RECONSTRUCTING COSMOLOGICAL MATTER PERTURBATIONS USING STANDARD CANDLES AND RULERS

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ABSTRACT

For a large class of dark energy (DE) models, for which the effective gravitational constant is a constant and there is no direct exchange of energy between DE and dark matter (DM), knowledge of the expansion history suffices to reconstruct the growth factor of linearized density perturbations in the non-relativistic matter component on scales much smaller than the Hubble distance. In this paper, we develop a non-parametric method for extracting information about the perturbative growth factor from data pertaining to the luminosity or angular size distances. A comparison of the reconstructed density contrast with observations of large-scale structure and gravitational lensing can help distinguish DE models such as the cosmological constant and quintessence from models based on modified gravity theories as well as models in which DE and DM are either unified or interact directly. We show that for current supernovae (SNe) data, the linear growth factor at \( z = 0.3 \) can be constrained to 5% and the linear growth rate to 6%. With future SNe data, such as expected from the Joint Dark Energy Mission, we may be able to constrain the growth factor to 2%–3% and the growth rate to 3%–4% at \( z = 0.3 \) with this unbiased, model-independent reconstruction method. For future baryon acoustic oscillation data which would deliver measurements of both the angular diameter distance and the Hubble parameter, it should be possible to constrain the growth factor at \( z = 2.5\%–9\% \). These constraints grow tighter with the errors on the data sets. With a large quantity of data expected in the next few years, this method can emerge as a competitive tool for distinguishing between different models of dark energy.

Key words: cosmological parameters – cosmology: theory – distance scale

Online-only material: color figures

1. INTRODUCTION

Over the last decade, observations of Type Ia supernovae (SNe) have shown that the expansion of the universe is currently accelerating (Perlmutter et al. 1998, 1999; Riess et al. 1998, 2005, 2007; Tonry et al. 2003; Astier et al. 2005; Wood-Vasey et al. 2007; Kowalski et al. 2008). This remarkable discovery has led cosmologists to hypothesize the presence of dark energy (DE), a negative pressure energy component which dominates the energy content of the universe at present. Many theories have been propounded to explain this phenomenon, the simplest of which is the cosmological constant \( \Lambda \), with a constant energy density and the equation of state \( w = -1 \). Although \( \Lambda \) appears to explain all current observations satisfactorily, to do so its value must necessarily be very small \( \Lambda / 8 \pi G \approx 10^{-47} \text{ GeV}^4 \). So, it represents a new small constant of nature in addition to those known from elementary particle physics, many of them being very small if expressed in the Planck units. However, since it is not known at present how to derive \( \Lambda \) from these small constants and it is also unclear if DE is in fact time-independent, other phenomenological explanations for cosmic acceleration have been suggested (see reviews Sahni & Storobinsky 2000, 2006; Carroll 2001; Peebles & Ratra 2003; Padmanabhan 2003; Copeland et al. 2006; Nojiri & Odintsov 2007). These are based either on the introduction of new physical fields (quintessence models, Chaplygin gas, etc.) or on modifying the laws of gravity and therefore the geometry of the universe (scalar–tensor gravity, \( f(R) \) gravity, higher dimensional "braneworld" models, etc.). The plethora of competing DE models has led to the development of parametric and non-parametric methods as a means of obtaining model-independent information about the nature of DE directly from observations (see Starobinsky 1998; Huterer & Turner 1999; Huterer & Starkman 2003; Corasaniti et al. 2003; Alam et al. 2004, 2007; Saini et al. 2004; Jassal et al. 2005; Wang & Mukherjee 2006; Lazkoz et al. 2006; Sarkar et al. 2008; Sahni et al. 2008; Sahni & Storobinsky 2006, and references therein).

The next decade will see the emergence of many new cosmological probes. A large number of these are likely to make important contributions to the field of DE. The Sloan Digital Sky Survey (SDSS) began its stage III observations in 2008, and its Baryon Oscillation Spectroscopic Survey (BOSS) is expected to map the spatial distribution of luminous galaxies and quasars and detect the characteristic scale imprinted by baryon acoustic oscillations (BAOs) in the early universe (SDSS Collaboration). The Joint Dark Energy Mission (JDEM) is expected to discover a large number of SNe and also provide important data on weak lensing and BAOs (JDEM Collaboration). The Square Kilometer Array (SKA) will map out over a billion galaxies to a redshift of about 1.5, and is expected to determine the power spectrum of dark matter fluctuations as well as its growth as a function of cosmic epoch (Blake et al. 2004). Important clues to the growth of structure will also come from current and future weak-lensing surveys (Canada–France–Hawaii Telescope Legacy Survey, Dark Energy Survey, JDEM, Euclid, SKA, Large Scale Structure Telescope), galaxy redshift-space distortions (Guzzo 2008; Song & Percival 2008; Percival

[References and further details are provided in the full paper.]

http://www.sdss3.org; http://cosmology.lbl.gov/BOSS; http://www-physics.lbl.gov/physdiv/div-office/2006_dir_review/Schlegel.pdf

http://universe.nasa.gov/program/probes/jdem.html
are linked via the linearized Poisson equation

\[ k^2 \phi = -4\pi Ga^2 \rho_m \delta_m. \]  

(3)

If the DE energy–momentum tensor is covariantly conserved, then \( \rho_{DE} \propto a^{-3} \). In this case, it is straightforward to show (see, e.g., Wang & Steinhardt 1998; Starobinsky 1998) that on scales much smaller than the effective Jeans scale for DE, \( \lambda_J \sim c_s H^{-1} \), where \( H(z) \equiv \dot{a}/a \) is the Hubble parameter and \( c_s \) is the effective DE sound velocity (\( c_s = 1 \) for standard quintessence), linearized matter density perturbations in a FRW universe containing DE with an arbitrary effective equation of state \( w(t) \equiv p_{DE}/\rho_{DE} \) satisfy the same equation as in the case of a standard FRW model driven by dust and a cosmological constant (Peebles 1980):

\[ \ddot{\delta}_m + 2H \dot{\delta}_m - 4\pi G \rho_m \delta_m = 0 \]  

(4)

(we ignore the subscript in \( \delta_m \) in the ensuing discussion). However, the generic textbook solution (Peebles 1980; Sahni & Coles 1995)

\[ \delta \propto H(z) \int_{z_0}^z \frac{1 + z}{H^2(z_1)} dz_1 \]  

(5)

is not applicable now, apart from the following cases: dust-like matter, a non-zero spatial curvature (and/or a tangled network of cosmic strings) and a cosmological constant, for which \( H^2(z) = C_1 + C_2(1 + z)^2 + C_3(1 + z)^3 \). The same refers to the other well-known expression valid only for dust and a cosmological constant:

\[ \delta \propto a(t) - H(t) \int_{t_1}^t a(t_1) dt_1 \]  

(6)

(see, e.g., Kofman & Starobinsky 1985; Bertschinger 2006).7 Thus, for an arbitrary physical DE, Equation (4) has to be solved numerically. Since there are no terms depending on the perturbation wave vector \( k \) in it, \( \delta(z)/\delta(0) \) will be \( k \)-independent, too. We will also suppose that \( c_s \) is not too small, so that \( k \gg a/\lambda_J \) for all scales of interest, in particular \( c_s \gg 0.01 \) if we consider scales up to 100\((1 + z)^{-1} \) Mpc.

The dimensionless physical distance

\[ E \equiv a(t_0)H_0 \int_{t}^{t_0} \frac{dt}{a(t)} = H_0 \int_{z_0}^{z_1} \frac{dz}{H(z_1)}, \]  

(7)

where \( t_0 \) is the present moment, plays a key role in measurements of the background universe using standard rulers and candles. \( E \) is proportional to the conformal time measured from the present

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7 Expression (6) is, in fact, the first term in the long-wave (super-Hubble) expansion of the adiabatic mode of a comoving density perturbation if the perturbed matter pressure tensor is proportional to the unit one and the spatial curvature may be neglected. Then \( \phi = \psi \equiv -\xi(1 + a^2) \), where the gauge-invariant curvature perturbation \( \xi \) does not depend on time for the growing adiabatic mode (see, e.g., Polarski & Starobinsky 1992; Bertschinger 2006). The quantity \( r_t \) is free and may be chosen to coincide with the moment of the first Hubble radius crossing during inflation (another choice would correspond to adding a decaying adiabatic mode with an arbitrary amplitude). Then, using Equation (3) and the fact that \( \rho_{DE} \propto a^{-3} \), formula (6) follows. Since DE is practically unclustered at sub-Hubble scales, it is tempting to try to use this formula for \( \lambda_J \ll \lambda \ll H^{-1} \), too. However, as pointed above, this works only if DE is a cosmological constant.
to a moment in the past. It is related to the luminosity distance, \( d_L \), via
\[
\frac{H_0 d_L(z)}{1 + z} = \frac{1}{\sqrt{|\Omega_K|}} \sinh \left( \sqrt{|\Omega_K|} E(z) \right), \quad \Omega_K < 0 \quad (8)
\]
\[
\frac{H_0 d_L(z)}{1 + z} = E(z), \quad \Omega_K = 0 \quad (9)
\]
\[
\frac{H_0 d_L(z)}{1 + z} = \frac{1}{\sqrt{|\Omega_K|}} \sinh \left( \sqrt{|\Omega_K|} E(z) \right), \quad \Omega_K > 0, \quad (10)
\]
where \( \Omega_K \equiv 1 - \Omega_{\text{total}} \). The following relationship between the luminosity distance \( d_L \) and the angular size distance \( d_A \) holds in a metric theory of gravity: \( d_L = (1 + z)^3 d_A \). Rewriting Equation (4) in terms of Equation (7) and using the fact that \( \rho_m \propto (1 + z)^3 \), we obtain
\[
\left( \frac{\delta'}{1 + z(E)} \right)' = \frac{3}{2} \Omega_{0m} \delta, \quad (11)
\]
where the prime denotes a derivative with respect to \( E \). It is straightforward to transform Equation (11) into the following set of integral equations for \( \delta(E) \) and its first derivative (Sahni & Starobinsky 2006):
\[
\delta(E) = 1 + \delta_0 \int_0^E \left[ 1 + z(E_1) \right] dE_1 + \frac{3}{2} \Omega_{0m} \int_0^E \delta(E_2) dE_2 \quad (12)
\]
\[
\delta'(E) = \delta_0' [1 + z(E)] + \frac{3}{2} \Omega_{0m} [1 + z(E)] \int_0^E \delta(E_1) dE_1, \quad (13)
\]
where \( \delta \) is normalized to \( \delta_0 \equiv \delta(z = 0) = 1 \). The remarkable fact that in contrast to formulae used in the reconstruction of \( H(z) \) from \( d_L(z) \) (Starobinsky 1998; Huterer & Turner 1999) or \( \delta(z) \) (Starobinsky 1998) which require taking a derivative of observational data with respect to the redshift, this formula contains integrations of observational data only, which is a sound operation for noisy data. By solving the above equations, we can calculate the linear growth factor:
\[
g(z) \equiv (1 + z) \delta(z), \quad (14)
\]
which represents the ratio of \( \delta(z) \) in the presence of DE to that in Standard Cold Dark Matter (CDM) without a cosmological constant. Another quantity of interest is the growth rate:
\[
f(z) = \frac{d \ln \delta}{d \ln a} = -\frac{1 + z}{H(z)} \frac{\delta'(E)}{\delta(E)}. \quad (15)
\]
To solve Equation (12) we start with initial guess values for \( \delta(E) \) and \( \delta'(E) \) and iteratively solve for \( \delta(E) \), calculating \( \delta'(E) \) in the successive iterations as the difference between adjacent values of \( \delta(E) \), i.e., \( \delta_i' = \delta_i - \Delta E_i \). For calculating \( f(z) \), we require to estimate \( H(z) \) as well. We obtain this quantity by differentiating the noisy data \( E(z) \) using a finite differencing method. This naturally amplifies the noise in the final results, so we expect the results for \( f(z) \) to be slightly noisier. However, typically the difference in \( f(z) \) between two models of DE is greater than the difference in \( g(z) \), so despite the greater noise, we expect \( f(z) \) to be useful for discriminating DE models. This method does not require prior knowledge of the parameter \( \delta_0' \), is robust to changes in the initial guess values and gives exact results for \( g(E) \) and \( f(E) \) for noiseless data. For data with errors, naturally the result is noisier; however, in the succeeding sections we will be able to put reasonable constraints on \( g \) and \( f \) using this method.

Data noise can also be decreased using smoothing techniques. In what follows, we shall use the lognormal smoothing scheme proposed in Shafieloo et al. (2006) which has been shown to be reasonably unbiased and efficient. It constructs a smooth quantity, \( E^\sigma \), from a noisy one, \( E(z_i) \), via the ansatz (Shafieloo et al. 2006):
\[
E^\sigma(z) = \sum_i E(z_i) \exp \left( \frac{-\ln^2 \frac{1+z_i}{1+z}}{2 \Delta^2} \right) \sum_i \exp \left( \frac{-\ln^2 \frac{1+z_i}{1+z}}{2 \Delta^2} \right), \quad (16)
\]
where \( \Delta \) is the smoothing scale (see also Shafieloo 2007). We take \( \Delta \approx 1/N \), where \( N \) is the total number of observations. Choosing this small value of \( \Delta \) leaves the results unbiased.

From the manner in which Equations (11)–(13) have been obtained, reasons as to why the linearized growth function \( \delta_{\text{obs}}(z) \) determined from actual observations of large-scale structure may differ from \( \delta(z) \) reconstructed using our method follow immediately.

1. \( \rho_m \) is not proportional to \( (1 + z)^3 \). This happens even in GR if the energy–momentum tensor of physical DE is not covariantly conserved separately, either due to the existence of a direct non-gravitational interaction between DE and DM (see Amendola 1999; Billyard & Coley 2000; Zimdahl & Pavon 2001; Caldera-Cabral et al. 2009, and references therein) or because DE and DM constitute some unique entity as occurs in unified DM–DE models such as the Chaplygin gas (Kamenshchik et al. 2001) and its generalizations. In such models DE is partially clustered with DM on small scales, and this can result in the appearance of significant \( k \)-dependent terms in the equation for \( \delta \), so that the growth factor \( g(z) \) becomes \( k \)-dependent. In particular, the latter effect is especially crucial for the generalized Chaplygin gas model (see the recent paper Gorini et al. 2008, and references to previous papers therein).

2. GR is modified and DE is geometrical. Then \( G \) in Equation (4) becomes some effective quantity \( G_{\text{eff}} \) which may be both time- and scale-dependent. A noticeable value of \( \phi - \psi \) may also arise even in the absence of free-streaming particles. However, on small scales, this value is strongly restricted by solar system tests of gravity (which do not suggest any such effect). Geometrical models of DE, which include braneworld models and models using scalar–tensor and \( f(R) \) gravity, have more degrees of freedom than GR, so it is natural that in this case, the linearized perturbation equation for \( \delta \) shows a departure from the Newtonian form (Equation (4)). For instance in extra-dimensional scenarios (Dvali et al. 2000; Sahni & Shtanov 2003), the presence of the fifth dimension (the bulk) can influence the behavior of perturbations residing on the brane (Lue et al. 2004; Koyama & Maartens 2006; Sawicki et al. 2007; Shtanov et al. 2007) making them significantly \( k \)-dependent even on scales much smaller than the Hubble distance (Shtanov et al. 2007). The same effect arises in viable DE models in \( f(R) \) gravity (Hu & Sawicki 2007; Starobinsky 2007; Tsujikawa et al. 2008; see the recent review Sotiriou & Faraoni 2008 for numerous papers on \( f(R) \) gravity as a whole). On the other hand, in some
cases $G_{\text{eff}}$ and $g(z)$ may remain scale-independent even on small scales, though they acquire a non-trivial, non-GR, time dependence. This occurs, for instance, in the Dvali–Gabadadze–Porrati (DGP) extra-dimensional model considered below as well as in scalar–tensor DE models with a large current value of the Brans–Dicke parameter $\omega_{\text{BD}}$ (Boisseau et al. 2000; Gannouji & Polarski 2008).

Thus, a comparison of the observed and reconstructed density contrast could help shed light on the nature of DE. While it is encouraging that future observations (Blake et al. 2004; Guzzo 2008; Song & Percival 2008; Cinotti et al. 2009) of large-scale structure may make possible the determination of $\delta_{\text{obs}}(z)$, in this paper we focus on reconstructing $\delta$ using observations of high redshift Type Ia SNe and BAOs.

### 2.1. Data Used

The method outlined in the previous section would be applicable to any observation which contains a measurement of $E(z)$, for example, measurements of luminosity distance or angular diameter distance. We shall use real data and mock data based on simulations of SNe Type Ia data and the angular diameter distance from BAOs, to test this method.

**Supernova data.** The light curves of Type Ia SNe show them to be "calibrated candles"; therefore, they are of enormous significance in cosmology today. The luminosity distance of Type Ia SNe provides us with a direct measurement of the acceleration of the universe, thus leading to constraints on the DE parameters. SNe data are in the form $(m_B, z, \sigma_{m_B}, \sigma_z)$, where the magnitude $m_B$ is related to $d_L(z)$ as

$$m_B = 5 \log_{10}[H_0 d_L(z)] + \mathcal{M},$$

$\mathcal{M}$ being a noise parameter usually marginalized over.

It should be noted that SNe data, on its own, are unable to break the degeneracy between DE and spatial curvature. The CMB, on the other hand, places stringent constraints on $\Omega_K$ and strongly suggests that the universe is spatially flat, in agreement with predictions made by the inflationary scenario. In this paper we shall work under the assumption that $\Omega_K = 0$ and use Equation (9) to relate $d_L \to E(z)$, with the latter playing the key role in our reconstruction exercise (12).

Currently, there are around 300 published SNe with the furthest observed one at a redshift of $z = 1.7$ (Kowalski et al. 2008) and an average error of $\sigma_{m_B} \simeq 0.15$. Future space-based projects such as the JDEM Collaboration are expected to observe about 2000 SNe with $\sigma_{m_B} = 0.07$. To date, SNe are the most direct evidence for DE, and in this paper we shall primarily use SNe data to constrain the growth parameters for different DE models.

**BAO data.** At present, BAOs are believed to be the method least plagued by systematic uncertainties; therefore, the detection of the first BAO scale (Eisenstein et al. 2005) has led to the speculation that BAO may in future become a potent discriminator for DE. Standing sound waves that propagate in the opaque early universe imprint a characteristic scale in the clustering of matter, providing a "standard ruler." Since the sound horizon is tightly constrained by cosmic microwave background (CMB) observations, measuring the angle subtended by this scale determines a distance to that redshift and constrains the expansion rate. The radial and transverse scales give measurements of $[r_s H(z)]/c$ and $[r_s/(1 + z) d_A(z)]$, respectively, where $r_s$ is the sound horizon obtained from the CMB. These quantities are correlated, and the present BAO data are not sensitive enough to measure both quantities independently (see however the recent papers Benitez et al. 2009; Gaztanaga et al. 2009), but future surveys are expected to give independent measurements of $d_A(z)$ and $H(z)$ (Seo & Eisenstein 2003). Future BAO surveys such as BOSS (SDSS Collaboration) should therefore place tighter constraints on DE parameters.

### 3. RESULTS

We first use SNe data to reconstruct the growth parameters. We simulate data according to two theoretical models as follows.

1. **Model 1.** A cosmological constant model with $w = -1$, $\Omega_\text{m} = 0.27$, $H_0 = 72$ km s$^{-1}$ Mpc$^{-1}$.
2. **Model 2.** A variable DE model with the equation of state given by

$$w(z) = w_0 + \frac{w_a z}{1 + z}, \quad w_0 = -0.9, \; w_a = 0.3,$$

and with the same values of $\Omega_\text{m}$ and $H_0$ as Model 1. Note that Models 1 and 2 provide excellent agreement with the current CMB+BAO+SNe data (Komatsu et al. 2009).

Model 2 has $w > -1$ everywhere, so it can be realized by quintessence with some potential for which Equation (4) is valid.

Two different data distributions are used: Set A resembles the quality of data available at present and Set B is modeled on expected future surveys.

1. **Set A.** $\sim 300$ SNe, with the redshift distribution and errors of the Union data set (Kowalski et al. 2008). For this data set, on average, $\sigma_{m_B} \simeq 0.15$, but a few SNe have very high errors of the order of unity. Since the method of integration would not work very well for very noisy data, and a single data point with large noise would affect the results of all data points after it, we restrict the analysis to SNe with $\sigma_{m_B} < 0.7$. By rejecting only 10 data points with this criterion, we enhance the results by a significant amount.

2. **Set B.** $\sim 2000$ SNe, with the redshift distribution and errors ($\sigma_{m_B} \sim 0.07$) expected from future surveys such as the JDEM (Aldering et al. 2004). The data cover a redshift range of $z = 0.1–1.7$ with a larger concentration of SNe on the mid-range redshifts ($z = 0.4–1.1$). The errors considered here are statistical only; we do not consider systematic errors, which are expected to be better controlled in the future with larger data sets.

For both cases, we marginalize over $\Omega_\text{m} = 0.27 \pm 0.03$. SNe data are unable to break the degeneracy between DE and curvature of the universe. In order to measure the growth parameters, we therefore consider only a flat universe, which is the preferred model from current CMB observations.

Figure 1 shows the results for the linear growth factor $g(z)$ for both data sets and for the two different cosmological models. We see that for both models, Set A results in rather noisy reconstruction (left panel), since the errors on the SNe are quite high. This is especially true at high redshifts ($z > 0.7$) where the sparse sampling affects the integral reconstruction scheme adversely. At $z = 0.3$, $g(z)$ is constrained to $\sim 5\%$. For JDEM-like data (Set B) however, $g(z)$ is reconstructed more accurately and has low errors at low redshifts (right panel). At $z = 0.3$, $g(z)$ is constrained accurately to $\sim 2\%$ for both models for Set B, while at $z = 1$, $g(z)$ is constrained to $\sim 4\%$. 
Figure 1. Reconstructed linear growth factor $g(z)$ for different data sets. The top panels show the results for Model 1 ($\Lambda$CDM) using Union-like (Set A, left panel) and JDEM-like (Set B, right panel) SNe data sets, while the bottom panels show results for Model 2 (variable $w$, Equation (18)) using Set A (left panel) and Set B (right panel). In each figure, the black dotted line represents the true model, while the green dashed line represents the other model. The red solid lines show the $1\sigma$ error bars for the integral reconstruction using Equation (12).

(A color version of this figure is available in the online journal.)

Figure 2 shows the reconstruction of the growth rate $f(z)$. As before, the results for Set A are poor, with $f(z)$ constrained to $\sim 6\%$ at $z = 0.3$. The results for Set B are reasonable; however, the errors are slightly larger in this case than for $g(z)$, since there is an additional error from the calculation of $H(z)$ from $E(z)$. At $z = 0.3$, $f(z)$ is constrained accurately to $\sim 3\%$ for both models for Set B, while at $z = 1$, $f(z)$ is constrained to $\sim 8\%$. We also note that the quantity $f(z)$ has slightly greater discriminatory power than $g(z)$, since typically the growth factor shows rather less variation between different DE models as compared to the growth rate at any given redshift. Therefore, even though $f(z)$ is slightly noisier, for Set B, Models 1 and 2 can be discriminated at $1\sigma$ using $f(z)$.

If the data are first smoothed with the smoothing scheme (Equation (16)) the results improve, especially for Set A which has much noisier data, as seen in Figure 3. The results for $f(z)$ improve markedly for both data sets. This is because an additional quantity $H(z)$ is required for obtaining $f(z)$, and a smoother $E(z)$ leads to a much more accurate estimation of $H(z)$. Errors on $g(z)$ and $f(z)$ are $\sim 1\%$ and $\sim 1.5\%$, respectively, at $z = 0.3$ and $\sim 3\%$ and $\sim 6\%$, respectively, at $z = 1$ for Model 1 with JDEM-like data. Model 2 gives similar constraints. The results for the growth parameters are summarized in Table 1 for Model 1 and in Table 2 for Model 2. We see that this method obtains quite reasonable constraints on the growth parameters at low redshifts for Set B; therefore, it can be used successfully to constrain growth parameters from future SNe data. It should be noted that for future SNe data to accurately constrain the growth parameters, it is important to keep the SNe systematics under control ($\sigma_{\text{sys}} < \sim 0.05$). A systematic error of $\sigma_{\text{sys}} = 0.1$ (as on the current data) would weaken all constraints significantly.

3.1. Dependence on the Nature of Data

We now check how the results change if the redshift distribution or error distribution is changed. To study the dependence on the number of SNe, we use three redshift distributions: (1) Set A ($\sim 300$ SNe) with double the number of SNe at low ($z < 0.3$) and high ($z > 0.7$) redshifts, (2) Set A with double the SNe at mid-range ($0.3 < z < 0.7$) redshifts, and (3) a distribution with the JDEM (Set B) redshift distribution ($\sim 2000$ SNe) with errors of the order of the Union (Set A) SNe. The results for Model 1 are shown in Figure 4. We see that doubling the number of SNe in a particular redshift bin changes the results very slightly. This is to be expected because when integrating noisy data, having a larger number of points with the same amount of noise does not improve results significantly. Increasing the total number of SNe by a significant amount (nearly 7 times, as in the right panel) does improve the scatter, but the results still do not
Figure 2. Reconstructed growth rate $f(z)$ for different data sets. The top panels show the results for Model 1 ($\Lambda$CDM) using Union-like (Set A, left panel) and JDEM-like (Set B, right panel) SNe data sets, while the bottom panels show results for Model 2 (variable $w$, Equation (18)) using Set A (left panel) and Set B (right panel). In each figure, the thick black dotted line represents the true model, while the green dashed line represents the other model. The red solid line show the $1\sigma$ error bars for the integral reconstruction using Equation (12). The blue vertical lines in the right panel show the expected observational constraints from Euclid (Cimatti et al. 2009). (A color version of this figure is available in the online journal.)

We now study the effect of the errors. Once again we study three distributions: (1) Set A with the errors halved for $z < 0.3$ and $z > 0.7$ redshift bins; (2) Set A with errors halved in the $0.3 < z < 0.7$ redshift bin; and (3) Set A with errors replaced by JDEM-like errors on all SNe. The results for Model 1 are shown in Figure 5. We see that in this case, decreasing the errors at low redshift or high redshift changes the results very slightly. This is because there are very few points at low redshift so they do not affect the integration process strongly, while the high redshift points cannot affect the low redshift points. The results in the redshift range $0.3 < z < 0.7$ become better if the mid-range SNe have lower errors.

Since the high errors of Set A make it unsuitable for this reconstruction approach, in the following sections we will use Set B to study the robustness of the results to various other factors.

3.2. Growth Rate from $w(z)$

We may also calculate the growth rate $f$ from the SNe data via the equation of state using the following approximation (Wang & Steinhardt 1998):

$$f(z) \simeq \Omega_m(z)^\gamma = \left[ \frac{\Omega_m(1+z)^3}{H^2(z)} \right]^{\gamma}$$

(19)

$$\gamma(z) = \frac{3}{5} - \frac{w}{1+w} + \frac{3}{125} \left( 1 - w \right) \left( 1 - \frac{3}{5} w \right) \left( 1 - \Omega_m(z) \right) + O((1 - \Omega_m(z))^2),$$

(20)

where the equation of state $w(z)$ may be calculated using a likelihood parameter estimation from the luminosity distance. This approximation works quite well for a large number of physical DE models with a constant or slowly changing $w$ including $\Lambda$CDM, for which $\gamma \simeq 0.55$ (Wang & Steinhardt 1998; Linder 2005). We use the familiar CPL fit (Chevallier & Polarski 2001; Linder 2003)

$$w(z) = w_0 + \frac{w_a z}{1+z},$$

(21)

$$H^2(z) = H_0^2 \left[ \Omega_m(1+z)^3 \left( 1 + \frac{w_0 + w_a}{1+w} e^{3 w_a (1/(1+z)-1)} \right) + (1 - \Omega_m(1+z)^3 \left( 1 + \frac{w_0 + w_a}{1+w} e^{3 w_a (1/(1+z)-1)} \right) \right].$$

(22)
A likelihood parameter estimation is expected to lead to smaller errors, but the drawback of this method is that the result may be biased due to the parameterization. Also, the errors on $w$ may propagate extremely nonlinearly to $f$ and therefore the result for $f(z)$ would be much less trustworthy.

Figure 6 shows the reconstructed $f(z)$ for Models 1 and 2 for Set B. As expected, the errors are lower than those for our reconstruction method. However, it is also noteworthy that the resulting confidence levels are not symmetric around the true value; in fact at higher redshifts, the true model appears to be on the verge of being ruled out. These results are commensurate with those found in Mortonson et al. (2009), where reconstruction of the growth parameters through $w$ leads to biases in the growth parameter results even though $w$ is recovered accurately. This is due to the fact that errors propagate nonlinearly from $w$ to $f(z)$. We therefore conclude that,

$$g(z)$$ for Model 1 ($\Lambda$CDM) for the growth factor $g(z)$ (left panel) and the growth rate $f(z)$ (right panel). The bottom panels show the results for Model 2 (variable $w$, Equation (18)) for $g(z)$ (left panel) and $f(z)$ (right panel). In each figure, the black dotted line represents the true model, while the pink dashed line represents the other model. The green dashed shaded area represents the $1\sigma$ errors for the integral reconstruction of Set A (Union-like), while the green hatched shaded area represents the reconstruction for Set B (JDEM-like). The blue vertical lines in the right panel show the expected observational constraints from Euclid (Cimatti et al. 2009).

(A color version of this figure is available in the online journal.)

Table 1

| Data Sets | $z$  | $g(z)$    | $g_{\text{smooth}}(z)$ | $g_{\text{exact}}(z)$ | $f(z)$    | $f_{\text{smooth}}(z)$ | $f_{\text{exact}}(z)$ |
|----------|-----|-----------|------------------------|------------------------|-----------|------------------------|------------------------|
| A (Union SNe) | 0.3 | 1.11 ± 0.04 | 1.12 ± 0.02            | 1.12                   | 0.65 ± 0.04 | 0.63 ± 0.03            | 0.64                   |
|           | 0.6 | 1.18 ± 0.05 | 1.18 ± 0.03            | 1.19                   | 0.77 ± 0.06 | 0.75 ± 0.04            | 0.76                   |
|           | 1.0 | 1.27 ± 0.06 | 1.26 ± 0.05            | 1.25                   | 0.82 ± 0.09 | 0.83 ± 0.06            | 0.85                   |
|           | 1.5 | 1.24 ± 0.15 | 1.26 ± 0.09            | 1.28                   | 0.97 ± 0.21 | 0.94 ± 0.10            | 0.92                   |
| B (JDEM SNe) | 0.3 | 1.13 ± 0.02 | 1.12 ± 0.01            | 1.12                   | 0.64 ± 0.02 | 0.63 ± 0.01            | 0.64                   |
|           | 0.6 | 1.21 ± 0.03 | 1.20 ± 0.02            | 1.19                   | 0.75 ± 0.04 | 0.76 ± 0.02            | 0.76                   |
|           | 1.0 | 1.24 ± 0.05 | 1.23 ± 0.03            | 1.25                   | 0.86 ± 0.07 | 0.84 ± 0.03            | 0.85                   |
|           | 1.5 | 1.25 ± 0.08 | 1.26 ± 0.09            | 1.28                   | 0.93 ± 0.11 | 0.92 ± 0.06            | 0.92                   |
| C (BOSS BAO) | 2.5 | 1.28 ± 0.13 | 1.29 ± 0.11            | 1.30                   | 1.01 ± 0.09 | 1.00 ± 0.07            | 0.97                   |
when reconstructing the growth parameters from SNe data, it is better to reconstruct the quantities directly rather than reconstructing them indirectly from the energy density or equation of state.

3.3. Dependence on $\Omega_{\text{om}}$

SNe data do not simultaneously constrain information on $\Omega_{\text{om}}$ and DE parameters. To reconstruct DE parameters, it is necessary to place constraints on $\Omega_{\text{om}}$ from other observations. In the calculations so far, we have marginalized over the true fiducial value for $\Omega_{\text{om}}$. However, since there is considerable uncertainty as to the real value of the matter density, we check how using incorrect values of $\Omega_{\text{om}}$ may bias our analysis. (It is well known that an incorrect value of $\Omega_{\text{om}}$ can significantly bias the results for DE; Shafieloo et al. 2006; Sahni et al. 2008.) The fiducial universe for Model 1 contains $\Omega_{\text{om}} = 0.27$. We now choose a different, incorrect, value of $\Omega_{\text{om}} = 0.3$ for marginalization and proceed to analyze the data using both the integral reconstruction method and the likelihood parameter estimation.
Figure 6. Reconstructed growth rate $f(z)$ for Model 1 (left panel) and Model 2 (right panel) using Set B (JDEM-like) with different reconstruction methods. The red solid lines show the 1σ limits for reconstructed $f(z)$ using the integral reconstruction method (Equation (12)) while the green hatched region shows the 1σ limits for $f(z)$ using $w$ parameterization (Equations (19) and (21)). The black dotted line represents the true model.

Figure 7. Reconstructed growth rate $f(z)$ for Model 1 (left panel) and Model 2 (right panel) for Set B (JDEM-like) with different reconstruction methods, using $\Omega_{0m} = \Omega_{0m}(\text{true}) + 0.03$. The red solid lines show the 1σ limits for reconstructed $f(z)$ using the integral reconstruction method (Equation (12)) while the green hatched region shows the 1σ limits for $f(z)$ using $w$ parameterization (Equations (19) and (21)). The black dotted line represents the true model. Note that the results for the two reconstructions lie on opposite sides of the true value of $f(z)$.

(A color version of this figure is available in the online journal.)

of $w$ outlined in the previous section. The results are shown in Figure 7. We see that choosing a higher value of $\Omega_{0m}$ gives biased results in both methods, but interestingly enough, the biases are in opposite directions. In the case of the integral reconstruction method, a higher value of $\Omega_{0m}$ leads to a lower value of $f(z)$ at high redshifts, whereas for the $w$ parameterization, a higher value of $\Omega_{0m}$ leads to a higher value of $f(z)$.

These results may be understood as follows. For the reconstruction from $w$, we see from Equation (19) that $f(z)$ changes primarily due to the change in the matter density $\Omega_{0m}(1 + z)^3$, since the value of $\gamma$ does not vary very strongly with $w$ and $H^2(z)$ is constrained by the data. Choosing a higher value of $\Omega_{0m}$ would result in the choice of a different $w(z)$ which would lead to nearly the same $H(z)$ as that for a lower value of $\Omega_{0m}$, and $\gamma$ would also not change by much. However, the quantity $\Omega_{0m}(1 + z)^3$ would increase proportionate to $\Omega_{0m}$. Therefore, a higher value of $\Omega_{0m}$ would simply result in a higher value of $f(z)$. In the case of the integral reconstruction however, we see from Equation (12) that both $\delta$ and $\delta'$ depend on $\Omega_{0m}$. In $\delta$ the leading term is unity and the other two terms containing $\delta'_0$ and $\Omega_{0m}$ are at about an order of magnitude smaller. In $\delta'$ the two terms containing $\delta'_0$ and $\Omega_{0m}$ are of the same order and opposite sign. The $\Omega_{0m}$ term contributes by making $\delta'$ less negative. Therefore increasing $\Omega_{0m}$ increases $\delta$ slightly and decreases the absolute value of $\delta'$ by a larger amount, so that the ratio of $\delta$ to $\delta'$ becomes a smaller negative quantity. Since $f(z)$ is essentially this ratio, this means that $f(z)$ also decreases with increasing $\Omega_{0m}$. Therefore, choosing a wrong value of $\Omega_{0m}$ causes the two different methods of reconstruction to be biased in opposite directions. This leads to the interesting conclusion that, provided other systematics are under control, comparing the integral reconstruction method with the standard likelihood estimation would give us a valuable consistency check on the accuracy of the prior chosen for $\Omega_{0m}$. 
3.4. Reconstruction for a Toy-modified Gravity Model

An influential braneworld model was suggested by DGP (Dvali et al. 2000). The expansion history for this model is given by

\[
H(z) = H_0 \left[ \frac{1 - \Omega_{0m}}{2} + \sqrt{\Omega_{0m}(1+z)^3 + \left( \frac{1 - \Omega_{0m}}{2} \right)^2} \right].
\] (23)

For physical models of DE, the growth rate is well approximated by Equation (19), for instance \( y \approx 0.55 \) for ΛCDM (Wang & Steinhardt 1998; Linder 2005; Peacock et al. 2001; Thomas et al. 2009; Acquaviva et al. 2008). This equation is not valid however if the observed acceleration originates from a modification of the equations of general theory of relativity; in the DGP braneworld theory, the growth rate is approximated by (Lue et al. 2004)

\[
f(z) \simeq \Omega_m(z)^{0.68}. \] (24)

This is the growth rate which would be measured through galaxy redshift distortions or weak gravitational lensing, whereas any analysis from the expansion history would obtain a growth rate commensurate with Equation (19).

Therefore, if the growth rate for this model is reconstructed using SNe data, on the one hand, and galaxy redshift distortions, on the other hand, we expect the results to be different. We reconstruct the growth rate using the JDEM-like SNe distribution for this modified gravity model by substituting Equation (23) into the integral reconstruction method described by Equation (12). The result is shown in Figure 8. For comparison, we have also plotted the expected observational constraints from galaxy redshift distortions for the future Euclid mission (Cimatti et al. 2009). We see that the two results are strongly discrepant, especially at low redshifts. If the origin of DE were indeed geometrical in nature, comparisons of this sort would provide crucial evidence for it.

Despite the popularity of the DGP model, it is currently facing several difficulties both of an observational and of a theoretical nature: Tension between this model and observational data sets has been pointed out in Fairbairn & Goobar (2006), Alam & Sahni (2006), Alam & Sahni (2002) and Maartens & Majoretto (2006), and the presence of a ghost in DGP gravity (Charmousis et al. 2006; Gregory et al. 2007; Deffayet et al. 2006; Koyama 2007) may be even more problematic. Consequently our purpose in this section has been to treat DGP cosmology as a toy model, used to demonstrate the utility of the reconstruction approach developed in this paper. (Note however the existence of other braneworld models which are ghost free (Sahni & Shtanov 2003; Shtanov et al. 2009) and agree well with observations (Alam & Sahni 2006).)

3.5. Current Supernovae Data

In Figure 9, we show the reconstructed growth parameters for the currently available SNe data—the Union data set (Kowalski et al. 2008). The results are marginalized over \( \Omega_{0m} = 0.26 \pm 0.03 \), the currently accepted value of \( \Omega_{0m} \) (Dunkley et al. 2009). The nuisance parameter \( \mathcal{M} \) which contains information on \( H_0 \) is also marginalized over. For the non-smoothed method, since errors are quite large, it is difficult to put any constraints on the growth parameters. If the smoothing scheme is used, \( f(z) \) may be constrained to \( \sim \)6% at \( z = 0.3 \). At this redshift, the growth
factor $g(z)$ would be constrained to $\sim$5%. The reconstructed $f(z)$ is commensurate with the cosmological constant model as well as Model 2 (variable $w$, Equation (18)) used in this paper. We also show the three current observations of $f(z)$ from galaxy redshift-space distortions (Guzzo et al. 2008; Verde et al. 2002; Ross et al. 2007). The error bars on these observations are at present quite large, but it is expected that future data in this field would be comparable with our results from SNe; thus we would be able to discern physical and geometrical DE using these different techniques (as shown in Section 3.4). Table 3 shows the 1σ limits on the growth parameters for the reconstruction.

3.6. Data Expected from Future BAO Experiments

We now check the method with BAO data. The SDSS BAO survey of BOSS is expected to measure the BAO power spectrum very accurately. The expected accuracy on the angular diameter distance $d_A$ is of the order of 1.0% at $z = 0.35$, 1.1% at $z = 0.6$, and 1.5% at $z = 2.5$, with errors of 1.8%, 1.7%, and 1.5% on $H(z)$ at the same redshifts (SDSS Collaboration5). We populate a redshift range of $z = 0.2–2.5$ with 20 data points with errors based on these numbers and use this data set to reconstruct the growth parameters. Since there are only 20 points in the data set, and not many at very low redshifts, the integration is not very accurate even though the errors on $d_A$ and $H$ are small. We find that for this data set, $g(z)$ and $f(z)$ are both constrained to $\sim$9% at $z = 2.5$ (see Tables 1 and 2, bottom row). Although these errors appear to be large compared to those from the SNe data, for a high redshift of $z = 2.5$, these errors are actually commensurate to the errors from SNe. The advantage of using the BAO is that we obtain the growth parameters at a higher redshift, which is complementary to the SNe results. In the future, if systematics are controlled, and probes such as JDEM are able to measure both SNe and BAO data, we should be able to obtain independent estimates of the growth parameters at both very low and very high redshifts from this method.

4. CONCLUSIONS

In this paper, we have proposed a method for extracting growth parameters for DE models (within the spatially flat FRW universe) from observations that map the background universe, such as measures of the luminosity distance or the angular diameter distance. The method is model independent and unbiased. For current data, the growth factor $g(z)$ may be constrained to $\sim$5% at $z = 0.3$, while the growth rate $f(z)$ is constrained to $\sim$6%. For future JDEM SNe data, we will be able to put constraints of the order of a few percent on the growth parameters, for example, 2% on the growth factor and 3% on the growth rate at a redshift of 0.3, and 4% on the growth factor and 8% on the growth rate at a redshift of unity. In conjunction with the likelihood parameter estimation method, this method acts as an important consistency check on the accuracy of the priors on $\Omega_{m}$ for SNe. With future probes such as JDEM and BOSS taken in conjunction, it will lead to an unbiased estimation of the growth parameters up to a redshift of $z = 2.5$.

It is well known that, in GR and for most DE models, the expansion history completely determines the linearized growth rate of density perturbations (Starobinsky 1998; Sahni & Starobinsky 2006; the exact conditions for this are formulated at the beginning of Section 2). Consequently, a comparison of the density contrast reconstructed from the expansion history would provide one more important consistency check for a large variety of DE models including the cosmological constant and quintessence. On the other hand, as explained in more detail at the beginning of Section 2, any departure of the observed density contrast from that reconstructed using standard candles and rulers would almost certainly indicate that either there is an exchange of energy between DE and DM (so that the effective energy–momentum tensor of DE is not on its own covariantly conserved) or cosmic acceleration is a consequence of modified, non-Einsteinian gravity. In modified gravity theories, such as braneworld models, scalar–tensor and $f(R)$ gravity, etc., the linearized perturbation equation for $\delta$ does not follow the Newtonian form (Equation (4); Lue et al. 2004; Koyama & Maartens 2006; Shtanov et al. 2007; Hu & Sawicki 2007; Boisseau et al. 2000; Jain & Zhang 2007; Song & Koyama 2009; Zhao et al. 2009; Song & Dorè 2009). Hence, the density contrast reconstructed using observations of standard candles/rulers via Equation (12) and the density contrast determined directly from observations of large-scale structure, say, by weak lensing, galaxy redshift distortions or cluster abundances at different $z$ (Guzzo 2008; Song & Percival 2008; Vikhlinin et al. 2009; Cimatti et al. 2009), are likely to differ.

In Section 3.4, we show that one can obtain a strong signature of modified gravity by comparing results of this reconstruction method with future observations of galaxy redshift distortions using the DGP model as a toy example of modified gravity, where the growth factor $g(z)$ is scale-independent on small scales. However, as discussed in Section 2, $g(z)$ often becomes scale-dependent both in modified gravity and in the case of direct DE–DM interaction (or their unification). Therefore, for further discrimination of DE models alternative to quintessence and the cosmological constant, measurement of $\delta$ at different comoving scales is required to determine if $g(z)$ is scale-dependent or not.

Future surveys such as JDEM are expected to deliver high quality data for both SNe and weak lensing. Using such surveys, it would then be possible to compare the reconstructed density contrast from standard candles (SNe) with the density contrast observed from gravitational clustering (lensing). Therefore, we hope that the techniques developed in this paper, combined with future observations, will help unravel the nature of that most enigmatic quantity—dark energy.

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