Adversarial Bandits Robust to \( S \)-Switch Regret

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Abstract

We study the adversarial bandit problem under \( S \) number of switching best arms for unknown \( S \). For handling this problem, we adopt the master-base framework using the online mirror descent method (OMD). We first provide a master-base algorithm with basic OMD, achieving \( \tilde{O}(S^{1/2}K^{1/3}T^{2/3}) \). For improving the regret bound with respect to \( T \), we propose to use adaptive learning rates for OMD to control variance of loss estimators, and achieve \( \tilde{O}(\min\{\mathbb{E}[\sqrt{SKT\rho_T(h^*)}], S\sqrt{KT}\}) \), where \( \rho_T(h^*) \) is a variance term for loss estimators.

1 Introduction

The bandit problem is a fundamental decision-making problem for dealing with the exploration-exploitation trade-off. In this problem, an agent plays an action, “arm”, at a time and receives loss or reward feedback for the option. The arm might be a choice of an item of a user in recommendation systems. In practice, it is often required to consider switching user preferences for items as time passes. This can be modeled by switching best arms.

In this paper, we focus on the oblivious adversarial (a.k.a. non-stochastic) bandit problem where losses for each arm at each time may be arbitrarily determined before starting. We also consider that the best arm in hindsight is changed several times over a time horizon. Therefore, regret is measured by competing with not a single best arm but switching best arms in hindsight as time passes. Importantly, we consider that the number of best arm switching in hindsight is not provided to an agent in prior.

The switching best arm has been widely studied. In the expert setting with full information Cesa-Bianchi et al. [1997], there are several algorithms Daniely et al. [2015], Jun et al. [2017] that achieve near-optimal \( \tilde{O}(\sqrt{ST}) \) regret bound for \( S \)-switch regret (which is defined later) without information of switch parameter \( S \). However, in the bandit problems, an agent cannot observe full information of loss at each time, which makes the problem more challenging compared with the full information setting. For stochastic bandit settings where each arm has switching reward distribution over time steps, referred to as non-stationary bandit problems, has been studied by Garivier and Moulines [2008], Auer et al. [2019], Russac et al. [2019]. Especially Auer et al. [2019] achieved near-optimal regret \( \tilde{O}(\sqrt{SKT}) \) without information of \( S \) in
prior. However, we cannot apply this method to the adversarial setting where losses may be determined arbitrarily. For the oblivious adversarial bandit setting, EXP3.S \cite{auer2002nonstochastic} achieved $\tilde{O}(\sqrt{SKT})$ with information $S$ and $\tilde{O}(S\sqrt{KT})$ without information of $S$. It was also shown that $\Omega(\sqrt{SKT})$ is a lower bound of the problem. When $S$ is not known, it is known that the Bandit-over-Bandit approach achieved $\tilde{O}(\sqrt{SKT} + T^{3/4})$ \cite{cheung2019optimal},\cite{foster2020bandit}. In the adaptive adversarial bandits, achieving the same result for all $S$ simultaneously is known to be impossible \cite{marinov2021bandits}. Recently, \cite{luo2022switching} studied switching adversarial linear bandits and achieved $\tilde{O}(\sqrt{dST})$ with known $S$.

In this paper, we study the adversarial bandit problems with switching best arms without information of $S$. To handle this problem, we adopt the master-base framework with the online linear descent method (OMD) which has been widely utilized for model selection problems \cite{agarwal2017optimism},\cite{pacchiano2020model},\cite{luo2022switching}. We first provide a master-base algorithm with basic OMD using negative entropy regularizers and analyze the regret of the algorithm achieving $\tilde{O}(S^{1/2}K^{1/3}T^{2/3})$. Based on the analysis, for improving the regret bound with respect to $T$, we propose to use adaptive learning rates for OMD with log-barrier and negative entropy regularizers to control variance of loss estimators and achieve $\tilde{O}(\max\{\mathbb{E}[\sqrt{SKT\rho_T(h^*)}], S\sqrt{KT}\})$, where $\rho_T(h^*)$ is a variance term for loss estimators.

Lastly, we compare the regret bounds of our algorithms with the previously achieved regret bounds from \cite{auer2002nonstochastic} and \cite{cheung2019optimal}.

\section{Problem statements}

Here we describe our problem settings. We let $\mathcal{K} = [K]$ be the set of arms and $I_t \in [0, 1]^K$ be a loss vector at time $t$ in which $l_t(a)$ is the loss value of arm $a$ at time $t$. The adversarial environment is given by an arbitrary sequence of reward vectors $l_1,l_2,\ldots,l_T \in [0, 1]^K$ over the horizon time $T$. At each time $t$, an agent selects an arm $a_t \in [K]$, after which one observes partial feedback $l_t(a_t) \in [0, 1]$. In this adversarial bandit setting, we aim to reduce the $S$-switch regret which is defined as follows. Let $\sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_T\} \in [K]^T$ be a sequence of actions. For a positive integer $S < T$, the set of sequence of actions with $S$ switches is defined as

$$B_S = \left\{ \sigma \in \mathcal{K}^T : \sum_{t=1}^{T-1} 1\{\sigma_t \neq \sigma_{t+1}\} \leq S \right\}.$$

Then, we define the $S$-switch regret as

$$R_S(T) = \max_{\sigma \in B_S} \sum_{t=1}^{T} \mathbb{E}[l_t(a_t)] - l_t(\sigma_t).$$

We assume that $S$ is not known to the agent. Indeed, the switching value $S$ may not be determined by the environment but by the regret measure. Therefore, the main objective of this paper is to design universal algorithms that achieve tight regret bounds for any unknown $S \in [T - 1]$. This problem is more general than the non-stationary stochastic bandit problems without knowing a switching parameter.
3 Algorithms and regret analysis

To handle this problem, we suggest to use OMD based on the master-base framework Agarwal et al. [2017], Pacchiano et al. [2020], Luo et al. [2022]. In the framework, at each time, a master algorithm selects a base and the selected base selects an arm. For the unknown switch value $S$, we suggest tuning each base algorithm using a candidate of $S$ as follows.

Let $\mathcal{H}$ represent the set of candidates of the switch parameter $S \in [T - 1]$ for the bases such that:

$$\mathcal{H} = \{T^0, T^1, T^2, \ldots, T\}.$$  

Then, each base adopts one of the candidate parameters in $\mathcal{H}$ for tuning its learning rate. For simplicity, let $H = |\mathcal{H}|$ such that $H = O(\log(T))$ and let base $h$ represent the base having the candidate parameter $h \in \mathcal{H}$ if there is no confusion. Also, let $h^\dagger$ be the largest value $h \leq S$ among $h \in \mathcal{H}$, which indicates the near-optimal parameter for $S$. Then we can observe that

$$e^{-1}S \leq h^\dagger \leq S.$$  

Here we describe OMD. For a regularizer function $F : \mathbb{R}^d \rightarrow \mathbb{R}$ and $p, q \in \mathbb{R}^d$, we define Bregman divergence as

$$D_F(p, q) = F(p) - F(q) - \langle \nabla F(q), p - q \rangle.$$  

Let $p_t$ be the distribution for selecting an action at time $t$ and $P_{d-1}$ be the probability simplex with dimension $d$. Then with a loss vector $l$, using the online mirror descent we can get $p_{t+1}$ as follows:

$$\tilde{p}_{t+1} = \arg \min_{p \in \mathbb{R}^d} \langle p, l \rangle + D_F(p, p_t),$$

$$p_{t+1} = \arg \min_{p \in P_{d-1}} D_F(p, \tilde{p}_{t+1}).$$  

(1)

We use a regularizer $F$ that contains a learning rate (to be specified later). We note that, in the bandit setting, we cannot observe full information of loss at time $t$, but get partial feedback based on a selected action. Therefore, it is required to use an estimator loss vector for OMD.

3.1 Master-base OMD

We first provide a simple master-base OMD algorithm (Algorithm 1) with the negative entropy regularizer defined as

$$F_\eta(p) = (1/\eta) \sum_{i=1}^d (p(i) \log p(i) - p(i)),$$

where $p \in \mathbb{R}^d$, $p(i)$ denotes the $i$-index entry for $p$, and $\eta$ is a learning rate. We note that well-known EXP 3 Auer et al. [2002] for the adversarial bandits is also based on the negative entropy function.
In Algorithm 1, at each time, the master selects a base \( h_t \) from distribution \( p_t \). Then following distribution \( p_{t,h_t} \) for selecting an arm, the base \( h_t \) selects an arm \( a_t \) and receive a corresponding loss \( l_t(a_t) \). Using the loss feedback, it gets unbiased estimators \( l_t'(h) \) and \( l_{t,h}(a) \) for a loss from selecting each base \( h \in \mathcal{H} \) and each arm \( a \in [K] \), respectively. Then using OMD with the estimators, it updates the distributions \( p_{t+1} \) and \( p_{t+1,h} \) for selecting a base and an arm from base \( h \), respectively.

For getting \( p_{t+1} \), it uses the negative entropy regularizer with learning rate \( \eta \). The domain for updating the distribution for selecting a base is defined as a clipped probability simplex such that \( \mathcal{A}_H = \mathcal{P}_{H-1} \cap [\alpha,1]^H \) for \( \alpha > 0 \). By introducing \( \alpha \), it can control the variance of \( l_t'(h) = l_t(a_t)\mathbb{1}(h = h_t)/p_t(h) \) by restricting the minimum value for \( p_t(h) \). For getting \( p_{t+1,h} \), it also uses the negative entropy regularizer with learning rates depending on a value of \( h \) for each base. The learning rate \( \eta(h) \) is tuned by using a candidate value \( h \) for \( S \) in the base \( h \) to control adaptation for switching such that

\[
\eta(h) = h^{1/2}/(K^{1/3}T^{2/3}).
\]

The domain for the distribution is also defined as a clipped probability simplex such that \( \mathcal{A}_K = \mathcal{P}_{K-1} \cap [\beta,1]^K \) for \( \beta > 0 \). The purpose of \( \beta \) is to introduce some regularization in learning \( p_{t+1,h} \) for dealing with switching best arms in hindsight, which is slightly different from the purpose of \( \alpha \). This technique was introduced in Herbster and Warmuth [2001], Lattimore and Szepesvári [2020] for dealing with the switching.

Now we provide a regret bound for the algorithm in the following theorem.

**Algorithm 1** Master-base OMD for switching targets

1: Given: \( T, K, \mathcal{H} \).
2: Initialization: \( \alpha = K^{1/3}/(T^{1/3}H^{1/2}), \beta = 1/(KT), \eta = 1/\sqrt{TK}, p_1(h) = 2H, \eta(h) = h^{1/2}/(K^{1/3}T^{2/3}), p_t(h) = 1/H, p_{t,h}(a) = 1/K \) for \( h \in \mathcal{H} \) and \( a \in [K] \).
3: for \( t = 1, \ldots, T \) do
4: Select a base and an arm:
5: Draw \( h_t \sim \text{probabilities} \ p_t(h) \) for \( h \in \mathcal{H} \).
6: Draw \( a_t \sim \text{probabilities} \ p_{t,h_t}(a) \) for \( a \in [K] \).
7: Receive \( l_t(a_t) \in [0,1] \).
8: Obtain loss estimators:
9: \( l_t'(h) = l_t(a_t)\mathbb{1}(h = h_t) \) for \( h \in \mathcal{H} \).
10: \( l_{t,h}(a) = l_t'(h)\mathbb{1}(a = a_t) \) for \( h \in \mathcal{H}, a \in [K] \).
11: Update distributions:
12: \( p_{t+1} = \arg\min_{p \in \mathcal{P}_H} \langle p, l_t' \rangle + D_{F_\eta}(p,p_t) \)
13: \( p_{t+1,h} = \arg\min_{p \in \mathcal{A}_K} \langle p, l_{t,h}' \rangle + D_{F_\eta(h)}(p,p_{t,h}) \) for all \( h \in \mathcal{H} \)
end for

**Theorem 1.** For any switch number \( S \in [T-1] \), Algorithm 1 achieves a regret bound of

\[
R_S(T) = \tilde{O}(S^{1/2}K^{1/3}T^{2/3})
\]
Then, by solving the optimization problem, we can get
\[
R_S(T) = \sum_{t=1}^{T} \mathbb{E} \left[ l_t(a_t) \right] - \sum_{s=0}^{S} \min_{k_s \in [K]} \sum_{t=t_s}^{t_{s+1} - 1} l_t(k_s)
\]
\[
= \sum_{t=1}^{T} \mathbb{E} \left[ l_t(a_t) \right] - \sum_{t=1}^{T} \mathbb{E} \left[ l_t(a_t(h^1)) \right] + \sum_{t=1}^{T} \mathbb{E} \left[ l_t(a_t(h^1)) \right] - \sum_{s=0}^{S} \min_{k_s \in [K]} \sum_{t=t_s}^{t_{s+1} - 1} l_t(k_s),
\]
(2)
in which the first two terms are closely related with the regret from the master algorithm against the near optimal base \(h^1\), and the remaining terms are related with the regret from \(h^1\) base algorithm against the best arms in hindsight. We note that the algorithm does not need to know \(h^1\) in prior and \(h^1\) is brought here only for regret analysis.

First we provide a bound for the following regret from base \(h^1\):
\[
\sum_{t=1}^{T} \mathbb{E} \left[ l_t(a_t(h^1)) \right] - \sum_{s=0}^{S} \min_{k_s \in [K]} \sum_{t=t_s}^{t_{s+1} - 1} l_t(k_s).
\]

Let \(k_s^* = \arg\min_{k_s \in [K]} \sum_{t=t_s}^{t_{s+1} - 1} l_t(k)\) and \(e_j\) denote the unit vector with 1 at \(j\)-index and 0 elsewhere. By following the proof of Theorem 31.1 in Lattimore and Szepesvári [2020] we have
\[
\sum_{t=t_s}^{t_{s+1} - 1} \mathbb{E} \left[ l_t(a_t(h^1)) \right] - l_t(k_s^*) = \sum_{t=t_s}^{t_{s+1} - 1} \mathbb{E} \left[ p_{t,h^1} \right] - \mathbb{E} \left[ p_{t,h^1} \right] - \sum_{t=t_s}^{t_{s+1} - 1} \mathbb{E} \left[ p_{t,h^1} \right] - \mathbb{E} \left[ p_{t,h^1} \right]
\]
\[
\leq \beta T_s K + \sum_{t=t_s}^{t_{s+1} - 1} \mathbb{E} \left[ p_{t,h^1} \right] - \mathbb{E} \left[ p_{t,h^1} \right]
\]
(3)
where the first term in the last inequality is obtained using the clipped domain \(\mathcal{A}_K\) and the second term is obtained from the unbiased estimator \(l_t^\prime_{t,h^1}\) such that \(\mathbb{E}[l_t^\prime_{t,h^1}] = \mathbb{E}[l_t(h)]\). We can observe that the clipped domain controls the distance between the initial distribution at \(t_s\) and the best arm unit vector for the time steps over \([t_s, t_{s+1} - 1]\). Let
\[
\tilde{p}_{t+1,h^1} = \arg\min_{p \in \mathbb{R}^K} \mathbb{E} \left[ p_{t,h^1} \right] \exp(-\eta(h^1)l_t^\prime_{t,h^1}(k)),
\]
for all \(k \in [K]\).

For the second term of the last inequality in (3), we provide a lemma in the following.
Lemma 1 (Theorem 28.4 in Lattimore and Szepesvári [2020]). For any \( p \in \mathcal{A}_K \) we have

\[
\begin{align*}
&\sum_{t=t_{s-1}}^{t_{s+1}-1} \langle \hat{p}_{t,h^1} - p, \hat{l}_t' \rangle \\
&\leq D_{F_{\eta(h^1)}}(p, \hat{p}_{t,h^1}) + \sum_{t=t_{s-1}}^{t_{s+1}-1} D_{F_{\eta(h^1)}}(\hat{p}_{t,h^1}, \hat{p}_{t+1,h^1}).
\end{align*}
\]

In Lemma 1, the first term is for the initial point diameter at time \( t_s \) and the second term is for the divergence of the updated policy. Using the definition of the Bregman divergence and the fact that \( p_{t_s,h^1}(k) \geq \beta \), the initial point diameter term can be shown to be bounded as follows:

\[
D_{F_{\eta(h^1)}}(\hat{p}_{t,h^1}, \hat{p}_{t+1,h^1}) \leq \frac{1}{\eta(h^1)} \sum_{k \in [K]} p(k) \log(1/p_{t_s,h^1}(k)) \\
\leq \frac{\log(1/\beta)}{\eta(h^1)}.
\]

Next, for the updated policy divergence term, using \( \tilde{p}_{t+1,h^1}(k) = p_{t,h^1}(k) \exp(-\eta(h^1)l''_{t,h^1}(k)) \) for all \( k \in [K] \), we have

\[
\begin{align*}
&\sum_{t=t_{s-1}}^{t_{s+1}-1} \mathbb{E} \left[ D_{F_{\eta(h^1)}}(p_{t,h^1}, \tilde{p}_{t+1,h^1}) \right] \\
&= \sum_{t=t_{s-1}}^{t_{s+1}-1} \sum_{k=1}^K \mathbb{E} \left[ \frac{1}{\eta(h^1)} p_{t,h^1}(k) \left( \exp(-\eta(h^1)l''_{t,h^1}(k)) - 1 + \eta(h^1)l''_{t,h^1}(k) \right) \right] \\
&\leq \sum_{t=t_{s-1}}^{t_{s+1}-1} \sum_{k=1}^K \mathbb{E} \left[ \frac{\eta(h^1)}{2} p_{t,h^1}(k) l''_{t,h^1}(k)^2 \right] \\
&\leq \sum_{t=t_{s-1}}^{t_{s+1}-1} \sum_{k=1}^K \mathbb{E} \left[ \frac{\eta(h^1)}{2p_t(h^1)} \right] \leq \frac{\eta(h^1)KT_s}{2\alpha},
\end{align*}
\]

where the first inequality comes from \( \exp(-x) \leq 1 - x + x^2/2 \) for all \( x \geq 0 \), the second inequality comes from \( \mathbb{E}[l''_{t,h^1}(k)^2 \mid p_{t,h^1}(k), p_t(h^1)] \leq 1/(p_t(h^1)p_{t,h^1}(k)) \), and the last inequality is obtained from \( p_t(h^1) \geq \alpha \) from the clipped domain. We can observe that the clipped domain controls the variance of estimators. Then from (3), Lemma 1, (4), and (5), by summing up over \( s \in [S] \), we have

\[
\sum_{t=1}^T \mathbb{E} \left[ l_t(a_t(h^1)) \right] - \sum_{s=0}^S \min_{k_s \in [K]} \sum_{t=t_{s-1}}^{t_{s+1}-1} l_t(k_s) \leq \beta T(K - 1) + \frac{S \log(1/\beta)}{\eta(h^1)} + \frac{\eta(h^1)KT}{2\alpha}.
\]

Next, we provide a bound for the following regret from the master:

\[
\sum_{t=1}^T \mathbb{E} \left[ l_t(a_t(h^1)) \right] - \sum_{t=1}^T \mathbb{E} \left[ l_t(a_t(h^1)) \right].
\]
From Theorem 28.4 and problem 28.10 in Lattimore and Szepesvári [2020] we can show that
\[
\sum_{t=1}^{T} \mathbb{E} \left[ l_t(a_t(h_t)) \right] - \sum_{t=1}^{T} \mathbb{E} \left[ l_t(a_t(h^\dagger)) \right] \leq \alpha TH + \frac{\log(1/\alpha)}{\eta} + \frac{\eta TK}{2} \tag{7}
\]
Therefore, putting (2), (6), and (7) altogether, we have
\[
R_S(T) = \sum_{t=1}^{T} \mathbb{E} \left[ l_t(a_t) \right] - \sum_{s=0}^{S} \min_{1 \leq k_s \leq K} \sum_{t=T_s}^{T_{s+1}-1} l_t(k_s)
\leq \alpha TH + \frac{\log(1/\alpha)}{\eta} + \frac{\eta TK}{2} + \beta T(K - 1) + \frac{S \log(1/\beta)}{\eta(h^\dagger)} + \frac{\eta(h^\dagger)KT}{2\alpha}
= \tilde{O}(S^{1/2}T^{2/3}K^{1/3}),
\]
where \( \alpha = K^{1/3}/(T^{1/3}H^{1/2}) \), \( \beta = 1/(KT) \), \( \eta = 1/\sqrt{TK} \), \( \eta(h^\dagger) = h^{1/2}/(K^{1/3}T^{2/3}) \), \( h^\dagger = \Theta(S) \), and \( H = \log(T) \). This concludes the proof.

From Theorem 1, the regret bound has tightness with respect to \( S \) compared with \( \text{EXP3.S} \) having \( \tilde{O}(S\sqrt{KT}) \). Therefore, when \( S \) is large with \( S = \omega((T/K)^{1/3}) \), Algorithm 1 performs better than \( \text{EXP3.S} \). Also, compared with previous bandit-over-bandit approach Cheung et al. [2019] having \( \tilde{O}(\sqrt{SKT} + T^{3/4}) \), our algorithm has a tighter regret bound with respect to \( T \). Therefore, when \( T \) is large with \( T = \omega(S^6K^4) \), Algorithm 1 achieves a better regret bound than the bandit-over-bandit approach.

However, the achieved regret bound from Algorithm 1 has \( O(T^{2/3}) \) instead of \( O(\sqrt{T}) \) because of the large variance of loss estimators from sampling twice at each time for a base and an arm. In the following, we provide an algorithm utilizing adaptive learning rates to control the variance of estimators.

### 3.2 Adaptive master-base OMD

Here we propose Algorithm 2, which utilizes adaptive learning rates to control variance of estimators. For the master algorithm, we adopt the method of Corral Agarwal et al. [2017], in which by using a log-barrier regularizer with increasing learning rates, it introduces a negative bias term to cancel a large variance from bases. The log-barrier regularizer is defined as:
\[
F_{\xi_t}(p) = -\sum_{i=1}^{d} \frac{\log p(i)}{\xi_t(i)}
\]
with adaptive learning rates \( \xi_t \) for the master algorithm. For the base algorithm, we use the negative entropy regularizer with adaptive learning rate \( \eta_t(h) \) such that
\[
F_{\eta_t(h)}(p) = (1/\eta_t(h)) \sum_{i=1}^{d} (p(i) \log p(i) - p(i)).
\]
The adaptive learning rate $\eta_t(h)$ is optimized using variance information for loss estimators at each time $t$ to control the variance such that

$$\eta_t(h) = \sqrt{h/(KT \rho_t(h))},$$

where $\rho_t(h)$ is a variance threshold term (to be specified later).

Here we describe the update learning rates procedure for the master and bases in Algorithm 2; the other parts are similar with Algorithm 1. The variance of the loss estimator $l'_t(h)$ for base $h$ is $1/p_{t+1}(h)$. If the variance $1/p_{t+1}(h)$ for base $h$ is larger than a threshold $\rho_t(h)$, then it increases learning rate as $\xi_{t+1}(h) = \gamma \xi_t(h)$ with $\gamma > 1$ and the threshold is updated as $\rho_{t+1}(h) = 2/p_{t+1}(h)$, which is also used for tuning the learning rate $\eta_t(h)$. Otherwise, it keeps the learning rate and threshold the same with the previous time step.

**Algorithm 2** Adaptive master-base OMD for switching targets

1: Given: $T, K, \mathcal{H}, \rho_T(h)$ for all $h \in \mathcal{H}$.
2: **Initialization:** $\alpha = 1/(TH), \beta = 1/(TK), \gamma = e^{\sqrt{T}}, \eta = \sqrt{H/T}, \eta_t(h) = \sqrt{h/(KT \rho_t(h))}, \rho_1(h) = 2H, \xi_1(h) = \eta, p_1(h) = 1/H, p_{1,h}(a) = 1/K$ for $h \in \mathcal{H}$ and $a \in [K]$.
3: for $t = 1, \ldots, T$ do
4: Select a base and an arm:
5: Draw $h_t \sim$ probabilities $p_t(h)$ for $h \in \mathcal{H}$.
6: Draw $a_t \sim$ probabilities $p_{t,h_t}(a)$ for $a \in [K]$.
7: Receive $l_t(a_t) \in [0, 1]$.
8: **Update loss estimators:**
9: $l'_t(h) = \frac{l_t(a_t)}{p_t(h)} \mathbb{1}(h = h_t)$ for $h \in \mathcal{H}$.
10: $l'_{t,h}(a) = \frac{l'_t(h)}{p_{t,h}(a)} \mathbb{1}(a = a_t)$ for $h \in \mathcal{H}, a \in [K]$.
11: **Update distributions:**
12: $p_{t+1} = \arg \min_{p \in \mathcal{A}_K} \{ p, l'_t \} + D_{\mathcal{F}_t}^p(p, p_t)$
13: $p_{t+1,h} = \arg \min_{p \in \mathcal{A}_K} \{ p, l'_{t,h} \} + D_{\mathcal{F}_t}^p(p, p_{t,h})$ for $h \in \mathcal{H}$
14: **Update learning rates:**
15: for $h \in \mathcal{H}$
16: If $\frac{1}{p_{t+1}(h)} > \rho_t(h)$, then
17: $\rho_{t+1}(h) = \frac{2}{p_{t+1}(h)}$, $\xi_{t+1}(h) = \gamma \xi_t(h)$.
18: Else, $\rho_{t+1}(h) = \rho_t(h), \xi_{t+1}(h) = \xi_t(h)$.
19: end for

In the following theorem, we provide a regret bound of Algorithm 2.

**Theorem 2.** For any switch number $S \in [T - 1]$, Algorithm 2 achives a regret bound of

$$R_S(T) = \tilde{O} \left( \min \left\{ E \left[ \sqrt{SKTp_T(h^l)} \right], S\sqrt{KT} \right\} \right).$$
Proof. Let $t_s$ be the time when the $s$-th switch of the best arm happens and $t_{S+1} - 1 = T$, $t_0 = 1$. Also let $t_{s+1} - t_s = T_s$. For any $t_s$ for all $s \in [0, S]$, the $S$-switch regret can be expressed as

$$R_S(T) = \sum_{t=1}^{T} \mathbb{E} [l_t(a_t)] - \sum_{s=0}^{S} \min_{k_s \in [K]} \sum_{t=t_s}^{t_{s+1}-1} l_t(k_s)$$

$$= \sum_{t=1}^{T} \mathbb{E} [l_t(a_t(h_t))] - \sum_{t=1}^{T} \mathbb{E} [l_t(a_t(h^\dagger))] + \sum_{t=1}^{T} \mathbb{E} [l_t(a_t(h^\dagger))] - \sum_{s=0}^{S} \min_{k_s \in [K]} \sum_{t=t_s}^{t_{s+1}-1} l_t(k_s),$$

in which the first two terms are closely related with the regret from the master algorithm against the near optimal base $h^\dagger$, and the remaining terms are related with the regret from $h^\dagger$ base algorithm against the best arms in hindsight.

First we provide a bound for the following regret from base $h^\dagger$. From (3), we can obtain

$$\sum_{t=t_s}^{t_{s+1}-1} \mathbb{E} [l_t(a_t(h^\dagger))] - \sum_{s=0}^{S} \min_{k_s \in [K]} \sum_{t=t_s}^{t_{s+1}-1} l_t(k_s) \leq \beta T_s K + \mathbb{E} \left[ \max_{p \in \mathcal{A}_K} \sum_{t=t_s}^{t_{s+1}-1} \langle p_{t,h} - p, l''_{t,h^\dagger} \rangle \right].$$

(8)

Then for the second term of the last inequality in (9), we provide a following lemma.

**Lemma 2.** For any $p \in \mathcal{A}_K$ we can show that

$$\sum_{t=t_s}^{t_{s+1}-1} \mathbb{E} [\langle p_{t,h} - p, l''_{t,h^\dagger} \rangle] \leq \mathbb{E} \left[ 2 \log(1/\beta) \sqrt{KT \rho_T(h^\dagger)} + \frac{T_s}{2} \sqrt{SK \rho_T(h^\dagger)} \right].$$

(9)

**Proof.** The proof is deferred to Appendix A.1

Then from (9) and Lemma 2, we have

$$\sum_{t=1}^{T} \mathbb{E} [l_t(a_t(h^\dagger))] - \sum_{s=0}^{S} \min_{1 \leq k_s \leq K} \sum_{t=T_s}^{T_{s+1}-1} l_t(k_s)$$

$$\leq \beta T(K - 1) + \mathbb{E} \left[ 2S \log(1/\beta) \sqrt{KT \rho_T(h^\dagger)} + \frac{1}{2} \sqrt{SK \rho_T(h^\dagger)} \right].$$

(10)

Next, we provide a bound for the regret from the master in the following lemma.
Lemma 3 (Lemma 13 in Agarwal et al. [2017]).

\[
\sum_{t=1}^{T} \mathbb{E}[l_t(a_t(h_t))] - \sum_{t=1}^{T} \mathbb{E}[l_t(a_t(h^1))] \\
\leq O \left( \frac{H \log(T)}{\eta} + T\eta \right) - \mathbb{E}\left[ \frac{\rho_T(h^1)}{40\eta \log T} \right] + \alpha T(H - 1).
\]

The negative bias term in Lemma 3 is derived from the log-barrier regularizer and increasing learning rates \(\xi_t(h)\). This term is critical to bound the worst case regret which will be shown soon. Also, \(H \log(T)/\eta\) is obtained from \(H \log(1/(H\alpha))/\eta\) considering the clipped domain.

Then, putting (8) and Lemmas 2 and 3 altogether, we have

\[
R_S(T) = \sum_{t=1}^{T} \mathbb{E}[l_t(a_t)] - \sum_{s=0}^{S} \min_{1 \leq k_s \leq K} \sum_{t=T_s}^{T_{s+1}-1} l_t(k_s) \\
\leq O \left( \frac{H \log(T)}{\eta} + T\eta \right) - \mathbb{E}\left[ \frac{\rho_T(h^1)}{40\eta \log T} \right] + \alpha T(H - 1) + \beta T(K - 1) \\
+ \mathbb{E}\left[ 2S \log(1/\beta) \sqrt{\frac{KT\rho_T(h^1)}{h^1}} + \frac{1}{2} \sqrt{SKT\rho_T(h^1)} \right] \\
= \tilde{O} \left( \mathbb{E}\left[ \sqrt{SKT\rho_T(h^1)} \right] \right) - \mathbb{E}\left[ \frac{\rho_T(h^1)\sqrt{TK}}{40\sqrt{H \log(T)}} \right],
\]

where \(\alpha = 1/(TH)\), \(\beta = 1/(TK)\), \(\eta = \sqrt{HT}/T, h^1 = h^1/(KT\rho_T(h)), H = \log(T)\), and \(h^1 = \Theta(S)\). Then we can obtain

\[
R_S(T) = \tilde{O} \left( \min \left\{ \mathbb{E}\left[ \sqrt{SKT\rho_T(h^1)} \right], S\sqrt{KT} \right\} \right),
\]

where \(\tilde{O}(S\sqrt{KT})\) is obtained from the worst case of \(\rho_T(h^1)\). The worst case can be found by considering a maximum value of the concave bound of the last equality in (11) with variable \(\rho_T(h^1) > 0\) such that \(\rho_T(h^1) = \Theta(S)\). This concludes the proof.

Here we provide regret bound comparison with other approaches. For simplicity in the comparison, we use the fact that \(\mathbb{E}[\sqrt{\rho_T(h^1)}] \leq \sqrt{\mathbb{E}[\rho_T(h^1)]}\) for the regret bound in Theorem 2 such that

\[
R_S(T) = \tilde{O} \left( \min \left\{ \sqrt{SKT \mathbb{E}[\rho_T(h^1)]}, S\sqrt{KT} \right\} \right).
\]

The regret bound in Theorem 2 depends on \(\rho_T(h^1)\) which is closely related with variance of loss estimators \(l'_t(h^1)\) for \(t \in [T - 1]\). Even though the regret bound depends on the variance
term, it is of interest that the worst case bound is always bounded by $\tilde{O}(S\sqrt{KT})$, which implies that the regret bound of Algorithm 2 is always smaller than or equal to that of EXP3.S, $O(S\sqrt{KT})$. Algorithm 2 has a tight regret bound $O(\sqrt{T})$ with respect to $T$. Therefore, when $T$ is large such that $T = \omega(\min\{S^2K^2E[\rho_T(h)]^2, S^4K^2\})$, Algorithm 2 shows a better regret bound compared with the bandit-over-bandit approach, $\tilde{O}(\sqrt{SKT + T^{3/4}})$. Also when $T = \omega(\min\{E[\rho_T(h)]^3K, S^3K\})$, Algorithm 2 shows a better regret bound than Algorithm 1.

Remark 1. For implementation of our algorithms, we describe how to update policy $p_t$ using OMD in general. Let $\hat{l}_s(a)$ be a loss estimator for action $a \in [d]$. For the negative entropy regularizer, by solving the optimization in (1),

$$p_{t+1}(a) = \frac{\exp\left(-\eta \sum_{s=1}^{t} \hat{l}_s(a)\right)}{\sum_{b \in [d]} \exp\left(-\eta \sum_{s=1}^{t} \hat{l}_s(b)\right)}.$$

In the case of the log-barrier regularizer, we have $p_{t+1}(a) = (\eta \sum_{s=1}^{t} \hat{l}_s(a) + Z)^{-1}$, where $Z$ is a normalization factor for a probability distribution. Also, a clipped domain with $0 < \epsilon < 1$ in the distribution can be implemented by adding a uniform probability to the policy such that $p_{t+1}(a) \leftarrow (1 - \epsilon)p_{t+1}(a) + \epsilon/d$.

Remark 2. The regret bounds of Theorems 1 and 2 can be achieved by our algorithms for the non-stationary stochastic bandit problems without knowing a switching parameter where reward distributions are switching over time steps. This is because the adversarial switching bandit settings are more general than the non-stationary stochastic bandit settings.

4 Conclusion

In this paper, we studied adversarial bandits with $S$-switch regret for unknown $S$. We proposed two algorithms that are based on a master-base framework with the OMD method. We showed that Algorithm 1 achieved $\tilde{O}(S^{1/2}K^{1/3}T^{2/3})$. Then by using adaptive learning rates for improving the regret bound with respect to $T$, Algorithm 2 achieved $\tilde{O}(\min\{E[\sqrt{SKT\rho_T(h)}], S\sqrt{KT}\})$. It is still an open problem to achieve the optimal regret bound for the worst case.

References

Alekh Agarwal, Haipeng Luo, Behnam Neyshabur, and Robert E Schapire. Corralling a band of bandit algorithms. In Conference on Learning Theory, pages 12–38. PMLR, 2017.

Peter Auer, Nicolo Cesa-Bianchi, Yoav Freund, and Robert E Schapire. The nonstochastic multiarmed bandit problem. SIAM journal on computing, 32(1):48–77, 2002.

Peter Auer, Pratik Gajane, and Ronald Ortner. Adaptively tracking the best bandit arm with an unknown number of distribution changes. In Conference on Learning Theory, pages 138–158, 2019.
Nicolo Cesa-Bianchi, Yoav Freund, David Haussler, David P Helmbold, Robert E Schapire, and Manfred K Warmuth. How to use expert advice. *Journal of the ACM (JACM)*, 44(3):427–485, 1997.

Wang Chi Cheung, David Simchi-Levi, and Ruihao Zhu. Learning to optimize under non-stationarity. In *The 22nd International Conference on Artificial Intelligence and Statistics*, pages 1079–1087, 2019.

Amit Daniely, Alon Gonen, and Shai Shalev-Shwartz. Strongly adaptive online learning. In *International Conference on Machine Learning*, pages 1405–1411, 2015.

Dylan J Foster, Akshay Krishnamurthy, and Haipeng Luo. Open problem: Model selection for contextual bandits. In *Conference on Learning Theory*, pages 3842–3846. PMLR, 2020.

Aurélien Garivier and Eric Moulines. On upper-confidence bound policies for non-stationary bandit problems, 2008.

Mark Herbster and Manfred K Warmuth. Tracking the best linear predictor. *Journal of Machine Learning Research*, 1(281-309):10–1162, 2001.

Kwang-Sung Jun, Francesco Orabona, Stephen Wright, and Rebecca Willett. Improved strongly adaptive online learning using coin betting. In *Artificial Intelligence and Statistics*, pages 943–951. PMLR, 2017.

Tor Lattimore and Csaba Szepesvári. *Bandit algorithms*. Cambridge University Press, 2020.

Haipeng Luo, Mengxiao Zhang, Peng Zhao, and Zhi-Hua Zhou. Corralling a larger band of bandits: A case study on switching regret for linear bandits. *arXiv preprint arXiv:2202.06151*, 2022.

Teodor Vanislavov Marinov and Julian Zimmert. The pareto frontier of model selection for general contextual bandits. *Advances in Neural Information Processing Systems*, 34, 2021.

Aldo Pacchiano, My Phan, Yasin Abbasi-Yadkori, Anup Rao, Julian Zimmert, Tor Lattimore, and Csaba Szepesvari. Model selection in contextual stochastic bandit problems. *arXiv preprint arXiv:2003.01704*, 2020.

Yoan Russac, Claire Vernade, and Olivier Cappé. Weighted linear bandits for non-stationary environments. In *Advances in Neural Information Processing Systems*, pages 12017–12026, 2019.
A Appendix

A.1 Proof of Lemma 2

For ease of presentation, we define the negative entropy regularizer without a learning rate as

\[ F(p) = \sum_{i=1}^{K} (p(i) \log p(i) - p(i)) \]

and define learning rate \( \eta_t(h^t) = \infty \). From the first-order optimality condition for \( p_{t+1,h^t} \) and using the definition of the Bregman divergence,

\[
\langle p_{t+1,h^t} - p_t, l'_t \rangle \leq \frac{1}{\eta_t(h^t)} \langle p - p_{t+1,h^t}, \nabla F(p_{t+1,h^t}) - \nabla F(p_{t,h^t}) \rangle = \frac{1}{\eta_t(h^t)} \left( D_F(p,p_{t,h^t}) - D_F(p_{t+1,h^t},p_{t,h^t}) \right). \tag{12}
\]

Also, we have

\[
\langle p_{t,h^t} - p_{t+1,h^t}, l''_t \rangle = \frac{1}{\eta_t(h^t)} \langle p_{t,h^t} - p_{t+1,h^t}, \nabla F(p_{t,h^t}) - \nabla F(p_{t+1,h^t}) \rangle = \frac{1}{\eta_t(h^t)} \left( D(p_{t+1,h^t},p_{t,h^t}) + D(p_{t,h^t},\hat{p}_{t+1,h^t}) - D(p_{t+1,h^t},\hat{p}_{t+1,h^t}) \right) \leq \frac{1}{\eta_t(h^t)} \left( D(p_{t+1,h^t},p_{t,h^t}) + D(p_{t,h^t},\hat{p}_{t+1,h^t}) \right). \tag{13}
\]

Then, we can obtain

\[
\sum_{t=t_s}^{t_{s+1} - 1} \langle p_{t,h^t} - p_t, l'_t \rangle \leq \sum_{t=t_s}^{t_{s+1} - 1} \langle p_{t,h^t} - p_{t+1,h^t}, l''_t \rangle + \sum_{t=t_s}^{t_{s+1} - 1} \frac{1}{\eta_t(h^t)} \left( D(p,p_{t,h^t}) - D(p_{t+1,h^t},p_{t,h^t}) \right) \]

\[ = \sum_{t=t_s}^{t_{s+1} - 1} \langle p_{t,h^t} - p_{t+1,h^t}, l''_t \rangle + \sum_{t=t_s+1}^{t_{s+1}} D_F(p_{t+1,h^t},p_{t,h^t}) \left( \frac{1}{\eta_t(h^t)} - \frac{1}{\eta_{t-1}(h^t)} \right) \]

\[ + \frac{1}{\eta_{t_s}(h^t)} D_F(p_{t+1,h^t},p_{t,h^t}) = \frac{2 \log(1/\beta)}{\eta_T(h)} + \sum_{t=t_s}^{t_{s+1} - 1} \frac{D_F(p_{t,h^t},\hat{p}_{t+1,h^t})}{\eta_t(h^t)} \]

\[ = 2 \log(1/\beta) \sqrt{\frac{KT}{h} p_T(h^t) / \eta_t(h^t)} \]

\[ + \sum_{t=t_s}^{t_{s+1} - 1} \frac{D_F(p_{t,h^t},\hat{p}_{t+1,h^t})}{\eta_t(h^t)}, \tag{14}
\]
where the first inequality is obtained from (12) and the last inequality is obtained from (13), 
\[ D(p_t, p_{t,h^†}) \leq \log(1/\beta), \]  
and \( \eta_t(h^†) \geq \eta_T(h^†) \) from non-decreasing \( \rho_t(h^†) \).

For the second term in the last inequality in (14), using \( \tilde{p}_{t+1,h^†}(k) = p_{t,h^†}(k) \exp(-\eta_t(h^†)l''_{t,h^†}(k)) \) for all \( k \in [K] \), we have

\[
\sum_{t=s}^{t+1-1} \mathbb{E} \left[ \frac{D_F(p_t,h^†,\tilde{p}_{t+1,h^†})}{\eta_t(h^†)} \right] = \sum_{t=s}^{t+1-1} \sum_{k=1}^{K} \mathbb{E} \left[ \frac{1}{\eta_t(h^†)} p_{t,h^†}(k) \left( \exp(-\eta_t(h^†)l''_{t,h^†}(k)) - 1 + \eta_t(h^†)l''_{t,h^†}(k) \right) \right] \\
\leq \sum_{t=s}^{t+1-1} \sum_{k=1}^{K} \mathbb{E} \left[ \frac{\eta_t(h^†)}{2} p_{t,h^†}(k) l''_{t,h^†}(k)^2 \right] \\
\leq \sum_{t=s}^{t+1-1} \sum_{k=1}^{K} \mathbb{E} \left[ \frac{\eta_t(h^†) \rho_t(h^†)}{2p_t(h^†)} \right] \\
\leq \sum_{t=s}^{t+1-1} \sum_{k=1}^{K} \mathbb{E} \left[ \frac{1}{2} \frac{h^† \rho_t(h^†)}{KT} \right] \\
\leq T_s \sqrt{\frac{h^† K}{T}} \mathbb{E} \left[ \frac{\rho_T(h^†)^{1/2}}{2} \right],
\]

(15)

where the first inequality comes from \( \exp(-x) \leq 1 - x + x^2/2 \) for all \( x \geq 0 \), the second inequality comes from \( \mathbb{E}[l''_{t,h^†}(k)^2 \mid p_{t,h^†}(k), p_t(h^†)] \leq 1/(p_t(h^†)p_{t,h^†}(k)) \), and the last inequality is obtained from \( 1/p_t(h^†) \leq \rho_t(h^†) \).