Notes on Moduli theory, Stacks and 2-Yoneda’s Lemma

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Abstract

This note is a survey on the basic aspects of moduli theory along with some examples. In that respect, one of the purposes of this current document is to understand how the introduction of stacks circumvents the non-representability problem of the corresponding moduli functor \( F \) by using the 2-category of stacks. To this end, we shall briefly revisit the basics of 2-category theory and present a 2-categorical version of Yoneda’s lemma for the "refined" moduli functor \( F \). Most of the material below are standard and can be found elsewhere in the literature. For an accessible introduction to moduli theory and stacks, we refer to [1, 3]. For an extensive treatment to the case of moduli of curves, see [5, 6, 8]. Basics of 2-category theory and further discussions can be found in [7].

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1 Functor of points, representable functors, and Yoneda’s Lemma

Main aspects of a moduli problem of interest can be encoded by a certain functor, namely a moduli functor of the form

\[ \mathcal{F} : \mathcal{C}^{\text{op}} \rightarrow \text{Sets} \quad (1.1) \]

where \( \mathcal{C}^{\text{op}} \) is the opposite category of the category \( \mathcal{C} \), and \( \text{Sets} \) denotes the category of sets. Equivalently, it is just a contravariant functor from the category \( \mathcal{C} \) to \( \text{Sets} \). The existence of a fine moduli space corresponds to the representability of this moduli functor. More details are to be discussed below.

Definition 1.0.1. Let \( \mathcal{C} \) be a category. For any object \( U \) in \( \mathcal{C} \) we define a functor

\[ h_U : \mathcal{C}^{\text{op}} \rightarrow \text{Sets} \quad (1.2) \]

as follows:

1. For each object \( X \in \text{Ob}(\mathcal{C}), \quad X \mapsto h_U(X) := \text{Mor}_\mathcal{C}(X,U) \)

2. For each morphism \( X \xrightarrow{f} Y, \)

\[ \left( X \xrightarrow{f} Y \right) \mapsto \left( \text{Mor}_\mathcal{C}(Y,U) \xrightarrow{f^*} \text{Mor}_\mathcal{C}(X,U), \quad g \mapsto g \circ f \right). \]

This functor \( h_U \) is called "the Yoneda functor" or "the functor of points".
As we shall see below, this functor can be used to recover any object $U$ of a category $\mathcal{C}$ by understanding morphisms into it via Yoneda’s Lemma. Let $\text{Fun}(\mathcal{C}^{\text{op}}, \text{Sets})$ denote the category of functors from $\mathcal{C}^{\text{op}}$ to $\text{Sets}$ with objects being functors $\mathcal{F} : \mathcal{C}^{\text{op}} \rightarrow \text{Sets}$ and morphisms being natural transformations between functors, then using the definition of $h_U$, one can introduce the following functor as well:

$$h : \mathcal{C} \rightarrow \text{Fun}(\mathcal{C}^{\text{op}}, \text{Sets})$$

(1.3)

where

1. For each object $U \in \text{Ob}(\mathcal{C})$, $U \mapsto h_U = \text{Mor}_\mathcal{C}(\cdot, U)$

2. To each morphism $U \xrightarrow{f} V$, $h$ assigns a natural transformation

$$\begin{array}{ccc}
\mathcal{C}^{\text{op}} & \xrightarrow{h_f} & \text{Sets}.
\end{array}$$

(1.4)

Here $h_f$ is defined as follows:

(a) For each object $X$ in $\mathcal{C}$, we set

$$h_f(X) : h_U(X) \rightarrow h_V(X), \ g \mapsto f \circ g.$$  

(1.5)

(b) Given a morphism $X \xrightarrow{\eta} Y$ in $\mathcal{C}$, from the associativity property of the composition map, the diagram

$$\begin{array}{ccc}
& & h_U(X) \\
& f \circ \eta & \downarrow \circ \eta \\
h_U(Y) & \xrightarrow{f \circ} & h_V(Y) \\
\circ \eta & & \circ \eta \\
& & h_U(X)
\end{array}$$

(1.6)

commutes.

**Definition 1.0.2.** [4] Let $\mathcal{C}, \mathcal{D}$ be two categories.

1. A functor $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{D}$ is called **fully faithful** if for any objects $A, B \in \mathcal{C}$, the map

$$\text{Hom}_\mathcal{C}(A, B) \rightarrow \text{Hom}_\mathcal{D}(\mathcal{F}(A), \mathcal{F}(B))$$

(1.7)

is a bijection of sets.

2. A functor $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{D}$ is called **essentially surjective** if for any objects $D \in \mathcal{D}$, there exists an object $A$ in $\mathcal{C}$ such that one has an isomorphism of objects

$$\mathcal{F}(A) \sim \rightarrow D.$$  

(1.8)

**Lemma 1.0.1.** *(Yoneda’s Lemma)* The functor $h$ above is fully faithful.

**Remark 1.0.1.**

1. Yoneda’s lemma implies that the functor $h$ serves as an embedding (sometimes it is also called *Yoneda’s embedding*), and hence $h_U$ determines $U$ up to a unique isomorphism. Therefore, one can recover any object $U$ in $\mathcal{C}$ by just knowing all possible morphisms into $U$. In the case of the category $\text{Sch}_\mathcal{C}$ of $\mathcal{C}$-schemes, for instance, it is enough to study the restriction of this functor to the full subcategory $\text{Aff}_\mathcal{C}$ of affine $\mathcal{C}$-schemes, in order to recover the scheme $U$.

2. Thanks to the Yoneda’s embedding, one can also realize some algebro-geometric objects (like schemes, stacks, derived "spaces", etc...) as a certain functor in addition to standard ringed-space formulation. We have the following enlightening diagram [2] encoding such a functorial interpretation:
One way of interpreting this diagram is as follows: In the case of schemes (stacks resp.), for instance, such a functorial description implies that points of a scheme (a stack resp.) X form a set (a groupoid resp.). These kind of interpretations, in fact, suggest the name "functor of points".

3. The bad news is that not all functors $F : \mathcal{C}^{op} \to \text{Sets}$ are of the form $h_U$ for some $U$ in a general set-up. In other words, $h$ is not essentially surjective in general. This in fact leads to the following definition:

**Definition 1.0.3.** A functor $F : \mathcal{C}^{op} \to \text{Sets}$ is called representable if there exists $M \in \text{Ob}(\mathcal{C})$ such that we have a natural isomorphism $F \cong h_M$. That is,

$$F = \text{Mor}_{\mathcal{C}}(\cdot, M)$$ for some $M \in \text{Ob}(\mathcal{C})$. (1.9)

If this is the case, then we say that $F$ is represented by $M$. In the case of moduli theory, $M$ is then called a fine moduli space. In the next section, we shall investigate the properties of $M$.

## 2 Moduli theory in functorial perspective

A moduli problem is a problem of constructing a classifying space (or a moduli space $\mathcal{M}$) for certain geometric objects (such as manifolds, algebraic varieties, vector bundles etc...) up to their intrinsic symmetries. In other words, a moduli space serves as a solution space of a given moduli problem of interest. In general, the set of isomorphism classes of objects that we would like to classify may not be able to provide a sufficient information to encode geometric properties of the moduli space itself. Therefore, we expect a moduli space to behave well enough to capture the underlying geometry. Thus, this expectation leads to the following wish-list for $\mathcal{M}$ to be declared as a "fine" moduli space:

1. $\mathcal{M}$ is supposed to serve as a parameter space in a sense that there must be a one-to-one correspondence between the points of $\mathcal{M}$ and the set of isomorphism classes of objects to be classified:

$$\{\text{points of } \mathcal{M}\} \leftrightarrow \{\text{isomorphism classes of objects in } \mathcal{C}\}$$ (2.1)

2. One ensures the existence of universal classifying object, say $\mathcal{T}$, through which all other objects parametrized by $\mathcal{M}$ can also be reconstructed. This, in fact, makes the moduli space $\mathcal{M}$ even more sensitive to the behavior of "families" of objects on any base object $B$. It is manifested by a certain representative morphism $B \to \mathcal{M}$. That is, for any family

$$\pi : X \to B$$ (2.2)

parametrized by some base scheme $B$ where

$$X := \{X_b \in \text{Ob}(\mathcal{C}) : \pi^{-1}(b) = X_b, \ b \in B\},$$

there exits a unique morphism $f : B \to \mathcal{M}$ such that one has the following fibered product diagram:

$$\begin{array}{ccc}
X & \to & \mathcal{T} \\
\pi \downarrow & & \downarrow \\
B & \to & \mathcal{M}
\end{array}$$ (2.3)

where $X = B \times_{\mathcal{M}} \mathcal{T}$. That is, the family $X$ can be uniquely obtained by pulling back the universal object $\mathcal{T}$ along the morphism $f$. 

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For an accessible overview, see [1]. Relatively complete treatments can be found in [9, 3].

Remark 2.0.1. More formally, a family over a base $B$ is a scheme $X$ together with a morphism $\pi : X \to B$ of schemes where for each (closed point) $b \in B$ the fiber $X_b$ is defined as fibered product

$$X_b = \{b\} \times_B X \xrightarrow{\pi} X$$

$$\{b\} \xrightarrow{\imath} B$$

where $\imath : \{b\} \hookrightarrow B$ is the usual inclusion map.

In the language of category theory, on the other hand, a moduli problem can be formalized as a certain functor

$$\mathcal{F} : \mathcal{C}^{\text{op}} \to \text{Sets},$$

which is called a moduli functor where $\mathcal{C}^{\text{op}}$ denotes the opposite category of the category $\mathcal{C}$, and $\text{Sets}$ is the category of sets. In other words, it is just a contravariant functor from the category $\mathcal{C}$ to $\text{Sets}$. In order to make the argument more transparent, we take $\mathcal{C}$ to be the category $\text{Sch}_\mathcal{C}$ of $\mathcal{C}$-schemes unless otherwise stated. Note that for each $\mathcal{C}$-scheme $U \in \text{Sch}_\mathcal{C}$, $\mathcal{F}(U)$ is the set of isomorphism classes (of families) parametrized by $U$. For each morphism $f : U \to V$ of schemes, we have a morphism $\mathcal{F}(f) : \mathcal{F}(V) \to \mathcal{F}(U)$ of sets.

Example 2.0.1. Given a scheme $U$, one can define $\mathcal{F}(U) := S(U)/\sim$ where $S(U)$ is the set of families over the base scheme $U$

$$S(U) := \left\{ X \to U : X \text{ is a scheme over } U, \text{each fiber } X_u \text{ is } C_g \forall u \in U \right\}$$

where $C_g$ is a smooth, projective, algebraic curve of genus $g$. We say that two families $\pi : X \to U$ and $\pi' : Y \to U$ over $U$ are equivalent if and only if there exists an isomorphism $f : X \simeq Y$ of schemes such that the following diagram commutes:

$$\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{\pi} & & \downarrow{\pi'} \\
U & \xrightarrow{\imath} & U
\end{array}$$

On morphisms $\phi : U \to V$, on the other hand, we have

$$\mathcal{F}(\phi) : \mathcal{F}(V) \to \mathcal{F}(U), \quad [X \to V] \mapsto [U \times_V X \to U]$$

where $U \times_V X$ is the fibered product given by pulling back the family $X \to V$ along the morphism $\phi : U \to V$:

$$\begin{array}{ccc}
U \times_V X & \xrightarrow{\phi} & X \\
\downarrow{\imath} & & \downarrow{\pi} \\
U & \xrightarrow{f} & V
\end{array}$$

With the above formalism in hand, the existence of a fine moduli space, therefore, corresponds to the representability of the moduli functor $\mathcal{F}$ in the sense that

$$\mathcal{F} = \text{Mor}_{\text{Sch}_\mathcal{C}}(\cdot, \mathcal{M}) \text{ for some } \mathcal{M} \in \text{Sch}_\mathcal{C}.$$ (2.9)

If this is the case, then we say that $\mathcal{F}$ is represented by $\mathcal{M}$.
Remark 2.0.2. Let \( F : \mathcal{C}^{op} \to \text{Sets} \) be a moduli functor represented by an object \( M \), then one can recast the desired properties of being a "fine" moduli space as follows:

1. Take \( B := \text{spec}(\mathbb{C}) = \{\ast\} \), then from the representability we have
   \[
   F(\{\ast\}) \cong h_M(\{\ast\}) = \text{Mor}_\mathcal{C}(\{\ast\}, M). \tag{2.10}
   \]
   Note that the RHS is just the set of (closed) points of \( M \), and LHS is the set of corresponding isomorphism classes.

2. When \( B := M \), then we get an isomorphism
   \[
   F(M) \cong h_M(M) = \text{Mor}_\mathcal{C}(M, M), \tag{2.11}
   \]
   which allows us to define the universal object \( T \) to be the object corresponding to the identity morphism \( \text{id}_M \in \text{Mor}_\mathcal{C}(M, M) \).

These observations yield the following corollary:

Corollary 2.0.1. If \( F : \mathcal{C}^{op} \to \text{Sets} \) is a moduli functor represented by an object \( M \) in \( \mathcal{C} \), then there exists an one-to-one correspondence between the set \( \{ X \to B \} \) of (equivalence classes of) families and \( \text{Mor}_\mathcal{C}(B, M) \). That is,

\[
\{ X \to B \}/ \sim \longleftrightarrow \text{Mor}_\mathcal{C}(B, M). \tag{2.12}
\]

Furthermore, for a morphism \( f : B \to M \) corresponding to the equivalence class \([ X \to B ]\) of the family \( X \to B \), we have

\[
[X \to B] = [B \times_M T \to B]. \tag{2.13}
\]

In many cases, however, a moduli functor is not representable in the category \( \text{Sch} \) of schemes. This is the place where the notion of a stack comes into play. In that situation, one can still make sense of the notion of a moduli space in a weaker sense. This version, namely a coarse moduli space, is still efficient enough to encode the isomorphism classes of points. That is, it has the correct points, and captures the geometry of moduli space. However, the sensitivity on the behavior of arbitrary families is no longer available. In other words, a coarse moduli space may not be able to distinguish two non-isomorphic families in many cases. Hence, the classification in this "family-wise" level is by no means possible. To elaborate the last statement, we first introduce the formal definition of a so-called coarse moduli space, and then we shall provide two important examples: (i) the moduli problem of classifying vector bundles of fixed rank over an algebraic curve over a field \( k \) [3], and (ii) the moduli of elliptic curves [1, 5].

Definition 2.0.1. Let \( \mathcal{C} := \text{Sch}_\mathbb{C} \) for the sake of simplicity. A coarse moduli space for a moduli functor \( F : \mathcal{C}^{op} \to \text{Sets} \) consists of a pair \((M, \psi)\) where \( M \) is an object in \( \mathcal{C} \), and \( \psi : F \to h_M \) is a natural transformation such that

1. \( \psi_{\text{spec}(\mathbb{C})} : F(\text{spec}(\mathbb{C})) \to h_M(\text{spec}(\mathbb{C})) \) is a bijection of sets.

2. Such a pair \((M, \psi)\) satisfies the following universal property: For any scheme \( N \) and any natural transformation \( \phi : F \to h_N \), there exists a unique morphism \( f : M \to N \) such that the following diagram commutes:

\[
\begin{array}{ccc}
F & \xrightarrow{\psi} & h_M \\
\phi \downarrow & & \downarrow \exists h_f \\
h_N & & h_N
\end{array}
\]

(2.14)

Remark 2.0.3. Here, \( h_f : h_M \to h_N \) is the associated natural transformation of functors 1.3. Second condition also implies that if it exists, a coarse moduli space \( M \) for a moduli functor \( F \) is unique up to a unique isomorphism.
Proposition 2.0.1. [9] Let $(M, \psi)$ be a coarse moduli space for a moduli functor $F : \mathcal{C}^{op} \rightarrow \text{Sets}$ where $M$ is a scheme and $\psi : F \rightarrow h_M$ is the corresponding natural transformation. Then $(M, \psi)$ is a fine moduli space if and only if the following conditions hold:

1. There exists a family $T \rightarrow M$ such that $\psi_M(T) = \text{id}_M \in \text{Mor}_\mathcal{C}(M, M)$.
2. For families $X \rightarrow B$ and $Y \rightarrow B$ on a base scheme $B$,
   \[ [X \rightarrow B] = [Y \rightarrow B] \iff \psi_B(X) = \psi_B(Y). \] (2.15)

Proof. It follows directly from the definition of a fine moduli space. \(\square\)

2.1 Moduli of vector bundles of fixed rank

We would like to investigate the moduli problem of classifying vector bundles of fixed rank over a smooth, projective algebraic curve $X$ of genus $g$ over a field $k$ with $\text{char } k = 0$. We define the corresponding moduli functor $F_X^n : \text{Sch}_{\mathcal{C}}^{op} \rightarrow \text{Sets}$ (2.16) as follows:

To each object $U$ in $\text{Sch}_{\mathcal{C}}$, $F_X^n$ assigns the set $F_X^n(U)$ of isomorphism classes of families of vector bundles of rank $n$ on $X$ parametrized by $U$. That is,

\[ F_X^n(U) = \{ E \rightarrow U \times_{\text{spec} \mathbb{C}} X : E \text{ is a vector bundle of rank } n \} / \sim \]

Here, we say that two vector bundles $\pi : E \rightarrow U \times_{\text{spec} \mathbb{C}} X$ and $\pi' : E' \rightarrow U \times_{\text{spec} \mathbb{C}} X$ over $U \times_{\text{spec} \mathbb{C}} X$ are equivalent if and only if there exists an isomorphism $f : E \rightarrow E'$ of vector bundles such that the following diagram commutes:

\[ \begin{array}{ccc} E & \xrightarrow{f} & E' \\ \downarrow & & \downarrow \pi' \\ U \times_{\text{spec} \mathbb{C}} X & \xrightarrow{\pi} & X \end{array} \] (2.17)

To each morphism $f : U \rightarrow V$ in $\text{Sch}_{\mathcal{C}}$, $F_X^n$ assigns the map of vector bundles

\[ F_X^n(f) : F_X^n(V) \rightarrow F_X^n(U) \] (2.18)

which is defined by pulling back of the vector bundle $E \rightarrow V \times_{\text{spec} \mathbb{C}} X$ along the morphism $f \times \text{id}_X$. Note that $U \times_{\text{spec} \mathbb{C}} X$ is just the usual direct product $U \times X$ with the projection maps $\text{pr}_1 : U \times X \rightarrow U$ and $\text{pr}_2 : U \times X \rightarrow X$ such that

\[ \begin{array}{ccc} U \times_{\text{spec} \mathbb{C}} X & \xrightarrow{\text{pr}_2} & X \\ \downarrow \text{pr}_1 & & \downarrow \\ U & \rightarrow & \text{spec} \mathbb{C} \end{array} \] (2.19)

Now, we would like to show that $F_X^n$ can not be representable by some scheme $M$.

Claim: $F_X^n$ is not representable in $\text{Sch}_{\mathcal{C}}$.

Proof. Assume that $F_X^n$ is representable by a scheme $M$. That is, we have a natural isomorphism

\[ F_X^n \cong h_M. \] (2.20)

Let $U$ be a scheme and $E \in F_X^n(U)$. Then, we have a vector bundle $E \rightarrow U \times X$ of rank $n$. Let $L$ be a line bundle over $U$, then we can define the induced bundle $\text{pr}_1^* L$ on $U \times X$ by pulling back $L$ along the projection map $\text{pr}_1$. Therefore, we obtain a particular vector bundle

\[ E' := E \otimes \text{pr}_1^* L \rightarrow U \times X. \] (2.21)
Indeed, $E'$ is a twisted bundle where each fiber of $E'$ is obtained by multiplying each fiber of $E$ by a scalar. Hence, we produce a new vector bundle $E'$ by twisting $E$ such that $E'$ is not (globally) isomorphic to $E$. Let $\{U_i\}$ be a local trivializing cover of $U$ such that $L|_{U_i}$ is trivial. Then it follows from the definition of $E'$ that

$$E'|_{U_i \times X} \cong E|_{U_i \times X} \quad \forall i. \quad (2.22)$$

As $\mathcal{F}_X^\circ$ is representable by a scheme $M$, there are morphisms $f_E : U \to M$ and $f_{E'} : U \to M$ in $Mor_{Sch_c}(U, M)$ corresponding to $E$ and $E'$ respectively such that

$$f_{E'}|_{U_i \times X} = f_E|_{U_i \times X} \quad \forall i. \quad (2.23)$$

Since $h_M$ is a sheaf, it follows from the gluing axiom that all such morphisms are glued together nicely such that

$$f_E|_{U \times X} = f_{E'}|_{U \times X}. \quad (2.24)$$

But, from the representability of the functor $\mathcal{F}_X^\circ$, it implies that

$$E'|_{U \times X} \cong E|_{U \times X}. \quad (2.25)$$

This yields a contradiction to $E \not\cong E'$. □

**Remark 2.1.1.** The reason behind the failure of the representability of $\mathcal{F}_X^\circ$ is that vector bundles have a number of non-trivial automorphisms, for instance, induced by a scalar multiplication as above. This example, in fact, may provide an important insight why a generic moduli problem is destined to be non-representable in the category of schemes. In many cases, the main source of non-representability problem turns out to be the existence of non-trivial automorphisms for the moduli problem of interest.

### 2.2 Moduli of elliptic curves

In this section, we study the moduli space of elliptic curves, and try to show how the existence of non-trivial automorphisms again leads to the non-representability of the corresponding moduli functor. The study of such a moduli problem is indeed a classical topic, and further discussion can be found elsewhere. For an introduction, we refer to [1]. More detailed treatment (possibly with different approaches) can be found, for instance, in [8, 5, 6].

**Definition 2.2.1.** We first recall that one can define the notion of an elliptic curve over $\mathbb{C}$ in a number of equivalent ways. An elliptic curve over $\mathbb{C}$ is defined to be either of the following objects:

1. A Riemannian surface $\Sigma$ of genus 1 with a choice of a point $p \in \Sigma$.
2. A quotient space $\mathbb{C}/\Lambda$ where $\Lambda = \omega_1 \cdot \mathbb{Z} \oplus \omega_2 \cdot \mathbb{Z}$ is a rank 2 lattice in $\mathbb{C}$ for each $\omega_i \in \mathbb{C}$.
3. A smooth algebraic curve of genus 1 and degree 3 in $\mathbb{P}^2_\mathbb{C}$.

We actually use the second characterization of an elliptic curve, namely the one given in terms of lattices. With this interpretation in hand, the study of the moduli of elliptic curves boils down to the study of integer lattices of full rank in $\mathbb{C}$.

**$SL_2(\mathbb{Z})$-action on the upper half plane and the fundamental domain.** Recall that the Lie group $SL_2(\mathbb{Z})$ acts on the upper half plane $\mathbb{H} = \{ z \in \mathbb{C} : \text{Im}(z) > 0 \}$ as follows:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot z := \frac{az + b}{cz + d} \quad \text{for all} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}) \quad \text{and} \quad \forall z \in \mathbb{H}. \quad (2.25)$$

It is clear from the definition of the action that both $\pm I$ act on $\mathbb{H}$ in the same way, and hence we will concentrate on the action of $PSL_2(\mathbb{Z}) := SL_2(\mathbb{Z})/\{\pm I\}$ on $\mathbb{H}$. Then the fundamental domain $\Gamma := \mathbb{H}/PSL_2(\mathbb{Z})$ of this action turns out to be the set

$$\Gamma = \left\{ z : -\frac{1}{2} \leq \text{Re}(z) < \frac{1}{2}, \ |z| > 1 \right\} \cup \left\{ z : -\frac{1}{2} \leq \text{Re}(z) \leq 0, \ |z| = 1 \right\}.$$
It is very well known that $\Gamma$ is in fact a Riemann surface whose points correspond to the isomorphism classes of lattices of full rank in $\mathbb{C}$ up to homotheties. Note also that any lattice $\Lambda = \omega_1 \mathbb{Z} \oplus \omega_2 \mathbb{Z}$ is isomorphic to a "normalized" lattice

$$\Lambda_{\tau} := 1 \cdot \mathbb{Z} \oplus \tau \cdot \mathbb{Z} \quad \text{for some } \tau \in \mathbb{H}. \quad (2.26)$$

We say that two lattices $\Lambda_{\tau_1} = 1 \cdot \mathbb{Z} \oplus \tau_1 \cdot \mathbb{Z}$ and $\Lambda_{\tau_2} = 1 \cdot \mathbb{Z} \oplus \tau_2 \cdot \mathbb{Z}$ with $\tau_1 \in \mathbb{H}$ are homothetic if there exits $g \in PSL_2(\mathbb{Z})$ such that $\tau_2 = g \cdot \tau_1$. In other words, $\mathbb{H}/PSL_2(\mathbb{Z})$ serves as a coarse moduli space for isomorphism classes of elliptic curves $\mathbb{C}/\Lambda_{\tau}$ with $\tau \in \Gamma$. As we shall see soon, however, it turns out that the space $\Gamma$ is not sensitive enough to parametrize certain families of elliptic curves. This amounts to say that not all families of elliptic curves over some base $B$ correspond to morphisms $B \to \mathbb{H}/PSL_2(\mathbb{Z})$. Hence, $\Gamma$ fails to become a fine moduli space.

Remark 2.2.1.

1. The fundamental domain $\Gamma$ can also be represented as a free product of finite groups $\mathbb{Z}_2$ and $\mathbb{Z}_3$ as follows:

$$\Gamma = \langle S := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, T := \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \in PSL_2(\mathbb{Z}) \mid S^2 = (ST)^3 = I \rangle \cong \mathbb{Z}_2 \star \mathbb{Z}_3.$$ (2.27)

2. The action of $PSL_2(\mathbb{Z})$ on $\mathbb{H}$ is not free. Indeed, some routine computations show that

$$Stab_{PSL_2(\mathbb{Z})}(\tau) \cong \begin{cases} \mathbb{Z}_2, & \tau = i \\ \mathbb{Z}_3, & \tau = \mu \text{ or } \rho \\ \{e\}, & \text{else.} \end{cases} \quad (2.27)$$

3. It follows from the non-freeness of the action that one has special types of lattices, namely the square lattice $\Lambda_i$ and the hexagonal lattice $\Lambda_\mu$ (or $\Lambda_\rho$) such that

$$\text{Aut}(\Lambda_{\tau}) = \begin{cases} \mathbb{Z}_4, & \tau = i \\ \mathbb{Z}_6, & \tau = \mu \text{ or } \rho. \end{cases} \quad (2.28)$$

This gives rise to non-trivial automorphisms for the corresponding elliptic curves $\mathbb{C}/\Lambda_i$ and $\mathbb{C}/\Lambda_\mu$ by using, for instance, rotational symmetries of a square and that of a hexagon respectively. As before, the existence of non-trivial automorphisms allows us to produce some examples which eventually show that the corresponding moduli problem of elliptic curves cannot be represented by the space $\mathbb{H}/PSL_2(\mathbb{Z})$. But, we should keep in mind that it becomes the coarse moduli space.
Moduli functor for the families of elliptic curves. We define the corresponding moduli functor

$$\mathcal{F}_{\text{ell}} : \text{Sch}_{\mathbb{C}}^{op} \to \text{Sets}, \ U \mapsto \mathcal{F}_{\text{ell}}(U) \quad (2.29)$$

as follows: Given a scheme $U$, one can define $\mathcal{F}_{\text{ell}}(U) := S_{\text{ell}}(U)/\sim$ where $S_{\text{ell}}(U)$ is the set of (continuous) families of elliptic curves over the base scheme $U$:

$$S_{\text{ell}}(U) := \left\{ E \to U : \text{each fiber } E_u \text{ is } \mathbb{C}/\Lambda_{\tau(u)} \ \forall u \in U \right\} \quad (2.30)$$

where $\mathbb{C}/\Lambda_{\tau(u)}$ is an elliptic curve with $\tau : U \to \mathbb{H}/\text{PSL}_2(\mathbb{Z})$, $u \mapsto \tau(u)$, and $E = \bigsqcup_{u \in U} E_u$. We say that two families $\pi_E : E \to U$ and $\pi_{E'} : E' \to U$ over $U$ are equivalent if and only if there exists an isomorphism $f : E \isom E'$ of families such that on each fiber $E_u = \pi^{-1}_E(u)$, $f$ restricts to an automorphism of elliptic curves

$$f|_{E_u} : E_u \isom E'_u, \quad (2.31)$$

and the following diagram commutes:

$$\begin{array}{ccc}
E & \xrightarrow{f} & E' \\
\pi_E & & \pi_{E'} \\
\downarrow & & \downarrow \\
U & \xrightarrow{\pi_E} & U
\end{array} \quad (2.32)$$

Note that from the previous discussion, it is not hard to observe that the space $\mathbb{H}/\text{PSL}_2(\mathbb{Z})$ in fact serves as the desired coarse moduli space for the moduli functor above. We now would like to show that the moduli functor $\mathcal{F}_{\text{ell}}$ is not representable by $\mathbb{H}/\text{PSL}_2(\mathbb{Z})$.

**Claim:** The moduli functor $\mathcal{F}_{\text{ell}}$ is not representable by $\mathbb{H}/\text{PSL}_2(\mathbb{Z})$.

**Proof.** Assume the contrary. Then, we have the following one-to-one correspondence between two sets:

$$\text{Mor}_{\text{Sch}}(U, \mathbb{H}/\text{PSL}_2(\mathbb{Z})) \cong \mathcal{F}_{\text{ell}}(U). \quad (2.33)$$

We first consider a "constant" family $E$ of elliptic curves on the interval $[0, 1]$ where each fiber $E_x$ is of the form

$$E_x := \mathbb{C}/\Lambda_i \quad \text{for all } x. \quad (2.34)$$

Recall that $\text{Aut}(\mathbb{C}/\Lambda_i) \cong \mathbb{Z}_4$. Let $f$ be a non-trivial automorphism of $\mathbb{C}/\Lambda_i$ given as a multiplication by $i$,

$$f : \mathbb{C}/\Lambda_i \to \mathbb{C}/\Lambda_i, \ z \mapsto iz. \quad (2.35)$$

Then one can identify the fibers $E_0$ and $E_1$ along the morphism $f$ so that a particular family $E$ of elliptic curves over $S^1$ can be obtained. Similarly, one can construct another family $E'$ of elliptic curves over $S^1$ by gluing the fibers $E_0$ and $E_1$ via the identity morphism. We then obtain two non-isomorphic families $E$ and $E'$ of elliptic curves over $S^1$ with the generic fibers being all isomorphic. That is, $[E] \neq [E']$ such that

$$E_x \cong E'_x \cong E_x \quad \text{for all } x \in (0, 1),$$

where $E_x$ and $E'_x$ denote the fibers of "twisted" and "trivial" families respectively. See Figure 2.

As $\mathcal{F}_{\text{ell}}$ is representable by $\mathbb{H}/\text{PSL}_2(\mathbb{Z})$, there are corresponding morphisms $f_E$, $f_{E'} : S^1 \to \Gamma$ for $E$ and $E'$ respectively such that

$$f_E, f_{E'} : S^1 \to \mathbb{H}/\text{PSL}_2(\mathbb{Z}), \ s \mapsto [i]. \quad (2.36)$$

That is, each representing morphism is just the constant map. Let $\{U_k\}$ be a local trivializing cover for $S^1$ such that

$$\pi_{E}^{-1}(U_k) \cong (\mathbb{C}/\Lambda_i) \times U_k \cong \pi_{E'}^{-1}(U_k). \quad (2.37)$$

Then the representability condition implies that the representing morphisms are locally the same as well. That is,

$$f_E|_{U_k} = f_{E'}|_{U_k} \ \forall k. \quad (2.38)$$
Figure 2: The "trivial" and "twisted" families of elliptic curves over $\mathbb{S}^1$ with generically isomorphic fibers.

Since $h_{\mathcal{H}/PSL_2(\mathbb{Z})} = \text{Mor}_{\text{Schc}}(-, \mathcal{H}/PSL_2(\mathbb{Z}))$ is a sheaf, it follows from the gluing axiom that all such morphisms are glued together nicely. That is,

$$f_{\mathcal{E}} = f_{\mathcal{E}'} \text{ on } \mathbb{S}^1.$$  \hspace{1cm} (2.39)

But, it follows from the representability of the functor $\mathcal{F}_{\text{ell}}$ that we must have

$$[\mathcal{E}] = [\mathcal{E}'],$$  \hspace{1cm} (2.40)

which is a contradiction.

\[ \square \]

**Remark 2.2.2.**

1. The construction above shows that the correspondence 2.33 is not good enough to distinguish the "trivial" and "twisted" families of isomorphism classes of elliptic curves over $\mathbb{S}^1$ with generically isomorphic fibers. As before, the main source of this failure is due to the existence of non-trivial automorphism group for the fibers of the form $\mathbb{C}/\Lambda_i$. The existence of non-trivial automorphisms, on the other hand, is due to the fact that $PSL_2(\mathbb{Z})$ acts on $\mathbb{H}$ non-freely.

2. One way of circumventing this sort of problem is to change the way of organizing the moduli data. For instance, we can use the language of stacks, and redefine the moduli problem as a certain groupoid-valued "functor"

$$\mathcal{F} : \mathcal{C}^{\text{op}} \rightarrow \text{Grpds}$$  \hspace{1cm} (2.41)

where $\text{Grpds}$ denotes the 2-category of groupoids with objects being categories $\mathcal{C}$ in which all morphisms are isomorphisms (these sorts of categories are called groupoids), 1-morphisms being functors $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{D}$ between groupoids, and 2-morphisms being natural transformations $\psi : \mathcal{F} \Rightarrow \mathcal{F}'$ between two functors. Note that the groupoid structure, in fact, allows us to keep track of the non-trivial automorphisms as a part of the moduli data.

3. When we go back the example above, one can easily check that the space $\mathcal{H}/PSL_2(\mathbb{Z})$ can, in fact, be regarded as a groupoid where objects are the elements of $\mathbb{H}$, and the set of morphisms is the set $PSL_2(\mathbb{Z}) \times \mathbb{H}$. In fact,

$$\text{Mor}_{\mathcal{H}/PSL_2(\mathbb{Z})}(x,y) := \{ g \in PSL_2(\mathbb{Z}) : y = g \cdot x \} \cong PSL_2(\mathbb{Z}) \times \mathbb{H}.$$  \hspace{1cm} (2.42)
Denote a morphism by $(g, x): x \mapsto g \cdot x$. Note that two morphisms $(g, x)$ and $(h, y)$ are composable if $x = h \cdot y$. Then we have

$$(g, h \cdot y) \circ (h, y) = (gh, y).$$

Furthermore, the inverse of the morphism $(g, x)$ is $(g^{-1}, g \cdot x)$. Informally speaking, two non-isomorphic families $E$ and $E'$ above can be represented by points like $[i, f]$ and $[i, id]$ via suitable constant representing morphisms as above where the second slots in the parenthesis are to keep track of possible automorphisms distinguishing these families. The last statement will be elaborated in the next section. In literature, the space $\mathbb{H}/\text{PSL}_2(\mathbb{Z})$ is an example of an orbifold.

3 2-categories and Stacks

Stacks and 2-categories serve as motivating/prototype conceptual examples before introducing the notions like $\infty$-categories, derived schemes, higher stacks and derived stacks [2]. By using 2-categorical version of the Yoneda lemma [3], one can show that the "refined" moduli functor

$$F : C \to \text{Grpds}$$

turns out to be representable in the 2-category $\text{Stks}$ of stacks. The price we have to pay is to adopt higher categorical dictionary, which leads to the change in the level of abstraction in a way that objects under consideration may become rather counter-intuitive. We first briefly recall the basics of 2-category theory. For details, we refer to [7, 3].

3.1 A digression on 2-categories

Definition 3.1.1. A 2-category $\mathcal{C}$ consists of the following data:

1. A collection of objects: $\text{Ob}(\mathcal{C})$.

2. For each pair $x, y$ of objects, a category $\text{Mor}_\mathcal{C}(x, y)$. Here, objects of the category $\text{Hom}_\mathcal{C}(x, y)$ are called 1-morphisms and are denoted either by $f : x \to y$ or $x \xrightarrow{f} y$. The morphisms of $\text{Mor}_\mathcal{C}(x, y)$ are called 2-morphisms and are denoted either by $\phi : f \Rightarrow g$ or

$$
\begin{array}{ccc}
  x & \xrightarrow{\phi} & y \\
  f \downarrow & & \downarrow g \\
  x & \xrightarrow{\phi} & y
\end{array}
$$

(3.2)

The composition of two 2-morphisms $\alpha : f \Rightarrow g$ and $\beta : g \Rightarrow h$ in $\text{Mor}_\mathcal{C}(x, y)$ is called a vertical composition, and denoted by $\beta \circ \alpha : f \Rightarrow h$ or

$$
\begin{array}{ccc}
  x & \xrightarrow{\alpha} & y \\
  f \downarrow & & \downarrow g \\
  x & \xrightarrow{\alpha} & y
\end{array} \quad \begin{array}{ccc}
  x & \xrightarrow{\beta} & y \\
  f \downarrow & & \downarrow g \\
  x & \xrightarrow{\beta} & y
\end{array}
$$

(3.3)

A 2-morphism $\alpha : f \Rightarrow g$ is invertible if there exits a 2-morphism $\beta : g \Rightarrow f$ such that $\beta \circ \alpha = id_f$ and $\alpha \circ \beta = id_g$. Furthermore, an invertible 2-morphism $\alpha : f \Rightarrow g$ is called a 2-isomorphism. It is sometimes denoted by $\alpha : f \Leftrightarrow g$.

3. For each triple $x, y, z$ of objects in $\mathcal{C}$, there is a composition functor

$$
\mu_{x, y, z} : \text{Mor}_\mathcal{C}(x, y) \times \text{Mor}_\mathcal{C}(y, z) \to \text{Mor}_\mathcal{C}(x, z)
$$

(3.4)

which is defined as follows:
(a) On 1-morphisms, it acts as the usual composition of morphisms in $\mathcal{C}$:

$$\mu_{x,y,z} : (x \xrightarrow{f} y, y \xrightarrow{g} z) \mapsto (y \xrightarrow{gf} z)$$

(b) On 2-morphisms, it acts as a horizontal composition, denoted by $\star$:

$$\mu_{x,y,z} : (\alpha \xrightarrow{f} f', \alpha' \xrightarrow{g} g') \mapsto \left(g \circ f \xrightarrow{\alpha \circ \alpha'} g' \circ f'\right) \quad (3.5)$$

That is, given two 2-morphisms

$$\begin{array}{ccc}
  x & \xrightarrow{f} & y \\
  \downarrow{\alpha} & & \downarrow{\alpha'} \\
  f' & \xrightarrow{g'} & z
\end{array}$$

$\mu_{x,y,z}$ maps the pair $(f \xrightarrow{\alpha} f', g \xrightarrow{\alpha'} g')$ of 2-morphisms to a 2-morphism

$$\begin{array}{ccc}
  x & \xrightarrow{\alpha \circ \alpha} & z \\
  \downarrow{g \circ f} & & \\
  f' & \xrightarrow{g' \circ f'} &
\end{array} \quad (3.6)$$

These data must satisfy the following conditions:

- For each object $X$ of $\mathcal{C}$ and each 1-morphism $f : A \to B$, we have an identity 1-morphism $\text{id}_X : X \to X$ and an identity 2-morphism $\text{id}_f : f \Rightarrow f$ respectively.

- The composition of 1-morphisms (2-morphisms respectively) is associative.

- Horizontal and vertical compositions of 2-morphisms are "compatible" in the following sense. For a composition diagram

$$\begin{array}{ccc}
  x & \xrightarrow{f} & y \\
  \downarrow{\alpha} & \downarrow{\beta} & \downarrow{\alpha'} \\
  f'' & \xrightarrow{g''} & z
\end{array}$$

we have

$$(\beta' \circ \beta) \star (\alpha' \circ \alpha) = (\beta' \star \alpha') \circ (\beta \star \alpha). \quad (3.8)$$

**Example 3.1.1.** Every category can be realized as a 2-category. Indeed, let $\mathcal{C}$ be a category and $\text{Mor}_C(A, B)$ denote the set of morphisms between two objects $A, B$. Then it is clear to observe that for any pair $(A, B)$ of objects in $\mathcal{C}$ the set $\text{Mor}_C(A, B)$ defines a category $\text{Mor}_C(A, B)$ whose objects (1-morphisms) are just morphisms $A \to B$ in $\mathcal{C}$, and morphisms (2-morphisms) are just identities. That is, there are no non-trivial higher morphisms in this realization. A category is sometimes called a 1-category.

**Remark 3.1.1.** Given a 2-category $\mathcal{C}$, one can obtain a category $\mathcal{C}_0$ by defining $\text{Ob}(\mathcal{C}_0) := \text{Ob}(\mathcal{C})$ and the "set" $\text{Mor}_{\mathcal{C}_0}(A, B)$ of morphisms in $\mathcal{C}_0$ to be

$$\text{Mor}_{\mathcal{C}_0}(A, B) := \text{Mor}_C(A, B)/\sim \quad (3.10)$$

where $f \sim g$ if there exits a 2-isomorphism $\alpha : f \Leftrightarrow g$. That is, $\text{Mor}_{\mathcal{C}_0}(A, B)$ is just the set of isomorphism classes of 1-morphisms in $\mathcal{C}$.
Example 3.1.2. A collection of categories forms a 2-category, namely the 2-category \( \text{Cat} \) of categories. Here, objects of \( \text{Cat} \) are just categories \( C \), 1-morphisms in \( \text{Cat} \) are functors \( F : C \to D \) between two categories, and 2-morphisms are natural transformations \( \eta : F \Rightarrow G \) of functors. In this example, there are no non-trivial higher \( n \)-morphisms for \( n > 2 \). Once we allow such types of morphisms, we land in the territory of higher categories.

Definition 3.1.2. Let \( C \) be a 2-category. Two objects \( X, Y \) in \( C \) are said to be equivalent if there exist a pair \((X \xrightarrow{f} Y, Y \xrightarrow{g} X)\) of 1-morphisms, and two 2-isomorphisms \( \alpha : g \circ f \Rightarrow \text{id}_X \) and \( \alpha' : f \circ g \Rightarrow \text{id}_Y \).

Definition 3.1.3. A pseudo-functor \( F : C \to D \) between two 2-categories \( C, D \) consists of the following data:

1. For each object \( A \) in \( C \), an object \( F(A) \) in \( D \),
2. For each 1-morphism \( A \xrightarrow{f} B \) in \( C \), a 1-morphism \( F(f) : F(A) \to F(B) \) in \( D \),
3. For each 2-morphism \( \alpha : f \Rightarrow g \) in \( C \), a 2-morphism \( F(\alpha) : F(f) \Rightarrow F(g) \) in \( D \) satisfying the following conditions:
   
   (a) \( F \) respects 1- and 2-identities: \( F(\text{id}_A) = \text{id}_{F(A)} \) and \( F(\text{id}_f) = \text{id}_{F(f)} \) for all \( A \in \text{Ob}(C) \) and \( f \in \text{Mor}_C(X, Y) \).
   
   (b) \( F \) respects a composition of 1-morphisms up to a 2-isomorphism: Given a composition of 1-morphisms \( A \xrightarrow{f} B \xrightarrow{g} C \) in \( C \), there is a 2-isomorphism
   
   \[
   \phi_{g,f}^{F} : F(g \circ f) \Rightarrow F(g) \circ F(f)
   \]
   such that the following diagram commutes (encoding the associativity):
   
   \[
   \begin{array}{ccc}
   F(h \circ g \circ f) & \xrightarrow{\phi_{h\circ g,f}^{F}} & F(h) \circ F(g \circ f) \\
   \downarrow_{\phi_{h\circ g,f}^{F}} & & \downarrow_{\phi_{h,f}^{F} \circ \phi_{g,f}^{F}} \\
   F(h \circ g) \circ F(f) & \xrightarrow{\phi_{h,g}^{F} \circ \text{id}_{F(f)}} & F(h) \circ F(g) \circ F(f)
   \end{array}
   \]
   (3.11)
   
   such that \( \phi_{f,\text{id}_A}^{F} = \phi_{\text{id}_B,f}^{F} = \text{id}_{F(f)} \).

   (c) \( F \) respects both vertical and horizontal compositions: Given a vertical composition \( \beta \circ \alpha : f \Rightarrow h \) with

   \[
   \xymatrix{ x \ar@<0.5ex>[r]^{\alpha} \ar@<0.5ex>[d]_{\beta} & y \ar@<0.5ex>[d]_{\delta} \\
    h &}
   \]
   (3.13)
   
   we have \( F(\beta \circ \alpha) = F(\beta) \circ F(\alpha) \). Given a horizontal composition

   \[
   \xymatrix{ x \ar@<0.5ex>[r]^{\alpha'} \ar@<0.5ex>[d]_{\alpha} & z \ar@<0.5ex>[d]_{\delta'} \\
    g' \circ f' &}
   \]
   (3.14)
   
   with \( \alpha : f \Rightarrow f' \) and \( \alpha' : g \Rightarrow g' \), we have the following commutative diagram:

   \[
   \begin{array}{ccc}
   F(g) \circ F(f) & \xrightarrow{\phi_{g,f}^{F}} & F(g') \circ F(f') \\
   \downarrow_{\phi_{g,f}^{F}} & & \downarrow_{\phi_{g',f'}^{F}} \\
   F(g \circ f) & \xrightarrow{\phi_{g',f'}^{F} \circ \text{id}_{F(f)}} & F(g' \circ f')
   \end{array}
   \]
   (3.15)
Definition 3.1.4.

1. A prestack on \( C \) is defined to be a particular (contravariant) pseudo-functor \( F : C \to D \) where \( C \) is an ordinary category and \( D \) is the 2-category \( \text{Grpds} \) of groupoids. That is, it is a pseudo-functor \( F : C^{\text{op}} \to \text{Grpds} \) for some category \( C \). Recall that, in the 2-category \( \text{Grpds} \) of groupoids, objects are categories \( C \) in which all morphisms are isomorphisms (these sorts of categories are called groupoids), 1-morphisms are functors \( F : C \to D \) between groupoids, and 2-morphisms are natural transformations \( \psi : F \Rightarrow F' \) between two functors.

2. Let \( C_\tau \) be a site. That is, \( C_\tau \) is a category \( C \) endowed with a Grothendieck topology \( \tau \). Then, a stack on \( C_\tau \) is just a prestack with local-to-global properties w.r.t. \( \tau \). So, it can be considered as "a sheaf of groupoids" in a suitable sense.

3.2 2-category of Stacks and 2-Yoneda’s Lemma

We like to present how stacks over a site \( C_\tau \) form a 2-category. To this end, we need to introduce the notions of 1- and 2-morphisms between two stacks.

Definition 3.2.1. Let \( C \) be a category, and \( X, Y : C^{\text{op}} \to \text{Grpds} \) be two prestacks. A 1-morphism \( F : X \to Y \) of prestacks consists of the following data:

1. For each object \( A \) in \( C \), a functor \( F_A : X(A) \to Y(A) \),
2. For each morphism \( f : A \to B \) in \( C \), a 2-isomorphism \( X(B) \xrightarrow{F_f} Y(A) \) \( \xrightarrow{X(f) \circ F_B} Y(f) \circ F_B \)

such that the following diagram commutes up to a 2-isomorphism

\[
\begin{array}{ccc}
X(B) & \xrightarrow{F_f} & Y(A) \\
\downarrow & & \downarrow \\
Y(f) & \xrightarrow{F_B} & Y(f) \\
\end{array}
\]

(3.16)

in \( \text{Grpds} \):

\[
\begin{array}{ccc}
X(B) & \xrightarrow{F_B} & Y(B) \\
\downarrow & & \downarrow \\
X(A) & \xrightarrow{F_A} & Y(A) \\
\end{array}
\]

(3.17)

3. Given a composition of 1-morphisms \( A \xrightarrow{f} B \xrightarrow{g} C \) in \( C \), the corresponding 2-isomorphisms \( F_f \) and \( F_g \) define \( F_{g \circ f} \) in compatible with the natural 2-isomorphisms \( \phi^X_{g,f} \) and \( \phi^Y_{g,f} \). Indeed, using horizontal and vertical compositions of 2-morphisms, \( F_{g \circ f} : F_A \circ X(g \circ f) \Rightarrow Y(g \circ f) \circ F_C \) is given by

\[
F_{g \circ f} = (\phi^Y_{g,f} \circ \psi F_C) \circ (\psi Y_{(g)} \circ F_g) \circ (\psi F_f \circ F_A \circ \phi^X_{g,f}).
\]

Such a 1-morphism is just a natural transformation between two pseudo-functors of 2-categories.

Definition 3.2.2. Let \( C \) be a category, and \( F, G : X \to Y \) be a pair of 1-morphisms of prestacks. A 2-morphism of between \( F \) and \( G \)

\[
\begin{array}{ccc}
X & \xrightarrow{\Psi} & Y \\
\downarrow & & \downarrow \\
G & & F \\
\end{array}
\]

(3.18)

is a collection \( \{ \Psi_A : F_A \Rightarrow G_A : A \in \text{Ob}(C) \} \) of invertible natural transformations of functors.
Remark 3.2.1. A 1-morphism (2-morphism respectively) of stacks is defined as a 1-morphism (2-morphism resp.) of the underlying prestacks.

Proposition 3.2.1. [3]

1. Given a category $\mathcal{C}$, prestacks over $\mathcal{C}$ form a 2-category $\text{PreStk}_\mathcal{C}$ of prestacks where 1- and 2-morphisms are defined as above.

2. Stacks over a site $\mathcal{C}$ form a 2-category $\text{Stk}_\mathcal{C}$ of stacks with 1- and 2-morphisms being as above.

Furthermore, it follows from the construction that we have the following observations [3].

Proposition 3.2.2. Let $\mathcal{C}$ be a category. Then the (1-) category $\text{PreShv}_\mathcal{C}$ of presheaves (of sets) over $\mathcal{C}$ is a full 2-subcategory of $\text{PreStk}_\mathcal{C}$. In addition, if $\mathcal{C}$ admits a site structure, then the (1-) category $\text{Shv}_\mathcal{C}$ of sheaves (of sets) over $\mathcal{C}$ is a full 2-subcategory of $\text{Stk}_\mathcal{C}$.

Remark 3.2.2. As we have already discussed in Remark 1.0.1, Yoneda’s lemma 1.0.1 implies that any object $X$ in a category $\mathcal{C}$ can be understood by studying all morphisms into it. In other words, for any $X \in \text{Ob}(\mathcal{C})$ one has a particular sheaf of sets

$$h_X : \mathcal{C}^{op} \to \text{Sets}, \quad Y \mapsto h_X(Y) := \text{Mor}_\mathcal{C}(Y, X)$$  \hspace{1cm} (3.19)

which uniquely determines $X$. If $\mathcal{C}$ admits a suitable site structure, then it follows from Proposition 3.2.2 that the sheaf $\text{Mor}_\mathcal{C}(\cdot, X)$ can be considered as a stack with trivial 2-morphisms. We denote this stack by $\underline{X}$.

Lemma 3.2.1. (2-categorical Yoneda’s Lemma for prestacks) Let $\mathcal{Y} : \mathcal{C}^{op} \to \text{Grpds}$ be a prestack over a category $\mathcal{C}$. Then for each object $X$ in $\mathcal{C}$, there exits an equivalence of categories

$$\mathcal{Y}(X) \cong \text{Mor}_{\text{PreStk}_\mathcal{C}}(\underline{X}, \mathcal{Y}).$$  \hspace{1cm} (3.20)

Proof. First, let us try to understand the objects of interests in the statement. On the left hand side, $\mathcal{Y}(X)$ is a groupoid, i.e. a category for which all morphisms are isomorphisms. On the right hand side, $\text{Mor}_{\text{PreStk}_\mathcal{C}}(\underline{X}, \mathcal{Y})$ is the category of morphisms in the 2-category $\text{PreStk}_\mathcal{C}$. 1-morphisms are a collection

$$\{ F_A : \underline{X}(A) \to \mathcal{Y}(A) : A \in \text{Ob}(\mathcal{C}) \}$$

of functors with some compatibility conditions as above. Such collection is denoted by $F : \underline{X} \to \mathcal{Y}$. On the other hand, 2-morphisms are of the form

$$\begin{array}{ccc}
F & \cong \\
\Phi & \downarrow \\
\mathcal{Y} & \sim \\
G
\end{array}$$  \hspace{1cm} (3.21)

where for each object $Z$, $\Phi_Z : F_Z \Rightarrow G_Z$ is an invertible natural transformation. Therefore, for any $f \in \underline{X}(Z)$, we have an isomorphism $\Psi_{Z,f} : F_Z(f) \Rightarrow G_Z(f)$ in $\mathcal{Y}(Z)$. To show the desired equivalence, we introduce the following functors:

1. We define the functor $\Theta : \text{Mor}_{\text{PreStk}_\mathcal{C}}(\underline{X}, \mathcal{Y}) \to \mathcal{Y}(X)$ as follows:
   (a) On objects (1-morphisms), $(F : \underline{X} \to \mathcal{Y}) \mapsto F_X(id_{\underline{X}})$
   (b) On morphisms (2-morphisms), $\psi : F \Rightarrow G \mapsto (\Phi_{X, id_{\underline{X}}} : F_X(id_{\underline{X}}) \Rightarrow G_X(id_{\underline{X}}))$

2. Define the functor $\eta : \mathcal{Y}(X) \to \text{Mor}_{\text{PreStk}_\mathcal{C}}(\underline{X}, \mathcal{Y})$ as follows:
   (a) On objects $A$ in $\mathcal{Y}(X)$, $A \mapsto (F^{(A)} : \underline{X} \to \mathcal{Y})$. Here $F^{(A)}$ is given as a collection
   $$\{ F^{(A)}_U : U \in \text{Ob}(\mathcal{C}) \}$$
   of functors such that
   $$F^{(A)}_U : \underline{X}(U) \to \mathcal{Y}(U), \quad (U \downarrow_{x} X) \mapsto \mathcal{Y}(f)(A)$$  \hspace{1cm} (3.22)
   where $\mathcal{Y}(f) : \mathcal{Y}(X) \to \mathcal{Y}(U)$. Notice that $\underline{X}(U) = \text{Mor}_\mathcal{C}(U, X)$ is just a set in the first place, but, as we remarked before, it can be viewed as a category for which all morphisms are identities. Therefore, $F^{(A)}_U$ acts on morphisms of $\underline{X}(U)$ trivially. That is, it maps $id_f$ to $id_{F^{(A)}_U(f)}$.
(b) η sends morphisms \( \varphi : A \rightarrow B \) in \( \mathcal{Y}(X) \) to 2-morphisms

\[
\begin{array}{ccc}
X & \xrightarrow{\Psi(\varphi)} & \mathcal{Y} \\
\downarrow & & \downarrow \\
F(B) & \xrightarrow{F(A)} & F(A)
\end{array}
\]

where \( \Psi(\varphi) \) is a collection \( \{ \Psi(\varphi)_U : F^A_U \Rightarrow F^B_U : U \in \text{Ob}(\mathcal{C}) \} \) of invertible natural transformations of functors where for each object \( f : U \rightarrow X \) in \( \mathcal{X}(U) \), there is an isomorphism

\[
\Psi(\varphi)_U(f) : F^A_U(f) \Rightarrow F^B_U(f), \quad \Psi(\varphi)_U(f) := \mathcal{Y}(f)(\varphi).
\]

Notice that \( \mathcal{Y}(f) : \mathcal{Y}(X) \rightarrow \mathcal{Y}(U) \) is a functor of groupoids, and hence it maps \( \varphi \) to an isomorphism.

3. Now, we like to show that the compositions of η and Θ are identities up to 2-isomorphisms.

(a) Let \( A \) be an object in \( \mathcal{Y}(X) \). Then we have

\[
(\Theta \circ \eta)(A) = \Theta(F^A(A)) = \mathcal{Y}(id_X)(A)
\]

where \( \mathcal{Y}(id_X)(A) = A \) since \( \mathcal{Y}(id_X) \) is a functor of groupoids.

(b) Let \( F : X \rightarrow Y \) be an object in \( \text{Mor}_{\mathcal{H}_{\text{pre-stk}}}^\mathcal{C}(X, Y) \). Then we get

\[
(\eta \circ \Theta)(F) = \eta(F_X(id_X)) = F(F_X(id_X))
\]

where the 1-morphism \( F(F_X(id_X)) \) is defined by, for all \( U \in \text{Ob}(\mathcal{C}) \),

\[
F(F_X(id_X)) : \mathcal{X}(U) \rightarrow \mathcal{Y}(U), \quad (U \xrightarrow{f} X) \mapsto \mathcal{Y}(f)(F_X(id_X))
\]

Claim. \( \mathcal{Y}(f)(F_X(id_X)) \cong F_U(f) \).

Proof. It follows directly from the definition of a 2-morphism that for a morphism \( U \xrightarrow{f} X \), there exists

\[
F_f : F_U \circ X(f) \Rightarrow \mathcal{Y}(f) \circ F_X
\]

such that the following diagram commutes up to a 2-isomorphism \( F_f \):

\[
\begin{array}{ccc}
X(X) & \xrightarrow{F_X} & \mathcal{Y}(X) \\
\downarrow & & \downarrow \\
X(U) & \xrightarrow{F_U} & \mathcal{Y}(U)
\end{array}
\]

where \( X(f) : X(X) \rightarrow X(U) \), \( g \mapsto g \circ f \). Therefore, we have

\[
(\mathcal{Y}(f) \circ F_X)(id_X) \cong F_U(id_X \circ f) = F_U(f),
\]

which proves the claim.

Therefore, since the claim holds for all \( U \), we conclude that

\[
(\eta \circ \Theta)(F) = F
\]

As a result, we get the desired equivalence of categories

\[
\eta : \mathcal{Y}(X) \xrightarrow{\sim} \text{Mor}_{\mathcal{H}_{\text{pre-stk}}}^\mathcal{C}(X, \mathcal{Y}) : \Theta.
\]

\[\square\]

Remark 3.2.3. 2-categorical Yoneda’s lemma implies that if \( \mathcal{Y} : \mathcal{C}^{op} \rightarrow \mathcal{Grpd} \) is a moduli functor for some moduli problem, then it is always representable by \( \mathcal{Y} \) in the 2-category of (pre-)stacks.
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