Average Number of Coherent Modes for Pulse Random Fields

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Abstract
Some consequences of spatio-temporal symmetry for the deterministic decomposition of complex light fields into factorized components are considered. This enables to reveal interrelations between spatial and temporal coherence properties of wave. An estimation of average number of the decomposition terms is obtained in the case of statistical ensemble of light pulses.

Keywords: partial coherence, modal decompositions

1. INTRODUCTION
Modal description of light coherence, being a multidimensional generalisation of the well-known Karhunen-Loéve expansion, was first introduced in optics by Gamo[1]. In short, the essence of the approach lies in the fact that any correlation function of a field $E(r, t)$ — in particular the transverse beam coherence $\Gamma_S(r, r')$ — can be expressed as a superposition of factorized components

$$\Gamma_S(r, r') = \int dt E(r, t)E^*(r', t) = \sum_n u_n \mathcal{E}_n(r) \mathcal{E}^*_n(r'),$$

where each term of the sum represents a completely coherent partial wave $\mathcal{E}_n(r)$. The decomposition basic functions and modal energies $u_n$ are eigenvectors and eigenvalues of Fredholm’s integral equation

$$u_n \mathcal{E}_n(r) = \int d^2r' \Gamma_S(r, r') \mathcal{E}_n(r')$$

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and, since the kernel of (2) is Hermitian, the set of functions $E_n(r)$ is orthonormal

$$\int d^2r E_n(r) E^*_m(r) = \delta_{n,m}. \quad (3)$$

In (1) the transverse correlation function $\Gamma_S(r, r')$ is determined as a time average (over the pulse duration or time of registration) and, in this sense, it is a deterministic characteristic of any particular wave. In the case of statistical ensemble of similar pulses the time averaging can be replaced by statistical one or both types of averaging may be combined. Formally it has no effect on relations (1) – (3), but, as it will be seen later, changes their physical meaning.

The modal expansion (1) – (3), as well as its modification for space-frequency domain\cite{2, 3}, is broadly used in coherence theory as a convenient tool for estimation of entropy and informational capacity of light beams\cite{4}, as the best method for modelling of coherence properties of a complex light wave with a finite set of simple mutually incoherent waves\cite{4}, and so on. These relations constitute the mathematical basis for proof of various types of uncertainty inequalities\cite{5}.

The next principal step in the development of the modal formalism was made by Pasmanik and Sidorovich\cite{6}. They demonstrated the spatio-temporal symmetry for decomposition (1) – (3), that leads to some important relations between spatial and temporal coherence characteristics of light waves. The discussion of them both in deterministic form and under application of the ensemble averaging is a main goal of the present paper.

2. DUAL APPROACH OVERVIEW

Let us suppose that for some wavefield $E(r, t)$, where $r$ is a two-dimensional radius-vector at a plane $z = const$, the modal expansion (1) – (3) is known. So far the solutions of the integral equation (2) form a complete functional basis (when including the functions, corresponding to zero eigenvalues\cite{7}), one can define a set of projections of the initial field onto this basis

$$\sqrt{u_n} e_n(t) = \int d^2r E(r, t) E^*_n(r), \quad (4)$$

that, in turn, allows to build up a modal representation of the field itself

$$E(r, t) = \sum_n \sqrt{u_n} E_n(r) e_n(t). \quad (5)$$

The most important point here is that for pulses of finite total energy

$$U = \int d^2r \int dt |E(r, t)|^2 < \infty$$

the projections $e_n(t)$ also constitute the complete orthonormal set of basic functions

$$\int dt e_n(t) e^*_m(t) = \delta_{n,m}. \quad (6)$$
Last relation can be proved by direct substitution of definition (4) into (6) and accounting (3). It is just this mutual orthogonality of temporal functions \( e_n(t) \), that leads to absence of any interference between different terms of spatial basis.

Another approach to evaluation of temporal basis (4) lies in use of dual integral equation

\[
 u_n e_n(t) = \int dt' \Gamma_T(t, t') e_n(t'),
\]

where \( \Gamma_T(t, t') \) is a global temporal correlation function of the field \( E(r, t) \)

\[
 \Gamma_T(t, t') = \int d^2 r E(r, t) E^*(r, t') = \sum_n u_n e_n(t) e_n^*(t').
\]

In contrast to the standard definition[2] (with averaging over the time or ensemble of pulses) the averaging procedure in (8) is carried out over the beam cross-section. Hence, the function \( \Gamma_T(t, t') \) expresses the overall correlation the wavefront patterns and is closely related to the degree of similarity[8, 9] \( H(t, t') \) of the wavefield for consequent time moments

\[
 H(t, t') = \left| \int d^2 r E(r, t) E^*(r, t') \right|^2 \left( \int d^2 r |E(r, t)|^2 \right) \left( \int d^2 r |E(r, t')|^2 \right).
\]

One can easily see that equations (2) and (7) make up two equivalent dual variants for evaluation the decomposition (5). Both equations have identical spectra of eigenvalues and for complete description of modal structure of field one needs to know only one set of basic functions \( E_n(r) \) or \( e_n(t) \). The second can be immediately determined through projection (4) or via its dual equivalent

\[
 \sqrt{u_n} E_n(r) = \int dt E(r, t) e_n^*(t).
\]

The last variant (6) – (9) has an advantage of dealing with 1-D task. One more exact consequence of the dual formalism is that two functions \( \Gamma_S(r, r') \) and \( \Gamma_T(t, t') \) in the case of no degeneracy (all \( u_n \) are different) allow one to completely reconstruct the form of field. Under degeneration (e.g. when \( u_i = u_j \)) the ambiguity arises from the fact that two different wave structures \( e_i(t)E_i(r) + e_j(t)E_j(r) \) and \( e_i(t)E_i(r) + e_j(t)E_j(r) \) produce the same correlation functions (1), (8).

Strictly speaking, the modal structure of the field does not remain constant under the wave propagation, but mode mixing is comparatively low for quasi-monochromatic beams with small divergence[3]. That is why the deterministic dual decomposition (5) is inherently aimed to description of laser pulses and has been first applied in nonlinear optics[9], where the partial coherence just means a high complexity of interacting waves.

In practice the complete modal description can be fulfilled only for very few classes of models[10, 11], what is, first of all, related with intricacy of integral equations (2), (7) solving. Therefore those consequences of the method are taking the special significance, for which one does not need to know the exact basic functions \( e_n(t), E_n(r) \).
3. EFFECTIVE NUMBER OF MODES

So far as the mode number $n$ in general cannot be univalently associated with any other parameter of partial wave (except its energy), the only natural way to restore distribution of $u_n$, without solving (2), (7), is evaluation of nonlinear $k$-order moments of modal spectra $\sum_n u_n^k$. It can be done with use of iterated kernels theorem via sequential integration of functions $\Gamma_S(r, r')$ or $\Gamma_T(t, t')$. The moment of 1st order has a trivial meaning of total field energy

$$U = \sum_n u_n = \int d^2r \Gamma_S(r, r) = \int dt \Gamma_T(t, t).$$

As it shown in Ref. 12, the higher moments determine a probability distribution of wave amplitude under conditions of wave mixing at a strong scatterer. The most important characteristic of field structure is an effective number of terms in decompositions (1), (5), (8)\cite{1, 6, 13}, which is expressed through 2nd order moment

$$N_{eff} = \left(\sum_n u_n^2\right) / \left(\sum_n u_n^2\right) = U^2 \left(\int d^2r \int d^2r' |\Gamma_S(r, r')|^2\right) = U^2 \left(\int dt \int dt' |\Gamma_T(t, t')|^2\right).$$

(10)

The value of $N_{eff}$ specifies the ability of the total field to produce interference effects between two arbitrary separate points of the beam cross-section\cite{3, 12} and changes from unity for spatial coherent one-mode wave to infinity for completely incoherent field.

Two equivalent forms of (10) reflect real interconnection between spatial and temporal parameters of a beam. If one determines effective area of beam cross-section $- S_{eff}$, area of spatial coherence $- \sigma_c$, pulse duration $- T_{eff}$ and correlation time $- \tau_c$ in the form

$$S_{eff} = \left(\int d^2r \Gamma_S(r, r)\right)^2 / \left(\int d^2r \Gamma_S^2(r, r)\right),$$

(11.a)

$$T_{eff} = \left(\int dt \Gamma_T(t, t)\right)^2 / \left(\int dt \Gamma_T^2(t, t)\right),$$

(11.b)

$$\sigma_c = \left(\int d^2r \int d^2\rho |\Gamma_S(r + \rho/2, r - \rho/2)|^2\right) / \left(\int d^2r \Gamma_S^2(r, r)\right),$$

(11.c)

$$\tau_c = \left(\int dt \int dt' |\Gamma_S(t + \tau/2, t - \tau/2)|^2\right) / \left(\int dt \Gamma_S^2(t, t)\right).$$

(11.d)

then relation (10) takes the form of equality for spatial and temporal degrees of freedom of wavefield

$$\frac{S_{eff}}{\sigma_c} = \frac{T_{eff}}{\tau_c}.$$  

(12)
It means that number of coherence zones per beam cross-section is equal to number of different spatial patterns over the pulse duration.

Three of introduced in (11) parameters — $T_{\text{eff}}, S_{\text{eff}}, \sigma_c$ — have quite traditional meaning[13] and need no special remarks. The averaged over beam cross-section coherence time $\tau_c$ describes time of global changing of field structure or, in other words, characteristic width by $t - t'$ of the degree of similarity $H(t, t')$ of spatial wave patterns. Definitions (10), (11) have no sensitivity to overall phase modulation of the field

$$E(r, t) \iff E(r, t) \exp(i\phi(r) + i\psi(t))$$

and, therefore, value of $\tau_c$ can rather significantly differ from local correlation time, which is defined in signal theory.

4. APPLICATION OF STATISTICAL AVERAGING

Till this point the basic formalism has dealt with a wave-field $E(r, t)$ as with the deterministic one. At the same time classical coherence theory usually operates with radiation characteristics, averaged over ensemble of similar fields, because in the majority of cases the individual pulse parameters are not of interest. Hence, the natural question arises — how can such stochastic hypotheses influence on the results of previous analysis?

As a first step let us consider what statistical averaging gives at the stage of the basic integral equations (2), (7) formulation. Just as in the standard approach[1], one can substitute the kernels $\Gamma_S(r, r')$ and $\Gamma_T(t, t')$ with their averages $\langle \Gamma_S(r, r') \rangle$ and $\langle \Gamma_T(t, t') \rangle$. However it is evident, that transversal correlation function (1), averaged over time interval only, contains much more information about spatial structure of a beam, than similar value of $\langle \Gamma_S(r, r') \rangle$ does. The approximate equality can take place only in the limit of infinite pulse duration and under quasiergodicity of the ensemble. Exactly the same with appropriate changing of words can be stated about temporal correlation function $\Gamma_T(t, t')$. It is easy to see that such a lack of information breaks the main property of the present formalism — its spatio-temporal duality.

As a result of this procedure the interpretation of the relations (1) – (4) and (6) – (9) must be changed. All other conclusions of the above sections (with exception of equality (12), which disappears) remain valid if taking into account that now we talk about two different and complementary means for describing of coherence of random pulse ensemble (but not for a particular wave). For transversal coherence this will be nonstationary variant of Gamo’s treatment[1] and for temporal correlation function it will have the form of modified Karhunen-Loéve expansion[14] with double averaging — over ensemble and beam cross-section.
In the subsequent discussion we shall utilise the fact that for each decomposition one can introduce its own number of degrees of freedom (10) — spatial $N_S$ and temporal $N_T$

$$N_S = \frac{\langle U^2 \rangle}{\left( \int d^2r \int d^2r' \left| \langle \Gamma_S(r,r') \rangle \right|^2 \right)} = \lim_{N_T,T\to\infty} N_{\text{eff}}, \quad (13.a)$$

$$N_T = \frac{\langle U^2 \rangle}{\left( \int dt \int dt' \left| \langle \Gamma_T(t,t') \rangle \right|^2 \right)} = \lim_{N_S,S\to\infty} N_{\text{eff}}. \quad (13.b)$$

Now it is clear that in order to preserve spatio-temporal duality of the formalism, the ensemble averaging should be applied at the later stages of consideration. As an example of such approach let us estimate the number of terms in the modal decomposition of a mean light pulse from the ensemble. The simplest and the most popular type of field statistics is Gaussian. In this case the coarse estimation can be done by averaging of integrals in definition of $N_{\text{eff}}$ (10), that, accounting the splitting of higher correlations and (13), gives a very simple formula

$$N_{\text{eff}} = \frac{N_S N_T}{N_S + N_T}. \quad (14)$$

On the basis of general reasons one can formulate some more requirements, which a priori should be satisfied by any admissible dependence $N_{\text{eff}}(N_S, N_T)$. Thus, the function $N_{\text{eff}}(N_S, N_T)$ must be symmetrical about permutation of its arguments because of the dual status of spatial and temporal degrees of freedom

$$N_{\text{eff}}(N_S, N_T) = N_{\text{eff}}(N_T, N_S). \quad (15)$$

The value of $N_{\text{eff}}$ must be a non-decreasing function of its arguments, that with (13) leads to conclusion

$$N_{\text{eff}} \leq N_S, N_T$$

and in asymptotics $N_S = \text{const} \cdot N_T \gg 1$ the average number of modes will be linear with respect to any of arguments, in particular,

$$N_{\text{eff}}(N_S = N_T = N \gg 1) \propto N.$$

Ensemble with only one degree of freedom in any of the basic subspaces corresponds to coherent (in terms of (5)) field

$$N_{\text{eff}}(N_S = 1, N_T) = N_{\text{eff}}(N_S, N_T = 1) = 1.$$

It is easy to see that estimation (14) obeys all above requirements but the last one, i.e. it poorly describes the region of small numbers of degrees of freedom (it is the consequence of approximate way of averaging $\langle N_{\text{eff}} \rangle$). The situation can be improved by taking into account the fluctuations not only of the denominator, but also of the numerator (energy of light pulses) of expression (10)

$$\langle U^2 \rangle = \langle U \rangle^2 \left( 1 + \varepsilon(N_S, N_T) \right),$$
\[ \varepsilon = \frac{1}{\langle U \rangle^2} \int d^2r \int d^2r' \int dt \int dt' |\langle E(r, t)E^*(r', t') \rangle|^2. \]

Correction \( \varepsilon \) must satisfy the condition (15) and have the order of magnitude \( \varepsilon \propto 1/(N_SN_T) \), that can be confirmed by consideration cross-spectrally pure light [15], when correlation function factorizes. Hence, the refined estimation of the number of modes in the mean pulse may be written as following

\[ N_{eff} = \frac{N_SN_T + 1}{N_S + N_T}. \]  

(16)

For the first time estimation like (16) was given without a proof in paper [16] for a system of several identical, statistically independent emitters with drifting phase. At limit \( N_S, N_T \gg 1 \) appropriate formula from Ref. 16 converts to (14). One can point out some more cases, which asymptotically lead to the same dependence. All this allows to say that area of applicability of relation (14) as estimation of \( N_{eff} \) is much wider than above assumptions.

In order to illustrate the consequences from relation (14) we can consider a very vivid example of the ensemble of Schell-model fields [10, 17]

\[ \langle E(r, t)E^*(r', t') \rangle = \sqrt{I(r, t)I(r', t')\gamma(r - r', t - t')} \]  

(17)

One of the possible interpretations of model (17) corresponds to illustrative situation when a fast shutter cuts off a pulse of radiation from a primary steady-state uniform partially coherent source. Then just beyond the shutter the degree of coherence \( \gamma(r, t) \) is specified by statistical parameters of the source only (say, with \( \sigma, \tau \) being an area and a time of correlation, respectively), while \( I(r, t) \) is (within a factor) a deterministic function of the shutter transmittance \( S, T \) — the shutter aperture area and the time it is opened). On substituting (17) into (13) and accounting (11), (14) one can assure that the effective area of coherence (in the mean pulse) depends not only on spatial parameters, but as well on ratio between temporal characteristics \( T/\tau \) of the primary source and the shutter. And vice versa, the lifetime of a particular wavefront structure in the mean pulse is also determined by the ratio of \( S/\sigma \). It explains the significance of the discussed modal formalism for the nonlinear optical and laser beam problems.

5. DISCUSSION

In conclusion it is worth to point out the resemblance of the considered modal technique with bi-orthogonal decompositions used in other branches of physics — e. g. turbulence theory [4] and pattern recognition [14]. Such tie is based on the common concept of complex process representation. By this analogy, the spatial partial coherence may be described as a sequence of more or less similar frames (instant wavefield structures) replacing each other. From this viewpoint coherent modes specify the feature basis of wavefronts evolution.
According to the general concept the application of global ensemble averaging procedure is efficient (it gives results with comparatively small relative variance) when the number of modes in the mean pulse is high. Nevertheless, there are situations where under small $N_{\text{eff}}$ the number of the ensemble degrees of freedom in one of the subspaces is much more than it ($N_T \gg N_{\text{eff}}$ or $N_S \gg N_{\text{eff}}$). In this case the statistical averaging over the corresponding complex substructure of the field may be useful.

How it is seen from (5), each single mode in deterministic decomposition produces factorized correlation function, i.e. corresponds to cross-spectral pure light\(^1\). But if we go to the whole multimode field, the spectral purity vanishes. Moreover, for the statistical ensemble even one-mode field will, in general, not be cross-spectral pure.

Besides the discussed manifestations of spatio-temporal symmetry for modal decomposition, there exists a wide class of uncertainty relations\(^5\), where it must also appear. They should have the form of inequalities bounding modal characteristics with such parameters of wave as angular divergence and spectral bandwidth. This statement leans against the fact that proof of uncertainty relations for correlation functions does not depend on the type of averaging used.

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