Lateral Control of Functionally Graded Composite Beam under Static Load by using Piezoelectric Layer

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Abstract. Controlling the deflection of a composite cantilever beam under concentrate static force at the free end is investigated. The composite beam is made of ceramic and metal, which gradually change from ceramic to the metal in thickness coordinate. This kind of composite is called Functionally Graded Material (FGM). In FGM, material properties are presumed to differ as an exponential function. The governing equation of the rectangular composite beam is based on Euler-Bernoulli deformation theory. Two control models are considered in this article. First, two piezoelectric actuators are attached to the two sides of the composite beam which each of these actuators is applied the voltage separately. To control the deflection of each point of the composite beam, the difference between the two voltages is considered constant. Second, a piezoelectric actuator is attached to the free end of the composite beam, which also provides a relation to control any desired point of the composite beam. By comparing the two models, the superiority of the second model is proven.

Nomenclature

\( l \) = Length of beam \\
\( b \) = Width of beam \\
\( D_f \) = Flexural rigidity of FGM beam \\
\( D_{eq} \) = Flexural rigidity of FGM beam with piezoelectric layer \\
\( h_u \) = Thickness of actuator layer \\
\( h \) = Thickness of FGM layer \\
\( E_{fe} \) = Modulus elasticity of FGM \\
\( E_c \) = Modulus elasticity of ceramic \\
\( E_m \) = Modulus elasticity of metal \\
\( E_u \) = Modulus elasticity of actuator \\
\( x, z \) = axes coordinates \\
\( u \) = Displacement in x-axis \\
\( w \) = Displacement in z-axis \\
\( Q \) = Elastic coefficients matrix \\
\( e \) = Piezoelectric coefficients matrix \\
\( V_s \) = Shear force \\
\( d_u \) = Distance between actuator center to center of whole area \\
\( V_t \) = Operating voltage of top actuator \\
\( V_b \) = Operating voltage of bottom actuator \\
\( q \) = Uniformly distributed force \\
\( V \) = Operating voltage \\
\( M \) = Bending moment \\
\( M_y \) = Bending moment about y-axis \\
\( M_u \) = Bending moment of actuator \\
\( M_{y}^a \) = Bending moment of actuator on top \\
\( M_{y}^b \) = Bending moment of actuator on bottom
\( w_0 \) = Displacement in mid-plane

\( E \) = Electric field vector

\( \nu \) = Poisson ratio

\( I_f \) = Second moment of FGM area

\( \sigma_{xx}, \sigma_{yy} \) = Normal stress in x, y axes

\( \varepsilon_{xx}, \varepsilon_{yy} \) = Normal strain in x, y axes

\( x_c \) = Length of the beam without actuator

\( E_x, E_y, E_z \) = Electric field components

\( I_a \) = Second moment of actuator area

\( F \) = Concentrated force

\( \sigma_{xy} \) = Shear stress in x-y plane

\( \varepsilon_{xy} \) = Shear strain in x-y plane

1. Introduction

In the past two decades, the use of a distributed piezoelectric layer as an actuator and sensor has been extensively studied by researchers to control or damping the elastic deformation of structures and also energy harvesting [1-4]. Piezoelectric materials under a mechanical deformation generate the electric charge. This phenomenon is called the direct piezoelectric effect. Conversely, the electric field causes stress and strain, which is called the converse effect of piezoelectric [5]. This property of piezoelectric materials with other attractive advantages such as inexpensive, easily shaped, lightweight and very good attached to structures makes an excellent choice for static and dynamic control of structures as actuators and sensors. Piezoelectric layers together with non-piezoelectric layers of structures can be considered as composite structures [6]. Tylitkowski and Hetnarski pioneered the study of the effect of the piezoelectric layer on transverse vibration control of Euler-Bernoulli two sides join beam under bending moment. They reduced the deflection of the beam by placing a piezoelectric layer in the middle of the beam and by applying appropriate voltage [7]. Tauchert et al. [8] examined a beam with piezoelectric layers on top and bottom which subjected to distributed loading and thermal loading. They analyzed this beam under dynamic loading by using the Fourier transform method. In order to find the appropriate voltage to reduce the deflection of the beam, they used the trial and error method. Donthireddy and Chandrashekhara developed a mathematical model of laminated composite beams with piezoelectric actuators by using a finite element method [9]. Then several researchers studied piezoelectric laminated beam and plate under mechanical and thermo-mechanical loads based on the finite element method [10-12]. Thomas Bailey et al. [13] studied design and analysis of the damping effect of distributed piezoelectric polymer layer on a thin cantilever beam by using an active damping model. Fumihiro Ashida et al. [14] examined the damping effect of piezothermoelastic layers on the vibration response of a composite beam under thermal loading by using a speed feedback control method. Osama J. Aldraihem et al. [15] compared control dynamic response of two different mathematical models (Euler-Bernoulli vs. Timoshenko) of cantilever composite beam based on Lyapunov control law. Although numerous researchers have studied different aspects such as mathematical model, kind of structure (beam, plate, shell, and so on), solving method, controlling model, loading and etc. of piezoelectric materials [16, 17].

Like piezoelectric materials, Functionally Graded Materials (FGMs) have received much attention from researchers over the past two decades [18, 19]. In FGMs, the volume fraction of composite in a specific direction is changing continuously [20]. To show the mechanical properties of FGM in the direction that volume fraction changes, several models and functions are provided [20-22]. However, three functions such as exponential, power-law, and sigmoid are the most notable. Due to characteristics of FGMs (material properties gradually change), disadvantages such as delamination, residual stress, and stress concentration under thermo-mechanical stresses are dramatically reduced [23]. B. L. Wang and N. Noda [24] are analyzed the stress discontinuity and thermal deformation of the FGM beam with two piezoelectric layers as actuator and sensor by using a finite element method solution. They revealed that the stress singularity and thermal deflection at the interface of the FGM layer and Piezoelectric layers can be controlled by using an exponential function for FGM material properties. Y. Kiani et al. [25] studied buckling analysis of FGM beam with piezoelectric layers under thermo-mechanical loading.
They considered the power-law function to show material properties of the FGM layer. The results of their work showed the effect of actuator voltage on buckling behavior of the Euler-Bernoulli beam. The present work examines two models of static control of an FGM cantilever beam under concentrated load at the free end of the beam. In the first model, an FGM beam containing distributed piezoelectric actuators on top and bottom is developed. The second model containing a cantilever FGM beam with one-layer piezoelectric actuator at the free end of the beam is considered. The both models are based on the classical beam theory and using integration method to solve. The material properties of the FGM layer is considered as an exponential function. Poisson’s ratio is assumed to be constant. In order to control the deflection of the composite beam, applying a voltage to piezoelectric actuators to control the cantilever beam deflection is provided a suitable relationship for the two model beams which are introduced.

2. Extracting Governing Equations

The displacement field of components of Euler-Bernoulli beam (EBT) are considered in the form below [26]:

\[ u(x, z) = -z \frac{dw_0(x)}{dx} \]

\[ w(x, z) = w_0(x) \]

Therefore, the axial strain component on the neutral axis of the EBT beam is

\[ \varepsilon_{xx} = \frac{du}{dx} = -z \frac{d^2w_0}{dx^2} \]

The stress resultant on a cross-section of the beam by using the linear elastic constitutive relation for an FGM as

\[ \sigma_{xx} = E_{j_0} \varepsilon_{xx} = E_{j_0} \left( -z \frac{d^2w_0}{dx^2} \right) \]

2.1. Material properties of FGM

The material properties of the FGM layer varying continuously based on exponential function across the thickness. Thus, the exponential modulus of elasticity of the FGM layer can be written as [27, 28]:

\[ E_{j_0}(z) = E_e e^{\left[ \frac{1}{h} \left( \frac{E}{E_e} - \frac{1}{2} \right) \right]} \]

2.2. Piezoelectric Equations

The converse equation of piezoelectric [29] is

\[
\begin{bmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{xy}
\end{bmatrix} =
\begin{bmatrix}
Q_{11} & Q_{12} & Q_{16} \\
Q_{21} & Q_{22} & Q_{26} \\
Q_{61} & Q_{62} & Q_{66}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_{xx} \\
\varepsilon_{yy} \\
\gamma_{xy}
\end{bmatrix}
- \begin{bmatrix}
0 & 0 & e_{31} \\
0 & 0 & e_{32} \\
0 & 0 & e_{36}
\end{bmatrix}
\begin{bmatrix}
E_x \\
E_y \\
E_z
\end{bmatrix}
\]

Where \( \sigma_j, Q_i, \varepsilon_j, \gamma_j, \) and \( E_i \) are the stress vector, the elastic coefficients, the strain vector, the piezoelectric coefficients, and the electric field vector respectively. The applied voltages (V) on the actuator layers are considered along their thicknesses (\( h_i \)). Therefore, the electric field vectors are defined as follows:
\[
\{ E \} = \begin{bmatrix} 0 & 0 & \frac{1}{h_i} \end{bmatrix}^T V
\]

Substituting Equations (3) and (7) into Equation (6), the converse piezoelectric equation of Euler-Bernoulli beam is as below:

\[ \sigma_{xx} = Q_{11} \varepsilon_{xx} + \epsilon_{31} E_z \]

The elastic coefficient is given as

\[ Q_{11} = \frac{E_p}{1 - \nu_1 \nu_2} \]

2.3. Governing Equation of beam

Equilibrium equations of an element of Euler-Bernoulli beam under transverse distributed load is [30]

\[
\begin{align*}
\sum F_z &= 0 \quad \Rightarrow \quad -\frac{dV_z}{dx} = q(x) \quad \text{(10)} \\
\sum M_y &= 0 \quad \Rightarrow \quad V_i - \frac{dM(x)}{dx} = 0 \quad \text{(11)}
\end{align*}
\]

\( M(x) \) is the bending moment and it can be defined in term of the stress (Equation (4)) on a cross section of the beam.

\[
M(x) = \int_A \sigma_{xx} z dA = -D_{eq} \frac{d^2 w_0}{dx^2} \quad \text{(12)}
\]

Substituting Equation (12) into Equation (11) and then substituting Equation (11) into Equation (10), the fourth order governing equation of FGM beam based on Euler-Bernoulli assumption is

\[
D_{eq} \frac{d^4 w_0}{dx^4} = q(x) \quad \text{(13)}
\]

In order to analyze the classical FGM beam with the embedded piezoelectric actuator, two models are considered:

**The Model I**: FGM cantilever beam as shown in Figure 1 with distributed piezoelectric actuator on top and bottom along the length of the beam subjected to concentrated force on the free end. Therefore, the governing equation of this beam is

\[
D_{eq} \frac{d^4 w_0}{dx^4} = 0 \quad \text{(14)}
\]

\[
D_{eq} = E_p I_f + 2E_s I_u
\]

By integrating four times from Equation (14), the governing equation is as follows:

\[
D_{eq} w(x) = \frac{c_1}{6} x^6 + \frac{c_2}{2} x^4 + c_3 x + c_4
\]
Boundary conditions of case I are as follow:

\[ w(0) = 0, \quad \frac{dw}{dx}(0) = 0 \]
\[ \frac{d^2 w}{dx^2}(l) = \frac{M_r^a + M_b^a}{D_{eq}}, \quad \frac{d^3 w}{dx^3}(l) = \frac{F}{D_{eq}} \]

The values of \( M_r^a \) and \( M_b^a \) can be obtained by using Equation (8) are

\[ M_r^a = -be_1a_iV_i \]
\[ M_b^a = -be_1a_iV_b \]

Using the boundary conditions in the Equation (15), the following equation is obtained.

\[ \Delta V = V_r - V_b = \frac{2}{be_1a_i} \left( -D_{eq}w(x) - \frac{F}{6}x^3 + \frac{Fl}{2}x^2 \right) \]

To find the voltage difference needed to set zero deflection of FGM beam at any desired point, in the Equation (18) have to substitute zero deflection and desired length of the beam respectively.

![Figure 1. FGM cantilever beam with piezoelectric actuators under concentrated force.](image)

The Model II: FGM cantilever beam as shown in Figure 2 with piezoelectric actuator on the bottom and the free end of the beam subjected to concentrated force on free end. Therefore, the governing equation of this beam is

\[ D[w(x)] = \frac{c_1}{6}x^3 + \frac{c_2}{2}x^2 + c_3x + c_4 \quad 0 \leq x \leq x_i \]
\[ D_j = E_{ji}I_j \]
\[ D_{eq}[w(x)] = \frac{c_1}{6}x^3 + \frac{c_2}{2}x^2 + c_3x + c_4 \quad x_i \leq x \leq l \]
\[ D_{eq} = E_{ji}I_j + E_{ai}I_a \]

Boundary conditions of Equations (19) and (20) are as follow:
The conditions of continuity of two Equations (19) and (20) are

\[
\begin{align*}
& w(x^-) = w(x^+) = 0, \quad \frac{dw}{dx}(x^-) - \frac{dw}{dx}(x^+) = 0 \\
& \frac{d^2 w}{dx^2}(x^-) - \frac{d^2 w}{dx^2}(x^+) = \frac{M^e}{D_{eq}}
\end{align*}
\]

By substituting the boundary conditions (Equation (21)) and the conditions of continuity (Equation (22)) into Equations (19) and (20), the voltage applied to the actuator to control the deflection of any arbitrary point of the FGM beam are as follow:

\[
V = \frac{-1}{b e_{st} d_a x^2} \left( D_f w(x) - \frac{F}{6} x^3 + \frac{F}{2} x^2 \right) \quad 0 \leq x \leq x_i
\]

\[
V = \frac{-1}{b e_{st} d_a} \left( \frac{x^2}{2} + x_i x - \frac{x_i^2}{2} \right) \left( D_{eq} w(x) - \frac{F}{6} x^3 + \frac{F}{2} x^2 \right) \quad x_s \leq x \leq l
\]

Figure 2. FGM cantilever beam with piezoelectric actuator at free end under concentrated force.

3. Results and Discussions
Two control models are presented in section 2 for controlling the deflection of composite cantilever beam (Equations (18) and (23, 24)). Based on these two models, it is possible to control the deflection of the FGM beam at any specific point. The geometrical dimensions and material properties of the piezoelectric and FGM layers used in static analysis of the composite beam are presented in Table 1. Figures 3 and 4 show the deflections of FGM beam with and without control model I. Since this model consist of two piezoelectric layers as actuators on top and bottom of the FGM beam, therefore, the
voltage difference between these two layers is used to control the deflection of a specific point of the beam. Figures 3 and 4 show the deflections of free end and point $x = 0.5$ of the FGM beam which are zero for the voltage difference $\Delta V = -530.4266$ and $\Delta V = -663.0332$ respectively. Figures 3 and 4 also demonstrate the control deflection at the free end and point $x = 0.5$ of the FGM beam for different voltages respectively.

**Table 1.** Dimensions and properties of FGM and piezoelectric materials

| Parameter | Quantity | Parameter | Quantity | Parameter | Quantity |
|-----------|----------|-----------|----------|-----------|----------|
| $a_h$ (m) | 0.0002   | $V_{12} = V_{21}$ | 0.3      | $x_f$ (m) | 0.7      |
| $h$ (m)   | 0.004    | $E_a$ (GPa) | 63       | $e_{31}$ (m/V) | $-190 \times 10^{-12}$ |
| $l$ (m)   | 1        | $E_m$ (GPa) | 10       | $F$ (N)   | 1        |
| $b$ (m)   | 0.05     | $E_c$ (GPa) | 100      |           |          |

**Figure 3.** Control the free end of the beam in the model I for different voltages.

**Figure 4.** Control the deflection of point ($x = 0.5$) in model I for different voltages.
Figures 5 and 6 show the deflections of the FGM beam with and without control model II. Since this model consists of one piezoelectric layer as an actuator at the free end of the FGM beam, therefore, the FGM beam has two segments. The first segment \( (x < x_f) \) of the FGM beam that has no piezoelectric layer and the second segment \( (x \geq x_f) \) has a piezoelectric layer. In order to control the deflection of the FGM beam, if the desired point is in segment one the Equation (23) is used. If the specific point is in the second segment the Equation (24) is used. Figures 5 and 6 show the deflections of free end and point \( x = 0.5 \) of the FGM beam which are zero for the voltage \( V = -277.7103 \) and \( V = -331.5166 \) respectively. Figures 5 and 6 also demonstrate the control deflection at the free end and point \( x = 0.5 \) of the FGM beam for different voltages respectively.

By comparing the figures of the model I with those of model II, it is clear that the model II is much more efficient. As the second model requires less applied voltage and the much smaller piezoelectric layer. In this study, only mainly about the static control effect of two piezoelectric models on the FGM beam has been investigated and the advantages of using FGM in control of structural behavior with piezoelectric materials have not been addressed. Therefore, the author intends to research this area in the future.
4. Conclusion
The piezoelectric effect on a functionally graded composite beam with cantilever condition subjected to a concentrated force at the free end is studied. The material properties of the FGM beam were considered based on exponential function. This study proposes two models to control the static deflections of thin composite beams with piezoelectric layers. In the first model, the composite beam has two distributed piezoelectric layers along the length on top and bottom of it. The second model, the composite beam has one layer piezoelectric on the free end. By considering the inverse piezoelectric effect on the thin composite beam, the following items are observed that,

- The converse equation of piezoelectric can be used to control any point of the beam in static conditions.
- In the first control model, the deflection of each point of the beam is directly related to the difference in applied voltage to the actuators on top and bottom.
- The applied voltage to the actuator is directly related to the deflection at each point of the beam and inversely related to squared distance the deflection point to the fixed edge of the beam.
- The second control model is much more efficient than the first control model.

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