Bayesian Estimation for the Parameters and Reliability Function of Basic Gompertz Distribution under Squared Log Error Loss Function

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Abstract
In this paper, some estimators for the unknown shape parameters and reliability function of Basic Gompertz distribution were obtained, such as Maximum likelihood estimator and some Bayesian estimators under Squared log error loss function by using Gamma and Jefferys priors. Monte-Carlo simulation was conducted to compare the performance of all estimates of the shape parameter and Reliability function, based on mean squared errors (MSE) and integrated mean squared errors (IMSE’s), respectively. Finally, the discussion is provided to illustrate the results that are summarized in tables.

Keywords: Basic Gompertz distribution, Maximum likelihood estimator, Bayes estimator, Squared log error loss function, Reliability function, Mean squared errors.

1. Introduction
The Gompertz distribution plays an important role in modeling survival times, human mortality and actuarial data. It was formulated by Gompertz (1825) to fit mortality tables[1].

The probability density function of the Gompertz distribution is defined as follows [2]:

\[ f(t; \lambda) = \lambda \exp \left[ ct + \frac{\lambda}{c} (1 - \exp(t)) \right] ; \quad t \geq 0 \quad c, \lambda > 0 \]

where \( c \) is the scale parameter and \( \lambda \) is the shape parameter of the Gompertz distribution.

In this paper, we’ll assume that \( c=1 \), which is a special case of Gompertz distribution known as Basic Gompertz distribution with the following probability density function [3]

\[ f(t; \lambda) = \lambda \exp[t + \lambda(1 - \exp(t))]; \quad t \geq 0 \quad \lambda > 0 \]

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The corresponding cumulative distribution function $F(t)$ and reliability or survival function $R(t)$ of Basic Gompertz distribution are given by

\[
F(t) = 1 - \exp[\lambda(1 - e^t)] \quad ; \quad t \geq 0
\]

\[
R(t) = F(t) = \exp[\lambda(1 - e^t)] \quad ; \quad t \geq 0
\]

2. Maximum likelihood Estimator of the Shape Parameter ($\lambda$)

Assume that $t_1, t_2, \ldots, t_n$ are a random sample of size $n$ from the Basic Gompertz distribution defined by eq.(1), then the likelihood function for the sample observation will be as follows [4]

\[
L(t_1, t_2, \ldots, t_n; \lambda) = \prod_{i=1}^{n} f(t_i; \lambda) = \lambda^n \exp\left[\sum_{i=1}^{n} t_i + \lambda \sum_{i=1}^{n} (1 - e^{t_i})\right] \quad (3)
\]

By letting $\frac{\partial}{\partial \lambda} \ln L(t_i; \lambda) = 0$, the MLE of $\lambda$ becomes

\[
\hat{\lambda}_{ML} = -\frac{n}{T}
\]

where $T = \sum_{i=1}^{n} (1 - e^{t_i})$

Based on the invariant property of the MLE, the MLE for $R(t)$ will be as follows

\[
\hat{R}(t)_{ML} = \exp[\hat{\lambda}_{ML}(1 - e^t)]
\]

3. Bayesian Estimation

We provide Bayesian estimation method for estimating $\lambda$ and $R(t)$ of Basic Gompertz distribution, including informative and non-informative priors.

3.1 Posterior Density Functions Using Gamma Distribution

In this subsection, we assumed that $\lambda$ is distributed Gamma as a prior distribution with density [5].

\[
g_1(\lambda) = \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta \lambda} \quad ; \quad \lambda > 0, \quad \alpha, \beta > 0
\]

In general, the posterior probability density function of unknown parameter $\lambda$ with prior $g(\lambda)$ can be expressed as

\[
\pi(\lambda|t) = \frac{L(t_1, t_2, \ldots, t_n; \lambda) g(\lambda)}{\int_{\lambda} L(t_1, t_2, \ldots, t_n; \lambda) g(\lambda) d\lambda}
\]

Now, combining eq. (3) with eq. (5) in eq. (6) yields:

\[
\pi_1(\lambda|t) = \frac{\lambda^{n+a-1} e^{-\lambda[\beta - \sum_{i=1}^{n} (1 - e^{t_i})]}}{\int_{\lambda} \lambda^{n+a-1} e^{-\lambda[\beta - \sum_{i=1}^{n} (1 - e^{t_i})]} d\lambda}
\]

After simplification, we get

\[
\pi_1(\lambda|t) = \frac{(\beta - T)^{n+a} \lambda^{n+a-1} e^{-\lambda(\beta - T)}}{\Gamma(n+a) \lambda^{n+a} e^{-\lambda(\beta - T)}}
\]

\[
\pi_2(\lambda|t) = \frac{(\beta - T)^{n+a} \lambda^{n+a-1} e^{-\lambda(\beta - T)}}{\Gamma(n+a) \lambda^{n+a} e^{-\lambda(\beta - T)}}
\]

Notice that, the posterior p.d.f. of the parameter $\lambda$ is obviously Gamma distribution, i.e.

$\lambda|t \sim$ Gamma $(n + \alpha, \beta - T)$, with: $E(\lambda|t) = \frac{n+a}{\beta-T}$, $Var(\lambda|t) = \frac{n+a}{(\beta-T)^2}$

3.2 Posterior Density Functions Using Jeffrey's Prior

Assume that $\lambda$ has a non-informative prior density defined, using Jeffrey's prior information $g_2(\lambda)$, as follows [6]

\[
g_2(\lambda) \propto \sqrt{I(\lambda)}
\]

where $I(\lambda)$ is Fisher information which is defined as
\[ I(\lambda) = -nE\left(\frac{\partial^2 \ln f}{\partial \lambda^2}\right) \]

Hence,

\[ g_2(\lambda) = k \sqrt{-nE\left(\frac{\partial^2 \ln f(t; \lambda)}{\partial \lambda^2}\right)} \tag{7} \]

where \( k \) is a constant.

By taking the natural logarithm for p.d.f of Basic Gompertz distribution and taking the second partial derivative with respect to \( \lambda \), we get

\[ E\left(\frac{\partial^2 \ln f(t; \lambda)}{\partial \lambda^2}\right) = -\frac{1}{\lambda^2} \]

After substitution into eq. (7), we have

\[ g_2(\lambda) = \frac{k}{\lambda} \sqrt{n}, \quad \lambda > 0 \]

After substituting in eq. (6), the posterior density function based on Jeffreys prior can be written as

\[ \pi_2(\lambda|t) = \frac{\lambda^{n-1} e^{-\lambda \sum_{i=1}^{n}(e^{t_i} - 1)}}{\int_{0}^{\infty} \lambda^{n-1} e^{-\lambda \sum_{i=1}^{n}(e^{t_i} - 1)} d\lambda} \]

\[ = \frac{p^n \lambda^{n-1} e^{-\lambda p}}{\Gamma(n)} \]

where \( p = \sum_{i=1}^{n}(e^{t_i} - 1) = -T \)

The posterior density \( \pi_2(\lambda|t) \) is recognized as the density of the Gamma distribution, i.e.

\( (\lambda|t_1, ..., t_n) \sim \text{Gamma} (n, p) \), with \( E(\lambda|t_1, ..., t_n) = \frac{n}{p} \quad \text{and} \quad \text{var}(\lambda|t_1, ..., t_n) = \frac{n}{p^2} \)

### 3.3 Bayes Estimation under Squared Log Error Loss Function

This loss function was used by Brown in 1968 and takes the following formula [7]

\[ L(\hat{\lambda}, \lambda) = (\ln \hat{\lambda} - \ln \lambda)^2 = \left(\ln \frac{\hat{\lambda}}{\lambda}\right)^2 \]

This is coordinated with \( \lim L(\hat{\lambda}, \lambda) \to \infty \) as \( \hat{\lambda} \to 0 \) or \( \infty \).

Any equivalent loss function considers the estimation error and the convenience quality, however, the unequal loss function simply gets the estimation error. This loss function is convex toward \( \frac{\hat{\lambda}}{\lambda} \) and otherwise is concave, yet its risk function has a unique minimum with \( \hat{\lambda} \) [8].

According to the above mentioned loss function, we drive the corresponding Bayes estimators for \( \lambda \) using Risk function \( R(\hat{\lambda}, \lambda) \), which minimizes the posterior risk,

\[ R(\hat{\lambda}, \lambda) = E[L(\hat{\lambda}, \lambda)] = \int_{0}^{\infty} (\ln \hat{\lambda} - \ln \lambda)^2 \pi(\lambda|t_1, ..., t_n) d\lambda \]

\[ = (\ln \hat{\lambda})^2 - 2(\ln \hat{\lambda}) E(\ln \lambda|t) + E((\ln \lambda)^2|t) \]

Taking the partial derivative for \( R(\hat{\lambda}, \lambda) \) with respect to \( \hat{\lambda} \) and setting it equal to zero, gives

\[ \ln \hat{\lambda} = E(\ln \lambda|t) \]

Hence,

\[ \hat{\lambda} = \exp(E(\ln \lambda|t)) \tag{8} \]

### 3.3.1 Bayes Estimation under Squared Log Error Loss Function with Gamma Prior

Bayes estimator relative to Squared log error loss function based on Gamma prior can be derived as follows

\[ E(\ln \lambda|t) = \int_{0}^{\infty} \ln \lambda \frac{(\beta - T)^{n+a}}{\Gamma(n + a)} \lambda^{n+a-1} e^{-\lambda(\beta - T)} d\lambda \tag{9} \]

By using the transformation technique by assuming that,
Recall that $\Gamma(n + \alpha)$ is a Gamma function which is defined as
\[
\int_0^{\infty} y^{n+\alpha-1} e^{-y} dy = \frac{d}{d(n + \alpha)} \ln \Gamma(n + \alpha) = \psi(n + \alpha)
\]
\[\int_0^{\infty} y^{n+\alpha-1} e^{-y} dy = d(n + \alpha) \ln \Gamma(n + \alpha) = \psi(n + \alpha)
\]

Combining equations \((14)\) and \((15)\) gives
\[
\hat{R}(t)_{BG} = \text{Exp}\left(\frac{n + \alpha}{\beta - T} (1 - e^t)\right)
\]
Where, $\hat{R}(t)_{BG}$ represents Bayesian estimation for R(t) under Squared log error loss function with Gamma prior.

### 3.3.2 Bayes Estimation under Squared Log Error Loss Function with Jeffreys Prior

Similarly, we can obtain the Bayes estimator for the shape parameter $\lambda$ under Jeffreys prior by using eq. \((8)\) as follows
\[
E(\ln \lambda | t) = \int_0^{\infty} \ln \lambda \frac{p^n}{\Gamma(n)} \lambda^{n-1} e^{-\lambda p} d\lambda
\]

By letting $y = \lambda p$ which implies that,
$\lambda = \frac{y}{p}$ and $d\lambda = \frac{dy}{p}$

After substituting into eq.\((16)\), we get
\[
E(ln \lambda |t) = \int_0^\infty \ln \left( \frac{y}{p} \right) \frac{p^n}{\Gamma(n)} \left( \frac{y}{p} \right)^{n-1} e^{-y} \frac{dy}{p}
\]

\[
= \int_0^\infty \ln y y^{n-1} e^{-y} \frac{dy}{\Gamma(n)} - \ln(p) \int_0^\infty y^{n-1} e^{-y} dy
\]

\[
E(ln \lambda |t) = \psi(n) - \ln P
\]

So, from eq. (8) and eq.(18), we get the Bayes estimator of parameter \( \lambda \) under Squared log error loss function, based on Jefferys prior, as the following form

\[
\lambda_{BJ} = \exp(\psi(n) - \ln P)
\]

Similarly, we can find the corresponding estimator for \( R(t) \) by using eq. (14) where

\[
E(ln R(t) |t) = \int_0^\infty ln R(t) \pi_2(\lambda |t) d\lambda
\]

\[
E(ln R(t) |t) = \frac{p^n}{\Gamma(n)} \int_0^\infty \lambda (1 - e^t) \lambda^{n-1} e^{-\lambda p} d\lambda
\]

\[
= n \int_0^\infty \frac{p^{n+1}}{\Gamma(n+1)} \lambda^n e^{-\lambda p} d\lambda - e^t n \int_0^\infty \frac{p^{n+1}}{\Gamma(n+1)} \lambda^{n-1} e^{-\lambda p} d\lambda
\]

\[
E(ln R(t) |t) = \frac{n}{p} (1 - e^t)
\]

(20)

Substitute eq. (20) into eq. (14) gives

\[
\hat{R}(t)_{BJ} = \exp \left( \frac{n(1 - e^t)}{p} \right)
\]

Recall that \( P = -T \), thus, \( \hat{R}(t)_{BJ} \) is equivalent to MLE for \( \hat{R}(t)_{ML} \)

4. Simulation Study

In this section, a Monte Carlo simulation was performed to compare the performance of the different estimators of the unknown shape parameter \( \lambda \) and Reliability function \( R(t) \) for Basic Gompertz distribution. The process was repeated 5000(L=5000) times with different sample sizes \( (n = 15, 50, \text{ and } 100) \).

The default values of the shape parameter \( \lambda \) and two values of the Gamma prior parameters were chosen to be less than and greater than one, as \( \lambda = 0.5, 3; \alpha = 0.8, 3; \beta = 0.5, 3. \)

All estimators for \( \lambda \) that were derived in the previous section are evaluated based on their mean squared errors (MSE’s), where,

\[
\text{MSE}(\lambda) = \frac{1}{L} \sum_{i=1}^{L} (\hat{\lambda}_i - \lambda)^2
\]

The integrated mean squared error (IMSE) was employed to compare the performance of the Bayesian estimators for \( R(t) \). IMSE is an important global measure and more accurate than MSE, which is defined as the distance between the estimated value and actual value of reliability function given by equation, where

\[
\text{IMSE}(\hat{R}(t)) = \frac{1}{L} \sum_{i=1}^{L} \left( \frac{1}{n_i} \sum_{j=1}^{n_i} (\hat{R}_i(t_j) - R(t_j)) \right)^2
\]

\[
= \frac{1}{n_\ell} \sum_{j=1}^{n_\ell} \text{MSE}(\hat{R}_i(t_j))
\]

where \( \ell = 1, \ldots, L, n_\ell \) the random limits of \( t_\ell \).

In this paper, we chose \( t = 0.1, 0.2, 0.3, 0.4, 0.5 \).

The expected values (EXP) and mean squared errors (MSEs) of all estimates of the shape parameter of Basic Gompertz distribution \( \lambda \) were tabulated in Tables-(1-2) for all sample sizes. The integrated mean squared error (IMSE) values for the estimates of \( R(t) \) were tabulated in Tables- (3-4).
Table 1- Expected values (EXP) and MSEs of the different estimates of the shape parameter $\lambda$ of Basic Gompertz distribution when $\lambda = 0.5$

| n  | Estimate Criteria | MLE          | Jeffreys prior | Gamma prior α=0.8 | α=3     |
|----|-------------------|--------------|----------------|-------------------|---------|
|    |                    | β=0.5        | β=3            | β=0.5             | β=3     |
| 15 | EXP               | 0.5353497    | 0.5169348      | 0.5351083         | 0.4893851 |
|    | MSE               | 0.0225615    | 0.0210955      | 0.0219541         | 0.0403994 |
| 50 | EXP               | 0.5026217    | 0.5026106      | 0.5031548         | 0.5167699 |
|    | MSE               | 0.0056782    | 0.0054159      | 0.0055315         | 0.0068623 |
| 100| EXP               | 0.5053751    | 0.4978777      | 0.505705          | 0.514384 |
|    | MSE               | 0.0026505    | 0.0025493      | 0.0025594         | 0.0025636 |

Table 2-Expected values (EXP) and MSEs of the different estimates of the shape parameter $\lambda$ of Basic Gompertz distribution when $\lambda = 3$

| n  | Estimate Criteria | MLE          | Jeffreys prior | Gamma prior α=0.8 | α=3     |
|----|-------------------|--------------|----------------|-------------------|---------|
|    |                    | β=0.5        | β=3            | β=0.5             | β=3     |
| 15 | EXP               | 3.2120990    | 3.1016020      | 2.936160          | 2.2394540 |
|    | MSE               | 0.8484274    | 0.7594381      | 0.5280226         | 0.707329 |
| 50 | EXP               | 3.0612750    | 3.0156660      | 2.9713070         | 2.6938410 |
|    | MSE               | 0.2044161    | 0.1949720      | 0.1780807         | 0.2026478 |
| 100| EXP               | 3.0321400    | 2.9872730      | 2.965800          | 2.8177440 |
|    | MSE               | 0.0954172    | 0.0917731      | 0.0887214         | 0.1014135 |

Table 3- IMSEs of the different estimates for reliability function R(t) of Basic Gompertz distribution when $\lambda = 0.5$

| n  | MLE                  | Jeffreys prior | Gamma prior α=0.8 | α=3     |
|----|----------------------|----------------|-------------------|---------|
|    |                      | β=0.5          | β=3               | β=0.5   | β=3     |
| 15 | 0.0014595            | 0.0014595      | 0.0021712         | 0.0021770 | 0.0007793 |
| 50 | 0.0003847            | 0.0003847      | 0.0003079         | 0.0004564 | 0.0003111 |
| 100| 0.0001869            | 0.0001869      | 0.0001837         | 0.0002053 | 0.0001678 |

Table 4-IMSEs of the different estimates for reliability function R(t) of Basic Gompertz distribution when $\lambda = 3$

| n  | MLE                  | Jeffreys prior | Gamma prior α=0.8 | α=3     |
|----|----------------------|----------------|-------------------|---------|
|    |                      | β=0.5          | β=3               | β=0.5   | β=3     |
| 15 | 0.0069471            | 0.0069471      | 0.0008101         | 0.0055115 | 0.0629822 |
| 50 | 0.0021114            | 0.0021114      | 0.0022277         | 0.0019502 | 0.0145029 |
| 100| 0.0010669            | 0.0010669      | 0.0010965         | 0.0010238 | 0.0050060 |

4. Results, Discussion and Analysis

The discussion of the results obtained from applying the simulation study can be summarized as follows:
1. When the shape parameter $\lambda=0.5$,
   - The best estimator for $\lambda$ is Bayes estimator under Squared log error loss function based on Gamma prior, with $\alpha=0.8$ and $\beta=3$ for all sample sizes (see Table-1).
   - From Table-3, it is clear that the best estimator for R(t) is Bayes estimator under Squared log error loss function with Gamma prior, when $\alpha=3$ and $\beta=3$ for all sample sizes.
2. When the shape parameter $\lambda=3$,  

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• From Table-2, notice that the performance of Bayes estimator under Squared log error loss function based on Gamma prior, is the best with $\alpha=0.8$ and $\beta=0.5$ for all sample sizes.
• Table-4 shows that the best estimate for $R(t)$ is Bayes estimator under the Squared log error loss function based on Gamma prior with $\alpha=3$ and $\beta=0.5$ for all sample sizes.

3. In general, MSEs and IMSEs are increasing with the increase of the shape parameter value.
4. From Tables-(3, 4), it is observed that IMSE values for Maximum likelihood and Bayes estimates under the Squared log error loss function with Jefferys prior are the same.

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