SINGULARITY OF PROJECTIONS OF
2-DIMENSIONAL MEASURES INVARIANT UNDER
THE GEODESIC FLOW

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Abstract. We show that on any compact Riemann surface with
variable negative curvature there exists a measure which is invari-
ant and ergodic under the geodesic flow and whose projection to
the base manifold is 2-dimensional and singular with respect to the
2-dimensional Lebesgue measure.

1. Introduction

Let $S$ be a compact surface with possibly variable negative curva-
ture and let $\varphi = \varphi_t$, $t \in \mathbb{R}$, be the geodesic flow on the unit tangent
bundle $T^1 S$. In this note we are interested in the projections of $\varphi$-
invariant measures to $S$ by the canonical projection $\Pi : T^1 S \to S$, where
$\Pi(x, v) = x$. In particular, we prove:

Theorem 1.1. For any compact surface $S$ whose curvature is every-
where negative, there exists an ergodic $\varphi$-invariant measure $m$ on $T^1 S$
such that $\Pi_* m$ has Hausdorff dimension equal to 2 and is singular with
respect to the Lebesgue measure on $S$.

The image of a measure $\mu$ under a map $F$ is denoted by $F_* \mu$ and the
Hausdorff dimension of a measure $\mu$ is defined as follows:

$$\dim_H \mu = \inf \{ \dim_H A \mid \mu(A) > 0, A \text{ is a Borel set} \}.$$  

It was shown by Ledrappier and Lindenstrauss \cite{LL} that the canoni-
cal projection of a $\varphi$-invariant measure of dimension greater than 2
is absolutely continuous with respect to the Lebesgue measure. Our
result shows that this property does not hold at the threshold 2. Moti-
vation for this study comes from Quantum Unique Ergodicity (QUE).

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on $S$. The associated eigenvalues converge to infinity and the problem of QUE is to describe the possible weak* limits of the probability measures with density $|\psi_n|^2$ as $n$ tends to infinity. This was solved by Lindenstrauss for arithmetic hyperbolic surfaces in the case when $\psi_n$’s are also eigenfunctions of the Hecke operators – the only limit is the normalized Lebesgue measure (see [Lin]). In the case of a general hyperbolic surface, or of more general eigenfunctions, if any, on an arithmetic surface, Anantharaman and Nonnenmacher [AN] proved that any limit is the projection to $S$ of a $\varphi$-invariant measure with dimension at least 2. Rivière [Ri] showed that this property is still true on surfaces with variable negative curvature. Our note shows that one still cannot conclude from their results that any weak* limit is nonsingular, which is a weak form of the QUE conjecture.

In order to prove Theorem 1.1, it suffices to construct an ergodic $\varphi$-invariant measure $m$ with dimension 2 and a measurable set $A \subset T^1 S$ with the properties that $m(A^c) = 0$ and $H^2(A) = 0$, where $A^c = T^1 S \setminus A$ and $H^2$ is the 2-dimensional Hausdorff measure. Since the Hausdorff measure cannot increase under the canonical projection, we have that $\Pi_* m$ is singular with respect to the Lebesgue measure $L^2$ on $S$. But, since the Hausdorff dimension of a $\varphi$-invariant measure is preserved under the projection by [LL], the Hausdorff dimension of $\Pi_* m$ is 2, as claimed. For 3-dimensional Anosov flows, metric balls can be approximated by dynamical ones on the manifold, so it is possible, by controlling entropy and exponent, to construct many invariant Gibbs measures with dimension exactly 2, see Section 2. What we have to ensure is that the fluctuations of the measures of the dynamical balls are large enough so that for $m$-almost every point $(x,v) \in T^1 S$, we have

$$\limsup_{\varepsilon \to 0} \frac{m(B((x,v),\varepsilon))}{\varepsilon^2} = +\infty.$$  

As shown in Section 3 this property follows from a vector valued almost sure invariance principle for hyperbolic dynamical systems proved in [MN2].

2. INVARIANT MEASURES OF DIMENSION 2

We start by recalling some well-known facts of geodesic flows on negatively curved surfaces (see for example [KH]). Let $m$ be a $\varphi$-invariant ergodic measure on $T^1 S$. The (largest Lyapunov) exponent $\lambda(m)$ is defined by

$$\lambda(m) = \lim_{t \to \infty} \frac{1}{t} \ln \|D\varphi_t(x,v)\|.$$  

The limit exists for $m$-almost all $(x,v) \in T^1 S$ by the invariance and it is $m$-almost surely independent of $(x,v)$ by the ergodicity. In the case of constant curvature $-1$ we have $\lambda = 1$. The entropy $h_m(\varphi)$
is a number, \(0 \leq h_m(\varphi) \leq \lambda(m)\), which measures the randomness of typical trajectories (see e.g. \([W]\) for the precise definition). The Hausdorff dimension of \(m\) is given by

\[
dim_H m = 1 + 2 \frac{h_m(\varphi)}{\lambda(m)}
\]

(see \([PS]\)). It follows that \(m\) has dimension 2, if and only if we have \(h_m(\varphi) = \lambda(m)/2 = 1/2\) in the case of constant curvature \(-1\). On any family for which the ratio \(h_m(\varphi)/\lambda(m)\) varies continuously from 0 to 1, there will be measures with dimension 2. In order to have specific examples, we shall consider a special family: Markov measures in a symbolic coding of the geodesic flow.

Let \(A = A_{ij}\), \(1 \leq i, j \leq n\), be an \(n \times n\)-matrix with entries 0 or 1 and define a subshift of finite type \(\Sigma \subset \{1, \ldots, n\}^\mathbb{Z}\) as the set of sequences \(\omega = (\omega_k)_{k \in \mathbb{Z}}\) such that \(A_{\omega_{k+1} \omega_k} = 1\) for all \(k \in \mathbb{Z}\). The metric on \(\Sigma\) is given by some number \(\theta\) with \(0 < \theta < 1\): \(d(\omega, \omega') = 0\) and for \(\omega \neq \omega'\),

\[
d(\omega, \omega') = \theta^n(\omega, \omega')
\]

where \(n(\omega, \omega')\) is the largest number with \(\omega_j = \omega'_j\) for all \(j\) with \(|j| < n(\omega, \omega')\), using the interpretation \(n(\omega, \omega') = 0\) if \(\omega_0 \neq \omega'_0\). The left shift on \(\Sigma\) is denoted by \(\sigma\), that is, \(\sigma(\omega)_i = \omega_{i+1}\) for all \(i \in \mathbb{Z}\). If there is a positive \(p \in \mathbb{N}\) such that all the entries of the matrix \(A^p\) are positive, the shift \((\Sigma, \sigma)\) is topologically mixing. For a positive continuous function \(r\) on \(\Sigma\), we define the special flow \((\tilde{\Sigma}_r, \tilde{\sigma}_t)\), \(t \in \mathbb{R}\), by translation on the second coordinate, where

\[
\tilde{\Sigma}_r := \{(\omega, s) \mid \omega \in \Sigma, 0 \leq s \leq r(\omega)\}/(\sigma, r(\omega)) \sim (\sigma(\omega), 0),
\]

that is, for \(t \geq 0\) we have \(\tilde{\sigma}_t(\omega, s) = (\sigma^k(\omega), u)\) where \(u = t + s - \sum_{j=0}^{k-1} r(\sigma^j(\omega))\) and \(k\) is the unique natural number satisfying \(0 \leq u < r(\sigma^k(\omega))\), and similarly for \(t < 0\). The following result is due to Ratner \([Ra1]\) (see \([S]\) for a more geometric description of the space \(\Sigma\)).

**Proposition 2.1.** There exist a mixing subshift of finite type \((\Sigma, \sigma)\), a Hölder continuous function \(r\) on \(\Sigma\) and a Hölder continuous mapping \(\pi : \tilde{\Sigma}_r \to T^1S\) such that \(\pi \circ \tilde{\sigma}_t = \varphi_t \circ \pi\). The mapping \(\pi\) is finite-to-one and one-to-one outside a closed invariant set of smaller topological entropy.

We call an \(n \times n\)-matrix \(P = P_{ij}\), \(1 \leq i, j \leq n\), a Markov matrix on \(\Sigma\) if it is Markov (meaning that \(P_{ij} \geq 0\) for all \(i, j, 1, \ldots, n\) and \(\sum_{j=1}^{n} P_{ij} = 1\) for all \(i = 1, \ldots, n\)) and \(P_{ij} > 0\), if and only if \(A_{ij} = 1\). To a Markov matrix \(P\) on \(\Sigma\) is associated a unique \(\sigma\)-invariant probability measure \(\mu_P\) on \(\Sigma\) given by the formula

\[
\mu_P([\omega]_{n_2}^{n_1}) = v_{\omega_{n_2}} \prod_{i=n_2}^{n_1-1} P_{\omega_i \omega_{i+1}} = e^{\sum_{i=n_2}^{n_1-1} G(\sigma^i(\omega))},
\]

where \([\omega]_{n_2}^{n_1}\) is the cylinder of order \((n_2, n_1)\) containing \(\omega\), i.e.

\[
[\omega]_{n_2}^{n_1} := \{\omega' \in \Sigma \mid \omega'_i = \omega_i \text{ for } n_2 \leq i \leq n_1\},
\]
v is the unique left eigenvector with $\sum_{i=1}^{n} v_i = 1$ corresponding to the eigenvalue $1$ of $P$ and $G(\omega) = \ln P_{\omega,\omega}$. The system $(\Sigma, \sigma, \mu_P)$ is mixing for all such $P$. Define a $\sigma$-invariant probability measure $\tilde{\mu}_P$ on $\Sigma_r$ by

$$\tilde{\mu}_P = \frac{\int_{\Sigma_r} C_{[0,r]} d\mu_P}{\int_{\Sigma_r} r d\mu_P}$$

and set $m_P = \pi_* \tilde{\mu}_P$. The measure $m_P$ is ergodic for $\varphi$.

The entropy $h_{m_P}(\varphi)$ is given by Abramov formula (see [AB]):

$$h_{m_P}(\varphi) = h_{\tilde{\mu}_P}(\tilde{\sigma}) = \frac{h_{\mu_P}(\sigma)}{\int_{\Sigma_r} r d\mu_P}.$$  

The exponent $\lambda(m_P)$ is given by

$$\lambda(m_P) = \frac{\int_{\Sigma_r} F^u d\mu_P}{\int_{\Sigma_r} r d\mu_P},$$

where $F^u(\omega) = \ln \|D\varphi_r(\omega)(\pi(\omega, 0)) (v^u)\|$ and $v^u \in T^1_{\pi(\omega, 0)} S$ is tangent to the unstable manifold at $\pi(\omega, 0)$. Note that $F^u(\omega)$ is the expansion in the unstable direction from $\pi(\omega, 0)$ to $\pi(\omega, r(\omega))$ ($F^u = r$ in the constant curvature $-1$ case). The function $F^u$ is Hölder continuous and positive on $\Sigma$. In the same way one defines $F^s$ for the inverse flow $\varphi_{-t}$. Our aim is to verify that there are many Markov matrices $P$ on $\Sigma$ such that $h_{\mu_P}(\sigma)/\int_{\Sigma_r} F^u d\mu_P = 1/2$ giving $\dim_H m_P = 2$. The real analyticity of the mapping $P \mapsto h_{\mu_P}(\sigma)/\int_{\Sigma_r} F^u d\mu_P$ follows from [Ru, Corollary 7.10], but we do not know a priori that $1/2$ is a possible value for $h_{\mu_P}(\sigma)/\int_{\Sigma_r} F^u d\mu_P$. Nevertheless, we claim:

**Proposition 2.2.** There exist a mixing subshift of finite type $(\Sigma, \sigma)$, a Hölder continuous function $r$ on $\Sigma$, a Hölder continuous mapping $\pi : \Sigma_r \to T^1 S$ such that $\pi \circ \tilde{\sigma} = \varphi_t \circ \pi$ and the set of Markov matrices $P$ on $\Sigma$ for which $\dim_H m_P = 2$ contains a smooth submanifold having codimension equal to one. The mapping $\pi$ is finite-to-one and one-to-one outside a closed invariant set of smaller topological entropy.

**Proof.** To find infinitely many suitable Markov matrices, it is sufficient to find a coding system $\Sigma$ such that 1/2 is an interior point of the image set of the map $P \mapsto h_{\mu_P}(\sigma)/\int_{\Sigma_r} F^u d\mu_P$. Since the set of Markov matrices is connected, the image set of $h_{\mu_P}(\sigma)/\int_{\Sigma_r} F^u d\mu_P$ is an interval. Clearly, $\int_{\Sigma_r} F^u d\mu_P \geq \inf F^u$, whereas $h_{\mu_P}(\sigma)$ can be arbitrarily small. Therefore, the interval is $(0, a_1]$, where $a_1 := \max_P h_{\mu_P}(\sigma)/\int_{\Sigma_r} F^u d\mu_P$. If $a_1 \leq 1/2$, we can relabel $\Sigma$, using as alphabet those words $i_1 \cdots i_\ell$ of length $\ell$ for which $A_{i_1 \cdots i_\ell} = 1$ for all $k = 1, \ldots, \ell - 1$ and as a new matrix

$$A_{i_1 \cdots i_\ell} = 1 \iff j_1 = i_2, \ldots, j_{\ell-1} = i_\ell.$$ 

Markov matrices in this new presentation define a bigger family of invariant measures, the $\ell$-step Markov measures in the initial presentation. We set $a_\ell := \max_P h_{\mu_P}(\sigma)/\int_{\Sigma_r} F^u d\mu_P$, where $P_\ell$ varies among all
Markov measures in the alphabet of length $\ell$. We have to show that for $\ell$ large enough we have $a_\ell > 1/2$. Indeed, $a_\ell \to 1$ as $\ell \to \infty$. This is true because the Liouville measure $m_0$ on $T^1S$, for which $h_{m_0}(\varphi) = \lambda(m_0)$ and $\dim_H m_0 = 3$, is a Gibbs measure whose entropy can be approximated by the entropies of $\ell$-step Markov measures with $\ell$ going to infinity.

Finally, we prove that the set of Markov matrices giving dimension 2 contains a smooth submanifold having codimension equal to 1. Consider a line $L = [v, a]$ starting from a vertex $v$ of the simplex of all Markov matrices and ending at a maximum point $a$ of the dimension function $d$. Recall that $d(a) > 2$. Since $d(v) = 1$ (entropy equals 0) there are points on $L$ where $d$ equals 2. Denote this level set on $L$ by $L_2$. If $d'(x) \neq 0$ for some $x \in L_2$ we have found a regular point of the dimension function and we are done. Suppose $d'(x) = 0$ for all $x \in L_2$. All points $x \in L_2$ cannot be local maximum points since $d(a) > 2$. Let $x_0$ be the smallest point in $L_2$ which is not a local maximum point. Then $x_0$ is not a local minimum point since $d(y) \leq d(x_0)$ for all $y \leq x_0$. Thus for small $h > 0$ we have $d(x_0 + h) - d(x_0) = h^n + O(h^{n+1})$ for some odd $n$. Therefore, after a local coordinate transformation $x_0$ becomes a regular point. This coordinate transformation is smooth outside the hyperplane $V$ perpendicular to $L$ at $x_0$ and is an identity on $V$. If the level set $d = 2$ contains a point outside $V$, we are done. If the level set is contained in $V$, it is a piece of a hyperplane.

Observe in particular that, since $m_P$ is ergodic and the support of $m_P$ is the whole space $T^1S$, $\pi$ is invertible $m_P$-almost everywhere.

3. Fluctuations of measures of balls

Suppose we have chosen $P$ by Proposition 2.22 in such a way that $\dim_H m_P = 2$. Then

$$\lim_{\varepsilon \to 0} \frac{\ln m_P(B((x, v), \varepsilon))}{\ln \varepsilon} = 2$$

for $m_P$-almost all $(x, v) \in T^1S$. In this section we show that we can choose $P$ in such a way that equation (1.1) also holds for $m_P$-almost all $(x, v) \in T^1S$. Thus the upper derivative $D(m_P, H^2; (x, v)) = \infty$ for $m_P$-almost all $(x, v) \in T^1S$ implying that $m_P$ and $H^2$ are mutually singular (see [M, Theorem 2.12]). By the discussion in the introduction, this completes the proof of Theorem 1.1. We have to estimate $m_P(B((x, v), \varepsilon))$ from below.

Lemma 3.1. There exists a constant $C$ such that, for $\varepsilon$ small enough, if the functions $n_1 = n_1(\omega, \varepsilon)$ and $n_2 = n_2(\omega, \varepsilon)$ satisfy

$$\sum_{k=0}^{n_1} F^a(\sigma^k(\omega)) \geq -\ln \varepsilon + C \quad \text{and} \quad \sum_{k=0}^{n_2} F^a(\sigma^{-k}(\omega)) \geq -\ln \varepsilon + C,$$
then
\[ B(\pi(\omega, t), \varepsilon) \supset \pi([\omega]^{n_1}_{n_2} \times [t - \varepsilon/4, t + \varepsilon/4]) \]
for all \((\omega, t) \in \Sigma_r\).

**Proof.** Recall that \( \varphi \) is an Anosov flow on \( T^1S \) and the map \( \pi \), given by Proposition 2.1, is constructed using a Markov partition for the flow (see for example [B, Ra]). In particular, there is \( \delta > 0 \) such that at each point \( v \in T^1S \) (for notational simplicity we omit the base point \( x \in S \) from \((x, v) \in T^1S\) the sets
\[ W^s_\delta(v) := \{ w \in T^1S \mid d(\varphi_t(w), \varphi_t(v)) < \delta \ \forall t \geq 0 \text{ and} \lim_{t \to \infty} d(\varphi_t(w), \varphi_t(v)) = 0 \} \]
and
\[ W^u_\delta(v) := \{ w \in T^1S \mid d(\varphi_t(w), \varphi_t(v)) < \delta \ \forall t \leq 0 \text{ and} \lim_{t \to -\infty} d(\varphi_t(w), \varphi_t(v)) = 0 \} \]
are open 1-dimensional submanifolds, called a local stable and a local unstable manifold, respectively. Moreover, if \( w_1, w_2 \in \pi([\omega]^{n_1}_{n_2} \times \{0\}) \) there exist a unique \( t \) with \(|t| < \delta\) and a unique \( w_3 := [w_1, w_2] \in W^s_\delta(\varphi_t(w_1)) \cap W^u_\delta(w_2) \). Furthermore, the projection along the flow to \( \pi([\omega]^{n_1}_{n_2} \times \{0\}) \) is Lipschitz continuous when \(-r(\sigma^{-1}(\omega)) \leq t \leq r(\omega)\).

Let \( \omega_1, \omega_2 \in [\omega]^{n_1}_{n_2} \) and define \( w_1 = \pi(\omega_1, 0) \) and \( w_2 = \pi(\omega_2, 0) \). Choose \(-\delta < t < \delta\) such that \( w_3 = [w_1, w_2] \in W^s_\delta(\varphi_t(w_1)) \). Since \( F^u \) and \( F^s \) are Hölder continuous, the distance between \( w_2 \) and \( w_3 \) along the unstable leaf is, up to a bounded constant, \( \exp(-\sum_{k=0}^{n_1} F^u(\sigma^k(\omega))) \), and the distance between \( w_3 \) and \( \pi(\omega_1, t) \) along the stable leaf is, up to a bounded constant, \( \exp(-\sum_{k=0}^{n_2} F^s(\sigma^{-k}(\omega))) \). Using the Lipschitz continuity of the projection along the flow, one can choose a constant \( C \) in the assumptions such that \( d(w_1, w_2) < \varepsilon \). Since the flow \( \varphi_t \) is smooth there exists \( c' > 0 \) such that \( d(\varphi_u(w_1), \varphi_u(w_2)) \leq c'd(w_1, w_2) \) for all \( 0 \leq u \leq r(\omega) \). Finally, as \( d(\pi(\omega_2, t), \pi(\omega_1, u)) \leq |t - u| \), the claim follows by changing the constant \( C \) obtained above. \( \square \)

Let \((x, v) = \pi(\omega, t) \). Using Lemma 3.1 and (2.1), we find \( c > 0 \) such that
\[ m_P(B((x, v), \varepsilon)) \geq \frac{\varepsilon}{2} \mu_P([\omega]^{n_1}_{n_2}) \geq c\varepsilon e^{\sum_{k=-n_2}^{n_1} G(\sigma^k(\omega))}, \]
where \( n_1 \) and \( n_2 \) are as in Lemma 3.1. Define
\[ X^u_n(\omega) = -\sum_{i=0}^{n-1} G(\sigma^i(\omega)), \quad X^s_n(\omega) = -\sum_{i=1}^{n} G(\sigma^{-i}(\omega)), \]
\[ Y^u_n(\omega) = \sum_{i=0}^{n-1} F^u(\sigma^i(\omega)) \text{ and } Y^s_n(\omega) = \sum_{i=0}^{n-1} F^s(\sigma^{-i}(\omega)) \]
By the Shannon-McMillan-Breiman theorem (see for example [W, Remark p. 93]) and by the choice of \( P \), we have for \( \mu_P \)-almost all \( \omega \in \Sigma \)
that
\[(3.3) \lim_{n \to \infty} \frac{X_n^u}{Y_n^u} = \frac{1}{2} = \lim_{n \to \infty} \frac{X_n^s}{Y_n^s}.\]

Let \(a = -\int_{\Sigma} G \, d\mu_P\) and \(b = \int_{\Sigma} F^u \, d\mu_P = \int_{\Sigma} F^s \, d\mu_P\). Then for \(v \in \{u, s\}\)
\[(3.4) \lim_{n \to \infty} \frac{X_n^v(\omega)}{n} = a \quad \text{and} \quad \lim_{n \to \infty} \frac{Y_n^v(\omega)}{n} = b\]
for \(\mu_P\)-almost all \(\omega \in \Sigma\). Equation (3.3) gives \(b = 2a\).

Observe that \((X_n^s - na, Y_n^s - nb)\) and \((X_n^s - na, Y_n^s - nb)\) are not independent. However, using the fact that \(F^u\) and \(G\) are Hölder continuous, we can find \(K > 0\) such that for all \(n \in \mathbb{N}\) we have
\[(3.5) |X_n^u(\omega) - X_n^u(\omega')| < K \quad \text{and} \quad |Y_n^u(\omega) - Y_n^u(\omega')| < K\]
for all \(\omega, \omega' \in \Sigma\) such that \(\omega_i = \omega'_i\) for all \(i = 0, 1, \ldots\) Moreover, the same holds for \(X_n^u\) and \(Y_n^u\) with the condition \(\omega_i = \omega'_i\) for all \(i = -1, -2, \ldots\) Let \(\Sigma^c\) and \(\Sigma^c\) be the one-sided subshifts corresponding to the negative and nonnegative indices, respectively. Fix \(\xi^- \in \Sigma^c\) and \(\xi^+ \in \Sigma^c\) for \(i = 1, \ldots, n\) such that \((\xi^-)_i = i\) and \((\xi^+)_0 = i\).

Define \(\eta, \psi : \{1, \ldots, n\} \to \{1, \ldots, n\}\) such that \(A_{\eta(i)} = i\) and \(A_{\psi(i)} = i\). Setting \(X_n^u(\omega^\geq) := X_n^u(\xi^- \lor \omega^\geq)\), \(Y_n^u(\omega^\geq) := Y_n^u(\xi^- \lor \omega^\geq)\), \(X_n^s(\omega^\leq) := X_n^s(\omega^\leq \lor \xi^+ \lor \omega^\leq)\) and \(Y_n^s(\omega^\leq) := Y_n^s(\omega^\leq \lor \xi^+ \lor \omega^\leq)\), one can consider \(X_n^u\) and \(Y_n^u\) as functions on \(\Sigma^c\) and \(X_n^s\) and \(Y_n^s\) as functions on \(\Sigma^c\). By [Ru, Lemma 5.9] the measure \(\mu_P \times \tilde{\mu}_P\) restricted to the set \(\{\omega^\leq, \omega^\geq\} \in \Sigma^c \times \Sigma^c\) \(\omega^\leq \lor \omega^\geq \in \Sigma\) is equivalent with \(\mu_P\) and the Radon-Nikodym derivative is bounded from above and from below, where \(\mu_P\) and \(\tilde{\mu}_P\) are the Markov measures given by (2.1) on \(\Sigma^c\) and \(\Sigma^c\), respectively. In particular, there exists \(L > 0\) such that
\[(3.6) \mu_P([a]_{-n_1}^{n_1} \times [b]_{-n_2}^{n_2}) \geq L^{-1}(\mu_P \times \tilde{\mu}_P)([a]_{-n_1}^{-1} \times [b]_{-n_2}^{n_2})\]
for all \(n_1, n_2 \in \mathbb{N}\). Further, inequalities (3.5) imply that (3.1) is valid for any \(\omega \in \Sigma\) for which \(Y_n^u(\omega^\leq) \geq -\ln \varepsilon + C + K\) and \(Y_n^s(\omega^\leq) \geq -\ln \varepsilon + C + K\). Thus it is enough to consider \((X_n^u - na, Y_n^u - nb)\) and \((X_n^s - na, Y_n^s - nb)\) as independent observables.

The almost sure invariance principle [MN2, Theorem 3.6] (see also [MN2, Theorem 1.3]) implies that the observables \((X_n^u - na, Y_n^u - nb)\) and \((X_n^s - na, Y_n^s - nb)\) can be approximated by 2-dimensional Brownian motions. This means that for \(v \in \{u, s\}\) there exist \(\lambda > 0\) and a probability space \((\mathcal{X}, \mathcal{P})\) supporting a sequence of random variables \((\tilde{X}_n^u, \tilde{Y}_n^u)\) having the same distribution as \((X_n^u - na, Y_n^u - nb)\) and a 2-dimensional Brownian motion \(W^v\) with a covariance matrix \(Q^v\) such that
\[|(\tilde{X}_n^u, \tilde{Y}_n^u) - W^v(n)| \ll n^{\frac{1}{2} - \lambda}\]
\[\mathbb{P}\text{-almost surely for large } n. \text{ In fact, } Q^v \text{ does not depend on } v \text{ but this information is not necessary.}\]

To prove that the covariance matrix \(Q\) is nonsingular, we need a couple of lemmas. For a smooth flow on a compact manifold \(M\) generated by a vector field \(X\) and preserving a probability measure \(\mu\), one can define an element \(H\) of the first homology space \(H_1(M, \mathbb{R})\) in the following way: The mapping which associates to a closed 1-form \(\psi\) on \(M\) the integral \(\int_M \psi(X) \, d\mu\) is linear and, by invariance of \(\mu\), vanishes on exact 1-forms. This defines a linear real valued mapping on the first de Rham cohomology space \(H^1_{dR}(M, \mathbb{R})\). We denote it and the corresponding element of \(H_1(M, \mathbb{R})\) (given by de Rham’s theorem) by \(H\).

We need the following result of Arnol’d \cite[Lemma 23.2]{An}:

**Lemma 3.2.** With the above notation, for the geodesic flow on a compact manifold not homeomorphic to the two-torus and for the Liouville measure \(\mu\), we have \(H = 0\).

We will also use a variant of Livšic’s Theorem \cite{Liv}:

**Lemma 3.3.** Let \(F\) be a Hölder continuous positive function on \(T^1S\) and \(b\) a positive number such that \(\int_\gamma F \in b\mathbb{N}\) for all closed geodesics \(\gamma\). Then there is a Hölder continuous function \(f\) on \(T^1S\) with \(f(v)\) for all \(v \in T^1S\) and for all \(t \geq 0\)

\[
f(\varphi_t(v)) = e^{\frac{2\pi i}{T} \int_0^t F(\varphi_s(v)) \, ds}\]

for all \(v\) in the unit circle and satisfies equation \eqref{eq:3.7} for all \(v\) in the orbit of \(v_0\) and for all \(t \geq 0\). We claim that \(f\) is uniformly Hölder continuous on the orbit of \(v_0\). Since the orbit is dense, this implies that \(f\) extends to a Hölder continuous function on \(T^1S\).

Obviously this extension has the desired properties. Now we prove the uniform Hölder continuity of \(f\). If \(\varphi_t(v_0)\) and \(\varphi_{t'}(v_0)\) are close enough (assume that \(t < t'\)), there exist a uniform constant \(c\) and a closed geodesic \(\gamma\) of length \(\ell\) with \(|\ell - (t' - t)| < cd(\varphi_t(v_0), \varphi_{t'}(v_0))\) such that

\[
d(\varphi_{t+s}(v_0), \varphi_{t+s}(v_0)) \leq cd(\varphi_t(v_0), \varphi_{t'}(v_0))\]

for \(0 \leq s \leq \ell\) by Anosov Closing Lemma (see \cite[Theorem 6.4.15 and p. 548]{KH}). Moreover, since the two geodesics \(\gamma(0), 0 \leq s \leq \ell\), and \(\varphi_s(v_0), t \leq s \leq t'\), remain close, they have to get closer exponentially, i.e., there exist \(\tau\) and \(\tau'\) with \(|\tau|, |\tau'| < cd(\varphi_{t}(v_0), \varphi_{t'}(v_0))\) and \(\tilde{c}, \alpha > 0\) such that

\[
d(\varphi_{t+s}(v_0), \gamma(\tau + s)) \leq \tilde{c}e^{-\alpha s}d(\varphi_{t}(v_0), \varphi_{t'}(v_0))\]

for \(0 \leq s \leq (t' - t)/2\) and

\[
d(\varphi_{t+s}(v_0), \gamma(\tau' + s)) \leq \tilde{c}e^{-\alpha(t' - t - s)}d(\varphi_{t}(v_0), \varphi_{t'}(v_0))\]
for \((t' - t)/2 \leq s \leq t' - t\) (see [KH, Corollary 6.17]). This implies that
\[
\left| \int_t^{t'} F(\varphi_s(v_0)) \, ds - \int_0^\ell F(\gamma(s)) \, ds \right| \leq \bar{c}d(\varphi_1(v_0), \varphi_{t'}(v_0))^\alpha,
\]
where \(\alpha\) is the Hölder exponent of \(F\) and \(\bar{c} > 0\). Since \(\int_0^\ell F(\gamma(s)) \, ds\) is a multiple of \(b\), we have for some \(\hat{c} > 0\) that
\[
|f(\varphi_t(v_0)) - f(\varphi_{t'}(v_0))| = e^{\frac{2\pi b}{\beta}} \left| \int_t^{t'} F(\varphi_s(v_0)) \, ds - 1 \right| \leq \hat{c}d(\varphi_1(v_0), \varphi_{t'}(v_0))^\alpha.
\]

\[
\Box
\]

**Lemma 3.4.** There exists \(P\) such that \(\dim_H m_P = 2\) and the covariance matrix \(Q\) is nonsingular.

**Proof.** According to [MN2, Remark 1.2] (see also [HM, Section 4.3] and [MN1, Corollary 2.3]), the covariance matrix \(Q\) is nonsingular, if
\[-G - a + \alpha(F^v - b) \neq f \circ \sigma - f\]
and
\[-F^v - b \neq f \circ \sigma - f\]
for all \(\alpha \in \mathbb{R}\) and for all Lipschitz functions \(f\). To check the first claim, it is enough to find a periodic orbit \(\gamma = (\omega_1, \ldots, \omega_n)\) such that
\[
\sum_{i=1}^n (G - a + \alpha(F^v - b)))(\omega_i) \neq 0.
\]

Choose periodic orbits \(\gamma_j = (\omega_{j1}, \ldots, \omega_{jn_j})\) for \(j = 1, 2, 3\) such that each \(\gamma_j\) contains a step \(k_jl_j\) (that is, \((\omega_{i,j})_{s,s+1} = k_jl_j\) for some \(s\)) which is not included in the other orbits. Assume that one can vary \(P_{k_jl_j}\) without changing \(P_{mn}\) for any other step \(mn\) included in \(\gamma_j\) for \(j = 1, 2, 3\) such that (3.3) is valid. This is possible since by Proposition 2.2 the level set contains a submanifold having codimension equal to 1 and we see in the proof of Proposition 2.2 that we may choose a coding system \(\Sigma\) such that the dimension of the space of Markov matrices is as large as we wish. Write \(\beta = a + ab\) and consider the system of equations
\[
\begin{cases}
\sum_{i=1}^{n_1} (G - a + \alpha F^v)(\omega_{i1}^1) = n_1 \beta \\
\sum_{i=1}^{n_2} (G - a + \alpha F^v)(\omega_{i2}^2) = n_2 \beta \\
\sum_{i=1}^{n_3} (G - a + \alpha F^v)(\omega_{i3}^3) = n_3 \beta,
\end{cases}
\]
View for a moment \(\alpha\) and \(\beta\) as independent variables. Assume that this linear system has a unique solution \((\alpha, \beta)\). If \((\alpha, \beta)\) is not a solution of
\[
\sum_{i=1}^{n_3} (G - a + \alpha F^v)(\omega_{i3}^3) = n_3 \beta,
\]
we are done. If \((\alpha, \beta)\) is a solution for this third equation, we may change \(G\) in the third equation by varying \(P_{k_3l_3}\) without changing \(G\) in
If this change does not change $a$ and $b$, that is $\beta$, then the solution $\alpha$ for the third equation will change and we are done. If $\beta$ changes, then $(\alpha, \beta)$ is not a solution for (3.9) any more. The remaining case is that (3.9) has infinitely many solutions. Then the equations describe the same line, and therefore changing $G$, that is, the affine part in the second equation, implies that we have two different parallel lines and there is no solution for (3.9).

Finally, we have to show that there exists a periodic orbit $\gamma = (\omega_1, \ldots , \omega_n)$ such that

$$
\sum_{i=1}^{n} (F^v - b)(\omega_i) \neq 0.
$$

This follows from Lemma 3.2. Indeed, if this is not the case, let $f$ be the function given by Lemma 3.3. Then $\ln f$ is a purely imaginary multivalued function which is smooth in the direction of the flow and $X(\ln f) = \frac{2\pi i}{b} F$. Given $\varepsilon$, one can find a smooth function $g$ with values in the complex unit circle such that $|X(\ln g) - X(\ln f)| < \varepsilon$ (see [An, Proof of Lemma 23.1]). Then $\frac{1}{i} d\ln g$ is a real 1-form on $T^1 S$ such that

$$
H(\frac{1}{i} d\ln g) = \int_{M} \frac{1}{i} X(\ln g) d\mu \neq 0
$$
as soon as $\varepsilon < \frac{2\pi}{b} \int_{M} F d\mu$. \hfill $\square$

Fix $D > 1$ and let $\tilde{C} = C + K$, where $C$ is as in Lemma 3.1 and $K$ is as in (3.5). Define events $E^v_n$ for $v \in \{u, s\}$ by “$X^v_n \leq na - D\sqrt{n}$ and $Y^v_n \geq nb + C$” and let $E$ be the event “$E^u_n$ and $E^s_n$ happens for infinitely many $n$”. Then $E$ is a tail event, and therefore, it has probability 0 or 1. By independence,

$$
(\mu_P^* \times \mu_P^*)(E) \geq \limsup_{n \to \infty} (\mu_P^*(E^u_n))^2.
$$

The scale invariance [MP, Lemma 1.7] and Lemma 3.4 imply that

$$
\mathbb{P}(W(n) \in \{(x, y) \mid x < -2D\sqrt{n} \text{ and } y > n^{\frac{1}{2} - \lambda} + \tilde{C}\})
\geq \mathbb{P}(W(1) \in \{(x, y) \mid x < -2D \text{ and } y > 1\}) = \rho > 0
$$
for $n > 2\tilde{C}^2$. Choose $n$ large enough such that

$$
\mathbb{P}(|(\tilde{X}^u_n, \tilde{Y}^u_n) - W^u(n')| < (n')^{\frac{1}{2} - \lambda} \text{ for all } n' \geq n) > 1 - \frac{\rho}{2}.
$$

The almost sure invariance principle implies that $\mu_P^*(E^u_n) > \rho/2$ and thus the event $E$ has probability 1.

Write $\varepsilon(n) = e^{-nb}$. Recalling (3.2), one can find for $\mu_P$-almost all $\omega \in \Sigma$ a sequence $n$ tending to infinity such that

$$
m_P(B((x, v), \varepsilon(n))) \geq \tilde{c}\varepsilon(n)e^{-2an + 2D\sqrt{n}} = \tilde{c}\varepsilon(n)^2 e^{c'\sqrt{-\ln\varepsilon(n)}}
$$
for some constants $\tilde{c}$ and $c'$. This implies (1.1).
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