FREE BOUNDARY PROBLEM OF BARENBLATT EQUATION
IN STOCHASTIC CONTROL

XIAOSHAN CHEN
School of Mathematical Sciences
South China Normal University, Guangzhou 510631, China

FAHUAI YI*
School of Finance
Guangdong University of Foreign Studies, Guangzhou 510006, China

ABSTRACT. The following type of parabolic Barenblatt equations
\[ \min\{\partial_t V - L_1 V, \partial_t V - L_2 V\} = 0 \]

is studied, where \( L_1 \) and \( L_2 \) are different elliptic operators of second order. The (unknown) free boundary of the problem is a divisional curve, which is the optimal insured boundary in our stochastic control problem. It will be proved that the free boundary is a differentiable curve.

To the best of our knowledge, this is the first result on free boundary for Barenblatt Equation. We will establish the model and verification theorem by the use of stochastic analysis. The existence of classical solution to the HJB equation and the differentiability of free boundary are obtained by PDE techniques.

1. Introduction. A general form of parabolic Barenblatt equation is
\[ \partial_t u(x,t) - \inf_{\alpha \in A} \{L_{\alpha}u(x,t) + f_{\alpha}\} = 0, \]

where \( A \) is a finite or infinite set, \( L_{\alpha} \) is a set of elliptic operators of second order. In Lieberman’s book [8] this fully nonlinear equation is called as parabolic Bellman equation. In physics and mechanics it is called as Barenblatt equation [3] and in mathematical finance it is called as Black-Scholes-Barenblatt equation [4, 5, 12]. Even if \( A \) is a infinite set, Krylov [6] proves this equation always has a classical solution in any bounded domain. In our problem \( \alpha \) only takes two values, i.e., \( V(x,t) \) satisfies
\[
\begin{align*}
&\min\{\partial_t V - L_1 V, \partial_t V - L_2 V\} = 0, \quad x > 0, \quad 0 < t \leq T, \\
&V(0,t) = 0, \quad t \in (0,T], \\
&V(x,0) = x, \quad x > 0,
\end{align*}
\]

\( 2010 \) Mathematics Subject Classification. 35R35, 60G40, 91B70, 93E20.

Key words and phrases. Free boundary problem, Barenblatt equation, optimal control, stochastic control, HJB equation.

* Corresponding author. The project is supported by NNSF of China (No.11271143, No.11371155, No.11471276, No.71471045 and No.11526090), University Special Research Fund for Ph.D. Program of China (20124407110001), NSF of Guangdong Province of China (No.2015A030313574 and No.2016A030313448) and The Humanities and Social Science Research Foundation of the National Ministry of Education of China (No.15YJAZH051).
where
\[
\mathcal{L}_1 V = \frac{1}{2} \sigma^2 \partial_{xx} V + \mu \partial_x V - (r + \lambda)V,
\]
\[
\mathcal{L}_2 V = \frac{1}{2} \sigma^2 \partial_{xx} V + (\mu - \lambda L) \partial_x V - rV,
\]
\[\sigma, \mu, r, \lambda, L \text{ are positive constants with } \mu > \lambda L.\]

This problem arises naturally from mathematical finance [10]: where a firm’s asset follows standard Ito process and may suffer from a Poisson loss of given magnitude \(L\) (accident). However the firm can choose to insure part of this loss at a fair market premium each unit of time. The firm chooses the amount of coverage at each moment of time, thus the problem becomes a standard stochastic control problem. It has been shown that the value function for the problem satisfies the Barenblatt equation.

Explicit solution to the equation can be found in closed form when insurance contracts can be written with infinite maturity [10]. In this paper, we generalize the model by considering insurance contracts with finite maturity. We found that with finite maturity insurance contracts, the problem boils down to a free boundary problem of PDE instead of ODE. We justify our generalization as all insurance contracts in reality have finite maturity instead of infinite maturity.

The rest of the paper is organized as following, Section 2 formulates the model as an optimal exit time problem. Section 3 derives some properties of the classical solution to the Barenblatt equation. One of our main contributions lies in Section 4 which proves the differentiability of the free boundary. Finally, in Section 5, we present a verification theorem, derive the optimal insurance strategy and give an financial interpretation of the free boundary.

To the best of our knowledge, this is the first result on the free boundary for the Barenblatt equation. We establish monotonicity and concavity of value function, and the verification theorem through stochastic analysis. Then we prove existence of classical solution for HJB equation and differentiability of free boundary using PDE techniques.

2. Model formulation. We fix a filtered probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with a filtration \((\mathcal{F}_t)_{t \geq 0}\) that satisfies the usual conditions, the process \(W_t\) is a standard Brownian motion, \(P_t\) is a Poisson process with intensity \(\lambda\).

Suppose \(\mu\) is the expected return of the cash reserve, while the Poisson risk can be insured for a fair premium \(\lambda L\) per unit of time, we assume that \(\mu > \lambda L\) (otherwise the technology would not be profitable [10]). The dynamics of the cash reserve \(X_s\) is now given by
\[
\begin{cases}
    dX_s = (\mu - \lambda L i_s) ds + \sigma dW_s - (1 - i_s)LdP_s, & s > t, \\
    X_t = x.
\end{cases}
\]

The control variable of the firm is represented by an adapted process \(i_s \in [0, 1]\), which is the fraction of the Poisson loss that is insured. \(\sigma\) is the volatility of \(W_s\). We denote by \(\mathcal{A}_{t,x}\) the set of all admissible control processes \(i_s\) from time \(t\) and \(X_t = x\), where \(i_s\) is a progressively measurable (with respect to the filtration \((\mathcal{F}_t)_{t \geq 0}\)) process satisfying
\[
E\left[ \int_0^T i_s^2 ds \right] \leq +\infty.
\]
For all $i_s \in \mathcal{A}_{t,x}$, the above condition ensures the existence and uniqueness of a strong solution to the SDE (2.1).

Suppose that the company will distribute all the cash reserve as dividend at terminal time $T$. The objective is to choose the optimal control so as to maximize the expected discounted value of the cash reserve at time $T$ if it is not bankrupt at $[t, T]$. We define the bankruptcy time $\tau$ as

$$\tau = \inf \{ s > t : X_s \leq 0 \}.$$ 

The value function is defined as

$$U(x, t) = \sup_{A_{t,x}} E_{tx} \left[ e^{-r(T-t)} X_T \chi(\tau > T) \right], \quad (2.2)$$

where $E_{tx}[\cdot] = E[\cdot | X_t = x]$. 

We formulate the problem as an exit time problem from $(0, \alpha_1)$, where $\alpha_1$ is the bankruptcy point. Take $B_{t,x} \subseteq \mathcal{A}_{t,x}$ the set of all admissible control $i_t \in \mathcal{A}_{t,x}$ with initial value $X_t = x$, such that the solution to the stochastic differential equation (2.1) satisfies

$$X_s > 0, \quad s \in [t, T].$$

Hence the definition of $U(x, t)$ in (2.2) is equivalent to the following definition

$$U(x, t) = \sup_{B_{t,x}} E_{tx} \left[ e^{-r(T-t)} X_T \right]. \quad (2.3)$$

**Theorem 2.1.** The value function $U(x, t)$ defined in (2.2) is increasing and concave in $x$ on $(0, +\infty)$.

**Proof.** First we prove the monotonicity of $U(x, t)$ w.r.t. $x$. Fix some arbitrary $0 < x \leq y$, and $i_s \in \mathcal{B}_{t,x} \subseteq \mathcal{B}_{t,y}$. Suppose $X_{s}^{t,x}, X_{s}^{t,y}$ are two processes of SDE (2.1) related to the initial values $X_t = x$ and $X_t = y$, respectively. By comparison principle of stochastic differential equation, we have

$$X_{s}^{t,x} \leq X_{s}^{t,y}, \quad a.s., s \in [t, T]. \quad (2.4)$$

Thus by the equivalent definition of the value function (2.3), we obtain

$$E_{tx} \left[ e^{-r(T-t)} X_{T}\right] \leq E_{ty} \left[ e^{-r(T-t)} X_{T}\right] \leq U(y, t).$$

By the arbitrary of $i_s \in \mathcal{B}_{t,x}$ we obtain $U(x, t) \leq U(y, t)$.

On the other hand for any $0 < x_1 < x_2, i_{s}^{k} \in \mathcal{B}_{t,x_k}, k = 1, 2$, and $X_{s}^{k}$ is the solution to the stochastic differential equation

$$\begin{cases}
  dX_{s}^{k} = (\mu - \lambda L i_{s}^{k}) dt + \sigma dW_s - (1 - i_{s}^{k}) L dP_s, \\
  X_{t}^{k} = x_k.
\end{cases}$$

For any $\alpha \in [0, 1]$, denote $x_\alpha = \alpha x_1 + (1 - \alpha) x_2, i_{s}^{\alpha} = \alpha i_{s}^{1} + (1 - \alpha) i_{s}^{2}$, from the linear dynamics of (2.1), we see that the solution to the following SDE

$$\begin{cases}
  dX_{s}^{\alpha} = [\mu - \lambda L (\alpha i_{s}^{1} + (1 - \alpha) i_{s}^{2})] dt + \sigma dW_s - [1 - (\alpha i_{s}^{1} + (1 - \alpha) i_{s}^{2})] L dP_s, \\
  X_{t}^{\alpha} = x_\alpha.
\end{cases}$$

is

$$X_{s}^{\alpha} = \alpha X_{s}^{1} + (1 - \alpha) X_{s}^{2}, \quad t \leq s \leq T.$$
The above equality also shows $X^a_t > 0$ by $X^1_s > 0$ and $X^2_s > 0$, thus $i^a_s \in \mathcal{B}_{t,x}$. Hence by the equivalent definition of the value function (2.3),
\[
U(x_0, t) \geq \alpha E_t \left[ e^{-r(T-t)}X^1_T \right] + (1 - \alpha) E_t \left[ e^{-r(T-t)}X^2_T \right].
\]
Since the above inequality holds true for any $i^1_s \in \mathcal{B}_{t,x_1}$, $i^2_s \in \mathcal{B}_{t,x_2}$, we have that
\[
U(x_0, t) \geq \alpha U(x_1, t) + (1 - \alpha) U(x_2, t).
\]
So we obtain the concavity of the value function.

Now we will establish HJB equation of the problem. Suppose $U(x, t) \in C^{2,1}((0, +\infty) \times (0, T))$ (we will prove this in Theorem 3.2), applying dynamic programming principle [9] to (2.3), for any $t < \delta < T$,
\[
U(x, t) = \sup_{\mathcal{B}_{t,x}} \left[ e^{-r(\delta-t)}U(X_\delta, \delta) \right], \quad (x, t) \in (0, +\infty) \times [0, T).
\] (2.5)
Applying Itô’s formula with jump,
\[
e^{-r(\delta-t)}U(X_\delta, \delta) - U(x, t) = \int_t^\delta e^{-r(s-t)} \left[ \partial_t U + (\mu - \lambda L_i)s \partial_x U + \frac{1}{2} \sigma^2 \partial_{xx} U - rU \right] ds + \sum_{t \leq s \leq \delta} e^{-r(s-t)}[U(X_s, s) - U(X_{s-}, s)] + \int_t^\delta e^{-r(s-t)}\sigma \partial_x U dW_s.
\]
Suppose $\partial_x U(x, t)$ is uniformly bounded (it will be proved, see (3.13)), then the last term in the above equality is a square integral martingale. Taking the expectation in above equality,
\[
E_t \left[ e^{-r(\delta-t)}U(X_\delta, \delta) \right] = U(x, t) + E_t \left[ \sum_{t \leq s \leq \delta} e^{-r(s-t)}(U(X_s, s) - U(X_{s-}, s)) \right] + E_t \left\{ \int_t^\delta e^{-r(s-t)} \left[ \partial_t U + (\mu - \lambda L_i)s \partial_x U + \frac{1}{2} \sigma^2 \partial_{xx} U - rU \right] ds \right\}.
\] (2.6)
Since $X_t - X_{t-} = -(1 - i_s)L \Delta P_t$, where $\Delta P_t = P_t - P_{t-} = 1$ or 0, then
\[
\sum_{t \leq s \leq \delta} e^{-r(s-t)}[U(X_s, s) - U(X_{s-}, s)] = \sum_{t \leq s \leq \delta} e^{-r(s-t)}[U(X_s, s) - (1 - i_s)L, s) - U(X_{s-}, s)]\Delta P_s = \int_t^\delta e^{-r(s-t)}[U(X_s, s) - (1 - i_s)L, s) - U(X_{s-}, s)]dP_s = \int_t^\delta e^{-r(s-t)}[U(X_s, s) - (1 - i_s)L, s) - U(X_{s-}, s)]d(P_s - \lambda s) + \int_t^\delta e^{-r(s-t)}[U(X_s, s) - (1 - i_s)L, s) - U(X_{s-}, s)]\lambda ds,$
Thus the control is the bang-bang type, and where compensated Poisson process \( P_s - \lambda s \) is a martingale ([11], Th. 11.2.4). So (2.6) becomes 

\[
E_{tx} \left[ e^{-r(\delta-t)}U(X_\delta, \delta) \right]
= U(x, t) + E_{tx} \left\{ \int_t^\delta e^{-r(s-t)}[U(X_s^- - (1 - i_s)L, s) - U(X_s^-, s)]\lambda ds \right\}
+ E_{tx} \left\{ \int_t^\delta e^{-r(s-t)} [\partial_t U + (\mu - \lambda i_s)\partial_x U + \frac{1}{2}\sigma^2\partial_{xx}U - rU] ds \right\}.
\]

Plugging it back into (2.5), dividing the equation in (2.5) by \( \delta - t \) and letting \( \delta - t \to 0^+ \), we have

\[
\sup_{i \in [0,1]} \left\{ \partial_t U + (\mu - \lambda i)\partial_x U + \frac{1}{2}\sigma^2\partial_{xx}U - rU + \lambda[U(x - (1 - i)L, t) - U(x, t)] \right\} = 0,
\]

i.e.,

\[
\partial_t U + \frac{1}{2}\sigma^2\partial_{xx}U + \mu\partial_x U - (r + \lambda)U + \sup_{i \in [0,1]} \left\{ \lambda U(x - (1 - i)L, t) - i\lambda\partial_x U \right\} = 0. \tag{2.8}
\]

Notice that, when \( x \leq 0 \), then the firm is liquidated, thus

\[
U(x, t) = 0, \quad x \leq 0. \tag{2.9}
\]

Denote \( g(i) = \lambda U(x - (1 - i)L, t) - i\lambda\partial_x U(x, t) \), then

- When \( x - L \geq 0 \), with the concavity of \( U(x, t) \) in \((0, +\infty)\), we obtain

\[
g'(i) = \lambda\partial_x U(x - (1 - i)L, t) - \lambda\partial_x U(x, t) \geq 0, \quad i \in [0,1],
\]

which implies

\[
\sup_{i \in [0,1]} g(i) = g(1); \tag{2.10}
\]

- when \( x - L < 0 \), from (2.9) and the monotonicity and concavity of \( U(x, t) \) in \((0, +\infty)\) to obtain

\[
g'(i) = \begin{cases} 
\lambda\partial_x U(x - (1 - i)L, t) - \lambda\partial_x U(x, t) \geq 0, & i \in [1 - \frac{L}{x}, 1], \\
-\lambda\partial_x U(x, t) \leq 0, & i \in [0, 1 - \frac{L}{x}].
\end{cases}
\]

Thus the control is the bang-bang type, and

\[
\sup_{i \in [0,1]} g(i) = \max\{g(0), g(1)\} = \max \{\lambda U(x - L, t), \lambda U(x, t) - \lambda\partial_x U(x, t)\}.
\]

We claim that

\[
\max \{\lambda U(x - L, t), \lambda U(x, t) - \lambda\partial_x U(x, t)\} = \max \{0, \lambda U(x, t) - \lambda\partial_x U(x, t)\} \tag{2.11}
\]

In fact, if \( x - L \geq 0 \), applying (2.10),

\[
\lambda U(x, t) - \lambda\partial_x U(x, t) = g(1) \geq g(0) = \lambda U(x - L, t) \geq 0,
\]

so (2.11) is true. And if \( x - L < 0 \), \( U(x - L, t) = 0 \). Hence (2.11) holds.
Denote
\[ \mathcal{L}_1 U = \frac{1}{2} \sigma^2 \partial_{xx} U + \mu \partial_x U - (r + \lambda) U, \] (2.12)
\[ \mathcal{L}_2 U = \frac{1}{2} \sigma^2 \partial_{xx} U + (\mu - \lambda L) \partial_x U - r U, \] (2.13)

(2.8) and (2.11) show that \( U(x,t) \) satisfies
\[ \max\{ \partial_t U + \mathcal{L}_1 U, \partial_t U + \mathcal{L}_2 U \} = 0. \] (2.14)

At the terminal time \( T \), since there is no time left for the firm to become liquidated, and we want to maximize the expected discounted value of dividends, then the firm would distribute \( X_T \) as dividends, hence
\[ U(x,T) = x. \] (2.15)

Hence \( U(x,t) \) satisfies
\[
\begin{cases}
\max\{ \partial_t U + \mathcal{L}_1 U, \partial_t U + \mathcal{L}_2 U \} = 0, & (x,t) \in \mathbb{R}^+ \times (0,T), \\
U(x,t) = 0, & (x,t) \in (-\infty,0] \times [0,T], \\
U(x,T) = x, & x \in \mathbb{R}^+.
\end{cases}
\] (2.16)

Especially, (2.10) shows that
\[ \partial_t U + \mathcal{L}_2 U = 0, \quad x > L, \quad 0 \leq t < T. \] (2.17)

We summarize the above facts as the following lemma.

**Lemma 2.2.** The value function \( U(x,t) \) defined in (2.2) satisfies the following HJB equation with boundary and terminal conditions
\[
\begin{cases}
\max\{ \partial_t U + \mathcal{L}_1 U, \partial_t U + \mathcal{L}_2 U \} = 0, & (x,t) \in \mathbb{R}^+ \times (0,T), \\
U(x,t) = 0, & (x,t) \in (-\infty,0] \times [0,T], \\
U(x,T) = x, & x \in \mathbb{R}^+,
\end{cases}
\] (2.18)

where \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) are defined in (2.12) and (2.13). Especially,
\[ \partial_t U + \mathcal{L}_2 U = 0, \quad x > L, \quad 0 \leq t < T. \] (2.19)

3. **Properties of the value function.** Denote
\[ V(x,t) = U(x,T-t), \]
where \( U(x,t) \) is the solution of problem (2.18), then \( V(x,t) \) satisfies (1.1). Set
\[
\mathcal{L} V := \frac{1}{2} (\mathcal{L}_1 V + \mathcal{L}_2 V) = \frac{1}{2} \sigma^2 \partial_{xx} V + \left( \mu - \frac{1}{2} \lambda L \right) \partial_x V - \left( r + \frac{1}{2} \lambda \right) V, \] (3.1)
\[ \mathcal{F} V := \frac{1}{2} (\mathcal{L}_1 V - \mathcal{L}_2 V) = \frac{1}{2} \lambda (L \partial_x V - V) \] (3.2)
where the operators \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) are defined in (1.2) and (1.3). Hence
\[ \mathcal{L}_1 V = \mathcal{L} V + \mathcal{F} V, \] (3.3)
\[ \mathcal{L}_2 V = \mathcal{L} V - \mathcal{F} V. \] (3.4)

We can rewrite the problem (1.1) as
\[
\begin{cases}
\partial_t V - \mathcal{L} V - |\mathcal{F} V| = 0, & x > 0, \quad 0 < t \leq T, \\
V(0,t) = 0, & t \in (0,T], \\
V(x,0) = x, & x > 0.
\end{cases}
\] (3.5)
Since $Q := (0, +\infty) \times (0, T)$ is unbounded, we first consider the problem in a bounded domain $Q_n := (0, n) \times (0, T)$, suppose $V_n(x, t)$ satisfies
\[
\begin{aligned}
\partial_t V_n - \frac{1}{2} \sigma^2 \partial_{xx} V_n - (\mu - \frac{1}{2} \lambda L) \partial_x V_n + \left( r + \frac{1}{2} \lambda \right) V_n &= \frac{1}{2} \lambda |L| L \partial_x V_n - V_n, \quad (x, t) \in Q_n, \\
V_n(0, t) &= 0, \quad \partial_x V_n(n, t) = 1, \quad t \in (0, T], \\
V_n(x, 0) &= x, \quad x \in (0, n).
\end{aligned}
\]
(3.6)

**Lemma 3.1.** There exists a solution $V_n(x, t) \in C(Q_n) \cap C^{2,1}(Q_n \setminus \{(0, 0)\})$ to the problem (3.6). Moreover, $\partial_x V_n(x, t) \in C(Q_n)$ and
\[
e^{- (r + \lambda) t} x \leq V_n(x, t) \leq x + \frac{\mu}{r},
\]
(3.7)
\[
e^{- (r + \lambda) t} \leq \partial_x V_n(x, t) \leq k_1 k_2,
\]
(3.8)
where $k_2 = \frac{2 \mu}{\sigma^2}$, $k_1 = \max \left\{ \frac{\epsilon e^2}{\epsilon^2}, \frac{(\frac{\epsilon}{2} + 1) \epsilon e^2}{\epsilon^2 - 1} \right\}$.

**Proof.** Leray-Schauder fixed point theorem [2] and the theory of linear parabolic equation [7] imply the existence of $W^{2,1}_p(Q_n)$ (for any $p$, $3 < p < +\infty$) solution to the nonlinear problem (3.6), where
\[
W^{2,1}_p(Q_n) = \left\{ u(x, t) : u, \partial_x u, \partial_{xx} u, \partial_t u \in L^p(Q_n) \right\}.
\]
Thus $\partial_x V_n(x, t) \in C(Q_n)$ by embedding theorem and $p > 3$ ([7] P.80, Lemma 3.3). The theory of regularity for parabolic equation [7] shows that $V_n(x, t) \in C^{2,1}(Q_n \setminus \{(0, 0)\})$. It means that $V_n(x, t)$ is a classical solution of (3.6).

Now we prove (3.7). Denote $v_1(x, t) := e^{- (r + \lambda) t} x$, then
\[
\begin{aligned}
\partial_t v_1 - \frac{1}{2} \sigma^2 \partial_{xx} v_1 - (\mu - \frac{1}{2} \lambda L) \partial_x v_1 + \left( r + \frac{1}{2} \lambda \right) v_1 - \frac{1}{2} \lambda |L| \partial_x v_1 - v_1 &= e^{- (r + \lambda) t} \left[ - (r + \lambda) x - \left( \mu - \frac{1}{2} \lambda L \right) x - \frac{1}{2} \lambda |L| L - x \right] \\
&= \begin{cases}
- \mu e^{- (r + \lambda) t} \leq 0, & x \leq L, \\
(-\lambda x - (\mu - \lambda L)) e^{- (r + \lambda) t} \leq 0, & x > L,
\end{cases} \\
v_1(0, t) &= 0, \quad \partial_x v_1(n, t) = e^{- (r + \lambda) t} \leq 1 = \partial_x V_n(n, t), \\
v_1(x, 0) &= x = V_n(x, 0).
\end{aligned}
\]
Applying comparison principle we get
\[
v_1(x, t) \leq V_n(x, t).
\]
Denote $v_2(x, t) := x + \frac{\mu}{r}$, then
\[
\begin{aligned}
\partial_t v_2 - \frac{1}{2} \sigma^2 \partial_{xx} v_2 - (\mu - \frac{1}{2} \lambda L) \partial_x v_2 + \left( r + \frac{1}{2} \lambda \right) v_2 - \frac{1}{2} \lambda |L| \partial_x v_2 - v_2 &= \begin{cases}
- \mu + (r + \lambda)(x + \frac{\mu}{r}) \geq 0, & 0 < x < L - \frac{\mu}{r}, \\
(\mu - \lambda L) + r(x + \frac{\mu}{r}) \geq 0, & x \geq (L - \frac{\mu}{r}) \land 0,
\end{cases} \\
v_2(0, t) &= \frac{\mu}{r} \geq 0, \quad \partial_x v_2(n, t) = 1 = \partial_x V_n(n, t), \\
v_2(x, 0) &= x + \frac{\mu}{r} \geq V_n(x, 0).
\end{aligned}
\]
Applying comparison principle we obtain (3.7).
In order to prove (3.8), we first show

\[ V_n(x, t) \leq k_1(1 - e^{-k_2x}), \quad 0 \leq x \leq 1, \tag{3.9} \]

where \( k_1, k_2 \) are defined in Lemma 3.1. Denote \( v_3(x, t) := k_1(1 - e^{-k_2x}) \), then

\[
\begin{align*}
&v_3(0, t) = 0 = V_n(0, t), \\
&v_3(1, t) = k_1(1 - e^{-k_2}) \geq \frac{\mu}{r} + 1 \geq V_n(1, t), \quad (by \ (3.7)) \\
&v_3(x, 0) = k_1(1 - e^{-k_2x}) \geq x = V_n(x, 0), \quad 0 \leq x \leq 1,
\end{align*}
\]

\[
\partial_t v_3 - \frac{1}{2} \sigma^2 \partial_{xx} v_3 - \left( \mu - \frac{1}{2} \lambda L \right) \partial_x v_3 + \left( r + \frac{1}{2} \lambda \right) v_3 - \frac{1}{2} \lambda |L \partial_x v_3 - v_3| = k_1 \left[ \frac{1}{2} \sigma^2 k_2^2 e^{-k_2x} - \left( \mu - \frac{1}{2} \lambda L \right) k_2 e^{-k_2x} + \left( r + \frac{1}{2} \lambda \right)(1 - e^{-k_2x}) \\
- \frac{1}{2} \lambda |L k_2 e^{-k_2x} + e^{-k_2x} - 1| \right] = \begin{cases} k_1 \left[ e^{-k_2x} \left( \frac{1}{2} \sigma^2 k_2^2 - \mu k_2 - (r + \lambda) \right) + (r + \lambda) \right] \geq 0, & Lk_2 + 1 \geq e^{k_2x}, \\
& k_1 \left[ e^{-k_2x} \left( \frac{1}{2} \sigma^2 k_2^2 - (\mu - \lambda L) k_2 - r \right) + r \right] \geq 0, & Lk_2 + 1 < e^{k_2x}.
\end{cases}
\]

In terms of comparison principle, we obtain (3.9). Hence owing to (3.9), we have

\[
\partial_x V_n(0, t) = \lim_{x \to 0^+} \frac{V_n(x, t) - V_n(0, t)}{x} \leq \lim_{x \to 0^+} \frac{k_1(1 - e^{-k_2x})}{x} = k_1 k_2.
\]

Thanks to the left hand side of (3.7), we also have

\[
\partial_x V_n(0, t) = \lim_{x \to 0^+} \frac{V_n(x, t) - V_n(0, t)}{x} \geq \lim_{x \to 0^+} \frac{e^{-(r+\lambda)t}x}{x} = e^{-(r+\lambda)t}.
\]

Set \( v(x, t) := \partial_x V_n(x, t) \), then \( v(x, t) \) satisfies

\[
\begin{cases}
\partial_t v - \frac{1}{2} \sigma^2 \partial_{xx} v - \left( \mu - \frac{1}{2} \lambda L + \frac{1}{2} \lambda LH(L \partial_x V_n - V_n) \right) \partial_x v \\
\quad + \left( r + \frac{1}{2} \lambda + \frac{1}{2} \lambda H(L \partial_x V_n - V_n) \right) v = 0, \quad (x, t) \in Q_n, \\
e^{-\lambda (r+\lambda)t} \leq v(0, t) \leq k_1 k_2, \quad v(n, t) = 1 \quad t \in (0, T), \\
v(x, 0) = 1, \quad x \in (0, n),
\end{cases}
\]  

\[
\begin{cases}
1, & \xi > 0, \\
0, & \xi = 0, \\
-1, & \xi < 0.
\end{cases}
\]

It is obviously \( k_1 k_2 \) is a supersolution of (3.10), hence \( v(x, t) \leq k_1 k_2 \).
Set \( v_4(x, t) := e^{-(r+\lambda)t} \), then \( v_4(x, t) \) satisfies
\[
\partial_t v_4 - \frac{1}{2} \partial_{xx} v_4 - \left( \mu - \frac{1}{2} \lambda L + \frac{1}{2} \lambda H(L \partial_x V_n - V_n) \right) \partial_x v_4 \\
+ \left( r + \frac{1}{2} \lambda + \frac{1}{2} \lambda H(L \partial_x V_n - V_n) \right) v_4
\]
which, combining with the initial and boundary conditions, applying comparison principle to \( v_4 \) and \( v \) we get the left hand side of (3.8).

**Theorem 3.2.** There exists a solution \( V(x, t) \in C(\overline{Q}) \cap C^{2,1}(\overline{Q}\setminus\{(0,0)\}) \) to the problem (3.5), and \( \partial_x V(x, t) \in C(\overline{Q}) \) with estimates
\[
e^{-(r+\lambda)t} x \leq V(x, t) \leq x + \frac{\mu}{r}, \quad (3.12)
\]
\[
e^{-(r+\lambda)t} \leq \partial_x V(x, t) \leq k_1 k_2, \quad (3.13)
\]
where \( k_1, k_2 \) are defined in Lemma 3.1.

Moreover, the classical solution of problem (3.5) satisfying estimate (3.12) is unique.

**Proof.** In view of (3.7) and (3.8) we have
\[-x - \frac{\mu}{r} \leq L \partial_x V_n - V_n \leq L k_1 k_2,
\]
hence for any \( m < n \)
\[|L \partial_x V_n - V_n|_{L^p(Q_m)} \leq C_m,
\]
where \( C_m \) is independent of \( n \). Thus for any \( p > 2, \)
\[|V_n|_{W^{2,1}_p(Q_m)} \leq C_m, \quad n = m + 1, m + 2, \cdots
\]
where \( C_m \) is also independent of \( n \). Letting \( n \to \infty \), there exists \( V^{(m)}(x, t) \in W^{2,1}_p(Q_m) \) and a subsequence of \( \{V_n\} \) (still denoted by \( \{V_n\} \)), such that
\[V_n(x, t) \rightharpoonup V^{(m)}(x, t) \quad \text{weakly in } W^{2,1}_p(Q_m),
\]
\[\partial_x V_n(x, t) \to \partial_x V^{(m)}(x, t) \quad \text{strongly in } C(\overline{Q}_m).
\]
Define
\[V(x, t) := V^{(m)}(x, t), \quad (x, t) \in Q_m, \quad m = 1, 2, \cdots
\]
then \( V(x, t) \) is well defined on \( Q \) and for each fixed \( n \), \( V(x, t) \in W^{2,1}_p(Q_n) \). Sending \( n \to \infty \) in (3.6), we see that \( V(x, t) \) is a solution of problem (3.5).

By embedding theorem, we know \( V(x, t) \in C^{1+\alpha, (1+\alpha)/2}(\overline{Q}_n) \), and then \( |F V| \in C^{\alpha, \alpha/2}(\overline{Q}_n) \), applying Schauder interior estimate to the equation in (3.5), we have that \( V(x, t) \in C^{2+\alpha, 1+\alpha/2}(\overline{Q}_n \setminus \{(0,0)\}) \). Since \( n \) is arbitrary, so \( V(x, t) \in C^{2,1}(\overline{Q}\setminus\{(0,0)\}) \) with \( V(x, t) \in C^1(\overline{Q}) \). Moreover (3.12) and (3.13) are obtained by letting \( n \to \infty \) in (3.7) and (3.8).
Finally we prove the uniqueness. Suppose that $V_1$, $V_2$ are two classical solutions satisfying (3.12) to the problem (3.5). Then
\[
\begin{aligned}
\partial_t V_1 - LV_1 - |FV_1| &= 0, & (x, t) &\in Q, \\
\partial_t V_2 - LV_2 - |FV_2| &= 0, & (x, t) &\in Q, \\
V_1(x, t) &= V_2(x, t), & (x, t) &\in \partial_p Q,
\end{aligned}
\]
where $\partial_p Q$ is parabolic boundary of $Q$, thus
\[
\begin{aligned}
\partial_t (V_1 - V_2) - L(V_1 - V_2) &= |FV_1| - |FV_2| \leq |FV_1 - FV_2|, & (x, t) &\in Q, \\
(V_1 - V_2)(x, t) &= 0, & (x, t) &\in \partial_p Q,
\end{aligned}
\]
note that
\[
|FV_1 - FV_2| = H(FV_1 - FV_2)(FV_1 - FV_2)
= \frac{1}{2} \lambda H(FV_1 - FV_2)[L\partial_x(V_1 - V_2) - (V_1 - V_2)],
\]
(3.14) implies that $(V_1 - V_2)$ satisfies
\[
\begin{aligned}
\partial_t (V_1 - V_2) - \frac{1}{2}\sigma^2\partial_{xx}(V_1 - V_2) &- (\mu - \frac{1}{2}\lambda L + \frac{1}{2}\lambda LH(\cdot))\partial_x(V_1 - V_2) \\
&+ (r + \frac{1}{2}\lambda + \frac{1}{2}\lambda H(\cdot))(V_1 - V_2) \leq 0, & (x, t) &\in Q, \\
(V_1 - V_2)(x, t) &= 0, & (x, t) &\in \partial_p Q.
\end{aligned}
\]
Since $V_1$ and $V_2$ satisfy estimate (3.12), we can apply the maximum principle [1] to know
\[
V_1(x, t) - V_2(x, t) \leq 0, & (x, t) &\in Q.
\]
Similarly, we can deduce from
\[
\begin{aligned}
\partial_t (V_1 - V_2) - L(V_1 - V_2) &= |FV_1| - |FV_2| \geq -|FV_1 - FV_2|, & (x, t) &\in Q, \\
(V_1 - V_2)(x, t) &= 0, & (x, t) &\in \partial_p Q,
\end{aligned}
\]
that
\[
V_1(x, t) - V_2(x, t) \geq 0, & (x, t) &\in Q.
\]
Hence we proved the uniqueness of the problem.

4. Differentiability of the free boundary. Denote
\[
w(x, t) := L\partial_x V(x, t) - V(x, t).
\]
The equation in (3.5) can be rewritten as
\[
\partial_t V - \frac{1}{2}\sigma^2\partial_{xx}V - \left(\mu - \frac{1}{2}\lambda L\right)\partial_x V + \left(r + \frac{1}{2}\lambda\right)V - \frac{1}{2}\lambda|w| = 0.
\]
Differentiating (4.2) with respect to $x$ yields
\[
\partial_t (\partial_x V) - \frac{1}{2}\sigma^2 \partial_{xx}(\partial_x V) - \left(\mu - \frac{1}{2}\lambda L\right)\partial_x (\partial_x V) + \left(r + \frac{1}{2}\lambda\right)(\partial_x V) - \frac{1}{2}\lambda H(w)\partial_x w = 0,
\]
(4.3)
where $H(\cdot)$ was defined in (3.11).

Multiplying (4.3) by $L$ and abstracting (4.2) from the resulting equation, we obtain
\[
\partial_t w - \frac{1}{2}\sigma^2\partial_{xx}w - \left(\mu - \frac{1}{2}\lambda L + \frac{1}{2}\lambda LH(w)\right)\partial_x w + \left(r + \frac{1}{2}\lambda\right)w + \frac{1}{2}\lambda|w| = 0.
\]
the initial and boundary values of $w$ are
\[ w(x, 0) = L - x, \quad x > 0, \]  
\[ w(0, t) = L\partial_x V(0, t), \quad 0 < t \leq T. \tag{4.6} \]

**Lemma 4.1.** $w(x, t)$ satisfies
\[ w(0, t) > 0, \quad t \in [0, T], \]  
\[ \partial_x w(x, t) < 0, \quad x > 0, \quad t \in [0, T]. \tag{4.8} \]

**Proof.** According to (4.6) and (3.13), we have
\[ w(0, t) = L\partial_x V(0, t) \geq Le^{-(r+\lambda)t} > 0. \]
Moreover, Theorem 2.1 reveals $\partial_{xx} V(x, t) = \partial_{xx} U(x, T-t) \leq 0$, thus
\[ \partial_x w = L\partial_{xx} V - \partial_x V \leq -\partial_x V \leq e^{-(r+\lambda)t} < 0, \]
by (3.13), so (4.8) holds. \hfill \Box

Now we prove main theorem.

**Theorem 4.2.** There exists a function $S(t) \in C[0, T] \cap C^1(0, T]$ with $S(0) = L$, such that
\[ w(S(t), t) = 0, \quad 0 \leq t \leq T, \]
moreover,
\[ w(x, t) > 0, \quad x < S(t), \]  
\[ w(x, t) < 0, \quad x > S(t), \]  
and
\[ 0 < S(t) \leq L. \tag{4.11} \]

**Proof.** From Theorem 3.2 we know that
\[ w = L\partial_x V - V \in C(\overline{Q}), \]  
\[ \partial_x w = L\partial_{xx} V - \partial_x V \in C(\overline{Q}\setminus\{(0, 0)\}). \]
Note that $w(L, 0) = 0$, applying implicit function theorem and (4.8), there exists a continuous function $S(t) \in C[0, T] \cap C^1(0, T]$, such that $S(0) = L$ and
\[ w(S(t), t) = 0, \quad t \in [0, T]. \]
(4.8) means that $w(x, t)$ is strictly monotonic decreasing, so (4.9) and (4.10) hold. $S(t) > 0$ follows from (4.7). On the other hand from (2.19),
\[ \partial_t V - \mathcal{L}_x V = 0, \quad x > L, \quad 0 \leq t < T. \]
Applying (3.4), we see that
\[ \partial_t V - \mathcal{L} V + \mathcal{F} V = 0, \quad x > L, \quad 0 \leq t < T. \]
Comparing this with the equation in (3.5), we have that
\[ \mathcal{F} V < 0, \quad x > L, \quad 0 \leq t < T. \]
From the definition (3.2),
\[ w < 0, \quad x > L, \quad 0 \leq t < T. \]
It means that $S(t) \leq L$. At last we prove $S(t) \in C^1(0, T]$. If fact if $\partial_t w(S(t), t)$ is continuous in $\overline{Q}$, then the implicit function Theorem claims that $S(t)$ is continuous.
differentiable in \((0, T]\). In fact, differentiating (3.1) with respect to \(t\), then \(\partial_t V(x, t)\) satisfies
\[
\partial_t(\partial_t V) - \frac{1}{2} \sigma^2 \partial_{xx}(\partial_t V) - \left(\mu - \frac{1}{2} \lambda L + \frac{1}{2} \lambda L H(w)\right) \partial_x(\partial_t V) \\
+ \left(r + \frac{1}{2} \lambda + \frac{1}{2} \lambda H(w)\right)(\partial_t V) = 0, \quad (x, t) \in Q,
\]
Applying \(W^{p,1}_\text{loc}\) interior estimate, we know \(\partial_t V(x, t) \in W^{2,1}_{p,\text{loc}}(Q)\) for any \(p > 3\), where
\[
W^{2,1}_{p,\text{loc}}(Q) = \left\{ u(x, t) : u, \partial_x u, \partial_{xx} u, \partial_t u \in L^p(Q'), \forall Q' \subset Q \right\}.
\]
And then the embedding theorem tells us that \(\partial_{xx} V(x, t) \in C(Q)\), therefore
\[
\partial_t w(x, t) = (L \partial_{xx} V - \partial_t V)(x, t) \in C(Q).
\]
Hence \(S(t)\) is continuous differentiable and
\[
S'(t) = - \frac{\partial w(S(t), t)}{\partial_x w(S(t), t)}.
\]

5. Verification theorem and optimal strategy. First we present a version of the verification theorem for the value function \(U(x, t)\) defined in (2.3) in this section.

**Theorem 5.1.** Consider a function \(u(x, t) : \mathbb{R} \times [0, T] \to \mathbb{R}\), and \(u(x, t) \in C^{2,1}(Q) \cap C^1(\overline{Q})\) with \(\partial_x u\) is uniformly bounded, satisfying (2.16) and \(u(x, t)\) is concave in \(x > 0\), then \(u(x, t) \geq U(x, t)\).

If, in addition, let \(i^* \in \mathcal{A}_{t,x}\) be the measurable function defined by
\[
i^*_s = \chi_{\{X^*_s > S(T-t)\}} = \chi_{\{\partial_t u + \sigma \partial_{xx} u > \partial_t u + \lambda L u\}}, \tag{5.1}
\]
where \(X^*_s\) be the solution to the following SDE
\[
dX^*_s = (\mu - \lambda L i^*_s)ds + \sigma dW_s - (1 - i^*_s)LdP_s, \quad s > t
\]
with \(X^*_t = x > 0\), we then have
\[
U(x, t) = E_{t,x}[e^{-r(T-t)}X^*_T],
\]
and \(u(x, t) = U(x, t)\).

**Proof.** For any fixed \(x > 0\), an admissible control process \(i_s \in \mathcal{A}_{t,x}\). By the general Itô’s formula, it yields
\[
e^{-r(T-t)}u(X_T, T) = u(x, t) + \sum_{t \leq s < T} e^{-r(T-t)}[u(X_s, s) - u(X_{s-}, s)] \\
+ \int_t^T e^{-r(s-t)}\left[\partial_t u + \frac{1}{2} \sigma^2 \partial_{xx} u + (\mu - \lambda L i_s)\partial_x u - ru\right] ds \\
+ \int_t^T e^{-r(s-t)}\sigma \partial_x u dW_s.
\]
Since \(\partial_x u(x, t)\) is uniformly bounded, the last term in the above formula is a square integrable martingale. Observe that \(\Delta X_s := X_s - X_{s-} = -(1 - i_s)L \Delta P_s\), where \(\Delta P_s = P_s - P_{s-}\), then
\[
u(X_s, s) - u(X_{s-}, s) = u(X_{s-} + \Delta X_s, s) - u(X_{s-}, s).
\]
Thus
\[ E_{tx} \left[ e^{-r(T-t)} u(X_T, T) \right] = u(x, t) + E_{tx} \left[ \int_t^T e^{-r(s-t)} (\mathcal{L}^i u)(X_s, s) ds \right] \\
+ E_{tx} \left[ \sum_{t \leq s < T} e^{-r(s-t)} (u(X_{s-} + \Delta X_s, s) - u(X_{s-}, s)) \right] \]
\[ = u(x, t) + E_{tx} \left[ \int_t^T e^{-r(s-t)} (\mathcal{L}^i u)(X_s, s) ds \right] \\
+ E_{tx} \left[ \int_t^T e^{-r(s-t)} \lambda [u(X_{s-} - (1-i_s)L, s) - u(X_{s-}, s)] ds \right]. \]

where \( \mathcal{L}^i u = \partial_t u + \frac{1}{2} \sigma^2 \partial_{xx} u + (\mu - \lambda L i_s) \partial_x u - ru. \)

Since \( u(x, t) \) is concave on \( x \in (0, +\infty) \) and \( u(x, t) = 0, \ x \in (-\infty, 0] \), then
\[ \max_{i \in [0, 1]} [\lambda u(x - (1 - i)L, t) - i\lambda L \partial_x u(x, t)] \]
\[ \leq \max \left\{ \lambda u(x - L, t), \lambda u(x, t) - \lambda L \partial_x u(x, t) \right\} \]
\[ = \max \left\{ 0, \lambda u(x, t) - \lambda L \partial_x u(x, t) \right\}, \]
by (2.11). Thus
\[ (\mathcal{L}^i u)(X_s, s) + \lambda [u(X_{s-} - (1 - i_s)L, s) - u(X_{s-}, s)] \leq \max \left\{ \partial_t u + \mathcal{L}^1 u, \partial_t u + \mathcal{L}^2 u \right\} = 0. \]

Hence, we have
\[ u(x, t) \geq E_{tx} \left[ e^{-r(T-t)} u(X_T, T) \right] = E_{tx} \left[ e^{-r(T-t)} X_T \right]. \]

By the arbitrary of \( i_s \in \mathcal{A}_{i,x} \), we know \( u(x, t) \geq U(x, t). \)

On the other hand, from the definitions of \( i^*_s \) and \( X^*_s \),
\[ (\mathcal{L}^i u)(X^*_s, s) + \lambda [u(X^*_{s-} - (1 - i^*_s)L, s) - u(X^*_{s-}, s)] = 0, \quad t \leq s \leq T; \]
then
\[ u(x, t) \]
\[ = E_{tx} \left[ e^{-r(T-t)} u(X_T^*, T) \right] \\
- E_{tx} \left\{ \int_t^T e^{-r(s-t)} \left[ (\mathcal{L}^i u)(X^*_s, s) + \lambda [u(X^*_{s-} - (1 - i^*_s)L, s) - u(X^*_{s-}, s)] \right] ds \right\} \\
= E_{tx} \left[ e^{-r(T-t)} X_T^* \right] \leq U(x, t). \]
Hence we obtain \( u(x, t) = E_{tx} \left[ e^{-r(T-t)} X_T^* \right] = U(x, t). \)

Recalling \( U(x, t) = V(x, T - t) \), denote \( f(t) = S(T - t) \), definition (5.1) is equivalent to
\[ i^*_s = X(X^*_s > f(s)), \]
and \( x = f(t) \) is the optimal insurance boundary (Fig. 1). When \( x \) lies on the right hand side of the boundary \( x = f(t) \), the optimal insurance of the Poisson loss is to have full coverage, when \( x \) lies on the left hand side of the boundary, the optimal strategy is to have no coverage at all.
At any given time $t$, if the cash reserves of the firm $X_t$ falls in the region $\{X_t > f(t)\}$, then the firm should fully insure the Poisson risk at once; otherwise, the firm should not insure the Poisson risk.

REFERENCES

[1] A. Friedman, *Partial Differential Equations of Parabolic Type*, Prentice-Hall Inc., Englewood Cliffs, N.J., 1964.

[2] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, New York, 1983.

[3] S. Kamin, L. A. Peletier and J. L. Vazquez, On the Barenblatt equation of elasto-plastic filtration, *Indiana Math. J.*, 40 (1991), 1333–1362.

[4] D. Kelome and A. Swiech, Viscosity solutions of an infinite-dimensional Black-Scholes-Barenblatt equation, *Appl Math Optim.*, 47 (2003), 253–278.

[5] A. Kolesnichenko and G. Shopina, Valuation of Portfolios Under Uncertain Volatility: Black-Scholes Barenblatt Equations and the Static Hedging, Technical report, IDE0739, November 14, 2007.

[6] N. V. Krylov, Boundedly inhomogeneous elliptic and parabolic equations in a domain, *Izv. Akad. Nauk SSSR*, 47 (1983), 75–108.

[7] O. A. Ladyženskaja, V. A. Solonnikov and N. N. Ural’ceva, *Linear and Quasilinear Equations of Parabolic Type*, Translated from the Russian by S. Smith. Translations of Mathematical Monographs, Vol. 23. American Mathematical Society, Providence, R.I., 1967.

[8] G. M. Lieberman, *Second Order Parabolic Differential Equations*, World Scientific, 1996.

[9] H. Pham, *Continuous-time Stochastic Control and Optimization with Financial Applications*, Springer-Verlag, Berlin, 2009.

[10] J. Rochet and S. Villeneuve, Liquidity management and corporate demand for hedging and insurance, *J. Finan. Intermediation*, 20 (2011), 303–323.

[11] S. E. Shreve, *Stochastic Calculus for Finance II*, Springer, 2004.

[12] T. Vargiolu, Existence, Uniqueness and Smoothness for the Black-Scholes-Barenblatt Equation, Technical Report of the Department of Pure and Appl. Math. of the University of Padava, 2001.

Received January 2014; 1st revision February 2014; 2nd revision March 2014.

E-mail address: xchen53@gmail.com
E-mail address: fhyi@scnu.edu.cn