FRACTIONAL BLOOM BOUNDEDNESS AND COMPACTNESS OF COMMUTATORS

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ABSTRACT. Let $T$ be a non-degenerate Calderón-Zygmund operator and let $b : \mathbb{R}^d \to \mathbb{C}$ be locally integrable. Let $1 < p \leq q < \infty$ and let $\mu^p \in A_p$ and $\lambda^q \in A_q$, where $A_p$ denotes the usual class of Muckenhoupt weights. We show that

$$\| [b, T] \|_{L^p \mu \to L^q \lambda} \sim \| b \|_{BMO^{\alpha \nu}},$$

where $L^p_\mu = L^p(\mu^p)$ and $\alpha/d = 1/p - 1/q$, the symbol $K$ stands for the class of compact operators between the given spaces, and the fractional weighted $BMO^\alpha_\nu$ and $VMO^\alpha_\nu$ spaces are defined through the following fractional oscillation and Bloom weight

$$O^\alpha_\nu(b; Q) = \nu(Q)^{-\alpha/d} \left( \frac{1}{\nu(Q)} \int_Q |b - \langle b \rangle_Q| \right), \quad \nu = \left( \frac{\nu}{\lambda} \right)^\beta, \quad \beta = (1 + \alpha/d)^{-1}.$$

The key novelty is dealing with the off-diagonal range $p < q$, whereas the case $p = q$ was previously studied by Lacey and Li. However, another novelty in both cases is that our approach allows complex-valued functions $b$, while other arguments based on the median of $b$ on a set are inherently real-valued.

1. INTRODUCTION AND BACKGROUND

1.1. Unweighted commutator theory. The study of commutators of singular integrals have their roots in the work of Nehari [19], where the boundedness of the commutator of the Hilbert transform and a multiplication by $b$ (the symbol of the commutator) was characterized through a connection with Hankel operators. Later, in 1976, Coifman, Rochberg and Weiss [3] developed real-analytic methods and extended Nehari’s result by providing the following commutator lower- and upper bounds\(^1\)

$$\| b \|_{BMO(\mathbb{R}^d)} \lesssim \sum_{j=1}^d \| [b, R_j] \|_{L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)} \lesssim \| b \|_{BMO(\mathbb{R}^d)},$$

where $p \in (1, \infty)$, $BMO = BMO^0$, and we denote

$$\| b \|_{BMO^\alpha} := \sup_{Q \in \mathcal{Q}} O^\alpha(b; Q), \quad O^\alpha(b; Q) = \ell(Q)^{-\alpha} \int_Q |b - \langle b \rangle_Q|,$$

where $\mathcal{Q}$ stands for the collection of all cubes. When $\alpha = 0$, we drop the superscript. The upper bound (1.1) was proved in [3] for a wide class of bounded singular integrals with

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\(^1\)For notation see Section 2.2 below.
convolution kernels, while the lower bound especially concerned the Riesz transforms
\[ R_j f(x) = p.v. \int_{\mathbb{R}^d} \frac{x_j - y_j}{|x - y|^{d+1}} f(y) \, dy. \]

The lower bound in (1.1) was improved separately by both Janson [10] and Uchiyama [23] to \( \|b\|_{\text{BMO}} \lesssim \|b, T\|_{L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)} \) for a wider class of singular integrals that includes any single Riesz transform (in contrast with (1.1) involving all the \( d \) Riesz transforms). Janson’s proof also gives the following off-diagonal extension of the boundedness of the commutator in terms of the homogeneous Hölder space; when \( 1 < p < q < \infty \), there holds that
\[
\| [b, T] \|_{L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)} \sim \|b\|_{C^{0,\alpha}(\mathbb{R}^d)} := \sup_{x \neq y} \frac{|b(x) - b(y)|}{|x - y|^{\alpha}}, \quad \frac{1}{d} = \frac{1}{p} - \frac{1}{q}.
\]

When \( \alpha > 0 \), an elementary argument shows that \( \|b\|_{C^{0,\alpha}} \sim \|b\|_{\text{BMO}^\alpha} \), and hence, for \( 1 < p \leq q < \infty \), we have a unified characterization of the boundedness of the commutator in terms of a local oscillatory testing condition on the symbol of the commutator. The second off-diagonal case \( 1 < q < p < \infty \) was characterised by one of us [7] through a global oscillatory condition as
\[
\| [b, T] \|_{L^p \to L^q} \sim \|b\|_{L^r} := \inf_{c \in \mathbb{C}} \|b - c\|_{L^r}, \quad \frac{1}{q} = \frac{1}{r} + \frac{1}{p}.
\]

The commutator lower bounds of (1.1), (1.3) and (1.4) are currently available by two different methods, the first being the approximate weak factorization (awf) argument, the second being the median method. The advantage of both is that they provide a uniform approach to all of the three lower bounds in (1.1), (1.3) and (1.4). As discovered in [7], at their common core lies a minimal notion of non-degeneracy of kernels of singular integrals. Let us next briefly recall the appropriate definitions, thus also fixing the operators of interest in this article.

1.5. Definition. A singular integral operator (SIO) \( T \) is a linear operator \( T: S \to L^1_{\text{loc}} \) on the class of Schwartz functions that has the off-support representation
\[
T f(x) = \int_{\mathbb{R}^d} K(x, y) f(y) \, dy, \quad x \notin \text{spt}(f),
\]
where the kernel \( K: \mathbb{R}^d \times \mathbb{R}^d \setminus \{x = y\} \rightarrow \mathbb{C} \) satisfies the size estimate
\[
|K(x, y)| \leq C|x - y|^{-d},\]
and the smoothness estimates
\[
|K(x', y) - K(x, y)| + |K(y, x') - K(y, x)| \leq \omega \left( \frac{|x - x'|}{|x - y|} \right) |x - y|^{-d},
\]
whenever \( |x - x'| \leq \frac{1}{2} |x - y| \), and where the modulus \( \omega \) is increasing and sub-additive, and satisfies \( \omega(0) = 0 \) and
\[
\|\omega\|_{\text{Dini}} = \int_0^1 \omega(t) \frac{dt}{t} < \infty.
\]
1.7. Definition. A Calderón-Zygmund operator (CZO) is a singular integral operator that is bounded on \( L^2(\mathbb{R}^d) \). We also denote
\[
\|T\|_{\text{CZO}} = \|T\|_{L^2 \to L^2} + \|\omega\|_{\text{Dini}} + C,
\]
where $C$ is the smallest admissible constant in (1.6).

1.8. **Definition.** A kernel $K(x, y)$ is said to be non-degenerate, if for all $y$ and $r > 0$ there exists $x$ such that

$$|x - y| \geq r, \quad |K(x, y)| \gtrsim r^{-d}.$$  

1.9. **Definition.** An SIO is said to be non-degenerate if its kernel is non-degenerate, and a CZO is said to be non-degenerate if it is a non-degenerate SIO.

This notion of non-degeneracy was introduced in [7] (Definition 2.1.1). It generalizes similar notions in the earlier literature, in particular one due to Stein [22] (IV.4.6). See [7], Remark 2.1.2 and Examples 2.1.3 to 2.1.6, for a detailed comparison of these notions of non-degeneracy.

It is typical that commutator upper bounds are valid for all Calderón-Zygmund operators, while the lower bounds require some non-degeneracy from the singular integral. We gather everything from the above into the following theorem that fully characterizes, for $p, q \in (1, \infty)$, the boundedness of the commutator between unweighted spaces.

1.10. **Theorem** ([3, 7, 10]). Let $1 < p, q < \infty$, $b \in L^1_{\text{loc}}(\mathbb{R}^d; \mathbb{C})$ and $T$ be a non-degenerate Calderón-Zygmund operator. Then, there holds that

$$\| [b, T] \|_{L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)} \sim \begin{cases} \|b\|_{bMO(\mathbb{R}^d)}, & q = p, \quad (1976) \ [3], \\ \|b\|_{C^{\alpha, \nu}(\mathbb{R}^d)}, & \frac{1}{q} = \frac{1}{p} - \frac{1}{\alpha}, \quad q > p, \quad (1978) \ [10], \\ \|b\|_{L^s(\mathbb{R}^d)}, & \frac{1}{q} = \frac{1}{s} + \frac{1}{p}, \quad q < p, \quad (2021) \ [7]. \end{cases}$$

In addition to boundedness, we also study the compactness of commutators.

1.11. **Definition.** A linear operator $T : X \to Y$ between two Banach spaces is said to be compact, provided that for each bounded set $A \subset X$, the image $TA \subset Y$ is relatively compact, i.e. the closure $\overline{TA}$ is compact in $Y$. We denote by $\mathcal{K}(X, Y)$ the collection of all compact operators from $X$ to $Y$.

Recall that all compact operators are bounded. Already Uchiyama [23] in 1978 provided, for a wide class of singular integrals, a characterization of compactness of their commutators in terms of the symbol belonging to the space $\text{VMO} \subset \text{BMO}$ of vanishing mean oscillation (see Definition 2.1 and set $\alpha = 0, \nu = 1$). Without providing the whole background, a special case of Uchiyama’s result is the following

1.12. **Theorem** ([23]). Let $b \in L^1_{\text{loc}}$ and $p \in (1, \infty)$. Then, for each $j = 1, \ldots, d$, there holds that

$$(1.13) \quad [b, \mathcal{R}_j] \in \mathcal{K}(L^p(\mathbb{R}^d), L^p(\mathbb{R}^d)) \iff b \in \text{VMO}(\mathbb{R}^d).$$

The main purpose of this article is to provide two-weight extensions of Theorems 1.10 and 1.12 that fully cover the case $1 < p \leq q < \infty$.

1.2. **Setup for two-weight bounds.** For integrability parameters $1 < p, q < \infty$ and weights $\mu, \lambda : \mathbb{R}^d \to \mathbb{R}_+$ (positive locally integrable functions) our goal is to obtain a characterization of both the boundedness and the compactness of the commutator as a mapping

$$[b, T] : L^p_\mu \to L^q_\lambda, \quad \|f\|_{L^q_\lambda} = \left( \int |f\sigma|^q \right)^{\frac{1}{q}}.$$
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This convention of treating the weight as a multiplier goes hand in hand with using the following rescaled weight characteristics

\[
[\sigma, \omega]_{A_{p,q}} = \sup_{Q \in \mathcal{Q}} \langle \sigma^q \rangle_{Q}^\frac{1}{q} \langle \omega^{-p'} \rangle_{Q}^\frac{1}{p'}, \quad [\mu]_{A_{p,q}} = [\mu, \mu]_{A_{p,q}}, \quad [\mu^p]_{A_p} = [\mu]_{A_{p,p}}.
\]

We say that \( \mu \in A_{p,p} \), provided that \([\mu]_{A_{p,p}} < \infty\), and \( \mu \in A_{p} \), provided that \([\mu]_{A_p} < \infty\).

In 1985, under the assumption that \( \mu, \lambda \in A_{p,p} \), Bloom [1] characterized the two-weight boundedness of the Hilbert commutator on the line by providing the two-sided estimate

\[
\|b\|_{BMO_{\nu}} \lesssim \| [b, H] \|_{L_p^\mu \rightarrow L_p^\lambda} \lesssim \| b \|_{BMO_{\nu}},
\]

where we refer to the left and right estimates as the Bloom lower and upper bounds, respectively, and where

\[
\|b\|_{BMO_{\nu}} = \sup_{Q \in \mathcal{Q}} \frac{1}{\nu(Q)} \int_Q |b - \langle b \rangle_Q|, \quad \nu = \mu/\lambda.
\]

Recall that both estimates on the line (1.15) depend on the weight characteristics \([\mu]_{A_{p,p}}\), \([\lambda]_{A_{p,p}}\), a fact we will suppress in our notation. This seminal work of Bloom on the Hilbert transform was extended to general Calderón-Zygmund operators by Segovia and Torrea [21].

A renewed interest into such estimates, now commonly called “of Bloom type”, was sparked by the works of Holmes, Lacey and Wick [5, 6] who revisited both results with the recent technology of dyadic representation theorems, due to Petermichl [20] for the Hilbert transform, and due to one of us [9] for general Calderón-Zygmund operators. Only shortly after, Lerner, Ombrosi and Rivera-Ríos [14] obtained a sparse domination of commutators that yielded a simpler proof and also the sharp upper bound, in the sense that the sharp one-weight estimate was reproduced. At this point, the Bloom lower bound was checked for the vector of the Riesz transform in [5] (involving all the \( d \) Riesz transforms, similarly as above on the line (1.1)). The extension to real-valued homogeneous symbols that do not change sign on some open subset of \( S^{d-1} \) was obtained by Lerner et al. [15], and the Bloom lower bound for general SIOs of either variable or rough homogeneous kernel, and for complex symbols, was subsequently proved by one of us [7]. To complete the picture, recently K. Li [16] sketched the proof of the Bloom upper bound for rough homogeneous CZOs. All in all, the state-of-the-art qualitative two-weight Bloom boundedness of commutators is recorded as the following

1.17. Theorem ((5, 7, 14, 16)). Let \( 1 < p < \infty \), let \( \mu, \lambda \in A_{p,p} \), let \( T \) be a non-degenerate Calderón-Zygmund operator and \( b \in L_{1_{loc}}^1(\mathbb{R}^d; \mathbb{C}) \). Then, there holds that

\[
\| [b, T] \|_{L_p^\mu \rightarrow L_p^\lambda} \sim \| b \|_{BMO_{\nu}}.
\]

The corresponding characterization of compactness for CZOs of variable kernel was recently obtained by Lacey and J. Li [11], where they proved the following

1.19. Theorem ([11]). Let \( 1 < p < \infty \) and \( \mu, \lambda \in A_{p,p} \), let \( T \) be a non-degenerate Calderón-Zygmund operator with variable kernel, as in Definition 1.5, and let \( b \in L_{1_{loc}}^1(\mathbb{R}^d; \mathbb{R}) \). Then, there holds that

\[
[b, T] \in \mathcal{K}(L_p^\mu, L_p^\lambda) \iff b \in \text{VMO}_{\nu}.
\]
Their proof of the “if” part refers to “the proof of the upper bound for the commutators in [5]”, which in turn is based on the dyadic representation theorem [9]. In another paper of the same authors with Chen and Vempati [2], they reprove this result via sparse domination and extend it to spaces of homogeneous type in place of \(R^d\). The “if” part is also partially recovered via an abstract method of extrapolation of compactness by Liu, Wu and Yang [17]; however, this method leads to assumptions that appear to be stronger than \(b \in VMO_\nu\) in dimensions \(d \geq 2\), although they can be shown to equivalent in the special dimension \(d = 1\).

Both [2, 11] prove the “only if” direction of Theorem 1.19 via the median method, which imposes the restriction to real-valued symbols \(b\). (See also [12], where the contain-

In the fractional setting thus far only one-weight commutator estimates have been studied and only for homogeneous SIOs

\[
T_\Omega f(x) = \int_{\mathbb{R}^d} \frac{\Omega(x-y)}{|x-y|^d} f(y) \, dy, \quad \Omega(u) = \Omega\left(\frac{u}{|u|}\right),
\]

for which Guo, He, Wu and Yang [4] proved the following

1.21. Theorem ([4]). Let \(T_\Omega\) be a homogeneous SIO with \(\Omega \in L^\infty(\mathbb{S}^{d-1}; \mathbb{C})\) that does not change sign on some open subset of the unit sphere \(\mathbb{S}^{d-1}\). Let \(b \in L^1_{loc}(\mathbb{R}^d; \mathbb{R})\). Let \(1 < p < q < \infty\) with \(\alpha/d = 1/p - 1/q\) and suppose that \([u]_{A_{p,q}} < \infty\). Then, there holds that

\[
[b, T_\Omega] \in \mathcal{K}(L^p_\nu, L^q_\nu) \iff b \in VMO^\alpha,
\]

where \(VMO^\alpha = VMO^\alpha_1\), see Definition 2.1 below.

(The paper [4] also deals with \(\Omega \in L^r(\mathbb{S}^{d-1}; \mathbb{C})\) for finite \(r\), but in this case the range of admissible exponent \(p, q\) and weights \(\nu\) need to be further restricted; see [4, Theorem 1.8] for details.)

The weighted compactness of commutators has also been considered for bilinear op-
erators, see e.g Wang, Xue [24]. It would be interesting to know if our methods can be used to extend the results of [24] to the multilinear Bloom-type setting, see e.g. [16].

The recent work of Wen [25] also addresses the \(L^p_\nu\)-to-\(L^q_\nu\) compactness of certain commutators, but the fractional nature of these commutators is in the operator (a fractional integral instead of a CZO) rather than in the symbol; in [25], the characterizing condition for the said compactness is simply \(b \in VMO_\nu\), with the usual Bloom-type VMO\(_\nu\) space. In contrast, we will deal with \(L^p_\nu\)-to-\(L^q_\nu\) boundedness and compactness of \([b, T]\), where \(T\) is a (non-degenerate) CZO, and the characterizing condition will be the containment of \(b\) in a suitable fractional BMO\(_\nu\) (for boundedness) or VMO\(^\alpha\) (for compactness). We will give the precise statement in the following section after introducing the necessary notation.
2. Main Definitions and Results

2.1. Definition. For a weight $\nu$ and a parameter $\alpha \in \mathbb{R}$, define the space of weighted fractional bounded mean oscillation $\text{BMO}_\nu^\alpha$ through

\begin{equation}
\|b\|_{\text{BMO}_\nu^\alpha} = \sup_{Q \in \mathcal{Q}} \mathcal{O}_\nu^\alpha(b; Q), \quad \mathcal{O}_\nu^\alpha(b; Q) = \nu(Q)^{-\alpha/d} \left( \frac{1}{\nu(Q)} \int_Q |b - \langle b \rangle_Q| \right).
\end{equation}

Similarly, we define the space $\text{VMO}_\nu^\alpha \subset \text{BMO}_\nu^\alpha$ of weighted fractional vanishing mean oscillation through

\begin{align}
\lim_{r \to 0^+} \sup_{\ell(Q) \leq r} \mathcal{O}_\nu^\alpha(b; Q) &= 0 \\
\lim_{r \to \infty} \sup_{\ell(Q) \geq r} \mathcal{O}_\nu^\alpha(b; Q) &= 0 \\
\sup_{Q \in \mathcal{Q}} \lim_{|x| \to \infty} \mathcal{O}_\nu^\alpha(b; Q + x) &= 0.
\end{align}

This particular form of the oscillation $\mathcal{O}_\nu^\alpha(b; Q)$ is related to the fact that it is notionally amenable to a weighted fractional John-Nirenberg inequality, see Appendix A.

2.6. Remark. Sometimes $(2.5)$ is replaced with the following a priori stronger uniform condition

\begin{equation}
\lim_{r \to \infty} \sup_{Q, \text{dist}(Q, 0) > r} \mathcal{O}_\nu^\alpha(b; Q) = 0.
\end{equation}

In the presence of the conditions $(2.3)$, $(2.4)$ and under the mild assumption that $\nu$ is doubling, the conditions $(2.5)$ and $(2.7)$ are in fact equivalent.

Indeed, by the conditions $(2.3)$ and $(2.4)$, there exists $k > 0$ such that $\ell(Q) \in \mathbb{R}_+ \setminus [2^{-k}, 2^k]$ implies $\mathcal{O}_\nu^\alpha(b; Q) \leq \varepsilon$. Hence it is enough to check $(2.7)$ for cubes with $\ell(Q) \in (2^{-k}, 2^k)$, see the left-hand side of the estimate $(2.9)$ below. We now invoke the method of adjacent dyadic systems, also known as the $\frac{1}{3}$-trick: there exist $3^d$ dyadic grids $\mathcal{D}_i$, where $i = 1, \ldots, 3^d$, such that each cube $Q$ is contained in some $Q' \in \bigcup_{i=1}^{3^d} \mathcal{D}_i$ with $\ell(Q') \leq 3\ell(Q)$; see e.g. [8, Lemma 3.2.26]. From this and the fact that $\nu$ is doubling, for each $Q$ with $\ell(Q) \in (2^{-k}, 2^k)$ there exist $Q \in \bigcup_{j=1}^{3^d} \mathcal{D}_i^{k+10}$ such that

\begin{equation}
Q \subset \hat{Q}, \quad |Q| \sim_k |\hat{Q}|, \quad \mathcal{O}_\nu^\alpha(b; Q) \lesssim_{\nu, \alpha, k} \mathcal{O}_\nu^\alpha(b; \hat{Q}).
\end{equation}

Let $\{c_j\}$ enumerate the centre points of $\bigcup_{i=1}^{3^d} \mathcal{D}_i^{k+10}$ and denote $Q_k = [-2^{k+9}, 2^{k+9}]^d$. Then, by the right-most estimate on line $(2.8)$ and $(2.5)$, we find

\begin{align}
\lim_{r \to \infty} \sup_{Q, \text{dist}(Q, 0) > r} \mathcal{O}_\nu^\alpha(b; Q) &\lesssim_{\nu, \alpha, k} \lim_{r \to \infty} \sup_{Q, \text{dist}(Q, 0) > r} \mathcal{O}_\nu^\alpha(b; \hat{Q}) \\
&= \lim_{j \to \infty} \mathcal{O}_\nu^\alpha(b; Q_k + c_j) = 0,
\end{align}

and this concludes the proof that $(2.5)$ implies $(2.7)$.

2.10. Definition. Given two weights $\mu, \lambda$ and exponents $1 < p, q < \infty$, we define the Bloom weight

\begin{equation}
\nu = \nu_{p, q} = \left( \frac{\mu}{\lambda} \right)^{\frac{1}{p(1+[1/p])}} = \left( \frac{\mu}{\lambda} \right)^{\frac{1}{\alpha/d}} \quad \alpha/d = 1/p - 1/q.
\end{equation}
Our main result is the following.

2.12. Theorem. Let $T$ be a non-degenerate Calderón-Zygmund operator let $b \in L^1_{\text{loc}}(\mathbb{R}^d; \mathbb{C})$. Let $1 < p \leq q < \infty$, let $\alpha/d = 1/p - 1/q$, let $\mu \in A_{p,p}$ and $\lambda \in A_{q,q}$ and let $\nu = \nu_{p,q}$ be the fractional Bloom weight as in Definition 2.10. Then, there holds that

$$
\|[b, T]\|_{L^p_{\mu} \rightarrow L^\lambda_{\lambda}} \sim \|b\|_{\text{BMO}_C} \tag{2.13}
$$

and

$$
[b, T] \in \mathcal{K}(L^p_{\mu}, L^q_{\lambda}) \iff b \in \text{VMO}_\nu. \tag{2.14}
$$

2.1. Interpretation in the one weight setting. Since we are assuming $1 < p \leq q$, it especially follows directly from Hörder’s inequality that $[w]_{A_{q,q}}, [w]_{A_{p,p}} \lesssim [w]_{A_{p,q}}$, hence $A_{p,q} \subset A_{p,p} \cap A_{q,q}$, and we extend Theorem 1.21 to Calderón-Zygmund operators with variable kernel.

2.15. Corollary. Let $T$ be a non-degenerate Calderón-Zygmund operator and $b \in L^1_{\text{loc}}(\mathbb{R}^d; \mathbb{C})$. Let $1 < p \leq q < \infty$ with $\alpha/d = 1/p - 1/q$ and suppose that $w \in A_{p,p} \cap A_{q,q}$. Then, there holds that

$$
\|[b, T]\|_{L^p_{\mu} \rightarrow L^\lambda_{\lambda}} \sim \|b\|_{\text{BMO}_C} \tag{2.13}
$$

and

$$
[b, T] \in \mathcal{K}(L^p_{w}, L^q_{w}) \iff b \in \text{VMO}_\alpha. \tag{2.14}
$$

2.2. Notation.

- Whenever we have fixed exponents $p, q$ and weights $\mu, \lambda$, we will always without exception denote $\alpha/d = 1/p - 1/q$ and $\nu = \nu_{p,q}$, as in Definition 2.10.
- When $p = q$ and $\alpha = 0$ we drop the superscript $\alpha$.
- When $w = 1$ is the Lebesgue measure, we do not mark it.
- We denote $L^1_{\text{loc}}(\mathbb{R}^d; \mathbb{C}) = L^1_{\text{loc}}(\mathbb{R}^d) = \int_{\mathbb{R}^d} f$, and so on, mostly leaving out the ambient space whenever this information is obvious or unimportant.
- We denote averages with $\langle f \rangle_A = \frac{1}{|A|} \int_A f$, where $|A|$ denotes the Lebesgue measure of the set $A$. The indicator function of a set $A$ is denoted by $1_A$.
- We denote various dyadic grids with the symbol $\mathcal{D}$.
- We denote various sparse collections with the symbol $\mathcal{S}$.
- We denote $A \lesssim B$, if $A \leq CB$ for some constant $C > 0$ depending only on the dimension of the underlying space, on integration exponents, on sparse constants, on constants from the kernel estimates, on constants depending on the weights $\mu, \lambda$, and on other absolute constants appearing in the assumptions that we do not care about. Then $A \sim B$, if $A \lesssim B$ and $B \lesssim A$. Moreover, subscripts on constants $(C_{a,b,c,...})$ and quantifiers $(\lesssim_{a,b,c,...})$ signify their dependence on those subscripts.
- We again emphasize that the implicit constants in the previous point are allowed to depend on the weights $\mu$ and $\lambda$, hence our theorem statements appear e.g. as Theorem 2.12 and Corollary 2.15 do. Sometimes in order to make the proofs easier to follow we indicate the weight dependence in the implicit constant as $\lesssim_{[\mu]_{A_{p,p}}, [\lambda]_{A_{q,q}}}$, for example.
3. ON THE BLOOM WEIGHT

In the following Proposition 3.1 we gather the basic properties of the fractional Bloom weight of Definition 2.10.

3.1. Proposition. Suppose that $1 < p, q < \infty$, $\mu \in A_{p,p}$ and $\lambda \in A_{q,q}$. Then, there holds that

\begin{equation}
1 \leq \frac{\mu^p(Q)^{\frac{1}{p}} \lambda^{-q} (Q)^{\frac{1}{q}}}{\nu(Q)^{\frac{1}{p}+\frac{1}{q}}} \leq [\mu]_{A_{p,p}} [\lambda]_{A_{q,q}}
\end{equation}

and

\begin{equation}
[\nu]_{A_{s(p,q)}}^{2/s(p,q)} \leq [\mu]_{A_{p,p}} [\lambda]_{A_{q,q}}, \quad \nu^{1/s(p,q)} = \left(\frac{\mu}{\lambda}\right)^{\frac{1}{s}};
\end{equation}

where

\begin{equation}
\frac{1}{s(p,q)} = 1 + \frac{1/q + 1/p'}{1/p + 1/q'} = \frac{2}{1 + \alpha/d}.
\end{equation}

3.5. Remark. There holds that $s(p,q) < 2$, if $p < q$; $s(p,q) = 2$, if $p = q$; $s(p,q) > 2$, if $p > q$.

and especially for each $1 < p, q < \infty$ that $\nu = \nu_{p,q} \in A_{\infty}$.

Proof. The first identity on line (3.4) will be automatically checked while verifying the left claim on line (3.3), while the second is simple algebra. Next, we check (3.2). By Hölder’s inequality applied with the exponents

\[ \frac{1/p}{1/p + 1/q'} + \frac{1/q'}{1/p + 1/q'} = 1 \]

we have

\[ \langle \nu\rangle_{Q}^{\frac{1}{p}+\frac{1}{q'}} = \left(\frac{\mu}{\lambda} \langle \nu^{1/p+1/q'} \rangle_{Q} \right)^{\frac{1}{p}+\frac{1}{q'}} \leq \left(\langle \mu^p(Q)^{\frac{1}{p}} \lambda^{-q} (Q)^{\frac{1}{q}} \rangle_{Q} \right)^{\frac{1}{p}+\frac{1}{q'}} = \langle \mu^p(Q)^{\frac{1}{p}} \lambda^{-q} (Q)^{\frac{1}{q}} \rangle_{Q}^{\frac{1}{p}+\frac{1}{q'}}. \]

For the other direction, suppose that $\mu \in A_{p,p}$, $\lambda \in A_{q,q}$. Then, there holds that

\[ \langle \mu^p(Q)^{\frac{1}{p}} \lambda^{-q} (Q)^{\frac{1}{q}} \rangle_{Q} \leq [\mu]_{A_{p,p}} [\lambda]_{A_{q,q}} (\langle \nu^{-p}(Q)^{\frac{1}{p}} \lambda^{-q} (Q)^{\frac{1}{q}} \rangle_{Q}^{-1}, \]

and by Hölder’s inequality

\[ \left(\langle \mu^{-p}(Q)^{\frac{1}{p}} \lambda^{-q} (Q)^{\frac{1}{q}} \rangle_{Q}^{-1}\right)^{\frac{1}{p}+\frac{1}{q'}} = \left(\langle \nu^{-t(p,q)}(Q)^{-t(p,q)}\rangle_{Q}^{-1}\right)^{\frac{1}{p}+\frac{1}{q'}}, \]

where we denote $t(p,q) = \frac{1/p+1/q'}{1/p+1/q'}$. Since $x \mapsto x^{-\beta}$ is convex, for any $\beta \geq 0$, by Jensen’s inequality

\[ (\langle \nu^{-t(p,q)}(Q)^{-t(p,q)}\rangle_{Q}^{-1})^{\frac{1}{p}+\frac{1}{q'}} \leq \langle \nu\rangle_{Q}^{\frac{1}{p}+\frac{1}{q'}} \]

and we conclude that

\[ \langle \mu^p(Q)^{\frac{1}{p}} \lambda^{-q} (Q)^{\frac{1}{q}} \rangle_{Q} \leq [\mu]_{A_{p,p}} [\lambda]_{A_{q,q}} \langle \nu\rangle_{Q}^{\frac{1}{p}+\frac{1}{q'}}. \]

Hence, we have checked both sides of (3.2).
We already saw above that
\[
\langle \nu \rangle_Q^{1/p+1/q'} \leq \langle \mu^p \rangle_Q^{\frac{1}{p}} \langle \lambda^{-q'} \rangle_Q^\frac{1}{q'}, \quad \langle \nu^{-t(p,q)} \rangle_Q^{\frac{1}{t(p,q)}} \leq \langle \mu^{-p'} \rangle_Q^{\frac{1}{p'}} \langle \lambda^q \rangle_Q^{\frac{1}{q}}
\]
and multiplying these estimates together, then raising to the power \((1/p + 1/q')^{-1}\), gives
\[
(3.6) \quad \langle \nu \rangle_Q^{t(p,q)} \leq \langle \mu \rangle_Q^{(1/p+1/q')^{-1}} < \infty.
\]
Since \((s \mapsto s'/s) : (1, \infty) \to (0, \infty)\) is bijective, \((3.6)\) shows that \(\nu \in A_s(p,q)\) for the unique \(s(p,q) \in (1, \infty)\) such that \(\frac{sp}{s'} = t(p,q)\). Solving for \(s(p,q)\), we find
\[
(3.7) \quad s(p,q) = 1 + t(p,q)^{-1} = 1 + \frac{1}{q} + \frac{1}{p' q},
\]
and by \((3.6)\) that \([\nu]_{A_{s(p,q)}} \leq \langle [\mu]_{A_{p,p}} [\lambda]_{A_{q,q}} \rangle^{1/(1/p+1/q')}\). By \((s(p,q))(1/p + 1/q') = 2\), we have
\[
(3.8) \quad \langle \mu/\lambda \rangle^{\frac{1}{2}} = \langle \mu/\lambda \rangle^{(1/p+1/q')^{-1} s(p,q)^{-1}} = \nu^{1/s(p,q)}.
\]
Denoting \(s = s(p,q)\) and using that \([\nu]_{A_s} = [\nu^{1/s}]_{A_{s,s}}\), we find that
\[
\langle (\mu/\lambda)^{\frac{1}{2}} \rangle_{A_s,s} = \langle [\nu^{1/s}]_{A_{s,s}} \rangle_{A_s,s} = \langle \nu \rangle_{A_s} \leq \langle \mu \rangle_{A_{p,p}} [\lambda]_{A_{q,q}} \rangle^{(1/p+1/q')^{-1} s^{-1}} = \langle \mu \rangle_{A_{p,p}} [\lambda]_{A_{q,q}} \rangle^{\frac{1}{2}},
\]
which gives \((3.3)\). \(\square\)

3.9. Definition. Let \(\mu, \lambda\) be weights and \(b \in L^1_{\text{loc}}(\mathbb{R}^d; \mathbb{C})\) and define
\[
(3.10) \quad \|b\|_{\text{BMO}^{p,q}_{\mu,\lambda}} = \sup_{Q \in \mathcal{Q}} \frac{1}{\mu^p(Q)^{\frac{1}{p}} \lambda^{-q'}(Q)^{\frac{1}{q'}}} \int_Q |b - \langle b \rangle_Q|.
\]

As an immediate corollary of line \((3.2)\) we get the following

3.11. Proposition. Suppose that \(1 < p, q < \infty\) and \(\mu \in A_{p,p}\), \(\lambda \in A_{q,q}\) and let \(\alpha/d = 1/p - 1/q\). Let \(b \in L^1_{\text{loc}}(\mathbb{R}^d; \mathbb{C})\). Then, there holds that
\[
(3.12) \quad \|b\|_{\text{BMO}^p_{\nu}} \sim \|b\|_{\text{BMO}^{p,q}_{\mu,\lambda}}.
\]

4. BOUNDEDNESS

4.1. Lower bound. Different versions of the following Proposition 4.2 appear throughout the literature, the one with complex valued symbols attained through the approximate weak factorization argument is from [7]. Before formulating it, let us fix the following convention. If a function \(\psi\) has support on \(Q\), then we write \(\psi = \psi_Q\), in particular if \(\psi\) is supported on a major subset \(E \subset Q\), we write \(\psi = \psi_E = \psi_{E_Q}\).

4.1. Definition. A collection of sets \(\mathscr{S}\) is said to be \(\gamma\)-sparse, for \(\gamma \in (0, 1)\), if there exists a pairwise disjoint collection \(\mathscr{S}_E = \{ U_E : U_E \subset U \in \mathscr{S}, |U_E| > \gamma |U| \}\). Furthermore, when \(|U_E| > \gamma |U|\), we speak of \(\gamma\)-major subsets. When the parameter \(\gamma\) is of no consequence, we speak of major subsets and sparse collections.
4.2. Proposition ([7]). Let $T$ be a non-degenerate singular integral and $b \in L^1_{\text{loc}}(\mathbb{R}^d; \mathbb{C})$. Let $Q$ be a fixed cube. Then, there exists a cube $\tilde{Q}$ such that $\text{dist}(Q, \tilde{Q}) \sim \ell(Q) = \ell(\tilde{Q})$ and for any $\gamma$-major subsets $E \subset Q$ and $E \subset \tilde{Q}$ we have

$$\int_E |b - \langle b \rangle_E| \lesssim \left| \langle [b, T]g_{E}, h_{E} \rangle \right| + \left| \langle [b, T]h_{E}, g_{E} \rangle \right|,$$

where the auxiliary functions satisfy

$$g_E = 1_E, \quad g_{\tilde{E}} = 1_{\tilde{E}}, \quad |h_E| \lesssim 1_E, \quad |h_{\tilde{E}}| \lesssim 1_{\tilde{E}},$$

and all the above implicit constants depend only on the kernel of $T$ and the parameter $\gamma$.

4.4. Proposition. Let $1 < p, q < \infty$, let $\mu \in A_{p,p}$, $\lambda \in A_{q,q}$, let $\alpha/d = 1/p - 1/q$, let $T$ be a non-degenerate singular integral and $b \in L^1_{\text{loc}}(\mathbb{R}^d; \mathbb{C})$. Then, there holds that

$$\|b\|_{\text{BMO}^p} \lesssim \|b, T\|_{L^p \to L^q},$$

where the implicit constant depends only on the weights and the kernel of $T$.

Proof. By Proposition 3.11 it is enough to check the lower bound with $\|b\|_{\text{BMO}^p}$ in place of $\|b\|_{\text{BMO}^p}$. In Proposition 4.2 we let $E = Q$ and $\tilde{E} = \tilde{Q}$ and then estimate the right-hand side of (4.3) as

$$\left| \langle [b, T]g_{\tilde{Q}}, h_{Q} \rangle \right| = \left| \langle \lambda [b, T]g_{\tilde{Q}}, h_{Q} \lambda^{-1} \rangle \right| \lesssim \| [b, T]g_{\tilde{Q}} \|_{L^p_{\lambda}} \| h_{Q} \|_{L^q_{\lambda^{-1}}},$$

$$\leq \| [b, T] \|_{L^p \to L^q} \| \tilde{g} \|_{L^p_{\lambda}} \| h_{Q} \|_{L^q_{\lambda^{-1}}} \lesssim \| [b, T] \|_{L^p \to L^q} (\mu^p(\tilde{Q}))^{1/2} (\lambda^{\alpha/q}(Q))^{1/2}.$$

The second term is estimated identically. To conclude it remains to show that

$$\mu^p(\tilde{Q}) \lesssim [\mu]_{A_{p,p}} \mu^p(Q),$$

which follows easily from the definition of $A_{p,p}$ weights, or from the doubling property. $\square$

4.2. Upper bound. We turn to the upper bound. We will involve the unweighted sparse operator and the two-weight fractional sparse operator

$$A_{\mathcal{V}} f = A(f; \mathcal{V}) = \sum_{Q \in \mathcal{V}} (f)_Q 1_Q, \quad A_{\mu,\lambda}^{p,q}(f; \mathcal{V}) = \sum_{P \in \mathcal{V}} \frac{\mu^p(P)^{1/2} \lambda^{-\alpha/q}(P)^{1/2}}{|P|} (f)_P 1_P.$$

The following sparse domination of commutators was first obtained by Lerner, Ombrosi, Rivera-Ríos [14] as a step towards a Bloom type upper bound for commutators.

4.6. Lemma. Let $T$ be a Calderón-Zygmund operator. Suppose that $f \in L^1_{\text{loc}}$. Then, there exist sparse collections $\mathcal{V}_j \subset 2^d$, where $j = 1, \ldots, 3^d$, such that

$$[b, T]f \lesssim \sum_{j=1}^{3^d} \left( A_{b,\mathcal{V}_j} f + A_{b,\mathcal{V}_j}^s f \right),$$

$$A_{b,\mathcal{V}_j} f = \sum_{Q \in \mathcal{V}_j} \langle [b, \mathcal{V}_j] f \rangle_Q 1_Q.$$
4.8. Lemma. Let $\mathcal{F} \subset \mathcal{D}$ be an arbitrary sparse collection of cubes. Then, for each function $b \in L^1_{\text{loc}}$, there exists a sparse collection sandwiched as $\mathcal{D} \supset \mathcal{F} \supset \mathcal{I}$ such that for each $Q \in \mathcal{F}$

$$|b - \langle b \rangle_Q| 1_Q \lesssim \sum_{P \in \mathcal{F} \cap Q} \langle |b - \langle b \rangle_P| \rangle_p 1_P.$$  

4.9. Theorem. Let $T$ be a Calderón-Zygmund operator, let $1 < p \leq q < \infty$ and $\mu \in A_{p,p}$ and $\lambda \in A_{q,q}$. Then, there holds that

$$\|[b, T]\|_{L^p_\mu \to L^q_\lambda \mathcal{H}} \lesssim \|[b]\|_{\text{BMO}_{p,q}}.$$  

Proof. By Lemma 4.6

$$\|[b, T]\|_{L^p_\mu \to L^q_\lambda \mathcal{H}} \lesssim \|A_{p,q,b}^*\|_{L^p_\mu \to L^q_\lambda} + \|A_{q,q,b}^*\|_{L^p_\mu \to L^q_\lambda}$$

and by $\|A_{p,q,b}^*\|_{L^p_\mu \to L^q_\lambda} = \|A_{q,q,b}^*\|_{L^p_\mu \to L^q_\lambda}$, it is enough to show the upper bound for $A_{p,q,b}^*$. By Lemma 4.8 and Proposition 3.11 we have

$$\langle |b - \langle b \rangle_Q| f \rangle_Q \lesssim \left\langle \sum_{P \in \mathcal{F} \cap Q} \langle |b - \langle b \rangle_P| \rangle_p f 1_P \right\rangle_Q = \left\langle \sum_{P \in \mathcal{F} \cap Q} \langle |b - \langle b \rangle_P| \rangle_p \langle f \rangle_p 1_P \right\rangle_Q$$  

(4.10)

$$\lesssim_{[\mu]_{A_{p,p}},[\lambda]_{A_{q,q}}} \|[b]\|_{\text{BMO}_{p,q}} \left\langle \sum_{P \in \mathcal{F} \cap Q} \frac{\mu^p(P)^{\frac{1}{2}}_q \lambda^{-q}(P)_{q}^{\frac{1}{2}}}{|P|} \langle f \rangle_p 1_P \right\rangle_Q$$

and hence by the boundedness of the sparse operator (see e.g. [13]) we find

$$\|A_{p,q,b}^*\|_{L^p_\mu \to L^q_\lambda} \lesssim_{[\mu]_{A_{p,p}},[\lambda]_{A_{q,q}}} \|[b]\|_{\text{BMO}_{p,q}} \|f \mapsto A_{p,q}(f; \mathcal{F})\|_{L^p_\mu \to L^q_\lambda}.$$  

Now what remains is to estimate as follows

$$\left| \langle A_{p,q}^*, \lambda (f; \mathcal{F}), g \rangle \right| \leq \sum_{Q \in \mathcal{F}} \mu^p(Q)^{\frac{1}{2}}_q \langle f \rangle_Q \lambda^{-q}(Q)^{\frac{1}{2}} \langle g \rangle_Q$$

$$\leq \left( \sum_{Q \in \mathcal{F}} \langle f \rangle_Q^p \mu^p(Q)^{\frac{1}{2}} \right)^{\frac{1}{2}} \left( \sum_{Q \in \mathcal{F}} \langle g \rangle_Q^q \lambda^{-q}(Q)^{\frac{1}{2}} \right)^{\frac{1}{2}}$$

$$\leq \left( \sum_{Q \in \mathcal{F}} \langle f \rangle_Q^p \mu^p(Q) \right)^{\frac{1}{2}} \left( \sum_{Q \in \mathcal{F}} \langle g \rangle_Q^q \lambda^{-q}(Q) \right)^{\frac{1}{2}}$$

$$\lesssim_{[\mu]_{A_{p,p}},[\lambda]_{A_{q,q}}} \|[f]_{L^p_\mu,q} \|_{L^q_\lambda}^{\frac{1}{2}}.$$  

where we used $\| \cdot \|_{p,q} \leq \| \cdot \|_{p,r}$ (by $p \leq q$) and Lemma A.13.

5. Compactness

5.1. Sufficiency. Let $T$ be a Calderón-Zygmund operator. In this section we show that $b \in \text{VMO}_p^q$ implies $[b, T] \in \mathcal{K}(L^p_\mu, L^q_\lambda)$. We mimic the proof from Lacey and Li in [11] and we provide all details, some of which are different. Formally the idea is to show that

$$T = T_c + T_e, \quad [b, T_c] \in \mathcal{K}(L^p_\mu, L^q_\lambda), \quad \lim_{\varepsilon \to 0} \|[b, T_c]\|_{L^p_\mu \to L^q_\lambda} = 0.$$  

(5.1)
Then, we would be done by the fact that compact linear operators form a closed subspace of all bounded linear operators.

We select a bump such that

\[ \varphi \in C^\infty_c(\mathbb{R}^d, [0, 1]), \quad \varphi(x) = \begin{cases} 1, & |x| \leq \frac{1}{2}, \\ 0, & |x| \geq 1. \end{cases} \]

We define

\[ \varphi^0(y) = \varphi(\frac{x - y}{r}) \]

and with 0 < r < R < \infty partition unity as

\[ 1 = \varphi^0_0 + \varphi^r_R + \varphi^r_\infty, \quad \varphi^r_R = \varphi^0_0 - \varphi^0_R, \quad \varphi^R_\infty = 1 - \varphi^0_R. \]

We also denote

\[ \varphi^{a,b} = \varphi^a, \quad a, b \in [0, \infty]. \]

Then, we decompose

\[ T = \left( \varphi^0 + \varphi^R_\infty \right) T \left( \varphi^0 + \varphi^R_\infty \right) \]

and the other we decompose further:

\[ T^0_R = T^0_R \left( \varphi^0_0 + \varphi^R_\infty \right) = T^0_R \varphi^0_0 + T^0_R \varphi^R_\infty, \]

where we noted that T^0_R \varphi_x^{10} = 0. In total, our decomposition is the following

\[ T = T_c + T_z, \quad T_c = T^0_R \varphi^r_\infty, \quad T_z = T^0_R \varphi^r_0 + T^R_\infty, \]

where we now take the convention of writing \( \varepsilon = r = R^{-1} \).

The rest of this section is devoted to proving the two claims on line (5.1), and we begin with

5.3. Proposition. Let 1 < p \leq q < \infty, \mu \in A_{p,q} and \lambda \in A_{q,q}. Suppose that \( b \in B{\text{MO}}_\nu^\mu \). Then

\[ [b, T_c] \in K(L^p_\mu, L^q_\lambda). \]

Proof. Let us denote \( Q = [-R, R]^d, S = 10R \) and

\[ T_L f(x) = \int L(x, y) f(y) dy, \quad L(x, y) = K(x, y) \varphi^r_\infty(y). \]

so that

\[ T_c = T^0_R \varphi^r_\infty = \varphi^0_R \circ T_L \circ \varphi^R_\infty. \]

We view the commutator as

\[ [b, T_c] = \varphi^0_R \circ [b, T_L] \circ \varphi^0_R \]

and show that both terms separately are compact. More precisely, in the diagrams

\[ L^p_\mu \xrightarrow{\varphi^0_R (b - \langle b \rangle_Q)} L^1(Q) \xrightarrow{T_L} C(Q; \| \cdot \|_\infty) \xrightarrow{\varphi^0_R} L^q_\lambda \]
we show that the first and the last maps are bounded and that the one in the middle is compact. We only provide details for the diagram (5.4) with (5.5) being completely analogous. We first show the compactness of $T_L$. Let there be a family $\{f_j\}_j$ such that $\sup_j \|f_j\|_{L^1(Q)} \lesssim 1$; then we need to show that the collection $\{T_L f_j\}_j \subset C(Q; \| \cdot \|_\infty)$ has a converging subsequence. By the Arzela-Ascoli theorem, it is enough to check equicontinuity and equiboundedness of $\{T_L f_j\}_j \subset C(Q; \| \cdot \|_\infty)$. Equiboundedness is immediate by the simple estimate

$$
\left| T_L f(x) \right| \leq \int_Q |K(x, y)\varphi_{x,S}^p(y)|f(y)| dy \lesssim r^{-d}\|f\|_{L^1(Q)}.
$$

For equicontinuity, we first check that $L(x, y) = \varphi_{x,S}^p(x)K(x, y)$ is a CZ-kernel, which can be seen as follows. Recall that $L(x, y) = (\varphi_{x,S}^0 - \varphi_{x,S}^0)K(x, y)$ and combine this with the fact that $\varphi_{x,S}^0(y)K(x, y)$ is a CZ-kernel, for $t > 0$, for this see the estimate (5.19) below, and that CZ-kernels are closed under summation. Let $x, x' \in Q$ be such that $|x - x'| < sr$, for $0 < s < \frac{1}{4}$. Then, we have

$$
|T_L f(x) - T_L f(x')| \leq \sup_{y \in Q} |L(x, y) - L(x', y)|\|f\|_{L^1(Q)}
$$

and using the regularity estimate of $L$ for the factor in front,

$$
|L(x, y) - L(x', y)| = |L(x, y) - L(x', y)|1_{B(x,r/2)^c}(y)
\lesssim \omega\left(\frac{|x - x'|}{|x - y|}\right) |x - y|^{-d}1_{B(x,r/2)^c}(y) \lesssim \omega(s)r^{-d}.
$$

This shows equicontinuity.

The rightmost map of diagram (5.4) is bounded:

$$
\|\varphi^{0,R}f\|_{L^q_{\lambda}} \leq \left( \int_Q |f|^q\lambda^q \right)^{1/q} \leq \|f\|_{L^\infty} \left( \int_Q \lambda^q \right)^{1/q} \lesssim \|f\|_{L^\infty}.
$$

For the first map, we have

$$
\|\varphi^{0,R}(b - \langle b \rangle)qf\|_{L^1(Q)} \leq \|b - \langle b \rangle q\|_{L^{p-1}_{\mu}(Q)}\|f\|_{L^p_{\mu}}
$$

and it remains to show the finiteness of the term in front. Using Lemma 4.8 with a single cube $Q$, then Lemma A.9 and the fact that $b \in \text{BMO}^{\alpha}_{\mu} = \text{BMO}^{p,q}_{\mu,\lambda}$, we find

$$
\|b - \langle b \rangle q\|_{L^{p-1}_{\mu}(Q)} \lesssim \sum_{P \in \mathcal{F}(Q)} \|b - \langle b \rangle q\|_{L^{p-1}_{\mu}(Q)} \approx P \approx \left( \sum_{P \in \mathcal{F}(Q)} \|b - \langle b \rangle q\|_{L^{p-1}_{\mu}(Q)} \right)^{\frac{1}{p}}
$$

and

$$
\|b\|_{\text{BMO}^{p,q}_{\mu,\lambda}} \left( \sum_{P \in \mathcal{F}(Q)} \frac{\|b - \langle b \rangle q\|_{L^{p-1}_{\mu}(Q)}}{|P|} \right)^{\frac{1}{p}} \lesssim \left( \sum_{P \in \mathcal{F}(Q)} \frac{|P|}{\mu(P)^\frac{1}{p}} \lambda^{-q'}(P) \right)^{\frac{1}{q'}}.
$$
It remains to show that the inside sum is finite. For this, we have
\[
\sum_{P \in \mathcal{F}(Q)} \left\| \frac{\mu^p(\lambda^q(\mathcal{F}|_Q))^{1/p}}{|P|} 1_P \right\|_{L_{\nu}^{\mu}}^{p'} \leq \left[ \mu^p \right]_{A_{p,\mu}}^{0} \sum_{P \in \mathcal{F}(Q)} \lambda^{-q}(\mathcal{F}|_Q)^{\frac{q}{p}} \lesssim \left( \sum_{P \in \mathcal{F}(Q)} \lambda^{-q}(\mathcal{F}|_Q)^{\frac{q}{p}} \right)^{\frac{q}{q}} \lesssim \lambda^{-q}(\mathcal{F}|_Q)^{\frac{q}{p}} < \infty,
\]
which concludes the proof. \(\square\)

The first step to checking the right-most claim on the line (5.1) is to prove the following Proposition 5.6, which is interesting by itself.

**5.6. Proposition.** Let \(1 < p \leq q < \infty\), \(\mu \in A_{p,\mu}\), \(\lambda \in A_{q,\lambda}\) and \(b \in \text{VMO}^\mu_0\). Let \(\mathcal{F}\) be sparse and denote
\[
(5.7) \quad \mathcal{F}_k = \mathcal{F} \setminus \mathcal{F}_k^0, \quad \mathcal{F}_k^0 = \left\{ Q \in \mathcal{F} : \ell(Q) \in [k^{-1}, k], \ \text{dist}(Q, 0) \leq k \right\}, \quad k > 0.
\]
Then, given any \(\varepsilon > 0\), there exists a large \(k = k_\varepsilon > 0\) such that
\[
\| A_{b,\mathcal{F}_k} f \|_{L_\mu^p} + \| A_{b,\mathcal{F}_k}^* f \|_{L_\lambda^1} \leq \varepsilon \| f \|_{L_\mu^p}.
\]

**Proof.** We only show the claim for \(A_{b,\mathcal{F}_k}^* f\). We cover \(\mathcal{F}_k\) with the union of
\[
\mathcal{F}_{kd} = \left\{ Q \in \mathcal{F} : \text{dist}(Q, 0) > k \right\}, \quad \mathcal{F}_{ks} = \left\{ Q \in \mathcal{F} : \ell(Q) < k \right\}, \quad \mathcal{F}_{kb} = \left\{ Q \in \mathcal{F} : \ell(Q) > k \right\},
\]
and estimate
\[
\| A_{b,\mathcal{F}_k}^* f \|_{L_\lambda^1} \leq \sum_{w=s,b,d} \| A_{b,\mathcal{F}_k}^* f \|_{L_\lambda^1}.
\]

We first consider \(w \in \{s, d\}\). By conditions (2.3) and (2.5) (also, see Remark 2.6), let \(k\) be so large that if \(\ell(Q) < k\) or \(\text{dist}(Q, 0) > k\), then \(O^w_{b} (b; Q) \leq \varepsilon\). Especially, if \(P \subset Q \in \mathcal{F}_{kw}\), then either \(\ell(P) < k\) or \(\text{dist}(P, 0) > k\), depending on \(w = s, d\). In any case, following the proof of Theorem 4.9, the factor \(\| b \|_{\text{BMO}_0^\mu}\) in the estimate (4.10) can be replaced with \(\varepsilon\), giving
\[
\left( \langle b - \langle b \rangle_Q \rangle f \right)_Q \leq \varepsilon \left( \sum_{P \subseteq Q \subset \mathcal{F}_{kw}} \left( \frac{\mu^p(\lambda^{-q}(\mathcal{F}|_Q)^{1/p})^{1/p}}{|P|} \langle f \rangle_P 1_P \right)_Q \right) \leq \varepsilon \left( \sum_{P \subseteq Q \subset \mathcal{F}_{kw}} \left( \frac{\mu^p(\lambda^{-q}(\mathcal{F}|_Q)^{1/p})^{1/p}}{|P|} \langle f \rangle_P 1_P \right)_Q \right),
\]
while the remaining estimate is identical, in total yielding:
\[
\sum_{w=s,d} \| A_{b,\mathcal{F}_k}^* f \|_{L_\lambda^1} \leq \varepsilon \| f \|_{L_\mu^p}.
\]

Then, we turn to the case \(\mathcal{F}_{kb}^d\) of big cubes. Without loss of generality we may assume that \(\mathcal{F} \subset \mathcal{D}\). By conditions (2.4) and (2.5), we find cubes \(R_i \in \mathcal{D}\), where \(i = 1, \ldots, 2^d\),
such that if \( \ell(P) \geq \ell(R_1) \), or \( P \subset \mathbb{R}^d \setminus \bigcup_{i=1}^{d} R_i \), then \( O^o_p(b; P) \leq \varepsilon \). We consider these cubes henceforth fixed throughout the rest of the argument.

We cover
\[
\mathcal{S}_k^b \subset \bigcup_{i=1}^{d} \mathcal{S}_{k,i}^b \cup (\mathcal{S}_k^b \setminus \bigcup_{i=1}^{d} \mathcal{S}_{k,i}^b), \quad \mathcal{S}_{k,i}^b = \{ Q \in \mathcal{S}_k^b : Q \cap R_i \neq \emptyset \}
\]
and estimate
\[
\| A_{b,\mathcal{S}_{k,i}^b} f \|_{L^p} \leq \sum_{i=1}^{2^d} \| A_{b,\mathcal{S}_{k,i}^b} f \|_{L^p} + \| A_{b,\mathcal{S}_k^b \setminus \bigcup_{i=1}^{d} \mathcal{S}_{k,i}^b} f \|_{L^p}. \]

The right-most term is handled as the case of \( \mathcal{S}_k^d \). For each fixed \( i \), we estimate
\[
\| A_{b,\mathcal{S}_{k,i}^b} f \|_{L^p} \leq \sum_{Q \in \mathcal{S}_{k,i}^b} \left( \sum_{P \in \mathcal{S}_{k,i}^b \setminus P \supset R_j} \langle | b - \langle b \rangle_P | P \| f \| P 1_P \rangle_Q \right) 1_Q \| f \|_{L^p}.
\]

and further split and estimate the interior sum as
\[
\sum_{P \in \mathcal{S}_{k,i}^b} = \sum_{j=1}^{2^d} \left( \sum_{P \in \mathcal{S}_{k,i}^b \setminus P \supset R_j} + \sum_{P \in \mathcal{S}_{k,i}^b \setminus P \supset \bigcup_{j=1}^{d} R_j} \right) = \sum_{j=1}^{2^d} (I_{i,j} + II_{i,j}) + III_i.
\]

The term corresponding to the sum \( III_i \) is estimated as in the case \( \mathcal{S}_k^d \) of the distant cubes; while for \( II_{i,j} \), we have \( R_j \subset P \), hence \( \ell(P) \geq \ell(R_1) \), hence \( O^o_p(b; P) \leq \varepsilon \) and the rest of the estimate is as before. It remains to handle the term corresponding to \( I_{i,j} \), where we rewrite the condition \( Q \cap R_i \neq \emptyset \) as \( Q \supset R_i \) under the assumption that \( Q \in \mathcal{S}_k^b \) is bigger than \( R_i \):

\[
\left\| \sum_{Q \supset R_i} \left( \sum_{P \in \mathcal{S}_{k,i}^b : P \subset R_j} \langle | b - \langle b \rangle_P | P \| f \| P 1_P \rangle_Q \right) 1_Q \| f \|_{L^p}. \right.
\]

We estimate the interior of (5.10) as
\[
\left\langle \sum_{P \in \mathcal{S}_{k,i}^b \setminus P \subset R_j} \langle | b - \langle b \rangle_P | P \| f \| P 1_P \rangle_Q \right\rangle \leq \| b \|_{\text{BMO}} \left\langle \sum_{P \in \mathcal{S}_{k,i}^b \setminus P \subset R_j} \mu^p(P)^{\frac{1}{p'} \lambda^{-d} (P)^{\frac{1}{p}} \langle | f \| P 1_P \rangle_Q \right\rangle,
\]

and further
\[
\left\langle \sum_{P \in \mathcal{S}_{k,i}^b \setminus P \subset R_j} \frac{\mu^p(P)^{\frac{1}{p'} \lambda^{-d} (P)^{\frac{1}{p}} \langle | f \| P 1_P \rangle_Q}{|P|} \langle | f \| P 1_P \rangle_Q \right\rangle = \frac{|R_j|}{|Q|} \left\| f \| f \|_{L^p} \right\|_{L^p} \| f \|_{L^p} \| f \|_{L^p}. \]
where in the last estimate we used the estimates \( \ell(R_i) \sim \ell(R_j) \), and \( \|1_{R_i}\|_{L^q_\lambda} \sim 1 \), recalling that the boundedly many cubes \( R_i \) are considered fixed, and hence dependence on them will be suppressed. Substituting, we find

\[
(5.12) \quad \|b\|_{\text{BMO}_0} \|f\|_{L^p_\mu} \| \sum_{Q \in \mathcal{S}_k} \frac{|R_i|}{|Q|} 1_Q \|_{L^q_\lambda}.
\]

It remains to show that the last term can be made small. Let \( \delta > 0 \). Provided that \( k = k_\delta \gg \ell(R_i) \) is taken sufficiently large, by summing a geometric series we find

\[
\sum_{Q \in \mathcal{S}_k} \frac{1_Q}{|Q|} |R_i| \leq \delta.
\]

On the other hand

\[
\sum_{Q \in \mathcal{S}_k \setminus \mathcal{R}_i} \frac{1_Q}{|Q|} |R_i| \leq \sum_{Q \in \mathcal{S}_k \setminus \mathcal{R}_i} \frac{|R_i|}{|Q|} 1_Q \lesssim M(1_{R_i}).
\]

Hence, we find

\[
(5.13) \quad \left\| \sum_{Q \in \mathcal{S}_k \setminus \mathcal{R}_i} \frac{1_Q}{|Q|} |R_i| \right\|_{L^q_\lambda} \lesssim \min(\delta, M(1_{R_i})) \|_{L^q_\lambda}.
\]

Since \( M \) is a bounded operator on \( L^q_\lambda \) (by \( \lambda \in A_{p,q} \)), by dominated convergence the right-hand side of (5.13) can be made smaller than \( \varepsilon \), by choosing \( \delta \) small (\( k_\delta \) large) enough. \( \square \)

**5.14. Corollary.** Let \( 1 < p \leq q < \infty, \mu \in A_{p,p}, \lambda \in A_{q,q} \) and \( b \in \text{VMO}_0^\alpha \). Let \( \mathscr{S} \) be sparse. Then,

\[
[b, A_{\mathscr{S}}], A_{b, \mathscr{S}} \in \mathcal{K}(L^p_\mu, L^q_\lambda).
\]

**Proof.** By Proposition 5.6, we can approximate both by finite rank operators to arbitrary precision. \( \square \)

**5.15. Proposition.** Suppose that \( b \in \text{VMO}_0^\alpha \). Then,

\[
\lim_{\varepsilon \to 0} \left\| [b, T_\varepsilon] \right\|_{L^p_\mu \to L^q_\lambda} = 0.
\]

**Proof.** By Proposition 5.6 it is enough to show that for each \( k > 0 \), the following holds: for all \( \varepsilon > 0 \) small enough (depending on \( k \)) there exists a sparse collection \( \mathscr{S} \) such that

\[
(5.16) \quad |\langle [b, T_\varepsilon]f, g \rangle| \lesssim |\langle A_{b, \mathscr{S}}f, |g| \rangle| + |\langle A_{b, \mathscr{S}}*f, |g| \rangle|, \quad \mathscr{S} = \mathscr{S}_k,
\]

where \( \mathscr{S}_k \) is as in (5.7). Recall that \( \varepsilon = r = R^{-1} \). As

\[
(5.17) \quad T_\varepsilon = \varphi^{0,R}T \varphi^{0,R} + \varphi^{R,\infty}T \varphi^{0,R} + \varphi^{R,\infty}T R \varphi^{0,R} + \varphi^{0,R}T \varphi^{R,\infty},
\]

it is enough to show that claim (5.16) is satisfied for each of the four pieces of \( T_\varepsilon \). Each of the three pieces inside the brackets is handled in the same way. For example, directly from the sparse domination of \( [b, T] \) we acquire a sparse collection \( \mathscr{S} \) such that

\[
(5.18) \quad |\langle [b, \varphi^{0,R}T \varphi^{R,\infty}]f, g \rangle| \lesssim |\langle A_{b, \mathscr{S}}f, |g\varphi^{0,R}| \rangle| + |\langle A_{b, \mathscr{S}}*f, |g\varphi^{0,R} \rangle|.
\]
Considering the first term on the right-hand side of (5.18), we have
\[ |\langle A_0, \mathcal{S} f \varphi^{R, \infty}, g \varphi^{0, R} \rangle| \leq |\langle A_0 (f; \mathcal{S} R, \infty), |\varphi| \rangle|, \]
where \( \mathcal{S} R, \infty = \{ Q \in \mathcal{S} : Q \cap B(0, \frac{1}{2} R)^c \neq \emptyset \} \). Clearly \( \mathcal{S} R, \infty = \mathcal{S} R, k \) for sufficiently small \( \varepsilon \) (large \( R \)). The other bracketed pieces are handled identically.

It remains to handle the term \( \langle b, \varphi \varphi^{0, R} T \varphi^{0, R} \varphi^{0, R} f, g \rangle \), which we write as
\[ \langle [b, \varphi \varphi^{0, R} T \varphi^{0, R} \varphi^{0, R}] f, g \rangle = \langle [b, \tilde{T} \tilde{f}], \tilde{g} \rangle, \quad \tilde{T} r = T \varphi^{0, R}, \quad \tilde{f} = \varphi^{0, R} f, \quad \tilde{g} = \varphi^{0, R} g. \]

We express \( \mathbb{R}^d = \bigcup_j Q_j \) as a disjoint union of cubes with \( \text{diam}(Q_j) = r \). Then, we have
\[ \langle [b, \tilde{T} \tilde{f}], \tilde{g} \rangle = \sum_j \langle 1_{Q_j} [b, \tilde{T}] \tilde{f}, \tilde{g} \rangle \]
and it is enough to show that each \( \langle 1_{Q_j} [b, \tilde{T}] \tilde{f}, \tilde{g} \rangle \) admits a sparse domination localized to the cube \( Q_j \). Using
\[ 1_{Q_j}(x) \tilde{T} f(x) = 1_{Q_j}(x) \tilde{T} (1_{Q_j} f)(x), \quad Q_j^* = 3Q_j, \]
we write
\[ \langle 1_{Q_j} [b, \tilde{T}], \tilde{f}, \tilde{g} \rangle = \langle 1_{Q_j} [b, \tilde{T}](1_{Q_j} \tilde{f}), \tilde{g} \rangle. \]

Next we argue that \( \tilde{T} \) is a CZO with \( \| \tilde{T} \|_{\text{CZO}} \lesssim 1 \), with constant independent of \( r > 0 \). Since
\[ \tilde{T} r = T - T \varphi^{0, \infty}, \]
for the uniform \( L^2 \)-to-\( L^2 \)-boundedness it is enough to show that the \( T \varphi^{0, \infty} \) are uniformly \( L^2 \)-to-\( L^2 \)-bounded. By Cotlar’s inequality, the truncated operators \( T \varphi^{0, \infty} \) are uniformly bounded and hence it is enough to give the following uniform bound for the difference:
\[ |(T r - T \varphi^{0, \infty}) f(x)| \leq \int_{B(x, r) \setminus B(x, \frac{r}{2})} |K(x, y)(1 - \varphi^{0, \infty}(y))| f(y) \, dy \lesssim M f(x). \]

Next we check that \( \varphi^{0, r} K(x, y) \) is a CZ kernel with uniform constants. The size estimate is immediate by \( \| \varphi^{0, r} K(x, y) \| \lesssim \| 1_{B(x, r)} K(x, y) \| \). For the regularity estimate, provided that \( |x - x'| \leq \frac{1}{2} |x - y| \), we have
\begin{equation}
|\varphi^{0, r}(y) K(x', y) - \varphi^{0, r}(y) K(x, y)| \leq |\varphi^{0, r}(y) (K(x', y) - K(x, y))| + |(\varphi^{0, r}(y) - \varphi^{0, r}(y)) K(x, y)| \lesssim \| \varphi^{0, r} \|_{L^\infty} \omega \left( \frac{|x - x'|}{|x - y|} \right) |x - y|^{-d} + |\varphi^{0, r}(y) - \varphi^{0, r}(y)| |x - y|^{-d}. \end{equation}

The remaining term we estimate, using \( |x - x'| \leq \frac{1}{2} |x - y| \) in the first identity, as
\[ |\varphi^{0, r}(y) - \varphi^{0, r}(y)| = |\varphi^{0, r}(y) - \varphi^{0, r}(y)| 1_{B(x,2r)}(y) \lesssim \| \nabla \varphi^{0, r} \|_{L^\infty} |x - x'| 1_{B(x,2r)}(y) \sim \frac{1}{r} |x - x'| 1_{B(x,2r)}(y) \lesssim \frac{|x - x'|}{|x - y|}. \]

The other estimate is completely symmetric. We have shown that \( \| \tilde{T} \|_{\text{CZO}} \lesssim 1 \), independently of \( r > 0 \).
Now, by the standard proof of the sparse domination of the commutator, see e.g. \cite{14}, we conclude that
\begin{equation}
\left|\langle 1_{Q_j} [b, \tilde{T}_r] (1_{Q_j^c} \tilde{f}), g \rangle\right| \lesssim \langle A_b(|f|; \mathcal{S}(Q_j)), |g| \rangle + \langle A_b(|f|; \mathcal{S}(Q_j)), |g| \rangle,
\end{equation}
where \( \mathcal{S}(Q_j) \) is a sparse collection inside \( Q_j^c \), and the implicit constant in (5.20) is independent of \( r \). All in all, we have shown that
\[ |\langle b, \varphi_0^R T \varphi_0^R f, g \rangle| \lesssim \langle A_b(|f|; \mathcal{S}), |g| \rangle + \langle A_b(|f|; \mathcal{S}), |g| \rangle, \]
where \( \mathcal{S} = \bigcup_j \mathcal{S}(Q_j) \). By \( \ell(Q_j) = r \), for all \( j \), it follows that \( \mathcal{S} = \mathcal{S}_k \) for a choice of \( \varepsilon = r \) sufficiently small. \( \square \)

5.2. Necessity. Let \( T \) be a non-degenerate singular integral. In this section we show that \([b, T] \in K(L_p^\mu, L_{q,\lambda}^q)\) implies \( b \in \text{VMO}_p^\mu \).

The following Lemma is originally implied in Uchiyama \cite{23} and also in several later works.

5.21. Lemma. Let \( 1 < p, q < \infty \), let \( \mu \) and \( \lambda \) be arbitrary weights, and let \( U : L_p^\mu \to L_{q,\lambda}^q \) be a bounded linear operator. Then, there does not exist a sequence \( \{u_i\} = \{u_i\}_{i=1}^\infty \) with the following properties:

\begin{enumerate}[(i)]
  \item \( \sup_{i \in \mathbb{N}} \|u_i\|_{L_p^\mu} \lesssim 1 \),
  \item \( \{x : u_i(x) \neq 0\} \cap \{x : u_j(x) \neq 0\} = \emptyset \), whenever \( i \neq j \), and
  \item there exists an \( \Phi \in L_{\lambda}^\lambda \) such that
    \[ \lim_{i \to \infty} \|\Phi - U(u_i)\|_{L_{\lambda}^q} = 0, \quad \|\Phi\|_{L_{\lambda}^\lambda} > 0. \]
\end{enumerate}

Proof. We show that the existence of such a sequence contradicts the boundedness of \( U \). By the point (iii), and by passing to a subsequence if needed, we can assume that \( \|\Phi - U(u_i)\|_{L_{\lambda}^q} \leq 2^{-i} \). By \( p, q' > 1 \), let us choose a sequence \( \{a_i\} \in (L^p \cap L^{p'}) \setminus \ell^1 \) of positive numbers. We define \( g_k = \sum_{i=1}^{\infty} a_i u_i \) and next show that \( \{g_k\} \) is a Cauchy sequence in \( L_p^\mu \). Suppose that \( m \leq k \). By assumptions (i) and (ii) we find that
\[ \|g_k - g_m\|_{L_p^\mu} = \| \sum_{i=m+1}^{k} a_i u_i \|_{L_p^\mu} = \left( \sum_{i=m+1}^{k} |a_i|^p \|u_i\|_{L_p^\mu}^p \right)^{\frac{1}{p}} \lesssim \left( \sum_{i=m+1}^{k} |a_i|^p \right)^{\frac{1}{p}}, \]
from which by \( \{a_i\} \in L^p \) we see that \( \{g_k\} \) is Cauchy. By completeness of \( L_p^\mu \), the sequence of functions \( g_k \) converges in \( L_p^\mu \); and since \( U \) is continuous, \( U(g_k) \) converges in \( L_{q,\lambda}^q \). Let us define \( h_k := \sum_{i=1}^{k} a_i \Phi \). Then, we have
\[ \|h_k - U(g_k)\|_{L_{\lambda}^q} \leq \sup_{i \in \mathbb{N}} \|h_i\|_{L_{\lambda}^q} \lesssim 1 \]
\[ \leq \left( \sum_{i=1}^{\infty} |a_i|^q \|U(u_i)\|_{L_{\lambda}^q}^q \right)^{\frac{1}{q}} \leq \left( \sum_{i=1}^{\infty} 2^{-iq} \right)^{\frac{1}{q}} \leq 1 \]
uniformly in \( k \), and as \( U(g_k) \) converges in \( L_{q,\lambda}^q \), it follows that \( \sup_{k \in \mathbb{N}} \|h_k\|_{L_{\lambda}^q} \lesssim 1 \) and
\[ 1 \geq \sup_{k \in \mathbb{N}} \|h_k\|_{L_{\lambda}^q} = \sup_{k \in \mathbb{N}} \sum_{i=1}^{k} |a_i|^q \|\Phi\|_{L_{\lambda}^q} \sim \|\{a_i\}\|_{\ell^q}, \]
which contradicts \( \{a_i\} \notin \ell^1 \). \( \square \)
5.22. Lemma. Let $\nu \in A_\infty$ and suppose that $b \in \text{BMO}^\alpha_\nu \setminus \text{VMO}^\alpha_\nu$ with $\alpha > -d$. Then, there exists a sparse collection of cubes $\mathcal{S} = \{Q, E_Q\}_{Q \in \mathcal{S}}$ such that $\mathcal{O}_\nu^\alpha(b; E_Q) \gtrsim 1$, uniformly.

Proof. If the VMO$^\alpha_\nu$ condition fails via (2.5), then the construction of $\mathcal{S}$ is immediate; we can guarantee that all the cubes are disjoint by choosing cubes ever farther away from the origin and we simply let $E_Q = Q$.

For the duration of this proof let us denote $\beta = 1 + \alpha/d > 0$. If the VMO$^\alpha_\nu$ condition fails via (2.3), then we are guaranteed a sequence of cubes $\{Q_j\}$ such that
\[
\mathcal{O}_\nu^\alpha(b; Q_j) \gtrsim 1, \quad \lim_{j \to \infty} \ell(Q_j) = 0.
\]

By passing to a subsequence, we may assume that $\sum_{j=1}^\infty |Q_j| < \infty$.

In order to construct the disjoint subsets sets $E_j = E_{Q_j}$, we make the following observation:
\[
(5.23) \quad \forall Q \quad \exists \theta_Q \in (0, \frac{1}{2}) \quad \forall E \subset Q : \quad |E| \geq (1 - \theta_Q)|Q| \quad \Rightarrow \quad \mathcal{O}_\nu^\alpha(b; E) \geq \frac{1}{2} \mathcal{O}_\nu^\alpha(b; Q).
\]

Indeed, if not, then we can find a sequence of $E_n \subset Q$ with $\mathcal{O}_\nu^\alpha(b; E_n) \leq \frac{1}{2} \mathcal{O}_\nu^\alpha(b; Q)$, while $|E_n| \to |Q|$, and a subsequence will satisfy $1_{E_n} \to 1_Q$ almost everywhere. Then dominated convergence shows that $\nu(E_n) = \int 1_{E_n} \nu \to \nu(Q)$ and $\langle b \rangle_{E_n} = |E_n|^{-1} \int_Q 1_{E_n} b \to \langle b \rangle_Q$, and finally
\[
\int_{E_n} |b - \langle b \rangle_{E_n}| = \int_Q 1_{E_n} |b - \langle b \rangle_Q + (\langle b \rangle_Q - \langle b \rangle_{E_n})| \to \int_Q |b - \langle b \rangle_Q|,
\]
so that $\mathcal{O}_\nu^\alpha(b; E_n) \to \mathcal{O}_\nu^\alpha(b; Q)$, contradicting $\mathcal{O}_\nu^\alpha(b; E_n) \leq \frac{1}{2} \mathcal{O}_\nu^\alpha(b; Q)$, thus proving (5.23).

We now pick a subsequence $\{Q'_j = Q_{k(j)}\}_{j}$ of the original sequence $\{Q_j\}_{j}$ and as follows. Let $Q'_1 = Q_1$. Assuming that $Q'_1, \ldots, Q'_j = Q_{k(j)}$ have already been chosen, we choose $k(j+1) > k(j)$ large enough so that
\[
\sum_{i=k(j+1)}^{\infty} |Q_i| \leq \theta_j |Q'_j|, \quad \theta_j = \theta_{Q'_j},
\]
and then set $Q'_{j+1} = Q_{k(j+1)}$. For the sequence thus constructed, we hence have
\[
\sum_{j=1}^{\infty} |Q'_j| \leq \sum_{j=k(j+1)}^{\infty} |Q_i| \leq \theta_j |Q'_j|,
\]
and thus $E_j = Q'_j \setminus \bigcup_{i=j+1}^{\infty} Q'_i$ satisfies $|E_j| \geq (1 - \theta_{Q'_j}) |Q'_j|$. By (5.23), this implies that $\mathcal{O}_\nu^\alpha(b; E_j) \geq \frac{1}{2} \mathcal{O}_\nu^\alpha(b; Q'_j) \gtrsim 1$. This gives the required sparse collection $\{Q, E_Q\}_{Q \in \mathcal{S}} = \{Q'_j, E_{Q'_j}\}_{j=1}^{\infty}$ in this case.

Lastly, suppose that the VMO$^\alpha_\nu$ condition fails via (2.4), i.e. we find a sequence of cubes $\{Q_j\}_{j=0}^{\infty}$ such that
\[
\mathcal{O}_\nu^\alpha(b; Q_j) \gtrsim 1, \quad \lim_{j \to \infty} \ell(Q_j) = \infty.
\]

We only show first step of the iterative construction of $\mathcal{S}$, as the subsequent inductive steps are entirely analogous. We begin with setting $\mathcal{S}_0 = \{Q_0, E_{Q_0}\}$ with $E_{Q_0} = Q_0$ and show how to construct $\mathcal{S}_1 \supset \mathcal{S}_0$. Let us denote $P_k = Q_j$ for some $j$ such that $\ell(Q_j) \geq 2^k$ and $\ell(P_k) \geq \ell(Q_0)$. If $P_k \cap Q_0 = \emptyset$, we choose $Q_1 = P_k$ and $E_{Q_1} = Q_1$ and with $\mathcal{S}_1 = \{Q_i, E_{Q_i}\}_{i=0,1}$ we are done. If $P_k \cap Q_0 \neq \emptyset$, by $\ell(P_k) \geq \ell(Q_0)$ there exists
a cube $\hat{P}_k \supset P_k \cup Q_0$ such that $\ell(P_k) \sim \ell(\hat{P}_k)$; and by $\nu$ being doubling, we know that $O^0_\nu(\hat{P}_k) \gtrsim O^0_\nu(P_k) \gtrsim 1$. Hence, without loss of generality, let us denote $P_k = \hat{P}_k$ and assume that $Q_0 \subset P_k$. We will next show that if $k$ is sufficiently large, then $E_{P_k} = P_k \setminus Q_0$ satisfies $O^0_\nu(b; E_{P_k}) \gtrsim 1$, and then we set $Q_1 = P_k$ and $E_{Q_1} = E_{P_k}$.

We first estimate

$$
\int_{P_k} |b - \langle b \rangle_{E_{P_k}}| \leq \int_{Q_0} |b - \langle b \rangle_{Q_0}| + \int_{Q_0} |\langle b \rangle_{Q_0} - \langle b \rangle_{E_{P_k}}| + \int_{E_{P_k}} |b - \langle b \rangle_{E_{P_k}}|,
$$

where the last term is of the desired form. We will next show that

$$
\lim_{k \to \infty} \nu(P_k)^{-\beta} \int_{Q_0} |b - \langle b \rangle_{Q_0}| = 0, \quad \lim_{k \to \infty} \nu(P_k)^{-\beta} \int_{Q_0} |\langle b \rangle_{Q_0} - \langle b \rangle_{E_{P_k}}| = 0.
$$

The left claim on line (5.25) is immediate from $b \in L^1_{\text{loc}}$ and $\nu \in A_\infty$ and we provide details only for the right limit. Let $\{R_k\}_{k=0}^m$ (we have $m \sim \log_2(|P_k|/|Q_0|)$ but the exact value of $m$ will not play a further role in the argument) be a sequence of cubes such that

$$R_0 = Q_0, \quad R_m = P_k, \quad R_k \subset R_{k+1}, \quad \ell(R_{k+1}) \sim \ell(R_k),$$

also denote $R_{m+1} = E_{P_k}$. Then, we estimate

$$
|\langle b \rangle_{Q_0} - \langle b \rangle_{E_{P_k}}| \leq \sum_{j=0}^m |\langle b \rangle_{R_{j+1}} - \langle b \rangle_{R_j}| \lesssim \|b\|_{\text{BMO}_\nu} \sum_{j=0}^m \frac{\nu(R_j)^\beta}{|R_j|}
$$

and hence we find that

$$
\nu(P_k)^{-\beta} \int_{Q_0} |\langle b \rangle_{Q_0} - \langle b \rangle_{E_{P_k}}| \lesssim \|b\|_{\text{BMO}_\nu} \sum_{j=0}^m \frac{|Q_0|}{|R_j|} \left( \frac{\nu(R_j)}{\nu(P_k)} \right)^\beta.
$$

Next we show that the sum tends to zero as $k$ (equivalently $m$) tends to infinity. Since $\nu \in A_\infty$ we know that for some $\delta > 0$ there holds that $\nu(E_Q)/\nu(Q) \lesssim (|E_Q|/|Q|)^\delta$ for all $E_Q \subset Q$. Hence,

$$
\sum_{j=0}^m \frac{|Q_0|}{|R_j|} \left( \frac{\nu(R_j)}{\nu(P_k)} \right)^\beta \lesssim \sum_{j=0}^m \frac{|Q_0|}{|R_j|} \left( \frac{|R_j|}{|P_k|} \right)^\delta \lesssim \sum_{j=0}^m 2^{-jd}2^{-(m-j)d\delta}
$$

and clearly the right-hand side tends to zero by $\beta > 0$; we conclude the right limit on line (5.25). All in all, using

$$
\int_{P_k} |b - \langle b \rangle_{P_k}| = \int_{P_k} |b - c - \langle b - c \rangle_{P_k}| \leq 2 \int_{P_k} |b - c|$$
for any constant $c$, and then lines (5.24) and (5.25), we have shown that for any $\varepsilon > 0$ there exists $k$ large enough so that

$$1 \lesssim O_\nu(b; P_k) = \nu(P_k)^{-\beta} \int_{P_k} \|b - \langle b \rangle_{P_k}\| \lesssim \nu(P_k)^{-\beta} \int_{P_k} \|b - \langle b \rangle_{P_k}\|$$

(5.28)

$$\lesssim \varepsilon + \nu(P_k)^{-\beta} \int_{P_k} \|b - \langle b \rangle_{P_k}\|$$

Choosing $\varepsilon > 0$ small enough, we conclude that $O_\nu(b; P_k) \gtrsim 1$. 

5.29. Proposition. Let $1 < p, q < \infty$, $\mu \in A_{p, p}$ and $\lambda \in A_{q, q}$. Let $\nu$ be the fractional Bloom weight of Definition 2.10, let $T$ be a non-degenerate SIO and $b \in L^1_{\text{loc}}(\mathbb{R}^d, \mathbb{C})$. Then, $[b, T] \in \mathcal{K}(L^p_{\mu}, L^q_{\lambda})$ implies that

$$\|b, T\|_{L^p_{\mu}, L^q_{\lambda}} \lesssim \|b, T\|_{L^p_{\mu}, L^q_{\lambda}}$$

for any constant $c$ for any constant $c$. Let us assume for a contradiction that $b \notin \text{VMO}_\nu^\alpha$, i.e. $b \in \text{BMO}_\nu^\alpha \setminus \text{VMO}_\nu^\alpha$, and by Proposition 5.22 find a sparse sequence $(Q_j, E_j)_{j=1}^\infty$ of cubes such that $O_\nu(b; E_j) \gtrsim 1$. By Proposition 4.2 applied with $g_{E_j} = 1_{E_j}$ and $g_{E_j} = 1_{Q_j}$, we have

$$\int_{E_j} \|b - \langle b \rangle_{E_j}\| \lesssim \|\langle [b, T]g_{E_j}, h_{Q_j}\rangle + \langle [b, T]h_{E_j}, g_{Q_j}\rangle\|$$

(5.30)

$$\sim \max \left\{ \|\langle [b, T]g_{E_j}, h_{Q_j}\rangle\|, \|\langle [b, T]h_{E_j}, g_{Q_j}\rangle\| \right\} \lesssim \|b, T\|_{\phi_{E_j}, \psi_{Q_j}}$$

for the choice of $\phi_{E_j} \in \{g_{E_j}, h_{E_j}\}$ and $\psi_{Q_j} \in \{h_{Q_j}, g_{Q_j}\}$ that achieve the maximum. By Proposition 4.2 we have

$$\|\phi_{E_j}\| \lesssim 1_{E_j}, \quad \|\psi_{Q_j}\| \lesssim 1_{Q_j}.$$  

Recall that the implicit constants above are independent of $j$ (as are the implicit constants later on in this proof as well). Combining the above information, we have

$$1 \lesssim O_\nu(b; E_j) = \frac{1}{\nu(E_j)^{1+\alpha/d}} \int_{E_j} \|b - \langle b \rangle_{E_j}\|$$

(5.32)

$$\lesssim \frac{1}{\nu(E_j)^{1+\alpha/d}} \|\langle [b, T]\phi_{E_j}, \psi_{Q_j}\rangle\| \lesssim \frac{1}{\nu(E_j)^{1+\alpha/d}} \|\langle [b, T]\phi_{E_j}, \psi_{Q_j}\rangle\|$$

where we denote $u_{E_j} := \mu^p(Q_j)^{-1/p} \phi_{E_j}$. By the uniform bound

$$\|u_{E_j}\|_{L^p_{\mu}} = \mu^p(Q_j)^{-1} \int_{Q_j} |\phi_{E_j}|^p \mu^p \leq \|\phi_{E_j}\|_{L^p_{\mu}} \lesssim 1.$$
the compactness of $[b, T]$ gives a subsequence of $\{[b, T]u_{E_j}\}_{j=1}^{\infty}$ with a limit $\Phi$ in $L^q$, and by (5.32) we have $\|\Phi\|_{L^q} > 0$. Furthermore, the functions $u_{E_j}$ are disjointly supported. Concluding, we have constructed a sequence of functions just as in Lemma 5.21, hence $[b, T]$ is not bounded, a contradiction, and thus necessarily $b \in \text{VMO}^\alpha_{\nu}(\mathbb{R}^d)$. □

APPENDIX A. WEIGHTED FRACTIONAL JOHN-NIRENBERG INEQUALITY FOR VMO

Let us define the weighted $\text{BMO}^r w, \alpha$ and $\text{VMO}^r w, \alpha \subset \text{BMO}^r w$ similarly as in Definition 2.1 but with the oscillation

\[
\mathcal{O}^r w, \alpha (f; Q) = w(Q)^{-\alpha/d} \left( \frac{1}{w(Q)} \int_Q (|f - \langle f \rangle_Q|^r)^{r/d} \, dw \right)^{\frac{1}{r}} ,
\]

note that $\mathcal{O}^w w, \alpha (f; Q) = \mathcal{O}^w 1, \alpha (f; Q)$. When $\alpha = 0$, the following Theorem A.2 is a classical result of Muckenhoupt and Wheeden [18].

A.2. Theorem. Let $1 \leq p < \infty$ and suppose that $w \in A_p$. Let $\alpha \in [0, \infty)$. Then, there holds that

\[
\|b\|_{\text{BMO}^r w, \alpha} \sim \|b\|_{\text{BMO}^1 w, \alpha} ,
\]

whenever $1 \leq r \leq p'$ and $r < \infty$.

Here we give a short proof of Theorem A.2 and extend it to weighted fractional VMO as

A.4. Theorem. Let $1 \leq p < \infty$ and suppose that $w \in A_p$. Let $\alpha \in [0, \infty)$. Then, there holds that

\[
\text{VMO}^r w, \alpha = \text{VMO}^1 w, \alpha ,
\]

whenever $1 \leq r \leq p'$ and $r < \infty$.

A.6. Remark. If $w \in A_p$, then by the reverse Hölder property $w \in A_{p-\delta}$ for some $\delta > 0$. Hence, the conclusions (A.5) and (A.3) hold with $1 \leq r < (p - \delta)'$, where $(p - \delta)' > p'$. Restating, for each $w \in A_p$ there exists some $\varepsilon > 0$ such that the conclusions (A.5) and (A.3) hold for all $1 \leq r < p' + \varepsilon$.

Both Theorems A.2 and A.4 follow almost immediately from the following

A.7. Proposition. Let $1 \leq p < \infty$ and suppose that $w \in A_p$, and let $\alpha \in [0, \infty)$. Then, for each $f \in L^1_{\text{loc}}$ and a cube $Q_0$, there exists a sparse collection $\mathcal{F}(Q_0) \subset \mathcal{D}(Q_0)$ such that

\[
\mathcal{O}^r w, \alpha (f; Q_0) \lesssim \left( \frac{1}{w(Q_0)^{1+\frac{\alpha}{d}}} \sum_{Q \in \mathcal{F}(Q_0)} \mathcal{O}^1 w, \alpha (f; Q)^r w(Q)^{1+\frac{\alpha}{d}} \right)^{\frac{1}{r}} ,
\]

whenever $1 \leq r \leq p'$ and $r < \infty$. 

Proof of Theorem A.2. Suppose first that \( b \in \text{BMO}\,_{w}^{r,\alpha} \) for some \( r > 1 \). Then, by Hölder’s inequality we find

\[
\int_{Q} |b - \langle b \rangle_{Q}| \leq \left( \int_{Q} |b - \langle b \rangle_{Q}|^{r} w^{-\frac{r}{p}} \right)^{\frac{1}{r}} \left( \int_{Q} w \right)^{\frac{1}{p}} = \left( \frac{1}{w(Q)} \right) \int_{Q} \left| b - \langle b \rangle_{Q} \right| \frac{1}{w} \, d\mu \left( \int_{Q} w \right)^{\frac{1}{p}} \left( \int_{Q} w \right)^{\frac{1}{p}} \leq \|b\|_{\text{BMO}\,_{w}^{r,\alpha}} \, w(Q)^{\alpha/d} \, w(Q).
\]

For the other direction, by Proposition (A.7) we estimate

\[
\mathcal{O}_{w}^{r,\alpha}(f; Q) \lesssim \left( \frac{1}{w(Q)} \right)^{1+\frac{r}{p}} \sum_{Q \in \mathcal{S}(Q_0)} \mathcal{O}_{w}^{1,\alpha}(f; Q)^{r} \, w(Q)^{1+\frac{r}{p}} \lesssim \|b\|_{\text{BMO}\,_{w}^{1,\alpha}},
\]

where we used sparseness in the last estimate; recall that sparseness with respect to any measure in \( A_{\infty} \) is equivalent with sparseness with respect to the Lebesgue measure. \( \Box \)

Proof of Theorem A.4. Suppose that \( b \in \text{VMO}\,_{w}^{1,\alpha} \) and we will show that \( b \in \text{VMO}\,_{w}^{r,\alpha} \). Let us consider the different kinds of cubes as in the conditions (2.3), (2.4), (2.5). If a cube is sufficiently small, or sufficiently far away from the origin, as respectively in the conditions (2.3) and (2.5), then all the subcubes are also, and it is immediate from Proposition A.7 that the conditions (2.3) and (2.5) hold with all \( 1 \leq r \leq p' \). Let us then turn to check the condition (2.4).

By conditions (2.3) and (2.5) there exists a cube \( P_0 \) such that if \( \ell(Q) \geq \ell(P_0) \) or \( Q \cap P_0 \neq \emptyset \), then \( \mathcal{O}_{w}^{1,\alpha}(b; Q) \leq \varepsilon \), and let \( Q \) be exactly such a cube. We show that \( \mathcal{O}_{w}^{r,\alpha}(b; Q) \lesssim \varepsilon \), provided that \( Q \) is taken sufficiently large. Let \( Q_0 \supset Q \) be a cube such that \( \ell(Q_0) \lesssim \ell(Q) \) and \( P_0 \in \mathcal{S}(Q_0) \), notice that we can always arrange this with the implicit constant independent of \( Q \). By \( w \) being doubling, we know that \( \mathcal{O}_{w}^{r,\alpha}(b; Q) \lesssim \mathcal{O}_{w}^{r,\alpha}(b; P_0) \). Let \( \mathcal{S}(Q_0) \subset \mathcal{D}(Q_0) \) be the sparse collection as in Proposition A.7 and we estimate

\[
\left( \sum_{L \in \mathcal{S}(Q_0)} \mathcal{O}_{w}^{1,\alpha}(b; L)^{r} w(L)^{1+\frac{r}{p}} \right)^{\frac{1}{r}} \lesssim \|b\|_{\text{BMO}\,_{w}^{1,\alpha}} \, w(Q_0)^{1+\frac{r}{p}}.
\]

The first sum on the right is controlled by the fact that \( \text{VMO}\,_{w}^{1,\alpha} \subset \text{BMO}\,_{w}^{1,\alpha} \) as

\[
\sum_{L \in \mathcal{S}(Q_0)} \mathcal{O}_{w}^{1,\alpha}(b; L)^{r} w(L)^{1+\frac{r}{p}} \lesssim \|b\|_{\text{BMO}\,_{w}^{1,\alpha}} \, \sum_{L \in \mathcal{S}(Q_0)} w(L)^{1+\frac{r}{p}} \lesssim \|b\|_{\text{BMO}\,_{w}^{1,\alpha}} \, w(P_0)^{1+\frac{r}{p}},
\]

where we used that \( \| \cdot \|_{w} \leq \| \cdot \|_{\alpha} \) for \( s \geq 1 \), and sparsity; while for the second sum if \( L \cap P_0^c \neq \emptyset \), then by \( L, P_0 \in \mathcal{D}(Q_0) \) either \( P_0 \subset L \) or \( L \subset P_0^c \), and in both cases by the choice of \( P_0 \) we know that \( \mathcal{O}_{w}^{1,\alpha}(b; L) \leq \varepsilon \). Hence, we find that

\[
\left( \sum_{L \in \mathcal{S}(Q_0)} \mathcal{O}_{w}^{r,\alpha}(b; Q_0)^{r} \right)^{\frac{1}{r}} \lesssim \|b\|_{\text{BMO}\,_{w}^{1,\alpha}} \, w(P_0)^{1+\frac{r}{p}} + \varepsilon^{r} w(Q_0)^{1+\frac{r}{p}},
\]

from which the claim follows by choosing \( Q \) (hence \( Q_0 \)) sufficiently large. \( \Box \)
In proving Proposition A.7 we can use the following almost orthogonality principle.

A.9. Lemma. Let \( 1 < p < \infty \) and \( w \in A_p \), let \( \mathcal{S} \subset \mathcal{D} \) be a sparse collection of cubes. For each \( Q \in \mathcal{S} \), let \( f_Q \) be a function supported on \( Q \) that is constant on each \( P \in \text{ch}(Q) \), where \( \text{ch}(Q) \) denotes the maximal elements of \( \{ P \in \mathcal{S} : P \subsetneq Q \} \). Then, there holds that

\[
\| \sum_{Q \in \mathcal{S}} f_Q \|_{L^p(w)} \lesssim \left( \sum_{Q \in \mathcal{S}} \| f_Q \|_{L^p(w)}^p \right)^{\frac{1}{p}}.
\]

Lemma A.9 is certainly known but we could not find a reference, hence we recall it here.

Proof. By \( f_Q \) being constant on \( \text{ch}(Q) \) we can write

\[
\| \sum_{Q \in \mathcal{S}} f_Q \|_{L^p(w)} \leq \| \sum_{Q \in \mathcal{S}} f_Q 1_{E_Q} \|_{L^p(w)} + \| \sum_{Q \in \mathcal{S}} \sum_{P \in \text{ch}(Q)} \langle f_Q \rangle P 1_P \|_{L^p(w)}
\]

and by disjointness of \( E_Q \subset Q \) the left sum on the right clearly satisfies the desired estimate. For the second one, by duality, it is enough to estimate as follows. We have

\[
\int \sum_{Q \in \mathcal{S}} \sum_{P \in \text{ch}(Q)} \langle f_Q \rangle P 1_P \cdot g = \sum_{Q \in \mathcal{S}} \sum_{P \in \text{ch}(Q)} \langle f_Q \rangle P w(P)^{\frac{1}{p}} (g)_P |P|^p |w(P)|^{-\frac{1}{p}} \leq \left( \sum_{Q \in \mathcal{S}} \sum_{P \in \text{ch}(Q)} \| \langle f_Q \rangle P w(P) \|_{p'}^{p'} \right)^{\frac{1}{p'}} \left( \sum_{Q \in \mathcal{S}} \sum_{P \in \text{ch}(Q)} \| g \|_{p'} |P|^{p'} w(P)^{-\frac{1}{p'}} \right)^{\frac{1}{p'}}.
\]

By \( f_Q \) being constant on \( \text{ch}(Q) \) we have

\[
\left( \sum_{Q \in \mathcal{S}} \sum_{P \in \text{ch}(Q)} \| \langle f_Q \rangle P w(P) \|_{p'}^{p'} \right)^{\frac{1}{p'}} \leq \left( \sum_{Q \in \mathcal{S}} \| f_Q \|_{L^p(w)}^p \right)^{\frac{1}{p'}}
\]

and it remains to estimate the term with the function \( g \). By Hölder’s inequality, we know that \( |P|^{p'} w(P)^{-\frac{1}{p'}} = |P| w^{-\frac{1}{p'}} (P) \leq w^{-\frac{1}{p'}} (P) \), and hence

\[
\left( \sum_{Q \in \mathcal{S}} \sum_{P \in \text{ch}(Q)} \| g \|_{p'} |P|^{p'} w(P)^{-\frac{1}{p'}} \right)^{\frac{1}{p'}} \leq \left( \sum_{Q \in \mathcal{S}} \sum_{P \in \text{ch}(Q)} \| g \|_{p'} w^{-\frac{1}{p'}} (P) \right)^{\frac{1}{p'}} \lesssim [w^{-\frac{1}{p'}}]_{A_p} \| g \|_{L^{p'}(w^{-\frac{1}{p'}})} \sim [w]_{A_p} \| g \|_{L^{p'}(w^{-\frac{1}{p'}})},
\]

where in the second estimate we used Lemma A.13 below. \( \Box \)

A.13. Lemma. Suppose that \( 1 < p < \infty \), that \( w \in A_p \) and that \( \mathcal{S} \) is sparse. Then, there holds that

\[
\left( \sum_{Q \in \mathcal{S}} \langle f \rangle Q w(Q) \right)^{\frac{1}{p}} \lesssim \| f \|_{L^p(w)}.
\]

Proof. By sparseness and the boundedness of the maximal function on \( L^p(w) \), we find that

\[
\left( \sum_{Q \in \mathcal{S}} \langle f \rangle Q w(Q) \right)^{\frac{1}{p}} \lesssim [w]_{A_p} \left( \int \left( \sum_{Q \in \mathcal{S}} \langle f \rangle Q 1_{E_Q} \right) \, dw \right)^{\frac{1}{p}} \leq \left( \int (Mf)^p \, dw \right)^{\frac{1}{p}} \lesssim [w]_{A_p} \| f \|_{L^p(w)},
\]

where we used the boundedness of the maximal operator in the last step. \( \Box \)
Proof of Proposition A.7. Given that $f \in L^1_{\text{loc}}$, by an iterated Calderón-Zygmund decomposition, see e.g. [14] and this is also a special case of Lemma 4.8, there exists a sparse collection $\mathcal{S}(Q_0) \subset 2^\mathcal{S}(Q_0)$ so that

$$|f - \langle f \rangle_{Q_0}|1_{Q_0} \lesssim \sum_{Q \in \mathcal{S}(Q_0)} \langle |f - \langle f \rangle_Q|1_Q \rangle_{Q}.$$  

Note that $p \leq r'$ (by $r \leq p'$) and hence $w^{1-r} \in A_r$ (by $w \in A_p \subset A_r'$). Hence, by Lemma A.9 we estimate

$$w(Q_0)^{1+\frac{\alpha}{r'}}O^{\alpha}_{w}(f; Q_0)^{r} \lesssim \int_{Q_0} \left( \sum_{Q \in \mathcal{S}(Q_0)} \langle |f - \langle f \rangle_Q|1_Q \rangle_{Q} \right)^{r} \, dw$$

$$= \left\| \sum_{Q \in \mathcal{S}(Q_0)} w(Q)^{\frac{\alpha}{r'}} \langle w \rangle_{Q} O^{\alpha}_{w}(f; Q) 1_{Q} \right\|_{L^r(w^{1-r})}^{r}$$

$$\lesssim [w^{1-r}]_{A_r'} \sum_{Q \in \mathcal{S}(Q_0)} \left\| w(Q)^{\frac{\alpha}{r'}} \langle w \rangle_{Q} O^{\alpha}_{w}(f; Q) 1_{Q} \right\|_{L^r(w^{1-r})}^{r}$$

$$\sim [w]_{A_r'} \sum_{Q \in \mathcal{S}(Q_0)} w(Q)^{\frac{\alpha}{r'}} O^{\alpha}_{w}(f; Q)^{r} w(Q)^{1+\frac{\alpha}{r'}}.$$  

where in the last estimate we used that $\langle w \rangle_{Q}^{r}w^{1-r}(Q) \leq [w]_{A_r'}^{r}w(Q)$. Now the proof is concluded after dividing with $w(Q_0)^{1+\frac{\alpha}{r'}}$ and taking the $r$th root. □

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