Duality for Modules and Applications to Decoding Linear Codes over Finite Commutative Rings

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Abstract

Using linear functional-based duality of modules, we generalize the syndrome decoding algorithm of linear codes over finite fields to those over finite commutative rings. Moreover, if the ring is local the algorithm is simplified by introducing the control matrix.

Keywords. Control matrix, Dual code, Finite ring, Linear code, Syndrome decoding.

1 Introduction

Syndrome decoding is a more efficient method of decoding linear codes over finite fields over a noisy channel [5]. Thus, in this paper we investigate the generalization of the syndrome decoding to linear codes over finite commutative rings. A first generalization was given in [1] via Pontryagin duality. In the same direction we give another generalization using linear functional-based duality. In general, linear functional-based duality and character-based (or Pontryagin) duality are not equivalent (for more details see [8]).

Syndrome decoding of linear codes over finite fields is based on the two following famous results in linear algebra [9]:

\[ C^⊥⊥ = C \]

\[ \dim(C^⊥) = n - \dim(C) \]

where \( C \) is a subspace of \( K^n \) and \( K \) is a field. These properties are not always valid in \( A^n \) with \( A \) is a ring. Wood in [10] has shown the property (1) for any submodule of \( A^n \) with \( A \) is a finite quasi-Frobenius ring. Afterwards, Mittelholzer in [8] extends the class of rings for which the property (1) holds for projective submodules of \( A^n \) to artinian rings. In this work we present a detailed proof of (1) for free submodules of \( A^n \) with \( A \) is a finite ring. In first, we prove in proposition 3 the property (1) on local finite ring using the existence of free direct summand of free submodule of \( A^n \). The decomposition of any finite commutative ring as a direct sum of local rings allows us to generalize this property to any finite ring in theorem 1.

This article is organized as follows. Section 2 begins by recalls the notions of dual module, orthogonal and bi-orthogonal of submodules in the framework of linear functional-based duality. the following of this section is devoted on the proof of (1). The coding theory begins in section 3 with a review of essential definitions of linear codes over rings. After introducing the concept of dual code, we prove that every dual code of a free
code over local ring is also free and its rank satisfies the property (2). Based on results of previous sections especially on theorem 2, we present in section 4 the syndrome decoding algorithm. Computing the syndrome is simplified by introducing a control matrix for linear code over local ring.

2 Duality - Orthogonality

Throughout this paper, $A$ denotes a finite commutative ring with identity and $M$ an $A$-module.

**Definition 1**
The $A$-module $\text{Hom}_A(M, A)$ of linear functionals of $M$ is called the dual module of $M$ and denoted $M^*$.

**Proposition 1** ([2], Proposition 6.1.5)
If $M$ is a free module of finite rank $n$, i.e., $M \cong A^n$ as $A$-modules. Then $M^*$ is free of finite rank $n$ too.

**Definition 2**
Let $N$ be a submodule of $M$.
1. The orthogonal of $N$ is the submodule of $M^*$ :
   $$N^o = \{ f \in M^* : f(x) = 0, \forall x \in N \}$$
2. The bi-orthogonal of $N$ is the submodule of $M^*$ :
   $$N^{oo} = \{ x \in M : f(x) = 0, \forall f \in N^o \}$$

**Proposition 2** ([9], Proposition 8.7.8)
If $N$ is a submodule of $M$, then the $A$-modules $(M/ N)^*$ and $N^o$ are isomorphic.

The aim of the following is to show that every free submodule $N$ of $A^n$ satisfies $N^{oo} = N$. We begin by establishing this result on a local ring using the following lemma. This last, appears in Appendix II of [6], is valid on artinian rings, in particular on finite rings.

**Lemma 1**
If $A$ is local and $F$ is a free submodule of $A^n$. Then there exists a free submodule $Q$ of $A^n$ such that $F \oplus Q = A^n$ and $A^n/ F$ is free.

**Proposition 3**
Suppose that $A$ is local. Let $N$ be a free submodule of $A^n$ and $x \in A^n$.
1. $x = 0$ if and only if for all $f \in (A^n)^*$, $f(x) = 0$.
2. $x \in N$ if and only if for all $f \in N^o$, $f(x) = 0$.

**Proof**
1. The necessary condition is trivial. Conversely, suppose that $\forall f \in (A^n)^* f(x) = 0$. Let $(e_i)_{1 \leq i \leq n}$ be a basis of $A^n$ and $(e^*_i)_{1 \leq i \leq n}$ its dual basis in $(A^n)^*$. If $x = \sum_{i=1}^{n} x_i e_i$ with $(x_i)_{1 \leq i \leq n} \in A^n$, then $e^*_i(x) = x_i = 0 \ \forall 1 \leq i \leq n$. Therefore $x = 0$.

2. The necessary condition is a consequence of the orthogonal of $N$. Conversely, suppose that $\forall f \in N^o f(x) = 0$. Let $\varphi \in (A^n/ N)^*$, the map $f : A^n \to A, y \mapsto f(y) = \varphi(\tilde{y})$ is linear and $f \in N^o$. Then $f(x) = 0$. Therefore $\varphi(\tilde{x}) = 0$ for all $\varphi \in (A^n/ N)^*$ and $A^n/ N$ is free by Lemma 1. Thus $\tilde{x} = 0$ and $x \in N$. Consequently, $N^{oo} = N$. 


The ring $A$ is finite commutative. According to the structure theorem for such rings \[7\], $A$ can be written as a finite direct sum of local rings $A_i$ i.e.,

$$A = \bigoplus_{i=1}^{l} A_i. \quad (3)$$

where $A_i \cong Ae_i \forall i = 1, ..., l$ and $(e_i)_{1 \leq i \leq l}$ is a complete system of orthogonal idempotents of $A$, i.e.,

$$e_i^2 = e_i, \ e_ie_j = 0 \text{ for } i \neq j \text{ and } \sum_{i=1}^{l} e_i = 1.$$

For all $i = 1, ..., l$, $M_i = e_iM$ is a submodule of $M$ that can provide a structure of $A_i$-module \[3\]. Moreover,

$$M = \bigoplus_{i=1}^{l} M_i. \quad (4)$$

**Lemma 2**

Suppose that $M$ is a finitely generated module over $A$. If $M$ is $A$-free then $M_i$ is $A_i$-free for all $i = 1, ..., l$.

**Proof**

Suppose that $M$ is free over $A$. Let $(s_1, ..., s_n)$ be a basis of $M$. We show that $(e_is_1, ..., e_is_n)$ is a basis of $M_i$.

- Let $e_is_i \in M_i$ such that $x = \sum_{j=1}^{n} x_sj \in M$ and $(x_1, ..., x_n) \in A^n$. $e_is_i = \sum_{j=1}^{n} (x_j e_i) s_j = \sum_{j=1}^{n} (x_j e_i)(e_is_j) = \sum_{j=1}^{n} \alpha_j(e_is_j)$ with $\alpha_j \in A_i$. Therefore $(e_is_j)_{1 \leq j \leq n}$ generates $M_i$.

- Let $(\alpha_1, ..., \alpha_n) \in A^n$ such that $\sum_{j=1}^{n} \alpha_j(e_is_j) = 0$. Then $\forall j = 1, ..., n, \ e_is_j = a_j e_i \in Ae_i$ and $\sum_{j=1}^{n} (a_j e_i)s_j = 0$. Since $(s_i)_{1 \leq i \leq n}$ is free over $A$, then $a_j e_i = \alpha_j = 0 \forall j$. So $(e_is_j)_{1 \leq j \leq n}$ is linearly independent. \[\square\]

Let $f \in Hom_A(M, A)$. For all $i = 1, ..., l$, the map $f_i : e_i x \mapsto e_if(x)$ of $M_i$ to $A_i$ is $A_i$-linear. Furthermore, the map

$$Hom_A(M, A) \longrightarrow \bigoplus_{i=1}^{l} Hom_{A_i}(M_i, A_i)$$

$$f \longrightarrow (f_1, ..., f_l) \quad (5)$$

is bijective. Indeed, if $(f_1, ..., f_l) \in \bigoplus_{i=1}^{l} Hom_{A_i}(M_i, A_i)$. Then $g = \sum_{i=1}^{l} f_i \circ pr_i$, where $pr_i : M \rightarrow M_i, \ x \mapsto e_ix$ is the canonical projection, is a linear functional of $M$ because if $a \in A$ and $x \in M$ then

$$g(ax) = \sum_{i=1}^{l} f_i(e_i ax)$$

$$= \sum_{i=1}^{l} ae_i f_i(e_i x)$$

$$= a \sum_{i=1}^{l} f_i(e_i x)$$

$$= ag(x).$$

And $g$ is the unique element of $Hom_A(M, A)$ satisfying for all $x \in M$ $f_i(e_ix) = e_ig(x)$. 3
Theorem 1
Suppose that $M \cong A^n$. If $N$ is a free submodule of $M$, then $N^\circ\circ = N$.

Proof
$N$ is on the form $\bigoplus_{i=1}^l N_i$ with $N_i = e_i N$. For all $i = 1,...,l$, $N_i$ is a submodule of the $A_i$-module $M_i$. Let $f \in \text{Hom}_A(M, A)$. Using (5), we have $f |_{N_i} = 0$ iff $f_i |_{N_i} = 0 \forall i = 1,...,l$. Then $N^\circ\circ = \bigoplus_{i=1}^l N_i^\circ$. Thus $N^\circ\circ = \bigoplus_{i=1}^l N_i = N$.

3 Linear Codes

Definition 3
1. A linear code $C$ over $A$ of length $n$ is a submodule of $A^n$. If $C$ is free over $A$ of rank $k$, $C$ is said an $(n,k)$-code over $A$. The elements of $C$ are called codewords.

2. Let $C$ be an $(n,k)$-code over $A$. The matrix $G \in M_{k,n}(A)$ whose rows form a basis of $C$ is said to be a generator matrix of $C$.

3. The Hamming distance between $x = (x_1,...,x_n)$ and $y = (y_1,...,y_n)$ in $A^n$ is
   $$d(x,y) = |\{i \in \{1,...,n\} : x_i \neq y_i\}|.
   $$

4. The Hamming weight of $x = (x_1,...,x_n) \in A^n$ is
   $$w(x) = d(x,0) = |\{1 \leq i \leq n : x_i \neq 0\}|.
   $$

5. The minimal distance of a linear code $C$ is :
   $$d(C) = \min\{d(x,y) : x \neq y \in C\} = \min\{w(x) : x \in C - \{0\}\}.$$

   The space $A^n$ with the Hamming distance is a metric space.

Definition 4 (Dual Code)
Let $C$ be a linear code over $A$ of length $n$. We define the dual code of $C$ by :
   $$C^\perp = \{y \in A^n : <x,y> = 0, \forall x \in C\}$$
where $<,>$ is the symmetric bilinear form defined for all $x = (x_1,...,x_n)$ and all $y = (y_1,...,y_n)$ in $A^n$ by :
   $$<x,y> = \sum_{i=1}^n x_i y_i.$$

Proposition 4
Let $C$ be a linear code over $A$ of length $n$. Then $C^\perp$ and $C^\circ$ are isomorphic.
Proof
Let
\[ \varphi : A^n \rightarrow (A^n)^* \]
\[ x \mapsto \varphi(x) \]

with \( \varphi(x)(y) = < x, y > \) for all \( y \in A^n \). We show that \( \varphi \) is an isomorphism.

- It is easy to check that \( \varphi \) is linear.
- \( \varphi \) is surjective: Let \( f \in (A^n)^* \) and let \((e_1, ..., e_n)\) be the canonical basis of \( A^n \). If \( y = \sum_{i=1}^{n} y_ie_i \in A^n \), then \( f(y) = \sum_{i=1}^{n} y_i f(e_i) = < x, y > \) with \( x = \sum_{i=1}^{n} f(e_i)e_i \). Therefore \( f = < x, . > = \varphi(x) \).
- \( \varphi \) is injective: If \( x \in \ker \varphi \). Then for all \( y \in A^n \), \( \sum_{i=1}^{n} x_i y_i = 0 \). Especially for \( y = e_i \) we have \( x_i = 0 \), for all \( i = 1, ..., n \). Therefore \( x = 0 \).

\( \varphi \) induces an isomorphism \( C^\perp \cong \varphi(C^\perp) = C^o \)

Theorem 2
Let \( C \) be an \((n, k)\)-code over \( A \).

1. \( C^{\perp \perp} = C \).
2. If \( A \) is local, then \( C^\perp \) is free of rank \( n - k \).

Proof
1. It is clear that \( C \subseteq C^{\perp \perp} \). Let \( x \in C^{\perp \perp} \), then for all \( y \in C^\perp \), \( < x, y > = 0 \). By the previous isomorphism between \( C^\perp \) and \( C^o \) we have for all \( f \in C^o \), \( f(x) = 0 \). Thus \( x \in C^{oo} = C \), by Theorem 1.

2. By Proposition \[4\] we have \( C^\perp \cong C^o \) and by Proposition 2, we have \( C^o \cong (A^n/C)^* \). The ring \( A \) is local then by Lemma \[1\] we have that \( A^n/C \) is free and \( \text{rank} \left( A^n/C \right) = n - k \). Therefore \( (A^n/C)^* \) is free and \( \text{rank} \left( A^n/C \right)^* = \text{rank} \left( A^n/C \right) \) by Proposition 1. Thus \( C^\perp \) is free and \( \text{rank}(C^\perp) = n - k \).

4 Decoding linear codes
4.1 Syndrome decoding
The principle of this method is to associate each received word after transmission, a quantity \( S \) called syndrome. If the error is lightweight this one is uniquely determined by \( S \).

Let \( C \) be an \((n, k)\)-code over \( A \), of minimal distance \( d \) and \( t = \left\lfloor \frac{d-1}{2} \right\rfloor \) the correction capacity of \( C \). Hence \( C \) can detect \((d - 1)\)−errors and correct \( t \)−errors \[5\].

Definition 5
Let \( x \in A^n \). The syndrome of \( x \) is
\[ S(x) = ( < x, y > )_{y \in C^\perp} \]
We note that the map $S$ is additive. For all $x$ and $y$ in $A^n$, $S(x + y) = S(x) + S(y)$.

The following result, which generalize the similar fact on fields, is the main tool that allows the code to detect errors.

**Proposition 5**

Let $x \in A^n$. Then $x \in C$ if and only if $S(x) = 0$.

**Proof**

Suppose that $x \in C$. Then $x \in C^\perp$ and for all $y \in C^\perp$, $<x, y> = 0$, which implies that $S(x) = 0$. Conversely, if $S(x) = 0$, then for all $y \in C^\perp$, $<x, y> = 0$. Hence $x \in C^\perp = C$ by Theorem 2.

As in the case of linear codes on fields, two vectors have the same syndrome if and only if they have the same coset modulo $C$:

**Proposition 6**

Let $x, y \in A^n$. Then $\bar{x} = \bar{y}$ in $A^n/C$ if and only if $S(x) = S(y)$.

**Proof**

\[
\bar{x} = \bar{y} \iff x - y \in C
\iff S(x - y) = 0
\iff S(x) - S(y) = 0
\iff S(x) = S(y).
\]

**Corollary 1**

If $r$ is the received word and $e$ the associated error vector. Then $c = r - e \in C$ and $S(r) = S(e)$.

**Proposition 7**

Let $e$ be the error vector. If $w(e) \leq t$ and $S(e) = S$. Then $e$ is the unique vector of weight $\leq t$ having syndrome $S$.

**Proof**

Let $e' \in A^n$ such that $w(e') \leq t$ and $S(e) = S(e')$ then $e - e' \in C$.

\[
w(e - e') = d(e - e', 0)
= d(e, e')
\leq d(e, 0) + d(0, e')
\leq w(e) + w(e')
\leq 2t < d
\]

So $e - e' = 0$ and $e = e'$.

We are, now, able to present the syndrome decoding algorithm.
Algorithm 1 The Syndrome Decoding Algorithm

1: Compute the syndrome $S$ of the received word $r$
2: Compute the syndrome of all vectors $e$ of weight $\leq t$
3: if there exists no vector $e$ of weight $\leq t$ and syndrome $S$ then
4: The algorithm fails /*more than $t$-errors occur*/
5: else
6: Determine the unique vector $e$ of weight $\leq t$ with syndrome $S$
7: Decode $r$ by $c = r - e \in C$
8: end if

4.2 Control Matrix

Throughout this part we assume that $A$ is local. Let $C$ be an $(n, k)$-code over $A$ of generator matrix $G$. By Theorem 2, we have that $C^\perp$ is free and $\text{rank}(C^\perp) = n - k$.

Definition 6

A generator matrix of $C^\perp$ is called control matrix of $C$.

Let $H$ be a control matrix of $C$. The row vectors of $G$ form a basis of $C$ and the column vectors of $H^t$ form a basis of $C^\perp$, thus $GH^t = 0$. The following theorem gives a necessary and sufficient condition so that a matrix $H \in M_{n-k,n}(A)$ is a control matrix of $C$.

Theorem 3

Let $H \in M_{n-k,n}(A)$. Then $H$ is a control matrix of $C$ if and only if $GH^t = 0$ and row vectors of $H$ are linearly independent.

Proof

The necessary condition is a consequence of the definition of control matrix. Conversely, suppose that $GH^t = 0$ and $(e_1, ..., e_{n-k})$, the row vectors of $H$, are linearly independent. Then for all $1 \leq i \leq n-k$, $e_i \in C^\perp$.

Hence $\bigoplus_{i=1}^{n-k} A e_i \subseteq C^\perp$ and $\text{rg}\left(\bigoplus_{i=1}^{n-k} A e_i\right) = n - k = \text{rg}(C^\perp)$. Therefore $\bigoplus_{i=1}^{n-k} A e_i \cong C^\perp$ and $\left|\bigoplus_{i=1}^{n-k} A e_i\right| = |C^\perp|$. Thus $\bigoplus_{i=1}^{n-k} A e_i = C^\perp$ and $(e_i)_{1 \leq i \leq n-k}$ is a basis of $C^\perp$.

Corollary 2

If the generator matrix of $C$ is in standard form (i.e) $G = (I_k, P)$. Then $H = (-P^t, I_{n-k})$ is a control matrix of $C$, with $P \in M_{k,n-k}(A)$ and $I_k$ denotes the identity matrix of order $k$.

Proof

We have $GH^t = -P + P = 0$ and it is clear that the lines of $H$ are linearly independent.

Proposition 8

Let $H$ be a matrix control of $C$. Then $\forall x, y \in A^n$ :

1. $S(x) = 0$ if and only if $Hx^t = 0$
2. $S(x) = S(y)$ if and only if $Hx^t = Hy^t$.
Proof

Let

\[ H = \begin{pmatrix}
  e_{i1} & \cdots & \cdots & e_{in} \\
  \vdots & \ddots & \ddots & \vdots \\
  e_{n-k,1} & \cdots & \cdots & e_{n-k,n}
\end{pmatrix} \]

be a control matrix of \( C \), with \( e_i = (e_{i1}, \ldots, e_{in}) \) in \( C^\perp \) for all \( i = 1, \ldots, n - k \)

If \( x = (x_1, \ldots, x_n) \in A^n \) then \( Hx^t = (\sum_{j=1}^{n-k} x_je_{ij})_{1 \leq i \leq n-k} = \langle x, e_i \rangle_{1 \leq i \leq n-k} \).

1. If \( S(x) = 0 \) then for all \( y \in C^\perp \), \( \langle x, y \rangle = 0 \). Hence for all \( i = 1, \ldots, n - k \), \( \langle x, e_i \rangle = 0 \) and \( Hx^t = 0 \).

Conversely, suppose that \( Hx^t = 0 \). Let \( z = \sum_{i=1}^{n-k} a_i e_i \in C^\perp \) Then

\[
\langle x, z \rangle = \langle x, \sum_{i=1}^{n-k} a_i e_i \rangle = \sum_{i=1}^{n-k} a_i \langle x, e_i \rangle = 0.
\]

Therefore \( S(x) = 0 \).

2. Straightforward from (1).

The minimal distance of a code \( C \) is an important factor in the decoding algorithm of linear codes. It allows us to determine the correction capability of the code. The following proposition gives us a way to find the minimal distance by using the control matrix.

Proposition 9 ([1], Proposition 7)

Let \( H \) be a control matrix of \( C \). Then the minimal distance of \( C \) is the minimal number of dependent columns of \( H \).

Example 1

We give an example illustrating the concepts studied above. The computations are simple but tedious, therefore we have used the computer algebra system Maple to verify the calculations [4].

Let \( A \) be the finite commutative local ring \( \mathbb{Z}/4\mathbb{Z} \). Let \( C \) be the \((20,10)\)-linear code over \( A \) of generator matrix

\[
G = (I_{10}, P)
\]

with

\[
P = \begin{pmatrix}
  1 & 0 & 3 & 0 & 1 & 3 & 0 & 2 & 2 & 0 \\
  1 & 3 & 0 & 3 & 2 & 1 & 1 & 3 & 3 & 3 \\
  3 & 1 & 2 & 0 & 1 & 1 & 3 & 2 & 3 & 0 \\
  2 & 0 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 3 \\
  0 & 0 & 3 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\
  3 & 1 & 3 & 2 & 3 & 3 & 3 & 1 & 2 & 2 \\
  2 & 0 & 0 & 1 & 2 & 1 & 0 & 1 & 1 & 2 \\
  2 & 0 & 1 & 3 & 1 & 1 & 1 & 0 & 3 & 1 \\
  0 & 2 & 1 & 1 & 2 & 2 & 1 & 3 & 0 & 3 \\
  0 & 0 & 1 & 2 & 1 & 2 & 2 & 0 & 1 & 1
\end{pmatrix}
\]

We begin by computing the control matrix \( H \) for the code \( C \).
\[ P := \begin{bmatrix}
1,1,3,2,0,3,2,2,0,0 \\
0,3,1,0,0,1,0,0,2,0 \\
3,0,2,2,3,0,1,1,1,1 \\
0,3,0,2,0,2,1,3,1,2 \\
1,2,1,2,0,3,2,1,2,1 \\
3,1,1,3,0,3,1,1,2,2 \\
0,1,3,3,2,3,0,1,1,2 \\
2,3,2,3,0,1,1,0,3,0 \\
2,3,3,3,0,2,1,3,0,1 \\
0,3,0,3,0,2,2,1,3,1
\end{bmatrix};
\]
\[ H := \begin{bmatrix}\text{Transpose}(P) | \text{IdentityMatrix}(10)\end{bmatrix} \mod 4; \]
\[ \text{evalm}(H); \]

By proposition \[d\] the minimal distance of \( C \) is \( d = 3 \) and this permits to detect 2 errors and correct \( t = \left\lfloor \frac{d - 1}{2} \right\rfloor = 1 \) error.

Let \( c = 1020230013001002303 \in C \) the transmitted codeword and \( r = 10202130013001002303 \) the received noisy word. To determine if an error exists in this received word, we compute the syndrome of \( r \).

Firstly, we define a procedure that returns the syndrome of a vector

\[
\text{Syndrome} := \text{proc}(A,x): \text{return MatrixVectorMultiply}(A,x) \mod 4; \text{end proc};
\]

We compute the syndrome of \( r \)

\[ \text{Syndrome}(H,r); \]

Because this syndrome is nonzero, we know \( r \) is erroneous. To find the error in \( r \) we must Compute the syndrome of all vectors of \( A^{20} \) of weight 1 and compare them with the syndrome of \( r \). Therefore, We collect these vectors in a matrix \( E \).

\[
E := \begin{bmatrix}\text{IdentityMatrix}(20) | 2\times\text{IdentityMatrix}(20) | 3\times\text{IdentityMatrix}(20)\end{bmatrix}; \]
\[ \text{evalm}(E); \]
We use the following commands to find the column in $E$ that matches the error.

```math
fc := 0:
cn := 0:
while (fc <> 1) and (cn < 60)do
    cn := cn+1;
    if Equal(Syndrome(H,Column(E,cn)), Syndrome(H,r)) = true then
        fc := 1;
    fi;
od:

cn:
```

This value for $cn$ indicates that the error is the 46th column. We can then see that the error vector that corresponds to $r$ as follows

```math
error := Column(E,cn):
evalm(error);
```

(0 0 0 0 0 3 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0)

The error is $e = 00000300000000000000$. Thus the transmitted codeword is $c = r - e = 10202230013001002303$.

**Acknowledgments**

The authors would like to thank the group of algebra and geometry of Moulay Ismal university, especially M. Ait Ben Haddou. Thanks also to M. E. Charkani from university of Fes for his help in commutative algebra.
References

[1] K. Abdelmoumen, M. Najmeddine et H. Ben-Azza. Pontrjagin Duality and Codes over Finite Commutative Rings. World Academy of Science, Engineering and Technology 56 2011, pp. 999-1003.

[2] W.A. Adkins and S.H. Weintraub. Algebra : an Approach via Module Theory. Springer, 1992.

[3] N. Bourbaki. Algèbre chapitres 1 à 3. Springer, 1970.

[4] R-E. Kilima, N. Sigmon, E. Stitzinger. Applications of abstract algebra with Maple. CRC Press LLC; 1999.

[5] S.Ling and C.Xing. Coding Theory : A First Course. Cambridge University Press, 2004.

[6] H.-A. Loeliger and T. Mittelholzer. Convolutional Codes over Groups. IEEE Trans. Information Th., Vol. 42(6), Nov. 1996, pp. 1660-1686.

[7] B.R. McDonald. Finite Rings with Identity. Marcel Dekker, 1974.

[8] T. Mittelholzer. Linear Codes and Their Duals over Artinian Rings. Codes, Systems, and Graphical Models, Eds. B. Marcus and J. Rosenthal. The IMA Volumes in Mathematics and its Applications, pp. 361-379, Springer, 2001.

[9] P. Tauvel. Mathématiques Générales pour l’Agrégation. Masson, Paris, 1992.

[10] J.A. Wood. Duality for Modules over Finite Rings and Applications to Coding Theory. American J. of Math., Vol 121.3, June 1999, pp. 555-575.