AUTOMORPHIC $L$-FUNCTIONS, INTERTWINING OPERATORS, AND THE IRREDUCIBLE TEMPERED REPRESENTATIONS OF $p$-ADIC GROUPS

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Introduction. The problem of identifying the tempered, or admissible dual of a reductive group consists of two problems. Determine the discrete series representations of Levi subgroups of the group, and decompose the resulting parabolically induced representations. Neither problem is resolved in any generality, and while [11] discusses the first, here we concentrate on the second and point out recent developments on the subject.

For the tempered dual, we will mainly discuss our approach, which is based on the theory of $R$-groups, a method with application to the theory of automorphic forms [1,2]. Roughly speaking, $R$-groups are finite groups whose duals parameterize irreducible constituents of representations parabolically induced from discrete series, i.e., the non-discrete tempered spectrum. To determine the $R$-group, one needs to determine the zeros of the Plancherel measure, a measure supported on the tempered spectrum whose restriction to the discrete part gives their formal degrees. On the other hand, a conjecture of Langlands relates the Plancherel measures to certain objects of arithmetic significance. This has played an important role in the recent progress. Our goal is to describe this crucial relationship between arithmetic and harmonic analysis.

There are several reasonable expositions of the Langlands program. For the questions that we are considering, one should consult [22]. We also call attention to [19] for its clarity, as well as its annotated bibliography.

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§1 L-functions and the Langlands program. We give a brief introduction to
the Langlands program, with an eye toward our goal of describing the classification
of irreducible tempered representations of reductive groups over local fields. For
much more comprehensive introductions, and more motivation, one should see [7, 18, 19, 22, 69]. For our purposes, it is enough to remark that the motivation for the
Langlands program is local and global class field theory. Namely, the remarkable
fact that the characters (i.e. one dimensional representations) of the multiplica-
tive group of a local (or global) field $F$ are naturally parameterized by characters
of the Galois group of $\bar{F}/F$. Viewing $F^\times$ as $GL_1(F)$, one hopes to describe the
representations of $G(F)$ through some kind of Galois representation theory.

(a) $L$–groups. Let $F$ be a local non–archimedean field, of characteristic zero and
let $O$ be the ring of integers of $F$. We denote by $\mathfrak{P}$, the prime ideal of $F$, and set
$q = |O/\mathfrak{P}|$. Choose a non-trivial unramified character $\psi$ of $F$. Let $G$ be a connected,
reductive, algebraic group, defined over $F$ [57]. We will assume that

1. $G$ is quasi–split, i.e. there exists a Borel subgroup $B = TU$ defined over $F$.
2. $G$ splits over an unramified extension $L/F$, i.e., $T(L) \simeq (L^\times)^{\dim T}$.

We recall from [57] that $G$ is given by a based root datum:

$$\Psi = (X^*(T), \Delta, X_*(T), \tilde{\Delta}),$$

where $X^*(T)$ is the group of rational characters of $T$, $\Delta$ a choice of simple roots,
$X_*(T)$ the cocharacters, and $\tilde{\Delta}$ the simple dual roots. Let $\tilde{\Psi}$ be the based root
datum given by $\tilde{\Psi} = (X_*(T), \tilde{\Delta}, X^*(T), \Delta)$. Let $^L G^0$ be the complex group with
root datum $\tilde{\Psi}$ (see [57, 4.11]). Let $\Gamma_{L/F}$ be the Galois group of $L/F$. Define

$$^L G = ^L G^0 \rtimes \Gamma_{L/F}.$$ 

Here, $\Gamma_{L/F}$ acts on $^L G^0$ by its action on the root datum $\tilde{\Psi}$. Note that whenever
$G$ is a split group over $F$, then $\Gamma_{L/F}$ acts trivially on the root datum $\Psi$, and hence,

$$^L G = ^L G^0 \rtimes \Gamma_{L/F}.$$ 

Remark. In general $\Gamma_{L/F}$ must be replaced by the Weil group (at least) but, for
the moment, we choose this form for simplicity. Later, we will use the Weil group
[80].
Examples:

1. \(G = GL_n, \quad \Psi = \bar{\Psi}, \) so \(L^0G = GL_n(\mathbb{C}),\) and thus, \(L^0G = GL_n(\mathbb{C}) \times \Gamma_{L/F} \). Note that if \(n = 1,\) then \(L^0G = \mathbb{C}^\times.\) Thus, local class field theory says that \(\hat{G}(F)\) is parameterized by the homomorphisms from \(\Gamma_{F/F} \) to \(L^0G.\)

2. \(G = Sp_{2n} = \{g \in GL_{2n}|^t gJg = J\}, \) where \(J,\) a symplectic form. For the purpose of this example, we choose \(J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.\)

\[T = \left\{ \begin{pmatrix} x_1 & & & \\ & x_2 & & \\ & & \ddots & 0 \\ & & & x_n \end{pmatrix} \begin{pmatrix} 0 & & & 0 \\ & 0 & & x_1^{-1} \\ & \ddots & \ddots & \ddots \\ & & & x_n^{-1} \end{pmatrix} x_i \in G_m \right\} \]

We denote a typical element of \(T\) by \(t(\{x_i\}).\) The simple roots are given by \(\{e_i - e_{i+1}\}_{i=1}^{n-1} \cup \{2e_n\}, \) where \(e_i(t(\{x_i\})) = x_i, \) \(2e_i(t) = x_i^2.\) This root system is of type \(C_n.\) For the coroots, we note that \(\check{e}_i(x) = (2e_i)^\vee(x) = t(\{1,1,\ldots,x,1,\ldots,1\}),\) where the \(x\) appears in the \(i\)-th position. Similarly,

\[(e_i - e_j)^\vee(x) = t(\{1,\ldots,x,\ldots,x^{-1},1,\ldots,1\}),\]

where \(x\) appears in the \(i\)-th position, and \(x^{-1}\) is in the \(j\)-th position. So, \(\bar{\Psi}\) is of type \(B_n.\) Therefore,

\[L^0G = SO(2n + 1, \mathbb{C}) = \{g \in GL_{2n+1}(\mathbb{C})|^t gJ'g = J'\}
\]

\[J' = \begin{pmatrix} 0 & 0 & I_n \\ 0 & 1 & 0 \\ I_n & 0 & 0 \end{pmatrix} \]

Since \(G\) is a split group, \(L^0G = G \\times \Gamma_{L/F} \)

3. \(G = SO_{2n+1}, \quad \bar{L}G = Sp_{2n}(\mathbb{C}) \times \Gamma_{L/F} \)

4. \(G = SO_{2n}, \quad \bar{L}G = Spin(2n) \times \Gamma_{L/F} \)

5. \(G = U_{n,n}. \) Let \(E/F\) be a quadratic extension, with \(\sigma : x \mapsto \overline{x}\) the Galois automorphism. Let \(E = F(\beta),\) with \(\overline{\beta} = -\beta.\) Set \(J = \begin{pmatrix} 0 & \beta I \\ -\beta I & 0 \end{pmatrix}.\) Then \(G = \{g \in GL_{2n}|^t gJg = J\}.\) Looking at the maximal torus of diagonal elements, we
see that $G$ is quasi-split, but not split. One can see that $G(E) = GL_{2n}(E)$ and thus, $L^G_0 = GL_{2n}$. Let $x \in L^G_0$. Then $\sigma(x) = \Phi_n^{-1}x^{-1}\Phi_n^{-1}$, where

$$\Phi_n = \begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix}$$

(see [17]). Then this gives the action of $\Gamma_{E/F}$ on $L^G_0$, and thus defines $L^G$.

(b) **The first conjecture.** The heart of the Langlands program is the philosophy that harmonic analysis, number theory, and geometry should be connected within the theory of automorphic forms. The formalization of this philosophy lies in the very deep conjectures of Langlands. We give a rough sketch of one of these conjectures below.

Let $G$ be as in Section 1, and set $G = G(F)$. We denote by $P$ the projection of $L^G$ onto $\Gamma_{L/F}$. Let $\varphi : \Gamma_{L/F} \rightarrow L^G$ be a homomorphism. We say that $\varphi$ is admissible if $P \circ \varphi$ is the identity map. We say that two admissible homomorphisms are equivalent, if they differ by an inner automorphism of $L^G$. Notice that if $G = GL_1 = G_m$, then an admissible homomorphism is just a character of $\Gamma_{L/F}$.

**Conjecture 1.1 (Langlands).** The equivalence classes of irreducible admissible representations of $G$ should be parameterized by the equivalence classes of admissible homomorphisms $\varphi : \Gamma_{L/F} \rightarrow L^G$. (In fact we should replace $\Gamma_{L/F}$ by the Weil–Deligne group $W'_F$ [7,80]. Of course, in this context, admissible is a somewhat more technical concept.)

**Remarks.** When $G = G_m = GL_1$, we see that, for a fixed $L$, the admissible homomorphisms $\varphi : \Gamma_{L/F} \rightarrow L^G$ parameterize only those characters which “factor” through $L$. Thus, we must consider all fields $L$ over which $G$ splits. This is why one must replace $\Gamma_{L/F}$ with a larger group. However, even with the Weil–Deligne group, Conjecture 1 is too much to ask for. For instance, for $G = SL_2$, Labesse and Langlands, [50], showed that sometimes inequivalent representations must be parameterized by the same admissible homomorphism. So, one can only
expect to partition the tempered representations of $G$ into finite subsets, called $L$-packets, such that these $L$-packets are parameterized by admissible homomorphisms, modulo conjugacy in $^LG$. The representations in a given $L$-packet are said to be $L$-indistinguishable.

**Example.** If $\pi$ is a tempered representation of $GL_n(F)$, then $\{\pi\}$ is an $L$-packet [7]. Suppose $\pi|_{SL_n(F)} = \pi_1 \oplus \ldots \oplus \pi_k$. Then [78] the $\pi_i$ are distinct and Gelbart and Knapp, [21], showed that $\{\pi_1, \ldots, \pi_k\}$ is an $L$-packet for $SL_n(F)$. Note $^L(SL_n(F))^0 = PGL_n(\mathbb{C})$. Suppose $\varphi : \Gamma_{L/F} \rightarrow GL_n(\mathbb{C}) \times \Gamma_{L/F}$ is a parameter for $\pi$. Composing with the projection $\eta : GL_n(\mathbb{C}) \rightarrow PGL_n(\mathbb{C})$, should give a parameter for an $L$-packet of $SL_n(F)$. What Gelbart and Knapp showed was that, assuming the Langlands correspondence is understood for $GL_n$, then $\eta \circ \varphi$ must be the parameter for $\{\pi_1, \ldots, \pi_k\}$.

**(c) Unramified representations.** Since $G$ splits over an unramified extension, $L$ of $F$, we can take the $O_F$-points of $G$. Let $K = G(O_F)$. Then, $K$ is a “good” maximal compact subgroup of $G$ [11]. A representation $(\pi, V)$ of $G$ is unramified, or class 1, if there is a $v \in V$ with $\pi(k)v = v$ for all $k \in K$.

**Lemma 1.2.** (See [11].) If $(\pi, V)$ is an admissible irreducible representation of $G = G(F)$, then $\dim_{\mathbb{C}}(V^K) \leq 1$.

Suppose that $\pi$ is class 1. Note that $(\pi(\mathcal{H}(G//K)), V^K)$ is a character, $\chi$, and $f \mapsto \chi(f)$ determines a semi-simple conjugacy class $\{A\}$ in $^LT^0$, via the Satake isomorphism [14], unique up to the action of the Weyl group $W(^LG^0, L^T^0)$.

**Example.** Suppose $G = GL_n$ and $\pi = \text{Ind}_B^G(\omega_1, \ldots, \omega_n)$, with $\omega_1, \ldots, \omega_n$ unramified characters, then

$$A = \begin{pmatrix} \omega_1(\varpi) & & \\ & \ddots & \\ & & \omega_n(\varpi) \end{pmatrix},$$

and $A$ determines $\pi$ (up to permutation of $(\omega_1, \ldots, \omega_n)$).

Let $\tau$ be the Frobenius class in $\Gamma_{L/F}$. Suppose that $\pi$ is unramified. Suppose $r$ is a finite dimensional representation of $^LG$, i.e., $r : ^LG \rightarrow GL_n(\mathbb{C})$ is a homomorphism,
with $r|_{L^G}$ a complex analytic representation of $L^G$. Let $\tilde{r}$ be its contragredient. For $s \in \mathbb{C}$, let

$$L(s, \pi, r) = \det(I - r(A \times \tau)q^{-s})^{-1}.$$  

In trying to understand the reason for assigning this value to $\pi$, $r$, and $s$, one should keep in mind the concept of an Artin $L$-function [19,32]. If we have a representation $r : \Gamma_{L/F} \to GL_n(\mathbb{C})$, then the attached local Artin $L$-function is $L(s, r) = \det(I - r(\tau)q^{-s})^{-1}$ and we hope this determines the splitting of the prime ideal in $L$.

(d) Automorphic Representations on $G$. Let $F$ be a number field, and let $A_F$ be the adeles of $F$. Let $\pi = \bigotimes_v \pi_v$ be a cuspidal automorphic representation of $G(A_F)$. We refer to [8,54] for the definition of the space $L^2_0(G(F) \backslash G(A_F), \omega)$ of cuspidal representations whose central character is $\omega$.

Note that $G_v = G \times F_v$ gives the group $G$ over $F_v$, and we have $L^G$ ($L$-group of $G$) and $L^G_v$, the $L$-group of $G_v$. For almost all $v$, $G \times F_v$ is unramified, and $\pi_v$ is $K_v$ unramified. Suppose $r$ is a representation of $L^G$, and let $r_v$ be defined by $r_v : L^G_v \to L^G \to GL_n(\mathbb{C})$. Then we have a local $L$-function, $L(s, \pi_v, r_v)$, whenever $\pi_v$ and $G_v$ are unramified.

(e) The Main Conjecture. We now give Langlands conjecture on the existence of global $L$-functions [51,52]. Let $\psi$ be a character of $A_F$ which is trivial on $F$, and suppose $\psi = \prod_v \psi_v$. Let $S$ be a finite set of places so that if $v \notin S$, $\pi_v$ and $G \times F_v$ are unramified. Let

$$L_S(s, \pi, r) = \prod_{v \notin S} L(s, \pi_v, r_v).$$

**Theorem 1.3.** ([7]). $L_S(s, \pi, r)$ converges for $\text{Re } s >> 0$.

**Conjecture 1.4 (Langlands).** For $v \in S$ it is possible to define a local $L$-function $L(s, \pi_v, r_v)$, so that $L(s, \pi_v, r_v) = (P_v(q_v^{-s}))^{-1}$, with $P_v(t)$ a polynomial whose constant term is 1, and a local root number $\varepsilon(s, \pi_v, r_v, \psi_v)$, (a monomial in $q_v^{-s}$) so that

$$L(s, \pi, r) = \prod_v L(s, \pi_v, r_v)$$
has a meromorphic continuation to $\mathbb{C}$, with finitely many poles, and

$$L(s, \pi, r) = \varepsilon(s, \pi, r)L(1 - s, \pi, \bar{r}),$$

with

$$\varepsilon(s, \pi, r) = \prod_v \varepsilon(s, \pi_v, r_v, \psi_v)$$

Moreover, if $v \in S$, then $\varepsilon(s, \pi_v, r_v, \psi_v) = 1$, and $L(s, \pi_v, r_v)$ is a $s$ in subsection (d).

Remarks.

(1) Note that the local root numbers $\varepsilon(s, \pi_v, r_v, \psi_v)$ depend on the choice of the character $\psi$, but that the global root number $\varepsilon(\pi, s, r)$ does not.

(2) Suppose that $\varphi_v : \Gamma_{\tilde{F}_v/F_v} \rightarrow LG_v$ is the admissible homomorphism corresponding to $\pi_v$. Then $L(s, \pi_v, r_v)$ should be $L(s, r_v \circ \varphi_v)$, where this last object is the Artin $L$–function.

(3) One must be careful not to read too much into this conjecture. In particular, it does not formally describe the nature of a global Langlands parameter. Such a parameterization requires an object (not yet known) much larger than the global Weil-Deligne group.

(4) Because of the relationship between automorphic forms and automorphic representations, $L$-functions, which are arithmetic in nature, must play a fundamental role in the harmonic analysis of reductive groups over both local and global fields.

While this exposition of the Langlands program is far from complete, we hope that it will give the reader enough background to see how the theory of the next section fits into this philosophy.

§2 Intertwining operators and reducibility of induced representations.

(a) Preliminaries. If $X$ is a totally disconnected space, and $Y$ is a complex vector space, then we let $C^\infty(X, Y)$ be the space of functions $f : X \rightarrow Y$ which are locally constant. We let $C^\infty_c(X, Y)$ be the subspace of $f \in C^\infty(X, Y)$ which are compactly supported. For a totally disconnected group $G$, we let $\mathcal{E}_c(G)$ be the collection of equivalence classes of irreducible admissible representations of $G$. We denote by
The (pre)-unitary classes in $\mathcal{E}_c(G)$. The supercuspidal classes are denoted by $\circ \mathcal{E}_c(G)$, and $\circ \mathcal{E}(G)$ denotes $\mathcal{E}(G) \cap \circ \mathcal{E}_c(G)$. We let $\mathcal{E}_2(G)$ denote the discrete series, and $\mathcal{E}_t(G)$ the tempered classes in $\mathcal{E}_c(G)$. (See [11] for details on these definitions.)

Let $F$ be a local field of characteristic zero. Suppose $G$ is a connected reductive, quasi–split algebraic group, defined over $F$. Let $G = G(F)$. Suppose $P$ is a parabolic subgroup of $G$, and let $P = MN$ be the Levi decomposition of $P$. We call $M$ the Levi component of $P$, and $N$ the unipotent radical of $P$. Let $P = P(F) = MN = M(F)N(F)$. Suppose that $(\sigma, V)$ is an admissible complex representation of $M$. We denote the contragredient representation by $(\tilde{\sigma}, \tilde{V})$. We let $\delta_P$ be the modular function of $P$. With this data, we set

$$V(\sigma) = \{ f \in C^\infty(G, V) | f(mng) = \sigma(m)\delta_P(m)^{1/2}f(g), \forall m \in M, n \in N, g \in G\}.$$

Then $G$ acts on $V(\sigma)$ by right translation, and this action is called the representation of $G$ unitarily induced from $\sigma$. We denote the induced representation by $\text{Ind}_P^G(\sigma)$. The factor $\delta_P^{1/2}$ is there to ensure that $\text{Ind}_P^G(\sigma)$ is unitary if $\sigma$ is. (Hence the term unitary induction.)

The classification theorems of Jacquet, [31], and Langlands, [9], indicate the importance of studying these induced representations. When $\sigma$ is an irreducible discrete series representation, then the components of $\text{Ind}_P^G(\sigma)$ are tempered. Furthermore, every irreducible tempered representation of $G$ is a component of $\text{Ind}_P^G(\sigma)$ for some $P$, and some discrete series $\sigma$ of $M$. In this section we are interested in two aspects of induced representations. The first is the classification of the tempered spectrum of $G$, i.e., determining the structure of $\text{Ind}_P^G(\sigma)$ for every choice of $P = MN$, and every irreducible discrete series representation $\sigma$. That is, we wish to determine for which $\sigma$ the representation $\text{Ind}_P^G(\sigma)$ is reducible. Furthermore, we want to know how many components $\text{Ind}_P^G(\sigma)$ has, what are the multiplicities with which these components appear, and how are the characters of these components related. The second point of interest is determining the arithmetic properties of $\text{Ind}_P^G(\sigma)$, and its components. In particular one wants to know how the $L$-functions for $\sigma$ and those for the components of $\text{Ind}_P^G(\sigma)$ are related. The fact that the answers to these two questions are related is quite deep, [67], and is in fact what has allowed significant progress to be made recently.
Suppose that \( \varphi : W_F \rightarrow L M \) is a conjectural parameter for the discrete series \( L \)-packet \( \{ \sigma \} \) which contains \( \sigma \). Then \( W_F \xrightarrow{\varphi} L M \xrightarrow{i} L G \), should define an \( L \)-packet for \( G \). If \( \Pi_G(\sigma) \) is the collection of components of \( \text{Ind}_{P}^{G}(\sigma) \), then \( \Pi_\varphi(G) = \bigcup_{\tau \in \{\sigma\}} \Pi_G(\tau) \), should be this \( L \)-packet. Given the conjectural properties of the Langlands \( L \)-functions given in Section 1, it makes heuristic sense to believe that the \( L \)-functions for elements of \( \Pi_\varphi(G) \) are “induced” from those of \( \{\sigma\} \). That is, for \( \pi \in \Pi_\varphi(G) \), and a complex representation \( \rho \) of \( L G \), we expect that \( L(\nu, \pi, \rho) = L(\nu, \rho \circ i \circ \varphi) \), where the last object is the Artin \( L \)-function.

Let \( A \) be the split component of \( M \), i.e., maximal torus in the center of \( M \). Then \( M = Z_G(A) \). Let \( W = N_G(A)/M \). Then \( W \) is called the Weyl group of \( G \) with respect to \( A \), and we may denote this group by \( W(G, A) \), if there is any ambiguity about \( G \) and \( A \).

Let \( \tilde{w} \in W \). Choose \( w \in N_G(A) \) representing \( \tilde{w} \). Let \( w\sigma(m) = \sigma(w^{-1}mw) \). The class of \( w\sigma \) is independent of the choice of representative \( w \) for \( \tilde{w} \). We write \( W(\sigma) \) for the subgroup of \( \tilde{w} \in W \) which fix the class of \( \sigma \). Let \( C(\sigma) \) be the commuting algebra of \( \text{Ind}_{P}^{G}(\sigma) \).

**Theorem 2.1 (Bruhat [10]).** For \( \sigma \in \mathcal{E}_2(M) \), we have \( \dim_{\mathbb{C}}(C(\sigma)) \leq |W(\sigma)|. \)

One wishes to use the group \( W(\sigma) \) to decompose \( \text{Ind}_{P}^{G}(\sigma) \). This led to the development of the theory of the standard intertwining operators. Namely, one can attach to each element \( w \) of \( W(\sigma) \) a self intertwining operator \( \mathcal{A}(w, \sigma) \) for \( \text{Ind}_{P}^{G}(\sigma) \), and a complete understanding of these operators determines the algebra \( C(\sigma) \). We will outline the theory of these operators.

Set \( a = \text{Hom}(X(M)_F, \mathbb{R}) \), where \( X(M)_F \) is the set of \( F \)-rational characters of \( M \). Let \( a^* = X(M)_F \otimes_{\mathbb{Z}} \mathbb{R} \). Let \( a^*_C = a^* \otimes_{\mathbb{R}} \mathbb{C} \). Then \( a^*_C \) is a complex manifold.

There is a homomorphism, \( H_P : X(M)_F \rightarrow a \), so that, \( |\chi(m)|_F = q^{\langle \chi, H_P(m) \rangle} \), for all \( \chi \in X(M)_F, m \in M \) [31]. Let \( \nu \in a^*_C \), and \( I(\nu, \sigma) = \text{Ind}_{P}^{G}(\sigma \otimes q^{\langle \nu, H_P(\cdot) \rangle}) \). We denote by \( V(\nu, \sigma) \) be the space of functions on which we realize \( I(\nu, \sigma) \).

Let \( K \) be a good maximal compact (say \( G(O_F) \)). (See [12, 56] for a definition of a good maximal compact.) Then, for \( k \in M \cap K \), and all \( \nu \), we have \( \langle \nu, H_P(k) \rangle = 1. \)
Thus, we suppose that \( f \in C^\infty(K,V) \) is such that

\[
f(m_K n_K, k') = \sigma(m_K)\delta_P^{1/2}(m_K)f(k'), \quad \text{for all } m_K n_K \in P \cap K, \; k' \in K.
\]

By the Iwasawa decomposition, \([11,56]\), \( G = PK \). So, for each \( \nu \), we set \( f_\nu(mnk) = \sigma(m)q^{(\nu,H_P(m))}\delta_P^{1/2}(m)f(k) \in V(\nu,\sigma) \). Note \( f = f_\nu|_K \). Thus, we have a natural isomorphism between \( V(\nu,\sigma) \) and \( V(\nu',\sigma) \) given by \( f_{\nu'} \leftrightarrow f_\nu \).

Fix \( \nu \in \mathfrak{a}_K^* \), and \( \sigma \in \mathcal{E}_2(M) \). Let \( \bar{w} \in W \), and choose \( w \) representing \( \bar{w} \). Fix \( f \in I(\nu,\sigma) \). Define

\[
A(\nu,\sigma,w)f(g) = \int_{N_w} f(w^{-1}ng) \, dn,
\]

where \( N_w = U \cap w^{-1}N^-w \), and \( N^- \) is opposite to \( N \). Let \( m_1 \in M, \; n_1 \in N \). Then,

\[
A(\nu,\sigma,w)f(m_1n_1g) = \int_{N_w} f(w^{-1}nm_1n_1g) \, dn
\]

\[
= \int_{N_w} f(w^{-1}m_1ww^{-1}m_1^{-1}nm_1n_1g) \, dn
\]

\[
= \sigma(w^{-1}m_1w)\delta_P^{1/2}(w^{-1}m_1w)q^{(\nu,H_P(w^{-1}m_1w))} \int_{N_w} f(w^{-1}m_1^{-1}nm_1n_1g) \, dn.
\]

Now we note that, on \( N_w \), the measures \( d(m_1^{-1}nm_1) \), and \( dn \) are related by

\[
d(m_1^{-1}nm_1) = (\delta_P(m_1)/w\delta_P(m_1))^{1/2}
\]

\[
= (\delta_P(m_1)/\delta_P(w^{-1}m_1w))^{1/2},
\]

(cf [62]). Therefore, making the substitution \( n' = m_1^{-1}nm_1n \), we have

\[
A(\nu,\sigma,w)f(m_1n_1g) = w\sigma(m_1)\delta_P^{1/2}(m_1)q^{(\nu,w,H_P(m_1))} \int_{N_w} f(w^{-1}n'g) \, dn'
\]

\[
= w\sigma(m_1)\delta_P^{1/2}(m_1)q^{(\nu,w,H_P(m_1))} A(\nu,\sigma,w)f(g),
\]

which implies \( A(\nu,\sigma,w)f \in V(\nu w, w\sigma) \).

Note that we have only defined \( A(\nu,\sigma,w) \) formally, in that we have said nothing about the convergence of \( A(\nu,\sigma,w)f(g) \). However, the above argument shows that, if \( A(\nu,\sigma,f)(g) \) converges for all \( f \in V(\nu,\sigma) \), and \( g \in G \), then \( A(\nu,\sigma,w) \) defines an intertwining operator between \( I(\nu,\sigma) \) and \( I(\nu w, w\sigma) \).
Theorem 2.2 (Harish–Chandra). Let $\sigma \in \mathcal{E}_2(M)$. Then, for $\Re \nu > 0$, the operator $A(\nu, \sigma, w)$ converges absolutely, with a meromorphic continuation to all of $a_\mathbb{C}^*$. So, fixing $f \mapsto f_{\nu}$, as above, then $\nu \mapsto \langle \tilde{v}, A(\nu, \sigma, w) f_{\nu}(g) \rangle$ is meromorphic for all $f, \tilde{v},$ and $g$. □

Suppose $w\sigma \simeq \sigma$. Then $A(0, \sigma, w)$ gives an intertwining operator between $\text{Ind}^G_P(\sigma)$ and $\text{Ind}^G_P(w\sigma)$, which are isomorphic representations. Of course, $A(\nu, \sigma, w)$ may have a pole at $\nu = 0$, and this question of analyticity at $\nu = 0$ is in fact crucial to determining the structure of $\text{Ind}^G_P(\sigma)$.

Theorem 2.3 (Harish–Chandra). There is a meromorphic, complex valued function $\nu \mapsto \mu(\nu, \sigma, w)$ so that

$$A(\nu, \sigma, w)A(w\nu, w\sigma, w^{-1}) = \gamma_w(G/P)^2 \mu(\nu, \sigma, w)^{-1},$$

where

$$\gamma_w(G/P) = \int_{N_w} q^{\langle \nu, H_P(n) \rangle} dn.$$ 

Furthermore, $\mu(\nu, \sigma, w)$ is holomorphic and non-negative on $i\mathfrak{a}^*$. □

Let $\Phi(P, A)$ be the reduced roots of $A$ in $P$. The length of $\tilde{w} \in W$, is given by $\ell(\tilde{w}) = |\{\alpha \in \Phi(P, A) | \tilde{w} \alpha < 0\}|$. There is a longest element $\tilde{w}_0 \in W$ [15]. We write $\mu(\nu, \sigma)$ for $\mu(\nu, \sigma, \tilde{w}_0)$, and $\mu(\sigma)$ for $\mu(0, \sigma)$. We call $\mu(\nu, \sigma)$ the Plancherel measure of $(\nu, \sigma)$.

Theorem 2.4 (Harish–Chandra [72]). Suppose $P$ is maximal and proper, and there is some $\tilde{w} \neq 1$ in $W$, with $\tilde{w}\sigma \simeq \sigma$. Then $\text{Ind}^G_P(\sigma)$ is reducible if and only if $\mu(\sigma) \neq 0$. □

(b) Arithmetic Considerations.

We now examine the arithmetic properties of the function $\mu$. It turns out that the Plancherel measure is directly related to the theory of $L$-functions. Suppose, for the moment that $P$ is a maximal proper parabolic subgroup of $G$. Then $N_{\tilde{w}} = N$, and $\mathfrak{a}_\mathbb{C}^*/\mathfrak{j} \simeq \mathbb{C}$. (Here $\mathfrak{j}$ is the Lie algebra of the split component of $G$.) Let $\rho$ be a representation of $^L\mathbb{M}$, and set

$$\gamma(s, \sigma, \rho, \psi_F) = \varepsilon(s, \sigma, \rho, \psi_F)L(1 - s, \tilde{\sigma}, \rho)/L(s, \sigma, \rho),$$
where $L(s, \sigma, \rho)$ is the conjectural Langlands $L$-function attached to $\sigma$ and $\rho$. By [7] there is a parabolic subgroup $^LG$ of $^LG$, with $^LP = ^LM^LN$, for some unipotent subgroup $^LN$. Let $^Ln$ be the Lie algebra of $^LN$. Denote by $r$ the adjoint representation of $^LM$ on $^Ln$.

**Conjecture 2.5 (Langlands [53]).**

If $\sigma \in E_2(M)$, then $\mu(\sigma)\gamma_{\tilde{w}_0}(G/P)^2 = \gamma(0, \sigma, r, \overline{\psi}_F)\gamma(0, \tilde{\sigma}, r, \psi_F)$. □

To proceed, we need the notion of a generic representation. Suppose that $G$ is a connected reductive quasi-split algebraic group, defined over $F$. Let $\Phi$ be the roots of $T$ in $G$, and $\Delta$ the set of simple roots given by our choice of a Borel subgroup. For $\alpha \in \Delta$, let $U_\alpha$ be the subgroup of $U$ whose Lie algebra is $g_\alpha \oplus g_{2\alpha}$ [6,35,57]. The subgroup

$$U' = \prod_{\alpha \in \Phi^+ \setminus \Delta} U_\alpha$$

is normal in $U$. Furthermore,

$$U/U' \simeq \prod_{\alpha \in \Delta} U_\alpha/U_{2\alpha}.$$

Let $\alpha \in \Delta$ and let $\psi_\alpha$ be a character of $U_\alpha/U_{2\alpha}$. The character $\psi$ of $U$, trivial on $U'$, and given by

$$\psi = \prod_{\alpha \in \Delta} \psi_\alpha,$$

is called **non-degenerate** if $\psi_\alpha$ is non-trivial for each $\alpha$. An admissible representation $(\pi, V)$ of $G$ is called **non-degenerate**, or **generic** if there is a non-degenerate character $\psi$ of $U$, and a linear functional $\lambda$ on $V$, such that

$$\lambda(\pi(u)v) = \psi(u)\lambda(v), \text{ for all } u \in U, v \in V.$$

Such a functional is called a **Whittaker functional**. Let $V_\psi^*$ be the complex vector space of Whittaker functionals on $V$ (with respect to a fixed $\psi$).

**Example.** Let $G = GL_n$, so $G(F) = GL_n(F)$. Suppose that $B = TU$ is the Borel subgroup of upper triangular matrices. For $t \in F$, we denote by $E_{ij}(t)$ the $n \times n$ matrix whose $ij$-th entry is $t$, and all other entries are zero. Let $\alpha = e_i - e_{i+1}$.
Then $U_\alpha = \{ I + E_{i(i+1)} | t \in F \}$. Recall that $\psi_F$ is a fixed non-trivial character of $F^+$. Let $\psi_\alpha(I + E_{i(i+1)}(t)) = \psi_F(t)$. Setting $\psi = \prod_{\alpha \in \Delta} \psi_\alpha$, we have

$$
\psi \left( \begin{pmatrix} 1 & x_{12} & & \ & 1 & x_{23} & \ & & \cdots & \ & & 1 & x_{(n-1)n} \end{pmatrix} \right) = \psi_F(x_{12} + x_{13} + \cdots + x_{(n-1)n}).
$$

Then $\psi$ is non-degenerate, and in fact any non-degenerate character of $U$ is conjugate to $\psi$.

It is a result of Gelfand and Kazhdan that every irreducible discrete series representation of $GL_n(F)$ is generic [23]. Jacquet, [37], showed that every irreducible tempered representation of $GL_n(F)$ is generic. For other classical groups, there are examples of discrete series representations which fail to be generic [34].

**Theorem 2.6 (Shalika [70]).** Let $(\pi, V)$ be an irreducible admissible representation of $G$, then $\dim_{\mathbb{C}} V_\psi^* \leq 1$. □

Rodier showed that the dimension of $V_\psi^*$ is preserved under parabolic induction, which we state below.

**Theorem 2.7 (Rodier [59]).** Let $G$ be quasi-split, and suppose that $P = MN$ is a parabolic subgroup of $G$. If $(\sigma, V)$ is an irreducible admissible representation of $M$, and $(\pi, W) \simeq \text{Ind}_M^G(\sigma)$, then $\dim_{\mathbb{C}} W_\psi^* = \dim_{\mathbb{C}} V_\psi^*$. □

Suppose $(\sigma, V)$ is an irreducible generic discrete series representation of $M$. Let $\lambda \in V_\psi^*$ be non-zero. For each $v \in V$, let $W_v(m) = \lambda(\sigma(m)v)$. Then $W_v$ is called the **Whittaker function** attached to $\lambda$ and $v$. Note that, for $u \in U \cap M$, we have $W_v(ug) = \lambda(\sigma(ug)v) = \psi(u)W_v(g)$. Thus, $W_\sigma = \{ W_v | v \in V \}$ is a subspace of $\text{Ind}_U^M(\psi)$. For $m_1 \in M$, we see that

$$
m_1 \cdot W_v(m) = W_v(mm_1) = \lambda(\sigma(mm_1)v) = W_{\sigma(m_1)}v(m).
$$

So the restriction of $\text{Ind}_U^M(\psi)$ to $W_\sigma$ is isomorphic to $\sigma$, and we call this the **Whittaker model** for $\sigma$. 

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*L-FUNCTIONS AND INTERTWINING OPERATORS*
Let $\nu \in a^*_C$. Consider $V(\nu, \sigma)$ as a space of functions from $G$ to $W_\sigma$. For each $f \in V(\nu, \sigma)$, and each $g \in G$, we denote by $(f(g), m)$ the value of $f(g)$ at $m$. There is a natural way to define a Whittaker functional $\lambda(\nu, \sigma, \psi)$ on $V(\nu, \sigma)$. Namely, let

$$
\lambda(\nu, \sigma, \psi)(f) = \int_{N'} f(w_0^{-1}n', e)\psi(n') \, dn',
$$

where $N' = w_0Nw_0^{-1}$. Casselman and Shalika, [16], showed that $\lambda(\nu, \sigma, \psi)$, which converges absolutely in a right half plane, has an analytic continuation to all of $a^*_C$. Moreover, $\lambda(\nu, \sigma, \psi)$ defines an element of $V(\nu, \sigma)^*_\psi$. Note that if $f \in V(\nu, \sigma)$, then, for each $u \in N,

$$
\lambda(w\nu, w\sigma, \psi)A(\nu, \sigma, w)(I(\nu, \sigma)(u)f) = \lambda(w\nu, w\sigma, \psi)I(\nu, \sigma)(u)A(\nu, \sigma, w)f = \psi(u)\lambda(w\nu, w\sigma, \psi)A(\nu, \sigma, w)f,
$$

so $\lambda(w\nu, w\sigma, \psi)A(\nu, \sigma, w) \in V(\nu, \sigma)^*_\psi$. Rodier’s Theorem gives rise to the following result.

**Proposition 2.8 ([64]).** There is a complex number $C_\psi(\nu, \sigma, w)$, such that

$$
\lambda(\nu, \sigma, \psi) = C_\psi(\nu, \sigma, w)\lambda(w\nu, w\sigma, \psi)A(\nu, \sigma, w).
$$

Furthermore, $\nu \mapsto C_\psi(\nu, \sigma, w)$ is meromorphic on $a^*_C$. □

We call $C_\psi(\nu, \sigma, w)$ the **local coefficient** attached to $\psi, \nu, \sigma$, and $w$.

**Proposition 2.9 ([64]).**

1. For all $\nu \in a^*_C$, we have the identity

$$
C_\psi(w\nu, w\sigma, w^{-1}) = \overline{C_\psi(-\nu, \sigma, w)}.
$$

2. If $\nu = -\bar{\nu}$, and $\sigma$ is unitary, then

$$
|C_\psi(\nu, \sigma, w)|^2 = \gamma^{-2}(G/P)\mu(\nu, \sigma). \quad \Box
$$

By studying the relationship of local coefficients to Plancherel measures, Shahidi was able to prove Langlands’s conjecture on Plancherel measures for generic representations. He has also used this method to derive estimates toward the Ramanujan conjecture [66].
Theorem 2.10 (Shahidi [67, Theorem 3.5]). Let $\sigma$ be irreducible admissible and generic. Then, for each $r$ and $\psi_F$, there exists a unique arithmetic function, $\gamma(s, \sigma, r, \psi_F)$, (cf Theorem 3.5 of [67] for its uniquely defining properties) such that

$$\mu(\sigma)\gamma_{w_0}(G/P)^2 = \gamma(0, \sigma, r, \overline{\psi_F})\gamma(0, \overline{\sigma}, r, \psi_F).$$

Moreover, if we accept two conjectures in harmonic analysis, we can remove the assumption that $\sigma$ is generic. □

(c) $R$-groups. We now move toward our goal of describing the intertwining algebra $C(\sigma)$ of $\text{Ind}^G_P(\sigma)$. The idea is to use Plancherel measures, and the operators $A(\nu, \sigma, w)$, to construct normalized intertwining operators which have no poles on the unitary axis $ia^*$. From these we construct a collection of self-intertwining operators for $\text{Ind}^G_P(\sigma)$, and determine how to construct a basis of $C(\sigma)$ from among these. We describe this below.

Theorem 2.11. There is a meromorphic normalizing factor, $r(\nu, \sigma, w)$, so that

$$A(\nu, \sigma, w) = r(\nu, \sigma, w)A(\nu, \sigma, w)$$

is holomorphic on $ia^*$. □

Theorem 2.12 (Shahidi) [67]. Let $\sigma$ be irreducible admissible and generic. Then $r(\nu, \sigma, w)$ can be chosen so that

$$r(0, \sigma, w) = \varepsilon(0, \sigma, \overline{\tau}, \psi_F)L(1, \sigma, \overline{\tau})/L(0, \sigma, \overline{\tau}).$$

Now we see that reducibility of induced representations has an arithmetic interpretation. Conversely, if we can determine when $\text{Ind}^G_P(\sigma)$ is reducible, then we can determine the poles of $L(s, \sigma, r)$.

Let $A(\sigma, w) = A(0, \sigma, w)$, and $A(\sigma, w) = A(0, \sigma, w)$.

Proposition 2.13. Let $w_1, w_2 \in W$

(1) If $\ell(w_1w_2) = \ell(w_1) + \ell(w_2)$, then

$$A(\sigma, w_1w_2) = A(w_2\sigma, w_1)A(\sigma, w_2)$$
We call (2) the cocycle relation for the normalized intertwining operators $A(\sigma, w)$. □

We wish to determine the structure of $\text{Ind}^G_P(\sigma)$ for arbitrary $P$. So far we have Harish-Chandra’s theorem for maximal parabolic subgroups. We want to know how to utilize this theorem in the more general case. Harish-Chandra proved a product formula for Plancherel measures, which reduces the computation of Plancherel measure to the case of maximal parabolic subgroups.

Let $\Phi(P, A)$ be the reduced roots of $A$ in $P$. For $\beta \in \Phi(P, A)$, let $A_\beta = (A \cap \ker \chi_\beta)^0$ and $M_\beta = Z_G(A_\beta)$. (Here $\chi_\beta$ is the root character attached to $\beta$.) Let $N_\beta = M_\beta \cap N$. Then $^*P_\beta = MN_\beta$ is a maximal parabolic subgroup of $M_\beta$. So, there is a Plancherel measure $\mu_\beta(\nu, \sigma)$ attached to $\beta, \nu$, and $\sigma$. Note that $\mu_\beta(\sigma) = 0$ if and only if $W(M_\beta, A) \cap W(\sigma) \neq \{1\}$ and $\text{Ind}_{^*P_\beta}^{M_\beta}(\sigma)$ is irreducible.

**Theorem 2.14 (Harish–Chandra [31]) Product formula for Plancherel Measures.**

$$\gamma^{-2}(G/P)\mu(\nu, \sigma) = \prod_{\beta \in \Phi(P, A)} \gamma^{-2}(M_\beta/^{*}P_\beta)\mu_\beta(\nu, \sigma). \quad \square$$

Notice that this gives us an inductive formula for the $\gamma$-factors, $\gamma(\nu, \sigma, w)$, which is one of their fundamental (conjectural) properties. That such an inductive formula for $\gamma$-factors exists is part of Shahidi’s result [67].

For $\beta \in \Phi(P, A)$, let $\tilde{w}_\beta$ be the reflection in the reduced root $\beta$. If $\tilde{w}_\beta \in W(\sigma)$, then $N_{\tilde{w}_\beta} = N_\beta$, and thus

$$A(\nu, \sigma, w_\beta)f(g) = \int_{N_\beta} f(w_\beta^{-1}ng) \, dn,$$

which, for $g \in M_\beta$ is the intertwining operator that determines the reducibility of $\text{Ind}_{^*P_\beta}^{M_\beta}(\sigma)$. Furthermore, the product formula for the intertwining operators shows that every $A(\nu, \sigma, w)$ can be written as the composition of operators of the form (2.1) [64].

**Example.** Suppose that $G = Sp_{2n}$, and $B = TU$ is the Borel subgroup of upper triangular matrices in $G$. For the purposes of this discussion, we assume that $G$ is
defined with respect to the form
\[
\begin{pmatrix}
-1 & 1 \\
& \\
& \\
& \\
& \\
1 & \\
-1 & \\
\end{pmatrix}
\]

Then
\[
T = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ x_n^{-1} \\ \cdots \\ x_2^{-1} \\ x_1^{-1} \end{pmatrix} \bigg| x_i \in \mathbb{G}_m \right\}.
\]

We might denote a typical element of \( T \) by \( t(\{x_i\}) \). The root system \( \Phi(\mathbb{G}, T) \) is of type \( C_n \), with simple roots \( \{e_i - e_{i+1}\} \) for \( i = 1, \ldots, n - 1 \), and \( 2e_n \). If \( \chi \) is a character of \( T = T(F) \), then \( \chi \) is of the form \( \chi = \chi_1 \otimes \cdots \otimes \chi_n \), for a collection of characters \( \{\chi_i\} \subset \overline{F}^\times \). That is, \( \chi(t(\{x_i\})) = \prod_i \chi_i(x_i) \).

Note that if \( \beta = e_1 - e_2 \), then \( A_\beta = \{t(\{x_i\})| x_1 = x_2 \} \). In this case
\[
w_\beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ & & I_{2n-4} \\ & & & 0 & 1 \\ & & & & 1 & 0 \end{pmatrix}.
\]

So
\[
M_\beta = \left\{ \begin{pmatrix} g_1 \\ x_3 \\ \vdots \\ x_3^{-1} \\ \tau g_1^{-1} \end{pmatrix} \bigg| g_1 \in GL_2, x_i \in \mathbb{G}_m \right\}.
\]

Here \( \tau g_1 \) is the transpose of \( g_1 \) with respect to the off diagonal. Note that
\[
N_\beta = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \\ & & \ddots \\ & & & 1 \\ & & & & 1 & -x \\ & & & & 0 & 1 \end{pmatrix} \bigg| x \in F \right\}.
\]
Set $G_1 = GL_2(F)$, and let $B_1$ be its its Borel subgroup. Then, the analytic behavior of intertwining operator $A(\nu, \sigma, w_\beta)$ at $\nu = 0$ detects the reducibility of $\text{Ind}_{B_1}^{G_1}(\chi_1 \otimes \chi_2)$. The other roots of the form $e_i - e_j$ are treated in a similar fashion.

Suppose that $\alpha = 2e_n$. Then

$$M_\alpha = \begin{cases} \begin{pmatrix} x_1 & x_2 & \cdots & h \\ x_2 & \cdots & x_1^{-1} \\ \vdots & \ddots & \vdots \\ h & \cdots & x_2^{-1} \end{pmatrix} & | \hspace{1em} h \in Sp_2 = SL_2, x_i \in G_m \end{cases}.$$ 

Let $H = SL_2(F)$, and $B'$ be its Borel subgroup. Then, $A(\nu, \sigma, w_\alpha)$ can be considered as the intertwining operator which determines the reducibility of $\text{Ind}_{B'}^{H}(\chi_n)$.

As a consequence of the above computations, we see that the Plancherel measure $\mu(\nu, \chi)$ can be computed if we understand $\mu_{G_1}(\nu_1, \chi_i \otimes \chi_j)$, and $\mu_H(\nu_2, \chi_i)$, where $\mu_{G_1}$ and $\mu_H$ have the obvious meanings.

Suppose $w\sigma \simeq \sigma$. Let $T_w : V \rightarrow V$ be an isomorphism between $w\sigma$ and $\sigma$, i.e., $T_w w\sigma = \sigma T_w$. Since $A(\sigma, w) : \text{Ind}_{P}^{G}(\sigma) \rightarrow \text{Ind}_{P}^{G}(w\sigma)$, we see that

$$A'(\sigma, w) = T_w A(\sigma, w) : \text{Ind}_{P}^{G}(\sigma) \rightarrow \text{Ind}_{P}^{G}(\sigma).$$

We have $A'(\sigma, w_1 w_2) = \eta(w_1, w_2) A'(\sigma, w_1) A'(\sigma, w_2)$, where $\eta(w_1, w_2)$ is given by $T_{w_1 w_2} = \eta(w_1, w_2) T_{w_1} T_{w_2}$.

**Theorem 2.15 (Harish–Chandra) Commuting Algebra Theorem.**

*The collection $\{A'(\sigma, w) | w \in W(\sigma)\}$ spans $C(\sigma)$. □* 

Harish-Chandra’s Commuting Algebra Theorem justifies our concentration on the theory of the operators $A(\nu, \sigma, w)$. We need to determine which of the operators $A'(\sigma, w)$ are scalar, and which are not. In a series of papers, beginning in the early 1960’s, Kunze and Stein computed the poles of intertwining operators for groups over $\mathbb{C}$ [46,47,48,49]. This work was extended by Knapp and Stein [40,41]. Finally, Knapp and Zuckerman used these results to classify the irreducible tempered representations of Semisimple Lie Groups [42]. Knapp and Stein described
an algorithm for determining a basis of \( C(\sigma) \). Silberger [71,73] showed that this construction is valid for \( p \)-adic groups. Let \( \beta \in \Phi(\mathbf{P}, \mathbf{A}) \), and let \( \mathbf{M}_\beta, \mathbf{P}_\beta, \) and \( \mathbf{N}_\beta \) be as before. Let \( \Delta' = \{ \beta \in \Phi(\mathbf{P}, \mathbf{A}) | \mu_\beta(\sigma) = 0 \} \). The following lemma is quite important, and its proof is non-trivial.

**Lemma 2.16 (Knapp–Stein).** \( \Delta' \) is a sub–root system of \( \Phi(\mathbf{P}, \mathbf{A}) \). \( \square \)

Let \( W' = \langle w_\beta | \beta \in \Delta' \rangle \). The lemma guarantees that this is well defined. Let \( R(\sigma) = \{ w \in W(\sigma) | w\Delta' = \Delta' \} = \{ w | w\beta > 0, \text{ for all } \beta \in \Delta' \} \). We sometimes denote \( R(\sigma) \) by \( R \). If \( \beta \in \Delta' \), then \( w_\beta \sigma \simeq \sigma \), and \( \text{Ind}_{\mathbf{P}_\beta}^{\mathbf{M}_\beta}(\sigma) \) is irreducible. Therefore, by Schur’s Lemma, the normalized operator \( A'(\sigma, w_\beta) \) is a scalar. By the cocycle relation, we see that \( A'(\sigma, w) \) is scalar for every \( w \in W' \). On the other hand, if \( r \in R \), then \( N_r \cap N_{w_\beta} = \{ I \} \), for each \( \beta \in W' \). Suppose that \( r = w_\alpha \) for some \( \alpha \in \Phi(\mathbf{P}, \mathbf{A}) \). Then, since \( w_\alpha \in W(\sigma) \), and \( \alpha \notin \Delta' \), we see that \( \text{Ind}_{\mathbf{P}_\alpha}^{\mathbf{M}_\alpha}(\sigma) \) is reducible. Thus, by the Commuting Algebra Theorem, \( A'(\sigma, w_\alpha) \) is non-scalar. Now, if \( r = r'w_\alpha \), then, by the cocycle relation, \( A'(\sigma, r) \) is non-scalar.

**Theorem 2.17 (Knapp–Stein, Silberger).** For every \( \sigma \in \mathcal{E}_2(M) \), \( W(\sigma) = R \ltimes W' \), and \( W' = \{ w | A'(\sigma, w) \text{ is scalar} \} \). \( \square \)

So \( \{ A'(\sigma, r) | r \in R \} \), gives a basis for \( C(\sigma) \). Note that \( \eta : R \times R \to \mathbb{C}^\times \) is a 2–cocycle of \( R \). Moreover, we have

\[
A'(\sigma, w_1w_2) = \eta(w_1, w_2)A'(\sigma, w_1)A'(\sigma, w_2).
\]

Thus, \( C(\sigma) \simeq \mathbb{C}[R]_\eta \), where \( \mathbb{C}[R]_\eta \) is the complex group algebra of \( R \), with multiplication twisted by \( \eta \). If \( \sigma \) is generic, then \( \eta \) splits [39]. For simplicity, we assume \( \eta \) splits. Let \( \rho \) be an irreducible representation of \( R \), and suppose \( \chi_\rho \) is it’s character. Set

\[
A_\rho = \frac{1}{|R|} \dim \rho \sum_{r} \chi_\rho(r)A'(\sigma, r).
\]

If \( \rho \) and \( \rho' \) are irreducible representations of \( R \), we have

\[
A_\rho A_{\rho'} = \frac{1}{|R|^2} \dim \rho \dim \rho' \sum_{r, r'} \chi_\rho(r)\overline{\chi_{\rho'}(r')}A'(\sigma, rr')
\]
\[
= \frac{1}{|R|^2} \dim \rho \dim \rho' \sum_{w \in R} \left( \sum_{rr' = w} \overline{\chi_{\rho}(r)} \overline{\chi_{\rho'}(r')} \right) A'(\sigma, w)
\]
\[
= \frac{1}{|R|^2} \dim \rho \dim \rho' \sum_{w \in R} \overline{\chi_{\rho}(w)} \left( \sum_{r \in R} \overline{\chi_{\rho}(r)} \overline{\chi_{\rho'}(r)^{-1}} \right) A'(\sigma, w).
\]

By the Schur orthogonality relations, we see that this is 0 if \( \rho \not\cong \rho' \), and is \( A_{\rho} \) if \( \rho \cong \rho' \). So the \( A_{\rho} \) are orthogonal projections. Note that

\[
A'(\sigma, w)A_{\rho} = \frac{1}{|R|} \dim \rho \sum_{r} \overline{\chi_{\rho}(r)} A'(\sigma, wr)
\]
\[
= \frac{1}{|R|} \dim \rho \sum_{r} \overline{\chi_{\rho}(w^{-1}rw)} A'(\sigma, ww^{-1}rw)
\]
\[
= \frac{1}{|R|} \dim \rho \sum_{r} \overline{\chi_{\rho}(r)} A'(\sigma, rw)
\]
\[
= A_{\rho} A'(\sigma, w).
\]

So each \( A_{\rho} \) is in the center of \( C(\sigma) \).

Suppose \( \text{Ind}_{P}^{G}(\sigma) = m_{1} \pi_{1} \oplus \ldots \oplus m_{n} \pi_{n} \). Then \( \dim C(\sigma) = m_{1}^{2} + \ldots + m_{n}^{2} \). But \( \dim C[R] = |R| = \sum_{\rho \in \hat{R}} (\dim \rho)^2 \). Moreover, \( \dim Z(C[R]) = |\hat{R}| \). Thus, if \( V_{\rho} = \text{Im}(A_{\rho}) \), then \( V_{\rho} \) must be an isotypic subspace.

**Theorem 2.18 (Keys [39]).** Assume that \( \eta \) splits.

(1) The inequivalent components of \( \text{Ind}_{P}^{G}(\sigma) \) are parameterized by the irreducible representations \( \rho \) of \( R \).

(2) \( \dim \text{Hom}_{G}(\pi_{\rho}, \text{Ind}_{P}^{G}(\sigma)) = \dim \rho. \) \( \square \)

**d) Computations.** In order to compute the poles of \( A(\nu, \sigma, w) \), it is helpful to know the following result, which follows from the fact that \( PN^{-} \) is dense in \( G \), and that there is a non–degenerate pairing between \( V(\nu, \sigma) \) and \( V(-\nu, \sigma) \).

**Lemma 2.19 (Rallis [68]).** Suppose \( P = MN \) is a maximal proper parabolic subgroup of \( G \). Let \( N^{-} \) be the unipotent radical opposed to \( N \). Then every pole of \( \nu \mapsto A(\nu, \sigma, w) \) is a pole of \( \nu \mapsto A(\nu, \sigma, w) f(e) \), for some \( f \in V(\nu, \sigma) \) with \( \text{supp} f \) contained in \( PN^{-} \) modulo \( P \). \( \square \)

**Example 1.** Our first example is the computation of the pole of the intertwining operator for \( G = SL_{2}(F) \). For a complete treatment of this case, one should consult
Let $G = SL_2$, and take $B$ to be the Borel subgroup of upper triangular matrices of determinant one. Let $\chi: F^\times \to \mathbb{C}^x$, be a character. We also denote by $\chi$ the character of $B = B(F)$ given by $\chi\left(\begin{array}{cc} a & b \\ 0 & a^{-1} \end{array}\right) = \chi(a)$. Moreover, every character of $B$ is of this form, for some $\chi \in \hat{F}$. We have $\mathfrak{a}_c^* / \mathfrak{j} \simeq \mathbb{C}$, and we choose the isomorphism given by $s \mapsto |s|$. Note that $\delta_B\left(\begin{array}{cc} a & 0 \\ 0 & a^{-1} \end{array}\right) = |a|^2$. Let $I(s, \chi) = \text{Ind}^G_B(\chi \otimes |s|)$. Suppose $C$ is a compact neighborhood of 0 in $F$, and let

$$f\left(\begin{array}{cc} 1 & 0 \\ x & 1 \end{array}\right) = \begin{cases} 1 & \text{if } x \in C \\ 0 & \text{if } x \notin C. \end{cases}$$

Now extend $f$ to $BU^-$, by $f\left(\begin{array}{cc} a & b \\ 0 & a^{-1} \end{array}\right)\left(\begin{array}{cc} 1 & 0 \\ x & 1 \end{array}\right) = \chi(a)|a|^{s+1}f\left(\begin{array}{cc} 1 & 0 \\ x & 1 \end{array}\right)$.

We choose $w = \left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right)$ to represent the non-trivial Weyl group element. Then

$$A(s, \chi, w)f(e) = \int_U f(w^{-1}u)du.$$

If $u = \left(\begin{array}{cc} 1 & x \\ 0 & 1 \end{array}\right)$, then $w^{-1}u \in BU^-$ if, and only if $x \in F^\times$, in which case

$$w^{-1}u = \left(\begin{array}{cc} x^{-1} & -1 \\ 0 & x \end{array}\right)\left(\begin{array}{cc} 1 & 0 \\ x^{-1} & 1 \end{array}\right).$$

So

$$A(s, \chi, w)f(e) = \int_{F^\times} \chi(x)^{-1}|x|^{-s+1}f\left(\begin{array}{cc} 1 & 0 \\ x^{-1} & 1 \end{array}\right) dx = \int_C \chi(x)|x|^{s-1} dx.$$

The poles of this last expression come from $L$-function $L(\chi, s)$ of Tate’s thesis [79]. We have $L(\chi, s) = 1$, unless $\chi$ is unramified, in which case

$$L(\chi, s) = \frac{1}{1 - \chi(\infty)q^{-s}}.$$

So, $A(s, \chi, w)$ has a pole at $s = 0$ if and only if $L(\chi, s)$ has a pole at $s = 0$, which occurs if and only if $\chi = 1$.

Note that

$$w\chi\left(\begin{array}{cc} a & * \\ 0 & a^{-1} \end{array}\right) = \chi\left(w^{-1}\left(\begin{array}{cc} a & * \\ 0 & a^{-1} \end{array}\right)w\right) = \chi\left(\begin{array}{cc} a^{-1} & * \\ 0 & a \end{array}\right) = \chi(a^{-1}) = \chi^{-1}(a).$$

So $w\chi \simeq \chi$ if and only if $\chi = \chi^{-1}$. Therefore, we have the following result.
Theorem 2.20 (Sally [61]). Let $\chi \in \widehat{F^\times}$, and extend $\chi$ to a character of $B$. Then $\text{Ind}^G_P(\chi)$ is reducible if, and only if, $\chi^2 = 1$, $\chi \neq 1$. □

If $G$ is a Chevalley group [6,74] and $B$ is a minimal parabolic subgroup, then Winarsky, [81], computed the poles of the intertwining operators $A(\nu, \chi, w)$ for any $\chi \in \widehat{B}$, and Weyl group element $w$. Winarsky’s computation reduces the computation of Plancherel measures to the case of $SL_2$, and then uses the techniques of [61].

Example 2. We next consider the case $G = GL_n$. This case was studied by Jacquet, [36], Jacquet and Godement, [24], Ol’šanskiĭ, [58], Bernstein and Zelevinsky, [4,5], and Shahidi [65]. The derivation of the pole of the intertwining operator is due to Ol’šanskiĭ. This result was duplicated by Bernstein and Zelevinsky by different methods. In [37], Jacquet showed that every tempered representation of $GL_n$ is generic. Shahidi used the local coefficient to derive an explicit formula for the Plancherel measure. Bushnell and Kutzko [13] have recently classified the admissible dual of $GL_n$, using the theory of types. Their techniques also give many of the above results.

Let $B$ be the Borel subgroup of non-singular upper triangular matrices. We will consider maximal parabolic subgroups. Suppose that $n = m + k$, and that

$$A = \left\{ \begin{pmatrix} \lambda I_k & 0 \\ 0 & \eta I_m \end{pmatrix} \bigg| \lambda, \eta \in \mathbb{G}_m^\times \right\}.$$ 

Let

$$M = Z_G(A) = \left\{ \begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix} \bigg| g \in GL_k, h \in GL_m \right\} \simeq GL_k \times GL_m.$$ 

We take $N = \left\{ \begin{pmatrix} I_k & * \\ 0 & I_m \end{pmatrix} \right\}$. Then $P = MN$ is a maximal parabolic subgroup of $G$, and every standard maximal parabolic subgroup of $G$ is of this form.

Let $\sigma = \sigma_1 \otimes \sigma_2$ be an irreducible unitary supercuspidal representation of $M$. Let $\omega_{\sigma_i}$ be the central character of $\sigma_i$. Note that, if $m \neq k$, then $W(G, A) = \{1\}$, so, by Bruhat’s Theorem, $\text{Ind}^G_P(\sigma)$ is irreducible. Suppose $m = k$. Then $w = \begin{pmatrix} 0 & I_k \\ I_k & 0 \end{pmatrix}$ represents the unique nontrivial element of $W(G, A)$. If $(g, h) \in M$, then $w((g, h)) = (h, g)$. So, $w\sigma \simeq \sigma_2 \otimes \sigma_1$, and $w\sigma \simeq \sigma$ if and only if $\sigma_1 \simeq \sigma_2$. Let $V$ be the space on which $\sigma_1$ acts, and suppose that $\sigma_2 = \sigma_1$. For $v \in V$ and $\tilde{v} \in \tilde{V}$ we denote...
by $\varphi_{v,\tilde{v}}$ the associated matrix coefficient, i.e., $\varphi_{v,\tilde{v}}(x) = \langle \tilde{v}, \sigma_1(x)v \rangle$. Choose a compact subset $L$ of $0$ in $M_k(F)$, and vectors $v_1, v_2 \in V$. Let $f \left( \begin{pmatrix} I & X \\ 0 & 0 \end{pmatrix} \right) = \xi_L(X)(v_1 \otimes v_2)$, where $\xi_L$ is the characteristic function of $L$. A straightforward matrix computation shows that, if $n = \begin{pmatrix} I & X \\ 0 & I \end{pmatrix}$, then $w^{-1}n \in PN^-$ if and only if $X \in GL_k$. In this case,

$$w^{-1}n = \begin{pmatrix} -X^{-1} & I \\ 0 & X \end{pmatrix} \begin{pmatrix} I & 0 \\ X & I \end{pmatrix}.$$

Fix $\tilde{v}_1, \tilde{v}_2 \in \tilde{V}$. Note that $a^*_C/3 \simeq \mathbb{C}$, via $s : (g, h) \mapsto |\det gh^{-1}|^s/2$. We choose this normalization because it is the one used in [65] and [67]. This allows us to easily describe the complementary series (see subsection (e).) Let $dX$ be a Haar measure on $M_k(F)$, and $d^\times X$ the associated Haar measure on $GL_k(F)$.

We have

$$< \tilde{v}_1 \otimes \tilde{v}_2, A(s, \sigma, w)f(e) > = \int_N f(w^{-1}n) \, dn,$$

$$= \int_{GL_k(F)} < \tilde{v}_1 \otimes \tilde{v}_2, f \left( \begin{pmatrix} -X^{-1} & I \\ 0 & X \end{pmatrix} \begin{pmatrix} I & 0 \\ X & I \end{pmatrix} \right) > \, dX,$$

$$= \omega_{\sigma_1}(-1) \int_{GL_k(F)} < \tilde{v}_1 \otimes \tilde{v}_2, \sigma_1(X)^{-1}v_1 \otimes \sigma_1(X)v_2 > |\det X|^{-s+2k}\xi_L(X^{-1}) \, dX.$$  

(2.2)

$$= \int_{GL_k(F)} \varphi_{v_1,\tilde{v}_1}(X)\varphi_{v_2,\tilde{v}_2}(X^{-1})\xi_L(X)|\det X|^s d^\times X.$$

Now, (2.2) always has a pole at $s=0$, for some choice of $v_i, \tilde{v}_i$, and $L$. For example, choose $v_1 = v_2 \neq 0$, and $\tilde{v}_1 = \tilde{v}_2$, with $< \tilde{v}_1, v_1 > = 1$, and $L = K_0$, an open compact subgroup with $v_1 \in V^{K_0}$. Then

$$(2.2) \sim \int_{K_0} |\det X|^s \, d^\times X,$$

has a pole at $s = 0$.

**Example 3.** We compute the $R$-group in a specific instance. For Chevalley groups, Keys, [38], used the results of [81] to compute the $R$-groups when $P$ is minimal. We examine an example when the minimal parabolic subgroup is of $p$-rank two.
Let 

$G = S_{p_4} = \left\{ g \in GL_4 \mid ^t g \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ -1 & 1 \\ \end{pmatrix} g = \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ -1 & 1 \\ \end{pmatrix} \right\}$. 

Let 

$T = \left\{ \begin{pmatrix} x & \alpha x^{-1} \\ \gamma x^{-1} & 1 \\ \end{pmatrix} : x, y \in G_m \right\}$. 

We denote a typical element of $T$ as $t(x, y)$. $\Phi(G, T)$ is of type $C_2$. Let $\alpha = e_1 - e_2$, and $\beta = 2e_2$ be the simple roots given by 

$\alpha(t(x, y)) = xy^{-1}$, and $\beta(t(x, y)) = y^2$. 

Note that $W(G, T) \cong \mathbb{Z}_2 \times (\mathbb{Z}_2 \times \mathbb{Z}_2)$, with generators 

- $w_\alpha: (t(x, y)) \mapsto (y, x)$, 
- $w_\beta: t((x, y)) \mapsto t(x, y^{-1})$, and 
- $w_\gamma: t((x, y)) \mapsto t(x^{-1}, y)$. 

Note that $w_\alpha w_\beta w_\alpha^{-1} = w_\gamma$. 

Let $T = T(F)$, and suppose that $\chi \in \hat{T}$. Then, for some $\chi_1, \chi_2 \in \hat{F}^\times$, we have $\chi(t(x, y)) = \chi_1(x)\chi_2(y)$. Therefore, we write $\chi = (\chi_1, \chi_2)$. Computing directly, we see that $w_\alpha \chi = \chi$ if and only if $\chi_1 = \chi_2$, and $w_\beta \chi = \chi$ if and only if $\chi_2^2 = 1$. 

We have $A_\alpha = \{t(x, x) : x \in G_m \}$, and $M_\alpha = \left\{ \begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix} : g, h \in GL_2 \right\} \cap G$. Since 

$\begin{pmatrix} t g & 0 \\ 0 & t h \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ -1 & 1 \\ \end{pmatrix} \begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ -1 & 1 \\ \end{pmatrix}$, 

we have 

$^t g \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} h = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and thus, $h = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ^t g^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = ^\tau g^{-1}$, 

where $^\tau \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & b \\ c & a \end{pmatrix}$. So $M_\alpha \cong GL_2$. Let $G' = GL_2$, and denote its Borel subgroup by $B'$. Recall that $W(M_\alpha, T) = \{1, w_\alpha\}$. So $\chi$ ramifies in $M_\alpha$ if and only
if $\chi_1 = \chi_2$. By example 2, $[5,58]$, $\text{Ind}_{\beta \cdot R_\alpha}^M(\chi) \simeq \text{Ind}_{B'}^G(\chi_1 \otimes \chi_2)$ is always irreducible. Thus, $\mu_\alpha(\chi) = 0$ if and only if $\chi_1 = \chi_2$.

We next consider the simple root $\beta$. We have $A_\beta = \{t(x,1) | x \in G_m\}$. Thus,

$$M_\beta = \left\{ \left( \begin{array}{ccc} x & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & x^{-1} \end{array} \right) \ | \ x \in G_m, h \in GL_2 \right\} \cap G.$$ 

Note that we must have $t^h\left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right)$, i.e. $h \in Sp_2 = SL_2$. So $M_\beta \simeq G_m \times SL_2$. Let $G'' = SL_2$, and denote its Borel subgroup by $B''$. We have $W_\beta(M_\beta, T) \simeq \langle w_\beta \rangle$, and $\text{Ind}^M_{\beta \cdot B_\beta}(\chi) \simeq \chi_1 \otimes \text{Ind}_{B'}^{G''}(\chi_2)$ is reducible if and only if $\chi_2^2 = 1$, $\chi_2 \neq 1$ (example 1). So $\mu_\beta(\chi) = 0$ if and only if $\chi_2 = 1$. Similarly, since $\gamma = 2e_1 = w_\alpha(\gamma)$, we have $\mu_\gamma(\chi) = 0$ if and only if $\chi_1 = 1$.

We can now compute the $R$-groups. The result we state is Theorem $C_n$ of $[38]$ for $n = 2$.

**Theorem 2.21 (Keys).** Let $R$ be the $R$-group attached to $\chi$. Then $R \simeq \mathbb{Z}_2^d$, where $d$ is the number of unequal elements of $\{\chi_1, \chi_2\}$ for which $\chi_i^2 = 1$, with $\chi_i \neq 1$.

**Proof.** We first claim that $w_\alpha \not\in R$ for any $\chi$. Suppose $w_\alpha \in W(\chi)$. Then, $\chi_1 = \chi_2$, and thus, $\mu_\alpha(\chi) = 0$, i.e. $\alpha \in \Delta'$. Since $w_\alpha(\alpha) = -\alpha < 0$, we have $w_\alpha \not\in R$, as claimed. Therefore, $R \subseteq \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle w_\gamma, w_\beta \rangle$. For $w \in W$, we let $R(w) = \{\delta > 0 | w\delta < 0\}$. Then $R(w_\beta) = \{\beta\}$, $R(w_\gamma) = \{\alpha, \gamma\}$, and $R(w_\beta w_\gamma) = \{\alpha, \beta, \gamma\}$. Since $R(w_\beta w_\gamma) = \Phi^+(G, T)$, we see that $w_\beta w_\gamma \in R$ implies $\Delta' = \emptyset$. However, $w_\beta w_\gamma \in W(\chi)$ also implies that $\chi_1^2 = \chi_2^2 = 1$, and thus, $w_\gamma, w_\beta \in W(\chi)$. Therefore, we see that $w_\beta w_\gamma \in R$ implies $w_\beta$ and $w_\gamma \in R$. $w_\gamma, w_\beta \in R$. Since $R(w_\beta) = \{\beta\}$, we have $w_\beta \in R$ if and only if $w_\beta \in W(\chi)$, and $\mu_\beta(\chi) \neq 0$, i.e. $\chi_2^2 = 1$, $\chi_2 \neq 1$. If $\chi_1 = \chi_2$, and $w_\gamma \in W(\chi)$, we have $\alpha \in \Delta'$, and thus $w_\gamma \not\in R$. Therefore, $\gamma_1 \in R$ if and only if $\chi_1^2 = 1$, $\chi_1 \neq 1$, and $\chi_1 \neq \chi_2$. This completes the proof \(\square\)

Knapp and Zuckerman, $[43]$, were first to find an example of a non-abelian $R$-group, showing that sometimes $\text{Ind}_F^G(\sigma)$ has components which appear with multiplicity larger than 1. Keys found many more examples in $[38,39]$. For $G = Sp_{2n}(F)$, or $SO_n(F)$, Goldberg computed the $R$-groups for all parabolic subgroups in $[26, 27]$. For $SL_n$ the possible $R$-groups are computed in $[29]$, which builds upon the
case of the minimal parabolic, where the $R$-groups were known from $[20,21,38]$. For the quasi-split unitary groups $U_n$, Goldberg computed the $R$-groups in $[25]$.

(e) **Elliptic Representations.** The collection of irreducible elliptic tempered representations played a key role in determining the tempered spectrum of real reductive groups $[42]$. While their analog in the $p$-adic case are important, understanding their characters will not be enough to determine the characters of all tempered representations. We describe what is known about such representations.

A regular element $[57]$ of $G$ is said to be **elliptic** if its centralizer is compact modulo the center if $G$. We write $G^e$ for the set of regular elliptic elements of $G$. An irreducible admissible representation $\pi$ of $G$ is said to be **elliptic** if $\chi_\pi$ is not identically zero on $G^e$. Here, $\chi_\pi$ is the distribution character of $\pi$ $[11]$. The following result exhibits the importance of elliptic representations.

**Theorem 2.22 (Knapp-Zuckerman $[42]$).** Suppose that $G$ is a connected reductive algebraic group defined over $\mathbb{R}$, and set $G = G(\mathbb{R})$. Let $\pi$ be an irreducible tempered representation of $G$. If $\pi$ is not elliptic, then there is some proper parabolic subgroup $P = MN$ of $G$, and an irreducible elliptic tempered representation $\sigma$ of $M$ for which $\pi = \text{Ind}_P^G(\sigma)$. □

We now consider the case where $F$ is a $p$-adic field of characteristic zero. Arthur, $[3]$, has given a necessary and sufficient condition, in terms of $R$-groups, for a tempered representation to be elliptic. For simplicity, we assume that the $R$-group in question is abelian, and the cocycle $\eta$ is a coboundary. (Arthur makes no such assumptions, and his result becomes slightly more technical.) For $H \in \mathfrak{a}$, we let $w \cdot H$ denote the image of $H$ under the action of $W(G, \mathfrak{a})$ on $\mathfrak{a}$. Let

$$a_w = \{H \in \mathfrak{a} | w \cdot H = H\}.$$ 

Suppose $\sigma \in \mathcal{E}_2(M)$, and suppose that $R = R(\sigma)$ is the $R$-group attached to $\text{Ind}_P^G(\sigma)$. Then we let $a_R = \bigcap_{w \in R} a_w$.

**Theorem 2.23 (Arthur $[3]$).** Suppose that $\sigma \in \mathcal{E}_2(M)$, and that $R = R(\sigma)$ is abelian. We further assume that the 2-cocycle $\eta$ attacher to $\sigma$ and $R$ splits. Then the following are equivalent:

(a) $\text{Ind}_P^G(\sigma)$ has an elliptic component;
(b) Every component of $\text{Ind}^G_P(\sigma)$ is elliptic;
(c) There is a $w \in R$, with $a_w = 1$. □

In view of the theorem of Knapp and Zuckerman, it is reasonable to ask which irreducible tempered representations of $G$ are irreducibly induced from elliptic representations of proper Levi subgroups. In [33] Herb gives a description of such representations, within the constraints of the previous theorem.

**Theorem 2.24 (Herb [33]).** Let $\sigma$, $R$, and $\eta$ be as above. Let $\pi$ be a subrepresentation of $\text{Ind}^G_P(\sigma)$. Then there is a parabolic subgroup $P' = M'N'$, and an irreducible elliptic $\tau \in \mathcal{E}_i(M')$ with $\pi = \text{Ind}^G_{P'}(\tau)$, if and only if there is a $w \in R$ with $a_w = a_R$. □

Herb was able to use these results to describe the elliptic tempered representations of $\text{Sp}_{2n}$ and $\text{SO}_n$ [33]. For $\text{Sp}_{2n}(F)$ and $\text{SO}_{2n+1}(F)$ the analog of Theorem 2.22 holds. That is, an irreducible tempered representation is either elliptic, or is irreducibly induced from an elliptic representation of a proper parabolic subgroup.

For $\text{SO}_{2n}(F)$, this statement is false. We give a simple example due to Herb [33].

Let $G = \text{SO}_6$. We define $G$ with respect to the form

$$
\begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{bmatrix}
$$

Then

$$
T = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_3^{-1} \\ x_2^{-1} \\ x_1^{-1} \end{pmatrix} \left| x_i \in G_m \right. \right\}.
$$

The root system $\Phi(G, T)$ is of type $D_3$, and the Weyl group is isomorphic to $S_3 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$. Here $S_3$ acts on the indices of the $x_i$'s. The subgroup $\{1\} \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$
is given by \(\{1, c_1c_2, c_2c_3, c_1c_3\}\), where

\[
c_1 = \begin{pmatrix} 1 & 1 \\ 1 & I_4 \end{pmatrix},
\]

\[
c_2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix},
\]

and

\[
c_3 = \begin{pmatrix} I_2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & I_2 \end{pmatrix}.
\]

Let \(\chi \in \hat{T}\). Then for some \(\chi_1, \chi_2, \chi_3 \in \hat{G}^\times\), we have \(\chi = \chi_1 \otimes \chi_2 \otimes \chi_3\). We need the following lemma from [38]. This result follows from example (2), and its proof is similar to the proof that \(w_\alpha \not\in R\) for any \(\chi\) in example 3.

**Lemma 2.25 (Keys).** If \(w = sc \in R\), with \(s \in S_3\) and \(c \in \mathbb{Z}_2 \times \mathbb{Z}_2\), then \(s = 1\). \(\square\)

Therefore, \(R \subset \mathbb{Z}_2 \times \mathbb{Z}_2\) is abelian. Let \(\chi = \chi_1 \otimes \chi_2 \otimes \chi_3 \in \hat{T}\). Then \(c_i c_j \in W(\chi)\) if and only if \(\chi_i^2 = 1\), and \(\chi_j^2 = 1\). Let \(d\) be the number of unequal elements of \(\{\chi_1, \chi_2, \chi_3\}\) which satisfy \(\chi_i^2 = 1\). If \(d = 0\), then \(W(\chi) = \{1\}\), so \(R = \{1\}\). Otherwise, \(\mathbb{R} \simeq \mathbb{Z}_2^{d-1}\). In particular, if \(\chi_i^2 = 1\), for \(i = 1, 2, 3\), and \(\chi_i \neq \chi_j\), for \(i \neq j\), then \(R = \mathbb{Z}_2 \times \mathbb{Z}_2\). Suppose that this is the case. Then \(\text{Ind}^G_B(\chi)\) has four inequivalent components. Note that \(a = \{\text{diag}\{a, b, c, -c, -b, -a\} | a, b, c \in \mathbb{R}\}\). Furthermore, \(3 = \{0\}\). We denote a typical element of \(a\) by \(t(a, b, c)\). Direct computation shows that \(a_{c_1c_2} = \{t(0, 0, c) | c \in \mathbb{R}\}\), \(a_{c_1c_3} = \{t(0, b, 0) | b \in \mathbb{R}\}\), and \(a_{c_2c_3} = \{t(a, 0, 0) | a \in \mathbb{R}\}\). Therefore, \(a_w \neq 3\), for all \(w \in R\). Consequently, \(\text{Ind}^G_B(\chi)\) does not have elliptic constituents. Moreover, since \(a_R = \{0\}\), these components cannot be irreducibly induced from an elliptic tempered representation of a proper parabolic subgroup.

In fact, what happens is that whenever \(\chi\) is induced to a rank one parabolic subgroup, \(P = MN\), the induced representation breaks into two elliptic components. Since there are four components of \(\text{Ind}^G_B(\chi)\), each of the components of \(\text{Ind}^M_B(\chi)\)
induces to \( G \) reducibly. Since \( SO_6 \) is locally homeomorphic to \( SL_4 \), such a construction is valid for the Borel subgroup of \( SL_4 \) as well. \( \square \)

Herb classified all the irreducible tempered representations of \( SO_{2n}(F) \) which are non-elliptic, and cannot be irreducibly induced from elliptic representations. Goldberg classified all the possible \( R \)-groups for \( SL_n \), and, motivated by [33], determined all irreducible tempered representations which are non-elliptic, and cannot be irreducibly induced from elliptic representations [29]. For \( U_n \) Goldberg showed that such representations do not exist [25].

**Complementary series.** So far we have discussed the structure of \( \text{Ind}^G_P(\sigma) \), with \( \sigma \in E_2(M) \). However, one would like to know the structure of the representations \( I(\nu, \sigma) \). For instance, one would like to know when \( I(\nu, \sigma) \) is irreducible and unitarizable. Such representations are said to be in the complementary series. Also, one would like to know the points of reducibility for \( I(\nu, \sigma) \), and what properties its subrepresentations and subquotients possess. We outline now what is known. For \( G = GL_n \), and \( P \) maximal, Bernstein and Zelevinsky [5,82] showed that \( \text{Ind}^G_P(\sigma_1 \otimes \sigma_2) \) is reducible if and only if \( \sigma_2 \cong \sigma_1 \otimes | |^{\pm 1} \). Bernstein and Zelevinsky showed that at the points of reducibility, there is a unique non-supercuspidal discrete series representations. These representations are often referred to as special representations. Shahidi [65] proved that the local coefficients satisfy the relation

\[
C_\psi(s, \sigma_1 \otimes \tilde{\sigma}_2)C_\psi(1 - s, \tilde{\sigma}_1 \otimes \sigma_2) = \omega_{\sigma_1}^k \omega_{\sigma_2}^k (-1).
\]

If \( \sigma_1 = \sigma_2 \) is unitary, then \( A(s, \sigma, w) \) has a pole at \( s = 0 \), so \( \mu(s, \sigma_1 \otimes \sigma_1) \) has a zero at \( s = 0 \). Thus, by Shahidi’s theorem on local coefficients, and (2.3), we see that \( \mu(s, \sigma_1 \otimes \sigma_1) \) must have a pole at \( s = 1 \). Thus, \( I(1, \sigma_1 \times \sigma_1) \) is reducible, duplicating the results of Bernstein and Zelevinsky for maximal parabolic subgroups.

Shahidi was able to prove a general result about the complementary series when \( P = MN \) is maximal. Suppose that \( LP = LM^L N \) is the standard parabolic subgroup of \( LG \) with Levi subgroup \( LM \) [7]. Let \( L_n \) be the real Lie algebra of \( L N \). Then \( LM \) acts on \( LN \) by the adjoint representation. We denote this representation by \( r \). There is a particular way of ordering the components of \( r \), as described in [67], and we write \( r = r_1 \oplus \cdots \oplus r_m \), accordingly. We choose the isomorphism \( a^*_C / J \cong \mathbb{C} \) as in [67]. Let \( \sigma \) be an irreducible unitary generic supercuspidal representation of
We let \( P_{\sigma,i} \) be the unique polynomial satisfying \( P_{\sigma,i}(0) = 1 \), and \( P_{\sigma,i}(q^{-s}) \) is the numerator of \( \gamma(s, \sigma, r_i, \psi) \). We set \( L(s, \sigma, r_i) = P_{\sigma,i}(q^{-s})^{-1} \).

We now describe a convenient parameterization of \( a^*_C \). Let \( \alpha \) be the unique simple root in \( N \), and let \( \rho_P \) be half the sum of the positive roots in \( N \). Let

\[
<\rho_P, \alpha> = 2(\rho_P, \alpha)/(\alpha, \alpha),
\]

where \(( , )\) is the standard euclidean inner product on \( \Phi(G, T) \). We define an element \( \tilde{\alpha} \) of \( a^*_C \) by \( \tilde{\alpha} = <\rho_P, \alpha>^{-1} \rho_P \). We let \( I(s, \sigma) = I(s\tilde{\alpha}, \sigma) \).

**Theorem 2.26 (Shahidi [67]).** Let \( P = MN \) be a maximal parabolic subgroup of \( G \). Assume that \( W(G, A) \neq \{1\} \), and let \( w_0 \) represent the unique non-trivial Weyl group element. Suppose \( \sigma \in \mathcal{O}_E(M) \) is generic.

1. For \( 3 \leq i \leq m \), we have \( L(s, \sigma, r_i) = 1 \).
2. The following are equivalent
   a) \( s = 0 \) is a pole of \( A(s, \sigma, w_0) \).
   b) \( P_{\sigma,i}(1) = 0 \) for either \( i = 1 \), or \( 2 \), and only for one of them.
   c) \( w_0\sigma \simeq \sigma \), and \( \text{Ind}^G_P(\sigma) \) is irreducible.
3. Suppose (2a) is satisfied. Moreover assume that in (2b), \( P_{\sigma,i}(1) = 0 \). Then:
   a) For \( 0 < s < 1/i \), The representation \( I(s, \sigma) \) is irreducible and unitarizable, i.e., is in the complementary series.
   b) The representation \( I(1/i, \sigma) \) is reducible, with a unique generic special sub-representation. Its Langlands quotient is non-generic, unitarizable, and non-tempered.
   c) For \( s > 1/i \), the representation \( I(s, \sigma) \) is always irreducible and never unitarizable.
4. If \( \text{Ind}^G_P(\sigma) \) is reducible, then for all \( s > 0 \), the representation \( I(s, \sigma) \) is irreducible, and is never unitarizable. \( \square \)

Shahidi’s result points out the power of Langlands’s conjectures, because of the connection they give between number theory and harmonic analysis. Tadic has been able to derive many results, complementary to Shahidi’s, for classical groups via the theory of Jacquet modules [75,76,77].
§3 Endoscopy. The theory of twisted endoscopy has proved very useful in the computation of the poles of intertwining operators. The theory itself is quite technical, and therefore, beyond the scope of these lectures. We will attempt to illustrate its power by working an example. For more details on the general theory one should consult [44,45,55].

Let $n \geq 1$, and $G = Sp_{2n}, SO_{2n}$, or $SO_{2n+1}$. In each case, $G$ has a maximal parabolic subgroup (called the Siegel parabolic subgroup) with $M \cong GL_n$. More precisely, if $g \in GL_n(F)$, then

\[
m(g) = \begin{cases}
\begin{pmatrix} g & 0 \\ 0 & tg^{-1} \end{pmatrix} & \text{if } G = Sp_{2n} \text{ or } SO_{2n}, \\
\begin{pmatrix} g & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & t g^{-1} \end{pmatrix} & \text{if } G = SO_{2n+1},
\end{cases}
\]

is the corresponding element of $M = M(F)$. Let $(\sigma, V) \in E_2(GL_n(F))$. For $s \in \mathbb{C}$, we set $I(s, \sigma) = \text{Ind}_G^H(\sigma \otimes |^s) = \{ f : G \rightarrow V | f(m(g_0)ng) = \sigma(g_0) | \det g_0 |^{s+\delta} f(g) \}$,

where $\delta = (n+1)/2$, $(n-1)/2$, or $n/2$ respectively. In each case we take $w_0$ to be the longest element of the Weyl group.

Consider the standard representation $\rho_n : GL_n(\mathbb{C}) \rightarrow GL_n(\mathbb{C})$. This is just the natural action of $GL_n(\mathbb{C})$ on $\mathbb{C}^n$. We denote the exterior square of $\rho_n$ by $\wedge^2 \rho_n$. If $v, u \in \mathbb{C}^n$, then $\wedge^2 \rho_n(g)(v \wedge u) = g(v \wedge u)$. Let $G = Sp_{2n}$. Then $L^0G^0 = SO_{2n+1}(\mathbb{C})$, and the adjoint representation $r$ of $L^0 M$ on $L^0 n$ is isomorphic to $\rho_n \oplus \wedge^2 \rho_n$. By [24], $L(s, \sigma, \rho_n) = 1$. So we must examine $L(s, \sigma, \wedge^2 \rho_n)$. More specifically, we need to find the polynomial $P(t)$ such that $P(0) = 1$ and $P(q^{-2s})A(s, \sigma, w_0)$ is holomorphic and non-zero. Then, by [67],

\[ L(s, \sigma, \wedge^2 \rho_n) = P(q^{-s})^{-1}. \]

If $G = SO_{2n}$, then $r = \wedge^2 \rho_n$, and if $G = SO_{2n+1}$, then $r = \text{Sym}^2 \rho_n$, the symmetric square of $\rho_n$.

Suppose that $\sigma$ is supercuspidal, and $\varphi : W_F \rightarrow GL_n(\mathbb{C}) = LM_0$, is the (conjectural) Langlands parameter for $\sigma$. Let $N = n(n-1)/2$, and consider $\wedge^2 \rho_n \circ \varphi :
We must have $L(s, \sigma, \wedge^2 \rho_n) = L(s, \wedge^2 \rho_n \circ \varphi)$, where this last object is the Artin $L$-function. Now, $P(1) = 0$ if and only is $L(s, \wedge^2 \rho_n \circ \varphi)$ has a pole at $s = 0$, which can only occur if the trivial representation appears in $\wedge^2 \rho_n \circ \varphi$. Thus, $\text{Im} \varphi$ must fix a vector in the alternating space $\wedge^2 \mathbb{C}^n$. By duality, there must be some $B \in (\wedge^2 \mathbb{C}^n)^*$, which is fixed by $\text{Im} \varphi$, i.e. the image of $\varphi$ must fix a skew symmetric form in $n$ variables. Since $\sigma$ is supercuspidal, $\text{Im} \varphi$ must be irreducible [7], and hence $B$ must be non-degenerate. Thus, $n$ must be even, and $\varphi$ should factor through $Sp_n(\mathbb{C})$. Dually, one expects $\sigma$ to “come from” $H = SO_{n+1}(F)$, since $L_H^0 = Sp_n(\mathbb{C})$.

Note that we must have the following result.

**Proposition 3.1 (Shahidi).** If $n$ is odd, then $L(s, \sigma, \wedge^2 \rho_n) = 1$.

**Proof.** Consider $G = SO_{2n}$ and $M = GL_n$. Then the Weyl group $W(G, A) = \{1\}$, so, by [67], $P(t) = 1$. □

If $G = Sp_{2n}$, then $W(G, A) \cong \mathbb{Z}_2$, and $w_0(m(g_0)) = m(t g_0^{-1})$. Thus, $w_0 \sigma \simeq \sigma$ if and only if $\sigma \simeq \bar{\sigma}$ [4, §7]. When the residual characteristic of $F$ is 2, then such representations always exist.

**Corollary 3.2 (Shahidi).** If $n$ is odd and $\sigma \simeq \bar{\sigma}$, then $\text{Ind}_F^G(\sigma)$ is reducible, and for all $s > 0$, the representation $I(s, \sigma)$ is irreducible. □

So, we need to concentrate on the case where $n$ is even. Suppose $G = SO_{2n}$, with $n$ even. Then

$$N = \left\{ \begin{pmatrix} I & X \\ 0 & I \end{pmatrix} \mid t^X = -X \right\}.$$ 

We have $w_0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$. If $n = \begin{pmatrix} I & X \\ 0 & I \end{pmatrix} \in N$, then $w_0^{-1} n \in PN$ if and only if $X \in GL_n(F)$. In this case

$$w_0^{-1} n = \begin{pmatrix} -X^{-1} & I \\ 0 & X \end{pmatrix} \begin{pmatrix} I & 0 \\ X^{-1} & I \end{pmatrix}.$$ 

Let $L$ be compact and $h(\tilde{n}) = \xi_L(X)v$, for some $v \in V$. Taking $\tilde{v} \in \tilde{V}$, we have

$$< \tilde{v}, A(s, \sigma, w_0)f(e) > = \int_{\text{det } X \neq 0} < \tilde{v}, \sigma(-X)^{-1}v > |\det X|^{-s+(n-1)/2} \xi_L(X^{-1})dX,$$
which we rewrite as
\[
(3.1) \quad \omega_\sigma(-1) \int_{\frac{t_X}{\det X \neq 0}} \varphi_{v,\bar{v}}(X) \det X |^s \xi_L(X) \, d^\times X.
\]

We can choose \( f \in C_c^\infty(\text{GL}_n(F)) \) so that
\[
\varphi_{v,\bar{v}}(g) = \int_{Z_n(F)} f(zg) \omega_\sigma^{-1}(z) \, dz,
\]
where \( Z_n(F) \) is the center of \( \text{GL}_n(F) \).

Let \( \gamma_0 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \). Since all symplectic forms are equivalent, any \( X \) with \( tX = -X \) and \( \det X \neq 0 \), is of the form \( t^g \gamma_0 g \), for some \( g \in \text{GL}_n(F) \). For technical reasons we take \( \theta^*(g) = w^{-1} t^g w^{-1}, \) with
\[
w = \begin{pmatrix}
1 & & \\
& -1 & \\
& & 1 \\
-1 & & \\
& & & \\
& & & & 1 \\
& & & & & \\
& & & & & & \ddots
\end{pmatrix}.
\]

For \( \gamma \in \text{GL}_n(F) \), we let \( \mathbf{G}_{\theta^*,\gamma} = \{ g \in \text{GL}_n \mid g^{-1} \gamma \theta^*(g) = \gamma \} \). Now (3.1) is proportional to
\[
(3.2) \quad \int_{\mathbf{G}_{\theta^*,\gamma}(F)^0 \backslash \text{GL}_n(F)} f(g^{-1} z \gamma_0 \theta^*(g)) |\det(g^{-1} z \gamma_0 \theta^*(g))|^s \xi_L(g^{-1} z \gamma_0 \theta^*(g)) \, dz \, dg.
\]

We now define some terms. For \( \gamma \in \text{GL}_n(F) \), and \( f \in C_c^\infty(\text{GL}_n(F)) \), let
\[
\Phi_{\theta^*}(\gamma, f) = \int_{\mathbf{G}_{\theta^*,\gamma}(F) \backslash \text{GL}_n(F)} f(g^{-1} \gamma \theta^*(g)) \, dg.
\]

We say that \( \gamma \) is \( \theta^* \)-semisimple if \( (g, \theta) \) is semisimple in the disconnected algebraic group \( \text{GL}_n \times < \theta^* > \). We say that such a \( \gamma \) is strongly \( \theta^* \)-regular if \( \mathbf{G}_{\theta^*,\gamma} \) is abelian.

We say that \( \gamma \) and \( \gamma' \) are \( \theta^* \)-conjugate if, for some \( g \in \text{GL}_n(F) \), \( \gamma' = g^{-1} \gamma \theta^*(g) \). Finally, \( \gamma \) and \( \gamma' \) are stably \( \theta^* \)-conjugate if, for some \( g \in \text{GL}_n(F) \), \( \gamma' = g^{-1} \gamma \theta^*(g) \).

We write \( \gamma \sim \gamma' \) for stable \( \theta^* \)-conjugacy. If \( \gamma \sim \gamma' \) are strongly \( \theta^* \)-regular, then \( \mathbf{G}_{\theta^*,\gamma} \) and \( \mathbf{G}_{\theta^*,\gamma'} \) are inner. We can therefore transfer measures. Set
\[
\Phi_{\theta^*}^{\gamma,f}(\gamma, f) = \sum_{\gamma' \sim \gamma} \Phi_{\theta^*}(\gamma', f).
\]
This is called the $\theta^*$-stable twisted orbital integral of $f$ at $\gamma$.

Let $H = SO_{n+1}$, and let $T = T_H$ be a Cartan subgroup of $H$ defined over $F$. Suppose that $T'$ is a $\theta^*$-stable Cartan of $GL_n$, defined over $F$. Let

$$T'_{\theta^*} = T'/(1 - \theta^*)T'.$$

There exists $T = T_H$, such that there is an isomorphism between $T$ and $T'_{\theta^*}$ defined over $F$. Thus, we get a one-to-one map $A$ between semisimple conjugacy classes of $H(\bar{F})$, and $\theta^*$-semisimple $\theta^*$-conjugacy classes of $GL_n(\bar{F})$.

**Definition 3.3.** We say that $\delta \in H$ is a norm of $\gamma \in GL_n(F)$ if the $GL_n(F) - \theta^*$-conjugacy class of $\gamma$ is the image of $\delta$ under $A$. We write $\delta = N\gamma$.

If $\gamma$ is strongly $\theta^*$-regular, then $\delta = N\gamma$ is strongly regular. Suppose $\psi \in C^\infty_c(H)$.

Let

$$\Phi(\delta, \psi) = \int_{H_\theta^*(F) \setminus H} \psi(h^{-1}\delta h) \, dh,$$

and

$$\Phi^{st}(\delta, \psi) = \sum_{\delta' \sim \delta} \Phi(\delta', \psi),$$

for $\delta$ strongly regular.

**Assumption 3.4.** For every $f \in C^\infty_c(GL_n(F))$, there exists a $f^H \in C^\infty_c(H)$, so that

$$\Phi^{st}_\theta(\gamma, f) = \Phi^{st}(N\gamma, f^H),$$

for every strongly $\theta$-regular $\gamma \in GL_n(F)$, and $\Phi^{st}(\delta, f^H) = 0$, if $\delta$ is not a norm.

**Proposition 3.5.** Up to a non-zero constant,

$$\int_{Sp_n(F) \setminus GL_n(F)} f(t^g w^{-1} gw) \, dg = f^H(e),$$

for $f \in C^\infty_c(GL_n(F))$. □

**Definition 3.6.** An irreducible supercuspidal representation $\sigma$ of $GL_n(F)$ is said to “come from” $SO_{n+1}(F)$ if, $n$ is even, and $f^H(e) \neq 0$, for some $f \in C^\infty_c(GL_n(F))$ defining a matrix coefficient of $\sigma$. 
Definition 3.7. We say \( \sigma \simeq \tilde{\sigma} \) comes from \( SO^*_n(F) \) if \( n \) is even, and \( \sigma \) does not come from \( SO_{n+1}(F) \). We say \( \sigma \) comes from \( Sp_{n-1}(F) \) if \( n \) is odd.

Theorem 3.8. The residue of \( A(s, \sigma, w_0) \) at \( s = 0 \), given by (3.2), is proportional to
\[
\int_{Sp_n(F) \setminus GL_n(F)} f(t^tgw^{-1}gw) \, dg.
\]
Therefore, \( A(s, \sigma, w_0) \) has a pole at \( s = 0 \) if and only if, for some choice of \( f \), defining a matrix coefficient of \( \sigma \),
\[
\int_{Sp_n(F) \setminus GL_n(F)} f(t^tgw^{-1}gw) \, dg \neq 0.
\]
It follows that \( \sigma \simeq \tilde{\sigma} \), and \( \omega_{\sigma} = 1 \). \( \square \)

Theorem 3.9 (Shahidi [68]). Suppose that \( \sigma \) is an irreducible supercuspidal representation of \( GL_n(F) \), with \( \sigma \simeq \tilde{\sigma} \).

(a) If \( G = SO_{2n+1} \), then \( \text{Ind}^G_P(\sigma) \) is irreducible if and only if \( \sigma \) comes from \( SO^*_n(F) \), or \( Sp_{n-1}(F) \).

(b) If \( G = Sp_{2n} \), then \( \text{Ind}^G_P(\sigma) \) is irreducible if and only if \( \sigma \) comes from \( SO_{n+1}(F) \).

(c) If \( G = SO_{2n} \), then \( \text{Ind}^G_P(\sigma) \) is irreducible if and only if \( \sigma \) comes from \( SO_{n+1}(F) \) or \( Sp_{n-1}(F) \). \( \square \)

The content of Shahidi’s Theorem should not be overstated. It says that if the correspondence \( f \mapsto f^H \) exists, with the desired properties, then \( f^H(e) \neq 0 \) if and only if
\[
\int_{Sp_n(F) \setminus GL_n(F)} f(t^gw^{-1}gw) \, dg \neq 0.
\]
In particular, it does not guarantee the existence of the correspondence \( f \mapsto f^H \).

In [28], Goldberg used Shahidi’s method, and the explicit matching between \( GL_2(E) \) and \( U(2) \), [60], to describe the poles of \( A(s, \sigma, w) \) when \( G = U(2, 2) \). He was able to show that the poles of the intertwining operator distinguish the images of the two base change maps described in [60]. In [30] Goldberg carried out a similar computation for \( G = U(n, n) \), and gave a relation between reducibility and
lifting. Shahidi, [63], has studied the residues of the intertwining operators for more maximal parabolic subgroups of $SO_{2n}(F)$. The case of a general classical group is the subject of a work in progress by Goldberg and Shahidi.

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