Higher-loop anomalies in chiral gravities

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Abstract

The one-loop anomalies for chiral $W_3$ gravity are derived using the Fujikawa regularisation method. The expected two-loop anomalies are then obtained by imposing the Wess-Zumino consistency conditions on the one-loop results. The anomalies found in this way agree with those already known from explicit Feynman diagram calculations. We then directly verify that the order $\hbar^2$ non-local BRST Ward identity anomalies, arising from the “dressing” of the one-loop results, satisfy Lam’s theorem. It is also shown that in a rigorous calculation of $Q^2$ anomaly for the BRST charge, one recovers both the non-local as well as the local anomalies. We further verify that, in chiral gravities, the non-local anomalies in the BRST Ward identity can be obtained by the application of the anomalous operator $Q^2$, calculated using operator products, to an appropriately defined gauge fermion. Finally, we give arguments to show why this relation should hold generally in reparametrisation-invariant theories.
1 Introduction

The rôles that anomalies take in different theories can vary substantially according to the physical content of a given theory. The existence of anomalies in four-dimensional theories can render them non-renormalizable and non-unitary, while in two dimensions, because of the softer ultraviolet behaviour, anomalies may prove not to be disastrous. Furthermore, the renormalizability of two-dimensional theories, the relative simplicity of the forms of propagators and consequently of operator products and also the extent to which the BRST formalism can be used, have helped to reveal new features of anomalies.

There exist two basic approaches to the treatment of anomalies. One standard approach is to anticipate the occurrence of the anomalies already at the classical level by introducing classically-decoupling compensating fields. A well-known illustration of this would be when anomalies give rise to propagating Liouville modes. An alternative approach, to be adopted here, is to extract the anomaly-induced dynamics directly from the Ward identities of the worldsheet symmetries.

It is known that the two-dimensional $W$ gravity theories have anomalies beyond the one-loop order. Anomalies occur at the two-loop order for $W_3$ and at all higher-loop orders for $W_\infty$ and $w_\infty$ gravities. The presence of such higher-loop anomalies make these theories particularly interesting to study. In this paper we investigate the anomalies in the fully quantised chiral $W_3$ theory, considering specifically those appearing up to order $\hbar^2$. This formalism is outlined in the following section.

It turns out that discussing anomalies beyond the one-loop order is not simply a matter of extending existing methods. Some techniques employed to evaluate one-loop anomalies are restricted in applicability to that order alone. For instance, in Section 3, the one-loop anomalies shall be calculated using the path-integral method initiated by Fujikawa [1][2]. This is a purely one-loop approach and it is still unknown how the two-loop anomalies can arise in such a path-integral formalism. In Section 4, we shall show that this deficiency of the Fujikawa method can be overcome for our purposes by using the Wess-Zumino consistency condition to derive the two-loop anomalies.

A noteworthy distinction between the one- and two-loop anomalies is that, while those at the one-loop order are necessarily local, those at higher-loop orders are not. The non-local order $\hbar^2$ anomalies arising in our theory can be used to make a non-trivial verification of Lam’s theorem [3]. This theorem states that all the non-local contributions to the Ward identity anomaly can be obtained via the “dressing” of lower order local anomalies. The explicit verification of this is the subject of Section 5.

Anomalies can also be studied in the context of the BRST charge $Q$. The non-nilpotence of the BRST charge at the full quantum level is another expression of an anomaly. Although it is known how local anomalies are obtained in the BRST anomalous algebra [4], little attention has been paid to the non-local anomalies. In Section 6, both local and non-local anomalies shall be derived from the expectation value of the square of the BRST charge.

It has recently been noted that the BRST algebra anomalies in chiral gravities
can be related to those of the BRST Ward identity via a construction involving an appropriately defined gauge fermion \[4\]. Specifically, the Ward-identity anomalies are obtained by the application of the anomalous operator \(Q^2\) to this gauge fermion. This relation has as yet been discussed only for local anomaly contributions. In Section 7, we shall show that the anomaly relation also holds for the non-local terms. In Section 8, we shall show that this relationship is valid for a more general gauge-fixing condition. Lastly, Section 9 considers the general origin of this anomaly relation.

2 BRST Quantisation of \(W_3\) gravity

The classical action for chiral \(W_3\) gravity is \[5\]

\[
I_{\text{class}} = \frac{1}{\pi} \int d^2 z \left( -\frac{1}{2} \bar{\partial} \phi^i \partial \phi^i + \frac{1}{2} h \partial \phi^i \partial \phi^i + \frac{1}{3} B d_{ijk} \partial \phi^i \partial \phi^j \partial \phi^k \right),
\]

where \(\phi^i (i = 1 \cdots n)\) are scalar fields, \(h\) and \(B\) are the spin-2 and spin-3 gauge fields respectively and \(d_{ijk}\) is a constant symmetric traceless tensor. Also, \(\bar{\partial} = \partial / \partial \bar{z}\) and Einstein summation is assumed throughout.

This action is invariant under the following nonlinear gauge symmetries:

\[
\delta \phi^i = \epsilon \partial \phi^i + \nu d_{ijk} \partial \phi^j \partial \phi^k,
\]

\[
\delta h = \bar{\partial} \epsilon + \epsilon \partial h - \partial \epsilon h - \frac{1}{2} \left( \nu \partial B - \partial \nu B \right) a \partial \phi^i \partial \phi^i,
\]

\[
\delta B = \bar{\partial} \nu + \epsilon \partial B - 2 \partial \epsilon B + 2 \nu \partial h - \partial \nu h,
\]

where \(\epsilon\) is the spin-2, \(\nu\) is the spin-3 infinitesimal parameter and \(a\) is a constant.

Imposing derivative gauge-fixing conditions, \(\bar{\partial} h = 0\) and \(\bar{\partial} B = 0\), we necessarily extend the phase space to include spin-2 Faddeev-Popov ghost and antighost fields \(c, b\), and, likewise, in the spin-3 sector, \(\gamma, \beta\). The resulting BRST-transformations are not nilpotent off-shell (i.e. without imposing the equations of motion) \[6\] \[7\]. Learning a lesson from supergravity \[8\], we see that this difficulty can be overcome by the introduction of auxiliary fields. For \(W_3\) gravity we find that the momenta conjugate to the ghosts and anti-ghosts naturally play the role of such auxiliary fields. Hence, we introduce these momenta \(\pi\) and reduce the action to first-order form.\[9\] Finally, requiring renormalisation terms to achieve the cancellation of matter-dependent anomalies, we arrive at the extended quantum action of chiral \(W_3\) gravity in the derivative gauge and in the first order form \[10\],

\[
S = \frac{1}{\pi} \int \left[ -\frac{1}{2} \bar{\partial} \phi^i \partial \phi^i + \pi_h \bar{\partial} h + \pi_B \bar{\partial} B - \pi_b \bar{\partial} b - \pi_c \bar{\partial} c - \pi_\beta \bar{\partial} \beta - \pi_\gamma \bar{\partial} \gamma 
\]

\[
- \delta \left( h \pi_c + B \pi_\gamma \right) + K_{\phi^i} \delta \phi^i \right],
\]

\[1\]This nilpotency can equally well be achieved in the conformal gauge, by eliminating the gauge fields from the gauge-fixed action as seen in the Batalin-Vilkovisky formalism \[11\].
where $\delta$ indicates a BRST variation, $\varphi^i$ denotes all fields and momenta and $K_{\varphi^i}$ are the corresponding sources for their variations. The action is expressed in this form for brevity as well as to make the classical BRST invariance manifest. A peculiarity of this “canonical” action, which derives from its origin in a fully reparametrisation-invariant theory, is that the “Hamiltonian” $\delta(h\pi_c + B\pi_\gamma)$ is BRST exact. We shall see that this has important consequences later.

The full BRST symmetries of the action are given by:

$$\delta \phi^i = c \partial \phi^i + d_{ijk} \gamma \partial \phi^j \partial \phi^k + a \pi_c \gamma \partial \phi^i + \sqrt{h} \left( -a \alpha_i \partial (\pi_c \gamma \partial \gamma) ight)$$

$$\delta h = \pi_b,$$

$$\delta B = \pi_\beta,$$

$$\delta c = c \partial c - \frac{a}{2} \gamma \partial \gamma \partial \phi^i \partial \phi^i - a\sqrt{h} \alpha_i \gamma \partial \gamma \partial^2 \phi^i + \frac{1 - 17a}{30} h(2 \gamma \partial^3 \gamma - 3 \partial \gamma \partial^2 \gamma),$$

$$\delta \gamma = c \partial \gamma - 2 \partial c \gamma,$$

$$\delta \beta = \pi_B, \quad \delta \pi_c = T_{\text{mat}} + T_{\text{gh}}, \quad \delta \pi_\gamma = W_{\text{mat}} + W_{\text{gh}},$$

$$\delta \pi_h = 0, \quad \delta \pi_B = 0, \quad \delta \pi_b = 0, \quad \delta \pi_\beta = 0,$$

$$\delta K_{\varphi^i} = 0,$$

(4)

where the spin-2 and spin-3 matter currents are

$$T_{\text{mat}} = -\frac{1}{2} \partial \phi^i \partial \phi^i - \sqrt{h} \alpha_i \partial^2 \phi^i,$$

$$W_{\text{mat}} = -\frac{1}{3} d_{ijk} \partial \phi^i \partial \phi^j \partial \phi^k - \sqrt{h} e_{ij} \partial \phi^j \partial^2 \phi^i - \hbar f_i \partial^3 \phi^i,$$

(5)

and the corresponding ghost currents are given by:

$$T_{\text{gh}} = -2 \pi_c \partial c - 3 \pi_\gamma \partial \gamma - 2 \partial \pi_\gamma \gamma,$$

$$W_{\text{gh}} = -\partial \pi_c \partial c - 3 \pi_\gamma \partial \gamma - a(\partial (\pi_c \gamma T_{\text{mat}}) + \pi_c \partial \gamma T_{\text{mat}})$$

$$+ \frac{(1 - 17a)}{30} h(2 \gamma \partial^3 \pi_c + 9 \partial \gamma \partial^2 \pi_c + 15 \partial^2 \gamma \partial \pi_c + 10 \partial^3 \gamma \pi_c).$$

(6)

To insure the cancellation of all the matter-dependent anomalies that arise in this theory, the symmetric structure constant $d_{ijk}$, the background-charge $\alpha_i$ and the renormalisation constants $e_{ij}$ and $f_i$ must satisfy the following conditions [7]:

$$d_{ijj} - 6e_{ij} \alpha_j + 6f_i = 0,$$

$$e_{(ij)} - d_{ijk} \alpha_k = 0,$$

$$3f_i - \alpha_j e_{ji} = 0,$$

$$d_{kli} d_{jkl} + 6d_{ijk} f_k - 3e_{ik} e_{jk} = \frac{1}{2} \delta_{ij},$$

$$d_{(ij} m d_{kl) m} = \frac{1}{2} a \delta_{(ij} \delta_{kl)},$$

$$d_{ijk}(e_{ik} - e_{kl}) + 2e_{(i} l d_{j)kl} = a \alpha_k \delta_{ij},$$

(7)
where the constant \( a \) is equal to \( 16/(22 + 5C_{\text{mat}}) \). (Note that at the quantum level, \( d_{ijk} \) is no longer traceless.) The central charge \( C_{\text{mat}} \) is given in terms of the number of scalar fields and the background charge by the following expression:

\[
C_{\text{mat}} = n + 12\alpha_i\alpha_i, \tag{8}
\]

or, equivalently, it can be shown that:

\[
C_{\text{mat}} = 2d_{ijk}d_{ijk} - 18e_{ij}e_{ij} - 12e_{ij}e_{ji} - 360f_i^2. \tag{9}
\]

Note that, although usually one would expect in field theory that the BRST variations of the original fields are just gauge transformations with the gauge parameters replaced by the corresponding ghosts, this simple replacement fails for \( W_3 \) gravity. The root of this feature can be seen even at the level of the gauge transformation of the spin-2 gauge field \( (2) \), which transforms non-linearly into the scalar fields. The original gauge-invariant part of the action is not invariant under the BRST symmetries. To compensate for this non-invariance, one needs to modify the transformation of the scalar fields \( \phi \) by the additional term \( a\pi_{\gamma\gamma}\partial\gamma\partial\phi \).

3 The derivative-gauge anomalies of \( W_3 \) gravity

The presence of anomalies is expressed in the Ward identity, which in turn results from requiring the invariance of the partition function under the symmetries of the action. The partition function corresponding to the action \( S \) \( (3) \) is given by a path integral over all fields and their conjugate momenta, weighted by the exponential of the extended action, namely:

\[
Z = \int \mathcal{D}\varphi^i \exp S_{\text{ext}}, \tag{10}
\]

where the extended action is obtained by introducing sources \( J_{\varphi^i} \) for all the fields, i.e.

\[
S_{\text{ext}} = S + \int J_{\varphi^i}\varphi^i. \]

The non-invariance of the measure of this path integral, together with the variation of the renormalized extended action leads to an expression of the anomalous Ward identity.

Under the symmetries \( (4) \) the partition function transforms as follows:

\[
Z \rightarrow Z' = \int \mathcal{D}\varphi^i \exp (S_{\text{ext}} + \int J_{\varphi^i}\delta\varphi^i + \Delta), \tag{11}
\]

where the anomaly \( \Delta \) includes a contribution \( \mathcal{A} \) from varying the measure and a contribution \( \delta S \) from varying the renormalised action. Expanding the exponential and using the following Legendre transform:

\[
\Gamma = \ln Z - \int J_{\varphi^i}\varphi^i, \tag{12}
\]

Interestingly, these conditions are in fact identical to those of Romans, designed to obtain quantum \( W_3 \) symmetry \cite{10}. This identification also holds in \( W_\infty \) \cite{11} although there is as yet no proof that it must hold generally.
we obtain the Ward identity in terms of the effective action $\Gamma$. Explicitly, one has the relation

$$\frac{\delta \Gamma}{\delta K^{\phi^i}} \frac{\delta \Gamma}{\delta \phi^i} = \triangle \cdot \Gamma,$$

(13)

where $\triangle \cdot \Gamma$ denotes all one-particle-irreducible graphs with precisely one insertion of the composite anomaly operator $\triangle$. The local chiral $W_3$ anomalies are already known from the explicit Feynman-diagram calculation of the left-hand side of expression (13) [4]. In the following, we shall derive them by an alternative path-integral method.

In evaluating $\triangle$, we first evaluate the terms $A$, stemming from the measure. Under the symmetries (4), the path-integral measure transforms through a Jacobian factor which, when properly regularized, leads to expressions that contribute to the anomalies. The part of the Jacobian $J_1$ which contributes to the anomalies at order $\bar{\hbar}$ is given by:

$$J_1 = \exp \text{Tr} \left[ \frac{\partial \delta \phi^i}{\partial \phi^j} \right]_0 = e^{A_1},$$

(14)

where the subscript “0” indicates terms of zeroth order in $\bar{\hbar}$ and the trace includes an integration over space-time.

The above Jacobian is singular, as it consists of products of delta functions. These delta-functions arise both through taking derivatives as well as through taking the subsequent trace in the above expression. This divergent Jacobian can, however, be regularized to give well-defined expressions which contribute to the local anomalies at one-loop order [3]. The regularized expression is:

$$A_1 = \lim_{M \to \infty} \text{Tr} \left( \left[ \frac{\partial \delta \phi^i}{\partial \phi^j} \right]_0 e^{-H_{\phi}/M^2} + \left[ \frac{\partial \delta c}{\partial c} \right]_0 e^{-H_c/M^2} ight.$$

$$+ \left[ \frac{\partial \delta \gamma}{\partial \gamma} \right]_0 e^{-H_{\gamma}/M^2} + \left[ \frac{\partial \delta \pi_c}{\partial \pi_c} \right]_0 e^{-H_{\pi_c}/M^2} + \left[ \frac{\partial \delta \pi_{\gamma}}{\partial \pi_{\gamma}} \right]_0 e^{-H_{\pi_{\gamma}}/M^2} \left. \right)$$

(15)

where the delta-functions that appear when evaluating the square brackets are replaced by Gaussian regulators

$$H_\phi = \frac{\partial^2 S}{\partial \phi^i \partial \phi^i}$$

(16)

for the scalar fields and

$$H_c = \left( \left. \frac{\partial^2 S}{\partial c \partial \pi_c} \right)^\dagger \frac{\partial^2 S}{\partial c \partial \pi_c} \right|_{\text{chiral}}$$

(17)

for the spin-2 ghost and likewise for the rest of the fields [2][12]. In the above expressions, the subscript “chiral” means we are restricting to terms chiral in gauge fields.

Evaluating the trace in (15) in the plane-wave basis, isolating the infinite part of the resulting expression and also calculating the contribution arising from the

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3 It is well-known how to construct regulators which lead to consistent anomalies (i.e. anomalies that satisfy the Wess-Zumino consistency condition). For example, one may use the Pauli-Villars scheme [2].
variation of the action itself, one is left with the following expression for the anomaly at order $\bar{\hbar}$:

$$\triangle_1 = A_1 + (\delta S)_1$$

$$= \bar{\hbar} \left( \frac{16}{30\pi} (1 - 17a) - \frac{a}{12\pi} C_{\text{mat}} \right) \int d^2 z \left( \gamma \pi_c \partial \gamma \left( \partial^3 h - \partial^3 K_{\pi_c} \right) + \partial^3 c \left( \gamma K_c \partial \gamma - \pi_c (\partial \gamma B - \gamma \partial B - \partial \gamma K_{\pi_c} + \gamma \partial K_{\pi_c}) \right) \right)$$

$$- \frac{\bar{\hbar}}{12\pi} (100 - C_{\text{mat}}) \int d^2 z \left( \partial^3 h - \partial^3 K_{\pi_c} \right).$$

(18)

In the same manner, we can evaluate the anomalies at order $\bar{\hbar}^2$ arising from the variation of the renormalised action and also from the Jacobian for the measure of the path integral. Putting all terms together, we obtain:

$$\triangle'_2 = A_2 + (\delta S)_2$$

$$= \bar{\hbar}^2 \left( - \frac{29}{50\pi} (1 - 17a) + \frac{C_{\text{mat}} - 2 d_{ijk} d_{ijk}}{360\pi} \right) \int d^2 z \gamma (\partial^3 B - \partial^3 K_{\pi_c}).$$

(19)

In evaluating the preceding order-$\hbar$ and $-\hbar^2$ anomalies, we have found all the local anomaly contributions that are accessible from knowledge of the one-loop result.

Nevertheless, there is still further information about the anomalies that can be deduced from the above. Note that in expression (18) there are ‘off-diagonal’ terms mixing the spin-2 and spin-3 sectors. The presence of such off-diagonal anomalies is particularly significant for the structure of the anomalies at the next order in $\hbar$, since further contractions are possible. In fact, the dressing of these off-diagonal terms in $\triangle_1$ produces the following non-local anomaly contribution, $A_{2,\text{nl}}$, at order $\hbar^2$:

$$A_{2,\text{nl}} = \triangle_1 \cdot S_0 = \bar{\hbar}^2 \left( \frac{16}{30\pi} (1 - 17a) - \frac{a}{12\pi} C_{\text{mat}} \right)$$

$$\times \int d^2 z \left[ \left( \partial^3 h - \partial^3 K_{\pi_c} \right) \left( 2 \partial \gamma \frac{\partial^2}{\partial} - \frac{5}{6} \gamma \frac{\partial^3}{\partial} \right) (K_{\pi_c} - B) + \partial^3 c \left( \frac{5}{6} (B - K_{\pi_c}) \frac{\partial^3}{\partial} - 2 (\partial B - \partial K_{\pi_c}) \frac{\partial^2}{\partial} \right) (K_{\pi_c} - B) \right.$$  

$$\left. - \frac{3}{10} \left( \partial \gamma (B - K_{\pi_c}) - \gamma (\partial B - \partial K_{\pi_c}) \right) \frac{\partial^3}{\partial} (K_{\pi_c} - h) \right],$$

(20)

where $S_0$ is the classical part of the renormalised action (3). We have now evaluated both the local and non-local parts of the one-loop order $W_3$ anomalies, to order $\hbar^2$.

### 4 Consistency of the $W_3$ anomalies

In the preceding section we derived one-loop anomalies for chiral $W_3$ gravity in a derivative gauge. We now wish to consider the consistency of these derived anomalies and begin by looking at the Ward identity.

\footnote{Note that the prime on $\triangle'_2$ in eqn.(19) indicates that since this contribution is obtained using only one-loop method, it might yet be accompanied by genuine two-loop contributions, as we shall see.}
At order $\hbar$ the Ward identity (13) reads

$$\triangle_1 = \frac{\delta S_0}{\delta \varphi^i} \frac{\delta \Gamma_1}{\delta K_{\varphi^i}} + \frac{\delta S_0}{\delta K_{\varphi^i}} \frac{\delta \Gamma_1}{\delta \varphi^i} + \frac{\delta S_{1/2}}{\delta \varphi^i} \frac{\delta S_{1/2}}{\delta K_{\varphi^i}}$$

$$= (S_0, \Gamma_1) + \frac{1}{2} (S_{1/2}, S_{1/2}), \quad (21)$$

where the second line is written in the more compact anti-bracket notation, defined by

$$(M, N) \equiv \frac{\delta M}{\delta \varphi^i} \frac{\delta N}{\delta K_{\varphi^i}} + \frac{\delta M}{\delta K_{\varphi^i}} \frac{\delta N}{\delta \varphi^i}. \quad (22)$$

In order to obtain the well-known order $\hbar$ Wess-Zumino consistency condition [13], we take the anti-bracket of $S_0$ with the above expression. Simple manipulations using the Jacobi identity for the anti-bracket, together with following low-order relations

$$(S_0, S_0) = 0, \quad (S_0, S_{1/2}) = 0, \quad (23)$$

lead to the desired result, namely:

$$(S_0, \triangle_1) = 0. \quad (24)$$

If we insert the order $\hbar$ anomaly derived in the previous section (eqn. 18), a straightforward calculation verifies that it indeed satisfy this consistency condition to this order.

Next, we move on to consider the consistency of the anomalies at order $\hbar^2$. To obtain the order $\hbar^2$ Wess-Zumino consistency condition, we once again begin with the Ward identity (13) at the appropriate order:

$$(S_0, \Gamma_2) + (S_{1/2}, \Gamma_{3/2}) + \frac{1}{2} (\Gamma_1, \Gamma_1) = \triangle_2 + A_{2, nl}. \quad (25)$$

The consistency condition is then derived as in the order $\hbar$ case, noting also now that the local contribution to the anomaly at order $\hbar^{3/2}$ is zero. The resulting local condition is given by

$$(S_0, \triangle_2) + (S_0, A_{2, nl})_{\text{loc}} + (S_1, \triangle_1) = 0, \quad (26)$$

where the second term is restricted to local contributions after taking the antibracket. \[5\] Any non-local condition would have to hold separately.

On insertion of the calculated results (18, 19, 20) and after a relatively tedious calculation, we discover that this condition only holds if there is an extra contribution to the local order-$\hbar^2$ anomaly. This contribution is given by

$$\triangle_2 - \triangle_2' = \hbar^2 \left( \frac{2d_{ijk}d_{ijk}}{360\pi} \right) \int d^2 z \gamma (\partial^5 B - \partial^5 K_{\pi_c}) \quad (27)$$

Note that the only non-local terms that can contribute to this local Wess-Zumino consistency condition, are those that are of first-order in non-locality (i.e. with just one factor of $1/\partial\bar{\partial}$).
and stems from a genuine two-loop diagram. Recall that the techniques employed in evaluating the anomalies in the preceding section are necessarily restricted to the one-loop order. Thus, having used the consistency conditions to overcome the deficiencies of the Fujikawa method, we find that the anomalies now agree with those already known from explicit Feynman-diagram calculations.

5 Lam’s theorem and non-local anomalies

So far, we have derived the non-local anomalies at order $\hbar^2$ simply via the dressing of the local one-loop anomalies. In this section, we will go on to derive these non-local anomalies directly from the left-hand side of the Ward identity (13). Comparison of the two results will then form an explicit check of Lam’s theorem, which states that they should be identical. There has been to date no direct check of this theorem in an anomalous system as complex as that of chiral $W_3$ gravity.

In checking Lam’s theorem we are looking for the terms that make non-local contributions to the left-hand side of the Ward identity (13), at order $\hbar^2$. Note that, for our purposes, we need only consider terms that are of first order in non-locality, for comparison with $A_{2,nl}$ (20).

Since this is a somewhat tedious calculation, details of the relevant diagrams together with their contributions are contained in Appendix A.

It is, however, worth commenting on the following point. At order $\hbar$, a simple “buck-passing” mechanism exists. As an illustration, consider the non-local terms arising from the spin-2 gauge field self-energy diagram, shown in Fig.1(a). These are precisely cancelled by the purely non-local diagram, Fig.1(b), of higher order in the spin-2 gauge field. What remains after this cancellation is simply the well-known Virasoro anomaly.

![Figure 1: “Buck-passing” at order $\hbar$](image)

We might expect that this mechanism also applies at order $\hbar^2$. If this were the case, the non-local terms, (which we must anticipate if Lam’s theorem is to hold), would have to arise entirely from other types of diagrams.

At order $\hbar^2$, the illustration analogous to that above is to consider the spin-3 gauge-field self-energy diagram shown in Fig.2, since this is where the local anomaly arises. By comparison with the order-$\hbar$ phenomenon, we might expect that the non-local terms arising from this diagram are precisely cancelled by contributions from

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6 This approach to overcoming the deficiencies of the Fujikawa method has also been used in $W_\infty$. 

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diagrams of higher order in the spin-2 gauge field. In fact, taking into account all loop orders that might feasibly give a cancellation of the non-local terms resulting from this diagram, (see Appendix A), we find that no such cancellation occurs. No such “buck-passing” mechanism exists at order \( \hbar^2 \) and so no non-local term can be ignored.

The complete first-order non-local contributions to the left-hand side of the Ward identity (13) at order \( \hbar^2 \), having examined all possible contributory diagrams, is given by:

\[
(\Gamma, \Gamma)_{\hbar^2, \text{nl}} = \hbar^2 \left( \frac{16(1 - 17a)}{30\pi} - \frac{a C_{\text{mat}}}{12\pi} \right) \times \int \left[ \partial^3 c \left( -\frac{5}{6}(K_{\pi,\gamma} - B) \frac{\partial^3}{\partial} + 2(\partial K_{\pi,\gamma} - \partial B) \frac{\partial^2}{\partial} \right) (K_{\pi,\gamma} - B) \right. \\
+ \left. \partial^3 (K_{\pi,\gamma} - h) \left( 5\gamma \frac{\partial^3}{\partial} (K_{\pi,\gamma} - B) - 2\partial \gamma \frac{\partial^2}{\partial} (K_{\pi,\gamma} - B) \right) \right] \\
+ \left( \frac{16(1 - 17a)}{30\pi} - \frac{a n}{12\pi} - \frac{10 a \alpha_i \alpha_i}{9\pi} \right) \times \int \left[ \frac{3}{10} \left( \partial \gamma (K_{\pi,\gamma} - B) - \gamma (\partial K_{\pi,\gamma} - \partial B) \right) \frac{\partial^5}{\partial} (K_{\pi,\gamma} - h) \right] \\
+ \int \left( - \frac{n}{360} + \frac{d_{ijk} d_{ijk} - 9 e_{ij} e_{ij} - 6 e_{ij} e_{ji}}{180} \right) \times \frac{\partial^2}{\partial} (K_{\pi,\gamma} - h) \left( \partial \gamma \partial^3 \gamma \partial + 2 \partial^4 \gamma \gamma - 2 \gamma \partial^4 \right) (K_{\pi,\gamma} - B) \\
- \int \frac{e_{ij} (e_{ij} + e_{ji})}{12} \left( \gamma (\partial^4 K_{\pi,\gamma} - \partial^4 B) - \partial^4 \gamma (K_{\pi,\gamma} - B) \right) \frac{\partial^2}{\partial} (K_{\pi,\gamma} - h) \\
- \int \frac{d_{ij} f_{ij}}{6} \frac{\partial^2}{\partial} (K_{\pi,\gamma} - h) \left( \partial \gamma \partial^3 + \gamma \partial^4 - \partial^4 \gamma - \partial^3 \gamma \partial \right) (K_{\pi,\gamma} - B). \tag{28} \]

After invoking the conditions (7), we indeed find that the above Ward identity calculation yields the expected result of \( A_{2,\text{nl}} \). With this, we have verified that Lam’s theorem holds for the non-trivial example of fully quantised chiral \( W_3 \) gravity.

6 Non-local anomalies from the BRST charge

The failure of the BRST charge \( Q \) to be nilpotent also gives an expression for the anomaly. Whether we investigate the anomalies of a theory through the BRST
charge or the Ward identity, compatible results should be expected. Since non-local anomalies are found when considering the Ward identity, such terms should also be found in the BRST charge approach. However, as the BRST charge contains a contour integral, it might naively be considered to be a purely local expression. Here, we show how the non-local features do indeed appear in the BRST-charge approach to the anomalies.

The full quantum level BRST charge in the derivative gauge is given by

\[ Q = \oint c(T_{\text{mat}} + \frac{1}{2}T_{\text{gh}}) + \gamma(W_{\text{mat}} + \frac{1}{2}W_{\text{gh}}) + \pi_h \pi_b + \pi_B \pi_\beta. \]  

(29)

If we interpret the BRST-charge as the integral of a normal-ordered operator current:

\[ Q = \oint \frac{dz}{2\pi i} J(z), \]  

(30)

where the integral in complex world-sheet coordinates is taken to be a closed loop around the origin, then we may use standard operator-product techniques to obtain the expectation value of \( Q^2 \). To do this, one would consider correlators such as

\[ \langle Q^2 \cdots \rangle = \int D\varphi^i(w) \int \frac{dz}{2\pi i} < JJ >_1(z) \exp S(w) \cdots, \]  

(31)

where \( \varphi^i \) stands generically for all the fields that are contained in the BRST action (3), the subscript “1” in \( < JJ >_1 \) denotes taking the residue of the first-order pole \( (z - w)^{-1} \), in the Laurent series resulting from the operator-product \( J(z)J(w) \) and \( \cdots \) stands for any other operator insertion.

One may replace correlator relations like (31) by equivalent operator relations. Expanding the exponential, the known expression [4] for the local part of \( Q^2 \) arises from the first term

\[ Q^2_{\text{loc}} = \oint \left( \frac{100 - C_{\text{mat}}}{6} \right) c \partial^3 c + \left( -\frac{16}{15} (1 - 17a) + \frac{a}{6} C_{\text{mat}} \right) \gamma \pi_c \partial \gamma \partial^5 c \]

\[ + \hbar \left( \frac{29}{25} (1 - 17a) - \frac{C_{\text{mat}}}{180} \right) \gamma \partial^5 \gamma, \]  

(32)

whilst the following terms, of first order in non-locality, stem from the second

\[ Q^2_{\text{nl}} = \oint \frac{dz}{2\pi i} < JJ >_1(z), S >\]

\[ = \oint \hbar \left( -\frac{16}{15} (1 - 17a) + \frac{a}{6} C_{\text{mat}} \right) \left( \frac{3}{10} \gamma \partial \gamma \partial^5 \right) (K_{\pi_c} - \hbar) \]

\[ - 2\partial^3 c \partial \gamma \partial^2 (K_{\pi \gamma} - B) + \frac{5}{6} \partial^3 c \gamma \partial^3 (K_{\pi \gamma} - B). \]  

(33)

Thus we see that non-local terms arise similarly in the BRST approach and have their roots in the off-diagonal part of the local anomaly.

\(^7\)Since Cauchy’s theorem is used to collect simple poles, the factor of \((2\pi i)^{-1}\) appears in the measure so as to make contact with the standard formulae.
7 Relating the BRST algebra anomaly to the BRST Ward identity anomaly

In ref.[4], a relationship was proposed between the Ward-identity anomalies and those obtained from the square of the BRST charge. It was shown that the local parts of the one- and two-loop $W_3$ anomalies arising in the BRST Ward identity can be obtained by the action of the anomalous operator $Q^2$ on the gauge fermion. Explicitly, one has the relation

$$\Delta_1 + \Delta_2 = -\frac{1}{2\pi} \int d^2v \langle <JJ>_1, \Psi(v) \rangle_1.$$  \hspace{1cm} (34)

In this section, we explore whether or not this relationship can be extended to include the non-local anomalies.

From the action in “canonical BRST” form $\pi^a \tilde{\partial}q^a - \delta \Psi$ (3) we can see that the gauge fermion $\Psi$ for our action (3) is given by:

$$\Psi = \pi_c h + \pi_\gamma B + \sum_i (-1)^{[i]} \phi^i K_{\phi^i},$$  \hspace{1cm} (35)

where $[i]$ takes the values (0, 1) for (bose, fermi) variables. Taking the operator product of the non-local part of $Q^2$ with this gauge fermion, relatively straightforward algebra verifies that:

$$A_{2, nl} = -\frac{1}{2\pi} \int d^2v \langle <JJ>_1, S \rangle, \Psi(v) \rangle_1.$$  \hspace{1cm} (36)

Thus, the non-local Ward-identity anomalies can indeed be obtained from the action of the non-local part of $Q^2$ acting on the gauge fermion.

8 Gauge-independence of the anomaly relation

In the preceding section we saw that the relationship between the BRST algebra anomalies (32, 33) and those in the BRST Ward identity (18, 19, 20, 27), originally proposed for local anomalies, also holds for the non-local terms. In this section, we investigate the gauge dependence of this anomaly relation by choosing a more general gauge-fixing condition.

Let us consider chiral $W_3$ gravity in a parametrised gauge, specifically $\zeta \tilde{\partial}h + \mu h = 0$ and for the spin-3 sector $\zeta' \tilde{\partial}B + \mu' B = 0$. The action in this gauge is

$$S = \frac{1}{\pi} \int \left[ -\frac{1}{2} \tilde{\partial} \phi^i \tilde{\partial} \phi^j + \pi_b \tilde{\partial} h + \pi_B \tilde{\partial} B - \pi_b \tilde{\partial} b - \pi_c \tilde{\partial} c - \pi_\beta \tilde{\partial} \beta - \pi_\gamma \tilde{\partial} \gamma - \delta \Psi \right],$$  \hspace{1cm} (37)

where the gauge fermion $\Psi$ is $(h \pi_c - \mu b + B \pi_\gamma - \mu' B \beta + \sum_i (-1)^{[i]} \phi^i K_{\phi^i})$ and the variations are as in (4) apart from

$$\delta h = \frac{\pi_b}{\zeta} ; \quad \delta B = \frac{\pi_\beta}{\zeta'}.$$  \hspace{1cm} (38)
The Ward-identity anomalies remain as in the derivative gauge (18, 19, 20, 27), since the new terms arising in the action (37) in this parametrised gauge make no additional contribution. The BRST charge similarly remains unchanged from the derivative gauge expression (29).

If we note the following modified operator-product expansions:
\[
\pi_h(z)h(w) \sim \frac{h}{\zeta(z-w)}; \quad \pi_B(z)B(w) \sim \frac{h}{\zeta'(z-w)},
\]
(39)
then it is simple to check that the anomaly relations (34, 36) hold independently of the gauge choice.

It is worth emphasising a curious feature of this result: in a theory with gauge anomalies, the anomalous operator \(Q^2\) annihilates a certain part of the gauge fermion \(\Psi\) to give a gauge-independent result despite the fact that it is \(\Psi\) that establishes the choice of gauge.

9 General validity of the Anomaly Relation

So far we have seen that the anomaly relation holds for local and non-local chiral \(W_3\) anomalies, under a general gauge-fixing condition. This motivates seeking an underlying reason for this relation, independent of the theory concerned. In this section we give a heuristic argument for the anomaly relation.

In conventional field theory the anomaly is explored through the Ward identity. To obtain a mathematical expression for the Ward identity we can take either of two approaches. The Ward identity may be expressed in terms of a classically-conserved Noether current or in terms of the effective action after introducing sources into the theory (see Section 2).

In either case, one starts with the partition function \(Z\) and checks that it is invariant under the relevant field transformations. In the first approach, the Ward identity is obtained by considering the variation of the partition function under a change of integration variables of the form of classical symmetry transformation, but with a space-time dependent parameter \(\alpha\), as follows:
\[
Z \rightarrow Z' = \int \mathcal{D}\phi^i \exp\left(\int (\mathcal{L} + K_{\phi^i} \delta \phi^i) - \int \alpha \partial_\mu J^\mu + \alpha A\right),
\]
(40)
where \(A\) is the anomaly, \(\alpha \partial_\mu J^\mu\) is the variation of the Lagrangian, \(J^\mu\) being the Noether current and \(K_{\phi^i}\) are the sources for the variations of the fields \(\phi^i\). Expanding the last two terms in the exponent and requiring the invariance of the partition function under changes of integration variables, we obtain
\[
\int \partial_\mu J^\mu = A.
\]
(41)
This is a statement of the Ward identity in terms of the Noether current.

In the second approach, we introduce sources \(J_{\phi^i}\) for the fields \(\phi^i\). The variation of the partition function, under a global symmetry, in terms of the extended Lagrangian
\[ \mathcal{L}_{\text{ext}} = \mathcal{L} + J_{\varphi^i} \varphi^i + K_{\varphi^i} \delta \varphi^i \] is given by

\[ Z \rightarrow Z' = \int \mathcal{D} \varphi^i \exp \left( \int \mathcal{L}_{\text{ext}} + \int J_{\varphi^i} \delta \varphi^i + A \right). \tag{42} \]

Expanding the last two terms in this exponent, and using a Legendre transformation (12), to obtain the effective action \( \Gamma \) from the partition function, we find the well-known expression of the Ward identity,

\[ \frac{\delta \Gamma}{\delta \varphi^i} \frac{\delta \Gamma}{\delta K_{\varphi^i}} = A. \tag{43} \]

Hence, we can see that the Ward identity anomaly can equally well be expressed either way.

Now, we can use this information to consider the general validity of our anomaly relation. Since the BRST charge \( Q \) is the spatial integral of the time component of the Noether current \( J \), we have

\[ A = \frac{\delta \Gamma}{\delta \varphi^i} \frac{\delta \Gamma}{\delta K_{\varphi^i}} = \frac{\partial Q}{\partial t} = -[Q, H], \tag{44} \]

where \( H \) is the Hamiltonian. In all reparametrisation-invariant theories written in “canonical” form similar to (3) the Hamiltonian can be expressed as the variation of a gauge fermion \( \Psi \) [16], hence,

\[ A = -[Q, \{Q, \Psi\}]. \tag{45} \]

Subsequent use of the Jacobi identity in the right hand side then yields,

\[ A = -\frac{1}{2} \{Q, \{Q, \Psi\}\}. \tag{46} \]

The above relation is the same as that discussed in Sections 7 and 8, although there we used operator-product language rather than commutators.

### 10 Conclusion and Outlook

In this paper, we have made a detailed survey of the structure of \( W_3 \) anomalies. In our work, we used the Romans’ realisation for a fully quantised \( W_3 \) algebra. Using the Romans’ renormalisation conditions ensures exactly the cancellation of matter-dependent anomalies. We investigated the remaining anomalies up to order \( \bar{\hbar}^2 \), using several different approaches.

Firstly, the Fujikawa method of anomaly derivation was used to find the local order \( \hbar \) and \( \hbar^2 \) anomaly contributions. Non-local order \( \hbar^2 \) terms were then found by dressing the off-diagonal order \( \hbar \) anomaly. The Fujikawa path-integral approach is a purely one-loop technique. We showed that genuine two-loop contributions can nevertheless be accessed by demanding that the Wess-Zumino consistency conditions hold. It remains an interesting problem to discover how the two-loop anomalies may
be handled directly in the path-integral formalism. This might be an equivalent problem to that of finding how the Pauli-Villars regularisation can be extended to the regularisation of diagrams beyond one-loop.

Next, the non-local anomalies at order $h^2$ were calculated again, this time directly from Feynman diagrams. It was shown that these non-local contributions to the Ward-identity precisely match those from the dressing of the one-loop anomaly. This constitutes a non-trivial check of Lam’s theorem. We also noted that the order-$\hbar$ “buck-passing” mechanism, which is usually taken for granted at order $\hbar$, does not apply at order $h^2$. Explicitly, the non-local anomaly contributions from the spin-3 gauge-field self-energy diagram are not simply cancelled by terms from diagrams of higher order in the gauge fields.

We further considered the $Q^2$ anomaly for the BRST algebra. Although it is well known how to obtain local terms this way, non-local anomalies have rarely been discussed. Thus, we looked closely at the expectation value of the square of the BRST charge as a means of calculating all the anomalies. Again, both the local and non-local contributions were found up to order $h^2$.

We also discussed the relationship between the anomalies found in the two approaches. The anomaly relation proposed in ref.[4] requires that the BRST Ward identity anomalies be proportional to the application of the square of the BRST charge to the gauge fermion. This relation, previously discussed for local anomaly contributions in chiral gravities, was shown to apply not only to the non-local terms but also to hold under a more general gauge-fixing condition and furthermore to be valid for all reparameterisation-invariant theories. It remains an open problem to see if the anomaly relation holds for a more general class of theories, for example Yang-Mills theory.

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Appendix A

In the following we use the Ward identity (13) to evaluate the non-local anomalies at order $h^2$.

For simplicity, we list only the possible contributions that would result in terms with no explicit factors of $\phi^i, b, \beta$, any of the momenta or any of the sources except for $K_{\pi, c}, K_{\pi, a}$. They should also be of no more than second order in the gauge fields. This is to facilitate comparison with the result obtained via the dressing of the one-loop anomaly (20).

The following operator-product expansions obtained from the action (3) [7] are
used in the evaluation of the diagrams presented in this appendix:

\[
\partial \phi^i(z) \partial \phi^j(w) \sim \frac{-\hbar}{(z - w)^2},
\]

\[
\pi_h(z) h(w) \sim \frac{\hbar}{z - w}; \quad \pi_B(z) B(w) \sim \frac{\hbar}{z - w},
\]

\[
c(z) \pi_c(w) \sim \frac{\hbar}{z - w}; \quad b(z) \pi_b(w) \sim \frac{\hbar}{z - w},
\]

\[
\gamma(z) \pi_\gamma(w) \sim \frac{\hbar}{z - w}; \quad \beta(z) \pi_\beta(w) \sim \frac{\hbar}{z - w}.
\] (47)

Let us firstly consider the simple loop:

\[
(\hbar - K_{\phi}) \quad \text{\circle{\hbar - K_{\phi}}}
\]

which contributes as a \( \phi \) loop, a spin-2 ghost loop and a spin-3 ghost loop, giving:

\[
\frac{\delta \Gamma_1}{\delta K_{\phi \phi}} \frac{\delta S_1}{\delta \pi_c} = \frac{\hbar^2(100 - n)(1 - 17a)}{360\pi} \times \int \left[ \frac{2}{3} \partial^3 \gamma(K_{\pi\gamma} - B) - 3 \partial^2 \gamma \left( \partial K_{\pi\gamma} - \partial B \right) \right. \\
\left. + 3 \partial \gamma \left( \partial^2 K_{\pi\gamma} - \partial^2 B \right) - 2 \gamma \left( \partial^3 K_{\pi\gamma} - \partial^3 B \right) \right] \frac{\partial^3}{\partial (K_{\pi_c} - \hbar)}.
\] (48)

Next, we consider contributory diagrams with external \( \phi \) lines. The first in this series of diagrams is the following \( \phi \) loop:

\[
(\hbar - K_{\phi}) \quad \text{\circle{\hbar - K_{\phi}}} \quad \phi
\]

which contributes:

\[
\frac{\delta \Gamma_1}{\delta \phi^i} \frac{\delta S_1}{\delta K_{\phi'}} = -\frac{\hbar^2 d_{i j j} f_j}{6\pi} \int (K_{\pi\gamma} - B) \partial^3 \gamma \frac{\partial^3}{\partial (K_{\pi_c} - \hbar)}.
\] (49)
The second is the mixed spin-2 and spin-3 ghost loop:

\[
\begin{array}{c}
\phi \\
(B-K_{\gamma}) \\ (B-K_{\gamma})
\end{array}
\]

which contributes:

\[
\frac{\delta \Gamma_{3/2}}{\delta \phi^{i}} \delta \phi_{1/2}^{j} = \frac{h^{2} a \alpha_{i} \alpha_{i}}{\pi} \int \partial^{3} c \left[ \frac{5}{6} (K_{\pi, \gamma} - B) \frac{\partial^{3}}{\partial} - 2(\partial K_{\pi, \gamma} - \partial B) \frac{\partial^{2}}{\partial} \right] (K_{\pi, \gamma} - B).
\] (50)

The final external \( \phi \) contribution is the \( \phi \) loop corresponding to the above diagram, giving:

\[
\frac{\delta \Gamma_{3/2}}{\delta \phi^{i}} \delta \phi_{1/2}^{j} = -\frac{h^{2} e_{ij} d_{ijk} \alpha_{k}}{6\pi} \int \partial^{2} c (K_{\pi, \gamma} - B) \frac{\partial^{1}}{\partial} (K_{\pi, \gamma} - B).
\] (51)

The next set of diagrams have external \( \gamma \) lines. The first is the \( \phi \) loop:

\[
\begin{array}{c}
\gamma \\
(h-K_{\gamma}) \\
K_{\phi}
\end{array}
\]

which contributes:

\[
\frac{\delta \Gamma_{1}}{\delta K_{\phi^{i}}} \frac{\delta S_{1}}{\delta \phi^{i}} = \frac{h^{2} d_{ij} f_{j}}{6\pi} \int \gamma (\partial^{3} K_{\pi, \gamma} - \partial^{3} B) \frac{\partial^{3}}{\partial} (K_{\pi, \gamma} - h).
\] (52)

The mixed spin-2 ghost, \( \phi \) loop:

\[
\begin{array}{c}
\gamma \\
K_{\phi} \\
(B-K_{\gamma})
\end{array}
\]

gives:

\[
\frac{\delta \Gamma_{3/2}}{\delta K_{\phi^{i}}} \frac{\delta S_{1/2}}{\delta \phi^{i}} = -\frac{h^{2} a \alpha_{i} \alpha_{i}}{\pi} \int \left[ \partial \gamma (K_{\pi, \gamma} - B) - \gamma (\partial K_{\pi, \gamma} - \partial B) \right] \frac{\partial^{5}}{\partial} (K_{\pi, \gamma} - h).
\] (53)
The mixed spin-2, spin-3 ghost loop:

\[ (B-K_\pi) \bigg/ K_\phi \]

contributes:

\[ \delta \Gamma_{3/2} \delta S_{1/2} \delta K \phi / \delta \phi^i \delta \pi / \delta \pi = -\frac{h^2 a \alpha_i \alpha_i}{\pi} \int \left[ 2 \partial \gamma \frac{\partial^2}{\partial \gamma} (K_{\pi \gamma} - B) - \frac{5}{6} \gamma \frac{\partial^3}{\partial \gamma} (K_{\pi \gamma} - B) \right] (\partial^3 K_{\pi e} - \partial^3 h). \]  

The next contribution is from the \( \phi \) loop corresponding to the previous diagram, which gives:

\[ \delta \Gamma_{3/2} \delta S_{1/2} \delta K \phi / \delta \phi^i \delta \phi^i = \frac{h^2 e_{ij} d_{ijk} \alpha_k}{6\pi} \int \gamma (\partial^2 K_{\pi e} - \partial^2 h) \frac{\partial^4}{\partial \gamma} (K_{\pi \gamma} - B). \]  

In the following figures (a) is a \( \phi \) loop and (b) has contributions from a mixed spin-2, spin-3 ghost loop as well as a pure \( \phi \) loop.

Together they contribute:

\[ \frac{\delta \Gamma_2 \delta S_0}{\delta K_{\pi \gamma} \delta \pi \gamma} = h^2 \left( -\frac{29(1-17a)}{50\pi} + \frac{(d_{ijk} d_{ijk} - 9 e_{ij} e_{ij} - 6 e_{ij} e_{ji})}{180\pi} \right) \int \frac{\partial^5}{\partial \gamma} (K_{\pi \gamma} - B) \]

\[ \times \left[ 2(\partial K_{\pi e} - \partial h) \gamma - (K_{\pi e} - h) \partial \gamma + (\partial K_{\pi \gamma} - \partial B) c - 2(K_{\pi \gamma} - B) \partial c \right]. \]  

This concludes the external \( \gamma \) diagrams.

The bow-tie diagram,
with one $\phi$ loop and one mixed spin-2, spin-3 ghost loop, contributes:

$$
\frac{\delta \Gamma_2}{\delta K_{\pi\gamma}} \frac{\delta S_0}{\delta \pi c} + \frac{\delta \Gamma_2}{\delta K_{\pi\gamma}} \frac{\delta S_0}{\delta \pi \gamma} =
- \frac{\hbar^2 \alpha_0}{12\pi} \left\{ \int \partial^3c \left[ 2(\partial K_{\pi\gamma} - \partial B) \frac{\partial^2}{\partial (K_{\pi\gamma} - B)} - \frac{5}{6}(K_{\pi\gamma} - B) \frac{\partial^3}{\partial (K_{\pi\gamma} - B)} \right] + 
\left[ \frac{5}{6} \gamma \frac{\partial^3}{\partial (K_{\pi\gamma} - B)} - 2\partial \gamma \frac{\partial^2}{\partial (K_{\pi\gamma} - B)} \right] \partial^3 K_{\pi c} - \partial^3 h \right\}.
$$

(57)

There are also contributions from some three-point functions besides the bow-tie diagram. Consider the three-vertex diagram:

\[
\begin{array}{c}
\begin{array}{c}
\text{(h-K}_{\pi c}\text{)}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{(B-K}_{\pi q}\text{)}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{(B-K}_{\pi q}\text{)}
\end{array}
\end{array}
\]

This $\phi$ loop gives:

$$
\frac{\delta \Gamma_2}{\delta K_{\pi\gamma}} \frac{\delta S_0}{\delta \pi c} + \frac{\delta \Gamma_2}{\delta K_{\pi\gamma}} \frac{\delta S_0}{\delta \pi \gamma} =
- \frac{\hbar^2 \alpha_0}{360\pi} \left\{ \int \partial^5 \gamma \left[ 2(\partial K_{\pi\gamma} - \partial B) \frac{\partial}{\partial (K_{\pi\gamma} - B)} - 4(K_{\pi\gamma} - B) \frac{\partial^2}{\partial (K_{\pi\gamma} - B)} \right] (K_{\pi c} - h) + 
(\partial^5 K_{\pi\gamma} - \partial^5 B) \left[ 4\gamma \frac{\partial}{\partial (K_{\pi\gamma} - B)} - 2\partial \gamma \frac{1}{\partial (K_{\pi\gamma} - B)} \right] (K_{\pi c} - h) + 
\frac{\partial^5}{\partial (K_{\pi\gamma} - B)} \left[ 2(\partial K_{\pi\gamma} - \partial B) c - 4(K_{\pi\gamma} - B) \partial c + 4\gamma (\partial K_{\pi c} - \partial h) - 2\partial \gamma (K_{\pi c} - h) \right] \right\}.
$$

(58)
There is also a three-point diagram generated by the counterterms:

\[ \begin{align*}
\text{(h-K}_c) \\
\text{(B-K}_q) & \quad \text{(B-K}_q)
\end{align*} \]

This $\phi$ loop contributes:

\[
\frac{\delta \Gamma_2}{\delta K_{\pi_\gamma}} \frac{\delta S_0}{\delta \pi_{\gamma}} + \frac{\delta \Gamma_2}{\delta K_{\pi_\gamma}} \frac{\delta S_0}{\delta \pi_{c}} = - \frac{h^2(e_{ij}e_{ij} + 2e_{ij}e_{ji})}{60\pi} \\
\times \int \left\{ \left[ \partial \gamma (K_{\pi_\gamma} - h) + 2(K_{\pi_\gamma} - B) \partial c - c(\partial K_{\pi_\gamma} - \partial B) - 2\gamma (\partial K_{\pi_\gamma} - \partial h) \right] \frac{\partial^5}{\partial} (K_{\pi_\gamma} - B) \\
+ \left[ \frac{\partial}{\partial} (K_{\pi_\gamma} - h) - 2\gamma \frac{\partial}{\partial} (K_{\pi_\gamma} - h) \right] (\partial^5 K_{\pi_\gamma} - \partial^5 B) \\
+ \left[ 2(K_{\pi_\gamma} - B) \frac{\partial}{\partial} (K_{\pi_\gamma} - h) - (\partial K_{\pi_\gamma} - \partial B) \frac{1}{\partial} (K_{\pi_\gamma} - h) \right] \partial^5 \gamma \right\} \\
- \frac{h^2(e_{ij}e_{ij} + 2e_{ij}e_{ji})}{12\pi} \int \left[ \gamma (\partial^2 K_{\pi_\gamma} - \partial^2 h) \frac{\partial^4}{\partial}(K_{\pi_\gamma} - B) \\
+ \gamma \frac{\partial^2}{\partial} (K_{\pi_\gamma} - B) - (K_{\pi_\gamma} - B) \partial^2 c \frac{\partial^4}{\partial}(K_{\pi_\gamma} - B) \\
- (K_{\pi_\gamma} - B) \frac{\partial^2}{\partial}(K_{\pi_\gamma} - h) \partial^4 \gamma \right]. \tag{59}
\]

Finally, there are also ghost-loop contributions from the preceding three-point diagram, contributing to the anomaly as follows:

\[
\frac{h^2(1 - 17a)}{30\pi} \left[ \frac{-644}{30} \left( \partial^6 \gamma (K_{\pi_\gamma} - B) + \gamma \partial^6 (K_{\pi_\gamma} - B) \right) \\
- 24 \left( \partial^4 \gamma \partial^2 (K_{\pi_\gamma} - B) + \partial^2 \gamma \partial^4 (K_{\pi_\gamma} - B) \right) \\
- \frac{101}{2} \left( \partial^5 \gamma \partial (K_{\pi_\gamma} - B) - \partial \gamma \partial^5 (K_{\pi_\gamma} - B) \right) \right] \frac{1}{\partial} (K_{\pi_\gamma} - h) \\
+ \frac{h^2(1 - 17a)}{30\pi} (K_{\pi_\gamma} - h) \left[ \frac{322}{15} \frac{\partial^6}{\partial} + \frac{221}{5} \partial^5 \gamma \frac{\partial^5}{\partial} \right] \\
+ \frac{744}{9} \partial^3 \gamma \frac{\partial^3}{\partial} + 56 \partial^2 \gamma \frac{\partial^4}{\partial} + 32 \partial^4 \gamma \frac{\partial^2}{\partial}(K_{\pi_\gamma} - B) \\
+ \frac{h^2(1 - 17a)}{30\pi} c \left[ \frac{322}{15} (K_{\pi_\gamma} - B) \frac{\partial^6}{\partial} + \frac{221}{5} \partial(K_{\pi_\gamma} - B) \frac{\partial^5}{\partial} \right] \\
+ \frac{744}{9} \partial^3 (K_{\pi_\gamma} - B) \frac{\partial^3}{\partial} + 56 \partial^2 (K_{\pi_\gamma} - B) \frac{\partial^4}{\partial} + 32 \partial^4 (K_{\pi_\gamma} - B) \frac{\partial^2}{\partial}(K_{\pi_\gamma} - B). \tag{60}
\]
This concludes all the non-local order-$h^2$ contributions that are relevant for comparison with the result obtained from the dressing of one-loop anomaly. This is indeed the complete result since all other contributions cancel amongst themselves.

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