Existence, covolumes, and infinite generation of lattices for Davis complexes

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Abstract

Let \( \Sigma \) be the Davis complex for a Coxeter system \((W, S)\). The automorphism group \( G \) of \( \Sigma \) is naturally a locally compact group, and a simple combinatorial condition due to Haglund–Paulin determines when \( G \) is nondiscrete. The Coxeter group \( W \) may be regarded as a uniform lattice in \( G \). We show that many such \( G \) also admit a nonuniform lattice \( \Gamma \), and an infinite family of uniform lattices with covolumes converging to that of \( \Gamma \). It follows that the set of covolumes of lattices in \( G \) is nondiscrete. We also show that the nonuniform lattice \( \Gamma \) is not finitely generated. Examples of \( \Sigma \) to which our results apply include buildings and non-buildings, and many complexes of dimension greater than 2. To prove these results, we introduce a new tool, that of “group actions on complexes of groups”, and use this to construct our lattices as fundamental groups of complexes of groups with universal cover \( \Sigma \).

1 Introduction

Let \( G \) be a locally compact topological group, with Haar measure \( \mu \). A discrete subgroup \( \Gamma \leq G \) is a lattice if \( \Gamma \backslash G \) carries a finite \( G \)-invariant measure, and is uniform if \( \Gamma \backslash G \) is compact. Some basic questions are:

1. Does \( G \) admit a (uniform or nonuniform) lattice?

2. What is the set of covolumes of lattices in \( G \), that is, the set of positive reals

\[ \mathcal{V}(G) := \{ \mu(\Gamma \backslash G) \mid \Gamma < G \text{ is a lattice} \} \]

3. Are lattices in \( G \) finitely generated?

These questions have been well-studied in classical cases. For example, suppose \( G \) is a reductive algebraic group over a local field \( K \) of characteristic 0. Then \( G \) admits a uniform lattice, constructed by arithmetic means (Borel–Harder [BHar]), and a nonuniform lattice only if \( K \) is archimedean (Tamagawa [La]). If \( G \) is a semisimple real Lie group, the set \( \mathcal{V}(G) \) is in most cases discrete (see [Lu] and its references). If in addition \( G \) is higher-rank, then \( G \) and hence its lattices have Kazhdan’s Property (T) (see, for example, [Ma]). Since countable groups with Property (T) are finitely generated, it follows that all lattices in \( G \) are finitely generated.

A nonclassical case is \( G = \text{Aut}(T) \) the automorphism group of a locally finite tree \( T \). The study of lattices in \( G \) was initiated by Bass and Lubotzky, and has yielded many surprising differences

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from classical results (see the survey \cite{Lu} and the reference \cite{BL}). For example, the set $\mathcal{V}(G)$ is in many cases nondiscrete, and nonuniform tree lattices are never finitely generated.

In fact, the automorphism group $G$ of any locally finite polyhedral complex $X$ is naturally a locally compact group (see Section \ref{sec:aut}). For many $X$ with $\dim(X) \geq 2$, there is greater rigidity than for trees, as might be expected in higher dimensions. For instance, Burger–Mozes \cite{BM} proved a ‘Normal Subgroup Theorem’ for products of trees (parallel to that of Margulis \cite{Ma} for higher-rank semisimple Lie groups), and Bourdon–Pajot \cite{BP} and Xie \cite{Xie} established quasi-isometric rigidity for certain Fuchsian buildings. Apart from right-angled buildings, very little is known for $X$ of dimension $> 2$. Almost nothing is known for $X$ not a building.

In this paper we consider Questions (1)–(3) above for lattices in $G = \text{Aut}(\Sigma)$, where $\Sigma$ is the Davis complex for a Coxeter system $(W,S)$ (see \cite{D} and Section \ref{sec:davis} below). The Davis complex is a locally finite, piecewise Euclidean CAT(0) polyhedral complex, and the Coxeter group $W$ may be regarded as a uniform lattice in $G$. Our results are the Main Theorem and its Corollaries \ref{cor:1.1} and \ref{cor:1.2} below, which establish tree-like properties for lattices in many such $G$. After stating these results, we discuss how they apply to (barycentric subdivisions of) buildings and non-buildings, and to many Davis complexes $\Sigma$ with $\dim(\Sigma) > 2$.

To state the Main Theorem, recall that for a Coxeter system $(W,S)$ with $W = \langle S \mid (st)^{m_{s,t}} \rangle$, and any $T \subset S$, the special subgroup $W_T$ is the subgroup of $W$ generated by the elements $s \in T$. A special subgroup $W_T$ is spherical if it is finite, and the set of spherical special subgroups of $W$ is partially ordered by inclusion. The poset of nontrivial spherical special subgroups is an abstract simplicial complex $L$, called the nerve of $(W,S)$. We identify each generator $s \in S$ with the corresponding vertex $W_s = \langle s \rangle$ of $L$, and denote by $A$ the group of label-preserving automorphisms of $L$, that is, the group of automorphisms $\alpha$ of $L$ such that $m_{s,t} = m_{\alpha(s)\alpha(t)}$ for all $s,t \in S$. The group $G = \text{Aut}(\Sigma)$ is nondiscrete if and only if there is an $\alpha \in A$ such that $\alpha$ fixes the star in $L$ of some vertex $s$, and $\alpha \neq 1$ (Haglund–Paulin \cite{HP1}).

**Main Theorem.** Let $(W,S)$ be a Coxeter system, with nerve $L$ and Davis complex $\Sigma$. Let $A$ be the group of label-preserving automorphisms of $L$. Assume that there are vertices $s_1$ and $s_2$ of $L$, and elements $\alpha_1, \alpha_2 \in A$ of prime order, such that for $i = 1,2$:

1. $\alpha_i$ fixes the star of $s_{3-i}$ in $L$, and $\alpha_i(s_i) \neq s_i$;
2. for all $t_i \neq s_i$ such that $t_i$ is the image of $s_i$ under some power of $\alpha_i$, $m_{s_i,t_i} = \infty$; and
3. all spherical special subgroups $W_T$ with $s_i \in T$ are halvable along $s_i$ (see Definition \ref{def:halvable} below).

Then $G = \text{Aut}(\Sigma)$ admits:

- a nonuniform lattice $\Gamma$; and
- an infinite family of uniform lattices $(\Gamma_n)$, such that $\mu(\Gamma_n \setminus G) \to \mu(\Gamma \setminus G)$, where $\mu$ is Haar measure on $G$.

**Corollary \ref{cor:1.1}**. The set of covolumes of lattices in $G$ is nondiscrete.

**Corollary \ref{cor:1.2}**. The group $G$ admits a lattice which is not finitely generated.

Corollary \ref{cor:1.2} follows from the proof of the Main Theorem and Theorem \ref{thm:1.3} below. By the discussion above, Corollary \ref{cor:1.2} implies that the group $G$ in the Main Theorem does not have Property (T). This was already known for $G = \text{Aut}(\Sigma)$, where $\Sigma$ is any Davis complex (Haglund–Paulin \cite{HP1}); our results thus provide an alternative proof of this fact in some cases.
We describe several infinite families of examples of Davis complexes Σ to which our results apply in Section 5 below. To establish these applications, we use properties of spherical building (in [R]), and some results of graph theory [DM]. In two dimensions, examples include the Fuchsian buildings considered in [T2], and some of the highly symmetric Platonic polygonal complexes investigated by Świątkowski [Sw]. Platonic polygonal complexes are not in general buildings, and even the existence of lattices (other than \( W \)) in their automorphism groups was not previously known. An example of a Platonic polygonal complex is the (unique) CAT(0) 2–complex with all 2–cells squares, and the link of every vertex the Petersen graph (Figure 1 below). The Main Theorem and its corollaries also apply to many higher-dimensional Σ, including both buildings and non-buildings.

To prove the Main Theorem, we construct the lattices \( \Gamma_n \) and \( \Gamma \) as fundamental groups of complexes of groups with universal covers Σ (see [BH] and Section 2.3 below). The construction is given in Section 4 below, where we also prove Corollary 1.2.

Complexes of groups are a generalisation to higher dimensions of graphs of groups. Briefly, given a polyhedral complex \( Y \), a (simple) complex of groups \( G(\Sigma) \) over \( Y \) is an assignment of a local group \( G_\sigma \) to each cell \( \sigma \) of \( Y \), with monomorphisms \( G_\sigma \to G_\tau \) whenever \( \tau \subset \sigma \), so that the obvious diagrams commute. The action of a group \( G \) on a polyhedral complex \( X \) induces a complex of groups \( G(Y) \) over \( Y = G \setminus X \). A complex of groups is developable if it is isomorphic to a complex of groups constructed in this way. A developable complex of groups \( G(Y) \) has a simply-connected universal cover \( \tilde{G}(\Sigma) \), equipped with a canonical action of the fundamental group of the complex of groups \( \pi_1(G(Y)) \).

A key difference from graphs of groups is that complexes of groups are not in general developable. In addition, even if \( G(Y) \) is developable, with universal cover say \( X \), if \( \dim(X) \geq 2 \) it may be impossible to identify \( X \) using only local data such as the links of its vertices (see Ballmann–Brin [BB] and Haglund [H1]). To ensure that our complexes of groups are developable with universal covers \( \Sigma \), we use covering theory for complexes of groups (see [BH] and [H1], and Section 3.1 below). The main result needed is that if there is a covering of complexes of groups \( G(Y) \to H(Z) \), then \( G(Y) \) is developable if and only if \( H(Z) \) is developable, and the universal covers of \( G(Y) \) and \( H(Z) \) are isometric (see Theorem 3.2 below).

The other main ingredient in the proof of the Main Theorem is Theorem 1.3 below, which introduces a theory of “group actions on complexes of groups”. This is a method of manufacturing new complexes of groups with a given universal cover, by acting on previously-constructed complexes of groups. Given a complex of groups \( G(Y) \), and the action of a group \( H \) on \( Y \), the \( H \)-action extends to an action on \( G(Y) \) if there is a homomorphism from \( H \) to \( \text{Aut}(G(Y)) \). Roughly, this means that for each cell \( \sigma \) of \( Y \), each \( h \in H \) induces a group isomorphism \( G_\sigma \to G_{h \cdot \sigma} \), so that the obvious

![Figure 1: Petersen graph](image-url)
diagrams commute (see Section 3.1 below for definitions). In Section 3 below we prove:

**Theorem 1.3.** Let $G(Y)$ be a (simple) complex of groups over $Y$, and suppose that the action of a group $H$ on $Y$ extends to an action on $G(Y)$. Then the $H$-action induces a complex of groups $H(Z)$ over $Z = H \setminus Y$ such that there is a covering of complexes of groups $G(Y) \to H(Z)$. Moreover the fundamental group $\pi_1(H(Z))$ splits as $\pi_1(H(Z)) \cong \pi_1(G(Y)) \rtimes H$.

Theorem 1.3 is also used in [KBT], and we expect this result to be of independent interest. To our knowledge, automorphisms of complexes of groups have not previously been considered. In [B3], BASS–JIANG determined the structure of the full automorphism group of a graph of groups, but did not define or study the graph of groups induced by a group action on a graph of groups. A more precise statement of Theorem 1.3, including some additional facts about $H(Z)$, is given as Theorem 3.1 below.

The Main Theorem is then proved as follows. The action of the Coxeter group $W$ on $\Sigma$ induces a complex of groups $G(Y_1)$ over $Y_1 = W \setminus \Sigma$, with local groups the spherical special subgroups of $W$. We construct a family of finite complexes of groups $G(Y_n)$ and $H(Z_n)$, and infinite complexes of groups $G(Y_\infty)$ and $H(Z_\infty)$, so that there are coverings of complexes of groups as sketched in Figure 2 below.

![Figure 2: Coverings of complexes of groups](image)

The fundamental groups of $H(Z_n)$ and $H(Z_\infty)$ are, respectively, the uniform lattices $\Gamma_n$, and the nonuniform lattice $\Gamma$, in $G = \text{Aut}(\Sigma)$. For the local groups of $G(Y_n)$ and $G(Y_\infty)$, we use Condition 3 in the Main Theorem to replace certain spherical special subgroups $W_T$ by the subgroup half$_s(W_T)$, defined as follows:

**Definition 1.4.** Let $W_T$ be a spherical special subgroup of $W$, and suppose $s \in T$. Then $W_T$ is halvable along $s$ if the union

$$ (T - \{s\}) \cup \{sts \mid t \in T - \{s\} \} $$

generates an index 2 subgroup, denoted half$_s(W_T)$, of $W_T$.

The complexes of groups $H(Z_n)$ and $H(Z_\infty)$ are induced by group actions on, respectively, $G(Y_n)$ and $G(Y_\infty)$. To construct these group actions, we use the automorphisms $\alpha_1$ and $\alpha_2$ of $L$.

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2 Background

In this section we present brief background. In Section 2.1 we describe the natural topology on $G$, the group of automorphisms of a locally finite polyhedral complex $X$, and characterise uniform and nonuniform lattices in $G$. Section 2.2 constructs the Davis complex $\Sigma$ for a Coxeter system $(W, S)$, following [D]. In Section 2.3 we recall the basics of Haefliger’s theory of complexes of groups (see [BH]), and describe the complex of groups $G(Y_1)$ induced by the action of $W$ on $\Sigma$.

2.1 Lattices in automorphism groups of polyhedral complexes

Let $G$ be a locally compact topological group. We will use the following normalisation of Haar measure $\mu$ on $G$.

**Theorem 2.1** (Serre, [S]). Suppose that a locally compact group $G$ acts on a set $V$ with compact open stabilisers and a finite quotient $G\backslash V$. Then there is a normalisation of the Haar measure $\mu$ on $G$, depending only on the choice of $G$–set $V$ in Theorem 2.1 above. By the same arguments as for tree lattices ([BL], Chapter 1), it can be shown (for any suitable set $V$) that a discrete subgroup $\Gamma \leq G$ is a lattice if and only if the series $\text{Vol}(\Gamma \backslash V)$ converges, and $\Gamma$ is uniform if and only if this is a sum with finitely many terms. In Section 2.2 below we describe our choice of $G$–set $V$ when $G$ is the group of automorphisms of a Davis complex $\Sigma$.

2.2 Davis complexes

In this section we recall the construction of the Davis complex for a Coxeter system. We follow the reference [D].

A *Coxeter group* is a group $W$ with a finite generating set $S$ and presentation of the form

$$W = \langle s \in S \mid s^2 = 1 \ \forall s \in S, (st)^{m_{st}} = 1 \ \forall s, t \in S \text{ with } s \neq t \rangle$$

with $m_{st}$ an integer $\geq 2$ or $m_{st} = \infty$, meaning that $st$ has infinite order. The pair $(W, S)$ is called a *Coxeter system*.

**Example 1**: This example will be followed throughout this section, and also referred to in Sections 2.3 and 4 below. Let

$$W = \langle s_1, s_2, s_3, s_4, s_5 \mid s_1^2 = 1, (s_1s_4)^m = (s_2s_4)^m = (s_3s_4)^m = 1, \quad (s_1s_5)^{m'} = (s_2s_5)^{m'} = (s_3s_5)^{m'} = 1 \rangle$$

where $m$ and $m'$ are integers $\geq 2$.

Let $(W, S)$ be a Coxeter system. A subset $T$ of $S$ is *spherical* if the corresponding special subgroup $W_T$ is spherical, that is, $W_T$ is finite. By convention, $W_{\emptyset}$ is the trivial group. Denote by $S$ the set of all spherical subsets of $S$. The set $S$ is partially ordered by inclusion, and the poset $S_{\geq \emptyset}$ is the nerve $L$ of $(W, S)$ (this is equivalent to the description of $L$ in the introduction). By definition,
nonempty set $T$ of vertices of $L$ spans a simplex $\sigma_T$ in $L$ if and only if $T$ is spherical. We identify the generator $s \in S$ with the vertex $\{s\}$ of $L$. The vertices $s$ and $t$ of $L$ are joined by an edge in $L$ if and only if $m_{st}$ is finite, in which case we label this edge by the integer $m_{st}$. The nerve $L$ of Example 1 above, with its edge labels, is sketched in Figure 3 below.

![Figure 3: Nerve $L$ of Example 1, with edge labels](image)

We denote by $K$ the geometric realisation of the poset $S$. Equivalently, $K$ is the cone on the barycentric subdivision of the nerve $L$ of $(W, S)$. Note that $K$ is compact and contractible, since it is the cone on a finite simplicial complex. Each vertex of $K$ has type a spherical subset of $S$, with the cone point having type $\emptyset$. For Example 1 above, $K$ and the types of its vertices are sketched on the left of Figure 4.

![Figure 4: $K$ and types of vertices (left) and mirrors (right) for Example 1](image)

For each $s \in S$ let $K_s$ be the union of the (closed) simplices in $K$ which contain the vertex $s$
but not the cone point. In other words, $K_s$ is the closed star of the vertex $s$ in the barycentric subdivision of $L$. Note that $K_s$ and $K_t$ intersect if and only if $m_{st}$ is finite. The family $(K_s)_{s \in S}$ is a mirror structure on $K$, meaning that $(K_s)_{s \in S}$ is a family of closed subspaces of $K$, called mirrors. We call $K_s$ the $s$–mirror of $K$. For Example 1 above, the mirrors $K_s = K_{s_i}$ are shown in heavy lines on the right of Figure 4.

For each $x \in K$, put

$$S(x) := \{ s \in S \mid x \in K_s \}.$$ 

Now define an equivalence relation $\sim$ on the set $W \times K$ by $(w, x) \sim (w', x')$ if and only if $x = x'$ and $w^{-1}w' \in W_{S(x)}$. The Davis complex $\Sigma$ for $(W, S)$ is then the quotient space:

$$\Sigma := (W \times K)/\sim.$$ 

The types of vertices of $K$ induce types of the vertices of $\Sigma$, and the natural $W$–action on $W \times K$ descends to a type-preserving action on $\Sigma$, with compact quotient $K$, so that the $W$–stabiliser of a vertex of $\Sigma$ of type $T \in S$ is a conjugate of the spherical special subgroup $W_T$.

We identify $K$ with the subcomplex $(1, K)$ of $\Sigma$, and write $wK$ for the translate $(w, K)$, where $w \in W$. Any $wK$ is called a chamber of $\Sigma$. The mirrors $K_s$ of $K$, or any of their translates by elements of $W$, are called the mirrors of $\Sigma$. Two distinct chambers of $\Sigma$ are $s$–adjacent if their intersection is an $s$–mirror, and are adjacent if their intersection is an $s$–mirror for some $s \in S$. Note that the chambers $wK$ and $w'K$ are $s$–adjacent if and only if $w^{-1}w' = s$, equivalently $w' = ws$ and $w's = w$. For Example 1 above, part of the Davis complex $\Sigma$ for $(W, S)$ is shown in Figure 5. There are $2m$ copies of $K$ glued around the vertices of types $\{s_i, s_5\}$, for $i = 1, 2, 3$, since $W_{\{s_i, s_5\}}$ has order $2m$. Similarly, there are $2m'$ copies of $K$ glued around the vertices of types $\{s_i, s_5\}$, for $i = 1, 2, 3$.

The Davis complex $\Sigma$ may be metrised with a piecewise Euclidean structure, such that $\Sigma$ is a complete CAT(0) space (Moussong, see Theorem 12.3.3 of [D]). From now on we will assume that $\Sigma$ is equipped with this metric.

Suppose that $G = \text{Aut}(\Sigma)$ is the group of automorphisms of a Davis complex $\Sigma$. Since $W$ acts cocompactly on $\Sigma$, with finite stabilisers, it may be regarded as a uniform lattice in $G$. We take as the set $V$ in Theorem 2.1 above the set of vertices of $\Sigma$ of type $\emptyset$. Then the covolume of $W$ is 1, since $W$ acts simply transitively on $V$.

### 2.3 Complexes of groups

We now outline the basic theory of complexes of groups. The definitions of the more involved notions of morphisms and coverings of complexes of groups are postponed to Section 2.3 below. All references to [BH] in this section are to Chapter III.C.

In the literature, a complex of groups $G(Y)$ is constructed over a space or set $Y$ belonging to various different categories, including simplicial complexes, polyhedral complexes, or, most generally, scwols (small categories without loops). In each case there is a set of vertices, and a set of oriented edges with a composition law. The formal definition is:

**Definition 2.2.** A scwol $X$ is the disjoint union of a set $V(X)$ of vertices and a set $E(X)$ of edges, with each edge $a$ oriented from its initial vertex $i(a)$ to its terminal vertex $t(a)$, such that $i(a) \neq t(a)$. A pair of edges $(a, b)$ is composable if $i(a) = t(b)$, in which case there is a third edge $ab$, called the composition of $a$ and $b$, such that $i(ab) = i(b)$, $t(ab) = t(a)$, and if $i(a) = t(b)$ and $i(b) = t(c)$ then $(ab)c = a(bc)$ (associativity).

We will always assume scwols are connected (see Section 1.1, [BH]).

**Definition 2.3.** An action of a group $G$ on a scwol $X$ is a homomorphism from $G$ to the group of automorphisms of the scwol (see Section 1.5 of [BH]) such that for all $a \in E(X)$ and all $g \in G$:
1. \( g.i(a) \neq t(a) \); and

2. if \( g.i(a) = i(a) \) then \( g.a = a \).

Condition (2) in this definition ensures that if the scwol \( X \) is the barycentric subdivision of a polyhedral complex \( Y \), the action of \( G \) on \( X \) corresponds to an action \textit{without inversions} on \( Y \), meaning that if \( g \in G \) fixes a cell \( \sigma \) of \( Y \) setwise, then \( g \) fixes \( \sigma \) pointwise.

Suppose \( \Sigma \) is the Davis complex for a Coxeter system \((W, S)\), as defined in Section 2.2 above. Recall that each vertex \( \sigma \in V(\Sigma) \) has type \( T \) a spherical subset of \( S \). The edges \( E(\Sigma) \) are then naturally oriented by inclusion of type. That is, if the edge \( a \) joins the vertex \( \sigma \) of type \( T \) to the vertex \( \sigma' \) of type \( T' \), then \( i(a) = \sigma \) and \( t(a) = \sigma' \) exactly when \( T \subset T' \). It is clear that the sets \( V(\Sigma) \) and \( E(\Sigma) \) satisfy the properties of a scwol. Moreover, if \( Y \) is a subcomplex of \( \Sigma \), then the sets \( V(Y) \) and \( E(Y) \) also satisfy Definition 2.2 above. By abuse of notation, we identify \( \Sigma \) and \( Y \) with the associated scwols.

We now define complexes of groups over scwols.

**Definition 2.4.** A complex of groups \( G(Y) = (G_\sigma, \psi_a, g_{a,b}) \) over a scwol \( Y \) is given by:

1. a group \( G_\sigma \) for each \( \sigma \in V(Y) \), called the local group at \( \sigma \);

2. a monomorphism \( \psi_a : G_{i(a)} \to G_{t(a)} \) along the edge \( a \) for each \( a \in E(Y) \); and
3. for each pair of composable edges, a twisting element \( g_{a,b} \in G_{t(a)} \), such that

\[
\text{Ad}(g_{a,b}) \circ \psi_{ab} = \psi_a \circ \psi_b
\]

where \( \text{Ad}(g_{a,b}) \) is conjugation by \( g_{a,b} \) in \( G_{t(a)} \), and for each triple of composable edges \( a, b, c \) the following cocycle condition holds

\[
\psi_a(g_{b,c}) g_{a,bc} = g_{a,b} g_{ab,c}.
\]

A complex of groups is simple if each \( g_{a,b} \) is trivial.

**Example:** This example will be followed throughout this section, and used in the proof of the Main Theorem in Section 4 below. Let \((W, S)\) be a Coxeter system with nerve \( L \) and let \( K \) be the cone on the barycentric subdivision of \( L \), as in Section 2.2 above. Put \( Y_1 = K \), with the orientations on edges discussed above. We construct a simple complex of groups \( G(Y_1) \) over \( Y_1 \) as follows. Let \( \sigma \in V(Y_1) \). Then \( \sigma \) has type a spherical subset \( T \) of \( S \), and we define \( G_{\sigma} = W_T \). All monomorphisms along edges of \( Y_1 \) are then natural inclusions, and all \( g_{a,b} \) are trivial. For \((W, S)\) as in Example 1 of Section 2.2 above, the complex of groups \( G(Y_1) \) is shown in Figure 6 below. In this figure, \( D_{2m} \) and \( D_{2m'} \) are the dihedral groups of orders \( 2m \) and \( 2m' \) respectively, and \( C_2 \) is the cyclic group of order 2.

![Figure 6: The complex of groups \( G(Y_1) \), for Example 1 of Section 2.2](image)

Suppose a group \( G \) acts on a scwol \( X \), as in Definition 2.3 above. Then the quotient \( Y = G \setminus X \) also has the structure of a scwol, and the action of \( G \) induces a complex of groups \( G(Y) \) over \( Y \) (this construction is generalised in Section 3.2 below). Let \( Y = G \setminus X \) with \( p : X \to Y \) the natural projection. For each \( \sigma \in V(Y) \), choose a lift \( \overline{\sigma} \in V(X) \) such that \( p(\overline{\sigma}) = \sigma \). The local group \( G_{\overline{\sigma}} \) of \( G(Y) \) is then defined to be the stabiliser of \( \overline{\sigma} \) in \( G \), and the monomorphisms \( \psi_a \) and elements \( g_{a,b} \) are
defined using further choices. The resulting complex of groups $G(Y)$ is unique up to isomorphism (see Definition 3.3 below).

A complex of groups is developable if it is isomorphic to a complex of groups $G(Y)$ induced, as just described, by such an action. Complexes of groups, unlike graphs of groups, are not in general developable. The complex of groups $G(Y_1)$ above is developable, since it is the complex of groups induced by the action of $W$ on $\Sigma$, where for each $\sigma \in V(Y_1)$, with $\sigma$ of type $T$, we choose $\sigma$ in $\Sigma$ to be the vertex of $(1, K) = K \subset \Sigma$ of type $T$.

Let $G(Y)$ be a complex of groups. The fundamental group of the complex of groups $\pi_1(G(Y))$ is defined so that if $Y$ is simply connected and $G(Y)$ is simple, $\pi_1(G(Y))$ is isomorphic to the direct limit of the family of groups $G_\sigma$ and monomorphisms $\psi_\sigma$. For example, since $Y_1 = K$ is contractible and $G(Y_1)$ is a simple complex of groups, the fundamental group of $G(Y_1)$ is $W$.

If $G(Y)$ is a developable complex of groups, then it has a universal cover $\widetilde{G(Y)}$. This is a connected, simply-connected scwol, equipped with an action of $\pi_1(G(Y))$, so that the complex of groups induced by the action of the fundamental group on the universal cover is isomorphic to $G(Y)$. For example, the universal cover of $G(Y_1)$ is $\Sigma$.

Let $G(Y)$ be a developable complex of groups over $Y$, with universal cover $X$ and fundamental group $\Gamma$. We say that $G(Y)$ is faithful if the action of $\Gamma$ on $X$ is faithful, in which case we may identify $\Gamma$ with its image in $\text{Aut}(X)$. If $X$ is locally finite, then with the compact-open topology on $\text{Aut}(X)$, by the discussion in Section 2.1 above the subgroup $\Gamma \leq \text{Aut}(X)$ is discrete if and only if all local groups of $G(Y)$ are finite. Moreover, if $\text{Aut}(X)$ acts cocompactly on $X$, a discrete $\Gamma \leq \text{Aut}(X)$ is a uniform lattice in $\text{Aut}(X)$ if and only if $Y \cong \Gamma \backslash X$ is finite, and is a nonuniform lattice if and only if $Y$ is infinite and the series $\text{Vol}(\Gamma \backslash V)$ converges, for some $V \subset X$ on which $G = \text{Aut}(X)$ acts according to the hypotheses of Theorem 2.1 above.

3 Group actions on complexes of groups

In this section we introduce a theory of group actions on complexes of groups. The main result is Theorem 3.1 below, which makes precise and expands upon Theorem 1.3 of the introduction. The terms used in Theorem 3.1 which were not discussed in Section 2.1 above are defined in Section 3.1 below, where we also introduce some notation. In Section 3.2 we construct the complex of groups induced by a group action on a complex of groups, and in Section 3.3 we construct the induced covering. Using these results, in Section 3.4 we consider the structure of the fundamental group of the induced complex of groups. As in Section 2.2 above, all references to [BH] are to Chapter III.C.

We will need only actions on simple complexes of groups $G(Y)$ by simple morphisms; this case is already substantially technical. If in addition the action has a strict fundamental domain in $Y$, it is possible to make choices so that the induced complex of groups is also simple, and many of the proofs in this section become much easier. However, in our applications, the group action will not necessarily have a strict fundamental domain.

**Theorem 3.1.** Let $G(Y)$ be a (simple) complex of groups over a connected scwol $Y$, and suppose that the action of a group $H$ on $Y$ extends to an action (by simple morphisms) on $G(Y)$. Then the $H$–action induces a complex of groups $H(Z)$ over $Z = H \backslash Y$, with $H(Z)$ well-defined up to isomorphism of complexes of groups, such that there is a covering of complexes of groups

$$G(Y) \to H(Z).$$

Moreover the fundamental group $\pi_1(H(Z))$ splits as

$$\pi_1(H(Z)) \cong \pi_1(G(Y)) \rtimes H.$$

Finally, if $G(Y)$ is faithful and the $H$–action on $G(Y)$ is faithful then $H(Z)$ is faithful.
We will need the following general result on functoriality of coverings (see [BH] and [LT]).

**Theorem 3.2.** Let \( G(Y) \) and \( H(Z) \) be complexes of groups over scwols \( Y \) and \( Z \). Suppose there is a covering of complexes of groups \( \Phi : G(Y) \to H(Z) \). Then \( G(Y) \) is developable if and only if \( H(Z) \) is developable. Moreover, \( \Phi \) induces a monomorphism of fundamental groups

\[
\pi_1(G(Y)) \hookrightarrow \pi_1(H(Z))
\]

and an equivariant isomorphism of universal covers

\[
\widetilde{G(Y)} \cong \widetilde{H(Z)}.
\]

### 3.1 Definitions and notation

We gather here the definitions and notation needed for the statement and proof of Theorem 3.1 above. Throughout this section, \( Y \) and \( Z \) are scwols, \( G(Y) = (G_\sigma, \psi_\alpha) \) is a simple complex of groups over \( Y \), and \( H(Z) = (H_\tau, \theta_\alpha, h_{a,b}) \) is a complex of groups over \( Z \).

**Definition 3.3.** Let \( f : Y \to Z \) be a morphism of scwols (see Section 1.5 of [BH]). A morphism \( \Phi : G(Y) \to H(Z) \) over \( f \) consists of:

1. a homomorphism \( \phi_\sigma : G_\sigma \to H_{f(\sigma)} \) for each \( \sigma \in V(Y) \), called the local map at \( \sigma \); and
2. an element \( \phi(a) \in H_{t(f(a))} \) for each \( a \in E(Y) \), such that the following diagram commutes

\[
\begin{array}{ccc}
G_{t(a)} & \xrightarrow{\psi_a} & G_{t(a)} \\
\downarrow{\phi_{t(a)}} & & \downarrow{\phi_{t(a)}} \\
H_{f(t(a))} & \xrightarrow{Ad(\phi(a)) \circ f(a)} & H_{f(t(a))}
\end{array}
\]

and for all pairs of composable edges \((a, b)\) in \( E(Y)\),

\[
\phi(ab) = \phi(a) \psi_a(\phi(b)) h_{f(a), f(b)}.
\]

A morphism is simple if each element \( \phi(a) \) is trivial. If \( f \) is an isomorphism of scwols, and each \( \phi_\sigma \) an isomorphism of the local groups, then \( \Phi \) is an isomorphism of complexes of groups.

We introduce the following, expected, definitions. An automorphism of \( G(Y) \) is an isomorphism \( \Phi : G(Y) \to G(Y) \). It is not hard to check that the set of automorphisms of \( G(Y) \) forms a group under composition, which we denote \( \text{Aut}(G(Y)) \) (see Section 2.4 of [BH] for the definition of composition of morphisms).

**Definition 3.4.** A group \( H \) acts on \( G(Y) \) if there is a homomorphism

\[
\rho : H \to \text{Aut}(G(Y)).
\]

Our notation is as follows. Suppose \( H \) acts on \( G(Y) \). Then in particular, \( H \) acts on the scwol \( Y \) in the sense of Definition 2.3 above. We write the action of \( H \) on \( Y \) as \( \sigma \mapsto h.\sigma \) and \( a \mapsto h.a \), for \( h \in H \), \( \sigma \in V(Y) \) and \( a \in E(Y) \). The element \( h \in H \) induces the automorphism \( \Phi^h \) of \( G(Y) \). The data for \( \Phi^h \) is a family \((\phi^h_\sigma)_{\sigma \in V(Y)}\) of group isomorphisms \( \phi^h_\sigma : G_\sigma \to G_{h.\sigma} \), and a family of elements \((\phi^h(a))_{a \in E(Y)}\) with \( \phi^h(a) \in G_{t(h.a)} \), satisfying the definition of morphism above (Definition 3.3).
We say that the $H$-action is by simple morphisms if each $\Phi^h$ is simple, that is, if each $\phi^h(a) \in G_{\tau(h,a)}$ is the trivial element. Explicitly, for each $a \in E(Y)$ and each $h \in H$, the following diagram commutes.

$$
\begin{array}{ccc}
G_{h,i(a)} & \xrightarrow{\psi_{h,a}} & G_{h,i(a)} \\
\phi_{h,i(a)} & & \phi_{h,i(a)} \\
G_{i(a)} & \xrightarrow{\psi_a} & G_{i(a)}
\end{array}
$$

We note also that the composition of simple morphisms $\Phi^{h'} \circ \Phi^h$ is the simple morphism $\Phi^{h'h}$ with local maps

$$
\phi_{h'h} = \phi_{h',\sigma} \circ \phi_{h}.
$$

Finally we recall the definition of a covering of complexes of groups.

**Definition 3.5.** A morphism $\Phi : G(Y) \to H(Z)$ over a nondegenerate morphism of scwols $f : Y \to Z$ (see Section 1.5 of [BH]) is a covering of complexes of groups if further:

1. each $\phi_\sigma$ is injective; and
2. for each $\sigma \in V(Y)$ and $b \in E(Z)$ such that $t(b) = f(\sigma)$, the map on cosets

$$
\Phi_{\sigma/b} : \prod_{a \in f^{-1}(b)} G_\sigma / \psi_a(G_{i(a)}) \to H_{f(\sigma)} / \theta_b(H_{i(b)})
$$

induced by $g \mapsto \phi_\sigma(g)\phi(a)$ is a bijection.

### 3.2 The induced complex of groups and its properties

Suppose that a group $H$ acts by simple morphisms on a simple complex of groups $G(Y) = (G_\sigma, \psi_a)$. In this section we construct the complex of groups $H(Z)$ induced by this action, prove that $H(Z)$ is well-defined up to isomorphism of complexes of groups, and discuss faithfulness.

Let $Z$ be the quotient scw $Z = H \setminus Y$ and let $p : Y \to Z$ be the natural projection. For each vertex $\tau \in V(Z)$ choose a representative $\tau' \in V(Y)$ such that $p(\tau') = \tau$. Let $\text{Stab}_H(\tau')$ be the subgroup of $H$ fixing $\tau'$ and let $G_{\tau'}$ be the local group of $G(Y)$ at $\tau'$. Since the $H$-action is by simple morphisms, by Equation (1) above there is a group homomorphism $\zeta : \text{Stab}_H(\tau') \to \text{Aut}(G_{\tau'})$ given by $\zeta(h) = \phi^h_{\tau'}$. For each $\tau \in V(Z)$ we then define the local group $H_\tau$ to be the corresponding semidirect product of $G_{\tau'}$ by $\text{Stab}_H(\tau')$, that is,

$$
H_\tau := G_{\tau'} \rtimes \zeta \text{Stab}_H(\tau') = G_{\tau'} \rtimes \text{Stab}_H(\tau').
$$

For each edge $a \in E(Z)$ with $i(a) = \tau$ there is, since $H$ acts on $Y$ in the sense of Definition 2.3 above, a unique edge $\tau' \in E(Y)$ such that $p(\tau') = \tau$ and $i(\tau') = i(a) = \tau$. For each $a \in E(Z)$ choose an element $h_a \in H$ such that $h_a.t(\bar{a}) = t(a)$.

**Lemma 3.6.** Let $g \in G_{i(\bar{a})} = G_{i(a)}$ and $h \in \text{Stab}_H(\bar{a})$. Then the map

$$
\theta_a : (g, h) \mapsto (\phi^h_{i(\bar{a})} \circ \psi_\bar{a}(g), h_a h h_a^{-1})
$$

is a monomorphism $H_{i(\bar{a})} \to H_{i(a)}$. 

12
Proof. We will show that \( \theta_a \) is a group homomorphism. Since \( \phi_{t_{(\bar{a})}}^{h_a}, \psi_{\bar{t}} \) and the conjugation \( h \mapsto h_a h h_a^{-1} \) are all injective, the conclusion follows.

Let \( g, g' \in G_{t_{(\bar{a})}} \) and \( h, h' \in \text{Stab}_{H}(t_{{(\bar{a})}}) \). Note that since \( h \) and \( h' \) fix \( i(\bar{a}) = i(\bar{t}) \), they fix the edge \( \bar{a} \) and hence fix the vertex \( t(\bar{a}) \) as well. We have

\[
\theta_a((g, h)(g', h')) = \theta_a(g \phi_{t_{(\bar{a})}}^{h_a}(g'), h h_a^{-1})
\]

while

\[
\theta_a(g, h) \theta_a(g', h') = (\phi_{t_{(\bar{a})}}^{h_a} \circ \psi_{\bar{t}}(g), h a h a^{-1})(\phi_{t_{(\bar{a})}}^{h_a} \circ \psi_{\bar{t}}(g'), h a h a^{-1})
\]

After applying Equation (1) above to the map \( \phi_{a, h}^{h a, h a^{-1}} \), and some cancellations, it remains to show that

\[
\psi_{\bar{t}} \circ \phi_{t_{(\bar{a})}}^{h_a}(g') = \phi_{t_{(\bar{a})}}^{h_a} \circ \psi_{\bar{t}}(g').
\]

This follows from the fact that \( \Phi^h \) is a simple morphism with \( h \bar{a} = \bar{a} \).

To complete the construction of \( H(Z) \), for each composable pair of edges \((a, b)\) in \( E(Z) \), define

\[ h_{a,b} = h_a h_b h_{a,b}^{-1}. \]

One checks that \( h_{a,b} \in \text{Stab}_{H}(t_{(\bar{a})}) \) hence \( (1, h_{a,b}) \in H_{t_{(a)}} \). By abuse of notation we write \( h_{a,b} \) for \((1, h_{a,b})\).

Proposition 3.7. The data \( H(Z) = (H_\sigma, \theta_a, h_{a,b}) \) is a complex of groups.

Proof. Given Lemma 3.6 above, it remains to show that for each pair of composable edges \((a, b)\) in \( E(Z) \),

\[ \text{Ad}(h_{a,b}) \circ \theta_{ab} = \theta_a \circ \theta_b, \quad (2) \]

and that the cocycle condition holds. Let \((g, h) \in H_{t_{(b)}} = G_{t_{(b)}} \rtimes \text{Stab}_{H}(t_{(b)}) \). We compute

\[ \text{Ad}(h_{a,b}) \circ \theta_{ab}(g, h) = (\phi_{t_{(\bar{a})}}^{h_{a,b}} \circ \phi_{t_{(\bar{a})}}^{h_{a,b}} \circ \psi_{\bar{a}b}(g), h_{a,b} h a h b h_{a,b}^{-1} h_{a,b}^{-1}) \]

while

\[ \theta_a \circ \theta_b(g, h) = (\phi_{t_{(\bar{a})}}^{h_{a,b}} \circ \psi_{\bar{a}b} \circ \phi_{t_{(\bar{b})}}^{h_{a,b}} \circ \psi_{\bar{a}b}(g), h_{a,b} h b h b^{-1} h_{a,b}^{-1}). \]

By definition of \( h_{a,b} \) it remains to show equality in the first component. By Equation (1) and the definition of \( h_{a,b} \),

\[ \phi_{t_{(\bar{a})}}^{h_{a,b}} \circ \psi_{\bar{a}b} = \psi_{\bar{a}b} \circ \phi_{t_{(\bar{b})}}^{h_{a,b}} \circ \psi_{\bar{a}b}. \]

Hence it suffices to prove

\[ \phi_{t_{(\bar{a})}}^{h_{a,b}} \circ \psi_{\bar{a}b} = \psi_{\bar{a}b} \circ \phi_{t_{(\bar{b})}}^{h_{a,b}} \circ \psi_{\bar{a}b}. \quad (3) \]

Since \( G(Y) \) is a simple complex of groups, and \( \bar{ab} \) is the composition of the edges \( h_{u}^{-1} \bar{a} \) and \( \bar{b} \), we have

\[ \psi_{\bar{a}b} = \psi_{h_{u}^{-1} \bar{a}} \circ \psi_{\bar{b}}. \]
Applying this, and the fact that \( \phi_{t(ab)}^{h_t} \) is a simple morphism on the edge \( h_t^{-1} \tau \), we have

\[
\phi_{t(ab)}^{h_t} \circ \psi_{a} = \phi_{t(ab)}^{h_t} \circ \psi_{h_t^{-1} \tau} = \psi_{\tau} \circ \phi_{t(b)}^{h_t} \circ \psi_{b}.
\]

Hence Equation 3 holds.

The cococycle condition follows from the definition of \( h_{a,b} \). We conclude that \( H(Z) \) is a complex of groups. \( \square \)

We now have a complex of groups \( H(Z) \) induced by the action of \( H \) on \( G(Y) \). This construction depended on choices of lifts \( \tau \) and of elements \( h_a \in H \). We next show (in a generalisation of Section 2.9(2) of [BH]) that:

**Lemma 3.8.** The complex of groups \( H(Z) \) is well-defined up to isomorphism of complexes of groups.

**Proof.** Suppose we made a different choice of lifts \( \tau \) and elements \( h'_a \), resulting in a complex of groups \( H'(Z) = (H'_a, \theta'_a, h'_a, b) \). An isomorphism \( \Lambda = (\lambda, \lambda(a)) \) from \( H(Z) \) to \( H'(Z) \) over the identity map \( Z \rightarrow Z \) is constructed as follows. For each \( \tau \in V(Z) \), choose an element \( k_\tau \in H \) such that \( k_\tau \tau = \tau' \), and define a group isomorphism \( \lambda_\tau : H_\tau \rightarrow H'_\tau \) by

\[
\lambda_\tau(g, h) = (\phi_{k_\tau}^h(g), k_\tau h k_\tau^{-1}).
\]

For each \( a \in E(Z) \), define \( \lambda(a) = (1, k_{t(a)} h_{t(a)} k_{t(a)}^{-1} h_{t(a)}^{-1}) \). Note that by 2.9(2) of [BH], \( \lambda(a) \in H_{t(a)} \).

The verification that \( \Lambda = (\lambda, \lambda(a)) \) is an isomorphism of complexes of groups is straightforward. \( \square \)

**Lemma 3.9.** If \( G(Y) \) is faithful and the \( H \)-action on \( Y \) is faithful then \( H(Z) \) is faithful.

**Proof.** This follows from the construction of \( H(Z) \), and the characterisation of faithful complexes of groups in Proposition 38 of [LT]. \( \square \)

### 3.3 The induced covering

Suppose \( H \) acts by simple morphisms on a simple complex of groups \( G(Y) \), inducing a complex of groups \( H(Z) \) as in Section 2.2 above. In this section we construct a covering of complexes of groups \( \Lambda : G(Y) \rightarrow H(Z) \) over the quotient map \( p : Y \rightarrow Z \).

For \( \sigma \in V(Y) \), the local maps \( \lambda_\sigma : G_\sigma \rightarrow H_{p(\sigma)} \) are defined as follows. Recall that for each vertex \( \tau \in V(Z) \) we chose a lift \( \tau \in V(Y) \). Now for each \( \sigma \in V(Y) \), we choose \( k_\sigma \in H \) such that \( k_\sigma \sigma = p(\sigma) \). Hence \( \phi_\sigma^k \) is an isomorphism \( G_\sigma \rightarrow G_{p(\sigma)} \). The local map \( \lambda_\sigma : G_\sigma \rightarrow H_{p(\sigma)} \) is then defined by

\[
\lambda_\sigma : g \mapsto (\phi_\sigma^k(g), 1).
\]

Note that each \( \lambda_\sigma \) is injective.

For each edge \( a \in E(Y) \), define

\[
\lambda(a) = (1, k_{t(a)}^{-1} h_{t(a)} h_b^{-1})
\]

where \( p(a) = b \in E(Z) \). Note that, since \( H \) acts on \( Y \) in the sense of Definition 2.3 above, we have \( k_{t(a)} a = b \) hence \( k_{t(a)}^{-1} h_{t(b)} \) fixes \( t(b) \). Thus \( \lambda(a) \in H_{t(b)} \) as required.

**Proposition 3.10.** The map \( \Lambda = (\lambda_\sigma, \lambda(a)) \) is a covering of complexes of groups.
Proof. It may be checked that $\Lambda$ is a morphism of complexes of groups. As noted, each of the local maps $\lambda_\sigma$ is injective. It remains to show that for each $\sigma \in V(Y)$ and $b \in E(Z)$ such that $t(b) = p(\sigma) = \tau$, the map on cosets

$$\Lambda_{\sigma/b} : \left( \prod_{a \in p^{-1}(b)} G_\sigma/\psi_a(G_i(a)) \right) \to H_\tau/\theta_b(H_t(b))$$

induced by $g \mapsto \lambda_\sigma(g)\lambda(a) = (\phi_\sigma^{k_\sigma}(g), k_\sigma k^{-1}_{i(a)} h^{-1}_b)$ is a bijection.

We first show that $\Lambda_{\sigma/b}$ is injective. Suppose $a$ and $a'$ are in $p^{-1}(b)$ with $t(a) = t(a') = \sigma$, and suppose $g, g' \in G_\sigma$ with $g$ representing a coset of $\psi_a(G_i(a))$ in $G_\sigma$ and $g'$ a coset of $\psi_a'(G_i(a'))$ in $G_\sigma$. Assume that $\lambda_\sigma(g)\lambda(a)$ and $\lambda_\sigma(g')\lambda(a')$ belong to the same coset of $\theta_b(H_i(b))$ in $H_\tau$.

Looking at the second component of the semidirect product $H_\tau$, it follows from the definition of $\theta_b$ (Lemma 3.4 above) that for some $h \in \text{Stab}_H(i(b))$, $k_\sigma k^{-1}_{i(a)} h^{-1}_b = \left( k_\sigma k^{-1}_{i(a')} h^{-1}_b \right) \left( h_b h^{-1}_b \right) = k_\sigma k^{-1}_{i(a')} h^{-1}_b$.

Thus $k_{i(a')}k^{-1}_{i(a)} = h$ fixes $i(b)$. Hence $k^{-1}_{i(a')}k^{-1}_{i(a)}$ fixes $k^{-1}_{i(a')}i(b) = i(a)$, and so $k^{-1}_{i(a')}k^{-1}_{i(a)}$ fixes $a$. Thus $k_{i(a')}a = k_{i(a')}a = \overline{b} = k_{i(a')}a'$, hence $a = a'$.

Looking now at the first component of $\Lambda_{\sigma/b}(g)\lambda(a)$ and $\Lambda_{\sigma/b}(g')\lambda(a') = \Lambda_{\sigma/b}(g')\lambda(a)$ in the semidirect product $H_\tau$, by definition of $\theta_b$, for some $x \in G_{t(b)}$ we have

$$\phi_\sigma^{k_\sigma}(g) = \phi_\sigma^{k_\sigma}(g') \phi_{t(b)}^{k^{-1}_{i(a)} h^{-1}_b} \circ \phi_{t(b)}^{h_b h^{-1}_b} \circ \psi(x)$$

$$= \phi_\sigma^{k_\sigma}(g') \phi_\sigma^{k_\sigma} \circ \phi_{t(b)}^{k^{-1}_{i(a)} h^{-1}_b} \circ \psi(x).$$

Since $\phi_\sigma^{k_\sigma}$ is an isomorphism, and $k^{-1}_{i(a)} \overline{b} = a$, this implies

$$(g')^{-1}g = \phi_{t(b)}^{k_{i(a)} h^{-1}_b} \circ \psi(x) = \psi_a \circ \phi_{t(b)}^{k^{-1}_{i(a)} h^{-1}_b} \circ \psi(x) \in \psi_a(G_{i(a')})$$

as required. Thus the map $\Lambda_{\sigma/b}$ is injective.

To show that $\Lambda_{\sigma/b}$ is surjective, let $g \in G_\sigma$ and $h \in \text{Stab}_H(i(b))$, so that $(g, h) \in H_\tau$. Let $a$ be the unique edge of $Y$ with $t(a) = \sigma$ and such that $k_\sigma a = hh_b \overline{b}$. Let $g'$ be the unique element of $G_\sigma$ such that $\phi_\sigma^{k_\sigma}(g') = g \in G_{t(b)}$. We claim that $\lambda_\sigma(g')\lambda(a)$ lies in the same coset as $(g, h)$. Now

$$\lambda_\sigma(g')\lambda(a) = (\phi_\sigma^{k_\sigma}(g'), k_\sigma k^{-1}_{i(a)} h^{-1}_b) = (g, k_\sigma k^{-1}_{i(a)} h^{-1}_b)$$

so it suffices to show that $k_\sigma k^{-1}_{i(a)} h^{-1}_b \in hh_b \text{Stab}_H(i(b)) h^{-1}_b$. Equivalently, we wish to show that $h^{-1}_b h^{-1}_b k_\sigma k^{-1}_{i(a)}$ fixes $i(b)$. We have $k_{i(a), i}(a) = \overline{i(b)}$ by definition, and the result follows by our choice of $a$. Thus $\Lambda_{\sigma/b}$ is surjective.

Hence $\Lambda$ is a covering of complexes of groups. $\square$

### 3.4 The fundamental group

Suppose $H$ acts by simple morphisms on a simple complex of groups $G(Y)$, inducing a complex of groups $H(Z)$ as in Section 3.2 above. In this section we show that the fundamental group of $H(Z)$ is the semidirect product of the fundamental group of $G(Y)$ by $H$ (Proposition 3.11 below).
Let $\sigma_0$ be a vertex of $Y$ and let $p : Y \to Z$ be the natural projection. We refer the reader to Section 3 of [BH] for the definition of the fundamental group of $G(Y)$ at $\sigma_0$, denoted $\pi_1(G(Y), \sigma_0)$. We will use notation and results from that section in the following proof. Let $\pi_1(H(Z), p(\sigma_0))$ be the fundamental group of $H(Z)$ at $p(\sigma_0)$.

**Proposition 3.11.** There is a group isomorphism

$$\pi_1(H(Z), p(\sigma_0)) \cong \pi_1(G(Y), \sigma_0) \rtimes H.$$  

**Proof.** We will construct a short exact sequence which splits

$$1 \to \pi_1(G(Y), \sigma_0) \to \pi_1(H(Z), p(\sigma_0)) \to H \to 1.$$  

To obtain a monomorphism $\pi_1(G(Y), \sigma_0) \to \pi_1(H(Z), p(\sigma_0))$, we use the morphism of complexes of groups $\Lambda : G(Y) \to H(Z)$ defined in Section 3.3 above. By Proposition 3.6 of [BH], $\Lambda$ induces a natural homomorphism

$$\pi_1(\Lambda, \sigma_0) : \pi_1(G(Y), \sigma_0) \to \pi_1(H(Z), p(\sigma_0)).$$

Since $\Lambda$ is a covering (Proposition 3.10 above), Theorem 3.2 above implies that this map $\pi_1(\Lambda, \sigma_0)$ is in fact injective.

We next define a surjection $\pi_1(H(Z), p(\sigma_0)) \to H$. The group $H$ may be regarded as a complex of groups over a single vertex. There is then a canonical morphism of complexes of groups $\Phi : H(Z) \to H$, defined as follows. For each $\tau \in V(Z)$, the local map $\phi_\tau : H_\tau \to H$ is trivial on $G_\tau \leq H_\tau$, and is inclusion on $\text{Stab}_H(\tau) \leq H_\tau \leq H$. For each edge $b$ of $Z$, $\phi(b) = h_b$. By Proposition 3.6 of [BH], the morphism $\Phi$ induces a homomorphism of fundamental groups

$$\pi_1(\Phi, p(\sigma_0)) : \pi_1(H(Z), p(\sigma_0)) \to H.$$  

By 3.14 and Corollary 3.15 of [BH], if $G(Y)$ were a complex of trivial groups, this map would be surjective. Since the image of $\pi_1(\Phi, p(\sigma_0))$ does not in fact depend on the local groups of $G(Y)$, we have that in all cases, $\pi_1(\Phi, p(\sigma_0))$ is surjective, as required.

It follows from definitions that the image of the monomorphism $\pi_1(\Lambda, \sigma_0)$ is the kernel of the surjection $\pi_1(\Phi, p(\sigma_0))$, hence the sequence above is exact. Given the definition of $\Phi$ above, a section $\iota : H \to \pi_1(H(Z), p(\sigma_0))$ is not hard to construct.

This completes the proof of Theorem 3.1.

## 4 Proof of the Main Theorem

We now prove the Main Theorem and Corollary 1.2 stated in the introduction. Throughout this section, we adopt the notation of the Main Theorem, and assume that the vertices $s_1$ and $s_2$ of the nerve $L$, and the elements $\alpha_1$ and $\alpha_2$ of the group $A$ of label-preserving automorphisms of $L$, satisfy Conditions 1, 2 of its statement. In Section 4.1 we introduce notation, and construct a family of finite polyhedral complexes $Y_n$, for $n \geq 1$, and an infinite polyhedral complex $Y_\infty$. We then in Section 4.2 construct complexes of groups $G(Y_n)$ and $G(Y_\infty)$ over these spaces, and show that there are coverings of complexes of groups $G(Y_n) \to G(Y_1)$ and $G(Y_\infty) \to G(Y_1)$. In Section 4.3 we define the action of a finite group $H_n$ on $Y_n$, and of an infinite group $H_\infty$ on $Y_\infty$, and then in Section 4.4 we show that these actions extend to actions on the complexes of groups $G(Y_n)$ and $G(Y_\infty)$. In Section 4.5 we combine these results with Theorem 3.1 above to complete the proof of the Main Theorem. Corollary 1.2 is proved in Section 4.6.
4.1 The spaces $Y_n$ and $Y_\infty$

In this section we construct a family of finite polyhedral complexes $Y_n$ and an infinite polyhedral complex $Y_\infty$.

We first set up some notation. For $i = 1, 2$, let $q_i$ be the order of $\alpha_i$, assumed prime. It will be convenient to put, for all $k \geq 0$, $s_{2k+1} = s_1$ and $s_{2k+2} = q_2$, and similarly $\alpha_{2k+1} = \alpha_1$, $\alpha_{2k+2} = \alpha_2$, $q_{2k+1} = q_1$ and $q_{2k+2} = q_2$. Conditions (1)–(3) of the Main Theorem then become:

1. for all $n \geq 1$, $\alpha_n$ fixes the star of $s_{n+1}$ in $L$, and $\alpha_n(s_n) \neq s_n$;
2. for all $n \geq 1$, and all $t_n \neq s_n$ such that $t_n$ is the image of $s_n$ under some power of $\alpha_n$, $m_{s_nt_n} = \infty$; and
3. for all $n \geq 1$, all spherical special subgroups of $W$ which contain $s_n$ are halvable along $s_n$.

We now use the sequences $\{s_n\}$ and $\{\alpha_n\}$ to define certain elements and subsets of $W$. Let $w_1$ be the trivial element of $W$ and for $n \geq 2$ let $w_n$ be the product

$$w_n = s_1s_2 \cdots s_{n-1} \in W.$$ 

Denote by $W_{n,n}$ the one-element set $\{w_n\}$. For $n \geq 2$, and $1 \leq k < n$, in order to simplify notation, write $\alpha_{n-1}^{j_1} \cdots \alpha_k^{j_k}$ for the composition of automorphisms

$$\alpha_{n-1}^{j_1} \cdots \alpha_k^{j_k} = \alpha_{n-1}^{j_1} \cdots \alpha_k^{j_k},$$

where $0 \leq j_i < q_i$ for $k \leq i < n$. Let $w_{j_{n-1}, \ldots, j_k}$ be the element of $W$:

$$w_{j_{n-1}, \ldots, j_k} = w_n \alpha_{n-1}^{j_{n-1}}(s_{n-1}) \alpha_{n-2}^{j_{n-2}}(s_{n-2}) \cdots \alpha_{k+1}^{j_{k+1}}(s_{k+1}) \alpha_{k}^{j_k}(s_k).$$

(4)

Now for $n \geq 2$ and $1 \leq k < n$, define

$$W_{k,n} = \{w_{j_{n-1}, \ldots, j_k} \in W \mid 0 \leq j_i < q_i \text{ for } k \leq i < n\}.$$

Note that if $j_{n-1} = 0$ then $w_{j_{n-1}, \ldots, j_k} \in W_{k,n-1}$.

**Example:** Let $(W, S)$ be the Coxeter system in Example 1 of Section 2.2 above, with nerve $L$ shown in Figure 3 above. For $i = 1, 2$, let $\alpha_i \in A$ be the automorphism of $L$ which fixes the star of $s_{3-i}$ in $L$ and interchanges $s_i$ and $s_3$. Then if $m$ and $m'$ are both even, the Main Theorem applies to this example. (If $T = \{s\}$ then $W_T$ is halvable along $s$ with half$_s(W_T)$ the trivial group. If $T = \{s, t\}$ then $W_T$ is the dihedral group of order $2m_{st}$, and $W_T$ is halvable along $s$ if and only if $m_{st}$ is even, in which case half$_s(W_T)$ is the dihedral group of order $m_{st}$.) Note that $q_1 = q_2 = 2$, and so, for instance,

$$W_{1,3} = \{1, s_1\alpha_1(s_1), s_1s_2\alpha_2(s_2)\alpha_2(s_1), s_1s_2\alpha_2(s_2)\alpha_2\alpha_1(s_1)\}$$

$$W_{2,3} = \{s_1, s_1s_2\alpha_2(s_2)\}$$

$$W_{3,3} = \{s_1s_2\}.$$

The following lemma establishes key properties of the sets $W_{k,n}$.

**Lemma 4.1.** For all $n \geq 1$:

1. the sets $W_{1,n}$, $W_{2,n}$, $\ldots$, $W_{n,n}$ are pairwise disjoint; and
2. for all $1 \leq k < n$, if

$$w_{j_{n-1}, \ldots, j_k} = w_{j'_{n-1}, \ldots, j'_k}$$

(where $0 \leq j_i < q_i$ for $k \leq i < n$) then $j_k = j'_k$, $j_{k+1} = j'_{k+1}$, $\ldots$, and $j_{n-1} = j'_{n-1}$.
Proof. Given \(1 \leq k \leq k' < n\), with \(0 \leq j_i < q_i\) for \(k \leq i < n\) and \(0 \leq j'_i < q_i\) for \(k' \leq i < n\), suppose
\[
 w_{j_n-1 \ldots j_k} = w_{j'_n-1 \ldots j'_{k'}}. \tag{5}
\]
Then
\[
\alpha^{j_n-1}(s_{n-1})\alpha^{j_{n-1}j_{n-2}}(s_{n-2}) \ldots \alpha^{j_{k+1}j_k}(s_k) = \alpha^{j'_n-1}(s_{n-1})\alpha^{j'_{n-1}j'_{n-2}}(s_{n-2}) \ldots \alpha^{j'_{k+1}j'_k}(s_k).
\]
By Condition (1) above, for each \(k \leq i < n\), the automorphism \(\alpha_i\) fixes \(s_{i+1}\), thus
\[
\alpha^{j_n-1 \ldots j_{i+1}}(s_{i+1})\alpha^{j_{i+1} \ldots j_i}(s_i) = \alpha^{j_n-1 \ldots j_{i+1}}(s_{i+1})\alpha^{j_{i+1} \ldots j_i}(s_i) = \alpha^{j_n-1 \ldots j_i}(s_{i+1}s_i).
\]
Also by Condition (1), since \(\alpha_i\) fixes the star of \(s_{i+1}\) but \(\alpha_i(s_i) \neq s_i\), we have \(m_{s_{i+1}s_i} = \infty\). Since \(\alpha^{j_n-1 \ldots j_i}\) is a label-preserving automorphism, it follows that the product of the two generators
\[
\alpha^{j_n-1 \ldots j_{i+1}}(s_{i+1})\alpha^{j_{i+1} \ldots j_i}(s_i)
\]
has infinite order, for each \(k \leq i < n\). Similarly for each \(k' \leq i < n\). Thus the only way for Equation (5) to hold is if \(k = k'\), and for each \(k \leq i < n\), \(\alpha^i_j(s_i) = \alpha^i_{j'}(s_i)\). Since \(\alpha_i\) is of prime order \(q_i\) and we specified \(0 \leq j_i < q_i\), the result follows. 

For \(n \geq 1\), and \(1 \leq k \leq n\), define \(Y_{k,n}\) to be the set of chambers
\[
Y_{k,n} := \{ wK \mid w \in W_{k,n} \}. \]
Recall that we are writing \(wK\) for the pair \((w, K)\). By Lemma 4.1 above, for fixed \(n\), the sets \(Y_{1,n}, \ldots, Y_{n,n}\) are pairwise disjoint. We now define \(Y_n\) to be the polyhedral complex obtained by “gluing together” the chambers in \(Y_{1,n}, \ldots, Y_{n,n}\), using the same relation \(\sim\) as in the Davis complex \(\Sigma\) for \((W, S)\). More precisely,
\[
Y_n := \left( \prod_{k=1}^{n} Y_{k,n} \right) / \sim
\]
where, for \(x, x' \in K\), we have \((w, x) \sim (w', x')\) if and only if \(x = x'\) and \(w^{-1}w' \in W_{s(x)}\). Note that \(Y_1 = Y_{1,1} = K\). To define \(Y_\infty\), for each \(k \geq 1\), noting that \(W_{k,n}\) is only defined for \(1 \leq k \leq n\), put
\[
W_{k,\infty} := \bigcup_{n=k}^{\infty} W_{k,n}.
\]
Then \(Y_{k,\infty}\) is the set of chambers
\[
Y_{k,\infty} := \{ wK \mid w \in W_{k,\infty} \}.
\]
Similarly to the finite case, the sets \(Y_{1,\infty}, Y_{2,\infty}, \ldots\) are pairwise disjoint, and we define
\[
Y_\infty := \left( \prod_{k=1}^{\infty} Y_{k,\infty} \right) / \sim
\]
for the same relation \(\sim\). Note that there are natural strict inclusions as subcomplexes
\[
Y_1 \subset Y_2 \subset \cdots \subset Y_n \subset \cdots Y_\infty.
\]
(In fact, \(Y_n\) and \(Y_\infty\) are subcomplexes of the Davis complex \(\Sigma\), but we will not adopt this point of view.) We define a mirror of \(Y_n\) or \(Y_\infty\) to be an interior mirror if it is contained in more than one chamber.
Example: Let \((W, S)\), \(\alpha_1\) and \(\alpha_2\) be as in the previous example of this section. To indicate the construction of \(Y_n\) and \(Y_\infty\) in this case, Figure 7 below depicts the dual graph for \(Y_4\), that is, the graph with vertices the chambers of \(Y_4\), and edges joining adjacent chambers. The edges are labelled with the type of the corresponding interior mirror. Figure 8 sketches the dual graph for \(Y_\infty\).

We now describe features of \(Y_n\) and \(Y_\infty\) which will be needed below. The first lemma follows from the construction of \(Y_n\) and \(Y_\infty\) and Lemma 4.1 above.

Lemma 4.2. Let \(w = w_{j_{n-1}, \ldots, j_k} \in W_{k,n}\). All of the chambers of \(Y_n\) to which \(wK \in Y_{k,n}\) is adjacent are described by the following.

1. For \(n \geq 1\) and \(1 \leq k < n\), the chamber \(wK\) is adjacent to exactly one chamber of \(Y_{k+1,n}\), namely it is \(\alpha^{j_{n-1}, \ldots, j_k(s_k)}\)-adjacent to the chamber \(w_{j_{n-1}, \ldots, j_{k+1}, n}\) of \(Y_{k+1,n}\).

2. For \(n \geq 2\) and \(1 \leq k \leq n\), the chamber \(wK\) is adjacent to exactly \(q_{k-1}\) distinct chambers of \(Y_{k-1,n}\), namely for each \(0 \leq j_{k-1} < q_{k-1}\), the chamber \(wK\) is \(\alpha^{j_{n-1}, \ldots, j_{k}, j_{k-1}(s_{k-1})}\)-adjacent to the chamber \(w_{j_{n-1}, \ldots, j_{k-1}, j_{k-1}, j_{k-1}, n}\) of \(Y_{k-1,n}\).

Similarly for \(Y_\infty\).

Corollary 4.3. 1. Any vertex of \(Y_n\) is contained in at most two distinct chambers of \(Y_n\), and similarly for \(Y_\infty\).

2. Any two interior mirrors of \(Y_n\) or \(Y_\infty\) are disjoint.

Proof. Suppose \(\sigma\) is a vertex of \(Y_n\), contained in the chamber \(wK\), where \(w\) is as in Lemma 4.2 above. If \(\sigma\) is contained in more than one chamber of \(Y_n\) or \(Y_\infty\), then \(\sigma\) is contained in an interior
Proof. The vertex of is given by sending the chamber \( wK \) and \( wsK \). Now suppose \( s \) is one of the \( q_{k-1} \) images of \( s_{k-1} \) under some element of \( A \). Suppose \( s \) is in the image of \( s_k \). Condition 1 of the Main Theorem implies that \( m_{s_k,s_{k-1}} = \infty \). Hence the mirror \( K_s \) is disjoint from each of the \( q_{k-1} \) mirrors of types the \( q_{k-1} \) images of \( s_{k-1} \). Therefore the only chambers of \( Y_n \) which contain \( \sigma \) are the two chambers \( wK \) and \( wsK \). Now suppose \( s \) is one of the \( q_{k-1} \) images of \( s_{k-1} \) under some element of \( A \). Condition 2 of the Main Theorem implies that the mirrors of types each of these images are pairwise disjoint, and so again \( \sigma \) is contained in only two distinct chambers of \( Y_n \). Similarly, any two interior mirrors of \( Y_n \) or \( Y_\infty \) are disjoint. 

Corollary 4.4. For all \( n \geq 2 \), there are \( q_{n-1} \) disjoint subcomplexes of \( Y_n \), denoted \( Y_{j_{n-1}} \) for \( 0 \leq j_{n-1} < q_{n-1} \), each isomorphic to \( Y_{n-1} \), and with \( Y_{j_{n-1}} = Y_{n-1} \). For each \( 0 \leq j_{n-1} < q_{n-1} \), the subcomplex \( Y_{j_{n-1}} \) is attached to the chamber \( w_n K = s_1 s_2 \cdots s_{n-1} K \) of \( Y_n \) along its mirror of type \( \alpha^{j_{n-1}}(s_{n-1}) \). An isomorphism 

\[
F_{j_{n-1}} : Y_{n-1} \to Y_{j_{n-1}}
\]

is given by sending the chamber \( w_{j_{n-2},...,j_k} K \) in \( Y_{k,n-1} \) to the chamber \( w_{j_{n-1},j_{n-2},...,j_k} K \) in \( Y_{k,n} \), and the vertex of \( w_{j_{n-2},...,j_k} K \) of type \( T \) to the vertex of \( w_{j_{n-1},j_{n-2},...,j_k} K \) of type \( \alpha^{j_{n-1}}(T) \), for each spherical subset \( T \) of \( S \).

Proof. By induction on \( n \), using Lemma 12 and Corollary 13 above.

4.2 Complexes of groups \( G(Y_n) \) and \( G(Y_\infty) \)

We now construct complexes of groups \( G(Y_n) \) over each \( Y_n \), and \( G(Y_\infty) \) over \( Y_\infty \), and show that there are coverings \( G(Y_n) \to G(Y_1) \) and \( G(Y_\infty) \to G(Y_1) \). To simplify notation, write \( Y \) for \( Y_n \) or \( Y_\infty \).

To define the local groups of \( G(Y) \), let \( \sigma \) be a vertex of \( Y \), of type \( T \). By Corollary 13 above, \( \sigma \) is contained in at most two distinct chambers of \( Y \). If \( \sigma \) is only contained in one chamber of \( Y \), put \( G_\sigma = W_T \). If \( \sigma \) is contained in two distinct chambers of \( Y \), then by Corollary 13 above \( \sigma \) is contained in a unique interior mirror \( K_s \), with \( s \in T \). By the construction of \( Y \), \( s \) is in the \( A \)-orbit of some \( s_n \), \( n \geq 1 \). By Condition 3 of the Main Theorem, it follows that the group \( W_T \) is halvable along \( s \). We define the local group at \( \sigma \) to be \( G_\sigma = \text{half}_s(W_T) \).

The monomorphisms between local groups are defined as follows. Let \( a \) be an edge of \( Y \), with \( i(a) \) of type \( T' \) and \( t(a) \) of type \( T' \), so that \( T \subseteq T' \). If both of the vertices \( i(a) \) and \( t(a) \) are contained in a unique chamber of \( Y \), then the monomorphism \( \psi_a \) along this edge is defined to be the natural inclusion \( W_T \to W_{T'} \). If \( i(a) \) is contained in two distinct chambers, then \( i(a) \) is contained in a unique interior mirror \( K_s \), with \( s \in T \). Thus \( s \in T' \) as well, and so \( t(a) \) is also contained in the mirror...
$K_s$. From the definitions of $\text{half}_s(W_T)$ and $\text{half}_s(W_T')$, it follows that there is a natural inclusion $\text{half}_s(W_T) \hookrightarrow \text{half}_s(W_T')$, and we define $\psi_a$ to be this inclusion. Finally suppose $i(a)$ is contained in a unique chamber of $Y$ but $t(a)$ is contained in two distinct chambers of $Y$. Then for some $k \geq 1$, $i(a)$ is in a chamber of $Y_{k,n}$ (respectively, $Y_{k,\infty}$), and $t(a)$ is either in $Y_{k-1,n}$ or in $Y_{k+1,n}$ (respectively, in $Y_{k-1,\infty}$ or $Y_{k+1,\infty}$). Moreover $t(a)$ is contained in a unique interior mirror $K_s$, with $s \in T' - T$. If $t(a)$ is in $Y_{k-1,n}$ (respectively, $Y_{k-1,\infty}$), then we define $\psi_a$ to be the natural inclusion $W_T \hookrightarrow \text{half}_s(W_T')$. If $t(a)$ is in $Y_{k+1,n}$ (respectively, $Y_{k+1,\infty}$), then we define $\psi_a$ to be the monomorphism defined on the generators $t \in T$ of $W_T$ by $\psi_a(t) := sts \in \text{half}_s(W_T')$, that is, $\psi_a = \text{Ad}(s)$.

It is not hard to verify that for all pairs of composable edges $(a,b)$ in $Y$, $\psi_{ab} = \psi_a \circ \psi_b$. Hence we have constructed simple complexes of groups $G(Y_n)$ and $G(Y_{\infty})$ over $Y_n$ and $Y_{\infty}$ respectively. Note that these complexes of groups are faithful, since by construction the local group at each vertex of type $\emptyset$ is trivial. Note also that $G(Y_1)$ is the same complex of groups as constructed in Section 2.8 above, which has fundamental group $W$ and universal cover $\Sigma$.

**Example:** Let $(W, S)$, $a_1$ and $a_2$ be as in the examples in Section 4.1 above. The complex of groups $G(Y_2)$ is sketched in Figure 9. From left to right, the three chambers here are $K$, $s_1K$ and $s_1a_1(s_1K) = s_1s_3K$. We denote by $D_{2m}$ the dihedral group of order $2m$, with $D_m$ the dihedral group of order $m$, and similarly for $D_{2m'}$ and $D_{m'}$ (recall that $m$ and $m'$ are even).

![Figure 9: Complex of groups $G(Y_2)$](image-url)

**Proposition 4.5.** There are coverings of complexes of groups $G(Y_n) \to G(Y_1)$ and $G(Y_{\infty}) \to G(Y_1)$.

**Proof.** Let $f_n : Y_n \to Y_1$ and $f_{\infty} : Y_{\infty} \to Y_1$ be the maps sending each vertex of $Y_n$ or $Y_{\infty}$ of type $T$ to the unique vertex of $Y_1 = K$ of type $T$. Then by construction of $Y_n$ and $Y_{\infty}$, the maps $f_n$ and $f_{\infty}$ are nondegenerate morphisms of scows. We define coverings $\Phi_n : G(Y_n) \to G(Y_1)$ and $\Phi_{\infty} : G(Y_{\infty}) \to G(Y_1)$ over $f_n$ and $f_{\infty}$ respectively. To simplify notation, write $Y$ for respectively $Y_n$ or $Y_{\infty}$, $f$ for respectively $f_n$ or $f_{\infty}$, and $\Phi$ for respectively $\Phi_n$ or $\Phi_{\infty}$.

Let $\sigma$ be a vertex of $Y$, of type $T$. If the local group at $\sigma$ is $G_\sigma = W_T$ then the map of local groups $\phi_\sigma : G_\sigma \to W_T$ is the identity map. If the local group at $\sigma$ is half$_s(W_T)$, for some $s \in T$, then $\phi_\sigma : \text{half}_s(W_T) \to W_T$ is the natural inclusion as an index 2 subgroup. To define elements $\phi(a)$,
it is enough to verify that the inclusion half
half
F or this, suppose that
w, w
there is a unique
inclusion
W
K
unique edge
a
G
φ
H
and that of an infinite group
σ/b
Y
4.3 Group actions on
if the monomorphism ψα in G(Y) is natural inclusion, define φ(a) = 1. If ψα is Ad(s), then define
φσ = s. It is then easy to check that, by construction, Φ is a morphism of complexes of groups.

To show that Φ is a covering of complexes of groups, we first observe that each of the local maps
φσ is injective. Now fix σ a vertex of Y, of type T', and b an edge of Y1 = K such that t(b) = f(σ),
with i(b) of type T (hence T ⊆ T'). We must show that the map
Φσ/h = \coprod_{a \in f^{-1}(b)} Gσ/ψα(Gi(a)) → W_T/W_T
induced by g → φσ(g)φ(a) is a bijection, where Gσ and Gi(a) are the local groups of G(Y).

First suppose that σ is contained in a unique chamber of Y. Then by construction, there is a
unique edge a of Y with i(a) of type T and t(a) = σ, hence a unique edge a ∈ f^{-1}(b) with t(a) = σ.
Moreover, Gσ = W_T, Gi(a) = W, the monomorphism ψα is natural inclusion hence φ(a) = 1, and
φσ : Gσ → W_T is the identity map. Hence Φσ/h is a bijection in this case.

Now suppose that σ is contained in two distinct chambers of Y. Then σ is contained in a unique
interior mirror Kσ of Y, with s ∈ T'. Assume first that s ∈ T as well. Then there is a unique
dege a of Y with i(a) of type T and t(a) = σ. This edge is also contained in the mirror Kσ. Hence
there is a unique a ∈ f^{-1}(b) with t(a) = σ. By construction, we have Gσ = half_s(W_T), the map
φσ : Gσ → W_T is natural inclusion as an index 2 subgroup, Gi(a) = half_s(W), the map ψα is
natural inclusion, and φ(a) trivial. Since the index [W_T : W] = [half_s(W_T)] is finite, it is enough to verify that the inclusion half_s(W_T) → W_T induces an injective map on cosets
half_s(W_T)/half_s(W) → W_T/W.

For this, suppose that w, w' ∈ half_s(W_T) and that wW_T = w'W_T in W_T. Then w^{-1}w' ∈ W_T ∩ half_s(W_T). By definitions, it follows that w^{-1}w' ∈ half_s(W), as required.

Now assume that σ is contained in the interior mirror Kσ, with s /∈ T. There are then two
deges a_1, a_2 ∈ f^{-1}(b) such that t(a_1) = t(a_2) = σ. Without loss of generality, ψa_1 is natural
inclusion W_T → half_s(W_T) and φ(a_1) = 1, while ψa_2(g) = sgs with φ(a_2) = s. Since the index
half_s(W_T) : W_T = 1/2[W_T : W] is finite, it is enough to show that the map on cosets Φσ/h is
surjective. Let w ∈ W_T. If w ∈ half_s(W_T) ≤ W_T, then the image of the coset wψa_1(Gi(a_1)) = wW_T
in Gσ is the coset wW_T in W_T. If w /∈ half_s(W_T), then since half_s(W_T) has index 2 in W_T, and
s /∈ half_s(W), there is a w' ∈ half_s(W_T) ≤ W_T such that w = sW. The image of the coset
w'ψa_2(Gi(a_2)) = w'(sW) is then the coset w'φ(a_2)W_T = w'sW_T = wW_T in W_T. Thus Φσ/h is surjective, as required.

We conclude that Φ is a covering of complexes of groups.

4.3 Group actions on Y_n and Y_∞
In this section we construct the action of a finite group H_n on Y_n in the sense of Definition 2.8.3 above,
and that of an infinite group H_∞ on Y_∞.

We first define the groups H_n and H_∞. For each n ≥ 1, let C_{q_n} denote the cyclic group of order
q_n. We define H_1 to be the trivial group and H_2 = C_{q_1}. For n ≥ 3, we define H_n to be the wreath product
H_n = H_{n-1} \wr C_{q_{n-1}} = \cdots ((C_{q_1} \wr (C_{q_2} \wr C_{q_3})) \cdots ) \wr C_{q_{n-1}} = C_{q_1} \wr C_{q_2} \wr \cdots \wr C_{q_{n-1}},
that is, H_n is the semidirect product by C_{q_{n-1}} of the direct product of q_{n-1} copies of H_{n-1}, where
C_{q_{n-1}} acts on this direct product by cyclic permutation of coordinates. Note that H_n is a finite
We define $H_\infty$ to be the infinite iterated (unrestricted) wreath product

$$H_\infty := C_{q_1} \wr C_{q_2} \wr \cdots \wr C_{q_{n-1}} \wr \cdots$$

We then have natural inclusions

$$H_1 < H_2 < \cdots < H_n < \cdots < H_\infty.$$ 

The following lemma will be needed for the proof of Corollary 1.2 in Section 4.6 below.

**Lemma 4.6.** The group $H_\infty$ is not finitely generated.

**Proof.** By definition of $H_\infty$, for any nontrivial $h \in H_\infty$ there is an $n \geq 1$ such that $h \in H_n$.  

We now define the actions of $H_n$ and $H_\infty$ on $Y_n$ and $Y_\infty$ respectively. This uses the label-preserving automorphisms $\alpha_n \in A$. Note that the action of $A$ on the nerve $L$ extends to the chamber $K$, fixing the vertex of type $\emptyset$. This action does not in general have a strict fundamental domain. Inconveniently, this action also does not satisfy Condition (3) of Definition 2.3 above, since for any nontrivial $\alpha \in A$, there is an edge $a$ of $K$ with $i(a)$ of type $\emptyset$ but $\alpha(a) \neq a$. However, to satisfy Definition 2.3 it suffices to define actions on $Y_n$ and $Y_\infty$, and then extend in the obvious way to the scwols which are the barycentric subdivisions of these spaces, with naturally oriented edges.

For each $n \geq 1$ fix a generator $a_n$ for the cyclic group $C_{q_n}$. Recall that $\alpha_n \in A$ has order $q_n$. Thus for any $\alpha \in A$, there is a faithful representation $C_{q_n} \to A$, given by $a_n \mapsto \alpha \alpha_n \alpha^{-1}$. Recall also that $\alpha_n$ fixes the star in $L$ of the vertex $s_{n+1}$, and that $\alpha_n(s_n) \neq s_n$. Hence $a_n \mapsto \alpha \alpha_n \alpha^{-1}$ induces an action of $C_{q_n}$ on the chamber $K$, which fixes pointwise the mirror of type $\alpha(s_{n+1})$, and permutes cyclically the set of mirrors of types $\alpha \alpha_n^{j+1}(s_n)$, for $0 \leq j < q_n$.

We define the action of $H_n$ on $Y_n$ inductively, as follows. The group $H_1$ is trivial. For $n \geq 2$, assume that the action of $H_{n-1}$ on $Y_{n-1}$ has been given. The subgroup $C_{q_{n-1}}$ of $H_n$ then fixes the chamber $w_n K = s_1 s_2 \cdots s_{n-1} K$ of $Y_{n-1}$ setwise, and acts on this chamber via $a_{n-1} \mapsto \alpha_{n-1}$. By the discussion above, this action fixes pointwise the mirror of type $s_n$ of $w_n K$, and permutes cyclically the $q_{n-1}$ mirrors of types $\alpha_{n-1}^{j+1}(s_{n-1})$, with $0 \leq j < q_{n-1}$, along which (by Lemma 4.3 above), $q_{n-1}$ disjoint subcomplexes of $Y_n$, each isomorphic to $Y_{n-1}$, are attached.

By induction, a copy of $H_{n-1}$ in $H_n$ acts on each of these copies of $Y_{n-1}$ in $Y_n$. More precisely, for $0 \leq j_{n-1} < q_{n-1}$, the $j_{n-1}$-st copy of $H_{n-1}$ in $H_n$ acts on the subcomplex $Y_{n-1}^{j_{n-1}}$ of $Y_{n-1}$ of Lemma 4.3 above. This action is given by conjugating the (inductively defined) action of $H_{n-1}$ on $Y_{n-1} \subset Y_n$ by the isomorphism $F^n_{n-1} : Y_{n-1} \to Y_{n-1}^{j_{n-1}}$ in Lemma 4.3. By definition, the action of $C_{q_{n-1}}$ cyclically permutes the subcomplexes $Y_{n-1}^{j_{n-1}}$, and so we have defined an action of $H_n$ on $Y_n$. The action of $H_\infty$ on $Y_\infty$ is similar.

We now describe the fundamental domains for these actions. For each $n \geq 1$ and each $1 \leq k \leq n$, observe that $H_n$ acts transitively on the set of chambers $Y_k$. Let $K_1 := K$, and for $n \geq 2$ let $K_n$ be the quotient of the chamber $w_n K = s_1 s_2 \cdots s_{n-1} K$ by the action of $C_{q_{n-1}} \leq H_n$ as defined above. In $K_n$, the mirrors of types $\alpha_{n-1}^{j+1}(s_{n-1})$, for $0 \leq j < q_{n-1}$, have been identified. By abuse of notation, we refer to these identified mirrors as the mirror of type $s_{n-1}$ of $K_n$. Note also that $C_{q_{n-1}} \leq H_n$ fixes pointwise the mirror of type $s_n$ of $w_n K$, and so we may speak of the mirror of type $s_n$ of $K_n$. Then a fundamental domain for the action of $H_n$ on $Y_n$ is the finite complex

$$Z_n := (K_1 \cup K_2 \cup \cdots \cup K_n) / \sim,$$

where $\sim$ means we identify the $s_{1-1}$-mirrors of $K_{i-1}$ and $K_i$, for $1 \leq i < n$. Similarly, a fundamental domain for the action of $H_\infty$ on $Y_\infty$ is the infinite complex

$$Z_\infty := (K_1 \cup K_2 \cup \cdots \cup K_n \cup \cdots) / \sim.$$
Finally we describe the stabilisers in \( H_n \) and \( H_\infty \) of the vertices of \( Y_n \) and \( Y_\infty \). Let \( wK \) be a chamber of \( Y_n \) or \( Y_\infty \). Then there is a smallest \( k \geq 1 \) such that \( wK \in Y_k \). By construction, it follows that the stabiliser in \( H_n \) or \( H_\infty \) of any vertex in the chamber \( wK \) is a subgroup of the finite group \( H_k \). Hence \( H_n \) and \( H_\infty \) act with finite stabilisers. Note also that for every \( n \geq 1 \), the action of \( H_n \) fixes the vertex of type \( \emptyset \) in the chamber \( w_nK \). We may thus speak of the vertex of type \( \emptyset \) in the quotient \( K_n \) defined above. In fact, in the fundamental domains \( Z_n \) and \( Z_\infty \) defined above, the vertex \( \emptyset \) in \( K_n \), for \( n \geq 1 \), has a lift in \( Y_n \) or \( Y_\infty \) with stabiliser the finite group \( H_n \). We observe also that the actions of \( H_n \) and \( H_\infty \) are faithful, since the stabiliser of the vertex of type \( \emptyset \) of \( K_1 = K \) is the trivial group \( H_1 \). Figure 10 shows \( Z_\infty \) and the stabilisers of (lifts of) its vertices of type \( \emptyset \) for the example in Section 4.1 above.

![Figure 10: Fundamental domain \( Z_\infty \)](image)

### 4.4 Group actions on \( G(Y_n) \) and \( G(Y_\infty) \)

In this section we show that the actions of \( H_n \) and \( H_\infty \) on \( Y_n \) and \( Y_\infty \), defined in Section 4.3 above, extend to actions (by simple morphisms) on the complexes of groups \( G(Y_n) \) and \( G(Y_\infty) \). To simplify notation, write \( H \) for \( H_n \) or \( H_\infty \), \( Y \) for \( Y_n \) or \( Y_\infty \), and \( Z \) for \( Z_n \) or \( Z_\infty \). Technically, instead of working with \( G(Y) \), we work with the corresponding naturally defined complex of groups over the barycentric subdivision of \( Y \), so that the action of \( H \) satisfies Definition 2.3 above. By abuse of notation we will however continue to write \( G(Y) \).

Recall that for \( \sigma \) a vertex of \( Y \) of type \( T \), the local group \( G_\sigma \) is either \( W_T \) or \( \text{half}_s(W_T) \), and the latter occurs if and only if \( \sigma \) is contained in an interior \( s \)–mirror of \( Y \) with \( s \in T \). Let \( wK \) be a chamber of \( Y \) and let \( h \in H \). By definition of the \( H \)–action, there is an \( \alpha \in A \) such that for each vertex \( \sigma \) in \( wK \), with \( \sigma \) of type \( T \), the vertex \( h.\sigma \) of \( h.wK \) has type \( \alpha(T) \). Moreover, if \( \sigma \) is contained in an interior \( s \)–mirror then \( h.\sigma \) is contained in an interior \( \alpha(s) \)–mirror. We may thus define the local map \( \phi_\alpha^h : G_\sigma \to G_{h.\sigma} \) by \( \phi_\alpha^h(t) = \alpha(t) \) for each \( t \in T \), and (if \( G_\sigma = \text{half}_s(W_T) \)), \( \phi_\alpha^h(sts) = \alpha(s)\alpha(t)\alpha(s) \). Then \( \phi_\alpha^h \) is an isomorphism either \( W_T \to W_{\alpha(T)} \), or \( \text{half}_s(W_T) \to \text{half}_{\alpha(s)}(W_{\alpha(T)}) \), as appropriate. It is not hard to verify that these local maps define an action of \( H \) on \( G(Y) \) by simple morphisms.

### 4.5 Conclusion

In this section we combine the results of Sections 4.1, 4.4 above to complete the proof of the Main Theorem.

Recall that \( G(Y_1) \) is developable with universal cover \( \Sigma \) (see Section 2.2). By Proposition 3.10 and Theorem 4.2 above, it follows that the complexes of groups \( G(Y_n) \) and \( G(Y_\infty) \) are developable with universal cover \( \Sigma \). Let \( H(Z_n) \) be the complex of groups induced by \( H_n \) acting on \( G(Y_n) \), and \( H(Z_\infty) \) that induced by \( H_\infty \) acting on \( G(Y_\infty) \). By Theorem 3.3 above, there are coverings of complexes of
groups $G(Y_n) \to H(Z_n)$ and $G(Y_\infty) \to H(Z_\infty)$. Hence (by Theorem 4.22 above) each $H(Z_n)$ and $H(Z_\infty)$ is developable with universal cover $\Sigma$.

Let $\Gamma_n$ be the fundamental group of $H(Z_n)$ and $\Gamma$ the fundamental group of $H(Z_\infty)$. Since the complexes of groups $G(Y_n)$ and $G(Y_\infty)$ are faithful, and the actions of $H_n$ and $H_\infty$ are faithful, Theorem 5.1 above implies that $H(Z_n)$ and $H(Z_\infty)$ are faithful complexes of groups. Thus $\Gamma_n$ and $\Gamma$ may be identified with subgroups of $G = \text{Aut}(\Sigma)$. Now $G(Y_n)$ and $G(Y_\infty)$ are complexes of finite groups, and the $H_n$– and $H_\infty$–actions have finite vertex stabilisers. Hence by construction, $H(Z_n)$ and $H(Z_\infty)$ are complexes of finite groups. Therefore $\Gamma_n$ and $\Gamma$ are discrete subgroups of $G$. Since the fundamental domain $Z_n$ is finite, it follows that each $\Gamma_n$ is a uniform lattice. To show that $\Gamma$ is a nonuniform lattice, we use the normalisation of Haar measure $\mu$ on $G = \text{Aut}(\Sigma)$ defined in Section 2.1 above, with the $G$–set $V$ the set of vertices of $\Sigma$ of type $\emptyset$. Since the local groups of $H(Z_\infty)$ at the vertices of type $\emptyset$ in $Z_\infty$ are $H_1$, $H_2$, . . . , we have

$$\mu(\Gamma\backslash G) = \sum_{n=1}^{\infty} \frac{1}{|H_n|}.$$ 

This series converges (see Equation 4.1 above for the order of $H_n$, and note that each $q_n \geq 2$). We conclude that $\Gamma$ is a nonuniform lattice in $G$. Moreover, as the covolumes of the uniform lattices $\Gamma_n$ are the partial sums of this series, we have $\mu(\Gamma_n\backslash G) \to \mu(\Gamma\backslash G)$, as required. This completes the proof of the Main Theorem.

4.6 Proof of Corollary 1.2

The nonuniform lattice $\Gamma$ is the fundamental group of the complex of groups $H(Z_\infty)$ induced by the action of $H_\infty$ on $G(Y_\infty)$. By Theorem 5.1 above, there is an isomorphism $\Gamma \cong \pi_1(G(Y_\infty)) \rtimes H_\infty$. Hence there is a surjective homomorphism $\Gamma \to H_\infty$. Since $H_\infty$ is not finitely generated (Lemma 4.10 above), we conclude that $\Gamma$ is not finitely generated.

5 Examples

In this section we describe several infinite families of examples to which the Main Theorem applies. By the dimension of the Davis complex $\Sigma$ for a Coxeter system $(W, S)$, we mean the maximum cardinality of a spherical subset of $S$. We note that there may be maximal spherical special subgroups $W_T$ with $|T|$ strictly less than $\text{dim}(\Sigma)$.

5.1 Two-dimensional examples

If $\text{dim}(\Sigma) = 2$ then the nerve of the Coxeter system $(W, S)$ is a graph $L$ with vertex set $S$ and two vertices $s$ and $t$ joined by an edge if and only if $m_{st}$ is finite. Assume for simplicity that for some integer $m \geq 2$ all finite $m_{st} = m$. Then $\Sigma$ is the barycentric subdivision of a polygonal complex $X$, with all 2–cells of $X$ regular Euclidean $2m$–gons, and the link of every vertex of $X$ the graph $L$. Such an $X$ is called a $(2m, L)$–complex. Condition (3) of the Main Theorem can hold only if $m$ is even, and so we also assume this. It is then not hard to find graphs $L$ so that, for some pair $s_1$ and $s_2$ of non-adjacent vertices of $L$, and for some nontrivial elements $\alpha_1, \alpha_2 \in \text{Aut}(L)$, Conditions (1) and (2) of the Main Theorem also hold. We present three infinite families of examples.

5.1.1 Buildings with complete bipartite links

Let $L$ be the complete bipartite graph $K_{q, q'}$, with $q, q' \geq 2$. If $q \geq 3$ then there is a vertex $s_1$ of $L$ so that some element $\alpha_2 \in \text{Aut}(L)$, without loss of generality of prime order, fixes the star of $s_1$
in $L$, but $\alpha_2 \neq 1$. Let $s_2$ be a vertex of $L$ such that $\alpha_2(s_2) \neq s_2$. The Main Theorem then applies provided $m$ is even.

If $m = 2$ then $\Sigma$ is the barycentric subdivision of the product of trees $T_q \times T_{q'}$, where $T_q$ is the $q$–regular tree. In particular, if $m = m' = 2$ in Example 1 of Section 2.2 above, then $\Sigma$ is the barycentric subdivision of $T_3 \times T_2$. If $m \geq 4$, then by Theorem 12.6.1 of [D] the complex $\Sigma$ may be metrised as a piecewise hyperbolic CAT($-1$) polygonal complex. With this metric, if $p = 2m$ and $q = q'$ then $\Sigma$ is the barycentric subdivision of Bourdon’s building $I_{p,q}$ (studied in, for example, [B1] and [BP]), which is the unique 2–complex with all 2–cells regular right-angled hyperbolic $p$–gons $P$, and the link of every vertex the complete bipartite graph $K_{q,q}$. Bourdon’s building is a right-angled hyperbolic building, of type $(W', S')$ where $W'$ is the Coxeter group generated by reflections in the sides of $P$.

5.1.2 Fuchsian buildings

A Fuchsian building is a 2–dimensional hyperbolic building. Bourdon’s building $I_{p,q}$ is a (right-angled) Fuchsian building. For Fuchsian buildings which are not right-angled see, for example, [B2] and [GP].

To show that the Main Theorem applies to certain Fuchsian buildings which are not right-angled, let $L$ be the finite building of rank 2 associated to a Chevalley group $G$ (see [R]). Then $L$ is a bipartite graph, with vertex set say $S = S_1 \sqcup S_2$, and for some $k \in \{3, 4, 6, 8\}$, $L$ has girth $2k$ and diameter $k$. Figure 11 depicts the building $L$ for the group $G = GL(3, \mathbb{F}_2) = GL(3, 2)$, for which $k = 3$. The white vertices of this building may be identified with the set of one-dimensional subspaces of the vector space $V = \mathbb{F}_2 \times \mathbb{F}_2 \times \mathbb{F}_2$, and the black vertices with the set of two-dimensional subspaces of $V$. Two vertices are joined by an edge if those two subspaces are incident.

![Figure 11: The building $L$ for $G = GL(3, 2)$](image)

The group $G$ acts on $L$, preserving the type of vertices, with quotient an edge. Suppose $s_1 \in S_1$, and let $s_2 \in S_2$ be a vertex at distance $k$ from $s_1$. Since $L$ is a thick building, there is more than one such vertex $s_2$. For $i = 1, 2$, the stabiliser $P_i$ of $s_i$ in $G$ acts transitively on the set of vertices of $L$ at distance $k$ from $s_i$. Now, by Theorem 6.18 of [R], $P_i$ has a Levi decomposition

$$P_i = U_i \rtimes L_i$$

where $L_i$ is the subgroup of $P_i$ fixing the vertex $s_{3-i}$. Moreover, by Lemma 6.5 of [R], $U_i$ fixes the star of $s_i$ in $L$. Hence, we may find elements $\alpha_{3-i} \in U_i$, without loss of generality of prime order, for which Condition (1) of the Main Theorem holds. Condition (2) of the Main Theorem follows since $L$ is bipartite and the action of $G$ preserves the type of vertices. For example, for $L$ as in Figure 11.
if \(s_1\) is the vertex \(\{(1, 0, 0)\}\), we may choose \(s_2\) to be the vertex \(\{(0, 1, 0), (0, 0, 1), (0, 1, 1)\}\), and then choose

\[
\alpha_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \alpha_2 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]

Suppose now that \(L\) as above is the nerve of a Coxeter system \((W, S)\). By Theorem 12.6.1 of [D], since \(L\) has girth \(\geq 6\), the corresponding Davis complex \(\Sigma\) may also be metrised as a piecewise hyperbolic CAT(−1) polygonal complex. With this metrisation, \(\Sigma\) is then the barycentric subdivision of a Fuchsian building, with the link of every vertex \(L\) and all 2–cells regular hyperbolic 2\(m\)–gons (of vertex angle \(\frac{2\pi}{m}\)). We call such a building a \((2m, L)\)-building. In general, there may be uncountably many isomorphism classes of \((2m, L)\)-buildings (see for instance [GP]). In fact, the Davis complex \(\Sigma\) is the barycentric subdivision of the unique locally reflexive \((2m, L)\)-building with trivial holonomy (see Haglund [H2]).

### 5.1.3 Platonic polygonal complexes

A polygonal complex \(X\) is Platonic if \(\text{Aut}(X)\) acts transitively on the set of flags (vertex, edge, face) in \(X\). Any Platonic polygonal complex is a \((k, L)\)-complex, with \(k \geq 3\) and \(L\) a graph such that \(\text{Aut}(L)\) acts transitively on the set of oriented edges in \(L\). In [Sw], Świątkowski studied CAT(0) Platonic polygonal complexes \(X\), where \(L\) is a trivalent graph. Such complexes are not in general buildings.

A graph \(L\) is said to be \(n\)-arc regular, for some \(n \geq 1\), if \(\text{Aut}(L)\) acts simply transitively on the set of edge paths of length \(n\) in \(L\). For example, the Petersen graph in Figure 1 above is 3–arc regular. Any finite, connected, trivalent graph \(L\), with \(\text{Aut}(L)\) transitive on the set of oriented edges of \(L\), is \(n\)-arc regular for some \(n \in \{1, 2, 3, 4, 5\}\) (Tutte [T]). Świątkowski [Sw] showed that if \(n \in \{3, 4, 5\}\), then for all \(k \geq 4\) there is a unique \((k, L)\)-complex \(X\), with \(X\) Platonic. Thus if \(k = 2m\) is even, the barycentric subdivision of \(X\) is the Davis complex \(\Sigma\) for \((W, S)\), where \((W, S)\) has nerve \(L\) and all finite \(m_{st} = m\).

Now suppose \(L\) is a finite, connected, trivalent, \(n\)-arc regular graph with \(n \in \{3, 4, 5\}\). Choose vertices \(s_1\) and \(s_2\) of \(L\) at distance two in \(L\) if \(n = 3, 4\), and at distance three in \(L\) if \(n = 5\). Then by Propositions 3–5 of Djoković–Miller [DM], for \(i = 1, 2\) there are involutions \(\alpha_i \in \text{Aut}(L)\) such that \(\alpha_i\) fixes the star of \(s_{3-i}\) in \(L\), and \(\alpha_i(s_i) \neq s_i\) is not adjacent to \(s_i\). Thus if \(m\) is even, the Main Theorem applies to \(G = \text{Aut}(\Sigma)\).

### 5.2 Higher-dimensional examples

We first discuss when Condition [3] in the Main Theorem can hold. Suppose \(W_T\) is a spherical special subgroup of \(W\), with \(k = |T| > 2\). If \(W_T\) is irreducible, then from the classification of spherical Coxeter groups (see, for example, [D]), it is not hard to verify that \(W_T\) is halvable along \(s\) if and only if \(W_T\) is of type \(B_k\), with \(s \in T\) the unique generator so that \(m_{st} \in \{2, 4\}\) for all \(t \in T - \{s\}\); in this case \(\text{half}_s(W_T)\) is of type \(D_k\).

Many reducible spherical special subgroups \(W_T\) are also halvable along \(s\), for example \(W_T \cong W_{T - \{s\}} \times \langle s \rangle\). Thus an easy source of examples in dimension \(\geq 2\) is right-angled Coxeter systems \((W, S)\) (meaning that \(m_{st} \in \{2, \infty\}\) for all \(s, t \in S\) with \(s \neq t\)). The Davis complex \(\Sigma\) will then be the barycentric subdivision of a cubical complex.

In both of the following examples, since \(L\) is a building, \(\Sigma\) is the barycentric subdivision of a building.

1. Let \(L\) be the join of \(k\) sets of points, of cardinalities respectively \(n_1, \ldots, n_k\). Then \(\Sigma\) is the barycentric subdivision of the product of trees \(T_{n_1} \times \cdots \times T_{n_k}\). Provided at least one of
n_1, \ldots, n_k is greater than 2, vertices s_1 and s_2 in L and elements \alpha_1, \alpha_2 \in \text{Aut}(L) can be found so that the Main Theorem applies.

2. Let L be a finite building of rank \( k \geq 3 \) associated to a Chevalley group \( G \), and choose vertices \( s_1 \) and \( s_2 \) in \( L \) which are opposite (see [R]). By the same arguments as in Section 5.1.2 above, there are elements \( \alpha_1, \alpha_2 \in \text{Aut}(L) \) such that Conditions (1) and (2) of the Main Theorem hold.

We do not know of any hyperbolic buildings of dimension > 2 to which the Main Theorem applies (for the 3–dimensional constructions of Haglund–Paulin in [HP2], certain of the \( m_{st} \) must be equal to 3, so Condition (3) of the Main Theorem will not hold). Slight modifications of the above examples, for example by adding a vertex \( s \) to \( L \) with \( m_{st} = \infty \) for all \( t \in S - \{s\} \), produce nerves which are not buildings, hence examples of \( \Sigma \) which are not buildings.

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