Order-Preservation for Multidimensional Stochastic Functional Differential Equations with Jump*

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Abstract

Sufficient and necessary conditions are presented for the order-preservation of stochastic functional differential equations on $\mathbb{R}^d$ with non-Lipschitzian coefficients driven by the Brownian motion and Poisson processes. The sufficiency of the conditions extends and improves some known comparison theorems derived recently for one-dimensional equations and multidimensional equations without delay, and the necessity is new even in these special situations.

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1 Introduction

The order-preservation of stochastic processes is crucial since it enables one to control complicated processes by using simpler ones. For a large class of diffusion-jump type Markov processes on $\mathbb{R}^d$, the order-preservation property has been well described in the distribution sense (see [14], 15 and references therein), see also [14] for a study of super processes. To derive the pathwise order-preservation, one establishes the comparison theorem for stochastic

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differential equations (SDEs), which goes back to [12, 16]. The study of comparison theorem for one-dimensional SDEs is now very complete, see e.g. [3, 5, 6, 7, 8, 9, 10, 18] and references within. Equations considered in these references include forward or backward SDEs with jump and with delay. The aim of this note is to provide a sharp criterion on the comparison theorem for multidimensional stochastic functional stochastic differential equations (SFDEs), which is yet unknown in the literature.

Throughout the paper, we fix a constant \( r_0 \geq 0 \) and a natural number \( d \geq 1 \). Let
\[
\mathcal{C} = \{ \xi = (\xi^1, \cdots, \xi^d) : [-r_0, 0] \to \mathbb{R}^d \text{ is cadlag} \}.
\]

Recall that a path is called cadlag, if it is right-continuous having finite left limits. For any \( \xi \in \mathcal{C} \), we have
\[
\|\xi\|_\infty = \sum_{i=1}^d \sup_{s \in [-r_0, 0]} |\xi^i(s)| < \infty.
\]

Then under the uniform norm \( \|\cdot\|_\infty \) the space \( \mathcal{C} \) is complete but not separable. To make \( \mathcal{C} \) a Polish space, we take the Skorohod metric rather than the uniform metric.

For any cadlag \( f : [-r_0, \infty) \to \mathbb{R}^d \) and \( t \geq 0 \), we let \( f_t \in \mathcal{C} \) be such that \( f_t(\theta) = f(\theta + t) \) for \( \theta \in [-r_0, 0] \), and define \( f_{t-} \in \mathcal{C} \) for \( t > 0 \) such that \( f_{t-}(\theta) = f((t + \theta)-) := \lim_{s\uparrow t+} f(s) \) for \( \theta \in [-r_0, 0] \). We call \( (f_t)_{t \geq 0} \) the segment of \( f(t) \) for \( t \geq -r_0 \).

Now, let \( B(t) \) be an \( m \)-dimensional Brownian motion, and let \( N(ds, dz) \) be a Poisson counting process with characteristic measure \( \nu \) on a measurable space \((E, \mathcal{E})\), with respect to a complete filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\). We assume that \( B \) and \( N \) are independent. We will consider the order-preservation of SFDEs driven by \( B \) and \( N \). To characterize the non-Lipshitz regularity of coefficients in the SDDEs, we introduce the following class of control functions:
\[
\mathcal{U} = \left\{ u \in C^1((0, \infty); [1, \infty)) : \int_0^1 \frac{ds}{su(s)} = \infty, \lim_{s \downarrow 0} su(s)^2 = 0, \right. \\
\left. \text{and } s \mapsto su(s) \text{ is increasing and concave} \right\}.
\]

Typical elements in this class are \( u(s) = 1 \) and \( u(s) = \log(1 + s^{-1}) \).

Consider the following SFDEs on \( \mathbb{R}^d \):
\[
\begin{align*}
\text{(1.1) } & \begin{cases}
    \text{d}X(t) = b(t, X_t)\text{d}t + \sigma(t, X_t)\text{d}B(t) + \int_E \gamma(t, X_{t-}, z)N(\text{d}t, \text{d}z), \\
    \text{d}\bar{X}(t) = \bar{b}(t, \bar{X}_t)\text{d}t + \sigma(t, \bar{X}_t)\text{d}B(t) + \int_E \bar{\gamma}(t, \bar{X}_{t-}, z)N(\text{d}t, \text{d}z),
\end{cases}
\end{align*}
\]

where
\[
\begin{align*}
b, \bar{b} : [0, \infty) \times \mathcal{C} \times \Omega \to \mathbb{R}^d, \quad \sigma, \bar{\sigma} : [0, \infty) \times \mathcal{C} \times \Omega \to \mathbb{R}^d \otimes \mathbb{R}^m, \\
\gamma, \bar{\gamma} : [0, \infty) \times \mathcal{C} \times E \times \Omega \to \mathbb{R}^d
\end{align*}
\]

are progressively measurable.
For any $s \geq 0$ and $\mathcal{F}_s$-measurable random variables $\xi, \bar{\xi}$ on $\mathcal{G}$, a solution to (1.1) for $t \geq s$ with $(X_s, \bar{X}_s) = (\xi, \bar{\xi})$ is a cadlag adapted process $(X(t), \bar{X}(t))_{t \geq s}$ such that $\mathbb{P}$-a.s. for all $t \geq s$

$$X(t) = \xi(0) + \int_s^t b(r, X_r) dr + \int_s^t \sigma(r, X_r) dB(r) + \int_{[s,t] \times E} \gamma(r, X_r, z) N(dr, dz),$$

$$\bar{X}(t) = \bar{\xi}(0) + \int_s^t \bar{b}(r, \bar{X}_r) dr + \int_s^t \bar{\sigma}(r, \bar{X}_r) dB(r) + \int_{[s,t] \times E} \bar{\gamma}(r, \bar{X}_r, z) N(dr, dz),$$

where, according to the initial condition $(X_s, \bar{X}_s) = (\xi, \bar{\xi})$, $X_r$ and $\bar{X}_r$ for $r \geq s$ are well defined.

To ensure the existence and uniqueness of solutions, we make use of the following assumptions:

(A1) There exist some positive $K \in C([0, \infty))$ and $u \in \mathcal{U}$ such that $\mathbb{P}$-a.s.

$$|b(t, \xi) - b(t, \eta)| + |\bar{b}(t, \xi) - \bar{b}(t, \eta)| + \|\sigma(t, \xi) - \sigma(t, \eta)\|_{HS} + \|\bar{\sigma}(t, \xi) - \bar{\sigma}(t, \eta)\|_{HS}$$

$$+ \int_E \left( |\gamma(t, \xi, z) - \gamma(t, \eta, z)| + |\bar{\gamma}(t, \xi, z) - \bar{\gamma}(t, \eta, z)| \right) \nu(dz)$$

$$\leq K(t) \|\xi - \eta\|_\infty u(\|\xi - \eta\|_\infty), \quad \xi, \eta \in \mathcal{G}, t \geq 0.$$

(A2) For any $T > 0$ there exists a constant $C(T) > 0$ such that $\mathbb{P}$-a.s.

$$\sup_{t \in [0, T]} \left( |b(t, 0)| + |\bar{b}(t, 0)| + \|\sigma(t, 0)\|_{HS} + \|\bar{\sigma}(t, 0)\|_{HS} \right)$$

$$+ \int_{[0,T] \times E} \left( |\gamma(t, 0, z)| + |\bar{\gamma}(t, 0, z)| \right) d\nu(dz) \leq C(T).$$

When $u \equiv 1$, (A1) reduces to the usual Lipschitz condition. In general, (A1) allows the coefficients to be non-Lipschitzian.

According to Theorem 3.1 below, for any $s \geq 0$ and $\mathcal{F}_s$-measurable random variables $\xi, \bar{\xi}$ on $\mathcal{G}$, the equation (1.1) has a unique solution for $t \geq s$ with $X_s = \xi$ and $\bar{X}_s = \bar{\xi}$, and the solution is non-explosive. We denote the solution by $\{X(s, \xi; t), \bar{X}(s, \bar{\xi}; t)\}_{t \geq s}$.

To introduce the notion of order-preservation of the solutions, we take the usual partial-order on $\mathbb{R}^d$, i.e. for $x = (x^1, \ldots, x^d), y = (y^1, \ldots, y^d) \in \mathbb{R}^d$, we write $x \leq y$ if $x^i \leq y^i$ holds for all $1 \leq i \leq d$. Similarly, for $\xi = (\xi^1, \ldots, \xi^d), \eta = (\eta^1, \ldots, \eta^d) \in \mathcal{G}$, we write $\xi \leq \eta$ if $\xi^i(\theta) \leq \eta^i(\theta)$ holds for all $\theta \in [-r_0, 0]$ and $1 \leq i \leq d$. Moreover, for any $\xi, \eta \in \mathcal{G}$, let $\xi \wedge \eta \in \mathcal{G}$ be such that $(\xi \wedge \eta)^i = \min\{\xi^i, \eta^i\}, 1 \leq i \leq d$; and let $\xi \vee \eta = -\{(-\xi) \wedge (-\eta)\}$.

**Definition 1.1.** The solutions of (1.1) are called order-preserving if for any $s \geq 0$ and $\mathcal{F}_s$-measurable random variables $\xi, \bar{\xi}$ on $\mathcal{G}$ with $\mathbb{P}$-a.s. $\xi \leq \bar{\xi}$, $\mathbb{P}$-a.s. $X(s, \xi; t) \leq \bar{X}(s, \bar{\xi}; t)$ holds for all $t \geq s$.

**Theorem 1.1.** Assume (A1) and (A2). The solutions to (1.1) are order-preserving if the following conditions are satisfied:
Proof of Theorem 1.1.

2 Proofs of Theorems 1.1 and 1.2

jumps.

a result on the existence and uniqueness of solutions to stochastic functional equations with \( \xi, \sigma \), \( \nu \) variables when \( C \) the sequel the continuity on \( \nu \) recently in [19] for \( 0 \leq 296 \) where the condition 2

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Theorem 1.2.

Comparing with existing comparison theorems derived in the above mentioned references

for one-dimensional equations, Theorem 1.1 has a rather broad range of applications. Next,

a multidimensional comparison theorem without delay has been presented in [13, Theorem

296] where the condition 2 imply that \( \bar{b}(t, x) \) is and is thus much stronger than the condition (1) in Theorem 1.1 with \( r_0 = 0 \) (i.e. the case without delay). Moreover, when \( r_0 = 0 \) (i.e. without delay) Theorem 1.1 also covers the comparison theorem derived recently in [10] for \( \nu(\bar{E}) < \infty \) and Lipschitzian coefficients.

On the other hand, our next result shows that the conditions in Theorem 1.1 are also

necessary for the order-preservation when the coefficients are continuous and either \( \nu \) is finite or \( \gamma(t, \xi, \cdot), \gamma(t, \xi, \cdot) \) are integrable with respect to \( \nu \) locally uniformly in \( (t, \xi) \), where and in the sequel the continuity on \( \mathcal{C} \) is with respect to the the Skorohod metric. This result is new even for the case without delay.

**Theorem 1.2.** Assume (A1), (A2) and that the solutions to (1.1) are order-preserving.

(I) If \( \mathbb{P} \)-a.s. \( b, \bar{b} \in C([0, \infty) \times \mathcal{C}; \mathbb{R}^d) \) and for any \( n \geq 1 \)

\[
\lim_{\epsilon \downarrow 0} \sup_{t \in [0, n]} \int_E \epsilon \wedge (|\gamma(t, \xi, z)| + |\gamma(t, \xi, z)|) \nu(dz) = 0,
\]

then condition (1) holds.

(II) If \( \mathbb{P} \)-a.s. \( \sigma, \bar{\sigma} \in C([0, \infty) \times \mathcal{C}; \mathbb{R}^d \otimes \mathbb{R}^m) \), then condition (2) holds.

(III) If \( \mathbb{P} \times \nu \)-a.e. \( \gamma, \bar{\gamma} \in C([0, \infty) \times \mathcal{C}; \mathbb{R}^d) \), then condition (3) holds.

Note that condition (1.2) holds if either \( \nu \) is finite or \( \gamma(t, \xi, \cdot), \gamma(t, \xi, \cdot) \) are integrable with respect to \( \nu \) locally uniformly in \( (t, \xi) \).

In the next section we present proofs of the above two theorems. In Section 3, we present

a result on the existence and uniqueness of solutions to stochastic functional equations with

jumps.

2 Proofs of Theorems 1.1 and 1.2

**Proof of Theorem 1.1.** Assume that (1)–(3) hold. For any \( t_0 \geq 0 \) and \( \mathcal{F}_{t_0} \)-measurable random variables \( \xi, \bar{\xi} \) on \( \mathcal{C} \) such that \( \xi \leq \bar{\xi} \), we aim to prove that for any \( T > t_0 \),

\[
\mathbb{E} \sup_{r \in [t_0, T]} (X^i(t_0, \xi; r) - \bar{X}^i(t_0, \bar{\xi}; r))^+ = 0, \quad 1 \leq i \leq d.
\]
For equations without delay, this can be done by using a Tanaka type formula for \((X^i(t) - X^i(t))^+\) (see [13, 153]). Below, we shall make use an approximation argument which plays the same role as the Tanaka type formula. For simplicity, we will simply denote \(X(t) = X(t_0, \xi; t)\) and \(\bar{X}(t) = \bar{X}(t_0, \bar{\xi}; t)\) for \(t \geq t_0 - r_0\). Recall that \(X(t_0, \xi; t) = \xi(t - t_0)\) and \(\bar{X}(t_0, \bar{\xi}; t) = \bar{\xi}(t - t_0)\) for \(t \in [t_0 - r_0, t_0]\).

For any \(n \geq 1\) let \(\psi_n : \mathbb{R} \to [0, \infty)\) be constructed as follows: \(\psi_n(s) = \psi'_n(s) = 0\) for \(s \in (-\infty, 0]\), and

\[
\psi''_n(s) = \begin{cases} 
4n^2s, & s \in [0, \frac{1}{2n}], \\
-4n^2(s - \frac{1}{n}), & s \in [\frac{1}{2n}, \frac{1}{n}], \\
0, & \text{otherwise}.
\end{cases}
\]

We have

\[
0 \leq \psi'_n \leq 1, \quad \text{and as } n \uparrow \infty : 0 \leq \psi_n(s) \uparrow s^+, \quad s\psi''_n(s) \leq 21(0, \frac{1}{n})(s) \downarrow 0.
\]

Let

\[
\tau_k = \inf\{t \geq t_0 : |X(t) - X(t) \wedge \bar{X}(t)| \geq k\}, \quad k \geq 1.
\]

Since by (2) we have \(\sigma = \bar{\sigma}\) and

\[
\psi_n(X^i(t_0) - \bar{X}^i(t_0)) = \psi_n(\xi^i(0) - \bar{\xi}^i(0)) = 0,
\]

the Itô formula yields

\[
\psi_n(X^i(t \wedge \tau_k) - \bar{X}^i(t \wedge \tau_k)) = M(t \wedge \tau_k) + \int_{t_0}^{t \wedge \tau_k} \left( b^i(s, X_s) - \bar{b}^i(s, \bar{X}_s) \right) \psi'_n(X^i(s) - \bar{X}^i(s)) ds
\]

\[
\quad + \frac{1}{2} \sum_{j=1}^{m} \int_{t_0}^{t \wedge \tau_k} (\sigma^{ij}(s, X_s) - \sigma^{ij}(s, \bar{X}_s))^2 \psi''_n(X^i(s) - \bar{X}^i(s)) ds
\]

\[
\quad + \int_{[t_0, t \wedge \tau_k] \times \mathbb{E}} \left( \psi_n(X^i(s^-) - \bar{X}^i(s^-)) - \psi_n(X^i(s^-) - \bar{X}^i(s^-)) - \gamma^i(s, X_{s^-}, z) + \bar{\gamma}^i(s, \bar{X}_{s^-}, z) \right) N(ds, dz)
\]

for any \(k, n \geq 1, 1 \leq i \leq d\) and \(t \geq t_0\), where

\[
M(t) := \sum_{j=1}^{m} \int_{t_0}^{t} (\sigma^{ij}(s, X_s) - \sigma^{ij}(s, \bar{X}_s)) \psi'_n(X^i(s) - \bar{X}^i(s)) dB^j(s).
\]

Noting that \(0 \leq \psi'_n(X^i(s) - \bar{X}^i(s)) \leq 1_{\{X^i(s) > \bar{X}^i(s)\}}\) and when \(X^i(s) > \bar{X}^i(s)\) one has \((X_s \wedge \bar{X}_s)^i(0) = (\bar{X}_s)^i(0)\), it follows from (1) that \(\mathbb{P}\)-a.s.

\[
(b^i(s, X_s \wedge \bar{X}_s) - \bar{b}^i(s, \bar{X}_s)) \psi'_n(X^i(s) - \bar{X}^i(s)) \leq 0, \quad n \geq 1, s \in [t_0, T].
\]
Combining this with (A1) and $0 \leq \psi'_n \leq 1$, we obtain $\mathbb{P}$-a.s.

\begin{equation}
(b'(s, X_s) - \bar{b}'(s, \bar{X}_s))\psi'_n(X^i(s) - \bar{X}^i(s))
\leq C(T)||X_s - X_s \wedge \bar{X}_s||_\infty u(||X_s - X_s \wedge \bar{X}_s||_\infty) \quad n \geq 1, s \in [t_0, T]
\end{equation}

for some constant $C(T) > 0$. Next, by (A1), \[(2.2)\] and (2), we have $\mathbb{P}$-a.s.

\begin{equation}
\begin{aligned}
\sum_{i=1}^{m} |\sigma^{ij}(s, X_s) - \tilde{\sigma}^{ij}(s, \bar{X}_s)|^2 \psi''_n(X^i(s) - \bar{X}^i(s)) \\
\leq C(T)1_{\{X^i(s) - \bar{X}^i(s) \in (0, \frac{1}{n})\}}|X^i(s) - \bar{X}^i(s)|^2 u(|X^i(s) - \bar{X}^i(s)|)^2 \\
\leq C(T)\varepsilon(n), \quad n \geq 1, s \in [t_0, T]
\end{aligned}
\end{equation}

for some constant $C(T) > 0$, where since $u \in \mathcal{U}$,

$$
\varepsilon(n) := \sup_{s \in (0, \frac{1}{n})} su(s)^2 \downarrow 0 \text{ as } n \uparrow \infty.
$$

Moreover, (A1), \[(2.2)\] and (2) also imply $\mathbb{P}$-a.s.

\begin{equation}
\begin{aligned}
\sum_{j=1}^{m} |\sigma^{ij}(s, X_s) - \tilde{\sigma}^{ij}(s, \bar{X}_s)| \psi'_n(X^i(s) - \bar{X}^i(s)) \\
\leq C(T)(X^i(s) - \bar{X}^i(s))^+ u((X^i(s) - \bar{X}^i(s))^+), \quad n \geq 1, s \in [t_0, T]
\end{aligned}
\end{equation}

for some constant $C(T) > 0$. Finally, by (3) we have

\begin{equation}
X^i(s) \wedge \bar{X}^i(s) + \gamma^i(s, X_s \wedge \bar{X}_s, \cdot) \leq \bar{X}^i(s) + \bar{\gamma}^i(s, \bar{X}_s, \cdot), \quad \nu \times \mathbb{P}\text{-a.e.}
\end{equation}

If $X^i(s) \leq \bar{X}^i(s)$, then \[(2.7)\] becomes

$$
X^i(s) + \gamma^i(s, X_s \wedge \bar{X}_s, \cdot) \leq \bar{X}^i(s) + \bar{\gamma}^i(s, \bar{X}_s, \cdot), \quad \nu \times \mathbb{P}\text{-a.e.,}
$$

so that $0 \leq \psi'_n \leq 1$ and $\psi_n(s) = 0$ for $s \leq 0$ imply

\begin{align*}
\psi'_n(X^i(s) - \bar{X}^i(s)) &= \psi'_n(X^i(s) - \bar{X}^i(s)) \\
&= \psi'_n(X^i(s) - \bar{X}^i(s) + \gamma^i(s, X_s, \cdot) + \bar{\gamma}^i(s, \bar{X}_s, \cdot) - \gamma^i(s, X_s, \cdot) - \bar{\gamma}^i(s, \bar{X}_s, \cdot)) \\
&= \psi'_n(\gamma^i(s, X_s, \cdot) - \gamma^i(s, X_s \wedge \bar{X}_s, \cdot) + (X^i(s) + \gamma^i(s, X_s \wedge \bar{X}_s, \cdot)) - (\bar{X}^i(s) + \bar{\gamma}^i(s, \bar{X}_s, \cdot))) \\
&\leq |\gamma^i(s, X_s, \cdot) - \gamma^i(s, X_s \wedge \bar{X}_s, \cdot)|, \quad \nu \times \mathbb{P}\text{-a.e.}
\end{align*}
Similarly, if $X^i(s) \geq \bar{X}^i(s)$ then (2.7) becomes
\[
\gamma^i(s, X_s \land \bar{X}_s, \cdot) \leq \bar{\gamma}^i(s, \bar{X}_s, \cdot), \quad \nu \times \mathbb{P}\text{-a.e.,}
\]
so that $0 \leq \psi'_n \leq 1$ implies
\[
\psi_n \left( X^i(s) - \bar{X}^i(s) + \gamma^i(s, X_s, \cdot) - \bar{\gamma}^i(s, \bar{X}_s, \cdot) \right) - \psi_n \left( X^i(s) - \bar{X}^i(s) \right) \leq \psi_n \left( X^i(s) - \bar{X}^i(s) + \gamma^i(s, X_s, \cdot) - \bar{\gamma}^i(s, \bar{X}_s, \cdot) \right) - \psi_n \left( X^i(s) - \bar{X}^i(s) \right) \leq \left| \gamma^i(s, X_s, \cdot) - \bar{\gamma}^i(s, X_s, \cdot) \right|, \quad \nu \times \mathbb{P}\text{-a.e.}
\]
Combining these with (A1) we obtain
\[
\mathbb{E} \int_{[t_0, t \land T_k] \times E} \left\{ \psi_n \left( X^i(s) - \bar{X}^i(s) + \gamma^i(s, X_s, z) - \bar{\gamma}^i(s, \bar{X}_s, z) \right) - \psi_n \left( X^i(s) - \bar{X}^i(s) \right) \right\} N(ds, dz) = \mathbb{E} \int_{[t_0, t \land T_k] \times E} \left\{ \psi_n \left( X^i(s) - \bar{X}^i(s) + \gamma^i(s, X_s, z) - \bar{\gamma}^i(s, \bar{X}_s, z) \right) - \psi_n \left( X^i(s) - \bar{X}^i(s) \right) \right\} ds \nu(dz) \leq 2 \mathbb{E} \int_{t_0}^{t \land T_k} ds \int_{E} |\gamma^i(s, X_s, z) - \bar{\gamma}^i(s, X_s, z)| \nu(dz) \leq C(T) \mathbb{E} \int_{t_0}^{t \land T_k} \left\| X_s - X_s \land \bar{X}_s \right\|_{\infty} u(\left\| X_s - X_s \land \bar{X}_s \right\|_{\infty}) ds
\]
for some constant $C(T) > 0$ and all $n \geq 1, t \in [t_0, T]$.

Let
\[
\phi_k(s) = \sup_{r \in [t_0 - r_0, s \land T_k]} \left| X(r) - X(r) \land \bar{X}(r) \right|, \quad s \geq t_0.
\]
By combining (2.3)-(2.8) with $X_{t_0} \leq \bar{X}_{t_0}$ and using the Burkholder-Davis-Gundy inequality, we obtain that for some constant $C(T) > 0$ and all $t \in [t_0, T]$,
\[
\sum_{i=1}^{d} \mathbb{E} \sup_{r \in [t_0 - r_0, t \land T_k]} \psi_n \left( X^i(r) - \bar{X}^i(r) \right) \leq C(T) \int_{t_0}^{t} \mathbb{E} \left\{ \phi_k(s) u(\phi_k(s)) \right\} ds + C(T) \varepsilon(n), \quad k, n \geq 1.
\]
Letting $n \uparrow \infty$ using the Jensen inequality, we derive
\[
\mathbb{E} \phi_k(t) \leq C(T) \int_{t_0}^{t} \left\{ \mathbb{E} \phi_k(s) \right\} u(\mathbb{E} \phi_k(s)) ds, \quad t \in [t_0, T], k \geq 1.
\]
Since $\int_0^1 \frac{1}{\sigma_u(s)}ds = \infty$, by the Bihari inequality this implies that (see e.g. the end of the proof of Theorem 4.2 in [11])

$$
\mathbb{E}\phi_k(T) = 0, \ k \geq 1.
$$

Letting $k \uparrow \infty$ we prove (2.10). \hfill \Box

To prove Theorem 1.2 we need the following Lemma 2.1. For any $h \in C^2_b(\mathbb{R}^d)$, let

$$(Lh)(t, \xi) = \sum_{i=1}^d b_i(t, \xi) \partial_i h(\xi(0)) + \frac{1}{2} \sum_{i,j=1}^d (\sigma \sigma^*)(t, \xi) \partial_i \partial_j h(\xi(0))$$

$$+ \int_E \{h(\xi(0) + \gamma(t, \xi, z)) - h(\xi(0))\} \nu(dz),$$

$$(\bar{L}h)(t, \xi) = \sum_{i=1}^d \bar{b}_i(t, \xi) \partial_i h(\xi(0)) + \frac{1}{2} \sum_{i,j=1}^d (\bar{\sigma} \bar{\sigma}^*)(t, \xi) \partial_i \partial_j h(\xi(0))$$

$$+ \int_E \{h(\xi(0) + \bar{\gamma}(t, \xi, z)) - h(\xi(0))\} \nu(dz), \ t \geq 0, \ \xi \in \mathcal{C},$$

where $\partial_i (1 \leq i \leq d)$ is the derivative with respect to the $i$-th component in $\mathbb{R}^d$. By (A1) and (A2), $Lh$ and $\bar{L}h$ are locally bounded with respect to the usual metric on $[0, \infty)$ and the uniform norm on $\mathcal{C}$.

Let $\mathcal{M}$ be the class of all increasing functions on $\mathcal{C}$, where a function $h$ on $\mathcal{C}$ is called increasing if $h(\xi) \leq h(\eta)$ holds for all $\xi \leq \eta$.

**Lemma 2.1.** Assume (A1) and (A2). If the solutions to (1.1) are order-preserving, then for any $s \geq 0$ and $\mathcal{F}_s$-measurable random variables $\xi, \bar{\xi}$ on $\mathcal{C}$ with $\xi \leq \bar{\xi}$, and any $h \in \mathcal{M} \cap C^2_b(\mathbb{R}^d)$ with $h(\xi(0)) = h(\bar{\xi}(0))$, there holds $\mathbb{P}$-a.s.

$$
\mathbb{E}\left( \liminf_{t \downarrow s} (Lh)(t, X_t(s, \xi)) \big| \mathcal{F}_s \right) \leq \mathbb{E}\left( \limsup_{t \downarrow s} (\bar{L}h)(t, \bar{X}_t(s, \bar{\xi})) \big| \mathcal{F}_s \right),
$$

where $X.(s, \xi)$ and $\bar{X}.(s, \bar{\xi})$ are the segment processes of $X(s, \xi; \cdot)$ and $\bar{X}(s, \bar{\xi}; \cdot)$ respectively.

**Proof.** Simply denote $X(t) = X(s, \xi; t), \bar{X}(t) = X(s, \bar{\xi}; t)$ for $t \geq s - r_0$. Let

$$
\tau = \inf\{t \geq s : |X(t)| + |\bar{X}(t)| \geq 1 + \|\xi\|_\infty + \|\bar{\xi}\|_\infty\}.
$$

Since $h(\xi(0)) = h(\bar{\xi}(0))$ and $X(t) \leq \bar{X}(t)$ for all $t \geq s$, we have

$$(2.9) \quad \mathbb{E}(h(X(t \wedge \tau))) \big| \mathcal{F}_s) - h(\xi(0)) \leq \mathbb{E}(h(\bar{X}(t \wedge \tau))) \big| \mathcal{F}_s) - h(\bar{\xi}(0)), \ t \geq s.$$

By the Itô formula and the Fatou lemma, it is easy to see that

$$(2.10) \quad \liminf_{t \downarrow s} \frac{\mathbb{E}(h(X(t \wedge \tau))) \big| \mathcal{F}_s) - h(\xi(0))}{t - s} \geq \mathbb{E}\left( \liminf_{t \downarrow s} (Lh)(t, X_t) \big| \mathcal{F}_s \right),$$

$$\limsup_{t \downarrow s} \frac{\mathbb{E}(h(\bar{X}(t \wedge \tau))) \big| \mathcal{F}_s) - h(\bar{\xi}(0))}{t - s} \leq \mathbb{E}\left( \limsup_{t \downarrow s} (\bar{L}h)(t, \bar{X}_t) \big| \mathcal{F}_s \right).$$
Combining this with (2.9) we finish the proof.

Below we only prove the first formula in (2.10), as the proof of the second is completely similar. By the Itô formula, for \( t \in (s, s+1] \) we have

\[
\frac{\mathbb{E}(h(X(t \wedge \tau)) \mid \mathcal{F}_s) - h(\xi(0))}{t - s} = \frac{\mathbb{E}(\int_s^{t \wedge \tau} (Lh)(r, X_r) \mid \mathcal{F}_s)}{t - s} \\
\geq \mathbb{E}\left( \mathbb{1}_{\{\tau > t\}} \inf_{r \in (s, t \wedge \tau]} (Lh)(r, X_r) \mid \mathcal{F}_s \right) - C \mathbb{P}(\tau < t \mid \mathcal{F}_s),
\]

where, due to (A1) and (A2),

\[
C := \sup_{\Omega} \{ |Lh|(r, \eta) : r \in [s, s+1], |\eta|_\infty \leq 1 + |\xi|_\infty + |\bar{\xi}|_\infty \} < \infty.
\]

Since \( \tau > s \) due to the right-continuity of the solution, and since \( Lh \) is locally bounded, by letting \( t \downarrow s \) we obtain the first formula in (2.10) from the Fatou lemma.

**Proof of Theorem 1.2** Let 1 \( \leq i \leq d \) and \( t_0 \geq 0 \) be fixed. For any \( \xi, \bar{\xi} \in \mathcal{C} \), let \( X(t) = X(t_0, \xi, ; t), \bar{X}(t) = X(t_0, \bar{\xi}, t) \).

(a) Proof of (III). Let \( \xi \leq \bar{\xi} \). (A1) and (A2) imply that \( b, \sigma \) and \( \int_E (|\gamma(\cdot, \cdot, z)| + |\bar{\gamma}(\cdot, \cdot, z)|) \nu(dz) \) are locally bounded. We aim to prove

\[
(2.11) \quad \xi^i(0) + \gamma^i(t_0, \xi, \cdot) \leq \xi^i(0) + \bar{\gamma}^i(t_0, \bar{\xi}, \cdot), \quad \nu \times \mathbb{P}\text{-}a.e.
\]

Due to the continuity of \( \gamma \) and \( \bar{\gamma} \) in the first two variables and the separability of \([0, \infty) \times \mathcal{C}\) (recall that we use the Skorohod metric on \( \mathcal{C} \)), this implies condition (3).

Let \( \tau = \inf\{ t \geq t_0 : \| X_t - \xi \|_\infty + \| \bar{X}_t - \bar{\xi} \|_\infty \geq 1 \} \). By the Itô formula and the local boundedness of the coefficients and the right-continuity of \( X(s) \), for any \( t > t_0 \) we have

\[
\begin{align*}
\mathbb{E} \psi_n & \{ X^i(t \wedge \tau) - \bar{X}^i(t \wedge \tau) \} \\
& = \mathbb{E} \int_{t_0}^{t \wedge \tau} \left\{ (b^i(s, X_s) - \bar{b}^i(s, \bar{X}_s)) \psi_n'(X^i(s) - \bar{X}^i(s)) \\
& + \sum_{j=1}^m (\sigma^{ij}(s, X_s) - \bar{\sigma}^{ij}(s, \bar{X}_s))^2 \psi_n''(X^i(s) - \bar{X}^i(s)) \\
& + \int_E [\psi_n(X^i(s) - \bar{X}^i(s) + \gamma^i(s, X_s, z) - \bar{\gamma}^i(s, \bar{X}_s, z)) - \psi_n(X^i(s) - \bar{X}^i(s))] \nu(dz) \right\} ds.
\end{align*}
\]

Since \( X^i(s) \leq \bar{X}^i(s), X^i(t \wedge \tau) \leq \bar{X}^i(t \wedge \tau) \) and \( \psi_n'(s) = \psi_n''(s) = 0 \) for \( s \leq 0 \), this implies

\[
\mathbb{E} \int_{t_0}^{t \wedge \tau} \left\{ \int_E [\psi_n(X^i(s) - \bar{X}^i(s) + \gamma^i(s, X_s, z) - \bar{\gamma}^i(s, \bar{X}_s, z)) \nu(dz)] \right\} ds = 0 \quad t \geq t_0.
\]

By \( \tau > t_0 \), the right-continuity of the solutions, and the joint-continuity of \( \gamma^i \) and \( \bar{\gamma}^i \) in the first two variables, from this and the Fatou lemma we conclude that

\[
\mathbb{E} \int_E [\xi^i(0) - \bar{\xi}^i(0) + \gamma^i(t_0, \xi, z) - \bar{\gamma}^i(t_0, \bar{\xi}, z)] \nu(dz) = 0.
\]
Since $\psi_n(s) \uparrow s^+$ as $n \uparrow \infty$, by letting $n \uparrow \infty$ we arrive at

$$
\mathbb{E} \int_{E} (\xi^i(0) - \bar{\xi}^i(0) + \gamma^i(t_0, \xi, z) - \bar{\gamma}^i(t_0, \bar{\xi}, z))^+ \nu(dz) = 0,
$$

and hence (2.11) holds.

(b) Proof of (I). Let $\xi \leq \bar{\xi}$ and $\xi^i(0) = \bar{\xi}^i(0)$. For any $\varepsilon \in (0, 1)$, let $\phi_\varepsilon \in C_0^\infty(\mathbb{R})$ such that

$$
0 \leq \phi_\varepsilon \leq 1, \quad \phi_\varepsilon|_{[-\varepsilon, \varepsilon]} = 1, \quad \phi_\varepsilon|_{[-2\varepsilon, 2\varepsilon]} = 0.
$$

Take

$$
h_\varepsilon(x) = \int_0^{x-\xi^i(0)} \phi_\varepsilon(s)ds, \quad x \in \mathbb{R}^d.
$$

Then $h_\varepsilon \in \mathcal{M} \cap C^2_b(\mathbb{R}^d)$ and by the continuity of $b, \bar{b}$ and the right continuity of the solutions,

$$
\lim_{t \downarrow t_0} \langle b(t, X_t), \nabla h_\varepsilon(X(t)) \rangle = b_0(t_0, \xi), \quad \lim_{t \downarrow t_0} \langle \bar{b}(t, \bar{X}_t), \nabla h_\varepsilon(\bar{X}(t)) \rangle = \bar{b}_0(t_0, \bar{\xi}),
$$

$$
\lim_{t \downarrow t_0} \nabla^2 h_\varepsilon(X(t)) = \lim_{t \downarrow t_0} \nabla^2 h_\varepsilon(\bar{X}(t)) = 0,
$$

$$
|h_\varepsilon(X(t) + \gamma(t, X_t, z)) - h_\varepsilon(X(t))| + |h_\varepsilon(\bar{X}(t) + \bar{\gamma}(t, \bar{X}_t, z)) - h_\varepsilon(\bar{X}(t))| \leq (4\varepsilon) \wedge (|\gamma(t, X_t, z)| + |\gamma(t, \bar{X}_t, z)|).
$$

Combining this with Lemma 2.11 we obtain $\mathbb{P}$-a.s.

$$
b^i(t_0, \xi) \leq \bar{b}^i(t_0, \bar{\xi}) + \sup_{t \in [t_0, t_0 + 1], \|\eta\|_\infty \vee \|\xi\|_\infty \leq 1 + \|\xi\|_\infty \vee \|\bar{\xi}\|_\infty} \int_E \left\{ (4\varepsilon) \wedge (|\gamma(t, \eta, z)| + |\gamma(t, \bar{\eta}, \bar{z})|) \right\} \nu(dz).
$$

Letting $\varepsilon \to 0$ and using (1.2) we prove $b^i(t_0, \xi) \leq \bar{b}^i(t_0, \bar{\xi}), \mathbb{P}$-a.s. This implies condition (1) by the continuity of $b, \bar{b}$ and the separability of $(0, \infty) \times \mathcal{G}$.

(c) Proof (II). If condition (2) does not hold, then there exist $\xi, \bar{\xi} \in \mathcal{G}$ with $\xi^i(0) = \bar{\xi}^i(0)$ such that for some $1 \leq j \leq m$ one has $\mathbb{P}(\sigma^j(t_0, \xi) \neq \bar{\sigma}^j(t_0, \bar{\xi})) > 0$. Since $\sigma$ and $\bar{\sigma}$ are continuous, there exists a constant $\varepsilon \in (0, 1)$ such that $\mathbb{P}(A_\varepsilon) > 0$, where

$$
A_\varepsilon := \left\{ |\sigma^j(t, \eta) - \bar{\sigma}^j(t, \bar{\eta})| \geq \varepsilon \text{ for } t \in [t_0, t_0 + \varepsilon], \|\eta - \xi\|_\infty + \|\bar{\eta} - \bar{\xi}\|_\infty \leq \varepsilon \right\}.
$$

Let

$$
\tilde{\tau} = \inf\{t \geq t_0 : |\sigma^j(t, X_t) - \bar{\sigma}^j(t, \bar{X}_t)| \leq \varepsilon\},
$$

$$
\tau = \inf\{t \geq t_0 : \|X_t - \xi\|_\infty + \|\bar{X}_t - \bar{\xi}\|_\infty \geq \varepsilon\},
$$

$$
\tau_n = \inf\{t \geq t_0 : \|X^i(t) - \bar{X}^i(t)\| \geq n^{-1}\}, \quad n \geq 1.
$$

Let $g_n(s) = e^{ns} - 1$. Since $X^i_s \leq \bar{X}^i_s$, and due to (a)

$$
X^i(s) - \bar{X}^i(s) + \gamma^i(s, X_s, z) - \bar{\gamma}^i(s, \bar{X}_s, z) \leq 0, \quad s \geq t_0,
$$

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by the Itô formula we obtain
\[ 0 \geq \mathbb{E} g_n((X^i - \bar{X}^i)(t_0 + \varepsilon) \wedge \bar{\tau} \wedge \tau \wedge \tau_n)) \]
\[ = \mathbb{E} \int_{t_0}^{(t_0 + \varepsilon) \wedge \bar{\tau} \wedge \tau \wedge \tau_n} \left\{ g'_n((X^i - \bar{X}^i)(s))(b'(s, X_s) - \bar{b}'(s, \bar{X}_s)) + \frac{g''_n(X^i(s) - \bar{X}^i(s))}{2} \sum_{j=1}^{m} (\sigma_{ij}(s, X_s) - \bar{\sigma}_{ij}(s, \bar{X}_s))^2 \right\} \, ds \]
\[ + \int_E \left\{ g_n(X^i(s) - \bar{X}^i(s) + \gamma^i(s, X_s, z) - \bar{\gamma}^i(s, \bar{X}_s, z)) - g_n(X^i(s) - \bar{X}^i(s)) \right\} \nu(dz) \, ds \]
\[ \geq \left( \frac{n^2 \varepsilon^2}{2e} - C n \right) \mathbb{E} \left\{ \varepsilon \wedge (\bar{\tau} - t_0) \wedge (\tau - t_0) \wedge (\tau_n - t_0) \right\}, \quad n \geq 1, \]
where, according to (A1) and (A2),
\[ C := \sup \left\{ |b'(t, \eta) - \bar{b}'(t, \bar{\eta})| + \int_E |\gamma^i(t, \eta, z) - \bar{\gamma}^i(t, \bar{\eta}, z)| \nu(dz) : \right. \]
\[ t \in [t_0, t_0 + \varepsilon], \|\eta - \xi\|_{\infty} + \|\bar{\eta} - \bar{\xi}\|_{\infty} \leq \varepsilon \} < \infty. \]
This implies \( \mathbb{E}((\bar{\tau} - t_0) \wedge (\tau - t_0) \wedge (\tau_n - t_0)) = 0 \) for large \( n \), which is impossible since \( \mathbb{P}(A_{\varepsilon}) > 0 \) and due to the right-continuity of the solutions \( \bar{\tau} \wedge \tau_n \wedge \tau > t_0 \) holds on the set \( A_{\varepsilon} \).

3 Existence and uniqueness of solutions

When \( N = 0 \) and \( b, \sigma \) are deterministic, the following result is included in [11] Theorem 4.2. The appearance of \( N \) makes the solution discontinuous, so that the argument in the proof of [11] Theorem 4.2] leading to the existence of weak solutions by proving the tightness of the approximating solutions is no longer valid. Moreover, since the coefficients are now random, the Yamada-Watanabe principle used there is invalid neither. Due to (A1) and (A2), the proof of the uniqueness and non-explosion is standard. To prove the existence, we approximate the original equation by those with Lipschitz coefficients, and construct a strong solution to the original equation by proving that the approximating solutions form a Cauchy sequence under the topology of locally uniform convergence.

**Theorem 3.1.** Let \( b, \sigma, \gamma \) satisfy (A1) and (A2) with \( \bar{b} = 0, \bar{\sigma} = 0 \) and \( \bar{\gamma} = 0 \). Then for any \( s \geq 0 \) and \( \mathcal{F}_s \)-measurable random variable \( \xi \) on \( \mathcal{G} \), the equation
\[ dX(t) = b(t, X_t)dt + \sigma(t, X_t)dB(t) + \int_E \gamma(t, X_t, z)N(dt, dz), \quad t \geq s, X_s = \xi \]
has a unique solution which satisfies
\[ \mathbb{E} \left( \sup_{t \in [s - \bar{\tau}_0, T]} |X(t)| \left| \mathcal{F}_s \right\rangle < \infty, \quad T > s. \]
Proof. Without loss of generality, we only prove for $s = 0$ and simply denote $E_0 = E(\cdot | \mathcal{F}_0)$.

(a) $E_0 \sup_{t \leq T} |X(t)| < \infty$ for any $T > 0$. Let $X(t)$ be a solution to the equation. Let

$$\tau_n = \inf \{ t \geq 0 : |X(t)| \geq \| \xi \|_\infty + n \}, \quad n \geq 1.$$ 

By the Itô formula, the Burkholder inequality, (A1), (A2), and the Jensen inequality, for any $T > 0$ we may find a constant $C(T) > 0$ such that for any $n \geq 1$, the process $\phi_n(t) := E_0 \sup_{s \leq t} |X(s \wedge \tau_n)|$ satisfies

$$E_0 \phi_n(t) \leq C(T) + \| \xi \|_\infty + C(T) \int_0^t E_0 \phi_n(s) u(\mathbb{E}_0 \phi_n(s)) ds, \quad t \in [0, T].$$

Let $G(s) = \int_1^s \frac{1}{r_\nu(r)} dr, s > 0$. By the Bihari inequality we have

$$E_0 \phi_n(t) \leq G^{-1}(G(C(T) + \| \xi \|_\infty + C(T)t) < \infty, \quad t \in [0, T].$$

Letting $n \uparrow \infty$, we conclude that $\tau_n \uparrow \infty$ and $E_0 \sup_{t \leq T} |X(t)| < \infty$.

(b) The uniqueness of the solution. Let $X(t)$ and $\tilde{X}(t)$ be two solutions to the equation with the same initial data $X_0$. By the Itô formula, the Burkholder inequality, (A1) and (A2), and the Jensen inequality, for any $T > 0$ we may find a constant $C(T) > 0$ such that the process $\phi(t) := \sup_{s \leq t} |X(s) - \tilde{X}(s)|$ satisfies

$$E_0 \phi(t) \leq C(T) \int_0^t E_0 \phi(s) u(\mathbb{E}_0 \phi(s)) ds, \quad t \in [0, T].$$

Since $\int_0^1 \frac{1}{r_\nu(s)} ds = \infty$, by the Bihari inequality we conclude that $E_0 \phi(t) = 0$ for $t \in [0, T]$. This implies that $X(t) = \tilde{X}(t)$ for all $t \geq 0$ since $T > 0$ is arbitrary.

(c) Existence of the solution for bounded $b, \sigma$ and $\theta := \int_E |\gamma(\cdot, \cdot, z)| \nu(dz)$. If $u \equiv 1$, i.e. the coefficients are Lipschitz continuous in $\xi \in \mathcal{C}$ with respect to the uniform norm, then the existence and uniqueness of the solutions can be proved by a standard argument (cf. [13]). To prove the existence of the solution, we approximate the coefficients by using Lipschitz ones as follows. Let $\mu$ be the distribution of the $\mathcal{C}$-valued random variable $B$ with $B(s) := \tilde{B}(r_0 + 1 + s), s \in [-r_0, 0]$, where $\tilde{B}(s)$ is a $d$-dimensional Brownian motion with $\tilde{B}(0) = 0$. For any $n \geq 1$, let

$$b_n(t, \xi) = \int_{\mathcal{C}} b(t, \xi + n^{-1} \eta) \mu(d\eta), \quad \sigma_n(t, \xi) = \int_{\mathcal{C}} \sigma(t, \xi + n^{-1} \eta) \mu(d\eta),$$

$$\gamma_n(t, \xi, z) = \int_E \gamma(t, \xi + n^{-1} \eta, z) \mu(dz), \quad t \geq 0, \xi \in \mathcal{C}, z \in E.$$

Since $b, \sigma$ and $\theta$ are bounded, applying [2, Corollary 1.3] for $\sigma = \frac{1}{n} I_d \times d, m = 0, Z = b = 0$ and $T = 1 + r_0$, we conclude that for any $n \geq 1$,

$$|b_n(t, \xi) - b_n(t, \eta)| + \|\sigma_n(t, \xi) - \sigma_n(t, \eta)\|_{HS} + \int_E |\gamma_n(t, \xi, z) - \gamma_n(t, \eta, z)| \nu(dz) \leq K_n(t) \|\xi - \eta\|_\infty.$$
holds for some positive $K_n \in C([0, \infty))$. Therefore, the equation

$$
(3.1) \quad dX^{(n)}(t) = b_n(t, X^{(n)}_t)dt + \sigma_n(t, X^{(n)}_t)dB(t) + \int_E \gamma_n(t, X^{(n)}_t, z)N(dt, dz)
$$

for $X^{(n)}_0 = \xi$ has a unique solution. Moreover, by the Jensen inequality, we see that (A1) and (A2) hold for $b_n, \sigma_n$ and $\gamma_n$ uniformly in $n \geq 1$.

Next, by (A1) we may find a positive function $K \in C([0, \infty))$ such that

$$
|b_n(t, \xi) - b(t, \xi)| + \|\sigma_n(t, \xi) - \sigma(t, \xi)\|_{HS} + \int_E |\gamma_n(t, \xi, z) - \gamma(t, \xi, z)|\nu(dz) \leq K(\varepsilon_{n,t}),
$$

where, according to $\mu(\cdot; \cdot) < \infty$, $su(s) \leq c(1 + s)$ for some constant $c > 0$, and $su(s) \to 0$ as $s \to 0$,

$$
\varepsilon_{n,t} := \int_E \|(n^{-1} - l^{-1})\eta\|_{\infty}u(|(n^{-1} - l^{-1})\eta|_{\infty})\mu(d\eta) \to 0 \quad \text{as} \quad n, l \to \infty.
$$

Combining this with (A1) we obtain

$$
(3.2) \quad |b_n(t, \xi) - b(t, \eta)| + \|\sigma_n(t, \xi) - \sigma(t, \eta)\|_{HS} + \int_E |\gamma_n(t, \xi, z) - \gamma(t, \eta, z)|\nu(dz)
$$

$$
\leq K(t)\varepsilon_{n,t} + |b_n(t, \xi) - b_n(t, \eta)| + \|\sigma_n(t, \xi) - \sigma_n(t, \eta)\|_{HS}
$$

$$
+ \int_E |\gamma_n(t, \xi, z) - \gamma_n(t, \eta, z)|\nu(dz)
$$

$$
\leq K(t)\varepsilon_{n,t} + K(t)\|\xi - \eta\|_{\infty}u(|\xi - \eta|_{\infty}), \quad t \geq 0, \xi, \eta \in \mathcal{C}
$$

for some positive $K \in C([0, \infty))$. Moreover, since (A1) and (A2) hold for $b_n, \sigma_n$ and $\gamma_n$ uniformly in $n$, by (a) we have

$$
(3.3) \quad \sup_{n \geq 1} \sup_{t \leq T} |X^{(n)}(t)| < \infty, \quad T > 0.
$$

Now, as in (a) and (b), by the Itô formula, the Burkholder inequality, (A1) and (A2) holding for $b_n, \sigma_n$ and $\gamma_n$ uniformly in $n$, the Jensen inequality and (3.2), for any $T > 0$ we may find a constant $C(T) > 0$ such that the process $\phi_{n,t}(t) := \sup_{s \leq t} |X^{(n)}(s) - X^{(l)}(s)|$ satisfies

$$
\mathbb{E} \phi_{n,t}(t) \leq C(T) \int_0^t \mathbb{E} \phi_{n,t}(s)u(\mathbb{E} \phi_{n,t}(s))ds + C(T)\varepsilon(n, l), \quad t \in [0, T].
$$

Since $\varepsilon(n, l) \to 0$ as $n, l \to \infty$, by the Bihari inequality and $\int_0^1 \frac{1}{su(s)}ds = \infty$, we obtain

$$
\lim_{n, l \to \infty} \mathbb{E} \sup_{t \leq T} |X^{(n)}(t) - X^{(l)}(t)| = 0, \quad T > 0.
$$

Therefore, as $n \to \infty$ the process $X^{(n)}$ converges locally uniformly to a process $X$, which solves the first equation in (1.1) according to (A1), (3.3) and the facts that $su(s) \leq c(1 + s)$ for some constant $c > 0$ and $su(s) \to 0$ as $s \to 0$. 

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(d) Existence of the solution for unbounded $b, \sigma$ and $\theta$. For any $n \geq 1$, let $\vec{n} = (n, \cdots, n) \in \mathbb{R}^d$. Define
\[ \alpha_n(\xi) = (\xi \wedge \vec{n}) \vee (-\vec{n}), \quad n \geq 1, \xi \in \mathcal{C}. \]
Let
\[ b_n(t, \xi) = b(t \wedge n, \alpha_n(\xi)), \quad \sigma_n(t, \xi) = \sigma(t \wedge n, \alpha_n(\xi)), \quad \gamma_n(t, \xi, z) = \gamma(t \wedge n, \alpha_n(\xi), z). \]
Then $b_n, \sigma_n$ and $\theta_n := \int_E |\gamma_n(\cdot, \cdot, z)|\nu(dz)$ are bounded and satisfy (A1) and (A2). Thus, according to (a)-(c), the equation (3.1) with $X_0^{(n)} = \xi$ has a unique solution $X^{(n)}(t), t \geq 0$. Since for any $l \geq n \geq 1$ and $\xi \in \mathcal{C}$ with $\|\xi\|_{\infty} \leq n$ we have
\[ b_n(t, \xi) = b_l(t, \xi), \quad \sigma_n(t, \xi) = \sigma_l(t, \xi), \quad \gamma_n(t, \xi, z) = \gamma_l(t, \xi, z), \quad t \in [0, n], \]
by the uniqueness one has $X^{(n)}(t) = X^{(l)}(t)$ for $t \leq \tau_n$, where
\[ \tau_n := n \wedge \inf\{t \geq 0 : \|X_t^{(n)}\|_{\infty} \geq n\}. \]
Moreover, as in (a) we can prove that $\tau_n \uparrow \infty$ as $n \uparrow \infty$. Therefore, $X(t) := X^{(n)}(t)$ if $t < \tau_n$ gives rise to a solution of the original equation. \qed

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