The cone Moser-Trudinger inequalities and their applications

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Abstract: In this article, we firstly study the cone Moser-Trudinger inequalities and their
best exponents $\alpha_2$ on both bounded and unbounded domains $\mathbb{R}^2_+$. Then, using the cone
Moser-Trudinger inequalities, we study the existence of weak solutions to the nonlinear
equation
\[
\begin{cases}
-\Delta_B u = f(x, u), & \text{in } x \in \text{int}(B), \\
u = 0, & \text{on } \partial B,
\end{cases}
\]
where $\Delta_B$ is Fuchsian type Laplace operator investigated with totally characteristic de-
generacy on the boundary $x_1 = 0$, and the nonlinearity $f$ has the subcritical exponential
growth or the critical exponential growth.

Keywords: Cone Moser-Trudinger inequalities, Mellin transform, mountain pass theo-
rem, weak solution.

1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be an open set, $N \geq 2$. It is well-known that $W^{1,p}_0(\Omega) \subset L^{N/p}_\infty$ if $1 \leq p < N$, and $W^{1,p}_0(\Omega) \subset L^\infty(\Omega)$, if $p > N$. The case $p = N$ is the limit case of these imbeddings and it is known that $W^{1,N}_0(\Omega) \subset L^q(\Omega)$ for $N \leq q < \infty$ and $W^{1,N}_0(\Omega) \not\subset L^\infty(\Omega)$.

Trudinger [35] and Pohozaev [37] found independently that the maximal growth is of
exponential type. More precisely, there exist two positive constants $\alpha$ and $C$ depending
only on $N$ such that
\[
\int_{\Omega} e^{\alpha \left( \frac{|u(x)|}{\|u(x)\|_{\infty}} \right)^{|\Omega|^N}} dx \leq C|\Omega|
\] (1.1)
for $u \in W^{1,N}_0(\Omega) \setminus \{0\}$, where the constants $\alpha, C$ are independent of $u$ and $\Omega$. In order
to prove (1.1), Trudinger in [35] used a combination of the power series expansion of the

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exponential function and sharp multiplicative inequalities

$$|u|_q \leq C(N, q)|\nabla u|_N^{1-\frac{N}{q}}.$$  

There are many types of extensions for the Trudinger-Moser inequality. The first one is to find the best exponents in (1.1). Moser [27] (see also [22]) showed that (1.1) holds for $\alpha \leq \alpha_N$ but not for $\alpha > \alpha_N$, where

$$\alpha_N = N \omega_{N-1}^{-\frac{1}{N}}$$

and $\omega_{N-1}$ is the surface area of the unit sphere in $\mathbb{R}^N$. Moser used symmetrization of functions and reduce (1.1) to one-dimensional inequality. And the reader can be referred to [7, 18, 26, 33] for the attainability of

$$\sup \left\{ \int_{\Omega} e^{\alpha_N \left( \frac{|u(x)|}{|\nabla u(x)|_N} \right)^{\frac{N}{N-1}}} \, dx : u \in W^{1,N}_0(\Omega) \setminus \{0\} \right\}. \quad (1.2)$$

The second direction is to extend Trudinger’s result for unbounded domains and for Sobolev spaces of higher order and fractional order (see [1, 3, 6, 28, 29, 32]). In [2, 28, 29], the following Trudinger Type inequality was studied without best exponents

$$\int_{\mathbb{R}^N} \left( e^{\alpha \left( \frac{|u(x)|}{|\nabla u(x)|_N} \right)^{\frac{N}{N-1}}} - \sum_{j=0}^{N-2} \frac{1}{j!} \left( \alpha \left( \frac{|u(x)|}{|\nabla u(x)|_N} \right)^{\frac{N}{N-1}} \right)^j \right) \, dx \leq C \left( \frac{|u(x)|_N^N}{|\nabla u(x)|_N^N} \right) \quad (1.3)$$

for $u \in W^{1,N}_0(\mathbb{R}^N) \setminus \{0\}$. In [1], the best exponents $\alpha$ in (1.3) was obtained, moreover, by Moser’s idea, a simplified proof for (1.3) was given. With regard to the case of higher order derivatives, since the symmetrization is not available, D. Adams [3] proposed a new idea to find the sharp constants for higher order Moser’s type inequality, that is, to express $u$ as the Riesz potential of its gradient of order $m$, and then apply O’Neil’s result on the rearrangement of convolution functions and use techniques of symmetric decreasing rearrangements.

In [8], the authors proved an affine Moser-Trudinger inequality. The authors of [20] proved the sharp singular affine Moser-Trudinger inequalities on both bounded and unbounded domains in $\mathbb{R}^N$ and they improved Adams type inequality in the spirit of Lions [24].

Another extension is to establish the Trudinger-Moser inequality and the Adams inequality on compact Riemannian manifolds and noncompact Riemannian manifolds (see [23, 36]).

Moser-Trudinger inequalities have played important roles and have been widely used in geometric analysis and PDEs, see for example [9, 16, 15, 21, 34], and references therein. The main purpose of this paper is to study the cone Moser-Trudinger inequalities and their applications. To the best of our knowledge, the related research is rare and

The outline of this paper are as follows. In Section 2 we give some preliminaries, such as the definition of the cone Sobolev spaces and some lemmas which will be used in the later sections. In Section 3, we give the cone Moser-Trudinger inequalities and their proofs. In Section 4, as the applications of the cone Moser-Trudinger inequalities, the existence of multiple solutions to the degenerate elliptic equations with the subcritical
exponential growth or the critical exponential growth will be discussed. Our main results are Theorem 3.1, Theorem 3.2, Theorem 3.3, Theorem 4.3 and Theorem 4.4.

2 Cone Sobolev spaces

In this section we introduce the manifold with conical singularities and the corresponding cone Sobolev spaces.

Let $X$ be a closed, compact, $C^\infty$ manifold. We set $X^\Delta = (\mathbb{R}_+ \times X)/\{\{0\} \times X\}$ as a local model interpreted as a cone with the base $X$. Next, we denote $X^\wedge = \mathbb{R}_+ \times X$ as the corresponding open stretched cone with the base $X$.

A $n$-dimensional manifold $B$ with conical singularities is a topological space with a finite subset $B_0 = \{b_1, \cdots , b_M\} \subset B$ of conical singularities, with the following two properties:

1. $B \setminus B_0$ is a $C^\infty$ manifold.
2. Each $b \in B_0$ has an open neighbourhood $U$ in $B$ such that there is a homeomorphism $\phi : U \rightarrow X^\Delta$ for some closed compact $C^\infty$ manifold $X = X(b)$, and $\phi$ restricts to a diffeomorphism $\phi^1 : U \setminus \{b\} \rightarrow X^\wedge$.

For such a manifold, let $n \geq 2$ and $X \subset S^{n-1}$ be a bounded open set in the unit sphere of $\mathbb{R}^n$. The set $B := \{x \in \mathbb{R}^n \setminus \{0\} : \frac{x}{|x|} \in X\} \cup \{0\}$ is an infinite cone with the base $X$ and the conical point $\{0\}$. Using the polar coordinates, one can get a description of $B \setminus \{0\}$ in the form $X^\wedge = \mathbb{R}_+^* \times X$, which is called the open stretched cone with the base $X$, and $\{0\} \times X$ is the boundary of $X^\wedge$.

Now, we assume that the manifold $B$ is paracompact and of dimension $n$. By this assumption we can define the stretched manifold associated with $B$. Let $\mathbb{B}$ be a $C^\infty$ manifold with compact $C^\infty$ boundary $\partial \mathbb{B} = \bigcup_{x \in B_0} X(x)$ for which there exists a diffeomorphism $B \setminus B_0 = \mathbb{B} \setminus \partial \mathbb{B} = 'int\mathbb{B}$, the restriction of which to $G_1 \setminus B_0 = U_1 \setminus \partial \mathbb{B}$ for an open neighborhood $G_1 \subset B$ near the points of $B_0$ and a collar neighborhood $U_1 \subset \mathbb{B}$ with $U_1 = \bigcup_{x \in B_0} \{[0,1) \times X(x)\}$.

The typical differential operators on a manifold with conical singularities, called Fuchs type, are operators that are in a neighborhood of $x_1 = 0$ of the following form

$$A = x_1^{-m} \sum_{k=0}^{m} a_k(x_1) (-x_1 \partial x_1)^k$$

with $(x_1, x) \in X^\wedge$ and $a_k(x_1) \in C^\infty(\mathbb{R}_+, \text{Diff}^{m-k}(X))$ (see [17, 25, 31]). The differential $x_1 \partial x_1$ in Fuchs type operators provokes us to apply the Mellin transform $M$ (see Definition 2.1).

**Definition 2.1.** Let $u(t) \in C^\infty_0(\mathbb{R}_+)$, $Z \in \mathbb{C}$. The Mellin transform is defined by the formula

$$Mu(z) = \int_0^{+\infty} t^z u(t) \frac{dt}{t},$$

and

$$M : C^\infty_0(\mathbb{R}_+) \rightarrow A(\mathbb{C}),$$

where $A(\mathbb{C})$ denotes the spaces of entire functions.

**Proposition 2.2.** The Mellin transform satisfies the following identities
(1) \( M((-t\partial_t)u)(z) = zM(z) \),
(2) \( M(t^{−p}u)(z) = (Mu)(z−p) \),
(3) \( M((\log t)u)(z) = (\partial_z Mu)(z) \),
(4) \( M(u(t^β))(z) = β^{-1}(Mu)(β^{-1}z) \),

for \( t \in \mathbb{R}_+ \), \( z, p \in \mathbb{C} \), \( β \in \mathbb{R} \setminus \{0\} \), and \( u \in C_0^∞(\mathbb{R}_+) \).

Now let \( Γ_β = \{ z \in \mathbb{C} : \text{Re } z = β \} \). We define the weighted Mellin transform with weight data \( γ \) as follows

\[
M_γ u := Mu|_{Γ_{\frac{1}{2}−γ}} = \int_0^∞ t^{\frac{1}{2}−γ+iτ} u(t) \frac{dt}{t},
\]

and the inverse weighted Mellin transform is defined as

\[
(M_γ^{-1}g)(t) = \frac{1}{2πi} \int_{γ−i∞}^{γ+i∞} t^z g(z) dz.
\]

For \( u(t) \in C_0^∞(\mathbb{R}_+) \), set \( S_γ u(r) = e^{-\left(\frac{1}{2}−γ\right)r} u(e^{-r}) \), then we obtain

\[
(M_γ u) \left( \frac{1}{2} − γ + iτ \right) = (\mathcal{F} S_γ u(τ)), \quad (2.1)
\]

where \( \mathcal{F} \) is the 1-dimensional Fourier transform corresponding to \( t \). Indeed, by changing variables \( t = e^{-r} \) and set \( z = \frac{1}{2} − γ + iτ \in \mathbb{C} \), it is easy to see that

\[
(\mathcal{F} S_γ u)(τ) = \int_{−∞}^{+∞} e^{-iτr} e^{-\left(\frac{1}{2}−γ\right)r} u(e^{-r}) dr = \int_{−∞}^{+∞} e^{-\left(\frac{1}{2}−γ+iτ\right)r} u(e^{-r}) dr
= \int_0^∞ t^z u(t) \frac{dt}{t} = (M_γ u) \left( \frac{1}{2} − γ + iτ \right).
\]

Then, we have the following result.

**Lemma 2.3** ([31]). The operator \( M_γ : C_0^∞(\mathbb{R}_+) \to S(Γ_{\frac{1}{2}−γ}) \) extends by continuity to an isomorphism

\[
M_γ : L_2^r(\mathbb{R}_+) \to L^2(Γ_{\frac{1}{2}−γ})
\]

for all \( γ \in \mathbb{R} \) and \( L_2^r(\mathbb{R}_+) = t^r L_2(\mathbb{R}_+) \), where

\[
|u|_{L_2^r(\mathbb{R}_+)} = (2π)^{-\frac{1}{2}} |M_γ u|_{L^2(Γ_{\frac{1}{2}−γ})}.
\]

**Definition 2.4.** For \( s, γ \in \mathbb{R} \), we denote by \( \mathcal{H}_2^{s, γ}(\mathbb{R}_+^{n+1}) \) the space of all \( u \in D'(\mathbb{R}_+^{n+1}) \) such that

\[
\frac{1}{2πi} \int_{Γ_{\frac{n+1}{2}−γ}} \int_{\mathbb{R}_n} (1 + |z|^2 + |ξ|^2)^s \left| \left( M_{γ−\frac{n+1}{2}, t→z} \mathcal{F}_{x→ξ} u \right) (z, ξ) \right|^2 dzdξ < +∞,
\]

where \( M_{γ−\frac{n+1}{2}} \) is the weighted Mellin transform and \( \mathcal{F}_{x→ξ} \) the \( n \)-dimensional Fourier trans-
form. Naturally, the space $H^s_{2}((R^+\times R^n))$ admits a norm

$$\|u\|_{H^s_{2}(R^+\times R^n)} = \left[ \frac{1}{2\pi i} \int_{\Gamma_{\alpha+1,\gamma}} \int_{R^n} (1 + |z|^2 + |\xi|^2)^s \left| \left( M_{\gamma+\alpha, t \to \gamma} F_{x \to \xi} u \right)(z, \xi) \right|^2 dz d\xi \right]^{\frac{1}{2}}.$$ 

Then we easily obtain the weighted Mellin Sobolev space of integer smoothness.

**Definition 2.5.** Let $L_2(R^{n+1})$ be the space of square integrable functions on $R^{n+1}$, with respect to $dt dx$, and $(t, x) \in R_+ \times R^n$. For $m \in N$, and $\gamma \in R$, we define

$$H^{m,\gamma}_{2}(R^{n+1}) = \{ u \in D' (R^{n+1}) : (t \partial_t)^{\alpha} \partial_x^3 u \in L^{p}(R^{n+1}, dt dx) \},$$

for arbitrary $\alpha \in N, \beta \in N^n$, and $|\alpha| + |\beta| \leq m$. Then $H^{m,\gamma}_{2}(R^{n+1})$ is a Hilbert space with the norm

$$\|u\|_{H^{m,\gamma}_{2}(R^{n+1})} = \sum_{|\alpha| + |\beta| \leq m} \left[ \int_{R_+ \times R^n} |(t \partial_t)^{\alpha} \partial_x^3 u(t, x)|^2 dt dx \right]^{\frac{1}{2}}.$$ 

If we denote by $L_2(R^{n+1})$ the space of square integrable functions with respect to the measure $\frac{dt}{t} dx$, we can write (2.2) as follows:

$$H^{m,\gamma}_{2}(R^{n+1}) = \bigg\{ u \in D' (R^{n+1}) : (t^{\alpha+1}) (t \partial_t)^{\alpha} \partial_x^3 u \in L^{p}(R^{n+1}, \frac{dt}{t} dx) \bigg\},$$

for all $\alpha \in N, \beta \in N^n$, and $|\alpha| + |\beta| \leq m$. Here $m \in N$ is called the smoothness of Sobolev spaces, and $\gamma \in R$ the flatness of $t$-variable. Next, we introduce a map

$$\left( S^{\alpha+1,\gamma}_{2} u \right) (r, x) = e^{-\frac{(t+1)}{2} \gamma} u \left( e^{-r} x \right)$$

for $u(t, x) \in C_0^\infty(R^{n+1})$, which is a continuous map $S^{\alpha+1,\gamma}_{2} : C_0^\infty(R^{n+1}) \to C_0^\infty(R^{n+1})$.

Similar to (2.1), we can extend (2.4) to an isomorphism

$$\left( S^{\alpha+1,\gamma}_{2} : H^{m,\gamma}_{2}(R^{n+1}) \to H^{m}_{2}(R^{n+1}) \right),$$

this isomorphism can also be said the norm $\|u\|_{H^{m,\gamma}_{2}(R^{n+1})}$ is equivalent to the norm $\|S^{\alpha+1,\gamma}_{2} u\|_{H^{m}_{2}(R^{n+1})}$, where $H^{m}_{2}(R^{n+1})$ denotes the distribution space for $(r, x) \in R^{n+1}$ such that

$$H^{m,\gamma}_{2}(R^{n+1}) = \bigg\{ v(r, x) \in D' (R_{n+1}) : \partial_r^\alpha \partial_x^3 v(r, x) \in L^{2}(R^{n+1}, dr dx) \bigg\},$$

for all $\alpha \in N, \beta \in N^n$ and $|\alpha| + |\beta| \leq m$. The readers can be referred more details and information on Fuchs type operators and the weighted Mellin Sobolev spaces in [17, 31].

The space $H^{m,\gamma}_{2}(R^{n+1})$ can be extended to more general cases $H^{m,\gamma}_{p}(R^{n+1})$ for $1 \leq p < +\infty$ and $H^{m,\gamma}_{p}(\mathbb{B})$ (the cone Sobolev spaces on manifolds with conical singularities).

**Definition 2.6.** For $(t, x) \in R_+ \times R^n$, we say that $u(t, x) \in L_{p}(R^{n+1}, \frac{dt}{t} dx)$ if

$$|u|_{L_{p}} = \left[ \int_{R_+ \times R^n} t^{n+1} |u(t, x)|^p \frac{dt}{t} dx \right]^{\frac{1}{p}} < +\infty.$$
Furthermore, the weighted $L^p$-spaces with weight data $\gamma \in \mathbb{R}$ is denoted by $L^\gamma_p(\mathbb{R}^{n+1}, \frac{dt}{t} dx)$, that is, if $u(t, x) \in L^\gamma_p(\mathbb{R}^{n+1}, \frac{dt}{t} dx)$, then $t^{-\gamma}u(t, x) \in L^p_p(\mathbb{R}^{n+1}, \frac{dt}{t} dx)$, and

$$|u|_{L^\gamma_p} = \left[ \int_{\mathbb{R}_+ \times \mathbb{R}^n} t^{n+1}|t^{-\gamma}u(t, x)|^{\frac{p}{\gamma}t} \frac{dt}{t} dx \right]^{\frac{\gamma}{p}} < +\infty.$$ 

The weighted Sobolev space for all $1 \leq p < +\infty$ can be defined as

**Definition 2.7.** For $m \in \mathbb{N}$, the spaces

$$\mathcal{H}_p^{m, \gamma}(\mathbb{R}^{n+1}) = \left\{ u \in D'(\mathbb{R}^{n+1}) : t^{-\gamma}(t\partial_t)^\alpha \partial_x^\beta u \in L^p_p(\mathbb{R}^{n+1}, \frac{dt}{t} dx) \right\}, \quad (2.6)$$

for all $\alpha \in \mathbb{N}$, $\beta \in \mathbb{N}^n$ and $|\alpha| + |\beta| \leq m$. In other words, if $u(t, x) \in \mathcal{H}_p^{m, \gamma}(\mathbb{R}^{n+1})$, then $(t\partial_t)^\alpha \partial_x^\beta u \in L^\gamma_p(\mathbb{R}^{n+1}, \frac{dt}{t} dx)$.

It is easy to see that $\mathcal{H}_p^{m, \gamma}(\mathbb{R}^{n+1})$ is a Banach space with norm

$$||u||_{\mathcal{H}_p^{m, \gamma}(\mathbb{R}^{n+1})} = \sum_{|\alpha| + |\beta| \leq m} \left[ \int_{\mathbb{R}_+ \times \mathbb{R}^n} t^{n+1}|t^{-\gamma}(t\partial_t)^\alpha \partial_x^\beta u(t, x)|^{\frac{p}{\gamma}t} \frac{dt}{t} dx \right]^{\frac{\gamma}{p}}.$$

Similarly (see [14]), the weighted Sobolev spaces $\mathcal{H}_p^{m, \gamma}(X^\gamma)$ with $1 \leq p < \infty$ can be defined on manifolds with conical singularities. Let $X$ be a closed compact $C^\infty$ manifold, and $U = \{U_1, \cdots, U_N\}$ an open covering of $X$ by coordinate neighborhoods. If we fix a subordinate partition of unity $\{\phi_1, \cdots, \phi_N\}$ and charts $\chi_j : U_j \rightarrow \mathbb{R}^n$, $j = 1, 2, \cdots, N$, then $u \in \mathcal{H}_p^{m, \gamma}(X^\gamma)$ if and only if $u \in D'(X^\gamma)$ with the norm

$$||u||_{\mathcal{H}_p^{m, \gamma}(X^\gamma)} = \left[ \sum_{j=1}^N \left\| (1 \times \chi_j^*)^{-1} \phi_ju \right\|_{\mathcal{H}_p^{m, \gamma}(\mathbb{R}^{n+1})}^p \right]^{\frac{1}{p}} < +\infty,$$

where $1 \times \chi_j^* : C^\infty_0(\mathbb{R}_+ \times \mathbb{R}^n) \rightarrow C^\infty_0(\mathbb{R}_+ \times U_j)$ is the pull-back function with respect to $1 \times \chi_j : \mathbb{R}_+ \times U_j \rightarrow \mathbb{R}_+ \times \mathbb{R}^n$. We denote the closure of $C^\infty_0(X^\gamma)$ with respect to the norm $\| \cdot \|_{\mathcal{H}_p^{m, \gamma}(X^\gamma)}$ by $\mathcal{H}_p^{m, \gamma}(X^\gamma)$.

**Lemma 2.8 (See[30]).** For all $m \in \mathbb{N}$, $\gamma \in \mathbb{R}$, we have $\mathcal{H}_p^{m, \gamma}(X^\gamma) \subset W^{m,p}_{loc}(X^\gamma)$, where $W^{m,p}_{loc}(X^\gamma)$ denotes the subspace of all $u \in D'(X^\gamma)$ such that $\phi u \in W^{m,p}(X^\gamma)$ for each $\phi \in C^\infty_0(X^\gamma)$.

Let $\mathbb{B}$ be the stretched manifold of $B$, we will always denote $\omega(t) \in C^\infty(\mathbb{B})$ as a real-valued cut-off function which equals 1 near $\{0\} \times \partial B$.

**Definition 2.9.** Let $\mathbb{B}$ be the stretched manifold to a manifold $B$ with conical singularities. Then $\mathcal{H}_p^{m, \gamma}(\mathbb{B})$ for $m \in \mathbb{N}, \gamma \in \mathbb{R}$ denotes the subspace of all $u \in W^{m,p}_{loc}(\text{int}(\mathbb{B}))$ such that

$$\mathcal{H}_p^{m, \gamma}(\mathbb{B}) = \{ u \in W^{m,p}_{loc}(\text{int}(\mathbb{B})) : \omega u \in \mathcal{H}_p^{m, \gamma}(X^\gamma) \}$$

for any cut-off function $\omega$, supported by a collar neighbourhood of $[0, 1) \times \partial \mathbb{B}$, where $\text{int}(\mathbb{B}) = \mathbb{B} \setminus \partial \mathbb{B}$. Moreover, the subspace $\mathcal{H}_p^{m, \gamma}(\mathbb{B})$ is defined as follows:

$$\mathcal{H}_p^{m, \gamma}(\mathbb{B}) = [\omega]\mathcal{H}_p^{m, \gamma}(X^\gamma) + [1-\omega]W^{m,p}_0(\text{int}(\mathbb{B}))$$
where $W_0^{m,p}(\text{int}(\mathbb{B}))$ denotes the closure of $C_0^\infty(\text{int}(\mathbb{B}))$ in Sobolev spaces $W^{m,p}(\tilde{X})$ when $\tilde{X}$ is a closed compact $C^\infty$ manifold of dimension $n + 1$ that contains $\mathbb{B}$ as a submanifold with boundary.

Lemma 2.10 (See[30]). We have the following properties:

1. $\mathcal{H}_p^{m,\gamma}(\mathbb{B})$ is Banach space for $1 \leq p < +\infty$, and is Hilbert space for $p = 2$.
2. $L_p^\gamma(\mathbb{B}) = \mathcal{H}_p^{0,\gamma}(\mathbb{B})$.
3. $L_p^\gamma(\mathbb{B}) = \mathcal{H}_p^{0,0}(\mathbb{B})$.
4. $t^{2\gamma}\mathcal{H}_p^{m,\gamma}(\mathbb{B}) = \mathcal{H}_p^{m,\gamma+\gamma_0}(\mathbb{B})$.
5. The embedding $\mathcal{H}_p^{m,\gamma}(\mathbb{B}) \hookrightarrow \mathcal{H}_p^{m',\gamma'}(\mathbb{B})$ is continuous if $m \geq m'$, $\gamma \geq \gamma'$ and is compact if $m > m'$, $\gamma > \gamma'$.

3 Cone Moser-Trudinger inequalities

Let $P : \mathbb{R}_+ \to \mathbb{R}_+$ be an $N$-function, that is, $P$ is continuous, convex, with $P(t) > 0$ for $t > 0$, $\frac{P(0)}{t} \to 0$ as $t \to 0$, and $\frac{P(t)}{t} \to \infty$ as $t \to \infty$. Equivalently, $P$ admits the representation $P(s) = \int_s^\infty p(\tau)d\tau$ where $p : \mathbb{R}_+ \to \mathbb{R}_+$ is non-decreasing, right-continuous, with $p(0) = 0$, $p(t) > 0$ for $t > 0$, and $p(t) \to \infty$ as $t \to \infty$.

The $N$-function $\tilde{P}$ conjugate to $P$ is defined by $\tilde{P}(t) = \int_0^t \tilde{p}(\tau)d\tau$, where $\tilde{p} : \mathbb{R}_+ \to \mathbb{R}_+$ is given by $\tilde{p}(t) = \sup\{s : p(s) \leq t\}$ (see [2]). It is easy to see that these $N$-functions can be extended into even functions on all $\mathbb{R}$.

The $N$-function $P$ is said to satisfy the $\Delta_2$ condition if, for some $k > 0$,

$$P(2t) \leq kP(t), \quad \forall t > 0.$$ (3.1)

When (3.1) holds only for $t$ at least some $t_0 > 0$, then $P$ is said to satisfy the $\Delta_2$ condition near infinity.

Furthermore, the weighted Orlicz spaces with weight data $\gamma \in \mathbb{R}$ is denoted by $L_p^\gamma(\mathbb{R}^{n+1}_+; \frac{dt}{t})$, that is, if $u(t, x) \in L_p^\gamma(\mathbb{R}^{n+1}_+; \frac{dt}{t})$, then $t^{-\gamma}u(t, x) \in L_p(\mathbb{R}^{n+1}_+; \frac{dt}{t})$, and

$$\left[\int_{\mathbb{R}_+} \int_{\mathbb{R}^n} t^{n+1}P(\|t^{-\gamma}u(t, x)\|) \frac{dt}{t} dx\right] < +\infty.$$

We easily know that $L_p^\gamma(\mathbb{R}^{n+1}_+; \frac{dt}{t})$ is a Banach space under the norm

$$|u|_{L_p^\gamma} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}_+} \int_{\mathbb{R}^n} t^{n+1}P\left(\frac{|t^{-\gamma}u(t, x)|}{\lambda}\right) \frac{dt}{t} dx < +\infty \right\}.$$

$L_p^\gamma(\mathbb{B}, \frac{dt}{t})$ is a convex subset of $L_p^\gamma(\mathbb{R}^{n+1}_+; \frac{dt}{t})$.

The closure in $L_p^\gamma(\mathbb{B}, \frac{dt}{t})$ of the set of bounded measurable functions with compact support in $\mathbb{B}$ is denoted by $E_P(\mathbb{B})$. The equality $E_P(\mathbb{B}) = L_p^\gamma(\mathbb{B}, \frac{dt}{t})$ holds if and only if $P(s)$ satisfies the $\Delta_2$ condition, for all $s$ or for $s$ large according to whether $\mathbb{B}$ has infinite measure or not. The dual of $E_P(\mathbb{B})$ can be identified with $L_{\tilde{P}}(\mathbb{B}, \frac{dt}{t})$ by means of the pairing $\int_{\mathbb{B}} u(x)v(x)\frac{dt}{t} dx$, and the dual norm on $L_{\tilde{P}}(\mathbb{B}, \frac{dt}{t})$ is equivalent to $|\cdot|_{L_{\tilde{P}}}.

The space $L_{\tilde{P}}(\mathbb{B}, \frac{dt}{t})$ is reflexive if and only if $P$ and $\tilde{P}$ satisfy the $\Delta_2$ condition (near infinity only if $\mathbb{B}$ has finite measure). $P \ll M$ means that $P$ grows essentially less rapidly than $M$, that is, for each $\varepsilon > 0$, $P(t)/(M(\varepsilon t)) \to 0$ as $t \to \infty$. This is the case if and only if $M^{-1}(t)/P^{-1}(t) \to 0$ as $t \to \infty$. Therefore, we have the continuous imbedding $L_M(\mathcal{B}) \subset E_P(\mathcal{B})$ when $\mathcal{B}$ has finite measure.
Let $B$ be a $n$-dimensional compact manifold with conical singularity at the point $b \in \partial B$, and $\mathbb{B}$ be the stretched manifold of $B$, i.e. without loss of generality, we suppose $\mathbb{B} = [0, 1) \times (-1, 1)$, $X$ is a closed compact manifold of dimension $n - 1$, $\partial \mathbb{B} = 0 \times X$.

Next we denote $\| \cdot \|_{L^p}$, $\| \cdot \|_{H^m, p(\mathbb{B})}$ by $\| \cdot \|_p$, $\| \cdot \|_{m,p}$, respectively.

**Theorem 3.1.** Let $B \in \mathbb{R}^2_+$, $u \in H^{1, 1}_2(\mathbb{B})$ and $|\nabla_B u|^2 \leq 1$. Then there exists a constant $C > 0$ such that

$$
\int_{\mathbb{B}} e^{\alpha u^2} \frac{dx_1}{x_1} dx_2 \leq C(\mathbb{B}),
$$

where $\alpha \leq \alpha_2 = 2\omega_1 = 4\pi$, $\nabla_B = (x_1 \partial_{x_1}, \partial_{x_2})$, and $\omega_1 = 2\pi$ is the perimeter of the unit sphere.

**Proof.** Let $B_r(1, 0)$ be a ball in $\mathbb{R}^2_+$ with radius $R$, that is, $B_R(1, 0) = \{(x_1, x_2) \in \mathbb{R}^2_+ : |x|^2 = |\ln x_1|^2 + x_2^2 \leq R^2\}$.

We may assume that $u \geq 0$ since we can replace $u$ by $|u|$ without increasing the integral of the gradient. Also, it suffices to prove the statement for a set of functions $u$ which is dense in the unit ball of $H^{1, 1}_2(\mathbb{B})$. For example, we may assume that $u$ has compact support and is in $C^\infty_0(\mathbb{B})$, or instead of the last requirement, is piecewise linear.

We use symmetrization: with $u(x) > 0$ we associate a function $u^*(x)$ depending on $|x|$ only by the requirement

$$
|\{x : u^* > \rho\}| = |\{x \in \mathbb{B} : u > \rho \text{ for each } \rho \geq 0\}|.
$$

Clearly, $u^*$ is a decreasing function of $|x|$ which is 0 for $|x| > R$ where $R$ is the radius of the sphere whose volume is

$$
|(B_R(0, 1))| = \int_{|x| \leq R} \frac{dx_1}{x_1} dx_2.
$$

We define $\mathbb{B}^*$ as $|x| \leq R$. Similar to the Laplace operator, we can build heat kernel theory for the operator $\Delta_{\mathbb{B}}$. Hence we easily obtain the following Pólya-Szegö inequality

$$
|\nabla_B u^*|^2 \leq |\nabla_B v|^2,
$$

while,

$$
\int_{\mathbb{B}^*} e^{\alpha(u^*)^p} \frac{dx_1}{x_1} dx_2 = \int_{\mathbb{B}} e^{\alpha u^*} \frac{dx_1}{x_1} dx_2.
$$

This reduces the problem at once to a one dimensional one. For convenience we introduce the variable $t$ by

$$
\frac{|x|^2}{R^2} = e^{-t},
$$

and set

$$
w(t) = (2\omega_1)^{\frac{1}{2}} u^*(x).
$$

Since spherical coordinate integral formula still holds in this sense, $w$ is monotone increasing and

$$
\int_0^\infty w^2 dt = \int_{\mathbb{B}^*} |\nabla_B u^*|^2 \frac{dx_1}{x_1} dx_2,
$$

$$
\int_0^\infty e^{\beta w^p - t} dt = \frac{1}{|\mathbb{B}^*|} \int_{\mathbb{B}^*} e^{\alpha(u^*)^p} \frac{dx_1}{x_1} dx_2,
$$

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where $\beta = \alpha / \alpha_2$. Thus it is sufficient to prove:

If $q \geq 2$ and $w(t)$ is a $C^1$-function and $0 \leq t < \infty$ satisfying

$$w(0) = 0, \dot{w} \geq 0, \int_0^\infty \dot{w}^q(t)dt \leq 1,$$

then

$$\int_0^\infty e^{\beta w^p} - t dt \leq C_1, \quad \text{if } \beta \leq 1, \frac{1}{p} + \frac{1}{q} = 1,$$

where the constant $C_1$ depends on $q$ only.

From Hölder’s inequality

$$w(t) = \int_0^t \dot{w}(t)dt \leq t^{\frac{1}{p}} \left( \int_0^t \dot{w}^q(t)dt \right)^{\frac{1}{q}} \leq t^{\frac{1}{p}};$$

it is clear that

$$\int_0^\infty e^{\beta w^p} - t dt \leq \int_0^\infty e^{(\beta - 1) t} = \frac{1}{1 - \beta}, \quad \text{for } \beta < 1. \quad (3.4)$$

But for $\beta = 1$ we have to proceed more carefully. The same simple device allows to show that the integral in $(3.3)$ exists for any positive $\beta$. Indeed, given any $\varepsilon > 0$ there exists $T = T(\varepsilon)$ such that

$$\int_T^\infty \dot{w}^q dt < \varepsilon,$$

from which we conclude, again by Hölder inequality, that

$$w(t) \leq w(T) + \varepsilon^{\frac{1}{q}}(t - T)^{\frac{1}{p}}, \quad \text{for } t \geq T,$$

hence

$$\lim_{t \to \infty} \frac{w(t)}{t^{\frac{1}{p}}} = 0.$$
Remark: Note that the inequality (3.5) is scale invariant, that is, if for $r > 0$ we define the rescaled function

$$u_r(x_1, x_2) = u(rx_1, rx_2), \ (x_1, x_2) \in \mathbb{R}^2_+,$$

then

$$\int_{\mathbb{R}^2_+} u^2(x_1, rx_2) \frac{dx_1}{x_1} dx_2 = \frac{1}{r^2} \int_{\mathbb{R}^2_+} u^2(y_1, y_2) \frac{dy_1}{y_1} dy_2,$$

$$\int_{\mathbb{R}^2_+} |\nabla_B u_r|^2 \frac{dx_1}{x_1} dx_2 = \int_{\mathbb{R}^2_+} |\nabla_B u|^2 \frac{dx_1}{x_1} dx_2.$$

and

$$\int_{\mathbb{R}^2_+} \left( e^{\frac{|u(x)|^2}{|\nabla_B u(x)|^2}} - 1 \right) \frac{dx_1}{x_1} dx_2 = \frac{1}{r^2} \int_{\mathbb{R}^2_+} \left( e^{\frac{|u_r(x)|^2}{|\nabla_B u_r(x)|^2}} - 1 \right) \frac{dx_1}{x_1} dx_2.$$

Proof. Fix $u \in \mathcal{H}_2^{1,1}(\mathbb{R}^2_+) \setminus \{0\}$ and define

$$v(x) = \frac{|u(x)|}{|\nabla_B u(x)|}.$$

Then $\|\nabla v\|_2 = 1$ and (3.5) reduces to

$$\int_{\mathbb{R}^2_+} e^{\alpha^2(x)} \frac{dx_1}{x_1} dx_2 \leq C_\alpha |v(x)|^2_2. \tag{3.7}$$

Let $v^*$ be the spherically symmetric rearrangement of $v$. Then $v^*(x) = w(|x|)$, where $w$ is nonnegative, decreasing, and locally absolutely continuous. Hence,

$$\nabla_B v^* = w(|x|) \frac{x}{|x|}.$$

Similar to the Laplace operator, we can build heat kernel theory for the operator $\Delta_B$. Hence we easily obtain the following Pólya-Szegő inequality

$$|\nabla_B v^*|^2_2 \leq |\nabla_B v|^2_2.$$

Now using spherical coordinates, if follows that

$$\omega_1 \int_0^\infty |w'(r)|^2 r dr = |\nabla_B v^*|^2_2 \leq |\nabla_B v|^2_2 = 1. \tag{3.8}$$

$$\int_{\mathbb{R}^2_+} (v^*)^2 \frac{dx_1}{x_1} dx_2 = \int_{\mathbb{R}^2_+} v^2 \frac{dx_1}{x_1} dx_2. \tag{3.9}$$

Define

$$r_0 := \inf\{r \geq 0 : w(r) \leq 1\}. \tag{3.10}$$

Since $w(r) \to 0$ as $r \to \infty$, we have that $r_0$ must be finite.
Using spherical coordinates, we have that
\[
\int_{\mathbb{R}^2} e^{\alpha^2} \frac{dx_1}{x_1} \frac{dx_2}{x_2} = \int_{\mathbb{R}^2} e^{\alpha v^2} \frac{dx_1}{x_1} \frac{dx_2}{x_2} = \omega_1 \int_0^\infty e^{\alpha w^2(r)} r \, dr
\]
\[
= \omega_1 \int_0^{r_0} e^{\alpha w^2(r)} r \, dr + \omega_1 \int_{r_0}^\infty e^{\alpha w^2(r)} r \, dr =: I + II. \tag{3.11}
\]

To estimate \(I\), it is enough to consider the case that \(r_0 > 0\), so that \(w(r_0) = 1\) by (3.10). Since \(w\) is locally absolutely continuous, by the fundamental theorem of calculus, Hölder’s inequality, (3.8) and (3.10), for \(0 < r < r_0\), we have that
\[
w(r) = w(r_0) - \int_r^{r_0} w'(\tau) d\tau \leq 1 + \int_r^{r_0} |w'(\tau)| \frac{\tau^2}{\tau^2} d\tau
\]
\[
\leq 1 + \left( \int_0^{\infty} |w'(\tau)|^2 \, d\tau \right)^{\frac{1}{2}} \left( \ln \left( \frac{r_0}{r} \right) \right)^{\frac{1}{2}}
\]
\[
\leq 1 + \omega_1 \left( \ln \left( \frac{r_0}{r} \right) \right)^{\frac{1}{2}}. \tag{3.12}
\]
By the convexity of the function \(s^2\), for every \(\varepsilon > 0\) we may find a constant \(C_\varepsilon > 0\) such that
\[(1 + s)^2 \leq (1 + \varepsilon) s^2 + C_\varepsilon\]
for all \(s \geq 0\). Hence,
\[w^2(r) \leq (1 + \varepsilon) \omega_1^{-1} \ln \left( \frac{r_0}{r} \right) + C_\varepsilon \tag{3.13}\]
for all \(0 < r < r_0\). Since \(0 < \alpha < \alpha_2\), we may take \(\varepsilon > 0\) so small that
\[\alpha(1 + \varepsilon) < \alpha_2 = 2\omega_1,\]
and so \(\alpha(1 + \varepsilon) \beta^{-1} < 2\). Hence, by (3.13),
\[
I \leq \omega_1 e^{\alpha C_\varepsilon} \int_0^{r_0} e^{\alpha(1+\varepsilon) \omega_1^{-1} \ln \left( \frac{r_0}{r} \right)} r \, dr
\]
\[
= \omega_1 e^{\alpha C_\varepsilon} r_0^{\alpha(1+\varepsilon) \beta^{-1}} \int_0^{r_0} r^{1-\gamma(1+\varepsilon) \omega_1^{-1}} \, dr
\]
\[
= \frac{\omega_1 e^{\alpha C_\varepsilon} r_0^{\alpha(1+\varepsilon) \beta^{-1}}}{1 - \gamma(1+\varepsilon) \omega_1^{-1}} \omega_1 r_0^2 =: C_1(2, \alpha) r_0^2. \tag{3.14}
\]
On the other hand, by the Lebesgue monotone convergence theorem and the fact that \(w(r) < 1\) for all \(r > r_0\), by (3.10), we have that
\[
II \leq \omega_1 \sum_{n=1}^{\infty} \frac{\alpha^n}{n!} \int_{r_0}^{\infty} w^{2n}(r) \, dr
\]
\[
\leq \omega_1 \sum_{n=1}^{\infty} \frac{\alpha^n}{n!} \int_{r_0}^{\infty} w^2(r) \, dr
\]
\[ = \omega_1 e^{\alpha} \int_{r_0}^{\infty} w^2(r) dr. \quad (3.15) \]

Combining this estimate with (3.11) and (3.14), we get
\[ \int_{\mathbb{R}_+^2} e^{\alpha w} \frac{dx_1}{x_1} dx_2 \leq C_1 (2, \alpha) r_0^2 + \omega_1 e^{\alpha} \int_{r_0}^{\infty} w^2(r) dr. \quad (3.16) \]

Using spherical coordinates and (3.9), one has that
\[ e^{\alpha} \omega_1 \int_{r_0}^{\infty} w^2(r) dr \leq \int_{\mathbb{R}_+^2} (v^*)^2 \frac{dx_1}{x_1} dx_2 = \int_{\mathbb{R}_+^2} v^2 dx_1 dx_2. \quad (3.17) \]

Thus, to obtain (3.7), it remains to estimate in the case that \( r_0 > 0 \). By (3.10) and the fact that \( w \) is decreasing, we have that \( w(r) > w(r_0) = 1 \) if and only if \( 0 < r < r_0 \). Hence,
\[ \{ x \in \mathbb{R}_+^2 : v^*(x) > 1 \} = \{ x \in \mathbb{R}_+^2 : w(|x|) > 1 \} = B_{r_0}(0, 1). \]

Furthermore, we have
\[ \alpha_2 r_0^2 = \left| \{ x \in \mathbb{R}_+^2 : v^*(x) > 1 \} \right| = \left| \{ x \in \mathbb{R}_+^2 : v(|x|) > 1 \} \right| \leq \int_{\{ v > 1 \}} v^2 \frac{dx_1}{x_1} dx_2, \quad (3.18) \]
and so we have proved (3.7), and in turn, (3.5).

By (3.5), the number \( s = |\nabla u|^2 \) is admissible in the definition of \( |u|_A \), and so (3.6) follows.

**Theorem 3.3.** For \( \alpha \geq \alpha_2 \), there exists a sequence \( \{ u_k(x) \} \subset H_{\alpha}^{1,1}(\mathbb{R}_+^2) \) such that \( |\nabla u|^2 = 1 \) and
\[ \frac{1}{|u_k|^2} \int_{\mathbb{R}_+^2} \left( e^{\alpha \frac{|u_k(x)|^2}{|\nabla u_k(x)|^2}} - 1 \right) \frac{dx_1}{x_1} dx_2 \geq \frac{1}{|u_k|^2} \int_{\mathbb{R}_+^2} \left( e^{\alpha \frac{|u_k(x)|^2}{|\nabla u_k(x)|^2}} - 1 \right) \frac{dx_1}{x_1} dx_2 \rightarrow \infty, \quad (3.19) \]
as \( k \rightarrow \infty \).

**Proof.** We shall take similar argument in the proof of Theorem 3.2. It suffices to find a sequence of functions \( w(r) = u_k(|x|) \) which satisfies
\[ \omega_1 \int_0^{\infty} |w_k'(r)|^2 r dr = |\nabla u_k|^2 = 1, \quad (3.20) \]
and
\[ \int_{\mathbb{R}_+^2} u_k^2(|x|) \frac{dx_1}{x_1} dx_2 = \omega_1 \int_0^{\infty} w_k^2(r) r dr \rightarrow 0, \quad (3.21) \]
\[ \int_{\mathbb{R}_+^2} e^{\alpha u_k^2(|x|)} \frac{dx_1}{x_1} dx_2 = \omega_1 \int_0^{\infty} e^{\alpha w_k^2(r)} r dr \geq \frac{1}{2}, \quad (3.22) \]
Here we give an example of $u_k(r)$ explicitly. We set

$$u_k(r) = \begin{cases} 
0, & \text{if } r \geq 0, \\
-2 \ln r \frac{k^{-\frac{1}{2}}}{\sqrt{2\omega_1}}, & \text{if } e^{-\frac{k}{2}} < r \leq 1, \\
\frac{1}{\sqrt{2\omega_1}k^{\frac{1}{2}}}, & \text{if } 0 < r \leq e^{-\frac{k}{2}}.
\end{cases}$$

(3.23)

It is easily see that $u_k$ satisfies (3.20)–(3.22).

\[\square\]

## 4 Nonlinear Dirichlet boundary value problems

In this section, we consider the following Dirichlet boundary value problems

$$\begin{cases} 
-\Delta_B u = f(x, u), & \text{in } x \in \text{int}(B), \\
u = 0, & \text{on } \partial B,
\end{cases}$$

(4.1)

where, $-\Delta_B = (x_1 \partial_{x_1})^2 + (\partial_{x_2})^2$, $f$ is a continuous real function and satisfies the following assumptions:

(\textit{f}_1) $f \in C(\mathbb{B} \times \mathbb{R})$ with $f(x, 0) = 0$ and $f(x, t)$ has subcritical exponential growth on $\mathbb{B}$, i.e.,

$$\lim_{t \to +\infty} \frac{|f(x, t)|}{e^{\alpha t^2}} = 0, \text{ uniformly on } x \in \mathbb{B} \text{ for all } \alpha > 0,$$

(\textit{f}_1') $f \in C(\overline{\mathbb{B}} \times \mathbb{R})$ with $f(x, 0) = 0$ and $f(x, t)$ critical exponential growth on $\mathbb{B}$, i.e, there exists $\alpha_0 > 0$ such that

$$\lim_{t \to +\infty} \frac{|f(x, t)|}{e^{\alpha t^2}} = 0, \text{ uniformly on } x \in \mathbb{B} \text{ for all } \alpha > \alpha_0,$$

and

$$\lim_{t \to +\infty} \frac{|f(x, t)|}{e^{\alpha t^2}} = +\infty, \text{ uniformly on } x \in \mathbb{B} \text{ for all } \alpha < \alpha_0.$$  

(\textit{f}_2) $\lim_{|t| \to \infty} \frac{F(x, t)}{|t|^2} = +\infty, \text{ uniformly on } x \in \mathbb{B}.$

(\textit{f}_3) there exists $\theta \geq 1$ such that $\theta F(x, t) \geq F(x, st)$ for $(x, t) \in \mathbb{B} \times \mathbb{R}$ and $s \in [0, 1]$,

where, $F(x, t) := \int_0^t f(x, s)ds$, $\mathcal{F}(x, t) := f(x, t)t - 2F(x, t)$,

(\textit{f}_4) $\limsup_{t \to 0^+} \frac{|2F(x, t)|}{|t|^2} < \lambda_1$, uniformly on $x \in \mathbb{B}$.

where $\lambda_1$ is the first eigenvalue of $-\Delta_{\mathbb{B}}$ with Dirichlet problem (see [12]).

We define the functional

$$I(u) = \frac{1}{2} \int_{\mathbb{B}} |\nabla_B u|^2 \frac{dx_1}{x_1} dx_2 - \int_{\mathbb{B}} F(x, u) \frac{dx_1}{x_1} dx_2, \ u \in H_{2,0}^{1,1}(\mathbb{B}).$$
It is easy to check that \( I \in C^1(\mathcal{H}_{2,0}^{1,1}(\mathbb{R}), \mathbb{R}) \), and the critical point of \( I \) are precisely the weak solutions of problem (4.1). We will prove the existence of such critical points by the Mountain Pass Theorem. Recently, there are some interesting results about nonlinear differential equations on manifolds with conical singularities (see [4, 10, 11, 12, 13]).

**Definition 4.1.** Let \((X, \| \cdot \|_X)\) be a reflexive Banach space with its dual space \((X^*, \| \cdot \|_{X^*})\) and \(I \in C^1(X, \mathbb{R})\). For \(c \in \mathbb{R}\), we say that \(I\) satisfies the \((C)_c\) condition if for any sequence \(\{x_n\} \subset X\) with

\[
I(x_n) \to c, \quad (1 + \|x_n\|)\|I'(x_n)\| \to 0 \quad \text{in} \quad X^*,
\]

there is a subsequence \(\{x_{n_k}\}\) such that \(\{x_{n_k}\}\) converges strongly in \(X\).

**Proposition 4.2** (See [5], Mountain Pass Theorem). Let \((X, \| \cdot \|_X)\) be a reflexive Banach space, \(I \in C^1(X, \mathbb{R})\) satisfies the \((C)_c\) condition for any \(c \in \mathbb{R}\), \(I(0) = 0\) and

1. There are constants \(\rho, \alpha > 0\) such that \(I|_{\partial B_\rho} \geq \alpha\);
2. There exists \(e \in X \setminus B_\rho\) such that \(I(e) \leq 0\).

Then \(c = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} I(\gamma(t)) \geq \alpha\) is a critical point of \(I\) where
\[
\Gamma = \{ \gamma \in C^0([0, 1], X), \gamma(0) = 0, \gamma(1) = e \}.
\]

Next we denote \(\| \cdot \|_{\mathcal{H}_{2,0}^{1,1}(\mathbb{R})}\) by \(\| \cdot \|\), and we can prove the following results:

**Theorem 4.3.** Assume that \((f_1)\)–\((f_4)\) are satisfied, then problem (4.1) has a nontrivial solution in \(\mathcal{H}_{2,0}^{1,1}(\mathbb{R})\).

**Theorem 4.4.** Assume that \((f'_1)\), \((f'_2)\)–\((f_4)\) and

\[
(f_5) \lim_{t \to +\infty} f(x, t) e^{-\alpha_0 t^2} \geq \beta > \left( \frac{2}{d} \right)^2 \frac{1}{M\alpha_0}, \quad \text{uniformly in} \ (x, t) \quad \text{where} \ d \ \text{is the inner radius of} \ \mathbb{B}, \ i.e. \ d := \text{radius of the largest open ball} \subset \mathbb{B},
\]

\[
M = \lim_{n \to \infty} n \int_0^1 e^{\alpha_0 (t^2 - t)} dt \ (\geq 2),
\]

and

\[
(f_6) \ f \ \text{is class} \ (A_0), \ i.e. \ for \ any \ \{u_n\} \ \text{in} \ \mathcal{H}_{2,0}^{1,1}(\mathbb{B}), \ if \ \left\{ \begin{array}{l}
u_n \to 0 \quad \text{in} \ \mathcal{H}_{2,0}^{1,1}(\mathbb{B}), \\
f(x, u_n) \to 0 \quad \text{in} \ L_1^1(\mathbb{B}) \end{array} \right., \ \text{then}
\]

\[
F(x, u_n) \to 0 \ \text{in} \ L_1^1(\mathbb{B}) \ \text{(up to a subsequence),}
\]

are satisfied, then problem (4.1) has a nontrivial solution in \(\mathcal{H}_{2,0}^{1,1}(\mathbb{B})\).

The following lemmas will be used for proving our problems.

**Lemma 4.5.** Let \(f\) satisfy \((f_2)\). Then \(I(tu) \to -\infty\) as \(t \to \infty\) for all nonnegative function \(u \in \mathcal{H}_{2,0}^{1,1}(\mathbb{B}) \setminus \{0\}\).

**Proof.** By the condition \((f_2)\), there exist constants \(C_1, C_2\) such that

\[
F(x, t) \geq C_1 |t|^\theta - C_2.
\]

Then

\[
I(tu) \leq \frac{t^2}{2} \|u\|^2 - C_1 t^\theta \int_{\mathbb{B}} |u|^\theta \frac{dx_1}{x_1} dx_2 + C_2
\]
By Hölder’s inequality and the cone Moser-Trudinger embedding, we have
\[
\int_{\mathbb{B}} |u|^q \frac{dx_1}{x_1} dx_2 \leq \frac{t^2}{2} \left( \|u\|^2 - C_1 \int_{\mathbb{B}} |u|^q \frac{dx_1}{x_1} dx_2 \right) + C_2. \tag{4.2}
\]

Now, choose \( M > \frac{\|u\|^2}{2\|u\|^2} \), we have \( J(tu) \to \infty \) as \( t \to \infty \), so \( I \) satisfies (ii) of Proposition 4.2.

\[\square\]

**Lemma 4.6.** Let \( f \) satisfy \( (f_1) \) and \( (f_4) \). Then there exists \( \delta, \rho > 0 \) such that
\[
I(u) \geq \delta, \quad \text{if } \|u\| = \rho.
\]

**Proof.** Using \( (f_1) \) and \( (f_4) \), there exists \( k, \tau > 0 \) and \( q > 2 \) such that
\[
F(x, s) \leq \frac{1}{2} (\lambda_1 - \tau) |s|^2 + C |s|^q e^{ks^2}, \quad \text{for all } (x, s) \in \mathbb{B} \times \mathbb{R}.
\]

By Hölder’s inequality and the cone Moser-Trudinger embedding, we have
\[
\int_{\mathbb{B}} |u|^q e^{ks^2} \frac{dx_1}{x_1} dx_2 \leq \left( \int_{\mathbb{B}} e^{kr\|u\|^2} \frac{dx_1}{x_1} dx_2 \right)^{\frac{1}{r}} \cdot \left( \int_{\mathbb{B}} |u|^q \frac{dx_1}{x_1} dx_2 \right)^{\frac{1}{q}} \leq C \left( \int_{\mathbb{B}} |u|^q \frac{dx_1}{x_1} dx_2 \right)^{\frac{1}{q}}, \tag{4.3}
\]
if \( r > 1 \) sufficiently close to 1 and \( \|u\| \leq \sigma \), where \( r\sigma^2 < \alpha_2 \). Thus by the definition of \( \lambda_1 \) and the Sobolev embedding:
\[
I(u) \geq \frac{1}{2} \left( 1 - \frac{(\lambda_1 - \tau)}{\lambda} \right) \|u\|^2 - C \|u\|^q.
\]

Since \( \tau > 0 \) and \( q > 2 \), we may choose \( \rho, \delta > 0 \) such that \( I(u) \geq \delta \) if \( \|u\| = \rho \).

\[\square\]

**Lemma 4.7.** Let \( f \) satisfy \( (f_1)-(f_3) \). Then the functional satisfies \( (C)_c \) condition for \( c \in \mathbb{R} \).

**Proof.** Let \( \{u_n\} \) be a \( (C)_c \) sequence of \( I \). We first show that \( \{u_n\} \) is bounded. If \( \{u_n\} \) is unbounded, up to a subsequence we may assume that for some \( c \in \mathbb{R} \),
\[
I(u_n) \to c, \quad \|u_n\| \to \infty, \quad \|I'(u_n)\| \cdot \|u_n\| \to 0. \tag{4.4}
\]
So we have
\[
\lim_{n \to \infty} \left( \int_{\Omega} \frac{1}{2} \int_{\Omega} F(x, u_n) dx \right) = \lim_{n \to \infty} \left\{ I(u_n) - \frac{1}{2} \langle I'(u_n), u_n \rangle \right\} = c, \tag{4.5}
\]
Let \( w_n = \frac{u_n}{\|u_n\|} \), up to a subsequence we may assume that
\[
w_n \rightharpoonup w \text{ in } H^{1,1}_{2,0}(\mathbb{B}), \quad w_n \to w \text{ in } L^1_{\text{loc}}(\mathbb{B}), \quad w_n \to w \text{ a.e. } x \in \mathbb{B}. \tag{4.6}
\]
We may similarly show that \( w_n^+ \rightharpoonup w^+ \text{ in } H^{1,1}_{2,0}(\mathbb{B}) \), where \( w^+ = \max\{w, 0\} \). If \( w = 0 \), similar to \( p \)-Laplacian case in [19, 38], we can choose a sequence \( \{t_n\} \subset \mathbb{R} \) such that
\[
I(t_n u_n) = \max_{t \in [0, 1]} I(t u_n). \tag{4.7}
\]
For any given $R > 0$, by $(f_1)$, there exists $C = C(m) > 0$ such that
\[ F(x, s) \leq C|s| + e^{\frac{m}{2}s^2}, \text{ for all } (x, s) \in \mathbb{B} \times \mathbb{R}. \tag{4.8} \]

Also since $\|u_n\| \to \infty$, we have
\[ I(t_n u_n) \geq I \left( \frac{m}{\|u_n\|} u_n \right) = I(mw_n) \tag{4.9} \]

and by (4.8) and the fact $\int_{\mathbb{B}} F(x, w_n) \frac{dx_1}{x_1}dx_2 = \int_{\mathbb{B}} F(x, w_n) \frac{dx_1}{x_1}dx_2$, we obtain
\[ 2I(mw_n) \geq m^2 - 2cm \int_{\mathbb{B}} |w_n^+| \frac{dx_1}{x_1}dx_2 - 2 \int_{\mathbb{B}} e^{\alpha_2 |w_n^+|^2} \frac{dx_1}{x_1}dx_2 \]
\[ \geq m^2 - 2cm \int_{\mathbb{B}} |w_n^+| \frac{dx_1}{x_1}dx_2 - 2 \int_{\mathbb{B}} e^{\alpha_2 |w_n^+|^2} \frac{dx_1}{x_1}dx_2. \tag{4.10} \]

Since $\|w_n\| = 1$, we have that $\int_{\mathbb{B}} e^{\alpha_2 |w_n|^2} \frac{dx_1}{x_1}dx_2$ is bounded by a universal constant $C(\mathbb{B}) > 0$ by the Moser-Trudinger inequality. Also, since $w_n^+ \to 0$ in $H_{2,0}^{1,1}(\mathbb{B})$, we have that $\int_{\mathbb{B}} |w_n^+| \frac{dx_1}{x_1}dx_2 \to 0$. Thus using (4.9) and letting $n \to \infty$ in (4.10), and then letting $m \to \infty$, we get
\[ I(t_n u_n) \to \infty. \]

Note that $I(0) = 0$, $I(u_n) \to c$, we see that $t_n \in (0, 1)$ and
\[ \int_{\mathbb{B}} |\nabla t_n u_n|^2 \frac{dx_1}{x_1}dx_2 - \int_{\mathbb{B}} f(x, t_n u_n) t_n u_n \frac{dx_1}{x_1}dx_2 = \langle I'(t_n u_n), t_n u_n \rangle \]
\[ = t_n \frac{d}{dt} \bigg|_{t=t_n} I(t u_n) = 0. \tag{4.11} \]

Therefore, by the condition $(f_3)$,
\[ \frac{1}{2} \int_{\mathbb{B}} F(x, u_n) \frac{dx_1}{x_1}dx_2 \geq \frac{1}{2} \int_{\mathbb{B}} \frac{F(x, t_n u_n)}{\theta} \frac{dx_1}{x_1}dx_2 \]
\[ \geq \frac{1}{\theta} I(t_n u_n) - \frac{1}{2} \langle I'(t_n u_n), t_n u_n \rangle \]
\[ = \frac{1}{\theta} I(t_n u_n) \to \infty. \tag{4.12} \]

This contradicts with (4.5).

Now from the first limit in (4.4), when $\|u_n\| \geq 1$ we obtain
\[ \frac{1}{2} \|u_n\|^2 - (c + o(1)) \geq \int_{\mathbb{B}} F(x, u_n) \frac{dx_1}{x_1}dx_2. \tag{4.13} \]

Using (4.13) and the condition $(f_2)$ we deduce
\[ \frac{1}{2} - \frac{c + o(1)}{\|u_n\|^2} \geq \int_{\mathbb{B}} \frac{F(x, u_n^+)}{\|u_n\|^2} \frac{dx_1}{x_1}dx_2 \]
\[ = \left( \int_{w=0} + \int_{w \neq 0} \right) \frac{F(x, u_n^+)}{|u_n^+|^2} |w_n^+|^2 \frac{dx_1}{x_1}dx_2. \]
\[ \geq \int_{w \neq 0} \frac{F(x, u_n^+)}{|u_n^+|^2} |w_n^+|^2 - \Lambda \int_{w = 0} |w_n^+|^2 \, dx_1 \, dx_2. \] (4.14)

For \( x \in \Theta := \{ x \in \mathbb{B} : w^+(x) \neq 0 \} \), we have \( |u_n^+(x)| \to +\infty \). By the condition \((f_2)\) we have
\[ \frac{f(x, u_n^+)}{|u_n^+|^2} |w_n^+|^2 \to +\infty. \] (4.15)

Note that the Lebesgue measure of \( \Theta \) is positive, using the Fatou Lemma we deduce
\[ \int_{w^+ \neq 0} \frac{f(x, u_n^+)}{|u_n^+|^2} |w_n^+|^2 \, dx_1 \, dx_2 \to +\infty. \] (4.16)

This contradicts with (4.14).

This proves that \( \{u_n\} \) is bounded in \( \mathcal{H}_{2,0}^{1,1}(\mathbb{B}) \). Without loss of generality, suppose that
\[ \begin{cases} \|u_n\| \leq K, \\ u_n \rightharpoonup u \text{ in } \mathcal{H}_{2,0}^{1,1}(\mathbb{B}), \\ u_n \to u \text{ a.e. } \mathbb{B}, \\ u_n \to u \text{ in } L_p^1(\mathbb{B}), \text{ for all } p > 1. \end{cases} \] (4.17)

Now, since \( f \) has the subcritical exponential growth on \( \mathbb{B} \), we can find a constant \( c_K > 0 \) such that
\[ f(x, s) \leq c_K e^{\frac{\alpha s^2}{R^2}}, \text{ for all } (x, s) \in \mathbb{B} \times \mathbb{R}. \]

Then from the cone Moser-Trudinger inequality, we deduce
\[ \left| \int_{\mathbb{B}} f(x, u_n)(u_n - u) \frac{dx_1}{x_1} \, dx_2 \right| \leq \int_{\mathbb{B}} |f(x, u_n)(u_n - u)| \frac{dx_1}{x_1} \, dx_2 \]
\[ \leq \left( \int_{\mathbb{B}} |f(x, u_n)|^2 \frac{dx_1}{x_1} \, dx_2 \right)^{\frac{1}{2}} \cdot \left( \int_{\mathbb{B}} |u_n - u|^2 \frac{dx_1}{x_1} \, dx_2 \right)^{\frac{1}{2}} \]
\[ \leq C \left( \int_{\mathbb{B}} e^{\frac{\alpha x^2}{R^2}} \frac{dx_1}{x_1} \, dx_2 \right)^{\frac{1}{2}} \cdot \|u_n - u\|_2 \]
\[ \leq C \left( \int_{\mathbb{B}} e^{\frac{\alpha x^2}{R^2}} |u_n|^2 \frac{dx_1}{x_1} \, dx_2 \right)^{\frac{1}{2}} \cdot \|u_n - u\|_2 \]
\[ \leq C \|u_n - u\|_2 \to 0, \text{ as } n \to \infty. \] (4.18)

Similarly, since \( u_n \rightharpoonup u \) \( \text{in } \mathcal{H}_{2,0}^{1,1}(\mathbb{B}) \), \( \int_{\mathbb{B}} f(x, u)(u_n - u) \frac{dx_1}{x_1} \, dx_2 \to 0 \). Thus we can conclude that
\[ \int_{\mathbb{B}} f(x, u_n) - f(x, u)(u_n - u) \frac{dx_1}{x_1} \, dx_2 \to 0, \text{ as } n \to \infty. \] (4.19)

Moreover, by (4.4)
\[ (I'(u_n)) - I'(u), u_n - u) \to 0, \text{ as } n \to \infty. \] (4.20)

From (4.19) and (4.20), we get
\[ \int_{\mathbb{B}} |\nabla u_n - \nabla u|^2 \frac{dx_1}{x_1} \, dx_2 \to 0, \text{ as } n \to \infty. \]

So we have \( u_n \to u \) strongly in \( \mathcal{H}_{2,0}^{1,1}(\mathbb{B}) \) which shows that \( I \) satisfies \((PS)_c\) condition. \( \square \)
**Proof of Theorem 4.3.** By Lemma 4.5– Lemma 4.7 and Mountain Pass Theorem (Proposition 4.2), it is clear that we can deduce that the problem (4.1) has a nontrivial weak solution.

**Proof of Theorem 4.4.** Similar to the proof of Theorem 4.3, by our conditions, we see that the functional \( I \) satisfies \((C)_c\) condition. Now we consider the Moser functions

\[
\tilde{M}_2(x) = \begin{cases} 
\frac{1}{\omega_1^2} \left( \sqrt{\ln 2}, \; 0 \leq |x| \leq \frac{1}{2}, \right) \\
\frac{\ln(1/|x|)}{\sqrt{\ln 2}}, \; \frac{1}{2} \leq |x| \leq 1, \\
0, \; |x| \geq 1.
\end{cases}
\] (4.21)

Obviously, \( \tilde{M}_2(x) \in \mathcal{H}^{1,1}_{2,0}(B_1(1,0)) \) and \( \|M_n\| = 1 \), for all \( n \in \mathbb{N} \). Since \( d \) is the inner radius of \( \mathbb{B} \), we can find \( x_0 \in \mathbb{B} \), such that \( B_d(x_0) \subseteq \mathbb{B} \). Moreover, we set \( M_2(x) = \frac{\tilde{M}_2(x - x_0)}{2} \). And we see that \( M_2(x) \in \mathcal{H}^{1,1}_{2,0}(B_1(1,0)) \), \( \|M_2\| = 1 \) and \( \text{supp} M_2 = B_d(x_0) \). As in proof Theorem 1.3 in [15], we can deduce that

\[
\max \{ I(tM_2) : t \geq 0 \} < \frac{1}{2} \left( \frac{\alpha_2}{\alpha_0} \right). 
\]

It is easy to show that \( I \) satisfy the mountain pass geometry. Hence, we can find a Cerami sequence \( \{u_n\} \) such that

\[
I(u_n) \to C_M < \frac{1}{2} \left( \frac{\alpha_2}{\alpha_0} \right), \quad \|I'(u_n)\| \cdot \|u_n\| \to 0. \quad (4.22)
\]

We shall prove that \( \{u_n\} \) is bounded in \( \mathcal{H}^{1,1}_{2,0}(\mathbb{B}) \). In fact, if we suppose that \( \{u_n\} \) is unbounded, let \( w_n = \frac{u_n}{\|u_n\|} \), up to a subsequence, and we may assume that

\[
w_n \rightharpoonup w \text{ in } \mathcal{H}^{1,1}_{2,0}(\mathbb{B}), \quad w_n \to w \text{ in } L^1_p(\mathbb{B}), \quad w_n \to w \text{ a.e. } x \in \mathbb{B}. \quad (4.23)
\]

We may similarly show that \( w_n^+ \to w^+ \) in \( \mathcal{H}^{1,1}_{2,0}(\mathbb{B}) \), where \( w^+ = \max\{w, 0\} \). Let \( t_n \in [0, 1] \) such that

\[
I(t_n u_n) = \max_{t \in [0, 1]} I(tu_n),
\]

and \( m \in \left( 0, \frac{1}{2} \left( \frac{\alpha_2}{\alpha_0} \right)^{1/2} \right) \). Choose \( \varepsilon = \frac{\alpha_2}{m^2} - \alpha_0 > 0 \), according to the condition \((f_1)\), there exists \( C > 0 \) such that

\[
F(x, s) \leq C|s| + \left| \frac{\alpha_2}{m^2} - \alpha_0 \right| e^{(\alpha_0 + \varepsilon)s^2}, \quad \text{for all } (x, s) \in \mathbb{B} \times \mathbb{R}. \quad (4.24)
\]

Since \( \|u_n\| \to \infty \), we deduce

\[
I(t_n u_n) \geq I \left( \frac{m}{\|u_n\|} u_n \right) = I(m w_n),
\]

and by (4.24) and \( \|w_n\| = 1 \), it follows that

\[
2I(mw_n) \geq m^2 - 2cm \int_\mathbb{B} |w_n^+| \frac{dx_1}{x_1} dx_2 - 2 \left| \frac{\alpha_2}{m^2} - \alpha_0 \right| \int_\mathbb{B} e^{(\alpha_0 + \varepsilon)m^2 w_n^2} \frac{dx_1}{x_1} dx_2. \quad (4.26)
\]
From the cone Moser-Trudinger inequality (Lemma 2.3), we know that
\[
\int_B e^{(\alpha_0 + \varepsilon)m^2 w_n^2} \frac{dx_1}{x_1} dx_2 = \int_B e^{\alpha_2 m^2} \frac{dx_1}{x_1} dx_2
\]
is bounded by an universal constant \( C(B) > 0 \) thanks to the choice of \( \varepsilon \). Also, since \( w_n^+ \to 0 \) in \( H^{1,1}_{2,0}(B) \), we have that \( \int_B |w_n^+| \frac{dx_1}{x_1} dx_2 \to 0 \). Thus if we let \( n \to \infty \) in (4.26), and then let \( m \to \left[ \left( \frac{\alpha_2}{\alpha_0} \right)^{\frac{1}{2}} \right]^{-1} \) and using (4.25), we obtain
\[
\liminf_{n \to \infty} I(t_n u_n) \geq \frac{1}{2} \left( \frac{\alpha_2}{\alpha_0} \right) > C_M.
\] (4.27)

Now note that \( I(0) = 0 \) and \( I(u_n) \to C_M \), we can assume that \( t_n \in (0, 1) \). Since \( I'(t_n u_n) t_n u_n = 0 \), we get
\[
t_n^2 \| u_n \|^2 = \int_B f(x, t_n u_n) t_n u_n \frac{dx_1}{x_1} dx_2.
\]
Also, (4.22) implies that
\[
\int_B [f(x, u_n) u_n - 2F(x, u_n)] \frac{dx_1}{x_1} dx_2 = \| u_n \|^2 + 2C_M - \| u_n \|^2 + o(1) = 2C_M + o(1).
\]
According to the condition \( (f_3) \), we know that
\[
2I(t_n u_n) = t_n^2 \| u_n \|^2 - \int_B 2F(x, t_n u_n) \frac{dx_1}{x_1} dx_2
\]
\[
= \int_B [f(x, t_n u_n) t_n u_n - 2F(x, t_n u_n)] \frac{dx_1}{x_1} dx_2
\]
\[
\leq \int_B [f(x, u_n) u_n - 2F(x, u_n)] \frac{dx_1}{x_1} dx_2
\]
\[
= 2C_M + o(1),
\] (4.28)
which contradicts with (4.27). Therefore, \( \{u_n\} \) is bounded in \( H^{1,1}_{2,0}(B) \). Then, up to a subsequence, we can suppose that \( u_n \to u \) in \( H^{1,1}_{2,0}(B) \). Now, following the proof of Lemma 4 in [16], we know that \( u \) is a weak solution of (4.1). So we only need to show that \( u \neq 0 \). Indeed, if \( u = 0 \), as in [16], we have \( f(x, u_n) \to 0 \) in \( L^1_1(B) \). The condition \( (f_6) \) implies that \( F(x, u_n) \to 0 \) in \( L^1_1(B) \) and we can get
\[
\lim_{n \to \infty} \| u_n \|^2 = 2C_M < \frac{\alpha_2}{\alpha_0}
\] (4.29)
and again, following the proof in [16], we have a contradiction.

The proof is completed.
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