The Tidal Clock Effects of the Lunisolar Gravitational Field and the Earth’s Tidal Deformation

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1. Introduction

Quantum mechanical absorbers have well-defined transition frequencies, and exist in a mass of identical copies (Hinkley et al. 2014). They can provide the best available references for time and frequency. Based on the microwave transitions, atomic clocks such as the Cs-fountain have defined Système International d’Unités (SI) second. Optical lattice clocks oscillate at the petahertz ($10^{15}$) regime that divides time into finer intervals. With 60 years of development, these clocks have reached the unprecedented instability and uncertainty of the level of $10^{-18}$, even $10^{-19}$ (Bloom et al. 2014; Nicholson et al. 2015; Sanner et al. 2019). They have a great potential to redefine the SI second based on optical frequency standards (Riehle et al. 2018; Lodewyck 2019). At the same time, the stability of clock comparison with the 1840 km optical fiber link has also been demonstrated within the order of $10^{-18}$ (Droste et al. 2013). Due to the unprecedented performance, atomic and optical clocks have widely been applied in the domain of technology and science: the Global Positioning System (GPS; Nelson 2011; Guang et al. 2018; Weyers et al. 2018), relativistic geodesy and gravimetry (Müller et al. 2008, 2018; Grotti et al. 2018; Kopeikin et al. 2018; Mehlstäuber et al. 2018), searching for dark matter (Derevianko & Pospelov 2014; Roberts et al. 2017), detecting possible variation of fundamental constant variation (Godun et al. 2014; Huntemann et al. 2014; Safronova et al. 2018), tests of the Lorentz invariance (LJ: Pihan-Le Bars et al. 2017; Shaniv et al. 2018), etc. The GPS satellites carrying onboard cesium or rubidium atomic clocks can make precise determinations of positions in the vicinity of Earth. Their navigation solutions have to take into account the relativistic corrections. The atomic clock experiments about hunting for dark matter require an accurate knowledge of gravitational effects. In other applications, the relativistic effects in clocks are also significant. Therefore, it is indispensable to perform rigorous calculations in the framework of general relativity (GR) for frequency shift.

Einstein’s equivalence principle (EEP) is the foundation of GR. It distinguishes between the metric theories and non-metric theories, and plays a nontrivial role in theories of gravitation. EEP includes three subprinciples: local Lorentz invariance (LLI), local position invariance (LPI), and universality of freefall also known as the weak equivalence principle (WEP). From these subprinciples, in a local reference system that can be constructed by methods of Fermi coordinates (Poisson & Will 2014), the influence of external matter should be given by the form of tidal potentials. The intrinsic effects of external masses come from their inhomogeneous gravitational field. The well-known absence of the noon–midnight redshift has also no frequency shift due to the Sun’s or Moon’s gravitational potential at the first order in $\Delta U/s/m/c^2$ when comparing two clocks fixed on the surface of the Earth (Hoffmann 1961; Wolf & Blanchet 2016). This implies that the measurements of the lunisolar gravitational effect in clocks can provide a test of EEP.

The tidal field has influences on the light propagation (Qin et al. 2019) and gravitational waves (Abbott et al. 2017, 2018; Raithel et al. 2018). Analogous to the gravitational redshift, the tidal potentials would also slightly change the rate of clocks, which we call the tidal clock effect. Although it is weak compared with Earth’s gravitational redshift, the frequency shift due to the Sun’s and Moon’s tidal potentials is observable for the best clocks currently. The tidal clock effects are barely measurable when comparing two clocks with distances about 1000 km (Qin et al. 2019). In addition to external matter, the tidal deformation of internal matter also leads to a contribution
in the tidal clock effect. This effect is important for the onboard clocks of low-orbit satellites. It has different properties from that due to external matter. With the developing performance, these tidal clock effects for future clock experiments would be non-negligible. The measurable tidal clock effects and their possible applications urgently demand further study of the position and distance dependences, as well as periodic features.

The realization of coordinate timescales in the vicinity of Earth requires the syntonization of clocks with respect to geocentric coordinate time (TCG). The syntonization of clocks operating on Earth and on satellites has to be taken into account at an accuracy level below the clock stability. An example is Terrestrial Time (TT). It is the ideal form of International Atomic Time (TAI) and related to TCG linearly by a defined constant \((\lambda - L_C)\). TAI is based on more than 250 atomic clocks distributed in 50 worldwide laboratories that should be syntonized to satisfy the TT definition. Generally, the constant part of syntonization is relatively easy to investigate. Here, we are interested in the time-varying part that includes the effects of the Earth’s rotation, tidal potential, etc. The effects of the irregularities in Earth’s rotation, tidal potential, etc. The effects of the tidal clock effect. This effect is important for the onboard clocks present the same periods. In particular, tidal clock effects due to the lunisolar gravitational field. When we consider the effects in clock comparison experiments, the frequency observables can be appropriately described in the GCRS. We consider two clocks, \(A\) and \(B\), with each other using light signal. Clock \(A\) sends the light signal with the proper frequency \(\nu_A\), and then this signal is received by clock \(B\) with the proper frequency \(\nu_B\). General relativistic formalism to calculate the frequency shift between the two clocks linked by the electromagnetic signal has been presented in Brumberg (1991) and Kopeikin et al. (2011). The one-way frequency shift between the two clocks is given by

\[
\frac{\nu_A}{\nu_B} = \left( \frac{d\tau_B}{d\tau_A} \right) \left( \frac{d\tau_A}{d\tau_B} \right),
\]

where \(\tau_A\) and \(\tau_B\) are the proper times of clocks \(A\) and \(B\), respectively. \(t_A\) and \(t_B\) are the coordinate times corresponding to proper times \(\tau_A\) and \(\tau_B\), respectively.

The derivatives \(d\tau_A/dt_A\) and \(d\tau_B/dt_B\) depend on the state of the clocks containing the gravitational redshift and the second-order Doppler effect. They may be calculated using the invariance of the Riemannian spacetime interval. Considering a transported clock, the invariable spacetime interval is given by

\[
d^2 = g_{00} c^2 dt^2 + 2g_{0i} c dt dx^i + g_{ij} dx^i dx^j = -c^2 d\tau^2.
\]

where \(g_{\mu\nu}\) is the metric tensor. When considering clock effects in the experiments, this equation allows us to calculate the gravitational contributions due to Earth’s Newtonian gravitational potential, Earth’s mass multipole moments, and lunisolar gravitational field.

The term \(d\tau_A/dt_B\) is dependent on the light trajectory of the signal. For example, for the clock comparison experiments with optical fiber links, light trajectory is determined by the optical fiber. Clearly, this term includes influences of changing refractive index, changing fiber length due to temperature, or effects due to fiber motion, etc. These influences have been investigated in detail (Cohen & Fleming 1979; Geršl et al. 2005). It is not our purpose in this paper to study them.

To develop Equation (1), we start with discussions of the GCRS. For most of the experiments in the vicinity of Earth, GCRS is more convenient for the calculations of observable modeling than other optional reference systems. With the recommendations of the IAU resolutions (IAU Resolution B1.3 (2000); Soffel et al. 2003), the metric tensor in the GCRS (coordinates \(ct, x'\)) can be written in the two-potential form

\[
g_{00} = -1 + \frac{2\omega}{c^2} - \frac{2\omega^2}{c^4} + O(c^{-5}),
\]

\[
g_{0i} = -\frac{4\delta_{ij}\omega^j}{c^2} + O(c^{-5}),
\]

\[
g_{ij} = \delta_{ij} \left(1 + \frac{2\omega}{c^2}\right) + O(c^{-4}).
\]
where \( w \) and \( w^i \) are scalar harmonic and vector harmonic potentials, respectively, which generally are written as

\[
w = w_e + w_{\text{ext}} + O(c^{-4}),
\]

and

\[
w^i = w^i_e + w^i_{\text{ext}} + O(c^{-2}),
\]

where the subscript “e” represents Earth’s contribution, and subscript “ext” represents the external potential. Usually, it is advantageous to express Earth’s gravitational potential \( U_e \) as a multipole series that usually called potential coefficients or Blanchet–Damour (B-D) moments (Damour et al. 1991). Using these B-D moments, we can obtain a simple form for the multipole moments of Earth’s gravitational potential in the first post-Newtonian approximation. Theoretically, these B-D moments can be determined by the distribution of mass and matter currents. However, they may be presented as some parameters that can be evaluated by numerically fitting to various kinds of experimental data, such as the satellite motion tracking, geodetic measurements, gravimetry, etc. An approximate expansion of Earth’s gravitational potential is the spherical harmonic expansion, which is given by Soffel et al. (2003) and Pavlis et al. (2012) as

\[
w_e(t, \mathbf{x}) = \frac{GM_e}{r} \left( 1 + \sum_{l=2}^{\infty} \sum_{k=0}^{\infty} \frac{r_{\text{ke}}^l}{r} \right) P_l^k(\cos \theta) \\
\times \left( C_l^k(t, r) \cos(k\phi) + S_l^k(t, r) \sin(k\phi) \right) + O(c^{-4}),
\]

with

\[
C_l^k(t, r) = C_{lk}(t) - \frac{r^2}{4l - 2} \frac{d^2}{dt^2} C_{lk},
\]

\[
S_l^k(t, r) = S_{lk}(t) - \frac{r^2}{4l - 2} \frac{d^2}{dt^2} S_{lk},
\]

where \( r = |\mathbf{x}|, \theta \) and \( \phi \) are the polar angles corresponding to the spatial coordinates \( x^k \) of GCRS, \( r_{\text{ke}} \) is Earth’s equatorial radius, and \( P_l^k \) are associated with the Legendre polynomials. \( C_{lk} \) and \( S_{lk} \) relate to approximately constant potential coefficients in the terrestrial system corotating with Earth. For \( C_{lk} \) or \( S_{lk} \), the order of magnitude of the second time derivative term is \((\Omega_c r_{\text{ke}}/c)^2 \approx 10^{-12}\) times smaller than that of the first term. For the case \( l = 2, k = 0 \), the order of magnitude of \( C_{30} \) is \( 10^{-3} \), which leads to a contribution of the order of \( 10^{-13} \) in \( g_{00} \). The second time derivative term has a contribution of the order of \( 10^{-25} \) in \( g_{00} \). Therefore, these second time derivative terms can be neglected.

However, it is clear that this formalism (Damour–Soffel–Xu formalism) does not take into account the tidal interactions between the bodies. Since Earth is an extended body, the tidal forces caused by inhomogeneities of the Sun’s and Moon’s gravitational field lead to a tidal deformation. This deformation produces a time-varying potential, which contributes a correction in the metric tensor. This correction can result in observable perturbations in the orbital motion of the satellite and clock rate. The relativistic tidal properties are generally determined by Love numbers, such as neutron stars (Damour & Nagar 2009). For Earth, the measurements from gravimeters can be used to determine Love numbers, which reveal information of tidal deformation. By introducing the Love number, we restructure Earth’s gravitational potential \( w_e^i \) as

\[
w_e^i = w_e^i + k_2 \sum_{b=e} \frac{G M_b}{r_b^3} \left[ 3(n_{be} \cdot \mathbf{n})^2 - 1 \right],
\]

where the latter term comes from the tidal deformation of Earth (Poisson & Will 2014), \( n = x/r, k_2 \) is the Love number of Earth, and \( n_{be} \) is the unit vector pointing to the center of Earth from that of body \( b \). The Sun and Moon are the main sources of Earth’s tidal deformation. Since the Moon is close enough to Earth, it leads to a greater deformation than the Sun. The contribution of the tidal deformation decreases with distance and its influence is greatest on the surface of Earth. It is worth noting that the influences of analogous deformations are significant in many close binary-star systems.

As mentioned in Soffel et al. (2003), the external potentials can be split into tidal contribution and internal contribution. Tidal terms are at least quadratic in the \( x^i \) and generated by all external bodies. The inertial terms are linear in \( x^i \) and in the four-acceleration of the geocenter. The four-acceleration term is a result of the coupling between Earth’s higher-order multipole moments and the external tidal field, which vanishes for a purely spherical and nonrotating Earth. The external scalar potential \( w_{\text{ext}} \) is recommended in the form Soffel et al. (2003),

\[
w_{\text{ext}} = w_{\text{tidal}} + w_{\text{iner}}.
\]

The magnitude of the \( w_{\text{iner}} \) term due to the Moon is about \( 2.5 \times 10^{-4} \text{ m}^2 \text{s}^{-2} \). It is negligible for our calculation in clock effects. The tidal potential term \( w_{\text{tidal}} \) is a generalization of the Newtonian tidal potential of external bodies (primarily due to the Sun and Moon). It’s post-Newtonian expression can be found in Damour et al. (1992). For our calculation, \( w_{\text{tidal}} \) can be expressed as

\[
w_{\text{tidal}} = \sum_{b=e} \left[ U_b(r_{be} + \mathbf{x}) - U_b(r_{be}) - \mathbf{x} \cdot \nabla U_b(r_{be}) \right]
\]

\[
= \sum_{b=e} \frac{GM_b}{2r_{be}^3} \left[ 3(n_{be} \cdot \mathbf{x})^2 - \mathbf{x}^2 \right] + O(r_{be}^{-4}),
\]

where \( r_{be} \) is the vector from the center of body \( b \) to that of Earth. From Equations (9) and (11), tidal potentials from the external and internal matter have different properties. With increasing distance, tidal contribution due to external matter becomes more significant.

The vector potential contributed by Earth’s angular momentum is approximately expressed as

\[
w_e^i = - \frac{GM_e}{2r^3}(\mathbf{x} \times \mathbf{s_e})',
\]

where \( \mathbf{s_e} \) is the angular momentum per unit of mass of Earth. On the surface of Earth, \( w_e^i \) results in a contribution of the order of \( 10^{-16} \) in \( g_{00} \). Generally, considering clock effects, the vector potential coupled with the clock’s velocity leads to a level much smaller than \( 10^{-18} \). Another term, \( w_{\text{ext}} \), is negligible.
From Equation (2), the proper time of clock A located on the position \( x_A(t) \) is given by Brumberg & Kopeikin (1990) as
\[
\frac{dr_A}{dt} = 1 - \frac{1}{c^2} \left( w' + \frac{v^2}{2} \right)_A - \frac{1}{c^4} \left( \frac{v^4}{8} + \frac{3v^2w'_e}{2} - \frac{(w'_e)^2}{2} - 4w'v') \right)_A, \tag{13}
\]
where \( v \) is the velocity of clock A, and subscript A indicates that all quantities are expressed for clock A.

Then, by introducing an instantaneous coordinate distance of \( r_{AB} = x_B(t_s) - x_A(t_s) \), the term \( dA_B/dt_A \) is given in the form for using the quantities measured at emission instant \( t_A \),
\[
\frac{dA_B}{dt_A} = 1 + dA_B \left( \frac{r_{AB}}{c} + \frac{r_{AB} \cdot v_B}{c^2} \right) + \frac{r_{AB}}{2c^3} \left[ r_{AB} \cdot a_B + v_B^2 + \frac{(r_{AB} \cdot v_B)^2}{r_{AB}^2} \right]
+ \frac{1}{6c^4} [r_{AB}^2 (r_{AB} \cdot b_B + 3a_B \cdot v_B)
+ 6(r_{AB} \cdot v_B)(v_B^2 + r_{AB} \cdot a_B) ], \tag{14}
\]
where \( v_B \) is the velocity of clock B, \( a_B \) is the acceleration of clock B, and \( dA_B = dA_B/\text{dt} \) (all three quantities are expressed at emission time \( t_A \)). This equation is useful since the time of signal is generally recorded in one clock.

3. The Tidal Clock Effect

In an \( N \)-body system, where bodies are elastic, the gravitational interactions could cause their deformations. As the response, a corresponding variation of the potential is generated. Therefore, we consider that the tidal potentials arise from both the external and internal matter. All tidal potentials are able to change the clock’s rate, the so-called tidal clock effect. It is essentially gravitational redshift caused by tidal potentials. Considering the case in the vicinity of Earth, the external sources are mainly the Sun and the Moon, whereas the tidal deformation of Earth is an internal source.

The External Matter. We start the discussion with the external bodies. The influences of the Sun and Moon come from the inhomogeneities of their gravitational field. When we focus on the particular experiments of clock comparison, their effects are to change the clock’s rate. Although the changes in the clock’s rate are weak, they are barely measurable with the best current clocks. From Equations (1) and (11), the tidal clock effect due to the external masses is obtained as
\[
\left( \frac{\delta v}{v} \right)_{\text{ext}} = \sum_{b=e} \frac{GM_b r^2}{c^2 \ell_b^3} \left[ P_2(n_{be} \cdot n) + \frac{r}{\ell_b} P_3(n_{be} \cdot n) + ... \right], \tag{15}
\]
where \( P_2 \) and \( P_3 \) are Legendre polynomials, and the apostrophe represents higher-order Legendre polynomials, which are negligible here. This effect grows quadratically with distance from Earth’s center of mass. Its influence is minimal for ground-based clocks. By setting \( r \) to Earth’s radius, this equation gives the solution for these ground-based clocks. A priori calculations demonstrate that the solution of the equation leads to a small contribution with the order of \( 10^{-17} \).

This magnitude corresponds to the changes in the altitude of the order of \( 10^{-13} \) m on the surface of Earth.

For the space-based clocks, tidal clock effect is much more significant, which can reach the level of \( 10^{-16} \), or even \( 10^{-15} \). To develop this effect, we assume that a clock is on the satellite with the Keplerian orbit \( r = a(1 - e^2)/(1 + e \cos f) \), where \( a \) is the semimajor axis, \( e \) is the eccentricity, and \( f \) is the true anomaly. Other useful parameters are the longitude of the ascending node \( \Omega \), the argument of latitude \( \lambda \), and the inclination \( i \) of the satellite orbit to the equator. When only the \( P_2 \) term is included, for this clock on the satellite, the clock tidal effect can be expressed in the orbit-element form of Kozai (1973)
\[
\left( \frac{\delta v}{v} \right)_{\text{ext}} = \sum_{b,e} \frac{GM_b r^2}{c^2 \ell_b^3} \left\{ \left[ \frac{3}{4}(A_b^2 + B_b^2) - 2 \right] + \frac{3}{4}(A_b^2 - B_b^2) \cos 2L + 2A_b B_b \sin 2L \right\}, \tag{16}
\]
where
\[
A_b = \cos \delta_b \cos (\Omega - a_b), \tag{17}
\]
\[
B_b = -\cos \delta_b \cos \Omega \sin (\Omega - a_b) + \sin \delta_b \sin i. \tag{18}
\]
\( \delta_b \) and \( a_b \) are the decl. and R.A. of body \( b \), respectively. \( A_b \) and \( B_b \) depend on the orbital elements of satellite and position of body \( b \) and they do not include the short-period variations. In Equation (16), the second term has an obvious dependence on the orbital period of the satellite.

For the influences of external masses, there are two types of effects for onboard clocks. The superscript \( s \) represents the clocks on the satellite. The first is the secular and long-period terms that have steady drift in several orbital periods. Such parts are typically much more interesting and important. They accumulate over time and have a significant influence after a complete orbit. Considering the average with respect to \( M \) (mean anomaly), we have the following relations:
\[
\frac{1}{2\pi} \int_0^{2\pi} \left( \frac{r}{a} \right)^2 dM = \frac{2 + 3e^2}{2}, \tag{19}
\]
\[
\frac{1}{2\pi} \int_0^{2\pi} \left( \frac{r}{a} \right)^2 \cos 2L \cdot dM = \frac{5e^2 \cos 2\omega}{2(1 + e)^{1/2}}, \tag{20}
\]
\[
\frac{1}{2\pi} \int_0^{2\pi} \left( \frac{r}{a} \right)^2 \sin 2L \cdot dM = \frac{5e^2 \sin 2\omega}{2(1 + e)^{1/2}}, \tag{21}
\]
where \( \omega \) is the argument of perigee of the satellite. After averaging out short-period terms, the secular and long-period effects are given by
\[
\left( \frac{\delta v}{v} \right)_{\text{ext,sl}} = \sum_{b,e} \frac{GM_b a^2}{c^2 \ell_b^3} \left\{ \frac{1}{8}(3A_b^2 + B_b^2) - 2 \right\}(2 + 3e^2)
+ \frac{15e^2}{8(1 + e)^{1/2}} \left[ (A_b^2 - B_b^2) \cos 2\omega + 2A_b B_b \sin 2\omega \right]. \tag{22}
\]

The second type is the short-period terms that oscillate with orbital period. The short-period terms should be included for some space missions that require the consideration of precise observables. Another example is the case of high-orbit
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satellites. For the order of \( e^2 \), they are given by

\[
\frac{\delta v}{v}_{\text{ext,sp}} = \sum_{b \neq e} \frac{GM_b a^2}{c^2 r_{be}} \left\{ \frac{1}{4} \left[ 3(A_b^2 + B_b^2) - 2 \right] \times \left( \left( \frac{r}{a} \right)^2 - \frac{3 e^2 + 2}{2} \right) \right. \\
+ \frac{3}{4} (A_b^2 - B_b^2) \left( \left( \frac{r}{a} \right)^2 \cos 2L - \frac{5 e^2}{2} \cos 2\omega \right) \\
+ \frac{3}{2} A_b B_b \left( \left( \frac{r}{a} \right)^2 \sin 2L - \frac{5 e^2}{2} \sin 2\omega \right) \bigg\}.
\]

(23)

In addition, the Moon is close to Earth with a distance of about \( 3.8 \times 10^8 \) m. Equation (16) does not calculate Moon’s contributions for the high-orbit satellites well since \( r/r_{me} \) is not a small parameter. It is necessary to take the \( P_3 \) term of the Moon into account. Analogously, we take the \( P_3 \)-term average with respect to \( M \). Then, the secular and long-period effects \((\delta v/v)_{\text{int,sp}}\) are written as

\[
\frac{\delta v}{v}_{\text{int,sp}} = -\frac{GM_m a^3}{c^2 r_{me}^3} \left\{ \frac{15 e}{64} \left[ 5(A^2 + B^2) - 4 \right] \times \left( A \cos \omega + B \sin \omega \right) (4 + 3 e^2) \\
+ \frac{175 e^3}{64} \left[ (A^2 - 3B^2)A \cos 3\omega + (3A^2 - B^2)B \sin 3\omega \right] \bigg\}.
\]

(24)

The short-period terms are

\[
\frac{\delta v}{v}_{\text{int,sp}} = -\frac{GM_m a^3}{2c^2 r_{me}^3} \left\{ \frac{3}{8} \left[ 5(A^2 + B^2) - 4 \right] \left( \frac{r}{a} \right)^3 A \cos L \\
+ \frac{5 e (4 + 3 e^2)}{8} A \cos \omega + \left( \frac{r}{a} \right)^3 B \sin L \\
+ \frac{5 e (4 + 3 e^2)}{8} B \sin \omega \bigg\} \\
+ \frac{5}{8} \left[ (A^2 - 3B^2)A \left( \frac{r}{a} \right)^3 \cos 3L + \frac{35 e^3}{8} \cos 3\omega \right] \\
+ (3A^2 - B^2)B \left( \frac{r}{a} \right)^3 \sin 3L + \frac{35 e^3}{8} \sin 3\omega \bigg\}.
\]

(25)

The Internal Matter. Subsequently, the tidal deformation of Earth is taken into account. From Equation (9), the tidal clock effect caused by internal matter is obtained as

\[
\frac{\delta v}{v}_{\text{int}} = k_2 \frac{r_{oe}^5}{c^2 r_e^3} \sum_{b \neq e} \frac{GM_b}{2r_{be}} \left[ 3(n_{be} \cdot n)^2 - 1 \right].
\]

(26)

\( n_{be} \cdot n \) indicates that it periodically varies with time. This effect decays cubically with distance from Earth’s center of mass. For the satellite clocks at low altitudes, it reaches a few parts in \( 10^{18} \). With increasing latitudes bigger than \( 10^7 \) m, the effects do not exceed \( 1 \times 10^{-18} \). This property is different from that of external masses. As an estimation for tidal influences, Table 1 shows the effects of the syntonization of clocks on Earth’s surface and on the satellites with respect to TCG. When \( r = r_{oe} \), it becomes the situation in which clocks are on Earth’s surface. This effect reaches the biggest contribution. In addition, the height’s variation due to deformation should be considered in this case. Another Love number \( h_2 \) can describe the radial variation of Earth’s surface. By using \( h_2 \), we obtain this change in the clock’s rate:

\[
\frac{\delta v}{v}_h = -h_2 \sum_{b \neq e} \frac{GM_b}{2c^2 r_{be}^3} \left[ 3(n_{be} \cdot x)^2 - x^2 \right].
\]

(27)

It is understandable that Equation (27) has the same form as Equation (11). As the response of Earth to the tidal forces \( f_{tid} \), the radial variation is characterized as the form of \( f_{tid} \cdot n \) \((\sim [3(n_{be} \cdot n)^2 - 1])\). Hence the corresponding effect in the clock’s rate is the same as with tidal potentials. Finally, we obtain the effective tidal clock effect, \((1 + k_2 - h_2)w_{tidal}/c^2\), for clocks on Earth’s surface. It is sufficient to take \((1 + k_2 - h_2) = 0.69\) for most Earth models (Wolf & Petit 1995). It is noticeable that this effective coefficient is smaller than 1. For clocks on Earth’s surface, the total effects of Earth’s tidal response can soften the influences of external masses.

Another example is Jupiter. We only consider tidal deformation caused by the Sun. As a gaseous planet, its tidal deformation is much more prominent. The Love number of Jupiter recently reported as \( k_2 = 0.59 \) is 10% larger than previous estimates (Wahl et al. 2016). With greater mass and a greater value of the Love number, the clock tidal effects of Jupiter’s deformation are three orders of magnitude greater than that of Earth’s deformation.

For the internal matter, we also consider the effects on the onboard clocks described above. Similarly to the expression of Equation (16), it is given by

\[
\frac{\delta v}{v}_{\text{int}} = \sum_{b \neq e} \frac{k_2 GM_b r_{oe}^5}{c^2 r_{be}^3} \left[ \frac{1}{4} [3(A_b^2 + B_b^2) - 2] \\
+ \frac{3}{4} (A_b^2 - B_b^2) \cos 2L + 2A_b B_b \sin 2L \right].
\]

(28)

It represents the tidal clock effect due to Earth’s tidal deformation. Increasing orbital altitudes, it decreases with cubic form. After averaging Equation (28) with respect to \( M \),

\[ \text{Table 1} \]

| Effect | Order of Magnitude | Order of Magnitude |
|--------|--------------------|--------------------|
| Earth’s Gravitational Potential | \( 7 \times 10^{-10} \) | \(< 2 \times 10^{-10} \) |
| The Second-order Doppler | \( 1 \times 10^{-12} \) | \( 8 \times 10^{-11} \) |
| Earth’s Tidal Deformation | \( 10^{-17} \) | \( 10^{-19} \) |
| External Masses (Sun and Moon) | \( 10^{-17} \) | \( 10^{-16} \) |

\textbf{Note.} The orders of magnitude are estimated for clocks \( C_{OE} \) and \( C_{OS} \). The altitude of the satellites is about 20,000 km, close to the GPS.
Moreover, we can also easily obtain the short-period terms

\[
\left( \frac{\delta u}{v} \right)_{\text{int,ls}} = k G M_{\text{be}} r_{\text{be}}^5 \left( 1 - e^2 \right)^{3/2} \frac{1}{4} \left[ 3(A_b^2 + B_b^2) - 2 \right].
\]

For small eccentricities, the secular and long-period effects are independent to the eccentricity of the orbit. However, they have a dependence with \( e \) in the cases of high eccentricities. When \( e \to 1 \), Equation (29) diverges, which is not surprising as \( r_p \gg r_{be} \) makes Equation (29) insufficient for calculations. Moreover, we can also easily obtain the short-period terms

\[
\left( \frac{\delta u}{v} \right)_{\text{int,sp}} = \sum_{b \neq e} k G M_{\text{be}} r_{\text{be}}^5 \left( \frac{1}{a^3} \right) \left( \frac{1}{1 - e^2} \right)^{3/2} \left[ 3(A_b^2 + B_b^2) - 2 \right]
\]

\[\times \left( \frac{a}{r} \right)^3 \left( \frac{1}{1 - e^2} \right)^{3/2} + \frac{3a^3}{4r^3} \left( A_b^2 - B_b^2 \right) \cos 2L + 2A_b B_b \sin 2L \right\}.
\]

(30)

Taking the Moon as an example, we illustrated in Figure 1 the secular and long-period effects of external and internal masses on the satellite clocks. The purple, blue, and black lines represent satellites with orbital semimajor axes of 1, 10, and 100 km, respectively. The solid lines are for external masses with the order of 10^7, and 10^8 m, and 2.6 × 10^8 km, respectively. The solid lines are for external masses with the order of 10^16. The order of magnitude of the dashed lines are 10^{-19} for internal mass. The order of magnitude of the frequency shift grows with the semimajor axis for solid lines, and decays with the semimajor axis for the dashed lines. This is consistent with the above discussion. Two types of lines have the same tendency with \( i \) varying and reach zero for the same orbital inclination. Additionally, we also display short-period effects of internal masses on the onboard clocks. For a fixed value of eccentricity, the effect for different orbital inclinations is visualized in Figure 2. At a fixed inclination, we change the orbital eccentricities to show the variation of short-period effects in Figure 3. In Figure 3, a peak is visible for a big eccentricity and a bigger eccentricity value results in a higher peak. The features of the short-period effects of the external masses are similar to the characteristics in Figures 2 and 3.

In addition, we consider clock comparisons in laboratories again. The secular and long-period effects play a crucial role in the syntonization of ground-based clocks with respect to TCG or TT. To discuss them, it is convenient to parameterize the tidal clock effect in the form of clock positions. A method is to express them in the celestial coordinate system. Considering the coordinate relationships, \( \mathbf{n}_{\text{be}} \cdot \mathbf{n}_A \) can be written in the form of celestial coordinates,

\[
\mathbf{n}_{\text{be}} \cdot \mathbf{n}_A = \sin \varphi_A \sin \delta_b + \cos \varphi_A \cos \delta_b \cos H_b,
\]

(31)

where \( \varphi_A \) is the latitude of position \( A \), and \( H_b \) and \( \delta_b \) are the local hour angle and decl. of body \( b \), respectively. By using the relationship, the tidal clock effect for the laboratory’s clocks becomes

\[
\left( \frac{\delta u}{v} \right)_{\text{lab}} = \sum_{b \neq e} \frac{3 \eta}{2} G M_{\text{be}} r_{\text{be}}^2 \left[ \cos^2 \varphi \cos^2 \delta_b \cos 2H_b \right.
\]

\[+ \sin 2 \varphi \sin 2 \delta_b \cos H_b \]

\[+ \left( \sin^2 \varphi - \frac{1}{3} \right) \left( \sin^2 \delta_b - \frac{1}{3} \right) \]

(32)

where \( \eta = 1 + k_2 - h_2 \). On the right-hand side of this equation, the first two terms are short-period terms, and the third term represents secular and long-period terms. \( \cos H_b \) and \( \cos 2H_b \) represent variations with the diurnal and semi-diurnal periods, respectively. Their magnitudes reach the order of 10^{-17}. Evaluation of Equation (32) gives the diurnal and semi-diurnal contributions in the 100 km clock comparisons whose amplitudes reach the level of order 10^{-18}. These short periods should be treated carefully since the temperature effects may have same periodic effects, as well as the possible violation effects of LI (Nicholson et al. 2015). For exploring possible effects beyond GR, these periodic effects should be cautiously removed in the remote-distance clock comparison experiments. Taking the Sun and Moon into calculations, the terms involving \( \delta_b \) and \( \delta_m \) appear in the last term, which represent the long-period variations with periods of half a year and half a month, respectively. The last term also includes the latitude \( \varphi \), which implies that the tidal clock effect depends on the clock’s position on Earth. Equation (32) provides a straightforward method to study the position dependence.

To demonstrate the position dependence, we independently consider the differences in altitude, longitude, and latitude, respectively. Only considering the difference of height \( \Delta H \), the
tidal clock effect on the clock comparison can be written as

$$\left( \frac{\delta \nu}{\nu} \right)_{\text{tidal}} = \frac{2 \eta \Delta H}{n_0 e^2} w_{\text{tidal}}(x)$$  \hspace{1cm} (33)

where $w_{\text{tidal}}(x)$ represents the tidal potential of the clock position in the laboratory. Equation (33) retains the first order of the altitude differences with respect to the geoid. Estimation of Equation (33) reveals that the hundreds-of-kilometer altitude difference $\Delta H$ makes an observable effect of the order of $10^{-18}$. This means that the height difference is negligible for laboratory experiments. Whereas for clock comparisons between geostationary satellites and ground, it should be taken into account.

Then, we consider the tidal clock effect due to position differences that are only caused by longitude differences $\Delta l$. For the position dependence of the tidal clock effects, the correction of the long-term average is more interesting. From the evaluation of longitude differences $\Delta l$ in Equation (32), we find that it averages out after a long time. Therefore, the longitude difference does not contribute any long-term correction in the tidal clock effect for clock comparisons.

Further, we consider the position differences only due to latitude differences $\Delta \varphi$ (comparisons between $\varphi + \Delta \varphi$ and $\varphi$). Analogously, we focus on the long-term average. By using the relationship of Equation (31) and after a long-term interval, the average in the tidal clock effect due to a latitude difference...
is

\[
\left( \frac{\delta \nu}{\nu} \right)_{t \rightarrow \infty} = \sum_{b \in E} D_b(r)(3\sin^2 \delta_b - 1)\sin(2\varphi - \Delta \varphi) \sin \Delta \varphi
\]

(34)

with

\[
D_b(r) = \frac{\eta D_b}{c^2} \left( \frac{r}{r_{0b}} \right)^2 \left( \frac{c_b}{r_b} \right)^3,
\]

(35)

where \(D_b\) is the Doodson constant of body \(b\), the overline symbol represents the expected value for the corresponding time interval, and \(c_b\) and \(r_b\) are the average distance and practical distance from body \(b\) to the center of Earth’s mass, respectively. This equation describes the long-term corrections in the tidal clock effect of clock comparisons caused by latitude differences. Clearly, the value of the long-term corrections is not zero but is dependent on their latitudes and latitude differences. The position differences can be decomposed into the differences of height, longitude, and latitude, which implies that we could give position dependence by these three components. For ground-based clock experiments, the long-term tidal clock effects should be considered if experimental geographic positions have a latitude difference, which is given by Equation (34). Whereas for the longitude, long-term corrections can be omitted. However, the periodic variations should be taken into account for short time intervals.
To demonstrate tidal clock effects, we give the calculation for the frequency comparison between two locations, $A$ and $B$ (Figure 4(a)). The east longitude and north latitude of $A$ are E114° and N30° (Wuhan), and that of $B$ are E120° and N30° (Shanghai), respectively. We shall use nominal planetary ephemeris, astronomy parameters, general gravitational constants of Earth, Sun, and Moon, etc. Figure 4(a) shows the calculated frequency shift due to the tidal potentials between $A$ and $B$ with dates from 2020 January 1 to 14. From the figure, the amplitudes can reach $5 \times 10^{-18}$ for the clock distance of about $7 \times 10^3 \text{km}$ that is achievable in modern experiments. The diurnal and semidiurnal variations are clearly visible having the same characteristics of gravity tides. For a long-term average, this effect between $A$ and $B$ almost is zero. This is consistent with the aforementioned. By building the global network of atomic clocks, we may obtain a global network of tidal clock effects. Tidal clock effects and gravity tides complement each other, which may have the potential to provide an insight for gravity.

To show the latitude dependence, we additionally calculate the frequency comparison between $A$ and $C$ as shown in Figure 4(b). We choose the position of $C$ as E114° and N36° so that the distances of $A - B$ and $A - C$ are the same but their directions are perpendicular. Similar to the characteristic in Figure 4(a), tidal clock effects between $A$ and $C$ have diurnal and semidiurnal variations with the amplitude of several $10^{-18}$. The long-term average of the tidal clock effect between $A$ and $C$ is about $2 \times 10^{-18}$, much bigger than that of $A - B$. In a mathematical point of view, a direct calculation of Equation (34) gives a long-term average of about $2 \times 10^{-18}$ for the tidal clock effect between $A$ and $C$, which is in accord with Figure 4(b).

A further study demonstrates the special positions located at the south or north latitude of 35°.27. The long-term average of the tidal clock effects is zero between the clock rate at these positions and the geocentric coordinate time. This characteristic means that we may apply a north latitude of N35°.27 as the baseline for a global clock network. Based on the baseline of N35°.27, Figure 5 gives a global long-term average of the tidal clock effects for clock comparison of the global clock network. This correction may be taken into account for the measurement of the global geoid. Most of the clock experiments are implemented in laboratories. The calculated result of Figure 5 is based on an ansatz that all clock positions are on the ground. From the figure, there is a symmetrical distribution about the Equator. The latitude lower than baseline is the positive long-term correction and the higher latitude gives the negative value. The biggest difference between North Pole and Equator is about $2 \times 10^{-17}$.

4. Conclusion

We present a relativistic procedure to rigorously deduce the algorithms of frequency shift in the framework of GR. Time-varying parts of frequency shift are very interesting in testing physical problems and we focus on the tidal effects on clocks operating on Earth’s surface and in satellites. In our calculations, Earth’s elasticity is introduced into the GCRS. By using the Love numbers, the tidal potential caused by Earth’s tidal deformation is separated from Earth’s gravitational potential. The total tidal potentials are divided into external masses (the Sun and Moon) and internal mass (Earth’s tidal response). All of them can change the clock rates, but with different properties. The external-mass effect on clocks grows quadratically with distance from Earth’s center of mass. On the contrary, the internal-mass effect decays with a cubic form. Although the internal tidal deformation and lunisolar gravitational field paly the same important role on clock experiments on the ground, when clock experiments are on space the effects of the lunisolar gravitational field are more significant. Moreover, the calculations are given in parameter-dependent forms, which can be used to straightforwardly estimate effects from orbital elements and geographic positions.

The influences of tidal potentials are given in period-dependent forms for ground- and space-based clocks. For clocks located on board satellites, we use orbital elements to study two types of effects. The short-period terms are dependent on the orbital periods of satellites. The secular and long-period terms have stable drifts after a complete orbit. With the increasing satellite altitudes, internal and external tidal clock effects exhibit suppressive and enhanced tendencies, respectively. However, when changing orbital inclination, their variation forms are the same and they reach zero for a special inclination (Figure 1). The short-period effects on space-based clocks are demonstrated in Figures 2 and 3 where variations follow the orbital period that is apparent. When orbital eccentricity is big enough, short-period effects deviate significantly from the sine function. Moreover the Moon’s third-order tidal potential presents non-negligible effects on high-altitude satellite clocks. For clocks fixed on Earth’s surface, we employ a celestial coordinate system to manifest position dependence and distance dependence of the tidal effects. When distances are about 1000 km, tidal effects on clock comparisons reach the level of $10^{-17}$ precision. The sidereal-day and sidereal-year periodic effects of tidal potentials on remote-distance clock comparisons must be considered for testing fundamental physical assumptions. Benefiting from the high-precision atomic clocks and optical fiber links, the tidal clock effects are measurable reaching the level of $10^{-18}$ for clock comparisons with the distance of hundreds of kilometers. Figure 4 reveals the position dependence of the tidal clock effect that latitude differences in positions could contribute a long-term frequency shift whereas for longitude differences they are almost zero. N35°.27 latitude is proposed as a baseline for the global clock network. Based on this line, we give the global long-term corrections of the tidal clock effects for clock comparison in the global clock network (Figure 5). The symmetrical characteristic about the Equator implies that the biggest difference is between the two poles and Equator. Furthermore, the tidal clock effect complements the gravity tides. The global clock tidal effect network provided by the global clock network also complements the gravity tides network, which has a great potential in science and geophysics.

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