TRANSITION TORI NEAR AN ELLIPTIC-FIXED POINT

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Abstract. Let $F : (M, \omega) \rightarrow (M, \omega)$ be a smooth symplectic diffeomorphism with a fixed point $a$ and a heteroclinic orbit in the sense of been in the intersection of the central stable and the central unstable manifolds of the fixed point. It is studied the case when the tangent space of a point in the heteroclinic orbit is the direct sum of three subspaces. The first one is the characteristic bundle of the central stable manifold of $a$; the second one is the characteristic bundle of the central unstable manifold of $a$; and the third one is tangent to the intersection of the central stable and unstable manifolds.

In this situation, the homoclinic map $\Lambda$ is a smooth and symplectic diffeomorphism of open subsets of the central manifold of $a$.

Moreover, if an invariant circle intersects the domain of definition of $\Lambda$ and its image intersects other circle, there are orbits that wander from one circle to the other. This phenomenon is similar to the Arnold diffusion.

The Melnikov Method gives sufficient conditions for the existence of homoclinic maps, and non identity homoclinic maps in a perturbation of a Hamiltonian system.

1. introduction. Arnold diffusion is a chaotic phenomenon related to perturbations of integrable Hamiltonian systems of three or more degrees of freedom. The first example was given by V. I. Arnold in [1]. In it, he conjectured the generic character of this phenomenon in perturbations of quasiperiodic Hamiltonians and its presence in the three body problem. V. I. Arnold never defined the “Arnold diffusion”, and it is understood in many different ways. This work is related with the line followed by the papers [7, 8, 17, 10].

The Arnold’s example contains a normally hyperbolic manifold $K$, foliated by a family of invariant tori, such that their action varies diffeomorphically; and the stable manifold of an invariant torus intersects transversally the unstable manifold of the nearby tori. Two corollaries of this phenomenon are (1) the existence of orbits that wander between different tori and (2) the non integrability of the Hamiltonian system.

The Arnold’s construction starts with a Hamiltonian of three degrees of freedom and a two-parameter perturbation, let $\epsilon$ and $\mu$ be the parameters. The analysis of V. I. Arnold goes through the cases: $\epsilon = 0$ and $\mu = 0$, $\epsilon > 0$ and $\mu = 0$, and finally gets the Arnold Diffusion when $\epsilon > 0$ and $\mu > 0$, each case is considered a perturbation of the previous one and the first two are straightforward.

When $\epsilon = 0$ and $\mu = 0$ the Hamiltonian is quasiperiodic. Thus the Hamiltonian, in action-angle coordinates, has the form: $H(I, \theta) = H(I)$ and the derivative $DH(I)$ is invertible for all $I$ in the domain.

When $\epsilon > 0$ and $\mu = 0$, a normally hyperbolic manifold $K$ of dimension four and its homoclinic manifold of dimension five are formed. They are foliated by a family

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of invariant tori of dimension two and their homoclinic manifolds of dimension three respectively, they are called the whiskered tori and their whiskers. The motion inside $K$ does not have expansion or contraction directions. In fact, when $\epsilon > 0$ and $\mu = 0$, the Hamiltonian system has two independent subsystems; the first one is a Hamiltonian of one degree of freedom that has a hyperbolic fixed point and a homoclinic orbit, the second one is a quasiperiodic Hamiltonian of two degrees of freedom. The system has 2 independent integrals in involution; and each whiskered torus is the product of the hyperbolic fixed point with a torus, and each whisker is the product of the homoclinic orbit with a torus.

When $\epsilon > 0$ and $\mu > 0$, the perturbation does not affect $K$, it continues to be a hyperbolic manifold, but the whole system is coupled and not integrable. In fact, the whiskers break into stable and unstable manifolds, and they intersect transversally each other. To accomplish this step, V. I. Arnold used a technique of H. Poincaré and V. K. Melnikov [9] called the Melnikov method; it consists of the implicit function theorem applied to the variation with respect to $\mu$ of the two independent integrals along the homoclinic manifold to break the whiskers, with a way to compute the Jacobian. Since $\mu$ is used to change the homoclinic manifolds, a structure defined by $\epsilon$ it is natural for $\mu$ to be smaller than $\epsilon$, in fact in the Arnold example $\mu$ must be exponentially smaller than $\epsilon$.

Therefore, when $0 < \epsilon$ and $0 < \mu << \epsilon$, there is a sequence of tori with irrational rotation number contained in $K$, where the position of the first and the last torus are arbitrary, and the unstable manifold manifold of each torus intersects the stable manifold of the next one. This phenomenon is the Arnold diffusion.

Using the $\lambda$–lemma it is possible to show that all the stable manifolds and all the unstable manifolds of all the tori accumulate each other forming a complex and not very well understood tangle, and we can find nearby orbits that wander arbitrarily among the tori, this fact implies the nonexistence of integrals for the perturbed Hamiltonian.

P. J. Holmes and J. E. Marsden [7, 8] extended the Arnold example in several ways: first, the degrees of freedom can be bigger than three, and $\mathbb{R}^n$ can be replaced by a manifold. Second and more important, the perturbation can act in the normally hyperbolic manifold $K$, and some of the whiskered tori can disappear creating gaps inside $K$. Using KAM theory, it should be possible to maintain control of the size of the gaps. They used Poincaré maps (time-one maps) to clarify the meaning of the whiskerer tori and their whiskers. They also gave a systematic analysis of the Melnikov Method, study continued by C. Robinson [11] and others.

In [17], Z. Xia proved the existence of the Arnold diffusion in the three-body problem in the sense that it is presented before. He started with a system similar to the case $\epsilon > 0$ and $\mu = 0$ in the Arnold’s example: an integrable Hamiltonian system with two independent subsystems; one has a (degenerate) hyperbolic fixed point with a homoclinic orbit, the other one is quasiperiodic in $T^2 \times \mathbb{R}^2$; its homoclinic manifold is the Cartesian product of the homoclinic orbit with $T^2 \times \mathbb{R}^2$. As before, $K$ and its homoclinic manifold are foliated by a family of tori and their homoclinic manifolds. In a first perturbation of this Hamiltonian, Z. Xia preserved the independence of the two subsystems and made a transversal breaking of the homoclinic manifold of the first subsystem. It is well known that a Smale’s horseshoe arises; and with it, a non numerable family of hyperbolic periodic points. In the second subsystem, the KAM-theory implies the persistence of most of the invariant tori of $K$. A complicated structure is formed, the manifold $K$ persists, but the homoclinic manifold is broken (instead of the whiskers as in [1, 7, 8]). We also have a family of normally hyperbolic manifolds, formed by the Cartesian product
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of the hyperbolic periodic points with $\mathbb{T}^2$. $K$ and this family are foliated by the
invariant circles formed by the Cartesian product of the specific points in $\mathbb{R}^2$ in the
first component and the invariant circles of the second component. Now he defines
the map that assigns $p$ to $q$, where $p$ and $q$ are two points in $K$ such that the
unstable manifold of $p$ intersects the stable manifold of $q$; this map is well defined
in a small neighborhood, contains some information related to the transversal in-
tersection of the stable and unstable manifolds and reduces the dimension of the
problem. We are going to call it the homoclinic map. At this step, the homoclinic
map is the identity. Making a further perturbation, Z. Xia obtained a non identity
homoclinic map, to do this he used KAM-theory and the previous step to show the
transversal intersection of many invariant and persistent tori with their homoclinic
image, obtaining as a corollary the Arnold diffusion.

The role of the symplectic geometry is fundamental; the Poincar’e map of the
flow and the homoclinic map are symplectic diffeomorphisms, and the invariant
manifolds have symplectic properties; $K$ and the total manifold are symplectic, the
whiskers are Lagrangian and the stable and unstable manifolds of $K$ are contact
manifolds, so the intersection of the whiskers can not be arbitrary.

This paper is intended to study homoclinic maps, and their relation with the
symplectic geometry and Melnikov functions. To avoid technical difficulties we
restrict ourselves to symplectic diffeomorphisms with an elliptic-hyperbolic fixed
point $a$ and a heteroclinic orbit in the sense of been in the intersection of the central
stable manifold of $a$ and the central unstable manifold of $a$, but the geometric
techniques can be used in more general settings. The proofs are postponed to
section (4), with the exception of theorems 3.5, 3.6, 3.8 and 3.10 that can be
proved using the standard techniques (See [12]).

2. elliptic hyperbolic fixed points.

2.1. Definitions and local structure. Let $(M, \Omega)$ be a symplectic manifold of
dimension $2n + 2 > 2$. All the submanifolds of $M$ have the form induced by the
inclusion. Let $F : (M, \Omega) \to (M, \Omega)$ be a symplectic diffeomorphism with a fixed
point $a$. The derivative $DF(a)$ induce a decomposition of the tangent space in three
subspaces: $T_a M = E^s \oplus E^c \oplus E^u$, which correspond to the eigenvalues of
$DP(a)$ of norm greater than one, equal to one and less than one. The symplectic
structure of $M$ implies $\dim E^s = \dim E^c$, and $\dim E^u$ is even, we will assume that
the last one is 2. The theory of normally hyperbolic manifolds (see [6, 13, 15] or
the series [2, 3, 4]), implies the existence of $0 < \mu < 1 < \lambda$, and a neighborhood of
$a$: $V$ such that the following sets are invariant smooth manifolds, with the given
dimensions at $a$.

$$W^s(a, V) = \{ q \in V/ d(F^j(q), a) < \mu^j d(q, a), j \geq 0 \} \quad [n]$$
$$W^u(a, V) = \{ q \in V/ d(F^{-j}(q), a) < \lambda^{-j} d(q, a), j \geq 0 \} \quad [n]$$
$$W^{cs}(a, V) = \{ q \in V/ d(F^j(q), a) \mu^j \to 0 \text{ as } j \to \infty \} \quad [n + 2]$$
$$W^{cu}(a, V) = \{ q \in V/ d(F^{-j}(q), a) \lambda^{-j} \to 0 \text{ as } j \to \infty \} \quad [n + 2]$$

They are called the local stable, unstable, central stable and the central unstable
manifolds of $a$. Define the central manifold of $a$ as

$$W^c(a) = W^{cs}(a, V) \cap W^{cu}(a, V), \quad (2.1)$$

the tangent space of $W^k(a, V)$ at $a$ is $\mathbb{R}^k$, for $k = s, u, cs, cu$.

For $x \in W^c(a)$ define

$$W^s(x, V) = \{ q \in V/ d(F^j(q), F^j(x)) < \mu^j d(q, x), j \geq 0 \}$$

$$W^u(x, V) = \{ q \in V/ d(F^{-j}(q), F^{-j}(x)) < \lambda^{-j} d(q, x), j \geq 0 \}$$

However, the dimension of $W^c(a, V)$ is $2n$ and $\mu > 1$, so, according to
the definition of $W^c$ and the invariant properties of $W^k$, we deduce the
following results:

1. $W^s(a, V)$ is $\mathbb{R}^n$.
2. $W^u(a, V)$ is $\mathbb{R}^n$.
3. $W^{cs}(a, V)$ is $\mathbb{R}^{n+1}$.
4. $W^{cu}(a, V)$ is $\mathbb{R}^{n+1}$.

$$W^c(a, V) = \{ x \in V/ d(F^j(x), F^j(a)) < \mu^j d(x, a), j \geq 0 \} \quad [n]$$

$$W^c(a, V) = \{ x \in V/ d(F^{-j}(x), F^{-j}(a)) < \lambda^{-j} d(x, a), j \geq 0 \} \quad [n]$$

$$W^c(a, V) = \{ x = F^j(a) \} \quad [n]$$

$$W^c(a, V) = \{ x = F^{-j}(a) \} \quad [n]$$

Therefore, the central manifold of $a$ is a submanifold of $V$ with the dimension $2n$.
These sets are submanifolds of $W^{cs}(a, V)$, they define a smooth foliation as follows

$$W^{cs}(a, V) = \bigcup_{x \in W^s(a)} W^s(x, V).$$

(2.2)

And the following function is a smooth submersion:

$$\pi^s : W^{cs}(a, V) \rightarrow W^s(a)$$

$$q \in W^s(x, V) \mapsto x.$$

(2.3)

Therefore, given any invariant submanifold $C$ of $W^c(a)$, $(\pi^s)^{-1}(C)$ is an invariant submanifold of $W^{cs}(a, V)$ of dimension $\dim C + n$; it is called the stable manifold of $C$; we only consider the case when $C$ is an invariant circle with irrational rotation number.

**Definition 2.1.** Let $F : (M, \Omega) \rightarrow (M, \Omega)$ be a symplectic diffeomorphism with a fixed point $a$. $a \in M$ is elliptic-hyperbolic fixed point if $\dim W^s(a) = 2$ and $F|_{W^s(a)}$ is a twist map around $a$.

**Remark 2.2.** The first condition implies the existence of only two eigenvalues of $DF(a)$ of modulus $1$, write them as $\lambda = e^{\pm \omega_i}$. The theorem (2.3) prove that $W^c(a)$ is a symplectic manifold. The second condition and theorem (2.3) imply that $\omega \notin \frac{1}{2}Z + \frac{3}{4}Z$ and (in action-angle coordinates):

$$F|_{W^c(a)}(I, \theta) = (I + c(I, \theta), \theta + \omega + \beta I + d(I, \theta))$$

(2.4)

where $c(I, \theta)$ and $d(I, \theta) = O(I^{1/2})$ and $\beta \neq 0$. By KAM theorem most of the points in $W^c(a)$ near $a$ are in invariant circles with irrational rotation number.

The local symplectic properties of the invariant manifolds are given by the following theorem, which proof is given in section (4).

**Theorem 2.3.** Let $F : (M, \Omega) \rightarrow (M, \Omega)$ be a symplectic diffeomorphism and let $a \in M$ be a elliptic-hyperbolic fixed point. Then:

1. $W^c(a)$ is a symplectic manifold of dimension two, and $a$ is an elliptic fixed point of $F|_{W^c(a)}$. Most of the points in small neighborhoods of $a$ are in invariant circles with irrational rotation number.

2. For all $x \in W^c(a)$, $W^s(x)$ is an isotropic manifold of dimension $n$.

3. $W^{cs}(a)$ is a manifold of dimension $n + 2$, and

$$W^{cs}(a) = \bigcup_{x \in W^s(a)} W^s(x).$$

(2.5)

Its characteristic bundle at $q \in W^s(x)$ is $T_qW^s(x)$.

4. If $C \subset W^c(a)$ is an invariant curve thus the stable manifold of $C$ is a Lagrangian manifold.

The same follows for $W^{cu}(a)$, $W^u(C)$ and $W^u(x)$ with obvious modifications.

2.2. Homoclinic intersections. Our main interest is the study of the intersections of the stable and unstable manifolds outside $W^c(a)$; let $x$ and $y$ be points, $C$ be an invariant circle contained in $W^c(a)$ and $p$ be a point in $W^u(x) \cap W^u(y)$ not in $W^c(a)$. The most natural conditions to study are the following transversal intersections:

1. $W^s(y) \pitchfork W^{cu}(a)$.

2. $W^u(x) \pitchfork W^{cs}(a)$.

3. $W^s(C) \pitchfork W^u(C)$.

4. $W^{cs}(a) \pitchfork W^{cu}(a)$.
Let $WZ. Xia used this map to prove the existence of Arnold diffusion in
The main questions in the definition of the homoclinic maps are related
The map $WZ \in \pi_{\Psi} \in \omega_{\Psi}(a)$; and let $x$ and $y \in W^s(a)$ such that $p \in W^u(x)$ and $p \in W^s(y)$. Then the following
four properties are equivalent:

a. $W^c(a) \cap W^c(a)$ is locally a symplectic manifold near $p$. [There exists an
open connected manifold of dimension 2, that we call $K$ such that $p \in K$,
$K \subset W^c(a) \cap W^c(a)$ and $K$ is symplectic].
b. $W^u(x) \cap W^c(a)$.
c. $W^s(y) \cap W^c(a)$.
d. $W^c(a)$ is transverse to $W^c(a)$ and

$$T_p W^s(y) \bigoplus T_p \left( W^c(a) \cap W^c(a) \right) \bigoplus T_p W^u(x) = T_p M.$$ 

And they imply:

e. The map $f^u : V^u \subset W^c(a) \mapsto K$, such that $z \mapsto q \in W^u(z) \cap K$ is a well
defined smooth symplectic diffeomorphism. $z$ goes to the only element in
the leaf $W^u(y)$ that is in $K$.
f. The map $f^s : K \mapsto V^s \subset W^c(a)$, such that $q \in W^s(z) \cap K \mapsto z$ is a well
defined smooth symplectic diffeomorphism.
g. $\psi = f^s \circ f^u : V^u \subset W^c(a) \mapsto W^c(a)$ is a smooth symplectic diffeomorphism,
$\psi(x) = y$.
h. If in addition, there exist $C_1$ and $C_2$ invariant circles contained in $W^c(a)$,
such that $x \in C_1$, $y \in C_2$ and $W^s(C_1) \cap W^u(C_2)$, then $\psi$ is not the identity
and $\psi(C_1) \cap W^u(C_2)$.

Remark 2.5. Note that in [e] and [f], the maps are in opposite directions. In fact
$f^u = [\pi_\psi][\kappa]^{-1}$ and $f^s = \pi^s[\kappa]$.

2.3. Homoclinic maps.

Definition 2.6. Let $F : (M, \omega) \mapsto (M, \omega)$ be a symplectic diffeomorphism and
$a \in M$ an elliptic-hyperbolic fixed point of $F$. The triple $(x, p, y)$ of points in
$M$ is a homoclinic triple if $x$ and $y$ are in $W^c(a)$ and $W^s(x) \cap p W^c(a)$ and
$W^s(y) \cap p W^c(a)$. The map $\Psi$ of the theorem (2.4) is called the homoclinic map
for this triple.

Remark 2.7. Z. Xia used this map to prove the existence of Arnold diffusion in
the three body problem, his main difficulty was to prove that this map is not the
identity, see [16, 17]. Condition [h] is easier to use.

Remark 2.8. If $(x, p, y)$ is a homoclinic triple then $(F(x), F(p), F(y))$, and in general,
$(F^n(x), F^n(p), F^n(y))$ are homoclinic triples. In addition, if $\overline{\Psi}$ is the homoclinic map for one of these triples then $\overline{\Psi} \circ F = F \circ \Psi$. Moreover the domain of $\overline{\Psi}$ is $F^n(V^u)$, and its image is $F^n(V^s)$.

Remark 2.9. The main questions in the definition of the homoclinic maps are related
to the interaction of the homoclinic maps and $F$; and the geometry of the homoclinic
maps. In particular, to determine the domain and the image of the homoclinic maps
and when the homoclinic map is not the identity.
Let $H : M \mapsto M$ be a symplectic diffeomorphism and $a$ be a elliptic-hyperbolic fixed point, and suppose there are invariant circles $C_1$ and $C_2 \subset W^c(a)$ of irrational rotation number, for which $\Psi(C_1 \cap V^u) \cap C_2 \neq \emptyset$. Then for any $x_1 \in C_1$, $x_2 \in C_2$, $N \in \mathbb{N}$ and $\epsilon > 0$ there exists a point $q \in M$ and there exists $n > N$ such that $d(q, x_1) < \epsilon$ and $d(F^n(q), x_2) < \epsilon$.

**Corollary 2.11.** Let $F : M \mapsto M$ be a symplectic diffeomorphism and $a$ be a elliptic-hyperbolic fixed point, with a family of invariant circles $\{ \ldots, C_1, C_2, \ldots \} \subset W^c(a)$ of irrational rotation number, and with a family of homoclinic maps $\Psi_i : V^s_i \mapsto V^s_i$ for which $\Psi(C_i \cap V^u) \cap C_{i+1} \neq \emptyset$. Then all the points in $\cup_i C_i$ are in the same chain component.

**Definition 2.12.** Assume any of the conditions [a]-[d] of the theorem (2.4) and $C \subset W^c(a)$ be an invariant circle with irrational rotation number such that $C \cap V^u \neq \emptyset$, $C$ has the transition circle condition if:

a. There exists $x \in C \cap \Psi(C \cap V^u)$, and $T_xC \neq T_x\Psi(C \cap V^u)$.

b. There is a sequence of invariant circles $C_n$ with irrational rotation number such that the distance from $C$ to $C_n$ goes to zero.

**Remark 2.13.** If $C$ has a transition circle condition then $\Psi(C_n \cap V^u) \cap C_m \neq \emptyset$ for infinite number of $n$ and $m$ in $\mathbb{N}$. So by Theorem (2.10), there are orbits that starts close to $C_n$ go away from $W^c(a)$ and come back close to $C_m$.

**Remark 2.14.** By KAM-theory (see [14]), most of the points near $a$ are in invariant circles with irrational rotation number, therefore the condition b) is satisfied for most of those points.

### 3. Hamiltonian Systems with Elliptic-Hyperbolic Fixed Points.

#### 3.1. Definitions.

Consider the Hamiltonian:

$$H(p, t) = F(p^1) + G(p^2) + \mu K(p, t),$$

where $p = (p^1, p^2) \in \mathbb{R}^4$, $F, G : \mathbb{R}^2 \mapsto \mathbb{R}$, and $K : \mathbb{R}^2 \mapsto \mathbb{R}$ are smooth functions, and $\mu \geq 0$ is a small perturbation parameter.

The associated time dependent Hamiltonian differential equation is:

$$\dot{p}^1 = J\nabla F(p^1) + \mu J\nabla^1 K(p, t),$$  \hspace{1cm} (3.2a)

$$\dot{p}^2 = J\nabla G(p^2) + \mu J\nabla^2 K(p, t).$$  \hspace{1cm} (3.2b)

Where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\nabla^j K(p, t)$ is the transpose of the $j$-partial derivative of $K$. If $\phi(p, t)$ is the solution of (3.2) that satisfies $\phi(p, 0) = p$, then the associated Poincare map is the map $\Lambda : \mathbb{R}^4 \mapsto \mathbb{R}^4$ defined by $\Lambda(p) = \phi(p, 1)$. $\Lambda$ is a symplectic diffeomorphism with respect to the form $\Omega = \frac{1}{2} (dz^1 \wedge dz^3 + dz^2 \wedge dz^4)$. The map $\phi(p, t)$ will be studied using $\Lambda$.

We are going to use an upper index 1 or 2 to denote the first or the second component in the flow of (3.2), the Poincare map and so on. We are also going to use a subindex $\mu$ or any specific value to emphasize the dependence on $\mu$. $0^2 = (0, 0)$ is the null vector in $\mathbb{R}^2$, $0 = 0^1 = (0^2, 0)$ is the null vector in $\mathbb{R}^4$.

When $\mu = 0$, the system (3.2) is called unperturbed and is decoupled and integrable, $F$ and $G$ are integrals.

**Hypothesis 3.1.** Conditions on $F$:

$$F(p^1) = F(x, y) = -\left(\omega \left(\frac{x^2 + y^2}{2}\right) + \beta \left(\frac{x^2 + y^2}{2}\right)^2 + \text{h. o. t.}\right),$$  \hspace{1cm} (3.3)
where \( \beta \neq 0 \), or in action-angle coordinates \([x = \sqrt{2I} \cos \theta, y = \sqrt{2I} \sin \theta]\):

\[
F(I, \theta) = -\omega I - \frac{\beta}{2} r^2 + \text{h. o. t.},
\]

where \( 0 \leq I \) and \( \theta \in S^1 \). So the motion on \( p^1 \) is an integrable twist map, and \( 0 \) is an elliptic fixed point for \( \Lambda^1 \). \( C_I = \{I\} \times S^1 \) is an invariant circle with rotation number: \( \omega + \beta I \).

**Hypothesis 3.2.** Conditions on \( G \): (3.2b) has a hyperbolic fixed point with a homoclinic orbit. Examples are Duffing or pendulum equations.

**Hypothesis 3.3.** Conditions on \( K \): \( K \) is 1-periodic on \( t \), \( \nabla K(0, t) = 0 \). So \( 0 \) is always a fixed point of \( \Lambda \). \( D_1 D_2 K(p, t) \neq 0 \) except in a set of zero measure.

We are going to give conditions on \( K \), the coupling, to ensure the existence of homoclinic maps for the perturbed Hamiltonians.

### 3.2. Analysis of the flow in the unperturbed Hamiltonian

When \( \mu = 0 \), (3.1) implies that \( \Lambda^1(0^2) \) is an elliptic fixed point: So \( D \Lambda^1(0^2) \) has as eigenvalue \( \lambda = \epsilon \omega \), where \( |\lambda| = 1 \) and \( \lambda^2, \lambda^3, \lambda^4 \neq 1 \); therefore it has the following normal form:

\[
\Lambda^1(I, \theta) = (I, \theta + \omega + \beta I + \text{h. o. t.}),
\]

we assume that \( \beta > 0 \).

Since (3.2b) has a hyperbolic fixed point \( 0^2 \) with an homoclinic orbit \( \Gamma \) the same is true for \( \Lambda^2 \).

An elliptic-hyperbolic fixed points for \( \Lambda \) is \( 0^4 \) but its invariant manifolds (stable, central, central stable, unstable, central unstable) are very degenerate, in fact they are products of sets, for example:

\[
W^s(0) = W^u(0) = 0^2 \times \Gamma,
\]

\[
W^c(0) = \mathbb{R}^2 \times 0^2,
\]

\[
W^{cs}(0) = W^{cu}(0) = \mathbb{R}^2 \times \Gamma.
\]

and if \((p^1, 0^2) \in W^c(0)\) then

\[
W^s(p^1, 0^2) = W^u(p^1, 0^2) = p^1 \times \Gamma.
\]

Moreover these sets define a \( C^k \)-foliation of \( W^{cu}(0) \). So

\[
W^{cu}(0) = \bigcup_{p \in W^s(0)} W^u(p)
\]

and the map

\[
\Pi^u: W^{cu}(0) \mapsto W^c(0)
\]

\((p^1, p^2) \mapsto (p^1, 0^2)\)

is a submersion.

If \( C \) is an invariant subset of \( W^c(0) \) then:

\[
W^u(C) = W^s(C) = C \times \Gamma \quad \text{Note the identification } C = C \times 0^2
\]

These manifolds are Lagrangian and invariant. For any point \( p = (p_1, p_2) \) in \( W^s(C) \), the manifolds \( W^s(p_1, 0) \) and \( C \times \{p_2\} \) have dimension 1, and they are transversal in \( W^s(C) \); clearly it has dimension 2 and is contained in \( W^{cs}(0) \), the last set has dimension 3.

To study the tangent spaces of the different manifolds we define the vectors \( u = (J \nabla F(p), 0), v = (0, J \nabla G(p)) = (0, \Gamma'(0)), w = (\nabla F(p), 0), \) and \( y = (0, \nabla G(p)) \), where \( \Gamma(0) = p_2 \). They are related to the tangent spaces as follows:
The following functions are called Melnikov functions:

\[ T_p W^s(p_1, 0) = \langle v \rangle \quad T_p(C \times \{p_2\}) = \langle u \rangle \]
\[ < v > \oplus < u > = T_p W^s(C) \subset T_p W^{cs}(0) = < v > \oplus < u > \oplus < w > \]

Note that \( y \) and \( w \) are not parallel to \( W^s(C) \), but only \( y \) is not parallel to \( W^{cs}(0) \); therefore, it is possible to measure the movement of \( W^{cs}(0) \) in the level sets of \( G \), and the movement of \( W^s(C) \) on the level sets of \( F \) and \( G \).

By normally hyperbolic manifold theory and symplectic geometry, the elliptic-hyperbolic fixed point is preserved after a small perturbation along with the invariant manifolds, inclusions and foliations associated with it. In fact, \( W^s(0) \) continues to exits and so by Theorem (2.3), it is symplectic. By the Darboux’s theorem, there exists a family of symplectic maps \( \Lambda_\mu : W^s_c(0) \mapsto W^s_c(0) \) into a family: \( \Lambda_\mu : \mathbb{R}^2 \mapsto \mathbb{R}^2 \); where \( 0 \) is an elliptic fixed point and, using action-coordinates, \( \Lambda \) has the form:

\[
\Lambda_\mu : (0, b) \times S^1 \mapsto (0, \infty) \times S^1
\]

\[
(I, \theta) = (I + \mu \epsilon(I, \theta, \mu), \theta + \omega + \beta I + \mu d(I, \theta, \mu)) \tag{3.12}
\]

Where \( \epsilon(I, \theta, \mu) \) and \( d(I, \theta, \mu) = O(I^{3/2}) \) are smooth and, if \( I = 0 \) or \( \mu = 0 \) we have \( \epsilon(I, \theta, \mu) = 0 \). So the circle \{0\} \times \mathbb{S}^1 is \( \Lambda_\mu \)-invariant, and the circles \{I\} \times \mathbb{S}^1 are \( \Lambda_0 \)-invariant.

The KAM theorem implies that the invariant circles in \( W^s(0) \) with Diophantine rotation number are continued in a smooth way to \( C^k \)-close circles.

All these perturbed sets and maps are no longer products of sets or projections in some component, but sets with a rather complicated structure, in general they intersect transversally, so we can apply the theorem (2.4). In the next two subsections we are going to study conditions where this is the case.

### 3.3. Melnikov Method

Let \( C \) be a circle in \( W^c(0) \) that persists under small perturbations. Using the Tubular Neighborhood Theorem for \( Y = W^s_0(C) \) we have a neighborhood \( K \) foliated by sets \( \Phi_p, p \in W^s_0(C) \), and \( \Phi_p \) is transversal to \( W^s_0(C) \) at \( z \in W^s_0(C) \). So \( \Phi_p \) is transversal to \( W^s(C) \) and \( W^u(C) \) if \( \mu \) is small. Since \( \text{codim} \Phi_p = \dim W^s(C) \) there are unique points: \( p^s(p, \mu) \in W^s(C) \cap \Phi_p \) and \( p^u(p, \mu) \in W^u(C) \cap \Phi_p \). These points depend smoothly on \( p \) and \( \mu \).

More over if \( C' \) is another invariant and persistent circle close to \( C \) then we can use the same \( K \) and its foliation to extend \( p^s \) and \( p^u \) to the nearby points in the stable manifold of invariant and persistent circles in a smooth way.

We want to measure the splitting of \( W^s(C) \) with \( W^u(C) \), using \( p^s \) and \( p^u \). The first problem is that the direct computation of these functions is almost impossible, but we can use the level sets of \( F \) and \( G \) to measure the change of these functions on different directions on \( \Phi_p \). In fact, 2 points in \( \Phi_p \) are equal if and only if they lie in the same level set of \( F \) and \( G \). \( F \) is a good measure of the variation in the elliptic part, that is inside of \( W^{cs}(0) \). And \( G \) is a good measure of the variation in the hyperbolic part that is outside of \( W^{cs}(0) \).

**Definition 3.4.** The following functions are called Melnikov functions:

\[
M_1(p) = \frac{\partial}{\partial \mu}(F(p^s(p, \mu)) - F(p^u(p, \mu)))_{\mu=0}
\]
\[
M_2(p) = \frac{\partial}{\partial \mu}(G(p^s(p, \mu)) - G(p^u(p, \mu)))_{\mu=0}
\]

Where \( p \in W^s_0(C) \), the vector \( M = (M_1, M_2) \) is called the Melnikov vector.
A natural question to ask is the possible extension of $M_1$ and $M_2$ to all $W^{cs}_\mu(0)$. We are going to discuss this issue later, now we are going to discuss their meaning.

Define the functions:

$$
M_F(p, \mu) = F(p^s(p, \mu)) - F(p^u(p, \mu))
$$

$$
M_G(p, \mu) = G(p^s(p, \mu)) - G(p^u(p, \mu))
$$

Thus $p^s(p, \mu) = p^u(p, \mu)$ if and only if $M_F(p, \mu) = M_G(p, \mu) = 0$. Using the Taylor expansion and $p^s(p, 0) = p^u(p, 0) = p$ we have:

$$
M_F(p, \mu) = \mu M_2(p) + h.o.t., \quad M_G(p, \mu) = \mu M_1(p) + h.o.t.
$$

Therefore $M_1$ and $M_2$ measure the first variation of the functions $M_F, M_G$ at $\mu = 0$. Although they are similar in their definition and hard to compute directly, there are important differences in their computation and role in the symplectic structure. The next theorem gives a easier way to compute $M_1$ and $M_2$.

**Theorem 3.5.**

$$
M_1(p) = \lim_{n \to \infty} \int_{A_n}^{B_n} \{F, K\}_{\phi(t, p, 0)} dt
$$

$$
M_2(p) = \int_{-\infty}^{\infty} \{G, K\}_{\phi(t, p, 0)} dt
$$

Where $\{G, K\} = (\nabla V)^T J (\nabla K)$ is the Poisson bracket of $G$ and $K$ and $\phi(t, p, \mu)$ is the solution of (3.2) that satisfies $z(0) = p$, and $A_n \to -\infty$, $\phi(A_n, p, 0) \to p_\infty$. $B_n \to \infty$ and $\phi(B_n, 0) \to p_\infty$ as $n \to \infty$. The second integral converges absolutely.

It is not true that the definition of $M_1$ and $M_2$ depends on the particular choice of $p^s$ and $p^u$; we can choose arbitrary functions that satisfy: $p^s(0) = p^u(0) = p$ and $p^s(\mu) \in W^s_\mu(C)$ and $p^u(\mu) \in W^u_\mu(C)$ to obtain the same Melnikov functions.

In the other hand, $M_2$ is measuring the speed of the breaking not only of $W^s_\mu(C)$ and $W^u_\mu(C)$, but also the speed of the breaking of $W^{cs}_\mu(0)$ and $W^{cu}_\mu(0)$. This is related to the fact that the integral in (3.17) is absolutely convergent. The function $M_1$ is measuring the speed of the breaking of $W^s_\mu(C)$ and $W^u_\mu(C)$ in the intersection of $W^{cs}_\mu(0)$ and $W^{cu}_\mu(0)$, this is why the integral in (3.16) is not absolutely convergent. In the next section we are going to extend the domain of definition of the Melnikov function associated with $G$ to the set $W^{cs}_0(0)$. This is possible due to the underlying hyperbolic structure. As a result we are going to obtain a sufficient condition to use Theorem (2.4). In particular we can define the homoclinic map in a neighborhood of the perturbation of $p_0$. The extension of $M_1$ to a larger domain depends on the dynamics of the flow inside $W^c(0)$.

The following theorem is the essence of the Arnold diffusion, it is proved by Arnold in [1] for a very particular case, and extended by P. J. Holmes and J. E. Marsden in [7]. It implies that the stable and the unstable manifolds of nearby persistent tori intersect each other forming a complicated tangle, very similar to our transition circle condition.

**Theorem 3.6.** *Let $0$ be a elliptic-hyperbolic fixed point for the Hamiltonian system (3.1) that satisfies the conditions (3.1), (3.2) and (3.3). Let $C$ be a circle in $W^s_0(0)$ with irrational rotation number that persists after small perturbations, and $p_0$ is a point in $W^s_\mu(C)$ and the restriction of $M$ to $W^s_\mu(0)$ has a non singular zero at $p_0$. Then $W^s_\mu(C)$ is transversal to $W^u_\mu(C)$ for small $\mu$.***
3.4. Homoclinic maps. Assume \( p = (x_0, \Gamma(0)) \) is in the intersection of \( W^c_{0}(0) \), and \( W^u_{0}(z) \), where \( z = (x_0, 0) \). Let \( p_\mu, z_\mu \) and \( q_\mu \) points in \( \mathbb{R}^3 \) that satisfy:
\[
\begin{align*}
p_\mu, z_\mu & \text{ and } q_\mu \text{ depend smoothly on } \mu, \\
z_\mu & \in W^c_\mu (0) \\
p_\mu & \in W^u_\mu (z_\mu) \\
q_\mu & \in W^cs_\mu (0) \\
q_0 & = p_0 = p \text{ and } z = z_0.
\end{align*}
\]

**Definition 3.7.** The *-Melnikov function is
\[
M^*_\mu(p) = \frac{\partial}{\partial \mu} (G(q_\mu) - G(p_\mu))_{\mu=0}
\]

\( M^*_\mu \) measures the speed of the splitting of \( W^u_\mu (z_\mu) \) and \( W^cs_\mu (0) \) in \( p \) in the transversal direction to the level sets of \( G \).

**Theorem 3.8.** \( M^*_\mu \) does not depend on the particular choice of \( p_\mu, z_\mu \) and \( q_\mu \), as long as they satisfy (3.18). Moreover
\[
M^*_\mu(p) = \int_{-\infty}^{\infty} \{G, K\}_\phi(t, p, 0) \ dt
\]

So if \( p \) is in a persistent circle then \( M^*_\mu(p) = M_2(p) \).

**Definition 3.9.** For all the points \( p \in W^s_0(0) \) where this integral converges absolutely.
\[
M^*_\mu(p) = \int_{-\infty}^{\infty} \{F, K\}_\phi(t, p, 0) \ dt
\]

The Melnikov vector is \( M^* = (M^*_1, M^*_2) \).

\( v \) is a tangent vector in \( W^s_\mu (z_0) \), therefore \( \partial_\mu M^*_\mu(p_0) \) measures the speed of the breaking of \( W^c_\mu (0) \) and \( W^u_\mu (z_\mu) \) in the direction of \( \nabla G \), when \( p_0 \) moves in \( W^u_\mu (z_0) \).

The following theorem gives a sufficient condition for the existence of the homoclinic map, compare with theorem (2.4).

**Theorem 3.10.** Let 0 be an elliptic-hyperbolic fixed point for the Hamiltonian system (3.1) that satisfies the conditions (3.1), (3.2) and (3.3).

a) Assume \( p_0 \in W^s_0(0) \) satisfies \( M^*_1(p_0) = 0 \) and \( \partial_\mu M^*_2(p_0) := D M^*_2(p_0)v \neq 0 \), where \( v = (0, J \nabla G(p_0)) \). Then given any smooth curve \( z_\mu \) in \( \mathbb{R}^4 \) that satisfies \( z_\mu \in W^s_\mu (0) \) and \( p_0^u(z_\mu) = p_0 \), there exist \( \mu_0 > 0 \) and \( p_\mu \in W^s_\mu (z_\mu) \) for \( 0 < \mu < \mu_0 \) such that \( W^u_\mu (z_\mu) \cap p_\mu \notin W^{cs}_\mu (0) \). Hence the homoclinic map is well defined.

b) If in addition, there exist \( C_1 \) and \( C_2 \) invariant circles contained in \( W^c (0) \), such that \( x \in C_1, y \in C_2 \) where \( C \) is an invariant and persistent circle, and \( p_0 \) is a non singular zero of the the Melnikov vector \( M \); thus for \( \mu > 0 \) small, the homoclinic map \( \Psi \) is not the identity. In fact, \( C \) and \( \Psi (C) \) are transversal in \( W^c (0) \) and \( C \) has the transition circle condition.

**Remark 3.11.** The above theorem implies that all the perturbed unstable manifolds of the points in a certain neighborhood of \( z_0 \) contained in \( W^c_\mu (0) \) intersect transversally \( W^{cs}_\mu (0) \). The size of this neighborhood depends on \( \mu \).

**Remark 3.12.** A natural question is how far two circles can be, possibly separated by gaps, but connected with homoclinic maps, some results in this direction can be found in [18]).

4. proofs.
4.1. Proof of theorem (2.3). It follows from the next lemma, whose proof is omitted, applied to \( A = DF(a) \) and the smoothness of \( \Omega \).

**Lemma 4.1.** Let \( V \) be a vector space of even dimension, \( \Omega \) be a linear symplectic form and let \( A : V \to V \) be a linear and symplectic isomorphism. Assume \( V^u \oplus V^s \oplus V^\perp \) is the induced decomposition of \( V \) by the eigenvalues of \( A \) of norm greater than one, equal to one and less than one, let \( V^{cu} = V^u \oplus V^s \), \( V^{cs} = V^s \oplus V^\perp \) and; for \( \alpha \in V^c \), let \( V^{\alpha} = V^s \ominus \alpha > 0 \); let \( i_r : V^r \to V \), for \( r = u, c, s, cs, cu, s \alpha \) be the canonical inclusion and \( A|V^c \) be diagonalizable, then:

1. \((V^c, i^*_c\Omega)\) is symplectic linear space.
2. \(i^*_c\Omega = 0\) and \(i^*_s\Omega = 0\).
3. The characteristic bundle of \( i^*_\alpha\) is \( V^s\).
4. \( V^{\alpha}\) is a Lagrangian subspace of \( V \).

4.2. Proof of theorem (2.4). [d] implies [b]. Since

\[
T_p W^{cs}(a) = T_p W^s(y) \bigoplus T_p \left( W^{cs}(a) \cap W^{cu}(a) \right)
\]

Thus:

\[
T_p M = T_p W^s(y) \bigoplus T_p \left( W^{cs}(a) \cap W^{cu}(a) \right) \bigoplus T_p W^u(x) \subset T_p W^{cs}(a) + T_p W^u(x),
\]

which implies that \( W^u(x) \cap T_p W^{cs}(a) \).

[b] implies [a]. [b] or [c] imply that \( W^{cs}(a) \cap T_p W^{cu}(a) \). From transversality theory (see [5]), \( W^{cs}(a) \cap W^{cu}(a) \) is locally a smooth manifold.

It remains to prove that \( \omega|_K \) is non degenerate. Let \( 0 \neq \alpha \in T_K \cap T_p W^{cs}(a) \), thus there exists \( \beta \in T_M \) with \( \omega(\alpha, \beta) \neq 0 \). By [b]: \( \beta = \beta_1 + \beta_2 \), where \( \beta_1 \in T_p W^u(x) \) and \( \beta_2 \in T_p W^{cs}(a) \). By Theorem (2.3), we have \( \omega(\alpha, \beta_1) = 0 \) so \( \omega(\alpha, \beta_2) \neq 0 \). Therefore \( \alpha \) is not in the characteristic bundle of \( W^{cs}(a) \) which is \( T_p W^s(y) \). Hence \( T_p K \bigoplus T_p W^s(y) = T_p W^{cs}(a) \).

In addition, \( \beta_2 = \beta_3 + \beta_4 \), where the first term is in \( T_p W^s(a) \), and the second is in \( T_p K \), by Lemma (2.3) again, we have \( \omega(\alpha, \beta_3) = 0 \). Therefore: \( \omega(\alpha, \beta_4) = \omega(\alpha, \beta) \neq 0 \).

From here we have that \( \omega|_K \) is not degenerate.

[a] implies [d]. \( W^{cu}(a) \) is a submanifold of \( M \) with characteristic bundle at \( p \) given by \( T_p W^u(x) \), since \( K \) is symplectic, \( T_p K \cap T_p W^u(x) \) can only have one element: \( 0 \). It follows that \( K \cap_p W^u(x) \) in \( W^{cu}(a) \). Analogously: \( K \cap_p W^s(y) \) in \( W^{cs}(a) \).

Balancing dimensions:

\[
T_p K \bigoplus T_p W^u(x) = T_p W^{cu}(a) \quad \text{and} \quad T_p K \bigoplus T_p W^s(y) = T_p W^{cs}(a)
\]

Since \( W^{cs}(a) \cap T_p W^{cu}(a) \), thus \( T_p W^{cs}(a) \oplus T_p W^{cu}(a) = T_p M \). Using the decomposition of \( T_p W^{cs}(a) \) and \( T_p W^{cu}(a) \) in the previous equation we get:

\[
T_p W^u(x) \bigoplus T_p K \bigoplus T_p W^s(y) = T_p M \tag{4.1}
\]

[a] and [b] imply [e]. For \( n = 1 \), we can use Stokes Theorem. In general, let \( x' \) be a point in \( W^{cs}(a) \) close to \( x \); by the normally hyperbolic manifold theory, \( W^u(x') \) is \( C^r \)-close to \( W^u(x') \), and \( W^u(x') \) is transverse to \( W^{cs}(a) \) in a point \( p' \) \( C^r \)-close to \( p \). It means that \( f^u \) is well defined and smooth. Now, from the definition of \( f^u \) we have that \( f^u \circ \pi^u = Id_K \) and \( \pi^u \circ f^u = Id_{W^{cu}(a)} \), where \( \pi^u \) is defined in (2.3). So \( f^u \) is a diffeomorphism.

Let \( \alpha \) and \( \beta \) be in \( T_p K \), define \( \alpha^* = D \pi^u(p)\alpha \) and \( \beta^* = D \pi^u(p)\beta \). We need to prove: \( \omega_p(\alpha, \beta) = \omega_x(\alpha^*, \beta^*) \).
\[
\omega_p(\alpha, \beta) = \omega_{F^n(p)}(D F^n(p)\alpha, D F^n(p)\beta) \quad \text{for all } n \in \mathbb{N}
\]

and
\[
\omega_p(\alpha^*, \beta^*) = \omega_{F^n(x)}(D F^n(x)\alpha^*, D F^n(x)\beta^*) \quad \text{for all } n \in \mathbb{N}
\]

\[
= \omega_{F^n(x)}(D F^n(x)D \pi^u(p)\alpha, D F^n(x)D \pi^u(p)\beta)
\]

\[
= \omega_{F^n(x)}(D (F^n \circ \pi^u)(p)\alpha, D (F^n \circ \pi^u)(p)\beta)
\]

\[
= \omega_{F^n(x)}(D (\pi^u \circ F^n)(p)\alpha, D (\pi^u \circ F^n)(p)\beta)
\]

The last equation is implied from \(\pi^u \circ F^n = F^n \circ \pi^u\).

Now, the distances between \(F^n(p)\) and \(F^n(x)\), \(D (\pi^u \circ F^n)(p)\alpha\) and \(D F^n(p)\alpha\), and finally \(D (\pi^u \circ F^n)(p)\beta\) and \(D F^n(p)\beta\) go to zero. Thus \(\omega_p(\alpha, \beta) = \omega_p(\alpha^*, \beta^*)\).

Now, the propositions [a] and [c] imply [f], [c] implies [a], and [d] implies [e] are analogous to previous cases and we omit the proof.

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