BOTT-CHERN COHOMOLOGY OF SOLVMANIFOLDS

DANIELE ANGELLA AND HISASHI KASUYA

Abstract. We study conditions under which sub-complexes of a double complex of vector spaces allow to compute the Bott-Chern cohomology. We are especially aimed at studying the Bott-Chern cohomology of a special class of solvmanifolds.

Introduction

Given a double complex \((A^\bullet, \partial, \overline{\partial})\) of vector spaces, both the cohomology of the associated total complex \((\bigoplus_{p+q=\bullet} A^{p,q}, \partial + \overline{\partial})\) and the cohomologies of the rows \((A^\bullet, q, \overline{\partial})\) and of the columns \((A^p, \bullet, \partial)\) have been widely studied. Two other interesting cohomologies are the Bott-Chern cohomology, namely, the cohomology of the complex

\[
A^{p-1,q-1} \xrightarrow{\partial} A^{p,q} \xrightarrow{\partial + \overline{\partial}} A^{p+1,q} \oplus A^{p,q+1},
\]

and the Aeppli cohomology, namely, the cohomology of the complex

\[
A^{p-1,q} \oplus A^{p,q-1} \xrightarrow{(\partial, \overline{\partial})} A^{p,q} \xrightarrow{\partial + \overline{\partial}} A^{p,q+1}.
\]

For a compact complex manifold \(X\), the Bott-Chern and the Aeppli cohomologies of the double complex \((\wedge^\bullet X, \partial, \overline{\partial})\) have been studied by many authors in several contexts, see, e.g., [1, 19, 16, 29, 33, 34, 17, 16, 63, 11, 10]. They appear to be a completing useful tool besides the de Rham and the Dolbeault cohomologies. In this spirit, in [10], it is shown that an inequality à la Frölicher, involving just the dimensions of the Bott-Chern cohomology and of the de Rham cohomology, holds true on any compact complex manifold, and further allows to characterize the validity of the \(\partial\overline{\partial}\)-Lemma (namely, the very special cohomological property that every \(\partial\)-closed \(\overline{\partial}\)-closed \(\partial\)-exact form is \(\partial\overline{\partial}\)-exact too, see, e.g., [29]).

A compact manifold satisfies the \(\partial\overline{\partial}\)-Lemma if and only if the Bott-Chern cohomology is naturally isomorphic to the Dolbeault cohomology, [29, Remark 5.16]. Therefore, since compact Kähler manifolds satisfy the \(\partial\overline{\partial}\)-Lemma because of the Kähler identities, [29, Lemma 5.11], the Bott-Chern cohomology is particularly interesting in studying complex non-Kähler manifolds.

In non-Kähler geometry, a very fruitful source of counter-examples is provided by the class of nilmanifolds and solvmanifolds, namely, compact quotients of connected simply-connected nilpotent, respectively solvable, Lie groups by co-compact discrete subgroups. For instance, the geometry of nilmanifolds can be often reduced to the study of the associated Lie algebras, [21, 60, 14]. On the other hand, nilmanifolds do not admit too strong geometric structures, [15, 35]. More precisely, on a nilmanifold, the finite-dimensional sub-complex of left-invariant forms (namely, the forms being invariant for the action of the Lie group on itself given by left-translations) suffices in computing the de Rham cohomology, [55, 37]. Whenever the nilmanifold is endowed with a suitable left-invariant complex structure, also the Dolbeault cohomology, [61, 25, 22, 59, 60], and the Bott-Chern cohomology, [4], can be computed by means of just left-invariant forms.

Instead, for solvmanifolds, the left-invariant forms are usually not enough to recover the whole de Rham cohomology: an example is the non-completely-solvable solvmanifold provided in [27, Corollary 4.2]. The de Rham cohomology of solvmanifolds has been studied by several authors, e.g., A. Hattori [37], G. D. Mostow [53], S. Console and A. Fino [23], and the second author [39, 43]. Several results concerning the Dolbeault cohomology have been proven by the second author, [40, 43]; such results

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allow to study Hodge symmetry, Hodge decomposition, formality, and the Hodge and Frölicher spectral sequence on solvmanifolds, [111213].

In this note, we study the Bott-Chern cohomology of a certain class of solvmanifolds. This is done with the scope to further investigate the complex geometry of non-Kähler manifolds and especially its cohomological aspects. More precisely, we start by studying conditions under which the Bott-Chern cohomology of a double complex can be completely recovered by a suitable sub-complex; see Theorem 1.2 and Theorem 1.4. As an application, we get the following result. (For further applications to the study of the symplectic cohomologies studied by L.-S. Tseng and S.-T. Yau in [66, 67], see [8].)

Theorem (see Theorem 2.16 and Theorem 2.25). Let $G$ be a connected simply-connected solvable Lie group admitting a co-compact discrete subgroup $\Gamma$ and endowed with a $G$-left-invariant complex structure. If

- either $G$ is a semidirect product $\mathbb{C}^n \ltimes N$ of $\mathbb{C}^n$ and a connected simply-connected nilpotent Lie group $N$ endowed with an $N$-left-invariant complex structure satisfying some conditions (see Assumption 2.11),
- or $G$ is a complex Lie group,

then there is an explicit finite-dimensional sub-complex $C^{\bullet,\bullet}$ of the double complex $(\wedge^{\bullet,\bullet} \Gamma \backslash G, \partial, \overline{\partial})$ which computes the Bott-Chern cohomology of the solvmanifold $\Gamma \backslash G$.

As an application, we explicitly compute the Bott-Chern cohomology of the completely-solvable Nakamura manifold and of the complex parallelizable Nakamura manifold. This gives us, as a corollary, the following result.

Theorem (see Theorem 2.20). Satisfying the $\overline{\partial} \partial$-Lemma is not a strongly-closed property under small deformations of the complex structure.

In fact, in [7], we prove that satisfying the $\overline{\partial} \partial$-Lemma is not a (Zariski-)closed property.

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1. Computing the cohomologies of double complexes by means of sub-complexes

In this section, we study several cohomologies associated to a bounded double complex of $\mathbb{C}$-vector spaces; in particular, we are interested in studying when such cohomologies can be recovered by means of a suitable (possibly finite-dimensional) sub-complex.

1.1. The cohomology of the associated total complex. Let $(\mathcal{A}^{*,*}, \partial, \overline{\partial})$ be a bounded double complex of $\mathbb{C}$-vector spaces, namely, $\partial \in \text{End}^1(\mathcal{A}^{*,*})$ and $\overline{\partial} \in \text{End}^0(\mathcal{A}^{*,*})$ are such that $\partial^2 = \overline{\partial}^2 = [\partial, \overline{\partial}] = 0$, and $\mathcal{A}^{p,q} = \{0\}$ but for finitely-many $(p, q) \in \mathbb{Z}^2$. Denote by

$$
\text{Tot}^* (\mathcal{A}^{*,*}) := \bigoplus_{p+q = \bullet} \mathcal{A}^{p,q}, \ d := \partial + \overline{\partial}
$$

the total complex associated to $(\mathcal{A}^{*,*}, \partial, \overline{\partial})$. The bi-grading of $(\mathcal{A}^{*,*}, \partial, \overline{\partial})$ induces two natural bounded filtrations of $(\text{Tot}^* (\mathcal{A}^{*,*}), d)$, namely,

$$
\begin{align*}
\left\{ \left. \partial_p \text{Tot}^* (\mathcal{A}^{*,*}) := \bigoplus_{r+s = \bullet} \mathcal{A}^{r,s}, \ d \mid \partial_p \text{Tot}^* (\mathcal{A}^{*,*}) \right| p \in \mathbb{Z} \right\} & \hookrightarrow (\text{Tot}^* (\mathcal{A}^{*,*}), d) \\
\left\{ \left. \overline{\partial}_q \text{Tot}^* (\mathcal{A}^{*,*}) := \bigoplus_{r+s = \bullet} \mathcal{A}^{r,s}, \ d \mid \overline{\partial}_q \text{Tot}^* (\mathcal{A}^{*,*}) \right| q \in \mathbb{Z} \right\} & \hookrightarrow (\text{Tot}^* (\mathcal{A}^{*,*}), d)
\end{align*}
$$

Such filtrations induce naturally two spectral sequences, respectively,

$$
\left\{ \left. (\partial_r^*, \mathcal{A}^{*,*}, \partial, \overline{\partial}), \ (\partial_r) \right| r \in \mathbb{Z} \right\} \text{ and } \left\{ \left. (\overline{\partial}_r^*, \mathcal{A}^{*,*}, \partial, \overline{\partial}), \ (\overline{\partial}_r) \right| r \in \mathbb{Z} \right\}
$$
such that

\[ E_1^{p,q} (A^{\bullet}, \partial, \overline{\partial}) \cong H^2 (A^{\bullet}, \overline{\partial}) \Rightarrow H^{p+1,q} (\text{Tot}^* (A^{\bullet}), d), \]

and

\[ E_1^{p,q} (A^{\bullet}, \partial, \overline{\partial}) \cong H^2 (A^{\bullet}, \partial) \Rightarrow H^{p+1,q} (\text{Tot}^* (A^{\bullet}), d), \]

see, e.g., \[51, \text{Theorem 3.5}\]; in particular, the induced map

Lemma 1.2. sub-complex, we provide the following lemma.

One gets straightforwardly the following result, providing a sufficient condition under which a sub-complex \((C^{\bullet}, \partial, \overline{\partial}) \hookrightarrow (A^{\bullet}, \partial, \overline{\partial})\) allows to recover the cohomology of \((\text{Tot}^* (A^{\bullet}), d)\).

**Proposition 1.1.** Let \((A^{\bullet}, \partial, \overline{\partial})\) be a bounded double complex of \(\mathbb{C}\)-vector spaces, and let \((C^{\bullet}, \partial, \overline{\partial}) \hookrightarrow (A^{\bullet}, \partial, \overline{\partial})\) be a sub-complex. If, for every \(p \in \mathbb{Z}\), the induced map \((\text{Tot}^* (C^{\bullet}), \partial, \overline{\partial}) \hookrightarrow (\text{Tot}^* (A^{\bullet}), \partial, \overline{\partial})\) of complexes is a quasi-isomorphism, then the induced map

\[ (\text{Tot}^* (C^{\bullet}), d) \hookrightarrow (\text{Tot}^* (A^{\bullet}), d) \]

of complexes is a quasi-isomorphism.

**Proof.** The inclusion \((C^{\bullet}, \partial, \overline{\partial}) \hookrightarrow (A^{\bullet}, \partial, \overline{\partial})\) induces a morphism

\[ \{ ('F^p \text{Tot}^* (C^{\bullet}), d) \}_{p \in \mathbb{Z}} \to \{ ('F^p \text{Tot}^* (A^{\bullet}), d) \}_{p \in \mathbb{Z}} \]

of the associated bounded filtrations, and hence in particular a morphism

\[ \{ ('E^{p,*}_r (C^{\bullet}, \partial, \overline{\partial}), 'd_r) \}_{r \in \mathbb{Z}} \to \{ ('E^{p,*}_r (A^{\bullet}, \partial, \overline{\partial}), 'd_r) \}_{r \in \mathbb{Z}} \]

of the associated spectral sequences.

By the hypothesis, the inclusion induces an isomorphism at the first level,

\[ 'E_1^{p,q} (C^{\bullet}, \partial, \overline{\partial}) \cong 'E_1^{p,q} (A^{\bullet}, \partial, \overline{\partial}) \]

and hence, \(A^{\bullet}\) being bounded, also an isomorphism

\[ H^* (\text{Tot}^* (C^{\bullet}), d) \cong H^* (\text{Tot}^* (A^{\bullet}), d) \]

see, e.g., \[51, \text{Theorem 3.5}\]; in particular, the induced map

\[ (\text{Tot}^* (C^{\bullet}), d) \hookrightarrow (\text{Tot}^* (A^{\bullet}), d) \]

is a quasi-isomorphism. \(\square\)

1.2. **The Bott-Chern cohomology.** For any \((p, q) \in \mathbb{Z}^2\), other than the cohomologies of \((\text{Tot}^* (A^{\bullet}), d)\), of \((A^{\bullet}, \partial)\), and of \((A^{\bullet}, \partial)\), one can consider also the Bott-Chern cohomology, \[10\], namely, the cohomology of the complex

\[ A^{p-1,q-1} \xrightarrow{\partial} A^{p,q} \xrightarrow{\partial + \overline{\partial}} A^{p+1,q} \oplus A^{p,q+1}, \]

and the Aeppli cohomology, \[1\], namely, the cohomology of the complex

\[ A^{p-1,q} \oplus A^{p,q-1} \xrightarrow{\partial + \overline{\partial}} A^{p,q} \xrightarrow{\partial + \overline{\partial}} A^{p+1,q+1}. \]

1.2.1. **Conditions yielding a surjective map in Bott-Chern cohomology.** In order to study conditions under which the Bott-Chern cohomology of a double complex can be recovered by means of a suitable sub-complex, we provide the following lemma.

**Lemma 1.2.** Let \((A^{\bullet}, \partial, \overline{\partial})\) be a bounded double complex of \(\mathbb{C}\)-vector spaces, and let \((C^{\bullet}, \partial, \overline{\partial}) \hookrightarrow (A^{\bullet}, \partial, \overline{\partial})\) be a sub-complex. Suppose that, for every \(p \in \mathbb{Z}\), the induced map \((\text{Tot}^* (C^{\bullet}), \partial, \overline{\partial}) \hookrightarrow (\text{Tot}^* (A^{\bullet}, \partial, \overline{\partial})\) of complexes is a quasi-isomorphism. If \(\phi \in A^{p,q}\) is such that \(\overline{\partial} \phi \in A^{p,q+1}\), then there exist \(\hat{\phi} \in C^{p,q}\) and \(\hat{\phi} \in A^{p,q+1}\) such that \(\phi = \hat{\phi} + \overline{\partial} \hat{\phi}\).
Let \( \theta \) denote the map \( \theta : H^q \rightarrow H^q \).

**Theorem 1.3.** Let \( (A^{\bullet, \bullet}, \partial, \delta) \) be a bounded double complex of \( \mathbb{C} \)-vector spaces, and let \((C^{\bullet, \bullet}, \partial, \delta)\) be a sub-complex. Fix \((p, q) \in \mathbb{Z}^2\). Suppose that:

(i) for every \( r \in \mathbb{Z} \), the induced map \( (C^{r, \bullet}, \delta) \rightarrow (A^{r, \bullet}, \delta) \) of complexes is a quasi-isomorphism,

(ii) for every \( s \in \mathbb{Z} \), the induced map \( (C^{\bullet, s}, \partial) \rightarrow (A^{\bullet, s}, \partial) \) of complexes is a quasi-isomorphism,

(iii) the induced map

\[
\frac{\ker (\delta : \text{Tot}^{p+q} (A^{\bullet, \bullet}) \rightarrow \text{Tot}^{p+q+1} (A^{\bullet, \bullet})) \cap C^{p,q}}{\text{im} (\delta : \text{Tot}^{p+q+1} (A^{\bullet, \bullet}) \rightarrow \text{Tot}^{p+q} (C^{\bullet, \bullet}))}
\]

is surjective.

Then the induced map

\[
\left( C^{p-1,q-1,1} \oplus C^{p,q} \oplus C^{p+1,q+1} \right) \rightarrow \left( A^{p-1,q-1} \oplus A^{p,q} \oplus A^{p+1,q+1} \right)
\]

of complexes induces a surjective map in cohomology.

**Proof.** Up to shifting, assume that \( A^{r,s} = \{0\} \) whenever \((r, s) \notin \mathbb{N}^2\).

**Step 1** - Firstly, we prove that, under the hypotheses \[(i)\] and \[(ii)\] the inclusion \( (C^{\bullet, \bullet}, \partial, \delta) \hookrightarrow (A^{\bullet, \bullet}, \partial, \delta) \) induces, for every \((r, s) \in \mathbb{Z}^2\), a surjective map

\[
\frac{\text{im} (\delta : \text{Tot}^{r+s} (C^{\bullet, \bullet}) \rightarrow \text{Tot}^{r+s+1} (C^{\bullet, \bullet})) \cap C^{r,s}}{\text{im} (\delta : \text{Tot}^{r+s+1} (A^{\bullet, \bullet}) \rightarrow \text{Tot}^{r+s} (A^{\bullet, \bullet})) \cap A^{r,s}}
\]

Indeed, let

\[
\left( \omega^{r,s} \mod \text{im} (\delta : A^{r-1,s-1} \rightarrow A^{r,s}) \right) := \left( \text{im} (\delta : \text{Tot}^{r+s+1} (A^{\bullet, \bullet}) \rightarrow \text{Tot}^{r+s} (A^{\bullet, \bullet})) \cap A^{r,s} \right)
\]

Consider the bi-degree decomposition \( \eta := \sum_{(a,b) \in \mathbb{Z}^2} \eta^{a,b} \) where \( \eta^{a,b} \in A^{a,b} \), for \((a, b) \in \mathbb{Z}^2\). Hence, consider the system

\[
\begin{align*}
\partial \eta^{r+s-1,0} &= 0 \\
\partial \eta^{r+s-\ell,\ell-1} + \partial \eta^{r+s-\ell-1,\ell} &= 0 \\
\partial \eta^{r,s-1} + \partial \eta^{r-1,s} &= \omega^{r,s} \mod \text{im} (\delta : A^{r-1,s-1} \rightarrow A^{r,s}) \\
\partial \eta^{r+s-\ell,\ell-1} + \partial \eta^{r-1,s-\ell} &= 0 \\
\partial \eta^{0,r+s-1} &= 0
\end{align*}
\]

for \( \ell \in \{1, \ldots, s-1\} \) and \( \ell \in \{1, \ldots, r-1\} \).
Set $\eta^{r+s-2,-1} := 0$, and consider the equation
\[
\partial \eta^{r+s-\ell-1,\ell} + \partial \eta^{r+s-\ell-1,\ell} = 0 \mod \text{im } (\partial \overline{\partial}: A^{r+s-\ell-1,\ell-1} \to A^{r+s-\ell,\ell}) \quad \text{for } \ell \in \{0, \ldots, s-1\}.
\]

If $\eta^{r+s-\ell,\ell-1} \in C^{r+s-\ell,\ell-1}$ for some $\ell \in \{0, \ldots, s-1\}$, then, by applying Lemma 1.2 to the double complex $(A^{\bullet, \bullet}, \partial, \overline{\partial})$, one gets that there exist $\tilde{\eta}^{r+s-\ell-1,\ell} \in C^{r+s-\ell-1,\ell}$ and $\eta^{r+s-\ell-2,\ell} \in A^{r+s-\ell-2,\ell}$ such that
\[
\eta^{r+s-\ell-1,\ell} = \tilde{\eta}^{r+s-\ell-1,\ell} + \partial \eta^{r+s-\ell-2,\ell},
\]
therefore, when $\ell \leq s-2$, one gets the system
\[
\begin{cases}
\partial \eta^{r+s-\ell,0} = 0 \\
\partial \eta^{r+s-\ell-1,\ell} + \partial \eta^{r+s-\ell-1,\ell} = 0 \\
\partial \eta^{r+s-\ell-1,\ell} + \partial \eta^{r+s-\ell-1,\ell} = 0 \\
\partial \eta^{r+s-\ell-1,\ell} + \partial \eta^{r+s-\ell-1,\ell} = 0 \\
\partial \eta^{r+s-\ell,1} + \partial \eta^{r+s-\ell,1} = 0 \\
\partial \eta^{r+s-\ell+1,2} + \partial \eta^{r+s-\ell+1,2} = 0 \\
\partial \eta^{r+s-\ell,1} + \partial \eta^{r+s-\ell,1} = 0 \\
\partial \eta^{r+s-\ell-1,0} = 0
\end{cases}
\]
where $\eta^{r+s-\ell-1,\ell-1} \in C^{r+s-\ell-1,\ell-1}$, and when $\ell = s-1$, one gets the system
\[
\begin{cases}
\partial \eta^{r+s-\ell,0} = 0 \\
\partial \eta^{r+s-\ell-1,\ell} + \partial \eta^{r+s-\ell-1,\ell} = 0 \\
\partial \eta^{r+s-\ell-1,\ell} + \partial \eta^{r+s-\ell-1,\ell} = 0 \\
\partial \eta^{r+s-\ell,1} + \partial \eta^{r+s-\ell,1} = 0 \\
\partial \eta^{r+s-\ell+1,2} + \partial \eta^{r+s-\ell+1,2} = 0 \\
\partial \eta^{r+s-\ell,1} + \partial \eta^{r+s-\ell,1} = 0 \\
\partial \eta^{r+s-\ell-1,0} = 0
\end{cases}
\]
where $\eta^{r+s-\ell,1} \in C^{r,s-1}$.

In particular, since $\eta^{r+s-2,-1} = 0 \in C^{r+s-2,-1}$, we may assume that $\eta^{r,s-1} \in C^{r,s-1}$.

Analogously, by applying Lemma 1.3 to the double complex $(A^{\bullet, \bullet}, \partial, \overline{\partial})$, we may assume that $\eta^{r-1,s} \in C^{r-1,s}$.

Therefore
\[
\omega^{r,s} \mod \text{im } (\partial \overline{\partial}: A^{r-1,s-1} \to A^{r,s}) = (\partial \eta^{r,s-1} + \partial \eta^{r-1,s}) \mod \text{im } (\partial \overline{\partial}: A^{r-1,s-1} \to A^{r,s}) \in \frac{\text{im } (d: \text{Tot}^{r+s-1}(C^{\bullet, \bullet}) \to \text{Tot}^{r+s}(C^{\bullet, \bullet})) \cap C^{r,s}}{\text{im } (\partial \overline{\partial}: A^{r-1,s-1} \to A^{r,s})},
\]
that is, the induced map
\[
\frac{\text{im } (d: \text{Tot}^{r+s-1}(C^{\bullet, \bullet}) \to \text{Tot}^{r+s}(C^{\bullet, \bullet})) \cap C^{r,s}}{\text{im } (\partial \overline{\partial}: C^{r-1,s-1} \to C^{r,s})} \to \frac{\text{im } (d: \text{Tot}^{r+s-1}(A^{\bullet, \bullet}) \to \text{Tot}^{r+s}(A^{\bullet, \bullet})) \cap A^{r,s}}{\text{im } (\partial \overline{\partial}: A^{r-1,s-1} \to A^{r,s})}
\]
is surjective.

**Step 2** – Now, we prove that, under the additional assumption (iii), the induced map
\[
\frac{\ker (\partial: C^{p,q} \to C^{p+1,q}) \cap \ker (\partial: C^{p,q} \to C^{p,q+1})}{\text{im } (\partial \overline{\partial}: C^{p-1,q-1} \to C^{p,q})} \to \frac{\ker (\partial: A^{p,q} \to A^{p+1,q}) \cap \ker (\partial: A^{p,q} \to A^{p,q+1})}{\text{im } (\partial \overline{\partial}: A^{p-1,q-1} \to A^{p,q})}
\]
is surjective.
Indeed, consider the commutative diagram

\[
\begin{array}{ccc}
0 & \xrightarrow{\text{im}(d)} & \text{Tot}^p+q-1(C^{\bullet, \cdot}) \cap \text{Tot}^p+q(C^{\bullet, \cdot}) \cap \text{Tot}^p+q(A^{\bullet, \cdot}) \cap \text{AP}^q \\
& & \text{im}(d) : C^{p+1,q-1} \rightarrow C^{p,q} \rightarrow 0 \\
& & \text{im}(d) : A^{p+1,q-1} \rightarrow A^{p,q} \rightarrow 0 \\
\end{array}
\]

whose rows and columns are exact. By the Five Lemma, see, e.g., \([51, page 26]\), the map

\[
\ker(\partial : C^{p,q} \rightarrow C^{p+1,q}) \cap \ker(\overline{\partial} : C^{p,q} \rightarrow C^{p+1,q}) \rightarrow \ker(\partial : A^{p,q} \rightarrow A^{p+1,q}) \cap \ker(\overline{\partial} : A^{p,q} \rightarrow A^{p+1,q})
\]

is surjective, completing the proof. \(\square\)

1.2.2. Conditions yielding an injective map in Bott-Chern cohomology.
In order to provide conditions under which the inclusion of a suitable sub-complex induces an injective map in Bott-Chern cohomology, we consider a further structure of Hilbert space on the double complex. (For similar results in the case of solvmanifolds, see \([22, Lemma 9]\), \([4, Lemma 3.6]\).)

Let \(A\) be a Hilbert space, with inner product \(\langle \cdot | \cdot \rangle : A \times A \rightarrow C\). Denote by \(\|\cdot\| := (\langle \cdot | \cdot \rangle)^{1/2}\) the associated norm.

Given a densely-defined linear operator \(L : A \supseteq \text{dom}(L) \rightarrow A\) on \(A\), denote by

\[
L^*_{\langle \cdot | \cdot \rangle} : \text{dom} \left( L^*_{\langle \cdot | \cdot \rangle} \right) \rightarrow A
\]

its \(\langle \cdot | \cdot \rangle\)-adjoint operator, that is, the unique linear operator with domain

\[
\text{dom} \left( L^*_{\langle \cdot | \cdot \rangle} \right) := \{ y \in A : \langle L \cdot | y \rangle : \text{dom}(L) \rightarrow C \text{ is continuous} \}
\]

and defined by

\[
\forall x \in \text{dom}(L), \forall y \in \text{dom} \left( L^*_{\langle \cdot | \cdot \rangle} \right), \quad \langle Lx | y \rangle = \langle x | L^*_{\langle \cdot | \cdot \rangle}y \rangle.
\]

Given a closed sub-space \(C\) of \(A\), denote the induced inner product on \(C\) by \(\langle \cdot | \cdot \rangle_{\langle \cdot | \cdot \rangle} := \langle \cdot | \cdot \rangle |_{C \times C} : C \times C \rightarrow C\), and the orthogonal projection onto \(C\) by \(\pi^C_{\langle \cdot | \cdot \rangle} : A \rightarrow C \subseteq A\). One has that

\[
\pi^C_{\langle \cdot | \cdot \rangle}|_C = \text{id} C \quad \text{and} \quad \langle C \left( \text{id}_A - \pi^C_{\langle \cdot | \cdot \rangle} \right) (A) \rangle = \{0\}.
\]

(To simplify notations, we do not specify the inner product \(\langle \cdot | \cdot \rangle\) in writing the projection or the adjoint, whenever it is clear from the context.)

We firstly record the following lemma, stating that, if \(L\) commutes with \(\pi^C\), then also \(L^*\) does.

**Lemma 1.4.** Let \(A\) be a Hilbert space, with inner product \(\langle \cdot | \cdot \rangle\). Let \(L : A \supseteq \text{dom}(L) \rightarrow A\) be a densely-defined linear operator on \(A\). Let \(C\) be a closed sub-space of \(A\) contained in \(\text{dom}(L)\) and in \(\text{dom} \left( L^*_{\langle \cdot | \cdot \rangle} \right)\). Suppose that

\[
\pi^C_{\langle \cdot | \cdot \rangle} \circ L = L \circ \pi^C_{\langle \cdot | \cdot \rangle} : \text{dom}(L) \rightarrow C.
\]

Then

\[
\pi^C_{\langle \cdot | \cdot \rangle} \circ L^*_{\langle \cdot | \cdot \rangle} = L^*_{\langle \cdot | \cdot \rangle} \circ \pi^C_{\langle \cdot | \cdot \rangle} : \text{dom} \left( L^*_{\langle \cdot | \cdot \rangle} \right) \rightarrow C;
\]

in particular, \(L^*_{\langle \cdot | \cdot \rangle} |_C : C \rightarrow C\), and hence \((L|_C)_{\langle \cdot | \cdot \rangle}_C = L^*_{\langle \cdot | \cdot \rangle} |_C\).
Proof. It suffices to note that \( \pi^C : A \to C \subseteq A \) is self-(\( \cdot | \cdot \))-adjoint: for any \( \alpha, \beta \in A \),
\[
\langle \pi^C \alpha \mid \beta \rangle = \langle \pi^C \alpha \mid (\beta - \pi^C \beta) \rangle = \langle \pi^C \alpha \mid \pi^C \beta \rangle = \langle \pi^C \alpha + (\alpha - \pi^C \alpha) \mid \pi^C \beta \rangle = \langle \alpha \mid \pi^C \beta \rangle.
\]
It follows straightforwardly that \( \pi^C \circ L^* = L^* \circ \pi^C : \text{dom}(L^*) \to C \). In particular, since \( \pi^C|_C = id_C \) and \( C \subseteq \text{dom}(L^*) \), it follows that \( L^*(C) = \langle L^* \circ \pi^C \rangle (C) = (\pi^C \circ L^*) \circ (C) \subseteq C \), and hence \( L^*|_C : (L|_C)_{\{c \}} : C \to C \). \( \square \)

Now, let \( A_* \) be a bounded \( \mathbb{Z}^2 \)-graded vector space with a structure of Hilbert space, with inner product \( \langle \cdot | \cdot \rangle \) such that \( \langle A^{p,q} | A^{p',q'} \rangle = \{0\} \) for every \( (p,q) \neq (p',q') \). Let \( \partial : A_* \to A_+ \) be densely-defined linear operators yielding a structure \( \left((\text{dom}(\partial) \cap \text{dom}(\partial))^{(\cdot | \cdot)}, \partial, \partial\right) \) of bounded double complex of \( \mathbb{C} \)-vector spaces. Denote by \( \partial^* := \partial^*_\{\cdot \} : A_* \to A_- \) and \( \overline{\partial} := \overline{\partial}_\{\cdot \} : A_* \to A_+ \) the \( (\cdot | \cdot) \)-adjoint operators of \( \partial \) and, respectively, \( \overline{\partial} \).

Following [145], Proposition 5, see also [143], 3.2.b, 3.2.c, define the (densely-defined) self-(\( \cdot | \cdot \))-adjoint operator
\[
\Delta^{BC} := \Delta^{BC}_{\{\cdot \}} := \left( (\partial \overline{\partial})(\partial \overline{\partial})^* + (\partial \overline{\partial})(\partial \overline{\partial}) + (\overline{\partial} \partial)(\overline{\partial} \partial)^* + (\overline{\partial} \partial)(\overline{\partial} \partial) + (\partial \overline{\partial} \overline{\partial} \partial + \partial^* \partial \right) \in \text{Hom}^{0,0}(\text{dom}(\Delta^{BC})^{(\cdot | \cdot)}, A_*^{\cdot \cdot})\]

The following lemma states that, under a suitable decomposition hypothesis, the Bott-Chern cohomology of \( (A_*^{\cdot \cdot}, \partial, \overline{\partial}) \) is isomorphic to \( \ker \Delta^{BC} \).

**Lemma 1.5.** Let \( A_* \) be a bounded \( \mathbb{Z}^2 \)-graded vector space with a structure of Hilbert space, with inner product \( \langle \cdot | \cdot \rangle \) such that \( \langle A^{p,q} | A^{p',q'} \rangle = \{0\} \) for every \( (p,q) \neq (p',q') \). Let \( \partial : A_* \to A_+ \) and \( \overline{\partial} : A_* \to A_- \) be densely-defined linear operators yielding a structure \( \left((\text{dom}(\partial) \cap \text{dom}(\partial))^{(\cdot | \cdot)}, \partial, \partial\right) \) of bounded double complex of \( \mathbb{C} \)-vector spaces. Suppose that the operator \( \Delta^{BC}_{\{\cdot \}} \in \text{Hom}^{0,0}(\text{dom}(\Delta^{BC})^{(\cdot | \cdot)}, A_*^{\cdot \cdot}) \) induces the decomposition
\[
\text{dom}(\Delta^{BC}_{\{\cdot \}}) = \ker \Delta^{BC}_{\{\cdot \}} \oplus \text{im} \Delta^{BC}_{\{\cdot \}}.
\]

Then, for every \( (p,q) \in \mathbb{Z}^2 \), the induced map
\[
\left(0 \to \ker \Delta^{BC}_{\{\cdot \}} \cap A^{p,q} \to 0\right) \leftarrow \left(A^{p-1,q-1} \to A^{p,q} \to A^{p+1,q} \oplus A^{p,q+1}\right)
\]
is a quasi-isomorphism.

**Proof.** Note that, for every \( \eta \in \text{dom}(\Delta^{BC}) \), one has
\[
\langle \Delta^{BC} \eta \mid \eta \rangle = \left\| (\partial \overline{\partial}) \eta \right\|^2 + \left\| (\partial \overline{\partial}) \eta \right\|^2 + \left\| (\partial \overline{\partial}) \eta \right\|^2 + \left\| (\partial \overline{\partial}) \eta \right\|^2 + \left\| (\partial \overline{\partial}) \eta \right\|^2,
\]
hence
\[
\ker \Delta^{BC} = \ker \partial \cap \ker \overline{\partial} \cap \ker (\partial \overline{\partial})^*.
\]

On the other hand, since \( \text{im} \Delta^{BC} \subseteq \text{im} \partial \overline{\partial} \oplus \left( \text{im} \partial^* + \text{im} \overline{\partial} \right) \) and \( \left( \text{im} \partial^* + \text{im} \overline{\partial} \right) \cap (\ker \partial \cap \ker \overline{\partial}) = \{0\} \), one has
\[
\text{im} \Delta^{BC} \cap (\ker \partial \cap \ker \overline{\partial}) = \text{im} \overline{\partial}.
\]
It follows that
\[
\ker \Delta^{BC} \cap A^{p,q} \cong \frac{\ker \Delta^{BC} \cap A^{p,q} + \text{im} \partial \overline{\partial} \cap A^{p,q}}{\text{im} \partial \overline{\partial} : A^{p-1,q-1} \to A^{p,q}} \cong \frac{\ker (\partial + \overline{\partial} : A^{p,q} \to A^{p+1,q} \oplus A^{p,q+1})}{\text{im} \partial \overline{\partial} : A^{p-1,q-1} \to A^{p,q}}
\]
completing the proof. \( \square \)

We have now the following result.
Theorem 1.6. Let $A^{**}$ be a bounded $\mathbb{Z}^2$-graded vector space with a structure of Hilbert space, with inner product $\langle \cdot | \cdot \rangle$ such that $\langle A^p,q | A^{p',q'} \rangle = \{0\}$ for every $(p,q) \neq (p',q')$. Let $\partial : A^{**} \supseteq \text{dom}(\partial)^{**} \to A^{**+1}$ and $\overline{\partial} : A^{**} \supseteq \text{dom}(\overline{\partial})^{**} \to A^{**+1}$ be densely-defined linear operators yielding a structure $\left( \text{dom}(\partial) \cap \text{dom}(\overline{\partial}) \right)^{**}, \partial, \overline{\partial} \right)$ of bounded double complex of $\mathbb{C}$-vector spaces. Let 

\[ j : (C^{**}, \partial, \overline{\partial}) \leftrightarrow \left( \left( \text{dom}(\partial) \cap \text{dom}(\overline{\partial}) \right)^{**}, \partial, \overline{\partial} \right) \]

be a sub-complex. Suppose that:

(i) the operator $\delta_{(\cdot \cdot \cdot)}^{BC} \in \text{Hom}^0 \left( \text{dom} \left( \delta_{(\cdot \cdot \cdot)}^{BC} \right)^{**} ; A^{**} \right)$ induces the decomposition

\[ \text{dom} \left( \delta_{(\cdot \cdot \cdot)}^{BC} \right) = \ker \delta_{(\cdot \cdot \cdot)}^{BC} \oplus \text{im} \delta_{(\cdot \cdot \cdot)}^{BC} ; \]

(ii) it holds that

\[ \delta_{(\cdot \cdot \cdot)}^{BC} \mid C^{**} = \left( \delta_{(\cdot \cdot \cdot)}^{C^{**}} \right)^* \mid C^{**} : \text{dom} \left( \delta_{(\cdot \cdot \cdot)}^{C^{**}} \right)^* \to C^{*-1} ; \]

and

\[ \overline{\delta}_{(\cdot \cdot \cdot)}^{BC} \mid C^{**} = \left( \overline{\delta}_{(\cdot \cdot \cdot)}^{C^{**}} \right)^* \mid C^{**} : \text{dom} \left( \overline{\delta}_{(\cdot \cdot \cdot)}^{C^{**}} \right)^* \to C^{*-1} ; \]

in particular, it follows that

\[ \delta_{(\cdot \cdot \cdot)}^{BC} \mid C^{**} = \delta_{(\cdot \cdot \cdot)}^{BC} \mid C^{**} \in \text{Hom}^0 \left( \text{dom} \left( \delta_{(\cdot \cdot \cdot)}^{BC} \right)^{**} ; C^{**} \right) ; \]

(iii) the operator $\overline{\delta}_{(\cdot \cdot \cdot)}^{BC} \mid C^{**} \in \text{Hom}^0 \left( \text{dom} \left( \overline{\delta}_{(\cdot \cdot \cdot)}^{BC} \right)^{**} ; C^{**} \right)$ induces the decomposition

\[ \text{dom} \left( \overline{\delta}_{(\cdot \cdot \cdot)}^{BC} \right)^{**} \mid C^{**} = \ker \overline{\delta}_{(\cdot \cdot \cdot)}^{BC} \oplus \text{im} \overline{\delta}_{(\cdot \cdot \cdot)}^{BC} \mid C^{**} . \]

Then, for every $(p,q) \in \mathbb{Z}^2$, the induced map

\[ j : \left( C^{p-1,q-1} \oplus A^p,q \oplus C^{p+1,q} \oplus A^{p,q+1} \right) \leftrightarrow \left( A^{p-1,q-1} \oplus A^p,q \oplus A^{p+1,q} \oplus A^{p,q+1} \right) \]

of complexes induces an injective map $j^*$ in cohomology.

Proof. By Lemma 1.5 and under the hypotheses [6] [7] and [11], one gets that both

\[ \left( 0 \to \ker \delta_{(\cdot \cdot \cdot)}^{BC} \cap A^p,q \to 0 \right) \leftrightarrow \left( A^{p-1,q-1} \text{ dom } A^p,q \text{ dom } A^{p+1,q} \text{ dom } A^{p,q+1} \right) \]

and

\[ \left( 0 \to \ker \delta_{(\cdot \cdot \cdot)}^{BC} \mid C^{**} \cap C^{p,q} = \ker \delta_{(\cdot \cdot \cdot)}^{BC} \mid C^{**} \cap C^{p,q} \to 0 \right) \leftrightarrow \left( C^{p-1,q-1} \text{ dom } C^p,q \text{ dom } C^{p+1,q} \text{ dom } C^{p,q+1} \right) \]

are quasi-isomorphisms.

Hence, one has the commutative diagram

\[ \begin{array}{ccc} \ker \delta_{(\cdot \cdot \cdot)}^{BC} \mid C^{**} \cap C^{p,q} & \cong & \ker \left( \partial + \overline{\partial} : C^{p,q} \to C^{p+1,q} \oplus C^{p,q+1} \right) \\
\downarrow & & \downarrow \\
\ker \delta_{(\cdot \cdot \cdot)}^{BC} \cap A^p,q & \cong & \ker \left( \partial + \overline{\partial} : A^p,q \to A^{p+1,q} \oplus A^{p,q+1} \right) \end{array} \]

getting that $j^*$ is injective. \qed

By using Lemma 1.4 one gets the following corollary of Theorem 1.6 concerning closed sub-complexes.

Corollary 1.7. Let $A^{**}$ be a bounded $\mathbb{Z}^2$-graded vector space with a structure of Hilbert space, with inner product $\langle \cdot | \cdot \rangle$ such that $\langle A^p,q | A^{p',q'} \rangle = \{0\}$ for every $(p,q) \neq (p',q')$. Let $\partial : A^{**} \supseteq \text{dom}(\partial)^{**} \to A^{**+1}$ and $\overline{\partial} : A^{**} \supseteq \text{dom}(\overline{\partial})^{**} \to A^{**+1}$ be densely-defined linear operators yielding a structure $\left( \left( \text{dom}(\partial) \cap \text{dom}(\overline{\partial}) \right)^{**}, \partial, \overline{\partial} \right)$ of bounded double complex of $\mathbb{C}$-vector spaces. Let $j : (C^{**}, \partial, \overline{\partial}) \leftrightarrow \left( \left( \text{dom}(\partial) \cap \text{dom}(\overline{\partial}) \right)^{**}, \partial, \overline{\partial} \right)$ be a closed sub-complex. Suppose that:
Corollary 1.8. \[ \text{dom} \left( \Delta_{\{1\ldots\} BC} \right) \subseteq \ker \Delta_{\{1\ldots\} BC} \oplus \im \Delta_{\{1\ldots\} BC} ; \]

Proof. By Lemma 1.3 one has \( \pi_{C^{\cdots}} \circ \partial = \partial \circ \pi_{C^{\cdots}} : \text{dom} (\partial)^{\cdots} \rightarrow C^{\cdots+1} \) and \( \pi_{C^{\cdots}} \circ \overline{\partial} = \overline{\partial} \circ \pi_{C^{\cdots}} : \text{dom}(\overline{\partial})^{\cdots} \rightarrow C^{\cdots+1} \), and hence one gets the decomposition

\[ \text{dom}(\partial)^{\cdots} \cap \text{dom}(\overline{\partial})^{\cdots} \cap \text{dom} \left( \partial_{1\ldots} \right), \quad \text{and} \quad \pi_{C^{\cdots}} \circ \partial = \partial \circ \pi_{C^{\cdots}} : \text{dom}(\partial)^{\cdots} \rightarrow C^{\cdots+1}. \]

Then, for every \((p,q) \in \mathbb{Z}^2\), the induced map

\[ j: \left( C^{p-1,q-1} \oplus C^{p,q} \oplus C^{p+1,q} \oplus C^{p,q+1} \right) \rightarrow \left( A^{p-1,q-1} \oplus A^{p,q} \oplus A^{p+1,q} \oplus A^{p,q+1} \right) \]

of complexes induces an injective map \( j^* \) in cohomology.

Note that hypothesis (iii) in Theorem 1.3 is satisfied whenever the sub-complex \( C^{\cdots} \) is finite-dimensional.

Corollary 1.8. Let \( A^{\cdots} \) be a bounded \( \mathbb{Z}^2 \)-graded vector space with a structure of Hilbert space, with inner product \( \langle \cdot | \cdot \rangle \) such that \( \langle A^p,q | A^p',q' \rangle = \{0\} \) for every \((p,q) \neq (p',q')\). Let \( \partial: A^{\cdots} \supseteq \text{dom}(\partial)^{\cdots} \rightarrow A^{\cdots+1} \) and \( \overline{\partial}: A^{\cdots} \supseteq \text{dom}(\overline{\partial})^{\cdots} \rightarrow A^{\cdots+1} \) be densely-defined linear operators yielding a structure \((\text{dom}(\partial) \cap \text{dom}(\overline{\partial}))^{\cdots}, \partial, \overline{\partial}) \) of bounded double complex of \( \mathbb{C} \)-vector spaces. Let \( j: (C^{\cdots}, \partial, \overline{\partial}) \rightarrow (\text{dom}(\partial) \cap \text{dom}(\overline{\partial}))^{\cdots}, \partial, \overline{\partial}) \) be a sub-complex. Suppose that:

(i) the operator \( \Delta_{\{1\ldots\} BC} \in \text{Hom}^{0,0} \left( \text{dom} \left( \Delta_{\{1\ldots\} BC} \right)^{\cdots}; A^{\cdots} \right) \) induces the decomposition

\[ \text{dom}(\Delta_{\{1\ldots\} BC})^{\cdots} = \ker \Delta_{\{1\ldots\} BC} \oplus \im \Delta_{\{1\ldots\} BC} ; \]

(ii) \( C^{\cdots} \) is finite-dimensional;

(iii) it holds that

\[ \partial_{\{1\ldots\}}^{\cdots} | C^{\cdots} = (\partial | C^{\cdots})_{\{1\ldots\}c^{\cdots}} : C^{\cdots} \rightarrow C^{\cdots+1}; \]

and

\[ \overline{\partial}_{\{1\ldots\}}^{\cdots} | C^{\cdots} = (\overline{\partial} | C^{\cdots})_{\{1\ldots\}c^{\cdots}} : C^{\cdots} \rightarrow C^{\cdots+1}. \]

Then, for every \((p,q) \in \mathbb{Z}^2\), the induced map

\[ j: \left( C^{p-1,q-1} \oplus C^{p,q} \oplus C^{p+1,q} \oplus C^{p,q+1} \right) \rightarrow \left( A^{p-1,q-1} \oplus A^{p,q} \oplus A^{p+1,q} \oplus A^{p,q+1} \right) \]

of complexes induces an injective map \( j^* \) in cohomology.

Proof. Note that, if \( C^{\cdots} \subseteq \text{dom} \partial \cap \text{dom} \overline{\partial} \) is finite-dimensional, as in (ii) then the C-linear operators \( \partial | C^{\cdots}, C^{\cdots} \rightarrow C^{\cdots+1} \) and \( \overline{\partial} | C^{\cdots}, C^{\cdots} \rightarrow C^{\cdots+1} \) are continuous, and hence dom \( (\partial | C^{\cdots})_{\{1\ldots\}c^{\cdots}} = \text{dom}(\partial)^{\cdots} \subseteq C^{\cdots} \) and dom \( (\overline{\partial} | C^{\cdots})_{\{1\ldots\}c^{\cdots}} = \text{dom}(\overline{\partial})^{\cdots} \subseteq C^{\cdots} \). By hypothesis (iii) it follows that \( \Delta_{\{1\ldots\} BC} | C^{\cdots} = \Delta_{\{1\ldots\} BC}^{\cdots} \in \text{End}^{0,0}(C^{\cdots}) \). In particular, \( \Delta_{\{1\ldots\} BC} | C^{\cdots} = \Delta_{\{1\ldots\} BC}^{\cdots} = C^{\cdots} \).
Hence, in order to apply Theorem 1.6, it suffices to show that, given a finite-dimensional $C$-vector space $C$ endowed with an inner product $\langle \cdot | \cdot \rangle$, any self-$\langle \cdot | \cdot \rangle$-adjoint endomorphism $L \in \text{Hom}(C)$ yields a decomposition
\[ C = \ker L \oplus \text{im } L. \]
Indeed, take $\ker L \subseteq C$ and let $V \subseteq C$ be the $C$-vector sub-space of $C$ being $\langle \cdot | \cdot \rangle$-orthogonal to $\ker L$; in particular, $C = \ker L \oplus V$. It suffices to show that $V = \text{im } L$. Since $L$ is self-$\langle \cdot | \cdot \rangle$-adjoint, then $(\ker L \cap \text{im } L) = \{0\}$, and hence $\text{im } L \subseteq V$. Since $\dim_C C = \dim_C L + \dim_C \ker L < +\infty$, it follows that $V = \text{im } L$. 

Remark 1.9. Obviously, Theorem 1.6 as well as its corollaries, holds, with straightforward modifications, also for the cohomologies associated to the operators $\Delta_{\langle \cdot | \cdot \rangle} := [d, d^*]$, and $\square_{\langle \cdot | \cdot \rangle} := [\partial, \partial^*]$, and $\square_{\langle \cdot | \cdot \rangle} := [\overline{\partial}, \overline{\partial}^*]$, and $\Delta_{\langle \cdot | \cdot \rangle} := \partial^* + \overline{\partial}^* + (\partial \overline{\partial})^* (\partial \overline{\partial}) + (\partial \overline{\partial})^* (\partial \overline{\partial})^*$. 

2. Applications

We are now interested in applying the general results of the previous section to suitable sub-complexes of the double complex $(\Lambda^\bullet \Lambda^\bullet, X, \partial, \overline{\partial})$, where $X$ is a compact complex manifold. We are especially interested in the case when $X$ is a solvmanifold.

2.1. Complexes of PD-type. Let $(A^\bullet \bullet, \partial, \overline{\partial})$ be a double complex of $C$-vector spaces. Suppose that $A^\bullet \bullet$ have a structure $\wedge$ of $C$-algebra being compatible with the $\mathbb{Z}^2$-grading (namely, $A^{p,q} \wedge A^{p',q'} \subseteq A^{p+p',q+q'}$ for every $(p,q), (p',q') \in \mathbb{Z}^2$), and with respect to which $d := \partial + \overline{\partial}$ satisfies the Leibniz rule, namely,
\[
\text{for every } a \in \text{Tot}^0 A^\bullet \bullet, \quad [d, a \wedge \cdot] = d a \wedge \cdot \in \text{End}^{\infty+1}(\text{Tot}^\bullet A^\bullet \bullet).
\]
Following the notation introduced in [44, §2] by the second author, $(A^\bullet \bullet, \partial, \overline{\partial})$ is said to be a bi-differential $\mathbb{Z}^2$-graded algebra of PD-type if

(i) whenever $p < 0$ or $q < 0$, then $A^{p,q} = \{0\}$, and $A^{0,0} = C \langle 1 \rangle$;

(ii) there exists $n \in \mathbb{N}$ such that, whenever $p > n$ or $q > n$, then $A^{p,q} = \{0\}$, and $A^{n,n} = C \langle n \rangle$; (call $n$ the PD-dimension of $A^\bullet \bullet$);

(iii) for every $(h,k) \in \{(0, \ldots, n)^2 \}$, the bi-$C$-linear map $A^h,k \times A^n-h,n-k \rightarrow A^n \cong C$ induced by $\wedge$ is non-degenerate;

(iv) $d \text{Tot}^0 A^\bullet \bullet = \{0\}$ and $d \text{Tot}^{2n-1} A^\bullet \bullet = \{0\}$.

Given a bi-differential $\mathbb{Z}^2$-graded algebra $(A^\bullet \bullet, \partial, \overline{\partial})$ of PD-type, let $\langle \cdot | \cdot \rangle$ be an inner product on $A^\bullet \bullet$ being compatible with the $\mathbb{Z}^2$-grading, namely, $\langle A^{p,q} | A^{p',q'} \rangle = \{0\}$ whenever $(p,q) \neq (p',q')$, and being compatible with the PD-type structure, namely, $(\langle \cdot | \cdot \rangle \text{anti-linear map})$
\[
\bar{\bar{\sigma}}_{\langle \cdot | \cdot \rangle} : A^\bullet \bullet \rightarrow A^{n-n,n-n} \quad \text{such that for every } \alpha, \beta \in A^\bullet \bullet, \quad \alpha \wedge \bar{\bar{\sigma}}_{\langle \cdot | \cdot \rangle} \beta = (\alpha | \beta) \cdot v
\]
(as above, we will understand the scalar product $\langle \cdot | \cdot \rangle$ whenever it is clear from the context).

By considering the Hilbert space given by the $\langle \cdot | \cdot \rangle$-completion of $A^\bullet \bullet$, one has that the operators
\[
\partial^* := -\bar{\bar{\sigma}}_{\langle \cdot | \cdot \rangle} \partial \bar{\bar{\sigma}}_{\langle \cdot | \cdot \rangle} : A^\bullet \bullet \rightarrow A^{\bullet+1,\bullet} \quad \text{and} \quad \overline{\partial} := -\bar{\bar{\sigma}}_{\langle \cdot | \cdot \rangle} \overline{\partial} \bar{\bar{\sigma}}_{\langle \cdot | \cdot \rangle} : A^\bullet \bullet \rightarrow A^{\bullet-1,\bullet}
\]
are in fact the $\langle \cdot | \cdot \rangle$-adjoint operators $\partial^*_{\langle \cdot | \cdot \rangle}$, respectively $\overline{\partial}^*_{\langle \cdot | \cdot \rangle}$, of $\partial : A^\bullet \bullet \rightarrow A^{\bullet+1,\bullet}$, respectively $\overline{\partial} : A^\bullet \bullet \rightarrow A^{\bullet-1,\bullet}$, and the operators
\[
d^* := -\bar{\bar{\sigma}}_{\langle \cdot | \cdot \rangle} d \bar{\bar{\sigma}}_{\langle \cdot | \cdot \rangle} = \partial^* + \overline{\partial} : \text{Tot}^\bullet A^\bullet \bullet \rightarrow \text{Tot}^{\bullet-1} A^\bullet \bullet
\]
is in fact the $\langle \cdot | \cdot \rangle$-adjoint operator $d^*_{\langle \cdot | \cdot \rangle}$ of $d := \partial + \overline{\partial} : \text{Tot}^\bullet A^\bullet \bullet \rightarrow \text{Tot}^{\bullet+1} A^\bullet \bullet$, [44, Lemma 2.4].

The following result is an application of Corollary 1.8 to the case of bi-differential $\mathbb{Z}^2$-graded algebras of PD-type.

Proposition 2.1. Let $(A^\bullet \bullet, \partial, \overline{\partial})$ be a bi-differential $\mathbb{Z}^2$-graded algebra of PD-type of PD-dimension $n$. Let $\langle \cdot | \cdot \rangle$ be an inner product on $A^\bullet \bullet$ being compatible with the $\mathbb{Z}^2$-grading and with the PD-type structure. Consider the Hilbert space given by the $\langle \cdot | \cdot \rangle$-completion of $A^\bullet \bullet$, and suppose that the operator $\Delta_{\langle \cdot | \cdot \rangle}^{\overline{BC}} \in \text{End}^{0,0}(A^\bullet \bullet)$ induces the decomposition
\[
A^\bullet \bullet = \ker \Delta_{\langle \cdot | \cdot \rangle}^{BC} \oplus \text{im } \Delta_{\langle \cdot | \cdot \rangle}^{BC}.
\]
Let \((C^{\bullet, \bullet}, \partial, \overline{\partial}) \hookrightarrow (A^{\bullet, \bullet}, \partial, \overline{\partial})\) be a finite-dimensional sub-complex of \((A^{\bullet, \bullet}, \partial, \overline{\partial})\) having a structure of bi-differential \(Z^2\)-graded algebra of PD-type of PD-dimension \(n\) induced by \(A^{\bullet, \bullet}\). Suppose that
\[
\overline{\partial}_{(\cdot \mid \cdot)}|_{C^{\bullet, \bullet}} : C^{\bullet, \bullet} \to C^{n-\bullet, n-\bullet}.
\]

Then, for any \((p, q) \in \mathbb{Z}^2\), the induced inclusions
\[
(\text{Tot}^* (C^{\bullet, \bullet}), \partial + \overline{\partial}) \hookrightarrow (\text{Tot}^* A^{\bullet, \bullet}, \partial + \overline{\partial})
\]
and
\[
(C^{p, q}, \partial) \hookrightarrow (A^{p, q}, \partial)
\]
and
\[
(C^{p-1, q}, \partial + \overline{\partial}) \hookrightarrow (A^{p-1, q-1}, \partial + \overline{\partial})
\]
and
\[
(C^{p, q-1} \oplus C^{p+1, q+1}, \partial) \hookrightarrow (A^{p, q-1} \oplus A^{p, q}, \partial)
\]
to the \(\overline{\partial}\)-adjoint elliptic differential operators
\[
\text{Tot}^\ast \overline{\partial}
\]
\[
(A^{p, q-1} \oplus A^{p, q}, \partial)
\]
induce injective maps in cohomology.

**Proof.** By the hypothesis that
\[
\overline{\partial}_{(\cdot \mid \cdot)}|_{C^{\bullet, \bullet}} : C^{\bullet, \bullet} \to C^{n-\bullet, n-\bullet},
\]
then, for any \(\beta \in C^{\bullet, \bullet}\), it holds that
\[
(\overline{\partial}_{(\cdot \mid \cdot)}|_{C^{\bullet, \bullet}} \alpha - \overline{\partial}_{(\cdot \mid \cdot)}|_{C^{\bullet, \bullet}} \beta = 0; \text{ by taking }
\]
\[
(\overline{\partial}_{(\cdot \mid \cdot)}|_{C^{\bullet, \bullet}} \alpha - \overline{\partial}_{(\cdot \mid \cdot)}|_{C^{\bullet, \bullet}} \alpha = \overline{\partial}_{(\cdot \mid \cdot)}|_{C^{\bullet, \bullet}} \alpha),
\]
In particular, it follows that
\[
\partial_{(\cdot \mid \cdot)}|_{C^{\bullet, \bullet}} \partial_{(\cdot \mid \cdot)}|_{C^{\bullet, \bullet}} \partial_{(\cdot \mid \cdot)}|_{C^{\bullet, \bullet}} \partial_{(\cdot \mid \cdot)}|_{C^{\bullet, \bullet}} = (\partial_{(\cdot \mid \cdot)}|_{C^{\bullet, \bullet}} \partial_{(\cdot \mid \cdot)}|_{C^{\bullet, \bullet}} \partial_{(\cdot \mid \cdot)}|_{C^{\bullet, \bullet}} \partial_{(\cdot \mid \cdot)}|_{C^{\bullet, \bullet}})
\]
\[
\text{and }
\]
\[
\overline{\partial}_{(\cdot \mid \cdot)}|_{C^{\bullet, \bullet}} \overline{\partial}_{(\cdot \mid \cdot)}|_{C^{\bullet, \bullet}} \overline{\partial}_{(\cdot \mid \cdot)}|_{C^{\bullet, \bullet}} \overline{\partial}_{(\cdot \mid \cdot)}|_{C^{\bullet, \bullet}} = (\overline{\partial}_{(\cdot \mid \cdot)}|_{C^{\bullet, \bullet}} \overline{\partial}_{(\cdot \mid \cdot)}|_{C^{\bullet, \bullet}} \overline{\partial}_{(\cdot \mid \cdot)}|_{C^{\bullet, \bullet}} \overline{\partial}_{(\cdot \mid \cdot)}|_{C^{\bullet, \bullet}}).
\]

Hence Corollary \([13]\) see also Remark \([14]\) applies. \(\square\)

2.2. **Compact complex manifolds.** Let \(X\) be a compact complex manifold of complex dimension \(n\) endowed with a Hermitian metric \(g\). (Note that all manifolds are assumed to have no boundary.)

By considering the (\(C\)-anti-linear) Hodge-\(s\)-operator
\[
\tilde{\ast}_g : \wedge^{\ast, \ast} X \to \wedge^{n-\ast, n-\ast} X
\]
and the inner product
\[
\langle \cdot \mid \cdot \rangle := \int_X \langle \cdot \rangle \wedge \tilde{\ast}_g \langle \cdot \rangle ,
\]
one gets that the double complex \((\wedge^{\ast, \ast} X, \partial, \overline{\partial})\) has a structure of bi-differential \(Z^2\)-graded algebra of PD-type of PD-dimension \(n\), such that \((\cdot \mid \cdot)\) is compatible with the \(Z^2\)-grading and with the PD-type structure of \(\wedge^{\ast, \ast} X\).

The 2\text{nd} order self-\((\cdot \mid \cdot)\)-adjoint elliptic differential operators
\[
\Delta_g := [d, d^\ast] \in \operatorname{End}^0 (\wedge^{\ast, \ast} X \otimes \mathbb{C}) ,
\]
and
\[
\square_g := [\partial, \partial^\ast] \in \operatorname{End}^{0,0} (\wedge^{\ast, \ast} X) , \quad \square_g := \left( \overline{\partial}, \overline{\partial} \right) \in \operatorname{End}^{0,0} (\wedge^{\ast, \ast} X) ,
\]
and the 4\text{th} order self-\((\cdot \mid \cdot)\)-adjoint elliptic differential operators, \([16]\) Proposition \(5\), \([16]\) \(\S 2.6, \S 2.6\),
\[
\overline{\Delta}_g^{BC} := \left( \overline{\partial} \partial \right)^\ast + \left( \partial \overline{\partial} \right)^\ast + \left( \overline{\partial} \overline{\partial} \right)^\ast + \left( \overline{\partial} \overline{\partial} \right)^\ast + \left( \overline{\partial} \overline{\partial} \right)^\ast + \left( \overline{\partial} \overline{\partial} \right)^\ast + \left( \overline{\partial} \overline{\partial} \right)^\ast + \left( \overline{\partial} \overline{\partial} \right)^\ast
\]
and
\[
\Delta_g^{A} := \partial \partial^\ast + \overline{\partial} \overline{\partial}^\ast + \left( \partial \overline{\partial} \right)^\ast + \left( \overline{\partial} \partial \right)^\ast + \left( \partial \overline{\partial} \right)^\ast + \left( \overline{\partial} \partial \right)^\ast + \left( \partial \overline{\partial} \right)^\ast + \left( \overline{\partial} \partial \right)^\ast
\]
and
\[
, \text{from now on, the metric } g \text{ will be understood whenever it is clear from the context,) induce the } (\cdot \mid \cdot)\text{-orthogonal decompositions, \([15]\) page 450},
\]
\[
\wedge^{\ast, \ast} X \otimes \mathbb{R} \mathbb{C} = \ker \Delta \oplus \im \Delta = \ker \Delta \oplus \im d \oplus \im d^\ast
\]
and
\[
\wedge^{\ast, \ast} X = \ker \square \oplus \im \square = \ker \square \oplus \im \partial \oplus \im \partial^\ast
\]
\[
= \ker \square \oplus \im \overline{\partial} \oplus \im \overline{\partial}^\ast ,
\]
and, \[ \text{[63 Théorème 2.2, §2.c]}, \]

\[
\wedge \bullet X = \ker \Delta_{BC} \oplus \im \Delta_{BC} = \ker \Delta_{BC} \oplus \im \partial \oplus \left( \im \partial^* + \im \overline{\partial} \right)
\]

\[
= \ker \Delta^A \oplus \im \Delta^A = \ker \Delta^A \oplus \left( \im \partial + \im \overline{\partial} \right) \oplus \im \left( \partial \overline{\partial} \right)^*.
\]

In particular, by arguing as in Lemma \[ \text{[63 Corollaire 2.3, §2.c]}, \]

\[
H_{BC}^{•,•}(X; \mathbb{C}) := \frac{\ker \partial}{\im \partial} \simeq \ker \Delta, \quad H_{\partial}^{•,•}(X) := \frac{\ker \partial}{\im \partial} \simeq \ker \Box, \quad H_{\overline{\partial}}^{•,•}(X) := \frac{\ker \overline{\partial}}{\im \overline{\partial}} \simeq \ker \Box,
\]

and, \[ \text{[63 Corollaire 2.3, §2.c]}, \]

\[
H_{BC}^{•,•}(X) := \frac{\ker \partial \cap \ker \overline{\partial}}{\im \partial \cap \im \overline{\partial}} \simeq \ker \Delta_{BC}, \quad H_{A}^{•,•}(X) := \frac{\ker \partial \cap \ker \overline{\partial}}{\im \partial + \im \overline{\partial}} \simeq \ker \Delta^A.
\]

Note that \( \bar{s}_g \circ \Delta_{BC} = \Delta^A \circ \bar{s}_g \), and hence the Hodge-\( s \)-operator induces the isomorphism

\[
H_{BC}^{•,•}(X) \cong H_{A}^{•,•,\text{odd}}(X).
\]

In particular, by Proposition \[ \text{[2.1]} \] one gets straightforwardly the following result, which provides a condition under which the Bott-Chern cohomology of a finite-dimensional sub-complex \( \wedge \bullet X \) is a subgroup of \( H_{BC}^{•,•}(X) \). Such a result will be applied in the next section with the aim to study the Bott-Chern cohomology of a certain class of solvmanifolds.

**Proposition 2.2.** Let \( X \) be a compact complex manifold of complex dimension \( n \) endowed with a Hermitian metric \( g \). Let \( (\wedge\bullet X, \partial, \overline{\partial}) \) be a finite-dimensional sub-complex of \( (\wedge \bullet X, \partial, \overline{\partial}) \) having a structure of bi-differential \( \mathbb{Z}^2 \)-graded algebra of PD-type of PD-dimension \( n \) induced by \( \wedge \bullet X \). Suppose that

\[
\bar{s}_g \mid_{C^{•,•}} : C^{•,•} \to C^{n-\bullet,n-•}.
\]

Then, for any \( (p,q) \in \mathbb{Z}^2 \), the induced inclusions

\[
(\text{Tot}^•(C^{p,•}), \partial + \overline{\partial}) \hookrightarrow (\wedge p X \otimes_\mathbb{R} \mathbb{C}, \partial), \quad (C^{•,q}, \partial) \hookrightarrow (\wedge • q X, \partial), \quad (C^{p,•}, \overline{\partial}) \hookrightarrow (\wedge p • X, \overline{\partial}),
\]

and

\[
\left( C^{p-1,q-1} \otimes \overline{\partial} \right) \subset C^{p,q} \otimes (\partial X) \quad \hookrightarrow \quad \left( \wedge p,q-1 X \otimes \overline{\partial} \right) \subset \wedge p,q X \otimes (\partial X)
\]

and

\[
\left( C^{p-1,q} \oplus C^{p,q-1} \right) \otimes (\partial X) \quad \hookrightarrow \quad \left( \wedge p,q-1 X \oplus \wedge p,q X \right) \otimes (\partial X)
\]

induce injective maps in cohomology.

**Proof.** The proof follows straightforwardly by \[ \text{[61 Théorème 2.2, §2.c]} \] and \[ \text{[63, page 450]} \], and by Proposition \[ \text{[2.1]} \].

\[ \square \]

**Remark 2.3.** By applying Corollary \[ \text{[17]} \] to the \( \langle \cdot, \cdot \rangle \)-completion of \( \wedge \bullet X \), the same conclusion of Proposition \[ \text{[2.2]} \] holds true for a (possibly non-finite-dimensional) closed sub-complex \( (C^{•,•}, \partial, \overline{\partial}) \hookrightarrow (\wedge \bullet X, \partial, \overline{\partial}) \) such that \( \pi C^{•,•} \circ \partial = \partial \circ \pi C^{•,•} : \wedge • X \to C^{•,•} \) and \( \pi C^{•,•} \circ \overline{\partial} = \overline{\partial} \circ \pi C^{•,•} : \wedge • X \to C^{•,•} \).

In order to study cohomologies of solvmanifolds, we need also the following result.

To simplify the notation, we say that a sub-complex \( (C^{•,•}, \partial, \overline{\partial}) \hookrightarrow (\wedge \bullet X, \partial, \overline{\partial}) \) suffices in computing the de Rham, respectively conjugate Dolbeault, respectively Dolbeault, respectively Bott-Chern, respectively Aeppli cohomology of \( X \) if the induced inclusion

\[
(\text{Tot}^•(C^{p,•}), \partial + \overline{\partial}) \hookrightarrow (\wedge p X \otimes_\mathbb{R} \mathbb{C}, \partial),
\]

respectively, for any \( q \in \mathbb{N} \),

\[
(C^{•,q}, \partial) \hookrightarrow (\wedge • q X, \partial),
\]

respectively, for any \( p \in \mathbb{N} \),

\[
(C^{p,•}, \overline{\partial}) \hookrightarrow (\wedge p • X, \overline{\partial}),
\]

respectively, for any \( (p,q) \in \mathbb{Z}^2 \),

\[
\left( C^{p-1,q-1} \otimes \overline{\partial} \right) \subset C^{p,q} \otimes (\partial X) \quad \hookrightarrow \quad \left( \wedge p,q-1 X \otimes \overline{\partial} \right) \subset \wedge p,q X \otimes (\partial X)
\]

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Lemma 2.6 (see [11, Proposition 5.4]). Easily check that the lemma holds in the general case.

Lemma 2.5. Let \( \mathfrak{g} \) be a solvmanifold, and consider the F. A. Belgun symmetrization map \( \varepsilon : \mathfrak{g} \to \mathfrak{g} \). Let \( (B^{\bullet, \bullet}, \partial, \overline{\partial}) \) be a sub-complex of \((C^{\bullet, \bullet}, \partial, \overline{\partial})\) having a structure of bi-differential type of \( Z^2 \)-graded algebra of \( \mathfrak{p} \mathfrak{d} \)-type of \( \mathfrak{p} \mathfrak{d} \)-dimension \( n \) induced by \( C^{\bullet, \bullet} \) and such that

\[
\varepsilon_g \left[ C^{\bullet, \bullet} : C^{\bullet, \bullet} \to C^{\bullet, \bullet} \right].
\]

If \((B^{\bullet, \bullet}, \partial, \overline{\partial})\) satisfies in computing the cohomologies of \( X \), then also \((C^{\bullet, \bullet}, \partial, \overline{\partial})\) satisfies in computing the corresponding cohomologies of \( X \).

Proof. By Proposition 2.4 and Proposition 2.2 both the inclusions \( B^{\bullet, \bullet} \hookrightarrow C^{\bullet, \bullet} \) and \( C^{\bullet, \bullet} \hookrightarrow \Lambda^{\bullet, \bullet} X \) induce injective maps in cohomology, whose composition is an isomorphism by the hypothesis.

2.3. Complex nilmanifolds. Let \( X = \Gamma \backslash G \) be a solvmanifold (respectively, a nilmanifold), namely, a compact quotient of a connected simply-connected solvable (respectively, nilpotent) Lie group \( G \) by a co-compact discrete subgroup \( \Gamma \), endowed with a \( G \)-left-invariant (almost-)complex structure \( J \). We recall that a solvmanifold is called completely-solvable if, for any \( g \in G \), all the eigenvalues of \( \text{Ad}_g := d (\psi)_g \in \text{Aut}(\mathfrak{g}) \) are real, equivalently, for any \( X \in \mathfrak{g} \), all the eigenvalues of \( \text{ad}_X := [X, \cdot] \in \text{End}(\mathfrak{g}) \) are real, where \( \psi : g \mapsto (\psi_g : h \mapsto ghg^{-1}) \in \text{Aut}(G) \) and \( e \) is the identity element of \( G \).

Recall that, by J. Milnor’s Lemma [52, Lemma 6.2], \( G \) is unimodular (that is, \( \det(\text{Ad}_g) = 1 \) for any \( g \in G \)), and hence, in particular, there exists a \( G \)-bi-invariant volume form \( \eta \) on \( X \) such that \( \int_X \eta = 1 \). Therefore, consider the F. A. Belgun symmetrization map in [14] Theorem 7, namely,

\[
\mu : \Lambda^{\bullet} X \otimes_{\mathbb{R}} \mathbb{C} \to \Lambda^{\bullet} (\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C})^*, \quad \mu(\alpha) := \int_X \alpha|_x \eta(x).
\]

Note, [14] Theorem 7, that \( \mu \) commutes with \( d \) and with \( J \), and hence also with \( \partial \) and \( \overline{\partial} \), and that \( \mu|_{\Lambda^{\bullet} (\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C})^*} = \text{id}_{\Lambda^{\bullet} (\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C})^*} \).

Lemma 2.5. Let \( \Gamma \backslash G \) be a solvmanifold, and consider the F. A. Belgun symmetrization map \( \mu : \Lambda^{\bullet} X \otimes_{\mathbb{R}} \mathbb{C} \to \Lambda^{\bullet} (\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C})^* \) in [14] Theorem 7. For a \( G \)-left-invariant differential form \( \theta \) on \( \Gamma \backslash G \) and for a differential form \( \omega \) on \( \Gamma \backslash G \), we have

\[
\mu(\theta \wedge \omega) = \theta \wedge \mu(\omega).
\]

Proof. Suppose that \( \theta \) is a \( G \)-left-invariant 1-form on \( \Gamma \backslash G \). Let \( \omega \) be a \( p \)-form on \( \Gamma \backslash G \). Then for \( X_1, \ldots, X_{p+1} \in \mathfrak{g} \), since \( \theta(X_j) \) is constant for every \( j \in \{1, \ldots, p+1\} \), we have

\[
\mu(\theta \wedge \omega)(X_1, \ldots, X_{p+1}) = \int_{\Gamma \backslash G} \sum_{\sigma \in \mathcal{S}_{p+1}} \theta_x (X_{\sigma(1)}) \cdot \omega (X_{\sigma(2)}, \ldots, X_{\sigma(p+1)}) \eta(x)
\]

\[
= \sum_{\sigma \in \mathcal{S}_{p+1}} \theta (X_{\sigma(1)}) \cdot \int_{\Gamma \backslash G} \omega_x (X_{\sigma(2)}, \ldots, X_{\sigma(p+1)}) \eta(x)
\]

\[
= (\theta \wedge \mu(\omega))(X_1, \ldots, X_{p+1}),
\]

where \( \mathcal{S}_{p+1} \) is the set of permutations of \( p+1 \) elements. Hence, in this case, the lemma holds. We can easily check that the lemma holds in the general case.

Lemma 2.6 (see [17] Proposition 5.4). Let \( X = \Gamma \backslash G \) be a completely-solvable solvmanifold endowed with a \( G \)-left-invariant complex structure \( J \). Consider the sub-complex

\[
j : (\Lambda^{\bullet} (\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C})^*, d) \hookrightarrow (\Lambda^{\bullet} X \otimes_{\mathbb{R}} \mathbb{C}, d),
\]
which is a quasi-isomorphism by A. Hattori’s theorem [37, Corollary 4.2]. The induced map
\[
j : \ker (d : \wedge^{p+q} (g \otimes \mathbb{R} C)^* \to \wedge^{p+q+1} (g \otimes \mathbb{R} C)^*) \cap \wedge^{p,q} g^* \to \ker (d : \wedge^{p+q-1} (g \otimes \mathbb{R} C)^* \to \wedge^{p+q} (g \otimes \mathbb{R} C)^*) \cap \wedge^{p,q} X\]
is an isomorphism.

**Proof.** For the sake of completeness, we recall here the argument of the proof (note that the statement holds, more in general, in the almost-complex setting).

The F. A. Belgun symmetrization map \(\mu : \wedge^* X \otimes \mathbb{R} C \to \wedge^* (g \otimes \mathbb{R} C)^*\) induces the map
\[
\mu : \ker (d : \wedge^{p+q} X \otimes \mathbb{R} C \to \wedge^{p+q+1} X \otimes \mathbb{R} C) \cap \wedge^{p,q} X \to \ker (d : \wedge^{p+q-1} (g \otimes \mathbb{R} C)^* \to \wedge^{p+q} (g \otimes \mathbb{R} C)^*) \cap \wedge^{p,q} X.
\]
Hence, one gets the commutative diagram
\[
\begin{array}{ccc}
\ker (d : \wedge^{p+q} (g \otimes \mathbb{R} C)^* \to \wedge^{p+q+1} (g \otimes \mathbb{R} C)^*) \cap \wedge^{p,q} (g \otimes \mathbb{R} C)^* & \overset{(j)}{\longrightarrow} & \ker (d : \wedge^{p+q-1} (g \otimes \mathbb{R} C)^* \to \wedge^{p+q} (g \otimes \mathbb{R} C)^*) \cap \wedge^{p,q} X \\
\overset{\mu}{\downarrow} & & \downarrow \mu \\
\ker (d : \wedge^{p+q} X \otimes \mathbb{R} C \to \wedge^{p+q+1} X \otimes \mathbb{R} C) \cap \wedge^{p,q} X & \overset{(id)}{\longrightarrow} & \ker (d : \wedge^{p+q-1} (g \otimes \mathbb{R} C)^* \to \wedge^{p+q} (g \otimes \mathbb{R} C)^*) \cap \wedge^{p,q} X
\end{array}
\]
from which one gets that
\[
j : \ker (d : \wedge^{p+q} X \otimes \mathbb{R} C \to \wedge^{p+q+1} X \otimes \mathbb{R} C) \cap \wedge^{p,q} X \to \ker (d : \wedge^{p+q-1} (g \otimes \mathbb{R} C)^* \to \wedge^{p+q} (g \otimes \mathbb{R} C)^*) \cap \wedge^{p,q} X
\]
is injective, and that
\[
\mu : \ker (d : \wedge^{p+q} X \otimes \mathbb{R} C \to \wedge^{p+q+1} X \otimes \mathbb{R} C) \cap \wedge^{p,q} X \to \ker (d : \wedge^{p+q-1} (g \otimes \mathbb{R} C)^* \to \wedge^{p+q} (g \otimes \mathbb{R} C)^*) \cap \wedge^{p,q} X
\]
is surjective.

Moreover, since \(j : (\wedge^* (g \otimes \mathbb{R} C)^*, d) \to (\wedge^* X \otimes \mathbb{R} C, d)\) is a quasi-isomorphism by A. Hattori’s theorem [37, Theorem 4.2], one gets that \(\mu : H^*_d X(\mathbb{C}) \to H^* (\wedge^* (g \otimes \mathbb{R} C)^*, d)\) is in fact the identity map, and hence
\[
\mu : \ker (d : \wedge^{p+q} X \otimes \mathbb{R} C \to \wedge^{p+q+1} X \otimes \mathbb{R} C) \cap \wedge^{p,q} X \to \ker (d : \wedge^{p+q-1} (g \otimes \mathbb{R} C)^* \to \wedge^{p+q} (g \otimes \mathbb{R} C)^*) \cap \wedge^{p,q} X
\]
is also injective.

Since \(X\) is compact, the dimension of \(H^*_d X(\mathbb{C})\) is finite, and hence \(\mu\) is in fact an isomorphism. □

As an application of Theorem 1.3 and Proposition 2.2, one recovers the following results, concerning the Bott-Chern cohomology of nilmanifolds. (We refer to [71, 51, 13, 3, 25, 22, 50, 62] for definitions and notation.)
Corollary 2.7 ([4, Theorem 3.8]). Let $X = \Gamma \backslash G$ be a nilmanifold endowed with a $G$-left-invariant complex structure $J$, and denote the Lie algebra naturally associated to $G$ by $\mathfrak{g}$. Suppose that one of the following conditions holds:

- $X$ is complex parallelizable;
- $J$ is an Abelian complex structure;
- $J$ is a nilpotent complex structure;
- $J$ is a rational complex structure;
- $\mathfrak{g}$ admits a torus-bundle series compatible with $J$ and with the rational structure induced by $\mathfrak{g}$;
- $\dim_{\mathbb{R}} \mathfrak{g} = 6$ and $\mathfrak{g}$ is not isomorphic to $\mathfrak{g}_7 := (\mathbb{R}^3, 12, 13, 23)$.

Then the inclusion $j: \left(\Lambda^{\bullet,\bullet} (\mathfrak{g} \otimes \mathbb{C})^*, \partial, \bar{\partial} \right) \hookrightarrow \left(\Lambda^{\bullet,\bullet} X, \partial, \bar{\partial} \right)$ induces the isomorphisms

$$H_{B\mathfrak{C}}^{\bullet,\bullet}(X) \cong \ker \left( d: \Lambda^{\bullet,\bullet} (\mathfrak{g} \otimes \mathbb{C})^* \to \Lambda^{\bullet+1,\bullet+1} (\mathfrak{g} \otimes \mathbb{C})^* \right) \text{ and }$$

$$H_A^{\bullet,\bullet}(X) \cong \ker \left( \partial: \Lambda^{\bullet,\bullet} (\mathfrak{g} \otimes \mathbb{C})^* \to \Lambda^{\bullet+1,\bullet+1} (\mathfrak{g} \otimes \mathbb{C})^* \right) + \text{im} \left( \bar{\partial}: \Lambda^{\bullet+1,\bullet+1} (\mathfrak{g} \otimes \mathbb{C})^* \to \Lambda^{\bullet,\bullet} (\mathfrak{g} \otimes \mathbb{C})^* \right).$$

Proof. Choose a $G$-left-invariant Hermitian metric $\mathfrak{g}$ on $X$. The sub-complex $\left(\Lambda^{\bullet,\bullet} (\mathfrak{g} \otimes \mathbb{C})^*, \partial, \bar{\partial} \right)$ being finite-dimensional, the induced maps in Bott-Chern, respectively Aeppli cohomologies are injective by Proposition 2.2.

Under the hypothesis, by [51, Theorem 1], [25, Main Theorem], [22, Theorem 2, Remark 4], [50, Theorem 1.10], and [60, Corollary 3.10], one has that, for any fixed $p \in \mathbb{N}$, the induced map

$$j: \left(\Lambda^{p,\bullet} (\mathfrak{g} \otimes \mathbb{C})^*, \partial \right) \to \left(\Lambda^{p,\bullet} X, \partial \right)$$

is a quasi-isomorphism. By conjugation, one has also that, for any fixed $q \in \mathbb{N}$, the induced map

$$j: \left(\Lambda^{\bullet,q} (\mathfrak{g} \otimes \mathbb{C})^*, \partial \right) \to \left(\Lambda^{\bullet,q} X, \partial \right)$$

is a quasi-isomorphism. Lastly, condition (iii) in Theorem 1.3 is satisfied by Lemma 2.6. Hence, by Theorem 1.3, the induced map in Bott-Chern cohomology is surjective.

As regards Aeppli cohomologies, it suffices to note that the Hodge-$*$-operator $\ast_{\mathfrak{g}}$ induces the isomorphisms $H_{B\mathfrak{C}}^{\bullet,\bullet}(X) \cong H_A^{\bullet,\bullet}(X)$ and $\frac{\ker d \circ \ast_{\mathfrak{g}} (\mathfrak{g} \otimes \mathbb{C})^*}{\text{im} \partial} \cong \frac{\ker \partial \circ \ast_{\mathfrak{g}} (\mathfrak{g} \otimes \mathbb{C})^*}{\text{im} \partial + \text{im} \bar{\partial}}$, where $n$ is the complex dimension of $X$. 

The previous result can be used to compute the cohomology of the left-invariant complex structures classified by M. Ceballos, A. Otal, L. Ugarte, and R. Villacampa in [20], as in [6] and [44].

2.4. Complex solvmanifolds. Let $G$ be a connected simply-connected $n$-dimensional solvable Lie group admitting a discrete co-compact subgroup $\Gamma$, and denote by $\mathfrak{g}$ the (solvable) Lie algebra of $G$. Set $\mathfrak{g}_C := \mathfrak{g} \otimes \mathbb{C}$.

Consider the adjoint action

$$\text{ad}: \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}) \; ; \; \text{ad}_X := [X, \cdot] ;$$

by denoting by $\text{Der}(\mathfrak{g}) := \{ D \in \mathfrak{gl}(\mathfrak{g}) : \forall X \in \mathfrak{g}, [D, \text{ad}_X] = \text{ad}_DX \}$ the $\mathbb{R}$-vector space of derivations of $\mathfrak{g}$, one has that $\text{ad}(\mathfrak{g}) \subseteq \text{Der}(\mathfrak{g})$. One has that every derivation $\text{ad}_X$, for $X \in \mathfrak{g}$, admits a unique Jordan decomposition, see, e.g., [32, 2.1.10], namely,

$$\text{ad}_X = (\text{ad}_X)_s + (\text{ad}_X)_n \; ,$$

where $(\text{ad}_X)_s \in \mathfrak{gl}(\mathfrak{g})$ is semi-simple (that is, each $(\text{ad}_X)_s$-invariant sub-space of $\mathfrak{g}$ admits an $(\text{ad}_X)_s$-invariant complementary sub-space in $\mathfrak{g}$), and $(\text{ad}_X)_n \in \mathfrak{gl}(\mathfrak{g})$ is nilpotent (that is, there exists $N \in \mathbb{N}$ such that $(\text{ad}_X)_n^N = 0$).

Let $\mathfrak{n}$ be the nilradical of $\mathfrak{g}$, that is, the maximal nilpotent ideal in $\mathfrak{g}$. Since $\mathfrak{g}$ is solvable, there exists an $\mathbb{R}$-vector sub-space $V$ (which is not necessarily a Lie algebra) of $\mathfrak{g}$ so that (i) $\mathfrak{g} = V \oplus \mathfrak{n}$ as the direct sum of $\mathbb{R}$-vector spaces, and, (ii) for any $A, B \in V$, it holds that $(\text{ad}_A)_n(B) = 0$, see, e.g., [32, Proposition II 1.1.1]. Hence, one can define the map

$$\text{ad}_n: \mathfrak{g} \to \text{Der}(\mathfrak{g}) \; ; \; \mathfrak{g} = V \oplus \mathfrak{n} \ni (A, X) \mapsto (\text{ad}_n)_A + X := (\text{ad}_A)_s \in \text{Der}(\mathfrak{g}) .$$

Moreover, one has that $\langle \text{ad}_n(\mathfrak{g}), \text{ad}_n(\mathfrak{g}) \rangle = \{0\}$, and (iv) $\text{ad}_n: \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is $\mathbb{R}$-linear, see, e.g., [32, Proposition III 1.1].
Since we have \([\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{n}\), see, e.g., [22 II.1.9], and \(ad_\mathfrak{n}(\mathfrak{n}) = \{0\}\), the map \(ad_\mathfrak{g}: \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})\) is a representation of \(\mathfrak{g}\), whose image \(ad_\mathfrak{g}(\mathfrak{g})\) is Abelian and consists of semi-simple elements. Hence, denote by
\[ Ad_\mathfrak{g}: G \to \text{Aut}(\mathfrak{g}), \quad \text{respectively } Ad_\mathfrak{g}: G \to \text{Aut}(\mathfrak{g}_C), \]
the unique representation which lifts \(ad_\mathfrak{g}: \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})\), see, e.g., [22 Theorem 3.27], respectively the natural \(C\)-linear extension.

Let \(T\) be the Zariski-closure of \(Ad_\mathfrak{g}(G)\) in \(\text{Aut}(\mathfrak{g}_C)\). Denote by \(\text{Char}(T) := \text{Hom}(T; \mathbb{C}^*)\) the set of all 1-dimensional algebraic group representations of \(T\). Set
\[ C_T := \{\beta \circ Ad_\mathfrak{g} \in \text{Hom}(G; \mathbb{C}^*) : \beta \in \text{Char}(T), (\beta \circ Ad_\mathfrak{g})|_{r=1}\}. \]

We consider the differential graded sub-algebra
\[ \bigoplus_{\alpha \in C_T} \mathfrak{g}_C^\alpha \]
of \(\bigwedge^\ast \Gamma \setminus G \otimes_R \mathbb{C}\). (Note that we have used left-translations on \(G\) to identify the elements of \(\bigwedge^\ast \mathfrak{g}_C^\alpha\) with the \(G\)-left-invariant complex forms in \(\bigwedge^\ast \Gamma \setminus G \otimes_R \mathbb{C}\), namely, the complex forms being invariant for the action of the Lie group \(G\) on \(\Gamma \setminus G\) given by left-translations.) By \(\text{Ad}_\mathfrak{g}(G) \subseteq \text{Aut}(\mathfrak{g}_C)\) we have the \(Ad_\mathfrak{g}(G)\)-action on the differential graded algebra \(\bigoplus_{\alpha \in C_T} \mathfrak{g}_C^\alpha\). We denote by \(A^\ast_T\) the space consisting of the \(\text{Ad}_\mathfrak{g}(G)\)-invariant elements of \(\bigoplus_{\alpha \in C_T} \mathfrak{g}_C^\alpha\), namely,
\[ (1) \quad A^\ast_T := \left\{ \varphi \in \bigoplus_{\alpha \in C_T} \mathfrak{g}_C^\alpha : (Ad_\mathfrak{g})_g(\varphi) = \varphi \text{ for every } g \in G \right\}. \]

Now we consider the inclusion
\[ A^\ast_T \subseteq \bigwedge^\ast \Gamma \setminus G \otimes_R \mathbb{C} \]
of differential graded algebras. We have the following result.

**Theorem 2.8** ([33 Corollary 7.6]). Let \(\Gamma \setminus G\) be a solvmanifold, and consider \(A^\ast_T\) as defined in (1).

Then the inclusion
\[ (A^\ast_T, d) \hookrightarrow (\bigwedge^\ast \Gamma \setminus G \otimes_R \mathbb{C}, d) \]
of differential graded algebras induces an isomorphism in cohomology.

Note that \(\text{Ad}_\mathfrak{g}(G) \subseteq \text{Aut}(\mathfrak{g}_C)\) consists of simultaneously diagonalizable elements. Let \(\{X_1, \ldots, X_n\}\) be a basis of \(\mathfrak{g}_C\) with respect to which
\[ \text{Ad}_\mathfrak{g} = \text{diag}(\alpha_1, \ldots, \alpha_n): G \to \text{Aut}(\mathfrak{g}_C) \]
for some characters
\[ \alpha_1 \in \text{Hom}(G; \mathbb{C}^*), \ldots, \alpha_n \in \text{Hom}(G; \mathbb{C}^*). \]

Let \(\{x_1, \ldots, x_n\}\) be the dual basis of \(\mathfrak{g}_C^\ast\) of \(\{X_1, \ldots, X_n\}\). For the basis \(\{x_1 \wedge \cdots \wedge x_p\}_{1 \leq i_1 < \cdots < i_p \leq n}\) of \(\bigwedge^p \mathfrak{g}_C^\ast\), for \(\alpha \in C_T\), we have
\[ (Ad_\mathfrak{g})_g(\alpha x_1 \wedge \cdots \wedge x_p) = \alpha(g) \alpha_1^{-1}(g) \alpha x_1 \wedge \cdots \wedge x_p, \]
where we have shortened \(\alpha_{i_1 \cdots i_p} := \alpha_{i_1} \cdots \alpha_{i_p} \in \text{Hom}(G; \mathbb{C}^*)\). Then the basis
\[ \{\alpha x_1 \wedge \cdots \wedge x_p : 1 \leq i_1 < i_2 < \cdots < i_p \leq n \text{ and } \alpha \in C_T\} \]
of \(\bigoplus_{\alpha \in C_T} \alpha \cdot \bigwedge^p \mathfrak{g}_C^\ast\) diagonalizes the \(\text{Ad}_\mathfrak{g}(G)\)-action, and \(\alpha x_1 \wedge \cdots \wedge x_p \in A^p_T\) if and only if \(\alpha = \alpha_{i_1 \cdots i_p}\) and \(\alpha_{i_1 \cdots i_p}|_{r=1}\). Hence the differential graded algebra \(A^p_T\) is written as
\[ (2) \quad A^p_T := \bigoplus_{\alpha \in C_T} \langle \alpha_{i_1 \cdots i_p} x_{i_1} \wedge \cdots \wedge x_{i_p} : 1 \leq i_1 < i_2 < \cdots < i_p \leq n \text{ such that } \alpha_{i_1 \cdots i_p}|_{r=1}\rangle. \]

In fact, the following result holds.

**Theorem 2.9.** Let \(\Gamma \setminus G\) be a solvmanifold. Let \(\{X_1, \ldots, X_n\}\) be a basis of the \(\mathbb{C}\)-vector space \(\mathfrak{g}_C\) with respect to which \(\text{Ad}_\mathfrak{g} = \text{diag}(\alpha_1, \ldots, \alpha_n)\) for some characters \(\alpha_1, \ldots, \alpha_n \in \text{Hom}(G; \mathbb{C}^*)\). Consider the finite set of characters
\[ \mathcal{A}_T := \{\alpha_{i_1 \cdots i_p} \in \text{Hom}(G; \mathbb{C}^*) : 1 \leq i_1 < i_2 < \cdots < i_p \leq n \text{ such that } \alpha_{i_1 \cdots i_p}|_{r=1}\}. \]

Then the sub-complex
\[ \iota: \left( \bigoplus_{\alpha \in \mathcal{A}_T} \alpha \cdot \bigwedge^\ast \mathfrak{g}_C^\ast, d \right) \hookrightarrow (\bigwedge^\ast \Gamma \setminus G \otimes_R \mathbb{C}, d) \]
induces an isomorphism in cohomology.

Suppose furthermore that $G$ is endowed with a $G$-left-invariant complex structure. Consider the bi-

graded $\mathbb{C}$-vector sub-space 

$$
\iota: \bigoplus_{\alpha \in \mathcal{A}_G} \alpha \cdot \Lambda^{p,q}_G \ni \alpha \cdot \Lambda^{p,q}_G \Gamma \setminus G ;
$$

then $\iota$ induces, for any $(p,q) \in \mathbb{Z}^2$, the isomorphism

$$
\iota^*: \frac{\ker d|_{\bigoplus_{\alpha \in \mathcal{A}_G} \alpha \cdot \Lambda^{p,q}_G}}{d \bigg( \bigoplus_{\alpha \in \mathcal{A}_G} \alpha \cdot \Lambda^{p,q-1}_G \bigg)} \cong \frac{\ker d|_{\Lambda^{p,q-1}_G \Gamma \setminus G}}{d \big( \Lambda^{p,q-1}_G \Gamma \setminus G \otimes \mathbb{R} \big)} .
$$

Proof. Consider the $G$-left-invariant Hermitian metric

$$
g := \sum_{j=1}^n x_j \otimes \bar{x}_j
$$
on $\Gamma \setminus G$, and the associated $\mathbb{C}$-anti-linear Hodge-$*$-operator $\tilde{\ast}_\rho: \Lambda^* \Gamma \setminus G \otimes \mathbb{C} \to \Lambda^{n-*} \Gamma \setminus G \otimes \mathbb{C}$, where $n$ is the dimension of $\Gamma \setminus G$. If the restriction of a character $\alpha$ of $G$ on $\Gamma$ is trivial, then $\alpha$ induces a function on $\Gamma \setminus G$ and the image $\alpha(G)$ is a compact subgroup of $\mathbb{C}^*$, and hence $\alpha$ is unitary. For $\alpha_1, \ldots, \alpha_p \in \mathcal{A}_G$, since $G$ is unimodular, $[22$ Lemma 6.2$]$, for the complement $\{j_1, \ldots, j_{n-p}\} := \{1, \ldots, n\} \setminus \{i_1, \ldots, i_p\}$ we have

$$
\tilde{\ast}_\rho (\alpha_1, \ldots, \alpha_p, x_{i_1} \wedge \cdots \wedge x_{i_p}) = \alpha_{j_1}, \ldots, \alpha_{j_{n-p}}.
$$

By this, we have

$$
\tilde{\ast}_\rho (\alpha_i, \ldots, \alpha_{i_p}, x_{i_1} \wedge \cdots \wedge x_{i_p}) = \alpha_{j_1}, \ldots, \alpha_{j_{n-p}}, x_{j_1}, \ldots, x_{j_{n-p}} \in A_{h}^{p-q}.
$$

Hence the sub-complexes

$$(A_{h}^*, d) \hookrightarrow \left( \bigoplus_{\alpha \in \mathcal{A}_G} \alpha \cdot \Lambda^* g^*_G, d \right) \hookrightarrow (\Lambda^* \Gamma \setminus G \otimes \mathbb{C}, d)$$

are such that

$$
\tilde{\ast}_\rho|_{A_{h}^*}: A_{h}^* \to A_{h}^{p-q} \quad \text{and} \quad \tilde{\ast}_\rho|_{\bigoplus_{\alpha \in \mathcal{A}_G} \alpha \cdot \Lambda^* g^*_G}: \bigoplus_{\alpha \in \mathcal{A}_G} \alpha \cdot \Lambda^* g^*_G \to \bigoplus_{\alpha \in \mathcal{A}_G} \alpha \cdot \Lambda^{n-*} g^*_G ,
$$

therefore the first assertion follows from Theorem 2.8 and Proposition 2.3.

Consider the F. A. Belgun symmetrization map $\mu: \Lambda^* X \otimes \mathbb{C} \to \Lambda^* g^*_G$, $[14$ Theorem 7$]$. For $\alpha \in \mathcal{A}_G$, we define the map

$$
\varphi_\alpha: \Lambda^* \Gamma \setminus G \otimes \mathbb{C} \to \alpha \cdot \Lambda^* g^*_G , \quad \varphi_\alpha(\omega) := \alpha \cdot \mu \left( \frac{\omega}{\alpha} \right) .
$$

By the definition of $\mu$, for a $G$-left-invariant differential form $\theta$ on $\Gamma \setminus G$ and for a differential form $\omega$ on $\Gamma \setminus G$, we have $\mu(\theta \wedge \omega) = \theta \wedge \mu(\omega)$, see Lemma 2.13. By this we have, for any $\alpha \in \mathcal{A}_G$,

$$
\varphi_\alpha(d\omega) = \alpha \cdot \mu \left( \frac{d\omega}{\alpha} \right) = \alpha \cdot \mu \left( d \left( \frac{\omega}{\alpha} \right) + d\frac{\alpha}{\alpha} \wedge \frac{\omega}{\alpha} \right)
$$

$$
= \alpha \cdot \mu \left( \frac{\omega}{\alpha} \right) + d\alpha \wedge \mu \left( \frac{\omega}{\alpha} \right) = d \left( \alpha \cdot \mu \left( \frac{\omega}{\alpha} \right) \right)
$$

$$
= d \varphi_\alpha(\omega) ,
$$

and hence $\varphi_\alpha$ is a morphism of cochain complexes. Furthermore, for $\alpha \in \mathcal{A}_G$, by considering the inclusion

$$
\iota_\alpha: \alpha \cdot \Lambda^* g^*_G \hookrightarrow \Lambda^* \Gamma \setminus G \otimes \mathbb{C} ,
$$

we have that

$$
\varphi_\alpha \circ \iota_\alpha = \text{id}_{\alpha \cdot \Lambda^* g^*_G} .
$$

For distinct characters $\alpha, \alpha' \in \mathcal{A}_G$, for the $G$-left-invariant form $\alpha' \frac{\omega}{\alpha} d \left( \frac{\omega}{\alpha} \right)$, since $\eta$ is a $G$-left-invariant volume form, we can choose $\lambda \in \Lambda^{16} \otimes \mathbb{R}$ such that $\alpha' \frac{\omega}{\alpha} d \left( \frac{\omega}{\alpha} \right) \wedge \lambda = \eta$. Then we have

$$
d \left( \frac{\alpha}{\alpha'} \lambda \right) = \frac{\alpha}{\alpha'} \frac{\alpha'}{\alpha} d \left( \frac{\alpha}{\alpha'} \right) \wedge \lambda = \frac{\alpha}{\alpha' \eta}.
By this, using Stokes’ theorem, for $\alpha \omega \in \alpha \cdot \Lambda^q g^*_C$ and for $X_1, \ldots, X_p \in g \otimes_R \mathbb{C}$, we have
\[
\mu \left( \frac{\alpha}{\alpha'} \omega \right) (X_1, \ldots, X_p) = \int_{\Gamma \setminus G} \frac{\alpha(x)}{\alpha'(x)} \omega \lvert_x X_1 \arccurlyvee \ldots \arccurlyvee X_p \lvert_x \eta(x) = \omega (X_1, \ldots, X_p) \int_{\Gamma \setminus G} \frac{\alpha(x)}{\alpha'(x)} \eta(x)
\]
and hence we have
\[
\varphi_{\alpha'} \circ \iota_{\alpha} = 0.
\]
By the definition and since the complex structure on $\Gamma \setminus G$ is $G$-left-invariant, we have that, for any $\alpha \in A_\Gamma$, for any $(p, q) \in \mathbb{Z}^2$,
\[
\varphi_{\alpha} (\Lambda^{p,q} \Gamma \setminus G) \subseteq \alpha \cdot \Lambda^{p,q} g^*_C.
\]
By noting that the set $A_\Gamma$ is finite, we define the map
\[
\Phi := \sum_{\alpha \in A_\Gamma} \varphi_{\alpha} : \Lambda^{*,*} \Gamma \setminus G \to \bigoplus_{\alpha \in A_\Gamma} \alpha \cdot \Lambda^{*,*} g^*_C;
\]
note that $\Phi$ is a morphism of cochain complexes and we have, for any $(p, q) \in \mathbb{Z}^2$,
\[
\Phi (\Lambda^{p,q} \Gamma \setminus G) \subseteq \bigoplus_{\alpha \in A_\Gamma} \alpha \cdot \Lambda^{p,q} g^*_C \quad \text{and} \quad \Phi \circ \iota = \text{id}_{\bigoplus_{\alpha \in A_\Gamma} \alpha \cdot \Lambda^{p,q} g^*_C},
\]
where $\iota$ denotes the inclusion $\iota : \bigoplus_{\alpha \in A_\Gamma} \alpha \cdot \Lambda^{*,*} g^*_C \to \Lambda^{*,*} \Gamma \setminus G$. Consider the induced maps
\[
\iota^* : H^* \left( \text{Tot}^* \bigoplus_{\alpha \in A_\Gamma} \alpha \cdot \Lambda^{*,*} g^*_C, d \right) \to H^*_{dR} (\Gamma \setminus G ; \mathbb{C})
\]
and
\[
\Phi^* : H^*_{dR} (\Gamma \setminus G ; \mathbb{C}) \to H^* \left( \text{Tot}^* \bigoplus_{\alpha \in A_\Gamma} \alpha \cdot \Lambda^{*,*} g^*_C, d \right).
\]
Since $\iota^*$ is an isomorphism by the first assertion and $\Phi^* \circ \iota^* = \text{id}$, then $\Phi^*$ is the inverse of $\iota^*$. By $\Phi (\Lambda^{p,q} \Gamma \setminus G) \subseteq \bigoplus_{\alpha \in A_\Gamma} \alpha \cdot \Lambda^{p,q} g^*_C$, we have
\[
\Phi^* \left( \frac{\ker d|_{\Lambda^{p,q} \Gamma \setminus G}}{d (\Lambda^{p,q-1} \Gamma \setminus G \otimes_R \mathbb{C})} \right) \subseteq \frac{\ker d|_{\bigoplus_{\alpha \in A_\Gamma} \alpha \cdot \Lambda^{p,q} g^*_C}}{d (\bigoplus_{\alpha \in A_\Gamma} \alpha \cdot \Lambda^{p,q-1} g^*_C)}.
\]
Hence the restriction of $\Phi^*$ to $\frac{\ker d|_{\Lambda^{p,q} \Gamma \setminus G}}{d (\Lambda^{p,q-1} \Gamma \setminus G \otimes_R \mathbb{C})}$ is the inverse of the restriction of $\iota^*$ to $\frac{\ker d|_{\bigoplus_{\alpha \in A_\Gamma} \alpha \cdot \Lambda^{p,q} g^*_C}}{d (\bigoplus_{\alpha \in A_\Gamma} \alpha \cdot \Lambda^{p,q-1} g^*_C)}$, which is hence an isomorphism. Therefore the second assertion follows.

**Corollary 2.10.** Let $\Gamma \setminus G$ be a solvmanifold. Let $J$ be a $G$-left-invariant complex structure on $G$ satisfying, for all $g \in G$,
\[
J \circ (\text{Ad}_g) = (\text{Ad}_g) \circ J.
\]
Then, by setting $A^p_{\Gamma} := A^p \cap \Lambda^{p,q} \Gamma \setminus G$ for any $(p, q) \in \mathbb{Z}^2$, we have that the differential graded subalgebra $(A^p_{\Gamma}, d)$ defined in (11) is actually $\mathbb{Z}^2$-graded,
\[
A^p_{\Gamma} = \bigoplus_{p+q=\gamma} A^{p,q}_{\Gamma},
\]
and the inclusion $A^{*,*}_{\Gamma} \subseteq \Lambda^{p,q} \Gamma \setminus G$ induces the isomorphism
\[
\frac{\ker d|_{A^p_{\Gamma,q}}}{d (A^{p,q-1}_{\Gamma})} \cong \frac{\ker d|_{\Lambda^{p,q} \Gamma \setminus G}}{d (\Lambda^{p,q-1} \Gamma \setminus G \otimes_R \mathbb{C})}.
\]

**Proof.** Consider the Ad$_g(G)$-action on $\bigoplus_{\alpha \in A_\Gamma} \alpha \cdot \Lambda^{*,*} g^*_C$. Then $A^{*,*}_{\Gamma}$ is the sub-complex that consists of the elements of $\bigoplus_{\alpha \in A_\Gamma} \alpha \cdot \Lambda^{*,*} g^*_C$ fixed by this action. Since Ad$_g$ is diagonalizable, we have the decomposition
\[
\bigoplus_{\alpha \in A_\Gamma} \alpha \cdot \Lambda^{*,*} g^*_C = A^{*,*}_{\Gamma} \oplus D^*.
such that $D^\bullet$ is a sub-complex and this decomposition is a direct sum of cochain complexes. By the assumption $J \circ (\text{Ad}_s) = (\text{Ad}_s) \circ J$ for any $g \in G$, the $\text{Ad}_s(G)$-action is compatible with the bi-grading $\bigoplus_{\alpha \in A_r} \alpha \cdot \wedge^s g_s \mathbb{C}$. Hence we have in fact
\[
\bigoplus_{\alpha \in A_r} \alpha \cdot \wedge^s g_s \mathbb{C} = A^s \oplus D^s.
\]
Consider the projection $p: \bigoplus_{\alpha \in A_r} \alpha \cdot \wedge^s g_s \mathbb{C} \to A^s \oplus$ and the inclusion $\iota: A^s \oplus \bigoplus_{\alpha \in A_r} \alpha \cdot \wedge^s g_s \mathbb{C}$. Then we have $p \circ \iota = \text{id}_{A^s \oplus}$. As similar to the proof of Theorem 2.4.1, we have that $\iota$ induces, for any $(p, q) \in \mathbb{Z}^2$, the isomorphism
\[
\iota^*: \ker d|_{A^s \oplus} \cong \ker d|_{\bigoplus_{\alpha \in A_r} \alpha \cdot \wedge^s g_s \mathbb{C}}.
\]
Hence the corollary follows from Theorem 2.4.1.  

2.4.1. Complex solvmanifolds of splitting type. We consider now solvmanifolds of the following type.

**Assumption 2.11.** Consider a solvmanifold $X = \Gamma \backslash G$ endowed with a $G$-left-invariant complex structure $J$. Assume that $G$ is the semi-direct product $\mathbb{C}^n \ltimes \phi N$ so that:

(i) $N$ is a connected simply-connected $2m$-dimensional nilpotent Lie group endowed with an $N$-left-invariant complex structure $J_N$ (denote the Lie algebras of $\mathbb{C}^n$ and $N$ by $\mathfrak{a}$ and, respectively, $\mathfrak{n}$);

(ii) for any $t \in \mathbb{C}^n$, it holds that $\phi(t) \in \text{GL}(N)$ is a holomorphic automorphism of $N$ with respect to $J_N$;

(iii) $\phi$ induces a semi-simple action on $\mathfrak{n}$;

(iv) $G$ has a lattice $\Gamma$; (then $\Gamma$ can be written as $\Gamma = \Gamma_{\mathbb{C}} \ltimes \phi \Gamma_N$ such that $\Gamma_{\mathbb{C}}$ and $\Gamma_N$ are lattices of $\mathbb{C}^n$ and, respectively, $N$, and, for any $t \in \Gamma'$, it holds $\phi(t)(\Gamma_N) \subseteq \Gamma_N$);

(v) the inclusion $\wedge^{n \bullet} (\mathbb{C} \otimes \mathbb{C})^* \hookrightarrow \wedge^{n \bullet} (\Gamma_N \backslash N)$ induces the isomorphism
\[
H^\bullet (\wedge^{n \bullet} (\mathbb{C} \otimes \mathbb{C})^*, \mathfrak{g}) \cong H^\bullet (\Gamma_N \backslash N).
\]

Consider the standard basis $\{X_1, \ldots, X_n\}$ of $\mathbb{C}^n$. Consider the decomposition $\mathfrak{n} \otimes \mathbb{C} = \mathfrak{n}^{1.0} \oplus \mathfrak{n}^{0.1}$ induced by $J_N$. By the condition (i) this decomposition is a direct sum of $\mathfrak{C}^n$-modules. By the condition (iii) we have a basis $\{Y_1, \ldots, Y_m\}$ of $\mathfrak{n}^{1.0}$ and characters $\alpha_1, \ldots, \alpha_m \in \text{Hom}(\mathbb{C}^n; \mathbb{C}^\ast)$ such that the induced action $\phi$ on $\mathfrak{n}^{1.0}$ is represented by
\[
\mathbb{C}^n \ni \mathbf{t} \mapsto \phi(t) = \text{diag} (\alpha_1(t), \ldots, \alpha_m(t)) \in \text{GL}(\mathbb{C}^{1.0}).
\]

For any $j \in \{1, \ldots, m\}$, since $Y_j$ is an $N$-left-invariant $(1, 0)$-vector field on $N$, the $(1, 0)$-vector field $\alpha_j Y_j$ on $\mathbb{C}^n \ltimes \phi N$ is $(\mathbb{C}^n \ltimes \phi N)$-left-invariant. Consider the Lie algebra $\mathfrak{g}$ of $G$ and the decomposition $\mathfrak{g}^{\mathfrak{c}} := \mathfrak{g} \otimes \mathbb{C} = \mathfrak{g}^{1.0} \oplus \mathfrak{g}^{0.1}$ induced by $J$. Hence we have a basis $\{X_1, \ldots, X_n, \alpha_1 Y_1, \ldots, \alpha_m Y_m\}$ of $\mathfrak{g}^{1.0}$, and let $\{x_1, \ldots, x_n, \alpha_1 y_1, \ldots, \alpha_m y_m\}$ be its dual basis of $\Lambda^{1,0} \mathfrak{g}^{\mathfrak{c}}$. Then we have
\[
\Lambda^{1,0} \mathfrak{g}^{\mathfrak{c}} = \wedge^1 \langle x_1, \ldots, x_n, \alpha_1 y_1, \ldots, \alpha_m y_m \rangle \otimes \Lambda^0 \langle x_1, \ldots, x_n, \alpha_1 y_1, \ldots, \alpha_m y_m \rangle.
\]

The following lemma holds.

**Lemma 2.12** (Lemma 2.2). Let $X = \Gamma \backslash G$ be a solvmanifold endowed with a $G$-left-invariant complex structure $J$ as in Assumption 2.11. Consider a basis $\{Y_1, \ldots, Y_m\}$ of $\mathbb{C}^{1.0}$ such that the induced action $\phi$ on $\mathbb{C}^{1.0}$ is represented by $\phi(t) = \text{diag} (\alpha_1(t), \ldots, \alpha_m(t))$ for $\alpha_1, \ldots, \alpha_m \in \text{Hom}(\mathbb{C}^n; \mathbb{C}^\ast)$ characters of $\mathbb{C}^n$. For any $j \in \{1, \ldots, m\}$, there exist unique unitary characters $\beta_j \in \text{Hom}(\mathbb{C}^n; \mathbb{C}^\ast)$ and $\gamma_j \in \text{Hom}(\mathbb{C}^n; \mathbb{C}^\ast)$ on $\mathbb{C}^n$ such that $\alpha_j \beta_j^{-1}$ and $\alpha_j \gamma_j^{-1}$ are holomorphic.

We recall the following result by the second author.

**Theorem 2.13.** (Corollary 4.2) Let $X = \Gamma \backslash G$ be a solvmanifold endowed with a $G$-left-invariant complex structure $J$ as in Assumption 2.11. Consider the standard basis $\{X_1, \ldots, X_n\}$ of $\mathbb{C}^n$. Consider a basis $\{Y_1, \ldots, Y_m\}$ of $\mathbb{C}^{1.0}$ such that the induced action $\phi$ on $\mathbb{C}^{1.0}$ is represented by $\phi(t) = \text{diag} (\alpha_1(t), \ldots, \alpha_m(t))$ for $\alpha_1, \ldots, \alpha_m \in \text{Hom}(\mathbb{C}^n; \mathbb{C}^\ast)$ characters of $\mathbb{C}^n$. Let $\{x_1, \ldots, x_n, \alpha_1 y_1, \ldots, \alpha_m y_m\}$ be the basis of $\Lambda^{1,0} \mathfrak{g}^{\mathfrak{c}}$ which is dual to $\{X_1, \ldots, X_n, \alpha_1 Y_1, \ldots, \alpha_m Y_m\}$. For any $j \in \{1, \ldots, m\}$, let $\beta_j$ and $\gamma_j$ be the unique unitary characters on $\mathbb{C}^n$ such that $\alpha_j \beta_j^{-1}$ and $\alpha_j \gamma_j^{-1}$
are holomorphic, as in Lemma \ref{lem:holomorphic}. Define the differential bi-graded sub-algebra \( B^{\bullet \bullet}_r \subset \wedge^{\bullet \bullet} \Gamma \backslash G \), for \((p,q) \in \mathbb{Z}^2\), as
\[
(3) \quad B^{p,q}_r := \mathbb{C} \langle x_I \wedge (\alpha^{-1}_J J) y_J \wedge \overset{\circ}x_K \wedge (\overset{\circ}\alpha^{-1}_L \gamma_L) y^L \mid |I| + |J| = p \quad \text{and} \quad |K| + |L| = q \n\]
such that \((\beta_J \gamma_L)_{[r]} = 1\).

Then the inclusion \( B^{\bullet \bullet}_r \subset \wedge^{\bullet \bullet} \Gamma \backslash G \) induces the cohomology isomorphism
\[
H^{\bullet \bullet} (B^{\bullet \bullet}_r, \partial) \cong H^{\bullet \bullet}_r (\Gamma \backslash G) .
\]

As a straightforward consequence, by means of conjugation, we get the following result.

Corollary 2.14. Let \( X = \Gamma \backslash G \) be a solvmanifold endowed with a \( G \)-left-invariant complex structure \( J \) as in Assumption \ref{ass:complex}. Consider \( B^{\bullet \bullet}_r \) as in \ref{def:sub-complex}, and let
\[
(4) \quad B^{\bullet \bullet}_r := \{ \omega \in \wedge^{\bullet \bullet} \Gamma \backslash G : \omega \in B^{\bullet \bullet}_r \} .
\]

The inclusion \( B^{\bullet \bullet}_r \hookrightarrow \wedge^{\bullet \bullet} \Gamma \backslash G \) induces the cohomology isomorphism
\[
H^{\bullet \bullet} (B^{\bullet \bullet}_r, \partial) \cong H^{\bullet \bullet}_r (\Gamma \backslash G) .
\]

Hence we get the following result.

Corollary 2.15. Let \( \Gamma \backslash G \) be a solvmanifold endowed with a \( G \)-left-invariant complex structure \( J \) as in Assumption \ref{ass:complex}. Consider \( B^{\bullet \bullet}_r \) as in \ref{def:sub-complex}, and \( B^{\bullet \bullet}_r \) as in \ref{def:sub-complex}. Let
\[
(5) \quad C^{\bullet \bullet}_r := B^{\bullet \bullet}_r + B^{\bullet \bullet}_r .
\]

Then we have
\begin{enumerate}[(i)]
\item the inclusion \( C^{\bullet \bullet}_r \hookrightarrow \wedge^{\bullet \bullet} \Gamma \backslash G \) induces the cohomology isomorphism
\[
H^{\bullet \bullet} (C^{\bullet \bullet}_r, \partial) \cong H^{\bullet \bullet}_r (\Gamma \backslash G) ;
\]
\item the inclusion \( C^{\bullet \bullet}_r \hookrightarrow \wedge^{\bullet \bullet} \Gamma \backslash G \) induces the cohomology isomorphism
\[
H^{\bullet \bullet} (C^{\bullet \bullet}_r, \partial) \cong H^{\bullet \bullet}_r (\Gamma \backslash G) ;
\]
\item for any \((p,q) \in \mathbb{Z}^2\), the inclusion \( C^{\bullet \bullet}_r \hookrightarrow \wedge^{\bullet \bullet} \Gamma \backslash G \) induces the surjective map
\[
\frac{\ker d|_{C^{p,q}_r}}{d \left( \text{Tot}^{p+q-1} C^{p,q}_r \right)} \to \frac{\ker d|_{\wedge^{p+q-1} \Gamma \backslash G \otimes \mathbb{C}}}{d \left( \wedge^{p+q-1} \Gamma \backslash G \otimes \mathbb{C} \right)} .
\]
\end{enumerate}

Proof. Let \( g \) be the \( G \)-left-invariant Hermitian metric on \( G \) defined by
\[
g := \sum_{j=1}^{n} x_j \circ \overset{\circ}x_j + \sum_{k=1}^{m} \alpha_{k}^{-1} \alpha_{k}^{-1} y_k \circ \overset{\circ}y_k ,
\]
and consider its associated \( \mathbb{C} \)-anti-linear Hodge-\( + \)-operator \( \tilde{\ast}_g : \wedge^* \Gamma \backslash G \to \wedge^{2n-*} \Gamma \backslash G \), where \( 2N := 2n + 2m = \dim \Gamma \backslash G \). Then for multi-indices \( I, J \subset \{1, \ldots, n\} \) and \( K, L \subset \{1, \ldots, m\} \), and their complements \( I', J' \subset \{1, \ldots, n\} \) and \( K', L' \subset \{1, \ldots, m\} \), we have
\[
\tilde{\ast}_g (x_I \wedge (\alpha^{-1}_J J) y_J \wedge \overset{\circ}x_K \wedge (\overset{\circ}\alpha^{-1}_L \gamma_L) y^L ) = x_{I'} \wedge (\alpha^{-1}_{J'} J') y_{J'} \wedge \overset{\circ}x_{K'} \wedge (\overset{\circ}\alpha^{-1}_{L'} \gamma_{L'}) y^{L'} .
\]
Since \( G \) is unimodular by the existence of a lattice, \ref{lem:unimodular}, we have \( \alpha_{J} \alpha_{J'} \overset{\circ}\alpha_{L} \overset{\circ}\alpha_{L'} = 1 \) and so we have \( \beta_{J} \beta_{J'} \gamma_{L} \gamma_{L'} = \beta_{J'} \beta_{J} \gamma_{L} \gamma_{L'}^{-1} = \beta_{J} \beta_{J'} \gamma_{L} \gamma_{L'}^{-1} \). This implies
\[
x_{I'} \wedge (\alpha^{-1}_{J'} J') y_{J'} \wedge \overset{\circ}x_{K'} \wedge (\overset{\circ}\alpha^{-1}_{L'} \gamma_{L'}) y^{L'} = x_{I'} \wedge (\alpha^{-1}_{J'} J') y_{J'} \wedge \overset{\circ}x_{K'} \wedge (\overset{\circ}\alpha^{-1}_{L'} \gamma_{L'}) y^{L'} \in B^{\bullet \bullet}_r .
\]

Then we have \( \tilde{\ast}_g (B^{\bullet \bullet}_r) \subset B^{N-* \cdot N-*} \) and so also
\[
\tilde{\ast}_g (C^{\bullet \bullet}_r) \subset C^{N-* \cdot N-*} .
\]

Hence \ref{thm:betti} respectively \ref{thm:cohomology} follows from Theorem \ref{thm:betti} respectively Corollary \ref{cor:betti} and Proposition \ref{prop:cohomology}.
respect to the basis \( \{ X_1, \ldots, X_n, \bar{X}_1, \ldots, \bar{X}_n, \alpha_1 Y_1, \ldots, \alpha_m Y_m, \bar{\alpha}_1 \bar{Y}_1, \ldots, \bar{\alpha}_m \bar{Y}_m \} \) of \( g_C = g^{1,0} \oplus g^{0,1} \), we have, for \((t, n) \in G = C^n \ltimes \phi N\),
\[
(Ad_\phi)_{(t, n)} = \left( \begin{array}{cc}
\text{id}_{(C^n)^{t,g(n)_{p,1}}} & 0 \\
0 & \phi_\star|_{\alpha_1 \alpha_0 \gamma^{0,1}}(t)
\end{array} \right)
\]
\[
= \text{diag}(1, \ldots, 1, \alpha_1(t), \ldots, \alpha_m(t), \bar{\alpha}_1(t), \ldots, \bar{\alpha}_m(t)).
\]

Hence we have \( J \circ (Ad_\phi)_{(t, n)} = (Ad_\phi)_{(t, n)} \circ J \), and we can easily see that \( A_{t}^{\star} \subseteq C_{t}^{\star} \subseteq \wedge^{\star} \Gamma \backslash G \). Since the composition
\[
\ker d|_{A_{t}^{\star-p,q}} \to \ker d|_{C_{t}^{\star-p,q}} \to \ker d|_{\wedge^{\star-p,q-1} \Gamma \backslash G} \oplus \mathbb{R} C
\]
is an isomorphism, then \((iii)\) of the corollary follows. \(\square\)

Finally we get the following theorem.

**Theorem 2.16.** Let \( \Gamma \backslash G \) be a solvmanifold endowed with a \( G \)-left-invariant complex structure \( J \) as in Assumption 2.1. Consider \( C_{t}^{\star} \) as in \([3]\). For any \((p, q) \in \mathbb{Z}^2\), the inclusion \( C_{t}^{\star} \subseteq \wedge^{\star} \Gamma \backslash G \) induces the isomorphism
\[
H \left( C_{t}^{\star-p,q-1} \Gamma \backslash G \right) \cong H^{p,q}_{BC} \left( \Gamma \backslash G \right).
\]

**Proof.** By Corollary 2.15 the surjectivity follows from Theorem 1.3. The injectivity follows from Proposition 2.2. \(\square\)

**Example 2.17** (The completely-solvable Nakamura manifold, [10] Example 1]). The completely-solvable Nakamura manifold, firstly studied by I. Nakamura in \([34]\) page 90], is an example of a cohomologically Kähler non-Kähler solvmanifold, in Example 3.1, \([26, 33]\) Example 3.1, \([26, 33]\) [3].

Let \( G := C \ltimes \phi C^2 \), where
\[
\phi \left( x + \sqrt{-1} y \right) := \left( \begin{array}{cc}
e^x & 0 \\
0 & e^{-x}
\end{array} \right) \in \text{GL} \left( \mathbb{C}^2 \right).
\]

Then for some \( a \in \mathbb{R} \) the matrix \( \left( \begin{array}{cc}
e^x & 0 \\
0 & e^{-x}
\end{array} \right) \) is conjugate to an element of \( \text{SL}(2; \mathbb{Z}) \). We have a lattice \( \Gamma := \left( a Z + b \sqrt{-1} Z \right) \ltimes \phi \Gamma'' \) such that \( \Gamma'' \) is a lattice of \( \mathbb{C}^2 \). Consider the completely-solvable solvmanifold \( \Gamma \backslash G \).

As a matter of notation, we choose holomorphic coordinates \( \{ z_1, z_2, z_3 \} \), where \( \{ z_1 := x + \sqrt{-1} y \} \) is the holomorphic coordinate on \( C \), and we short form, for example, \( e^{-z_1} \text{d} z_{123} := e^{-z_1} \text{d} z_1 \wedge \text{d} z_2 \wedge \text{d} z_3 \).

By A. Hattori’s theorem, \([37]\) Corollary 4.2, the de Rham cohomology of \( \Gamma \backslash G \) does not depend on \( \Gamma \) and can be computed using just \( G \)-left-invariant forms on \( G \); more precisely, one gets
\[
\begin{align*}
H^0_{dR} \left( \Gamma \backslash G ; \mathbb{R} \right) &= \mathbb{R} (1), \\
H^1_{dR} \left( \Gamma \backslash G ; \mathbb{R} \right) &= \mathbb{R} (\text{d} z_1, \text{d} \bar{z}_1), \\
H^2_{dR} \left( \Gamma \backslash G ; \mathbb{R} \right) &= \mathbb{R} (\text{d} z_{23}, \text{d} z_{12}, \text{d} z_{13}, \text{d} z_{123}), \\
H^3_{dR} \left( \Gamma \backslash G ; \mathbb{R} \right) &= \mathbb{R} (\text{d} z_{123}, \text{d} z_{123}, \text{d} z_{123}, \text{d} z_{123}), \\
H^4_{dR} \left( \Gamma \backslash G ; \mathbb{R} \right) &= \mathbb{R} (\text{d} z_{1234}, \text{d} z_{1234}, \text{d} z_{1234}), \\
H^5_{dR} \left( \Gamma \backslash G ; \mathbb{R} \right) &= \mathbb{R} (\text{d} z_{12345}, \text{d} z_{12345}), \\
H^6_{dR} \left( \Gamma \backslash G ; \mathbb{R} \right) &= \mathbb{R} (\text{d} z_{123456}),
\end{align*}
\]
where we have listed the harmonic representatives with respect to the \( G \)-left-invariant Hermitian metric
\[
g := \text{d} z_1 \wedge \text{d} \bar{z}_1 + e^{-z_1} \text{d} z_2 \wedge \text{d} \bar{z}_2 + e^{z_1} \text{d} z_3 \wedge \text{d} \bar{z}_3 \] instead of their cohomology classes.

We consider \( C_{t}^{\star} \) as in \([3]\). The bi-differential bi-graded algebra \( B_{t}^{\star, \star} \) varies for a choice of \( b \). By using Theorem 2.10 we compute \( H^{p,q}_{BC} \left( \Gamma \backslash G \right) \cong H^{p,q}_{BC} \left( C_{t}^{\star} \right) \), case by case:

(i) \( b = 2m \pi \) for some integer \( m \in \mathbb{Z} \); (ii) \( b = (2m + 1) \pi \) for some integer \( m \in \mathbb{Z} \);
\( (iii) \) \( b \neq m\pi \) for any integer \( m \in \mathbb{Z} \).

Firstly, we write down \( C_T^{\bullet,\bullet} \) case by case in Table 1, Table 2, and Table 3.

| Case | \( C_T^{\bullet,\bullet} \) |
|------|------------------|
| (0, 0) | \( \mathbb{C} \langle 1 \rangle \) |
| (1, 0) | \( \mathbb{C} \langle d \, z_1, e^{-z_1} \, d \, z_2, e^{z_1} \, d \, z_3, e^{-z_1} \, d \, z_2, e^{z_1} \, d \, z_3 \rangle \) |
| (0, 1) | \( \mathbb{C} \langle d \, z_1, e^{-z_1} \, d \, z_2, e^{z_1} \, d \, z_3, e^{-z_1} \, d \, z_2, e^{z_1} \, d \, z_3 \rangle \) |
| (2, 0) | \( \mathbb{C} \langle d \, z_{12}, e^{-z_1} \, d \, z_{12}, e^{z_1} \, d \, z_{12}, e^{-z_1} \, d \, z_{12}, e^{z_1} \, d \, z_{12} \rangle \) |
| (1, 1) | \( \mathbb{C} \langle d \, z_{12}, e^{-z_1} \, d \, z_{12}, e^{z_1} \, d \, z_{12}, e^{-z_1} \, d \, z_{12}, e^{z_1} \, d \, z_{12}, e^{-z_1} \, d \, z_{12}, e^{z_1} \, d \, z_{12}, e^{-z_1} \, d \, z_{12}, e^{z_1} \, d \, z_{12}, e^{-z_1} \, d \, z_{12}, e^{z_1} \, d \, z_{12} \rangle \) |
| (0, 2) | \( \mathbb{C} \langle e^{-z_1} \, d \, z_{12}, e^{z_1} \, d \, z_{12}, e^{-z_1} \, d \, z_{12}, e^{z_1} \, d \, z_{12} \rangle \) |
| (3, 0) | \( \mathbb{C} \langle d \, z_{123} \rangle \) |
| (2, 1) | \( \mathbb{C} \langle e^{-z_1} \, d \, z_{123}, e^{-z_1} \, d \, z_{123}, e^{z_1} \, d \, z_{123}, e^{-z_1} \, d \, z_{123}, e^{z_1} \, d \, z_{123}, e^{-z_1} \, d \, z_{123}, e^{z_1} \, d \, z_{123}, e^{-z_1} \, d \, z_{123}, e^{z_1} \, d \, z_{123}, e^{-z_1} \, d \, z_{123}, e^{z_1} \, d \, z_{123} \rangle \) |
| (1, 2) | \( \mathbb{C} \langle e^{-z_1} \, d \, z_{123}, e^{-z_1} \, d \, z_{123}, e^{z_1} \, d \, z_{123}, e^{-z_1} \, d \, z_{123}, e^{z_1} \, d \, z_{123}, e^{-z_1} \, d \, z_{123}, e^{z_1} \, d \, z_{123}, e^{-z_1} \, d \, z_{123}, e^{z_1} \, d \, z_{123}, e^{-z_1} \, d \, z_{123}, e^{z_1} \, d \, z_{123} \rangle \) |
| (0, 3) | \( \mathbb{C} \langle d \, z_{123} \rangle \) |
| (3, 1) | \( \mathbb{C} \langle d \, z_{123}, e^{-z_1} \, d \, z_{123}, e^{z_1} \, d \, z_{123}, e^{-z_1} \, d \, z_{123}, e^{z_1} \, d \, z_{123} \rangle \) |
| (2, 2) | \( \mathbb{C} \langle e^{-z_1} \, d \, z_{123}, e^{-z_1} \, d \, z_{123}, e^{z_1} \, d \, z_{123}, e^{-z_1} \, d \, z_{123}, e^{z_1} \, d \, z_{123}, e^{-z_1} \, d \, z_{123}, e^{z_1} \, d \, z_{123}, e^{-z_1} \, d \, z_{123}, e^{z_1} \, d \, z_{123}, e^{-z_1} \, d \, z_{123}, e^{z_1} \, d \, z_{123} \rangle \) |
| (1, 3) | \( \mathbb{C} \langle d \, z_{123}, e^{-z_1} \, d \, z_{123}, e^{z_1} \, d \, z_{123}, e^{-z_1} \, d \, z_{123}, e^{z_1} \, d \, z_{123} \rangle \) |
| (3, 2) | \( \mathbb{C} \langle e^{-z_1} \, d \, z_{123}, e^{z_1} \, d \, z_{123}, e^{-z_1} \, d \, z_{123}, e^{z_1} \, d \, z_{123} \rangle \) |
| (2, 3) | \( \mathbb{C} \langle d \, z_{123}, e^{-z_1} \, d \, z_{123}, e^{z_1} \, d \, z_{123} \rangle \) |
| (3, 3) | \( \mathbb{C} \langle d \, z_{123} \rangle \) |

Table 1. The double complex \( C_T^{\bullet,\bullet} \) for the completely-solvable Nakamura manifold in case \([6]\).

Note that, since \( \partial \overline{\partial} (C_T^{\bullet,\bullet}) = \{0\} \) for each case, we have, by using Theorem 2.16

\[
H_T^{\bullet,\bullet}(\Gamma/\mathbb{G}) \simeq H_T^{\bullet,\bullet}(C_T^{\bullet,\bullet}) \Rightarrow \ker d|_{C_T^{\bullet,\bullet}}.
\]

Hence, we compute the Bott-Chern cohomology of the Nakamura manifold case by case in Table 4 and Table 5, note that, in the case \([iii]\), simply we have:

\[
H_T^{\bullet,\bullet}(\Gamma/\mathbb{G}) \simeq C_T^{\bullet,\bullet} \quad \text{in case \([iii]\)}.\]

We summarize in Table 6 the results of the computations of the Bott-Chern cohomology as done in Table 4 and Table 5, and of the Dolbeault cohomology, as done in 10 Example 1.

**Remark 2.18.** Note that in any case the canonical map \( \text{Tot}^\bullet H_T^{\bullet,\bullet}(\Gamma/\mathbb{G}) \rightarrow H_T^{\bullet,\bullet}(\Gamma/\mathbb{G}) \) is surjective. (With the notation of 13 13, this means that, in any case, \( \Gamma/\mathbb{G} \) is complex-C^\infty-pure-and-full at every stage, namely, the de Rham cohomology admits a decomposition in pure-type subgroups with respect to the complex structure.) In the case \([iii]\) by Proposition 2.14 we have \( H_T^{\bullet,\bullet}(\Gamma/\mathbb{G}) \simeq H^\bullet(\text{Tot}^\bullet C_T^{\bullet,\bullet}) = \)
\[ C_{\Gamma}^{\bullet, \bullet} \]

| case | \( C_{\Gamma}^{\bullet, \bullet} \) |
|------|------------------|
| (0, 0) | \( \mathbb{C} \langle d z_1 \rangle \) |
| (1, 0) | \( \mathbb{C} \langle d z_1 \rangle \) |
| (2, 0) | \( \mathbb{C} \langle dz_{23} \rangle \) |
| (1, 1) | \( \mathbb{C} \langle dz_{11}, e^{-2z_{11}} d z_{22}, e^{-2z_{11}} d z_{22}, e^{2z_{11}} d z_{33}, e^{2z_{11}} d z_{33}, d z_{23}, d z_{32} \rangle \) |
| (0, 2) | \( \mathbb{C} \langle dz_{23} \rangle \) |
| (3, 0) | \( \mathbb{C} \langle dz_{123} \rangle \) |
| (2, 1) | \( \mathbb{C} \langle dz_{231}, e^{-2z_{11}} d z_{122}, e^{-2z_{11}} d z_{122}, e^{2z_{11}} d z_{133}, e^{2z_{11}} d z_{133}, d z_{123}, d z_{132} \rangle \) |
| (1, 2) | \( \mathbb{C} \langle dz_{123}, e^{-2z_{11}} d z_{212}, e^{-2z_{11}} d z_{212}, e^{2z_{11}} d z_{313}, e^{2z_{11}} d z_{313}, d z_{213}, d z_{312} \rangle \) |
| (0, 3) | \( \mathbb{C} \langle dz_{123} \rangle \) |
| (3, 1) | \( \mathbb{C} \langle dz_{1231} \rangle \) |
| (2, 2) | \( \mathbb{C} \langle dz_{1231}, e^{-2z_{11}} d z_{1213}, e^{-2z_{11}} d z_{1213}, e^{2z_{11}} d z_{1313}, e^{2z_{11}} d z_{1313}, d z_{2323}, d z_{3112} \rangle \) |
| (1, 3) | \( \mathbb{C} \langle dz_{1231} \rangle \) |
| (3, 2) | \( \mathbb{C} \langle dz_{1232} \rangle \) |
| (2, 3) | \( \mathbb{C} \langle dz_{2312} \rangle \) |
| (3, 3) | \( \mathbb{C} \langle dz_{123123} \rangle \) |

Table 2. The double complex \( C_{\Gamma}^{\bullet, \bullet} \) for the completely-solvable Nakamura manifold in case \([iii]\).

Tot\(^*\) \( C_{\Gamma}^{\bullet, \bullet} \) and hence the canonical map \( \text{Tot}^* H_{BC}^{\bullet, \bullet} (\Gamma \backslash G) \to H_{dR}^{\bullet, \bullet} (\Gamma \backslash G) \) induced by the identity is in fact an isomorphism: this implies that \( \Gamma \backslash G \) in case \([iii]\) satisfies the \( \partial \bar{\partial} \)-Lemma (namely, every \( \partial \)-closed \( \bar{\partial} \)-exact form is \( \partial \bar{\partial} \)-exact too, see \([20]\)). In \([10]\), it is shown that for some left-invariant Hermitian metric the space of harmonic forms admits the Hodge decomposition and symmetry (see also \([11]\) for higher dimensional examples with the Hodge decomposition and symmetry).

**Remark 2.19.** In view of \([10]\) Theorem A, Theorem B], stating that, for every compact complex manifold \( X \), for any \( k \in \mathbb{Z} \), the inequality

\[ \sum_{p+q=k} (\dim_{\mathbb{C}} H_{BC}^{p,q}(X) + \dim_{\mathbb{C}} H_{A}^{p,q}(X)) \geq \sum_{p+q=k} \left( \dim_{\mathbb{C}} H_{\partial}^{p,q}(X) + \dim_{\mathbb{C}} H_{\bar{\partial}}^{p,q}(X) \right) \geq 2 \dim_{\mathbb{C}} H_{dR}^{k}(X; \mathbb{C}) \]

holds, and that equalities hold for any \( k \in \mathbb{Z} \) if and only if \( X \) satisfies the \( \partial \bar{\partial} \)-Lemma, one gets that the non-negative integer numbers \( \sum_{p+q=k} (\dim_{\mathbb{C}} H_{BC}^{p,q}(X) + \dim_{\mathbb{C}} H_{A}^{p,q}(X)) - 2 \dim_{\mathbb{C}} H_{dR}^{k}(X; \mathbb{C}) \in \mathbb{N} \), varying \( k \in \mathbb{Z} \), provide a “measure” of the non-Kählerianity of \( X \).

Note that, for the completely-solvable Nakamura manifold, in any case, one has

\[ \dim_{\mathbb{C}} H_{BC}^{p,q}(X) + \dim_{\mathbb{C}} H_{A}^{p,q}(X) = \dim_{\mathbb{C}} H_{\bar{\partial}}^{p,q}(X) + \dim_{\mathbb{C}} H_{\partial}^{p,q}(X) \]

for any \((p, q) \in \mathbb{Z}^2\). On the other hand,

\[ \sum_{p+q=k} (\dim_{\mathbb{C}} H_{BC}^{p,q}(X) + \dim_{\mathbb{C}} H_{A}^{p,q}(X)) - 2 \dim_{\mathbb{C}} H_{dR}^{k}(X; \mathbb{C}) = \begin{cases} 8 & \text{for } k \in \{1, 5\} \\ 20 & \text{for } k \in \{2, 4\} \\ 24 & \text{for } k = 3 \\ 0 & \text{otherwise} \end{cases} \]

in case \([iii]\).
Table 3. The double complex $C^{*,*}_\Gamma$ for the completely-solvable Nakamura manifold in case (iii).

\[
\begin{array}{c|c}
\text{case (iii)} & C^{*,*}_\Gamma \\
(0, 0) & \mathbb{C} \langle d z_1 \rangle \\
(1, 0) & \mathbb{C} \langle d z_1, d z_2 \rangle \\
(0, 1) & \mathbb{C} \langle d z_2 \rangle \\
(2, 0) & \mathbb{C} \langle d z_{23} \rangle \\
(1, 1) & \mathbb{C} \langle d z_{11}, d z_{23}, d z_{32} \rangle \\
(0, 2) & \mathbb{C} \langle d z_{23} \rangle \\
(3, 0) & \mathbb{C} \langle d z_{123} \rangle \\
(2, 1) & \mathbb{C} \langle d z_{231}, d z_{123}, d z_{412} \rangle \\
(1, 2) & \mathbb{C} \langle d z_{123}, d z_{213}, d z_{312} \rangle \\
(0, 3) & \mathbb{C} \langle d z_{123} \rangle \\
(3, 1) & \mathbb{C} \langle d z_{1231} \rangle \\
(2, 2) & \mathbb{C} \langle d z_{1213}, d z_{2323}, d z_{1312} \rangle \\
(1, 3) & \mathbb{C} \langle d z_{1123} \rangle \\
(3, 2) & \mathbb{C} \langle d z_{12132} \rangle \\
(2, 3) & \mathbb{C} \langle d z_{23123} \rangle \\
(3, 3) & \mathbb{C} \langle d z_{123123} \rangle \\
\end{array}
\]

and

\[
\sum_{p+q=k} (\dim_{\mathbb{C}} H_{BC}^{p,q}(X) + \dim_{\mathbb{C}} H_{A}^{p,q}(X)) - 2 \dim_{\mathbb{C}} H_{dR}^{k}(X; \mathbb{C}) = \begin{cases} 
0 & \text{for } k \in \{1, 5\} \\
4 & \text{for } k \in \{2, 4\} \\
8 & \text{for } k = 3 \\
0 & \text{otherwise}
\end{cases} \quad \text{in case (ii)},
\]

and

\[
\sum_{p+q=k} (\dim_{\mathbb{C}} H_{BC}^{p,q}(X) + \dim_{\mathbb{C}} H_{A}^{p,q}(X)) - 2 \dim_{\mathbb{C}} H_{dR}^{k}(X; \mathbb{C}) = \begin{cases} 
0 & \text{for } k \in \{1, 5\} \\
0 & \text{for } k \in \{2, 4\} \\
0 & \text{for } k = 3 \\
0 & \text{otherwise}
\end{cases} \quad \text{in case (iii)}.
\]

In particular, by [10, Theorem B], one gets that $\Gamma \setminus G$ in case (iii) satisfies the $\partial \bar{\partial}$-Lemma, as noticed also in Remark 2.18.

Given a property depending on the complex structure, one says that it is open under small deformations (respectively, strongly-closed under small deformations) if, for any complex-analytic families of complex compact manifolds parametrized by $\mathcal{B}$, the set of parameters for which the property holds is open (respectively, closed) in the strong topology of $\mathcal{B}$.

We recall that satisfying the $\partial \bar{\partial}$-Lemma is an open property under small deformations, see [70, Proposition 9.21], [73, Theorem 5.12], [65, §1B], [10, Corollary 2.7]. On the other hand, as pointed out by Luis Ugarte, the completely-solvable Nakamura manifold provides a counterexample to the strongly-closedness of the property of satisfying the $\partial \bar{\partial}$-Lemma: indeed, complex structures in class (iii) satisfy the $\partial \bar{\partial}$-Lemma while complex structures in classes (i) and (ii) do not. We have hence the following theorem.

**Theorem 2.20.** Satisfying the $\partial \bar{\partial}$-Lemma is not a strongly-closed property under small deformations of the complex structure.
Let $\Gamma$ be a lattice of a Lie group $G$, and denote by $\mathcal{P}$ the set of all connected simply-connected complex solvable Lie groups admitting a lattice $\Gamma$. Actually, as remarked by Luis Ugarte, in defining closedness for deformations, one should consider the Zariski topology, see, e.g., [56], namely, a property of the lemma in the case $\Gamma$. Since we have $\alpha_1 = \alpha_2 = \alpha_3 = 1$, we can re-examine the lemma in order to prove that satisfying the $\partial\overline{\partial}$-Lemma is also non-(Zariski-)closed.

### 2.4. Complex parallelizable solvmanifolds.

Let $G$ be a connected simply-connected complex solvable Lie group admitting a lattice $\Gamma$, and denote by $2n$ the real dimension of $G$. Denote the Lie algebra naturally associated to $G$ by $\mathfrak{g}$. We use the following lemma.

**Lemma 2.22.** Let $\alpha$, $\beta$ be holomorphic characters of a connected simply-connected complex solvable Lie group $G$. If $\alpha\beta$ is a unitary character, then $\alpha = \beta^{-1}$.

**Proof.** Since we have $\alpha([G,G]) = [\alpha(G),\alpha(G)] = [\beta(G),\beta(G)] = 1$, we can regard $\alpha$ and $\beta$ as characters of $G/[G,G]$. Since $G$ is connected simply-connected, $G/[G,G]$ is also connected simply-connected, see [28] Theorem 3.5. Since $G/[G,G]$ is Abelian, it is sufficient to show the lemma in the case $G = \mathbb{C}^n$. For the coordinate set $(z_1,\ldots,z_n)$ of $\mathbb{C}^n$, we write $\alpha = \exp\left(\sum_{j=1}^{n} a_j z_j\right)$ and $\beta = \exp\left(\sum_{j=1}^{n} b_j z_j\right)$, for some $a_1,\ldots,a_n,b_1,\ldots,b_n \in \mathbb{C}$. If $\alpha\beta$ is unitary, then we have $\Re\left(\sum_{j=1}^{n} (a_j z_j + b_j \overline{z}_j)\right) = 0$. By simple computations, we have $a_j = -b_j$ for any $j \in \{1,\ldots,n\}$. Hence the lemma follows.

Denote by $\mathfrak{g}_+$ (respectively, $\mathfrak{g}_-$) the Lie algebra of the $G$-left-invariant holomorphic (respectively, anti-holomorphic) vector fields on $G$. As a (real) Lie algebra, we have an isomorphism $\mathfrak{g}_+ \simeq \mathfrak{g}_-$ by means of the complex conjugation.
Let $N$ be the nilradical of $G$. We can take a connected simply-connected complex nilpotent subgroup $C \subseteq G$ such that $G = C \cdot N$, see, e.g., [28, Proposition 3.3]. Since $C$ is nilpotent, the map

$$C \ni c \mapsto (\operatorname{Ad}_c)_s \in \operatorname{Aut}(\mathfrak{g}_+^-)$$

is a homomorphism, where $(\operatorname{Ad}_c)_s$ is the semi-simple part of the Jordan decomposition of $\operatorname{Ad}_c$. Let $\mathfrak{c}$ be the Lie algebra of $C$; we take a subspace $V \subseteq \mathfrak{c}$ such that $\mathfrak{g} = V \oplus \mathfrak{n}$. Then the diagonalizable representation $\operatorname{Ad}_s$ constructed above, [24], is identified with the map

$$G = C \cdot N \ni c \cdot n \mapsto (\operatorname{Ad}_c)_s \in \operatorname{Aut}(\mathfrak{g}),$$

see [33, Remark 4].

We have a basis $\{X_1, \ldots, X_n\}$ of $\mathfrak{g}_+$ such that, for $c \in C$,

$$(\operatorname{Ad}_c)_s = \operatorname{diag}(\alpha_1(c), \ldots, \alpha_n(c)), $$

for some characters $\alpha_1, \ldots, \alpha_n$ of $C$. By $G = C \cdot N$, we have $G/N = C/C \cap N$ and regard $\alpha_1, \ldots, \alpha_n$ as characters of $G$. Let $\{x_1, \ldots, x_n\}$ be the basis of $\mathfrak{g}_+^*$ which is dual to $\{X_1, \ldots, X_n\}$.

**Theorem 2.23.** ([33, Corollary 6.2 and its proof]) Let $G$ be a connected simply-connected complex solvable Lie group admitting a lattice $\Gamma$. Denote the Lie algebra naturally associated to $G$ by $\mathfrak{g}$. Consider a basis $\{X_1, \ldots, X_n\}$ of the Lie algebra $\mathfrak{g}_+$ of the $G$-left-invariant holomorphic vector fields on $G$ with respect to which $(\operatorname{Ad}_c)_s = \operatorname{diag}(\alpha_1(c), \ldots, \alpha_n(c))$ for some characters $\alpha_1, \ldots, \alpha_n$ of $C$. Regard $\alpha_1, \ldots, \alpha_n$ as characters of $G$. Let $B^*_{\Gamma}$ be the sub-complex of $(\wedge^{0, \bullet} \Gamma \backslash G, \partial)$ defined as

$$B^*_{\Gamma} := \left\langle \frac{\partial_I}{\alpha_I} \right| I \subseteq \{1, \ldots, n\} \text{ such that } \left( \frac{\partial_I}{\alpha_I} \right)_\Gamma = 1 \right\rangle,$$

(there we shorten, e.g., $\alpha_I := \alpha_{i_1} \cdots \alpha_{i_k}$ for a multi-index $I = (i_1, \ldots, i_k)$). Then the inclusion $B^*_{\Gamma} \hookrightarrow \wedge^{0, \bullet} \Gamma \backslash G$ induces the isomorphism

$$H^* \left( B^*_{\Gamma}, \partial \right) \cong H^0_{\partial} \left( \Gamma \backslash G \right).$$
By this theorem, since $\Gamma \setminus G$ is complex parallelizable, for the differential bi-graded algebra $(\wedge^* g^+ \otimes C B^\bullet_\Gamma, \partial)$, the inclusion $\wedge^* g^+ \otimes C B^\bullet_\Gamma \hookrightarrow \wedge^* \Gamma \setminus G$ induces the isomorphism

$$\wedge^* g^+ \otimes C H_{\partial}^{\bullet, \bullet} (\Gamma \setminus G) \cong H_{\partial}^{\bullet, \bullet} (\Gamma \setminus G).$$

Consider the $G$-left-invariant Hermitian metric

$$g := \sum_{j=1}^n x_j \otimes \bar{x}_j.$$

Then, for $x_I \wedge \bar{\alpha}_K \bar{x}_K \in \wedge^{|I|} g^+ \otimes C B^{|K|}_\Gamma$, since $G$ is unimodular, [52] Lemma 6.2], we have

$$s_g (x_I \wedge \bar{\alpha}_K \bar{x}_K) = x_{I'} \wedge \frac{\bar{\alpha}_K}{\bar{\alpha}_K} \bar{x}_{K'}, = x_{I'} \wedge \bar{\alpha}_{K'} \bar{x}_{K'} \in \wedge^{n-|I|} g^+ \otimes C B^{|K'-|}_{\Gamma},$$

where $I' := \{1, \ldots, n\} \setminus I$ and $K' := \{1, \ldots, n\} \setminus K$ are the complements of $I$ and $K$ respectively. Hence we have $s_g (\wedge^* g^+ \otimes C B^\bullet_\Gamma) \subseteq \wedge^{n-} g^+ \otimes C B^\bullet_\Gamma$.

We consider the space

$$B^\bullet_\Gamma = \left\{ \frac{x_I}{\alpha_I} \left| I \subseteq \{1, \ldots, n\} \text{ such that } \left( \frac{x_I}{\alpha_I} \right)_\Gamma = 1 \right. \right\}.$$

Then the inclusion $B^\bullet_\Gamma \otimes C \wedge^* g^\perp \subseteq \wedge^* \Gamma \setminus G$ induces the isomorphism in $\partial$-cohomology

$$H^{\bullet, \bullet} (B^\bullet_\Gamma \otimes C \wedge^* g^\perp, \partial) \cong H_{\partial}^{\bullet, \bullet} (\Gamma \setminus G).$$

Consider

$$C^{\bullet, \bullet} := \wedge^* g^+ \otimes C B^\bullet_\Gamma + B^\bullet_\Gamma \otimes C \wedge^* g^\perp. \tag{8}$$

Then we have $s_g (C^{\bullet, \bullet}) \subseteq C^{\bullet, n-\bullet}$.

As similar to Corollary 2.15, we can show the following result.

---

Table 6. The dimensions of the de Rham, Dolbeault, and Bott-Chern cohomologies of the completely-solvable Nakamura manifold.
Corollary 2.24. Let $G$ be a connected simply-connected complex solvable Lie group admitting a lattice $\Gamma$. Denote the Lie algebra naturally associated to $G$ by $\mathfrak{g}$. Consider the sub-complex $C_T^{*,*} \subseteq \wedge^{*,*} \Gamma \backslash G$ as defined in \cite{3}. 

(i) The inclusion $C_T^{*,*} \hookrightarrow \wedge^{*,*} \Gamma \backslash G$ induces the $\partial$-cohomology isomorphism

$$H^{*,*}(C_T^{*,*}, \partial) \xrightarrow{\cong} H^{*,*}_X(\Gamma \backslash G).$$

(ii) The inclusion $C_T^{*,*} \hookrightarrow \wedge^{*,*} \Gamma \backslash G$ induces the $\overline{\partial}$-cohomology isomorphism

$$H^{*,*}(C_T^{*,*}, \overline{\partial}) \xrightarrow{\cong} H^{*,*}_X(\Gamma \backslash G).$$

(iii) The inclusion $C_T^{*,*} \hookrightarrow \wedge^{*,*} \Gamma \backslash G$ induces, for any $(p, q) \in \mathbb{Z}^2$, the surjection

$$\ker d|_{\bigwedge^{p,q} C_T^{*,*}} \to \ker d|_{\bigwedge^{p,q} \Gamma \backslash G} \to \ker d|_{A_T^{*,*}} \to \ker d|_{A_T^{*,*} \Gamma \backslash G}.$$

Proof. By $\tilde{\gamma}_g(C^{*,*}) \subseteq C^{n-1,n-2}$, the first and second assertions follow as similar to the proof of Corollary 2.22.

By denoting the complex structure by $J$, for any $c \in C$, since we have $\text{Ad}_c \circ J = J \circ \text{Ad}_c$, we have $(\text{Ad}_c)_J \circ J = J \circ (\text{Ad}_c)_J$, and hence we have $(\text{Ad}_c)_g \circ J = J \circ (\text{Ad}_c)_g$ for any $g \in G$. We consider the sub-complex $A_T^{*,*} \subseteq \wedge^* \Gamma \backslash G \otimes \mathbb{C}$ as in \cite{4}, see Theorem 2.8. By Corollary 2.10 the inclusion $A_T^{*,*} \hookrightarrow \bigwedge^{*,*} \Gamma \backslash G$ induces the isomorphism

$$\ker d|_{A_T^{*,*}} \xrightarrow{\cong} \ker d|_{A_T^{*,*} \Gamma \backslash G} \to \ker d|_{\bigwedge^{p,q} \Gamma \backslash G} \to \ker d|_{\bigwedge^{p,q} \Gamma \backslash G}.$$

We have

$$A_T^{*,*} = \langle \alpha_I \alpha_J x_I x_J | I, J \subseteq \{1, \ldots, n\} \text{ such that } (\alpha_I \alpha_J)|_{1} = 1 \rangle.$$

For $(\alpha_I \alpha_J)|_{1} = 1$, since we can regard $\alpha_I \alpha_J$ as a function on $\Gamma \backslash G$, the image of $\alpha_I \alpha_J$ is compact and hence it is unitary. By Lemma 2.22 we have $\alpha_I \alpha_J = \frac{\alpha_I \alpha_J}{\alpha_I \alpha_J}$, hence we have the inclusion $A_T^{*,*} \subseteq \text{Tot}^* \wedge^* \mathfrak{g}_+^* \otimes B^*$ and so we have the inclusion $A_T^{*,*} \subseteq C_T^{*,*} \subseteq \wedge^{*,*} \Gamma \backslash G$. Since the composition

$$\ker d|_{A_T^{*,*}} \xrightarrow{\cong} \ker d|_{\bigwedge^{p,q} \Gamma \backslash G} \to \ker d|_{\bigwedge^{p,q} \Gamma \backslash G}$$

is an isomorphism, then the third assertion of the corollary follows. \qed

By this, we get the following result.

Theorem 2.25. Let $G$ be a connected simply-connected complex solvable Lie group admitting a lattice $\Gamma$. Consider the sub-complex $C_T^{*,*} \subseteq \wedge^{*,*} \Gamma \backslash G$ as defined in \cite{3}. The inclusion $C_T^{*,*} \hookrightarrow \wedge^{*,*} \Gamma \backslash G$ induces the isomorphism

$$H^\ast \left( C_T^{*,*} \otimes \bigwedge C_T^{*,*} \otimes C_T^{*,*} \right) \xrightarrow{\cong} H^\ast_X(\Gamma \backslash G).$$

Example 2.26 (The complex parallelizable Nakamura manifold). Let $G = \mathbb{C} \ltimes \mathbb{C}^2$ be such that

$$\phi(z) = \begin{pmatrix} e^z & 0 \\ 0 & e^{-z} \end{pmatrix}.$$

Then there exist $a + \sqrt{-1} b \in \mathbb{C}$ and $c + \sqrt{-1} d \in \mathbb{C}$ such that $\mathbb{Z}(a + \sqrt{-1} b) + \mathbb{Z}(c + \sqrt{-1} d)$ is a lattice in $\mathbb{C}$ and $\phi(a + \sqrt{-1} b)$ and $\phi(c + \sqrt{-1} d)$ are conjugate to elements of $\text{SL}(4; \mathbb{Z})$, where we regard $\text{SL}(2; \mathbb{C}) \subset \text{SL}(4; \mathbb{R})$, see \cite{5}. Hence we have a lattice $\Gamma := \mathbb{Z}(a + \sqrt{-1} b) + \mathbb{Z}(c + \sqrt{-1} d)$ \ltimes $\Gamma''$ of $G$ such that $\Gamma''$ is a lattice of $\mathbb{C}^2$. Let $X := \Gamma \backslash G$ be the complex parallelizable Nakamura manifold, \cite{5} (2).

We take the connected simply-connected complex nilpotent subgroup $C := \mathbb{C} \subseteq G$ such that $G = C \cdot N$, where $N$ is the nilradical of $G$. Recall that $\mathfrak{g}_+$ denotes the Lie algebra of the $G$-left-invariant holomorphic vector fields on $G$. For a coordinate set $(z_1, z_2, z_3)$ of $\mathbb{C} \ltimes \mathbb{C}^2$, we have the basis $\left\{ \frac{\partial}{\partial z_1}, e^{z_1} \frac{\partial}{\partial z_2}, e^{-z_1} \frac{\partial}{\partial z_2} \right\}$ of $\mathfrak{g}_+$ such that

$$(\text{Ad}(z_1, z_2, z_3))_r = \text{diag}(1, e^{z_1}, e^{-z_1}) \in \text{Aut}(\mathfrak{g}_+).$$


(a) If \( b \in \pi \mathbb{Z} \) and \( d \in \pi \mathbb{Z} \), then, for \( z \in \left( a + \sqrt{-1} b \right) \mathbb{Z} + \left( c + \sqrt{-1} d \right) \mathbb{Z} \), we have \( \phi(z) \in \text{SL}(2; \mathbb{R}) \). Since 
\[
\left( \frac{e^z}{\sqrt{2\pi}} \right)_X = (e^{z_1 - z_2})|_X = 1
\]
we have 
\[
B^*_{\Gamma} = \wedge^* C \langle d \, z_1, e^{z_1} \, d \, z_2, e^{z_1} \, d \, z_3 \rangle.
\]

Hence the double complex \( C^{*,*}_{\Gamma} \) in case \((\mathfrak{m})\) is the one in Table 7 (We recall that, in order to shorten the notation, we write, for example, \( e^{z_1} \, d \, z_{13} := e^{z_1} \, d \, z_1 \land d \, z_3 \)).

| case \((\mathfrak{m})\) | \( C^{*,*}_{\Gamma} \) |
|----------------------|-----------------|
| (0, 0) | \( \mathbb{C} \langle 1 \rangle \) |
| (0, 1) | \( \mathbb{C} \langle d \, z_1, e^{-z_1} \, d \, z_2, e^{z_1} \, d \, z_3 \rangle \) |
| (0, 2) | \( \mathbb{C} \langle e^{-z_1} \, d \, z_{12}, e^{z_1} \, d \, z_{13}, d \, z_{23}, e^{z_1} \, d \, z_{12} \rangle \) |
| (1, 0) | \( \mathbb{C} \langle d \, z_{11}, e^{-z_1} \, d \, z_{12}, e^{z_1} \, d \, z_{13}, e^{-2z_1} \, d \, z_{22}, d \, z_{23}, d \, z_{31}, d \, z_{32}, e^{2z_1} \, d \, z_{33} \rangle \) |
| (1, 1) | \( \mathbb{C} \langle d \, z_{11}, e^{-z_1} \, d \, z_{12}, e^{z_1} \, d \, z_{13}, e^{-2z_1} \, d \, z_{22}, d \, z_{23}, d \, z_{31}, e^{2z_1} \, d \, z_{32}, e^{2z_1} \, d \, z_{33} \rangle \) |
| (1, 2) | \( \mathbb{C} \langle e^{-z_1} \, d \, z_{112}, e^{-2z_1} \, d \, z_{122}, d \, z_{132}, e^{2z_1} \, d \, z_{133}, e^{-z_1} \, d \, z_{232}, e^{z_1} \, d \, z_{233} \rangle \) |
| (1, 3) | \( \mathbb{C} \langle d \, z_{123} \rangle \) |
| (2, 0) | \( \mathbb{C} \langle e^{-z_1} \, d \, z_{12}, e^{z_1} \, d \, z_{13}, e^{-z_1} \, d \, z_{12}, d \, z_{13} \rangle \) |
| (2, 1) | \( \mathbb{C} \langle e^{-z_1} \, d \, z_{12}, e^{-z_1} \, d \, z_{13}, d \, z_{132}, e^{z_1} \, d \, z_{133} \rangle \) |
| (2, 2) | \( \mathbb{C} \langle e^{-z_1} \, d \, z_{12}, e^{-z_1} \, d \, z_{13}, d \, z_{132}, e^{z_1} \, d \, z_{133} \rangle \) |
| (2, 3) | \( \mathbb{C} \langle e^{-z_1} \, d \, z_{12}, e^{z_1} \, d \, z_{13}, d \, z_{132}, e^{z_1} \, d \, z_{133} \rangle \) |
| (3, 0) | \( \mathbb{C} \langle d \, z_{123}, e^{-z_1} \, d \, z_{1232}, e^{z_1} \, d \, z_{1233}, e^{-z_1} \, d \, z_{1233} \rangle \) |
| (3, 1) | \( \mathbb{C} \langle e^{-z_1} \, d \, z_{12}, e^{-z_1} \, d \, z_{13}, e^{z_1} \, d \, z_{12}, d \, z_{13} \rangle \) |
| (3, 2) | \( \mathbb{C} \langle e^{-z_1} \, d \, z_{12}, e^{z_1} \, d \, z_{13}, d \, z_{132}, e^{z_1} \, d \, z_{133} \rangle \) |
| (3, 3) | \( \mathbb{C} \langle d \, z_{123}, e^{-z_1} \, d \, z_{1232}, e^{z_1} \, d \, z_{1233}, e^{-z_1} \, d \, z_{1233} \rangle \) |

Table 7. The double complex \( C^{*,*}_{\Gamma} \) in \((\mathfrak{m})\) for the complex parallelizable Nakamura manifold in case \((\mathfrak{m})\).

We compute the Bott-Chern cohomology for the complex parallelizable Nakamura manifold in case \((\mathfrak{m})\) in Table 8.

The differential algebra \( A^* \) for the complex parallelizable Nakamura manifold in case \((\mathfrak{m})\) is summarized in Table 9.

**Remark 2.27.** Suppose \( b \in 2\pi \mathbb{Z} \) and \( d \in 2\pi \mathbb{Z} \). Considering another Lie group \( H := \mathbb{C} \ltimes \psi \mathbb{C}^2 \) such that
\[
\psi(z) := \begin{pmatrix} e^{\frac{1}{2}(z_1 + z_3)} & 0 \\ 0 & e^{-\frac{1}{2}(z_1 + z_3)} \end{pmatrix},
\]
the correspondence \( G \in (z_1, z_2, z_3) \mapsto (z_1, z_2, z_3) \in H \) gives an embedding \( \Gamma \hookrightarrow H \) as a lattice and hence we can identify \( \Gamma \backslash G \) with \( \Gamma \backslash H \), see [74, Section 3]. Since \( H \) is equal to the solvable completely-solvable Lie group in Example 2.17 this case is identified with case \((\mathfrak{m})\) in Example 2.17. Note that \( A^* \) is not \( G \)-left-invariant in this case (for example the 2-form \( d \, z_{23} \) is not \( G \)-left-invariant).
and hence $H^\ast (\wedge \ast \mathfrak{g}^*; d) \not\cong H^\ast_{dR} (\Gamma \backslash G ; \mathbb{R})$, see also [27, Corollary 4.2]. On the other hand, we have $H^\ast (\wedge \ast \mathfrak{h}^*; d) \simeq H^\ast_{dR} (\Gamma \backslash H ; \mathbb{R})$, where $\mathfrak{h}$ is the Lie algebra of $H$. In [23, Main Theorem], it is proven that, for any solvmanifold $\Gamma \backslash G$, there exist a connected simply-connected solvable Lie group $\tilde{G}$ and a finite index subgroup $\tilde{\Gamma} \subseteq \Gamma$ such that $H^\ast (\wedge \ast \mathfrak{g}^*; d) \simeq H^\ast_{dR} (\tilde{\Gamma} \backslash \tilde{G} ; \mathbb{R})$, where $\mathfrak{g}$ is the Lie algebra of $\tilde{G}$.

(b) If $b \not\in \pi \mathbb{Z}$ or $d \not\in \pi \mathbb{Z}$, then the sub-complex $B^\ast_{\Gamma}$ defined in (1) is

$$B^1_{\Gamma} = \mathbb{C} \langle d \bar{z}_1 \rangle, \quad B^2_{\Gamma} = \mathbb{C} \langle d \bar{z}_2 \wedge d \bar{z}_3 \rangle,$$
Remark 2.28. Note that, for any \((p, q) \in \mathbb{Z}^2\),
\[
\dim_C H_{BC}^{p,q}(X) + \dim_C H_{A}^{p,q}(X) = \dim_C H_{dR}^{p,q}(X) + \dim_C H_{\bar{\partial}}^{p,q}(X)
\]
in both case \((a)\) and case \((b)\); note also that
\[
\sum_{p+q=k} (\dim_C H_{BC}^{p,q}(X) + \dim_C H_{A}^{p,q}(X)) - 2 \dim_C H_{dR}^{k}(X; \mathbb{C}) = \begin{cases} 
8 & \text{for } k \in \{1, 5\} \\
20 & \text{for } k \in \{2, 4\} \\
24 & \text{for } k = 3 \\
0 & \text{otherwise}
\end{cases} \text{ in case } \((a)\),
\]
and
\[
\sum_{p+q=k} (\dim_C H_{BC}^{p,q}(X) + \dim_C H_{A}^{p,q}(X)) - 2 \dim_C H_{\bar{\partial}dR}^{k}(X; \mathbb{C}) = \begin{cases} 
4 & \text{for } k \in \{1, 5\} \\
8 & \text{for } k \in \{2, 4\} \\
8 & \text{for } k = 3 \\
0 & \text{otherwise}
\end{cases} \text{ in case } \((b)\).
2.5. Currents. Let $X$ be a compact complex manifold, of complex dimension $n$. Denote the space of currents on $X$ by $D^{\bullet,\bullet}X := D_{n-\bullet,\bullet}X$, namely, the topological dual space of $\wedge^{n-\bullet,\bullet}X$; endow $D^{\bullet,\bullet}X$ with a structure of double complex, by defining $\partial : D^{\bullet,\bullet}X \to D^{\bullet+1,\bullet}X$ and $\bar{\partial} : D^{\bullet,\bullet}X \to D^{\bullet,\bullet+1}X$ by duality.

By means of the injective operator

$$T : \wedge^{\bullet,\bullet}X \to D^{\bullet,\bullet}X, \quad T_\eta := \int_X \eta \wedge \cdot, $$

which satisfies $T \circ \partial = \partial \circ T$ and $T \circ \bar{\partial} = \bar{\partial} \circ T$, consider the de Rham double complex $(\wedge^{\bullet,\bullet}X, \partial, \bar{\partial})$ as a double sub-complex of $(D^{\bullet,\bullet}, \partial, \bar{\partial})$.

For $(p, q) \in \mathbb{Z}^2$, denote the sheaf of $p$-holomorphic forms on $X$ by $\Omega^p_X$, denote the sheaf of $(p, q)$-forms on $X$ by $A^{p,q}_X$, and denote the sheaf of bi-degree $(p, q)$-currents by $D^{p,q}_X$. Recall that, for any fixed $p \in \mathbb{Z}$, both

$$0 \to \Omega^p_X \to (A^{\bullet}_X, \bar{\partial}) \quad \text{and} \quad 0 \to \Omega^p_X \to (D^{\bullet,\bullet}_X, \bar{\partial})$$

2.5. Currents. Let $X$ be a compact complex manifold, of complex dimension $n$. Denote the space of currents on $X$ by $D^{\bullet,\bullet}X := D_{n-\bullet,\bullet}X$, namely, the topological dual space of $\wedge^{n-\bullet,\bullet}X$; endow $D^{\bullet,\bullet}X$ with a structure of double complex, by defining $\partial : D^{\bullet,\bullet}X \to D^{\bullet+1,\bullet}X$ and $\bar{\partial} : D^{\bullet,\bullet}X \to D^{\bullet,\bullet+1}X$ by duality.

By means of the injective operator

$$T : \wedge^{\bullet,\bullet}X \to D^{\bullet,\bullet}X, \quad T_\eta := \int_X \eta \wedge \cdot, $$

which satisfies $T \circ \partial = \partial \circ T$ and $T \circ \bar{\partial} = \bar{\partial} \circ T$, consider the de Rham double complex $(\wedge^{\bullet,\bullet}X, \partial, \bar{\partial})$ as a double sub-complex of $(D^{\bullet,\bullet}, \partial, \bar{\partial})$.

For $(p, q) \in \mathbb{Z}^2$, denote the sheaf of $p$-holomorphic forms on $X$ by $\Omega^p_X$, denote the sheaf of $(p, q)$-forms on $X$ by $A^{p,q}_X$, and denote the sheaf of bi-degree $(p, q)$-currents by $D^{p,q}_X$. Recall that, for any fixed $p \in \mathbb{Z}$, both

$$0 \to \Omega^p_X \to (A^{\bullet}_X, \bar{\partial}) \quad \text{and} \quad 0 \to \Omega^p_X \to (D^{\bullet,\bullet}_X, \bar{\partial})$$
Remark 2.29. More precisely, given $X$ a compact complex manifold, for any $p \in \mathbb{Z}$ and for any $q \in \mathbb{Z}$, the maps $T: (\wedge^{p\cdot} X, \partial) \to (D^{p\cdot} X, \partial)$ and $T: (\wedge^{p\cdot} X, \overline{\partial}) \to (D^{p\cdot} X, \overline{\partial})$ are quasi-isomorphisms.

Indeed, firstly, we show that $T: (\wedge^{p\cdot} X, \partial) \to (D^{p\cdot} X, \partial)$ induces an injective map in cohomology. Fix $g$ a Hermitian metric on $X$. If $T|_{\alpha} = [\overline{\partial}S] = [0] \in H^{\bullet}(D^{p\cdot} X, \overline{\partial})$ with $\alpha$ the $\overline{\partial}$-harmonic representative of $[\alpha] \in H^{\bullet}(\wedge^{p\cdot} X, \overline{\partial})$ and $S \in D^{p\cdot -1} X$, then in particular $T|_{\alpha} \mid_{\ker \overline{\partial}} = 0$. Since $\tilde{s}_g \alpha \in \ker \overline{\partial}$, it follows that $0 = T_{\alpha}(\tilde{s}_g \alpha) = \int_{X} \alpha \wedge \tilde{s}_g \alpha$ and hence $\alpha = 0$. Now, since $\ker(\overline{\partial}: \wedge^{p\cdot} X \to \wedge^{p\cdot +1} X)$ and $\ker(\overline{\partial}: \wedge^{p\cdot} X \to \overline{\partial}\text{-currents})$ are isomorphic $\mathbb{C}$-vector spaces of finite dimension, it follows that $T: (\wedge^{p\cdot} X, \overline{\partial}) \to (D^{p\cdot} X, \overline{\partial})$ is actually a quasi-isomorphism. By conjugation, also $T: (\wedge^{q\cdot} X, \partial) \to (D^{q\cdot} X, \partial)$ is a quasi-isomorphism.

By applying Proposition [14] to $(\wedge^{p\cdot} X, \overline{\partial}) \to (D^{p\cdot} X, \overline{\partial})$, or by noting that both $0 \to \underline{\mathcal{C}}_{X} \to (\mathcal{A}_{X}^\bullet \otimes \mathbb{C}, d)$ and $0 \to \underline{\mathcal{C}}_{X} \to (\mathcal{D}_{X}^\bullet \otimes \mathbb{C}, d)$ are acyclic resolutions of the constant sheaf $\underline{\mathcal{C}}_{X}$ over $X$ (where, for $k \in \mathbb{Z}$, the sheaf of $k$-forms on $X$ is denoted by $\mathcal{A}_{X}^{\bullet}$, and the sheaf of degree $k$-currents is denoted by $\mathcal{D}_{X}^{\bullet}$), one gets that
\[
\frac{\ker (d: \wedge^{k} X \otimes \mathbb{R} \to \wedge^{k+1} X \otimes \mathbb{R})}{\ker (\overline{\partial}: \wedge^{k} X \otimes \mathbb{R} \to \overline{\partial}\text{-currents})} \simeq H^{\bullet}(X; \underline{\mathcal{C}}_{X}) \simeq \frac{\ker (d: D^{k} X \otimes \mathbb{R} \to D^{k+1} X \otimes \mathbb{R})}{\ker (d: D^{k} X \otimes \mathbb{R} \to D^{k+1} X \otimes \mathbb{R})}.
\]

Lemma 2.30. Let $X$ be a compact complex manifold. For any $(p,q) \in \mathbb{Z}^2$, the map $T: \wedge^{p\cdot q} X \to D^{p\cdot q} X$ induces the isomorphism
\[
\frac{\ker (d: \wedge^{p\cdot q} X \to \wedge^{p\cdot q+1} X \otimes \mathbb{R})}{\ker (d: D^{p\cdot q+1} X \otimes \mathbb{R} \to D^{p\cdot q+1} X \otimes \math{C})} \simeq \frac{\ker (d: D^{p\cdot q} X \to D^{p\cdot q+1} X \otimes \mathbb{R})}{\ker (d: D^{p\cdot q} X \otimes \mathbb{R} \to D^{p\cdot q+1} X \otimes \mathbb{R})}.
\]

| dim$_{\mathbb{C}} H^{\bullet \bullet}_{\mathbb{C}}(\Gamma \setminus G)$ | case (1) $\overline{\partial}$ BC | case (2) $\partial$ BC |
|-----------------|------------------|------------------|
| (0, 0) | 1 | 1 | 1 | 1 | 1 |
| (1, 0) | 2 | 3 | 3 | 1 | 2 | 3 |
| (0, 1) | 3 | 3 | 1 | 1 | 2 | 3 |
| (2, 0) | 5 | 3 | 3 | 3 | 3 |
| (1, 1) | 9 | 7 | 3 | 3 | 1 | 1 |
| (0, 2) | 3 | 3 | 1 | 1 |
| (3, 0) | 1 | 1 | 1 | 1 |
| (2, 1) | 8 | 9 | 9 | 4 | 3 | 3 |
| (1, 2) | 8 | 9 | 9 | 3 | 3 |
| (0, 3) | 1 | 1 | 1 |
| (3, 1) | 3 | 3 | 3 | 3 |
| (2, 2) | 5 | 9 | 11 | 3 | 3 | 5 |
| (1, 3) | 3 | 3 | 3 | 1 |
| (3, 2) | 2 | 3 | 5 | 2 | 1 | 3 |
| (2, 3) | 3 | 5 | 3 | 3 |
| (3, 3) | 1 | 1 | 1 | 1 | 1 | 1 |
Proof. Consider the regularization process in [31 Theorem III.12]: there exist \( R: D^{\bullet, \bullet} X \to \wedge^{\bullet, \bullet} X \) and \( A: D^{\bullet} X \otimes_{\mathbb{C}} \mathbb{C} \to D^{\bullet+1} X \otimes_{\mathbb{C}} \mathbb{C} \) linear operators such that
\[
\text{id}_{D^{\bullet, \bullet} X} = R + d A + A d, \quad \text{and} \quad R|_{\wedge^{\bullet, \bullet} X} = \text{id}_{\wedge^{\bullet, \bullet} X} \quad \text{and} \quad A|_{\wedge^{\bullet, \bullet} X} = 0.
\]
Take \( S \in \ker (d: D^{p,q} X \to D^{p+1,q} X \otimes_{\mathbb{C}} \mathbb{C}) \). Since the map \( T: \wedge^{\bullet, \bullet} X \to D^{\bullet, \bullet} X \) is a quasi-isomorphism, then there exist \( \eta \in \ker d \cap \wedge^{p,q} X \) and \( U \in D^{p,q-1} X \otimes_{\mathbb{C}} \mathbb{C} \) such that
\[
S = T \eta + d U;
\]
hence one gets
\[
RS = T \eta + d (U - AS),
\]
and hence the lemma follows. \( \square \)

As a consequence, by using Theorem [163] we get another proof of the following result by M. Schweitzer: see [64], and also [47 §3.4], where it is noticed as a consequence of the hypercohomological interpretation of the Bott-Chern cohomology, see also [30 IV.12.1].

Corollary 2.31 (see [64 §4.d]). Let \( X \) be a compact complex manifold. Then, for any \((p,q) \in \mathbb{Z}^2\), the natural map
\[
T: \frac{\ker (\partial + \overline{\partial}: \wedge^{p,q} X \to \wedge^{p+1,q} X \oplus \wedge^{p,q+1} X)}{\ker (\partial + \overline{\partial}: \wedge^{p,q} X \to \wedge^{p+1,q} X \oplus \wedge^{p,q+1} X)} \to \frac{\ker (\partial + \overline{\partial}: D^{p,q} X \to D^{p+1,q} X \oplus D^{p,q+1} X)}{\ker (\partial + \overline{\partial}: D^{p,q} X \to D^{p,q+1} X)}
\]
induced by \( T: \wedge^{\bullet, \bullet} X \ni \eta \mapsto T \eta := \int_X \eta \wedge \cdot \in D^{\bullet, \bullet} X \) is an isomorphism.

Proof. We firstly prove that \( T \) induces an injective map in Bott-Chern cohomology. Indeed, let \( \alpha = [\alpha] \in H^{p,q}_{BC}(X) \) be such that \( |T \alpha| = 0 \in \ker (\partial + \overline{\partial}: D^{p,q} X \to D^{p+1,q} X \oplus D^{p,q+1} X) \). Choose \( g \) a Hermitian metric on \( X \), and let \( \alpha \in \wedge^{p,q} X \) be the \( \overline{\Delta}^{BC} \)-harmonic representative of \( \alpha \) with respect to \( g \). Therefore, there exists \( S \in D^{p,q-1} X \) such that \( T \eta = \partial \overline{\partial} S \). In particular, \( T \eta \mid_{\ker \partial \overline{\partial}} = 0 \). Since \( \ast \eta \alpha \in \ker \partial \overline{\partial} \), it follows that \( 0 = T \eta (\ast \eta \alpha) = \int_X \alpha \wedge \cdot \), and hence \( \alpha = [\alpha] = 0 \).

We prove now that \( T \) induces a surjective map in Bott-Chern cohomology. Firstly, by Remark [2.29] for any \( p \in \mathbb{Z} \) and for any \( q \in \mathbb{Z} \), the maps \( (\wedge^{p,q} X, \partial) \to (D^{p,q} X, \partial) \) and \( T: (\wedge^{p,q} X, \overline{\partial}) \to (D^{p,q} X, \overline{\partial}) \) are quasi-isomorphisms. Furthermore, by Lemma [2.30] the induced map
\[
T: \frac{\ker (d: \wedge^{\bullet} X \otimes \mathbb{C} \to \wedge^{\bullet+1} X \otimes \mathbb{C}) \cap \wedge^{p,q} X}{\ker (d: D^{\bullet} X \otimes \mathbb{C} \to D^{\bullet+1} X \otimes \mathbb{C}) \cap D^{p,q} X} \to \frac{\ker (d: D^{\bullet} X \otimes \mathbb{C} \to D^{\bullet+1} X \otimes \mathbb{C}) \cap D^{p,q} X}{\ker (d: D^{\bullet} X \otimes \mathbb{C} \to D^{\bullet} X \otimes \mathbb{C})}
\]
is surjective. Hence, Theorem [1.15] applies, yielding that the map \( T \) induces a surjective map in Bott-Chern cohomology. \( \square \)

Remark 2.32. Given \( X \) a compact manifold of complex dimension \( n \) and \( A \) a finite group of biholomorphisms of \( X \), consider the compact complex orbifold \( \tilde{X} := X/G \) of complex dimension \( n \) (namely, \[63\] Definition 2), \( \tilde{X} \) is a singular complex space whose singularities are locally isomorphic to quotient singularities \( \mathbb{C}^n/G \) with \( G \subset \text{GL}(\mathbb{C}^n) \) finite; see \[15\] Theorem 1, see also \[57\] Theorem 1.7.2).

By extending the action of \( G \) on \( X \) to \( \wedge^{\bullet} X \), respectively \( \wedge^{\bullet, \bullet} X \), set \( \wedge^{\bullet} \tilde{X} \) the space of \( G \)-invariant forms on \( \wedge^{\bullet} X \), respectively \( \wedge^{\bullet, \bullet} \tilde{X} \) the space of \( G \)-invariant forms in \( \wedge^{\bullet, \bullet} X \). Analogously, consider \( D^{\bullet} \tilde{X} \) the space of \( G \)-invariant currents in \( D^{\bullet} \tilde{X} \), respectively \( D^{\bullet, \bullet} \tilde{X} \) the space of \( G \)-invariant currents in \( D^{\bullet, \bullet} \tilde{X} \).

Consider the sub-complex \( T: (\wedge^{\bullet, \bullet} \tilde{X}, \partial, \overline{\partial}) \to (D^{\bullet, \bullet} \tilde{X}, \partial, \overline{\partial}) \). By W. L. Baily’s result [12] page 807, and arguing as in Remark [1.15] by means of a Hermitian metric on \( \tilde{X} \), namely, a \( G \)-invariant Hermitian metric on \( X \), it follows that, for any \( p \in \mathbb{Z} \), the induced inclusion \( T: (\wedge^{p,\bullet} \tilde{X}, \overline{\partial}) \to (D^{p,\bullet} \tilde{X}, \overline{\partial}) \) is a quasi-isomorphism; by conjugation, it follows also that, for any \( q \in \mathbb{Z} \), the induced inclusion \( T: (\wedge^{\bullet, q} \tilde{X}, \partial) \to (D^{\bullet, q} \tilde{X}, \partial) \) is a quasi-isomorphism. In particular, by using Proposition [1.11] one recovers that the induced inclusion \( T: (\wedge^{\bullet, \bullet} \tilde{X}, d) \to (D^{\bullet, \bullet} \tilde{X}, d) \) is a quasi-isomorphism, as proved also by I. Satake. \[63\] Theorem 1.]
We note that the inclusion $T: \mathcal{A}^\bullet \times \mathcal{A}^\bullet \to \mathcal{D}^\bullet \times \mathcal{D}^\bullet$ induces the surjective map

$$\ker \left( d: \mathcal{A}^{p+q} \otimes \mathcal{C} \to \mathcal{A}^{p+q+1} \otimes \mathcal{C} \right) \cap \mathcal{A}^p \mathcal{A}^q \to \im \left( d: \mathcal{D}^{p+q-1} \otimes \mathcal{C} \to \mathcal{D}^{p+q} \otimes \mathcal{C} \right),$$

indeed, since $g^* \circ T \circ g^* = T$ for any $g \in G$, the regularization (see [31 Theorem III.12]) of a $G$-invariant current of bidegree $(p, q)$ gives a $G$-invariant $(p, q)$-form.

Hence, Theorem 1.3 applies, yielding that, for any $(p, q) \in \mathbb{Z}^2$, the inclusion $T$ induces an isomorphism

$$\ker \left( d: \mathcal{A}^{p,q} \to \mathcal{A}^{p,q+1} \otimes \mathcal{A}^{p,q+1} \right) \cong \ker \left( d: \mathcal{D}^{p,q-1} \to \mathcal{D}^{p,q} \right),$$

as proved also in [5].

Note that one can argue also by means of the sheaf-theoretic interpretation of the Bott-Chern and Aeppli cohomologies, developed by J.-P. Demailly, [30, §12], as proved also in [5, Theorem 1].

Remark 2.33 ([5]). We note that the results in Section 1 can be used also to investigate the symplectic Bott-Chern and Aeppli cohomologies, as introduced and studied by L.-S. Tseng and S.-T. Yau in [66, §4], see also [47, §3.2].

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