N=2 Superconformal Affine Liouville Theory

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Abstract

We present a new supersymmetric integrable model: the $N=2$ superconformal affine Liouville theory. It interpolates between the $N=2$ super Liouville and $N=2$ super sine-Gordon theories and possesses a Lax representation on the complex affine Kac-Moody superalgebra $\hat{sl}(2|2)^{(1)}$. We show that the higher spin $W_{1+\infty}$-type symmetry algebra of ordinary conformal affine Liouville theory extends to a $N=2 W_{1/2+\infty}$-type superalgebra.
1 Introduction

It is a well-established fact [1-4] that an important class of conformally invariant integrable models can be produced starting from constrained WZNW theories based on affine Kac-Moody algebras (Conformally Affine Toda or CAT theories). The standard massive affine Toda theories can be recovered as a special limit of these models.

At the bosonic level any given simple Lie algebra admits an affine extension and the associate CAT model can be defined, with a Lax pair given on this affine extension.

In the supersymmetric case this is not always possible: to obtain a manifestly supersymmetric integrable model one needs to start from superalgebras admitting a set of purely fermionic simple roots, the reason for that being the fact that the Lax pair operators should be fermionic objects in this case. So, to learn which superalgebras admit an $N = 1$ superextended integrable CAT model it is sufficient to look at the classification of superalgebras and their root systems as is given in [5]. For instance, the $N = 1$ superconformal affine Liouville (super CAL) theory [4] which generalizes the massive super sinh-Gordon model [6] is associated with the twisted affine superalgebra $osp(2|2)^{(2)}$ having two fermionic simple roots.

As for the $N = 2$ case, the demand that a superalgebra admits a set of purely fermionic simple roots is necessary but not sufficient; a detailed discussion of which superalgebras can give rise to integrable models possessing a second supersymmetry has been given in [7-9]. The simplest $N = 2$ superconformal integrable model is the $N = 2$ super Liouville theory [10] associated with the superalgebra $sl(2|1)$ and the simplest $N = 2$ massive integrable model is the $N = 2$ super sine-Gordon theory associated with the loop superalgebra $sl(2|2)^{(1)}$ [9, 11].

In this letter we present an integrable $N = 2$ superconformal affine Liouville model which reduces to the $N = 2$ super sine-Gordon and the $N = 2$ super Liouville models in two special limits. We will work in a manifestly supersymmetric $N = 2$ superfield approach. At first we will introduce our model in the lagrangian formulation and then we will discuss it in a Lax pair context [12]. We will also construct supercurrents which generate an infinite-dimensional higher spin symmetry superalgebra generalizing the $W_{1+\infty}$ type symmetry algebra of the ordinary CAT models [13].

2 The action and equations of motion

We will work in a manifestly supersymmetric formalism.

Let $x^{\pm \pm}$ denote the bosonic $2D$ light-cone coordinates, then the $N = 2 2D$ superspace is parametrized by $x^{\pm \pm}$ and the complex fermionic coordinates $\theta^{\pm}$ (and their conjugate $\bar{\theta}^{\pm}$). The $N = 1$ superspace is recovered by letting $\theta^{\pm} = \bar{\theta}^{\pm}$. The $N = 2$ spinor derivatives $D_{\pm}, \bar{D}_{\pm}$ are defined as:

\[
D_{\pm} = \frac{\partial}{\partial x^{\pm \pm}} - i\theta^{\pm} \partial_{\pm \pm},
\]

\[
\bar{D}_{\pm} = -\frac{\partial}{\partial \theta^{\pm}} + i\theta^{\pm} \partial_{\pm \pm}.
\]

The only non-vanishing bracket between them is given by

\[
\{D_{\pm}, \bar{D}_{\pm}\} = 2i \partial_{\pm \pm}.
\]
In particular we have
\[ D_\pm^2 = \mathcal{D}_\pm^2 = 0. \]

An \( N = 2 \) chiral superfield \( \Psi \), the simplest matter \( N = 2 \) supermultiplet, can be defined by the constraint
\[ D_\pm \Psi = 0, \]
while its conjugate satisfies
\[ \mathcal{D}_\pm \Psi^\dagger = 0 \]
and so it is an anti-chiral \( N = 2 \) superfield.

The bosonic conformal affine Liouville theory is formulated in terms of three real bosonic fields [1]. A natural way to promote it to a \( N = 2 \) supersymmetric theory is to put these fields in proper minimal \( N = 2 \) supermultiplets. So we will define the \( N = 2 \) superconformal affine Liouville theory via three \( N = 2 \) chiral superfields \( \Phi, \Lambda, \Sigma \). By analogy with the \( N = 0 \) and \( N = 1 \) [1, 4] cases we choose the action \( S \) to be
\[ S = \frac{1}{4} \int d^2 x d^2 \theta_+ d^2 \theta_- \{ \Phi \Phi^\dagger + \Sigma \Lambda^\dagger + \Lambda \Sigma^\dagger + \alpha e^{\Phi_+ \theta_-} + \beta e^{\Lambda_- \Phi_+ \theta_-} + \alpha^\dagger e^{\Phi_- \overline{\theta}_+ \overline{\theta}_-} + \beta^\dagger e^{\Lambda^\dagger_+ - \Phi_- \theta_+} \}. \] (2)

Without loss of generality, the constants \( \alpha, \beta \) can be chosen equal unity, \( \alpha, \beta = 1 \) (by means of proper constant shifts of the superfields \( \Phi, \Lambda \) and their conjugates).

From the above action the following superfield equations of motion can be derived:
\[ \mathcal{D}_+ \mathcal{D}_- \Phi = -e^{\Phi_+ \theta_+} + e^{\Lambda^\dagger_+ - \Phi_- \theta_+} \]
\[ \mathcal{D}_+ \mathcal{D}_- \Sigma = -e^{\Lambda^\dagger_+ - \Phi_- \theta_+} \]
\[ \mathcal{D}_+ \mathcal{D}_- \Lambda = 0 \] (3)
(and their conjugates). The \( N = 2 \) super sine-Gordon theory [3, 11] is recovered as the special solution \( \Lambda = \Lambda^\dagger = 0 \). On the other hand, redefining \( \Lambda \) as \( \Lambda \to a \Lambda \) and letting \( a \to \infty \), one recovers the \( N = 2 \) super Liouville theory [10]. In the bosonic limit the theory constructed becomes a conformal affine generalization of both the sine- and sinh-Gordon theories, as it is evident from the equations of motion written in terms of the component fields (see below).

Let us introduce the component fields as follows:
\[ \varphi \equiv \Phi| \]
\[ \psi_+ \equiv \mathcal{D}_+ \Phi| \]
\[ F \equiv \mathcal{D}_+ \mathcal{D}_- \Phi| \]
\[ \lambda \equiv \Lambda| \]
\[ \mu_+ \equiv \mathcal{D}_+ \Lambda| \]
\[ L \equiv \mathcal{D}_+ \mathcal{D}_- \Lambda| \]
\[ \sigma \equiv \Sigma| \]
\[ \rho_+ \equiv \mathcal{D}_+ \Sigma| \]
\[ S \equiv \mathcal{D}_+ \mathcal{D}_- \Sigma|. \] (4)
All these fields are complex, \( \varphi, \lambda, \sigma \) being bosonic, \( \psi_+, \mu_+ \), \( \rho_+ \) fermionic and \( F, L, S \) auxiliary.

After eliminating the auxiliary fields by their equations of motion we are led to the following expression for the action \( S \) in terms of component fields:

\[
S = \int d^2x \left\{ \partial_{++}\varphi\partial_{-}\varphi^\dagger + \partial_{++}\sigma\partial_{-}\lambda^\dagger + \partial_{++}\lambda\partial_{-}\sigma^\dagger \\
-\frac{i}{2}\partial_{-}\psi_+\overline{\psi}_+ - \frac{i}{2}\partial_{++}\psi_-\overline{\psi}_- - \frac{i}{2}\partial_{-}\rho_+\overline{\rho}_+ - \frac{i}{2}\partial_{++}\rho_-\overline{\rho}_- \\
-\frac{i}{2}\partial_{-}\mu_+\overline{\mu}_+ - \frac{i}{2}\partial_{++}\rho_-\overline{\rho}_- - \frac{i}{2}\psi_+\varphi^\dagger e^\varphi + \frac{i}{2}\overline{\psi}_+\varphi^\dagger e^{\varphi^\dagger} \\
-\frac{i}{2}(\mu_- - \rho_-)(\mu_+ - \rho_+)e^{\lambda^\dagger - \lambda} + \frac{i}{4}(\overline{\mu}_- - \overline{\rho}_-)(\overline{\mu}_+ - \overline{\rho}_+)e^{\lambda^\dagger - \lambda^\dagger} \\
-\frac{i}{4}e^{\varphi + \varphi^\dagger} - \frac{i}{4}e^{\lambda^\dagger - \lambda}e^{\lambda^\dagger - \lambda^\dagger} + \frac{i}{4}e^{\lambda^\dagger - \lambda^\dagger}e^{\varphi + \varphi^\dagger} \right\} .
\]

The equations of motion for the bosonic fields are:

\[
\begin{align*}
\partial_{++}\partial_{-}\varphi - \frac{1}{4}\overline{\psi}_-\psi_+e^{\varphi^\dagger} + \frac{1}{4}(\overline{\mu}_- - \overline{\rho}_-)(\overline{\mu}_+ - \overline{\rho}_+)e^{\lambda^\dagger - \lambda} \\
+\frac{1}{2}e^{\varphi^\dagger} - \frac{1}{4}e^{\lambda^\dagger + \lambda^\dagger}e^{\varphi + \varphi^\dagger} - \frac{1}{4}e^{\lambda^\dagger + \lambda^\dagger - \varphi - \varphi^\dagger} + \frac{1}{4}e^{\lambda^\dagger + \lambda^\dagger - \varphi - \varphi^\dagger} = 0
\end{align*}
\]

\[
\begin{align*}
\partial_{-}\partial_{++}\sigma - \frac{1}{4}(\overline{\mu}_- - \overline{\rho}_-)(\overline{\mu}_+ - \overline{\rho}_+)e^{\lambda^\dagger - \lambda^\dagger} + \frac{1}{4}e^{\lambda^\dagger + \lambda^\dagger - \varphi - \varphi^\dagger} - \frac{1}{4}e^{\lambda^\dagger + \lambda^\dagger - \varphi - \varphi^\dagger} = 0 \\
\partial_{--}\partial_{++}\lambda = 0 .
\end{align*}
\]

In the bosonic limit, with all fermions discarded, they are reduced to the system

\[
\begin{align*}
\partial_{--}\partial_{++}(Re\varphi) + \frac{1}{4}e^{2Re\varphi} - \frac{1}{4}e^{2Re\lambda - 2Re\varphi} &= 0 \\
\partial_{-}\partial_{+}(Im\varphi) - \frac{1}{2}e^{Re\lambda}sin(Im\lambda - 2Im\varphi) &= 0 \\
\partial_{-}\partial_{++}(Re\sigma) + \frac{1}{4}e^{2Re\lambda}cos(2Im\varphi) - \frac{1}{4}e^{Re\lambda}cos(Im\lambda - 2Im\varphi) &= 0 \\
\partial_{-}\partial_{++}(Im\sigma) + \frac{1}{4}e^{2Re\lambda}sin(2Im\varphi) + \frac{1}{4}e^{Re\lambda}sin(Im\lambda - 2Im\varphi) &= 0 \\
\partial_{--}\partial_{++}(Re\lambda) = \partial_{--}\partial_{++}(Im\lambda) &= 0 .
\end{align*}
\]

As was mentioned above, this set is a conformally invariant extension of both the sinh- and sine-Gordon equations (for the fields \( Re\varphi \) and \( Im\varphi \), respectively) which are restored in the limit \( \lambda = 0 \).

We end this section by giving the equations of motion for the fermionic fields

\[
\begin{align*}
\partial_{--}\psi_+ + \frac{i}{2}\overline{\psi}_-e^{\varphi^\dagger} - \frac{i}{2}(\overline{\mu}_- - \overline{\rho}_-)e^{\lambda^\dagger - \lambda^\dagger} &= 0 \\
\partial_{++}\psi_- - \frac{i}{2}\overline{\psi}_+e^{\varphi^\dagger} + \frac{i}{2}(\overline{\mu}_+ - \overline{\rho}_+)e^{\lambda^\dagger - \lambda} &= 0 \\
\partial_{-}\partial_{+}\rho_+ + \frac{i}{2}(\overline{\rho}_- - \overline{\rho}_-)e^{\lambda^\dagger - \lambda^\dagger} &= 0 \\
\partial_{++}\rho_- + \frac{i}{2}(\overline{\rho}_+ - \overline{\rho}_+)e^{\lambda^\dagger - \lambda} &= 0 \\
\partial_{--}\mu_+ = \partial_{++}\mu_- &= 0 .
\end{align*}
\]

### 3 Lax pair formulation

The model defined in the previous section is the conformal affine extension of the \( N = 2 \) super sine-Gordon theory considered in ref. \[\square\]. The latter theory is integrable:
its equations can be cast in a Lax form with the Lax connections valued in the loop superalgebra \( sl(2|2) \). In this section we will make explicit the integrability properties of our model by writing down its Lax pair formulation, which turns out to be based on an affine extension of \( sl(2|2) \).

Before going on, let us first recall some basic facts about \( sl(2|2) \) \cite{5, 9}. This superalgebra contains four bosonic generators \( h_i \) in the Cartan subalgebra and a set of four simple roots generators \( e^\pm_i \) (these all are fermionic). The (anti)commutation relations between the Cartan and simple roots generators are given by

\[
[h_i, e^\pm_j] = \pm a_{ij} e^\pm_j \\
\{e^\pm_i, e^\mp_j\} = \delta_{ij} h_j,
\]

with \( i, j = 1, \ldots, 4 \) and \( a_{ij} \) being the Cartan matrix

\[
a_{ij} = \begin{pmatrix}
0 & -1 & 0 & 1 \\
-1 & 0 & 1 & 0 \\
0 & 1 & 0 & -1 \\
1 & 0 & -1 & 0
\end{pmatrix}.
\]

The above commutation relations are the only ones which are actually needed to completely define the massive integrable theory corresponding to \( sl(2|2) \).

The key property of the above Cartan matrix is that it is degenerate, having rank \( r = 2 \): the Cartan elements \( h_1 + h_3 \) and \( h_2 + h_4 \) are \( c \)-numbers which commute with all other generators.

This fact is an indication that the integrable theory associated with this superalgebra, i.e. the \( N = 2 \) super sine-Gordon theory, is not conformally invariant: it is already known from the bosonic \cite{1, 2} and \( N = 1 \) supersymmetric \cite{4} cases that conformally invariant integrable models can be produced only starting from (super)algebras which have enough Cartan generators to remove the degeneracy among different roots. The best example is supplied by the sinh-Gordon theory related to the \( sl(2) \)-loop algebra. A conformally invariant generalization of such a model can be constructed by enlarging the \( sl(2) \)-loop algebra with additional new generators in its Cartan sector (a central extension and a derivative operator counting the powers of the loop spectral parameter): as a result one goes from the \( sl(2) \)-loop algebra to its affine extension.

A naive transfer of this procedure to the case of \( N = 2 \) super CAL theory does not work basically for two reasons: first, because the introduction of the derivative operator cannot completely remove the degeneracy among roots (an additional operator is needed for this purpose) and, second, because such a derivative is a real operator while we are dealing with complex superfields in the \( N = 2 \) case.

An essentially new feature of the superalgebra \( sl(2|2) \) compared to the (super)algebras relevant to the previously known conformal affine theories consists in the existence of an involution between its generators. Just this feature allows for the second supersymmetry. One should therefore expect that the extension of \( sl(2|2) \) appropriate to the present case is realized via the introduction of a pair of operators in involution. Indeed, our model involves chiral superfields and their conjugates.

Let us discuss in detail how such an extension can be constructed.
At first we notice that the involution just mentioned is realized, in the particular case of Cartan and simple roots generators, as

\[ h_1 \leftrightarrow h_2 \quad h_3 \leftrightarrow h_4 \]
\[ e_1^+ \leftrightarrow e_2^+ \quad e_3^+ \leftrightarrow e_4^+ \quad . \]  
\[ (11) \]

Since no confusion will arise, let us call for simplicity from now on the above involution as “conjugation” operation.

To enlarge the superalgebra we introduce a couple of conjugate bosonic operators \( d, \bar{d} \). These are defined as follows: the commutator between them and a generator \( e_\alpha \) corresponding to the root \( \alpha \), with the decomposition \( \alpha = \sum_i n_i \alpha_i \) for \( i = 1, \ldots, 4 \), \( n_i \) integer and \( \alpha_i \) denoting the simple roots, is given by:

\[
[d, e_\alpha] = -n_4 e_\alpha \\
[\bar{d}, e_\alpha] = n_3 e_\alpha ,
\]
\[ (12) \]

while their commutators with themselves and with the Cartan generators \( h_i \) are assumed to vanish.

Due to the properties of the root decomposition the above relations are consistent with the Jacobi identities.

The operators \( \bar{d}, d \) are grading operators with respect to the roots \( e_3^+, e_4^+ \), respectively. The choice of grading with respect to just these roots is to some extent arbitrary, the important point is that the root operators should be conjugate. For standard integrable theories associated with non-singular Cartan matrices such grading operators are not too useful notion since they can be re-expressed as combinations of the original Cartan generators. The importance of \( d, \bar{d} \) in the case at hand is that they allow to remove the degeneracy of the Cartan subalgebra.

We will denote the algebra enlarged by \( d, \bar{d} \) as \( \hat{sl}(2|2)^{(1)} \):

\[ sl(2|2)^{(1)} = sl(2|2)^{(1)} \oplus C d \oplus C \bar{d} \]

Clearly, \( d, \bar{d} \) belong to the Cartan sector of \( sl(2|2)^{(1)} \).

It should be mentioned that, in the bosonic and \( N = 1 \) supersymmetric cases \([1,4]\), in order to obtain a conformally invariant theory one is forced to introduce not only the derivative operator, but also the central extension. In the present case we do not need to further enlarge the algebra with central extensions, because we have already at our disposal a couple of (conjugate) \( c \)-numbers given by \( h_1 + h_3, h_2 + h_4 \) which play such a role.

To express our formulas in a concise notation, it is convenient to split the Cartan subalgebra in the two conjugate parts

\[
\mathcal{H} = \{ h_1, h_3, d \} \\
\overline{\mathcal{H}} = \{ h_2, h_4, \bar{d} \}
\]
\[ (13) \]

and to denote the sum over the sets of two conjugate simple roots by

\[ \mathcal{E}^\pm = e_1^+ + e_3^+ \quad \overline{\mathcal{E}}^\pm = e_2^+ + e_4^+ . \]
\[ (14) \]
Further, let us denote with \( \Psi, \overline{\Psi} \) two conjugate superfields taking values in the Cartan subsectors \( \mathcal{H}, \overline{\mathcal{H}} \), respectively:
\[
\Psi = A^\dagger h_1 + B^\dagger h_3 + C^\dagger d
\overline{\Psi} = Ah_2 + Bh_4 + C\overline{d}.
\]
(15)

We possess now all the necessary ingredients to express the integrability properties of our system by introducing two conjugate linear systems of the Lax pair type:
\[
(D_\pm + L_\pm)T = 0
\]
(16)
and
\[
(D_\pm + \overline{L}_\pm)\overline{T} = 0,
\]
(17)
where \( L_\pm, \overline{L}_\pm \) belong to the superalgebra \( \mathfrak{sl}(2|2)^{(1)} \) and \( T, \overline{T} \) to its associate affine KM supergroup.

The zero-curvature condition is the compatibility condition of the above linear systems and it is provided by the relations
\[
D_+ L_- + D_- L_+ + \{L_+, L_-\} = 0
\overline{D}_+ \overline{L}_- + \overline{D}_- \overline{L}_+ + \{\overline{L}_+, \overline{L}_-\} = 0.
\]
(18)

To ensure integrability, the equations of motion of the lagrangian formulation (3) should be recovered from the above zero-curvature relations. This is indeed the case.

Let us define
\[
L_+ = D_+ \Psi + e^{ad \Psi} \mathcal{E}^+
L_- = -\mathcal{E}^-
\]
(19)
for one copy of the Lax pair, and
\[
\overline{L}_+ = -\overline{\mathcal{E}}^+
\overline{L}_- = \overline{D}_- \overline{\Psi} + e^{ad \overline{\Psi}} \overline{\mathcal{E}}^-
\]
(20)
for the conjugate copy. Explicitly we have, in the former case,
\[
L_+ = (D_+ A^\dagger)h_1 + (D_+ B^\dagger)h_3 + (D_+ C^\dagger)d + e^{(B-A)}e^+_1 + e^{(C-(B-A))}e^+_3
L_- = -(e^-_1 + e^-_3)
\]
(21)
(and similar expressions in the latter case). It is a straightforward exercise to check that the zero-curvature condition is satisfied, once provided that the superfields \( A, B, C \) and their conjugates satisfy the following relations
\[
D_+ D_- A^\dagger = -e^{(B-A)}
D_+ D_- B^\dagger = -e^{(C-(B-A))}
D_+ D_- C^\dagger = 0
\]
(22)
together with their conjugate counterparts.

It is worth mentioning that the zero appearing in the r.h.s. of the equation for the superfield $C^\dagger$ is due to the fact that the $d$ generator never appears in the commutators of generators of $sl(2|2)^1$. Moreover, in the r.h.s. of all equations the superfields $A, B$ come out only in the combination $B - A$ since $h_1 + h_3$ is a $c$-number.

The Lax-pair version of the theory involves 3 complex superfields just like the lagrangian formulation of the previous section. The zero-curvature equations (22) reproduce just the equations of motion of the lagrangian theory under the following identification:

\[
\Phi \equiv B - A, \quad \Lambda \equiv C, \quad \Sigma \equiv -B
\]  
(23)

(the resulting equations are conjugate of eqs. (3)).

4  Superconformal properties and higher spin primary fields

A characteristic feature of the model we have constructed is its invariance under $N = 2$ superconformal transformations which are infinitesimally given by

\[
\begin{align*}
\delta \Phi &= -D_+ \delta \theta^+ \\
\delta \Lambda &= -2D_+ \delta \theta^+ \\
\delta \Sigma &= \alpha D_+ \delta \theta^+ \\
\mathcal{D}_- \delta \theta^+ &= 0
\end{align*}
\]  
(24)

and by their conjugate counterparts. Here $\alpha$ is an arbitrary real parameter. It is straightforward to check that the action (2) and the equations of motion (3) are invariant with respect to these transformations. Similar relations hold for the superconformal transformations acting in the opposite light-cone sector of $N = 2$ superspace (on the coordinates $x^{--}, \theta^-, \bar{\theta}^-$).

The presence of extra superfields $\Lambda, \Sigma$ allows to reestablish the superconformal invariance which is spoiled in the corresponding massive model ($N = 2$ super sine-Gordon theory). Due to these superfields the constructed model turns out to have a richer algebraic structure than both its special limits, the $N = 2$ super Liouville and super sine-Gordon theories. In particular, the action (2) is invariant under the shifts

\[
\Sigma \to \Sigma + \delta \Sigma, \quad \Sigma^\dagger \to \Sigma^\dagger + \delta \Sigma^\dagger,
\]

with $\delta \Sigma$ ($\delta \Sigma^\dagger$) being a sum of two arbitrary chiral (anti-chiral) superfunctions living in the left and right light-cone sectors of $N = 2$ superspace. By the way, using this freedom, one may always redefine the superconformal transformation of $\Sigma$ so as to make it homogeneous (this amounts to choosing $\alpha = 0$ in eqs. (24)).

Recall that the bosonic analog of our model, the CAL theory (as well as its Toda generalizations), exhibits a $W_{1+\infty}$ type symmetry algebra generated by an infinite set of higher spin primary fields together with the spin 2 conformal stress-tensor and a quasi-primary spin 1 field [13]. We will repeat in our case the procedure developed in [13] and will find an infinite set of supercurrents generating a $N = 2$ superextension of the
algebra present in the CAL theory. Like in most (super)conformally invariant theories, this superalgebra splits into two commuting light-cone copies. So, without loosing generality, we will limit our study to one of them corresponding to the light-cone direction specified by \(x^{++}, \theta^+, \bar{\theta}^+\).

Let us first recall the definition of primary \(N = 2\) superfields which are a natural generalization of the spin \(s\) primary fields used in the construction of ref. [13]. These are denoted by \(T^{(s_1 s_2)}\), with \(s_1, s_2\) being arbitrary integers, and transform under the \(N = 2\) superconformal group (24) according to the rule

\[
\delta T^{(s_1 s_2)} = (s_1 \overline{D}_+ \delta \theta^+ - s_2 D_+ \delta \theta^+) \cdot T^{(s_1 s_2)} .
\] (25)

The external conformal spin of \(T\) is defined as \(s \equiv \frac{1}{2}(s_1 + s_2)\) while \(h \equiv s_1 - s_2\) has the meaning of the external \(U(1)\) charge of \(T\) (under the convention that the \(U(1)\) charge of \(\theta^\pm\) equals 1).

The basic ingredient of the construction of ref. [13] is the anomaly free spin 2 conserved current \(T^{+++}\). Its analog in the \(N = 2\) case is the anomaly free conserved spin 1 supercurrent \(\tilde{J}^{(11)}_{++}\). In order to find it we proceed as follows.

As a first step we define the most general spin 1 real conserved supercurrent which turns out to be

\[
\begin{align*}
J_{++} &= \overline{D}_+ \Phi D_+ \Phi^\dagger + 2i \partial_{++}(\Phi - \Phi^\dagger) + \\
& \quad \overline{D}_+ \Lambda D_+ \Sigma^\dagger + \overline{D}_+ \Sigma D_+ \Lambda^\dagger + 4i \partial_{++}(\Sigma - \Sigma^\dagger) + \\
& \quad a \overline{D}_+ \Lambda D_+ \Lambda^\dagger + ib \partial_{++}(\Lambda - \Lambda^\dagger)，
\end{align*}
\] (26)

with \(a, b\) being arbitrary real constants.

The parameters \(a, b\) in (26) are uniquely fixed via the parameter \(\alpha\) defined in (24) if we further demand \(J_{++}\) to transform as a primary \((1, 1)\) \(N = 2\) superfield, i.e. require the absence of an inhomogeneous piece in its superconformal transformation law. In this way we recover the expression for the anomaly-free supercurrent \(\tilde{J}^{(11)}_{++}\) which is given by eq. (26) with the following fixed values of the involved parameters

\[
a = \alpha - \frac{1}{4} \quad b = 2\alpha - 1.
\] (27)

Note that the canonical \(N = 2\) conformal supercurrent generating, via appropriate super Poisson brackets, superconformal transformations of the superfields \(\Phi, \Lambda\) and \(\Sigma\) in eqs. (26) corresponds to a different choice of the parameters, namely,

\[
a = 0 \quad b = -2\alpha .
\] (28)

It is worth mentioning that it is anomaly-free at \(\alpha = \frac{1}{4}\).

One more entity used in [13] to construct an infinite sequence of the primary higher spin currents generating a \(W\) type symmetry algebra is a conserved spin 1 current. It

\footnote{A supercurrent \(J\) is called conserved if, by using the equations of motion for the involved superfields, it satisfies the relations \(D_- J = \overline{D}_- J = 0\) which imply in particular the condition of light-cone chirality with respect to the ordinary space-time variables, \(\partial_{--} J = 0\).}
is quasi-primary, i.e. inhomogeneously transforms under the conformal group. \( N = 2 \)

analogs of this current are two conjugate spin \( \frac{1}{2} \) conserved supercurrents

\[
J_+ = D_+ \Lambda^+ , \quad \bar{J}_+ = \bar{D}_+ \Lambda .
\]

These are quasi-primary with respect to superconformal transformations, e.g.,

\[
\delta J_+ = -D_+ \delta \theta^+ J_+ + 4i \partial_+ \delta \theta^+ .
\]

Now we are ready to generalize to our case the procedure employed in [13]. We introduce the operators \( I_1, I_2 \) which count, respectively, the weights \( s_1 \) and \( s_2 \) of the primary superfields

\[
I_{1,2} T^{(s_1, s_2)} = s_{1,2} T^{(s_1, s_2)}
\]

and define two supercovariant derivatives

\[
D_+ = D_+ - \frac{1}{2} D_+ \Lambda^+ I_1 , \quad \bar{D}_+ = \bar{D}_+ - \frac{1}{2} \bar{D}_+ \Lambda I_2 .
\]

They have the following properties: applied to a superfield of weights \( (s_1, s_2) \),

\[
T^{(s_1, s_2)} ,
\]

they send it, respectively, into superfields of weights \( (s_1, s_2 + 1) \) and \( (s_1 + 1, s_2) \). Indeed, it is a simple exercise to check that the superfields

\[
D_+ T^{(s_1, s_2)} , \quad \bar{D}_+ T^{(s_1, s_2)}
\]

transform according to the generic transformation law (25) with the aforementioned weights. Note that the supercovariant derivatives \( D_+ , \bar{D}_+ \) satisfy the following anticommutation relation:

\[
\{ D_+ , \bar{D}_+ \} = 2i \partial_+ + \left( \frac{1}{4} \bar{D}_+ \Lambda D_+ \Lambda^+ - i \partial_+ \Lambda^+ \right) I_1 + \left( \frac{1}{4} D_+ \Lambda^+ \bar{D}_+ \Lambda - i \partial_+ \Lambda^+ \right) I_2 \\
- \frac{1}{2} D_+ \Lambda^+ \bar{D}_+ (I_1 + 1) - \frac{1}{2} \bar{D}_+ \Lambda D_+ (I_2 + 1)
\]

\[
D_+ \bar{D}_+ = \bar{D}_+ D_+ = 0 .
\]

Now, starting with the anomaly-free spin 1 supercurrent \( \tilde{J}_{++}^{(1)} \) and acting on it successively by \( D_+ \) and \( \bar{D}_+ \), we may construct an infinite tower of conserved supercurrents with higher \( s_1 \) and \( s_2 \) which is a genuine \( N = 2 \) generalization of the set of currents of the bosonic CAL model. All these are primary with respect to the canonical \( N = 2 \) conformal supercurrent, i.e. have no anomalous terms in their superconformal transformations. The generic form of the basic supercurrents is as follows (taking account of the relations (30))

\[
J^{(n, n)}_{++} = \bar{D}_+ D_+ \ldots \bar{D}_+ D_+ \tilde{J}_{++}^{(1)} ,
\]

\[
J^{(n, n+1)}_{++} = D_+ J^{(n, n)}_{++} .
\]

The remaining primary supercurrents are obtained from these two basic sequences via complex conjugation and permutation \( s_1 \leftrightarrow s_2 \).

Together with the spin \( \frac{1}{2} \) conserved supercurrents and the canonical conformal supercurrent, the primary supercurrents defined above form a set which is nonlinearly closed under the super Poisson brackets and so is recognized as a kind of infinite-dimensional nonlinear \( N = 2 \) \( W_{1/2+\infty} \) superalgebra. Based on analogy with the bosonic case [13], in the limit of “large” \( \Lambda \) these supercurrents are expected to close on a \( N = 2 \) supersymmetric extension of the linear area-preserving \( w_{1+\infty} \) algebra. The detailed discussion will be reported elsewhere.
5 Concluding remarks

To summarize, in this paper we have constructed the $N = 2$ conformal affine super Liouville theory, shown its integrability by defining the appropriate superfield Lax pair (zero curvature) representation and found an infinite set of primary $N = 2$ supercurrents which are counterparts of the analogous currents of the bosonic CAT theories and form a nonlinear $N = 2 W_{1/2+\infty}$ type symmetry algebra of the model. It is straightforward to extend this construction to arbitrary $N = 2$ super Toda theory as like this has been done for the bosonic case in ref. [7]. An interesting question is how to reproduce $N = 2$ CAT theories from the appropriate supergroup $N = 2$ supersymmetric WZNW sigma models via hamiltonian reduction (for the bosonic and $N = 1$ supersymmetric CAT theories this has been done in [3] and [4]). A trouble here is that up to now a manifestly $N = 2$ supersymmetric superfield formulation of such sigma models for general target (super)groups is lacking. One more interesting problem we are planning to address in the nearest future is the construction of $N = 4$ supersymmetric conformal affine Liouville theory which should be an extension of $N = 4$ super Liouville theory [14].

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