§1. Introduction. Suppose $M$ is an oriented 3-manifold. A Dehn surgery on $M$ (defined below) is a process by which $M$ is altered by deleting a tubular neighborhood of an embedded circle and replacing it again via some diffeomorphism of the boundary torus. It was shown by W.B.R. Lickorish [Li] and A. Wallace [Wa] that any closed oriented connected 3-manifold can be obtained from any other such manifold by a finite sequence of Dehn surgeries. Thus under this equivalence relation all closed oriented 3-manifolds are equivalent. We shall investigate this same question for more restricted classes of surgeries. In particular we shall insist that our Dehn surgeries preserve the integral (or rational) homology groups. Specifically, if $M_0$ and $M_1$ have isomorphic integral (respectively rational) homology groups, is there a sequence of Dehn surgeries, each of which preserves integral (respectively rational) homology, that transforms $M_0$ to $M_1$? What is the situation if we further restrict the Dehn surgeries to preserve more of the fundamental group? Is there a difference if we require “integral” surgeries? We also show that these Dehn surgery relations are strongly connected to the following questions concerning another point of view towards understanding 3-manifolds. Is there a Heegard splitting of $M_0$, $M_0 = H_1 \cup_f H_2$ ($H_i$ are handlebodies of genus $g$ and $f$ is a homeomorphism of their common boundary

*partially supported by the National Science Foundation
surface), and a homeomorphism \( g \) of \( \partial H_1 \) such that \( M_1 \) has a Heegard splitting using \( g \circ f \) as the identification? Since there are many natural subgroups of the mapping class group, such as the Torelli subgroup and the “Johnson subgroup,” one can ask the same question where \( g \) is restricted to lie in one of these subgroups. This is related to work of Morita on Casson’s invariant for homology 3-spheres [Mo]. Even under these restrictions it has been known for some time that any homology 3-sphere is related to \( S^3 \). This fact has been used to define, calculate and understand invariants of homology 3-spheres (such as Casson’s invariant) by choosing such a “path to \( S^3 \)” in the “space” of 3-manifolds.

We shall show that, in general, the equivalence relations among 3-manifolds thus induced are non-trivial but most can be beautifully characterized in terms of classical invariants such as the linking form, the cohomology ring and Massey products. This precise and beautiful correspondence between the geometric equivalence relation of Dehn surgery and the algebraic equivalence of classical invariants will form the philosophical basis of a new theory of finite type invariants for 3-manifolds by T.D. Cochran and P. Melvin [CM].

Before stating the main theorems let us be more specific. We suppose throughout that \( M \) is a compact oriented 3-manifold. Suppose that \( \gamma \) is a smoothly embedded oriented circle in \( M \) which is of finite order in \( H_1(M; \mathbb{Z}) \). Let \( N(\gamma) \) denote a regular neighborhood of \( \gamma \). We define several (isotopy classes of) embedded closed curves on \( \partial N(\gamma) \). The meridian of \( \gamma \), \( \mu(\gamma) \), is given by the boundary of a meridional disk \( * \times D^2 \subset S^1 \times D^2 \equiv N(\gamma) \). It is easy to see that \( \mu(\gamma) \) is unique and is of infinite order in \( H_1(M - N(\gamma); \mathbb{Z}) \). A parallel of \( \gamma \), denoted \( \rho(\gamma) \), is any simple closed curve which is homologous to \( \gamma \) in \( N(\gamma) \) and intersects \( \mu(\gamma) \) precisely once. Any two parallels “differ” by some number of meridians. The longitude of \( \gamma \), \( \ell(\gamma) \), is a simple closed curve which is homotopic in \( N(\gamma) \) to some positive integral multiple of \( \gamma \) and which is of finite order in \( H_1(M - N(\gamma); \mathbb{Z}) \). The longitude is unique and, in the case that \( \gamma \) is null-homologous, constitutes a preferred parallel. However in general \( \ell(\gamma) \cdot \mu(\gamma) \) is not 1 and hence the longitude (unfortunately)
may not be used as a parallel. With respect to some choice of parallel (the preferred parallel if $\gamma$ is null-homologous) one defines $p/q$-Dehn surgery on $M$ along $\gamma$, denoted $M_\gamma$, to be $[M - \dot{N}(\gamma)] \cup_\phi (S^1 \times D^2)$ where $\phi : S^1 \times \partial D^2 \to \partial N(\gamma)$ is an orientation-reversing homeomorphism which sends the curve $* \times \partial D^2$ to a curve representing $p\mu(\gamma) + q\rho(\gamma)$ in $H_1(\partial N)$. The number $p/q$ is called the framing of $\gamma$. We assume $(p, q) = 1$ and do not allow $q = 0$. We call the surgery integral if $q = \pm 1$ and note that this is independent of the choice of parallel. The surgery is longitudinal surgery if $p\mu + q\rho$ coincides with the longitude up to sign. If $\gamma$ is null-homologous this is equivalent to $p = 0$. The resulting Dehn surgery $M_\gamma$ does not depend on the orientation of $\gamma$. Then we can ask whether or not two 3-manifolds are related by Dehn surgeries on curves $\gamma$ which are not arbitrary, but are restricted to lie in some subgroup $N$ of $\pi_1(M)$. In addition we shall restrict the “framing” $p/q$ of the surgery in order that the integral (respectively rational) homology groups are preserved. Specifically, suppose $N$ is the normal closure of a finite number of elements of $G = \pi_1(M)$ and also suppose that the elements of $N$ are trivial in $H_1(G; \mathbb{Z})$ (or $H_1(G; \mathbb{Q})$ in the $\mathbb{Q}$-case). We shall be concerned primarily with these examples:

i) $N = G_k$, the $k$-th term of the lower central series of $G$, where $G_{k+1} = [G, G_{k-1}]$ and $G_1 = G$;

ii) $N = \{e\}$, the trivial group;

iii) $N = G''$, the second derived group of $G$, that is $G'' = [G_2, G_2]$;

iv) $N = G_2^\mathbb{Q} = \{x \in G \mid \exists n, x^n \in G_2\}$ which is the set of elements which are torsion in $H_1(M; \mathbb{Z})$ (this would be in the rational case).

v) $N = G_k^\mathbb{Q}$ the $k^{th}$ term of the rational lower central series where $G_{k+1}^\mathbb{Q}$ is generated by $[G, G_k^\mathbb{Q}]$ and elements of $G$ for which some power lies in $[G, G_k^\mathbb{Q}]$ [Stallings].

In all these cases $N$ is a characteristic verbal subgroup and we can view $N$ as a functor from groups to normal subgroups. That is, given a space $X$ we can speak of $N(\pi_1(X))$ in each of these cases.
Definition 1.1: $M_1$ is $N$-surgery related to $M_0$ if there is a finite sequence $M_0 = X_0, X_1, \ldots, X_m = M_1$ where $X_{i+1}$ is obtained from $X_i$ by $p_i/q_i$ Dehn surgery along $\gamma_i$ with $\gamma_i \in N(\pi_1(X_i))$ and $p_i = \pm 1$ (in the $\mathbb{Z}$ case where $\gamma_i$ is null-homologous and $p_i$ is well-defined) and merely non-longitudinal surgery (in the $\mathbb{Q}$-case).

In particular, if $N$ is the “$k$th lower central series subgroup” as in $i$), then we will say that $M_1$ is $k$-surgery equivalent to $M_0$. In the next section we will see that, in fact, the $k$-surgery relation is an equivalence relation, and indeed preserves $\pi_1(M)/N_0$.

The case $k = 2$, perhaps most fundamental, we call “integral homology surgery equivalence” or sometimes merely “surgery equivalence”. The question of characterizing this equivalence relation was posed by G. Kuperberg via a newsgroup posting and was partially answered by the Ph.D. thesis of the second author Gerges. A more precise answer is given in this paper. We find (see Theorem 3.1) that this equivalence relation is completely controlled by not just by $H_1(M)$ but by the full triple cup product structures and by the linking form on the torsion subgroup of $H_1(M)$. For example, we deduce that any 2 closed, connected, oriented 3-manifolds with isomorphic $H_1 \cong \mathbb{Z}^m$ ($m < 3$) are (integral homology) surgery equivalent. This generalizes the well-known result for homology spheres ($m = 0$). The latter certainly appeared in public lectures by Andrew Casson in 1985, and we are informed that it was known earlier. This appeared in a 1987 paper by S.V. Matveev, where it was also announced that two 3-manifolds have isomorphic $H_1$ and linking forms if and only if they are related by “Borromean surgeries” [Ma, Theorem 2 and Remark 2].

Other sample results concerning integral homology surgery equivalence are:

**Corollary 3.5.** Let $\mathcal{S}_m$ be the set of surgery equivalence classes of closed oriented 3-manifolds with $H_1 \cong \mathbb{Z}^m$. Then there is a bijection $\psi : \mathcal{S}_m \to \Lambda^3(\mathbb{Z}^m)/\text{GL}_m(\mathbb{Z})$ from $\mathcal{S}_m$ to the set of orbits of the third exterior power of $\mathbb{Z}^m$ under the action induced by $\text{GL}_m(\mathbb{Z})$ on $\mathbb{Z}^m$. Any such manifold is surgery equivalent to one which is the result of 0-framed
surgery on a m component link in $S^3$ which is obtained from the trivial link by replacing a number of trivial 3-string braids by a 3-string braid whose closure is the Borromean rings.

**Corollary 3.6.** A 3-manifold with $H_1 \cong \mathbb{Z}^m$ is surgery equivalent to $\#_{i=1}^m S^1 \times S^2$ if and only if its integral triple cup product form $H^1 \oplus H^1 \oplus H^1 \rightarrow \mathbb{Z}$ vanishes identically.

**Corollary 3.8.** If $H_1(M)$ is torsion-free then $M$ is (integral homology) surgery equivalent to $-M$.

Corollary 3.8 is interesting because it is not obvious from a geometric viewpoint how to construct such a path of surgeries, and the result fails in general for manifolds with torsion in $H_1$.

**Corollary 3.9.** The set $S(\mathbb{Z}_n)$ of (integral homology) surgery equivalence classes of closed oriented 3-manifolds with $H_1 \cong \mathbb{Z}_n$ is in bijection with the set of equivalence classes of units of $\mathbb{Z}_n$, modulo squares of units, the correspondence being given by the image of the fundamental class of the manifold in $H_3(\mathbb{Z}_n) \cong \mathbb{Z}_n$. The correspondence is also given by the self-linking linking number of a generator of $H_1$ ($\lambda(1,1) = q$ and $q$ is viewed as an element of $\mathbb{Z}_n$). The equivalence class of $q \in \mathbb{Z}_n$ contains the lens space $L(n,q)$.

**Corollary 3.10.** ($H_1 \cong \mathbb{Z}_p$, $p$ prime): Any 3-manifold with $H_1 \cong \mathbb{Z}_2$ is surgery equivalent to $\mathbb{R}P(3)$. For any odd prime there are precisely two surgery equivalence classes represented by $L(p,1)$ and $L(p,q)$ where $q \not\equiv k^2 \mod p$. Hence if $p \equiv 3 \mod 4$ then we may take $q = -1$ and we see that $L(p,1)$ and $-L(p,1)$ are not surgery equivalent. If $p \equiv 1 \mod 4$ then $-L(p,1)$ is surgery equivalent to $L(p,1)$.

**Corollary 3.12.** If $\pi_1(M_0) \cong \pi_1(M_1)$ is abelian then $M_0$ is (integral homology) surgery equivalent to $M_1$ if and only if $M_0$ is orientation-preserving homotopy equivalent to $M_1$.

**Proposition 3.17.** Let $M_0$, $M_1$ be oriented, connected 3-manifolds. Then the following are equivalent:

A. $M_0$ is 2-surgery equivalent to $M_1$
B. There is a framed boundary link $L$ in $M_0$ such that Dehn surgery (with framings $\pm 1$) on $L$ yields $M_1$.

C. $M_0$ and $M_1$ have isomorphic linking forms and triple cup product forms (in the sense of 3.1).

Once again the result for homology spheres, where $C$ is vacuous, was known. The earliest reference we can find is S.V. Matveev [Ma; Theorem A]. This result was reproved and used by S. Garoufalidis [Ga].

Finally let $K$ be the subgroup of the mapping class group generated by Dehn twists along simple closed curves which bound a subsurface. Then, building on the observations of [GL] we have the following generalization of a theorem of Morita for homology spheres [Mo1; Proposition 2.3] [Jo1] [Mo2].

**Theorem 3.18.** Let $M_0$, $M_1$ be closed oriented 3-manifolds. Then the following are equivalent.

A. $M_0$ and $M_1$ are 2-surgery equivalent.

B. There exist Heegard splittings $M_0 = H_1 \cup_f H_2$, $M_1 = H_1 \cup_{\psi \circ f} H_2$ where $\psi \in K$.

C. $M_0$ and $M_1$ have isomorphic linking forms and triple cup product forms (as in 3.1).

The rational case for $k = 2$ is controlled by $H_1(M; \mathbb{Z})$/Torsion and the integral triple cup product form (Theorem 5.1). Here are a few sample results concerning rational homology surgery equivalence.

**Corollary 5.2.** If $m < 3$ any 2 closed, oriented 3-manifolds with identical first Betti number $m$ are rational homology surgery equivalent.

**Corollary 5.3.** There is a bijection $S_m^Q \to \Lambda^3(\mathbb{Z}^m)/\text{GL}_m(\mathbb{Z})$ given by the integral triple cup product form. Hence if $M_0$, $M_1$ have torsion-free homology groups then they are
rational homology surgery equivalent if and only if they are integral homology surgery equivalent.

The situation for higher $k$ is controlled by the above and also higher order Massey products (see §6).

Cases ii and iii above do not lead to equivalence relations and will be considered in a future paper.

This paper is organized as follows:

§1. Introduction

§2. Preliminaries

§3. Integral Homology Surgery Equivalence of Closed 3-Manifolds

§4. Proofs of Theorem 3.1 and other Basic Theorems

§5. Rational Homology Surgery Equivalence of Closed 3-Manifolds

§6. Surgery Equivalence Preserving Lower Central Series Quotients

§2. Preliminaries. In this section we prove several important technical results. In particular we show that the surgery relations having to do with the lower central series are in fact equivalence relations, but that symmetry fails in general. We also show that in these cases we may safely assume our surgeries are “integral” surgeries.

**Proposition 2.1.** Suppose $\gamma$ is a simple closed curve in $M_0$ which is null-homologous (or merely zero in $H_1(M_0; \mathbb{Q})$ in the $\mathbb{Q}$ case). Suppose $M_1$ is the result of $\pm 1/q$ surgery along $\gamma$ (non-longitudinal surgery in the $\mathbb{Q}$ case) and $\gamma'$ is the meridian of $\gamma$ viewed as a curve in $\pi_1(M_1)$. If $\gamma \in (\pi_1(M_0))_k$ for some $2 \leq k < \omega$ then $\gamma' \in (\pi_1(M_1))_k$ (in the $\mathbb{Q}$ case the same holds using the rational lower central series).

**Proof of 2.1.** Let $N = N(\gamma)$ be the solid torus regular neighborhood of $\gamma$ and $T$ its boundary. First we treat the integral case. Let $G = \pi_1(M_0), P = \pi_1(M_1)$, and suppose $\gamma \in G_k$. Then the longitude $\ell(\gamma)$ also lies in $G_k$ and we know that there exists an immersed
(k − 1)-stage half grope \( f : (S_1, S_2, \ldots, S_{k-1}) \cong M_0 \) whose boundary is \( \ell(\gamma)[FQ] \). Here we mean (as usual) that \( S_1 \) is a connected surface with \( f \mid_{\partial S_1} = \gamma \) and that \( S_i \) is a collection of connected surfaces \( S_{ij} \) such that \( f \mid_{\partial S_{ij}} \) is \( f(a_{ij}) \) where \( a_{ij} \) is a simple closed curve on \( S_{i-1} \). Moreover each stage \( S_{i-1} \) has a 1/2-rank system of such curves \( a_{ij} \) which occur as boundaries of \( S_i \). We assume that the immersion of \( S_1 \), when restricted to a collar of \( S_1 \), is an embedding whose image lies in \( M_0 − \hat{N}(\gamma) \). It follows that \( S_1 \) intersects \( \gamma \) transversely an algebraically zero number of times since \( \ell(\gamma) = k\mu(\gamma) \) in \( H_1(M_0 − \hat{N}(\gamma)) \) has only the solution \( k = 0 \). Recall that \( \gamma \in G_2 \) implies \( \ell(\gamma) \) is null-homologous in \( M_0 − \hat{N}(\gamma) \) and hence \( \ell(\gamma) \in P_2 \).

By general position we also assume that all \( a_{ij} \) lie in \( M_0 − N \) and that each \( S_i \) meets \( \gamma \) transversely. Since \( \gamma' = \mu(\gamma) \) is freely homotopic to \( \pm q\ell(\gamma) \) in \( M_1 \), it follows that \( \gamma' \) lies in \( P_2 \) as well. Suppose now, by induction, that \( \ell(\gamma) \in P_{n-1} \) for some 3 ≤ \( n \) ≤ \( k \). We shall show that \( \ell(\gamma) \in P_n \) which will complete the proof in the integral case. Note that the induction hypotheses implies that \( \mu(\gamma) \in P_{n-1} \). It follows that \( \partial S_{2j} \), for any stage 2 surface, lies in \( P_{n-1} \), because \( \partial S_{2j} \) bounds a \( (k - 2) \)-stage half grope in \( M_0 \) which lies completely in \( M_0 − \text{int } N \subseteq M_1 \) except for a collection of small 2-disks corresponding to the transverse intersections with \( \gamma \). Hence \( \partial S_{2j} \) lies in \( P_{k-1} \) modulo a product of conjugates of \( \mu(\gamma) \), which itself lies in \( P_{n-1} \).

Now delete the algebraically-zero number of 2-disks of intersection of \( S_1 \) with \( N \) and tube along \( T \) to get \( S_1^* \) in \( M_1 − \text{int } N \). Then we see that \( \ell(\gamma) \) is congruent, modulo \( P_n \), to a product of conjugates of elements of the form \( [x_i, \mu(\gamma)]^{\pm 1} \). Since \( \mu(\gamma) \in P_{n-1} \), \( \ell(\gamma) \in P_n \).

Now we address the “rational case.” We suppose that \( \gamma \in G_k^Q \) and hence \( \ell(\gamma) \in G_k^Q \). Then there exists a “rational” \( (k − 1) \)-stage half-grope whose “boundary” is \( \ell(\gamma) \). By this we mean that \( \partial S_1 = n_1\ell(\gamma) \) for some positive integer \( n_1 \) and similarly \( \partial S_{ij} = n_{ij}a_{ij} \). Again we conclude that \( S_1 \cdot \gamma \) is algebraically zero since the equation \( n_1\ell(\gamma) = m\mu(\gamma) \) in \( H_1(M_0 − \hat{N}(\gamma); \mathbb{Z}) \) has only the solution \( m = 0 \) since \( \ell(\gamma) \) is torsion while \( \mu(\gamma) \) is not. We
claim that $\gamma' = \mu(\gamma)$ is “rationally related” to $\ell(\gamma)$ in $P$, that is that there exist integers $x, y$ such that $(\mu(\gamma))^x = (\ell(\gamma))^y$ where $x \neq 0$. To see this, consider the (abelian) subgroup $T$ of $P$ generated by $\mu$ and the parallel $\rho$. Suppose $\ell(\gamma) = a\mu + b\rho$ where $(a, b) = 1$. Then $T$ is a quotient of the abelian group $A = \langle \mu, \rho | p\mu + q\rho = 0 \rangle$ and contains $\ell(\gamma)$. The vectors $(p, q)$ and $(a, b)$ are linearly independent in $\mathbb{Z} \times \mathbb{Z}$ since they are primitive and $(p, q) \neq \pm (a, b)$ since our surgery is not longitudinal. Hence $A/\langle \ell(\gamma) \rangle$ is a finite group and thus there are integers $x, y$ $x \neq 0$ such that $x\mu = y\ell(\gamma)$ in $A$ and hence in $T$. Using this relation, the proof now proceeds as in the integral case. □

**Corollary 2.2.** The relation of $k$-surgery equivalence is an equivalence relation on the set of oriented 3-manifolds. The relation of rational $k$-surgery equivalence is also an equivalence relation (Here we mean the rational lower central series and non-longitudinal surgery as in example v).

**Proof of 2.2.** Reflexivity and transitivity are obvious and symmetry is guaranteed by 2.1.

**Proposition 2.3.** If $M_0$ and $M_1$ are $k$-surgery equivalent (respectively rationally $k$-surgery equivalent) then they are so equivalent using only integral surgeries, that is $\pm 1$ surgeries (respectively integral non-longitudinal surgeries). In generality, if $M_1$ is $N$-surgery related to $M_0$ then there is an epimorphism $\pi_1(M_1) \twoheadrightarrow \pi_1(M_0)/N(\pi_1(M_0))$ (in either $\mathbb{Z}$ or $\mathbb{Q}$ case of Definition 1.1). Consequently if $M_1$ is rationally 2-surgery equivalent to $M_0$ then $\beta_1(M_0) = \beta_1(M_1)$. If $M_1$ is integrally 2-surgery equivalent to $M_0$ then $H_1(M_0; \mathbb{Z}) \cong H_1(M_1; \mathbb{Z})$.

**Proof of 2.3.** The sequence of homeomorphisms shown in Figure 2.4 using the “Rolfsen-Kirby” calculus is well-known (see [CG; p. 501] [R; p. ]). This shows that $1/n$ surgery on $\gamma$ is the same as a sequence of $\pm 1$ surgeries on parallel copies of $\gamma$, denoted $\gamma_1, \gamma_2, \ldots, \gamma_n$. Let $M_i^*$ be the result of surgery on $\{\gamma_1, \gamma_2, \ldots, \gamma_i\}$. We may view $\gamma_{i+1}$ as the longitude of
\( \gamma_i \) and 2.1 guarantees that if \( \gamma_i \) lies in the \( k \)-th term of the lower central series of \( \pi_1(M_{i-1}^*) \)
then \( \mu(\gamma_i) \) and \( \ell(\gamma_i) \) lie in \( (\pi_1(M_{i}^*))_k \). Hence \( M_n^* \) is \( k \)-surgery equivalent to \( M_0 \) via \(+1\) surgeries as claimed.

\textbf{Figure 2.4}

Now consider the case that \( M_1 \) is rationally \( k \)-surgery equivalent to \( M_0 \) via a single surgery on \( \gamma \) in \( M_0 \). Since \( M_0 = S^3_j \) for some framed link \( J \) in \( S^3 \), which may be assumed to be disjoint from \( \gamma \), we may consider \( \gamma \subset S^3 \) with framing \( p/q \) with respect to the longitude of \( \gamma \) in \( S^3 \). Suppose the surgery is \textit{not} integral, i.e., \( q \neq 1 \) (we assume \( q > 0 \)). Then \( p/q = \pm (m + \frac{r}{q}) = \pm ((m + 1) - \frac{q-r}{q}) \) where \( m = \lfloor p/q \rfloor \) and \( 0 < r < q \). Then it is well known that the 3 pictures of Figure 2.5 are homeomorphic [Rolfsen, ], where the upper sign is used if \( p \geq 0 \). Here the framings are all relative to \( S^3 \). Since \( 0 < r < q \) and \( 0 < q - r < q \), we may use 2.5 as the basis of an induction to reduce all surgeries in a daisy-chain of circles to integers (get \( q = 1 \)). Since the first circle \( \gamma \) lies in \( (\pi_1(M_0))^Q_k \), it suffices to show that the second curve in 2.5, say \( \gamma_2 \), lies in \( G^Q_k \) where \( G = \pi_1(M^*) \), \( M^* \) being the result of \( \pm m \) or \( \pm (m + 1) \) surgery on \( \gamma \). In addition we must show that the \( \pm m \) (or \( \pm (m + 1) \)) surgery is integral and non-longitudinal and show the surgery on \( \gamma_2 \) is also non-longitudinal. Firstly, \textit{at most one} of \( \pm m \) or \( \pm (m + 1) \) could be longitudinal with respect to \( M_0 \) so we may choose the non-longitudinal one. Moreover the surgery on \( \gamma \) is integral means that the defining torus homeomorphism extends over a solid torus. But this is independent of coordinate system so the fact that \( \pm m, \pm (m + 1) \) are integers suffices to show these surgeries are integral relative to \( M^* \).
Now, as has been mentioned, and is an immediate consequence of the second part of 2.3 (which is proved below), rational $k$-surgery equivalence preserves $H_1/(\text{torsion})$. Thus $\beta_1(M_0) = \beta_1(M^*) = \beta_1(M_1)$. This implies that the framing on $\gamma_2$ relative to $M^*$ is non-longitudinal since it is precisely the longitudinal surgery which changes $\beta_1$.

In general, if $M_1$ is $N$-surgery related to $M_0$ via a single $\pm 1/n$ surgery then Figure 2.4 shows that there is a link $L$ of $n$ components in $M_0$, each component of which lies in $N(\pi_1(M_0))$, such that adding $n$ two-handles to $M_0 \times [0,1]$ along $L$ yields a cobordism $W$ between $M_0$ and $M_1$ rel $\partial M_0 = \partial M_1$. But then both inclusion maps induce epimorphisms on $\pi_1$ and the kernel of $\pi_1(M_0) \to \pi_1(W)$ lies in $N$. Thus $\pi_1(W)/N(\pi_1(W)) \cong \pi_1(M_0)/N(\pi_1(M_0))$ and the second claimed result follows easily in the $\mathbb{Z}$ case. In the $\mathbb{Q}$-case, if $M_1$ is $N$-related to $M_0$ via a single non-longitudinal surgery then 2.5 shows that there is a cobordism $W$ as above and a link $L$ each of whose components lies in $N(\pi_1(M_0))$ (in fact most are null-homotopic!). Then the argument above for the $\mathbb{Z}$-case holds. Note that if $M_1$ is rationally 2-surgery equivalent to $M_0$ then $\pi_1(M_1)$ maps onto the free abelian group $\mathbb{Z}^{\beta_1(M_0)} = \pi_1(M_0)/(\pi_1(M_0))_2^\mathbb{Q}$. Hence $\beta_1(M_1) \geq \beta_1(M_0)$. But by symmetry (2.1) $\beta_1(M_0) = \beta_1(M_1)$ and consequently $\pi_1(M_1)/(\pi_1(M_1))_2^\mathbb{Q} \cong \mathbb{Z}^{\beta_1(M_0)}$. \qed

**Example 2.6:** It is important to note that “$N$-surgery related” is not a symmetric relation if $N = \{e\}$ or $N = G''$. In a later paper we will consider strengthening the $N$-surgery relation to force symmetry. Figure 2.7a shows a “Kirby calculus” description of $M_0 = S^1 \times S^2$ with a dashed curve $\gamma$ which is clearly null-homotopic in $M_0$. Yet $+1$ surgery
along $\gamma$ yields $M_1$ which (since the Whitehead link is symmetric) is homeomorphic to the manifolds of Figure 2.7b and 2.7c. Hence $M_1$ is 0-surgery on a left-handed trefoil knot and the loop $\gamma'$ is neither null-homotopic in $M_1$ nor even in $(\pi_1(M_1))''$. If $M_0$ were $N$-surgery related to $M_1$ for $N = G''$ (or $\{e\}$) then by 2.3 there would exist an epimorphism from $Z = \pi_1(M_0)$ to $\pi_1(M_1)/N(\pi_1(M_1))$. Since the trefoil knot has non-trivial Alexander module, this is not possible. One also sees that, in the case $N = \{e\}$, forcing symmetry would force $\pi_1$ itself to be preserved (since 3-manifold groups are Hopfian) and perhaps this is too strong to be of interest.

We have defined our “equivalence” relations to be generated by single Dehn surgeries. It is also possible to define relations generated by surgeries on certain types of links. These are sometimes equivalent notions as the following show. The proof of the first is elementary and left to the reader.

**Proposition 2.8.** The following are equivalent.

1. $M_0$ and $M_1$ are 2-surgery equivalent.

2. There is a link $L = \{L_1, \ldots, L_m\}$ in $M_0$ with null-homologous components, each framed $\pm 1$, with $\ell k(L_i, L_j) = 0$ so that $M_1$ is the result of surgery along $L$. In other words, the “linking matrix” of $L$ is invertible over $\mathbb{Z}$ and diagonal.

**Proposition 2.9.** The following are equivalent.

1. $M_0$ and $M_1$ are rationally 2-surgery equivalent.

2. There is a framed link $L = \{L_1, \ldots, L_m\}$ in $M_0$, each component of which is
rationally null-homologous in $M_0$, such that the “linking matrix” of $L$ is non-singular over $\mathbb{Q}$ and such that $M_1$ is obtained by surgery on $L$.

Here, by “linking matrix” of the framed link $L = \{\gamma_1, \ldots, \gamma_m\}$ we mean the matrix over $\mathbb{Q}$ given by $v_{ij} = \ell k(\rho_i, \gamma_j)$ where $\rho_i$ here is the circle on $\partial N(\gamma_i)$ which bounds the meridional disk in the surgery solid torus ($p_i\mu_i + q_i\rho_i$ in the notation of §1). The proof of 2.9 is given after the proof of Theorem 4.2.

§3. Integral Homology Surgery Equivalence of Closed 3-manifolds. In this chapter we give a comprehensive treatment of 2-surgery equivalence of closed 3-manifolds. This is precisely the equivalence relation generated by Dehn surgeries which preserve integral homology and thus was called \textit{HTS-equivalence} (homologically trivial surgery) by Kuperberg [Ku] and Gerges [Ge]. This equivalence relation is perhaps the most basic and important. It forms the basis of the philosophy of Cochran and P. Melvin in their theory of finite type invariants for arbitrary 3-manifolds [CM]. The question of characterizing 2-equivalence was asked by Kuperberg and answered by Gerges in his Ph.D. thesis. Here we prove a sharper theorem. Our characterization theorem says that $M_0$ and $M_1$ are HTS equivalent precisely when they have the same $H_1$ and some isomorphism induces an isomorphism of $\mathbb{Q}/\mathbb{Z}$ linking forms and that part of the cohomology ring coming from triple cup products. In the next chapter we will prove the characterization theorem, which appears in Gerges [Ge] without the relation to the linking form. In this chapter we will discuss examples, invariants and representatives for the 2-equivalence classes.

Before stating the theorem, we set up some notation. We let $K(H_1(M_0),1)$ be the usual Eilenberg-Maclane space with fundamental group $H_1(M_0)$. We can build this space from $M_0$ by adding cells of dimension greater than 1 and we let $f_0 : M_0 \rightarrow K(H_1(M_0),1)$ denote this inclusion. Then if $\phi_1 : H_1(M_1) \rightarrow H_1(M_0)$ is any isomorphism, there is a unique homotopy class $f_1 : M_1 \rightarrow K(H_1(M_0),1)$ inducing $\phi_1$ on $H_1$. Let $B : H^1(\_; \mathbb{Z}_n) \rightarrow H^2(\_; \mathbb{Z})$ denote the Bockstein operator associated with the short exact sequence $0 \rightarrow \cdots$
The following theorem in the case of homology 3-spheres was certainly known to and used by Andrew Casson in public lectures at M.S.R.I. in 1985. We are informed that it was known even earlier. In this case it says merely that any two oriented homology 3-spheres are 2-surgery equivalent. This case also appeared in a 1987 paper of S.V. Matveev. In the latter, moreover, it is proved that two 3-manifolds have isomorphic $H_1$ and linking forms if and only if they are related by “Borromean surgeries,” a result clearly close in spirit to our final one [Ma, Theorem 2 and Remark 2].

The equivalence of B and (a slightly stronger version of) D is claimed in passing in [Tu1], but no proof is offered.

**Theorem 3.1.** (see [Ge]) Suppose $M_0$ and $M_1$ are closed, oriented, connected 3-manifolds. The following 4 conditions are equivalent.

A) $M_0$ and $M_1$ are 2-surgery equivalent, i.e., each can be obtained from the other by a sequence of $\pm 1$ surgeries (equivalently $\pm 1/q$ surgeries) or null-homologous circles.

B) There exists an isomorphism $\phi_1 : H_1(M_1) \to H_1(M_0)$ such that $(f_0)_*([M_0]) = (f_1)_*([M_1])$ in $H_3(H_1(M_0); \mathbb{Z})$ where $f_0$, $f_1$ are as above. In brief one could also say that $M_0$ and $M_1$ have the same homology and are bordant over $K(H_1(M_0), 1)$ for some $f_i$.

C) There exists an isomorphism $\phi_1 : H_1(M_1) \to H_1(M_0)$ such that the set of induced maps $\phi_n^1 : H^1(M_0; \mathbb{Z}/n\mathbb{Z}) \to H^1(M_1; \mathbb{Z}/n\mathbb{Z})$ for $n = 0$ and for each $n = p^r$ where $p^r$ is the exponent of the $p$-torsion subgroup of $H_1(M_0; \mathbb{Z})$ (all elements of order some power of the prime $p$) satisfies the following:

   a) $\langle \alpha \cup \beta \cup \gamma, [M_0] \rangle = \langle \phi_n^1(\alpha) \cup \phi_n^1(\beta) \cup \phi_n^1(\gamma), [M_1] \rangle$ where $\alpha, \beta, \gamma \in H^1(M_0; \mathbb{Z}/n\mathbb{Z})$ and $[M_i]$ denotes the fundamental class in $H_3(M_i; \mathbb{Z}/n\mathbb{Z})$,

   b) $\langle \alpha \cup \tau_*B(\gamma), [M_0] \rangle = \langle \phi_n^1(\alpha) \cup \tau_*B(\phi_n^1(\gamma))\rangle, [M_1] \rangle$ where $\alpha, \gamma, B$ are as above, but $n \neq 0$, and $\tau : H^2(M_i; \mathbb{Z}) \to H^2(M_i; \mathbb{Z}_n)$. 

D) The same condition as C with b) replaced by c) If \( \lambda_i \) represent the \( \mathbb{Q}/\mathbb{Z} \) linking forms on \( T(H_1(M_i)) \) then \( \lambda_1(x, y) = \lambda_0(\phi_1(x), \phi_1(y)) \) for all \( x, y \in T(H_1(M_1)) \), that is to say that \( \phi_1 \) induces an isomorphism between \( \lambda_0 \) and \( \lambda_1 \).

Let \( A \) be a finitely generated abelian group and \( A^*_n = \text{Hom}(A; \mathbb{Z}_n) \equiv H^1(A; \mathbb{Z}_n) \) for \( n = 0 \) or \( n = p^r \) (the exponent of the \( p \)-torsion subgroup of \( A \)). Consider a set of skew-symmetric trilinear forms \( u_n : A^*_n \times A^*_n \times A^*_n \to \mathbb{Z}_n \), where \( n \) ranges over \( \{0, p^r\} \) as above, which are compatible in the sense of [Tu2; Definition 1.2]. Let \( \lambda : \text{Torsion } A \times \text{Torsion } A \to \mathbb{Q}/\mathbb{BZ} \) be a non-degenerate symmetric bilinear form. Any automorphism \( \phi : A \to A \) induces isomorphic forms \( \{\phi^*(u_n)\} \) and \( \{\phi_*\lambda\} \) given by \( \phi^*(u_n)(\alpha, \beta, \gamma) = u_n(\phi^*_n(\alpha), \phi^*_n(\beta), \phi^*_n(\gamma)) \) where \( \phi^*_n : A^*_n \to A^*_n \) and \( \phi_*\lambda(x, y) = \lambda(\phi^{-1}x, \phi^{-1}y) \). Given any oriented 3-manifold with \( H_1 \cong A \), the triple cup product forms and linking form yield a pair \( \{u_n\}, \lambda \) which is well-defined up to isomorphism. Let \( S(A) \) be the set of isomorphism classes of such pairs which are realizable by a 3-manifold. In fact by [Tu2; Theorem 1] and [KK], any pair is realizable if \( A \) has no 2-torsion. In general there is a mild compatibility condition between \( u_{2r} \) and \( \lambda \). Hence Theorem 3.1 may be restated as follows.

**Theorem 3.1 (Restatement).** Let \( A \) be a finitely generated abelian group. The set of surgery equivalence classes of closed, oriented 3-manifolds with \( H_1 \cong A \) is in bijection with \( S(A) \).

**Example 3.2** \( (H_1 \cong \mathbb{Z}^m \ m < 3) \): Any two homology 3-spheres are surgery equivalent since \( B \) is trivially satisfied in this case. Indeed since \( H_3(\mathbb{Z}^m) = 0 \) if \( m < 3 \), any two 3-manifolds \( M_0, M_1 \) with \( H_1 \cong \mathbb{Z}^m \) are surgery equivalent if \( m < 3 \).

**Example 3.3** \( H_1 \cong \mathbb{Z}^3 \): If \( M_0 = \#_{i=1}^3 S^1 \times S^2 \) and \( M_1 = S^1 \times S^1 \times S^1 \) then \( M_0 \) is not surgery equivalent to \( M_1 \) because the image of \( M_0 \) in \( H_3(H_1(M_0)) \) is zero since
it factors through $H_3(\pi_1(M_0))$. But for any automorphism of $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$, the induced map $M_1 \to S^1 \times S^1 \times S^1 = K(H_1(M_0), 1)$ is of degree $\pm 1$ since the identity map is degree 1 and $\text{Aut}(H_3(\mathbb{Z}^3)) = \{\pm 1\}$. Hence condition B fails (equivalently condition C part a with $n = 0$). More generally, let $M_n$ be the 3-manifold shown in figure 3.4 as zero surgery on a link with $\mu(123) = n$. Then $M_n$ is surgery equivalent to $M_m$ if and only if $|n| = |m|$ since the triple cup product of the Hom-duals of the meridians is $n$ times the fundamental class in $H^3$. Any element of $\text{Aut}(\mathbb{Z}^3)$ induces an element of $\text{Aut}(H_3(\mathbb{Z}^3))$ and this correspondence is $P \to \det P$. Since $\det P = \pm 1$, the classes $n, m$ in $H_3(\mathbb{Z}^3)$ are equivalent under the action of $\text{GL}(3, \mathbb{Z})$ if and only if $n = \pm m$. Note that this implies that $M_0$ and $M_1$ are surgery equivalent if and only if they have identical Lescop invariant [Les]. This generalizes to give the following.

**Corollary 3.5.** Let $S_m$ be the set of surgery equivalence classes of closed oriented 3-manifolds with $H_1 \cong \mathbb{Z}^m$. Then there is a bijection $\psi_* : S_m \to \Lambda^3(\mathbb{Z}^m)/\text{GL}_m(\mathbb{Z})$ from $S_m$ to the set of orbits of the third exterior power of $\mathbb{Z}^m$ under the action induced by $\text{GL}_m(\mathbb{Z})$ on $\mathbb{Z}^m$. Any such manifold is surgery equivalent to one which is the result of 0-framed surgery on a $m$ component link in $S^3$ which is obtained from the trivial link by replacing a number of trivial 3-string braids by a 3-string braid whose closure is the Borromean rings.

**Proof of 3.5.** Fix an identification $H_3(\pi_1(M))^m \equiv H_3(\mathbb{Z}^m) \equiv \Lambda^3 \mathbb{Z}^m$. For any $M$ with $H_1 \cong \mathbb{Z}^m$ choose an isomorphism $\phi : H_1(M) \to \mathbb{Z}^m$. This induces a unique homotopy
class of maps $\psi: M \to (S^1)^m$. The image of the fundamental class of $M$ is the desired element $\psi_*([M])$. All possible isomorphisms $\phi$ may be achieved by post-composing a fixed $\phi$ with elements of $GL_m(\mathbb{Z})$. Then $\psi_*(M)$ is well-defined in the orbit space. If $M$ and $M'$ are surgery equivalent then condition B of 3.1 guarantees their images are the same. Hence $\psi_*$ is well-defined. $B \Rightarrow A$ implies that $\psi_*$ is injective. The surjectivity of $\psi_*$ follows from work of D. Sullivan [Su]. Alternatively, for any set of $\binom{m}{3}$ integers $\{a_{ijk} \mid 1 \leq i < j < k \leq m\}$ is it easy to construct an ordered link of $m$ components in $S^3$ such that $\mu(ijk) = a_{ijk}$ by the procedure described in the last sentence of the Corollary. If $M$ is zero surgery on this link then $\psi_*(M) = \sum a_{ijk}(e_i \wedge e_j \wedge e_k)$ with respect to a basis induced by the meridians (see Lemma 4.2 of [Tu2]). □

The structure of the set $\Lambda^3(\mathbb{Z}^m)/GL_m$ seems to be quite complicated for large $m$ and so the general decidability question for whether or not two 3-manifolds are surgery equivalent may not be easy. However since the 0 element is the only element in its orbit we have:

**Corollary 3.6.** A 3-manifold with $H_1 \cong \mathbb{Z}^m$ is surgery equivalent to $\#_{i=1}^m S^1 \times S^2$ if and only if its integral triple cup product form $H^1 \oplus H^1 \oplus H^1 \to \mathbb{Z}$ vanishes identically.

We also observe the following surprising result.

**Corollary 3.7.** The map $S_3 \xrightarrow{f} S_4$ given by $M \to M \# S^1 \times S^2$ is a bijection. Thus any $N$ with $H_1(N) \cong \mathbb{Z}^4$ is surgery equivalent to precisely one of $M_n \# S^1 \times S^2$ where $n \geq 0$ (see Figure 3.4).

**Proof of 3.7.** $\Lambda^3 \mathbb{Z}^4 \cong \Lambda^1 \mathbb{Z}^4 \cong \mathbb{Z}^4$ by duality. Thus $\Lambda^3 \mathbb{Z}^4/GL_4 \cong \mathbb{Z}^4/GL_4 \cong \mathbb{Z}_+ \cup \{0\}$. Under this bijection, $n(e_1 \wedge e_2 \wedge e_3)$ goes to $ne_4$ and the former is $\psi_*(M_n \# S^1 \times S^2)$. □

**Corollary 3.8.** If $H_1(M)$ is torsion-free then $M$ is surgery equivalent to $-M$.

**Proof of 3.8.** The element of $GL_m$ which reverses the order of a basis $\{e_1, \ldots, e_m\} \to \{e_m, \ldots, e_1\}$ induces $-1$ on $\Lambda^3 \mathbb{Z}^m$. □
Now consider that \( H_1 \cong \mathbb{Z}_n \). First consider the general question of which classes \( \mu \in H_3(A) \) can be realized as the image of the fundamental class of a 3-manifold \( M_0 \) under some map \( \pi_1(M_0) \to H_1(M_0) \xrightarrow{f} A \) where \( f \) is an isomorphism. This question has been answered by Turaev in great generality. The answer is that \( \mu \) is realizable if and only if \( x \mapsto x \cap \mu \) is an isomorphism \( \text{Tors} H^2(A) \to \text{Tors} H_1(A) \). Both groups are \( \mathbb{Z}_n \) in the case at hand. If we denote by \( \mu = 1 \in H_3(\mathbb{Z}_n) = \mathbb{Z}_n \) the image of the class of \( L(n, 1) \) under some map then certainly \( x \mapsto x \cap 1 \) is an isomorphism. A general class \( \mu = k \cdot 1 \) will induce the map \( x \mapsto k(x \cap 1) \) which is the composition of the isomorphism \( x \mapsto x \cap 1 \) with multiplication by \( k \) on \( \mathbb{Z}_n \). Hence \( k \in H_3(\mathbb{Z}_n) \) is realizable if and only if \( k \) is a unit in \( \mathbb{Z}_n \). Moreover by 3.1 B, two 3-manifolds \( M_0, M_1 \) with \( H_1 \cong \mathbb{Z}_n \) representing classes \( k_0, k_1 \) in \( H_3(\mathbb{Z}_n) \) with respect to some identifications \( H_1(M_i) \cong \mathbb{Z}_n \), will be surgery equivalent if and only if there is an automorphism of \( \mathbb{Z}_n \) which induces an automorphism of \( H_3(\mathbb{Z}_n) \) sending \( k \) to \( k_1 \). Multiplication by (a unit) \( m \) on \( \mathbb{Z}_n \) induces multiplication by \( m^2 \) on \( H_3(\mathbb{Z}_n) \) (see Proposition 3 of [Ru] and Theorem 29.5 of [Co]). Therefore we have derived the following.

**Corollary 3.9.** The set \( S(\mathbb{Z}_n) \) of surgery equivalence classes of closed oriented 3-manifolds with \( H_1 \cong \mathbb{Z}_n \) is in bijection with the set of equivalence classes of units of \( \mathbb{Z}_n \), modulo squares of units, the correspondence being given by the image of the fundamental class of the manifold in \( H_3(\mathbb{Z}_n) \equiv \mathbb{Z}_n \). The correspondence is also given by the self-linking linking number of a generator of \( H_1 \) (\( \lambda(1, 1) = q \) and \( q \) is viewed as an element of \( \mathbb{Z}_n \)). The equivalence class of \( q \in \mathbb{Z}_n \) contains the lens space \( L(n, q) \).

**Proof of 3.9.** The first statement is proved above using 3.1 B. For the second statement, use 3.1 D. Note that since \( H_1 \) is cyclic, the cup products \( H^1 \oplus H^1 \to H^2 \) must vanish by anticommutativity with any coefficients unless \( n \) is even. Therefore if \( n \) is odd, part a) of 3.1 D is vacuous. For even \( n \) the triple cup product form on a cyclic group is determined by the linking form [Tu2; Theorem 1], so in any case we need only consider condition c)
of 3.1 D. Clearly the linking form \( \lambda \) on a cyclic group is determined by \( \lambda(1,1) = \frac{a}{n} \) and \( a \in \mathbb{Z}_n \) is well-defined modulo squares of units. Moreover two such forms \( \lambda \) and \( \lambda' \) are isomorphic if and only if \( a \equiv a' \) modulo squares. For \( L(n,q) \), \( \lambda(1,1) = \frac{q}{n} \).

**Corollary 3.10.** \((H_1 \cong \mathbb{Z}_p, p \text{ prime}): \) Any 3-manifold with \( H_1 \cong \mathbb{Z}_2 \) is surgery equivalent to \( \mathbb{R}P(3) \). For any odd prime there are precisely two surgery equivalence class represented by \( L(p,1) \) and \( L(p,q) \) where \( q \not\equiv k^2 \mod p \). Hence if \( p \equiv 3 \mod 4 \) then we may take \( q = -1 \) and we see that \( L(p,1) \) and \( -L(p,1) \) are not surgery equivalent. If \( p \equiv 1 \mod 4 \) then \( -L(p,1) \) is surgery equivalent to \( L(p,1) \).

More generally we see that:

**Corollary 3.11.** If \( M_0 \) and \( M_1 \) are orientation-preserving homotopy equivalent then they are surgery equivalent.

**Proof of 3.11.** This is immediate from 3.1 B.

**Corollary 3.12.** If \( \pi_1(M_0) \cong \pi_1(M_1) \) is abelian then \( M_0 \) is surgery equivalent to \( M_1 \) if and only if \( M_0 \) is orientation-preserving homotopy equivalent to \( M_1 \).

**Proof of 3.12.** One implication follows from 3.11. Suppose \( M_1 \) is surgery equivalent to \( M_0 \). By Theorem 4.2 there is cobordism \( W \) from \( M_0 \) to \( M_1 \) built from \( M_0 \times [0,1] \) by adding two-handles attached along curves being in \( [\pi_1(M_0), \pi_1(M_0)] = 0 \). Hence \( W \simeq M_0 \vee S^2 \cdots \vee S^2 \), and there is a retraction \( r : W \to M_0 \). The inclusion \( M_1 \to W \) followed by \( r \) is a degree 1 map \( M_1 \to M_0 \) inducing an isomorphism on \( \pi_1 \) and all homology groups. We may assume the manifolds contain no fake 3-cells since these are irrelevant to the question of being homotopy equivalent. Since \( \pi_1 \) is abelian it is not a non-trivial free product so we may assume that \( \pi_2(M_0) = \pi_2(M_1) = 0 \) or that \( M_0 \cong M_1 \cong S^1 \times S^2 \). In the first case \( \pi_1 \) must be finite cyclic and then it is easy to see that \( f \) induces an isomorphism on \( \pi_3 \) by considering the universal cover of \( W \). Hence \( f \) is a degree 1 homotopy equivalence. \( \square \)
**Corollary 3.13.** \( L(n, q) \) is surgery equivalent to \( L(n, q') \) if and only if they are orientation-preserving homotopy equivalent, that is if \( qq' \equiv k^2 \mod n \) for some unit \( k \).

**Example 3.14** \((H_1 \cong \mathbb{Z} \times \mathbb{Z}_n)\): Since \( H_3(\mathbb{Z} \times \mathbb{Z}_n) \to H_3(\mathbb{Z}_n) \) is an isomorphism, the surgery equivalence class depends only on the linking form. From another point of view, since \( H^1(\mathbb{Z} \times \mathbb{Z}_n; \mathbb{Z}_m) \) is generated by 2-elements, the triple cup product forms vanish if \( m \) is odd and are determined by the linking form if \( m = 2^r \). Therefore each surgery equivalence class contains a representative of the form \( S^1 \times S^2 \# L(n, q) \) and the self-linking number of an element of order \( n \) in \( H_1(M) \), \( \lambda(1, 1) = \frac{a}{n} \) viewed as an element \( a \in \mathbb{Z}_n \) will distinguish the classes when viewed in the group of units modulo squares (as in the case \( H_1 \cong \mathbb{Z}_n \)).

**Example 3.15:** Some words of caution are in order. One must be careful in applying 3.1. There exist 3-manifolds which have isomorphic \( H_1 \), linking forms and integral triple cup product forms but are not surgery equivalent as detected by a \( \mathbb{Z}_p \) triple cup product form. Namely, let \( M_0 \) be \( \#_{i=1}^3 L(5, 1) \) and let \( M_1 \) be 5/1-surgery on each component of a Borromean Rings.

It is even possible that \( M_0 \) and \( M_1 \) have isomorphic linking forms and isomorphic triple cup product forms with all coefficients, yet *not* be surgery equivalent because the isomorphisms are not induced by the same isomorphism \( \phi \) on \( H_1 \)! Consider the manifolds in Figure 3.16. Since they have the same linking matrix (expand the 5/2 to a chain of integral surgeries if you like), their linking forms are isomorphic to \((1/5) \oplus (2/5)\) on \( \mathbb{Z}_5 \times \mathbb{Z}_5 \). The triple cup product form on integral \( H^1 \) is zero while the triple cup product forms on \( H^1(\_; \mathbb{Z}_5) \equiv (\mathbb{Z}_5)^4 \) are isomorphic by “swapping the meridians of the 5 and 5/2 knots.” But these isomorphisms are incompatible. We do not provide details.

We can relate surgery equivalence to two other geometric equivalence relations which have appeared in the literature. The first is concerned with Dehn surgery on links; the
second is concerned with Heegard splittings and mapping class groups.

Recall that, given $M_0$, any closed, oriented 3-manifold $M_1$ can be obtained by integral surgery on some framed link in $M_0$. If the links are restricted what can be said? Recall that a boundary link is a very special link with all linking numbers zero, namely one whose components bound disjoint Seifert surfaces. This makes sense in any 3-manifold. The following is mildly surprising.

**Proposition 3.17.** Let $M_0$, $M_1$ be oriented, connected 3-manifolds. Then the following are equivalent:

A. $M_0$ is 2-surgery equivalent to $M_1$

B. There is a framed boundary link $L$ in $M_0$ such that Dehn surgery (with framings $\pm 1$) on $L$ yields $M_1$.

C. $M_0$ and $M_1$ have isomorphic linking forms and triple cup product forms (in the sense of 3.1).

Once again the result for homology spheres, where C is vacuous, was known. The earliest reference we can find is S.V. Matveev [Ma; Theorem A]. This result was reproved and used by S. Garoufalidis [Ga].

**Proof of 3.17.** The equivalence of A and C is part of 3.1. B$\Rightarrow$A is almost immediate. One need only note that the remaining components of a boundary link remain null-homologous (use the same Seifert surface) after performing Dehn surgery on some
of its components. Similarly the framings remain $\pm 1$ because the longitude remains the same. Thus we need only establish $A \Rightarrow B$. By 2.8 we can assume that $M_1$ is the result of $\pm 1$ surgeries on an link in $M_0$ whose pairwise linking numbers are zero. By induction, suppose $M_1$ is $\pm 1$ surgery on a null-homologous knot $K$ in $S(M_0, \{L_1, \ldots, L_n\})$, the result of $\pm 1$-framed surgery on the boundary link $L$ in $M_0$, where $\ell k(K, L_i) = 0$ for all $i$. The following type of argument has been used by others to prove the case of homology spheres. It serves equally well in general. Let $L' \subseteq S$ be the link consisting of the cores of the surgery solid torii (so $S - L' \equiv M_0 - L$). We shall describe an isotopy of $K$ to $K'$ in $S$ (passing through $L'$) such that $L \cup K'$ is a boundary link in $M_0$. Consider a set $W = \{W_1, \ldots, W_n\}$ of disjoint Seifert surfaces for $L$ in $M_0 - L$ whose boundaries are longitudes. Since we may actually assume the homeomorphisms defining the Dehn surgeries carry longitudes to longitudes, these surfaces can be extended by adding annuli to $\hat{W} = \{\hat{W}_1, \ldots, \hat{W}_n\}$, disjoint Seifert surfaces in $S$ for the components of $L'$. Now $K$ bounds a surface $V$ in $M_0 - L \equiv S - L'$ because of the hypothesis on linking numbers. Hence $\partial \hat{W}_i \cap V = \phi$. $V$ has a 1-dimensional spine whose transverse intersections with $\hat{W}$ may be removed by isotopy merely by pushing over $\partial \hat{W}$. This isotopy extends to $V$. The resulting $K' = \partial V'$ forms a boundary link with $L$ since $V' \cap W \subseteq V' \cap \hat{W} = \phi$. Note that, in the presence of other components $K_2, K_3, \ldots$ such that $\ell k(K, K_i) = 0$, the isotopy can be chosen so as to preserve that latter fact. \[\square\]

The reader might find it interesting to compare this with Theorem B of [Ma] which maintains that $M_0$ and $M_1$ have isomorphic homology groups and linking forms if and only if $M_1$ can be obtained from $M_0$ by surgery on a “$T_0$-boundary link” (recently re-introduced by Garoufalidis and Levine who used the term “blink” [GL]). This was an improvement on a theorem of Hilden who showed that any homology 3-sphere can be obtained from $S^3$ by surgery on a blink [H].

Another way to describe 3-manifolds is by their Heegard splittings. Given $M_0$, by
choosing a Heegard splitting $M_0 = H_1 \cup_f H_2$, one can vary $f$ by composing with another homeomorphism $g$ and in doing so change the three-manifold. One can then ask if an arbitrary $M_1$ may be obtained in such a manner (it can by Lickorish’s theorem), or if it can be obtained using only $g$ taken from some subgroup of the group of homeomorphisms. In particular, if $\Gamma$ is the mapping class group for the closed, orientable surface of genus $g$ then we can consider the Torelli group $T$, and the Johnson group $K$ of $\Gamma$. The Torelli group is the subgroup generated by homeomorphisms inducing the identity map on $H_1$. The Johnson subgroup $K \subseteq T$ is the subgroup of $\Gamma$ generated by Dehn twists along simple closed curves in the surface which bound a subsurface [Jo1] [Mo1] [Mo2] [GL]. Then, building on the observations of [GL] and our own work we have the following generalization of the theorem of Morita for homology spheres [Mo1; Prop. 2.3].

**Theorem 3.18.** Let $M_0, M_1$ be closed oriented 3-manifolds. Then the following are equivalent.

A. $M_0$ and $M_1$ are 2-surgery equivalent.

B. There exist Heegard splittings $M_0 = H_1 \cup_f H_2$, $M_1 = H_1 \cup_{\psi \circ f} H_2$ where $\psi \in K$.

C. $M_0$ and $M_1$ have isomorphic linking forms and triple cup product forms (as in 3.1).

**Proof of 3.18.** The arguments in the first 3 paragraphs of section 2.3 of [GL], although given for homology spheres, suffice to show that 3.18B is equivalent to 3.17B. □

**Remark 3.19:** We have shown that $K$ corresponds to boundary links which, in turn, corresponds to homology surgery equivalence. Strangely, the seemingly more natural Torelli group corresponds to “blinks” (see [Ma] and [GL]) which corresponds to preserving only $H_1$ and the linking form. Combining the work of [Ma], [GL] and our present work yields an analogous theorem to this effect, with “homology surgery equivalence” changed so that the relation is generated by surgery on a 2-component blink rather than a knot.
§4. Proofs of Theorem 3.1 and other Basic Theorems. In this section we prove Theorem 3.1. Several major components of the proof are derived in much greater generality so that they can be employed in later sections.

A ⇒ B: We will prove a more general result which will be useful later.

Proposition 4.1. Suppose $M_1$ is obtained from $M_0$ by ±1 surgeries on a link $\{\gamma_1, \ldots, \gamma_n\}$ where $\gamma_i \in N(\pi_1(M_0))$ and the meridians $\gamma'_i \in N(\pi_1(M_1))$. Then there exists an isomorphism $\phi : \pi_1(M_1)/N(\pi_1(M_1)) \to \pi_1(M_0)/N(\pi_1(M_0))$ such that $(f_0)_*([M_0]) = (f_1)_*([M_1])$ in $H_3(\pi_1(M_0)/N(\pi_1(M_0)); \mathbb{Z})$ where $f_1$ induces $\phi$ on $\pi_1$ and $f_0$ is the “natural inclusion.” Equivalently, there exists an $f_1$, inducing an isomorphism $\phi$, such that $(M_0, f_0)$ and $(M_1, f_1)$ are bordant over $K(\pi_1(M_0)/N(\pi_1(M_0)), 1)$.

Proof of 4.1. Let $W$ be the usual cobordism from $M_0$ to $M_1$ obtained by attaching 2-handles to the $\gamma_i$ in $M_0 \times \{1\} \subset M_0 \times [0, 1]$. The curves $\gamma'_i$ are the attaching circles of the dual 2-handles attached to $M_1$. Hence the inclusions $j_0, j_1$ induce isomorphisms $(j_0)_*, (j_1)_*$ on $\pi_1$ “modulo the $N$-subgroup.” Let $\phi = (j_0)^{-1} \circ (j_1)_*$. Let $\psi : \pi_1(W)/N(\pi_1(W)) \to \pi_1(M_0)/N(\pi_1(M_0))$ be $(j_0)^{-1}$. Then there are continuous maps $f_0, F$ and $f_1$ from $M_0, W, M_1$ respectively inducing the projection, $\psi$ and $\phi$ respectively on $\pi_1$ and such that $F$ extends $f_i$. The result follows. □

To see that 4.1 implies $A \Rightarrow B$, we apply 2.3 to reduce to the case of ±1 surgeries, we note that it suffices to prove $A \Rightarrow B$ for the case of a single ±1 surgery, then appeal to 2.1 to see that the hypotheses of 4.1 are satisfies.

A ⇒ D: It suffices to consider the case that $M_0$ and $M_1$ are cobordant via a single 2-handle attached with ±1 framing along a circle $\gamma$ which is null-homologous in $M_0$. Let $\phi_1 = (j_0)^{-1} \circ (j_1)_*$ be the isomorphism on $H_1$ induced by the inclusions. Then $\phi_1$ algebraically induces $\phi^1_n$ in $C$ and the key point is to observe that in this case $\phi^1_n$ equals
$j_1^* \circ (j_0^*)^{-1}$ where $j_1^*$ are the isomorphisms on $H^1(\_; \mathbb{Z}_n)$ induced by the inclusions. Hence for any $\alpha, \beta, \gamma$ in $H^1(M_0; \mathbb{Z}_n)$, we have

$$\langle \phi_1^*(\alpha) \cup \phi_1^*(\beta) \cup \phi_1^*(\gamma), [M_1] \rangle = \langle (j_0^*)^{-1}(\alpha) \cup (j_0^*)^{-1}(\beta) \cup (j_0^*)^{-1}(\gamma), (j_1)_*(\langle [M_1] \rangle) \rangle$$

since $j_1$ is a continuous map. But $(j_0)_*(\langle [M_0] \rangle) = (j_1)_*(\langle [M_1] \rangle)$ for any coefficients so the above expression equals $\langle \alpha \cup \beta \cup \gamma, [M_0] \rangle$ as desired. This shows condition a) of D and C. Now we demonstrate that $\lambda_1(\phi_1^{-1}x, \phi_1^{-1}y) = \lambda_0(x, y)$ for all torsion classes $x, y$ in $H_1(M_0; \mathbb{Z})$.

We may choose circles $\tilde{x}, \tilde{y}$ in $M_0$ to represent these classes so that they are disjoint from $\gamma$ and are in fact disjoint from a Seifert surface $S$ for $\ell(\gamma)$. This is true because if $\tilde{x}$ hits $S$, we are free to isotope $\tilde{x}$ “through $\gamma$” to achieve that the algebraic number of such intersections is 0. Then modify $S$ to miss $\tilde{x}$. Choose an integer $n$ and a 2-chain $d$ of $M_0$ such that $\partial d = n\tilde{x}$ and such that $d$ meets $\tilde{y}$ and $\gamma$ transversely. Then $\lambda_0(x, y)$ equals $\frac{1}{n}$ times $\#(\tilde{y} \cdot d)$. Now note that $\tilde{x}$ and $\tilde{y}$ are perfectly good representatives of $\phi_1^{-1}(x)$ and $\phi_1^{-1}(y)$ since $\phi_1$ is induced by the inclusions. We construct $d'$, a 2-chain in $M_1$ such that $\partial d' = \partial d = nx'$. For each intersection of $d$ with $\gamma$, delete the 2-disk $d \cap N(\gamma)$ and replace it by a copy of the annulus in the surgery solid torus which expresses the fact that $\mu(\gamma)$ is isotopic to $\pm \ell(\gamma)$ after surgery, and a copy of $\pm S$. This 2-chain $d'$ lies in $M_1$, has the same boundary as $d$ and $d' \cdot \tilde{y} = d \cdot \tilde{y}$ since $\tilde{y} \subseteq M_0 - N(\gamma)$ and $\tilde{y}$ is disjoint from $S$. Hence $\lambda_1(\phi_1^{-1}(x), \phi_1^{-1}(y)) = \lambda_0(x, y)$. □

B ⇒ A: We will prove a significantly broader result than is necessary in order to use it in later sections.

**Theorem 4.2.** Suppose $M_0$ and $M_1$ are closed, oriented 3-manifolds. Suppose $N \trianglelefteq \pi_1(M_0)$ is the normal closure of a finite number of elements and that $N$ is contained in the commutator subgroup of $\pi_1(M_0)$ (or merely in $(\pi_1(M_0))^Q$ in the rational case). Suppose there exists an epimorphism $\phi : \pi_1(M_1) \to \pi_1(M_0)/N$ which induces an isomorphism on $H_1(\_; \mathbb{Z})$ (or merely $H_1(\_; Q)$ in the rational case) such that $(f_0)_*(\langle [M_0] \rangle) = (f_1)_*(\langle [M_0] \rangle)$ in

25
\[ H_3(\pi_1(M_0)/N; \mathbb{Z}) \] (here \( f_1 \) is induced by \( \phi \) as usual, and \( f_0 \) induced by the inclusion into \( K(\pi_1(M_0)/N, 1) \)). Then \( (M_0, f_0) = (M_1, f_1) \) in \( \Omega_3(K(\pi_1(M_0)/N, 1)) \) via a 4-manifold with only 2-handles (rel \( M_0 \)) whose attaching circles lie in \( N \) and whose linking matrix with respect to \( M_0 \) is diagonal and invertible over \( \mathbb{Z} \) (merely invertible over \( \mathbb{Q} \) in the rational case). Consequently \( M_0 \) is obtained from \( M_1 \) by \( \pm 1 \) surgeries (integral surgeries in the rational case) on a link \( \{ \gamma_1, \ldots, \gamma_n \} \) such that \( [\gamma_i] \in N \). This link may be ordered in such a way that the sequence of these surgeries exhibits that \( M_1 \) is \( N \)-surgery related to \( M_0 \). Conversely \( M_1 \) is obtainable from \( M_0 \) in a similar manner by surgery on a link \( \{ \gamma'_i \} \) and that \( \ker \phi \) is the normal subgroup of \( \pi_1(M_1) \) generated by \( \{ \gamma'_i \} \). Example 2.6 shows that this cannot, in general, be strengthened.

Before proving 4.2, we should check that it implies \( B \Rightarrow A \). Apply 4.2 with \( N = (\pi_1(M_0))_2 \) to get the link \( \{ \gamma_1, \ldots, \gamma_n \} \). Apply 2.8. Hence \( M_1 \) is 2-equivalent via integral surgeries to \( M_0 \).

**Proof of 4.2.** Let \( X = K(\pi_1(M_0)/N, 1) \) which we may think of as constructed by adjoining cells to \( M_0 \). Then we have natural maps \( f_0 : M_0 \to X \) and \( f_1 : M_1 \to X \) such that \( (f_1)_* = \phi \). It is well known that the map from \( \Omega_3(X) \to H_3(X) \) is an isomorphism given by the image of the fundamental class. Thus the hypotheses guarantee that there is a compact oriented 4-manifold \( W \) and a map \( F : W \to X \) such that \( \partial(W, F) = (M_1, f_1) \amalg (-M_0, f_0) \). Since \( F_* \) is necessarily surjective on \( \pi_1 \), we may perform surgery on circles in \( W \) and assume \( F_* \) is an isomorphism.

Choose a handlebody structure of \( W \) rel \( M_0 \) with no handles of index 0 or 4. We may then proceed to “trade” 1-handles for 2-handles as in [Ki2, pp. 6–7, p. 247]. This may also be thought of as performing a surgery on the interior of \( W \) along a circle \( c \) passing over the 1-handle. Since \( (f_0)_* \) is an epimorphism on \( \pi_1 \), these circles may be altered by loops in \( M_0 \) so that \( F_*(c) = 0 \) (then in fact these loops are null-homotopic) and hence the map \( F \) extends to the “new” \( W \). Since \( \phi_* = (f_1)_* \) is surjective we may trade all 3-handles for
2-handles, by viewing then as 1-handles attached to \( M_1 \).

Now let \( V \) be the “linking matrix” of the attaching maps of the 2-handles rel \( M_0 \). By this we mean the following. If \( \{ \gamma_1, \ldots, \gamma_m \} \) denote the attaching circles and \( \{ \rho_1, \ldots, \rho_m \} \) are the surgery circles on \( \partial N(\gamma_i) \) which are null-homotopic in \( M_1 \), then let \( v_{ij} = \ell k(\rho_i, \gamma_j) \). Since \( \rho_i, \gamma_j \) are disjoint oriented circles in \( M_0 \) which are null-homologous (torsion in the \( \mathbb{Q} \)-case) \( v_{ij} \) is a well-defined integer (rational number in the \( \mathbb{Q} \)-case). We shall show that \( V \) is invertible over \( \mathbb{Z} \) (respectively over \( \mathbb{Q} \)).

We treat the \( \mathbb{Q} \) case first. Consider the long exact sequence in rational homology for the pair \((M_0, M_0 - \mathring{N}(L))\) where \( L = \{ \gamma_1, \ldots, \gamma_m \} \):

\[
\rightarrow H_2 \left(M_0, M_0 - \mathring{N}(L)\right) \xrightarrow{\partial_*} H_1 \left(M_0 - \mathring{N}(L)\right) \xrightarrow{\pi_0} H_1(M_0) \rightarrow 0.
\]

Since the first term is \( \mathbb{Q}^m \) generated by the meridional disks we get \( \mathbb{Q}^m \xrightarrow{i} H_1(M_0 - \mathring{N}(L)) \xrightarrow{\pi_0} H_1(M_0) \rightarrow 0 \) where \( i(e_i) = \mu_i = \mu(\gamma_i) \). It is well-known that \( i \) is injective when \( \gamma_i \) are zero in \( H_1(M; \mathbb{Q}) \). A splitting of \( i \) is given by \( \phi(x) = \sum_{j=1}^m \ell k(x, \gamma_j) e_j \). Hence \( \psi : H_1(M - L) \rightarrow \mathbb{Q}^m \oplus H_1(M_0) \) by \( x \rightarrow (\phi(x), \pi_0(x)) \) is an isomorphism. Note that \( \psi(\rho_i) = \left( \sum_{j=1}^m \ell k(\rho_i, \gamma_j) e_j, 0 \right) \) since \( \rho_i = 0 \) in \( H_1(M_0; \mathbb{Q}) \). Since \( H_1(M_1) \cong H_1(M_0 - L)/\langle \rho_i \rangle \), \( H_1(M_1) \cong H_1(M_0) \oplus \mathbb{Q}^{m-\text{rank } V} \). But \( \phi \) is an isomorphism on \( H_1(M; \mathbb{Q}) \) so \( \text{rank } V = m \).

In the integral case we have the exact sequence in integral homology:

\[
\mathbb{Z}^m \xrightarrow{i} H_1 \left(M_0 - \mathring{N}(L)\right) \xrightarrow{\pi_0} H_1(M_0) \rightarrow 0
\]

where \( i \) is injective because it is injective with rational coefficients. Here \( i(e_i) = \mu_i \) and there is an isomorphism \( \phi : \ker \pi_0 \rightarrow \mathbb{Z}^m \) given by \( \phi(x) = \sum_{j=1}^m \ell k(x, \gamma_j) e_j \) such that \( \phi \circ i = \text{identity} \). In particular \( \phi(\rho_i) = \sum_{j=1}^m \ell k(\rho_i, \gamma_j) e_j = \sum_{j=1}^m v_{ij} e_j \) so \( i \left( \sum_{j=1}^m v_{ij} e_j \right) = \rho_i \). Since \( H_1(M_1; \mathbb{Z}) \cong H_1(M_0 - \mathring{N}(L))/\langle \rho_i \rangle \), we see that the cokernel of \( \mathbb{Z}^m \xrightarrow{V} \mathbb{Z}^m \) embeds in \( H_1(M_1; \mathbb{Z}) \) via the map \( i \). If this cokernel is non-zero then there is a class...
$x \in \text{ker} \, \pi_0$ which is non-zero under $\pi_1 : H_1(M_0 - \mathcal{N}(L)) \to H_1(M_1)$. This implies that $\pi_1(x)$ is a non-trivial element in the kernel of the inclusion map $H_1(M_1) \to H_1(W)$ which is a contradiction. Hence $V$ is invertible over $\mathbb{Z}$.

Now that we have established that the linking matrix is invertible, in the integral case, we appeal to the classification of symmetric bilinear forms. We can change $W$ by adding a single $\pm 1$ framed 2-handle attached along a trivial circle in order to assume $V$ is an indefinite, odd form. Such a form has a $\mathbb{Z}$-basis for which $V$ is diagonal with $\pm 1$'s on the diagonal. This basis change can be realized geometrically by handle slides (see [Ki2; Chapter 2]). Thus $M_1$ is obtained from $M_0$ by $\pm 1$ surgeries on a link $\{\gamma_1, \ldots, \gamma_n\}$ such that each $[\gamma_i] \in N(\pi_1(M_0))$ and $\ell k(\gamma_i, \gamma_j) = 0$. Conversely $M_0$ is obtainable from $\pm 1$ surgery on the dual link $\{\gamma_1', \ldots, \gamma_n'\}$ where, in general, all we know is that $[\gamma_i'] \in \text{kernel} \, \phi$ which in turn lies in $[\pi_1(M_1), \pi_1(M_1)]$.

For the rational case we need the following Lemma, which may be well known. The proof was suggested to me by Richard Stong.

**Lemma 4.3.** If $q : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Q}$ is a symmetric, bilinear, non-singular form then there is a basis $e_1, \ldots, e_n$ for $\mathbb{Z}^n$ such that $q$ restricted to $\langle e_1, \ldots, e_i \rangle$ is non-singular for each $i \leq i \leq n$.

**Proof.** First we claim that for any such form there is a basis such that $q(e_1, e_1) \neq 0$. For an arbitrary basis $\{e_1, \ldots, e_n\}$, if some $j$ has $q(e_j, e_j) \neq 0$ then we are done by re-ordering. If all $q(e_j, e_j) = 0$ then, by non-singularity, there is some $j$ such that $q(e_1, e_j) \neq 0$. Then the basis $\{e_1 + e_j, e_2, \ldots, e_n\}$ works.

Now we proceed by induction. Suppose we have a basis $\{e_1, \ldots, e_n\}$ such that $q|\langle e_1, \ldots, e_j \rangle$ is non-singular for each $j < i$. We shall re-choose $\{e_i, \ldots, e_n\}$ such that $q|\langle e_1, \ldots, e_i \rangle$ is also non-singular. To do so write $q$ in our basis as $q = \begin{pmatrix} A & B \\ B' & C \end{pmatrix}$ where $A$ is $(k - 1)$ by $(k - 1)$ and $B'$ is the transpose of $B$. We can make a rational change of basis to replace
this matrix by \( q^\wedge = \begin{pmatrix} A & O \\ O & D \end{pmatrix} \) where \( D = C - B'A^{-1}B \). Since \( q^\wedge \) is non-singular (since \( q \) is), \( D \) is a non-singular matrix. As in the first step of our proof, there is an integral invertible matrix \( P \) such that the \((1, 1)\) entry of \( P'DP \) is non-zero. Use this matrix \( P \) to change our basis \( \{e_1, \ldots, e_{i-1}, e_i, \ldots, e_n\} \) to \( \{e_1, \ldots, e_{i-1}, e'_{i}, \ldots, e'_{n}\} \). In this new basis the matrix for \( q \) is obtained by conjugating the \( q \) matrix by \( \begin{pmatrix} I & O \\ O & P \end{pmatrix} \), which yields \( q = \begin{pmatrix} A & BP \\ P'B' & P'CP \end{pmatrix} \). We claim that \( q \) restricted to the subspace spanned by \( \{e_1, \ldots, e_i\} \) is non-singular. To verify this it suffices to apply the “same” rational change of basis we used above but only to the first \( i \times i \) submatrix. This yields \( \begin{pmatrix} A & O \\ O & (P'DP)_{11} \end{pmatrix} \) where 11 means the \((1, 1)\) entry. Since this matrix is non-singular, the original \( q \) restricted to the span of \( \{e_1, \ldots, e_i\} \) is non-singular. □

Now apply 4.3 to the linking matrix \( V \). The change of bases can be achieved by re-ordering the handles and by “handle slides” in \( M_0 \). Thus we may assume that \( V_i \), the linking matrix of \( \{\gamma_1, \ldots, \gamma_i\} \) with respect to \( M_0 \), is non-singular for each \( 1 \leq i \leq n \). Let \( M^*_i \) be the result of the surgeries on \( \{\gamma_1, \ldots, \gamma_i\} \). We can assume by induction that the cobordism \( W^* \) from \( M_0 \) to \( M^*_i \) is a product on \( H_1 \) modulo torsion. Thus \( [\gamma_{i+1}] \) is of finite order in \( H_1(M^*_i; \mathbb{Z}) \) since it is of finite order in \( H_1(M_0; \mathbb{Z}) \). By the argument of the proof of 4.2, \( H_1(M^*_i; \mathbb{Q}) \cong H_1(M_0; \mathbb{Q}) \) since \( V_{i+1} \) is non-singular. But then \( H_1(M^*_i; \mathbb{Q}) \cong H_1(M^*_i; \mathbb{Q}) \) so the surgery on \( \gamma_{i+1} \subseteq M^*_i \) is non-longitudinal with respect to \( M^*_i \). Thus \( M_1 \) is \( N \)-surgery related to \( M_0 \).

This concludes the proof of Theorem 4.2. □

We can now prove 2.9.

**Proof of 2.9.** Suppose \( M_0 \) and \( M_1 \) are rationally 2-surgery equivalent. By 2.3 we may assume that there is a sequence \( M_0 = X_0 \to X_1 \to \cdots \to X_m = M_1 \) where \( X_{i+1} \) is obtained from \( X_i \) by a single integral non-longitudinal surgery on a rationally null-homologous circle \( \gamma_{i+1} \). We may assume \( \{\gamma_i\} \) are disjoint in \( M_0 \). Consider the induced
cobordism $W$ from $M_0$ to $X_i$ discussed in the proof of 2.3. The proof of 2.3 shows that $W$ is a product on $H_1$ modulo torsion. Since $[\gamma_{i+1}]$ is trivial in $H_1(X_i; \mathbb{Q})$, it is also trivial in $H_1(M_0; \mathbb{Q})$. Moreover the argument above in the proof of 4.2 shows that the linking matrix of $\{\gamma_1, \ldots, \gamma_m\}$ in $M_0$ is non-singular over $\mathbb{Q}$. \qed

We want to show that there exist maps $f_i : M_i \to X$ which induces isomorphisms on the first integer homology group and such that $(f_0)_*([M_0]) = (f_1)_*([M_1]) \in H_3(X)$. We may assume as before that $f_0$ is the inclusion map. Here $X = K(H_1(M_0), 1)$. Of course, since $X$ is aspherical, there exists a map $f_1$ induced by $\phi_1$.

First we note that it would suffice to show that $(f_0)_*([M_0]) = (f_1)_*([M_1])$ for certain $h \in H^3(X; \mathbb{Z}_n)$. For if $\alpha = (f_0)_*([M_0]) - (f_1)_*([M_1])$ is not zero in $H_3(X; \mathbb{Z})$ then there is an element of $\text{Hom}(H_3(X); \mathbb{Z}_n)$ which detects it, since $H_3(X)$ is a finitely generated abelian group which has a element of order $n$ only if $H_1(M_0)$ has an element of order $n$. More precisely, suppose $H_1(M_0) \cong \mathbb{Z}^m \times \mathbb{Z}_{n_1} \times \ldots \times \mathbb{Z}_{n_k}$ where each $n_i$ is a prime power. Then the torsion-free summand of $H_3(H_1(M_0); \mathbb{Z})$ is merely $H_3(\mathbb{Z}^m; \mathbb{Z}) \cong H_3(S^1 \times \ldots \times S^1; \mathbb{Z})$. If $\alpha$ lies in this summand then it can be detected by an element $h$ of the subgroup $H^3(\mathbb{Z}^m; \mathbb{Z})$. On the other hand if $\alpha$ is of finite order, then it can be detected by some $h \in H^3(X; \mathbb{Z}_n)$ where $n = p^r$ where $p^r$ is the maximal order of all elements in $H_1(M_0)$ which have order a power of $p$. This is true since $\mathbb{Z}_{p^i}$ injects into $\mathbb{Z}_{p^r}$ if $r \geq i$. Therefore these are the only types of elements $h$ we need consider.

Next we need to understand the cohomology rings $H^*(X; \mathbb{Z}_n)$.

**Proposition 4.4.** Suppose $X$ is a finitely generated abelian group and $n$ is the exponent of the $p$-torsion subgroup of $X$ (elements of order $p^r$). Then the ring $H^3(X; \mathbb{Z}_n)$ is generated by elements of the form $\alpha \cup \beta \cup \gamma$ and $\alpha \cup \tau_s B(\gamma)$ where $\alpha, \beta, \gamma \in H^1(X; \mathbb{Z}_n)$, $B$ is the Bockstein associated to $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_n \to 0$, and $\tau_s : H^2(X; \mathbb{Z}) \to H^2(X; \mathbb{Z}_n)$.

Before proving 4.4, we finish the proof that $C \Rightarrow B$. First consider the case that
$h = \alpha \cup \beta \cup \gamma$. Note that this includes the case $n = 0$ since $H^3(Z^n; \mathbb{Z})$ is generated by such elements. Then we have (using C b)),

$$\langle f_0^*(h), [M_0] \rangle = \langle f_0^*(\alpha) \cup f_0^*(\beta) \cup f_0^*(\gamma), [M_0] \rangle$$

$$= \langle \phi_n^1 \circ f_0^*(\alpha) \cup \phi_n^1 \circ f_0^*(\beta) \cup \phi_n^1 \circ f_0^*(\gamma), [M_1] \rangle$$

$$= \langle f_1^*(h), [M_1] \rangle$$

since $\phi_n^1 \equiv f_1^* \circ (f_0^*)^{-1}$. Now suppose $h = \alpha \cup \tau_B(\gamma)$. $f_0^*(\alpha \cup \tau_B(\gamma)) = f_0^*(\alpha) \cup f_0^*\tau_B(\gamma) = f_0^*(\alpha) \cup \tau_B f_0^*(\gamma)$ since $f_0$ is a continuous map. Hence, using condition c of C we have that:

$$\langle f_0^*(h), [M_0] \rangle = \langle \phi_n^1 \circ f_0^*(\alpha) \cup \tau_B(\phi_n^1 \circ f_0^*(\gamma)), [M_1] \rangle$$

$$= \langle f_1^*(\alpha) \cup \tau_B f_1^*(\gamma)), [M_1] \rangle$$

$$= \langle f_1^*(h), [M_1] \rangle .$$

Thus 4.4 will complete $C \Rightarrow B$.

**Proof of 4.4.** First we need the following:

**Lemma 4.5.** Suppose $X$ and $Y$ are spaces whose homology is finitely generated in each dimension. Suppose $n$ is a prime power. Then the cohomology cross product induces an isomorphism:

$$\Theta_n : \sum_{p+q=3} H^p(X; \mathbb{Z}_n) \otimes \mathbb{Z}_n \otimes H^q(Y; \mathbb{Z}_n) \longrightarrow H^3(X \times Y; \mathbb{Z}_n).$$

**Proof of 4.5.** $\Theta_n$ is a monomorphism by [Mu; Theorem 61.6]. Since the homology groups of $X$ and $Y$ are finitely generated the domain and range of $\Theta_n$ are finite groups. Thus it will suffice to show that they are abstractly isomorphic. First we list some abbreviations: $x_p \equiv H_p(X; \mathbb{Z}), y_q \equiv H_q(Y; \mathbb{Z}), x_p^t \equiv \mathbb{Z}_n$-torsion subgroup of $x_p$, $e_p^t = \text{Ext}(x_p; \mathbb{Z}_n)$, $\otimes \equiv \otimes \mathbb{Z}$, $\otimes_n \equiv \otimes \mathbb{Z}_n$. If $A, B$ are finitely generated abelian groups, then the following are easily verified: $\text{Hom}(A; \mathbb{Z}_n) \cong A \otimes \mathbb{Z}_n$, $e_p^t = x_p^t \otimes \mathbb{Z}_n \cong x_p^t$, $A \ast B \cong A^t \ast B^t,$
\((A \otimes \mathbb{Z}_n) \otimes_n (B \otimes \mathbb{Z}_n) \cong (A \otimes B) \otimes \mathbb{Z}_n\). Expanding \(H^3(X \times Y; \mathbb{Z}_n)\) using the Universal Coefficient Theorem for cohomology and then the Kunnet Theorem for homology and applying the above, we get \(\bigoplus_{p+q=3} (x_p \oplus y_q) \oplus (x_1^t \ast y_1^t) \oplus x_2^t \oplus y_2^t \oplus \text{Ext}(x_1 \oplus y_1; \mathbb{Z}_n)\) all tensored with \(\mathbb{Z}_n\). On the other hand the domain of \(\Theta_n\) may be expanded as \(\bigoplus_{p+q=3} [(x_p \oplus \mathbb{Z}_n \oplus x_{p-1}^t) \oplus_n (y_q \oplus \mathbb{Z}_n \oplus y_{q-1}^t)]\). Expanding and comparing terms shows that these expressions are isomorphic, using the fact that \(\text{Ext}(x_1 \oplus y_1; \mathbb{Z}_n) \oplus x_1^t \oplus y_1^t \cong (x_1^t \oplus_n (y_1 \oplus \mathbb{Z}_n)) \oplus ((x_1 \oplus \mathbb{Z}_n) \oplus_n y_1^t)\) which is easily seen by expressing \(x_1, y_1\) as direct sums of their “torsion and torsion-free parts.”

Now we show that 4.5 implies 4.4. Since \(X\) is a finitely-generated abelian group, it is a product \(\times_{i=1}^k X_i\) cyclic groups of infinite or prime-power order. Now apply 4.5 inductively. Recall that if \(\alpha \in H^p(X; \mathbb{Z}_n), \beta \in H^q(Y; \mathbb{Z}_n)\) then \(\alpha \times \beta = \pi_1^*(\alpha) \cup \pi_2^*(\beta)\) where \(\pi_i\) are the projections to the factors. Then 4.5 implies that \(H^3(X; \mathbb{Z}_n)\) is generated by elements of the form \(\pi^*(\alpha) \cup \pi^*(\beta) \cup \pi^*(\gamma), \pi^*(\alpha) \cup \pi^*(\Delta), \) and \(\pi^*(\Gamma)\) where \(\alpha \in H^1(X_i; \mathbb{Z}_n), \beta \in H^1(X_j; \mathbb{Z}_n), \gamma \in H^1(X_k; \mathbb{Z}_n), \Delta \in H^2(X_s; \mathbb{Z}_n), \Gamma \in H^3(X_m; \mathbb{Z}_n)\). The only cases where \(H^2(X_s; \mathbb{Z}_n)\) is non-zero are \(H^2(Z_{p^r}; Z_{p^r})\) where \(s \leq r\) since \(n\) is the exponent. Consider the coefficient sequence \(0 \rightarrow Z_{p^{r'}} \xrightarrow{i} Z_{p^{2r'}} \xrightarrow{\pi} Z_{p^r} \rightarrow 1\). Then the induced map \(H^1(Z_{p^{r'}}; Z_{p^{r'}}) \rightarrow H^1(Z_{p^r}; Z_{p^r})\) is zero since the composition \(Z_{p^{r'}} \xrightarrow{\phi} Z_{p^{2r'}} \xrightarrow{\pi} Z_{p^r}\) is zero for any \(\phi\). Hence the Bockstein \(\widetilde{B}: H^1(Z_{p^r}; Z_{p^r}) \rightarrow H^2(Z_{p^r}; Z_{p^r})\) is an isomorphism and consequently \(\Delta = \widetilde{B}(\gamma)\) for some \(\gamma \in H^1(X_s; \mathbb{Z}_n)\) and \(\pi^*(\alpha) \cup \pi^*(\Delta) = \pi^*(\alpha) \cup \widetilde{B}\pi^*(\gamma)\).

But in this case \(\tau_s: H^2(Z_{p^r}; Z) \rightarrow H^2(Z_{p^r}; Z_{p^r})\) is an isomorphism and so one sees that \(\widetilde{B} = \tau_s B\) as desired.

The only cases where \(H^3(X_m; \mathbb{Z}_n)\) is non-zero are \(H^3(Z_{p^m}; Z_{p^r})\) where \(m \leq r\). We shall show \(\Gamma = \alpha \cup \Delta\) reducing to the case above. Let \(L = L(p^m, 1)\) be the 3-dimensional lens space. The map \(L \xrightarrow{i} K(Z_{p^m}, 1)\) can be constructed by adding cells of dimension 4 and higher to \(L\) and thus induces isomorphisms on first and second cohomology and a monomorphism on \(H^3\). By Poincaré Duality for \(L\), \(i^*(\Gamma) = i^*(\alpha) \cup i^*(\Delta)\) for some
\[ \alpha \in H^1(\mathbb{Z}_p^m; \mathbb{Z}_p^r) \text{ and } \Delta \in H^2(\mathbb{Z}_p^m; \mathbb{Z}_p^r). \text{ Hence } \Gamma = \alpha \cup \Delta \text{ as claimed. This completes the verification of our second claim.} \]

\[ \square \]

**D \Rightarrow C:** For brevity let \( \phi_* \) denote the isomorphism \( \phi_1 \) and \( \phi^* \) denote its induced adjoint \( \phi_n^1 \). Let \( i \) denote the inclusion \( \mathbb{Z}_n \rightarrow \mathbb{Q}/\mathbb{Z} \) where \( 1 \rightarrow \frac{1}{n} \). Let \( T_i(M) \) denote the torsion subgroup of \( H_i(M; \mathbb{Z}) \). Since a linking pairing \( \lambda \) is non-singular, \( \lambda(-, a) \) is an isomorphism \( T_1(M) \rightarrow \text{Hom}(T_1(M); \mathbb{Q}/\mathbb{Z}) \). For any \( \gamma \in H^1(M; \mathbb{Z}_n) \), \( i \langle \gamma, - \rangle \in \text{Hom}(T_1(M); \mathbb{Q}/\mathbb{Z}) \) and we let \( D(\gamma) \) be its inverse under the above isomorphism. Thus \( D: H^1(M; \mathbb{Z}_n) \rightarrow T_1(M) \) and \( \lambda(D(\gamma), a) = i \langle \gamma, a \rangle \) for all \( a \in T_1(M) \).

We claim that if \( \gamma \in H^1(M_1; \mathbb{Z}_n) \) then \( \phi_* D_0 \phi^*(\gamma) = D_1(\gamma) \) where \( D_0, D_1 \) correspond to \( M_0, M_1 \) respectively. For if \( a \in T_1(M_1) \) then \( a = \phi_*(b) \) for some \( b \in T_1(M_0) \). So \( \lambda_1(\phi_* D_0 \phi^*(\gamma), a) = \lambda_1(\phi_* D_0 \phi^*(\gamma), \phi_*(b)) = \lambda_0(D_0 \phi^*(\gamma), b) \), by hypothesis c) 4.1 D. Continuing, \( \lambda_0(D_0 \phi^*(\gamma), b) = i \langle \phi^*(\gamma), b \rangle = i \langle \gamma, \phi_*(b) \rangle = i \langle \gamma, a \rangle \). Hence \( \phi_* D_0 \phi^*(\gamma) = D_1(\gamma) \).

Next we claim that the Poincaré dual of \( B(\gamma) \) is \( D_1(\gamma) \) where \( B: H^1(M_1; \mathbb{Z}_n) \rightarrow H^2(M_1; \mathbb{Z}) \) is the Bockstein associated to \( 0 \rightarrow \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \rightarrow \mathbb{Z}_n \rightarrow 0 \). We need to show \( \lambda_1(B(\gamma) \cap [M_1], a) = i \langle \gamma, a \rangle \) for each \( a \in T_1(M_1) \). Suppose \( g \) is 1-cochain with coefficients in \( \mathbb{Z}_n \) representing \( \gamma \) and \( \tilde{g} \) is an integral 1-cochain reducing to \( g \). Then \( B(\gamma) \) is represented by \( \frac{1}{n} \delta \tilde{g} \), a 2-cochain with integral values. Note that \( B(\gamma) \) is \( n \)-torsion so its Poincaré dual lies in \( T_1(M_1) \). If \( \Sigma \) is a chain representing the orientation class \([M_1]\) then the Poincaré dual is represented by \( \frac{1}{n} \delta \tilde{g} \cap \Sigma = \frac{1}{n} \partial(\tilde{g} \cap \Sigma) \). Thus \( \lambda_1(B(\gamma) \cap [M], a) \) is given by \( \frac{1}{n} \cdot \#((\tilde{g} \cap \Sigma) \cdot a') \) where \( a' \) is a chain representing \( a \) and \( \# \) is the number of signed intersection points modulo \( n \). But this is also a calculation of \( i \langle \gamma, a \rangle \), finishing the verification of our second claim.
Now we can finish the proof that $D \Rightarrow C$. If $\alpha, \gamma \in H^1(M_1; \mathbb{Z}_n)$ then
\[
i \langle \alpha \cup \pi^* B \gamma, [M_1] \rangle = i \langle \alpha, \pi^* B \gamma \cap [M_1] \rangle
= i \langle \alpha, \pi_* (B(\gamma) \cap [M_1]) \rangle
= i \langle \alpha, \pi_* D_1(\gamma) \rangle
= i \langle \alpha, D_1(\gamma) \rangle
= \lambda_1 (D_1(\alpha), D_1(\gamma))
\]
and similarly
\[
i \langle \phi^* \alpha \cup \pi^* B \phi^* \gamma, [M_0] \rangle = \lambda_0 (D_0(\phi^* \alpha), D_0(\phi^* (\gamma)))
= \lambda_1 (\phi_* D_0(\phi^* \alpha), \phi_* D_0(\phi^* \gamma))
\]
by hypothesis D. By our first claim this equals $\lambda_1 (D_1(\alpha), D_1(\gamma))$ as above. Since $i$ is injective, condition b) of C is established. □

§5. Rational Homology Surgery Equivalence. In this chapter we address the question of when two 3-manifolds are related by a sequence of Dehn surgeries on rationally null-homologous curves which preserve $H_1(\_; \mathbb{Q})$. In the language of §2, this is case iv) where $N = G^Q_2 = \{ x \in G | \exists n, x^n \in G_2 \}$, or rational 2-surgery equivalence. Note that $G/G^Q_2 = H_1(G)/T_1(G)$. By 2.2 this is an equivalence relation and by 2.3 it is sufficient to consider non-zero integral framings. We find that this is completely controlled by the isomorphism class of the integral cup product form. Beware that, because we restrict to rationally null-homologous curves, a surgery which preserves $\beta_1$ will necessarily preserve $H_1/T_1$ and consequently $H^1(\_; \mathbb{Z})$. Therefore it is not possible to have, for example, $H_1(M_0) \cong \mathbb{Z}$, $H_1(M_1) \cong \mathbb{Z}$ with the natural map between them being “times 2.” This would be possible to achieve by allowing certain surgeries on curves in $M_0$ which are essential in $H_1(M_0; \mathbb{Z})$ but not primitive. Hence rational 2-equivalence is NOT the relation generated by Dehn surgeries which preserve $H_1(\_; \mathbb{Q})$ but rather those which preserve $H_1(\_; \mathbb{Z})/T_1$. 34
Theorem 5.1. Suppose $M_0$ and $M_1$ are closed, oriented connected 3-manifolds. The following 3 conditions are equivalent.

QA) $M_0$ and $M_1$ are rationally 2-surgery equivalent; that is, each may be obtained from the other by a sequence of non-longitudinal Dehn surgeries on circles which are zero in $H_1(\_; \mathbb{Q})$ (equivalently integral non-longitudinal surgeries.

QB) There exists an isomorphism $\phi_1 : H_1(M_1)/T_1(M_1) \rightarrow H_1(M_0)/T_1(M_0)$ such that $(f_0)_*([M_0]) = (f_1)_*([M_1])$ in $H_3(H_1(M_0)/T_1(M_0); \mathbb{Z}) \cong H_3((S^1)^{\beta_1(M_0)}; \mathbb{Z})$ where $f_0$ is induced by “inclusion” and $f_1$ is induced by $\phi_1$. That is, $M_0$ and $M_1$ are bordant over $(S^1)^{\beta_1(M_0)}$.

QC) There exists an isomorphism $\phi_1$ as in the first line of QB such that $\langle \alpha \cup \beta \cup \gamma, [M_0] \rangle = \langle \phi^1(\alpha) \cup \phi^1(\beta) \cup \phi^1(\gamma), [M_1] \rangle$ for all $\alpha, \beta, \gamma \in H^1(M_0; \mathbb{Z})$ and $\phi^1$ is the adjoint (Hom-Dual) of $\phi_1$. That is, the integral cup product forms of $M_0$ and $M_1$ are isomorphic.

Proof of 5.1. QA $\Rightarrow$ QB: By 2.3 with $N = (\pi_1(M_0))^Q_2$, we may reduce to the case of a single integral non-longitudinal surgery on $\gamma \subset M_0$ such that $[\gamma] \in N$. If $W$ is the cobordism corresponding to this surgery, then $\beta_1(W) = \beta_1(M_0)$ and the inclusion map induces an isomorphism modulo the $N$-subgroups (see last paragraph of proof of 2.3), which, in the case at hand, means it induces an epimorphism on $H_1$ modulo torsion. But by the symmetry of 2.1 and 2.2 the same may be said of $M_1$. Hence we may let $\phi_1 = (j_0)^{-1} \circ (j_1)_*$ as in the proof of 4.1. Then, as in 4.1, there are continuous maps $f_0$, $F$ and $f_1$ from $M_0$, $W$, $M_1$ respectively, to $K(\mathbb{Z}^{\beta_1(M_0)}, 1)$ inducing the obvious maps on $\pi_1$ and the result follows. □
QA ⇒ QC: Note that the inclusion maps \( j_0, j_1 \) as defined above induce isomorphisms on \( H^1(\_; \mathbb{Z}) \) since \( \text{Hom}(H_1; \mathbb{Z}) \cong \text{Hom}(H_1/T_1; \mathbb{Z}) \). The proof is now the same as the first part of the proof of \( A \Rightarrow D \) in \( \S 4 \).

QC ⇒ QB: Note that all the maps \( j_0^*, j_1^*, \phi^1, f_0^*, f_1^* \) are isomorphisms on \( H^1(\_; \mathbb{Z}) \). Since the cohomology ring of \( K(\mathbb{Z}^m, 1) \) is well-known to be generated by triple cup products, the easy part of the proof of \( C \Rightarrow B \) in \( \S 4 \) applies word for word. \( \square \)

QB ⇒ QA: Apply 4.2 with \( N = (\pi_1(M_0))^Q_2 \). \( \square \)

Let us denote by \( S_m^Q \) the set of rational homology surgery equivalence classes of closed oriented 3-manifolds with \( \beta_1 = m \). By 5.1, if \( m < 3 \) then \( S_m^Q \) contains a single element.

Corollary 5.2. If \( m < 3 \) any 2 closed, oriented 3-manifolds with identical first Betti number \( m \) are rational homology surgery equivalent.

Corollary 5.3. There is a bijection \( S_m^Q \rightarrow \Lambda^3(\mathbb{Z}^m)/\text{GL}_m(\mathbb{Z}) \) given by the integral triple cup product form. Hence if \( M_0, M_1 \) have torsion-free homology groups then they are rational homology surgery equivalent if and only if they are integral homology surgery equivalent.

Proof of 5.3. See the proof of 3.5 and use Sullivan’s work [Su].

Example 5.4: It is possible for \( M_0 \) and \( M_1 \) to be rational homology surgery equivalent, have isomorphic first homology and linking forms, yet not be integral homology surgery equivalent (see Example 3.15).

Corollary 5.5. For any closed, oriented 3-manifold \( M \), \( M \) is rational homology surgery equivalent to \(-M\).
§6. Surgery Equivalence Preserving Lower Central Series Quotients. We have seen that the relation generated by $\pm 1/n$ Dehn surgery on circles which lie in $(\pi_1(M))_k$ is an equivalence relation which we called $k$-surgery equivalence. The equivalence relation generated by non-longitudinal surgeries on circles lying in the $k^{th}$ term of the rational lower central series, we call rational $k$-surgery equivalence. Just as 2-equivalence was controlled by $G/G_2$ and the cup products (and linking form), we shall see that $k$-equivalence is controlled by $G/G_k$ and higher Massey products (and the linking form). We only attempt a complete algebraic characterization of $k$-surgery equivalence to “the zero element,” i.e. $\#_{i=1}^m S^1 \times S^2$. Here we see that $k$-equivalence is controlled by Massey products of length less than $2k-1$, or equivalently by the isomorphism class of $G/G_{2k-1}$. A similar characterization for the general case is made difficult by the ill-definedness of Massey products and our ignorance of $H_3$ of torsion-free nilpotent groups. It may well be, however, that there is sufficient information in the literature to complete the general characterization.

Theorem 6.1. Suppose $M_0$ and $M_1$ are closed, oriented, connected 3-manifolds. The following are equivalent.

A) $M_0$ and $M_1$ are $k$-surgery equivalent.

B) There exists an isomorphism $\phi : \pi_1(M_1)/(\pi_1(M_1))_k \to \pi_1(M_0)/(\pi_1(M_0))_k$ such that $\phi_*([M_1]^k) = [M_0]^k$ where $[M_i]^k$ means the image, in $H_3(\pi_1(M_i)/(\pi_1(M_i))_k; \mathbb{Z})$, of the fundamental class of $M_i$ under some map $f_i : M_i \to K(\pi_1(M_i)/(\pi_1(M_i))_k, 1)$ inducing the obvious quotient on $\pi_1$.

Corollary 6.2. The set of $k$-surgery equivalence classes of closed, oriented 3-manifolds $M$ with $\pi_1(M)/(\pi_1(M))_k \cong G$ is in bijection with the subset of $H_3(G)/\text{Aut}(G)$ consisting of those elements which are “realizable”, that is which can arise as $[M]^k$ for some closed 3-manifold. The correspondence is given by the fundamental class (see [Tu1] for an analysis of this realizable set.)
**Proof of 6.1.** $A \Rightarrow B$ is implied by Propositions 2.3, 2.1 and 4.1.

$B \Rightarrow A$ is implied by Theorem 4.2. $\Box$

**Proof of 6.2.** Merely note that $[M_i]^k$ is only well-defined up to the action of Aut$(G_i)$ on $H_3(G_i)$.

**Theorem 6.3.** Suppose $M_0$ and $M_1$ are closed, oriented, connected 3-manifolds. The following are equivalent.

QA) $M_0$ and $M_1$ are rationally $k$-surgery equivalent.

QB) Same condition as 6.1B with rational lower central series replacing the integral one.

**Corollary 6.4.** The set of rational $k$-surgery equivalence classes of closed, oriented 3-manifolds with $\pi_1(M)/(\pi_1(M))^Q_k \cong G$ is in bijection with the subset of $H_3(G)/\text{Aut}(G)$ corresponding to realizable classes (see [Tu1]).

**Proof of 6.4.** The argument for QA $\Rightarrow$ QB in the proof of 5.1 works here (using that finitely generated nilpotent groups are Hopfian). For QB $\Rightarrow$ QA apply 4.2 with $N = (\pi_1(M_0))^Q_k$. $\Box$

**Corollary 6.5.** Any two 3-manifolds with $\beta_1 = 0$ (or any two with $\beta_1 = 1$) are rationally $k$-surgery equivalent for each $k$.

**Proof of 6.5.** Suppose $\beta_1(M_0) = 1$. Then the epimorphism $\pi_1(M_0) \twoheadrightarrow \mathbb{Z}$ induces isomorphisms modulo any term of the rational lower central series [St], so $\pi_1(M_0)/(\pi_1(M))^Q_k$ is $\mathbb{Z}$. Since $H_3(\mathbb{Z}) = 0$, the result follows from 6.4. $\Box$

**Example 6.6:** Let $M_0 = S^1 \times S^2 \# S^1 \times S^2$ and let $M_1$ be the manifold obtained by 0-surgery on each component of a Whitehead link in $S^3$, as shown by the solid lines in Figure 6.7a. Performing a $+1$ surgery on the dashed circle $\gamma$ in Figure 6.7b transforms $M_0$...
to $M_1$, and $\gamma$ is clearly null-homologous in $M_0$ so $M_0$ and $M_1$ are 2-surgery equivalent (as they must be since $H_3(\mathbb{Z} \times \mathbb{Z}) = 0$). However $\gamma \notin (\pi_1(M_0))_3$ and so it is not clear whether or not $M_0$ and $M_1$ are 3-surgery equivalent. In this case $M_1$ is known to be the “Heisenberg manifold” (Euler class $\pm 1$ circle bundle over the torus) whose fundamental group is $F/F_3$ where $F$ is the free group on $\{x, y\}$. Hence $\pi_1(M_0)/(\pi_1(M_0))_3 \cong \pi_1(M_1)/(\pi_1(M_1))_3 \cong F/F_3$. But 6.1B is not satisfied since $[M_0]$ represents the trivial class in $H_3(F/F_3)$ since it factors through $H_3(F)$, whereas $M_1$ is a $K(F/F_3, 1)$ and so $[M_1]$ represents a generator of $H_3(F/F_3) \cong \mathbb{Z}$. Thus the manifolds are not 3-surgery equivalent (nor rationally 3-surgery equivalent).

We shall now show that $k$-surgery equivalence is related to higher order Massey products and that this is the correct generalization of the triple cup product form. However, Massey products may not be uniquely defined and this makes statements of results difficult. For this reason we shall restrict our focus to situations where the Massey products are uniquely defined. In general if $M_0$ and $M_1$ are $k$-surgery equivalent then their lower central series quotients $G/G_j$ are isomorphic for $1 \leq j \leq k$ and this is known to entail a “correspondence” between order $k - 1$ Massey products with any abelian coefficients [Dw; Corollary 2.7]. We shall state this only for a restricted case.

**Proposition 6.8.** Suppose $M_0$ and $M_1$ are $k$-surgery equivalent, and that all $j$th order Massey products vanish for $M_0$ for $2 \leq j \leq (k - 2)$. Then there is an isomorphism $\phi : H_1(M_1) \to H_1(M_0)$ such that for all abelian groups $A$ and $\alpha_i \in H^1(M_0; A)$, $1 \leq i \leq k$, $\langle \alpha_1 \cup \langle \alpha_2, \ldots, \alpha_k \rangle, [M_0] \rangle = \langle \phi^*\alpha_1 \cup \langle \phi^*\alpha_2, \ldots, \phi^*\alpha_k \rangle, [M_1] \rangle$ where $\langle \alpha_2, \ldots, \alpha_k \rangle$
is the Massey product in $H^2(M_0; A)$. In fact, $\pi_1(M_0)/(\pi_1(M_0))_k \cong F/F_k$ ($F$ a free group) if and only if $H_1(M_0)$ is torsion free and all Massey products of length less than $k$ vanish for $M_0$. More precisely, the latter conditions imply that any given isomorphism $\pi_1(M_0)/(\pi_1(M_0))_{k-1} \cong F/F_{k-1}$ can be extended.

\textbf{Example 6.9}: Let $M_0 = \#^4_{i=1} S^1 \times S^2$ and let $M_1$ be the manifold shown in Figure 6.10a. as 0-surgery on the 4-component link $L$. Then $M_0$ is 2-surgery equivalent to $M_1$ as shown by the 3 circles labelled $\pm 1$ in 6.10b.. Note these circles lie in the $G_2 - G_3$ where $G = \pi_1(M_0)$. It is known that the link in 6.10a. has $\mathfrak{m}(1234) = \pm 1$, so for $M_0$ all Massey products vanish, but for $M_1$, $\langle x_1 \cup \langle x_2, x_3, x_4 \rangle, [M_1] \rangle = \pm 1$ where $x_i$ are the Hom-duals of the meridians [Ko; Theorem 3]. Hence $M_0$ and $M_1$ are not 4 surgery equivalent, by 6.8. In fact they are not even 3-surgery equivalent but 6.8 is too weak to show this.

\textbf{Proof of 6.8}. Apply [Dw; Corollary 2.7] to the maps $j_0 : M_0 \to W$ and $j_1 : M_1 \to W$ where $W$ is the cobordism over $\pi_1(M_0)/(\pi_1(M_0))_k$ guaranteed by 6.1. Then use naturality of Massey products and $(j_0)_*([M_0]) = (j_1)_*([M_1])$ to get the first claimed result. Details are left for the reader.

We consider the last claim of 6.8. Suppose $\pi_1(M_0)/(\pi_1(M_0))_k \cong F/F_k$, $k \geq 2$. Then there is a map $f : M_0 \to K(F/F_k, 1)$ which induces an isomorphism on first cohomology with any abelian coefficients. It is known that $F/F_k$ has vanishing Massey products of length less than $k$ and that $H^2(F/F_k)$ is generated by Massey products of length $k$ [O2;
Lemma 16. By naturality, $f^*(\langle x_1, \ldots, x_m \rangle) \subseteq \langle f^*x_1, \ldots, f^*x_m \rangle$. Hence, if $m < k$ then $0 \in \langle f^*x_1, \ldots, f^*x_m \rangle$. But the first non-vanishing level of Massey products are uniquely defined, so by induction, $0 = \langle f^*x_1, \ldots, f^*x_m \rangle$ for $m < k$. This implies that all Massey products of length less than $k$ vanish for $M_0$.

Now suppose $H_1(M_0)$ is torsion-free and all Massey products of length less than $k$ vanish for $M_0$. By induction we can assume $\pi_1(M_0)/\langle \pi_1(M_0) \rangle_{k-1} \cong F/F_{k-1}$ so there is a map $g: F \to G$ ($\pi_1(M_0) = G$) inducing this isomorphism. It would then suffice to prove

$$
\begin{array}{cccc}
0 & \longrightarrow & H_2(F/F_{k-1}) & \longrightarrow & F_{k-1}/F_k & \longrightarrow & 0 \\
\downarrow & & \cong & & \downarrow & g_* & \downarrow & g_* \\
H_2(G) & \longrightarrow & H_2(G/G_{k-1}) & \longrightarrow & G_{k-1}/G_k & \longrightarrow & 0
\end{array}
$$

that $F_{k-1}/F_{k-2} \cong G_{k-1}/G_k$ and the diagram above shows that this is equivalent to showing $\pi_*$ is the zero map. Since $H_2(M_0)$ maps surjectively to $H_2(G)$ it suffices to show $H_2(M_0) \xrightarrow{\pi_*} H_2(G/G_{k-1})$ is the zero map. But $H^2(G/G_{k-1}) \cong H^2(F/F_{k-1})$ is generated by $\langle x_1, \ldots, x_{k-1} \rangle$ so $\pi^*\langle x_1, \ldots, x_{k-1} \rangle = \langle \pi^*x_1, \ldots, \pi^*x_{k-1} \rangle = 0$. Thus $\pi^*$ and $\pi_*$ are zero. Note that we have actually proved that any isomorphism $g_* : G/G_{k-1} \longrightarrow F/F_{k-1}$ can be extended to $g_* : G/G_k \longrightarrow F/F_k$. □

Example 6.9 indicates that 6.8 is too weak. Indeed, 6.1B should be seen as two conditions, and 6.8 says that the first of these conditions controls Massey products of lengths up to $k - 1$. We shall see that the second conditions controls lengths up to $2k - 2$. This is analagous to the Cochran-Orr conjecture that a link in $S^3$ is “null $k$-cobordant” if and only if its Milnor $\overline{\mu}$-invariants of length $j$, $1 \leq j \leq 2k$ are zero. This has been positively resolved by X.S. Lin and Orr-Igusa [L] [IO]. Based on techniques of the latter, we shall now discuss an algebraic characterization of $k$-surgery equivalence to $\#S^1 \times S^2$.

**Theorem 6.10.** For any integer $k \geq 2$, $M$ is $k$-surgery equivalent to $\#_{i=1}^{m}S^1 \times S^2$ if and only if $H_1(M) \cong \mathbb{Z}^m$ and all Massey products of order less than $2k - 1$ vanish for $M$. In
particular if $M$ is zero surgery on an $m$-component link $L$ in a homology 3-sphere then $M$ is $k$-surgery equivalent to $\#_{i=1}^{m}S^1 \times S^2$ if and only if Milnor’s $\mu$-invariants of length less than $2k$ vanish for $L$.

**Proof of 6.10.** Let $G = \pi_1(M)$. Suppose $H_1(M) \cong \mathbb{Z}^m$ and all Massey products of length less than $2k - 1$ vanish for $M$. By the proof of the last part of 6.8, any isomorphism $g_* : G/G_k \to F/F_k$ extends to an isomorphism $h_* : G/G_{2k-1} \to F/F_{2k-1}$. It follows that “the” natural map $M \to K(G/G_k, 1) \to K(F/F_k, 1)$ factors through $K(F/F_{2k-1}, 1)$ and thus that $[M]^k = 0$ in $H_3(F/F_k; \mathbb{Z})$ since Igusa and Orr have shown that the map $H_3(F/F_{2k-1}) \to H_3(F/F_k)$ is zero [IO]. Hence, by 6.1, $M$ is $k$-surgery equivalent to $\#_{i=1}^{m}S^1 \times S^2$.

Now suppose $M$ is $k$-surgery equivalent to $\#_{i=1}^{m}S^1 \times S^2$. Then $[M]^k = 0$ in $H_3(F/F_k)$. First we show that this implies that $G/G_{k+1} \cong F/F_{k+1}$. This follows from this more general result.

**Lemma 6.11.** Suppose $\pi_1(M_0) = G_0$, $\pi_1(M_1) = G_1$, $G_0/(G_0)_k \cong F/F_k$, $G_1/(G_1)_{k+1} \cong F/F_{k+1}$. If $M_0$ is $k$-surgery equivalent to $M_1$ then $G_0/(G_0)_{k+1} \cong F/F_{k+1}$ by an isomorphism extending $f$.

**Proof of 6.11.** Consider a cobordism $W$ from $M_0$ to $M_1$ which contains only 2-handles and is an $F/F_k$-cobordism (see 4.2). Let $F$, $F_0$, $F_1$ denote the maps to $K(F/F_k, 1)$ from $W$, $M_0$, $M_1$ respectively. For any cohomology classes $\{x_1, \ldots, x_k\} \subset H^1(M)$, choose $y_i \in H^1(F/F_k)$ so $(F_0)^*(y_i) = x_i$. Then $\langle x_1, \ldots, x_k \rangle = \langle F_0^*y_1, \ldots, F_0^*y_k \rangle = j_0^*(F^*y_1, \ldots, F^*y_k)$, where the Massey products are uniquely defined since products of lesser length vanish since all spaces have $G/G_k \cong F/F_k$ (see 6.9). By 6.8, it will suffice to show $\langle F^*y_1, \ldots, F^*y_k \rangle = 0$. Certainly $j_1^*\langle F^*y_1, \ldots, F^*y_k \rangle = 0$ since all Massey products of length $k$ vanish for $M_1$. To finish we will show that $j_1^* : H^2(W) \to H^2(M_1)$ is injective on the image of $H^2(F/F_k)$. Recall that we can assume that $W$ is built from
$M_1 \times [0, 1]$ by adding 2-handles whose attaching circles lie in $F_k$. Thus $H_2(W)$ splits as $H_2(M_1) \oplus H_2(W, M_1)$ where the latter is a free abelian group generated by the cores of the 2-handles “capped-off” by surfaces in $M_1$. But these latter classes clearly become spherical in $K(F/F_k, 1)$ and hence vanish in $H_2(F/F_k, 1)$. Considering the dual splitting of $H^2(W)$, this implies that the image of $H^2(F/F_k)$ lies in the summand $H^2(M_1) \hookrightarrow H^2(W)$. It follows that $j_1^*$ is injective on this image. □

Suppose $\pi_1(M) \cong G$ and $G/G_k \cong F/F_k$ for some free group $F$. Suppose also that $f$ is a specific such isomorphism. Then we can define $\theta_k(M, f) \in H^3(F/F_k)$ to be the image of the fundamental class under the map $M \to K(F/F_k, 1)$ induced by $f$.

**Lemma 6.12.** $\theta_k(M, f) \in \text{Image}(H^3(F/F_{k+1}) \xrightarrow{\pi_*} H^3(F/F_k))$ if and only if there is some isomorphism $\tilde{f} : G/G_{k+1} \to F/F_{k+1}$ extending $f$ such that $\pi_*(\theta_{k+1}(M, \tilde{f})) = \theta_k(M, f)$.

**Proof of 6.12.** Suppose $\theta_k(M, f) = \pi_*(x) \in H^3(F/F_{k+1})$. By [O1; Theorem 4], there exists a 3-manifold $M_1$ with $\pi_1(M_1) \cong G_1$ and $G_1/(G_1)_k \cong F/F_{k+1}$ such that $\theta_{k+1}(M_1, g) = x$. Since $\pi_*(x) = \theta_k(M_1, \pi \circ g) = \theta_k(M, f)$, it follows from 6.1 that $(M, f)$ and $(M_1, \pi \circ g)$ are cobordant over $F/F_k$ and are $k$-surgery equivalent. From 6.11 we can conclude that there is an isomorphism $\tilde{f}$ as desired. Since $\tilde{f}$ extends $f$, $\pi_*(\theta_{k+1}(M, \tilde{f})) = \theta_k(M, f)$, by definition. The other implication of 6.12 is immediate. □

To finish the proof of 6.10 we need the following theorem from [IO].

**Theorem 6.13 (Igusa-Orr [IO]).** Suppose $F$ is the free group of rank $m$. Let $N_i = \text{rank} \; H_2(F/F_i)$. Then $H_3(F/F_k; \mathbb{Z})$ is $\bigoplus_{i=k}^{2k-2} \mathbb{Z}^{m_{N_i-N_{i+1}}}$. If we define the **weight** of the summand $\mathbb{Z}^{m_{N_i-N_{i+1}}}$ to be $i + 1$ then the natural maps $H_3(F/F_{k+1}) \to H_3(F/F_k)$ have the property that they preserve weight whenever possible and map each weight summand injectively and by the zero map otherwise.
Corollary 6.14. The map $H_3(F/F_{2k-1}) \to H_3(F/F_k)$ is zero. Any element in the kernel of $H_3(F/F_{k+j}) \to H_3(F/F_k)$, $j \leq k - 1$, lies in the image of $H_3(F/F_{2k-1}) \to H_3(F/F_{k+j})$.

Proof of 6.14. An element of the kernel of the above map has weight greater than $2k - 1$ and at most $2k + 2j - 1$ i.e. at most $4k - 3$, which is precisely the range of weights possible for an element of $H_3(F/F_{2k-1})$. □

Finally, if $\theta_k(M, f) = 0$ for some $f$ then assume by induction (use 6.12 for the start of induction) that $f$ extends to $\tilde{f} : G/G_{k+j} \to F/F_k$ so that $\theta_{k+j}(M, \tilde{f})$ is defined and is a lift of $\theta_k(M, f)$. But then as long as $j \leq k - 1$, 6.14 guarantees that $\theta_{k+j}(M, \tilde{f}) \in \text{Image}(H_3(F/F_{k+j+1}) \to H_3(F/F_{k+j}))$ and 6.12 then implies $\theta_{k+j+1}$ is defined. Hence we can lift to $\theta_{2k-1}$ and $G/G_{2k-1} \cong F/F_{2k-1}$ implying that all Massey products of length less than $2k - 1$ are zero for $M$. This concludes the proof of the first sentence of 6.10. The second statement follows immediately from [Ko; Theorem 2] relating Massey products of 0-surgery on a link to $\pi$-invariants of a link. □

It follows from 6.10 that the two manifolds in Example 6.9 are not 3-surgery equivalent.

Example 6.15: The example $M_1$ in Figure 6.7a (the Heisenberg manifold) is 2-surgery equivalent to $S^1 \times S^2 \# S^1 \times S^2$ since the first non-zero Milnor invariant of the Whitehead link, $\pi(1122)$, is of length 4, but for the same reason the Whitehead link is NOT null 2-cobordant. In general, if a link is null $k$-cobordant then its 0-surgery is $k$-surgery equivalent to $\# S^1 \times S^2$ but the converse is false, requiring in addition that the $\pi$-invariants of length $2k$ vanish.

References

[CG] Cochran, T.D. and Gompf, R.E., Applications of Donaldson's Theorems to Classical Knot Concordance, Homology 3-Spheres and Property P, Topology 27 (1988), 495–512.
[CM] Cochran, T.D. and Melvin, P., *Finite Type Invariants for 3-Manifolds*, preprint 1997, Rice University, available at [http://math.rice.edu/~cochran](http://math.rice.edu/~cochran).

[Co] Cohen, M., *A Course in Simple Homotopy Theory*, Springer-Verlag, New York, 1973.

[Dw] Dwyer, W.G., *Homology, Massey Products and Maps Between Groups*, J. Pure and Applied Algebra 6 (1975), 177–190.

[Ga] Garoufalidis, S., *On Finite Type 3-manifold Invariants I*, to appear, Journal of Knot Theory and its Ramifications.

[GL] Garoufalidis, S. and Levine, J., *Finite Type 3-manifold Invariants, the Mapping Class Group and Blanks*, preprint, Brandeis Univ., 1996.

[Ge] Gerges, Amir, *Surgery, Bordism and Equivalence of 3-manifolds*, Ph.D. Thesis, 1996, Rice University, preprint available.

[H] Hilden, H.M., *Representations of Homology 3-spheres*, Pac. J. Math 94 (1981), no. 1, 125–129.

[IO] Igusa, K. and Orr, K.E., *Milnor Invariants and Homology of Free Nilpotent Groups*, preprint 1997, Indiana University.

[Jo1] Johnson, D., *Homeomorphisms of a Surface Which act Trivially on Homology*, Proc. Amer. Math. Soc. 75 (1979), 118–125.

[KK] Kawauchi, Akio and Kojima, S., *Algebraic Classification of Linking Pairings on 3-manifolds*, Math. Annalen 253 (1980), 29–42.

[Ki2] Kirby, R., *The Topology of 4-Manifolds*, Springer-Verlag, 1989.

[Ko] Kojima, Sadoyoshi, *Milnor’s µ-invariants, Massey products and Whitney’s Trick in 4 dimensions*, Top. and Applns. 16 (1983), 43–60.

[Ku] Kuperberg, Greg, *private communication*.

[Les] Lescop, C., *Global Surgery Formula for the Casson-Walker Invariant*, Annals of Math. Studies #140, Princeton Univ. Press, Princeton, N.J., 1996.

[Li] Lickorish, W.B.R., *A representation of orientable combinatorial 3-manifolds*, Annals of Math. 76 (1962), 531–540.

[L] Lin, X-S., *Null k-cobordant links in S^3*, Comment. Math. Helv. 66 (1991), 333–339.

[Ma] Matveev, S.V., *Generalized surgery of three-dimensional manifolds and representations of homology spheres*.

[Mo1] Morita, S., *Casson’s Invariant for Homology 3-spheres and Characteristic Classes of Vector Bundles*, I, Topology 28 (1989), 305–323.

[Mo2] ________, *On the Structure of the Torelli Group and Casson’s Invariant*, Topology 30 (1991), 603–621.

[Mu] Munkres, James, *Elements of Algebraic Topology*, Addison-Wesley, Menlo Park California, 1984.

[O1] Orr, K.E., *Homotopy Invariants of Links*, Inventiones Math. 95 (1989), 379–394.

[O2] ________, *Link Concordance Invariants and Massey Products*, Topology 30 (1991), no. 4, 699–710.

[Ru] Ruberman, D., *Imbedding Punctured Lens Spaces and Connected Sums*, Pacific Journal of Mathematics 113 (1984), no. 2, 481–491.

[St] Stallings, J., *Homology and Central Series of Groups*, J. Algebra 2 (1965), 170–181.

[Su] Sullivan, Dennis, *On the intersection ring of compact 3-manifolds*, Topology 14 (1975), 275–277.

[Tu1] Turaev, V., *Nilpotent homotopy types of closed 3-manifolds*, Proceedings of a Topology Conference in Leningrad, Springer Lecture Notes in Mathematics,, vol. 1060, Springer-Verlag, 1982.

[Tu2] ________, *Cohomology rings, linking forms and invariants of Spin Structures on three-dimensional manifolds*, Math. USSR Sbornik 48 (1984), no. 1, 65–79.

[Wa] Wallace, A.H., *Modifications and Cobounding Manifolds*, Canadian Journal of Math. (1960), 503–528.

Tim Cochran
Mathematics Department

45
Rice University
6100 Main Street
Houston, TX 77005–1892
cochran@rice.edu
Related papers available also at http://math.rice.edu/~cochran/