A new solution for inflation

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Abstract

Many pedagogical introductions of inflation are effective due to the simplicity of the relevant equations. Here an analytic solution of the cosmological equations is presented and used as an example to discuss fundamental aspects of the inflationary paradigm.

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1 Introduction

The standard big bang cosmology\(^1\)-\(^4\) is a very successful theory of the universe with certain drawbacks. They include the horizon, flatness and monopole problems\(^5\),\(^6\) or, in short, the ability to explain the observed universe only when the initial conditions are extremely fine-tuned. Cosmic inflation solves the problems of big bang cosmology and was introduced\(^7\) at the beginning of the 80’s. Today, almost 20 years later, inflation is incorporated into modern cosmology and is a mature topic for inclusion in introductory courses in gravitation and cosmology.

Inflation consists of a short period of accelerated superluminal expansion of the early universe, at the end of which the description of the standard big bang model is applied. During the inflationary epoch, the matter content of the universe has an equation of state very close to that of the quantum vacuum, \(P = -\rho\) (where \(\rho\) and \(P\) are, respectively, the energy density and pressure of matter). Inflation also provides a mechanism for the generation of density perturbations through quantum fluctuations of the scalar field which is supposed to drive the cosmic expansion\(^4\),\(^8\)-\(^12\).

Although the original scenario of inflation and many others proposed to date are based on specific particle physics theories, the point of view of modern cosmology has shifted: inflation is currently regarded as a paradigm, a general idea that can be implemented in a variety of ways to describe the early universe. There are many inflationary scenarios in the literature, but none is accepted as compelling and there is no “standard model” of inflation. In this phenomenological point of view it is possible to present cosmological inflation in the classroom in an easy way, without previous knowledge of advanced particle physics. The task is made feasible by the simplicity of the equations of inflation, ordinary differential equations which are solved in the slow rollover approximation, a simplification used in most inflationary theories. Inflation is the subject of numerous textbooks and introductory review papers\(^4\),\(^8\),\(^10\)-\(^13\) as well as science popularization articles and books\(^14\).

Important issues in understanding inflation are the slow rollover approximation, the equivalence between a constant scalar field potential and the cosmological constant, and the correspondence between the equation of state and the scalar field potential. This paper discusses these aspects of inflation in detail, through the example of an exact analytic solution of the dynamical equations. The latter is derived using elementary calculus and is used to clarify the issues mentioned above, with the added virtue of being useful for introducing the cosmic no-hair theorems.
2 The slow rollover approximation

In this section we recall the basics about the cosmological constant, inflation, and the slow rollover approximation used for simplifying and solving the equations of inflation.

In general relativity a spatially homogeneous and isotropic universe is described by a metric of the Friedmann-Lemaitre-Robertson-Walker class. For simplicity, we set our discussion in such a universe with flat spatial sections, described by the line element

\[ ds^2 = -dt^2 + a^2(t) \left( dx^2 + dy^2 + dz^2 \right), \]  

(2.1)

in comoving coordinates \((t, x, y, z)\), where \(a(t)\) is the scale factor of the universe. For this metric, the Einstein equations of general relativity reduce to

\[ \ddot{a} - \frac{4\pi G}{3} (\rho_m + 3P_m) + \frac{\Lambda}{3} = 0, \]  

(2.2)

\[ H^2 \equiv \left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho + \frac{\Lambda}{3}, \]  

(2.3)

where an overdot denotes differentiation with respect to the comoving time \(t\) and \(\rho_m\) and \(P_m\) are, respectively, the energy density and pressure of the material content of the universe, which is assumed to be a perfect fluid. \(H \equiv \dot{a}/a\) is the Hubble parameter, \(\Lambda\) is the cosmological constant, \(G\) is Newton’s constant and units are used in which the speed of light in vacuum assumes the value unity.

As is clear from the inspection of Eqs. (2.2) and (2.3), it is possible to describe the contribution of the cosmological constant as a fluid with energy density and pressure given by

\[ \rho_\Lambda = \frac{\Lambda}{8\pi G}, \quad P_\Lambda = -\frac{\Lambda}{8\pi G}, \]  

(2.4)

respectively, and equation of state \(P = -\rho\). Accordingly, one can rewrite Eqs. (2.2) and (2.3) as

\[ \ddot{a} - \frac{4\pi G}{3} (\rho + 3P) = 0, \]  

(2.5)

\[ \left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G \rho}{3}, \]  

(2.6)

where \(\rho \equiv \rho_m + \rho_\Lambda\) and \(P \equiv P_m + P_\Lambda\).

Inflation is defined as an epoch in the history of the universe during which the cosmic expansion is accelerated, \(\ddot{a} > 0\). Eq. (2.3) shows that acceleration is equivalent
to a negative pressure satisfying \( P < -\rho/3 \). It turns out that an inflationary period in which the universe expands by the factor \( e^{70} \) solves the fine-tuning problems of the standard big bang cosmology\(^{18}\).

In the original model and in most scenarios, inflation is obtained by assuming that at an early time of the order of \( 10^{-34} \) seconds the energy density of the cosmological fluid was dominated by a scalar field called \textit{inflaton}. Scalar fields are ever present in particle physics, and it is natural that they played a role when the universe had the size of a subatomic particle. Inflation can then be seen as scalar field dynamics; the energy density and pressure of a scalar field \( \phi \) minimally coupled to gravity are given by\(^{19,20}\)

\[
\rho = \frac{(\dot{\phi})^2}{2} + V ,
\]

\[
P = \frac{(\dot{\phi})^2}{2} - V ,
\]

where \( V(\phi) \) and \( (\dot{\phi})^2/2 \) are, respectively, the potential and kinetic energy densities of the scalar field. The scalar \( \phi \) only depends on the time coordinate, due to the assumption of spatial homogeneity, and satisfies the well known Klein-Gordon equation\(^{2,4}\)

\[
\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0 .
\]

By inserting Eqs. (2.7) and (2.8) into Eqs. (2.5) and (2.6) one obtains

\[
\frac{\dot{a}}{a} = \frac{8\pi G}{3} \left[ V(\phi) - \dot{\phi}^2 \right] ,
\]

\[
H^2 = \frac{8\pi G}{3} \left[ V(\phi) + \frac{\dot{\phi}^2}{2} \right] ,
\]

which, together with Eq. (2.9), constitute the \textit{equations of inflation}. Note that only two of the three equations (2.9), (2.10) and (2.11) are independent; when \( \dot{\phi} \neq 0 \), Eq. (2.9) follows from the energy conservation equation\(^{21}\)

\[
\dot{\rho} + 3H (\rho + P) = 0 .
\]

Different inflationary scenarios correspond to different choices of the form of the scalar field potential \( V(\phi) \), which are usually motivated by particle physics arguments. Certain scenarios are set in theories of gravity alternative to general relativity\(^{22}\), and will not be considered here.
From the didactical point of view, it is interesting that the dynamics of the inflaton field can be viewed as the motion of a ball with unit mass, position $\phi$ and speed $\dot{\phi}$ rolling on a hill which has a profile given by the shape of $V(\phi)$.

The equations of inflation are usually solved in the slow rollover approximation, which assumes that the inflaton’s speed $\dot{\phi}$ is small, that $V(\phi) \gg \dot{\phi}^2/2$, and that the inflaton’s motion (described by Eq. (2.9)) is friction-dominated. Then, the acceleration term $\ddot{\phi}$ can be neglected in comparison with the force term $V' \equiv dV/d\phi$ and with the friction term $3H\dot{\phi}$ in Eq. (2.9). A necessary condition for this to occur is that the inequalities

$$|V'| \ll 4\sqrt{\pi G}, \quad (2.13)$$
$$|V''| \ll 8\pi G, \quad (2.14)$$

hold. In other words, in the slow rollover approximation the slow roll parameters

$$\epsilon \equiv \frac{1}{16\pi G} \left(\frac{V'}{V}\right)^2, \quad (2.15)$$
$$\eta \equiv \frac{1}{8\pi G} \frac{V''}{V}, \quad (2.16)$$

are small, $\epsilon, |\eta| \ll 1$. $\epsilon$ and $\eta$ are related, respectively, to the slope and the curvature of the scalar field potential $V(\phi)$; the smallness of $\epsilon$ and $|\eta|$ means that the potential $V(\phi)$ is relatively flat during slow roll inflation. The equations of inflation reduce to the two first order equations

$$H^2 = \frac{8\pi G}{3} V, \quad (2.17)$$
$$3H\dot{\phi} = -\frac{dV}{d\phi}. \quad (2.18)$$

It would be misleading to put excessive emphasis on the requirement that $\epsilon$ and $\eta$ be small and on the existence of an almost flat section of the potential $V(\phi)$ on which the scalar field $\phi$ can slowly roll (i.e. with small speed $\dot{\phi}$). Indeed, Eqs. (2.13) and (2.14) are necessary but not sufficient conditions for neglecting the acceleration term in Eq. (2.9); a solution of the field equations may well satisfy Eqs. (2.13) and (2.14), but still have a large speed $\dot{\phi}$ and therefore not be in slow-rollover; this is the case, at early times, for the exact solution presented in Sec. 4.
3 Equivalence between a constant potential and the cosmological constant

In this section we discuss the de Sitter solution viewed as the prototype of inflation; the similarity between slow roll inflation and the de Sitter-like expansion of the universe is emphasized.

The de Sitter solution of the Einstein equations corresponds to exponential cosmic expansion,

\[ a(t) = a_0 e^{Ht}, \]

\[ H = \left( \frac{\Lambda}{3} \right)^{1/2}, \]

and is obtained by setting \( \rho_m = p_m = 0 \) in Eqs. (2.2) and (2.3) and by allowing only the cosmological constant in the right hand side of the field equations. Historically, the de Sitter solution has been known since the beginnings of general relativistic cosmology, long before the idea of inflation. As seen in the previous section, the cosmological constant is equivalent to a fluid with the equation of state of quantum vacuum

\[ P = -\rho, \]

and therefore cosmological constant and vacuum energy are synonyms in modern cosmology\(^4\). The equation of state (3.3) uniquely leads to the solution (3.1). The idea that quantum vacuum should be considered as a form of matter and hence as a source of gravity in the Einstein equations arose in the Russian school\(^{24}\).

While the original de Sitter solution (3.1) is associated to a geometrical cosmological constant, matter can generate an effective cosmological constant without the need of introducing a geometrical \( \Lambda \). In fact, the vacuum energy of a scalar field mimics a geometric cosmological constant and the de Sitter expansion of the universe (3.1) can be obtained as the result of scalar field dynamics; the latter are rather trivial. A constant scalar field

\[ \phi = \phi_0, \]

with the potential \( V = V_0 = \text{constant} \) solves the equations of inflation (2.10), (2.11), and (2.9) and

\[ a = a_0 \exp \left( \sqrt{\frac{8\pi G V_0}{3}} t \right) \]
for this solution. The idea that a constant, or almost constant, scalar field appearing in grand unified theories of particle physics plays the role of a vacuum state contributed to the development of the idea of inflation\textsuperscript{24}.

The de Sitter solution was the prototype of inflation\textsuperscript{4,8,11,12,14}, but its importance for inflation is not merely historical. In fact, most of the inflationary scenarios proposed thus far share the common feature of being solved in the slow rollover approximation (2.13) and (2.14), which implies that the expansion of the universe is quasi-exponential during the slow rollover phase. In fact, when the scalar field is in slow rollover, the dominance of the potential over the kinetic energy density, $(\dot{\phi})^2/2 \ll V$, implies that $\rho \approx V$ and $P \approx -V$, and the equation of state is approximately $\dot{V} = \frac{1}{3} V$, which is equivalent to say that the scale factor is approximated by (3.1). More precisely, during slow roll inflation, the scale factor is given by

$$ a(t) = a_0 \exp \left( \int H(t) dt \right) \simeq a_0 \exp \left( H_0 t + \frac{H_1}{2} t^2 \right), \quad (3.6) $$

where the constant term $H_0$ in the expansion of $H(t)$ is dominant.

In the rest of this paper we set the geometric cosmological constant to zero and we only consider the effective cosmological constant due to the scalar field.

4 A new analytic solution

A new analytic solution can be derived in the classroom using only elementary calculus, and is presented in this section. It differs from the de Sitter solution, but it is derived from Eqs. (2.2), (2.3), (2.7), and (2.8) under the same assumption of constant scalar field potential that led to the de Sitter solution (3.1).

One begins by noting that during slow roll inflation the nearly flat section of the scalar field potential plays the role of an effective cosmological constant. One then sets out to determine all the solutions of the exact equations of inflation (2.9)-(2.11) corresponding to a constant scalar field potential

$$ V = V_0 = \text{const.} \quad (4.1) $$

The Klein-Gordon equation (2.9) reduces to

$$ \ddot{\phi} + 3H\dot{\phi} = 0, \quad (4.2) $$

a trivial solution of which is given by $\dot{\phi} = 0$, and corresponds to the de Sitter solution discussed in the previous section. Another solution is possible when $\dot{\phi} \neq 0$; in this case
Eq. (4.2) can be divided by \( \dot{\phi} \) and immediately integrated to yield
\[
\dot{\phi} = \pm \frac{C}{a^3} ,
\] (4.3)
where \( C \) is a positive integration constant. By substituting Eq. (4.3) into Eq. (2.7) and the resulting expression of the energy density into Eq. (2.3) with \( \Lambda = 0 \), one obtains
\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \left( \frac{C^2}{2a^6} + V_0 \right) .
\] (4.4)
Upon use of the auxiliary variable \( y \equiv \ln a \), Eq. (4.4) can be reduced to a quadrature,
\[
\int \frac{dy}{\sqrt{1 + \alpha \exp (-6y)}} = \pm \sqrt{\frac{8\pi GV_0}{3} (t - t_0)} ,
\] (4.5)
where \( \alpha = C^2/(2V_0) \) and \( t_0 \) is an integration constant. By using
\[
\int \frac{dz}{z\sqrt{1 + z}} = \ln \left( \frac{z + 1}{z + 1 + 1} \right) ,
\] (4.6)
where \( z = \alpha e^{-6y} = \alpha a^{-6} \), one easily obtains
\[
\frac{\sqrt{a^6 + \alpha} - a^3}{\sqrt{a^6 + \alpha} + a^3} = \exp \left( \pm 12 \sqrt{\frac{2\pi GV_0}{3}} t \right) ,
\] (4.7)
where the boundary condition \( a(t = 0) = 0 \) has been imposed. The function
\[
f(x) = \frac{(x^6 + \alpha)^{1/2} - x^3}{(x^6 + \alpha)^{1/2} + x^3}
\] (4.8)
is monotonically decreasing for \( x \geq 0 \) and, since \( f(0) = 1 \), it is \( 0 < f < 1 \) for any \( x > 0 \). The right hand side of Eq. (4.7) is always greater than unity when the positive sign is adopted, hence Eq. (4.7) has no solutions in this case. By contrast, adopting the negative sign in Eq. (4.7), one obtains after straightforward calculations
\[
a(t) = a_1 \sinh^{1/3} \left( \sqrt{3\Lambda} t \right) ,
\] (4.9)
where \( a_1 = (4\pi G C^2/\Lambda)^{1/6} \) and \( \Lambda = 8\pi G V_0 \). Eq. (4.3) yields the corresponding scalar field
\[
\phi(t) = \phi_0 \ln \left[ \tanh \left( \frac{\sqrt{3\Lambda} t}{2} \right) \right] + \phi_1 ,
\] (4.10)
where \( \phi_0 = \pm (12\pi G)^{-1/2} \) and \( \phi_1 \) is an integration constant with the dimensions of a mass. It is straightforward to check that the solution (4.9) and (4.10) satisfies Eqs. (2.9) and (2.11); to check that Eq. (2.10) is also satisfied, it is sufficient to note that, given Eqs. (2.9) and (2.11), Eq. (2.10) follows from the conservation equation (2.12).

The solution given by Eqs. (4.9) and (4.10) has a big bang singularity at \( t = 0 \), with the asymptotic behavior
\[
a(t) \approx t^{1/3} , \quad \phi \approx \phi_0 \ln \left( \frac{\sqrt{3\Lambda} t}{2} \right) + \phi_1 \] (4.11)
as \( t \to 0 \). At \( t = 0 \) the scale factor vanishes while \( \phi, \rho \) and \( P \) diverge. The initial speed \( \dot{\phi} \) also diverges and the solution is definitely not in slow roll over, despite the fact that the potential is flat (in fact, constant). However, a slow roll over regime is approached as the universe evolves: at late times \( t \to +\infty \) the solution (4.9) and (4.10) is asymptotic to the de Sitter solution (3.1). This is in agreement with the cosmic no-hair theorems, as explained later.

5 Equation of state and scalar field potential

We will now return to the exact solution (4.9) and (4.10) and interpret them in terms of the equation of state relating pressure and energy density.

The effective equation of state of the universe corresponding to Eqs. (4.9) and (4.10) is given by
\[
\frac{P}{\rho} = 1 - 2 \tanh^2 \left( \sqrt{3\Lambda} t \right) ;
\] (5.1)

it changes with time and interpolates between the equation of state \( P = \rho \) at early times and the vacuum equation of state \( P = -\rho \) at late times. This feature of the solution cautions against a possible misunderstanding, i.e. the belief that fixing the scalar field potential \( V(\phi) \) is equivalent to prescribing the equation of state. This belief would be false: in fact, by fixing the scalar field potential to be constant, \( V = V_0 \), one does not uniquely determine the evolution of the universe: the solutions (4.9), (4.10) and (3.1), (3.4) correspond to very different physical situations and equations of state. In order
to completely specify the microphysics, it is not sufficient to prescribe the scalar field potential, but one must provide complete information on the state of the scalar field (i.e. also the field’s “speed” $\dot{\phi}$ in our example).

For a general potential $V$, the effective equation of state of the universe dominated by a scalar field is given by

$$\frac{P}{\rho} = \frac{\dot{\phi}^2 - 2V}{\dot{\phi}^2 + 2V} \equiv w(x), \quad (5.2)$$

where $x \equiv \dot{\phi}^2/2V$ is the ratio between the kinetic and the potential energy densities of the scalar $\phi$. Under the usual assumption $V \geq 0$ (which guarantees that the energy density $\rho$ is non-negative when $\dot{\phi} = 0$), one has that, for $x \geq 0$, the function $w(x) = (x^2 - 1)(x^2 + 1)^{-1}$ increases monotonically from its minimum $w_{\text{min}} = -1$ attained at $x = 0$ to the horizontal asymptote $+1$ as $x \to +\infty$. The slow rollover regime corresponds to the region $|x| \ll 1$ and to $w$ near its minimum, where the kinetic energy density of $\phi$ is negligible in comparison to its potential energy density. As the kinetic energy density $\dot{\phi}^2/2$ increases, the equation of state progressively deviates from $P = -\rho$ and the pressure becomes less and less negative; the system gradually moves away from the slow rollover regime. At the equipartition between the kinetic and the potential energy densities ($x = 1$), one has the “dust” equation of state $P = 0$. The pressure becomes positive as $x$ increases and, when the kinetic energy density completely dominates the potential energy density ($x \gg 1$), one finally reaches the equation of state $P = \rho$. The solution (4.9) and (4.10) for $V = \text{const.}$ spans the entire possible range for the equation of state during the evolution of the universe, starting from $x = +\infty$ at early times and asymptotically evolving towards $x = 0$ at late times.

6 Discussion and conclusions

We now comment on the exact solution described by Eqs. (4.9) and (4.10) and compare it to the de Sitter solution. In doing so, we will sharpen our understanding of the slow rollover approximation and introduce an interesting cosmic no-hair theorem.

Both solutions (4.9), (4.10) and (3.1), (3.4) of the Einstein-Friedmann equations correspond to a constant scalar field potential $V = V_0$ and exactly satisfy Eqs. (2.13) and (2.14), which are necessary, but not sufficient, conditions for slow roll inflation. While the de Sitter solution (3.1) corresponds to perfect slow roll ($\dot{\phi} = 0$) and to exact exponential expansion of the universe, the solution (4.9), (4.10) exhibits significant differences. It has a big bang singularity at $t = 0$, while the de Sitter universe with infinite age has existed
forever and the timescale over which the latter changes, \(\sqrt{3/\Lambda}\), is constant (see Ref. 25 for a pedagogical discussion of the self-similarity properties of the de Sitter solution).

By contrast, the solution given by Eqs. (4.9) and (4.10) has a timescale given by the Hubble time

\[ H^{-1} = \sqrt{\frac{3}{\Lambda}} \tanh \left( \sqrt{3\Lambda} t \right), \tag{6.1} \]

which is time-dependent and becomes nearly constant (with value \(\sqrt{3/\Lambda}\)) only for \(t \to +\infty\). Near the big bang singularity the universe obeys the equation of state \(P = \rho\). In this region, not only the solution fails to be in slow roll over the flat potential, but it is not even inflationary; even though the slow roll parameters \(\epsilon\) and \(\eta\) exactly vanish, the speed \(\dot{\phi}\) of the scalar field is large. This clearly shows that the slow rollover approximation must be formulated by means of conditions on the solutions of the field equations, not only as a set of conditions on the scalar field potential \(V(\phi)\). The relative flatness of the potential is not all there is to the slow rollover approximation.

Finally, the solution described by Eqs. (4.9) and (4.10) constitutes a useful example to introduce the cosmic no-hair theorems\(^\text{26}\). The latter state that, when a positive cosmological constant (or vacuum energy) \(\Lambda\) is present, the de Sitter solution (3.1), (3.4) behaves as an attractor for the other solutions\(^\text{26}\). The example solution (4.9) and (4.10) clearly illustrates this behavior and is particularly useful to introduce the cosmic no-hair theorems (the proof of which requires more sophisticated mathematics than the ones used in this paper\(^\text{26}\)). In spite of the fact that the solution (4.9) and (4.10) begins at early times very differently from a de Sitter solution, it converges exponentially fast to the latter as time progresses. In fact, the ratio of the (suitably normalized) scale factors (4.9) and (3.1) is given by \(\left[1 - \exp \left(-2\sqrt{3\Lambda} t\right)\right]^{1/3}\), which tends to unity at large times \(t\), while the scalar field (4.10) asymptotically converges to the constant \(\phi_1\). This happens because the de Sitter solution (3.1) is an attractor point that captures the orbits of the solutions in the phase space, including the orbit of the exact solution (4.9) and (4.10). Indeed, the cosmic no-hair theorems are more general, stating that the de Sitter space is approached even starting from an anisotropic spacetime\(^\text{26}\).

As a conclusion, cosmological inflation is described by simple ordinary differential equations and its basic features can be discussed in the classroom without the need of complicated mathematical tools. The phenomenological approach to inflation adopted in recent years by the community of cosmologists supports the introduction of the inflationary paradigm without the need of a lengthy premise about advanced particle physics to justify it. Indeed, the basic features of inflation can be grasped without the knowledge of high energy theories.
The increasing number of pedagogical introductions to inflation seems to reflect this point of view. The exact solution shown in this paper should help the circulation of simple but significant ideas of modern cosmology.
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