The gravitational light shift and the Sachs-Wolfe effect

Cesar Merlín∗ and Marcelo Salgado†

Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México, A.P. 70-543, México D.F. 04510, México.

Abstract

Using a 3+1 decomposition of spacetime, we derive a new formula to compute the gravitational light shifts as measured by two observers which are normal to the spacelike hypersurfaces defining the foliation. This formula is quite general and is also independent of the existence of Killing fields. Known examples are considered to illustrate the usefulness of the formula. In particular, we focus on the Sachs-Wolfe effect that arises in a perturbed Friedman-Robertson-Walker cosmology.

I. INTRODUCTION

In a flat spacetime the frequency shifts of light (FSL) or Doppler effect, arises due to the relative motion between the source and the detector (see Ref. [1] for rotational FSL as opposed to translational FSL), while in a curved spacetime, the FSL can be due to several factors. The more striking one is perhaps the FSL associated with the time dilation due to the presence of a (strong) gravitational field in the neighborhood of a clock. Clearly if a source and a detector are moving relative to each other in a curved spacetime, kinematical and gravitational FLS can be combined into a non trivial fashion. This is precisely what happens in the detection of light coming from many astrophysical sources. If the source is far away from us, then the FLS can have cosmological contributions in addition to the kinematical, thermal and local gravity contributions. The flat rotation curves of (spiral) galaxies [2] is a notable example of the kind of results that can be obtained by measuring the FSL. Spectroscopy is usually the basic tool to measure the FSL.

One of the most important effects that allows one to test the predictions of general relativity (or any other metric theory of gravity) are the gravitational FSL (redshift or blue shift) [3]. In particular, these shifts can be easily computed when the spacetime possesses several symmetries. The fact that the projection of a Killing vector field along a geodesic tangent vector is constant along that geodesic, simplifies considerably the calculation of such shifts. When a priori Killing fields are absent or when the spacetime has only approximate symmetries (as in the case of a perturbed symmetric spacetime), this calculation might not be straightforward. The so called Sachs-Wolfe effect [4] that arises in a perturbed Friedman-Robertson-Walker (FRW) cosmology and that we discuss below (Sec. IV) represents a prominent example of such a case.

The observation of light shifts provide some information about the geometry or strength of the gravitational field hosting the light source and might also help for distinguishing between different gravity theories in the strong field regime [5]. In the case of cosmology, the Sachs-Wolfe effect can help to distinguish between alternative metric theories since for instance the potentials \( \psi \) and \( \phi \) used in the Newtonian gauge for scalar perturbations [6,7] might differ (not only in sign) at the time of decoupling, but also in magnitude depending on the gravitational theory one is dealing with [8]. For example, under the standard linear perturbation theory in pure general relativity, when one discards the anisotropic stresses of matter at the time of decoupling one has \( |\psi| \approx |\phi| \), and the Sachs-Wolfe effect then depends only on one of these potentials (cf. Eq. [4.4]). However, in alternative theories of gravity like scalar-tensor theories and modified \( f(R) \) theories one has \textit{a priori} \( |\psi| \neq |\phi| \) [6,8] since the effective energy-momentum tensor associated with those theories is not necessarily isotropic \( i.e. \) it might include non-diagonal terms at the time of decoupling. So the Sachs-Wolfe effect can be a useful tool to validate, bound or even rule out alternative theories.

The Sachs-Wolfe effect has two contributions: one which is produced at the last scattering surface (the primary or primordial Sachs-Wolfe effect) and another one which is due to the effective FSL produced when the photons encounter local gravitational potentials that evolve in time during their trip towards our detectors (the integrated Sachs-Wolfe or ISW). Both effects are reported in the most recent Cosmic-Background-Radiation (CBR) observations and appear at large angular scales in the sky (\( \gtrsim 10^\circ \)). In particular, the primordial Sachs-Wolfe effect (the quadrupole contributions to the CBR) has to be contrasted with the contributions due to the relative motion of our galaxy with respect to the CBR frame (the dipole contributions) or due to the plasma oscillations (FSL noticeable at small angular scales). The

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*Electronic address: cesar.merlin@nucleares.unam.mx
†Electronic address: marcelo@nucleares.unam.mx
primordial effect is almost scale invariant and so it is often referred to as the Sachs-Wolfe plateau in the CBR power spectrum.

In this report we obtain a novel formula for computing the FSL associated with a certain family of observers. In the context of the 3+1 decomposition of spacetime, such observers are hypersurface orthogonal and are usually called Eulerian. In this context, the interpretation of the origin of FSL will depend on the identification or not of the Eulerian observers with the observers associated with the underlying symmetries of a spacetime. For instance, in a flat spacetime the formula leads to zero FSL if the observers are inertial (see below). However, if one chooses non inertial observers for which the metric components are not canonical, then the formula can lead to non trivial FSL. In this sense, the formula presented below does not really distinguish between FSL coming from observers in a real gravitational field (i.e. observers whose world lines live in a curved spacetime) and FSL coming from accelerated observers in a flat spacetime. Here we see one clear illustration of the Einstein’s equivalence principle. Two specific examples of this situation are to be found in the Rindler spacetime (a portion of Minkowski spacetime as viewed by accelerated observers whose hypersurfaces of simultaneity have hyperbolic topology) and in the Milne universe (a portion of Minkowski spacetime as viewed by expanding observers whose hypersurfaces of simultaneity have hyperbolic topology). One should then be careful in the interpretation of FSL outcome and not to confuse results which are coordinate dependent with results that are observer dependent (see Refs. [3, 10] for a discussion on this issue and also for the interpretation of cosmological FSL versus Doppler effect and the Milne example as well).

The formula does not rely on the existence of Killing vector fields nor on a specific metric theory of gravitation. However, when the former are present, the formula naturally reproduce the well known results (see Sec III). On the other hand, when Killing fields are absent, we show the usefulness of the formula by using as model example the case of a linearly perturbed FRW spacetime which leads to the Sachs-Wolfe effect.

II. THE FORMULA

We shall assume that a spacetime \( (M, g_{ab}) \), is globally hyperbolic and thus that it admits a foliation by a family of spacelike Cauchy hypersurfaces \( \Sigma_t \), parametrized by a global time function, \( t \). The normal \( n^a \) to these hypersurfaces is a future pointing time-like vector field (i.e. \( n_a \gamma^a = -1 \)) which defines a family of observers called Eulerian (see Refs. [11, 12] for a thorough introduction to the 3+1 formalism).

On the spacetime manifold \( M \), one is given a local coordinate system \( x^\mu \) such that \( x^\mu = (t, x^i) \) is a local coordinate system adapted to the foliation of \( M \) where \( x^i \) is a local spatial coordinate system of the embedded manifold \( \Sigma_t \). Therefore, with respect to these coordinates \( n^a = (1/N)(\partial/\partial t)^a + (N^i/N)(\partial/\partial x^i)^a \) or simply \( n^\mu = (1/N, N^i/N) \) and \( n_a = -N \nabla_a t \) (i.e. \( n_\mu = (-N, 0, 0, 0) \)), where \( N \) is the lapse function and \( N^a \) is the shift vector. This latter is orthogonal to \( n^a \) and has only spatial contravariant components. In terms of the 3+1 decomposition the metric reads

\[
ds^2 = - (N^2 - N_i N^j) \, dt^2 - 2N_i dx^i dt + h_{ij} dx^i dx^j,
\]

where \( h_{ij} \) stands for the 3-metric components (see below).

Now, consider two Eulerian observers (one associated with the source of light and the other one associated with the detector) located on \( \Sigma_t \) so that their point location \( p_e \) and \( p_d \) have spacetime coordinates \( (t, x^i_e) \) and \( (t, x^i_d) \). A light signal is emitted from \( p_e \) which is detected at \( p_d \) at time \( t + \Delta t \). The photon’s null geodesic four vector \( k^\mu \) is such that \( \frac{dx^\mu}{d\lambda} = k^\mu \), where \( x^\mu(\lambda) = (t(\lambda), x^i(\lambda)) \) provides the path of the photon in terms of the local coordinate system and \( \lambda \) is an affine parameter. Now, the light frequency measured by any of the Eulerian observers at some point \( p \) is given by \( \omega = -k^an_a \vert_{p} \). So \( \omega(x^\mu(\lambda)) \) is a scalar field that changes smoothly along the photon’s path. We consider then that at point \( p_e \) the photon’s path has coordinates given by \( x^\mu(\lambda) \) while at the detection point \( p_d \) has coordinates \( x^\mu(\lambda + \Delta \lambda) \). In other words, we consider that along the photon’s path there is always an Eulerian observer measuring its frequency (at the intersection point between the photon null geodesic and the observer path). That is, a family of Eulerian observers can be represented locally by a congruence of lines or orbits that intersects the photon’s path at different points. The frequency shift between the emitted and detected light signal is then given by

\[
\omega_d - \omega_e = \omega(x^\mu(\lambda + \Delta \lambda)) - \omega(x^\mu(\lambda)).
\]

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\(^1\) Here we use the Wald’s convention [11] where Latin indices \( a, b, c \) refer to abstract indices, while Greek indices refer to components and run \( 0 \) – \( 3 \). We use Latin indices \((i, j = 1, 2, 3)\) to denote spatial components.
The “instantaneous” frequency shift along the photon’s path is then given by

\[
\lim_{\Delta \lambda \to 0} \frac{\omega(x^\mu(\lambda + \Delta \lambda)) - \omega(x^\mu(\lambda))}{\Delta \lambda} = \frac{d\omega}{d\lambda} = \frac{d\omega^\mu}{d\lambda} \nabla_\mu \omega .
\]

(2.3)

This local FSL is associated with the light frequency measured by Eulerian observers which are infinitesimally closed to each other. From the above definition we have

\[
\frac{d\omega}{d\lambda} = k^a \nabla_a \omega = -k^a \nabla_a (k^b n_b) = -k^a k^b \nabla_a n_b .
\]

(2.4)

where in the last step we used the fact that photons follow geodesics \(k^a \nabla_a k^b = 0\). On the other hand, the term \(\nabla_a n_b\) is related to the extrinsic curvature \(K_{ab}\) (or second fundamental form) of \(\Sigma_t\) by \(\nabla_a n_b = -K_{ab} - n_a n_b\) \([12]\), where \(a_a = n^c \nabla_c n_a\) is the four acceleration of the normal observers. We remind the reader that the extrinsic curvature is given by \(\nabla_a n_b\), where \(\nabla_n\) stands for the Lie derivative along \(n^a\), and \(h_{ab} = g_{ab} + n_a n_b\) is the induced metric (or 3-metric) on \(\Sigma_t\) \([11\,12]\). In fact it is not difficult to prove that the four acceleration \(a_a\) can be written in terms of the lapse function \(N\) as follows \(a_a = h^b_n \nabla_b \ln N = D_a \ln N\), where \(D_a\) is the covariant derivative compatible with the 3-metric \([11\,12]\). By using these results in Eq. (2.4) we obtain

\[
\frac{d\omega}{d\lambda} = k^a k^b K_{ab} - \omega k^a D_a \ln N .
\]

(2.5)

Furthermore, the projector \(h^a_n = \delta^a_n + n^a n_b\) can be used to define a 3-vector \(\hat{3}k^a := h^a_n k^b\) which is tangent to \(\Sigma_t\) and orthogonal to \(n^a\). Moreover we can define \(\hat{3}k^a = \frac{3k^a}{\omega}\) which is normalized (i.e. \(\hat{3}k^a \hat{3}k^b h_{ab} = 1\)) since \(k^a\) is null. The vector \(k^a\) can then be decomposed as follows \(k^a = \hat{3}k^a + \omega n^a = \omega \left(\hat{3}k^a + n^a\right)\). Inserting this expression for \(k^a\) in Eq. (2.5) and using the fact that \(n^a K_{ab} \equiv 0 \equiv n^a D_a \ln N\) we obtain

\[
\frac{d\omega}{d\lambda} = \hat{3}k^a \hat{3}k^b K_{ab} - \omega \hat{3}k^a D_a \ln N
\]

\(= \omega^2 \left(\hat{3}k^a \hat{3}k^b K_{ab} - \hat{3}k^a D_a \ln N\right)\).

(2.6)

An alternative way of writing Eqs. (2.6) and (2.7) is as follows

\[
\frac{1}{\omega} \frac{d\omega}{d\lambda} = \omega \left(\hat{3}k^a \hat{3}k^b K_{ab} + \nabla_n \ln N\right) - \frac{d \ln N}{d\lambda} .
\]

(2.7)

where we used \(D_a \ln N = \nabla_a \ln N + n_a n^b \nabla_b \ln N\) and then \(k^a D_a \ln N = d \ln N/d\lambda - \omega \nabla_n \ln N\).

In this way Eq. (2.7) provides the most general formula for the local gravitational light shift. In order to obtain a finite frequency shift (i.e. a frequency shift as measured by two Eulerian observers separated by a finite spatial distance) one needs to integrate the previous equation along the photon’s path. Now, a physical interpretation to the previous equation can be given, but before we proceed, it will be useful to write the following explicit expression for the spatial components of the extrinsic curvature in terms of the 3-metric and the shift vector \(N^a\) \([12]\)

\[
K_{ij} = -\frac{1}{2} N \left(\partial_i h_{ij} + D_i N_j + D_j N_i\right) .
\]

(2.8)

We are now in position to give some insight about the different terms entering in Eq. (2.7). In static situations where one usually identifies the function \(f\) defining the spacelike hypersurfaces \(\Sigma_t\) with the parameter associated with the hypersurface orthogonal Killing vector field \(\xi^a = \left(\partial/\partial t\right)^a\), it turns \(n^a = \xi^a / N\) and the shift vector is identically null. In this case the term within parenthesis in Eq. (2.7) also vanishes identically [cf. Eq. (2.8)]. Therefore when the spacetime is static the Eulerian observers are naturally identified with static observers and the FSL measured by them reduces to the relationship \(\omega/\omega_\infty = N(x^i_\lambda)/N(x^i_\infty)\), where \(x^i_\lambda, x^i_\infty\) refer to the spatial coordinates of the points where the light was emitted and detected respectively. As a result of this, the lapse function is often referred to as the redshift factor. Note that in general \(N \neq 1\), which implies that the Eulerian observers are not necessarily geodesic (i.e. the four acceleration \(a_a = D_a \ln N\) is not null in general). In the next section we use the FRW cosmology as an example where the natural Eulerian observers are geodesic, in which case, \(a_a\) vanishes identically. On the other hand,

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2 It is to note that in Wald’s book \([13]\), the extrinsic curvature is defined with the opposite sign.
the term within parenthesis in Eq. (2.7) provides additional frequency shifts when the spacetime changes in time (cf. Eq. 2.8) or when it is stationary. Indeed, in a stationary situation the timelike Killing field is not hypersurface orthogonal but the metric is time independent. Therefore, in this situation and when one identifies the Eulerian observers’ coordinates with those associated with the symmetries of a stationary spacetime, it turns that the formula Eq. (2.7) accounts for two contributions to the FSL: one is due to the fact that the Eulerian observers are not geodesic and therefore are accelerating (this contribution is again due to the last term at the r.h.s of Eq. 2.7); the second contribution arises from the terms which involve the shift vector \( N^a \) [cf. Eq. (2.8)]. This additional contribution to the FSL is due to the so called dragging of inertial frames. In the next section we give a specific example that illustrates these interpretations. Another simple but non trivial situation arises in FRW cosmology where \( N = 1 \) and \( N^a \equiv 0 \). In this case the FSL is only due to time variations of the gravitational field (i.e., due to the expansion of the Universe; see next section). Finally, we shall consider the Sachs-Wolfe effect (see Sec. IV) where the time and spatial variations of the gravitational field combine to give a FSL.

One last comment is in order. This is in fact related to a potential confusion between coordinate and observer dependent effects (see Ref. [1] for a further discussion on this issue). One could in fact be surprised in giving a physical interpretation to the terms appearing in Eq. (2.7), since after all, one thing that one learns in the analysis of the 3+1 decomposition of spacetime is that both \( N \) and \( N_i \) define the coordinate gauge and therefore that their meaning is not physical. However, precisely different choices of \( N \) and \( N_i \) define the kind of observers we use to “coordinateitize” the spacetime and, in the present context, the kind of observers we use to compute FLS as well. As mentioned before, the Rindler spacetime [13] is a simple example of this situation. In Rindler coordinates the lapse function is given by \( N = (1 + gx) \) (where \( g \) is the proper acceleration of the observer whose worldline is associated with the coordinate \( x = 0 \)), while the rest of the metric components are trivial. Therefore one finds a non zero FSL due to the term with \( d\ln N / d\lambda \) in Eq. (2.7). This FSL is due to the fact that two Rindler observers accelerate in a different way depending on their relative positions, and not due to the curvature of spacetime\(^3\). Actually the Rindler spacetime represents only a portion of Minkowski spacetime. Note however that in ordinary Minkowski Cartesian coordinates, \( N = 1 \), \( N_i \equiv 0 \) and \( h_{ij} = \delta_{ij} \). Then the FSL is exactly zero. This means that the Eulerian observers in question are inertial. That is, they are in relative rest and so there are no FSL whatsoever. Moreover, if one consider a boosted family of observers, these also are inertial and the metric components for them takes exactly the same form as the metric for the other family of observers. Then for this second family of boosted observers there are no FSL either. So the formula given by Eq. (2.7) is by construction unable to account for the FSL (i.e., Doppler shifts) due to a relative motion of (local) inertial observers.

### III. EXAMPLES

#### Static and spherically symmetric spacetime

Consider the line element \( ds^2 = -N^2(r)dt^2 + A^2(r)dr^2 + r^2d\Omega^2 \). In this case the normal observers to \( \Sigma \) are static, and so \( n^a = (1/N)\xi^a \); where \( \xi^a = (\partial/\partial t)^a \) is the timelike hypersurface orthogonal Killing field. Moreover, for this metric \( K_{ij} \equiv 0 \equiv \mathcal{L}_n \ln N \). From Eq. (2.7) one easily finds:

\[
\frac{\omega_c}{\omega_d} = \frac{N(r_d)}{N(r_c)}.
\]

In the case of the Schwarzschild solution we recover the usual expression\(^4\)

\[
\frac{\omega_c}{\omega_d} = \sqrt{\frac{1 - 2M}{1 - 2M/r_c}}.
\]

#### FRW spacetime

The line element is given by

\[
ds^2 = -dt^2 + a^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2d\Omega^2 \right].
\]
In this case the normal observers to $\Sigma_t$ are comoving, and so $n^a = (\partial/\partial t)^a$ (i.e. $N \equiv 1$). The 3-metric $h_{ij}$ can be read off from Eq. (2.3), and from Eq. (2.8) we find $K_{ij} = -\frac{1}{2}h_{ij}$ and $L_{ab}\ln N \equiv 0 \equiv d\ln N/d\lambda$. Replacing these expressions in Eq. (2.7) and using $d/d\lambda = \omega_d/dt$ one easily finds the familiar expression

$$\frac{\omega_e}{\omega_d} = a(t_d)/a(t_e).$$

(3.4)

Depending on the cosmological model (e.g. matter content) and the theory at hand, one has explicit solutions for $a(t)$.

**Stationary and axisymmetric spacetime:** This case is perhaps a little more interesting than the previous ones, because the metric induces dragging effects on the light shifts. Let us then consider the following line element:

$$ds^2 = -(N^2 - N_\varphi N^\varphi) dt^2 - 2N_\varphi dt d\varphi + h_{ij} dx^i dx^j.$$  

(3.5)

where all the metric components are time and $\varphi$ independent but depend on the two coordinates $(x^1, x^2)$ which can be chosen to be of spherical or cylindrical type. Apart from the fact that $h_{\varphi i} \equiv 0$ (for $i = 1, 2$), the explicit form of the 3-metric $h_{ij}$ does not concern us since it will not be necessary in the calculation of the FSL. In this example, the Eulerian observers have four velocity given by $n^a = (1/N)(\partial/\partial t)^a + (N_\varphi/N)(\partial/\partial \varphi)^a$, where in fact, $\xi^a = (\partial/\partial t)^a$ and $\psi^a = (\partial/\partial \varphi)^a$ are the timelike and the spacelike Killing fields, respectively, which are associated with the time and axial symmetries.

A straightforward calculation leads to

$$3k^a k^b k_{ab} = -\frac{3k_\varphi}{N} k^\mu \partial_\mu N^\varphi = -\frac{3k_\varphi}{N} \frac{dN^\varphi}{d\lambda}.$$  

(3.6)

where $3k_a := h_{ab} k^b$ and in the first equality we used $3k^i = k^i$ (for $i \neq \varphi$) since $N^\varphi$ is the only non-null component of the shift vector. In fact since $k^\mu \partial_\mu N^\varphi \equiv 0$ (for $\mu = t, \varphi$) by the stationary and axisymmetry conditions, those terms do not contribute to $k^\mu \partial_\mu N^\varphi$, but we have retained them in order to explicitly obtain the last equality. Moreover, since $3k_a = k_a - \omega_n a = g_{ab} k^b - \omega n_a$, and using $n_i \equiv 0$ (for $i = 1, 2, 3$) then $3k_\varphi = g_{\varphi b} k^b \equiv k_\varphi$. The quantity $L := g_{ab} \psi^a k^b = g_{\varphi b} k^b$ which is conserved along the photon’s path is identified with the photon’s angular momentum. In this way Eq. (2.7) leads to the following differential equation

$$\frac{d\omega}{\omega d\lambda} = -\frac{L}{\omega N} \frac{dN^\varphi}{d\lambda} - \frac{d\ln N}{d\lambda}.$$  

(3.7)

This equation can be easily integrated and when evaluated at the emission and detection points we find

$$\frac{\omega_e}{\omega_d} = \frac{N_d}{N_e} \left[1 - \frac{L}{E} N_\varphi^e\right]/\left[1 - \frac{L}{E} N_\varphi^d\right],$$  

(3.8)

where $E = -k^a \xi_a$ is an integration constant which is identified with the photon’s energy and which is also conserved along the photon’s path. Here the subscripts $e$ and $d$ at the r.h.s mean that the quantities have to be computed at the points of emission and detection with coordinates $(x_1^e, x_2^e)$ and $(x_1^d, x_2^d)$ for any $t, \varphi$. Of course this calculation can be done in a few steps using the Killing vector fields from the start.

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5 It is interesting to mention that for $a(t) = t$ and $k = -1$, the metric (2.3) corresponds actually to a flat spacetime (without matter) [10, 12]. The Eulerian observers are expanding in a non trivial fashion and the spacelike hypersurfaces $\Sigma_t$ have hyperbolic topology. This spacetime is referred to as the *Misner Universe* [10] and the comoving coordinates cover only a portion of Minkowski spacetime. In fact the hypersurfaces $\Sigma_t$ are not really Cauchy surfaces of the whole Minkowski spacetime but only of that portion covered by the comoving coordinates. Therefore it is only that portion that can be foliated by the hypersurfaces $\Sigma_t$.

6 If the spacetime is asymptotically flat $E$ is the photon’s energy as measured by an Eulerian observer at spatial infinity.

7 We have $\omega = -k_\varphi n^a$ where $n^a = \xi^a/N + N^\varphi \psi^a/N$ then $\omega = (E/N) \left[1 - \frac{L}{E} N^\varphi\right]$ where $E = -k_\varphi \xi_a$ and $L = k_\alpha \psi^a$ are constants along the photon’s path. Evaluating $\omega$ at the points of emission and detection we recover Eq. (3.8). It is interesting to mention that for this kind of spacetimes the Eulerian observers are also called ZAMOs (Zero Angular Momentum Observers) [12] since $L_\varphi := n^a \psi_a \equiv 0$, as one can check.
IV. SACHS-WOLFE EFFECT

Let us considered scalar linear perturbations of the FRW metric in the Newtonian gauge \[ds^2 = -(1 + 2\phi)dt^2 + (1 + 2\psi)\hat{h}_{ij} dx^i dx^j ,\]
where \(\hat{h}_{ij}\) corresponds to the unperturbed FRW 3-metric of Eq. (2.3) and \(|\phi|, |\psi| \ll 1\). Up to first order, the lapse function is \(N = 1 + \phi\), and the 3-metric \(h_{ij} = (1 + 2\psi)\hat{h}_{ij}\). So from Eq. (2.8) the perturbed extrinsic curvature reads \(K_{ij} = -\frac{\dot{a}}{a} \hat{h}_{ij} + \left(-\partial_i \psi + \frac{\dot{a}}{a} \frac{\partial}{\partial t}\right) \hat{h}_{ij}\), where the first term provides the zero-order contribution as we saw in the second example. Moreover, we can write \(3\hat{k}^a = \hat{k}^a_0 + 3\hat{k}^a_1\) where the lower indices 0, 1 refer to zero and first order respectively, so that the normalization condition is verified at zero order and perturbatively as well: \(3\hat{k}^a 3\hat{k}^b h_{ab} = 1 = 3\hat{k}^0_0 3\hat{k}^1_0 \hat{h}_{ab}\). In this way, up to first order, \(3\hat{k}^a 3\hat{k}^b \hat{h}_{ab} = 1 - 2\psi\). These preliminary results allow us to find \(3\hat{k}^a 3\hat{k}^b K_{ij} = -\frac{\dot{a}}{a} + \frac{\dot{\alpha}}{\alpha} - \partial_i \psi\). On the other hand, \(d\ln N/d\lambda = d\phi/d\lambda\) and \(\mathcal{L}_n \ln N = n^i \nabla_i \ln N = \partial_i \phi\). Collecting all these partial results into Eq. (2.7) we obtain:

\[
\frac{1}{\omega} \frac{d\omega}{d\lambda} = \omega \left(-\frac{\dot{\alpha}}{\alpha} - \frac{\partial \psi}{\partial t} + \frac{\dot{\alpha}}{\alpha} \frac{\partial \phi}{\partial t}\right) - \frac{d\phi}{d\lambda}. 
\]

Now, since \(\omega = -n_a k^a = Nk^t\), then up to first order \(k^t/\omega = 1/N = 1 - \phi\), therefore, \(\frac{d\omega}{d\lambda} = \frac{d\omega}{dt} k^t = \frac{d\omega}{dt} \omega (1 - \phi)\), and similarly \(\frac{d\phi}{d\lambda} = \frac{d\phi}{dt} \omega (1 - \phi)\).

In this way we obtain the following expression valid up to first order,

\[
\frac{d\omega}{dt} = -\frac{\dot{\alpha}}{\alpha} - \frac{\partial \psi}{\partial t} - \frac{\partial \phi}{\partial t} - \frac{d\phi}{dt}. 
\]

Note the cancellation of the term \(\frac{\dot{\alpha}}{\alpha}\). The first term at the r.h.s of Eq. (4.3) is associated with the unperturbed FSF. Finally, we can write the perturbed frequency as \(\omega = \omega_0 + \omega_1\), and so \(\frac{d\omega}{d\lambda} = \frac{d\omega}{dt} + \frac{d\omega}{d\lambda} (1 + \omega_1/\omega_0)\). Now, for the unperturbed frequency we have \(\frac{d\omega_0}{dt} = -\frac{d\phi}{dt}\). On the other hand, up to first order \(\frac{d\omega_0}{d\lambda} = \frac{d\omega}{dt} (\frac{d\phi}{d\lambda})\).

Eq. (4.3) can now be integrated with respect to \(t\) to obtain the usual Sachs-Wolfe effect expression:

\[
\frac{\delta T}{T_0} \Bigr|_{t_e}^{t_d} = \phi(\vec{x}_e, t_e) - \phi(\vec{x}_d, t_d) + \int_{t_e}^{t_d} \frac{\partial D(\vec{x}(t), t)}{\partial t} dt 
\]

where we defined \(\frac{\delta T}{T_0} = \frac{\delta T}{\omega_0}\), as the relative temperature perturbations \((T_0 \sim \omega_0\) is the unperturbed temperature of the FRW Universe, and \(\delta T \sim \omega_1\) is the temperature perturbation) and \(D(\vec{x}, t) := \phi(\vec{x}, t) - \psi(\vec{x}, t)\). The last integral has to be evaluated along the photon’s path \(^8\).

\(^8\) One can take into account not only scalar perturbations but also first-order vector and tensor perturbations as well. In order to do so a convenient and simple gauge which generalizes the Newtonian gauge is the Poisson gauge [18, 19]. In such a gauge the perturbed metric reads:

\[
ds^2 = -(1 + 2\phi)dt^2 - 2N_i^T dx^i dx^j + \left[(1 + 2\psi)\hat{h}_{ij} + 2H_i^T T_j^T\right] dx^i dx^j ,
\]

where \(N_i^T\) means that the shift perturbation is transverse \((\vec{D} N_i^T = 0)\), where \(\vec{D}\) stands for the 3-covariant derivative compatible with to the non-perturbed metric \(\hat{h}_{ij}\) and \(H_i^T T_j^T\) is transverse and traceless \((\vec{D} H_i^T T_j^T = 0 = h_i^T \hat{h}_{ij} T_j^T)\). Notice that in this case we have six physical perturbations (two scalars \(\phi\) and \(\psi\), two associated with the transverse vector \(N_i^T\), and two which provide the two polarization modes of the gravitational waves associated with the transverse traceless tensor \(H_i^T T_j^T\)). The contribution of the vector and tensor perturbations to the Sachs-Wolfe effect is straightforward. In this case and up to first order, \(K_{ij} = K_{ij}^S - \partial_i \hat{h}_{ij} T_j^T - \tilde{D}_i N_j^T\), where \(K_{ij}^S\) is the extrinsic curvature up to first order which includes only the scalar perturbations as in the main text. So using Eq. (2.8) and the results of the main text one obtains

\[
\frac{\delta T}{T_0} \Bigr|_{t_e}^{t_d} = \left(\frac{\delta T}{T_0}\right) \Bigr|_{t_e}^{t_d} - \int_{t_e}^{t_d} \frac{d\phi}{dt} \left[\frac{\partial H_i^T T_j^T}{\partial t}(\vec{x}(t), t) + \partial_i N_j^T(\vec{x}(t), t)\right] dt ,
\]

where the first term at the r.h.s is given by the r.h.s of Eq. (4.4) (see Ref. [20, 21] for an alternative derivation which includes scalar, vector and tensor perturbations using a gauge-invariant and a general-gauge formalisms, respectively).
The primordial temperature fluctuations associated with the potential $\phi$ give rise to the ordinary Sachs-Wolfe effect, which corresponds to the redshift of light due to the “climbing” of photons through the potential $\phi(\vec{x}, t)$ at the last scattering surface. The term $\phi(\vec{x}_d, t_d)$ does not really contribute to the anisotropies since it is associated with the local gravitational field around the detector which contributes isotropically to the temperature perturbations. The last term which involves the integral is associated with the temperature perturbations due to the time variations of the potentials along the photon’s path. It is called the integrated Sachs-Wolfe (ISW) effect. Notice that in Eq. (4.4) temperature fluctuations due to peculiar velocities (Doppler shifts) are absent due to the limitations of the formula (2.7), as we have stressed before.

In general relativity and in absence of anisotropic stresses, the Einstein equations imply $\psi = -\phi$, and then $D(\vec{x}, t) = 2\phi(\vec{x}, t)$ which leads to the usual expression for the Sachs-Wolfe effect. In fact since the l.h.s of Eq. (4.4) is $(\delta T/T_0)_{t_d} - (\delta T/T_0)_{t_e}$ one can show $(\delta T/T_0)_{t_e} \approx -2\phi/3$ (for adiabatic perturbations in a matter dominated epoch, and $k = 0$ universe) [6, 21–23] and so $(\delta T/T_0)_{t_e} = \phi(\vec{x}_e, t_e)/3 + \text{ISW}$. The actual primordial temperature anisotropies measured today between photons coming from two different points (angles) at the last scattering surface is

$$\Delta T(\vec{x}, \vec{x}') := \frac{1}{3} \left[ \phi(\vec{x}_e, t_e) - \phi(\vec{x}'_e, t_e) \right] + \text{ISW}_1.$$

As mentioned in the introduction, in alternative metric theories of gravity the effective energy-momentum tensor associated with these theories is not a priori isotropic (i.e., it is spatially non-diagonal) at the last scattering surface, and therefore $|\psi| \neq |\phi|$. So, this can have observational consequences in the CBR angular power spectrum of temperature anisotropies [8, 24].

V. DISCUSSION

Based on a 3+1 decomposition of spacetime, we have presented a novel formula to compute the frequency shifts of light between two observers which are in general non geodesic. The formula does not account for the Doppler (kinematical) effects which arise when one of the observers is geodesic and the other is not or when the two observers are connected by Lorentz transformations (notably in flat spacetime). That is, when one considers two observers (kinematical) effects which arises when one of the observers is geodesic and the other is not or when the two observers of light between two observers which are in general non geodesic. The formula does not account for the Doppler effects due to the relative motion of observers (e.g., one of whom is geodesic and the other is not). Using Killing vector fields FLS formulae are obtained in Ref. [26]. Ellis [27] obtained a FSL formula using a 1+3 congruence formalism (as opposed to the 3+1 employed here) (see also Ref. [28] for a further review). In all such treatments, the Sachs-Wolfe effect is not computed whether because the authors focus only on FRW cosmology or because their formalism applies only to stationary situations. On the other hand, Dunsby [29] based on the 1+3 covariant approach [30] does analyze the Sachs-Wolfe effect for scalar perturbations (see also Refs. [31, 32]).

In our case, we showed that our formula leads in a more straightforward and geometrical fashion to the Sachs-Wolfe effect. Thanks to its generality and simplicity we consider that the formula can be useful in several situations of physical interest.

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