Viscosity solutions for a system of PDEs and optimal switching

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In this paper, we study the existence and uniqueness of viscosity solutions for a system of \( m \) variational partial differential inequalities with interconnected obstacles. A particular case of this system is the deterministic version of the Verification Theorem of the Markovian optimal \( m \)-states optimal switching problem in finite horizon. The switching cost functions are arbitrary and can be positive or negative. This has an economic incentive in terms of central valuation in cases where such organizations or states give grants or financial assistance to power plants that promote green energy in their production activity or that uses less polluting modes in their production. Our main tools is an approximation scheme and the notion of systems of reflected backward stochastic differential equations.

Keywords: real options; backward stochastic differential equations; Snell envelope; stopping times; switching; viscosity solution of PDEs; variational inequalities.

1. Introduction

In this paper, we consider the optimal \( m \)-states switching problem in finite horizon when the switching costs are arbitrary and not necessarily positive, which is the novelty of this paper. This has an economical motivation in terms of firms valuation.

In order to introduce the problem, let us deal with an example. Assume a power plant which produces electricity and which has several modes of production, therefore it is put in the instantaneous most profitable one which is affected by the price \((X_t)_{t \geq 0}\) of electricity in the market that fluctuates in reaction to many factors such as demand level, weather conditions, unexpected outages etc., in addition to the fact that electricity is non-storable, so once produced, it should be immediately consumed. Thus, the manager of the plant aims at maximizing its global profit. For this objective, she implements an optimal strategy which is a pair of two sequences \((\tau_n)_{n \geq 1}\) and \((\epsilon_n)_{n \geq 1}\) describing, respectively, the optimal successive switching times and modes. When the plant is in mode \( i \in I \), it provides a profit \( \psi_i(t, X_t) dt \) which depends on that mode. However, this gain also incorporates a switching cost \( g_{ij}(t, X_t) \), which could be positive or negative, when switching the plant from the mode \( i \) to another one. This means that when \( g_{ij}(t, X_t) > 0 \), then switching is not free and generates expenditures; on the other hand, when \( g_{ij}(t, X_t) \leq 0 \), it is the case when the state and environmental organizations provide grants and financial aid to power plants that use green energy in their production activities or methods of cleaner production, which emit less carbon into the air.
The switching from one regime to another one is realized sequentially at random times which are part of the decisions. So the manager of the power plant faces two main issues:

(i) when should she decide to switch the production from its current mode to another one?

(ii) to which mode the production has to be switched when the decision of switching is made?

Optimal switching problems were studied by several authors (see, e.g. Brennan & Schwartz, 1985; Tang & Yong, 1993; Brekke & Xksendal, 1994; Dixit & Pindyck, 1994; Duckworth & Zervos, 2001; Zervos, 2003; Hamadène & Jeanblanc, 2007; LyVath & Pham, 2007; Carmona & Ludkovski, 2008; Bouchard, 2009; Djehiche & Hamadène, 2009; Djehiche et al., 2009; El Asri & Hamadène, 2009; Bayraktar & Egami, 2010; El Asri, 2013a,b and the references therein). The motivations are mainly related to decision-making in the economic sphere. In order to tackle those problems, authors use mainly two approaches: Either a probabilistic one (Hamadène & Jeanblanc, 2007; Djehiche & Hamadène, 2009; Djehiche et al., 2009; Hamadène & Zhang, 2010) or an approach which uses partial differential inequalities (PDIs for short) (Tang & Yong, 1993; Brekke & Xksendal, 1994; Duckworth & Zervos, 2001; Zervos, 2003; Carmona & Ludkovski, 2008; El Asri & Hamadène, 2009; Hamadène & Morlais, 2013).

In the finite horizon framework, Djehiche et al. (2009) have studied the multi-modes switching problem when the profit and the switching costs only depend on \( t \), by using probabilistic tools. They proved existence of a solution and found an optimal strategy when the switching costs from state \( i \) to state \( j \) is strictly non-negative (\( g_{ij}(t) > \alpha > 0 \)). The partial differential equation approach (PDE in short) of this work has been carried out by El Asri & Hamadène (2009) when \( g_{ij}(t, X_t) > \alpha > 0 \). They showed that when the price process \( (X_t : t \geq 0) \) is the solution of a Markovian stochastic differential equation, then this problem is associated to a system of variational inequalities with interconnected obstacles for which they provided a solution in the viscosity sense. This solution turns out to be the value function of the system.

In the same spirit El Asri (2013a) studied the problem when \( g_{ij}(t, X_t) \geq 0 \), he showed the existence of the optimal strategy and uniqueness of the solution in viscosity sense of the problem. Nevertheless, those papers (El Asri, 2013a) and (El Asri & Hamadène, 2009) suffer from two drawbacks: (i) the switching cost functions \( g_{ij} \) are non-negative; (ii) in the Markovian case the optimal strategy satisfies \( \forall n \geq 1, \ p[\tau_n < T] < C/n \) in El Asri & Hamadène (2009); this property is the basis of the proof of existence of a solution in the viscosity sense, but in the general case where the costs of switching could be negative, this property is not satisfied. In the PDE approach, we also mention the recent result of Hamadène & Morlais (2013) that deals with existence and uniqueness, when the switching cost functions are positive and arbitrary, in the viscosity sense of a solution for an \( m \) system of variational PDIs with interconnected obstacles which is the deterministic version of the Verification Theorem of the Markovian optimal switching problem.

The novelty of this paper lies in the fact that we investigate the solution to the optimal multiple switching problem when the switching costs could be positive or negative, using probabilistic tools as the Snell envelope of processes, backward stochastic differential equations (BSDEs for short) and the PDE approach.

We prove existence and uniqueness of the vector of value functions and provide a characterization of an optimal strategy of this problem when the payoff rates \( \psi_i \) and the switching costs \( g_{ij} \) (positive or negative) are adapted only to the filtration generated by a Brownian motion. Later on, in the Markovian framework, we show that the value function of the problem is associated to an uplet of deterministic
functions \((v_1, \ldots, v_m)\) which is the unique solution of the following system of PDEs:

\[
\begin{cases}
\min \left\{ v_i(t,x) - \max_{j \in \mathcal{I}^{-i}} \{-g_{ij}(t,x) + h_j(t,x)\}, -\partial_i v_i(t,x) - A v_i(t,x) - \psi_i(t,x) \right\} = 0 \\
\forall (t,x) \in [0,T] \times \mathbb{R}^k, \quad i \in \mathcal{I} = \{1,\ldots,m\}, \quad \text{and} \quad v_i(T,x) = 0,
\end{cases}
\]

where \(A\) is an operator associated with a diffusion process and \(\mathcal{I}^{-i} := \mathcal{I} \setminus \{i\}\). It turns out that this system is the deterministic version of the Verification Theorem of the optimal multi-modes switching problem in finite horizon.

This paper is organized as follows: In Section 2, we formulate the problem and give the related definitions. In Section 3, we introduce the optimal switching problem under consideration and give its probabilistic Verification Theorem. It is expressed by means of the Snell envelope of processes. Then we introduce the approximating scheme which enables us to construct a solution for the Verification Theorem. Section 4 is devoted to the connection between the optimal switching problem, the Verification Theorem and the associated system of PDEs. This connection is made through BSDEs with one reflecting obstacle in the Markovian case. Further, we show the existence and continuity of a solution for the system of PDEs. Finally, in Section 5, we show that the solution of PDEs is unique in the class of continuous functions which satisfy a polynomial growth condition.

2. Formulation of the problem and assumptions

2.1 Setting of the problem

The finite horizon multiple switching problem can be formulated as follows. Let \(\mathcal{I}\) be the set of all possible activity modes of the production of a power plant. A management strategy of the plant consists, on the one hand, of the choice of a sequence of non-decreasing stopping times \((\tau_n)_{n \geq 1}\) (i.e. \(\tau_n \leq \tau_{n+1}\), \(\tau_0 = 0\) and \(\tau_n \to T\) when \(n \to +\infty\)) where the manager decides to switch the activity from its current mode to another one. On the other hand, it consists of the choice of the mode \(\xi_n\), which is an \(\mathcal{F}_{\tau_n}\)-measurable random variable taking values in \(\mathcal{I}\), to which the production is switched at \(\tau_n\) from its current mode. Therefore, the admissible management strategies of the plant are the pairs \((\delta, \xi) := ((\tau_n)_{n \geq 1}, (\xi_n)_{n \geq 1})\) and the set of these strategies is denoted by \(\mathcal{D}\).

Let \(X := (X_t)_{0 \leq t \leq T}\) be an adapted continuous \(\mathbb{R}^k\)-valued stochastic process, which stands for the market price of \(k\) factors which determine the market price of the commodity. Assuming that the production activity is in mode 1 at the initial time \(t = 0\), let \((u_t)_{0 \leq t \leq T}\) denote the indicator of the production activity’s mode at time \(t \in [0, T]\):

\[
u_t = 1_{[0,\tau_1]}(t) + \sum_{n \geq 1} \xi_n 1_{[\tau_n,\tau_{n+1}]}(t).
\]

Then, for any \(t \leq T\), the state of the whole economic system related to the project at time \(t\) is represented by the vector

\[
(t, X_t, u_t) \in [0, T] \times \mathbb{R}^k \times \mathcal{I}.
\]

Finally, let \(\psi_t(t, X_t)\) be the instantaneous profit when the system is in state \((t, X_t, i)\), and for \(i, j \in I\) \(i \neq j\), let \(g_{ij}(t, X_t)\) denote the switching cost of the production at time \(t\) from current mode \(i\) to another mode \(j\).
Then if the plant is run under the strategy \((\delta, \xi) = ((\tau_n)_{n \geq 1}, (\xi_n)_{n \geq 1})\), the expected total profit is given by

\[
J(\delta, \xi) = E \left[ \int_0^T \psi_{w}(s, X_s) \, ds - \sum_{n \geq 1} g_{u_{n-1} u_n} (\tau_n, X_{\tau_n}) 1_{[\tau_n < T]} \right].
\]

Therefore, the problem we are interested in is to find an optimal strategy, i.e. a strategy \((\delta^*, \xi^*)\) such that \(J(\delta^*, \xi^*) \geq J(\delta, \xi)\) for any \((\delta, \xi) \in \mathcal{D}\).

We now consider the following system of \(m\) variational inequalities with interconnected obstacles: \(\forall i \in \mathcal{I}\)

\[
\begin{cases}
\min \left\{ v_i(t, x) - \max_{j \in \mathcal{I}^{-i}} \{-g_{ij}(t, x) + v_j(t, x)\}, -\partial_i v_i(t, x) - \mathcal{A}v_i(t, x) - \psi_i(t, x) \right\} = 0, \\
v_i(T, x) = 0,
\end{cases}
\]

where \(\mathcal{A}\) is given by

\[
\mathcal{A} = \frac{1}{2} \sum_{i,j=1}^{m} (\sigma \sigma^*)_{ij}(t, x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{m} b_i(t, x) \frac{\partial}{\partial x_i};
\]

hereafter the superscript \((^*)\) stands for the transpose, \(\text{Tr}\) is the trace operator and finally \((x, y)\) is the inner product of \(x, y \in \mathbb{R}^k\).

The main objective of this paper is to focus on the existence and uniqueness of the solution in the viscosity sense of (2.3). This system is the deterministic version of the optimal \(m\)-states switching problem when we assume that the market price process \(X\) of the commodity is an Itô diffusion.

Recall the notion of viscosity solution of the system (2.3).

**Definition 1** Let \((v_1, \ldots, v_m)\) be a \(m\)-uplet of continuous functions defined on \([0, T] \times \mathbb{R}^k\), \(\mathbb{R}\)-valued and such that \(v_i(T, x) = 0\) for any \(x \in \mathbb{R}^k\) and \(i \in \mathcal{I}\). The \(m\)-uplet \((v_1, \ldots, v_m)\) is called:

(i) a viscosity supersolution (respectively, subsolution) of the system (2.3) if, for each fixed \(i \in \mathcal{I}\), for any \((t_0, x_0) \in [0, T] \times \mathbb{R}^k\) and any function \(\varphi_i \in C^{1,2}([0, T] \times \mathbb{R}^k)\) such that \(\varphi_i(t_0, x_0) = v_i(t_0, x_0)\) and \((t_0, x_0)\) is a local maximum of \(\varphi_i - v_i\) (respectively, minimum), we have

\[
\min \left\{ v_i(t_0, x_0) - \max_{j \in \mathcal{I}^{-i}} \{-g_{ij}(t_0, x_0) + v_j(t_0, x_0)\}, -\partial_i v_i(t_0, x_0) - \mathcal{A}v_i(t_0, x_0) - \psi_i(t_0, x_0) \right\} \geq 0 \quad \text{(respectively,} \leq 0)\;
\]

(ii) a viscosity solution if it is both a viscosity supersolution and subsolution.

There is an equivalent formulation of this definition (see, e.g. Crandall et al., 1992) which we give because it will be useful later. So firstly we define the notions of superjet and subjet of a continuous function \(v\).

**Definition 2** Let \(v \in C((0, T) \times \mathbb{R}^k)\), \((t, x)\) be an element of \((0, T) \times \mathbb{R}^k\) and finally \(S_k\) be the set of \(k \times k\) symmetric matrices. We denote by \(J^{2,+}v(t, x)\) (respectively, \(J^{2,-}v(t, x)\)) the superjets
Throughout this paper we consider viscosity solutions of the following system of PDEs:

\[ \frac{\partial v}{\partial t} + \mathcal{A}(D^2 v) + f(x, v, \nabla v) = 0, \quad x \in \Omega, \; t > 0, \]

where \( \mathcal{A} \) is a second-order differential operator, \( f(x, v, \nabla v) \) is a function depending on the state variable \( x \), the value \( v \), and its gradient \( \nabla v \), and \( \Omega \) is a bounded domain in \( \mathbb{R}^k \).

A function \( v \) is called a viscosity solution if it is both a viscosity subsolution and supersolution.

2.2 Assumptions

Throughout this paper \( T \) (respectively, \( k, d \)) is a fixed real (respectively, integer) positive numbers. Let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a fixed probability space on which is defined a standard \( d \)-dimensional Brownian motion \( B = (B_t)_{0 \leq t \leq T} \) whose natural filtration is \( (\mathcal{F}_t^0 := \sigma(B_s, s \leq t))_{0 \leq t \leq T} \). Let \( F = (\mathcal{F}_t)_{0 \leq t \leq T} \) be the completed filtration of \( (\mathcal{F}_t^0)_{0 \leq t \leq T} \) with the \( \mathbb{P} \)-null sets of \( \mathcal{F} \).

Furthermore, let:

- \( \mathcal{P} \) be the \( \sigma \)-algebra on \([0, T] \times \Omega\) of \( F \)-progressively measurable sets;
- \( \mathcal{M}^2 \) be the set of \( \mathcal{P} \)-measurable and \( \mathbb{R}^k \)-valued processes \( w = (w_t)_{t \leq T} \) such that \( E[\int_0^T |w_t|^2 \, dt] < \infty \) and \( S^2 \) be the set of \( \mathcal{P} \)-measurable, continuous processes \( w = (w_t)_{t \leq T} \) such that \( E[\sup_{t \leq T} |w_t|^2] < \infty \);
- for any stopping time \( \tau \in [0, T] \), \( \mathcal{T}_\tau \) denote the set of all stopping times \( \theta \) such that \( \tau \leq \theta \leq T \);
- \( \Pi \) be the class of functions with polynomial growth, defined as follows:

\[ \Pi := \{ \varphi : (t, x) \in [0, T] \times \mathbb{R}^k \rightarrow \varphi(t, x) \in \mathbb{R}, \text{ such that } |\varphi(t, x)| \leq C(1 + |x|^\gamma) \text{ for some non-negative real constants } C \text{ and } \gamma \} \].

Note that if \( \varphi - v \) has a local maximum (respectively, minimum) at \((t, x)\), then we obviously have

\[ (D_i \varphi(t, x), D_i v(t, x), D_{ik}^2 \varphi(t, x)) \in J^2 - v(t, x) \quad \text{(respectively, } J^{2+} v(t, x)) \].

We now give an equivalent definition of a viscosity solution of the parabolic system with interconnected obstacles.

**Definition 3** Let \((v_1, \ldots, v_m)\) be a \( m \)-uplet of continuous functions defined on \([0, T] \times \mathbb{R}^k \), \( \mathbb{R} \)-valued and such that \((v_1, \ldots, v_m)(T, x) = 0\) for any \( x \in \mathbb{R}^k \). The \( m \)-uplet \((v_1, \ldots, v_m)\) is called a viscosity supersolution (respectively, subsolution) of (2.3) if, for any \( i \in \mathcal{I} \), \((t, x) \in (0, T) \times \mathbb{R}^k \) and \((p, q, X) \in J^2 - v_i(t, x) \) (respectively, \( J^{2+} v_i(t, x) \)),

\[ \min \left\{ v_i(t, x) - \max_{j \in \mathcal{I}^i} (-g_{ij}(t, x) + v_j(t, x)), -p - \frac{1}{2} \operatorname{Tr}[\sigma \sigma^T] - \langle b, q \rangle - \psi_i(t, x) \right\} \geq 0 \quad \text{(respectively, } \leq 0) \].

It is called a viscosity solution if it is both a viscosity subsolution and supersolution.
We now make the following assumptions on the data:

[H1]: \( b : [0, T] \times \mathbb{R}^k \to \mathbb{R}^k \) and \( \sigma : [0, T] \times \mathbb{R}^k \to \mathbb{R}^{k \times d} \) are two continuous functions for which there exists a constant \( C > 0 \) such that, for any \( t \in [0, T] \) and \( x, x' \in \mathbb{R}^k \),

\[
|\sigma (t, x)| + |b(t, x)| \leq C(1 + |x|) \quad \text{and} \quad |\sigma (t, x) - \sigma (t, x')| + |b(t, x) - b(t, x')| \leq C|x - x'|. 
\] (2.6)

Throughout this paper, we assume that assumption [H1] holds.

[H2]: For \( i \in \mathcal{I} \), \( \psi_i : [0, T] \times \mathbb{R}^k \to \mathbb{R} \) is continuous and belongs to \( \Pi \).

[H3]: For any \( i, j \in \mathcal{I} \) and \( (t, x) \in [0, T] \times \mathbb{R}^k \):

(i) \( g_{ij} : [0, T] \times \mathbb{R}^k \to \mathbb{R} \) is jointly continuous in \( (t, x) \), could be positive or negative and belongs to \( \Pi \). Moreover, as a convention we assume that \( g_{ii}(t, x) = 0 \).

(ii) For any sequence of indices \( i_1, \ldots, i_k \in \mathcal{I} \) such that \( i_1 = i_k \) and \( \text{card}\{i_1, \ldots, i_k\} = k - 1 \), we have

\[
g_{i_1i_1}(t, x) + g_{i_2i_1}(t, x) + \cdots + g_{i_{k-1}i_1}(t, x) + g_{i_1i_1}(t, x) > 0. \] (2.7)

(iii) For any \( i, j \in \mathcal{I} \), if \( g_{ij} \) is non-positive, we assume that

\[
g_{ij}(T, x) = 0, \quad j \neq i. \] (2.8)

3. The Verification Theorem and existence of the processes \( Y^i, i = 1, \ldots, m \)

Note that in order that the quantity \( J(\delta, \xi) \) makes sense, we assume throughout this paper that, for any \( i, j \in \mathcal{I} \), the processes \( (\psi_i(t, X_t) \}_{t \leq T} \) and \( (g_{ij}(t, X_t) \}_{t \leq T} \) belong to \( \mathcal{M}^{2,1} \) and \( \mathcal{S}^2 \), respectively. There is one-to-one correspondence between the pairs \( (\delta, \xi) \) and the pairs \( (\delta, u) \). Therefore, throughout this paper one refers indifferently to \( (\delta, \xi) \) or \( (\delta, u) \).

3.1 The Verification Theorem

To tackle the problem described above, Djehiche et al. (2009) have introduced a Verification Theorem which is expressed by means of the Snell envelope of processes. The Snell envelope of a stochastic process \( (\eta_t)_{0 \leq t \leq T} \) of \( \mathcal{S}^2 \) (with a possible positive jump at \( T \)) is the lowest supermartingale \( R(\eta) := (R(\eta_t))_{0 \leq t \leq T} \) of \( \mathcal{S}^2 \) such that, for any \( t < T, R(\eta_t) \geq \eta_t \). It has the following expression:

\[
\forall t \leq T, \; R(\eta)_t = \text{ess sup}_{\tau \in \mathcal{T}_t} E[\eta_\tau | \mathcal{F}_t] \quad \text{and satisfies} \; R(\eta)_T = \eta_T. 
\]

For more details, we refer the reader to Cvitanic & Karatzas (1996) and El Karoui (1980).

The following is the Verification Theorem for the \( m \)-states optimal switching problem.

**Theorem 3.1** (Verification Theorem) Assume that, for any \( i, j \in \mathcal{I} \), the following hold:

(a) \( \psi_i \) satisfies (H2);

(b) \( g_{ij} \) satisfies (H3)(i) and (ii);
(c) there exist \( m \) processes \( (Y_i^j := (Y_i^j)_{0 \leq i \leq T}, i = 1, \ldots, m) \) of \( \mathcal{S}^2 \) such that

\[
\forall t \leq T, \quad Y_t^j = \text{ess sup}_{\tau \geq t} E \left[ \int_t^\tau \psi_i(s, X_s) \, ds + \max_{j \in \mathcal{I}} (-g_j(\tau, X_\tau) + Y_j^j) 1_{[\tau < T]} |\mathcal{F}_\tau \right],
\]

\[
Y_T^j = 0.
\]

Then, the following conditions are satisfied:

(i) \( Y_0^1 = \sup_{(\delta, u) \in D} J(\delta, u) \).

(ii) Define the sequence of \( \mathbf{F} \)-stopping times \( \delta = (\tau_n^*)_{n \geq 1} \) as follows:

\[
\tau_1^* = \inf \left\{ s \geq 0, \ Y_s^1 = \max_{j \in \mathcal{I}^{-1}} (-g_j(s, X_s) + Y_j^j), s \leq T \right\} \wedge T.
\]

For \( n \geq 2 \),

\[
\tau_n^* = \inf \left\{ s \geq \tau_{n-1}^*, \ Y_{s-}^n = \max_{k \in \mathcal{I}^{n-1}} \left( -g_{u_{n-1}^k}^*(\tau_{n-1}^*, X_{\tau_{n-1}^*}) + Y_{\tau_{n-1}^*}^k \right), s \leq T \right\} \wedge T,
\]

where

- \( u_{\tau_1^*} = \sum_{j \in \mathcal{I}} 1_{\{u_{\tau_1^*} = j\}} Y_j^j \);
- for any \( n \geq 1 \) and \( t \geq \tau_n^* \), \( Y_t^\mu_{u_{\tau_n^*}} = \sum_{j \in \mathcal{I}} 1_{\{u_{\tau_n^*} = j\}} Y_j^j \);
- for any \( n \geq 2 \), \( u_{\tau_n^*} = l \) on the set

\[
\begin{cases}
\max_{k \in \mathcal{I}^{n-1}} \left( -g_{u_{n-1}^k}^*(\tau_{n-1}^*, X_{\tau_{n-1}^*}) + Y_{\tau_{n-1}^*}^k \right) = -g_{u_{n-1}^k}^*(\tau_{n-1}^*, X_{\tau_{n-1}^*}) + Y_{\tau_{n-1}^*}^k \\
\end{cases}
\]

with \( g_{u_{n-1}^k}^*(\tau_{n-1}^*, X_{\tau_{n-1}^*}) = \sum_{j \in \mathcal{I}} 1_{\{u_{n-1}^k = j\}} g_{jk}^*(\tau_{n-1}^*, X_{\tau_{n-1}^*}) \) and \( \mathcal{I}^{n-1} = \sum_{j \in \mathcal{I}} 1_{\{u_{n-1}^k = j\}} \mathcal{I}^{-j} \).

Then the strategy \( (\delta^*, u^*) \) is optimal and admissible, i.e. it satisfies

\[
-\infty < - \infty \left[ \sum_{k \geq 1} g_{u_{\tau_n^*}^k}^*(\tau_{\tau_n^*}^k, X_{\tau_n^*}^k) 1_{[\tau_n^* < T]} \right] < +\infty \quad \text{and} \quad \mathbb{P}[\tau_n^* < T, \forall n \geq 0] = 0.
\]

**Proof.** The proofs of (i) and (ii) are omitted since they are similar to the ones in Djeihiche et al. (2009) and El Asri (2013a), except for the property \( \mathbb{P}[\tau_n^* < T, \forall n \geq 0] = 0 \) the proof of which we are going to give below.

Let us show now that the optimal strategy \( (\delta^*, u^*) = (\tau_n^*, u_{\tau_n^*})_{n \geq 0} \) is admissible, that is we should prove that \( \mathbb{P}[\tau_n^* < T, \forall n \geq 0] = 0 \). Let us assume the contrary, i.e. \( \mathbb{P}[\tau_n^* < T, \forall n \geq 0] > 0 \) and show by contradiction that is impossible.
Let \( \tau_0^* = 0 \) and \( u_0^* = i \). Owing to the definition of \( \tau_n^* \), we have
\[
\mathbb{P}[Y_{n+1}^{u_n^*} = -g_{u_n^* u_{n+1}^*}(\tau_{n+1}^*, X_{n+1}^{\tau_{n+1}^*}) + Y_{n+1}^{u_{n+1}^*}, u_{n+1}^* \in \mathcal{I}^{-u_n^*}, \forall n \geq 1] > 0.
\]
Since \( \mathcal{I} \) is finite, there is a state \( i_0 \in \mathcal{I} \) and a loop \( i_0, i_1, \ldots, i_k, i_0 \) of elements of \( \mathcal{I} \) such that \( \text{card}\{i_0, i_1, \ldots, i_k\} = k + 1 \) and
\[
\mathbb{P}[Y_{n+1}^{i_{k+1}} = -g_{i_{k+1}}(\tau_{n+1}^*, X_{n+1}^{\tau_{n+1}^*}) + Y_{n+1}^{i_{k+1}}, l = 0, \ldots, k, (i_{k+1} = i_0) \forall n \geq 1] > 0.
\]
Therefore taking the limit w.r.t. \( n \) we obtain
\[
\mathbb{P}[Y_{\tau}^i = -g_{i_0}(\tau, X_\tau) + Y_{\tau}^{i_0}, \quad l = 0, \ldots, k, (i_{k+1} = i_0)] > 0,
\]
where \( \tau := \lim_{n \to \infty} \tau_n^* \). But this implies that
\[
\mathbb{P}[g_{i_0}(\tau, X_\tau) + \cdots + g_{i_k}(\tau, X_\tau) = 0] > 0,
\]
which contradicts (2.7). Therefore \( \mathbb{P}[\tau_n^* < T, \forall n > 0] = 0 \).

Then the optimal strategy is admissible. \( \square \)

**Remark 1** The condition \( \mathbb{P}[\tau_n^* < T, \forall n > 0] = 0 \) means that the sequence \( (\tau_n^*(\omega))_{n \geq 0} \) is stationary; moreover, in economic terms it signifies that the manager is allowed to make only a finite number of decisions during time interval \([0, T]\); otherwise the switching costs would be infinite, and then \( J(\delta, \xi) \) would go to 0.

### 3.2 Existence of the processes \( Y^i, i = 1, \ldots, m \)

We will now establish the existence of the processes \( Y^1, \ldots, Y^m \). They will be obtained as a limit of a sequence of processes \( (Y^{1,n}, \ldots, Y^{m,n}) \) defined recursively by means of the Snell envelope notion as follows:

For \( i \in \mathcal{I} \), we set, for any \( 0 \leq t \leq T \),
\[
Y_{t}^{i,0} = E \left[ \int_t^T \psi_i(s, X_s) \, ds \mid \mathcal{F}_t \right], \quad 0 \leq t \leq T, \tag{3.2}
\]
and, for \( n \geq 1 \),
\[
Y_{t}^{i,n} = \text{ess sup}_{\tau \in \mathcal{T}_t} E \left[ \int_t^\tau \psi_i(s, X_s) \, ds + \max_{k \in \mathcal{I}^{-i}} (-g_{ik}(\tau, X_\tau) + Y_{\tau}^{k,n-1}) 1_{[\tau < T]} \mid \mathcal{F}_t \right], \quad 0 \leq t \leq T. \tag{3.3}
\]

Next we will give some useful properties of \( Y^{1,n}, \ldots, Y^{m,n} \).

**Lemma 3.1** Assume that, for any \( i, j \in \mathcal{I} \),

(i) \( \psi_i \) satisfies (H2);

(ii) \( g_{ij} \) satisfies (H3)(i) and (iii).

Then, for any \( n \geq 0 \) the processes \( Y^{1,n}, \ldots, Y^{m,n} \) are continuous and belong to \( S^2 \).
Proof. Let us show by induction that, for any \( n \geq 0 \) and every \( i \in \mathcal{I} \), the \( Y_{i,n} \)'s are continuous and belong to \( S^2 \).

For \( n = 0 \), the property holds true since we can write \( Y_{i,0} \) as the sum of a continuous process and a martingale w.r.t to the Brownian filtration which is continuous, therefore \( Y_{i,0} \) is continuous and since the process \((\psi_t(s,X_s))_{0 \leq s \leq T}\) belongs to \( \mathcal{M}^2 \), if follows that by using Doob's inequality we obtain that \( Y_{i,0} \) belong to \( S^2 \).

Suppose now that the property is satisfied for some \( n \):

For every \( i \in \mathcal{I} \) and up to a term, \( Y_{i,n+1} \) is the Snell envelope of the process: \( (\int_0^t \psi(s,X_s) \, ds + \max_{k \in \mathcal{I}_{-1}} (-g_k(t,X_t) + Y^{k,n}_t)1_{t < T})_{0 \leq i \leq T} \) and verifies \( Y_{i,n+1} = 0 \).

The process \( \max_{k \in \mathcal{I}_{-1}} (-g_k(t,X_t) + Y^{k,n}_t)1_{t < T} \) is continuous on \([0, T] \) owing to the continuity of \( Y_{i,n} \), and at \( T \) we have two cases: for every \( i, k \in \mathcal{I} \)

(a) If \( g_k(T,X_T) \) is positive, then \( \max_{k \in \mathcal{I}_{-1}} (-g_k(t,X_t) + Y^{k,n}_t)_{t = T} < 0 \). Thus, the process \( \max_{k \in \mathcal{I}_{-1}} (-g_k(t,X_t) + Y^{k,n}_t) \) is continuous on \([0, T] \) and has a positive jump at \( T \) since \( Y_{i,n+1} = 0 \). Then we deduce by Proposition 2 (iii) in Djejiche et al. (2009) that \( Y_{i,n+1} \) is continuous on \([0, T] \).

(b) If \( g_k(T,X_T) \) is negative, then \( \max_{k \in \mathcal{I}_{-1}} (-g_k(t,X_t) + Y^{k,n}_t)_{t = T} = 0 \) since we have, by assumption (H3)-(iii), that \( g_k(T,X_T) = 0 \) for \( k \neq i \). Thus, \( Y_{i,n+1} \) is continuous on \([0, T] \).

Therefore, we deduce that \( Y_{i,n+1} \) is continuous on \([0, T] \) and belongs to \( S^2 \).

This shows that, for any \( n \geq 0 \) and every \( i \in \mathcal{I} \), the \( Y_{i,n} \)'s are continuous and belong to \( S^2 \).

In the Proposition 3.1, we will show that the sequence of processes \((Y_{1,n}, \ldots, Y_{m,n})\) converge increasingly and pointwisely P-a.s. in \( \mathcal{M}^{2,1} \). But to do so, we will need an additional assumption on the negative switching costs:

[H4]: For any \( n \geq 1 \), for any \( \tau_n \in \mathcal{T}_0, \xi_n \in \mathcal{I} \) and \( x \in \mathbb{R}^k \) there exists some real \( K > 0 \) such that

\[
\text{Card}\{0 \leq \tau_n \leq T, n \geq 1 \text{ such that } g_{i_n-1,i_n}(\tau_n, x) < 0\} \leq K. \tag{3.4}
\]

This assumption means that the number of stopping times where we can get negative switching costs is bounded, which means in particular that the number of negative switching costs is also bounded. Actually economically speaking this assumption is realistic since the negative switching costs could be seen as a kind of reward in the form of grants or financial aid given, for example, to power plants using green energy as explained in the introduction. Note that this financial aid could not be infinite (it should be bounded), otherwise that would mean that we could earn money by switching modes of production over and over but this will lead to an infinite value. \( \square \)

We will discuss in the remark below from a mathematical point of view the reason why we impose this assumption.

Remark 2 We need to control the following quantity:

\[
E \left[ \sum_{j=1}^{n} (-g_{i_j-1,i_j}(\tau_j, X_{\tau_j}))1_{[\tau_j < T]} \right].
\]
In fact, we only need to control that quantity over the negative switching costs, since we have the following:

\[
E \left[ \sum_{j=1}^{n} (-g_{u_{j-1}u_{j}} (\tau_j, X_{\tau_j})) 1_{[\tau_j < T]} \right]
\]

\[
\leq E \left[ \sum_{j=1}^{n} (-g_{u_{j-1}u_{j}} (\tau_j, X_{\tau_j})) 1_{[g_{u_{j-1}u_{j}} (\tau_j, X_{\tau_j}) \geq 0]} 1_{[\tau_j < T]} \right]
\]

\[
+ E \left[ \sum_{j=1}^{n} (-g_{u_{j-1}u_{j}} (\tau_j, X_{\tau_j})) 1_{[g_{u_{j-1}u_{j}} (\tau_j, X_{\tau_j}) < 0]} 1_{[\tau_j < T]} \right]
\]

\[
\leq E \left[ \sum_{j=1}^{n} (-g_{u_{j-1}u_{j}} (\tau_j, X_{\tau_j})) 1_{[g_{u_{j-1}u_{j}} (\tau_j, X_{\tau_j}) < 0]} 1_{[\tau_j < T]} \right].
\]

Now applying assumption (H4) on the last term of the previous inequality, we obtain that

\[
E \left[ \sum_{j=1}^{n} (-g_{u_{j-1}u_{j}} (\tau_j, X_{\tau_j})) 1_{[\tau_j < T]} \right] \leq KE \left[ \max_{l\neq j, j\neq l} \left\{ \sup_{s \leq T} (-g_{ij}(s, X_s)) \right\} \right]. \quad (3.5)
\]

Note that without assumption (H4) we can only get that

\[
E \left[ \sum_{j=1}^{n} (-g_{u_{j-1}u_{j}} (\tau_j, X_{\tau_j})) 1_{[\tau_j < T]} \right] < \infty.
\]

**Proposition 3.1** Assume that (H2), (H3) and (H4) are fulfilled. Then, for any \( i \in \mathcal{I} \), the sequence \((Y_{i,n})_{n \geq 0}\) converges increasingly and pointwisely P-a.s. for any \( 0 \leq t \leq T \) and in \( \mathcal{M}^{2,1} \) to càdlàg processes \( \tilde{Y}_{i} \).

Moreover, these limit processes \( \tilde{Y}_{i} = (\tilde{Y}_{i})_{0 \leq t \leq T}, i = 1, \ldots, m \), satisfy the following:

(a) \( E[\sup_{0 \leq t \leq T} |\tilde{Y}_{i}|^2] < +\infty, \ i \in \mathcal{I}. \)

(b) For any \( 0 \leq t \leq T \), we have

\[
\tilde{Y}_{i} = \text{ess sup}_{r \in \mathcal{T}_i} E \left[ \int_{t}^{r} \psi_{i}(s, X_{s}) \ ds + \max_{k \in \mathcal{I}_i} (-g_{ik}(\tau, X_{\tau}) + \tilde{Y}_{k}) 1_{[\tau < T]}|\mathcal{F}_{\tau} \right]. \quad (3.6)
\]

**Proof.** Let us now set \( D_{i}^{n} = \{ u \in \mathcal{D} \text{ such that } u_0 = i, \ \tau_1 \geq t \text{ and } \tau_{n+1} = T \}. \)

Using the same arguments as the ones of the Verification Theorem, the following characterization of the processes \( Y_{i,n} \) holds true:

\[
Y_{i,n} = \text{ess sup}_{u \in D_{i}^{n}} E \left[ \int_{t}^{T} \psi_{u}(s, X_{s}) \ ds - \sum_{j=1}^{n} g_{u_{j-1}u_{j}} (\tau_j, X_{\tau_j}) 1_{[\tau_j < T]}|\mathcal{F}_{\tau} \right].
\]
Since $D^{i,n}_t \subset D^{i,n+1}_t$, we have P-a.s. for all $t \in [0,T]$, $Y^{i,n}_t \leq Y^{i,n+1}_t$ owing to the continuity of $Y^{i,n}$. Moreover, we have

$$Y^{i,n}_t = \text{ess sup}_{u \in D^{i,n}_t} \left[ \int_t^T \psi_u(s, X_s) \, ds - \sum_{j=1}^n g_{u_{j-1}u_j}(\tau_j, X_{\tau_j}) 1_{[\tau_j < T]} | \mathcal{F}_t \right]$$

$$\leq \text{ess sup}_{u \in D^{i,n}_t} \left[ \int_t^T \psi_u(s, X_s) \, ds | \mathcal{F}_t \right] + \text{ess sup}_{u \in D^{i,n}_t} \left[ \sum_{j=1}^n (-g_{u_{j-1}u_j}(\tau_j, X_{\tau_j}) 1_{[\tau_j < T]} \right] | \mathcal{F}_t \right]$$

$$\leq E \left[ \int_t^T \max_{i \in \mathcal{I}} |\psi_i(s, X_s)| \, ds | \mathcal{F}_t \right] + \text{ess sup}_{u \in D^{i,n}_t} \left[ \sum_{j=1}^n (-g_{u_{j-1}u_j}(\tau_j, X_{\tau_j}) 1_{[\tau_j < T]} | \mathcal{F}_t \right].$$

(3.7)

Now using assumption (H4) and Remark 2 (inequality (3.5)), we get that

$$Y^{i,n}_t \leq E \left( \int_t^T \max_{i \in \mathcal{I}} |\psi_i(s, X_s)| \, ds | \mathcal{F}_t \right) + KE \left[ \max_{l \in \mathcal{I}, j \neq l} \left\{ \sup_{s \leq T} (-g_{l,j}(s, X_s)) \right\} | \mathcal{F}_t \right].$$

Therefore, for every $i \in \mathcal{I}$ the sequence $(Y^{i,n})_{n \geq 0}$ satisfies

$$Y^{i,n}_t \leq Y^{i,n+1}_t \leq E \left( \int_t^T \max_{i \in \mathcal{I}} |\psi_i(s, X_s)| \, ds | \mathcal{F}_t \right) + KE \left[ \max_{l \in \mathcal{I}, j \neq l} \left\{ \sup_{s \leq T} (-g_{l,j}(s, X_s)) \right\} | \mathcal{F}_t \right] \quad \forall t \in [0,T].$$

(3.8)

Now since $\psi_i$ and $g_{ij}$ belong to $\mathcal{M}^{2,1}$ and $\mathcal{S}^2$, respectively. Then $(Y^{i,n})_{n \geq 0}$ converges to some limit $\tilde{Y}^i_t := \lim_{n \to \infty} Y^{i,n}_t$ that satisfies: $\forall i \in \mathcal{I}$

$$Y^{i,0}_t \leq \tilde{Y}^i_t \leq E \left( \int_t^T \max_{i \in \mathcal{I}} |\psi_i(s, X_s)| \, ds | \mathcal{F}_t \right) + KE \left[ \max_{l \in \mathcal{I}, j \neq l} \left\{ \sup_{s \leq T} (-g_{l,j}(s, X_s)) \right\} | \mathcal{F}_t \right] \quad \forall t \in [0,T].$$

(3.9)

Next, using (3.9), the fact that $\psi_i$ and $g_{ij}$ belong to $\mathcal{M}^{2,1}$ and $\mathcal{S}^2$, respectively, and Doob’s Maximal Inequality yield, for each $i \in \mathcal{I}$,

$$E \left( \sup_{t \leq T} |\tilde{Y}^i_t|^2 \right) < +\infty.$$

By the Lebesgue Dominated Convergence Theorem, the sequence $(Y^{i,n})_{n \geq 0}$ also converges to $\tilde{Y}^i$ in $\mathcal{M}^{2,1}$.

Let us now show that $\tilde{Y}^i$ is càdlàg. Actually, for each $i \in \mathcal{I}$ and $n \geq 1$, by (3.3) the process $(Y^{i,n} + \int_0^t \psi(s, X_s) \, ds)_{0 \leq t \leq T}$ is a continuous supermartingale. Hence its limit process $(\tilde{Y}^i + \int_0^t \psi(s, X_s) \, ds)_{0 \leq t \leq T}$ is càdlàg as a limit of increasing sequence of continuous supermartingales (see Dellacherie & Meyer, 1980, p.86). Therefore, $\tilde{Y}^i$ is càdlàg.

Finally, since the càdlàg processes $\tilde{Y}^1, \ldots, \tilde{Y}^m$ are limits of the sequence of increasing continuous processes $Y^{i,n}, i \in \mathcal{I}$ that satisfy (3.3), it follows that by Snell envelope properties, the processes...
\[ \tilde{Y}^1, \ldots, \tilde{Y}^m \] satisfy the following: for any \( 0 \leq t \leq T, i = 1, \ldots, m \)

\[
\tilde{Y}_t^i = \text{ess sup}_{r \in \mathcal{I}_i} E \left[ \int_t^T \psi_i(s, X_s) \, ds + \max_{k \in \mathcal{I}^{-i}} \left( -g_{ik}(t, X_t) + \tilde{Y}_t^k \right) \mathbf{1}_{[t < T]} | \mathcal{F}_t \right],
\]

which is the desired result. \( \square \)

We will now prove that the processes \( \tilde{Y}^1, \ldots, \tilde{Y}^m \) are continuous and satisfy the Verification Theorem (Theorem 3.1).

**Theorem 3.2** Assume that (H2), (H3) and (H4) hold. Then the limit processes \( \tilde{Y}^1, \ldots, \tilde{Y}^m \) satisfy the Verification Theorem (Theorem 3.1).

**Proof.** Recall from Proposition 3.1 that the processes \( \tilde{Y}^1, \ldots, \tilde{Y}^m \) are càdlàg and uniformly \( L^2 \)-integrable and satisfy (3.6). It remains to prove that they are continuous.

Indeed, note that, for \( t_i \in \mathcal{I} \), the process \( \left( \tilde{Y}_t^i + \int_{t_0}^t \psi_i(s, X_s) \, ds \right)_{0 \leq t \leq T} \) is the Snell envelope of

\[
\left( \int_{t_0}^t \psi_i(s, X_s) \, ds + \max_{k \in \mathcal{I}^{-i}} \left( -g_{ik}(t, X_t) + \tilde{Y}_t^k \right) \mathbf{1}_{[t < T]} \right)_{0 \leq t \leq T}
\]

since the processes \( \left( \int_{t_0}^t \psi_i(s, X_s) \, ds \right)_{0 \leq t \leq T} \) are continuous. Therefore, from the property of the jumps of the Snell envelope (see Djejiche et al., 2009; Proposition 2 (ii)), when there is a (necessarily negative) jump of \( \tilde{Y}_t^i \) at \( t \), there is a jump, at the same time \( t \), of the process \( \left( \max_{k \in \mathcal{I}^{-i}} \left( -g_{ik}(t, X_t) + \tilde{Y}_t^k \right) \right)_{0 \leq t \leq T} \). Since \( g_{ij} \) are continuous, there is \( j \in \mathcal{I}^{-i} \) such that \( \Delta_t \tilde{Y}_t^j < 0 \), and \( \tilde{Y}_t^j = -g_{ij}(t, X_t) + \tilde{Y}_t^i \).

Suppose now there is an index \( i_1 \in \mathcal{I} \) for which there exists \( t \in [0, T] \) such that \( \Delta_t \tilde{Y}_t^{i_1} < 0 \). This implies that there exists another index \( i_2 \in \mathcal{I}^{-i_1} \) such that \( \Delta_t \tilde{Y}_t^{i_2} < 0 \) and \( \tilde{Y}_t^{i_2} = -g_{i_2 i_1}(t, X_t) + \tilde{Y}_t^{i_1} \). But given \( i_2 \), there exists an index \( i_3 \in \mathcal{I}^{-i_2} \) such that \( \Delta_t \tilde{Y}_t^{i_3} < 0 \) and \( \tilde{Y}_t^{i_3} = -g_{i_3 i_2}(t, X_t) + \tilde{Y}_t^{i_2} \). Repeating this argument many times, we get a sequence of indices \( i_1, \ldots, i_r, \ldots \in \mathcal{I} \) that have the property that \( i_k \in \mathcal{I}^{-i_{k-1}}, \Delta_t \tilde{Y}_t^{i_k} < 0 \) and \( \tilde{Y}_t^{i_k} = -g_{i_k i_{k-1}}(t, X_t) + \tilde{Y}_t^{i_{k-1}} \).

Since \( \mathcal{I} \) is finite, there exist two indices \( q < r \) such that \( i_q = i_r \) and \( i_q, i_{q+1}, \ldots, i_{r-1} \) are mutually different. It follows that

\[
\tilde{Y}_t^{i_q} = -g_{i_q i_{q+1}}(t, X_t) + \tilde{Y}_t^{i_{q+1}} = -g_{i_q i_{q+1}}(t, X_t) - g_{i_{q+1} i_{q+2}}(t, X_t) + \tilde{Y}_t^{i_{q+2}} = \cdots = -g_{i_q i_{q+1}}(t, X_t) - \cdots - g_{i_{r-1} i_r}(t, X_t) + \tilde{Y}_t^{i_r}.
\]

As \( i_q = i_r \), we get

\[
-g_{i_q i_{q+1}}(t, X_t) - \cdots - g_{i_{r-1} i_r}(t, X_t) = 0,
\]

which contradicts assumption (2.7). Therefore, there is no \( i \in \mathcal{I} \) for which there is a \( t \in [0, T] \) such that \( \Delta_t \tilde{Y}_t^i < 0 \). This means that the processes \( \tilde{Y}^1, \ldots, \tilde{Y}^m \) are continuous. Since they satisfy (3.6), by uniqueness \( \tilde{Y}_t^i = \tilde{Y}_t^i \) for any \( i \in \mathcal{I} \).

Thus the Verification Theorem (Theorem 3.1) is satisfied by \( Y^1, \ldots, Y^m \). \( \square \)
4. Existence of a solution for the system of variational inequalities

In this section, we will address the question of existence of a solution for the system of variational inequalities (2.3). But first let is make the link between those solutions and BSDEs with one reflecting barrier in the Markovian framework.

4.1 Connection with BSDEs with one reflecting barrier

Let \((t,x) \in [0,T] \times \mathbb{R}^k\) and let \((X^{tx}_s)_{s \leq T}\) be the solution of the following standard SDE:

\[
dX^{tx}_s = b(s, X^{tx}_s) \, ds + \sigma (s, X^{tx}_s) \, dB_s \quad \text{for } t \leq s \leq T \quad \text{and} \quad X^{tx}_T = x \quad \text{for } s \leq t
\]

where the functions \(b\) and \(\sigma\) are the ones of (2.6). These properties of \(\sigma\) and \(b\) imply in particular that the process \((X^{tx}_s)_{0 \leq s \leq T}\) is a solution of the standard SDE (4.1) exists and is unique for any \(t \in [0,T]\) and \(x \in \mathbb{R}^k\).

The operator \(A\) that appears in (2.4) is the infinitesimal generator associated with \(X^{tx}\). In the following result, we collect some properties of \(X^{tx}\).

**Proposition 4.1** (see, e.g. Revuz & Yor, 1991) The process \(X^{tx}\) satisfies the following estimates:

(i) For any \(q \geq 2\), there exists a constant \(C\) such that

\[
E \left[ \sup_{0 \leq s \leq T} |X^{tx}_s|^q \right] \leq C(1 + |x|^q). \tag{4.2}
\]

(ii) There exists a constant \(C\) such that, for any \(t, t' \in [0,T]\) and \(x, x' \in \mathbb{R}^k\),

\[
E \left[ \sup_{0 \leq s \leq T} |X^{tx}_s - X^{tx'}_{s}|^2 \right] \leq C(1 + |x|^2)(|x - x'|^2 + |t - t'|). \tag{4.3}
\]

We are going now to introduce the notion of a BSDE with one reflecting barrier introduced in El Karoui et al. (1997). This notion will allow us to make the connection between the variational inequalities system and the \(m\)-states optimal switching problem described in the previous section.

So let us introduce the deterministic functions \(f : [0,T] \times \mathbb{R}^{k+1+d} \to \mathbb{R}\), \(h : [0,T] \times \mathbb{R}^k \to \mathbb{R}\) and \(g : \mathbb{R}^k \to \mathbb{R}\) continuous, of polynomial growth and such that \(h(x,T) \leq g(x)\). Moreover, we assume that, for any \((t,x) \in [0,T] \times \mathbb{R}^k\), the mapping \((y,z) \in \mathbb{R}^{1+d} \mapsto f(t,x,y,z)\) is uniformly Lipschitz. Then we have the following result related to BSDEs with one reflecting barrier.

**Theorem 4.1** (El Karoui et al., 1997, Theorems 5.2 and 8.5) For any \((t,x) \in [0,T] \times \mathbb{R}^k\) there exits a unique triple of processes \((Y^{tx}, Z^{tx}, K^{tx})\) such that

\[
\begin{align*}
Y^{tx}_s &= g(X^{tx}_T) + \int_s^T f(r, X^{tx}_r, Y^{tx}_r, Z^{tx}_r) \, dr - \int_s^T Z^{tx}_r \, dB_r + K^{tx}_T - K^{tx}_s, \quad s \leq T, \\
Y^{tx}_s &\geq h(s, X^{tx}_s), \quad \forall s \leq T \quad \text{and} \quad \int_0^T (Y^{tx}_r - h(r, X^{tx}_r)) \, dK^{tx}_r = 0.
\end{align*}
\]
Moreover, the following characterization of $Y^{ix}$ as a Snell envelope holds true:

$$\forall s \leq T, \quad Y^{ix}_s = \text{ess sup}_{r \in \mathcal{T}_s} E \left[ \int_t^T f(r, X^ix_r, Y^ix_r, Z^ix_r) \, dr + h(t, X^ix_t) \mathbf{1}_{[t \leq T]} + g(X^ix_T) \mathbf{1}_{[t = T]} \mid \mathcal{F}_s \right]. \tag{4.5}$$

On the other hand, there exists a deterministic continuous with polynomial growth function $u : [0, T] \times \mathbb{R}^k \to \mathbb{R}$ such that

$$\forall s \in [t, T], \quad Y^{ix}_s = u(s, X^{ix}_s).$$

Moreover, the function $u$ is the unique viscosity solution in the class of continuous functions with polynomial growth of the following PDE with obstacle:

$$\begin{cases}
\min\{u(t, x) - h(t, x), -\partial_t u(t, x) - Au(t, x) - f(t, x, u(t, x), \sigma(t, x)^* \nabla u(t, x))\} = 0, \\
u(T, x) = g(x).
\end{cases}$$

4.2 Existence of a solution for the system of variational inequalities

Let $(Y^{i,ix}_1, \ldots, Y^{n,ix}_n)_{0 \leq s \leq T}$ be the processes which satisfy the Verification Theorem 3.1 in the case when the process $X \equiv X^{ix}$. Therefore, using the characterization (4.5), there exist processes $K^{i,ix}$ and $Z^{i,ix}$, $i \in I$, such that the triples $(Y^{i,ix}, Z^{i,ix}, K^{i,ix})$ are unique solutions of the following reflected BSDEs: for any $i = 1, \ldots, m$ we have,

$$\begin{cases}
Y^{i,ix} \in S^2 \text{ and } Z^{i,ix} \in \mathcal{M}^{2, d}, \quad K^{i,ix}_0 = 0, \\
Y^{i,ix}_s = \int_s^T \psi_i(r, X^{ix}_r) \, dr - \int_s^T Z^{i,ix}_r \, dB_r + K^{i,ix}_r - K^{i,ix}_s, \quad 0 \leq s \leq T, \quad Y^{i,ix}_T = 0, \\
Y^{i,ix}_s \geq \max_{j \in I^{-i}} (-g_{ij}(s, X^{ix}_s) + Y^{j,ix}_s), \quad 0 \leq s \leq T, \\
\int_0^T \left( Y^{i,ix}_r - \max_{j \in I^{-i}} (-g_{ij}(r, X^{ix}_r) + Y^{j,ix}_r) \right) \, dK^{i,ix}_r = 0.
\end{cases} \tag{4.6}$$

Moreover, we have the following result.

**Proposition 4.2** Assume that (H2), (H3) and (H4) hold. Then there are deterministic functions $v^1, \ldots, v^m : [0, T] \times \mathbb{R}^k \to \mathbb{R}$ such that

$$\forall (t, x) \in [0, T] \times \mathbb{R}^k, \forall s \in [t, T], \quad Y^{i,ix}_s = v_i(s, X^{ix}_s), \quad i = 1, \ldots, m.$$

Moreover, the functions $v_i, i = 1, \ldots, m$, are of polynomial growth.

**Proof.** For $n \geq 0$ let $(Y^{n,1,ix}_1, \ldots, Y^{n,m,ix}_n)_{0 \leq s \leq T}$ be the processes constructed in (3.2)–(3.3). Therefore, using an induction argument and Theorem 4.1 there exist deterministic continuous with polynomial growth functions $v_{ij,n} (i = 1, \ldots, m)$ such that, for any $(t, x) \in [0, T] \times \mathbb{R}^k, \forall s \in [t, T], Y^{i,ix}_s = v_{ij,n}(s, X^{ix}_s)$. Next, since we have that

$$Y^{i,ix}_t = \text{ess sup}_{u \in \mathcal{D}^{ix}_t} E \left[ \int_t^T \psi_{ij}(s, X^{ix}_s) \, ds + \sum_{j=1}^n (-g_{ij} v_{ij,n}(\tau_j, X^{ix}_{\tau_j})) \mathbf{1}_{[\tau_j < T]} \mid \mathcal{F}_t \right],$$
Proof. Let us show that, for every 

\[ Y^{i,n \, t^x}_t \leq Y^{i,n+1 \, t^x}_t \leq E \left[ \int_t^T \max_{i \in I} |\psi_i(s, X^{t^x}_s)| \, ds \big| \mathcal{F}_t \right] + KE \left[ \max_{l,j \in \mathcal{E}, \ l \neq j} \left( \sup_{s \leq T} (-g_{ij}(s, X^{t^x}_s)) \right) \right]. \]

Moreover, 

\[ Y^{i,n \, t^x}_t \leq Y^{i,n+1 \, t^x}_t \leq E \left[ \int_t^T \max_{i \in I} |\psi_i(s, X^{t^x}_s)| \, ds \big| \mathcal{F}_t \right] + KE \left[ \max_{l,j \in \mathcal{E}, \ l \neq j} \left( \sup_{s \leq T} (-g_{ij}(s, X^{t^x}_s)) \right) \right]. \]

Therefore, combining the polynomial growth of \( \psi_i \) and \( g_{ij} \) and estimating (4.2) for \( X^{t^x} \), we obtain 

\[ v_{i,n}(t,x) \leq v_{i,n+1}(t,x) \leq C_K (1 + |x|^\gamma), \]

for some constants \( C_K \) and \( \gamma \) independent of \( n \). In order to complete the proof, it is enough now to set 

\[ v_i(t,x) := \lim_{n \to \infty} v_{i,n}(t,x), \quad (t,x) \in [0,T] \times \mathbb{R}^k \] since \( Y^{i,n \, t^x}_t \nrightarrow Y^{i \, t^x}_t \) as \( n \to \infty \).

We are now going to focus on the continuity of the functions \( v_1, \ldots, v_m \).

**Theorem 4.2** Assume that (H2), (H3) and (H4) are fulfilled. Then the functions \( (v_1, \ldots, v_m)(t,x) : [0,T] \times \mathbb{R}^k \to \mathbb{R} \) are continuous and solutions in the viscosity sense of the system of variational inequalities with interconnected obstacles (2.3).

**Proof.** Let us show that, for every \( i \in \mathcal{I} \), \( v_i \) is continuous. First, we have that \( v_i(t,x) = Y^{i,t^x}_t \); then

\[ |Y^{i,n \, t^x}_t - Y^{i,n' \, t^x}_t| \leq |Y^{i,n \, t^x}_t - Y^{i,n \, t^x}_t| + |Y^{i,n \, t^x}_t - Y^{i,n \, t^x}_t| + |Y^{i,n \, t^x}_t - Y^{i,n \, t^x}_t|. \] (4.7)

Moreover,

\[ |Y^{i,n \, t^x}_t - Y^{i,n \, t^x}_t| \leq \sup_{0 \leq s \leq T} |Y^{i,n \, t^x}_s - Y^{i,n \, t^x}_s| + |Y^{i,n \, t^x}_s - Y^{i,n \, t^x}_s| \]

\[ \leq \sup_{0 \leq s \leq T} |Y^{i,n \, t^x}_s - Y^{i,n \, t^x}_s| + |Y^{i,n \, t^x}_s - Y^{i,n \, t^x}_s|. \] (4.8)

Next we will show the \( L^2 \)-continuity of the value functions \( (t,x) \to Y^{i,n \, t^x}_t \). Recall that 

\[ Y^{i,n \, t^x}_s = \operatorname{ess} \sup_{u \in \mathcal{D}_{t,s}^{n \, t^x}} \left[ \int_s^T \psi_u(r, X^{t^x}_r) \, dr - \sum_{j=1}^n g_{a_{n-1 \to n}} (r_j, X^{t^x}_{t_j}) 1_{[r_j < T]} \big| \mathcal{F}_s \right]. \]
where $\mathcal{D}_{s}^{i,n} = \{ u \in \mathcal{D} \text{ such that } u_{0} = i, \tau_{1} \geq s, \text{ and } \tau_{n+1} = T \}$. Therefore

\[
|Y_{s}^{i,n,tx} - Y_{s}^{i,n,t'x'}| = \sup_{u \in \mathcal{D}_{s}^{i,n}} E \left[ \int_{s}^{T} \psi_{u_{r}}(r, X_{r}^{tx}) \, dr - \sum_{j=1}^{n} g_{u_{r-1}}(\tau_{j}, X_{\tau_{j}}^{tx}) 1_{[\tau_{j} < T]} \mid \mathcal{F}_{s} \right] 
\]

\[
- \sup_{u \in \mathcal{D}_{s}^{i,n}} E \left[ \int_{s}^{T} \psi_{u_{r}}(r, X_{r}^{t'x'}) \, dr - \sum_{j=1}^{n} g_{u_{r-1}}(\tau_{j}, X_{\tau_{j}}^{t'x'}) 1_{[\tau_{j} < T]} \mid \mathcal{F}_{s} \right] 
\]

\[
\leq \sup_{u \in \mathcal{D}_{s}^{i,n}} E \left[ \int_{s}^{T} |\psi_{u_{r}}(r, X_{r}^{tx}) - \psi_{u_{r}}(r, X_{r}^{t'x'})| \, dr 
+ \sum_{j=1}^{n} |(g_{u_{r-1}}(\tau_{j}, X_{\tau_{j}}^{tx}) - g_{u_{r-1}}(\tau_{j}, X_{\tau_{j}}^{t'x'})) 1_{[\tau_{j} < T]}| \mid \mathcal{F}_{s} \right] 
\]

\[
\leq E \left[ \int_{0}^{T} \max_{i \in I} \left| \psi_{i}(r, X_{r}^{tx}) - \psi_{i}(r, X_{r}^{t'x'}) \right| \, dr 
+ n \max_{i \in I, j \neq i} \left\{ \sup_{0 \leq r \leq T} \left| g_{ij}(r, X_{r}^{tx}) - g_{ij}(r, X_{r}^{t'x'}) \right| \right\} \mid \mathcal{F}_{s} \right] .
\]

Now using Doob's Maximal Inequality and taking expectation, there exists a constant $C > 0$ such that

\[
E \left[ \sup_{0 \leq s \leq T} \left| Y_{s}^{i,n,tx} - Y_{s}^{i,n,t'x'} \right|^2 \right] 
\]

\[
\leq CE \left[ \int_{0}^{T} \max_{i \in I} \left| \psi_{i}(r, X_{r}^{tx}) - \psi_{i}(r, X_{r}^{t'x'}) \right|^2 \, dr 
+ n \max_{i \in I, j \neq i} \left\{ \sup_{0 \leq r \leq T} \left| g_{ij}(r, X_{r}^{tx}) - g_{ij}(r, X_{r}^{t'x'}) \right| \right\}^2 \right] .
\]

(4.9)

In the right-hand side of (4.9) the first term converges to 0 as $(t', x') \to (t, x)$. Indeed, for any $\rho > 0$ it holds true that

\[
E \left[ \int_{0}^{T} \max_{i \in I} \left| \psi_{i}(r, X_{r}^{tx}) - \psi_{i}(r, X_{r}^{t'x'}) \right|^2 \, dr \right] 
\]

\[
\leq E \left[ \int_{0}^{T} \max_{i \in I} \left| \psi_{i}(r, X_{r}^{tx}) - \psi_{i}(r, X_{r}^{t'x'}) \right|^2 1_{|X_{r}^{tx}| + |X_{r}^{t'x'}| \leq \rho} \, dr \right] 
\]

\[
+ E \left[ \int_{0}^{T} \max_{i \in I} \left| \psi_{i}(r, X_{r}^{tx}) - \psi_{i}(r, X_{r}^{t'x'}) \right|^2 1_{|X_{r}^{tx}| + |X_{r}^{t'x'}| > \rho} \, dr \right] .
\]

By the Lebesgue Dominated Convergence Theorem, the continuity of $\psi_{i}$ and estimates (4.3), the first term of the right-hand side of this inequality converges to 0 as $(t', x') \to (t, x)$.
The second term satisfies
\[
E \left[ \int_0^T \max_{i \in \mathcal{I}} |\psi_i(r, X_{i}^{\text{fs}}) - \psi_i(r, X_{r}^{\text{fs}})|^2 1_{|X_{r}^{\text{fs}}| + |X_{i}^{\text{fs}}| > \rho} \, dr \right]
\]
\[
\leq \left\{ E \left[ \int_0^T \max_{i \in \mathcal{I}} |\psi_i(r, X_{i}^{\text{fs}}) - \psi_i(r, X_{r}^{\text{fs}})|^4 \, dr \right] \right\}^{1/2} \left\{ E \left[ \int_0^T 1_{|X_{r}^{\text{fs}}| + |X_{i}^{\text{fs}}| > \rho} \, dr \right] \right\}^{1/2}
\]
\[
\leq \left\{ E \left[ \int_0^T \max_{i \in \mathcal{I}} |\psi_i(r, X_{i}^{\text{fs}}) - \psi_i(r, X_{r}^{\text{fs}})|^4 \, dr \right] \right\}^{1/2} \left\{ \rho^{-1} E \left[ \int_0^T (|X_{r}^{\text{fs}}| + |X_{i}^{\text{fs}}|) \, dr \right] \right\}^{1/2}.
\]
Using estimates (4.2) and the polynomial growth of \( \psi_i \), then we have
\[
E \left[ \int_0^T \max_{i \in \mathcal{I}} |\psi_i(r, X_{i}^{\text{fs}}) - \psi_i(r, X_{r}^{\text{fs}})|^2 1_{|X_{r}^{\text{fs}}| + |X_{i}^{\text{fs}}| > \rho} \, dr \right] \leq C(1 + |x|^\gamma + |x'|^\gamma) \rho^{-1/2},
\]
where \( C \) and \( \gamma \) are real constants which are bound to the polynomial growth of \( \psi_i \) and estimate (4.2). As \( \rho \) is arbitrary, then making \( \rho \to +\infty \) we obtain that
\[
\lim_{(t,x) \to (t',x')} E \left[ \int_0^T \max_{i \in \mathcal{I}} |\psi_i(r, X_{i}^{\text{fs}}) - \psi_i(r, X_{r}^{\text{fs}})|^2 1_{|X_{r}^{\text{fs}}| + |X_{i}^{\text{fs}}| > \rho} \, dr \right] \to 0.
\]
Thus the claim is proved.

In the same way, we have
\[
\lim_{(t,x) \to (t',x')} E \left[ \max_{i \in \mathcal{I}, j \in I_s} \sup_{0 \leq r \leq T} |g_{ij}(r, X_{i}^{\text{fs}}) - g_{ij}(r, X_{r}^{\text{fs}})|^2 \right] \to 0.
\]
Then the right-hand side of (4.9) converges to 0 as \((t',x') \to (t,x)\). Thus, we obtain that
\[
E \left[ \sup_{0 \leq s \leq T} |Y_{s}^{i,n,t,x} - Y_{s}^{i,n,t',x'}|^2 \right] \to 0 \quad \text{as} \quad (t',x') \to (t,x). \tag{4.10}
\]
Then the function \((s,t,x) \to Y_{s}^{i,n,t,x}\) is continuous from \([0,T]^2 \times \mathbb{R}^k\) into \(L^2\), which is the desired result.

Next, recall that, from (4.7) and (4.8), we have
\[
|Y_{t}^{i,t,x} - Y_{t}^{i,t',x'}| \leq |Y_{t}^{i,t,x} - Y_{t}^{i,n,t,x}| + \sup_{0 \leq s \leq T} |Y_{s}^{i,n,t,x} - Y_{s}^{i,n,t',x'}| + |Y_{t}^{i,n,t,x} - Y_{t}^{i,n,t',x'}| + |Y_{t}^{i,t,n,x'} - Y_{t}^{i,t',x'}|. \tag{4.11}
\]
We put \( n \to +\infty \) and using the fact that \( Y_{t}^{i,n,t,x} \) is deterministic, \( Y_{t}^{i,n,t,x} \) converges to \( Y_{t}^{i,t,x} \) locally uniformly and the continuity of \( Y_{t}^{i,n,t,x} \) in \( t \), together with (4.10) we get that the right-hand side terms of (4.11) converge to 0 as \((t',x') \to (t,x)\). Therefore, \( Y_{t}^{i,t',x'} \to Y_{t}^{i,t,x} \) as \((t',x') \to (t,x)\). Next, since by Proposition 4.2, we have \forall i \in \mathcal{I} \ Y_{t}^{i,t,x} = v_i(t,x) \), the deterministic functions \( v_1, \ldots, v_m \) are continuous in \((t,x)\), moreover they are of polynomial growth. Then taking into account Theorem 4.1 implies that \((v_1, \ldots, v_m)\) is a viscosity solution for the system of variational inequalities with interconnected obstacles (2.3). The proof of Theorem 4.2 is now complete. \(\square\)
5. Uniqueness of the solution of the system of Variational Inequalities

In this section, we are going to show the uniqueness of the viscosity solution of the system (2.3). We first need the following lemma.

**Lemma 5.1** Let \((v_i)_{i=1,\ldots,m}\) be a supersolution of the system (2.3); then, for any \(\gamma \geq 0\), there exists \(\alpha > 0\) such that, for any \(\lambda \geq \alpha\) and \(\theta > 0\), the \(m\)-uplet \((v_i(t,x) + \theta e^{-\lambda t}|x|^{2\gamma + 2})_{i=1,\ldots,m}\) is a supersolution for (2.3).

**Proof.** We assume w.l.o.g. that the functions \((v_i(t,x))_{i=1,\ldots,m}\) are lsc. Let \(i \in \mathcal{I}\) be fixed and let \(\varphi \in C^{1,2}\) be such that the function \(\varphi - (v_i + \theta e^{-\lambda t}|x|^{2\gamma + 2})\) has a local maximum in \((t,x)\) which is equal to 0. Since \((v_i(t,x))_{i=1,\ldots,m}\) is a supersolution for (2.3), it follows that we have \(\forall i \in \mathcal{I},\)

\[
\min \left\{ v_i(t,x) - \max_{j \in \mathcal{I} - i} [-g_{ij}(t,x) + v_j(t,x)], -\partial_t (\varphi(t,x) - \theta e^{-\lambda t}|x|^{2\gamma + 2}) - \frac{1}{2} \text{Tr} \left[ \sigma \sigma^*(t,x) D^2_x (\varphi(t,x) - \theta e^{-\lambda t}|x|^{2\gamma + 2}) \right] - D_x (\varphi(t,x) - \theta e^{-\lambda t}|x|^{2\gamma + 2}) b(t,x) - \phi_i(t,x) \right\} \geq 0.
\]

Hence

\[
(v_i(t,x) + \theta e^{-\lambda t}|x|^{2\gamma + 2}) - \max_{j \in \mathcal{I} - i} (-g_{ij}(t,x) + (v_j(t,x) + \theta e^{-\lambda t}|x|^{2\gamma + 2})) = v_i(t,x) - \max_{j \in \mathcal{I} - i} (-g_{ij}(t,x) + v_j(t,x)) \geq 0. \tag{5.1}
\]

On the other hand,

\[
-\partial_t (\varphi(t,x) - \theta e^{-\lambda t}|x|^{2\gamma + 2}) - \frac{1}{2} \text{Tr} \left[ \sigma \sigma^*(t,x) D^2_x (\varphi(t,x) - \theta e^{-\lambda t}|x|^{2\gamma + 2}) \right] - D_x (\varphi(t,x) - \theta e^{-\lambda t}|x|^{2\gamma + 2}) b(t,x) - \phi_i(t,x) \geq 0.
\]

Thus

\[
-\partial_t \varphi(t,x) - \frac{1}{2} \text{Tr} \left[ \sigma \sigma^*(t,x) D^2_x \varphi(t,x) \right] - D_x (\varphi(t,x)) b(t,x) - \phi_i(t,x) \geq \theta \lambda e^{-\lambda t}|x|^{2\gamma + 2} - \frac{1}{2} \theta e^{-\lambda t} \text{Tr} [\sigma \sigma^*(t,x) D^2_x |x|^{2\gamma + 2}] - \theta e^{-\lambda t} D_x (|x|^{2\gamma + 2}) b(t,x). \tag{5.2}
\]

Therefore, taking into account the growth conditions on \(b\) and \(\sigma\) and setting \(\theta > 0\), there exists two positive constants \(C_1\) and \(C_2\) such that \(\frac{1}{2} \theta e^{-\lambda t} \text{Tr} [\sigma \sigma^*(t,x) D^2_x |x|^{2\gamma + 2}] \leq C_1 |x|^{2\gamma + 2}\) and \(D_x (|x|^{2\gamma + 2}) b(t,x) \leq C_2 |x|^{2\gamma + 2}\).

Then by (5.2) we get

\[
-\partial_t \varphi(t,x) - \frac{1}{2} \text{Tr} [\sigma \sigma^*(t,x) D^2_x \varphi(t,x)] - D_x (\varphi(t,x)) b(t,x) - \phi_i(t,x) \geq \theta (\lambda - \alpha) e^{-\lambda t}|x|^{2\gamma + 2}, \tag{5.3}
\]

where \(\alpha = C_1 + C_2\).

Now since \(\theta > 0\), we conclude that, for \(\lambda \geq \alpha\), the right-hand side of (5.3) is non-negative.

Finally, noting that \(i\) is arbitrary in \(\mathcal{I}\) together with (5.1), we obtain that \((v_i(t,x) + \theta e^{-\lambda t}|x|^{2\gamma + 2})_{i=1,\ldots,m}\) is a viscosity supersolution for (2.3). \(\square\)
Now we give an equivalent form of the quasi-variational inequality (2.3). In this section, we consider the new function $I_i$ given by the classical change of variable $I_i(t, x) = \exp(t)v_i(t, x)$ for any $t \in [0, T]$ and $x \in \mathbb{R}^k$. Of course, the function $I_i$ is continuous and of polynomial growth with respect to its second argument.

A second property is given by the following proposition.

**Proposition 5.1** $v_i$ is a viscosity solution of (2.3) if and only if $I_i$ is a viscosity solution to the following quasi-variational inequality in $[0, T] \times \mathbb{R}^k$,

$$
\begin{cases}
\min\{I_i(t, x) - \max \left( -e^j g_{ij}(t, x) + I_j(t, x) \right) , \\
I_i(t, x) - \partial_t I_i(t, x) - A I_i(t, x) - e^j \psi_i(t, x) \} = 0, \\
I_i(T, x) = e^T v_i(T, x) = 0.
\end{cases}
$$

(5.4)

We are now going to address the question of uniqueness of the viscosity solution of the system (2.3). We note that assumption (H4) will not be used in the proof of the uniqueness in the theorem below, and there are no restrictions on the negative switching costs.

We now give the main result of this section.

**Theorem 5.1** Assume that (H2) and (H3)(i)–(ii) hold. Then the viscosity solution of the system of variational inequalities with interconnected obstacles (2.3) is unique in the space of continuous functions on $[0, T] \times \mathbb{R}^k$ which satisfy a polynomial growth condition, i.e. in the space $C := \{ \varphi : [0, T] \times \mathbb{R}^k \to \mathbb{R}, \text{ continuous and for any} \\
(t, x), |\varphi(t, x)| \leq C(1 + |x|^\gamma) \text{ for some constants } C \text{ and } \gamma \}.$

**Proof.** We will show by contradiction that if $u_1, \ldots, u_m$ and $v_1, \ldots, v_m$ are a subsolution and a supersolution, respectively, for (5.4), then, for any $i = 1, \ldots, m$, $u_i \leq v_i$. Therefore, if we have two solutions of (5.4), then they are obviously equal. Next, according to Lemma 5.1, it is enough to show that, for any $i \in \mathcal{I}$, we have

$$
\forall (t, x) \in [0, T] \times \mathbb{R}^k, \quad u_i(t, x) \leq v_i(t, x) + \theta e^{-\lambda t} |x|^{2\gamma + 2},
$$

since in taking the limit as $\theta \to 0$, we get the desired result. So let us set $w_i(t, x) = v_i(t, x) + \theta e^{-\lambda t} |x|^{2\gamma + 2}$, $(t, x) \in [0, T] \times \mathbb{R}^k$. Next, assume that there exists a point $(\tilde{t}, \tilde{x}) \in [0, T] \times \mathbb{R}^k$ such that, for $i \in \mathcal{I}$, $\max_{i \in \mathcal{I}}(u_i(\tilde{t}, \tilde{x}) - w_i(\tilde{t}, \tilde{x})) > 0$. Then, using the growth condition, there exists $R > 0$ such that

$$
\forall (t, x) \in [0, T] \times \mathbb{R}^k \text{ s.t. } |x| \geq R, \quad u_i(t, x) - w_i(t, x) < 0.
$$

Since $u_i(T, x) = v_i(T, x) = 0$, it implies that

$$
0 < \max_{(t, x) \in [0, T] \times \mathbb{R}^k} \max_{i \in \mathcal{I}}(u_i(t, x) - w_i(t, x)) = \max_{(t, x) \in [0, T] \times \bar{B}_R} \max_{i \in \mathcal{I}}(u_i(t, x) - w_i(t, x))
$$

$$
= \max_{i \in \mathcal{I}}(u_i(\hat{t}, \hat{x}) - w_i(\hat{t}, \hat{x})),
$$

(5.5)

where $\bar{B}_R := \{ x \in \mathbb{R}^k; |x| < R \}$ and $(\hat{t}, \hat{x}) \in [0, T] \times \bar{B}_R$. 
Now let us define $\tilde{I}$ as

$$\tilde{I} := \left\{ j \in I, u_j(\hat{t}, \hat{x}) - w_j(\hat{t}, \hat{x}) = \max_{i \in I} (u_i(\hat{t}, \hat{x}) - w_i(\hat{t}, \hat{x})) \right\}.$$

First note that $\tilde{I}$ is not empty. Here $\gamma$ is the growth exponent of the functions which w.l.o.g we assume integer and $\geq 2$. Next, for a small $\epsilon > 0$ and $j \in \tilde{I}$, let us set, for $(t, x, y) \in [0, T] \times B_R \times B_R$,

$$\phi^j_\epsilon(t, x, y) = u_j(t, x) - w_j(t, y) - \varphi_\epsilon(t, x, y),$$

where

$$\varphi_\epsilon(t, x, y) = \frac{1}{2\epsilon} |x - y|^{2\gamma} + \eta \left( |x - \hat{x}|^{2\gamma + 2} + |y - \hat{x}|^{2\gamma + 2} \right) + \beta(t - \hat{t})^2$$

and $\beta, \eta > 0$. Now let $(t_e, x_e, y_e) \in [0, T] \times B_R \times B_R$ be such that

$$\phi^j_\epsilon(t_e, x_e, y_e) = \max_{(t, x, y) \in [0, T] \times B_R \times B_R} \phi^j_\epsilon(t, x, y),$$

which exists since $\phi^j_\epsilon$ is continuous. On the other hand, from $2\phi^j_\epsilon(t_e, x_e, y_e) \geq \phi^j_\epsilon(t_e, x_e, x_e) + \phi^j_\epsilon(t_e, y_e, y_e)$, we have

$$\frac{1}{2\epsilon} |x_e - y_e|^{2\gamma} \leq (u_j(t_e, x_e) - u_j(t_e, y_e)) + (w_j(t_e, x_e) - w_j(t_e, y_e)), \quad (5.6)$$

and consequently $(1/2\epsilon)|x_e - y_e|^{2\gamma}$ is bounded, and as $\epsilon \to 0$, $|x_e - y_e| \to 0$. Since $u_j$ and $w_j$ are uniformly continuous on $[0, T] \times B_R$, it follows that $(1/2\epsilon)|x_e - y_e|^{2\gamma} \to 0$ as $\epsilon \to 0$.

Since

$$u_j(\hat{t}, \hat{x}) - w_j(\hat{t}, \hat{x}) = \phi^j_\epsilon(\hat{t}, \hat{x}, \hat{x}) \leq \phi^j_\epsilon(t_e, x_e, y_e) \leq u_j(t_e, x_e) - w_j(t_e, y_e), \quad (5.7)$$

it follows, as $\epsilon \to 0$ and by the continuity of $u$ and $w$, that, up to a subsequence,

$$(t_e, x_e, y_e) \to (\hat{t}, \hat{x}, \hat{x}). \quad (5.8)$$

We now claim that, for some $k \in \tilde{I}$, we have

$$u_k(\hat{t}, \hat{x}) > \max_{j \in \tilde{I}} (u_j(\hat{t}, \hat{x}) - e^j g_{kj}(\hat{t}, \hat{x})).$$

Indeed if, for any $k \in \tilde{I}$, we have

$$u_k(\hat{t}, \hat{x}) \leq \max_{j \in \tilde{I}} (u_j(\hat{t}, \hat{x}) - e^j g_{kj}(\hat{t}, \hat{x})), \quad (5.9)$$

then there exists $j \in \tilde{I}^{-k}$ such that

$$u_k(\hat{t}, \hat{x}) - u_j(\hat{t}, \hat{x}) \leq -e^j g_{kj}(\hat{t}, \hat{x}).$$

From the supersolution property of $w_j$, we have

$$w_k(\hat{t}, \hat{x}) \geq \max_{j \in \tilde{I}} (w_j(\hat{t}, \hat{x}) - e^j g_{kj}(\hat{t}, \hat{x})).$$
Then
\[ w_k(\hat{t}, \hat{x}) - w_j(\hat{t}, \hat{x}) \geq -e^t g_{kj}(\hat{t}, \hat{x}). \]

It follows that
\[ u_k(\hat{t}, \hat{x}) - u_j(\hat{t}, \hat{x}) \leq -e^t g_{kj}(\hat{t}, \hat{x}) \leq w_k(\hat{t}, \hat{x}) - w_j(\hat{t}, \hat{x}). \]

Hence
\[ u_k(\hat{t}, \hat{x}) - w_k(\hat{t}, \hat{x}) \leq -e^t g_{kj}(\hat{t}, \hat{x}) + u_j(\hat{t}, \hat{x}) - w_k(\hat{t}, \hat{x}) \leq u_j(\hat{t}, \hat{x}) - w_j(\hat{t}, \hat{x}). \]

But since \( k \in \tilde{I} \), it follows that
\[ u_k(\hat{t}, \hat{x}) - w_k(\hat{t}, \hat{x}) = u_j(\hat{t}, \hat{x}) - w_j(\hat{t}, \hat{x}) = -e^t g_{kj}(\hat{t}, \hat{x}) + u_j(\hat{t}, \hat{x}) - w_k(\hat{t}, \hat{x}), \]

which implies that \( j \) belongs also to \( \tilde{I} \) and
\[ u_k(\hat{t}, \hat{x}) - u_j(\hat{t}, \hat{x}) = -e^t g_{kj}(\hat{t}, \hat{x}). \]

Repeating this procedure as many times as necessary and since \( \tilde{I} \) is finite, we get the existence of a loop of indices \( i_1, \ldots, i_p, i_{p+1} \) of \( \tilde{I} \) such that \( i_{p+1} = i_1 \) and
\[ g_{i_1,i_2} (\hat{t}, \hat{x}) + \cdots + g_{i_p,i_{p+1}} (\hat{t}, \hat{x}) = 0. \]

But this contradicts the assumption (2.7), whence the claim holds.

To proceed, let us consider \( k \in \tilde{I} \) such that
\[ u_k (\hat{t}, \hat{x}) > \max_{j \in \tilde{I}} (u_j(\hat{t}, \hat{x}) - g_{kj}(\hat{t}, \hat{x})). \] (5.9)

By the continuity of \( u_j \) and \( g_{ij} \) and since \( (te, xe, uk(te, xe)) \to (\hat{t}, \hat{x}, uk(\hat{t}, \hat{x})) \), for \( \epsilon \) small enough we have
\[ u_k(te, xe) > \max_{j \in \tilde{I}} (u_j(te, xe) - g_{kj}(te, xe)). \] (5.10)

Next we have
\[ \begin{cases} D_t \varphi_e (t, x, y) = 2\beta(t - \hat{t}), \\ D_x \varphi_e (t, x, y) = \frac{\gamma}{\epsilon} (x - y)|x - y|^{2\gamma - 2} + \eta (2\gamma + 2)(x - \hat{x})|x - \hat{x}|^{2\gamma}, \\ D_y \varphi_e (t, x, y) = -\frac{\gamma}{\epsilon} (x - y)|x - y|^{2\gamma - 2} + \eta (2\gamma + 2)(y - \hat{x})|y - \hat{x}|^{2\gamma}, \\ B(t, x, y) = D^2_{x,y} \varphi_e (t, x, y) = \frac{1}{\epsilon} \begin{pmatrix} a_1(x, y) & -a_1(x, y) \\ -a_1(x, y) & a_1(x, y) \end{pmatrix} + \begin{pmatrix} a_2(x) & 0 \\ 0 & a_2(y) \end{pmatrix}, \end{cases} \] (5.11)

with \( a_1(x, y) = \gamma |x - y|^{2\gamma - 2} I + \gamma (2\gamma - 2)(x - y)(x - y)^* |x - y|^{2\gamma - 4} \) and \( a_2(x) = \eta (2\gamma + 2)|x - \hat{x}|^{2\gamma} I + \eta \gamma (2\gamma + 2)(x - \hat{x})(x - \hat{x})^* |x - \hat{x}|^{2\gamma - 2}. \)

Taking into account (5.10) and then applying the result by Crandall et al. (1992, Theorem 8.3) to the function
\[ u_k(t, x) - w_k(t, y) - \varphi_e (t, x, y) \]
at the point \((t_{\epsilon}, x_{\epsilon}, y_{\epsilon})\), for any \(\epsilon_1 > 0\), we can find \(c, d \in \mathbb{R}\) and \(X, Y \in S_k\), such that
\[
\begin{cases}
(c, \frac{\gamma}{\epsilon}(x_{\epsilon} - y_{\epsilon})|x_{\epsilon} - y_{\epsilon}|^{2\gamma-2} + \eta(2\gamma + 2)(x_{\epsilon} - \hat{x})|x_{\epsilon} - \hat{x}|^{2\gamma}, X) \\
(-d, \frac{\gamma}{\epsilon}(x_{\epsilon} - y_{\epsilon})|x_{\epsilon} - y_{\epsilon}|^{2\gamma-2} - \eta(2\gamma + 2)(y_{\epsilon} - \hat{y})|y_{\epsilon} - \hat{y}|^{2\gamma}, Y)
\end{cases}
\in J^{2+}(u_k(t_{\epsilon}, x_{\epsilon})),
\]
\[
c + d = D_{\epsilon} \varphi_k(t_{\epsilon}, x_{\epsilon}, y_{\epsilon}) = 2\beta(t_{\epsilon} - \hat{t})
\text{ and finally}
\]
\[
- \left( \frac{1}{\epsilon_1} + \|B(t_{\epsilon}, x_{\epsilon}, y_{\epsilon})\| \right) I \leq \left( \begin{array}{cc} X & 0 \\ 0 & -Y \end{array} \right) \leq B(t_{\epsilon}, x_{\epsilon}, y_{\epsilon}) + \epsilon_1 B(t_{\epsilon}, x_{\epsilon}, y_{\epsilon})^2.
\]

Taking now into account (5.10), and the definition of viscosity solution, we get
\[
-c + u_k(t_{\epsilon}, x_{\epsilon}) - \frac{1}{2} \text{Tr}[\sigma^*(t_{\epsilon}, x_{\epsilon})X\sigma(t_{\epsilon}, x_{\epsilon})] - \frac{\gamma}{\epsilon}(x_{\epsilon} - y_{\epsilon})|x_{\epsilon} - y_{\epsilon}|^{2\gamma-2}
+ \eta(2\gamma + 2)(x_{\epsilon} - \hat{x})|x_{\epsilon} - \hat{x}|^{2\gamma}, b(t_{\epsilon}, x_{\epsilon}) \leq 0
\text{ and }
\]
\[
d + w_k(t_{\epsilon}, y_{\epsilon}) - \frac{1}{2} \text{Tr}[\sigma^*(t_{\epsilon}, y_{\epsilon})Y\sigma(t_{\epsilon}, y_{\epsilon})] - \frac{\gamma}{\epsilon}(x_{\epsilon} - y_{\epsilon})|x_{\epsilon} - y_{\epsilon}|^{2\gamma-2}
- \eta(2\gamma + 2)(y_{\epsilon} - \hat{y})|y_{\epsilon} - \hat{y}|^{2\gamma}, b(t_{\epsilon}, y_{\epsilon}) \geq 0
\]
which implies that
\[
-c - d + u_k(t_{\epsilon}, x_{\epsilon}) - w_k(t_{\epsilon}, y_{\epsilon}) \leq \frac{1}{2} \text{Tr}[\sigma^*(t_{\epsilon}, x_{\epsilon})X\sigma(t_{\epsilon}, x_{\epsilon}) - \sigma^*(t_{\epsilon}, y_{\epsilon})Y\sigma(t_{\epsilon}, y_{\epsilon})]
+ \left( \frac{\gamma}{\epsilon}(x_{\epsilon} - y_{\epsilon})|x_{\epsilon} - y_{\epsilon}|^{2\gamma-2}, b(t_{\epsilon}, x_{\epsilon}) - b(t_{\epsilon}, y_{\epsilon}) \right)
+ \eta(2\gamma + 2)(x_{\epsilon} - \hat{x})|x_{\epsilon} - \hat{x}|^{2\gamma}, b(t_{\epsilon}, y_{\epsilon})
+ \eta(2\gamma + 2)(y_{\epsilon} - \hat{y})|y_{\epsilon} - \hat{y}|^{2\gamma}, b(t_{\epsilon}, y_{\epsilon})
+ e^\epsilon \psi_k(t_{\epsilon}, x_{\epsilon}) - e^\epsilon \psi_k(t_{\epsilon}, y_{\epsilon}).
\]

But from (5.11) there exist two constants \(C\) and \(C_1\) such that
\[
\|a_1(x_{\epsilon}, y_{\epsilon})\| \leq C|x_{\epsilon} - y_{\epsilon}|^{2\gamma-2} \text{ and } (\|a_2(x_{\epsilon})\| \vee \|a_2(y_{\epsilon})\|) \leq C_1 \eta.
\]
As
\[
B = B(t_{\epsilon}, x_{\epsilon}, y_{\epsilon}) = \frac{1}{\epsilon} \left( \begin{array}{cc} a_1(x_{\epsilon}, y_{\epsilon}) & -a_1(x_{\epsilon}, y_{\epsilon}) \\ -a_1(x_{\epsilon}, y_{\epsilon}) & a_1(x_{\epsilon}, y_{\epsilon}) \end{array} \right) + \left( \begin{array}{cc} a_2(x_{\epsilon}) & 0 \\ 0 & a_2(y_{\epsilon}) \end{array} \right),
\]
then
\[
B \leq \frac{C}{\epsilon} |x_{\epsilon} - y_{\epsilon}|^{2\gamma-2} \left( \begin{array}{cc} I & -I \\ -I & I \end{array} \right) + C_1 \eta I.
\]
It follows that
\[
B + \epsilon_1 B^2 \leq C \left( \frac{1}{\epsilon} |x_{\epsilon} - y_{\epsilon}|^{2\gamma-2} + \frac{\epsilon_1}{\epsilon^2} |x_{\epsilon} - y_{\epsilon}|^{4\gamma-4} \right) \left( \begin{array}{cc} I & -I \\ -I & I \end{array} \right) + C_1 \eta I.
\]
where \( C \) and \( C_1 \) hereafter may change from line to line. Choosing now \( \epsilon_1 = \epsilon \) yields the relation
\[
B + \epsilon_1 B^2 \leq \frac{C}{\epsilon} (|x_e - y_e|^{2\gamma} - 2 + |x_e - y_e|^{4\gamma - 2}) \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + C_1 \eta I.
\] (5.15)

Now, from (2.6), (5.12) and (5.15) we get
\[
\frac{1}{2} \text{Tr}[\sigma^*(t_e, x_e)X\sigma(t_e, x_e) - \sigma^*(t_e, y_e)Y\sigma(t_e, y_e)] \\
\leq \frac{C}{\epsilon} (|x_e - y_e|^{2\gamma} - 2 + |x_e - y_e|^{4\gamma - 2}) + C_1 \eta (1 + |x_e|^2 + |y_e|^2).
\]

Next
\[
\left( \frac{\dot{Y}}{\epsilon} (x_e - y_e)|x_e - y_e|^{2\gamma - 2}, b(t_e, x_e) - b(t_e, y_e) \right) \leq \frac{C^2}{\epsilon} |x_e - y_e|^{2\gamma},
\]
and finally,
\[
(\eta(2\gamma + 2)(x_e - \hat{x})|x_e - \hat{x}|^{2\gamma}, b(t_e, x_e)) + (\eta(2\gamma + 2)(y_e - \hat{x})|y_e - \hat{x}|^{2\gamma}, b(t_e, y_e)) \\
\leq C\eta(1 + |x_e| |x_e - \hat{x}|^{2\gamma + 1} + |y_e| |y_e - \hat{x}|^{2\gamma + 1}),
\]
so that, by plugging into (5.13), we obtain
\[
-2\beta(t_e - \hat{t}) + u_k(t_e, x_e) - w_k(t_e, y_e) \\
\leq \frac{C}{\epsilon} (|x_e - y_e|^{2\gamma} - 2 + |x_e - y_e|^{4\gamma - 2}) + C_1 \eta (1 + |x_e|^2 + |y_e|^2) + \frac{C^2}{\epsilon} |x_e - y_e|^{2\gamma} \\
+ C\eta(1 + |x_e| |x_e - \hat{x}|^{2\gamma + 1} + |y_e| |y_e - \hat{x}|^{2\gamma + 1}) + e^{\epsilon t} \psi_k(t_e, x_e) - e^{\epsilon t} \psi_k(t_e, y_e).
\]
By sending \( \epsilon \to 0, \eta \to 0 \) and taking into account the continuity of \( \psi_k \) and \( \gamma \geq 2 \), we obtain
\[
u_k(\hat{t}, \hat{x}) - w_k(\hat{t}, \hat{x}) \leq 0,
\]
which contradicts (5.5). The proof of Theorem 5.1 is now complete. \( \square \)

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