$V_{ub}$ from the Hadronic Invariant Mass Spectrum in Semileptonic $B$ Decay

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Abstract

The hadronic invariant mass spectrum for the inclusive charmless semileptonic decay $B \to X_u e \bar{v}_e$ is studied. Particular attention is paid to the region $s_H < m_D^2$, which may be useful for extracting the value of $|V_{ub}|$. The sensitivity of the spectrum to the parameter $\bar{\Lambda} \equiv m_B - m_b$ is explored. Perturbative QCD corrections to $d\Gamma/ds_H$ of order $\alpha_s^2 \beta_0$ are calculated. For $s_H \sim \bar{\Lambda} m_b$ nonperturbative QCD effects are important and the shape of the invariant mass spectrum is controlled by the $B$ meson matrix element of an infinite sum of local operators. The utility of the hadronic mass spectrum for extracting $|V_{ub}|$ is explored.
The traditional method for extracting $|V_{ub}|$ from experimental data involves a study of the electron energy spectrum in inclusive charmless semileptonic $B$ decay [1]. For a particular hadronic final state $X$ the maximum electron energy is $E_{e}^{\text{(max)}} = (m_{B}^{2} - m_{X}^{2})/2m_{B}$ (in the $B$ rest frame), and consequently electrons with energies in the endpoint region $m_{B}/2 > E_{e} > (m_{B}^{2} - m_{B}^{2})/2m_{B}$ (neglecting the pion mass) must arise from the $b \to u$ transition. A determination of $|V_{ub}|$ from experimental data on the electron spectrum in the endpoint region is possible, provided a theoretical prediction for the electron spectrum can be made.

Recently there has been considerable theoretical progress in our understanding of inclusive semileptonic $B$ decay [2,3,4]. It is based on the use of the operator product expansion (OPE) and heavy quark effective theory (HQET) to include in the differential decay rate nonperturbative effects suppressed by powers of $\Lambda_{\text{QCD}}/m_{b}$. At leading order in the $\Lambda_{\text{QCD}}/m_{b}$ expansion the $B$ meson decay rate is equal to the $b$ quark decay rate. There are no nonperturbative corrections of order $\Lambda_{\text{QCD}}/m_{b}$. At order $\Lambda_{\text{QCD}}^{2}/m_{b}^{2}$ [3,4] the nonperturbative corrections are characterized by two HQET matrix elements $\lambda_{1,2}$, which are defined by

$$\lambda_{1} = \langle B(v)| \bar{h}_{v}^{(b)} (iD)^{2} h_{v}^{(b)} |B(v)\rangle/2m_{B},$$  

$$\lambda_{2} = \langle B(v)| \frac{g_{s}}{2} \bar{h}_{v}^{(b)} \sigma_{\mu \nu} G^{\mu \nu} h_{v}^{(b)} |B(v)\rangle/6m_{B}.$$  

(1)

These matrix elements also occur in the expansion of the $B$ and $B^{*}$ masses in powers of $\Lambda_{\text{QCD}}/m_{b}$,

$$m_{B} = m_{b} + \bar{\Lambda} - (\lambda_{1} + 3\lambda_{2})/2m_{b} + \ldots ,$$  

$$m_{B^{*}} = m_{b} + \bar{\Lambda} - (\lambda_{1} - \lambda_{2})/2m_{b} + \ldots .$$  

(2)

Similar formulae hold for the $D$ and $D^{*}$ masses. The parameters $\lambda_{1}$ and $\lambda_{2}$ are independent of the heavy $b$ quark mass (there is a weak logarithmic dependence in $\lambda_{2}$) and are of order $\Lambda_{\text{QCD}}^{2}$. The measured $B^{*} - B$ mass splitting fixes $\lambda_{2} = 0.12 \text{GeV}^{2}$. The mass formulae for the $B$ and $B^{*}$ mesons involve not only $\lambda_{1,2}$ but also a parameter $\bar{\Lambda}$, which is the difference between the $B$ meson mass and the $b$ quark mass in the $m_{b} \to \infty$ limit. The measured
$B$ semileptonic decay spectrum in the region $E_e \geq 1.5 \text{ GeV}$ has been used to determine $\bar{\Lambda} \simeq 0.4 \text{ GeV}$ and $\lambda_1 \simeq -0.2 \text{ GeV}^2$ \cite{4}. Unfortunately the uncertainties from terms of order $(\Lambda_{QCD}/m_b)^3$ are quite large \cite{6}. (A linear combination of $\bar{\Lambda}$ and $\lambda_1$ is rather well constrained, but the individual values are more uncertain.)

The maximum electron energy in semileptonic $b$ quark decay is $m_b/2$. This is less than the physical endpoint by $\bar{\Lambda}/2$, which is comparable in size to the endpoint region $\Delta E_e^{(\text{endpoint})} = m_D^2/2mb \simeq 0.33 \text{ GeV}$. Using the operator product expansion and HQET, the effects which extend the electron spectrum beyond its partonic value appear as singular terms in the prediction for $d\Gamma/dE_e$ involving derivatives of delta functions, $\delta^{(n)}(E_e - m_b/2)$. Near the endpoint the electron spectrum must be smeared over a region of energies $\Delta E_e$ before theory can be compared with experiment. If the smearing region $\Delta E_e$ is much smaller than $\Lambda_{QCD}$, then higher dimension operators in the OPE become successively more important and the OPE is not useful for describing the electron energy spectrum. For $\Delta E_e$ much greater than $\Lambda_{QCD}$, higher dimension operators become successively less important and a useful prediction for the electron spectrum can be made using the first few terms in the OPE. When $\Delta E_e \sim \Lambda_{QCD}$ there is an infinite series of terms in the OPE which are all equally important. Since $\Delta E_e^{(\text{endpoint})}$ is about $\Lambda_{QCD}$, it seems unlikely that predictions based on a few low dimension operators in the OPE can successfully determine the electron spectrum in this region.

In the future, another possibility for determining $|V_{ub}|$ may come from a comparison of the measured hadronic invariant mass spectrum in the region $s_H < m_D^2$ with theoretical predictions. Here $s_H = (p_B - q)^2$, where $p_B$ is the $B$ meson four-momentum, and $q = p_e + p_{\bar{\nu}e}$ is the sum of the lepton four-momenta. An obvious advantage to studying this quantity rather than the lepton energy spectrum is that most of the $B \to X_u e \bar{\nu}_e$ decays are expected to lie in the region $s_H < m_D^2$, while only a small fraction of the $B \to X_u e \bar{\nu}_e$ decays have electron energies in the endpoint region. Both the invariant mass region, $s_H < m_D^2$, and the electron endpoint region, $m_B/2 > E_e > (m_B^2 - m_D^2)/2m_B$, receive contributions from hadronic final states with invariant masses that range up to $m_D$. However, for the electron endpoint
region the contribution of the states with masses nearer to \( m_D \) is kinematically suppressed since they typically decay to lower energy electrons. In fact, in the ISGW model \([7]\) the electron endpoint region is dominated by the \( \pi \) and the \( \rho \), with higher mass states making only a small contribution. The situation is very different for the low invariant mass region, \( s_H < m_D^2 \), with no cut on the electron energy. Now all states with invariant masses up to \( m_D \) contribute without any preferential weighting towards the lowest mass ones. In the ISGW model the \( \pi \) and the \( \rho \) mesons comprise only about a quarter of the \( B \) semileptonic decays to states with masses less than \( m_D \). Consequently, it is much more likely that the first few terms in the OPE will provide an accurate description \( B \) semileptonic decay in the region \( s_H < m_D^2 \) than in the endpoint region of the electron energy spectrum. Combining the invariant mass constraint, \( s_H < m_D^2 \), with a modest cut on the electron energy will not destroy this conclusion. (Such a cut will probably be required experimentally for the direct measurement of \( s_H \) via the neutrino reconstruction technique.) We also expect that the \( B \to X_u e \bar{\nu}_e \) rate in the invariant mass region \( s_H < m_D^2 \) is less sensitive to nonperturbative effects than is the rate in the hadron energy region \( E_H < m_D \) (in the \( B \) rest frame) \([8]\), since the hadron energy constraint cuts out more of the phase space for states with mass near \( m_D \) than for the lower mass states. In this letter we explore the utility of the hadronic invariant mass spectrum \([9]\) for determining the magnitude of \( V_{ub} \). The possibility of using the hadronic invariant mass spectrum in \( B \to X_c e \bar{\nu}_e \) to determine \( \bar{\Lambda} \) and \( \lambda_1 \) was discussed in Ref. \([10]\). The technique is promising but awaits better data on \( d\Gamma/ds_H \).

To begin with, consider the contribution of dimension three operators in the OPE to the hadronic mass squared spectrum in \( B \to X_u e \bar{\nu}_e \) decay. This is equivalent to \( b \) quark decay and implies a result for \( d\Gamma/dE_0 ds_0 \) (where \( E_0 = p_b \cdot (p_b - q)/m_b \) and \( s_0 = (p_b - q)^2 \) are the energy and invariant mass of the strongly interacting partons arising from the \( b \) quark decay) that can easily be calculated using perturbative QCD up to order \( \alpha_s^2/\beta_0 \). Even at this leading order in the OPE there are important nonperturbative effects that come from the relation between the \( b \) quark mass and the \( B \) meson mass in Eqs. \([2]\). The most significant effect comes from \( \bar{\Lambda} \), and including only it (\textit{i.e.}, neglecting the effect of \( \lambda_{1,2} \)), the hadronic
invariant mass $s_H$ is related to $s_0$ and $E_0$ by

$$ s_H = s_0 + 2\bar{\Lambda}E_0 + \bar{\Lambda}^2. \quad (3) $$

Changing variables from $(s_0, E_0)$ to $(s_H, E_0)$ and integrating $E_0$ over the range

$$ \sqrt{s_H} - \bar{\Lambda} < E_0 < \frac{1}{2m_B} (s_H - 2\bar{\Lambda}m_B + m_B^2), \quad (4) $$

gives $d\Gamma/ds_H$, where $\bar{\Lambda}^2 < s_H < m_B^2$. Feynman diagrams with only a $u$-quark in the final state contribute at $s_0 = 0$, which corresponds to the region $\bar{\Lambda}^2 < s_H < \bar{\Lambda}m_B$.

Although $d\Gamma/ds_H$ is integrable in perturbation theory, it has a double logarithmic singularity at $s_H = \bar{\Lambda}m_B$. At higher orders in perturbation theory, increasing powers of $\alpha_s \ln^2((s_H - \bar{\Lambda}m_B)/m_B^2)$ appear in the invariant mass spectrum. Therefore, $d\Gamma/ds_H$ in the vicinity of $s_H = \bar{\Lambda}m_B$ is hard to predict reliably even in perturbation theory. (In the region $s_H \lesssim \bar{\Lambda}m_B$ nonperturbative effects, which we discuss later, are also important.) The behavior of the spectrum near $s_H = \bar{\Lambda}m_B$ becomes less important for observables that average over larger regions of the spectrum, such as $d\Gamma/ds_H$ integrated over $s_H < \Delta^2$, with $\Delta^2$ significantly greater than $\bar{\Lambda}m_B$. Therefore, we present results for $d\Gamma/ds_H$ in the region $s_H > \bar{\Lambda}m_B$, where only the bremsstrahlung Feynman diagrams contribute. Calculating these Feynman diagrams gives the differential decay rate

$$ \frac{d\Gamma(B \to X_u e \bar{\nu}_e)}{d s_H} = \frac{G_F^2 m^3_B}{192\pi^3} |V_{ub}|^2 \left(1 - \frac{\bar{\Lambda}}{m_B}\right)^3 \times \left[ \frac{\alpha_s(\sqrt{s_H})}{\pi} X(s_H, \bar{\Lambda}) + \left(\frac{\alpha_s(\sqrt{s_H})}{\pi}\right)^2 \beta_0 Y(s_H, \bar{\Lambda}) + \ldots \right], \quad (5) $$

where $\beta_0 = 11 - 2n_f/3$ is the one-loop beta function of QCD.

In Figs. 1 we plot $X(s_H, \bar{\Lambda})$ and $Y(s_H, \bar{\Lambda})$ as functions of $s_H$ for $\bar{\Lambda} = 0.2, 0.4$ and $0.6$ GeV. The $\overline{\text{MS}}$ scheme is used for the strong coupling, and we choose to evaluate $\alpha_s$ at the scale $\sqrt{s_H}$. While $Y(s_H, \bar{\Lambda})$ is sensitive to this choice, the sum of the two terms in the square brackets in Eq. (5) has only a weak scale-dependence. Even though the $\alpha_s^2 \beta_0$ correction is



*For recent discussions of a similar phenomenon in the electron energy spectrum, see Ref. [11].
FIG. 1. The functions \( X(s_H, \bar{\Lambda}) \) and \( Y(s_H, \bar{\Lambda}) \) defined in Eq. (5) for \( \bar{\Lambda} = 0.2 \) GeV (dotted curve), 0.4 GeV (solid curve), and 0.6 GeV (dashed curve).

as large as the \( \alpha_s \) term, this does not necessarily imply a problem with the perturbative corrections, since there is a renormalon ambiguity of order \( \Lambda_{\text{QCD}} \) in \( \bar{\Lambda} \) which cancels a renormalon ambiguity in the perturbative QCD corrections.

To examine the sensitivity to \( \bar{\Lambda} \) of an extracted value of \(|V_{ub}|\) from the number of events in a region \( s_H < \Delta^2 \), we define the dimensionless quantity \( \hat{\Gamma}(\Delta^2, \bar{\Lambda}) \) by

\[
\int_0^{\Delta^2} \frac{d\Gamma(B \to X_u e \bar{\nu}_e)}{ds_H} = \frac{G_F^2 m_B^5}{192\pi^3} |V_{ub}|^2 \left(1 - \frac{\bar{\Lambda}}{m_B}\right)^5 \hat{\Gamma}(\Delta^2, \bar{\Lambda}).
\]

In Fig. 2 we plot \( \hat{\Gamma}(\Delta^2, \bar{\Lambda}) \) as a function of \( \Delta^2 \) for \( \bar{\Lambda} = 0.2, 0.4 \) and 0.6 GeV in the region \( \Lambda m_B < \Delta^2 < 4.5 \) GeV\(^2\), using \( \alpha_s(m_b) = 0.2 \). These curves approach \( \hat{\Gamma}(m_B^2, \bar{\Lambda}) \simeq 0.73 \) as \( \Delta^2 \to m_B^2 \) \[12\]. The spread of the curves in Fig. 2 together with the \( (1 - \bar{\Lambda}/m_B)^5 \) dependence factored out in Eq. (6) suggest that an accurate value of \(|V_{ub}|\) can be obtained from the number of events in a region \( s_H < \Delta^2 \) if \( \Delta^2 \) is not much below \( m_B^2 = 3.5 \) GeV\(^2\), and if a reasonably precise determination of \( \bar{\Lambda} \) is available. For example, with \( \Delta^2 = 3.5 \) GeV\(^2\) and \( \bar{\Lambda} = 0.4 \pm 0.1 \) GeV, the uncertainty arising from the error in \( \bar{\Lambda} \) in the extracted value of \(|V_{ub}|\) is only 8%. So far nonperturbative corrections from higher dimension operators in the OPE have been neglected. We discuss their influence on the extraction of \(|V_{ub}|\) later.
FIG. 2. The function $\hat{\Gamma}(\Delta^2, \Lambda)$ defined in Eq. (6) as a function of $\Delta^2$ for $\Lambda = 0.2$ GeV (dotted curve), 0.4 GeV (solid curve), and 0.6 GeV (dashed curve).

Experimental uncertainties will cause some of the $B \to X_c e \bar{\nu}_e$ events to appear to have $s_H < m^2_D$. If experimental $s_H$ resolution forces $\Delta^2$ to be much below $m^2_D$, the uncertainties increase significantly. It is not clear at the present time how, for example, $\Delta^2 = (1.5$ GeV$)^2$ compares to $\bar{\Lambda}m_B$. For such a small value of $\Delta^2$, our results for $\hat{\Gamma}(\Delta^2, \Lambda)$ in Fig. 2 are only reliable if $\bar{\Lambda}$ has a small value, below 0.4 GeV. With $\bar{\Lambda} = 0.4$ GeV, one should worry about the reliability of an extraction of $|V_{ub}|$ based on $\Delta^2 = (1.5$ GeV$)^2$, since higher order perturbative corrections and nonperturbative effects (which we discuss next) are likely to be important.†

In the low mass region $s_H \lesssim \bar{\Lambda}m_B$, nonperturbative corrections from higher dimension

†For example, the order $\alpha_s^2\beta_0$ result predicts for $\bar{\Lambda} = 0.4$ GeV that a large fraction (about 40%) of the $B \to X_u e \bar{\nu}_e$ events have $s_H > (1.5$ GeV$)^2$. Taking Fig. 2 literally and assuming $\bar{\Lambda} = 0.3 \pm 0.1$ GeV, the uncertainty in $|V_{ub}|$ would be 17%, but the sensitivity to uncalculated higher order perturbative and nonperturbative effects could be significant.
operators in the OPE are very important. Just as in the case of the electron spectrum in the endpoint region \[13\], the most singular terms can be identified and summed into a shape function, \(S(s_H)\). Neglecting perturbative QCD corrections, we write

\[
\frac{d\Gamma}{ds_H} = \frac{G_F^2 m_b^5}{192\pi^3} |V_{ub}|^2 S(s_H).
\]

(7)

It is convenient to introduce the scaled variable \(y = s_H/\bar{\Lambda} m_b\) and define a dimensionless shape function \(\hat{S}(y) = \bar{\Lambda} m_b S(s_H)\). Then

\[
\hat{S}(y) = \sum_{n=0}^{\infty} (-1)^n \frac{2A_n}{n! \bar{\Lambda}^n} \frac{d^n}{dy^n} \left[ y^{n+2} (3 - 2y) \theta(1 - y) \right].
\]

(8)

The matrix elements \(A_n\) are the same ones that determine the shape functions for the semileptonic \(B\) decay electron energy spectrum in the endpoint region and the endpoint photon energy region in weak radiative \(B\) decay. Explicitly,

\[
\langle B(v)| \bar{h}_v^{(b)} iD_{\mu_1} \ldots iD_{\mu_n} h_v^{(b)} |B(v)\rangle/2m_B = A_n v_{\mu_1} \ldots v_{\mu_n} + \text{terms involving the metric tensor}.
\]

(9)

The \(A_n\)'s have dimension of \([mass]^n\), and hence the coefficients \(A_n/\bar{\Lambda}^n\) are dimensionless numbers of order one. The first few \(A_n\)'s are \(A_0 = 1\), \(A_1 = 0\), \(A_2 = -\lambda_1/3\), and \(A_3 = -\rho_1/3\). Using the equations of motion, \(\rho_1\) can be related to the matrix element of a four-quark operator. In the vacuum saturation approximation, \(\rho_1 = (2\pi\alpha_s/9) m_B f_B^2 \bar{\Lambda}^2 \) \[5,14\]. Unfortunately, the scale-dependence of this result leaves the value of \(\rho_1\) highly uncertain \[6\].

The shape function \(\hat{S}(y)\) is an infinite sum of singular terms which gives an invariant mass spectrum that leaks out beyond \(y = 1\) (\(i.e., s_H = \bar{\Lambda} m_b\)). For \(y \sim 1\) (\(i.e., s_H \sim \bar{\Lambda} m_b\)) all terms in Eq. (8) are formally of equal importance. Since \(\bar{\Lambda} m_b \approx 2\ \text{GeV}^2\) is not too far from \(m_D^2\), it is necessary to estimate the influence of the nonperturbative effects on the fraction of \(B\) decays with invariant hadronic mass squared less than \(m_D^2\). It is difficult to obtain a model-independent estimate of the leakage of events above an experimental cutoff \(s_H = \Delta^2\), given that we can estimate only the first few moments, \(A_n\). Upper bounds on this leakage can be obtained if \(\hat{S}(y)\) is assumed to be positive definite. This is consistent with the naive
interpretation of the leading singularities as constituting a nonperturbative smearing of the rate beyond the $b$ quark decay endpoint \[13\]; however, there is no proof that this property actually holds. The differential decay rate is positive, but for $s_H$ comparable with $m_b^2$ there are other nonperturbative terms which are equally important. Furthermore, perturbative QCD corrections have been neglected. Models, such as the ACCMM model \[15\], do give a positive shape function.

The fraction of events with $s_H < \Delta^2$ is given by

\[ F(\Delta) = \int_0^{\epsilon(\Delta)} dy \hat{S}(y), \]

where $\epsilon(\Delta) = \Delta^2/\bar{\Lambda}m_b$. Recall that the kinematic point $s_H = \Delta^2$ corresponds to $y = \epsilon$. Assuming a positive shape function, $F(\Delta)$ is greater than

\[ F_P(\Delta) = \int_0^{m_b/\bar{\Lambda}} dy P(y, \epsilon(\Delta)) \hat{S}(y), \]

provided $P(y, \epsilon)$ satisfies the following properties: (i) $P(y, \epsilon) < 1$ for $y < \epsilon$; (ii) $P(y, \epsilon) < 0$ for $y > \epsilon$. The lower bound $F(\Delta) > F_P(\Delta)$ holds for any such $P(y, \epsilon)$. Furthermore, if $P(y, \epsilon) = P_k(y, \epsilon)$ is a polynomial of degree $k$ in $y$, then only the first $k$ moments, $A_k$, appear in the bound. Setting $P_k(y, \epsilon) = \sum_{\ell=0}^k a_{\ell}(\epsilon) y^\ell$, and integrating by parts $n$ times yields

\[ F_{P_k}(\Delta) = \sum_{\ell=0}^k \sum_{n=0}^\ell a_{\ell}(\epsilon) \frac{A_n}{\bar{\Lambda}^n} \binom{\ell}{n} \frac{2(\ell + 6)}{(\ell + 3)(\ell + 4)}. \]  

As an illustration of the utility of this bound, consider first the simple quadratic polynomial $P_2(y) = 1 - y^2/\epsilon^2$. This leads to the lower bound

\[ F(\Delta) > 1 - \frac{8}{15\epsilon^2} + \frac{8\lambda_1}{45\epsilon^2\bar{\Lambda}^2}, \quad \epsilon = \Delta^2/\bar{\Lambda}m_b. \]

For $\bar{\Lambda} = 0.4$ GeV and $\lambda_1 = -0.2$ GeV$^2$, the bound is $F(m_D) > 76\%$. With larger $\bar{\Lambda}$, the bound weakens dramatically. For $\bar{\Lambda} = 0.6$ GeV and $\lambda_1 = -0.2$ GeV$^2$, it is only $F(m_D) > 59\%$. Once again, an independent determination of $\bar{\Lambda}$ and $\lambda_1$ is necessary for these bounds to become useful. For a cubic polynomial, we also need to know $\rho_1$. For example, consider $P_3(y) = 1 - y^3/\epsilon^3$. Then the lower bound is
\[
F(\Delta) > 1 - \frac{3}{7c^3} + \frac{3\lambda_1}{7c^3\Lambda^2} + \frac{\rho_1}{7c^3\Lambda^3}, \quad \epsilon = \Delta^2/\bar{\Lambda}m_b. \tag{14}
\]

For \( \bar{\Lambda} = 0.4 \text{ GeV} \) and \( \lambda_1 = -0.2 \text{ GeV}^2 \), \( F(m_D) > 83\% \) if \( \rho_1 = 0 \) and \( F(m_D) > 87\% \) if \( \rho_1 = 0.1 \text{ GeV}^3 \). The dependence on \( \bar{\Lambda} \) is still the most important, as the latter bound falls to \( F(m_D) > 69\% \) for \( \bar{\Lambda} = 0.6 \text{ GeV} \). We could improve these bounds by optimizing the coefficients in the polynomial \( P_k \). Since the optimization itself will depend on \( \bar{\Lambda}, \lambda_1 \) and \( \rho_1 \), it does not seem worth while to proceed along this line at the present time.

If due to experimental resolution one can only use \( s_H < (1.5 \text{ GeV})^2 \), then the bounds become much weaker. For example, using the cubic polynomial above with \( \bar{\Lambda} = 0.4 \text{ GeV} \) and \( \lambda_1 = -0.2 \text{ GeV}^2 \), the bound is \( F(1.5 \text{ GeV}) > 36\% \) for \( \rho_1 = 0 \) and \( F(1.5 \text{ GeV}) > 51\% \) for \( \rho_1 = 0.1 \text{ GeV}^3 \).

An alternative is to resort to models for an estimate of the effect of high order terms in the sum in Eq. (8). As an example, consider the ACCMM model [15,16], where the \( B \) meson is modeled by a spectator quark with mass \( m_{sp} \) and momentum \( \vec{p} \), and a \( b \) quark with momentum \( -\vec{p} \) and effective mass \( m_b^{(\text{eff})} = m_B - \sqrt{m_{sp}^2 + \vec{p}^2} \). The probability that the spectator quark momentum takes the value \( \vec{p} \) is \( \Phi(\vec{p}) \). In this model \( \bar{\Lambda} = \int d^3p \Phi(\vec{p}) \sqrt{m_{sp}^2 + \vec{p}^2} \), and \( \lambda_1 = -\int d^3p \Phi(\vec{p}) |\vec{p}|^2 \).

To plot the shape function \( S(s_H) \) in the ACCMM model, we neglect the boost from the \( b \) quark into the \( B \) meson rest-frame (such affects are subleading in the \( m_b \to \infty \) limit for all values of \( s_H \)). In Fig. 3 we plot the shape function for three different cases that give \( \bar{\Lambda} = 0.4 \text{ GeV} \). They are \( \Phi(\vec{p}) \propto e^{-|\vec{p}|/p_F} \) with \( p_F = 0.13 \text{ GeV} \) and \( m_{sp} = 0 \) (dotted curve); \( \Phi(\vec{p}) \propto e^{-|\vec{p}|^2/p_F^2} \) with \( p_F = 0.35 \text{ GeV} \) and \( m_{sp} = 0 \) (dashed curve); and \( p_F = 0.3 \text{ GeV} \) and \( m_{sp} = 0.2 \text{ GeV} \) (solid curve). In these cases only 4.4%, 4.1% and 2.8%, respectively, of the \( B \to X_u \epsilon \bar{\nu}_e \) decays have \( s_H \geq m_{D_b}^2 \). Even if the leakage into the region \( s_H > m_{D_b}^2 \) were a factor of two or three greater than this (a possibility which is not at all unlikely), unknown nonperturbative effects characterized by the \( A_n \)'s only give rise to about a 10% uncertainty in the fraction of \( B \) semileptonic decays with \( s_H < m_{D_b}^2 \). On the other hand, if due to charm contamination one can only use events with \( s_H < (1.5 \text{ GeV})^2 \), then
FIG. 3. The shape function $S(s_H)$ in the ACCMM model. $\Phi(\vec{p}) \propto e^{-|\vec{p}|/p_F}$ with $p_F = 0.13$ GeV and $m_{sp} = 0$ (dotted curve); $\Phi(\vec{p}) \propto e^{-|\vec{p}|^2/p_F^2}$ with $p_F = 0.35$ GeV and $m_{sp} = 0$ (dashed curve); and $p_F = 0.3$ GeV and $m_{sp} = 0.2$ GeV (solid curve).

the sensitivity to the shape function is much greater. For the three models in Fig. 3, the fraction of $B \rightarrow X_u e \bar{\nu}_e$ decays with $s_H > (1.5 \text{ GeV})^2$ is 15%, 16%, and 15%, respectively. One should not conclude from the approximate agreement between these models that the uncertainty in these predictions for the leakage is less than a factor of two.

The larger the value of $\bar{\Lambda}$, the larger the fraction of $B \rightarrow X_u e \bar{\nu}_e$ decays that leak out beyond $s_H = m_D^2$. In Fig. 4 we plot the model $\Phi \propto e^{-|\vec{p}|^2/p_F^2}$ with $m_{sp} = 0.2$ GeV and $p_F$ taking three values ($p_F = 0$, 0.3, and 0.5 GeV) corresponding to the choices $\bar{\Lambda} = 0.2$, 0.4 and 0.6 GeV. For these values of $p_F$, the model gives $\lambda_1 = 0$, $-0.14$, and $-0.38 \text{ GeV}^2$ respectively, in qualitative agreement with the correlation between $\bar{\Lambda}$ and $\lambda_1$ from the electron energy spectrum in semileptonic $B$ decay \[3,4\]. The $p_F = 0$ (dotted) curve is given analytically by the $n = 0$ term in Eq. (8) with $\bar{\Lambda} = 0.2$ GeV. The fraction of $B \rightarrow X_u e \bar{\nu}_e$ decays with $s_H \geq m_D^2$ is 0, 2.8%, and 12%, respectively. The fraction of events with $s_H > (1.5 \text{ GeV})^2$ is 0, 15%, and 31%, respectively. The rapid variation of these values with $\bar{\Lambda}$ shows again that
FIG. 4. The shape function $S(s_H)$ in the ACCMM model, with $m_{sp} = 0.2 \text{GeV}$. The dotted curve corresponds to $\bar{\Lambda} = 0.2 \text{GeV}$ ($p_F = 0$), the solid curve is $\bar{\Lambda} = 0.4 \text{GeV}$ ($p_F = 0.3 \text{GeV}$), and the dashed curve is $\bar{\Lambda} = 0.6 \text{GeV}$ ($p_F = 0.5 \text{GeV}$).

A reliable determination of $|V_{ub}|$ from the number of events in the region $s_H < \Delta^2$ is only possible if $\bar{\Lambda}$ does not have too large a value. This is especially true if experimental issues force $\Delta^2$ to be significantly smaller than $m_D^2$.

We have investigated the utility of the hadronic invariant mass spectrum in $B \to X_u e^+ \bar{\nu}_e$ decay in the region $s_H < m_D^2$ for a possible model-independent determination of $|V_{ub}|$. Perturbative QCD corrections to $d\Gamma/ds_H$ of order $\alpha_s^2\beta_0$ were calculated paying particular attention to kinematic effects arising from $\bar{\Lambda}$. A measurement of $\int_0^{\Delta^2} ds_H (d\Gamma/ds_H)$ can be translated into a value of $|V_{ub}|$ using Fig. 2, and the result can then be corrected (with some model dependence) for nonperturbative effects coming from operators with dimension five and higher in the OPE. If $\Delta^2 \simeq m_D^2$ and $\bar{\Lambda} = 0.4 \pm 0.1 \text{GeV}$ are experimentally feasible, then $|V_{ub}|$ can be extracted model-independently with about 10% theoretical uncertainty from the error in the value of $\bar{\Lambda}$. In this case uncertainties associated with higher dimension operators in the OPE are likely to be small. A determination of $\bar{\Lambda}$ with a precision of
±0.1 GeV from experimental information on semileptonic $B$ decay and weak radiative $B$ decay seems possible [5,10,17,18].

It is possible that the experimental invariant mass resolution will necessitate an upper cut on the hadronic invariant mass squared $\Delta^2$ which is somewhat below $m_D^2$, in a region where nonperturbative effects that make the spectrum leak beyond $\Delta^2$ are not negligible. For $s_H \sim \bar{\Lambda} m_B$, both nonperturbative strong interaction effects and higher order perturbative corrections become important. If $\Delta^2$ has to be substantially smaller than $m_D^2$, then $\bar{\Lambda}$ cannot be too large for this method of extracting $|V_{ub}|$ to remain viable. In this case, the theoretical uncertainty in $|V_{ub}|$ will depend sensitively on both the value of $\Delta^2$ and the value and uncertainty in $\bar{\Lambda}$ at the time when a measurement of $\int_0^{\Delta^2} ds_H (d\Gamma/ds_H)$ is available. In addition, since the nonperturbative effects introduce a certain level of model-dependence, it will be important to compare the extracted value of $|V_{ub}|$ from the hadronic invariant mass spectrum with its value from other determinations, such as from exclusive decays [19].

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REFERENCES

[1] F. Bartelt et al., CLEO Collaboration, Phys. Rev. Lett. 71 (1993) 4111; H. Albrecht et al., Argus Collaboration, Phys. Lett. B255 (1991) 297.

[2] J. Chay et al., Phys. Lett. B247 (1990) 399.

[3] I.I. Bigi et al., Phys. Lett. B293 (1992) 430, [(E) ibid. B297 (1993) 477]; I.I. Bigi et al., Phys. Rev. Lett. 71 (1993) 496.

[4] A.V. Manohar and M.B. Wise, Phys. Rev. D49 (1994) 1310; B. Blok et al., Phys. Rev. D49 (1994) 3356 [(E) ibid. D50 (1994) 3572]; T. Mannel, Nucl. Phys. B413 (1994) 396.

[5] M. Gremm et al., Phys. Rev. Lett. 77 (1996) 20.

[6] M. Gremm and I. Stewart, Phys. Rev. D55 (1997) 1226.

[7] N. Isgur et al., Phys. Rev. D39 (1989) 799; N. Isgur and D. Scora, Phys. Rev. D52 (1995) 2783.

[8] C. Greub and S.-J. Rey, SLAC-PUB-7245 [hep-ph/9608247]; A.O. Bouzas and D. Zappala, Phys. Lett. B333 (1994) 215.

[9] V. Barger et al., Phys. Lett. B251 (1990) 629; J. Dai, Phys. Lett. B333 (1994) 212.

[10] A.F. Falk et al., Phys. Rev. D53 (1996) 2491; D53 (1996) 6316.

[11] G.P. Korchemsky and G. Sterman, Phys. Lett. B340 (1994) 96; R. Akhoury and I.Z. Rothstein, Phys. Rev. D54 (1996) 2349.

[12] M. Luke et al., Phys. Lett. B343 (1995) 329.

[13] M. Neubert, Phys. Rev. D49 (1994) 3392; D49 (1994) 4623; I.I. Bigi et al., Int. J. Mod. Phys. A9 (1994) 2467.

[14] T. Mannel, Phys. Rev. D50 (1994) 428; I.I. Bigi et al., Phys. Rev. D52 (1995) 196.

[15] G. Altarelli et al., Nucl. Phys. B208 (1982) 365; A. Ali and I. Pietarinen, Nucl. Phys. B154 (1979) 519.

[16] I. Bigi et al., Phys. Lett. B328 (1994) 431.

[17] A. Kapustin and Z. Ligeti, Phys. Lett. B355 (1995) 318; R.D. Dikeman et al., Int. J. Mod. Phys. A11 (1996) 571.

[18] Z. Ligeti and Y. Nir, Phys. Rev. D49 (1994) 4331; M. Luke and M.J. Savage, Phys. Lett. B321 (1994) 88; M.B. Voloshin, Phys. Rev. D51 (1995) 4934.

[19] N. Isgur and M.B. Wise, Phys. Rev. D42 (1990) 2388; Z. Ligeti and M.B. Wise, Phys. Rev. D53 (1996) 4937; J. Flynn, Nucl. Phys. B (Proc. Suppl.) 53 (1997) 168; and references therein.

[20] R.D. Dikeman and N.G. Uraltsev, TPI-MINN-97/06-T [hep-ph/9703437].