NECESSARY CONDITION FOR OPTIMAL CONTROL OF DOUBLY STOCHASTIC SYSTEMS

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Abstract. The aim of this paper is to establish a necessary condition for optimal stochastic controls where the systems governed by forward-backward doubly stochastic differential equations (FBDSDEs in short). The control constraints need not to be convex. This condition is described by two kinds of new adjoint processes containing two Brownian motions, corresponding to the forward and backward components and a maximum condition on the Hamiltonian. The proof of the main result is based on spike’s variational principle, duality technique and delicate estimates on the state and the adjoint processes with respect to the control variable. An example is provided for illustration.

1. Introduction. The theory of backward stochastic differential equations (BSDEs in short) can be traced back to Bismut [2, 3] who studied linear BSDEs motivated by stochastic control problems. Pardoux and Peng 1990 [26] proved the well-posedness for nonlinear BSDEs. Since then, BSDEs have been extensively studied and used in the areas of applied probability and optimal stochastic controls, particularly in financial engineering (see [19]). In order to provide a probabilistic interpretation for the solutions of a class of semilinear stochastic partial differential equations (SPDEs in short), Pardoux and Peng 1994 [27] introduced the following backward doubly stochastic differential equation (BDSDE in short):

\[
y(t) = \xi + \int_t^T f(s, y(s), z(s)) \, ds + \int_t^T g(s, y(s), z(s)) \, d\overrightarrow{B}(s) - \int_t^T z(s) \, dW(s).
\]

Note that the integral with respect to \(\overrightarrow{B}\) is a “backward Itô integral” and the integral with respect to \(W\) is a standard forward Itô’s integral. These two types of integrals are particular cases of the Itô’s-Skorohod integral. In other words,

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both the forward equation and the backward one are types of BDSDE (1) with different directions of stochastic integrals. So (1) provides a very general framework of backward stochastic systems, which is more extensive than the one in [26]. Eq. (1) can provide a probabilistic interpretation for the solutions of a class of quasilinear SPDEs (see [27]).

As a matter of fact, backward stochastic partial differential equations (BSPDEs) can be encountered in many applications of probability theory and stochastic processes, for instance in the optimal control of processes with incomplete information, as well as an adjoint equation of the Duncan-Mortensen-Zakai filtration equation (for instance, see [12, 15, 33, 40]). Besides, whenever studying non-Markovian control problems via dynamic programming principle (see Englezos and Karatzas [13], Peng [29]), one can naturally derive the backward stochastic Hamilton-Jacobi-Bellman equations. As mentioned before, BDSDEs are related to a class of SPDEs. There are huge literature to discuss the connections between BDSDEs and SPDEs. Buckdahn and Ma [6, 7] established a stochastic viscosity solution theory for SPDEs which can be formualted into a class of generalized BDSDEs; Boufoussi, Casteren and Mrhardy [5] provided a probability representation for the stochastic viscosity solution of SPDEs with nonlinear Neumann boundary conditions; Zhang and Zhao [35] applied the extended Feymann-Kac formula to study the stationary solutions of SPDEs; Ichihara [17] discussed the homogenization problem for SPDEs of Zakai type via the BDSDE theory; Matoussi and Stoica [22] obtained the existence and uniqueness result for the obstacle problem of quasilinear parabolic PDEs by using the BDSDEs. For the infinite dimensional case, BDSPDEs are first introduced and studied in Tang [34] by using the method of stochastic flows, while the generalized solution theory for BDSPDEs is can be found in Qiu and Tang (see [31]).

Naturally, one could not help thinking about the optimal control problems for FBDSDEs. Motivated by this question, we study the necessary condition of optimal control and the stochastic maximum principle for forward backward doubly stochastic optimal control systems. It will lead us to understand the real world much better such as in a derivative security market.

Specifically, in this paper the control problem for the state equation is driven by the following type:

\[
\begin{align*}
\mathrm{d}x(t) &= b(t, x(t), u(t)) \mathrm{d}t + \sigma(t, x(t), u(t)) \mathrm{d}W(t), \\
\mathrm{d}y(t) &= -f((t, x(t), y(t), z(t), u(t))) \mathrm{d}t - g((t, x(t), y(t))) \mathrm{d}\widetilde{B}(t) + z(t) \mathrm{d}W(t), \\
x(0) &= x_0 \in \mathbb{R}^n, \quad y(T) = \Phi(x(T)).
\end{align*}
\]

In Eq. (2), this forward equation is “forward” with respect to a standard stochastic integral \(\mathrm{d}W(t)\), as well as “backward” with respect to a backward stochastic integral \(\mathrm{d}\widetilde{B}(t)\); In other words, both the forward equation and the backward one are types of BDSDEs (2) with different directions of stochastic integrals. So Eq. (2) provides a very general framework of BDSDEs, which is more extensive than the one in Wu [36].

As for the classical control problems, Han, Peng and Wu [14] first established a Pontryagin type maximum principle for the optimal control problems for the state process driven by BDSDEs under convex control domain. Later, Zhang and Shi [42] considered the maximum principle for fully coupled forward-backward doubly stochastic control system, under the assumptions that the diffusion coefficient does not contain the control variable with non-convex control constraints. A similar
work can be seen in Ji, Wei and Zhang [18] and Bahlali, Gherbal [8] for convex control domains (see references for more details). Subsequently, Wang and Liu [38, 39] established the necessary conditions for only backward doubly stochastic control system, via the second-order Taylor expansion under no restriction on the convexity of control domain and the diffusion coefficient does not contain the control variable. However, as the authors claimed in their papers, necessary condition (22) or (26) is not very perfect. In contrast to the classical forward stochastic differential equation, a solution to a backward stochastic differential equation admits a pair of adapted solution, not only a stochastic process. They stated that “This is the reason that $r_1(\cdot)$ and $r_2(\cdot)$ appear in the above. And this causes some difficulty to our problem”. Indeed, to overcome this gap, one must display certain appropriate adjoint equations. Recently, Hu [16] introduces a direct method for treating this similar control problem for FBSDEs, namely considering the second-order terms in the Taylor expansion of the variation for the BSDE, furthermore derives the first and second order variational equations. Inspired a series of work mentioned above, in this paper, we will improve the previous results in the following aspects and owning new feature itself: First, we shall establish two kinds of new adjoint equations, which is crucial to deal with variational inequality; Second, we shall provide a pointwise second order necessary condition for optimal controls, which actually extends the work by Zhang and Shi [42]. A range of interesting phenomena in the framework of doubly stochastic systems will be raised as well.

The rest of this paper is organized as follows: After some preliminaries in the second section, we are devoted the third section to the maximum principle for the optimal control of FBDSDEs with the help of two kinds of new adjoint equations. Then, in Section 4, we verify our main results via an example. Basing on previous results in Section 3, we study the constraint problem for doubly stochastic systems. Some discussions are scheduled on Section 6.

2. Preliminaries and Notations. Let $(\Omega, \mathcal{F}, P)$ be a complete filtered probability space. Let $\{W_t, 0 \leq t \leq T\}$ and $\{\widetilde{B}_t, 0 \leq t \leq T\}$, for a fixed time $T > 0$, be two mutually independent standard Brownian motions taking values in $\mathbb{R}^d$, respectively. Denote by $\mathcal{N}$ the class of $P$-null sets of $\mathcal{F}$. Define $\mathcal{F}_t^X := \mathcal{F}_t^W \vee \mathcal{F}_t^B$ for $t \in [0, T]$, where $\mathcal{F}_s^X := \sigma\{X(r) - X(s), s \leq r \leq t\} \vee \mathcal{N}$ for any stochastic process $\{X(t)\}$, and $\mathcal{F}_t^X := \mathcal{F}_{0,t}^X$. Note that the collection $\{\mathcal{F}_t, t \in [0, T]\}$ is neither increasing nor decreasing, and it does not constitute a classical filtration. $\top$ appearing in this paper as superscript denotes the transpose of a matrix.

In FBDSDEs (2), there exist two independent Brownian motions $W(t)$ and $\widetilde{B}(t)$, where the $dW$-integral is a forward Itô’s integral and the $d\widetilde{B}$-integral is a backward Itô’s integral. The extra noise $\widetilde{B}$ in the equation can be thought of as some extra information that cannot be detected in practice, such as in a derivative security market, but is valuable to the partial investor.

We now introduce the following spaces of processes:

$$\mathcal{S}^2(0, T; \mathbb{R}) \triangleq \left\{ \text{\emph{R}}^n\text{-valued \emph{F}_t measurable } \phi(t); \mathbb{E}\left[ \sup_{0 \leq t \leq T} |\phi(t)|^2 \right] < \infty \right\},$$

$$\mathcal{M}^2(0, T; \mathbb{R}) \triangleq \left\{ \text{\emph{R}}^n\text{-valued \emph{F}_t measurable } \varphi(t); \mathbb{E}\left[ \int_0^T |\varphi(t)|^2 \, dt \right] < \infty \right\},$$

and denote
The performance functional is assumed as
\[ J(u(\cdot)) = \mathbb{E} [ h(y(0))] . \]

The objective of our problem is to minimize \( J(u(\cdot)) \) by finding an optimal control \( \bar{u} \) such that
\[ J(\bar{u}(\cdot)) = \inf_{u \in \mathcal{U}(0,T)} J(u(\cdot)). \]
(A1): The coefficients $b : [0, T] \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ and $\sigma : [0, T] \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^{n \times d}$ are twice continuously differentiable with respect to $x$; $b_x, b_{xx}, \sigma_x, \sigma_{xx}$ are continuous in $(x, u)$; $b_x, b_{xx}, \sigma_x, \sigma_{xx}$ are bounded $b, \sigma$ are bounded by $C(1 + |x| + |u|)$ for some positive constant $C$.

(A2): The coefficients $f : [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}$, $g : [0, T] \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, and $\Phi : \mathbb{R}^n \to \mathbb{R}$ are twice continuously differentiable with respect to $(x, y, z)$. $f, Df, D^2f, g, Dg, D^2g$ are continuous in $(x, y, z, u)$. $Df, D^2f, Dg, D^2g, \Phi_{xx}$ are bounded. $f, g$ are bounded by $C(1 + |x| + |y| + |z| + |u|)$, where $Df(Dg)$ is the gradient of $f(g)$ with respect to $(x, y, z)$, $D^2f$ is the Hessian matrix of $f$ with respect to $(x, y, z)$.

(A3): For any $(t, x, y, z) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}$, $u_1, u_2 \in U$, there exists a constant $C > 0$ such that

$$
|b(t, x, u_1) - b(t, x, u_2)|^2 + |\sigma(t, x, u_1) - \sigma(t, x, u_2)|^2 \\
+ |f(t, x, y, z, u_1) - f(t, x, y, z, u_2)|^2 \\
\leq C |u_1 - u_2|^2.
$$

Given any $u(\cdot) \in U(0, T)$, by Theorem 1.1 in [27], there exists a unique triple solution

$$(x(\cdot), y(\cdot), z(\cdot)) \in \mathcal{S}^2([0, T]; \mathbb{R}^n) \times \mathcal{S}^2([0, T]; \mathbb{R}) \times \mathcal{M}^2([0, T]; \mathbb{R}^d)$$

which solves FBDSDEs (2).

We need a further assumption as follows:

(A4): $b$ is continuously differentiable with respect to $y$. They and all their derivatives are bounded.

Denote $\bar{u}(\cdot)$ by the optimal control for the cost function defined in (4) and let $\bar{x}(\cdot)$ be the corresponding solution of the first equation of (2). A argument for deriving the maximum principle is with the help of variational principle. When $U$ is not convex, we use the spike variation method. More precisely, let $\varepsilon > 0$ and $E_\varepsilon \subset [0, T]$ be a Borel set with Borel measure $|E_\varepsilon| = \varepsilon$, define

$$
u^\varepsilon(t) = \bar{u}(t) I_{E_\varepsilon}(t) + u(t) I_{E_\varepsilon^c}(t),
$$

where $u \in U(0, T)$. Here $u^\varepsilon$ is called a spike variation of the optimal control $\bar{u}$. For deriving the maximum principle, we only need to use $E_\varepsilon = [s, s + \varepsilon]$ for $s \in [0, T - \varepsilon]$ and $\varepsilon > 0$. Similarly, we define $(x^\varepsilon(\cdot), u^\varepsilon(\cdot))$ for driven control $u^\varepsilon(\cdot)$.

We put

$$
b(t) = b(t, \bar{x}(t), \bar{u}(t)), \quad \delta b(t) = b(t, \bar{x}(t), u) - b(t, \bar{x}(t), \bar{u}(t)),
$$
similar for $b_x(\cdot), b_{xx}(\cdot)$, $\delta b_x(\cdot), \delta b_{xx}(\cdot), \sigma(\cdot), \sigma_x(\cdot), \sigma_{xx}(\cdot), \delta\sigma(\cdot), \delta\sigma_x(\cdot), \delta\sigma_{xx}(\cdot)$. We now introduce the following variational equations taken from Peng [28]:

\[
\begin{cases}
\quad dx_1(t) = b_x(t) x_1(t) \, dt + [\sigma_x(t) x_1(t) + \delta\sigma(t) I_{E_\varepsilon}(t)] \, dW(t), \\
\quad x_1(0) = 0,
\end{cases}
\]

and

\[
\begin{cases}
\quad dx_2(t) = \left[ b_x(t) x_2(t) + \delta b(t) I_{E_\varepsilon}(t) + \frac{1}{2} b_{xx}(t) x_1(t) x_1(t) \right] \, dt \\
\quad + [\sigma_x(t) x_2(t) + \delta\sigma(t) x_1(t) I_{E_\varepsilon}(t)] \, dW(t), \\
\quad x_2(0) = 0,
\end{cases}
\]
where
\[ b_{xx}(t)x_1(t)x_1(t) = \left(\text{tr} \left[ b^1_{xx}(t)x_1(t)x_1(t)^\top\right], \ldots, \text{tr} \left[ b^n_{xx}(t)x_1(t)x_1(t)^\top\right]\right)^\top, \]
the same to \( \sigma_{xx}(t)x_1(t)x_1(t) \).

**Lemma 2.2.** Under the assumption (A1), we have, for any \( \beta \geq 1 \),
\[
\mathbb{E} \left[ \sup_{t \in [0,T]} |x^\varepsilon(t) - \bar{x}(t)|^{2\beta} \right] = O(\varepsilon^\beta),
\]
\[
\mathbb{E} \left[ \sup_{t \in [0,T]} |x_1(t)|^{2\beta} \right] = O(\varepsilon^\beta),
\]
\[
\mathbb{E} \left[ \sup_{t \in [0,T]} |x_2(t)|^{2\beta} \right] = O(\varepsilon^{2\beta}),
\]
\[
\mathbb{E} \left[ \sup_{t \in [0,T]} |x^\varepsilon(t) - \bar{x}(t) - x_1(t)|^{2\beta} \right] = O(\varepsilon^{2\beta}),
\]
\[
\mathbb{E} \left[ \sup_{t \in [0,T]} |x^\varepsilon(t) - \bar{x}(t) - x_1(t) - x_2(t)|^{2\beta} \right] = o(\varepsilon^{2\beta}).
\]
Furthermore, for any \( \varphi \in C^2(\mathbb{R}^n) \) such that \( \varphi_{xx} \) is bounded, we have
\[
\mathbb{E}[\varphi(x^\varepsilon(T))] - \mathbb{E}[\varphi(\bar{x}(T))] = \mathbb{E}[\langle \varphi_x(\bar{x}(T)), x_1(T) + x_2(T) \rangle] + \frac{1}{2}\mathbb{E}[\langle \varphi_{xx}(\bar{x}(T))x_1(T), x_1(T) \rangle] + o(\varepsilon).
\]
The proof can be seen in Peng [28]. Particularly, one may regard \( x_1(t) = O(\sqrt{\varepsilon}) \), \( x_2(t) = O(\varepsilon) \).

**Lemma 2.3.** Under the assumptions (A1)-(A3), FBDSDEs (2) admit a unique triple solution. Let \( (y^i, z^i), i = 1, 2, \) be the solution to the following
\[
y^i(t) = \xi^i + \int_t^T f^i(s, y^i(s), z^i(s)) \, ds + \int_t^T g^i(s, y^i(s)) \, dB(s)
- \int_t^T z^i(s) \, dW(s),
\]
where \( \mathbb{E} \left[ |\xi^i|^{\beta} \right] < \infty, f^i(s, y^i, z^i) \) satisfies the conditions (A2)-(A3), and
\[
\mathbb{E} \left[ \left( \int_t^T |f^i(s, y^i(s), z^i(s))|^{\beta} \, ds \right)^\frac{\beta}{\beta} \right] < \infty;
\]
\( g^i(s, y^i, z^i) \) satisfies the conditions (A2)-(A3), and
\[
\mathbb{E} \left[ \left( \int_t^T |g^i(s, y^i(s))|^{2} \, ds \right)^\frac{4}{2} \right] < \infty.
Then, for some $\beta \geq 2$, there exists a positive constant $C_\beta$ such that

$$
\mathbb{E} \left[ \sup_{0 \leq t \leq T} |y^1(t) - y^2(t)|^\beta + \left( \int_0^T |z^1(s) - z^2(s)|^2 \, ds \right)^{\frac{2}{\beta}} \right] 
\leq C_\beta \mathbb{E} \left[ |\xi^1 - \xi^2|^\beta + \left( \int_0^T |f^1(s, y^1(s), z^1(s)) - f^2(s, y^2(s), z^2(s))|^2 \, ds \right)^{\frac{2}{\beta}} \right]
\quad + \left( \int_0^T |g^1(s, y^1(s)) - g^2(s, y^2(s))|^2 \, ds \right)^{\frac{2}{\beta}}.
$$

Particularly, whenever putting $\xi^2 = 0$, $f^2 = 0$, and $g^2 = 0$, one has

$$
\mathbb{E} \left[ \sup_{0 \leq t \leq T} |y^1(t)|^\beta + \left( \int_0^T |z^1(s)|^2 \, ds \right)^{\frac{2}{\beta}} \right] 
\leq C_\beta \mathbb{E} \left[ |\xi^1|^\beta + \left( \int_0^T |f^1(s, 0, 0)| \, ds \right)^{\frac{2}{\beta}} \right]
\quad + \left( \int_0^T |g^1(s, 0)|^2 \, ds \right)^{\frac{2}{\beta}}.
$$

The proof can be seen in Aman [1].

3. Optimal control of FBDSDEs. In this section, we study the optimal control for systems driven by Eq. (2) under the types of Pontryagin, namely, necessary maximum principles for optimal control.

We shall introduce the so called variational equations for FBDSDE (2) beginning from the following two adjoint equations:

$$
\left\{ \begin{array}{lcl}
-dp(t) &=& \Gamma_1(t) \, dt - \sum_{j=1}^d q^j(t) \, dW^j(t) + \sum_{j=1}^d \Gamma_2^j(t) \, d\hat{B}^j(t), \\
p(T) &=& \Phi_x(\bar{x}(T)),
\end{array} \right.
$$

and

$$
\left\{ \begin{array}{lcl}
-dP(t) &=& \Pi_1(t) \, dt - \sum_{j=1}^d Q^j(t) \, dW^j(t) + \sum_{j=1}^d \Pi_2^j(t) \, d\hat{B}^j(t), \\
P(T) &=& \Phi_{xx}(\bar{x}(T)),
\end{array} \right.
$$

where $\Gamma_1(\cdot)$, $\Pi_1(\cdot)$, $\Gamma_2(\cdot)$, and $\Pi_2(\cdot)$ are unknown four processes to be determined.

By virtue of Itô’s formula to $p^\top(t) (x_1(t) + x_2(t)) + \frac{1}{2} x_1^\top(t) P(t) x_1(t)$ on $[0, T]$, it follows that

$$
p^\top(T) (x_1(T) + x_2(T)) + \frac{1}{2} x_1^\top(T) P(T) x_1(T)
= p^\top(t) (x_1(t) + x_2(t)) + \frac{1}{2} x_1^\top(t) P(t) x_1(t)
\quad + \int_t^T \left[ \Lambda_1(s) I_{E_1}(s) + \Lambda_2^\top(s) (x_1(s) + x_2(s)) + \frac{1}{2} x_1^\top(s) \Lambda_3(s) x_1(s) \\
+ \Lambda_1^\top(s) x_1(s) I_{E_1}(s) \right] ds
\quad + \int_t^T \left[ p^\top(s) \delta \sigma(s) I_{E_1}(s) + (x_1(s) + x_2(s))^\top \Lambda_5(s) + \frac{1}{2} x_1^\top(s) \Lambda_6(s) x_1(s) \\
+ x_1^\top(s) \Lambda_7(s) I_{E_1}(s) \right] dW(s)
$$
\[ - \int_t^T \left[ (x_1(s) + x_2(s))^\top \Gamma_2(t) + \frac{1}{2} x_1^\top(s) \Pi_2(s) x_1(s) \right] \, d\hat{B}(s), \]

where

\[
\begin{align*}
\Lambda_1(s) & = p^\top(s) \delta b(s) + \sum_{j=1}^d \left[ (q_j^i(s))^\top \delta \sigma^j(s) + \frac{1}{2} (\delta \sigma(s))^\top P(s) \delta \sigma(s) \right], \\
\Lambda_2(s) & = b_x p(s) + \sigma_x(s) q(s) - \Gamma_1(s), \\
\Lambda_3(s) & = \left[ b_{xx}^1(s) p(s), \ldots, b_{xx}^n(s) p(s) \right] + \sum_{j=1}^d \left[ (\sigma_{xx}^j(s))^\top q^j(s) + 2P(s) b_x(s) \right. \\
& \quad + Q^j(t) \sigma_x^j(t) + (\sigma_x^j(t))^\top Q^j(t) + (\sigma_x^j(s))^\top P(s) \sigma_x^j(s) \bigg] - \Pi_1(s), \\
\Lambda_4(s) & = \sum_{j=1}^d \left[ (\delta \sigma_x^j(s))^\top q^j(s) + Q^j(s) \delta \sigma^j(s) + P(s) \sigma_x^j(s) \delta \sigma^j(s) \right], \\
\Lambda_5(s) & = \sum_{j=1}^d (\sigma_x^j(s))^\top p^j(s) + q(s), \\
\Lambda_6(s) & = \left[ \sum_{i=1}^n (\sigma_{xx}^{i1}(s))^\top p^i(s), \ldots, \sum_{i=1}^n (\sigma_{xx}^{id}(s))^\top p^i(s) \right] \\
& \quad + 2 \left[ P(s) \sigma_x^1(s), \ldots, P(s) \sigma_x^d(s) \bigg] + Q(s), \\
\Lambda_7(s) & = [\delta \sigma_x^1(s) p(s), \ldots, \delta \sigma_x^d(s) p(s)] + P(s) \delta \sigma(s).
\end{align*}
\]

Define

\[
\begin{cases}
\frac{d\gamma(t)}{dt} = -f(left(t, x^\varepsilon(t), y^\varepsilon(t), z^\varepsilon(t), w^\varepsilon(t)) \right) \, dt \\
\gamma(T) = \Phi(x^\varepsilon(T)).
\end{cases}
\]

Let

\[
\begin{align*}
\bar{y}^\varepsilon(t) & = y^\varepsilon(t) - \left[ p^\top(t) (x_1(t) + x_2(t)) + \frac{1}{2} x_1^\top(t) P(t) x_1(t) \right], \\
\bar{z}^\varepsilon(t) & = z^\varepsilon(t) - \left[ p(s) \delta \sigma(s) I_{E_x}(s) + (x_1(s) + x_2(s))^\top \Lambda_5(s) \\
& \quad + \frac{1}{2} x_1^\top(s) \Lambda_6(s) x_1(s) + x_1^\top(s) \Lambda_7(s) I_{E_x}(s) \right].
\end{align*}
\]

After some tedious computations, we have

\[
\begin{align*}
\bar{y}^\varepsilon(t) & = \Phi(\bar{x}(T)) + \int_t^T \left[ f(s, x^\varepsilon(s), y^\varepsilon(s), z^\varepsilon(s), w^\varepsilon(s)) \right. \\
& \quad + \Lambda_1(s) I_{E_x}(s) + \Lambda_2^2(s) (x_1(s) + x_2(s)) + \frac{1}{2} x_1^\top(s) \Lambda_3(s) x_1(s) \bigg] ds \\
& \quad + \int_t^T \left[ g(s, x^\varepsilon(s), y^\varepsilon(s)) - (x_1(s) + x_2(s))^\top \Gamma_2(s) \\
& \quad - \frac{1}{2} x_1^\top(s) \Pi_2(s) x_1(s) \right] \, d\hat{B}(s) - \int_t^T \bar{z}^\varepsilon(s) dW(s) + o(\varepsilon).
\end{align*}
\]
Put \( \hat{y}^\varepsilon (t) = \bar{y}^\varepsilon (t) - \bar{y} (t) \), \( \hat{z}^\varepsilon (t) = \bar{z}^\varepsilon (t) - \bar{z} (t) \), then we attain

\[
\hat{y}^\varepsilon (t) = \int_t^T \left[ f (s, x^\varepsilon (s), y^\varepsilon (s), z^\varepsilon (s), u^\varepsilon (s)) - f (s, \bar{x} (s), \bar{y} (s), \bar{z} (s), \bar{u} (s)) \right] ds + \Lambda_1 (s) I_{E_\varepsilon} (s) + \Lambda_2^\top (s) (x_1 (s) + x_2 (s)) + \frac{1}{2} x_1^\top (s) \Lambda_3 (s) x_1 (s) \right] ds
+ \int_t^T \left[ g (s, x^\varepsilon (s), y^\varepsilon (s)) - g (s, \bar{x} (s), \bar{y} (s)) \right. \\
- (x_1 (s) + x_2 (s))^\top \Gamma_2 (s) - \frac{1}{2} x_1^\top (s) \Pi_2 (s) x_1 (s) \right] d \bar{B} (s)
- \int_t^T \hat{z}^\varepsilon (s) \, d W (s) + o (\varepsilon).
\]

Next, we deal with

\[
\int_t^T \left[ f (s, x^\varepsilon (s), y^\varepsilon (s), z^\varepsilon (s), u^\varepsilon (s)) - f (s, \bar{x} (s), \bar{y} (s), \bar{z} (s), \bar{u} (s)) \right] ds
= \int_t^T \left[ f (s, \bar{x} (s), \bar{y} (s), \bar{z} (s) + p (s) \delta \sigma (s), u (s)) \right. \\
- f (s, \bar{x} (s), \bar{y} (s), \bar{z} (s), \bar{u} (s)) I_{E_\varepsilon} (s) + f (s, x^\varepsilon (s), y^\varepsilon (s), z^\varepsilon (s), u^\varepsilon (s)) \right] ds
+ \left. \int_t^T \left[ f (s, \bar{x} (s), \bar{y} (s), \bar{z} (s) + p (s) \delta \sigma (s) I_{E_\varepsilon} (s), u^\varepsilon (s)) \right] ds + o (\varepsilon), \right.
\]

where

\[
\Lambda_8 (s) = p^\top (t) (x_1 (t) + x_2 (t)) + \frac{1}{2} x_1^\top (t) P (t) x_1 (t),
\]

\[
\Lambda_9 (s) = (x_1 (s) + x_2 (s))^\top \Lambda_5 (s) + \frac{1}{2} x_1^\top (s) \Lambda_6 (s) x_1 (s) + x_1^\top (s) \Lambda_7 (s) I_{E_\varepsilon} (s).
\]

Indeed,

\[
\mathbb{E} \left( \int_t^T \left[ f (s, x^\varepsilon (s), y^\varepsilon (s), z^\varepsilon (s), u^\varepsilon (s)) \right. \\
- f (s, \bar{x} (s) + x_1 (s) + x_2 (s), \bar{y} (s) + \hat{y}^\varepsilon (s) + \Lambda_8 (s), \bar{z} (s) + \hat{z}^\varepsilon (s) + \Lambda_9 (s), \bar{u} (s)) \right. \\
+ f (s, \bar{x} (s), \bar{y} (s), \bar{z} (s)+ p (s) \delta \sigma (s) I_{E_\varepsilon} (s), u^\varepsilon (s)) \right. \\
- f (s, \bar{x} (s), \bar{y} (s), \bar{z} (s), \bar{u} (s)) \right] ds \right)^2
= C \left( \mathbb{E} \int_t^T | x^\varepsilon (s) - \bar{x} (s) + x_1 (s) + x_2 (s) | ds \right)^2
\]
\[ + C \left( \mathbb{E} \int_t^T |p(s) \delta(s) J_{E_{g}}(s)| \, ds \right)^2 + C \left( \mathbb{E} \int_t^T |u^g(s) - \bar{u}(s)| \, ds \right)^2 \]

\[ = o(\varepsilon^2) \, . \]

For \( g \), together with Lemma 2.2, we have

\[ g(s, x^\tau (s), y^\tau (s)) - g(s, \bar{x}(s), \bar{y}(s)) \]

\[ = g(s, \bar{x}(s) + x_1(s) + x_2(s), \bar{y}(s) + \bar{y}^\tau (s) + \Lambda_8(s)) - g(s, \bar{x}(s), \bar{y}(s)) + o(\varepsilon) \, . \]

Next we will find \( \Gamma_1, \Gamma_2, \Pi_1, \Pi_2 \), which are determined by the optimal quadruple \((\bar{x}(\cdot), \bar{y}(\cdot), \bar{z}(\cdot), \bar{u}(\cdot))\), such that

\[ f(s, \bar{x}(s) + x_1(s) + x_2(s), \bar{y}(s) + \Lambda_8(s), \bar{z}(s) + \Lambda_9(s), \bar{u}(s)) \]

\[ - f(s, \bar{x}(s), \bar{y}(s), \bar{z}(s), \bar{u}(s)) + \Lambda_2^T(s) (x_1(s) + x_2(s)) + \frac{1}{2} x_1^T(s) \Lambda_3(s) x_1(s) \]

\[ = o(\varepsilon) \, , \]

in which \( o(\varepsilon) \) does not involve the terms \( x_1(\cdot) \) and \( x_2(\cdot) \). Note that in BDSDE (10), there appears the term \( \frac{1}{2} x_1^T(s) \Lambda_3(s) x_1(s) \). Hence, we make use of Taylor’s expansion to

\[ f(s, \bar{x}(s) + x_1(s) + x_2(s), \bar{y}(s) + \Lambda_8(s), \bar{z}(s) + \Lambda_9(s), \bar{u}(s)) \]

\[ - f(s, \bar{x}(s), \bar{y}(s), \bar{z}(s), \bar{u}(s)) \]

and

\[ g(s, \bar{x}(s) + x_1(s) + x_2(s), \bar{y}(s) + \Lambda_8(s)) - g(s, \bar{x}(s), \bar{y}(s)) \, , \]

respectively, and then obtain

\[ \Gamma_1(t) = f_y(t) p(t) + \sum_{j=1}^d f_{x_j}(t) \sigma_x^j(t) \top p(t) + b_x(t) p(t) + g(t) f_x(t) \]

\[ + \sum_{j=1}^d (\sigma_x^j(t) \top q_j(t) + f_x(t) \, , \]

\[ \Gamma_2(t) = g_y(t) p(t) + g_x(t) \, , \]

\[ \Pi_1(t) = P(t) b_x(t) + (b_x(t)) \top P(t) + f_y(t) P(t) + \sum_{j=1}^d \left[ Q_j(t) \sigma_x^j(t) \right. \]

\[ + (\sigma_x^j(t) \top Q_j(t) + (\sigma_x^j(t) \top q_j(t) + (\sigma_x^j(t) \top P(t) \sigma_x^j(t) \]

\[ + \sum_{j=1}^d \left[ f_{x_j}(t) P(t) \sigma_x^j(t) + f_{x_j}(t) \sigma_x^j(t) \top P(t) + f_{x_j} Q_j(t) \right. \]

\[ + f_{x_j}(t) \sigma_x^j(t) \top p(t) \right) \]

\[ + (b_{xx}(t)) \top p(t) \]
Proof. To get (14) and (15), we reformulate (11) as follows:

\[
\Pi_2(t) = \left[ I_{n \times n}, p(t) \cdot (\sigma_x(t))^\top p(t) + q(t) \right] \cdot D^2f(t) \cdot \left[ I_{n \times n}, p(t) \right]^\top + g_y(t) P(t). \tag{12}
\]

Now consider the following BDSDE:

\[
\hat{\gamma}(t) = \int_t^T \left[ f_y(s) \hat{\gamma}(s) + \sum_{j=1}^d f_{x_j}(s) \hat{\beta}(s) + \begin{pmatrix} p^T(s) \delta b(s) \\
+ \sum_{j=1}^d \left[ (q^j(s))^T \delta \sigma^j(s) + \frac{1}{2} (\delta \sigma^j(s))^T P(s) \delta \sigma^j(s) \right] \\
+ f(s, \bar{x}(s), \hat{\gamma}(s), \hat{\beta}(s) + p^T(s) \delta \sigma(s), u) \\
- f(s, \bar{x}(s), y(s), \bar{\beta}(s), \bar{u}(s)) \right] \delta \sigma^j(s) ds \\
+ \sum_{j=1}^d \int_t^T g_y^j(s) \hat{\gamma}(s) d\hat{B}(s) - \sum_{j=1}^d \int_t^T \hat{\beta}(s) dW^j(s). \tag{13}
\]

**Lemma 3.1.** Assume that assumptions (A1)-(A3) are in force, we have the following estimates:

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} |\hat{\gamma}(t)|^2 + \int_0^T |\hat{\beta}(t)|^2 ds \right] = O(\varepsilon^2), \tag{14}
\]

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} |\hat{\gamma}(t)|^2 + \int_0^T |\hat{\beta}(s)|^2 ds \right] = O(\varepsilon^2), \tag{15}
\]

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} |\hat{\gamma}(t) - \hat{\gamma}(t)|^2 + \int_0^T |\hat{\beta}(s) - \hat{\beta}(s)|^2 ds \right] = o(\varepsilon^2). \tag{16}
\]

**Proof.** To get (14) and (15), we reformulate (11) as follows:

\[
\hat{\gamma}(t) = \int_t^T \left[ A_1(s) I_E(s) + \mathcal{L}_1(s) + \hat{f}_y(s) \hat{\gamma}(s) + \hat{f}_\beta(s) \hat{\beta}(s) \\
+ \frac{1}{2} [x_1(s) + x_2(s), A_8(s), A_9(s)] \hat{D} f(s) \cdot \\
[ x_1(s) + x_2(s), A_8(s), A_9(s) ]^T \\
+ \mathcal{L}_2(s) x_1(s) I_E(s) - \frac{1}{2} x_1(s) \mathcal{L}_3(s) D^2 f(s) (x_1(s) \mathcal{L}_3(s))^T ds \\
+ \int_t^T \left[ \mathcal{L}_1(s) + \mathcal{L}_2(s) \hat{\gamma}(s) \\
+ \frac{1}{2} [x_1(s) + x_2(s), A_8(s) + A_9(s)] \hat{D}^2 g(s) [ x_1(s) + x_2(s), A_8(s) ]^T \\
- \frac{1}{2} x_1(s) \mathcal{L}_4(s) D^2 g(s) (x_1(s) \mathcal{L}_4(s))^T \right] d\hat{B}(s)
\]
\[
- \int_t^T \hat{z}^\varepsilon (s) dW(s) + o(\varepsilon),
\]
where
\[
\begin{align*}
\mathcal{I}_1 (s) &= f (s, x^\varepsilon (s), y^\varepsilon (s), z^\varepsilon (s), u^\varepsilon (s)) \\
&\quad - f (s, \bar{x} (s) + x_1 (s) + x_2 (s), \bar{y}^\varepsilon (s) + \bar{z}^\varepsilon (s) + \bar{\lambda}_8 (s), \bar{u} (s)), \\
\tilde{D}^2 f (s) &= 2 \int_t^T \int_t^T \lambda \tilde{D} f(s, \bar{x} (s) + \lambda \mu (x_1 (s) + x_2 (s)), \bar{y} + \lambda \mu \bar{\lambda}_8 (s) , \\
&\quad \bar{z} (s) + \lambda \mu \bar{\lambda}_9 (s), \bar{u} (s)) ds, \\
\tilde{D}^2 g (s) &= 2 \int_t^T \int_t^T \lambda \tilde{D} g(s, \bar{x} (s) + \lambda \mu (x_1 (s) + x_2 (s)), \bar{y} (s) + \lambda \mu \bar{\lambda}_8 (s) ds
\end{align*}
\]
whilst
\[
\begin{align*}
f (s, \bar{x} (s) + x_1 (s) + x_2 (s), \bar{y}^\varepsilon (s) + \bar{z}^\varepsilon (s) + \bar{\lambda}_8 (s), \bar{u} (s)) \\
&\quad - f (s, \bar{x} (s) + x_1 (s) + x_2 (s), \bar{y} (s) + \bar{\lambda}_8 (s), \bar{z} (s) + \lambda \mu \bar{\lambda}_9 (s), \bar{u} (s)) \\
&\quad = f (s, \bar{x} (s) + x_1 (s) + x_2 (s), \bar{y}^\varepsilon (s) + \bar{\lambda}_8 (s), \bar{z}^\varepsilon (s) + \bar{\lambda}_9 (s), \bar{u} (s)) \\
&\quad - f (s, \bar{x} (s) + x_1 (s) + x_2 (s), \bar{y} (s) + \bar{\lambda}_8 (s), \bar{z}^\varepsilon (s) + \lambda \mu \bar{\lambda}_9 (s), \bar{u} (s)) \\
&\quad + f (s, \bar{x} (s) + x_1 (s) + x_2 (s), \bar{y} (s) + \bar{\lambda}_8 (s), \bar{z}^\varepsilon (s) + \lambda \mu \bar{\lambda}_9 (s), \bar{u} (s)) \\
&\quad - f (s, \bar{x} (s) + x_1 (s) + x_2 (s), \bar{y} (s) + \bar{\lambda}_8 (s), \bar{z} (s) + \lambda \mu \bar{\lambda}_9 (s), \bar{u} (s)) \\
&\quad = \tilde{f}_y (s) \bar{y}^\varepsilon (s) + \tilde{f}_z (s) \bar{z}^\varepsilon (s),
\end{align*}
\]
where
\[
\begin{align*}
\tilde{f}_y (s) &= \int_0^1 f_y (s, \bar{x} (s) + x_1 (s) + x_2 (s), \alpha (\bar{y}^\varepsilon (s) + \bar{\lambda}_8 (s)) \\
&\quad + (1 - \alpha) (\bar{y} (s) + \bar{\lambda}_8 (s)), \bar{z}^\varepsilon (s) + \bar{\lambda}_9 (s), \bar{y} (s)) ds, \\
\tilde{f}_z (s) &= \int_0^1 f_z (s, \bar{x} (s) + x_1 (s) + x_2 (s), \bar{y}^\varepsilon (s) + \bar{\lambda}_8 (s), \alpha (\bar{z}^\varepsilon (s) + \bar{\lambda}_9 (s)) \\
&\quad + (1 - \alpha) (\bar{z} (s) + \bar{\lambda}_9 (s)), \bar{y} (s)) ds.
\end{align*}
\]
Besides
\[
\begin{align*}
f (s, \bar{x} (s) + x_1 (s) + x_2 (s), \bar{y} (s) + \bar{\lambda}_8 (s), \bar{z} (s) + \bar{\lambda}_9 (s), \bar{u} (s)) \\
&\quad - f (s, \bar{x} (s), \bar{y} (s), \bar{z} (s), \bar{u} (s)) \\
&\quad = f_x (s) [x_1 (s) + x_2 (s)] + f_y (s) \bar{\lambda}_8 (s) + f_z (s) \bar{\lambda}_9 (s) \\
&\quad + \frac{1}{2} [x_1 (s) + x_2 (s), \bar{\lambda}_8 (s), \bar{\lambda}_9 (s)] \tilde{D}^2 (s) f (s) [x_1 (s) + x_2 (s), \bar{\lambda}_8 (s), \bar{\lambda}_9 (s)]^T,
\end{align*}
\]
where
\[
\begin{align*}
\tilde{D}^2 (s) f (s) &= 2 \int_0^1 \int_0^1 \lambda \tilde{D} f(s, \bar{x} (s) + \lambda \mu (x_1 (s) + x_2 (s)), \bar{y} (s) + \lambda \mu \bar{\lambda}_8, \\
&\quad \bar{z} (s) + \lambda \mu \bar{\lambda}_9 (s), \bar{u} (s)) d\lambda d\mu \\
\mathcal{I}_2 (s) &= f_z (s) [p (s) \delta \sigma_x (s) + P (s) \delta \sigma (s)], \\
\mathcal{I}_3 (s) &= \left[1, p (s), \sigma_x (s) p (s) + q (s)\right], \\
\mathcal{I}_4 (s) &= \left[1, p (s)\right].
\end{align*}
\]
For \( g \), we have
\[
\mathcal{L}_1 (s) = g (s, x^\varepsilon (s), y^\varepsilon (s)) - g (s, \bar{x} (s) + x_1 (s) + x_2 (s), \bar{y}^\varepsilon (s) + \bar{\lambda}_8 (s)),
\]
whilst
\[ g(s, x(s) + x_1(s) + x_2(s), \bar{y}^\tau(s) + \Lambda_8(s)) \]
\[ -g(s, \bar{x}(s) + x_1(s) + x_2(s), \bar{y}(s) + \Lambda_8(s)) \]
\[ = \hat{g}_y(s) \bar{y}^\tau(s), \]
where
\[ \hat{g}_y(s) = \int_0^1 g_y(s, \bar{x}(s) + x_1(s) + x_2(s), \alpha (\bar{y}^\tau(s) + \Lambda_8(s)) + (1 - \alpha) (\bar{y}(s) + \Lambda_8(s)), \bar{z}^\tau(s) + \Lambda_9(s)) \bar{y}^\tau(s) \, ds. \]

From assumptions (A1)-(A3), one can check that the adjoint equations (8) and (9) have a unique solution, respectively. Furthermore, by classical approach, we are able to get the following estimates for \( \beta \geq 2, \)
\[ \mathbb{E} \left[ \sup_{0 \leq t \leq T} \left( |p(t)|^\beta + |P(t)|^\beta \right) + \left( \int_0^T \left( |q(t)|^2 + |Q(t)|^2 \right) \, dt \right)^{\frac{\beta}{2}} \right] < +\infty. \quad (18) \]

From Lemma 2.2, it is fairly easy to check that
\[ \mathbb{E} \left[ \int_t^T \left[ \frac{1}{2} [x_1(s) + x_2(s), \Lambda_8(s)] \hat{D}^2 g(s) [x_1(s) + x_2(s), \Lambda_8(s)]^T \right. \right. \]
\[ \left. \left. + \mathcal{L}_1(s) - \frac{1}{2} x_1(s) \mathcal{L}_4(s) \hat{D}^2 g(s) (x_1(s) \mathcal{L}_4(s))^T \right] \, ds \right] \]
\[ \leq O(\varepsilon^2). \]

Then, from Eqs. (8), (17) and Lemma 2.3, we get (14) immediately. Applying Lemma 2.3 to (13) and observing that the integral term with \( d\bar{B}(s) \) disappears, we instantly obtain (15). Next we deal with (16).

Put
\[ \bar{x}^\tau(t) = x^\tau(t) - \bar{x}(t) - x_1(t) - x_2(t), \]
\[ \bar{y}^\tau(t) = \hat{y}^\tau(t) - \hat{y}(t), \]
\[ \bar{z}^\tau(t) = \hat{z}^\tau(t) - \hat{z}(t). \quad (19) \]

Then, we have, from BDSDEs (11) and (13)
\[ \hat{y}^\tau(t) = \int_t^T \left[ \mathcal{L}_1(s) + \hat{f}_y(s) \bar{y}^\tau(s) + \hat{f}_z(s) \bar{z}^\tau(s) \right. \]
\[ + ([\hat{f}_y(s) - f_y(s)] \hat{y}(s) + (\hat{f}_z(s) - f_z(s)) \hat{z}(s) \]
\[ - [f(s, \bar{x}(s), \hat{y}(s), \bar{z}(s) + p(s) \delta \sigma(s), u(s)] \]
\[ - f(s, \bar{x}(s), \bar{y}(s), \bar{z}(s), \bar{u}(s))] \mathcal{L}_6(s) \]
\[ + \frac{1}{2} [x_1(s) + x_2(s), \Lambda_8(s), \Lambda_9(s)] \hat{D}^2 f(s) [x_1(s) + x_2(s), \Lambda_8(s), \Lambda_9(s)]^T \]
\[ + \mathcal{L}_2(s) x_1(s) \mathcal{L}_6(s) - \frac{1}{2} x_1(s) \mathcal{L}_3(s) \hat{D}^2 f(s) (x_1(s) \mathcal{L}_3(s))^T \] \[ \int_t^T \] \[ + \mathcal{L}_1(s) + \hat{g}_y(s) \bar{z}^\tau(s) + (\hat{g}_y(s) - g_y(s)) \hat{y}(s) \]
\[ + \frac{1}{2} [x_1(s) + x_2(s), \Lambda_8(s)] \hat{D}^2 g(s) [x_1(s) + x_2(s), \Lambda_8(s)]^T \]
and

\[
\begin{align*}
&-\frac{1}{2}x_1(s)\mathcal{I}_4(s)D^2g(s)(x_1(s)\mathcal{I}_4(s))^\top\right]d\tilde{B}(s) \\
&-\int_t^T \tilde{z}(s)dW(s) + o(\varepsilon),
\end{align*}
\]

(20)

We are going to prove the following results:

\[
\mathbb{E}\left[\left(\int_0^T \left|\left(f(s, \tilde{x}(s), \tilde{y}(s), \tilde{z}(s) + p(s)\delta\sigma(s), u(s)\right) - f(s)\right| I_{E^c}(s) - \mathcal{I}_1(s)\right|ds\right)^2\right] = o(\varepsilon^2),
\]

(21)

\[
\mathbb{E}\left[\left(\int_0^T \left|\left(\tilde{f}_y(s) - f_y(s)\right)\tilde{y}(s) + \left(\tilde{f}_z(s) - f_z(s)\right)\tilde{z}(s)\right| ds\right)^2\right] = o(\varepsilon^2),
\]

(22)

\[
\mathbb{E}\left[\left(\int_0^T x_1(s)\mathcal{I}_3(s)\left(D^2f(s) - D^2f(s)\right)(x_1(s)\mathcal{I}_3(s))^\top ds\right)^2\right] = o(\varepsilon^2),
\]

(23)

and

\[
\mathbb{E}\left[\int_0^T |\mathcal{I}_1(s)|^2 ds\right] = o(\varepsilon^2),
\]

(24)

\[
\mathbb{E}\left[\int_0^T |(\tilde{g}_y(s) - g_y(s))\tilde{y}(s)|^2 ds\right] = o(\varepsilon^2),
\]

(25)

\[
\mathbb{E}\left[\int_0^T x_1(s)\mathcal{I}_3(s)\left(D^2g(s) - D^2g(s)\right)(x_1(s)\mathcal{I}_3(s))^\top ds\right] = o(\varepsilon^2).
\]

(26)

Let us first prove (21). We consider

\[
|\mathcal{I}_1(s) - [f(s, x, \tilde{x}(s), \tilde{y}(s), \tilde{z}(s) + p(s)\delta\sigma(s), u(s)) - f(s)]I_{E^c}(s)|
\]

\[
\leq C\left(|\tilde{z}(s)|
\right.
\]

\[
+ \left(|x_1(s) + x_2(s)| + |\tilde{y}(t)| + |\tilde{z}(t)| + |\Lambda_8(s)| + |\Lambda_9(s)|I_{E^c}(s)\right).
\]

(27)

Note that

\[
\mathbb{E}\left[\left(\int_0^T |q(s)x_1(s)| I_{E^c}(s)\right|ds\right)^2\right] \leq \mathbb{E}\left[\left(\int_{E^c} |q(s)x_1(s)| ds\right)^2\right] \varepsilon^2
\]

\[
\leq \mathbb{E}\left[\sup_{0 \leq s \leq T} |x_1(s)|^2 \int_{E^c} |q(s)|^2 ds\right] \varepsilon^2
\]

\[
\leq \left(\mathbb{E}\left[\sup_{0 \leq s \leq T} |x_1(s)|^2\right]\right)^{\frac{1}{2}} \left(\mathbb{E}\left[\int_{E^c} |q(s)|^2 ds\right]\right)^{\frac{1}{2}} \varepsilon^2
\]

\[
= o(\varepsilon^2).
\]

(28)
Combing (27) and (28), we get (21). We now deal with (22). Clearly,
\[
|\left(\tilde{y} (s) - f_y (s)\right)| + |\left(\tilde{z} (s) - f_z (s)\right)| \leq |\tilde{y} (s) - f_y (s)| + |\tilde{z} (s) - f_z (s)| + \hat{y} (s)
\]
\[
\leq C \left( |x_1 (s) + x_2 (s)| + |\Lambda_8 (s)| + |\Lambda_9 (s)|
\right)
\[
+ |\hat{y} (s)| + |\hat{z} (s)| \right).
\]
(29)

Besides,
\[
\mathbb{E} \left( \int_0^T |q (s) x_1 (s)| \hat{z} (s) \, ds \right)^2 
\]
\[
\leq \mathbb{E} \left( \sup_{0 \leq s \leq T} \left| x_1 (s) \right|^2 \int_0^T |q (s)|^2 \, ds \int_0^T |\hat{z} (s)|^2 \, ds \right)
\]
\[
\leq \left( \mathbb{E} \left( \sup_{0 \leq s \leq T} \left| x_1 (s) \right|^2 \right)^2 \right)^{\frac{1}{2}} \left( \mathbb{E} \left[ \left( \int_0^T |q (s)|^2 \, ds \right)^2 \right] \right)^{\frac{1}{2}}.
\]
(30)

From Lemma 2.2 and Lemma 3.1, (29) and (30), we obtain (22). Note that $D^2 f$ is bounded, we have for each $\beta \geq 2$,
\[
\mathbb{E} \left[ \left( \int_0^T \left| D^2 f (s) - D^2 f (s) \right|^2 |q (s)| \, ds \right)^\beta \right] \rightarrow 0, \text{ as } \varepsilon \rightarrow 0,
\]
by which it is easy to prove
\[
\mathbb{E} \left[ \left( \int_0^T \left| x_1 (s) I_3 (s) \left( D^2 f (s) - D^2 f (s) \right) I_3^T (s) x_1^T (s) \right| \, ds \right)^2 \right] = o (\varepsilon^2)
\]

Note that $D^2 g$ is bounded, we have for each $\beta \geq 2$,
\[
\mathbb{E} \left[ \left( \int_0^T \left| D^2 g (s) - D^2 g (s) \right|^2 |q (s)| \, ds \right)^\beta \right] \rightarrow 0, \text{ as } \varepsilon \rightarrow 0,
\]
by which it is also easy to prove
\[
\mathbb{E} \left[ \left( \int_0^T \left| x_1 (s) I_4 (s) \left( D^2 g (s) - D^2 g (s) \right) I_4^T (s) x_1^T (s) \right| \, ds \right)^2 \right] = o (\varepsilon^2)
\]

One can prove (24), (25) in the same way. Therefore, from (14) and (15), we have
\[
\mathbb{E} \left[ \int_t^T \left[ \mathcal{L}_1 (s) + (\hat{y} (s) - g_y (s)) \hat{y} (s)
\right.
\]
\[
+ \left. \frac{1}{2} \left[ x_1 (s) + x_2 (s) \right] \Lambda_8 (s) \right] \right] \rightarrow 0, \text{ as } \varepsilon \rightarrow 0,
\]
by which it is also easy to prove
\[
\mathbb{E} \left[ \left( \int_0^T \left| x_1 (s) I_4 (s) \left( D^2 g (s) - D^2 g (s) \right) I_4^T (s) x_1^T (s) \right| \, ds \right)^2 \right] = o (\varepsilon^2)
\]

One can prove (24), (25) in the same way. Therefore, from (14) and (15), we have
We thus complete the proof. \hfill \Box

We now give the adjoint equation for BDSDE (13) as follows:

\[
\begin{align*}
\frac{d\chi(t)}{dt} &= [f_x(t)\chi(t) + g_y(t)k(t)]dt + f_z(t)\chi(t)dW(t) + k(t)d\overline{B}(t), \\
\chi(0) &= h_y(\bar{y}(0)).
\end{align*}
\tag{31}
\]

From assumptions (A1)-(A3), it is fairly easy to check that BDSDEs (31) admit a unique solution pair \((\chi(t), k(t)) \in \mathcal{M}^2(0,T;\mathbb{R}) \times \mathcal{M}^2(0,T;\mathbb{R}^n)\).

We shall introduce the following variational inequality which is crucial to establish the necessary condition for optimal control.

**Lemma 3.2** (Variational inequality). *Under the assumptions (A1)-(A4), it holds that*

\[\mathbb{E}[h_y(y(0))\hat{y}(0)] \geq o(\varepsilon).\]

**Proof.** According to the definition of \(u^\varepsilon\), we have

\[J(u^\varepsilon) - J(\bar{u}) \geq 0,
\]

moreover,

\[\mathbb{E}[h(y^\varepsilon(0)) - h(\bar{y}(0))] = \mathbb{E}[h_y(\bar{y}(0))\hat{y}(0)] + o(\varepsilon),
\]

since the relation (19). We thus complete the proof. \hfill \Box

Now we define the Hamilton function as follows:

\[
H(t,x,y,z,u,p,q,k,P) = q^\top \sigma(t,x,y,z,u,p,q,k) + f(t,x,y,z,p(\sigma(t,x,u) - \sigma(t,x,u)),u) + \frac{1}{2}((\sigma(t,x,u) - \sigma(t,x,u)), P(\sigma(t,x,u) - \sigma(t,x,u)) + o(\varepsilon),
\]

where

\[H : [0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}.
\]

**Remark 1.** The Hamilton function (32) is completely different from the one in Han, Peng and Wu [14], Zhang and Shi [42]. Specifically, in (32), \(f\) depends on \(z + p(\sigma(t,x,u) - \sigma(t,x,u))\) instead of \(z\) only. We shall discuss the case, namely, \(z\) and control variable enter in \(g\) in Section 6.

We are now in the position to assert the main result.

**Theorem 3.3.** Assume that the conditions (A1)-(A4) hold. Let \((p(t), q(t))\) and \((P(t), Q(t))\) be an admissible pair of the solutions to (8) and (9), respectively, corresponding to \((\bar{x}(t), \bar{y}(t), \bar{z}(t), \bar{u}(t))\). Then, the following maximum principle holds:

\[
\begin{align*}
&\left< \chi(t), H(t,\bar{x}(t), \bar{y}(t), \bar{z}(t), \bar{u}(t), p(t), q(t), k(t), P(t)) \right> \\
&- H(t,\bar{x}(t), \bar{y}(t), \bar{z}(t), \bar{u}(t), p(t), q(t), k(t), P(t)) \geq 0, \quad \forall u \in \mathcal{U}, \text{ a.s., a.e.,}
\end{align*}
\]

where \((\chi(\cdot), k(\cdot))\) is the solution to (31) and \(H(\cdot)\) is defined in (32).
Proof. By virtue of Itô’s formula to \( \langle \chi (t), \tilde{y} (t) \rangle \) on \([0, T]\), it follows that

\[
E \left[ h_y (\tilde{y} (0)) \tilde{y} (0) \right] = E \left[ \int_0^T \left( \chi (t), H (t, \tilde{x} (t), \tilde{y} (t), \tilde{z} (t), u, p (t), q (t), k (t), P (t)) + \tilde{H} (t, \tilde{x} (t), \tilde{y} (t), \tilde{z} (t), \tilde{u} (t), p (t), q (t), k (t), P (t)) \right) I_{E_k} (t) \, dt \right].
\]

From Lemma 3.2, we get the desired result. \( \square \)

**Remark 2.** Theorem 3.3 actually extends the result obtained in [42], namely, the diffusion term \( \sigma \) contains the control variable under non-convex constraints. A natural and interesting question raises: How to establish the maximum principle for systems (2) whenever the coefficient \( q \) depends on \( z \) and the control variable? Note that the approach developed in this paper, can not solve this issue since the integral in the term with \( dB (t) \) makes the order of the variational equation (13) never reach to high order \( o (\varepsilon) \). We shall discuss this issue in Section 6. Besides, in contrast to Hu [16], in general, it is not possible to derive \( \chi > 0 \) in (31).

To sum up, from (12), (19) and Theorem 3.3, it is easy to derive the following variational equations for FBDSDEs (2):

\[
y^\varepsilon (t) = \tilde{y} (t) + p^\top (t) (x_1 (t) + x_2 (t)) + \frac{1}{2} x_1^\top (t) P (t) x_1 (t) + \tilde{y} (t) + o (\varepsilon) , \tag{33}
\]

\[
z^{j, \varepsilon} (t) = p (t) \delta \sigma^j (t) I_{E_k} (t) + \tilde{z}^{j, \varepsilon} (t) + \left( \frac{1}{2} \delta \sigma^j (t) + \frac{1}{2} P (t) \delta \sigma^j (t) \right) p (t) + q^j (t) x_1 (t) + x_2 (t) \tag{34}
\]

\[
\quad + \left( \frac{1}{2} \left( \sigma^j_{xx} (t) \right)^\top p (t) + P (t) \left( \sigma^j_x (t) + \sigma^j_y (t) \right)^\top p (t) + Q^j (t) \right) z_1 (t), \quad \text{for } j = 1, 2, \ldots, n.
\]

We define

\[
y_1 (t) = p^\top (t) x_1 (t),
\]

\[
z_1 (t) = p (t) \delta \sigma^j (t) I_{E_k} (t) + \langle (\sigma^j_x (t) \rangle^\top p (t) + q^j (t), x_1 (t) \rangle,
\]

and

\[
y_2 (t) = \langle p (t), x_2 (t) + \frac{1}{2} x_1^\top (t) P (t) x_1 (t) + \tilde{y} (t)
\]

\[
z^{j, \varepsilon} (t) = p (t) \delta \sigma^j (t) I_{E_k} (t) + \tilde{z}^{j, \varepsilon} (t) + \left( \frac{1}{2} \delta \sigma^j (t) + \frac{1}{2} P (t) \delta \sigma^j (t) \right) p (t) + q^j (t) x_1 (t) + x_2 (t) \tag{34}
\]

\[
+ \left( \frac{1}{2} \left( \sigma^j_{xx} (t) \right)^\top p (t) + P (t) \left( \sigma^j_x (t) + \sigma^j_y (t) \right)^\top p (t) + Q^j (t) \right) z_1 (t),
\]

\[
+ \langle p (t), x_2 (t) \rangle + \tilde{z}^j (t).
\]
Thus, by Lemma 2.1, we are able to the first order and second order variational equations. Indeed, applying Itô’s formula to $\langle p(t), x_1(t) \rangle$ and $\langle p(t), x_2(t) \rangle + \frac{1}{2} x_1^2(t) P(t) x_1(t)$ yield

$$
\begin{align*}
- dy_1(t) &= \left\{ f_x^T x_1(t) + f_y(t) y_1(t) + f_z^T z_1(t) \\
&\quad - \left[ f_z(t) p(t) \delta \sigma(t) + q(t) \delta \sigma(t) \right] I_{E_x}(t) \right\} dt - z_1(t) dW(t) \\
- g_y(t) y_1(t) d\widehat{B}(s),
\end{align*}
$$

and

$$
\begin{align*}
- dy_2(t) &= \left\{ f_x(t) x_2(t) + f_y(t) y_2(t) + f_z(t) z_2(t) \\
&\quad + \frac{1}{2} \left[ x_1(t), y_1(t), z_1(t) \right] D^2 f(t) \left[ x_1(t), y_1(t), z_1(t) \right]^T \\
&\quad + q(t) \delta \sigma(t) - \frac{1}{2} \left( p(t) \delta \sigma(t) \right)^T f_{zz}(p(t) \delta \sigma(t)) \\
&\quad - f(t, \bar{x}(t), \bar{y}(t), \bar{z}(t)) I_{E_x}(t) + L(t) x_1(t) I_{E_x}(t) \\
&\quad + f(t, \bar{x}(t), \bar{y}(t), \bar{z}(t) + p(t) \delta \sigma(t), u(t)) \right\} dt - z_2(t) dW(t) \\
&\quad + \left[ x_2^T(t) g_y(t) y(t) + \sum_{j=1}^{m} g_y^j(t) \bar{y}(t) \\
&\quad + \left[I_{n \times n}(p(t)) \cdot D^2 g(t) \cdot \left[I_{n \times n}(p(t)) \right]^T + g_y(t) P(t) \right] d\widehat{B}(s),
\end{align*}
$$

(35)

where $L(t) x_1(t) I_{E_x}(t) = o(\varepsilon)$, hence we do not give the explicit formula for $L(t)$.

4. Example. In this section, we provide a concrete example to illustrate our theoretical result.

**Example 1.** Consider the following example with $n = 1$ with $U = \{0, 1\}$.

$$
\begin{align*}
        dx(t) &= u(t) dW(t), \\
- dy(t) &= u(t) dt + \frac{1}{2} d\widehat{B}(t) - z(t) dW(t), \\
        x(0) &= 0, \quad y(T) = x(T).
\end{align*}
$$

(36)

Let $h(y(0)) = y(0)$. Up to now, one can easily get the solutions to (31), (8) and (9), namely, $\chi(t) = 1$, $k(t) = 0$, $p(t) = 1$, $q(t) = 0$, $P(t) = 0$ and $Q(t) = 0$, a.s. Hence, from Theorem 3.3, we are able to establish the maximum principle for optimal control of systems (36) as follows:

$$ u - \bar{u}(t) \geq 0, \quad \forall u \in U, \text{ a.s., a.e.} $$

from which one can immediately derive that the optimal control $\bar{u}(t) \equiv 0$ with the associated state trajectories $(\bar{x}(t), \bar{y}(t), \bar{z}(t)) = (0, 0, 0)$.

5. Constraint problem. In this section, we employ the celebrated Ekeland variational principle to attack the constraint problem. The technique has been used largely in Tang and Li [32] and Wu [37]. For simplicity, let $n = 1$, the multi-dimensional case can be treated analogously.

Apart from the cost function $J(u(\cdot))$ defined in (3), we add the following state constraint

$$ E\left[ \Psi(x(T), y(0)) \right] = 0, $$

(37)

where $\Psi : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$.

Assume that
(A5): The function $\Psi$ is twice continuously differentiable with respect to $(x, y)$; $D^2\Psi$ is bounded.

Lemma 5.1 (Ekeland Principle). Let $(S, d)$ be a complete metric space and $\rho : S \to R \cup \{\infty\}$ be a lower semicontinuous function, bounded from below. If for each $\epsilon > 0$, there exists $u^\epsilon \in S$ such that $\rho(u^\epsilon) \leq \inf_{u \in S} \rho(u) + \epsilon$. Then for any $\lambda > 0$, there exists $u_\lambda \in S$ such that

$$
\begin{align*}
&i) \quad \rho(u_\lambda) \leq \rho(u^\epsilon), \\
&ii) \quad d(u_\lambda, u^\epsilon) \leq \lambda, \\
&iii) \quad \rho(u_\lambda) \leq \rho(u) + \frac{\lambda}{\epsilon}d(u, u^\epsilon), \quad \text{for all } u \in S.
\end{align*}
$$

We define $U_C(0, T)$ as follows:

$$U_C(0, T) = \{ u(\cdot) \in U_C(0, T) : E[\Psi(x(T), y(0))] = 0 \}$$

with the following metric on $U_C(0, T)$:

$$d(u(\cdot), v(\cdot)) = dt \otimes P \{ (t, \omega) \in [0, T] \times \Omega : u(t, \omega) \neq v(t, \omega) \},$$

where $dt \otimes P$ is the product measure of the Lebesgue measure $dt$ with the probability measure $P$. It is well known that $(U_C(0, T), d)$ is a complete metric space (see Yong and Zhou [41]).

Set $\bar{u}(\cdot) \in U_C(0, T)$ be an optimal control and $(\bar{x}(\cdot), \bar{y}(\cdot), \bar{z}(\cdot))$ be the corresponding solution to Eqs. (2). For any $\lambda > 0$, define the following cost functional on $U_C(0, T)$:

$$J_\lambda(u(\cdot)) = \left[ h(y(0)) - h(\bar{y}(0)) + \lambda^2 + E[\Psi(x(T), y(0))] \right]^{\frac{1}{2}},$$

which is continuous. Clearly, it is easy to check that

$$\begin{align*}
&J_\lambda(u(\cdot)) \geq 0, \quad \forall u(\cdot) \in U_C(0, T), \\
&J_\lambda(\bar{u}(\cdot)) = \lambda \leq \inf_{u(\cdot) \in U_C(0, T)} J_\lambda(u(\cdot)) + \lambda.
\end{align*}$$

By Lemma 5.1, there exists a $u_\lambda(\cdot) \in U_C(0, T)$ such that

$$\begin{align*}
&J_\lambda(u_\lambda(\cdot)) \leq \lambda, \quad d(u_\lambda(\cdot), \bar{u}(\cdot)) \leq \lambda^{\frac{1}{2}}, \\
&J_\lambda(u_\lambda(\cdot)) - J_\lambda(u_\lambda(\cdot)) + \lambda^2 d(u_\lambda(\cdot), u(\cdot)) \geq 0, \quad \forall u(\cdot) \in U_C(0, T).
\end{align*}$$

For any $\epsilon > 0$, consider

$$u_\lambda^\epsilon(t) = u_\lambda(t) I_{E_C}(t) + u(t) I_{E_\epsilon}(t), \quad u(\cdot) \in U_C(0, T).$$

Obviously, we have

$$d(u_\lambda(\cdot), u_\lambda^\epsilon(\cdot)) \leq \epsilon.$$  

Let $(x_\lambda(\cdot), y_\lambda(\cdot), z_\lambda(\cdot)) ((x_\lambda^\epsilon(\cdot), y_\lambda^\epsilon(\cdot), z_\lambda^\epsilon(\cdot)))$ be the trajectory driven by $u_\lambda(\cdot) (u_\lambda^\epsilon(\cdot))$, respectively. After some tedious calculations, we have

$$\begin{align*}
0 &\leq J_\lambda(u_\lambda^\epsilon(\cdot)) - J_\lambda(u_\lambda(\cdot)) + \lambda^{\frac{1}{2}} \epsilon \\
&\leq \alpha_\lambda \left[ y_\lambda^\epsilon(0) - y_\lambda(0) \right] + \beta_\lambda \left[ E[\Psi(x_\lambda^\epsilon(0), y_\lambda^\epsilon(0))] - E[\Psi(x_\lambda(0), y_\lambda(0))] \right] \\
&\quad + \lambda^{\frac{1}{2}} \epsilon + o(\epsilon),
\end{align*}$$

where

$$\alpha_\lambda = \frac{y_\lambda(0) - \bar{y}(0) + \lambda}{J_\lambda(u_\lambda(\cdot))}, \quad \beta_\lambda = \frac{E[\Psi(x_\lambda(T), y_\lambda(0))]}{J_\lambda(u_\lambda(\cdot))}.$$ 

Clearly,

$$\alpha_\lambda^2 + \beta_\lambda^2 = 1. \quad (38)$$
Let \((p^\lambda (-), q^\lambda (-))\) and \((P^\lambda (-), Q^\lambda (-))\) be respectively the solutions to Eqs. (8) and (9) endowed with \(\lambda\). We have

\[
\mathbb{E} \left[ h (y^\lambda_0) - h (y_\lambda (0)) \right] = \mathbb{E} \left[ h_y (y_\lambda (0)) \tilde{y}_\lambda (0) \right] + o (\varepsilon),
\]

where

\[
g^\lambda (t) = \int_t^T \left[ f^\lambda_y (s) \tilde{g}_\lambda (s) + \sum_{j=1}^d f^\lambda_z (s) \tilde{z}^j_\lambda (s) + \left\{ \left( p^\lambda (s) \right)^\top \delta b^\lambda (s) \right. \right.
\]

\[
+ \sum_{j=1}^d \left[ \left( q^\lambda_j (s) \right)^\top \delta \sigma^\lambda-j (s) + \frac{1}{2} \left( \delta \sigma^\lambda-j (s) \right)^\top P^\lambda (s) \delta \sigma^\lambda-j (s) \right] b^\lambda_j (s)
\]

\[
+ f \left( s, x^\lambda (s), y_\lambda (s), z_\lambda (s) + p^\lambda (s)^\top \delta \sigma^\lambda (s), u \right) - f \left( s, x^\lambda (s), y_\lambda (s), z_\lambda (s), u_\lambda (s) \right) \right\} I_{E^\varepsilon} (s) \bigg] ds
\]

\[
+ \sum_{j=1}^d \int_t^T g^\lambda_j (s) \tilde{y}_\lambda (s) \lambda \bigg]  \int_t^T \tilde{z}^j_\lambda (s) \lambda dW_j (s).
\]

In order to deal with the first part of \(\Psi\), we introduce the following adjoint equations\(^1\):

\[
\begin{align*}
- d^\lambda \gamma (t) &= \left[ \beta^\lambda_y (t)^\top \gamma^\lambda (t) + \sigma^\lambda_x (t)^\top \kappa^\lambda_x (t) \right] dt - \kappa^\lambda (t) dW (t), \\
\gamma^\lambda (T) &= \beta^\lambda \Psi_x (x^\lambda (T), y_\lambda (0))
\end{align*}
\]

and

\[
\begin{align*}
- d^\lambda \chi (t) &= \left[ \beta^\lambda_y (t)^\top \chi^\lambda (t) + \sigma^\lambda_x (t)^\top \chi^\lambda (t) \right] dt - \chi^\lambda (t) dW (t), \\
\chi^\lambda (0) &= \alpha^\lambda + \beta^\lambda \mathbb{E} \left[ \Psi_x (x^\lambda (T), y_\lambda (0)) \right] \tilde{y}_\lambda (0) + o (\varepsilon). \quad (42)
\end{align*}
\]

Then,

\[
\beta^\lambda \mathbb{E} \left[ \Psi_x (x^\lambda (T), y^\lambda_0) - \Psi_x (x^\lambda (T), y_\lambda (0)) \right]
\]

\[
= \mathbb{E} \left[ \int_0^T \left\{ \langle \gamma^\lambda (s), \delta b^\lambda (s) \rangle + \langle \kappa^\lambda (s), \delta \sigma^\lambda (s) \rangle \right. \right.
\]

\[
+ \frac{1}{2} \left\langle \chi^\lambda (t), \delta \sigma^\lambda (s) \right. \bigg] ds \bigg] + \beta^\lambda \mathbb{E} \left[ \Psi_x (x^\lambda (T), y_\lambda (0)) \right] \tilde{y}_\lambda (0) + o (\varepsilon). \quad (43)
\]

We now give the adjoint equation (forward):

\[
\begin{align*}
\chi^\lambda (0) &= \alpha^\lambda + \beta^\lambda \mathbb{E} \left[ \Psi_y (x^\lambda (T), y_\lambda (0)) \right] \tilde{y}_\lambda (0).
\end{align*}
\]

\(^1\) Actually, these adjoint equations are classical ones employed by Peng [28] and Yong and Zhou [41] to derive the necessary condition for classical forward stochastic systems. Note that in Eq. (2), the first equation is classical one. Nevertheless, whenever the forward equation contains the “backward” Brownian motion, this situation becomes much harder since one must derive the first and second order adjoint equations respectively, but also belonging to doubly stochastic systems.
By Itô’s formula, it follows
\[
\{ \alpha_\lambda + \beta_\lambda \mathbb{E} [ \Psi_y (x_\lambda (T), y_\lambda (0)) ] \} \hat{y}_\lambda (0)
= \mathbb{E} \left[ \int_0^T \chi^\lambda (s) \left\{ \left( \gamma^\lambda (s), \delta b^\lambda (s) \right) + \left( \kappa^\lambda (s), \delta \sigma^\lambda (s) \right) \right\}
+ \frac{1}{2} \left( \Upsilon^\lambda (t) \delta \sigma^\lambda (s), \delta \sigma^\lambda (s) \right) \\
+ f \left( s, x_\lambda (s), y_\lambda (s), z_\lambda (s) + p^\lambda (s) \delta \sigma^\lambda (s), u \right) \\
- f \left( s, x_\lambda (s), y_\lambda (s), z_\lambda (s), u_\lambda (s) \right) \right\} I_{E_\varepsilon} (s)
\]
\[ + k^\lambda (s) \left[ g (s, x_\lambda (s), y_\lambda (s)) \right] \] ds.
(45)

The Hamiltonian function is defined by
\[
\mathbb{H} (t, x, y, z, u, x^1, u^1, \gamma, \kappa, \Upsilon, p, q, P, \chi, k)
= \langle \gamma + \chi P, b (t, x, u) \rangle + \langle \kappa + \chi q, \sigma (t, x, u) \rangle \\
+ \frac{1}{2} \left( (\Upsilon + \chi P) \left( \sigma (t, x, u) - \sigma (t, x^1, u^1) \right), \sigma (t, x, u) - \sigma (t, x^1, u^1) \right) \\
+ \chi f \left( t, x, y, z + \langle p, \sigma (t, x, u) - \sigma (t, x^1, u^1) \rangle, u \right) + kg (t, x, y).
\] (46)

Combining (39)-(45) together, we get
\[
0 \leq \mathbb{E} \left[ \int_0^T \left\{ \mathbb{H} (s, x_\lambda (s), y_\lambda (s), z_\lambda (s), u, x_\lambda (s), u_\lambda (s), \gamma^\lambda (s), \kappa^\lambda (s), \chi^\lambda (s), k^\lambda (s)) \\
+ \Upsilon^\lambda (s), p^\lambda (s), q^\lambda (s), P^\lambda (s), \chi^\lambda (s), k^\lambda (s) \right\} I_{E_\varepsilon} (s) \right] ds + \lambda \frac{2}{\varepsilon} + o (\varepsilon).
\]

By classical approach, we derive that
\[
0 \leq \mathbb{E} \left[ \int_0^T \left\{ \mathbb{H} (s, x_\lambda (s), y_\lambda (s), z_\lambda (s), u, x_\lambda (s), u_\lambda (s), \gamma^\lambda (s), \kappa^\lambda (s), \chi^\lambda (s), k^\lambda (s)) \\
+ \Upsilon^\lambda (s), p^\lambda (s), q^\lambda (s), P^\lambda (s), \chi^\lambda (s), k^\lambda (s) \right\} I_{E_\varepsilon} (s) \right] ds + \lambda \frac{2}{\varepsilon} + o (\varepsilon) \\
- \lambda \frac{2}{\varepsilon} \chi \varepsilon \geq \mathbb{H} (s, x_\lambda (s), y_\lambda (s), z_\lambda (s), u_\lambda (s), x_\lambda (s), u_\lambda (s), \gamma^\lambda (s), \kappa^\lambda (s), \chi^\lambda (s), k^\lambda (s)) \\
- \lambda \frac{2}{\varepsilon} \chi \varepsilon, \forall u \in U, \text{ a.e., a.s.}
\]

In light of (38), we claim that there exists subsequence (at least) also expressed as \((\alpha_\lambda, \beta_\lambda)\) approaches to certain \((\alpha, \beta)\) as \(\lambda \to 0\). By property of dependence of parameters on BDSDE, it is easy to attain
\[
\left( s, x_\lambda (s), y_\lambda (s), z_\lambda (s), u_\lambda (s), \gamma^\lambda (s), \kappa^\lambda (s), \chi^\lambda (s), k^\lambda (s) \right) \to
\]
where
\[
\begin{aligned}
&\begin{cases}
-d\gamma(t) = \left[b_x(t)^T \gamma(t) + \sigma_x(t)^T \kappa_x(t)\right] dt - \kappa(t) dW(t), \\
\gamma(T) = \beta \Psi_x(x(T), y(0)),
\end{cases}
\end{aligned}
\]
with the help of (47)-(49), we are able to assert the main result.

**Theorem 5.2.** Assume the assumptions (A1)-(A5) are in force. Let \( \bar{u}(\cdot) \) be an optimal control associated with the constraint (37). Then, there exist two positive constants \( (\alpha, \beta) \) such that \( |\alpha|^2 + |\beta|^2 = 1 \), and

\[
\begin{aligned}
&\begin{cases}
d\chi(t) = [f_y(t) \chi(t) + g_y(t) k(t)] dt + f_z(t) \chi(t) dW(t) \\
+ k(t) d\mathcal{B}(t), \\
\chi(0) = \alpha + \beta \mathbb{E}[\Psi_y(x(T), y(0))].
\end{cases}
\end{aligned}
\]

With the help of (47)-(49), we are able to assert the main result.

**Remark 3.** Theorem 5.2 involves two unknown parameters \( (\alpha, \beta) \), which actually are not displayed in \( H \) but on the solution \( (\gamma(\cdot), \kappa(\cdot), \Upsilon(\cdot)) \). From (38), it indicates that it is not easy to use in practice. We shall consider this topic in our future work.

6. Discussion. In this paper, we have established a necessary condition for optimal control of forward backward doubly stochastic differential equations with compact control domain by means of the variation equations and two new adjoint equations, which is actually extends the work by Zhang and Shi [42] to general case. Comparing with the existing literature, there are some interesting issues to be stated as follows:

- As observed in systems (2), the coefficient \( g \) is independent on \( z \) and control variable simultaneously. Further discussion displays here. The function \( g \) enters in the integral term of Brownian motion, which will lower the order of the variational inequality. Letting \( \varepsilon \rightarrow 0 \), it is impossible to get the maximum principle because of the B-D-G inequality. More precisely, If \( g \) contains \( z \) and \( u \), repeating the argument developed in this paper, there appears a term like:
\[
\int_T^t \left[ g \left( s, \bar{x}(s), \bar{y}(s), \bar{z}(s) + p^\top(s) \delta \sigma(s), u^\varepsilon(s) \right) \\
- g \left( s, \bar{\Theta}(s) \right) I_{E_\varepsilon}(s) \right] \cdot d\bar{B}(s),
\]

where \( \bar{\Theta}(s) = (\bar{x}(s), \bar{y}(s), \bar{z}(s), \bar{u}(s)) \). By virtue of B-D-G inequality, one can easily get

\[
E \left[ \int_T^t \left[ g \left( s, \bar{x}(s), \bar{y}(s), \bar{z}(s) + p^\top(s) \delta \sigma(s), u^\varepsilon(s) \right) \\
- g \left( s, \bar{\Theta}(s) \right) \right]^2 I_{E_\varepsilon}(s) \cdot ds \right],
\]

from which, it is impossible to get the variational inequality with order \( o(\varepsilon) \).

Hence, the methodology developed in this paper fails. To overcome this difficulty, one may suspect the variational equation for \( y \) involving second order of, and introducing two groups of adjoint equations with four equations. This topic will be under consideration in our future work.

• In classical stochastic control problem, the cost function should be involved, namely, the performance functional is assumed as

\[
J(u(\cdot)) = E \left[ \int_0^T l \left( s, x(s), y(s), z(s), u(s) \right) \cdot ds + h \left( y(0) \right) \right].
\]

(50)

Following the idea in Lemma 3.6 in [42], one expects the variational inequality to be

\[
0 \leq E \left[ \int_0^T l \left( s, \Theta^\varepsilon(s) \right) - l \left( s, \bar{\Theta}(s) \right) \cdot ds + h \left( y^\varepsilon(0) \right) - h \left( y(0) \right) \right],
\]

where \( \Theta^\varepsilon(s) = (x^\varepsilon(s), y^\varepsilon(s), z^\varepsilon(s), u^\varepsilon(s)) \).

However, by (33) and (34), one needs to deal with the extra term

\[
p^\top(t) \left( x_1(t) + x_2(t) \right) + \frac{1}{2} P(t) x_1(t).
\]

This topic is also interesting for our future work.

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