Random walk models approximating symmetric space-fractional diffusion processes

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Abstract

For the symmetric case of space-fractional diffusion processes (whose basic analytic theory has been developed in 1952 by Feller via inversion of Riesz potential operators) we present three random walk models discrete in space and time. We show that for properly scaled transition to vanishing space and time steps these models converge in distribution to the corresponding time-parameterized stable probability distribution. Finally, we analyze in detail a model, discrete in time but continuous in space, recently proposed by Chechkin and Gonchar.

Remark: Concerning the inversion of the Riesz potential operator \(I_0^\alpha\) let us point out that its common hyper-singular integral representation fails for \(\alpha = 1\). In our Section 2 we have shown that the corresponding hyper-singular representation for the inverse operator \(D_0^\alpha\) can be obtained also in the critical (often excluded) case \(\alpha = 1\), by analytic continuation.

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1 Introduction: concepts and notations

By a "space-fractional" diffusion process (or Lévy-Feller diffusion process) we mean a process of diffusion of an extensive quantity with density \( u(x, t) \) governed by an evolution equation

\[
\frac{\partial u(x, t)}{\partial t} = D_\theta^\alpha u(x, t) \quad (x \in \mathbb{R}, t > 0)
\]

with an initial condition

\[
u(x, 0) = f(x) \quad (f \in L_1(\mathbb{R})).
\]

We interpret \( x \) as space, \( t \) as time variable. \( D_\theta^\alpha \) is a pseudo-differential operator acting with respect to the space variable \( x \), its symbol being

\[
\hat{D}_\theta^\alpha(\kappa) = -|\kappa|^{\alpha} e^{i(\text{sign} \kappa) \theta \pi/2} = -|\kappa|^\alpha e^{i(\text{sign} \kappa) \theta},
\]

and the real parameters \( \alpha \) and \( \theta \) are restricted by the inequalities

\[
0 < \alpha \leq 2, \quad |\theta| \leq \begin{cases} \alpha & \text{if } 0 < \alpha \leq 1, \\ 2 - \alpha & \text{if } 1 < \alpha \leq 2. \end{cases}
\]

For a sufficiently well-behaved function or generalized function \( \phi \) defined on \( \mathbb{R} \) we denote by \( \hat{\phi} \) its Fourier transform:

\[
\hat{\phi}(\kappa) = \int_{-\infty}^{+\infty} e^{ikx} \phi(x) \, dx \quad (k \in \mathbb{R}).
\]

Then, for a generic linear pseudo-differential operator \( A \) acting on these functions, its symbol \( \hat{A}(\kappa) \) turns out to be defined through the Fourier representation of \( (A\phi)(x) \), namely \( \hat{A}\phi(\kappa) = \hat{A}(\kappa) \hat{\phi}(\kappa) \). An often applicable practical rule is

\[
\hat{A}(\kappa) = (Ae^{-ikx})e^{ikx}, \quad (k \in \mathbb{R}).
\]

If \( B \) is another pseudo-differential operator, then we have \( \hat{A}\hat{B}(\kappa) = \hat{A}(\kappa) \hat{B}(\kappa) \).

Let us remark, that we chose (1.5) to define the Fourier transform in agreement with the common terminology of probability theory.

Introducing the stable probability density \( p_\alpha(x; \theta) \) whose characteristic function (Fourier transform) is

\[
\hat{p}_\alpha(\kappa; \theta) = \exp\left(-|\kappa|^\alpha e^{i(\text{sign} \kappa) \theta \pi/2}\right) \quad (\kappa \in \mathbb{R})
\]
and rescaling $p_\alpha(x; \theta)$ for $x \in \mathbb{R}$, $t > 0$ by the similarity variable $xt^{-\frac{1}{\alpha}}$ we obtain the time-dependent stable probability density

\begin{equation}
(1.8) \quad g_\alpha(x, t; \theta) = t^{-\frac{1}{\alpha}}p_\alpha(xt^{-\frac{1}{\alpha}}; \theta) \quad (x \in \mathbb{R}, t > 0)
\end{equation}

with which we can write the solution to (1.1), (1.2) in the form

\begin{equation}
(1.9) \quad u(x, t) = \int_{-\infty}^{+\infty} g_\alpha(x - \xi, t; \theta)f(\xi)\, d\xi.
\end{equation}

Then, for all $t > 0$, we have

\begin{equation}
(1.10) \quad \left\{ \begin{array}{l}
  u(\cdot, t) \in C^\infty \cap L_1(\mathbb{R}), \\
  \int_{-\infty}^{+\infty} u(x, t)\, dx = \int_{-\infty}^{+\infty} f(x)\, dx \\
  f(x) \geq 0 \quad \text{for all} \quad x \in \mathbb{R} \Rightarrow u(x, t) \geq 0 \quad \text{for all} \quad x \in \mathbb{R}.
\end{array} \right.
\end{equation}

If the initial function is a probability density then so is also the function $u(x, t)$ and we have in (1.1), (1.2) the description of a Markov process.

For orientation on the general theory of stable probability distributions we recommend in particular [F71] and [F52], Feller’s parameterization being close to ours. In our foregoing considerations we essentially have surveyed results of Feller’s pioneering paper [F52]. For a few parameters pairs $(\alpha, \theta)$ representations of $p_\alpha(x; \theta)$ in terms of elementary or well-investigated special functions are available in the literature but in other parameterization (see e.g. [Zo], [Sc], [SaT]). We here content ourselves with recognizing the classical Gauss process and the Cauchy process, respectively, in

\begin{equation}
(1.11) \quad g_2(x, t; 0) = \frac{1}{2\sqrt{\pi}t^{-\frac{3}{2}}\exp\left(-\frac{x^2}{4t}\right)}, \quad g_1(x, t; 0) = \frac{1}{\pi} \frac{t}{x^2 + t^2}.
\end{equation}

Feller in [F52] has shown that $D_0^\alpha$ (in case $\alpha \neq 1$) can be viewed as inverse to the (later in [SKM] so called) Feller potential operator which is a linear combination of two Weyl integral operators. In the sequel we will restrict attention to the symmetric case $\theta = 0$ retaining however, the index 0 in order to be in concordance with the notation of our previous papers [GM98], [GFM], [GM99]. In [GM98] we have discussed for $\alpha \neq 1$ a random walk model for the whole range of values $\theta$; here we now concentrate on models.
not treated there, requiring however $\theta = 0$. This means that henceforth we will treat the evolution equation

\begin{equation}
\frac{\partial u(x, t)}{\partial t} = D_0^\alpha u(x, t) \quad (x \in \mathbb{R}, \ t > 0)
\end{equation}

with an initial condition

\begin{equation}
u(x, 0) = f(x) \quad (f \in L_1(\mathbb{R})).
\end{equation}

The symbol of the pseudo-differential operator $D_0^\alpha$ is

\begin{equation}
\hat{D}_0^\alpha(\kappa) = -|\kappa|^\alpha,
\end{equation}

and the fundamental solution to (1.11) (namely, for $u(x, 0) = \delta(x)$ = Dirac’s delta function) is the function $u(x, t) = g_\alpha(x, t; 0)$ whose Fourier transform is

\begin{equation}
\hat{g}_\alpha(\kappa, t; 0) = \exp(-t|\kappa|^\alpha) \quad (\kappa \in \mathbb{R}, \ t > 0).
\end{equation}

Then, see (1.7) and (1.8), we get by the Fourier inversion formula

\begin{equation}
g_\alpha(x, t; 0) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ikx} \exp(-t|\kappa|^\alpha) \ d\kappa = t^{-\frac{\alpha}{2}} p_\alpha(x t^{-\frac{\alpha}{2}}; 0).
\end{equation}

We will denote by $S(t)$ for $t > 0$ the random variable whose probability density is given by $g_\alpha(x, t; 0)$.

**Remark 1.1.** We call a process described by (1.1), (1.2) a ”space fractional” diffusion process because it is a ”fractional” generalization of the classical diffusion process which is recovered by taking $\alpha = 2$, $\theta = 0$. Writing (1.13) in the form $\hat{D}_0^\alpha(\kappa) = -(\kappa^2)^{\alpha/2}$ and observing that the operator $D^2$, defined by $(D^2 \phi)(x) = \frac{d^2 \phi(x)}{dx^2}$, has the symbol $-\kappa^2$, we see that $D_0^\alpha = -(-D^2)^{\alpha/2}$, hence the operator $D_0^\alpha$ is the negative of a fractional power of the (positive definite) operator $-D^2$. By calling a process described by (1.1), (1.2) also a Lévy-Feller diffusion process we honour both Lévy and Feller for their essential contributions [L25], [L54], [F52].

Our aim in the following sections is to derive discrete-space discrete-time random walk models approximating the space-fractional diffusion process (henceforth considered as a Markov process) described by (1.11) with $u(x, 0) =$
δ(x). We shall show that for properly scaled transition to vanishing space and time steps there is convergence in distribution to the probability distribution whose density is \( g_\alpha(x, t; 0) \). We shall give heuristic motivations for choosing our concrete models, using in a formal way calculations with symbols. The lack of rigour in these derivations will hopefully not be too annoying to the pure analyst, it will be remedied in the final end by rigorous proofs of convergence.

2 Operators and symbols

In this section we give a survey on the relevant operators and their symbols thereby always assuming

\[(2.1) \quad 0 < \alpha \leq 2.\]

For general orientation and more rigorous treatment we refer to [SKM], [R] and [F52]. We need the operator \( D \) of differentiation, the Weyl operators \( I_+^\alpha, I_-^\alpha \) and their (formal) inverses \( I_+^{-\alpha}, I_-^{-\alpha} \), the Riesz potential operator \( I_0^\alpha \) whose negative inverse \( -I_0^{-\alpha} \) is our pseudo-differential operator \( D_0^\alpha \) (if \( \alpha \neq 1 \)), and the Hilbert transform operator \( H \). For sufficiently well behaved functions \( \phi \) defined on \( \mathbb{R} \) and if required with appropriate understanding of the occurring integrals as Cauchy principal values we have (for \( x \in \mathbb{R} \))

\[(2.2) \quad (D \phi)(x) = \frac{d}{dx} \phi(x) = \phi'(x),\]

\[(2.3) \quad \begin{cases} (I_+^\alpha \phi)(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} (x - \xi)^{\alpha-1} \phi(\xi) \, d\xi, \\ (I_-^\alpha \phi)(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{\infty} (\xi - x)^{\alpha-1} \phi(\xi) \, d\xi, \end{cases}\]

\[(2.4) \quad I_\pm^{-\alpha} = \begin{cases} \pm DI_\pm^{1-\alpha} & \text{if } 0 < \alpha < 1, \\ D^2I_\pm^{2-\alpha} & \text{if } 1 < \alpha \leq 2, \end{cases}\]

\[(2.5) \quad (I_0^\alpha \phi)(x) = \frac{(I_+^\alpha \phi)(x) + (I_-^\alpha \phi)(x)}{2 \cos(\alpha \pi/2)} = \frac{\int_{-\infty}^{\infty} |x - \xi|^{\alpha-1} \phi(\xi) \, d\xi}{2\Gamma(\alpha) \cos(\alpha \pi/2)} \quad \text{for } \alpha \neq 1,\]

\[\frac{\int_{-\infty}^{\infty} |x - \xi|^{\alpha-1} \phi(\xi) \, d\xi}{2\Gamma(\alpha) \cos(\alpha \pi/2)} \quad \text{for } \alpha \neq 1,\]
\( (H\phi)(x) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\phi(\xi)}{x - \xi} d\xi. \)

Recalling formula (1.6) for calculation of symbols we get by direct computation

\[(2.7) \quad \hat{\mathcal{D}} = -i\kappa, \]

\[(2.8) \quad \hat{\mathcal{H}} = i \text{sign} \kappa. \]

We take the symbols of the Weyl operators and the Riesz potential operators from [R, Theorem 4.10] as

\[(2.9) \quad \hat{I}^\alpha_\pm(\kappa) = (\mp i\kappa)^{-\alpha} = |\kappa|^{-\alpha} e^{\pm i (\text{sign} \kappa) \alpha \pi/2} = |\kappa|^{-\alpha} i^{\pm \alpha \text{ sign} \kappa}, \]

from which by addition we get

\[(2.10) \quad \hat{I}^\alpha_0(\kappa) = |\kappa|^{-\alpha}. \]

As already remarked and as used in [F52]

\[(2.11) \quad D^\alpha_0 = -I^{-\alpha}_0 \quad \text{for} \quad \alpha \neq 1 \]

in agreement with the property of symbols

\[(2.12) \quad \widehat{D^\alpha_0}(\kappa) = -|\kappa|^\alpha = -\left(\hat{I}^\alpha_0(\kappa)\right)^{-1} \quad \text{for} \quad \alpha \neq 1. \]

In the special case \( \alpha = 1 \) we observe that by (2.7) and (2.8)

\[(2.13) \quad \widehat{D^1_0}(\kappa) = -|\kappa| = -\hat{D}(\kappa) \hat{\mathcal{H}}(\kappa), \]

hence as already observed in [F52] (with a sign-modified version of the Hilbert transform)

\[(2.14) \quad D^1_0 = -D \mathcal{H}. \]

Let us yet exhibit another representation of our pseudo-differential operator \( D^\alpha_0 \). Via the semi-group property (see [F52] or [SKM])

\[ I^\alpha_0 I^\beta_0 = I^{\alpha+\beta}_0 \quad \text{if} \quad \alpha, \beta \in (0, 1), \quad \alpha + \beta < 1, \]
analytic continuation to negative exponents can be justified and thus from (2.5) and (2.11) the formula

\begin{equation}
D_{\alpha}^{0} = \frac{-1}{2 \cos (\alpha \pi /2)}(I_{+}^{-\alpha} + I_{-}^{-\alpha}) \quad \text{for} \quad \alpha \neq 1.
\end{equation}

In Section 4 we shall find random walk schemes in the case \( \alpha \neq 1 \) by approximating in (2.15) the operators \( I_{+}^{-\alpha} \) and \( I_{-}^{-\alpha} \) with the Grünwald-Letnikov discretization. In Section 5 a second random walk scheme, for the whole range \( 0 < \alpha \leq 2 \), will be obtained by a straightforward discrete approximation of hypersingular integrals for \( I_{0}^{-\alpha} \) and \( DH \). From [SKM, formula (12.1')] we take (for \( 0 < \alpha < 2 \), \( \alpha \neq 1 \))

\begin{equation}
(I_{0}^{-\alpha} \phi)(x) = \frac{1}{2\Gamma(-\alpha) \cos (\alpha \pi /2)} \int_{0}^{\infty} \phi(x + \xi) - 2\phi(x) + \phi(x - \xi) \frac{\xi^{\alpha+1}}{\xi^{\alpha+1}} d\xi.
\end{equation}

Quite formally we can obtain (2.16) by replacing in (2.5) the integrand by \( |\xi|^{\alpha-1} \phi(x - \xi) \), then replacing \( \alpha \) by \(-\alpha\), splitting \( \int_{-\infty}^{\infty} = \int_{-\infty}^{0} + \int_{0}^{\infty} \), here regularizing the right hand side hypersingular integrals by subtracting \( \phi(x) \) in the numerators, finally substituting \(-\xi \) for \( \xi \) in the first right hand side integral and then putting both integrals together.

For convenience we simplify the coefficient in (2.16), introducing

\begin{equation}
b(\alpha) := -\frac{1}{2\Gamma(-\alpha) \cos (\alpha \pi /2)} = \frac{1}{\pi} \frac{\Gamma(\alpha + 1) \sin (\alpha \pi /2)}{\Gamma(-\alpha) \Gamma(\alpha+1)} = -\frac{\pi}{\sin (\alpha \pi)}.
\end{equation}

where for the latter equality we have used \( \sin (\alpha \pi) = 2 \sin (\alpha \pi /2) \cos (\alpha \pi /2) \) and the reflection formula for the gamma function \( \Gamma(-\alpha) \Gamma(\alpha+1) = -\pi / \sin (\alpha \pi) \).

Then, for \( 0 < \alpha < 2 \), \( \alpha \neq 1 \),

\begin{equation}
(D_{\alpha}^{0} \phi)(x) = -(I_{0}^{-\alpha} \phi)(x) = b(\alpha) \int_{0}^{\infty} \frac{\phi(x + \xi) - 2\phi(x) + \phi(x - \xi)}{\xi^{\alpha+1}} d\xi.
\end{equation}

Note that for \( \phi \in C^{2}(\mathbb{R}) \) and \( \phi(x) \) bounded the integral is finite as an improper Riemann integral and observe that \( b(\alpha) > 0 \) for the admitted values of \( \alpha \).

Because (2.17) gives \( b(1) = 1/\pi \) we are tempted to believe the formula

\begin{equation}
(D_{0}^{1} \phi)(x) = \frac{1}{\pi} \int_{0}^{\infty} \frac{\phi(x + \xi) - 2\phi(x) + \phi(x - \xi)}{\xi^{2}} d\xi.
\end{equation}
We will use (2.18) and (2.19) as motivation for our second random walk scheme in which the parameter value $\alpha = 1$ does no longer play a special role. We can indeed obtain (2.19) by looking at (2.14), then formally differentiating (behind the integral sign) (2.6) and then splitting and regularizing the resulting hypersingular integral in the same way as we have done in the case $\alpha \neq 1$.

In view of (2.18) and (2.19) we now have, with $b(\alpha)$ given in (2.17),

$$
(2.20) \quad (D^\alpha_0 \phi)(x) = b(\alpha) \int_0^{\infty} \frac{\phi(x + \xi) - 2\phi(x) + \phi(x - \xi)}{\xi^{\alpha+1}} \, d\xi \quad \text{for } 0 < \alpha < 2.
$$

Unfortunately, this formula loses its meaning in the case $\alpha = 2$.

3 General structure of the random walk models

What do we mean by a random walk model, discrete in space and discrete in time, for a Markov process? Let us be given a random variable $Y$ taking its values in the set $\mathbb{Z}$ of integers according to the probabilities

$$(3.1) \quad P(Y = k) = p_k \quad (k \in \mathbb{Z})$$

with

$$(3.2) \quad \text{all } p_k \geq 0 \text{ and } \sum_{k \in \mathbb{Z}} p_k = 1.$$ 

We discretize the space variable $x \in \mathbb{R}$ and the time variable $t \geq 0$ by grid points $x_j = jh$ and instants $t_n = n\tau$, with $h > 0, \tau > 0, j \in \mathbb{Z}, n \in \mathbb{N}_0$. Then, defining random variables

$$(3.3) \quad S_n = \sum_{m=1}^{n} (h Y_m) = h \sum_{m=1}^{n} Y_m \quad (n \in \mathbb{N})$$

with the $Y_m$ as independent identically distributed random variables, all having the same probability distribution as the random variable $Y$, we interpret $S_n$ as the position at time $t_n$ of a random walker starting in the point $x = 0$. 


at $t = 0$. Denoting by $y_j(t_n)$ the probability of sojourn of the walker in point $x_j$ at instant $t_n$, the recursion $S_{n+1} = S_n + hY_n$ implies

$$y_j(t_{n+1}) = \sum_{k \in \mathbb{Z}} p_k y_{j-k}(t_n) \quad (j \in \mathbb{Z}, n \in \mathbb{N}_0), \tag{3.4}$$

and the walker starting at point $x_0 = 0$ means $y_0(0) = 1$ and $y_j(0) = 0$ for $j \neq 0$. However, in the recursion scheme (3.4) it is legitimate to use a more general initial sojourn probability distribution $(y_j(0) | j \in \mathbb{Z})$.

There is yet another possible interpretation of (3.4), namely as a scheme of redistribution of an extensive quantity (e.g. mass, charge, in the random walk interpretation probability), $y_j(t_n)$ being considered as a clump of this extensive quantity sitting in point $x_j$ at instant $t_n$. Then (3.4) describes a conservative and non-negativity preserving redistribution scheme. In fact, for all $n \in \mathbb{N}$ it follows from (3.2) that, in analogy to (1.10),

$$\sum_{j \in \mathbb{Z}} y_j(t_n) = \sum_{j \in \mathbb{Z}} y_j(0) \quad \text{if} \quad \sum_{j \in \mathbb{Z}} |y_j(0)| < \infty,$$

$$\quad \text{all} \quad y_j(t_n) \geq 0 \quad \text{if all} \quad y_j(0) \geq 0.$$

Such discrete redistribution schemes have been used by one of the authors in discretization of diffusion processes governed by second order linear parabolic differential equations ([G70], [G78], [GN]) as they discretely imitate essential properties of the continuous process.

We come nearer to the Cauchy problem (1.11), (1.12) by intending $y_j(t_n)$ as approximation to

$$\int_{x_j-h/2}^{x_j+h/2} u(x, t_n) \, dx$$

which, if $u(\cdot, t_n)$ is continuous, is $\approx hu(x_j, t_n)$ for small $h$.

We want to show that for proper choice of the probability distribution of the random variable $Y$ and well-scaled transition

$$\tau = \sigma(h), \quad \sigma \text{ strictly monotonic, } \sigma(h) \to 0 \text{ as } h \to 0 \tag{3.5}$$

the random walk ”converges” in some sense to the Markov process with density $u(x, t)$ described by (1.11), (1.12) in the case that the initial function is a probability density. More specifically, we will prove for fixed $t > 0$,
$t = t_n = n\tau$ with $N \ni n = t/\tau \to \infty$ (and proper scaling of $h$ and $\tau$) that the random variable $S_n$ of (3.3) converges in distribution (other terminology: in law) to the random variable $S(t)$ whose density is $g_\alpha(\cdot, t; 0)$, the fundamental solution (1.15) of (1.11). Observing that the distribution function $G_\alpha(x, t; 0) = \int_{-\infty}^{x} g_\alpha(\xi, t; 0) d\xi$ is continuous in $x$ (due to the fast decay in $|\kappa|$ of $\hat{g}_\alpha(\kappa, t; 0) = \exp(-t|\kappa|^\alpha)$ the density $g_\alpha(\cdot, t; 0)$ is in $C^\infty(\mathbb{R})$) and invoking the continuity theorem of probability theory (see, e.g., [B, Theorem 8.28]), all we have to do is to show that for all $\kappa \in \mathbb{R}$ the characteristic function $\hat{y}(\kappa, t; h)$ of the random variable $S_n$ tends to $\exp(-t|\kappa|^\alpha)$ as $h \to 0$. Note the equivalences following from $t = t_n = n\tau$ and (3.5)

\begin{equation}
(3.6) \quad n \to \infty \iff h \to 0 \iff \tau \to 0.
\end{equation}

The general form of the characteristic function $\hat{y}(\kappa, t; h)$ can be found via the generating functions

\begin{equation}
(3.7) \quad \tilde{p}(z) = \sum_{j \in \mathbb{Z}} p_j z^j, \quad \tilde{y}(z, t_n) = \sum_{j \in \mathbb{Z}} y_j(t_n) z^j.
\end{equation}

As probabilities both the $p_j$ and $y_j(t_n)$ sum up to 1 if added over the index $j$, hence these series converge absolutely and uniformly on the periphery $|z| = 1$ of the unit circle, and so the functions $\tilde{p}$ and $\tilde{y}_n$ are there uniformly continuous. The random walk $S_n$ starting at $x = 0$, we have (using the Kronecker symbol) $y_j(0) = \delta_{j0}$, and the recursion (3.4) being a discrete convolution we get

\begin{equation}
(3.8) \quad \tilde{y}(z) = (\tilde{p}(z))^n.
\end{equation}

Replacing in (3.7) $z^j$ by $e^{i\kappa x_j} = e^{i\kappa jh}$ we obtain the corresponding characteristic functions ($\kappa \in \mathbb{R}$)

\begin{equation}
(3.9) \quad \tilde{p}(\kappa; h) = \tilde{p}(e^{i\kappa h}), \quad \tilde{y}(\kappa, t_n; h) = \tilde{y}(e^{i\kappa h}, t_n) = \left(\tilde{p}(e^{i\kappa h})\right)^n.
\end{equation}

Recalling our fixation of $t = t_n = n\tau = n\sigma(h) > 0$, the scaling relation (3.5) and the equivalences (3.6) we have to show that

\begin{equation}
(3.10) \quad \hat{y}(\kappa, t; h) \to \exp(-t|\kappa|^\alpha) \text{ for } n \to \infty,
\end{equation}

\begin{align*}
\text{(3.10)} & \quad \hat{y}(\kappa, t; h) \to \exp(-t|\kappa|^\alpha) \text{ for } n \to \infty,
\end{align*}
or, equivalently

\[ (3.11) \quad \frac{1}{\sigma(h)} \log \tilde{p}(e^{i\kappa h}) \to -|\kappa|^\alpha \quad \text{as} \quad h \to 0. \]

In the following sections we shall exhibit (3.10) as true for specific choices of the probabilities \( p_j \) and scalings \( \tau = \sigma(h) \). The fact that, strictly speaking, \( \tau = t/n = \sigma(h) \) and \( h \) in (3.6) and (3.11) are running through discrete sets will turn out as irrelevant for the proof of (3.11).

\section{The Grünwald-Letnikov random walk}

An idea suggesting itself is to discretize in (1.11) the time derivative \( \frac{\partial u}{\partial t} \) by a two-level difference quotient and the operators \( I_{\pm}^{-\alpha} \) and \( I_{\pm}^{-\alpha} \) (see (2.15)) by the Grünwald-Letnikov approximation (see, e.g., [SKM], [P]). This idea leads to

\[ (4.1) \quad \frac{y_j(t_{n+1}) - y_j(t_n)}{\tau} = hD_0^\alpha y_j(t_n) = -\frac{1}{2 \cos(\alpha \pi/2)}(hI_{+}^{-\alpha} + hI_{-}^{-\alpha}) y_j(t_n). \]

with the operators \( hI_{\pm}^{-\alpha} \) still to be specified. We must exclude the singular case \( \alpha = 1 \), hence will distinguish from now on the cases

\[ (a) \quad 0 < \alpha < 1 , \quad (b) \quad 1 < \alpha \leq 2. \]

We define

\[ (4.2) \quad hI_{\pm}^{-\alpha} y_j(t_n) = h^{-\alpha} \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} y_{j+\pm k}(t_n) \quad \text{in case (a)}, \]

\[ (4.3) \quad hI_{\pm}^{-\alpha} y_j(t_n) = h^{-\alpha} \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} y_{j+1+\pm k}(t_n) \quad \text{in case (b)}. \]

Note in case (b) the shift of the index, that is required in order to obtain non-negative values for all transition probabilities \( p_j \).

Solving (4.1) for \( y_j(t_{n+1}) \), thereby scaling by

\[ (4.4) \quad \tau = \mu h^\alpha =: \sigma(h), \]
gives (remember (3.4))

\[(4.5) \quad y_j(t_{n+1}) = \sum_{k \in \mathbb{Z}} p_k y_{j-k}(t_n)\]

with in case (a)

\[(4.6) \quad p_0 = 1 - \frac{\mu}{\cos (\alpha \pi / 2)}, \quad p_k = (-1)^{|k|+1} \frac{\mu}{2 \cos (\alpha \pi / 2)} \left( \frac{\alpha}{|k|} \right) \quad \text{for} \quad k \neq 0,\]

in case (b)

\[(4.7) \quad \begin{cases} 
  p_0 = 1 + \frac{\mu}{\cos (\alpha \pi / 2)} \left( \frac{\alpha}{1} \right), \\
  p_{\pm 1} = -\frac{\mu}{2 \cos (\alpha \pi / 2)} \left[ 1 + \left( \frac{\alpha}{2} \right) \right], \\
  p_{\pm k} = (-1)^k \frac{\mu}{2 \cos (\alpha \pi / 2)} \left( \frac{\alpha}{k+1} \right) \quad \text{for} \quad k = 2, 3, \ldots .
\end{cases}\]

Then, all \( p_k \geq 0 \) if

\[(4.8) \quad 0 < \mu \leq \cos(\alpha \pi / 2) \quad \text{in case (a)},\]

\[(4.9) \quad 0 < \mu \leq |\cos(\alpha \pi / 2)| / \alpha \quad \text{in case (b)}.\]

Note that \( \cos(\alpha \pi / 2) < 0 \) in case (b). In both cases, by rearrangement of series,

\[\sum_{k \in \mathbb{Z}} p_k = 1 - \frac{\mu}{\cos (\alpha \pi / 2)} \sum_{j=0}^{+\infty} (-1)^j \left( \frac{\alpha}{j} \right) = 1 - 0 = 1.\]

**Remark 4.1.** For all \( \alpha > 0 \) the series \( \sum_{j=0}^{+\infty} (-1)^j \left( \frac{\alpha}{j} \right) \) is absolutely convergent because \( \left( \frac{\alpha}{j} \right) = O(j^{-\alpha-1}) \) for \( j \to \infty \).

We see that, under the conditions (4.8) or (4.9), respectively, we can put

\[P(Y = k) = p_k \quad (k \in \mathbb{Z})\]

for the random variable \( Y \) of (3.1). Using (4.6) and (4.7) we identify the generating function \( \tilde{p} \) of (3.7) as

\[(4.10) \quad \tilde{p}(z) = 1 - \frac{\mu}{2 \cos (\alpha \pi / 2)} \left\{ (1 - z)^\alpha + (1 - z^{-1})^\alpha \right\} \quad \text{in case (a)},\]
$$\tilde{p}(z) = 1 - \frac{\mu}{2 \cos (\alpha \pi/2)} \{z^{-1}(1 - z)^\alpha + z(1 - z^{-1})^\alpha\} \text{ in case (b).}$$

Let us verify the limit relation (3.11) which implies (3.10). Because of the symmetry relation (for \( z = e^{i \kappa h}, \kappa \in \mathbb{R} \))

$$\tilde{p}(z) = \tilde{p}(z^{-1}) \text{ implying } \tilde{p}(e^{i \kappa h}) = \tilde{p}(e^{-i \kappa h})$$

it suffices to verify (3.11) for \( \kappa > 0 \) (the special case \( \kappa = 0 \) being trivial).

Let be \( \kappa > 0 \). Then in case (a) we have

$$\tilde{p}(z) = 1 - \frac{\mu}{\cos (\alpha \pi/2)} \Re(1 - z)^\alpha = 1 - \mu (\kappa h)^\alpha + o(h^\alpha) \text{ as } h \to 0,$$

since \((1 - z)^\alpha \sim (-i \kappa h)^\alpha = e^{-i \alpha \pi/2} (\kappa h)^\alpha \). With the scaling (4.4), namely

$$\sigma(h) = \mu h^\alpha,$$

follows (3.11).

In case (b) we have

$$\tilde{p}(z) = 1 - \frac{\mu}{\cos (\alpha \pi/2)} \Re(z^{-1}(1 - z)^\alpha),$$

and an analogous calculation gives

$$\frac{1}{\sigma(h)} \log \tilde{p}(e^{i \kappa h}) \sim -\kappa^\alpha \frac{\cos((\alpha \pi/2) + \kappa h)}{\cos(\alpha \pi/2)} \text{ as } h \to 0,$$

hence again (3.11). As result we have

**Theorem 4.2.** Distinguish the cases (a) 0 < \( \alpha < 1 \), (b) 1 < \( \alpha \leq 2 \). Define the probabilities \( p_k = P(Y = k) \) in case (a) by (4.6) with restriction (4.8), in case (b) by (4.7) with restriction (4.9). Let the scaling relation \( \tau = \mu h^\alpha = \sigma(h) \) hold and let for fixed \( t > 0 \) the index \( n = t/\tau \) run through \( \mathbb{N} \) towards \( \infty \). Then the random variable \( S_n \) of (3.3) converges in distribution to the random variable \( S(t) \) whose probability density is given by (1.15) as \( g_\alpha(x, t; 0) \).

**Remark 4.3.** In the special case \( \alpha = 2 \) the familiar explicit difference scheme

$$y_j(t_{n+1}) = (1 - 2\mu) y_j(t_n) + \mu y_j(t_{n-1}) + \mu y_j(t_{n+1})$$

is recovered from (4.7), and (4.9) goes over into the well-known stability condition 0 < \( \mu \leq 1/2 \).
Remark 4.4. The case \( \alpha = 1 \) is singular. For \( \alpha \to 1 \) both upper bounds in (4.8) and (4.9) tend to 0, and the denominators occurring in the definitions of the probabilities \( p_k \) tend to zero.

Remark 4.5. A motivation for the Gr"{u}nwald-Letnikov approximation of \( I_{\alpha}^- \) can be drawn from the fact that \( z = e^{ikh} \) is the symbol of the backward shift by a step \( h \): With

\[
(T_h \phi)(x) = \phi(x+h), \quad (T_{-h} \phi)(x) = \phi(x-h),
\]

we have

\[
\widehat{T_h}(\kappa) = e^{-ik(x+h)} e^{i\kappa x} = e^{-ikh}, \quad \widehat{T_{-h}}(\kappa) = e^{ikh}.
\]

From the symbol \( h^{-1}(1-z) = \widehat{hD_+}(\kappa) \) of the usual backward approximation

\[
(hD_+ \phi)(x) = h^{-1}(\phi(x) - \phi(x-h)) = h^{-1}(hI\phi)(x)
\]

we arrive by analogy at the symbol \( h^{-\alpha}(1-z)^\alpha \) as a candidate for the symbol of the operator \( hI_{\alpha}^- \). Analogously we get \( h^{-\alpha}(1-z^{-1})^\alpha \) as the symbol for the operator \( hI_{\alpha}^+ \). We use the corresponding approximations in case (a) \( 0 < \alpha < 1 \).

In case (b) \( 1 < \alpha \leq 2 \) we use the form \( D^2 I_{\alpha}^+ \) of the Riemann-Liouville left inverse of the operator \( I_{\alpha}^+ \), and put \( hI_{\alpha}^+ = hD^2 hI_{\alpha}^+ \). The corresponding symbol then is, with symmetrically \( (hD_+^2 \phi)(x) = h^{-2}(\phi(x+h) - 2\phi(x) + \phi(x-h)) \),

\[
hI_{\alpha}^+(\kappa) = hD^2(\kappa) hI_{\alpha}^+(\kappa) = h^{-2}(z^{-1} - 2 + z) h^{2-\alpha}(1-z)^{-(2-\alpha)}
\]

\[
= h^{-\alpha} z^{-1}(1 - 2z + z^2)(1-z)^{\alpha-2} = h^{-\alpha} z^{-1}(1-z)^\alpha.
\]

The symbol \( h^{2-\alpha}(1-z)^{\alpha-2} \) for \( hI_{\alpha}^+ \) here has been derived by the formal stipulation \( hI_{\alpha}^+ = hD_+^{-(2-\alpha)} \) using \( \widehat{hD_+}(\kappa) = h^{-1}(1-z) \).

Analogously we get \( h^{-\alpha} z (1-z^{-1})^\alpha \) as symbol of \( hI_{\alpha}^- \).

Remark 4.6. In [GM98] and [GM99] we have exploited the Gr"{u}nwald-Letnikov random walks in the more general setting of not necessarily symmetric Lévy-Feller diffusion (see Section 1). The proof for the case of symmetry \( (\theta = 0) \) given in the present paper is considerably simpler.
5 The Gillis-Weiss random walk

Gillis and Weiss in 1970 (see [GiW]) showed (we interpret one of their results in the language of probability theory) that every symmetric random variable \( Y \) with values in \( \mathbb{Z} \) and asymptotically \( P(Y = k) \sim c/|k|^{\alpha+1} \) (where \( c > 0 \)) lies in the domain of attraction of the corresponding symmetric Lévy distribution, hence can be used for an approximating random walk in the sense of Section 3. Only assuming their asymptotics they naturally cannot describe precisely how the coefficients \( \mu \) and \( \lambda \) of the scaling law appear in the transition probabilities. However, from their analysis we can deduce that the scaling law is of the form

\[
\tau = \sigma(h) = \mu h^\alpha \quad \text{if } 0 < \alpha < 2, \quad \tau = \sigma(h) = \lambda h^2 \log h \quad \text{if } \alpha = 2.
\]

Remarkably, the parameter value \( \alpha = 1 \) is not singular, but the scaling law becomes discontinuous at \( \alpha = 2 \), thus giving an example of a distribution with non-finite variance lying in the domain of attraction of the normal (Gauss) distribution.

We will now re-work and complement their analysis in the framework of our Section 3 for the special symmetric probability distribution \((p_k | k \in \mathbb{Z})\) with

\[
(5.1) \quad p_0 = 1 - 2\lambda \sum_{k=1}^{\infty} k^{-(\alpha+1)}, \quad p_k = \lambda|k|^{-(\alpha+1)} \quad \text{for } k \neq 0,
\]

where (so that \( p_0 \geq 0 \)) \( \lambda \) is restricted by

\[
(5.2) \quad 0 < \lambda \leq \left(2 \sum_{k=1}^{\infty} k^{-(\alpha+1)}\right)^{-1}.
\]

The parameter \( \alpha \) is only restricted as in (1.4) by \( 0 < \alpha \leq 2 \). Differently from Gillis and Weiss we motivate this choice of probabilities by (2.20), where the special character of the value \( \alpha = 2 \) already becomes visible. So, assume meanwhile \( 0 < \alpha < 2 \).

Discretizing \( D_0^\alpha u \) via a straightforward quadrature formula for the right hand side of (2.20) as

\[
(5.3) \quad D_0^\alpha y_j(t_n) = b(\alpha) h \sum_{k=1}^{\infty} \left( \frac{y_{j+k}(t_n) - 2y_j(t_n) + y_{j-k}(t_n)}{(kh)^{\alpha+1}} \right)
\]
and solving the equation 
\[
\frac{y_j(t_{n+1}) - y_j(t_n)}{\tau} = h D_0^\alpha y_j(t_n)
\]
for \(y_j(t_{n+1})\) we identify the transition probabilities \(p_k\) in (3.4) as
\[
(5.4) \quad p_0 = 1 - 2\mu b(\alpha) \zeta(\alpha + 1), \quad p_k = \mu b(\alpha) |k|^{-(\alpha+1)} \quad \text{for} \quad k \neq 0 ,
\]
with \(\mu = h^{-\alpha} \tau\), \(b(\alpha) = \Gamma(\alpha + 1) \sin (\alpha \pi / 2) / \pi\) and the Riemann \(\zeta\)-function
\[
(5.5) \quad \zeta(z) = \sum_{k=1}^{\infty} k^{-z} \quad \text{for} \quad \Re z > 1 .
\]
Obviously \(\sum_{k\in\mathbb{Z}} p_k = 1\), and the non-negativity condition in (3.2) requires
\[
(5.6) \quad 0 < \mu \leq \frac{1}{2 b(\alpha) \zeta(\alpha + 1)} = \frac{\pi}{2 \Gamma(\alpha + 1) \sin (\alpha \pi / 2) \zeta(\alpha + 1)} .
\]

We want to free the parameter value \(\alpha = 2\) from its singular character. Recalling (2.17) we see that \(b(2) = 0\), so that in (5.4) \(p_0 = 1\) and all \(p_k = 0\) for \(k \neq 0\) whereas the upper bound for \(\mu\) in (5.6) tends to \(\infty\) as \(\alpha \to 2^-\). This degenerate random walk obtained in (5.4) by formally setting \(\alpha = 2\) being neither interesting nor useful we replace \(\mu b(\alpha)\) by \(\lambda\) and obtain the transition probabilities in the form (5.1) with restriction (5.2). In (5.1) the special value \(\alpha = 2\) seems to be a quite regular value, and we shall see that we have a valid random walk model for all \(\alpha\) obeying \(0 < \alpha \leq 2\). However a price must be paid. Whereas for \(0 < \alpha < 2\) we can scale by \(\tau = \mu h^\alpha\) we can no longer do so in the case \(\alpha = 2\). So, assume henceforth (if not explicitly stated otherwise) the condition (5.2).

We have now the generating function
\[
(5.7) \quad \tilde{p}(z) = 1 - 2\lambda \zeta(\alpha + 1) + \lambda \sum_{k=1}^{\infty} k^{-(\alpha+1)} (z^k + z^{-k})
\]
with \(z = e^{i\kappa h}\), \(\kappa \in \mathbb{R}\). With the polylogarithmic function
\[
\Phi(z, \beta) = \sum_{k=1}^{\infty} \frac{z^k}{k^\beta} \quad (\beta \in \mathbb{R})
\]
we can write

\[(5.8) \quad \tilde{p}(z) = 1 - 2\lambda \zeta(\alpha + 1) + \lambda \left\{ \Phi(z, \alpha + 1) + \Phi(z^{-1}, \alpha + 1) \right\} \]

and could carry out the required asymptotic analysis by specializing some of the formulas in [T]. See also [EHTF] and [Le] for properties of the polylogarithmic function and the more general Lerch function. We prefer, however, the direct way to obtain (3.11). This asymptotic relation is trivial for \(\kappa = 0\), and because of \(\tilde{p}(e^{i\kappa h}) = \tilde{p}(e^{-i\kappa h})\), it suffices to treat the case \(\kappa > 0\) what we now will do.

From the common integral representation of the gamma function we take

\[k^{-(\alpha+1)} = \frac{1}{\Gamma(\alpha + 1)} \int_0^\infty u^\alpha e^{-ku} du\]

and using \(z^{-1} = \bar{z}\) we get

\[(5.9) \quad \tilde{p}(z) = 1 - 2\lambda \Re \gamma(z)\]

with

\[(5.10) \quad \Gamma(\alpha + 1) \gamma(z) = \int_0^\infty u^\alpha \sum_{k=1}^{\infty} e^{-ku}(1 - z^k) du = \int_0^\infty \frac{u^\alpha e^{-u}}{1 - e^{-u}(1 - z)} du.\]

The last equality in (5.10) has been obtained by summing the two involved geometric series.

In the Appendix we have performed in detail the required asymptotic analysis of \(\Re \gamma(z)\) for \(\nu = \kappa h \to 0+\ (\kappa \text{ fixed})\), which is resumed in formulas (A.6) and (A.7). Insertion of these asymptotic behaviours into (5.9) yields

\[(5.11) \quad \log \tilde{p}(e^{i\kappa h}) \sim -\frac{\lambda\pi}{\Gamma(\alpha + 1) \sin (\alpha\pi/2)} |\kappa|^\alpha h^\alpha \quad \text{if } 0 < \alpha < 2, \ \kappa \neq 0,\]

\[(5.12) \quad \log \tilde{p}(e^{i\kappa h}) \sim -\lambda\kappa^2 h^2 \log (1/(|\kappa|h)) \quad \text{if } \alpha = 2, \ \kappa \neq 0.\]

Recalling that it suffices to prove (3.11) for \(\kappa \neq 0\) and observing, that there the parameter \(\kappa\) can be treated like a constant, we see that \(\log(1/(|\kappa|h)) \sim \log (1/h)\), where, because \(h \to 0\), we can assume \(0 < h < 1\).
Hence we can replace (5.12) by

\[(5.13) \quad \log \tilde{p}(e^{i\kappa h}) \sim -\lambda \kappa^2 h^2 \log \frac{1}{h} \quad \text{if} \quad \alpha = 2, \kappa \neq 0.\]

Then the limit relation (3.11) (equivalently (3.10)) holds if we scale by

\[(5.14) \quad \tau = \sigma(h) = \frac{\lambda \pi}{\Gamma(\alpha + 1) \sin(\alpha \pi/2)} h^\alpha \quad \text{if} \quad 0 < \alpha < 2,\]

\[(5.15) \quad \tau = \sigma(h) = \lambda h^2 \log \frac{1}{h} \quad \text{if} \quad \alpha = 2.\]

Putting \(\mu = \lambda \pi/(\Gamma(\alpha + 1) \sin(\alpha \pi/2)) = \lambda/b(\alpha)\) in (5.9) with \(b(\alpha)\) defined in (2.17) we obtain from (5.4) the regular scaling law

\[(5.16) \quad \tau = \sigma(h) = \mu h^\alpha \quad \text{for} \quad 0 < \alpha < 2\]

with the restriction (5.6) for \(\mu\). As result we have

**Theorem 5.1.** Distinguish the cases (i) \(0 < \alpha < 2\), (ii) \(\alpha = 2\). Define the probabilities \(p_k = P(Y = k)\) in case (i) by (5.4) with restriction (5.6), in case (ii) by

\[p_0 = 1 - 2\lambda \zeta(3), \quad p_k = \lambda |k|^{-3} \quad \text{for} \quad k \neq 0\]

with restriction \(0 < \lambda \leq 1/(2\zeta(3))\). Let the scaling relation

\[(5.17) \quad \tau = \mu h^\alpha \quad \text{in case (i)}, \quad \tau = \lambda h^\alpha \log \frac{1}{h} \quad \text{in case (ii)}\]

hold and let for fixed \(t > 0\) the index \(n = t/\tau\) run through \(\mathbb{N}\) towards \(\infty\). Then the random variable \(S_n\) of (3.3) converges in distribution to the random variable \(S(t)\) whose probability density is given by (1.15) as \(g_\alpha(x,t;0)\).

**Remark 5.2.** We can use throughout \(0 < \lambda \leq 2\) the parameter \(\lambda\) and then have in (5.1) under the restriction (5.2) a unified representation of the transition probabilities. Here, in contrast to the Grünwald-Letnikov random walk, the value \(\alpha = 1\) does no longer play a special role. With \(\mu = \lambda/b(\alpha)\) we have for \(0 < \alpha < 2\) the regular scaling law \(\tau = \mu h^\alpha\). However, the price to be paid for this unified representation is the non-regular scaling \(\tau = h^2 \log(1/h)\) for \(\alpha = 2\). Another price is that the generating function \(\tilde{p}(z)\) in (5.8) is non-elementary, requiring considerable efforts in its asymptotic analysis.
6 A globally binomial random walk

The random walk model discussed in Section 4 has the disadvantage that the case \( \alpha = 1 \) is excluded and the representation of the transition probabilities \( p_k \) for \( 1 < \alpha \leq 2 \) is different from that for \( 0 < \alpha < 1 \). However, for all admissible values of \( \alpha \) we have the regular scaling law \( \tau = \mu h^\alpha \). The method treated in Section 5 has the advantage of a unified representation of the transition probabilities in the whole interval \( 0 < \alpha \leq 2 \), but the scaling law \( \tau = \mu h^\alpha \) holds only for \( 0 < \alpha < 2 \), it breaks down at \( \alpha = 2 \). In this section we present a model that in the whole interval \( 0 < \alpha \leq 2 \) admits a unified representation of the \( p_k \) via binomial coefficients and has there a scaling law of the form \( \tau = \mu h^\alpha \). Moreover, the generating function \( \tilde{p}(z) \) is elementary for all \( \alpha \in (0, 2] \).

The use of the binomial coefficients \( \binom{\alpha}{j} \) in the Grünwald-Letnikov random walk has caused singular behaviour for \( \alpha = 1 \). One reason for this sad fact is that \( \binom{1}{j} = 0 \) for integer \( j \geq 2 \). We can remove this singular behaviour by removing the factor \( \alpha - 1 \).

For \( 0 < \alpha \leq 2 \), \( \alpha \neq 1 \) let us define

\[
(6.1) \quad p_0 = 1 - 2\lambda, \quad p_k = (-1)^{k+1} \frac{\lambda}{\alpha - 1} \binom{\alpha}{|k| + 1} \text{ for } k \neq 0.
\]

Observing that here the singularity at \( \alpha = 1 \) is removable, let us for \( \alpha = 1 \) define (via \( \alpha \to 1 \) in (6.1))

\[
(6.2) \quad p_0 = 1 - 2\lambda, \quad p_k = \frac{\lambda}{|k|(|k| + 1)} \text{ for } k \neq 0.
\]

In (6.1) and (6.2) \( \sum_{k \in \mathbb{Z}} p_k = 1 \) and if \( 0 < \lambda \leq 1/2 \) all \( p_k \geq 0 \). In the special case \( \alpha = 2 \) we get

\[
p_0 = 1 - 2\lambda, \quad p_1 = p_{-1} = \lambda, \quad p_k = 0 \text{ for } |k| \geq 2,
\]

the familiar random walk for approximation of the classical process governed by the equation \( \frac{du}{dt} = \frac{d^2 u}{dx^2} \).

The generating function \( \tilde{p}(z) = \sum_{k \in \mathbb{Z}} p_k z^k \) has in the case \( \alpha \neq 1 \) the form

\[
(6.3) \quad \tilde{p}(z) = 1 - \lambda \{q(z) + q(z^{-1})\}
\]
with
\[
q(z) = \frac{1}{\alpha - 1} (1 - z^{-1}) \{ (1 - z)\alpha^{-1} - 1 \}.
\]

By passing here to the limit or directly from (6.2) we get for \(\alpha = 1\) the representation

(6.4) \(\tilde{p}(z) = 1 - \lambda \{(1 - z^{-1}) \log(1 - z) + (1 - z) \log(1 - z^{-1})\}, \quad \tilde{p}(1) = 1.\)

We have proposed and investigated the particular random walk so generated (its transition probabilities given in (6.2)) in [GM99, Section 5].

In the special case \(\alpha = 2\) we find

(6.5) \(\tilde{p}(z) = 1 + \lambda(z - 2 + z^{-1}).\)

We will now show that for all \(\alpha \in (0, 2]\) there exists a finite positive number \(c(\alpha)\) so that, with

(6.6) \(\mu = c(\alpha) \lambda,\)

we arrive for \(\kappa \in \mathbb{R} \setminus \{0\}\) at the small \(h\) asymptotics

(6.7) \(\tilde{p}(e^{ih\kappa}) = 1 - \mu(|\kappa|h)^\alpha + o\left((|\kappa|h)^\alpha\right)\)

which implies (3.11). As in Sections 4 and 5 we can ignore the value \(\kappa = 0\) as trivial.

Referring to [GM99] for detailed treatment of the case \(\alpha = 1\), let now be \(0 \neq \kappa \in \mathbb{R}\) and \(0 < \alpha \leq 2, \alpha \neq 1, z = e^{ih}\). In view of (6.3) we investigate the asymptotics of \(q(z) + q(z^{-1})\) for \(h \to 0\). From \(z^{-1} = \bar{z}\) and

\[
(1 - \alpha)q(z) = z^{-1}(1 - z)^\alpha - z^{-1} + 1 = e^{-i\kappa h}(1 - e^{i\kappa h})^\alpha - e^{-i\kappa h} + 1,
\]

we conclude on

(6.8) \(\psi(z) := (1 - \alpha)\{q(z) + q(z^{-1})\} = 2\Re \left\{e^{-i\kappa h}(1 - e^{i\kappa h})^\alpha\right\} + 2(1 - \cos(\kappa h)),\)

and here

(6.9) \(\Re \left\{e^{-i\kappa h}(1 - e^{i\kappa h})^\alpha\right\} \sim \Re \left((-i\kappa h)^\alpha\right) = (|\kappa|h)^\alpha \cos(\alpha \pi/2),\)
\[ 1 - \cos(\kappa h) \sim \frac{1}{2}(|\kappa| h)^2. \]

We distinguish three cases: (i) \(0 < \alpha < 1\), (ii) \(1 < \alpha < 2\), (iii) \(\alpha = 2\).

In cases (i) and (ii) the leading term in the asymptotics of \(\psi(z)\) turns out to be
\[
\psi(z) \sim 2 (|\kappa| h)^\alpha \cos(\alpha \pi/2).
\]

In case (iii) where \(\alpha = 2\) however, this term is matched in order of magnitude by \((6.10)\) so that we obtain
\[
\psi(z) \sim 2(|\kappa| h)^2(-1) + (|\kappa| h)^2 = -(|\kappa| h)^2.
\]

Collecting results and dividing \((6.8)\) by \(1 - \alpha\) we get (with \(z = e^{i\kappa h}\))
\[
\lambda \{q(z) + q(z^{-1})\} \sim \begin{cases} 
\frac{2 \cos(\alpha \pi/2)}{1 - \alpha} (|\kappa| h)^\alpha & \text{if } 0 < \alpha < 2, \ \alpha \neq 1, \\
\lambda (|\kappa| h)^2 & \text{if } \alpha = 2.
\end{cases}
\]

Hence, in view of \((6.3)\), we obtain \((6.7)\) with \((6.6)\) by putting
\[
c(\alpha) = \begin{cases} 
\frac{2 \cos(\alpha \pi/2)}{1 - \alpha} & \text{if } 0 < \alpha < 2, \ \alpha \neq 1, \\
1 & \text{if } \alpha = 2.
\end{cases}
\]

The scaling coefficient \(c(\alpha)\) allows continuous extension to the value \(\alpha = 1\), giving \(\lim_{\alpha \to 1} c(\alpha) = \pi\) in accordance with \([GM99, \text{formula (5.1)}]\). At \(\alpha = 2\), however, \(c(\alpha)\) is discontinuous. In fact
\[
c(2) = 1 \neq 2 = \lim_{\alpha \to 2} c(\alpha).
\]

Let us finally display the transition probabilities with \(\mu\) instead of \(\lambda\) as parameter.
For $0 < \alpha < 2$, $\alpha \neq 1$:

$$
\begin{align*}
\begin{cases}
p_0 &= 1 - 2\mu \frac{1 - \alpha}{2 \cos(\alpha \pi/2)}, \\
p_k &= \frac{(-1)^k}{2 \cos(\alpha \pi/2)} \left( \frac{\alpha}{|k| + 1} \right) \text{ for } k \neq 0, \\
0 < \mu \leq \frac{\cos(\alpha \pi/2)}{1 - \alpha},
\end{cases}
\end{align*}
$$

for $\alpha = 1$ (see [GM99, formula (5.1))):

$$
\begin{align*}
p_0 &= 1 - \frac{2\mu}{\pi}, \\
p_k &= \frac{\mu}{\pi |k|(|k| + 1)} \text{ for } k \neq 0, \quad 0 < \mu \leq \pi/2,
\end{align*}
$$

for $\alpha = 2$:

$$
\begin{align*}
p_0 &= 1 - 2\mu, \\
p_1 = p_{-1} &= \mu, \\
p_k &= 0 \text{ for } |k| \geq 2, \quad 0 < \mu \leq 1/2.
\end{align*}
$$

The discontinuity at $\alpha = 2$ has so been transferred to the upper bound for $\mu$.

We comprise the result in

**Theorem 6.1.** Take the probabilities $p_k = P(Y = k)$ and the restrictions for $\mu$ as in formulas (6.14), (6.15), (6.16), and use the scaling relation $\tau = \mu h^n$. Let for fixed $t > 0$ the index $n = t/\tau$ run through $\mathbb{N}$ towards $\infty$. Then the random variable $S_n$ of (3.3) converges in distribution to the random variable $S(t)$ whose probability density is given by (1.15) as $g_\alpha(x,t;0)$.

### 7 The Chechkin-Gonchar random walk

In this section we adopt to each other considerations of Chechkin and Gonchar [ChG] and the framework of our Section 3, restricting attention to the parameter range $0 < \alpha < 2$. So doing we exclude the well-known case of the classical Gaussian process. We will obtain a random walk, which is *discrete in time* but *continuous in space*, in more precise words: whose jumping width (in the instants $t_n = n\tau$) can assume any real number, having an everywhere positive probability density. We modify our theory of Section 3 by allowing the random variable $Y$ to have a strictly monotonic continuous distribution function $W(x) = P(Y < x)$ \((x \in \mathbb{R})\), that we furthermore require to be
symmetric in the sense \( W(x) + W(-x) = 1 \; (x \in \mathbb{R}) \), being only interested in the symmetric case \( \theta = 0 \) of Section 1. We then have in the sum

\[
S_n = \sum_{m=1}^{n} (hY_m) = h \sum_{m=1}^{n} Y_m \quad (n \in \mathbb{N})
\]

a description of a random walk, starting in the point \( x = 0 \). Here \( h > 0 \) is a scaling width that we let depend on the time-step \( \tau > 0 \) via a strictly monotonic scaling relation \( \tau = \sigma(h) \), with \( \sigma(h) \to 0 \) as \( h \to 0 \). We expect the scaling relation to have the form \( \tau = \mu h^\alpha \) with the positive coefficient \( \mu \) to be specified, by having found orientation in Gnedenko’s theorem on normal attraction (see [GnK], §35). It should be noted, however, that in this theorem the scaling constant \( C \) appearing there is given with a wrong value as has been remarked in [Ba].

As previously, we let the \( Y_m \) be independent identically distributed random variables, all having their distribution common with \( Y \). However, we now assume \( Y \) to have an everywhere positive (not necessarily bounded) probability density \( w = W' \) which is an even function \( w(x) = w(-x) \; (x \in \mathbb{R}) \). We will use the fact that \( w \) is normalized, \( \int_{-\infty}^{\infty} w(x) \, dx = 1 \).

Fixing a value \( t > 0 \) and again setting \( t = t_n = n\tau \) (equivalent to \( n = t/\tau \)) with \( n \in \mathbb{N} \) we want that the random variable \( S_n \) converges in distribution to the random variable \( S(t) \) whose density is given by (1.15). To this purpose we introduce a condition on the asymptotic behaviour of the density \( w \), namely

\[
w(x) = (b + \epsilon(|x|)) \, |x|^{-(\alpha+1)}, \quad |\epsilon(|x|)| \leq \min \left\{ K, E \, |x|^{-\gamma} \right\} \quad (x \in \mathbb{R}),
\]

with positive constants \( b, K, E \) and \( \gamma \).

With \( \hat{w}(\kappa) = \int_{-\infty}^{\infty} e^{i\kappa x} w(x) \, dx \) as characteristic function of the density \( w(x) \) we observe that the random variable \( hY \) has density \( w(x/h)/h \), hence the characteristic function \( \hat{w}(kh) \), and proceeding in analogy to the general method described in Section 3, replacing \( \hat{p}(\kappa, h) = \hat{p}(e^{i\kappa h}) \) in (3.9) by \( \hat{w}(\kappa h) \), we will find a scaling function \( \sigma(h) \) such that for all \( \kappa \in \mathbb{R} \), in analogy to (3.11),

\[
\frac{1}{\sigma(h)} \log \hat{w}(\kappa h) \to -|\kappa|^\alpha \quad \text{as} \quad h \to 0.
\]

Of course, (7.3) is trivial for \( \kappa = 0 \). Since \( \hat{w} \) like \( w \) is an even function it suffices to consider (7.3) for (fixed) values \( \kappa > 0 \). We see that (7.3) is
equivalent to

\begin{equation}
\hat{w}(\kappa h) = 1 - |\kappa|^\alpha \sigma(h) + o(\sigma(h)) \quad \text{as} \quad h \to 0.
\end{equation}

In view of the symmetry and normalization properties of \(w(x)\) and abbreviating \(\kappa h = \nu\) we find

\[
\hat{w}(\nu) - 1 = \int_0^\infty \left( e^{i\nu x} + e^{-i\nu x} - 2 \right) w(x) \, dx = -4 \int_0^\infty (\sin(\nu x/2))^2 \, w(x) \, dx
\]

so, using (7.2),

\[
\hat{w}(\nu) = 1 - 2^{-\alpha+2} b^{\alpha+2} \int_0^\infty \xi^{-\alpha-1} (\sin \xi)^2 \, d\xi - 4 \int_0^\infty \epsilon(x) x^{-\alpha-1} (\sin(\nu x/2))^2 \, dx.
\]

The first integral can be evaluated in terms of the gamma function. In fact, from [GR, (3.823)] we take

\[
\int_0^\infty \xi^{-\alpha-1} (\sin \xi)^2 \, d\xi = -\frac{\Gamma(-\alpha) \cos(\alpha \pi/2)}{2^{1-\alpha}} = \frac{\pi}{2^{2-\alpha} \Gamma(\alpha+1) \sin(\alpha \pi/2)}.
\]

The latter equality follows by the reflection formula for the gamma function.

We estimate the second integral via decomposition \(\int_0^\infty \ldots = \int_0^\eta \ldots + \int_\eta^\infty \ldots\), taking \(\eta = \nu^{-(2\alpha+\gamma)/(2\alpha+2\gamma)}\), using \(|\sin \xi| \leq \min \{\xi, 1\}\) for \(\xi \geq 0\) and the condition on \(\epsilon(|x|)\) of (7.2). By careful calculation we find that it behaves asymptotically as \(o(\nu^{\alpha}) = |\kappa|^\alpha o(h^{\alpha})\). Combining these results and recalling that \(\hat{w}\) is an even function, we obtain

\begin{equation}
\hat{w}(\kappa h) = 1 - |\kappa|^\alpha \frac{b\pi}{\Gamma(\alpha+1) \sin(\alpha \pi/2)} h^{\alpha} + |\kappa|^\alpha o(h^{\alpha}) \quad (h \to 0)
\end{equation}

as valid for all \(\kappa \in \mathbb{R}\). In view of (7.4) and the theory developed in Section 3 we thus arrive at the scaling relation

\begin{equation}
\tau = \sigma(h) = \mu h^{\alpha}, \quad \text{with} \quad \mu = \frac{b\pi}{\Gamma(\alpha+1) \sin(\alpha \pi/2)}.
\end{equation}

Now we are in the position to formulate

Theorem 7.1. Let \(0 < \alpha < 2\) and assume the random variable \(Y\) to have a probability density \(w\) of the form (7.2). Let the scaling relation (7.6) hold
and let for fixed \( t > 0 \) the index \( n = t/\tau \) run through \( \mathbb{N} \) towards \( \infty \). Then the random variable \( S_n \) of (7.1) converges in distribution to the random variable \( S(t) \) whose probability density is given by (1.15).

**Remark 7.2.** According to the well-known asymptotic expansions of the function \( p_\alpha(x;0) = g_\alpha(x,1;0) \) (see [F52], [F71], [Zo]) we have

\[
(7.7) \quad b = \frac{\Gamma(\alpha + 1) \sin(\alpha \pi/2)}{\pi}
\]

if we take \( w(x) = p_\alpha(x;0) \), hence in this case \( t = 1, \mu = 1 \) and \( h = \tau^{1/\alpha} = n^{-1/\alpha} \). If we require in (7.1) the \( Y_m \) to have this special density, then \( S_n \) for all \( t > 0 \) has the same probability distribution as \( S(t) \) whose characteristic function is \( \exp(-t|\kappa|^\alpha) \). We can here obtain the scaling relation also via the convolution theorem.

**Remark 7.3.** For actual simulation a random variable \( Y \) having the required properties is particularly useful if its distribution function \( W(x) = \int_{-\infty}^{x} w(\xi) d\xi \) is easily invertible. We can then generate a realization of \( Y \) by a standard Monte Carlo method (see [HH]). Generate a random number \( y \) uniformly distributed in the interval \([0,1)\). Then solve the equation \( y = W(x) \) for \( x \) and take \( x \) as a realization of \( Y \). Chechkin and Gonchar in [ChG] have proposed to use

\[
(7.8) \quad W(x) = \begin{cases} 
\frac{1}{2} (1 + |x|^\alpha)^{-1} & \text{for } x < 0, \\
1 - \frac{1}{2} (1 + x^\alpha)^{-1} & \text{for } x \geq 0,
\end{cases}
\]

a function easily invertible. The density

\[
(7.9) \quad w(x) = W'(x) = \frac{\alpha |x|^\alpha - 1}{2 (1 + |x|^\alpha)}
\]

has the property (7.2) with \( b = \alpha/2, \gamma = \alpha \), hence we get

\[
(7.10) \quad \mu = \frac{\pi}{2\Gamma(\alpha) \sin(\alpha \pi/2)}.
\]

The density (7.9) is unbounded at the origin if \( 0 < \alpha < 1 \). To avoid this we propose

\[
(7.11) \quad w(x) = \frac{\alpha}{2} (1 + |x|)^{-(\alpha+1)},
\]

25
which again satisfies the asymptotic condition (7.2). Then

\[
W(x) = \begin{cases} 
\frac{1}{2} (1 + |x|)^{-\alpha} & \text{for } x < 0, \\
1 - \frac{1}{2} (1 + x)^{-\alpha} & \text{for } x \geq 0
\end{cases}
\]

is also easily invertible, and (7.10) remains valid.

**Remark 7.4.** Among the symmetric densities \( p_\alpha(x; 0) \), only the Cauchy density \( w(x) = p_1(x; 0) = (1/\pi) (1 + x^2)^{-1} \) offers easy invertibility of the corresponding distribution function, namely of the function \( W(x) = 1/2 + (1/\pi) \arctan x \). Via random numbers \( y_m \) uniformly distributed in \([0,1)\) we can get realizations of the \( Y_m \) in \( \tau \tan(\pi(y_m - 1/2)) \) (here \( \tau = 1 h^1 = h \)) and so obtain in \( S_n \) a snapshot at instant \( t_n = n \tau \) of a true Cauchy process.

## 8 Conclusions

Anomalous diffusion processes have in recent years gained revived interest among physicists, and methods of fractional calculus have shown their usefulness for purposes of modelling. In the space-fractional case one is naturally led to a generalization of the classical diffusion equation with respect to the second-order spatial operator. One arrives in a natural way at the processes of Lévy-Feller type in which stable probability distributions play the essential role. Also among physicists and mathematicians who have found it rewarding to work in theory of finance, such processes are becoming more and more popular (see e.g. [M], [BoP], [MS]). So, it is no wonder that also in pure mathematics such types of processes are now investigated in great generality and analytical sophistication (see e.g. [J], [Be], [S], [Za]). From the more practical point of view discrete models are esteemed. They not only show that very different microscopic behaviour of particles can result in the same macroscopic behaviour but offer also possible visualizations of what is happening in such processes. Furthermore such discrete models can be used for simulation purposes, be it for simulation of particle paths via Monte Carlo methods (the microscopic view) or via solution of the underlying Cauchy problem for a pseudo-differential equation (the macroscopic view). And, last but not least, such models are fascinating as seen from the mathematical standpoint (or, more specifically, from the position of probability theory).

In our present investigation we first have given a survey on and drawn motivations from basic theory of fractional calculus and Lévy-Feller diffusion.
processes. Then we have obtained and rigorously analyzed (with respect to their convergence in distribution for passing to the limit of infinitely fine discretization) three models of random walk occurring on a regular spatio-temporal grid. The first model is devised from the Grünwald-Letnikov discretization of the two Weyl operators, the composition of which gives the inverse of the Riesz potential operator. The second model is an adaptation of ideas of Gillis and Weiss [GiW] to our framework. We have provided it with a new motivation, namely as obtainable from straightforward discretization of the hypersingular integral representation of the spatial pseudo-differential operator. The third model’s intention is to overcome peculiar deficiencies of the first two models. It is a modification and improvement of the first model, and again properties of the binomial coefficients are used.

Finally, to offer also a highly efficient method for numerical simulation, we have mutually adapted our theoretical frame to ideas of Chechkin and Gonchar [ChG]. We so obtain a random walk still proceeding in equidistant instants of time but allowing spatial jumps of arbitrary length in positive or negative direction.

Appendix A: Asymptotics of an integral

Abbreviating $\kappa h = \nu$ in $z = e^{i\kappa h}$ in the right hand side of (5.10), and keeping in mind $0 < \alpha \leq 2$, elementary calculation yields the equation

\[(A.1) \quad \Gamma(\alpha + 1) \Re \gamma(z) = \int_0^\infty u^\alpha e^{-u} \frac{(1 + e^{-u}) (1 - \cos \nu)}{(1 - e^{-u}) |1 - e^{-u} e^{i\nu}|^2} \, du\]

which we will treat asymptotically for $0 < \nu \to 0^+$ by the Laplace method for integrals (see [dB]), using the fact that the lower bound $u = 0$ is the critical one (the integrand tending to $\infty$ as $u \to 0$). We have $1 - \cos \nu = \nu^2/2 + O(\nu^4)$ and

$$|1 - e^{-u} e^{i\nu}|^2 = (1 - e^{-u})^2 + 2e^{-u}(1 - \cos \nu) = (1 - e^{-u})^2 + \nu^2 e^{-u} + O(\nu^4),$$

uniformly in $0 \leq u < \infty$, hence

$$\Gamma(\alpha + 1) \Re \gamma(z) \sim \frac{\nu^2}{2} \int_0^\infty \frac{u^\alpha e^{-u} (1 + e^{-u})}{(1 - e^{-u}) \{(1 - e^{-u})^2 + \nu^2 e^{-u}\}} \, du.$$
Because this integral diverges for $\nu = 0$ we can simplify the integrand (for small $u$) which, for small $\nu$, gives the essential contribution: $1 + e^{-u} \sim 2$, $1 - e^{-u} \sim u$, $e^{-u} \sim 1$. We obtain

$$\Gamma(\alpha + 1) \Re \gamma(z) \sim \nu^2 \int_0^\infty u^{\alpha-1} \frac{e^{-u}}{u^2 + \nu^2} du$$

and, by substituting $u = \nu w$,

$$(A.2) \quad \Gamma(\alpha + 1) \Re \gamma(z) = \nu^\alpha \int_0^\infty w^{\alpha-1} \frac{e^{-\nu w}}{w^2 + 1} dw := \nu^\alpha \rho(\nu).$$

In the investigation of the integral

$$(A.3) \quad \rho(\nu) = \int_0^\infty w^{\alpha-1} \frac{e^{-\nu w}}{w^2 + 1} dw$$

we distinguish the cases (i) $0 < \alpha < 2$, (ii) $\alpha = 2$. In the case (i) simply

$$\rho(\nu) \to \int_0^\infty \frac{w^{\alpha-1}}{w^2 + 1} dw \text{ for } \nu \to 0$$

and with $\beta = \alpha - 1$, hence $-1 < \beta < 1$, we have to determine the value of

$$q(\beta) = \int_0^\infty \frac{x^{\beta}}{x^2 + 1} dx.$$

Observing that $q(-\beta) = q(\beta)$ (substitute $\xi = 1/x$) we do this $0 \leq \beta < 1$. Complementation by (integrate along the upper edge of the negative real semi-axis)

$$\int_0^{-\infty} \frac{x^{\beta}}{x^2 + 1} dx = e^{i\beta \pi} \int_0^{+\infty} \frac{x^{\beta}}{x^2 + 1} dx$$

gives, via the residue theorem,

$$\left(1 + e^{i\beta \pi}\right) q(\beta) = \int_{-\infty}^{+\infty} \frac{x^{\beta}}{x^2 + 1} dx = \pi i \beta = \pi e^{i\beta \pi/2}.$$

So

$$q(\beta) = \frac{\pi}{2 \cos(\beta \pi/2)} = \frac{\pi}{2 \sin(\alpha \pi/2)},$$

and hence

$$(A.4) \quad \rho(\nu) \to \frac{\pi}{2 \sin(\alpha \pi/2)} \quad \text{if } 0 < \alpha < 2 \quad (\nu \to 0+).$$
In case (ii) the integral diverges for \( \nu = 0 \), so we must proceed in another way. Inserting \( \alpha = 2 \) in (A.3) and differentiating we obtain for \( \nu > 0 \)

\[
-\rho'(\nu) = \int_0^\infty \frac{w^2 e^{-\nu w}}{w^2 + 1} dw = \int_0^\infty e^{-\nu w} \left( 1 - \frac{1}{w^2 + 1} \right) dw = \frac{1}{\nu} - \frac{\pi}{2} + o(1),
\]

and then by integration

\[(A.5) \quad \rho(\nu) \sim -\log \nu = \log \frac{1}{\nu} \quad (\nu \to 0+).\]

Now we can collect results. From (A.1) - (A.5), using \( \nu = \kappa h \) which because of symmetry we can replace by \( |\kappa| h \) (admitting also negative values of \( \kappa \)) we deduce

\[(A.6) \quad \Re \gamma(z) \sim \frac{\pi}{2\Gamma(\alpha + 1) \sin (\alpha \pi/2)} |\kappa|^{\alpha} h^{\alpha} \quad \text{if} \quad 0 < \alpha < 2, \quad \kappa \neq 0, \quad \text{as} \quad h \to 0,\]

\[(A.7) \quad \Re \gamma(z) \sim \kappa^2 h^2 \log \frac{1}{|\kappa|h} \quad \text{if} \quad \alpha = 2, \quad \kappa \neq 0, \quad \text{as} \quad h \to 0.\]

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