Induced Quantized Spin Current in Vacuum

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We uncover a fundamental effect of the QED vacuum in an external electromagnetic (EM) field. We show that the quantized vacuum of electrons is spin polarized by the EM field and manifests as a vacuum spin current. An experiment is proposed to measure the spin torque exerted by the spin current by measuring the twisted angle of the director axis of a nematic liquid crystal.

Introduction. — The quantum nature of vacuum is a fascinating place to look for novel physical phenomena. For example, the large scale fluctuation of the Universe is supposed to emerge from the quantum ripples of the near de Sitter vacuum in the inflationary universe [1]. In high energy physics, the metastability of the Higgs vacuum [2] has nontrivial cosmological consequences and could provide a new observational window to particle physics well beyond what collider experiments can achieve.

Recently, it was found that the vacuum fluctuations of boundary QED system in the presence of an external magnetic field result in a magnetization current near its boundary [3, 4]. This nontrivial electromagnetic response of the vacuum is due to the electric charges carried by the virtual electrons and positrons of the theory. As these quantum fluctuations also carry spin, it is natural to ask if and how the quantum fluctuation of the spin degrees of freedom of the vacuum would manifest in observation. In this Letter, we are interested in the renormalization phenomena of the QED vacuum in the presence of an external electromagnetic field $A^\mu$. As the velocity is represented by $v^\mu = \beta^\gamma$ in the Dirac theory, we propose the following definition of spin current in quantum field theory:

$$J^{\mu\alpha}(x) := \bar{\psi}(x)S^{\mu\alpha}\psi(x),$$  \hspace{1cm} (4)

where

$$S^{\mu\alpha} := \frac{1}{2} [\gamma^\mu T^\alpha (x) + T^\alpha (x) \gamma^\mu]$$  \hspace{1cm} (5)

and $T^\alpha$ is the covariant Bergmann-Wigner spin operator in a background electromagnetic field:

$$T^i = \beta^\gamma \pi^i/m, \quad T^0 = \frac{1}{m} \Sigma \cdot \pi.$$  \hspace{1cm} (6)

Here $\pi^i = p^i - eA^i$ is the covariant momentum and $e > 0$ is the magnitude of the electric charge. Similar definitions [8, 9] have been considered before. Our definition is better justified since we have adopted a symmetrization prescription such that Eq. (4) reduces to the one (Eq. (1)) of Rashba in the nonrelativistic limit. A further justification of our definition (4) can be obtained by noticing that the current Eq. (4) satisfies the conservation law

$$\partial_\mu J^{\mu\alpha} = \frac{e}{m} S_\alpha F^{\alpha\beta},$$  \hspace{1cm} (7)
with a source term. Here $S^\alpha$ is the density defined by
\[ S^i := \psi^\dagger \sigma^i \psi, \quad S^0 := \psi^\dagger \gamma^5 \psi. \] (8)
We note in passing that $J^{\mu \alpha}$ is not a Noether current, but Eq. (7) is obtained from the fermion equation of motion. For the spatial directions $\alpha = i$, Eq. (7) gives explicitly
\[ \frac{\partial \rho^i}{\partial t} + \nabla \cdot J^i = \frac{e}{m} \left( (S \times B)^i + S^0 E^i \right). \] (9)
Here $\rho^i := J^{0i}$ and $J^i := e_h J^{ki}$. It is clear that $\rho^i$ gives the spin polarization density. As a result, $J^i$ does admit the correct interpretation as a current density for spin polarization in the $i$th direction. We also note that $-\nabla \cdot J^i$ and the EM terms on the right-hand side of Eq. (9) can be interpreted as the spin torque from the matters and the external EM fields. Classically in a vacuum, $S^0$ vanishes and the spin current respects the continuity equation
\[ \frac{\partial \rho^i}{\partial t} + \nabla \cdot J^i = 0. \] (10)
We will show in this Letter that, in the presence of a background electromagnetic field, the spin current $J^i$ and the spin density $\rho^i$ becomes nonzero due to the polarization effect of the electromagnetic field on the quantum fluctuations of the vacuum. Nevertheless the conservation law (10) is still satisfied at the quantum mechanical level.

We end this section with a couple of remarks. (i) Properly speaking, the current Eq. (4) measures the flow of spin polarization and should be called a spin polarization current. As a result, $J^i$ does admit the correct interpretation as a current density for spin polarization in the $i$th direction. We also note that $\nabla \cdot J^i$ and the EM terms on the right-hand side of Eq. (9) can be interpreted as the spin torque from the matters and the external EM fields. Classically in a vacuum, $S^0$ vanishes and the spin current respects the continuity equation
\[ \frac{\partial \rho^i}{\partial t} + \nabla \cdot J^i = 0. \] (10)

This current couples to gravity and is also sometimes referred to as a spin current in the respective community. However we emphasize that this is different from the spin current we introduced in this Letter.

**Vacuum expectation of spin current.** — In quantum field theory, the spin current (11) is a composite operator which needs to be renormalized. We are interested in the vacuum expectation value (vev) of the spin current in a background electromagnetic field $A_\mu$. This can be computed in perturbation theory as
\[ \langle J^{\mu \alpha}(x) \rangle_A = -ie \int d^4 y (J^{\mu \alpha}(x) J^{\beta}(y) A_\beta(y) + O(A^2), \] (13)
or, it can be written in the momentum space as
\[ \langle J^{\mu \alpha}(q) \rangle_A = e T^{\mu \alpha \beta}(q) A_\beta(q) + O(A^2), \] (14)
where $T^{\mu \alpha \beta}(q)$ is the Green’s function
\[ T^{\mu \alpha \beta}(q) := -i \int d^4 x e^{iq x} \langle J^{\mu \alpha}(x) J^\beta(0) \rangle. \] (15)
At 1-loop, $T^{\mu \alpha \beta}(q)$ is given by
\[ T^{\mu \alpha \beta}(q) = -i \int \frac{d^4 p}{(2\pi)^4} (-1) \text{tr} \left( S^{\mu \alpha} \frac{i}{p - m} \gamma^\beta \frac{i}{p + q - m} \right) \] (16)
where $S^{\mu \alpha}$ is given by Eq. (5). The trace of the gamma matrices can be simplified and we obtain
\[ T^{\mu \alpha \beta} = 0, \quad T^{\mu i \beta} = -4 m \epsilon^{\mu i \beta \gamma} q_\gamma I(q), \quad i = 1, 2, 3, \] (17)
where $I(q)$ is the momentum function defined by
\[ I(q) := \int \frac{d^4 p}{(2\pi)^4} \frac{1}{p^2 - m^2} \frac{1}{(p + q)^2 - m^2}. \] (18)
Note that $I(q)$ and hence the Green’s function is logarithmic divergent. This is due to the singular product of local quantum fields in the expression (11) for the spin current operator. In order to give a proper definition of this composite operator, one need to isolate the divergent terms with a regularization scheme and subtract it away with counterterms in the Lagrangian [10]. Regularizing Eq. (18), we obtain
\[ I(q) = \frac{i}{16\pi^2} \left( \log \frac{\Lambda^2}{\mu^2} - 1 + \log \frac{\mu^2}{m^2} - h \left( \frac{q^2}{m^2} \right) + \cdots \right) \] (19)
for a momentum cutoff regularization, and
\[ I(q) = \frac{i}{16\pi^2} \left( \frac{2}{\epsilon} - \gamma_E + \log \frac{4\pi\mu^2}{m^2} - h \left( \frac{q^2}{m^2} \right) + \cdots \right) \] (20)
for dimensional regularization to $d = 4 - \epsilon$ dimension. Here $\cdots$ denotes terms of order $O(1/\Lambda^2)$ or $O(\epsilon)$, and $h(x)$ is the function
\[ h(x) := \int_0^1 d\xi \ln[1 + \xi(1 - \xi) x] \] (21)
with $\hbar(0) = 0$. We can now subtract away the divergence with a counterterm and obtain

$$
\langle J^{\mu i}_R(x) \rangle_A = -\frac{emc}{8\pi^2\hbar^2} e^{i\mu\delta\beta} \left( \log \frac{m^2}{\mu^2} + a_J + \hbar \left( \square \right) \right) F_{\delta\beta}(x)
$$

(22)

and $\langle J^{\mu i}_R(0) \rangle_A = 0$. Here $\mu$ is the standard RG scale of the QFT and $a_J$ is an independent arbitrary constant that is due to the arbitrariness in the choice of the finite part of the counterterm for the composite operator $J^{\mu i}$. To uniquely fix the finite part (i.e., fixing $a_J$) and hence the definition of the renormalized spin current, a normalization condition is required. Physically, if the electron mass is sent to infinity, then the quantum loop effects are completely suppressed and the renormalized spin current must vanish. In a theory with a cutoff $\Lambda$, the fermion loop effects are now suppressed as $m$ approaches this scale. As a result, we have the decoupling condition

$$
\lim_{m \to \Lambda} \langle J^{\mu i}_R \rangle_A = O(\frac{1}{\Lambda}).
$$

(23)

This serves as a natural normalization condition for the renormalized spin current. However extra care is needed for QED where there is a Landau pole and the UV cutoff cannot be taken to exceed that. In fact, the positiveness of the beta function $\beta(\alpha) = 2\alpha^2/3\pi \left( \alpha = e^2/4\pi\hbar c \right)$ (the fine structure constant) goes up to the RG flow for the coupling

$$
\frac{1}{\alpha(\mu)} - \frac{1}{\alpha(\mu_0)} = -\frac{2}{3\pi} \log \frac{\mu}{\mu_0}. 
$$

(24)

This implies the presence of a cutoff scale $\Lambda_L = me^{3\pi/2\alpha}$, the Landau scale, where the bare coupling $\alpha(\Lambda_L)$ becomes infinite. The Landau scale represents the highest possible UV cutoff one may utilize in the renormalization program of QED. Taking $\Lambda = \Lambda_L$ in Eq. (23), the renormalization constant $a_J = -\log \Lambda_L^2/\mu^2$ is fixed up to $O(1/\Lambda_L)$ terms, which we will ignore. Restoring the units of $c/\hbar^2$ \[11\], we finally obtain

$$
\langle J^{\mu i}_R(x) \rangle_A = \frac{emc}{8\pi^2\hbar^2} \left( \frac{3\pi}{\alpha} - \hbar \left( \square \right) \right) e^{i\mu\delta\beta} F_{\delta\beta}.
$$

(25)

Note that the current Eq. (25) is conserved due to the Bianchi identity of the EM field. For a classical electromagnetic background in vacuum, it is $\square F_{\alpha\beta} = 0$ and we have

$$
\langle J^{\mu i}_R(x) \rangle_A = \frac{3emc}{8\pi^2\hbar^2} e^{i\mu\delta\beta} F_{\delta\beta}
$$

(26)

or, in terms of components explicitly

$$
\langle J^{\mu i}_R(x) \rangle_A = \frac{3emc}{4\pi\hbar^2} e^{i\mu k} F^k, 
$$

(27a)

$$
\langle J^{\mu i}_R(x) \rangle_A = \frac{3emc}{4\pi\hbar^2} e^{i\mu} B^i
$$

(27b)

The results (25), (26), (27) are the main results of this Letter. For the rest of the Letter, we will be focusing on the case (26) of a classical background. However for generality we present the result (25) to cover the situation where the background EM fields go beyond the Maxwell description; for example, if quantum nonlinear corrections of QED is included, or if the classical EM fields are coupled to other background fields (e.g., an axion background).

A couple of remarks are in order. (i) Note that our result is independent of the adopted regularization scheme. Apart from the cutoff regularization and the dimensional regularization, one can also do the heat-kernel regularization and obtain the same intermediate result (22), and the final result (25) after imposing the normalization condition. (ii) It is interesting that the normalization condition (23) fixes $a_J$ in terms of the renormalization scale $\mu$. In the end, $\mu$ remains free in the QFT and it is remarkable that the renormalized spin current is independent of it. (iii) We remark that the renormalization condition (23) is essentially nonperturbative in nature as the resulting prediction (25) scales inversely with $\alpha$ and is not smoothly connected with the free theory result.

It is instructive to compare our effect with the famous Schwinger effect \[12\] \[13\], which refers to the production of electron-positron pairs under the influence of an applied electric field. The Schwinger effect is nonperturbative and requires a strong electric field stronger than the critical field strength $E \geq E_{critical} = m^2c^3/(e\hbar)$ in order to produce an observable amount of particle pairs. Unlike the Schwinger effect, the spin current predicted in this Letter is a nonperturbative consequence of the nontrivial spin polarization of the vacuum. There is no real production of particles involved. Our result is also different from the analysis of Ref. \[9\] where the Schwinger-effect-produced electric current is acted on by a second electric field to produce a spin current. In this case, the spin current is generated in a similar manner as in an ordinary material sample except that the source electric current has a nonperturbative origin and so very small in magnitude.

In the above we have considered pure QED. In a more realistic setting where QED is embedded as the low energy part of a consistent high energy theory, e.g., a grand unified theory or string theory when gravity is included, the Landau pole would be replaced by the corresponding GUT or Planck scale. In this case, the 1-loop results (25) will have the factor $3\pi/\alpha$ replaced by $3\pi(1/\alpha - 1/\alpha_G)$ where $\alpha_G$ is the fine structure coupling at the unification scale. It is interesting that the vacuum expectation value of the spin current actually provides a probe to the UV physics. The value of $1/\alpha_G$ is model dependent. For example, for the MSSM GUT, one has $1/\alpha_G \approx 24.3$ and $M_G \approx 2 \times 10^{16}$ GeV. In any case, $1/\alpha_G$ is expected to be small compared to the observed $1/\alpha$ at the electron mass scale. In the following, we will continue to analyze the result (25) for pure QED, but keeping in mind the overall magnitude of the spin current may be different.
from Eq. (25) by a small fraction.

**Physical picture.** — It may appear strange that the switching on of an electric field in vacuum can produce an observable spin current [27]. However the physical origin of the spin current can be easily understood in terms of the spin-orbit coupling of the quantum fluctuation of the vacuum. To see this, let us consider the nonrelativistic expansion of Dirac’s Hamiltonian up to $O(p^4)$ where a spin-orbit coupling term arises, $H_{SO} = -\frac{i\hbar}{2m^2c^2}\Sigma \cdot \mathbf{E} \times \mathbf{p}$. It is well known that the spin-orbit coupling term allows impurity in a material to scatter the electrons in a spin-dependent way (skew scattering) and generates a spin current [15, 16]. The spin-orbit coupling also give rises to a side jump [17, 18] and condensation of the quantized vacuum as a result of external fields.

**Proposed experiment.** — The induced spin current may be observed by measuring the torque exerted by the spin current on a probe placed in the vacuum. Consider an infinitesimal volume element $\delta V = \delta x\delta y\delta z$ in the interior of a probe placed under the influence of an external EM field. The spin momentum torque acting on $\delta V$ is $\tau^i = -\hbar/2 \int_{\delta V} d^3x \nabla \cdot (\mathbf{J}^i)$. Using the result [27] for a classical EM field in vacuum, we have

$$\tau^i = \frac{3\epsilon mc}{8\pi\hbar\alpha} B^i \delta V,$$

(30)

where $B^i := \delta V^{-1} \int_{\delta V} \delta B^i / \partial t$ is the average rate of change of the magnetic field over the volume $\delta V$. We will be using the SI units from now on and hence a factor of $c$ appears in Eq. (30). For a $B$ field pointing in the $z$ direction described by a wave of the form

$$B_z = B_0 g \left( t - \frac{x}{c} \right),$$

(31)

where $g$ is as in Fig. [1] we obtain

$$\tau^z = \frac{3\epsilon mc}{8\pi\hbar\alpha} B_0 f \delta V,$$

(32)

where $f := 1/T_1 - 1/T_2$. For an external field with $B_0 = 100$ G, $f = 1000$ Hz, the quantum spin torque per unit volume is

$$\tau^z / \delta V = 6.5 \times 10^{-5} \text{ Nm}^{-2}.$$

(33)

Not all of the vacuum torque is transferred to the probe. Physically the angular momentum density $\mathcal{J}_0^i = (\hbar/2)\rho^i$ acquired by the vacuum generates a vacuum magnetization $\mathbf{M}_0 = (e/m)\mathcal{J}_0$ and this corresponds to quantum addition to the $B$ field, $\Delta \mathbf{B} = \mu_0 \mathbf{M}_0$. For a material probe with magnetic susceptibility $\chi$, the $B$ field generates a probe magnetization $\mathbf{M} = \chi\mathbf{M}_0/(1 + \chi)$. These are just the Barnett effect and the Einstein-de Haas effect for the interplay between angular momentum and magnetization [21]. This implies the probe receives a torque $\tau^z_p$,

$$\tau^z_p = \eta\tau^z,$$

(34)
where $\eta = 4\pi \chi / (1 + \chi) \approx \chi$ as $|\chi| \ll 1$. The spin torque on the probe is thus tiny. Nevertheless, the twisting effect may be observable by using a liquid crystal which is known to be exceptional in sensitivity for torque measurement. In fact, recently the Casimir torque exerted on the surface of a liquid crystal \cite{23} has been observed successfully \cite{23}. We propose here a similar setup to observe the spin torque arising from the spin current.

Consider a layer of nematic liquid crystal, e.g., 4-cyano-40-pentylbiphenyl (5CB), placed in a vacuum cavity, influenced by the external EM field; See Fig. 2. We place the liquid crystal such that its director axis is always in the $xy$ plane, i.e., $n(z) = [\cos \theta(z), \sin \theta(z), 0]$ where $\theta(z)$ describes the orientation of the liquid crystal molecules with respect to the $x$ axis. Because of the action of the spin torque \cite{32}, the molecules try to orient themselves correspondingly. An equilibrium configuration is attained when this torque is balanced by the restoring elastic torque of the liquid crystal.

Suppose there is no bend or splay of the liquid crystal, then only the twist contribute to distorting energy density $u_d = (k/2)(\partial \theta / \partial z)^2$, where $k$ is the twist elastic constant. For example, $k = 3.6$ pN for the crystal 5CB.

The elastic energy stored in $\delta V$ is thus

$$E_d[\theta(z)] = A \int_z^{z + \delta z} dz \frac{k}{2} \left( \frac{\partial \theta}{\partial z} \right)^2,$$

where $A = \delta x \delta y$ is the area of the element. This gives a restoring torque

$$\tau_d(z) = \frac{\delta E_d}{\delta \theta(z)} = -k \frac{\partial^2 \theta}{\partial z^2} \delta V.$$

This is in analogy with the Newton’s second law $F = m \ddot{y}$ for the inertia. The equilibrium is attained when Eq. \cite{34} is balanced out by the restoring torque \cite{24}. Note that the volume factor cancels out. This gives

$$\theta(z) = \frac{\beta}{2} z^2, \quad \beta := \frac{3\pi \chi}{2a} \frac{m f_B f}{k \lambda_C},$$

where $f_B := eB_0 / (2\pi m)$ is the cyclotron frequency and $\lambda_C$ is the Compton wavelength. In deriving Eq. \cite{37}, we have taken the boundary conditions $\theta = \partial_x \theta = 0$ at the glass contact $z = 0$.

To enhance the detection of the twisted angle, a thicker liquid crystal layer and a stronger magnetic field is preferred. However, both of these are limited by the properties of the liquid crystal. In order for the liquid crystal to be able to register the torque, the response time of the liquid crystal should be smaller than $T_1$ and $T_2$. This means $f$ is upper bounded by the response frequency $f_r$:

$$f < f_r.$$  \hspace{1cm}

However, the response frequency is limited by the thickness as generally the response time of the liquid crystal increases with the thickness, linearly or quadratically depending on the voltage \cite{25}. As for the EM field, the field strengths cannot be too strong as otherwise the crystal may be driven into a Frederick transition \cite{25}. Nematic liquid crystals are typically diamagnetic with $\chi$ of the order of $10^{-5}$ \cite{25}. Commercial nematic liquid crystal has a $f_r$ in the range of 100 Hz and thickness of the order of 10 m. As an estimate, consider a setup with a 100 G $B$ field and a nematic liquid crystal with an elastic constant $k = 3.6$ pN, thickness $L_z = 1$ mm, a response frequency of 1 kHz. The total twist angle accumulated over the thickness of the probe is

$$\Delta \theta = 0.005^\circ \cdot \frac{3.6 \text{ pN}}{k} \cdot \frac{B_0}{100 \text{ G}} \cdot \frac{f}{1 \text{ kHz}} \cdot \left( \frac{L_z}{1 \text{ mm}} \right)^2.$$

This looks feasible. It will be interesting to perform an experiment to make observation of the spin current predicted in this Letter. The quantum spin current could have a wide range of applications, from novel theoretical properties of physical system, e.g. dark energy, to practical effects on the workings of micro-machined device.

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