Reputational Bargaining with Unknown Values*

Harry Pei†  Maren Vairo‡

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Abstract: A buyer and a seller bargain over the price of an object. The buyer’s value is common knowledge and the seller has private information about his production cost. In the beginning of the game, each player proposes a price and becomes committed to it with small probability. We characterize the set of equilibria. We show that equilibria with inefficient delays exist if and only if the difference in cost between some pair of adjacent types is large enough and the probability of low-cost type seller is sufficiently high. When there are multiple equilibria, the buyer prefers the least efficient equilibrium and all types of the seller prefer the most efficient equilibrium. In an extension where the seller can decide whether to adopt a cost-saving technology before bargaining, we pin down the equilibrium adoption rate and provide conditions under which bargaining inefficiencies arise in all equilibria.

Keywords: bargaining, reputational bargaining, inefficient delay, inefficient investment.

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†Department of Economics, Northwestern University. Email: harrydp@northwestern.edu
‡Department of Economics, Northwestern University. Email: mvairo@u.northwestern.edu
1 Introduction

Whether trading parties can reach efficient agreements in decentralized markets is an important topic in modern economic theory. When the gains from trade, the losses from delays, and the trading parties’ rationality are commonly known, an efficient agreement will be reached without any delay (Rubinstein, 1982). When one or both parties have private information about their rationality, or their gains from trade, or their losses from delays, the existing literature has reached different conclusions in terms of the efficiency of bargaining as well as the division of surplus, depending on the nature of the trading parties’ private information, the bargaining protocols (e.g., who can make offers), the selection of equilibrium, and so on.

This paper contributes to the literature on bargaining with incomplete information by analyzing a model where both trading parties can build reputations for being obstinate and one of them (e.g., a seller) has private information about his gains from trade as well as his loss from delays (e.g., the seller has private information about his cost of production). Our results examine how bargaining inefficiencies depend on the set of plausible production costs. As an application, we analyze the seller’s incentive to adopt cost-saving technologies. We show that the equilibrium adoption rate can be lower when the social benefit from adoption increases. Similar to other reputational bargaining models such as Kambe (1999) and Abreu and Gul (2000), our conclusions are not sensitive to the bargaining protocols, in the sense that they are valid as long as the bargaining frictions are sufficiently small and players can make offers at comparable rates.

In our model, a buyer and a seller bargain over the price of an object. The buyer’s value is commonly known. The seller’s cost is strictly less than the buyer’s value, which is the seller’s private information (we call it the seller’s *type*) and is drawn from a finite set. In the beginning of the game, each player proposes a price and becomes committed to it with small probability. Players trade immediately if the buyer’s proposed price is no less than that of the seller’s. Otherwise, players engage in a war-of-attrition where a player never concedes if he is committed, and decides when to concede if he is not committed. Once a player concedes, trade happens immediately at his opponent’s proposed price.

Theorem 1 characterizes the set of equilibria when players’ commitment probabilities are close to zero. Under a generic condition on the seller’s cost distribution, each equilibrium is characterized by a cutoff type such that (i) the buyer and every type of the seller with cost no more than this cutoff propose the cutoff type’s Rubinstein bargaining price, and (ii) every type with cost greater than this cutoff demands the entire surplus and trades with a delay, with the expected loss in welfare equals half of the social surplus. A type can be a cutoff type in some equilibria if and only if (i) the difference between this type’s cost and that of

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1The Rubinstein bargaining price is a weighted average of the buyer’s value and the seller’s cost, with the weight depends on players’ discount factors.
the next highest type is large enough, and (ii) the probability with which the seller’s cost is no more than this type is large enough.

Theorem 1 implies that (i) efficient equilibria always exist (since the highest-cost type can always be a cutoff type) where players trade immediately at the highest-cost type’s Rubinstein bargaining price, and (ii) inefficient equilibria exist if and only if there are other cutoff types aside from the highest-cost type. When multiple equilibria exist, every type of the seller prefers the most efficient equilibrium, and the buyer prefers the least efficient equilibrium.

The presence of inefficient equilibria stands in contrast to the conclusion in Abreu, Pearce, and Stacchetti (2015), who study a reputational bargaining model where the total surplus is common knowledge and one of the players has private information about his discount rate. When every commitment type makes an offer in the beginning of the game, they show that all equilibria are efficient and players trade at the Rubinstein bargaining price of the most patient type.

The comparison between their result and ours unveils a conceptual difference between private information about discount rates and private information about the size of surplus. In our model, the seller’s private information affects both the gains from trade and his losses from delays. When the buyer faces uncertainty about the seller’s cost, he has an incentive to extract more surplus by offering a price that is lower than some types of the seller’s costs. For those high-cost types, they will never concede to the buyer’s low price since doing so leads to a negative payoff. On the other hand, the only way for the high-cost type to credibly signal his cost is by demanding the entire surplus, since the buyer must have an incentive not to concede in the beginning (in order to discourage the low-cost types from mimicking) and must have an incentive to concede after a significant delay (in order to discourage the high-cost types from deviating and demanding anything less). By contrast, when the seller has private information about his discount rate but not his cost, either all types obtain a positive surplus from conceding or no type obtains a positive surplus from conceding, and the driving forces behind our inefficient equilibria disappear.

The above logic also explains our conditions for the cutoff types. The gap between adjacent types must be large enough so that the high-cost seller can credibly commit not to concede after the buyer offers the cutoff type’s Rubinstein price, and the buyer’s belief must attach high probability to the low-cost types in order for him to have an incentive to offer a low price and to forego obtaining a positive surplus in the event that the seller’s type is high.

Motivated by the coexistence of efficient and inefficient equilibria in our bargaining model, we study

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2Abreu, Pearce, and Stacchetti (2015) also show that there are inefficient equilibria when there exist commitment types who make their initial offers with significant delays.
two extensions where the seller’s type is endogenous, and provide conditions under which bargaining inefficiencies arise in all equilibria.

First, suppose the seller decides whether to adopt a cost-saving technology before the bargaining stage. Adopting the new technology is costly to the seller and the adoption decision is not observed by the buyer. We show that bargaining inefficiencies arise in all equilibria when the cost of adoption is between half of the reduction in production cost and the reduction in production cost. Our result also uniquely pins down the equilibrium adoption rate, which is strictly less than one despite adoption is socially efficient. The equilibrium adoption rate is first decreasing and then increasing once we increase the social gains from adoption. This is because when the benefit from adoption increases, it also increases the extent to which the buyer can expropriate the gain from adoption, which discourages the seller from adopting the technology.

Next, suppose the seller decides whether to conduct costly research and learn about his cost of serving the buyer before the bargaining stage. This fits applications where different buyers are horizontally differentiated and the cost of serving a buyer depends on whether the buyer’s need matches the seller’s expertise, which is not known to the seller in advance before doing costly research. We show that bargaining inefficiencies arise in all equilibria when the seller’s cost of learning his production cost is low enough.

**Related Literature:** This paper contributes to the literature on bargaining pioneered by Rubinstein (1982), and in particular, the works of Chatterjee and Samuelson (1987), Myerson (1997), Kambe (1999), Abreu and Gul (2000), Abreu, Pearce, and Stacchetti (2015), and Fanning (2018) on reputational bargaining.

The inefficient equilibria in our model are reminiscent of the separating equilibria in Admati and Perry (1987), in which the strong type of the informed player signals their strength by a delayed response to their opponent’s offer. The same prediction applies when the informed player’s private information is about their discount factor. In our model, players’ ability to build a reputation for being obstinate implies that only the strongest type matters for the equilibrium outcome in the war-of-attrition stage. As a result, the informed player is willing to reveal information only when the uninformed player makes an offer such that some types of the informed player cannot benefit from concessions. Making low offers is incentive compatible when the buyer is unsure about the seller’s cost, but not when the buyer is unsure about the seller’s discount rate.

Our work also contributes to the literature on the efficiency of bargaining under incomplete information pioneered by Stokey (1981) and Gul, Sonnenschein, and Wilson (1986) and has been revisited recently by Strulovici (2017) and Liu, Mierendorff, Shi, and Zhong (2019). The driving forces behind the inefficient equilibria in our model differ from those identified in existing works, such as the gains from

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3 Fanning (2018) and Basak and Deb (2020) show that players have incentives to make incompatible offers when they expect new information about the cost of delay or about the cost of concession to arrive in the future.
trade can be arbitrarily close to zero (Ausubel and Denecker, 1989), players face higher order uncertainty (Feinberg and Skrzypacz, 2005), players’ values are interdependent (Deneckere and Liang, 2006), new players arrive over time (Fuchs and Skrzypacz, 2010), the seller has stochastic time-varying costs (Ortner, 2017), and players face uncertainty about the future costs of delay (Fanning, 2018).

Ortner (2017) shows that when a monopolist’s cost may decrease over time, the bargaining outcome is efficient if and only if the buyer’s value is drawn from an interval. Intuitively, the gap between types helps the uninformed player to overcome their lack-of-commitment problem since it gives them an incentive to delay lowering prices. Although our analysis reaches a similar conclusion that all equilibria are efficient if and only if the gap between different types’ costs are small enough, the underlying logic is different from Ortner (2017). In particular, the gap between types helps the informed player to credibly signal their type since the high-cost type has no incentive to accept the buyer’s low-price offer.

Kim (2009) studies a bargaining model in which the seller is either committed or rational, the buyer has private information about their valuation, and the seller makes all the offers. They show that all equilibria are efficient when the probability of commitment type is small enough, which stands in contrast to the prediction of our model that some equilibria are inefficient and only the least efficient equilibria are robust when there are inefficient equilibria. This is because only the uninformed player can make offers in Kim (2009), so the outcome of the bargaining game can be approximated by a continuous-time war of attrition where all types of the informed player offer their lowest value or their highest cost. By contrast, the informed player can also make offers in our model, so sellers with different costs (or buyers with different valuations) can offer different prices. This can result in inefficient delays due to the high-cost seller’s (or low-valuation buyer’s) incentives to separate from the low-cost types (or high-valuation types).

2 Baseline Model

A buyer and a seller bargain over the price of an object in continuous time over an infinite horizon. The buyer’s value for the object is common knowledge which we normalize to 1. The seller’s production cost takes a finite number of values \( \theta \in \Theta \equiv \{ \theta_1, \ldots, \theta_n \} \subset [0,1) \), with \( 0 \leq \theta_1 < \theta_2 < \ldots < \theta_n < 1 \). We assume that \( \theta \) is the seller’s private information (i.e., the seller’s type) and is drawn according to a full support distribution \( \pi \in \Delta(\Theta) \). We use \( \pi[\theta', \theta''] \) to denote the probability that \( \theta' \leq \theta \leq \theta'' \) and we use \( \pi(\theta') \) to denote the probability that \( \theta = \theta' \).

Time 0 consists of three phases: 0₀, 0₁, and 0₂. At time 0₀, the seller privately observes \( \theta \). At time 0₁, the seller announces the price he is willing to accept \( p_\theta \in P^N \) and the buyer announces the price he is willing
to pay \( p_b \in P^N \). We call \( p_s \) and \( p_b \) the seller’s and the buyer’s bargaining postures. We study both the case where players can only offer prices on a discrete grid, i.e., \( P^N = \{0, \frac{1}{N}, \frac{2}{N}, \ldots, \frac{N-1}{N}, 1\} \) for some large \( N \in \mathbb{N} \), and the case where players can offer any price on \([0, 1]\), i.e., \( N = +\infty \). The case with a finite \( N \) will be used to guarantee the existence of equilibrium when we apply our results in the reputational bargaining game to study the seller’s incentives to invest in cost-saving technologies (Section 4).\(^4\)

After making announcements, each player becomes committed with probability \( \varepsilon \) and is flexible otherwise. If a player is committed, then he cannot accept any offer that is inferior to his initial demand. That is, the committed seller cannot accept any price lower than \( p_s \) and the committed buyer cannot accept any price greater than \( p_b \). If a player is flexible, then he chooses whether and when to concede in order to maximize his expected payoff. Whether each player is committed or flexible is independent of \( \theta \) and of whether his opponent is committed. Our analysis focuses on the case where \( \varepsilon \) is close to 0 and \( N \) is large, i.e., players are rational with probability close to one and can fine-tune their offers.

If \( p_s \leq p_b \), then the game ends at 0 and trade happens at price \( \frac{p_s + p_b}{2} \). If \( p_s > p_b \), then at time 0 and at every \( t \in (0, +\infty) \), each player decides whether to concede. The game ends as soon as at least one player concedes. If a player concedes before his opponent does, then trade happens at his opponent’s offered price. If players concede simultaneously, then trade happens at price \( \frac{p_s + p_b}{2} \).

Let \( r > 0 \) be players’ discount rate.\(^5\) If trade happens at time \( \tau \) and at price \( p \), then the seller’s payoff is \( e^{-r\tau}(p - \theta) \) and the buyer’s payoff is \( e^{-r\tau}(1 - p) \). Both players get 0 if they never reach an agreement.

The public history consists of calendar time, players’ bargaining postures, and whether any player has conceded before. The buyer’s private history consists of the public history and whether he is committed. The seller’s private history consists of the public history, whether he is committed, and the realized \( \theta \).

The seller’s strategy consists of a mapping from his type to a distribution of bargaining postures \( \sigma_s : \Theta \rightarrow \Delta(P^N) \), and a mapping from his type and the realized bargaining postures \((p_s, p_b)\) to a distribution of concessions times \( \tau_s(\theta, p_s, p_b) \in \Delta(\mathbb{R}_+ \cup \{+\infty\}) \), where \( \tau_s = +\infty \) means that the seller never concedes. The buyer’s strategy consists of a distribution of bargaining postures \( \sigma_b \in \Delta(P^N) \) and a mapping from realized bargaining postures to a distribution of concession times \( \tau_b(p_s, p_b) \in \Delta(\mathbb{R}_+ \cup \{+\infty\}) \). By definition, players reach an agreement at time \( \min\{\tau_s, \tau_b\} \), and the commitment types of each player chooses \( \tau_s = +\infty \) and \( \tau_b = +\infty \), respectively.

The solution concept is Perfect Bayesian equilibrium (or equilibrium for short), which is a strategy

\(^4\)The technical assumption that players offer prices from a discrete grid is also made in other bargaining papers in order to guarantee equilibrium existence. See for example Schmidt (1993).

\(^5\)We assume that players share the same discount rate \( r \) and are committed with the same probability in order to simplify notation. Our results extend when players have different discount rates and different probabilities of being committed.
profile \((\sigma_s, \sigma_b, \tau_s, \tau_b)\) and a belief such that

1. Type \(\theta\) seller chooses \(p_s\) in order to maximize \(\varepsilon V_s(\theta, p_s, \sigma_b, \tau_b) + (1 - \varepsilon)V^*_s(\theta, p_s, \sigma_b, \tau_b)\) where \(V_s\) is the seller’s payoff when he is committed and \(V^*_s\) is his payoff when he is flexible. The buyer chooses \(p_b\) in order to maximize \(\varepsilon V_b(p_b, \sigma_s, \tau_s) + (1 - \varepsilon)V^*_b(p_b, \sigma_s, \tau_s)\) where \(V_b\) is the buyer’s payoff when he is committed and \(V^*_b\) is his payoff when he is flexible.

2. Every flexible player’s concession strategy is sequentially rational under his belief.

3. Players’ beliefs respect Bayes rule at every information set on the equilibrium path.

**Remark:** We assume that players do not know whether they are committed or flexible when choosing their bargaining postures, which is also assumed in \(\text{Crawford (1982)}\), \(\text{Kambe (1999)}\), and \(\text{Wolitzky (2012)}\).

For an economic interpretation, think about the two players as the managers of two firms: the buyer-firm needs to purchase an item from the seller-firm. The buyer-firm’s manager becomes committed when he faces pressure from the board, making it costly for him to accept any offer that is inferior to his initial demand. A similar story applies to the seller-firm’s manager. Whether a manager will face pressure from the board is not known to him before the bargaining stage. Under this interpretation, the probability with which a player is committed may depend on the bargaining postures (e.g., the seller is less likely to face pressure if the buyer offers to pay a high price). The qualitative features of our main results extend when players’ commitment probabilities are functions of \((p_s, p_b)\).

### 3 Characterization Result

Our main result characterizes the set of equilibrium outcomes when \(\varepsilon\) is close to 0 and \(N\) is arbitrarily large. Since the buyer’s value is 1 and players share the same discount rate \(r\), type \(\theta\) seller’s Rubinstein bargaining price is

\[
p_{\theta} \equiv \frac{1 + \theta}{2}.
\]

Recall that \(\Theta \equiv \{\theta_1, \ldots, \theta_n\}\) and \(0 \leq \theta_1 < \ldots < \theta_n < 1\). For convenience, let \(\theta_{n+1} \equiv +\infty\). Let

\[
\Theta' \equiv \left\{ \theta_i \in \Theta \mid \text{there exists } j > i \text{ such that } \frac{\pi[\theta_i, \theta_j]}{\pi[\theta_i, \theta_j]} = \frac{1 - \theta_j}{\theta_j - \theta_i} \right\};
\]

and

\[
\Theta^* \equiv \left\{ \theta_i \in \Theta \mid p_{\theta_i} \leq \theta_{i+1} \text{ and for every } j > i, \text{ we have } \frac{\pi[\theta_i, \theta_j]}{\pi[\theta_{i+1}, \theta_j]} \geq \frac{2(1 - \theta_j)}{\theta_j - p_{\theta_i}} \right\}.
\]
3 \ CHARACTERIZATION RESULT

According to (3.2), $\Theta^\prime$ is empty for generic $\pi \in \Delta(\Theta)$. According to (3.3), $\Theta^*$ is nonempty since the highest cost $\theta_b$ belongs to $\Theta^*$ if and only if $\theta \in \Theta^*$ if and only if $\pi(\theta_i) = \frac{2(1-\theta_i)}{\theta_i - p_{\theta_i}}$. For every $\theta_i \neq \theta_b$, $\theta_i \in \Theta^*$ if and only if $\pi(\theta_i) \leq \theta_2$ and $\forall \theta_i \in \Theta^*$ if and only if $\pi(\theta_i) \geq \frac{2(1-\theta_i)}{\theta_i - p_{\theta_i}}$.

1. $\Theta^* = \{\theta_1, \theta_2\}$ if and only if $\frac{1+\theta_i}{\theta_i} = p_{\theta_i} \leq \theta_2$ and $\forall \theta_i \in \Theta^*$ if and only if $\pi(\theta_i) \geq \frac{2(1-\theta_i)}{\theta_i - p_{\theta_i}}$.

2. $\Theta^\prime$ is empty if and only if $\frac{\pi(\theta_i)}{\pi(\theta_2)} \neq \frac{1-\theta_i}{\theta_i - p_{\theta_i}}$, which is satisfied generically.

For every $\theta^* \in \Theta^*$, let $\sigma_{0,\theta^*}$ be the Dirac measure on $p_{\theta^*}$ and let $\sigma_{s,\theta^*}(\theta)$ be the Dirac measure on $p_{\theta^*}$ if $\theta \leq \theta^*$, and be the Dirac measure on 1 if $\theta > \theta^*$. Let $\sigma_{s,\theta^*} = \{\sigma_{s,\theta^*}(\theta)\}_{\theta \in \Theta}$. For every $\theta \in [0, 1]$ and $N \in \mathbb{N}$, let $|p|_N \equiv \max \{\frac{1}{N} | j \in \mathbb{N} \text{ and } \frac{j}{N} \leq p \}$ be the highest price on the $N$-grid that is no more than $p$. Let $|p|_{+\infty} \equiv p$. Theorem 1 characterizes the set of equilibria when players are rational with probability close to 1.

**Theorem 1.** 1. For every $\theta^* \in \Theta^*$ and for every $\eta > 0$, there exist $\bar{\epsilon} > 0$ and a function $N : (0, 1) \to \mathbb{N}$ such that when $\epsilon < \bar{\epsilon}$ and $N \in [N(\epsilon), +\infty)$, there exists an equilibrium such that

- every type of seller with cost strictly greater than $\theta^*$ demands 1,
- the buyer and every type of seller with cost no more than $\theta^*$ demand $|p_{\theta^*}|_N$.

If the offered prices are $p_s = 1$ and $p_b = |p_{\theta^*}|_N$, then the seller never concedes and the flexible buyer concedes at some constant rate $\lambda$ over the interval $[T(\lambda), \infty)$, where $T(\lambda) \equiv \frac{\log \left( \frac{1}{1-2r+\epsilon} \right)}{r}$ and $\lambda > r$.

2. Suppose that $\Theta^\prime$ is empty. For every $\eta > 0$, there exist $\bar{\epsilon} > 0$ and $N \in \mathbb{N}$ such that, if $\epsilon < \bar{\epsilon}$ and $N \in [N, +\infty)$, for every equilibrium strategy profile $(\sigma_s, \sigma_b, \tau_s, \tau_b)$, there exists $\theta^* \in \Theta^*$ such that the distance between $\sigma_b$ and $\sigma_{b,\theta^*}$ and the distance between $\sigma_s(\theta)$ and $\sigma_{s,\theta^*}(\theta)$ for every $\theta \in \Theta$ are no more than $\eta$ under the Prokhorov metric. The expected loss from delay conditional on the seller’s type satisfies:

$$E \left[ 1 - e^{-r\min\{\tau_s, \tau_b\}} \right] \begin{cases} < \eta & \text{if } \theta \leq \theta^* \\ \in (\frac{1}{2} - \eta, \frac{1}{2} + \eta) & \text{if } \theta > \theta^*. \end{cases}$$

(3.4)

The first statement constructs a class of equilibria when the probability of commitment $\epsilon$ is small: Each equilibrium in this class is characterized by a cutoff type $\theta^* \in \Theta^*$ such that the buyer and every type of the seller with cost no more than $\theta^*$ demand (approximately) the Rubinstein bargaining price of type $\theta^*$ and reach an agreement without any delay. Every type of seller with cost strictly more than $\theta^*$ demands the
entire surplus and the buyer concedes to such an offer at a constant rate $\lambda$ after a delay $T(\lambda)$. Since

$$\lim_{\varepsilon \to 0} \int_{\tau = T(\lambda)}^{+\infty} e^{-r\tau} d(1 - e^{-\lambda(\tau - T(\lambda))}) = \frac{1}{2},$$

the expected loss from delay is approximately half of the surplus as $\varepsilon \to 0$.

The second statement shows that when $\varepsilon$ is small and under generic parameter values (i.e., $\Theta^o$ is empty), every equilibrium of the reputational bargaining game is close to one of the equilibrium constructed in the first statement. Hence, Theorem 1 characterizes the set of surplus divisions and expected losses from delay that can occur in equilibrium. Our theorem nests the result of Kambe (1999), who shows that when the seller’s cost is common knowledge, players trade immediately at the Rubinstein bargaining price.

In our follow-up discussions, we partition the set of equilibria into two classes, depending on whether they are socially efficient:

1. Every equilibrium with cutoff $\theta_n$ is efficient since players reach an agreement immediately.

2. Every equilibrium with cutoff lower than $\theta_n$ is inefficient since there is significant delay when $\theta > \theta^*$. The efficient equilibria in our model are reminiscent of those in Abreu, Pearce and Stacchetti (2015, APS). They study a reputational bargaining game where one player has private information about his discount rate. When players can only commit to constant demands, they show that all equilibria are efficient and players trade immediately at the Rubinstein bargaining price of the most patient type.

Similar to APS, every player’s payoff from conceding to his opponent is strictly positive in the efficient equilibrium of our model. As in Kambe (1999) and Abreu and Gul (2000), players engage in a war-of-attrition where the low-cost types concede first followed by the high-cost types. Since the buyer’s concession rate is lower when the seller’s cost is higher, most of the delay is incurred when the low-cost types have already conceded and the buyer engages in a war-of-attrition only with the highest-cost type. When $\varepsilon$ is close to 0, the highest-cost type offers his Rubinstein bargaining price from which he is guaranteed to receive half of the surplus from trade. By pooling with the highest-cost type, every low-cost type can trade instantaneously at the highest-cost type’s Rubinstein bargaining price, which is the highest payoff they can receive in equilibrium. This leads to our efficient equilibria where the seller’s proposed price reveals no information and players trade at the Rubinstein bargaining price of the highest-cost type without any delay.

The presence of inefficient equilibria is novel compared to APS. Such an inefficiency is driven by the

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6APS show that there can be substantial delays in reaching agreements when there exist commitment types that make initial offers with a delay. The source of inefficient delay in our model is different from theirs: it is caused by high-cost seller’s temptation to demand a large share of the surplus rather than rational types imitating commitment types who make offers with delays.
buyer’s incentive to offer prices that are lower than some types of the seller’s costs, so the high-cost type’s payoff from conceding is negative. As a result, when facing the high-cost types, the buyer is not playing a standard war-of-attrition game. This can happen when different types have different costs, but cannot happen when different types have different discount rates.

For some intuition, consider an example with two types $\theta_1 < \theta_2$. Suppose the buyer offers a price between $\theta_1$ and $\theta_2$. The high-cost seller’s payoff from conceding is negative, so he can credibly commit not to concede, and he has an incentive to exploit his commitment power by demanding a high price. The seller demanding a high price has two effects on the buyer’s payoff. First, it decreases the buyer’s profit when facing the high-cost type. Second, it allows the buyer to extract more surplus from the low-cost type. This is because the delay associated with demanding a high price makes it more costly for the low-cost type to imitate the high-cost type, which deters the low-cost type from extracting information rents.

The high-cost type can credibly signal his type by demanding the entire surplus, since the buyer is indifferent between conceding and waiting, so he has an incentive to concede after a delay and delays are more costly for the low-cost type. The high-cost type cannot signal his type by demanding anything lower. This is because the buyer will receive a strictly positive payoff from conceding, which means that the buyer will concede with positive probability at time 0. This provides the low-cost type a strict incentive to imitate the high-cost type by demanding the same price and conceding at time 0 (if the buyer hasn’t conceded yet).

In summary, the buyer offering a low price helps him to extract more surplus from the low-cost type at the expense of his profit from the high-cost type. He has an incentive to offer a low price when the probability of low-cost types is above some cutoff. The high-cost seller can credibly commit not to concede only if his cost is greater than the low-cost type’s Rubinstein bargaining price. This explains the two conditions in our definition of $\Theta^*$, i.e., the set of cutoff types that can arise in equilibrium.

Theorem 1 implies that when the seller demands the entire surplus, the expected loss in social welfare is approximately half of the surplus. This is because, on the one hand, the low-cost type obtains half of the surplus he generated in our inefficient equilibrium, so his payoff from demanding the entire surplus cannot lead to a higher payoff. Hence, the loss from delay is at least half of the surplus. On the other hand, the high-cost type can guarantee half of the surplus by demanding a price slightly less than one. This is because either the buyer concedes immediately (if he believes that the seller’s cost is high for sure), or the low-cost seller is indifferent between conceding immediately to the buyer’s offer $p\theta_i$ (after which he obtains payoff $\frac{1-\theta_i}{2}$) and waiting until the buyer concedes (after which he obtains payoff $1 - \theta_1$). The low-cost type’s indifference condition implies that the cost of delay is approximately half of the surplus, so the loss from delay is at most half of the surplus. This leads to the following corollary.
**Corollary 1.** When $\varepsilon$ is close to 0, in every equilibrium with cutoff type $\theta^* \in \Theta^*$, the expected social welfare is approximately:

$$\sum_{\theta \in \Theta} \pi(\theta)(1 - \theta) - \frac{1}{2} \sum_{\theta > \theta^*} \pi(\theta)(1 - \theta).$$

When there are equilibria with different cutoff types, the uninformed buyer prefers the least efficient equilibrium and every type of the informed player prefers the most efficient equilibrium.

**Corollary 2.** The expected social welfare is strictly increasing in the cutoff, every type of seller’s expected payoff is increasing in the cutoff, and is strictly increasing for the lowest-cost type, the buyer’s expected payoff is decreasing in the cutoff.

Intuitively, consider an example with two types and suppose $\Theta^* = \{\theta_1, \theta_2\}$. Compare the efficient equilibrium with the inefficient one, the high-cost type’s payoff is the same, the low-cost type prefers the efficient equilibrium since he receives a higher price, and the buyer prefers the inefficient equilibrium since when the seller plays the inefficient equilibrium, the buyer can guarantee his payoff in the efficient equilibrium by offering $p_{\theta_2}$, after which all types of the seller will concede at time 0 with probability close to 1.

**Remark:** The second statement of Theorem 1 leaves out a knife-edge case that $\Theta^* = \emptyset$, under which there exist equilibria where the buyer mixes between different bargaining postures. In an example with two types, the buyer mixes between $p_b$ and $p_s$, with $p_b < p_s$, the high-cost seller demands 1, and the low-cost seller demands $\pi_b$. In order for the buyer’s offers to be optimal when $\varepsilon$ is close to 0, it must be the case that $\pi_b \approx \theta_2$ so that the high-cost-type seller concedes immediately when the buyer demands $\pi_b$, and that $p_s \approx 1 + \theta_1 - \theta_2$ so that the low-cost-type seller concedes immediately when the buyer demands $p_s$. As $\varepsilon$ goes to zero, the buyer’s indifference condition implies that $\pi(\theta_1)(1 - p_b) = (\pi(\theta_1) + \pi(\theta_2))(1 - \pi_b)$, which is the equality in (3.2) when there are two types.

### 3.1 Proof Sketch of Theorem 1

We sketch a proof of Theorem 1. For illustration purposes, we focus on the case where $N = +\infty$, with technical details relegated to Appendix A and the case with a large but finite $N$ to the Online Appendix.

**Statement 1:** Fix $\theta^* \in \Theta^*$. Consider the strategy profile described in Theorem 1 together with buyer’s beliefs about the seller’s cost after observing the seller’s demand, $\pi(p_s) \in \Delta(\Theta)$ such that $\pi(p_s)$ assigns probability one to type $\theta^*$ when $p_s \not\in \{p_{\theta^*}, 1\}$.

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When players have heterogenous discount rates $r_b$ and $r_s$, the expected loss in welfare equals $\frac{1}{\frac{1}{2} - r_s} \sum_{\theta > \theta^*} \pi(\theta)(1 - \theta)$, that is, the buyer’s share of the surplus in a Rubinstein bargaining game when $\theta$ is commonly known and conditional on $\theta > \theta^*$. 

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3 CHARACTERIZATION RESULT

We start by verifying the buyer’s incentive constraint. The buyer’s equilibrium payoff is \((1 - p_{\theta^*})\pi[\theta_1, \theta^*]\). Every price below \(p_{\theta^*}\) is strictly dominated by \(p_{\theta^*}\), since the buyer concedes with positive probability at time 0 when the seller offers \(p_{\theta^*}\) and the seller never concedes when they offer 1. For every \(p > p_{\theta^*}\), the buyer’s expected payoff is at most:

\[
\pi(\theta^*, p)(1 - p) + \pi[\theta_1, \theta^*]
\left(1 - \frac{p_{\theta^*} + p}{2}\right)
\leq \max_{\theta > \theta^*} \left\{ \pi(\theta^*, \theta)(1 - \theta) + \pi[\theta_1, \theta^*]
\left(1 - \frac{p_{\theta^*} + \theta}{2}\right) \right\}
\tag{3.6}
\]

The right-hand side is lower than \((1 - p_{\theta^*})\pi[\theta_1, \theta^*]\) given that \(\theta^* \in \Theta^*\).

Next, consider the seller. If his type is \(\theta \leq \theta^*\), his payoff from offering \(p_{\theta^*}\) is \(p_{\theta^*} - \theta\). Type \(\theta\)’s payoff from offering \(p_s \in (p_{\theta^*}, 1)\) is strictly less than \(p_{\theta^*} - \theta\) since the buyer’s posterior belief assigns probability 1 to \(\theta^*\) and, hence, the buyer does not concede at time zero with positive probability whenever \(p_s > p_{\theta^*}\). Conditional on being rational, type \(\theta\)’s payoff from offering 1 and then waiting until \(T(\lambda) + \tau\) to concede is:

\[
W_\theta(\tau) = e^{-rT(\lambda)} \left[ (1 - \varepsilon) \int_0^\tau \lambda e^{-(r + \lambda) t} dt + (1 - (1 - \varepsilon)(1 - e^{-\lambda \tau})) e^{-r(\tau + T(\lambda) - \theta)} \right]
\tag{3.7}
\]

The gain from deviating, \(W_\theta(\tau) - (p_{\theta^*} - \theta)\), is increasing in \(\theta\) for every \(\tau\). It follows that, to check the incentive constraints of all sellers with type \(\theta \leq \theta^*\), it suffices to check that the constraint is satisfied for the \(\theta^*\)-type seller. Furthermore, the fact that \(\lambda > r\) ensures that \(W_{\theta^*}(\tau)\) is uniquely maximized at \(\tau^*\) such that \(e^{-\lambda \tau^*}(1 - \varepsilon)(\lambda - r) = r\varepsilon\). Using the fact that \(\tau^* \to \infty\) as \(\varepsilon \to 0\) and plugging in the expression for \(p_{\theta^*}\), we have that for every \(\eta > 0\):

\[
W_{\theta^*}^* - e^{-rT(\lambda)}(1 - \varepsilon) \frac{\lambda}{r + \lambda}(1 - \theta^*) < \eta,
\tag{3.8}
\]

when \(\varepsilon\) is small.

Hence, for small values of \(\varepsilon\), a sufficient condition for the incentive constraint of the type \(\theta \leq \theta^*\) seller is:

\[
e^{-rT(\lambda)}(1 - \varepsilon) \frac{\lambda}{r + \lambda}(1 - \theta^*) < \frac{1 - \theta^*}{2},
\tag{3.9}
\]

which is ensured by our definition of \(T(\lambda)\).

Otherwise, if the seller’s type is \(\theta > \theta^*\), his payoff from offering \(p_{\theta^*}\) is no more than 0 since \(p_{\theta^*} \leq \theta\) for all \(\theta > \theta^*\). Suppose type \(\theta\) offers \(p_s \in (p_{\theta^*}, 1)\), the buyer concedes with zero probability at time 0 (since...
he believes that the seller’s cost is $\theta^*$ and the seller’s offer is greater than $p_{\theta^*}$ and concedes at rate

$$\lambda_b \equiv \frac{r(p_{\theta^*} - \theta^*)}{p_s - p_{\theta^*}}. \quad (3.10)$$

The rational buyer finishes conceding at time $T' \equiv -\frac{\log(\varepsilon)(p_s - p_{\theta^*})}{r(p_{\theta^*} - \theta^*)}$. Since the seller gets weakly negative payoff from conceding to the buyer’s offer, the best the seller can do in this scenario is wait for the buyer to concede. Therefore, the seller with cost $\theta$ receives no more than the following payoff from deviating to $p_s$:

$$(p_s - \theta) \int_0^{T'} \lambda_b e^{-(\lambda_b + r)t} dt = \frac{(p_s - \theta)(1 - \varepsilon^{p_{\theta^*} - \theta^*})}{p_s - \theta^*}(p_{\theta^*} - \theta^*). \quad (3.11)$$

The above expression is increasing in $p_s$, so by proposing a price close to one the seller can guarantee a payoff arbitrarily close to:

$$W_\theta \equiv \frac{(1 - \theta)(1 - \varepsilon^2)}{2} \quad (3.12)$$

On the other hand, his equilibrium payoff is:

$$W^*_\theta \equiv e^{-rT}(1 - \varepsilon) \frac{\lambda}{r + \lambda}(1 - \theta) \quad (3.13)$$

Combining (3.12) and (3.13), the incentive compatibility condition for a seller of type $\theta > \theta^*$ is equivalent to:

$$e^{-rT}(1 - \varepsilon) \frac{\lambda}{r + \lambda} \geq \frac{1 - \varepsilon^2}{2} \quad (3.14)$$

Which does not depend on $\theta$ and is satisfied with equality when $T$ is defined as in Theorem I.

**Statement 2:** In order to illustrate the main ideas of the proof, in this section we show that all equilibria in which the buyer uses a deterministic bargaining posture must satisfy the conditions in the Theorem. In Appendix A we establish that, under generic parameter values (i.e. when $\Theta^\circ$ is empty), the buyer’s equilibrium strategy converges to a deterministic price as $\varepsilon \to 0$. Let $\sigma_\theta(\theta) \in \Delta(P^{+\infty})$ be the equilibrium distribution over bargaining postures assumed by a seller of type $\theta$, and let $p_s^* \in P^{+\infty}$ be the equilibrium deterministic bargaining posture of the buyer. Let $P_s \equiv \bigcup_{i=1}^n \text{supp}(\sigma_\theta(\theta_i))$ be the set of offers the seller makes with positive probability and let $\bar{\sigma}_s \equiv \sum_{i=1}^n \pi(\theta_i)\sigma_\theta(\theta_i)$. For every $p_s \in P^{+\infty}$, let $\pi(p_s) \in \Delta(\Theta)$ be the buyer’s belief about $\theta$ at time 0 after observing the seller demanded $p_s$. Let $\bar{\theta}(p_s)$ be the lowest $\theta$ in the support of $\pi(p_s)$. Let $\tau_b(p_s, p_b)$ and $(\tau_s(\theta, p_s, p_b))_{i=1}^n$ be the buyer and the seller’s equilibrium distributions over concession times following bargaining postures $(p_s, p_b)$. In what follows, for a given measure $\mu$ on the real
line and a measurable set $A \subset \mathbb{R}$, we use the notation $\mu A$ to denote the measure of $A$ under $\mu$. Let

$$t_s(p_s, p_b) \equiv \min_{i=1, \ldots, n} \left\{ \sup_{t \in \mathbb{R}_+} \{ \tau_s(\theta_i, p_s, p_b)[0, t] = 0 \} \right\}$$

be the first time instant at which the seller concedes. Let

$$q^*(p_s, p_b) \equiv \int_0^{t_s(p_s, p_b)} e^{-rt} d\tau_b(p_s, p_b)$$

be the discounted average probability with which the buyer concedes before the seller starts conceding.

First, we note that, if given a pair $p_b < p_s$ a seller of type $\theta_i$ does not concede to $p_b$ before time $t$ with positive probability, then neither does any seller with a higher type $\theta > \theta_i$. This is because the value of postponing an agreement is increasing in the seller’s cost.

**Lemma 1.** For any $t \in \mathbb{R}_+$ and $(p_s, p_b) \in (P^{\infty})^2$ such that $p_b < p_s$, $\tau_s(\theta_i, p_s, p_b)[0, t] = 0$ for some $i = 1, \ldots, n - 1$ implies $\tau_s(\theta_j, p_s, p_b)[0, t] = 0$ for all $j = i + 1, \ldots, n$.

**Proof.** For any $i = 1, \ldots, n$, $\tau_s(\theta_i, p_s, p_b)[0, t] = 0$ if and only if for every $t' < t$, seller $\theta_i$’s continuation payoff from waiting to concede until period $t$ weakly dominates the payoff from conceding at $t'$. That is:

$$p_b - \theta_i \leq (p_s - \theta_i) \int_{t'}^t e^{-(s-t')} \frac{d\tau_b(p_s, p_b)}{\tau_b(p_s, p_b)[0, t']} + e^{-r(t-t')} \frac{(1 - \tau_b(p_s, p_b)[t', t])}{\tau_b(p_s, p_b)[0, t']} (p_b - \theta_i)$$

(3.17)

The result then follows from the fact that if inequality (3.17) holds for $i = 1, \ldots, n - 1$, then it must also hold when evaluated at $j = i + 1, \ldots, n$.

For every $p_s < 1$, $q^*(p_s, p_b^*) > 0$ if and only if the buyer concedes with positive probability at time 0. When $p_s = 1$, $q^*(p_s, p_b^*)$ can be strictly positive even if the buyer concedes with zero probability at time 0.

Next, we show that the buyer concedes with strictly positive probability at time zero following $(p_s, p_b^*)$ for any $p_s \in P_s$ such that $p_b^* < p_s < 1$, and that, when $\varepsilon$ is small, the seller demands a price in an arbitrary small neighborhood of $p_b^*$ with strictly positive probability in any equilibrium.

**Lemma 2.** $q^*(p_s, p_b^*) > 0$ for every $p_s \in P_s \setminus \{p_b^*\}$ such that $p_s < 1$, and for every $\eta > 0$ there is $\bar{\varepsilon} > 0$ such that $\varepsilon < \bar{\varepsilon}$ implies that $|\inf P_s - p_b^*| < \eta$. 

Proof. First, consider $p_s \in P_s$ such that $q^*(p_s, p_b^*) = 0$ and $p_s > p_b^*$. If $p_s < 1$ and $q^*(p_s, p_b^*) = 0$, it must be that the seller of type $\theta(p_s)$ concedes with positive probability in period zero and, hence, $\theta(p_s) > p_b^*$. But then, the seller of type $\theta(p_s)$ is strictly better off by offering $p_b^*$ instead of $p_s$, a contradiction.

Second, we show that $\inf P_s \approx p_b^*$ when $\varepsilon$ is small. Clearly, $\inf P_s \geq p_b^*$. Let $\underline{p}_s = \inf P_s$ and suppose by contradiction that $\underline{p}_s > p_b^*$. By the result in the previous paragraph, the buyer has to concede with positive probability in period zero after any offer $p_s \in P_s \setminus \{1\}$. Moreover, if $1 \in P_s$, it must be that the buyer concedes with positive probability after bargaining postures $(1, p_b^*)$, for if not the seller of type $\theta(1)$ can do strictly better by demanding $p_s < 1$ and waiting for the buyer to concede. Hence, the buyer’s equilibrium payoff is:

$$W^* = \tilde{\sigma}_s(P_s \setminus \{1\}) \mathbb{E}[(1 - p_s)(1 - \varepsilon^2 e^{-rT(p_s, p_b^*)})|p_s \in (\underline{p}_s, \eta, 1)]$$

where $T(p_s, p_b^*)$ is the time at which players finish conceding in the equilibrium of the war of attrition that follows bargaining postures $(p_s, p_b^*)$ characterized in Appendix A. By deviating to $\underline{p}_s + \eta$ for small $\eta > 0$, the buyer gets at least:

$$\tilde{W} = \tilde{\sigma}_s(\underline{p}_s, \underline{p}_s + \eta)(1 - \underline{p}_s) + \tilde{\sigma}_s(\underline{p}_s + \eta, 1) \mathbb{E}[(1 - p_s)(1 - \varepsilon^2 e^{-rT(p_s, p_b^*)})|p_s \in (\underline{p}_s, \eta, 1)]$$

Note that $\tilde{W} > W^*$ if $\tilde{\sigma}_s(\underline{p}_s, \eta, 1) \approx 0$. Indeed, we now show that for every $\eta > 0$ there is $\varepsilon$ such that, when $\varepsilon < \bar{\varepsilon}$, then $\tilde{\sigma}_s(\underline{p}_s, \underline{p}_s + \eta, 1) = 1$. For $p_s \in P_s$, let $c_b(p_s, p_b^*) > 0$ be the probability that the buyer concedes at time zero after $(p_s, p_b^*)$. Suppose by contradiction that there is $p_s \in P_s \setminus \{\underline{p}_s, 1\}$ such that $p_s - \underline{p}_s > \eta$. Take any $p_s' \in P_s$ arbitrarily close to $\underline{p}_s$. The condition that guarantees that the seller of type $\theta(p_s')$ does not want to deviate to $p_s$ is:

$$p_b^* - \theta(p_s') + c_b(p_s', p_b^*)(p_s - p_b^*) - \varepsilon^2 e^{-rT(p_s', p_b^*)}(p_b^* - \theta(p_s')) \geq 0$$

$$p_b^* - \theta(p_s') + c_b(p_s, p_b^*)(p_s - p_b^*) - \varepsilon^2 e^{-rT(p_s, p_b^*)}(p_b^* - \theta(p_s')),$$

which is equivalent to:

$$p_s - \underline{p}_s \leq \frac{1}{c_b(p_s, p_b^*)}[(c_b(p_s, p_b^*) - c_b(p_s, p_b^*))\underline{p}_s + \varepsilon^2 e^{-rT(p_s, p_b^*)} - e^{-rT(\underline{p}_s, p_b^*)})(p_b^* - \theta(p_s'))]. \quad (3.18)$$

A necessary condition for (3.18) is:

$$p_s - \underline{p}_s \leq \frac{1}{c_b(p_s, p_b^*)}[(1 - c_b(p_s, p_b^*))\underline{p}_s + \varepsilon^2 (p_b^* - \theta(p_s'))].$$
The result then follows from Lemma 8 in Appendix A which establishes that $c_b(p_s, p_b^*) \to 1$ as $\varepsilon \to 0$. Hence, $\tilde{W} > W^*$ when $\varepsilon$ is small. This leads to a contradiction.

Our next result shows that, as $\varepsilon$ vanishes, every price in the support of the seller’s strategy is either close to $p_b^*$ or equal to 1.

**Lemma 3.** For every $\eta > 0$ and every $p_s \in P_s$, there is $\bar{\varepsilon} > 0$ such that $\varepsilon < \bar{\varepsilon}$ implies that either $p_s - p_b^* < \eta$ or $p_s = 1$.

**Proof.** Suppose there is $p_s \in P_s$ such that $p_b^* < p_s < 1$. By Lemma 2 when $\varepsilon < \bar{\varepsilon}$, there is $\theta \in \Theta$ such that $p_b^* \in \text{supp}(\sigma_\theta)$. The condition for type $\theta$ to be willing to offer $p_b^*$ instead of $p_s$ is:

$$p_b^* - \theta \geq p_b^* - \theta + c_b(p_s, p_b^*) (p_s - p_b^*) - \varepsilon^2 e^{-rT(p_s, \bar{p}_b^*)} (p_b^* - \theta)$$

$$\iff p_s - p_b^* \leq \frac{\varepsilon^2 e^{-rT(p_s, \bar{p}_b^*)} (p_b^* - \theta)}{c_b(p_s, p_b^*)} \leq \frac{\varepsilon^2 (p_b^* - \theta)}{c_b(p_s, p_b^*)} \to 0, \quad \text{as } \varepsilon \to 0$$

Because lower types prefer earlier agreement, Lemma 4 shows that every equilibrium where the buyer’s bargaining posture is deterministic is characterized by a cutoff $\theta^*$ such that all types weakly less than $\theta^*$ demand $p_b^*$ with probability one, and all types above $\theta^*$ demand 1 with probability one.

**Lemma 4.** In every equilibrium where the buyer uses a deterministic bargaining posture and for every $\eta > 0$, there exists $\theta^* \in \Theta$ and $\bar{\varepsilon} > 0$ such that, when $\varepsilon < \bar{\varepsilon}$, every type with $\theta \leq \theta^*$ adopts a bargaining posture in $[p_b^*, p_b^* + \eta]$ with probability 1, and every type with $\theta > \theta^*$ uses 1 with probability 1.

**Proof.** Fix $p_s' \approx p_b^*$ such that $p_s' \in P_s$ and let $\theta^*$ be the highest type that offers $p_s'$ with strictly positive probability. By Lemma 3 when $\varepsilon < \bar{\varepsilon}$, $\text{supp}(\sigma_{\theta^*}) \subset [p_b^*, p_b^* + \eta] \cup \{1\}$. Suppose by way of contradiction that type $\theta^*$ offers 1 with positive probability. Since $p_s' \in \text{supp}(\sigma_{\theta^*})$, it must be that $p_b^* > \theta^*$. This implies that the seller of type $\theta^*$ has to concede following bargaining postures $(1, p_b^*)$. Their payoff from offering 1 will then be strictly less than $p_b^* - \theta^*$, leading to a contradiction.

We now use the above results to establish convergence of equilibrium strategies to the strategies in Theorem 1. First, we show that $p_b^* \approx p_\theta$ for some $\theta \in \Theta$. Let $\theta^*$ be the highest type that proposes (approximately) $p_b^*$ with positive probability, which is well-defined given Lemma 4. If $p_b^* > p_{\theta^*}$, then, if $\varepsilon$ is small enough, the buyer can increase their payoff by proposing $p_{\theta^*}$. This is because the seller proposes $p_b^*$ only if $\theta \leq \theta^*$, and hence they will concede immediately to $p_{\theta^*}$. If $p_b^* < p_{\theta^*}$, then the seller of type $\theta^*$ can do strictly
better by either demanding $p_{\theta^*}$ if $\theta \in \text{supp}(\pi(p_{\theta^*}))$ for some $\theta \geq \theta^*$ since then the buyer would concede immediately to $p_{\theta^*}$, or by demanding something close to 1 and waiting for the buyer to concede if $\theta < \theta^*$ for all $\theta \in \text{supp}(\pi(p_{\theta^*}))$. This implies that $p^*_b$ is close to $p_{\theta^*}$ when $\varepsilon$ is close to 0.

Finally, we show that $\theta^* \in \Theta^*$. Let $i^*$ be such that $\theta^* = \theta_{i^*}$. Suppose, by way of contradiction, that $p_{\theta^*} > \theta_{i^*} + 1$. Because they get strictly positive payoff from conceding, the seller with type $\theta_{i^*} + 1$ has to accept the buyer’s offer with positive probability in period zero after demanding 1. But this is a contradiction, since they can do strictly better by deviating and demanding $p_{\theta^*}$ instead of 1. Hence, $p_{\theta^*} \leq \theta_{i^*} + 1$. Moreover, the incentive constraint of the buyer requires that offering $p_{\theta^*}$ is not dominated by making an offer slightly above $\theta$ for some $\theta > p_{\theta^*}$, to which all rational sellers with type $\theta' \leq \theta$ would concede almost immediately. This yields, in the limit as $\varepsilon \to 0$:

$$
\pi[\theta_1, \theta^*](1 - p_{\theta^*}) \geq \pi(\theta^*, \theta)(1 - \theta) + \pi[\theta_1, \theta^*]
\left(1 - \frac{p_{\theta^*} + \theta}{2}\right)
\forall \theta > \theta^*,
$$

(3.19)

which is equivalent to the second condition in the definition of $\Theta^*$. Hence, $\theta^* \in \Theta^*$.

In sum, in every equilibrium where the buyer uses a deterministic bargaining posture equal to $p^*_b$, there is $\theta^* \in \Theta^*$ such that, for every $\eta > 0$, there is $\tilde{\varepsilon} > 0$ such that when $\varepsilon < \tilde{\varepsilon}$: i) $|p - p_{\theta^*}| < \eta$ for $p = p^*_b$ and for all $p \in \text{supp}(\sigma_s(\theta))$ when $\theta \leq \theta^*$, and ii) $\sigma_s(\theta) = \sigma_{s,\theta^*}(\theta)$ for all $\theta > \theta^*$. This establishes convergence of equilibrium strategies to $\sigma_{b,\theta^*}, \sigma_{s,\theta^*}$ in the Prokhorov metric.

It remains to be shown that the expected loss from delay, conditional on the seller’s cost being greater than $\theta^*$, is $\eta$-close to one half in every equilibrium. Clearly, a seller of type $\theta' > \theta^*$ will never concede after the buyer demands $p_{\theta^*}$ since they get negative payoff from doing so. Moreover, after bargaining postures $(1, p_{\theta^*})$ are used, the buyer is always indifferent between conceding and maintaining their position. Hence, $\tau_b(1, p_{\theta^*})$ is not uniquely determined by the buyer’s incentive constraint. In Appendix A, we derive bounds on the discounted probability that the buyer concedes using the seller’s incentive constraint which imply the desired result.

## 4 Implications of Inefficient Equilibria

Although efficient equilibria always exist in the reputational bargaining game, we show that bargaining inefficiencies can arise in all equilibria when the seller’s type is endogenous. We consider two examples: an investment game where the seller can invest in a cost-saving technology, and an endogenous learning game where the seller can privately learn his cost of serving the buyer. Throughout this section, we focus
on games with a finite but large $N$, which ensures the existence of at least one equilibrium.

### 4.1 Endogenous Investment in Cost-Saving Technologies

Suppose the seller’s cost of production is determined by his investment decision before bargaining starts. The seller can either use the default technology where the cost of production is $\theta_2$, or he can adopt a new technology at a fixed cost $c$ after which his cost of production is reduced to $\theta_1$. The seller’s investment decision cannot be observed by the buyer, so the buyer faces uncertainty about the seller’s cost.

If players coordinate on the efficient equilibrium in the bargaining stage, then type $\theta_2$ seller’s payoff is $p\theta_2 - \theta_2$ and type $\theta_1$ seller’s payoff is $p\theta_2 - \theta_1$. The seller has an incentive to invest if and only if $c \leq \theta_2 - \theta_1$. If players coordinate on the inefficient equilibrium in the investment game, then type $\theta_2$’s payoff is approximately $\frac{1-\theta_2}{2}$ and type $\theta_1$’s payoff is approximately $\frac{1-\theta_1}{2}$. The seller has an incentive to invest if and only if $c \leq \frac{\theta_2 - \theta_1}{2}$ (depending on the equilibrium probability of investment).

The seller’s investment decision is socially efficient when $c \geq \theta_2 - \theta_1$ or $c \leq \frac{\theta_2 - \theta_1}{2}$, regardless of the equilibrium players coordinate on in the bargaining stage. In what follows, we focus on the interesting case where $c \in (\frac{\theta_2 - \theta_1}{2}, \theta_2 - \theta_1)$ and $\theta_2 > p_{\theta_1}$, i.e., it is socially efficient for the seller to invest and there may exist inefficient equilibria in the bargaining stage.

In equilibrium, the seller cannot invest with probability 1. This is because when the buyer knows that the seller’s cost is $\theta_1$ for sure, he will offer $p_{\theta_1}$, so type $\theta_1$’s payoff is $\frac{1-\theta_1}{2}$. If so and when $\varepsilon$ is small enough, then the high-cost seller can guarantee a payoff close to $\frac{1-\theta_1}{2}$ by demanding a price arbitrarily close to 1, which means that his benefit from investment is approximately $\frac{1-\theta_1}{2} - \frac{1-\theta_1}{2} = \frac{\theta_2 - \theta_1}{2}$, which is less than his cost of investment. This contradicts the seller’s incentive to invest. Similarly, the seller cannot invest with zero probability. This is because when the seller’s cost is $\theta_1$ with probability less than $\frac{2(1-\theta_1)}{1-p_{\theta_1}}$, Theorem [\ref{thm:equilibria}] implies that all equilibria are efficient in the bargaining stage and the seller will have a strict incentive to invest given that $c < \theta_2 - \theta_1$. Our next result characterizes the equilibrium probability of investment. Let

$$
\pi^* = \frac{1 - \theta_2}{\theta_2 - \theta_1}.
$$

**Proposition 1.** Suppose $\theta_2 > p_{\theta_1}$, and $c \in (\frac{\theta_2 - \theta_1}{2}, \theta_2 - \theta_1)$. For every $\eta > 0$, there exist $\bar{\varepsilon} > 0$ and $N \in \mathbb{N}$ such that for every $\varepsilon < \bar{\varepsilon}$ and $N \in [N, +\infty)$, the seller’s investment probability belongs to $(\pi^* - \eta, \pi^* + \eta)$ in every equilibrium of the game with endogenous investment.

Since $\pi^*$ is strictly less than 1 when $\theta_2 > p_{\theta_1}$, Proposition [\ref{prop:equilibria}] implies that when $c \in (\frac{\theta_2 - \theta_1}{2}, \theta_2 - \theta_1)$ and $p_{\theta_1} < \theta_2$, the seller’s investment decision is inefficient in every equilibrium. A corollary is that the outcome
in the bargaining stage is also inefficient conditional on the seller’s investment decision.

We pin down the equilibrium investment probability by ruling out probabilities that are bounded away from \( \pi^* \). According to Theorem 1, if the investment probability is bounded away from \( \pi^* \), only two classes of equilibria exist: efficient equilibria in which the seller’s return from investment is \( \theta_2 - \theta_1 \), and inefficient equilibria in which the seller’s return from investment is \( \frac{\theta_2 - \theta_1}{2} \). When the cost of investment is strictly between \( \frac{\theta_2 - \theta_1}{2} \) and \( \theta_2 - \theta_1 \), the seller cannot be indifferent between investing and not investing if players coordinate on the efficient equilibrium or the inefficient equilibrium in the bargaining stage. It follows that the investment probability must be close to \( \pi^* \) in order for the seller to be indifferent at the investment stage.

In Appendix B, we show that an equilibrium exists in the game with investment when \( N \) is finite, which together with Proposition 1 establishes the common properties of all equilibria in the investment game.

One caveat when applying Theorem 1 is that Theorem 1 characterizes the set of limit points for a fixed \( \pi \) when \( \varepsilon \) is small enough, while in the game with endogenous investment decisions, we need to fix a small enough \( \varepsilon \) while allowing the seller to choose the distribution of his types. This explains why we can only show that the investment probability is close to \( \pi^* \) when \( \varepsilon \) is close to 0, but we cannot exactly pin down the investment probability for any fixed \( \varepsilon \).

Implications: The right-hand-side of (4.1) decreases when \( \theta_2 \) increases (i.e., the status quo technology is less efficient) and when \( \theta_1 \) decreases (i.e., the new technology is more efficient).

Proposition 1 implies that if we fix the cost of investment \( c \in (\frac{\theta_2 - \theta_1}{2}, \theta_2 - \theta_1) \), when \( \theta_2 \) increases (or when \( \theta_1 \) decreases), the fraction of sellers who invest in the new technology first decreases and then reaches 1 when \( \theta_2 - \theta_1 \) is more than \( 2c \). Somewhat surprisingly, our result suggests that increasing the benefit from using the new technology can decrease the fraction of sellers who adopt it. This is because of the buyers’ incentives to extract surplus from the seller in the bargaining stage.

One can also consider a perturbed investment game where the seller is inept and does not have the ability to invest with probability \( \xi \in (0, 1) \), in which case his production cost is \( \theta_2 \). The result depends on the relative magnitude of \( \xi \) and players’ commitment probability \( \varepsilon \). When \( \xi \ll \varepsilon \), the seller’s investment probability is close to \( \pi^* \) in every equilibrium of the perturbed investment game. When \( \xi \gg \varepsilon \), there exists an equilibrium where the seller invests efficiently whenever he is not inept. Intuitively, the presence of inept sellers discourages buyers from offering low prices, since the seller’s cost is high with positive probability. When \( c < \theta_2 - \theta_1 \), there exists an efficient equilibrium where the seller invests whenever he can, players coordinate on the efficient equilibrium in the bargaining stage, and the seller’s return from investment equals \( \theta_2 - \theta_1 \). This is not an equilibrium when \( \xi = 0 \) or when \( \xi \ll \varepsilon \), since each seller has a strict incentive to
invest if his benefit from doing so equals $\theta_2 - \theta_1$, and a seller’s investment decision imposes a negative externality on the other sellers in the population by making them vulnerable to the buyer’s exploitation.

### 4.2 Endogenous Learning about Production Cost

We study situations in which the seller faces different costs when serving different types of buyers, and the seller can acquire costly information about the buyer’s type before the bargaining stage. This fits the custom software industry where the seller does not automatically know the cost of developing a custom software after a buyer describes her needs, so he needs to estimate his costs before giving the buyer a quote. Indeed, Banerjee and Duflo (2000) document large amounts of cost overruns in the Indian custom software industry, which highlights the difficulties faced by the seller in knowing his costs in advance.

We assume that the buyer does not know the seller’s cost. This assumption fits situations where different buyers are *horizontally differentiated* and the cost to fulfill a buyer’s needs depends mostly on the seller’s expertise (e.g., whether the seller has served similar customers before).

Formally, the buyer’s type $\theta$ is either $\theta_1$ or $\theta_2$, with $0 < \theta_1 < \theta_2 < 1$. The buyer’s value from obtaining the seller’s service is 1 regardless of his type. The seller’s cost of serving type $\theta$ buyer is $\theta$. The buyer does not know $\theta$. The seller can learn the buyer’s type at cost $c > 0$ before making his initial offer. Whether the seller learns $\theta$ is unbeknownst to the buyer. Let $\pi_1$ be the probability that the buyer is of type $\theta_1$. Let $\theta \equiv \pi_1 \theta_1 + (1 - \pi_1) \theta_2$ be the expected value of $\theta$. The seller’s *type* in the bargaining stage is his belief about the buyer’s type, which is either $\theta_1$, $\theta$, or $\theta_2$. Since trade is socially efficient regardless of $\theta$, information acquisition is wasteful and it is socially efficient to reach an agreement without learning about $\theta$.

If players coordinate on the efficient equilibrium in the bargaining stage, then the seller has no incentive to learn $\theta$. This is because the buyer and all types of the seller offer the same price in the efficient bargaining equilibrium, and this price is between every type of the seller’s cost and 1. Hence, the seller does not gain from learning $\theta$.

The seller may have an incentive to learn about $\theta$ when an inefficient equilibrium is played in the bargaining stage, since the price the buyer offers is strictly below some types of the seller’s cost. Proposition 2 provides sufficient conditions under which inefficient bargaining happens in all Perfect Bayesian equilibria of the bargaining game with endogenous learning, subject to a no-signaling-what-you-don’t-know refinement (Fudenberg and Tirole, 1991).\(^8\)

---

\(^8\)Since the seller may not know $\theta$, we need the no-signaling-what-you-don’t-know refinement to rule out equilibria where the uninformed seller’s belief about $\theta$ depends on the buyer’s initial offer.
Proposition 2. Suppose $\theta_2 > \frac{1 + \theta}{2}$ and $c \in \left(0, \frac{1}{2}(1 - \pi_1)(\theta_2 - \bar{\theta})\right)$. If $\varepsilon$ is small enough and $N$ is large enough, then bargaining is inefficient in every Perfect Bayesian equilibrium that satisfies no-signaling-what-you-don’t-know.

Proof. Suppose first that the seller learns $\theta$ with strictly positive probability. Then, by Theorem 1, the unique efficient equilibrium at the bargaining stage features the buyer and all types of seller offering a price close to $p_{\theta_2}$ and reaching an agreement with negligible delay. Hence, the seller’s equilibrium payoff is approximately $p_{\theta_2} - \bar{\theta} - c$. But then, the seller has no incentive to learn $\theta$, since she can guarantee an expected payoff of $p_{\theta_2} - \bar{\theta}$ by not learning $\theta$ and then offering $p_{\theta_2}$. This leads to a contradiction.

Second, suppose that in equilibrium the seller learns $\theta$ with zero probability. According to Kambe (1999), the buyer and the seller trade with negligible delay at price $\frac{\theta_1 + 1}{2}$, from which the seller’s expected payoff is $1 - \theta_2$. Consider the seller’s payoff from the following deviation:

- Learn $\theta$ at cost $c > 0$.
- If $\theta = \theta_1$, then offer $\frac{1 + \theta}{2}$ and on the equilibrium path, trade with the buyer immediately.
- If $\theta = \theta_2$, then offer a price $p'$ that is close to 1.

The seller’s payoff from this deviation depends on what is the buyer’s off-path belief after observing that the seller demanded $p'$, which we denote by $\pi(p') \in \Delta\{\theta_1, \bar{\theta}, \theta_2\}$. If $\pi(p')$ assigns zero probability to type $\theta_2$, then the buyer does not concede at time zero with positive probability and the seller has to wait until the buyer finishes conceding. Hence, the seller’s continuation payoff after learning that $\theta = \theta_2$ and demanding $p'$ is $\int_0^\infty e^{-\tau} d\tau_b(p', \frac{1 + \bar{\theta}}{2}) \times (p' - \theta_2)$. Using our characterization of the equilibrium of the war-of-attrition game in Appendix A, we know that when $\varepsilon$ is small,

$$\int_0^\infty e^{-\tau} d\tau_b(p', \frac{1 + \bar{\theta}}{2}) \geq \frac{1 + \bar{\theta} - \bar{\theta}}{p' - \theta}$$

Taking $p'$ to 1, the seller’s expected payoff under this deviation is at least:

$$\pi_1\left(\frac{1 + \bar{\theta}}{2} - \theta_1\right) + (1 - \pi_1)\frac{1 - \theta_2}{2} - c,$$

as $\varepsilon \to 0$. When

$$c < (1 - \pi_1)\left[\frac{1 - \theta_2}{2} - \left(\frac{1 + \bar{\theta}}{2} - \theta_2\right)\right] = (1 - \pi_1)\frac{\theta_2 - \bar{\theta}}{2},$$

See expressions A.15, A.16 and A.17 for a detailed discussion of this inequality.
the above deviation is strictly profitable for the seller, which leads to a contradiction.

If, otherwise, $\pi(p')$ assigns positive probability to $\theta_2$, then since $\theta_2 > \frac{1+\theta}{2}$ the buyer would have to concede at time zero with probability approaching one after the seller demands $p'$. This yields a deviation payoff for the seller that is strictly higher than the one considered above. The two cases together imply that there exists no equilibrium in which bargaining is efficient when $c \in \left(0, \frac{1}{2}(1 - \pi_1)(\theta_2 - \overline{\theta})\right)$.

\qed
A Proof of Theorem 1

In this section, we establish two main results that complete the argument of the proof of part two of Theorem 1 for the case in which \( N = +\infty \). We deal with the case \( N < +\infty \) in the Online Appendix. The first step is to show that the result holds beyond equilibria in which the buyer’s bargaining posture is deterministic, and the second is to prove the result on the expected loss from delay. In order to do so, we first lay out some useful results which characterize the equilibrium in the continuation game that arises after the buyer and the seller use bargaining postures \((p_s, p_b)\) with \( p_s > p_b \). To do this, let \( \pi(p_s) \in \Delta(\Theta) \) be the buyer’s posterior belief about the seller’s type after observing price \( p_s \) and let \( \bar{\Theta} \) be its support. We denote the resulting continuation game by \( \Gamma(p_b, p_s, \pi, r, \varepsilon) \), and a pair of equilibrium strategies for the seller of type \( \theta \) and the buyer by \( \tau_b \in \Delta(\mathbb{R}_+ \cup \{\infty\}) \) and \( \tau_s(\theta) : \Theta \to \Delta(\mathbb{R}_+ \cup \{\infty\}) \).

Since \( p_s \) is kept fixed for this part of the game, we omit the \( p_s \) argument when referring to the buyer’s belief \( \pi(p_s) \). Let

\[
\hat{\Theta} \equiv \{ \theta \in \bar{\Theta} : p_b > \theta \},
\]

and, if \( \hat{\Theta} \) is not empty, we denote its elements by \( \hat{\theta}_1, \ldots, \hat{\theta}_m \) with \( \hat{\theta}_1 < \cdots < \hat{\theta}_m \). Let

\[
\lambda_s \equiv \frac{r(1 - p_s)}{p_s - p_b}, \tag{A.1}
\]

which is the seller’s concession rate that keeps the buyer indifferent between conceding and waiting. For every \( j \in \{1, \ldots, m\} \), let

\[
\lambda_b^j \equiv \frac{r(p_b - \hat{\theta}_j)}{p_s - p_b}, \tag{A.2}
\]

which is the buyer’s concession rate that keeps type \( \hat{\theta}_j \) seller indifferent between conceding and waiting. For convenience, set \( \lambda_b^{m+1} = 0 \).

If the seller doesn’t concede at time zero with positive probability and their strategy, conditional on their type being \( \hat{\theta}_j \in \hat{\Theta} \), is to concede at a constant rate equal to \( \lambda_s \) over the interval \((T^{j-1}, T^j)\), with \( 0 = T^0 < T^1 < \cdots < T^m \), then the buyer’s belief that the seller is committed or has type strictly above \( \theta_j \) reaches one at:

\[
T_s^j = \frac{-\log(\varepsilon + \pi(\hat{\theta}_j, \theta_n)(1 - \varepsilon))}{\lambda_s}, \tag{A.3}
\]

Likewise, if the buyer doesn’t concede at time zero and concedes at a constant rate equal to \( \lambda_b^j \) over the
interval \((T^{j-1}, T^j)\), the seller’s belief that the buyer is committed reaches one at:

\[
T_b = \frac{-\log(\epsilon) - \sum_{j=1}^{m-1} (\lambda^j_b - \lambda^{j+1}_b) T^j}{\lambda^m_b}
\]

Lemma 5 characterizes the probability that players concede at time zero when \(p_s < 1\) and \(\Theta \neq \emptyset\). The proof is analogous to Abreu et al. (2015), so we omit it.

**Lemma 5.** Suppose that \(p_s < 1\) and \(\Theta \neq \emptyset\). Let

\[
L = \frac{-\lambda_s \log \epsilon + \sum_{j=1}^{m-1} (\lambda^j_b - \lambda^{j+1}_b) \log(\epsilon + \pi(\hat{\theta}_j, \theta_n)(1-\epsilon))}{-\lambda^m_b \log(\epsilon + \pi(\hat{\theta}_m, \theta_n)(1-\epsilon))}
\]

(A.4)

In any equilibrium of \(\Gamma(p_b, p_s, \pi, r, \epsilon)\), the buyer (seller) concedes with positive probability at time zero if and only if \(L > (\leq) 1\). The time-zero concession probabilities for the buyer and the seller are given by:

\[
c_b = \begin{cases} 
1 - \sum_{j=1}^m \lambda^j_b (T^j - T^{j-1}) \times \epsilon & \text{if } L > 1 \\
0 & \text{if } L \leq 1 
\end{cases} \quad c_s = \begin{cases} 
1 - \left( \epsilon^{-\lambda_s} \prod_{j=1}^m (\epsilon + \pi(\hat{\theta}_j, \theta_n)(1-\epsilon))^{\lambda^j_b - \lambda^{j+1}_b} \right)^{1/\lambda^1_b} & \text{if } L < 1 \\
0 & \text{if } L \geq 1 
\end{cases}
\]

(A.5)

When \(c_s > 0\), let \(j^* = \min\{j \in \{1, \ldots, m\} : c_s < \pi(\hat{\theta}_1, \hat{\theta}_j)\}\). When \(j^* > 1\), all rational sellers with type \(\hat{\theta}_1, \ldots, \hat{\theta}_{j^*-1}\) concede at time zero with probability one, and the seller with type \(\theta_{j^*}\) concedes with probability in \([0, 1)\). Hence, after observing no concession at time zero, the buyer’s belief that the seller is committed conditional on them having cost \(\theta \in \{\hat{\theta}_1, \ldots, \hat{\theta}_{j^*-1}\}\) reaches one. After no concession at time zero, the buyer behaves as if playing a war of attrition game against sellers of type in the set \(\{\hat{\theta}_{j^*}, \ldots, \hat{\theta}_m\}\). In this continuation game, the buyer and the seller finish conceding at the same time, given by

\[
T_m = \min\left\{ \frac{-\log(\epsilon) - \sum_{j=j^*}^{m-1} (\lambda^j_s - \lambda^{j+1}_s) T^j_s}{\lambda^m_s}, T^m_s \right\}
\]

(A.6)

Moreover, a seller of type \(\hat{\theta}_j\), with \(j \in \{j^*, \ldots, m-1\}\), finishes conceding at:

\[
T^j_s = T^j_s + \frac{\log(1 - c_s)}{\lambda_s}
\]

(A.7)

The following result completes the characterization of players’ strategies.
Lemma 6. In any equilibrium of $\Gamma(p_b, p_s, \pi, r, \varepsilon)$ where $\hat{\Theta}$ is non-empty and $p_s < 1$, $\tau_b$ and $\tau_s(\theta)$ satisfy:

1. For $j = j^*, \ldots, m$, the buyer concedes at a constant rate equal to $\lambda^j_b$ over the interval $(T^{j-1}, T^j)$ with $T^{j-1} = 0$.

2. The seller of type $\theta \in \{\hat{\theta}^j, \ldots, \hat{\theta}_m\}$ concedes at rate $\lambda_s$ over the interval $(T^{j-1}, T^j)$ with $T^{j-1} = 0$.

3. The seller of type $\theta > \hat{\theta}_m$ never concedes.

Finally, it will be useful to derive the following limiting results for the probability of concession at time zero as $\varepsilon \to 0$.

Lemma 7. Consider a sequence $(\varepsilon^k)_{k=1}^{\infty}$ such that $\varepsilon^k \to 0$ as $k \to \infty$. Let $(L^k, c_b^k, c_s^k)_{k=1}^{\infty}$ be given according to (A.4) and (A.5) with $\varepsilon = \varepsilon^k$, and let $(L^{\infty}, c_b^{\infty}, c_s^{\infty})$ be the limit as $k \to \infty$. Then,

$$L^{\infty} = \begin{cases} \frac{1}{\lambda_b} & \text{if } \pi(\hat{\theta}_m, \theta_n] = 0 \\ +\infty & \text{if } \pi(\hat{\theta}_m, \theta_n] > 0 \end{cases}$$

(A.8)

$$c_b^{\infty} = \begin{cases} 1 & \text{if } \pi(\hat{\theta}_m, \theta_n] > 0 \text{ or } \lambda_s > \lambda^m_b \\ 0 & \text{otherwise} \end{cases}$$

$$c_s^{\infty} = \begin{cases} 1 & \text{if } \pi(\hat{\theta}_n, \theta_0] = 0 \text{ and } \lambda_s < \lambda^m_b \\ 0 & \text{otherwise} \end{cases}$$

(A.9)

We now use these results to analyze equilibria where the buyer mixes over bargaining postures.

Equilibria with non-degenerate buyer’s bargaining postures: Let $(\sigma_b, \sigma_s(\theta)) \in [\Delta([0,1])]^{n+1}$ be the equilibrium bargaining postures of the buyer and the seller. Let $\sigma_s \equiv \sum_{j=1}^{n} \pi(\theta_j) \sigma_s(\theta_j)$ be the overall distribution of bargaining postures of the seller. Let $P_b \equiv \text{supp}(\sigma_b)$ and $P_s \equiv \bigcup_{i=1}^{n} \text{supp}(\sigma_b)$. For $i \in \{b,s\}$, let $p_i \equiv \inf P_i$ and $\bar{p}_i \equiv \sup P_i$. For every $p_s \in P_s$, let $\Theta(p_s) \equiv \min\{\theta \in \Theta : p_s \in \text{supp}(\sigma_b)\}$.

Lemma 8. Let $\sigma_b^k \in \Delta([0,1])$ be the buyer’s equilibrium strategy over bargaining postures when the commitment probability is $\varepsilon^k$ and suppose that $\Theta^o$ is empty. Then, there is $p_b^* \in [0,1]$ such that $\sigma_b^k$ converges to the Dirac measure on $p_b^*$ in the Prokhorov metric.

Proof. Let $\sigma_b^{\infty} \equiv \lim_{k \to +\infty} \sigma_b^k$. Suppose that the buyer’s limiting equilibrium strategy is mixed –i.e. for any $p_b \in [0,1]$, $\sigma_b^{\infty}$ is bounded away in the Prokhorov metric from the Dirac measure on $p_b$.

Step 1: We first consider the case in which $\bar{p}_s < 1$. First, suppose that $P_s \setminus \{\bar{p}_b\}$ is non-empty and that $q^*(p_s, \bar{p}_b) = 0$ for all $p_s \in P_s \setminus \{\bar{p}_b\}$. This implies that $P_s \cap (\bar{p}_b, 1]$ is empty, since the seller who always has to
concede when proposing \( p_s > \bar{p}_b \) can do strictly better by demanding \( \bar{p}_b \). The buyer’s payoff from offering \( \bar{p}_b \) is approximately \( 1 - \bar{p}_b \).

On the other hand, because \( q^s(\bar{p}_b, \bar{p}_b) = 0 \) implies that \( q^s(\bar{p}_b, p_b) = 0 \) for any \( p_b \) such that \( \bar{p}_b < p_b < p_s \), his payoff from offering \( p_b > \bar{p}_b \) is close to:

\[
\tilde{\sigma}_s[p_b, p_b] \left( 1 - \frac{\mathbb{E}[p_s | p_s \leq \bar{p}_b] + p_b}{2} \right) + \tilde{\sigma}_s(p_b, \bar{p}_b)(1 - p_b)
\]

(A.10)

Comparing (A.10) with \( 1 - \bar{p}_b \), it follows that the only way for the buyer to be indifferent between \( \bar{p}_b \) and \( p_b \) is if \( p_b - \bar{p}_b \to 0 \).

Second, suppose that there exists \( p_s \in P_s \setminus \{ \bar{p}_s \} \) such that the buyer concedes at time zero with probability \( c_b(p_s, \bar{p}_b) > 0 \) following bargaining postures \( (p_s, \bar{p}_b) \). If \( p_b = \bar{p}_s \), then fixing \( p'_s \in P_s \) such that \( p'_s \approx \bar{p}_s \), the equilibrium payoff of the seller with type \( \theta(p'_s) \) is approximately \( \frac{p^s + \mathbb{E}[p_b]}{2} - \theta(p'_s) \). By instead demanding \( p_s \), he gets at least

\[
\sigma_b[p_b, \bar{p}_b + \eta] c_b(p_s, \bar{p}_b)(p_s - \theta(p'_s)) + (1 - c_b(p_s, \bar{p}_b) - \varepsilon^2 e^{-\varepsilon T(p_s, \bar{p}_b)})(p_b - \theta(p'_s))|p_b \in [p_b, \bar{p}_b + \eta] \]

\[
+ \sigma_b(p_b + \eta, \bar{p}_b)|\mathbb{E}[(1 - \varepsilon^2 e^{-\varepsilon T(p_s, \bar{p}_b)})(p_b - \theta(p'_s))]|p_b > p_s + \eta],
\]

Where \( \eta > 0 \) is such that \( c_b(p_s, \bar{p}_b) > 0 \) for all \( p_b \in [p_b, \bar{p}_b + \eta] \). This goes to \( \sigma_b[p_b, \bar{p}_b + \eta](p_s - \theta(p'_s)) + \sigma_b(p_b + \eta, \bar{p}_b)|\mathbb{E}[p_b | p_b > \bar{p}_b + \eta] - \theta(p'_s) \). This is strictly higher than type \( \theta(p'_s) \)'s equilibrium payoff unless \( \bar{p}_b - \bar{p}_b \to 0 \).

If, otherwise, \( \bar{p}_b < p_s \), it must be that either the buyer’s strategy is close the Dirac measure on \( \bar{p}_s \), or that \( q^s(\bar{p}_b, \bar{p}_b) > 0 \) and \( q^s(p_s, \bar{p}_b) = 0 \) for all \( p_s > \bar{p}_b \), for if not the seller of type \( \theta(p'_s) \), where \( p'_s \in P_s \) satisfies \( p'_s \approx \bar{p}_s \), would deviate to \( p_s > \bar{p}_s \) by the same arguments as in the previous paragraph. This in turn implies again that \( P_s \cap (\bar{p}_b, 1) \) is empty.

Taking \( \eta > 0 \) to be small, the buyer’s payoff from offering \( \bar{p}_b \) is approximately

\[
\tilde{\sigma}_s[p_s, \bar{p}_b + \eta](1 - \bar{p}_b) + \tilde{\sigma}_s(p_s + \eta, \bar{p}_b)(1 - \bar{p}_b),
\]

Whereas when he offers \( \bar{p}_b \) he obtains

\[
\tilde{\sigma}_s[p_s, \bar{p}_b] \left( 1 - \frac{\mathbb{E}[p_s | p_s \leq \bar{p}_b] + \bar{p}_b}{2} \right)
\]
As before, the only way for the buyer to be indifferent between \( p_b \) and \( \bar{p}_b \) is if \( \bar{p}_b - p_b \to 0 \).

Finally, if \( P_s \setminus \{ p_s \} \) is empty, we have: \( \bar{p}_b \leq \bar{p}_s = p_b \leq \bar{p}_b \), so \( \bar{p}_b = p_b \).

**Step 2:** Next, consider the case in which \( \bar{p}_s = 1 \). It must be that \( q^*(1, p_b) > 0 \), for if not the seller of type \( \theta(1) \) always has to concede to the buyer’s offer and could do strictly better by offering \( \bar{p}_b \) instead of 1. This in turn implies that \( p_b \leq \theta(1) \).

If \( p_b = p_s \), then it must be that either \( q^*(p_s, p_b) = 0 \) for all \( p_s \in P_s \setminus \{ p_s \} \) or that \( \sigma^*_{\theta} \) is equal to the Dirac measure on \( p_b \). This is because, fixing \( p_s' = p_s \) such that \( p_s' \in P_s \), the payoff that type \( \theta(p_s') \) gets from offering \( p_s' \) is approximately \( \frac{p_s + E[p_b]}{2} - \theta(p_s') \). If the buyer concedes with positive probability following \( p_s > p_s' \), by instead demanding \( p_s \), their payoff is at least

\[
\sigma_b[p_b, p_b + \eta]E[c_b(p_s, p_b)(p_s - \theta(p_s'))] + (1 - c_b(p_s, p_b) - \varepsilon^2 e^{-rT(p_s, p_b)})(p_b - \theta(p_s'))|p_b \in [p_b, p_b + \eta] \\
+ \sigma_b[p_b + \eta, \bar{p}_b]E(1 - \varepsilon^2 e^{-rT(p_s, p_b)})(p_b - \theta(p_s'))|p_b > p_b + \eta],
\]

where \( \eta > 0 \) is such that \( c_b(p_s, p_b) > 0 \) for all \( p_s \in [p_b, p_b + \eta] \). This goes to \( \sigma_b[p_b, p_b + \eta](p_s - \theta(p_s')) + \sigma_b(p_b + \eta, \bar{p}_b][E[p_b | p_b > p_b + \eta] - \theta(p_s')) \). This is strictly higher than type \( \theta(p_s') \)'s equilibrium payoff unless \( \bar{p}_b - p_b \to 0 \).

If \( p_b < p_s \), then a very similar argument shows that we must have \( q^*(p_s, p_b) = 0 \) for all \( p_s > p_s' \) in order to prevent type \( \theta(p_s') \), with \( p_s' \in P_s \) close to \( p_s \), from deviating to \( p_s > p_s' \). This implies that \( P_s \cap (\bar{p}_b, 1) \) is empty, for if not the seller who always has to concede after demanding \( p_s > \bar{p}_b \) can do strictly better by deviating to \( \bar{p}_b \). Furthermore, it implies that all sellers with type \( \theta \geq \theta(1) \) demand 1 with probability one. If not, since \( p_b \leq \theta(1) \), our limiting result in Lemma \[\] implies that the buyer would have to concede with positive probability following bargaining postures \( (p_s, p_b) \) where \( p_s \in \text{supp}(\sigma_{\theta}) \) and \( p_s < 1 \), which would contradict our result that \( q^*(p_s, p_b) = 0 \) for all \( p_s > p_s' \).

Putting everything together, we can write the buyer’s payoff from offering \( p_s \), to be at least:

\[
\tilde{\sigma}_s[p_s, p_s + \eta](1 - p_s) + \tilde{\sigma}_s(p_s + \eta, \bar{p}_b)(1 - p_b),
\]

(8.11)

Where \( [p_s, p_s + \eta] \) is the arbitrarily small neighborhood of \( p_s \) in which the buyer may have to concede to the seller’s offer. On the other hand, his payoff from offering any \( p_b > p_b \) is approximately:

\[
\tilde{\sigma}_s[p_s, p_b](1 - E[p_s | p_s \leq p_b] + p_b) + \tilde{\sigma}_s(p_b, \bar{p}_b)(1 - p_b) + \tilde{\sigma}_s(1)\varepsilon(1, p_b)(1 - p_b),
\]

(8.12)
where \( \gamma_b(1,p_b) \equiv \mathbb{E} \left[ \int_0^\infty e^{-rt} d\tau_b(\theta,1,p_b) \right] \) is the expected discounted concession probability of the seller following bargaining postures \((1,p_b)\). Comparing the two expressions, it follows that the buyer can be indifferent between \( p_b \) and \( p_b > p_b \) only if \( p_b - p_b \approx 0 \) or if \( \bar{\sigma}_i(1) \gamma_b(1,p_b) > 0 \). So from now on we focus on the case in which \( \bar{p}_b - p_b \) is bounded away from zero and \( \gamma_b(1,p_b) > 0 \) for all \( p_b \in P_b \setminus \{ p_b \} \). The latter implies that \( p_b \geq \theta(1) \) for all \( p_b \in P_b \setminus \{ p_b \} \), since otherwise the seller would not want to concede after bargaining postures \((1,p_b)\). Moreover, it must be that, for all \( p_b \in P_b \setminus \{ p_b \} \), \( p_b \approx \theta \) for some \( \theta \geq \theta(1) \).

This is because, if \( p_b > \max \{ \theta : \sigma_i(1) > 0 \text{ and } \theta \leq p_b \} \), then the buyer can do strictly better by lowering the price which still gets all types below \( p_b \) to concede with probability close to one after they demand 1. Also, for all \( p_b \in P_b \setminus \{ p_b \} \), a rational seller with type less or equal than \( p_b \) must concede with probability close to one following bargaining postures \((p_b,1)\). If not, the buyer can do strictly better by offering \( p_b + \eta \) to which the seller has to concede immediately after he demands 1.

Let \( i^* \) be such that \( \theta(1) = \theta_{i^* + 1} \), and let \( j > i^* \) be such that \( \bar{p}_b = \theta_j \). Next, we show that when \( \varepsilon \) is small every \( p_s \in P_s \) must be arbitrarily close to an element of \( \{ p_s, \bar{p}_b, 1 \} \). To see this, observe that the payoff that type \( \theta < \theta(1) \) obtains from demanding \( \bar{p}_b \) and always conceding after the buyer offers something below \( \bar{p}_b \) is approximately \( \mathbb{E}[p_b] - \theta \). Given that \( q^*(p_s, p_b) = 0 \) for all \( p_b \in P_b \) and \( p_s > p_s \), by demanding \( p_b \) such that \( p_s < p < \bar{p}_b \) the seller of type \( \theta(p) \) obtains approximately:

\[
\sigma_b(p_s, p) \left( \mathbb{E}[p_b | p_b \leq p] - \theta(p) \right) + \sigma_b(p, \bar{p}_b) \left( \frac{p + \mathbb{E}[p_b | p_b > p]}{2} - \theta(p) \right),
\]

which is strictly less than the payoff from demanding \( \bar{p}_b \), unless \( \sigma_b(p, \bar{p}_b) \) is close to zero for every \( p < \bar{p}_b \) —i.e., the buyer’s strategy is approximately deterministic.

Finally, it must be that either \( p_s < \bar{p}_b \) and \( q^*(p_s, p_s) > 0 \), or that \( p_s \approx \bar{p}_b \). If not, the payoffs from demanding \( p_s \) are again strictly less than \( \mathbb{E}[p_b] - \theta \) which is what the seller of type \( \theta \) can attain by demanding \( \bar{p}_b \). Since \( q^*(\bar{p}_b, p_s) = 0 \), the seller concedes with positive probability at time zero following bargaining postures \((\bar{p}_b, p_s)\). Given this, and observing that the seller has to concede to \( p_b \) following bargaining postures \((p_b, p_b)\) for any \( p_b \) satisfying \( p_b > 1 + \theta_j - \bar{p}_b \), the buyer will want to offer \( p_b \approx 1 + \theta_j - \bar{p}_b \). By a similar argument, given that the buyer concedes after \((p_s, p_b)\) for any \( p_s \) satisfying \( p_s < 1 + \theta_j - p_b \), it must be that \( p_s \approx 1 + \theta_j - p_b \). Combining the two we obtain \( p_s \approx \bar{p}_b \) so all sellers of type \( \theta < \theta(1) \) demand approximately \( \bar{p}_b \) with probability close to one.

The condition that makes the buyer indifferent between offering \( \bar{p}_b \approx \theta_j \) and \( p_b \approx 1 + \theta_j - \theta_j \) is:

\[
\pi[\theta_1, \theta_j] (1 - \theta_j) \approx \pi[\theta_1, \theta_j] (\theta_j - \theta_j),
\]
Which can not be satisfied for small values of $\varepsilon$ under our assumption that $\Theta^o$ is empty.

Since in every equilibrium the buyer’s strategy over bargaining postures is approximately deterministic, one can apply the same arguments from Section 3.1 to establish convergence of equilibrium strategies to $\sigma^b, \theta^*$ and $\sigma^s, \theta^*$ for some $\theta^* \in \Theta^*$.

**Bounds on discounted concession probability:** Next, we establish condition 3.4. To do so, we use the seller’s incentive constraint to derive bounds on the discounted probability that the buyer concedes, after bargaining postures $(1, p_{\theta^*})$ are used. First, $p_{\theta^*}$ is incentive compatible for sellers of type $\theta \leq \theta^*$ only if the seller does not want to deviate to $p_s = 1$ and never concede after that. In particular, the following incentive constraint for type $\theta^*$ seller is necessary:

$$p_{\theta^*} - \theta^* \geq \int_0^\infty e^{-rt} d\tau_b(1, p_{\theta^*})(1 - \theta^*) \iff \int_0^\infty e^{-rt} d\tau_b(1, p_{\theta^*}) \leq \frac{1}{2}$$

(A.13)

Second, we derive a lower bound using the incentive constraints of sellers with cost strictly greater than $\theta^*$. Consider the payoff from deviating to $p_s < 1$. The condition that guarantees that the seller of type $\theta > \theta^*$ does not want to deviate to $p_s$ is:

$$\int_0^\infty e^{-rt} d\tau_b(1, p_{\theta^*})(1 - \theta) \geq \int_0^T e^{-rt} d\tau_b(p_s, p_{\theta^*})(p_s - \theta)$$

(A.14)

Where $T \in \mathbb{R}_+$ is the instant at which the rational buyer finishes conceding. Let $\pi(p_s) \in \Delta(\Theta)$ be the buyer’s belief after observing $p_s$ and let $\hat{\Theta}(p_s) \equiv \{ \theta \in \text{supp} \pi(p_s) : \theta < p_{\theta^*} \}$. Note that it cannot be the case that the buyer concedes with positive probability at time zero after the seller demands $p_s$ close to 1, for if he did the seller of type $\theta > \theta^*$ would want to deviate to $p_s$. This, in turn, implies that $\hat{\Theta}(p_s)$ is not empty and hence the buyer’s strategy after the seller demands $p_s$ is characterized by Lemma 6.

Now, let $\hat{\theta}_m \equiv \max \hat{\Theta}(p_s)$. By Lemma 6, we have that, for all $t < T$, $d\tau_b(p_s, p_{\theta^*}) \geq \lambda_m^b e^{-\lambda_m^b t} dt$. As a result we can write:

$$\int_0^T e^{-rt} d\tau_b(p_s, p_{\theta^*}) \geq \int_0^T \lambda_m^b e^{-\lambda_m^b t} dt = \frac{p_{\theta^*} - \hat{\theta}_m}{p_s - \hat{\theta}_m} (1 - e^{-\lambda_m^b T})$$

(A.15)

Moreover, since $\theta \geq p_{\theta^*}$ for all $\theta > \theta^*$, we have that $\hat{\theta}_m \leq \theta^*$ and hence:

$$\int_0^T e^{-rt} d\tau_b(p_s, p_{\theta^*}) \geq \frac{p_{\theta^*} - \theta^*}{p_s - \theta^*} (1 - e^{-\lambda_m^b T})$$

(A.16)
Putting everything together and letting \( p_s \to 1 \) gives the following necessary condition for the seller’s incentive constraint:

\[
\int_0^\infty e^{-rt} d\pi_b(1, p_{\theta^r}) \geq \frac{1}{2} (1 - e^{-(\lambda_b^b + \epsilon) T}).
\]

The right-hand-side of (A.14) converges to 1/2 as \( \epsilon \to 0 \).

## B Existence of Equilibrium in Games with Endogenous Types

Suppose that, prior to the bargaining game, the seller is able to take an action that affects the distribution of \( \theta \). Formally, we add a new stage to the game described in Section 2 in which, prior to time 0, the seller takes an action \( a \in A \) and, conditional on the chosen action, \( \theta \) is drawn at time zero from a distribution \( \pi_a \in \Delta(\Theta) \).

We refer to \( a \) as the seller’s *initial action*. Each initial action \( a \in A \) has an associated cost for the seller, denoted by \( c_a \in \mathbb{R} \). We assume that the set \( A \) is finite and that the buyer does not observe the initial action taken by the seller.

In terms of the applications discussed in Section 4, the investment game in Section 4.1 fits the setting if we let \( \Theta = \{ \theta_1, \theta_2 \}, A = \{ \text{invest, not invest} \}, \pi_{\text{invest}} \) be the Dirac measure on \( \theta_1 \) and \( \pi_{\text{not invest}} \) be the Dirac measure on \( \theta_2 \), and \( c_{\text{invest}} = c \) and \( c_{\text{not invest}} = 0 \). Likewise, the game in Section 4.2 is a special case of this environment if we set \( \Theta = \{ \theta_1, \bar{\theta}, \theta_2 \}, A = \{ \text{learn, not learn} \}, \pi_{\text{learn}} \) assign probability \( \pi_1 \) to \( \theta_1 \) and \( 1 - \pi_1 \) to \( \theta_2 \) and \( \pi_{\text{not learn}} \) be the Dirac measure on \( \bar{\theta} \), and \( c_{\text{learn}} = c \) and \( c_{\text{not learn}} = 0 \).

In order to establish existence of an equilibrium, we assume that players choose a price from a finite (but arbitrarily fine) partition of the unit interval. To do this, we define, for any \( N \in \{1, 2, \ldots \} \), a discrete price grid defined by \( P^N = \{0, 1/N, 2/N, \ldots, (N - 1)/N, 1\} \). Let \( \rho \in \Delta(A) \) be the seller’s strategy over initial actions, and let \( \sigma_b \in \Delta(P^N) \) and \( \sigma_s : A \times \Theta \to \Delta(P^N) \) be a strategy over bargaining postures for the buyer and the seller. Let \( \pi : P^N \to \Delta(\Theta) \) be a system of beliefs for the buyer, assigning a probability distribution over the seller’s cost after every demand by the seller \( p_s \in P^N \).

Fix a continuation equilibrium in the war of attrition game \( \Gamma(p_b, p_s, \hat{\pi}, r, \epsilon) \) that follows bargaining postures \( (p_b, p_s) \) with \( p_b < p_s \) and a belief by the buyer about the seller’s type given by \( \hat{\pi} \in \Delta(\Theta) \). For \( p_b < p_s \), let \( V_b(p_b, p_s, \hat{\pi}) \) and \( V_\theta(p_b, p_s, \hat{\pi}) \), \( \theta \in \Theta \), be the buyer’s and the seller’s equilibrium continuation payoffs in the game \( \Gamma(p_b, p_s, \hat{\pi}, r, \epsilon) \). For \( p_b \geq p_s \), we set \( V_b(p_b, p_s, \hat{\pi}) = 1 - \frac{p_s + p_b}{2} \) and \( V_\theta(p_b, p_s, \hat{\pi}) = \frac{p_s + p_b}{2} - \theta \), which are their payoffs when there is immediate agreement.

We can write the buyer’s expected payoff from demanding \( p_b \), when the seller’s strategy is \( (\rho, \sigma_s) \) and
the buyer’s system of beliefs is \( \pi \) as:

\[
U_b(p_b, \sigma_b, \rho, \pi) = \sum_{a \in A} \sum_{\theta \in \Theta} \rho(a) \pi_a(\theta) \mathbb{E}_{\sigma_b(\theta)}[V_b(p_b, p_s, \pi(p_s))]
\]

Likewise, we can write the seller’s expected payoff from offering \( p_s \) when their type is \( \theta \in \Theta \) as:

\[
U_\theta(p_s, \sigma_b, \pi) = \mathbb{E}_{\sigma_b}[V_\theta(p_b, p_s, \pi(p_s))]
\]

Finally, the seller’s expected payoff from choosing initial action \( a \in A \) is:

\[
U_s(a, \sigma_b, \sigma_s, \pi) = \sum_{\theta \in \Theta} \pi_a(\theta) \mathbb{E}_{\sigma_s(\theta)}[U_\theta(p_s, \sigma_b, \pi)] - c_a
\]

We sometimes abuse notation by substituting the first argument in the above utility functions with a *mixed* action, with the obvious interpretation in mind.

Using this notation, we define an equilibrium of the game with endogenous seller types to be an assessment \( (\pi, \rho, \sigma_b, \sigma_s) \) such that:

1. Strategies are *sequentially rational*:
   
   a) For all \( a' \in A \), \( U_s(\rho, \sigma_b, \sigma_s, \pi) \geq U_s(a', \sigma_b, \sigma_s, \pi) \),
   
   b) For all \( p'_b \in P^N \), \( U_b(\sigma_b, \sigma_s, \rho, \pi) \geq U_b(p'_b, \sigma_s, \rho, \pi) \),
   
   c) And for all \( p'_s \in P^N \) and \( (a, \theta) \in A \times \Theta \), \( U_\theta(\sigma_s(\theta), \sigma_b, \pi) \leq U_\theta(p'_s, \sigma_b, \pi) \).

2. Buyer’s beliefs \( \pi \) are *consistent*, as defined in Kreps and Wilson (1982).

**Lemma 9.** For every \( N < +\infty \), there exists an equilibrium in the game with endogenous types.

**Proof.** We start by deriving the continuation payoff functions \( V_b(p_b, p_s, \hat{\pi}), V_\theta(p_b, p_s, \hat{\pi}) \) and showing that they are continuous in \( \hat{\pi} \). This is clearly so when \( p_b \geq p_s \), since in that case there is immediate agreement and payoffs are constant in \( \hat{\pi} \). For \( p_b < p_s \), we consider the following cases separately:

1. Suppose that \( p_b > \theta_1 \) and \( p_s < 1 \). Let \( \theta^* \equiv \max\{ \theta \in \Theta : p_b > \theta \} \). If \( \hat{\pi}[\theta_1, \theta^*] > 0 \), the unique continuation equilibrium \( (\tau_b, \tau_s(\theta)) \) is described in Lemmas 5 and 6. Following the notation in that section, for \( j = 1, \ldots, m \), let \( T^j(\hat{\pi}), c_b(\hat{\pi}), c_s(\hat{\pi}) \) be the equilibrium objects defined in Lemmas 5 and 6 as a function of beliefs. Note that all four functions are continuous in \( \hat{\pi} \).
We can write expected equilibrium payoffs for $\hat{\pi} \in \{\pi \in \Delta(\Theta) : \pi[\theta_1, \theta^*] > 0\}$ as:

$$V_b(p_b, p_s, \hat{\pi}) = c_s(\hat{\pi})(1 - p_b) + (1 - \varepsilon)(1 - c_s(\hat{\pi}))(1 - p_s) + \varepsilon(1 - c_s(\hat{\pi})) \int_0^{T^m(\hat{\pi})} \lambda_t e^{-(r + \lambda_t)T} dt (1 - p_b)$$

$$V_\theta(p_b, p_s, \hat{\pi}) = \begin{cases} 
( c_b(\hat{\pi}) + (1 - c_b(\hat{\pi})) \sum_{j < \lambda} e^{(\lambda_j - \lambda_0 T)T} \int_{T^{j-1}(\hat{\pi})}^{T^j(\hat{\pi})} \lambda_t e^{-(r + \lambda_t)T} dt ) (p_s - \theta_j) \\
+ (1 - c_b(\hat{\pi})) e^{-\lambda_j T} \left( (1 - \varepsilon)(p_b - \theta_j) + \varepsilon \sum_{j \geq 1} e^{(\lambda_j - \lambda_0 T)T} \int_{T^{j-1}(\hat{\pi})}^{T^j(\hat{\pi})} \lambda_t e^{-(r + \lambda_t)T} dt (p_s - \theta_j) \right) & \text{if } \theta_j \leq \theta^*
\end{cases}$$

$$V_\theta(p_b, p_s, \hat{\pi}) = \begin{cases} 
( c_b(\hat{\pi}) + (1 - c_b(\hat{\pi})) \sum_{j = 1}^m e^{(\lambda_j - \lambda_0 T)T} \int_{T^{j-1}(\hat{\pi})}^{T^j(\hat{\pi})} \lambda_t e^{-(r + \lambda_t)T} dt ) (p_s - \theta_j) & \text{if } \theta_j > \theta^*
\end{cases}$$

All of which are continuous for all $\hat{\pi} \in \{\pi \in \Delta(\Theta) : \pi[\theta_1, \theta^*] > 0\}$, given continuity of the functions $T^j(\hat{\pi}), c_b(\hat{\pi}), c_s(\hat{\pi})$.

It remains to establish continuity at $\pi_0 \in \{\hat{\pi} \in \Delta(\Theta) : \hat{\pi}[\theta_1, \theta^*] = 0\}$. Taking limits to $\pi_0$ we have that:

$$\lim_{\hat{\pi} \to \pi_0} T^j(\hat{\pi}) = 0 \quad \forall j = 1, \ldots, m$$

$$\lim_{\hat{\pi} \to \pi_0} c_b(\hat{\pi}) = 1 - \varepsilon$$

$$\lim_{\hat{\pi} \to \pi_0} c_s(\hat{\pi}) = 0$$

Whereas, when $\hat{\pi} = \pi_0$, the buyer knows that the seller will never concede (since $p_b \leq \theta, \forall \theta \in \text{supp}(\hat{\pi})$), and given that $p_s < 1$ the rational buyer concedes with probability one at time zero. If the buyer doesn’t concede at time zero, the seller concludes that he is committed, and hence he will concede immediately if $\theta < p_b$ and he will never concede otherwise. Comparing this outcome with the above limits, we obtain that $V_b(p_b, p_s, \hat{\pi})$ and $V_\theta(p_b, p_s, \hat{\pi})$ are continuous in $\hat{\pi}$ for all $\hat{\pi} \in \Delta(\Theta)$.

2. In the remaining cases, either the buyer or the seller (of any type) makes weakly negative payoff by yielding to the opponent’s demand, and as a result, there may be multiple continuation equilibria at these nodes of the game. Whenever there is multiplicity, we select one such continuation equilibrium and payoffs.

i. If $p_s = 1$ and $\theta_1 \leq p_b$, there is an equilibrium where the seller of type $\theta \leq p_b$ concedes at time zero with probability one, and the seller of type $\theta > p_b$ and the buyer never concede. Payoffs
are:

\[
V_b(p_b, p_s, \hat{\pi}) = (1 - \varepsilon)(1 - p_b), \quad V_b(p_b, p_s, \hat{\pi}) = \begin{cases} 
(1 - \varepsilon)(p_b - \theta), & \text{if } \theta \leq p_b \\
0, & \text{if } \theta > p_b
\end{cases}
\]

ii. If \( p_s = 1 \) and \( p_b < \theta_1 \), no player concedes and everyone gets zero payoff.

iii. If \( p_s < 1 \) and \( p_b < \theta_1 \), the buyer concedes immediately and the seller never concedes. Players’ payoffs are:

\[
V_b(p_b, p_s, \hat{\pi}) = (1 - \varepsilon)(1 - p_s), \quad V_\theta(p_b, p_s, \hat{\pi}) = (1 - \varepsilon)(p_s - \theta), \forall \theta \in \Theta
\]

In all three cases, payoffs are continuous in \( \hat{\pi} \).

Having established that payoffs are continuous in the buyer’s beliefs, the rest of the proof follows from a fixed point argument, with the only caveat that we have to deal with the fact that final payoffs depend non-trivially on beliefs. We circumvent this by constructing a correspondence whose fixed point is an equilibrium assessment, specifying both strategies and beliefs. Then we show that such a fixed point exists.

First, for a given small \( \eta > 0 \), define \( \mathcal{P}^\eta \) and \( \mathcal{A}^\eta \) to be the set of probability distributions over \( P^N \) and \( A \), respectively, that assign probability at least \( \eta \) to every \( p \in P^N \) and every \( a \in A \). Let \( \mathcal{X}^\eta = \mathcal{A}^\eta \times (\mathcal{P}^\eta)^{n+1} \). Define \( \Pi^\eta : \mathcal{X}^\eta \rightarrow [\Delta(\Theta)]^{N+1} \) to be the correspondence mapping every action profile \((\rho, \sigma_b, \sigma_s)\) in \( \mathcal{X}^\eta \) to a system of beliefs for the buyer obtained at every information set using Bayes’ rule applied to the strategies \((\rho, \sigma_b, \sigma_s)\). Because \((\rho, \sigma_b, \sigma_s)\) assigns strictly positive probability to all pure actions, \( \Pi^\eta(\rho, \sigma_b, \sigma_s) \) is non-empty, single-valued (and hence compact and convex), and it’s upper hemi-continuous.

Next, we define the perturbed best response correspondence at every information set at which players move for a given action profile (being played at every other information set) and a given system of beliefs:

\[
BR_0^\eta(\sigma_b, \sigma_s, \rho, \pi) = \arg \max_{\rho \in \mathcal{P}^\eta} U_s(\rho, \sigma_b, \sigma_s, \pi) \\
BR_\theta^\eta(\sigma_b, \rho, \pi) = \arg \max_{\sigma_s \in \mathcal{A}^\eta} U_\theta(\sigma_s, \sigma_b, \pi), \quad \theta \in \Theta \\
BR_b^\eta(\sigma_s, \rho, \pi) = \arg \max_{\sigma_b \in \mathcal{P}^\eta} U_b(\sigma_b, \sigma_s, \rho, \pi)
\]

Each of these best response correspondences are non-empty, convex and compact valued. Moreover, the fact that payoffs are continuous in \( \pi \) ensures that they are upper hemi-continuous. Then, we can define the
correspondence $E^\eta : [\Delta(\Theta)]^{N+1} \times \mathcal{X}^\eta \Rightarrow [\Delta(\Theta)]^{N+1} \times \mathcal{X}^\eta$ to be:

$$E^\eta(\pi, \rho, \sigma_b, \sigma_s) = \Pi^\eta(\rho, \sigma_b, \sigma_s) \times BR_0^\eta(\sigma_b, \sigma_s, \pi) \times BR_b^\eta(\sigma_s, \rho, \pi) \times BR_b^\eta(\sigma_b, \rho, \pi) \times \ldots \times BR_b^{\eta_n}(\sigma_b, \rho, \pi)$$

According to the Kakutani’s fixed point theorem, $E^\eta$ has a fixed point. Let $(\pi^\eta, \rho^\eta, \sigma_b^\eta, \sigma_s^\eta)$ be a fixed point of $E^\eta$.

Now, take a sequence $\eta^k$ such that $\eta^k \to 0$ as $k \to \infty$. From the sequence $(\pi^\eta^k, \rho^\eta^k, \sigma_b^\eta^k, \sigma_s^\eta^k)_{k=1}^\infty$ pick a subsequence converging to some assessment $(\pi^*, \rho^*, \sigma_b^*, \sigma_s^*)$. This assessment is an equilibrium: consistency of $\pi^*$ holds by construction, and sequential rationality at every information set follows again from continuity of payoffs in $\pi$.

\[ \square \]

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