SUBSHEAVES OF A HERMITIAN TORSION FREE COHERENT SHEAF ON AN ARITHMETIC VARIETY

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INTRODUCTION

Let $K$ be a number field and $O_K$ the ring of integers of $K$. Let $(E,h)$ be a hermitian finitely generated flat $O_K$-module. For an $O_K$-submodule $F$ of $E$, let us denote by $h_{F→E}$ the submetric of $F$ induced by $h$. It is well known that the set of all saturated $O_K$-submodules $F$ with $\hat{\deg}(F,h_{F→E}) ≥ c$ is finite for any real numbers $c$ (for details, see [4, the proof of Proposition 3.5]).

In this note, we would like to give its generalization on a projective arithmetic variety. Let $X$ be a normal and projective arithmetic variety. Here we assume that $X$ is an arithmetic surface to avoid several complicated technical definitions on a higher dimensional arithmetic variety. Let us fix a nef and big $C^\infty$-hermitian invertible sheaf $H$ on $X$ as a polarization of $X$. Then we have the following finiteness of saturated subsheaves with bounded arithmetic degree, which is also a generalization of a partial result [5, Corollary 2.2].

Theorem A (cf. Theorem 3.1). Let $E$ be a torsion free coherent sheaf on $X$ and $h$ a $C^\infty$-hermitian metric of $E$ on $X(\mathbb{C})$. For any real number $c$, the set of all saturated $O_X$-subsheaves $F$ of $E$ with $\hat{\deg}(\hat{c}_1(H) \cdot \hat{c}_1(F,h_{F→E})) ≥ c$ is finite.

For a non-zero $C^\infty$-hermitian torsion free coherent sheaf $G$ on $X$, the arithmetic slope $\hat{\mu}_{\overline{\Pi}}(G)$ of $G$ with respect to $\overline{H}$ is defined by

$$\hat{\mu}_{\overline{\Pi}}(G) = \frac{\hat{\deg}(\hat{c}_1(\overline{H}) \cdot \hat{c}_1(G))}{\text{rk} G}.$$ 

As defined in the paper [5], $(E,h)$ is said to be arithmetically $\mu$-semistable with respect to $\overline{\Pi}$ if, for any non-zero saturated $O_X$-subsheaf $F$ of $E$,

$$\hat{\mu}_{\overline{\Pi}}(F,h_{F→E}) ≤ \hat{\mu}_{\overline{\Pi}}(E,h).$$

The above semistability yields an arithmetic analogue of the Harder-Narasimham filtration of a torsion free sheaf on an algebraic variety as follows: A filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_l = E$$

of $E$ is called an arithmetic Harder-Narasimham filtration of $(E,h)$ with respect to $\overline{\Pi}$ if

1. $E_i/E_{i-1}$ is torsion free for every $1 ≤ i ≤ l$.
2. Let $h_{E_i/E_{i-1}}$ be a $C^\infty$-hermitian metric of $E_i/E_{i-1}$ induced by $h$, that is,

$$h_{E_i/E_{i-1}} = (h_{E→E_i}E_i+E_{i-1},h_{E_i/E_{i-1}}E_i/E_{i-1}) = (h_{E→E_{i-1}}E_{i-1},h_{E_{i-1}→E/E_{i-1}})$$
(for details, see Proposition 1.1.1). Then \((E_i/E_{i-1}, h_{E_i/E_{i-1}})\) is arithmetically \(\mu\)-semistable with respect to \(\mathcal{F}\).

(3) \(\bar{\mu}_{\mathcal{F}}(E_1/E_0, h_{E_1/E_0}) > \bar{\mu}_{\mathcal{F}}(E_2/E_1, h_{E_2/E_1}) > \cdots > \bar{\mu}_{\mathcal{F}}(E_i/E_{i-1}, h_{E_i/E_{i-1}})\).

As a consequence of the above theorem, we can show the unique existence of an arithmetic Harder-Narasimhan filtration:

**Theorem B** (cf. Theorem 5.1). There is a unique arithmetic Harder-Narasimham filtration of \((E, h)\).

## 1. Preliminaries

### 1.1. Hermitian vector space.

In this subsection, let us recall several basic facts of hermitian complex vector spaces.

Let \((V, h)\) be a finite dimensional hermitian complex vector space, i.e., \(V\) is a finite dimensional vector space over \(\mathbb{C}\) and \(h\) is a hermitian metric of \(V\). Let \(\phi : V' \to V\) be an injective homomorphism of complex vector spaces. If we set \(h'(x, y) = h(\phi(x), \phi(y))\), then \(h'\) is a hermitian metric of \(V'\). This metric \(h'\) is called the **submetric of \(V'\ induced by \(h\ and \(V' \to V\)**, and it is denoted by \(h_{V' \to V}\).

Let \(\psi : V \to V''\) be a surjective homomorphism of complex vector spaces. Let \(W\) be the orthogonal complement of \(\text{Ker}(\psi)\) with respect to \(h\). Let \(h_{W \to V}\) be the submetric of \(W\) induced by \(h\) and \(W \to V\). Then there is a unique hermitian metric \(h''\) of \(V''\) such that the isomorphism \(\psi|_W : W \to V''\) gives rise to an isometry \((W, h_{W \to V}) \cong (V'', h'')\). The metric \(h''\) is called the **quotient metric of \(V''\ induced by \(h\ and \(V \to V''\)**, and it is denoted by \(h_{V \to V''}\).

For simplicity, the submetric \(h_{V' \to V}\) and the quotient metric \(h_{V \to V''}\) are often denoted by \(h_{V'}\) and \(h_{V''}\) respectively. It is easy to see the following proposition:

**Proposition 1.1.1.** Let \(V, V', V''\) be finite dimensional complex vector spaces with \(V'' \subseteq V' \subseteq V\). Let \(h\) be a hermitian metric of \(V\). Then

\[
(h_{V' \to V})_{V' \to V''} = (h_{V \to V''})_{V \to V''} \cdot (V' \to V'')
\]

as hermitian metrics of \(V'/V''\).

More generally, we have the following lemma:

**Lemma 1.1.2.** Let \((V, h)\) be a finite dimensional hermitian complex vector space. Let \(W\) and \(U\) be subspaces of \(V\). Let us consider a natural homomorphism

\[
\phi : W \to V \to V/\mathcal{U}
\]

of complex vector spaces. Let \(Q\) be the image of \(\phi\). Let us consider two natural hermitian metrics \(h_1\) and \(h_2\) of \(Q\) given by

\[
h_1 = (h_{W \to V})_{W \to Q}\quad \text{and} \quad h_2 = (h_{V \to V/\mathcal{U}})_{Q \to V/\mathcal{U}}.
\]

Then \(h_1(x, x) \geq h_2(x, x)\) for all \(x \in Q\). In particular, if \(\{x_1, \ldots, x_s\}\) is a basis of \(Q\), then \(\det(h_1(x_i, x_j)) \geq \det(h_2(x_i, x_j))\).

**Proof.** Let \(T\) be the orthogonal complement of \(\text{Ker}(\phi : W \to Q)\) with respect to \(h_{W \to V}\). Then \(h(v, v) = h_1(\phi(v), \phi(v))\) for all \(v \in T\). Let \(U^\perp\) be the orthogonal complement of \(U\) with respect to \(h\). Then, for \(v \in T\), we can set \(v = u + u'\) with \(u \in U\) and \(u' \in U^\perp\). Then \(h_2(\phi(v), \phi(v)) = h(u', u')\). Thus

\[
h_2(\phi(v), \phi(v)) = h(u', u') \leq h(v, v) = h_1(\phi(v), \phi(v)).
\]

For the last assertion, see [4, Lemma 3.4].
Let $e_1, \ldots, e_n$ be an orthonormal basis of $V$ with respect to $h$. Let $V^\vee$ be the dual space of $V$ and $e_1^{\vee}, \ldots, e_n^{\vee}$ the dual basis of $e_1, \ldots, e_n$. For $\phi, \psi \in V^\vee$, we set

$$h^\vee(\phi, \psi) = \sum_{i=1}^{n} a_i \bar{b}_i,$$

where $\phi = a_1 e_1^{\vee} + \cdots + a_n e_n^{\vee}$ and $\psi = b_1 e_1^{\vee} + \cdots + b_n e_n^{\vee}$. It is easy to see that $h^\vee$ does not depend on the choice of the orthonormal basis of $V$, so that the hermitian metric $h^\vee$ of $V^\vee$ is called the \textit{dual hermitian metric of} $h$. Moreover we can easily check the following facts:

\textbf{Proposition 1.1.3.} \hspace{1em} (1) $h^\vee(\phi, \phi) = \sup_{x \in V \setminus \{0\}} \frac{|\phi(x)|^2}{h(x,x)}$.

(2) Let $x_1, \ldots, x_n$ be a basis of $V$ and $x_1^{\vee}, \ldots, x_n^{\vee}$ be the dual basis of $V^\vee$. If we set $H = (h(x_i, x_j))$ and $H^\vee = (h^\vee(x_i^{\vee}, x_j^{\vee}))$, then $H^\vee = H^{-1}$.

(3) Let $0 \to V_1 \to V_2 \to V_3 \to 0$ be an exact sequence of finite dimensional complex vector spaces and $h_1, h_2, h_3$ hermitian metrics of $V_1, V_2, V_3$ respectively. We assume that $h_1 = (h_2)_{V_1 \to V_2}$ and $h_3 = (h_2)_{V_2 \to V_3}$. Let us consider the dual exact sequence $0 \to V_3^\vee \to V_2^\vee \to V_1^\vee \to 0$ of $0 \to V_1 \to V_2 \to V_3 \to 0$ and the dual hermitian metrics $h_1^\vee, h_2^\vee, h_3^\vee$ of $h_1, h_2, h_3$ respectively. Then $h_3^\vee = (h_2^\vee)_{V_3^\vee \to V_2^\vee}$ and $h_1^\vee = (h_2^\vee)_{V_2^\vee \to V_1^\vee}$.

Let $(U, h_U)$ and $(W, h_W)$ be finite dimensional hermitian vector spaces over $\mathbb{C}$. Then $U \otimes_{\mathbb{C}} W$ has the standard hermitian metric $h_{U \otimes W}$ defined by

$$(h_{U \otimes W})(u \otimes w, u' \otimes w') = h_U(u, u')h_W(w, w').$$

Thus the standard hermitian metric of $\bigotimes^r V$ is given by

$$(\bigotimes^r h)(v_1 \otimes \cdots v_r, v'_1 \otimes \cdots v'_r) = h(v_1, v'_1) \cdots h(v_r, v'_r).$$

Let $\pi : \bigotimes^r V \to \bigwedge^r V$ be the natural surjective homomorphism and $\bigwedge^r h$ a hermitian metric of $\bigwedge^r V$ given by

$$\bigwedge^r h = r!(\bigotimes h)_{\bigotimes^r V \to \bigwedge^r V}.$$

Then we have the following:

\textbf{Proposition 1.1.4.} $(\bigwedge^r h)(x_1 \wedge \cdots \wedge x_r, x_1 \wedge \cdots \wedge x_r) = \det(h(x_i, x_j))$.

\textbf{Proof.} For $a_1, \ldots, a_r \in V$, we set

$$\phi(a_1, \ldots, a_r) = \frac{1}{r!} \sum_{\sigma \in S_r} \text{sgn}(\sigma) a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(r)}.$$

Then, by an easy calculation, for $\sigma \in S_r$ and $a_1, \ldots, a_r, b_1, \ldots, b_r \in V$, we can see

\begin{equation}
(\bigotimes^r h)(a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(r)}, \phi(b_1, \ldots, b_r)) = \text{sgn}(\sigma)(\bigotimes^r h)(a_1 \otimes \cdots \otimes a_r, \phi(b_1, \ldots, b_r))
\end{equation}

Note that $\text{Ker}(\pi)$ is generated by elements of type

$$a_1 \otimes \cdots \otimes a_r,$$
where $a_i = a_j$ for some $i \neq j$. Therefore, by (1.1.4.1), $\phi(x_1, \ldots, x_r) \in \text{Ker}(\pi) \perp$ for all $x_1, \ldots, x_r \in V$. Thus, since
\[
\pi(\phi(x_1, \ldots, x_r)) = x_1 \wedge \cdots \wedge x_r,
\]
we have
\[
(\bigotimes_{i=1}^{r} h)_{V \to A} V(x_1 \wedge \cdots \wedge x_r, x_1 \wedge \cdots \wedge x_r) = (\bigotimes_{i=1}^{r} h)(\phi(x_1, \ldots, x_r), \phi(x_1, \ldots, x_r)).
\]
On the other hand, by using (1.1.4.1) again, we can check
\[
(\bigotimes_{i=1}^{r} h)(\phi(x_1, \ldots, x_r), \phi(x_1, \ldots, x_r)) = \frac{1}{r!} \det(h(x_i, x_j)).
\]
Therefore we get our assertion. \qed

1.2. Finitely generated modules over a 1-dimensional noetherian integral domain. Let $R$ be a noetherian integral domain with $\dim R = 1$, and $K$ the quotient field of $R$. For $a \in R \setminus \{0\}$, we set $\text{ord}_R(a) = \text{length}_R(R/aR)$, which yields a homomorphism $\text{ord}_R : R \setminus \{0\} \to \mathbb{Z}$, that is, $\text{ord}_R(ab) = \text{ord}_R(a) + \text{ord}_R(b)$ for $a, b \in R \setminus \{0\}$. Thus it extends to a homomorphism on $K^\times$ given by $\text{ord}_R(a/b) = \text{ord}_R(a) - \text{ord}_R(b)$.

Proposition 1.2.1. Let $E$ be a finitely generated $R$-module. Let $s_1, \ldots, s_r$ and $s'_1, \ldots, s'_r$ be sequences of elements of $E$ such that $s_1, \ldots, s_r$ and $s'_1, \ldots, s'_r$ form bases of $E \otimes_R K$ respectively. Let $A = (a_{ij})$ be an $r \times r$-matrix such that $a_{ij} \in K$ for all $i, j$ and $s'_i = \sum_{j=1}^{r} a_{ij} s_j$ in $E \otimes_R K$ for all $i$. Then
\[
\text{length}_R(E/Rs'_1 + \cdots + Rs'_r) = \text{length}_R(E/Rs_1 + \cdots + Rs_r) + \text{ord}_R(\det(A)).
\]

Proof. We set $M = Rs_1 + \cdots + Rs_r$ and $M' = Rs'_1 + \cdots + Rs'_r$. First we assume that $M' \subseteq M$. Then $a_{ij} \in R$. An exact sequence
\[
0 \to M/M' \to E/M' \to E/M \to 0.
\]
yields
\[
\text{length}_R(E/M') = \text{length}_R(E/M) + \text{length}_R(M/M').
\]
Note that $M$ is a free $R$-module. Let $\phi : M \to M$ be an endomorphism given by $\phi(s_i) = s'_i$. Then, by [EGA IV, Lemme 21.10.17.3], $\text{length}_R(M/\phi(M)) = \text{length}_R(R/\text{det}(\phi)R)$. Thus we get
\[
\text{length}_R(E/M') = \text{length}_R(E/M) + \text{length}_R(R/\text{det}(A)R).
\]
Next we consider a general case. Since $E/M$ is a torsion module, there is $b \in R \setminus \{0\}$ with $bM' \subseteq M$. Thus, by the previous observation,
\[
\text{length}_R(E/bM') = \text{length}_R(E/M) + \text{length}_R(R/\text{det}(bA)R)
\]
because $bs_i = \sum_{j=1}^{r} b a_{ij} s_j$ in $E \otimes_R K$ for all $i$. Moreover
\[
\text{length}_R(E/bM') = \text{length}_R(E/M') + \text{length}_R(R/b^r R).
\]
Hence the proposition follows. \qed
Lemma 1.3.2. Let \( \{x_1, \ldots, x_r\} \) be a basis of \( E \otimes_R K \). Let \( s_1, \ldots, s_r \in E \) and \( a \in R \setminus \{0\} \) such that \( ax_i = s_i \) in \( E \otimes_R K \) for all \( i \). Then the number
\[
\text{length}_R(E/\text{Rs}_1 + \cdots + \text{Rs}_r) - r \text{ord}_R(a)
\]
does not depend on the choice of \( s_1, \ldots, s_r \) and \( a \), so that it is denoted by \( \ell_R(E; x_1, \ldots, x_r) \).

(2) Let \( \{x_1, \ldots, x_r\} \) and \( \{x'_1, \ldots, x'_r\} \) be bases of \( E \otimes_R K \). Let \( B = (b_{ij}) \) be an \( r \times r \) matrix such that \( x'_i = \sum_{j=1}^r b_{ij}x_j \) for all \( i \). Then
\[
\ell_R(E; x'_1, \ldots, x'_r) = \ell_R(E; x_1, \ldots, x_r) + \text{ord}_R(\det(B)).
\]

Proof. (1) Let \( s'_1, \ldots, s'_r \in E \) and \( a' \in R \setminus \{0\} \) be another choice with \( a'x_i = s'_i \) in \( E \otimes_R K \) for all \( i \). Then \( s'_i = (a'/a)s_i \) in \( E \otimes_R K \). Thus, by the previous proposition,
\[
\text{length}_R(E/\text{Rs}'_1 + \cdots + \text{Rs}'_r) = \text{length}_R(E/\text{Rs}_1 + \cdots + \text{Rs}_r) + \text{ord}_R((a'/a)^r),
\]
which yields the assertion.

(2) Let us choose \( a, b \in R \setminus \{0\} \) and \( s_1, \ldots, s_r \in E \) such that \( ax_i = s_i \) in \( E \otimes_R K \) for all \( i \) and \( bb_{ij} \in R \) for all \( i, j \). If we set \( s'_i = \sum_{j=1}^r bb_{ij}s_i \), then \( abx'_i = s'_i \) in \( E \otimes_R K \) for all \( i \). Thus
\[
\ell_R(E; x_1, \ldots, x_r) = \text{length}_R(E/\text{Rs}_1 + \cdots + \text{Rs}_r) - r \text{ord}_R(a)
\]
\[
\ell_R(E; x'_1, \ldots, x'_r) = \text{length}_R(E/\text{Rs}'_1 + \cdots + \text{Rs}'_r) - r \text{ord}_R(ab).
\]
On the other hand, by the previous proposition,
\[
\text{length}_R(E/\text{Rs}'_1 + \cdots + \text{Rs}'_r) = \text{length}_R(E/\text{Rs}_1 + \cdots + \text{Rs}_r) + \text{ord}_R(\det(bB)).
\]
Hence we obtain (2). \( \square \)

1.3. Subsheaves of a torsion free coherent sheaf. In this subsection, we consider how we can get a saturated subsheaf.

Proposition 1.3.1. Let \( X \) be an irreducible noetherian integral scheme, \( \eta \) the generic point of \( X \), and \( K = \mathcal{O}_{X,\eta} \) the function field of \( X \). Let \( E \) be a torsion free coherent sheaf on \( X \). Let \( \Sigma(X, E) \) be the set of all saturated \( \mathcal{O}_X \)-subsheaves of \( E \) and \( \Sigma(K, E_\eta) \) the set of all vector subspaces of \( E_\eta \) over \( K \). Then the map \( \gamma : \Sigma(X, E) \to \Sigma(K, E_\eta) \) given by \( \gamma(F) = F_\eta \) is bijective. For a vector subspace \( W \) of \( E_\eta \) over \( K \), the subsheaf given by \( \gamma^{-1}(W) \) is called the saturated \( \mathcal{O}_X \)-subsheaf of \( E \) induced by \( W \) and is denoted by \( \mathcal{O}_X(W; E) \).

Proof. Let us begin with the following lemma:

Lemma 1.3.2. Let \( F, G \) be \( \mathcal{O}_X \)-subsheaves of \( E \) such that \( F \) is saturated in \( E \) and \( F_\eta = G_\eta \). Then \( F \supseteq G \).

Proof. Let us consider a homomorphism \( \phi : G \to E \to E/F \). Then \( \phi_\eta = 0 \). Since \( E/F \) is torsion free, we have \( \phi = 0 \), which means that \( G \subseteq F \). \( \square \)

The injectivity of \( \gamma \) is a consequence of the above lemma. Let \( W \) be a vector subspace of \( E_\eta \) over \( K \). We set \( F(U) = W \cap E(U) \) for any Zariski open set \( U \) of \( X \). Then \( F_\eta = W \). We need to see that \( F \) is saturated in \( E \). Since \( F \) is the kernel of the natural homomorphism \( E \to E_\eta \to E_\eta/W \), we have an injection \( E/F \to E_\eta/W \), so that \( E/F \) is torsion free. \( \square \)
Proposition 1.3.3. Let $X$ be a noetherian scheme and $E$ a locally free coherent sheaf on $X$. Let $\pi : P = \text{Proj}(\bigoplus_{d \geq 0} \text{Sym}^d(E^*)) \to X$ be the projective bundle and $\mathcal{O}_P(1)$ the tautological line bundle of $P \to X$. Let $\Gamma(X,P)$ be the set of all sections of $\pi : P \to X$. Moreover let $\Sigma'_1(X,E)$ be the set of all $\mathcal{O}_X$-subsheaves $L$ such that $L$ is invertible and $E/L$ is locally free. For $s \in \Gamma(X,P)$, let
\[ \phi_s : s^*(\mathcal{O}_P(-1)) \to s^*\pi^*(E) = E \]
be a homomorphism obtained from the dual homomorphism $\mathcal{O}_P(-1) \to \pi^*(E)$ of the natural homomorphism $\pi^*(E^*) \to \mathcal{O}_P(1)$ by applying $s^*$. We denote the image of $\phi_s : s^*(\mathcal{O}_P(-1)) \to E$ by $L(s)$. Then $L(s) \in \Sigma'_1(X,E)$ for all $s \in \Gamma(X,P)$ and a map
\[ \Gamma(X,P) \to \Sigma'_1(X,E) \]
given by $s \mapsto L(s)$ is bijective.

Proof. See [1, Theorem 7.1 and Proposition 7.12].

\[ \square \]

1.4. Hermitian locally free coherent sheaf on a smooth variety. Let $X$ be a smooth variety over $\mathbb{C}$, $\eta$ be the generic point of $X$, and $K = \mathcal{O}_{X,\eta}$ the function field of $X$.

Proposition 1.4.1. Let $(E,h)$ and $(E',h')$ be $C^\infty$-hermitian locally free coherent sheaves on $X$. If there is a dense Zariski open set $U$ of $X$ such that $(E,h)|_U$ is isometric to $(E',h')|_U$, then this isometry extends to an isometry over $X$.

Proof. Since $V = E_\eta$ is isomorphic to $E'_\eta$, we may assume that $E'$ is a subsheaf of $V$. Then $(E,h)|_U$ coincides with $(E',h')|_U$.

First let us see that $E = E'$. For this purpose, it is sufficient to see that $E_\gamma = E'_\gamma$ for all codimension one points $\gamma$. Let $\{\omega_1, \ldots, \omega_r\}$ and $\{\omega'_1, \ldots, \omega'_r\}$ be local bases of $E_\gamma$ and $E'_\gamma$ respectively. Then there are $r \times r$-matrices $(a_{ij})$ and $(b_{ij})$ such that $a_{ij}, b_{ij} \in K$ for all $i, j$ and
\[ \omega'_i = \sum_{j=1}^{r} a_{ij} \omega_j, \quad \omega_i = \sum_{j=1}^{r} b_{ij} \omega'_j \]
for all $i$. Clearly $(a_{ij})(b_{ij}) = (b_{ij})(a_{ij}) = (\delta_{ij})$.

Claim 1.4.1.1. $a_{ij}, b_{ij} \in \mathcal{O}_{X,\gamma}$ for all $i, j$.

For each $i$, we set $e_i = \min_{1 \leq j \leq r} \{\text{ord}_\gamma(a_{ij})\}$. We assume that $e_i < 0$. Let $t$ be a local parameter of $\mathcal{O}_{X,\gamma}$. Then $t^{-e_i}a_{ij} \in \mathcal{O}_{X,\gamma}$ for all $j$. Thus $t^{-e_i}\omega'_i \in E_\gamma$ and $t^{-e_i}\omega'_i \neq 0$ in $E_\gamma \otimes \kappa(\gamma)$. Let $\Gamma$ be the Zariski closure of $\{\gamma\}$. If we choose a general closed point $x_0$ of $\Gamma$, then $\omega'_i \neq 0$ in $E'_{x_0} \otimes \kappa(x_0)$ and $t^{-e_i}\omega'_i \neq 0$ in $E_{x_0} \otimes \kappa(x_0)$. On the other hand, there is an open neighborhood $U_{x_0}$ of $x_0$ such that
\[ h(t^{-e_i}\omega'_i, t^{-e_i}\omega'_i)(x) = h'(t^{-e_i}\omega'_i, t^{-e_i}\omega'_i)(x) \]
for $x \in U_{x_0} \cap U$. Thus if we set
\[ f(x) = h(t^{-e_i}\omega'_i, t^{-e_i}\omega'_i)(x) = |t|^{-2e_i} h'(\omega'_i, \omega'_i)(x) \]
on $U_{x_0} \cap U$, then $\lim_{x \to x_0} f(x) = h(t^{-e_i}\omega'_i, t^{-e_i}\omega'_i)(x_0) = 0$ because $t = 0$ at $x_0$. This is a contradiction because $t^{-e_i}\omega'_i \neq 0$ in $E_{x_0} \otimes \kappa(y)$. Therefore we can see that $a_{ij} \in \mathcal{O}_{X,\gamma}$ for all $i, j$. In the same way, $b_{ij} \in \mathcal{O}_{X,\gamma}$ for all $i, j$. 
By the above claim, \( \{\omega_1, \ldots, \omega_r\} \) and \( \{\omega'_1, \ldots, \omega'_{r'}\} \) generate the same \( \mathcal{O}_X, \gamma \)-module in \( V \). Thus \( E_\gamma = E'_\gamma \). Hence we get \( E = E' \).

Let \( x \) be an arbitrary closed point of \( X \). Let \( v, v' \in E_x \otimes \kappa(x) \). Choose \( \omega, \omega' \in E_x \) such that \( \omega \) and \( \omega' \) give rise to \( v \) and \( v' \) in \( E_x \otimes \kappa(x) \). Then there is a neighborhood \( U_x \) of \( x \) such that \( h(\omega, \omega')(y) = h'(\omega, \omega')(y) \) for all \( y \in U_x \). Thus
\[
\lim_{y \to x} h(\omega, \omega')(y) = \lim_{y \to x} h'(\omega, \omega')(y) = h(\omega, \omega')(x),
\]
which means that \( h_x(v, v') = h'_x(v, v') \).

\[\Box\]

**Proposition 1.4.2.** Let \((E, h)\) be a \(C^\infty\)-hermitian locally free coherent sheaf on \( X \). Let \( x_1, \ldots, x_r \) be a \( K \)-linearly independent elements of \( E_\eta \). Then \( \log(\det(h(x_i, x_j))) \) is a locally integrable function.

**Proof.** Let \( W \) be a vector subspace of \( E_\eta \) generated by \( x_1, \ldots, x_r \). By Proposition 1.3.1, there is a saturated \( \mathcal{O}_X \)-subsheaf \( F \) of \( E \) with \( F_\eta = W \). First we assume that \( F \) and \( E/F \) are locally free. For a closed point \( x \in X \), let \( \{\omega_1, \ldots, \omega_r\} \) be a local basis of \( F_\eta \). Then we can find a matrix \( A = (a_{ij}) \) such that \( a_{ij} \in K \) for all \( i \) and \( j \). Then
\[
\det(h(x_i, x_j)) = |\det(A)|^2 \det(h(\omega_i, \omega_j)).
\]
Since \( F \) and \( E/F \) are locally free, \( \det(h(\omega_i, \omega_j)) \) is a non-zero \( C^\infty \)-function around \( x \) and \( \det(A) \) is a non-zero rational function on \( X \). Thus \( \log(\det(h(x_i, x_j))) \) is locally integrable around \( x \).

In general, if we set \( Q = E/F \), then there is a proper birational morphism \( \mu : Y \to X \) of smooth algebraic varieties over \( \mathbb{C} \) such that
\[
\mu^*(Q)/(\text{the torsion part of } \mu^*(Q))
\]
is locally free. We set \( F' = \text{Ker}(\mu^*(E) \to \mu^*(Q)/(\text{the torsion part of } \mu^*(Q))) \). Then \( F' \) and \( \mu^*(E)/F' \) are locally free. Thus, since \( F'_\eta = W \),
\[
\log(\det(\mu^*(h)(x_i, x_j))) = \mu^*(\log(\det(h(x_i, x_j))))
\]
is a locally integrable function on \( Y \). Therefore so is \( \log(\det(h(x_i, x_j))) \) on \( X \) by virtue of [3, Proposition 1.2.5] \[\Box\]

1.5. **Arakelov geometry.** For basic definitions concerning Arakelov geometry, we refer to [6, Section 1]. Let \( X \) be a projective arithmetic variety. We use several kinds of positivity of a \( C^\infty \)-hermitian invertible sheaf on \( X \) (like ampleness, nefness and bigness) as defined in [6, Section 2]. Let \( \overline{H} = (\overline{H}_1, \ldots, \overline{H}_d) \) be a sequence of nef \( C^\infty \)-hermitian invertible sheaves on \( X \), where \( d = \dim X_0 \). Note that the sequence is empty in the case of \( d = 0 \). We say \( \overline{H} \) is fine if \((X, \overline{H}_1, \ldots, \overline{H}_d)\) gives rise to a fine polarization of the function field of \( X \) (for details, see [7, Section 6.1]). For example, if \( \overline{H}_i \)'s are nef and big, then \( \overline{H} \) is fine. Finally we consider the following lemma.

**Lemma 1.5.1.** Let \( X \) be a generically smooth arithmetic variety and \( U \) a Zariski open set of \( X \) with \( \text{codim}(X \setminus U) \geq 2 \). Then the natural homomorphism
\[
\widehat{\text{CH}}^1_D(X) \to \widehat{\text{CH}}^1_D(U)
\]
is injective.
Proposition 2.2. We assume that $g$ gives rise to a model of $(E, h)$ yielding models of $(F, h)$.

Then $V = \{ \eta \}$ in terms of $C$. We set $\mu$ of $X$. Then $\mu$.

of $(E, h)$ is called a model of $(E, h)$ in terms of $\mu : X' \to X$. Note that if $\mu' : X'' \to X'$ is a proper birational morphism of normal and generically smooth arithmetic varieties, then $\mu''(E', h')$ is also a model of $(E, h)$ in terms of $\mu \circ \mu' : X'' \to X$. For, let $X_0'$ be the maximal Zariski open set over which $\mu'$ is an isomorphism. Then $\text{codim}(X' \setminus X_0') \geq 2$. Thus if we set $V = \mu(U' \cap X_0')$, then we can see the above properties for $V$.

Proces 2.1. Let $X$ be a normal arithmetic variety and $(E, h)$ a birationally $C^\infty$-hermitian torsion free coherent sheaf on $X$. Let $F$ be a saturated $O_X$-subsheaf of $E$. Let $h_{F \to E}$ (resp. $h_{E \to E/F}$) be the submetric of $F$ induced by $F \to E$ and $h$ (resp. the quotient metric of $E/F$ induced by $E \to E/F$ and $h$) on a big Zariski open set of $X$, i.e., a Zariski open set whose complement has the codimension greater than or equal to 2. Then $(F, h_{F \to E})$ and $(E/F, h_{E \to E/F})$ are also a birationally $C^\infty$-hermitian torsion free coherent sheaf on $X$.

Proof. Let $\eta$ be the generic point of $X$. Let $(E', h')$ be a model of $(E, h)$ in terms of $\mu : X' \to X$. Let $F'$ be a saturated $O_X$-subsheaf $F'$ of $E'$ with $F'_\eta = F_\eta$ (cf. Proposition 1.3.1). We set $Q = E'/F'$. By [8, Theorem 1 in Chapter 4], there is a proper birational morphism $\mu' : X'' \to X'$ of normal and generically smooth arithmetic varieties such that $\mu''(Q)/(\text{torsion})$ is locally free. Let $F'' = \text{Ker}(\mu''(E') \to \mu''(Q)/(\text{torsion})).$

Then $F''$ and $\mu''(E')/F''$ are locally free. Thus

$(F'', \mu''(h')(F'' \to \mu''(E')))$ and $(\mu''(E')/F'', \mu''(h')\mu''(E')/\mu''(E'/F''))$

yield models of $(F, h_{F \to E})$ and $(E/F, h_{E \to E/F})$ respectively because $\mu''(E', h')$ gives rise to a model of $(E, h)$.

Proposition 2.2. We assume that $X$ is projective. Let $\overline{H} = (\overline{H}_1, \ldots, \overline{H}_d)$ be a sequence of nef $C^\infty$-hermitian invertible sheaves on $X$, where $d = \dim X_Q$. Then the quantity

$\overline{\deg}(c_1(\mu^*(\overline{H}_1))) \cdots c_1(\mu^*(\overline{H}_d)) \cdot 1(g, h')$
does not depend on the choice of a model \((E', h')\) in terms of \(\mu : X' \to X\). It is denoted by \(\deg_{\mathcal{H}}(E, h)\) and is called the arithmetic degree of \((E, h)\) with respect to \(\mathcal{H}\).

**Proof.** Let us begin with the following lemma.

**Lemma 2.3.** Let \(\nu : Y \to X\) be a birational morphism of normal and projective arithmetic varieties such that \(Y\) is generically smooth. Let \((E, h)\) and \((E', h')\) be \(C^\infty\)-hermitian locally free coherent sheaves on \(Y\). We assume that there is a Zariski open set \(U\) of \(X\) such that \(\text{codim}(X \setminus U) \geq 2\) and \(\nu\) is an isomorphism over \(U\), that is, if we set \(V = \nu^{-1}(U)\), then \(\nu|_V : V \simto U\). Let \(\mathcal{L}_1, \ldots, \mathcal{L}_d\) be \(C^\infty\)-hermitian invertible sheaves on \(X\), where \(d = \dim X_Q\). If \((E, h)|_V\) is isometric to \((E', h')|_V\), then

\[
\hat{\deg}(c_1(\nu^*(\mathcal{L}_1))) \cdots c_1(\nu^*(\mathcal{L}_d)) \cdot \hat{c}_1(E, h) = \hat{\deg}(c_1(\nu^*(\mathcal{L}_1))) \cdots c_1(\nu^*(\mathcal{L}_d)) \cdot \hat{c}_1(E', h').
\]

**Proof.** Let \(\eta\) be the generic point of \(Y\) and \(x_1, \ldots, x_r\) a basis of \(E_\eta\). Let \(x_1', \ldots, x_r'\) be the corresponding basis of \(E'_\eta\) with \(x_1, \ldots, x_r\). Let \(Y^{(1)}\) be the set of all codimension one points of \(Y\). Then \(\hat{c}_1(E, h)\) and \(\hat{c}_1(E', h')\) are represented by

\[
\sum_{\gamma \in Y^{(1)}} \ell_{\mathcal{O}_Y, \gamma}(E; x_1, \ldots, x_r) \cdot \left(\gamma, -\log(\det(h(x_i, x_j)))\right)
\]

and

\[
\sum_{\gamma \in Y^{(1)}} \ell_{\mathcal{O}_Y, \gamma}(E'; x_1', \ldots, x_r') \cdot \left(\gamma, -\log(\det(h'(x_i', x_j')))\right)
\]

respectively. By Proposition 1.4.1, we can see that

\[
\det(h(x_i, x_j)) = \det(h'(x_i', x_j'))
\]

on \(Y(\mathbb{C})\). Here

\[
\ell_{\mathcal{O}_Y, \gamma}(E; x_1, \ldots, x_r) = \ell_{\mathcal{O}_Y, \gamma}(E'; x_1', \ldots, x_r')
\]

for all \(\gamma \in V^{(1)}\). Moreover, for \(\gamma \in Y^{(1)} \setminus V^{(1)}\), since \(\text{codim}(\nu(\{\gamma\})) \geq 2\),

\[
\hat{\deg}(c_1(\nu^*(\mathcal{L}_1))) \cdots c_1(\nu^*(\mathcal{L}_d)) \cdot \left(\{\gamma\}, 0\right) = 0
\]

by the projection formula (cf. [6, Proposition 1.2 and Proposition 1.3]). Thus we have our lemma. \(\square\)

Let us go back to the proof of Proposition 2.2. Let \((E_1, h_1)\) and \((E_2, h_2)\) be two models of \((E, h)\) in terms of \(\mu_1 : X_1 \to X\) and \(\mu_2 : X_2 \to X\) respectively. We can choose a normal, projective and generically smooth arithmetic variety \(Y\) and birational morphisms \(\pi_1 : Y \to X_1\) and \(\pi_2 : Y \to X_2\) with \(\mu_1 \circ \pi_1 = \mu_2 \circ \pi_2\). We set \(\nu = \mu_1 \circ \pi_1 = \mu_2 \circ \pi_2\). First of all, by the projection formula, we have

\[
\hat{\deg}(c_1(\mu_1^*(\mathcal{H}_1))) \cdots c_1(\mu_1^*(\mathcal{H}_d)) \cdot \hat{c}_1(E_1, h_1) = \hat{\deg}(c_1(\nu^*(\mathcal{H}_1))) \cdots c_1(\nu^*(\mathcal{H}_d)) \cdot \hat{c}_1(\pi_1^*(E_1, h_1))
\]
and
\[ \overline{\deg}(\tilde{c}_1(\mu_2^*(\overline{H}_1))) \cdots \tilde{c}_1(\mu_2^*(\overline{H}_d)) \cdot \tilde{c}_1(E_2, h_2)) = \overline{\deg}(\tilde{c}_1(\nu^*(\overline{H}_1))) \cdots \tilde{c}_1(\nu^*(\overline{H}_d)) \cdot \tilde{c}_1(\pi_2^*(E_2, h_2)). \]

Moreover, by Lemma 2.3,
\[ \overline{\deg}(\tilde{c}_1(\nu^*(\overline{H}_1))) \cdots \tilde{c}_1(\nu^*(\overline{H}_d)) \cdot \tilde{c}_1(\pi_1^*(E_1, h_1)) = \overline{\deg}(\tilde{c}_1(\nu^*(\overline{H}_1))) \cdots \tilde{c}_1(\nu^*(\overline{H}_d)) \cdot \tilde{c}_1(\pi_2^*(E_2, h_2)). \]

Thus we get the assertion.

Let \( X \) be a normal arithmetic variety and \((E, h)\) a birationally \( C^\infty\)-hermitian torsion free sheaf on \( X \). Let \( \pi: X' \to X \) be a proper birational morphism of normal arithmetic varieties and \((E', h')\) a birationally \( C^\infty\)-hermitian torsion free sheaf on \( X' \). We say \((E, h)\) is birationally dominated by \((E', h')\) by means of \( \pi: X' \to X \) if there is a Zariski open set \( U \) of \( X \) with the following properties:

1. \( \text{codim}(X \setminus U) \geq 2 \) and \( U \) is generically smooth.
2. \((E, h)\) is a \( C^\infty\)-hermitian locally free sheaf over \( U \).
3. If we set \( U' = \pi^{-1}(U) \), then \( \pi|_{U'}: U' \sim U \).
4. \( (\pi|_{U'})^*((E, h)|_{U'}) \) is isomorphic to \((E', h')|_{U'}\).

Then we have the following:

**Proposition 2.4.** The notation is the same as above. We assume that \((E, h)\) is birationally dominated by \((E', h')\) by means of \( \pi: X' \to X \).

1. Let \( F \) be a saturated \( \mathcal{O}_X\)-subsheaf of \( E \) and \( F' \) the corresponding saturated \( \mathcal{O}_{X'}\)-subsheaf of \( E' \) with \( F \). Then \((F, h_{F \to E})\) and \((E/F, h_{E \to E/F})\) are birationally dominated by \((F', h'_{F' \to E'})\) and \((E'/F', h'_{F' \to E'/F'})\) respectively.

2. We assume that \( X \) and \( X' \) are projective. Let \( \overline{\mathcal{H}} = (\overline{H}_1, \ldots, \overline{H}_d) \) be a sequence of nef \( C^\infty\)-hermitian invertible sheaves on \( X \), where \( d = \dim X_\mathbb{Q} \).

Then \( \overline{\deg}_{\overline{\mathcal{H}}}(E, h) = \overline{\deg}_{\overline{\mathcal{H}}}(E', h') \).

**Proof.** (1) There is a Zariski open set \( U_1 \) such that \( U_1 \subseteq U \), \( \text{codim}(X \setminus U_1) \geq 2 \) and that \( E|_{U_1} \) and \( E/F|_{U_1} \) are locally free. We set \( U_1' = \pi^{-1}(U_1) \). Then \((\pi|_{U'})^*((F, h_{F \to E})|_{U_1'}) \) is isometric to \((F', h'_{F' \to E'})|_{U_1'} \). Thus our assertions follow.

(2) Let \((E'', h'')\) be a model of \((E', h')\) in terms of a birational morphism \( \mu: Y \to X' \). Then it is easy to see that \((E'', h'')\) is a model of \((E, h)\) in terms of \( \pi \circ \mu: Y \to X \). Thus we have (2) by Proposition 2.2.

3. **Finiteness of subsheaves with bounded arithmetic degree**

In this section, we would like to give the proof of the main theorem of this note.

**Theorem 3.1.** Let \( X \) be a normal projective arithmetic variety and \((E, h)\) a birationally \( C^\infty\)-hermitian torsion free coherent sheaf on \( X \). Let \( \overline{\mathcal{H}} = (\overline{H}_1, \ldots, \overline{H}_d) \) be a fine sequence of nef \( C^\infty\)-hermitian invertible sheaves on \( X \), where \( d = \dim X_\mathbb{Q} \). For any real number \( c \), the set of all non-zero saturated \( \mathcal{O}_X\)-subsheaf \( F \) of \( E \) with
\[ \overline{\deg}_{\overline{\mathcal{H}}}(\tilde{c}_1(F, h_{F \to E})) \geq c \]
is finite, where \( h_{F \to E} \) is the submetric of \( F \) induced by \( h \) over a big open set.
Proof. Let \((E', h')\) be a model of \((E, h)\) in terms of \(\mu : X' \to X\). Let \(\eta\) be the generic point of \(X\). For each vector subspace \(W\) of \(E_\eta\), let \(F\) (resp. \(F'\)) be a saturated \(\mathcal{O}_X\)-subsheaf of \(E\) (resp. \(\mathcal{O}_X\)-subsheaf of \(E'\)) induced by \(W\). Then, by Proposition 2.4,
\[
\widetilde{\deg}_{\overline{\mathbb{P}}}(F, h_{F \to E}) = \widetilde{\deg}_{\mathbb{P}^1}(F', h_{F' \to E'}).
\]
Therefore we may assume that \(X\) is generically smooth, \(E\) is locally free and \(h\) is a \(C^\infty\)-hermitian metric of \(E\).

For each \(0 < s < \text{rk}\,E\), let \(\Sigma_s(X, E)\) be the set of all saturated rank \(s\) \(\mathcal{O}_X\)-subsheaves of \(E\). First let us see that, for any real number \(c\), the set
\[
\{ L \in \Sigma_1(X, E) \mid \widetilde{\deg}_{\overline{\mathbb{P}}}(F, h_{F \to E}) \geq c \}
\]
is finite. Let \(\pi : P = \text{Proj}(\bigoplus_{d \geq 0} \text{Sym}^d(E^\vee)) \to X\) be the projective bundle and \(\mathcal{O}_P(1)\) the tautological line bundle of \(P\). Let \(h_P\) be the quotient hermitian metric of \(\mathcal{O}_P(1)\) by using the surjective homomorphism \(\pi^*(E^\vee) \to \mathcal{O}_P(1)\) and the hermitian metric \(\pi^*(h^\vee)\). In other words, the metric \(h_P^{-1}\) of \(\mathcal{O}_P(-1)\) is the submetric induced by the injective homomorphism \(\mathcal{O}_P(-1) \to \pi^*(E)\) and \(\pi^*(h)\) (cf. (3) of Proposition 1.1.3). Let \(P_\eta\) be the generic fiber of \(\pi : P \to X\), and \(K\) the function field of \(X\).

For a \(K\)-rational point \(x\) of \(P_\eta\), let us introduce \(\Delta_x\), \(U_x\), \(V_x\) and \(s_x\) as follows:\n\(\Delta_x\) is the Zariski closure of \(x\) in \(P\) and \(U_x\) is the maximal open set of \(X\) over which \(\pi|_{\Delta_x} : \Delta_x \to X\) is an isomorphism. Further \(V_x = (\pi|_{\Delta_x})^{-1}(U_x)\) and \(s_x : U_x \to P\) is the section induced by the isomorphism \(\pi|_{V_x} : V_x \to U_x\).

Let \(\Sigma_1(K, E_\eta)\) be the set of all 1-dimensional vector subspaces of \(E_\eta\) over \(K\). Then, by Proposition 1.3.3, there is a natural bijection
\[
P_\eta(K) \to \Sigma_1(K, E_\eta).
\]
Moreover let \(\Sigma_1(X, E)\) be the set of all saturated rank one \(\mathcal{O}_X\)-subsheaves of \(E\).

By Proposition 1.3.1, we have a bijective map \(\Sigma_1(X, E) \to \Sigma_1(K, E_\eta)\).

Therefore there is a natural bijection between \(P_\eta(K)\) and \(\Sigma_1(X, E)\). For a \(K\)-rational point \(x\) of \(P_\eta\), the corresponding saturated rank one \(\mathcal{O}_X\)-subsheaf of \(E\) is denoted by \(L(x)\). Then, by using Proposition 1.3.3, we can see that \(L(x)\) has the following property: Let \(s_x^*\mathcal{O}_P(-1) \to s_x^*\pi^*(E) = \mathcal{E}|_{U_x}\) be the homomorphism from the natural homomorphism \(\mathcal{O}_P(-1) \to \pi^*(E)\) by applying \(s_x^*\). Then the image of \(s_x^*\mathcal{O}_P(-1) \to \mathcal{E}|_{U_x}\) is \(L(x)|_{U_x}\). Let \(h_x\) be the submetric of \(L(x)\) induced by \(h\).

Claim 3.1.1. \(\hat{c}_1(L(x), h_x) = (\pi|_{\Delta_x})_* \left( \hat{c}_1 \left( (\mathcal{O}_P(-1), h_P^{-1})|_{\Delta_x} \right) \right)\).

Since the metric \(h_P^{-1}\) is the submetric of \(\mathcal{O}_P(-1)\) induced by \(\pi^*(h)\), we can see that \(s_x^*\mathcal{O}_P(-1, h_P^{-1})\) is isometric to \((L(x), h_x)|_{U_x}\). Thus \((\mathcal{O}_P(-1, h_P^{-1})|_{U_x})\) is isometric to \((\pi|_{V_x})^*((L(x), h_x)|_{U_x})\), which implies that
\[
(\pi|_{V_x})_* \left( \hat{c}_1 \left( (\mathcal{O}_P(-1), h_P^{-1})|_{V_x} \right) \right) = (\pi|_{V_x})_* \left( \hat{c}_1 \left( ((\pi|_{V_x})^*((L(x), h_x)|_{U_x})) \right) \right) = \hat{c}_1((L(x), h_x)|_{U_x}).
\]

This means that the assertion of the claim holds over \(U_x\). Thus so does over \(X\) by Lemma 1.5.1.
For a \( K \)-rational point \( x \) of \( P_\eta \), the height \( h_{\mathcal{O}(1)}(x) \) with respect to \( \mathcal{O}_P(1) \) and \( (X, \overline{\mathcal{H}}) \) is given by

\[
h_{\mathcal{O}(1)}(x) = \widehat{\deg} \left( \hat{c}_1 \left( (\pi|_{\Delta_x})^*(\mathcal{H}_1) \right) \cdots \hat{c}_1 \left( (\pi|_{\Delta_x})^*(\mathcal{H}_d) \right) \cdot \hat{c}_1 \left( \left( (\mathcal{O}_P(1), h_P)|_{\Delta_x} \right) \right) \right).
\]

By using the above claim and the projection formula,

\[
-h_{\mathcal{O}(1)}(x) = \widehat{\deg} \left( \hat{c}_1 \left( (\pi|_{\Delta_x})^*(\mathcal{H}_1) \right) \cdots \hat{c}_1 \left( (\pi|_{\Delta_x})^*(\mathcal{H}_d) \right) \cdot \hat{c}_1 \left( \left( (\mathcal{O}_P(-1), h_P^{-1})|_{\Delta_x} \right) \right) \right)
= \widehat{\deg} \left( \hat{c}_1(\mathcal{H}_1) \cdots \hat{c}_1(\mathcal{H}_d) \cdot \hat{c}_1(L(x), h_x) \right) = \widehat{\deg}_{\overline{\mathcal{H}}}(L(x), h_x).
\]

Thus we have a bijective correspondence between

\[
\{ L \in \Sigma_1(X, E) \mid \widehat{\deg}_{\overline{\mathcal{H}}}(F, h_{F \rightarrow E}) \geq c \}
\]

and

\[
\{ x \in P_\eta(K) \mid h(x) \leq -c \}.
\]

On the other hand, by virtue of Northcott’s theorem over finitely generated field (cf. [6, Theorem 4.3]), \( \{ x \in P_\eta(K) \mid h(x) \leq -c \} \) is a finite set. Therefore we get the case where \( s = 1 \).

For \( F \in \Sigma_d(X, E) \), let \( \lambda(F) \) be the saturation of

\[
\bigwedge^s F/\text{(the torsion part of } \bigwedge^s F)\text{ in } \bigwedge^s E.
\]

**Claim 3.1.2.** If \( \lambda(F) = \lambda(F') \), then \( F = F' \).

We assume that \( \lambda(F) = \lambda(F') \). Let \( K \) be the function field of \( X \). Then, using Plücker coordinates over \( K \), we can see that \( F \otimes K = F' \otimes K \). Thus, by Lemma 1.3.2, \( F' = F \).

Let \( h_{\lambda(F)} = (\bigwedge^s h)_{\lambda(F)} \rightarrow \bigwedge^s E \). Then, by Proposition 1.1.4,

\[
\hat{c}_1(F, h_F) = \hat{c}_1(\lambda(F), h_{\lambda(F)}).
\]

Therefore, by using the above claim and the case where \( s = 1 \), our theorem follows. \( \square \)

Let \( X \) be a normal and projective arithmetic variety and \((E, h)\) a birationally \( C^\infty\)-hermitian torsion free coherent sheaf on \( X \). Let \( \overline{\mathcal{H}} = (\mathcal{H}_1, \ldots, \mathcal{H}_d) \) be a fine sequence of nef \( C^\infty\)-hermitian invertible sheaves on \( X \). For a non-zero saturated \( \mathcal{O}_X \)-subsheaf \( G \) of \( E \), we set

\[
\hat{\mu}_{\overline{\mathcal{H}}}(G, h_{G \rightarrow E}) = \frac{\hat{\deg}_{\overline{\mathcal{H}}}(G, h_{G \rightarrow E})}{\text{rk } G}.
\]

A saturated \( \mathcal{O}_X \)-subsheaf \( F \) of \( E \) is called a **maximal slope sheaf of \((E, h)\)** with respect to \( \overline{\mathcal{H}} \) if \( \hat{\mu}_{\overline{\mathcal{H}}}(F, h_{F \rightarrow E}) \) gives rise to the maximal value of the set

\[
\{ \hat{\mu}_{\overline{\mathcal{H}}}(G, h_{G \rightarrow E}) \mid G \text{ is a non-zero saturated } \mathcal{O}_X \text{-subsheaf of } E \}.
\]

Moreover a maximal slope sheaf \( F \) of \((E, h)\) is called a **maximal destabilizing sheaf of \((E, h)\)** with respect to \( \overline{\mathcal{H}} \) if \( \text{rk } F \) is maximal among all maximal slope sheaves of \((E, h)\). As a corollary of Theorem 3.1, we have the following:

**Corollary 3.2.** There is a maximal destabilizing sheaf of \((E, h)\) with respect to \( \overline{\mathcal{H}} \).
4. Arithmetic first Chern class of a subsheaf

Let $X$ be a normal and generically smooth arithmetic variety and $\eta$ the generic point of $X$. Let $(E, h)$ be a $C^\infty$-hermitian locally free sheaf on $X$. Let $F$ be an $\mathcal{O}_X$-subsheaf of $E$. Let $x_1, \ldots, x_r$ be a basis of $F_\eta$. Let us consider an arithmetic codimension one cycle $z(F; x_1, \ldots, x_r)$ (i.e., an element of $\mathbb{Z}^1_D(X)$) given by

$$z(F; x_1, \ldots, x_r) = \left( \sum_{\Gamma} \ell_{\mathcal{O}_X, \Gamma}(F_\Gamma; x_1, \ldots, x_r) \Gamma, -\log \det(h(x_i, x_j)) \right).$$

Note that $\log \det(h(x_i, x_j))$ is locally integrable on $X(\mathbb{C})$ by Proposition 1.4.2. Let $x'_1, \ldots, x'_r$ be another basis of $F_\eta$. There is an $r \times r$-matrix $A = (a_{ij})$ with $x'_i = \sum_{j=1}^r a_{ij} x_j$. Using (2) of Corollary 1.2.2, we can see that

$$z(F; x'_1, \ldots, x'_r) = z(F; x_1, \ldots, x_r) + (\det(A)).$$

Therefore the class of $z(F; x_1, \ldots, x_r)$ in $\mathbb{CH}_D^1(X)$ does not depend on the choice of $x_1, \ldots, x_r$. We denote the class of $z(F; x_1, \ldots, x_r)$ in $\mathbb{CH}_D^1(X)$ by $\hat{c}_1(F \hookrightarrow E, h)$.

If $F = E$, then $\hat{c}_1(E \hookrightarrow E, h)$ is equal to the usual $\hat{c}_1(E, h)$. Note that

$$\hat{c}_1(F \hookrightarrow E, h) = \hat{c}_1(F, h_{F \hookrightarrow E})$$

if $F$ is saturated in $E$. More generally, we have the following:

**Proposition 4.1.** Let $F$ be an $\mathcal{O}_X$-subsheaf of $E$ and $\bar{F}$ the saturation of $F$ in $E$. Then $\hat{c}_1(\bar{F}, h_{\bar{F} \hookrightarrow E}) - \hat{c}_1(F' \hookrightarrow E, h)$ is represented by an arithmetic divisor

$$\left( \sum_{\Gamma : \text{prime divisor}} \text{length}_{\mathcal{O}_{X, \Gamma}}(\bar{F}_\Gamma/F_\Gamma) \Gamma, 0 \right).$$

In particular, if $\overline{\mathcal{H}} = (\overline{\mathcal{H}}_1, \ldots, \overline{\mathcal{H}}_d)$ is a sequence of nef $C^\infty$-hermitian invertible sheaves on $X$, then

$$\hat{\text{deg}}(\hat{c}_1(\overline{\mathcal{H}}_1) \cdots \hat{c}_1(\overline{\mathcal{H}}_d) \cdot \hat{c}_1(F \hookrightarrow E, h)) \leq \hat{\text{deg}}(\hat{c}_1(\overline{\mathcal{H}}_1) \cdots \hat{c}_1(\overline{\mathcal{H}}_d) \cdot \hat{c}_1(\bar{F}, h_{\bar{F} \hookrightarrow E})).$$

**Proof.** Let $\eta$ be the generic point of $X$. Let $\{x_1, \ldots, x_r\}$ be a basis of $F_\eta$. Then $\{x_1, \ldots, x_r\}$ also gives rise to a basis of $\bar{F}_\eta$. Thus $\hat{c}_1(\bar{F}, h_{\bar{F} \hookrightarrow E}) - \hat{c}_1(F \hookrightarrow E, h)$ is represented by

$$\left( \sum_{\Gamma} (\ell_{\mathcal{O}_X, \Gamma}(\bar{F}_\Gamma; x_1, \ldots, x_r) - \ell_{\mathcal{O}_X, \Gamma}(F_\Gamma; x_1, \ldots, x_r)) \Gamma, 0 \right).$$

Hence it is sufficient to see that

$$\ell_{\mathcal{O}_X, \Gamma}(\bar{F}_\Gamma; x_1, \ldots, x_r) - \ell_{\mathcal{O}_X, \Gamma}(F_\Gamma; x_1, \ldots, x_r) = \text{length}_{\mathcal{O}_X, \Gamma}(\bar{F}_\Gamma/F_\Gamma)$$

for all $\Gamma$. Let $a$ be an element of $\mathcal{O}_{X, \Gamma} \setminus \{0\}$ such that $ax_i \in \mathcal{O}_{X, \Gamma}$ for all $i$. Then

$$\ell_{\mathcal{O}_X, \Gamma}(\bar{F}_\Gamma; x_1, \ldots, x_r) = \text{length}_{\mathcal{O}_X, \Gamma}(\bar{F}_\Gamma/\mathcal{O}_{X, \Gamma}ax_1 + \cdots + \mathcal{O}_{X, \Gamma}ax_r) - r \text{ ord}_\Gamma(a),$$

$$\ell_{\mathcal{O}_X, \Gamma}(F_\Gamma; x_1, \ldots, x_r) = \text{length}_{\mathcal{O}_X, \Gamma}(F_\Gamma/\mathcal{O}_{X, \Gamma}ax_1 + \cdots + \mathcal{O}_{X, \Gamma}ax_r) - r \text{ ord}_\Gamma(a).$$

Therefore we get our proposition.  \( \square \)
5. Arithmetic Harder-Narasimham filtration

Let $X$ be a normal and projective arithmetic variety and $\mathcal{H} = (\mathcal{H}_1, \ldots, \mathcal{H}_d)$ a fine sequence of nef $C^\infty$-hermitian invertible sheaves. Let $(E, h)$ be a birationally $C^\infty$-hermitian torsion free coherent sheaf on $X$. $(E, h)$ is said to be arithmetically $\mu$-semistable with respect to $\mathcal{H}$ if, for any non-zero saturated $\mathcal{O}_X$-subsheaf $F$ of $E$, 

$$\hat{\mu}_\mathcal{H}(F, h_{\mathcal{F}\to E}) \leq \hat{\mu}_\mathcal{H}(E, h).$$

A filtration 

$$0 = E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_l = E$$

of $\mathcal{O}_X$-subsheaves of $E$ is called a saturated filtration of $E$ if $E_i/E_{i-1}$ is torsion free for every $1 \leq i \leq l$. Moreover we say a saturated filtration $0 = E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_l = E$ of $E$ is an arithmetic Harder-Narasimham filtration of $(E, h)$ with respect to $\mathcal{H}$ if

1. Let $h_{E_i/E_{i-1}}$ be a $C^\infty$-hermitian metric of $E_i/E_{i-1}$ induced by $h$, that is,

$$h_{E_i/E_{i-1}} = (h_{E_i\to E})_{E_i\to E_{i-1}} = (h_{E\to E/E_{i-1}})_{E_i\to E_{i-1}}.$$

Then $(E_i/E_{i-1}, h_{E_i/E_{i-1}})$ is arithmetically $\mu$-semistable with respect to $\mathcal{H}$.

2. $\hat{\mu}_\mathcal{H}(E_i/E_{i-1}, h_{E_i/E_{i-1}}) > \hat{\mu}_\mathcal{H}(E_i/E_{i-1}, h_{E_i/E_{i-1}})$.

In the case where $X$ is generically smooth and $(E, h)$ is a $C^\infty$-hermitian locally free coherent sheaf on $X$, for a non-zero $\mathcal{O}_X$-subsheaf $G$ of $E$, we set

$$\hat{\mu}_\mathcal{H}(G \hookrightarrow E, h) = \frac{\deg(\hat{c}_1(\mathcal{H}_1) \cdots \hat{c}_1(\mathcal{H}_d) \cdot \hat{c}_1(G \hookrightarrow E, h))}{\rk G}.$$

The purpose of this section is to prove the following unique existence of an arithmetic Harder-Narasimham filtration:

**Theorem 5.1.** Let $X$ be a normal and projective arithmetic variety. Let $(E, h)$ be a birationally $C^\infty$-hermitian torsion free coherent sheaf on $X$. Let $\mathcal{H} = (\mathcal{H}_1, \ldots, \mathcal{H}_d)$ be a fine sequence of nef $C^\infty$-hermitian invertible sheaves. Then there exists uniquely an arithmetic Harder-Narasimham filtration of $(E, h)$ with respect to $\mathcal{H}$. Moreover, if $(E, h)$ is not arithmetically $\mu$-semistable with respect to $\mathcal{H}$, then a maximal destabilizing sheaf of $(E, h)$ is unique.

We need several lemmas to prove the above theorem.

**Lemma 5.2.** Let $(E, h)$ and $(E', h')$ be birationally $C^\infty$-hermitian torsion free coherent sheaves on normal projective arithmetic varieties $X$ and $X'$ respectively. Let $\mathcal{H} = (\mathcal{H}_1, \ldots, \mathcal{H}_d)$ be a fine sequence of nef $C^\infty$-hermitian invertible sheaves on $X$. We assume that there is a birational morphism $\pi : X' \to X$ and $(E, h)$ is dominated by $(E', h')$ by means of $\pi : X' \to X$. Then we have the followings:

1. $(E, h)$ is arithmetically $\mu$-semistable with respect to $\mathcal{H}$ if and only if so is $(E', h')$ with respect to $\pi^*(\mathcal{H})$.

2. Let $F$ be a saturated $\mathcal{O}_X$-subsheaf of $E$ and $F'$ the corresponding saturated $\mathcal{O}_{X'}$-subsheaf of $E'$. Then $F$ is a maximal destabilizing sheaf of $(E, h)$ with respect to $\mathcal{H}$ if and only if so is $F'$ with respect to $\pi^*(\mathcal{H})$. 
(3) Let \(0 = E_0 \subset E_1 \subset \cdots \subset E_l = E\) be a saturated filtration of \(E\) and \(0 = E'_0 \subset E'_1 \subset \cdots \subset E'_l = E'\) the corresponding saturated filtration of \(E'\). Then \(0 = E_0 \subset E_1 \subset \cdots \subset E_l = E\) is a Harder-Narasimham filtration with respect to \(\mathcal{H}\) if and only if so is \(0 = E'_0 \subset E'_1 \subset \cdots \subset E'_l = E'\) with respect to \(\pi^*(\mathcal{H})\).

Proof. This is a consequence of Proposition 2.4. \(\square\)

**Lemma 5.3.** Let \((E, h)\) be a birationally \(C^\infty\)-hermitian torsion free coherent sheaf on a normal projective arithmetic variety \(X\). If \((E, h)\) is not arithmetically \(\mu\)-semistable with respect to \(\mathcal{H}\) and \(F\) is a maximal slope sheaf of \((E, h)\), then

\[
\hat{\mu}_{\mathcal{H}}(F, h_{F \to E}) > \hat{\mu}_{\mathcal{H}}(E/F, h_{E \to E/F}).
\]

Proof. We set \(a = \text{rk}(F)\) and \(b = \text{rk}(E/F)\). Then

\[
\hat{\mu}_{\mathcal{H}}(E, h) = \frac{a}{a+b}\hat{\mu}_{\mathcal{H}}(F, h_{F \to E}) + \frac{b}{a+b}\hat{\mu}_{\mathcal{H}}(E/F, h_{E \to E/F}).
\]

Thus, since \(\hat{\mu}_{\mathcal{H}}(F, h_{F \to E}) > \hat{\mu}_{\mathcal{H}}(E, h)\), we get our lemma. \(\square\)

**Lemma 5.4.** Let \((E, h)\) be a birationally \(C^\infty\)-hermitian torsion free coherent sheaf on a normal projective arithmetic variety \(X\). Let \(\mathcal{H} = (\mathcal{H}_1, \ldots, \mathcal{H}_d)\) be a fine sequence of nef \(C^\infty\)-hermitian invertible sheaves. Then there are a model \((E', h')\) of \((E, h)\) in terms of a birational morphism \(\mu : Y \to X\) of normal projective arithmetic varieties and a Harder-Narasimham filtration

\[
0 = E'_0 \subset E'_1 \subset \cdots \subset E'_l = E'
\]

of \((E', h')\) with respect to \(\mu^*(\mathcal{H})\) such that \(E'_i/E'_{i-1}\) is locally free for every \(i = 1, \ldots, l\).

Proof. Let \((E', h')\) be a model of \((E, h)\) in terms of \(\mu : Y \to X\). By Proposition 2.4, \((E, h)\) is arithmetically \(\mu\)-semistable with respect to \(\mathcal{H}\) if and only if so is \((E', h')\) with respect to \(\mu^*(\mathcal{H})\). Thus we may assume that \((E, h)\) is not arithmetically \(\mu\)-semistable with respect to \(\mathcal{H}\). Let \(E'_1\) be a maximal destabilizing sheaf of \((E', h')\). Considering Proposition 2.4 and a suitable birational morphism \(\mu' : Y' \to Y\) of normal, projective and generically smooth arithmetic varieties to remove the pinching points of \(E'/E'_1\), we may assume that \(E'_1\) and \(E'/E'_1\) are locally free. If \((E'/E'_1, h'_{E'/E'_1})\) is arithmetically \(\mu\)-semistable, then we are done. Otherwise, let \(E'_2\) be a saturated \(\mathcal{O}_Y\)-subsheaf of \(E'\) such that \(E'_1 \subset E'_2\) and \(E'_2/E'_1\) is a maximal destabilizing sheaf of \((E'/E'_1, h'_{E'/E'_1})\). Changing \(Y\) as before, we may assume that \(E'_2\) and \(E'/E'_2\) are locally free. Moreover, by Lemma 5.3,

\[
\hat{\mu}_{\mu'*(\mathcal{H})}(E'_1, h_{E'_1 \to E'}) = \hat{\mu}_{\mu'*(\mathcal{H})}(E'_1, (h_{E'_2 \to E})E'_2 \to E'_1) > \hat{\mu}_{\mu'*(\mathcal{H})}(E'_2/E'_1, (h_{E'_2 \to E})E'_2 \to E'_1).
\]

Thus, continuing this construction, we have our lemma. \(\square\)

**Lemma 5.5.** Let \((E, h)\) be a \(C^\infty\)-hermitian locally free coherent sheaf on a normal projective and generically smooth arithmetic variety \(X\). Let \(\mathcal{H} = (\mathcal{H}_1, \ldots, \mathcal{H}_d)\) be a fine sequence of nef \(C^\infty\)-hermitian invertible sheaves. Let \(0 = E_0 \subset E_1 \subset \cdots \subset E_l = E\) be an arithmetic Harder-Narasimham filtration of \((E, h)\) such that \(E_i/E_{i-1}\) is locally free for every \(i = 1, \ldots, l\). If \(F\) is a maximal slope sheaf of \((E, h)\), then \(F \subset E_1\) and \(\hat{\mu}_{\mathcal{H}}(F \to E, h) = \hat{\mu}_{\mathcal{H}}(E_1 \to E, h)\).
Proof. We choose $i$ such that $F \subseteq E_i$ and $F \not\subseteq E_{i-1}$. We assume that $i \geq 2$. Let $Q$ be the image of $F \rightarrow E_i/E_{i-1}$. Let $h_Q$ be the quotient metric of $Q$ induced by $h_{F \rightarrow E}$ and $F \rightarrow Q$, that is, $h_Q = (h_{F \rightarrow E})_{F \rightarrow Q}$. Then, by virtue of Lemma 1.1.2,
\[
\hat{\mu}(Q, h_Q) \leq \hat{\mu}(Q \hookleftarrow E_i/E_{i-1}, h_{E_i/E_{i-1}}).
\]
On the other hand, since $(F, h_{F \rightarrow E})$ and $(E_i/E_{i-1}, h_{E_i/E_{i-1}})$ are arithmetically $\mu$-semistable,
\[
\hat{\mu}(F, h_{F \rightarrow E}) \leq \hat{\mu}(Q, h_Q)
\]
and
\[
\hat{\mu}(Q \hookleftarrow E_i/E_{i-1}, h_{E_i/E_{i-1}}) \leq \hat{\mu}(E_i/E_{i-1}, h_{E_i/E_{i-1}}).
\]
Therefore,
\[
\hat{\mu}(F, h_{F \rightarrow E}) \leq \hat{\mu}(E_i/E_{i-1}, h_{E_i/E_{i-1}}) < \hat{\mu}(E_i, h_{E_i \rightarrow E}).
\]
which contradicts to the maximality of $\hat{\mu}(F, h_{F \rightarrow E})$. Thus $F \subseteq E_1$. Moreover, since $(E_i, h_{E_i \rightarrow E})$ is arithmetically $\mu$-semistable, $\hat{\mu}(F, h_{F \rightarrow E}) \leq \hat{\mu}(E_i, h_{E_i \rightarrow E})$. Therefore $\hat{\mu}(F, h_{F \rightarrow E}) = \hat{\mu}(E_i, h_{E_i \rightarrow E})$ by the maximality of $\hat{\mu}(F, h_{F \rightarrow E})$.

Let us start the proof of Theorem 5.1. The existence of a Harder-Narasimham filtration is a consequence of Lemma 5.4 and Proposition 2.4. Let us see the uniqueness of a Harder-Narasimham filtration. Clearly we may assume that $(E, h)$ is not arithmetically $\mu$-semistable. Let $0 = E_0 \subseteq E_1 \subseteq \cdots \subseteq E_l = E$ and $0 = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_l' = E$ be Harder-Narasimham filtration of $(E, h)$. Let $(E', h')$ be a model of $(E, h)$ in terms of $\mu : Y \rightarrow X$. Let $0 = E'_0 \subseteq E'_1 \subseteq \cdots \subseteq E'_l = E'$ and $0 = G'_0 \subseteq G'_1 \subseteq \cdots \subseteq G'_l = E'$ be corresponding Harder-Narasimham filtration of $(E', h')$. Let $0 = E_0 \subseteq E_1 \subseteq \cdots \subseteq E_l = E$ and $0 = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_l' = E$ respectively. By taking a birational morphism $\mu' : Y' \rightarrow Y$, we may assume that $E'_i/E'_{i-1}$ and $G'_j/G'_{j-1}$ are locally free for all $i = 1, \ldots, l$ and $j = 1, \ldots, l'$. Let $F'$ be a maximal destabilizing sheaf of $(E', h')$. Then, by Lemma 5.5, $F' \subseteq E'_1$ and $\hat{\mu}(F', h_{F' \rightarrow E'}) = \hat{\mu}(E'_1, h_{E'_1 \rightarrow E'})$. Thus $F' = E'_1$. In the same way, $F' = G'_1$. Hence, by considering a Harder-Narasimham filtration of $(E'/F', h_{E'/F' \rightarrow E'})$ and induction on the rank, we have $l = l'$ and $E'_i = G'_i$ for all $i$.

The above observation also show the uniqueness of a maximal destabilizing sheaf.

\[\Box\]

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