POINCARÉ SERIES OF CHARACTER VARIETIES FOR NILPOTENT GROUPS

MENTOR STAF A

ABSTRACT. For any compact and connected Lie group $G$ and any free abelian or free nilpotent group $\Gamma$, we determine the cohomology of the path component of the trivial representation of the representation space (character variety) $\text{Rep}(\Gamma, G)_{1}$, with coefficients in a field $F$ with char$(F)$ either 0 or relatively prime to the order of the Weyl group $W$. We give explicit formulas for the Poincaré series. In addition we study $G$-equivariant stable decompositions of subspaces $X(q, G)$ of the free monoid $J(G)$ generated by the Lie group $G$, obtained from finitely generated free nilpotent group representations.

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1. Introduction

Let $G$ be a compact and connected Lie group and $\Gamma$ be a finitely generated discrete group. In this article we study representation spaces given by orbit spaces $\text{Rep}(\Gamma, G) = \text{Hom}(\Gamma, G)/G$ obtained from spaces of homomorphisms $\text{Hom}(\Gamma, G) \subseteq G^n$ endowed with the subspace topology of $G^n$, modulo the conjugation action of $G$. If $G$ is compact the representation space $\text{Rep}(\Gamma, G)$ coincides with the character variety $\text{Hom}(\Gamma, G)//G$, sometimes also denoted by $X_G(\Gamma)$. On the other hand, if $G$ is the group of complex or real points of a reductive linear algebraic group and
$C \subseteq G$ is a maximal compact subgroup, Bergeron [6, Theorem II] showed that for $\Gamma$ finitely generated nilpotent, there is a homotopy equivalence $\text{Hom}(\Gamma, G)/G \simeq \text{Hom}(\Gamma, C)/C$ (for non-examples see [2]). Of particular interest here are the cases when $\Gamma$ is either a free abelian or free nilpotent group. For these groups we give a complete answer for the Poincaré series of the connected components of the trivial representation $\text{Rep}(\Gamma, G)_1 \subseteq \text{Rep}(\Gamma, G)$.

The answer depends only on the Weyl group $W$ of $G$ and its action on the maximal torus $T$. Ramras and Stafa [20] give an analogous complete answer for the Poincaré series of the connected components of the trivial representation $\text{Hom}(\Gamma, G)_1 \subseteq \text{Hom}(\Gamma, G)$. The answer for $\text{Hom}(\Gamma, G)_1$ is governed by the maximal torus $T$ of rank $r$, the Weyl group $W$ and its corresponding characteristic degrees $d_1, \ldots, d_r$ as follows

$$P(\text{Hom}(\mathbb{Z}^n, G)_1; q) = |W|^{-1} \prod_{i=1}^r (1 - q^{2d_i}) \sum_{w \in W} \frac{\det(1 + qw)^n}{\det(1 - q^2w)}.$$ 

In our case characteristic degrees do not appear in the Poincaré series of $\text{Rep}(\Gamma, G)_1$.

The topology of the spaces $\text{Hom}(\Gamma, G)$ and $\text{Rep}(\Gamma, G)$ for free abelian and nilpotent groups (and certainly other discrete groups) has attracted considerable attention recently [13]. The space of homomorphisms $\text{Hom}(\mathbb{Z}^n, G)$ is known as the space of ordered pairwise commuting $n$-tuples in $G$, and the representation space $\text{Rep}(\mathbb{Z}^n, G)$ can be identified with the moduli space of isomorphism classes of flat connections on principal $G$-bundles over the $n$-torus. These spaces and their variations including the space of almost commuting elements [9], have also been studied in various settings outside topology, most notably including work of Witten [25, 26] and Kac–Smilga [18] on supersymmetric Yang-Mills theory.

**Statements of main results.** Let $F_n$ be the free group on $n$ letters, with descending central series given by $\cdots \subseteq \Gamma_3 \subseteq \Gamma_2 \subseteq \Gamma_1 = F_n$. Then there is a filtration of $G^n$ by spaces of homomorphisms

$$\text{Hom}(F_n/\Gamma^2, G) \subseteq \text{Hom}(F_n/\Gamma^3, G) \subseteq \text{Hom}(F_n/\Gamma^4, G) \subseteq \cdots \subseteq G^n.$$ 

The main variation of the spaces $\text{Hom}(F_n/\Gamma^n, G)$ studied in this paper are spaces of almost commuting or almost nilpotent $n$-tuples in $G$, which are defined in Section 3. Let $J(G)$ be the free monoid generated by $G$, also known as the James reduced product on $G$. The descending central series above can be used to define a sequence of spaces $X(q, G)$ that filter $J(G)$

$$X(2, G) \subseteq X(3, G) \subseteq X(4, G) \subseteq \cdots \subseteq J(G)$$

also defined in Section 3, previously defined and used in work of Cohen–Stafa [14] and Ramras–Stafa [20]. In particular, we denote the space $X(2, G)$ by $\text{Comm}(G)$. $G$ acts by conjugation on the elements in $\text{Hom}(F_n/\Gamma^n, G)$ and elements (or words) in $X(q, G)$. We show that after one suspension the orbit spaces $X(q, G)/G$ decompose into infinite wedge sums of smaller spaces.

**Theorem 1.1.** Let $G$ be a compact and connected Lie group. For each $q \geq 2$ there is a homotopy equivalence

$$\Sigma(X(q, G)/G) \simeq \bigvee_{n \geq 1} \Sigma \widehat{\text{Hom}}(F_n/\Gamma^n, G)/G.$$ 

Here the space $\widehat{\text{Hom}}(F_n/\Gamma^n, G)$ is the quotient of the space of homomorphisms $\text{Hom}(F_n/\Gamma^n, G)$ by the subspace consisting of $n$-tuples with at least one coordinate
the identity. Note that these decompositions hold also for the component of the trivial representation, which we denote by $X(q, G)_1$ and $\text{Comm}(G)_1$, as observed for instance in [20]. This is explained in more detail below. The main theorem that makes it possible for the calculation of the Poincaré series is the following.

**Theorem 1.2.** Let $G$ be a compact and connected Lie group with maximal torus $T$ and Weyl group $W$. Then there is a homeomorphism

$$\text{Comm}(G)_1 / G \approx J(T) / W$$

and a homotopy equivalence

$$\Sigma(\text{Comm}(G)_1 / G) \simeq \bigvee_{n \geq 1} \left( \hat{T}^n / W \right).$$

Using this theorem and a few other observations, we describe the Poincaré series for representation spaces and their infinite dimensional analogues.

**Theorem 1.3.** Let $G$ be a compact and connected Lie group with maximal torus $T$ and Weyl group $W$. Then the Poincaré series of $\text{Comm}(G)_1 / G$ is given by

$$P(\text{Comm}(G)_1 / G; s) = \frac{1}{|W|} \sum_{w \in W} \left( \sum_{k \geq 0} (\det(1 + sw) - 1)^k \right).$$

Moreover, we have $P(X(q, G)_1 / G; s) = P(\text{Comm}(G)_1 / G; s)$ for all $q \geq 2$.

From this theorem and the stable decompositions in Theorem 1.1 of the spaces $X(q, G)_1 / G$ and their corresponding Poincaré series in Theorem 1.3, we finally show the main theorem.

**Theorem 1.4.** Let $G$ be a compact and connected Lie group with maximal torus $T$ and Weyl group $W$. Then the Poincaré series of $\text{Rep}(\mathbb{Z}^n, G)_1$ is given by

$$P(\text{Rep}(\mathbb{Z}^n, G)_1; s) = \frac{1}{|W|} \sum_{w \in W} \det(1 + sw)^n.$$

Moreover, we have $P(\text{Rep}(F_n / \Gamma^q, G)_1; s) = P(\text{Rep}(\mathbb{Z}^n, G)_1; s)$ for all $q \geq 2$.

Bergeron and Silberman [7] show that if $\Gamma$ is a finitely generated nilpotent group, then $\text{Rep}(\Gamma, G)_1$ has the same homology as $\text{Rep}(\Gamma / \Gamma^q, G)_1$ with coefficients in a field with characteristic not dividing $|W|$. Therefore, Theorem 1.4 applies also to all such $\Gamma$, in particular to free nilpotent groups $F_n / \Gamma^q$.

**Structure of the paper.** In Section 3 we first define and study topological properties of a family of spaces $\mathcal{B}_n(q, G, K) \subseteq G^n$ and their infinite dimensional analogue $X(q, G, K) \subset J(G)$. In Section 4 we show that after one suspension we obtain a $G$-equivariant stable decomposition for each of these spaces, proving also Theorem 1.1. In section 5 we describe the stable decomposition and the homeomorphism in Theorem 1.2. We describe the Hilbert-Poincaré series for these representation spaces and prove Theorem 1.3 and Theorem 1.4 in Section 6. The last section is devoted to examples.

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2. Character varieties and representation spaces

In this section we mention some standard notions about character varieties and representation spaces. Let $G$ be a compact Lie group. By Peter-Weyl theorem there is a faithful embedding $G \hookrightarrow \text{GL}(n, \mathbb{R})$ for sufficiently large $n$. This gives $G$ the structure of a linear algebraic group. The complexification of $G$ is the group $G' = G_{\mathbb{C}}$ given by the zero locus in $\text{GL}(n, \mathbb{C})$ of the defining ideal of $G$. The group $G'$ is called a complex algebraic group and is independent of the embedding $G \hookrightarrow \text{GL}(n, \mathbb{R})$ given by Peter-Weyl. For example the compact Lie group $\text{SU}(n)$ can be seen as a subgroup of $\text{GL}(2n, \mathbb{R})$ with complexification $\text{SL}(n, \mathbb{C}) \subset \text{GL}(2n, \mathbb{C})$. It is a well-known fact that the inclusion $G \hookrightarrow G'$ of $G$ to its complexification $G'$ is a homotopy equivalence. Another fact that we should mention is the relation between complex reductive algebraic groups and compact Lie groups. Namely, a complex linear algebraic group is reductive if and only if it is the complexification of a compact Lie group. For details see for instance Onischick and Vinberg [19].

Let $V$ be a complex vector space, $G \subset \text{GL}(V)$ a complex reductive algebraic group, and $X \subset V$ a $G$-variety upon which $G$ acts algebraically. This endows the ring of regular functions $\mathbb{C}[X]$ with an action of $G$, with corresponding ring of $G$-invariant regular functions denoted $\mathbb{C}[X]^G$. The affine geometric invariant theory quotient, or GIT quotient of $X$ is defined by

$$X//G := \text{Spec}(\mathbb{C}[X]^G)$$

consisting of the closed $G$-orbits.

If $G$ is a complex reductive linear algebraic group and $\Gamma$ is finitely generated, then the space of homomorphisms $\text{Hom}(\Gamma, G)$ has the structure of an affine algebraic variety endowed with an action of $G$ by conjugation. If $\Gamma$ has $n$-generators, then $\text{Hom}(\Gamma, G)$ can be seen as a subspace of $G^n$, by identifying each homomorphism $f : \Gamma \to G$ with the $n$-tuple given by the image of the generators of $\Gamma$ under $f$. The space $\text{Hom}(\Gamma, G) \subset G^n$ can then be given the subspace topology. Moreover, the action of $G$ on $\text{Hom}(\Gamma, G)$ corresponds to the restriction of the diagonal conjugation action of $G$ on the product $G^n$. The GIT quotient

$$X_G(\Gamma) := \text{Hom}(\Gamma, G) // G$$

is called the $G$-character variety of $\Gamma$ and the topological orbit space

$$\text{Rep}(\Gamma, G) := \text{Hom}(\Gamma, G) / G$$

is called the representation space of $\Gamma$ into $G$. If $G$ is compact then $X_G(\Gamma) = \text{Rep}(\Gamma, G)$. Moreover, it was shown by Florentino and Lawton [15] that if $K \leq G$ is a maximal compact subgroup, then $\text{Hom}(\mathbb{Z}^n, G)/G$ deformation retracts onto $\text{Hom}(\mathbb{Z}^n, K)/K$. In this article we always assume that $G$ is compact and connected, but this discussion shows that results about representation spaces will imply the same results about the corresponding character varieties.

3. Spaces of homomorphisms and James construction

For a group $Q$ define the subgroups $\Gamma^k(Q) \leq Q$ inductively by setting $\Gamma^1(Q) = Q$, and for all $i > 0$ define $\Gamma^{i+1}(Q) := [\Gamma^i(Q), Q]$. This way we obtain a normal series

$$\cdots \leq \Gamma^{i+1}(Q) \leq \Gamma^i(Q) \leq \cdots \leq \Gamma^2(Q) \leq \Gamma^1(Q) = Q$$

called the descending central series of $Q$. If $Q$ is the the free group $F_n$ we write $\Gamma^q = \Gamma^q(F_n)$. Using this normal series we define filtrations of $G^n$ by spaces of
homomorphisms, and related filtrations of the infinite dimensional analogue space \( J(G) \), which can be seen as a free monoid with prescribed topology, also known as the James reduced product on \( G \).

3.1. **Subspaces of \( G^n \).** Suppose \( K \leq G \) is a closed central subgroup of the compact and connected Lie group \( G \), that is \( K \leq Z(G) \). Borel–Friedman–Morgan [9] and Adem–Cohen–Gomez [3] studied the spaces of almost commuting tuples in \( G \) to study the spaces of commuting tuples. They are defined as follows. A \( K \)-almost commuting \( n \)-tuple in \( G \) is an \( n \)-tuple \( (g_1, \ldots, g_n) \in G^n \) such that \([g_i, g_j] \in K\). Therefore, the space of \( K \)-almost commuting \( n \)-tuples in \( G \) is given by

\[
\mathcal{B}_n(G, K) := \{(g_1, \ldots, g_n) \in G^n : [g_i, g_j] \in K \text{ for all } i, j \}
\]

with the subspace topology of \( G^n \). This space can also be seen as the space of homomorphisms \( f : F_n \to G \) such that \( f(\Gamma^2) \subseteq K \). This point of view will allow us to define other variations of this space. In particular, if \( K = 1_G \) then \( \mathcal{B}_n(G, K) = \text{Hom}(Z^n, G) \). Moreover, as noted for instance in [3], the quotient of Lie groups \( G \to G/K \) gives a principal \( K^n \)-bundle

\[
K^n \to G^n \to (G/K)^n
\]

which restricts to \( K \)-almost commuting tuples to give a principal \( K^n \) bundle

\[
K^n \to \mathcal{B}_n(G, K) \to \text{Hom}(Z^n, G/K),
\]

since a homomorphism satisfying \( f(\Gamma^2) \subseteq \{1_G\} \) is equivalent to a homomorphism \( f : Z^n \to G/K \). Similarly, define the spaces \( \mathcal{B}_n(q, G, K) \subset G^n \) by

\[
\mathcal{B}_n(q, G, K) := \{f : F_n \to G \mid f(\Gamma^q) \subseteq K\} \subset G^n,
\]

where \( q = 2 \) gives \( \mathcal{B}_n(2, G, K) = \mathcal{B}_n(G, K) \subset G^n \). For fixed \( n \) and varying values of \( q \), we get a filtration of \( G^n \)

\[
\mathcal{B}_n(2, G, K) \subset \mathcal{B}_n(3, G, K) \subset \mathcal{B}_n(4, G, K) \subset \cdots \subset G^n,
\]

and for \( K = 1_G \) we obtain the ordinary filtration of \( G^n \)

\[
\text{Hom}(F_n/\Gamma^2, G) \subset \text{Hom}(F_n/\Gamma^3, G) \subset \text{Hom}(F_n/\Gamma^4, G) \subset \cdots \subset G^n.
\]

A special version of the following lemma was shown in [3, Lemma 2.3].

**Lemma 3.1.** Let \( G \) be a compact and connected Lie group and \( K \leq Z(G) \) a closed subgroup. Then for any \( q \geq 2 \) the bundle \( G^n \to (G/K)^n \) induces a principal \( K^n \)-bundle

\[
\varphi_n : \mathcal{B}_n(q, G, K) \to \text{Hom}(F_n/\Gamma^q, G/K).
\]

**Proof.** The case when \( q = 2 \) was proved in [3, Lemma 2.3]. The proof for other values of \( q \) is the same by noting that a homomorphism \( f : F_n \to G \) satisfying \( f(\Gamma^q) = \{1_G\} \) is equivalent to a homomorphism \( f : F_n/\Gamma^q \to G/K \).

The spaces \( \mathcal{B}_n(q, G, K) \) have the structure of simplicial spaces as follows. The Cartesian products \( \mathcal{B}_nG := G^n \), for varying \( n \), have the structure of simplicial spaces. That is, there are standard face and degeneracy maps

\[
\sigma_i(g_1, \ldots, g_n) = (g_1, \ldots, g_i, 1_G, g_{i+1}, \ldots, g_n), \text{ and}
\]

\[
\delta_j(g_1, \ldots, g_n) = \begin{cases} 
(g_2, \ldots, g_n) & \text{if } j = 0, \\
(g_1, \ldots, g_jg_{j+1}, \ldots, g_n) & \text{if } 0 < j < n, \\
(g_1, \ldots, g_{n-1}) & \text{if } j = n,
\end{cases}
\]
that satisfy the simplicial relations. Recall that the geometric realization of $B_*G$ is the classifying space of $G$. The same face and degeneracy maps together with the relations restricted to $B_*(q, G, K)$ make the spaces $B_*(q, G, K)$ into $G$-simplicial spaces since

- the maps $\sigma_i$ and $\delta_j$ preserve the spaces $B_*(q, G, K)$ as they are induced by homomorphisms between free groups,
- these group homomorphisms satisfy the property $\sigma_i(\Gamma^q(F_n)) \subset \Gamma^q(F_{n+1})$ and $\delta_i(\Gamma^q(F_n)) \subset \Gamma^q(F_{n-1})$, and
- conjugation by $G$ leaves the spaces $B_n(q, G, K)$ invariant.

3.2. Subspaces of $J(G)$. The James reduced product $J(X)$ is a classical construction introduced by James [17] and can be defined for any CW-complex $X$ with a good basepoint $\ast$. In our paper $X$ is mainly a compact Lie group with basepoint the identity element $1_G$. In general, $J(X)$ is defined as the topological space

$$J(X) := \left( \bigcup_{n \geq 0} X^n \right) \sim$$

where $\sim$ is the relation $(\ldots, \ast, \ldots) \sim (\ldots, \hat{\ast}, \ldots)$ omitting the coordinates equal to the basepoint. Alternatively this has the structure of the free monoid generated by the elements of $X$ with the basepoint acting as the identity element. The space $J(X)$ is weakly homotopy equivalent to $\Omega \Sigma X$, the loops on the suspension of $X$ and the suspension of $J(X)$ is given by

$$\Sigma J(X) \simeq \Sigma \bigvee_{k \geq 1} \tilde{X}^n,$$

where $\tilde{X}^n$ is the $n$-fold smash product of $X$. This fact will be important below in our computations of the Poincaré series. Let $R$ be a commutative ring with 1. Bott and Samelson [10] showed that if the homology of $X$ is a free $R$-module, then the homology of $J(X)$ with $R$ coefficients is isomorphic as an algebra to the tensor algebra $\mathcal{T}[\tilde{H}_*; R]$ generated by the reduced homology of $X$, another crucial fact that will be used below.

For compact and connected $G$ with maximal torus $T$ we want to study the spaces $J(T) \subset J(G)$ and the subspaces of $J(G)$ defined next, by assembling all the spaces $B_n(q, G, K)$ into a single infinite dimensional space. Define

$$X(q, G, K) := \left( \bigcup_{n \geq 1} B_n(q, G, K) \right) \sim,$$

where $\sim$ is the relation in the definition of the James reduced product. For fixed $K$ we obtain a filtration

$$J(T) \subset X(2, G, K) \subset X(3, G, K) \subset X(4, G, K) \subset \cdots \subset J(G).$$

When $K = 1_G$ we obtain the corresponding filtration studied in [14, 20], namely

$$J(T) \subset X(2, G) \subset X(3, G) \subset X(4, G) \subset \cdots \subset X(\infty, G) = J(G).$$

It is important to single out the space $\text{Comm}(G) = X(2, G)$, which will play a more significant role here. Each of the spaces $X(q, G, K)$ plays an important role in the
remark 3.2. the spaces \( \mathcal{B}_n(q, G, K) \) and \( X(q, G, K) \) are not path connected in general. for example sjerve and torres-giese \[22\] showed that \( \text{Hom}(\mathbb{Z}^n, \text{SO}(3)) \) has several path components, including the path component of the trivial representation denoted by \( \text{Hom}(\mathbb{Z}^n, \text{SO}(3))_1 \), and a finite number of other path component all homeomorphic to \( S^3/\mathbb{Q}_8 \), the three sphere modulo quaternions. it was then shown in \[23\] that \( X(2, \text{SO}(3)) \) also consists of the path component of the trivial representation, denoted \( X(2, \text{SO}(3))_1 \), and an infinite number of copies of \( S^3/\mathbb{Q}_8 \). on the other hand, if \( G = \text{SU}(m) \), then both spaces \( \text{Hom}(\mathbb{Z}^n, \text{SU}(m)) \) and \( X(2, \text{SU}(m)) \) are path connected. we define the space

\[
\mathcal{B}_n(q, G, K)_1 \subseteq \mathcal{B}_n(q, G, K)
\]
to be the path component of the trivial representation, which corresponds to the \( n \)-tuple \( (1_G, \ldots, 1_G) \). similarly we define the space

\[
X(q, G, K)_1 = \left( \bigsqcup_{n \geq 1} \mathcal{B}_n(q, G, K)_1 \right) / \sim \subseteq X(q, G, K)
\]
to be the path component of the class of the trivial representation.

remark 3.3. it is important to distinguish the cases when \( K = 1_G \), since then \( \mathcal{B}_n(q, G, 1) = \text{Hom}(F_n/\Gamma^q, G) \) and \( X(q, G, 1) = X(q, G) \) are the only spaces of interest, for which we will give explicit answers for the poincaré series in section 6.

there is also an analogous \( p \)-descending central series of \( Q \) as follows. suppose \( Q \) is a group and \( p \) is a prime number. define the subgroups \( \Gamma^p_i(Q) \leq Q \) inductively by setting \( \Gamma^1_p(Q) = Q \), and for all \( i > 0 \) define \( \Gamma^{i+1}_p(Q) := [\Gamma^i_p(Q), Q]/(\Gamma^i(Q))^p \). we obtain a normal series for \( Q \)

\[
\cdots \leq \Gamma^{i+1}_p(Q) \leq \Gamma^i_p(Q) \leq \cdots \leq \Gamma^2_p(Q) \leq \Gamma^1_p(Q) = Q.
\]

by abuse of terminology one can think of the ordinary descending central series of \( Q \) as the case \( p = 0 \). if \( Q \) is the free group \( F_n \) and \( \Gamma^p_n = \Gamma^p_p(F_n) \), for fixed \( p \) define the spaces \( \text{Hom}(F_n/\Gamma^q_p, G) \subseteq G^n \), which for varying \( n \) have the structure of simplicial spaces with geometric realization \( B(q, G, p) \subset BG \). these spaces have very interesting features. for instance, if \( G \) is finite the spaces \( B(2, G, p) \subset BG \) is a space assembled from the \( p \)-elementary abelian subgroups in \( G \) and the inclusion \( B(2, G, p) \hookrightarrow BG \) can detect mod \( p \) cohomology of \( G \), see \[4, proposition 3.4\]. it would be compelling to understand the constructions in this paper using the \( p \)-descending central series.

4. \( G \)-equivariant stable decompositions

there is a filtration associated to the spaces \( \mathcal{B}_n(q, G, K) \) defined above given by

\[
F^n_k(\mathcal{B}_n(q, G, K)) \subset F^{n-1}_k(\mathcal{B}_n(q, G, K)) \subset \cdots \subset F^1_k(\mathcal{B}_n(q, G, K)) \subset \mathcal{B}_n(q, G, K),
\]

where \( F^i_k(\mathcal{B}_n(q, G, K)) \subset \mathcal{B}_n(q, G, K) \) is the subspace consisting of \( n \)-tuples with at least \( i \) coordinates the identity element \( 1_G \). the pairs \( (F^k_n, F^{k+1}_n) \) are ndr-pairs, by work of adem–cohen \[2\], adem–cohen–gomez \[3\] and villarreal \[24\]. to be more precise, the filtrations \( F^{k+1}_n \) have the homotopy type of a \( G \)-cw complex and it
can be seen as a $G$-subcomplex of $F^k_n$. Moreover, the simplicial spaces $\mathcal{B}_n(q, G, K)$ are simplicially $G$-NDR as explained in [3, §4] and filtration quotients are given by

$$F^k_n(\mathcal{B}_n(q, G, K))/F^k_{n+1}(\mathcal{B}_n(q, G, K)) \approx \bigvee_{i \geq k} \mathcal{B}_{n-k}(q, G, K)/(F^1_{n-k}(\mathcal{B}_k(q, G, K)).$$

For details see [2, §5] or in [3, §5]. Therefore, we obtain the following theorem.

**Theorem 4.1.** Let $G$ be a compact and connected Lie group and $K$ a connected central subgroup. Then there is a $G$-equivariant homotopy equivalence

$$\Sigma(\mathcal{B}_n(q, G, K)) \simeq \Sigma \bigvee_{1 \leq k \leq n} \mathcal{B}_k(q, G, K)/F^1_k(\mathcal{B}_k(q, G, K)),$$

and hence a homotopy equivalence

$$\Sigma(\mathcal{B}_n(q, G, K))/G \simeq \Sigma \bigvee_{1 \leq k \leq n} \mathcal{B}_k(q, G, K)/G[F^1_k(\mathcal{B}_k(q, G, K))/G].$$

**Proof.** The case when $q = 2$ follows from [3, Theorem 1.1]. The same arguments show that this works for $q > 2$, or alternatively by [24, Theorem 1.3] and [1, Theorem 1.6].

Note that the first result of this nature was proved by Adem and Cohen [2], where they write the decomposition of the spaces Hom($\mathbb{Z}^n$, $G$) into wedge sums after one suspension for any closed subgroup of $GL(n, \mathbb{C})$.

Similarly, the spaces $X(q, G, K)$ can also be filtered by setting

$$F_1(X(q, G, K)) \subset F_2(X(q, G, K)) \subset \cdots \subset F_\infty(X(q, G, K)) = X(q, G, K),$$

where for each $k$, we define $F_k(X(q, G, K))$ as the image of

$$X_k(q, G, K) = \left( \bigcup_{k \geq n \geq 1} \mathcal{B}_n(q, G, K)_1 \right)/\sim$$

in $X(q, G, K)$. The pairs $(F_{n-k}(X(q, G, K)), F_{n-k+1}(X(q, G, K)))$ are also $G$-NDR-pairs, and the filtration quotients are clearly homeomorphic to

$$(F_{n-k}(X(q, G, K)), F_{n-k+1}(X(q, G, K))) \approx \mathcal{B}_k(q, G, K)/(F^1_k(\mathcal{B}_k(q, G, K))).$$

Therefore, we obtain the following theorem.

**Theorem 4.2.** Let $G$ be a compact and connected Lie group and $K$ a connected central subgroup. Then there is a $G$-equivariant homotopy equivalence

$$\Sigma(X(q, G, K)) \simeq \Sigma \bigvee_{k \geq 1} \mathcal{B}_k(q, G, K)/F^1_k(\mathcal{B}_k(q, G, K)),$$

and hence a homotopy equivalence

$$\Sigma(X(q, G, K))/G \simeq \Sigma \bigvee_{k \geq 1} \mathcal{B}_k(q, G, K)/G[F^1_k(\mathcal{B}_k(q, G, K))/G].$$

**Proof.** The pairs $(F_{n-k}(X(q, G, K)), F_{n-k+1}(X(q, G, K)))$ are $G$-NDR-pairs with filtration quotient given by $\mathcal{B}_k(q, G, K)/F^1_k$ as above. The James reduced product $J(G)$ splits as a wedge sum after one suspension

$$\Sigma J(G) \simeq \bigvee_{k \geq 1} \hat{G}^k$$
and the splitting for the suspension of \( X(q, G, K) \subseteq J(G) \) then follows by inspection.

\[ \Box \]

In particular, we obtain the following useful theorem.

**Theorem 4.3.** Let \( G \) be a compact and connected Lie group. Then there is a homotopy equivalence

\[
\Sigma(X(q, G)/G) \simeq \bigvee_{n \geq 1} \widehat{\text{Hom}}(F_n/\Gamma^q, G)/G,
\]

and in particular

\[
\Sigma(\text{Comm}(G)/G) \simeq \bigvee_{n \geq 1} \widehat{\text{Hom}}(\mathbb{Z}^n, G)/G.
\]

Note that the first decomposition holds also for the component of the trivial representation, as observed also in [20]. The conjugation action of \( G \) preserves the component \( X(q, G)_1 \), giving \( G \)-equivariant decompositions of the suspension of \( X(q, G)_1 \). In particular, we are interested in the following special case.

**Theorem 4.4.** Let \( G \) be a compact and connected Lie group. Then there is a homotopy equivalence

\[
\Sigma(X(q, G)_1/G) \simeq \bigvee_{n \geq 1} \widehat{\text{Hom}}(F_n/\Gamma^q, G)_1/G,
\]

5. **The spaces \( \text{Comm}(G)_1/G \) and \( \text{Rep}(\mathbb{Z}^n, G)_1 \)**

We start with the following theorem, which was mentioned in [5, Remark 4, p. 746] and in [3, Theorem 1.3]. We give a sketch of the proof here.

**Theorem 5.1.** Let \( G \) be a compact and connected Lie group with maximal torus \( T \) and Weyl group \( W \). Then there is a homeomorphism

\[
\text{Rep}(\mathbb{Z}^n, G)_1 \simeq T^n/W
\]

and a homotopy equivalence

\[
\Sigma(\text{Rep}(\mathbb{Z}^n, G)_1) \simeq \bigvee_{1 \leq k \leq n} \bigvee_{\binom{n}{k}} \widehat{T}^k/W,
\]

where \( W \) acts diagonally on \( T^n \) and \( \widehat{T}^k \).

**Proof.** There is a map

\[
G \times T^n \rightarrow \text{Hom}(\mathbb{Z}^n, G)_1
\]

\[
(g, t_1, \ldots, t_n) \mapsto (gt_1g^{-1}, \ldots, gt_ng^{-1}),
\]

which is a surjection by Baird [5]. This map factors through the quotient by the normalizer of \( T \)

\[
\phi_n: G \times_{NT} T^n \rightarrow \text{Hom}(\mathbb{Z}^n, G)_1,
\]

where \( NT \) acts by left multiplication on \( G \) and diagonally by conjugation on \( T^n \). This map is \( G \)-equivariant, where \( G \) acts by left multiplication on \( G \), trivially on \( T^n \) and by conjugation on \( \text{Hom}(\mathbb{Z}^n, G)_1 \) [3]. Therefore, we get a commutative diagram
Injectivity of the map $T^n/W \to \text{Rep}(\mathbb{Z}^n, G)_1$ follows by Sikora [21]. This gives the homeomorphism $T^n/W \approx \text{Rep}(\mathbb{Z}^n, G)_1$. Moreover, we have 

$$\hat{T}^n/W \approx \text{Rep}(\mathbb{Z}^n, G)_1/(F_1^n(\text{Hom}(\mathbb{Z}^n, G)_1)/G).$$

The following theorem is the analogue of the above theorem, for the subspace $\text{Comm}(G) \subseteq J(G)$.

**Theorem 5.2.** Let $G$ be a compact and connected Lie group with maximal torus $T$ and Weyl group $W$. Then there is a homeomorphism 

$$\text{Comm}(G)_1/G \approx J(T)/W$$

and a homotopy equivalence 

$$\Sigma(\text{Comm}(G)_1/G) \approx \bigvee_{n \geq 1} \hat{T}^n/W.$$

**Proof.** The maps 

$$\phi_n: G \times_{NT} T^n \to \text{Hom}(\mathbb{Z}^n, G)_1,$$

in the proof of Theorem 5.1 combine to give a map 

$$\Phi: G \times_{NT} J(T) \to \text{Comm}(G)_1.$$

The map $\Phi$ is $G$-equivariant, where $G$ acts by left multiplication on the left on the first factor and by conjugation on $\text{Comm}(G)_1$. Therefore, we get a homeomorphism 

$$\text{Comm}(G)_1/G \approx J(T)/W.$$ 

The decomposition follows easily from the following 

$$\Sigma(\text{Comm}(G)_1/G) \simeq \Sigma J(T)/W.$$

6. **Poincaré series of $\text{Comm}(G)_1/G$ and $\text{Rep}(\mathbb{Z}^n, G)_1$**

The Poincaré series of a space $X$ is the series 

$$P(X; t) := \sum_{k \geq 0} \text{rank}_\mathbb{Q}(H_k(X; \mathbb{Q}))t^k.$$ 

In this section we describe the Poincaré series of $\text{Comm}(G)_1/G$ and $\text{Rep}(\mathbb{Z}^n, G)_1$. Using this and results by others we obtain Poincaré series for their nilpotent versions. We refine $P(X; t)$ to have a bi-graded Hilbert–Poincaré series as follows.
First note that by Bott–Samelson [10] the homology of the James reduced product \( J(T) \), where \( T \) is the maximal torus of \( G \), is the tensor algebra \( \mathcal{T}[\tilde{H}_s(T)] \) generated by the reduced homology of \( T \). This tensor algebra has a bigrading

\[
\mathcal{T}[\tilde{H}_s(T)] = \bigoplus_{i,j} \mathcal{T}[\tilde{H}_s(T)]_{i,j},
\]

where \( \mathcal{T}[\tilde{H}_s(T)]_{i,j} \) is generated by \( j \)-fold tensors of total (co)homological degree \( i \). The action of the Weyl group \( W \) preserves this bigrading. Note that the cohomology of \( J(T) \) is the dual \( \mathcal{T}^*[\tilde{H}_s(T)] \) of the tensor algebra, which is isomorphic to the tensor algebra since the homology of \( T \) is torsion free. Below we find the bigraded version of the Poincaré series of the \( W \)-invariant subalgebra

\[
P(\mathcal{T}[\tilde{H}_s(T)]^W; s, t) = \sum_{i,j} \text{rank}_q(\mathcal{T}[\tilde{H}_s(T)]_{i,j})^W, s^i t^j,
\]

which we call a Hilbert–Poincaré series because of the cohomology and tensor gradings. To recover the ordinary Poincaré series we set \( t = 1 \), since the tensor degree does not contribute to the cohomological degree. We begin with a proposition.

**Proposition 6.1.** Let \( G \) be a compact and connected Lie group with maximal torus \( T \) and Weyl group \( W \). Then

\[
P(\text{Comm}(G)_1/G; s) \cong P(J(T)/W; s) \cong P(\mathcal{T}[\tilde{H}_s(T)]^W; s).
\]

**Proof.** The first isomorphism follows from the Theorem 5.2. The second homomorphism follows from the fact that

\[
H_*(J(T); \mathbb{Q}) \cong \mathcal{T}[\tilde{H}_s(T)],
\]

and a result of Grothendieck [16] that if \( \Gamma \) is a finite group acting (not necessarily freely) on a \( CW \) complex \( X \), then \( H^*(X/W; \mathbb{F}) \cong H^*(X; \mathbb{F})^W \), where \( \text{char}(\mathbb{F}) = 0 \) or relatively prime to \( |\Gamma| \). \( \square \)

First we determine the Hilbert–Poincaré series \( P(\text{Comm}(G)_1/G; s, t) \), where the bigrading comes from Proposition 6.1.

**Theorem 6.2.** Let \( G \) be a compact and connected Lie group with maximal torus \( T \) and Weyl group \( W \). Then the Hilbert–Poincaré series of \( \text{Comm}(G)_1/G \) is given by

\[
P(\text{Comm}(G)_1/G; s, t) = \frac{1}{|W|} \sum_{w \in W} \left( \sum_{k \geq 0} (\det(1 + sw) - 1)^k \right)^t.
\]

**Proof.** For simplicity set \( X_{i,j} = \mathcal{T}[\tilde{H}_s(T)]_{i,j} \) so that \( X = \bigoplus_{i,j} X_{i,j} \). The action of the Weyl group \( W \) preserves both tensor and cohomological degree, so it preserves each \( X_{i,j} \), hence the direct sum decomposition \( \bigoplus_{i,j} X_{i,j} \). We want to determine

\[
P(\text{Comm}(G)_1/G; s, t) = P(\mathcal{T}[\tilde{H}_s(T)]^W; s, t) = P(X^W; s, t).
\]

As shown in [14, Appendix p.406] we have

\[
P(X^W; s, t) = \frac{1}{|W|} \sum_{w \in W} \left( \sum_{i,j=0}^{\infty} \text{Trace}(w|_{X_{i,j}}) s^i t^j \right).
\]

(7)
From [11, Example 3.25, p. 49], we have
\[ \sum_{i=0}^{n} \text{Trace}(w|_{\lambda_i \mathbb{R}^n}) s^i = \det(1 + sw) \]
and
\[ \sum_{i,j \geq 0} \text{Trace}(w|_{X_{i,j}}) s^i t^j = \frac{1}{1 - t(\det(1 + sw) - 1)}. \]
Therefore,
\[ P(X^W; s, t) = \frac{1}{|W|} \sum_{w \in W} \left( \frac{1}{1 - t(\det(1 + sw) - 1)} \right) \]
\[ = \frac{1}{|W|} \sum_{w \in W} \left( \sum_{k \geq 0} (\det(1 + sw) - 1)^k t^k \right). \]

Now using Theorem 6.2 we would like to prove Theorem 1.4. We begin by proving the following proposition.

**Proposition 6.3.** The Poincaré series of $\hat{T}^k / W$ is given by
\[ P(\hat{T}^k / W; s) = \frac{1}{|W|} \sum_{w \in W} (\det(1 + sw) - 1)^k. \]

**Proof.** We first rearrange the terms in the Hilbert–Poincaré series and write
\[ P(\text{Comm}(G)_1 / G; s, t) = \frac{1}{|W|} \sum_{w \in W} \left( \sum_{k \geq 0} (\det(1 + sw) - 1)^k t^k \right) \]
\[ = \frac{1}{|W|} \sum_{k=0}^{\infty} \left( \frac{1}{|W|} \sum_{w \in W} (\det(1 + sw) - 1)^k \right) t^k. \]
The bigrading in the (co)homology of $\text{Comm}(G)_1 / G$ comes from the homeomorphism $\text{Comm}(G)_1 / G \approx J(T)/W$. There is an induced homotopy equivalence
\[ \Sigma(\text{Comm}(G)_1 / G) \simeq \Sigma J(T)/W \simeq \bigvee_{k \geq 1} \hat{T}^k / W \]
in Theorem 5.2, and the proposition follows since the cohomology of $\hat{T}^k$ is concentrated in tensor degree $k$, and $W$ preserves tensor degree. \qed

Now we can prove the main theorem, which is only a corollary at this point.

**Theorem 6.4.** Let $G$ be a compact and connected Lie group with maximal torus $T$ and Weyl group $W$. Then the Hilbert-Poincaré series of $\text{Rep}(\mathbb{Z}^n, G)_1$ is given by
\[ P(\text{Rep}(\mathbb{Z}^n, G)_1; s, t) = \frac{1}{|W|} \sum_{w \in W} \left( \sum_{k=0}^{n} \binom{n}{k} (\det(1 + sw) - 1)^k t^k \right), \]
and the ordinary Poincaré series is given by
\[ P(\text{Rep}(\mathbb{Z}^n, G)_1; s) = \frac{1}{|W|} \sum_{w \in W} \det(1 + sw)^n. \]
Proof. From Theorem 5.1 and Proposition 6.3 we have the first part of the theorem

\[ P(\text{Rep}(\mathbb{Z}^n, G)_1; s, t) = \frac{1}{|W|} \sum_{w \in W} \left( \sum_{k=0}^{n} \binom{n}{k} (\det(1 + sw) - 1)^k t^k \right). \]

For the ordinary Poincaré series we set \( t = 1 \). The second part then follows from

\[ \sum_{k=0}^{n} \binom{n}{k} x^k = (x + 1)^n \]

by letting \( x = \det(1 + sw) - 1 \). □

Remark 6.5. In the examples in Section 7, one can easily verify that for the specific Lie groups there the Euler characteristic of \( \text{Rep}(\mathbb{Z}^n, G)_1 \) is 0.

Now we turn to finitely generated nilpotent groups. The following was shown by Bergeron and Silberman [7, Proposition 3.1].

**Proposition 6.6.** Let \( G \) be a compact connected Lie group. If \( Q \) is a finitely generated nilpotent group, then, for all \( i \geq 2 \), the inclusion

\[ \text{Hom}(Q/\Gamma_i(Q), G) \to \text{Hom}(Q, G) \]

is a homeomorphism onto the union of those components of the target intersecting the image of \( i \).

In particular, since \( F_n/\Gamma^2 = \mathbb{Z}^n \) the inclusion

\[ \text{Hom}(\mathbb{Z}^n, G)_1 \to \text{Hom}(F_n/\Gamma^2, G)_1 \]

is a homeomorphism, and we obtain

\[ \text{Hom}(\mathbb{Z}^n, G)_1/G \simeq \text{Hom}(F_n/\Gamma^i, G)_1/G. \]

**Corollary 6.7.** Let \( \Gamma \) be a finitely generated nilpotent group, with \( \text{rank } H_1(\Gamma) = N \), and \( G \) a reductive algebraic group. Then the Hilbert–Poincaré series of \( \text{Rep}(\Gamma, G)_1 \) is given by

\[ P(\text{Rep}(\Gamma, G)_1; s, t) = \frac{1}{|W|} \sum_{w \in W} \det(1 + sw)^N. \]

In particular, this is true for

- the free nilpotent group \( F_n/\Gamma^q \) with rank \( n \),
- the surface groups \( S_g \) modulo \( \Gamma^q(S_g) \), with rank \( 2g \),
- the braid groups \( B_k \) modulo \( \Gamma^q(B_k) \), with rank \( 1 \).

**Remark 6.8.** If we consider the descending central series of \( S_g \), then there are inclusion maps

\[ \text{Rep}(\mathbb{Z}^{2g}, G)_1 \to \text{Rep}(S_g/\Gamma^3, G)_1 \to \text{Rep}(S_g/\Gamma^3, G)_1 \to \cdots \to \text{Rep}(S_g, G) \]

each of them inducing isomorphisms in cohomology with field coefficients with characteristic 0 or relatively prime to \( |W| \). It is known that \( \text{Rep}(S_g, G) = \text{Rep}(S_g, G)_1 \) is path connected. It would be interesting to compare the answer to the cohomology of \( \text{Rep}(S_g, G) \) with the same coefficients, given in a result of Cappell, Lee and Miller [12, Theorem 2.2].

Finally we can determine also the Hilbert–Poincaré series of \( X(q, G)_1/G \).
**Theorem 6.9.** Let \( G \) be a compact and connected Lie group with maximal torus \( T \) and Weyl group \( W \). For all \( q \geq 2 \) there is a homotopy equivalence

\[
\Sigma X(q, G) / G \simeq \Sigma X(q + 1, G) / G.
\]

In particular, for all \( q \geq 2 \), the Hilbert-Poincaré series of \( X(q, G) / G \) is given by

\[
P(X(q, G) / G; s, t) = \frac{1}{|W|} \sum_{w \in W} \left( \sum_{k \geq 0} (\det(1 + sw) - 1)^k t^k \right).
\]

**Proof.** Consider the following commutative diagram of cofibrations

\[
\begin{array}{ccc}
S_n,2(G) & \xrightarrow{i} & \text{Hom}(\mathbb{Z}^n, G)_1 \\
\downarrow{i} & & \downarrow{i} \\
S_n,q(G) & \xrightarrow{i} & \text{Hom}(F_n/\Gamma^q_n, G)_1
\end{array}
\]

considered also in [20], where \( S_n,q(G) \) is the subspace of \( \text{Hom}(F_n/\Gamma^q_n, G)_1 \) consisting of \( n \)-tuples with at least one coordinate the identity, and \( q \geq 2 \). The first two vertical maps are inclusions and are homotopy equivalences by the gluing lemma and [7]. It follows that the third vertical map is also a homotopy equivalence. The first vertical map and the second vertical maps are \( G \)-equivariant by results of Bergeron and Silberman [7]. Therefore, obtain a commutative diagram

\[
\begin{array}{ccc}
S_n,2(G) / G & \xrightarrow{i} & \text{Hom}(\mathbb{Z}^n, G)_1 / G \\
\downarrow{i} & & \downarrow{i} \\
S_n,q(G) / G & \xrightarrow{i} & \text{Hom}(F_n/\Gamma^q_n, G)_1 / G
\end{array}
\]

It follows that the right vertical map is also a homotopy equivalence

\[
\tilde{T}^n / W \simeq \text{Hom}(F_n/\Gamma^q_n, G)_1 / G.
\]

The rest follows from the decomposition in Proposition 6.3 and Theorem 4.4. \( \square \)

### 7. Examples of Poincaré series for \( \text{Rep}(\mathbb{Z}^n, G)_1 \)

We give examples of Poincaré series for \( G \) one of the Lie groups \( \text{SU}(2) \), \( \text{U}(2) \), \( \text{U}(3) \), \( \text{U}(4) \), and \( G_2 \). Biswas, Lawton and Ramras [8] show that the fundamental group of the representation space is given by

\[
\pi_1(\text{Rep}(\mathbb{Z}^n, G)_1) \cong \pi_1((G, G) / [G,G])^n.
\]

In degree 1, the formulas below recover the rank of this group. Also \( \det(1 + sw) \) is invariant under conjugation, so we only need to know the decomposition of the Weyl group into conjugacy classes. The bigraded Hilbert–Poincaré series are left to the reader as they can be easily obtained from Theorem 6.4.
Example 7.1. The group $G = SU(2)$ has maximal torus of rank 1 and Weyl group $\Sigma_2 = \{1, -1\}$ as a subgroup of $GL(t^*)$. The representation space $\text{Rep}(\mathbb{Z}^n, SU(n))$ is path connected. Therefore, after setting $t = 1$ in Theorem 6.4, the Poincaré series is given by

$$P(\text{Rep}(\mathbb{Z}^n, SU(2)); s) = \frac{1}{2} \left( (1 + s)^n + (1 - s)^n \right).$$

Example 7.2. The group $G = U(2)$ has maximal torus of rank 2 and Weyl group $\Sigma_2 = \{1, w = (\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})\}$, and $\text{Rep}(\mathbb{Z}^n, U(2))$ is path connected. Therefore, we get the following Poincaré polynomial

$$P(\text{Rep}(\mathbb{Z}^n, U(2)); s) = \frac{1}{2} \left( (1 + s)^2n + (1 - s^2)^n \right).$$

Example 7.3. For $G = U(3)$ the rank is 3 and the Weyl group is the symmetric group on 3 letters

$$W = \Sigma_3 = \{c, (12), (13), (23), (123), (132)\}.$$  

The matrix representations $W \subseteq GL(t^*)$ can be obtained by applying each permutation in $\Sigma_3$ to the rows of the $3 \times 3$ identity matrix. This can be done in general for the Weyl group $\Sigma_n$ of $U(n)$. There are three conjugacy classes of elements in $\Sigma_3$, depending on the unordered partitions of 3, that is the class of the trivial element, the class of $(12) \in \Sigma_3$ containing three elements, and the class of $(123) \in \Sigma_3$ containing two elements. Therefore, the Poincaré series equals

$$P(\text{Rep}(\mathbb{Z}^n, U(3)); s) = \frac{1}{6} \left( (1 + s)^{3n} + 3(1 - s^2)^n(1 + s)^n + 2(1 + s^3)^n \right).$$

Example 7.4. $G = U(4)$ has rank 4 and Weyl group the symmetric group on 4 letters. The matrix representations $W \subseteq GL(t^*)$ can be obtained by applying each permutation in $\Sigma_4$ to the rows of the $4 \times 4$ identity matrix. There are five conjugacy classes of elements in $\Sigma_4$, depending on the unordered partitions of 4, namely $[1], [(12)], [(123)], [(12)(34)], [(1234)]$, each having 1, 6, 8, 3, 6 elements, and $\det(1 + sw)$ equal to $(1 + s)^4, (1 - s^2)^4, (1 + s^2)^4, (1 + s^3)^4, (1 - s^2)^4, 1 - s^4$, respectively. Therefore, the Poincaré series is given by

$$P(\text{Rep}(\mathbb{Z}^n, U(4)); s) = \frac{1}{24} \left( (1 + s)^{4n} + 6(1 - s^2)^n(1 + s)^{2n} + 8(1 + s^3)^n(1 + s)^n + 3(1 - s^2)^{2n} + 6(1 - s^4)^n \right).$$

Example 7.5. Now consider the exceptional Lie group $G_2$, which has rank 2 and Weyl group the dihedral group $W = D_{12}$ of order 12, with presentation

$$W = D_{12} = \langle s, t | s^2, t^2, (st)^3 \rangle = \{1, t, t^2, t^3, t^4, t^5, s, st, st^2, st^3, st^4, st^5\}.$$  

As a subgroup of $GL(t^*)$ we set

$$s = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \text{ and } t = \frac{1}{2} \begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}.$$ 

$G_2$ has a non-toral elementary abelian 2–subgroup of rank 3, so $\text{Rep}(\mathbb{Z}^n, G_2)$ is not path-connected. Setting $t = 1$ and after simplifying, the Poincaré series of
\[ \text{Rep}(\mathbb{Z}^n, G_2)_1 \text{ is} \]
\[ P(\text{Rep}(\mathbb{Z}^n, G_2)_1; s) = \frac{1}{12} \left( (1 + s)^{2n} + 6(-s^2 + 1)^n + (-1 + s)^{2n} \right. \]
\[ \left. + 2(s^2 + s + 1)^n + 2(s^2 - s + 1)^n \right). \]

**Example 7.6.** The Poincaré series of \( \text{Comm}(U(3))/U(3) \) can be given by Theorem 1.3
\[ P(\text{Comm}(U(3))/U(3); s) = \frac{1}{6} \sum_{w \in \Sigma_n} \sum_{k \geq 0} (\det(1 + sw) - 1)^k. \]

Therefore, using the data in the previous examples we have
\[ P(\text{Comm}(U(3))/U(3); s) = \frac{1}{6} \sum_{k \geq 0} \left( (3s + 3s^2 + s^3)^k + 3(s - s^2 - s^3)^k + 2s^{3k} \right). \]

**Remark 7.7.** The same information can be used to obtain the same Poincaré series for \( \text{Rep}(F_n/\Gamma_q, G) \) and \( \text{Rep}(S_g/\Gamma_q, G) \) (and other examples) as shown above. Moreover, the same information can be used to obtain Hilbert–Poincaré series for \( X(m, G)_1/G \) for all \( m \geq 2 \).

Another observation is that our calculations in this section indicate that for even \( n \), and \( G \) as above, the spaces \( \text{Rep}(\mathbb{Z}^n, G)_1 \) are rational Poincaré duality spaces, contrary to \( \text{Hom}(\mathbb{Z}^n, G)_1 \), for which this happens when \( n \) is odd [20].

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