Local automorphisms of finite dimensional simple Lie algebras

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Abstract

Let \( g \) be a finite dimensional simple Lie algebra over an algebraically closed field \( K \) of characteristic 0. A linear map \( \varphi : g \to g \) is called a local automorphism if for every \( x \) in \( g \) there is an automorphism \( \varphi_x \) of \( g \) such that \( \varphi(x) = \varphi_x(x) \). We prove that a linear map \( \varphi : g \to g \) is local automorphism if and only if it is an automorphism or an anti-automorphism.

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1 Introduction

Mappings which are close to automorphisms and derivations of algebras have been extensively investigated: in particular, since the 1990s (see [17], [18], [19]), the description of local and 2-local automorphisms (respectively, local and 2-local derivations) of algebras has been deeply studied by many authors.

Given an algebra \( A \) over a field \( k \), a \textit{local automorphism} (respectively, \textit{local derivation}) of \( A \) is a \( k \)-linear map \( \varphi : A \to A \) such that for each \( a \in A \) there exists an automorphism (respectively, a derivation) \( \varphi_a \) of \( A \) such that \( \varphi(a) = \varphi_a(a) \). A map \( \varphi : A \to A \) (not \( k \)-linear in general) is called a \textit{2-local automorphism} (respectively, a \textit{2-local derivation}) if for every \( x, y \in A \), there exists an automorphism (respectively, a derivation) \( \varphi_{x,y} \) of \( A \) such that \( \varphi(x) = \varphi_{x,y}(x) \) and \( \varphi(y) = \varphi_{x,y}(y) \).

In [18] the author proves that the automorphisms and the anti-automorphisms of the associative algebra \( M_n(\mathbb{C}) \) of complex \( n \times n \) matrices exhaust all its local automorphisms. On the other hand,
it is proven in [10] that a certain commutative subalgebra of $M_3(\mathbb{C})$ has a local automorphism which is not an automorphism.

Among other results (see the Introduction of [4] for a detailed historical account), assuming the field $k$ is algebraically closed of characteristic zero, in [1] the authors proved that every 2-local derivation of a finite dimensional semisimple Lie algebra is a derivation; in [2] it is proved that every local derivation of a finite dimensional semisimple Lie algebra is a derivation. As far as automorphisms are concerned, in [9] the authors proved that if $g$ is a finite dimensional simple Lie algebra of type $A_\ell (\ell \geq 1)$, $D_\ell (\ell \geq 4)$, or $E_i (i = 6, 7, 8)$, then every 2-local automorphism of $g$ is an automorphism. This result was extended to any finite dimensional semisimple Lie algebra in [3]. On the other hand, for local automorphisms of simple Lie algebras it is only known that the automorphisms and the anti-automorphisms of the special linear algebra $\mathfrak{sl}(n)$ exhaust all its local automorphisms ([4, Theorem 2.3]).

The main purpose of this paper is to extend this result to any finite dimensional simple Lie algebra: namely we prove that a $K$-linear endomorphism of a finite dimensional simple Lie algebra $g$ over the algebraically closed field $K$ of characteristic zero is a local automorphism if and only if it is an automorphism or an anti-automorphism of $g$.

Let $G$ be the connected component of the automorphism group of $g$: then $G$ is the adjoint simple algebraic group over $K$ with the same Dynkin diagram as $g$. It is clear that every automorphism of $g$ is a local automorphism: we show that every anti-automorphism of $g$ is a local automorphism too. For this purpose we make use of the Bala-Carter theory for the classification of nilpotent elements in $g$.

To show that a local automorphism of $g$ is an automorphisms or an anti-automorphisms, we make use of the Tits’ Building $\Delta(G)$ of $G$ (as defined in [22, Chap. 5.3]) and the classification theorem [22, Theorem 5.8] which in particular describes the automorphisms of $\Delta(G)$.

2 Preliminaries

Throughout the paper $K$ is an algebraically closed field of characteristic zero. We denote by $\mathbb{R}$ the reals, by $\mathbb{Z}$ the integers.

Let $A = (a_{ij})$ be a finite indecomposable Cartan matrix of rank $n$. To $A$ there is associated a root system $\Phi$, a simple Lie algebra $g$ and a simple adjoint algebraic group $G$ over $K$. We
fix a maximal torus $T$ of $G$, and a Borel subgroup $B$ containing $T$: $B^-$ is the Borel subgroup opposite to $B$, $U$ (respectively $U^-$) is the unipotent radical of $B$ (respectively of $B^-$). We denote by $\mathfrak{h}$, $\mathfrak{n}$, $\mathfrak{n}^-$ the Lie algebra of $T$, $U$, $U^-$ respectively. Then $\Phi$ is the set of roots relative to $T$, and $B$ determines the set of positive roots $\Phi^+$, and the simple roots $\Delta = \{\alpha_1, \ldots, \alpha_n\}$. The real space $E = \mathbb{R}\Phi$ is a Euclidean space, endowed with the scalar product $(\alpha_i, \alpha_j) = d_i a_{ij}$. Here $\{d_1, \ldots, d_n\}$ are relatively prime positive integers such that if $D$ is the diagonal matrix with entries $d_1, \ldots, d_n$, then $DA$ is symmetric. For $\beta = m_1\alpha_1 + \cdots + m_n\alpha_n$, the height of $\beta$ is $m_1 + \cdots + m_n$.

For $\alpha, \beta \in \Phi$, we put $\langle \beta, \alpha \rangle = \frac{2(\beta, \alpha)}{\langle \alpha, \alpha \rangle}$.

We denote by $W$ the Weyl group; $s_\alpha$ is the reflection associated to $\alpha \in \Phi$, we write for short $s_i$ for the simple reflection associated to $\alpha_i$, $w_0$ is the longest element of $W$. We put $\Pi = \{1, \ldots, n\}$, $\vartheta$ the symmetry (called the opposite involution) of $\Pi$ induced by $-w_0$ and we fix a Chevalley basis $\{h_i, i \in \Pi; e_\alpha, \alpha \in \Phi\}$ of $\mathfrak{g}$ (see [7, Chap. 4.2]). We put $h_\beta = [e_\beta, e_{-\beta}]$ for $\beta \in \Phi$ (hence $h_i = h_{\alpha_i}$ for $i \in \Pi$).

We use the notation $x_\alpha(\xi)$, for $\alpha \in \Phi$, $\xi \in K$, as in [7], [21]. For $\alpha \in \Phi$ we put $X_\alpha = \{x_\alpha(\xi) \mid \xi \in K\}$, the root-subgroup corresponding to $\alpha$. We identify $W$ with $N/T$, where $N$ is the normalizer of $T$.

We choose the $x_\alpha$’s so that, for all $\alpha \in \Phi$, $n_\alpha = x_\alpha(1)x_{-\alpha}(-1)x_\alpha(1)$ lies in $N$ and has image the reflection $s_\alpha$ in $W$. The family $(x_\alpha)_{\alpha \in \Phi}$ is called a realization of $\Phi$ in $G$.

Given an element $w \in W$ we shall denote a representative of $w$ in $N$ by $\dot{w}$. We can, and shall, take $\dot{w}$ defined over $\mathbb{Z}$.

For algebraic groups we use the notation in [14], [8]. In particular, for $J \subseteq \Pi$, $\Delta_J = \{\alpha_j \mid j \in J\}$, $\Phi_J$ is the corresponding root system, $W_J$ the Weyl group, $P_J$ the standard parabolic subgroup of $G$, $L_J = T\langle X_\alpha \mid \alpha \in \Phi_J \rangle$ the standard Levi subgroup of $P_J$. For $w \in W$ we have

$$\dot{w}U^-\dot{w}^{-1} \cap U = \prod_{\alpha > 0} \prod_{\alpha < 0} X_\alpha.$$

If $x$ is an element of $\mathfrak{g}$, $C_G(x)$ is the centralizer of $x$ in $G$.

We denote by $GL(\mathfrak{g})$ the group of automorphisms of $\mathfrak{g}$ as a $K$-vector space. The group $\text{AUT}(\mathfrak{g})$ of automorphisms of $\mathfrak{g}$ as a Lie algebra is completely described in [15, Chap. IX], [13, 16.5].
We denote by $NB(\mathfrak{g})$ the set of the nilradicals of Borel subalgebras of $\mathfrak{g}$. This is a unique orbit under $G$: if $n_1 \in NB(\mathfrak{g})$ then, by the Bruhat decompositon in $G$, there exists a unique $w \in W$ and a unique $u \in \tilde{w}U\tilde{w}^{-1} \cap U$ such that $n_1 = \text{Ad } uw.n$.

3 The main result

We recall that a parabolic subgroup $P$ is called distinguished if $\dim P/R_uP = \dim R_uP/(R_uP)'$. Here $R_uP$ is the unipotent radical of $P$ and $(R_uP)'$ is the derived subgroup of $R_uP$ (see [8, p. 167]). Two parabolic subgroups are said to be opposite if their intersection is a common Levi subgroup (see [5, 14.20]). If $P$ is a parabolic subgroup and $L$ is a Levi subgroup of $P$, then there exists a unique parabolic subgroup opposite to $P$ containing $L$. Any two opposite parabolic subgroups of $P$ are conjugate by a unique element of $R_uP$ ([5, Proposition 14.21]).

Lemma 3.1 Let $P$ be a distinguished parabolic subgroup of a semisimple algebraic group $R$ and let $P^\text{op}$ be an opposite parabolic subgroup of $P$. Then $P$ and $P^\text{op}$ are conjugate in $R$.

Proof. It is enough to assume $R$ simple, $P = P_J = \langle B, X_{-\alpha_i} \mid i \in J \rangle$, $P^\text{op} = \langle B^-, X_{\alpha_i} \mid i \in J \rangle$ for a certain $J \subseteq \Pi$. If $w_0 = -1$, then $P^\text{op} = \tilde{w}_0P\tilde{w}_0^{-1}$. We are left with the cases where $R$ is of type $A_n$, $n \geq 2$, $D_n$ with $n \geq 5$, $n$ odd, $E_6$. From the tables in [8], p. 174 - 176, one checks that again $P^\text{op} = \tilde{w}_0P\tilde{w}_0^{-1}$, since in each case the diagram of $P$ is invariant under the opposite involution $\vartheta$ of the Dynkin diagram. □

Theorem 3.2 The anti-automorphism $-i_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{g}$, $x \mapsto -x$ is a local automorphism of $\mathfrak{g}$.

Proof. Let $x \in \mathfrak{g}$. We have to show that there exists $\alpha \in \text{AUT}(\mathfrak{g})$ such that $\alpha(x) = -x$. Let $O$ be the $G$-orbit of $x$: it is enough to show that this holds for a certain $y \in O$. In fact, if $x = \text{Ad } g.y$ and $\beta(y) = -y$ for certain $g \in G$, $\beta \in \text{AUT}(\mathfrak{g})$, then $\alpha(x) = -x$, where $\alpha$ is the automorphism of $\mathfrak{g}$ given by $\alpha = (\text{Ad } g)\beta(\text{Ad } g)^{-1}$.

Let $x = s + e$ be the Jordan-Chevalley decomposition of $x$, i.e. $s$ is semisimple, $e$ is nilpotent, with $[s, e] = 0$. Let $H = C_G(s)$. This is a Levi subgroup of $G$ and, up to conjugacy in $G$, we may assume that $H$ is the standard Levi subgroup $L_J$ of $G$. Moreover the centralizer of $s$ in $\mathfrak{g}$ is the Lie algebra $l_J$ of $L_J$, $e$ lies in $l_J$, and $s$ lies in the center $Z(l_J) \subseteq \mathfrak{h}$. Let $m$ be a minimal Levi subalgebra of $l_J$ containing $e$. Let $M$ be the Levi subgroup of $H$ such that $m = \text{Lie}(M)$, and let
Let \( M' \) be the semisimple part of \( M \) and \( m' = \text{Lie}(M') \). Then \( e \) lies in \( m' \) and \( e \) is distinguished in \( m' \). There exists a distinguished parabolic subgroup \( P_{M'} \) of \( M' \) such that \( e \) lies in the dense orbit of \( P_{M'} \) on the Lie algebra \( u_{P_{M'}} \) of its unipotent radical. Up to conjugation by an element of \( H \), we may assume that \( P_{M'} = \langle T_1, X_\alpha, X_{-\alpha}, X_\delta \mid \alpha \in \Psi_1, \delta \in \Psi_2 \rangle \) for \( T_1 \) a certain subtorus of \( T \) and \( \Psi_1, \Psi_2 \) certain disjoint subsets of \( \Phi^+ \).

Now we consider an automomorphism \( \iota \) of \( G \) satisfying
\[
\iota(t) = t^{-1} \quad \text{for every } t \in T \quad , \quad \iota(X_\alpha) = X_{-\alpha} \quad \text{for every } \alpha \in \Phi
\]
[16, proof of Corollary 1.16, p. 189]. Then the differential \( d\iota \) is an automorphism of \( g \) satisfying
\[
d\iota(h) = -h \text{ for every } h \in \mathfrak{h}, \text{ in particular } d\iota(s) = -s.\]
It is enough, by [20, Lemma 2.2.1], to show that \( d\iota(e) \) and \( -e \) are conjugate by an element of \( H \). But \( \iota(P_{M'}) = \langle T_1, X_\alpha, X_{-\alpha}, X_\delta \mid \alpha \in \Psi_1, \delta \in \Psi_2 \rangle \) is opposite to \( P_{M'} \) (since \( P_{M'} \cap \iota(P_{M'}) = \langle T_1, X_\alpha, X_{-\alpha} \mid \alpha \in \Psi_1 \rangle \), a Levi subgroup of \( M' \)). Since a parabolic subgroup has a unique dense orbit on the Lie algebra of its unipotent radical, and clearly \( -e \) lies in the dense orbit of \( P_{M'} \) on \( u_{P_{M'}} \), and \( d\iota(N) \) lies in the dense orbit of \( \iota(P_{M'}) \) on \( d\iota(u_{P_{M'}}) \), it is enough to show that \( P_{M'} \) and \( \iota(P_{M'}) \) are conjugate in \( H \).

From Lemma 3.1 it follows that \( P_{M'} \) and \( \iota(P_{M'}) \) are already conjugate in \( M' \), and we are done.\( \square \)

We denote by \( \text{AUT}^*(g) \) the group of automorphisms of the \( K \)-vector space \( g \) which are either automorphisms or anti-automorphisms of the Lie algebra \( g \). Then \( \text{AUT}^*(g) = \text{AUT}(g) \rtimes \langle -i_g \rangle \).

We observe that if \( \varphi \) is a local automorphism of \( g \), then \( \varphi \) is invertible and its inverse is a local automorphism. It is also clear that the composite of local automorphisms is a local automorphism, therefore the set \( \text{LAut}(g) \) of local automorphisms of \( g \) is a subgroup of \( GL(g) \). By Theorem 3.2 we have

\textbf{Corollary 3.3} Every anti-automorphism of \( g \) is a local automorphism, i.e. \( \text{AUT}^*(g) \leq \text{LAut}(g) \).
\( \square \)

We shall prove that \( \text{LAut}(g) = \text{AUT}^*(g) \).

\textbf{Lemma 3.4} Let \( \varphi \) be in \( \text{LAut}(g) \). Then \( \varphi \) leaves invariant the set \( N \) of nilpotent elements and the set \( S \) of semisimple elements of \( g \).
Proof. Let $x \in \mathfrak{g}$. There exists $\varphi_x \in \text{AUT}(\mathfrak{g})$ such that $\varphi_x(x) = \varphi(x)$. Since automorphisms map nilpotent (respectively, semisimple) elements to nilpotent (respectively, semisimple) elements, it follows that $\varphi(N) \subseteq N$ and $\varphi(S) \subseteq S$. Since $\varphi^{-1}$ is also a local automorphism, we conclude that $\varphi(N) = N$ and $\varphi(S) = S$. □

A classical theorem of Gerstenhaber [12] states that any vector space consisting of nilpotent $n \times n$ matrices has dimension at most $\frac{1}{2}n(n-1)$, and that any such space attaining this maximal possible dimension is conjugate to the space of upper triangular matrices. In [11] the authors generalized this result to the Lie algebra of any reductive algebraic group over any algebraically closed field, under certain conditions in case the characteristic of the field is 2 or 3. We restate this generalization for our purposes. For short we say that a subspace $V$ of $\mathfrak{g}$ is nilpotent, if $V$ consists of nilpotent elements.

**Theorem 3.5** ([11, Theorem 1]) Let $V$ be a nilpotent subspace of a finite dimensional semisimple Lie algebra $\mathfrak{g}$ over $K$. Then $\dim V \leq \frac{1}{2}(\dim \mathfrak{g} - \text{rk} \mathfrak{g})$ and, if equality holds, $V$ is the nilradical of a Borel subalgebra of $\mathfrak{g}$.

In particular the nilpotent subspaces of maximal dimension are the maximal nilpotent subalgebras $\mathfrak{g}$: they constitute the set $\mathcal{N}\mathcal{B}(\mathfrak{g})$ defined in the Preliminaries.

**Proposition 3.6** Let $\varphi$ be in $\text{LAut}(\mathfrak{g})$. Then $\varphi$ induces a permutation of the set $\mathcal{N}\mathcal{B}(\mathfrak{g})$.

**Proof.** Let $V$ be any nilpotent subspace of $\mathfrak{g}$. By Lemma 3.4 $\varphi(V)$ and $\varphi^{-1}(V)$ are nilpotent subspaces of $\mathfrak{g}$. Therefore $\varphi$ induces a permutation $V \mapsto \varphi(V)$ of the set of all nilpotent subspaces of $\mathfrak{g}$. In particular $\varphi$ induces a permutation of $\mathcal{N}\mathcal{B}(\mathfrak{g})$. □

We introduce the canonical Tits’ Building $\Delta(G)$ associated to $G$.

**Definition 3.7** [22, Chap. 5.3] The building $\Delta(G)$ of $G$ is the set of all parabolic subgroups of $G$, partially ordered by reverse of inclusion.

The maximal elements of $\Delta(G)$ (called chambers) are the Borel subgroups of $G$. The set of Borel subgroups of $G$ is in canonical bijection with the set of Lie algebras of Borel subgroups of $G$ (i.e. the Borel subalgebras of $\mathfrak{g}$, [5, 14.25]), and this set is in canonical bijection with the set
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By Proposition 3.6, a local automorphism \( \varphi \) of \( g \) induces a permutation of \( \mathcal{NB}(g) \), and therefore a permutation \( \rho_{\varphi} \) of the set of chambers of \( \Delta(G) \). Let \( B_1, B_2 \) be adjacent chambers: this means that the codimension (as algebraic varieties) of \( B_1 \cap B_2 \) in \( B_1 \) (and \( B_2 \)) is 1. Since \( B_1 \cap B_2 \) always contains a maximal torus of \( G \), this is equivalent to the condition that the codimension (as \( k \)-vector spaces) of \( n_1 \cap n_2 \) in \( n_1 \) (and \( n_2 \)) is 1, where \( n_i \) is the nilradical of the Lie algebra of \( B_i \) for \( i = 1, 2 \).

**Proposition 3.8** Let \( \varphi \) be in \( \text{LAut}(g) \). Then \( \rho_{\varphi} \) can be (uniquely) extended to an automorphism of \( \Delta(G) \).

**Proof.** By the previous discussion, this follows from [22, Theorem 3.21, Corollary 3.26]. \( \square \)

We shall still denote by \( \rho_{\varphi} \) the automorphism of \( \Delta(G) \) induced by \( \varphi \).

A symmetry of the Dynkin diagram of \( G \) is a permutation \( \delta \) of the nodes of the diagram such that \( \langle \alpha_{\delta(i)}, \alpha_{\delta(j)} \rangle = \langle \alpha_i, \alpha_j \rangle \) for all \( i, j \in \Pi \) ([15, p. 277]. Note that in [7, p. 200] the definition is different, in order to deal also with fields of characteristic 2 or 3). We denote the group of symmetries of the Dynkin diagram by Diagr.

**Definition 3.9** Let \( \delta \) be a symmetry of the Dynkin diagram of \( g \). We denote by \( d_{\delta} \) both the isometry of \( E \) and the graph automorphism of \( g \) defined respectively by

\[
d_{\delta}(\alpha_i) = \alpha_{\delta(i)} \quad \text{for every } i \in \Pi
\]

\[
d_{\delta}(e_{\alpha_i}) = e_{\alpha_{\delta(i)}}, \quad d_{\delta}(e_{-\alpha_i}) = e_{-\alpha_{\delta(i)}}, \quad d_{\delta}(h_{\alpha_i}) = h_{\alpha_{\delta(i)}} \quad \text{for every } i \in \Pi
\]

**Proposition 3.10** Let \( \varphi = c \iota_g \), for a certain \( c \in K^* \). Then \( \varphi \in \text{LAut}(g) \) if and only if \( c = \pm 1 \).

**Proof.** We only need to show that if \( \varphi = c \iota_g \) is a local automorphism, then \( c = \pm 1 \). By [7, Proposition 6.4.2] we have

\[
\text{Ad } n_{\alpha} h_{\beta} = h_{\alpha_{\delta}(\beta)}
\]

for every \( \alpha, \beta \in \Phi \), so that

\[
\text{Ad } \hat{w} h_{\beta} = h_{\delta(\beta)}
\]

for every \( w \in W, \beta \in \Phi \). Now fix any \( \alpha \in \Phi, h \in \mathfrak{h} \). There exists \( g \in G, \delta \in \text{Diagr} \) such that

\[
c h_{\alpha} = \varphi(h_{\alpha}) = d_{\delta} \text{Ad } g h_{\alpha}
\]
Hence $\text{Ad } g.h.\alpha = cd^{-1}\delta h.\alpha \in \mathfrak{h}$, which means that the elements $h.\alpha$ and $cd^{-1}\delta h.\alpha$ of $\mathfrak{h}$ are conjugate under $G$, and therefore they are conjugate under $W$, i.e. there exists $w \in W$ such that $\text{Ad } g.h.\alpha = \text{Ad } w.h.\alpha = h.w(\alpha)$. Hence $cd^{-1}\delta h.\alpha = h.w(\alpha)$, $c h.\alpha = d\delta h.w(\alpha) = h.\delta w(\alpha) = h.\beta$, for $\beta = \delta w(\alpha) \in \Phi$. It follows that $\beta = \pm \alpha$, i.e. $c = \pm 1$. □

A semilinear isomorphism between two Lie algebras is a bijective semilinear mapping of the underlying vector spaces which respects Lie multiplication.

**Definition 3.11** Let $f \in \text{Aut } K$. We denote by $a_f$ both the field automorphism of $G$ (as an abstract group) and the $f$-semilinear automorphism of $\mathfrak{g}$ defined respectively by

$$a_f(x.\alpha(k)) = x.\alpha(f(k)) \quad \text{for every } \alpha \in \Phi, k \in K$$

$$a_f(k e.\alpha) = f(k) e.\alpha \quad \text{for every } \alpha \in \Phi, k \in K$$

**Remark 3.12** Note that we also have $a_f(k h.\alpha) = f(k) h.\alpha$ for every $\alpha \in \Phi$, $k \in K$, since $h.\alpha = [e.\alpha, e.-\alpha]$ for every $\alpha \in \Phi$. Moreover, for every $g \in G$, $x \in \mathfrak{g}$ we have $a_f(\text{Ad } g.x) = \text{Ad } (a_f(g)).a_f(x)$.

**Proposition 3.13** Let $\varphi \in \text{GL}(\mathfrak{g})$ and $f \in \text{Aut } K$ be such that $\varphi(X) = a_f(X)$ for every $X \in \mathcal{N}\mathcal{B}(\mathfrak{g})$. Then $f = i_K$ and there is $c \in K^*$ such that $\varphi = c i_\mathfrak{g}$.

**Proof.** We have $a_f(n) = n$ and $a_f(n^-) = n^-$. It follows that

$$a_f(\text{Ad } x.\alpha(k)\dot{w}.n) = \text{Ad } x.\alpha(f(k))\dot{w}.n \quad a_f(\text{Ad } x.\alpha(k)\dot{w}.n^-) = \text{Ad } x.\alpha(f(k))\dot{w}.n^-$$

for every $\alpha \in \Phi$, $k \in K$, since we fixed the representatives $\dot{w}$ over $\mathbb{Z}$, and therefore $a_f(\dot{w}) = \dot{w}$ for every $w \in W$.

We shall repeatedly use the fact that if $n_1, n_2 \in \mathcal{N}\mathcal{B}(\mathfrak{g})$ are such that $n_1 \cap n_2 = \langle v \rangle$ with $v \neq 0$, then

$$\langle \varphi(v) \rangle = \varphi(n_1) \cap \varphi(n_2) = a_f(n_1) \cap a_f(n_2) = \langle a_f(v) \rangle$$

For every $i \in \Pi$ we have

$$\text{Ad } \dot{s}_i.n^- \cap n = \langle e.\alpha_i \rangle \quad \text{Ad } \dot{s}_i.n \cap n^- = \langle e.-\alpha_i \rangle$$
hence
\[ \langle \varphi(e_{\alpha_i}) \rangle = \langle af(e_{\alpha_i}) \rangle = \langle e_{\alpha_i} \rangle, \quad \langle \varphi(e_{-\alpha_i}) \rangle = \langle af(e_{-\alpha_i}) \rangle = \langle e_{-\alpha_i} \rangle \]

Let \( \alpha \in \Phi \). There exists \( w \in W, i \in \Pi \) such that \( w(\alpha_i) = \alpha \). Then
\[ \langle e_{\alpha_i} \rangle = \text{Ad } \hat{w} \cdot \langle e_{\alpha_i} \rangle = \text{Ad } \hat{w} \cdot \langle e_{\alpha_i} \rangle \cap \text{Ad } \hat{w} \cdot n \]
so that
\[ \langle \varphi(e_{\alpha_i}) \rangle = \langle af(e_{\alpha_i}) \rangle = \langle e_{\alpha_i} \rangle \]

Hence, for every \( \alpha \in \Phi \) there exists \( c_\alpha \in K^* \) such that \( \varphi(e_{\alpha_i}) = c_\alpha e_{\alpha_i} \).

By [7, p. 64], for every \( \alpha \in \Phi, k \in K \) we have
\[ \text{Ad } x_\alpha(k)e_{\alpha_i} = e_{\alpha_i}, \quad \text{Ad } x_\alpha(k)e_{-\alpha} = e_{-\alpha} + kh_\alpha - k^2e_{\alpha_i} \]
Let us fix \( i \) in \( \Pi \). From \( \text{Ad } \hat{s}_i \cdot n \cap n^- = \langle e_{-\alpha_i} \rangle \) we get
\[ \text{Ad } x_{\alpha_i}(k) \cdot \text{Ad } \hat{s}_i \cdot n \cap \text{Ad } x_{\alpha_i}(k) \cdot n^- = \text{Ad } x_{\alpha_i}(k) \cdot e_{-\alpha_i} = \langle e_{-\alpha_i} + kh_\alpha - k^2e_{\alpha_i} \rangle \]
so that
\[ \langle \varphi(e_{-\alpha_i} + kh_\alpha - k^2e_{\alpha_i}) \rangle = \langle af(e_{-\alpha_i} + kh_\alpha - k^2e_{\alpha_i}) \rangle = \langle e_{-\alpha_i} + f(k)h_\alpha - f(k)^2e_{\alpha_i} \rangle \]
(3.1)
In particular, for \( k = 1 \) we get
\[ \langle \varphi(e_{-\alpha_i} + h_\alpha - e_{\alpha_i}) \rangle = \langle e_{-\alpha_i} + h_\alpha - e_{\alpha_i} \rangle \]
hence \( \varphi(h_{\alpha_i}) = d_i h_{\alpha_i} + x_i e_{\alpha_i} + y_i e_{-\alpha_i} \) for certain \( d_i, x_i, y_i \in K, i = 1, \ldots, n \). From (3.1), for every \( k \in K \) there exits \( p_k \in K^* \) such that
\[ c_{-\alpha_i} e_{-\alpha_i} + k(d_i h_{\alpha_i} + x_i e_{\alpha_i} + y_i e_{-\alpha_i}) - k^2c_{\alpha_i} e_{\alpha_i} = p_k(e_{-\alpha_i} + f(k)h_\alpha - f(k)^2e_{\alpha_i}) \]
(3.2)
hence \( k d_i = p_k f(k) \) for every \( k \in K \) and in particular, for \( k = 1, d_i = p_1 \). But then \( p_k = f(k)^{-1} p_1 \) for every \( k \in K^* \), so that \( p_k = p_1 \) for every \( k \) in the prime field \( \mathbb{Q} \) of \( K, k \neq 0 \). From (3.2) we obtain \( c_{-\alpha_i} e_{-\alpha_i} + ky_i e_{-\alpha_i} = p_1 e_{-\alpha_i} \) and \( kx_i e_{\alpha_i} - k^2c_{\alpha_i} e_{\alpha_i} = -p_1 k^2 e_{\alpha_i} \) for every \( k \in \mathbb{Q}^* \), so that \( y_i = 0, c_{-\alpha_i} = p_1, x_i = 0 \) and \( c_{\alpha_i} = p_1 \). We have proved that
\[ \varphi(h_{\alpha_i}) = c_{\alpha_i} h_{\alpha_i}, c_{-\alpha_i} = c_{\alpha_i}, \quad \varphi(e_{\alpha_i}) = c_{\alpha_i} e_{\alpha_i}, c_{-\alpha_i} = c_{\alpha_i} \]
Moreover, from (3.2) it follows that
\[ c_\alpha e_{-\alpha_i} + k c_\alpha h_{\alpha_i} - k^2 c_\alpha e_{\alpha_i} = p_k (e_{-\alpha_i} + f(k) h_{\alpha_i} - f(k)^2 e_{\alpha_i}) \]
for every \( k \in K \), hence \( p_k = c_\alpha \) and \( f(k) = k \) for every \( k \in K \), i.e. \( f = i_K \).

So far we have proved that \( f = i_K \), and that for every \( i = 1, \ldots, n \) we have \( \varphi(e_{\alpha_i}) = c_\alpha e_{\alpha_i} \), \( \varphi(e_{-\alpha_i}) = c_\alpha e_{-\alpha_i} \) and \( \varphi(h_i) = c_\alpha h_i \). Our aim is to show that \( c_\alpha = c_\beta \) for every \( \alpha, \beta \in \Phi \). We prove that \( c_\alpha = c_\beta \) for every \( \alpha, \beta \in \Phi^+ \). With a similar procedure it will follow that \( c_\alpha = c_\beta \) for every \( \alpha, \beta \in \Phi^- \), so that \( c_\alpha = c_\beta \) for every \( \alpha, \beta \in \Phi \) by (3.3).

By [7, p. 64], for linearly independent roots \( \alpha, \beta \) we have
\[ \text{Ad } x(t).e_\beta = \sum_{r=0}^{q} M_{\alpha,\beta,r} t^r e_{r\alpha+\beta} \]
where \( M_{\alpha,\beta,0} = 1 \), \( M_{\alpha,\beta,r} = \pm \left( \frac{p+r}{r} \right) \) for \( r \geq 1 \), \( -p\alpha + \beta, \ldots, \beta, \ldots, q\alpha + \beta \) is the \( \alpha \)-chain through \( \beta \) with \( p \) and \( q \) non negative integers. In particular, for \( t = 1 \) we get
\[ \text{(3.4)} \quad \text{Ad } x(1).e_\beta = \sum_{r=0}^{q} M_{\alpha,\beta,r} e_{r\alpha+\beta} \]

We begin by showing that \( c_{\alpha_i} = c_{\alpha_j} \) for every \( i, j \in \Pi \). Assume \( \alpha_i + \alpha_j \in \Phi \). Then
\[ \text{Ad } x_{\alpha_i}(1).e_{\alpha_j} = \sum_{r=0}^{q} M_{\alpha_i,\alpha_j,r} e_{r\alpha_i+\alpha_j} \]
with \( q \geq 1 \). From \( \text{Ad } \dot{s}_j.n^- \cap n = \langle e_{\alpha_j} \rangle \) we get
\[ \langle \text{Ad } x_{\alpha_i}(1).e_{\alpha_j} \rangle = \text{Ad } x_{\alpha_i}(1) \text{Ad } \dot{s}_j.n^- \cap \text{Ad } x_{\alpha_i}(1).n \]
so that
\[ \langle \varphi(\text{Ad } x_{\alpha_i}(1).e_{\alpha_j}) \rangle = \langle a_f(\text{Ad } x_{\alpha_i}(1).e_{\alpha_j}) \rangle = \langle \text{Ad } x_{\alpha_i}(1).e_{\alpha_j} \rangle \]

There exists \( c \in K^* \) such that
\[ \varphi(\sum_{r=0}^{q} M_{\alpha_i,\alpha_j,r} e_{r\alpha_i+\alpha_j}) = c \left( \sum_{r=0}^{q} M_{\alpha_i,\alpha_j,r} e_{r\alpha_i+\alpha_j} \right) \]
Since \( M_{\alpha_i,\alpha_j,r} \neq 0 \) for every \( r = 0, \ldots, q \), we get
\[ c_{r\alpha_i+\alpha_j} = c \]
for every \( r = 0, \ldots, q \), and in particular \( c_{\alpha_j} = c, c_{\alpha_i + \alpha_j} = c \), so that \( c_{\alpha_j} = c_{\alpha_i + \alpha_j} = c \). Similarly, by considering \( \text{Ad} \, x_{\alpha_j} \cdot e_{\alpha_i} \), we obtain \( c_{\alpha_i} = c_{\alpha_j + \alpha_i} \); hence \( c_{\alpha_i} = c_{\alpha_j} = c \). Since the Dynkin diagram is connected, we get \( c_{\alpha_i} = c_{\alpha_j} = c \) for every \( i, j \in \Pi \) (incidentally, the previous argument shows that \( c_\alpha = c_\beta = c \) for positive roots \( \alpha, \beta \) of height at most 2).

Assume that \( \beta \) is a positive root of height \( m \) with \( m \geq 2 \). Then we may write \( \beta = \gamma + \alpha_i \), for a certain \( \gamma \in \Phi^+ \) (of height \( m - 1 \)) and a certain \( i \in \Pi \). Then

\[
\text{Ad} \, x_\gamma \cdot e_{\alpha_i} = \sum_{r=0}^{q} M_{\gamma, \alpha_i, r} \cdot e_{r\gamma + \alpha_i}
\]

with \( q \geq 1 \). From \( \text{Ad} \, s_i \cdot \mathfrak{n}^- \cap \mathfrak{n} = \langle e_{\alpha_i} \rangle \) we get

\[
\langle \text{Ad} \, x_\gamma \cdot e_{\alpha_i} \rangle = \langle \text{Ad} \, x_\gamma \cdot \text{Ad} \, s_i \cdot \mathfrak{n}^- \cap \text{Ad} \, x_\gamma \cdot \mathfrak{n} \rangle
\]

so that

\[
\langle \varphi(\text{Ad} \, x_\gamma \cdot e_{\alpha_i}) \rangle = \langle a_f(\text{Ad} \, x_\gamma \cdot e_{\alpha_i}) \rangle = \langle \text{Ad} \, x_\gamma \cdot e_{\alpha_i} \rangle
\]

There exists \( d \in K^* \) such that

\[
\varphi \left( \sum_{r=0}^{q} M_{\gamma, \alpha_i, r} \cdot e_{r\gamma + \alpha_i} \right) = d \left( \sum_{r=0}^{q} M_{\gamma, \alpha_i, r} \cdot e_{r\gamma + \alpha_i} \right)
\]

Since \( M_{\gamma, \alpha_i, r} \neq 0 \) for every \( r = 0, \ldots, q \), we get

\[
c_{r\gamma + \alpha_i} = d
\]

for every \( r = 0, \ldots, q \), and in particular \( c_{\alpha_i} = d, c_\beta = c_{\gamma + \alpha_i} = d \), so that \( c_\beta = c_{\alpha_i} = c \). We have therefore proved that \( c_\alpha = c \) for every \( \alpha \in \Phi^+ \). Similarly one can prove that \( c_\alpha = c' \) for every \( \alpha \in \Phi^- \) for a certain \( c' \in K^* \). Since by (3.3) we have \( c_{-\alpha_i} = c_{\alpha_i} \) we get \( c' = c \), i.e. \( c_\alpha = c \) for every \( \alpha \in \Phi \). But we also have \( \varphi(h_i) = c \, h_i \) for every \( i \in \Pi \), we conclude that \( \varphi = c \, i_g \). \( \square \)

**Theorem 3.14** Let \( \mathfrak{g} \) be a finite dimensional simple Lie algebra over the algebraically closed field \( K \) of characteristic zero. Then a linear map \( \varphi : \mathfrak{g} \to \mathfrak{g} \) is local automorphism if and only if it is an automorphism or an anti-automorphism, i.e. \( \text{LAut}(\mathfrak{g}) = \text{AUT}^*(\mathfrak{g}) \).

**Proof.** The case when \( \mathfrak{g} \) is of type \( A_n, n \geq 1 \), is dealt with in [4]. For completeness, here we give a proof also for this case. By Corollary 3.3 we have \( \text{AUT}^*(\mathfrak{g}) \leq \text{LAut}(\mathfrak{g}) \). Let \( \varphi \) be a local
automorphism of $\mathfrak{g}$. We show that there exists an automorphism $\beta$ of $\mathfrak{g}$ and $c \in K^*$ such that $\beta^{-1}\varphi = c i_{\mathfrak{g}}$.

Assume first that $\mathfrak{g}$ has rank 1, i.e. $\mathfrak{g} = \mathfrak{sl}(2)$. Then the result follows from the main theorem in [6] (see Remark on page 45). So assume $\text{rk } \mathfrak{g} \geq 2$. By Proposition 3.8, $\varphi$ induces an automorphism $\rho_\varphi$ of the building $\Delta(G)$ of $G$. By the structure theorem on isomorphisms of buildings ([22, Theorem 5.8]), there exists an automorphism $\alpha$ of $G$ (as an algebraic group) and a field automorphism $a_f$ of $G$ such that $\rho_\varphi(P) = \alpha a_f(P)$ for every parabolic subgroup $P$ of $G$. It follows that, for $\beta = d\alpha$, the differential of $\alpha$, we get
\[
\beta^{-1}\varphi(X) = a_f(X)
\]
for every $X$ in $\mathcal{N}\mathcal{B}(\mathfrak{g})$. By Proposition 3.13, $\beta^{-1}\varphi = c i_{\mathfrak{g}}$ for a certain $c \in K^*$.

Finally, from Proposition 3.10, we get $c = \pm 1$, and $\varphi = \pm \beta \in \text{AUT}^*(\mathfrak{g})$. □

**Remark 3.15** From the structure of the automorphism group of $\mathfrak{g}$, it follows that any $\varphi \in \text{LAut}(\mathfrak{g})$ is of the form $\varphi = \pm d_\delta(\text{Ad } g)$ for a unique $g \in G$ and a unique graph automorphism $d_\delta$.

**References**

[1] S. Ayupov, K. Kudaybergenov, I. Rakhimov, *2-local derivations on finite-dimensional Lie algebras*, Linear Algebra Appl. 474, 1–11 (2015).

[2] S. Ayupov, K. Kudaybergenov, *Local derivations on finite dimensional Lie algebras*, Linear Algebra Appl. 493, 381–398 (2016).

[3] S. Ayupov, K. Kudaybergenov, *2-local automorphisms on finite-dimensional Lie algebras*, Linear Algebra Appl. 507, 121–131 (2016).

[4] S. Ayupov, K. Kudaybergenov, *Local automorphisms on finite-dimensional Lie and Leibniz algebras*, preprint arXiv:1803.03142v2.

[5] A. Borel, *Linear Algebraic Groups*, Second enlarged edition, Springer-Verlag, New York (1991).

[6] P. Botta, S. Pierce, W. Watkins, *Linear transformations that preserve the nilpotent matrices*, Pacific J. Math. 104(1), 39–46 (1983)
[7] R. W. CARTER, *Simple Groups of Lie Type*, John Wiley (1989).

[8] R. W. CARTER, *Finite Groups of Lie Type*, John Wiley (1985).

[9] Z. CHEN, D. WANG, *2-Local automorphisms of finite-dimensional simple Lie algebras*, Linear Algebra Appl. 486, 335–344 (2015).

[10] R. CRIST, *Local Automorphisms*, Proc. Amer. Math. Soc. 128, 1409–1414 (2000).

[11] J. DRAISMA, H. KRAFT, J. KUTTLER, *Nilpotent subspaces of maximal dimension in semi-simple Lie algebras*, Compos. Math. 142(2), 464–476 (2006).

[12] M. GERSTENHABER, *On nilalgebras and linear varieties of nilpotent matrices, I*, Amer. J. Math. 80, 614–622 (1958).

[13] J.E. HUMPHREYS, *Introduction to Lie algebras and representation theory*, Third printing, Graduate Texts in Mathematics, No. 9, Springer-Verlag, New York-Heidelberg (1980).

[14] J.E. HUMPHREYS, *Linear Algebraic Groups*, Third printing, Graduate Texts in Mathematics, No. 21, Springer-Verlag, New York-Heidelberg (1987).

[15] N. JACOBSON, *Lie Algebras*, Republication of the 1962 original, Dover Publications, Inc., New York (1979).

[16] J.C. JANTZEN, *Representations of algebraic groups*, Pure and Applied Mathematics 131, Academic Press, Inc., Boston, MA (1987).

[17] R.V. KADISON, *Local derivations*, J. Algebra 130, 494–509 (1990).

[18] D.R. LARSON, A.R. SOUROUR, *Local Derivations and Local Automorphisms of B(H)*, In: “Operator theory: operator algebras and applications, Part 2” (Durham, NH, 1988), Proc. Sympos. Pure Math. 51, 187–194 (1990).

[19] P. ŠEMRL, *Local automorphisms and derivations on B(H)*, Proc. Amer. Math. Soc. 125, 2677–2680 (1997).

[20] A. SINGH, M. THAKUR, *Reality properties of conjugacy classes in algebraic groups*, Israel J. Math. 165, 1–27 (2008).
[21] T.A. Springer, *Linear Algebraic Groups*, Second Edition, Progress in Mathematics 9, Birkhäuser (1998).

[22] J. Tits, *Buildings of spherical type and finite BN-pairs*, Lecture Notes in Mathematics, Vol. 386, Springer-Verlag, Berlin-New York (1974).