On the Convergence of Alternating Direction Lagrangian Methods for Nonconvex Structured Optimization Problems

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Abstract—This paper investigates convergence properties of scalable algorithms for nonconvex and structured optimization. We focus on two methods that combine the fast convergence properties of augmented Lagrangian-based methods with the separability properties of alternating optimization. The first method is adapted from the classic quadratic penalty function method and is called the Alternating Direction Penalty Method (ADPM). Unlike the original quadratic penalty function method, in which single-step optimizations are adopted, ADPM uses alternating optimization, which in turn is exploited to enable scalability of the algorithm. The second method is the well-known Alternating Direction Method of Multipliers (ADMM). We show that the ADPM asymptotically converges to a primal feasible point under mild conditions. Moreover, we give numerical evidence to demonstrate the potentials of the ADPM for computing a good objective value. In the case of the ADMM, we give sufficient conditions under which the algorithm asymptotically reaches the standard first order necessary conditions for local optimality. Throughout the paper, we substantiate the theory with numerical examples and finally demonstrate possible applications of ADPM and ADMM to a nonconvex localization problem in wireless sensor networks.

Index Terms—Nonconvex Optimization, ADMM, Localization

I. INTRODUCTION

T HE last few decades’ increasingly rapid technological developments have resulted in vast amounts of dispersed data. Optimization techniques have played a central role in transforming the vast data sets into usable information. However, due to the increasing size of the related optimization problems, it is essential that these optimization techniques scale with data size. Fortunately, many large scale optimization problems in real world applications possess appealing structural properties. Thus, increasing research efforts have been devoted to the investigation of how these structural properties can be exploited in the algorithm design to achieve scalability. The focal point of these efforts has been on “well-behaved” convex problems, rather than more challenging nonconvex problems. Nevertheless, large scale nonconvex problems arise in many real world applications. Examples include, matrix factorization techniques for recommender systems (the Netflix challenge) [1], localization in wireless sensor networks [2], and optimal power flow in smart grids [3], [4], among others. Interestingly, these large scale nonconvex applications tend to have the structural advantages that are commonly exploited to design scalable algorithms for their convex counterparts. This suggests that the algorithms used for large scale convex problems can potentially be applied to nonconvex problems as well. However, theoretical guarantees for these algorithms in the nonconvex regime have not yet been established. This paper investigates theoretical convergence properties of class of scalable algorithms when applied to nonconvex problems.

A. Related Literature

Many recent studies on large scale optimization have focused on distributed subgradient methods in the context of multi-agent networks [5]–[11]. There, multiple agents, each with a private objective function, cooperatively minimize the aggregate objective function by communicating over the network. In contrast to [5]–[9], the papers [10] and [11] consider nonconvex multi-agent problems. Specifically, [10] employs distributed subgradient methods to the (convex) dual problem and investigates sufficient conditions under which the approach converges to a pair of optimal primal/dual variables. On the other hand, [11] studies the convergence of stochastic subgradient methods to a point satisfying the first order necessary conditions for local optimality with probability one. A main drawback of these gradient based approaches is their poor convergence rate [12].

Another widely used approach for structured convex optimization is the Alternating Direction Method of Multipliers (ADMM) [12]–[14]. ADMM is a variant of the classical method of multipliers (MM) [15] Chapter 2, [16] Chapter 4.2, where the primal variable update of the MM is split into subproblems, whenever the objective is separable. This structure is common to large scale optimization problems that arise in practice [12]. Even problems that do not possess such a structure can often be posed equivalently in a form appropriate for ADMM by introducing auxiliary variables and linear constraints. These techniques have been employed in many recent works when designing distributed algorithms for convex, as well as nonconvex problems [12], [17]–[26]. A key property of ADMM compared with other existing scalable approaches, such as subgradient and dual decent methods (mentioned above) is its superior convergence behavior, see [12], [18] for empirical results. Characterizing the exact convergence rate of
ADMM is still an ongoing research topic \cite{21, 27}. Many recent papers have also numerically demonstrated the fast and appealing convergence behavior of ADMM even on nonconvex problems \cite{22–25}. Despite these encouraging observations, there are still no theoretical guarantees for ADMM’s convergence in the nonconvex regime. Therefore, investigating convergence properties of the ADMM and related algorithms in nonconvex settings is of great importance in theory as well as in practice, and is motivated by the many emerging large scale nonconvex applications.

B. Contribution of the Paper

The key contribution of the paper is to analyse the convergence properties of two scalable algorithms for nonconvex and structured optimization problems. To this end, we consider a mathematical optimization problem with 1) a separable objective function, 2) separable constraint functions, and 3) coupled affine constraint functions, where the related functions can possibly be nonconvex. In connection with the considered problem, we study two methods that combine the fast convergence properties of augmented Lagrangian-based methods with the separability properties of alternating optimization.

The first method, the Alternating Direction Penalty Method (ADPM), is a variant of the classic quadratic penalty function method \cite{16} Chapter 4.2.1. As opposed to the single-step minimizations used in the original quadratic penalty function method, in the case of the ADPM, we adopt alternating minimizations over the subsets of variables, which in turn enables scalability of the ADPM. We show under mild conditions that the ADPM converges to primal feasible point. Numerical evidence is provided to demonstrate the potency of ADPM for computing a good objective value.

The second method is the classic Alternating Direction Method of Multipliers (ADMM) \cite{12, 18}. We give sufficient conditions under which the ADMM yields limit points which satisfy the standard first order necessary conditions for local optimality of the considered nonconvex problem. Numerical experiments are given to validate the theoretical developments.

Finally, we illustrate how the considered methods can be applied to design distributed algorithms for cooperative localization in wireless sensor networks.

C. Notation and Definitions

Vectors and matrices are represented by boldface lower and upper case letters, respectively. The set of real and natural numbers (excluding zero) are denoted by \(\mathbb{R}\) and \(\mathbb{N}\), respectively. The set of real \(n\) vectors and \(n \times m\) matrices are denoted by \(\mathbb{R}^n\) and \(\mathbb{R}^{n \times m}\), respectively. The \(i\)th component of the vector \(x\) is denoted by \(x_i\). The superscript \((\cdot)\text{T}\) stands for transpose. All vectors \(x\) are viewed as column vectors and we write row vectors as the transpose of column vectors, i.e., \(x^\text{T}\). We use parentheses to construct vectors and matrices from comma separated lists as \((x_1, \cdots, x_n) = [x_1^\text{T}, \cdots, x_n^\text{T}]^\text{T}\) and \((A_1, \cdots, A_n) = [A_1^\text{T}, \cdots, A_n^\text{T}]^\text{T}\), respectively. We denote the diagonal block matrix with \(A_1, \cdots, A_n\) on the diagonal by \(\text{diag}(A_1, \cdots, A_n)\). For a square matrix \(A\), we let \(A \succeq 0\) and \(A \succ 0\) indicate that \(A\) is positive definite and positive semidefinite, respectively.

The following definition will be useful.

**Definition 1.** Consider the following mathematical optimization problem:

\[
\begin{align*}
\text{minimize} & \quad f(x) \\
\text{subject to} & \quad \phi(x) = 0, \quad \psi(x) \leq 0
\end{align*}
\]

where \(\phi : \mathbb{R}^p \to \mathbb{R}^q\) and \(\psi : \mathbb{R}^p \to \mathbb{R}^q\) are continuously differentiable functions. We say that the variables \(x^\ast \in \mathbb{R}^p\) and \((\lambda^\ast, \mu^\ast) \in \mathbb{R}^{n_1 + n_2}\) satisfy the first order necessary conditions for problem (1), if following holds 1) Primal feasibility: \(\phi(x^\ast) = 0\) and \(\psi(x^\ast) \leq 0\). 2) Dual feasibility: \(\mu^\ast \geq 0\). 3) Complementary slackness: \((\mu^\ast), \psi(x^\ast) = 0\), \(i = 1, \cdots, q_2\). 4) Lagrangian vanishes: \(\nabla f(x^\ast) = \nabla \phi(x^\ast) \lambda^\ast + \nabla \psi(x^\ast) \mu^\ast\). We refer to \(x^\ast\) and \((\lambda^\ast, \mu^\ast)\) as the primal and dual variables, respectively.

We will sometimes say that a primal variable \(x^\ast\) satisfies the first order necessary conditions of Problem (1) without specifying the dual variables \((\lambda^\ast, \mu^\ast)\).

II. PROBLEM STATEMENT AND RELATED BACKGROUND

We consider the following optimization problem

\[
\begin{align*}
\text{minimize} & \quad f(x) + g(z) \\
\text{subject to} & \quad (x, z) \in X \times Z
\end{align*}
\]

where \(A \in \mathbb{R}^{q_1 \times p_1}\), \(B \in \mathbb{R}^{q_2 \times p_2}\), and \(c \in \mathbb{R}^q\). The use of the variable notation \(x\) and \(z\) is consistent with the literature \cite{12}. The functions \(f : X \to \mathbb{R}\) and \(g : Z \to \mathbb{R}\) are continuously differentiable on \(\mathbb{R}^{p_1}\) and \(\mathbb{R}^{p_2}\), respectively, and may be nonconvex. Moreover, the sets \(X \subseteq \mathbb{R}^{p_1}\) and \(Z \subseteq \mathbb{R}^{p_2}\) are closed and can be specified by a finite number of equality and inequality constraints of continuously differentiable functions. We refer to the affine constraint \(Ax + Bz = c\) as the coupling constraint. We assume that Problem (2) is feasible. Note that problem (2) is general in the sense that many interesting large scale problems, including consensus, and sharing \cite{12} Section 7], among others can be equivalently posed in its form. Therefore, our analytical results in subsequent sections apply to a broad class of problems of practical importance.

In the sequel, we discuss the key related research methodologies for addressing Problem (2).

A. Penalty and Augmented Lagrangian Methods

Nonconvex problems of the form (2) can be gracefully handled by penalty and augmented Lagrangian methods, such as the quadratic penalty function method and method of multipliers, \cite{15} Chapter 2 \cite{16} Chapter 4.2. The main ingredient of these methods is the augmented Lagrangian, given by

\[
L_p(x, z, y) = f(x) + g(z) + y^\text{T}(Ax + Bz - c) + (\rho/2)\|Ax + Bz - c\|^2.
\]
Here $x$ and $z$ are the primal variables of Problem (2) and $y \in \mathbb{R}^n$ and $\rho \in \mathbb{R}$ are referred to as the multiplier vector and the penalty parameter, respectively.

The penalty and augmented Lagrangians methods consist of iteratively updating the variables $x$, $z$, $y$, and $\rho$. An update common to all the methods is the primal variable update, i.e.,

$$(x(t+1), z(t+1)) = \arg\min_{(x,z) \in X \times Z} L_{\rho(t)}(x, z, y(t)),$$  \hspace{1cm} (3)

where $t \in \mathbb{N}$ is the iteration index. The main difference between the two methods lies in the $y$ and $\rho$ updates. For example, in the case of the quadratic penalty method, the penalty parameter $\rho(t)$ is chosen to be nondecreasing and diverging, i.e., $\rho(t+1) \geq \rho(t)$ for all $t \in \mathbb{N}$ and $\lim_{t \to \infty} \rho(t) = \infty$. Moreover, the Lagrangian multipliers are chosen to be bounded, i.e., there exists $M \in \mathbb{R}$ such that $||y(t)||_2 < M$ for all $t \in \mathbb{N}$. With this setting, every limit point of the sequence $\{(x(t), z(t))\}_{t \in \mathbb{N}}$ is a global minimum of problem (2) \cite{15} Proposition 2.1.

Unlike the quadratic penalty method, in which $\{\rho(t)\}_{t \in \mathbb{N}}$ is increasing and divergent, in the original method of multipliers \cite{15} Chapter 2, the sequence $\{\rho(t)\}_{t \in \mathbb{N}}$ is chosen to be nondecreasing but possibly convergent. In addition, the multiplier vectors are updated according to the recursion: $y(t+1) = y(t) + \rho(x(t+1) + z(t+1))$. Under mild conditions, the method of multipliers converges to a local optimal point $(x^*, z^*)$ and to a corresponding optimal Lagrangian multiplier $y^*$ \cite{15} Proposition 2.4. When $f$ and $g$ are not convex, then the primal update step (3) need not be solved to global optimality; rather convergence of the pair $(x(t), z(t))$ to a local optimal solution of (2) is guaranteed when the updates are taken to be locally optimal solutions of the subproblems (3).

In general, the penalty and augmented Lagrangian methods mentioned above are very reliable and effective for handling problems of the form (2). However, these methods entail centralized solvers, especially in the $(x,z)$-update (3), even if the objective function of problem (2) has a desirable separable structure in $x$ and $z$. More specifically, these methods do not allow the possibility of performing the $(x,z)$-update in two steps: first $x$-update and then $z$-update. Otherwise, the assertions on the convergence of the algorithms do not hold anymore. Therefore, the penalty and augmented Lagrangian methods are not generally applied in distributed settings, whenever the problems possess decomposition structures. Such restrictions have motivated an adaptation of the classical penalty and augmented Lagrangian methods that has excellent potential for a parallel/distributed implementation.

Let us refer to these approaches as Alternating Direction Lagrangian Methods (ADLM).

Convergence results for ADLM variants under strong assumptions on Problem (2), such as convexity, have been heavily investigated, e.g., Alternating Direction Method of Multipliers (ADMM) \cite{12}. However, convergence results for ADLM variants when Problem (2) is nonconvex have not been explored. To this end, we consider two variants of ADLM. The first variant is analogous to the quadratic penalty approach, where the sequence of penalty parameters $\{\rho(t)\}_{t \in \mathbb{N}}$ and the multiplier vectors $\{y(t)\}_{t \in \mathbb{N}}$ are taken to be nondecreasing/divergent and bounded, respectively. We refer to this approach as the Alternating Direction Penalty Method (ADPM). The second variant is the classic ADMM itself, the analog of the method of multipliers. We now pose the question: can the convergence of the considered ADLM variants, ADPM and ADMM, still be guaranteed when Problem (2) is nonconvex?

In general, the convergence results of the classical penalty and augmented Lagrangian methods do not directly apply in the case of the considered ADLM variants. This is because the ADLM uses the two step optimization (4) and (5), instead of the one step (3). However, we still find that the ADLM methods possess desirable convergence properties, as investigated in next two sections.

### III. ALTERNATING DIRECTION PENALTY METHOD

In this section we study convergence properties of the ADPM for addressing Problem (2). In Section III-A we give an explicit algorithm description, in Section III-B we discuss properties of the algorithm, and in Section III-C we provide numerical examples.

#### A. Algorithm Description

The steps of ADPM are shown in Algorithm 1.

Algorithm 1: The Alternating Direction Penalty Method (ADPM)

1) Initialization: Set $t = 0$ and initialize $z(0), y(0)$, and $\rho(0)$.
2) $x$-update: $x(t+1) = \arg\min_{x \in X} L_{\rho(t)}(x, z(t), y(t))$.
3) $z$-update: $z(t+1) = \arg\min_{z \in Z} L_{\rho(t)}(x(t+1), z, y(t))$.
4) $\rho/y$-update: Update $\rho(t+1)$ and $y(t+1)$.
5) Stopping criterion: If stopping criterion is met terminate, otherwise set $t = t + 1$ and go to step 2.

The algorithm parameters $\rho(t)$ and $y(t)$ are chosen such that $\rho(t+1) \geq \rho(t)$, $\lim_{t \to \infty} \rho(t) = \infty$, and the sequence $\{y(t)\}_{t \in \mathbb{N}}$ is taken to be bounded. In the initialization (step 1), it is beneficial to consider all prior information when setting $z(t), y(t)$, and $\rho(t)$. The $x$- and $z$- updates (steps 2 and 3) are the main steps of the algorithm where the augmented Lagrangian is minimized in two steps.

Nonconvexities of $f$ and $g$ suggest potential difficulties in the implementation of the $x$- and $z$- updates (see steps 2 and 3). However, it is worth noting that despite the nonconvexities, the special structures of the problems considered can often be
exploited to successfully implement the x- and z-updates. Several examples are given below.

**Example 1.** Let $\mathcal{X}$ (or $\mathcal{Z}$) be compact and convex and let $f$ (or $g$) be twice continuously differentiable. Moreover, suppose $A$ (or $B$) has full column rank. Then the optimization problem in the x-update (or y-update) becomes strongly convex for sufficiently large $\rho(t)$. This can be seen from the Hessian $\nabla^2 f(x, z(t), y(t)) = \nabla^2 f(x) + \rho(t) A^T A$. Because $\nabla^2 f(x)$ is continuous and $\mathcal{X}$ is compact, there exists a scalar $\alpha$ such that $\nabla^2 f(x) \succ \alpha I$. Moreover, since $A$ has full column rank, $A^T A$ is positive definite and it follows that $\nabla^2 L_{\rho(t)}(x, z(t), y(t)) \succ (\alpha + \rho(t) \lambda_{\min}(A^T A)) I$. An illustrative example where the augmented Lagrangian becomes strongly convex in $x$ and $z$ coordinate is given Fig. 1.

**Example 2.** Let $f(x) = x^T Q x + q^T x$ where $Q \in \mathbb{R}^{p_1 \times p_1}$ is a symmetric indefinite matrix. Then if $x^T Q x > 0$ for all $x \in \mathbb{R}^{p_1} \setminus \{0\}$ in the null space of $A$, then there exists $\bar{\rho} \in \mathbb{R}$ such that $L_{\rho(t)}(\cdot, z(t), y(t))$ is convex in $x$ for all $\rho(t) \geq \bar{\rho}$, see [16] Lemma 3.2.1 and Figure 3.2.1.

**Example 3.** A potential feature of the multi-agent setting is that the $x$-update is separable into low dimensional problems. More specifically, suppose the variable $x$ is partitioned into low dimensional subvectors as $x = (x_1, \cdots, x_N)$, where there is no coupling between $x_i$ and $x_j$ in the constraints, for all $i, j = 1, \cdots, N$ such that $i \neq j$. Suppose also that the objective function is separable with respect to the partition, i.e., $f(x) = \sum_{i=1}^{N} f_i(x_i)$. Then the objective function in the $x$-update is also separable with respect to the partition. Thus, provided that each subvector $x_i$ is of low dimension, global methods such as branch and bound can be efficiently used to optimally solve the optimization problem in the $x$-update.

### B. Algorithm properties

In this section, we derive the convergence properties of ADPM algorithm. Our convergence results assert that the primal feasibility of problem (2), which is a necessary optimality condition, is achieved as ADPM proceeds. More specifically, we show that regardless of whether $f, g$ are convex or nonconvex, whenever $\mathcal{X}$ and $\mathcal{Z}$ are convex, the primal residual at iteration $t$ of the ADPM (i.e., $Ax(t) + Bz(t) - c$) converges to zero as ADPM proceeds. To establish this result precisely, let us first make the following assumptions:

**Assumption 1.** The functions $f$ and $g$ of problem (2) are continuously differentiable.

**Assumption 2.** The sets $\mathcal{X}$ and $\mathcal{Z}$ of problem (2) are convex and compact.

**Assumption 3.** Slater’s condition [16] holds individually for $\mathcal{X}$ and $\mathcal{Z}$. In particular, there exists a $x \in \mathcal{X}$ (respectively, $z \in \mathcal{Z}$) such that all the inequality constraints characterizing $\mathcal{X}$ (respectively, $\mathcal{Z}$) are inactive at $x$ (respectively, $z$).

**Assumption 4.** The matrices $A$ and $B$ of problem (2) have full column rank.

Note that we make no convexity assumptions on $f$ and $g$. However, the convexity assumption on $\mathcal{X}$ and $\mathcal{Z}$ is essential. Otherwise, primal feasibility is not guaranteed in general as we show by Example 4 later in this section. Assumption 3 is an additional technical condition, similar to the constraint qualifications usually used in convex analysis. The last assumption is technically necessary to ensure that both $A^T A$ and $B^T B$ are positive definite. It is quite common in practice that this assumption holds, as desired, see Section V. The following proposition establishes the convergence of ADPM:

**Proposition 1.** Suppose assumptions [4] hold. Let $\{\rho(t)\}_{t \in \mathbb{N}}$ be a sequence of penalty parameters used in the ADPM algorithm, where $\rho(t + 1) \geq \rho(t)$ for all $t$ and suppose there exists an integer $\kappa > 0$ and a scalar $\Delta > 0$ such that $\rho(t + \kappa) \geq \Delta \rho(t)$ for all $t$. Let $r(t)$ be the residual at iteration $t$ of the ADPM defined as $r(t) = \|Ax(t) + Bz(t) - c\|^2$. Then

$$\lim_{t \to \infty} r(t) = 0.$$  

(6)
Proof: Recall that \( \{ y(t) \}_{t \in \mathbb{N}} \) is a bounded sequence. Thus, there exists \( M_0 > 0 \) such that \( ||y(t)||_2 \leq M_0 \), for all \( t \in \mathbb{N} \). We denote by \( \mathcal{Y} \) the closed ball with radius \( M_0 \) centered at the origin 0, i.e., \( \mathcal{Y} = \{ y \in \mathbb{R}^n | ||y||_2 \leq M_0 \} \).

Since \( f \) and \( g \) are continuous and the sets \( \mathcal{X} \) and \( \mathcal{Z} \) are compact there exists a scalar \( M_1 > 0 \) such that
\[
M_1 = \max_{(x,z,y) \in \mathcal{X} \times \mathcal{Z} \times \mathcal{Y}} |f(x)+g(z)+y^T(Ax+Bz-c)|.
\]
(7)

In addition, \( \hat{x} : \mathbb{R}^{p_2} \rightarrow \mathbb{R} \) and \( \hat{z} : \mathbb{R}^{p_1} \rightarrow \mathbb{R} \), defined as
\[
\hat{x}(z) = \arg\min_{x \in \mathcal{X}} ||Ax + Bz - c||_2^2,
\]
\[
\hat{z}(x) = \arg\min_{z \in \mathcal{Z}} ||Ax + Bz - c||_2^2,
\]
(8) (9)
are well-defined continuous functions [compare with Assumption 4]. By definition, \( x(t+1) \) is a solution of the optimization problem in x-update of the ADPM. This, together with \( \hat{a}(l+1) \) yields
\[
L_{\rho(t)}(x(t+1), z(t), y(t)) \leq M_1 + (\rho(t)/2)||Ax(t+1) + Bz(t) - c||_2^2.
\]
(10)

Similarly, we get
\[
L_{\rho(t)}(x(t+1), z(t+1), y(t)) \leq M_1 + (\rho(t)/2)||Ax(t+1) + Bz(t+1) - c||_2^2.
\]
(11)

Let us first use \( \hat{a}(t+1) \) and \( \hat{a}(t) \) to derive a recursive relation for \( r(t) \). By rearranging the terms of (10) and by using that \( ||M_1 - (f(x) + g(z) + y^T(Ax+Bz-c))|| \leq 2M_1 \) for all \( (x,z,y) \in \mathcal{X} \times \mathcal{Z} \times \mathcal{Y} \), we have for all \( t \in \mathbb{N} \),
\[
||Ax(t+1) + Bz(t) - c||_2^2 \leq \frac{4M_1}{\rho(t)} + ||Ax(t) + Bz(t) - c||_2^2.
\]
(12)

Moreover, we have for all \( t \in \mathbb{N} \),
\[
r(t+1) \leq \frac{4M_1}{\rho(t)} + ||Ax(t+1) + Bz(t+1) - c||_2^2 \leq \frac{8M_1}{\rho(t)} + ||Ax(t) + Bz(t) - c||_2^2
\]
(13)
\[
\leq \frac{8M_1}{\rho(t)} + r(t),
\]
(14)
where (13) follows similarly by rearranging the terms of (11) and by using that \( ||M_1 - (f(x) + g(z) + y^T(Ax+Bz-c))|| \leq 2M_1 \) for all \( (x,z,y) \in \mathcal{X} \times \mathcal{Z} \times \mathcal{Y} \). (14) follows from combining the inequalities (12) and (13), together with the definition of \( \hat{x} \) and \( \hat{z} \), and (15) follows by the definition of \( \hat{x} \).

Let us next use the recursive inequality (15) above to show that \( \{ r(t) \}_{t \in \mathbb{N}} \) converges to a finite value. The inequality (15) implies for all \( t, n \geq 0 \),
\[
r(t+n) \leq 8M_1 \sum_{i=0}^{n-1} \frac{1}{\rho(t+i)} + r(t).
\]
(16)

From the definition of \( \{ \rho(t) \}_{t \in \mathbb{N}} \), we get
\[
\sum_{i=0}^{n} \frac{1}{\rho(t+i)} \leq \sum_{i=0}^{n} \frac{1}{\rho(t+i)} \leq \sum_{i=0}^{n} \frac{\kappa}{\Delta t \rho(t)} \leq \frac{\kappa}{\rho(t)} \sum_{i=0}^{\infty} \frac{1}{\Delta t i}
\]
(17) \( \leq \frac{\kappa}{\rho(t)} \sum_{i=0}^{\infty} \frac{1}{\Delta t i}
\]
(18) \( \leq \frac{\kappa}{\rho(t)} \sum_{i=0}^{\infty} \frac{1}{\Delta t i}
\]
(19)
where (17) follows because the sum on the right contains all the terms of the sum on the left (and possibly more) and all the terms are positive, (18) follows because \( 1/\rho(t+i+j) \leq 1/(\Delta t \rho(t)) \) for all \( 0 \leq j \leq \kappa - 1 \), and (19) trivially follows from the nonnegativity of summands. Since \( \Delta > 1 \), \( \sum_{i=0}^{\infty} 1/\Delta t i \) is a convergent geometric series, and thus let \( \sum_{i=0}^{\infty} \kappa/\Delta t = M_2 \). This, together with (16)-(19) implies that for all integers \( t, n \geq 0 \),
\[
r(t+n) \leq 8M_1M_2/\rho(t) + r(t).
\]
(20)

Now note that \( \{ r(t) \}_{t \in \mathbb{N}} \) is bounded. Moreover, because \( \{ \rho(t) \}_{t \in \mathbb{N}} \) is an increasing sequence, it follows that for all \( \epsilon > 0 \), there exists a \( T \) such that \( (8M_1M_2/\rho(t)) \leq \epsilon \). These, taken together with (20) and Lemma 1 (see p. 9), ensure that the sequence \( \{ r(t) \}_{t \in \mathbb{N}} \) converges to a finite value, denoted by \( R \), i.e., \( R = \lim_{t \to \infty} r(t) \).

Let us finally show that \( R = 0 \). Since the set \( \mathcal{X} \times \mathcal{Z} \) is compact, the sequence \( \{ x(t), z(t) \}_{t \in \mathbb{N}} \) has a limit point, say \( (\hat{x}, \hat{z}) \in \mathcal{X} \times \mathcal{Z} \). Moreover, note that the function \( ||Ax + Bz - c||_2^2 \) is continuous on \( \mathcal{X} \times \mathcal{Z} \). Therefore, taking limits as \( t \to \infty \) in \( r(t) = ||Ax(t) + Bz(t) - c||_2^2 \), we have
\[
R = \lim_{t \to \infty} ||Ax(t) + Bz(t) - c||_2^2 = ||A\hat{x} + B\hat{z} - c||_2^2.
\]
(21)

Let us now consider the limits in the inequality (15) as \( t \to \infty \). Since \( \lim_{t \to \infty} r(t+1) = \lim_{t \to \infty} ((8M_1M_2)/\rho(t) + r(t)) = R \), from (13), (14), and the squeezing lemma, together with the continuity of functions \( \hat{x} \) and \( \hat{z} \) it follows that
\[
R = ||A\hat{x} + B\hat{z} - c||_2^2 = ||A\hat{x} + B\hat{z} - c||_2^2.
\]
(22)

By combining (21) and (22), together with the definitions (8) and (9), we get
\[
\hat{x} = \arg\min_{x \in \mathcal{X}} ||Ax + Bz - c||_2^2,
\]
(23)
\[
\hat{z} = \arg\min_{z \in \mathcal{Z}} ||Ax + Bz - c||_2^2.
\]
(24)

Since Slater’s constraint qualifications condition is satisfied for both sets \( \mathcal{X} \) and \( \mathcal{Z} \) (Assumption 3), \( \hat{x} \) and \( \hat{z} \) satisfy the first order necessary conditions for problems (23) and (24), respectively. By combining these first order necessary conditions and (24), it follows that \( (\hat{x}, \hat{z}) \) satisfies the first order necessary conditions for the problem
\[
\begin{align*}
\text{minimize} & \quad ||Ax + Bz - c||_2^2 \\
\text{subject to} & \quad (x, z) \in \mathcal{X} \times \mathcal{Z}.
\end{align*}
\]
(25)
Since problem (25) is convex and the constraints sets satisfy Slater’s constraint qualifications condition, we conclude that
Note that our Assumption [1] is a weaker condition than assuming that $f$ and/or $g$ are convex. As a result, characterizing generally the properties of the objective value of ADPM after the convergence is technically challenging. Nevertheless, ADPM appears to resemble a sequential optimization approach, which provides degrees of freedom to hover over the true objective function for locating a good objective value. In the sequel, we give some experiments to numerically show these appealing aspects of the ADPM, besides those ensured by Proposition [1].

C. Numerical evaluations

Let us start by examining two small dimensional examples, where the convergence of ADPM can be visualized, see Fig. 3 (a) and (b).

Fig 3(a) corresponds to a problem of the form (2), with $f(x) = x^2$, $g(z) = -(z - 2)^2$, $\mathcal{X} \times \mathcal{Z} = [-1, 3] \times [-1, 3]$, and with the coupling constraint $x - z = 0$. Note that this problem is nonconvex. However, the associated optimization problems at $x$- and $z$- updates of ADPM become strongly convex if $\rho(0)>2$ (compare with Example [1]). The nonlinear contours of $f(x) + g(z)$ and the points associated with the coupling constraint $x - z = 0$ are plotted for clarity.

Fig 3(b) depicts the sequences of $\{x(t), z(t)\}_{t \in \mathbb{N}}$ generated by ADPM for different values of $\Delta$ and $\kappa$, with the initialization $z(0) = 3$, $g(t) = 0$ for all $t \in \mathbb{N}$, and $\rho(0) = 1.1$. Note that $\Delta$ is a measure of how aggressively the penalty parameter $\rho$ is increased. Moreover, $\kappa$ is a measure of how frequently the penalty parameter $\rho$ is increased by the factor $\Delta$. For example, $(\Delta, \kappa) = (1.5, 1)$ corresponds to a case where $\rho$ is increased less aggressively (i.e., by a factor 1.5) at the fastest rate (i.e., at every iteration) and $(\Delta, \kappa) = (5, 3)$ corresponds to a case with more aggressively and slow rate increase of $\rho$. On the other hand, $(\Delta, \kappa) = (2, 1)$ corresponds to a case where $\rho$ is increased aggressively, as well as at a fastest rate. We refer to such cases as restrictive settings.

As established by the Proposition [1], numerical results show that for all the considered cases the corresponding sequences of $\{x(t), z(t)\}_{t \in \mathbb{N}}$ achieve primal feasibility of the problem. Results further show that the ADPM with the two cases $(\Delta, \kappa) = (1.5, 1)$ and $(\Delta, \kappa) = (5, 3)$, achieves global optimal solution of the nonconvex problem. Having similar results in these two cases is consistent with our intuition that increasing the penalty parameter $\rho$ less aggressively at a fast rate [i.e., $(\Delta, \kappa) = (1.5, 1)$] resembles an increasing $\rho$ more aggressive with a slower rate [i.e., $(\Delta, \kappa) = (5, 3)$]. The choice $(\Delta, \kappa) = (2, 1)$ yields neither a global nor local optimal point. This result is intuitively expected, because restrictive settings limit the algorithm’s degrees of freedom to hover over the objective function for locating a better point. Nevertheless, even with such restrictive settings [e.g., $(\Delta, \kappa) = (2, 1)$], the algorithm manages to drive the latter points of the sequence $\{x(t), z(t)\}_{t \in \mathbb{N}}$ along a decreasing trajectory until the primal feasibility is achieved. Thus, although there are no analytical guarantees, numerical results suggest that, while the primal variables converge to a feasible solution (see, Proposition [1]), ADPM can also produce solutions with near optimal objective value. Moreover, our extensive numerical experience with

| Example 2. Consider the problem |
|---------------------------------|
| $\min_{x,z} \quad x^2 + z^2$ |
| subject to $\quad -2x + z = 0.1$, $x \in [-1,0] \cup [1,2]$, $z \in [0,3]$ |
| (26) |

The feasibility set and contours of the objective function are given in Fig. 2. It can be observed that if $z(0) = 0$ and $y(t) = 0$ for all $t \in \mathbb{N}$, then the optimal solution of both the $x$- and $z$- updates is 0 for all $t \in \mathbb{N}$, i.e., $lim_{t \rightarrow \infty} x(t) = 0$ and $lim_{t \rightarrow \infty} z(t) = 0$. This means that the algorithm converges to $(0,0)$, which is an infeasible point.
IV. ALTERNATING DIRECTION METHOD OF MULTIPLIERS

In this section we investigate some new general properties of the ADMM in a nonconvex setting. In particular, we give an algorithm description in section IV-A, study analytic properties in section IV-B, and give numerical evaluations in section IV-C.

A. Algorithm Description

The ADMM can explicitly be stated as follows.

Algorithm 2: THE ALTERNATING DIRECTION METHOD OF MULTIPLIERS (ADMM)

1) Initialization: Set $t = 0$ and put initial values to $z(t)$, $y(t)$, and $\rho$.
2) $x$-update: $x(t+1) = \arg\min_{x \in \mathcal{X}} L_\rho(x, z(t), y(t))$.
3) $z$-update: $z(t+1) = \arg\min_{z \in \mathcal{Z}} L_\rho(x(t+1), z, y(t))$.
4) $y$-update: $y(t+1) = y(t) + \rho(Ax(t+1) + Bz(t+1) - c)$.
5) Stopping criterion: If stopping criterion is met terminate, otherwise set $t = t + 1$ and go to step 2.

The first step is the initialization (step 1). If there is any prior knowledge it is beneficial to exploit it during the initialization. After the initialization, the algorithm continues by iteratively performing the $x$-, $z$-, and $y$- updates, until a stopping criterion is fulfilled. As presented above, the $x$- and $z$- updates require a solution of an optimization problem. This is not as restrictive as it may seem, since under mild conditions such requirements are accomplished, see Examples 12. However, we note that no such global optimality requirement of $x(t+1)$ and $z(t+1)$ is necessary in our convergence assertions, as we will show in subsequent sections. More specifically, our convergence results apply as long as $x(t+1)$ [respectively, $z(t+1)$] is a local minimum.

B. Algorithm Properties

In this section, we show that, under mild assumptions, if the sequence $\{y(t)\} \in \mathbb{N}$ converges to some $\bar{y}$, then any limit point of the sequence $\{x(t), z(t)\} \in \mathbb{N}$, together with $\bar{y}$, satisfy first order necessary optimality conditions for problem (2) (compare with Definition 11). It is worth noting that these results hold regardless of whether $f$, $g$, $\mathcal{X}$, and $\mathcal{Z}$ are convex or nonconvex.

Let us now scrutinize the above assertion precisely. The analysis is based on the following assumption which can be
expected to hold for many problems of practical interest:

**Assumption 5.** The sets $\mathcal{X}$ and $\mathcal{Z}$ of problem \(2\) are compact and can be expressed in terms of a finite number of equality and inequality constraints. In particular,

\[
\mathcal{X} = \{ x \in \mathbb{R}^p_+ \mid \psi(x) = 0, \phi(x) \leq 0 \},
\]

\[
\mathcal{Z} = \{ z \in \mathbb{R}^p_+ \mid \theta(z) = 0, \sigma(z) \leq 0 \},
\]

where $\psi: \mathbb{R}^p_+ \to \mathbb{R}^q_+$, $\phi: \mathbb{R}^p_+ \to \mathbb{R}^q_+$, $\theta: \mathbb{R}^p_+ \to \mathbb{R}^q$, and $\sigma: \mathbb{R}^p_+ \to \mathbb{R}^q$ are continuously differentiable functions.

**Assumption 6.** For every $t \in \mathbb{N}$, $x(t)$ [respectively, $z(t)$] computed at step 2 (respectively, step 3) of the ADMM algorithm is locally or globally optimal.

**Assumption 7.** Let $L$ denote the set of limit points of the sequence $\{(x(t), z(t))\}_{t \in \mathbb{N}}$ and let $(x, z) \in L$. The set of constraint gradient vectors at $x$,

\[
C_X(x) = \{ \nabla \psi_i(x) | i = 1, \ldots, q_1 \} \cup \{ \nabla \phi_i(x) | i \in A_X(x) \}
\]

associated with the set $\mathcal{X}$ is linearly independent, where $A_X(x) = \{ i \mid \phi_i(x) = 0 \}$. Similarly, the corresponding set of constraint gradient vectors $C_Z$ associated with the set $\mathcal{Z}$ is linearly independent.

Assumption 5 is self-explanatory. Note that steps 2 and 3 of the algorithm involve nonconvex optimization problems, where the computational cost of finding the solutions $x(t+1)$ and $z(t+1)$, in general, can be entirely prohibitive. However, Assumption 6 indicates that the solution $x(t+1)$ [respectively, $z(t+1)$] of the optimization problem associated with the steps 2 (respectively, 3) of the ADMM should only be a local minimum and not necessarily a global minimum. In other words, minimizations in steps 2 and 3 of the algorithm should be performed only sub-optimally (or approximately). Thus, Assumption 6 can usually be accomplished by employing efficient local optimization methods (see [32, Section 1.4.1]).

In the literature, Assumption 7 is called the “regularity assumption” and is usually satisfied in practice. Moreover, any point that complies with the assumption is called regular, see [16, p. 269]. Let us next document two results that will be important later.

**Remark 1.** Suppose Assumptions 5 and 7 hold. Let $\{(x(t), z(t))\}_{t \in \mathbb{N}}$ be a subsequence of $\{(x(t), z(t))\}_{t \in \mathbb{N}}$ with $\lim_{t \to \infty} (x(t), z(t)) = (x, z)$. Then there exists $K$ such that the sets of vectors $C_X(x(t))$ and $C_Z(z(t))$ are each linearly independent for all $k \geq K$.

**Proof:** First note that if $i \notin A(x)$, then $\phi_i(x(t)) < 0$ [or $i \notin A(x(t))$] for all sufficiently large $k$, since $\phi_i$ is continuous and the set $\{x \in \mathbb{R} | x \neq 0\}$ is open. Therefore, it suffices to show that the columns of the matrix $D(x(t)) \in \mathbb{R}^{p_1 \times (q_1 + |A(x)|)}$ are linearly independent for all sufficiently large $k$, where

\[
D(x) = [(\nabla \psi_i(x))_{i=1, \ldots, q_1}, (\nabla \phi_i(x))_{i \in A_X(x)}].
\]

Since $\text{Det}(D(x)^T D(x))$ is continuous (see Assumption 5), $\text{Det}(D(x(t))^T D(x(t)))$ can be made arbitrarily close to $\text{Det}(D(x)^T D(x))$, which is nonzero, see Assumption 7. Equivalently, there exists $K \in \mathbb{N}$ such that $\det(D(x(t))^T D(x(t)))$ is nonzero for all $k \geq K$, which in turn ensures that $C_X(x(t))$ is a linearly independent set for $k \geq K$. The linear independence of $C_Z(z(t))$ for all sufficiently large $k$ can be proved similarly.

**Remark 2.** Suppose Assumptions 7 and 7 hold. Let $\{(x(t), z(t))\}_{t \in \mathbb{N}}$ be a subsequence of $\{(x(t), z(t))\}_{t \in \mathbb{N}}$ with $\lim_{t \to \infty} (x(t), z(t)) = (x, z)$. Then for sufficiently large $k$, there exist Lagrange multipliers $(\lambda(t), \gamma(t)) \in \mathbb{R}^{q_1} \times \mathbb{R}^{q_2}$ such that $\mu(t), \omega(t) \in \mathbb{R}^{q_1} \times \mathbb{R}^{q_2}$ satisfy the first order necessary conditions to the optimization problem in the $x$- (respectively, $z$-) update of the ADMM algorithm (compare with Definition 1).

**Proof:** From Remark 1, we have that $x(t) = z(t)$ are regular for sufficiently large $k$. This combined with the assumptions yields the result, which is an immediate consequence of Proposition 3.3.1.

Remarks 1 and 2 play a central role when deriving our convergence results, as we will show in the sequel. The following proposition establishes the convergence results of the ADMM algorithm:

**Proposition 2.** Suppose the Assumptions 7 and 7 hold and the sequence $y(t)$ converges to a point, i.e., $\lim_{t \to \infty} y(t) = y$ for some $y$. Then every limit point of the sequence $\{(x(t), z(t))\}_{t \in \mathbb{N}}$ together with $y$ and some $\lambda \in \mathbb{R}^{q_1}$, $\gamma \in \mathbb{R}^{q_2}$, $\mu \in \mathbb{R}^{q_1}$, and $\omega \in \mathbb{R}^{q_2}$ satisfy the first order necessary conditions for problem 2.

**Proof:** Let $(x, z)$ be a limit point of $\{(x(t), z(t))\}_{t \in \mathbb{N}}$ and $\{(x(t), z(t))\}_{t \in \mathbb{N}}$ be a subsequence such that $\lim_{t \to \infty} (x(t), z(t)) = (x, z)$. We show that the primal variables $x$, $z$ and the Lagrangian multipliers $y$, $\lambda$, $\gamma$, $\mu$, and $\omega$ satisfy the first order necessary conditions, where $\lambda$, $\gamma$, $\mu$, and $\omega$ are chosen as in Lemma 2.

In the sequel, we show that the four conditions of Definition 1 (First order necessary condition) are all satisfied.

1) **Primal feasibility:** Since $(x(t), z(t)) \in \mathcal{X} \times \mathcal{Z}$ and the set $\mathcal{X} \times \mathcal{Z}$ is closed it follows that $(x, z) \in \mathcal{X} \times \mathcal{Z}$. Since $y = y(0) + \sum_{i=1}^{\infty} (Ax(t) + Bz(t) - c)$, we must have $\lim_{t \to \infty} \|Ax(t) + Bz(t) - c\| = 0$, or $Ax + Bz = c$.

2) **Dual feasibility:** It holds for $\gamma(t)$ and $\omega(t)$ from Remark 2 that $\gamma(t) \geq 0$ and $\omega(t) \geq 0$ (compare with Definition 1). Hence, since the closed right half-plane is a closed set, it follows that $\gamma \geq 0$ and $\omega \geq 0$.

3) **Complementary slackness:** If $\phi_i(x) = 0$ then $\gamma_i \phi_i(x) = 0$ trivially holds. On the other hand, if $\phi_i(x) < 0$ then we showed in the proof of lemma 2 that $\gamma_i = 0$. Hence, it follows that $\gamma_i \phi_i(x) = 0$.

4) **Lagrangian vanishes:** We need to show that

\[
\nabla_x f(x) + A^T y + \nabla_x \psi(x) \lambda + \nabla_x \phi(x) \gamma = 0, \quad (28)
\]

\[
\nabla_z g(z) + B^T y + \nabla_z \theta(z) \mu + \nabla_z \sigma(z) \omega = 0. \quad (29)
\]

Let us start by showing (29). From Remark 2 we get for all sufficiently large $k$ [compare with Definition 1]:

\[
\nabla_x L(x(t), z, y(t-1)) + \nabla_x \theta(z(t)) \mu(t) + \nabla_x \sigma(z(t)) \omega(t) = 0. \quad (30)
\]
By using \( y(t_k - 1) = y(t_k) - \rho(Ax(t_k) + Bz(t_k) - c) \) in equation (29) and rearranging the terms we get that
\[
\nabla_x g(z(t_k)) + B^T y(t_k) + \nabla_x \theta(z(t_k)) \mu(t_k) + \nabla_x \sigma(x(t_k)) \omega(t_k) = 0. \tag{31}
\]
By using that \( \lim_{k \to \infty} (x(t_k), z(t_k), y(t_k)) = (\bar{x}, \bar{z}, \bar{y}) \) and \( \lim_{k \to \infty} (\lambda(t_k), \gamma(t_k), \mu(t_k), \omega(t_k)) = (\Lambda, \Gamma, \mu, \omega) \), we conclude that equation (29) holds. By using the same arguments as above we get for all sufficiently large \( k \)
\[
\nabla_x f(x(t_k)) + A^T y(t_k) + \nabla_x \psi(x(t_k)) \lambda(t_k) + \nabla_x \phi(x(t_k)) \gamma(t_k) = \rho A^T B(z(t_k) - x(t_k) - 1). \tag{32}
\]
Therefore, by the arguments above, if we can show that
\[
\lim_{t \to \infty} \rho A^T B(z(t+1) - z(t)) = 0,
\]
then equation (28) holds. The assumption \( \bar{y} = \lim_{t \to \infty} y(t) \) together with the relation
\[
y(t + 1) = y(t) + \rho \sum_{i=1}^{t+1} Ax(l) + Bz(l) - c
\]
can be used to show that the series
\[
\sum_{t=1}^{\infty} (Ax(t) + Bz(t+1) - c), \quad \sum_{t=1}^{\infty} (Ax(t) + Bz(t) - c),
\]
are convergent. By taking the difference of the two series and using that the sum of convergent series is a convergent series, we get that
\[
\sum_{t=1}^{\infty} A(z(t+1) - z(t)) = 0
\]
implies that the set \( \lim_{t \to \infty} B(z(t+1) - z(t)) = 0 \). By multiplying \( \rho A^T \) from the left side we get that
\[
\lim_{t \to \infty} \rho A^T B(z(t+1) - z(t)) = 0.
\]

**Lemma 2.** Let \( \{t_k\}_{k \in \mathbb{N}} \) be a sequence such that \( \lim_{k \to \infty} (x(t_k), z(t_k)) = (\bar{x}, \bar{z}) \). Then the limits: \( \lim_{k \to \infty} \lambda(t_k), \lim_{k \to \infty} \gamma(t_k), \lim_{k \to \infty} \mu(t_k), \) and \( \lim_{k \to \infty} \omega(t_k) \) exist, where \( \lambda(t_k), \gamma(t_k), \mu(t_k), \omega(t_k) \) are chosen as in Remark 2.

**Proof:** We prove the existence of the first two limits. The proof of the existence of the latter two limits follows similarly.

Since \( \nabla f, \nabla \psi, \) and \( \nabla \phi \) are continuous functions (see Assumption 1), we have
\[
\lim_{k \to \infty} \nabla f(x(t_k)) = \nabla f(\bar{x}),
\]
\[
\lim_{k \to \infty} \nabla \psi(x(t_k)) = \nabla \psi(\bar{x}),
\]
\[
\lim_{k \to \infty} \nabla \phi(x(t_k)) = \nabla \phi(\bar{x}).
\]
This, together with Remark 1, implies that there exists \( K \) such that \( D(x(t_k))^T D(x(t_k)) \) (see Eq. (27)) is invertible for all \( k \geq K \). Hence, it follows that for all \( k \geq K \), we have
\[
(D(x(t_k))^T D(x(t_k)) - D(x(t_k))^T D(x(t_k))\lambda(t_k)) = D(x(t_k))^T D(x(t_k))\gamma(t_k).
\]
Since \( D(t_k) \) and \( \nabla f(x(t_k)) \) converge when \( k \to \infty \) it follows that
\[
\lim_{k \to \infty} (\lambda(t_k), \gamma(t_k)) = (\bar{x}, \bar{z}).
\]
Next we show that \( \lim_{t \to \infty} \gamma_i(t_k) = 0 \) if \( i \not\in \mathcal{A}_X(\bar{x}) \). Since \( \phi_i(\bar{x}) \), there exists an open set \( U \subseteq \mathbb{R}^p \) containing \( \bar{x} \) such that \( \phi_i(x) < 0 \) for all \( x \in U \). In particular, there exists \( K \in \mathbb{N} \) such that \( \phi_i(x(t_k)) < 0 \) for all \( k \geq K \). Therefore, there must exist \( K \in \mathbb{N} \) such that \( \gamma_i(t_k) = 0 \) for all \( k \geq K \), since complementary slackness \( \gamma_i(t_k) \phi_i(x(t_k)) = 0 \) holds for all sufficiently large \( k \) [compare with Remark 2].

A stronger version of Proposition 2 is shown in the following corollary:

**Corollary 1.** If \( \lim_{t \to \infty} (x(t), z(t), y(t)) \) = \((\bar{x}, \bar{z}, \bar{y})\), then \( \bar{x} \) and \( \bar{z} \) satisfy the first order necessary conditions.

The corollary follows immediately because the hypothesis implies that the set \( \mathcal{L} \) defined in Assumption 7 is a singleton.

Technically, Proposition 2 characterizes the solution of the ADMM algorithm applied on the possibly nonconvex problem 3. More specifically, the proposition claims that under mild assumptions the solutions computed by ADMM satisfy the first order necessary conditions for problem 2, if at every iteration, the subproblems are locally (or globally) solved and if the dual variables of ADMM converge. In addition to its technical relevance, Proposition 2 can serve as a practical criterion for numerically determining the characteristics of the solution of ADMM, which would have been impossible otherwise. In particular, by observing the sequence \( \{y(t)\}_{t \in \mathbb{N}} \) as the ADMM proceeds (e.g., a plot of \( y(t) \) versus \( t \)), one can realize the “practical certainty” or even “practical impossibility” of the convergence of the sequence. If the convergence of \( y(t) \) is observed, then from the proposition, one can conclude that the solution of ADMM satisfies the first order necessary conditions. This will be further exemplified in subsequent sections.

**C. Numerical Evaluations**

Fig. 4 illustrates the theoretical assertions for ADMM in Section IV-B with the 2 dimensional examples introduced in Section III-C (compare with Fig. 3). Fig. 4(a) and Fig. 4(c) show the sequences \( \{x(t), z(t)\}_{t \in \mathbb{N}} \), analogously to Fig. 3. Fig. 4(b) and Fig. 4(d) display the objective value \( f(x(t)) + g(z(t)) \) and the multiplier \( y(t) \) at every iteration \( t \), respectively. Note that Assumption 7 is trivially satisfied, because the sets \( \mathcal{X} \) and \( \mathcal{Z} \) are boxes.

Fig. 4(a) and Fig. 4(b) depict the results for a nonconvex quadratic example with the fixed initialization \((x(0), z(0)) = (-1, 3)\) and different \( \rho \) values [compare with Example 1]. Results show that the ADMM yields \((-1, -1)\) for all the considered cases. Note from Fig. 4(b) that \( y(t) \) converges (numerically) to \( y^* = 5.33, 6, 5.24, \) and \( 5.42 \) for the cases \( \rho = 2.1, 3, 5, \) and 10, respectively. Therefore, from Proposition 2 we can conclude that \((x^*, z^*) = (-1, -1)\) satisfies the first order necessary conditions for the problem. This can be visually verified by inspecting the contour and the constraints plotted in the Fig. 4(a).

Fig. 4(c) and Fig. 4(d) depict the results for the problem with multiple local minimas with \( \rho = 1.1 \) and different initializations. Results suggest that the different initializations correspond to different limit points \((5\pi/4, 5\pi/4), (-3\pi/4, 3\pi/4),\) and \((-8, -8)\). Moreover, Fig. 4(d) shows that \( y(t) \) converges for all considered cases. Thus, from Proposition 2 we conclude that the resulting limit points of \( \{(x(t), z(t), y(t))\}_{t \in \mathbb{N}} \) satisfy the first order necessary conditions for the problem.

The next section demonstrates the potential of the proposed ADMM approaches (see Sections III and IV) in a problem of great practical relevance.
V. APPLICATION: COOPERATIVE LOCALIZATION IN WIRELESS SENSOR NETWORKS

In this section, we use the ADLM methods to design distributed algorithms for Cooperative Localization (CL) in wireless sensor networks.

Consider an undirected graph \( \mathcal{N}, \mathcal{E} \), where \( \mathcal{N} = \{1, \ldots, N\} \) is a set of nodes embedded into a convex and compact set \( \mathcal{R} \subseteq \mathbb{R}^2 \). Let \( \mathcal{N} = S \cup A \), where \( S = \{1, \ldots, S\} \) is the set of sensors with unknown locations and \( A = \{S+1, \ldots, N\} \) is the set of anchors with known locations. We denote the location of node \( n \in A \) by \( a_n \), and an estimate of the location of node \( n \in S \) by \( z_n \).

Suppose the measurements of the squared\(^4\) distance between two nodes \( n, m \in \mathcal{N} \), denoted by \( d_{n,m}^2 \), are available if and only if \( (n, m) \in \mathcal{E} \). The additive measurement errors are assumed to be independent and Gaussian distributed with zeros mean and variance \( \sigma^2 \). Then the CL problem consists of finding the maximum likelihood estimate of \( (z_n)_{n \in S} \) by solving the following problem:

\[
\begin{align*}
\text{minimize} & \quad \sum_{n \in S} \left( \sum_{m \in S_n} |d_{n,m}^2 - |z_n - z_m||^2 \right)^2 \\
& \quad + 2 \sum_{m \in A_n} |d_{n,m}^2 - |z_n - a_m||^2 \\
\text{subject to} & \quad z_i \in \mathcal{R}, \text{ for all } i \in S,
\end{align*}
\tag{33}
\]

where \( S_n = \{m \in S | (n, m) \in \mathcal{E}\} \) and \( A_n = \{m \in A | (n, m) \in \mathcal{E}\} \). Note that Problem (33) is NP-hard\(^{[29]}\).

To enable distributed implementation (among the nodes) of the proposed ADLM approaches, let us first equivalently reformulate problem (33) into a general consensus form\(^{[12]}\) (Section 7.2). We start by introducing at each node \( n \in \mathcal{N} \), a local copy \( x_n \) of \( (z_n)_{m \in S_n} \), where \( S_n = S_n \cup \{n\} \). More specifically, we let \( x_n = (x_{n,m})_{m \in S_n} \), where \( x_{n,m} \in \mathbb{R}^2 \) denotes the local copy of \( z_m \) at node \( n \). To formally express
the consistency between $x_n$ and $z = (z_1, \cdots, z_S)$, we introduce the matrix $E_n \in \mathbb{R}^{2|S_n| \times 2S}$, which is a $|S_n| \times S$ block matrix of $2 \times 2$ blocks. In particular, the $i$-th, $j$-th block of $E_n$ is given by $(E_n)_{i,j} = I_2$, if $x_{n,j}$ is the $i$-th block of the vector $x_n$ and $(E_n)_{i,j} = 0$ otherwise. Then Problem (33) is equivalently given by

$$
\begin{align*}
\text{minimize} & \quad \sum_{n \in \mathcal{N}} f_n(x_n), \\
\text{subject to} & \quad x_n \in \mathcal{R}^{|S_n|}, \text{ for all } n \in \mathcal{N}, \\
& \quad x_n = E_n z, \text{ for all } n \in \mathcal{N},
\end{align*}
$$

where $x = (x_1, \cdots, x_N) \in \mathbb{R}^{\sum_{n \in \mathcal{N}} |S_n|}$, $z \in \mathbb{R}^{2S}$, and

$$
f_n(x_n) = \begin{cases} 
\sum_{m \in S_n} |d_{n,m}^2 - ||x_{n,m} - a_n||^2|^2 & n \in \mathcal{S} \\
+ \sum_{m \in A_n} |d_{n,m}^2 - ||x_{n,m} - a_n||^2|^2 & n \in \mathcal{A}.
\end{cases}
$$

Problem (34) fits the form of Problem (2) and proposed ADLM approaches can readily be applied. The augmented Lagrangian of problem (34) can be written as

$$L_\rho(x, z, y) = \sum_{n \in \mathcal{N}} f_n(x_n) + y_n^T(x_n - E_n z) + \frac{\rho}{2} ||x_n - E_n z||^2, $$

where $y = (y_1, \cdots, y_n)$ is the Lagrangian multiplier. Note that the variables $x$ and $y$ are separable among $n \in \mathcal{N}$. The resulting distributed-ADLM is as follows.

**Algorithm 3: DISTRIBUTED ALTERNATING DIRECTION ALTERNATING DIRECTION ALGORITHM (D-ADLM)**

1) **Initialization:** Set $t = 0$ and put initial values to $z(t)$, $y(t)$, and $\rho(t)$.

2) **Subproblem:** Each node $n \in \mathcal{N}$ solves

$$x_n(t+1) = \text{argmin}_{x_n \in \mathcal{R}^{|S_n|}} L_\rho(x_n, z(t), y(t)).$$

3) **Communication/Averaging:** $z(t+1)$ is given by

$$z_{n}(t+1) = \frac{1}{|S_n|} \sum_{i \in S_n} E_{i,n}^T(x_i(t+1) + y_i(t)), \quad (35)$$

for $n \in S$, where $E_{i,n}$ is the column $n$ of the block matrix $E_i$.

4) **Local parameter update:** Each node $n \in \mathcal{N}$ updates its local parameters $\rho(t)$ and $y(t)$ accordingly.

5) **Stopping criterion:** If stopping criterion is met terminate, otherwise set $t = t + 1$ and go to step 2.

Note that the D-ADLM can be carried out either as an ADPM or as an ADMM by performing $\rho(t)$ and $y(t)$ updates at step 4 accordingly (compare with step 4 of ADPM and ADMM algorithms).

As indicated in the first step, the initial setting of the algorithm should be agreed on among the nodes. Other steps can be carried out in a distributed manner with local message exchanges. Note that (35) is simply the average of the local copies of $z_n$ and the corresponding dual variables [scaled by $\rho(t)$], which can be performed by employing standard gossiping algorithms, e.g., [30]. Moreover, the last step requires a mechanism to terminate the algorithm. A natural stopping criterion is to fix the number of iterations, which requires no coordination among the nodes except at the beginning. In order to control the accuracy level $\epsilon$ of the coupling constraints, one can, for example, terminate the algorithm when $\max_{n \in \mathcal{N}} ||x_n(t) - E_n z(t)||_2 < \epsilon$. This can be accomplished with an additional coordination among the nodes.

Let us next test the D-ADLM on CL problems and interpret the results together with Proposition 1 and 2.

### A. Numerical Results

Ten random networks, each with $S = 10$, $A = 4$, and $\mathcal{R} = [0,1]^2$ are considered. In each graph, the 4 anchors are located in the corners of the box $\mathcal{R}$, i.e., at $(0,0)$, $(0,1)$, $(1,0)$, $(1,1)$.
When implementing D-ADLM with ADPM (D-ADLM:ADPM), we use $\Delta = 1.2$, $\kappa = 15$, and consider two dual variable updates: a) $y(t) = 0$, b) $y_n(t+1) = y_n(t) + \rho(t)(x_n(t+1) - E_n z(t+1))$. In the case of the D-ADLM:ADMM we consider two penalty parameters: a) $\rho = 1$, b) $\rho = 10$. For every simulation setting, we consider one hundred $z(0)$ initializations, uniformly distributed on the unit box $\mathcal{R}$. The dual variable is always initialized at zero, i.e., $y(0) = 0$. The subproblems in the $x$-update are solved by using a local optimal solver, e.g., fmincon in MATLAB. We note that the Assumptions 1 and 7 hold in all the considered simulation settings, and therefore the assertions of Proposition 1 and Proposition 2 are readily validated.

Fig. 5 depicts the residuals $r(t) = \sum_{n \in \mathcal{N}} \| x_n(t) - E_n z(t) \|_2$ versus $t$, for all executions of the D-ADLM with $\sigma^2 = 0$, i.e., scenario 1. The results show that $r(t)$ reaches 0 (to MATLAB precision) in a finite number of iterations in all the considered simulation settings. In the case of D-ADLM:ADPM, the results agree with Proposition 1 that $r(t)$ will be arbitrarily small as the algorithm proceeds. In case of D-ADLM:ADMM, we have $r(t) = (1/\rho^2) \| y(t+1) - y(t) \|_2^2$. The results shows that $r(t)$ or equivalently $\| y(t+1) - y(t) \|_2$ is becoming arbitrarily small. This suggests that $\{y(t)\}_{t \in \mathcal{N}}$ converges (compare with a Cauchy sequence). Therefore, from Proposition 2 we can conclude that the limit points of sequence $\{x(t), z(t)\}_{t \in \mathcal{N}}$ generated by D-ADLM:ADMM satisfy the first order necessary conditions for Problem (34). Interestingly, our extensive numerical simulations showed that
for each considered case the limit points of \( \{ x(t), z(t) \} \) are unique. The results were almost identical in the case of non zero measurement errors.

Fig. 6 depicts the estimated localizations obtained by the D-ADLM algorithm for one of the test networks with measurement errors \( \sigma^2 = 0.05 D \), i.e., scenario 2. The gray circles in the corners represent the anchors. The large black markers represent the true locations of each sensor. The smaller colored markers are the estimated locations of the sensors returned by D-ADLM after the convergence of the algorithm, where each marker corresponds to one of the 100 different initializations. In the case of ADMM and ADPM with dual update, the mean square errors (MSE) between the true and estimated locations of the sensors are always below 0.009. However, in the case of ADPM without dual update, the MSE are relatively higher or 0.017 in the worst case. Therefore the results suggest that the ADPM with dual updates and the ADMM algorithms are more robust than the ADPM without dual updates.

VI. CONCLUSIONS

We investigated the convergence behavior of scalable variants of two standard nonconvex optimization methods: a novel method we call Alternating Direction Penalty Method and the well known Alternating Direction Method of Multipliers, variants of the Quadratic Penalty Method and the Method of Multipliers, respectively. Our theoretical results showed the ADPM asymptotically reaches primal feasibility under assumptions that hold widely in practice. Furthermore, we provided sufficient conditions for the asymptotically convergence of ADMM to the first order necessary condition for local optimality. These results on the behavior of the ADMM hold even when the problem considered has nonconvex constraints and when the subproblems are only solved to local optimum. The theoretical results were substantiated by illustrative examples. Finally, we demonstrated how the methods can be used to design distributed algorithms for nonconvex cooperative localization in wireless sensor networks.

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