Dynamical Systems on Bundles

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Abstract

In this study, Hamiltonian and Lagrangian theories, which are mathematical models of mechanical systems, are structured on the horizontal and the vertical distributions of tangent and cotangent bundles. In the end, the geometrical and physical results related to Hamiltonian and Lagrangian dynamical systems are concluded.

Keywords: Tangent and Cotangent Geometry, Lagrangian-Hamiltonian theories.

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1 Introduction

The tangent bundle $TM$ and cotangent bundle $T^*M$ of manifold $M$ are in good condition to be phase-spaces of velocity and momentum of a given configuration space. Then modern differential geometry explains the Lagrangians and Hamiltonian theories in classical mechanics. For example, the tangent bundle $TM$ carries some natural object fields, as: Liouville vector field $V$, tangent structure $J$, almost product structure $P$, the vertical distribution $V$, the horizontal distribution $H$, semispray $X$. Therefore these structures have an important role in physical fields. The following symbolical equation expresses the dynamical equations for both theories:

\[ i_X \Phi = F \]  

(1)

If one studies the Hamiltonian theory then equation (1) is the intrinsical form of Hamiltonian equations, where $\Phi = \phi_H = -d\lambda$, $F = dH$ and $\lambda$ is Liouville form canonically constructed on the cotangent bundle $T^*M$ of the configuration manifold $M$, and $H$ is a $C^\infty$ function on $T^*M$ such that $H : T^*M \to \mathbb{R}$. If one studies the Lagrangian theory then equation (1) is the intrinsical form of Lagrangian equations, where $\Phi = \Phi_L = -ddpL$ such that $L : TM \to \mathbb{R}$. Lagrangian function, $F = dE_L$, $E_L = V(L) - L$, $V$ the Liouville vertical vector field on $TM$, $E_L$ the energy associated to the function $L$ and $X$ is Liouville vertical vector field canonically constructed on the tangent bundle $TM$ of the configuration manifold $M$. Mathematical expressions of mechanical systems are always determined by the Hamiltonian and Lagrangian systems. These expressions, in particular geometric expressions in mechanics and dynamics, are given in some studies [1, 2, 3, 4]. Klein introduced that the geometric study of the Lagrangian theory admits an alternative approach without the use of the regular condition on $L$ [5]. Para-complex analogues of the Lagrangians and Hamiltonians were obtained in the framework of Kählerian
manifold and the geometric conclusions on a para-complex dynamical systems were obtained [6]. As well-known from before works, Lagrangian distribution on symplectic manifolds are used in geometric quantization and a connection on a symplectic manifold is an important structure to obtain a deformation quantization [7]. In the before studies; although real/(para)complex geometry and real/(para)complex mechanical-dynamical systems were analyzed successfully, they have not dealt with dynamical systems on horizontal distribution $HTM$ and vertical distribution $VTM$ of tangent bundle $TM$ of manifold $M$. In this Letter, therefore, Euler-Lagrange and Hamiltonian equations related to dynamical systems on the distributions used in obtaining geometric quantization have been given.

2 Preliminaries

In this paper all geometrical object fields and all mappings are considered of the class $C^\infty$, expressed by the words ”differentiate” or ”smooth”. The indices $i, j, ...$ run over set $\{1, ..., n\}$ and Einstein convention of summarizing is adopted over all this paper. $R, \mathcal{F}(TM), \chi(TM), \chi(T^*M)$ denote the set of real numbers, the set of real functions on $TM$, the set of vector fields on $TM$ and the set of 1-forms on $T^*M$.

2.1 Manifold, Bundle and Distributions

In this subsection, some definitions were derived and taken from [9]. Let $TM$ be tangent bundle of a real $n$-dimensional differentiable manifold $M$. Then it will be denoted a point of $M$ by $x$ and its local coordinate system by $(U, \varphi)$ such that $\varphi(x) = (x^i)$. Such that the projection $\pi : TM \rightarrow M$, $\pi(u) = x$, a point $u \in TM$ will be denoted by $(x, y)$, its local coordinates
being \((x^i, y^j)\). There are the natural basis \((\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^j})\) and dual basis \((dx^i, dy^j)\) of the tangent space \(T_u TM\) and the cotangent space \(T^*_u(TM)\) at the point \(u \in TM\), respectively. Consider the \(\mathcal{F}(TM)\)– and \(\mathcal{F}(T^*M)\)– linear mappings (also named to be almost tangent structures) 

\[ J : \chi(TM) \to \chi(TM) \text{ and } J^* : \chi(TM) \to \chi(TM) \]

given by

\[ J(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial y^i}, \quad J(\frac{\partial}{\partial y^i}) = 0, \]

and

\[ J^*(dx^i) = dy^i, \quad J^*(dy^i) = 0. \]

The tangent space \(V_u\) to the fibre \(\pi^{-1}(x)\) in the point \(u \in TM\) is locally spanned by \(\{\frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^n}\}\). Therefore, the mapping \(V : u \in TM \to V_u \subset T_u TM\) provides a regular distribution generated by the adapted basis \(\{\frac{\partial}{\partial y^i}\}\). Consequently, \(V\) is an integrable distribution on \(TM\). \(V\) is called the vertical distribution on \(TM\).

Let \(N\) be a nonlinear connection on \(TM\). \(N\) is characterized by \(v, h\) vertical and horizontal projectors. We consider the vertical projector \(v : \chi(TM) \to \chi(TM)\) defined by \(v(X) = X, \forall X \in \chi(VTM)\); \(v(X) = 0, \forall X \in \chi(HTM)\). Similarly, the mapping \(H : u \in TM \to H_u \subset T_u TM\) provides a regular distribution determined by the adapted basis \(\{\frac{\partial}{\partial x^i}\}\). Consequently, \(H\) is an integrable distribution on \(TM\). \(H\) is called the horizontal distribution on \(TM\).

There is a \(\mathcal{F}(TM)\)–linear mapping \(h : \chi(TM) \to \chi(TM)\), for which \(h^2 = h\), \(\text{Ker } h = \chi(VTM)\). Any vector field \(X \in \chi(TM)\) can be uniquely written as follows

\[ X = hX + vX = X^H + X^V. \]

Therefore \(X^H\) and \(X^V\) are horizontal and vertical components of vector field \(X\). Therefore, any vector field \(X\) can be uniquely written in the form

\[ X = X^H + X^V \]

such that

\[ X^H = X^i(\frac{\partial}{\partial x^i} - N^i_j(x, y)\frac{\partial}{\partial y^j}), \quad X^V = X^iN^i_j(x, y)\frac{\partial}{\partial y^j}. \]
where \(N^i_j\) are local coefficients of nonlinear connection \(N\) on \(TM\); \(P\) is an almost product structure on \(TM\). \((\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j})\) is a local basis adapted to the horizontal distribution \(HTM\) and the vertical distribution \(VTM\). Then \((dx^i, \delta y^i)\) is dual basis of \((\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j})\) basis. We have
\[
P(X) = X, \forall X \in \chi(HTM); \quad P(X) = -X, \forall X \in \chi(VTM)
\]
\[
P^*(\omega) = \omega, \forall \omega \in \chi(HT^*M); \quad P^*(\omega) = -\omega, \forall \omega \in \chi(VT^*M),
\]
where \(P^*\) is the dual structure of \(P\). For the \((\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j})\) basis and \((dx^i, \delta y^i)\) dual basis we have
\[
\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^i_j(x, y)\frac{\partial}{\partial y^j},
\]
and
\[
\delta y^i = dy^i + N^i_j(x, y) dx^j.
\]

For the operators \(h, v, P, P^*, J, J^*\) we get
\[
h + v = I; \quad P = 2h - I; \quad P = h - v; \quad P = I - 2v,
\]
\[
JP = J; \quad PJ = -J; \quad J^*P^* = J^*; \quad P^*J^* = -J^*,
\]
\[
h(\frac{\delta}{\delta x^i}) = \delta x^i; \quad h\frac{\partial}{\partial y^i} = 0; \quad v\frac{\delta}{\delta x^i} = 0; \quad v\frac{\partial}{\partial y^i} = \frac{\partial}{\partial y^i},
\]
\[
P(\frac{\delta}{\delta x^i}) = \frac{\delta}{\delta x^i}; \quad P\frac{\partial}{\partial y^i} = -\frac{\partial}{\partial y^i},
\]
\[
P^*(dx^i) = dx^i; \quad P^*(\delta y^i) = -\delta y^i.
\]

### 3 Lagrangian Dynamical Systems

In this section, Euler-Lagrange equations for classical mechanics are structured by means of almost product structure \(P\) under the consideration of the basis \((\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j})\) on distributions \(HTM\) and \(VTM\) of tangent bundle \(TM\) of manifold \(M\). Let \((x^i, y^i)\) be its local coordinates. Let semispray be the vector field \(X\) given by
\[
X = X^i \frac{\delta}{\delta x^i} + \dot{X}^i \frac{\partial}{\partial y^i}, \quad \dot{X}^i = X^i N^i_j
\]
where the dot indicates the derivative with respect to time \( t \). The vector field denoted by \( V = P(X) \) and expressed by

\[
V = X^i \frac{\delta}{\delta x^i} - X^i \frac{\partial}{\partial y^i}
\]

is called Liouville vector field on the bundle \( TM \). The maps given by \( T, P : TM \to \mathbb{R} \) such that \( T = \frac{1}{2}m_i(x^i)^2, P = m_i gh \) are called the kinetic energy and the potential energy of the mechanical system, respectively. Here \( m_i, g \) and \( h \) stand for mass of a mechanical system having \( m \) particles, the gravity acceleration and distance to the origin of a mechanical system on the tangent bundle \( TM \), respectively. Then \( L : TM \to \mathbb{R} \) is a map that satisfies the conditions; i) \( L = T - P \) is a Lagrangian function, ii) the function given by \( E_L = V(L) - L \) is a Lagrangian energy. The operator \( i_P \) induced by \( P \) and shown by

\[
i_P \omega(X_1, X_2, ..., X_r) = \sum_{i=1}^r \omega(X_1, ..., P(X_i), ..., X_r)
\]

is said to be vertical derivation, where \( \omega \in \wedge^r TM, X_i \in \chi(TM) \). The vertical differentiation \( d_P \) is defined by

\[
d_P = [i_P, d] = i_P d - d i_P
\]

where \( d \) is the usual exterior derivation. For an almost product structure \( P \), the closed fundamental form is the closed 2-form given by \( \Phi_L = -dd_P L \) such that

\[
d_P : \mathcal{F}(TM) \to T^* M
\]

Then we have

\[
\Phi_L = -(\frac{\delta}{\delta x^j} dx^j + \frac{\partial}{\partial y^j} \delta y^j)(\frac{\delta L}{\delta x^i} dx^i - \frac{\partial L}{\partial y^i} \delta y^i)
\]

\[
= \frac{\partial^2 L}{\delta x^j \delta x^i} dx^j \wedge dx^i - \frac{\delta (\delta L)}{\delta x^j \delta y^i} dx^i \wedge \delta y^j - \frac{\partial (\delta L)}{\delta y^j \delta x^i} \delta y^j \wedge dx^i + \frac{\partial^2 L}{\delta y^j \delta y^i} \delta y^j \wedge \delta y^i.
\]
Let \( X \) be the second order differential equation (semispray) determined by Eq. (1) and

\[
\begin{align*}
i_X \Phi_L &= \Phi_L(X) = -X^i \frac{\delta^2 L}{\delta x^i \delta x^j} \delta^j dx^i + X^i \frac{\delta L}{\delta x^i} dx^j + X^i \frac{\delta(\partial L)}{\delta y^i} \delta^j dy^i - \dot{X}^i \frac{\delta(\partial L)}{\delta x^i} dx^i \\
-\dot{X}^i \frac{\partial(\partial L)}{\partial y^i} \delta^j dx^i + X^i \frac{\partial(\partial L)}{\delta y^i} \delta^j dy^i + \ddot{X}^i \frac{\partial^2 L}{\delta y^i \delta y^j} \delta^j dy^j - \dot{X}^i \frac{\partial^2 L}{\delta y^i \delta y^j} dy^j.
\end{align*}
\]  

(8)

Since the closed 2-form \( \Phi_L \) on \( TM \) is in the symplectic structure, it is found

\[
E_L = V(L) - L = X^i \frac{\delta L}{\delta x^i} - \dot{X}^i \frac{\partial L}{\partial y^i} - L
\]

and hence

\[
dE_L = X^i \frac{\delta^2 L}{\delta x^i \delta x^j} dx^j - \dot{X}^i \frac{\delta L}{\delta x^i} dx^j - X^i \frac{\delta(\partial L)}{\delta y^i} dx^j + X^i \frac{\delta(\partial L)}{\delta y^i} \delta^j dy^i - \dot{X}^i \frac{\partial^2 L}{\delta y^i \delta y^j} \delta^j dy^j
\]

(9)

With the use of Eq. (1), considering (8) and (10) we get

\[
\begin{align*}
-X^i \frac{\delta^2 L}{\delta x^i \delta x^j} \delta^j dx^i + X^i \frac{\delta^2 L}{\delta x^i \delta x^j} \delta^j dx^j + X^i \frac{\delta(\partial L)}{\delta y^i} \delta^j dy^i - \dot{X}^i \frac{\delta(\partial L)}{\delta x^i} \delta^j dx^j \\
-\dot{X}^i \frac{\partial(\partial L)}{\partial y^i} \delta^j dx^i + X^i \frac{\partial(\partial L)}{\delta y^i \delta x^j} \delta^j dy^i + \ddot{X}^i \frac{\partial^2 L}{\delta y^i \delta y^j} \delta^j dy^j - \dot{X}^i \frac{\partial^2 L}{\delta y^i \delta y^j} \delta^j dy^j
\end{align*}
\]

(10)

or

\[
\begin{align*}
-X^i \frac{\delta^2 L}{\delta x^i \delta x^j} dx^j - \dot{X}^i \frac{\delta(\partial L)}{\delta y^i \delta x^j} dx^j + \delta^j L dx^j + X^i \frac{\delta(\partial L)}{\delta y^i \delta x^j} \delta^j dy^j + \ddot{X}^i \frac{\partial^2 L}{\delta y^i \delta y^j} \delta^j dy^j + \dot{X}^i \frac{\partial^2 L}{\delta y^i \delta y^j} \delta^j dy^j = 0,
\end{align*}
\]

(11)

If a curve denoted by \( \alpha : \mathbb{R} \rightarrow TM \) is considered to be an integral curve of \( X \), i.e. \( X(\alpha(t)) = \frac{d\alpha(t)}{dt} \)

then

\[
-\frac{d}{dt} \left( \frac{\delta L}{\delta x^i} \right) + \frac{\delta L}{\delta x^i} + \frac{d}{dt} \left( \frac{\partial L}{\partial y^i} \right) + \frac{\partial L}{\partial y^i} = 0.
\]

(12)

or

\[
\frac{d}{dt} \left( \frac{\delta L}{\delta x^i} \right) - \frac{\partial L}{\partial y^i} = 0, \quad \frac{d}{dt} \left( \frac{\partial L}{\partial y^i} \right) + \frac{\delta L}{\delta x^i} = 0.
\]

(13)

Thus the equations given by (13) are seen to be a Euler-Lagrange equations on \( HTM \) horizontal and \( VTM \) vertical distributions, and then the triple \((TM, \Phi_L, X)\) is seen to be a mechanical system with taking into account the basis \( \{ \frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i} \} \) on the distributions \( HTM \) and \( VTM \).
4 Hamiltonian Dynamical Systems

In this section, Hamiltonian equations for classical mechanics are structured on the distributions \(HT^*M\) and \(VT^*M\) of \(T^*M\). Suppose that an almost product structure, a Liouville form and a 1-form on \(T^*M\) are shown by \(P^*\), \(\lambda\) and \(\omega\), respectively. Then we hold

\[
\omega = \frac{1}{2} (y^i dx^i + x^i \delta y^i) \tag{14}
\]

and

\[
\lambda = P^*(\omega) = \frac{1}{2} (y^i dx^i - x^i \delta y^i). \tag{15}
\]

It is concluded that if \(\phi\) is a closed 2-form on \(T^*M\), then \(\phi_H\) is also a symplectic structure on \(T^*M\). If Hamiltonian vector field \(X_H\) associated with Hamiltonian energy \(H\) is given by

\[
X_H = X^i \frac{\delta}{\delta x^i} + Y^i \frac{\partial}{\partial y^i}, \tag{16}
\]

then

\[
\phi_H = -d\lambda = -\delta y^i \wedge dx^i \tag{17}
\]

and

\[
i_{X_H} \phi = -Y^i dx^i + X^i \delta y^i. \tag{18}
\]

Moreover, the differential of Hamiltonian energy is written as follows:

\[
dH = \frac{\delta H}{\delta x^i} dx^i + \frac{\partial H}{\partial y^i} \delta y^i. \tag{19}
\]

By means of Eq. (1), using Eq. (18) and Eq. (19), the Hamiltonian vector field is calculated to be

\[
X_H = \frac{\partial H}{\partial y^i} \frac{\delta}{\delta x^i} - \frac{\delta H}{\delta x^i} \frac{\partial}{\partial y^i}. \tag{20}
\]
Suppose that a curve
\[ \alpha: I \subset \mathbb{R} \rightarrow T^*M \] (21)
be an integral curve of the Hamiltonian vector field \( X_H \), i.e.,
\[ X_H(\alpha(t)) = \frac{d\alpha(t)}{dt}, \quad t \in I. \] (22)

In the local coordinates, it is concluded that
\[ \alpha(t) = (x^i(t), y^i(t)) \] (23)
and
\[ \frac{d\alpha(t)}{dt} = \frac{dx^i}{dt} \frac{\delta}{\delta x^i} + \frac{dy^i}{dt} \frac{\partial}{\partial y^i}. \] (24)

Taking into consideration Eqs. (21), (19), (23), the result equations can be found
\[ \frac{dx^i}{dt} = \frac{\partial H}{\partial y^i}, \quad \frac{dy^i}{dt} = -\frac{\delta H}{\delta x^i}. \] (25)

Thus, the equations (25) are seen to be Hamiltonian equations on the horizontal distribution \( HT^*M \) and vertical distribution \( VT^*M \), and then the triple \((T^*M, \phi_H, X_H)\) is seen to be a Hamiltonian mechanical system with the use of basis \( \{ \frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i} \} \) on the distributions \( HT^*M \) and \( VT^*M \).

5 Conclusions

This paper has obtained to exist physical proof of the both mathematical equality given by \( TM = HTM \oplus VTM \) and its dual equality. Lagrangian and Hamiltonian dynamics have intrinsically been described with taking into account the basis \( \{ \frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i} \} \) and dual basis \( (dx^i, \delta y^i) \) on distributions of tangent and cotangent bundles \( TM \) and \( T^*M \) of manifold \( M \).
6 Discussions

As known, geometry of Lagrangians and Hamiltonians give a model for Relativity, Gauge Theory and Electromagnetism in a very natural blending of the geometrical structures of the space with the characteristics properties of these physical fields. Therefore we consider that the equations (13) and (25) especially can be used in fields determined the above of physical.

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