Safe Search for Stackelberg Equilibria in Extensive-Form Games

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Abstract

Stackelberg equilibrium is a solution concept in two-player games where the leader has commitment rights over the follower. In recent years, it has become a cornerstone of many security applications, including airport patrolling and wildlife poaching prevention. Even though many of these settings are sequential in nature, existing techniques pre-compute the entire solution ahead of time. In this paper, we present a theoretically sound and empirically effective way to apply search, which leverages extra online computation to improve a solution, to the computation of Stackelberg equilibria in general-sum games. Instead of the leader attempting to solve the full game upfront, an approximate “blueprint” solution is first computed offline and is then improved online for the particular subgames encountered in actual play. We prove that our search technique is guaranteed to perform no worse than the pre-computed blueprint strategy, and empirically demonstrate that it enables approximately solving significantly larger games compared to purely offline methods. We also show that our search operation may be cast as a smaller Stackelberg problem, making our method complementary to existing algorithms based on strategy generation.

1 Introduction

Strong Stackelberg equilibria (SSE) have found many uses in security domains, such as wildlife poaching protection (Fang et al. 2017) and airport patrols (Pita et al. 2008). Many of these settings, particularly those involving patrolling, are sequential by nature and are best represented as extensive-form games (EFGs). Finding a SSE in general EFGs is provably intractable (Letchford and Conitzer 2010). Existing methods convert the problem into a normal-form game and apply column or constraint generation techniques to handle the exponential blowup in the size of the normal-form game (Jain, Kiekintveld, and Tambs 2011). More recent methods cast the problem as a mixed integer linear program (MILP) (Bosansky and Cermak 2015). Current state-of-the-art methods build upon this by heuristically generating strategies, and thus avoid considering all possible strategies (Cerny, Bosansky, and Kiekintveld 2018).

All existing approaches for computing SSE are entirely offline. That is, they compute a solution for the entire game ahead of time and always play according to that offline solution. In contrast, search additionally leverages online computation to improve the strategy for the specific situations that come up during play. Search has been a key component for AI in single-agent settings (Lin 1965, Hart, Nilsson, and Raphael 1968), perfect-information games (Tesauro 1995, Campbell, Hoane Jr, and Hsu 2002, Silver et al. 2016, 2018), and zero-sum imperfect-information games (Moravcik et al. 2017, Brown and Sandholm 2017b, 2019). In order to apply search to two-player zero-sum imperfect-information games in a way that would not do worse than simply playing an offline strategy, safe search techniques were developed (Burch, Johanson, and Bowling 2014, Moravcik et al. 2016, Brown and Sandholm 2017). Safe search begins with a blueprint strategy that is computed offline. The search algorithm then adds extra constraints to ensure that its solution is no worse than the blueprint (that is, that it approximates an equilibrium at least as closely as the blueprint). However, safe search algorithms have so far only been developed for two-player zero-sum games.

In this paper, we extend safe search to SSE computation in general-sum games. We begin with a blueprint strategy for the leader, which is typically some solution (computed offline) of a simpler abstraction of the original game. The leader follows the blueprint strategy for the initial stages of the game, but upon reaching particular subgames of the game tree, computes a refinement of the blueprint strategy online, which is then adopted for the rest of the game.

We show that with search, one can approximate SSEs in games much larger than purely offline methods. We also show that our search operation is itself solving a smaller SSE, thus making our method complementary to other methods based on strategy generation. We evaluate our method on a two-stage matrix game, the classic game of Goofspiel, and a larger, general-sum variant of Leduc hold’em. We demonstrate that in large games our search algorithm outperforms offline methods while requiring significantly less computation, and that this improvement increases with the size of the game. Our implementation is publicly available online: https://github.com/lingchunkai/SafeSearchSSE.

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2 Background and Related Work

As is standard in game theory, we assume that the strategies of all players, including the algorithms used to compute those strategies, are common knowledge. However, the outcomes of stochastic variables are not known ahead of time.

EFGs model sequential interactions between players, and are typically represented as game trees in which each node specifies a state of the game where one player acts (except terminal nodes where no player acts). In two-player EFGs, there are two players, \( P = \{1, 2\} \). \( H \) is the set of all possible nodes \( h \) in the game tree, which are represented as sequences of actions (possibly chance). \( A(h) \) is the set of actions available in node \( h \) and \( P(h) \in P \cup \{c\} \) is the player acting at that node, where \( c \) is the chance player. If a sequence of actions leads from \( h \) to \( h' \), then we write \( h \subseteq h' \). We denote \( Z \subseteq H \) to be the set of all terminal nodes in the game tree. For each terminal node \( z \), we associate a payoff for each player \( i \in P \), i.e., \( \{h \in H, P(h) = i\} \) are partitioned into information sets \( I_i \). All nodes \( h \) belonging to the same information set \( I_i \), \( I_i \subseteq H \), are indistinguishable and players must behave the same way for all nodes in \( I_i \). Furthermore, all nodes in the same information set are required to have the same actions, if \( h, h' \in I_i \), then \( A(h) = A(h') \). Thus, we overload \( A(I_i) \) to define the set of actions in \( I_i \). We assume that the game exhibits perfect recall, i.e., players do not ‘forget’ past observations or own actions; for each player \( i \), the information set \( I_i \) is preceded by a unique series of actions and information sets of \( i \).

Sequence Form Representation. Strategies in games with perfect recall may be compactly represented in sequence-form (Von Stengel 1996). A sequence \( \sigma_i \) is an (ordered) list of actions taken by a single player \( i \) in order to reach node \( h \). The empty sequence \( \emptyset \) is the sequence without any actions. The set of all possible sequences achievable by player \( i \) is given by \( \Sigma_i \). We write \( \sigma_i \bigcup = \sigma'_i \) if a sequence \( \sigma'_i \in \Sigma_i \) may be obtained by appending an action \( a \) to \( \sigma_i \). With perfect recall, all nodes \( h \) in information sets \( I_i \subseteq H \) may be reached by a unique sequence \( \sigma_i \), which we denote by \( \text{Seq}(I_i) \) or \( \text{Seq}(h) \). Conversely, \( \text{Inf}(\sigma'_i) \) denotes the information set containing the last action taken in \( \sigma'_i \). Using the sequence form, mixed strategies are given by realization plans, \( r_i : \Sigma_i \rightarrow \mathbb{R} \), which are distributions over sequences. Realization plans for sequences \( \sigma_i \) give the probability that this sequence of moves will be played, assuming all other players played such as to reach \( \text{Inf}(\sigma_i) \). Mixed strategies obey the sequence form constraints, \( \forall h, r_i(\emptyset) = 1 \), \( \forall I_i \subseteq H, r_i(\sigma_i) = \sum_{a \in A(I_i)} r_i(\sigma_i(a)) \) and \( \sigma_i \in \text{Seq}(I_i) \).

Sequence forms may be visualized using treeplexes (Hoda et al. 2010), one per player. Informally, a treplex is a tree rooted at \( \emptyset \) with subsequent nodes alternating between information sets and sequences, and are operationally useful for providing recursive implementations for common operations in EFGs such as finding best responses. Since understanding treeplexes is helpful in understanding our method, we provide a brief introduction in the Appendix.

Stackelberg Equilibria in EFGs. Strong Stackelberg Equilibria (SSE) describe games in which there is asymmetry in the commitment powers of players. Here, players 1 and 2 play the role of leader and follower, respectively. The leader is able to commit to a (potentially mixed) strategy and the follower best-responds to this strategy, while breaking ties by favoring the leader. By carefully committing to a mixed strategy, the leader implicitly issues threats, and followers are made to best-respond in a manner favorable to the leader. SSE are guaranteed to exist and the value of the game for each player is unique. In one-shot games, a polynomial-time algorithm for finding a SSE is given by the multiple-LP approach (Conitzer and Sandholm 2006).

However, solving for SSE in EFGs in general-sum games with either chance or imperfect information is known to be NP-hard in the size of the game tree (Letchford and Conitzer 2010) due to the combinatorial number of pure strategies. Bosansky and Cermak (2015) avoid transformation to normal form and formulate a compact mixed-integer linear program (MILP) which uses a binary sequence-form follower best response variable to modestly-sized problems. More recently, Cerny, Bosansky, and Kiekintveld (2018) propose heuristically guided incremental strategy generation.

Safe Search. For this paper, we adopt the role of the leader and seek to maximize his expected payoff under the SSE. We assume that the game tree may be broken into several disjoint subgames. For this paper, a subgame is defined as a set of states \( H_{sub} \subseteq H \) such that (a) if \( h \subseteq h' \) and \( h \in H_{sub} \) then \( h' \in H_{sub} \), and (b) if \( h \in I_i \) and \( h \in H_{sub} \), then for all \( h' \in I_i \), \( h' \in H_{sub} \). Condition (a) implies that one cannot leave a subgame after entering it, while (b) ensures that information sets are ‘contained’ within subgames—if any history in an information set belongs to a subgame, then every history in that information set belongs to that subgame. For the \( j \)-th subgame \( H_{sub} \), \( I_{i, sub} \subseteq I_i \) is the set of information sets belonging to player \( i \) within subgame \( j \). Furthermore, let \( I_{i, head} \subseteq I_{i,j} \) be the ‘head’ information sets of player \( i \) in subgame \( j \), i.e., \( I_i \in I_{i, head} \) if and only if \( \text{Inf}_i(\text{Seq}(I_i)) \) does not exist or does not belong to \( I_{i,j} \). With a slight abuse of notation, let \( I_{i, head}(z) \) be the (unique, if existent) information set in \( I_{i, head} \) preceding leaf \( z \).

At the beginning, we are given a blueprint strategy for the leader, typically the solution of a smaller abstracted game. The leader follows the blueprint strategy in actual play until reaching some subgame. Upon reaching the subgame, the leader computes a refined strategy and follows it thereafter. The pseudocode is given in Algorithm 1. The goal of the paper is to develop effective algorithms for the refinement step (*). Algorithm 1 implicitly defines a leader strategy distinct from the blueprint. Crucially, this implies that the follower responds to this implicit strategy and not the blueprint. Search is said to be safe when the leader applies Algorithm 1 such that its expected payoff is no less than the blueprint, supposing the follower best responds to the algorithm.
These changes cause the leader’s EV to drop from
behavior may arise.
To motivate our algorithm, we first explore how unsafe be-
anteeing no change of follower strategies after refinement.
under na¨ıve search is shown in the box, as are bounds guar-
regions denote subgames. Expected values for each player un-
game with initial states in obeying this distribution.

Consider the 2-player EFG in Figure 1, which begins with
the follower chooses to stay(exit) on the left(right) branches,
sponding to the blueprint. Thus, under the blueprint strate-
when best re-

Figure 1: Unsafe naïve search and its game tree. Boxed re-
denote subgames. Expected values for each player under
under naïve search is shown in the box, as are bounds guar-
ening no change of follower strategies after refinement.

3 Unsafe Search

To motivate our algorithm, we first explore how unsafe be-
Naïve search assumes that prior to entering a subgame, the follower plays the best-response to
For each subgame, the leader computes a normalized distri-
itial (subgame) states and solves a new game with initial states in obeying this distribution.
Consider the 2-player EFG in Figure 1 which begins with chance choosing each branch with equal probability. The follower then decides to e(X)it, or (S)tay, where the lat-
er brings the game into a subgame, denoted by the dotted box. Upon reaching A, the follower recieves an expected value (EV) of 1 when best responding to the blueprint. Upon reaching B, the follower recieves an EV of 0 when best re-
sponding to the blueprint. Thus, under the blueprint strategy, the follower chooses to stay(exit) on the left(right) branches, and the expected payoff per player is (1.5, 1.5).

Example 1. Suppose the leader performs naïve search in Figure 1 which improves the leader’s EV in A from 1 to 2 but reduces the follower’s EV in A from 1 to −1. The fol-
ower is aware that the leader will perform this search and thus chooses X1 out S1 even before entering A, since ex-
iting gives a payoff of 0. Conversely, suppose this search improves the leader’s EV in B from 0 to 1 and also improves the follower’s EV from 0 to 4. Then the higher post-search payoff in B causes the follower to switch from X2 to S2. These changes cause the leader’s EV to drop from 1.5 to 0.5.

Thus, sticking to the blueprint is preferable to naïve search, which means naïve search is unsafe.

Insight: Naïve search may induce changes in the follower’s strategy before the subgame, which adjusts the probability of entering each state within the subgame. If one could enforce that in the refined subgame, payoffs to the follower in A remain no less than 0, then the follower would continue to stay, but possibly with leader payoffs greater than the blueprint.
Similarly, we may avoid entering B by enforcing that follower payoff in B not exceed 2.

Example 2. Consider the game in Figure 2. Here, the fol-
lower chooses to exit or stay before the chance node is reached. If the follower chooses stay, then the chance node determines which of two identical subgames is entered. Un-
der the blueprint, the follower receives an EV of 1 for choosing stay and an EV of 0 for choosing exit.

Suppose search is performed only in the left subgame, which decreases the follower’s EV in that subgame from 1 to −1. Then, the expected payoff for staying is (1.5, 0). The follower continues to favor staying (breaking ties in favor of the leader) and the leader’s EV increases from 1.0 to 1.5.

Now suppose search is performed on whichever subgame is encountered during play. Then the follower knows that his EV for staying will be −1 regardless of which subgame is reached, and thus will exit. Exiting decreases the leader’s payoff to 0 compared to the blueprint value of 1, and thus the search is unsafe.

Insight: Performing search using Algorithm 1 is equivalent to performing search for all subgames. Even if conducting search only in a single subgame does not cause a shift in the follower’s strategy, the combined effect of applying search to multiple subgames may. Again, one could remedy this by carefully selecting constraints. If we bound the follower post-search EVs for each of the 2 subgames to be ≥ 0, then we can guarantee that X would never be chosen. Note that this is not the only scheme which ensures safety, e.g., a lower bound of 1 and −1 for the left and right subgame is safe too.

4 Safe Search for SSE

The crux of our method is to modify naïve search such that the follower’s best response remains the same even when

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1This counterexample is because of the general-sum nature of this game, and does not occur in zero-sum games.
2This issue occurs in zero-sum nature of this game, and does not occur in zero-sum games.
search is applied. This, in turn, can be achieved by enforcing bounds on the follower’s EV in any subgame strategies computed via search. Concretely, our search method comprises 3 steps, (i) preprocess the follower’s best response to the blueprint and its values, (ii) identify a set of non-trivial safety bounds on follower payoffs \( c \), and (iii) solving for the SSE in the subgame reached constrained to respect the bounds computed in (ii).

**Preprocessing of Blueprint.** Denote the leader’s sequence form blueprint strategy as \( r^\text{bp}_1 \). We will assume that the game is small enough such that the follower’s (pure, tiebreaks leader-favored) best response to the blueprint may be computed—denote it by \( r^\text{bp}_2 \). We call the set of information sets which, based on \( r^\text{bp}_2 \) have non-zero probability of being reached the *trunk*, \( T \subseteq \mathcal{I}_2 : r^\text{bp}_2(\text{Seq}_2(T)) = 1 \). Next, we traverse the follower’s treeplex bottom up and compute the payoffs at each information set and sequence (accounting for chance factors \( C(z) \) for each leaf). We term these as best-response values (BRVs) under the blueprint. These are recursively computed for both \( \sigma_2 \in \Sigma_2 \) and \( I_2 \in \mathcal{I}_2 \) recursively; (i) \( \text{BRV}(I_2) = \max_{\sigma_2 \in A(I_2)} \text{BRV}(\sigma_2) \), and (ii) \( \text{BRV}(\sigma_2) = \sum_{I' \in \mathcal{I}_2} \text{Seq}_2(I') = \sigma_2 \text{BRV}(I') + \sum_{\sigma_1 \in \Sigma_1} r_1(\sigma_1) g_2(\sigma_1, \sigma_2) \), where \( g_i(\sigma_1, \sigma_{-i}) \) is the expected utility of player \( i \) over all nodes reached when executing the sequence pair \( (\sigma_i, \sigma_{-i}) \), \( g_i(\sigma_1, \sigma_{-i}) = \sum_{h \in Z; \sigma_i = \text{Seq}_i(h)} u_i(h) \cdot C(h) \). This processing step involves just a single traversal of the game tree.

**Generating Safety Bounds.** Loosely speaking, we traverse the follower’s treeplex top down while propagating follower payoffs bounds which guarantee that the follower’s best response remains \( r^\text{bp}_2 \). This is recursively done until we reach an information set \( I \) belonging to some subgame \( j \). The EV of \( I \) is then required to satisfy its associated bound for future steps of the algorithm. We illustrate the bounds generation process using the worked example in Figure 3. Values of information sets and sequences are in blue and annotated in order of traversal alongside their bounds, whose computation is as follows.

- The empty sequence \( \emptyset \) requires a value greater than \(-\infty\).
- For each information set (in this case, B) which follows \( \emptyset \), we require (vacuously) for their values to be \( \geq -\infty \).
- We want the sequence C to be chosen. Hence, the value of C has to be \( \geq 3 \), which, with the lower bound of \(-\infty\) gives a final bound of \( \geq 3 \).
- Sum of values for parallel information sets D and H must be greater than C. Under the blueprint, their sum is 5. This gives a ‘slack’ of 2, split evenly between D and H, yielding bounds of \( 2 - 1 = 1 \) and \( 3 - 1 = 2 \) respectively.
- Sequence E requires a value no smaller than F, G, and the bound for by the D, which contains it. Other actions have follower payoffs smaller than 1. We set a lower bound of 1 for E and an upper bound of 1 for F and G.
- Sequence I should be chosen over J. Furthermore, the value of sequence I should be \( \geq 2 \)—this was the bound propagated into H. We choose the tighter of the J’s blueprint value and the propagated bound of 2, yielding a bound of \( \geq 2.5 \) for I and a bound of \( \leq 2.5 \) for J.
- Sequences K and L should not be reached if the follower’s best response to the blueprint is followed—we cannot make this portion too appealing. Hence, we apply upper bounds of 1.5 for sequences K and L.

A formal description for bounds generation is deferred to the Appendix. The procedure is recursive and identical to the worked example. It takes as input the game, blueprint, best response \( r^\text{bp}_2 \), and follower BRVs and returns upper and lower bounds \( \text{BRV}(I) \) for all head information sets of subgame \( j \). Since the blueprint strategy and its best response satisfies these bounds, feasibility is guaranteed. By construction, lower and upper bounds are obtained for information sets within and outside the trunk respectively. Note also that bounds computation requires only a single traversal of the follower’s treeplex, which is smaller than the game tree.

![Figure 3: Example of bounds computation.](attachment:BoundsComputation.png)

The bounds generated are not unique. (i) Suppose we are splitting lower bounds at an information set \( I \) between child sequences (e.g., the way bounds for sequences E, F, G under information set D were computed). Let \( I \) have a lower bound of \( \text{BRV}(I) \) and the best and second best actions \( \sigma^* \) and \( \sigma' \) under the blueprint is \( v^* \) and \( v' \) respectively. Our implementation sets lower and upper bounds for \( \sigma^* \) and \( \sigma' \) to be \( \max \{ (\alpha \cdot v^* + (1 - \alpha) \cdot v'), \text{BRV}(I) \} \), \( \alpha \in [0, 1] \). Achieves safety. (ii) Splitting lower bounds at sequences \( \sigma \) between

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1 One could trivially achieve safety by sticking to the blueprint.
parallel information sets under \( \sigma \) (e.g., when splitting the slack at \( C \) between \( D \) and \( H \), or in Example 2). Our implementation splits slack evenly though any non-negative split suffices. We explore these issues in our experiments.

**MILP formulation for constrained SSE.** Once safety bounds are generated, we can include them in a MILP similar to that of Bosansky and Cermak (2013). The solution of this MILP is the strategy of the leader, normalized such that \( r^\text{bp}_{\text{seq}}(I_i) \) for all \( I_i \in \mathcal{I}_{i, \text{head}} \) is equal to 1. Let \( Z^j \) be the set of terminal states which lie within subgame \( j \), \( Z^j = Z \cap H^j_{\text{sub}} \). Let \( C^j(z) \) be the new chance probability when all actions taken prior to the subgame are converted to strategies, \( v \)

\[
\max_{p,r,v,s} \sum_{z \in Z^j} p(z)u_1(z)C^j(z)
\]

\[
v_{\text{Inf}}(\sigma_2) = s_{\sigma_2} + \sum_{i \in \mathcal{I}_{2,\text{head}}} v_{I_i} + \sum_{\sigma_1 \in \Sigma_2} r_1(\sigma_1, \sigma_2) g^1_2(\sigma_1, \sigma_2) \quad \forall \sigma_2 \in \Sigma_2^j
\]

\[
r_1(\sigma_i) = 1 \quad \forall i \in \{1,2\}; I_i \in \mathcal{I}_{i, \text{head}} : \text{Seq}_i(I_i) = \sigma_i
\]

\[
r_i(\sigma_i) = \sum_{a \in A_i(I_i)} r_i(\sigma_i, a)
\]

\[
\forall i \in \{1,2\} \quad \forall I_i \in \mathcal{I}_i, \sigma_i = \text{Seq}_i(I_i)
\]

\[
0 \leq s_{\sigma_2} \leq (1 - r_2(\sigma_2)) : M \quad \forall \sigma_2 \in \Sigma_2^j
\]

\[
0 \leq p(z) \leq r_2(\text{Seq}_2(z)) \quad \forall z \in Z^j
\]

\[
0 \leq p(z) \leq r_2(\text{Seq}_2(z)) \quad \forall z \in Z^j
\]

\[
\sum_{z \in Z^j} p(z)C^j(z) = M(j)
\]

\[
v_{I_2} \geq B(I_2) \quad \forall I_2 \in \mathcal{I}_{2,\text{head}} \cap T
\]

\[
v_{I_2} \geq B(I_2) \quad \forall I_2 \in \mathcal{I}_{2,\text{head}} \cap T
\]

\[
r_2(\sigma_2) \in \{0,1\} \quad \forall \sigma_2 \in \Sigma_2^j
\]

\[
0 \leq r_1(\sigma_i) \leq 1 \quad \forall \sigma_1 \in \Sigma_1^j
\]

Conceptually, \( p(z) \) is such that the probability of reaching \( z \) is \( p(z)C(z) \). \( r_1 \) and \( r_2 \) are the leader and follower sequence form constraints, \( v \) is the value of information set when \( r \) is adopted and \( s \) is the slack for each sequence.

**Objective** \( 1 \) is the expected payoff in the full game that the leader gets from subgame \( j \). \( 3 \) and \( 4 \) are sequence form constraints. \( 5 \) and \( 6 \) ensure the follower is best responding, and \( 8 \) ensures that the probability mass entering \( j \) is identical to the blueprint. Constraints \( 9 \) and \( 10 \) are bounds previously generated and ensure the follower does not deviate from \( r^\text{bp}_2 \) after refinement. We discuss more details of the MILP in the appendix.

**Safe Search as SSE solutions.** One is not restricted to using a MILP to enforce these safety bounds. Here we show that the constrained SSE to be solved may be the solution to another SSE problem. This implies that we can employ other SSE solvers, such as those involving strategy generation (Cerny, Bosansky, and Kiekintveld 2018). We briefly describe how transformation is performed on the \( j \)-th subgame, under the mild assumption that follower head information sets \( I^j_{2, \text{head}} \) are the initial states in \( H^j_{\text{sub}} \). More detail is provided in the Appendix. Figure 4 shows an example construction based on the game in Figure 1.

For every state \( h \in I^j_{2, \text{head}} \), we compute the probability \( \omega_h \) of reaching \( h \) under the blueprint, assuming the follower plays to reach it. The transformed game begins with chance leading to a normalized distribution of \( \omega \) over these states. Now, recall that we need to enforce bounds on follower payoffs for head information sets \( I_2 \in \mathcal{I}^j_{2, \text{head}} \). To enforce a lower bound \( BV(I_2) \geq B(I_2) \), we use a technique similar to subgame resolving (Burch, Johanson, and Bowling 2014). Before each state \( h \in I_2 \), insert an auxiliary state \( h' \) belonging to a new information set \( I_2' \), where the follower may opt to terminate the game with a payoff of \( (-\infty, B(I_2)/|\omega_h(I_2)|) \) or continue to \( h \), whose subsequent states are unchanged. If the leader’s strategy has \( BV'(I_2) < B(I_2) \), the follower would do better by terminating the game, leaving the leader with \( -\infty \) payoff.

Enforcing upper bounds \( BV'(I_2) \leq B(I_2) \) may be done analogously. First, we reduce the payoffs to the leader for all leaves underneath \( I_2 \) to \( -\infty \). Second, the follower has an additional action at \( I_2 \) to terminate the game with a payoff of \( (0, B(I_2)/|\omega_h(I_2)|) \). If the the follower’s response to the leader’s strategy gives \( BV'(I_2) > B(I_2) \), then the follower would choose some action other than to terminate the game, which nets the leader \( -\infty \). If the bounds are satisfied, then the leader gets a payoff of 0, which is expected given that an upper bound implies that \( I_2 \) is not part of the trunk.

**5 Experiments**

In this section we show experimental results for our search algorithm (based on the MILP in Section 4) in synthetic 2-stage games, Goofspiel and Leduc hold’em poker (modified

\[\text{Figure 4: The transformed tree for solving the constrained SSE with the safety bounds of Figure 1. A’ and B’ are auxiliary states introduced for the follower. B’ is identical to B, except that leader payoffs are } -\infty.\]
simplex. Here, \( \kappa \) strategy has on the next stage.

Given that the leader played action pair \((a_1, b_1)\), the follower's best response is computed and used to evaluate the leader's payoff. Note by Algorithm 1. The follower's best response to this strategy is approximates a SSE and the prize is discarded. Hence, Goofspiel is not zero-sum, and the prize is discarded. Hence, Goofspiel (Ross 1971) is a game where players simultaneously bid over a sequence of \( n \) prizes, valued at \(0, \ldots, n-1\). Each player owns cards worth \(1, \ldots, n\), which are used in closed bids for prizes auctioned over a span of \( n \) rounds. Bids are public after each round. Cards bid are discarded regardless of the auction outcome. The player with the higher bid wins the prize. In a tie, neither player wins and the prize is discarded. Hence, Goofspiel is not zero-sum, players can benefit by coordinating to avoid ties.

In our setting, the \( n \) prizes are ordered uniformly in an order unknown to players. Subgames are selected to be all states which have the same bids and prizes after first \( m \) rounds are resolved. As \( m \) grows, there are fewer but larger subgames. When \( m = n \), the only subgame is the entire game. The blueprint was chosen to be the NE under a zero price subgame. To properly evaluate the benefits of search, we perform search on every subgame and combine the resulting subgame strategies to obtain the implicit full-game strategy prescribed by Algorithm 1. The follower’s best response to this strategy is computed and used to evaluate the leader’s payoff. Note that this is only done to measure how closely \( \kappa \) approximates a SSE—in practice, search is applied only to the subgame reached in actual play and is performed just once. Hence, the worst-case time for a single playthrough is no worse than the longest time required for search over a single subgame (and not the sum over all subgames).

We compare our method against the MILP proposed by Bosansky and Cermak (2015) rather than the more recent incremental strategy generation method proposed by Černý, Bošanský, and Kiekintveld (2018). The former is more flexible and applies to all EFGs with perfect recall, while the latter involves the Stackelberg Extensive Form Correlated Equilibrium (SEFCE) as a subroutine for strategy generation. Computing an SEFCE is itself computationally difficult except in games with no chance, in which case finding an SEFCE can be written as a linear program.

Two-Stage Games. The two-stage game closely resembles a 2-step Markov game. In the first stage, both players play a general-sum matrix game \( G_{\text{main}} \) of size \( n \times n \), after which, actions are made public. In the second stage, one out of \( M \) secondary games \( \{G_{\text{sec}}\} \), each general-sum and of size \( m \times m \) is chosen and played. Each player obtains payoffs equal to the sum of their payoffs for each stage. Given that the leader played action \( a_1 \), the probability of transitioning to game \( j \) is given by the mixture, \( P(G_{\text{sec}}(a_1) = \kappa \cdot X_{j,a_1} + (1 - \kappa) \cdot q_j) \), where \( X_{j,a_1} \) is a \( M \times n \) transition matrix non-negative entries and columns summing to 1 and \( q_j \) lies on the \( M \) dimensional probability simplex. Here, \( \kappa \) governs the level of influence the leader’s strategy has on the next stage.\(^1\)

The columns of \( X \) are chosen by independently drawing weights uniformly from \([0, 1]\) and re-normalizing, while \( q \) is uniform. We generate 10 games each for different settings of \( M, m, n \) and \( \kappa \). A subgame was defined for each action pair played in the first stage, together with the secondary game transitioned into. The blueprint was chosen to be the SSE of the first stage alone, with actions chosen uniformly at random for the second stage. The SSE for the first stage was solved using the multiple LP method and runs in negligible time (< 5 seconds). For full-game solving, we allowed Gurobi to run for a maximum of 100s. For search, we allowed 100s—in practice, this never exceeds more than 20 seconds for any subgame.

We report the average quality of solutions in Table 1. The full-game solver reports the optimal solution if converges. This occurs in the smaller game settings where \((M = m \leq 10)\). In these cases search performs near-optimally. In larger games \((M = m \geq 100)\), full-game search fails to converge and barely outperforms the blueprint strategy. In fact, in the largest setting only 2 out of 10 cases resulted in any improvement from the blueprint, and even so, still performed worse than our method. Our method yields substantial improvements from the blueprint regardless of \( \kappa \).

| \( n \) | \( M \) | \( m \) | \( \kappa \) | Blueprint | Ours | Full-game |
|-------|-------|-------|-------|--------|------|--------|
| 2     | 2     | 2     | 0.1   | 1.2945 | 1.4778 |
|       | 0.9   | 1.2951| 1.4519| 1.4790 |
| 2     | 10    | 10    | 0.1   | 1.1684 | 1.6179| 1.6186 |
|       | 0.9   | 1.1723| 1.6183| 1.6190 |
| 2     | 100   | 100   | 0.1   | 1.1730 | 1.6696| 1.3125 |
|       | 0.9   | 1.1722| 1.6696| 1.2652 |
| 2     | 100   | 100   | 0.1   | 1.3756 | 1.8722| 1.4073 |
|       | 0.9   | 1.3752| 1.8723| 1.4534 |

Table 1: Average leader payoffs for two-stage games.

The columns of \( X \) are chosen by independently drawing weights uniformly from \([0, 1]\) and re-normalizing, while \( q \) is uniform. We generate 10 games each for different settings of \( M, m, n \) and \( \kappa \). A subgame was defined for each action pair played in the first stage, together with the secondary game transitioned into. The blueprint was chosen to be the SSE of the first stage alone, with actions chosen uniformly at random for the second stage. The SSE for the first stage was solved using the multiple LP method and runs in negligible time (< 5 seconds). For full-game solving, we allowed Gurobi to run for a maximum of 100s. For search, we allowed 100s—in practice, this never exceeds more than 20 seconds for any subgame.

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Goofspiel. Goofspiel (Ross 1971) is a game where 2 players simultaneously bid over a sequence of \( n \) prizes, valued at \(0, \ldots, n-1\). Each player owns cards worth \(1, \ldots, n\), which are used in closed bids for prizes auctioned over a span of \( n \) rounds. Bids are public after each round. Cards bid are discarded regardless of the auction outcome. The player with the higher bid wins the prize. In a tie, neither player wins and the prize is discarded. Hence, Goofspiel is not zero-sum, players can benefit by coordinating to avoid ties.

In our setting, the \( n \) prizes are ordered uniformly in an order unknown to players. Subgames are selected to be all states which have the same bids and prizes after first \( m \) rounds are resolved. As \( m \) grows, there are fewer but larger subgames. When \( m = n \), the only subgame is the entire game. The blueprint was chosen to be the NE under a zero first stage with payoffs adjusted for the second. This intuition is incorrect—the leader can issue non-credible threats in the second stage, inducing the follower to behave favorably in the first.

\(^1\)One may be tempted to first solve the \( M \) Stackelberg games independently, and then apply backward induction, solving the

\[\begin{align*}
\text{Table 1: Average leader payoffs for two-stage games.}
\end{align*}\]
Table 2: Results for Goofspiel. §This is the earliest time that the incumbent solution achieves the given utility. †This is equivalent to full-game search.

| n     | (|Σ|, |I|)       | m  | Num. of subgames | Max. time per subgame (s) | Leader utility |
|-------|-------------|-----|------------------|---------------------------|----------------|
|       |             | 2   | 1728             | 5                         | 3.02           |
|       |             | 3   | 64               | 5                         | 3.07           |
| 4     | (2.1, 1.7) · 10^4 | 4†  | 1                | 5                         | 4.06           |
|       |             |     | 5.5 · 10^2       |                           |                |
| 5     | (2.7, 2.2) · 10^6 | 3   | 8000             | 1.0 · 10^2                | 5.19           |
|       |             | 4   | 125              | 1.0 · 10^2                | 5.29           |
|       |             | 5†  | 1                | 1.0 · 10^4                | 5.03           |
|       |             |     | 1.8 · 10^4†      |                           | 5.65           |

Table 3: Leader payoffs for Leduc hold’em with n cards.

| α    | Goofspiel | Leduc |
|------|-----------|-------|
| 0.1  | -0.1609   | -0.1686 |
| 0.25 | -0.182   | -0.1862 |
| 0.5† | -0.2028   | -0.2028 |
| 0.75 | -0.212   | -0.212 |
| 0.9  | -0.203   | -0.203 |

Table 4: Leader payoffs for varying α and β. We consider Goofspiel with n = 5, m = 4 and Leduc Hold’em with n = 4. Time constraints are the same as previous experiments. ††These are the default values for α and β.

| β    | Goofspiel | Leduc |
|------|-----------|-------|
| 1††  | -0.182   | -0.1862 |
| 2     | -0.178   | -0.1832 |
| 4     | -0.1670  | -0.1609 |
| 8     | N/A       | -0.1690 |
| 16    | -0.1957  | -0.203 |

Leduc Hold’em. Leduc hold’em (Southey et al. 2012) is a simplified form of Texas hold’em. Players are dealt a single card in the beginning. In our variant there are n cards with 2 suits, 2 betting rounds, an initial bet of 1 per player, and a maximum of 5 bets per round. The bet sizes for the first and second round are 2 and 4. In the second round, a public card is revealed. If a player’s card matches the number of the public card, then he/she wins in a showdown, else the higher card wins (a tie is also possible).

Our variant of Leduc includes rake, which is a commission fee to the house. We assume for simplicity a fixed rake ρ = 0.1. This means that the winner receives a payoff of (1 − ρ)x instead of x. The loser still receives a payoff of −x. When ρ > 0, the game is not zero-sum. Player 1 assumes the role of leader. Subgames are defined to be all states with the same public information from the second round onward. The blueprint strategy was obtained using the unraked (ρ = 0, zero-sum) variant and is solved efficiently using a linear program. We limited the full-game method to a maximum of 5000 seconds and 200 seconds per subgame for our method. We reiterate that since we perform search only on subgames encountered in actual play, 200 seconds is an upper bound on the time taken for a single playthrough when employing search (some SSE are easier than others to solve).

The results are summarized in Table 3. For large games, the full-game method struggles with improving on the blueprint. In fact, when n = 8 the number of terminal states is so large that the Gurobi model could not be created even after 3 hours. Even when n = 6, model construction took an hour—it had near 7 · 10^9 constraints and 4 · 10^9 variables, of which 2.3 · 10^4 are binary. Even when the model was successfully built, no progress beyond the blueprint was made.

Varying Bound Generation Parameters. We now explore how varying α affects solution quality. Furthermore, we experiment with multiplying the slack (see information sets D and H in Section 4) by a constant β ≥ 1. This results in weaker but potentially unsafe bounds. Results on Goofspiel and Leduc are summarized in Figure 4. We observe that lower values of α yield slightly better performance in Leduc, but did not see any clear trend for Goofspiel. As β increases, we observe significant improvements initially. However, when β is too large, performance suffers and even becomes unsafe in the case of Leduc. These results suggest that search may be more effective with principled selections of α and β, which we leave for future work.

6 Conclusion

In this paper, we have extended safe search to the realm of SSE in EFGs. We show that safety may be achieved by adding a few straightforward bounds on the value of follower information sets. We showed it is possible to cast the bounded search problem as another SSE, which makes our approach complementary to other offline methods. Our ex-
experimental results on Leduc hold’em demonstrate the ability of our method to scale to large games beyond those which MILPs may solve. Future work includes relaxing constraints on subgames and extension to other equilibrium concepts.

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A Treeplexes

Algorithms utilizing the sequence form may often be better understood when visualized as treeplexes (Hoda et al. 2010; Kroer, Farina, and Sandholm 2018), with one treeplex defined for each player. Informally, a treeplex may be visualized as a tree with adjacent nodes alternating between information sets and sequences (actions), with the empty sequence forming the root of the treeplex. An example of a treeplex for Kuhn poker (Kuhn 1950) is given in Figure 5. In this example, a valid (pure) realization plan would be to raise when one obtains a King, or Jack, and when dealt a Queen, call, and follow by folding if the opponent raises thereafter. Mixed strategies in sequence form may be represented by 'flow' conservation constraints at infosets, with this flow being duplicated for parallel information sets.

Operations such as best responses have easy interpretations when visualized as treeplexes. When one player’s strategy is fixed, the expected values of all leaves may be determined (multiplied by P(h) and the probability that the other player selects his required sequence). From there, the value of each information set and sequence may be computed via a bottom-up traversal of the treeplex; when parallel information sets are encountered, their values are summed, and when an information set is reached, we select the action with the highest value. After the treeplex is traversed, actions chosen in each information set describe the behavioral strategy of the best response.

Figure 5: Treeplex of player 1 in Kuhn Poker. Filled squares represent information sets, circled nodes are terminal payoffs, hollow squares are points which lead to parallel information sets, which are preceded by dashed lines. Actions/sequences are given by full lines, and information from the second player is in square brackets. The treeplex is ‘rooted’ at the empty sequence. Subtreeplexes for the J and K outcomes are identical and thus omitted.

B Algorithm for Computing Bounds

In Section 4 we provided a worked example of how one could compute a set of non-trivial follower bounds which guarantee safety. Algorithm 2 provides an algorithmic description of how this could be done.

```
Function COMPUTEBOUNDS
Input : EFG specification, Blueprint and its BRVs
Output: Bounds B(I) for all I ∈ TJ1.head
EXPSEQTRUNK(∅, −∞)
end
Function EXPSEQTRUNK(σ, lb)
for I2 ∈ {I2 | SeqI2(I2) = σ} do
| slack ← (BRV(σ) − lb) / |{I2 | SeqI2(I2) = σ}|
| EXPINFTRUNK(I2, BRV(I2)-slack)
end
end
Function EXPINFTRUNK(I, lb)
if I ∈ T2.head then
| B(I) ← lb
return
end
for σ∗, σ′ ∈ best, second best actions in I under blueprint v∗, v′ ∈ BRV(σ∗), BRV(σ′)
bound ← max (v∗ − a∗), lb)
for σ ∈ {Σ2 | InfI2(σ) = I} do
if σ = σ∗ is in best response then
| EXPSEQTRUNK(σ, bound)
else
| EXPSEQNONTRUNK(σ, bound)
end
end
Function EXPSEQNONTRUNK(σ, ub)
for I2 ∈ {I2 | SeqI2(I2) = σ} do
| slack ← (ub − BRV(σ)) / |{I2 | SeqI2(I2) = σ}|
| EXPINFNONTRUNK(I2, BRV(I2)+slack)
end
end
Function EXPINFNONTRUNK(I, ub)
if I ∈ T2.head then
| B(I) ← ub; return
end
for σ ∈ {Σ2 | InfI2(σ) = I} do
| EXPSEQNONTRUNK(σ, ub)
end
end
Algorithm 2: Bounds generation procedure.
```

The COMPUTEBOUNDS function is the starting point of the bounds generation algorithm. It takes in an EFG specification, a blueprint given, and the BRVs (of sequences σ ∈ Σ2 and information sets I ∈ T2 computed while preprocessing the blueprint). Our goal is to populate the function B(I) which maps information sets I ∈ T2.head to upper/lower bounds on their values. We begin the recursive procedure by calling EXPSEQTRUNK on the empty sequence ∅ and vacuous lower bound −∞. Note that ∅ is always in the trunk (by definition).

Specifically, EXPSEQTRUNK takes in some sequence σ ∈ Σ2 and a lower bound lb. Note that we are guaranteed that lb ≤ BRV(σ). The function compute a set of lower bounds on payoffs of information sets I2 following σ such that (a) the best response to the blueprint satisfies these suggested
bounds and (b) under the given bounds on values of $I_2$, the follower can at least expect a payoff of $lb$. This is achieved by computing the slack, the excess of the blueprint with respect to $lb$ split equally between all $I_2$ following $\sigma$. For each of these $I_2$, we require their value be no smaller than the lower bound given by their BRVs minus the slack. Naturally, this bound is weaker than the BRV itself.

Now, the function EXPINFTRUNK does the same bound generation process for a given information set $I$ inside the trunk, given a lower bound $lb$. First, if $I$ is part of the head of a game in subgame $j$, then we simply store $lb$ into $B(I)$. If not, then we look at all sequences immediately following $I$—specifically, we compare the best and second best sequences, given by $\sigma^*$ and $\sigma'$. To ensure that the best sequence still remains the best response, we need to decide on a threshold bound such that (i) all sequences other than $\sigma^*$ does not exceed bound, and the value of $\sigma^*$ is no less than bound and (ii) the blueprint itself must obey bound. One way to specify bound is to take the average of the BRVs of $\sigma^*$ and $\sigma'$. For $\sigma^*$ we recursively compute bounds by calling EXPINFTRUNK. For all other sequences, we enter a new recursive procedure which generates upper bounds.

The function EXPSEQNONTRUNK is similar in implementation to its counterpart EXPSEQTRUNK, except that we compute upper instead of lower bounds. Likewise, EXPINFONONTRUNK stores an upper bound if $I$ is in the head of subgame $j$, otherwise, it uses recursive calls to EXPINFONONTRUNK to make sure that all immediate sequences following $I$ does not have value greater than $lb$.

In Section 4, we remarked how bounds could be generated in alternative ways, for example, by varying $\alpha$. This would alter the computation of bound in EXPINFTRUNK. In Section 5 we experiment with increasing the slack by some factor $\beta \geq 1$. That is, we alter the computation of slack in EXPSEQTRUNK by multiplying it by $\beta$. Note that this can potentially lead to unsafe behavior, since the follower’s payoff under this sequence may possibly be strictly less than $lb$.

C Details of MILP formulation for SSE

First, we review the MILP of Bosansky and Cermak (2015).

$$\max \sum_{p,r,u,v,s} p(z) u_1(z) C(z)$$  \hspace{1cm} (13)

$$v_{\text{inl}}(\sigma_2) = s_{\sigma_2} + \sum_{I' \in I_2: \text{Seq}(I')=\sigma_2} v_{I'} + \sum_{\sigma_1 \in \Sigma_2} r_1(\sigma_1) g_2(\sigma_1, \sigma_2) \quad \forall \sigma_2 \in \Sigma_2$$  \hspace{1cm} (14)

$$r_i(\emptyset) = 1 \quad \forall i \in \{1, 2\}$$  \hspace{1cm} (15)

$$r_i(\sigma_i) = \sum_{a \in A_i(I_i)} r_i(\sigma_i a) \quad \forall i \in \{1, 2\}, \forall I_i \in I_i, \sigma_i = \text{Seq}_i(I_i)$$  \hspace{1cm} (16)

$$0 \leq s_{\sigma_2} \leq (1 - r_2(\sigma_2)) \cdot M \quad \forall \sigma_2 \in \Sigma_2$$  \hspace{1cm} (17)

$$0 \leq p(z) \leq r_2(\text{Seq}_2(z)) \quad \forall z \in Z$$  \hspace{1cm} (18)

$$0 \leq p(z) \leq r_1(\text{Seq}_1(z)) \quad \forall z \in Z$$  \hspace{1cm} (19)

$$\sum_{z \in Z} p(z) C(z) = 1$$  \hspace{1cm} (20)

$$r_2(\sigma_2) \in \{0, 1\} \quad \forall \sigma_2 \in \Sigma_2$$  \hspace{1cm} (21)

$$0 \leq r_1(\sigma_1) \leq 1 \quad \forall \sigma_1 \in \Sigma_1$$  \hspace{1cm} (22)

Conceptually, $p(z)$ is the product of player probabilities to reach leaf $z$, such that the probability of reaching $z$ is $p(z) C(z)$. The variables $r_1$ and $r_2$ are the leader and follower strategies in sequence form respectively, while $v$ is the EV of each follower information set when $r_1$ and $r_2$ are adopted. $s$ is the (non-negative) slack for each sequence/action in each information set, i.e., the difference of the value of an information set and the value of a particular sequence/action within that information set. The term $g_i(\sigma_i, \sigma_{-i})$ is the EV of player $i$ over all nodes reached when executing a pair of sequences $(\sigma_i, \sigma_{-i})$. This is achieved by multiplying it by $\alpha_i(h)$. $C(h)$.

Constraint (14) ties in the values of the information set $v$ to the slack variables $s$ and payoffs. That is, for every sequence $\sigma_2$ of the follower, the value of its preceding information set is equal to the EV of all information sets $I'$ immediately following $\sigma_2$ (second term) added with the payoffs from all leaf sequences terminating with $\sigma_2$ (third term), compensated by the slack of $\sigma_2$. Constraints (15) and (16) are the sequence form constraints (Von Stengel 1996). Constraint (17) ensures that, for large enough values of $M$, if the follower’s sequence form strategy is 1 for some sequence, then the slack for that sequence cannot be positive, i.e., the follower must be choosing the best action for himself. Constraints (18), (19), and (20) ensure that $p(z) C(z)$ is indeed the probability of reaching each leaf. Constraints (21) and (22) enforce that the follower’s best response is pure, and that sequence form strategies must lie in $[0, 1]$ for all sequences. The objective (13) is the expected utility of the leader, which is linear in $p(z)$.

The MILP we propose for solving the constrained subgame is similar in spirit. Constraints (23)-(29), (31) and (32) are analogous to constraints (14)-(20), (21), (22) except that they apply to the subgame $j$ instead of the full game. Similarly, the objective (13) is to maximize the payoffs from within subgame $j$. The key addition is constraint (26) and (27), which are precisely the bounds computed earlier when traversing the treemplex.

D Transformation of Safe Search into SSE

Solutions

We provide more details on how the constrained SSE can be cast as another SSE problem. The general idea is loosely related to the subgame resolving method of Burch, Johnson, and Bowling (2014), although our method extends to general sum games, and allows for the inclusion of both upper and lower bounds as is needed for our search operation.

The broad idea behind Burch, Johnson, and Bowling (2014) is to (i) create an initial chance node leading to all leading states in the subgame (i.e., all states $h \in H_{\text{sub}}^j$ such that there are no states $h' \in H_{\text{sub}}^j$ such that $h' \cap h$) based on the normalized probability of encountering those states
under \( r^{bp}_i \) and (ii) enforce the constraints using a gadget, specifically, by adding a small number of auxiliary information sets/actions to help coax the solution to obey the required bounds.

**Restricted case: initial states \( h \) in head information sets**

We make the assumption that the \( \mathcal{I}^j_{2,\text{head}} \) is a subset of the initial states in subgame \( j \).

**Preliminaries** For some sequence form strategy pair \( r_1, r_2 \) for leader and follower respectively, the expected payoff to player \( i \) is given by \( \sum_{z \in Z; \sigma_i} p_i(z) r_1(\sigma_1) r_2(\sigma_2) u_i(z) \cdot C(z) \), i.e., the summation of the utilities \( u_i(z) \) of each leaf of the game, multiplied by the probability that both players play the required sequences \( r_i \) and the chance factor \( C(z) \). That is, the utility from each leaf is weighed by the probability of reaching it \( r_1(\sigma_1) r_2(\sigma_2) C(z) \). The value of an information set \( I \in \mathcal{I}^j_2 \) is the contribution from all leaves under \( I \), i.e., \( V(I) = \sum_{h \in H, h' \in H, h = \text{Seq}_h(h)} r_1(\sigma_1) r_2(\sigma_2) u_i(h) C(h) \), taking into account the effect of chance for each leaf.

Now let \( b_i(\sigma_i) \) be the behavioral strategy associated with \( \sigma_i \), i.e., \( b_i(\sigma_i) = r_i(\sigma_i) / r_i(\text{Inf} (\text{Seq}(\sigma_i))) \) if \( r_i > 0, b_i = 0 \) otherwise. The sequence form \( r_i(\sigma_i) \) is the product of behavioral strategies in previous information sets. Hence, each of these terms in \( V \) (be it from leader, follower, or chance) can be separated into products involving those before or after subgame \( j \). That is, for a leaf \( z \in Z^j \), the probability of reaching it can be written as \( \omega(z) = \omega^l(z) \cdot \omega^r(z) \cdot \omega^c(z) \), where \( \omega^l \) and \( \omega^r \) represent probabilities accrued before and after subgame \( j \) respectively.

**The original game.** Figure 6 illustrates our setting. For head information set \( h \in \mathcal{I}^j_{2,\text{head}} \), define \( \omega^{bp}_h \) to be the probability of reaching \( h \) following the leader’s blueprint assuming the follower plays to reach \( h \). Now denote by \( h_1^{2} \) the first state in subgame \( j \) leading to leaf \( z \) such that \( \omega^{bp}_{h_1^2} = r^{bp}_1(z) \cdot C^j(z) \) is the product of the contributions from the leader and chance, but not the follower. Then, the probability of reaching leaf \( z \) is given by \( \omega(z) = \omega^{bp}_{h_1^2} \cdot \omega^r(z) \cdot \omega^c(z) C^j(z) \).

Observe that if \( z^j \) lies beneath an infoset \( I^j_{2,\text{head}} \cap T \) (i.e., it lies in the trunk and the follower under the blueprint plays to \( I_1 \), \( \omega^{bp}_{h_1^2} = r^{bp}_1(z) \) (since \( r^{bp}_2(\text{Seq}(I_2)) = 1 \)). Conversely, if \( z^j \) lies under \( I^j_{2,\text{head}} \cap T \), i.e., not part of the trunk, then \( \omega^{bp}_{h_1^2} = 0 \), and the probability of reaching the leaf (under \( r^{bp}_1 \) is 0. From now onward, we will drop the superscript \( \cdot \) from \( \omega \) when it is clear we are basing it on the blueprint strategy. This is consistent with the notation used in in Section 4.

We want to find a strategy \( r^j_1 \) such that for every information state \( I_2 \in \mathcal{I}^j_{2,\text{head}} \), when \( r_i = r^{bp}_i \), the best response \( r^*_2 \) ensures that \( V^j(I_2) \) obeys some upper or lower bounds. That is,

\[
V^j(I_2) = \sum_{z \in Z, h \in I_2, h_1^2 \in z} \omega^{h_1^2} \cdot r^j_1(z) \cdot r^j_2(z) \cdot C^j(z) \cdot u_i(z),
\]

should be no greater/less than some \( B(I_2) \).

**The transformed game.** Now consider the transformed subgame described in Section 4. Figure 7 illustrates how this transformation may look like and the corresponding probabilities. We look at all possible initial states in subgame \( j \), and start the game with chance leading to head states \( h \) with a distribution proportional to \( \omega_h \). For subgame \( j \), let the normalizing constant over initial states be \( \eta^j > 0 \). Note that since we are including states outside of the trunk, \( \eta^j \) may be greater or less than 1. We duplicate every initial state and head information set, giving the follower an option of terminating or continuing on with the game, where terminating yields an immediate payoff of \( -\infty \cdot B(I_2) / \omega_h(I_2) \) when the information set containing \( h \) belongs to the trunk, and \( 0 \cdot B(I_2) / \omega_h(I_2) \) otherwise. For leaves which are descendants of non-trunk information sets, i.e., \( z \in Z, h \in I_2 \in \mathcal{I}^j_{2,\text{head}} \cap T \), \( h \in z \), the payoffs for the leaders are adjusted to \(-\infty \). There is a one-to-one correspondence between the behavioral strategies in the modified subgame and the original game simply by using \( r^j_1 \) interchangeably.
Next, we show that (i) the bounds $B$ are satisfied by the solution to the transformed game, (ii) for head information sets in the trunk, any solution satisfying $B$ will never achieve a higher payoff by selecting an auxiliary action, and (iii) for head information sets outside of the trunk, the solutions satisfying $B$ will, by selecting the auxiliary action, achieve a payoff greater or equal to continuing with the game. For (i), we first consider information set $I_2^t$, which is a head info set also within the trunk. The terminate action will result in a follower payoff (taking into account the initial chance node which was added) independent of the leader’s subgame strategy $\tilde{r}_1$:

$$\sum_{h \in I_2^t} \frac{B(I_2^t)}{\omega_{h,t} |I_2^t|} \cdot \eta' \omega_{h,t} = \eta' \mathcal{B}(I_2^t).$$  \hspace{1cm} (24)

If the follower chooses to continue the game, then his payoff (now dependent on the leader’s refined strategy $\tilde{r}_1$) is obtained by performing weighted sums over leaves

$$\eta' \sum_{z \in Z, h \in I_2^t, h \subseteq z} \omega_{h,t} \cdot \tilde{r}_1(z) \cdot \tilde{r}_2(z) \cdot C_j(z) \cdot u_2(z).$$  \hspace{1cm} (25)

If the leader is to avoid obtaining $-\infty$, then the follower must choose to remain in the game, which will only happen when (25) $\geq$ (24), i.e.,

$$\sum_{z \in Z, h \in I_2^t, h \subseteq z} \omega_{h,t} \cdot \tilde{r}_1(z) \cdot \tilde{r}_2(z) \cdot C_j(z) \cdot u_2(z) \geq B(I_2^t).$$  \hspace{1cm} (26)

The expression on the left hand side of the inequality is precisely the expression in (25). Since the leader can always avoid the $-\infty$ payoff by selecting $\tilde{r}$ in accordance with the blueprint, the auxiliary action is never chosen and hence, the lower bounds for the value of trunk information sets is always satisfied.

Similar expressions can be found for non-trunk head info sets. (24) holds completely analogously. The solution to the transformed game needs to make sure the follower always selects the auxiliary action is always chosen for information sets not belonging to the trunk, so as to avoid $-\infty$ payoffs from continuing. Therefore, the solution to the transformed game guarantees that (26) holds, except that the direction of the inequality is reversed. Again, the left hand side of the expression corresponds to (23). Hence, the SSE for the transformed game satisfies our required bounds. Furthermore, by starting from (23) and working backward, we can also show that any solution $\tilde{r}_1$ satisfying the constrained SSE does not lead to a best response of $-\infty$ for the leader.

Finally, we show that the objective function of the game is identical up to a positive constant. In the original constrained SSE problem, we sum over all leaf descendants of the trunk and compute the leader’s utilities weighed by the probability of reaching those leaves.

$$\sum_{I_2 \in I_2^{\text{head}}} \sum_{T \subseteq I_2} \omega_{h,t} \cdot \tilde{r}_1(z) \cdot \tilde{r}_2(z) \cdot C_j(z) \cdot u_1(z)$$

the same expression, except for an additional factor of $\eta$. Unlike the constrained SSE setting, the initial distribution has a non-zero probability of starting in a non-trunk state $h \in I_2 \in I_2^{\text{head}} \cap T$. However, since the auxiliary actions are always taken under optimality, the leader payoffs from those branches will be 0.

**The general case**

The general case is slightly more complicated. Now, the initial states in subgames may not belong to the follower. The issue with trying to add auxiliary states the same way as before is that there could be leader actions lying between the start of the subgame and the (follower) head information set. These leader actions have probabilities which are not yet fixed during the start of the search process. To overcome this, instead of enforcing bounds on information sets, we enforce bounds on parts of their parent sequences (which will lie outside the subgame).

We first partition the head information sets into groups based on their parent sequence. Groups can contain singletons. Observe that information sets in the same group are either all in the trunk or all are not. Let the groups be $G_k = \{I_{2,1}^k, I_{2,2}^k, \ldots, I_{2,m_k}^k\}$, $I_{2,q}^k \in I_{2,q}^{\text{head}}$, and the group’s heads be the initial states which contain a path to some state in the group, i.e., they are: $G_k^{\text{head}} = \{h | h' \subseteq h, h' \in I_{2,q}^k \}$.
\( I^j_{\text{head}} \) and \( \not\exists h'' \in H^j_{\text{sub}}, h'' \subset h \). Crucially, note that for two distinct groups \( G_i, G_k, i \neq k \), their heads \( G_{i, \text{head}} \) and \( G_{k, \text{head}} \) are disjoint. This because (i) there must be some difference in prior actions that player 2 took (prior to reaching the head information sets) that caused them to be in different groups, and (ii) this action must be taken prior to the subgame by the definition of a head information set.

If 2 information sets \( I_{2,1}, I_{2,2} \in I^j_{\text{head}} \) have the same parent sequence \( \sigma_2 = \text{Seq}_2(I_{2,1}) = \text{Seq}_2(I_{2,2}) \), i.e., they belong to the same group, \( G_k \), it follows that their individual bounds \( B(I_{2,1}), B(I_{2,2}) \) must have come from some split on some bound (upper or lower) on the value of \( \sigma_2 \). Instead of trying to enforce that the bounds for \( I_{2,1} \) and \( I_{2,2} \) are satisfied, we try to enforce bounds on the sum of the values of \( I_{2,1}, I_{2,2} \), since the sum is what is truly important in the bounds for \( \sigma_2 \) when we perform the bounds generation procedure.

The transformation then proceeds in the same way as the restricted case, except that we operate on the heads of each group, rather than on the head information sets. The bounds for heads of each group is the sum of the bounds of head information sets in that group, and that the factor containing the size of the head information set is replaced by the number of heads for that group.

Upper and lower bounds are enforced using the same gadget as the restricted case, depending on whether the bound is an upper or lower bound. Figure 8 shows an example of a lower bound in group \( G_k \). Note that the follower payoff in the auxiliary actions contains a sum over bounds over all information sets belonging to the group. Technically, we are performing safe search while respecting weaker (but still safe) bounds. Upper bounds are done the same way analogous to the restricted case.

Figure 8: An example of a general transformation. Brown dashed lines are the heads of individual groups. \( I^k_{2,1} \) and \( I^k_{2,2} \) belong to the same group \( G_k \) with the heads \( G_{k, \text{head}} \). The newly created auxiliary states are in the new information set \( I'_{2,k} \). In this case, \( G_k \) is in the trunk, hence we enforce a lower bound being enforced for \( G_k \).