A simple finite element method for Reissner–Mindlin plate equations using the Crouzeix-Raviart element and the standard linear finite element

Bishnu P. Lamichhane *

January 31, 2014

Abstract

We present a simple finite element method for the discretization of Reissner–Mindlin plate equations. The finite element method is based on using the non-conforming Crouzeix-Raviart finite element space for the transverse displacement, and the standard linear finite element space for the rotation of the transverse normal vector. We also present two examples for the discrete Lagrange multiplier space for the proposed formulation.

Key words Reissner–Mindlin plate, Lagrange multiplier, biorthogonal system, Crouzeix-Raviart element, a priori error estimates

AMS subject classification. 65N30, 74K20

1 Introduction

It is a challenge to design a simple finite element scheme for Reissner–Mindlin plate equations, which does not lock when the plate thickness becomes close to zero. A standard discretization normally does not provide a uniform convergence with respect to the plate thickness. This problem is often referred to as locking. Many finite element techniques are developed over the past twenty years to avoid locking and obtain a uniform convergence with respect to the plate thickness [AF89, BF91, AB93, CL95, Lov96, AF97, CS98, Bra96, FT00, Bra01, ACPC02, Lov05]. Most of these finite element methods are either too complicated or too expensive to implement.

*School of Mathematical & Physical Sciences, Mathematics Building - V127, University of Newcastle, University Drive, Callaghan, NSW 2308, Australia, Bishnu.Lamichhane@newcastle.edu.au
In this paper, we present a very simple finite element method for Reissner–Mindlin plate equations providing a uniform convergence with respect to the plate thickness. We consider both simply supported and clamped boundary condition. Previously, a simple finite element method for Reissner–Mindlin plate equations is presented [Lam13b] for the case of clamped boundary condition, where we have enriched the standard linear finite element space with element-wise bubble functions for the approximation of the transverse displacement to ensure the stability of the system. In this paper, we show that the stability is ensured if we use the nonconforming Crouzeix-Raviart finite element space to approximate the transverse displacement, whereas other variables are discretized as in [Lam13b]. That means each component of the rotation of the transverse normal vector is approximated by the standard linear finite element, whereas we present two examples of the discrete Lagrange multiplier space. The first one is based on the standard linear finite element space, whereas the second one is based on a dual Lagrange multiplier space proposed in [Woh01, KLPV01]. The main advantage of using a dual Lagrange multiplier space is that it allows an efficient static condensation of the degrees of freedom associated with the Lagrange multiplier space. This leads to a positive-definite system. An iterative solver performs better for a positive-definite system than for a saddle point system. Hence the dual Lagrange multiplier space leads to a more efficient numerical scheme. The case of clamped boundary condition is treated by using the idea of mortar finite elements for the boundary modification [BD98, Lam13b].

We now want to point out some links of this present work with some previously presented nonconforming finite element schemes for Reissner-Mindlin plate equations [AF89, AF97, Lov05]. For example, the finite element scheme presented in [AF89] uses the nonconforming Crouzeix-Raviart finite element for the transverse displacement, but each component of the rotation of the transverse normal vector is approximated by the standard linear finite element space enriched with element-wise bubble functions, and the Lagrange multiplier space is discretized by the space of piecewise constant functions. We do not need to use bubble functions in our formulation, and hence our finite element method is more efficient than this finite element scheme.

The finite element scheme in [Lov05] uses the nonconforming Crouzeix-Raviart finite element for the transverse displacement and each component of the rotation of the transverse normal vector, whereas the Lagrange multiplier is approximated by the space of piecewise constant functions. Since the Crouzeix-Raviart element is used for each component of the rotation of the transverse normal vector, a stabilization is introduced in order to achieve Korn’s inequality. Since we use a conforming approach for the rotation of the transverse normal vector we do not need the stabilization for our finite element scheme.

The rest of the paper is planned as follows. The next section briefly recalls the Reissner–Mindlin plate equations in a modified form as given in [AB93]. We describe our finite element method and present assumptions on the discrete Lagrange
multiplier space in Section 3. Section 4 is devoted to the presentation of two examples of the discrete Lagrange multiplier space for the *simply supported boundary condition*, and we show the modification of the discrete Lagrange multiplier space for the *clamped boundary condition* in Section 5. Finally, we apply static condensation of the Lagrange multiplier in Section 6 before drawing a conclusion in the last section.

### 2 A mixed formulation of Reissner–Mindlin plate

Let $\Omega \subset \mathbb{R}^2$ be a bounded region with polygonal boundary. We need the following Sobolev spaces for the variational formulation of the Reissner–Mindlin plate with the plate thickness $t$:

$$H^1(\Omega) = [H^1_0(\Omega)]^2, \quad H^1_0(\Omega) = [H^1_0(\Omega)]^2, \quad \text{and} \quad L^2(\Omega) = [L^2(\Omega)]^2.$$  

Since the regularity of the shear stress depends on the plate thickness $t$, we use the Hilbert space for the shear stress depending on $t$. Let $(H^1_0(\Omega))'$ and $(H^1(\Omega))'$ be the dual spaces of $H^1_0(\Omega)$ and $H^1(\Omega)$, respectively. Now we define the Hilbert space for the shear stress as

$$M_t := \begin{cases} \{ \zeta \in (H^1_0(\Omega))' : \||\zeta||_t < \infty \}, & \text{for the clamped boundary}, \\ \{ \zeta \in (H^1(\Omega))' : \||\zeta||_t < \infty \}, & \text{for the simply supported boundary}, \end{cases}$$

where the norm $||| \cdot |||_t$ is defined as

$$|||\zeta||_t = \begin{cases} \|\zeta\|_{(H^1_0(\Omega))'} + \|\nabla \cdot \zeta\|_{H^{-1}(\Omega)} + t\|\zeta\|_{L^2(\Omega)} & \text{for the clamped boundary}, \\ \|\zeta\|_{(H^1(\Omega))'} + \|\nabla \cdot \zeta\|_{H^{-1}(\Omega)} + t\|\zeta\|_{L^2(\Omega)} & \text{for the simply supported boundary}. \end{cases}$$

In order to get a unified framework for the clamped and simply supported boundary of Reissner-Mindlin plate we define the space $V$ for the transverse displacement as

$$V := \begin{cases} H^1_0(\Omega), & \text{for the clamped boundary}, \\ H^1(\Omega), & \text{for the simply supported boundary.} \end{cases}$$

We consider the following modified mixed formulation of Reissner–Mindlin plate equations proposed in [AB93]. The mixed formulation is to find $(\phi, u, \zeta) \in V \times H^1_0(\Omega) \times M_t$ such that

$$a(\phi, u; \psi, v) + b(\psi, v; \zeta) = \ell(v), \quad (\psi, v) \in V \times H^1_0(\Omega),$$

$$b(\phi, u; \eta) - \frac{t^2}{\lambda(1-\nu^2)}(\zeta, \eta) = 0, \quad \eta \in M_t,$$

where $\lambda$ is a material constant depending on Young’s modulus $E$ and Poisson ratio $\nu$, and

$$a(\phi, u; \psi, v) = \int_{\Omega} C\varepsilon(\phi) : \varepsilon(\psi) \, dx + \lambda \int_{\Omega} (\phi - \nabla u) \cdot (\psi - \nabla v) \, dx,$$

$$b(\psi, v; \eta) = \int_{\Omega} (\psi - \nabla v) \cdot \eta \, dx,$$

$$\ell(v) = \int_{\Omega} g v \, dx.$$
Here $g$ is the body force, $u$ is the transverse displacement or normal deflection of the mid-plane section of $\Omega$, $\phi$ is the rotation of the transverse normal vector, $\zeta$ is the Lagrange multiplier, $C$ is the fourth order tensor, and $\varepsilon(\phi)$ is the symmetric part of the gradient of $\phi$. In fact, $\zeta$ is the scaled shear stress defined by

$$\zeta = \frac{\lambda(1 - t^2)}{t^2} (\phi - \nabla u).$$

## 3 A finite element discretization

We consider a quasi-uniform triangulation $T_h$ of the polygonal domain $\Omega$, where $T_h$ consists of triangles where $h$ denotes the mesh-size. Note that $T_h$ denotes the set of elements. For an element $T \in T_h$, let $P_n(T)$ be the set of all polynomials of degree less than or equal to $n \in \mathbb{N} \cup \{0\}$ in $T$. Let $\{x_i\}_{i=1}^N$ be the set of all vertices of the triangulation $T_h$, and $\mathcal{N}_h = \{i\}_{i=1}^N$. Let $\mathcal{E}_h$ be the set of all edges of elements in $T_h$, and $[v_h]_e$ the jump of the function $v_h$ across the edge $e$.

We consider a nonconforming finite element space $S_h$ for the transverse displacement, where the continuity of a function $v_h \in S_h$ across an edge $e \in \mathcal{E}_h$ will be enforced according to

$$J_e(v_h) := \int_e [v_h]_e \, d\sigma = 0.$$  

This is the standard nonconforming Crouzeix-Raviart finite element space $S_h$ defined as

$$S_h := \{v_h \in L^2(\Omega) : v_h|_K \in P_1(K), \ K \in T_h, \ J_e(v_h) = 0, \ e \in \mathcal{E}_h\}.$$  

The finite element basis functions of $S_h$ are associated with the mid-points of the edges of triangles. To impose the homogeneous Dirichlet boundary condition on $\Gamma$ we define $W_h$ as a subset of $S_h$ where

$$W_h := \{v_h \in S_h : \int_e v_h \, d\sigma = 0, \ e \in \mathcal{E}_h \cap \Gamma\}.$$  

As $W_h \not\subset H^1_0(\Omega)$, we cannot use the standard $H^1$-norm for an element in $W_h$. So we define a broken norm on $W_h$ as

$$\|v_h\|_{1,h} := \sqrt{\sum_{T \in T_h} \|v_h\|^2_{1,T}}, \quad v_h \in W_h,$$

and an element-wise defined gradient $\nabla_h$ and divergence $\nabla_h \cdot$ as

$$\nabla_h u_h |_T = \nabla (u_h|_T), \quad \nabla_h \cdot u_h |_T = \nabla \cdot (u_h|_T), \quad \text{on} \quad T, \quad T \in T_h.$$
We note that the standard linear finite element space enriched with element-wise defined bubble functions to approximate the transverse displacement is used in [AB93, Lam13b], which is a conforming approach. Our approach here is nonconforming for the transverse displacement since $W_h \not\subset H^1_0(\Omega)$.

Each component of the rotation of the transverse normal vector is discretized by using the standard linear finite element space

$$K_h := \{ q_h \in H^1(\Omega) : q_h|_K = P_1(K), \; K \in \mathcal{T}_h \}, \quad K^0_h := K_h \cap H^1_0(\Omega).$$

The finite element space for the rotation of the transverse normal vector is

$$V_h := \begin{cases} [K_h]^2, & \text{for the clamped boundary}, \\ [K^0_h]^2, & \text{for the simply supported boundary}. \end{cases}$$

### 3.1 Discrete Lagrange multiplier spaces

Let $M_h \subset L^2(\Omega)$ be a piecewise polynomial space with respect to the mesh $\mathcal{T}_h$ used to discretize each component of the Lagrange multiplier $\zeta \in M_t$. The discrete Lagrange multiplier space is defined as $M_h := [M_h]^2$. The Lagrange multiplier $\zeta \in M_t$ is the shear stress, and the discrete space for the shear stress should have the approximation property in the $L^2$-norm. Hence we need

$$\inf_{\mu_h \in M_h} \| v - \mu_h \|_{L^2(\Omega)} \leq C h |v|_{1,\Omega}, \quad v \in H^1(\Omega).$$

![Figure 1: Degrees of freedom for the finite element spaces](image)

The finite element formulation is to find $(\phi_h, u_h, \zeta_h) \in V_h \times W_h \times M_h$ such that

$$a_h(\phi_h, u_h; \psi_h, v_h) + b_h(\psi_h, v_h; \zeta_h) = \ell(v_h), \quad (\psi_h, v_h) \in V_h \times W_h, \quad (\zeta_h, \eta_h) \in M_h,$$

where

$$a_h(\phi_h, u_h; \psi_h, v_h) = \int_\Omega \mathbf{C} : \mathbf{e}(\phi_h) : \mathbf{e}(\psi_h) \, dx + \lambda \int_\Omega (\phi_h - \nabla_h u_h) \cdot (\psi_h - \nabla_h v) \, dx,$$

$$b_h(\psi_h, v_h; \eta_h) = \int_\Omega (\psi_h - \nabla_h v_h) \cdot \eta_h \, dx.$$
In order to get stability and optimality of our finite element scheme we impose
the following assumptions on the discrete Lagrange multiplier space as in [AB93,
Lam13b].

Assumption 1. [I(i)] \( \dim M_h = \dim V_h \).

[I(ii)] There is a constant \( \beta > 0 \) independent of the triangulation \( T_h \) such that

\[
\| \phi_h \|_{L^2(\Omega)} \leq \beta \sup_{\mu_h \in M_h \setminus \{0\}} \frac{\int_{\Omega} \mu_h \cdot \phi_h \, dx}{\| \mu_h \|_{L^2(\Omega)}}, \quad \phi_h \in V_h.
\]  

(Iii) The space \( M_h \) has the approximation property:

\[
\inf_{\mu_h \in M_h} \| \mu - \mu_h \|_{L^2(\Omega)} \leq Ch|\mu|_{1,\Omega}, \quad \mu \in H^1(\Omega).
\]  

[I(iv)] There exist two bounded linear projectors \( Q_h : H^1_0(\Omega) \to V_h \) and \( \Pi_h : H^1_0(\Omega) \to W_h \) for which

\[
b_h(Q_h \psi, \Pi_h v; \eta_h) = b(\psi, v; \eta_h), \quad \eta_h \in M_h.
\]

If these assumptions are satisfied, we obtain an optimal error estimate for the finite
element approximation, see [AB93]. We immediately see that the bilinear forms
\( a_h(\cdot, \cdot) \), \( b_h(\cdot, \cdot) \) and the linear form \( \ell(\cdot) \) are continuous with respect to the spaces
\( V_h \times W_h, \ V_h \times W_h \times M_h \) and \( W_h \), respectively, where the broken norm \( \| \cdot \|_{1,h} \) is
used for functions in \( W_h \). Similarly, the coercivity of the bilinear form \( a_h(\cdot, \cdot) \) over
the space \( V_h \times W_h \) also holds due to the Korn’s and Poincaré inequality. Note that
the use of Crouzeix-Raviart element does not cause problem here as it is only used to
discretize the transverse displacement. Then under above assumptions we have the
following theorem from the theory of saddle point problems [BF91, Bra01, AB93].
The proof of the following theorem is quite similar to the convergence result in
Lov05.

Theorem 1. Let \((\phi, u, \zeta) \in V \times W \times M\) be the solution \([1]\) and \((\phi_h, u_h, \zeta_h) \in V_h \times W_h \times M_h\) of \([2]\). Then under Assumptions [I(i)]–(iv) there exists a constant \( C \) independent of \( t \) and \( h \) such that

\[
\| \phi - \phi_h \|_{1,\Omega} + \| u - u_h \|_{1,h} + \| | \zeta - \zeta_h | | \leq Ch \left( \| \phi \|_{H^2(\Omega)} + \| u \|_{H^2(\Omega)} + \| \zeta \|_{H(\text{div},\Omega)} + t \| \zeta \|_{1,\Omega}\right),
\]

where, we assume that \( \phi \in H^2(\Omega), \ u \in H^2(\Omega) \) and \( \zeta \in H^1(\Omega) \).

4 Simply supported boundary condition

We first consider the case of simply supported boundary condition. We consider two
examples of the discrete Lagrange multiplier space satisfying Assumptions [I(i)]–(iv).
4.1 First example for $M_h$

Let $M_h := [K_h]^2$. We can see that this example satisfies Assumptions (i)–(iii). Now we prove that it also satisfies Assumptions (iv).

**Theorem 2.** There exist two bounded linear projectors $Q_h : \mathbb{H}^1(\Omega) \to V_h$ and $\Pi_h : H_0^1(\Omega) \to W_h$ for which

$$ b(Q_h \psi, \Pi_h v; \eta_h) = b(\psi, v; \eta_h), \quad \eta_h \in M_h. \quad (5) $$

In order to prove Theorem 2 we use the following result proved in [Lam13a].

**Lemma 3.** There exists a constant $\beta > 0$ independent of the mesh-size $h$ such that

$$ \sup_{v_h \in V_h} \frac{\int_{\Omega} \nabla_h \cdot v_h q_h \, dx}{\|v_h\|_{1,h}} \geq \beta \|q_h\|_{0,\Omega}, \quad q_h \in K_h. $$

From the standard theory of saddle point problems this lemma implies the following lemma [BF91, Bra01].

**Lemma 4.** Since the two spaces $\mathbb{H}_0^1(\Omega)$ and $L_0^2(\Omega)$ satisfy the inf-sup condition

$$ \sup_{u \in \mathbb{H}_0^1(\Omega)} \frac{\int_{\Omega} \nabla \cdot u \mu \, dx}{\|u\|_{1,\Omega}} \geq \beta \|\mu\|_{L^2(\Omega)}, \quad \mu \in L_0^2(\Omega), $$

where

$$ L_0^2(\Omega) = \left\{ v \in L^2(\Omega) : \int_{\Omega} v \, dx = 0 \right\}, $$

there exists a bounded linear projector $\Pi_h : \mathbb{H}_0^1(\Omega) \to [W_h]^2$ such that

$$ \int_{\Omega} \nabla_h \cdot \Pi_h u \mu_h \, dx = \int_{\Omega} \nabla \cdot u \mu_h \, dx, \quad \mu_h \in K_h \cap L_0^2(\Omega). $$

The boundedness means that there exists a constant $C$ independent of $h$ such that

$$ \|\Pi_h u\|_{1,h} \leq C \|u\|_{1,\Omega}. $$

Let $\Pi_h : H_0^1(\Omega) \to W_h$ be the scalar version of the projector $\Pi_h$. Then this projector $\Pi_h$ is bounded and has the following property [Lam13b].

**Lemma 5.** Let $v \in H_0^1(\Omega)$. The interpolation operator $\Pi_h$ satisfies

$$ \int_{\Omega} \nabla \Pi_h v \cdot \eta_h \, dx = \int_{\Omega} \nabla v \cdot \eta_h \, dx, \quad \eta_h \in M_h. $$
Now we present the proof of Theorem 2.

**Proof.** Let $Q_h : V \rightarrow V_h$ be the orthogonal projection. Then we have

\[
\int_{\Omega} Q_h v \cdot w_h \, dx = \int_{\Omega} v \cdot w_h \, dx, \quad w_h \in M_h.
\]

Moreover, from Lemma 5 we have

\[
\int_{\Omega} \nabla h \Pi_h v \cdot \eta_h \, dx = \int_{\Omega} \nabla v \cdot \eta_h \, dx, \quad \eta_h \in M_h.
\]

Hence we have (5):

\[b(Q_h \psi, \Pi_h v; \eta_h) = b(\psi, v; \eta_h), \quad \eta_h \in M_h,\]

since

\[b(Q_h \psi, \Pi_h v; \eta_h) = \int_{\Omega} (Q_h \psi - \nabla h \Pi_h v) \cdot \eta_h \, dx = \int_{\Omega} (\psi - \nabla v) \cdot \eta_h \, dx.\]

Thus the theorem is proved. \(\square\)

### 4.2 Second example for \(M_h\)

Our second example of the discrete Lagrange space is based on a biorthogonal system [Woh01, KLPV01, Lam06]. Let \(N\) be the number of vertices in the finite element mesh, and \(\{\varphi_1, \ldots, \varphi_N\}\) be the finite element basis of \(K_h\). We construct a space \(M_h\) spanned by the basis \(\{\xi_1, \ldots, \xi_N\}\), where the basis functions of \(K_h\) and \(M_h\) satisfy a condition of biorthogonality relation

\[
\int_{\Omega} \xi_i \varphi_j \, dx = c_j \delta_{ij}, \quad c_j \neq 0, \quad 1 \leq i, j \leq N,
\]

where \(\delta_{ij}\) is the Kronecker symbol, and \(c_j\) a scaling factor. The scaling factor can be chosen so that \(\int_{T} \xi_i \, dx = \int_{T} \varphi_i \, dx\).

The basis functions of \(M_h\) are also associated with the vertices of the finite element mesh \(T_h\), and they are constructed locally on a reference element \(\hat{T}\). The local basis functions of \(M_h\) on the reference triangle \(\hat{T} := \{(x, y) : 0 \leq x, 0 \leq y, x + y \leq 1\}\) are given by

\[
\hat{\xi}_1 := 3 - 4x - 4y, \quad \hat{\xi}_2 := 4x - 1, \quad \text{and} \quad \hat{\xi}_3 := 4y - 1,
\]

associated with its three vertices \((0, 0), (1, 0)\) and \((0, 1)\), respectively. Note that the sum of all basis functions is one.
The global basis functions for the space $M_h$ are constructed by gluing the local basis functions together in the same way as the global basis functions of $K_h$ are constructed. Such a biorthogonal system is first used in the context of mortar finite elements [Woh01, KLPV01, Lam06]. Construction of basis functions of $M_h$ satisfying the biorthogonality and an optimal approximation property for a higher order finite element space is considered in [Lam06]. We now prove Theorem 2 using the discrete Lagrange multiplier space $M_h := [M_h]^2$. First we prove the following lemma.

**Lemma 6.** There exists a constant $\beta > 0$ independent of the mesh-size $h$ such that

$$\sup_{v_h \in V_h} \frac{\int_\Omega \nabla_h \cdot v_h q_h \, dx}{\|v_h\|_{1,h}} \geq \beta \|q_h\|_{0,\Omega}, \quad q_h \in M_h.$$

**Proof.** We consider an operator $I_h : M_h \to K_h$ such that

$$I_h \mu_h = \sum_{i=1}^N c_i \varphi_i \quad \text{for} \quad \mu_h = \sum_{i=1}^N c_i \xi_i.$$ 

We first note that the basis functions of $M_h$ are constructed in such a way that

$$\int_T \xi_i \, dx = \int_T \varphi_i \, dx, \quad 1 \leq i \leq N.$$ 

Let $\mu_h \in M_h$, and $I_h \mu_h \in K_h$. Then

$$\sup_{v_h \in V_h} \frac{\int_\Omega \nabla_h \cdot v_h \mu_h \, dx}{\|v_h\|_{1,h}} = \sup_{v_h \in V_h} \frac{\int_\Omega \nabla_h \cdot v_h I_h \mu_h \, dx}{\|v_h\|_{1,h}} \geq \beta \|I_h \mu_h\|_{0,\Omega}.$$ 

The result follows by using the fact that $\|I_h \mu_h\|_0^2$, $\|\mu_h\|_0^2$ and $\sum_{i=1}^N c_i^2 h_i^2$ are equivalent, where $h_i$ is the local mesh-size at the $i$th node of $T_h$. \hfill \qed

We can apply the standard theory of saddle point problems as in Lemma 4 to get the following result.

**Lemma 7.** There exists a bounded linear projector $\Pi_h : H^1_0(\Omega) \to [W_h]^2$ such that

$$\int_\Omega \nabla_h \cdot \Pi_h u \mu_h \, dx = \int_\Omega \nabla \cdot u \mu_h \, dx, \quad \mu_h \in K_h \cap L^2_0(\Omega).$$

Thus we have the existence of a bounded projector $\Pi_h : H^1_0(\Omega) \to W_h$ as in our first example as the scalar version of $\Pi_h$.

**Lemma 8.** Let $v \in H^1_0(\Omega)$. The interpolation operator $\Pi_h$ satisfies

$$\int_\Omega \nabla_h \Pi_h v \cdot \eta_h \, dx = \int_\Omega \nabla v \cdot \eta_h \, dx, \quad \eta_h \in M_h.$$
Now we present the proof of Theorem 2 for the second example of the discrete Lagrange multiplier space.

Proof. Let $Q_h : V \to V_h$ be a quasi-projection defined as

$$
\int_\Omega Q_h v \cdot w_h \, dx = \int_\Omega v \cdot w_h \, dx, \quad w_h \in M_h.
$$

(7)

This quasi-projection is well-defined due to Assumptions 1(i)–(ii). We note that for the projection $\Pi_h$ from Lemma 8 we have

$$
b(Q_h \psi, \Pi_h v; \eta_h) = \int_\Omega (Q_h \psi - \nabla_h \Pi_h v) \cdot \eta_h \, dx = \int_\Omega (\psi - \nabla v) \cdot \eta_h \, dx,
$$

which yields (5):

$$
b(Q_h \psi, \Pi_h v; \eta_h) = b(\psi, v; \eta_h), \quad \eta_h \in M_h.
$$

Hence the theorem is proved. \hfill \Box

5 Clamped boundary condition

It is more difficult to construct a discrete Lagrange multiplier space for the clamped boundary condition. For example, for the discrete Lagrange multiplier space in the first example above if we choose $M_h = [K_h]^2$ Assumption 1(i) is violated, and if we choose $M_h = [K_0]^2$, Assumption 1(iii) is violated leading to a sub-optimal approximation property. In order to satisfy Assumption 1(iii), the discrete Lagrange multiplier space should contain constants in $\Omega$, and this does not happen if we choose $M_h = [K_0]^2$. While it may be possible to work around without Assumption 1(i) we see two difficulties if we remove this assumption. The first difficulty is that the analysis will be much more difficult. The second difficulty is that the Gram matrix between the basis functions of $M_h$ and $V_h$ will not be a square matrix. If the Gram matrix is not square, we cannot statically condense out the degrees of freedom corresponding the Lagrange multiplier from the algebraic system.

We now propose a modification of the discrete Lagrange multiplier space [Lam13b] to adapt to the situation of clamped boundary condition, which combines the idea of mortar finite element techniques [BD98] with that of [AB93] to satisfy Assumptions 1(i) and 1(iii). The presented modification is exactly as in [Lam13b]. We repeat the approach here for completeness.

We start with splitting the basis functions of $M_h$ to two groups: basis functions associated with the inner vertices of $T_h$ and basis functions associated with the
boundary vertices in $T_h$. Let $L \in \mathbb{N}$ with $L < N$ be the number of inner vertices in $T_h$. Let 
\[ B_1^h = \{ \varphi_1, \varphi_2, \ldots, \varphi_L \}, \quad \text{and} \quad B_2^h = \{ \varphi_{L+1}, \ldots, \varphi_N \} \]
be the two sets of basis functions of $K_h$ associated with the inner and boundary vertices in $T_h$, respectively. In the following, we assume that each triangle has at least one interior vertex. A necessary modification for the case where a triangle has all its vertices on the boundary is given in [BD98]. Let $\mathcal{N}$, $\mathcal{N}_0$ and $\partial \mathcal{N}$ be the set of all vertices of $T_h$, the vertices of $T_h$ interior to $\Omega$, and the vertices of $T_h$ on the boundary of $\Omega$, respectively. We define the set of all vertices which share a common edge with the vertex $i \in \mathcal{N}$ as 
\[ S_i = \{ j : i \text{ and } j \text{ share a common edge} \}, \]
and the set of neighbouring vertices of $i \in \mathcal{N}_0$ as 
\[ I_i = \{ j \in \mathcal{N}_0 : j \in S_i \}. \]

Then the set of all those interior vertices which have a neighbour on the boundary of $\Omega$ is defined as 
\[ \mathcal{I} = \bigcup_{i \in \partial \mathcal{N}} I_i, \quad \text{See Figure 5} \]

\[ \mathcal{S}_h = \{ i_1, i_2, i_3, i_4, i_5, i_6 \}, \quad \mathcal{I}_h = \{ i_1, i_2 \} \]

Figure 2: Examples for $S_i$ and $I_i$

The finite element basis functions \{ $\phi_1, \phi_2, \ldots, \phi_L$ \} for $\widetilde{M}_h$ are defined as 
\[ \phi_i = \begin{cases} \varphi_i, & i \notin \mathcal{N}_0 \setminus \mathcal{I} \\ \varphi_i + \sum_{j \in \partial \mathcal{N} \cap S_i} A_{j,i} \varphi_j, & A_{j,i} \geq 0, \quad i \in \mathcal{I} \end{cases} \]

We can immediately see that $\dim \mathcal{M}_h = \dim \mathcal{V}_h$. Moreover, if the coefficients $A_{i,j}$ are chosen to satisfy 
\[ \sum_{j \in S_i} A_{i,j} = 1, \quad i \in \mathcal{I}, \]
Assumptions 1(ii) and 1(iii) are also satisfied, see [BD98] for a proof. The vector Lagrange multiplier space is defined as $\tilde{M}_h = [M_h]^2$. Since $\tilde{M}_h \subset M_h$ for both examples we have the following theorem.

**Theorem 9.** There exist two bounded linear projectors $Q_h : H^1_0(\Omega) \to V_h$ and $\Pi_h : H^1_0(\Omega) \to W_h$ for which

$$b(Q_h \psi, \Pi_h v; \eta_h) = b(\psi, v; \eta_h), \quad \eta_h \in \tilde{M}_h.$$ 

### 6 Positive-definite formulation

Now we give a positive definition formulation for our finite element scheme. Let $R_h : L^2(\Omega) \to M_h$ be the orthogonal projection defined as

$$\int_{\Omega} R_h v \cdot \eta_h \, dx = \int_{\Omega} v \cdot \eta_h \, dx, \quad v \in L^2(\Omega)$$

for both examples of the discrete Lagrange multiplier space and both types of boundary condition. Then the second equation of the discrete saddle point problem (2) can be written as

$$\zeta_h = \lambda (1 - t^2) R_h (\phi_h - \nabla_h u_h).$$

Using this result in the first equation of (2) the positive-definite formulation is to find $(\phi_h, u_h) \in V_h \times W_h$ such that

$$A(\phi_h, u_h; \psi_h, v_h) = \ell(v_h), \quad (\psi_h, v_h) \in V_h \times W_h,$$

where

$$A(\phi_h, u_h; \psi_h, v_h) = \int_{\Omega} C \varepsilon(\phi_h) : \varepsilon(\psi_h) \, dx + \lambda \int_{\Omega} (\phi_h - \nabla u_h) \cdot (\psi_h - \nabla_v v_h) \, dx + \frac{\lambda (1 - t^2)}{t^2} \int_{\Omega} (\psi_h - \nabla v_h) \cdot R_h (\phi_h - \nabla_h u_h) \, dx.$$ 

The disadvantage of this positive-definite system is that the action of $R_h$ cannot be efficiently computed. Now we present another way of getting a positive-definite form for the second example of the discrete Lagrange multiplier space, which can be efficiently computed. Note that the two sets of basis functions of $M_h$ and $V_h$ form a biorthogonal system for the second example. Then the Gram matrix $D$ associated with these two sets of basis functions will be diagonal. Then putting $v_h = 0$ in the first equation of the saddle point system (2), we have

$$\int_{\Omega} C \varepsilon(\phi_h) : \varepsilon(\psi_h) \, dx + \lambda \int_{\Omega} (\phi_h - \nabla_h u_h) \cdot \psi_h \, dx + \lambda \int_{\Omega} \psi_h \cdot \zeta_h \, dx = 0.$$
Note that the Gram matrix \( D \) is associated with the inner product \( \int_{\Omega} \psi_h \cdot \zeta_h \, dx \). Thus with suitable choices of matrices \( A \) and \( B \), the algebraic form of this equation becomes

\[
A \phi_h + B^T u_h + D \zeta_h = 0.
\]

This equation can be solved for \( \zeta_h \) as

\[
\zeta_h = -D^{-1} \left( A \phi_h + B^T u_h \right).
\]

Thus we can statically condense out the Lagrange multiplier \( \zeta_h \) from the saddle point system. This leads to a reduced and positive definite system. Hence an efficient solution technique can be applied to solve the arising linear system.

7 Conclusion

We have presented a new finite element method for Reissner-Mindlin plate equations using nonconforming Crouzeix-Raviart finite element basis functions for the transverse displacement, and the standard linear finite element for the rotation of the transverse normal vector. We have also shown two examples of the discrete Lagrange multiplier space for the presented finite element approach. We note that the second example for the discrete Lagrange multiplier space provides a more efficient numerical method as the degrees of freedom corresponding to the Lagrange multiplier can be statically condensed out from the system in this case just by inverting a diagonal matrix.

References

[AB93] D.N. Arnold and F. Brezzi. Some new elements for the Reissner–Mindlin plate model. In Boundary Value Problems for Partial Differential Equations and Applications, pages 287–292. Masson, Paris, 1993.

[ACPC02] M. Amara, D. Capatina-Papaghiuc, and A. Chatti. New locking-free mixed method for the Reissner–Mindlin thin plate model. SIAM Journal on Numerical Analysis, 40:1561–1582, 2002.

[AF89] D.N. Arnold and R.S. Falk. A uniformly accurate finite element method for the Reissner–Mindlin plate. SIAM Journal on Numerical Analysis, 26:1276–1290, 1989.

[AF97] D.N. Arnold and R.S. Falk. Analysis of a linear-linear finite element for the Reissner–Mindlin plate model. Mathematical Models and Methods in Applied Science, 7:217–238, 1997.
[BD98] D. Braess and W. Dahmen. Stability estimates of the mortar finite element method for 3–dimensional problems. *East–West J. Numer. Math.*, 6:249–264, 1998.

[BF91] F. Brezzi and M. Fortin. *Mixed and hybrid finite element methods*. Springer–Verlag, New York, 1991.

[Bra96] D. Braess. Stability of saddle point problems with penalty. *M$^2$AN*, 30:731–742, 1996.

[Bra01] D. Braess. *Finite Elements. Theory, Fast Solver, and Applications in Solid Mechanics*. Cambridge Univ. Press, Second Edition, Cambridge, 2001.

[CL95] C. Chinosi and C. Lovadina. Numerical analysis of some mixed finite element methods for Reissner-Mindlin plates. *Computational Mechanics*, 16:36–44, 1995.

[CR73] M. Crouzeix and P.A. Raviart. Conforming and nonconforming finite element methods for solving the stationary stokes equations. *RAIRO Anal. Numér.*, 7:33–36, 1973.

[CS98] D. Chapelle and R. Stenberg. An optimal low-order locking-free finite element method for Reissner-Mindlin plates. *M$^3$AS*, 8:407–430, 1998.

[FT00] R.S. Falk and T. Tu. Locking-free finite elements for the Reissner-Mindlin plate. *Mathematics of Computation*, 69:911–928, 2000.

[KLPV01] C. Kim, R.D. Lazarov, J.E. Pasciak, and P.S. Vassilevski. Multiplier spaces for the mortar finite element method in three dimensions. *SIAM Journal on Numerical Analysis*, 39:519–538, 2001.

[Lam06] B.P. Lamichhane. *Higher Order Mortar Finite Elements with Dual Lagrange Multiplier Spaces and Applications*. PhD thesis, University of Stuttgart, 2006.

[Lam13a] B.P. Lamichhane. A nonconforming finite element method for the Stokes equations using the Crouzeix-Raviart element for the velocity and the standard linear element for the pressure. [http://arxiv.org/abs/1306.6112](http://arxiv.org/abs/1306.6112), 2013. Submitted to International Journal for Numerical Methods in Fluids.

[Lam13b] B.P. Lamichhane. Two simple finite element methods for Reissner–Mindlin plates with clamped boundary condition. *Applied Numerical Mathematics*, 2013. Published online.

[Lov96] C. Lovadina. A new class of mixed finite element methods for Reissner-Mindlin plates. *SIAM Journal on Numerical Analysis*, pages 2456–2467, 1996.
[Lov05] C. Lovadina. A low-order nonconforming finite element for Reissner–Mindlin plates. *SIAM Journal on Numerical Analysis*, 42:2688–2705, 2005.

[Woh01] B.I. Wohlmuth. *Discretization Methods and Iterative Solvers Based on Domain Decomposition*, volume 17 of LNCS. Springer, Heidelberg, 2001.