An extension to the complex plane of the Riemann-Siegel Z function.

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Abstract

The usual Riemann-Siegel Z(t) is a real-valued function. We construct a complex function depending from t and from distance from critical line. It is linked to Riemann Xi(s) function by the same real scaling factor of the usual Riemann-Siegel Z(t) on critical line. Errors are not greater than the errors of Riemann-Siegel Z(t) on the critical line, while this result covers at least the whole critical strip.

MSC-Class: 11M06, 11M26, 11M99

Keywords: Riemann Hypothesis; Generalized Riemann Hypothesis; Dirichlet L function; Riemann Z functions.

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1 Introduction

In 1859 Bernhard Riemann in an outstanding paper [2] [3, p. 299] [1] suggested that the non-trivial zeros of the analytic continuation of the function:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \Re(s) > 1$$  \hspace{1cm} (1.1)

to the so-called critical strip $0 < \Re(s) < 1$ must all lie on the line $\Re(s) = 1/2$ (the critical line). The complex argument $s$ is expressed throughout as:

$$s = \frac{1}{2} + \epsilon + it$$  \hspace{1cm} (1.2)

so that $\epsilon$ is the distance from critical line parallel to the real axis and $t$ is the imaginary coordinate. Until now, this conjecture has remained unproven, and is referred to as the Riemann Hypothesis (RH) [1]. A key finding in this effort has been the introduction of the function $\xi(s)$ [9] [3, p. 16], that is real on the critical line and has the same zeros of $\zeta(s)$ in the critical strip:

$$\xi(s) = \Gamma\left(\frac{s}{2} + 1\right) (s - 1)\pi^{-s/2} \zeta(s).$$  \hspace{1cm} (1.3)
In 1932 C. L. Siegel crucially succeeded in mastering the information contained in Riemann’s private notes about the values of the ζ function on the critical line [3, p. 136] [5, p. 3]. His main result was the approximate formula:

$$Z(t) = 2 \sum_{n=1}^{n=N} \cos \left( \frac{t \ln \left( \frac{n}{2\pi n^2} \right) - \frac{\pi}{8} }{\sqrt{n}} \right) + R_0(t) \sim -\frac{\xi(1/2 + it)}{f(t)} ; \quad N = \left\lceil \sqrt{\frac{t}{2\pi}} \right\rceil$$ (1.4)

where $\sim$ means asymptotically equal for $t \to \infty$ and $f(t)$ is the scale factor used in [3, p. 176], reported in eq. (3.33). Equation (1.4) has been the core of large scale computations of the zeros of $\zeta(s)$ as described in [4] and [5, p. 4].

The purpose of this paper is to develop an extension of the Riemann-Siegel function (1.4) that provides a means of expressing $\xi(1/2 + \epsilon + it)$ at least for $-1 < \epsilon < 1$ using a scale factor defined in eq. (3.31):

$$F(t) = \left( \frac{\pi}{2} \right)^{0.25} t^{2} e^{-\frac{\pi}{4} t}$$

and hyperbolic functions. For ease of reading, we present our final result, given in eq. (4.8):

$$Z(t, \epsilon) = 2 \sum_{n=1}^{N} \cosh \left[ \epsilon \ln \left( \frac{t}{2\pi n^2} \right) \right] \cos \left( \frac{t \ln \left( \frac{n}{2\pi n^2} \right) - \frac{\pi}{8} }{\sqrt{n}} \right) +$$

$$+ 2t \sum_{n=1}^{N} \sinh \left[ \epsilon \ln \left( \frac{t}{2\pi n^2} \right) \right] \sin \left( \frac{t \ln \left( \frac{n}{2\pi n^2} \right) - \frac{\pi}{8} }{\sqrt{n}} \right) + R(t, \epsilon) + \text{Err}(t, \epsilon) = -\frac{\xi(1/2 + \epsilon + it)}{F(t)e^{\frac{\pi}{8} t}}$$

where $N = \left\lceil \sqrt{\frac{t}{2\pi}} \right\rceil$, $F(t) = \left( \frac{\pi}{2} \right)^{0.25} t^{2} e^{-\frac{\pi}{4} t}$, [3, p. 119] for $t$ big, $|\text{Err}(t, \epsilon)| < e^{-0.1t}$ at least for $|t| > 100$ [3, p. 144], see (B.25), (B.26), and (B.27). $R_1(t)$ is given by eq. (4.4) specialized below for $M = 1$:

$$R_1(t, \epsilon) = (-1)^{N-1} \left( \frac{2\pi}{t} \right)^{1/4} \left[ C_0(p) + C_1(p, \epsilon) \left( \frac{2\pi}{t} \right)^{1/2} \right] ; \quad p = \sqrt{\frac{t}{2\pi}} - N$$ (1.6)

where $C_0(p) = \cos(2\pi(p^2 - p - 1/16))/\cos(2\pi p)$. Using $C_1(p, \epsilon)$ the $Z$ function given in eq. (1.5) is almost holomorphic for $t >> 1$ and small $\left| \frac{1}{2} \right|$, as shown in eq. (4.13). From a computational point of view it can be meaningful to use eq. (1.5) with only $C_0(p)$, that is disregarding $C_1(p, \epsilon)$, since the errors of (1.5) with $R_0(t)$ are not higher than the errors of the original Riemann-Siegel (1.4), as shown in Figure 3.

We report in tables 1 and 2 the result of a numerical comparison with Wolfram Mathematica Riemann-SiegelZ function, referred to as $Z_M(t - i\epsilon)$, evaluated in same points outside critical line.

Using (4.8) it is easy, for example, to compute points of a topographical surface defined by $\Re[e^{i\pi/2}Z(t, \epsilon)]$ above the $(t, \epsilon)$ plane, or to plot in the $(t, \epsilon)$ plane the zero-height points (i.e $(t, \epsilon) : \Re[e^{i\pi/2}Z(t, \epsilon)] = 0$). The same can be done with $\Im[e^{i\pi/2}Z(t, \epsilon)]$; see Figures 1 and 2. In fact the real or imaginary zero-condition computed for $e^{i\pi/2}Z(t, \epsilon)$, applies to $\xi(1/2 + \epsilon + it)$ as well. The original reason of the present work was precisely to plot such curves for the ξ function, as is done for the ζ function in [2, p. 342].

The paper is an exercise on [3, p. 136-155]. The reading will be easier with a copy of chapter 7 of [3] within reach.
ε | ℜ[\(Z(t, \epsilon)\)]; see 1.5 | ℑ[\(Z(t, \epsilon)\)]; see 1.5 | ℜ[\(Z_M(t - i \epsilon)\)] | ℑ[\(Z_M(t - i \epsilon)\)] | ||\(Z - Z_M||\) |
---|---|---|---|---|---|
0.1 | 3.241730475804 | -0.5787044368126 | 3.241771462370 | -0.578683059836 | 4.62E-005 |
0.2 | 3.744381644160 | -1.199282431530 | 3.744419881769 | -1.199241919295 | 5.57E-005 |
0.3 | 4.642337399179 | -1.907135487575 | 4.642369954238 | -1.907080635877 | 6.37E-005 |
0.4 | 6.033267420500 | -2.755039800815 | 6.033289474138 | -2.754980028392 | 6.36E-005 |
0.5 | 8.069550741786 | -3.807665184897 | 8.069554328448 | -3.807617504896 | 4.78E-005 |

Table 1: Comparison at \(t = 7000\) between \(Z(t, \epsilon)\) (with only \(C_0(p)\)) in 1.5 and Wolfram Mathematica RiemannSiegelZ function \(Z_M(t - i \epsilon)\). Both are evaluated with a precision of 16 digits.

ε | ℜ[\(Z(t, \epsilon)\)]; see 1.5 | ℑ[\(Z(t, \epsilon)\)]; see 1.5 | ℜ[\(Z_M(t - i \epsilon)\)] | ℑ[\(Z_M(t - i \epsilon)\)] | ||\(Z - Z_M||\) |
---|---|---|---|---|---|
0.1 | -0.9050244263086 | 0.1402585183494 | -0.9050238453328 | 0.1402587799723 | 6.37E-007 |
0.2 | -1.308045529855 | 0.2878855199489 | -1.3080449625265 | 0.2878860590214 | 7.83E-007 |
0.3 | -2.143656564420 | 0.441671103478 | -2.143656009082 | 0.441671983111 | 1.04E-006 |
0.4 | -3.715336354275 | 0.578566730780 | -3.715335798319 | 0.578568148509 | 1.52E-006 |
0.5 | -6.582066395361 | 0.629253718227 | -6.582065829497 | 0.629256211048 | 2.56E-006 |

Table 2: Comparison at \(t = 250000\) between \(Z(t, \epsilon)\) (with only \(C_0(p)\)) in 1.5 and Wolfram Mathematica RiemannSiegelZ function \(Z_M(t - i \epsilon)\). Both are evaluated with a precision of 16 digits.

2 Starting point

In [9] [3, p. 137] a contour integral is given for \(\zeta(s)\), \(s \neq 1\), \(s \in \mathbb{C}\):

\[
\zeta(s) = \frac{\Gamma(1-s)}{2\pi i} \int_C \frac{(-x)^s}{(e^x - 1)x} dx
\]

(2.1)

where \(C\) is the contour described in the positive sense starting at \(+\infty\), encircling the origin and returning to \(+\infty\) without crossing the positive real axis. Afterwards this path \(C\) is deformed continuously in \(\mathbb{C}_N\), encircling \(2N+1\) poles of the integral in eq. (2.1). Evaluating the integral and putting the result in the definition of \(\xi(s)\) in eq. (1.3), one finds (see [3, p. 138], [7, p. 20]):

\[
-\xi(s) = (1-s)\Gamma\left(\frac{s}{2} + 1\right) \pi^{\frac{s}{2}} \left(\sum_{n=1}^{n=N} n^{-s}\right) + (s)\Gamma\left(\frac{1-s}{2} + 1\right) \pi^{-\frac{1-s}{2}} \left(\sum_{n=1}^{n=N} n^{-(1-s)}\right) + \frac{(-s)^{s-1}}{(2\pi)^{s-1} 2 \sin(\pi s/2) 2\pi i} \int_{C_N} \frac{(-x)^{s-1} e^{-Nx} dx}{e^x - 1}
\]

(2.2)

where the sign has been changed for later convenience and:

\[
N = \left\lfloor \frac{t}{2\pi} \right\rfloor.
\]

(2.3)
Figure 1: The points of $\Im[Z(t, \epsilon)] = 0$ and $\Re[Z(t, \epsilon)] = 0$ are computed on equispaced lines at constant $t = \Delta t \times m : m = 1, 2, 3,...$. At the intersection between the curve $\Im[\xi(t, \epsilon)] \sim \Im[Z(t, \epsilon)] = 0$ with the critical line, at $\epsilon = 0$, the conformality of the $\xi(t, \epsilon)$, seen as a complex transformation, is lost. These points are extremal points for the amplitude of $\xi(1/2+it)$. At low $t$ values this is not true for $Z(t, \epsilon)$ because of the distorting effect of the scale factor $F(t)$. Note the first seven zeros at $t = 14.13...; 21.02...; 25.01...; 30.42...; 32.93...; 37.58...; 40.91...$

We now denote by $L_N$ is the usual broken line whose path segments $L_0$, $L_1$, $L_2$ and $L_3$ are defined in [3, p. 138] [7, p. 20] in order to apply the steepest descent method. In Appendix B it is shown that the dominant contribution to the integral in (2.2) comes from $L_1$ which extends from $a + \frac{1}{2} e^{i \frac{\pi}{4}}|a|$ to $a - \frac{1}{2} e^{i \frac{\pi}{4}}|a|$, see (B.1) and (B.17), where:

$$a = i\sqrt{2\pi t}$$

(2.4)

is the saddle point for the evaluation of main integral in (2.2), see Appendix A for the details. We also use the notation of [3, p. 139], where:

$$e^{\phi(x)} = (-x)^{s-1} e^{-N x}$$

The saddle point occurs when $\phi'(x) = 0$ with $\phi(x) = \Re[(s - 1) \ln(-x) - N x]$ to yield:

$$\alpha = -\frac{1}{2} + it \approx \frac{-\frac{1}{2} + it}{\sqrt{\frac{t}{2\pi}}} \approx 2\pi i N \approx i\sqrt{2\pi t} = a \quad \text{for} \quad t >> 1$$

(2.5)

so that $a$ in (2.4), is an approximation for the value $\alpha$ used in [3, p. 140]. Notice that in (2.5), $N$ comes from the numerator in $\frac{-\frac{1}{2} + it}{N}$, while in $2\pi i N$ it comes from intergrand denominator (i.e. $e^x - 1$).
Figure 2: See [3, p. 176] and [5, p. 6]. First Lehmer phenomenon is at $t \approx 7005$. Note the much simpler plot of $\Re[\xi] = 0$ and $\Im[\xi] = 0$ curves with respect to the plot of $\Re[\zeta] = 0$ and $\Im[\zeta] = 0$ curves in $(t, \epsilon)$ plane shown in [2, p. 342].

Nothing changes in the saddle point integral evaluation procedure with the replacement:

$$-\frac{1}{2} + it \rightarrow -\frac{1}{2} + \epsilon + it \text{ with } 0 < \epsilon < 1$$

as shown in Appendix A. Notice that for $\epsilon = \frac{1}{2}$ the approximation (2.5) becomes exact and we have $\alpha = a$.

We now make use of the Stirling series [8, p. 30] that allows us to write:

$$\ln(\Gamma(z + 1)) = \ln \left(e^{-z}z^{z+\frac{1}{2}}(2\pi)^{\frac{1}{2}}\right) + \sum_{k=1}^{K-1} \frac{B_{2k}}{2k(2k-1)z^{2k-1}} + R_{2K}(z) \quad (2.6)$$

Although the expression (2.6) is a non-convergent asymptotic expansion, it can be used to estimate the size of the error $|R_{2K}(z)|$ [3, p. 112]. The $B_i$ are the Bernoulli numbers that vanish for odd $i$ while (see for example [6, p. 114]):

$$B_2 = \frac{1}{6} \; ; \; B_4 = -\frac{1}{30} \; ; \; B_6 = \frac{1}{42} \; ; \; B_8 = -\frac{1}{30} \; ; \; B_{10} = \frac{5}{66} \; ; \; B_{12} = -\frac{691}{2730} \; ...$$

The modulus of the error term $|R_{2K}(z)|$ is bounded by:

$$|R_{2K}(z)| < \left(\frac{B_{2K}}{2K(2k-1)z^{2K-1}}\right)^{2K} \cdot \frac{1}{\cos\left(\frac{\text{arg}(z)}{2}\right)} \quad (2.7)$$

where $\text{arg}(z)$ is taken in the interval: $-\pi < \text{arg}(z) < \pi$ (see [3, p. 112] and theorem 2.3 in [8, p. 40] due to Stieltjes).
3 Splitting in simple pieces

To evaluate the expression in (2.2) it is convenient to rewrite $s$ in terms of the variable $z$ defined by:
\[
z = \frac{s}{2} = \frac{1 + 2\epsilon}{4} + \frac{it}{2}
\]
(3.1)
in the first sum, so that we have:
\[
\pi^{\frac{s}{2}}(1 - s) = \pi^{-s}(1 - 2z),
\]
while in the second sum and in the third term we use:
\[
z = \frac{1 - s}{2} = \frac{1 - 2\epsilon}{4} - \frac{it}{2}
\]
(3.2)
so that:
\[
\pi^{\frac{(1 - s)}{2}}(s) = \pi^{-s}(1 - 2z)
\]

In order to manage the computation and also to exploit intermediate results, it has been devised the notation $A_p^s$ where:

"A " can be: $A = \Re$ or $A = \Im$,

"s= +" means substitution (3.1) while "s= -" means substitution (3.2).

"p" is for part ($p=1,2$ or $3$)

Part 1 (i.e $\Re_1^\pm$ or $\Im_1^\pm$) is the asymptotic part (as $t \to \infty$) of $\ln (e^{-z}z^{\pm \frac{1}{2}}(2\pi)^{\frac{1}{2}})$

Part 2 (i.e $\Re_2^\pm$ or $\Im_2^\pm$) refers always to $\ln (e^{-z}z^{\pm \frac{1}{2}}(2\pi)^{\frac{1}{2}})$, but as $t \to \infty$ goes to zero

Part 3 (i.e $\Re_3^\pm$ or $\Im_3^\pm$) refers to:
\[
\left( \sum_{k=1}^{K-1} \frac{B_{2k}}{2k(2k - 1)z^{2k-1}} \right)
\]
(3.3)

For example for $K = 3$, setting $\arg(z) \approx \frac{\pi}{2}$ and $B_{2K} = 1/42$ as first erased term in (2.6), the error $|R_{2K}(z)|$ is bounded by (2.7) and using (3.1) and (3.2) we have, for the error term:
\[
|R_{2K}(z)^\pm| = |R_{error}^\pm + i \ Im_{error}^\pm| = \left| R_6 \left( \frac{1/2 \pm \epsilon \pm it}{2} \right) \right| \leq \left| \frac{B_6}{6 \times 5 \times \left( \frac{1/2 \pm \epsilon \pm it}{2} \right)^5 \times \cos \left( \arg \left( \frac{1/2 \pm \epsilon \pm it}{2} \right) \right)^6} \right| \leq \left| \frac{1}{157 \left( \frac{1/2 \pm \epsilon \pm it}{2} \right)^5} \right| < \frac{1}{4.9t^5}
\]
(3.4)

It is possible to improve the precision by increasing $K$ but, as we will see in the following, if we content ourselves with an error $< t^{-1}$ then we can completely ignore the Bernoulli sum in (2.6).
3.1 First substitution

If we take the first case \([3.1]\), to be used in first sum of \([2.2]\), we have for \([2.6]\):

\[
\ln \left( e^{-z} z^{\frac{1}{2}} (2\pi)^{\frac{1}{2}} \pi^{-z}(1 - 2z) \right)_{z = \frac{1+i2\epsilon}{2}} = \Re_1^+ + i\Im_1^+ + \Re_2^+ + i\Im_2^+
\]

\[
= \left( 3 + 2\epsilon \frac{i}{2} \right) \left[ \ln \left( \frac{1 + 2\epsilon}{4} \right)^2 + \frac{t^2}{4} + i \arctan \left( \frac{2t}{1 + 2\epsilon} \right) \right] -
\]

\[
- \left( 1 + 2\epsilon \frac{i}{2} \right) - \frac{\ln(2\pi)}{2} - \frac{1}{2} + \epsilon + it \ln(\pi) + \ln \left( \epsilon - \frac{1}{2} \right)^2 + t^2 + i \arctan \left( \frac{-t}{1 + 2\epsilon} \right)
\]

(3.5)

We have:

\[
\arctan \left( \frac{-t}{1 + 1/2 - \epsilon} \right) = -\frac{\pi}{2} + \arctan \left( \frac{1 - 2\epsilon}{2t} \right) \quad \text{and} \quad \arctan \left( \frac{2t}{1 + 2\epsilon} \right) = \frac{\pi}{2} - \arctan \left( \frac{1 + 2\epsilon}{2t} \right)
\]

so separating real part from imaginary part we have:

\[
\Re \left( \ln \left( e^{-z} z^{\frac{1}{2}} (2\pi)^{\frac{1}{2}} \pi^{-z}(1 - 2z) \right)_{z = \frac{1+i2\epsilon}{2}} \right) = \Re_1^+ + \Re_2^+ =
\]

\[
= \left( 3 + 2\epsilon \frac{i}{2} \right) \left[ \ln \left( \frac{t}{2} \right) + \frac{1}{2} \ln \left( 1 + \left( \frac{1 + 2\epsilon}{2t} \right)^2 \right) \right] -
\]

\[
- \frac{t}{2} \left[ \frac{\pi}{2} - \arctan \left( \frac{1 + 2\epsilon}{2t} \right) \right] - \left( 1 + 2\epsilon \frac{i}{2} \right) - \frac{\ln(2\pi)}{2} - \frac{\ln(\pi)(1 + 2\epsilon)}{4} + \ln(t) + \frac{1}{2} \ln \left[ 1 + \left( \frac{1 - 2\epsilon}{2t} \right)^2 \right]
\]

(3.6)

For \( t >> \epsilon \) we have

\[
- \frac{t}{2} \left[ - \arctan \left( \frac{1 + 2\epsilon}{2t} \right) \right] = \frac{1 + 2\epsilon}{4} \frac{2t}{1 + 2\epsilon} \arctan \left( \frac{1 + 2\epsilon}{2t} \right) \to \frac{1}{4} + \frac{\epsilon}{2}; \quad t \to \infty
\]

(3.7)

And we can write the asymptotic part not leading to zero as:

\[
- \frac{t}{2} \left[ - \arctan \left( \frac{1 + 2\epsilon}{2t} \right) \right] = \frac{1 + 2\epsilon}{4} \left[ \frac{2t}{1 + 2\epsilon} \arctan \left( \frac{1 + 2\epsilon}{2t} \right) + 1 - 1 \right] = \frac{1 + 2\epsilon}{4}
\]

(3.8)

And we put it in first piece \( \Re_1^+ \). While:

\[
\frac{1 + 2\epsilon}{4} \left[ \frac{2t}{1 + 2\epsilon} \arctan \left( \frac{1 + 2\epsilon}{2t} \right) - 1 \right]
\]

is put in second piece (i.e. \( \Re_2^+ \)), because as \( t \to \infty \) it goes to zero.

Using the notation \( A_p^s \), we try afterward to put together all pieces.

\( \Re_1^+ + \Re_2^+ \) and \( \Im_1^+ + \Im_2^+ \) are developed in powers of \( t^{-1}, t^{-2}, t^{-3} \ldots \) in subsection \( 3.14 \); see \( 3.25 \).
3.2 Computation of $\Re_1^+$

So $\Re_1^+$ first piece of Real part with first substitution (3.1) is:

$$\Re_1^+ = \frac{3}{4} \ln \left( \frac{t}{2} \right) - \frac{1}{4} + \frac{\ln(2\pi)}{2} - \frac{\ln(\pi)}{4} + \frac{1}{4} + \frac{\epsilon}{2} + \frac{\epsilon}{2} \left( \ln \left( \frac{t}{2} \right) - \ln(\pi) \right) -$$

$$- \frac{\epsilon}{2} - \frac{\pi}{4} t + \ln(t) = \left( \frac{3}{4} \ln \left( \frac{t}{2} \right) + \frac{\ln(2\pi)}{2} - \frac{\ln(\pi)}{4} - \frac{\pi}{4} t + \ln(t) \right) + \ln \left( \sqrt{\frac{t}{2\pi}} \right)^\epsilon$$

(3.9)

3.3 Computation of $\Re_2^+$

While $\Re_2^+$, second piece of Real part for first substitution (3.1), is:

$$\Re_2^+ = \left( \frac{3 + 2\epsilon}{4} \right) \left[ \frac{1}{2} \ln \left( 1 + \left( \frac{1 + 2\epsilon}{2t} \right)^2 \right) \right] +$$

$$+ \frac{1 + 2\epsilon}{4} \left[ \frac{2t}{1 + 2\epsilon} \arctan \left( \frac{1 + 2\epsilon}{2t} \right) - 1 \right] + \frac{1}{2} \ln \left[ 1 + \left( \frac{1 - 2\epsilon}{2t} \right)^2 \right] =$$

(3.10)

$$= \left( \frac{3 + 2\epsilon}{4} \right) \left[ \frac{1}{2} \left( \frac{1 + 2\epsilon}{2t} \right)^2 - \frac{1}{4} \left( \frac{1 + 2\epsilon}{2t} \right)^4 + \ldots \right] + \frac{1 + 2\epsilon}{4} \left[ 1 - \frac{1}{3} \left( \frac{1 + 2\epsilon}{2t} \right)^2 - 1 + \ldots \right] +$$

$$+ \frac{1}{2} \left[ \left( \frac{1 - 2\epsilon}{2t} \right)^2 - \frac{1}{2} \left( \frac{1 - 2\epsilon}{2t} \right)^4 + \ldots \right] =$$

Note that the most significant powers are $t^{-2}$ so that, up to the terms of order $t^{-2}$

$$\Re_2^+ = \left( \frac{3 + 2\epsilon}{4} \right) \left[ \frac{1}{2} \left( \frac{1 + 2\epsilon}{2t} \right)^2 - \ldots \right] + \frac{1 + 2\epsilon}{4} \left[ -\frac{1}{3} \left( \frac{1 + 2\epsilon}{2t} \right)^2 + \ldots \right] + \frac{1}{2} \left[ \left( \frac{1 - 2\epsilon}{2t} \right)^2 + \ldots \right]$$

It is apparent that all three terms of $\Re_2^+$ go to zero as $t \to \infty$.

3.4 Computation of $\Im_1^+$

For the imaginary part we have:

$$\Im \left( e^{-z} e^{z^+ 2\pi i} (2\pi i)^{1/2} e^{-z} (1 - 2z) \right) \bigg|_{z = \frac{1 + 2\epsilon}{2} + \frac{i}{2}} \right) = \Im_1^+ + \Im_2^+ =$$

$$= \left( \frac{t}{2} \right) \left[ \ln \left( \frac{t}{2} \right) + \frac{1}{2} \ln \left( 1 + \left( \frac{1 + 2\epsilon}{2t} \right)^2 \right) \right] +$$

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\[ + \left( \frac{3 + 2\epsilon}{4} \right) \left[ \frac{\pi}{2} - \arctan \left( \frac{1 + 2\epsilon}{2t} \right) \right] - \frac{t}{2} - \frac{t}{2} \ln(\pi) - \frac{\pi}{8} + \arctan \left( \frac{1 - 2\epsilon}{2t} \right) \]  (3.11)

For \( t >> \epsilon \)

\[ \Im_1^+ = \frac{t}{2} \ln \frac{t}{2\pi} - \frac{t}{2} - \frac{\pi}{8} + \frac{\pi \epsilon}{4} ; \quad t >> \epsilon \]  (3.12)

### 3.5 Computation of \( \Im_2^+ \)

\[ \Im_2^+ = \frac{t}{4} \ln \left( 1 + \left( \frac{1 + 2\epsilon}{2t} \right)^2 \right) - \left( \frac{3 + 2\epsilon}{4} \right) \left[ \arctan \left( \frac{1 + 2\epsilon}{2t} \right) \right] + \arctan \left( \frac{1 - 2\epsilon}{2t} \right) = \]  (3.13)

\[ = \frac{t}{4} \left( \left( \frac{1 + 2\epsilon}{2t} + ... \right)^2 \right) - \left( \frac{3 + 2\epsilon}{4} \right) \left[ \left( \frac{1 + 2\epsilon}{2t} \right) + ... \right] + \left( \frac{1 - 2\epsilon}{2t} \right) \]

up to terms of order \( t^{-1} \). Notice that \( \Im_2^+ \) goes to zero for \( t \rightarrow \infty \).

### 3.6 Computation of \( \Re_3^+ \) and \( \Im_3^+ \)

Let us take now the sum in (2.6), always with substitution (3.1), for \( K=3 \).

Here too it is useful to consider real part:

\[ \Re_3^+ (\text{only till } t^{-2} \text{ terms}) = \text{Re} \left( \sum_{k=1}^{K-1} \frac{B_{2k}}{2k(2k - 1)z^{2k-1}} \right) \bigg|_{z = \frac{1 + 2\epsilon}{4t + \frac{\eta}{2}}} ; \quad K=3 \]

\[ = \frac{1 + 2\epsilon}{48 \left[ 1 + \left( \frac{1 + 2\epsilon}{2t} \right)^2 \right] \left( \frac{t}{2} \right)^2} - \frac{(1 + 2\epsilon)^3}{360t^6 \left( 1 + \left( \frac{1 + 2\epsilon}{2t} \right)^2 \right)^3} + \frac{(1 + 2\epsilon)^2}{30t^4 \left( 1 + \left( \frac{1 + 2\epsilon}{2t} \right)^2 \right)^2} = \]  (3.14)

\[ = \frac{1 + 2\epsilon}{12t^2} - ... \]

and imaginary part:

\[ \Im_3^+ (\text{only till } t^{-1} \text{ terms}) = \Im \left( \sum_{k=1}^{K-1} \frac{B_{2k}}{2k(2k - 1)z^{2k-1}} \right) \bigg|_{z = \frac{1 + 2\epsilon}{4t + \frac{\eta}{2}}} ; \quad K=3 \]

\[ = -\frac{1}{6t \left[ 1 + \left( \frac{1 + 2\epsilon}{2t} \right)^2 \right]} - \frac{1}{45t^3 \left( 1 + \left( \frac{1 + 2\epsilon}{2t} \right)^2 \right)^3} + \frac{(1 + 2\epsilon)^2}{60t^5 \left( 1 + \left( \frac{1 + 2\epsilon}{2t} \right)^2 \right)^2} = \]  (3.15)

\[ = -\frac{1}{6t} + ... \]
3.7 Second substitution

Second substitution (3.2) applies to second sum ant remainder (i.e. 3rd term) in (2.2) . So we have, likewise:

\[
\ln \left( e^{-z}z^{+\frac{1}{2}}(2\pi)^{\frac{1}{2}}\pi^{-z}(1 - 2z) \right)_{z = \frac{-1}{2} - \frac{u}{2}} = \Re R_1^- + \Re R_2^- + i\Im S_1^- + i\Im S_2^- =
\]

\[
= \left( \frac{3}{4} - \frac{e}{2} - \frac{it}{2} \right) \ln \sqrt{\left( \frac{1}{4} - \frac{e}{2} \right)^2 + \frac{t^2}{4}} + i \tan \left( \frac{-t}{1/2 - e} \right) +
\]

\[
- \left( \frac{1 - 2\epsilon}{4} - \frac{it}{2} \right) + \ln \left( \frac{2\pi}{2} \right) -
\]

\[
- \frac{\pi}{2} - \ln \left( \frac{\pi}{2} \right) + \ln \left[ \left( \frac{1}{2} + \frac{1 + 2\epsilon}{2t} \right)^2 \right] + i \tan \left( \frac{+t}{1/2 + \epsilon} \right)
\]

Referring to (3.6), we separate again the real from the imaginary part. We find:

\[
\Re \left( \ln \left( e^{-z}z^{+\frac{1}{2}}(2\pi)^{\frac{1}{2}}\pi^{-z}(1 - 2z) \right)_{z = \frac{-1}{2} - \frac{u}{2}} \right) = \Re R_1^- + \Re R_2^- =
\]

\[
= \left( \frac{3}{4} - \frac{e}{2} \right) \ln \left( \frac{t}{2} \right) + \ln \left( \frac{1 + \left( \frac{1 + 2\epsilon}{2t} \right)^2}{2} \right) +
\]

\[
+ \frac{t}{2} \left[ \frac{\pi}{2} + \tan \left( \frac{1 - 2\epsilon}{2t} \right) \right] - \left( \frac{1 - 2\epsilon}{2} - \frac{t}{2} \right) + \ln \left( \frac{2\pi}{2} \right) - \left( \frac{1}{2} - \epsilon \right) \ln \left( \frac{\pi}{2} \right) +
\]

\[
+ \ln \left( t \right) + \frac{1}{2} \ln \left( 1 + \left[ \frac{1 + 2\epsilon}{2t} \right]^2 \right)
\]

For \( t \gg \epsilon \) we have

\[
\frac{t}{2} \left[ \tan \left( \frac{1 - 2\epsilon}{2t} \right) \right] = \frac{1 - 2\epsilon}{4} \tan \left( \frac{1 - 2t}{2t} \right) \rightarrow \frac{1}{4} - \frac{\epsilon}{2}; \quad t \rightarrow \infty \quad (3.16)
\]

And we can write the asymptotic part not leading to zero (contained in \( \Re R_1^- \)) as:

\[
\frac{t}{2} \left[ \tan \left( \frac{1 - 2\epsilon}{2t} \right) \right] = \frac{1 - 2\epsilon}{4} \left[ \tan \left( \frac{2t}{1 - 2\epsilon} \right) + 1 - 1 \right] = \frac{1 - 2\epsilon}{4} \quad (3.17)
\]

and

\[
\frac{1 - 2\epsilon}{4} \left[ \tan \left( \frac{2t}{1 - 2\epsilon} \right) - 1 \right]
\]

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which is contained in \( \Re^{-2} \), because as \( t \to \infty \) it goes to zero. Thus for \( t >> \epsilon \) and \( t >> 1 \):

\[
\Re \left( \ln \left( e^{-z} z^{1/2} (2\pi)^{1/2} \pi^{-z} (1 - 2z) \right) \right)_{z = \frac{\epsilon}{4} - \frac{1}{2} t} = \Re_1^- + \Re_2^-
\]

### 3.8 Computation of \( \Re_1^- \)

We can write:

\[
\Re_1^- = \frac{3}{4} \ln \left( \frac{t}{2} \right) - \frac{1}{4} + \frac{1}{4} + \frac{\ln(2\pi)}{2} - \frac{\ln(\pi)}{4} + \frac{\epsilon}{2} \left( \ln(\pi) - \ln \left( \frac{t}{2} \right) \right) + \frac{\epsilon}{2} - \frac{\pi t}{4} + \ln(t) + \frac{1}{4} - \frac{\epsilon}{2} = \\
= \frac{3}{4} \ln \left( \frac{t}{2} \right) + \frac{\ln(2\pi)}{2} - \frac{\ln(\pi)}{4} - \frac{\pi t}{4} + \ln(t) + \ln \left( \sqrt{\frac{2\pi}{t}} \right) ^\epsilon
\]

### 3.9 Computation of \( \Re_2^- \)

While \( \Re_2^- \), the second piece of the Real part using eq. (3.2), is:

\[
\Re_2^- (only \ till \ t^{-2} \ terms) = \left( \frac{3 - 2\epsilon}{4} \right) \left[ \frac{1}{2} \ln \left( 1 + \left( \frac{1 + 2\epsilon}{2t} \right)^2 \right) \right] + \\
+ \frac{1 - 2\epsilon}{4} \left[ \frac{2t}{1 - 2\epsilon} \arctan \left( \frac{1 - 2\epsilon}{2t} \right) - 1 \right] + \frac{1}{2} \ln \left[ 1 + \left( \frac{1 + 2\epsilon}{2t} \right)^2 \right] = \\
\left( \frac{3 - 2\epsilon}{4} \right) \left[ \frac{1}{2} \left( \left( \frac{1 + 2\epsilon}{2t} \right)^2 \right) \right] + \frac{1 - 2\epsilon}{4} \left[ - \frac{1}{3} \left( \frac{1 - 2\epsilon}{2t} \right)^2 \right] + \frac{1}{2} \left( \left( \frac{1 + 2\epsilon}{2t} \right)^2 \right) + \ldots
\]

Clearly all three terms of \( \Re_2^- \) go to zero as \( t \to \infty \).

### 3.10 Computation of \( \Im_1^- \)

For the imaginary part we have:

\[
\Im \left( \ln \left( e^{-z} z^{1/2} (2\pi)^{1/2} \pi^{-z} (1 - 2z) \right) \right)_{z = \frac{\epsilon}{4} - \frac{1}{2} t} = \Im_1^- + \Im_2^- = \\
= - \left( \frac{t}{2} \right) \left[ \ln \left( \frac{t}{2} \right) + \frac{1}{2} \ln \left( 1 + \left( \frac{1 + 2\epsilon}{2t} \right)^2 \right) \right] + \\
+ \frac{3 - 2\epsilon}{4} \left[ - \frac{\pi}{2} + \arctan \left( \frac{1 - 2\epsilon}{2t} \right) \right] + \frac{t}{2} + \frac{t}{2} \ln(\pi) + \frac{\pi}{2} - \arctan \left( \frac{1 + 2\epsilon}{2t} \right)
\]

(3.20)
Likewise for \( t >> \epsilon \) separating real and imaginary parts in a single expression we can write:

\[
\Re_1^- + i\Im_1^- = \frac{3}{4} \ln \left( \frac{t}{2} \right) + \frac{\ln(2\pi)}{2} - \frac{\ln(\pi)}{4} \\
+ \ln \left( \sqrt{\frac{2\pi}{t}} \right)^\epsilon - \frac{\pi}{4} t + \ln(t) + i \left( -\frac{t}{2} \ln \frac{t}{2\pi} + \frac{t}{2} + \frac{\pi}{8} + \frac{\pi}{4} \epsilon \right)
\]  
(3.21)

3.11 Computation of \( \Im_2^- \)

\( \Im_2^- \text{ (only till } t^{-1} \text{ terms)} = -\frac{t}{4} \ln \left( 1 + \left( \frac{1 + 2\epsilon}{2t} \right)^2 \right) + \frac{3 - 2\epsilon}{4} \left[ \arctan \left( \frac{1 - 2\epsilon}{2t} \right) \right] - \arctan \left( \frac{1 + 2\epsilon}{2t} \right) 
\)  
(3.22)

\[= -\frac{t}{4} \left( \frac{1 + 2\epsilon}{2t} \right)^2 + \frac{3 - 2\epsilon}{4} \left( \frac{1 - 2\epsilon}{2t} \right) - \frac{1 + 2\epsilon}{2t} + ... \]

3.12 Computation of \( \Re_3^- \)

Let us consider the sum in (2.6), with substitution (3.2), for \( K=3 \).

Here too it is useful to consider the real part:

\[
\Re_3^- \text{ (only till } t^{-1} \text{ terms)} = \Re \left( \sum_{k=1}^{K-1} \frac{B_{2k}}{2k(2k-1)z^{2k-1}} \right)_{z=\frac{1-2\epsilon}{4} - \frac{\epsilon}{2}} ; \quad K=3
\]

\[
= \frac{1 - 2\epsilon}{48 \left[ 1 + \left( \frac{1 - 2\epsilon}{2t} \right)^2 \right] \left( \frac{t}{4} \right)^2} - \frac{(1 - 2\epsilon)^3}{360t^6 \left( 1 + \left( \frac{1 - 2\epsilon}{2t} \right)^2 \right)^3} + \frac{(1 - 2\epsilon)}{304 \left( 1 + \left( \frac{1 - 2\epsilon}{2t} \right)^2 \right)^2} = (3.23)
\]

\[= \frac{1 - 2\epsilon}{12t^2} - ... \]

3.13 Computation of \( \Im_3^- \)

For the imaginary part:

\[
\Im_3^- \text{ (only till } t^{-1} \text{ terms)} = \Im \left( \sum_{k=1}^{K-1} \frac{B_{2k}}{2k(2k-1)z^{2k-1}} \right)_{z=\frac{1-2\epsilon}{4} - \frac{\epsilon}{2}} ; \quad K=3
\]

\[
= + \frac{1}{6t \left( 1 + \left( \frac{1 - 2\epsilon}{2t} \right)^2 \right)} + \frac{1}{45t^3 \left( 1 + \left( \frac{1 - 2\epsilon}{2t} \right)^2 \right)^3} - \frac{(1 - 2\epsilon)^2}{60t^5 \left( 1 + \left( \frac{1 - 2\epsilon}{2t} \right)^2 \right)^3} = (3.24)
\]

\[= - \frac{1}{6t} + ... \]
3.14 Error merging

Let us merge the more significant errors (i.e. second and third part).

For imaginary part we take only till $t^{-1}$ term (for $\epsilon = 0$ we get $\pm \frac{1}{48}$ as in [p. 120 (1)]

$$\Im \pm 2 + \Im \pm 3 = \frac{t}{2} - 1 + \frac{(1 + 2\epsilon)^2}{2} + \frac{3 + 2\epsilon}{8} \pm \frac{1 + 2\epsilon}{2} \pm \frac{1}{2} (1 + 2\epsilon)^2 + \frac{1}{2} \frac{1 + 2\epsilon}{12} \pm \frac{1}{2} \frac{1 + 2\epsilon}{12} \pm \frac{1}{2} (1 + 2\epsilon)^2 + \frac{1}{2} \frac{1 + 2\epsilon}{12} \pm \frac{1}{2} \frac{1 + 2\epsilon}{12} + ... = (3.25)$$

For real part we take only till $t^{-2}$ term:

$$\Re \pm 2 + \Re \pm 3 = \frac{t}{2} \left[ \frac{3 + 2\epsilon}{4} - \frac{1 + 2\epsilon}{12} \right] + \frac{1 + 2\epsilon}{2} \left[ \frac{1 + 2\epsilon}{2} \right] + \frac{1 + 2\epsilon}{12} \pm \frac{1}{2} \frac{1 + 2\epsilon}{12} \pm \frac{1}{2} \frac{1 + 2\epsilon}{12} + ... = (3.26)$$

The error (3.4) contribution, if we limit to $t^{-1}$ for $\Im$ and $t^{-2}$ for $\Re$, does not appear.

So we have:

$$\Im \pm 2 + \Im \pm 3 = \frac{1 - 84\epsilon + 10\epsilon^2}{48t} + \left[ t^{-3} + t^{-5} + ... \right] \quad \Im \pm 2 + \Im \pm 3 = - \frac{1 + 108\epsilon - 12\epsilon^2}{48t} + \left[ t^{-3} + t^{-5} + ... \right]$$

and

$$\Re \pm 2 + \Re \pm 3 = \frac{27 + 94\epsilon + 84\epsilon^2 + 8\epsilon^3}{96t^2} + \left[ t^{-4} + ... \right] \quad \Re \pm 2 + \Re \pm 3 = \frac{27 - 22\epsilon + 36\epsilon^2 - 8\epsilon^3}{96t^2} + \left[ t^{-4} + ... \right]$$

In $\Im_3$ we could include also the error whose we know only the bound (2.7).

3.15 Putting pieces together

Let us try to summarize.

The coefficient of first sum of (2.2) is:

$$\ln(\Gamma(z + 1)) = \ln \left( e^{-\frac{1}{2} \left( 2\pi \right)^{\frac{1}{2}}} \right) + \sum_{k=1}^{K-1} \frac{B_{2k}}{2k(2k - 1)z^{2k-1}} = \Re_1 + \Re_2 + \Re_3 + i(\Im_1 + \Im_2 + \Im_3) \quad (3.29)$$

while for the second sum and the remainder of (2.2):

$$\ln(\Gamma(z + 1)) = \ln \left( e^{-\frac{1}{2} \left( 2\pi \right)^{\frac{1}{2}}} \right) + \sum_{k=1}^{K-1} \frac{B_{2k}}{2k(2k - 1)z^{2k-1}} = \Re_1 + \Re_2 + \Re_3 + i(\Im_1 + \Im_2 + \Im_3) \quad (3.30)$$
Let us neglect now \( (\Re_j^+) \) and \( (\Re_j^-) \) with \( j > 1 \), namely second and third piece.

It is useful to isolate real terms that depend only on \( t \).

We define:

\[
\ln[F(t)] := \frac{3}{4} \ln\left(\frac{t}{2}\right) + \frac{\ln(2\pi)}{2} - \frac{\ln(\pi)}{4} - \frac{\pi}{4} t + \ln(t)
\]

(3.31)

\( F(t) \) is the scale factor :

\[
F(t)e^{i\frac{\pi}{4} \epsilon} = e^{\frac{\ln(2\pi)}{2} - \frac{\ln(\pi)}{4} - \frac{\pi}{4} t} e^{i\frac{\pi}{4} (\epsilon + it)} = \left(\frac{\pi}{2}\right)^{0.25} t^{\frac{\pi}{4}} e^{i\frac{\pi}{4} (\epsilon + it)}
\]

(3.32)

In [3, p. 176], for \( \epsilon = 0 \) and in [5, p. 5]

\[
f(t) = \frac{1}{2} \pi^{-1/4} (t^2 + 1/4) |\Gamma(1/4 + it/2)| = e^{Re[\ln\Gamma(s/2)]} -1/4 t^2 - 1/4
\]

(3.33)

While

\[
F(t) = \Re\left[(s-1)\Gamma\left(\frac{s}{2} + 1\right) \pi^{-\frac{s}{2}}\right]_{\epsilon=0} = f(t)
\]

So \( F(t) \) used here and \( f(t) \) in in [3, p. 176] and in [5, p. 5] are the same.

Note that same scale factor \( F(t) \), applies to (3.9) (i.e. \( \Re_1^+ \)), as well as to (3.21) (i.e. \( \Re_1^- \)).

Looking at the imaginary terms and we define:

For (3.12)

\[
i\Im_1^+ = i\left(\frac{t}{2} \ln\frac{t}{2\pi} - \frac{t}{2} - \frac{\pi}{8} + \frac{\pi}{4} \epsilon\right) = i\left(\theta_1(t) + \frac{\pi}{4} \epsilon\right)
\]

For (3.21)

\[
i\Im_1^- = i\left(-\frac{t}{2} \ln\frac{t}{2\pi} + \frac{t}{2} + \frac{\pi}{8} - \frac{\pi}{4} \epsilon\right) = i\left(-\theta_1(t) + \frac{\pi}{4} \epsilon\right)
\]

Where

\[
\theta_1(t) := \frac{t}{2} \ln\frac{t}{2\pi} - \frac{t}{2} - \frac{\pi}{8} = \frac{t}{2} \ln\left(\frac{t}{2\pi}\right) - \frac{\pi}{8}
\]

(3.34)

Meanwhile referring to (1) [3, p. 120] we have coherently:

\[
\theta(t) = -[-\theta_1(t) + \Im_2^+ (\epsilon = 0) + \Im_3^- (\epsilon = 0)] = \theta_1(t) + \Im_3^+ (\epsilon = 0) + \Im_3^- (\epsilon = 0)
\]

Looking at the real terms that depend on \( \epsilon \) and on \( t \) at the same time we have:

For (3.9)

\[
\frac{\epsilon}{2} \left(\ln\frac{t}{2} - \ln(\pi)\right) = \ln\left(\frac{t}{2\pi}\right)^{\frac{\epsilon}{2}}
\]

For (3.21)

\[
\frac{\epsilon}{2} \left(\ln(\pi) - \ln\left(\frac{t}{2}\right)\right) = \ln\left(\frac{2\pi}{t}\right)^{\frac{\epsilon}{2}}
\]

Note that also this apply to 3.9 (i.e. \( \Re_1^+ \)), as well as to 3.21 (i.e. \( \Re_1^- \)). So we could write:
\[ \Re_1^+ = \ln[F(t)] + \ln \left( \frac{t}{2\pi} \right)^{\epsilon/2} \]

\[ \Re_1^- = \ln[F(t)] + \ln \left( \frac{2\pi}{t} \right)^{\epsilon/2} \]

Thus (for \( t >> 1 \)) we can use only the first piece (\( \Re_1 \) and \( \Im_1 \)).

So we have:

\[ \ln \left[ \Gamma \left( \frac{S}{2} + 1 \right) (1 - s)\pi^{-\frac{s}{2}} \right] \]

\[ = \ln \left( e^{-z} z^{-\frac{1}{2}} (2\pi)^{\frac{1}{2}} \pi^{-\frac{1}{2}} (1 - 2z) \right)_{z = \frac{1 + 2s - \sqrt{t^2}}{2}} \approx \Re_1^+ + i\Im_1^+ = \ln[F(t)] + \ln \left[ \left( \frac{t}{2\pi} \right)^{\epsilon/2} \right] + i \left( \theta(t) + \frac{\pi}{4} \epsilon \right) \]

and

\[ \ln \left[ \Gamma \left( \frac{1 - S}{2} + 1 \right) (+s)\pi^{-\frac{1-s}{2}} \right] \]

\[ = \ln \left( e^{-z} z^{-\frac{1}{2}} (2\pi)^{\frac{1}{2}} \pi^{-\frac{1}{2}} (1 - 2z) \right)_{z = \frac{1 - 2s - \sqrt{t^2}}{2}} \approx \Re_1^- + i\Im_1^- = \ln[F(t)] + \ln \left[ \left( \frac{2\pi}{t} \right)^{\epsilon/2} \right] + i \left( -\theta(t) + \frac{\pi}{4} \epsilon \right) \]

(3.35)

\[ \text{and} \]

\[ \ln \left[ \Gamma \left( \frac{S}{2} + 1 \right) (1 - s)\pi^{-\frac{s}{2}} \right] \]

\[ = \ln \left( e^{-z} z^{-\frac{1}{2}} (2\pi)^{\frac{1}{2}} \pi^{-\frac{1}{2}} (1 - 2z) \right)_{z = \frac{1 + 2s - \sqrt{t^2}}{2}} \approx \Re_1^- + i\Im_1^- = \ln[F(t)] + \ln \left[ \left( \frac{2\pi}{t} \right)^{\epsilon/2} \right] + i \left( -\theta(t) + \frac{\pi}{4} \epsilon \right) \]

(3.36)

\section{Asymptotic expressions}

Putting together all the terms in (2.2) and setting \( \theta(t) \approx \theta_1(t) \) we have:

\[ -\xi \left( \frac{1}{2} + \epsilon + it \right) \sim F(t)e^{i\pi/4} \sum_{n=1}^{N} \left( \frac{\sqrt{\frac{t}{2\pi}}}{n} \right)^{\epsilon} e^{i(\theta(t) - t \ln(n))} \sqrt{n} \]

\[ + F(t)e^{i\pi/4} \left\{ \sum_{n=1}^{N} \left( \frac{n}{\sqrt{\frac{t}{2\pi}}} \right)^{\epsilon} e^{-i(\theta(t) - t \ln(n))} \sqrt{n} + R(t) \right\} \quad ; \quad t >> \epsilon \]

(4.1)

where \( F(t) = (\pi/2)^{0.25} t^{\frac{3}{2}} e^{-\frac{\pi}{4} t} \), see (3.31).

Let us look for an exact expression. Using (3.29) and (3.30), and using:

\[ A^\pm = e^{\Re_2^\pm + i\Im_2^\pm} + e^{i(\Im_2^\pm + \Im_\text{error})} \]

we obtain:

\[ -\xi \left( \frac{1}{2} + \epsilon + it \right) = F(t)e^{i\pi/4}. \]

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Let us define a useful function strictly related to $R$ in (6) [3, p. 147] and [3, p. 154].

The remainder $R$ is the same quantity computed in Appendix A also for $\epsilon \neq 0$.

Note however that (1.3) cannot be actually computed because of the errors in (4.2) and in (B.28), we only have an upper bound.

In Appendix A the coefficients $C_n(p, \epsilon)$ for $\epsilon > 0$ are generalized.

$$R_M(t, \epsilon) \approx (-1)^{N-1} \left( \frac{2\pi}{t} \right)^{1/4} \sum_{j=0}^{M} C_j(p, \epsilon) \left( \frac{2\pi}{t} \right)^{j/2} \quad \text{with:}$$

$$C_0(0.5) = 0.382683, \quad \leq C_0(p) = \frac{\cos(2\pi(p^2 - p - 1/16))}{\cos(2\pi p)} \leq \cos(\pi/8) \approx 0.923879.$$  

where $C_0(p, \epsilon) \equiv C_0(p)$ is independent of $\epsilon$; see Appendix A

So we can put $\forall \epsilon$:

$$R_0(t) = (-1)^{N-1} \left( \frac{2\pi}{t} \right)^{1/4} C_0(p) \quad \forall \epsilon$$

In the following we use $R(t)$ in the meaning of (4.4) with sum in square brackets only till $C_i$. We use simply $R(t)$ instead of $R_{\infty}(t)$.

Let us define an useful function strictly related to $\xi(s)$:

$$Z(t, \epsilon) := \sum_{n=1}^{N} \left( \frac{\sqrt{2\pi}}{n} \right)^{\epsilon} e^{i(\theta(t) - \ln(n))} \frac{e^{-i(\theta(t) - \ln(n))}}{\sqrt{n}} + R(t) \sim -\xi(\frac{1}{2} + \epsilon + it) \frac{F(t)e^{\frac{\pi}{4}t}}{}$$

By the way: as $A^\pm \rightarrow 1$ (see (4.2)), for $\epsilon \rightarrow 0$, then (4.3) or (4.1), lead us to the classical $Z$ of Riemann Siegel [3, p. 139].

Note that $Z(t, \epsilon)$ has “almost” the same phase of $-\xi(\frac{1}{2} + \epsilon + it)$. So now it is clear why we changed the sign in (2.2) with respect to (5) [3, p. 138].

This choice is congruent with $\theta(t)$ in [3, p. 119-120] where $\theta(t)$ is developed in powers of $1/t$ and $(1/t)^3$.  

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In fact, from (3.11), we have

$$\theta(t, \epsilon = 0) = 3 \left( \ln \left[ \Gamma \left( \frac{s}{2} + 1 \right) \pi^{\frac{s}{2}} (1 - s) \right] \right)_{s=1/2+i\epsilon} = \frac{t}{2} \ln \frac{t}{2\pi} - \frac{t}{2} - \frac{\pi}{8} + \frac{1}{48t} + \frac{7}{5760t^3} = \theta(t)$$

like in (1) [3, p. 120].

An interesting way to express (4.7) is using hyperbolic functions:

$$-\xi(t, \epsilon) \sim Z(t, \epsilon) = 2 \sum_{n=1}^{N} \cosh \left[ \frac{\epsilon}{\sqrt{n}} \ln \left( \sqrt{\frac{t}{2\pi n^2}} \right) \right] \cos \left( t \ln \left( \sqrt{\frac{t}{2\epsilon n^2}} \right) - \frac{\pi}{8} \right) + .. + 2i \sum_{n=1}^{N} \sinh \left[ \frac{\epsilon}{\sqrt{n}} \ln \left( \sqrt{\frac{t}{2\pi n^2}} \right) \right] \sin \left( t \ln \left( \sqrt{\frac{t}{2\epsilon n^2}} \right) - \frac{\pi}{8} \right) + R(t, \epsilon)$$

(4.8)

Note that, as \(\partial \frac{\partial}{\partial t} \ln \left( \sqrt{\frac{t}{2\pi n^2}} \right) = \frac{\epsilon}{\sqrt{2\pi t}}\), each addend of the sum in (4.8) verifies for small \(\epsilon\):

$$\partial \text{Im} \frac{\partial}{\partial \epsilon} = -\partial \text{Re} \frac{\partial}{\partial t} ; \quad \partial \text{Re} \frac{\partial}{\partial \epsilon} = \partial \text{Im} \frac{\partial}{\partial t}$$

(4.9)

For reminder term we have:

$$\frac{\partial R_0(t)}{\partial t} = \partial \left\{ (1)^{N-1} \left( \frac{2\pi}{t} \right)^{1/4} [C_0(p) + ...] \right\}$$

where:

$$p = \sqrt{\frac{t}{2\pi}} - N \rightarrow \frac{\partial p}{\partial t} = \frac{1}{2\sqrt{2\pi t}}$$

So we have:

$$\frac{\partial R_0(t)}{\partial t} = (1)^{N-1} \left\{ -\frac{1}{4t} \left( \frac{2\pi}{t} \right)^{1/4} C_0(p) + \frac{\partial C_0(p)}{\partial p} \left( \frac{2\pi}{t} \right)^{1/4} \frac{1}{2\sqrt{2\pi t}} \right\} \approx_{t>1} ...$$

(4.10)

$$\approx_{t>1} (1)^{N-1} \frac{\partial C_0(p)}{\partial p} \left( \frac{2\pi}{t} \right)^{1/4} \frac{1}{2\sqrt{2\pi t}}$$

in (A.30)

$$C_1(p, \epsilon) \omega^1 = -\epsilon \psi^{(1)}(p) \frac{i\omega}{4\pi} - \psi^{(3)}(p) \frac{\omega}{2^5\pi^23}$$

(4.11)

as \(\omega = \sqrt{\frac{2\pi}{t}}\); see (A.21).

As \(\psi^{(1)}(p) = \frac{\partial C_0(p)}{\partial p}\); see (A.28). Besides \(\frac{\partial \psi^{(3)}(p)}{\partial t} < (t>1) \frac{\partial C_0(p)}{\partial p} \frac{1}{2\sqrt{2\pi t}}\).

So taking only \(C_1(p, \epsilon)\), after \(C_0(p)\), in (4.4):
$$R_1(t, \epsilon) = (-1)^{N-1} \left( \frac{2\pi}{t} \right)^{1/4} \left[ C_0(p) + C_1(p, \epsilon) \left( \frac{2\pi}{t} \right)^{1/2} \right]$$  \hspace{1cm} (4.12)$$

if \( t \gg 1 \), we almost match, also for the remainder \( R_1(t, \epsilon) \), the first equation of (4.9). From (4.10) the main part of \( \frac{\partial \text{Re}[R(t, \epsilon)]}{\partial t} \) is:

$$\frac{\partial \text{Re}[R(t, \epsilon)]}{\partial t} \approx _{t>\gg 1} (-1)^{N-1} \frac{\partial C_0(p)}{\partial p} \left( \frac{2\pi}{t} \right)^{1/4} \frac{1}{2\sqrt{2\pi t}}$$

While

$$\frac{\partial \text{Im}[R(t, \epsilon)]}{\partial \epsilon} \approx _{t>\gg 1} -(-1)^{N-1} \frac{\omega}{4\pi} \frac{\partial C_0(p)}{\partial p} = -(-1)^{N-1} \frac{\partial C_0(p)}{\partial p} \left( \frac{2\pi}{t} \right)^{1/4} \frac{1}{2\sqrt{2\pi t}}$$  \hspace{1cm} (4.13)

So the first of the (4.9) is almost verified also for \( R_1(t, \epsilon) \); see (4.12).

Note that the second of ( 4.9) is identically zero for \( \epsilon \to 0 \).

With reminder \( R_1(t, \epsilon) \) which uses (4.4) with only \( C_0(p) \), and \( C_1(p, \epsilon) \), we can then say that (4.8), for \( t \gg 1 \) and for small \( \frac{\epsilon}{t} \), is almost holomorphic as the RiemannCauchy (4.9) holds only with small discrepancies that tend to zero as \( t \to \infty \).

A Main integral in 2.2 along path \( L_1 \) with offset from critical line \((\epsilon > 0)\)

The aim of this appendix is to compute the main integral (A.2) following the same logical path of [3, p. 138-155] with the only addition of \( \epsilon \neq 0 \).

First footnote in [3, p. 138] suggests exactly this. A brief summary shows the path to follow.

A.1 Summary of logical path

Let us summarize the logical path of the following evaluation.

- To ignore the integral contribution on \( L_0, L_2, L_3 \); see (A.5).
- To develop \((-x)^{\epsilon-1/2+it}e^{-Nx}\) as a power series centered at \( a = i\sqrt{2\pi t} \); see (A.9).
- To use a “matching factor” (A.11) (rewritten here for easy reading)

$$g(x-a) = \sum_{n=0}^{n=\infty} b_n(x-a)^n = e^{\left[-\frac{i(x-a)^2}{4\epsilon}-(p+N)(x-a)+(-\frac{1}{2}+\epsilon+it) \ln(1+x-x/a)\right]}$$  \hspace{1cm} (A.1)

To compute the \( b_n(\omega, \epsilon) \) for a perfect matching of all like powers of \( \omega = \sqrt{\frac{2\pi}{t}} \) obtained by iteration (A.23); see also (A.22).
With (4.3) and (4.2) we can write:

\[ L \]

Some note on (A.2).

The \((-1)\) factor at the integrand numerator stems for the gathering of:

- Suppose now (3.27) and (3.28):

  \[
  \text{To expand the expression obtained by the matching (i.e. } \sum_{n=0}^{\infty} b_n(x - a)^n, \text{ on all } L_1 \text{ line, irrespective of the radius of convergence of the power series centered in } x = i\sqrt{2/\pi} \text{ (A.20) by a wise contour integration [3 p. 147-148] due to Riemann. This allows us to compute exactly, on complex plane } u = x - 2\pi Ni \text{ [3 p. 147-148], the main integral for } b_0 i.e. \ C_0(p) = \psi^{(0)}(p) \text{ (A.20). In order to compute } C_n(p, \epsilon) \text{ by formula (A.25) we need till } b_{3n} \text{ see (A.29).}
  \]

- through (A.29) all pieces are joined to compute } C_i(p, \epsilon) \text{ of (4.4).

  The computation has been actually carried on only for } i = 1, 2; \text{ see (A.30) and (A.35).}

### A.2 Main integral over } L_0 + L_1 + L_2 + L_3

With (4.3) and (4.2) we can write:

\[
F(t)e^{\frac{t}{2}x}A^-R(t, \epsilon) = \left(\frac{2i}{\sqrt{\pi}}\right)^{-\frac{x-1}{2}} \frac{\Gamma\left(\frac{1-s}{2}\right)}{(2\pi)^{s-1}} \frac{1}{\sin(\pi s/2)} \frac{1}{2\pi i} \int_{L_0, L_1, L_2, L_3} (-1)^{s-1} e^{-N_x} dx \ e^{\frac{s-1}{2}x - 1} \]

\[
e^{\Re_2 + \Re_3 + \Re_{error} + i(3\Re_2 + 3\Re_3 + 3\Re_{error})} \]

\[
\cdots \left(\frac{2i}{\sqrt{\pi}}\right)^{\frac{s-1}{2}} e^{-\frac{t}{2}x} e^{-i\theta_1(t)} \left(\sqrt{\frac{2\pi}{t}}\right)^{\epsilon} \int_{L_0, L_1, L_2, L_3} (-x)^{s-1} e^{-N_x} dx \ e^{\frac{s-1}{2}x - 1} \]

(A.2)

Some note on (A.2).

- Suppose now (3.27) and (3.28): \( e^{\Re_2 + \Re_3 + \Re_{error} + i(3\Re_2 + 3\Re_3 + 3\Re_{error})} \approx 1 \).

Below in (A.25) for higher order approximation \( e^{it(3\Re_2 + 3\Re_3)} \) is taken in the evaluation of \( C_2(p, \epsilon) \); see (A.4).

- From (3.36) \( (s)\Gamma\left(\frac{1-s}{2}\right) \approx e^{\Re_2 + i\Im_1} = F(t)e^{\frac{t}{2}x} e^{-i\theta_1(t)} \left(\sqrt{\frac{2\pi}{t}}\right)^{\epsilon} \).

- The \((-1)\) factor at the integrand numerator stems for the gathering of \( x, \), under \((s - 1)\) exponent: \( \frac{(-x)^{s-1} dx}{(e^{s-1} - 1) x} = \frac{(-1)^{s-1} dx}{e^{s-1} - 1} \).

- This \((-1)\) at numerator integrand is, afterward, joined to \( 2i \sin(\pi s/2) \) development [3 p. 139] in the denominator of external coefficient:

\[
(-1)2i \sin(\pi s/2) = -\left[ e^{\frac{2\pi}{t} (\frac{1}{s} + \epsilon + it)} - e^{-\frac{2\pi}{t} (\frac{1}{s} + \epsilon + it)} \right] = -e^{-\frac{2\pi}{t}} \left[ e^{\frac{2\pi}{t} - \frac{2\pi}{t} + \frac{2\pi}{t} - e^{-\frac{2\pi}{t}} e^{\frac{2\pi}{t}} + \frac{2\pi}{t}} \right]
\]

\[
e^{-\frac{2\pi}{t}} e^{\frac{2\pi}{t}} \left[ e^{-\frac{2\pi}{t}} - ie^{\frac{2\pi}{t}} e^{-\epsilon t} \right] = e^{-\frac{2\pi}{t}} \left[ e^{-\frac{2\pi}{t}} - ie^{\frac{2\pi}{t}} e^{-\epsilon t} \right] e^{\frac{2\pi}{t}}
\]

- So in (A.2) the integral is multiplied by:

\[
\frac{1}{(2\pi)^{s}} \left( F(t)e^{\frac{t}{2}x} e^{-i\theta_1(t)} \left(\sqrt{\frac{2\pi}{t}}\right)^{\epsilon} \right) \left( e^{-\frac{2\pi}{t}} \left[ e^{-\frac{2\pi}{t}} - ie^{\frac{2\pi}{t}} e^{-\epsilon t} \right] e^{\frac{2\pi}{t}} \right)
\]
If we put [3, p. 139]:

\[ P_0 = \frac{e^{-i \theta_1(t)}e^{-t \pi /2}}{(2\pi)^{1/2}e^{-i \pi /4}(1 - ie^{-t \pi})} \quad \text{for} \quad \epsilon = 0 \quad (A.3) \]

then, for \( \forall \epsilon \) we may write:

\[ P_{\epsilon} = \left\{ \frac{e^{-i \theta_1(t)}e^{-t \pi /2}}{(2\pi)^{1/2}e^{-i \pi /4}(1 - ie^{-t \pi})} \right\} \left[ \frac{(1 - ie^{-t \pi})}{(2\pi)^{1/2}e^{-i \pi /2 - i \epsilon e^{-t \pi}}} \right] \quad \epsilon > 0 \quad (A.4) \]

As from Appendix B contributions outside \( L_1 \) are very small (see fig. 3) then \( (A.2) \) can be written:

\[ P_{\epsilon} \int_{L_0,L_1,L_2,L_3} \frac{(-x)^{s-1}e^{-N_x}dx}{e^{x} - 1} \approx P_{\epsilon} \int_{L_1} \frac{(-x)^{s-1}e^{-N_x}dx}{e^{x} - 1} \quad (A.5) \]

### A.3 Evaluation of main integral on \( L_1 \)

We follow the path of [3, p. 145] with the only difference of introducing \( \forall \epsilon \).

It is useful also [7, p. 14-28]. We sum and subtract \( a \).

\[ -x - a + a = -a \left( 1 + \frac{a - x}{-a} \right) = -a \left( 1 + \frac{x - a}{a} \right) \quad (A.6) \]

Where \( a = i\sqrt{2\pi t} \) is the point chosen to expand (A.10) in powers of \( (x - a) \).

\[ (-x)^{-\frac{1}{2} + \epsilon + it}e^{-N x} = e^{(\epsilon - 1/2 + it)} \ln(-a) + \ln(1 + \frac{x - a}{a}) e^{-Na - N(x - a)} \quad (A.7) \]

for \( \left| \left( \frac{x - a}{a} \right) \right| < 1 \)

\[ \ln \left( 1 + \frac{x - a}{a} \right) \approx \frac{x - a}{a} - \frac{(x - a)^2}{2a^2} + \frac{(x - a)^3}{3a^3} - .... \quad (A.8) \]

So:

\[ (-x)^{-\frac{1}{2} + \epsilon + it}e^{-N x} = (-a)^{-\frac{1}{2} + \epsilon + it}e^{-Na} e^{\left( \frac{-\frac{1}{2} + \epsilon + it}{a} - N \right)(x - a) + \left( -\frac{\epsilon}{2a^2} + \frac{(x - a)^3}{3a^3} - .... \right)} \quad (A.9) \]

We have:

\[ \frac{-\frac{1}{2} + \epsilon + it}{a} = \frac{-\frac{1}{2} + \epsilon + it}{i\sqrt{2\pi t}} \approx \sqrt{\frac{t}{2\pi}} \]

and:

\[ \frac{1 - \frac{1}{2} + \epsilon + it}{a^2} = \frac{1 - \frac{1}{2} + \epsilon + it}{2 \left( i\sqrt{2\pi t} \right)^2} \approx \frac{i}{4\pi} \]

The numerator of the integral can be written (with \( a = i\sqrt{2\pi t} \) [3, p. 145] bottom):

\[ (-x)^{-\frac{1}{2} + \epsilon + it}e^{-N x} \approx (-a)^{-\frac{1}{2} + \epsilon + it}e^{-Na} e^{\rho(x - a) e^{i(x - a)^2/4\pi}} \]

We look for a factor that transform the above approximate relation in a true equality:
\[-x^{-\frac{1}{2}+\epsilon+it}e^{-N\epsilon x} = (-a)^{-\frac{1}{2}+\epsilon+it}e^{-Na\epsilon p(x-a)}e^{\frac{i(x-a)^2}{4\pi}}g(x-a) = (A.10)\]

\[= (-a)^{-\frac{1}{2}+\epsilon+it}e^{-Na\epsilon p(x-a)}e^{\frac{i(x-a)^2}{4\pi}}\sum_{n=0}^{\infty}b_n(x-a)^n\]

Where \(a = i\sqrt{2\pi t}\) is the point chosen to expand \([A.10]\) in powers of \((x-a)\).

\[g(x-a) \text{ defined in [3, p. 145] bottom and in [7, p. 24] is:}\]

\[g(x-a) = \sum_{n=0}^{\infty} b_n(x-a)^n = e^{\frac{-(x-a)^2}{4\pi} - (p+N)(x-a)+(-\frac{1}{2} + \epsilon + it) \ln(1+\frac{e-x}{a})}\]

\[(A.11)\]

It is apparent that \(b_0 = 1\) even when \(\epsilon \neq 0\).
Because \([A.11]\) for \(x = a\) is bound to be \(\epsilon\) independent as \(\epsilon\) effects are multiplied by \(\ln(1)\).
For a given \(\epsilon\), with the power series \([A.1]\) “a” centered the reminder \(R(t, \epsilon)\) is:

\[R(t, \epsilon) = P_\epsilon \int_{L_1} (-a)^{-\frac{1}{2}+\epsilon+it}e^{-Na\epsilon p(x-a)}e^{\frac{i(x-a)^2}{4\pi}}g(x-a)dx = \frac{e^{-it\sqrt{N\epsilon}}}{e^x-1}\]

\[= P_\epsilon (-a)^\epsilon \int_{L_1} (-a)^{-\frac{1}{2}+it}e^{-Na\epsilon p(x-a)}e^{\frac{i(x-a)^2}{4\pi}}g(x-a)dx = . . .\]

\[(A.12)\]

We have eliminated \(F(t)e^{ix\epsilon}\) as we are looking for \([4,7]\) i.e. \(Z(t, \epsilon) \sim \frac{-\xi(\frac{1}{2}+\epsilon+it)}{F(t)e^{ix\epsilon}}\).
If we ignore \(e^{-t\pi}\) with respect to 1 ( note that for \(t > 14\) : \(e^{-t\pi} < e^{-14\times\pi} \approx 10^{-19}\) we get:

\[P_0 \approx \frac{e^{-it\sqrt{N\epsilon}}}{(2\pi)^{\frac{1}{2}}(2\pi)^{it}e^{-it\pi/4}(1-i\times 0)} \text{ for } \epsilon = 0\]

\[(A.13)\]

and for same reason we have:

\[P_\epsilon \approx \left\{\frac{e^{-it\sqrt{N\epsilon}}}{(2\pi)^{\frac{1}{2}}e^{-it\pi/4}(2\pi)^{it}(1-i\times 0)}\right\} \left[\frac{(1-i\times 0)}{(2\pi)^{it}e^{-it\pi/4}(1-i\times 0)}\right] \left[\left(\frac{2\pi}{t}\right)^{\epsilon}\right] \text{ for } t > > \epsilon > 0\]

\[(A.14)\]

So \(R(t, \epsilon)\) in \((A.12)\) becomes:

\[= P_0 \int_{L_1} \frac{(-a)^{-\frac{1}{2}+\epsilon+it}e^{-Na\epsilon p(x-a)}e^{\frac{i(x-a)^2}{4\pi}}g(x-a)dx}{e^x-1} \text{ for } t > > \epsilon > 0\]

\[(A.15)\]

because:

\[P_\epsilon (-a)^\epsilon = P_0 \frac{1}{(2\pi)^{it}e^{-it\pi}}\left(\sqrt{\frac{2\pi}{t}}\right)^{\epsilon} (-a)^\epsilon = \left(\sqrt{\frac{2\pi}{t}}\right)^{\epsilon} e^{it\frac{\pi}{2}}(-i)^\epsilon P_0 = P_0 \text{ for } t > > \epsilon > 0\]

What happens is that main integral in \((A.2)\) gives rise to a factor \((-a)^\epsilon = (-i\sqrt{2\pi})^\epsilon\) [ (2) p. 146 [3] ] who keeps invariant the coefficient \(P_0\) before integral \(\forall \epsilon\).
We can fetch out from integral in (A.15) the expression:

\[-a^{-\frac{1}{2} + it} e^{Na} = -i\sqrt{2\pi t}^{-\frac{1}{2} + it} e^{-N\sqrt{2\pi t}} = [-i\sqrt{2\pi t}]^{-\frac{1}{2} + it} e^{-Ni2\pi\sqrt{\frac{t}{\pi}}}\]

\[= [-i\sqrt{2\pi t}]^{-\frac{1}{2} + it} e^{-Ni2\pi\sqrt{\frac{t}{\pi}}} = [-i\sqrt{2\pi t}]^{-\frac{1}{2} + it} e^{-Ni2\pi(N+p)}\]

So (A.15) becomes:

\[= P_0 \ (a)^{-\frac{1}{2} + it} e^{-Na} \int_{L_1} \ \frac{e^{p(x-a)} e^{i(x-a)^2} g(x-a) dx}{e^x - 1} \ ; \ \quad t >> \epsilon \quad (A.16)\]

Developing the coefficient taking into account (A.14) gives the same result of [3, p. 146]:

\[P_0 \ (a)^{-\frac{1}{2} + it} e^{-Na} = \frac{e^{-\theta_1(t)} e^{-i\pi/2} [-i\sqrt{2\pi t}]^{-\frac{1}{2} + it} e^{-Ni2\pi(N+p)}}{(2\pi)^{1/2} (2\pi)^i e^{-i\pi/4}}\]

\[= \frac{e^{-\theta_1(t)} (\sqrt{t/2\pi})^{-\frac{1}{2} + it} e^{-Ni2\pi(N+p)}}{(2\pi)^{1} e^{-Ni2\pi(N+p)}}\]

\[= \frac{e^{-\theta_1(t)} (\sqrt{t/2\pi})^{-\frac{1}{2} + it} e^{-Ni2\pi(N+p)}}{2\pi i}\]

Which is the coefficient of \(\int_{L_1} e^{p(x-a)} e^{i(x-a)^2} g(x-a) dx\) in [3, p. 146] (apart \(1 - ie^{-\pi t}\) at denominator just reduced to 1).

We can change variable accordingly to [3, p. 146]:

\[x = u + 2\pi i N, \quad x - a = u + 2\pi i N - 2\pi i \sqrt{\frac{t}{2\pi}} = u - 2\pi ip \quad (A.17)\]

So numerator integrand change in :

\[e^{p(x-a)} e^{i(x-a)^2 / 4\pi} \rightarrow e^{pu-2\pi ip^2} e^{i^{2\pi^2-2\pi ip^2} / 4\pi} = e^{i^{2\pi^2+2\pi ip^2-4\pi i/p^2\pi}} \quad (A.18)\]

We can now take away from integral the objects: \(-2\pi ip^2 - \pi ip^2\)

Elaborating we get:

\[-Ni2\pi(N+p) = -2\pi i N^2 - 2\pi i Np \rightarrow -2\pi i N^2 - 2\pi i Np - 2\pi ip^2 \quad \pi ip^2 = \]

which is the same as :

\[-i\pi(N + p)^2 - i\pi N^2 - 2\pi ip^2 = -i\pi N^2 - i\pi p^2 - i\pi 2Np - i\pi N^2 - 2\pi ip^2\]

Besides \(-i\pi(N + p)^2 - i\pi N^2 - 2\pi ip^2 = -i\pi N^2 - i\pi \frac{t}{2\pi} - 2\pi ip^2\)

but:
\[-\theta(t) = -\left[\frac{t}{2} \ln \left(\frac{t}{2\pi e}\right) - \frac{\pi}{8}\right]\]

so we have:

\[e^{-i\left[\frac{t}{2} \ln \left(\frac{t}{2\pi}\right)\right] + it} e^{-i\pi \frac{t}{4\pi}} = e^{i\frac{\pi}{8}}\]

\[e^{-i\pi N^2} = (-1)^{N-1}\]

So like in (2) [3 p. 147] we can write \(R(t, \epsilon)\) in (A.15) as:

\[R(t, \epsilon) = (-1)^{N-1} \left(\sqrt{\frac{t}{2\pi}}\right)^{-\frac{1}{2}} \frac{e^{i\frac{\pi}{8}} e^{-2\pi i p^2}}{2\pi i} \int_{L_1} \frac{e^{i\frac{u^2}{4\pi} + 2\pi u}{g(u - 2\pi i p)}}{e^u - 1} du \quad (A.19)\]

The usual notation is:

\[C_0(p) = \psi(p) = \frac{e^{i\frac{\pi}{8}} e^{-2\pi i p^2}}{2\pi i} \int_{\text{whole } L_1} \frac{e^{i\frac{u^2}{4\pi} + 2\pi u}{g(u - 2\pi i p)}}{e^u - 1} du \quad (A.20)\]

Details of contour integration used to get (A.20) are in [3 p. 147-148].

### A.4 Dependence from \(\epsilon\)

So from (5) [3 p. 145] and (3) [3 p. 150] it is obvious that \(C_0(p)\), defined in [3 p. 154] is \(\epsilon\) independent; see (4.4). For \(C_i(p, \epsilon)\) with \(i > 0\) the usefulness of following manipulation is apparent when logarithmic derivative of \(g(x - a)\) with exponential and with the sum: \(\sum_{n=0}^{\infty} b_n(x - a)^n\) are equated; see (A.23).

Putting

\[\omega = \sqrt{\frac{2\pi}{t}} \quad (A.21)\]

for [3 p. 148] (bottom page) we have:

\[\frac{\partial}{\partial x} \left[-\frac{i(x - a)^2}{4\pi} - (p + N)(x - a) + \left(-\frac{1}{2} + \epsilon + it\right) \ln \left(1 + \frac{x - a}{\epsilon}\right)\right]\]

\[= \frac{x - a}{2\pi i} - \omega^{-1} + \frac{\epsilon - \frac{1}{2} + 2\pi i\omega^{-2}}{2\pi i\omega^{-1} + (x - a)} = \frac{(x - a)^2 + 2\pi i(\epsilon - \frac{1}{2})}{2\pi i[2\pi i\omega^{-1} + (x - a)]} = \sum_{n=0}^{\infty} b_n n(x - a)^{n-1} \quad (A.22)\]

where, equating like powers of \((x - a)\), it is easy to get \(b_k\) recursively. So we get the generalization with \(\epsilon > 0\) of [(5) [3 p. 152]:

24
\[ b_{n+1} = \frac{\omega}{4\pi^2(n+1)} \left( 2\pi i(n + 0.5 - \epsilon)b_n - b_{n-2} \right) \rightarrow b_n = \left( \frac{i}{2\pi} \right)^n \omega \frac{P_n(n, n)b_{n+1} + \left( \frac{i}{2\pi} \right)^2 \omega}{n} b_{n-3} \] (A.23)

where \( P_n(n, n) \) is defined as:

\[ P_n(n, n) = \prod_{n=n+1}^{n=n^2} \left( n - \frac{1}{2} - \epsilon \right) \] (A.24)

with \( b_0 = 1 \); \( b_{-1} = b_{-2} = 0 \)

For higher order approximation the expression (3) [3, p. 150] is exactly:

\[ \left[ F(t) e^{i\frac{n\pi}{4}} \right]^{-1} = \left( +s \right) \Gamma \left( \frac{1-s}{2} + 1 \right) \pi^{-\frac{1-s}{2}} \int_{C_{h,N}} \frac{(-x)^{s-1} e^{-Nx} dx}{e^x - 1} \]

\[ = R(t, \epsilon) \approx (-1)^{N-1} \left( \sqrt{\frac{2\pi}{t}} \right)^{0.25} e^{i(\Re - \Im)} \left[ \sum_{k=0}^{K} b_k(\omega)c_k \right] \] (A.25)

where \( b_k(\omega, \epsilon) \) is a polynomial of max degree \( n \) in \( \omega \), while \( c_k \) (“\( \epsilon \) independent”) is a linear combination of even, if \( n \) is even, or odd, if \( n \) is odd, derivatives with respect to \( p \) of \( C_0(p) = \psi(p) \); see (A.20).

And where the recovered \( \Re - \Im \) terms in \( t^{-1} \) is given in (3.25). While \( c_k \) is from (A.26) and \( F(t) \) is defined in (3.31).

With variable change \( x - a = u - 2\pi p \); see (A.17) and (A.10).

The \( c_n \) are in (3) [3, p. 150] which is the same of (A.25) here.

They are computed in (A.26) which derives from (4) [3, p. 150] given below:

\[ e^{2\pi i j} \sum_{m=0}^{\infty} \frac{\psi^m(p)}{m!} y^m = \sum_{n=0}^{\infty} \frac{(2y)^n}{n!} c_n \] (4) [3, p. 150]

equating homogeneous coefficients in \( y \) of left and right series.

Result is the following:

\[ c_n = \frac{e^{\pi/8} e^{-2\pi p^2}}{2\pi i} \int_{\Gamma} e^{iu^2/4\pi e^{2\pi pu}} (u - 2\pi ip)^n du = \frac{n}{2^n} \sum_{j=0}^{[n/2]} \frac{(2\pi i)^j \psi(n-2j)(p)}{j! (n-2j)!} \] (A.26)

\[ = \sum_{j=0}^{[n/2]} c_{n,n-2j} \psi(n-2j) = c_n \] (A.27)

So \( c_{k,j} \) is the coefficient of contribution at \( c_k \) of derivative \( \psi^{(j)} \).

Remind that

\[ \psi^{(0)}(p) = \frac{\cos[2\pi(p^2 - p - 1/16)]}{\cos(2\pi p)} \] (A.28)
From iteration (A.23) we have:

\[ b_{-2}(\omega) = 0 \quad , \quad b_{-1}(\omega) = 0 \quad , \quad b_0(\omega) = 1 \quad , \quad b_1(\omega) = B_{1,1} \omega^1 \]

\[ b_2(\omega) = B_{2,2} \omega^2 \quad , \quad b_3 = B_{3,3} \omega^3 + B_{3,1} \omega^1 \quad , \quad b_4(\omega) = B_{4,4} \omega^4 + B_{4,2} \omega^2 \quad , \quad b_5(\omega) = B_{5,5} \omega^5 + B_{5,3} \omega^3 \]

\[ b_6(\omega) = B_{6,6} \omega^6 + B_{6,4} \omega^4 + B_{6,2} \omega^2 \quad , \quad b_7(\omega) = B_{7,7} \omega^7 + B_{7,5} \omega^5 + B_{7,3} \omega^3 \quad , \quad b_8(\omega) = B_{8,8} \omega^8 + B_{8,6} \omega^6 + B_{8,4} \omega^4 \]

\[ b_9(\omega) = B_{9,9} \omega^9 + B_{9,7} \omega^7 + B_{9,5} \omega^5 + B_{9,3} \omega^3 \]

Where \( B_{m,k}(\epsilon) \) is the coefficient of \( \omega^k \) in the polynomial \( b_m(\omega, \epsilon) \). They contain the dependence from \( \epsilon \) throughout (A.24) factors.

The polynomial \( b_m(\omega, \epsilon) \) max degree is \( n \).

While the minimum degree is:

\[ \left\lfloor \frac{n}{3} \right\rfloor + n - 3 \times \left\lfloor \frac{n}{3} \right\rfloor \]

The formula connecting big single indexed \( C_i(p, \epsilon) \) in (4.4), small single indexed \( c_k \), from (A.26), small doubly indexed \( c : c_{k,j} \), from (A.27) and doubly indexed big B above i.e. \( B_{m,k}(\epsilon) \) is:

\[ C_k \omega^k = \sum_{i \geq 3k - 2j \leq 3k} c_i B_{i,k} \omega^k = \sum_{i \geq 3k - 2h \leq 3k} \sum_{j=0,1,2...} \left\lceil \frac{n}{2} \right\rceil \sum_{j=0,1,2...} \left\lfloor \frac{n}{2} \right\rfloor c_{i,i-2j} \psi^{(i-2j)} \right \} B_{i,k} \omega^k \]

(A.29)

Once fixed \( k \) of \( C_k \omega^k \) in (4.4), the index \( i \) can varies between \( 3k \) and \( 3k - 2h \geq k \), with \( h = 0, 1, 2,... \)

### A.5 Evaluation of \( C_1(p, \epsilon) \)

From (A.24) we have:

\[ B_{1,1} = \frac{i \omega}{2\pi} \frac{P_\epsilon(1,1)}{1!} \]

\[ B_{2,2} = \left( \frac{i \omega}{2\pi} \right)^2 \frac{P_\epsilon(1,2)}{2!} \]

\[ B_{3,3} = \left( \frac{i \omega}{2\pi} \right)^3 \frac{P_\epsilon(1,3)}{3!} \quad , \quad B_{3,1} = \left( \frac{\omega}{2\pi} \right) \frac{i^2}{2\pi 3} \]
\[ C_1 \omega^1 = c_1 B_{1,1} \omega^1 + c_3 B_{3,1} \omega^1 \]

where
\[
c_1 = \frac{1! (2\pi i)^0 \psi^{1-0}}{2^1 0! 0!}
\]
and
\[
c_3 = \frac{3!}{2^3} \left[ \frac{(2\pi i)^0 \psi^{3-0}}{0! 3!} + \frac{(2\pi i)^1 \psi^{3-2}}{1! 1!} \right] = c_{3,3} \psi^{(3)} + c_{3,1} \psi^{(1)}
\]

Terms like \( \omega^1 \) have “b” produced by \( b_1 \) and by \( b_3 \), so:

\[
C_1(p, \epsilon) \omega^1 = \frac{i \omega}{2 \pi} P_t(1, 1) \left[ \frac{1}{2^1} (2\pi i)^0 \psi^{1-0} \frac{1}{0!} + \frac{3!}{2^3} \left[ \frac{(2\pi i)^0 \psi^{3-0}}{0! 3!} + \frac{(2\pi i)^1 \psi^{3-2}}{1! 1!} \right] \left[ -\frac{\omega}{2 \pi} \frac{1}{2 \pi 3} \right] \right)
\]
\[
= \psi^{(1)}(p) \frac{i \omega}{4 \pi} \left( \left[ \frac{1}{2} - \epsilon \right] - \frac{3!}{2^3} \frac{2\pi}{2 \pi 3} \right) - \psi^{(3)}(p) \frac{\omega}{2^5 \pi^2 3} = -\epsilon \psi^{(1)}(p) \frac{i \omega}{4 \pi} - \psi^{(3)}(p) \frac{\omega}{2^5 \pi^2 3} \quad (A.30)
\]

Putting \( \epsilon = 0 \), then the imaginary part is zero and the real part happens to be like in [3, p. 153-154].

### A.6 Evaluation of \( C_2(p, \epsilon) \)

Consider now:
\[
C_2 \omega^2 = c_2 B_{2,2} \omega^2 + c_4 B_{4,2} \omega^2 + c_6 B_{6,2} \omega^2
\]

Terms like \( \omega^2 \) stem from \( b_{3 \times 2} \) till \( b_2 \): \( b_2(\omega), b_4(\omega) \) e \( b_6(\omega) \), so may be involved only the even derivatives:

\[
\psi^{(m)}(p) \quad ; \quad m = 0, 2, 4, 6
\]

\( B_{2,2} \) is computed above.

\[
B_{4,2} = \left( \frac{\omega}{2 \pi} \right)^2 \frac{(i)^{4-1}}{\pi 4!} \left[ P_t(4, 4) + \frac{3 \times 2}{2} P_t(1, 1) \right]
\]
and

\[
B_{6,2} = \left( \frac{\omega}{2 \pi} \right)^2 \frac{(i)^{6-2} 5 \times 4}{\pi 6!} \frac{1}{2}
\]

Besides

\[
c_2 = \frac{2!}{2^2} \left[ \frac{\psi^{(2)}}{2!} + \frac{\psi^{(0)} (2\pi i)^1}{0! 1!} \right] = c_{2,2} \psi^{(2)} + c_{2,0} \psi^{(0)}
\]

with:

\[
c_4 = \frac{4!}{2^4} \left[ \frac{\psi^{(4)}}{4!} + \frac{\psi^{(2)} (2\pi i)^1}{2! 1!} + \frac{\psi^{(0)} (2\pi i)^2}{0! 2!} \right] = c_{4,4} \psi^{(4)} + c_{4,2} \psi^{(2)} + c_{4,0} \psi^{(0)}
\]
and:

\[
c_6 = \frac{6!}{2^6} \left[ \frac{\psi^{(6)}}{6!} + \frac{\psi^{(4)}}{4!} \frac{(2\pi i)^1}{1!} + \frac{\psi^{(2)}}{2!} \frac{(2\pi i)^2}{2!} + \frac{\psi^{(0)}}{0!} \frac{(2\pi i)^3}{3!} \right] = c_{6,6}\psi^{(6)} + c_{6,4}\psi^{(4)} + c_{6,2}\psi^{(2)} + c_{6,0}\psi^{(0)}
\]

Let us compute separately terms like \( \omega^2 \) produced by \( \psi^{(0)}(p), \psi^{(2)}(p), \psi^{(4)}(p), \psi^{(6)}(p) \):

**A.6.1 Factors of \( \psi^{(0)}(p) \)**

\[
\psi^{(0)}(p) \left\{ \frac{2!}{2^2} \frac{(2\pi i)^1}{2!} \left( \frac{i\omega}{2\pi} \right)^2 P_\epsilon(1, 2) \right\} + \\
+ \psi^{(0)}(p) \left\{ \frac{4!}{2^4} \frac{(2\pi i)^2}{2!} \frac{(\omega)}{2\pi}^2 \left( \frac{(i)^{4-1}}{\pi4!} P_\epsilon(4, 4) + \frac{3 \times 2}{2} P_\epsilon(1, 1) \right) + \frac{6!}{2^6} \frac{(2\pi i)^3}{3!} \frac{(\omega)}{2\pi}^2 \left( \frac{(i)^{6-2}}{\pi^26!} 5 \times 4 \right) \right\} = \\
\psi^{(0)}(p)\pi\omega^23 \quad \text{[A.31]}
\]

dropping powers of \( \epsilon \) greater that 1

Consider now the phase shift due the multiplication by \( A^- \) of the reminder integral in 4.3. Let us limit to the \( t^{-1} \) term, then from (3.25) we have:

\[
\theta^+(t, \epsilon) = \theta_1(t) - \frac{-1 + 84\epsilon + 10\epsilon^2}{48t} \quad ; \quad \theta^-(t, \epsilon) = -\theta_1(t) - \frac{1 + 108\epsilon - 12\epsilon^2}{48t}
\]

from (4) in [3 p. 147], we can argue that \( C_n(p, \epsilon) \) terms are multiplied by:

\[
e^{i(\frac{3\pi}{2} + 3\pi)} = e^{i\left(\frac{-11 + 108\epsilon - 12\epsilon^2}{48t}\right)} \approx 1 - \frac{i\omega^2(1 + 108\epsilon - 12\epsilon^2)}{2^5\pi^3} \quad \text{[A.32]}
\]

If we multiply main integral in \( \text{(A.2)} \) by \( \text{(A.32)} \), then the term in \( \omega^0 : \psi^{(0)}(p) \), will be translated to \( \omega^2 \) summing up with homogeneous term stemming by other source.

In this case (sum of \( \text{(A.31)} \) with \( \text{(A.32)} \) at \( \epsilon = 0 \)) we have a cancellation, so for \( \epsilon = 0 \), \( \psi^{(0)}(p) \) does not contribute to \( C_2(p, \epsilon = 0) \).

**A.6.2 Factors of \( \psi^{(2)}(p) \)**

\[
\psi^{(2)}(p) \left\{ \frac{2!}{2^22!} \left( \frac{i\omega}{2\pi} \right)^2 P_\epsilon(1, 2) + \frac{4!}{2^42!} \frac{(2\pi i)^2}{2!} \frac{(\omega)}{2\pi}^2 \left( \frac{(i)^{4-1}}{\pi4!} P_\epsilon(4, 4) + \frac{3 \times 2}{2} P_\epsilon(1, 1) \right) \right\} + \\
+ \psi^{(2)}(p) \left\{ \frac{6!}{2^62!} \frac{(2\pi i)^3}{2!} \frac{(\omega)}{2\pi}^2 \left( \frac{(i)^{6-2}}{\pi^26!} 5 \times 4 \right) \right\} = \\
\psi^{(2)}(p)\frac{5}{2^6\pi^2} \left( 1 - \frac{8\epsilon}{10} \right) - \frac{\psi^{(2)}(p)3}{2^7\pi^2} \left[ \left( 1 - \frac{2\epsilon}{3} \right) \left( 1 - 2\epsilon \right) \right] - \frac{\psi^{(2)}(p)5}{2^7\pi^2}
\]

which is \( \psi^{(2)}(p)\frac{5}{2^6\pi^2} \) for \( \epsilon = 0 \).
A.6.3 Factors of $\psi^{(4)}(p)$

\[
\psi^{(4)}(p) \left\{ \frac{4!}{2^44!} \left( \frac{\omega}{2\pi} \right)^2 \left( \frac{i}{4!} \right)^{4-1} P_4(4, 4) + \frac{3 \times 2}{2} P_2(1, 1) \right\} + \frac{6!}{2^64!} \frac{5 \times 4}{1!} \left( \frac{\omega}{2\pi} \right)^2 \left( \frac{i}{\pi^26!} \right)^{6-2} \left( \frac{5 \times 4}{2} \right)
\]

\[= \frac{-i\psi^{(4)}(p)(5 - 8\epsilon)}{2^9\pi^33} + \frac{i\psi^{(4)}(p)5}{2^9\pi^33} \]

which is 0 for $\epsilon = 0$. So $\psi^{(4)}(p)$ gives no contribution to $C_2(p, \epsilon = 0)$.

A.6.4 Factors of $\psi^{(6)}(p)$

\[
\frac{6!}{2^6} \left[ \frac{(2\pi i)^0}{0!} \frac{\psi^{(6)}(p)}{6!} \right] \left( \frac{\omega}{2\pi} \right)^2 \frac{2}{\pi^6} \frac{5(i)^{6-2}}{\pi^6} = \frac{\psi^{(6)}(p)5\omega^2}{2^7\pi^46!} = \frac{\psi^{(6)}(p)5}{2^7\pi^46!}
\]

So it does not depend on $\epsilon$ accordingly with third row in [3, p. 153] bottom.

In summary for $\epsilon = 0$ the contribution to $\omega^2$ term simplify drastically [3, p. 154] bottom:

\[C_2(p, \epsilon = 0) = \frac{\psi^{(2)}(p)}{2^6\pi^2} + \frac{\psi^{(6)}(p)5}{2^7\pi^46!} \quad (A.34)\]

Similarly we can built all the $C_i(p, \epsilon)$ of (4.4):

\[C_2(p, \epsilon) = \frac{i\psi^{(0)}(p)}{2^0\pi3} \left[ 15 \left( 1 - \frac{2\epsilon}{3} \right) (1 - 2\epsilon) - 5 \left( 1 - \frac{8\epsilon}{10} \right) \right] - \frac{i\psi^{(0)}(p)(1 + 108\epsilon - 12\epsilon^2)}{2^5\pi3} + \frac{\psi^{(2)}(p)5}{2^6\pi^2} \left( 1 - \frac{8\epsilon}{10} \right) - \frac{\psi^{(2)}(p)3}{2^7\pi^2} \left[ \left( 1 - \frac{2\epsilon}{3} \right) (1 - 2\epsilon) \right] - \frac{\psi^{(2)}(p)5}{2^7\pi^2} + \frac{-i\psi^{(4)}(p)(5 - 8\epsilon)}{2^9\pi^33} + \frac{i\psi^{(4)}(p)5}{2^9\pi^33} + \frac{\psi^{(6)}(p)5}{2^7\pi^46!} \quad (A.35)\]
B  Bounds for Main Integral on paths:  $L_0, L_1, L_2, L_3$. With offset from critical line ($\epsilon > 0$)

B.1 Upperbound of Main integral on $L_0$ path

We follow [3, p. 141-142] “Estimation of the integral away from saddle point”, first case $j=0$, with the only difference of introducing $\epsilon > 0$.

We move in the complex plane $x$ on line: $x = a + ke^{i\pi/4}$ with real $k \geq 1$, and, we choose as boundary between $L_0$ and $L_1$ point:

$$\text{Boundary}(L_0, L_1) = a + \frac{|a|}{r} e^{i\pi/4} ; \quad 1 < r \leq 2$$ (B.1)

Where $a$ is defined in (2.4).

In [3, p. 141-142] $r = 2$, but, if $r > 1$ it is enough for the logarithmic series (A.8) to converge.

If we write the numerator module of the integrand in (A.2) as $|(-x)^{s-1} e^{-Nx}| = e^{\phi(k)}$ where

$$\phi(k) = Re[(\epsilon - 1/2 + it) \ln(a + ke^{i\pi/4}) - N(a + ke^{i\pi/4})]$$

then

$$\frac{d\phi(k)}{dk} = Re\left[ Ne^{i\pi/4} \left( \frac{-1/2 + \epsilon + it}{N(a + ke^{i\pi/4})} - 1 \right) \right] = Re\left[ Ne^{i\pi/4} \left( \left[ \frac{-1/2 + \epsilon}{N} + \frac{it}{N} \right] \frac{i\sqrt{2\pi t + \frac{k}{\sqrt{2}}} + \frac{1}{\sqrt{2}} + i\frac{k}{\sqrt{2}}}{1} \right) \right] =$$

$$Re\left[ Ne^{i\pi/4} \left( \left[ \frac{-1/2 + \epsilon - t}{N} + i\frac{1/2 + \epsilon}{N} \right] \frac{k/\sqrt{2} - i\sqrt{2\pi t + \frac{k}{\sqrt{2}}} + \frac{1}{\sqrt{2}}}{(2\pi t + \frac{k}{\sqrt{2}})^2 + \frac{k^2}{2}} \right) \right] =$$

$$= \frac{N}{\sqrt{2}} \left[ Re \left( \frac{-1/2 + \epsilon - t}{N} + i\frac{1/2 + \epsilon}{N} \right) \frac{k/\sqrt{2} - i\sqrt{2\pi t + \frac{k}{\sqrt{2}}} + \frac{1}{\sqrt{2}}}{(2\pi t + \frac{k}{\sqrt{2}})^2 + \frac{k^2}{2}} \right] - 1 \right] =$$

$$= \frac{N}{\sqrt{2}} \left[ \frac{-1/2 + \epsilon - t}{N} \frac{k/\sqrt{2} - i\sqrt{2\pi t + \frac{k}{\sqrt{2}}} + \frac{1}{\sqrt{2}}}{(2\pi t + \frac{k}{\sqrt{2}})^2 + \frac{k^2}{2}} \right] + \frac{(t - 1/2 + \epsilon) \left( \sqrt{2\pi t + \frac{k}{\sqrt{2}}} \right)}{N \left( (2\pi t + \frac{k}{\sqrt{2}})^2 + \frac{k^2}{2} \right)} - 1 \right] = ..$$

$$= \frac{N}{\sqrt{2}} \left[ \frac{-1/2 + \epsilon - t}{N} \frac{2k/\sqrt{2}}{(2\pi t + \frac{k}{\sqrt{2}})^2 + \frac{k^2}{2}} + \frac{(t - 1/2 + \epsilon) \left( \sqrt{2\pi t} \right)}{N \left( (2\pi t + \frac{k}{\sqrt{2}})^2 + \frac{k^2}{2} \right)} - 1 \right] < 0$$ (B.2)

Where to be $.. < 0$ is justified by:

(1) $N = \sqrt{\frac{1}{2\pi}}$ ,

(2) Both numerator and denominator have: $\sqrt{\frac{t}{2\pi}}(2\pi t)$.

At numerator, for $|\epsilon| < 1/2$ and $k \geq 1$ we have to add negative quantities, while at denominator we add all positive quantities .

In [3 p. 141-142] it is chosen $r = 2$, but , in order (A.8) to converge $r > 1$ is enough.
So the numerator module of the integrand in (A.2) on \( L_0 \), is max in \( x = \text{Boundary}(L_0, L_1) = a + \frac{|a|}{r} e^{i \frac{\pi}{4}} \); see B.1.

As \( \ln(w) = \ln |w| + i \angle[w] \) and \( |e^w| = e^{Re[w]} \), with complex \( w \), So

\[
e^{(\frac{-t}{2} + it) \ln|a - \frac{|a|}{r} e^{i \frac{\pi}{4}}|} = \left|\sqrt{2\pi t} e^{\frac{-t}{2} - i \left(1 + \frac{1}{r \sqrt{2}}\right)}\right|
\]

then the numerator module of the integrand in (A.2) is at most:

\[
e^{\Re\left[\left(-\frac{t}{2} + it\right) \ln\left(-a - \frac{|a|}{r} e^{i \frac{\pi}{4}}\right)\right] - N \frac{|a|}{r \sqrt{2}}} = \left|\sqrt{2\pi t} e^{\frac{-t}{2} - i \left(1 + \frac{1}{r \sqrt{2}}\right)}\right|
\]

remembering \( a = i \sqrt{2\pi t} \) and \( N + p = \sqrt{\frac{t}{2\pi}} \), we have:

\[
-N \frac{|a|}{r \sqrt{2}} = - \left(\sqrt{\frac{t}{2\pi}} - p\right) \frac{\sqrt{2\pi t}}{r \sqrt{2}} = - \frac{t}{r \sqrt{2}} + p \frac{\sqrt{\pi t}}{r} \quad 0 \leq p < 1
\]

Or in other words: we can elaborate \(-N \frac{\sqrt{\pi t}}{r}\) to get:

\[
-N \frac{\sqrt{\pi t}}{r} < -(N - 1) \frac{\sqrt{\pi t}}{r} \leq - \left(\sqrt{\frac{t}{2\pi}} - 1\right) \frac{\sqrt{\pi t}}{r} = - \frac{t}{r \sqrt{2}} + \frac{\sqrt{\pi t}}{r}
\]

besides

\[
\Im\left[\ln\left(-a - \frac{|a|}{r} e^{i \frac{\pi}{4}}\right)\right] = \Im\left[\ln(a) + \Im\left[-1 + \frac{i}{r} e^{i \frac{\pi}{4}}\right]\right] = -\frac{\pi}{2} + \Im\left[-1 + \frac{i}{r} \left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)\right]
\]

so we can write:

\[
e^{-t \Im\left[\ln\left(-a - \frac{|a|}{r} e^{i \frac{\pi}{4}}\right)\right] - N \frac{|a|}{r \sqrt{2}}} = e^{-t \left(-\frac{\pi}{2} - \arctan\left(\frac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}} + 1}\right)\right)}\left(-N \frac{\sqrt{\pi t}}{r}\right)
\]

defining further:

\[
K_0(r) := \arctan\left(\frac{1}{r \sqrt{2} + 1}\right) - \frac{1}{r \sqrt{2}} \quad 1 < r \leq 2
\]

we can write (B.3) as:

\[
\left|\sqrt{2\pi t} e^{\frac{-t}{2} - i \left(1 + \frac{1}{r \sqrt{2}}\right)}\right| e^{-t \left(-\frac{\pi}{2} - \arctan\left(\frac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}} + 1}\right)\right)}\left(-N \frac{\sqrt{\pi t}}{r}\right)
\]

taking \( p = 1 \), we can bound the numerator module of the integrand in (A.2) i.e. (B.3) with:

\[
< \left|\sqrt{2\pi t} e^{\frac{-t}{2} - i \left(1 + \frac{1}{r \sqrt{2}}\right)}\right| e^{-t \left(-\frac{\pi}{2} - \arctan\left(\frac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}} + 1}\right)\right)}\left(-N \frac{\sqrt{\pi t}}{r}\right)\left(1 \leq r \leq 2\right)
\]

because
\[ e^{-N \sqrt{\pi t}} < e^{-\frac{1}{r \sqrt{2}} \frac{\sqrt{\pi t}}{r^2}} \]

but:

\[ 1 < \left| \left( 1 + \frac{1}{r \sqrt{2}} \right) - i \frac{1}{r \sqrt{2}} \right| < 2 \quad \forall r : 1 < r \leq 2 \]

We can bound, by (B.5), the numerator value on \( L_0 \) with the value that \( |(-x)^{s-1}e^{-Nx}| \) assumes on \( \text{Boundary}(L_0, L_1) \), so the integral on \( L_0 \) is bounded by:

\[ u = \Re(x) \text{ from } u = \frac{|u|}{r \sqrt{2}} = \frac{\sqrt{\pi t}}{r} \text{ to } \infty \]

\[ \int_{L_0} \frac{|dx|}{|e^x - 1|} \leq \int_{L_0} e^{-u} \frac{\sqrt{2} du}{1 - e^{-u}} \quad u = \Re(x) \quad x = i\sqrt{2\pi} + \frac{\sqrt{2t\pi}}{r} e^{i\pi/4} k \quad |dx| = \sqrt{2} du \]  

(B.6)

because: \( |1 - e^{-x}| = \sqrt{(1 - e^{-\Re(x)} \cos(-\Im(x)))^2 + (e^{-\Re(x)} \sin(-\Im(x)))^2} \geq |1 - e^{-\Re(x)} \cos(-\Im(x))| \geq |1 - e^{-\Re(x)}| = 1 - e^{-u} \). And we e have \( \Re(x) > 0 \)

In other words we change an integral on “x” complex variable with an integral on real variable \( u \)

which gives us a bound of module.

And we can write

\[ \int_{L_0}^\infty \frac{|dx|}{|e^x - 1|} \leq \int_{L_0}^\infty e^{-u} \frac{\sqrt{2} du}{1 - e^{-u}} < \sqrt{2} \frac{|-e^{-u}|}{\sqrt{\pi t}} = \frac{\sqrt{2} e^{-\frac{\sqrt{\pi t}}{r}}}{1 - e^{-\frac{\sqrt{\pi t}}{r}}} \]

(B.7)

Afterword \( \int_{L_0}^\infty \frac{|dx|}{|e^x - 1|} \) is multiplied by B.5

Then the \( A' R(t, \epsilon)e^{\frac{j\pi t}{4}} \) in A.2 module ( \( \text{divided by } F(t) \text{ for } 4.7 \) ) is bounded above ( because of (B.2) by):

\[ \left| e^{Re_2 + Re_3 + i(3_2 + \bar{3}_3)} \right| e^{\frac{j\pi}{4} e^{-\pi t/2} e^{-i\theta_4(t)}} \left( \frac{2\pi}{t} \right)^{\frac{1}{2} + \epsilon} |e^{-\frac{j\pi}{4} - ie^{\frac{i\pi}{4}} e^{-t\pi}}| \left( \sqrt{\frac{2\pi}{t}} \right)^{\epsilon} \left( \frac{2\pi}{t} \right)^{-\frac{1}{2}} 2 e^{K_0}(t) e^{\frac{\sqrt{\pi t}}{r}} \int_{L_0} |dx| \frac{\sqrt{2\pi t}}{r^{\epsilon - 1}} \right| \]

\[ = \left( \frac{2\pi}{t} \right)^{\frac{1}{2} + \epsilon} |e^{-\frac{j\pi}{4} - ie^{\frac{i\pi}{4}} e^{-t\pi}}| \left( \sqrt{\frac{2\pi}{t}} \right)^{-\frac{1}{2}} 2 e^{K_0}(t) e^{\frac{\sqrt{\pi t}}{r}} \int_{L_0} |dx| \frac{\sqrt{2\pi t}}{r^{\epsilon - 1}} \right| \]

simplifies with \( \frac{1}{(2\pi)^{\frac{1}{2} + \epsilon}} \) resulting in :

\[ = \left( \frac{2\pi}{t} \right)^{\frac{1}{2} + \epsilon} |e^{-\frac{j\pi}{4} - ie^{\frac{i\pi}{4}} e^{-t\pi}}| \left( \sqrt{\frac{2\pi}{t}} \right)^{-\frac{1}{2}} 2 e^{K_0}(t) e^{\frac{\sqrt{\pi t}}{r}} \int_{L_0} \frac{|dx|}{|e^x - 1|} \approx .. \]

\[ t > 10 \quad \forall \epsilon \quad |e^{-\frac{j\pi}{4} - ie^{\frac{i\pi}{4}} e^{-t\pi}}| \to 1 \]

\[ .. \approx \left( \frac{2\pi}{t} \right)^{\frac{1}{2} + \epsilon} |e^{-\frac{j\pi}{4} - ie^{\frac{i\pi}{4}} e^{-t\pi}}| \left( \sqrt{\frac{2\pi}{t}} \right)^{-\frac{1}{2}} 2 e^{K_0}(t) e^{\frac{\sqrt{\pi t}}{r}} \int_{L_0} \frac{|dx|}{|e^x - 1|} < ... \]

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See (3.28) and (3.27) \( \Re_2 + \Re_3 = \left( \frac{27}{96} + f(\epsilon) \right) \frac{1}{t^2} + (\ldots) \frac{1}{t} \ldots \) so they can be neglected, for \( t > 10 \) and applying [B.7] we have

\[
.. < \left( \frac{27}{96} + f(\epsilon) \right) \frac{1}{t^2} + (\ldots) \frac{1}{t} < \frac{27}{96} + f(\epsilon) \frac{1}{t^2} + (\ldots) \frac{1}{t} = \text{UpperBound}_L(t)
\]

So we reach an stricter upper bound with respect to [3, p. 143]; see (A.2). So we have:

\[
\left( +s \right) \Gamma \left( \frac{1-s}{2} + 1 \right) \frac{\pi^{(1-s)/2}}{(2\pi)^{s-1} \sin(\pi s/2) 2\pi i} \int_{L_0} (-x)^{s-1} e^{-Nx} dx < \ldots
\]

\[
... < \left( \frac{27}{96} + f(\epsilon) \right) \frac{1}{t^2} + (\ldots) \frac{1}{t} < \frac{27}{96} + f(\epsilon) \frac{1}{t^2} + (\ldots) \frac{1}{t} = \text{UpperBound}_L(t)
\]

Note that upper bound on \( L_0 \) ([B.8]) is decreasing while \( \epsilon \) increasing.

### B.2 \( L_1 \) spurious contribution

To compute (A.16) in Appendix A, we used, like [3, p. 147], a contour integral which gives an exact result (see (4.5) and (A.19)) on entire line \( L \supset L_1 \).

We are interested in the evaluation between \( \text{Boundary}(L_0, L_1) = a + |a| e^{i\pi/4} ; 1 < r \leq 2 \) and \( \text{Boundary}(L_1, L_2) = a - |a| e^{i\pi/4} ; 1 < r \leq 2 \), where \( a \) is the saddle point given in (2.4).

Here we evaluate the spurious contribution inserted with this contour integral in \( L - L_1 \), that in (4.3), is referred as :

\[
\gamma_{L - L_1} \quad \text{(B.9)}
\]

For standard Gaussian formulas see [6, p. 183].

\[
G(x) = \left( \frac{c}{\pi} \right)^{1/2} e^{-cx^2} = \frac{1}{\sqrt{2\pi \sigma}} e^{-x^2/2 \sigma^2} ; \quad c = \frac{1}{2\sigma^2}
\]

An approximate expression for large values of argument \( x > 2\sigma \) can be obtained from :

\[
\int_{-\infty}^{\infty} G(x) dx = \text{Erf} \left( y = \frac{x}{\sigma \sqrt{2}} \right) \approx 1 - \frac{e^{-y^2}}{\sqrt{\pi} y} \left( 1 - \frac{1}{2y^2} + \frac{1 \times 3}{(2y^2)^2} - \frac{1 \times 3 \times 5}{(2y^2)^3} + \ldots \right) \quad \text{for} \quad y > \sqrt{\frac{2}{\sigma}}
\]

(B.10)

Now let us focus (A.16) on :

\[
\int_{L \to L_1} e^{p(x-a)} \frac{e^{(x-a)^2/4}}{e^x - 1} g(x-a) dx
\]

(B.11)

See also [3, p. 146].

Putting:

\[
x - a = \sqrt{2} y = y \left( \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \right)
\]

(B.12)
namely:

\[ x = a + \sqrt{i}y = i\sqrt{2\pi}t + \sqrt{i}y = 2 \left( \frac{y}{2\sqrt{2}} + i \left( \sqrt{\frac{\pi t}{2}} + \frac{y}{2\sqrt{2}} \right) \right) \]  \hspace{1cm} (B.13)

As \((\sqrt{i})^2 = i\) we have:

\[ e^{i(x-a)^2} = e^{i(\sqrt{i})^2} = e^{-(y)^2} \]

and:

\[ e^{p(x-a)} = e^{p(\sqrt{\frac{\pi}{2}} + \sqrt{\frac{\pi}{2}})} \]

Let us pose \(g(x - a) = 1\) and compute:

\[ \frac{|e^{p(x-a)} e^{i(x-a)^2}|}{e^x - 1} \]

for \(\frac{1}{e^x - 1}\) on the line \(L_1\) we have

\[ \frac{1}{e^x - 1} = \frac{e^{(-\frac{1}{2})x}}{e^{x/2} - e^{-x/2}} = 2 e^{(-1)\left(\frac{y}{2\sqrt{2}} + i(\sqrt{\frac{\pi t}{2}} + \frac{y}{2\sqrt{2}})\right)} \]

\[ \sinh \left( \frac{y}{2\sqrt{2}} + i \left( \sqrt{\frac{\pi t}{2}} + \frac{y}{2\sqrt{2}} \right) \right) \]

\[ = \sinh \left( \frac{y}{2\sqrt{2}} \right) \cos \left( \sqrt{\frac{\pi t}{2}} + \frac{y}{2\sqrt{2}} \right) + i \cosh \left( \frac{y}{2\sqrt{2}} \right) \sin \left( \sqrt{\frac{\pi t}{2}} + \frac{y}{2\sqrt{2}} \right) \]

and taking the module:

\[ 2 \frac{e^{-\frac{y}{2\sqrt{2}} - i(\sqrt{\frac{\pi t}{2}} + \frac{y}{2\sqrt{2}})}}{\sinh \left( \frac{y}{2\sqrt{2}} + i \left( \sqrt{\frac{\pi t}{2}} + \frac{y}{2\sqrt{2}} \right) \right)} \leq 2 \frac{e^{-\frac{y}{2\sqrt{2}}}}{\sqrt{\sinh^2 \left( \frac{y}{2\sqrt{2}} \right) \cos^2 \left( \sqrt{\frac{\pi t}{2}} + \frac{y}{2\sqrt{2}} \right) + \cosh^2 \left( \frac{y}{2\sqrt{2}} \right) \sin^2 \left( \sqrt{\frac{\pi t}{2}} + \frac{y}{2\sqrt{2}} \right)}} \]

If we avoid \(y = 0\) and the points \(\in L_1\), as :

\[ \sinh[\alpha] < \cosh[\alpha] \quad \forall \alpha \in \mathbb{R}, \]

\[ \left\{ \begin{array}{c}
2 \frac{e^{-\frac{y}{2\sqrt{2}} - i(\sqrt{\frac{\pi t}{2}} + \frac{y}{2\sqrt{2}})}}{\sinh \left( \frac{y}{2\sqrt{2}} + i \left( \sqrt{\frac{\pi t}{2}} + \frac{y}{2\sqrt{2}} \right) \right)}_{y \notin L_1} < 2 \frac{e^{-\frac{y}{2\sqrt{2}}}}{\sinh \left( \frac{y}{2\sqrt{2}} \right)}_{y \notin L_1} \quad \{ \begin{array}{c}
< 2 \quad \text{for } y < 0 \\
\approx 2e^{-\frac{y}{\sqrt{2}}} \quad \text{for } y > 0
\end{array} \}
\]

For others factors of integrand, by \(B.12\):

\[ \text{...} \]

---

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\[
|e^{p(x-a)} e^{i(x-a)^2/4\pi^2}| = |e^{py((1/\sqrt{2}) + i/\sqrt{2}) e^{-y^2/4\pi}}| = e^{(\sqrt{2}y - y^2/4\pi)}
\] (B.15)

The module \( (B.15) \) has a Gaussian shape, with \( \sigma = \sqrt{2\pi} \approx 2.5066 \), and, the max at \( y_{\text{max}} = \frac{2\pi}{\sqrt{2}} \approx p \times 4.4428 \approx p\sigma \times 1.772 \); \( 0 \leq p \leq 1 \)

While \( (r \approx 1) : -|a| < y < |a| = \sqrt{2\pi t} \approx \sqrt{t} \times 2.5066 = \sqrt{t} \times \sigma \)

Or, if we follow [3, p. 141] i.e. \( r = 2 : -\frac{|a|}{2} < y < \frac{|a|}{2} \).

If \( t = 20, p = \sqrt{t/(2\pi)} - \left| \sqrt{t/(2\pi)} \right| = 0.7841 \).

So we have the maximum of Gaussian shape at:

\[ p\sigma \times 1.772 = \sigma \times (0.7841 \times 1.772) = 1.3894 \times \sigma \]

See (2.4): \( |a| = \sqrt{2\pi t} \approx \sqrt{t} \times \sigma = 4.472\sigma \).

So for \( r \geq 1 \) we have a distance of Gaussian maximum to the boundary given by:

\[ (4.472 - 1.38)\sigma = 3.19\sigma \]

So let us call

\[ \text{Contribution}_{y \notin L_1} = \int_{y \notin L_1} e^{(\frac{ty}{\sqrt{2}} - t^2/4\pi)} dy \]

the additional spurious contribution that enters in contour integration of \( (B.11) \), both for positive or negative \( y \), taking into account \( (B.14) \).

And let us call

\[ \text{Contribution}_{y \in L_1} = \int_{y \in L_1} e^{(\frac{ty}{\sqrt{2}} - t^2/4\pi)} dy \]

the contribution of the integral in \(-|a| < y < |a|\) for \( t = 20 \).

Then we have:

\[ |\text{Contribution}_{y \notin L_1}| \times 0.003 > |\text{Contribution}_{y \in L_1}| \] (B.16)

because \( (B.11) \), Gaussian beyond \( 3\sigma \). If we take \( r = 2 \) like in [3, p. 141] the boundary is at \( \frac{|a|}{2} = 2.237\sigma \) and we ought to choose a \( t > 20 \) in order to reach the same \( 3\sigma \) distance of border from Gaussian maximum .

Anyhow with \( t > 100 \) \( (B.16) \) holds with huge margin .

This analysis together with fig. [3] and \( (B.28) \), explains why the case study of \( \xi(t, \epsilon = 0) \), in [3, p. 155] presents a very good match with Haselgrove tabel [3, p. 122] at least from \( t > 18 \). Besides we have to take into account \( (B.14) \) which lowers further the contributions at right. Instead spurious contribution at left are multiplied by \( \approx 2 \), but distance from \( \text{Boundary}(L_1, L_2) \), in worst case \( (p = 0) \), is, for \( t > 20 \): \( |a| = \sqrt{2\pi t} \approx \sqrt{t} \times \sigma > 4.472\sigma \). With \( (B.10) \) we can appreciate the corresponding \( \Delta R_R \) in \( (B.28) \) and compare with data in fig. 3.
B.3 Upper bound of Main integral on \(L_2\) path

We follow [3, p. 142] “Estimation of the integral away from saddle point”, second case \(j=2\), with the only difference of introducing \(\epsilon \neq 0\). The boundary of \(L_2\) segment are:

\[
\text{Boundary}(L_1, L_2) = a - \frac{|a|}{r} e^{i\frac{\pi}{2}} \quad ; \quad 1 < r \leq 2
\]

\[
\text{Boundary}(L_2, L_3) = (\text{Re}[a - \frac{|a|}{r} e^{i\frac{\pi}{2}}], -\pi(2N + 1)) \quad ; \quad 1 < r \leq 2
\]

Where \(a\) is defined in (2.4).

So, in the integration on \(L_2\), always holds:

\[
\Re[x] = -\frac{|a|}{\sqrt{2}r} = -\frac{\sqrt{\pi}t}{r} = -b
\]

The integrand denominator of (A.2) is:

\[
|e^x - 1| \geq |e^{-b} - 1| > 0.5 \quad \text{for} \quad t > 10 \quad 1 < r \leq 2
\]

As \(-N(-b) = (-\sqrt{\frac{t}{2\pi}} + p)(-b) < \sqrt{\frac{t}{2\pi}}b\), the integrand numerator of A.2 is at most:

\[
|(-x)^{\epsilon - \frac{1}{2} + i\epsilon N}e^{-Nz}| \leq \max |x|^{\epsilon - \frac{1}{2} + i\epsilon N} e^{-\epsilon \Im[\ln(-x)]}e^{\frac{\sqrt{\pi}t}{2r}b}
\]

Because \(\epsilon - 1/2 \leq 0\) the \(\max |x|^{\epsilon - \frac{1}{2}}\) is positioned where \(L_2\) crosses real axis, and, it is:

\[
\max |x|^{\epsilon - \frac{1}{2}} = b^{\epsilon - \frac{1}{2}}
\]

the max value of \(e^{-t \Im[\ln(-x)]}\) happens to be where the exponent is max, i.e. where \(\Im[\ln(-x)]\) is min, namely in point \(\text{Boundary}(L_1, L_2)\) where -x phase has a minimum. We have (\(a\) is defined in [2.4]):

\[
\Im[\ln(-x)] = \Im[\ln(-a - \frac{|a|}{r} e^{i\frac{\pi}{2}})] = \Im \ln \left[ -i + \frac{1}{r \sqrt{2}} + i \frac{1}{r \sqrt{2}} \right] = \Im \ln \left[ (1 - \frac{1}{r \sqrt{2}} + i \frac{1}{r \sqrt{2}}) \right] = -\frac{\pi}{2} + \arctan \left( \frac{1}{1 - \frac{1}{r \sqrt{2}}} \right) = -\frac{\pi}{2} + \arctan \left( \frac{1}{\sqrt{2r} - 1} \right)
\]

So:

\[
\max_{on \ L_2} \left\{ e^{-t \Im[\ln(-x)]} \right\} = e^{\frac{t}{2}} e^{-t \arctan \left( \frac{1}{\sqrt{2r} - 1} \right)}
\]

while

\[
e^{-N(-b)} < e^{\frac{t}{\sqrt{2r}}}
\]

Then the numerator integrand module in (A.2) is at most:

\[
b^{\epsilon - \frac{1}{2}} e^{\frac{t}{2}} e^{-t \left( \arctan \left( \frac{1}{\sqrt{2r} - 1} \right) - \frac{\sqrt{\pi}t}{2r} \right)} = b^{\epsilon - \frac{1}{2}} e^{\frac{t}{2}} e^{-t(K_2(r))}
\]
Where:

\[ K_2(r) := - \left( \arctan \left( \frac{1}{\sqrt{2r^2 - 1}} \right) - \frac{1}{\sqrt{2r^2}} \right) \]  

(B.19)

So reminding that (B.18):

\[ \left( \frac{1}{e^{x}} - 1 \right)_{L_2} < 2 \]

The length of segment \( L_2 \) is about 2|a| = 2\sqrt{2\pi t} = 2r\sqrt{2\sqrt{\pi t}} = 2r\sqrt{2b}, \) then, after clearing the common factor \( F(t) \) and for \( t > 10 \) (see also (A.2)) we have:

\[
\left| e^{R_{\xi_2}^2 + R_{\xi_3}^2} \right| \frac{e^{-\pi t/2}}{(2\pi)^{\frac{3}{2}+\epsilon}} \left( e^{-\frac{\pi}{2} \epsilon} - ie^{\frac{\pi}{2} \epsilon} e^{-t\pi} \right) \left( \sqrt{\frac{2\pi}{t}} \right)^\epsilon \int_{L_2} \frac{(\pi e^{x})^{\epsilon - 1} e^{-N_x \pi x} dx}{e^x - 1} < \\
< \frac{e^{-\pi t/2}}{(2\pi)^{\frac{3}{2}+\epsilon}} \left( \sqrt{\frac{2\pi}{t}} \right)^\epsilon b^{-\frac{1}{2}} e^{\frac{1}{2} \epsilon} e^{K_2(r)t} 2r\sqrt{2b} = \\
\quad = \frac{b^{-\frac{1}{2}}}{(2\pi)^{\frac{3}{2}+\epsilon}} \left( \sqrt{\frac{2\pi}{t}} \right)^\epsilon e^{K_2(r)t} 2r\sqrt{2b} = \frac{2\epsilon \pi t e^{-t(k)2r\sqrt{2}}}{(2\pi)^{\frac{3}{2}+\epsilon}} \left( \sqrt{\frac{2\pi}{t}} \right)^\epsilon e^{-t(k)2r\sqrt{2}} \left( \sqrt{\frac{\pi t}{r}} \right) = \\
\quad = \frac{(\frac{1}{r})^\epsilon}{(2\pi)^{\frac{3}{2}+\epsilon}} \left( \sqrt{2\pi} \right)^\epsilon e^{K_2(r)t} 2r\sqrt{2} \left( \sqrt{\frac{\pi t}{r}} \right)^{\frac{1}{2}} = \ldots \frac{(\frac{1}{r})^\epsilon}{(2\pi)^{\frac{3}{2}+\epsilon}} \left( \sqrt{2} \right)^{\frac{1}{2}} e^{-t(k)} 2r\sqrt{2} \left( \sqrt{\frac{\pi t}{r}} \right)^{\frac{1}{2}} = \\
\quad = \frac{(\frac{1}{r})^\epsilon}{(2\pi)^{\frac{3}{2}+\epsilon}} \left( \sqrt{2} \right)^{\frac{1}{2}} e^{K_2(r)t} 2\sqrt{r} \sqrt{2} \left( \sqrt{\frac{\pi t}{r}} \right)^{\frac{1}{2}} = \frac{(\frac{1}{r})^\epsilon}{(2\pi)^{\frac{3}{2}+\epsilon}} e^{-t(k)2\sqrt{r} \sqrt{2}} \left( \sqrt{\frac{\pi t}{r}} \right)^{\frac{1}{2}} = \\
\quad = \frac{(\frac{1}{r})^\epsilon}{(2\pi)^{\frac{3}{2}+\epsilon}} \left( \sqrt{2} \right)^{\frac{1}{2}} e^{K_2(r)t} \sqrt{2} \left( \sqrt{\frac{\pi t}{r}} \right)^{\frac{1}{2}} = \frac{(\frac{1}{r})^\epsilon}{(2\pi)^{\frac{3}{2}+\epsilon}} e^{-t(k)2\sqrt{r} \sqrt{2}} \sqrt{2} \left( \sqrt{\frac{\pi t}{r}} \right)^{\frac{1}{2}} = \\
\quad = \frac{(\frac{1}{r})^\epsilon}{(2\pi)^{\frac{3}{2}+\epsilon}} \left( \sqrt{2} \right)^{\frac{1}{2}} e^{K_2(r)t} 2\sqrt{r} \left( \sqrt{\frac{\pi t}{r}} \right)^{\frac{1}{2}} = \frac{(\frac{1}{r})^\epsilon}{(2\pi)^{\frac{3}{2}+\epsilon}} e^{-t(k)2\sqrt{r} \sqrt{2}} \sqrt{2} \left( \sqrt{\frac{\pi t}{r}} \right)^{\frac{1}{2}} = \frac{(\frac{1}{r})^\epsilon}{(2\pi)^{\frac{3}{2}+\epsilon}} e^{K_2(r)t} 2\sqrt{r} \left( \sqrt{\frac{\pi t}{r}} \right)^{\frac{1}{2}} = \frac{(\frac{1}{r})^\epsilon}{(2\pi)^{\frac{3}{2}+\epsilon}} e^{-t(k)2\sqrt{r} \sqrt{2}} \sqrt{2} \left( \sqrt{\frac{\pi t}{r}} \right)^{\frac{1}{2}} = \\
\quad = \frac{(\frac{1}{r})^\epsilon}{(2\pi)^{\frac{3}{2}+\epsilon}} \left( \sqrt{2} \right)^{\frac{1}{2}} e^{K_2(r)t} 2\sqrt{r} \left( \sqrt{\frac{\pi t}{r}} \right)^{\frac{1}{2}} = \frac{(\frac{1}{r})^\epsilon}{(2\pi)^{\frac{3}{2}+\epsilon}} e^{-t(k)2\sqrt{r} \sqrt{2}} \sqrt{2} \left( \sqrt{\frac{\pi t}{r}} \right)^{\frac{1}{2}} = \\
\quad = \frac{(\frac{1}{r})^\epsilon}{(2\pi)^{\frac{3}{2}+\epsilon}} \left( \sqrt{2} \right)^{\frac{1}{2}} e^{K_2(r)t} 2\sqrt{r} \left( \sqrt{\frac{\pi t}{r}} \right)^{\frac{1}{2}} = \frac{(\frac{1}{r})^\epsilon}{(2\pi)^{\frac{3}{2}+\epsilon}} e^{-t(k)2\sqrt{r} \sqrt{2}} \sqrt{2} \left( \sqrt{\frac{\pi t}{r}} \right)^{\frac{1}{2}} = (B.20)

\[
\left( \frac{1}{r} \sqrt{2} \right)^{\frac{1}{2}} e^{K_2(r)t} 2\sqrt{r} \left( \sqrt{\frac{\pi t}{r}} \right)^{\frac{1}{2}} = Upper Bound_{L_2}(t)
\]

Note that for \( \epsilon > 0 \) the upper bound (B.20) decreases.
B.4 Upperbound of Main integral on $L_3$ path

We follow [3, p. 144] “Estimation of the integral away from saddle point”, third case j=3, with the only difference of introducing $\epsilon > 0$.

$$Boundary(L_2, L_3) = (-b, -\pi i(2N + 1)) = \left( -\frac{\sqrt{\pi t}}{r}, -\pi i(2N + 1) \right)$$

Denominator is:

$$e^x - 1 = e^{\Re(x) - \pi i(2N+1)} - 1 = -e^{\Re(x)} - 1 \rightarrow | -e^{\Re(x)} - 1 | > 1$$

so module of denominator is always $> 1$. We can write integral numerator as:

$$(-x)^{\epsilon-1/2}(-x)^{\epsilon} e^{-Nx} \quad (B.21)$$

Because $\epsilon - 0.5 \leq 0$ the max of $|x|^{\epsilon-1/2}$, on $L_3$, is: $|\pi(2N+1)|^{\epsilon-1/2}$, attained when $\Re(x) = 0$.

$$|(-x)^\epsilon| = |e^{(\Re[\ln(-x)]+i\Im[\ln(-x)])\epsilon}| = e^{-t \Im[\ln(-x)]}$$

What about max reached by $-t \Im[\ln(-x)]$ on $L_3$? Namely which is the smallest phase of $-x$ on $L_3$?

Note that $\angle x = \angle[-(-x)]$ is negative and then increases, so $\angle[-x]$ is positive in $Boundary(L_2, L_3)$ and then decreases (i.e. $\Re(x) \rightarrow +\infty$ namely $\Re(-x) \rightarrow -\infty$).

Answer: the phase assumed in $Boundary(L_2, L_3)$:

$$\arctan\left( \frac{\pi(2N + 1)}{b} \right) \approx \arctan\left( \frac{\pi \frac{2\sqrt{\frac{t}{2\pi}} + 1}{\sqrt{2\pi} r}}{2\sqrt{2\pi} r} \right) > \frac{\pi}{4}$$

because $|b| = \frac{\sqrt{\pi t}}{r}$ and:

$$\frac{\pi \frac{2\sqrt{\frac{t}{2\pi}} + 1}{\sqrt{2\pi} r}}{2\sqrt{2\pi} r} = \frac{r}{\sqrt{t\pi}} \left( \sqrt{2\pi} + \pi \right) = r \left( \sqrt{2} + \sqrt{\frac{\pi}{t}} \right) > 1 \; ; \text{because} \; 1 < r \leq 2$$

So

$$|(-x)^\epsilon| < e^{-t\frac{\pi}{4}} \quad (B.22)$$

So the module of $A.2$ on $L_3$ ($t > 10$), putting $u = \Re(x)$ and taking off $F(t)$ factor, for $B.22$ is less then:

$$\left[ \left( \frac{1}{2\pi t} \right)^{\epsilon/2} e^{-t\pi/2} \right] e^{-\pi/4}[(2N + 1)\pi]^{\epsilon-1/2} \int_{-b}^{\infty} e^{-Nu} du$$

$$= \left( \frac{\sqrt{2\pi t} + \pi}{\sqrt{2\pi t}} \right)^\epsilon e^{-\frac{\pi}{4} t} \frac{e^{t/\sqrt{2\pi r}}}{\sqrt{2\pi N t} + \frac{\pi}{\sqrt{2\pi t}}} < \left( \frac{\sqrt{2\pi t} + \pi}{\sqrt{2\pi t}} \right)^\epsilon e^{-t} = UpperBound_{L_3}(t) \quad (B.23)$$
because
\[ \forall t > 10 \quad (e^{-t \pi/4} - ie^{-t \pi/4}) \approx e^{-t \pi/4} \]

And considering that
\[ \left| \frac{e^{i \pi t/4} e^{-i \theta_1(t)}}{(2\pi)^{1/4} e^{-i \pi/4}} \right| = 1 \]

then we have:
\[ \left| \frac{e^{i \pi t/4} e^{-\pi t/2} e^{-i \theta_1(t)}}{(2\pi)^{1/2} e^{-i \pi/4}} \right| = e^{-\pi t/2} \left( \frac{2\pi}{t} \right)^{\epsilon} = \left( \frac{1}{2\pi} \right)^{\epsilon} e^{-\pi t/2} \]

While, for \( B.22 \) and \( B.21 \)
\[ \int_{L_3} (-x)^{-1/2 + \epsilon} e^{-N \epsilon} \, dx \left/ e^x - 1 \right| < e^{-\pi/4} [(2N + 1) \pi]^{1/2} \int_{-b}^{\infty} e^{-N u} \, du \]
besides:
\[ N[-(-b)] \approx \sqrt{\frac{t}{2\pi \epsilon}} \frac{t}{r} = \frac{t}{\sqrt{2r}} \]

and:
\[ -3t \pi / 4 + t / (\sqrt{2} r) = t \left( \frac{3\pi}{4} + \frac{1}{\sqrt{2r}} \right) < t \]

Note that upper bound on \( L_3 \) \( B.23 \) \( e^{-t} \) increases with \( \epsilon \) but it is irrelevant in front of \( B.8 \) and \( B.20 \) see fig 3.

### B.5 Conclusions on Upperbounds

Upper bounds can be summed up and compared with minimum value of \( R_0(t) \) (i.e. when \( p = 0.5 \), see (4.5) and (4.6)).

Summarizing results we have:

\[ \text{(B.24)} \]

(i.e. \( A.2 \) along \( L_0 \) upperbound) is decreasing if \( \epsilon \) grows. \( K_0(r) \) is defined in \( B.4 \) \( K_0(r) \leq K_0(2) \approx -0.1 \quad 1 < r \leq 2 \).

\[ \int_{L_0} \frac{(x)^{t-1} e^{-N \epsilon} \, dx}{e^{x-1}} < \frac{2\sqrt{2}}{\sqrt{\sqrt{2\pi}}} e^{K_0(r) t} = \text{Upper Bound}_{L_0}(t) \quad \text{(B.25)} \]

\[ \text{B.20} \] (i.e. \( A.2 \) along \( L_2 \) upperbound) is decreasing if \( \epsilon \) grows. \( K_2(r) \) is defined in \( B.19 \) \( K_2(r) \leq K_2(2) \approx -0.2 \quad 1 < r \leq 2 \).

\[ \int_{L_2} \frac{(-x)^{t-1} e^{-N \epsilon} \, dx}{e^{x-1}} < \frac{2\sqrt{2}}{\sqrt{\sqrt{\sqrt{2\pi}}}} \left( \frac{t}{\pi} \right)^{1/4} e^{K_2(r) t} = \text{Upper Bound}_{L_2}(t) \quad \text{(B.26)} \]
Figure 3: Comparison of overall Upperbounds ratio $\frac{\Delta R}{R}$ (B.28) with parameters $r$ (B.1) and $\epsilon$. With $(r, \epsilon) = (1.05, \pm 0.5)$ we have the two lower curves, and, $\epsilon = +0.5$ is the lower one between the two. For upper curves we have $(r, \epsilon) = (2, \pm 0.5)$, and, the lower curve is always with $\epsilon = +0.5$. Of course the case $\epsilon = 0$ is midway between found bounds for $\epsilon = \pm 1/2$. Due to the conjugate symmetry of $\xi(1/2 + \epsilon + it)$ with respect to $\epsilon$, it is the lower UpperBound that applies. So errors in (4.1) , and similar formulas, is maximum for $\epsilon = 0$. The aim of error upperbounds in extended $Z(t, \epsilon)$ less or equal to the same in $Z(t)$ is so met Here, differently as in [3, p. 141], (B.1) and (B.17), in $a \pm \frac{|a|}{r} e^{\pm \pi \frac{3}{4}}$ the parameter $r$ has been left free to change in the interval $1 < r \leq 2$. This choice gives a more realistic evaluation of upperbounds error of (4.1), and similar formulas, at low $t$ values. In [3, p. 144] it is suggested the value $t \geq 100$ to get a practically error free expression. Considering the figure, it appears pessimistic, and, only with $r$ close to 2 (as chosen in [3]) is justified. The only restriction on $r$ is linked to convergence of (A.8). The convergence speed is not an issue. By the way this explains the good match noted in [3, p. 155] between $Z(t > 18, \epsilon = 0)$ and Haselgrove table [3, p. 122].
\( B.23 \) (i.e. \( A.2 \) along \( L_3 \) upperbound) increases with \( \epsilon \) but the factor \( e^{-t} \) cancel the contribution.

\[
\int_{L_3} \frac{(-x)^{s-1}e^{-N\pi dx}}{R_0(t)_{p=0.5}} < \frac{\left( \frac{\sqrt{2\pi t+\pi}}{\sqrt{2\pi t}} \right)^{\epsilon}}{e^{-t}} = UpperBound_{L_3}(t)
\]

Of course the upper bound of the sum is less than the sum of the upper bounds, so we have an overall upperbound ratio of:

\[
\left| \int_{L_0,L_2,L_3} \frac{(-x)^{s-1}e^{-N\pi dx}}{e^{\pi-1}} \right| = \frac{\Delta R}{R} < UpperBound_{L_0}(t) + UpperBound_{L_2}(t) + UpperBound_{L_3}(t)
\]

Of course the case \( \epsilon = 0 \) is between \( \epsilon = \pm 1/2 \). Here, in \( (B.1) \) and in \( (B.17) \), differently that in \([3, p. 141]\) the parameter \( r \) has been left free to change in the interval \( 1 < r \leq 2 \). This choice gives a more realistic evaluation of upper-bounds error of \( (4.1) \) and similar formulas at low \( t \) values. In \([3, p. 144]\) it is suggested the value \( t \geq 100 \) to get a practically error free expression.

In light of above consideration, this appears pessimistic because only with \( r \) close to 2 is justified. The only restriction on \( r \) is linked to convergence of \([A.8]\). The convergence speed is not an issue. By the way, this explains the good match noted in \([3, p. 155]\) between \( Z(t > 18, \epsilon = 0) \) and Haselgrove table \([3, p. 122]\). The spurious contribution \((B.9)\) is an insignificant fraction of \( R_0(t) \) at least for \( t > 20 \).

Besides the bound with \( \epsilon > 0 \) (and so also for \( \epsilon < 0 \)) is lower than with \( \epsilon = 0 \).

We can affirm that \((4.8)\) can be used profitably from \( t = 20 \) with \( \epsilon \) in critical strip and beyond.

Acknowledgments

I thank professor Richard B. Paris for his kind support and for useful discussions. I also thank Paolo Lodone for useful discussions and for contributing in some points of this work.

Bibliography

References

[1] Bombieri E. The Riemann Hypothesis”, Official Problem Description, Clay foundation, www.claymath.org

[2] J. Brian Conrey “The Riemann Hypothesis” 2003 notices of the AMS

[3] Edwards H.M. “Riemann Zeta Function” Academic Press 1974

[4] Gourdon Xavier “ The 10^{13} first zeros of the Riemann Zeta function, and zeros computation at very large height “ October 24th 2004

[5] Ivi´c Aleksandar arXiv:math/0311162v1 [math.NT] 11 Nov 2003
[6] Murray R, Spiegel Manuale di Matematica ETAS Libri 1974

[7] Pugh Glendon Ralph “ The Riemann -Siegel formula and large scale computation of the Riemann zeta function ” B.Sc., University of New Brunswick, 1992

[8] Pugh Glendon Ralph “ An analysis of the Lanczos gamma approximation ”1999

[9] Riemann Bernhard ,“Uber die Anzahl der Primzahlen unter einer gegebenen Grosse”,Monatsberichte der Berliner Akademie 1859. (an English translation is included in [Reference 3] pag. 299] and available on www.claymath.org)