Phase transitions at strong coupling in the 2+1-d abelian Higgs model

R. B. MacKenzie
Groupe de physique des particules, Université de Montréal, C.P. 6128, succ. centre-ville, Montréal, Québec, Canada, H3C 3J7
E-mail: rbmack@lps.umontreal.ca

Faïza Nebia-Rahal
Groupe de physique des particules, Université de Montréal, C.P. 6128, succ. centre-ville, Montréal, Québec, Canada, H3C 3J7
E-mail: faiza@lps.umontreal.ca

M. B. Paranjape
Groupe de physique des particules, Université de Montréal, C.P. 6128, succ. centre-ville, Montréal, Québec, Canada, H3C 3J7
E-mail: paranj@lps.umontreal.ca

Abstract. We study, using numerical Monte-Carlo simulations, an effective description of the 2+1 dimensional Abelian Higgs model which is valid at strong coupling, in the broken symmetry sector. In this limit, the massive gauge boson and the massive neutral Higgs decouple leaving only the massive vortices. The vortices have no long range interactions. We find a phase transition as the mass of the vortices is made lighter and lighter. At the transition, the contributions to the functional integral come from a so-called infinite vortex anti-vortex loop. Adding the Chern-Simons term simply counts the linking number between the vortices. We find that the Wilson loop exhibits perimeter law behaviour in both phases, although the polarization cloud increases by an order of magnitude at the transition. We also study the ’t Hooft loop. We find the ’t Hooft loop exhibits perimeter law behaviour in the presence of the Chern-Simons term but is trivial in its absence. Thus we have a theory with perimeter law for both the Wilson loop and the ’t Hooft loop, but contains no massless particles.

1. Abelian Higgs model at strong coupling

The abelian Higgs model corresponds to the Lagrangian:

\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} D_\mu \phi (D^\mu \phi)^* - \frac{\lambda}{4} (|\phi|^2 - \eta^2)^2. \] (1)

At strong coupling, \( \lambda, e \to \infty \), keeping \( \lambda/e^2 \) fixed, the neutral scalar mass \( \sqrt{2\lambda \eta} \) and the gauge boson mass \( e \eta \) diverge but the vortex mass \( \mu = \eta^2/f(2\lambda/e^2) \) can be held finite and variable, see [1] for details.
1.1. Evidence for a phase change

We calculate the Euclidian vacuum persistence amplitude using the functional integral and lattice Monte Carlo simulations. The field configurations that contribute to this amplitude correspond to closed vortex loops, which describe the virtual process of the creation of a vortex anti-vortex pair and its subsequent annihilation. The Euclidian action for such a configuration is, to first approximation, simply the vortex mass multiplied by the length of the vortex loop [2]. Thus we generate configurations of closed loops on a 3 dimensional lattice, and weigh each such configuration using the Boltzmann factor corresponding to the total length of the loops multiplied by $\mu$. For large $\mu$ only small minimal loops enter the calculation, however as $\mu$ is lowered, larger and larger loops become important. After a specific value, $\mu = 0.152$, there is a sudden transition where about half of the loops reorganize and form a so-called infinite loops. For smaller values of $\mu$, the total length of the vortex loops increases, however this occurs simply by appending to the length of the infinite loop, the number of finite loops nor their total length do not appreciably change.

An infinite loop corresponds to a loop that has a length substantially larger than what a loop should have if it were of a linear dimension equal to the size of the lattice. The loops are closed non-self intersecting random walks. On a cubic lattice their linear dimension should be approximately $N^{3/5}$ [3]. Our lattice is actually a much more intricate lattice, where it is not known what the size of an infinite loop should be. We consider any loop longer than 10000 steps to be infinite.

Clear evidence for a change of phase is seen in Figure (1) of the average length of the loops versus the logarithm of the mass of the vortices, $\ln \mu$. We see that as $\mu$ is diminished, the average length of the loops increases dramatically about six-fold.

![Figure 1](image-url) (color online) Average length of the loops as a function of $\ln(\mu)$.
2. Wilson loop
We compute the expectation value of the Wilson loop [4], using our Monte Carlo simulation. We place the Wilson loop along a prescribed rectangular contour. The Wilson loop operator simply measures the total linking number of the Wilson loop contour with the dynamical vortex loops.

\[ W(L, T) = \langle e^{-i \frac{q}{e} \oint A_\mu dx_\mu} \rangle, \]  

(2)

We then evaluate this for various different spatial sizes, we find for example:

\[ \text{Figure 2.} \] (color online) Plot of \(-\ln(W)\) for \(\mu = .13\). The dashed lines are a linear fit, we find \(-\ln(W) = 0.009280(\pm 4 \times 10^{-6}) \cdot (L + T) + 0.00350(\pm 5 \times 10^{-4})\) 

perimeter law. The perimeter law exists on both sides of the transition, only the slope changes by about 10 fold [1].

3. Chern-Simons term
It is possible to add the Chern-Simons term to the action:

\[ L_{CS} = 16\pi^3 \kappa \int d^3 x \epsilon^{\mu \nu \lambda} A_\mu F_{\nu \lambda} \]  

(3)

The Chern-simons term converts charged particles or magnetic flux tubes into anyons. Hence the vortices behave as fractional statistics particles. With this normalization, the term is just equal to \(2\kappa \times L_T\) where \(L_T\) is the total linking number of all the vortex loops with each other. The important distinction is that in Euclidean space this term, being t-odd remains imaginary [5] in Euclidean space. Hence it cannot be used to define the Boltzmann factor in the Monte Carlo simulation. However the real part of the action provides a perfectly good measure on the space.
of vortex loops, which respects all the symmetries of the theory with the Chern-Simons term added. Hence we can use the ensemble obtained by the Monte Carlo procedure to determine the expectation values with the Chern-Simons term added, then:

$$\langle O \rangle \approx \langle O \rangle_N = \frac{\sum_{i=1}^{N} O(C_i)e^{iS_{CS}}}{\sum_i e^{iS_{CS}}}$$

(4)

In principle, the set of Green’s functions obtained from this measure will satisfy the Wightman axioms [6], and can be used to reconstruct [7] the full Chern-Simons added quantum field theory.

4. Wilson loop and ’t Hooft loop in the Chern-Simons theory

The calculation of these two order parameters follows in a straightforward fashion in the Chern-Simons theory, for details see [8]. The Wilson loop operator is unchanged since the Chern-simons term has no import on the distribution of the linking number of the Wilson loop. Hence, at fixed total linking number, the average of the Wilson loop is equal to its average without the restriction of fixed total linking number and the effect of the Chern-Simons term cancels in Eqn.(4). The Wilson loop continues to display perimeter law.

The ’t Hooft loop [9] on the other hand is completely dependent on the Chern-Simons term. The ’t Hooft loop corresponds to the insertion of a singular flux tube into the functional integral, along a fixed rectangular path. The Chern-Simons term simply measures the linking number of this flux tube with the rest of the dynamical vortex loops. It is evident that the ’t Hooft loop is trivial in the absence of the Chern-Simons term. However, for fixed value of the Chern-Simons coefficient, the ’t Hooft loop behaves exactly as the Wilson loop for changes in the dimensions of rectangular path. since it is $2\kappa$ times the linking number of the rectangular loop, it behaves exactly as the Wilson loop as a function of the parameters of the rectangular path. Therefore it displays the perimeter law.

5. Conclusions

We have computed using Monte Carlo simulations in an effective description of the 2+1 dimensional Abelian Higgs model, properties of the vacuum. Although we find a phase change, the expectation value of the Wilson loop and the ’t Hooft loop does not change. The ’t Hooft loop is trivial without the Chern-simons term, the Wilson loop displays perimeter law behaviour in both phases. The two order parameters continue to display the perimeter law in both phases, even in the presence of the Chern-Simons term. This is a counter example to the understanding that such behaviour necessitates the existence of massless particles, [10]. Here we have no massless particles, however the Chern-Simons term exercises a subtle long range, statistics changing interaction, which is able to permit the perimeter law behaviour for both order parameters.

6. Acknowledgments

We thank Al Shapere and Sumit Das for a wonderful, well organized QTS6 conference and Jacques Richer of the Réseau Québécois de Calcul de Haute Performance for many discussions and help with the calculations. We also thank NSERC of Canada for financial support.

References

[1] R. MacKenzie, F. Nebia-Rahal and M. B. Paranjape, “Phase transitions in a 3-d lattice loop gas,”
arXiv:0710.3236 [hep-lat].
[2] H.B. Nielsen and P.Olesen, Nucl. Phys. B61, 45 (1973).
[3] Madras, N. and Slade, G. ‘The Self-Avoiding Walk’, Birkhäuser, Boston, (1993).
[4] K. G. Wilson, “Confinement of quarks,” Phys. Rev. D 10, 2445 (1974).
[5] G. Alexanian, R. MacKenzie, M. B. Paranjape and J. Ruel, “Path integration and perturbation theory with complex Euclidean actions,” Phys. Rev. D 77, 105014 (2008) [arXiv:0802.0354 [hep-th]].
[6] A. S. Wightman, “Quantum Field Theory in Terms of Vacuum Expectation Values,” Phys. Rev. 101, 860 (1956).
[7] K. Osterwalder and R. Schrader, “Axioms For Euclidean Green's Functions” Commun. Math. Phys. 31, 83 (1973); “Axioms For Euclidean Green's Functions 2,” Commun. Math. Phys. 42, 281 (1975).
[8] R. MacKenzie, F. Nebia-Rahal, M. B. Paranjape and J. Richer, "Phase transitions in a theory of anyons", in preparation.
[9] G. t’Hooft, Nucl. Phys. B138, 1 (1978).
[10] A. Ukawa, P. Windey and A. H. Guth, “Dual Variables For Lattice Gauge Theories And The Phase Structure Of Z(N)Systems,” Phys. Rev. D 21, 1013 (1980).