Scattering by $\mathcal{PT}$-symmetric Non-local Potentials

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Abstract

A general formalism is worked out for the description of one-dimensional scattering by non-local separable potentials and constraints on transmission and reflection coefficients are derived in the cases of $\mathcal{P}$, $\mathcal{T}$ or $\mathcal{PT}$ invariance of the Hamiltonian. The case of a solvable Yamaguchi potential is discussed in detail.

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1 Introduction

Non local potentials (see, e.g., Ref.[1]) play an important role in several applications of quantum scattering theory. In nuclear physics, for instance, they naturally arise from convolution of an effective nucleon-nucleon interaction with the density of a target nucleus. In the present note, focused on one-dimensional scattering, we study the behaviour of reflection and transmission
coefficients of a non-local solvable potential with separable kernel in connection with the characteristics of the kernel itself, by considering, in particular, the cases where the kernel is real symmetric, hermitian or \( PT \)-symmetric.

2 Formalism

The formalism adopted in the present work is consistent with recent general references on one-dimensional scattering by complex potentials, such as Ref.[2], and \( PT \)-symmetric potentials, such as Ref.[3].

Let us introduce the one-dimensional Schrödinger equation for a monochromatic wave of energy \( E = k^2 \) scattered by a non-local potential with kernel \( K \), written in units \( \hbar = 2m = 1 \)

\[
-\frac{d^2}{dx^2}\Psi(x) + \lambda \int K(x, y)\Psi(y)dy = k^2\Psi(x),
\]

where the potential strength, \( \lambda \), is a real number.

It is easy to check, by calculating scalar products, that the kernel of a hermitian non-local potential satisfies the condition

\[
K(x, y) = K^*(y, x).
\]

Parity \((P)\) invariance of the potential could be similarly checked to imply

\[
K(x, y) = K(-x, -y).
\]

The condition of time reversal \((T)\) invariance of \( K \) can be written in the form

\[
K(x, y) = K^*(x, y),
\]

while \( PT \) invariance corresponds to the condition

\[
K(x, y) = K^*(-x, -y),
\]

in agreement with formula (3) of Ref.[4], which corrects a misprint in the corresponding formula (113) of Ref.[2].

In order to deal with a solvable \( PT \)-symmetric potential, we consider only separable kernels of the kind

\[
K(x, y) = g(x)e^{i\alpha x}h(y)e^{i\beta y},
\]
Table 1: Possible symmetries of the separable kernel (6)

| Symmetry            | Condition                                      |
|---------------------|------------------------------------------------|
| Reality             | $\alpha = \beta = 0$                           |
| Symmetry under $x \leftrightarrow y$ | $\alpha = \beta, g = h$                      |
| Hermiticity         | $\alpha = -\beta, g = h$                      |
| $\mathcal{P}$ Invariance | $\alpha = \beta = 0, g(x) = g(-x), h(y) = h(-y)$ |
| $\mathcal{T}$ Invariance | $\alpha = \beta = 0$                          |
| $\mathcal{PT}$ Invariance | $g(x) = g(-x), h(y) = h(-y)$                |

where $\alpha$ and $\beta$ are real numbers, and $g(x)$ and $h(y)$ are real functions of their argument, suitably vanishing at $\pm \infty$.

For this kind of kernel, the hermiticity condition (2) implies $\alpha = -\beta$ and $g = h$. Parity invariance (3) requires $\alpha = \beta = 0$ and $g(x) = g(-x)$, $h(y) = h(-y)$. Time reversal invariance (4) requires $\alpha = \beta = 0$, but does not impose conditions on $g$ and $h$.

The various conditions that can be imposed on kernel (6) are summarized in Table 1. Finally, $\mathcal{PT}$ invariance (5) does not impose conditions on $\alpha$ and $\beta$, but requires $g(x) = g(-x)$, $h(y) = h(-y)$. As an important consequence, their Fourier transforms, $\tilde{g}(q)$ and $\tilde{h}(q')$, are real even functions, too.

In order to solve eq. (1), we resort to the Green’s function method. As is known, the Green’s function of the problem is a solution to Eq. (1) with the potential term replaced with a Dirac delta function

$$\frac{d^2}{dx^2}G_\pm(x, y) + (k^2 \pm i\varepsilon)G_\pm(x, y) = \delta(x - y). \quad (7)$$

Here, we introduce the infinitesimal positive number $\varepsilon$ in order to shift upwards, or downwards in the complex momentum plane the singularities of the Fourier transform of the Green’s function, $G_\pm(q, q')$, lying on the real axis.

In fact, after defining the Fourier transform, $\tilde{f}(q)$, of a generic function $f(x)$ as

$$\tilde{f}(q) = \int_{-\infty}^{+\infty} f(x)e^{-iqx}dx \quad \leftrightarrow \quad f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{f}(q)e^{iqx}dq,$$

and expressing $G_\pm(x, y)$ and $\delta(x - y)$ in terms of their Fourier transforms, we quickly solve eq. (7) for $G_\pm$

$$G_\pm(q, q') = \frac{2\pi\delta(q + q')}{-q^2 + k^2 \pm i\varepsilon}.$$
Therefore, the Green’s function in coordinate space is
\[
G_\pm(x, y) = -\frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{1}{q^2 - k^2 \mp i\varepsilon} e^{iq(x-y)} dq = G_\pm(x - y) .
\] (8)

The integral (8) is easily computed by the method of residues. In fact, the integrand in \(G_+(x - y)\) has two first order poles at \(q_1 = k + i\varepsilon'\) and \(q_2 = -k - i\varepsilon'\), where \(\varepsilon' = \varepsilon/(2k)\): the integral is thus computed along a contour made of the real \(q\) axis and of a half-circle of infinite radius in the upper half-plane for \(x - y > 0\), on which the integrand vanishes, thus enclosing the pole at \(q = q_1\), and in the lower half-plane for \(x - y < 0\), enclosing the pole at \(q = q_2\), for the same reason. Notice that the \(G_+\) contour integral is done in the counterclockwise direction for \(x - y > 0\), while it is done in the clockwise direction for \(x - y < 0\), so that the latter acquires a global sign opposite to the former. The result is
\[
G_+(x - y) = -\frac{i}{2k} \left[ e^{ik(x-y)}\theta(x - y) + e^{-ik(x-y)}\theta(y - x) \right] = -\frac{i}{2k} e^{ik|x-y|} , \quad \text{(9)}
\]
where \(\theta(x)\) is the step function, equal to 1 for \(x > 0\) and 0 otherwise.

The second Green’s function, \(G_-(x - y)\), is the complex conjugate of \(G_+(x - y)\)
\[
G_-(x - y) = \frac{i}{2k} \left[ e^{-ik(x-y)}\theta(x - y) + e^{ik(x-y)}\theta(y - x) \right] = \frac{i}{2k} e^{-ik|x-y|} . \quad \text{(10)}
\]

Now, we go back to eq. (1) with kernel (6), call \(\Psi_\pm(x)\) two linearly independent solutions, for a reason that will become clear in the next few lines, and define the following integral depending on \(\Psi_\pm\)
\[
I_\pm(\beta, k) = \int_{-\infty}^{+\infty} e^{i\beta y} h(y) \Psi_\pm(y) dy .
\]
It is easy to show that \(I_\pm(\beta, k)\) can be written as a convolution of the Fourier transforms of \(h(y)\) and \(\Psi_\pm(y)\).

The general solution to eq. (1) is thus implicitly written as
\[
\Psi_\pm(x) = c_\pm e^{ikx} + d_\pm e^{-ikx} + \lambda I_\pm(\beta, k) \int_{-\infty}^{+\infty} G_\pm(x - y) g(y) e^{i\alpha y} dy . \quad \text{(11)}
\]

Eq. (11) allows us to express \(I_\pm(\beta, k)\) in terms of the constants \(c_\pm\) and \(d_\pm\) and of Fourier transforms of known functions: in fact, by multiplying both sides by \(h(x) e^{i\beta x}\) and integrating over \(x\), we obtain
\[
I_\pm(\beta, k) = c_\pm \tilde{h}(k + \beta) + d_\pm \tilde{h}(k - \beta) + \lambda N_\pm(\alpha, \beta, k) I_\pm(\beta, k) , \quad \text{(12)}
\]
where we have exploited the symmetry $\tilde{h}(-k - \beta) = \tilde{h}(k + \beta)$ and $N_\pm$ is defined as

$$N_\pm(\alpha, \beta, k) = \int_{-\infty}^{+\infty} h(x) e^{i\beta x} G_\pm(x - y) g(y) e^{i\alpha y} dx dy$$

(13)

$$= \pm \frac{i}{2k} \int_{-\infty}^{+\infty} h(x) e^{i\beta x} e^{\pm ik|x-y|} g(y) e^{i\alpha y} dx dy,$$

so that

$$I_\pm(\beta, k) = \frac{c_\pm \tilde{h}(k + \beta) + d_\pm \tilde{h}(k - \beta)}{1 - \lambda N_\pm(\alpha, \beta, k)} = (c_\pm \tilde{h}(k + \beta) + d_\pm \tilde{h}(k - \beta)) D_\pm,$$  

(14)

where

$$D_\pm(\alpha, \beta, k) \equiv \frac{1}{1 - \lambda N_\pm(\alpha, \beta, k)}.$$

Let us examine now the asymptotic behaviour of the two independent solutions, starting from $\Psi_+(x)$

$$\Psi_+(x) = c_+ e^{ikx} + d_+ e^{-ikx} + \lambda I_+(\beta, k) \int_{-\infty}^{+\infty} G_+(x - y) g(y) e^{i\alpha y} dy.$$  

(15)

The asymptotic behaviour of the integral on the r. h. s. of eq. (15) is promptly evaluated by observing that, according to eq. (9),

$$\lim_{x \to \pm\infty} G_+(x - y) = -\frac{i}{2k} e^{\pm ik(x-y)},$$

so that

$$\lim_{x \to \pm\infty} \Psi_+(x) = c_+ e^{ikx} + d_+ e^{-ikx} - i \omega I_+(\beta, k) \tilde{g}(k \mp \alpha) e^{\pm ikx},$$

where we have put $\omega = \lambda/(2k)$.

Remembering the expression (14) of $I_+$, we finally obtain

$$\lim_{x \to -\infty} \Psi_+(x) = c_+ e^{ikx} + \left\{ d_+ - i \omega \tilde{g}(k + \alpha) \left[ c_+ \tilde{h}(k + \beta) + d_+ \tilde{h}(k - \beta) \right] D_+ \right\} e^{-ikx},$$

$$\lim_{x \to +\infty} \Psi_+(x) = \left\{ c_+ - i \omega \tilde{g}(k - \alpha) \left[ c_+ \tilde{h}(k + \beta) + d_+ \tilde{h}(k - \beta) \right] D_+ \right\} e^{ikx} + d_+ e^{-ikx}.$$  

The constants $c_+$ and $d_+$ are fixed by initial conditions: if we impose that $\Psi_+(x)$ represents a wave travelling from left to right,

$$\lim_{x \to -\infty} \Psi_+(x) = e^{ikx} + R_{L\to R} e^{-ikx},$$

$$\lim_{x \to +\infty} \Psi_+(x) = T_{L\to R} e^{ikx},$$

5
we immediately have $c_+ = 1$, $d_+ = 0$ and the transmission and reflection coefficients turn out to be, respectively

$$
T_{L \to R} = 1 - i\omega \tilde{g}(k - \alpha)\tilde{h}(k + \beta)D_+(\alpha, \beta, k),
$$

$$
R_{L \to R} = -i\omega \tilde{g}(k + \alpha)\tilde{h}(k + \beta)D_+(\alpha, \beta, k).
$$

It is worthwhile to point out that the above expressions break unitarity, i.e. $|T_{L \to R}|^2 + |R_{L \to R}|^2 \neq 1$, because probability flux is not conserved in general.

We come now to the second solution, $\Psi_-(x)$, written in the form

$$
\Psi_-(x) = c_- e^{ikx} + d_- e^{-ikx} + \lambda I_- (\beta, k) \int_{-\infty}^{+\infty} G_-(x - y) g(y) e^{i\alpha y} dy.
$$

The asymptotic behaviour of the Green’s function, $G_-(x)$, is now

$$
\lim_{x \to \pm \infty} G_-(x - y) = \frac{i}{2k} e^{\mp ik(x-y)},
$$

so that

$$
\lim_{x \to \pm \infty} \Psi_-(x) = c_- e^{ikx} + d_- e^{-ikx} + i\omega I_- (\beta, k) \tilde{g}(k \pm \alpha) e^{\mp ikx},
$$

or, using the explicit expression (14) of $I_-$,

$$
\lim_{x \to -\infty} \Psi_-(x) = d_- e^{-ikx} + \left\{ c_- + i\omega \tilde{g}(k - \alpha) \left[ c_- \tilde{h}(k + \beta) + d_- \tilde{h}(k - \beta) \right] \right\} e^{ikx},
$$

$$
\lim_{x \to +\infty} \Psi_-(x) = c_- e^{ikx} + \left\{ d_- + i\omega \tilde{g}(k + \alpha) \left[ c_- \tilde{h}(k + \beta) + d_- \tilde{h}(k - \beta) \right] \right\} e^{-ikx}.
$$

Since $\Psi_-(x)$ and $\Psi_+(x)$ are linearly independent, we can impose that $\Psi_-(x)$ is a wave travelling from right to left,

$$
\lim_{x \to -\infty} \Psi_-(x) = T_{R \to L} e^{-ikx},
$$

$$
\lim_{x \to +\infty} \Psi_-(x) = e^{-ikx} + R_{R \to L} e^{ikx}.
$$

The initial conditions now are

$$
c_- + i\omega \tilde{g}(k - \alpha)(c_- \tilde{h}(k + \beta) + d_- \tilde{h}(k - \beta)) D_- (\alpha, \beta, k) = 0,
$$

$$
d_- + i\omega \tilde{g}(k + \alpha)(c_- \tilde{h}(k + \beta) + d_- \tilde{h}(k - \beta)) D_- (\alpha, \beta, k) = 1,
$$
where
\[ d_- = T_{R \to L}, \quad c_- = R_{R \to L}. \] (18)

We then obtain
\[
T_{R \to L} = 1 - i\omega \tilde{g}(k + \alpha)\tilde{h}(k - \beta)D_-(\alpha, \beta, k), \\
R_{R \to L} = -i\omega \tilde{g}(k - \alpha)\tilde{h}(k - \beta)D_-(\alpha, \beta, k).
\] (19)

where
\[
D_-(\alpha, \beta, k) = \frac{1}{1 - \lambda N_- + i\omega(\tilde{g}(k + \alpha)\tilde{h}(k - \beta) + \tilde{g}(k - \alpha)\tilde{h}(k + \beta))}.
\]

Formulae (16-18) show that, in general, for a \(\mathcal{PT}\)-symmetric non-local potential, \(T_{R \to L} \neq T_{L \to R}\). In fact, from the quoted formulae,
\[
T_{R \to L} - T_{L \to R} = i\omega \Delta D_+ (\alpha, \beta, k) D_-(\alpha, \beta, k),
\]
where
\[
\Delta = \tilde{g}(k - \alpha)\tilde{h}(k + \beta) - \tilde{g}(k + \alpha)\tilde{h}(k - \beta) + \lambda (N_+ \tilde{g}(k + \alpha)\tilde{h}(k - \beta) - N_- \tilde{g}(k - \alpha)\tilde{h}(k + \beta)) + i\omega(\tilde{g}(k - \alpha)\tilde{h}(k + \beta)\tilde{g}(k + \alpha)\tilde{h}(k - \beta) + \tilde{g}(k - \alpha)\tilde{h}(k + \beta)).
\]

Computation of the \(N\) integrals yields the general forms
\[
N_+ (\alpha, \beta, k) = \frac{-i}{4k} \left[ \tilde{g}(k - \alpha)\tilde{h}(k + \beta) + \tilde{g}(k + \alpha)\tilde{h}(k - \beta) \right] + Q (\alpha, \beta, k), \\
N_- (\alpha, \beta, k) = \frac{i}{4k} \left[ \tilde{g}(k - \alpha)\tilde{h}(k + \beta) + \tilde{g}(k + \alpha)\tilde{h}(k - \beta) \right] + Q (\alpha, \beta, k),
\]

where the function \(Q (\alpha, \beta, k)\) is real.

If we now make the additional assumption that our kernel is symmetric, \(K(x, y) = K(y, x)\), i.e. \(g = h\) and \(\alpha = \beta\), we obtain \(T_{R \to L} = T_{L \to R}\). It is worthwhile to stress that imposing the symmetry of the kernel is equivalent to imposing the intertwining condition \(T K T^{-1} = K^\dagger\), which ensures the equality of the two transmission coefficients.

Furthermore, when \(\alpha = \beta = 0\), the symmetric kernel becomes real, and exhibits both hermiticity and time reversal invariance. One can then show that \(Q (\alpha = 0, \beta = 0, k)\) is a real function of \(k\) and that unitarity holds, i.e. \(|T|^2 + |R|^2 = 1\).
In order to obtain a complete solution of the scattering problem, \textit{i.e.} the explicit form of the $Q$ function, we now consider the one-dimensional $\mathcal{PT}$-symmetric version of a Yamaguchi potential\cite{5}, with

$$g (x) = \exp (-\gamma |x|), \quad h (y) = \exp (-\delta |y|), \quad (-\infty < x, y < +\infty)$$ \hspace{1cm} (20)

where $\gamma$ and $\delta$ are positive numbers. The Fourier transforms of $g$ and $h$ are, respectively

$$\tilde{g} (q) = \frac{2\gamma}{q^2 + \gamma^2}, \quad \tilde{h} (q) = \frac{2\delta}{q^2 + \delta^2}.$$ \hspace{1cm} (21)

In this case, the $N_{\pm}$ integrals can be computed by elementary methods, without making use of the Parseval-Plancherel relation and of the convolution theorem: the $Q$ function can be written in the form

$$Q (\alpha, \beta, \gamma, \delta, k) = \frac{2 (\gamma^2 - \alpha^2 + k^2) (\gamma + \delta) - 4\alpha \gamma (\alpha + \beta)}{(\gamma + \delta)^2 + (\alpha + \beta)^2} \left[ (\gamma^2 - \alpha^2 + k^2)^2 + 4\alpha^2\gamma^2 \right]$$

$$+ \frac{1}{4\delta k} \cdot \left[ (k - \beta) \tilde{g} (k + \alpha) \tilde{h} (k - \beta) + (k + \beta) \tilde{g} (k - \alpha) \tilde{h} (k + \beta) \right].$$ \hspace{1cm} (22)

By inserting the expression of the $Q$ function given above into the $N_{\pm}$ integrals, we obtain the complete analytic expressions of the two transmission coefficients, $T_{L\rightarrow R}$ and $T_{R\rightarrow L}$, as well as the two reflection coefficients, $R_{L\rightarrow R}$ and $R_{R\rightarrow L}$, respectively.

Let us put $L \rightarrow R = a$ and $R \rightarrow L = b$ for brevity’s sake, and indicate with $\varphi (z)$ the phase of the complex number $z = |z| e^{i\varphi}$, where $z$ represents either a transmission, or a reflection coefficient. Direct inspection of the formulae allows us to characterize the behaviour of the coefficients when the kernel is real, hermitian, or $\mathcal{PT}$-symmetric, summarized in table 2.

It is worthwhile to point out that in the real and hermitian cases, when $|T_a| = |T_b| = |T|$ and $|R_a| = |R_b| = |R|$, unitarity is conserved ($|T|^2 + |R|^2 = 1$), while in the $\mathcal{PT}$-symmetric cases ($|R_a| \neq |R_b|$) unitarity is broken ($|T_i|^2 + |R_i|^2 \neq 1$, with $i = a, b$).

In the case of local $\mathcal{PT}$-symmetric potentials, starting from flux considerations, a left-right asymmetry in unitarity breaking for a wave entering the interaction region from the absorptive side ($\Im V < 0$), or from the emissive side ($\Im V > 0$) was noticed in Ref.\cite{6} : while one set of transmission and reflection coefficients obeys the inequality $|T_i|^2 + |R_i|^2 \leq 1$ for all values of the momentum, $k$, the second set can have $|T_j|^2 + |R_j|^2 > 1$ for some values
of $k$. Changing the sign of the imaginary part of the $\mathcal{PT}$-symmetric potential while leaving the real part unchanged is equivalent to a parity transformation, which exchanges left with right and, consequently, the two sets of coefficients with their asymmetric unitarity breaking: this property is called handedness in Ref.\[6\].

A parity transformation of the $\mathcal{PT}$-symmetric Yamaguchi potential \[6\] could be obtained either by applying the definition of $\mathcal{P}$ ($x \rightarrow -x$, $y \rightarrow -y$), or, equivalently, by the reflection $\alpha \rightarrow -\alpha$, $\beta \rightarrow -\beta$. Thus, one can expect that the latter transformation induces some kind of left-right transformation of scattering observables. Therefore, one can assert for the $\mathcal{PT}$-symmetric Yamaguchi potential with real $\lambda$, $\alpha$, $\beta$ and positive $\gamma$, $\delta$ the validity of the relations

$$T_{L \rightarrow R} (\alpha, \beta) = T_{R \rightarrow L} (-\alpha, -\beta) ,$$
$$R_{L \rightarrow R} (\alpha, \beta) = R_{R \rightarrow L} (-\alpha, -\beta) .$$  \[(23)\]

In particular, when $\alpha = \beta = 0$, one recovers parity invariance of the potential. In general, under the $\alpha \rightarrow -\alpha$, $\beta \rightarrow -\beta$ transformation, the two transmission coefficients exchange their phases, while the two reflection coefficients exchange their moduli. The asymmetry in unitarity breaking is no more valid, \textit{a priori}, for non-local potentials, and for the Yamaguchi potential in particular. In fact, (\textit{i}) the probability flux does not obey the standard continuity equation\[1\] and (\textit{ii}) it is not trivial to identify an absorptive and an emissive side of the interaction region unambiguously.

Numerical evaluation of the transmission and reflection coefficients for a Yamaguchi potential barrier ($\lambda > 0$) shows that, when $\alpha$ and $\beta$ have opposite sign, $|T_i|^2 + |R_i|^2$ may be $\leq 1$ in a range of $k$ values, and $\geq 1$ in another range for $i = L \rightarrow R$ and $R \rightarrow L$ simultaneously. In this case, we do not distinguish an absorptive side and an emissive side any more.

When both $\alpha$ and $\beta$ are positive, we obtain $|T_{L \rightarrow R}|^2 + |R_{L \rightarrow R}|^2 \leq 1$ and $|T_{R \rightarrow L}|^2 + |R_{R \rightarrow L}|^2 \geq 1$ for all values of $k$. Left and right are, obviously, exchanged under $\alpha \rightarrow -\alpha$, $\beta \rightarrow -\beta$, as expected from the parity transformation of the potential. Relations \[(23)\] are, of course, rigorously valid for all values of $\alpha$ and $\beta$. 
\[ \alpha = \beta = 0 \quad |T_a| = |T_b| \quad \varphi(T_a) = \varphi(T_b) \quad |R_a| = |R_b| \quad \varphi(R_a) = \phi(R_b) \]

\[ \alpha = -\beta, \gamma = \delta \quad |T_a| = |T_b| \quad \varphi(T_a) \neq \varphi(T_b) \quad |R_a| = |R_b| \quad \varphi(R_a) = \phi(R_b) \]

\[ \alpha = \beta \neq 0, \gamma = \delta \quad |T_a| = |T_b| \quad \varphi(T_a) = \varphi(T_b) \quad |R_a| \neq |R_b| \quad \varphi(R_a) = \phi(R_b) \]

\[ \alpha \neq \beta, \gamma \neq \delta \quad |T_a| = |T_b| \quad \varphi(T_a) \neq \varphi(T_b) \quad |R_a| \neq |R_b| \quad \varphi(R_a) = \phi(R_b) \]

Table 2: Properties of transmission and reflection coefficients for (1) real, (2) hermitian, (3) symmetric-$\mathcal{PT}$-symmetric and (4)-$\mathcal{PT}$-symmetric kernels.

3 Conclusions

An explicit construction of a solvable separable complex potential has been presented and worked out in detail. A particularly notable difference between local and non-local $\mathcal{PT}$-symmetric potentials is the non-equality of the phases of the two transmission coefficients in the non-local case.

While the Yamaguchi kind of kernel we have constructed has a cusp at the origin ($x = y = 0$), it has the merit of permitting analytic calculations of scattering observables in one dimension. Thus, it provides a very convenient solvable model for $\mathcal{PT}$-symmetric non-local interactions. A non-trivial feature of this model is the violation of the handedness in unitarity breaking.

As an outlook on future work, we mention some topics not discussed in detail in the present work: (i) the study of the Yamaguchi potential well ($\lambda < 0$) and of the corresponding bound states; (ii) the analytic structure of the transmission coefficients \[10\], which appear to be rational functions of $k$.

Particular attention should be paid to the zeros of the denominator of $T_i(k)$ connected with bound states (imaginary $k$) and to the zeros of the numerator. If the latter occur at positive values of $k$, they may produce exotic anomalies in the $k$-dependence, i.e. the vanishing of $T_i(k)$ at isolated values of $k$. In our understanding, the anomalies are connected with the asymptotic vanishing of the Wronskian of the solutions, $\Psi_+$ and $\Psi_-$. The physical interpretation of such a pathological effect deserves further investigation.

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