Flux Compactifications of String Theory on Twisted Tori

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Abstract

Global aspects of Scherk-Schwarz dimensional reduction are discussed and it is shown that it can usually be viewed as arising from a compactification on the compact space obtained by identifying a (possibly non-compact) group manifold $G$ under a discrete subgroup $\Gamma$, followed by a truncation. This allows a generalisation of Scherk-Schwarz reductions to string theory or M-theory as compactifications on $G/\Gamma$, but only in those cases in which there is a suitable discrete subgroup of $G$. We analyse such compactifications with flux and investigate the gauge symmetry and its spontaneous breaking. We discuss the covariance under $O(d,d)$, where $d$ is the dimension of the group $G$, and the relation to reductions with duality twists. The compactified theories promote a subgroup of the $O(d,d)$ that would arise from a toroidal reduction to a gauge symmetry, and we discuss the interplay between the gauge symmetry and the $O(d,d,\mathbb{Z})$ T-duality group, suggesting the role that T-duality should play in such compactifications.

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1 Introduction

In [1], Scherk and Schwarz proposed two related forms of dimensional reduction of field theories, both of which led to non-abelian gauge symmetries, a scalar potential and mass terms. Somewhat confusingly, both have been referred to as Scherk-Schwarz reductions in the literature. In one type, a theory with a global duality symmetry is reduced on a circle or torus with a duality twist or monodromy around each circle. Following [2], we will refer to these as reductions with a duality twist.

In the other type of reduction introduced in [1], the dependence of fields on the internal coordinates $y^i$ is through a matrix $\sigma^i_m(y)$, so that for example the internal components of the metric $g_{ij}(x, y)$ lead to scalar fields $\phi_{mn}$ depending only on the remaining external coordinates $x$ through the ansatz

$$g_{ij}(x, y) = \phi_{mn}(x)\sigma^m_i(y)\sigma^n_j(y)$$

where $\sigma^m_i(y)$ is the inverse of $\sigma^i_m(y)$. This leads to a reduced theory in which the $y$-dependence drops out completely provided the matrices $\sigma^m_i(y)$ satisfy the constraint that the coefficients

$$f^m_{np} = -\sigma^i_n\sigma^j_p(\partial_i\sigma^m_j - \partial_j\sigma^m_i)$$

are constant. Then the one-forms $\sigma^m = \sigma^m_i(y)dy^i$ satisfy the structure equation

$$d\sigma^m + \frac{1}{2}f^m_{np}\sigma^n \wedge \sigma^p = 0$$

and the integrability condition for this is that the constants $f^m_{np}$ satisfy the Jacobi identity and so are the structure constants for a Lie group $G$. In (1.1), the ansatz $g_{ij}(x, y) = \phi_{ij}(x)$ that would be used for a toroidal reduction is ‘twisted’ by the matrices $\sigma^i_m(y)$ and so the reduction is sometimes referred to as reduction on a ‘twisted torus’, and we will use this terminology here. However, we will be particularly interested in the global structure and we will see that although the internal space looks like a torus locally, the global structure can be quite different, and so this terminology can be rather misleading. In many standard cases, such as those discussed in section 3, the internal space is in fact a torus bundle over a circle or torus, so that the name is appropriate, but other examples include those in which the internal space is a compact group manifold, so that the internal space is very different from a torus, twisted or otherwise.

We will be interested here in flux compactifications in which the $(p+1)$-form field strength $\hat{G}$ for a $p$-form gauge field has a flux of the form

$$\frac{1}{(p+1)!}K_{mn...p+1}\sigma^m \wedge \sigma^n \wedge ... \wedge \sigma^{p+1} + ...$$

where $K^m_{m_1m_2...m_{p+1}}$ are constant coefficients, satisfying constraints that ensure that this form is closed.
In [1], reductions with duality twists arose as particular examples of reductions on twisted tori, but this is not true in general, as will be discussed below. The reductions of [1] are of field theories such as supergravities, with a truncation to a lower-dimensional field theory that is independent of the extra coordinates \( y \). An important feature is that both kinds of Scherk-Schwarz reductions allow consistent truncations [3, 4, 5, 6], in the sense that it is consistent with the full higher-dimensional field equations to set all the massive Kaluza-Klein modes to zero while keeping a finite number of light or massless fields non-zero (this is not true for generic compactifications; for example, Calabi-Yau compactifications are not consistent in this sense).

A key question is whether such reductions can be extended to the full Kaluza-Klein theory or string theory in a way that gives sensible lower-dimensional physics with a mass-gap. This is a non-trivial question as there are many cases where such extensions do not work. For example, there are gauged supergravities with non-compact gauge group whose lift to higher dimensions is to a background with non-compact ‘internal’ space [7]. In such cases, there can be a consistent truncation to a lower-dimensional supergravity, but if the Kaluza-Klein spectrum of modes depending on the internal coordinates are included, one finds a continuous spectrum without mass gap, so that the theory cannot be properly regarded as a lower-dimensional theory at all, but is best interpreted in the full higher-dimensional space-time [8, 9, 10]. As the gauge groups arising from Scherk-Schwarz reductions are typically non-compact, there is a danger that the full lift of a Scherk-Schwarz reduction could be to such a ‘non-compactification’. If a reduction scheme can be regarded as a compactification on a compact internal space, then it can be extended to the full Kaluza-Klein theory or string theory with a mass gap. It is to this question of whether Scherk-Schwarz reductions can be viewed as reductions on compact spaces that we now turn.

Consider first the reduction on the twisted torus. The simplest way of realising this is if the internal manifold is the group manifold of \( \mathcal{G} \) with \( \sigma^m \) the left-invariant Maurer-Cartan forms, which automatically satisfy (1.3). For compact groups, this of course leads to a compactification, although the reduction ansatz is not the usual one. For a non-compact group, the internal group manifold is non-compact, but one can still consistently truncate to the light sector that is independent of the internal group-manifold coordinates, recovering the lower-dimensional field theory of the Scherk-Schwarz reduction. However, in this case there would be no mass-gap and so no satisfactory way of extending to the full theory. As was pointed out in [1], the group \( \mathcal{G} \) being non-compact does not necessarily imply that the internal manifold is non-compact, so that it is possible for the internal space to be compact so that there is a mass-gap and a well-defined Kaluza-Klein reduction. For non-compact groups, if the internal space is compact, then it cannot be the group manifold. Nonetheless, for the reduction to be well-defined, the matrices \( \sigma_m^i(y) \) should exist globally on the internal manifold, so that there are globally defined one-forms \( \sigma^m \). This implies that the internal manifold is parallelisable, and so locally must be a group manifold. Thus the
internal manifold must be a group manifold identified under the action of some freely acting
discrete group $\Gamma$. The group manifold admits a natural left action $G_L$ and a right action $G_R$,
but only the left action preserves the one-forms $\sigma^m$ appearing in the ansatz, so the discrete
group $\Gamma$ must be a subgroup of $G_L$. Then the internal space must be of the form $G/G'$ for
some discrete left-acting $\Gamma \subset G_L$, and we are particularly interested in the cases in which
$\Gamma$ can be chosen so that $G/G'$ is compact. Such a $\Gamma$ is said to be cocompact, and not all
groups have cocompact discrete subgroups. Groups without a cocompact discrete subgroup
give Scherk-Schwarz reductions of supergravity that cannot be extended to compactifications
of string theory in this way. Note that in general $\Gamma$ need not be unique, and we can also
consider discrete quotients $G/G'$ for compact $G$.

One of the aims of this paper is to study reductions on twisted tori from the global
viewpoint, showing that they can be regarded as dimensional reductions on compact internal
spaces of the form $G/G'$ (when $\Gamma$ can be chosen so that $G/G'$ is compact) and so can be extended
to string theory. There has been much interest in applying Scherk-Schwarz-type reductions
to supergravity (including [2,11-25]) and to string theory (including [2, 24, 26, 27, 28, 29]),
but there are important issues as to how to properly define the full string theory (as opposed
to its supergravity limit). Regarding the reduction as a compactification on $G/G'$ allows a
proper definition of the full string theory. It is important that the ansatz for the metric,
gauge fields and fluxes of the background is invariant under rigid $G_L$, so that one can identify
under a discrete subgroup of $G_L$.

Identifying the theory under the action of a discrete subgroup of the isometry group
will break part of the isometry symmetry, and so affect the gauge symmetry of the reduced
theory. There has been some confusion in the literature as to the way that gauge symmetry
works in these compactifications, and our viewpoint clarifies some of the issues involved. We
give a careful treatment of the gauge symmetry, and its breaking.

The simplest case is that in which the internal space is the group manifold for a compact
group $G$. The conventional Kaluza-Klein ansatz for compactification on the group manifold
introduces gauge fields for $G_L \times G_R$ and has $G_L \times G_R$ local gauge symmetry. The ground state
in which the internal metric is the bi-invariant Killing metric $\delta_{mn}\sigma^m_i(y)\sigma^n_j(y)$ has isometry
$G_L \times G_R$ and so preserves the full gauge group. The Scherk-Schwarz ansatz for the same
compact group manifold is a truncation of this to the sector invariant under $G_L$ and only
has gauge fields for $G_R$. This ansatz is invariant under local $G_R$ transformations but only
under rigid $G_L$ transformations, and the Killing metric gives a vacuum preserving all these
symmetries. The conventional $G_L \times G_R$ ansatz does not allow a consistent truncation, but
the Scherk-Schwarz one does. Since only invariance under $G_L$ is required, a more general
ansatz for the ground state metric with $g_{ij} = \phi_{mn}\sigma^m_i(y)\sigma^n_j(y)$ for any positive definite matrix
$\phi_{mn}$ is permissible, and this breaks $G_R$ to the subgroup preserving this background value for
the scalars $\phi_{mn}$. 
Consider now the case of non-compact $G$. In order to be able to factor by a discrete subgroup of $G_L$, one must use the Scherk-Schwarz ansatz which is invariant under rigid $G_L$ symmetry, and the resulting theory has $G_R$ gauge symmetry, but the identification under $\Gamma$ breaks the $G_L$ symmetry. The space $G/\Gamma$ still has an action of $G_R$, with infinitesimal transformations given by well-defined vector fields on $G/\Gamma$, and introducing gauge fields in the adjoint of $G$ gives a theory with local $G_R$ symmetry. However, this non-compact gauge symmetry is necessarily non-linearly realised and so is broken to a linearly realised compact subgroup. (A non-compact symmetry can have no unitary representation on a finite number of fields, and so must be non-linearly realised.) The scalar fields $\phi_{mn}$ define a positive definite metric on the internal space, and the expectation value of this for any ground state will not in general be invariant under the full non-compact $G_R$ symmetry but will spontaneously break this to a subgroup.

We turn now to consider reductions with duality twists, and their implementation in string theory. A theory in $D+1$ dimensions with a global symmetry $G$ is reduced on a circle with the dependence of fields $\Psi(x, y)$ (in some representation of $G$) on the circle coordinate $y$ given by a $y$-dependent $G$ transformation $\Psi(x, y) = \exp(My)\psi(x)$ where $M$ is a generator of $G$ in the appropriate representation. However, in the full theory in which all massive modes are kept, $G$ is typically broken to a discrete subgroup $G(\mathbb{Z})$.

For the reduction with duality twist, on going round the circle, $y \rightarrow y + 2\pi$ there is a monodromy $\mathcal{M} = \exp(2\pi M)$. In order for $\Psi(x, y + 2\pi) = \mathcal{M}\Psi(x, y)$ to be well-defined, the monodromy $\mathcal{M}$ must be in the symmetry group $G(\mathbb{Z})$ [30]. However, if only $G(\mathbb{Z})$ is a symmetry of the full theory, it is not immediately clear how to extend the continuous action of $G$ that is needed in the ansatz from the low-energy sector to the full theory with massive modes and only $G(\mathbb{Z})$ symmetry, and care is needed in the full definition of the theory. For the case of the toroidal reduction on $T^d$ followed by a reduction on a circle with a $Gl(d, \mathbb{Z})$ twist, this was resolved in [26]. There it was shown that the reduction can be viewed as a reduction on a $d+1$-dimensional compact space that is a $T^d$ bundle over a circle. As this is a compactification, it can be used in string theory. Moreover, this compact space is locally a group manifold for a non-semi-simple lie group, as we shall show in section 3. Then the torus bundle is locally the group manifold for a group $G$, and globally is of the form $G/\Gamma$ for a discrete subgroup $\Gamma$, and so this is a reduction on a twisted torus. In this case, the twisted torus is a torus bundle over a circle, so that in this case it is actually a topological twisting of a torus.

In this example, the twist is by a duality group that has a geometric realisation in the higher dimensional space – in this case $GL(\mathbb{Z})$ is the group of large diffeomorphisms of $T^d$. However, the T-duality or U-duality groups that arise in string theory have many elements that are not geometric in this sense, and such non-geometric twists have been considered in [2, 31]. In such cases, there is not in general any way of realising such reductions as
reductions on a geometric background, and these cannot be realised as reductions on twisted tori, and any such lift would lead to what might be called a non-geometric background; such backgrounds have been discussed in [32, 33]. For such duality twist reductions corresponding to non-geometric backgrounds, one needs to check that these really are consistent string backgrounds. However, in some cases, these can be realised as ‘compactifications’ of F-theory (or one of its generalisations) on a twisted torus [26]. Here, we will restrict ourselves to geometric backgrounds that can be realised as twisted tori.

In this paper, we will consider generalised Scherk-Schwarz compactifications on $d$-dimensional twisted tori $G/\Gamma$ with flux for the 3-form field strength $H$ (the NS-NS 3-form in type II or heterotic superstring theory); other fluxes will be considered elsewhere. The flux will be taken to be of the form

$$K = \frac{1}{6} K_{mnp} \sigma^m \wedge \sigma^n \wedge \sigma^p$$

(1.5)

for some constant coefficients $K_{mnp}$ and so is manifestly invariant under $G_L$. The gauge group contains the isometry group $G$ with $d$ gauge generators $Z_m$ corresponding to the Killing vectors of the internal space. For string theory, there are, as we shall see, an additional $d$ generators $X^m$ associated with the gauge fields arising from the reduction of the 2-form gauge field, and the structure constants of the $2d$ dimensional gauge group generated by $(Z_m, X^m)$ depend on the 3-form flux. It was shown in [15] that the final theory has an elegant formulation that is covariant under the action of $O(d,d)$, with the gauge generators $(Z_m, X^m)$ combining to form a vector of $O(d,d)$, and manifestly invariant under the gauge group $G$, which can be viewed as a subgroup of $O(d,d)$. Indeed, the low energy field theory can be thought of as arising from a gauging of a subgroup of the $O(d,d)$ symmetry that would arise in a torus reduction. For the heterotic string, the $O(d,d)$ symmetry is contained in $O(d,d+16)$ while for type II strings it is contained in the U-duality group. We review the construction of [15] for the common sector of the type II and heterotic theories, focusing on the $O(d,d)$ subgroup of the full symmetry.

In [6], the case of compact $G$ was considered and there it was found that field redefinitions simplified the structure of the theory, so that the resulting gauge algebra $G$ simplified to the semi-direct product of $G$ with $U(1)^d$. We find that in the non-compact case, such redefinitions are not possible in general, and the presence of flux necessarily leads to non-trivial structure of the gauge group $G$.

The structure of the paper is as follows. In section 2 we discuss the global structure of twisted torus reductions, showing that they are reductions on discrete quotients of group manifolds. In section 3 we discuss reductions with duality twists and show that they are twisted torus compactifications when the duality symmetry is of geometrical origin. In section 4 we discuss dimensional reduction on twisted tori with fluxes and in section 5 apply this to string theory, and discuss the gauge symmetry and $O(d,d)$ covariance that were found by Kaloper and Myers [15]. In Section 6 we analyse the breaking of the gauge symmetry.
The final section discusses the generalisations of our results to M-theory compactifications, and addresses the role of the \( O(d,d) \) covariance. On the one hand, a subgroup of the \( O(d,d) \) symmetry of the toroidal reduction of the low-energy field theory has been promoted to a gauge symmetry, but in the full string theory one expects the \( O(d,d) \) symmetry to be broken to a discrete T-duality subgroup, so that the issue of whether a subgroup of \( O(d,d) \) can be gauged in the full string theory arises. We resolve the issue of the relation between the discrete symmetry and the local gauge symmetry, and discuss the status of T-duality in such compactifications.

2 Twisted Tori and Group Manifolds

In this section we review Scherk-Schwarz reduction of field theory and show that in many cases it can be viewed as a compactification on a compact twisted torus, and so can be extended to string theory.

2.1 Scherk-Schwarz Reduction

We shall consider Scherk-Schwarz dimensional reduction [1] in which the internal space is a \( d \) dimensional manifold \( \mathcal{X} \) with coordinates \( y^i \) and a basis of nowhere-vanishing one-forms \( \sigma^m \) specified by a vielbein \( \sigma^m_i(y) \)

\[
\sigma^m = \sigma^m_i(y) dy^i
\]  

(2.1)

In the ansatz of [1], the internal components \( T_{ij...k} \) of a tensor field \( T_{MN...P} \) are taken to have \( y \) dependence given only by the frame fields

\[
T_{ij...k}(x,y) = T_{mn...p}(x) \sigma^m_i \sigma^n_j ... \sigma^p_k
\]  

(2.2)

defining scalar fields \( T_{mn...p}(x) \) in the reduced theory, so that for example the internal metric takes the form (1.1). The frame fields satisfy the structure equation

\[
d\sigma^m + \frac{1}{2} f^m_{np} \sigma^n \wedge \sigma^p = 0
\]  

(2.3)

where the coefficients \( f^m_{np} \) are the structure constants for some Lie group \( \mathcal{G} \), satisfying the Jacobi identity

\[
f^q_{[mn} f^r_{p]q} = 0
\]  

(2.4)

Such manifolds \( \mathcal{X} \) are sometimes referred to as twisted tori in the cases in which the coordinates \( y^i \) satisfy periodicity conditions, and the matrix \( \sigma^m_i(y) \) can be thought of as defining the twisting of the frames with respect to the coordinate basis. The structure equation implies that \( \mathcal{X} \) is locally isomorphic to the group manifold \( \mathcal{G} \), but this need not be true globally; in general, \( \mathcal{X} = \mathcal{G}/\Gamma \) where \( \Gamma \) is a discrete subgroup of \( \mathcal{G} \), and \( \mathcal{X} \) can be compact even if \( \mathcal{G} \) is.
non-compact. Then Scherk-Schwarz reduction can be viewed as compactification on a group manifold or \( \mathcal{X} = \mathcal{G}/\Gamma \) (if this is compact). Note that in general \( \mathcal{G}/\Gamma \) will not be a group manifold.

### 2.2 Compact Groups

The simplest case is that in which \( \mathcal{X} \) is the group manifold for a compact group \( \mathcal{G} \), with \( \sigma^m \) the left-invariant Maurer-Cartan one-forms, so that if \( t_m \) are Lie algebra generators \( \sigma^m t_m = g^{-1}dg \) for some \( g \in \mathcal{G} \). It will be useful to introduce the Cartan-Killing metric for \( \mathcal{G} \) given by \( \eta_{mn} = \frac{1}{2}f^q_{mp}f^p_{nq} \) (which is proportional to \( \delta_{mn} \) for compact \( \mathcal{G} \)). Then the metric

\[
ds^2 = \eta_{mn}\sigma^m\sigma^n \tag{2.5}
\]

is invariant under the isometry group \( \mathcal{G}_L \times \mathcal{G}_R \), where \( \mathcal{G}_L \) is the left action \( g \to k_Lg \) and \( \mathcal{G}_R \) is the right action \( g \to gk_R \) for \( k_L, k_R \in \mathcal{G} \). The inverse vielbein \( \sigma^i_m \) can be used to define the left-invariant vector fields

\[
Z_m = \sigma^i_m \frac{\partial}{\partial y^i} \tag{2.6}
\]

with Lie bracket

\[
[Z_m, Z_n] = f^p_{mn}Z_p \tag{2.7}
\]

There are also right-invariant one-forms \( \tilde{\sigma}^m \) with \( \tilde{\sigma}^m t_m = dg g^{-1} \) satisfying

\[
d\tilde{\sigma}^m - \frac{1}{2}f^m_{np}\tilde{\sigma}^n \wedge \tilde{\sigma}^p = 0 \tag{2.8}
\]

and right-invariant vector fields \( \tilde{Z}_m \) given by

\[
\tilde{Z}_m = \tilde{\sigma}^i_m \frac{\partial}{\partial y^i} \tag{2.9}
\]

The Killing vectors \( Z_m \) generate \( \mathcal{G}_R \) which leaves \( \tilde{\sigma}^m \) invariant while the Killing vectors \( \tilde{Z}_m \) generate \( \mathcal{G}_L \) which leaves \( \sigma^m \) invariant.

A conventional Kaluza-Klein reduction on a group manifold would take the bi-invariant metric (2.5) as the vacuum and would introduce Kaluza-Klein gauge fields for the full isometry group \( \mathcal{G}_L \times \mathcal{G}_R \). The gauge group for the dimensionally reduced theory would then be \( \mathcal{G}_L \times \mathcal{G}_R \) with vector fields \( A^m_\mu \) for \( \mathcal{G}_R \) and \( \tilde{A}^m_\mu \) for \( \mathcal{G}_L \). The coordinates of the spacetime are \( \{x^\mu, y^i\} \) where \( x^\mu \) are the coordinates of the uncompactified part of spacetime. However, for generic theories there is no consistent truncation to a dimensionally reduced theory with a finite number of fields and it is necessary to keep the full Kaluza-Klein tower of states [4]. This can be easily seen from the Einstein equations for the metric \( g_{\mu\nu} \), the right-hand-side of which includes a term \( (Z_m \cdot \tilde{Z}_n)F^m_{\mu\nu}\tilde{F}^n_{\nu\rho} \) where \( F^m_{\mu\nu} \) and \( \tilde{F}^m_{\mu\nu} \) are the field strengths of \( A^m_\mu \) and \( \tilde{A}^m_\mu \) respectively. The fact that \( (Z_m \cdot \tilde{Z}_n) \) depends non-trivially on \( y \) means that \( g_{\mu\nu} \)
must be taken to be a function of both \( x \) and \( y \), and a truncation to a finite set of fields depending only on \( x \) would be inconsistent with the field equations; it is necessary to keep all the massive Kaluza-Klein modes \( g^N_{\mu\nu}(x) \) from the expansion of \( g_{\mu\nu}(x,y) \). However, for the theories with a metric, 2-form and dilaton that arise in string theory, there is evidence that there is a consistent truncation to a theory with gauge group \( G_L \times G_R \) \([5, 6]\).

In the Scherk-Schwarz reduction, by contrast, the ansatz introduces only Kaluza-Klein gauge fields \( A^m_\mu \) for the isometry group \( G_R \), and truncates the full Kaluza-Klein spectrum to the sector of singlets under \( G_L \). This does allow a consistent truncation to a finite number of fields, as the Einstein equations now only involve source terms such as \((Z_m \cdot Z_n) F^m_{\mu\nu} F^n_{\nu\lambda}\) and the fact that \((Z_m \cdot Z_n) = \eta_{mn}\) is independent of \( y \) allows a truncation to a metric \( g_{\mu\nu}(x) \) that is independent of \( y \). The ansatz for the metric is

\[
ds_D^2 = e^{2\alpha \varphi} g_{\mu\nu} dx^\mu dx^\nu + e^{2\beta \varphi} g_{mn} \nu^m \nu^n
\]

where the one-forms \( \nu^m \) are

\[
\nu^m = \sigma^m - A^m
\]

and \( \alpha, \beta \) are constants. Dimensional reduction gives rise to a metric \( g_{\mu\nu}(x) \), \( d \) Kaluza-Klein one-form gauge fields \( A^m_\mu(x) \), and \( d(d+1)/2 \) scalars \( \varphi(x) \) and \( g_{mn}(x) \), where \( g_{mn}(x) \) is a positive definite symmetric matrix with unit determinant. This ansatz is invariant under rigid \( G_L \) transformations, and under local \( G_R \) transformations in which the parameters depend on \( x^\mu \) and the \( A^m \) transform as gauge fields, while the scalar fields \( g_{mn}(x) \) transform as the bi-adjoint. A vacuum in which the scalars have the expectation value \( \bar{g}_{mn} = \eta_{mn} \) will be invariant under \( G_R \) while any other expectation value \( \bar{g}_{mn} \) will break the gauge symmetry to the subgroup preserving \( \bar{g}_{mn} \).

The ansatz for antisymmetric tensor gauge fields is the most general one that is invariant under \( G_L \), so that for a \( p \)-form potential

\[
\bar{B}_p = B_p + B_{(p-1)m} \wedge \nu^m + \frac{1}{2!} B_{(p-2)mn} \wedge \nu^m \wedge \nu^n + \ldots + \frac{1}{p!} B_{(0)m_1m_2m_3\ldots m_p} \nu^{m_1} \wedge \ldots \wedge \nu^{m_p} + \varpi(p)
\]

where \( B_p \) is an \( p \)-form gauge field on \( M_d \) and a flux term \( \varpi(p) \) is included (see section 3.2). Again, the invariance under \( G_L \) guarantees consistency \([3]\). We will be particularly interested here in the case in which \( p = 2 \), in which case the reduction gives a 2-form, \( d \) vector fields \( B_{(1)m} \) and scalars \( B_{(0)m_1m_2} \). The reduction of antisymmetric tensors, and the constraints that must be imposed on the flux, will be discussed further in section 4.2.

### 2.3 Non-Compact Groups

Consider now the case of non-compact \( G \), so that \( \eta_{mn} \) is no longer positive definite. If the group is non-semi-simple, \( \eta_{mn} \) will be non-invertible. For a spacetime which is a (possibly warped) product of a spacetime \( M \) and the non-compact group manifold \( G \), then one
can consider $G$ as an internal space and attempt to expand in terms of modes on $G$, but the resulting theory has no mass-gap in general and cannot be properly regarded as a $d$-dimensional theory. Nonetheless, there is still a consistent truncation as above to a finite set of fields in $M$, using the same ansatz (2.10). In this case, there is usually no $G_R$-invariant ground state. This is because the internal metric $g_{mn}$ of the ground state is required to be positive-definite and usually there is no positive definite invariant metric for a non-compact group. The Cartan-Killing metric is invariant but not positive-definite. For semi-simple non-compact groups, the only invariant metric is the Cartan-Killing metric, but non-semi-simple groups sometimes have positive invariant metrics [34]. One cannot set $g_{mn} = \eta_{mn}$, and any expectation value for $g_{mn}$ will break the gauge symmetry to the subgroup of $G_R$ preserving the expectation value $\bar{g}_{mn}$.

As the ansatz (2.10),(2.12) is invariant under rigid $G_L$ transformations, one can identify the internal space under the action of a discrete subgroup $\Gamma$ of $G_L$ so that the internal space is the left coset $\mathcal{X} = G/\Gamma$. If the discrete subgroup is chosen so that $\mathcal{X} = G/\Gamma$ is compact, then one can perform a compactification with this internal space and there will be a Kaluza-Klein spectrum with a mass-gap governed by the size of $\mathcal{X}$. Then $\mathcal{X}$ is locally isomorphic to the group manifold $G$, and much of the structure will be the same. In particular, the left-invariant one-forms $\sigma^m$ are well-defined on $\mathcal{X}$ and satisfy the structure equation (2.3) so that this is a Scherk-Schwarz compactification, and any such compactification must be of this type. The low-energy effective physics in $M$ only depends on the local structure of $\mathcal{X}$, and so must contain the consistent truncation of the theory on $G$ with gauge symmetry $G_R$ described in the previous paragraph. This must then be a consistent truncation of the theory on $G/\Gamma$ also.

Importantly, not every group $G$ has a cocompact discrete subgroup $\Gamma$ that gives a compact space $G/\Gamma$. Only if there is such a $\Gamma$ can the Scherk-Schwarz reduction and truncation of the low-energy field theory be promoted to a Kaluza-Klein compactification or a compactification of the field theory. If there is such a $\Gamma$, it may not be unique and compactifications on $G/\Gamma$ or $G/\Gamma'$ will give different theories that have the same effective low-energy theory. In particular, for a compact group $G$, we could consider either compactification on $G$ itself, or a quotient $G/\Gamma$.

The right-action of $G_R$ is well-defined on the left-coset $G/\Gamma$, so that the vector fields $Z_m$ that generate this action are well-defined on the quotient space. For any given expectation value $\bar{g}_{mn}$ of the internal metric, only a subset of the $Z_m$ will be Killing vectors, and the gauge symmetry is spontaneously broken to the compact subgroup of $G_R$ generated by the $Z_m$ which are Killing vectors for $\bar{g}_{mn}$. The compactified theory has local gauge symmetry under the full non-compact gauge group $G_R$, even though there are no vacua with a full set of Killing vectors $Z_m$ generating $G_R$, and this is always broken to a compact subgroup in any solution.


3 Reductions With Duality Twists

3.1 Geometric Twists

In this section we discuss reductions with duality twists and show that in a large class of cases they are equivalent to compactifications on twisted tori.

Consider a $D+d+1$ dimensional field theory coupled to gravity. We reduce the theory on a $d$-dimensional torus $T^d$, with real coordinates $z^a \sim z^a + 1$ where $a = 1, 2...d$. This produces a theory in $D+1$ dimensions with scalar fields that include those in the coset $GL(d, \mathbb{R})/SO(d)$ arising from the torus moduli. Truncating to the $z^a$ independent zero mode sector, this theory has a global $G$ symmetry that contains the $GL(d, \mathbb{R})$ arising from diffeomorphisms of the torus, while in the full Kaluza-Klein theory this is broken to the $GL(d, \mathbb{Z})$ that acts as large diffeomorphisms on the $d$-torus. In string theory $G$ is typically broken to a discrete subgroup $G(\mathbb{Z})$. For supergravity theories with sufficient supersymmetry, the full set of scalar fields typically take values in the coset $G/K$, where $K \subset G$ is the maximal compact subgroup of $G$. We denote the action of $G$ on fields $\psi$ of the reduced theory in some representation of $G$ as $\psi \rightarrow \gamma[\psi]$.

We reduce to $D$ dimensions on a further circle with periodic coordinate $y \sim y+1$, twisting the fields over the circle by an element of $G$ using the ansatz [26, 24, 16, 2]

$$\psi(x^\mu, y) = \gamma_y[\psi(x^\mu)]$$

(3.1)

where $x^\mu$ are the $D$ spacetime coordinates. Consistency of the reduction requires the reduced theory to be independent of $y$, which is achieved by choosing the form of $\gamma$ to be

$$\gamma(y) = \exp(My)$$

(3.2)

for some mass matrix $M$ in the Lie algebra of $G$. (The masses of the reduced theory are given by the matrix $M$.) The map $\gamma(y)$ is not periodic, but has monodromy $\mathcal{M}(\gamma) = \gamma(0)\gamma(1)^{-1} = e^M$ in $G$. The physically distinct reductions are classified by the conjugacy class of the monodromy [26].

We now focus on the case in which the monodromy is in the geometrical $GL(d, \mathbb{R})$ subgroup of $G$. If it is in fact in $G(\mathbb{Z})$, then the reduction is equivalent to the compactification on a $T^d$ bundle over a circle, with monodromy $\mathcal{M}$ [26]. We will see that this compact space is locally a group manifold, i.e. it is of the form $G/\Gamma$.

Let

$$ds^2 = H(\tau)_{ab}dz^a dz^b$$

(3.3)

be the metric on the $d$-torus, depending on the moduli $\tau$, which take values in the coset $GL(d, \mathbb{R})/SO(d)$. There is a natural action of $GL(d, \mathbb{R})$ on the metric and coordinates $z^a$ in
which
\[ H_{ab} \to (U^t)_a^c H_{cd} U^d_b \quad z^a \to (U^{-1})^a_b z^b \] (3.4)
where \( U^b_a \in GL(d, \mathbb{R}) \). This defines the transformation \( \tau \to \tau' \) of the moduli through
\[ H_{ab}(\tau') = (U^t)_a^c H_{cd}(\tau) U^d_b \] (3.5)

In the twisted reduction, we introduce dependence on the circle coordinate \( y \) through a \( GL(d, \mathbb{R}) \) transformation \( U = \gamma(y) \) where \( \gamma(y) = \exp(My) \). This defines the \( y \)-dependence of \( \tau \) through
\[ H(\tau(y))_{ab} = (\gamma(y)^t)_a^c H(\tau_0)_{cd} \gamma(y)^d_c \] (3.6)
for some arbitrary choice of \( \tau_0 \). If the monodromy is in \( SL(d, \mathbb{Z}) \), which we now assume, then the twisted reduction is equivalent to the reduction on a \( T^d \) bundle over \( S^1 \) with metric
\[ ds^2_{d+1} = dy^2 + H(\tau(y))_{ab} dz^a dz^b = (\sigma^y)^2 + H(\tau_0)_{ab} \sigma^a \sigma^b \] (3.7)
where
\[ \sigma^y = dy \quad \sigma(y)^a = \gamma(y)^a_b dz^b \] (3.8)

We now consider the group structure of this space. The forms (3.8) are globally defined on the torus bundle, and satisfy
\[ d\sigma^a + M^a_b \sigma^y \wedge \sigma^b = 0 \] (3.9)
The space is then parallelisable, and locally looks like a group manifold \( \mathcal{G} \) with Maurer-Cartan forms \( \sigma \) associated with the Lie algebra
\[ [t_a, t_y] = M^b_a t_b, \quad [t_a, t_b] = 0 \] (3.10)
This algebra can be represented by the \((d + 1) \times (d + 1)\) matrices
\[ t_y = \begin{pmatrix} -M^a_b & 0 \\ 0 & 0 \end{pmatrix}, \quad t_a = \begin{pmatrix} 0 & e_a \\ 0 & 0 \end{pmatrix} \] (3.11)
where \( e_a \) is the \( d \)-dimensional column vector with a 1 in the \( a \)'th position and zeros everywhere else. Coordinates \( y, z^a \) can be introduced for the group manifold, with the group element \( g = g(y, z^a) \in \mathcal{G} \) given by
\[ g = \begin{pmatrix} \gamma^{-1}(y) & z \\ 0 & 1 \end{pmatrix} \] (3.12)
Then the left-invariant Maurer-Cartan forms are given by
\[ g^{-1} dg = \begin{pmatrix} -M^a_b \sigma^y & \sigma^a \\ 0 & 0 \end{pmatrix} = \sigma^m t_m \] (3.13)
in agreement with (3.8).

The left action of

$$h(\alpha, \beta^a) = \begin{pmatrix} \gamma^{-1}(\alpha) & \beta \\ 0 & 1 \end{pmatrix}$$

(3.14)

is

$$g(y, z^a) \rightarrow h(\alpha, \beta^a) \cdot g(y, z^a)$$

(3.15)

and acts on the coordinates through

$$y \rightarrow y + \alpha \quad z^a \rightarrow (e^{-M\alpha})^a_b z^b + \beta^a$$

(3.16)

The $h(\alpha, \beta^a)$ with $\alpha, \beta^a \in \mathbb{Z}$ can be written as

$$h(\alpha, \beta^a) = \begin{pmatrix} \mathcal{M}^{-a} & \beta \\ 0 & 1 \end{pmatrix}$$

(3.17)

and form a discrete subgroup $\Gamma$ and we can identify $\mathcal{G}$ under the action of $\Gamma$, so that the coordinates $y, z^a$ are subject to the identifications

$$y \sim y + \alpha \quad z^a \sim (\mathcal{M}^{-a})^a_b z^b + \beta^a$$

(3.18)

for $\alpha, \beta^a \in \mathbb{Z}$. This in general gives a compact space, and is the required twisted torus construction.

### 3.2 Non-Geometric Twists and F-Theory

In some cases, the discussion of geometric twists can be extended to non-geometric twists, i.e. twists by duality transformations that do not arise from higher-dimensional diffeomorphisms. Consider for example the $SL(2, \mathbb{Z})$ U-duality of the IIB string theory [30]. Reducing from 10 to 9 dimensions on a circle with monodromy in $SL(2, \mathbb{Z})$ was investigated in [16, 17, 26, 35]. As the $SL(2, \mathbb{Z})$ symmetry is not geometric, this cannot be realised as a compactification on a twisted torus in the usual way. However, it can be realised as a ‘compactification’ of F-theory on the twisted torus corresponding to a $T^2$ bundle over $S^1$ with $SL(2, \mathbb{Z})$ monodromy [26]. Many other examples can be thought of as compactifications of F-theory [36] or its generalisations [37]. For example, the reduction of M-theory to 7 dimensions, followed by a reduction on a further circle with a twist by an $SL(5, \mathbb{Z})$ U-duality transformation can be viewed as a compactification of the $F'$ theory of [37] on a twisted torus constructed as a $T^5$ bundle over $S^1$ [26].

### 3.3 Examples with $SL(2)$ Twists

We now consider the example of $d = 2$ in more detail. Reducing from $D + 3$ dimensions on $T^2$, and then on a further circle with an $SL(2, \mathbb{Z})$ twist is equivalent to reducing on a $T^2$
bundle over \(S^1\) [26] with monodromy \(\mathcal{M} = e^M \in SL(2, \mathbb{Z})\). The \(T^2\) has moduli \(A, \tau = \tau_1 + i\tau_2\) where \(A\) is the area of the torus and \(\tau\) its complex structure modulus, and the metric is

\[
H(\tau) = A \frac{1}{\tau_2} \begin{pmatrix}
1 & \tau_1 \\
\tau_1 & |\tau|^2
\end{pmatrix}
\tag{3.19}
\]

There is an action of \(SL(2, \mathbb{R})\) under which an element \(g \in SL(2, \mathbb{R})\) given by

\[
g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc = 1 \tag{3.20}
\]

acts on the torus modulus as

\[
\tau \to \frac{a\tau + b}{c\tau + d} \equiv \gamma[\tau] \tag{3.21}
\]

In the \(T^2\) bundle over \(S^1\), the torus modulus \(\tau\) varies with the circle coordinate \(y\), with the \(y\) dependence given by the \(SL(2, \mathbb{R})\) transformation \(\gamma(y) = exp(My)\), so that

\[
\tau(y) = \gamma(y)[\tau_0] \tag{3.22}
\]

for some fixed \(\tau_0\). The metric on the bundle is then (3.7), which can be rewritten as [26]

\[
ds^2 = R^2dy^2 + \frac{A}{Im(\tau)}|dz|^2 + \tau|dz|^2 \tag{3.23}
\]

with \(\tau(y) = \gamma(y)[\tau_0]\) and constant \(A\).

### 3.3.1 Conjugacy Classes of \(SL(2, \mathbb{R})\) and \(SL(2, \mathbb{Z})\)

For the reduced theory truncated to the sector independent of the internal coordinates, the monodromy can be in \(SL(2, \mathbb{R})\), and the distinct theories are classified by \(SL(2, \mathbb{R})\) conjugacy classes and there are three distinct theories corresponding to the three conjugacy classes of \(SL(2, \mathbb{R})\) [26]. If the massive Kaluza-Klein modes are kept, then the monodromy must be in \(SL(2, \mathbb{Z})\), and there is a richer class of theories corresponding to the conjugacy classes of \(SL(2, \mathbb{Z})\).

For \(SL(2, \mathbb{R})\) monodromy, the three conjugacy classes are the elliptic, the parabolic and the hyperbolic classes which have \(|Tr(e^{M_e})| < 2\), \(|Tr(e^{M_p})| = 2\) and \(|Tr(e^{M_h})| > 2\) respectively. The three conjugacy classes of \(SL(2, \mathbb{R})\) are

\[
\mathcal{M}_e = U e^{N_e} U^{-1}, \quad \mathcal{M}_h = U e^{N_h} U^{-1}, \quad \mathcal{M}_p = U e^{N_p} U^{-1} \tag{3.24}
\]

where the mass matrices are

\[
N_p = \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}, \quad N_h = \begin{pmatrix} m & 0 \\ 0 & -m \end{pmatrix}, \quad N_e = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix} \tag{3.25}
\]
where $\theta$ takes values in the range $[0, 2\pi]$, $m$ is real and $U$ is an arbitrary matrix of $SL(2, \mathbb{R})$.

For the monodromy $M$ to be in $SL(2, \mathbb{Z})$ requires ‘quantization conditions’ on the parameters $m, \theta$ and restrictions on $U$. For the parabolic class, the restriction to $SL(2, \mathbb{Z})$ requires $m \in \mathbb{Z}$ and the conjugacy classes are represented by

$$M_m = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$$

(3.26)

where $m \in \mathbb{Z}$ give a distinct conjugacy class for each integer $m$.

For the elliptic class, if $U = 1$ then $\theta = m\pi/2$ where $m$ is an integer, giving two classes represented by

$$M_2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad M_4 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(3.27)

which generate the groups $\mathbb{Z}_2$ and $\mathbb{Z}_4$ respectively. There are two more elliptic $SL(2, \mathbb{Z})$ conjugacy classes (with $U \neq 1$), represented by

$$M_3 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \quad M_6 = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$$

(3.28)

which generate the $\mathbb{Z}_3$ and $\mathbb{Z}_6$ groups respectively. These are of the form

$$M_3 = U N_e(\theta = 2\pi/3)U^{-1} \quad M_6 = U N_e(\theta = \pi/3)U^{-1}$$

(3.29)

for certain $U$.

There is an infinite family of hyperbolic conjugacy classes with $U \neq 1$ given by

$$M_m = \begin{pmatrix} m & 1 \\ -1 & 0 \end{pmatrix}$$

(3.30)

where $m \in \mathbb{Z}$ and $m > 2$. In addition there are an infinite number of sporadic monodromies $M(t)$ where $t$ denotes the trace of the matrix, again with $U \neq 1$ [38]. The first five are:

$$M(8) = \begin{pmatrix} 1 & 2 \\ 3 & 7 \end{pmatrix} \quad M(10) = \begin{pmatrix} 1 & 4 \\ 2 & 9 \end{pmatrix} \quad M(12) = \begin{pmatrix} 1 & 2 \\ 5 & 11 \end{pmatrix}$$

$$M(13) = \begin{pmatrix} 2 & 3 \\ 7 & 11 \end{pmatrix} \quad M(14) = \begin{pmatrix} 1 & 1 \\ 6 & 13 \end{pmatrix} \ldots$$

(3.31)

3.3.2 Parabolic Twist

We shall consider the application of these monodromy matrices to Scherk-Schwarz reductions. First, we consider a twist by an element of the parabolic conjugacy class of $SL(2, \mathbb{R})$ given by

$$M_p = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$$

(3.32)
for some fixed $m \in \mathbb{R}$. This will be in $SL(2, \mathbb{Z})$ if the mass parameter $m$ is quantized, $m \in \mathbb{Z}$. The Scherk-Schwarz ansatz gives

\[
\gamma_p(y) = \begin{pmatrix} 1 & my \\ 0 & 1 \end{pmatrix}, \quad \tau(y) = \tau_0 + my
\]

(3.33)

where the mass matrix is

\[
M_p = \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}
\]

(3.34)

and $\tau_0 = \tau_1 + i\tau_2$ is some constant modulus. For simplicity we shall choose $A = R = 1$ in the metric (3.23).

If $m \in \mathbb{Z}$, this reduction may be thought of as a twisted torus reduction on $G/\Gamma$ where $G$ is the group manifold for the Heisenberg group [27], as we shall now review. This space is sometimes called the nilmanifold and the metric is given by (3.23) with $\tau(y)$ given by (3.33).

The generators satisfy the Heisenberg algebra

\[
[t_2, t_y] = mt_1 \quad [t_1, t_y] = 0 \quad [t_1, t_2] = 0
\]

(3.35)

and the group element $g \in G$ corresponding to the coordinates $y, z^a$ is

\[
g(y, z^a) = \begin{pmatrix} 1 & -my & z^1 \\ 0 & 1 & z^2 \\ 0 & 0 & 1 \end{pmatrix}
\]

(3.36)

The Heisenberg group is non-compact and the compact nilmanifold is obtained by identifying the coordinates under

\[
(y, z^1, z^2) \sim (y + \alpha, z^1 - m\alpha z^2 + \beta^1, z^2 + \beta^2)
\]

(3.37)

with $\alpha, \beta^a \in \mathbb{Z}$. This can be understood as a quotient by a discrete subgroup $\Gamma \subset G_L$. This identification may be written as the left quotient $g \sim h \cdot g$ where $h \in \Gamma$ and $\Gamma$ is the discrete subgroup of matrices of the form

\[
h(\alpha, \beta^a) = \begin{pmatrix} 1 & -m\alpha & \beta^1 \\ 0 & 1 & \beta^2 \\ 0 & 0 & 1 \end{pmatrix} \in \Gamma
\]

(3.38)

with integer $\alpha, \beta^a$. The left action $g \rightarrow h \cdot g$ leaves the metric invariant, so the identification is consistent with the ansatz.

### 3.3.3 Elliptic Twist

As a second example, we take $M$ to lie in the elliptic conjugacy class of $SL(2, \mathbb{Z})$ where

\[
\gamma_e(y) = U \begin{pmatrix} \cos(\theta y) & \sin(\theta y) \\ -\sin(\theta y) & \cos(\theta y) \end{pmatrix} U^{-1}, \quad M_e = U \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix} U^{-1}
\]

(3.39)
then the group $G$ generated by $\{t_a, t_y\}$ is $ISO(2)$, the group of isometries of the Euclidean plane and $\theta$ takes values between 0 and $2\pi$. The monodromy is

$$M_e = U \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} U^{-1}$$

and if $U = 1$ the complex structure is

$$\tau = \frac{\tau_0 \cos(\theta y) + \sin(\theta y)}{-\tau_0 \sin(\theta y) + \cos(\theta y)}$$

Note that if $\tau_0 = i$, then $\tau = i$ is independent of $\theta$.

The metric of the internal space of the reduced theory is $ds^2 = (\sigma^y)^2 + H(y)_{ab} \sigma^a \sigma^b$ where the left-invariant one-forms are $\sigma^m = (\sigma^y, \gamma_e(y)^a dz^b)$. The group manifold of $ISO(2)$ has topology $S^1 \times \mathbb{R}^2$ and is parameterised by matrices of the form

$$g(y, z^a) = U \begin{pmatrix} \cos(\theta y) & -\sin(\theta y) & z^1 \\ \sin(\theta y) & \cos(\theta y) & z^2 \\ 0 & 0 & 1 \end{pmatrix} U^{-1}$$

where

$$U = \begin{pmatrix} U & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\tilde{z}^a = (U^{-1})^a_b z^b$$

The space is compactified under the left action of

$$h(\alpha, \beta^a) = \begin{pmatrix} 1 & 0 & \beta^1 \\ 0 & 1 & \beta^2 \\ 0 & 0 & 1 \end{pmatrix}$$

where we require $\beta^a \in \mathbb{Z}$.

The Cartan-Killing metric is degenerate for this case; $\eta = diag\{1, 0, 0\}$ but there is an invariant metric of the form $diag\{a, b, b\}$ for any $a, b$. For any choice of metric $g_{mn}$ on $X = T^3$, the vector fields $Z_1, Z_2$ generating the non-compact part of $G$ will not be Killing vectors and the gauge group $ISO(2)$ will be broken to at most the compact subgroup $SO(2)$. Indeed, for this reduction the scalar potential has a minimum at $g_{mn} = \delta_{mn}$ [2] and in this vacuum the gauge group $ISO(2)$ is spontaneously broken to the $SO(2)$ generated by $Z_y$.

As a final comment, we note that if $\tau_0 = i$ the $y$-dependence of $\tau$ cancels out in the above ansatz and $\tau = \tau_0 = i$ as $\tau_0 = i$ is a fixed point of the $SL(2, \mathbb{R})$ transformation generated by $M$. This fixed point of the T-duality twist is a minimum of the Scherk-Schwarz potential $V(\tau, A)$ at $\tau = i$, and the theory at the minimum corresponds to to an orbifold reduction for which there is an exact conformal field theory description [2].
3.3.4 Hyperbolic Twist

The third and final conjugacy class is the hyperbolic with mass matrix $M$ and monodromy $\mathcal{M}$ given by

$$M_h = U \begin{pmatrix} m & 0 \\ 0 & -m \end{pmatrix} U^{-1} \quad \mathcal{M}_h = U \begin{pmatrix} e^m & 0 \\ 0 & e^{-m} \end{pmatrix} U^{-1}$$

(3.46)

for some $U$. The Scherk-Schwarz ansatz gives dependence on the circle coordinate $y$ through

$$\gamma_h(y) = U \begin{pmatrix} e^{my} & 0 \\ 0 & e^{-my} \end{pmatrix} U^{-1}$$

(3.47)

The Lie algebra is that of $ISO(1,1)$ and the group elements are

$$g(y, z^a) = U \begin{pmatrix} e^{-my} & 0 & z^1 \\ 0 & e^{my} & z^2 \\ 0 & 0 & 1 \end{pmatrix} U^{-1}$$

(3.48)

and a compact space is obtained by identifying under the left action of a discrete subgroup $\Gamma \subset \mathcal{G}$ of matrices of the form

$$h(\alpha, \beta^a) = \begin{pmatrix} \mathcal{M}^{-a} & \beta \\ 0 & 1 \end{pmatrix}$$

(3.49)

where $\alpha, \beta^a$ are integers. The left-invariant Maurer-Cartan forms and generators of the right action are well defined on the compact space.

4 Scherk-Schwarz Dimensional Reduction

In this section, we review the results of Scherk-Schwarz dimensional reduction [1] in field theory, truncating to the zero-mode sector, and include fluxes, generalising the results of [15]. It can be viewed as a compactification on a $d$-dimensional compact space $\mathcal{X} = \mathcal{G}/\Gamma$ given by the left-coset of a group manifold $\mathcal{G}$ by a discrete subgroup $\Gamma$, as in the last section, and much of the structure of the theory reduced on $\mathcal{X}$ is the same as it would be for the reduction on the group manifold $\mathcal{G}$, and in particular there is a consistent truncation to a finite set of fields in the reduced theory. The ansatz uses the one-forms $\sigma^m = \sigma^m_i dy^i$ satisfying the structure equation

$$d\sigma^m + \frac{1}{2} f^m_{np} \sigma^n \wedge \sigma^p = 0$$

(4.1)
4.1 Gravity Reduction

We shall reduce the $D$-dimensional Einstein-Hilbert Lagrangian

$$\mathcal{L}_D = \hat{R} \ast 1 \quad (4.2)$$

on a $d$-dimensional manifold $\mathcal{X} = \mathcal{G}/\Gamma$, following the notation of [6]. The spacetime has coordinates $\{x^\mu, y^i\}$, where the $y^i$ are coordinates for the internal space $\mathcal{X}$ and the $x^\mu$ are the coordinates for the reduced spacetime. The most general left-invariant Einstein frame reduction ansatz is

$$ds^2_D = e^{2\alpha \varphi} ds^2_d + e^{2\beta \varphi} g_{mn} \nu^m \nu^n \quad (4.3)$$

where the one-forms

$$\nu^m = \sigma^m - A^m \quad (4.4)$$

introduce the Kaluza-Klein gauge fields $A^m_\mu$, which have two-form field strength

$$F^m = dA^m + \frac{1}{2} f^m_{np} A^n \wedge A^p \quad (4.5)$$

We have retained the $d \mathcal{G}_L$ invariant Kaluza-Klein gauge fields (graviphotons) $A^m(x) = A^m_\mu(x) dx^\mu$ and the $d(d+1)/2$ scalars $g_{mn}(x)$ and $\varphi$, where $g_{mn}(x)$ has unit determinant. All other fields in the gravity sector are truncated out in the Scherk-Schwarz ansatz. Note that the ansatz breaks the $\mathcal{G}_L \times \mathcal{G}_R$ symmetry of $\mathcal{G}$ down to $\mathcal{G}_L$ unless $g_{mn}(x)$ is an invariant metric, which for semi-simple groups requires it to be proportional to $\eta_{mn}$.

The condition for the volume element $(\sqrt{g})$ to be invariant under the left action $\mathcal{G}_L$ is

$$f^m_{nm} = 0 \quad (4.6)$$

which is the condition that the group be unimodular (i.e. the adjoint action on the Lie algebra is trace-free). If this is satisfied, then the action can be dimensionally reduced to give an action that is independent of the internal coordinates [1]. If this condition is not satisfied, one can instead dimensionally reduce the field equations, giving a set of field equations that are independent of the internal coordinates, but which in general cannot be derived from an action that is independent of the internal coordinates [39]. Thus for unimodular groups, there is a consistent truncation of the action, while for groups which are not unimodular, there is a consistent truncation of the field equations, but in general not of the action. If one keeps the full Kaluza-Klein theory without truncation, there is no need to apply this condition and, for compact internal space, there will still be a mass gap, and there will an infinite tower of massive field equations on the reduced space. Here we will present results for the reduction of the action for unimodular $\mathcal{G}$. There is a generalisation of our results

$^1$Note $\varphi$ introduced in the ansatz is distinct from the dilaton $\phi$ that will be introduced when we come to consider the string frame Lagrangian.
to the reduction of the field equations in the non-unimodular case (see [39] for examples). For reductions with duality twists for which $G$ has Lie algebra (3.10), the group will be unimodular if the mass matrix is traceless, $M_{ii} = 0$.

The reduced Lagrangian is [1, 6, 15]

$$L_D = R \ast 1 - \frac{1}{2} * d\varphi \wedge d\varphi - \frac{1}{2} g^{mp} g^{nq} * Dg_{mn} \wedge Dg_{pq} - \frac{1}{2} \epsilon^{2(\beta - \alpha)} g_{mn} * F^m \wedge F^n$$

$$- \frac{1}{2} \epsilon^{2(\beta - \alpha)} (g_{mn} g^{pq} g^{rs} f_{pq} f_{rs} + 2 g^{mn} f^p_{qm} f^q_{pn}) * 1$$

(4.7)

where

$$\alpha = - \left( \frac{D - d}{2(d - 2)(D - 2)} \right)^{\frac{1}{2}} \quad \beta = \left( \frac{d - 2}{2(D - d)(D - 2)} \right)^{\frac{1}{2}}$$

(4.8)

and

$$Dg_{mn} = dg_{mn} + g_{mp} f^p_{nq} A^q + g_{np} f^p_{mq} A^q$$

(4.9)

### 4.2 Antisymmetric Tensor Gauge Field Reduction with Flux

In this section we consider the reduction of a p-form gauge field $\hat{B}(p)$ with $p + 1$-form field strength $\hat{G}_{(p+1)} = d\hat{B}(p)$. We include the most general $G_L$-invariant flux for the field strength $\hat{G}_{(p+1)}$,

$$\hat{G}_{(p+1)} = \frac{1}{(p+1)!} K_{m_1m_2...m_{p+1}} \sigma^m \wedge \sigma^n \wedge \ldots \sigma^{p+1} + \ldots$$

(4.10)

where $K_{m_1m_2...m_{p+1}}$ are constant coefficients. The Bianchi identity $d\hat{G}_{(p+1)} = 0$, and therefore $d(K_{m_1m_2...m_{p+1}} \sigma^m \wedge \sigma^n \wedge \ldots \sigma^{p+1}) = 0$, imposes the integrability condition

$$K_{[m_1m_2m_3...m_{p+1}]} = 0$$

(4.11)

and we require the constant $K_{m_1m_2...m_{p+1}}$ to satisfy this algebraic constraint. We use the most general $G_L$-invariant reduction ansatz for $\hat{B}(p)$, which is

$$\hat{B}(p) = B(p) + B(p-1)m \wedge \nu^m + \frac{1}{2!} B(p-2)mn \wedge \nu^m \wedge \nu^n + \ldots$$

$$... + \frac{1}{p!} B(0)m_1m_2...m_p \nu^{m_1} \wedge \ldots \nu^{m_p} + \varpi(p)$$

(4.12)

where we have included the flux $\hat{G}_{m_1m_2...m_{p+1}} = (-)^p K_{m_1m_2...m_{p+1}}$ through $\varpi(p)$, which satisfies

$$d\varpi(p) = \frac{1}{(p+1)!} K_{m_1m_2...m_{p+1}} \sigma^{m_1} \wedge \ldots \wedge \sigma^{m_{p+1}}$$

(4.13)

We use this notation, even though the flux $\varpi(p)$ may not be defined globally, to emphasise the requirement $d(K_{m_1m_2...m_{p+1}} \sigma^m \wedge \sigma^n \wedge \ldots \wedge \sigma^{p+1}) = 0$. Note that the spectrum is the same as for a toroidal reduction, with one $p$-form, $d - p - 1$-forms, $d(d-1)/2 - p - 2$-forms etc, but these are now charged under the gauge group of the reduced theory in general.
The field strength \( \hat{G}_{(p+1)} \) can then be decomposed as

\[
\hat{G}_{(p+1)} = G_{(p+1)} + G_{(p)m} \wedge \nu^m + \frac{1}{2!} G_{(p-1)mn} \wedge \nu^m \wedge \nu^n + \ldots \\
\ldots + \frac{1}{(p+1)!} G_{(0)m_1m_2\ldots m_{p+1}} \nu^{m_1} \wedge \ldots \wedge \nu^{m_{p+1}}
\]

(4.14)

where, the \( i \)-form field strengths are

\[
G_{(i)m_1m_2\ldots m_{p+1-i}} = DB_{(i-1)m_1m_2\ldots m_{p+1-i}} + (-)^p B_{(i-2)m_1m_2\ldots m_{p+1-i}} \wedge F^n \\
+ (-)^p c(i, p) f_{m_1m_2} B_{(i)m_3m_4\ldots m_{p+1-i}} \wedge F^n \\
+ \frac{(-)^p}{i!} K_{m_1m_2\ldots m_{p+1}} A^{m_{p+1-i}} \wedge A^{m_{p+2-i}} \wedge \ldots \wedge A^{m_{p+1}}
\]

(4.15)

and the coefficient \( c(i, p) \) is given by

\[
c(i, p) = \frac{(p + 1 - i)!}{2(p - 1 - i)!}
\]

(4.16)

The \( G_L \)-covariant derivatives are

\[
DB_{(i-1)m_1m_2\ldots m_{p+1-i}} = dB_{(i-1)m_1m_2\ldots m_{p+1-i}} + (-)^i c(i, p) B_{(i-1)m_1m_2\ldots m_{p-i}} f_{m_{p-i+1}} [g] \wedge A^q
\]

(4.17)

The generic antisymmetric tensor Lagrangian

\[
L_G = -\frac{1}{2} \star \hat{G}_{(p+1)} \wedge \hat{G}_{(p+1)}
\]

(4.18)

is therefore reduced to

\[
L_G = -\frac{1}{2} \sum_{i=0}^{p+1} g^{m_1m_2} g^{m_3m_4} \ldots g^{m_{2(p+i-1)}} m_{2(p+i-1)} \star G_{(i)m_1m_2\ldots m_{2(p+i-1)}} \wedge G_{(i)m_2m_4\ldots m_{2(p+1-i)}}
\]

(4.19)

5 String Theory Compactifications on Twisted Tori

If a group \( \mathcal{G} \) has a cocompact discrete subgroup \( \Gamma \), then we have seen that the Scherk-Schwarz reduction of a field theory using the structure constants of \( \mathcal{G} \) is equivalent to the compactification on a compact twisted torus \( \mathcal{G}/\Gamma \) (or on the group manifold \( \mathcal{G} \) itself, if this is compact) with \( \mathcal{G}_L \)-invariant ansatz for the ground state fields, followed by a consistent truncation to a \( \mathcal{G}_L \)-invariant sector of the spectrum that is independent of the internal coordinates. The resulting theory has a local gauge symmetry that includes the isometry group \( \mathcal{G}_R \). As the internal space is compact, this can be extended to the full Kaluza-Klein theory by compactifying on \( \mathcal{G}/\Gamma \) with \( \mathcal{G}_L \)-invariant metric and fluxes, but keeping the full Kaluza-Klein spectrum, which will have a tower of massive states separated by a mass-gap.
This can now be extended to string theory or M-theory by considering compactification on the compact space $G/\Gamma$ with a $G_L$-invariant ansatz for the metric and anti-symmetric tensor gauge fields. Again there is a well-defined mass-gap, and the theory can be truncated to the Kaluza-Klein theory, which in turn can be truncated to the $G_L$-invariant sector and to the fields independent of the internal coordinates, corresponding to the Scherk-Schwarz reduction of the low-energy field theory. This effective field theory is a consistent truncation and for generic vacua includes all the light fields. However, for special vacua, the symmetry is enhanced and there are extra light fields. For example, for generic vacua, the metric gives gauge fields for the gauge group $G_R$, but if the metric is chosen to be a $G_L \times G_R$-invariant metric (the Cartan-Killing metric for a compact group) then the metric gives massless gauge fields for the gauge group $G_L \times G_R$. In the following subsections, we will describe the Scherk-Schwarz reduction of the low-energy field theory.

Thus a Scherk-Schwarz reduction of a low-energy field theory associated with a group $G$ can be extended to a Kaluza-Klein compactification or a compactification of string theory if there is a discrete subgroup $\Gamma \subset G_L$ such that $G/\Gamma$ is compact, i.e. if there is a cocompact discrete subgroup. A necessary condition for this is that the group be unimodular, i.e. that the structure constants satisfy $f_{nm}^m = 0$. Note that semi-simple groups are unimodular, and the groups with algebra (3.10) arising in reductions with duality twists are unimodular if the mass matrix is traceless, $M_i = 0$. For non-unimodular groups $G$, as discussed in section 4, although there is no Scherk-Schwarz reduction of the action, there is a Scherk-Schwarz reduction of the field equations, but this cannot be extended to a compactification of string theory.

### 5.1 Scherk-Schwarz Reduction of Low Energy Field Theory

We shall apply the results of the last section to the reduction of the $D$-dimensional Lagrangian

$$\mathcal{L}_S = e^{-\hat{\Phi}} \left( \hat{\mathcal{R}} \ast 1 + \ast d\hat{\Phi} \wedge d\hat{\Phi} - \frac{1}{2} \ast \hat{G}_{(3)} \wedge \hat{G}_{(3)} \right)$$

(5.1)

governing the massless fields of the bosonic string, or a subset of the massless bosonic fields of the various superstrings. It is related to the Einstein frame Lagrangian by a conformal scaling

$$ds^2_{\text{String}} = e^{-\hat{\Phi}} ds^2_{\text{Einstein}}$$

(5.2)

where $\hat{\Phi} = -\frac{1}{2a^2} \phi$, $a^2 = 8/(D - 2)$. The Lagrangian in the Einstein frame is

$$\mathcal{L}_E = \hat{\mathcal{R}} \ast 1 - \frac{1}{2} \ast d\phi \wedge d\phi - \frac{1}{2} e^{a\phi} \ast \hat{G} \wedge \hat{G}$$

(5.3)

where $\hat{G} = d\hat{B}$ and $\hat{B}$ is the Kalb-Ramond form.
The Scherk-Schwarz reduction of this theory was given in [1], and the generalization to include flux was considered in [6, 15]. We introduce a flux \( \hat{G} = K + \ldots \) where

\[
K = \frac{1}{6} K_{mnp} \sigma^m \wedge \sigma^n \wedge \sigma^p
\]  
(5.4)

with constant coefficients \( K_{mnp} \), and requiring \( dK = 0 \) gives the integrability condition

\[
K_{\ell[mn} f_{pq]}^\ell = 0
\]  
(5.5)

which will later arise as part of the Jacobi identities for the gauge algebra of this theory. As \( K \) is closed, there is locally a 2-form potential \( \varpi_{(2)} \) for the flux such that

\[
d\varpi_{(2)} = K
\]  
(5.6)

so that the ansatz for the potential is of the form \( \hat{B} = \varpi_{(2)} + \ldots \). The ansatz for the reduction of the potential is

\[
\hat{B} = B_{(2)} + B_{(1)m} \wedge \nu^m + \frac{1}{2} B_{(0)mn} \nu^m \wedge \nu^n + \varpi_{(2)}
\]  
(5.7)

giving the field strength

\[
\hat{G} = G_{(3)} + G_{(2)m} \wedge \nu^m + \frac{1}{2} G_{(1)mn} \wedge \nu^m \wedge \nu^n + \frac{1}{6} G_{(0)mnp} \nu^m \wedge \nu^n \wedge \nu^p
\]  
(5.8)

where

\[

g_{(3)} = dB_{(2)} + B_{(1)m} \wedge F^m + \frac{1}{6} K_{mnp} A^m \wedge A^n \wedge A^p
\]

\[
g_{(2)m} = DB_{(1)m} + B_{(0)mn} F^n + \frac{1}{2} K_{mnp} A^n \wedge A^p
\]

\[
g_{(1)mn} = DB_{(0)mn} + f^n_{mn} B_{(1)p} + K_{mnp} A^p
\]

\[
g_{(0)mnp} = 3B_{(0)[m|q|f_{np}|]} + K_{mnp}
\]  
(5.9)

and

\[
DB_{(1)m} = dB_{(1)m} - B_{(1)n} f^n_{mp} \wedge A^p
\]

\[
DB_{(0)mn} = dB_{(0)mn} + 2B_{(0)[m|p|f_{n|q|]} A^q}
\]  
(5.10)

The Lagrangian of the Kalb-Ramond sector of the reduced theory, in the Einstein frame, is then

\[
\mathcal{L}_D = e^{\phi - 4\alpha\varphi} \left( -\frac{1}{2} * G_{(3)} \wedge G_{(3)} - \frac{1}{2} e^{-2(\beta - \alpha)\varphi} g_{m}^{mn} * G_{(2)m} \wedge G_{(2)n} \\
- \frac{1}{2} e^{-4(\beta - \alpha)\varphi} g_{mn}^{pq} * G_{(1)mp} \wedge G_{(1)np} \\
- \frac{1}{2} e^{-6(\beta - \alpha)\varphi} g_{mn}^{pq} g^{ts} * G_{(0)mpt} \wedge G_{(0)nts} \right)
\]  
(5.11)
It will be convenient to work in the string frame, where the metric reduction ansatz is
\[ ds_D^2 = g_{\mu\nu} dx^\mu dx^\nu + g_{mn} \nu^m \nu^n \] (5.12)

Reducing (5.1) on the spacetime of (5.12) gives
\[ \mathcal{L}_D = e^{-\phi} \left( R * 1 + *d\phi \wedge d\phi + \frac{1}{2} * Dg^{mn} \wedge Dg_{mn} - \frac{1}{2} g_{mn} * F^m \wedge F^n - \frac{1}{2} * G(3) \wedge G(3) 
- \frac{1}{2} g^{mn} G(2)_m \wedge G(2)_n - \frac{1}{2} g^{mn} g^{pq} G(1)m_p \wedge G(1)n_q + V * 1 \right) \] (5.13)

where the potential \( V \) is
\[ V = -\frac{1}{4} g_{mn} g^{pq} g^{st} f^m_p f^s_t - \frac{1}{2} g^{mn} f^p_q f^q_p - \frac{1}{2} g^{mn} g^{pq} g^{st} G(0)m_p G(0)n_q \] (5.14)

Here \( \phi \) is the shifted dilaton
\[ \phi = \hat{\Phi} - \ln(\sqrt{g}) \] (5.15)

where \( g = \det(g_{mn}) \).

### 5.2 Gauge algebra

In this section we consider the gauge algebra of the reduced Lagrangian (5.13). The gauge fields of the reduced theory are the 2-form \( B(2) \), the \( d \) vector fields \( B(1)_m \) and the \( d \) vector fields \( A^m \) and we denote the generators of the corresponding gauge transformations \( W, X, Z \) and the parameters \( \Lambda(1) = \Lambda(1)(x^\mu) \), \( \lambda(0)_m = \lambda(0)_m(x^\mu) \) and \( \omega^m = \omega^m(x^\mu) \) respectively, so that the gauge transformations include

\[ \delta_W(\Lambda(1))B(2) = d\Lambda(1) \]
\[ \delta_X(\lambda(0)_m)B(1)_m = d\lambda(0)_m + \ldots \]
\[ \delta_Z(\omega)A^m = -d\omega^m + \ldots \] (5.16)

The \( W \) and \( X \) symmetries come from the \( D \)-dimensional gauge transformations \( \hat{B} \to \hat{B} + d\hat{\lambda} \) on reducing the parameter
\[ \hat{\lambda} = \Lambda(1) + \lambda(0)_m \nu^m \]
\[ d\hat{\lambda} = d\Lambda(1) - \lambda(0)_m F^m + (d\lambda(0)_m + \lambda(0)_n f^m_n A^p) \wedge \nu^m - \frac{1}{2} \lambda(0)_p f^p_m \nu^m \wedge \nu^n \] (5.17)

The resulting gauge transformations of the reduced potentials are
\[ \delta_W(\Lambda(1))B(2) = d\Lambda(1) \]
\[ \delta_X(\lambda(0)_m)B(1)_m = -\lambda(0)_m F^m \]
\[ \delta_X(\lambda(0)_m)B(1)_m = d\lambda(0)_m + \lambda(0)_n f^m_n A^p = D\lambda(0)_m \]
\[ \delta_X(\lambda(0)_m)B(0)_{mn} = -\lambda(0)_p f^p_{mn} \] (5.18)
The $Z$ symmetries arise from the diffeomorphism symmetry of the higher dimensional theory. Under a diffeomorphism of the internal space with parameter $\omega^m(x)$ the basis forms transform as $\delta(\omega) \nu^m = \mathcal{L}_\omega \nu^m = (\iota_\omega d + d\iota_\omega) \nu^m = -\nu^m f^m_{np} \omega^p$. The requirement $\delta(\omega) \hat{B} = 0$ that the ansatz (5.7) is invariant under these general coordinate transformations induces the following transformations on the reduced potentials

$$
\delta(\omega) B_{(2)} = \frac{1}{2} K_{mn \rho} \omega^p A^m \wedge A^n + \left( d\Xi_{(1)} - \Xi_{(0)m} F^m \right),
$$

$$
\delta(\omega) B_{(1)m} = B_{(1)n} f^n_{mp} \omega^p - K_{mn \rho} \omega^p A^n + D\Xi_{(0)m},
$$

$$
\delta(\omega) B_{(0)mn} = 2 B_{(0)[mpl]q} \omega^q + K_{mn \rho} \omega^p - \Xi_{(0)p} f^p_{mn}
$$

(5.19)

where $\hat{\Xi} = \Xi_{(1)} + \Xi_{(0)m} \nu^m = t_\omega \overline{\omega}_{(2)}$ has explicit internal coordinate dependence. We remove the internal dependence by a gauge transformation $\hat{B} \to \hat{B} + d\hat{\lambda}$ with parameter $\hat{\lambda} = -\hat{\Xi}$, yielding the gauge transformations

$$
\delta Z(\omega) B_{(2)} = \frac{1}{2} K_{mn \rho} \omega^p A^m \wedge A^n
$$

$$
\delta Z(\omega) B_{(1)m} = B_{(1)n} f^n_{mp} \omega^p - K_{mn \rho} \omega^p A^n
$$

$$
\delta Z(\omega) B_{(0)mn} = 2 B_{(0)[mpl]q} \omega^q + K_{mn \rho} \omega^p
$$

$$
\delta Z(\omega) A^m = -d\omega^m - f^m_{np} \omega^p A^n = -D\omega^m
$$

$$
\delta Z(\omega) g_{mn} = 2 g_{(m|p} f^p_{|n)q} \omega^q
$$

(5.20)

It is straightforward to calculate the gauge algebra of the reduced theory using the Jacobi identity $f^q_{[mn} f^t_{pq]} = 0$ and the integrability condition $K_{t[mn} f^t_{pq]} = 0$, giving

$$
[\delta Z(\omega), \delta Z(\omega)] = \delta Z(f^m_{np} \omega^n \overline{\omega}^p) - \delta_X(K_{mn \rho} \omega^n \overline{\omega}^p) - \delta_W(K_{mn \rho} \omega^n \overline{\omega}^p A^m)
$$

$$
[\delta X(\lambda), \delta Z(\omega)] = -\delta_X(\lambda_m f^m_{np} \omega^p)
$$

(5.21)

All other commutators vanish. Note the appearance of a field-dependent parameter $\Lambda_{(1)} = K_{mn \rho} \omega^n \overline{\omega}^p A^m$ for the gauge transformation on the right hand side in (5.21). This extra term involving an anti-symmetric tensor gauge transformation only occurs when acting on the anti-symmetric tensor potential $B_{(2)}$, and is absent when acting on the anti-symmetric tensor field strength $G_{(3)}$. Such a field-dependent term in the algebra is a natural feature of theories with Chern-Simons terms, as we shall discuss in the next section.

The Lie algebra underlying this field-dependent gauge algebra is the algebra found in [15]

$$
[Z_m, Z_n] = f^p_{mn} Z_p - K_{mn \rho} X^p
$$

$$
[X^m, Z_n] = -f^m_{np} X^p
$$

$$
[X^m, X^n] = 0
$$

(5.22)

where, following [15], we have defined the generators $Z_m, X^m$ of the spin-one transformations $\delta Z(\omega) = \omega^m Z_m$ and $\delta_X(\lambda_m) = \lambda_m X^m$. The Jacobi identity of this algebra is equivalent to
the integrability conditions $f_{[mn}^{q}f_{pq]}^{t} = 0$ and $K_{t[mn}f_{pq]}^{t} = 0$. Note that in the absence of flux, this contains the algebra of the group $G_d$ generated by the right action vector fields $Z_m$ of section 2 and the $2d$ dimensional gauge group is $G_{2d} \simeq G_d \times U(1)^d$. However, the presence of flux modifies the commutator of the $Z_m$ and can lead to non-trivial algebras, although, as we shall see in section 5, in some cases the algebra simplifies.

5.3 $O(d, d)$ Covariant Formulation

Remarkably, the theory can be written in an $O(d, d)$ covariant form [15, 40, 41]. The Lie algebra (5.22) may be written as

$$[T_A, T_B] = t_{AB}^C T_C$$

(5.23)

where the generators $Z_m, X^m (m = 1, 2, 3...d)$ are combined into an $O(d, d)$ vector

$$T_A = \begin{pmatrix} Z_m & X^m \end{pmatrix}$$

(5.24)

with $A = 1, 2, 3...2d$. Defining $t_{ABC} = L_{AD} t_{BC} D$ where $L_{AB}$ is the $O(d, d)$ invariant matrix

$$L_{AB} = \begin{pmatrix} 0 & \mathbb{I}_d \\ \mathbb{I}_d & 0 \end{pmatrix}$$

(5.25)

the structure constants are $t_{np}^m = f_{np}^m$ and $t_{mnp} = K_{mnp}$. $\mathbb{I}_d$ is the $d$-dimensional identity matrix $\delta_{mn}$.

To write the full Lagrangian in a manifestly $O(d, d)$covariant form we define [15, 40]

$$B_{(2)} = B_{(2)} - \frac{1}{2} B_{(1)m} \wedge A^m$$

(5.26)

and

$$A^A = \begin{pmatrix} A^m \\ B_{(1)m} \end{pmatrix} \quad \mathcal{F}^A = \begin{pmatrix} F^m \\ G_{(2)m} - B_{(0)mn} F^n \end{pmatrix}$$

(5.27)

The scalars take values in the coset $O(d, d)/O(d) \times O(d)$ and can be parameterised by a coset metric

$$\mathcal{M}^{AB} \equiv \begin{pmatrix} \mathcal{M}^{mn} & \mathcal{M}^{m} \\ \mathcal{M}^{m} & \mathcal{M}_{mn} \end{pmatrix} = \begin{pmatrix} g^{mn} & -B_{(0)np}g^{pm} \\ -B_{(0)mq}g_{np} & g_{mn} + g^{pq}B_{(0)mp}B_{(0)aq} \end{pmatrix}$$

(5.28)

The reduced Lagrangian is then

$$L_D = e^{-\phi} \left( R + 1 + d\phi \wedge d\phi + \frac{1}{2} G_{(3)} \wedge G_{(3)} + \frac{1}{4} L_{AC} L_{BD} \mathcal{M}^{AB} \wedge D \mathcal{M}^{CD} 
- \frac{1}{2} L_{AC} L_{BD} \mathcal{M}^{AB} \mathcal{F}^C \wedge \mathcal{F}^D - \frac{1}{12} \mathcal{M}^{AD} \mathcal{M}^{BE} \mathcal{M}^{CF} t_{ABCD} t_{DEF} 
+ \frac{1}{4} \mathcal{M}^{AD} \mathcal{M}^{BE} \mathcal{L}^{CF} t_{ABCD} t_{DEF} \right)$$

(5.29)
where

\[ G(3) = dB(2) + \frac{1}{2} \left( L_{AB} A^A \wedge F^B - \frac{1}{3!} t_{ABC} A^A \wedge A^B \wedge A^C \right) \quad (5.30) \]

and

\[ DM^{AB} = dM^{AB} + M^{AC} t_{CD} B^D + M^{BC} t_{CD} A^D \quad (5.31) \]

Note that

\[ DM_n^m = g^{mp} f_{np} B_{(1)q} + g^{mp} K_{mpq} A^q + ... \quad (5.32) \]

and the scalar kinetic term \((DM)^2\) term gives mass terms for the one-form fields \(B_{(1)m}\) and \(A^m\).

This form of the theory is invariant under \(O(d, d)\) transformations, provided the structure constants are taken to change under \(O(d, d)\), so that the \(t_{AB}^C\) transform as a tensor under \(O(d, d)\).

This Lagrangian is invariant under the gauge symmetry generated by \(\delta_Z(\omega), \delta_X(\lambda)\) and \(\delta_W(\Lambda)\). Combining the gauge parameters \(\omega^m, \lambda_m\) into an \(O(d, d)\) vector \(\alpha^A = (\omega^m, -\lambda_m)^T\), the transformation of the one-form gauge fields is

\[ \delta(\alpha) A^A = \alpha^B T_B : A^A = -d\alpha^A - t_{BC} A^C A^B \]

\[ = \begin{pmatrix} -d\omega^m - f_{np} \omega^p A^n \\ d\lambda_m + \lambda_n f_{np} A^p + B_{(1)n} f_{np} \omega^p A^n - K_{mpq} \omega^p A^n \end{pmatrix} \quad (5.33) \]

The 3-form field strength

\[ G(3) = dB(2) + \frac{1}{2} \Omega(A, F) \quad (5.34) \]

has a Chern-Simons form

\[ \Omega(A, F) = L_{AB} A^A \wedge F^B - \frac{1}{3!} t_{ABC} A^A \wedge A^B \wedge A^C \quad (5.35) \]

satisfying

\[ d\Omega = L_{AB} F^A \wedge F^B \quad (5.36) \]

As usual for field strengths with Chern-Simons terms, for \(G(3)\) to be gauge invariant under the infinitesimal gauge transformations \(\delta(\alpha)\), it is necessary that \(B(2)\) transforms non-trivially. The required \(B(2)\) transformation is, up to a total derivative,

\[ \delta(\alpha) B(2) = \frac{1}{2} L_{AB} A^A dA^B \quad (5.37) \]

The algebra of these transformations is field-dependent:

\[ [\delta(\tilde{\alpha}), \delta(\alpha)] = \delta(t_{BC} A^A \alpha^B \alpha^C) - \delta_W (t_{[ABC]} A^{B \alpha^C}) \quad (5.38) \]

This is simply the \(O(d, d)\) covariant form of the algebra (5.21). All reductions with non-trivial fluxes and twists may be written in this form so one expects such a field dependent algebra
in any such reduction where $t_{[ABC]} \neq 0$. As usual, we shall refer to the Lie algebra (5.23) as the gauge algebra, even though strictly speaking the symmetry algebra is the algebra (5.38) with field-dependent structure functions.

In the abelian limit in which the structure constants $t_{AB}^C$ are set to zero, this reduces to the standard reduction on a $d$-torus, giving a field theory with manifest global $O(d,d)$ symmetry. The non-abelian reduction discussed here can be thought of as a gauging of a $2d$-dimensional subgroup $G_{2d}$ of the $O(d,d)$ symmetry. In particular, when a supergravity theory is dimensionally reduced in this way, the result is a gauged supergravity theory with gauge group $G_{2d}$.

6 Symmetry Breaking and Examples of Flux Reductions

6.1 Symmetry Breaking

The twisted reduction with flux gives rise to a compactified theory with $2d$-dimensional gauge group $G_{2d}$ with gauge algebra (5.22), and this symmetry is in general spontaneously broken. First, some of the gauge symmetry is non-linearly realised, and as a non-linearly realised transformation acts as a shift on certain scalar $\phi$, $\delta \phi = \alpha + O(\phi)$, it cannot be preserved by any vacuum expectation value of $\phi$ and so is necessarily broken by any vacuum, so that the gauge group is necessarily broken down to its linearly realised subgroup. Then any given vacuum solution (e.g. one arising from a critical point of the scalar potential) can then break the linearly realised subgroup further to the subgroup preserving that vacuum.

In this section, we will discuss the first stage of symmetry breaking down to the linearly realised subgroup that is generic for any solution. For vacua with vanishing scalar expectation value, this is the complete breaking, but for non-trivial scalar expectation values there will be further breaking through the standard Higgs mechanism. The transformation for the scalar fields $B_{(0)mn}$ is

$$\delta B_{(0)mn} = -\lambda_{(0)p} f_{mn}^p + K_{mnp} \omega^p + 2B_{(0)[m|p} f_{|n|q}^p \omega^q$$

(6.1)

and from this one can find the non-linearly realised symmetries, i.e. the ones realised as shifts of scalar fields.

Consider first the case without flux, $K_{mnp} = 0$, so that the gauge group with algebra (5.22) is $G_{2d} = G_d \ltimes U(1)^d$ where $G_d$ is the isometry group with structure constants $f_{mn}^p$. The transformation

$$\delta B_{(0)mn} = -f_{mn}^p \lambda_{(0)p} + O(B_{(0)mn})$$

(6.2)
implies that the subgroup of $Z$ transformations $G_d$ is linearly realised, but that some of the $U(1)^d$ $X$ transformations with parameter $\lambda$ are non-linearly realised and so broken. If $G_d$ is semi-simple, then one can raise and lower group indices using the Cartan-Killing metric and define the Goldstone field

$$\chi_p = \frac{1}{2} f^m_n B_{(0)mn}$$

which transforms as

$$\delta_X (\lambda_{(0)m}) \chi_m = -\lambda_{(0)m}$$

and the remaining scalar fields

$$\tilde{B}_{(0)mn} = B_{(0)mn} - f^p_{mn} \chi_p$$

transform linearly. Then the subgroup $U(1)^d$ of $X$-transformations are non-linearly realised and so spontaneously broken, with the Goldstone fields $\chi_p$ eaten by the gauge fields $B_{(1)p}$ which all become massive. The group $G_{2d} = G_d \ltimes U(1)^d$ is broken to the subgroup $G_d$, and this in turn may be further broken in vacua in which the remaining scalars $\tilde{B}_{(0)mn}, g_{mn}$ have a non-trivial vacuum expectation value. If the group is not semi-simple, then not all of $U(1)^d$ is broken in general, and the linearly realised subgroup is unbroken. In the trivial case in which $G_d$ is abelian, there are no shifts and the full $U(1)^d$ symmetry is unbroken, while in the general case it will be broken to the subgroup for which the infinitesimal parameters $\lambda$ satisfy

$$f^p_{mn} \lambda_{(0)p} = 0$$

For the abelian case ($f^m_{np} = 0$) the whole group remains unbroken, for the semi-simple case there are no solutions and $G_{2d}$ is broken to $G_d$, and for non-semi-simple cases there in general will be solutions in the kernel of the map $\mathbb{R}^d \to \mathbb{R}^{d(d-1)/2}$ defined by $\alpha_p \to \alpha_p f^p_{mn}$, resulting in a partial breaking. This case will be analysed as a particular case of a more general construction in section 6.5.2.

Consider next the case with flux, but with a toroidal reduction with $f^m_{np} = 0$, so that $G_d = U(1)^d$. Then

$$\delta B_{(0)mn} = K_{mnp} \omega^p$$

and the $X$-transformations are linearly realised but a subgroup of the group $G_d = U(1)^d$ of $Z$-transformations is non-linearly realised and so spontaneously broken. The breaking depends on the form of the flux $K_{mnp}$. The $Z$ transformations will be broken to the subset for which the parameters satisfy $K_{mnp} \omega^p = 0$. The remaining $Z$-transformations will be broken, with the corresponding scalars $B_{(0)mn}$ eaten by gauge fields $A^m$, which become massive. In this case, the unbroken generators correspond to the kernel of the map $\mathbb{R}^d \to \mathbb{R}^{d(d-1)/2}$ defined by $\beta^p \to \beta^p K_{mnp}$. This case will be analysed in section 6.5.1.

We will now consider further examples of twisted reductions with fluxes and discuss their symmetry breaking. In 6.2 we consider a flux that may be removed by a simple field
redefinition. Section 6.3 discusses a theory with a flux constructed from an invariant metric, in which a linear combination of $B_{(1)m}$ and $A^m$ gauge fields becomes massive. Section 6.4 considers the general case and finally, in section 6.5, we illustrate the general approach with two examples.

### 6.2 Trivial Flux

The flux

$$K_{mnp} = \zeta_{mq} f^q_{np} + \zeta_{nq} f^q_{pm} + \zeta_{pq} f^q_{pm}$$

satisfies (5.5) for any constant antisymmetric $\zeta_{nm} = -\zeta_{mn}$. The physical effect of this flux, its appearance in the gauge algebra and in the Lagrangian, may be removed entirely by the field redefinitions.

$$\tilde{B}_{(2)} = B_{(2)} + \frac{1}{2} \zeta_{mn} A^m \wedge A^n$$

$$\tilde{B}_{(1)m} = B_{(1)m} - \zeta_{mn} A^n$$

$$\tilde{B}_{(0)mn} = B_{(0)mn} + \zeta_{mn}$$

(6.9)

Fluxes of this form are therefore not physically significant.

### 6.3 Flux Constructed from Invariant Metric

If the group $G_d$ has an invariant metric $h_{mn} = h_{nm}$ satisfying

$$h_{[m|p} f^p_{|[n]q]} = 0$$

(6.10)

then

$$f_{mnp} = h_{mq} f^q_{np}$$

(6.11)

is totally antisymmetric, $f_{mnp} = f_{[mnp]}$, and the flux

$$K_{mnp} = f_{mnp}$$

(6.12)

satisfies the integrability constraint (5.5) by virtue of the Jacobi identity (2.4). If $G_d$ is semi-simple then any invariant metric is proportional to the Cartan-Killing metric $\eta_{mn}$, so that

$$h_{mn} = \mu \eta_{mn}, \quad \eta_{mn} = \frac{1}{2} f^q_{mp} f^p_{nq}$$

(6.13)

for some parameter $\mu$, which plays the role of a mass parameter in the reduced theory. This metric is invertible, and is the case considered for compact groups in [6]. The special case $\mu = 0$ gives vanishing flux $K = 0$ and so gives the reduction for a semi-simple group without flux discussed in Section 6.1 as a special case. For a general non-semi-simple group, $h_{mn}$ need not be related to the Cartan-Killing metric and need not be invertible.
We shall see that theories with this type of flux admit various field redefinitions which simplify the gauge algebra of the theory. The first step involves a change of basis which brings the gauge group to a semi-direct product form. If the metric $h_{mn}$ is invertible, then a further redefinition is possible that brings the gauge group to a direct product form $G_d \times U(1)^d$ and this is spontaneously broken to $G_d$.

### 6.3.1 General Case with Flux from Invariant Metric

If $h_{mn}$ is an invariant metric, the flux $K_{mnp} = f_{mnp}$ is a non-trivial flux in that its effect cannot be removed by a field redefinition of the kind (6.9). The gauge algebra is (5.22) with

\[ [Z_m, Z_n] = f^p_{mn} Z_p - f_{mnp} X^p \]
\[ [X^m, Z_n] = -f^m_{np} X^p \]
\[ [X^m, X^n] = 0 \] (6.14)

With this choice of flux, the algebra may be simplified by the change of basis

\[ \hat{Z}_m = Z_m + h_{mn} X^n \] (6.15)

so that

\[ \delta \hat{Z}(\omega^m) = \delta Z(\omega^m) + \delta X(\lambda_{(0)m} \equiv h_{mn} \omega^n) \] (6.16)

The gauge algebra is then isomorphic to that of the standard Scherk-Schwarz reduction in the absence of flux [1]

\[ [\hat{Z}_m, \hat{Z}_n] = f^p_{mn} \hat{Z}_p \]
\[ [X^m, \hat{Z}_n] = -f^m_{np} X^p \]
\[ [X^m, X^n] = 0 \] (6.17)

which generates the gauge group

\[ G_d \ltimes U(1)^d \] (6.18)

The $\hat{Z}$ transformations are

\[ \delta \hat{Z}(\omega) B_{(2)} = -h_{mn} \omega^m dA^n \]
\[ \delta \hat{Z}(\omega) B_{(1)m} = h_{mn} d\omega^n + B_{(1)n} f^p_{mp} \omega^p \]
\[ \delta \hat{Z}(\omega) B_{(0)mn} = B_{(0)np} f^p_{mq} \omega^q - B_{(0)np} f^p_{mq} \omega^q \]
\[ \delta \hat{Z}(\omega) A^m = -d\omega^m - f^m_{np} \omega^p A^n \] (6.19)

Note that $\delta \hat{Z}(\omega) A^m = \delta Z(\omega) A^m$ since the $A^m$ fields are singlets of the $X^m$ transformations. The similarity of the $A^m$ and $B_{(1)m}$ transformations may be exploited to define a field $C_{(1)m}$ which transforms covariantly under the semi-simple subgroup $\hat{G}_d$ generated by $\delta \hat{Z}(\omega)$

\[ C_{(1)m} = B_{(1)m} + h_{mn} A^n \] (6.20)
where

\[ \delta \hat{Z}(\omega) C_{(1)m} = C_{(1)n} f_{mp}^n \omega^p \]
\[ \delta X(\lambda) C_{(1)m} = D \lambda_{(0)m} \] (6.21)

The field strengths (5.9) then become

\[ G_{(3)} = dB_{(2)} + C_{(1)m} \wedge F^m - \Omega_3(h) \]
\[ G_{(2)m} = DC_{(1)m} + B_{(0)mn} F^n - h_{mn} F^n \]
\[ G_{(1)mn} = DB_{(0)mn} + f_{mn} C_{(1)p} \]
\[ G_{(0)mpn} = -3B_{(0)[m|q|} f_{|np]}^q + f_{mpn} \] (6.22)

where

\[ \Omega_3(h) = h_{mn} \left( A^m \wedge dA^n + \frac{1}{3} f_{mn} A^n \wedge A^p \wedge A^q \right) \] (6.23)

is a generalised Chern-Simons term satisfying

\[ d\Omega_3(h) = h_{mn} F^m \wedge F^n \] (6.24)

These field redefinitions give a formulation with gauge fields \((C_{(1)m}, A^m)\) and gauge group \(G_d \rtimes U(1)^d\) with \(A^m\) the gauge fields for the \(\hat{Z}\) transformations generating \(G_d\) with the \(U(1)^d\) gauge fields \(C_{(1)m}\) transforming covariantly under \(G_d\). The field redefinition (6.20) brings the vector mass term in the Lagrangian to the form

\[ L_D = -\frac{1}{2} e^{-\phi} g^{mn} g^{pq} * G_{(1)mp} \wedge G_{(1)nq} + ... \]
\[ = -\frac{1}{2} e^{-\phi} g^{mn} g^{pq} f_{mp}^t f_{nq}^s * C_{(1)t} \wedge C_{(1)s} + ... \] (6.25)

so that only the \(C_{(1)m}\) can become massive (which ones do so depends on the structure constants) while the \(A^m\) remain massless.

### 6.3.2 Case with Invertible Metric

To go further, we now restrict ourselves to the case in which \(h_{mn}\) is non-degenerate, with inverse \(h^{mn}\). In this case the gauge group may be simplified further. An important class of examples is that in which \(G_d\) is a semi-simple group, and the invariant metric \(h_{mn} = \mu \eta_{mn}\) is proportional to the Cartan-Killing metric \(\eta_{mn}\). Then further redefinitions are possible that simplify the gauge group to a direct product \(G_d \times U(1)^d\) which is spontaneously broken to \(G_d\) with the gauge fields \(C_{(1)m}\) becoming massive [6].

The gauge fields \(C_{(1)m}\) become massive by eating the Goldstone fields

\[ \chi_p = \frac{1}{2} f_{pm}^n B_{(0)mn} \] (6.26)
where \( f_{pm}^{mn} = h^{mq} f_{q}^{n} \). This transforms as
\[
\delta_{\widehat{Z}}(\omega_{m})\chi_{m} = \chi_{n} f_{mp}^{n} \omega^{p} \\
\delta_{X}(\lambda_{(0)m})\chi_{m} = -\lambda_{(0)m}
\]
so that the \( \chi \) are Stuckelberg fields shifting under the \( X \) transformations (if \( \mu \neq 0 \)).

The gauge algebra can be further simplified to a direct product structure by defining massive fields which are gauge singlets under \( \delta_{X}(\lambda_{(0)m}) \), as follows
\[
\check{B}_{(2)} = B_{(2)} - \chi_{m} F_{m}^{n} \\
\check{C}_{(1)m} = C_{(1)m} + D\chi_{m} \\
\check{B}_{(0)mn} = B_{(0)mn} - f_{pm}^{n} \chi_{p}
\]
(6.28)
The Goldstone bosons \( \chi_{m} \) of the broken \( U(1)^{d} \) symmetry \( \delta_{X}(\lambda_{(0)m}) \) are eaten by the \( C_{(1)m} \) which become the massive vector fields \( \check{C}_{(1)m} \). These redefinitions bring the \( \widehat{Z} \) transformations for \( \check{C}_{(1)m} \) and \( \check{B}_{(0)mn} \) to the canonical form,
\[
\delta_{\widehat{Z}}(\omega_{m})\check{B}_{(2)} = -h_{mn} \omega^{m} dA^{n} \\
\delta_{\widehat{Z}}(\omega_{m})\check{C}_{(1)m} = \check{C}_{(1)p} f_{mp}^{n} \omega^{p} \\
\delta_{\widehat{Z}}(\omega_{m})\check{B}_{(0)mn} = \check{B}_{(0)pq} f_{pq}^{m} \omega^{q} - \check{B}_{(0)np} f_{pq}^{m} \omega^{q} \quad (6.29)
\]

The gauge algebra realised on \( \check{B}_{(0)mn} \) and \( \check{C}_{(1)m} \) is
\[
[\delta_{\widehat{Z}}(\tilde{\omega}), \delta_{\widehat{Z}}(\omega)] = \delta_{\widehat{Z}}(f_{mp}^{n} \omega^{n} \tilde{\omega}^{p})
\]
(6.30)
Since the massive potentials are singlets of \( \delta_{X}(\lambda_{(0)m}) \) this is the complete algebra. The gauge algebra is then
\[
\left[\widehat{Z}_{m}, \widehat{Z}_{n}\right] = f_{mn}^{p} \widehat{Z}_{p} \\
\left[X^{m}, \widehat{Z}_{n}\right] = 0 \\
\left[X^{m}, X^{n}\right] = 0
\]
(6.31)
which generates the gauge group
\[
\mathcal{G}_{d} \times U(1)^{d}
\]
(6.32)
The action of \( \delta_{\widehat{Z}}(\omega) \) on the \( \check{B}_{(0)mn} \) and \( \check{C}_{(1)m} \) fields is equivalent to that of \( \delta_{Z}(\omega) \) since these fields are singlets of the \( U(1)^{d} \) subgroup. The \( \check{B}_{(2)} \) field is still massless and transforms as \( \delta_{W}(\Lambda_{(1)})\check{B}_{(2)} = d\Lambda_{(1)} \). The full gauge algebra is
\[
[\delta_{\widehat{Z}}(\tilde{\omega}), \delta_{\widehat{Z}}(\omega)] = \delta_{\widehat{Z}}(f_{mp}^{n} \omega^{n} \tilde{\omega}^{p}) - \delta_{W}(f_{mp}^{n} \omega^{n} \tilde{\omega}^{p} A^{m})
\]
(6.33)
with all other commutators vanishing.

The field redefinitions (6.28) are of the form of the infinitesimal gauge transformations generated by \( \delta_{X}(\lambda) \), although they are not infinitesimal. Since the field strengths are invariant under transformations generated by \( \delta_{X}(\lambda) \) the field strengths take the same form as (6.22), with \( B_{(2)}, C_{(1)m} \) and \( B_{(0)mn} \) replaced by \( \check{B}_{(2)}, \check{C}_{(1)m} \) and \( \check{B}_{(0)mn} \) respectively.
6.4 General Case

In this section we shall consider a reduction on a general twisted torus with arbitrary flux, i.e. we shall allow the $d$-dimensional group $G_d$, upon which the compactification is based, to be non-semi-simple and the flux $K$ may vanish along certain directions. We shall analyse the symmetry breaking of the gauge symmetry $G_{2d} \simeq G_d \ltimes U(1)^d$ to a linearly realised subgroup.

The gauge fields $A^m, B^{(1)}_m$ become massive by eating a subset of the scalar fields $B^{(0)}_{mn}$, so it is on these fields that we shall focus. The infinitesimal transformation of the scalar $B$-fields under the gauge symmetry $G_{2d}$ is

$$\delta B^{(0)}_M = -\lambda^{(0)m}_M f^m_M + \omega^m K_{mM} + O(B^{(0)}_M)$$

(6.34)

where we have defined the compound index $M = [mn], M = 1, 2, \ldots D$, where $D = d(d-1)/2$. It will be useful to write this variation in terms of the $O(d,d)$ covariant structure constants $t^{ABC}_{AB}$ of $G_{2d}$ and the gauge parameter $\alpha_A = L_{AB}\alpha^B$ introduced in section 5.3 where $A = 1, 2, \ldots 2d$.

$$\delta B^{(0)}_M = (-\lambda^m, \omega^m) \begin{pmatrix} f^m_M \\ K_{mM} \end{pmatrix} + O(B^{(0)}_M)$$

$$= \alpha_A t^A_M + O(B^{(0)}_M)$$

(6.35)

The structure constants $t^A_M$ may be thought of as defining a map $\alpha_A \rightarrow \alpha_A t^A_M$ from $\mathbb{R}^{2d}$ to $\mathbb{R}^D$, which is non-invertible for $d > 5$ (and may be for $d \leq 5$), and part of our analysis will be concerned with finding a generalised inverse of this map.

First we identify the set of gauge fields which become massive. The map $t : \alpha_A \rightarrow \alpha_A t^A_M$ will have a kernel of dimension $2d - d'$ for some $d'$, and it is useful to choose a basis $\{e_A\} = \{e_{A'}, e_A\}$ for $\mathbb{R}^{2d}$ consisting of a basis $\{e_A\}$ for the kernel of the map $t$ with $A = d' + 1, \ldots 2d$ together with its complement, a basis $\{e_{A'}\}$ for the cokernel, with $A' = 1, 2, \ldots d'$. Then

$$t^A_M = 0$$

(6.36)

for all $M$ while $t^{A'}_M$ has no zero eigenvectors and so for each $A'$, there is some $M$ such that $t^{A'}_M \neq 0$. Then in this basis the structure constants $t^A_M$ are a $2d \times D$ matrix

$$t^A_M = \begin{pmatrix} t^{A'}_M \\ 0 \end{pmatrix}$$

(6.37)

where 0 is the $2d - d' \times D$ zero matrix.

In this basis the $B^{(0)}_M$ scalar field transforms as

$$\delta B^{(0)}_M = \alpha_{A'} t^{A'}_M + O(B^{(0)}_M)$$

(6.38)

so that the symmetry $G_{2d}$ is broken to the $2d - d'$ dimensional subgroup generated by the $T^A$ with parameters $\alpha_A$ and gauge bosons $A_A$, while the remaining symmetries generated
by $T^{A'}$ are all broken and the vector fields $A_{A'}$ are massive. Indeed, the mass term of the Lagrangian 5.29 for the gauge fields $A_{A'}$ is

$$
L_D = \frac{1}{4} e^{-\phi} M^{AB} M^{CD} t_{AC}^{E'} t_{BD}^{F'} A_{E'} \wedge A_{F'} + \ldots
$$

(6.39)

and it is clear that fields $A_{A'}$ are massive whilst the $A_{\bar{A}}$ remain massless, as the latter do not appear in the mass term. In general, $A_{A'}$ and $A_{\bar{A}}$ will be linear combinations of the $A^m$ and $B_{(1)m}$ fields. This highlights one effect of introducing flux into such twisted reductions: if $K = 0$, then only a subset of the $B_{(1)m}$ fields become massive whilst the $A^m$ remain massless. By introducing fluxes, the fields which become massive are linear combinations of the $A^m$ and $B_{(1)m}$ fields.

We now turn to finding the Goldstone fields that are eaten by the massive one-form fields $A_{A'}$. The structure constants also give a map from $\mathbb{R}^D$ to $\mathbb{R}^{2d}$ defined by $\beta^M \rightarrow t^A_M \beta^M$. As before we choose a basis for $\mathbb{R}^D$ so that the index $M$ splits into $(M', \bar{M})$ where $\bar{M} = d' + 1, \ldots D$ label a basis for the kernel of this map, and $M' = 1, 2, \ldots d'$ labels a basis for the cokernel. Then $t^A_M$ takes the form of a $2d \times D$ matrix

$$
t^A_M = \begin{pmatrix} t^{A' M'} & 0 \\ 0 & 0 \end{pmatrix}
$$

(6.40)

The $2d \times D$ matrix $t^A_M$ is non-invertible, but the $d' \times d'$ matrix $t^{A' M'}_M$ is non-degenerate by construction, so we may define its inverse $\tilde{t}^{M' A'}$ (with $\tilde{t}^{M' A'} t^{A' N'} = \delta^{M' N'}$).

Now

$$
\delta B_{(0)\bar{M}} = O(B_{(0)}) \quad \delta B_{(0)M'} = \alpha_{A'} t^{A' M'} + O(B_{(0)}),
$$

(6.41)

so that we can define Goldstone bosons $\chi_{A'}$ by $\chi_{A'} = B_{(0)M'} \tilde{t}^{M' A'}$ so that they transform as

$$
\delta \chi_{A'} = \alpha_{A'} + O(B_{(0)})
$$

(6.42)

We may then define massive gauge bosons as

$$
\bar{A}_{A'} = A_{A'} + D\chi_{A'}
$$

(6.43)

so that the massive gauge fields $\bar{A}_{A'}$ transform in a linear realisation of the unbroken gauge group generated by $T^A$.

### 6.5 Examples

In the following we illustrate this general method with two examples.
6.5.1 Toroidal Reduction with Flux

If \( f^p_{mn} = 0 \), then the group \( G_d \) is abelian and the internal manifold (after discrete identifications to compactify, if necessary) is a torus and we take \( G_d = U(1)^d \). With flux \( K \), the gauge algebra is

\[
[Z_m, Z_n] = -K_{mnp} X^p
\]

with all other commutators vanishing. The internal index \( m \) can be split into \( (m', \bar{m}) \), so that \( \bar{m} \) labels the \( d - d' \) dimensional kernel of the map \( \alpha^m \to \alpha^m K_{mnp} \), and \( m' \) labels the cokernel, so that

\[
K_{mnp} = 0 \quad K_{m'n'p'} \neq 0 \quad (6.45)
\]

Then the transformation of the \( B_{(0)} \) scalars is

\[
\delta B_{(0)n'p'} = \omega^{m'} K_{m'n'p'}, \quad \delta B_{(0)m\bar{m}} = 0 \quad (6.46)
\]

The transformations generated by \( Z_{m'} \) with parameters \( \omega^{m'} \) are spontaneously broken, with \( B_{(0)m'n'} \) the Goldstone fields that are eaten by the gauge fields \( A^{m'} \). The \( A^{m'} \) fields have mass term in the Lagrangian

\[
L_D = -\frac{1}{2} e^{-\phi} g^{mn} g^{pq} K_{mpt} K_{npq} * A^t \wedge A^p + \ldots \quad (6.47)
\]

The 2\( d \) dimensional gauge group is broken to the 2\( d - d' \) dimensional abelian subgroup \( U(1)^{2d-d'} \) generated by \( Z_{\bar{m}} \) and \( X^m \) with parameters \( \omega_{\bar{m}} \) and \( \lambda_m \) respectively.

Let \( \tilde{K}^{m'n'p'} \) be any constants satisfying \( \tilde{K}^{m'n'p'} K_{n'p'q'} = \delta^{m'}_{~q'} \). Then we can define Goldstone fields \( \chi^{m'} \) by

\[
\chi^{m'} = \tilde{K}^{m'n'p'} B_{n'p'} \quad (6.48)
\]

transforming as a shift the \( \omega^{m'} \) transformations

\[
\delta B_{(0)\bar{M}} = 0 \quad \delta \chi^{m'} = \omega^{m'} \quad (6.49)
\]

The remaining scalars are invariant, \( \delta B_{(0)m\bar{m}} = 0 \). We may then define the massive graviphotons \( \tilde{A}^{m'} = A^{m'} + d\chi^{m'} \) which are singlets of the gauge transformations.

6.5.2 Reduction with Non-semi-simple Twisted Tori

The next example we consider is that of a general non-semi-simple group \( G_d \) with structure constants \( f^p_{mn} \) but with zero flux, so that the analysis of subsection 6.3.1 applies with \( \mu = 0 \). Such groups arise for example in reductions with duality twists. The reduction is then on a compactification \( G_d/\Gamma \) of the non-compact group manifold for the non-semi-simple group \( G_d \).
The map $\mathbb{R}^d \to \mathbb{R}^{d(d-1)/2}$ defined by $V_m \to V_m f_{np}^n$ will have a kernel of dimension $d - d'$, say, which we label by $\tilde{m} = d' + 1, ..., d$, while the cokernel is labelled by $m' = 1, ..., d'$, so that the index $m$ is split into $(m', \tilde{m})$. Then

$$f_{np}^{m} = 0 \quad \forall n, p$$

and the map can be written as $V_{m'} \to V_{m'} f_{np}^{m'}$.

The transformation of the $B_{(0)mn}$ scalars under $G_{2d}$ is

$$\delta B_{(0)np} = -\lambda_{m'} f_{np} f_{m'n} + O(B_{(0)})$$

so that the transformations generated by the $X^{m'}$ are broken, with the vector fields $B_{(1)m'}$ becoming massive. Indeed, the fields $B_{(1)m'}$ have a mass term in the Lagrangian

$$\mathcal{L}_D = -\frac{1}{2} e^{-\frac{1}{2} g^{mn} g^{pq} f_{mp} f_{nq} \wedge B_{(1)t'}} \wedge B_{(1)t'} + ...$$

The gauge group is then broken to the $2d - d'$ dimensional subgroup generated by $Z_m$ and $X^m$ with parameters $\omega^m$ and $\lambda_m$, with massless gauge fields $A^m, B_{\tilde{m}}$.

To proceed with the analysis, we define a compound index $M = [mn]$ $M = 1, ..., D$, where $D = d(d-1)/2$, labelling the space of 2-forms $\mathbb{R}^D$, so that the structure constants define a $d \times D$ matrix $f_{m'n}$. We then split this into indices $M = (M', \tilde{M})$, $M' = 1, 2, ..., d'$, $\tilde{M} = d' + 1, ..., D$ with $M'$ labelling the cokernel of the map defined by $\beta^M \to f_{m'n}^{M'} \beta^M$ and $\tilde{M} = d' + 1, ..., D$ labelling the kernel. Then $f_{m'n}^{M'}$ is an invertible $d' \times d'$ matrix with inverse $\tilde{f}_{M'n}$. The Goldstone bosons of the broken symmetry $\chi_{m'}$ are $\chi_{m'} = B_{(0)M'} f_{M'n}^{M'}$ and the scalar fields transforming linearly in the unbroken gauge symmetry $G_{d} \ltimes U(1)^{d-d'}$ are $B_{(0)\tilde{M}}$. The transformation properties of these fields under $X$ and $Z$ transformations following from $(6.51)$ are

$$\delta B_{(0)\tilde{M}} = O(B_{(0)}) \quad \delta \chi_{m'} = \lambda_{m'} + O(B_{(0)})$$

so that $\chi_{m'}$ is the Goldstone field for the $\lambda_{m'}$ transformations. We then define the massive gauge boson $\tilde{B}_{(1)m'} = B_{(1)m'} + D \chi_{m'}$ which transforms in a linear representation of the unbroken gauge group $G_{d} \ltimes U(1)^{d-d'}$.

As an example, consider the case where $d = 3$, $m' = 1, 2$ and $\tilde{m} = 3$, with the only non-zero structure constants given by $f_{3n'}^{m'} = M_{n'm'}$ for some matrix $M_{n'm'}$, so that the gauge algebra, in the absence of flux, becomes

$$[Z_{m'}, Z_i] = M_{m'n'} Z_{n'}$$

$$[X^{m'}, Z_i] = -M_{m'n'} X^{n'}$$

(6.54)

with all other commutators vanishing. This is precisely the algebra $(3.10)$ of an $SL(2)$ twisted $T^2$ fibration over $S^1$ as discussed in section 3, with mass matrix $M_{n'm'}$. For example, in the case of an elliptic twist, the gauge group $ISO(2) \ltimes U(1)^3$ is broken to the linearly realised sub-group $ISO(2) \ltimes U(1)$.
7 Discussion

We have seen that a Scherk-Schwarz reduction of a low energy field theory on a ‘twisted torus’ associated with a group $\mathcal{G}$ can be extended to a string theory compactification provided the group $\mathcal{G}$ has a discrete cocompact subgroup $\Gamma$, and the string theory is then compactified on $\mathcal{G}/\Gamma$. This is a non-trivial restriction, as not all $\mathcal{G}$ have such subgroups. A large class (but not all) of reductions with duality twists can be viewed as compactifications on such $\mathcal{G}/\Gamma$, while others can be regarded as compactifications of F-theory, or one of its generalisations, on $\mathcal{G}/\Gamma$. This extends to M-theory: a Scherk-Schwarz reduction of 11-dimensional supergravity can be promoted to a compactification of M-theory on a compact $\mathcal{G}/\Gamma$, when this exists.

The $O(d,d)$ covariant formulation of the reduced theory is very suggestive, and it is natural to ask whether this generalises to M-theory or type II compactifications, and whether these can be written in a way that is covariant under the action of a duality group. The results for the heterotic string with Wilson line fluxes for the heterotic gauge fields were given in [15], and were found to be $O(d, d+16)$ covariant. The results are of the form given in section 5, but with the indices $A$ now running over $2d+16$ values and transforming under $O(d, d+16)$.

For general Scherk-Schwarz compactifications of string theory, the vector fields $B_{\mu m}$ arise from the 2-form gauge field $B_{MN}$. These correspond to the gauge group generators $X^m$ and couple to string winding modes. In generalising to M-theory, these are replaced by vector fields $C_{\mu mn} (C_{\mu mn} = -C_{\mu nm})$ arising from the 3-form gauge field $C_{MNP}$. These are associated with group generators $X^{mn} = -X^{nm}$ and couple to membrane wrapping modes. This gives a gauge group whose generators include $Z_m, X^{mn}$, with algebra

\[
\begin{align*}
[Z_m, Z_n] &= f_{mnp}^p Z_p - K_{mnpq} X^{pq} \\
[X^{mn}, Z_p] &= 2 f_{p[q}^m X^{n]q} \\
[X^{mn}, X^{pq}] &= 0
\end{align*}
\]

where $K_{mnpq}$ are the constants defining the 4-form flux. This algebra was also found by [25] for Scherk-Schwarz reductions of 11-dimensional supergravity to 4 dimensions. One might have expected that the $O(d,d)$ covariance reviewed here would extend to a covariance under the appropriate U-duality group. However, we find that in general fluxes provide obstructions to the dualisations needed to bring the theory to a more symmetric form with U-duality covariance, but nonetheless, an elegant and suggestive structure emerges, with covariance under the ‘electric subgroup’ of the U-duality group (i.e. the subgroup that can be realised on the gauge potentials). The details will be given in [42].

The toroidal reduction of the field theory action gives a theory with a rigid $O(d,d)$ duality symmetry, and the twisted torus reduction with flux gives a theory in which a $2d$ dimensional subgroup $\mathcal{G}_{2d}$ of $O(d,d)$ is promoted to a gauge group. The gauge fields are in the vector
representation of $O(d,d)$, so this must become the adjoint of $G_{2d} \subset O(d,d)$. If the original theory is a supergravity theory, the result is a gauged supergravity theory. Consider now the lifting of this to string theory. The $O(d,d)$ duality symmetry of the toroidal reduction is broken to the T-duality group $O(d,d;\mathbb{Z})$ yet a subgroup of the continuous group $O(d,d)$ is meant to be a gauge symmetry. This raises the issue of how the two symmetries are related. This was addressed in [15], where it was suggested that they were distinct and that there is an $O(d,d;\mathbb{Z}) \times G_{2d}$ gauge symmetry. However, we will see that the situation is rather subtle, and that although they are distinct, they do not commute.

The structure constants $t_{AB}^C$ specify the embedding of $G_{2d} \subset O(d,d)$. In the field theory reduction of section 5, there is covariance under $O(d,d)$ which acts non-trivially on the structure constants $t_{AB}^C$, as they transform covariantly. Then an $O(d,d)$ transformation acts not just on the fields, but on the coupling constants $t_{AB}^C$ and hence changes the form of the mass terms and potential. The gauge group is still $G_{2d}$, but after the transformation it is embedded differently in $O(d,d)$. The new form of the theory is related to the original one by field redefinitions, so it is physically equivalent. A similar situation was discussed in [43, 45, 44], where the action of $SL(8,\mathbb{R}) \subset E_7$ duality symmetries was considered on $N = 8$ gauged supergravity in $D = 4$, giving equivalent gaugings (although singular limits gave new gaugings). Thus there is still an action of the rigid $O(d,d)$, but it does not leave the gauged theory invariant, but changes it to a physically equivalent theory, with the gauge group $G_{2d}$ transformed to a conjugate gauge group $G_{2d}$ embedded differently in $O(d,d)$.

In the compactifications of string theory discussed here, there is a $G_{2d}$ gauge symmetry, but one might expect that the $O(d,d)$ covariance should be broken to a discrete subgroup. A simple case in which this should happen is that of toroidal reductions with $H$-flux. The $O(d,d)$ contains a geometric subgroup $GL(d,\mathbb{R}) \ltimes \mathbb{R}^{d(d-2)/2}$, acting through diffeomorphisms and shifts of the $B$ field, and these are broken to $GL(d,\mathbb{Z}) \ltimes \mathbb{Z}^{d(d-2)/2}$ by the torus boundary conditions and the requirement that the quantum theory be invariant under $B$-shifts. Then the $O(d,d)$ covariance should be broken to a subgroup containing $GL(d,\mathbb{Z}) \ltimes \mathbb{Z}^{d(d-2)/2}$, so that a natural conjecture would be that the theory should have $O(d,d;\mathbb{Z})$ covariance, and this is precisely what the low-energy field theory suggests. However, the status of the T-duality transformations in $O(d,d;\mathbb{Z})$ is not clear. The transformation corresponding to T-duality in one torus direction removes the flux and turns on a twist, giving some structure constants $f^m_{np}$ and a twisted torus [26, 46]. However, a further T-duality would then take the background to a non-geometric background [32] and so an understanding of whether the conjecture that there should be $O(d,d;\mathbb{Z})$ covariance would require a generalisation of the compactifications considered here to non-geometric backgrounds, and a better understanding of T-duality in such cases in which the usual rules do not apply. However, our analysis makes it clear that if there is such a covariance under $O(d,d;\mathbb{Z})$ or some other discrete subgroup of $O(d,d)$, it would not be a conventional symmetry acting on the fields alone, but must be duality that acts on the coupling constants $t_{AB}^C$ as well, just as S-duality in $N = 4$ Yang-Mills acts
on the coupling constant as well as the fields. The flux and twist then transform into each other and fit together into an irreducible representation. In particular, the $O(d, d)$ action does not commute with the gauge symmetry. Similar remarks should apply to the interplay between gauge symmetry and U-duality in compactifications of M-theory, as we will discuss elsewhere.
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