Polarization tensor vanishing structure of general shape: Existence for small perturbations of balls

Hyeonbae Kang† Xiaofei Li‡ Shigeru Sakaguchi§

January 20, 2020

Abstract

The polarization tensor is a geometric quantity associated with a domain. It is a signature of the small inclusion’s existence inside a domain and used in the small volume expansion method to reconstruct small inclusions by boundary measurements. In this paper, we consider the question of the polarization tensor vanishing structure of general shape. The only known examples of the polarization tensor vanishing structure are concentric disks and balls. We prove, by the implicit function theorem on Banach spaces, that a small perturbation of a ball can be enclosed by a domain so that the resulting inclusion of the core-shell structure becomes polarization tensor vanishing. The boundary of the enclosing domain is given by a sphere perturbed by spherical harmonics of degree zero and two. This is a continuation of the earlier work [15] for two dimensions.

AMS subject classifications. 35Q60 (primary); 31B10, 35B40, 35R30, 35R05 (secondary)

Key words. Polarization tensor, polarization tensor vanishing structure, weakly neutral inclusion, neutral inclusion, existence, perturbation of balls, implicit function theorem, invisibility cloaking

1 Introduction

In the inverse conductivity problem or the electrical impedance tomography, the measurement of boundary data is utilized to reconstruct inclusions buried inside the domain. When the inclusion is of small size, the small volume expansion shows that the leading order term of the boundary perturbation is expressed by the polarization tensor (abbreviated by PT afterwards) associated with the inclusion. Thus the polarization tensor is a signature of inclusion’s existence, which can be effectively used to reconstruct the inclusion (see, for example, [1, 3, 4, 6, 7, 8, 20]).

This paper is concerned with the problem of the opposite direction: hiding inclusions by making the PT vanish. Since the PT for simply connected homogeneous domain is either positive- or negative-definite, we consider the inclusions of core-shell structure. It is known that concentric...
disks and balls can be made to be PT vanishing (see (1.4) below), and these are the only known examples of the PT-vanishing inclusions.

We are concerned with the following question:

**Polarization Tensor Vanishing Structure.** Find a domain $\Omega$ enclosing the given domain $D$ of arbitrary shape so that polarization tensor of the resulting inclusion $(D, \Omega)$ of the core-shell structure vanishes.

The purpose of this paper is to prove that if the core $D$ is a small perturbation of a ball in three dimensions, then there is $\Omega$ enclosing $D$ such that the inclusion $(D, \Omega)$ becomes PT-vanishing. This is a continuation of the work [15], where a similar result is proved in two dimensions. So, we move directly to description of the problem and statement of the result leaving additional motivational remarks and historical accounts to that paper and references therein (see also recent survey article [12]).

To define the PT-vanishing structure of the core-shell shape, let $D$ and $\Omega$ be bounded simply connected domains in $\mathbb{R}^d$ ($d = 2, 3$) such that $D \subset \Omega$. The pair $(D, \Omega)$ of domains may be regarded as an inclusion of the core-shell structure where the core $D$ is coated by the shell $\Omega \setminus D$. Let $\sigma$ be a piecewise constant function, representing the conductivity distribution, defined by

$$\sigma = \begin{cases} \sigma_c & \text{in } D, \\ \sigma_s & \text{in } \Omega \setminus D, \\ \sigma_m & \text{in } \mathbb{R}^d \setminus \Omega, \end{cases}$$

(1.1)

where the conductivities $\sigma_c$, $\sigma_s$ and $\sigma_m$ of the core, the shell and the matrix are assumed to be isotropic (scalar). We then consider the following conductivity problem:

$$\begin{cases} \nabla \cdot \sigma \nabla u = 0 & \text{in } \mathbb{R}^d, \\ u(x) - a \cdot x = O(|x|^{-d+1}) & \text{as } |x| \to \infty, \end{cases}$$

(1.2)

where $a$ is a unit vector representing the background uniform field.

In absence of the inclusion $(D, \Omega)$ the field is uniform, i.e., $\nabla u = a$. This uniform field is perturbed by insertion of the inclusion and the perturbation is not zero in general. It is known that the solution $u$ to (1.2), or the perturbation $u - a \cdot x$, admits the following dipole asymptotic expansion:

$$u(x) - a \cdot x = \frac{1}{\omega_d} \frac{(Ma, x)}{|x|^d} + O(|x|^{-d}), \quad |x| \to \infty,$$

(1.3)

where $\omega_d$ is the surface area of $S^{d-1}$, the $(d-1)$-dimensional sphere, and $M$ is a $d \times d$ matrix and called the polarization tensor (PT), which is determined by the inclusion $(D, \Omega)$ and conductivity ratios $(\sigma_c/\sigma_m, \sigma_s/\sigma_m)$, namely,

$$M = M(D, \Omega) = M(D, \Omega; \sigma_c/\sigma_m, \sigma_s/\sigma_m)$$

(see, for example, [2, 17]). The question of the PT-vanishing structure can be rephrased as follows: Given $D$ of arbitrary shape, find $\Omega \supset D$ so that $M(D, \Omega) = 0$.

If $D$ is a disk or a ball, then one can choose $\Omega$ to be a concentric disk or ball and the conductivity parameters so that the perturbation $\nabla u - a$ of the uniform field $a$ is zero outside $\Omega$. In fact, if $D = \{|x| < r_D\}$ and $\Omega = \{|x| < r_\Omega\}$ in $\mathbb{R}^3$, and if the following relation among conductivities and the volume fractions holds:

$$(2\sigma + \sigma_c)(\sigma_m - \sigma_s) + \rho^2(\sigma_s - \sigma_c)(2\sigma_s + \sigma_m) = 0,$$

(1.4)
where \( \rho = r_i / r_e \) and \( \rho^3 \) is the volume fraction, then the solution \( u \) to (1.2) satisfies
\[
  u(x) - a \cdot x \equiv 0 \quad \text{for all } x \in \mathbb{R}^3 \setminus \Omega.
\] (1.5)

This discovery of Hashin and Shtrikman [10, 11] has laid significant implications in the theory of composite for which we refer to [17].

The inclusion \((D, \Omega)\) is said to be neutral to multiple uniform fields if (1.5) holds for all constant vector \( a \). However, a pair of concentric balls is the only structure neutral to multiple uniform fields as proved in [14]. It is worth mentioning that the problem (1.2) is well-posed even if \( \sigma_m \) is a positive-definite matrix. It is believed to be true, but has not been proved, that if \( \sigma_m \) is a positive-definite matrix, then the only inclusion neutral to multiple uniform fields is a pair of confocal ellipsoids whose common foci are determined by the eigenvalues of \( \sigma_m \). See [14] for descriptions of this problem and a related over-determined problem (see also [12]). The question in two dimensions has been solved [13, 18].

While the problem (1.2) requires \( u(x) - a \cdot x = O(|x|^{-d+1}) \) at \( \infty \) and the neutrality requires (1.5), the PT-vanishing property requires in-between them, namely,
\[
  u(x) - a \cdot x = O(|x|^{-d}) \quad \text{as } |x| \to \infty,
\] (1.6)
as one can see from (1.3). This is the reason why the PT-vanishing inclusion is also called the weakly neutral inclusion, as used in the title of the earlier version of the manuscript (arXiv:1911.07250v1). However, the name ‘PT-vanishing structure’ seems to convey the meaning more directly, and the title has been changed accordingly in this new version of the manuscript.

To present the main result of this paper in a precise manner, let \( S^2 \) be the unit sphere in \( \mathbb{R}^3 \) and let \( W^{2, \infty}(S^2) \) be the collection of all functions \( f \) on \( S^2 \) such that
\[
  \| f \|_{2, \infty} := \| f \|_{\infty} + \| \nabla_T f \|_{\infty} + \| \nabla_T^2 f \|_{\infty} < \infty,
\]
where \( \nabla_T \) and \( \nabla_T^2 \) be tangential gradient and Hessian on \( S^2 \), and \( \| \cdot \|_p \) denotes the usual \( L^p \) norm. The space \( W^{1, \infty}(S^2) \) with norm \( \| \cdot \|_{1, \infty} \) is defined similarly.

Let \( D_0 := \{ |x| < r_i \} \) for some radius \( r_i \). The core in this paper is defined to be a perturbation of \( D_0 \) by a function \( h \in W^{2, \infty}(S^2) \). Denoting it by \( D_h \), it is defined by
\[
  \partial D_h = \{ \, x \mid x = (r_i + h(\hat{x}))\hat{x}, \quad |\hat{x}| = 1 \, \}.
\] (1.7)
The shell is defined also to be a perturbation of a ball. Let \( \Omega_0 := \{ |x| < r_e \} \) (\( r_e > r_i \)) and define its perturbation by
\[
  \partial \Omega_b = \{ \, x \mid x = (r_e + b(\hat{x}))\hat{x}, \quad |\hat{x}| = 1 \, \}.
\] (1.8)
The perturbation function \( b \) for the shell is chosen from a subclass of \( W^{2, \infty}(S^2) \): Let
\[
  \{ Y_l \}_{l=1}^{6} := \left\{ \frac{1}{\sqrt{15}}, \hat{x}_1\hat{x}_2, \hat{x}_2\hat{x}_3, \frac{1}{2\sqrt{3}}(-\hat{x}_1^2 - \hat{x}_2^2 + 2\hat{x}_3^2), \hat{x}_1\hat{x}_3, \frac{1}{2}(\hat{x}_1^2 - \hat{x}_2^2) \right\}.
\] (1.9)

We mention that \( Y_1 \) is constant (a spherical harmonics of order 0) and \( Y_l, \, 2 \leq l \leq 6, \) is a spherical harmonics of order 2, and \( Y_l, \, 1 \leq l \leq 6, \) are mutually orthogonal and normalized so that the following holds for all \( l \):
\[
  \int_{S^2} |Y_l|^2 dS = \frac{4\pi}{15}.
\] (1.10)

We take this normalization just for ease of notation. Let \( W_6 \) be the space spanned by \( \{ Y_l \} \) and \( \Omega_b \) is defined for \( b \in W_6 \).
If \( h \) and \( b \) are sufficiently small, then \( D_h \subset \Omega_0 \) and hence the PT corresponding to \( (D_h, \Omega_0) \), which is denoted by \( M = M(h, b) \), is well-defined. We choose \( r_e \), the radius of \( \Omega_0 \), so that \( r_i \) and \( r_e \) satisfy neutrality condition (1.4) for given conductivities \( \sigma_c, \sigma_s \) and \( \sigma_m \). For that, \( \sigma_c, \sigma_s \) and \( \sigma_m \) need to satisfy
\[
0 < \frac{(2\sigma_s + \sigma_c)(\sigma_s - \sigma_m)}{(2\sigma_s + \sigma_m)(\sigma_s - \sigma_c)} < 1.
\] (1.11)

Then \( (D_0, \Omega_0) \) is neutral, namely, \( M(0, 0) = 0 \).

The following is the main result of this paper:

**Theorem 1.1.** Given \( r_i \), let \( r_e \) satisfy the neutrality condition (1.4). There is \( \varepsilon > 0 \) such that for each \( h \in W^{2,\infty}(S^2) \) with \( \| h \|_{2,\infty} < \varepsilon \) there is \( b = b(h) \in W_6 \) such that
\[
M(h, b(h)) = 0,
\] (1.12)

namely, the inclusion \( (D_h, \Omega_{b(h)}) \) of the core-shell structure is PT-vanishing. The mapping \( h \mapsto b(h) \) is continuous.

Let us briefly describe how Theorem 1.1 is proved. Since \( M = M(h, b) = (m_{ij})_{i,j=1}^3 \) is a symmetric matrix, we can identify \( M \) with \( (m_{11}, m_{22}, m_{33}, m_{12}, m_{13}, m_{23}) \). We then regard \( M \) as a function from \( U \times V \) into \( \mathbb{R}^6 \), where \( U \) is a small neighborhood of 0 in \( W^{2,\infty}(S^2) \) and \( V \) is a small neighborhood of 0 in \( W_6 \) identified with \( \mathbb{R}^6 \), i.e.,
\[
M : U \times V \subset W^{2,\infty}(S^2) \times \mathbb{R}^6 \to \mathbb{R}^6.
\]

Moreover, since \( (D_0, \Omega_0) \) is neutral to multiple fields, it is PT-vanishing, namely, \( M(0, 0) = 0 \). We then show the Jacobian determinant of \( M \) is non-zero, namely,
\[
\frac{\partial (m_{11}, m_{22}, m_{33}, m_{12}, m_{13}, m_{23})}{\partial (b_1, b_2, b_3, b_4, b_5, b_6)}(0, 0) \neq 0.
\] (1.13)

Then, Theorem 1.1 follows from the implicit function theorem (Theorem 4.1).

The idea and structure of the proof are the same as those in [15]. However, since we are dealing with spherical harmonics in three dimensions in this paper, details are much more involved.

By switching roles of \( h \) and \( b \), we obtain the following theorem:

**Theorem 1.2.** Given \( r_e \), let \( r_i \) satisfy the neutrality condition (1.4). There is \( \varepsilon > 0 \) such that for each \( h \in W^{2,\infty}(S^2) \) with \( \| h \|_{2,\infty} < \varepsilon \) there is \( b = b(h) \in W_6 \) such that the inclusion \( (D_{b(h)}, \Omega_h) \) of the core-shell structure is PT-vanishing. The mapping \( h \mapsto b(h) \) is continuous.

This paper is organized as follows. In section 2 we review the definition of the PT in terms of a system of integral equations, and prove continuity and differentiability of the relevant integral operator. Section 3 includes some preliminary computations of quantities to be used in proving Theorem 1.1 which is proved in section 4. This paper ends with a short conclusion.

## 2 The integral equations and its stability properties

### 2.1 Preliminary: layer potentials and PT

Let \( G(x) \) be the fundamental solution to the Laplacian, that is, \( G(x) = 1/(2\pi) \log |x| \) in two dimensions, and \( G(x) = -((4\pi|x|)^{-1} \) in three dimensions. Let \( D \) be a bounded simply connected
domain with Lipschitz continuous boundary. Let \( S_{\partial D} \) and \( K^*_{\partial D} \) be the single layer potential and the Neumann-Poincaré operator, respectively, namely, for a function \( \varphi \in L^2(\partial D) \)

\[
S_{\partial D}[\varphi](x) := \int_{\partial D} G(x - y)\varphi(y) \, dS(y), \quad x \in \mathbb{R}^3,
\]

and

\[
K^*_{\partial D}[\varphi](x) = \int_{\partial D} \partial_{\nu} G(x - y)\varphi(y) \, dS(y),
\]

where \( dS \) is the surface element on \( \partial D \) and \( \partial_{\nu} \) denotes the outward normal derivative on \( \partial D \). The relation between \( S_{\partial D} \) and \( K^*_{\partial D} \) is given by the following jump formula:

\[
\partial_{\nu} S_{\partial D}[\varphi](x)|_\pm = \left( \pm \frac{1}{2} I + K^*_{\partial D} \right) [\varphi](x), \quad \text{a.e. } x \in \partial D,
\]

where \( I \) is the identity operator and subscripts \( \pm \) denote the limits from outside and inside \( D \), respectively.

Let \( \Omega \) and \( D \) be two bounded domains such that \( \overline{D} \subset \Omega \subset \mathbb{R}^d \), whose boundaries are assumed to be Lipschitz continuous. The solution \( u_l \) (\( 1 \leq l \leq d \)) to \( (1.2) \) when \( a \cdot x = x_l \) is represented as

\[
u_{\partial D}^l \subset L^2(\partial D) \times L^2(\partial \Omega) \]

is the unique solution to the system of integral equations

\[
\left[ -\lambda I + K^*_{\partial D} \begin{array}{c}
\partial_{\nu} S_{\partial D} \\
\partial_{\nu} S_{\partial \Omega}
\end{array} \right] \begin{bmatrix} \varphi_1^{(l)} \\ \varphi_2^{(l)} \end{bmatrix} = - \begin{bmatrix} \nu_{\partial D}^l \\ \nu_{\partial \Omega}^l \end{bmatrix}.
\]

Here \( \nu_{\partial D}^l \) is the \( l \)-th component of the outward unit normal vector \( \nu_{\partial D} \) to \( \partial D \), \( \nu_{\partial \Omega}^l \) is defined likewise, and the numbers \( \lambda \) and \( \mu \) are given by

\[
\lambda = \frac{\sigma_c + \sigma_s}{2(\sigma_c - \sigma_s)} \quad \text{and} \quad \mu = \frac{\sigma_s + \sigma_m}{2(\sigma_s - \sigma_m)}.
\]

Here and afterwards, \( L^2(\partial D) \) denotes the collection of square integrable functions on \( \partial D \) with the mean zero. We refer to the discussion in [15] for a proof of unique solvability of \( (2.5) \) on \( L^2(\partial D) \times L^2(\partial \Omega) \).

The PT \( M = M(D, \Omega) = (m_{ll'})_{l,l'=1}^d \) of the core-shell structure \( (D, \Omega) \) is defined by

\[
m_{ll'} = \int_{\partial D} x_{ll'} \varphi_1^{(l')} \, dS + \int_{\partial \Omega} x_{ll'} \varphi_2^{(l')} \, dS, \quad l, l' = 1, \ldots, d.
\]

The expansion \( (1.3) \) of the solution \( u \) to \( (1.2) \) is valid with this PT.

### 2.2 Parametrizations of integral equations

We consider the system of integral equations \( (2.5) \) when \( D = D_h \) and \( \Omega = \Omega_b \) where \( D_h \) and \( \Omega_b \) are defined by \( (1.7) \) and \( (1.8) \), respectively:

\[
\begin{cases}
( -\lambda I + K^*_{\partial D_h} ) [\varphi_1] + \partial_{\nu} S_{\partial \Omega_b} [\varphi_2] = \psi_1 & \text{on } \partial D_h, \\
\partial_{\nu} S_{\partial D_h} [\varphi_1] + ( -\mu I + K^*_{\partial \Omega_b} ) [\varphi_2] = \psi_2 & \text{on } \partial \Omega_b,
\end{cases}
\]

(2.8)
on $L^2_0(\partial D_h) \times L^2_0(\partial \Omega_h)$. This system of equations admits a unique solution and there is a constant $C = C(h, b)$ such that
\[
\|\varphi_1\|_{L^2(\partial D_h)} + \|\varphi_2\|_{L^2(\partial \Omega_h)} \leq C(\|\psi_1\|_{L^2(\partial D_h)} + \|\psi_2\|_{L^2(\partial \Omega_h)}).
\] (2.9)

We now transform (2.8) in three dimensions to a system of integral equations on $L^2_0(S^2)^2$ where $S^2$ is the unit sphere. To do so, let
\[
x_{i,h}(x) := (r_i + h(x))x, \quad x \in S^2, \tag{2.10}
\]
which is a change of variables from $S^2$ onto $\partial D_h$. Then the unit normal vector $\nu(x_{i,h}(x)) =: \nu_{i,h}(x)$ on $\partial D_h$ is given by the relation
\[
J_{i,h}(x)\nu_{i,h}(x) = (r_i + h(x))[\nu_i + h(x)x - \nabla_T h(x)], \tag{2.11}
\]
where
\[
J_{i,h}(x) := (r_i + h(x))\sqrt{(r_i + h(x))^2 + |\nabla_T h(x)|^2}. \tag{2.12}
\]
The tangential gradient $\nabla_T h(x)$, which was already used in Introduction, is defined to be
\[
\nabla_T h(x) = \sum_{j=1}^2 (\nabla h(x), T_j(x))T_j(x), \tag{2.13}
\]
where $T_1$ and $T_2$ are two unit orthogonal tangent vector fields on $S^2$, and $\nabla h$ is defined after extending $h$ to a tubular neighborhood of $S^2$. Note that $J_{i,h}(x)$ is the Jacobian determinant of the change of variables $x_{i,h}(x)$, namely, the following formula holds:
\[
dS(x_{i,h}(x)) = J_{i,h}(x)dS(x), \quad x \in S^2. \tag{2.14}
\]
Likewise, let
\[
x_{e,b}(x) := (r_e + b(x))x, \quad x \in S^2, \tag{2.15}
\]
which is a change of variables from $S^2$ onto $\partial \Omega_b$. Then the normal vector $\nu(x_{e,b}(x)) =: \nu_{e,b}(x)$ on $\partial \Omega_b$ is given by
\[
J_{e,b}(x)\nu_{e,b}(x) = (r_e + b(x))[\nu_e + b(x)x - \nabla_T b(x)], \tag{2.16}
\]
where
\[
J_{e,b}(x) := (r_e + b(x))\sqrt{(r_e + b(x))^2 + |\nabla_T b(x)|^2}. \tag{2.17}
\]
Then it holds that
\[
dS(x_{e,b}(x)) = J_{e,h}(x)dS(x), \quad x \in S^2. \tag{2.18}
\]
Straight-forward calculations using (2.10)-(2.18) show that the following relations hold:

- Let $A(h)$ be the operator on $L^2(S^2)$ defined by the integral kernel
\[
A_h(x, y) = \frac{1}{4\pi} \frac{\langle x_{i,h}(x) - x_{i,h}(y), \nu_{i,h}(x) \rangle}{|x_{i,h}(x) - x_{i,h}(y)|^3}J_{i,h}(x). \tag{2.19}
\]
Then,
\[
A(h)[f_1](x) = J_{i,h}(x)K^*_{\partial D_h}[\varphi_1](x_{i,h}(x)), \tag{2.20}
\]
where $f_1(y) := \varphi_1(x_{i,h}(y))J_{i,h}(y)$. 

• Let $B(b)$ be defined by
\[
B_b(x, y) = \frac{1}{4\pi} \frac{\langle x_{e,b}(x) - x_{e,b}(y), \nu_{e,b}(x) \rangle}{|x_{e,b}(x) - x_{e,b}(y)|^3} J_{e,b}(x).
\] (2.21)
Then,
\[
B(b)[f_2](x) = J_{e,b}(x) K_{\partial \Omega_b}^2[\varphi_2](x_{e,b}(x)),
\] (2.22)
where $f_2(y) := \varphi_2(x_{e,b}(y)) J_{e,b}(y)$.

• Let $C(h, b)$ be defined by
\[
C_{h,b}(x, y) = \frac{1}{4\pi} \frac{\langle x_{i,h}(x) - x_{e,b}(y), \nu_{i,h}(x) \rangle}{|x_{i,h}(x) - x_{e,b}(y)|^3} J_{i,h}(x).
\] (2.23)
Then,
\[
C(h, b)[f_2](x) = J_{i,h}(x) \partial \nu S_{\partial \Omega_b}^2[\varphi_2](x_{i,h}(x)).
\] (2.24)

• Let $D(h, b)$ be defined by
\[
D_{h,b}(x, y) = \frac{1}{4\pi} \frac{\langle x_{e,b}(x) - x_{i,h}(y), \nu_{e,b}(x) \rangle}{|x_{e,b}(x) - x_{i,h}(y)|^3} J_{e,b}(x).
\] (2.25)
Then,
\[
D(h, b)[f_1](x) = J_{e,b}(x) \partial \nu S_{\partial D_h}^2[\varphi_1](x_{e,b}(x)).
\] (2.26)

Thanks to above formulae, the integral equation (2.8) now takes the form
\[
\begin{align*}
(−\lambda I + A(h)) [f_1] + C(h, b)[f_2] &= g_1, \\
D(h, b)[f_1] + (−\mu I + B(b)) [f_2] &= g_2,
\end{align*}
\] (2.27)
where
\[
g_1(x) := J_{i,h}(x) \psi_1(x_{i,h}(x)) \quad \text{and} \quad g_2(x) := J_{e,b}(x) \psi_2(x_{e,b}(x)).
\] (2.28)
Let
\[
A(h, b) := \begin{bmatrix} −\lambda I + A(h) & C(h, b) \\ D(h, b) & −\mu I + B(b) \end{bmatrix}
\] (2.29)
and $f = (f_1, f_2)^\top, \ g = (g_1, g_2)^\top$ (\(\top\) for transpose). Then (2.27) can be written in short as
\[
A(h, b) f = g
\] (2.30)
on $L^2_0(S^2)^2$. Moreover, (2.29) shows that there is a constant $K > 0$ depending on $h$ and $b$ such that
\[
\|A(h, b)^{-1} g\|_2 \leq K\|g\|_2,
\]
where $\| \cdot \|_2$ denotes the norm on $L^2(S^2)^2$. Here and throughout this paper $K$ denotes a positive constant which may differ at each appearance.
2.3 Continuity of the integral operator

We now consider the continuity of the operator \( A(h, b) \) with respect to \( h \) and \( b \). For that we assume that \( b \in W^{2,\infty}(S^2) \). We first obtain the following proposition for the continuity. This proposition for two dimensions was obtained in [15]. Even though the idea and the procedure of the proof are almost identical, the proof here and there are technically dissimilar because of the nature of the integral kernels. For example, the integral kernels \( A_h(x, y) \) and \( B_b(x, y) \) have singularities of order 1 at \( x = y \) in three dimensions, while there is no singularity in two dimensions. See the first paragraph of the following proof.

Proposition 2.1. There is \( \varepsilon > 0 \) such that if \( h, b \in W^{2,\infty}(S^2) \) and \( \| h \|_{2,\infty} + \| b \|_{2,\infty} \leq \varepsilon \), then

(i) \( A(h, b) \) is continuous at \( (h, b) = (0, 0) \) strongly, namely, there is a constant \( K > 0 \) such that

\[
\| (A(h, b) - A(0, 0))[f] \|_2 \leq K(\| h \|_{2,\infty} + \| b \|_{2,\infty})\| f \|_2
\]

(2.31)

for all \( f \in L^2(S^2)^2 \).

(ii) \( A(h, b) \) is continuous at \( (h, b) \neq (0, 0) \) weakly, namely, for each \( f \in L^2(S^2)^2 \)

\[
\| (A(k, d) - A(h, b))[f] \|_2 \to 0
\]

(2.32)
as \( \| k - h \|_{2,\infty} + \| d - b \|_{2,\infty} \to 0 \).

Proof. We first deal with the operator \( B(b) \) and include the proof in detail here since the proof is more involved than that for the two-dimensional case in [15] due to the singularity. The operator \( A(h) \) can be treated similarly. The operators \( C(h, b) \) and \( D(h, b) \) can be dealt with in the same way as in two dimensions since their integral kernels do not have singularities. However, we include a brief proof since the details presented in this proof will be used in the later part of the paper.

One can easily see from (2.15) and (2.16) that

\[
\langle x_{e, b}(x) - x_{e, b}(y), J_{e, b}(x)\nu_{e, b}(x) \rangle
\]

\[
= (r_e + b(x))^3 - (r_e + b(x))(r_e + b(y)) \left[ (r_e + b(x)) x \cdot y - y \cdot \nabla_T b(x) \right].
\]

Here, we used the fact that \( x \cdot \nabla_T b(x) = 0 \). Therefore, we have

\[
\langle x_{e, b}(x) - x_{e, b}(y), J_{e, b}(x)\nu_{e, b}(x) \rangle = \frac{1}{2}r_e^3|x - y|^2(1 + R_1),
\]

(2.33)

where

\[
R_1 = R_1(b; x, y) = \frac{2b(x) + b(y)}{r_e} + \frac{b(x)(b(x) + 2b(y))}{r_e^2} + \frac{b^2(x)b(y)}{r_e^3}
\]

\[
+ \frac{2(r_e + b(x))(b(x) - b(y))^2 + 2(r_e + b(x))(r_e + b(y))(b(x) - b(y) + y \cdot \nabla_T b(x))}{r_e^3|x - y|^2}.
\]

(2.34)

Let \( b \) be extended to \( \mathbb{R}^3 \setminus \{0\} \) by defining \( b(x) = b(x/|x|) \). Then, since

\[
b(x) - b(y) + y \cdot \nabla_T b(x) = -b(y) + b(x) + (y - x) \cdot \nabla b(x)
\]

for \( x, y \in S^2 \), we have from Taylor’s theorem

\[
|b(x) - b(y) + y \cdot \nabla_T b(x)| \leq K\| b \|_{2,\infty}|x - y|^2
\]

(2.35)
for some constant $K$. Thus,

$$|(r_e + b(x))(b(x) - b(y))^2 + (r_e + b(x))(r_e + b(y))[b(x) - b(y) + y \cdot \nabla_T b(x)]| \leq K\|b\|_{2,\infty}|x - y|^2.$$  \hfill (2.36)

The constant $K$ may differ at each occurrence. We then infer

$$\sup_{x,y \in S^2}|R_1(b; x, y)| \leq K\|b\|_{2,\infty}. \hfill (2.37)$$

One can also see that

$$|x_{e,b}(x) - x_{e,b}(y)|^2 = r_e^2|x - y|^2 + |b(x)x - b(y)y|^2 + 2r_e(x - y) \cdot (b(x)x - b(y)y).$$

Thus we have

$$|x_{e,b}(x) - x_{e,b}(y)| = r_e^3|x - y|^3(1 + R_2)^{3/2}, \hfill (2.38)$$

where

$$R_2 = R_2(b; x, y) = \frac{2(x - y) \cdot (b(x)x - b(y)y)}{r_e|x - y|^2} + \frac{|b(x)x - b(y)y|^2}{r_e^2|x - y|^2}.$$  \hfill (2.39)

Note that

$$\sup_{x,y \in S^2}|R_2(b; x, y)| \leq K\|b\|_{1,\infty}. \hfill (2.40)$$

We have from (2.21), (2.33) and (2.38) that

$$B_b(x, y) = \frac{1}{8\pi} \frac{1}{|x - y|} \frac{1}{\sqrt{1 + R_2}} \left[ 1 + \frac{R_1 - R_2}{1 + R_2} \right]. \hfill (2.41)$$

The singularity $|x - y|^{-1}$ on the righthand side above is specific to three dimensions, and does not appear in two dimensions. Since

$$B_b(x, y) - B_0(x, y) = \frac{1}{8\pi} \frac{1}{|x - y|} \frac{1}{\sqrt{1 + R_2}} \left[ 1 + \frac{R_1 - R_2}{1 + R_2} - \sqrt{1 + R_2} \right] \left[ 1 + \frac{R_1 - R_2}{1 + R_2} - \sqrt{1 + R_2} \right]$$

$$= \frac{1}{8\pi} \frac{1}{|x - y|} \frac{1}{\sqrt{1 + R_2}} \left[ \frac{R_1 - R_2}{1 + R_2} - \frac{R_2}{1 + \sqrt{1 + R_2}} \right],$$

one can see from (2.37) and (2.40) that

$$|B_b(x, y) - B_0(x, y)| \leq \frac{K\|b\|_{2,\infty}}{|x - y|}, \hfill (2.42)$$

provided that $\|b\|_{1,\infty}$ is sufficiently small (because of (2.40)). Thus we have

$$\| (B(b) - B(0)) [f_2] \|_2 \leq K\|b\|_{2,\infty}\|f_2\|_2$$  \hfill (2.43)

for all $f_2 \in L^2(S^2)$.

Let $f_2 \in L^2(S^2)$ and $\delta$ be an arbitrary but fixed positive small number. For each $x \in S^2$, we write

$$ (B(d) - B(b))[f_2](x) = \int_{S^2} (B_d(x, y) - B_b(x, y))f_2(y) dS(y)$$

$$= \int_{|y - x| \leq \delta} + \int_{|y - x| > \delta} =: I_\delta(x) + II_\delta(x).$$
If \( \|d - b\|_{2,\infty} \to 0 \), then \( B_d(x, y) - B_b(x, y) \to 0 \) unless \( x = y \). Moreover, we have from (2.42)

\[
|(B_d(x, y) - B_b(x, y))f_2(y)| \leq K \frac{|f_2(y)|}{|x - y|} \leq K \frac{1}{\delta} |f_2(y)|,
\]

provided that \(|x - y| > \delta\). Thus, by Lebesgue dominated convergence theorem, we infer that \( II_\delta(x) \to 0 \) for each \( x \) as \( d \to b \) in \( W^{2,\infty}(S^2) \). Further, we have

\[
|II_\delta(x)| \leq \frac{C}{\delta} \|f_2\|_2.
\]

We then apply Lebesgue dominated convergence theorem once more to infer that \( \|II_\delta\|_2 \to 0 \) as \( d \to b \) in \( W^{2,\infty}(S^2) \).

To handle \( I_\delta(x) \), we first observe that

\[
|I_\delta(x)| \leq C \int_{|y - x| \leq \delta} \frac{|f_2(y)|}{|y - x|} dS(y) = C \sum_{j=1}^\infty \int_{2^{-j} \delta < |y - x| \leq 2^{-j} \delta + 1} \frac{|f_2(y)|}{|y - x|} dS(y).
\]

Thus we have

\[
|I_\delta(x)| \leq C \sum_{j=1}^\infty \frac{1}{2^{-j} \delta} \int_{|y - x| \leq 2^{-j} \delta + 1} |f_2(y)| dS(y) \leq C \delta M(|f_2|)(x),
\]

where \( M \) is the maximal function, namely,

\[
M(|f_2|)(x) = \sup_r \frac{1}{r^2} \int_{|y - x| \leq r} |f_2(y)| dS(y)
\]

(up to some constant multiplication). Note that in the above the constant \( C \) differs at each occurrence. It then follows that

\[
\int_{S^2} |I_\delta(x)|^2 dS \leq C \delta^2 \int_{S^2} |M(|f_2|)(x)|^2 dS \leq C \delta^2 \|f_2\|^2_2,
\]

where the last inequality comes from the fact that the maximal function \( M \) is bounded on \( L^2 \), for which we refer to [19].

So far, we have shown that

\[
\|(B(d) - B(b))[f_2]\|_2 \leq C \delta \|f_2\|_2 + \|II_\delta\|_2.
\]

Since \( \lim_{d \to b} \|II_\delta\|_2 = 0 \),

\[
\limsup_{d \to b} \|(B(d) - B(b))[f_2]\|_2 \leq C \delta \|f_2\|_2,
\]

where \( d \to b \) in \( W^{2,\infty}(S^2) \). Since \( \delta \) is arbitrary, we conclude that \( \|(B(d) - B(b))[f_2]\|_2 \to 0 \) for each fixed \( f_2 \).

Similarly, one can show that

\[
\|(A(h) - A(0))[f_1]\|_2 \leq K \|h\|_{2,\infty} \|f_1\|_2
\]

(2.44)
for all $f_1 \in L^2(S^2)$, and
\[
\|(A(k) - A(h))[f_1]\|_2 \to 0
\] (2.45)
as $k \to h$ in $W^{2,\infty}(S^2)$ for each fixed $f_1$.

To handle the operator $C(h, b)$, let
\[
\alpha(h, b; x, y) := \langle x_{i,h}(x) - x_{e,b}(y), J_{i,h}(x)\nu_{i,h}(x) \rangle
\] (2.46)
and
\[
\beta(h, b; x, y) := |x_{i,h}(x) - x_{e,b}(y)|^3
\] (2.47)
so that
\[
C_{h,b}(x, y) = \frac{\alpha(h, b; x, y)}{4\pi \beta(h, b; x, y)}.
\] (2.48)

One can see that
\[
\sup_{x,y \in S^2} |\alpha(h, b; x, y) - \alpha(k, d; x, y)| \leq K(\|h - k\|_{1,\infty} + \|b - d\|_{1,\infty})
\] (2.49)
and
\[
\sup_{x,y \in S^2} |\beta(h, b; x, y) - \beta(k, d; x, y)| \leq K(\|h - k\|_{\infty} + \|b - d\|_{\infty}).
\] (2.50)

In fact, it is straight-forward to derive (2.49). To show (2.50), we see that
\[
|\beta(h, b; x, y)^{2/3} - \beta(k, d; x, y)^{2/3}| \leq K(\|h - k\|_{\infty} + \|b - d\|_{\infty})
\] for all $x, y \in S^2$. Furthermore, we have
\[
\beta(h, b; x, y) = |x_{i,h}(x) - x_{e,b}(y)|^3 \geq \frac{1}{8}(r_e - r_i)^3,
\] (2.51)
provided that $\|h\|_{\infty}$ and $\|b\|_{\infty}$ are sufficiently small. Thus we have (2.50) by using a simple identity for positive numbers $a$ and $b:
\[
a - b = \frac{(a^{2/3} - b^{2/3})(a^{4/3} + a^{2/3}b^{2/3} + b^{4/3})}{a + b}.
\]

It then follows from (2.49), (2.50) and (2.51) that
\[
\sup_{x,y \in S^2} |C_{h,b}(x, y) - C_{k,d}(x, y)| \leq K(\|h - k\|_{1,\infty} + \|b - d\|_{1,\infty}),
\]
from which we conclude that
\[
\|(C(h, b) - C(k, d))[f_2]\|_2 \leq K(\|h - k\|_{1,\infty} + \|b - d\|_{1,\infty})\|f_2\|_2
\] (2.52)
for all $f_2 \in L^2(S^2)$.

Similarly one can show that
\[
\|(D(h, b) - D(k, d))[f_1]\|_2 \leq K(\|h - k\|_{1,\infty} + \|b - d\|_{1,\infty})\|f_1\|_2
\] (2.53)
for all $f_1 \in L^2(S^2)$. Now (2.31) and (2.32) follow, and the proof is complete. \qed
Note that $A(0,0)$ is nothing but the operator appearing in (2.5), and so it is invertible. Note also that

$$A(h,b)^{-1} = (I + A(0,0)^{-1}(A(h,b) - A(0,0)))^{-1} A(0,0)^{-1}.$$}

Thanks to (2.31), the operator norm of $A(0,0)^{-1}(A(h,b) - A(0,0))$ is small if $\|h\|_{2,\infty} + \|b\|_{2,\infty}$ is small. Thus $(I + A(0,0)^{-1}(A(h,b) - A(0,0)))^{-1}$ exists. So, we have the following corollary.

**Corollary 2.2.** There is $\varepsilon > 0$ such that

$$\|A(h,b)^{-1}g\|_2 \leq K \|g\|_2$$

for all $g \in L^2_0(S^2)^2$ for some $K > 0$ independent of $h$ and $b$ satisfying $\|h\|_{2,\infty} + \|b\|_{2,\infty} < \varepsilon$.

### 2.4 Differentiability of the integral operator

We now look into differentiability of $A(h,b)$ with respect to $b$ when $b$ belongs to $W_6$, namely, $b$ is of the form

$$b = \sum_{l=1}^{6} b_l Y_l,$$

where $Y_l$ is given by (1.9). For such a $b$, $\|b\|_{2,\infty}$ is equivalent to

$$|b|_{\infty} := \max_{1 \leq j \leq 6} |b_j|.$$

For the rest of this paper, we assume that $b$ is of the form (2.54).

Let $\partial_j$ denote the partial derivative with respect to $b_j$ ($j = 1, \ldots, 6$). Since $\partial_j b = Y_j$, $\partial_j x_{e,b}(y) = Y_j(y) y$. Thus we see from the definitions (2.46) and (2.47) that

$$\partial_j \alpha(h,b;x,y) := -\langle Y_j(y)y, J_{i,h}(x) \nu_i,h(x) \rangle$$

and

$$\partial_j \beta(h,b;x,y) := -3Y_j(y)|x_{i,h}(x) - x_{e,b}(y)|(x_{i,h}(x) - x_{e,b}(y)) \cdot y.$$

We then see easily from (2.48) and (2.51) that

$$\sup_{x,y \in S^2} |\partial_j C_{h,b}(x,y)| \leq K$$

for some constant $K > 0$. Moreover, if $k \in W^{2,\infty}(S^2)$ and $d = \sum_{l=1}^{6} d_l Y_l$, then

$$\sup_{x,y \in S^2} |\partial_j C_{h,b}(x,y) - \partial_j C_{k,d}(x,y)| \leq K(\|h - k\|_{1,\infty} + \|b - d\|_{\infty}).$$

Thus we see that the operator $\partial_j C(h,b)$ is bounded on $L^2(S^2)$ and

$$\|((\partial_j C(h,b) - \partial_j C(k,d)) [f_2])_2 \|_2 \leq K(\|h - k\|_{1,\infty} + \|b - d\|_{\infty}) \|f_2\|_2$$

for all $f_2 \in L^2(S^2)$.

Similarly one can see that the operator $\partial_j D(h,b)$ is bounded on $L^2(S^2)$ and

$$\|((\partial_j D(h,b) - \partial_j D(k,d)) [f_1])_2 \|_2 \leq K(\|h - k\|_{1,\infty} + \|b - d\|_{\infty}) \|f_1\|_2$$

for all $f_1 \in L^2(S^2)$.
Let $R_l$ and $R_2$ be the quantities defined by (2.34) and (2.39), respectively, with $b$ and $d$ of the form (2.54). We claim that the following inequalities hold for $l = 1, 2$ and $j, k = 1, \ldots, 6$:

$$\sup_{x,y \in S^2} |\partial_j R_l(b, x, y)| \leq K, \quad (2.57)$$

$$\sup_{x,y \in S^2} |R_l(b, x, y) - R_l(d, x, y)| \leq K|b - d|_\infty, \quad (2.58)$$

$$\sup_{x,y \in S^2} |\partial_k \partial_j R_l(b, x, y)| \leq K, \quad (2.59)$$

$$\sup_{x,y \in S^2} |\partial_j R_l(b, x, y) - \partial_j R_l(d, x, y)| \leq K|b - d|_\infty. \quad (2.60)$$

In fact, since $\partial_j b = Y_j$, we obtain from (2.34)

$$\partial_j R_1(b, x, y) = \frac{2Y_j(x) + Y_j(y)}{r^2_e} + \frac{Y_j(x)(b(x) + 2b(y)) + b(x)(Y_j(x) + 2Y_j(y))}{r^3_e} + \frac{2}{r^3_e} \cdot \nabla T \cdot b(x), \quad (2.61)$$

where

$$I_b(x, y) := Y_j(x)(b(x) - b(y))^2 + 2(r_e + b(x))(b(x) - b(y))(Y_j(x) - Y_j(y)) + [r_e(Y_j(x) + Y_j(y)) + Y_j(x)b(y) + b(x)Y_j(y)][b(x) - b(y) + y \cdot \nabla T \cdot b(x)] + (r_e + b(x))(r_e + b(y))[Y_j(x) - Y_j(y) + y \cdot \nabla T \cdot y_j(x)].$$

Using (2.35), we see that

$$|I_b(x, y)| \leq K|x - y|^2$$

for some $K$. Thus we arrive at

$$\sup_{x,y \in S^2} |\partial_j R_1(b, x, y)| \leq K, \quad j = 1, \ldots, 6. \quad (2.62)$$

We then immediately obtain (2.57) and hence (2.58) when $l = 1$. By taking further derivatives in (2.61), one can also show (2.59) and (2.60). The case when $l = 2$ can be proved similarly using (2.39).

From (2.41) we have, for $j = 1, \ldots, 6$,

$$\partial_j B_0(x, y) = \frac{1}{8\pi} \frac{1}{|x - y|} \left[ - \frac{1}{2} (1 + R_2)^{-3/2} \partial_j R_2 \left( 1 + \frac{R_1 - R_2}{1 + R_2} \right) + (1 + R_2)^{-1/2} \left( \frac{\partial_j R_1 - \partial_j R_2}{1 + R_2} - \frac{(R_1 - R_2) \partial_j R_2}{(1 + R_2)^2} \right) \right]. \quad (2.63)$$

It then follows from (2.57), (2.40), and (2.57)-(2.60) that for $j, k = 1, \ldots, 6$

$$\sup_{x,y \in S^2} |\partial_k \partial_j B_0(x, y)| \leq K,$$

and hence

$$\sup_{x,y \in S^2} |\partial_j B_0(x, y) - \partial_j B_d(x, y)| \leq K|b - d|_\infty.$$

Thus we have

$$\|\partial_j B(b) - \partial_j B(d)\|_2 \leq K|b - d|_\infty \|f_2\|_2 \quad (2.64)$$

for all $f_2 \in L^2(S^2)$.

The following proposition is an immediate consequence of (2.55), (2.56) and (2.64).
Proposition 2.3. There is a constant \( \varepsilon > 0 \) such that if \( b \) is of the form \([2.34]\) and \( ||h||_{1,\infty} + ||b||_{\infty} < \varepsilon \), then \( \partial_j A(h,b) \) is bounded on \( L^2(S^2)^2 \) for \( j = 1, \ldots, 6 \). Moreover, there is \( K > 0 \) such that if \( d \) is of the form \([2.34]\) and \( ||k||_{1,\infty} + ||d||_{\infty} < \varepsilon \), then

\[
\|(\partial_j A(h,b) - \partial_j A(k,d)) [f]\|_2 \leq K(||h - k||_{1,\infty} + ||b - d||_{\infty})\|f\|_2
\]  

(2.65)

for all \( f \in L^2(S^2)^2 \).

3 Some computations

In this section we compute the quantities

\[
\langle x_l, \partial_j B(0)[x_i]\rangle, \quad \langle x_l, \partial_j C(0,0)[x_i] + \rho \partial_j D(0,0)[x_i]\rangle,
\]  

(3.1)

for \( 1 \leq l \leq l' \leq 3 \) and \( j = 1, \ldots, 6 \). Here \( \langle , \rangle \) denotes the inner product on \( L^2(S^2) \) and \( \rho = r_i/r_e \).

These quantities appear in computation of the Jacobian determinant of the PT at \((0,0)\) in the next section. Note that \( \partial_j A(h) = 0 \) since \( A(h) \) is independent of \( b \).

For computations in this section, the following three identities are useful:

\[
\int_{S^2} \frac{1}{|x - y|} dS(y) = 4\pi, \quad \int_{S^2} \frac{y_k}{|x - y|} dS(y) = \frac{4\pi}{3} x_k,
\]  

(3.2)

and

\[
\int_{S^2} \frac{y_i y_k}{|x - y|} dS(y) = \frac{16\pi}{15} \delta_{ik} + \frac{4\pi}{5} x_i x_k,
\]  

(3.3)

for \( x \in S^2 \) and for \( i, k = 1, 2, 3 \), where \( \delta_{ik} \) is the Kronecker delta.

These identities can be proved using the Funk-Hecke Formula \([5, \text{Theorem 2.22}]\): for \( f \in L^1(-1,1), x \in S^2 \) and for every homogeneous harmonic polynomial \( Y \) of degree \( n \), the following formula holds

\[
\int_{S^2} f(x \cdot y) Y(y) dS(y) = \lambda_n Y(x),
\]  

(3.4)

where the constants \( \lambda_n \) are given by

\[
\lambda_n = 2\pi \int_{-1}^1 P_{n,3}(t) f(t) dt,
\]  

(3.5)

where \( P_{n,3}(t) \) are the Legendre polynomial of degree \( n \) in three dimensions. Note that the constant \( \lambda_n \) depends only on degree \( n \). In fact, since

\[
1 - x \cdot y = \frac{1}{2} |x - y|^2, \quad x, y \in S^2,
\]  

(3.6)

the relevant function \( f \) for \((3.2)\) and \((3.3)\) is \( f(t) = (2(1 - t))^{-1/2} \). Since \( P_{0,3} = 1, P_{1,3} = t \) and \( P_{2,3} = 1/2(3t^2 - 1) \), we may apply \((3.4)\) and \((3.5)\) to derive \((3.2)\) and \((3.3)\). An additional remark may be required for the case when \( i = k \) in \((3.3)\). Even though \( y_i^2 \) is not harmonic, \( y_i^2 - 1/3 |y|^2 \) is.

So we may apply \((3.4)\) to this function and derive \((3.3)\) when \( i = k \).

Let \( U \) be the unit ball so that \( \partial U = S^2 \). Since both sides of the equality in \((3.2)\) are harmonic in \( x \in U \), \((3.2)\) holds for every \( x \in \overline{U} \). By the same reason, if \( i \neq k \), \((3.3)\) holds for every \( x \in \overline{U} \). Thus we have from \((3.2)\) and \((3.3)\) that for every \( i, k = 1, 2, 3 \), and for every \( x \in \overline{U} \)

\[
\int_{S^2} \frac{x_k - y_k}{|x - y|} dS(y) = \frac{8\pi}{3} x_k,
\]  

(3.7)
and
\[ \int_{S^2} \frac{(x - y_i)(x_k - y_k)}{|x - y|} dS(y) = \frac{32\pi}{15} x_i x_k \quad \text{if } i \neq k. \] (3.8)

By differentiating (3.7) in \( x \), with the aid of the first equality in (3.2), we have for every \( i, k = 1, 2, 3 \), and for every \( x \in \mathcal{U} \)
\[ \int_{S^2} \frac{(x - y_i)(x_k - y_k)}{|x - y|^3} dS(y) = \frac{4\pi}{3} \delta_{ik}. \] (3.9)

Similarly, by differentiating (3.8) in \( x \), with the aid of (3.7), we have for every \( i, k, l = 1, 2, 3 \), and for every \( x \in \mathcal{U} \)
\[ \int_{S^2} \frac{(x - y_i)(x_k - y_k)(x_l - y_l)}{|x - y|^3} dS(y) = \frac{8\pi}{15} (\delta_{ik} x_l + \delta_{kl} x_i + \delta_{li} x_k), \] (3.10)

unless \( i = k = l \). Even if \( i = k = l \), we can recover (3.10) by using (3.7) and \(|x - y|^2 = \sum_{i=1}^{3} (x_i - y_i)^2\).

We now compute the first quantity in (3.1). The following identities can be derived immediately from (2.34) and (2.61):
\[ R_1(0; x, y) = 0, \]
\[ \partial_j R_1(0; x, y) = \frac{2Y_j(x) + Y_j(y)}{r_e} + \frac{2(Y_j(x) - Y_j(y) + y \cdot \nabla T Y_j(x))}{r_e |x - y|^2}, \]

and the following from (2.39) (and by taking derivatives and using (3.9)):
\[ R_2(0; x, y) = 0, \]
\[ \partial_j R_2(0; x, y) = \frac{Y_j(x) + Y_j(y)}{r_e}. \]

Here and afterwards, we denote \( \partial_j B_0(x, y) := \partial_j B_0(x, y)|_{b=0} \) for \( j = 1, \ldots, 6 \), just for simplicity. We obtain from (2.63) and the above identities that
\[ \partial_j B_0(x, y) = \frac{1}{8\pi} \frac{1}{|x - y|} \left( \partial_j R_1(0; x, y) - \frac{3}{2} \partial_j R_2(0; x, y) \right) \]
\[ = \frac{1}{16\pi r_e} \frac{Y_j(x) - Y_j(y)}{|x - y|} + \frac{1}{4\pi r_e} \frac{Y_j(x) - Y_j(y) + y \cdot \nabla T Y_j(x)}{|x - y|^3}. \] (3.11)

Since \( Y_1 \) is constant, we see from (3.11) that
\[ \partial_1 B_0(x, y) = 0. \]

For \( j = 2, \ldots, 6 \), it is convenient to abuse notation and denote by \( Y_j(x) \) the homogenous harmonic polynomial of order 2 such that it is the spherical harmonic \( Y_j(x) \) when \(|x| = 1\). If we use such notation, then by Taylor expansion we have
\[ Y_j(y) = Y_j(x) + \nabla Y_j(x) \cdot (y - x) + \frac{1}{2} \sum_{i,k=1}^{3} G_{ik}^j (y_i - x_i)(y_k - x_k), \] (3.12)

where \( G_{ik}^j = \frac{\partial^2}{\partial x_i \partial x_k} Y_j(x) \), which are constants. Moreover, we have for \( x \in S^2 \)
\[ \nabla T Y_j(x) = \nabla Y_j(x) - (x \cdot \nabla Y_j(x)) x. \] (3.13)
Using these two identities, we obtain

\[ Y_j(x) - Y_j(y) + y \cdot \nabla_x Y_j(x) = Y_j(x) - Y_j(y) + y \cdot [\nabla Y_j(x) - (x \cdot \nabla Y_j(x))x] \]

\[ = (x \cdot \nabla Y_j(x))(1 - x \cdot y) - \frac{1}{2} \sum_{i,k=1}^{3} G^j_{ik}(y_i - x_i)(y_k - x_k). \]

Recall the identity

\[ x \cdot \nabla Y_j(x) = 2Y_j(x), \quad x \in S^2, \tag{3.14} \]

which is a special case (of order 2) of Euler’s theorem on homogeneous functions. Using this identity and (3.15), we have

\[ Y_j(x) - Y_j(y) + y \cdot \nabla_x Y_j(x) = Y_j(x)|x - y|^2 - \frac{1}{2} \sum_{i,k=1}^{3} G^j_{ik}(y_i - x_i)(y_k - x_k). \tag{3.15} \]

Plugging (3.15) into (3.11), we have

\[ \partial_j B_0(x, y) = \frac{1}{8\pi r_e} \left( \frac{\frac{5}{8} Y_j(x) - \frac{1}{8} Y_j(y)}{|x - y|} - \sum_{i,k=1}^{3} \frac{G^j_{ik}(y_i - x_i)(y_k - x_k)}{8\pi|x - y|^3} \right). \tag{3.16} \]

Then,

\[ \langle x'_r, \partial_j B(0)[x_1] \rangle \]

\[ = \frac{1}{r_e} \int_{S^2} x'_r \int_{S^2} \left( \frac{\frac{5}{8} Y_j(x) - \frac{1}{8} Y_j(y)}{8\pi|x - y|} - \sum_{i,k=1}^{3} \frac{G^j_{ik}(y_i - x_i)(y_k - x_k)}{8\pi|x - y|^3} \right) y dS(y) dS(x). \tag{3.17} \]

We now compute the right-hand side of (3.17). The second equality in (3.2) yields

\[ \int_{S^2} x'_r \int_{S^2} \frac{\frac{5}{8} Y_j(x) - \frac{1}{8} Y_j(y)}{8\pi|x - y|} y dS(y) dS(x) = \frac{1}{3} \int_{S^2} x'_r x_1 Y_j(x) dS(x). \tag{3.18} \]

Moreover, with the aid of (3.9) and (3.10), we compute

\[ \sum_{i,k=1}^{3} G^j_{ik} \int_{S^2} x'_r \int_{S^2} \frac{(y_i - x_i)(y_k - x_k)}{8\pi|x - y|^3} y dS(y) dS(x) \]

\[ = \sum_{i,k=1}^{3} G^j_{ik} \int_{S^2} x'_r \int_{S^2} \left\{ \frac{(y_i - x_i)(y_k - x_k)(y_l - x_l)}{8\pi|x - y|^3} + \frac{(y_i - x_i)(y_k - x_k)x_l}{8\pi|x - y|^3} \right\} dS(y) dS(x) \]

\[ = \frac{2}{15} \sum_{i=1}^{3} G^j_{i1} \int_{S^2} x'_r x_i dS(x) + \frac{1}{6} \sum_{i,k=1}^{3} G^j_{ik} \delta_{ik} \int_{S^2} x'_r x_i dS(x) \]

\[ = \frac{2}{15} \sum_{i=1}^{3} G^j_{i1} \frac{4\pi}{3} \delta_{i1} + \frac{1}{6} \sum_{i=1}^{3} G^j_{ii} \frac{4\pi}{3} \delta_{i1} \]

\[ = \frac{8\pi}{45} G^j_{11}, \tag{3.19} \]
where the last inequality holds because $\sum_{i=1}^{3} G_{ii}^j = \Delta Y_j = 0$. Then (3.17) together with (3.18) and (3.19) yields
\[ \langle x', \partial_j B(0) | x_l \rangle = \frac{1}{3\pi e} \int_{S^2} x' x_i Y_j(x) dS - \frac{8\pi}{45r_e} G_{lj}^j. \] (3.20)

Since the first term on the right-hand side above appears repeatedly, we write down the values here. Let
\[ C_{ll'}^j := \int_{S^2} x' x_i Y_j(x) dS. \] (3.21)

Then, $C_{ll'}^j$ is symmetric in $l$ and $l'$, namely, $C_{ll'}^j = C_{l'l}^j$, and for $1 \leq l \leq l' \leq 3$
\[ C_{ll'}^j = \frac{4\pi}{15} \times \begin{cases} \frac{\sqrt{2}}{3} & \text{if } (l, l', j) = (1, 1, 1), (2, 2, 1), (3, 3, 1), \\ -\frac{1}{\sqrt{3}} & \text{if } (1, 1, 4), (2, 2, 4), \\ 1 & \text{if } (1, 1, 6), (1, 2, 2), (1, 3, 5), (2, 3, 3), \\ -1 & \text{if } (2, 2, 6), \\ \frac{2}{\sqrt{3}} & \text{if } (3, 3, 4), \\ 0 & \text{otherwise}, \end{cases} \] (3.22)

which can be seen from (1.9) and (1.10). We then see that $\langle x', \partial_j B(0) | x_l \rangle$ is symmetric in $l$ and $l'$, and obtain for $1 \leq l \leq l' \leq 3$
\[ \langle x', \partial_j B(0) | x_l \rangle = \frac{4\pi}{45r_e} \times \begin{cases} \frac{1}{\sqrt{3}} & \text{if } (l, l', j) = (1, 1, 4), (2, 2, 4), \\ -\frac{2}{\sqrt{3}} & \text{if } (3, 3, 4), \\ 1 & \text{if } (1, 1, 6), (1, 2, 2), (2, 3, 3), (1, 3, 5), \\ -1 & \text{if } (2, 2, 6), \\ 0 & \text{otherwise}. \end{cases} \] (3.23)

To compute $\partial_j C_{h,b}(x, y)$ at point $(h, b) = (0, 0)$, we first observe that $\alpha$ and $\beta$ given by (2.46) and (2.47) take the form
\[ \alpha(0, b; x, y) = \frac{1}{2} r_i |r_i x - r_e y|^2 (1 + R_3), \]
where
\[ R_3 = R_3(b; x, y) = \frac{r_i^2 - r_e^2 - 2r_i b(y) (x \cdot y)}{|r_i x - r_e y|^2}, \]
and
\[ \beta(0, b; x, y) = |r_i x - r_e y|^3 (1 + R_4)^{3/2}, \]
where
\[ R_4 = R_4(b; x, y) = \frac{b(y) (b(y) + 2r_e - 2r_i x \cdot y)}{|r_i x - r_e y|^2}. \]

It then follows from (2.48) that
\[ C_{0,b}(x, y) = \frac{r_i}{8\pi |r_i x - r_e y|} \frac{1}{(1 + R_3)^{3/2}} \frac{1 + R_3}{(1 + R_4)^{3/2}}, \] (3.24)
and hence
\[ \partial_j C_{0,b}(x, y) = \frac{r_i}{8\pi |r_i x - r_e y|} \left[ \frac{\partial_j R_3}{(1 + R_4)^{3/2}} - \frac{3}{2} (1 + R_4)^{1/2} \frac{(1 + R_3) \partial_j R_4}{[1 + R_4]^3} \right]. \]
Since $R_4(0; x, y) = 0$ if $b = 0$, we have

$$
\partial_j C_{0, 0}(x, y) = \frac{r_i}{8\pi |r_i x - r_e y|} \left[ \partial_j R_3(0; x, y) - \frac{3}{2} (1 + R_3(0; x, y)) \partial_j R_4(0; x, y) \right]. \tag{3.25}
$$

Straightforward computations yield the following:

$$
R_3(0; x, y) = \frac{r_i^2 - r_e^2}{|r_i x - r_e y|^2}, \quad R_4(0; x, y) = 0,
$$

and for $j = 1, \ldots, 6$

$$
\partial_j R_3(0; x, y) = \frac{-2r_i x \cdot y}{|r_i x - r_e y|^2} Y_j(y), \quad \partial_j R_4(0; x, y) = \frac{2r_e - 2r_i x \cdot y}{|r_i x - r_e y|^2} Y_j(y).
$$

Plugging these terms into (3.25) we have

$$
\partial_j C_{0, 0}(x, y) = \frac{r_i}{8\pi} \left[ \frac{-3r_e + r_i x \cdot y}{|r_i x - r_e y|^3} \cdot \frac{3(r_i^2 - r_e^2)(r_e - r_i x \cdot y)}{|r_i x - r_e y|^5} \right] Y_j(y), \quad j = 1, \ldots, 6. \tag{3.26}
$$

Since $Y_1$ is constant, this can be written as

$$
\partial_j C_{0, 0}(x, y) = \frac{\partial_1 C_{0, 0}(x, y)}{Y_1} Y_j(y), \quad j = 1, \ldots, 6. \tag{3.27}
$$

To compute $\partial_j D_{h,b}(x, y)$ at $(h, b) = (0, 0)$, set

$$
\xi(h, b; x, y) := \langle x_{e,b}(x) - x_{i,h}(y), J_{e,b}(x) \nu_{e,b}(x) \rangle,
$$

and

$$
\zeta(h, b; x, y) := |x_{e,b}(x) - x_{i,h}(y)|^3.
$$

Then we have

$$
\xi(0, b; x, y) = \frac{1}{2} r_e |r_i x - r_e y|^2 (1 + R_5),
$$

where

$$
R_5 = R_5(b; x, y) = \frac{r_e (r_i^2 - r_e^2) + 2b(r_i^2 + (2r_e + b)(r_e + b - r_i x \cdot y)) + 2r_i (r_e + b) y \cdot \nabla r_i x}{r_e |r_i x - r_e y|^2}.
$$

We also have

$$
\zeta(0, b; x, y) = |r_i x - r_e y|^3 (1 + R_0)^{3/2},
$$

where

$$
R_6 = R_6(b; x, y) = \frac{b(b + 2r_e - 2r_i x \cdot y)}{|r_i x - r_e y|^2}.
$$

Then

$$
D_{0,b}(x, y) = \frac{\xi(0, b; x, y)}{4\pi \zeta(0, b; x, y)} = \frac{r_e}{8\pi |r_i x - r_e y|^2} \frac{1 + R_5}{(1 + R_0)^{3/2}}, \tag{3.28}
$$

and hence

$$
\partial_j D_{0,b}(x, y) = \frac{r_e}{8\pi |r_i x - r_e y|^2} \left[ \frac{\partial_j R_5}{(1 + R_6)^{3/2}} - \frac{3}{2} (1 + R_5) \frac{\partial_j R_6}{(1 + R_0)^{3/2}} \right].
$$
Since $R_6(0; x, y) = 0$ if $b = 0$, we have
\[
\partial_j D_{0,0}(x, y) = \frac{r_e}{8\pi |r_e y - r_e y|^2} \left[ \partial_j R_5(0; x, y) - \frac{3}{2} (1 + R_5(0; x, y)) \partial_j R_6(0; x, y) \right]. \tag{3.29}
\]
Straightforward computations yield the following:
\[
R_5(0; x, y) = \frac{r_e^2 - r_i^2}{|r_i x - r_e y|^2}, \quad R_6(0; x, y) = 0,
\]
and
\[
\partial_j R_5(0; x, y) = \frac{2(3r_e - 2r_i x \cdot y) Y_j(x)}{|r_i x - r_e y|^3} + \frac{2r_e y \cdot \nabla Y_j(x)}{|r_i x - r_e y|^2},
\]
\[
\partial_j R_6(0; x, y) = \frac{2r_e - 2r_i x \cdot y}{|r_i x - r_e y|^2} Y_j(x).
\]
Plugging these terms into (3.29) we have
\[
\partial_j D_{0,0}(x, y) = \frac{r_e^3 - 3r_e r_i x \cdot y}{8\pi |r_e y - r_e y|^3} Y_j(x) + \frac{r_e^2 Y_j(x)}{|r_i x - r_e y|^3} - \frac{3r_e^2 - r_i^2}{8\pi |r_i x - r_e y|^5} Y_j(x).
\]
Thanks to (3.27) and (3.30), this formula can be rephrased as
\[
\partial_j D_{0,0}(x, y) = -\frac{\partial_j C_{0,0}(x, y)}{\rho Y_1} Y_j(x) + E_j(x, y), \quad j = 1, \ldots, 6, \tag{3.30}
\]
where
\[
E_j(x, y) = \frac{r_e r_i y \cdot \nabla Y_j(x)}{4\pi |r_i x - r_e y|^3}. \tag{3.31}
\]
We now compute $\langle x_y, \partial_j C(0,0)[x_i] + \rho \partial_j D(0,0)[x_i] \rangle$. Thanks to (3.27) and (3.30), we have
\[
\langle x_y, \partial_j C(0,0)[x_i] + \rho \partial_j D(0,0)[x_i] \rangle \tag{3.32}
\]
\[
= \int_{S^2} x_y \int_{S^2} \left( \frac{\partial_1 C_{0,0}(x, y)}{Y_1} (Y_j(x) - Y_j(x)) + \rho E_j(x, y) \right) y_s dS(y) dS(x).
\]
Since $\partial_1 C_{0,0}(x, y)$ is a function of variable $x \cdot y$, we may apply Funk-Hecke formula (3.3) to see that
\[
\int_{S^2} x_y \int_{S^2} \frac{\partial_1 C_{0,0}(x, y)}{Y_1} (Y_j(x) - Y_j(x)) y_s dS(y) dS(x)
\]
\[
= \frac{1}{Y_1} \int_{S^2} x_y \int_{S^2} \partial_1 C_{0,0}(x, y) Y_j(x) y_s dS(y) dS(x) - \frac{1}{Y_1} \int_{S^2} y_s \partial_1 C_{0,0}(x, y) Y_j(x) x_y dS(x) dS(y)
\]
\[
= 0.
\]
Thus we have
\[
\langle x_y, \partial_j C(0,0)[x_i] + \rho \partial_j D(0,0)[x_i] \rangle = \rho \langle x_y, E_j[x_i] \rangle, \tag{3.32}
\]
where
\[
E_j[x_i](x) := \int_{S^2} E_j(x, y) y_s dS(y). \tag{3.33}
\]
We now compute the term $\langle x_y, E_j[x_i] \rangle$. Clearly $E_1[x_i] = 0$. For $j = 2, \ldots, 6$ and for $t = 1, 2, 3$, by (3.12) we have
\[
\frac{\partial Y_j}{\partial y_t}(y) = \frac{\partial Y_j}{\partial y_t}(x) + \sum_{k=1}^3 G_{jk}^j(y_k - x_k).
Then it follows from Euler’s theorem (3.14) that
\[
y \cdot \nabla Y_j(x) = 2Y_j(y) - \sum_{t,k=1}^3 G_{tk}^j(y_k - x_k)yt.
\]  
(3.34)

Hence, by (3.13) and (3.31), we have for \( j = 2, \ldots, 6 \)
\[
E_j(x, y) = \frac{r_e r_i}{4\pi} \left[ \frac{2Y_j(y) - 2Y_j(x) x \cdot y - \sum_{t,k=1}^3 G_{tk}^j(y_k - x_k)yt}{r_i x - r_e y} \right].
\]  
(3.35)

Therefore we have
\[
\langle x^j, E_j[x_i] \rangle = \int_{S^2} x^j \int_{S^2} E_j(x, y) y dS(y) dS(x) =
\]
\[
\frac{r_e r_i}{4\pi} \int_{S^2} \int_{S^2} \left\{ \frac{2x^j v_j(y) y t}{r_i x - r_e y^3} - \frac{2x^j v_j(x) y t x \cdot y}{r_i x - r_e y^3} - \sum_{t,k=1}^3 G_{tk}^j \frac{x^j y_k y_l y_l - x^j x_k y_l y_l}{r_i x - r_e y^3} \right\} dS(y) dS(x). 
\]  
(3.36)

Before calculating (3.36), we prepare several integral formulas. Since formula (3.2) holds for every \( x \in \mathbb{U} \), we infer that for every \( x \in \mathbb{U} \) and every \( k = 1, 2, 3 \),
\[
\int_{S^2} \frac{1}{r_i x - r_e y} dS(y) = \frac{4\pi}{r_e} , \quad \int_{S^2} \frac{y_k}{r_i x - r_e y} dS(y) = \frac{4\pi r_i}{3r_e^2} x_k. 
\]  
(3.37)

Moreover, by the Funk-Hecke formula, we have for every \( x, y \in S^2 \) and for \( k = 1, 2, 3 \)
\[
\int_{S^2} \frac{1}{r_i x - r_e y^3} dS(y) = \frac{4\pi r_i}{r_e^2(r_e^2 - r_i^2)} x_k, 
\]  
(3.38)

and
\[
\int_{S^2} \frac{y_k}{r_i x - r_e y^3} dS(y) = \frac{4\pi r_i}{r_e^2(r_e^2 - r_i^2)} x_k, \quad \int_{S^2} \frac{x_k}{r_i x - r_e y^3} dS(x) = - \frac{4\pi r_e}{r_i^2(r_i^2 - r_e^2)} y_k. 
\]  
(3.39)

Here we used the fact that the denominator of the integrands never vanish. The first equality in (3.39) can be also obtained by differentiating the first equality in (3.37) with respect to \( x \), with the aid of (3.38). Moreover, by differentiating the second equality in (3.37) with respect to \( x \), with the aid of the first equality of (3.39), we have for \( k, t = 1, 2, 3 \), and for every \( x \in \mathbb{U} \)
\[
\int_{S^2} \frac{y_k y_t}{r_i x - r_e y^3} dS(y) = \frac{4\pi}{3r_e^2} \delta_{kt} + \frac{4\pi r_i^2}{r_i^2(r_i^2 - r_e^2)} x_k x_t. 
\]  
(3.40)

Using (3.39) and (3.40), we have
\[
\int_{S^2} \int_{S^2} \frac{2x^j v_j(y) y t}{r_i x - r_e y^3} dS(y) dS(x) = - \frac{8\pi r_e}{r_i^2(r_e^2 - r_i^2)} \int_{S^2} y_t y_j(y) dS(y),
\]
\[
\int_{S^2} \int_{S^2} \frac{2x^j v_j(x) y t x \cdot y}{r_i x - r_e y^3} dS(y) dS(x) = \frac{8\pi(r_e^2 + 2r_i^2)}{3r_i^2(r_e^2 - r_i^2)} \int_{S^2} x_t x_j(x) dS(x),
\]
and
\[
\int_{S^2} \int_{S^2} \sum_{t,k=1}^3 G_{tk}^j \frac{x^j y_k y_l y_l - x^j x_k y_l y_l}{r_i x - r_e y^3} dS(y) dS(x)
\]
\[
= - \frac{8\pi(r_e^4 + r_i^4)}{r_i^2 r_e^2 (r_e^2 - r_i^2)} \int_{S^2} x_t x_j(x) dS(x) - \frac{16\pi^2}{9r_e^2} G_{tt}. 
\]
where we used the fact that $\sum_{l,k=1}^{3} G^{j}_{lk} x_l x_k = 2Y_j(x)$. Plugging these identities into (3.36) yields

$$\langle x_{l'}, E_j[x_l] \rangle = -\frac{2\rho_l}{3r_e^2} \int_{S^2} x_{l'} x_l Y_j(x) dS(x) + \frac{4\pi \rho_l}{9r_e^2} G^{j}_{ll}. \quad (3.41)$$

Thus we see from (3.32) that $\langle x_{l'}, \partial_j C(0,0)[x_l] + \rho\partial_j D(0,0)[x_l] \rangle$ is symmetric in $l, l'$ and for $1 \leq l \leq l' \leq 3$:

$$\langle x_{l'}, \partial_j C(0,0)[x_l] + \rho\partial_j D(0,0)[x_l] \rangle = \rho \langle x_{l'}, E_j[x_l] \rangle = \begin{cases} - \frac{1}{\sqrt{3}} & \text{if } (l, l', j) = (1, 1, 4), (2, 2, 4), \\ \frac{2}{\sqrt{3}} & \text{if } (3, 3, 4), \\ 1 & \text{if } (1, 1, 6), (1, 2, 2), (2, 3, 3), (1, 3, 5), \\ -1 & \text{if } (2, 2, 6), \\ 0 & \text{otherwise.} \end{cases} \quad (3.42)$$

### 4 Proof of Theorem 1.1

In this section we prove Theorem 1.1 by showing that $m_{l'l'}$ satisfies the hypothesis of the implicit function theorem: continuity in $(h, b)$, continuous differentiability in $b$, and (1.13). Here we recall the implicit function theorem in the following form [16]:

**Theorem 4.1.** Let $X$ be a Banach space. Let $U \times V$ be an open subset of $X \times \mathbb{R}^6$. Suppose that $F = (F_1, \ldots, F_6) : (x, y) \in U \times V \mapsto \mathbb{R}^6$ is continuous and has the property that the derivative of $F$ with respect to $y$ exists and is continuous at each point of $U \times V$. Further assume that at point $(x_0, y_0) \in U \times V$,

$$F(x_0, y_0) = 0 \quad \text{and} \quad \frac{\partial F}{\partial y}(x_0, y_0) \neq 0.$$

Then there exist neighborhood $N_1 \subset U$ of $x_0$ and neighborhood $N_2 \subset V$ of $y_0$ such that, for each $x$ in $N_1$, there is a unique $y \in N_2$ satisfying

$$F(x, y) = 0.$$

The function $\hat{y}$, thereby uniquely defined near $x_0$ by the condition $\hat{y}(x) = y$, is continuous.

Let $m_{l'l'}(h, b) := m_{l'l'}(D_h, \Omega_b)$ as before. By definition (2.7), $m_{l'l'}(h, b), l, l' = 1, 2, 3$, are given by

$$m_{l'l'}(h, b) = \int_{\partial D_h} x_{l'} \varphi^{(l)}_{1} dS + \int_{\partial \Omega_b} x_{l'} \varphi^{(l)}_{2} dS,$$

where $\varphi^{(l)} = (\varphi^{(l)}_1, \varphi^{(l)}_2) \in L^2(\partial D_h) \times L^2(\partial \Omega_b)$ is the unique solution to (2.5). Using changes of variables (2.10) and (2.15), we see that

$$m_{l'l'}(h, b) = \int_{S^2} (r_i + h(x)) x_{l'} J^{l'}_{h,1,b}(x) dS + \int_{S^2} (r_e + b(x)) x_{l'} J^{l'}_{h,2,b}(x) dS, \quad (4.1)$$

where

$$J^{l'}_{h,1,b}(x) := \varphi^{(l)}_1(x, h(x)) J_{i,h}(x), \quad J^{l'}_{h,2,b}(x) := \varphi^{(l)}_2(x, e_b(x)) J_{e,b}(x).$$
Let \( f_{h,b}^{(l)} = (f_{h,b,1}^{(l)}, f_{h,b,2}^{(l)})^{\top} \) and

\[
p(h, b) := (r_i + h(x), r_e + b(x))^{\top},
\]

where \( \top \) denotes the transpose. Then, we have

\[
m_{ll'}(h, b) = \langle x_{l'}p(h, b), f_{h,b}^{(l)} \rangle.
\]

Note that \( f_{h,b}^{(l)} \) is the solution of

\[
\mathcal{A}(h, b)[f_{h,b}^{(l)}] = g_{h,b}^{(l)} := - \left[ \begin{array}{c}
\nu_{\partial D_h}^{(l)}(x_i,h(x))J_i,h(x) \\
\nu_{\partial l_b}^{(l)}(x_e,b(x))J_e,b(x)
\end{array} \right].
\]

We see from (2.11) and (2.16) that \( g_{h,b}^{(l)}, l = 1, 2, 3, \) is given by

\[
g_{h,b}^{(l)} = - \left[ \begin{array}{c}
(r_i + h(x))[(r_i + h(x))x_l - \nabla_T h(x)_l] \\
(r_e + b(x))[(r_e + b(x))x_l - \nabla_T b(x)_l]
\end{array} \right].
\]

In what follows, we show that the mapping \( F := (m_{11}, m_{12}, m_{13}, m_{22}, m_{23}, m_{33}) \) satisfies hypothesis of Theorem 4.1.

**Continuity in \((h, b)\).** We only prove continuity of \( m_{11} \) since the others can be handled in the same way.

Suppose \( k \in W^{2,\infty}(S^2) \) and \( d \in W_6 \). Then we have

\[
\mathcal{A}(k, d)[f_{k,d}^{(1)}] = g_{k,d}^{(1)}.
\]

Thus,

\[
\mathcal{A}(k, d)[f_{k,d}^{(1)}] - f_{h,b}^{(1)} = - (\mathcal{A}(k, d) - \mathcal{A}(h, b))[f_{h,b}^{(1)}] + (g_{k,d}^{(1)} - g_{h,b}^{(1)}).
\]

We then infer using Corollary 2.2 that

\[
\left\| f_{k,d}^{(1)} - f_{h,b}^{(1)} \right\|_2 \leq K \left( \left\| (\mathcal{A}(k, d) - \mathcal{A}(h, b))[f_{h,b}^{(1)}] \right\|_2 + \left\| g_{k,d}^{(1)} - g_{h,b}^{(1)} \right\|_2 \right)
\]

for some constant \( K \) independent of \((k, d)\) as long as \( \|k\|_{2,\infty} \) and \( |d|_\infty \) are sufficiently small. We then infer from (2.32) that

\[
\left\| (\mathcal{A}(k, d) - \mathcal{A}(h, b))[f_{h,b}^{(1)}] \right\|_2 \to 0
\]

as \( \|k - h\|_{2,\infty} + |d - b|_\infty \to 0 \). It is obvious from (1.5) that \( \|g_{k,d}^{(1)} - g_{h,b}^{(1)}\|_2 \to 0 \). Thus we have

\[
\left\| f_{k,d}^{(1)} - f_{h,b}^{(1)} \right\|_2 \to 0.
\]

We then conclude using (4.3) that \( m_{11}(k, d) - m_{11}(h, b) \to 0 \) as \( \|k - h\|_{2,\infty} + |d - b|_\infty \to 0 \).

**Continuous differentiability in \( b \).** By differentiating (4.4) with respect to the \( b_j \)-variable, we have

\[
\mathcal{A}(h, b)[\partial_j f_{h,b}^{(1)}] = \partial_j g_{h,b}^{(1)} - \partial_j \mathcal{A}(h, b)[f_{h,b}^{(1)}],
\]

namely,

\[
\partial_j f_{h,b}^{(1)} = \mathcal{A}(h, b)^{-1} \left[ \partial_j g_{h,b}^{(1)} - \partial_j \mathcal{A}(h, b)[f_{h,b}^{(1)}] \right].
\]

We mention that this argument is formal since we take the derivative of \( f_{h,b}^{(1)} \) without proving its existence. However, this formal argument can be justified easily.
It is clear from (4.5) that $\partial_j g_{h,b}^{(1)}$ is continuous in $(h, b)$. Then Corollary 2.2, Proposition 2.3 and continuity of $f_{h,b}^{(1)}$ in $(h, b)$ imply that $\partial_j f_{h,b}^{(1)}$ is continuous in $(h, b)$. We then obtain from (4.3) that

$$
\partial_j m_{11}(h, b) = \left\langle x_1 \partial_j p(h, b), f_{h,b}^{(1)} \right\rangle + \left\langle x_1 p(h, b), \partial_j f_{h,b}^{(1)} \right\rangle,
$$

which shows that $\partial_j m_{11}(h, b)$ is continuous in $(h, b)$.

**Proof of (1.13).** For ease of notation we put

$$
\psi_l(x) := x_l, \quad l = 1, 2, 3.
$$

Then derivatives of $m_{1l}$ takes the following form

$$
\partial_j m_{1l}(0, 0) = \left\langle \psi_l \partial_j p(0, 0), f_{0,0}^{(l)} \right\rangle + \left\langle \psi_l p(0, 0), \partial_j f_{0,0}^{(l)} \right\rangle.
$$

(4.7)

To compute terms on the right-hand side above, we first show that $A(0, 0)$ preserves the space spanned by $\psi_l(1, 0)^\top$ and $\psi_l(0, 1)^\top$, $l = 1, 2, 3$, and $A(0, 0)^{-1}$ on that space is given by

$$
A(0, 0)^{-1} \begin{bmatrix} a\psi_l \\ b\psi_l \end{bmatrix} = \gamma_1 \begin{bmatrix} (-\mu + \frac{1}{6}) & \frac{1}{2} \rho^2 \\ -\frac{2}{3} \rho & \rho^3 (\mu + \frac{1}{6}) \end{bmatrix} \begin{bmatrix} a\psi_l \\ b\psi_l \end{bmatrix},
$$

(4.8)

where $\mu$ is the number defined in (2.6) and

$$
\gamma_1 = \frac{1}{\rho^3 (1/2 + \mu)(1/2 - \mu)}.
$$

(4.9)

To do so, we need to compute $A(0)[\psi_l]$, $B(0)[\psi_l]$, $C(0, 0)[\psi_l]$ and $D(0, 0)[\psi_l]$. We see from (2.19), (2.21) and (3.6) that

$$
A_0(x, y) = B_0(x, y) = \frac{1}{4\pi} \frac{(x - y, x)}{|x - y|^3} = \frac{1}{8\pi} \frac{1}{|x - y|^2}, \quad x, y \in S^2.
$$

Thus it follows from the second identity in (3.2) that

$$
A(0)[\psi_l] = B(0)[\psi_l] = \frac{1}{6} \psi_l.
$$

(4.10)

We see from (2.23) and (2.25) that

$$
C_{0,0}(x, y) = \frac{1}{4\pi} \frac{\langle r_i, x - r_i, y \rangle}{|r_i, x - r_i, y|^3} r_i^2 \quad \text{and} \quad D_{0,0}(x, y) = \frac{1}{4\pi} \frac{\langle r_3, x - r_3, y \rangle}{|r_3, x - r_3, y|^3} r_3^2.
$$

Thus, (3.38) and (3.39) yield

$$
C(0, 0)[\psi_l] = -\frac{\rho^2}{3} \psi_l \quad \text{and} \quad D(0, 0)[\psi_l] = \frac{2\rho}{3} \psi_l.
$$

(4.11)

Thus,

$$
A(0, 0) \begin{bmatrix} a\psi_l \\ b\psi_l \end{bmatrix} = \begin{bmatrix} -\lambda I + A(0) & C(0, 0) \\ -\mu I + B(0) \end{bmatrix} \begin{bmatrix} a\psi_l \\ b\psi_l \end{bmatrix} = \begin{bmatrix} -\lambda + 1/6 & -\rho^2/3 \\ 2\rho/3 & -\mu + 1/6 \end{bmatrix} \begin{bmatrix} a\psi_l \\ b\psi_l \end{bmatrix}.
$$

The desired formula (4.8) now follows thanks to the relation $\lambda = \frac{1}{6} - \rho^3 (\mu + \frac{1}{6})$, which comes from (1.4) (the neutrality condition) and (2.6) (definitions of $\lambda$ and $\mu$).
We now compute the first term on the right-hand side of (4.7). Since \( g_{0,0}^{(l)} = -\gamma l (r_e^2, r_e^2) \), we have
\[
f_{0,0}^{(l)} = A(0,0)^{-1} g_{0,0}^{(l)} = \psi_1 V_1, \tag{4.12}
\]
where the constant vector \( V_1 \) is defined by
\[
V_1 := \gamma_2 r_e^2 \begin{bmatrix} -1 \\ \rho \end{bmatrix} \quad \text{with} \quad \gamma_2 = \frac{1}{\rho(1/2 + \mu)} = \rho^2(1/2 - u) \gamma_1. \tag{4.13}
\]
By (4.2), \( \partial_j p(0, 0) = (0, Y_j)^T \), and hence \( \partial_j p(0, 0) \cdot V_1 = \gamma_2 r_e^2 \rho Y_j \). Therefore,
\[
\langle \psi_1 \partial_j p(0, 0), f_{0,0}^{(l)} \rangle = \gamma_2 r_e^2 \rho C_{l,l'}^{j}, \tag{4.14}
\]
where \( C_{l,l'}^{j} \) is defined and computed in (3.21) and (3.22).

To compute the second term on the right-hand side of (4.7), namely, \( \langle \psi_1 p(0, 0), \partial_j f_{0,0}^{(l)} \rangle \), we first observe from (4.6) that
\[
\partial_j f_{0,0}^{(l)} = A(0,0)^{-1} \left[ \partial_j g_{0,0}^{(l)} - \partial_j A(0,0) [f_{0,0}^{(l)}] \right].
\]
Thus we have
\[
\langle \psi_1 p(0, 0), \partial_j f_{0,0}^{(l)} \rangle = \langle (A(0,0)^{-1}^* [\psi_1 p(0, 0)], \partial_j g_{0,0}^{(l)} - \partial_j A(0,0) [f_{0,0}^{(l)}] \rangle,
\]
where \( (A(0,0)^{-1})^* \) is the adjoint operator of \( A(0,0)^{-1} \). In view of (4.8), we have
\[
(A(0,0)^{-1}^* [\psi_1 p(0, 0)] = \psi_1 \gamma_1 r_e \rho (1/2 + \mu) \begin{bmatrix} -1 \\ \rho^2 \end{bmatrix} =: \psi_1 V_2, \tag{4.15}
\]
and hence
\[
\langle \psi_1 p(0, 0), \partial_j f_{0,0}^{(l)} \rangle = \langle \psi_1 V_2, \partial_j g_{0,0}^{(l)} - \partial_j A(0,0) [f_{0,0}^{(l)}] \rangle. \tag{4.16}
\]

For ease of notation, let \( V_3 := (0,1)^T \). Then, one can see from (4.5) that \( \partial_j g_{0,0}^{(l)} \) are given by the following:
\[
\partial_j g_{0,0}^{(l)} = \begin{cases} -2 r_e \psi_1 Y_j V_3, & j = 1, \\ - \left( 4 r_e \psi_1 Y_j - r_e \frac{\partial \gamma}{\partial \sigma} \right) V_3, & j \neq 1. \end{cases} \tag{4.17}
\]
Note \( V_2 \cdot V_3 = \gamma_1 r_e \rho^3 (1/2 + \mu) \). Now straightforward but tedious computations yield for \( 1 \leq l \leq l' \leq 3 \)
\[
\langle \psi_1 V_2, \partial_j g_{0,0}^{(l)} \rangle = \frac{4 \pi}{15} \gamma_1 r_e^2 \rho^3 (1/2 + \mu) \times \begin{cases} -\frac{2 \sqrt{15}}{3} & \text{if } (l, l', j) = (1, 1, 1), (2, 2, 1), (3, 3, 1), \\ -\frac{1}{\sqrt{3}} & \text{if } (1, 1, 4), (2, 2, 4), \\ 1 & \text{if } (1, 1, 6), (1, 2, 2), (1, 3, 5), (2, 3, 3), \\ -1 & \text{if } (2, 2, 6), \\ \frac{2}{\sqrt{3}} & \text{if } (3, 3, 4), \\ 0 & \text{otherwise}. \end{cases} \tag{4.18}
\]
It now remains to calculate \( \langle \psi_{V_2} \partial_j A(0,0)[f_{0,0}^{(l)}] \rangle \). By (3.30) and (4.12), we have

\[
\begin{align*}
\langle \psi_{V_2} \partial_j A(0,0)[f_{0,0}^{(l)}] \rangle &= \gamma r_{\rho} r(1/2 + \mu) \gamma_2 r_e^2 \int_{S^2} \psi_{V} \left[ \frac{-1}{\rho^2} \right] \cdot \left[ -\partial_j C(0,0)[\psi_l] + \rho \partial_j B(0)[\psi_l] \right] dS \\
&= \gamma r_{\rho} r(1/2 + \mu) \gamma_2 \int_{S^2} \psi_{V} \left[ -\partial_j C(0,0)[\psi_l] - \rho \partial_j D(0,0)[\psi_l] + \rho^2 \partial_j B(0)[\psi_l] \right] dS.
\end{align*}
\]

It then follows from (3.33) and (3.42) that for \( 1 \leq l \leq l' \leq 3 \):

\[
\begin{align*}
\langle \psi_{V_2} \partial_j A(0,0)[f_{0,0}^{(l)}] \rangle &= \frac{16\pi}{45} \rho^3 r_e^2 r_1 \gamma_1 \times \begin{cases}
\frac{1}{\sqrt{3}} & \text{if } (l, l', j) = (1, 1, 4), (2, 2, 4), \\
-1 & \text{if } (1, 1, 6), (1, 2, 2), (1, 3, 5), (2, 3, 3), \\
1 & \text{if } (2, 2, 6), \\
-\frac{2}{3} & \text{if } (3, 3, 4), \\
0 & \text{otherwise.}
\end{cases}
\end{align*}
\]

We then have from (4.7), (4.14), (4.16), (4.18) and (4.19) that for \( 1 \leq l \leq l' \leq 3 \):

\[
\partial_j m_{V}(0,0) = \pi \rho^3 r_e^2 r_1 \gamma_1 \times \begin{cases}
-\frac{4}{3\sqrt{15}}(\frac{1}{2} + 3\mu) & \text{if } (l, l', j) = (1, 1, 1), (2, 2, 1), (3, 3, 1), \\
-\frac{28}{45} & \text{if } (1, 1, 4), (2, 2, 4), \\
-\frac{28}{45} & \text{if } (1, 1, 6), (1, 2, 2), (1, 3, 5), (2, 3, 3), \\
-\frac{56}{45} & \text{if } (2, 2, 6), \\
0 & \text{if } (3, 3, 4), \\
0 & \text{otherwise.}
\end{cases}
\]

Thus,

\[
\begin{align*}
\frac{\partial(m_{11}, m_{12}, m_{13}, m_{22}, m_{23}, m_{33})}{\partial(b_1, b_2, b_3, b_4, b_5, b_6)}(0,0) &= \left( \frac{4}{3\sqrt{15}} \left( \frac{1}{2} + 3\mu \right) \frac{28}{45} \right) \left( \frac{28}{45} \right)^4 \frac{28}{45\sqrt{3}} \neq 0.
\end{align*}
\]

Thus, (1.13) is proved.

**Remark 4.2.** By switching roles of \( h \) and \( b \), let \( M(b, h) \) be the polarization tensor associated with domain \( (\Omega_h, D_h) \). Similar computations yield

\[
\frac{\partial(m_{11}, m_{12}, m_{13}, m_{22}, m_{23}, m_{33})}{\partial(b_1, b_2, b_3, b_4, b_5, b_6)}(0,0) = \left( \frac{\rho r_e^2 \gamma_1 \pi}{\sqrt{15}} \right)^6 \frac{4}{3\sqrt{15}} \left( \frac{1}{2} + 3\mu \right) \left( \frac{28}{45} \right)^4 \frac{28}{45\sqrt{3}} \neq 0.
\]

Thus we have Theorem (1.7).
Conclusion

In this paper we consider the problem of the PT-vanishing inclusion (or the weakly neutral inclusion) of the core-shell structure: Given a domain of arbitrary shape find a domain enclosing the given domain so that the core-shell structure is PT-vanishing. We show that such a domain for shell exists if the given domain is a small perturbation of a ball. The result of this paper is a proof of existence. As far as we are aware of, there is no known method of constructing such domains for shells. Even shells for ellipses or ellipsoids are not known. Thus it is quite interesting to find a way to construct shells for the PT-vanishing structure. In this regard, we mention that there is a numerical attempt to construct the PT-vanishing structure using shape derivative [9].

References

[1] H. Ammari and H. Kang, Reconstruction of small inhomogeneities from boundary measurements, Lecture Notes in Math. 1846, Springer-Verlag, 2004.

[2] H. Ammari and H. Kang, Polarization and moment tensors, Applied Mathematical Sciences, 162, Springer, New York, 2007.

[3] H. Ammari and H. Kang, Expansion Methods, Handbook of Mathematical Mehtods of Imaging, 447–499, Springer, 2011.

[4] H. Ammari, H. Kang, E. Kim, and M. Lim, Reconstruction of Closely Spaced Small Inclusions, SIAM Journal on Numerical Analysis Vol 42, No. 6 (Mar, 2005), 2408-2428.

[5] K. Atkinson and W. Han, Spherical Harmonics and Approximations on the Unit Sphere: An Introduction, Lecture Notes in Math., 2044, Springer-Verlag, 2012.

[6] G. Bal and O. Pinaud, Small volume expansions for elliptic equations, Asymptotic Analysis 70 (2010), 13–50.

[7] Y. Capdeboscq and M.S. Vogelius, A general representation formula for boundary voltage perturbations caused by internal conductivity inhomogeneities of low volume fraction, Math. Model. Numer. Anal. 37(1) (2003), 159–174.

[8] D.J. Cedio-Fengya, S. Moskow and M.S. Vogelius, Identification of conductivity imperfections of small diameter by boundary measurements. Continuous dependence and computational reconstruction, Inverse Problems 14 (1998), 553–594.

[9] T. Feng, H. Kang and H. Lee, Construction of GPT-vanishing structures using shape derivative, J. Comp. Math. 35 (2017), 569–585.

[10] Z. Hashin, The elastic moduli of heterogeneous materials, J. Appl. Mech. 29 (1962), 143–150.

[11] Z. Hashin and S. Shtrikman, A variational approach to the theory of the effective magnetic permeability of multiphase materials, J. Appl. Phy. 33 (1962), 3125–3131.

[12] Y.-G. Ji, H. Kang, X. Li and S. Sakaguchi, Neutral inclusions, weakly neutral inclusions, and an over-determined problem for confocal ellipsoids, arXiv:2001.04610

[13] H. Kang and H. Lee, Coated inclusions of finite conductivity neutral to multiple fields in two dimensional conductivity or anti-plane elasticity, Euro. J. Appl. Math., 25 (3) (2014), 329–338.
[14] H. Kang, H. Lee and S. Sakaguchi, An over-determined boundary value problem arising from neutrally coated inclusions in three dimensions, Annali della Scuola Normale Superiore di Pisa, Classe di Scienze, Vol. XVI, issue 4 (2016), 1193–1208.

[15] H. Kang, X. Li, and S. Sakaguchi, Existence of coated inclusions of general shape weakly neutral to multiple fields in two dimensions, arXiv:1808.01096.

[16] S. G. Krantz and H. R. Parks, The implicit function theorem: History, Theory, and Applications, Springer Science+Business Media, New York, 2003.

[17] G. W. Milton, The theory of composites, Cambridge Monographs on Applied and Computational Mathematics, Cambridge University Press, 2002.

[18] G. W. Milton and S. K. Serkov, Neutral coated inclusions in conductivity and anti-plane elasticity, Proc. R. Soc. Lond. A 457 (2001), 1973–1997.

[19] E. M. Stein, Singular integrals and differentiability properties of functions, Princeton Univ. Press, Princeton, 1970.

[20] M.S. Vogelius and D. Volkov, Asymptotic formulas for perturbations in the electromagnetic fields due to the presence of inhomogeneities of small diameter, Math. Model. Numer. Anal. 34 (2000), 723748.