Cascades and Dissipative Anomalies in Compressible Fluid Turbulence

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(Dated: April 13, 2017)

We investigate dissipative anomalies in a turbulent fluid governed by the compressible Navier-Stokes equation. We follow an exact approach pioneered by Onsager, which we explain as a non-perturbative application of the principle of renormalization-group invariance. In the limit of high Reynolds and Péclet numbers, the flow realizations are found to be described as distributional or “coarse-grained” solutions of the compressible Euler equations, with standard conservation laws broken by turbulent anomalies. The anomalous dissipation of kinetic energy is shown to be due not only to local cascade, but also to a distinct mechanism called pressure-work defect. Irreversible heating in stationary, planar shocks with an ideal-gas equation of state exemplifies the second mechanism. Entropy conservation anomalies are also found to occur by two mechanisms: an anomalous input of negative entropy (negentropy) by pressure-work and a cascade of negentropy to small scales. We derive “4/5th-law”-type expressions for the anomalies, which allow us to characterize the singularities (structure-function scaling exponents) required to sustain the cascades. We compare our approach with alternative theories and empirical evidence. It is argued that the “Big Power-Law in the Sky” observed in electron density scintillations in the interstellar medium is a manifestation of a forward negentropy cascade, or an inverse cascade of usual thermodynamic entropy.

I. INTRODUCTION

Compressible fluids play a vital role in problems of astrophysics (interstellar medium [1], star-formation [2,3]), applied physics (inertial confinement fusion [4]), and engineering (high-temperature reactive flows [5], supersonic aircraft design [6]). Relativistic fluids are necessarily compressible, of course, and occur in astrophysical flows (pulsars [7], gamma-ray bursts [8]), high-energy physics (heavy-ion collisions [9]), and condensed matter physics (graphene [10–12], strange metals [13,14]). In many of the above examples the fluid is either directly observed or indirectly inferred to be in a turbulent state. The nature of compressible turbulence has been highly controversial, however. It is currently debated whether the notion of an “energy cascade”, as it was developed by Kolmogorov [15–17], Obukhov [18], Onsager [19,20], Heisenberg [21] and von Weizsäcker [22] to describe incompressible fluid turbulence, is applicable at all to turbulence in compressible fluids. On the one hand, some authors argue that, much the same as for incompressible turbulence, compressible fluids possesses a turbulent inertial range “which is immune from direct effects of viscosity and large scale forcing” [23] through which kinetic energy is transferred to small scales by a cascade process that is local in scale. On the other hand, exact statistical relations have been derived for non-relativistic compressible turbulence [24] and for relativistic turbulence [25], which do not involve kinetic energy and which have been invoked to argue that “the interpretation of the Kolmogorov relation for the incompressible turbulence in terms of the energy cascade may be misleading” [25]. This is a controversy whose resolution has profound consequences for all physical systems where compressible fluid turbulence manifests itself.

The primary physical effect of cascades in incompressible fluids are “dissipative anomalies,” in which ideal invariants of the fluid equations, such as kinetic energy, are non-conserved even in the inviscid or high Reynolds-number limit. This effect was deduced semi-phenomenologically from geophysical observations by Taylor [26] and confirmed in classical wind-tunnel experiments [27]. For modern evidence from numerical simulations and experiments, see [28–30]. This type of empirical evidence motivated the theories of Kolmogorov [15–17], Obukhov [18], Onsager [19,20], Heisenberg [21] and von Weizsäcker [22]. A particularly deep contribution was made by Onsager [20], who argued that turbulent fluids could be described by singular (weak) solutions of incompressible Euler equations whose kinetic energy balance equations would be afflicted with an anomaly due to the nonlinear cascade mechanism. Onsager’s derivation was by a (smoothed) version of a point-splitting regularization, which yielded for the anomaly an expression closely related to the Kolmogorov 4/5th-law but valid for individual flow realizations, without averaging over ensembles [31]. Kolmogorov’s weaker statistical relation is, of course, well-known to physicists, e.g. Polyakov has pointed out the formal analogy of Kolmogorov’s relation and its point-splitting derivation to axial anomalies in quantum gauge field theories [32,33]. Onsager’s deeper contribution has received little attention in the physics community, on the other hand, although the many predictions of Onsager’s analysis are consistent with all available experimental evidence. In particular, his prediction of 1/3 Hölder singularities for the velocity field has been confirmed experimentally (e.g. [34]). In fact, an entire multifractal spectrum of singularities has been measured, as in the later elaboration of Parisi-Frisch [35,36].
We show here that the Onsager theory carries over to compressible fluids, completing earlier work of Aluie \[38,39\]. There have been extensive further developments of Onsager’s ideas, which we shall exploit. In particular, we follow closely the approaches of Eyink \[38,39\], Constantin et al. \[40\], and Duchon-Robert \[41\], who derived necessary conditions for dissipative anomalies of kinetic energy in turbulent solutions of incompressible Euler equations. Subsequently, there has been very deep mathematical work constructing dissipative, Hölder-continuous Euler solutions for the incompressible case by “convex integration” methods, using ideas originating in the Nash-Kuiper theorem and Gromov’s h-principle (e.g. see DeLellis & Szkołek iedy \[12,13\]). This circle of ideas led recently to a proof that Onsager’s 1/3 Hölder exponent is sharp \[44\]. These remarks might suggest that a very high level of mathematical sophistication is necessary to grasp the essentials of Onsager’s ideas on turbulent weak solutions. This is not the case. As a matter of fact, Onsager’s ideas are very closely related to standard physical notions of spatial coarse-graining and renormalization-group invariance \[45,47\]. In addition to extending Onsager’s approach to compressible fluids and deriving many new testable predictions, we shall also explain carefully the connection to renormalization-group ideas. By means of this intuitive but rigorous approach, we shall resolve the controversies concerning nonrelativistic compressible fluid turbulence. In a companion paper, we further extend our analysis to relativistic fluid turbulence \[48\].

Turbulence is an essential strong-coupling problem to which perturbation theory does not apply, so that, as in Onsager’s original work, some mathematical tools of nonlinear analysis must be employed. The required background for full understanding of the finer points is mathematical analysis at a theoretical-physics level such as contained in \[49\], particularly standard spatial \(L^p\)-norms (§I.D.10) and basic theory of distributions/generalized functions (§VI.A-B). The tools employed are similar to those in the mathematical theory of fluid shock solutions. Most of our analysis can be grasped without even that technical background, but assuming just some familiarity with spatial coarse-graining and fluid turbulence.

### II. COMPRESSIBLE NAVIER-STOKES AND HYDRODYNAMIC SCALING

The model equations that we employ for (non-relativistic) compressible fluids in this paper are the standard Navier-Stokes equations in space dimension \(d\). These govern the evolution of the conserved densities (per volume) of mass \(\rho\), momentum \(j\), and total energy \(E = |j|^2/(2\rho) + u\), kinetic energy density \(|j|^2/(2\rho)\) plus internal energy density \(u\), by

\[
\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0, \tag{1}
\]

\[
\partial_t (\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v} \mathbf{v} + p \mathbf{I} - 2\eta \mathbf{S} - \zeta \Theta \mathbf{I}) = 0, \tag{2}
\]

where \(\mathbf{S} = \nabla \mathbf{v} + \nabla \mathbf{v}^T\) is the strain tensor and \(\zeta \Theta \mathbf{I}\) is the bulk viscosity, and \(\kappa(u, \rho)\) is the thermal conductivity, and

\[
\partial_t \left(\frac{1}{2} \rho v^2 + u\right) + \nabla \cdot \left((u + p + \frac{1}{2} \rho v^2) \mathbf{v}\right) - \kappa \nabla T - 2\eta S : \mathbf{v} - \zeta \Theta \mathbf{v} = 0. \tag{3}
\]

A fluid velocity \(\mathbf{v}\) has been defined conventionally by \(\mathbf{v} = j/\rho\), which is thus associated to the transport of mass. This is not the only possible choice of a fluid velocity. e.g. \[50,51\], but it is the most familiar one generally employed for a non-relativistic fluid. Above, \(\eta(u, \rho)\) is the shear viscosity, \(\zeta(u, \rho)\) is the bulk viscosity, and \(\kappa(u, \rho)\) is the thermal conductivity, and

\[
S_{ij} = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{1}{d} (\nabla \cdot \mathbf{v}) \delta_{ij} \right), \quad \Theta = \nabla \cdot \mathbf{v} \tag{4}
\]

are the strain tensor and dilatational field, respectively. We here admit any thermodynamically consistent relations for the pressure \(p(u, \rho)\) (the equation of state) and for the absolute temperature \(T(u, \rho)\). It should be pointed out that this set of equations has some well-known deficiencies in representing the internal structure of strong shocks, whose thickness is of the order of the mean-free-path length of the fluid \[52,53\]. This may cause concern, since compressible fluid turbulence is well-known to develop numerous small-scale “shocklets”. A more fundamental model for the dynamics of a neutral (non-ionized) gas would be the Boltzmann kinetic equation, whose solutions agree well with the experimental data for strong shocks. However, we expect that all of our conclusions below will still apply if such a kinetic description is employed, as we shall be concerned with length-scales much greater than the width of the shock front where, for both kinetic and fluid models, a similar description emerges as a discontinuous weak/distributional solution of the compressible Euler equations \[55\]. The use of a fluid description from the outset greatly simplifies our analysis, but similar arguments should carry over to kinetic theory. We also do not discuss here the effects of molecular noise, which for a thermodynamically consistent description of a compressible Navier-Stokes fluid requires stochastic PDE’s with suitable multiplicative noise given by a fluctuation-dissipation relation \[56,57\]. The effects of such noise are quite significant, presumably leading to a “stochastic anomaly” in addition to the dissipative anomaly already discussed \[58,59\]. This is an issue of fundamental importance for the problems of predicting, reproducing, or controlling turbulent flows \[22,63\] but, as argued further below, addition of thermal noise does not alter our conclusions in this paper on dissipative anomalies.

Compressible fluid turbulence is characterized by several dimensionless number groups which are revealed by a scaling of the fluid equations. There is more than one way to rescale the equations. Here we follow a very simple approach, introducing dimensionless variables

\[
\hat{\rho} = \rho/\rho_0, \quad \hat{\mathbf{v}} = \mathbf{v}/v_0, \quad \hat{u} = u/\rho_0 v_0^2 \tag{5}
\]

\[
\hat{x} = x/L_0, \quad \hat{t} = t/(L_0/v_0) \tag{6}
\]
\[ \hat{p} = \rho_0 \hat{v}_0^2, \quad \hat{T} = T/T_0. \] (7)

Above \( \rho_0 \) and \( v_0 \) are typical densities and velocities, such as spatial mean or r.m.s. values. \( L_0 \) is the large length-scale of the turbulent flow, e.g. the integral length, or another length characterizing the scale of injection of kinetic energy either by external stirring or by initial data. For temperature scale \( T_0 \) we may also take an r.m.s. value or, alternatively, \( T_0 = T(\rho_0 v_0^2, \rho_0) \). The non-dimensionalized equations of motion then become

\[ \partial_t \hat{\rho} + \nabla \cdot (\hat{\rho} \hat{v}) = 0, \] (8)

\[ \partial_t (\hat{\rho} \hat{v}) + \nabla \cdot (\hat{\rho} \hat{v} \hat{v} + \hat{\rho} \hat{I} - 2\hat{\eta} \hat{S} - \hat{\zeta} \hat{\Theta} \hat{I}) = 0, \] (9)

\[ \partial_t (\frac{1}{2} \hat{\rho} \hat{v}^2 + \hat{u}) + \nabla \cdot (\hat{u} + \hat{\rho} \hat{I} + \frac{1}{2} \hat{\rho} \hat{v}^2 \hat{\nabla} \hat{T} - 2\hat{\eta} \hat{S} \hat{v} - \hat{\zeta} \hat{\Theta} \hat{v}) = 0, \] (10)

where

\[ \hat{\eta}(\hat{u}, \hat{\rho}) = \frac{\eta}{\rho_0 v_0 L_0}, \quad \hat{\zeta}(\hat{u}, \hat{\rho}) = \frac{\zeta}{\rho_0 v_0 L_0} \] (11)

are inverse Reynolds numbers associated to the shear and bulk viscosities, respectively, and

\[ \hat{\kappa}(\hat{u}, \hat{\rho}) = \frac{\kappa T_0}{\rho_0 v_0^2 L_0}, \] (12)

is an inverse Péclet number. Fully-developed turbulent flow occurs when the Reynolds and Péclet numbers (as functions of \( \hat{u}, \hat{\rho} \)) are uniformly very large compared to unity, and \( \hat{\eta}, \hat{\zeta}, \hat{\kappa} \) are small. It should be borne in mind that these dimensionless transport quantities are generally made smaller, not by decreasing \( \eta, \zeta, \kappa \) but instead typically by increasing \( \rho_0, v_0 \) or especially \( L_0 \). We hereafter omit the hats “\( \hat{\cdot} \)” on all variables, but always assume that non-dimensionalization has been carried out. In particular, when we discuss below the ideal limit \( \eta, \zeta, \kappa \to 0 \) we always mean more literally that \( \hat{\eta}, \hat{\zeta}, \hat{\kappa} \ll 1 \).

There is one other dimensionless variable which plays a very important role in compressible turbulence that does not appear explicitly above. This is the Mach number or the inverse of the dimensionless sound speed:

\[ Ma = 1/c_s = v_0/c_s, \] (13)

with \( c_s = \sqrt{(\partial p/\partial \rho)_{s_0}} \) the adiabatic sound speed (the density derivative being taken at fixed entropy per particle \( s_0 \)). Of course, the Mach number as defined above is also a variable function of \( \hat{u}, \hat{\rho} \). The properties of compressible turbulence are very strongly dependent upon the Mach number. However, the main results of the present work are valid for any Mach number. We shall comment below on those parts of our analysis that make any assumptions depending on the Mach number, either explicitly or implicitly.

### III. DISSIPATIVE ANOMALIES

Immediate consequences of the compressible Navier-Stokes equations [1-3] are the kinetic energy balance

\[ \partial_t \left( \frac{1}{2} \rho v^2 \right) + \nabla \cdot \left[ (p + \frac{1}{2} \rho v^2) \nu - 2\eta S \nu - \zeta \Theta \nu \right] \]

\[ = p(\nabla \cdot \nu) - 2\eta |S|^2 - \zeta \Theta^2 \] (14)

and the internal energy balance:

\[ \partial_t u + \nabla \cdot [\nu v - \kappa \nabla T] = -p(\nabla \cdot \nu) + 2\eta |S|^2 + \zeta \Theta^2, \] (15)

where \( 2\eta |S|^2, \zeta \Theta^2 \) are the energy dissipation per volume due to shear and bulk viscosity, respectively. Because total energy is conserved, the kinetic energy lost is precisely equal to the internal energy gained.

The balance equations analogous to (14, 15) above hold also in the limit \( Ma \ll 1 \) of low Mach numbers. The incompressible Navier-Stokes equation then governs the velocity field, in which only the shear viscosity survives. The temperature field obeys a passive advection equation with viscous heating as a source term. The remarkable empirical fact for incompressible turbulence is that the viscous dissipation per mass \( Q_{\nu} = 2\nu |S|^2 \), where \( \nu = \eta/\rho \) is the kinematic viscosity, appears to have a non-vanishing limit as \( \nu \to 0 \). Full documentation of the relevant laboratory experiments and numerical simulations can be found in the references [25, 30]. A more precise statement is that a distributional limit [64] of the viscous dissipation field appears to exist

\[ Q = D^{-\lim_{\nu \to 0} Q_{\nu}} \] (16)

and, by experimental evidence, yields in the infinite Reynolds number limit a positive measure \( D \) with multifractal scaling properties [65, 66]. As shown in the work of Duchon & Robert [11], this same measure appears as an anomaly term in the kinetic energy balance equation

\[ \partial_t \left( \frac{1}{2} \nu^2 \right) + \nabla \cdot \left[ (p + \frac{1}{2} \nu^2) \nu \right] = -Q \] (17)

for weak solutions of the incompressible Euler equations which are obtained by strong \( L^3 \) limits of incompressible Navier-Stokes solutions as \( \nu \to 0 \). Duchon & Robert [11] also derived an inertial-range expression for \( Q \) closely related to the Kolmogorov “4/5th-law”. This allowed them to prove a refined version of the Onsager singularity theorem, namely, that \( p \)-th order scaling exponents \( \zeta_p^v \) of (absolute) velocity-increments must satisfy \( \zeta_p^v \leq p/3 \) for \( p \geq 3 \), or otherwise \( Q \equiv 0 \). See also [39, 40]. The empirical fact that kinetic energy dissipation has a non-vanishing limit for infinite Reynolds numbers, within the accuracy of current measurements, is so central to the modern understanding of incompressible fluid turbulence that it is sometimes called the “zeroth-law of turbulence”.

The fundamental hypothesis of the present work is that there shall similarly be a non-zero distributional limit

\[ Q = D^{-\lim_{\eta, \zeta, \kappa \to 0} [2\eta |S|^2 + \zeta \Theta^2]} = Q_\eta + Q_\zeta > 0 \] (18)
for viscous dissipation in compressible fluid turbulence. A number of previous works have investigated the statistical properties of the viscous dissipation in compressible turbulence, e.g. [67, 69]. In particular, [69] presents direct empirical evidence for the “zeroth-law” of compressible turbulence. Furthermore, there are simple shock solutions of the compressible Navier-Stokes solution which converge as $\eta, \zeta, \kappa \to 0$ strongly (and thus distributionally) to weak solutions of compressible Euler equations and for which $Q > 0$. For example, see Appendix A. In this respect, the theory of compressible fluids is better off than the incompressible theory, where there are still no rigorous examples of dissipative Euler solutions obtained by the physical limit of vanishing viscosity. The purpose of the present paper is to develop the consequences of hypothesis [18] for compressible turbulence.

Furthermore, we shall also consider in this work the balance equation for the entropy density per volume which is implied by the compressible Navier-Stokes equation. As is well-known (e.g. [70], Ch. XII, §49) this balance equation has the form

$$\partial s + \nabla \cdot \left( sv - \frac{\kappa \nabla T}{T} \right) = \frac{\kappa |\nabla T|^2}{T^2} + \frac{2\eta |S|^2}{T} + \frac{\zeta \Theta^2}{T},$$

(19)

where entropy production on the righthand side is positive, consistent with the second law of thermodynamics. Although fluid turbulence is a strongly dissipative macroscopic process, there seem to have been remarkably few attempts to understand its consistency with the thermodynamic second law. A pioneering work in this direction is a 1949 paper of Obukhov that considered the entropy balance for incompressible fluid turbulence in the low Mach-number limit [72]. Extending Obukhov’s theory to compressible fluid turbulence at arbitrary Mach numbers is one of the principal motivations of our paper.

While entropy is conserved for smooth solutions of the compressible Euler equations, it is very natural to hypothesize that the entropy balance will also be anomalous for compressible turbulent flow and that there will be a non-vanishing limiting measure

$$\Sigma = \lim_{\eta, \zeta, \kappa \to 0} \left[ \frac{\kappa |\nabla T|^2}{T^2} + \frac{2\eta |S|^2}{T} + \frac{\zeta \Theta^2}{T} \right],$$

(20)

describing anomalous entropy production. Simple shock solutions provide examples of such entropy anomalies for weak solutions of compressible Euler equations with step-discontinuities (Appendix A), but milder Hölder singularities typical of turbulent flow should suffice for anomalous entropy production. In this work we shall derive an inertial-range expression for $\Sigma$ which shows that an “inverse cascade” of entropy can provide a mechanism for an entropy anomaly and we characterize the type of Hölder singularities of the turbulent solutions required to sustain a non-vanishing entropy flux.

IV. THEORETICAL APPROACH

It is often assumed reflexively that fluid turbulence must be treated probabilistically. For some problems statistical ensembles are essential, for example, for predicting the future of a given turbulent flow [58, 60, 62]. For many problems, however, statistical methods are wholly inadequate, because one always observes in Nature a single turbulent flow realization. If one wants to understand the effects of turbulence in a specific solar flare event, one does not have the luxury of averaging over an ensemble of flares! An approach that is capable of treating individual flow realizations is intrinsically more fundamental than a probabilistic treatment, because statistical relations can always be obtained by subsequent averaging over ensembles. For these reasons, we shall make no use of statistical ensembles in the present paper. When we have occasion below to consider long-time steady states we shall employ time-averages and global space-averages, denoted by $\langle \cdot \rangle$, which may be operationally obtained in principle from a single flow realization.

Our analysis will be based not on ensemble-averaging but instead upon spatial and/or temporal coarse-graining, which we employ as a regularizer. Note that the existence of non-vanishing dissipative anomalies as in [18] and [20] requires that gradients of fluid variables must diverge, $|\nabla v|, |\nabla T| \to \infty$ as $\eta, \zeta, \kappa \to 0$. This is an ultraviolet divergence due the development of high-wavenumber excitations in the ideal limit, or, as described by Onsager [19], a “violet catastrophe”. It is a consequence of these divergences that the fluid equations [1–3] can no longer remain meaningful in the naive sense, because they involve the above diverging gradients. To make sense of the dynamics in the ideal limit, Onsager used a point-splitting regularization partially smoothed by a filtering kernel [31]. We employ an alternative approach [39, 40] more closely related to renormalization group (RG) methods, with fields $f(x,t)$ spatially coarse-grained as

$$\bar{f}_\ell(x,t) = \int d^d r \ G_\ell(r) f(x + r, t)$$

(21)

where $G_\ell(r) = \ell^{-d} G(r/\ell)$ and the filter kernel $G$ is non-negative, smooth, rapidly decreasing in space, and normalized so that $\int d^d r \ G(r) = 1$. This coarse-graining operation is a natural regularization which removes short-distance divergences. This can be seen from Cauchy-Schwarz bounds on the coarse-grained gradients:

$$|\nabla \bar{f}_\ell| \leq (1/\ell) \sqrt{\int d^d r \ |(\nabla G)_\ell(r)|^2 \cdot \int_{\text{supp}(G_\ell)} d^d r \ f^2(x + r),}$$

(22)

which are finite as long as the function $f$ is locally square-integrable. Such estimates are intuitively obvious, because high-wavenumbers $k \geq 1/\ell$ have been removed. As we shall see, the coarse-graining regularization in [21] is more powerful and more general than the point-splitting
An identical argument shows also that
\[ \langle \text{supp}(G) \rangle \leq \] and compactly supported, then so is \(|\eta(x+r,t)|\). The contribution of the shear viscosity can because coarse-graining commutes with space and time derivatives. The contribution of the shear viscosity can be bounded pointwise using the Cauchy-Schwartz inequality
\[ \langle \nabla \cdot [\mathbf{S}(x), \mathbf{v}(x) - \mathbf{v}(x_0)] \rangle \leq \frac{2}{\ell} \int d^3r \left| \left( \nabla G \right)_{x}(r) \cdot \eta(x+r,t) \mathbf{S}(x+r,t) \right| \]
\[ \leq \frac{2}{7} \left( \langle \eta(x,t) \rangle_{\ell} \times \int d^3r \left| \left( \nabla G \right)_{x}(r) \right|^2 \eta(x+r,t) |\mathbf{S}(x+r,t)|^2 \right) \]
with \( \langle \eta(x,t) \rangle_{\ell} = \int_{\text{supp}(G_{\ell})} d^3r \eta(x+r,t) \) and with \( \text{supp}(G_{\ell}) \) the compact support set of the function \( G_{\ell} \). The inverse power \( 1/\ell \), arose after using integration by parts to move the gradient to the filter function \( G \). It shows that this term is essentially “irrelevant” in the RG sense and is dropped out for increasing \( \ell \). When \( G \) is a test function in the Schwartz distribution theory (\( G \in C^\infty \) and compactly supported), then so is \( |\nabla G(x-x)|^2 \) and thus
\[ \lim_{\eta, \zeta, \kappa \to 0} \int d^3r \left| \left( \nabla G \right)_{x}(r) \right|^2 \eta(x+r,t) |\mathbf{S}(x+r,t)|^2 \]
\[ = \int |\left( \nabla G \right)_{x}(r-x)|^2 Q_\eta(dr) \]
by our fundamental hypothesis. On the other hand, \( \langle \eta(x) \rangle_{\ell} \to 0 \) whenever \( \eta \) tends to zero locally in \( L^1 \). An identical argument shows also that \( |\nabla \cdot [\mathbf{S}(x), \mathbf{v}(x) - \mathbf{v}(x_0)]| \to 0 \) pointwise for fixed length scale \( \ell \) when \( \zeta \) tends to zero locally in \( L^1 \). It follows that all of the molecular transport terms in the coarse-grained momentum balance become negligible in the limit of high Reynolds numbers.

This leads to the concept of the “inertial range”, or the length-scales \( \ell \) sufficiently large that the molecular transport can be ignored relative to the large-scale momentum transport \( \sim \rho_0 v_0^2 / L_0 \). The previous upper bound shows that this range extends down to at least \( \ell \sim L_0 / \sqrt{Re_s} \) with a Reynolds number defined by \( 1/Re_s = \eta_0 Q_{\eta}/\rho_0^2 v_0^4 \), which is analogous to the “Taylor microscale” of incompressible fluid turbulence. Here we have assumed that \( \zeta \sim \eta \) otherwise one must consider also the limit set by \( \ell \geq L_0 / \sqrt{Re_b} \) with another “bulk-viscosity Reynolds number” defined by \( 1/Re_b = \eta_0 Q_{\eta}/\rho_0^2 v_0^4 \). It should be emphasized that the above estimates of length-scales where viscosity effects become significant are expected to be over-estimates, because they are deduced from mathematical upper bounds on the molecular transport. The range of scales \( \ell \) where viscosity is significant is usually termed the “dissipation range” and extends down to scales of order the mean-free path length \( \lambda_{mf} \) of the fluid, where the hydrodynamic description breaks down.

The same arguments apply also to the energy balance, where the shear-viscosity contribution is bounded by
\[ |\nabla \cdot [\mathbf{S}(x), \mathbf{v}(x)]| \leq \frac{2}{\ell} \int \text{supp}(G_{\ell}) d^3r \left| \mathbf{v}(x+r,t) \right|^2 \eta(x+r,t) \]
\[ \times \left( \langle \nabla G \rangle_{x}(r) \right)^2 \eta(x+r,t) |\mathbf{S}(x+r,t)|^2 \]
and this vanishes at fixed \( \ell \), for example, if \( \mathbf{v} \) is locally \( L^2 \) and if \( \eta \) tends to zero locally in \( L^\infty \). The energy transfer by shear viscosity is negligible compared with large-scale advective transport \( \sim \rho_0 v_0^2 / L_0 \) again down to length scale \( \ell \sim L_0 / \sqrt{Re_b} \) (at least). Similar arguments apply to the other molecular contributions to energy transport. That from bulk viscosity tends to zero if \( \zeta \to 0 \) locally in \( L^\infty \) and is negligible down to at least the length-scale \( \ell \sim L_0 / \sqrt{Re_b} \). Finally, the contribution from thermal conductivity vanishes if temperature \( T \) is locally \( L^2 \) and \( \kappa \to 0 \) locally in \( L^\infty \), and it may be neglected down to at least length-scale \( \ell \sim L_0 / \sqrt{Re_b} \) for the thermal Péclet number defined by \( 1/Pe_s = \eta_0 \zeta \int \text{supp}(G_{\ell}) \).

The final conclusion of this argument is that for sufficiently large length-scales \( \ell \) (or for all \( \ell \) in the ideal limit \( \eta, \zeta, \kappa \to 0 \)) the following set of coarse-grained balance equations hold:
\[ \partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0, \quad \langle \eta(x,t) \rangle_{\ell} \quad \text{for all } \ell \]
compressible Euler equations in the usual naive sense \[78\]. This point can be made clearly by introducing the density-weighted Favre-average \[79, 80\].

\[ \tilde{f} = \rho f / \bar{\rho} \]  

(30)

and using the expansion of average products \( f_1 \cdots f_n \) into a finite sum of \( p \)-th order cumulants \( \tau(f_{i_1}, ..., f_{i_p}) \):

\[ f_1 \cdots f_n = \sum_{I} \prod_{r=1}^{r_I} \tau(f_{i^{(r)}_1}, ..., f_{i^{(r)}_{p_r}}) \]  

(31)

where the sum is over all distinct partitions \( I \) of \( \{1, ..., n\} \) into \( r_I \) disjoint subsets \( \{i^{(r)}_1, ..., i^{(r)}_{p_r}\} \), \( r = 1, ..., r_I \), so that \( \sum_{r=1}^{r_I} p_r = n \) for each partition \( I \) \[81, 82\]. One may likewise expand averaged products \( f_1 \cdots f_n \) into cumulants \( \tau(f_{i_1}, ..., f_{i_p}) \) for the original (non-density weighted) spatial coarse-graining. Exploiting these cumulant expansions in the ideal balance equations yields an equivalent set of equations

\[ \partial_t \bar{\rho} + \nabla \cdot (\bar{\rho} \bar{v}) = 0, \]  

(32)

\[ \partial_t (\rho \bar{v}) + \nabla \cdot (\rho \bar{v} \bar{v} + \bar{\rho} \bar{v} \tau(v, v) + \bar{p} \mathbf{I}) = 0, \]  

(33)

\[ \partial_t \left( \frac{1}{2} \bar{\rho} | \bar{v} |^2 + \bar{\rho} \bar{v} (v_1, v_1) + \bar{u} \right) + \nabla \cdot \left( [ (u + \bar{p}) \bar{v} + \bar{\rho} \bar{v} \tau(v, v) \right) \right] = 0. \]  

(34)

It is immediately clear that the coarse-grained fields \( \bar{\rho}, \bar{v}, \bar{u} \), although smooth and with all derivatives well-defined, do not satisfy the compressible Euler equations in the standard sense and that there are new transport terms at length-scale \( \ell \) which are introduced by the coarse-graining. It is, of course, not surprising that the effective equations for “block-spin” variables are renormalized and contain new, complex terms. Note, in particular, that the coarse-graining cumulants of second and higher orders are not simple closed functions of the “resolved” fields \( \rho, v, \bar{u} \). The cumulants are instead very complex functions of the resolved fields, with non-polynomial non-linearity and non-Markovian dependence on the past history. In fact, these cumulants cannot in principle be fixed, deterministic functions of the resolved fields, but must be considered stochastic \[83\] variables because of their dependence on the unknown degrees of freedom below length-scale \( \ell \) \[88\]. In the “large-eddy simulation” (LES) methodology of turbulence modelling, one seeks computationally tractable closed models of these cumulant terms as functions of the resolved fields (see \[83\] \[85\]). As we shall see, the cumulant terms that appear in these coarse-grained equations are the source of turbulent cascade and dissipative anomalies for weak Euler solutions \[39\].

The above description of weak solutions is somewhat novel and designed to make clear the close connection with renormalization-group methodology. A more traditional account follows by first taking the ideal limit \( \eta, \zeta, \kappa \to 0 \) of the coarse-grained conservation equations, just as above, and then followed by the limit \( \ell \to 0 \). The coarse-grained balance equations in this order of limits converge in the sense of distributions to

\[ \partial_t \rho + \nabla \cdot (\rho \bar{v}) = 0, \]  

(35)

\[ \partial_t (\rho \bar{v}) + \nabla \cdot (\rho \bar{v} \bar{v} + p \mathbf{I}) = 0, \]  

(36)

\[ \partial_t \frac{1}{2} \rho v^2 + u \right) + \nabla \cdot \left( [(u + p) \bar{v} + \rho \bar{v} \tau(v, v) \right) + \frac{1}{2} \rho \bar{v} \tau(v, v, v) \]  

(37)

This system follows because all space and time derivatives can be transferred to the test functions and all coarse-grained fields inside the derivatives converge to their fine-grained values under modest assumptions on the fields (e.g. if they are bounded, measurable functions). Equivalently, all of the coarse-graining cumulants of the fields converge to zero. In contrast to the regularized systems of equations \[27, 29\] or \[32, 34\], where all derivatives are taken in the classical sense, in the above set of \( \ell \to 0 \) limit equations \[35-37\] the derivatives must be interpreted distributionally, since the limit functions \( \rho, v, u \) are not generally even once-differentiable. (See further discussion on fluid singularities below.) This more conventional notion of weak solution is a suitable mathematical idealization of infinite Reynolds-number turbulence, where the inertial range extends to infinitesimally small scales. The concept goes back to Onsager \[20\], who termed it “ideal turbulence.”

As we now discuss, standard consequences of the Euler equations for smooth “strong” solutions do not generally hold for weak solutions, which are instead afflicted with dissipative anomalies due to turbulent cascade. First, we make an important comment on notations. Whenever coarse-grained quantities marked with \( \langle \cdots \rangle \) or \( \langle \cdots \rangle \) appear hereafter, we shall assume that the ideal limit \( \eta, \zeta, \kappa \to 0 \) has been taken, unless indicated otherwise (e.g. by explicitly taking this limit, or by retaining terms with explicit dependence on \( \eta, \zeta, \kappa \)). This convention for coarse-grained quantities simplifies the expressions involved by eliminating the terms which vanish in the ideal limit by the arguments given above.

V. ENERGY CASCADE

A. Kinetic Energy

We begin with kinetic energy cascade. Because \( |v|^2 \) is a convex function of \( v \), one has

\[ \frac{1}{2} \rho |\bar{v}|^2 \leq \frac{1}{2} \rho |\bar{v}|^2 = \frac{1}{2} \rho |\bar{v}|^2. \]  

(38)
Thus, the integral over space of \( \frac{1}{2} \bar{p} \rho \nabla v^2 \) is less than the total kinetic energy, and represents only the “resolved” kinetic energy, while the 2nd-order Favre cumulant
\[
\frac{1}{2} \bar{p} \tau(v_i, v_i) = \frac{1}{2} \bar{p} (\bar{v}_i^2 - \bar{v}_i^2) \geq 0
\]  
represents the “unresolved” or “subscale” kinetic energy. In a decaying turbulence without external forcing, the fine-grained kinetic energy integrated over space decreases because of the effect of viscosity. Since
\[
\frac{1}{2} \int d^dx \, \bar{p} \rho |\nabla |^2 \leq \frac{1}{2} \int d^dx \, \rho |\nabla v|^2 ,
\]  
this decrease must also occur with increasing time for
the resolved kinetic energy, despite the negligible effect of viscosities for \( \ell \) in the inertial range. Physically speaking, the kinetic energy will decay whether an observer is “wearing spectacles” or not. The question thus arises: how can the resolved kinetic energy decay, if not through the influence of viscosity? A similar question arises for forced-steady states. If the fluid is stirred by a large-scale acceleration field, then it is not hard to show that the input of resolved pressure-work \( \bar{p} \tau(\rho, v) \) represents pressure-work in the large-scales which contributes to kinetic energy cascade, with total inertial-range energy flux represented by the combination
\[
Q^\text{flux}_\ell = \nabla \bar{p} \cdot \bar{\tau}(\rho, v) - \bar{p} \nabla \cdot \bar{\tau}(\rho, v).
\]  
As we shall see presently, there are also compelling mathematical reasons to make the above separation of\ the pressure-work.

The cascade terms in the equation \( \text{(41)} \) are a possible source of the dissipative anomaly of kinetic energy for the weak solutions of Euler obtained in the limit first \( \eta, \zeta, \kappa \to 0 \) and then \( \ell \to 0 \). Taking the limit \( \ell \to 0 \) of the balance equation \( \text{(41)} \), one obtains
\[
\partial_t \left( \frac{1}{2} \rho v^2 \right) + \nabla \cdot [(p + \frac{1}{2} \rho v^2 \nabla ) v] = p \circ \Theta - Q^\text{flux}_\ell
\]  
which is the kinetic energy balance for the limiting weak Euler solution. Here we defined
\[
p \circ \Theta = D_{\ell \to 0} \bar{p} \cdot \Theta
\]  
with \( \Theta = \nabla \cdot v \) and
\[
Q^\text{flux}_\ell = D_{\ell \to 0} \lim Q^\text{flux}_\ell,
\]  
where \( D \)-lim denotes limit in the sense of distributions. We now discuss the physical meaning of these two terms.

Because \( p \) and \( v \) are not generally smooth functions, the divergence \( \Theta \) exists only as a distribution and its product with the non-smooth function \( p \) is thus ill-defined and ambiguous. The limit \( p \circ \Theta \) in \( \text{(45)} \) above is a standard approach to define a generalized product of distributions \( \text{[87]} \) and our circle notation “\( \circ \)” is meant to emphasize that this product must be carefully defined. Despite this subtlety, however, such a term is exactly the same as that which appears for a smooth Euler solution. It clearly represents pressure-work in the large-scales which converts energy from mechanical to internal, and vice-versa. Tendency to equipartition of total energy suggests that, when the turbulence is mechanically forced, the transfer will be on average from mechanical to internal. It was already argued in \( \text{[23, 37]} \) that the mean transfer \( \langle \bar{p} \bar{\Theta} \rangle \) at length-scales \( > \ell \) will saturate to a constant negative value as \( \ell \) decreases through the inertial range, and this saturation has been verified in numerical simulations of subsonic and transonic compressible turbulence \( \text{[88, 89]} \).

Our mathematical analysis implies that \( \langle p \circ \Theta \rangle < 0 \) will give the saturated level.

The additional term \( Q^\text{flux}_\ell \) that appears in \( \text{(44)} \) is, on the other hand, entirely missing for smooth Euler solutions and represents a kinetic energy anomaly. It is due
to the loss of kinetic energy by turbulent cascade to infinitesimally small scales. As shown by Onsager [20, 31] for the case of incompressible fluid turbulence, the non-vanishing of such a cascade term requires singularities of the fluid fields \( \rho, \mathbf{v}, \) and \( p. \) For a complete proof of the analogous result for compressible fluids, see the works of Aluie [23, 37, 90] and the companion paper [77]. Briefly, the result follows by expanding Favre averages into cumulants \( \tau(f_1, ..., f_n) \) of the original (non-density weighted) coarse-graining. A fundamental fact is that such cumulants and their spatial-gradients can be written entirely in terms of space-increments \( \delta f_i(r; x) = f_i(x + r) - f_i(x) \) of the fields \( f_i \) (for a proof see [70, Appendix B, or 91]).

From these basic identities one can derive estimates of the form

\[
\nabla \mathbf{v} = \frac{O(\delta v)}{\ell} \left[ 1 + O \left( \frac{\delta \rho}{\rho} \right) + O \left( \left( \frac{\delta \rho}{\rho} \right)^2 \right) \right], \quad (47)
\]

\[
\tau(\mathbf{v}, \mathbf{v}) = O(\delta v)^2 \left[ 1 + O \left( \frac{\delta \rho}{\rho} \right) + O \left( \left( \frac{\delta \rho}{\rho} \right)^2 \right) \right], \quad (48)
\]

where \( \delta v, \delta \rho \) denote increments over the length-scale \( \ell. \) For example, see eqs.(1)-(2) and intervening relations in the paper of Aluie [37]. Substituting such expressions into the formula for the deformation work yields an analogous result of the form [23, 37, 90].

\[\frac{1}{2} \mathbf{v} : \mathbf{\tau}(\mathbf{v}, \mathbf{v}) = \frac{1}{\rho} \mathbf{\tau}(\mathbf{v}, \mathbf{v}) \cdot \mathbf{\nabla} \mathbf{\tau}(\mathbf{v}, \mathbf{v})\]

One might naively conjecture that the latter quantity is the same as \( p * \Theta \) given by [15]. However, the general theory of distributional products makes this \( a \ priori \) highly unlikely. It is part of the definition of the product \( f * g = D * f \cdot g \) that the limiting distribution must be independent of precisely which filter \( G \) is employed, but it is generally not true that other regularizations \( f_\ast, g_\ast \) for which \( f_\ast \cdot g_\ast \neq f \cdot g \) will have same limiting product \( f_\ast \cdot g_\ast \neq f \cdot g \) [87]. Since viscosities and thermal conductivities are a different “regularization” of the Euler system than mere coarse-graining, one should expect that \( p * \Theta \neq p \Theta. \)

Nevertheless, the fine-grained/dissipation-range energy balance [52] must agree with the coarse-grained inertial-range balance [14] in the limit as \( \ell \to 0. \) Objective physical facts such as the rate of decay of energy or the rate of absorption of power input cannot depend on an arbitrary scale \( \ell \) of spatial resolution of observations. After taking first the limit \( \eta, \zeta, \kappa \to 0, \) we must then be able to take \( \ell \to 0 \) and reproduce the same result. Comparing [52] and [14], it follows necessarily that

\[
Q_{visc} = \tau(p, \Theta) + Q_{flux}, \quad (54)
\]

where we have defined the quantity

\[\tau(p, \Theta) = D \lim_{\ell \to 0} \lim_{\eta, \zeta, \kappa \to 0} \bar{\tau}(p, \Theta)\]
\[ p \sim \Theta - p \circ \Theta, \quad (55) \]

which we call the “pressure-dilatation defect”. It is non-vanishing when the joint limits \( \lim_{\ell \to 0} \) and \( \lim_{\nu,\zeta,\kappa \to 0} \) of the product \( \bar{\rho} \cdot \bar{\nabla} \) do not commute, but instead yield either \( p \circ \Theta \) or \( p \circ \Theta \) depending upon the order of the two limits. Unlike incompressible fluid turbulence where \( Q_{\text{visc}} = Q_{\text{flux}} \) \[ \frac{41}{41} \], we see that for compressible fluids the pressure-dilatation defect \( \tau(p, \Theta) \) can be another source of anomalous dissipation distinct from energy cascade. In fact, all stationary, planar shocks in fluids with an ideal-gas equation of state exhibit this mechanism in a pure form, because there \( Q_{\text{flux}} = 0 \) and \( Q_{\text{visc}} = \tau(p, \Theta) \geq 0 \). For a proof of this result, see Appendix \[ \frac{41}{41} \]. All of the anomalous dissipation in such shocks, or so-called “shock heating”, is due to the pressure-dilatation defect. In addition to the general inertial-range result that \( Q_{\text{flux}} = 0 \) for such shocks, we can also obtain exact dissipation-range limits at special values of the Prandtl number where analytical results are available: \( Pr = 3/4 \) \[ \frac{45}{45} \] and \( Pr = 0, \infty \) \[ \frac{46}{46} \]. For cases \( Pr = 3/4, \infty \), in particular, we show in Appendix \[ \frac{41}{41} \] that \( \tau(p, \Theta) > 0 \).

Our arguments show generally that \( Q_{\text{visc}} > 0 \) only if at least one of \( \tau(p, \Theta) \) or \( Q_{\text{flux}} \) is positive. For developed compressible turbulence one should expect that both of these mechanisms will contribute. At finite \( \ell \) we may rewrite the inertial-range kinetic energy balance \[ \frac{41}{41} \] as

\[
\begin{align*}
\partial_t \left( \frac{1}{2} \bar{\rho} \bar{\nabla}^2 \right) + \bar{\nabla} \cdot \left[ \frac{1}{2} \bar{\rho} \bar{\nabla}^2 \right] \bar{\nabla} \bar{\nabla}^2 + \bar{\rho} \bar{\nabla} \cdot \bar{\nabla} \bar{\nabla}^2 + \bar{\rho} \bar{\nabla} \cdot \bar{\nabla} \bar{\nabla}^2 &= \bar{\rho} \circ \Theta - Q_{\text{inert}}^\ell, \\
Q_{\text{inert}}^\ell &= \tau(p, \Theta) + Q_{\text{flux}}^\ell,
\end{align*}
\]

(56)

which is an effective “inertial dissipation” at scale \( \ell \), such that \( Q_{\text{inert}}^\ell := D \cdot \lim_{\ell \to 0} Q_{\text{inert}}^\ell = Q_{\text{visc}} \). The effective dissipation at each arbitrary scale \( \ell \) can agree with the fine-grained/viscous dissipation rate only if there is either nonlinear energy cascade \[ \frac{47}{47} \] with \( Q_{\text{flux}}^\ell > 0 \) or a positive defect \( \tau(p, \Theta) > 0 \) as \( \ell \to 0 \). The estimates of Aluie \[ \frac{23}{23} \] show that the fluid variables must be sufficiently rough in order to sustain energy cascade. The consequences \[ \frac{19}{19}, \frac{31}{31} \] for scaling exponents are directly testable predictions of the argument, which is an exact, non-perturbative application of the principle of renormalization-group invariance \[ \frac{45}{45}, \frac{47}{47} \]. Although it is not yet obvious, the condition that \( \tau(p, \Theta) > 0 \) as \( \ell \to 0 \) also requires these same exponent relations to hold. In order show this, we must develop a deeper understanding of the thermodynamics of compressible turbulence.

\section{Internal Energy}

The other half of the energy budget is internal energy. Numerical results \[ \frac{88}{88} \] show that up to 50% of the energy injected at large scales can be channeled into internal energy by the large-scale pressure work. We must therefore consider the inertial-range dynamics of internal energy. The simplest way to obtain an equation for the coarse-grained/resolved internal energy \( \hat{u} \) is to apply the coarse-graining operation to the equation \[ \frac{15}{15} \] for fine-grained internal energy and then to consider the ideal (infinite Reynolds and Pécelt number) limit. The first step yields

\[
\partial_t \hat{u} + \bar{\nabla} \cdot (\hat{u} \bar{\nabla} + \bar{\tau}(u, \bar{v})) = -\hat{p} \circ \Theta + \hat{Q}_{\text{visc}}. \quad (59)
\]

In the subsequent limit \( \ell \to 0 \) we get

\[
\partial_t u + \bar{\nabla} \cdot (u \bar{v}) = -p \circ \Theta + Q_{\text{visc}}, \quad (60)
\]

as the distributional balance of internal energy for the limiting weak Euler solution.

On the other hand, we can obtain another form of this equation by subtracting the balance equation \[ \frac{41}{41} \] for resolved kinetic energy from the coarse-graining of equation \[ \frac{47}{47} \] for conservation of total energy. This yields after some straightforward calculations the equation

\[
\partial_t \left( \pi + \frac{1}{2} \bar{\rho} \bar{\nabla}^2(v_i, v_i) \right) + \bar{\nabla} \cdot \left( \pi \bar{\nabla}^2(h, \bar{v}) + \frac{1}{2} \bar{\rho} \bar{\nabla}^2(v_i, v_i) \bar{\nabla}^2 \right) = -\hat{p} \circ \Theta + Q_{\text{flux}}^\ell. \quad (61)
\]

with \( h = u + p \) the enthalpy. There appears in this balance the quantity

\[
\pi^* := \pi + \frac{1}{2} \bar{\rho} \bar{\nabla} \bar{v}(v_i, v_i), \quad (62)
\]

which we shall call the *intrinsic large-scale/resolved internal energy*. It is a natural object because, based on coarse-grained observations alone, it is impossible to distinguish between energy in thermal fluctuations and in unresolved turbulent fluctuations. In contrast to the balance \[ \frac{59}{59} \] for resolved internal energy, the balance \[ \frac{61}{61} \] for intrinsic large-scale internal energy contains no direct contributions from microscopic dissipation and is a consequence solely of the limiting distributional Euler solution. In the limit as \( \ell \to 0 \) all of the cumulant terms in \[ \frac{61}{61} \] vanish distributionally and one obtains a second form of the internal energy balance:

\[
\partial_t u + \bar{\nabla} \cdot (u \bar{v}) = -p \circ \Theta + Q_{\text{flux}}. \quad (63)
\]

Using \[ \frac{47}{47} \], we may rewrite the right-hand side of \[ \frac{63}{63} \] as \( -p \circ \Theta + Q_{\text{inert}}^\ell \). The two equations \[ \frac{60}{60} \] and \[ \frac{63}{63} \] thus agree, since \( Q_{\text{visc}} = Q_{\text{inert}}^\ell = Q \). We see that the same \( Q \) which appears as a sink in the kinetic energy balance, \[ \frac{52}{52} \] or \[ \frac{56}{56} \], appears as a source in the balance of internal energy, \[ \frac{60}{60} \] or \[ \frac{63}{63} \].
One concern at this point is whether the equation \((50)\) for \(\bar{u}\) truly represents “inertial-range dynamics,” in contrast to equation \((61)\) for \(\bar{v}\) which is clearly an inertial-range balance. Quantities of the sort \(Q_{\text{visc}}\) have been much discussed for incompressible fluid turbulence in the context of the “Kolmogorov refined similarity hypothesis” \((98)\) and it has been a debated issue whether such coarse-grained dissipation fields for inertial-range lengths \(\ell\) should be considered inertial-range or dissipation-range. For example, Kraichnan \((99)\) concluded that \(Q_{\text{visc}}\) is not inertial-range and “Instead it is the integral of a dissipation-range quantity.” The question is hard to argue substantively, because there is no clear, accepted definition in the literature of what it means to be “inertial-range” or “dissipation-range”. We would like to offer a simple, precise definition of an “inertial-range quantity” as any field which exists as an ordinary function (as opposed to a distribution only) in the ideal limit \(\eta, \zeta, \kappa \to 0\). By this definition, \(Q_{\text{visc}}\) is clearly an inertial-range quantity and so is the pressure work \(p \ast \Theta\), although both involve effects of molecular dissipation which survive in the ideal limit \(\eta, \zeta, \kappa \to 0\).

The questions about inertial-range status of \(Q_{\text{visc}}\) and \(p \ast \Theta\) cannot however be legitimately answered by merely offering a definition. The more serious worry which underlies this question is whether these can be universal quantities independent of the particular micro-scale dissipation mechanism, or whether they shall be distinct for every particular fine-grained dissipation (e.g. ordinary viscosity vs. hyper-viscosity). As a matter of fact, the quantities \(Q_{\text{visc}}\) and \(p \ast \Theta\) probably cannot be completely universal in compressible fluid turbulence, as they can be shown to be Prandtl-number dependent \((100)\) even for planar, ideal-gas shocks (see Appendix A). On the other hand, it is a direct consequence of equation \((60)\) for the internal energy \(u\) that the combination \(-p \ast \Theta + Q_{\text{visc}}\) depends only upon the limiting Euler solution fields \(p, \nu, \nu\) and not upon the particular sequence \(\eta, \zeta, \kappa \to 0\) used to obtain that solution. It is explicitly verified for planar, ideal-gas shocks in Appendix A that the combined quantity is independent of Prandtl-number even though the quantities \(Q_{\text{visc}}\) and \(p \ast \Theta\) separately are \(Pr\)-dependent. This suggests that the combination \(Q_{\text{visc}} - p \ast \Theta\) for inertial-range length-scales \(\ell\) should be universal for a wide class of fine-grained dissipation mechanisms and determined only by fluid modes at scales comparable to \(\ell\).

We can make a substantive argument for this assertion based upon the following equation for sub-scale/unresolved kinetic energy in the ideal limit:

\[
\partial_t \left( \frac{1}{2} \bar{\rho} \bar{v}^2 + v \bar{\rho} \bar{v} \bar{v} \right) + \nabla \cdot \left[ \frac{1}{2} \bar{\rho} \bar{v}^2 (v_i, v_j) \bar{v} + \bar{\rho} \bar{\nu} + \frac{1}{2} \bar{\rho} \bar{v} (v_i, v_j, v_k) \right] = \bar{\rho} (p, \Theta) - Q_{\text{visc}} + Q_{\text{flux}}^{\ell}. \tag{64}
\]

This equation is straightforward to derive by considering the equations for \((1/2)\bar{\rho}|\bar{v}|^2\) and \((1/2)\bar{\rho}|\bar{v}|^2\), subtracting them, and taking the limit \(\eta, \zeta, \kappa \to 0\). A simple reorganization of this equation gives

\[
\partial_t \left( \frac{1}{2} \bar{\rho} \bar{v}^2 + v \bar{\rho} \bar{v} \bar{v} \right) + \nabla \cdot \left[ \bar{\rho} \bar{\nu} + \frac{1}{2} \bar{\rho} \bar{v} (v_i, v_j, v_k) \right] = \bar{\rho} (p, \Theta) - Q_{\text{visc}} + Q_{\text{flux}}^{\ell}. \tag{65}
\]

where \(\bar{D}_t = \partial_t + \bar{v} \cdot \nabla\) is the large-scale Lagrangian time-derivative. The important point is that all of the terms on the right-hand side of this equation are pure inertial-range quantities that are local-in-scale and thus determined only by fluid modes near the considered scale \(\ell\). The standard arguments for universality thus apply to the right-hand side and so we may argue that as well the lefthand side, the combination \(p \ast \Theta - Q_{\text{visc}}\), will be a universal, inertial-range quantity, independent of the particular microscale mechanism of dissipation. As we shall discuss further in the following section, the above considerations play an essential role in our proof of a complete Onsager theorem for compressible turbulence \((77)\). By this we mean the proof that a kinetic-energy dissipation anomaly \(Q \neq 0\) requires singularities in the fluid fields. As emphasized earlier, the arguments of Aluie \((23, 37, 90)\) imply that \(Q_{\text{flux}} \neq 0\) requires the inequalities \((49, 61)\) to hold, but it is still possible in principle that \(Q_{\text{visc}} = \tau(p, \Theta) > 0\) with milder singularities.

Let us close this section by briefly considering the energy balances that must exist in a long-time steady-state of mechanically forced compressible turbulence. In order for a steady state to exist one must take into account cooling mechanisms, such as electromagnetic radiation, otherwise the internal energy will continue to grow due to input from viscous dissipation and mechanical work. The situation may be modelled by the compressible Navier-Stokes equations modified to include an external acceleration field \(a_{\text{ext}}\) and a cooling function \(Q_{\text{cool}}:\)

\[
\partial_t \rho + \nabla \cdot (\rho \nu v) = 0, \tag{66}
\]

\[
\partial_t (\rho v) + \nabla \cdot (\rho v \nu v + p I - 2\eta S - \zeta \Theta I) = \rho a_{\text{ext}}, \tag{67}
\]

\[
\partial_t \left( \frac{1}{2} \rho v^2 + u \right) + \nabla \cdot \left[ (u + p + \frac{1}{2} \rho v^2) \nu - \kappa \nu T - 2\eta S \nu - \zeta \Theta \nu \right] = \rho v \cdot a_{\text{ext}} - Q_{\text{cool}}. \tag{68}
\]

The acceleration field is a source of mechanical input of kinetic energy \(Q_{\text{in}} = \rho v \cdot a_{\text{ext}}\), whereas the equation \((15)\) for internal energy now includes the cooling term \(-Q_{\text{cool}}\) on the right. When the forcing \(a_{\text{ext}}\) and cooling function \(Q_{\text{cool}}\) are large-scale (smooth) fields, then all of our previous considerations on the ideal limit apply. Steady-state kinetic energy balance gives \((Q_{\text{in}}) = \langle Q_{\text{trans}} \rangle + \langle Q \rangle\), where \(Q_{\text{trans}} = -p \ast \Theta\). From the fine-grained point of view, \(Q = Q_{\text{visc}}\) whereas in the inertial-range \(Q = Q_{\text{inert}}\) and \((Q_{\text{int}}) = \langle Q_{\text{int}} \rangle = \langle Q_{\text{trans}} \rangle + \langle Q \rangle\), exactly as argued earlier by Aluie \((23)\). The steady-state internal energy balance likewise gives \((Q_{\text{trans}} + \langle Q \rangle = \langle Q_{\text{cool}} \rangle\), so that \((Q_{\text{in}}) = \langle Q_{\text{cool}} \rangle\) in the steady-state. In decaying turbulence without external forcing such as we considered
throughout most of the paper, one expects a quasi-steady state with initial conditions supplying the reservoir of energy to drive the cascade and $-\langle \partial_t (\rho \nu^2/2) \rangle$ playing the role of $\langle Q_{in} \rangle$. Likewise, if cooling mechanisms are inefficient, then $\langle \partial_t u \rangle$ plays the role of $\langle Q_{cool} \rangle$.

VI. ENTROPY CASCADE

The energy transfer $\bar{p}\Theta$ from large-scale kinetic energy to large-scale internal energy $\bar{u}$ which was discussed in the previous section does not represent a global heating of the fluid resulting merely in a uniform increase in the internal energy. It is instead an “ordered” or “coherent” input of energy, which leads to large-scale structure of the internal energy field $\bar{u}$. One should thus expect this input to decrease the large-scale entropy of the system, which is maximum for a spatially homogeneous state. These considerations are one motivation to consider in detail the entropy balance of the turbulent flow, which allows us to verify the above expectations. Because of the constraints imposed by the second law of thermodynamics, the entropy in fact turns out to play a central role in the constraints imposed by the second law of thermodynamics.

We can now turn our discussion below the shorthand notations coarse-graining increases entropy. We use throughout our discussion here to single-phase flows. Also an analytic function, except at phase transitions, of thermodynamic equilibrium [103]. The entropy is limit [101, 102] and macroscopically from the stability to decrease the large-scale entropy of the system, which is maximum for a spatially homogeneous state. These considerations are one motivation to consider in detail the entropy balance of the turbulent flow, which allows us to verify the above expectations. Because of the constraints imposed by the second law of thermodynamics, the entropy in fact turns out to play a central role in the constraints imposed by the second law of thermodynamics.

We recall that the entropy per volume $s(u,n)$ is a concave function of the internal energy per volume $u$ and the particle number per volume $n$. This follows microscopically from the extensivity of the thermodynamic limit [101, 102] and macroscopically from the stability of thermodynamic equilibrium [103]. The entropy is also an analytic function, except at phase transitions, and we restrict our discussion here to single-phase flows. The quantity $s(\bar{u},\bar{n})$ naturally represents the “large-scale/resolved entropy” for a given length-scale $\ell$. We can in turn define the “small-scale/unresolved entropy”

$$\Delta s \equiv s(u,n) - s(\bar{u},\bar{n}) \leq 0.$$  \hspace{1cm} (69)

The non-positivity follows by concavity, so that spatial coarse-graining increases entropy. We use throughout our discussion below the shorthand notations

$$\bar{s} = s(u,n), \quad \underline{s} = s(\bar{u},\bar{n}),$$  \hspace{1cm} (70)

and likewise for other thermodynamic functions of $u,n$. Note that $\langle \bar{s} \rangle$ plays the role of a “cumulative entropy (co)spectrum” up to wave-number $\sim 1/\ell$ and $\langle \Delta s \rangle$ is analogous to a second-order “entropy structure function” at separation $\ell$. In many respects it is more natural to consider a quantity $s_{\text{max}} - s$ which is convex and decreasing in time rather than the traditional entropy, which is instead concave and increasing. The quantity $s_{\text{max}} - s$ was known variously as “capacity of entropy” by Gibbs [104], “deficiency of entropy” by Obukhov [72], and “negentropy” by Brillouin [105]. We use here the latter term.

We first derive the balance equation of large-scale entropy at finite $\eta$, $\zeta$, $\kappa$ using the equation (69) for $\bar{s}$ and (27) for $\delta s$, with $\bar{p} = \bar{p} \bar{v} + \bar{s}(\bar{p},\bar{v})$. Invoking the first law of thermodynamics in the form $du = T ds + p\gamma n$ for absolute temperature $T$ and chemical potential $\mu$, and denoting $D_t = \partial_t + \bar{v} \cdot \nabla$, we get from

$$T D_t \bar{s} = D_t \bar{u} - \mu D_t \bar{n}$$  \hspace{1cm} (71)

after some straightforward calculation that

$$\partial_t \bar{s} + \nabla \cdot [\bar{s} \bar{v} + \beta \bar{u} - \bar{v} \bar{a}(n,v)] = \Sigma^{\text{inert}},$$  \hspace{1cm} (72)

with inertial-range entropy production given by

$$\Sigma^{\text{inert}} = -I_{\text{mech}} + \Sigma^{\text{flux}} + \bar{Q}_{\text{visc}}/T,$$  \hspace{1cm} (73)

for mechanical input of negentropy

$$I_{\text{mech}} = \frac{\bar{p} - \bar{p}}{\bar{T}}$$  \hspace{1cm} (74)

and for negentropy flux

$$\Sigma^{\text{flux}} = \nabla \cdot [\beta \bar{u} - \bar{v} \bar{a}(n,v)] = \nabla \cdot \bar{v}.$$

The quantity $\beta = 1/T$ in the expressions above is inverse temperature and $\lambda = \mu/T$ is the thermodynamic potential entropically conjugate to particle number. We now discuss in detail the physical significance of each of these various contributions to the entropy balance.

First, $I_{\text{mech}}$ represents the net input of negentropy [106] into the large-scales from pressure work, where $-\bar{p} \bar{T}/T$ is the “coherent input” of negentropy at large-scales and $\bar{p} \bar{T}/T$ is the entropy production (destruction of negentropy) due to mechanical heating at all scales. There is competition between these two terms but, as anticipated, they will not cancel in general. We suggest that it is likely that $\langle \bar{p} \bar{T} / T \rangle < \langle \bar{p} \bar{T} / T \rangle < 0$, because of the greater coherence at larger scales and the near cancellations between positive and negative terms at small scales [88, 89]. An alternative decomposition to that above is

$$I_{\text{mech}} = \frac{\bar{p} \bar{T}}{T} + I^{\text{flux}} \quad I^{\text{flux}} = \frac{\Delta p \bar{T}}{T}$$  \hspace{1cm} (76)

with $\Delta p = \bar{p} - \bar{p}$. The first term is related to the pressure-dilatation defect and the second term is a “flux-like” contribution, in the sense that it represents an interaction between a large-scale dilatation $\Theta$ and a small-scale pressure $\Delta p$. There is a simple formula for the latter [77, 107] which provides an estimate

$$\Delta p = O \left( (\delta u)^2, (\delta u)(\delta p), (\delta p)^2 \right)$$  \hspace{1cm} (77)

yielding a “4/5th-law” type representation of the flux term. It is worth noting that for an ideal gas with adiabatic index $\gamma = c_p/c_v$, one has $p(u,n) = (\gamma - 1)u$ so that $\Delta p = 0$ exactly and the flux-like term is absent. Neither term is present in a naive fine-grained calculation.

Secondly, $\Sigma^{\text{flux}}$ represents negentropy flux to small scales, arising from small-scale turbulent transport of heat energy $\bar{v}(u,v)$ acting against large-scale temperature gradients $\nabla T$ and small-scale particle transport.
\( \tau(n,v) \) acting against large-scale \( \lambda \)-gradients. These contributions will be positive, indicating entropy production/inverse cascade of entropy/forward cascade of negentropy, when the turbulent transport is “down-gradient”, with heat-transport from higher to lower resolved temperatures \( \beta \) and particle-transport from higher to lower \( \lambda \)-potential. Because of lack of scale-separation of turbulent transport both positive and negative values will occur pointwise in space-time for finite \( \ell \), but one should expect that on average \( \langle \Sigma^{\text{flux}} \rangle > 0 \), consistent with the overall increase of entropy from the second law of thermodynamics. Of course, the term \( Q_{\text{visc}}/T \geq 0 \) is the resolved entropy production at large scales due to viscous dissipation.

We now consider the situation when there is anomalous entropy production in the ideal limit \( \eta, \zeta, \kappa \to 0 \), as hypothesized in (20). The same result must be obtained by considering either the fine-grained entropy balance (19) or the inertial-range balance (72). Indeed, because of concavity of the volumetric entropy density, the total entropy observed “without spectacles” at resolution \( \ell \) can only exceed the true entropy

\[
S_{\ell}(t) = \int d^3x \ s(\bar{u}_\ell, \bar{\rho}_\ell) \geq \int d^3x \ s(\rho, u) = S(t)
\]  

and in the limit \( \ell \to 0 \) they must agree. Thus, if entropy \( S(\ell) \) continues to grow over a finite time-interval in the limit \( \eta, \zeta, \kappa \to 0 \), then \( S_{\ell}(t) \) must also grow for the subsequent limit \( \ell \to 0 \) described by a weak Euler solution. Taking the limit \( \eta, \zeta, \kappa \to 0 \) of the fine-grained entropy balance (19), the anomaly is represented as

\[
\partial \cdot s + \nabla \cdot (sv) = \Sigma^{\text{diss}}
\]

with \( \Sigma^{\text{diss}} = \Sigma^{\text{therm}} + \Sigma^{\text{visc}} \), for viscous entropy production

\[
\Sigma^{\text{visc}} = \beta \ast Q_{\text{visc}} = D_s \lim_{\eta, \zeta, \kappa \to 0} \beta Q_{\text{visc}}
\]

and for entropy production through thermal conduction

\[
\Sigma^{\text{therm}} = D_s \lim_{\eta, \zeta, \kappa \to 0} \frac{\kappa}{T^2} |\nabla T|^2.
\]

The coarse-grained entropy balance (72) in the limit \( \eta, \zeta, \kappa \to 0 \) is unchanged, except that \( p\Theta \to p \ast \Theta \). In the subsequent limit \( \ell \to 0 \) the inertial-range entropy balance becomes

\[
\partial \cdot s + \nabla \cdot (sv) = \Sigma^{\text{inert}},
\]

where \( \Sigma^{\text{inert}} = -I_{\text{mech}} + \Sigma^{\text{flux}} + \beta \circ Q_{\text{visc}} \) with \( 108 \)

\[
\beta \circ Q_{\text{visc}} = D_s \lim_{\ell \to 0} \beta \tilde{Q}_{\text{visc}}.
\]

and where \( I_{\text{mech}} = I^{\text{flux}} + \beta \circ \tau(p, \Theta) \) with

\[
\beta \circ \tau(p, \Theta) = D_s \lim_{\ell \to 0} \beta \tilde{\tau}(p, \Theta).
\]

The expressions \( 82 \)-\( 84 \) provide an inertial-range representation of anomalous entropy production. Equating the two different expressions, \( \Sigma^{\text{inert}} = \Sigma^{\text{diss}} \).

The above general results are nicely illustrated by planar shocks in an ideal gas for the special value of Prandtl number \( Pr = 3/4 \). There is in fact a 2-parameter family of stationary shocks in ideal gases, labelled by the adiabatic index \( \gamma > 1 \) and by the pre-shock Mach number \( M_0 > 1 \), or, alternatively, the compression ratio \( R = (\gamma + 1)/((\gamma - 1)/2 + M_0^2) \). All of the anomalous quantities in the balances above are non-zero for \( Pr = 3/4 \) with the exception of \( I^{\text{flux}} \), since \( \Delta p \equiv 0 \) for an ideal gas and are proportional to Dirac delta functions at the location of the shock. For a shock situated at the origin there is a positive entropy production anomaly of the very simple form

\[
\Sigma^{\text{inert}} = \Sigma^{\text{diss}} = (\Delta s_m) j_s(\delta x),
\]

where \( \Delta s_m = s_{m,1} - s_{m,0} > 0 \) is the jump across the shock of the entropy per mass \( s_m = s/\rho \) (with “0” denoting the gas in front of the shock and “1” gas behind the shock) and \( j_s = j_0 = j_1 > 0 \) is the mass flux through the shock. Explicit expressions for all terms in the entropy balance are given in Appendix A. As expected, the inertial-range quantities \( \Sigma^{\text{flux}} \) and \( \beta \circ (Q_{\text{visc}} - \tau(p, \Theta)) \) in the infinite Reynolds-number limit are identical for all planar, ideal-gas shocks with the same values of \( \gamma \) and \( M_0 \), entirely independent of the precise molecular dissipation mechanism. In particular, these two particular quantities are Prandtl-number independent. It is interesting that the negentropy flux is non-zero for such shocks, even though the energy flux vanishes exactly. This shows clearly that the two cascades are distinct in general. In fact, it is the negentropy flux \( \Sigma^{\text{flux}} \) which inside the inertial-range of planar, ideal-gas shocks supplies the contribution \( (\Delta s_m) j_s(\delta x) \), arising from the particle transport term \(-\nabla \Delta \tau(\nu, \nu)\). All other inertial-range contributions cancel between \( I_{\text{mech}}, I^{\text{flux}}, \) and \( \beta \circ Q_{\text{visc}} \).

See Appendix A for details.

While Euler shock solutions with discontinuous fields provide a simple example where the negentropy flux is non-vanishing, more modest singularities are able to support a negentropy cascade. Note from the formula \( 77 \) for \( \Delta p \) and the definition \( 76 \) of \( I^{\text{flux}} \) that

\[
I^{\text{flux}}_\ell = O \left((\delta u)^2, (\delta u)(\delta \rho), (\delta \rho)^2\right) O \left(\frac{\delta v}{\ell}\right).
\]

Hence, this term may have a non-vanishing limit as \( \ell \to 0 \) whenever

\[
2 \min\{\zeta_q^u, \zeta_q^p\} + \zeta_q^q \leq q, \quad q \geq 3. \tag{87}
\]

Likewise, from the chain rule for gradients of the smooth functions \( \beta(u, \rho) \) and \( \lambda(u, \rho) \) one gets

\[
\nabla \beta, \nabla \lambda = \frac{O(\delta u, \delta \rho)}{\ell}. \tag{88}
\]
which, with the general result \( \tau(f, g) = O(\delta f \cdot \delta g) \) and the definition of \( \Sigma^{\text{flux}}_\ell \), gives the identical estimate

\[
\Sigma^{\text{flux}}_\ell = O \left( (\delta u)^2, (\delta u)(\delta \rho), (\delta \rho)^2 \right) O \left( \frac{\delta v}{\ell} \right). \tag{89}
\]

Thus, the inequality (87) again provides a necessary condition for a non-vanishing limit as \( \ell \to 0 \). The shock solutions with discontinuous fields have \( \zeta_n^\ell = \zeta_\ell^\rho = \zeta_\ell^v = 1 \) for \( q \geq 1 \) and thus satisfy these inequalities for all \( q \geq 3 \). However, multifractal fields \( u, \rho, v \) with positive Hölder exponents can also easily satisfy these inequalities. Thus, for compressible turbulent flow the anomalous entropy production should generally arise not just from shocks with zero Hölder exponents but also from the spectrum of milder Hölder singularities.

When the singularity conditions (49), (51) are not satisfied, then one expects that entropy will in fact be conserved. This statement is an analogue of the Onsager singularity theorem for a dissipative anomaly of negentropy. Such a result does not follow directly from the estimates (85), (88) on the fluxes, because of the additional terms contributing to the inertial-range entropy balance. However, such a result may be proved (77), by the following arguments. First, rewrite the inertial-range entropy balance as

\[
\partial_t \mathcal{S} + \nabla \cdot \left[ \mathcal{S} \vec{v} + \beta \mathcal{T}(u, v, \nabla) - \Lambda \mathcal{T}(n, v) \right] = \Sigma^{\text{flux}}_\ell - I^{\text{flux}}_\ell + \beta \left( \tilde{Q}_{\text{visc}} - \mathcal{T}(p, \Theta) \right) \tag{90}
\]

The first two terms on the right are those which have been shown to vanish as \( \ell \to 0 \) when (87) is not satisfied. To evaluate the last term, we use the equation (64) for the subscale kinetic energy. The first two terms on the right of (64) are exactly those appearing in the entropy balance, while the third is the energy flux. Thus, multiplying (64) by \( \beta \) gives

\[
\beta \left( \tilde{Q}_{\text{visc}} - \mathcal{T}(p, \Theta) \right) = \beta Q^{\text{flux}}_\ell + (\partial_t \beta) \frac{1}{2} \bar{\rho} \mathcal{T}(v_i, v_i) + \nabla \beta \cdot \left[ \frac{1}{2} \bar{\rho} \mathcal{T}(v_i, v_i) \vec{v} + \bar{\rho} \mathcal{T}(v, p, v) + \frac{1}{2} \bar{\rho} \mathcal{T}(v_i, v_i, v) \right] + (\cdots) \tag{91}
\]

where \((\cdots)\) denotes a total derivative term which vanishes distributionally in the limit \( \ell \to 0 \). The other three terms are all “flux-like”. The first of these contains a time-derivative, which is perhaps unexpected, but the physical meaning is clearly an entropy-production due to rate of change of large-scale inverse-temperature times subscale kinetic-energy. The term \( \beta Q^{\text{flux}}_\ell \) is an entropy-production due to kinetic energy cascade and the terms proportional to \( \nabla \beta \) are a correction to the turbulent internal energy transport \( \bar{\mathcal{T}}(u, v) \). These flux terms vanish as explicit power-laws in the limit \( \ell \to 0 \) for solutions that are not sufficiently singular. Precisely, at least one of the following conditions must be satisfied

\[
\begin{align*}
2 \min \{ \zeta_n^\ell, \zeta_\ell^v \} + \zeta_\ell^\rho &\leq q \quad q \geq 3 \tag{92} \\
\min \{ \zeta_n^\rho, \zeta_\ell^v \} + 2 \zeta_n^\rho &\leq q \quad q \geq 3 \tag{93} \\
3 \zeta_n^\rho &\leq q \quad q \geq 3, \tag{94}
\end{align*}
\]

if the fluxes are not to vanish. Here (92) is the same as (87), and (93) has also been replaced by (92), which implies vanishing of baroclinic work via (50).

There is, in fact, a much more fundamental way to reach the same conclusion. Let us define an intrinsic large-scale/resolved entropy density by

\[
\mathcal{S}^* = \mathcal{S} + \frac{1}{2} \beta \mathcal{T}(v_i, v_i). \tag{95}
\]

From the homogeneous Gibbs relation \( \mathcal{S} = \beta (\mathcal{T} + p) - \lambda \mathcal{M} \), it follows that \( \mathcal{S}^* = \beta (\mathcal{T} + p) - \lambda \mathcal{M} \), where \( \mathcal{M} \) is the “intrinsic large-scale internal energy” that was introduced in (62). Using the equation (61) for the intrinsic internal energy, the coarse-grained mass conservation equation (27), and the standard thermodynamic relation \( d(\beta p) = \left[ d \lambda / \lambda \right] \mathcal{M} \), it is then straightforward to verify the entropy balance equation

\[
\partial_t \mathcal{S}^* + \nabla \cdot \mathcal{S}^* = \Sigma^{\text{inert}}_\ell \tag{96}
\]

where

\[
\begin{align*}
\mathcal{S}^* &= \mathcal{S} + \beta \mathcal{T}(u, v) - \lambda \mathcal{T}(n, v) + \frac{1}{2} \beta \mathcal{T}(v_i, v_i) + \mathcal{T}(p, v), \tag{97}
\end{align*}
\]

is the spatial-current of intrinsic entropy and where \( \Sigma^{\text{inert}}_\ell = -I^{\text{flux}}_\ell + \Sigma^{\text{flux}}_\ell \), with

\[
\begin{align*}
\Sigma^{\text{flux}}_\ell &= \Sigma^{\text{inert}}_\ell + \beta Q^{\text{flux}}_\ell + (\partial_t \beta) \frac{1}{2} \bar{\rho} \mathcal{T}(v_i, v_i) + \nabla \beta \cdot \left[ \frac{1}{2} \bar{\rho} \mathcal{T}(v_i, v_i) \vec{v} + \frac{1}{2} \bar{\rho} \mathcal{T}(v, p, v) + \bar{\rho} \mathcal{T}(v_i, v_i, v) \right] + (\cdots)
\end{align*}
\]

\[
\begin{align*}
= \nabla \beta \cdot \left[ \frac{1}{2} \bar{\rho} \mathcal{T}(v_i, v_i) \vec{v} + \frac{1}{2} \bar{\rho} \mathcal{T}(v, p, v) + \bar{\rho} \mathcal{T}(v_i, v_i, v) \right] + (\cdots) \tag{98}
\end{align*}
\]

the flux of intrinsic inertial-range negentropy. Although this result is the same as that obtained by substituting (91) into (90), the present derivation is more general, because it makes no reference to any microscopic model. Thus, the equation (96) is seen to be valid for all distributional Euler solutions, including those derived from Boltzmann kinetic theory for example and not restricted to limits of compressible Navier-Stokes solutions.

Taking the limit as \( \ell \to 0 \) of the inertial-range balance equation (96) yields again the limiting balance (82) for the distributional solution of the compressible Euler equations. One concludes that any solution that is too
regular, satisfying none of the conditions [92]–[94], will obey local entropy conservation:
\[ \partial_t s + \nabla \cdot (s \mathbf{v}) = 0. \]  
(99)

Put another way, \( \Sigma \) dissatisfied. Because of the non-negativity of the separate dissipation anomalies must vanish when (92)–(94) are not satisfied. Of the non-negativity of the separate viscous and thermal conduction contributions to anomalous entropy production, \( \Sigma \) diss = 0 immediately implies that

\[ \Sigma_{\text{therm}} = 0, \quad \Sigma_{\text{visc}} = \beta \ast Q_{\text{visc}} = 0. \]  
(100)

It is not hard to see that the second condition implies that \( Q_{\text{visc}} = 0 \), since \( \beta \circ Q_{\text{visc}} \geq Q_{\text{visc}}/ |T| \| \geq 0 \). Coupled with \( Q_{\text{flux}} = 0 \) we obtain from (54), that \( \tau(p, \Theta) = 0 \) or \( p \circ \Theta = p \ast \Theta \). Thus, kinetic energy balance becomes

\[ \partial_t \left( \frac{1}{2} \rho \mathbf{v}^2 \right) + \nabla \cdot \left[ (p + \frac{1}{2} \rho \mathbf{v}^2) \mathbf{v} \right] = p \ast \Theta \]  
(101)

and internal energy balance becomes

\[ \partial_t u + \nabla \cdot (u \mathbf{v}) = -p \ast \Theta. \]  
(102)

In other words, kinetic energy and internal energy balances hold without anomalies for flows more regular than allowed by (92)–(94). It is interesting that the proof of a complete Onsager singularity theorem for energy dissipation anomalies in compressible turbulence seems to require an essential consideration of entropy.

We again close the section by considering steady-state compressible turbulence which is forced mechanically and also cooled, e.g. by radiation, governed by equations [66]–[68], this time for entropy balances. It is easy to see that the equation [61] for intrinsic resolved internal energy is modified by the addition of two terms:

\[ \partial_t \pi^* = \cdots + \tilde{\rho} \tilde{T}(v; a_{\text{ext}}) - \bar{Q}_{\text{cool}}. \]  
(103)

The first term \( \tilde{\rho} \tilde{T}(v; a_{\text{ext}}) \) is negligible when \( \ell \ll L \), where \( L \) is the length-scale of the smooth acceleration field \( a_{\text{ext}}. \) However, the second term \( -\bar{Q}_{\text{cool}} \) has a non-zero limit as \( \ell \to 0 \) and contributes to the balance of intrinsic resolved entropy an additional term \( -\beta \bar{Q}_{\text{cool}} \) on the right, which represents the removal of entropy at large scales by cooling. Likewise, the fine-grained entropy balance gains an additional term \( -\beta \bar{Q}_{\text{cool}} \) on the right. A statistically stationary and homogeneous steady-state must thus satisfy

\[ \langle \Sigma_{\text{diss}} \rangle = \beta Q_{\text{cool}} \]  
(104)

from the fine-grained balance and

\[ \langle \Sigma_{\text{inert}}^{\ell} \rangle = \beta \langle \bar{Q}_{\text{cool}}^{\ell} \rangle + \beta L \tilde{\rho} \tilde{T}(v; a_{\text{ext}}) \]

\[ \approx \langle \beta Q_{\text{cool}}^{\ell} \rangle, \quad \ell \ll L \]  
(105)

from the coarse-grained/inertial-range balance [109]. The physical picture is that the large-scale negentropy introduced by cooling cascades through an inertial-range down to small-scales where it is cancelled by microscopic entropy production. Alternatively, we may write as

\[ \langle \Sigma_{\text{flux}}^{\ell} \rangle = \langle I_{\text{flux}}^{\ell} \rangle + \beta \langle \bar{Q}_{\text{cool}}^{\ell} \rangle + \langle \beta \tilde{\rho} \tilde{T}(v; a_{\text{ext}}) \rangle \]

\[ \approx \langle I_{\text{flux}}^{\ell} \rangle + \beta Q_{\text{cool}}^{\ell}, \quad \ell \ll L \]  
(106)

so that the flux of intrinsic entropy through the initial-range includes also negentropy input from anomalous pressure-work, as well as from large-scale cooling. For an ideal gas equation of state \( I_{\text{flux}}^{\ell} = 0 \), of course, and \( \langle \Sigma_{\text{flux}}^{\ell} \rangle \approx \langle \Sigma_{\text{diss}} \rangle \) for \( \ell \ll L \).

In addition to these exact relations, some approximate balances are suggested by our results. For the purpose of a qualitative picture, let us assume an approximately constant temperature \( T \). This gives (on average) the fine-grained (dissipation-range) entropy balance

\[ \langle \Sigma_{\text{therm}} \rangle + \langle Q \rangle \bar{T} = \langle Q_{\text{cool}} \rangle \bar{T} \]  
(107)

and the coarse-grained (inertial-range) entropy balance

\[ \langle \Sigma_{\text{flux}} \rangle + \langle Q \rangle \bar{T} = \langle Q_{\text{cool}} \rangle \bar{T} + \langle I_{\text{mech}} \rangle. \]  
(108)

One also has the mean energy balance \( \langle Q_{\text{cool}} \rangle = \langle Q \rangle + \langle Q_{\text{trans}} \rangle \) with \( Q_{\text{trans}} = -p \ast \Theta \). Note that the fine-grained pressure-work \( Q_{\text{trans}} \) is an incoherent transfer of energy, which might be better denoted in this context as \( Q_{\text{incoh}} \).

The first equation (107) thus gives

\[ \langle \Sigma_{\text{therm}} \rangle = \langle Q_{\text{incoh}} \rangle \bar{T}. \]  
(109)

Decomposing \( \langle I_{\text{mech}} \rangle = (\langle Q_{\text{cool}} \rangle - \langle Q_{\text{incoh}} \rangle)/T \) with the definition \( \langle Q_{\text{cool}} \rangle = -\lim_{\ell \to 0} \langle Q \ast \Theta \rangle \) of coherent work input, substituting into the second equation (108), and using \( \langle Q_{\text{cool}} \rangle = \langle Q \rangle + \langle Q_{\text{incoh}} \rangle \) again yields

\[ \langle \Sigma_{\text{flux}} \rangle = \langle Q_{\text{cool}} \rangle \bar{T}. \]  
(110)

The above relations implicitly assume that \( \langle \beta \ast Q_{\text{cool}} \rangle \approx \langle \beta \circ Q_{\text{visc}} \rangle \approx \langle Q \rangle / T \). This heuristic argument suggests that the coherent input of negentropy at large scales by anomalous pressure work will be cascaded downscale, while the positive entropy input due to fine-grained (incoherent) transfer from mechanical to internal energy will go into entropy production due to thermal conduction. Needless to say, this is a non-rigorous mean-field argument ignoring temperature fluctuations and must be subject to empirical tests in order to determine its range of validity. It clearly requires a sufficiently large Mach number, since \( Q_{\text{cool}}, Q_{\text{incoh}} \) both vanish for incompressible flow. However, like our earlier arguments, it supports the conclusion that compressible turbulence will generally involve a forward cascade of negentropy, or, equivalently, an inverse cascade of the traditional entropy.
VII. RELATIONS TO OTHER APPROACHES

We now briefly discuss the relation of our analysis with other approaches to compressible fluid turbulence that have been proposed.

A. Barotropic Models

Barotropicity is a common assumption that is employed to simplify the description of compressible turbulence, with pressure \( p = p(\rho) \) taken to be a function of mass density only. Polytropic models with \( p(\rho) = K \rho^\gamma \) are a special case. This assumption has been employed in many classical theoretical papers \[110,111\] and also more recent theoretical works \[112,113\]. Barotropic models are often also employed for numerical simulations, especially in astrophysical applications \[113,110\]. No explicit equation for internal energy is solved in such models, and instead the internal energy per mass is obtained from

\[
e(\rho) = \int p(\rho) \frac{d\rho}{\rho^2}
\]

(111)

so that \( D_t \rho = -\rho (v \cdot v) \) implies \( \rho D_t e = -p (v \cdot v) \) and then \( u = pe \) is recovered satisfying

\[
\partial_t u + \nabla \cdot (u v) = -p (v \cdot v).
\]

(122)

The first law of thermodynamics in the form

\[
de = T ds_m + \frac{p}{\rho^2} d\rho,
\]

(13)

with \( s_m = s/\rho \) the entropy per mass, shows that \( u \) obtained as above can consistently represent internal energy only when either \( T = 0 \) (so that \( s = 0 \) by the Nernst law) or else entropy per particle \( s_n = s/n \) is constant. Barotropicity and \( \{111\} \) are thus equivalent to isentropy. It follows that the barotropic approximation rules out \textit{a priori} the entropy cascade central to the present theory.

The fundamental problem is that the assumption of isentropy is in conflict with viscous dissipative dynamics \[117\]. Whereas a smooth ideal Euler solution can be consistently taken to be isentropic, viscous barotropic models are physically inconsistent approximations to compressible Navier-Stokes, because they are in conflict with conservation of total energy for closed systems! The kinetic energy which is lost by viscous dissipation should reappear as internal energy of the fluid, but the equation \[112\] for the internal energy in barotropic models contains no viscous heating. Thus viscous barotropic models make the fallacious prediction that the total energy of the isolated fluid is non-conserved. While fundamentally defective as a model of compressible Navier-Stokes, the barotropic approximation is possibly adequate as a crude model for astrophysical fluids where kinetic energy lost by cascade is not acquired by the internal energy of the considered fluid, e.g. weakly collisional plasmas with negligible viscosities \[118\]. In addition to the above theoretical problems, the isentropic approximation is also observed empirically to be not well satisfied pointwise in subsonic and transonic turbulence for an ideal gas \[109,119\]. Substantial entropy fluctuations \( s_m = c_v \log(p/K \rho^\gamma) \) of the ideal gas are found, which are completely neglected by barotropic models and that are very physically significant (as discussed above and more below).

The polytropic model \( p = K \rho^\gamma \) with adiabatic index \( \gamma = 1 \) requires special discussion, because it is often interpreted by means of the ideal gas law \( p = n k_B T \) as an isothermal ideal gas, rather than as an isentropic fluid. With this interpretation the integral \( \int p(\rho) \frac{d\rho}{\rho^2} \) does not yield the internal energy per mass \( e \), but instead the chemical potential per mass \( \mu_m = \mu/m \). Indeed, the Gibbs-Duhem relation \( dp = s dT + \rho d\mu_m \) with \( dT = 0 \) and \( d(p/\rho) = 0 \) immediately yields \( d\mu_m = p d\rho/\rho^2 \). Consistently, substituting \( p = (k_B T/m) \rho \) gives

\[
\int p(\rho) \frac{d\rho}{\rho^2} = (k_B T/m) \log(\rho/\rho_0),
\]

(14)

which agrees with the chemical potential per mass of an ideal gas up to an additive constant depending only upon temperature. To obtain the internal energy consistently within this isothermal interpretation of the model one must rather than \( \{111\} \) instead use

\[
u = \alpha n k_B T = c_v \rho T, \quad \alpha = 1/(\gamma - 1)
\]

(15)

which implies that \( u \) is proportional to \( \rho \), just like \( p \), and satisfies the equation

\[
\partial_t u + \nabla \cdot (u v) = 0.
\]

(16)

Hence, a viscous isothermal gas model is also in conflict with conservation of total fluid energy, because kinetic energy lost by viscous dissipation is not transferred into internal energy. In addition, strict isothermality is not a dynamically consistent assumption for compressible Navier-Stokes. The temperature equation can be easily checked for a general equation of state to have the form

\[
\rho c_v D_t T = -T \left( \frac{\partial p}{\partial T} \right)_n \Theta + \nabla \cdot (k \nabla T) + 2\eta S^2 + \zeta \Theta^2,
\]

(17)

where \( c_v(n, T) \) and \( p(n, T) \) are taken to be functions of the two independent thermodynamic variables \( n, T \). For an ideal gas this simplifies further to

\[
\rho c_v D_t T = -p \Theta + \nabla \cdot (k \nabla T) + 2\eta S^2 + \zeta \Theta^2,
\]

(18)

with \( c_v = \alpha k_B/m \) constant. Clearly, pressure work and viscous heating will produce thermal inhomogeneities if none were present initially. At most it can be true in the presence of strong cooling, which adds a term \(-Q_{\text{cool}}\) to the right side of \( \{117\} \) and \( \{118\} \), that \( T'/T_0 \approx \rho'/\rho_0 \), i.e. that temperature fluctuations are much smaller than density fluctuations. However, as discussed in the previous section, strong cooling is also a strong source of negentropy and the standard isothermal models provide
no representation either of entropy production or of nonlinear dynamics of negentropy cascade to offset this. It thus highly questionable whether “isothermal” models produce a consistent and accurate representation of a strongly cooled compressible fluid with small temperature fluctuations.

B. Point-Splitting Regularization

As discussed in the Introduction, Onsager derived his result on dissipative anomalies for incompressible Euler equations in his original work using a point-splitting regularization [20, 31], very closely related to the methods used by Kolmogorov [17] to derive his statistical 4/5th-law for incompressible turbulence. Onsager’s argument was later made completely rigorous by Duchon-Robert [31] and a formal analogy to point-splitting derivations of anomalies in quantum gauge theories was pointed out by Polyakov [32, 33]. It is therefore natural to consider whether the point-splitting approach can be applied as well to compressible fluids. Two different groups of researchers have already attempted to obtain statistical relations analogous to the “4/5th-law” for compressible turbulence by distinct variations of point-splitting methods: Galtier & Banerjee [113, 114] and Falkovich-Fouxon-Oz [24]. We briefly discuss the works of both of these groups, in the light of our own results.

Galtier-Banerjee Relations: These authors have attempted to derive “4/5th-laws” for compressible turbulence within isothermal [113] and polytropic [114] fluid models, by a point-splitting of the total fluid energy (sum of kinetic and internal energy). It must be pointed out firstly that the quantity called “internal energy” for an isothermal fluid in [113], \( e = \frac{1}{2} \rho v^2 \), is in fact the chemical potential per mass. See previous section. However, much more seriously, we have demonstrated in section [IV] that there is no turbulent cascade/dissipative anomaly of total energy in a compressible Navier-Stokes fluid! As observed also in the previous section, the viscous isothermal/barotropic models studied in [113, 114] are inconsistent with conservation of fluid energy. For these mathematical models, [113, 114] derive a relation interpreted as describing a flux \( \varepsilon \) of total energy (kinetic + internal) through the inertial-range, which is finally dissipated by viscosity. However, in a Navier-Stokes fluid there is no viscous dissipation of total energy, which is a strictly conserved quantity, and the input \( \varepsilon \) of total energy from external sources (stirring, cooling, etc.) must identically vanish in a long-time steady state. The results of Galtier-Banerjee therefore have no validity for compressible turbulence in a Navier-Stokes fluid.

The possibility remains that the viscous isothermal/barotropic systems studied in [113, 114] may be acceptable as very crude models of weakly collisional astrophysical fluids, in which the viscosity is a proxy for some other physical mechanism which transforms the cascaded kinetic energy at small scales not into internal energy of the fluid in question but into some other form (e.g. internal energy of another fluid component, electromagnetic radiation, non-thermal particle acceleration, etc.). Note that the inertial-range analysis of the present paper and also of [23, 24] applies to the viscous barotropic model and also to the viscous “isothermal” model, if the later is interpreted as an isentropic ideal gas with adiabatic index \( \gamma = 1 \). The only change to our analysis of the inertial-range balance of \( u \) in section [V] is that terms \( Q_{	ext{visc}} \) must be set there to zero. To justify a fluid approximation for internal energy there must be some additional physical mechanism, not explicitly represented in eq. [112], which regularizes its solution. For example, this could be the tiny thermal conductivity of electrons due to rare collisions with ions [120, 121]. There may also be very weak collisional heating enhanced by kinetic mechanisms [113, 114]. Our analysis leads, however, to a very different picture than that of Galtier & Banerjee [113, 114] for their own models, where we predict no cascade of internal energy. Any “cascade of total energy” is only via kinetic energy cascade in our analysis.

It is also interesting to ask whether the results of the present paper on a kinetic energy anomaly for Euler equations might be alternatively derived by the Galtier-Banerjee point-splitting. The answer is no. The reason is that the point-splitting employed by Galtier-Banerjee is not a proper regularization of the kinetic energy equation and does not remove divergences in the infinite Reynolds-number limit. To see this, we note that the key identity in [113, 114] for the point-split kinetic energy evolved under isentropic Euler dynamics is

\[
\frac{\partial}{\partial t}(j\cdot v' + j'\cdot v) \equiv \nabla \cdot [(\delta j\cdot \delta v) + h_m(\nabla \cdot j) + h_m(\nabla \cdot j')] + (j\cdot v' - j \cdot v + p) \nabla \cdot v' + (j' \cdot v - j' \cdot v' + p') \nabla \cdot v.
\]

Here quantities marked with a prime “\( \prime \)” are evaluated at a space point \( x + r \), while unmarked quantities are evaluated at point \( x \), and \( \delta f \) is difference \( f(x + r) - f(x) \). The notation “\( \equiv \)” indicates equality up to overall space-gradient terms \( \nabla \cdot x(\ldots) \) which represent space-transport of kinetic energy. Finally, \( h_m = e + p/\rho \) is the enthalpy per mass, which satisfies \( dh_m = dp/\rho \) for isentropic flow. In the original work of Onsager [20, 31] and Duchon-Robert [41], a coarse-graining operation was applied to the separation-vector \( r \) to obtain a fully regulated expression. However, if the same approach is applied to the above identity, one gets terms that are ill-defined in the infinite-Reynolds number limit. For example, the last term on the right gives the contribution \( (j \cdot v - j \cdot v + p) \nabla \cdot v \) which involves a non-smooth function \( v \) multiplied with a distribution \( \nabla \cdot v \). Such terms are ill-defined at infinite Reynolds-number. Instead the coarse-graining approach of [23] and the present paper yields fully regularized expressions, as in our eq. [41]. Notice that the terms which cause trouble for point-splitting as a regularizer are absent in the incompressible case, because \( \nabla \cdot v = 0 \).

Just to be clear, we are not claiming that there is a mathematical mistake of a trivial sort in the analyses of
Galtier & Banerjee [113,114]. All of their calculations are meaningful and correct at finite Reynolds numbers. In fact, their mathematical relations have been checked to be true in numerical simulations of supersonic “isothermal” turbulence [116]. What we are claiming is that there are unphysical assumptions underlying the mathematical models employed by Galtier & Banerjee [113,114] and erroneous physical interpretations of the mathematical results. Their failure to regularize UV divergences associated to dissipative anomalies prevents them from drawing any conclusions on the infinite $Re$ limit.

Falkovich-Fouxon-Oz Relation: In the paper of Falkovich et al. [24] another generalization of the “4/5th-law” to compressible turbulence has been obtained for a barotropic fluid. This approach has also been applied to relativistic fluid turbulence by Fouxon & Oz [25] as we will discuss in a following paper [48]. Consideration of a point-split quantity $j \cdot j'$ allowed [24] to derive an exact relation for homogeneous, isotropic statistics, which reduces to the standard 4/5th-law in the incompressible limit. The quantity which is cascaded to small scales in their picture is the input of $(1/2)|j|^2$ by external forcing. The exact equation obeyed by this field for a smooth solution of compressible Euler equations (without need of any barotropic assumption) is

$$\partial_t \left( \frac{1}{2} |j|^2 \right) + \nabla \cdot \left( \frac{1}{2} |j|^2 \mathbf{v} + p \right) = -\frac{1}{2} |j|^2 (\nabla \cdot \mathbf{v}) + p (\nabla \cdot \mathbf{j}).$$

(120)

As a matter of fact, it is not hard to show that this balance equation may indeed be anomalous in a high-Reynolds-number compressible turbulence and to use a point-splitting regularization to derive the anomaly. With the same notations as in eq. (119), one easily finds

$$\partial_t (j \cdot j') \equiv \frac{1}{2} \nabla \cdot \left[ |\delta|^2 \delta \mathbf{v} \right] - \frac{1}{2} |j|^2 (\nabla \cdot \mathbf{v}') - \frac{1}{2} |j'|^2 (\nabla \cdot \mathbf{v}) + p (\nabla \cdot \mathbf{j}') + p' (\nabla \cdot \mathbf{j}).$$

(121)

Unlike the previous case, all terms are fully regularized after coarse-graining in the separation-vector $r$ and one obtains an anomaly term $-A$ appearing on the right side of eq. (120) for infinite Reynolds number, with

$$A = \mathcal{D} \lim_{\ell \to 0} \frac{1}{4\ell} \int d^d \ell (\nabla G)_\ell (r) \cdot \delta \mathbf{v}(r) |\delta j(r)|^2.$$

(122)

It is also straightforward to derive the anomalous balance equation for $(1/2)|j|^2$ by using the coarse-graining approach of the present paper, but we leave this as an exercise for the reader. Note that in this balance equation one faces the same issue of defining products like $\frac{1}{2} |j|^2 \circ \Theta$ and $\frac{1}{2} |j|^2 \ast \Theta$, similar to pressure-work in the energy balances in section IV of the present paper.

As with the previous point-splitting approach, we conclude that the result of Falkovich et al. [24] is mathematically correct and, even more, the derivation is valid in the infinite Reynolds-number limit. The statistical relation of [24] has also been verified in a numerical simulation of “isothermal” compressible turbulence [123] (although there are some subtle issues in the statistical evaluation of the external input). However, we disagree completely with the conclusion that the result of [24] “...indicates that the interpretation of the Kolmogorov relation for the incompressible turbulence in terms of the energy cascade may be misleading” (Fouxon & Oz, [25]). Such a conclusion could be justified if that relation were the only possible generalization of the 4/5th-law to compressible turbulence. However, the analysis of [23, 37] and ours in section IV A of the present paper fully support the existence of a kinetic energy cascade for compressible turbulence and yield the analogue of 4/5th-laws for the kinetic energy flux. A further issue with the result of Falkovich et al. [24] is that we see no compelling interest in the quantity $(1/2)|j|^2$ for compressible fluids. It is neither a conserved quantity nor any component of a conserved quantity, and it has no obvious dynamically important role in compressible turbulence. Just as in quantum field-theory, it is not hard to find infinitely many anomalous balance relations in the ideal limit of turbulence but most of them are not physically relevant and have no significant consequences. In our opinion, the deep importance of the 4/5th-law for incompressible turbulence arises from its connection to the dissipative anomaly for kinetic energy and its implication that fluid singularities of the type $C_q \leq q/3$ are required for such an anomaly. Our analysis shows that such a connection fully extends to compressible fluid turbulence.

As our final comments in this section, we would like to emphasize the general limitations of point-splitting in turbulence theory. It is not ruled out by the analysis in this paper that a clever point-splitting may someday be found for compressible turbulence which will yield the anomalous kinetic energy balance [52]. However, it is very hard to imagine that a point-splitting regularization will ever be found to yield the anomalous entropy balance [52] in this paper. The coarse-graining approach that we employ is a more powerful and general method than point-splitting. In addition to the Eulerian balances discussed here, coarse-graining can also be employed to obtain Lagrangian conservation-law anomalies, such as for fluid circulation in hydrodynamic turbulence [124] and magnetic flux conservation in MHD turbulence [76, 125].

C. Decomposition into Linear Wave Modes

Another common theoretical approach to compressible turbulence, which goes back to work of Kovácsnay [120] and Chu & Kovácsnay [127], is to expand compressible Navier-Stokes solutions into linear wave modes, based on an assumption of small perturbations around a homogeneous state and weak nonlinearity. This expansion identifies three ideal linear wave modes [128], the “sound mode” of frequency $c_s k$, for sound-speed $c_s$ and wave-number $k$, and two zero-frequency modes, the “vorticity
mode” and the “entropy mode.” An obvious question, which we address here, is how the “entropy mode” of Kovácsznay is related to our concept of an entropy cascade.

To briefly review the approach of Kovácsznay [126] and Chu & Kovácsznay [127], we recall that that it assumes an ideal gas equation of state, with pressure and entropy per particle given by

\[ p = nk_BT, \quad s_n = k_B \log(T^\alpha/Cn) \] (123)

as functions of \( n \) and \( T \), with \( \alpha = 1/(\gamma - 1) \). Linearization around a homogeneous state satisfying \( p_0 = n_0k_BT_0 \) yields for the fluctuations the linear relations

\[ \frac{p'}{p_0} = \frac{n'}{n_0} + \frac{T'}{T_0}, \quad \frac{s'_n}{k_B} = -\frac{n'}{n_0} + \alpha \frac{T'}{T_0}. \] (124)

Here we use the prime “\( \prime \)” to denote a putatively small fluctuation value. For ideal flow the “sound mode” has \( s'_n = 0 \) and the “entropy mode” has \( p' = 0 \). (For non-ideal flow Kovácsznay finds instead a small entropy \( s'_n \) associated to the “sound mode”, which is proportional to the molecular transport coefficients or dimensionless “Kundsen number” \( \epsilon \) and which is neglected at zeroth order in \( \epsilon \).) The zeroth-order dynamics of the fluctuations for ideal flow are found to be given by the linear equations

\[ \partial_t \omega' = 0, \quad \partial_t s'_n = 0, \quad \partial_s^2 p' - c_s^2 \nabla^2 p' = 0 \] (125)

with \( \omega' = \nabla \times \mathbf{v}' \) the vorticity fluctuation. See [127], eq. (6.5). Nonlinearity is recovered in the Kovácsznay approach by expansion to second-order in the nonlinearity, which yields mode-mode coupling terms, such as vortex self-stretching (a vorticity-vorticity mode coupling). See Table 1 of [127] for a complete tabulation of all second-order interactions. The only such couplings that contribute to entropy dynamics are entropy-vorticity and entropy-sound couplings of the form \(-\nabla \cdot \mathbf{v}' \nabla s'_n\), which describe advection of entropy by velocity fluctuations \( \mathbf{v}' \) due to vorticity and sound modes. Thus to quadratic order in nonlinearity, the entropy per particle \( s'_n \) appears as a passive scalar and entropy per volume \( s' = n s'_n \) as a passive density.

Independent of our work, there are a number of serious problems with the Kovácsznay modal expansion when considered as an \textit{a priori} theoretical approach. First and foremost, there is no small parameter on which to base such an expansion. Instead, fluctuations of thermodynamic variables in compressible flow can be very large relative to mean or r.m.s. values, as seen for example in [119], Fig. 4. This essential strong-coupling nature is, of course, the most well-known theoretical difficulty with the analysis of turbulent flow. A closely related problem is that solutions of the compressible Navier-Stokes equation cannot be consistently expanded into linear wave modes, because there is no superposition principle for such nonlinear dynamics. Even for an ideal gas, the thermodynamic relations [123] impose nonlinear constraints between \( p, n, T \) or \( s_n, n, T \), which will not be satisfied for superpositions of wave modes except in the very crude linear approximation \[124\]. For second-order moments of \( p'/p_0 \) and modest Mach numbers (0.1-0.6) the predictions of the linear approximation \[124\] are adequate to about 1% level (see data in Table 1 and Figs.2-3 in [119]), but the error grows with increasing Mach number and also for higher moments/larger fluctuations. It is worth noting that Kovácsznay himself was not attempting in his original works to develop a general theoretical approach for analysis of compressible fluid turbulence, but his goal was instead a more modest one of constructing a decomposition to assist in the interpretation of experimental measurements. Some later researchers have taken this type of modal decomposition much more literally than it was first intended.

Our analysis in this paper has shown that the entropy in high Reynolds-number compressible turbulence is not at all a passive scalar. Entropy is, of course, a nonlinear function of basic thermodynamic variables, e.g. \( s(u, n) \) taken as a function of internal energy density \( u \) and particle density \( n \). Its dynamics is completely determined by the dynamics of \( u \) and \( n \) and it is, in that sense, “passive”. However, the turbulent dynamics of entropy in the ideal limit of vanishing molecular transport is not that of a passive scalar. Comparing the weakly nonlinear expansion result

\[ \partial_t s' + \nabla \cdot (s' \mathbf{v}') = 0. \] (126)

with our own eq. (82), we see that, beyond passive advection, the inertial-range dynamics of entropy involves both anomalous input \( I_{\text{mech}} \) of negentropy from pressure-work and nonlinear entropy cascade \( \Sigma_{\text{flux}} \) (as well as entropy production by viscous heating). If one tried to interpret the mechanical input \( I_{\text{mech}} \) of entropy crudely within the Kovácsznay framework, it would have to be considered a turbulent “sound-sound” coupling which produces negentropy. It is completely missed by the Kovácsznay weakly nonlinear expansion which cannot detect such “anomalous” terms. Finally, the identification of \( s'_n \) as a passive scalar would imply that there is a forward cascade of \( |s'_n|^2 \), but our analysis instead predicts an inverse cascade of the entropy \( s \) as a nonlinear function of \( u \) and \( n \). Our predictions for entropy are thus fundamentally different from those obtained by treating the linear “entropy mode” as a passive scalar.

VIII. EMPIRICAL CONSEQUENCES AND EVIDENCE

Our analysis yields a great many predictions testable by laboratory experiments and numerical simulations, the two most novel being the pressure-dilatation defect \( \tau(p, \Theta) \) contribution to anomalous kinetic energy dissipation and the anomalous production of negentropy by pressure-work \( T_{\text{flux}} \) and nonlinear negentropy cascade \( \Sigma_{\text{flux}} \). These quantities are all straightforward to calculate in simulations of compressible turbulence, where
inertial-range contributions such as $\tilde{\tau}(p, \Theta)$ or negentropy flux $\Sigma_\ell^{\text{flux}}$ can be obtained by numerical implementation of the spatial filtering. Laboratory experiments can also measure such quantities using techniques such as holographic PIV \cite{129}. Our analysis yields as well testable predictions on scaling exponents through the inequalities \cite{92, 94}. Here we may note in particular the predictions for “roughness” of the internal energy density and mass density fields in order for an entropy cascade to exist, with structure-function exponents $C_\ell^q$ and $C_\ell^p$ required, essentially, to be less than or equal to K41 values for $q \geq 3$. Not only are these various predictions able to be checked in detail in future studies, but also many past works in retrospect provide supporting evidence. We next discuss some of this prior work.

First, the previous numerical studies of the pressure-work \cite{SS, SS9} provide evidence for a pressure-work defect $\tau(p, \Theta)$, although this was not clearly understood at the time. The main object of those studies was the saturation of $\langle \tilde{p}\Theta^\ell \rangle$ for $\ell$ decreasing through the inertial-range. This was demonstrated through study of the pressure-dilatation cospectrum

$$PD(k) = -\sum_{k': ||k'|-k|<0.5} \tilde{p}(k')\tilde{\Theta}(-k')$$ \hspace{1cm} (127)$$

and of the statistics of the pressure-dilatation residual $p\Theta - \tilde{p}\tilde{\Theta}$. In both studies \cite{SS, SS9} it was found that the cospectrum exhibited a power-law behavior $PD(k) \sim Ck^{-\beta}$ in the inertial range, crucially with $\beta > 1$ so that the integral over the range $k \in [0, \infty)$ would converge. However, in the finite Reynolds number simulations the power-law with exponent $\beta$ persists only over a finite range and in the dissipation-range the cospectrum was found to lie above the inertial-range power-law. See \cite{SS}, Fig. 2 and \cite{SS9}, Fig. 23. This is the signature to be expected from a positive mean defect $\langle \tau(p, \Theta) \rangle > 0$. Even more relevant are the previous numerical results for the pressure-dilatation residual, since it is directly related to the mean defect by

$$\langle \tilde{\tau}(p, \Theta) \rangle = \langle p\Theta - \tilde{p}\tilde{\Theta} \rangle.$$

The two previous studies both found that the residual for $\ell$ near the bottom of the inertial-range took on very large positive and negative values associated to shocks (small-scale shocklets or large-scale shocks, depending upon the compressibility of the forcing). See \cite{SS}, Fig. 4 and \cite{SS9}, Figs. 26,27. The large values nearly cancelled in a global space average, leaving only a small positive average $\langle \tau(p, \Theta) \rangle$, about 20 times smaller than the asymptotic value $\langle p \circ \Theta \rangle$. Aluie et al. \cite{SS9} considered this 5% contribution to be “negligible”. However, both the simulations \cite{SS, SS9} were for subsonic and transonic turbulence. If $\langle \tau(p, \Theta) \rangle$ arises mainly from shock heating, then it is reasonable to expect that this average will make an increasingly large contribution to the kinetic energy dissipation anomaly for increasing Mach numbers.

There is also evidence from prior studies for a negentropy cascade. Motivated by incompressible fluid turbulence where the temperature is a passive scalar, Ni et al. \cite{69} and Ni & Chen \cite{89} (see section 6 of both papers) have numerically studied “temperature cascade” in subsonic and transonic compressible turbulence of an ideal gas. Using the same coarse-graining approach as the present paper, those authors attempted to derive a balance equation for the quantity $G = (1/2)p\tilde{T}^2$. Their result \cite{89}, eqs.(6.3)-(6.8)) contains several errors \cite{130} but the expression for the subscale flux of $G$ that they obtained is the same as that for the correct equation derived from (118) and given here:

$$\partial_t \left( \frac{1}{2}p\tilde{T}^2 \right) + \nabla \cdot \left( \frac{1}{2}p\tilde{T}^2 \tilde{v} + \tilde{\rho}\tilde{T} - (\tilde{\kappa}\nabla T)\tilde{T}/c_P \right)$$

$$= -\Pi^\ell + (\bar{p}\Theta + C_{\text{visc}}) \tilde{T}/c_V - \tilde{\nabla} \cdot (\tilde{\kappa}\nabla T) \cdot \tilde{\nabla} \tilde{T}/c_V$$ \hspace{1cm} (129)$$

with

$$\Pi^\ell = -\bar{p} \tilde{\nabla} \cdot \tilde{T} \tilde{\theta}(T, \tilde{v})$$ \hspace{1cm} (130)$$

the subscale flux of $G$. Studies \cite{69, 89} have verified numerically that this quantity has a positive average $\langle \Pi^\ell \rangle > 0$ over a range of $\ell$, indicating a forward cascade of the quantity $G$ to small-scales. This is almost direct evidence for a forward negentropy cascade.

The quantity $G = (1/2)p\tilde{T}^2$ was, in fact, first introduced by Obukhov \cite{72} for incompressible fluid turbulence as an approximation to the “negentropy” or “information” introduced by an ordered temperature field, assuming an isobaric ideal gas and small amplitudes of temperature fluctuations \cite{151}. The concept of an “entropy cascade” was later invoked by L’vov \cite{132} for the cascade of $G$ within the Bolgiano-Obukhov picture of convective turbulence in a Boussinesq fluid. The ideas of Obukhov \cite{72} and L’vov \cite{132} are the closest analogue for an incompressible fluid of the entropy cascade proposed in this work. An important difference is that in the theories of \cite{72, 132} the flux $\Pi^\ell$ is in statistical balance with the “temperature dissipation” by thermal conductivity, or $\kappa|\nabla T|^2/c_P$, without any contribution from entropy production due to viscous heating and with no consideration of an explicit cooling mechanism. For compressible turbulence there is little reason to consider the approximation $G$ rather than the correct large-scale entropy $s(\bar{v}, \bar{p})$ and our balance equations \cite{72} for $s$ and \cite{96} for $s^\ell$ are more theoretically tractable than \cite{129} for $G$, because entropy is a conserved quantity for smooth solutions of compressible Euler equations whereas $G$ is not. However, the observation of \cite{69, 89} that $\langle \Pi^\ell \rangle > 0$ strongly suggests that $\langle \Sigma_\ell^{\text{flux}} \rangle > 0$ will hold over a similar range of $\ell$ and makes it vital to subject the latter prediction and the balance relations \cite{105-106} and \cite{109-110} to detailed empirical tests. Here we note that the spectra of density, temperature, and pressure (or, equivalently for an ideal gas, internal energy) in the simulations of \cite{119} and \cite{69} are consistent with the roughness expected for
negative entropy cascade. In particular, for transonic Mach numbers \((Ma \approx 0.6)\) all three thermodynamic variables have Fourier spectra close to \(k^{-5/3}\), scaling with the K41 exponent \([133]\).

The negative entropy cascade proposed here, if correct, must occur for compressible turbulent flows in Nature, with one of the most significant examples being turbulence in the interstellar medium (ISM). The electron density of the ISM exhibits a spectrum close to the Kolmogorov \(k^{-5/3}\) over a 13-decade range, as inferred from electron scintillation measurements over \(10^5-10^{10}\) km scales and from other observations over \(10^2-10^{15}\) km \([134,137]\). The spectacular extent of this scaling range has led the density spectrum to be dubbed the “Big Power Law in the Sky”. Because the ion mean free path in the ISM is \(\sim 10^7\) km, a fluid approximation is expected to be valid over the majority of this range. Magnetic fields also play a significant role in the dynamics of the ISM, so that the dynamics of the ISM at length scales above \(\sim 10^7\) km is expected to be that of a compressible magnetohydrodynamic (MHD) fluid with a Mach number of order unity. Our work suggests an identification of the “Big Power Law in the Sky” as resulting from a nonlinear inverse cascade of entropy (or forward negentropy cascade).

Note that all of our results in this paper extend straightforwardly to compressible MHD (for which see Landau & Lifschitz \[135\], Chapter VIII, \(\S 65-66\)). The only difference is that now there is a cascade of total mechanical energy (kinetic + magnetic) and the corresponding energy dissipation anomaly now contains a contribution from resistive heating

\[
\dot{Q} = \lim_{n,\zeta,\gamma \to 0} \frac{2\eta |\mathbf{S}|^2 + \zeta |\mathbf{E}|^2 + \gamma J^2}{4\pi}\tag{131}
\]

where \(\gamma = c^2/4\pi\sigma\) is the magnetic diffusivity and \(\mathbf{J} = \nabla \times \mathbf{B}\). In particular, our balance eq.\([72]\) for \(s\) remains valid for compressible MHD with the above change to \(Q_{\text{diss}}\), and the balance eq.\([96]\) for \(s^*\) now has contributions to \(Q_{\text{flux}}\) from the Lorentz force \([139]\). There is thus by our arguments a forward negentropy cascade in compressible MHD turbulence. We theorize that this nonlinear negentropy cascade is the origin of the plentiful density fluctuations in the large-scales of the ISM where compressible MHD is valid.

Most current theories of the electron density spectrum of the ISM, by contrast, have been developed within Kovácsznavy-type modal wave picture for compressible MHD, where the basic waves are now the “shear Alfén mode”, the “slow magnetosonic mode”, the “fast magnetosonic mode”, and the “entropy mode” (e.g. see \[140\], Ch.5). In particular, one popular theory of the power-law spectrum is that it results from a forward cascade of the “entropy mode” as a passive scalar \([141,142]\). However, the large scales of the ISM are believed to be nearly isothermal above a cooling scale \(L_{\text{cool}} \sim 10^{12}\) km, because of efficient radiative cooling (e.g. by electron impact excitation of metal line transitions). Referring to eq.\([124]\), one sees that there can then be no “entropy mode” with \(p'/p_0 = 0\) because \(T'/T_0 = 0\) and so cannot cancel the density fluctuation \(n'/n_0\) (or, more accurately, the entropy mode is extremely damped, because \(T'/T_0 \ll 1\)). In that case, the only remaining mode to carry density fluctuations is the isothermal sound mode (slow magnetosonic) with \(p'/p_0 = n'/n_0 = -s'_n/k_B\) (which, in contrast to the adiabatic sound mode, carries entropy fluctuations due to density changes) \([143]\). This linear analysis of the fluctuations leads to the so-called cooling catastrophe, which is concisely summarized in this quote:

“What, however, the entropy mode is rapidly damped in isothermal turbulence. As a consequence, small-scale density fluctuations may be significantly suppressed. There are two possible solutions to this “cooling catastrophe”: either (1) the outer scale is extremely small, small enough that the turbulence at the outer scale is nearly adiabatic; or (2) there are significant density fluctuations associated with the slow mode. However, in the latter case, the mean magnetic field must be amplified almost to equipartition with the gas pressure, so that \(\beta \sim 1\). Either of these two solutions would place stringent constraints on the nature of the turbulence that is responsible for observed density fluctuations.” — Lithwick & Goldreich \[142\]

Within a Kovácsznavy-type modal picture, the slow magnetosonic mode seems the most plausible source of the observed density fluctuations. However, in our nonlinear theory, there is no “cooling catastrophe” in the first place! Large-scale cooling adds excess negentropy (deficiency of entropy) that feeds the cascade of negentropy to small scales. See eq.\([106]\). This necessitates “rough” density and temperature fields with Kolmogorov-type spectra. In our view, the “cooling catastrophe” is an artifact of attempting to describe nonlinear compressible MHD turbulence in terms of linear wave modes. There is no sound theoretical basis for such a decomposition and, unsurprisingly, the Kovácsznavy mode-mode interactions lead to empirically wrong predictions for the problem \[144\].

A complete presentation of this theory of the ISM electron density spectrum will be given elsewhere, as it requires more specialized discussion of MHD turbulence and even plasma kinetics. A very interesting question is how our theorized negentropy cascade proceeds to smaller length-scales below the ion mean free path where a fluid approximation breaks down. As discussed earlier, the observed \(k^{-5/3}\) density spectrum in the ISM extends many decades below the ion mean-free path length. The key concept of plasma kinetic turbulence is the cascade of negative kinetic entropy or “free-energy” (electromagnetic energy minus kinetic entropy) to small scales of length and velocity in the 1-particle phase space \([118,122]\). The natural conjecture is that the negentropy cascade of compressible MHD turbulence merges with the
kinetic cascade at scales below the mean-free path, but details remain to be understood.

IX. DISCUSSION

The theory developed in this paper is based upon the hypothesis that compressible fluid turbulence should exhibit dissipative anomalies of energy and entropy, similar to those observed for incompressible fluids. From this hypothesis alone, we have argued that the high Reynolds- and Péclet-number limit should be governed by distributional or “coarse-grained” solutions of the compressible Euler equations. The argument closely follows that of Onsager [20, 31] for incompressible fluids, which we have explained as a non-perturbative application of the principle of renormalization-group invariance. The theory makes a great many predictions that are testable by experiment and simulations, in particular: (1) anomalous dissipation of kinetic energy by local energy cascade and by pressure-work defect; (2) anomalous input of negentropy into the inertial-range of compressible fluid turbulence by pressure-work, in addition to any external input by large-scale cooling mechanisms; (3) negentropy cascade to small-scales through a flux of intrinsic inertial-range entropy; and (4) fluid singularities required to sustain cascades of energy and entropy, so that at least one of (92)-(94), must hold.

It should be stressed that even for incompressible fluids, many difficult mathematical questions remain open concerning Onsager’s theory of “ideal turbulence” described by dissipative Euler solutions and its main support arises from successful agreement with a broad array of numerical simulations and laboratory experiments. The convex integration theory [42, 43] has revealed that the Cauchy problem for incompressible Euler equations has non-unique dissipative solutions with fixed initial data, suggesting that the infinite-Reynolds turbulent solutions are essentially unpredictable. So far, no dissipative Euler solutions of the type conjectured by Onsager have been mathematically derived from incompressible Navier-Stokes solutions by the physical limit of vanishing viscosity/infinite Reynolds-number. Work on toy “shell models” suggests that this limit will be very subtle and that the limiting Euler solutions will be non-unique and stochastic [59-61]. Further surprises and new insights are doubtless in store. However, Onsager’s theory for incompressible fluid turbulence has much more empirical support than many other highly-regarded physical theories, e.g. Einstein’s theory of general relativity.

For the compressible theory that we have developed here, further work is also clearly required on a few key issues. One of these is the Mach-number dependence of the various physical quantities in our theory. All of our derivations are formally independent of Mach number, but there is an implicit Mach-number dependence through the assumption that mass-density remains a bounded function in the ideal limit. Instead, there is empirical evidence from numerical simulations that for a sufficiently high Mach number the density is not even square-integrable and its ideal limit may exist only as a singular measure [145]. At small Mach numbers the anomalous negentropy input by pressure-work must tend to zero. If there is an external heating/cooling source to introduce internal energy (or temperature) inhomogeneities at large-scales, then the low Mach-number limit of our negentropy cascade must recover that long ago predicted by Obukhov [72] for incompressible fluids. However, if there is no such external source, then our predicted negentropy cascade presumably disappears for small Mach numbers, but the details are unclear. This is an urgent matter for evaluating the theory, since much empirical data exists for subsonic and transonic flows.

A second very important open issue has to do with the extension of our theory to kinetic regimes. Our theorem on turbulent entropy dissipation anomalies and entropy cascade applies to any distributional solution of compressible Euler equations, including those resulting from a kinetic equation. However, it is very unclear how our fluid negentropy cascade will merge into a kinetic description at scales much smaller than the mean-free-path of the fluid. This is a particularly important issue for plasma kinetics in astrophysics [118, 122], because the large mean-free-paths frequently encountered in astrophysical plasmas imply that long ranges of scales are described by Vlasov-Landau kinetic theory rather than a fluid description.

One strength of our theory is that it extends readily to relativistic fluid turbulence. This is the subject of our following paper [48].

ACKNOWLEDGMENTS

We thank Ethan Vishniac for very helpful discussions of the physics of the interstellar medium and Hussein Aluie for sharing with us his unpublished work.

Appendix A: Analytical Shock Solution

1. Model and Shock Solution

We shall consider a family of shock solutions derived by Becker [95] and Johnson [96, 146] for the 1D compressible Navier-Stokes system, obtained by reduction of the 3D equations to a single space dimension, with \( x \) the distance perpendicular to the planar shock and with \( v = v_x \) the corresponding velocity component:

\[
\partial_t \rho + \partial_x (\rho v) = 0, \tag{A1}
\]

\[
\partial_t (\rho v) + \partial_x (\rho v^2 + p - \eta \partial_x v) = 0, \tag{A2}
\]

\[
\partial_t (\frac{1}{2} \rho v^2 + u) + \partial_x (\rho v (\frac{1}{2} v^2 + h_m) - \eta \partial_x v - \kappa \partial_x T) = 0. \tag{A3}
\]

\( h_m \) denotes the energy per unit mass, and \( \kappa \) denotes the heat conductivity. Suppose \( \rho = \rho_0 \) and \( u = u_0 \) are constant across the shock. The shock speed is then given by

\[
v = \frac{u - u_0}{\eta}.
\]
Here $\eta = (4/3)\eta_{3D} + \zeta_{3D}$ for the 3D shear viscosity $\eta_{3D}$ and bulk viscosity $\zeta_{3D}$ [147]. An ideal-gas equation of state is assumed, with

$$p = (\gamma - 1)u, \quad u = c_v\rho T, \quad h_m = c_P T \quad (A4)$$

for any adiabatic index $\gamma = c_p/c_v > 1$. The solutions obtained are for the stationary equations, with all time-derivatives set to zero, and they reduce in the ideal limit ($\eta, \kappa \to 0$) to stationary shocks with discontinuous, step-function solution fields:

$$f(x) = \begin{cases} f_0 & x < 0 \\ f_1 & x > 0 \end{cases} = f_0 + (\Delta f)\theta(x) \quad (A5)$$

Here pre-shock values are labeled by 0 and post-shock values by 1, $\Delta f = f_1 - f_0$, and $\theta(x)$ is the Heaviside step function. We also denote $\bar{f} = \frac{1}{2}(f_0 + f_1)$. The values of the fields on the two sides of the shock are related by the Rankine-Hugoniot conditions:

$$\Delta(p \bar{v}) = 0, \quad \Delta\left(\frac{1}{2}\bar{v}^2 + h_m\right) = 0, \quad \Delta(p + \rho \bar{v}^2) = 0, \quad (A6)$$

with a mass flux $j_* = \rho_0 v_0 = \rho_1 v_1 > 0$. See [71], §84. The strength of the shock is characterized by the compression ratio $R = \rho_1/\rho_0 = v_0/v_1 > 1$ which, for an ideal gas, is given by

$$R = \frac{\gamma + 1}{(\gamma - 1) + 2/M_0^2} \quad (A7)$$

in terms of the pre-shock Mach number $M_0 = v_0/c_s > 1$. E.g., see Landau & Lifschitz (1987), §89. Note that, because of the ideal gas relation $p/\rho = (\gamma - 1)h_m/\gamma$, the second two Rankine-Hugoniot conditions determine the pre- and post-shock pressures by the formulas

$$p_i = \frac{j_*}{2\gamma} [(1 + \gamma)v_{1-i} + (1 - \gamma)v_i], \quad i = 0, 1 \quad (A8)$$

As was first noted by Becker [95], the stationary 1D Navier-Stokes equations of an ideal gas admit an exact integral for $\eta = \kappa/c_P$ or, assuming $\zeta_{3D} = 0$, for the 3D Prandtl number $Pr = c_P \kappa_{3D}/\kappa = 3/4$. This integral takes the form of a (non-ideal) Bernoulli equation which relates velocity and enthalpy per mass:

$$\frac{1}{2} \bar{v}^2 + h_m = \frac{1}{2} \bar{v}_0^2 + h_{m0} = \frac{\gamma + 1}{2(\gamma - 1)} v_0 v_1, \quad Pr = \frac{3}{4}. \quad (A9)$$

By means of this relation and the formula

$$\rho = j_*/v \quad (A10)$$

for the mass density, all thermodynamic variables can be related to the velocity. For example, using (A9) and the ideal gas relation $p/\rho = (\gamma - 1)h_m/\gamma$ gives for the pressure field

$$p = \frac{j_*}{2\gamma} [(\gamma + 1) \frac{v_0 v_1}{v} + (1 - \gamma)v], \quad Pr = \frac{3}{4}. \quad (A11)$$

Using $h_m = c_p T$ gives for the temperature field

$$T = \frac{1}{2c_P} \left[ \frac{\gamma + 1}{\gamma - 1} v_0 v_1 - v^2 \right], \quad Pr = 3/4, \quad (A12)$$

and likewise for other thermodynamic quantities. For a very clear discussion, see [93].

To obtain the velocity itself in the approach of [93] requires the evaluation of an integral involving the specific choice of dynamic viscosity $\eta(\rho, T)$ as a function of $\rho$ and $T$. This generally yields the velocity field in the implicit form $x(v)$. As pointed out by Johnson (2014), some choices of $\eta(\rho, T)$ permit one to invert the relation $x(v)$ to an explicit form $v(x)$. It turns out, however, that to evaluate the infinite Reynolds-number/Péclet-number limits, we need only the Bernoulli relation (A9) of [93] and its alternative forms (A11), (A12). We furthermore need one additional constraint which follows from the constancy of momentum flux:

$$j_* v + p - \eta \partial_x v \equiv \tau_*. \quad (A13)$$

The constant value $\tau_*$ can be evaluated far from the shock where the gradient vanishes, giving

$$\tau_* = j_* v_i + p_i, \quad i = 0, 1 \quad (A14)$$

or, using (A8) from the Rankine-Hugoniot conditions,

$$\tau_* = j_* v_{av} + p_{av} = \frac{1 + \gamma}{\gamma} j_* v_{av}. \quad (A15)$$

The equations (A13), (A15) and the Bernoulli relation (A11) for $p$ allow us to determine $\eta \partial_x v$ in terms of $v$ itself, yielding identical results for any choice of viscosity $\eta(\rho, T)$. As a consequence, all of our ideal limit results are independent of the details of the molecular transport coefficients, apart from the requirement that $Pr = 3/4$. Many inertial-range limit results hold with complete generality for all dissipative planar shocks in an ideal gas, and do not even depend upon Prandtl number $Pr$. Some inertial-range quantities do depend upon $Pr$, which we can explicitly verify for the cases $Pr = \infty (\kappa = 0)$ and $Pr = 0 (\eta = 0)$. As was noted by [136], there are Bernoulli-type relations also for those cases, which yield expressions for the pressure of the form

$$p = j_* \left[ -\frac{1 - \gamma}{2} v + \frac{1 - \gamma^2}{\gamma} v_{av} + \frac{1 + \gamma}{2} \frac{v_0 v_1}{v} \right], \quad Pr = \infty, \quad (A16)$$

and

$$p = j_* \left[ \frac{1 + \gamma}{\gamma} v_{av} - v \right], \quad Pr = 0. \quad (A17)$$

Employing these expressions for $p$ and eqs. (A13), (A15) we can also obtain formulas for $\eta \partial_x v$ with $Pr = 0, \infty$ which allow us to extend all of our results for $Pr = 3/4$ to those cases. Because the mathematical methods are essentially the same for all three cases, we shall below...
discuss explicitly only $Pr = 3/4$ and then just briefly mention some corresponding results for $Pr = 0, \infty$.

These solutions of Becker \cite{beck1} and Johnson \cite{john1, john2} are a nice example for our general mathematical framework, since they converge in $L^p$ norms for any $p \in [1, \infty)$ to a weak shock solution of 1D compressible Euler as $\nu, \kappa \to 0$.

We derive here all of the source terms which appear in the kinetic energy and the entropy balance for the shock solutions in the distributional limit as $\eta, \kappa \to 0$ for the fine-grained balances and as $\ell \to 0$ for the coarse-grained balances. A fact that we shall use frequently for ideal step-function fields below is

$$f(x) = f_0 + (\Delta f)\delta(x), \quad g(x) = g_0 + (\Delta g)\delta(x) \quad (A18)$$

and thus

$$g = g_0 + \frac{\Delta g}{\Delta f}(f - f_0), \quad \partial_x g = \frac{\Delta g}{\Delta f} \partial_x f. \quad (A19)$$

Furthermore,

$$\partial_x \tilde{f}(x) = (\Delta f)\delta(x). \quad (A20)$$

Similar results can be obtained from

$$f(x) = f_{av} + \frac{1}{2}(\Delta f)\text{sign}(x), \quad g(x) = g_{av} + \frac{1}{2}(\Delta g)\text{sign}(x) \quad (A21)$$

These relations are very helpful to derive inertial-range expressions for the shock solution.

2. Kinetic Energy Balance

a. Viscous Dissipation

Using in (A13) expression (A11) for $p$ with $Pr = 3/4$

$$\eta(\partial_x v) = \frac{\gamma + 1}{2\gamma} j_\ast \left[ v - 2v_{av} + \frac{\nu v_1}{v} \right]. \quad (A22)$$

Hence,

$$\eta(\partial_x v)^2 = \frac{\gamma + 1}{2\gamma} j_\ast \left[ \partial_x \left( \frac{1}{2} v^2 \right) - 2v_{av} \partial_x v + v_0 v_1 \partial_x (\ln v) \right]. \quad (A23)$$

Since

$$D\text{-lim}_{\eta, \kappa \to 0} \left[ \partial_x \left( \frac{1}{2} v^2 \right), \partial_x v, \partial_x (\ln v) \right] = \left[ \frac{1}{2}(v_1^2 - v_0^2), \Delta v, \ln(v_1/v_0) \right] \delta(x) \quad (A24)$$

one gets easily that

$$Q_{\text{visc}} \equiv D\text{-lim}_{\eta, \kappa \to 0} \eta(\partial_x v)^2 = \frac{\gamma + 1}{2\gamma} j_\ast \left[ v_1 v_0 \ln \left( \frac{v_1}{v_0} \right) - \frac{1}{2}(v_1^2 - v_0^2) \right] \delta(x) \quad (A25)$$

Note that $Q_{\text{visc}} \geq 0$ because

$$f(\theta) = \theta \ln \theta - \frac{1}{2} \theta^2 + \frac{1}{2} > 0, \quad 0 \leq \theta < 1 \quad (A26)$$

and $f(1) = 0$. This result is Prandtl-number dependent. In fact, $Q_{\text{visc}}$ for $Pr = \infty$ is larger by factor of $\gamma$ and for $Pr = 0$, obviously, $Q_{\text{visc}} = 0$.

b. Pressure-Dilatation Defect

From (A11) for $p$ with $Pr = 3/4$ we have

$$p(\partial_x v) = j_\ast \frac{\nu}{2\gamma} \left[ (\gamma + 1)v_1 (v_1 - 1) - \frac{1}{2} (\gamma - 1)(v_1^2 - v_0^2) \right] \delta(x) \quad (A27)$$

Thus,

$$p \circ \Theta \equiv D\text{-lim}_{\nu, \kappa \to 0} \ p(\partial_x v) = \frac{1}{2\gamma} j_\ast \left[ (\gamma + 1)v_1 v_0 \ln \left( \frac{v_1}{v_0} \right) - \frac{1}{2} (\gamma - 1)(v_1^2 - v_0^2) \right] \delta(x) \quad (A28)$$

This result is also Prandtl-number dependent (see below) and the above expression holds only for $Pr = 3/4$.

Next we calculate $p \circ \Theta$. Since $v$ and $p$ in the ideal limit are both step functions, $\partial_x \bar{v} = (\Delta \nu/\Delta p) \partial_x \bar{p}$ so that

$$\bar{p} \partial_x \bar{v} = \frac{\Delta v}{\Delta p} \partial_x \left( \frac{1}{2\gamma p^2} \right) \quad (A29)$$

and thus

$$p \circ \Theta \equiv D\text{-lim}_{\nu, \kappa \to 0} \ p(\partial_x \bar{v}) = (\Delta v)p_{av}(\delta(x)) = \frac{1}{2\gamma} j_\ast \left( v_1^2 - v_0^2 \right) \delta(x) \quad (A30)$$

using $p_{av} = j_\ast v_{av}/\gamma$ from eq. (A28). Note that this result is independent of the particular choice of filter kernel $G$, as required. It is also completely independent of the molecular dissipation, as it is determined solely from the limiting Euler solution fields. One finds by subtracting that

$$\tau(p, \Theta) \equiv p \circ \Theta - p \circ \Theta = \frac{1}{2\gamma} j_\ast \left[ v_1 v_0 \ln \left( \frac{v_1}{v_0} \right) - \frac{1}{2} (v_1^2 - v_0^2) \right] \delta(x) \quad (A31)$$

Clearly, $Q_{\text{visc}} = \tau(p, \Theta)$ for $Pr = 3/4$. This same identity in fact holds for all values of Prandtl number, allowing us to infer the $Pr$-dependence of $p \circ \Theta$ from that of $Q_{\text{visc}}$.

The underlying reason for this identity, which is valid for all planar shocks in an ideal gas, is explained in the next subsection on kinetic energy flux.
c. Kinetic Energy Flux

**Baropycnal work:** Using \( \bar{\tau}(\rho, v) = \bar{\rho}v - \bar{\rho}\bar{v} = j_* - \bar{\rho}\bar{v}, \)

\[
\frac{1}{\bar{\rho}} \bar{\tau}(\rho, v) = \frac{j_*}{\bar{\rho}} - \bar{v}.
\] (A32)

Using \( \partial_x \bar{\rho} = (\Delta p/\Delta \rho)\partial_x \bar{\rho} = (\Delta p/\Delta v)\partial_x \bar{v}, \)

\[
1 \bar{\tau}(\rho, v)\partial_x \bar{\rho} = j_* \frac{\Delta p}{\Delta \rho} \partial_x (\ln \bar{\rho}) - \frac{\Delta p}{\Delta v} \partial_x \left( \frac{1}{2} \bar{v}^2 \right).\] (A33)

Thus,

\[
Q_{\text{baro}} = D\lim_{\ell \to 0} \frac{\partial_x \bar{\rho}}{\bar{\rho}} \bar{\tau}(\rho, v)
\]

\[
= \Delta p \left[ \frac{j_*}{\Delta \rho} \ln \left( \frac{\rho}{\rho_0} \right) - v_{av} \right] \delta(x)
\]

\[
= -j_* \left[ v_{1v_0} \ln \left( \frac{v_1}{v_0} \right) - \frac{1}{2} (v_1^2 - v_0^2) \right] \delta(x)(A34)
\]

where the final line was obtained using \( \Delta p = -j_* \Delta v, \) which follows either from \( (A8) \) or directly from the Rankine-Hugoniot conditions \( (A9). \) Note that \( Q_{\text{baro}} \leq 0. \)

We see again that the limiting inertial-range result \( Q_{\text{baro}} \) is independent of the filter kernel \( G. \) This is true for all of the limits as \( \ell \to 0 \) of inertial-range expressions for the shock solutions that we obtain in this Appendix. Thus, we shall make no further note of this fact for the other limits derived below. Note that \( Q_{\text{baro}} \) is also completely independent of the molecular transport coefficients, as are all other quantities that are determined solely by the limiting Euler solution fields.

**Deformation Work:** Using \( \bar{\nu} = \bar{\rho}v/\bar{\rho} = j_*/\bar{\rho} \)

\[
\bar{\rho} \partial_x \bar{\nu} = -j_* \partial_x \bar{\rho}.
\] (A35)

Likewise from its definition and \( \rho v = j_* \) one gets

\[
\bar{\nu}(v, v) = \frac{j_*}{\bar{\rho}} - \frac{j_*^2}{\bar{\rho}^2}
\]

\[
= j_* \left[ \frac{\Delta v}{\Delta \rho} + \left( v_{av} - \frac{\Delta v}{\Delta \rho v_{av}} \right) \frac{1}{\bar{\rho}} - j_* \frac{1}{\bar{\rho}^2} \right]
\] (A36)

after substituting \( \bar{\nu} = v_{av} + \Delta v/\rho - \rho_{av} \) from \( (A21). \)

Thus,

\[
Q_{\text{defar}} = -D\lim_{\ell \to 0} \bar{\rho} \partial_x \bar{\nu}(v, v)
\]

\[
= j_* \left[ \frac{\Delta v}{\Delta \rho} (\ln \rho) - \left( v_{av} - \frac{\Delta v}{\Delta \rho v_{av}} \right) \frac{1}{\bar{\rho}} + j_* \frac{1}{\rho^{2}} \right] \delta(x)
\]

\[
= j_* \left[ v_{1v_0} \ln \left( \frac{v_1}{v_0} \right) - \frac{1}{2} (v_1^2 - v_0^2) \right] \delta(x) \geq 0
\] (A37)

upon simplification. Thus, \( Q_{\text{fux}} = Q_{\text{baro}} + Q_{\text{defar}} = 0. \)

Since \( Q_{\text{visc}} = \tau(p, \Theta) + Q_{\text{flux}} \) in general, this explains why \( Q_{\text{visc}} = \tau(p, \Theta) \) holds independent of the molecular transport coefficients for any planar, ideal-gas shock.

Note that the last identity can also be restated as \( -p \ast \Theta + Q_{\text{visc}} = -p \ast \Theta, \) which corroborates for these solutions the general argument in the text that the sum of \( -p \ast \Theta \) and \( Q_{\text{visc}} \) should be completely independent of the molecular dissipation, even though the two terms separately are Prandtl-number dependent.

In physical terms, there is a loss of kinetic energy \( -p \ast \Theta \) at the shock, and an equal gain \( -p \ast \Theta + Q_{\text{visc}} \) of internal energy. There is no external forcing to balance the kinetic energy loss and no cooling to balance the internal energy gain. While these shock solutions are stationary, they are not however homogeneous or isotropic.

Thus, the loss/gain is balanced by space-transport of kinetic/internal energy into/away from the shock. For example, the space flux of kinetic energy is

\[
J_{\text{kin}} = \left( \frac{1}{2} \rho v^2 + p \right) = j_* \left( \frac{1}{2} v^2 + \frac{p}{\rho} \right).
\] (A38)

One readily finds from eq. \( (A8) \) for \( p_i, i = 0, 1 \) that

\[
\Delta J_{\text{kin}} = j_* - \gamma^2 (v_1^2 - v_0^2) < 0
\] (A39)

so that more kinetic energy enters the shock than leaves it, and the difference is exactly the correct amount to offset the loss due to pressure-work. Similarly, more internal energy is transported away from the shock than enters it, balancing the gain from pressure-work and heating. This follows directly from the conservation of total energy, or else by using \( J_{\text{int}} = u\bar{v} \) for space-flux of internal energy and evaluating \( \Delta J_{\text{int}} = -\Delta J_{\text{kin}}. \)

3. Entropy Balance

a. Dissipation Range

**Viscous Heating** \( \beta \ast Q_{\text{visc}}: \) Using \( (A22) \) for \( \eta \partial_x v \) write

\[
\eta (\partial_x v)^2 = \frac{\gamma + 1}{2\gamma} j_* \left[ v - 2v_{av} + \frac{v_0 v_1}{v} \right] \partial_x v
\]

\[
= \frac{\gamma + 1}{2\gamma} j_* (1 + \frac{v_0 v_1}{v^2}) v_{av} \partial_x v - \frac{\gamma + 1}{\gamma} j_* v_{av} \partial_x \frac{v_{av}}{T}
\]

\[
= \frac{\gamma + 1}{\gamma} j_* \frac{v_0 v_1}{T} \partial_x \frac{v_{av}}{T}
\] (A40)

In the first term replace \( v^2 \) with \( T \) using both \( (A12) \) and its derivative \( v_{av} v = -c_p \partial_x T, \) while in the second term replace \( T \) with \( v^2 \) using \( (A12). \) Elementary anti-derivatives give for the first term

\[
-j_* c_v \partial_x (\ln T) + \frac{1}{2} j_* (\gamma - 1) c_v \partial_x \ln |a^2 - 2c_p T|
\]

with \( a^2 = \frac{\gamma + 1}{\gamma} v_0 v_1 \) and for the second term

\[
-2j_* c_v (\gamma + 1) \frac{v_{av}}{a} \partial_x \text{arctanh}(v/a)
\] (A42)

Noting that \( a^2 - 2c_p T = v^2 = j_*^2/\rho^2 \) gives finally that

\[
\beta \ast Q_{\text{visc}} = D\lim_{\ell \to 0} \frac{\eta (\partial_x v)^2}{T}
\]
Hence, taking the limit
\[ \eta \partial_v \] is an Euler shock. The result (A49) could have been an-
\[ \beta \]
\[ \Pr \]
This expression holds only for \( Pr = 3/4 \) and the quantity is generally Prandtl-number dependent. For example, \( \beta \ast Q_{\text{visc}} = 0 \) for \( Pr = 0 \).

**Thermal Conduction** \( \Sigma_{\text{therm}} \): Note since \( Pr = 3/4 \) and \( v \partial_x v = -c_v \partial_x T \) that
\[ \kappa \partial_x T = -\eta v \partial_x v, \] (A44)
and thus also using (A22) for \( \eta \partial_x v \) that
\[ \kappa \partial_x T = -j_s (\gamma + 1) \left( \frac{v_0 v_1}{\gamma - 1} - c_v T \right) + j_s (\gamma + 1) v_{\text{av}} v. \] (A45)

Hence,
\[ \frac{\kappa (\partial_x T)^2}{T^2} = -j_s (\gamma + 1) \left( \frac{v_0 v_1}{\gamma - 1} - c_v T \right) \frac{\partial_x T}{T^2} + 4 j_s c_v (\gamma + 1) v_{\text{av}} \frac{v^2 \partial_x v}{(a^2 - v^2)^2} \] (A46)

Elementary anti-derivatives and some lengthy algebraic simplifications give
\[ \frac{\kappa (\partial_x T)^2}{T^2} = j_s \gamma + 1 \left( \frac{v_0 v_1}{\gamma - 1} - c_v T \right) \frac{\partial_x T}{T^2} + j_s c_v (\gamma + 1) \partial_x (\ln T) \]
\[ - j_s \gamma + 1 \left( \frac{v_0 v_1}{\gamma - 1} - c_v T \right) \frac{v}{T} + 2 j_s c_v (\gamma + 1) \frac{v_{\text{av}}}{a} \partial_x \arctanh(v/a) \] (A47)

Taking the limit \( \eta, \kappa \rightarrow 0 \), one finds using the Bernoulli relation that the contributions of the first and third terms in the above expression cancel, giving the final result
\[ \Sigma_{\text{therm}} = D_{\ast} \lim_{\eta \rightarrow 0} \frac{\kappa (\partial_x T)^2}{T^2} = j_s c_v (\gamma + 1) \left[ \ln \left( \frac{T_1}{T_0} \right) + 2 \frac{v_{\text{av}}}{a} \Delta \arctanh(v/a) \right] \delta(x). \] (A48)

Once again, this quantity is Prandtl-number dependent and the above expression holds only for \( Pr = 3/4 \). Obviously \( \Sigma_{\text{therm}} = 0 \) for \( Pr = \infty \).

**Total Entropy Production** \( \Sigma_{\text{diss}} \): The inverse hyperbolic tangent terms cancel on addition, giving
\[ \Sigma_{\text{diss}} = \Sigma_{\text{therm}} + \beta \ast Q_{\text{visc}} \]
\[ = j_s c_v \left[ \ln \left( \frac{T_1}{T_0} \right) - (\gamma - 1) \ln \left( \frac{\rho_1}{\rho_0} \right) \right] \delta(x) \]
\[ = j_s \Delta s_m \delta(x), \] (A49)
using \( s_m = c_v \ln(T/C \rho^{\gamma - 1}) \). We see that \( \Sigma_{\text{diss}} > 0 \), since \( \Delta s_m > 0 \) is the standard entropy condition for an Euler shock. The result (A49) could have been anticipated on the basis of simple entropy balance, since \( J_{\text{ent}} = sv = s_m j_s \) is the space-flux of entropy and \( \Delta J_{\text{ent}} = (\Delta s_m) j_s \) is the net entropy transported away from the shock. Thus, the entropy production at the shock is balanced by transport of entropy to infinity. The result (A49) for \( \Sigma_{\text{diss}} \) is, for this reason, completely independent of the molecular dissipation. Note that \( \Sigma_{\text{therm}} = \Sigma_{\text{diss}} \) for \( Pr = 0 \) and that \( \beta \ast Q_{\text{visc}} = \Sigma_{\text{diss}} \) for \( Pr = \infty \). These results for \( Pr = 0, \infty \) can be obtained as well using the Bernoulli-type relations in [20] [140] and calculating in the same manner as for \( Pr = 3/4 \) above.

**b. Inertial-Range**

**Inertial-Range Viscous Heating** \( \beta \ast Q_{\text{visc}} \): From (A25),
\[ Q_{\text{visc}} = q_s (x), \]
so that
\[ Q_{\text{visc}} = q_s \delta(x). \] (A50)

On the other hand, for an ideal gas by definition of \( \beta \)
\[ \beta = c_v \frac{\rho}{\bar{u}}, \] (A51)
Because \( u, \rho \) are step-functions in the ideal limit,
\[ \bar{\rho} = \left( \rho_0 - \frac{\Delta \rho}{\Delta u} u_0 \right) + \frac{\Delta \rho}{\Delta u} \bar{u}, \quad \bar{\delta} = \frac{\partial_x \bar{u}}{\Delta u}. \] (A52)
Thus,
\[ \beta \bar{Q}_{\text{visc}} = c_v q_s \left[ \left( \rho_0 - \frac{\Delta \rho}{\Delta u} u_0 \right) \frac{\partial \bar{u}}{\Delta u} + \frac{\Delta \rho}{\Delta u} \delta(x) \right] \]
\[ = c_v q_s \left[ \left( \rho_0 \Delta u - u_0 \Delta \rho \right) \frac{\partial_x (\ln \bar{u})}{(\Delta u)^2} + \frac{\Delta \rho}{\Delta u} \delta(x) \right] \] (A53)
Hence
\[ \beta \ast Q_{\text{visc}} = D_{\ast} \lim_{t \rightarrow 0} \beta \bar{Q}_{\text{visc}} = \beta q_s \delta(x) \] (A54)
with
\[ \beta_s \equiv c_v \left[ \frac{\rho_0 u_1 - u_0 \rho_1}{(\Delta u)^2} \ln \left( \frac{u_1}{u_0} \right) + \frac{\Delta \rho}{\Delta u} \right] \] (A55)
This result is obviously independent of the filter-kernel \( G \) (as are all such limits of coarse-grained quantities), but the quantity \( q_s \) gives a Prandtl-number dependence.

**Pressure-Dilatation Defect** \( \beta \ast \tau(p, \Theta) \): Because of our earlier result \( p \ast \Theta = q_{PV} \delta(x) \), the same argument as above shows that
\[ D_{\ast} \lim_{t \rightarrow 0} \beta \bar{p} \ast \bar{u} = \beta q_{PV} \delta(x). \] (A56)
Next note using \( \bar{p} = (\gamma - 1) \bar{u} \) and (A51) for \( \beta \) that
\[ \beta \bar{p} \partial_x \bar{u} = c_v (\gamma - 1) \rho \partial_x \bar{v} = c_v (\gamma - 1) \frac{\Delta u}{\Delta \rho} \partial_x \left( \frac{1}{2} \bar{u}^2 \right), \] (A57)
where the last equality follows from $\partial_x \bar{v} = (\Delta v/\Delta p)\partial_x \bar{p}$. Thus,
\[
\mathcal{D} \lim_{\ell \to 0} \beta \, \bar{p} \, \partial_x \bar{v} = c_V (\gamma - 1) (\Delta v) \rho_0 \delta(x)
\]
\[
= c_V j_s (\gamma - 1) \left( \frac{\Delta v}{\nu_0} \right) \delta(x)
\]
(A58)
after using $\rho = j_s/\nu$. Finally,
\[
\beta \circ \tau(p, \Theta) = \left[ \beta_s q_{PV} - c_V j_s (\gamma - 1) \left( \frac{\Delta v}{\nu_0} \right) \right] \delta(x)
\]
(A59)
This quantity is of course Prandtl-number dependent through the coefficient $q_{PV}$.

**Combined Contribution $\beta \circ Q - \beta \circ \tau(p, \Theta)$**: Using the expression (A25) for $q_s$ and (A28) for $q_{PV}$, one can see that the log-term cancels in the difference and
\[
q_s - q_{PV} = -\frac{j_s}{\gamma} \left( \frac{1}{2} \nabla^2 \right) = j_s c_V (\Delta T)
\]
(A60)
In accord with our earlier remarks, this is the same as the coefficient of $-p \circ \Theta$ and is completely independent of choice of molecular transport coefficients. Thus,
\[
\beta \circ Q - \beta \circ \tau(p, \Theta) = c_V j_s \left[ \beta_s (\Delta T) + (\gamma - 1) \left( \frac{v_{uv} \Delta v}{\nu_0} \right) \right] \delta(x)
\]
(A61)
and is also independent of molecular dissipation.

**Negentropy Flux $\Sigma_{flux}$**: We first consider the contribution from $(\partial_x \beta) \bar{w}$. From $\beta = c_V \bar{p}/\bar{u}$ one gets
\[
\partial_x \beta = c_V \left( \frac{1}{\bar{u}} \partial_x \bar{p} - \frac{\bar{p}}{\bar{u}^2} \partial_x \bar{u} \right)
\]
(A62)
whereas
\[
\bar{w} = c_V \bar{p} T \bar{v} = c_V j_s T
\]
(A63)
Writing $\bar{p}$ in terms of $\bar{u}$ using (A19) and the similar relation for $T$ in terms of $\bar{u}$, one finds after some simplifications
\[
(\partial_x \beta) \bar{w} = -c_V j_s \left( \rho_0 - \Delta \rho \right) \left( \rho_0 - \frac{\Delta \rho}{\Delta U} u_0 \right) \left( T_0 - \frac{\Delta T}{\Delta U} u_0 \right) \left( \frac{\Delta U}{\Delta u} \frac{\partial_x \bar{u}}{\bar{u}} + \frac{\Delta T}{\Delta u} \frac{\partial_x \bar{u}}{\bar{u}} \right)
\]
(A64)
Thus,
\[
\mathcal{D} \lim_{\ell \to 0} (\partial_x \beta) \bar{w} = -c_V j_s \left( \rho_0 - \frac{\Delta \rho}{\Delta U} u_0 \right) \left( T_0 - \frac{\Delta T}{\Delta U} u_0 \right) \left( \frac{\Delta U}{\Delta u} \frac{\partial_x \bar{u}}{\bar{u}} + \frac{\Delta T}{\Delta u} \frac{\partial_x \bar{u}}{\bar{u}} \right) \delta(x)
\]
(A65)
Using the relation
\[
(T_0 u - T_1 u) (\rho_0 u_1 - \rho_1 u_0) = (\Delta T) (\Delta \rho) u_0 u_1
\]
(A66)
that follows from $u = c_V \rho T$, and the definition of $\beta_s$ from (A55), this reduces to
\[
\mathcal{D} \lim_{\ell \to 0} \partial_x \beta \, \bar{w} = -c_V (\Delta T) j_s \delta(x)
\]
(A67)
Next note that $\partial_x \Sigma_{m} \cdot \bar{w} = j_s \partial_x \Sigma_{m}$. Hence,
\[
\mathcal{D} \lim_{\ell \to 0} \partial_x \Sigma_{m} \cdot \bar{w} = -(\Delta \Sigma_{m}) j_s \delta(x)
\]
(A68)
For any equation of state, the Gibbs fundamental relation may be written as $s_m = h_m/T - \lambda_m$. For an ideal gas $h_m = c_p T$, so that $s_m = c_p - \lambda_m$ and $\Delta s_m = -(\Delta \lambda_m)$. Thus,
\[
\mathcal{D} \lim_{\ell \to 0} \partial_x \Sigma_{m} \cdot \bar{w} = -(\Delta s_m) j_s \delta(x)
\]
(A69)
Finally, consider the contributions from $\partial_x \beta (\bar{u}, \bar{v})$, $\partial_x \Sigma_{m} (\bar{\rho}, \bar{v})$. Using the equation (A62) for $\partial_x \beta$ one has
\[
\partial_x \beta (\bar{u}, \bar{v}) = c_V \bar{v} (\partial_x \bar{p}) - c_V \frac{\bar{p}}{\bar{u}} (\partial_x \bar{u})
\]
(A70)
Next use $\partial_x \Sigma_{m} = -\partial_x \Sigma_{m}$ and $s_m = c_V \ln(u/C \rho^\gamma)$ to obtain
\[
\partial_x \Sigma_{m} (\bar{\rho}, \bar{v}) = c_V \gamma \bar{v} (\partial_x \bar{p}) - c_V \frac{\bar{p}}{\bar{u}} (\partial_x \bar{u})
\]
(A71)
Subtracting these two expressions gives
\[
\partial_x \beta (\bar{u}, \bar{v}) - \partial_x \Sigma_{m} (\bar{\rho}, \bar{v}) = c_V (\gamma - 1) \bar{v} (\partial_x \bar{p}) - (\gamma - 1) c_V \frac{\Delta \rho}{\Delta u} \bar{u} (\partial_x \bar{u})
\]
(A72)
From this it follows easily that
\[
\mathcal{D} \lim_{\ell \to 0} [\partial_x \beta (\bar{u}, \bar{v}) - \partial_x \Sigma_{m} (\bar{\rho}, \bar{v})]
= c_V (\gamma - 1) j_s \left( \frac{v_{uv} \Delta v}{\nu_0} \right) \delta(x)
\]
(A73)
Putting together all of these results,\[
\Sigma_{flux} = \mathcal{D} \lim_{\ell \to 0} [\partial_x \beta (u, v) - \partial_x \Sigma_{m} (\rho, v)]
= j_s \left[ \Delta s_m - c_V \beta_s (\Delta T) - c_V (\gamma - 1) \left( \frac{v_{uv} \Delta v}{\nu_0} \right) \right] \delta(x)
\]
(A74)
which is valid in general for the limiting Euler solution independent of microscopic dissipation mechanism. Note that the first and third terms in the square bracket of the last expression are positive, while the second term is negative. We do not present details here, but it is possible to show that the sum of all three terms is strictly positive as a function of compression factor $R = \rho_1/\rho_0 = v_0/v_1$ and maximum compression factor $R_{\text{max}} = (\gamma + 1)/(\gamma - 1)$ over the allowed range $1 \leq R \leq R_{\text{max}}$.

**Total Entropy Production $\Sigma_{inert}$**: For the final inertial-range entropy production we get from (A61) and (A74) that
\[
\Sigma_{inert} = \Sigma_{flux} + \beta \circ (Q - \tau(p, \Theta)) = j_s \Delta s_m \delta(x)
\]
(A75)
which is independent of the molecular dissipation and in exact agreement with the net result of the dissipation-range/fine-grained calculation in section A 3 a. From relation (A90), it is also true that

$$\Sigma_{\text{flux}} = \Sigma_{\text{flux}} + \beta \circ (Q - \tau(p, \Theta)) = j_s \Delta s_m \delta(x). \quad (A76)$$

The intrinsic negentropy flux consistently gives the net entropy production for this problem, since $I_{\text{flux}} = 0$ for an ideal-gas equation of state.

[1] D. Falceta-Goncalves, G. Kowal, E. Fulgarone, and A.-L. Chian, Nonlinear Proc Geophys 21, 587 (2014).
[2] J. Ballesteros-Paredes, R. Klessen, M.-M. Mac Low, and E. Vazquez-Semadeni, Protostars and Planets V, 63 (2007).
[3] C. Federrath, in Journal of Physics: Conference Series, Vol. 719 (IOP Publishing, 2016) p. 012002.
[4] B. M. Haines, F. F. Grinstein, L. Welser-Sherrill, and J. R. Fincke, “Laser-driven turbulence in high energy density Physics and inertial confinement fusion experiments," in Coarse Grained Simulation and Turbulent Mixing, edited by F. F. Grinstein (Cambridge University Press, New York, 2016) pp. 232–281.
[5] M. F. Modest and D. C. Haworth, in Radiative Heat Transfer in Turbulent Combustion Systems (Springer, 2016) pp. 13–42.
[6] A. J. Smits and J.-P. Dussauge, Turbulent shear layers in supersonic flow (Springer Science & Business Media, 2006).
[7] N. Bucciantini, Astronomische Nachrichten 335, 234 (2014).
[8] R. Narayan and P. Kumar, Mon Not R Astron Soc 394, L117 (2009).
[9] R. D. de Souza, T. Koide, and T. Kodama, Prog Part Nucl Phys 86, 35 (2016).
[10] L. Fritz, J. Schmalian, M. Müller, and S. Sachdev, Phys Rev B 78, 085416 (2008).
[11] A. B. Kashuba, Phys Rev B 78, 085415 (2008).
[12] M. Müller, J. Schmalian, and L. Fritz, Phys Rev Lett 103, 025301 (2009).
[13] C. Hoyos, B. S. Kim, and Y. Oz, J High Energy Phys 2013, 145 (2013).
[14] R. A. Davison, K. Schalm, and J. Zaanen, Phys. Rev. B 89, 245116 (2014).
[15] A. N. Kolmogorov, Dokl. Akad. Nauk SSSR 30, 299 (1941).
[16] A. N. Kolmogorov, Dokl. Akad. Nauk SSSR 31, 319 (1941).
[17] A. N. Kolmogorov, Dokl. Akad. Nauk SSSR 32, 19 (1941).
[18] A. Obukhov, Dokl. Akad. Nauk SSSR 32, 22 (1941).
[19] L. Onsager, Phys Rev 68, 286 (1945).
[20] L. Onsager, Nuovo Cim. Suppl. 6, 279 (1949).
[21] W. Heisenberg, Z. Phys 124, 514 (1948).
[22] G. Von Weizsacker, Z. Phys 124, 614 (1948).
[23] H. A. Luttinger, Physica 247, 54 (1943).
[24] G. Falkovich, I. Fou oxon, and Y. Oz, J Fluid Mech 644, 465 (2010).
[25] I. Fou oxon and Y. Oz, Phys Lett B 694, 261 (2010).
[26] G. I. Taylor, “Observations and speculations on the nature of turbulent motion (1917),” in The Scientific Papers of Sir Geoffrey Ingram Taylor: Volume 2, Meteorology, Oceanography and Turbulent Flow, edited by G. K. Batchelor (Cambridge University Press, 1960) p. 6978.
[27] H. L. Dryden, Q Appl Math 1, 7 (1943).
[28] K. R. Sreenivasan, Phys Fluids 27, 1048 (1984).
[29] Y. Kaneda, T. Ishihara, M. Yokokawa, K. Itakura, and A. Uno, Phys Fluids 15, L21 (2003).
[30] B. Pearson, P.-Å. Krogstad, and W. Van De Water, Phys Fluids 14, 1288 (2002).
[31] G. L. Eyink and K. R. Sreenivasan, Rev Mod Phys 78, 87 (2006).
[32] A. Polyakov, arXiv preprint hep-th/9209046 (1992).
[33] A. Polyakov, Nucl Phys B 396, 367 (1993).
[34] P. Kestener and A. Arneodo, Phys Rev Lett 93, 044501 (2004).
[35] U. Frisch and G. Parisi, in Turbulence and Predictability in Geophysical Fluid Dynamics and Climate Dynamics (Amsterdam, North-Holland, Elsevier, 1985) pp. 84–88.
[36] U. Frisch, Turbulence: The Legacy of A. N. Kolmogorov (Cambridge University Press, 1995).
[37] H. Aluie, Phys. Rev. Lett. 106, 174502 (2011).
[38] G. L. Eyink, Physica D 78, 222 (1994).
[39] G. L. Eyink, J Stat Phys 78, 335 (1995).
[40] P. Constantin, E. Weinan, and E. S. Titi, Commun Math Phys 165, 207 (1994).
[41] J. Duchon and R. Robert, Nonlinearity 13, 249 (2000).
[42] C. De Lellis and L. Székelyhidi Jr, in European Congress of Mathematics: Kraków, 2-7 July, 2012 edited by R. Latala, A. Rucinska, P. Strzelecki, J. Swiatkowski, and D. Wrzosek (European Mathematical Society, 2013) pp. 13–30.
[43] C. De Lellis and L. Székelyhidi Jr, B Am Math Soc 49, 347 (2012).
[44] P. Isett, arXiv preprint arXiv:1608.08301 (2016).
[45] E. C. Stueckelberg and A. Petermann, Helvetica Physica Acta (Switzerland) 24 (1951).
[46] M. Gell-Mann and F. E. Low, Phys Rev 95, 1300 (1954).
[47] N. Bogolyubov and D. Shirkov, Dokl. Akad. Nauk SSSR 103, 391 (1955).
[48] G. L. Eyink and T. D. Drivas, “Cascades and dissipative anomalies in relativistic fluid turbulence,” submitted to Phys Rev X (2017).
[49] Y. Choquet-Bruhat, C. DeWitt-Morette, and M. Dillard-Bleick, Analysis, Manifolds, and Physics Analysis, Manifolds, and Physics No. pt. 1 (North-Holland Publishing Company, 1982).
[50] L. Onsager, Ann Phys New York 46, 241 (1945).
[51] H. Brenner, Physica A 349, 60 (2005).
[52] H. Mott-Smith, Phys Rev 82, 885 (1951).
[53] H. W. Liepmann, R. Narasimha, and M. T. Chahine, Analysis, Manifolds, and Physics (Springer, 1985) pp. 84–88.
[54] E. Salomon and M. Mareschal, Phys Rev Lett 69, 269 (1992).
[55] S.-H. Yu, Commun Pure Appl Math 58, 409 (2005).
The limit appears to exist in an even stronger sense than distributionally, which implies limits of $Q_{\epsilon}$ exist only when smeared with $C^{\infty}$, compactly supported test functions. Experimentally, the limit seems to exist even if $Q_{\epsilon}$ is smeared with bounded continuous functions (weak convergence of measures) or even when integrated over compact sets, such as balls or cubes of fixed size as $\nu \to 0$. See\cite{55,56}.

C. Meneveau and K. Sreenivasan, Nucl Phys B - Proc Supp 2, 49 (1987).

C. Meneveau and K. Sreenivasan, J Fluid Mech 224, 429 (1991).

S. Jagannathan and D. Donzis, in APS Meeting Abstracts (2011).

S. Jagannathan and D. A. Donzis, J Fluid Mech 789, 669 (2016).

Q. Ni, Y. Shi, and S. Chen, arXiv preprint arXiv:1505.02685 (2015).

S. De Groot and P. Mazur, Non-Equilibrium Thermodynamics (Dover Books on Physics (Dover Publications, 2013).

L. Landau and E. Lifshitz, Fluid Mechanics v. 6 (Elsevier Science, 2013).

A. Obukhov, Izv. Akad. Nauk. SSSR (Geogr. and Geo-Phys. series) 13, 58 (1949).

Here we may note that the coarse-graining regularization permits one to analyze dissipative anomalies as well in Lagrangian conservation laws such as fluid circulations \cite{121} or magnetic fluxes \cite{124} for advected loops, but it is very unclear how to apply point-splitting to Lagrangian invariants.

L. Kadanoff, Physics 2, 263 (1966).

K. G. Wilson, Phys Rev B 4, 3174 (1971).

G. L. Eyink, Astrophys J 807, 137 (2015).

T. D. Drivas and G. L. Eyink, “An Onsager singularity theorem for turbulent solutions of compressible Euler equations,” submitted to Commun Math Phys (2017).

The failure to appreciate this point has been the source of many misunderstandings. For example, consider the following typical quote:

“We therefore conclude that, for the large eddies which are the basis of any turbulent flow, the viscosity is unimportant and may be equated to zero, so that the motion of these eddies obeys Euler’s equation. In particular, it follows from this that there is no appreciable dissipation of energy in the large eddies.” – Landau & Lifshitz \cite{148}, §31.

This statement is correct, if one understands it to mean that the viscous dissipation is negligible for the large eddies. However, the coarse-graining which permits one to neglect viscosity at large-scales generates new stresses which do not conserve the energy of the large eddies! The use of Favre-average cumulants rather than cumulants for the original spatial coarse-graining is not essential. Their use does, however, reduce the number of additional cumulant terms that appear and permits a simple physical interpretation of each such term. For these reasons, Favre-averaging has been very popular in the practical engineering modelling of compressible turbulence\cite{55}.

A. Favre, in Problems of Hydrodynamics and Continuum Mechanics, edited by M. A. Lavrentiev (SIAM, Philadelphia, 1969) pp. 37–44.

K. Huang, Statistical mechanics (Wiley, 1987).

M. Germania, J Fluid Mech 238, 325 (1992).

One source of such stochasticity is thermal noise due to molecular degrees of freedom, so far neglected in our analysis, but other unknown perturbations can also lead to randomness. This stochasticity does not contradict our earlier claim of a “deterministic approach”. A given weak Euler solution $\rho, \mathbf{v}, u$ over a particular time-interval corresponds to a particular realization of these cumulants. It is only if one wants to predict or control the future behavior outside that particular time interval that one must recognize the intrinsic stochasticity. Here we may note that vanishingly small stochastic perturbations do not alter our conclusions in Section \cite{47} regarding the description of coarse-grained variables in the inertial-range by “weak Euler solutions”. This argument may be made precise for thermal noise by appealing to the Onsager fluctuation principle \cite{57,149,150}.

The latter principle states that the probability of observing a particular set of fields $\rho, \mathbf{v}, u$ as thermal fluctuations is related to the additional dissipation/entropy production required to produce the fluctuation. See\cite{57}, section 4, example 2, for compressible Navier-Stokes fluids. Even when this excess dissipation is non-vanishing, the noise terms vanish in the coarse-grained equations for the limit $\eta, \zeta \to 0$. Details will be given elsewhere, but the argument is essentially the same as for the deterministic fluid equations in Section \cite{47}. Thus, coarse-grained variables in the inertial-range range are described by weak Euler solutions even in the presence of thermal noise.

C. Meneveau and J. Katz, Annu Rev Fluid Mech 32, 1 (2000).

E. Garnier, N. Adams, and P. Sagaut, Large Eddy Simulation for Compressible Flows, Scientific Computation (Springer Netherlands, 2009).

W. Schmidt, Living Reviews in Computational Astrophysics 1, 1 (2015).

M. Oberhuggenberger, Multiplication of Distributions and Applications to Partial Differential Equations, Pitman research notes in mathematics series, Vol. 259 (Longman Scientific & Technical, 1992).

H. Aluie, S. Li, and H. Li, Astrophys J Lett 751, L29 (2012).

Q. Ni and S. Chen, arXiv preprint arXiv:1506.04179 (2015).

H. Aluie, “Generalizing the 4/5-th law to compressible turbulence,” unpublished (2012).

G. L. Eyink, “Turbulence theory, course notes,” http://www.ams.jhu.edu/~eyink/Turbulence/notes.html (2010).

In fact, unpublished work of Aluie\cite{90} shows how to recover the traditional 4/5-th law of Kolmogorov from such coarse-grained expressions for kinetic energy flux.
At least the results are distinct for $Pr = 0$, $Pr = \infty$, and $0 < Pr < \infty$. We have analytical results for only one finite positive value $Pr = 3/4$.

A caveat has to do with the contribution of the density. If, as we have assumed in this work, the density is a bounded function and Hoelder continuous in space, then density-increments $\delta \rho$ are scale-local. However, the coarse-grained density $\overline{\rho}$ in that case is dominated by energy-scale contributions and infrared-locality breaks down. On the other hand, there is evidence from numerical simulations that, for Mach numbers much larger than 1, the density in compressible fluid turbulence exists only as a distribution (measure) in the infinite Reynolds number limit [145]. In that case, $\overline{\rho}$ is scale-local, but $\delta \rho$ is dominated by dissipation-range contributions, and ultraviolet-locality breaks down. In either case, scale-locality through the dependence on density is always broken in one direction.

We are not being entirely consistent with our mathematical notations. If so, we should have defined $\beta \circ Q_{\text{visc}} = \lim_{\ell \to 0} \beta \cdot Q_{\text{visc}}$ and the quantities in equations (83) and (84) would require a new notation. To avoid a proliferation of new symbols, we use the same notation “$\circ$” everywhere for the relevant distributional products obtained by limits of $\ell$ through the inertial-range, whereas the notation “$\ast$” stands for the distributional product obtained by limits $\eta$, $\zeta$, $\kappa \to 0$ of fine-grained/dissipation-range quantities.

Because of the assumed smoothness of $Q_{\text{cool}}$, the product $\beta Q_{\text{cool}}$ is meaningful in the ordinary sense in the ideal limit and no special care is required for its definition.

The notion of “energy cascade” has sometimes been criticized as unphysical because it depends upon arbitrary scale decompositions. For example, consider the following quotes from one prominent critic:

“On the other hand, energy transfer, just like any physical process, should be invariant of particular decompositions/representations of a turbulent field. In this sense Kolmogorov’s choice of dissipation (and energy input) are well defined and decomposition independent quantities, whereas the energy flux is (generally) not, since it is decomposition dependent. After all Nature may and likely does not know about our decompositions.” – Tsinober [151]

and

“We have seen that there is an ambiguity in defining the meaning of the term ‘small scales’ (or more generally ‘scales’ or ‘eddies’, see appendix C) and consequently the meaning of the term ‘cascade.’.” – Tsinober [151]

One erroneous statement above is the claim that energy flux is “generally” decomposition dependent. In fact, the energy cascade rate over a long inertial range at high Reynolds numbers is demonstrably the same for any filter kernel satisfying very general, mild assumptions of smoothness and rapid spatial decay. However, the other remarks are correct and acute. Indeed, the physical process must be invariant of particular decompositions/representations and independent of the scale of observation. What the criticism is missing is that the requirement of such invariance is a positive principle which can be exploited to deduce exact consequences.

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“On the other hand, energy transfer, just like any physical process, should be invariant of particular decompositions/representations of a turbulent field. In this sense Kolmogorov’s choice of dissipation (and energy input) are well defined and decomposition independent quantities, whereas the energy flux is (generally) not, since it is decomposition dependent. After all Nature may and likely does not know about our decompositions.” – Tsinober [151]

and

“We have seen that there is an ambiguity in defining the meaning of the term ‘small scales’ (or more generally ‘scales’ or ‘eddies’, see appendix C) and consequently the meaning of the term ‘cascade.’.” – Tsinober [151]

One erroneous statement above is the claim that energy flux is “generally” decomposition dependent. In fact, the energy cascade rate over a long inertial range at high Reynolds numbers is demonstrably the same for any filter kernel satisfying very general, mild assumptions of smoothness and rapid spatial decay. However, the other remarks are correct and acute. Indeed, the physical process must be invariant of particular decompositions/representations and independent of the scale of observation. What the criticism is missing is that the requirement of such invariance is a positive principle which can be exploited to deduce exact consequences.
For finite values of $\eta, \zeta, \kappa$, the modes have complex frequencies with imaginary parts reflecting dissipative decay. For example, see [127], equations 6.1-3. Here we focus on the ideal nonlinear behavior at very high Reynolds and Péclet numbers, as in most of [127], section 6.

Alternatively, Obukhov [72] showed that the quantity $\Theta$ in their eq.(6.6) is factorized as $\eta \zeta^2 + \zeta \Theta^2$. This latter quantity vanishes in the ideal limit of high Reynolds numbers, whereas the quantity $\Theta$ is expected to have a non-zero value.

One example of a failure of the Kovácsznay mode-coupling theory which has already been considered is its prediction that the entropy per particle $s'_\text{ent}$ is a passive scalar. As another example, consider the estimate by [142], section 5.2, for the spectrum of density at $M_\text{a} = 1$ in a regime ("high-$\beta$") where magnetic pressure is small relative to thermal pressure. Invoking Kovácsznay’s quadratic “sound-sound” coupling $\partial^2 (v', v') / \partial x_i \partial x_j$, they predicted that density perturbations due to isothermal sound waves will lead to a $k^{-7/3}$ density spectrum. Instead, [152] in a simulation of “isothermal” compressible MHD at $M_\text{a} \sim 1$ have observed for large $\beta$ a $k^{-5/3}$ density spectrum, and [145] for “isothermal” hydrodynamic ($\beta = \infty$) turbulence at $M_\text{a} \sim 1$ found also a $k^{-5/3}$ density spectrum, contradicting mode-coupling predictions.

To make this argument correctly requires a discussion of the linear wave modes of compressible MHD. This problem is carefully treated by [142], Appendix A, including also the important effects of cooling. As discussed by those authors, the MHD entropy mode with cooling has at sufficiently small scales zero perturbation not of thermal pressure, but of total pressure (thermal + magnetic). Nevertheless, their analysis shows that the above argument based upon the “entropy mode” of a hydrodynamic (non-magnetized) fluid carries over to MHD with only minor changes. In fact, the hydrodynamic treatment of the entropy mode is exactly valid at sufficiently small-scales where slow magnetosonic waves can create pressure balance of the entropy mode, so that $p' = 0$. At larger scales the entropy modes carries a non-vanishing fluctuation $p' \neq 0$ of thermal pressure, which is balanced by a magnetic pressure fluctuation. In either case, the entropy mode is rapidly damped on a cooling time-scale $\tau_\text{cool}$, so that the amplitude of the entropy mode can be argued to be small above a cooling scale $L_\text{cool}$. See [142], Appendix A, for all details.