NON-INTEGRABLE DIMER MODELS: UNIVERSALITY AND SCALING RELATIONS

ALESSANDRO GIULIANI AND FABIO LUCIO TONINELLI

Abstract. In the last few years, the methods of constructive Fermionic Renormalization Group have been successfully applied to the study of the scaling limit of several two-dimensional statistical mechanics models at the critical point, including: weakly non-integrable 2D Ising models, Ashkin-Teller, 8-Vertex, and close-packed interacting dimer models. In this note, we focus on the illustrative example of the interacting dimer model and review some of the universality results derived in this context. In particular, we discuss the massless Gaussian free field (GFF) behavior of the height fluctuations. It turns out that GFF behavior is connected with a remarkable identity ('Haldane' or 'Kadanoff relation') between an amplitude and an anomalous critical exponent, characterizing the large distance behavior of the dimer-dimer correlations.

1. Introduction

The scaling limit of the Gibbs measure of a statistical mechanics model at a second order phase transition is expected to be universal, in particular, to be robust under ‘irrelevant’ perturbations and conformally invariant. Conceptually, the route towards universality is clear: one should first integrate out the small-scale degrees of freedom, then rescale the variables associated with the large-scale fluctuations, and show that the critical model reaches a fixed point, under iterations of this procedure (‘Wilsonian’ Renormalization Group (RG)). On general grounds, the fixed point is expected to be conformally invariant: therefore, in a second step, one can use the methods of Conformal Field Theory (CFT) to classify and characterize all the possible conformally invariant fixed points (CFT methods provide, in fact, a complete classification of such fixed point theories in two dimensions; remarkably, there has been recent progress in the characterization of three dimensional conformally invariant theories [37], too). The conformal fixed point theory of interest for the description of a given statistical mechanics system can often be identified by using specific information on the critical exponents, typically available from the RG construction.

Even though widely accepted and believed to be correct, there are only few cases, mostly in two dimensions, for which this procedure and/or its predictions can be rigorously confirmed.

(1) A first class of critical models for which the existence and conformal invariance of the scaling limit is rigorously known consists of two-dimensional, integrable Ising and dimer models on isoradial graphs [11, 14, 13, 27, 39]. The key technical tool used to prove conformal
invariance is discrete holomorphicity, which is a manifestation of integrability [2]. The method is flexible enough to prove robustness of the scaling limit under geometric deformations (e.g., of the domain, or of the underlying lattice), but it is not able to explain universality under perturbations of the microscopic Hamiltonian. [For a proof of conformal invariance of crossing probabilities via discrete holomorphicity methods in a non-integrable model, see [38].]

A second class of critical models for which several predictions of Wilsonian RG and CFT have been rigorously substantiated consists of non-integrable perturbations of determinantal models, such as interacting dimers [21, 22, 23], Ashkin-Teller and vertex models [7, 15, 20, 23]. These results are based on a constructive, fermionic, version of Wilsonian RG: they allow to construct the bulk scaling limit of ‘local fermionic observables’ and to prove scaling relations among critical exponents, but they are not flexible enough yet to compute ‘non-local’ observables (such as spin-spin correlations in perturbed Ising or monomer-monomer correlations in perturbed dimers) or to accommodate geometric deformations of the domain. [For a first result on perturbed Ising models, based on probabilistic tools, see [1].]

Not much is rigorously known about the existence and nature of the scaling limit of other critical models in two and three dimensions. It is a central challenge of mathematical physics for the coming years to extend and effectively combine the available techniques, in order to cover new models of physical interest and new instances of universality. In this paper, we review a few selected results on the universality of non-integrable two dimensional models, based on the fermionic RG methods mentioned above. For definiteness, we restrict our attention to ‘interacting dimer models’. We first introduce them informally, and give a first overview both of the ‘classical’ known results, and of our new results. Next, in the following sections, we introduce their definitions and state the relevant results on their critical behavior in a more precise way.

1.1. Model and results: an overview. Dimer models at close packing on planar, bipartite, graphs are highly simplified models of random surfaces or of liquids of anisotropic molecules at high density: the connection between these two apparently unrelated classes of systems is mediated by the notion of height function, which is in one-to-one correspondence with close-packed dimer configurations, as illustrated in Fig.1.

Integrable dimer models. A remarkable feature of close-packed dimer models is that there is a natural family of exactly solvable, critical, models, which exhibit a very rich and interesting behavior. This family is parametrized, on the one hand, by the specific periodic, infinite, bipartite planar lattice on which the dimer configuration lives; on the other hand, by the positive weights (or ‘activities’) associated to the edges of the graph. The exact solution is based on a determinantal representation of the partition function, due to Kasteleyn and Temperley-Fisher [26, 28, 40], valid for generic edge weights. The edge weights control the average slope of the height: in this
sense, the family of solvable dimer models describes discrete random surfaces with different average slopes (or ‘tilt’). The solution shows that there is an open set of edge weights for which the dimer-dimer correlations decay algebraically and, correspondingly, the height fluctuations are asymptotically described by a massless Gaussian Free Field (GFF) at large distances: this critical phase is called ‘liquid’ or ‘rough’, depending on whether one focuses on the behavior of the dimer correlations or of the height profiles. This critical phase displays a subtle form of universality, in that the variance of the GFF height fluctuations is asymptotically, at large distances, independent of the dimer weights (in particular, of the average slope of the height profile) and of the underlying graph.

Remark 1. In the case that the underlying lattice is the honeycomb lattice, as in Fig. 1, dimer configurations can be seen as stepped interfaces of minimal area (given the boundary height), i.e., they can be seen as domain walls for the 3D Ising model at zero temperature [4]. In fact, the probability measure on dimer configurations induced by setting all edge weights equal to 1 is the same as the zero-temperature measure of the 3D Ising model with suitable tilted Dobrushin-type boundary conditions. By using this connection, one recognizes that the GFF nature of the height fluctuations proves the existence of a rough phase in 3D Ising at $T = 0$ with such boundary conditions. It is an open problem to prove that the rough phase persists at small, positive, temperatures.

Non-integrable dimers: main results (in brief). Wilsonian RG and the bosonization method\(^1\) suggest that the GFF nature of the height fluctuations should be robust under non-integrable perturbations of the dimer model. In order to test this prediction in a concrete setting, we consider a class of ‘interacting’ dimer models, including the 6-vertex model (in its dimer representation [6, 15]) and non-integrable variants thereof. For weak enough interactions (weak but independent of the system size), we prove that the height fluctuations still converge to GFF, as in the integrable case. However, in this case, the variance appears to depend on the interaction and on the dimer weights, see Section 4. Therefore, the form of universality exhibited

---

\(^1\)See, e.g., [22, Sect.1.2] for a brief introduction to bosonization.
by the integrable dimer model seems to break down as soon as we move out of the exactly solvable case. Remarkably, a new form of universality emerges in the interacting case: the (pre-factor of the) variance equals the anomalous critical exponent of the dimer correlations. This is an instance of a ‘Kadanoff’ or ‘Haldane’ scaling relation, see [23, Sect.1] for a brief introduction to these ‘weak universality’ relations.

In the next sections, we will describe the models of interest and state our main results more precisely: in Section 2 we introduce the family of solvable dimer models and briefly review a selection of known results about its correlation functions; in Section 3 we introduce the class of non-integrable dimer models we consider, state our main results and give a brief sketch of the ideas involved in the proof (see [23] for a complete proof); in Section 4 we illustrate the Haldane scaling relation by a first order computation: we compute both the pre-factor of the variance and the critical exponent at linear order in the perturbation strength, and check that the two results agree (the computation shows how non-trivial and remarkable the result is: already at first order the validity of the Haldane relation requires very subtle cancellations). Our explicit computations proves, in particular, that the critical exponent is non-universal, i.e., it depends both on the interaction strength and on the dimer weights.

2. Non-interacting dimers

For simplicity, we restrict to dimers on the square lattice. A close-packed dimer configuration (or perfect matching) on \( \Lambda \subset \mathbb{Z}^2 \) is a collection of hard rods of length 1, which can be arranged on \( \Lambda \) in such a way that they cover all the vertices of \( \Lambda \) exactly once, see Fig. 2. It is important that \( \Lambda \) is bipartite: we emphasize this fact by coloring white and black the vertices of the two sublattices.

![Figure 2. A close-packed dimer configuration on a domain of the square grid and the corresponding height function (the height offset has been fixed to zero at the central face).](image)

Any dimer configuration is in one-to-one correspondence with a height profile, defined on the faces up to an overall additive constant. The height profile is defined by the differences between the values of the height at the face \( f \) and \( f' \):

\[
h(f') - h(f) = \sum_{b \in C_{f \rightarrow f'}} \sigma_b (\mathbb{1}_b - 1/4),
\]

(2.1)
where the sum runs over the bonds crossed by a lattice path from \( f \) to \( f' \), \( \sigma_b \) is equal to +1 or −1, depending on whether \( b \) crossed with the white site on the right or on the left and \( \mathbb{1}_b \) is the indicator function of having a dimer at \( b \). A key point of the definition of the height function is that the right side is independent of the choice of the lattice path from \( f \) to \( f' \) (independence follows from the close-packing condition).

The family of integrable, close-packed, dimer models that we informally introduced in the previous section is defined by the following partition function:

\[
Z^0_L = \sum_{M \in \Omega_L} p_{L,0}(M), \quad p_{L,0}(M) = \prod_{b \in M} t_{r(b)}, \tag{2.2}
\]
defined on a discrete torus \( T_L \) of side \( L \) (see [23, Sect.2.1]): here \( \Omega_L \) is the set of close-packed dimer configurations on the discrete torus, and \( r(b) \in \{1, 2, 3, 4\} \) label the ‘type’ of edge (we let \( r = 1 \) if the edge is horizontal with the white site on the right, \( r = 2 \) if it is vertical with the white site on the top, \( r = 3 \) if it is horizontal with the white site on the left, \( r = 4 \) if it is vertical with the white site on the bottom). The model is parametrized by the choice \( t_1, t_2, t_3, t_4 \); without loss of generality, we can set \( t_4 = 1 \), and we shall do so in the following. The dimer weights \( t_j \) play the role of chemical potentials, fixing in particular the average slope:

\[
E_L[\h(f + e_i) − h(f)] = \rho_i(t_1, t_2, t_3), \quad i = 1, 2,
\]
where \( E_L \) indicates the statistical average w.r.t. the probability measure \( \mathbb{P}_L \) induced by the weights \( p_{L,0}(M) \).

As anticipated above, this dimer model is exactly solvable [26, 40]: for example,

\[
Z^0_L = |\det K|,
\]
where \( K = K(t) \) is a complex adjacency matrix, known as the Kasteleyn matrix: its elements are labelled by a pair of sites of different color and are non-zero only if the pair forms a nearest-neighbor edge of the square lattice; the value of the element of \( K \) corresponding to an edge of type 1, 2, 3, 4 is, correspondingly,

\[
K_1 = t_1, \quad K_2 = it_2, \quad K_3 = −t_3, \quad K_4 = −i. \tag{2.3}
\]

Remarkably, also the multipoint dimer correlation functions can be explicitly computed. For instance, if \( b(x, j) \) is the bond of type \( j \) and black site \( x \), the two-point dimer correlation reads (in the special case of two dimers of type 1; similar formulas are valid for the other cases):

\[
\mathbb{E}[\mathbb{1}_{b(x,1)}; \mathbb{1}_{b(y,1)}] = −t_2^2 K^{-1}(x, y) K^{-1}(y, x),
\]
where: \( \mathbb{E} \) is the expectation w.r.t. \( \mathbb{P} \), the weak limit of \( \mathbb{P}_L \) as \( L \to \infty \); the semi-colon indicates ‘truncation’ (i.e., \( \mathbb{E}(A; B) = \mathbb{E}(A; B) − \mathbb{E}(A) \mathbb{E}(B) \)); and

\[
K^{-1}(x,y) = \frac{dk}{(2\pi)^2} \frac{e^{-ik(x−y)}}{\mu(k)}; \quad \mu(k) = t_1 + it_2 e^{ik_1} − t_3 e^{ik_1+ik_2} − ie^{ik_2}.
\]
Note that the zeros of $\mu(k)$ lie at the intersection of two circles in the complex plane, i.e., they are defined by the equation

$$e^{ik_2} = \frac{t_1 + it_2 e^{ik_1}}{i + t_3 e^{ik_1}}.$$ 

‘Generically’, these two circles intersect transversally in two points: in this situation $K^{-1}(x, y)$ decays to zero algebraically, like $|x - y|^{-1}$, as $|x - y| \to \infty$; this algebraic decay is often referred to by saying that the model is critical.

Once the two-point dimer correlations are explicitly known, one can compute the fluctuations of the height difference. In particular, the variance w.r.t. $\mathbb{P}$ of the height difference diverges logarithmically:

$$\text{Var}_\mathbb{P}[h(f) - h(f')] = \frac{1}{\pi^2} \log |f - f'| + O(1) \quad (2.4)$$

as the distance $|f - f'|$ tends to infinity [28, 29].

**Remark 2.** The pre-factor $1/\pi^2$ in front of the logarithm is independent of $t_1, t_2, t_3$: this ‘universality’ property is not accidental and is in fact related to a maximality (‘Harnack’) property of the spectral curve of integrable dimer models [29].

The proof of (2.4) is not trivial; it is based on a sufficiently smart combination of the following ingredients (the ‘smart’ part is in the use of the third ingredient): (1) the definition of height difference, see (2.1); (2) the explicitly formula of $\mathbb{E}[\mathbb{1}_b; \mathbb{1}_{b'}]$; (3) the path-independence of the height (in order to get a sensible expression in the large distance limit, one first needs to properly deform the two paths from $f$ to $f'$ involved in the computation of the variance; next, one can pass to a continuum limit, by replacing discrete sums with integrals and by replacing the finite distance dimer-dimer correlation with its large-distance asymptotic behavior, up to error terms that can be explicitly estimated [29]).

Building upon these ingredients one can refine (2.4) in various directions, in particular one can prove [27] that:

- height fluctuations converge to a massless GFF after proper coarse graining and rescaling,
- the scaling limit of the height field is conformally covariant.

It is very natural to ask whether these features are robust under perturbations of the Gibbs measure that break the determinant structure of Kasteleyn’s solution. This point is discussed in the following section.

### 3. Interacting Dimers: Model and Main Results

We consider a family of ‘interacting’ dimer models, defined by the following partition function:

$$Z_L^\lambda = \sum_{M \in \Omega_L} p_{L, \lambda}(M), \quad p_{L, \lambda}(M) = \left( \prod_{b \in M} t_{r(b)} \right) e^{\lambda \sum_{x \in \Lambda} f_{\tau_x M}}, \quad (3.1)$$

where: $\lambda$ is the interaction strength (to be thought of as ‘small’), $\Lambda$ is the lattice of black sites in $\mathbb{T}_L$, $f$ is a local function of the dimer configuration around the origin, and $\tau_x$ is the ‘translation operator’ by the lattice vector.
Theorem 1. [23] Let $t_1,t_2,t_3$ be such that $\mu(k)$ has two distinct simple zeros, $p^\pm$ (in particular, the ratio of $\alpha_\omega := \partial_k \mu(p^\omega)$ and $\beta_\omega := \partial_k \mu(p^\omega)$ is not real). Then, for $\lambda$ small enough,

$$E_\lambda[I_b(x,r); I_b(0,r')] = \frac{1}{4\pi^2} \sum_{\omega = \pm} \frac{K_{\omega,r}K_{\omega,r'}}{(\beta_\omega x_1 - \alpha_\omega x_2)^2}$$

$$+ \frac{1}{4\pi^2} \sum_{\omega = \pm} \hat{H}_{\omega,r} \hat{H}_{\omega,r'} e^{-i(\bar{p}^\omega - p^\omega)x} + \hat{R}_{r,r'}(x),$$

where\footnote{A warning on notation: given a quantity (such as $\alpha_\omega, p^\omega$) referring to the non-interacting model, the corresponding $\lambda$-dependent quantity for the interacting model will be distinguished by a bar, such as $\bar{\alpha}_\omega$, etc. On the other hand, we denote by $z^*$ the complex conjugate of a number $z$.} $\hat{K}_{\omega,r}, \hat{R}_{\omega,r}, \bar{\alpha}_\omega, \bar{\beta}_\omega, \bar{p}^\omega, \nu$ are analytic functions of $\lambda$, such that $\bar{\alpha}_\omega|_{\lambda=0} = \alpha_\omega, \bar{\beta}_\omega|_{\lambda=0} = \beta_\omega, \bar{p}^\omega|_{\lambda=0} = p^\omega$ and $\hat{K}_{\omega,r}|_{\lambda=0} = H_{\omega,r}|_{\lambda=0} = K_r e^{-\nu x}$. Recall that $K_r$ were defined in (2.3); moreover,

$$v_1 = (0,0), v_2 = (-1,0), v_3 = (-1,-1), v_4 = (0,-1).$$

These constants satisfy the following symmetry relations:

$$\bar{\alpha}_\omega = -\alpha_\omega, \quad \bar{\beta}_\omega = -\beta_\omega$$

$$\hat{K}^*_{\omega,r} = \hat{K}_{-\omega,r}, \quad \hat{H}^*_{\omega,r} = \hat{H}_{-\omega,r},$$

$$\bar{p}^+ + \bar{p}^- = (\pi, \pi).$$

x. In analogy with the non-interacting case, we let $P_{L,\lambda}$ be the finite volume Gibbs measure associated with weights $p_{L,\lambda}$ and $P_{\lambda}$ its infinite-volume limit (existence of the limit is part of the results in [23]).

Remark 3. For a special choice of $f$, the model reduces to the 6-vertex model, which is integrable by Bethe ansatz (but the solution is not determinantal), see [23, eq.(2.15)]. However, generically, the model is non-integrable.

Possibly, the simplest choice of $f$ that makes the model non-integrable is the ‘plaquette interaction’ previously considered in [3, 21, 22]:

$$f = \mathbb{I}_{e_1} \mathbb{I}_{e_2} + \mathbb{I}_{e_3} \mathbb{I}_{e_4} + \mathbb{I}_{e_5} \mathbb{I}_{e_6} + \mathbb{I}_{e_7}$$

where $e_1, \ldots, e_7$ are the edges in Fig. 3. Our main results do not depend on the specific choice of $f$. They concern the asymptotic behavior of dimer-dimer correlations and of the fluctuations of the height difference and are stated next.

FIGURE 3. The edges appearing in (3.2). $b$ is any fixed black vertex, say the one of coordinates $(0,0)$.
Moreover, 
\[ \nu(\lambda) = 1 + a\lambda + O(\lambda^2) \]
and, generically, \( a \neq 0 \). Finally, \( R_{r,r'}(x, x') = O(|x - x'|^{-5/2}) \) (the exponent 5/2 could be replaced by any \( \delta < 3 \) provided \( \lambda \) is small enough).

A few comments are in order:

- The proof provides a constructive algorithm for computing \( \bar{K}_{\omega,r}, \bar{H}_{\omega,r}, \) etc., at any desired precision. However, we do not have closed formulas for these quantities, and we do not expect that it is possible to obtain any by other methods.

- Generically, the anomalous exponent \( \nu \) has a non-zero first order contribution in \( \lambda \); therefore, it is larger or smaller than 1, depending on whether \( a\lambda \) is positive or negative. In particular, the asymptotic, large-distance, behavior of the dimer-dimer correlation is dominated by the first or second term in (3.3), depending on whether \( a\lambda \) is positive or negative.

- Once we have such a refined asymptotics as (3.3), we can plug it into the formula for the height variance,

\[ E_\lambda[h(f) - h(f'); h(f) - h(f')] = \sum_{b,b' \in C_{f \to f'}} \sigma_b \sigma_{b'} E_\lambda[1_b; 1_{b'}], \]
in order to understand its growth as \(|f - f'| \to \infty\). As anticipated above, one expects that the growth is still logarithmic, like in the non-interacting case. However, this is not completely straightforward: as remarked in the previous item, if \( a\lambda < 0 \) the asymptotic behavior of the dimer-dimer correlation is dominated by the second term in (3.3), which is characterized by the critical exponent \( \nu < 1 \). Therefore, after double-summation over \( b,b' \), one may fear that the interacting variance \( E_\lambda[h(f) - h(f'); h(f) - h(f')] \) grows like \(|f - f'|^{2(1-\nu)}\) at large distances, rather than logarithmically. Remarkably, this is not the case, thanks to the oscillating factor \( e^{-i(\bar{p}_\omega - \bar{p}_{\omega'})x} \) in front of the second term of (3.3).

**Theorem 2.** [23] Under the same hypothesis of Theorem 1,

\[ \text{Var}_{\mathbb{P}_\lambda}(h(f) - h(f')) = \frac{A(\lambda)}{\pi^2} \log |f - f'| + O(1) \]

where

\[ A(\lambda) = \left[ \frac{\bar{K}_{\omega,1} + \bar{K}_{\omega,2}}{\bar{\beta}_\omega} \right]^2 - \left[ \frac{\bar{K}_{\omega,1} + \bar{K}_{\omega,4}}{\bar{\alpha}_\omega} \right]^2. \]

In general, \( A(\lambda) \) depends explicitly on \( \lambda, t_1, t_2, t_3 \) and on the type of interaction potential in (3.1). Moreover,

\[ A(\lambda) = \nu(\lambda). \]

The proof of Theorem 2 provides two different algorithms for computing the coefficients of the analytic functions \( A \) and \( \nu \). At first order in \( \lambda \), one can check directly the validity of (3.8). The computation is very instructive and, as a by-product, it shows that \( A(\lambda) \) depends explicitly on \( \lambda, t_1, t_2, t_3; \)
see Sect.4 for a proof. However, a direct proof of the validity of \( A = \nu \) by
direct inspection and comparison of the two power series defining \( A \) and \( \nu \)
seems hopeless (the computation in the next section is a clear indication).
The proof of (3.8) is based on a subtle cancellation mechanism, which uses
a comparison between the lattice Ward Identities of the dimer model and
the emergent chiral Ward Identities of a ‘reference’ continuum model (the
‘infrared fixed point theory’), see [23, Sect.4 and 5].

The identity \( A(\lambda) = \nu(\lambda) \) is the analogue of one of the scaling relations
(the one between compressibility and the density-density critical exponent)
proposed by D. Haldane in the context of Luttinger liquids [24]. It is also
related to one of the scaling relations among the critical exponents of the 8-
vertex and of the Ashkin-Teller model, proposed by L. Kadanoff [25], see the
discussion right before eq.(1.2) of [23]. For this reason, we call it ‘Haldane’
or ‘Kadanoff’ relation.

There are not many previous examples of rigorously established Haldane
or Kadanoff relations: the known cases are mostly restricted to exactly
solved models of interacting fermions or quantum spins (Luttinger model,
or the XXZ chain [24]) and non-integrable perturbations thereof [9].

As stated, Theorem 2 concerns only the asymptotic growth of the variance
of the height difference. However, a slight extension of its proof, along the
same lines as [21, Sect.7.3], proves that, after coarse-graining, \( h(f) \) converges
to \( \phi \), the massless GFF of covariance

\[
E_{\lambda}(\phi(x)\phi(y)) = -\frac{A(\lambda)}{2\pi^2} \log |x - y|,
\]

in the same sense as [21, Theorem 3].

Theorem 2 and its extension mentioned in the previous item prove in a
very strong and sharp sense that the random surface associated with the
interacting dimer model is in a rough phase, characterized by logarithmic
fluctuations. In this sense, our theorem is of interest in the context of the
fluctuation theory of discrete random surface models, which is a classical
topic in probability and statistical mechanics. Previous, related, results
include the proofs of logarithmic height fluctuations in anharmonic crystals,
SOS model, 6-vertex and Ginzburg-Landau type models [10, 12, 15, 17, 19,
30, 34, 35, 36].

The proofs of Theorems 1 and 2 are hard and lengthy, and we refer the
reader to [23] for it. Here we limit ourselves to mention the main steps and
ideas involved in the proof:

1. The starting point is a representation of the non-interacting model,
characterized by Kasteleyn’s determinantal solution, in terms of a
free fermionic theory in dimension \( d = 1 + 1 \).

2. Next, we provide an exact representation of the interacting dimer
model in terms of an interacting fermionic theory in \( d = 1 + 1 \), the
interaction being (at dominant order) a quartic fermionic interac-
tion. In this sense, the interacting dimer model maps into a sort of
‘fermionic \( \phi^4 \) theory’.
The fermionic model, into which the original dimer model has been mapped, can be analyzed via standard fermionic multiscale cluster expansion methods (fermionic constructive RG) due, among others, to Gawedzki-Kupiainen [18], Lesniewski [31], Benfatto-Gallavotti [8], Feldman-Magnen-Rivasseau-Trubowitz [16].

The fermionic RG scheme turns out to be convergent if and only if the infrared flow of the ‘running coupling constants’, controlled by the so-called beta-function equation, is convergent in the limit of large scales. In order to control the RG flow, we compare it with the one of a reference continuum model: some of the ingredients involved in the comparison are the Ward Identities (both for the lattice dimer model and for the continuum reference one), the Schwinger-Dyson equation and the non-renormalization of the anomalies for the reference model [9].

In order to obtain the fine asymptotics of the dimer-dimer correlations, as well as the Haldane relation connecting $A$ and $\nu$, we need to compare the asymptotic, emergent, chiral Ward Identities of the reference model with the exact lattice Ward Identities of the dimer model, following from the local conservation law of the dimer number, $\sum_{b \sim x} 1 = 1$, with the sum running over edges incident to $x$. The comparison guarantees that the ratio $A/\nu$ is ‘protected’ by symmetry (there is no ‘dressing’ or ‘renormalization’ due to the interaction).

We conclude this section by mentioning a few open problems and perspectives:

- It would be interesting to study the critical theory in finite domains and, in perspective, prove its conformal covariance. From a technical point of view, going in this direction requires a non-trivial extension of the RG multiscale methods to the non-translationally-invariant setting, and a sharp control of the RG flow of the marginal boundary running coupling constants. Promising results in this direction have been recently obtained in the context of non-planar critical Ising models in the half-plane [5].

- So far, we can only compute the scaling limit of the height function after coarse graining, cf. for instance [21, Th. 3]. It would be very interesting to prove a central limit theorem on a more local scale; i.e., computing the average of $e^{i\omega(h(f) - h(f'))}$, instead of computing the characteristic function for the height function integrated against a test function. Similarly, it would be interesting to compute the scaling limit of the monomer-monomer correlations. This problem is not merely technical: it is strictly connected with the computation of the scaling limit of the spin-spin correlations in non-integrable, two-dimensional Ising models, which is currently out of reach of the current techniques.

- While we expect the analogs of Theorems 1 and 2 to be true also for the interacting dimer model on the honeycomb lattice of Fig. 1,
it is not obvious that the same qualitative behavior holds for interacting dimers on general $\mathbb{Z}^2$-periodic bipartite planar graphs, with an elementary cell containing more than two vertices. In the non-interacting case, using $\mathbb{Z}^2$ or any other $\mathbb{Z}^2$-periodic bipartite planar graph makes essentially no difference [29]. However, the effective fermionic theory is different for different periodic bipartite lattices: the larger the elementary cell, the larger the number of ‘colors’ of the fermionic field associated with the fermionic description of the system. In the presence of interactions, the number of colors is known to affect the qualitative behavior of the system (as an example, compare the behavior of the Luttinger model [33] with that of the 1D Hubbard model [32]): depending on it and on the specific form of the interaction, the system could display either an anomalous Fermi liquid behavior, or it could open a gap, entering more exotic quantum phases. It would be very interesting to see whether any of these exotic behaviors can arise in interacting dimer models on decorated periodic lattices.

• Finally, as suggested by the remarks in the introduction, it would be nice to see whether the current methods can be used to prove the existence of a rough, logarithmically correlated, low-temperature phase of the interface of 3D Ising with tilted Dobrushin boundary conditions. As observed above, the fluctuations of the surface at zero temperature can be mapped exactly into a problem of non-interacting dimers on the hexagonal lattice. It remains to be seen whether low-temperature fluctuations can be effectively described in terms of weakly interacting dimers on the hexagonal lattice.

4. First order computation

The goal of this section is to verify, at first order in $\lambda$, the validity of the ‘Haldane’, or ‘Kadanoff’, relation, (3.8), and to compute explicitly (to fix ideas, for the case of the plaquette interaction (3.2)) the constant $a$ in the expansion

$$\nu(\lambda) = A(\lambda) = 1 + a\lambda + O(\lambda^2).$$

The result is:

$$a = -\frac{4(t_1t_3 + t_2) \cos(p_1^+ \cos(p_2^+ \pi)}{\Im(\alpha_+\beta_-)},$$

where we recall that $p^\pm$ are the two zeros of $\mu(k)$, which are assumed to be non-degenerate, $\alpha_\pm = \partial_{k_1}\mu(p^\pm)$ and $\beta_\pm = \partial_{k_2}\mu(p^\pm)$. It is straightforward to check that the right side of (4.2) explicitly depends on the dimer weights: e.g., in the simple case $t_1 = t_3 = t$, $t_2 = 1$, in which $p^+ = 0$, $p^- = (\pi, \pi)$, one has $a = -2(1 + t^2)/(\pi t)$; in the case $t_1 = t_2 = t_3 = 1$, the result coincides with the one found in [22, App. B], $a = -4/\pi$.

We will use the same notations and conventions as in [23, 21]; we do not repeat here the definitions and we assume the reader has familiarity in particular with [23, Sec. 2 and 3] and [22, App. B]. We recall just that $\mathcal{E}_0$ denotes the Grassmann Gaussian measure with propagator (cf. (2.4))

$$g(x - y) := \mathcal{E}_0(\psi_x^- \psi_y^+) = K^{-1}(x, y)$$

(4.3)
For the following, it is convenient to introduce the rescaled coupling constant
\[ u := (t_1 t_3 + t_2) \lambda. \]

4.1. First-order computation of \( A \). The two-point dimer correlation function is given as
\[ G_{r,r'}^{(0,2)}(x,0) := \mathbb{E}_\lambda(1_e; 1_{e'}) = \lim_{L \to \infty} \partial_{A_e} \partial_{A_{e'}} \mathcal{W}_\lambda(A)|_{A \equiv 0}, \]
where \( e \) (resp. \( e' \)) is the edge of type \( r \) (resp. \( r' \)) with black endpoint of coordinates \( x \) (resp. 0) and \( \mathcal{W}_\lambda(A) \) is the moment generating function
\[ e^{\mathcal{W}_\lambda(A)} := \sum_{M \in \Omega_L} p_{L,\lambda}(M) \prod_e e^{A_e \mathbb{1}_e}. \]

Our convention for the Fourier transform of the two-point dimer correlation is that
\[ \hat{G}_{r,r'}^{(0,2)}(p) = \sum_x e^{-ipx} G_{r,r'}^{(0,2)}(x,0). \]

Since we are interested in the long-distance behavior of correlations, we will look at the small-\( p \) behavior and in particular at the discontinuity of \( \hat{G}_{r,r'}^{(0,2)}(p) \) at \( p = 0 \). We recall that, since we are working on the torus, \( \exp(\mathcal{W}_\lambda(A)) \) can be written as the linear combination of four Grassmann integrals with non-quadratic action, cf. [23, Eq. (3.16)]. For the computation of correlations in the \( L \to \infty \) limit and at finite order in perturbation theory, we can safely replace [23, Eq. (3.16)] with an expression involving a single Grassmann integral:
\[ e^{\mathcal{W}_\lambda(A)} = \int D\psi e^{S(\psi) + V(\psi,A)}, \]
with \( S \) and \( V \) as in [23, Eq. (3.16) and (3.17)]. In the case of the plaquette interaction (3.2), the potential \( V(\psi,A) \) equals (neglecting terms of order \( \lambda^2 \) and higher)
\[ V(\psi,A) = - \sum_e (e^{A_e} - 1) E_e + \lambda \sum_{\gamma = \{e_1, e_2\} \subseteq \Lambda} E_{e_1} E_{e_2} e^{A_{e_1} + A_{e_2}} \]
where the second sum runs over pairs of parallel edges \( \{e_1, e_2\} \) on the boundary of the same face. In particular, setting \( A \equiv 0 \) and using the definition [23, Eq. (3.13)] for \( E_e \) in terms of the Grassmann variables \( \psi^\pm \) associated to the endpoints of the edge \( e \), one finds that the potential is exactly quartic in the Grassmann fields:
\[ V_4(\psi) := V(\psi,0) = -u \sum_x \psi^+_x \psi^-_x \left[ \psi^+_x(0,1) \psi^-_x(-1,0) + \psi^+_x(1,0) \psi^-_x(-0,1) \right]. \]

From (4.5), (4.6) and (4.8) we see that the two-point dimer correlation function equals, at first order in \( \lambda \) and in the \( L \to \infty \) limit,
\[ G_{r,r'}^{(0,2)}(x,0) = \mathcal{E}_0(E_e; E_{e'}) - \lambda [\mathcal{E}_0(E_e; I_{0,r}^{(1)}) + \mathcal{E}_0(I_{x,r}^{(1)}; E_{e'})] + \mathcal{E}_0(E_e; E_{e'}; V_4) \]
where $\mathcal{E}_0(\ldots, \ldots)$ denotes truncated expectation and

$$J^{(1)}_{x,r} = \begin{cases} 
K_1K_3\psi^+_{x} \psi^-_{x} (\psi^+_{x+0(1)} \psi^-_{x-0(1)} + \psi^+_{x+0(1)} \psi^-_{x-0(1)}) & \text{if } r = 1 \\
K_2K_4\psi^+_{x} \psi^-_{x} \psi^+_{x+0(v)} (\psi^+_{x+0(1)} \psi^-_{x-0(1)} + \psi^+_{x-0(1)} \psi^-_{x-0(1)}) & \text{if } r = 2 \\
K_1K_3\psi^+_{x} \psi^-_{x} \psi^+_{x+0(v)} (\psi^+_{x+0(1)} \psi^-_{x-0(1)} + \psi^+_{x-0(1)} \psi^-_{x-0(1)}) & \text{if } r = 3 \\
K_2K_4\psi^+_{x} \psi^-_{x} \psi^+_{x+0(v)} (\psi^+_{x+0(1)} \psi^-_{x-0(1)} + \psi^+_{x-0(1)} \psi^-_{x-0(1)}) & \text{if } r = 4.
\end{cases}$$

(4.12)

For $\lambda = 0$ one finds from (4.11), (4.3) and from Lemma 1 below

$$\mathcal{G}^{(0,2)}_{r,r}(p) = -K_rK_{r'} \int_{[-\pi,\pi]^2} \frac{dk}{(2\pi)^2} \frac{e^{-ikv_r-i(k+p)v_r}}{\mu(k)\mu(k+p)}$$

$$= -\frac{i}{2\pi} \frac{K_rK_{r'}}{\alpha_+\beta_- - \alpha_-\beta_+} \sum_{\omega=\pm} \frac{D_{-\omega}(p)}{D_{\omega}(p)} e^{-i\omega(v_r+v_{r'})} + R(p).$$

Here and in the following, $R(p)$ denotes a function that is continuous at $p = 0$ (the precise value of $R(p)$ can change from line to line). In (4.13), $v_r \in \mathbb{Z}^2$, $r = 1, \ldots, 4$, is as in (3.4) while

$$D_{\omega}(p) = \alpha_+p_1 + \beta_\omega p_2, \quad \omega = \pm.$$  

(4.14)

Next we compute the first-order contribution

$$-\lambda [\mathcal{E}_0(E; I^{(1)}_{0,r}) + \mathcal{E}_0(I^{(1)}_{x,r}; E')$$

(4.15)

in (4.11). As explained at the beginning of [22, Sec. B.1], by the fermionic Wick theorem, we can replace $I^{(1)}_{x,r}$ by its “linearization” $\bar{I}^{(1)}_{x,r}$ obtained by contracting in all possible ways two of its four $\psi$ fields. For instance, with $g(\cdot)$ as in (4.3),

$$\bar{T}^{(1)}_{x,1} = K_1K_3 \left[ -g(v_1) \left( \psi^+_{x+0(1)} \psi^-_{x-0(1)} + \psi^+_{x+0(1)} \psi^-_{x-0(1)} \right) \\
+ g(v_2) \left( \psi^+_{x+0(1)} \psi^-_{x-0(1)} + \psi^+_{x-0(1)} \psi^-_{x-0(1)} \right) \\
+ g(v_4) \left( \psi^+_{x+0(1)} \psi^-_{x-0(1)} + \psi^+_{x-0(1)} \psi^-_{x-0(1)} \right) - 2g(v_3) \psi^+_{x} \psi^-_{x} \right].$$

(4.16)

In Fourier space (with the conventions of [23, Eq. (6.1)] for the Grassmann fields in momentum space) one has

$$\bar{T}^{(1)}_{x,r} = \int_{[-\pi,\pi]^2} \frac{dk}{(2\pi)^2} \int_{[-\pi,\pi]^2} \frac{dp}{(2\pi)^2} e^{ipx} \psi^+_{k+p} W_r(k,p) \psi^-_{k}$$

(4.17)

where

$$W_r(k,p) = K_1K_3 \left[ -g(v_1) \left( e^{i(k_1+k_2+p_1)} + e^{i(k_1+k_2+p_2)} \right) \\
+ g(v_2) \left( e^{i(k_2+p_2)} + e^{ik_2} \right) + g(v_4) \left( e^{i(k_3+p_1)} + e^{ik_1} \right) - 2g(v_3) \right].$$

(4.18)

Similar formulas hold for $r = 2, 3, 4$, with $W_1(k,p)$ replaced by $W_r(k,p)$. One easily checks (for $r = 1$ this can be verified immediately from (4.18))
that
\[ W_r(k,0) = 2t_r t_{r+2} W(k), \]
\[ W(k) = \left[ g(v_1)e^{i(k_1+k_2)} - g(v_2)e^{ik_2} - g(v_4)e^{ik_1} + g(v_3) \right] \]
\[ = \int_{[-\pi,\pi]^2} \frac{dk'}{(2\pi)^2} \frac{(\varepsilon^{ik_1} - \varepsilon^{ik'_1})(\varepsilon^{ik_2} - \varepsilon^{ik'_2})}{\mu(k')} \] (4.19)
with the convention that \( t_4 = 1 \) and that \( t_r := t_{r \mod 4} \) if \( r > 4 \). Note that
\[ W(p^+) = W(p^-), \] (4.20)
because \( p^+ + p^- = (\pi, \pi) \) and \( g(v_1), g(v_2) \in \mathbb{R} \), while \( g(v_2), g(v_4) \in i\mathbb{R} \). Using (4.19) and (4.21), together with
\[ E_e = K_r \int_{[-\pi,\pi]^2} \frac{dk}{(2\pi)^2} \int_{[-\pi,\pi]^2} \frac{dp}{(2\pi)^2} e^{ipx} \hat{\psi}^+_{k+p} \hat{\psi}^-_{k} e^{-ikv_r}, \] (4.21)
(if \( e \) is, as above, of type \( r \) and with black vertex of coordinates \( x \)) we see that
\[ -\lambda [\mathcal{E}_0(E_r; I_e^{(1)}; E_e)](p) = 2\lambda \int_{[-\pi,\pi]^2} \frac{1}{(2\pi)^2} \frac{\mu(k)\mu(k+p)}{\mu(k)} \times \]
\[ \times [K_r t_r t_{r+2} e^{-ikv_r} + K_r t_r t_{r+2} e^{-ikv_r}] W(k) + R(p). \] (4.22)
Thanks to Lemma 1, we can rewrite (4.22) as
\[ \frac{i\lambda}{\pi (\alpha_+ \beta_- - \alpha_- \beta_+)} \times \]
\[ \times \sum_{\omega = \pm} \frac{D_{-\omega}(p)}{D_{\omega}(p)} [K_r t_r t_{r+2} e^{-ip\omega v_r} + K_r t_r t_{r+2} e^{-ip\omega v_r}] W(p^\omega) + R(p). \]

It remains to compute the term \( \mathcal{E}_0(E_r; E_e; V_4) \) in (4.11). Applying Wick’s theorem, we see that either \( N = 0 \) or \( N = 2 \) of the four fields of \( V_4 \) are contracted among themselves, the remaining \( 4 - N \) ones being contracted with fields from \( E_e \) or \( E_e' \). Here we compute the contribution, call it \( a_{r_{1},r_{2}}^{(0,2)}(x,0) \), from those terms where \( N = 2 \). The contribution \( b_{r_{1},r_{2}}^{(0,2)}(x,0) \) from the terms with \( N = 0 \) is computed later. We note that
\[ \mathcal{E}_0(E_r; E_e') + \lambda a_{r_{1},r_{2}}^{(0,2)}(x,0) = \mathcal{E}_\lambda(E_r; E_e') + O(\lambda^2) \] (4.23)
where \( \mathcal{E}_\lambda(\cdot) \) is the Gaussian Grassmann measure where the action \( S(\psi) = -\sum E_e \) has been replaced by \( S(\psi) + \nabla_4 \), with \( \nabla_4 \) the linearization of \( V_4 \) (i.e. the bilinear operator obtained by contracting in every possible way two of the four fields of \( V_4 \), as above). In our case,
\[ \tilde{V}_4 = -2u \sum_z \left[ -g(v_1) \psi^+_z \psi^-_{z-(1,1)} + g(v_2) \psi^+_z \psi^-_{z-(0,1)} \right. \]
\[ -g(v_3) \psi^+_z \psi^-_{z} + g(v_4) \psi^+_z \psi^-_{z-(1,0)} \] (4.24)
\[ -g(v_3) \psi^+_z \psi^-_{z} + g(v_4) \psi^+_z \psi^-_{z-(1,0)} \] (4.25)
Then, we see that the measure $\mathcal{E}_\lambda(\cdot)$ is nothing but the Gaussian Grassmann measure where $\mu(k)$ is replaced by

$$\bar{\mu}(k) := \mu(k) - 2uW(k)$$  (4.26)

and $W(k)$ was defined in (4.19). As already mentioned in Theorem 1, the ratio of complex numbers $\alpha_\omega, \beta_\omega$ is not real [23]; therefore we can write uniquely $W(p^{\omega})$ as

$$W(p^{\omega}) = c_1^{\omega} \alpha_\omega + c_2^{\omega} \beta_\omega \text{ with } c_1^{\omega}, c_2^{\omega} \in \mathbb{R}.$$  (4.27)

Via Taylor expansion, we have then (always at first order in $\lambda$)

$$\bar{\mu}(k) = \bar{\alpha}_\omega(k_1 - p_1^{\omega}) + \bar{\beta}_\omega(k_2 - p_2^{\omega}) + O(|k - p^{\omega}|^2)$$  (4.28)

where the zeros of $\bar{\mu}(\cdot)$ are

$$p_j^{\omega} = p_j^{\omega} + 2uc_j^{\omega}, \quad j = 1, 2$$  (4.29)

and

$$\bar{\alpha}_\omega = \alpha_\omega + 2u[-\partial_{k_1}W(p^{\omega}) + c_1^{\omega} \partial_{k_1}^2 \mu(p^{\omega}) + c_2^{\omega} \partial_{k_1,k_2}^2 \mu(p^{\omega})]$$
$$\bar{\beta}_\omega = \beta_\omega + 2u[-\partial_{k_2}W(p^{\omega}) + c_2^{\omega} \partial_{k_2}^2 \mu(p^{\omega}) + c_1^{\omega} \partial_{k_1,k_2}^2 \mu(p^{\omega})].$$  (4.30)

Explicitly,

$$\partial_{k_1}W(p^{\omega}) = ig(v_1)e^{i(p_1^{\omega} + p_2^{\omega})} - g(v_4)e^{ip_1^{\omega}}$$
$$\partial_{k_2}W(p^{\omega}) = ig(v_1)e^{i(p_1^{\omega} + p_2^{\omega})} - g(v_4)e^{ip_2^{\omega}}$$
$$\partial_{k_1}^2 \mu(p^{\omega}) = i\alpha_\omega, \quad \partial_{k_2}^2 \mu(p^{\omega}) = i\beta_\omega, \quad \partial_{k_1,k_2}^2 \mu(p^{\omega}) = t_3e^{i(p_1^{\omega} + p_2^{\omega})}.\quad (4.31)$$

This implies in particular the symmetry (3.5) at first order in $\lambda$. Also, from (4.20) it follows that $c_j^{\omega} = -c_j^{-\omega}$ so that (3.7) holds a first order.

Altogether, (4.23) equals in Fourier space, for small $p$ and disregarding $O(\lambda^2)$ terms,

$$-\frac{i}{2\pi} \frac{K_r K_{r'}}{\alpha_+ \beta_- - \alpha_- \beta_+} \sum_{\omega = \pm} \frac{\bar{D}_{-\omega}(p)}{D_{\omega}(p)} e^{-i\varphi^{\omega}(v_r,v_{r'})} + R(p)$$  (4.32)

where $\bar{D}_\omega$ is defined as (4.14) with $\alpha_\omega, \beta_\omega$ replaced by $\bar{\alpha}_\omega, \bar{\beta}_\omega$.

Finally, we compute the contribution $b^{(0,2)}(\rho)$ to the Fourier transform of $\mathcal{E}_0(E_{x,r};E_{0,r'},V)$ where none of the fields of $V_4$ are contracted among themselves. First we write in Fourier space

$$V_4 = -u \int_{[-\pi,\pi]^{2}} \frac{dk}{(2\pi)^2} \int_{[-\pi,\pi]^{2}} \frac{dk'}{(2\pi)^2} \int_{[-\pi,\pi]^{2}} \frac{dp}{(2\pi)^2} \hat{\psi}_{k+p}^{+}\hat{\psi}_{k}^{-}\hat{\psi}_{k'}^{+}\hat{\psi}_{k'-p}^{-}$$
$$\times \left( e^{i(k_1+k_2 '-')} + e^{i(k_1'+k_2 - p)} \right)$$

$$= -\frac{u}{4} \int_{[-\pi,\pi]^{2}} \frac{dk}{(2\pi)^2} \int_{[-\pi,\pi]^{2}} \frac{dk'}{(2\pi)^2} \int_{[-\pi,\pi]^{2}} \frac{dp}{(2\pi)^2} \hat{\psi}_{k+p}^{+}\hat{\psi}_{k}^{-}\hat{\psi}_{k'}^{+}\hat{\psi}_{k'-p}^{-}W(k,k',p)$$  (4.33)

where

$$W(k,k',p) = e^{i(k_1'+k_2 - p)} + e^{i(k_1'+k_2 - p_1)} + e^{i(k_1+k_2+p)} + e^{i(k_1+k_2+p_1)}$$
$$- e^{i(k_2,k_1'+p)} - e^{i(k_1,k_2+p)} - e^{i(k_1',k_2+p)} - e^{i(k_1+k_2+p_1)}.\quad (4.34)$$
The second expression for $V_4$ is obtained by symmetrizing over the four possible ways of ordering the fields $\hat{\psi}_{k+p}^{\uparrow} \hat{\psi}_k^{\downarrow} \hat{\psi}_{k'}^{\uparrow} \hat{\psi}_{k''}^{\downarrow}$ in such a way that the order of the upper indices is $(+, -, +, -)$. Therefore, $b_{r,r'}^{(0,2)}(p)$ equals

$$-uK_rK_{r'} \int_{[-\pi,\pi]^2} \frac{dk}{(2\pi)^2} \int_{[-\pi,\pi]^2} \frac{dk'}{(2\pi)^2} W(k, k', p) \times$$

$$\times e^{-i(k'-p)v_r-i(k+p)v_{r'}} \mu(k)\mu(k+p)\mu(k')\mu(k'-p).$$

Note that

$$W(k, k', 0) = 2(e^{ik_1} - e^{ik'_1})(e^{ik_2} - e^{ik'_2}).$$

Then,

$$b_{r, r'}^{(0,2)}(p) = -2uK_rK_{r'} \int_{[-\pi,\pi]^2} \frac{dk}{(2\pi)^2} \int_{[-\pi,\pi]^2} \frac{dk'}{(2\pi)^2} \times$$

$$\times e^{-ik'v_r-ikv_r'}(e^{ik_1} - e^{ik'_1})(e^{ik_2} - e^{ik'_2}) \mu(k)\mu(k+p)\mu(k')\mu(k'-p) + R(p).$$

From this expression we want to extract the term that is discontinuous at $p = 0$. Expanding the product in (4.36) we see that the integral (4.37) is given by a sum of combinations of integrals of the type $I_a$ as in Lemma 1 below. Applying (4.62) to each of the integrals, it is easily checked that the terms $(D_\omega(p)/D_\omega(p))^2$ cancel and one is left with\(^3\)

$$-\frac{iuK_rK_{r'}}{\pi(\alpha_+\beta_- - \alpha_-\beta_+)} \sum_{\omega=\pm} \frac{D_{-\omega}(p)}{D_{\omega}(p)} (e^{-ip^{\omega}v_r}U^\omega_r + e^{-ip^{\omega}v_r}U^{-\omega}_r) + R(p),$$

$$U^\omega_r = e^{ip^{(1,1)}_\omega}C(-v_r) + C(-v_r + (1,1))$$

$$- e^{ip^{(1,0)}_\omega}C(-v_r + (0,1)) - e^{ip^{(0,1)}_\omega}C(-v_r + (1,0)).$$

Using (4.66) we see that

$$(U^\omega_r)^* = U^{-\omega}_r$$

\(^3\)Here we assume, without loss of generality, that $\cos(p^+_1) > 0$. Since $p^+_1 + p^-_1 = \pi$, we are just deciding which of the two zeros of $\mu(\cdot)$ we call $p^+$. 
Finally, using \((4.20), (4.40)\) and \((3.7)\), we see that the symmetry \((3.6)\) holds and from \((4.62)\) one checks that

\[
U_1^+ = \frac{e^{ip_2^+}}{2\pi} \int_{p_1^-} e^{ip_1^-} \frac{e^{ip_1^-} - e^{i\theta}}{(K_1 + K_2 e^{i\theta})^2} \frac{2i}{\pi (\alpha_+ \beta_- - \alpha_- \beta_+)} \beta_+ \cos(p_1^+) \cos(p_2^+),
\]

\[(4.41)\]

\[
U_2^+ = \frac{e^{ip_2^+}}{2\pi} \int_{p_1^-} e^{ip_1^-} \frac{e^{ip_1^-} - e^{i\theta}}{(K_1 + K_2 e^{i\theta})^2} \frac{2i}{\pi (\alpha_+ \beta_- - \alpha_- \beta_+)} e^{ip_1^-} \beta_+ \cos(p_1^+) \cos(p_2^+),
\]

\[(4.42)\]

\[
U_3^+ = \frac{1}{2\pi} \int_{p_1^-} e^{i\theta} \frac{e^{ip_1^-} - e^{i\theta}}{(-K_3 e^{i\theta} - K_4)^2} \frac{2i}{\pi (\alpha_+ \beta_- - \alpha_- \beta_+)} e^{ip_1^- + ip_2^+} \beta_+ \cos(p_1^+) \cos(p_2^+),
\]

\[(4.43)\]

\[
U_4^+ = \frac{1}{2\pi} \int_{p_1^-} e^{i\theta} \frac{e^{ip_1^-} - e^{i\theta}}{(K_3 e^{i\theta} + K_4)^2} \frac{2i}{\pi (\alpha_+ \beta_- - \alpha_- \beta_+)} e^{ip_2^+} \beta_+ \cos(p_1^+) \cos(p_2^+).
\]

\[(4.44)\]

Summarizing, we obtained \((4.45)\)

\[
\hat{G}_{r,r'}^{(0,2)}(p) = -\frac{i}{2\pi (\alpha_+ \beta_- - \alpha_- \beta_+)} \sum_{\omega = \pm} \frac{\tilde{D}_{-\omega}(p)}{\tilde{D}_\omega(p)} \tilde{K}_{\omega,r} \tilde{K}_{\omega,r'} e^{-ip_\omega(v_r + v_{r'})} + R(p),
\]

\[(4.45)\]

with

\[
\tilde{K}_{\omega,r} := K_r \left( e^{-ip_\omega v_r} - \frac{2\lambda_r t^2 + 2}{K_r} W(p_\omega) + 2uU_r\right).
\]

\[(4.46)\]

Thanks to \((3.5)\), the prefactor of \((4.47)\) is real and

\[
\left( \frac{\tilde{D}_-(p)}{\tilde{D}_+(p)} \right) = \frac{\tilde{D}_+(p)}{\tilde{D}_-(p)}.
\]

Finally, using \((4.20), (4.40)\) and \((3.7)\), we see that the symmetry \((3.6)\) holds at first order in \(\lambda\), so that \((4.45)\) reduces to

\[
\hat{G}_{r,r'}^{(0,2)}(p) = -\frac{i}{\pi (\alpha_+ \beta_- - \alpha_- \beta_+)} \mathbb{R} \left[ \frac{\tilde{D}_-(p)}{\tilde{D}_+(p)} \tilde{K}_{+,r} \tilde{K}_{+,r'} \right] + R(p).
\]

\[(4.47)\]

As discussed in [22, App. A], this asymptotic behavior of the Fourier transform of the dimer-dimer correlation for \(p \to 0\) is equivalent to the asymptotic behavior for large distances of the first line of \((3.3)\) \((\text{the part proportional to } H_{\omega,r})\), instead, in momentum space around \(p = 0\) can be absorbed into the error term \(R(p)\), because of the oscillating prefactor. In order to prove at first order that the variance of the height difference grows logarithmically at large distances, as stated in Theorem 2, we need to show that \((\text{see 23}, \ldots)\).
Theorem 2)
\[
\Delta_2 := \frac{\sum r \in \{1, 4\}}{\alpha_+} \frac{\tilde{K}^{+}}{\alpha_+} = \Delta_1 := \frac{\sum r \in \{1, 2\}}{\beta_+} \frac{\tilde{K}^{+}}{\beta_+}
\]
(4.48)
at order \(\lambda\). This ratio is nothing but the first-order expansion of \(i \sqrt{A(\lambda)}\), cf. Theorem 2. Since for \(\lambda = 0\) both ratios equal \(i\) we write, at order \(\lambda\),
\[
\frac{\bar{\alpha}_+}{\alpha_+} = \frac{\bar{\beta}_+}{\beta_+} = 2u \alpha_+^{(1)} + \frac{\alpha_+}{\beta_+} = \frac{\beta_+}{\alpha_+} = 2u \beta_+^{(1)},
\]
\[
\Delta_1 = i \beta_+ + 2u \Delta_1^{(1)}, \quad \Delta_2 = i \alpha_+ + 2u \Delta_2^{(1)},
\]
with \(\alpha_+^{(1)}, \beta_+^{(1)}, \Delta_1^{(1)}, \Delta_2^{(1)}\) four \(\lambda\)-independent constants. Then, we need to prove that
\[
(\Delta_2^{(1)} - i \alpha_+^{(1)}) \beta_+ - (\Delta_1^{(1)} - i \beta_+^{(1)}) \alpha_+ = 0.
\]
(4.49)
From (4.30) and (4.31) it follows that
\[
\alpha_+^{(1)} = -ig(v_1)e^{i(p_1^+ + p_2^+) + ig(v_4)e^{i(p_1^+ + p_2^+) + i\alpha_+} + c_1 \Delta_1 e^{i(p_1^+ + p_2^+) + i\beta_+ + c_2 \Delta_2 e^{i(p_1^+ + p_2^+)}},
\]
\[
\beta_+^{(1)} = -ig(v_1)e^{i(p_1^+ + p_2^+) + ig(v_2)e^{i(p_1^+ + p_2^+) + i\beta_+ + c_1 \Delta_1 e^{i(p_1^+ + p_2^+) + c_2 \Delta_2 e^{i(p_1^+ + p_2^+)}}}
\]
(4.50)
and from (4.29) and (4.46) we see that
\[
\Delta_2^{(1)} = -W(p^+) + K_1 U_1^+ + K_4 U_4^+ + i c_1 K_4 e^{i p_2^+}
\]
\[
\Delta_1^{(1)} = -W(p^+) + K_1 U_1^+ + K_2 U_2^+ + i c_1 K_2 e^{i p_2^+}.
\]
(4.51)
(4.52)
Therefore,
\[
\Delta_2^{(1)} - i \alpha_+^{(1)} = K_1 U_1^+ + K_4 U_4^+ - g(v_1)e^{i(p_1^+ + p_2^+)} + g(v_4)e^{i p_1^+}
\]
\[
\Delta_1^{(1)} - i \beta_+^{(1)} = K_1 U_1^+ + K_2 U_2^+ - g(v_1)e^{i(p_1^+ + p_2^+)} + g(v_2)e^{i p_2^+}
\]
(4.53)
where we used \(c_1^+ \alpha_+ + c_2^+ \beta_+ = W(p^+)\) as in (4.27). Then, using the definition of \(U_1^+, U_2^+\), the l.h.s. of (4.49) is
\[
-\frac{2}{\pi} \frac{\beta_+}{\bar{\beta}_-} \cos(p_1^+) \cos(p_2^+) - g(v_1)e^{i p_1^+ + p_2^+} \beta_+ - \alpha_+ + \beta_+ g(v_4)e^{i p_1^+} - g(v_4)e^{i p_2^+}
\]
\[
- \alpha_+ \frac{e^{i p_2^+}}{2\pi} \frac{\int_{p_1^+}^{p_1^+ + 2\pi}}{K_1 + K_2 e^{i\theta}} d\theta + \beta_+ \frac{e^{i p_2^+}}{2\pi} \frac{\int_{p_1^+}^{p_1^+ + 2\pi}}{K_1 + K_2 e^{i\theta}} d\theta
\]
\[
+ \frac{1}{K_1 + K_2 e^{i\theta}} d\theta. \quad (4.54)
\]
A simple application of the residue theorem shows that
\[
g(v_1) = \frac{1}{2\pi} \int_{p_1^+}^{p_1^+ + 2\pi} \frac{1}{K_1 + K_2 e^{i\theta}} d\theta
\]
(4.55)
\[
g(v_2) = \frac{1}{2\pi} \int_{p_1^+}^{p_1^+ + 2\pi} \frac{e^{i\theta}}{K_1 + K_2 e^{i\theta}} d\theta
\]
(4.56)
\[
g(v_4) = \frac{1}{2\pi} \int_{p_1^+}^{p_1^+ + 2\pi} \frac{1}{K_3 e^{i\theta} + K_4} d\theta. \quad (4.57)
\]
Therefore, the terms proportional to $\alpha_+$ in (4.54) cancel and one is left with

$$
- \frac{2}{\pi} \frac{\beta_+}{\beta_-} \cos(p_1^+) \cos(p_2^+)
- e^{i(p_1^++p_2^+)} \frac{1}{2\pi} \int_{p_1^-}^{p_1^+} \frac{1}{K_1 + K_2 e^{i\theta}} d\theta + \beta_+ e^{i p_1^+} \frac{1}{2\pi} \int_{p_1^-}^{p_1^+} \frac{1}{K_3 e^{i\theta} + K_4} d\theta
+ \beta_+ K_1 \frac{e^{ip_1^+}}{2\pi} \int_{p_1^-}^{p_1^+} \frac{1}{K_1 + K_2 e^{i\theta}} d\theta
+ \beta_+ K_4 \frac{e^{ip_1^+}}{2\pi} \int_{p_1^-}^{p_1^+} \frac{1}{K_3 e^{i\theta} + K_4} d\theta
= - \frac{2}{\pi} \frac{\beta_+}{\beta_-} \cos(p_1^+) \cos(p_2^+) + \beta_+ \frac{K_4 + K_3 e^{i p_1^+}}{2\pi i} \int_{e^{i p_1^+}}^{e^{i p_2^+}} \frac{dz}{(K_3 z + K_4)^2}
- \beta_+ (K_1 + K_2 e^{i p_1^+}) \frac{e^{i p_1^+}}{2\pi i} \int_{e^{i p_1^+}}^{e^{i p_2^+}} \frac{dz}{(K_1 + K_2 z)^2}.
$$

(4.58)

Computing the integrals in the complex plane and using the explicit expressions for $\alpha_\omega, \beta_\omega$, which follow from their definition (see the statement of Theorem 1), one finds that (4.58) is zero, so that (4.49) holds.

In addition, one sees that the ratio (4.48) is given by

$$
i \left( 1 + \frac{4u \cos(p_1^+) \cos(p_2^+)}{\alpha_+ \beta_- - \alpha_- \beta_+} \right) + O(\lambda^2).
$$

(4.59)

Recalling that this ratio equals $i \sqrt{A(\lambda)}$ and that $\alpha_+ = -\alpha_-, \beta_+ = -\beta_-$, we find that $A(\lambda) = 1 + a \lambda + O(\lambda^2)$, with $a$ as in (4.2).

4.2. **First-order computation of $\nu(\lambda)$**

By arguing like in [22, Appendix B.3], in order to compute the first order contribution to $\nu$, it is enough to extract the most divergent part, as $p \to p^- - p^+$, of the first order contribution to $G_{r,r'}^{(0,2)}(p)$. For definiteness, we assume, generically, that $p^+ - p^- \neq p^- - p^+ \mod 2\pi$ (the complementary case, that corresponds to zero average tilt, can be treated analogously). The coefficient $\nu_1$ in the expansion $\nu(\lambda) = 1 + \nu_1 \lambda + O(\lambda^2)$ can be read from:

$$
G_{r,r'}^{(0,2)}(p^- - p^+ + q) = \frac{-i \lambda \nu_1 t_1^2}{\pi (\alpha_+ \beta_- - \alpha_- \beta_+)} (\log |q|)^2 + \ldots.
$$

(4.60)

where the dots indicate lower order terms in $\lambda$ or in $q$, as $q \to 0$; i.e., they indicate terms that are either $O(\lambda^2)$, or less divergent than $(\log |q|)^2$ as $q \to 0$.

By inspection (see the analogous discussion a few lines before [22, eq.(B.29)]), the only term that diverges like $(\log |q|)^2$ as $q \to 0$ comes from $b_{r,r'}^{(0,2)}$, see (4.35); setting, e.g., $r = r' = 1$,

$$
G_{1,1}^{(0,2)}(p^- - p^+ + q) = -u t_1^2 \int_{[-\pi, \pi]^2} \frac{dk}{(2\pi)^2} \int_{[-\pi, \pi]^2} \frac{dk'}{(2\pi)^2} W(k, k', p^- - p^+ + q) \times
\frac{1}{\mu(k)\mu(k + p^- - p^+ + q)\mu(k')\mu(k' - p^- + p^+ - q)} + \ldots,
$$
where the dots indicate lower order terms, as explained above. The dominant contribution to the right side comes from the region where $k$ is close to $p^+$ and $k'$ is close to $p^-$. By explicitly computing the dominant contribution to the integrand in this region we find:

$$
\tilde{G}_{r,r'}^{(0,2)}(p^- - p^+ + q) = -\frac{ut^2}{16\pi^2} \int \int_D ds' W(p^+, p^-, p^- - p^+) \times \frac{1}{D_+(s)D_-(s + q)D_-(s')D_+(s' - q) + \ldots},
$$

where $\varepsilon$ is an arbitrary, small enough, positive number. Recalling (4.34) and the fact that $p^+ + p^- = (\pi, \pi)$, we find:

$$
W(p^+, p^-, p^- - p^+) = -2(e^{ip^+_1} - e^{ip^-_1})(e^{ip^+_2} - e^{ip^-_2}) = -8\cos(p^+_1)\cos(p^+_2).
$$

Moreover,

$$
\int [-\varepsilon, \varepsilon]^2 \frac{1}{D_+(s)D_-(s + q)} = \frac{4i\pi}{(\alpha_+\beta_+ - \alpha_-\beta_-)} \log |q| + \ldots
$$

Putting things together we find:

$$
\tilde{G}_{r,r'}^{(0,2)}(p^- - p^+ + q) = -\frac{8ut^2}{\pi^2} \frac{\cos(p^+_1)\cos(p^+_2)}{(\alpha_+\beta_+ - \alpha_-\beta_-)^2} \log |q|^2 + \ldots
$$

By comparing with (4.60), we find, as desired,

$$
\nu_1 = \frac{8(u/\lambda)}{\pi} \frac{1}{\alpha_+\beta_+ - \alpha_-\beta_-} \cos(p^+_1)\cos(p^+_2).
$$

4.3. A useful integral formula.

**Lemma 1.** Let $a = (a_1, a_2) \in \mathbb{Z}^2$ and

$$
\mathcal{I}_a(p) = \int_{[-\pi, \pi]^2} \frac{dk}{2\pi} \frac{e^{ika}}{\mu(k)\mu(k + p)}.
$$

Assume that $\cos(p^+_1) > 0$. Then, one has:

$$
\mathcal{I}_a(p) = \frac{i}{2\pi} \frac{1}{\alpha_+\beta_+ - \alpha_-\beta_-} \sum_{\omega=\pm} \frac{D^-\omega(p)}{D^\omega(p)} e^{i\omega a} + C(a) + R_0(p)
$$

where $R_0(p)$ tends continuously to zero as $p \to 0$ and

$$
C(a) = \frac{i}{2\pi} \frac{1}{\alpha_+\beta_+ - \alpha_-\beta_-} \sum_{\omega=\pm} \frac{\beta^-\omega e^{ia\omega}}{\beta^\omega}
$$

$$
+ \frac{(1 - a_2)}{2\pi} \int_0^{2\pi} d\theta e^{i\theta a_1} \left( K_1 + K_2 e^{i\theta} a_2^{-2} \right)
$$

$$
\times \left[ 1_{a_2 \leq 0} 1_{p^-_1 + 2\pi < \theta < p^+_1 + 2\pi} - 1_{a_2 \geq 1} 1_{p^-_1 < \theta < p^+_1} \right].
$$

The proof is obtained with a straightforward although a bit lengthy application of the residue theorem; we skip details. Note also that $(\mathcal{I}_a(p))^* = (-1)^{a_1 + a_2} \mathcal{I}_a(-p)$ so that

$$
C(a)^* = (-1)^{a_1 + a_2} C(a).
$$
Acknowledgements. This review is based on a longstanding collaboration with Vieri Mastropietro, whom we thank for countless inspiring discussions. This work has been supported by the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme (ERC CoG UniCoSM, grant agreement n.724939). F.T. was partially supported by the CNRS PICS grant 151933, by ANR-15-CE40-0020-03 Grant LSD, ANR-18-CE40-0033 Grant DIMERS and by Labex MiLyon (ANR-10-LABX-0070). This work was started during a long-term stay of A.G. at Univ. Lyon-1, co-funded by Amidex and CNRS.

References

[1] M. Aizenman, H. Duminil-Copin, V. Tassion, S. Warzel: Emergent Planarity in two-dimensional Ising Models with finite-range Interactions, to appear on Invent. Math., arXiv:1801.04960

[2] I.T. Alam, M.T. Batchelor: Integrability as a consequence of discrete holomorphy: loop models, J. Phys. A 47, 215201 (2014).

[3] F. Alet, J. Jacobsen, G. Misguich, V. Pasquier, F. Mila, M. Troyer: Interacting Classical Dimers on the Square Lattice, Phys. Rev. Lett. 94, 235702 (2005); F. Alet, Y. Ikhlef, J. Jacobsen, G. Misguich, V. Pasquier: Classical dimers with aligning interactions on the square lattice, Phys. Rev. E 74 (2006), 041124.

[4] R. Cerf, R. Kenyon: The Low-Temperature Expansion of the Wulff Crystal in the 3D Ising Model, Comm. Math. Phys. 222 (2001), 149-179.

[5] G. Antinucci, A. Giuliani, R. Greenblatt, in preparation.

[6] R. J. Baxter: Partition function of the Eight-Vertex lattice model, Ann. Phys. 70, 193-228 (1972).

[7] G. Benfatto, P. Falco, V. Mastropietro: Universal Relations for Nonsolvable Statistical Models, Phys. Rev. Lett. 104, 075701 (2010); and Extended Scaling Relations for Planar Lattice Models, Comm. Math. Phys. 292, 569-605 (2009).

[8] G. Benfatto, G. Gallavotti: Perturbation theory of the Fermi surface in a quantum liquid. A general quasiparticle formalism and one-dimensional systems, J. Stat. Phys. 59, 541-664 (1990).

[9] G. Benfatto, V. Mastropietro: Drude weight in non solvable quantum spin chains, J. Stat. Phys. 143, 251-260 (2011); G. Benfatto, V. Mastropietro: Universality relations in non-solvable quantum spin chains, J. Stat. Phys. 138, 1084-1108 (2010).

[10] H. J. Brascamp, E. H. Lieb, J. L. Lebowitz: The statistical mechanics of anharmonic lattices, in: Nachtergaele B., Solovej J.P., Yngvason J. (eds) Statistical Mechanics (1975), Springer, Berlin, Heidelberg.

[11] D. Chelkak, C. Hongler, K. Izyurov: Conformal invariance of spin correlations in the planar Ising model, Annals of mathematics 181 (2015), 1087-1138.

[12] J. G. Conlon, T. Spencer: A strong central limit theorem for a class of random surfaces, Commun. Math. Phys. 325 (2014), 1-15.

[13] J. Dubédat, Dimers and families of Cauchy-Riemann operators I, Journal of the American Mathematical Society 28 (2015), 1063-1167.

[14] H. Duminil-Copin, C. Hongler, P. Nolin: Connection probabilities and RSW-type bounds for the two-dimensional FK Ising model, Comm. Pure Appl. Math. 64, 1165-1198 (2011).

[15] P. Falco: Arrow-arrow correlations for the six-vertex model, Phys. Rev. E 88, 030103(R) (2013).

[16] J. Feldman, J. Magnen, V. Rivasseau, E. Trubowitz: An infinite volume expansion for many fermions Green functions, Helv. Phys. Acta 65, 679-721 (1992).

[17] J. Fröhlich, T. Spencer: The Kosterlitz-Thouless transition in two-dimensional Abelian spin systems and the Coulomb gas, Comm. Math. Phys. 81 (1981), 527-602.
[18] K. Gawedzki, A. Kupiainen: Gross-Neveu model through convergent perturbation expansions, Comm. Math. Phys. 102, 1-30 (1985).
[19] G. Giacomin, S. Olla, H. Spohn: Equilibrium uctuations for the $\nabla\phi$ interface model, Ann. Probab. 29, 1138-1172 (2001).
[20] A. Giuliani, V. Mastropietro: Anomalous critical exponents in the anisotropic Ashkin-Teller model, Phys. Rev. Lett. 93, 190603 (2004); and Anomalous universality in the anisotropic Ashkin-Teller model, Comm. Math. Phys. 256, 681-735 (2005).
[21] A. Giuliani, V. Mastropietro, F. Toninelli: Height fluctuations in interacting dimers, Ann. Inst. Henri Poincaré (Prob. Stat) 53 (2017), 98-168.
[22] A. Giuliani, V. Mastropietro, F. Toninelli: Haldane relation for interacting dimers, J. Stat. Mech. (2017) 034002.
[23] A. Giuliani, V. Mastropietro, F. Toninelli: Non-integrable dimers: Universal fluctuations of tilted height profiles, arXiv:1904.07526
[24] F. D. M. Haldane: General Relation of Correlation Exponents and Spectral Properties of One-Dimensional Fermi Systems: Application to the Anisotropic $S = 1/2$ Heisenberg Chain, Phys. Rev. Lett. 45, 1358-1362 (1980).
[25] L. P. Kadanoff: Connections between the Critical Behavior of the Planar Model and that of the Eight-Vertex Model, Phys. Rev. Lett. 39, 903-905 (1977).
[26] D. Poland, S. Rychkov, A. Vichi: The conformal bootstrap: Theory, numerical techniques, and applications, Rev. Mod. Phys. 91, 015002 (2019).
[27] S. Smirnov: Critical percolation in the plane: conformal invariance, Cardy’s formula, scaling limits, Comptes Rendus de l’Académie des Sciences-Series I-Mathematics, 333 (2001), 239-244.
