First passage under restart with branching

Arnab Pal1,2,3,† Iddo Eliazar1 and Shlomi Reuvuni1,2,‡

1School of Chemistry, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv 6997801, Israel
2Center for the Physics and Chemistry of Living Systems, Tel Aviv University, 6997801, Tel Aviv, Israel and
3The Sackler Center for Computational Molecular and Materials Science, Tel Aviv University, 6997801, Tel Aviv, Israel

(Dated: July 26, 2018)  

First passage under restart with branching is proposed as a generalization of first passage under restart. Strong motivation to study this generalization comes from the observation that restart with branching can expedite the completion of processes that cannot be expedited with simple restart; but a sharp and quantitative formulation of this statement is still lacking. We develop a comprehensive theory of first passage under restart with branching. This reveals that two widely applied measures of statistical dispersion—the coefficient of variation and the Gini index—come together to determine how restart with branching affects the mean completion time of an arbitrary stochastic process. The universality of this result is demonstrated, and its connection to extreme value theory is also pointed out and explored.

Restart can expedite the completion of a myriad of processes [1–13]. It works by taking advantage of stochastic fluctuations to trade long completion times with ones which are shorter on average. Thus, restart is particularly effective in expediting processes whose statistical completion time distributions are very broad. A classical example of this principle is the first passage time (FPT) [14,15] to a stationary target of a particle undergoing diffusion [16–18]. The distribution of this FPT is so broad that its mean diverges [16]. On average, a diffusive particle will thus take an infinite amount of time to reach a target; but restart can change this situation dramatically. Indeed, stopping the particle’s motion intermittently and placing it back at the origin to continue its motion from there regularizes the FPT distribution and renders its mean finite [1,2]. Remarkably, this ‘regularization by restart’ works for any process with divergent mean completion times – regardless of the specific details [4,5,9,10].

Some processes cannot be expedited with restart as this strategy crucially depends on the existence of stochastic fluctuations in completion times. Consider, as an illustrative example, a train which travels from point A to B at a constant velocity. Aside from small glitches here and there, such train takes a fixed amount of time to reach its final destination; in other words, the train’s FPT distribution is very sharply peaked around its mode. As it turns out, this punctuality is intimately related with the fact that taking the train, at any point along its route, and placing it back at the dock of departure will only prolong its journey. Indeed, whenever the coefficient of variation (CV) of the completion time is smaller than one – applying restart will increase the mean completion time [4,5,9,10]. As the CV is the ratio of the standard deviation to the mean, this criterion captures the need for relatively large stochastic fluctuations; also, this criterion sets a sharp boundary for the application of restart as a ‘speed-up’ mechanism.

Branching a process – so as to create several competing copies that run parallel to one another – is another way in which completion can, in principle, be expedited. Branching is a well established concept that has been extensively studied, e.g., in birth and death processes [19]. reaction-diffusion systems [20,21]. Brownian motion [22,23], and epidemic spread [24]. However, it was just earlier this year that branching was considered in the context of restart—coining the term ‘branching search’ [25].

The great advantages that restart brings to existing search strategies [26,27] were already discussed extensively [1,5,10,12,13,28,29]. Now, strong motivation to combine restart with branching comes from the observation that branching provides restart with the ability to expedite processes it would otherwise hinder. However, while our understanding of first-passage under restart enjoys the aforementioned CV criterion, a universal criterion that determines the effect of ‘restart-&-branching’ on the mean completion time of a general process is currently missing. Here we develop a general theory of first passage under restart with branching to address this important issue and others. To motivate the construction of this theory, we first consider an illustrative case study (Fig. 1).

**FIG. 1.** An illustration of diffusion under restart with branching. When restart occurs, particles are brought back to the origin and each particle is then branched into two identical copies of itself. Depending on the initial distance from the target, the diffusion coefficient, and drift velocity, this procedure could either increase or decrease the mean first passage time to the target.
velocity $V > 0$ it will eventually hit it at some random time $T$. This time is known to come from the Inverse-Gaussian distribution $f_T(t) = \frac{L}{\sqrt{4\pi D t^3}} e^{-\frac{(L - Vt)^2}{4Dt}} (t \geq 0)$, with $D > 0$ standing for the diffusion coefficient $[80]$. Defining the Péclet number $Pe = LV/2D$, one can then trivially observe that diffusion (drift) becomes negligible in the limit of $Pe \gg 1$ ($Pe \ll 1$). Indeed, the above FPT distribution becomes sharp for high Péclet numbers and resembles that of a particle undergoing pure diffusion.

Thus, by tuning the Péclet number, one can explore the wide range of cases which stand between the two extreme scenarios discussed above—simple motion at a constant velocity and pure diffusion.

Let us now consider the above process, but with the addition of restart and branching (Fig. 1). Specifically, assume that restart can occur with some probability at any given point in time; and that when this happens our particle is taken back to the origin and branched into two, or generally $m$, identical particles. These particles will start their motion anew, but if restart occurs again they will all be taken back to the origin and branched as before. This procedure repeats itself until one of the particles in the system hits the target.

To study the effect of restart with branching, we simulated the process illustrated in Fig. 1 [31]. First, we wanted to explore what happens when restart with branching is introduced into the system. Letting $r$ denote the constant rate at which restart with branching occurs, we looked at the logarithmic derivative of the mean FPT to the target with respect to $r$ at $r = 0$. This was plotted vs. the Péclet number for different values of the branching parameter $m$ (Fig. 2 markers). A negative derivative indicates that the introduction of restart with branching will, on average, expedite first passage to the target, and the converse is true when a positive derivative is found. Results coming from simulations were later corroborated by the general theory developed below (Fig. 2 dashed lines).

Analyzing the results presented in Fig. 2, we would first like to draw attention to the case $m = 1$. In this degenerate case restart is not accompanied by branching and one can observe that the derivative changes sign exactly when $Pe \approx 1$. This is no surprise. The mean of the Inverse-Gaussian distribution above is $L/V$ and its standard deviation is $\sqrt{2DL/V^2}$. Letting $CV$ stand for the coefficient of variation of this distribution, we observe that $CV^2 = 1/Pe$. Recalling that the theory of first passage under restart asserts that restart will expedite the completion of a process if and only if $CV > 1$ [4, 5, 9, 10], we conclude that the observed behaviour is a special case of a more general one.

For $m > 1$, we see that the critical Péclet number at which the derivative changes sign is always larger than unity. Moreover, this critical number grows as $m$ increases. Thus, in the case of diffusion with drift, branching allows restart to reduce the mean FPT even when the underlying FPT distribution is much narrower than the critical limit for simple restart ($Pe = 1$). However, and in contrast to the $m = 1$ case, we find the corresponding set of critical $CV$s (Fig. S2) to be particular to diffusion with drift, i.e., non-universal. Any general characterization of the effect that restart with branching has on the mean completion time of a generic process must thus rely on more than the first two moments of the underlying time distribution. To find this characterization, and further deepen our understanding, we now develop a general theory of first passage under restart with branching.

A theory of first passage under restart with branching.— Consider a generic process that starts at time zero and, if allowed to take place without interruptions, completes after a random time $T$. The process, however, can also be restarted after some random time $R$ and consecutively branched into $m$ daughter processes which are independent and identical copies of the parent process $(m = 1, 2, 3, \ldots)$. Thus, if restart and branching prevents the parent process from completing its course, $m$ daughter processes will start in its stead. The same procedure will then repeat itself: if one of the daughter processes is able to finish prior to restart completion will be declared. Otherwise, following a second restart event, each daughter process will, in itself, be branched into $m$ processes that will once again start their course afresh; and so on and so forth until one of the offspring processes is able to complete (Fig. 3).

To analyze the above scenario, we let $R'$ and $\{T_1, \cdots, T_m\}$ denote independent and identically distributed copies of $R$ and $T$ respectively. We then observe that $T_R$—the random completion time of a generic process under restart with branching—can be written as

$$T_R = \begin{cases} T & \text{if } T < R \\ R + [T^{(m)}]_R & \text{if } R \leq T \end{cases}$$

with $T^{(m)} = \min\{T_1, \cdots, T_m\}$. The rationale leading to Eq. (1) is simple. If the parent process is able to complete prior to restart the process there ends and $T_R = T$. Otherwise, after some random time $R$, restart with branching occurs. One could then ‘virtually’ join all the newly formed daughter processes into a single process whose uninterrupted completion time is $T^{(m)}$, i.e., the minimum over $m$ IID copies.
of $T$. However, since this joint process is also subject to restart with branching its actual completion time is given by $[T^{(m)}]_R$. Thus, when restart precedes completion, we have $T_R = R + [T^{(m)}]_R$.

Eq. (1) can be used to derive a formula for the mean FPT of a process that has been subjected to restart with branching. This is done in iterative steps. First, observe that Eq. (1) implies that $T_R = \min(T, R) + I(R \leq T) [T^{(m)}]_R$, where $\min(T, R)$ is the minimum of $T$ and $R$, and $I(R \leq T)$ is an indicator random variable which takes the value one when $R \leq T$ and zero otherwise. Taking expectations, and utilizing statistical independence, we obtain that

$$
\langle T_R \rangle = \langle \min(T, R) \rangle + \Pr(R \leq T) \langle [T^{(m)}]_R \rangle .
$$

(2)

The first term in Eq. (2) and the probability that $R \leq T$ can then be evaluated given the distribution functions of $T$ and $R$.

To further proceed, we utilize Eq. (1) one more time to obtain $[T^{(m)}]_R = \min(T^{(m)}, R) + I(R \leq T^{(m)}) [T^{(m)}]_R$. Taking expectations one arrives at an equation similar to Eq. (2) where $\min(T^{(m)}, R)$ and the probability that $R \leq T^{(m)}$ can once again be directly computed given the distribution functions of $T$ and $R$. As for the expectation of $[T^{(m)}]_R$, this is again evaluated by iterative use of Eq. (2).

Carried ad infinitum, the above procedure yields a formula that allows the evaluation of $\langle T_R \rangle$ for arbitrary choices of $T$ and $R$. Of particular interest is that regard is the case where restart is conducted at a constant rate $r$, i.e., when the random restart time $R$ is exponentially distributed with mean $1/r$. The first term in Eq. (2) and the probability that $R \leq T$ can then be evaluated given the distribution functions of $T$ and $R$. Thus, when restart precedes completion, we have $T_R = R + [T^{(m)}]_R$.

$$
\langle T_R \rangle = \frac{1}{r} \sum_{n=0}^{\infty} \prod_{k=0}^{n} \{1 - T^{(m)^k}(r)\} ,
$$

(3)

with $T^{(m)^k}(r)$ standing for the Laplace transform of $T^{(m)^k}$ evaluated at $r$. Equation (3) allows one to study the effect of restart with branching on the mean completion times of various process by systematic evaluation of $\langle T_R \rangle$ (Fig. 4). This shows that the effect of restart with branching can be varied. Depending on the process, and the branching index $m$, the introduction of restart with branching could either increase or decrease the mean completion time of a process. A universal and clear cut criterion to determine which of the two occurs is therefore in need. We will now derive this criterion and show that aside from the coefficient of variation it is also sensitive to another measure of statistical dispersion: the generalized Gini index.

The Gini index is widely applied in economics and in the social sciences as a measure of socioeconomic inequality. In general, the Gini index is a measure of statistical dispersion that is applicable to non-negative valued random variables with positive means. Here we apply the Gini index in the context of the random variable $T$. The Gini index has several equivalent representations, and one of them is

$$
GI_m = 1 - \langle \min \{T_1, \cdots, T_m\} \rangle / \langle T \rangle ,
$$

(4)

$(m = 1, 2, \ldots)$. Note that the index of order one vanishes $GI_1 = 0$; and that the index of order two is the Gini index $GI_2 = GI$.

With the generalized Gini index at hand, and the coefficient of variation $CV = \sigma(T) / \langle T \rangle$, we find that $d(T_k) / dr |_{r=0} < 0$ if and only if

$$
CV^2 + 2 \cdot GI_m > 1 .
$$

(5)

Equation (5) provides a universal criterion that determines how restart with branching affects the mean completion time of an arbitrary stochastic process. When $m = 1$, i.e., when there is no branching, Eq. (5) boils down to the known ‘pure restart’ criterion $CV > 1$. However, for $m > 1$, we see that the criterion in Eq. (5) also depends on the generalized Gini index. To better illustrate this we now consider a concrete example.

Consider the Weibull distribution $Pr[T \geq t] = e^{-(t/\alpha)^{\beta}} (t \geq 0)$; with $\alpha, \beta > 0$ standing respectively for the scale and shape parameters. This distribution is appropriate for modeling the time to failure of an item that has an initial defect which is exposed to an environment that accelerates the failure process. The Weibull distribution is characterized by two parameters, the scale parameter $\alpha$ and the shape parameter $\beta$. When $\beta = 1$, the Weibull distribution reduces to the exponential distribution. When $\beta > 1$, the Weibull distribution has a bathtub-shaped failure rate, indicating that the item is more likely to fail early in its life (infant mortality), or late in its life (wear-out failure). When $\beta < 1$, the Weibull distribution has a decreasing failure rate, indicating that the item is less likely to fail early in its life (infant mortality), or late in its life (wear-out failure).

The introduction of restart with branching on the mean completion time of the Weibull distribution can be studied by evaluating Eq. (2) for $T = T_k$. The first term in Eq. (2) and the probability that $R \leq T_k$ can then be evaluated given the distribution functions of $T_k$ and $R$. Thus, when restart precedes completion, we have $T_R = R + [T^{(m)}]_R$.
parameters. The coefficient of variation of this distribution is known to be given by $CV^2 = \Gamma(1 + 2/\beta)/\Gamma(1 + 1/\beta)^2 - 1$, where $\Gamma(x)$ is the Gamma function. To compute $G_m$, we first observe that if $T$ is a Weibull random variable with parameters $\alpha$ and $\beta$, then $T^{(m)} = \min\{T_1, \cdots, T_m\}$ is also a Weibull random variable with scale parameter $\alpha/m^{1/\beta}$ and shape parameter $\beta$ [31]. Eq. (4) then gives $G_m = 1 - m^{-1/\beta}$. Thus, both $CV^2$ and $G_m$ are uniquely determined by the shape parameter which provides a simple handle on the dispersion of the Weibull distribution—the larger $\beta$ the smaller $CV^2$ and $G_m$. It then becomes easy to explore the criterion in Eq. (5) both graphically (see Fig. 5) and algebraically as it can be re-arranged into an explicit form

$$m > \left[\frac{2\Gamma(1+1/\beta)^2}{\Gamma(1+2/\beta)}\right]^{\beta}. \quad (6)$$

Equation (6) relates the branching index $m$ to the shape parameter $\beta$ and provides a simple test to determine the effect of restart with branching. In particular, note that for $\beta < 1$, a.k.a. the stretched exponential regime [37, 38], this condition is always satisfied. In [31] we demonstrate that explicit forms of Eq. (5) can also be attained for distributions other than the Weibull, e.g., the uniform and Pareto.

Discussion and outlook.—Restart has recently attracted considerable attention [1, 3, 7–13, 25, 39–59] touching on subjects ranging from stochastic thermodynamics [60, 61] to optimization theory [62], and from quantum mechanics [63, 64] to biological physics [4–6, 65]. In this letter, we developed a comprehensive theory of first passage under restart and a universal criterion to determine the effect that restart with branching confers on a generic process. We now discuss several ramifications of this result.

The universal criterion in Eq. (5) can be re-written in the form

$$\frac{1}{2}\langle T^2 \rangle/\langle T \rangle > \langle \min\{T_1, \cdots, T_m\}\rangle. \quad (7)$$

This form has a probabilistic interpretation which becomes evident when contrasting two possible scenarios. Consider first a process (with no restart) that repeats itself indefinitely a-la Sisyphus. Renewal theory then asserts that if this process is probed at an arbitrary time epoch, $\frac{1}{2}\langle T^2 \rangle/\langle T \rangle$ units of time will pass, on average, before a completion event is first observed [66]. Now consider the introduction of an infinitesimally small restart rate. Properties of the Poisson process then assert that restart will occur at an arbitrary moment in time, thus replacing an expected mean completion time of $\frac{1}{2}\langle T^2 \rangle/\langle T \rangle$ by $\langle \min\{T_1, \cdots, T_m\}\rangle$. Comparing the two alternatives, Eq. (5) follows immediately.

Equation (7) invites a connection to Extreme Value Theory (EVT). A fundamental result of EVT, the Fisher-Tippett-Gnedenko theorem [67, 68], pinpoints the universal limits of the maxima and minima of IID random variables. In general, these could fall into the Gumbel, Fréchet or Weibull domains of attraction. However, the fact that $T$ is non-negative and hence bounded from below takes it out of the Gumbel’s and Fréchet’s domains of attraction, and leaves the Weibull as the only possible limit law for $\min\{T_1, \cdots, T_m\}$. This implies that — under certain conditions, and for $m \gg 1$ — the minimum can be approximated (in law) by $W/s_m$, where $s_m$ is a scale parameter that satisfies $\lim_{m \to \infty} s_m = \infty$, and $W$ is a Weibull-distributed limiting random variable [31]. Consequently, the right-hand side of Eq. (7) can be approximated by $\langle W \rangle/s_m$ – thus establishing an asymptotic version of the criterion in Eq. (7).

Finally, Eq. (5) can be expressed in terms of two equality indices. The first equality index is based on the coefficient of variation, and is given by $\mathcal{R}_2 = 1/(1+CV^2)$; this index belongs to a continuum of equality indices that are intimately related to the Renyi entropy [69]. The second equality index is the counterpart of the generalized Gini index of Eq. (4), and is given by $\mathcal{G}_m = 1 - GL_m$; this index belongs to a continuum of equality indices that extend the Gini index [36]. In terms of these two equality indices, Eq. (5) admits the following product representation

$$\mathcal{R}_2 \cdot \mathcal{G}_m < \frac{1}{2}. \quad (8)$$

Interestingly, Eq. (8) resembles an uncertainty principle which provides a fundamental bound on $\mathcal{G}_m$ given $\mathcal{R}_2$ and vice versa. Indeed, if the introduction of an infinitesimally small restart rate is observed to expedite the completion of a given process: $\mathcal{G}_m$ cannot exceed $\frac{1}{2}\mathcal{R}_2^{-1}$. Thus, high equality (less uncertainty) of one measure must be compensated by low equality (more uncertainty) of the other.

Acknowledgments.—We thank Dr. Somrita Ray and Dr. Debasis Mondal for commenting on early versions of this manuscript. A.P. acknowledges support from the Raymond and Beverly Sackler Post-Doctoral Scholarship. S.R. acknowledges support from the Azrieli Foundation.

* richard86arnab@gmail.com
† elaazar@post.tau.ac.il
‡ shlomire@tauex.tau.ac.il
[1] Evans, M.R. and Majumdar, S.N., 2011. Diffusion with stochastic resetting. Physical review letters, 106(16), p.160601.
[2] Evans, M.R. and Majumdar, S.N., 2011. Diffusion with optimal resetting. Journal of Physics A: Mathematical and Theoretical, 44(43), p.435001.
[3] Montero, M. and Villarroel, J., 2013. Monotonic continuous-time random walks with drift and stochastic reset events. Physical Review E, 87(1), p.012116.
[4] Reuveni, S., Urbakh, M. and Klafter, J., 2014. Role of substrate unbinding in Michaelis-Menten enzymatic reactions. Proceedings of the National Academy of Sciences, 111(12), pp.4391-4396.
[5] Rotbart, T., Reuveni, S. and Urbakh, M., 2015. Michaelis-Menten reaction scheme as a unified approach towards the optimal restart problem. Physical Review E, 92(6), p.060101.
[6] Robin, T., Reuveni, S. and Urbakh, M., 2018. Single-molecule theory of enzymatic inhibition. Nature communications, 9(1), p.779.
[7] Pal, A., Kundu, A. and Evans, M.R., 2016. Diffusion under time-dependent resetting. Journal of Physics A: Mathematical and Theoretical, 49(22), p.225001.
[8] Nagar, A. and Gupta, S., 2016. Diffusion with stochastic resetting at power-law times. Physical Review E, 93(6), p.060102.
[9] Reuveni, S., 2016. Optimal stochastic restart renders fluctuations in first passage times universal. Physical review letters, 116(17), p.170601.
[10] Pal, A. and Reuveni, S., 2017. First Passage under Restart. Physical review letters, 118(3), p.030603.
[11] Kusmierz, L., Majumdar, S.N., Sabhapandit, S. and Schehr, G., 2014. First order transition for the optimal search time of Lévy flights with resetting. Physical review letters, 113(22), p.220602.
[12] Kusmierz, L. and Gudowska-Nowak, E., 2015. Optimal first-arrival times in Lévy flights with resetting. Physical Review E, 92(5), p.052127.
[13] Bhat, U., De Bacco, C. and Redner, S., 2016. Stochastic search with Poisson and deterministic resetting. Journal of Statistical Mechanics: Theory and Experiment, 2016(8), p.083401.
[14] Condamin, S., Bénichou, O., Tejedor, V., Voituriez, R. and Klafter, J., 2007. First-passage times in complex scale-invariant media. Nature, 450(7166), pp.77-80.
[15] Guérit, T., Levernier, N., Bénichou, O. and Voituriez, R., 2016. Mean first-passage times of non-Markovian random walkers in confinement. Nature, 534(6607), p.356.
[16] Redner, S., 2007. A Guide to First-Passage Processes. A Guide to First-Passage Processes, by Sidney Redner, Cambridge, UK: Cambridge University Press, 2007.
[17] Metzler, R., Redner, S. and Oshanin, G., 2014. First-Passage Phenomena and Their Applications (Vol. 35). Singapore: World Scientific.
[18] Bray, A.J., Majumdar, S.N. and Schehr, G., 2013. Persistence and first-passage properties in nonequilibrium systems. Advances in Physics, 62(3), pp.225-361.
[19] T. E. Harris, The Theory of Branching Processes (Springer, Berlin, 1963).
[20] Bramson, M.D., 1978. Maximal displacement of branching Brownian motion. Communications on Pure and Applied Mathematics, 31(5), pp.531-581.
[21] Brunet, É. and Derrida, B., 2009. Statistics at the tip of a branching random walk and the delay of traveling waves. EPL (Europhysics Letters), 87(6), p.60010.
[22] Ramola, K., Majumdar, S.N. and Schehr, G., 2014. Universal order and gap statistics of critical branching Brownian motion. Physical Review Letters, 112(21), p.210602.
[23] Ramola, K., Majumdar, S.N. and Schehr, G., 2015. Spatial extent of branching Brownian motion. Physical Review E, 91(4), p.042131.
[24] Dumonteil, E., Majumdar, S.N., Rosso, A. and Zoia, A., 2013. Spatial extent of an outbreak in animal epidemics. Proceedings of the National Academy of Sciences, 110(11), pp.4239-4244.
[25] Eliazar, I., 2018. Branching Search. Europhysics Letters, 120(6), p.60008.
[26] Shlesinger, M.F., 2006. Mathematical physics: Search research. Nature, 443(7109), p.281.
[27] Bénichou, O., Loverdo, C., Moreau, M. and Voituriez, R., 2011. Intermittent search strategies. Reviews of Modern Physics, 83(1), p.81.
[28] Eliazar, I., Koren, T. and Klafter, J., 2007. Searching circular DNA strands. Journal of Physics: Condensed Matter, 19(6), p.065140.
[29] Chechkin A. and Sokolov I. M., Random search with resetting: A unified renewal approach, to appear in Phys. Rev. Lett.
[30] Cox, D.R., Miller, H.D., 2001. The Theory of Stochastic Processes. CRC Press. See Eq. (74) in Chapter 5.
[31] Pal, A., Eliazar, I. and Reuveni, S., see Supplementary Material.
[32] Chupeau, M., Bénichou, O. and Voituriez, R., 2015. Cover times of random searches. Nature Physics, 11(10), p.844.
[33] IyerBiswas, S. and Zilman, A., 2015. FirstPassage Processes in Cellular Biology. Advances in Chemical Physics, Volume 160, pp.261-306.
[34] Yitzhaki, S. and Schechtman, E., 2012. The Gini methodology: A primer on a statistical methodology (Vol. 272). Springer Science & Business Media.
[35] Eliazar, I., 2017. A tour of inequality. Annals of Physics.
[36] Eliazar, I., 2017. Inequality spectra. Physica A: Statistical Mechanics and its Applications, 469, pp.824-847.
[37] Klafter, J. and Shlesinger, M.F., 1986. On the relationship among three theories of relaxation in disordered systems. Proceedings of the National Academy of Sciences, 83(4), pp.848-851.
[38] Reuveni, S., Klafter, J. and Granek, R., 2012. Dynamic structure factor of vibrating fractals. Physical review letters, 108(6), p.068101.
[39] Gupta, S., Majumdar, S.N. and Schehr, G., 2014. Fluctuating interfaces subject to stochastic resetting. Physical review letters, 112(22), p.220601.
[40] Pal, A., 2015. Diffusion in a potential landscape with stochastic resetting. Physical Review E, 91(1), p.012113.
[41] Eule, S. and Metzger, J.J., 2016. Non-equilibrium steady states and gap statistics of critical branching Brownian motion. Physical Review E, 93(2), p.022106.
[42] Durang, X., Henkel, M. and Park, H., 2014. The statistical mechanics of the coagulationdiffusion process with a stochastic reset. Journal of Physics A: Mathematical and Theoretical, 47(4), p.045002.
[43] Majumdar, S.N., Sabhapandit, S. and Schehr, G., 2015. Dynamical transition in the temporal relaxation of stochastic processes under resetting. Physical Review E, 91(5), p.052131.
[44] Evans, M.R. and Majumdar, S.N., 2014. Diffusion with resetting in arbitrary spatial dimension. Journal of Physics A: Mathematical and Theoretical, 47(28), p.285001.
[45] Méndez, V. and Campos, D., 2016. Characterization of stationary states in random walks with stochastic resetting. Physical Review E, 93(2), p.022106.
[46] Christou, C. and Schadschneider, A., 2015. Diffusion with resetting in bounded domains. Journal of Physics A: Mathematical and Theoretical, 48(28), p.285003.
[47] Chatterjee, A., Christou, C. and Schadschneider, A., 2018. Diffusion with resetting inside a circle. Physical Review E, 97(6),
[48] Roldán, É. and Gupta, S., 2017. Path-integral formalism for stochastic resetting: Exactly solved examples and shortcuts to confinement. Physical Review E, 96(2), p.022130.
[49] Falcón-Cortés, A., Boyer, D., Giuggioli, L. and Majumdar, S.N., 2017. Localization transition induced by learning in random searches. Physical review letters, 119(14), p.140603.
[50] Falcao, R. and Evans, M.R., 2017. Interacting Brownian motion with resetting. Journal of Statistical Mechanics: Theory and Experiment, 2017(2), p.023204.
[51] Majumdar, S.N. and Oshanin, G., 2018. Spectral content of fractional Brownian motion with stochastic reset. arXiv preprint arXiv:1806.03435
[52] Hollander, F.D., Majumdar, S.N., Meylahn, J.M. and Touchette, H., 2018. Properties of additive functionals of Brownian motion with resetting. arXiv preprint arXiv:1801.09909
[53] Montero, M., Masó-Puigdellosas, A. and Villarroel, J., 2017. Continuous-time random walks with reset events: Historical background and new perspectives. J. Eur. Phys. J. B 90: 176.
[54] Majumdar, S.N., Sabhapandit, S. and Schehr, G., 2015. Random walk with random resetting to the maximum position. Physical Review E, 92(5), p.052126.
[55] Meylahn, J.M., Sabhapandit, S. and Touchette, H., 2015. Large deviations for Markov processes with resetting. Physical Review E, 92(6), p.062148.
[56] Boyer, D., Evans, M.R. and Majumdar, S.N., 2017. Long time scaling behaviour for diffusion with resetting and memory. Journal of Statistical Mechanics: Theory and Experiment, 2017(2), p.023208.
[57] Husain, K. and Krishna, S., 2016. Efficiency of a Stochastic Search with Punctual and Costly Restarts. arXiv preprint arXiv:1609.03754.
[58] Campos, D. and Méndez, V., 2015. Phase transitions in optimal search times: How random walkers should combine resetting and flight scales. Physical Review E, 92(6), p.062115.
[59] Shkilev, V.P., 2017. Continuous-time random walk under time-dependent resetting. Physical Review E, 91(1), p.012126.
[60] Fuchs, J., Goldt, S. and Seifert, U., 2016. Stochastic thermodynamics of resetting. EPL (Europhysics Letters), 113(6), p.60009.
[61] Pal, A. and Rahav, S., 2017. Integral fluctuation theorems for stochastic resetting systems. Physical Review E, 96(6), p.062135.
[62] Belan, S., 2018. Restart could optimize the probability of success in a Bernoulli trial. Physical review letters, 120(8), p.080601.
[63] Rose, D.C., Touchette, H., Lesanovsky, I. and Garrahan, J.P., 2018. Spectral properties of simple classical and quantum reset processes. arXiv preprint arXiv:1806.01298
[64] Mukherjee, B., Sengupta, K. and Majumdar, S.N., 2018. Quantum dynamics with stochastic reset. arXiv preprint arXiv:1806.00019
[65] Roldán, É., Lisica, A., Sánchez-Taltavull, D. and Grill, S.W., 2016. Stochastic resetting in backtrack recovery by RNA polymerases. Physical Review E, 93(6), p.062411.
[66] Gallager, R.G., 2013. Stochastic processes: theory for applications. Cambridge University Press.
[67] Reiss, R.D., Thomas, M., 2007. Statistical analysis of extreme values: with applications to insurance, finance, hydrology and other fields (Vol. 2). Basel: Birkhauser.
[68] Majumdar, S.N. and Pal, A., 2014. Extreme value statistics of correlated random variables. arXiv preprint arXiv:1406.6768
[69] Eliazar, I., 2017. Investigating equality: The Renyi spectrum. Physica A: Statistical Mechanics and its Applications, 481, pp.90-118.
FIG. S1. The Inverse-Gaussian first passage time distribution for diffusion with drift is plotted for two different values of the Péclet number (solid lines). Here, we fixed $L/D = 2.0$ so that $Pe = V$. It can be seen that when $Pe = 5$ the corresponding distribution is already sharply peaked around its mode. On the other extreme, when $Pe = 0.1$, the corresponding distribution closely resembles the one which is obtained in the pure diffusion limit (dashed line).

FIG. S2. Critical $CV$'s, as obtained in Fig. 2, from the relation $CV^2 = 1/Pe$, vs. the branching index $m$ for diffusion with drift under restart with branching. As $m$ increases, lower $CV$'s are found indicating that restart with branching can expedite completion even if first passage time distributions are narrow ($CV < 1$). However, while the critical $CV$ value found for $m = 1$ is universal (true for any process), the ones found for $m > 1$ are particular to diffusion with drift.
SUPPLEMENTARY TEXT

Details of the simulations in Fig. 2 and Fig. 4

In this subsection, we provide details of simulations done in Fig. 2 and Fig. 4. These were carried out using the following methodology. Based on the process at hand (see details below), we determined the distribution of the random completion time $T$ in the absence of restart and branching. Then, we set the distribution of the restart time $R$ to be exponential with mean $1/r$ (note, again, that this is equivalent to setting a constant restart rate $r$). After these distributions have been set, we simply followed Eq. (1) to gather statistics about the completion time of the process when it is subjected to restart with branching. This was done using the following algorithm: (i) draw two random numbers from the respective distributions of $T$ and $R$; (ii) If $T < R$, set the completion time $T_R$ to $T$ and end the program; (iii) Otherwise, when $R \leq T$, record the time $R$ that has passed, restart the parent process and branch it into $m$ daughter processes that are identical to it; (iv) Set $T_R = R + [T^{(m)}]_R$, where $T^{(m)}$ is a random variable distributed like the minimum of $m$ copies of $T$, and $R'$ is a random restart time taken from the same distribution as $R$; (v) Repeat steps (i)-(iv) with $T^{(m)}$ substituting for $T$ to determine $[T^{(m)}]_R$ and consecutively $T_R$. (vi) Repeat steps (i)-(v) many times to gather multiple samples of the random variable $T_R$. Use these to compute $\langle T_R \rangle$—the mean completion time of the process when subjected to restart with branching. Actual implementation of the above algorithm depends on the exact distribution of the random variable $T$ and on the value of the branching index $m$ used. These are specified below.

• Fig. 2 — To study the effect that the introduction of an infinitesimally small restart rate has on the mean FPT of a particle undergoing diffusion with drift, we simulated the process illustrated in Fig. 1 in the main text. For this, we let the random time $T$ come from the inverse Gaussian distribution [S4]

$$f_T(t) = \frac{L}{\sqrt{4\pi Dt^3}} e^{-\frac{(t+\nu)^2}{2Dt}} \quad (t \geq 0). \quad (S1)$$

Parameter values were set to: $D = 0.5$ and $L = 1.0$; and the value of $V$ was set to obey $Pe = LV/2D$. The value of the branching index was varied in the range $m = 1, 2, 3, 4$.

• Fig. 4 — Cover time of a random walker — Here, we let the random time $T$ be the normalized cover time of a random walker on a network of $N$ sites. The cover time $T(N)$ is defined as the time needed by the walker to visit all sites in the network. In [S3], the authors showed that after proper scaling this cover time admits a universal limit law. More specifically, letting $\langle \tau \rangle$ denote the global mean first passage time, defined as the mean first-passage time to a given target site averaged over all starting sites, it was shown that $X = T/\langle \tau \rangle - \ln N$ converges to the following asymptotic distribution

$$f_X(x) = \exp \left( -x - e^{-x} \right) \quad (\infty \leq x \leq \infty), \quad (S2)$$

in the limit of $N \to \infty$. Setting

$$\langle \tau \rangle \cdot [X + \ln N] / \langle T \rangle = \langle \tau \rangle / \langle T \rangle, \quad (S3)$$

with $N = 10^4$ and $\langle \tau \rangle = 4.1$, we generated samples from the random variable $T$ by drawing a random number from the distribution of $X$ in Eq. (S2) and then substituting it back in Eq. (S3).

• Fig. 4 — Diffusion with drift — Here we once again let the random time $T$ come from the inverse Gaussian distribution [S4]

$$f_T(t) = \frac{L}{\sqrt{4\pi Dt^3}} e^{-\frac{(t+\nu)^2}{2Dt}} \quad (t \geq 0). \quad (S4)$$

Parameters values were taken such that $L^2/2D = 10.0$ and $L/V = 1.0$ thus setting $\langle T \rangle = 1$.

• Fig. 4 — Cell growth model — Here we let the random time $T$ come from the Gamma distribution

$$f_T(t) = \frac{\kappa e^{-\kappa t}}{\Gamma(\kappa)} (\kappa t)^{-1+\theta} \quad (t \geq 0). \quad (S5)$$

Parameter values were set to $\kappa = \theta = 1/9$ so that $\langle T \rangle = \theta / \kappa = 1$. In [S5] this distribution was used to describe the time it takes a cell growing at a rate $\kappa$ to reach a certain threshold size $\theta$. 

Detailed derivation of Eq. (3)

Here we provide a detailed derivation of Eq. (3) in the main text. We start by letting $R', R'', R''' \ldots$ denote independent and identically distributed copies of the random restart time $R$. Iterative use of Eq. (3) in the main text then gives

$$T_R = \min(T, R) + I(R \leq T)|T^{(m)}|\epsilon',$$

$$[T^{(m)}]_{\epsilon'} = \min(T^{(m)}, R') + I(R' \leq T^{(m)})|T^{(m)}|\epsilon',$$

$$[T^{(m)}]_{\epsilon''} = \min(T^{(m)}, R'') + I(R'' \leq T^{(m)})|T^{(m)}|\epsilon'',$$

$$\ldots$$

(S6)

Taking expectations in Eq. (S6) and accommodating all terms in the series, we obtain

$$\langle T_R \rangle = \langle \min(T, R) \rangle + \Pr(R \leq T) \langle \min(T^{(m)}, R) \rangle + \Pr(R \leq T) \Pr(R \leq T^{(m)}) \langle \min(T^{(m)}, R) \rangle + \ldots,$$

where we have used statistical independence, the fact that $R', R'', R''' \ldots$ are independent and identically distributed copies of $R$, and the fact that $\langle I(\text{Event}) \rangle = \Pr(\text{Event occurred})$. It will be convenient to define

$$p_k = \Pr(R \leq T^{(m)}),$$

$$q_k = \langle \min(T^{(m)}, R) \rangle.$$  \hspace{1cm} (S8)

Using the above definitions, we can rewrite the expression for mean completion time from Eq. (S7) as

$$\langle T_R \rangle = q_0 + \sum_{n=0}^{\infty} q_{n+1} \prod_{k=0}^{n} p_k.$$ \hspace{1cm} (S9)

In particular, note that when $R$ is an exponentially distributed random variable with rate $r$, further simplifications can be made by making use of a relation between $p_k$ and $q_k$. In the following, we first derive this relation, and then show how Eq. (3) is obtained from Eq. (S9). First we observe that

$$p_k = \Pr(R \leq T^{(m)}),$$

$$= \int_0^m dt \ f_R(t) \Pr(t \leq T^{(m)}),$$

$$= r \int_0^\infty dt \ Pr(t \leq T^{(m)}) \ e^{-rt},$$ \hspace{1cm} (S10)

where we have used the fact that the restart time distribution is exponential i.e., $f_R(t) = re^{-rt}$. Making use of the same property, we find $\langle \min(T, R) \rangle = \int_0^m dt \ Pr(\min(T, R) \geq t) = \int_0^m dt \ Pr(T \geq t) \ Pr(R > t) = \int_0^\infty dt \ Pr(T \geq t) \ e^{-rt}$. Similarly, one can write

$$q_k = \langle \min(T^{(m)}, R) \rangle,$$

$$= \int_0^\infty dt \ Pr(T^{(m)} \geq t) \ e^{-rt}.$$ \hspace{1cm} (S11)

Examining Eq. (S10) and Eq. (S11), we establish the following relation $q_k = \frac{1}{r} p_k$. Using the above relation in Eq. (S9), we obtain the following expression for the mean completion time under restart with branching at a constant rate $r$

$$\langle T_R \rangle = \frac{1}{r} \ p_0 + \frac{1}{r} \ p_0 p_1 + \frac{1}{r} \ p_0 p_1 p_2 + \cdots$$

$$= \frac{1}{r} \ \sum_{n=0}^{\infty} \ \prod_{k=0}^{n} p_k.$$ \hspace{1cm} (S12)

Now using the definition of $p_k$ from Eq. (S8), we can write

$$p_k = \Pr(R \leq T^{(m)}),$$

$$= r \int_0^m dt \ Pr(t \leq T^{(m)}) \ e^{-rt}$$

$$= 1 - T^{(m)}(r).$$ \hspace{1cm} (S13)
where $T^{(m)}(r)$ is the Laplace transform of $T^{(m)}$ evaluated at $r$. To arrive at the final expression in Eq. (S13), we have used integration by parts in the second line. Substituting Eq. (S13) in Eq. (S12), we arrive at Eq. (3) in the main text

$$
\langle T_R \rangle = \frac{1}{r} \sum_{n=0}^{\infty} \prod_{k=0}^{n} \left[ 1 - \tilde{T}^{(m)}(r) \right].
$$

(S14)

In particular, for $m = 1$, one can sum the infinite series in Eq. (S14) to show that it boils down to the result derived in [S1, S2]

$$
\langle T_R \rangle = \frac{1}{r} \frac{1 - \tilde{T}(r)}{\tilde{T}(r)}
$$

(S15)

where $\tilde{T}(r)$ is the Laplace transform of $T$ evaluated at $r$.

**Detailed derivation of Eq. (5) in the main text**

Here we provide a detailed derivation of Eq. (5) in the main text. We start with the case of $m = 2$. Substituting the definition of $\rho_k$ in Eq. (S10) into Eq. (S12) we obtain

$$
\langle T_R \rangle = \int_0^\infty dt \Pr(T \geq t) e^{-rt} + r \int_0^\infty dt \Pr(T \geq t) \int_0^\infty dt \Pr(T^{(2)} \geq t) e^{-rt}
$$

$$
+ r^2 \int_0^\infty dt \Pr(T \geq t) \int_0^\infty dt \Pr(T^{(2)} \geq t) \int_0^\infty dt \Pr(T^{(4)} \geq t) e^{-rt} + \cdots
$$

(S16)

Differentiating both sides of Eq. (S16) with respect to $r$, we find that $\frac{d}{dr} \langle T_R \rangle \bigg|_{r=0} < 0$ if and only if

$$
\int_0^\infty dt \ t \Pr(T \geq t) > \int_0^\infty dt \Pr(T \geq t) \int_0^\infty dt \Pr(T^{(2)} \geq t).
$$

(S17)

We now observe that the first term on the right hand side of Eq. (S17) is nothing but $\langle T \rangle$. Indeed,

$$
\langle T \rangle = \int_0^\infty dt \ t f_T(t)
$$

$$
= - \int_0^\infty dt \ t \frac{d}{dt} \Pr(T \geq t)
$$

$$
= \int_0^\infty dt \Pr(T \geq t),
$$

(S18)

where to arrive at the last line we performed integration by parts in the second line. Similarly, we see that the second term on the right hand side of Eq. (S17) is given by

$$
\int_0^\infty dt \Pr(T^{(2)} \geq t) = \int_0^\infty dt \Pr(\min\{T_1, T_2\} \geq t) = \langle \min\{T_1, T_2\} \rangle.
$$

(S19)

As for the term on left hand side of Eq. (S17), we first note that

$$
\langle T^2 \rangle = \int_0^\infty dt \ t^2 f_T(t)
$$

$$
= - \int_0^\infty dt \ t^2 \frac{d}{dt} \Pr(T \geq t),
$$

(S20)

and integration by parts then gives

$$
\frac{1}{2} \langle T^2 \rangle = \int_0^\infty dt \ t \Pr(T \geq t).
$$

(S21)

Substituting Eq. (S18), Eq. (S19), and Eq. (S21) into Eq. (S17), we find that the condition in Eq. (S17) is equivalent to

$$
\frac{1}{2} \langle T^2 \rangle > \langle T \rangle \langle \min\{T_1, T_2\} \rangle.
$$

(S22)

Recalling that $CV^2 = \sigma^2(T) \langle T \rangle^2$ and the definition of the Gini index from Eq. (4) in the main text, we find that Eq. (S22) boils down to Eq. (5) in the main text for $m = 2$. 

We thus see that $T$ is then given by

\[ T \sim \text{Weibull}(\alpha, \beta) \]

as stated in the main text. Recalling that the coefficient of variation of the Weibull distribution is given by $CV^2 = \Gamma(1 + 2/\beta)/\Gamma(1 + 1/\beta)^2 - 1$, we conclude that for this distribution the criterion in Eq. (5) reads

\[ m > \left[ \frac{2\Gamma(1 + 1/\beta)^2}{\Gamma(1 + 2/\beta)} \right]^{\beta}. \]

**Explicit forms of the criterion in Eq. (5) in the main text**

In this section we consider three different distributions of the time $T$ to show how the criterion in Eq. (5) can be computed explicitly.

**Case A: Weibull distribution**

Consider the Weibull distribution $Pr(T \geq t) = e^{-(t/\alpha)^\beta}$ $(t > 0)$, and note that in this case we have

\[
Pr(T^{(m)} \geq t) = Pr(\min\{T_1, \cdots, T_m\} \geq t) \\
= Pr(T \geq t)^m \\
= e^{-m(t/\alpha)^\beta} \\
= e^{-(t/\bar{\alpha})\beta}, \quad \text{where } \bar{\alpha} = \alpha/m^{1/\beta}.
\]

We thus see that $T^{(m)}$ is also a Weibull random variable with scale parameter $\bar{\alpha}$ and shape parameter $\beta$. The expectation of $T^{(m)}$ is then given by $\langle T^{(m)} \rangle = \bar{\alpha}\Gamma(1 + 1/\beta)$, where $\Gamma(x)$ is the Gamma function. Plugging this expression in Eq. (4), we obtain

\[
GL_m = 1 - \frac{\langle T^{(m)} \rangle}{\langle T \rangle} \\
= 1 - \frac{1}{m^{1/\beta}},
\]

as stated in the main text. Recalling that the coefficient of variation of the Weibull distribution is given by $CV^2 = \Gamma(1 + 2/\beta)/\Gamma(1 + 1/\beta)^2 - 1$, we conclude that for this distribution the criterion in Eq. (5) reads

\[ m > \left[ \frac{2\Gamma(1 + 1/\beta)^2}{\Gamma(1 + 2/\beta)} \right]^{\beta}. \]

**Case B: Pareto distribution**

Consider the Pareto distribution $Pr(T \geq t) = \left( \frac{1}{1 + \alpha t} \right)^\alpha$ $(t \geq 0)$, with $\alpha > 0$ standing for the shape parameter. Recall that the expectation of $T$ diverges for $0 < \alpha \leq 1$, and that its variance diverges for $1 < \alpha \leq 2$. In both these cases, and as discussed in the main text, the introduction of restart will always reduce the mean completion time and the same can thus also be said for restart.
with branching. We therefore focus on the more interesting case of $\alpha > 2$. For this we observe that
\[
\Pr(T^{(m)} \geq t) = \Pr(\min \{T_1, \ldots, T_m\} \geq t)
\]
\[
= \Pr(T \geq t)^m
\]
\[
= \left(\frac{1}{1+t}\right)^{\alpha m}
\]
\[
= \left(\frac{1}{1+t}\right)^{\tilde{\alpha}}, \quad \text{where} \quad \tilde{\alpha} = \alpha m.
\] (S29)

From here we see that $T^{(m)}$ is also a Pareto random variable with shape parameter $\tilde{\alpha}$. The expectation of $T^{(m)}$ is then given by
\[
\langle T^{(m)} \rangle = 1/(\tilde{\alpha}-1).
\]
Substituting this expression into Eq. (4) in the main text, we find
\[
G_{1m} = 1 - \frac{\langle T^{(m)} \rangle}{\langle T \rangle} = \frac{\alpha m - \alpha}{\alpha m - 1}.
\] (S30)

For $\alpha > 2$ the coefficient of variation of the Pareto distribution is well defined and given by $CV^2 = \alpha/(\alpha - 2)$. Equation (5) in the main text boils down to
\[
m > \frac{\alpha - 1}{\alpha}.
\] (S31)

**Case C: Uniform distribution**

Consider the uniform distribution on the unit interval $[0, 1]$. For this distribution $\Pr(T \geq t) = (1-t) (0 \leq t \leq 1)$; and we thus have
\[
\Pr(T^{(m)} \geq t) = \Pr(\min \{T_1, \ldots, T_m\} \geq t)
\]
\[
= \Pr(T \geq t)^m
\]
\[
= (1-t)^m, \quad 0 \leq t \leq 1.
\] (S32)

It is then easy to show that $\langle T^{(m)} \rangle = 1/m\alpha$ and $G_{1m} = m-1/m\alpha$ follows immediately. In addition, the coefficient of variation for this distribution is simply given by $CV^2 = 1/3$. Equation (5) in the main text then boils down to a particularly simple condition
\[
m > 2.
\] (S33)

**Connection with extreme value theory**

In this section we derive an asymptotic version of Eq. (7) by use of extreme value theory. Denoting the cumulative distribution function of the random variable $T$ by $F(t) = \Pr(T \leq t)$ ($t \geq 0$), we assume that $F(t)$ is regularly varying at the origin, i.e., that
\[
\lim_{l \to 0} \frac{F(\varepsilon l)}{F(l)} = l^\varepsilon,
\] (S34)

for some $\varepsilon > 0$ (the regular-variation exponent). For example, it is easy to verify that the cumulative distribution functions of the Weibull, Pareto and Uniform distributions in the previous section all have this property. To see this, simply observe that: (i) for the Weibull—$\lim_{l \to 0} F(\varepsilon l)/F(l) = t^\beta$, so that $\varepsilon = \beta > 0$; (ii) for the Pareto—$\lim_{l \to 0} F(\varepsilon l)/F(l) = t$, so that $\varepsilon = 1$; and (iii) for the Uniform—$\lim_{l \to 0} F(\varepsilon l)/F(l) = t$, so that $\varepsilon = 1$.

Examine now the random variable $W_m = s_m \cdot \min \{T_1, \ldots, T_m\}$ (where $s_m$ is a positive scalar parameter) which satisfies
\[
\lim_{m \to \infty} m \cdot F\left(\frac{1}{s_m}\right) = 1,
\] (S35)
e.g., take $s_m$ such that $m \simeq 1/F(1/s_m)$ for $m \gg 1$. Extreme value theory then asserts that
\[
\lim_{m \to \infty} \Pr(W_m > t) = \exp(-t^\varepsilon) \quad (t \geq 0),
\] (S36)
i.e., $W_m$ converges, in law, to a limit governed by the Weibull distribution. One can then approximate $\langle W_m \rangle \sim \Gamma(1 + \frac{1}{\varepsilon})$. Recalling Eq. (7) in the main text we multiply and divide its right hand side by $s_m$ to obtain

$$\langle T^2 \rangle > \frac{1}{s_m} \langle s_m \cdot \min \{ T_1, \cdots, T_m \} \rangle = \frac{1}{s_m} \langle W_m \rangle.$$ (S37)

Substituting the expression for $\langle W_m \rangle$ we obtain an asymptotic version of Eq. (S37)

$$\langle T^2 \rangle > \frac{1}{s_m} \Gamma \left( 1 + \frac{1}{\varepsilon} \right).$$ (S38)

As the left-hand side of Eq. (S38) is fixed, and $\lim_{m \to \infty} s_m = \infty$ we conclude that, under the conditions specified above, the criterion in Eq. (S38) can always be met by taking $m$ to be large enough.

---

*S1* Reuveni, S., 2016. Optimal stochastic restart renders fluctuations in first passage times universal. Physical review letters, 116(17), p.170601.

*S2* Pal, A. and Reuveni, S., 2017. First Passage under Restart. Physical review letters, 118(3), p.030603.

*S3* Chupeau, M., Bénichou, O. and Voituriez, R., 2015. Cover times of random searches. Nature Physics, 11(10), p.844.

*S4* Cox, D.R., Miller, H.D., 2001. The Theory of Stochastic Processes. CRC Press. See Eq. (74) in Chapter 5.

*S5* IyerBiswas, S. and Zilman, A. 2015. First Passage Processes in Cellular Biology. Advances in Chemical Physics, Volume 160, pp.261-306.

*S6* Bingham, N.H., Goldie, C.M. and Teugels, J.L., 1989. Regular variation (Vol. 27). Cambridge university press.

*S7* Reiss, R.D., Thomas, M., 2007. Statistical analysis of extreme values: with applications to insurance, finance, hydrology and other fields (Vol. 2). Basel: Birkhauser.