A multiplicity result for the nonlinear Klein Gordon Maxwell equations *

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Abstract

In this paper we provide a new technique to find solutions to the Klein-Gordon-Maxwell system. The method, based on an iterative argument, permits to improve previous results where the reduction method was used.

We also show how this device permits to obtain a multiplicity result in the physically significant context known as “the positive potential case”.

Introduction

This paper is concerned with the following Klein-Gordon-Maxwell system

\[
\begin{aligned}
-\Delta u + [m^2 - (\omega - e\phi)^2]u - f(u) &= 0 \quad \text{in } \mathbb{R}^3 \\
-\Delta \phi &= e(\omega - e\phi)u^2 \quad \text{in } \mathbb{R}^3.
\end{aligned}
\]

This system was introduced in the pioneering work of Benci and Fortunato [5] in 2002 and represents a standing wave \( \psi = u(x) \exp[i\omega t] \) (charged matter field) in equilibrium with a purely electrostatic field \( E = -\nabla \phi(x) \).

The constant \( m \geq 0 \) represents the mass of the charged field and \( e \) is the coupling constant introduced in the minimal coupling rule [22].

It is immediately seen that (1) deserves some interest as system if and only if \( e \neq 0 \) and \( \omega \neq 0 \), otherwise we get \( \phi = 0 \). Through the paper we are looking for nontrivial solutions, that is solutions such that \( \phi \neq 0 \).

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Moreover we point out that the sign of $\omega$ is not relevant for the existence of solutions. Indeed if $(u, \phi)$ is a solution of (1) with a certain value of $\omega$, then $(u, -\phi)$ is a solution corresponding to $-\omega$. So, without loss of generality, we shall assume $\omega > 0$. Analogously the sign of $e$ is not relevant, so we assume $e > 0$.

Let us recall some previous results that led us to the present research. The first results are concerned with an homogeneous nonlinearity $f(t) = \frac{1}{p} |t|^p$. Therefore (1) becomes

$$\begin{cases}
-\Delta u + [m^2 - (e\phi - \omega)^2]u - |u|^{p-2}u = 0 & \text{in } \mathbb{R}^3 \\
-\Delta \phi = e(\omega - e\phi)u^2 & \text{in } \mathbb{R}^3.
\end{cases} \quad (2)$$

In [5] Benci and Fortunato showed the existence of infinitely many solutions whenever $p \in (4, 6)$ and $0 < \omega < m$.

In 2004 D’Aprile and Mugnai published two papers on this topic. In [18] they proved the existence of nontrivial solutions of (1) when $p \in (2, 4]$ and $\omega$ varies in a certain range depending on $p$:

$$0 < \omega < m g_0(p)$$

where

$$g_0(p) = \sqrt{\frac{p - 2}{2}}.$$  

Afterwards, in [19], the same authors showed that (2) has no nontrivial solutions if $p \geq 6$ and $\omega \in (0, m]$, or $p \leq 2$ (for nonexistence results see also [16]).

Recently in [1] the existence of a solution was proved in a larger range, precisely for any $0 < \omega < m$ when $p \in [3, 4]$ and

$$0 < \omega < m g_1(p)$$

where

$$g_1(p) = \sqrt{(p - 2)(4 - p)}$$

when $p \in ]2, 3[.$

The application of variational arguments to system (2) presents immediately the difficulty to handle with a strongly indefinite functional. In all the previous papers, the strategy used to overcome this difficulty was based on the application of the so called reduction method. The idea is: we solve the second equation with respect to $\phi$ for any fixed $u$ (the solution is unique), creating a dependence of one of the variable from the other. Then we look for critical points to the reduced functional, that is the
one-variable functional obtained from the original one replacing \( \phi \) with the unique solution of the second equation.

If from one hand this method allows to remove the strong indefiniteness of the original functional, on the other it leads to the technical difficulty of proving boundedness of Palais-Smale sequences of the new functional. Indeed, when \( p \) is too close to 2 and \( \omega \) too close to \( m \), standard estimates (Nehari and Pohozaev identity) are not sufficient to prove that the reduced functional computed on the set of the solutions is coercive with respect to the Sobolev norm of \( u \).

A way to overcome this difficulty is to make \( e \) play a role in the system. The key idea is the following: since for \( e = 0 \) the system reduces to the single equation

\[-\Delta u + (m^2 - \omega^2)u = |u|^{p-2}u\]

which, as well known, possesses infinitely many radial solutions if \( 0 < \omega < m \) and \( 2 < p < 6 \) then, if stability holds at some critical level, at least a solution could last for (2), provided that \( e \) is small enough.

Basing their arguments on this idea, Jeong and Seok [31] proved that, for a sufficiently small \( e \), there exists a solution to (1) under general assumptions on the function \( f \). In particular their result includes (2) for any \( \omega \in ]0, m[ \) and \( 2 < p < 6 \).

The aim of this paper is to understand how general the nonlinearity \( f \) could be in order to have an existence (and possibly a multiplicity) result which does not depend on the smallness of \( e > 0 \) and the distance \( m^2 - \omega^2 \).

First of all, we reformulate the system in the following form

\[
\begin{align*}
-\Delta u + e(2\omega - e\phi)\phi u - g(u) &= 0 \\
-\Delta \phi &= e(\omega - e\phi)u^2.
\end{align*}
\]  

(\textit{KGM})

We assume \( g \in C(\mathbb{R}, \mathbb{R}) \) is odd and \( G(s) = \int_0^s g(t) \, dt \). Moreover

\((g_1)\) \(-\infty < \liminf_{s \to 0^+} g(s)/s \leq \limsup_{s \to 0^+} g(s)/s = -\Omega < 0;\)

\((g_2)\) \(-\infty < \limsup_{s \to +\infty} g(s)/s^5 \leq 0;\)

\((g_3)\) there exists \( \zeta > 0 \) such that \( G(\zeta) > 0.\)

Now we state our main result

\textbf{Theorem 0.1.} Assume \( g \in C(\mathbb{R}, \mathbb{R}) \) is odd, \((g_1)\ldots(g_3)\) hold and set

\[\omega_0 := \sup \left\{ \omega > 0 \mid \sup_{s>0} \left( G(s) - \frac{1}{2} \omega^2 s^2 \right) > 0 \right\}.\]
If \( \omega \in (0, \omega_0) \) then there exist infinitely many radially symmetric solutions to \((KGM)\).

If \( \omega \geq \omega_0 \) then for any \( k \in \mathbb{N} \) there exists \( \bar{e} > 0 \) such that for any \( e \in (0, \bar{e}) \), the system \((KGM)\) has at least \( k \) solutions.

**Remark 0.2.** We point out that an existence result for \( \omega \geq \omega_0 \) with small \( e \) was already obtained in [31]. Here we provide a multiplicity result using a different approach.

A first consequence of Theorem 0.1 is the following Corollary which improves [1]

**Corollary 0.3.** If \( 0 < \omega < m \) and \( 2 < p < 6 \), system (2) has infinitely many radial solutions.

Assumptions \( g_1, g_2 \) and \( g_3 \) were introduced in the well known papers by Berestycki and Lions [12, 13] to prove the existence of a ground state and a multiplicity result for the equation

\[
-\Delta u = g(u).
\]  

In the same papers it was showed that these assumptions were in some sense almost optimal to have solutions for (3), so that their results turn out to be very general. Later, these and similar assumptions appeared in many papers, and the study of most general hypotheses needed to apply variational methods to elliptic equations became a subject of research. In particular the attention was focused on nonlinearities verifying the Berestycki-Lions condition \( g_3 \) which, differently from the well known Ambrosetti-Rabinowitz condition, prevents from the use of classical estimates of critical points theory to prove the boundedness of Palais-Smale sequences.

In [14], Brezis and Lieb used a constraint minimizing method to generalize [12] to the study of systems.

In [3, 26, 29] it was studied the Schrödinger equation

\[
-\Delta u + V(x)u = g(u)
\]

with general assumptions on \( g \), by means of the so called monotonicity trick (see [28, 35]) and the well known Pohozaev identity.

As to Klein-Gordon-Maxwell system, \( g_3 \) is in some sense related with the so called hylomorphic assumption introduced in recent papers by Benci and Fortunato [7, 10, 11] to find stable solitary waves of the hylomorphic type, in the physically consistent hypothesis of positive energy.

As showed in [7, 10, 11] (see also section 1), in order to have an a priori
estimate which guarantees positive energy, we need to study \((KG\mathcal{M})\) in the “positive potential case”. When the system is expressed in the general form \((KG\mathcal{M})\), positive potential case consists in requiring \(\omega \geq \omega_0\).

In section 1 we will show how Theorem 0.1 is related with the Klein-Gordon-Maxwell system with a positive potential.

Apart from the final result, we believe that also the device we use could be of interest for future developments on this and similar subjects. Indeed, differently from the past, here we do not use the reduction method but we develop an iterative argument which, up to our knowledge, is completely new. The scheme of this method is the following: we consider an arbitrary function \(u_0\) and find \(\phi_0\) solving the second equation. Then we solve the first equation with respect to \(u\) setting \(\phi = \phi_0\), and we call \(u_1\) a solution. Again we solve the second equation with respect to \(\phi\) with \(u = u_1\) and we find \(\phi_1\) which we insert in the first equation to find \(u_2\), and so on.

In this way we construct two sequences of functions, \((u_n)\) and \((\phi_n)\), which we hope converge to a solution. We remark that, for this technique to work, it is fundamental to have uniform a-priori estimates which guarantee the final convergence and that limit is not \((0,0)\). We will show that the key estimate is that proved firstly in D’Aprile and Mugnai [18] on the \(L^\infty\) norm of \(\phi\).

Section 2 and 3 are entirely devoted to the description of our iterative method and to the proof of the main Theorem and the Corollary.

1 Solitary waves in Abelian gauge theory

In the Abelian gauge theory the interaction between a matter field \(\psi\) obeying the nonlinear Klein-Gordon equation and the electromagnetic field represented by the gauge potentials \((A, \phi)\) is described by the equations obtained making the variation with respect to \(\psi, \phi\) and \(A\) of the action

\[
S(\psi, \phi, A) = \int (\mathcal{L}_0 + \mathcal{L}_1 - W(|\psi|)) \, dx \, dt
\]

where

\[
\mathcal{L}_0 = \frac{1}{2} \left[ |(\partial_t + ie \phi) \psi|^2 - |(\nabla - ie A) \psi|^2 \right]
\]

is the Klein-Gordon Lagrangian, and

\[
\mathcal{L}_1 = \frac{1}{2} \left[ |(\partial_t A + \nabla \phi)|^2 - |\nabla \times A|^2 \right]
\]
is the Maxwell Lagrangian. The system we get is

\[
(\partial_t + ie\phi)^2 \psi - (\nabla - ieA)^2 \psi + W'(|\psi|)\frac{\psi}{|\psi|} = 0
\]

\[
\nabla \cdot (\partial_t A + \nabla \phi) = e \left( \text{Im} \frac{\partial_t \psi}{\psi} + e\phi \right) |\psi|^2
\]

\[
\nabla \times (\nabla \times A) + \partial_t (\partial_t A + \nabla \phi) = e \left( \text{Im} \frac{\nabla \psi}{\psi} - eA \right) |\psi|^2.
\]

Previous system constitutes the Klein-Gordon-Maxwell equations which have been object of a deep study in recent years.

Stationary solutions in static situation are obtained imposing that potential \(A\) and \(\phi\) do not depend on time and \(\psi(x, t) = u(x) \exp[i(S(x) - \omega t)]\), with \(u > 0, \omega \in \mathbb{R} \) and \(S \in \mathbb{R}/2\pi\mathbb{Z} \). Replacing this expression of \(\psi\) in \(S\) and making the variation with respect to \(u, S, \phi\) and \(A\), we get the following system

\[
\begin{cases}
-\Delta u + [||\nabla S - eA||^2 - (\omega - e\phi)^2]u + W'(u) = 0 \\
-\nabla \cdot [(\nabla S - eA)u^2] = 0 \\
-\Delta \phi = e(\omega - e\phi)u^2 \\
\nabla \times (\nabla \times A) + \partial_t (\partial_t A + \nabla \phi) = e(\nabla S - eA)u^2.
\end{cases}
\] (4)

Now, there are various types of solutions to this system, according to the fact that one variable is assumed to be null.

When \(\phi = 0\) and \(A \neq 0\), solutions are said magnetostatic. It is obvious from the third equation that magnetostatic stationary solutions are actually static since \(\omega = 0\). Solutions of this type have been studied in dimension \(N = 2\) in [4].

When both \(\phi\) and \(A\) are not null, solutions are said electromagnetostatic. In particular, since these solutions have a non null angular momentum (since both \(\omega\) and \(\nabla S\) are different from zero), they are called vortices. Three dimensional vortex-solutions have been found in [8, 9].

When we try to find solutions with \(\phi \neq 0, A = 0\) and \(\nabla S = 0\), then we are looking for standing waves in electrostatic case.

The system arising is

\[
\begin{cases}
-\Delta u - (\omega - e\phi)^2 u + W'(u) = 0 \\
-\Delta \phi = e(\omega - e\phi)u^2.
\end{cases}
\] (5)

This is the most studied situation and presents an abundant literature (see [1, 2, 5, 6, 8, 9, 15, 16, 19, 32, 33, 36]). See also [17, 23, 24] for the analysis of the problem, possibly singularly perturbed, in a Riemannian manifold and [20, 21] for what concerns the problem in a bounded domain.
Most of the papers are concerned with nonlinearity $W$ of the type $W(s) = \frac{m^2}{2} s^2 - \frac{1}{p} |s|^p$ with $p > 2$.

Recently, Benci and Fortunato [6] observed that, since the energy of solutions to (4) is given by the expression

$$
E(u, A, \phi) = \frac{1}{2} \int \left[ |\nabla u|^2 + 2W(u) + \left| \frac{\partial S}{\partial t} + e\phi \right|^2 u^2 
+ |\nabla S - eA|^2 u^2 + \left| \frac{\partial A}{\partial t} + \nabla \phi \right|^2 + |\nabla \times A|^2 \right] dx,
$$

in order to have a fine apriori estimate for the energy of solitary waves, a positive potential $W$ should have been more appropriate. Precisely, in [6], it was assumed that

W1) $W \geq 0$, $W(0) = W'(0) = 0$,

W2) $W''(0) = m_0^2 > 0$,

W3) there exist $m_1, c > 0$ with $m_1 < m_0$ such that

$$W(s) \leq \frac{1}{2} m_1^2 s^2 + c \quad \text{for all } s \in \mathbb{R},$$

W4) for all $s \in \mathbb{R}$:

$$0 \leq \frac{1}{2} W''(s) s \leq W(s),$$

W5) there exist constants $c_1, c_2 > 0$ and $p < 4$ such that for all $s \in \mathbb{R}$

$$|W''(s)| \leq c_1 |s|^p + c_2,$$

and was proved the existence of a standing wave to (5) for small $e$. We remark that in this result $\omega$ is treated as a variable of the problem.

Relaxing assumptions W4 and W5, Mugnai in [33] obtained a similar result. In [9], a larger selection of potentials $W$ was allowed, replacing assumptions W3, W4 and W5 by

W3') if we set

$$W(s) = \frac{1}{2} m_0^2 s^2 + N(s)$$

then

$$\exists s_0 \in \mathbb{R}_+ \text{ such that } N(s_0) < 0,$$
for any $s \in \mathbb{R}$

$$|N'(s)| \leq c_1 |s|^{r-1} + c_2 |s|^{q-1}, \text{ for } q, r \text{ in } (2, 6)$$

and both vortices and standing waves were found. However again it was required $e$ to be sufficiently small and that $\omega$ was a variable. In [34] Mugnai and Rinaldi obtained a similar result as in [9] removing the dependence from $e$, but requiring something more than $W^3'$ on $N$.

By means of our main theorem, we can prove the following

**Corollary 1.1.** Assume that $W$ satisfies $W^1, W^2, W^3', W^4'$ and $N$ is even. Then for any $\omega \in (\sqrt{m_0^2 + 2Ns_0/s_0^2}, m_0)$ and any $k \in \mathbb{N}$, there exists $\bar{e} > 0$ such that for any $e \in (0, \bar{e})$ system (5) has $k$ positive solutions.

**Proof** First of all we point out the fact that, by $W^1$ and $W^3'$, certainly $m_0^2 + 2N(s_0/s_0^2) \in [0, m_0^2)$ and then the range where we choose $\omega$ is well defined.

By assumption $W^3'$, system (5) can be written as

$$\begin{cases}
-\Delta u + [m_0^2 - (\omega - e\phi)^2]u + N'(u) = 0 \\
-\Delta \phi = e(\omega - e\phi)u^2.
\end{cases}$$

which corresponds to system (KGM) with $g(s) = (\omega^2 - m_0^2)s - N'(s)$. So, by Theorem 0.1, we have just to show that $g$ satisfies $g_1, g_2$ and $g_3$ and $\omega \geq \omega_0$. As to the first two assumptions, it is easy to see that they are consequences of $W^2$ and $W^4'$. As to the third assumption, it comes from $W^3'$ and an easy computation. Finally, since by $W^1'$ for any $s > 0$ we have that $G(s) - \frac{1}{2}\omega^2s^2 = -W(s) \leq 0$, certainly $\omega \geq \omega_0$. \[\square\]

## 2 Proof of the main result: the case $\omega \in (0, \omega_0)$

First, some notations:

$H^1_r(\mathbb{R}^3)$ is the restriction to radial functions of the usual Sobolev space endowed with the norm

$$\|u\|^2 := \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) \, dx;$$

$\mathcal{D}^{1,2}_r(\mathbb{R}^3)$ is completion of radial functions in $C^\infty_0(\mathbb{R}^3)$ with respect to the norm

$$\|u\|^2_{\mathcal{D}^{1,2}_r(\mathbb{R}^3)} := \int_{\mathbb{R}^3} |\nabla u|^2 \, dx;$$
$L^p(\mathbb{R}^3)$ denote the usual Lebesgue space endowed with the norm
\[ \|u\|_p^p := \int_{\mathbb{R}^3} |u|^p \, dx; \]
$C^n(\mathbb{R}^3, \mathbb{R})$ is the space of continuous functions with continuous derivatives, till order $n$.

**Lemma 2.1.** If $u \in C^0(\mathbb{R}^3, \mathbb{R}) \cap H^1_r(\mathbb{R}^3)$ then there exists a unique $\phi \in \mathcal{D}^{1,2}_r(\mathbb{R}^3)$ solution of the equation
\[ -\Delta \phi = e(\omega - e\phi)u^2. \]
Moreover $\phi \in C^2(\mathbb{R}^3, \mathbb{R})$, it is radially decreasing and $0 < \phi < \frac{\omega}{e}$.

**Proof** The existence of a unique function $\phi \in \mathcal{D}^{1,2}_r(\mathbb{R}^3)$ which solves the equation has been proved for example in [5]. Since also $|\phi|$ solves the same equation, by uniqueness we have $\phi \geq 0$. Standard elliptic arguments prove that $\phi \in C^2(\mathbb{R}^3, \mathbb{R})$.

Now, by radial symmetry, if we set $\tilde{\phi}(r) = \phi(|x|)$ and $\tilde{u}(r) = u(|x|)$, then $\tilde{\phi} : \mathbb{R}_+ \to \mathbb{R}$ solves the Cauchy problem
\[
\begin{cases}
-\tilde{\phi}'' - \frac{2}{r}\tilde{\phi}' = e(\omega - e\tilde{\phi})\tilde{u}^2, & \text{for } r > 0, \\
\tilde{\phi}(0) = \eta, \\
\tilde{\phi}'(0) = 0,
\end{cases}
\]
for some value of $\eta \geq 0$.

From the equation we deduce that for any $r > 0$ we have
\[ -(r^2\tilde{\phi}')' = er^2(\omega - e\tilde{\phi})\tilde{u}^2. \]
Integrating in $(0, r)$ we get
\[ \tilde{\phi}'(r) = -\frac{e}{r^2} \int_0^r s^2(\omega - e\tilde{\phi}(s))\tilde{u}^2(s) \, ds, \text{ for any } r > 0. \tag{6} \]

We deduce that $\eta \leq \frac{\omega}{e}$, since otherwise $\tilde{\phi}'(r)$ should be positive for all $r > 0$ contradicting the fact that functions in $\mathcal{D}^{1,2}_r(\mathbb{R}^3)$ goes to zero at infinity. Of course $\eta \neq \frac{\omega}{e}$ since, otherwise, by uniqueness of solution of the Cauchy problem, we should have $\tilde{\phi} \equiv \frac{\omega}{e}$. Then, since $\eta < \frac{\omega}{e}$, certainly $\tilde{\phi}'(r) < 0$ for all $r > 0$ by (6), and as a consequence, $\tilde{\phi}$ is decreasing. Finally $\tilde{\phi} > 0$ since, if in some $R_0$ we had $\tilde{\phi}(R_0) = 0$, then by (6) there should exist $r > R_0$ such that $\tilde{\phi}(r) < 0$, contradicting the fact that $\tilde{\phi} \geq 0$.

**Remark 2.2.** The estimate on the $L^\infty$ norm of the solution to $-\Delta \phi = e(\omega - e\phi)u^2$ corresponding to an assigned $u \in H^1_r(\mathbb{R}^3)$ was firstly proved in [18] by means of arguments based on the maximum principle. Here we exploit continuity of $u$ and radial symmetry to provide an alternative proof “by hand”.
2.1 Existence of a positive solution

In this subsection we prove that there exists a solution \((\bar{\phi}, \bar{u})\) of \((KG\mathcal{M})\) such that \(\bar{u} > 0\).

Observe that, by the choice of \(\omega \in (0, \omega_0)\), certainly there exists \(s_0 > 0\) such that \(g(s_0) > 0\) and

\[
G(s_0) - \frac{\omega^2}{2}s_0^2 > 0. \tag{7}
\]

First we modify the function \(g\) according to the following two possibilities:

1st case: \(\lim \inf_{s \to +\infty} \frac{g(s)}{s^5} = 0\).

Then we define \(\tilde{g} = g_1 - g_2\) where

\[
g_1(s) = \begin{cases} 
(g(s) + ms)^+, & \text{if } s \geq 0, \\
0, & \text{if } s < 0,
\end{cases}
\]

and

\[
g_2(s) = \begin{cases} 
g_1(s) - g(s), & \text{if } s \geq 0, \\
-g_2(-s), & \text{if } s < 0.
\end{cases}
\]

2nd case: \(\lim \inf_{s \to +\infty} \frac{g(s)}{s^5} < 0\).

Then there exist \(\varepsilon > 0\) and an increasing diverging sequence of positive numbers \((s_n)_n\) such that \(g(s_n) \leq -\varepsilon s_n^5\). By (7) and continuity, certainly there exists \(\xi_0 > s_0\) such that \(g(\xi_0) + m\xi_0 = 0\). We set \(\tilde{g} = g_1 - g_2\) where

\[
g_1(s) = \begin{cases} 
(g(s) + ms)^+, & \text{if } s \in [0, \xi_0], \\
0, & \text{if } s \in [0, \xi_0]^c,
\end{cases}
\]

and

\[
g_2(s) = \begin{cases} 
g_1(s) - g(s), & \text{if } s \in [0, \xi_0], \\
ms, & \text{if } \xi_0 < s \\
-g_2(-s), & \text{if } s < 0.
\end{cases}
\]

It is standard to prove that if \((\phi, u)\) was a solution of \((KG\mathcal{M})\) with \(\tilde{g}\) in the place of \(g\), then \(u \geq 0\) and, if the second case occurred, we also should have \(u \leq \xi_0\). As a consequence \((\phi, u)\) should be a solution of the original problem. Then it is not restrictive assuming

\[
\lim_{s \to \pm\infty} \frac{g(s)}{s^5} = 0,
\]

up to replacing \(g\) with \(\tilde{g}\).

As a consequence, they are well defined and \(C^1\) in \(H^1_r(\mathbb{R}^3)\) the following
two functionals
\[ J_0(u) := \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 \, dx - \int_{\mathbb{R}^3} G(u) \, dx \]
and
\[ J_\omega(u) := \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + \omega^2 u^2) \, dx - \int_{\mathbb{R}^3} G(u) \, dx. \]   \tag{8}

Set
\[ c_0 := \inf_{\gamma \in \Gamma_0} \max_{t \in [0,1]} J_0(\gamma(t)) \]   \tag{9}
and
\[ c_\omega := \inf_{\gamma \in \Gamma_\omega} \max_{t \in [0,1]} J_\omega(\gamma(t)) \]
where \( \Gamma_i := \{ \gamma \in C([0,1], H^1_r(\mathbb{R}^3)) \mid \gamma(0) = 0, J_i(\gamma(1)) < 0 \} \), for \( i = 0, \omega \). Of course, by (7) the sets \( \Gamma_i \) are not empty, and the values \( c_i \) are well defined, for \( i = 0, \omega \) (see [30]).

**Lemma 2.3.** Pick any \( u_0 \in C^0(\mathbb{R}^3, \mathbb{R}) \cap H^1_r(\mathbb{R}^3) \) and set \( \phi_0 \in D_r(\mathbb{R}^3) \) the unique solution of
\[ -\Delta \phi = e(\omega - e\phi)u_0^2. \]
Then there exists a positive solution \( u_1 \in H^1_r(\mathbb{R}^3) \cap C^2(\mathbb{R}^3, \mathbb{R}) \) to the equation
\[ -\Delta u + e[2\omega - e\phi_0(x)]\phi_0(x)u - g(u) = 0, \quad \text{in } \mathbb{R}^3 \]
at the radial mountain pass level
\[ c_1 = \inf_{\gamma \in \Gamma_1} \max_{t \in [0,1]} J_1(\gamma(t)), \]
where \( \Gamma_1 := \{ \gamma \in C([0,1], H^1_r(\mathbb{R}^3)) \mid \gamma(0) = 0, J_1(\gamma(1)) < 0 \} \), and
\[ J_1(u) := \frac{1}{2} \int_{\mathbb{R}^3} [||\nabla u|^2 + e(2\omega - e\phi_0(x))\phi_0(x)u^2] \, dx - \int_{\mathbb{R}^3} G(u) \, dx. \]
Moreover \( c_0 \leqslant c_1 \leqslant c_\omega \).

**Proof** We write the equation in the following form
\[ -\Delta u + V_0(x)u - g(u) = 0 \quad \text{in } \mathbb{R}^3, \]
where \( V_0(x) = e[2\omega - e\phi_0(x)]\phi_0(x) \). By Lemma 2.1, it is easy to verify that \( V_0 \) is continuous together with its derivatives, nonnegative, radially decreasing and such that
\[ \lim_{x \to \infty} V_0(x) = 0. \]
Then the equation is of the type studied in [3] where it is proved the existence of a solution $u_1 \in H^1_0(\mathbb{R}^3)$ at the radial mountain pass level. Standard elliptic arguments prove that $u_1 \in C^2(\mathbb{R}^3, \mathbb{R})$. Moreover, since $u_1 \geq 0$ by our modification of $g$, the maximum principle implies that actually $u_1 > 0$. Finally, by Lemma 2.1, for any $u \in H^1_0(\mathbb{R}^3)$ we have $J_0(u) \leq J_1(u) \leq J_\omega(u)$ and, of course, $\Gamma_\omega \subset \Gamma_1 \subset \Gamma_0$. We deduce that obviously $c_0 \leq c_1 \leq c_\omega$. □

Now we iterate the process exposed in Lemma 2.3 considering

$$-\Delta \phi = e(\omega - e\phi)u_1^2$$

and finding $\phi_1 \in \mathcal{D}^{1,2}_r(\mathbb{R}^3) \cap C^2(\mathbb{R}^3, \mathbb{R})$ as in Lemma 2.1.

Again, as in Lemma 2.3 we find $u_2 \in H^1_0(\mathbb{R}^3) \cap C^2(\mathbb{R}^3, \mathbb{R})$ positive solution of the equation $-\Delta u + e[2\omega - e\phi_1(x)]\phi_1(x)u - g(u) = 0$ at the radial mountain pass level $c_2$ of the functional

$$J_2(u) := \frac{1}{2} \int_{\mathbb{R}^3} ||\nabla u|^2 + e(2\omega - e\phi_1(x))\phi_1(x)u^2| \, dx - \int_{\mathbb{R}^3} G(u) \, dx.$$

As before, we have $c_0 \leq c_2 \leq c_\omega$. Going on, we construct a couple of sequences, labeled $(u_n)_n$ and $(\phi_n)_n$, such that for any $n \geq 0$ we have $u_{n+1} > 0$ and

$$\begin{align*}
-\Delta u_{n+1} + e(2\omega - e\phi_n(x))\phi_n(x)u_{n+1} - g(u_{n+1}) &= 0 \\
-\Delta \phi_n &= e(\omega - e\phi_n)u_n^2
\end{align*}$$

and also

$$c_{n+1} = J_{n+1}(u_{n+1}) = \frac{1}{2} \int_{\mathbb{R}^3} ||\nabla u_{n+1}|^2 + e(2\omega - e\phi_n(x))\phi_n(x)u_{n+1}^2| \, dx - \int_{\mathbb{R}^3} G(u_{n+1}) \, dx,$$

with $c_0 \leq c_{n+1} \leq c_\omega$. Now, recall that, since $u_{n+1}$ is a solution of the first equation of (10), it satisfies the Pohozaev identity

$$\begin{align*}
\frac{1}{2} \int_{\mathbb{R}^3} |\nabla u_{n+1}|^2 \, dx + \frac{3}{2} \int_{\mathbb{R}^3} V_n(x)|u_{n+1}|^2 \, dx \\
+ \frac{1}{2} \int_{\mathbb{R}^3} (\nabla V_n(x) \cdot x)|u_{n+1}|^2 \, dx - 3 \int_{\mathbb{R}^3} G(u_{n+1}) \, dx &= 0
\end{align*}$$

where $V_n(x) = e(2\omega - e\phi_n(x))\phi_n(x)$. As a consequence, we have

$$J_{n+1}(u_{n+1}) = \frac{1}{3} \int_{\mathbb{R}^3} |\nabla u_{n+1}|^2 \, dx - \frac{1}{6} \int_{\mathbb{R}^3} (\nabla V_n(x) \cdot x)|u_{n+1}|^2 \, dx \quad (11)$$
and then
\[
c_0 \leq \frac{1}{3} \int_{\mathbb{R}^3} |\nabla u_{n+1}|^2 \, dx - \frac{1}{6} \int_{\mathbb{R}^3} (\nabla V_n(x) \cdot x)|u_{n+1}|^2 \, dx \leq c_\omega
\]
for any \( n \geq 0 \).

By Lemma 2.1 \( V_n \) is radially decreasing, so we deduce that \( (\|\nabla u_n\|_2) \) is bounded. By standard computations (see e.g. [3, Page 1364]) we have that \( (u_n) \) is bounded in \( H^1_r(\mathbb{R}^3) \) and then it possesses an extract (labeled at the same way) weakly and almost everywhere converging to \( \bar{u} \in H^1_r(\mathbb{R}^3) \).

From the second equation of (10), multiplying by \( \phi_n \) and integrating, we deduce that
\[
\|\phi_n\|_{D^{1,2}_r(\mathbb{R}^3)}^2 \leq e \int_{\mathbb{R}^3} (\omega - e\phi_n)\phi_n u_n^2 \, dx \leq c\|u_n\|_2^2
\]
which implies boundedness of \( (\phi_n) \). Up to subsequences, we can assume that \( (\phi_n) \) converges weakly and almost everywhere to \( \bar{\phi} \in D^{1,2}_r(\mathbb{R}^3) \).

Now, if we take \( \phi \) a \( C^\infty(\mathbb{R}^3,\mathbb{R}) \) function with compact support, by Holder and well known compact embedding theorems, we have (see e.g. computations in [2, Lemma 2.7])
\[
0 = J'_{n+1}(u_{n+1})[\phi] = \int_{\mathbb{R}^3} [(\nabla \bar{u} \cdot \nabla \phi) + e(2\omega - e\bar{\phi})\bar{\phi}\bar{u}\phi - g(\bar{u})\phi] \, dx
\]
and, from the second equation in (10)
\[
0 = \int_{\mathbb{R}^3} [(\nabla \phi_n \cdot \nabla \phi) - e(\omega - e\phi_n)u_n^2\phi] \, dx \rightarrow \int_{\mathbb{R}^3} [(\nabla \bar{\phi} \cdot \nabla \phi) - e(\omega - e\bar{\phi})\bar{u}^2\phi] \, dx.
\]

We conclude that the couple \( (\bar{\phi}, \bar{u}) \) solves system \( (KGM) \). Finally we are going to prove that this solution is not trivial. Indeed, we can write the first equation in (10) in the following way
\[
-\Delta u_{n+1} + e(2\omega - e\phi_n(x))\phi_n(x)u_{n+1} = g_1(u_{n+1}) - g_2(u_{n+1})
\]
where \( u \in H^1_r(\mathbb{R}^3) \mapsto \int_{\mathbb{R}^3} g_1(u)u \, dx \in \mathbb{R} \) is compact by Strauss radial Lemma.

On the other hand, we also know that
\[
-\Delta \bar{u} + e(2\omega - e\bar{\phi})\bar{\phi}\bar{u} = g_1(\bar{u}) - g_2(\bar{u}).
\]
By Fatou Lemma,

\[
\limsup_n \int_{\mathbb{R}^3} \left( |\nabla u_{n+1}|^2 + V_n(x) u_{n+1}^2 \right) dx = \limsup_n \int_{\mathbb{R}^3} \left[ g_1(u_{n+1}) - g_2(u_{n+1}) \right] u_{n+1} \, dx
\]

\[
= \int_{\mathbb{R}^3} g_1(\bar{u}) \, d\bar{u} \, dx - \liminf_n \int_{\mathbb{R}^3} g_2(u_{n+1}) u_{n+1} \, dx
\]

\[
\leq \int_{\mathbb{R}^3} g_1(\bar{u}) \, d\bar{u} \, dx - \int_{\mathbb{R}^3} g_2(\bar{u}) \, d\bar{u} \, dx
\]

\[
\leq \int_{\mathbb{R}^3} \left[ |\nabla \bar{u}|^2 + e(2\omega - e\bar{\phi})\bar{\phi}\bar{u}^2 \right] dx.
\]

On the other hand, by weak lower semicontinuity of the norms and Fatou Lemma,

\[
\int_{\mathbb{R}^3} |\nabla \bar{u}|^2 \, dx \leq \liminf_n \int_{\mathbb{R}^3} |\nabla u_{n+1}|^2 \, dx
\]

\[
\int_{\mathbb{R}^3} e(2\omega - e\bar{\phi})\bar{\phi}\bar{u}^2 \, dx \leq \liminf_n \int_{\mathbb{R}^3} V_n(x) u_{n+1}^2 \, dx.
\]

Then we deduce that

\[
\int_{\mathbb{R}^3} |\nabla \bar{u}|^2 \, dx = \lim_n \int_{\mathbb{R}^3} |\nabla u_{n+1}|^2 \, dx
\]

\[
\int_{\mathbb{R}^3} e(2\omega - e\bar{\phi})\bar{\phi}\bar{u}^2 \, dx = \lim_n \int_{\mathbb{R}^3} V_n(x) u_{n+1}^2 \, dx
\]

and, arguing as in [3], we prove also that \( u_n \to \bar{u} \) in \( L^2(\mathbb{R}^3) \) and then \( u_n \to \bar{u} \) in \( H^1_0(\mathbb{R}^3) \). Moreover, since the map \( \phi : H^1_0(\mathbb{R}^3) \to \mathcal{D}^{1,2}_r(\mathbb{R}^3) \) such that for any \( u \in H^1_0(\mathbb{R}^3) \) the function \( \phi(u) \) is the unique solution of second equation of (KGM) is continuous (see [5]), we deduce that \( \phi_n \to \bar{\phi} \) in \( \mathcal{D}^{1,2}_r(\mathbb{R}^3) \).

Finally, since \( 0 < c_0 \leq J_{n+1}(u_{n+1}) \), passing to the limit, by continuity we deduce that

\[
\frac{1}{2} \int_{\mathbb{R}^3} \left[ |\nabla \bar{u}|^2 + e(2\omega - e\bar{\phi})\bar{\phi}\bar{u}^2 \right] dx - \int_{\mathbb{R}^3} G(\bar{u}) \, dx \geq c_0,
\]

and then \( (\bar{\phi}, \bar{u}) \neq (0,0) \). Of course \( \bar{u} \geq 0 \) since it is the limit a.e. of a sequence of positive functions and \( \bar{u} > 0 \) by the maximum principle.

### 2.2 Multiplicity result

Here we exploit the symmetry property of the functionals to find a diverging sequence of critical levels. Since the proof is essentially the same of the
previous subsection, we just sketch it.
As previously, we need to modify the function $g$ in order to define a $C^1$ functional related with the problem. This time, we truncate $g$ as in [12] in order to preserve the oddness of $g$.
As before, we define the functionals $J_0$ and $J_{\omega}$. Now we go on as in [25] and define, for $i = 0, \omega$,

$$b_i^n = \inf_{\gamma \in \Gamma_n} \max_{\sigma \in D_n} J_i(\gamma(\sigma))$$

where $D_n = \{ \sigma = (\sigma_1, \ldots, \sigma_n) \in \mathbb{R}^n \mid |\sigma| \leq 1 \}$,

$$\Gamma_n = \left\{ \gamma \in C(D_n, H^1_r(\mathbb{R}^3)) \mid \begin{array}{l}
\gamma(-\sigma) = -\gamma(\sigma) \\
\gamma(\sigma) = \gamma_n(\sigma)
\end{array} \text{ for all } \sigma \in D_n \right\}$$

and $\gamma_n : \partial D_n \to H^1_r(\mathbb{R}^3)$ is the odd continuous map

$$\gamma_n : S^{n-1} = \{ \sigma = (\sigma_1, \ldots, \sigma_n) \in \mathbb{R}^n \mid |\sigma| = 1 \} \to H^1_r(\mathbb{R}^3),$$

such that

$$J_0(\gamma_n(\sigma)) \leq J_{\omega}(\gamma_n(\sigma)) < 0, \quad \text{for all } \sigma \in S^{n-1}.$$

For the existence of the map $\gamma_n$, see [13] and [25, Lemma 1.4].
By the geometry of the functionals, $0 < b_i^n$ for $i = 0, \omega$ and any $n \geq 1$.
Moreover, for any $n \geq 1$ we have $b_0^n \leq b_\omega^n$, so that they are well defined the intervals $I_n := [b_0^n, b_\omega^n]$. Since the sequence $(b_0^n)_n$ diverges (see [25]), up to subsequences we may assume that the intervals $I_n$ are disjoint.
As a consequence, to prove our result it is enough to show that for any $n \geq 1$ we can find a solution $(\phi^{(n)}, u^{(n)})$ of (KGMe) such that

$$\frac{1}{2} \int_{\mathbb{R}^3} |[\nabla u^{(n)}]^2 + e(2\omega - e\phi^{(n)})\phi^{(n)}(u^{(n)})^2| \, dx - \int_{\mathbb{R}^3} G(u^{(n)}) \, dx \in I_n.$$

Now, take $\tilde{n} \geq 1$ and set $I = I_{\tilde{n}}$ and $\Gamma = \Gamma_{\tilde{n}}$.
We repeat the iterative process starting from an arbitrary function $v_0 \in C^0(\mathbb{R}^3, \mathbb{R}) \cap H^1_r(\mathbb{R}^3)$ and setting $\psi_0 \in D^1_{r,2}(\mathbb{R}^3) \cap C^2(\mathbb{R}^3, \mathbb{R})$ the unique solution of

$$-\Delta \phi = e(\omega - e\phi)v_0^2.$$

Now, together with the functional $J_1$ defined as in previous subsection, following the method exposed in [25] and based on an idea in [27], we introduce the auxiliary functional $\tilde{J}_1 \in C^1(H^1_r(\mathbb{R}^3) \times \mathbb{R})$

$$\tilde{J}_1(u, \theta) := \frac{\exp[\theta]}{2} \int_{\mathbb{R}^3} |\nabla u|^2 \, dx$$

$$+ \frac{\exp[3\theta]}{2} \int_{\mathbb{R}^3} V_0(\exp[\theta]x) u^2 \, dx - \exp[3\theta] \int_{\mathbb{R}^3} G(u) \, dx,$$
where \( V_0(x) = e(2\omega - e\psi_0(x))\psi_0(x) \).
As in [25] we show that, if we define
\[
b_1 = b_1^\hat{n} := \inf_{\gamma \in \Gamma} \max_{\sigma \in D_n} J_1(\gamma(\sigma)),
\]
there exists a Palais Smale sequence \((u_m, \theta_m)_m\) for \( \tilde{J}_1\) at the level \( b_1 \), such that \( \theta_m \to 0 \). Observe that, since \( J_0 \leq J_1 \leq J_\omega \), certainly \( b_1 \in I \).
Since \( \tilde{J}_1(u_m, \theta_m) \to b_1 \) and \( \frac{\partial \tilde{J}_1}{\partial u}(u_m, \theta_m) \to 0 \), we deduce
\[
\frac{\exp[\theta_m]}{2} \int_{\mathbb{R}^3} |\nabla u_m|^2 \, dx + \frac{\exp[3\theta_m]}{2} \int_{\mathbb{R}^3} V_0(\exp[\theta_m]x)u_m^2 \, dx
\]
\[
- \exp[3\theta_m] \int_{\mathbb{R}^3} G(u_m) \, dx \to b_1 \tag{12}
\]
and
\[
\frac{\exp[\theta_m]}{2} \int_{\mathbb{R}^3} |\nabla u_m|^2 \, dx + \frac{3 \exp[3\theta_m]}{2} \int_{\mathbb{R}^3} V_0(\exp[\theta_m]x)u_m^2 \, dx
\]
\[
+ \frac{\exp[4\theta_m]}{2} \int_{\mathbb{R}^3} (\nabla V_0(\exp[\theta_m]x) \cdot x) |u_m|^2 \, dx
\]
\[
- 3 \exp[3\theta_m] \int_{\mathbb{R}^3} G(u_m) \, dx \to 0
\]
and then, comparing,
\[
\frac{\exp[\theta_m]}{3} \int_{\mathbb{R}^3} |\nabla u_m|^2 \, dx - \frac{\exp[4\theta_m]}{6} \int_{\mathbb{R}^3} (\nabla V_0(\exp[\theta_m]x) \cdot x) |u_m|^2 \, dx \to b_1.
\]
Since \( V_0 \) is radially decreasing and \( \theta_m \to 0 \), we deduce that \((\|\nabla u_m\|_2)_m \) is bounded.
Since \( \frac{\partial \tilde{J}_1}{\partial u}(u_m, \theta_m) \to 0 \) we have that
\[
- \exp[\theta_m] \Delta u_m + \exp[3\theta_m] V_0(\exp[\theta_m]x)u_m - \exp[3\theta_m]g(u_m) \to 0 \tag{13}
\]
in \((H^1_r(\mathbb{R}^3))'\).
From (13) we have
\[
\exp[\theta_m] \int_{\mathbb{R}^3} |\nabla u_m|^2 \, dx + \exp[3\theta_m] \int_{\mathbb{R}^3} V_0(\exp[\theta_m]x)|u_m|^2 \, dx
\]
\[
- \exp[3\theta_m] \int_{\mathbb{R}^3} g(u_m)u_m \, dx = o(1) \|u_m\|
which, taking into account that \((\|\nabla u_m\|_2)_m\) is bounded, by simple computations (see e.g. [3, Page 1364]) implies \((u_m)_m\) is bounded in \(H^1_r(\mathbb{R}^3)\). Since \(V_0(\exp[\theta_m] \cdot) \to V_0\) uniformly, by (12) and (13) we conclude that \((u_m)_m\) is a bounded Palais-Smale sequence for the functional \(J_1\) at the level \(b_1\).

By Strauss compactness radial Lemma and usual arguments we have that there exists \(v_1 \in H^1_r(\mathbb{R}^3)\) such that \(u_m \to v_1\). As a consequence \(v_1\) solves

\[-\Delta u + V_0(x)u = g(u)\]

with \(J_1(v_1) = b_1\). At this point we go on with the iterative process described in the previous subsection constructing the sequences \((\psi_j)_j\) and \((v_j)_j\) such that for any \(j \geq 0\)

\[
\begin{cases}
-\Delta v_{j+1} + e(2\omega - e\psi_j(x))\psi_j(x)v_{j+1} - g(v_{j+1}) = 0 \\
-\Delta \psi_j = e(\omega - e\psi_j)v_j^2
\end{cases}
\]

and also

\[
b_{j+1} = J_{j+1}(v_{j+1}) = \frac{1}{2} \int_{\mathbb{R}^3} [\|\nabla v_{j+1}\|^2 + e(2\omega - e\psi_j(x))\psi_j(x)v_{j+1}^2] \, dx - \int_{\mathbb{R}^3} G(v_{j+1}) \, dx;
\]

with \(b_{j+1} \in I\). Now, as in the previous section, we show that \((\psi_j, v_j)\) converges to a solution \((\psi, v)\) of \((KGM)\) and

\[
\frac{1}{2} \int_{\mathbb{R}^3} [\|\nabla v\|^2 + e(2\omega - e\psi(x))\psi(x)v^2] \, dx - \int_{\mathbb{R}^3} G(v) \, dx \in I.
\]

**Proof of Corollary 0.3** It is enough to observe that, is we set \(g(u) = (\omega^2 - m^2)u + |u|^{p-2}u\), then system (2) can be written as \((KGM)\). Now we conclude observing that \((g1), (g2)\) and \((g3)\) are satisfied and \(\omega_0 = +\infty\). □

### 3 Proof of the main result: the case \(\omega \geq \omega_0\)

First of all, we give the following

**Lemma 3.1.** There exists \(C > 0\) such that for any \(\tilde{u} \in H^1_1(\mathbb{R}^3)\), called \(\tilde{\phi} \in \mathcal{D}^{1,2}_r(\mathbb{R}^3)\) the solution of the equation

\[-\Delta \phi = e(\omega - e\phi)\tilde{u}^2\]

then

\[
\tilde{\phi}(x) \leq C\omega \frac{\|\tilde{u}\|_2}{2\sqrt{|x|}}. \tag{14}
\]
Proof By estimates on radial functions in $D^{1,2}(\mathbb{R}^3)$ (see [12, Lemma A.III.]), we know that there exists $C > 0$ such that for any $x \in \mathbb{R}^3$, 
\[
\tilde{\phi}(x) \leq C \frac{\|\tilde{\phi}\|_{D^{1,2}(\mathbb{R}^3)}}{\sqrt{|x|}}.
\]
On the other hand, since $-\Delta \tilde{\phi} = e(\omega - e\tilde{\phi})\tilde{u}^2$ and $0 \leq \tilde{\phi} \leq \frac{\omega}{e}$, we know that 
\[
\|\tilde{\phi}\|_{D^{1,2}(\mathbb{R}^3)}^2 = \int_{\mathbb{R}^3} e(\omega - e\tilde{\phi})\tilde{u}^2 \,dx \leq \left(\frac{\omega}{2}\|\tilde{u}\|_2\right)^2
\]
and then we conclude.

Consider 
\[
0 < \tilde{\omega} < \omega_0. 
\]
By definition of $\omega_0$, there exists $\tilde{s}$ such that 
\[
G(\tilde{s}) - \frac{\tilde{\omega}^2}{2}\tilde{s}^2 > 0. 
\]
We provide the same truncation of $g$ showed in the previous section. By [12] and (16), there exists $\tilde{z} \in H^1_r(\mathbb{R}^3)$ such that 
\[
\int_{\mathbb{R}^3} \left[G(\tilde{z}) - \frac{\tilde{\omega}^2}{2}\tilde{z}^2\right] \,dx > 0. 
\]
We apply our iterative argument starting from $u_0 \in C^0(\mathbb{R}^3, \mathbb{R}) \cap H^1_r(\mathbb{R}^3)$ and finding $\phi_0 \in C^2(\mathbb{R}^3, \mathbb{R}) \cap D^{1,2}(\mathbb{R}^3)$ the solution of 
\[-\Delta \phi = e(\omega - e\phi)u_0^2.\]
We define the functional $J_1$, the class of paths $\Gamma_1$ and the mountain pass level $c_1$ as in Lemma 2.3. Repeating the arguments of Lemma 2.3 we also prove the existence of a critical point $u_1$ for $J_1$ at the level $c_1$. Since for $\bar{t} > 0$ large enough the path 
\[
\gamma(t) = \begin{cases} 
\tilde{z}\left(\frac{\bar{t}}{t}\right) & \text{if } t \in (0,1] \\
0 & \text{if } t = 0 
\end{cases}
\]
belongs to $\Gamma_1$, we have that 
\[
c_1 \leq \max_{0 < t < 1} J_1(\gamma(t)) = \max_{0 < t < 1} \left[\frac{t}{2} \int_{\mathbb{R}^3} |\nabla \tilde{z}|^2 \,dx 
+ \frac{t^3}{2} \int_{\mathbb{R}^3} e(2\omega - e\phi_0(tx))\phi_0(tx)\tilde{z}^2 \,dx - t^3 \int_{\mathbb{R}^3} G(\tilde{z}) \,dx\right] 
\]
where we have set $\tilde{z} = \overline{z} (\cdot \overline{t})$.

By (14) and $0 \leq \phi_0 \leq \frac{\omega}{2}$, we have the following estimate

$$V_0(x) = e(2\omega - e\phi_0(x))\phi_0(x) \leq \begin{cases} \omega^2 & \text{if } |x| \geq \left( \frac{C\omega^2}{\omega^2} \right)^2 \|u_0\|_2^2 \\
\omega^2 & \text{if } |x| < \left( \frac{C\omega^2}{\omega^2} \right)^2 \|u_0\|_2^2 \end{cases}$$

so that, by (18),

$$c_1 \leq \max_{t>0} \left[ \frac{t}{2} \int_{\mathbb{R}^3} |\nabla \tilde{z}|^2 \, dx \\
+ \frac{t^3}{2} \int_{\mathbb{R}^3} e(2\omega - e\phi_0(tx))\phi_0(tx)\tilde{z}^2 \, dx - t^3 \int_{\mathbb{R}^3} G(\tilde{z}) \, dx \right]
\leq \max_{t>0} \left[ \frac{t}{2} \int_{\mathbb{R}^3} |\nabla \tilde{z}|^2 \, dx \\
+ \frac{t^3\omega^2}{2} \int_{|x|<\frac{\omega}{t} \left( \frac{\omega^2}{\omega} \right)^2 \|u_0\|_2^2} \tilde{z}^2 \, dx - t^3 \int_{\mathbb{R}^3} \left( G(\tilde{z}) - \frac{\omega^2}{2} \tilde{z}^2 \right) \, dx \right] := d_1. \quad (19)$$

On the other hand, as in (11), we have

$$c_1 = J_1(u_1) = \frac{1}{3} \int_{\mathbb{R}^3} |\nabla u_1|^2 \, dx - \frac{1}{6} \int_{\mathbb{R}^3} (\nabla V_0(x) \cdot x)|u_1|^2 \, dx$$

and then, $\|\nabla u_1\|_2^2 \leq 3c_1$. As in [3, Page 1364] we show that, for some $D > 0$ which depends only on the function $g$, any critical point $w$ of $J_1$ satisfies

$$\|w\|_2^2 \leq D\|\nabla w\|_2^6$$

so that

$$\|u_1\|_2^2 \leq 3c_1 + 27D(c_1)^3. \quad (20)$$

Now, using (19) and taking into account (17), we have that, if $\|u_0\|$ is sufficiently large and $\epsilon$ sufficiently small, then

$$3d_1 + 27D(d_1)^3 \leq \|u_0\|^2$$

which implies $\|u_1\| \leq \|u_0\|$.

Repeating the procedure, we find $\phi_1$ solution of

$$-\Delta \phi = e(\omega - e\phi)u_1^2,$$

define the functional $J_2$, the class of paths $\Gamma_2$ and the mountain pass level $c_2$. We obtain $u_2$ critical point of $J_2$ at the level $c_2$ and, proceeding as in (18)-(19), we arrive at the following estimate on $c_2$
\[ c_2 \leq \max_{t>0} \left[ \frac{t}{2} \int_{\mathbb{R}^3} |\nabla \bar{z}|^2 \, dx + \frac{t^3}{2} \int_{|x|<e^2 \frac{\omega^2}{2\gamma^2}} \omega^2 \bar{z}^2 \, dx - t^3 \int_{\mathbb{R}^3} \left( G(\bar{z}) - \frac{\omega^2}{2} \bar{z}^2 \right) \, dx \right] : = d_2. \]

Now observe that, since \( \|u_1\| \leq \|u_0\| \), we have \( d_2 \leq d_1 \). Then as in (20), we have

\[ \|u_2\|^2 \leq 3c_2 + 27D^3(c_2)^3 \leq 3d_2 + 27D^3(d_2)^3 \leq 3d_1 + 27D^3(d_1)^3 \leq \|u_0\|^2. \]

Going on, we build as usual the sequences \((u_n)_n\) in \( H_r^1(\mathbb{R}^3) \) and \((\phi_n)_n\) in \( D_r^{1.2}(\mathbb{R}^3) \) and, since for any \( n \geq 0 \) we have \( \|u_{n+1}\| \leq \|u_0\| \), the sequence \((u_n)_n\) is bounded and we can argue as in the previous section and find a solution \((\phi, u)\) such that

\[ c_0 \leq \frac{1}{2} \int_{\mathbb{R}^3} [\|\nabla u\|^2 + e(2\omega - e\phi(x))\phi(x)u^2] \, dx - \int_{\mathbb{R}^3} G(u) \, dx \leq d_1, \quad (21) \]

where \( c_0 \) is the same defined in (9).

Now we show how we find two solutions, being the generalization to \( k \) solutions easily deducible.

Define \( D_n \) as in subsection 2.2 and

\[ b_0^n = \inf_{\gamma \in \Gamma_n} \max_{\sigma \in D_n} J_0(\gamma(\sigma)) \]

where

\[ \Gamma_n = \left\{ \gamma \in C(D_n, H_r^1(\mathbb{R}^3)) \mid \begin{array}{ll} \gamma(-\sigma) = -\gamma(\sigma) & \text{for all } \sigma \in D_n \\ \gamma(\sigma) = \gamma_n(\sigma) & \text{for all } \sigma \in \partial D_n \end{array} \right\} \]

and \( \gamma_n : \partial D_n \to H_r^1(\mathbb{R}^3) \) is the odd continuous map

\[ \gamma_n : S^{n-1} = \{ \sigma = (\sigma_1, \cdots, \sigma_n) \in \mathbb{R}^n \mid |\sigma| = 1 \} \to H_r^1(\mathbb{R}^3), \]

such that, recalling the definition of \( \bar{\omega} \) in (15),

\[ J_0(\gamma_n(\sigma)) \leq J_\bar{\omega}(\gamma_n(\sigma)) < 0, \quad \text{for all } \sigma \in S^{n-1}. \]

(Here \( J_\bar{\omega} \) is defined as (8) replacing \( \omega \) with \( \bar{\omega} \). The existence of \( \gamma_n \) is guaranteed by [13]. Since \((b_0^n)_n\) is diverging, we can find \( \bar{n} \geq 0 \) such that \( b_0^{\bar{n}} > d_1 \).
Then we apply the iterative argument, starting from \( u_0 \in C^0(\mathbb{R}^3, \mathbb{R}) \cap H^1_\omega(\mathbb{R}^3) \) and calling as usual \( \phi_0 \) the corresponding solution to
\[
-\Delta \phi = e(\omega - e\phi)u_0^2.
\]

We define \( J_1 \). Up to rescaling, we can assume that
\[
J_1(\gamma_\delta(\sigma)) < 0, \quad \text{for all } \sigma \in S^{n-1},
\]
so that it is well defined the minmax value
\[
b_1^\delta = \inf_{\gamma \in \Gamma^\delta} \max_{\sigma \in D_\delta} J_1(\gamma(\sigma)).
\]

As in subsection 2.2 we find a critical point \( u_1 \) for \( J_1 \) at the level \( b_1^\delta \).

We construct the following map \( \bar{\eta}_n \in \Gamma^\delta \)
\[
\bar{\eta}_n(\sigma) = \begin{cases}
\pi_n(\sigma/|\sigma|) & \text{if } \sigma \in D_\delta \setminus (0,0,\ldots,0) \\
0 & \text{if } \sigma = (0,0,\ldots,0)
\end{cases}
\]
where the odd continuous map \( \pi_n : S^{n-1} \to H^1_\omega(\mathbb{R}^3) \) is defined in [13, Theorem 10] and \( \bar{t} \) is chosen such that \( \gamma_\delta(\sigma) := \pi_n(\sigma/\bar{t}) \) for every \( \sigma \in S^{n-1} \) (see [25, Lemma 1.4]). Without loss of generality, we can assume \( \bar{t} = 1 \), so that, by estimate (14), we have
\[
b_1^\delta \leq \max_{0 < t < 1} \max_{\sigma \in S^{n-1}} J_1(\bar{\eta}_n(\sigma)) = \max_{0 < t < 1} \max_{\sigma \in S^{n-1}} J_1(\pi_n(\sigma)(\cdot/t))
\]
\[
= \max_{0 < t \leq 1} \max_{\sigma \in S^{n-1}} \left[ \frac{t}{2} \int_{\mathbb{R}^3} |\nabla \pi_n(\sigma)|^2 \, dx + \frac{t^3}{2} \int_{\mathbb{R}^3} e(2\omega - e\phi_0(tx))\phi_0(tx)\pi_n(\sigma)^2 \, dx - \frac{t^3}{3} \int_{\mathbb{R}^3} G(\pi_n(\sigma)) \, dx \right]
\]
\[
\leq \max_{\sigma \in S^{n-1}} \left[ \frac{t}{2} \int_{\mathbb{R}^3} |\nabla \pi_n(\sigma)|^2 \, dx + \frac{t^3}{2} \int_{|x| < \omega (\bar{t} \omega)^2 \|u_0\|^2} \omega^2 |\pi_n(\sigma)|^2 \, dx - \frac{t^3}{3} \int_{\mathbb{R}^3} \left( G(\pi_n(\sigma)) - \frac{\bar{t}^2}{2} |\pi_n(\sigma)|^2 \right) \, dx \right] := h_1.
\]

We point out that, since \( J_2(\gamma_\delta(\sigma)) < 0 \), for all \( \sigma \in S^{n-1} \), we have \( h_1 \in \mathbb{R} \).

As before we can find \( \|u_0\| \) sufficiently large and \( e \) sufficiently small such that
\[
\|u_1\|^2 \leq 3b_1^\delta + 27D(b_1^\delta)^3 \leq 3h_1 + 27D(h_1)^3 \leq \|u_0\|^2.
\]
The iteration then guarantees the existence of a solution \((\phi^{(n)}, u^{(n)})\) such that
\[
b_0^{(n)} \leq \frac{1}{2} \int_{\mathbb{R}^3} \left[ |\nabla u^{(n)}|^2 + e(2\omega - e\phi^{(n)}(x))\phi^{(n)}(x)|u^{(n)}|^2 \right] dx - \int_{\mathbb{R}^3} G(u^{(n)}) \, dx \leq h_1,
\]
which, together with (21) and the fact that \(d_1 < b_0^{(n)}\), ensures \((\phi^{(n)}, u^{(n)}) \neq (\phi, u)\).

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