Reverse priority mechanisms in allocation of resources

L V Rossikhina¹, ², A V Kalach², S A Borsuchenko³, R B Golovkin⁴ and S N Mamedov⁴

¹ V A Trapeznikov Institute of Control Sciences of the Russian Academy of Sciences, 65 Profsoyuznaya St, Moscow, 117997, Russia
² Voronezh Institute of the Federal Penitentiary Service of the Russian Federation, 1a Irkutskaya St, Voronezh, 394072, Russia
³ The All-Russian State University of Justice (RLA of the Ministry of Justice of Russia), 2 Azovskaya St., Moscow, 117638, Russia
⁴ Institute of Law, Vladimir State University named after Alexander and Nikolay Stoletovs, 8 Studencheskaya St, Vladimir, 600005, Russia

E-mail: AVKalach@gmail.com

Abstract. The paper describes a problem of allocation of a limited resource under the conditions of probabilistic indeterminacy as related to the quantity of resource allocated. The active system consists of the Center and consumers (agents). The Center has a resource being allocated among the consumers based on their requests. When requests are announced, the consumers have information only on the function of allocation of resource being allocated by the Center. The priority mechanisms of resource allocation based on the reverse priority principle (the priority function is a decreasing request function) are studied for a deterministic case when the amount of resource available at the Center is known by the consumers as well as under the conditions of probabilistic indeterminacy. Nash Equilibria are determined for different priority functions provided the objective functions of consumers increase monotonically in relation to the resource quantity. A business game is proposed to study the mechanism of reverse priorities in resource allocation problems (deterministic and probabilistic cases). Informal and formal descriptions of games are included. Gaming algorithm is proposed. The results of experiments are presented for deterministic and probabilistic cases with different priority functions, which confirm the key theoretical conclusions.

1. Introduction

The problems of allocation of limited resources (financing of regional development programs, top-priority scientific and technological spheres) fall within the most common planning problems, which makes their consideration vitally important [1-4].

The classical resource allocation pattern looks as follows [5]. The Center has a certain quantity of resource R, which it allocates among the consumers (agents). Each consumer communicates a request for resource $s_i$, $i = 1, n$ to the Center. The Center allocates the resource based on the preset mechanism of resource allocation.

The theory of active systems suggests the priority mechanisms that can described by the following procedure of resource allocation [6]:

...
where \( x_i \) - quantity of resource received by the \( i \)th consumer;

\( \eta_i(S_i) \) - Priority function of the \( i \)th consumer depending on its request.

The minimization operation determines the condition when a consumer can receive the resource in the amount not exceeding the stated quantity.

Parameter \( \gamma \) based on the following condition

\[
\sum_{i=1}^{n} \min\{S_i, \gamma \eta_i(S_i)\} = R.
\]

Within the above mechanisms, the resource allocated in proportion to the values of consumer priority functions. Three types of priority mechanisms have been identified: the resource in the absolute priority mechanism is allocated in direct proportion to a preset priority value (but not exceeding the stated quantity); in the direct priority mechanisms, the priority function is an increasing function of consumer requests; in the reverse priority mechanisms, the priority function is a decreasing function of requests.

Let effect functions \( f_i(x_i) \) be strictly increasing functions of \( x_i \). In this case, the goal of consumers is to receive the maximum possible amount of resources, i.e. the maximum request for resource is the dominant strategy of any consumer in the direct priority mechanisms. That is the reason why the direct priority mechanisms generate a trend of resource request growth, which is a grave disadvantage.

The reverse priority mechanisms have certain advantages as compared to the direct priority mechanisms. It was the first time when the priority function in reverse priority mechanism studies had been proposed as follows \( \eta_i(S_i) = A_i / S_i, \ i = 1, n \), where \( A_i \) characterized the effect received by the \( i \)th consumer from the use of the resource. Then, \( \eta_i(S_i) \) determines the efficiency of resource usage.

The priority mechanisms were previously studied for a case when the Center’s resource was known [7, 8]. In practice, the quantity of resource is frequently unknown when it allocated.

The paper describes the analysis of the reverse priority mechanisms for a case when the consumers know the function of Center’s resource allocation.

2. Resource allocation model (deterministic case)

Setting of problem. Let us consider an active system consisting of the Center and \( n \) consumers. The Center has a resource being allocated among the consumers based on their requests. When the requests are being announced, the consumers have information on the quantity of resource \( R \), which is allocated by the Center. Objective functions \( f_i(x_i) \) of consumers are the increasing functions of received resource \( x_i \). Let us denote the resource request of the \( i \)th consumer by \( S_i \).

Let the priority function given by:

\[
\eta_i(S_i) = \frac{A_i}{S_i}, i = 1, n,
\]

where \( A_i \) – parameter limiting the priorities of consumers.

The resource allocation mechanism is given by:

\[
x_i(S) = \min\left\{ S_i, \frac{A_i \cdot R}{S_i \cdot Y} \right\}, \quad (2)
\]
where \( Y = \sum \frac{A_j}{S_j} \).

Let us take that the objective functions of agents are increasing in \( x_i \).

In equilibrium situation, it is obvious that

\[
S_i = \frac{A_i \cdot R}{S_i \cdot Y} \quad \text{or} \quad S_i = \sqrt[\lambda]{\frac{A_i \cdot R}{Y}}. 
\]

(3)

Based on conditions

\[
\frac{1}{S_i} = \sqrt[\lambda]{\frac{Y}{A_i \cdot R}}, \quad Y = \sum \frac{Y \cdot A_j}{R}
\]

we obtain the equilibrium situation

\[
Y = \frac{1}{R} \left( \sum \sqrt[\lambda]{A_j} \right)^2, \quad S_i = \sqrt[\lambda]{\frac{A_i \cdot R}{\sum \sqrt[\lambda]{A_j}}}. 
\]

(4)

Here, \( x_i = S_i \) for all \( i = 1, n \).

Let us consider another variant of the priority function

\[
\eta_i(S_i) = A_i - S_i, \quad i = 1, n. 
\]

(5)

The resource allocation mechanism is given by:

\[
x_i(S) = \min \left( S_i, \frac{A_i - S_i}{Y} - \frac{R}{Y} \right), 
\]

(6)

where \( Y = \sum (A_j - S_j) \).

In equilibrium situation

\[
S_i = \frac{A_i - S_i}{Y} \frac{R}{Y} \quad \text{or} \quad S_i = \frac{RA_i}{Y + R}, 
\]

(7)

\[
A_i - S_i = \frac{A_i Y}{Y + R}. 
\]

Based on condition \( Y = \sum \frac{A_j - S_j}{R + Y} = \frac{Y}{R} \sum A_j \)

we obtain \( Y^* = A - R, \quad S_i^* = \frac{A_i - R}{Y} \sum A_j \).

(8)

(9)

Condition \( A > R \) shall be fulfilled to make the equilibrium situation exist.

3. Resource allocation model (probabilistic case)

Setting of problem. Let us consider an active system consisting of the Center and \( n \) consumers. The Center has a resource being allocated among the consumers based on their requests. When the requests are being announced, the consumers have information only on allocation function \( F(R) \) of resource \( R \),
which is allocated by the Center. Objective functions \( f_i(x_i) \) of consumers are the increasing functions of received resource \( x_i \). Let us denote the resource request of the \( i^{th} \) consumer by \( S_i \).

Let us consider the priority function (1) \( \eta_i(S_i) = \frac{A_i}{S_i} \), and the resource allocation mechanism

\[
(2) \quad x_i(s) = \min \left( S_i, \frac{A_i \cdot R}{S_i \cdot Y} \right).
\]

Let us denote the resource quantity allocation function by \( F(R) \), which is a continuously differentiable function of \( R \).

The mathematical expectation of resource quantity of the \( i^{th} \) consumer (3) equals

\[
M(s) = \int_0^{R_i} \frac{A_i R}{S_i Y} dF(R) + S_i \left[ 1 - F(R_i) \right],
\]

where \( R_i = \frac{S_i^2 \cdot Y}{A_i} \), \( i = 1, n \).

Let us determine the maximum of this quantity as per \( S_i \), assuming that \( S_i \) estimate weakly influences \( Y \) value. That is, when the decision on the request quantity is made, the consumers do not take into account the influence of \( S_i \) on \( Y \), assuming that \( Y \) is simply a parameter.

Here,

\[
\int_0^{R_i} R dF(R) = R_i F(R_i) - \int_0^{R_i} F(R) dR.
\]

Let us calculate

\[
\frac{dM}{dS_i} = -\frac{A_i}{S_i^2 Y} \left[ R_i F(R_i) - \int_0^{R_i} F(R) dR \right] + \frac{A_i}{S_i Y} R_i F'(R_i) \frac{dR_i}{dS_i} + 1 - F(R_i) - S_i F'(R_i) \frac{dR_i}{dS_i}
\]

through simple transformations we obtain

\[
\frac{dM}{dS_i} = 1 - 2F(R_i) + \frac{1}{R_i} \int_0^{R_i} F(R) dR.
\]

The expression obtained does not depend on \( A_i \).

Let function \( M(S) \) be convex and has the maximum point. Equilibrium value \( R_i \) is the same for all consumers, i.e. \( R_i = R^* \) for all \( i \).

Based on condition \( 2F(R^*) - \frac{1}{R^*} \int_0^{R^*} F(R) dR = 1 \), let us determine maximum point \( R^* \) and, consequently,

\[
S_i^* = \sqrt{\frac{A_i \cdot R^*}{Y}}.
\]

Let us further calculate \( Y^* = \sum_j \frac{A_j}{S_j} \sqrt{\frac{Y}{R} \sum_j \sqrt{A_j}} \), or \( Y^* = \sqrt{\frac{\sum_j \sqrt{A_j}}{R^*}} \).

We finally obtain

\[
S_i^* = \sqrt{\frac{A_i}{\sum_j \sqrt{A_j}}} \cdot R^*.
\]
Let us consider the priority function (5) \( \eta_i (S_i) = A_i - S_i , \quad i = 1,n \) and the resource allocation mechanism (6) \( x_i (S) = \min (S, \frac{A_i - S_i}{Y} R) \).

By analogy with the previous case, we have

\[
M (S_i) = \frac{R_i}{Y} \frac{A_i - S_i}{R} dF(R) + S_i (1 - F(R_i)),
\]

where \( R_i = \frac{S_i Y}{A_i - S_i} \), \( \frac{dR}{dS_i} = \frac{YA_i}{(A_i - S_i)^2} \).

Let us calculate

\[
\frac{dM}{dS_i} = -\frac{1}{Y} \left[ R_i F(R_i) - \left( \frac{R_i}{Y} F(R) \right) dR \right] + 1 - F(R_i) = -\frac{1}{Y} \left( 1 + \frac{R_i}{Y} \right) F(R_i) + \frac{R_i}{Y} F(R)dR ,
\]

\[
\frac{d^2 M}{dR} = -\frac{1}{Y} F(R_i) - \left( 1 + \frac{R_i}{Y} \right) \frac{dF(R_i)}{dR} + \frac{1}{Y} F(R_i) = \left( 1 + \frac{R_i}{Y} \right) F'(R_i) < 0 .
\]

Therefore, dependence \( M (S_i) \) is convex, and the maximum point is determined based on the equation, which is the same for all consumers

\[
(R + Y) F(R) - \left[ F(x) \right]_0^R = Y .
\]

4. Resource allocation model (discrete variant)

Let the quantity of Center’s resource equals \( R_1 \) with a probability of \( p_1 \), and equals \( R_2 \) with a probability of \( p_2 = 1 - p_1 \).

Let us consider the reverse priority mechanism with priority function \( \eta_i (S_i) = \frac{A_i}{S_i} , \quad i = 1,n \).

Let us determine the dependence of an expected consumer’s resource quantity on the request quantity (the consumer number is omitted for simplicity).

Let us calculate \( d = \sqrt{\frac{AQ_1}{Y}} \), \( D = \sqrt{\frac{AQ_2}{Y}} \).

Let us consider three possible situations:

1) \( S \leq d \). In this case, the minimum is reached in the reverse priority mechanism for a request at both \( R = Q_1 \) and \( R = Q_2 \). Thus, \( M (S) = S \).

2) \( d < S \leq D \). In this case, at \( R = Q_1 \) the minimum is reached at \( x = \frac{AQ_1}{S \cdot Y} \), an at \( R = Q_2 \) the minimum is reached at \( x = S \). The expected resource quantity equals \( M (S) = P_1 \frac{AQ_1}{S \cdot Y} + P_2 \frac{AQ_2}{S \cdot Y} \).

3) \( S \geq D \). In this case, at \( R = Q_1 \) the minimum is reached at \( x = \frac{AQ_1}{S \cdot Y} \), and at \( R = Q_2 \) the minimum is reached at \( x = \frac{AQ_2}{S \cdot Y} \). The expected resource quantity equals \( M (S) = (P_1 Q_1 + P_2 Q_2) \frac{A}{Y \cdot S} = \frac{A}{Y \cdot S} R \).
The graph of function $M(s)$ is shown in figure 1.

![Graph of Function $M(s)$](image)

Figure 1. Graph of Function $M(s)$.

Within segment $[a, b]$, $M(s)$ is convex function of $s$. That is why, it reaches the maximum in points $a$ or $b$. Please note, that if $s = a$, then $M = b$, and at $s = b$ it also obtains $b$.

If $s = b$, then $M = A - b + bD$.

Maximum $M$ equals

$$
\max \left[ d; \frac{Q_1}{D \cdot Y} p_1 + p_2 D \right] = \frac{A}{Y} \max \left[ Q_1; \frac{Q_1}{Q_2} p_1 + \sqrt{Q_2} p_2 \right].
$$

If the maximum is reached at $\sqrt{Q_1}$, then the consumer selects strategy $d$, and if at $p_1 = \frac{Q_1}{\sqrt{Q_2}} + p_2 \sqrt{Q_2}$, then strategy $D$.

Let us determine $p_1^*$ from equation $\sqrt{Q_1} = \frac{Q_1}{\sqrt{Q_2}} p_1 + \sqrt{Q_2} (1 - p_2)$.

Through simple transformations we obtain

$$
p_1^* = \frac{\sqrt{Q_2}}{\sqrt{Q_1} + \sqrt{Q_2}}.
$$

Thus, if $p_1 > p^*$, then each consumer selects strategy $d$, and if $p_1 < p^*$, then strategy $D$.

In both cases, the allocation of resource in equilibrium is as follows $x_i = \frac{A_i}{B} Q$, where $B = \sum_i \sqrt{A_i}$.

Let us consider the priority function (5) $\eta_i(s_i) = A_i - s_i$, $i = 1, n$.

By analogy with the previous analysis, we have $x = \min \left( \frac{(A - s)Q}{Y} \right)$, where $Y = \sum_j (A_j - S_j)$.

From equation $S = \frac{(A - s)Q}{Y}$ we calculate $d = \frac{AO_1}{Y + Q_1}$, $D = \frac{AO_2}{Y + Q_2}$. If $s \leq d$, then $M[x] = d$. 

6
If \( d < S \leq D \), then
\[
M[x] = p_1 \left( \frac{A-S}{Y} Q_1 + p_2 S = R \frac{A}{Y} - S \left( \frac{p_1 Q_1}{Y} - p_2 \right) \right).
\]

If \( S \geq D \), then
\[
M[x] = \left( \frac{A-S}{Y} \right) \left( p_1 Q_1 + p_2 Q_2 \right).
\]

It is obvious than maximum \( M[x] \) is reached either in point \( d \) or in point \( D \). We have
\[
M[x] = \left\{ \begin{array}{ll}
d; p_1 \left( \frac{A-D}{Y} Q_1 + p_2 D \right) = A \max \left[ \frac{Q_1}{Q_1 + Y}, \frac{Q_1}{Q_1 + Y} \frac{Q_1}{Q_1 + Y} \frac{Q_1}{Q_1 + Y} \frac{Q_1}{Q_1 + Y} \frac{Q_1}{Q_1 + Y} \frac{Q_1}{Q_1 + Y} \right].
\end{array} \right.
\]

Let us determine boundary value \( p_1^* \) from equation
\[
\frac{Q_1}{Q_1 + Y} = \frac{Q_1}{Q_1 + Y} + \frac{Q_2}{Q_2 + Y} p_2.
\]

Through simple calculations we obtain
\[
p_1^* = \frac{Y}{Q_1 + Y}.
\]

Unlike the previous case, \( p_1^* \) depends on \( Y \).

Let \( S_j = d_j \). In this case \( Y = \sum_{j}(A_j - d_j) = B, Y = B - Q_1 \), \( p_1^* = 1 - \frac{Q_1}{B} \).

Let \( S_i = D_i, i = 1, n \). In this case \( p_1^* = 1 - \frac{Q_2}{B} \).

Let us consider three possible cases:

- \( p > 1 - \frac{Q_1}{B} \). In this case, all consumers select strategy \( d_i, i = 1, n \).
- \( p < 1 - \frac{Q_2}{B} \). In this case, all consumers select strategy \( D_i, i = 1, n \).
- \( 1 - \frac{Q_2}{B} < p < 1 - \frac{Q_1}{B} \).

An uncertain situation occurs. If all consumers have selected strategy \( d_i \), then \( p_1^* = 1 - \frac{Q_1}{B} \), and, consequently, \( p < p_1^* \), and strategy \( D_i \) is more profitable for them.

If they have selected strategy \( D_i \), then \( p_1^* = 1 - \frac{Q_2}{B} \) and, consequently, \( p > p_1^* \). So, strategy \( d_i \) is more profitable for all consumers. The situation is hard to predict.

5. Business (simulation) Game

To analyze the reverse priority mechanism, the resource allocation problems and experimental studies suggest business (simulation) games [9].

Informal description of game. The Resource business game is to be used for experimental validation of theoretical conclusions and training of employees when the reverse priority mechanism is deployed in practice. Several teams take part in the game. Each team shall obtain the required (optimum) quantity of resource. If the team obtains the resource in less-than-optimum quantity, then its payoff is directly
proportional to the quantity obtained. If the team obtains the resource in more-than-required quantity, then its payoff is decreased in proportion to the difference of the resource obtained and the optimum quantity. To obtain the resource, each team shall file a request indicating the desired quantity of resource. When all requests are received, the game master (the Center) allocates the resources available in accordance with the resource allocation mechanism accepted. The resource available at the Center is determined as follows: the game master tosses a hexagonal cube. The number, which turns out, corresponds to a certain quantity of Center’s resources. Having obtained the resource, the teams calculate their payoff. The play is repeated several times to get the steady results.

Formal description of game. There are \( n \) teams. Let us denote the request for resource of the \( i \)-th team by \( s_i \), quantity of resource obtained by the \( i \)-th team by \( x_i \). The team payoff \( f_i(x_i) \) is determined by the following formula:

\[
f_i(x_i) = \begin{cases} 
  k_i x_i, & if x_i \leq u_i, \\
  k_i u_i - a_i(x_i - u_i), & if x_i > u_i,
\end{cases}
\]  

(17)

where \( u_i \) – the optimum quantity of recourse for the \( i \)-th team, \( k_i > 0, a_i > 0, i = 1, \ldots, n \).

The Center allocates the resource as per the reverse priority mechanism

\[
x_i = \min \left[ s_i; \frac{\eta_i(s_i)}{Y(s)} R \right],
\]

(18)

where \( \eta_i(s_i) \) – priority function of the \( i \)-th team (decreasing function of \( s_i \)),
\( R \) – quantity of Center’s resource,
\( Y(s) = \sum_i \eta_i(s_i) \).

Gaming algorithm. The game master explains the game rules. Each play of the game contains 4 steps.

Step 1. Each team files request \( s_i \).

Step 2. The game master tosses a cube and determines the quantity of resource \( R \).

Step 3. The game master allocates the resource as per mechanism (18).

Step 4. Teams determine their payoffs as per formula (17).

When the game is over, the results are summed up, i.e. the sum of team payoffs in all plays is calculated.

Analysis of game results. The purpose of analysis of game results is to estimate the strategy of each team from the perspective of proximity to the theoretical estimate. It is advisable to leave out several first plays, since the teams need to adapt and understand the accepted mechanism of resource allocation.

Estimate \( \Delta \) of proximity of the announced values to theoretical equilibrium shall be determined as per the following formula for each team

\[
\Delta = \frac{1}{N \sum_{k \in P}} \left| \frac{s_k^*}{s_k} - 1 \right|
\]

where \( N \) – number of plays taken into account,
\( P \) – set of plays taken into account,
\( s_k^* \) – theoretical equilibrium strategy.

Six variants of games have been played with four teams.

The example of game results for a deterministic case of resource allocation with priority function \( \eta_i(s_i) = \frac{1}{s_i} \) is given in figures 2, 3.
Ten games, each consisting of ten plays, were carried out; after five-six plays, the team strategies oscillated about the equilibrium situation $s_i^* = 30$, $i = 1,4$. The relative deviation did not exceed 5%.

The result similar to the previous one was obtained for the deterministic case of resource allocation with priority function $\eta_i(s_i) = 60 - s_i$, $R = 120$.

For the probabilistic case of resource allocation with priority functions $\eta_i(s_i) = \frac{1}{s_i}$ and $\eta_i(s_i) = 60 - s_i$, $R = kQ$, where $k$ is the number on the cube tossed (discrete uniform distribution), $Q = 30$, after six-seven plays the requests in all games oscillated about equilibrium, however, with higher relative deviation as compared to the deterministic case (up to 20%). Since the discrete uniform distribution is close to uniform distribution, it can be suggested that in equilibrium $R^* = 120$, and request $s_i^* = 30$, $i = 1,4$.

For the discrete case of resource allocation with two statuses with priority functions $\eta_i(s_i) = \frac{1}{s_i}$ and $\eta_i(s_i) = 60 - s_i$, $R_1 = 60$, $R_2 = 120$, $p_1 = 0.8$, $p_1 = 0.2$ and $p_1 = 0.5$ at $p_1 = 0.8$ and $p_1 = 0.2$, after four plays the results stabilized about the equilibrium situation with the relative error of approximately 20%. At $p_1 = 0.5$, the game was chaotic and unpredictable.

The experimental studies based on the business (simulation) games developed to analyze the reverse priority mechanism in resource allocation problems have confirmed the key theoretical conclusions.

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