The Nyström method for elastic wave scattering by unbounded rough surfaces

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Abstract

We consider the numerical algorithm for the two-dimensional time-harmonic elastic wave scattering by unbounded rough surfaces with Dirichlet boundary condition. A Nyström method is proposed for the scattering problem based on the integral equation method. Convergence of the Nyström method is established with convergence rate depending on the smoothness of the rough surfaces. In doing so, a crucial role is played by analyzing the singularities of the kernels of the relevant boundary integral operators. Numerical experiments are presented to demonstrate the effectiveness of the method.

Keywords: elastic wave scattering, unbounded rough surface, Nyström method.

1 Introduction

We consider the two-dimensional time-harmonic elastic scattering problem for unbounded rough surfaces with Dirichlet boundary condition. This kind of problem has attracted a lot of attentions over the last decade since it has a wide range of applications in diverse scientific areas such as seismology and nondestructive testing.

The well-posedness of the direct scattering problems by rough surfaces has been extensively studied in the past thirty years. For acoustic case, the authors in [9, 10, 11, 23] first employed integral equation methods to prove that the scattering problem by an infinite rough surface with Dirichlet boundary condition is uniquely solvable. These works have been extended to other boundary conditions, see [25] for the

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impedance case and [22] for the penetrable case. By using variational formulation, the unique solvability of the sound-soft rough surface scattering problem has been established in [5, 7]. For the case of elastic scattering by Dirichlet rough surfaces, the uniqueness result was proved in [3], while the existence result was given in [4] using the boundary integral equation method (see also [2] for a comprehensive discussion). The authors in [12] studied the well-posedness of the elastic scattering problem by unbounded rough surfaces via the variational approach.

The computational aspect of the scattering problem by bounded obstacles has been extensively studied (see, e.g., [14, 21] for the finite elements method based on variational formulation and [6, 24] for the Nyström method based on boundary integral equations). For the acoustic scattering by unbounded rough surfaces, many numerical algorithms have been already developed. The numerical approach using finite elements method, combined with the perfectly matched layer technique, was presented in [8]. By dealing with a class of integral equations on the real line, the Nyström method has been applied to the acoustic scattering problem for the sound-soft rough surfaces [20, 18] and for the penetrable rough surfaces [17]. However, few works are available for the numerical solution of elastic scattering by unbounded rough surfaces.

In this paper, we propose the Nyström method for two-dimensional time-harmonic elastic scattering problem for unbounded rough surfaces with Dirichlet boundary condition. Our method is based on the integral equation formulations given in [2, 4], which can be reduced to a class of integral equations on the real line. A crucial role of our method is played by a thorough analysis on the singularities of the kernels in the relevant integral equations, which involves the Green tensor for Navier equation in the half-space. By splitting off the logarithmic singularity in the related kernels and using the asymptotic behavior of the Bessel functions, we obtain the convergence of the Nyström method with convergence rate depending on the smoothness of the rough surfaces. Several numerical examples are presented to verify our theoretical results and show the effectiveness of our method.

The paper is organized as follows. In Section 2, we give a brief introduction on the mathematical model of the scattering problem and present the existed well-posedness result using the integral equation method. Section 3 is devoted to analyzing the singularities for the relevant kernels included in the integral impression of the solution. In Section 4, we establish the convergence of the Nyström method. Numerical experiments are given to show the effectiveness of the proposed method in Section 5. Finally, we give a conclusion in Section 6.

2 The well-posedness of the scattering problem

In this section, we present the existed results for the well-posedness of the two-dimensional elastic wave scattering problem by unbounded rough surfaces. First, we introduce some basic notations and function spaces used in this paper. For $V \subset \mathbb{R}^m, m = 1, 2$, let $BC(V)$ represent the set of bounded and continuous complex-valued functions on $V$ under the norm $\|\varphi\|_{\infty, V} := \sup_{x \in V} |\varphi(x)|$. We denote by $BC^\infty(\mathbb{R}^m)$ the set
of all functions whose derivatives up to order $n$ are bounded and continuous on $\mathbb{R}^m$ with the norm
\[
\|\varphi\|_{C^n(R^m)} := \max_{l=0,1,\ldots,n} \max_{|\alpha|=l} \|\partial_{x_1}^{\alpha_1} \cdots \partial_{x_m}^{\alpha_m} \varphi\|_{C^0(R^m)},
\]
where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ and $|\alpha| := \sum_{i=1}^{m} \alpha_i$. We define $C^n_{0,\pi}(\mathbb{R}^2)$ and $BC^n_p(\mathbb{R}^2)$ as
\[
C^n_{0,\pi}(\mathbb{R}^2) := \{\varphi(s, t) \in BC^n(\mathbb{R}^2) : \varphi(s, t) = 0 \text{ for } |s - t| \geq \pi\},
\]
\[
BC^n_p(\mathbb{R}^2) := \{\varphi(s, t) \in BC^n(\mathbb{R}^2) : \|\varphi\|_{BC^n_p(\mathbb{R}^2)} < \infty\}
\]
with the norm
\[
\|\varphi\|_{BC^n_p(\mathbb{R}^2)} := \sup_{j,k=0,\ldots,\alpha_j+k \leq n} \left\| \frac{\partial^{j+k} \varphi(s, t)}{\partial s^j \partial t^k} \right\|_{C^0(R^2)},
\]
where the weight $w_p(s, t) := (1 + |s - t|)^p$ for some $p > 1$, which are closed subspaces of $BC^n(R^2)$. Let $H^1(V)$ and $H^{1/2}(\partial V)$ be the standard Sobolev spaces for any open set $V \subset \mathbb{R}^m$ provided the boundary of $V$ is smooth enough. The notations $H^1_{\text{loc}}(V)$ and $H^{1/2}_{\text{loc}}(V)$ stand for the set of functions which are elements of $H^1(V)$ and $H^{1/2}(V)$ for any $V \subset \subset V$, respectively. Here the notation $V \subset \subset V$ denotes that the closure of $V$ is a compact subset of $V$.

Throughout this paper, let $h$ be a real number with $h < \inf_{x_1 \in \mathbb{R}} f(x_1)$, where $f$ is the function of the rough surface which will be introduced later. We define the half-plane $U_h$ and its boundary $\Gamma_h$ as
\[
U_h := \{x \in \mathbb{R}^2 : x_2 > h\} \quad \text{and} \quad \Gamma_h := \{x \in \mathbb{R}^2 : x_2 = h\}.
\]
For $y = (y_1, y_2) \in U_h$, $y'$ is defined as
\[
y' := (y_1, 2h - y_2).
\]
The notations $J_n$ and $Y_n$ are Bessel functions and Neumann functions of order $n$, respectively. The linear combination
\[
H^{(1)}_n := J_n + i Y_n
\]
is known as the Hankel function of the first kind of order $n$.

As shown in Figure 1, the rough surface $\Gamma$ is described as
\[
\Gamma := \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = f(x_1)\},
\]
where $f \in B_{c,M}^{(n)}$ with
\[
B_{c,M}^{(n)} := \{f \in BC^{n+2}(\mathbb{R}) : \|f\|_{BC^{n+2}(\mathbb{R})} \leq M \quad \text{and} \quad \inf_{x_1 \in \mathbb{R}} f(x_1) \geq c\}
\]
for some nonnegative integer $n$, some constants $c > h$ and $M > 0$, which implies that the surface $\Gamma$ lies above the $x_1$-axis. The whole space is separated by $\Gamma$ into two unbounded half-spaces and the upper one is denoted by
\[
\Omega := \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > f(x_1)\}.
\]
Suppose an incoming field \( u^{\text{inc}} = (u_1^{\text{inc}}, u_2^{\text{inc}}) \) is incident on the infinite surface \( \Gamma \) from the upper region \( \Omega \). Then the scattering of \( u^{\text{inc}} \) by the infinite rough surface \( \Gamma \) can be modeled by the two-dimensional Navier equation

\[
\mu \Delta u^{sc} + (\lambda + \mu) \nabla \cdot u^{sc} + \omega^2 u^{sc} = 0 \quad \text{in} \quad \Omega, \tag{2.1}
\]

where \( u^{sc} = (u_1^{sc}, u_2^{sc}) \) is the scattered field, \( \omega > 0 \) represents the angular frequency, and \( \lambda \) and \( \mu \) are the Lamé constants satisfying \( \mu > 0 \) and \( \lambda + 2\mu > 0 \), which leads to that the second order partial differential operator \( \Delta^* := \mu \Delta + (\lambda + \mu) \nabla \cdot \) is strongly elliptic [19]. Let \( u := u^{\text{inc}} + u^{sc} \) denote the total field consisting of the incident field and the scattered field. Further, the Dirichlet boundary condition is imposed on \( \Gamma \), that is,

\[
u^{sc} = -u^{\text{inc}} \quad \text{on} \quad \Gamma. \tag{2.2}
\]

Since the Navier equation (2.1) is imposed in the unbounded region \( \Omega \), an appropriate radiation condition is needed for the considered scattering problem. In this paper, the scattered wave is assumed to satisfy the following upwards propagating radiation condition (UPRC) [4]:

\[
u^{sc}(x) = \int_{\Gamma_H} \Pi^{(2)}_{D,H}(x,y)\phi(y)ds(y), \quad x \in U_H, \tag{2.3}
\]

for some \( \phi \in [L^\infty(\Gamma_H)]^2 \) with \( H > \sup_{x_1 \in \mathbb{R}} f(x_1) \), and the kernel \( \Pi^{(2)}_{D,H}(x,y) = \left( \Pi^{(2)}_{D,H,jk}(x,y) \right)_{j,k=1,2} \) in (2.3) is a matrix function with the elements given by

\[
\Pi^{(2)}_{D,H,jk}(x,y) = \left( \mathbf{P}^{(0)} \left( G_{D,H,j}(x,y) \right)^\top \right)_k, \tag{2.4}
\]

where \( \mathbf{P} \) is the generalized stress vector defined on a curve \( \Lambda \in \mathbb{R}^2 \) with \( \nu \) being the unit normal on \( \Lambda \), that is,

\[
\mathbf{P} \varphi := (\mu + \widetilde{\mu}) \frac{\partial \varphi}{\partial \nu} + \widetilde{\lambda} \text{div} \varphi - \widetilde{\mu} \nu^+ \text{div}^+ \varphi \tag{2.5}
\]

with \( \widetilde{\lambda}, \widetilde{\mu} \in \mathbb{R} \) satisfying \( \widetilde{\lambda} + \widetilde{\mu} = \lambda + \mu, \nu^+ = (\nu_2, -\nu_1)^\top \) for \( \nu = (\nu_1, \nu_2)^\top \), and \( \text{div}^+ \varphi := \frac{\partial \varphi_2}{\partial x_2} - \frac{\partial \varphi_1}{\partial x_1} \) for \( \varphi = (\varphi_1, \varphi_2)^\top \). In (2.4), \( \nu \) is the unit normal on \( \Gamma_H \) pointing to the half-plane \( U_H \) and \( G_{D,H} \) is the Green’s
tensor of the Navier equation (2.1) in $U_H$ with Dirichlet boundary condition on $\Gamma_H$, which is given by

$$G_{D,H}(x,y) := G(x,y) - G(x,y') + U(x,y), \quad x,y \in U_H \quad \text{and} \quad x \neq y,$$  \hspace{1cm} (2.6)

where $G$ is the Green’s tensor for the Navier equation (2.1) in free space $\mathbb{R}^2$, which is defined by

$$G(x,y) = \frac{1}{\mu} \Phi(x,y,\kappa_1) + \frac{1}{\omega^2} \nabla_x \nabla_y^\top \left( \Phi(x,y,\kappa_2) - \Phi(x,y,\kappa_3) \right), \quad x,y \in \mathbb{R}^2 \quad \text{and} \quad x \neq y,$$  \hspace{1cm} (2.7)

with $\kappa_1$ and $\kappa_2$ being the shear and compressional wavenumbers defined by

$$\kappa_1 := c_s \omega, \quad \kappa_2 := c_p \omega \quad \text{with} \quad c_s = \mu^{-1/2}, \quad c_p := (\lambda + 2\mu)^{-1/2},$$

and

$$\Phi(x,y,\kappa) := \frac{i}{4} H_0^{(1)}(\kappa|x-y|)$$

being the fundamental solution for the two-dimensional Helmholtz equation, and $U(x,y)$ in (2.6) is a matrix function defined by

$$U(x,y) = -\frac{i}{2\pi \omega^2} \int_{\mathbb{R}} \left[ M_p(\tau, \gamma_p, \gamma_s; x_2, y_2) + M_s(\tau, \gamma_p, \gamma_s; x_2, y_2) \right] e^{-i(|x_1-y_1|)\tau} d\tau$$

with

$$\gamma_p := \sqrt{\kappa_s^2 - \tau^2}, \quad \gamma_s := \sqrt{\kappa_p^2 - \tau^2},$$

and

$$M_p(\tau, \gamma_p, \gamma_s; x_2, y_2) := \frac{e^{i\gamma_p (y_2 - y_1 - 2\tau)} - e^{i\gamma_s (y_2 - y_1 - 2\tau)}}{\gamma_p \gamma_s + \tau^2} \begin{bmatrix} -\tau^2 \gamma_s & \tau^3 \\ \tau \gamma_p \gamma_s & -\tau^2 \gamma_p \end{bmatrix},$$

$$M_s(\tau, \gamma_p, \gamma_s; x_2, y_2) := \frac{e^{i\gamma_p (y_2 - y_1 - 2\tau)} - e^{i\gamma_s (y_2 - y_1 - 2\tau)}}{\gamma_p \gamma_s + \tau^2} \begin{bmatrix} -\tau^2 \gamma_s & -\tau \gamma_p \gamma_s \\ -\tau^3 & -\tau^2 \gamma_p \end{bmatrix}.$$  

With the aid of [4, Theorem 2.1], we have $U(x,y) \in \left[ C^\infty(U_H) \cap C^1(\overline{U_H}) \right]^{2 \times 2}$, which will be used in the analysis on the convergence of the Nyström method. We refer to [2, 3] for more properties of the UPRC, and its relation to the Rayleigh expansion radiation condition for diffraction grating and the Kupradze’s radiation condition for the scattering by bounded obstacles.

To ensure the uniqueness of the scattering problem, we need the following vertical growth rate condition

$$\sup_{x \in \Omega} |x|^\beta |u^{sc}(x)| < \infty \quad \text{for some} \quad \beta \in \mathbb{R}.$$

(2.8)

In summary, the scattering problem (2.1)–(2.3) and (2.8) can be described by the following boundary value problem with $g = -u^{inc}$:

**Dirichlet Problem (DP):** Given $g \in \left[ BC(\Gamma) \cap H^{1/2}_{loc}(\Gamma) \right]^2$, find $u^{sc} \in \left[ C^2(\Omega) \cap C(\overline{\Omega}) \cap H^1_{loc}(\Omega) \right]^2$ such that
(i) \( u^{sc} \) is a solution of the Navier equation (2.1) in \( \Omega \),

(ii) \( u^{sc} \) satisfies the Dirichlet boundary condition \( u^{sc} = g \) on \( \Gamma \),

(iii) \( u^{sc} \) satisfies the UPRC (2.3),

(iv) \( u^{sc} \) satisfies the vertical growth rate condition (2.8).

The following uniqueness result has been proved in [3, Theorem 4.6] for the problem (DP).

**Theorem 2.1.** ([3, Theorem 4.6]) The problem (DP) has at most one solution.

The existence of the solution to the problem (DP) has been investigated in [4] by integral equation method. The main idea is to seek for a solution in the form of a combined single- and double-layer potential

\[
u^{sc}(x) = \int_{\Gamma} [\Pi^{(2)}_{D,h}(x,y) - i\eta G_{D,h}(x,y)\psi(y)] ds(y), \quad x \in \Omega, \tag{2.9}\]

where \( \Pi^{(2)}_{D,h}(x,y) \) is defined similar as \( \Pi^{(2)}_{D,h}(x,y) \) in (2.3) with \( \nu \) being the unit normal on \( \Gamma \) pointing to \( \Omega \), \( \psi \in [BC(\Gamma) \cap H^{1/2}_{loc}(\Gamma)]^2 \) and \( \eta \) is a complex number satisfying \( \text{Re}(\eta) > 0 \). Throughout this paper, to ensure that \( \Pi^{(2)}_{D,h}(x,y) \) has a weak singularity while \( |x - y| \to 0 \) for \( x, y \in \Gamma \), \( \tilde{\mu} \) and \( \tilde{\lambda} \) in (2.5) are chosen to be

\[
\tilde{\mu} = \frac{\mu(\lambda + \mu)}{\lambda + 3\mu}, \quad \tilde{\lambda} = \frac{(\lambda + \mu)(\lambda + 2\mu)}{\lambda + 3\mu}. \tag{2.10}\]

By doing so, it follows from Theorems 2.6 and 2.7, and Lemma 2.8 in [4] that \( u^{sc} \) given by (2.9) satisfies \( u^{sc} \in \left[ \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega}) \cap H^{1}_{loc}(\Omega) \right]^2 \). By Theorems 2.4 and 3.2 in [4], it can be deduced that \( u^{sc} \) given by (2.9) satisfies the Navier equation (2.1) and the UPRC (2.3). Further, as a consequence of Theorems 2.1 and 2.3 in [4], \( u^{sc} \) given by (2.9) satisfies the vertical growth rate condition (2.8) with \( \beta = -1/2 \).

According to the jump relations for elastic single- and double-layer potentials shown in Theorems 2.6 and 2.7 in [4], it is easy to see that \( u^{sc} \) given by (2.9) is a solution to the problem (DP) provided \( \psi \) is a solution to the following integral equation

\[
\frac{1}{2} \psi(x) + \int_{\Gamma} [\Pi^{(2)}_{D,h}(x,y) - i\eta G_{D,h}(x,y)\psi(y)] ds(y) = g(x), \quad x \in \Gamma, \tag{2.11}\]

which can be rewritten in the operator form

\[
(I + D_\Gamma - i\eta S_\Gamma)\psi = 2g \quad \text{on} \quad \Gamma, \tag{2.12}\]

where \( D_\Gamma \) and \( S_\Gamma \) are the elastic double-layer and single-layer operators given by

\[
(D_\Gamma \psi)(x) := 2 \int_{\Gamma} \Pi^{(2)}_{D,h}(x,y)\psi(y) ds(y), \quad x \in \Gamma,
\]

\[
(S_\Gamma \psi)(x) := 2 \int_{\Gamma} G_{D,h}(x,y)\psi(y) ds(y), \quad x \in \Gamma.
\]
For \( x, y \in \Gamma \), we denote \( x = x(s) = (s, f(s)), y = y(t) = (t, f(t)), \tilde{\psi}(t) := \psi(y(t)), \) and \( \tilde{g}(s) := g(x(s)) \). By changing the variables, we rewrite \( D \Gamma \) and \( S \Gamma \) as the following operators

\[
(D \tilde{\psi})(s) := 2 \int_{\mathbb{R}} \Pi_{D,\Gamma}^{(2)}(x(s), y(t)) \tilde{\psi}(t) \sqrt{1 + f'(t)^2} dt, \quad s \in \mathbb{R}, \tag{2.13}
\]

\[
(S \tilde{\psi})(s) := 2 \int_{\mathbb{R}} G_{D,\Gamma}(x(s), y(t)) \tilde{\psi}(t) \sqrt{1 + f'(t)^2} dt, \quad s \in \mathbb{R}. \tag{2.14}
\]

Then the solvability of (2.12) in \([BC(\Gamma)]^2\) is equivalent to finding the solution \( \tilde{\psi} \) to the integral equation

\[
(I + D - \eta S)\tilde{\psi} = 2\tilde{g} \tag{2.15}
\]

in \([BC(\mathbb{R})]^2\), which is given in the following theorem.

**Theorem 2.2.** ([4, Corollary 5.12]) For any \( f \in B_{c,M}^{(0)} \), the integral operator \( I + D - \eta S : [BC(\mathbb{R})]^2 \rightarrow [BC(\mathbb{R})]^2 \) is bijective (and so boundedly invertible) with

\[
\sup_{f \in B_{c,M}^{(0)}} \| (I + D - \eta S)^{-1} \|_{[BC(\mathbb{R})]^2 \rightarrow [BC(\mathbb{R})]^2} < \infty.
\]

Thus, the integral equation (2.11) and (2.15) have exactly one solution for every \( f \in B_{c,M}^{(0)} \) with

\[
\| u^{sc} \|_{\infty, \Gamma} \leq C \| g \|_{\infty, \Gamma}
\]

for some constants \( C > 0 \) depending only on \( B_{c,M}^{(0)} \) and \( \omega \).

**Remark 2.3.** By Theorems 2.1 and 2.2, the problem (DP) has a unique solution.

### 3 The singularity for the kernel of \( D - \eta S \)

This section is devoted to analyzing the singularity of the boundary integral equation (2.15), which will be used for the further investigation on the Nyström method in Section 4. The main idea is to write the kernel \( A(s, t) \) of the integral operator \( D - \eta S \) in (2.15) in the following form

\[
A(s, t) = \frac{1}{2\pi} B(s, t) \ln \left( 4 \sin^2 \frac{s - t}{2} \right) + C(s, t), \quad s, t \in \mathbb{R}, s \neq t, \tag{3.1}
\]

with smooth matrix functions \( B(s, t) \) and \( C(s, t) \) (see the formulas (3.24) and (3.25) below). For the details on the smoothness of \( B(s, t) \) and \( C(s, t) \), see Remark 4.2.

According to (2.4), (2.6), (2.13), and (2.14), the operators \( D \) and \( S \) can be decomposed into two parts as follows

\[
D = D_1 - D_2 \quad \text{and} \quad S = S_1 - S_2,
\]

where

\[
(D_1 \tilde{\psi})(s) := 2 \int_{\Gamma} \Pi_{1,\Gamma}^{(2)}(x(s), y(t)) \tilde{\psi}(t) \sqrt{1 + f'(t)^2} dt, \quad s \in \mathbb{R}, \tag{3.2}
\]
notations which will be used later. For \( x \)
This subsection is devoted to separating the logarithmic part of the operator
with the components of \( \Pi \)
Hence, the integral equation (2.15) can be rewritten as
\[
(S_1 \bar{\psi})(s) := 2 \int_\Gamma G(x(s), y(t)) \bar{\psi}(t) \sqrt{1 + f'(t)^2} dt, \quad s \in \mathbb{R},
\]
\[
(S_2 \bar{\psi})(s) := 2 \int_\Gamma [G(x(s), y(t')) - U(x(s), y(t))] \bar{\psi}(t) \sqrt{1 + f'(t)^2} dt, \quad s \in \mathbb{R},
\]
with the components of \( \Pi_1^{(2)}(x, y) \) and \( \Pi_2^{(2)}(x, y) \) given by
\[
\Pi_1^{(2)}(x, y) := \left( \mathbf{P}^{(2)}(G_j(x, y)) \right)_k \quad \text{and} \quad \Pi_2^{(2)}(x, y) := \left( \mathbf{P}^{(2)}(G_j(x, y') - U_j(x, y)) \right)_k.
\]
Hence, the integral equation (2.15) can be rewritten as
\[
[I + D_1 - i\eta S_1 - (D_2 - i\eta S_2)] \bar{\psi} = 2\mathbf{g} \quad \text{on} \quad \mathbb{R}.
\]
The remaining part of this section consists of three subsections, which focus on the singularity analysis of the kernels in the integral operators \( S_1, D_1, \) and \( D_2 - i\eta S_2, \) respectively.

### 3.1 Separating the logarithmic part of \( S_1 \)
This subsection is devoted to separating the logarithmic part of the operator \( S_1. \) We first introduce some notations which will be used later. For \( x(s), \ y(t) \in \Gamma, \) we define the distance between \( x(s) \) and \( y(t) \) as
\[
r = r(s, t) := |x(s) - y(t)| = \sqrt{(s - t)^2 + (f(s) - f(t))^2},
\]
and define the upward unit normal at \( x(s) \) and \( y(t) \) as
\[
\nu(s) = (\nu_1(s), \nu_2(s))^\top \quad \text{with} \quad \nu_1(s) = -\frac{f'(s)}{\sqrt{1 + f'(s)^2}} \quad \text{and} \quad \nu_2(s) = \frac{1}{\sqrt{1 + f'(s)^2}},
\]
\[
\nu(t) = (\nu_1(t), \nu_2(t))^\top \quad \text{with} \quad \nu_1(t) = -\frac{f'(t)}{\sqrt{1 + f'(t)^2}} \quad \text{and} \quad \nu_2(t) = \frac{1}{\sqrt{1 + f'(t)^2}}.
\]
Then for convenience, we define the vector \( l \) and \( l^\perp \) as
\[
l(s) = \sqrt{1 + f'(s)^2} \nu(s) \quad \text{and} \quad l^\perp(s) = \sqrt{1 + f'(s)^2} \nu^\perp(s),
\]
\[
l(t) = \sqrt{1 + f'(t)^2} \nu(t) \quad \text{and} \quad l^\perp(t) = \sqrt{1 + f'(t)^2} \nu^\perp(t).
\]
In terms of (2.7), a direct calculation shows that each element \( G_{jk} \) of the Green’s tensor \( G \) can be represented as
\[
G_{jk}(x, y) = \left[ \frac{i}{4\mu} \mathcal{H}_0^{(1)}(\kappa_s r) - \frac{i}{4\omega^2} \kappa_s \mathcal{H}_1^{(1)}(\kappa_s r) - \kappa_p \mathcal{H}_1^{(1)}(\kappa_p r) \right] \delta_{jk}
\]
\[
+ \frac{i}{4\omega^2} \kappa_s^2 \mathcal{H}_2^{(1)}(\kappa_s r) - \kappa_p^2 \mathcal{H}_2^{(1)}(\kappa_p r) \right] \frac{(x_j - y_j)(x_k - y_k)}{r^2}, \quad j, k = 1, 2,
\]
where \( x = (x_1, x_2), \ y = (y_1, y_2) \) and \( \delta_{jk} \) (\( j, k = 1, 2 \)) is the Kronecker delta function satisfying \( \delta_{jk} = 1 \) for \( j = k \) and \( \delta_{jk} = 0 \) for \( j \neq k \). Substituting \( x = x(s) := (x_1(s), x_2(s)) \) and \( y = y(t) := (y_1(t), y_2(t)) \) into (3.4) gives that \( S_1 \) can be rewritten as

\[
(S_1 \tilde{\psi})(s) = \int_{\mathbb{R}} A^{(1)}(s, t)\tilde{\psi}(t)dt, \quad s \in \mathbb{R},
\]

with the element of the matrix \( A^{(1)}(s, t) \) given by

\[
A^{(1)}_{jk}(s, t) = 2G_{jk}(x(s), y(t)) \sqrt{1 + f'(t)^2}
\]

\[
= \left\{ \frac{i}{2\mu} H_0^{(1)}(\kappa_s r) + \frac{\kappa_s J_1(\kappa_s r) - \kappa_p J_1(\kappa_p r)}{2\omega^2} \right\} \delta_{jk}
\]

\[
+ \frac{i}{2\omega^2} \frac{\kappa_p^2 J_2^2(\kappa_p r) - \kappa_s^2 J_2^2(\kappa_s r)}{r^2}(x_j(s) - y_j(t))(x_k(s) - y_k(t)) \sqrt{1 + f'(t)^2},
\]

for \( j, k = 1, 2 \). Based on the singularity of \( A^{(1)}_{jk}(s, t) \), we can separate the logarithmic part and decompose \( A^{(1)}_{jk}(s, t) \) as

\[
A^{(1)}_{jk}(s, t) = B^{(1)}_{jk}(s, t) \ln |s - t| + C^{(1)}_{jk}(s, t),
\]

where

\[
B^{(1)}_{jk}(s, t) = \frac{1}{\pi} \left\{ \frac{-1}{\mu} J_0(\kappa_s r) + \frac{\kappa_s J_1(\kappa_s r) - \kappa_p J_1(\kappa_p r)}{2\omega^2} \right\} \delta_{jk}
\]

\[
+ \frac{1}{2\omega^2} \frac{\kappa_p^2 J_2(\kappa_p r) - \kappa_s^2 J_2(\kappa_s r)}{r^2}(x_j(s) - y_j(t))(x_k(s) - y_k(t)) \sqrt{1 + f'(t)^2},
\]

(3.8)

\[
C^{(1)}_{jk}(s, t) = A^{(1)}_{jk}(s, t) - B^{(1)}_{jk}(s, t) \ln |s - t|.
\]

(3.9)

To get the exact expressions of \( B^{(1)}_{jk}(s, t) \) and \( C^{(1)}_{jk}(s, t) \) while \( s = t \) for numerical computation, we need to use the following asymptotic behavior of Bessel functions (see [15, (5.16.1)–(5.16.3)] and [16]): as \( r \to 0 \),

\[
Y_0(r) \approx -\frac{2}{\pi} \ln \frac{2}{r}, \quad H_0^{(1)}(r) \approx -\frac{2i}{\pi} \ln \frac{2}{r}, \quad (3.10)
\]

\[
J_n(r) \approx \frac{r^n}{2^n \Gamma(1 + n)}, \quad Y_n(r) \approx -\frac{\Gamma(n)}{\pi} \left( \frac{2}{r} \right)^n, \quad H_n^{(1)}(r) \approx -\frac{i\Gamma(n)}{\pi} \left( \frac{2}{r} \right)^n, \quad n > 0, \quad (3.11)
\]

\[
\kappa_s^2 H_2^{(1)}(\kappa_s r) - \kappa_p^2 H_2^{(1)}(\kappa_p r) = \frac{1}{\pi} (\kappa_p^2 - \kappa_s^2) + \frac{1}{4\pi} \left( \kappa_s^4 \ln \frac{\kappa_s r}{2} - \kappa_p^4 \ln \frac{\kappa_p r}{2} \right) r^2 + O(r^2), \quad (3.12)
\]

\[
\kappa_s^2 H_3^{(1)}(\kappa_s r) - \kappa_p^2 H_3^{(1)}(\kappa_p r) = \frac{2}{\pi} (\kappa_p^2 - \kappa_s^2) \frac{1}{r} - \frac{1}{4\pi} (\kappa_p^4 - \kappa_s^4) r + O(r^3 \ln \frac{1}{r}), \quad (3.13)
\]

where \( \Gamma(n) \) denotes the gamma function, and the notation \( \varphi_1(r) \approx \varphi_2(r) \) for functions \( \varphi_1 \) and \( \varphi_2 \) means that \( \lim_{r \to 0} \varphi_1(r)/\varphi_2(r) = 1 \). With the aid of \( H_0^{(1)}(r) = J_n(r) + iY_n(r) \) and the ascending series expansions of the Bessel functions (see [6, (3.97), (3.98)]), we can derive the following limits and asymptotic formulas

\[
\lim_{r \to 0} \left[ H_0^{(1)}(\kappa_s r) - \frac{2i}{\pi} J_0(\kappa_s r) \ln(\kappa_s r) \right] = \alpha,
\]
\[
\lim_{r \to 0} \frac{1}{r} \left\{ \kappa_s \left[ H_1^{(1)}(\kappa_s r) - \frac{2i}{\pi} J_1(\kappa_s r) \ln(\kappa_s r) \right] - \kappa_p \left[ H_1^{(1)}(\kappa_p r) - \frac{2i}{\pi} J_1(\kappa_p r) \ln(\kappa_p r) \right] \right\} = \frac{\alpha}{2} (\kappa_s^2 - \kappa_p^2),
\]
\[
\frac{1}{r^2} \left\{ \kappa_s^2 \left[ H_2^{(1)}(\kappa_s r) - \frac{2i}{\pi} J_2(\kappa_s r) \ln(\kappa_s r) \right] - \kappa_p^2 \left[ H_2^{(1)}(\kappa_p r) - \frac{2i}{\pi} J_2(\kappa_p r) \ln(\kappa_p r) \right] \right\} \approx \frac{\alpha}{8} (\kappa_s^4 - \kappa_p^4) - \frac{i}{\pi r^2} (\kappa_s^2 - \kappa_p^2),
\]
where \( \alpha = 1 + \frac{2i}{\pi} (C_E - \ln 2) \) with \( C_E \) standing for the Euler’s constant. The above asymptotic formulas imply that the diagonal terms are
\[
B_{jk}^{(1)}(s, s) = \frac{1}{\pi} \left[ -1 \cdot \frac{1}{\mu} + 1/2(c_s^2 - c_p^2) \right] \delta_{jk} \sqrt{1 + f'(s)^2}, \quad (3.14)
\]
\[
C_{jk}^{(2)}(s, s) = \sqrt{1 + f'(s)^2} \left\{ \left[ \frac{i}{4} \alpha (c_s^2 + c_p^2) - \frac{1}{2\pi} \left[ c_s^2 \ln(\kappa_s \sqrt{1 + f'(s)^2} + c_p^2 \ln(\kappa_p \sqrt{1 + f'(s)^2}) \right] \right] \delta_{jk}
+ \frac{1}{2\pi} (c_s^2 - c_p^2) \frac{l_j^2(s) + l_k^2(s)}{1 + f'(s)^2} \right\}, \quad (3.15)
\]
where \( l_j \) and \( l_k \) denote the components of the vector \( l^\perp \).

### 3.2 Separating the logarithmic part of \( D_1 \)

The purpose of this subsection is to separate the logarithmic part of the operator \( D_1 \). Due to (3.2), (3.6), (2.5), and (3.7), we can write \( D_1 \) as
\[
(D_1 \tilde{\psi})(s) = \int_\mathbb{R} A^{(2)}(s, t) \tilde{\psi}(t) dt, \quad s \in \mathbb{R},
\]
where the elements of the matrix \( A^{(2)}(s, t) \) are given by
\[
A^{(2)}_{jk}(s, t) := 2 \left[ P^{(s)} \left( G_j(x, y) \right) \right] \delta_{jk} \sqrt{1 + f'(t)^2}.
\]

Based on the singularity of \( A^{(2)}_{jk}(s, t) \), we can separate the logarithmic part of \( A^{(2)}_{jk}(s, t) \) as
\[
A^{(2)}_{jk}(s, t) = B^{(2)}_{jk}(s, t) \ln |s - t| + C^{(2)}_{jk}(s, t),
\]
where

\[
B_{jk}^{(2)}(s, t) := (\mu + \bar{\mu}) \left\{ \frac{1}{\pi \mu} \frac{\kappa_s J_1(\kappa_s r)}{r} - \frac{1}{\pi \omega^2} \frac{\kappa_{p}^2 J_2(\kappa_{p} r)}{r^2} \right\} \delta_{jk}
+ \frac{1}{\pi \omega^2} \frac{\kappa_s^2 J_3(\kappa_s r) - \kappa_{p}^3 J_3(\kappa_{p} r)}{r^2} \left[ (x_j(s) - y_j(t))(x_k(s) - y_k(t)) \right] \left\{ (s - t)f'(t) + f(t) - f(s) \right\}
+ (\mu + \bar{\mu}) \frac{1}{\pi \omega^2} \frac{\kappa_s^2 J_2(\kappa_s r) - \kappa_{p}^2 J_2(\kappa_{p} r)}{r^2} \left[ (x_j(s) - y_j(t))l_k(t) + (x_k(s) - y_k(t))l_j(t) \right]
+ \lambda \left\{ - \frac{1}{\pi \mu} \frac{\kappa_s J_1(\kappa_s r)}{r} + \frac{4}{\pi \omega^2} \frac{\kappa_s^2 J_2(\kappa_s r) - \kappa_{p}^2 J_2(\kappa_{p} r)}{r^2} \right\} (x_j(s) - y_j(t))l_k(t)
+ \frac{\mu - \bar{\mu}}{\pi \mu} \frac{\kappa_s J_1(\kappa_s r)}{r} \left\{ (f(s) - f(t))\delta_{j1} - (s - t)\delta_{j2} \right\} l_k^+(t),
\]

(3.16)

\[
C_{jk}^{(2)}(s, t) := A_{jk}^{(2)}(s, t) - B_{jk}^{(2)}(s, t) ln |s - t|.
(3.17)
\]

Similar as the previous subsection, we need to get the exact expressions of \(B_{jk}^{(2)}(s, t)\) and \(C_{jk}^{(2)}(s, t)\) while \(s = t\) for numerical computation. Using equations (3.10)–(3.13), choosing \(\bar{\mu}\) as in (2.10) and by a direct but lengthy calculation we obtain the diagonal terms

\[
B_{jk}^{(2)}(s, s) = 0,
\]

(3.18)

\[
C_{jk}^{(2)}(s, s) = -\frac{1}{2\pi} \frac{f''(s)}{1 + f'(s)^2} \left\{ \frac{1}{\pi \mu} \frac{\kappa_s J_1(\kappa_s r)}{r} + \frac{1}{\pi \omega^2} \frac{\kappa_s^2 J_2(\kappa_s r) - \kappa_{p}^2 J_2(\kappa_{p} r)}{r^2} \right\} \delta_{jk} + \left( c_{s}^2 - c_{p}^2 \right) \frac{l_k^+(t)l_j^+(s)}{1 + f'(s)^2},
\]

(3.19)

### 3.3 The computation of \(D_2 - i\eta S_2\)

Observing from (3.3), (3.5), and (3.6), the kernel of the integral operator \(D_2 - i\eta S_2\) is related to \(G(x, y')\) and \(U(x, y)\) with \(x = (s, f(s)) \in \Gamma, y = (t, f(t)) \in \Gamma,\) and \(y' = (t, 2h - f(t))\). Due to \(h < \inf_{x_1 \in \mathbb{R}} f(x_1)\), it is readily seen that there is a positive distance between \(x\) and \(y'\), which leads to that \(G(x, y')\) is smooth, and by [4, Theorem 2.1], we have \(U(x, y) \in \left[ C^{\nu}(U_h) \cap C^1(U_h) \right]^{2x2}\). It follows from (3.3), (3.5), (3.6), (2.5), (2.7) and a direct but lengthy calculation that the integral operator \(D_2 - i\eta S_2\) can be rewritten as integral on the real line, which reads

\[
\left[ (D_2 - i\eta S_2)\tilde{\psi} \right](s) = \int_{\mathbb{R}} A^{(3)}(s, t)\tilde{\psi}(t)dt, \quad s \in \mathbb{R},
\]

where

\[
A^{(3)}(s, t) = A^{(4)}(s, t) - A^{(5)}(s, t)
\]

(3.20)

with

\[
A_{jk}^{(4)}(s, t) := 2 \left\{ P^{(9)} \left[ G_j(x, y') \right]^T \right\} \delta_{jk} - \eta G_{jk}(x, y') \sqrt{1 + f'(t)^2} =: I_1 + I_2 + I_3 + I_4,
\]

and

\[
A_{jk}^{(5)}(s, t) := 2 \left\{ P^{(9)} \left[ G_j(x, y') \right]^T \right\} \delta_{jk} =: I_5 + I_6 + I_7 + I_8.
\]
\[ A_{jk}^{(5)}(s, t) := 2 \left\{ \begin{Bmatrix} \mathbf{P}(y) (U_j(x, y')) \end{Bmatrix}_k - i \eta U_{jk}(x, y') \right\} \sqrt{1 + f'(t)^2}. \]

Here, \( I_j \ (j = 1, 2, 3, 4) \) are defined by

\[
I_1 = (\mu + \bar{\mu}) \left\{ -i \frac{\kappa s H_1^{(1)}(\kappa s r')}{r'} \delta_{jk} + i \frac{\kappa^2 s H_2^{(1)}(\kappa s r') - \kappa^2 p H_2^{(1)}(\kappa p r')}{r'^2} \delta_{jk} \right. \\
- i \frac{\kappa^3 s H_3^{(1)}(\kappa s r') - \kappa^3 p H_3^{(1)}(\kappa p r')}{r'^3} (x_j(s) - y_j'(t))(x_k(s) - y_k'(t)) \left[(s - t)f'(t) + f(t) + f(s) - 2h \right] \\
\left. + i \frac{\kappa^2 s H_2^{(1)}(\kappa s r') - \kappa^2 p H_2^{(1)}(\kappa p r')}{r'^2} [(x_k(s) - y_k'(t))q_j l_j(t) + (x_j(s) - y_j'(t))q_k l_k(t)] \right\},
\]

\[
I_2 = -\lambda \left\{ -i \frac{\kappa s H_1^{(1)}(\kappa s r')}{r'} + i \frac{\kappa^2 s H_2^{(1)}(\kappa s r') - \kappa^2 p H_2^{(1)}(\kappa p r')}{r'^2} \right\} (y_j(s) - x_j'(t)) \\
+ i \frac{\kappa^3 s H_3^{(1)}(\kappa s r') - \kappa^3 p H_3^{(1)}(\kappa p r')}{r'^3} (x_j(s) - y_j'(t)) \left[(t - s)^2 - (f(t) + f(s) - 2h)^2 \right] \right\} l_k(t),
\]

\[
I_3 = -\mu \left\{ -i \frac{\kappa s H_1^{(1)}(\kappa s r')}{r'} + i \frac{\kappa^2 s H_2^{(1)}(\kappa s r') - \kappa^2 p H_2^{(1)}(\kappa p r')}{r'^2} \right\} \left[(s - t)\delta_{jk} + (f(s) + f(t) - 2h)\delta_{jk} \right] \\
- i \frac{\kappa^3 s H_3^{(1)}(\kappa s r') - \kappa^3 p H_3^{(1)}(\kappa p r')}{r'^3} (s - t)(f(s) + f(t) - 2h)(x_j(s) - y_j'(t)) \right\} l_k^{-1}(t),
\]

\[
I_4 = \eta \left\{ \frac{1}{2\mu} H_0^{(1)}(\kappa s r') - \frac{1}{2\omega^2} \kappa s H_1^{(1)}(\kappa s r') - \kappa p H_1^{(1)}(\kappa p r') \right\} \delta_{jk} \\
+ \frac{1}{2\omega^2} \kappa^2 s H_2^{(1)}(\kappa s r') - \kappa^2 p H_2^{(1)}(\kappa p r') \right\} (x_j(s) - y_j'(t))(x_k(s) - y_k'(t)) \right\} \sqrt{1 + f'(t)^2},
\]

where \( q = (q_1, q_2)^T = (-1, 1)^T \) and \( r' \) denotes the distance between \( x \) and \( y' \), that is,

\[ r' = r'(s, t) := \sqrt{(s - t)^2 + (f(s) + f(t) - 2h)^2}. \]

According to the above three subsections on the analysis of \( S_1, D_1, \) and \( D_2 - i \eta S_2 \), the integral operator \( D - i \eta S \) can be rewritten in the following form

\[
\left[ (D - i \eta S) \psi \right](s) = \int_{\mathbb{R}} A(s, t) \overline{\psi}(t) dt \quad (3.21)
\]

with

\[
A(s, t) = -i \eta A^{(1)}(s, t) + A^{(2)}(s, t) - A^{(3)}(s, t) := B^*(s, t) \ln |s - t| + C^*(s, t),
\]

where \( B^*(s, t) \) and \( C^*(s, t) \) are

\[
B^*(s, t) := -i \eta B^{(1)}(s, t) + B^{(2)}(s, t), \quad (3.22)
\]

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\[ C^*(s, t) := -i\eta C^{(1)}(s, t) + C^{(2)}(s, t) - A^{(3)}(s, t). \]  

(3.23)

In order to employ the Nyström method, we follow the ideas of [20, Theorem 2.1] and rewrite the integral kernel \( A(s, t) \) in the form (3.1) with \( B(s, t) \) and \( C(s, t) \) given by

\[ B(s, t) := \pi B^*(s, t) \chi(s - t), \quad \text{(3.24)} \]
\[ C(s, t) := B^*(s, t) \left[ (1 - \chi(s - t)) \ln |s - t| - \chi(s - t) \ln \left( \pi \frac{s - t}{\sqrt{t - i}} \right) \right] + C^*(s, t), \quad \text{(3.25)} \]

for \( s \neq t \), where \( \chi \in C^0(\mathbb{R}) \) is a cut-off function defined by

\[ \chi(s) = \begin{cases} 
1, & |s| \leq 1, \\
1 + \exp \left( \frac{1}{\pi - |s|} + \frac{1}{|s| - 1} \right), & 1 < |s| < \pi, \\
0, & |s| \geq \pi.
\end{cases} \]

It is easy to see that \( \chi \) satisfies \( \chi(s) \in [0, 1] \) for \( s \in \mathbb{R} \), \( \chi(s) = 0 \) for \( |s| \geq \pi \), \( \chi(s) = 1 \) for \( |s| \leq 1 \), and \( \chi(-s) = \chi(s) \) for \( s \in \mathbb{R} \).

Finally, with the help of (3.14), (3.15), (3.18), (3.19), (3.24) and (3.25), we obtain that

\[ B(s, s) := \pi \left[ -i\eta B^{(1)}(s, s) + B^{(2)}(s, s) \right], \]
\[ C(s, s) := -i\eta C^{(1)}(s, s) + C^{(2)}(s, s) - A^{(3)}(s, s). \]

4 Convergence analysis of the Nyström method

The goal of this section is to establish the convergence result of the Nyström method for the boundary integral equation (2.15). In views of (3.21), (2.15) can be rewritten in the following form

\[ \ddot{\psi}(s) + \frac{1}{2\pi} \int_{-\infty}^{+\infty} \ln \left( 4 \sin^2 \frac{s - t}{2} \right) B(s, t) \dot{\psi}(t) dt + \int_{-\infty}^{+\infty} C(s, t) \dot{\psi}(t) dt = 2\ddot{g}(s), \quad s \in \mathbb{R}. \]  

(4.1)

To get the numerical solution of (4.1), we truncate the infinite interval \((-\infty, +\infty)\) into a finite interval \((-cut, cut)\), and choose an equidistant set of knots \( t_j := -cut + j\pi/N \) for \( j = 0, 1, \ldots, 2Ncut/\pi \). If \( B(s, t) \in C_{0, n}^n(\mathbb{R}^2) \) and \( C(s, t) \in BC_p^n(\mathbb{R}^2) \) for some \( p > 1 \) and some positive integer \( n \), it follows from [20] that the two integrals in (4.1) can be approximated by

\[ \frac{1}{2\pi} \int_{-\infty}^{+\infty} \ln \left( 4 \sin^2 \frac{s - t}{2} \right) B(s, t) \dot{\psi}(t) dt \approx \sum_{j \in \mathbb{Z}} R^{(N)}_{j}(s) B(s, t_j) \dot{\psi}(t_j), \quad s \in \mathbb{R}, \]
\[ \int_{-\infty}^{+\infty} C(s, t) \dot{\psi}(t) dt \approx \frac{\pi}{N} \sum_{j \in \mathbb{Z}} C(s, t_j) \dot{\psi}(t_j), \quad s \in \mathbb{R}, \]

with the quadrature weights given by

\[ R^{(N)}_{j}(s) := -\frac{1}{N} \left\{ \sum_{m=1}^{N-1} \frac{1}{m} \cos m(s - t_j) + \frac{1}{2N} \cos N(s - t_j) \right\}. \]
Therefore, an approximated form of (4.1) is
\[
\tilde{\psi}_N(s) + \sum_{j \in \mathbb{Z}} \alpha_j^{(N)}(s)\tilde{\psi}_N(t_j) = 2\tilde{g}(s), \quad s \in \mathbb{R},
\]
with
\[
\alpha_j^{(N)}(s) := R_j^{(N)}(s)B(s, t_j) + \frac{\pi}{N}C(s, t_j).
\]

The remaining part of this section is to study the convergence result for \(\|\tilde{\psi} - \tilde{\psi}_N\|_{L^\infty(\mathbb{R})^2}\), which is presented in the following theorem.

**Theorem 4.1.** Let \(f \in B_{c,M}^{(n)}\) and \(\tilde{g} \in [BC^n(\mathbb{R})]^2\) for some \(n \in \mathbb{N}\) and \(c, M > 0\). There exists \(N_0 \in \mathbb{N}\) such that (4.2) admits a uniquely determined numerical solution \(\tilde{\psi}_N\) and
\[
\|\tilde{\psi} - \tilde{\psi}_N\|_{L^\infty(\mathbb{R})^2} \leq N^{-n}\|\tilde{g}\|_{BC^n(\mathbb{R})^2}
\]
for \(N > N_0\), where \(\tilde{\psi}\) is the unique solution of (2.15).

**Proof.** According to Theorem 2.2, the integral equation (2.15) has exactly one solution \(\tilde{\psi} \in [BC(\mathbb{R})]^2\) for every \(\tilde{g} \in [BC(\mathbb{R})]^2\) and there exists \(C_0 > 0\) such that \(\|(I + D - i\eta S)^{-1}\| \leq C_0\). Then by [20, Theorem 3.13], the statement of this theorem holds if \(B(s, t) \in C^1_{0,0}(\mathbb{R}^2)\) and \(C(s, t) \in BC^n_p(\mathbb{R}^2)\) for some \(p > 1\).

With the help of [20, Theorem 2.1], it is equivalent to showing the following three conditions: for all \(j, l \in \mathbb{N}\) with \(j + l \leq n\), there exists constants \(C > 0\) and \(p > 1\) such that

\begin{align*}
C1. \quad & \frac{\partial^{j+l}B^*(s, t)}{\partial s^j \partial t^l} \leq C, \quad s, t \in \mathbb{R}, \ |s - t| \leq \pi, \\
C2. \quad & \frac{\partial^{j+l}C^*(s, t)}{\partial s^j \partial t^l} \leq C, \quad s, t \in \mathbb{R}, \ |s - t| \leq \pi, \\
C3. \quad & \frac{\partial^{j+l}A(s, t)}{\partial s^j \partial t^l} \leq C(1 + |s - t|)^{-p}, \quad s, t \in \mathbb{R}, \ |s - t| \geq \pi,
\end{align*}

where \(B^*(s, t)\) and \(C^*(s, t)\) are defined by (3.22) and (3.23), respectively. Thus, it suffices to show that \(C1\)–\(C3\) hold.

For the condition \(C1\), we recall that \(B^*(s, t) = -i\eta B^{(1)}(s, t) + B^{(2)}(s, t)\), where \(B^{(1)}(s, t)\) and \(B^{(2)}(s, t)\) are defined by (3.8) and (3.16), respectively. By the asymptotic formulas (3.11) and the fact that \(J_n(r)\) is analytic for all \(r \in \mathbb{R}\), we obtain that \(J_n(r)/r^n\) is also analytic for all \(r \in \mathbb{R}\). Since \(f \in B_{c,M}^{(n)}\), it follows from (3.8) and (3.16) that \(B^{(1)}(s, t) \in BC^n(\mathbb{R}^2)\) and \(B^{(2)}(s, t) \in BC^n(\mathbb{R}^2)\), which implies that \(C1\) holds.

For the condition \(C2\), we recall that \(C^*(s, t) = -i\eta C^{(1)}(s, t) + C^{(2)}(s, t) - A^{(3)}(s, t)\), where \(C^{(1)}(s, t)\), \(C^{(2)}(s, t)\), and \(A^{(3)}(s, t)\) are defined by (3.9), (3.16), and (3.20), respectively. To prove this condition, we first introduce the following notations
\[
\rho_n(\kappa, s, t) := H_n^{(1)}(\kappa r) - \frac{2i}{\pi}J_n(\kappa r)\ln|s - t|, \quad y_n(s, t) := \frac{\kappa_1\rho_1(\kappa_1 s, t) - \kappa_2\rho_1(\kappa_2 s, t)}{r},
\]

\[14\]
\[ \gamma_2(s, t) := \frac{\kappa^2 \rho_2(\kappa_s, s, t) - \kappa^2 \rho_2(\kappa_p, s, t)}{r^2}, \quad \gamma_3(s, t) := r \left[ \kappa^3 \rho_3(\kappa_s, s, t) - \kappa^3 \rho_3(\kappa_p, s, t) \right], \]
\[ \xi(s, t) := \frac{(s - t)f'(t) + f(t) - f(s)}{r^2}, \]

and for \( j, k = 1, 2, \)
\[ \zeta_{jk}(s, t) := \frac{(x_j(s) - y_j(t))(\kappa_k(s) - y_k(t))}{r^2}, \]
\[ \sigma_{jk}(s, t) := -(\mu + \bar{\mu}) \frac{i}{2 \omega^2} \left( \frac{\gamma_2(s, t)}{r^2} \left[ (x_j(s) - y_j(t))\hat{k}(t) + (x_k(s) - y_k(t))\hat{l}(t) \right] + \hat{t} \left[ \frac{i}{2 \mu} \rho_1(\kappa_s, s, t) - \frac{i}{2 \omega^2} \frac{\gamma_2(s, t)}{r^2} \right] (x_j(s) - y_j(t))\hat{k}(t) \right) \]
\[ - \hat{t} \left( \frac{i}{2 \mu} \rho_1(\kappa_s, s, t) \right) \left[ \int (f(s) - f(t)) \delta \eta_j - (s - t) \delta \eta_j \right] \frac{\delta_j}{r^2} (t). \]

Using these notations and (3.23), (3.9), and (3.17), we can rewrite the elements of \( C^*(s, t) \) as
\[ C^*_{jk}(s, t) = \frac{\left( \rho_0(\kappa_s, s, t) - \frac{i}{2 \omega^2} \gamma_1(s, t) \right) \delta_{jk} + \frac{i}{2 \omega^2} \gamma_2(s, t) \zeta_{jk}(s, t)}{\sqrt{1 + f'(t)^2}} \]
\[ + (\mu + \bar{\mu}) \frac{i}{\omega^2} \kappa_t \rho_1(\kappa_s, s, t) + \frac{i}{2 \omega^2} \gamma_2(s, t) \delta_{jk} \frac{i}{2 \omega^2} \gamma_3(s, t) \zeta_{jk}(s, t) \xi(s, t) \]
\[ + \sigma_{jk}(s, t) - \bar{A}_{jk}^{(3)}(s, t), \quad j, k = 1, 2. \] (4.3)

With the help of \( H_n^{(1)}(z) = J_n(z) + iY_n(z) \) and the ascending series expansions of the Bessel functions (see [6, (3.97) and (3.98)]), we obtain that
\[ \rho_0(\kappa_s, s, t) = \left( 1 + \frac{2i}{\pi} C_E + \frac{2i}{\pi} \ln \frac{\kappa r}{2|s - \bar{t}|} \right) J_0(\kappa r) - \frac{2i}{\pi} \sum_{p=0}^{\infty} \frac{(-1)^p}{p!} \left( \frac{\kappa r}{2} \right)^{2p} \phi(p), \] (4.4)

\[ \kappa_t \rho_1(\kappa_s, s, t) = \left( 1 + \frac{2i}{\pi} C_E + \frac{2i}{\pi} \ln \frac{\kappa r}{2|s - \bar{t}|} \right) \frac{1}{\kappa_s} r J_1(\kappa_s r) - \frac{2i}{\pi} \sum_{p=0}^{\infty} \frac{(-1)^p}{p!} \frac{\left( \frac{\kappa_s r}{2} \right)^{2p+1}}{(p+1)!} \phi(p+1) + \phi(p)), \]

\[ \gamma_1(s, t) = \frac{\kappa_s \left( 1 + \frac{2i}{\pi} C_E + \frac{2i}{\pi} \ln \frac{\kappa r}{2|s - \bar{t}|} \right) J_1(\kappa_s r) - \kappa_p \left( 1 + \frac{2i}{\pi} C_E + \frac{2i}{\pi} \ln \frac{\kappa r}{2|s - \bar{t}|} \right) J_1(\kappa_p r)}{r} \]
\[ - \frac{i}{2 \pi} \sum_{p=0}^{\infty} \frac{(-1)^p}{p!(p+1)!} \left( \frac{r}{2} \right)^{2p} \left( \kappa_s^{2+2p} - \kappa_p^{2+2p} \right) (\phi(p) + \phi(p+1)), \] (4.5)

\[ \gamma_2(s, t) = \frac{\kappa_s^2 \left( 1 + \frac{2i}{\pi} C_E + \frac{2i}{\pi} \ln \frac{\kappa r}{2|s - \bar{t}|} \right) J_2(\kappa_s r) - \kappa_p^2 \left( 1 + \frac{2i}{\pi} C_E + \frac{2i}{\pi} \ln \frac{\kappa r}{2|s - \bar{t}|} \right) J_2(\kappa_p r)}{r^3} \]
\[ - \frac{i}{\pi} \left( \kappa_s^2 - \kappa_p^2 \right) - \frac{i}{2 \pi} \sum_{p=0}^{\infty} \frac{(-1)^p}{p!(p+2)!} \left( \frac{r}{2} \right)^{2p+2} \left( \kappa_s^{4+2p} - \kappa_p^{4+2p} \right) (\phi(p) + \phi(p+2)), \]

\[ \gamma_3(s, t) = \frac{\kappa_s^3 \left( 1 + \frac{2i}{\pi} C_E + \frac{2i}{\pi} \ln \frac{\kappa r}{2|s - \bar{t}|} \right) J_3(\kappa_s r) - \kappa_p^3 \left( 1 + \frac{2i}{\pi} C_E + \frac{2i}{\pi} \ln \frac{\kappa r}{2|s - \bar{t}|} \right) J_3(\kappa_p r)}{r^4} \]
\[ - \frac{2i}{\pi} \left( \kappa_s^2 - \kappa_p^2 \right) - \frac{1}{4 \pi} \left( \kappa_s^4 - \kappa_p^4 \right) \]
\[-\frac{i}{\pi} \sum_{p=0}^{+\infty} \frac{(-1)^p}{p!(p+3)!} \left( \frac{r}{2} \right)^{3+2p} (\kappa_s^{6+2p} - \kappa_p^{6+2p})(\phi(p) + \phi(p+3)), \tag{4.8} \]

and for $j, k = 1, 2,$

\[
\sigma_{jk}(s, t) = \frac{1}{\pi} \frac{\lambda + \mu}{\lambda + 3\mu} \xi(s, t) \delta_{jk} + \frac{i}{2\mu} \left[ \lambda (x_j(s) - y_j(t)) l_j(t) - \mu \left[ (f(s) - f(t)) \delta_{j1} - (s - t) \delta_{j2} \right] l_j(t) \right]
\]

\[
\times \left\{ \kappa_s \left( 1 + \frac{2i}{\pi} C_E + \frac{2i}{\pi} \ln \frac{\kappa_s r}{2|s - t|} \right) \frac{J_1(\kappa_s r)}{r} - \frac{2i}{\pi} \sum_{p=0}^{+\infty} \frac{(-1)^p}{p!(p+1)!} \left( \frac{\kappa_s r}{2(2s - t)} \right)^2 (\phi(p) + \phi(p+1)) \right\}
\]

\[
- \frac{i}{2\omega_0} \left[ (\mu + \overline{\mu}) \left[ (x_j(s) - y_j(t)) l_j(t) + (x_k(s) - y_k(t)) l_k(t) \right] + 4\lambda (x_j(s) - y_j(t)) l_j(t) \right]
\]

\[
\times \left\{ \kappa^2 \left( 1 + \frac{2i}{\pi} C_E + \frac{2i}{\pi} \ln \frac{\kappa_s r}{2|s - t|} \right) \frac{J_2(\kappa_s r)}{r^2} - \kappa^2 \left( 1 + \frac{2i}{\pi} C_E + \frac{2i}{\pi} \ln \frac{\kappa_p r}{2|s - t|} \right) \frac{J_2(\kappa_p r)}{r^2} \right\}
\]

\[
- \frac{i}{\pi} \left( \frac{\kappa^4}{4\pi} - \frac{\kappa^4}{4\pi} \right) - \frac{i}{\pi} \sum_{p=0}^{+\infty} \frac{(-1)^p}{p!(p+3)!} \left( \frac{r}{2} \right)^{3+2p} (\kappa_s^{6+2p} - \kappa_p^{6+2p})(\phi(p) + \phi(p+3)) \right\}, \tag{4.9} \]

where $\phi(0) := 0$ and $\phi(p) := \sum_{m=1}^{p} \frac{1}{m}$ for $p = 1, 2, 3, \ldots$. By a straightforward calculation and [1, Section 7.1.3], we have $\xi \in BC^n(\mathbb{R}^2), \sqrt{1 + f'(t)^2} \in BC^n(\mathbb{R}), \xi_j(s, t) \in BC^n(\mathbb{R}^2)$ and $\xi(s, t) \in BC^n(\mathbb{R}^2)$. Thus, using (4.4)–(4.8), we conclude that $\rho_0(\kappa, s, t) \in BC^n(\mathbb{R}^2), \kappa_s r p_1(\kappa_s, s, t) \in BC^n(\mathbb{R}^2), \gamma_1(s, t) \in BC^n(\mathbb{R}^2)$, $\gamma_2(s, t) \in BC^n(\mathbb{R}^2)$, and $\gamma_3(s, t) \in BC^n(\mathbb{R}^2)$. By a careful observation from (4.9), we have $\sigma_{jk}(s, t) \in BC^n(\mathbb{R}^2)$. Since $A^{(3)}(s, t) = A^{(4)}(s, t) - A^{(5)}(s, t)$, it follows from the smoothness of $G(x, y')$ and $U(x, y)$ for $x, y \in U_h$ that $A^{(3)}(s, t)$ is smooth. These, together with (4.3), imply that $C^{(3)}_{jk}(s, t) \in BC^n(\mathbb{R}^2)$. Thus, the condition C2 holds.

For the condition C3, it can be seen in [4, Theorem 2.1] that the elements of the Green’s tensor $G_{D, h}$ satisfies

\[
\max_{j,k=1,2} |G_{D, h, jk}(x, y)| \leq \frac{1 + (x_2 - h)(y_2 - h)}{|x_1 - y_1|^{3/2}}
\]

for $x, y \in U_h$ and $|x_1 - y_1| \geq \varepsilon > 0$. This, together with the regularity estimates for solutions to elliptic partial differential equations (see [13, Theorem 3.9]), implies that such estimates actually hold for partial derivatives of $G_{D, h}(x, y)$ of any order. Note that $A(s, t)$ is related to $G_{D, h}(x, y)$ and its derivatives, thus the condition C3 holds with $p = 3/2$ and $C > 0$ only depends on $M$. The proof is thus complete. \hfill \Box

Remark 4.2. From the proof of Theorem 4.1, it easily follows that $B(s, t) \in C^n_{0, \infty}(\mathbb{R}^2)$ and $C(s, t) \in BC^n(\mathbb{R}^2)$ with $p = 3/2$ if $f \in B^{(n)}_{c, M}$.  

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5 Numerical results

The purpose of this section is to illustrate the feasibility of the Nyström method by several numerical examples. As presented in (2.6), the expression of $G_{D,b}(x,y)$ involves the matrix function $U(x,y)$ which is smooth on the boundary $\Gamma$. However, $U(x,y)$ is given in terms of an improper integral on the infinite interval which is difficult to compute numerically. Thus in the numerical experiments, we replace $G_{D,b}(x,y)$ arising in (2.11) by $G(x,y) - G(x,y')$ to avoid the complicated computation of $U(x,y)$. It is shown that the numerical experiments are indeed satisfactory by using this replacement.

In the following examples, we assume that the Lamé constants $\lambda = 1$, $\mu = 1$ and the frequency $\omega = 20$. For the Nyström method of the integral equation (4.1), we choose $cut = 10\pi$. Setting $s = t_j$ for $j = 0, 1, 2, ..., 2cut/h$ in (4.2) gives a linear system of equations which can be solved to obtain the density $\overline{\psi}_N$, and then we can calculate the solution $u^{sc}$ through (2.9). In each example, we compute the scattered field at random points $z_i$, $i = 1, ..., Nb$ in the region $[-2.5, 2.5] \times [0.5, 1.5]$, where the number of random points $Nb = 101$. See the blue points in Figure 2 for the geometry profile. The elastic scattered field is a vector that can be written as $u^{sc} = (u_1^{sc}, u_2^{sc})$, we will compute the following error for this scattered field

$$E(v) := \frac{1}{Nb} \sum_{i=1}^{Nb} |v(z_i) - v^{app}(z_i)|^2. \quad (5.1)$$

Here $v$ is chosen to be $\text{Re} u_1^{sc}$, $\text{Im} u_1^{sc}$ or $|u_1^{sc}|$ for $i = 1, 2$ in our numerical implementation, $v^{app}$ is the corresponding value computed by our Nyström method.

![Figure 2: The solid lines in (a)–(c) represent the profile of the scattering interface for Examples 1–3, respectively, and the scattered field is computed on the blue random points in the region $[-2.5, 2.5] \times [0.5, 1.5]$.](image)

**Example 1.** We consider the elastic scattering by a planar $x_2 = 0$ with an incident plane wave. The profile of the flat surface is given in Figure 2(a). In general, an elastic plane wave can be written as a linear combination of a compressional plane wave $u_p^{inc}(x; \theta)$ and a shear plane wave $u_s^{inc}(x; \theta)$, that is,

$$u^{inc}(x; \theta) = \alpha u_p^{inc}(x; \theta) + \beta u_s^{inc}(x; \theta), \quad \alpha, \beta \in \mathbb{C}, \quad (5.2)$$

where $u_p^{inc}(x; \theta) = \theta e^{i\kappa_{p,\theta}x}$ and $u_s^{inc}(x; \theta) = \theta^j e^{i\kappa_{s,\theta}x}$ with $\theta \in S := \{ x \in \mathbb{R}^2 : |x| = 1 \}$ being an incident direction. In this example, we choose $\theta = (0, -1)^T$. We consider the cases $(\alpha, \beta) = (1, 0)$ and $(\alpha, \beta) =$
(0, 1) in (5.2), then the corresponding incident waves are
\[ u_1^{inc}(x) = u_p^{inc}(x) = (0, -e^{-i\kappa_0 z_2})^T \quad \text{and} \quad u_2^{inc}(x) = u_s^{inc}(x) = (-e^{-i\kappa_0 z_2}, 0)^T, \]
respectively. Since the rough surface is given by a planar \( x_2 = 0 \), it is easily seen that the corresponding scattered fields can be written explicitly as
\[ u_1^{sc}(x) = (u_{11}^{sc}, u_{12}^{sc})^T = (0, e^{i\kappa z_2})^T \quad \text{and} \quad u_2^{sc}(x) = (u_{21}^{sc}, u_{22}^{sc})^T = (e^{i\kappa z_2}, 0)^T, \]
respectively. Table 1 and Table 2 present the errors between the numerical results of the scattered fields and the exact solution computed by (5.1) for the cases \((\alpha, \beta) = (1, 0)\) and \((\alpha, \beta) = (0, 1)\), respectively. In each table, we give the errors that calculated by our Nyström method with \( N = 8, 16, 32, 64, 128 \), respectively. It can be seen from these two tables that the error between the numerical solution and the exact solution converges to 0 as \( N \) increases.

| N  | E(Re \( u_1^{sc} \)) | E(Im \( u_1^{sc} \)) | E(Re \( u_2^{sc} \)) | E(Im \( u_2^{sc} \)) | E(Re \( u_3^{sc} \)) | E(Im \( u_3^{sc} \)) |
|----|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|
| 8  | 0.1706786080         | 0.1604343838         | 0.3311134418         | 0.8032082301         | 0.6041408954         | 1.4073491255         |
| 16 | 0.0001175089         | 0.0000843658         | 0.0002018473         | 0.0001354001         | 0.0000366766         | 0.0001720767         |
| 32 | 0.0002007818         | 0.0001673652         | 0.0003681471         | 0.0001354001         | 0.0000366766         | 0.0001720767         |
| 64 | 0.0001618967         | 0.0001411445         | 0.0003030412         | 0.0001354001         | 0.0000366766         | 0.0001720767         |
| 128| 0.0001445112         | 0.0001313396         | 0.0002758508         | 0.0000699852         | 0.0000337565         | 0.0001037417         |

Table 2: Error against \( N \) for the incident wave \( u_2^{inc}(x) \) in Example 1 with a planar surface

| N  | E(Re \( u_1^{sc} \)) | E(Im \( u_1^{sc} \)) | E(Re \( u_2^{sc} \)) | E(Im \( u_2^{sc} \)) | E(Re \( u_3^{sc} \)) | E(Im \( u_3^{sc} \)) |
|----|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|
| 8  | 0.2047793681         | 0.2113488590         | 0.4161279271         | 0.3438342461         | 0.3232904684         | 0.6671247145         |
| 16 | 0.0007532224         | 0.001141582          | 0.0001894806         | 0.0000774519         | 0.0000768715         | 0.0001543233         |
| 32 | 0.000597451          | 0.000798512          | 0.0001395963         | 0.0000216754         | 0.0000316223         | 0.0000532977         |
| 64 | 0.000557738          | 0.000576602          | 0.000134340          | 0.0000130030         | 0.0000180143         | 0.0000310173         |
| 128| 0.000526921          | 0.000485121          | 0.0001012041         | 0.0000134101         | 0.0000154517         | 0.0000288618         |

**Example 2.** We consider the elastic scattering by a periodic unbounded rough surface with the periodic surface given by
\[ f(x_1) = 0.084 \sin(0.6\pi x_1) + 0.084 \sin(0.24\pi x_1) + 0.03 \sin(1.5\pi(x_1 - 1)). \]
See the profile of this periodic surface in Figure 2(b). The incident wave is chosen to be
\[ u_3^{inc}(x) = G(x, z)q \]
with the point \( z = (0, -3) \) and the polarization direction \( q = (0.6, 0.8)^T \). Due to the fact that the point \( z \) is below the surface \( \Gamma \), it follows from the well-posedness of the problem (DP) that the corresponding scattered field has the explicit expression
\[ u_3^{sc}(x) = (u_{31}^{sc}, u_{32}^{sc})^T = -G(x, z)q, \quad x \in \Omega. \]
Table 3 gives the errors between the numerical results of the scattered field calculated by our Nyström method with $N = 8, 16, 32, 64, 128$ and the exact solution, respectively. It can be seen from Table 3 that our Nyström method provides a satisfactory numerical results for this case.

| $N$ | $E(\Re u_{1,1}^{\text{inc}})$ | $E(\Im u_{1,1}^{\text{inc}})$ | $E(|u_{1,1}^{\text{inc}}|)$ | $E(\Re u_{1,2}^{\text{inc}})$ | $E(\Im u_{1,2}^{\text{inc}})$ | $E(|u_{1,2}^{\text{inc}}|)$ |
|-----|---------------------|---------------------|---------------------|---------------------|---------------------|---------------------|
| 8   | 0.0000729326        | 0.0000683190        | 0.0001412516        | 0.0000460758        | 0.0000434141        | 0.0000894899         |
| 16  | 0.000186844         | 0.000208419         | 0.000395263         | 0.0000323776        | 0.0000350254        | 0.0000674030         |
| 32  | 0.0000000333        | 0.0000000331        | 0.0000000664        | 0.0000000159        | 0.0000000179        | 0.0000000339         |
| 64  | 0.0000000344        | 0.0000000334        | 0.0000000678        | 0.0000000160        | 0.0000000173        | 0.0000000333         |
| 128 | 0.0000000346        | 0.0000000335        | 0.0000000681        | 0.0000000161        | 0.0000000173        | 0.0000000333         |

**Example 3.** We consider the elastic scattering by a non-periodic unbounded rough surface given by

$$f(x_1) = 0.1 \cos(0.1 x_1^2) e^{-\sin(x_1)}.$$  

See the profile of this rough surface in Figure 2(c). We choose the same incident wave $u_{3}^{\text{inc}}(x)$ as in (5.4). Similar as in Example 2, the corresponding scattered field has the form (5.5). Table 4 presents the error between our numerical results and the exact solution.

| $N$ | $E(\Re u_{1,1}^{\text{inc}})$ | $E(\Im u_{1,1}^{\text{inc}})$ | $E(|u_{1,1}^{\text{inc}}|)$ | $E(\Re u_{1,2}^{\text{inc}})$ | $E(\Im u_{1,2}^{\text{inc}})$ | $E(|u_{1,2}^{\text{inc}}|)$ |
|-----|---------------------|---------------------|---------------------|---------------------|---------------------|---------------------|
| 8   | 0.0000448509        | 0.0000367475        | 0.0000815984        | 0.0000443602        | 0.0000440444        | 0.0000884046         |
| 16  | 0.000098770         | 0.000073343         | 0.000172113         | 0.0000107694        | 0.0000074233        | 0.0000181927         |
| 32  | 0.000001459         | 0.000001046         | 0.000002505         | 0.000000564         | 0.000000511         | 0.000001075          |
| 64  | 0.000001320         | 0.000000788         | 0.000002108         | 0.000000510         | 0.000000474         | 0.000000985          |
| 128 | 0.000001320         | 0.000000782         | 0.000002102         | 0.000000506         | 0.000000479         | 0.000000985          |

### 6 Conclusion

In this paper, we present a Nyström method for the two-dimensional time-harmonic elastic scattering by unbounded rough surfaces with Dirichlet boundary condition. With the aid of the ascending series expansions of the Bessel functions, we analyze the singularities of the relevant integral kernels. Based on this, we obtain the superalgebraic convergence rate of the Nyström method depending on the smoothness of the rough surfaces. Several numerical examples demonstrate that the numerical solution converges when the number of quadrature points $N$ on the rough surface increases. A possible continuation is to extend the present work to the case of impedance boundary condition or the case of penetrable interface, which will be our future work.
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References

[1] K. Atkinson, *The Numerical Solution of Integral Equations of the Second Kind*, Cambridge University Press, UK, 1997.

[2] T. Arens, The scattering of elastic waves by rough surfaces, Ph.D. thesis, Brunel University, 2000.

[3] T. Arens, Uniqueness for elastic wave scattering by rough surfaces, *SIAM J. Math. Anal.* 33 (2001), 461-476.

[4] T. Arens, Existence of solution in elastic wave scattering by unbounded rough surface, *Math. Meth. Appl. Sci.* 25 (2002), 507-528.

[5] S.N. Chandler-Wilde and J. Elschner, Variational approach in weighted Sobolev spaces to scattering by unbounded rough surfaces, *SIAM J. Math. Anal.* 42 (2010), 2554-2580.

[6] D. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*, 4th edition, Springer, Berlin, 2019.

[7] S.N. Chandler-Wilde and P. Monk, Existence, uniqueness, and variational methods for scattering by unbounded rough surfaces, *SIAM J. Math. Anal.* 37 (2005), 598-618.

[8] S.N. Chandler-Wilde and P. Monk, The PML for rough surface scattering, *Appl. Numer. Math.* 59 (2009), 2131-2154.

[9] S.N. Chandler-Wilde and C.R. Ross, Scattering by rough surfaces: the Dirichlet problem for the Helmholtz equation in a non-locally perturbed half-plane, *Math. Methods Appl. Sci.* 19 (1996), 959-976.

[10] S.N. Chandler-Wilde, C.R. Ross and B. Zhang, Scattering by infinite one-dimensional rough surfaces, *Proc. Roy. Soc. Lond. A* 455 (1999), 3767-3787.

[11] S.N. Chandler-Wilde and B. Zhang, A uniqueness result for scattering by infinite rough surfaces, *SIAM J. Appl. Math.* 58 (1998), 1774-1790.

[12] J. Elschner and G. Hu, Elastic scattering by unbounded rough surfaces, *SIAM J. Math. Anal.* 44 (2012), 4101-4127.

[13] D. Gilbarg and N.S. Trudinger, *Elliptic Partial Differential Equations of Second-Order*, 2nd edition, Springer, Berlin, 1983.
[14] F. Ihlenburg, *Finite Element Analysis of Acoustic Scattering*, Springer, New York, 1998.

[15] N.N. Lebedev, *Special Functions and Their Applications*, Dover Publications, New York, 1972.

[16] J. Li and P. Li, Inverse elastic scattering for a random source, *SIAM J. Math. Anal.* 51 (2019), 4570-4603.

[17] J. Li, G. Sun and R. Zhang, The numerical solution of scattering by infinite rough interfaces based on the integral equation method, *Comput. Math. Appl.* 71 (2016), 1491-1502.

[18] J. Li and G. Sun, A nonlinear integral equation method for the inverse scattering problem by sound-soft rough surfaces, *Inverse Problems Sci. Eng.* 23 (2015), 557-577.

[19] W. McLean, *Strongly Elliptic Systems and Boundary Integral Equations*, Cambridge University Press, 2000.

[20] A. Meier, T. Arens, S.N. Chandler-Wilde and A. Kirsch, A Nyström method for a class of integral equations on the real line with applications to scattering by diffraction gratings and rough surfaces, *J. Integral Equations Appl.* 12 (2000), 281-321.

[21] P. Monk, *Finite Element Methods for Maxwell’s Equations*, Oxford University Press, New York, 2003.

[22] D. Natroshvili, T. Arens and S.N. Chandler-Wilde, Uniqueness, existence, and integral equation formulations for interface scattering problems, *Mem. Differential Equations Math. Phys.* 30 (2003), 105-146.

[23] C.R. Ross, Direct and inverse scattering by rough surfaces, Ph.D. Thesis, Brunel University, 1996.

[24] M.S. Tong and W.C. Chew, Nyström method for elastic wave scattering by three-dimensional obstacles, *J. Computat. Phys.* 226 (2007), 1845-1858.

[25] B. Zhang and S.N. Chandler-Wilde, Integral equation methods for scattering by infinite rough surfaces, *Math. Methods Appl. Sci.* 26 (2003), 463-488.