Entanglement-Entropy for Groundstates, Low-lying and Highly Excited Eigenstates of General (Lattice) Hamiltonians

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Abstract

We investigate the behavior of entanglement-entropy on a broad scale, that is, a large class of systems, Hamiltonians and states describing the interaction of many degrees of freedom. It is one of our aims to show which general characteristics are responsible for the different types of quantitative behavior of entanglement-entropy. Our main lesson is that what really matters is the degree of degeneracy of the spectrum of certain nearby reference Hamiltonians. For calculational convenience we study primarily systems defined on large but finite regions of regular lattices. We show that general vector states, being not related to some short-range Hamiltonian do not lead in the generic case to an area-like behavior of entanglement-entropy. The situation changes if eigenstates of a Hamiltonian with short-range interactions are studied. We found three broad classes of eigenstates. Global groundstates typically lead to entanglement-entropies of subvolumes proportional to the area of the dividing surface. Macroscopically excited (vector)states have in the generic case an entanglement-entropy which is proportional to the enclosed subvolume and, furthermore, display a certain Gibbsian behavior. Low-lying excited states, on the other hand, lead to an entanglement-entropy which is usually proportional to the logarithm of the enclosed subvolume times the area of the dividing surface. Our analysis is mainly based on a combination of concepts taken from the perturbation theory of Hamiltonians and certain insights coming from the foundations of quantum statistical mechanics.
1 Introduction

The microscopic origin of entropy in, for example, black hole physics and the so-called area law is still to some extent kind of a mystery. Or more precisely, there exist a variety of different explanations. For a nice discussion in form of a Galilean trialogue see [1]. Another very readable overview is given in [21], a collection of theses can also be found in [3]. Of perhaps even more interest is the proposed holographic principle (and the range of its validity), which discusses the emergence of entropy on a very fundamental level of physics, that is, “empty” space (-time) and its vacuum fluctuations. We do not attempt to give a complete list of references (as we focus in the following on a related but slightly different question), cf. for example the nice review [4] and the many references therein.

What is at least clear is that the black hole horizon divides space-time into exterior and interior regions. Hence, one might venture the idea that a basic role in this question should be played by some form of entanglement of two (in some macroscopic respect) separated regions. Such a form of entanglement can however be realized on different levels of our theoretical description, ranging from more ordinary ones (standard quantum mechanics or quantum field theory) to more pristine ones (e.g. the notorious Planck scale). On this point people widely disagree at the moment. As to our personal point of view, we think that the holographic principle indicates a drastic change of the statistical mechanical preassumptions usually taken for granted on the more ordinary scales as e.g. locality, good clustering conditions or sufficient decay of correlations etc., if one enters the fundamental regime of the Planck scale. Some very sketchy remarks can be found at the end of [5], a more detailed analysis is forthcoming ([6]). Be that as it may, one should emphasize the following:

Observation 1.1 The following two situations are different from a logical point of view, for one, entanglement-entropy induced by a fixed global state as e.g. the groundstate of a given Hamiltonian, for another, the maximally possible quantum of entropy or information, $S_{\text{max}}$, which can be stored in a given volume, the question which is addressed in the holographic principle. In the latter case also the highly excited states of a Hamiltonian should be investigated, a problem we explicitly address in the following sections.

In any case, in a first step, it seems to be useful to investigate ordinary models of quantum (field) theory and see to what extent their entropic behavior reflects some of the properties condensed in notions like area law and/or holographic principle. A certain kind of area-law-like behavior shows
up already in ordinary quantum field theory, see e.g. [7] or [8]. Similar phenomena were also observed in [9], [10] and more recently in [5]. As this is an interesting point, we will treat it in more detail elsewhere.

To make the problem more accessible, we start with the investigation of systems of many degrees of freedom living on a large regular lattice (of arbitrarily small but finite lattice constant). In [11] or [12] or the more recent [13] large arrays of coupled harmonic oscillators and their respective groundstates have been studied, with some sort of continuum limit performed in the end, yielding the vacuum state of e.g. a Klein-Gordon field theory. Note however that taking a continuum limit is quite delicate since, in order to get a finite result, some form of cut-off is necessary (as to other possible interesting applications to quantum field theory see for example [14]).

As everybody knows, the harmonic oscillator is an extremely well-behaved quantum system. The same holds for regular arrays of coupled harmonic oscillators. There exists a large arsenal of methods to extract useful information from these model systems. But nevertheless, the question of entanglement-entropy of e.g. the groundstate of this well-behaved model system turns already out to be quite intricate. This may, among other things, have its roots in the observation that the concept of entanglement-entropy and its quantitative behavior is not so easy to visualize, one reason being, that it is not a truly local concept.

It was found that if one divides a large volume, \( V \), into \( V_1 \cup V_2 \) and restricts the groundstate, \( \Psi_0 \), over \( V \) to the subvolumes \( V_1, V_2 \), the corresponding density matrices, \( W_1, W_2 \), have an entropy, \( S = -\sum w_i \ln w_i \), which is proportional to the area of the dividing surface between \( V_1 \) and \( V_2 \). One may therefore speculate that this observation has something to do with the area law in black hole physics and corresponding arguments were advanced in the two above cited papers.

If one pursues such an idea, various questions immediately suggest themselves.

i) In what respect is the groundstate of the Hamiltonian of a large system (i.e., many degrees of freedom being involved) a particular state in this special context. Put differently, what particular property of a global vectorstate (pure state) influences the entropic behavior of the partial traces, \( W_1, W_2 \)?

ii) In what respect does the picture change if we go over to lowly or highly excited eigenstates of such a Hamiltonian?

iii) Is the result, found for an array of harmonic oscillators, generic, that is, is it to a high degree independent of the particular model system being studied and what are the really important prerequisites?

These are the questions we will adress in the following sections. A prelimi-
nary qualitative answer may perhaps already be given. Perhaps a little bit surprisingly, what really seems to matter is the degree of degeneracy of the spectral values of certain nearby reference Hamiltonians. For groundstates, for example, this means that the area-behavior is a result of their usual non-degeneracy.

We begin our investigation with defining the general framework of systems, living on large regular lattices. We then discuss in section 3 several general properties of the notion of entanglement-entropy. We mention for example that a Gibbs state over $V_1$ can always be extended to a pure vector state, $\Psi$, on $V$ by enlarging the system on $V_1$ with $V = V_1 \cup V_2$. That is, the Gibbs state is now the partial trace of a global vectorstate. As, typically, such Gibbs states have an entropy which is proportional to the volume, $V_1$, this shows that, in general, entanglement-entropy can not be proportional to the area of the surface, dividing $V_1$ and $V_2$. We learn from this simple observation that, in order to get such a special result, we do need some more specifications.

This observation teaches us yet another lesson. One sometimes hears the argument that entanglement-entropy has to be proportional to the area of the dividing surface as this is the only quantity which is the same for both subvolumes, $V_1$ and $V_2$ while the volumes themselves can of course differ from each other (note that, perhaps a little bit surprisingly, entanglement-entropy is always the same for $V_1$ and $V_2$). This argument is obviously incorrect. What in general may happen is that we have something like the following

$$S(W_1) = \alpha_1 \cdot |V_1| = \alpha_2 \cdot |V_2| = S(W_2)$$

with $W_1, W_2$ the density matrices representing the total state, $\Psi$, on $V_1, V_2$ respectively. That is, both entropies may happen to depend linearly on the volume but with certain (volume-dependent) adjusting prefactors. As long as the volumes are finite, there is no unique answer as to the geometric dependencies. We need some kind of thermodynamic limit. In [13] one kind of such limit is discussed. One can, for example, keep the dividing surface fixed and let the subvolumes approach infinity in the orthogonal direction, or, on the other hand, it may become necessary, to scale them in a prescribed way with the prefactors remaining basically constant. In the former case we encounter a linear dependence on the area of the dividing surface if the limit is finite, in the latter (scaling) case we may get a linear asymptotic dependence of $S(W_i)$ on the volume.

At the end of that section we introduce a certain coarse-grained version or approximation of the notion of entanglement-entropy for the subvolumes
$V_1, V_2$, which, in some sense, resembles kind of a microcanonical ensemble version of entropy. In our particular case it is related to the logarithm of the dimension of certain Hilbert subspaces onto which the pure vectorstate happens to be essentially projected when calculating the relative trace. This is an important simplification: While the original standard notion of entropy, $S := -\sum w_i \cdot \ln w_i$, is deceptively simple at first glance, this is a very misleading impression. Even in the few cases where the Boltzmann weights, $w_i$, are known, what is usually more important is their (frequently huge) degeneracy and/or the local density of states, which in general are not at our disposal.

Remark: We recently learned that this approximation scheme we will discuss below has a certain resemblance to an approximation method used in the density matrix renormalisation group, DMRG and which can roughly be described as finding the essential subspace. It is also of relevance in data compressing and the like (see e.g. [31], [32], [33], [34]).

While we attempt to treat the problem of entanglement-entropy on quite a broad scale (as far as space dimension and class of hamiltonians are concerned), there are interesting topics we do not address. In our scenario we deal with a spatial tensor product structure of the underlying Hilbert space. Most notably in condensed matter physics there exist applications where other kinds of configurations prevail. One example is the case of fermionic quantum particles where the antisymmetrization postulate prevents such a representation. In that situation one can for example work in a mode representation; (basically Fock space) and discuss entanglement in such a mode space. Some work in this direction may be found in the review article [15], [16], [17] or [18]. We think that parts of our strategy can also be applied in this field of research but this is not done in the present paper.

Recently, Shi ([19]) attempted to extend such a mode-version of entanglement to relativistic quantum field theory. This is certainly an interesting programme but we would like to hint at a number of impending difficulties and obstacles. In free field theories and Fourier space, occupation number representations for example have a well-defined meaning. In coordinate space and, a fortiori, interacting theories such an approach is met with great difficulties due to the intricacies of the relativistic quantum regime. For one, expressions like $\phi^\dagger(x)\phi(x)$ are no longer operators but operator-valued distributions. The same applies to eigenstates, “localized at $x$”. What one can use instead are localized algebras of observables living in regions $O$ of spacetime. One may then restrict e.g. the vacuum state to such local algebras,
thus getting a mixed state.

It is however obvious that in general a naive splitting into, say, a bipartite system, having kind of a tensor-product structure, does not work (due to quantum non-locality and other problems; a catchword being the Reeh-Schlieder Property). In any case, the mathematics becomes extremely involved as most of the naive expressions become infinite (for example traces). Only for a few scenarios one can say more (like e.g. the Rindler-wedge). The interested reader may consult section V,5 of the book by Haag ([20],[14] or [21] for a general orientation.

Furthermore, our arguments rely to a certain extent (at least in the form presented here) on short-range correlation and short-range interactions. This is the field where one may expect that results are generic to some extent. Therefore our arguments do not (immediately) apply to quantum critical systems, i.e. systems displaying quantum phase transitions at temperature zero. These are presently mostly studied in low dimensions (space dimension frequently being one) by employing usually quite model-specific techniques. An incomplete selection of papers is [22],[23],[24],[25],[26],[27],[28].

Given the generality of our admitted class of models, it is not our aim and out of the question to really compute in every case the complete set of eigenvalues and their degeneracy or spectral density. This holds the more so for the v.Neumann entropy. We rather view the merit of our approach in providing a general scheme which only relies on a few characteristics of the respective models. As a consequence, our arguments and estimates do not yield precise numerical results but typically make statements about entanglement-entropy in some leading order (for more details see below).

For the same reason (a large class of different admissible Hamiltonians) we develop in section 4 in quite some detail various concepts belonging to the perturbational analysis of Hamiltonians. This turns out to be a very intricate subject matter in our context. The main reason is that, due to the dense distribution of eigenvalues typical for Hamiltonians of a large number of degrees of freedom, ordinary perturbation theory is only of a limited value.

In the last three sections we then deal with our main topic in a more quantitative way. In section 5 we treat the case of the groundstate of a Hamiltonian with finite range interactions. In the following section we analyse the situation for eigenstates which are highly excited, that is, for energies which are a macroscopic distance apart from the groundstate energy. In the last section we study, on the other hand, eigenvalues of the Hamiltonian which lie in the vicinity of the groundstate energy. We find the following:

i) Groundstates come with an entanglement-entropy which is proportional
to the area of the dividing surface.

ii) Highly (i.e. macroscopically) excited states lead in the generic case to entanglement-entropies which are proportional to the volume of the system. The notion generic is explained in the respective section. Furthermore, the restriction of the pure vectorstate to the respective subvolumes share a couple of properties with ordinary temperature states. As for more detailed qualifications see the corresponding section.

iii) Low-lying excited eigenstates states have an entanglement-entropy which is roughly proportional to the logarithm of the corresponding volume times the area of the dividing surface.

To sum up what we think have been important ingredients in our analysis, the arguments strongly rely on the finite range of the interactions, that is some version of locality. Furthermore, we assume the system to be away from a (quantum-)critical state where local degrees of freedom are typically long-range correlated. In this special regime results happen to be somewhat model dependent and our more general arguments need not automatically apply. As to the applied methods, two tools stand out. First, a combination of various perturbational arguments, second, a couple of technical ideas which have proved their worth already in the foundations of statistical mechanics.

We want to add a further remark. One might perhaps get the idea that high entanglement is necessarily related to long-range correlations. That this is not the case can be inferred from our following investigations. It is for example easy to construct a highly entangled eigenstate of a short-range Hamiltonian. What is only important is that the respective eigenvalue happens to be highly degenerate or that the full Hamiltonian is the perturbation of such a Hamiltonian with highly degenerate eigenstates. This is exploited in section 6. Note that, typically, most eigenvalues of Hamiltonians of many degrees of freedom are highly degenerate, an exception being usually the ground state. For general vector states there is even less correlation between entanglement and range of correlations. An interesting observation in this direction is for example also made in [29] and [30] where these two notions are compared with each other.

A related line of ideas is pursued in [35]. Again Shi concentrates mainly on mode expansions of many body systems and studies in particular the regime of phase transitions and spontaneous symmetry breaking. For example, in the case of Bose-Einstein condensation he remarks that a macroscopic occupation of the ground state leads to what he calls disentanglement with the rest of the modes. This is reasonable as the ground state occupation number operator becomes almost a classical observable in this situation. He
further addresses the relation between entanglement and range of correlations in this context. It conforms with our observations that long-range correlations make it in general difficult to divide a system into, for example, a weakly correlated bipartite system, while high entanglement does not necessarily imply long-range correlations.

2 Some General Framework and Estimates

Concerning this section a standard reference is for example [36]. To avoid irrelevant complications it is reasonable to choose as the general context discrete dynamical models living on a regular lattice, $\mathbb{Z}_a^d$, with lattice constant $a$ and space dimension $d$. The points of the lattice are denoted by $x_i$, the Hilbert space sitting at $x_i$ by $\mathcal{H}_i$. $\mathcal{H}_i$ may be finite or infinite dimensional (as is for example the case for an array of harmonic oscillators). If all the $\mathcal{H}_i$ have the same finite dimension it is denoted by $D$.

In the following we are interested in large but finite subsets of $\mathbb{Z}_a^d$, denoted by $V$. For convenience $V$ may be chosen as a $d$-dimensional cube of side length $L$, containing $N_V = (L/a)^d$ lattice points. The general case of regions of arbitrary shape can of course be treated in more or less the same way apart from some (irrelevant) numerical and technical details. The Hilbert space, $\mathcal{H}_V$, over $V$ is the tensor product

$$\mathcal{H}_V = \bigotimes_{x_i \in V} \mathcal{H}_i$$

spanned by tensor-monomials

$$v = v_1 \otimes \cdots \otimes v_{N_V}, \ v_i \in \mathcal{H}_i$$

If one chooses a basis, $\{e^{l_i}_i\}$, in each $\mathcal{H}_i$, general vectors in $\mathcal{H}_V$ are linear combinations of the basic vectors

$$e^{l_1}_1 \otimes \cdots \otimes e^{l_{N_V}}_{N_V}, \ 1 \leq l_i \leq D$$

(in case the dimension of all $\mathcal{H}_i$ is finite and equal). Then the dimension of $\mathcal{H}_V$ is $D^{N_V}$. The scalar product on $\mathcal{H}_V$ is given by

$$(v_1 \otimes \cdots \otimes v_{N_V} | v'_1 \otimes \cdots \otimes v'_{N_V}) := \prod_{x_i} (v_i | v'_i)$$

and (multi)linearly extended.
Hamiltonians are now defined as follows. If $A_i$ are operators (or matrices) on $\mathcal{H}_i$, operator monomials on $\mathcal{H}_V$ are given by

$$\bigotimes_{x_i} A_i : \bigotimes v_i \mapsto \bigotimes A_i \circ v_i$$

and (multi)linearly extended on more general vectors. Refraining from a too abstract approach we restrict ourselves to Hamiltonians given by a sum over one-point, two-point, ..., $k$-point-interactions. This means

$$\sum_{x_i} \Phi^1(x_i), \sum_{x_i,x_j} \Phi^2(x_i,x_j), \ldots$$

where

$$\Phi^1(x_i) = A^{(1)}_i, \Phi^2(x_i,x_j) = A^{(2)}_i \otimes A^{(2)}_j, \ldots$$

**Assumption 2.1** To avoid unneccessary complications we assume that only interaction terms up to a certain finite order occur in the Hamiltonian and that all the occurring operators $A_i$ are hermitean.

As all the $A_i,A_j$ do commute for $i \neq j$, the interaction terms defined above are hermitean as is their sum. Typical examples are spin systems. One of the simplest examples is

$$H = J \cdot \sum_i \vec{\sigma}_i \otimes \vec{\sigma}_{(i+1)} =: J \cdot \sum_i \vec{\sigma}_i \vec{\sigma}_{(i+1)}$$

with $\sigma$ denoting the usual Pauli-matrices and extension to higher dimensions being straightforward.

Usually one makes the additional assumption that the interaction has a finite range and is translation invariant.

**Definition 2.2** The interaction is called translation invariant if with $\Phi^k(x_{i1},\ldots,x_{ik})$ also $\Phi^k(x_{i1} + r,\ldots,x_{ik} + r)$ occurs in the Hamiltonian with $r$ some lattice vector and both being the same operators.

**Definition 2.3** The interaction is called to have a finite range, $\delta$, if with one lattice point fixed, e.g. $x_{i1}$, only finitely many members $\Phi^k(x_{i1},y_{i2},\ldots,y_{ik})$ different from zero exist in $H$ when the $y_{ij}$ vary over $\mathbb{Z}_d$ with all $y_j$ having a distance from $x_i$ which is $\leq \delta$. 
We not that we give these details only for consistency reasons, whereas only the general properties of such Hamiltonians will be of relevance in the following like their translation invariance and the finite range.

What will be of interest for the following discussion is the restriction of a general Hamiltonian, \( H \), to a certain subvolume \( V \subseteq \mathbb{Z}_d \) or the corresponding sub-Hilbert-space \( \mathcal{H}_V \).

**Definition 2.4** The restriction of a general Hamiltonian, \( H \), of the kind described above to a subvolume \( V \subseteq \mathbb{Z}_d \) or \( \mathcal{H}_V \) is the operator \( \mathcal{H}_V \) which consists of all the interaction terms which lie in \( V \), i.e., only \( k \)-tuples \( (x_{i_1}, \ldots, x_{i_k}) \) are admitted with all the \( x_{i_l} \in V \).

\[
H_V := \sum_{k; x_{i_l} \in V} \Phi^k(x_{i_1}, \ldots, x_{i_k}) =: \sum_{n(V)} \Phi^k \tag{10}
\]

(\( n(V) \), the number of terms in the sum being of order \( O(|V|) \)).

Another important part of the total \( H \) relative to a given volume \( V \) is the boundary contribution \( H_{V}^{bd} \).

**Definition 2.5** By \( H_{V}^{bd} \) we denote the part of the total \( H \) which consists of all interaction terms which have both lattice points in \( V \) and the dual set \( V' := \mathbb{Z}_d \setminus V \).

\[
H_{V}^{bd} := \sum_{k; x \in V, y \in V'} \Phi^k(x_{i_1}, \ldots; y_{i_1}, \ldots, y_{i_k}) \tag{11}
\]

**Observation 2.6** It will be important in the following that with the Hamiltonian having finite range and \( k \leq K \), the number of terms in \( H_{V}^{bd} \) is of order (area of boundary of \( V \)). We denote this number of terms by \( n(V,V') \).

We can make more detailed statements if we concentrate on the large subclass of models with all the \( \Phi^k \) bounded operators (which holds for example if all the \( \mathcal{H}_i \) are finite dimensional). This together with the assumed translation invariance and the above observation allow us to make the following important operator estimates.

**Proposition 2.7** Under the preceding assumptions we have

\[
\| H_V \| \leq C \cdot n(V), \quad \| H_{V}^{bd} \| \leq C' \cdot n(V,V') \tag{12}
\]

where in the generic case, which we usually assume to prevail, both norms are actually proportional to the volume, the area of the boundary, respectively.
In case the local Hilbert spaces \( \mathcal{H}_i \) are finite dimensional, all the occurring Hamiltonians are large but nevertheless finite hermitean matrices for finite \( V \), hence having a discrete spectrum, with the number of eigenvalues (counting multiplicity) being the same as the dimension of the respective Hilbert spaces (the eigenvalues being the zeros of the characteristic polynomial).

It is obvious that this latter observation greatly simplifies the quantitative analysis.

We will in the following mainly discuss the case where the local Hilbert spaces have finite dimension. As the extension of the results to the case where the local Hilbert spaces, \( \mathcal{H}_i \) have infinite dimension is not entirely straightforward (like e.g. arrays of coupled harmonic oscillators), we make some corresponding remarks in appendix A.

3 Some Properties of the Partial Trace and Entanglement Entropy

Let us take a vector state, \( \Phi \), from the Hilbert space \( \mathcal{H}_V \) with \( V \) large (or macroscopic). Let us divide \( V \) into \( V_1 \cup V_2 \) with the respective Hilbert spaces, \( \mathcal{H}_1, \mathcal{H}_2 \). We have

\[
\mathcal{H}_V = \mathcal{H}_1 \otimes \mathcal{H}_2, \quad \mathcal{A}_V = \mathcal{A}_1 \otimes \mathcal{A}_2
\]  

(13)

We can evaluate the pure state, \( \Phi \) on the restrictions \( \mathcal{H}_1, \mathcal{H}_2 \):

\[
\omega_1(A^{(1)}) := (\Phi| (A^{(1)} \otimes 1)\Phi) , \quad \omega_2(B^{(2)}) := (\Phi| (1 \otimes B^{(2)})\Phi)
\]  

(14)

which defines the states \( \omega_1, \omega_2 \) on \( \mathcal{H}_1, \mathcal{H}_2 \). Choosing a basis \( e_i \otimes e'_j \) in the tensor product \( \mathcal{H}_V = \mathcal{H}_1 \otimes \mathcal{H}_2 \) we get

\[
\Phi = \sum c_{ij} e_i \otimes e'_j
\]  

(15)

and

\[
(\Phi| (A \otimes 1)\Phi) = \sum_{m,i,j} \bar{c}_{im} c_{jm} (e_i | A e_j) =: \sum_{i,j} b_{ji} (e_i | A e_j)
\]  

(16)

with

\[
b_{ji} = \sum_{m=1}^{\text{dim}(\mathcal{H}_2)} c_{jm} \bar{c}_{im}
\]  

(17)

This can be rewritten as

\[
\omega_1(A) = Tr(W_1 \cdot A) \quad \text{with} \quad W_1 := \sum_{ij} b_{ji} |e_j > < e_i|
\]  

(18)
The following result is remarkable and leads to the definition of the notion of entanglement-entropy.

**Theorem 3.1** The partial traces $W_1, W_2$ have the same spectrum and the eigenvalues $\neq 0$ have the same multiplicity while the respective zero-eigenvalues may have different degeneracies, depending on the in general different dimensions of $\mathcal{H}_1, \mathcal{H}_2$.

**Corollary 3.2** The (entanglement-)entropies of the states $\omega_1, \omega_2$ or $W_1, W_2$ are the same, i.e.

$$S_2(W_2) = S_1(W_1) := -\sum \lambda_i \cdot \ln \lambda_i$$

while the respective volumes or dimension of Hilbert spaces can be very different.

As the full proof of this important result is frequently omitted in the literature or is incomplete we give, for the convenience of the reader, our own version in appendix B.

On the other hand we know that the usual thermodynamic entropy is an extensive quantity and depends in general linearly on the volume. So let us assume we have a Gibbs-state on the volume $V_1$ with Hamiltonian $H_1$, its eigenvalues and eigenstates being $E_i, \psi_i$. That means:

$$\omega_1(A) := \text{Tr}(e^{-\beta H_1} \cdot A)/\text{Tr}(e^{-\beta H_1})$$

with $\beta$ the inverse temperature $1/kT$. It follows

$$S_1 = -\sum p_i \ln p_i \cdot p_i = e^{-\beta E_i}/\sum e^{-\beta E_i}$$

and

$$S_1(V_1) \sim V_1$$

We now adjoin a volume $V_2$ with a Hilbert space $\mathcal{H}_2$ of sufficiently high dimension. We define the following vector, $\Psi$, in $\mathcal{H}_1 \otimes \mathcal{H}_2$.

$$\Psi := \sum \sqrt{p_i} \psi_i \otimes e_i$$

($e_i$ spanning a basis in $\mathcal{H}_2$ and degeneracies being included). We have

$$(\Psi|A^{(1)} \Psi) = \text{Tr}(e^{-\beta H_1} \cdot A^{(1)})/\text{Tr}(e^{-\beta H_1})$$

We conclude:
Observation 3.3  Every Gibbs state over $A_1$ can be represented by a vector state in a sufficiently large ambient Hilbert space. Restricted to $H_1$ this vector state is the partial trace, but we see from our above reasoning that the entanglement-entropy of $\Psi$ relative to $V_1$ is now proportional to the volume $V_1$.

It is important for the physical understanding and intuition to get a better feeling how entanglement-entropy is affected in both a quantitative and qualitative way by different physical conditions. It seems to be useful to introduce concepts which are, at least in a rough sense, of a similar character as entanglement-entropy because the latter can only be calculated in very few special cases. We think in this context of the relation between, for example, the canonical and the microcanonical ensemble in statistical mechanics.

For one, it is obvious that the entropy of a state, $\Psi_0$, reduced to, say, $V_1$ or $V_2$, does not depend on the choice of Hilbert space bases in the corresponding Hilbert spaces. So, in the following, we will frequently subdivide $V_1$ or $V_2$ further into, say, $V_1 = V_1' \cup V_1''$ with $V_1''$ a boundary layer in $V_1$ neighboring upon the interface, separating $V_1$ and $V_2$. One can then equally well choose a basis in $H_1$ by forming the tensor product of the respective bases in $H_1'$ and $H_1''$ etc.

For another, a famous theorem of E.Schmidt and von Neumann ([41],[42]) states that $\Psi_0$ in $H_1 \otimes H_2$ can be represented in the special form

$$\Psi_0 = \sum_i \sqrt{\lambda_i} \cdot \phi_{i,1} \otimes \phi_{i,2}$$

with $\phi_{i,1}$, $\phi_{i,2}$ particular orthonormal bases in $H_1$, $H_2$ (the so-called Schmidt-basis; this was also exploited in [43]).

Remark: In modern parlance this is nothing but the theorem that a compact operator can be put into such a canonical spectral form. Note in this context that $\Psi_0$, viewed as an operator from $H_2$ to $H_1$, belongs to the Hilbert-Schmidt-class as a consequence of its normalisation as a vector.

$\Psi_0$ reduced to $H_1$ then yields:

$$(\Psi_0 | A^{(1)} \Psi_0) = \sum_i \lambda_i \cdot (\phi_{i,1} | A \phi_{i,1}) = Tr(W_1 \cdot A)$$

What is important in this particular representation is that both systems of vectors are orthonormal. With $\Phi = \sum_{ij} c_{ij} e_i \otimes e'_j$ we can of course always
write
\[ \Phi = \sum_i e_i \otimes \phi_i = \sum_j \psi_j \otimes e_j' \] (27)

with
\[ \phi_i = \sum_j c_{ij} e_j', \quad \psi_j = \sum_i c_{ij} e_i \] (28)

or
\[ \Phi = \sum_i \|\phi_i\| \cdot e_i \otimes \hat{\phi}_i = \sum_j \|\psi_j\| \cdot e_j' \otimes \hat{\psi}_j \] (29)

\(\hat{\phi}_i, \hat{\psi}_j\) being the corresponding unit vectors.

The reduction to \(\mathcal{H}_1, \mathcal{H}_2\) then yields:
\[ (\Phi|A^{(1)}\Phi) = \sum_j (\psi_j|A\psi_j), \quad (\Phi|B^{(2)}\Phi) = \sum_i (\phi_i|B\phi_i) \] (30)

or
\[ (\Phi|A^{(1)}\Phi) = \sum_j \|\psi_j\|^2 \cdot (\hat{\psi}_j|A\hat{\psi}_j), \quad (\Phi|B^{(2)}\Phi) = \sum_i \|\phi_i\|^2 \cdot (\hat{\phi}_i|B\hat{\phi}_i) \] (31)

Remark: Note that the state, given as an expectation functional, is of course identical to the state in e.g. formula (18). But now the way the state is concentrated in a subspace (either spanned by the \(\hat{\psi}_j\) or the \(\hat{\phi}_i\)) of the total space is made transparent. Note that the subspace may depend on the chosen basis in contrast to its dimension (see below)

Trying to relate the canonical version of the notion of entropy, \(S(W) := -\sum_i p_i \ln p_i\) with some other (perhaps coarser) concept, we were inspired by the beautiful analysis of the entropy concept, as it is laid out in [44], sect.7 of chapt.1. It is shown there that the natural quantity which is relevant in this context is the number of microscopic quantum states, \(\Delta \Gamma\), a macrostate is smeared over or, in other words, the number of microstates which essentially contribute in a macrostate (taken with equal weights). It is then shown in [44] that in the regime of equilibrium statistical mechanics, the logarithm of this quantity coincides with the canonical notion of entropy given above, but this result is far from trivial. We see that \(\ln \Delta \Gamma\), giving equal apriori weight to the members of a certain selected sample of quantum states, implements the philosophy of the microcanonical ensemble picture.

In the following sections we are primarily interested in leading-order-estimates, that is, estimates of quantities on a scale given by macroscopic volumes or by the area of some bounding surface etc. So, by inspecting
the above formulas, we see that a rough notion which reflects the number of “different” states being involved in the reduction of a vector state to a partial trace over a certain subvolume, is given by the dimension of the respective Hilbert subspace the contributing vectors are lying in. This however needs more qualifications.

In a first step we regard the above representation (formulas (27), (28) of the vector $\Phi$ as a map from the Hilbert spaces $H_1, H_2$ to the corresponding subspaces spanned by the (in general non-orthogonal) $\phi_i, \psi_j$. These maps (more properly, the respective image spaces) are given by $C = (c_{ij})$ and the transposed matrix via

\[ \hat{C} : e'_i \rightarrow \sum_j c_{ij} e'_j = \phi_i \quad \text{etc.} \quad (32) \]

From linear algebra we know that the dimension of these subspaces is given by the rank of the matrices $C, C^T$, with the ranks of $C$ and its transpose $C^T$ being equal. Identifying the matrix $C$ with the corresponding abstract map $\hat{C}$, it is obvious that this dimension cannot depend on the chosen basis. Or put differently, while the subspaces themselves may change under a change of basis, the dimension is a unitary invariant. Without taking appropriate measures, these subspaces may (in general) have an infinite or “too large” dimension. What we are rather implying is the possibility to find lower-dimensional essential subspaces.

**Observation/Definition 3.4** The dimension of the subspaces, spanned by $\phi_i$ or $\psi_j$ as image vectors under the maps $\hat{C}, \hat{C}^T$, applied to certain bases are equal and independent of the chosen bases in $H_1, H_2$. Furthermore the logarithm of the dimension is representing a measure of the entanglement entropy in leading order under certain favorable conditions. These conditions may be: All image vectors contribute with roughly the same strength, or, rather, their respective weights are not too different. Or, some of the matrix elements happen to be very small compared with the rest so that one can go over to a purified matrix; in particular, some of the image vectors may be very small and can be neglected. In all these cases one will get some approximation of the original entanglement-entropy.

**Corollary 3.5** Note that in the Schmidt-basis the dimension of the essential subspace can particularly easily be read of. It is the dimension of the subspace spanned by the eigenvectors of $W_1$ or $W_2$ with non-marginal eigenvalues $\lambda_i$. It can of course happen that there is no natural division between marginal and non-marginal eigenvalues. This is for example the case when
the entanglement entropy is proportional to the volume, a situation we will observe in the following.

In the following sections we will apply this approximative concept of entanglement-entropy.

4 Perturbation Theory of Hamiltonians and Entanglement

The preceding section shows clearly that the details of the dependence of (entanglement-)entropy on volume and/or area as a function of the type of global vectorstate are presumably subtle and intricate and need more qualifications. This holds the more so if one wants to discuss whole classes of models. Therefore, it is a natural idea to study eigenstates of some Hamiltonian as described above. In most of the examples we are aware of, groundstates of certain physical models have been studied. It hence suggests itself to extend this investigation and inspect both ground and excited states more closely and try to infer characteristics of the respective entanglement entropy from the (general) properties of such eigenstates.

A general method which suggests itself is perturbation theory of the Hamiltonians under discussion. The strategy is the following. We start from a Hilbert space, \( \mathcal{H}_V \), over a macroscopic volume \( V \) and divide it into two connected subvolumes, \( V = V_1 \cup V_2 \), both \( V_1, V_2 \) still being macroscopic with

\[
\mathcal{H}_V = \mathcal{H}_1 \otimes \mathcal{H}_2
\]

We assume a Hamiltonian, \( H_V \), to be given on \( \mathcal{H}_V \) of the kind described above. This Hamiltonian can be written as

\[
H_V = H_{V_1} + H_{V_2} + H_{bd}
\]

with \( H_{V_i} =: H_i \) the commuting Hamiltonians of the regions \( V_1, V_2 \) and \( H_{bd} \) denoting the part of the interaction which comprises lattice points of both \( V_1 \) and \( V_2 \). That is, we have

\[
[H_1, H_2] = 0, \ [H_i, H_{bd}] \neq 0
\]

It sometimes happens that we have to discriminate between, for example, \( H_1 \) restricted to the subspace \( \mathcal{H}_1 \) and its embedded version, acting on the full Hilbert space by tensoring with the unit operator of the volume \( V_2 \).
Definition 4.1 We denote the embedded operators by $H_1$ etc. and the restricted versions by $H^r_1$ etc.

The idea now is to proceed in the following way (we explain it for the groundstate). We generally assume that our groundstates are not degenerate (this is frequently the case). We therefore have the unique groundstate, $\Psi_0^{(0)}$, of our reference Hamiltonian, $H_1 + H_2$, which is the tensor product of the groundstates of $H_i$, i.e.

$$\Psi_0^{(0)} = \psi_0^{(1)} \otimes \psi_0^{(2)}$$

One may now hope that one can get information about the structure of the groundstate of the full Hamiltonian, $H = H_1 + H_2 + H_{bd}$, by employing perturbation theory with respect to $H_1 + H_2$ around the reference groundstate $\Psi_0^{(0)}$. The problems and obstacles which show up if one wants to follow this route are described below. But anyway, as this strategy may not be entirely futile and, to our knowledge, has not been attempted in the past in this field, we describe now some of the necessary steps.

Generically $H_i$ are of “size” $V_i$ (e.g. their respective operator norms or (most of their) eigenvalues). That is

$$\|H_i\| \leq C_i \cdot n(V_i) = O(V_i), \quad \|H_{bd}\| \leq C' \cdot n(V_1, V_2) = O(boundary_{V_1, V_2})$$

Therefore one may have the idea to treat $H_{bd}$ as a relatively small perturbation of the operators $H_1$ or $H_2$.

The operators $H_1, H_2$, defined above over the regions $V_1, V_2$, commute. In the following we will encounter in various arguments such pieces of the total Hamiltonian which commute. Another example is the following. We subdivide $V_1$ further into regions $V_1', V_1''$ and correspondingly for $V_2$ with $V_1 = V_1' \cup V_1''$. $V_1''$ is the region in $V_1$ which lies within distance $d \geq \delta$ ($\delta$ the maximum over the ranges of the various interaction potentials ) of the common boundary with $V_2$. In certain calculations we choose $d$ macroscopic but $L \gg d \gg \delta$ (where, as usual, we take $\delta$ as a microscopic quantity). The respective Hilbert spaces are $\mathcal{H}_1', \mathcal{H}_1''$ with $\mathcal{H}_1 = \mathcal{H}_1' \otimes \mathcal{H}_1''$. We can now define another approximation of the total Hamiltonian $H$ in deleting the boundary terms with respect to the interfaces separating $V_1', V_1''$ on the one hand and $V_2', V_2''$ on the other hand,

$$H':= H_1 + H_1'' + H_{bd} + H_2' + H_2'' =: H_1' + H_{bd}' + H_2'$$

with

$$H_{bd}' := H_1'' + H_{bd} + H_2''$$

We now have
Observation 4.2 The operators $H'_1$, $H'_{bd}$, $H'_2$ commute and

$$H = H' + H_{bd(1)} + H_{bd(2)}$$

(40)

with the two boundary contributions describing the interaction through the interfaces between $V'_1$, $V''_1$ and $V'_2$, $V''_2$. The advantage is that we now still have included the interaction through the common interface between $V_1$, $V_2$ in $H'$, the interface we are originally interested in.

As to such commuting operators we have the following spectral result (cf. e.g. [47] or [50]) which goes back to v. Neumann.

Theorem 4.3 With (for simplicity reasons) $H_1$, $H_2$, ... commuting bounded selfadjoint operators, they all are functions of a common selfadjoint operator, $A$, i.e. $H_i = f_i(A)$. It follows in particular that, in case the spectra are discrete, it exists a complete set of common eigenvectors for this set of commuting Hamiltonians (including multiplicities).

One problem which however arises immediately if one wants to apply perturbation theory of operators is the following (if one is not entirely cavalier as to mathematical rigor). We know from almost every discussion of the foundations of statistical mechanics that for macroscopic volumes the spectrum of e.g. the corresponding Hamiltonians, while being frequently discrete, is nevertheless so extremely dense and/or highly degenerate that ordinary perturbation theory is practically useless. A rough estimate yields the following qualitative results. The number of eigenvalues (counting degeneracy) of a hermitean matrix is the same as the dimension of the underlying Hilbert space. This means in our case

$$\#(\text{eigenvalues of } H_V) = D^{N_V}$$

(41)

($D$ the dimension of the local Hilbert spaces $\mathcal{H}_x$, $N_V$ the number of sites in $V$). On the other hand, the spectrum of the corresponding Hamiltonian extends typically over an interval of order $|V|$. That is, whereas the higher excited states are typically much more degenerate and the spectrum is certainly not evenly distributed, a very crude estimate yields a typical density of states of the order $O(|V|^{-1} \cdot D^{\text{dim}})$. This prevents the immediate and naive application of ordinary (analytic) perturbation theory, which works well for perturbations which are small compared to the distance of neighboring eigenvalues of the unperturbed Hamiltonian. To be more precise, one knows from beautiful results derived by Rellich (see [45]), and in particular for the finite dimensional case, that for hermitean
perturbations the (discrete) eigenvalues and eigenstates are real-holomorphic functions in the coupling constant and that this does not only hold for very small values (see also [46], [50] or [48]).

**Theorem 4.4** With \( H_\varepsilon := H_0 + \varepsilon V \) selfadjoint for \( \varepsilon \in \mathbb{R}, H_0, V \) bounded (for simplicity reasons) and \( H_\varepsilon \) having purely discrete spectrum, the eigenvalues \( \lambda_i(\varepsilon) \) and eigenvectors \( \psi_i(\varepsilon) \) of \( H_\varepsilon \), with \( \lambda_i(0), \psi_i(0) \) the eigenvalues and eigenvectors of \( H_0 \), are real analytic functions of \( \varepsilon \). One can in particular choose \( \varepsilon = 1 \). It can however happen that eigenvalues cross (and hence degeneracies change in a superficial sense; see the following corollary).

**Corollary 4.5** This has the important consequence that multiplicities belonging to a fixed \( \lambda_i(\varepsilon) \) can only change at \( \varepsilon = 0 \) as analytic functions, being identical on a certain interval, are necessarily the same everywhere. The other \( \varepsilon \)-values where a singular behavior can occur lie in the complex plane away from the real axis. Note however that, as the eigenvalue functions can cross at some points, the counting of degeneracies at such points is a matter of taste. The dimension of the total eigenspace is of course the dimension of the union of the individual eigenspaces belonging to the different \( \lambda_i(\varepsilon) \) which meet at that point.

Remark: The deeper reason why the nasty Puiseux-series can be avoided derives from the fact that we have that \( H_\varepsilon \) is self-adjoint for real \( \varepsilon \) (see [45]).

On the other hand, convergence radii of the corresponding local power series expansions happen to be of the order of the distances between the points of the spectrum. This prevents to some extent concrete quantitative estimates. To see more clearly the true nature of the problem, we can for example start from the unperturbed groundstate, \( \Psi^{(0)}_0 \), of \( H^{(0)} := H_1 + H_2 \) and try to infer with the help of perturbation theory the structure of the corresponding groundstate \( \Psi_0 \) of \( H^{(0)} + H_{bd} \) as a power series expansion with respect to the eigenvectors of \( H^{(0)} \). That is,

\[
\Psi_0 = \sum c_m \cdot \Psi^{(0)}_m \tag{42}
\]

with

\[
c_m = c_m^{(0)} + c_m^{(1)} + \ldots, \quad c_0^{(0)} = 1, \quad c_m^{(0)} = 0 \text{ for } m \neq 0 \tag{43}
\]

The first order yields

\[
c_m^{(1)} = V_{m0}/(E_m^{(0)} - E_0^{(0)}), \quad m \neq 0, \quad \text{and} \quad V_{m0} = (\Psi^{(0)}_m | V \cdot \Psi^{(0)}_0) \tag{44}
\]
We see that for perturbation theory to make sense,

\[ |V_{m0}| < |E_m^{(0)} - E_0^{(0)}| \]  

(45)

**Observation 4.6** While \( H_{bd} \) is much smaller than \( H_1 \) or \( H_2 \) in general, it is still a macroscopic perturbation compared to the typically microscopic distances between the eigenvalues of \( H_i \). So ordinary perturbation theory is not immediately applicable.

Remark: There exists however a (complicated and tedious) way to deal with such problems to a certain extent (at least in the physics literature); see [49].

What will however better work is another important method of estimating eigenvalues and their behavior under perturbations which does not focus so much on the motion of individual eigenvalues under a perturbation but rather makes more global and qualitative statements. This method provides however no information about the respective eigenvectors, our main point of interest. The method is based on the so-called Rayleigh-Ritz-principle and/or the Poincare-Courant-Weyl estimates (see [50], [51] or [52]). All these statements are based on minimum-maximum- or maximum-minimum-estimates and the principle of stronger or weaker constraints on sets of comparison Hilbert space vectors.

A result, useful in our context, can e.g. be found in [50], p.224, called the Weyl-Courant-inequalities, which we reformulate here for bounded hermitean operators with discrete and only finitely degenerated spectrum (not having zero as an accumulation point).

Remark: For various reasons the numbering of eigenvalues is different in [50]. We start the counting, beginning with the groundstate.

**Theorem 4.7** With \( A_1, A_2 \) operators of the above kind, with sets of eigenvalues, chosen in increasing order (counting multiplicity),

\[ E_0^1 \leq E_1^1 \leq \ldots \; ; \; E_0^2 \leq E_1^2 \leq \ldots \]  

(46)

and

\[ E_0 \leq E_1 \leq \ldots \]  

(47)

the corresponding eigenvalues of \( A := A_1 + A_2 \), we have the estimates

\[ E_{p+q} \geq E_p^1 + E_q^2 \] , \( p,q = 0,1,2,\ldots \)  

(48)
Corollary 4.8 With \( H = H_0 + V \) and \( V \) a small perturbation of \( H_0 \), we have

\[
E_0^1 - \| V \| \leq E_0 \leq E_0^1 + \| V \|
\]

and more generally

\[
E_p^1 - \| V \| \leq E_p \leq E_p^1 + \| V \|
\]

\( E_0, E_0^1 \) the groundstates of \( H, H_0 \) respectively.

Proof: The lhs of the inequalities follow directly from the theorem and \( |E^2_q| \leq \| V \| \) for all the eigenvalues of \( V = A_2 \). The rhs follows from the theorem by interchanging the roles of the operators, that is

\[
A_1 = A - A_2
\]

and hence

\[
E_{p+q}^1 \geq E_p + E_q(-A_2) \geq E_p - \| A_2 \|
\]

i.e.

\[
E_p \leq E_{p+q}^1 + \| A_2 \|
\]

which yields the result by choosing \( q = 0 \). \( \square \)

5 The Groundstate of the Hamiltonian

We begin with the calculation of the entanglement-entropy of the ground state, \( \Psi_0 \) of the full Hamiltonian over \( V = V_1 \cup V_2 \). In the following we use the leading-order identification of entanglement-entropy made in observation/definition 3.4 and the perturbational results of the preceding section. In a first step we study the entanglement-entropy of the ground state, \( \Psi_0' \), of the approximate Hamiltonian, \( H' \), introduced in the preceding section (see formula (38)). We saw that \( H' \) can be written as

\[
H' = H_1' + H_2' + H_{bd}'
\]

with all the terms on the rhs commuting with each other. Assuming again that the ground states are not degenerate we infer from the results of the previous section that the ground state energy, \( E_0' \), of \( H' \) can be uniquely written as

\[
E_0' = E_{0,1}' + E_{0,2}' + E_{0,bd}'
\]

with the rhs the sum of the ground state energies of the terms occurring on the rhs of the previous equation.
Remark: Note that the embedded Hamiltonians always have full subspaces, belonging to an eigenvalue. For example, $H'_1$ has the eigenspace $\psi'_{0,1} \otimes \mathcal{H}''_1$ belonging to the ground state energy $E'_{0,1}$ in the Hilbert space $\mathcal{H}_1 = \mathcal{H}'_1 \otimes \mathcal{H}''_1$ and $\psi'_{0,1}$ the unique ground state of the restricted $H'_{1,(r)}$ acting on $\mathcal{H}'_1$.

The need to constantly make these distinctions is a bit nasty and we will be a little bit sloppy if no confusion can arise. We then have

**Observation 5.1** The ground state, $\Psi'_0$ of $H'$ can now be uniquely represented as the tensor product of the ground states of the restricted Hamiltonians, i.e.

$$\Psi'_0 = \psi'_{0,1} \otimes \psi'_{0,2} \otimes \psi'_{0,\text{bd}}$$  \hspace{1cm} (56)

where

$$\psi'_{0,\text{bd}} \in \mathcal{H}'_1 \otimes \mathcal{H}''_2$$  \hspace{1cm} (57)

In order to calculate the partial traces with respect to $\mathcal{H}_1$ or $\mathcal{H}_2$ we have in a first step to develop $\Psi'_0$ with respect to a basis of $\mathcal{H}_1 \otimes \mathcal{H}_2$ or, what amounts to the same, $\mathcal{H}'_1 \otimes \mathcal{H}'_2 \otimes \mathcal{H}''_1 \otimes \mathcal{H}''_2$. Choosing as bases in the subspaces the eigenvectors of the restricted Hamiltonians $H'_{1,(r)}$, $H'_{2,(r)}$, $H''_{1,(r)}$, $H''_{2,(r)}$, we can infer the following from the above observation.

**Conclusion 5.2** In the representation of $\Psi'_0$ with respect to the mentioned basis in

$$\mathcal{H}_1 \otimes \mathcal{H}_2 = \mathcal{H}'_1 \otimes \mathcal{H}'_2 \otimes \mathcal{H}''_1 \otimes \mathcal{H}''_2$$  \hspace{1cm} (58)

the only summation occurs in the boundary term, $\psi'_{0,\text{bd}}$, which is developed with respect to a basis in $\mathcal{H}'_1 \otimes \mathcal{H}''_2$. Taking for example the eigenvectors of $H''_{1,(r)}$, $H''_{2,(r)}$ we write

$$\psi'_{0,\text{bd}} = \sum c'_{i,j} \psi''_{i,1} \otimes \psi''_{j,2}$$  \hspace{1cm} (59)

and

$$\Psi'_0 = \psi'_{0,1} \otimes (\sum \ldots) \otimes \psi'_{0,2}$$  \hspace{1cm} (60)

We have

$$H'_{\text{bd}} = H''_1 + H''_2 + H_{\text{bd}}$$  \hspace{1cm} (61)

where the operators occurring on the rhs are all of roughly the same size, i.e. of order $O(\text{boundary}_{1,2})$. While $H''_1$, $H''_2$ commute, the support of $H_{\text{bd}}$ overlaps both with the support of $H''_1$ and $H''_2$ and the respective commutators are typically different from zero.
If we now view $\Psi'_{0}$ as a state over $A_{1}$, the algebra on $H_{1} = H'_{1} \otimes H''_{1}$ we get:
\[
(\Psi'_{0} | (A \otimes 1) \Psi'_{0}) = (\psi'_{0,1} \otimes \psi'_{0,bd} | (A \otimes 1) \psi'_{0,1} \otimes \psi'_{0,bd})
\]
where on the lhs $1$ is the unit operator on $H_{2}$, on the rhs it denotes the unit operator on $H''_{2}$. Inserting $\psi'_{0,bd} = \sum c'_{i,j} \psi''_{i,1} \otimes \psi''_{j,2}$ in the above expression we get
\[
(\Psi'_{0} | (A \otimes 1) \Psi'_{0}) = \sum_{ij} \sum_{l} c_{i,l} c'_{l,j} (\psi'_{0,1} \otimes \psi''_{i,1} | A \circ \psi'_{0,1} \otimes \psi''_{j,1})
\]
with $b_{ij} = \sum_{l} c_{i,l} c'_{l,j}$.

**Conclusion 5.3** The reduced state or density matrix on $H_{1}$, corresponding to the total vector state, $\Psi'_{0}$, is
\[
W'_{1} = |\psi'_{0,1} > < \psi'_{0,1}| \otimes W''_{1}
\]
with $W''_{1}$ the density matrix on $H''_{1}$ with matrix elements $b_{ij}$.

If the local Hilbert spaces have uniform dimension $D$ and with the assumed finite interaction distance $\delta$, we conclude that the dimension of the Hilbert space $H''_{1}$ is of order $O(D |bd_{1,2}|)$. From the preceding conclusion we infer that for the vector $\Psi'_{0}$ all perturbations are essentially restricted to the boundary region.

**Conclusion 5.4** For lattice Hamiltonians as we have introduced them, the groundstate of $H'$ is expected to contain or is scattered over a number of eigenstates of $H''_{1}$ of the order $O(D |bd_{1,2}|)$. Correspondingly we infer that its entanglement-entropy is of order $O(|bd_{1,2}|)$.

Remark: The deeper reason why we are able to infer such a general result for the approximate Hamiltonian $H'$ is, on the one hand, the sufficient localisation of the perturbation in a boundary layer of finite thickness and, on the other hand, the uniqueness properties of the groundstate as a tensor product of the corresponding groundstates of the Hamiltonians of the subvolumes.

Now we come to the groundstate of the full Hamiltonian
\[
H = H' + H_{bd(1)} + H_{bd(2)}
\]
The difference between $H$ and $H'$ is a small perturbation on the scale of $H$ or $H'$ as operators, but not on the scale defined by the difference between neighboring eigenvalues of $H$, $H'$. From the preceding section we know at least that

$$E'_p - \triangle \leq E_p \leq E'_p + \triangle$$ (66)

with

$$\triangle = \| H_{bd(1)} + H_{bd(2)} \| = O(|boundary_1,2|)$$ (67)

Our plan is to make an inference from the number of eigenstates of $H'_{1} + H''_{1}$ which essentially contribute in the representation of $\Psi'_0$ to the corresponding number which essentially contribute in the representation of $\Psi_0$, the groundstate of the full Hamiltonian. This number should be of the same order as the corresponding number of eigenstates of $H_1$, as both sets represent complete bases in $\mathcal{H}_1 = \mathcal{H}'_1 \otimes \mathcal{H}''_1$.

We do this in several steps employing the following reasoning. In a first step we add the boundary interaction $H_{bd(2)}$ to the start Hamiltonian $H'$ yielding the intermediate Hamiltonian $H'_{1} + H(V''_{1} \cup V_2)$. Its groundstate is $\psi'_{0,1} \otimes \phi'_0$ with $\phi'_0$ the groundstate of $H(V''_{1} \cup V_2)$. In $V_1$ we have more or less the same situation as before with possible perturbations again confined (by definition) to the region $V_1''$. The same argument as before yields an entropy for the reduced state over $V_1$ of order $O(|bd1,2|)$. Now we employ the fact that the entropies are necessarily the same on both sides. That is, we arrive at

**Observation 5.5** The entropy of the state $\phi'_0$ reduced to $V_2$ is of order $O(|bd1,2|)$.

Now we employ the localisation properties of $H_{bd(1)}$ about the interface $bd_1$ within a small strip of diameter $2\delta$, the interface itself having distance $d \gg \delta$ from the common boundary between $V_1$ and $V_2$. From general experience, drawn from the foundations of statistical mechanics and many-body-theory, we feel allowed to assume that deep inside the region $V_2'$, the groundstate, $\Psi_0$, of the full Hamiltonian $H$ should look similar to the groundstate, $\Phi'_0$ of the Hamiltonian $H(V''_{1} \cup V_2)$.

Remark: Note that we always make the assumption (cf. the introduction) that our system is not in a quantum-critical state, i.e. correlations do not extend to infinity. The latter case would need some extra discussion.

Concerning the groundstate of the latter Hamiltonian we learned that its restriction to $V_2$ has an entropy of order $O(bd_{12})$. From the above we again
conclude that, in \( V_2 \), \( \Psi_0 \) differs from \( \Psi'_0 \) (the groundstate of \( H' \)) essentially in a boundary layer about the interface \( bd_12 \). By symmetry we infer the same for the region \( V_1 \) and arrive at

**Conclusion 5.6** The above chain of reasoning leads to the conclusion that as a consequence of the spatial localisation properties of \( H_{bd} \), \( H_{bd(1)} \), \( H_{bd(2)} \) and certain natural assumptions about clustering or decay of influence and/or interactions, the groundstate of the full Hamiltonian, \( H \), has an entanglement-entropy of order \( O(bd_{12}) \).

Remark: This general conclusion is corroborated by exact results for (typically) low-dimensional models; cf. e.g. the literature mentioned in the introduction.

6 The Highly Excited Eigenstates

It turns out that for highly excited eigenstates it is more fruitful to adopt a completely different strategy, which is strongly inspired by ideas and methods taken from statistical mechanics. To simplify the discussion we treat the following system. We take a huge box of sidelength \( L \) as the total volume \( V \). We partition it by a lattice of small boxes, \( C_i \), of sidelength \( l \) with \( L \gg l \gg \delta \) (\( \delta \) the range of the interaction in the original Hamiltonian, \( H \)). I.e., we assume that \( l \) is small but still macroscopic; this is the usual assumption in statistical mechanics. As subvolumes, \( V_1 \), \( V_2 \) we take certain regions in \( V \) each of which contains an integer number of such small boxes. I.e., we assume (with \( N_l, N_{l,1}, N_{l,2} \) the respective numbers of boxes in \( V, V_i \))

\[
N_l = N_{l,1} + N_{l,2}, \quad N_i = L^3/l^3, \quad |V_i| = l^3 \cdot N_{l,i}
\]  

(68)

Each of the small boxes contains \( l^3/a^3 \) lattice sites of the original lattice. We assume of course that the interface, separating \( V_1 \) and \( V_2 \), is sufficiently regular, i.e. its area is assumed to be of order \( O(L^2) \).

In each of the small boxes, \( C_i \), we take as Hamiltonian, \( h_i \), the piece of our original total Hamiltonian \( H \) with interaction terms confined to \( C_i \), i.e., leaving out the interaction terms occurring in \( H \) between the different boxes. Note that, due to the assumed translation invariance of our interaction, all the \( h_i \) are equivalent as operators. As these small boxes still contain quite a few lattice sites, the spectrum of \( h = h_i \) may still be both complex and degenerated. As new reference Hamiltonians in \( V, V_j \), \( j = 1, 2 \), we take

\[
H' := H'_V := \sum_V h_i, \quad H'_j := \sum_{V_j} h_i
\]  

(69)
There is now, in contrast to the preceding section, no boundary term $H_{bd}$, operating in the vicinity of the boundary, $bd_{12}$, but the entanglement structure may still be quite complex as we will see below.

Remark: Such truncated systems are frequently discussed and their properties exploited in quantum statistical mechanics within the Gibbsian (ensemble) approach. See for example [53], [54], [55] or [56].

As all these $h_i$ commute (by construction), the eigenstates and eigenvalues of $H'$ or $H'_j$ can be built up from the more elementary components belonging to the $h_i$.

So, to begin with, let us start from some macroscopic volume, $V'$, of the above kind ($V' = V$ or $V_j$), the Hamiltonian, $H' = \sum_{V', h_i}$, and some eigenvalue, $E$, sufficiently far away from the groundstate energy $E_0 = \sum E^{(i)}_0$ with $E^{(i)}_j$ denoting the $j$-th energy level of the box Hamiltonian $h_i$. We have

$$E = \sum_{C_i} E^{(i)}_j$$

for certain appropriate combinations of energy levels, $E^{(i)}_j$, of the $h_i$. It is here where certain arguments of combinatorial statistical mechanics enter.

The problem can now be phrased a little bit differently. With $N$ boxes given and $h = h_i$ having the energy levels (counting multiplicity!) $E_1 \leq E_2 \leq \ldots \leq E_j \ldots$, we are interested in the number of ways of distributing the energies, $E_j$, over the $N$ boxes under the constraints

$$N = \sum N_j \quad , \quad E = \sum N_j \cdot E_j$$

with $N_j$ the number of boxes having energy $E_j$. Each such configuration is hence characterized by the sequence, $(N_1, N_2, \ldots, N_j, \ldots)$. We then have

**Observation 6.1** To each fixed configuration $(N_1, N_2, \ldots, N_j, \ldots)$ the number of ways of distributing the energies $E_j$ over the $N$ boxes under the above constraints is

$$W = (N!/N_1! \cdots N_j! \cdots) \quad , \quad N = \sum N_j \quad , \quad E = \sum N_j E_j$$

From the combinatorics of such expressions one knows that there exists a pronounced maximum of $W$ for a special configuration $(N_1, N_2, \ldots, N_j, \ldots)_{\text{max}}$ (cf. the above cited literature for more details). The constraints can be implemented via Lagrange multipliers with, in the end, the multiplier $\beta$,
belonging to the $E$-constraint, turning out to be something like an inverse temperature. It is however important (while usually not openly mentioned in the literature) that for this and other results to hold, $(E - E_0)$ has to be macroscopic, i.e. of order $O(|V|)$. This implies that, with the individual levels, $E_j$ of $h$, being microscopic, most of the occurring $N_j$ are sufficiently large so that Stirlings formula can be applied. After some calculations one winds up with the formula (for details see e.g. [54], chapt. 87, p. 364 ff, or [53], chapt. 46, p. 150 ff)

**Conclusion 6.2** Under the assumptions being made we have for the most probable configuration and the configuration entropy:

$$
N_j/N = e^{-\beta E_j} / \sum_j e^{-\beta E_j}, \quad \ln W_{\text{max}} = N \cdot \ln \left( \sum_j e^{-\beta E_j} + \beta \cdot E \right) \quad (73)
$$

(with $\beta$ only implicitly given by the first equation). In any case, $\ln W_{\text{max}}$ turns out to be in general proportional to the volume $|V|$ ($N \sim |V|$) for highly excited states.

These findings have the following consequences for our entanglement problem. With

$$
\Psi_E = \sum c_{ij} \phi_i^{(E_1)} \otimes \psi_j^{(E_2)}, \quad E_1 + E_2 = E \quad (74)
$$

and $\phi_i^{(E_1)}$, $\psi_j^{(E_2)}$ eigenvectors to the fixed energies $E_1$, $E_2$ of $H'_1$, $H'_2$, respectively, this is an eigenvector for $H'$ with energy $E$. We have just seen that the eigenvalues $E_1$ and $E_2$ are extremely degenerated so that the number of terms, occurring in the above sum, can in principle be chosen to be of order $O(e^V)$.

Note that, in addition, we could also sum over all possible combinations of $E_1$, $E_2$ with $E_1 + E_2 = E$ but this is not necessary for our argument. We can now make various choices. We can for example select a very special and simple eigenvector of product type (i.e. all $c_{ij} = 0$ except one):

$$
\Psi_E = \phi_i^{(E_1)} \otimes \psi_j^{(E_2)}, \quad E_1 + E_2 = E \quad (75)
$$

Its entanglement-entropy is of course zero.

On the other hand, due to the huge degeneracy of all macroscopic energy levels of $H'$, we can exploit our above conclusion [6.2] and what we said in the preceding sections about our coarse approximation of entanglement-entropy. It is easy to choose the $c_{ij}$ almost evenly spread over the full range of degenerated eigenstates belonging to $E_1$ and $E_2$ in such a way that a typical eigenvector to energy $E$ has an entanglement-entropy which is proportional to the volume $|V|$. That is
Conclusion 6.3 Due to the huge degeneracy of macroscopically excited energy levels of \( H' \), the typical eigenvector, belonging to the class of eigenvectors of such an energy level, has an entanglement-entropy of order \( O(|V|) \), more specifically

\[
S(W_1) = \alpha_1 \cdot |V_1| = \alpha_2 \cdot |V_2| = S(W_2)
\]

(76)

(a consequence of conclusion 6.2). Furthermore, our preceding discussion shows that these states, \( W_1, W_2 \) display features we know from statistical mechanics. By “typical” we mean, by randomly selecting one of the admissible eigenvectors from the huge class, we will get such a state with high probability as one knows that the combinatorial coefficients, calculated in conclusion 6.2, are extremely large for macroscopic volumes \( V, V_j \).

Observation 6.4 We remind the reader of our construction of a vector state belonging to a canonical equilibrium state of system (1) with the help of tensoring with a system (2). Our above findings on highly excited states represent, so to speak, the dual version of this observation. Highly excited states on \( V \) have, as we have seen, a tendency to resemble states on, say, \( V_1 \) which display a marked statistical mechanical behavior (they are of course not always true equilibrium states).

One can now go on and study the full untruncated Hamiltonian, \( H \), starting from such a reference Hamiltonian \( H' \). Assuming that by inserting the usually very small boundary terms between the blocks \( C_i \) and adding the boundary Hamiltonian, \( H_{bd} \), nothing spectacular will happen (the ordinary assumption in statistical mechanics), the degeneracy of eigenvalues and/or the density of states will remain essentially the same. This may be inferred from perturbation theory. We therefore arrive at the final conclusion

Conclusion 6.5 In the case of the full Hamiltonian \( H \), we have essentially the same result as in the preceding conclusion for the reference Hamiltonian \( H' \). Due to the expected huge density of states and/or the huge degeneracy of eigenvalues we have for macroscopically excited eigenvalues, \( E \), that a randomly selected (i.e., generic) eigenstate has an entanglement-entropy which is proportional to the volume in leading order.

7 Low-Lying Excited States

We now discuss the special case that the excited states lie in the vicinity of the groundstate, i.e. instead of energy levels fulfilling

\[
\Delta := (E - E_0) = O(|V|)
\]

(77)
we deal with excitation energies which are much smaller. It is perhaps surprising that in this case we have to use yet another strategy. The same general formula

\[ W = \frac{N!}{N_1! \cdots N_j! \cdots} \]  

holds of course also in this regime but for example Stirlings approximation is no longer applicable as the \( N_j \) are in general too small. Even if we would ignore this fact (which would presumably only affect the quantitative aspects of some estimates) there exists yet another more serious problem. The energy constraint (we now denote the energy levels of \( H' \) by \( E'_i \))

\[ \sum E_j \cdot N_j = E'_0 + \triangle = N \cdot E_0 + \triangle \]  

with \( \triangle \ll |V| \), is more difficult to implement in this regime. It is interesting to analyse the consequences of \( \triangle \ll |V| \).

At first glance it seems that we will get the same results as in the previous section by applying the same methods (and in the statistical mechanics literature known to us we have found almost no remark as to possible problems). The method we applied previously is indeed very general but there exists a subtle point. The Lagrange multiplier \( \beta \) is only implicitly defined via the constraint

\[ \frac{E'}{N} = \frac{\sum E_j \cdot e^{-\beta E_j}}{\sum e^{-\beta E_j}} \]  

i.e., it regulates the average energy per box Hamiltonian, \( h_i \), in form of a canonical distribution over the energy levels of \( h \). The \( E_j \) are in general not known in detail but one may infer that with \( (E' - E'_0) = O(|V|) \) both sides are of the same order for finite \( \beta \) so that it is reasonable that we can find some definite value for which the implicit equation for \( \beta \) can be fulfilled. But we now have \( (E' - E'_0)/N \ll 1 \) and we conjecture that in this regime the above implicit equation can only be fulfilled for \( \beta \gg 1 \) or \( \beta \to \infty \) (which seems to be quite natural, given the obvious similarities to statistical mechanics. A “thermal” state near the groundstate has by definition a low temperature).

**Observation 7.1** For \( (E' - E'_0) \) small, i.e. \( (E' - E'_0) \ll O(|V|) \), the parameter \( \beta \) becomes very large. For these values it becomes difficult to reliably estimate the terms in the occurring variational equations which are now combinations of very large and very small terms. Note in particular that for \( \beta \) large

\[ N_j = N \cdot \frac{e^{-\beta E_j}}{\sum e^{-\beta E_j}} \]  

becomes very small compared to \( N \).
Therefore we choose another strategy which is better adapted to this situation. We simply catalogue the low-lying excitations of $H'$ directly, beginning with the groundstate. We have

**Observation 7.2**

1) For the groundstate we have:

$$E'_0 = N \cdot E_0, \text{ no degeneration}$$  \hspace{1cm} (82)

2) For one box Hamiltonian excited we have:

$$E'_i = (N - 1)E_0 + E_i, \text{ } W(E'_i) = N$$  \hspace{1cm} (83)

3) Two levels excited; there are two possibilities, $E_i = E_j$ or $E_i \neq E_j$. We have

$$E'_{ii} = (N - 2)E_0 + 2E_i, \text{ } W(E'_{ii}) = (N \cdot (N - 1)/2)$$  \hspace{1cm} (84)

or

$$E'_{ij} = E_i + E_j + (N - 2)E_0, \text{ } W(E'_{ij}) = N(N - 1)$$  \hspace{1cm} (85)

etc.

Remark: Note that these results of course coincide with the general formula, if we insert the corresponding $N_i$.

We see the following. Already for the lowest excited levels of $H'$ we have a degeneracy, $W(E'_i) = O(|V|)$ or a small power of $|V|$. Repeating our previous arguments we infer

**Conclusion 7.3**

*Already the lowest excited levels of $H'$ have a degeneracy of order $O(|V|)$ or a small power of $|V|$, entailing that we can construct corresponding eigenstates having an entanglement-entropy of order $O(\ln |V|)$. Note that $H'$ does not contain the boundary term $H_{bd}$. For the full $H$ we hence expect a generic entanglement-entropy of order $O(\ln |V| \cdot |bd_{12}|)$ where the additional factor $|bd_{12}|$ comes from the term $H_{bd}$ as this term leads to a further splitting proportional to the area of the boundary as was for example the case for the groundstate.*

The situation changes slightly if we go over to higher excited levels. For, say, $k$ levels excited the two extreme cases are: 1) all $k$ levels identical or, 2) all levels being different. The intermediate class comprises cases where some of the $E_j$ coincide. We have the following estimate
Corollary 7.4 If \( k \) levels are excited, \( E_{i_1}, \ldots, E_{i_k} \), with repetitions allowed, we have the following estimate with respect to the degeneracies of the low-lying eigenvalues of \( H' \)

\[
\frac{(N!/(N-k)!) \leq W_k \leq (N \cdot (N - 1) \cdots (N - k + 1))}{(86)}
\]

For \( N \) still very large compared to \( k \), this entails

\[
W_k = O(N^k) \quad \text{and} \quad \ln W_k = O(k \cdot \ln N) = O(k \cdot \ln |V|) \quad (87)
\]

For the full Hamiltonian \( H \) we again get an additional multiplicative factor of order \( O(|bd_{12}|) \).

8 Appendix

A Infinite-Dimensional Local Hilbert-spaces

We assume that the spectra of the local and global hamiltonians are discrete but are now not necessarily bounded from above. We still assume that \( H_V \) is bounded from below (existence of a groundstate!). So let \( H \) be such a Hamiltonian on a separable Hilbert space, \( \mathcal{H} \). We select a certain (countable) basis, \( e_i \), and choose certain subspaces, \( \mathcal{H}_n \), spanned by the basis vectors, \( e_1, \ldots, e_n \). The projector on \( \mathcal{H}_n \) is denoted by \( P_n \). Then

\[
P_n \cdot H \cdot P_n
\]

is a bounded operator on \( \mathcal{H}_n \). Being a little bit more general, if \( H_V \) is the Hamiltonian on \( \mathcal{H}_V := \mathcal{H}_{i_1} \otimes \cdots \otimes \mathcal{H}_{i_N} \), we select in each \( \mathcal{H}_{i_\nu} \) the subspace \( \mathcal{H}_{i_\nu}^{(n)} \), spanned by \( e_{1(\nu)}, \ldots, e_{n(\nu)} \). From these local pieces we compose the subspace, \( \mathcal{H}_n \), in \( \mathcal{H}_V \), i.e.

\[
\mathcal{H}_n := \mathcal{H}_{i_1}^{(n)} \otimes \cdots \otimes \mathcal{H}_{i_N}^{(n)}
\]

and denote again the projector on this subspace by \( P_n \). In the same manner we take

\[
H_V^{(n)} := P_n \cdot H_V \cdot P_n
\]

as finite dimensional Hamiltonian on these \( \mathcal{H}_n \subset \mathcal{H}_V \).

Now we have to discuss what happens if we take the limits

\[
\mathcal{H}_n \to \mathcal{H}_V \text{, } H_V^{(n)} \to H_V \text{ etc.}
\]

(91)
Note that by construction all the $H^{(n)}$ are now bounded, even finite dimensional, but the limit Hamiltonians are in general unbounded. We do not want to be too tedious concerning technical details of functional analysis at this point. Suffice it to say that a reasonable concept of operator convergence in this context is convergence in the resolvent sense, i.e., instead of dealing with unbounded operators we deal with their bounded resolvents

$$(H - z)^{-1}, \text{Im}(z) \neq 0 \quad (92)$$

Under quite weak assumptions, which are in general fulfilled in our context, (strong) resolvent convergence can be assumed (cf. [37], sect. VIII.7). As a consequence we have the following result (discrete spectrum):

**Observation A.1** With $H$ having discrete spectrum and if $H_n \to H$ in strong resolvent sense, we can find to each eigenvalue $E$ of $H$ an interval $(a, b)$ so that $E$ is the only spectral value of $H$ in $(a, b)$. With $P_{(a,b)}^{(n)}$ the spectral projections of $H_n$ on the interval $(a, b)$ we have

$$P_{(a,b)}^{(n)} \psi \to P_{(a,b)} \psi = P_E \psi \quad (93)$$

for all $\psi \in \mathcal{H}$. We get even stronger results if we assume convergence in norm-resolvent sense.

We now sketch how results about entanglement-entropy, derived for large but nevertheless finite dimensional systems, could be transferred to the general case. But as there are several quite delicate technical steps involved, which to rigorously prove would need quite an amount of mathematical input, we refrain from giving all the intricate mathematical details at the moment. From a physical point of view the strategy seems to be quite reasonable.

In a first step we have to guarantee that for example in the case of groundstates

$$\Psi_0^{(n)} \to \Psi_0 \quad (94)$$

In the following sections we regard these vectorstates as states on a restricted region, $V_1$, and the corresponding Hilbert space or observable algebra. We then have for the respective density matrices over $\mathcal{H}_1$

$$W_1^{(n)} \to W_1 \quad (95)$$

in the form

$$Tr(W_1^{(n)} \cdot A) \to Tr(W_1 \cdot A) \quad (96)$$
What we need is a result like

\[ W_1^{(n)} \cdot \ln W_1^{(n)} \rightarrow W_1 \cdot \ln W_1 \quad (97) \]

in a suitable topology so that we may get in the end

\[ Tr(W_1^{(n)} \cdot \ln W_1^{(n)}) \rightarrow Tr(W_1 \cdot \ln W_1) \quad (98) \]

that is

\[ S(W_1^{(n)}) \rightarrow S(W_1) \quad (99) \]

Remark: Some of the necessary technical arsenal can be found for example in [38], see also the seminal paper by Wehrl ([39], in particular section IID about continuity properties of entropy as a function defined over the density matrices, or the more recent [40].

Note that in general the entropy is not continuous if the density matrices are equipped with the trace-norm

\[ \| \rho_1 - \rho_2 \|_{tr} := tr(|\rho_1 - \rho_2|) \quad (100) \]

It is an important observation that this property can be restored if one works with the subset of density matrices having finite energy. This means, one chooses a fixed reasonable Hamiltonian, \( H \), so that the canonical Gibbs state, \( \sigma_\beta \), at some inverse temperature, \( \beta \), has finite trace and finite mean energy

\[ tr(\sigma_\beta \cdot H) = E \quad (101) \]

The subset of admissible density matrices is then given by the condition

\[ tr(\rho \cdot H) \leq E \quad (102) \]

This is a reasonable condition for e.g. entanglement entropy. Taking for the bipartite system a Hamiltonian of the form \( \tilde{H} = H_1 + H_2 \) (i.e. no interaction across the common boundary) with reasonable \( H_i \), it suffices that the \( H_i \) are in the domain of definition of the pure state \( \Psi \) as we then have

\[ tr(W_1 \cdot H_i) = (\Psi, (H_i \otimes 1)\Psi) = E' \quad (103) \]

for some \( E' \). So, adapting \( \beta \), these particular density matrices always lie in appropriate subsets.
B Proof of Theorem 3.1

Viewing the matrices $C := (c_{jm})$, $C^* := (c_{jm}^T)$ as operators from $\mathcal{H}_2 \rightarrow \mathcal{H}_1$, $\mathcal{H}_1 \rightarrow \mathcal{H}_2$ respectively, we have

$$W_1 = C \cdot C^*, \quad W_2 = C^* \cdot C$$  \hspace{1cm} (104)

If $\psi_i$ is an eigenvector of $W_2 = C^* \cdot C$, i.e.

$$C^* \cdot C \circ \psi_i = \lambda_i \cdot \psi_i$$ \hspace{1cm} (105)$$

it follows that

$$CC^* \circ \psi_i = \lambda_i \cdot C \circ \psi$$ \hspace{1cm} (106)$$

i.e., $\lambda_i$ is eigenvalue of $W_1 = CC^*$ with eigenvector $C\psi_i$ and vice versa. Furthermore, if $\lambda_i \neq 0$, the degeneracy is the same with respect to $C^*C$ and $CC^*$. This follows from

$$(C\psi_1^1|C\psi_2^2) = (\psi_1^1|C^*C\psi_2^2) = \lambda_i(\psi_1^1|\psi_2^2)$$ \hspace{1cm} (107)$$

that is, with $\psi_1^1$, $\psi_2^2$ orthogonal eigenvectors to the eigenvalue $\lambda_i \neq 0$, $C\psi_1^1$, $C\psi_2^2$ are also orthogonal and non-vanishing and the same holds in the other direction.

By the same token we infer from the positive definiteness of $C^*C$ and $CC^*$ that all eigenvalues are $\geq 0$. The normalisation of the vector $\Phi$ as a state on $\mathcal{A}$ and $\mathcal{A}_i$ implies that the trace norm of $W_i$ is one. This proves the theorem.

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