedding by $(X, D)$. A finite group action $G \subset \text{Aut}(U \hookrightarrow X)$ is said to be toroidal if the stabilizer of every point is identified on the appropriate neighborhood with a subgroup of the torus $T$. We say that a toroidal action is $G$-strict if $X \setminus U$ is $G$-strict. In particular the toroidal embedding itself is said to be strict if $X \setminus U$ is strict. This is the same as the notion of toroidal embedding without self-intersections in [KKMS].

The fundamental theorem about toroidal embeddings we will exploit is the following:

**Theorem 0.4** ([KKMS], II §2, Theorem 11*, p. 94). For any strict toroidal embedding $U \hookrightarrow X$ there exists a canonical sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ such that the blowup $B_\mathcal{I}(X)$ is nonsingular.

(See [KKMS], II §2, definition 1 for the notion of canonical modification.)

1. **The proof**

1.1. **Reduction steps.** We argue by induction on $d = \dim X$. The case $d = 1$ is given by normalizing $X$.

As in [dJ], 3.6-3.10 we may assume that

1. $X$ is projective and normal, and
2. $Z$ is the support of a divisor.

Recall that according to [dJ], lemma 3.11 there is a modification $X' \to X$ and a generically smooth morphism $f : X' \to P = \mathbb{P}^{d-1}$ satisfying the following properties:

3. The modification $X' \to X$ is isomorphic in a neighborhood of $Z$,
4. every fiber of $f$ is of pure dimension 1,
5. the smooth locus of $f$ is dense in every fiber, and
6. the morphism $f|_Z : Z \to P$ is finite.
Since the construction in \[\mathcal{dJ}\] is obtained via a general projection, we can guarantee that

7. the generic fiber of \(f : X' \to P\) is a geometrically connected curve (this follows from \([\mathcal{J}], 6.3(4), \text{see } \mathcal{N}, 4.2\)).

We now replace \(X\) by \(X'\).

Applying Lemma 3.13 of \([\mathcal{dJ}]\), we may choose a divisor \(D \subset X\) mapping finitely to \(P\), which meets every component of every fiber of \(f\) in at least 3 smooth points. We may replace \(Z\) by \(Z \cup D\) (see \([\mathcal{dJ}, 3.9]\)), and therefore we have

8. \(Z\) meets every component of every fiber of \(f\) in at least 3 smooth points.

Let \(\Delta_X \subset P\) be the discriminant locus of the map \(X \to P\), and let \(\Delta_Z \subset P\) be the discriminant locus of the map \(Z \to P\). Let \(\Delta = \Delta_Z \cup \Delta_X\). We can think of \(\Delta\) as the discriminant locus of the pair \(Z \subset X\) → \(P\).

The inductive assumption gives us a resolution of singularities of \(\Delta \subset P\). Thus we may replace \(X, Z, P\) and assume that \(P\) is smooth and \(\Delta\) is a strict divisor of normal crossings. Let \(\nu : X^{\text{nor}} \to X\) be the normalization. By \([\mathcal{dJ-O}]\), the discriminant locus of \(\nu^{-1}Z \subset X^{\text{nor}} \to P\) is a divisor \(\Delta'\) contained in \(\Delta\). We replace \(X\) by \(X^{\text{nor}}\) and \(\Delta\) by \(\Delta'\). Thus we may assume in addition to 1 - 8, that

9. \(P\) is smooth, and
10. \(\Delta\) is a strict normal crossings divisor.

1.2. Stable reduction. We are now ready to perform stable reduction. We follow \([\mathcal{dJ}], 3.18-3.21\), but see Remark 1.2 below. Let \(j : U_0 = P \setminus \Delta \hookrightarrow P\). Let \(a_U : U'_0 \to U_0\) be an étale Galois cover which splits the projection \(Z_{U_0} = j^{-1}(U_0) \cap Z \to U_0\) into \(n\) sections, and trivializes the 3-torsion subgroup in the relative Jacobian of \(X_{U_0} \to U_0\). Let \(G\) be the Galois group of this cover. Let \(P^\sharp\) be the normalization of \(P\) in the function field of \(U'_0\). Let \(g\) be the genus of the generic fiber of \(X \to P\). Let us write \(\overline{3M}_{g,n}\) for the Deligne-Mumford compactification of the moduli scheme of \(n\)-pointed genus \(g\) curves with an abelian level 3 structure, see \([\mathcal{DM}], [\mathcal{PJ}, 2.3.7], [\mathcal{dJ}, 2.24]\) and references therein. There is a ‘universal’ \(\text{stable } n\text{-pointed curve over } \overline{3M}_{g,n}\). We take the closure of the graph of the morphism \(U'_0 \to \overline{3M}_{g,n}\) in \(P^\sharp \times \overline{3M}_{g,n}\) and obtain a modification \(P' \to P^\sharp\) and a family of \(\text{stable pointed curves } X' \to P'\). Note that \(P' \to P^\sharp\) blows up outside of \(U'_0\). We perform a \(G\)-equivariant blow up of \(P'\) to ensure that we have a morphism \(r : X' \to X \times_P P'\), see \([\mathcal{dJ}, 3.18, 3.19 \text{ and } 7.6]\). Again this blow up has center outside \(U'_0\). In summary:

**Situation 1.1.** There is a Galois alteration \(a : P' \to P\), with Galois group \(G\), and a modification \(r : X' \to X \times_P P'\) satisfying:

a. the morphism \(a\) is finite étale over \(U_0\),
b. the morphism \( r \) is an isomorphism over the open set \( U_0 \times_P P' \),
c. there are \( n \) sections \( \sigma_i : P' \to X' \) such that the proper transform \( Z' \) of \( Z \) is the union of their images, and
d. \( (X' \to P', \sigma_1, \ldots, \sigma_n) \) is a stable pointed curve of genus \( g \).

Remark 1.2. The results of [dJ-O] imply that the curve \( X' \) exists over the variety \( P' \). We suspect that the morphism \( r \) exists over \( P' \) as well. If this is true, the following step is redundant. See [K] for a similar construction using Kontsevich’s space of stable maps.

We want to change the situation such that we get \( P' = P'' \). This we can do as follows. Take a blow up \( \beta : P_1 \to P \) such that the strict transform \( P'_1 \) of \( P' \) with respect to \( \beta \) is finite flat over \( P_1 \), see [RG]. By our inductive assumption, we may assume that \( P_1 \) is nonsingular and that the inverse image \( \Delta_1 \) of \( \Delta \) in \( P_1 \) is a divisor with normal crossings. The Galois covering \( U'_0 \to U_0 \) pulls back to a Galois covering of \( P_1 \setminus \Delta_1 \), and we get a ramified normal covering \( P''_{P_1} \) of \( P_1 \). It follows that \( P''_{P_1} \) maps to the strict transform \( P'_1 \). Hence it is clear that if we replace \( P \) by \( P_1 \), \( \Delta \) by \( \Delta_1 \), \( X' \) by \( X' \times_P P''_{P_1} \), then the morphism \( r \) of 1.1 exists over \( P''_{P_1} \).

We now have in addition to (a) - (d),
e. \( P \) is smooth, \( \Delta \) is a strict divisor of normal crossings, and \( a : P' \to P \) is finite.

Since stable reduction over a normal base is unique (see [dJ-O], 2.3) we have that the action of \( G \) lifts to \( X' \). Note that the \( G \)-action does not preserve the order of the sections. Note also that since the local fundamental groups of \( P \setminus \Delta \) are abelian, so is the stabilizer in \( G \) of any point in \( P' \). The same is true for points on \( X' \). This property will therefore remain true for any equivariant modification of \( P' \), as well as for points on \( X' \).

1.3. Local description. Denote by \( q : P' \to P \) the quotient map. Let \( p \in P \) and let \( p' \in q^{-1}(p) \). Let \( s_1, \ldots, s_{d-1} \in \mathcal{O}_{P,p} \) be a regular system of parameters on \( P \) at \( p \) such that \( \Delta_x = V(s_1 \cdots s_r) \). Recall that the stabilizer of \( p' \) is abelian; this actually follows from (a)-(e) above as the morphism \( P' \to P \) is ramified only along the divisor of normal crossings \( \Delta \). Writing \( t_i^n = s_i \) for suitable \( n \), we identify a formal neighbourhood of \( p' \) in \( P' \) as a quotient of the smooth formal scheme \( P'' = \text{Spf} k[[t_1, \ldots, t_r, s_{r+1}, \ldots, s_{d-1}]] \) by the finite group \((\mathbb{Z}/n\mathbb{Z})^r\) acting by \( n \)-th roots of unity on the \( t_i \). Thus a formal neighbourhood of \( p' \) in \( P' \) is the quotient of \( P'' \) by a suitable subgroup \( H \subset (\mathbb{Z}/n\mathbb{Z})^r \). This implies that \((\mathbb{Z}/n\mathbb{Z})^r/H \) is identified with the stabilizer of \( p' \) in \( G \).

Denote by \( X'' = P'' \times_{P'} X' \to P'' \) the pulled back stable pointed curve. As \((\mathbb{Z}/n\mathbb{Z})^r/H \subset G \) acts on \( X' \) over \( P' \), we get an action of \((\mathbb{Z}/n\mathbb{Z})^r \) on \( X'' \) over
Let $x \in X''$ be a closed point lying over $p'$, and let $G_x$ be the stabilizer of $x$ in $(\mathbb{Z}/n\mathbb{Z})'$. There are two cases:

i. (Smooth case) Here $x$ is a smooth point of the morphism $X'' \to P''$. In this case the completion of $X''$ at $x$ is isomorphic to the formal spectrum of the $k[[t_1, \ldots, t_r, s_{r+1}, \ldots, s_{d-1}]]$-algebra

$$k[[t_1, \ldots, t_r, s_{r+1}, \ldots, s_{d-1}][[x]].$$

Here we chose some coordinate $x$ along the fiber such that $G_x$ acts by a character $\psi_x$ on $x$. There are two cases with respect to the position of the sections $Z'' \subset X''$:

(a) The point $x \in Z''$. In this case, since $Z''$ is invariant under the action of $G_x$, we can choose the coordinate $x$ so that $Z'' = V(x)$.

(b) The point $x \notin Z''$. Note that the coordinate $x$ is not uniquely chosen, and therefore the locus $x = 0$ is not uniquely determined. However, in case $\psi_x$ is nontrivial if we want to see $G_x$ as acting through the torus for some toroidal structure on $X''$ at $x$ it is necessary to include a locus like $V(x)$ in the boundary.

ii. (Node case) Here $x$ is a node of the fiber of $X'' \to P''$ over $p''$. In this case the completion of $X''$ at $x$ is isomorphic to the formal spectrum of the $k[[t_1, \ldots, t_r, s_{r+1}, \ldots, s_{d-1}]]$-algebra

$$k[[t_1, \ldots, t_r, s_{r+1}, \ldots, s_{d-1}]]/[x, y]/(xy - t_1^{k_1} \cdots t_r^{k_r}).$$

We may choose the coordinates $x, y$ such that there is a subgroup $G_d$ of $G_x$ of index at most 2 such that $G_d$ acts by characters on $x, y$, but elements of $G_x$ not in $G_d$ switch the fiber components $x = 0$ and $y = 0$.

We would like to have the stabilizers acting toroidally on $X''$ in such a way that the quotient $X$ becomes strict toroidal.

1.4. Making the group act toroidally. There are two issues we need to resolve: in case (ii) above, we want to modify so that $G_x/G_d$ disappears from the picture. In case (i) we need to modify so that the stratum $x = 0$ is not necessary for the toroidal description of the $G$-action.

In order to describe a global modification we go back to our stable pointed curve $f : X' \to P'$.

1.4.1. Separating branches along nodes. Let $S = \text{Sing } f$ be the singular scheme of the projection $f$. Let $Y' = B_S(X') \to X'$ be the blowup of $X'$ along $S$. Let $Y''$ be the fiber product $Y' \times_{P'} X''$, and let $y'' \in Y''$. We remark that neither $Y'$ nor $Y''$ is normal in general.

We want to give a local description of $Y'$ and $Y''$. We use the notation $X_{/x}$ to denote the completion of $X$ at the closed point $x \in X(k)$, and similar
for the other varieties occurring below. We have $P'/P' = \text{Spf } R$ where $R$ is the ring $(k[[t_1, \ldots, t_r, s_{r+1}, \ldots, s_{d-1}]])^H$, with $H$ as in 1.3 and we have $X'_{x'} = \text{Spf } R[[x,y]]/(xy - h)$ for some monomial $h \in R \subset k[[t_1, \ldots, t_r, s_{r+1}, \ldots, s_{d-1}]]$. Finally $S = V(x, y)$ scheme theoretically.

From this it is easy to read off the following local description, using that blowing up commutes with completion in a suitable manner.

i’. (Smooth case) $Y''_{y''} \simeq X''_{x''}$ as above.

ii’. (Node case) $Y''_{y''} \simeq \text{Spf } k[[t_1, \ldots, t_r, s_{r+1}, \ldots, s_{d-1}][x, z]/(xz^2 - t_1^l \cdots t_r^l)$.

The stabilizer $G_{y''}$ acts diagonally on $t_i, x, z$ (i.e., it acts via characters on these elements).

iii’. (Double case) $Y''_{y''} \simeq \text{Spf } k[[t_1, \ldots, t_r, s_{r+1}, \ldots, s_{d-1}][y, z]/(z^2 - t_1^l \cdots t_r^l)$.

The stabilizer $G_{y''}$ acts diagonally, but as in (i) the coordinate $x$ may not be determined.

The descriptions above determine the local structure of $Y'$ also. Indeed, $Y'_{y'}$ is simply the quotient of $Y''_{y''}$ by the group $H$ which acts trivially on the coordinate(s) $x$ (and $z$). We are actually interested in a local description of the normalization $Y = (Y')^{\text{nor}}$ of $Y'$.

Let $D \subset Y = (Y')^{\text{nor}}$ be the union of the inverse image of $\Delta$ and of $Z$ in $Y$. From the local descriptions given above it is already clear that $Y \setminus D \hookrightarrow Y$ is a strict toroidal embedding. Moreover, $D$ is $G$-strict, since $\Delta$ is strict and since the blowup $Y \to X'$ separates fiber components. However, the action of $G$ on the pair $(Y \setminus D, Y)$ is not yet toroidal: indeed, in case (i) and in case (iii) if the character $\psi_x$ is nontrivial, we have a problem. More precisely, this is the situation explained in (ii).

1.4.2. Torifying a pre-toroidal action. Here we show how to do one canonical blow up $Y_1 \to Y$ (analogously to [KKMS], II §2) which makes the action of $G$ toroidal. The situation $(Y, D, G)$ we reached at the end of Section 1.4.1 is summarized by the conditions in Definition 1.4 below.

Remark 1.3. We expect that this discussion should be of interest in a more general setting.

**Definition 1.4.** Let $U = Y \setminus D \subset Y$ be a toroidal embedding, $G \subset \text{Aut}(U \subset Y)$ a finite subgroup, such that $D$ is $G$-strict. For any point $y \in Y$ denote the stabilizer of $y$ by $G_y$. We say that the action of $G$ is **pre-toroidal** if at every point $y \in Y$, either $G_y$ acts toroidally at $y$, or the following situation holds:

- There exists an isomorphism $\epsilon : Y/y \cong \text{Spf } R[[x]]$,
- where $\text{Spf } R$ is the completion of a toroidal embedding $T_0 \hookrightarrow Y_0$ at a point $y_0 \in Y_0$,
• where $D/y$ corresponds to $(Y_0 \setminus T_0)/y_0 \times_{\text{Spf } k} \text{Spf } k[[x]]$,
• where $G_y$ acts toroidally on $T_0 \hookrightarrow Y_0$ fixing $y_0$ and
• $G_y$ acts on the coordinate $x$ via a character $\psi_x$ such that the isomorphism $\epsilon$ of completions is $G_y$-equivariant. In other words, $G_y$ acts toroidally on $$(T_0)/y_0 \times_{\text{Spf } k} \text{Spf } k[[x, x^{-1}]] \subset Y/y.$$ Analogously to definition 1 of [KKMS], II §2, we define pre-canonical ideals:

**Definition 1.5.** Let $G \subset \text{Aut}(U \subset Y)$ be a pre-toroidal action. A $G$-equivariant ideal sheaf $\mathcal{I} \subset \mathcal{O}_Y$ is said to be **pre-canonical** if the following holds:

For any $y, y'$ lying on the same stratum, and any isomorphism $\alpha : O_{y/y} \to O_{y'/y'}$ preserving the strata and inducing an isomorphism carrying $G_y$ to $G_{y'}$, we have $\alpha(\mathcal{I}/y) = \mathcal{I}/y'$.

If $\mathcal{I}$ is pre-canonical, we say that the normalized blowup $b : (B_\mathcal{I}Y)^{\text{nor}} \to Y$ is a pre-canonical blowup.

**Definition 1.6.** We say that a pre-canonical blowup $\widetilde{Y} \to Y$ **torifies** $Y$ if $G$ acts toroidally on $(b^{-1}U \subset \widetilde{Y})$.

**Theorem 1.7.** Let $G \subset \text{Aut}(U \subset Y)$ be a pre-toroidal action. Then there exists a canonical choice of a pre-canonical ideal sheaf $\mathcal{I}_G$ such that the pre-canonical blowup $b : (B_{\mathcal{I}_G}Y)^{\text{nor}} \to Y$ torifies $Y$.

The theorem follows immediately from the affine case below:

**Proposition 1.8.** Let $T_0 \subset X_0$ be an affine torus embedding, $X_0 = \text{Spec } R$. Let $G \subset T_0$ be a finite subgroup of $T_0$, let $p_0 \in X_0$ be a fixed point of the action of $G$, and let $\psi_x$ be a character of $G$. Consider the torus embedding of $T = T_0 \times \text{Spec } k[[x, x^{-1}]]$ into $X = X_0 \times \text{Spec } k[[x]]$, where we let $G$ act on $x$ via the character $\psi_x$. Assume that the map $G \to T$ induced from this is injective.

Write $p = (p_0, 0) \in X$ and write $D = (X_0 \setminus T_0) \times \text{Spec } k[[x]]$. There is a canonical $T$-equivariant ideal $I_G \subset R[[x]]$, satisfying the following:

Let $b : X_1 = (B_{I_G}X)^{\text{nor}} \to X$, the normalization of the blowup of $X$ along $I_G$. Let $U_1 = b^{-1}(T_0 \times \text{Spec } k[[x]])$. Then $U_1 \hookrightarrow X_1$ is toroidal and $G$ acts toroidally on $U_1 \hookrightarrow X_1$.

If $X'_0, T'_0, G', p'_0$ and $\psi'_x$ is a second set of such data, and if we have an isomorphism of completions

$$\varphi : X/p \cong X'/p',$$

which induces isomorphisms $G \cong G'$ and $D/p \cong D'/p'$, then $\varphi$ pulls back $I_G$ to the ideal $I_{G'}$.

Furthermore, if $q_0$ is any point of $X_0$ and if $G_q \subset G$ is the stabilizer of $q$ in $G$, then the stalk of $I_G$ at $q$ is the same as the stalk of $I_{G_q}$ at $q$.
Remark 1.9. The ideal $I_G$ is called the **torific ideal** of the situation $G \subset \text{Aut}(T_0 \times \text{Spec } k[x] \subset X)$.

**Proof.** For any monomial $t \in R[x]$ let $\chi_t$ be the associated character. We restrict these characters to $G$ and obtain a character $\psi_t : G \to k^*$.

Define $M_x = \{t|\psi_t = \psi_x\}$, the set of monomials on which $G$ acts as it acts on $x$. Notice that the character $\psi_x$ of $G$ is uniquely determined by the data $G \to \text{Aut}(X/p, D/p)$. Define the torific ideal $I_G = \langle M_x \rangle$, the ideal generated by $M_x$. We see that it satisfies the variance property of the proposition with respect to isomorphisms $\varphi$. We leave the localization property at the end of the proposition as an exercise to the reader.

Define $b : X_1 \to X$ as in the proposition.

Since $G$ is a subgroup of $T_0$, we have that $M_x$ contains a monomial in $R$. Therefore the ideal $I_G$ is generated by $x$ and a number of monomials $t_1, \ldots, t_m \in R \cap M_x$. The blow up has a chart associated to each of the generators $x, t_1, \ldots, t_m$. On the chart “$x \neq 0$” we have that the inverse image of $D$ contains the inverse image of $D \cup V(x)$, and hence the action of $G$ is toroidal, being toroidal with respect to $T \subset X$ on $X$. The other charts “$t_i \neq 0$” can be described as the spectra of the rings

\[
\tilde{R} = R[x][u, \{s_j\}_{j \neq i}]/(ut_i - x, s_j t_i - t_j, h_\alpha),
\]

that is $u = x/t_i$ and $s_j = t_j/t_i$. Since there are no relations between $x$ and $t_j$ we can take the $h_\alpha$ to be certain polynomials in $R[s_j]$ (the ideal generated by the $t_i$ is flat over $k[x]$). Note that $G$ fixes the element $u$ in this algebra.

Thus it follows that the normalization of the ring $\tilde{R}$ is of the form $R'[u]$, where $G$ acts trivially on $u$, $R'$ is the ring associated to an affine torus embedding and $G$ acts toroidally on Spec $R'$.

1.4.3. **Strictness.** We return to the triple $(Y, D, G)$ we obtained at the end of 1.4.1, in particular we have the $G$-equivariant map $Y \to P'$ and the Galois alteration $P' \to P$ with group $G$. Denote by $b : Y_1 \to Y$ the torifying blowup obtained by normalizing the blowup at the torific ideal of $Y$, as in Theorem 1.7. It remains to check that the divisor $b^{-1}(D) \subset Y_1$ is $G$-strict, so that the quotient is a strict toroidal embedding.

First, although a coordinate $x$ for the pre-toroidal action is not globally defined, we may always find such coordinates on Zariski neighborhoods of the relevant points. Let $y \in Y$ be such a point. Let $\epsilon : Y/y \cong \text{Spf } R[[x]]$ be as in Defintion 1.4. Let $x' \in O_{Y,y}$ be an element of the local ring of $Y$ at $y$ that is congruent to $\epsilon^{-1}(x)$ up to a high power of the maximal ideal. Then the element

\[
x'' = \sum_{g \in G_y} g(x') \psi_x^{-1}(g)/|G_y| \in O_{Y,y}
\]
transforms according to \( \psi_x \) under the action of \( G_y \) and is congruent to \( \epsilon^{-1}(x) \) up to a high power of the maximal ideal. We may then change the isomorphism \( \epsilon \) so that the element \( x'' \) will correspond to the coordinate \( x \). Therefore, we may assume that there is a \( G_y \)-invariant Zariski open neighbourhood \( W = W_y \) of \( y \) and a function \( x \in \Gamma(W, \mathcal{O}) \) that transforms according to the character \( \psi_x \) under the group \( G_y \), and giving rise to the local coordinate for a suitably chosen isomorphism \( \epsilon \) as in Definition 1.4. After possibly shrinking \( W \), we get that \( W \) is strict toroidal with respect to the divisor \( D_W = (W \cap D) \cup V(x) \). If we choose \( W \) sufficiently small, we may assume that the conical polyhedral complex of \( (W, D_W) \) a single cone \( \sigma_W \) (see [KKMS], II §1 definition 5, through p. 71).

Now, to show that \( b^{-1}(D) \) is strict, it suffices to show this on an affine neighborhood of any point. This follows immediately from theorem 1* of [KKMS], applied to \( (W, D_W) \).

At this point it should be remarked that we could get away without \( G \)-strictness: using the constructions in [KKMS], I §2 lemmas 1-3 on pages 33-35, and [14], 7.2, it is not difficult to construct a \( G \)-equivariant blowup which is a \( G \)-strict toroidal embedding. Still it is of interest to know that \( b^{-1}(D) \subset Y_1 \) is already \( G \)-strict.

Let \( E \subset b^{-1}(D) \) be a component and let \( g \in G \) be an element such that \( g(E) \cap E \neq \emptyset \). We have to show that \( g(E) = E \). In the case that \( b(E) \) is a component of \( D \), this follows immediately from the \( G \)-strictness of \( D \), see end of [14], the difficult case is that of a component that is collapsed under \( b \).

Let \( y \) be a general point of the intersection \( g(E) \cap E \), and let \( y = b(y) \). Let \( D_1, \ldots, D_s \) be the components of \( D \) at \( y \). The components \( g^{-1}(D_i) \) are the components of \( D \) at \( g^{-1}(y) \). After reordering, we may assume that \( D_1, \ldots, D_r \) are the components which contain \( b(E) \), for some \( r \leq s \). As \( y \in g(E) \cap E \) we get \( y \in g(b(E)) \cap b(E) \), hence \( y \in g(D_i) \cap D_i \) for all \( i = 1, \ldots, r \). By \( G \)-strictness of \( D \), we get \( g(D_i) = D_i \) for \( i = 1, \ldots, r \).

Let \( \sigma_y \) be the cone corresponding to the toroidal structure \( D_{W_y} \) at \( y \) of the second paragraph above. The divisors \( D_i \) correspond to rays \( \tau_{D_i, y} \) in the boundary of \( \sigma_y \). The divisor \( \bar{E} \) corresponds to a ray \( \tau_{E, y} \) in \( \sigma_y \), belonging to the polyhedral decomposition associated to the given blowup. For the Zariski open neighbourhood \( W_{g^{-1}(y)} \) we take \( g^{-1}(W_y) \) and for the divisor \( D_{W_{g^{-1}(y)}} \) we take \( g^{-1}(D_{W_y}) \). Hence we get an identification \( \sigma_y \cong \sigma_{g^{-1}(y)} \) given by \( g^{-1} \). Under this identification the ray \( \tau_{E, y} \) is mapped to the ray \( \tau_{g^{-1}(E), g^{-1}(y)} \). Note that \( E \) passes through \( g^{-1}(y) \), hence we have the ray \( \tau_{E, g^{-1}(y)} \) in \( \sigma_{g^{-1}(y)} \). We have to show that \( \tau_{E, g^{-1}(y)} = \tau_{g^{-1}(E), g^{-1}(y)} \).
Let \( M_y \) be the group of Cartier divisors supported along \( D_{W_y} \) (see [KKMS], II §1, definition 3). After shrinking \( W_y \) we may assume that \( M_y \) is the free abelian group generated by the divisors of rational functions \( f_j \in \Gamma(W_y, \mathcal{O}), j = 1, \ldots, m \) and the function \( x \) on \( W_y \). We may consider \( f_j \) and \( x \) as rational functions on \( Y_1 \) as well. The ray \( \tau_{E,y} \) is determined by the order of vanishing of the functions \( f_j \) and \( x \) along \( E \) on \( Y_1 \), let us call these orders \( n_j \) and \( n \). Similarly, \( \tau_{E,g^{-1}(y)} \) (resp. \( \tau_{g^{-1}(E),g^{-1}(y)} \)) is determined by the order of vanishing of the functions \( g^*(f_j) \) and \( g^*(x) \) along \( E \) (resp. \( g^{-1}(E) \)), let us call these orders \( n'_j \) and \( n' \) (resp. \( n''_j \) and \( n'' \)).

It is clear that \( n_j = n''_j \) and \( n = n'' \). Notice that \( g^*(f_j) \) are supported along \( g^{-1}(D_i) \) in \( W_{g^{-1}(y)} \). Here \( g^{-1}(D_i) = D_i \) for \( i \leq r \), so that \( f_j/g^*f_j \) has order 0 along \( D_i, i \leq r \). Since the vanishing along \( E \) may be computed on \( b^{-1}(W_y \cap W_{g^{-1}(y)}) \), we see that \( n_j = n'_j \). We are through if we show that \( n = n' \); this is equivalent to showing that the order of vanishing of \( x \) along \( E \) is the same as the order of vanishing of \( g^{-1}(x) \) along \( E \).

The component \( E \) is exceptional for the morphism \( b \). The torifying blowup \( b \) over \( W_y \) blows up inside \( V(x) \) only. Hence we see that \( b(E) \cap W_y \) is contained in \( V(x) \), i.e., \( n > 0 \). The same argument applied to \( g^{-1}(x) \) on \( W_{g^{-1}(y)} \) works to show that \( n' > 0 \). For a general point \( z \in b(E) \), we see that both \( x \) and \( g^{-1}(x) \) define a local coordinate that can be used to define the local toroidal structure around \( z \), as in Definition 1.4. Therefore, we need only to show that the valuation of \( x \) along \( E \) does not depend on the choice of the parameter \( x \) as in Definition 1.4.

Let \( \epsilon : Y_{/z} \cong \text{Spf } R[[x]] \) be as in Definition 1.4. Any other \( x' \in R[[x]] \) that gives a local coordinate for some pretoroidal structure is of the form

\[
x' = ux + r_0 + \sum_{j \geq 2} r_j x^j,
\]

with \( u, r_0, r_j \in R \) and where \( u \) is a unit and \( r_0 \) is in the maximal ideal of \( R \). The element \( u \) being a unit, we may divide by it without changing the orders of vanishing on \( Y_1 \). Thus we may assume \( u = 1 \). Consider the family of automorphisms \( \varphi_t, t \in [0, 1] \) of \( R[[x]] \) given by \( \varphi_t(r) = r, r \in R \) and

\[
\varphi_t(x) = x + t \left( r_0 + \sum_{j \geq 2} r_j x^j \right).
\]

These are automorphisms that occur in Proposition 1.8. Hence these act on the formal completion \( Y_1^{\wedge} \) of \( Y_1 \) along \( b^{-1}(z) \), in view of Proposition 1.8. Since they form a continuous family (acting continuously on the charts described in the proof of Proposition 1.8), they will fix (formal) components of \( b^{-1}(D_z) \) such as \( E^{\wedge} \) in \( Y_1^{\wedge} \). Hence the orders of vanishing of the functions \( \varphi_t(x) \) along the
(formal) component $E^\wedge$ are all the same. In particular, we get the equality for $x$ and $x'$, as desired.

1.5. **Conclusion of proof.** If we combine the results of 1.4.1, 1.4.2 and 1.4.3 then we see that we may assume our Galois alteration $(Y_1, D_1, G)$ of $(X, Z)$ is such that $Y \setminus D_1 \leftrightarrow Y_1$ is a $G$-strict toroidal embedding. Hence the quotient $(Y_1 \setminus D_1)/G \leftrightarrow Y_1/G$ is a strict toroidal embedding, and $Y_1/G \rightarrow X$ is a modification. Hence we may replace $(X, Z)$ by $(Y_1/G, D_1/G)$. By Theorem 0.4, there exists a (toroidal) resolution of singularities of this pair. This ends the proof of Theorem 0.1. 

**Remark 1.10.** If char $k = p > 0$ the proof goes through if it works on $P$, and if the exponent of the Galois group $G$ is small enough. Since $G$ can be taken as the Galois group of the torsion on a generalized Jacobian, we can bound it in terms of $g + n$, where $g$ is the relative genus and $n$ is the degree of the divisor $Z$. Given a family of varieties of finite type $X \rightarrow S$, we can make the constructions in the proof uniform over $S$. Roughly speaking, after replacing $S$ by a dense open, there is a sequence of projections $X \rightarrow X_1 \rightarrow \cdots \rightarrow S$ and relative divisors $D_i \subset X_i$ which will do the job. Therefore the order of all groups involved is bounded in terms of the relative genus of $X_i \rightarrow X_{i+1}$ and the degree of $D_i \rightarrow X_{i+1}$. Thus there is a “geometrically meaningful” function $M$, as described in 0.3.2.

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