THE NEHARI MANIFOLD FOR A FRACTIONAL LAPLACIAN EQUATION INVOLVING CRITICAL NONLINEARITIES

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Abstract. We study the combined effect of concave and convex nonlinearities on the numbers of positive solutions for a fractional equation involving critical Sobolev exponents. In this paper, we concerned with the following fractional equation
\[
\begin{aligned}
(-\Delta)^s u &= \lambda f(x)|u|^{q-2}u + g(x)|u|^{2^*-2}u, \quad x \in \Omega, \\
u &= 0, \quad x \in \partial \Omega,
\end{aligned}
\]
where \(0 < s < 1\), \(\lambda > 0\), \(1 \leq q < 2\), \(2^* = \frac{2N}{N - 2s}\), \(0 \in \Omega \subset \mathbb{R}^N(N > 4s)\) is a bounded domain with smooth boundary \(\partial \Omega\), and \(f, g\) are nonnegative continuous functions on \(\Omega\). Here \((-\Delta)^s\) denotes the fractional Laplace operator.

1. Introduction. This paper is concerned with the multiplicity of positive solutions for the following elliptic equation involving the fractional Laplacian
\[
\begin{aligned}
(-\Delta)^s u &= \lambda f(x)|u|^{q-2}u + g(x)|u|^{2^*-2}u, \quad x \in \Omega, \\
u &= 0, \quad x \in \partial \Omega,
\end{aligned}
\]
where \(0 < s < 1\), \(\lambda > 0\), \(1 \leq q < 2\), \(2^* = \frac{2N}{N - 2s}\), \(0 \in \Omega \subset \mathbb{R}^N(N > 4s)\) is a bounded domain with smooth boundary \(\partial \Omega\), and \(f, g\) are continuous functions on \(\Omega\).

In recent years, considerable attention has been given to nonlocal diffusion problems, in particular to the ones driven by the fractional Laplace operator. One of the reasons for this comes from the fact that this operator naturally arises in several physical phenomena like flames propagation and chemical reactions of liquids, and in population dynamics and geophysical fluid dynamics. It also provides a simple model to describe certain jump lévy processes in probability theory. For more details and applications, see [3, 4, 5] and the references therein.

For \(s = 1\), equation (2) become the following semi-linear elliptic problem involving concave-convex nonlinearities
\[
\begin{aligned}
-\Delta u + \mu u &= \lambda f(x)|u|^{q-2}u + |u|^{p-2}u, \quad x \in \Omega, \\
u &= 0, \quad x \in \partial \Omega,
\end{aligned}
\]
where \(\mu, \lambda \geq 0\), \(1 \leq q < 2\), and \(f\) is a continuous function on \(\Omega\). Ambrosetti, Brezis and Cerami in [2](\(\mu = 0, f \equiv 1, 2 < p \leq 2^* = \frac{2N}{N - 2}\)) has at least two

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positive solutions for $\lambda \in (0, \Lambda)$, and even more it has no positive solutions if $\lambda > \Lambda$. One can also see that the result from the bibliography of paper in [4, 3, 5]. Wu [29] ($\mu = 0$, $f \in C(\Omega)$ and changes sign, $2 < p < 2^*$) showed that equation (3) has at least two positive solutions for $\lambda$ sufficiently small. Lin in [20] show that for $\lambda \in (0, \Lambda)$, there exists a least one positive solution, and for different value of $q$, there is $k + 1$ positive solutions($k$ is integer). Any other related results about the subcritical case readers can refer [9, 18, 19, 31] and the references therein.

Especially, using the idea of category and Bahri-Li’s minimax argument, Adachi and Tanaka [1] consider the following equation

$$\left\{ \begin{array}{l}
-\Delta u + u = g(x)u^{p-1} + f(x), \quad x \in \mathbb{R}^N, \\
u \in H^1(\mathbb{R}^N),
\end{array} \right.$$  

where $g(x) \not= 1$, $g(x) \geq 1 - Ce^{-(2+\delta)|x|}$ for some $C$ and $\delta$, and sufficiently small $||f||_{H^{-1}} > 0$, they admit at least four positive solutions in $\mathbb{R}^N$.

For the semilinear fractional elliptic equation

$$\left\{ \begin{array}{l}
(-\Delta)^s u = \lambda |u|^{q-2}u + |u|^{2^*_s-2}u, \quad x \in \Omega, \\
u = 0, \quad x \in \partial \Omega.
\end{array} \right.$$  

(4)

This type of operators naturally arises in physical situations such as thin obstacle problem, optimization, water waves and so on, see [16]. We refer to [13, 22, 23, 21, 25, 28, 30] for the subcritical case and to [4, 3, 24, 27] for the critical case. In paper [8], Caffarelli and Silvestre gave a new formulation of the fractional Laplacian through Dirichlet-Neumann maps. This is extensively used in the recent literature since it allows to transform nonlocal problems to local ones, which permits using variational methods. For example, Barrios, Colorado, de Pablo and Sanchez [4] used the idea of the s-harmonic extension and studied the effect of lower order perturbations in the existence of positive solutions of (4). In [7], Cabre and Tan defined $(-\Delta)^{1/2}$ through the spectral decomposition of the Laplacian operator in $\Omega$ with zero Dirichlet boundary conditions, with classical local techniques, they established existence of positive solutions for problems with subcritical nonlinearities, regularity and $L^\infty$-estimates for weak solutions.

Very recently, Colorado, de Pablo, and Sanchez [15] studied the following non-homogeneous fractional equation involving critical Sobolev exponent

$$\left\{ \begin{array}{l}
(-\Delta)^su = |u|^{2^*_s-2}u + f(x)u^q, \quad x \in \Omega, \\
u = 0, \quad x \in \partial \Omega.
\end{array} \right.$$  

(5)

and proved the existence and multiplicity of solutions under right conditions on the size of $f(q = 0)$. For $q \not= 0$, Barrios e.t. in [3] show under the different boundary which $u = 0$ in $\mathbb{R}^N \setminus \Omega$, for $0 < q < 1$, the equation admits no solution, at least one solution and at least two solution in different condition of $\lambda$. On the other hand, for $1 < q < 2^*_s - 1$, equation (5) admits at least one solution. For $s = \frac{1}{2}$, Carboni and Mugnai consider the subcritical case, they show $\lambda < \lambda_*$, equation (5) does not admit non-trivial solution, and $\lambda > \lambda_*$, equation (5) has at least two non-trivial solutions, one is negative, one is positive. Also some relevant results about systems in [12, 17] and the references therein.

The purpose of this paper is to study equation (2) in the critical case. Using variational methods and Nehari manifold decomposition, we prove the equation admits at least a positive solution when $\lambda$ belongs to a certain subset of $\mathbb{R}$. We adopt the spectral definition of the fractional Laplacian in a bounded domain based upon a Caffarelli-Silvestre type extension, and not the integral definition. Here
we study a Dirichlet problem with critical growth, characterized by the following features: the equation contains a positive perturbation term with subcritical growth; moreover, a positive nonconstant coefficient $g(x)$ multiplies the term with critical exponent.

Assume that $f$ and $g$ satisfy the following hypotheses.

(h1) $f, g \in C(\Omega)$, $f \geq 0$, $f \not= 0$ and $g > 0$.

(h2) There exist $k$ points $a^1, a^2, \ldots, a^k$ in $\Omega$ such that

$$g(a^i) = \max_{z \in \Omega} g(z) = 1, \quad \text{for} \quad 1 \leq i \leq k$$

for some $\sigma > N$, $g(z) - g(a^i) = O(|z - a^i|^\sigma)$ as $z \to a^i$ uniformly in $i$.

(h3) Choosing $\rho_0 > 0$ such that

$$B_{\rho_0}(a^i) \cap B_{\rho_0}(a^j) = \emptyset, \quad \text{for} \quad 1 \leq i, j \leq k,$$

and $\bigcup_{i=1}^k B_{\rho_0}(a^i) \subset \Omega$, where $B_{\rho_0}(a^i) = \{ z \in \mathbb{R}^N | |z - a^i| \leq \rho_0 \}$. There exists a positive number $d_0$ such that $f(z) \geq d_0 > 0$ for any $z \in B_{\rho_0}(a^i)$.

To formulate the main result, we introduce

$$\Lambda = \frac{(2 - q)}{(2^*_s - q)} \frac{2 - q}{2^*_s - q} \frac{2 - q}{2^*_s - q} |\Omega| \frac{2 - q}{2^*_s - q} S^{\frac{2 - q}{2^*_s - q}} \frac{2 - q}{2^*_s - q} . \tag{6}$$

Then we have the following theorems.

**Theorem 1.1.** If $\lambda \in (0, \Lambda)$, there exists at least one positive ground state solution $u_\lambda$ of problem (2).

**Theorem 1.2.** Under the assumptions (h1)–(h3), and $\frac{N}{N - 2s} < q < 2$ and $N > 4s$, there exists a positive number $\Lambda^* \in (0, \Lambda)$ such that $\lambda \in (0, \Lambda^*)$, problem (2) has $k + 1$ positive solutions.

This paper is organized as follows. In Section 2, we introduce the variational setting of problem (2) and present some preliminary results. In Section 3, we give some properties about the Nehari manifold and prove Theorem 1.1. In Section 4, we construct the $k$-compact Palais-Smale sequences which are suitably localized in correspondence of the $k$ maximum points of $g$ and prove Theorem 1.2.

2. **Some preliminary facts.** In this section, we collect some preliminary facts in order to establish the functional setting. First of all, let us introduce the standard notations for future use in this paper. We denote the upper half-space in $\mathbb{R}^{N+1}$ by

$$\mathbb{R}^{N+1}_+ := \left\{ z = (x, y) = (x_1, \ldots, x_N, y) \in \mathbb{R}^{N+1}, y > 0 \right\}.$$

Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain. Denote

$$C_\Omega := \Omega \times (0, \infty) \subset \mathbb{R}^{N+1}_+,$$

the cylinder with base $\Omega$ and its lateral boundary by $\partial_{L} C_\Omega := \partial \Omega \times (0, \infty)$. The powers $(-\Delta)^p$ with zero Dirichlet boundary conditions are defined via its spectral decomposition, namely

$$(-\Delta)^p u(x) := \sum_{j=1}^{\infty} b_j \rho_j^p \varphi_j(x),$$

where $(\rho_j, \varphi_j)$ is the sequence of eigenvalues and eigenfunctions of operator $(-\Delta)^p$ in $\Omega$ under zero Dirichlet boundary data and $b_j$ are the coefficients of $u$ for the base
\{\varphi_j\}_{j=1}^\infty \text{ in } L^2(\Omega). \text{ In fact, the fractional Laplacian } (-\Delta)^s \text{ is well defined in the space of functions }
\begin{align*}
H^s_0(\Omega) := \left\{ u \in \sum_{j=1}^\infty b_j \varphi_j \in L^2(\Omega) : \|u\|_{H^s_0(\Omega)} = \left( \sum_{j=1}^\infty b_j^2 \rho_j^{2s} \right)^{\frac{1}{2}} < +\infty \right\},
\end{align*}

and \(\|u\|_{H^s_0(\Omega)} = \|(-\Delta)^s u\|_{L^2(\Omega)}\). The dual space \(H^{-s}(\Omega)\) is defined in the standard way, as well as the inverse operator \((-\Delta)^{-s}\).

**Definition 2.1.** We say that \(u \in H^s_0(\Omega)\) is a solution of (2) if the identity
\[
\int_\Omega (-\Delta)^s u(-\Delta)^s \varphi dx - \lambda \int_\Omega f(x)|u|^{q-2}u \varphi dx - \int_\Omega g(x)|u|^{2^*_s-2}u \varphi dx = 0
\]
holds for all \(\varphi \in H^s_0(\Omega)\).

Associated with problem (2), we consider the energy functional
\[
\tilde{J}_\lambda(u) := \frac{1}{2} \int_\Omega |(-\Delta)^s u|^2 dx - \lambda \int_\Omega f(x)|u|^{q} dx - \frac{1}{2^*_s} \int_\Omega g(x)|u|^{2^*_s} dx.
\]

The functional is well defined in \(H^s_0(\Omega)\), moreover, the critical points of the functional \(\tilde{J}_\lambda\) correspond to solutions of (2). We now conclude the main ingredients of a recently developed technique used in order to deal with fractional powers of the Laplacian. To treat the nonlocal problem (2), we shall study a corresponding extension problem, which allows us to investigate problem (2) by studying a local problem via classical variational methods. We first define the extension operator and fractional Laplacian for functions in \(H^s_0(\Omega)\). We refer the reader to [4, 3, 6, 10] and the references therein.

**Definition 2.2.** For a function \(u \in H^s_0(\Omega)\), we denote the s-harmonic extension \(w = E_s(u)\) to the cylinder \(C_\Omega\) as the solution of the problem
\[
\begin{cases}
\text{div}(y^{1-2s} \nabla w) = 0, & \text{in } C_\Omega, \\
w = 0, & \text{on } \partial_L C_\Omega, \\
w = u, & \text{on } \Omega \times \{0\},
\end{cases}
\]
and
\[
(-\Delta)^s u(x) = -k_s \lim_{y \to 0^+} y^{1-2s} \partial_y^s w(x, y),
\]
where \(k_s = 2^{1-2s} \Gamma(1-s)/\Gamma(s)\) is a normalization constant.

The extension function \(w(x, y)\) belongs to the space
\[
X^s_0(C_\Omega) := C^\infty_0(\Omega \times [0, \infty)) \|x^s(\cdot, \cdot)\|_X^s(\cdot, \cdot)
\]
edowed with the norm
\[
\|z\|_{X^s_0(C_\Omega)} := (k_s \int_{C_\Omega} y^{1-2s} |\nabla z|^2 dx dy)^{\frac{1}{2}}.
\]

The extension operator is an isometry between \(H^s_0(\Omega)\) and \(X^s_0(C_\Omega)\), namely
\[
\|u\|_{H^s_0(\Omega)} = \|E_s(u)\|_{X^s_0(C_\Omega)}, \text{ for any } u \in H^s_0(\Omega).
\]
With this extension we can reformulate (2) as the following local problem

\[
\begin{align*}
\text{div}(y^{1-2s}\nabla w) &= 0, & \text{in } C_\Omega \\
w &= 0, & \text{on } \partial IC_\Omega \\
w &= u, & \text{on } \Omega \times \{0\}, \\
\frac{\partial w}{\partial \nu^s} &= \lambda f(x)|w|^{q-2}w + g(x)|w|^{2^*-2}w, & \text{on } \Omega \times \{0\},
\end{align*}
\]  

(8)

where \( \frac{\partial w}{\partial \nu^s} := -k_s \lim_{y \to 0^+} y^{1-2s} \frac{\partial w}{\partial y}, \) and \( w \in X_0^s(C_\Omega) \) is the \( s \)-harmonic extension of \( u \in H_0^s(\Omega) \).

An energy solution to this problem is a function \( w \in E_0^s(C_\Omega) \) satisfying

\[
k_s \int_{C_\Omega} y^{1-2s} \nabla w \nabla \varphi \, dx \, dy = \lambda \int_{\Omega \times \{y=0\}} f(x)|w|^{q-2}w \varphi \, dx + \int_{\Omega \times \{y=0\}} g(x)|w|^{2^*-2}w \varphi \, dx
\]

for all \( \varphi \in X_0^s(C_\Omega) \). If \( w \in X_0^s(C_\Omega) \) satisfies (8), then \( u = w(\cdot, 0) \), defined in the sense of trace, belongs to the space \( H_0^s(\Omega) \) and it is a solution of the original problem (2). The associated energy functional to the problem (8) is denoted by

\[
J_\lambda(w) := \frac{k_s}{2} \int_{C_\Omega} y^{1-2s}|\nabla w|^2 \, dx \, dy - \frac{1}{q} \int_{\Omega \times \{y=0\}} \lambda f(x)|w|^q \, dx \\
- \frac{1}{2^*} \int_{\Omega \times \{y=0\}} g(x)|w|^{2^*} \, dx.
\]

Critical points of \( J_\lambda \) in \( X_0^s(C_\Omega) \) correspond to critical points of \( \tilde{J}_\lambda(u) : H_0^s(\Omega) \to \mathbb{R}^N \).

In the following lemma we list some relevant inequalities from [6].

**Lemma 2.3.** For any \( 1 \leq r \leq 2^*_s \) and any \( z \in X_0^s(C_\Omega) \), it holds

\[
(\int_{\Omega} |u(x)|^r \, dx)^{\frac{2}{r}} \leq C \int_{C_\Omega} y^{1-2s}|\nabla z(x, y)|^2 \, dx \, dy, \quad u := T \rho(z),
\]

(9)

for some positive constant \( C = C(r, s, N, \Omega) \). Furthermore, the space \( X_0^s(C_\Omega) \) is compactly embedded into \( L^r(\Omega) \), for every \( r < 2^*_s \).

**Remark 1.** When \( r = 2^*_s \), the best constant in (9) is denoted by \( \mathcal{S} \), that is

\[
\mathcal{S} := \inf_{z \in X^s_0(C_\Omega) \setminus \{0\}} \frac{\int_{C_\Omega} y^{1-2s}|\nabla z(x, y)|^2 \, dx \, dy}{(\int_{\Omega} |z(x, 0)|^{2^*_s} \, dx)^{\frac{2}{2^*_s}}}. \tag{10}
\]

It is not achieved in any bounded domain and for all \( z \in X^s(\mathbb{R}^{N+1}_+) \),

\[
\int_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla z(x, y)|^2 \, dx \, dy \geq \mathcal{S} \left( \int_{\mathbb{R}^N} |z(x, 0)|^{\frac{2^*_s}{N-2s}} \, dx \right)^{\frac{N-2s}{2}}. \tag{11}
\]

\( \mathcal{S} \) is achieved for \( \Omega = \mathbb{R}^N \) by function \( w_\varepsilon \), which are the \( s \)-harmonic extension of

\[
u_\varepsilon(x) := \frac{\varepsilon^{\frac{N-2s}{2}}}{(\varepsilon^2 + |x|^2)^{\frac{N-2s}{2}}}, \quad \varepsilon > 0, \quad x \in \mathbb{R}^N.
\]

Let \( U(x) = \frac{1}{(1+|x|^2)^{\frac{N+2s}{2}}} \) and let \( \mathcal{W} \) be the extension of \( U \) ([4, 6]). Then

\[
\mathcal{W}(x, y) = E_s(U) = c_{N,s} y^{2s} \int_{\mathbb{R}^N} \frac{U(z)}{|x-z|^2 + y^2}^{\frac{N+2s}{2}} \, dz,
\]
Hence, we have that 
may assume 
\( k \) is the extreme function for the fractional Sobolev inequality \((11)\). The constant \( S \) given in \((10)\) takes the exact value 
\[
S = \frac{2\pi^s \Gamma\left(\frac{N+2s}{2}\right) \Gamma(1-s) \Gamma\left(\frac{N}{2}\right)^{\frac{1}{2s}}}{\Gamma(s) \Gamma\left(\frac{N-2s}{2}\right) \Gamma(N)^s},
\]
and it is achieved for \( \Omega = \mathbb{R}^N \) by the functions \( w_x \). Without loss of generality, we may assume \( k_x = 1 \).

3. The Nehari manifold. It is useful to consider the functional on the Nehari manifold 
\[
\mathcal{M}_\lambda := \{ w \in X^*_\Omega(c_\Omega) \setminus \{0\} : \langle J'_\lambda(w), w \rangle = 0 \}.
\]
Thus \( w \in \mathcal{M}_\lambda \) if and only if 
\[
\langle J'_\lambda(w), w \rangle = \int_{\Omega} y^{1-2s}|\nabla w|^2 dxdy - \lambda \int_{\Omega \times \{y=0\}} f(x)|w|^q dx - \int_{\Omega \times \{y=0\}} g(x)|w|^{2^*} dx.
\]
It is clear that all critical points of \( J_\lambda \) must lie in \( \mathcal{M}_\lambda \), and as we will see below, local minimizers on \( \mathcal{M}_\lambda \) are actually critical points of \( J_\lambda \). We have the following results.

**Lemma 3.1.** The energy functional \( J_\lambda \) is coercive and bounded below on \( \mathcal{M}_\lambda \).

**Proof.** For \( w \in \mathcal{M}_\lambda \), by Hölder inequality and the Sobolev embedding theorem, we get 
\[
J_\lambda(w) = \frac{1}{2} \int_{\mathcal{C}_\Omega} y^{1-2s}|\nabla w|^2 dxdy - \lambda \int_{\Omega \times \{y=0\}} f(x)|w|^q dx - \frac{1}{2^*} \int_{\Omega \times \{y=0\}} g(x)|w|^{2^*} dx
= \frac{s}{N} \int_{\mathcal{C}_\Omega} y^{1-2s}|\nabla w|^2 dxdy - \lambda \left( \frac{1}{q} - \frac{1}{2^*} \right) \int_{\Omega \times \{y=0\}} f(x)|w|^q dx
\geq \frac{s}{N} \int_{\mathcal{C}_\Omega} y^{1-2s}|\nabla w|^2 dxdy - \lambda \left( \frac{1}{q} - \frac{1}{2^*} \right) |f|_\infty S^{-\frac{q}{2}} \| w \|_{X^*_\Omega(c_\Omega) \Omega}^{1-\frac{2}{2^*}}.
\]
(12)

Hence, we have that \( J_\lambda(w) \) is coercive and bounded below on \( \mathcal{M}_\lambda \). \( \square \)

Define \( \psi_\lambda(w) = \langle J'_\lambda(w), w \rangle \). Then for \( w \in \mathcal{M}_\lambda \), we get 
\[
\langle \psi'_\lambda(w), w \rangle = 2\| w \|_{X^*_\Omega(c_\Omega)}^2 - \lambda q \int_{\Omega \times \{y=0\}} f(x)|w|^q dx - 2^* \int_{\Omega \times \{y=0\}} g(x)|w|^{2^*} dx
= (2 - q)\| w \|_{X^*_\Omega(c_\Omega)}^2 - (2^* - q) \int_{\Omega \times \{y=0\}} g(x)|w|^{2^*} dx
= \lambda (2^* - q) \int_{\Omega \times \{y=0\}} f(x)|w|^q dx - (2^* - q)\| w \|_{X^*_\Omega(c_\Omega)}^2.
\]
(13)

We apply the method in [29], let 
\[
\mathcal{M}_\lambda^+ = \{ w \in \mathcal{M}_\lambda | \langle \psi'_\lambda(w), w \rangle > 0 \};
\mathcal{M}_\lambda^0 = \{ w \in \mathcal{M}_\lambda | \langle \psi'_\lambda(w), w \rangle = 0 \};
\mathcal{M}_\lambda^- = \{ w \in \mathcal{M}_\lambda | \langle \psi'_\lambda(w), w \rangle < 0 \}.
\]

By using the equality \((13)\), we get that \( \int_{\Omega \times \{y=0\}} f(x)|w|^q dx > 0 \) for \( w \in \mathcal{M}_\lambda^+ \). Moreover, we have the following results.
Lemma 3.2. If $z_0$ is a local minimizer for $J_\lambda$ on $M_\lambda$ and $z_0 \notin M_\lambda^0$, then $J'_\lambda(z_0) = 0$.

Proof. Let $z_0 \in M_\lambda$ be a local minimizer for $J_\lambda$ on $M_\lambda$. Then $z_0$ solves
\[
\min \{ J_\lambda(z) : \psi_\lambda(z) = 0 \}. \tag{14}
\]
Hence, there exists a Lagrange multiplier $\gamma \in \mathbb{R}$ such that $J'_\lambda(z_0) = \gamma \psi'_\lambda(z_0)$.

Thus
\[
J'_\lambda(z_0) = \gamma \langle \psi'_\lambda(z_0), z_0 \rangle = 0.
\tag{15}
\]
Since $z_0 \notin M_\lambda^0$, then $\langle \psi'_\lambda(z_0), z_0 \rangle \neq 0$, yielding $\gamma = 0$. This completes the proof. \hfill \square

Lemma 3.3. Let $\Lambda$ be a constant defined as in (6). If $0 < \lambda < \Lambda$, then $M_\lambda^0 = \emptyset$.

Proof. Assuming the contrary, there is a number $\lambda_0 \in (0, \Lambda)$ such that $M_{\lambda_0}^0 \neq \emptyset$.

Then for $w \in M_{\lambda_0}^0$, by (13), we have
\[
\|w\|_{X_0^s(\Omega)}^2 = \frac{2^*_s - q}{2 - q} \int_{\Omega \times \{y = 0\}} g(x) |w|^{2^*_s} dx = \lambda_0 \frac{2^*_s - q}{2 - q} \int_{\Omega \times \{y = 0\}} f(x) |w|^q dx.
\]
Using (h1), the Hölder and the Sobolev inequalities, we get
\[
\|w\|_{X_0^s(\Omega)}^2 \leq \frac{2^*_s - q}{2 - q} S^{-\frac{2^*_s}{q}} \|w\|^2_{X_0^s(\Omega)},
\]
then we get
\[
\|w\|_{X_0^s(\Omega)}^2 \geq \frac{2 - q}{2^*_s - q} S^{\frac{-2^*_s}{2}}. \tag{16}
\]
By direct computation, we obtain
\[
\|w\|_{X_0^s(\Omega)}^2 = \lambda_0 \frac{2^*_s - q}{2^*_s - 2} \int_{\Omega \times \{y = 0\}} f(x) |w|^q dx \leq \lambda_0 \frac{2^*_s - q}{2^*_s - 2} |f|_\infty \int_{\Omega \times \{y = 0\}} |w|^q dx \leq \lambda_0 \frac{2^*_s - q}{2^*_s - 2} |f|_\infty |\Omega|^{1 - \frac{2^*_s}{q}} S^{-\frac{2^*_s}{q}} \|w\|_{X_0^s(\Omega)}^q.
\]
then
\[
\|w\|_{X_0^s(\Omega)} \leq (\lambda_0 \frac{2^*_s - q}{2^*_s - 2} |f|_\infty |\Omega|^{\frac{2^*_s - 2}{q}} S^{-\frac{2^*_s}{q}})^{\frac{1}{q - 2}}. \tag{17}
\]
Combining (16) and (17), we have
\[
\lambda_0 \geq \frac{2^*_s - 2}{(2^*_s - q)/|f|_\infty} \left( \frac{2 - q}{2^*_s - q} \right)^{\frac{2^*_s - q}{2^*_s - 2}} |\Omega|^{\frac{q - 2}{2^*_s}} S^{\frac{q - 2}{2^*_s}} = \Lambda,
\]
which is a contradiction. \hfill \square

For $w \in H_+ = \{ w \in X_0^s(\Omega) | w \neq 0 \}$, let
\[
t_{\text{max}} = t_{\text{max}}(w) = \left( \frac{(2 - q)\|w\|_{X_0^s(\Omega)}^2}{2 - q} \right)^{\frac{1}{q - 2}} > 0.
\]
Lemma 3.4. For each \( w \in H_+ \), we have that
(i) if \( \int_{\Omega \times \{y = 0\}} |f(x)|^q dx = 0 \), then there exist unique positive number \( t^- = t^-(w) > t_{\max} \) such that \( t^- w \in \mathcal{M}_\Lambda^- \) and \( J_\Lambda(t^- w) = \sup_{t \geq t_{\max}} J_\Lambda(tw) \).
(ii) if \( 0 < \lambda < \Lambda \) and \( \int_{\Omega \times \{y = 0\}} |f(x)|^q dx > 0 \), then there exist unique numbers \( t^+ = t^+(w) < t_{\max} = t^-(w) \) such that \( t^+ w \in \mathcal{M}_\Lambda^+ \), \( t^- w \in \mathcal{M}_\Lambda^- \), and
\[
J_\Lambda(t^+ w) = \inf_{0 \leq t \leq t_{\max}} J_\Lambda(tu), \quad J_\Lambda(t^- w) = \sup_{t \geq t_{\max}} J_\Lambda(tw).
\]
Proof. For each \( w \in X_0^q(\Omega) \setminus \{0\} \) and \( w \neq 0 \), we define
\[
K(t) = K_w(t) = t^{2 - q} \|w\|^2_{X_0^q(\Omega)} - t^{\alpha + q} \int_{\Omega \times \{y = 0\}} g(x)|w|^2\, dx.
\]
Similar to Lemma 7 in [7], we get the conclusion.

Applying Lemma 3.3, \( \mathcal{M}_\Lambda^0 = \emptyset \) for \( 0 < \lambda < \Lambda \), we write \( \mathcal{M}_\Lambda = \mathcal{M}_\Lambda^+ \cup \mathcal{M}_\Lambda^- \) and define
\[
\alpha_\Lambda = \inf_{w \in \mathcal{M}_\Lambda} J_\Lambda(w), \quad \alpha_\Lambda^+ = \inf_{w \in \mathcal{M}_\Lambda^+} J_\Lambda(w), \quad \alpha_\Lambda^- = \inf_{w \in \mathcal{M}_\Lambda^-} J_\Lambda(w).
\]

Lemma 3.5.
(i) If \( \lambda \in (0, \Lambda) \), then \( \alpha_\Lambda \leq \alpha_\Lambda^+ < 0 \);
(ii) If \( \lambda \in (0, \frac{q\Lambda}{2}) \), then \( \alpha_\Lambda^- \geq d_0 > 0 \) for some constant \( d_0 = d_0(N, q, s, |\Omega|, \lambda, |f|_\infty) \).

Proof. (i) Let \( w \in \mathcal{M}_\Lambda^+ \), by (13), we get
\[
\frac{2 - q}{2s - q} \|w\|^2_{X_0^q(\Omega)} > \int_{\Omega \times \{y = 0\}} g(x)|w|^2\, dx.
\]
Then
\[
J_\Lambda(w) = \frac{1}{2} \int_{\Omega \times \{y = 0\}} \|\nabla w\|^2\, dx - \lambda \int_{\Omega \times \{y = 0\}} f(x)|w|^q\, dx - \frac{1}{2s} \int_{\Omega \times \{y = 0\}} g(x)|w|^2\, dx
\]
\[
= \left( \frac{1}{2} - \frac{1}{q} \right) \|w\|^2_{X_0^q(\Omega)} + \left( \frac{1}{q} - \frac{1}{2s} \right) \int_{\Omega \times \{y = 0\}} g(x)|w|^2\, dx
\]
\[
< \left[ \left( \frac{1}{2} - \frac{1}{q} \right) + \frac{2s - q}{2s q} \cdot \frac{2 - q}{2s - q} \right] \|w\|^2_{X_0^q(\Omega)}
\]
\[
= \frac{2 - q}{2s - q} \|w\|^2_{X_0^q(\Omega)} < 0.
\]
By the definition of \( \alpha_\Lambda \) and \( \alpha_\Lambda^+ \), we deduce that \( \alpha_\Lambda \leq \alpha_\Lambda^+ < 0 \).
(ii) Let \( w \in \mathcal{M}_\Lambda^- \), by (13) and the Sobolev embedding theorem, we get
\[
\frac{2 - q}{2s - q} \|w\|^2_{X_0^q(\Omega)} < \int_{\Omega \times \{y = 0\}} g(x)|w|^2\, dx < \int_{\Omega \times \{y = 0\}} |w|^2\, dx \leq S^{-\frac{2}{2s}} \|w\|^2_{X_0^q(\Omega)}.
\]
This implies
\[
\|w\|_{X_0^q(\Omega)} > \left( \frac{2 - q}{2s - q} \right)^{\frac{1}{s - q}} S^{\frac{2}{2s}}, \text{ for any } w \in \mathcal{M}_\Lambda^-.
\]
Using (12) and (13), we obtain that
\[
J_\Lambda(w) \geq \frac{s}{N} \|w\|^2_{X_0^q(\Omega)} - \lambda \left( \frac{1}{q} - \frac{1}{2s} \right) |f|_\infty S^{-\frac{2}{2s}} |\Omega|^{1 - \frac{2}{2s}} \|w\|^2_{X_0^q(\Omega)}.
\]
Hence, if $\lambda \in (0, \frac{2\lambda}{2})$, then for any $w \in \mathcal{M}_\lambda$,
\[ J_\lambda(w) \geq d_0(N, q, s, |\Omega|, \lambda, |f|_\infty) > 0. \]
\[ \square \]

Now, we use the idea in [13] to show the existence of a $(PS)_{\alpha\lambda}$-sequence and a $(PS)_{\alpha\lambda}$-sequence.

**Proposition 1.** We have the following

(i) for $\lambda \in (0, \Lambda)$, there is a $(PS)_{\alpha\lambda}$-sequence $\{u_n\} \subset \mathcal{M}_\lambda$ in $X_0^s(\Omega_1)$ for $J_\lambda$.

(ii) for $\lambda \in (0, \frac{\sqrt{2}}{2})$, we have that $(PS)_{\alpha\lambda}$-sequence $\{u_n\} \subset \mathcal{M}_\lambda$ in $X_0^s(\Omega_1)$ for $J_\lambda$.

**Proof.** The proof is similar to Wu [29].

**Proof of Theorem 1.1.** By proposition 3.1, there is a minimizing sequence $\{w_n\} \subset \mathcal{M}_\lambda$ for $J_\lambda$ such that
\[ J_\lambda(w_n) = \alpha_\lambda + o_1(1), \quad J'_\lambda(w_n) = o_1(1). \]
Since $J_\lambda$ is coercive on $\mathcal{M}_\lambda$, then $\{w_n\}$ is bounded in $X_0^s(\Omega)$. It follows that there exists a subsequence $\{w_n\}$ and $w_\lambda \in X_0^s(\Omega)$ such that, as $n \to \infty$,
\[ w_n(x, y) \rightharpoonup w_\lambda(x, y), \quad \text{weakly in } X_0^s(\Omega), \]
\[ w_n(x, 0) \to w_\lambda(x, 0), \quad \text{a.e in } \Omega, \]
\[ w_n(x, 0) \to w_\lambda(x, 0), \quad \text{strongly in } L^s(\Omega) (1 \leq s < 2^*_\lambda), \]
Since $\{w_n\} \subset M_\lambda$, we get
\[ \lambda \int_{\Omega \times \{0\}} f(x)w_n^q \, dx = \frac{q(2^*_\lambda - 2)}{2(2^*_\lambda - q)} \|w_n\|_{X_0^s(\Omega)}^2 - \frac{2^*_\lambda q}{2^*_\lambda - q} J_\lambda(w_n) \quad (19) \]
Let $n \to \infty$ in (19), we have
\[ \lambda \int_{\Omega \times \{0\}} f(x)w_\lambda^q \, dx \geq - \frac{2^*_\lambda q}{2^*_\lambda - q} \alpha_\lambda > 0, \quad (20) \]
therefore $w_\lambda \in M_\lambda$ is a nontrivial solution of (2). Next, we want to show that $w_n \to w_\lambda$ strongly in $X_0^s(\Omega)$ and $J_\lambda(w_\lambda) = \alpha_\lambda$. Since $w_\lambda \in M_\lambda$, combining (12) and Fatou Lemma, we have
\[ \alpha_\lambda \leq J_\lambda(w_\lambda) = \frac{s}{N} \|w_\lambda\|_{X_0^s(\Omega)}^2 - \lambda \left( \frac{2^*_\lambda - 2}{2^*_\lambda q} \right) \int_{\Omega \times \{y = 0\}} f(x) |w_\lambda|^q \, dx \]
\[ \leq \lim \inf_{n \to \infty} \frac{s}{N} \|w_n\|_{X_0^s(\Omega)}^2 - \lambda \left( \frac{2^*_\lambda - 2}{2^*_\lambda q} \right) \int_{\Omega \times \{y = 0\}} f(x) |w_n|^q \, dx \]
\[ \leq \lim \inf_{n \to \infty} J_\lambda(w_n) = \alpha_\lambda. \]
It follows that $J_\lambda(w_\lambda) = \alpha_\lambda$ and $\lim_{n \to \infty} \|w_n\|_{X_0^s(\Omega)}^2 = \|w_\lambda\|_{X_0^s(\Omega)}^2$, that is $w_n \to w_\lambda$ strongly in $X_0^s(\Omega)$. We claim that $w_\lambda \in \mathcal{M}_\lambda^\alpha$. On the contrary, assume that
$w_\lambda \in \mathcal{M}_\lambda$. Since $\lambda \int_{\Omega \times \{y=0\}} f(x)|w_\lambda|^q \, dx = 0$, by Lemma 3.4, then there exist positive numbers $t_0^+ < t_{\text{max}} < t_0^-$ = 1 such that $t_0^+ w_\lambda \in \mathcal{M}_\lambda^+$, $t_0^- w_\lambda \in \mathcal{M}_\lambda^-$ and $J_\lambda(t_0^- w_\lambda) < J_\lambda(t_0^+ w_\lambda) = J_\lambda(w_\lambda) = \alpha_\lambda$, which is a contradiction. Hence $w_\lambda \in \mathcal{M}_\lambda^+$ and $J_\lambda(w_\lambda) = \alpha_\lambda = \alpha_\lambda^+$. Since $J_\lambda(w_\lambda) = J_\lambda(|w_\lambda|)$ and $|w_\lambda| \in \mathcal{M}_\lambda$, by Lemma 3.2, we may assume that $w_\lambda$ is a nontrivial nonnegative solution of (8). Then by strong maximum principle([10]), we have $w_\lambda > 0$ in $\Omega$. Hence, $w_\lambda$ is a positive solution for (8).

4. Existence of $k+1$ solution. In this section, we shall detect the range of value $\beta$ for which $(PS)_\beta$ condition holds for the functional $J_{\lambda}$. First of all, we want to show that $J_{\lambda}$ satisfies the $(PS)_\beta$ condition in $X_{0}^\alpha(\Omega)$ for $\beta \in (-\infty, \frac{s}{N}S^\frac{s}{2} - C_0\lambda^{\frac{q}{2}})$. We shall have the following preliminary results.

**Lemma 4.1.** If $\{w_n\}$ is a $(PS)_\beta$-sequence in $X_{0}^\alpha(\Omega)$ for $J_{\lambda}$ with $w_n \rightharpoonup w$ weakly in $X_{0}^\alpha(\Omega)$. Then $J_{\lambda}(w) = 0$ in $(X_{0}^\alpha(\Omega))^{-1}$ and there exists a positive constant $C_0 = C_0(N, q, S, |\Omega|, |f|_{\infty}) > 0$ such that $J_{\lambda}(w) \geq -C_0\lambda^{\frac{q}{2}}$.

**Proof.** Since $\{w_n\}$ is a $(PS)_\beta$ sequence in $X_{0}^\alpha(\Omega)$ for $J_{\lambda}$. It is standard to check that $J_{\lambda}(w) = 0$. Then we have $(J_{\lambda}'(w), w) = 0$, namely

$$\|w\|_{X_{0}^\alpha(\Omega)}^2 - \lambda \int_{\Omega \times \{y=0\}} f(x)|w|^q \, dx = \int_{\Omega \times \{y=0\}} g(x)|w|^{2q} \, dx.$$ 

Consequently, we get

$$J_{\lambda}(w) = (\frac{1}{2} - \frac{1}{2^*_s}) \int_{\Omega} y^{1-2^*_s} |\nabla w|^2 \, dx \, dy - \lambda(\frac{1}{q} - \frac{1}{2^*_s}) \int_{\Omega \times \{y=0\}} f(x)|w|^q \, dx. \quad (21)$$

Then, combining (21), the Young inequality and the Sobolev embedding theorem, we obtain,

$$J_{\lambda}(w) \geq \frac{s}{N} \|w\|_{X_{0}^\alpha(\Omega)}^2 - \lambda(\frac{2^*_s-q}{2^*_sq})|\Omega|^{\frac{2^*_s-q}{2^*_s}} S^{-\frac{2^*_s}{2^*_s}} |f|_{\infty} \|w\|_{X_{0}^\alpha(\Omega)}^q \geq \frac{s}{N} \|w\|_{X_{0}^\alpha(\Omega)}^2 - C(\frac{2^*_s-q}{2^*_sq}) \lambda^{\frac{q}{2}} \|w\|_{X_{0}^\alpha(\Omega)}^q \geq \frac{s}{N} \|w\|_{X_{0}^\alpha(\Omega)}^2 - \frac{s}{N} \|w\|_{X_{0}^\alpha(\Omega)}^2 - C_0\lambda^{\frac{q}{2}} = -C_0\lambda^{\frac{q}{2}},$$

where $C_0 = C_0(N, q, S, |\Omega|, |f|_{\infty}) > 0$, $C = \frac{2^*_s-q}{2^*_sq} |\Omega|^{\frac{2^*_s-q}{2^*_s}} S^{-\frac{2}{2^*_s}}$, $\varepsilon = (\frac{NgC}{2^*_s})^{\frac{1}{2^*_s}}$, $C_0 = \frac{2^*_s-q}{2^*_sq}$. \qed

**Lemma 4.2.** $J_{\lambda}$ satisfies the $(PS)_\beta$-condition in $X_{0}^\alpha(\Omega)$ for $\beta \in (-\infty, \frac{s}{N}S^\frac{s}{2} - C_0\lambda^{\frac{q}{2}})$ where $C_0 > 0$ is given in Lemma 4.1.

**Proof.** Let $\{w_n\}$ be a $(PS)_\beta$ sequence in $X_{0}^\alpha(\Omega)$ for $J_{\lambda}$ such that

$$J_{\lambda}(w_n) = \beta + o_n(1), \quad J_{\lambda}'(w_n) = o_n(1).$$

Then

$$|\beta| + c_n + \frac{d_n\|w_n\|_{X_{0}^\alpha(\Omega)}}{2^*_s} \geq J_{\lambda}(w_n) - \frac{1}{2^*_s}(J_{\lambda}'(w_n), w_n) \geq \frac{s}{N} \|w_n\|_{X_{0}^\alpha(\Omega)}^2 - \lambda(\frac{2^*_s-q}{2^*_sq}) \int_{\Omega \times \{y=0\}} f(x)|w_n|^q \, dx$$
\[
\geq \frac{s}{N} \|w_n\|_{X_0^s(\mathcal{C}_\Omega)}^2 - \lambda \frac{2s - q}{2s q} \|f\|_\infty \Omega \frac{2^{s-q}}{2s} S^{-\frac{q}{2s}} \|w_n\|_{X_0^s(\mathcal{C}_\Omega)},
\]

where \(c_n = o_n(1), d_n = o_n(1)\) as \(n \to \infty\). It follows that \(\{w_n\}\) is bounded in \(X_0^s(\mathcal{C}_\Omega)\). Hence there exists \(w \in X_0^s(\mathcal{C}_\Omega)\) such that

\[
w_n(x, y) \rightharpoonup w(x, y), \quad \text{weakly in } X_0^s(\mathcal{C}_\Omega),
\]

\[
w_n(x, 0) \to w(x, 0), \quad \text{a.e in } \Omega,
\]

\[
w_n(x, 0) \to w(x, 0), \quad \text{strongly in } L^s(\Omega) \,(1 \leq s < 2^*_s).
\]

Then we obtain

\[
\lambda \int_{\Omega \times \{y = 0\}} f(x)\|w_n\|^q dx = \int_{\Omega \times \{y = 0\}} f(x)\|w\|^q dx + o_n(1),
\]

\[
\|w_n - w\|_{X_0^s(\mathcal{C}_\Omega)}^2 = \|w_n\|_{X_0^s(\mathcal{C}_\Omega)}^2 - \|w\|_{X_0^s(\mathcal{C}_\Omega)}^2 + o_n(1),
\]

\[
\int_{\Omega \times \{y = 0\}} g(x)\|w_n - w\|^2 dx = \int_{\Omega \times \{y = 0\}} g(x)\|w_n\|^2 dx - \int_{\Omega \times \{y = 0\}} g(x)\|w\|^2 dx + o_n(1).
\]

Since \(J_\lambda(w_n) = \beta + o_n(1)\) and \(J'_\lambda(w_n) = o_n(1)\), by \(J_\lambda(w) = 0\), we deduce that

\[
J_\lambda(w_n)
= \frac{1}{2} \int_{\mathcal{C}_\Omega} y^{-2s} |\nabla w_n|^2 dxdy - \frac{\lambda}{q} \int_{\Omega \times \{y = 0\}} f(x)\|w_n\|^q dx - \frac{1}{2s} \int_{\Omega \times \{y = 0\}} g(x)\|w_n\|^2 dx
\]

\[
= \frac{1}{2} \int_{\mathcal{C}_\Omega} y^{-2s} |\nabla (w_n - w)|^2 dxdy + \frac{1}{2} \int_{\mathcal{C}_\Omega} y^{-2s} |\nabla w|^2 dxdy - \frac{\lambda}{q} \int_{\Omega \times \{y = 0\}} f(x)\|w\|^q dx
\]

\[
- \frac{1}{2s} \int_{\Omega \times \{y = 0\}} g(x)\|w_n - w\|^2 dx - \frac{1}{2s} \int_{\Omega \times \{y = 0\}} g(x)\|w\|^2 dx
\]

\[
= J_\lambda(w) + \frac{1}{2} \int_{\mathcal{C}_\Omega} y^{-2s} |\nabla (w_n - w)|^2 dxdy - \frac{1}{2s} \int_{\Omega \times \{y = 0\}} g(x)\|w_n - w\|^2 dx,
\]

and

\[
\frac{1}{2} \int_{\mathcal{C}_\Omega} y^{-2s} |\nabla (w_n - w)|^2 dxdy - \int_{\Omega \times \{y = 0\}} g(x)\|w_n - w\|^2 dx = o_n(1).
\]

Now, we assume that

\[
\|w_n - w\|_{X_0^s(\mathcal{C}_\Omega)} \to l \quad \text{and} \quad \int_{\Omega \times \{y = 0\}} g(x)\|w_n - w\|^2 dx \to l \quad \text{as } n \to \infty.
\]

Applying the Sobolev inequality, we obtain

\[
\|w_n - w\|_{X_0^s(\mathcal{C}_\Omega)}^2 \geq S_\infty \|w_n(x, 0) - w(x, 0)\|_{L_{2^*_s}^*}^2.
\]

Then \(l \geq S_\infty \frac{s}{2}\). If \(l \neq 0\), then \(l \geq S_\infty \frac{s}{2}\), by Lemma 4.1, we have that

\[
\beta = \left(\frac{1}{2} - \frac{1}{2s}\right) l + J_\lambda(w) \geq \frac{s}{N} S_\infty - C_0 \lambda \frac{2^{s-q}}{2s},
\]

which is a contradiction. Hence, \(l = 0\), that is \(w_n \to w\) strongly in \(X_0^s(\mathcal{C}_\Omega)\). \(\square\)

Recall that the best Sobolev constant \(S\) is defined as

\[
S = \inf_{w \in X_0^s(\mathcal{C}_\Omega) \setminus \{0\}} \|k_s \int_{\mathcal{C}_\Omega} y^{-2s} |\nabla w|^2 dxdy\| \left(\int_{\Omega \times \{y = 0\}} |w|^2 dx\right)^{\frac{2}{2s}}.
\]
It is well known that $U(x) = \frac{1}{(1+|x|^2)^{\frac{N}{2}}}$ is a minimizer of $S$. Denote

$$U_{\varepsilon, a^i}(x) = \varepsilon^{\frac{N-2s}{2}} U\left(\frac{x - a^i}{\varepsilon}\right) = \frac{\varepsilon^{\frac{N-2s}{2}}}{(\varepsilon^2 + |x - a^i|^2)^{\frac{N}{2}}},$$

then the extension of $U_{\varepsilon, a^i}$ has the form

$$W_{\varepsilon}(x, y) = c_{N,s} y^{2s} \int_{\mathbb{R}^N} \frac{U_{\varepsilon, a^i}(z)dz}{(|x-z|^2 + y^2)^{\frac{N+2s}{2}}} = c_{N,s} y^{2s} \int_{\mathbb{R}^N} \frac{\varepsilon^{\frac{2s-N}{2}} U\left(\frac{z-a^i}{\varepsilon}\right)dz}{(|x-z|^2 + y^2)^{\frac{N+2s}{2}}},$$

$$= c_{N,s} y^{2s} \int_{\mathbb{R}^N} \frac{\varepsilon^{\frac{2s-N}{2}} \varepsilon^N U(m)}{|x-\varepsilon m - a^i|^2} dm$$

$$= \varepsilon^{\frac{2s-N}{2}} W(\frac{x-a^i}{\varepsilon}, \frac{y}{\varepsilon}),$$

where $W(x, y) = c_{N,s} y^{2s} \int_{\mathbb{R}^N} \frac{U(z)}{|(x-z|^2 + y^2)^{\frac{N+2s}{2}}} dz$.

Without loss of generality, we may assume that $0 \in \Omega$. We define the cut-off function $\phi \in C_0^\infty(\Omega)$, $0 \leq \phi \leq 1$ and for small fixed $\rho > 0$

$$\phi_i(x, y) = \begin{cases} 1, & \text{if } (x, y) \in B_\rho, \\ 0, & \text{if } (x, y) \notin B_{2\rho}, \end{cases} \quad (22)$$

where $B_\rho = \{(x, y) : |x - a^i|^2 + y^2 < \rho^2, y > 0\}$, we take $\rho$ so small that $B_{2\rho} \subset \overline{\Omega}$.

Define $w^i_\varepsilon = \phi W^i_\varepsilon \in X_0^s(\Omega)$ for $\varepsilon > 0$ small enough. From now on, we assume that $\frac{N}{2s-2} < q < 2$ and $N > 4s$.

**Lemma 4.3.** There exist $\tau > 0$ and $\Lambda_0 \in (0, \frac{2\Lambda}{q})$, then

$$\sup_{t \geq 0} J_\lambda(tu^i_\varepsilon) < \frac{s}{N} S_{\frac{N}{2}} - C_0 \lambda^{\frac{2}{2s-q}}, \text{ uniformly in } i$$

where $C_0 > 0$ is given in Lemma 4.1 and $\varepsilon = \lambda^{\frac{2}{q}}$. In particular, $0 < \alpha^i_\varepsilon < \frac{s}{N} S_{\frac{N}{2}} - C_0 \lambda^{\frac{2}{2s-q}}$ for any $\lambda \in (0, \Lambda_0)$.

**Proof.** By an argument similar to the proof of [4], we get

$$\|\phi W^i_\varepsilon\|_{X_0^s(\Omega)}^2 = \int_{B_{N+1}^+} y^{1-2s} |\nabla W^i_\varepsilon|^2 dy + O(\varepsilon^{N-2s})$$

$$= \int_{B_{N+1}^+} \varepsilon^{1-2s} \varepsilon^{2s-2sN} \varepsilon^{N+1} y \frac{1}{\varepsilon^2} (y \varepsilon^2)^{1-2s} |\nabla \varepsilon W|^2 dz_1 dz_2 + O(\varepsilon^{N-2s})$$

$$= \int_{B_{N+1}^+} z_2^{1-2s} |\nabla W(z_1, z_2)|^2 dz_1 dz_2 + O(\varepsilon^{N-2s}).$$

We notice that

$$\|\phi_i(x, 0)U_{\varepsilon, a^i}\|_{L^2_\varepsilon(\Omega)}^2 = \int_{\Omega} |\phi_i(x, 0)U_{\varepsilon, a^i}|^2 dx$$

$$= \int_{\Omega} \phi_i(x, 0)^2 \left(\frac{\varepsilon^{N-2s}}{(\varepsilon^2 + |x - a^i|^2)^{\frac{N}{2}}}\right)^\frac{2s}{N} dx$$

$$= \int_{\Omega} \varepsilon^N \phi_i(x, 0)^2 \left(\frac{\varepsilon^{N-2s}}{(\varepsilon^2 + |x - a^i|^2)^{\frac{N}{2}}}\right)^\frac{2s}{N} dx.$$
Then, we obtain
\[ \|U_{\varepsilon,a^i}\|_{L^2_2(\Omega)}^2 = \int_{\mathbb{R}^N} \frac{\varepsilon^N}{(\varepsilon^2 + |x-a^i|^2)^N} \, dx = \int_{\mathbb{R}^N} \frac{1}{(1 + |y|^2)^N} \, dy = \|U\|_{L^2_2(\Omega)}^2. \]

This implies that
\[ \|\phi_i(x,0)U_{\varepsilon,a^i}\|_{L^2_2(\Omega)}^2 - \|U_{\varepsilon,a^i}\|_{L^2_2(\Omega)}^2 \]
\[ = \int_{\Omega} \phi_i(x,0)^2 \left( \frac{\varepsilon^N}{(\varepsilon^2 + |x-a^i|^2)^N} - \frac{\varepsilon^N}{(\varepsilon^2 + |x-a^i|^2)} \right) \, dx \]
\[ \leq \int_{\Omega \setminus B(a^i,\varepsilon)} \frac{\varepsilon^N}{(\varepsilon^2 + |x-a^i|^2)^N} \, dx + \int_{\Omega \setminus \Omega} \frac{\varepsilon^N}{(\varepsilon^2 + |x-a^i|^2)} \, dx \]
\[ \leq \int_{\Omega \setminus B(a^i,\varepsilon)} \frac{1}{(1 + |y|^2)^N} \, dy \leq O(\varepsilon^N). \]

Taking \( \varepsilon \) so small that \( \varepsilon^N \|U\|_{L^2_2(\Omega)}^2 < 1 \), since \( \frac{N-2s}{2} = \frac{N-2s}{2} < 1 \), we obtain
\[ (1 - \varepsilon^N \|U\|_{L^2_2(\Omega)}^{-2}) \leq (\|\phi_i(x,0)U_{\varepsilon,a^i}\|_{L^2_2(\Omega)}^2 \|U\|_{L^2_2(\Omega)}^{-2}) \]
\[ \leq (1 + \varepsilon^N \|U\|_{L^2_2(\Omega)}^{-2}) \]
\[ \leq (1 + C\varepsilon^N \|U\|_{L^2_2(\Omega)}^{-2}). \]

Then, we obtain
\[ \|\phi(x,0)U_{\varepsilon,a^i}\|_{L^2_2(\Omega)}^2 = \|U\|_{L^2_2(\Omega)}^2 + O(\varepsilon^N). \quad (23) \]

For \( \frac{N}{N-2s} < q < 2, N > 4s \) and \( \varepsilon < \frac{\varepsilon_0}{2} \)
\[ \|U_{\varepsilon,a^i}\|_{L^q(\Omega)}^q = \int_{\mathcal{B}_R(0)} \left( \frac{\varepsilon^{N-2s}}{\varepsilon} U \left( \frac{x-a^i}{\varepsilon} \right) \right)^q \, dx + O(\varepsilon^{N-2s}) \]
\[ = \int_{\mathcal{B}_R(a^i)} \left( \frac{\varepsilon^{N-2s}}{1 + \left( \frac{x-a^i}{\varepsilon} \right)^2 \frac{N-2s}{2}} \right)^q \, dx + O(\varepsilon^{N-2s}) \]
\[ = \int_{\mathcal{B}_R(0)} \left( \frac{\varepsilon^{N-2s}}{1 + \left( \frac{x}{\varepsilon} \right)^2 \frac{N-2s}{2}} \right)^q \, dx + O(\varepsilon^{N-2s}) \]
\[ = \varepsilon^{N-2s} \int_{\mathcal{B}_R(0)} \left( \frac{\varepsilon^{N-2s}}{1 + \left( \frac{y}{\varepsilon} \right)^2 \frac{N-2s}{2}} \right)^q \, dy + O(\varepsilon^{N-2s}) \]
\[ = \varepsilon^{N-2s} \int_{0}^{\frac{R}{\varepsilon}} \varepsilon^{N-2s} \int_{0}^{\frac{R}{\varepsilon}} \varepsilon^{N-2s} \, dy \, dx + O(\varepsilon^{N-2s}) \]
\[ \geq C \frac{(N-2s)}{2} + O(\varepsilon^{N-2s}). \]
Next, we consider the functional $J_0 : X_0^s(\mathcal{C}_\Omega) \to \mathbb{R}^N$ defined by

$$J_0(w) = \frac{1}{2} \|w\|_{X_0^s(\mathcal{C}_\Omega)}^2 - \frac{1}{2^*} \int_{\Omega \times \{y = 0\}} g(x)|w|^2^* dx.$$ 

Step I. Show that $\sup_{t \geq 0} J_0(tw^i_\varepsilon) \leq \frac{N}{2^*} S^{\frac{N}{2^*}} + O(\varepsilon N^{-2s})$.

By $(h_2)$, we get that as $\varepsilon \to 0^+$

$$\left(\int_{\Omega \times \{y = 0\}} g(x)|w^i_\varepsilon|^2^* dx\right)^{\frac{N}{2^*}} = \left(\int_{\Omega \times \{y = 0\}} |\phi(x,0) U_{\varepsilon, a_i}|^2 L^2(\Omega)\right) = \|U\|^2_{L^2(\mathbb{R}^N)} + O(\varepsilon N).$$

Using (23) and (24), we obtain that

$$\left(\int_{\Omega \times \{y = 0\}} g(x)|w^i_\varepsilon|^2^* dx\right)^{\frac{N}{2^*}} = \frac{\int_{\mathbb{R}^N} y^{1-2s} |\nabla w(x, y)|^2 dx \, dy + O(\varepsilon N^{-2s})}{\|U\|^2_{L^2(\mathbb{R}^N)}} + O(\varepsilon N^N) = S + O(\varepsilon N^{-2s}).$$

Since

$$\max_{t \geq 0} \left(\frac{a}{2} t^2 - \frac{b}{2^*} t^{2^*}\right) = \frac{s}{N} \left(\frac{a}{b} \right)^{\frac{N}{2^*}}, \text{ for any } a > 0, b > 0,$$

by (25), we get that

$$\sup_{t \geq 0} J_0(tw^i_\varepsilon) = \frac{s}{N} \left(\frac{\int_{\Omega \times \{y = 0\}} g(x)|w^i_\varepsilon|^2^* dx}{\int_{\mathbb{R}^N} y^{1-2s} |\nabla w(x, y)|^2 dx \, dy + O(\varepsilon N^{-2s})} \right)^{\frac{N}{2^*}} \leq \frac{s}{N} S^{\frac{N}{2^*}} + O(\varepsilon N^{-2s}).$$

Step II. Choose a positive number $\Lambda_1 < \frac{q\lambda}{2}$ such that

$$\frac{2s}{N} S^{\frac{N}{2^*}} - C_0\lambda^{\frac{2}{2^*}} > 0, \text{ for any } \lambda \in (0, \Lambda_1).$$

By $(h_1)$, we get

$$J_\Lambda(tw^i_\varepsilon) \leq \frac{t^2}{2} \|w^i_\varepsilon\|_{X_0^s(\mathcal{C}_\Omega)}^2 \quad \text{for all } t \geq 0.$$

Since $J_\Lambda$ is continuous in $X_0^s(\mathcal{C}_\Omega)$, $J_\Lambda(0) = 0$, and $\{w^i_\varepsilon\}$ is uniformly bounded in $X_0^s(\mathcal{C}_\Omega)$ for $0 < \varepsilon < \min\{1, \frac{q}{4}\}$, by (26) and (27), there exists $t_0 > 0$ such that for any $\lambda \in (0, \Lambda_1)$

$$\sup_{0 \leq t \leq t_0} J_\Lambda(tw^i_\varepsilon) < \frac{s}{N} S^{\frac{N}{2^*}} - C_0\lambda^{\frac{2}{2^*}}, \text{ uniformly in } i.$$

Applying the results in step I and $(h_3)$, for $\frac{N}{N-2s} < q < 2$ and $N > 4s$ we have

$$\sup_{t \geq t_0} J_\Lambda(tw^i_\varepsilon) = \sup_{t \geq t_0} \left[ J_0(tw^i_\varepsilon) - \frac{tq\lambda}{q} \int_{\Omega \times \{y = 0\}} f(x)|w^i_\varepsilon|^q dx \right]$$

$$\leq \frac{s}{N} S^{\frac{N}{2^*}} + O(\varepsilon N^{-2s}) - \frac{tq}{q} \int_{B_{\rho_0}(a^i \times \{y = 0\})} |w^i_\varepsilon|^q dx$$

$$\leq \frac{s}{N} S^{\frac{N}{2^*}} + O(\varepsilon N^{-2s}) - \frac{tq}{q} C(N, \rho_0)\lambda d_0 \varepsilon^{\frac{(N-2)q}{2}}.$$
Denote $\theta = \frac{N-2s}{2} q > 0$. Let $\tau > 0$ be such that $\frac{\theta}{q} < \tau < N - 2s - \theta$. Then $\tau + \theta < N - 2s$ and $\tau + \theta < \frac{2\pi}{2^\frac{s}{q}}$. Fix $\varepsilon_0 > 0$ such that $\varepsilon_0 < \min\{1, \frac{\rho_0}{2}\}$, $\varepsilon_0^2 < \Lambda_1$ and

$$O(\varepsilon^{N-2s}) - \frac{t_0^2}{q} \varepsilon^{\tau+\theta} d_0 C(N, \rho_0) < -C_0 \varepsilon^{\frac{2}{2^\frac{s}{q}}}.$$

Set $\Lambda_0 = \varepsilon_0^2$, for any $\lambda \in (0, \Lambda_0)$, let $\varepsilon = \lambda \hat{\varepsilon} > 0$. Then for any $\lambda \in (0, \Lambda_0)$

$$\sup_{t \geq 0} J_\lambda(tw_\varepsilon^i) < \frac{s}{N} S^\frac{N}{2^\frac{s}{q}} - C_0 \lambda^\frac{2}{\hat{\varepsilon}}$$

uniformly in $i$.

Moreover, since $\int_{Q \times \{y = 0\}} f(x)(w_\varepsilon^i)^q dx > 0$, by lemma 3.4 (ii), there exists $(t_\varepsilon^i)^- > 0$ such that $(t_\varepsilon^i)^- w_\varepsilon^i \in \mathcal{M}_\lambda$, and

$$0 < \alpha^\lambda \leq J_\lambda((t_\varepsilon^i)^- w_\varepsilon^i) \leq \sup_{t \geq 0} J_\lambda(tw_\varepsilon^i) < \frac{s}{N} S^\frac{N}{2^\frac{s}{q}} - C_0 \lambda^\frac{2}{\hat{\varepsilon}}.$$

Then we complete the results. \qed

We know that

$$B_{\rho_0}(a^i) \cap B_{\rho_0}(a^j) = \emptyset, \quad \text{for } i \neq j \text{ and } 1 \leq i, j \leq k.$$ 

$$\bigcup_{i=1}^k B_{\rho_0}(a^i) \subset \Omega,$$ where $\rho_0 > 0$ and $g(a^i) = g_{\max}$. Define $K = \{a^i | 1 \leq i \leq k\}$ and $K_{2^\frac{s}{q}} = \bigcup_{i=1}^k B_{\rho_0}(a^i)$. Suppose $\bigcup_{i=1}^k B_{\rho_0}(a^i) \subset B_{r_0}(0)$ for some $r_0 > 0$. Let $Q : X_0(C_\Omega) \setminus \{0\} \to \mathbb{R}^N$ be given by

$$Q(u) = \frac{\int_{\Omega} \chi(z)|u|^{2^*} dz}{\int_{\Omega} |u|^{2^*} dz},$$

where $E_\chi(u) = w$ and $\chi : \mathbb{R}^N \to \mathbb{R}^N$

$$\chi(z) = \begin{cases} z, & \text{if } |z| \leq r_0, \\ r_0 \frac{z}{|z|}, & \text{if } |z| > r_0. \end{cases} \quad (28)$$

By Lemma 2.5, we have that $t_\varepsilon^i w_\varepsilon^i \in \mathcal{M}_\lambda$, then we have the following.

**Lemma 4.4.** $Q((t_\varepsilon^i)^- w_\varepsilon^i) \to a^i$ as $\varepsilon \to 0^+$. In particular, there exists $0 < \varepsilon^0 < \varepsilon_0$ such that if $\varepsilon < \varepsilon^0$, then $Q((t_\varepsilon^i)^- w_\varepsilon^i) \in K_{2^\frac{s}{q}}$ for $1 \leq i \leq k$.

**Proof.** Since

$$Q((t_\varepsilon^i)^- w_\varepsilon^i) = \frac{\int_{\Omega} \chi(z)|\varepsilon^\frac{2^*}{2^\frac{s}{q}} \phi_i(z, 0) U(\varepsilon^\frac{a^i}{\varepsilon})|^{2^*} dz}{\int_{\Omega} |\varepsilon^\frac{2^*}{2^\frac{s}{q}} \phi_i(z, 0) U(\varepsilon^\frac{a^i}{\varepsilon})|^{2^*} dz}$$

$$= \frac{\int_{\Omega} \chi(z)|\varepsilon z + a^i|U|^{2^*} dz}{\int_{\Omega} |U|^{2^*} dz} \to a^i, \quad \text{as } \varepsilon \to 0,$$

there exists $\varepsilon^0 > 0$ such that $Q((t_\varepsilon^i)^- w_\varepsilon^i) \in K_{2^\frac{s}{q}}$ for any $\varepsilon < \varepsilon^0$ and each $1 \leq i \leq k$. \qed

For each $1 \leq i \leq k$, define

$$O^i_\lambda = \left\{ w \in \mathcal{M}_\lambda^{-} | Q(w) - a^i < \rho_0 \right\},$$

$$\partial O^i_\lambda = \left\{ w \in \mathcal{M}_\lambda | Q(w) - a^i = \rho_0 \right\},$$
\[ \beta^i_\lambda = \inf_{w \in O^i_\lambda} J_\lambda(w) \text{ and } \tilde{\beta}^i_\lambda = \inf_{w \in \partial O^i_\lambda} J_\lambda(w). \]

Next, in order to show that \( \beta^i_\lambda \) are \((PS)\)-value. Associated with the critical problem, we define the energy functional \( I : X^s_0(\Omega) \to \mathbb{R} \),
\[
I(w) = \frac{1}{2} \int_{\mathbb{R}^{N+1}} y^{1-2s} \left| \nabla w \right|^2 dx dy - \frac{1}{2s} \int_{\Omega \times \{y = 0\}} |w|^{2^*_s} dx
\]
and define \( \gamma(\Omega) = \inf_{w \in \mathcal{M}(\Omega)} I(w) \) where
\[
\mathcal{M}(\Omega) = \left\{ w \in X^s_0(\mathbb{R}^N) \setminus \{0\} | w \neq 0, \langle I'(w), w \rangle = 0 \right\}.
\]

Now we state the following global compactness result for \( I \), which can be proved by the same argument in [26]. That is now standard, so we do not give a proof here.

**Lemma 4.5.** Let \( \{w_n\} \) be a \((PS)\)-sequence in \( X^s_0(\mathbb{R}^N) \), then there is a subsequence, still denoted by \( \{w_n\} \), a finite sequence \( \{W^j\}_{j=1}^k \subset X^s_0(\mathbb{R}^{N+1}) \), which are solutions of
\[
\begin{cases}
\text{div}(y^{1-2s} \nabla v) = 0, & \text{in } \mathbb{R}^{N+1}, \\
y^{1-2s} \frac{\partial v}{\partial y} = -|v(x,0)|^{2^*_s-2} v(x,0), & \text{in } \mathbb{R}^N,
\end{cases}
\]
and sequence \( \{x^j_n\}_{j=1}^k, \{\sigma^j_n\}_{j=1}^k \) satisfying \( \sigma^j_n > 0, x^j_n \in \Omega \) and \( n \to \infty \).

\[
\frac{1}{\varepsilon_n} \text{dist}(x^j_n, \partial \Omega) \to \infty, \quad \frac{\sigma^j_n}{\sigma^j_n} + \frac{\sigma^j_n}{\sigma^j_n} + |x^j_n - x^j_n| \to \infty,
\]

\[
\|w_n - w_0 - \sum_{j=1}^k \rho_{x^j_n, \varepsilon^j_n}(W^j)\|_{X^s_0(\mathbb{R}^N)} \to 0, \quad \rho_{x^j_n, \varepsilon^j_n}(W^j) = \varepsilon_n^{2s-N} U\left(\frac{x - x^j_n}{\varepsilon}\right),
\]

\[
I(w_n) = I(w_0) + \sum_{i=1}^\infty I_\infty(W^j) + o_n(1)
\]
where
\[
I_\infty(W^j) = \frac{1}{2} \int_{\mathbb{R}^{N+1}} y^{1-2s} \left| \nabla W^j \right|^2 dx dy - \frac{1}{2s} \int_{\Omega \times \{y = 0\}} |W^j|^{2^*_s} dx.
\]

It is well known that the best Sobolev constant \( S \) is independent of the domain and is never achieved except when \( \Omega = \mathbb{R}^N \). Moreover \( \gamma(\mathbb{R}^N) = \frac{\gamma}{\mathbb{R}^N} \).

**Lemma 4.6.** There exists a number \( \delta_0 > 0 \) such that if \( w \in \mathcal{M}_0 \) and \( J_0(w) \leq \alpha_0 + \delta_0 = \gamma(\Omega) + \delta_0 = \frac{\gamma}{\mathbb{R}^N} \), then \( Q(w) \in K_{2^*_s} \).

**Proof.** Assuming the contrary, there exists a sequence \( \{w_n\} \in \mathcal{M}_0 \) such that \( J_0(w_n) = \alpha_0 + o_n(1) \) as \( n \to \infty \) and \( Q(w_n) \notin K_{2^*_s} \) for all \( n \in \mathbb{N} \).

First of all, we want to show that \( \alpha_0 = \gamma(\Omega) \). Since \( g(z) \leq \max_{z \in \mathbb{R}^N} g(z) = 1 \), then \( \gamma(\Omega) \leq \alpha_0 \). From step 1 of Lemma 4.3 , sup \( t \geq 0 \) \( J_0(tw^i) \leq \gamma(\Omega) + O(\varepsilon^{N-2s}) \) uniformly in \( i \). Similarly to Lemma 3.5 , we have that there is a sequence \( \{t^i_n\} \subset \mathbb{R}^+ \) such that \( \alpha_0 \leq J_0(t^i_nw^i) = \sup_{t \geq 0} J_0(tw^i) \leq \gamma(\Omega) + O(\varepsilon^{N-2s}) \).

As \( \varepsilon \to 0^+ \), we get \( \alpha_0 \leq \gamma(\Omega) \), then \( \alpha_0 = \gamma(\Omega) \).
Next, there is a sequence \( \{s_n\} \subset \mathbb{R}^+ \) such that \( s_n w_n \in \mathcal{M}(\Omega) \),

\[
\gamma(\Omega) \leq I(s_n w_n) \leq J_0(s_n w_n) \leq \sup_{s \geq 0} J_0(sw_n) = J_0(w_n) = a_0 + o_n(1),
\]

then \( s_n = 1 + o_n(1) \) and \( I(s_n w_n) = \gamma(\Omega) + o_n(1) \). Using the same argument as in [14], every minimizing sequence in \( \mathcal{M}(\Omega) \) of \( \gamma(\Omega) \) is a \((PS)_{\gamma(\Omega)}\) sequence in \( X_0^0(\mathcal{C}_1) \) for \( I \). Since \( \inf_{v \in \mathcal{M}_0} I(v) = \gamma(\Omega) \) is not achieved, applying the Palais-Smale decomposition Lemma 4.5, we get that there exist sequences \( \varepsilon_n \to 0 \) and \( \{x_n\} \subset \Omega \) such that \( \frac{1}{\varepsilon_n} \text{dist}(x_n, \partial\Omega) \to \infty \) as \( n \to \infty \) as \( n \to \infty \) and

\[
s_n w_n = \varepsilon_n^{2\alpha-N} U\left(\frac{z - x_n}{\varepsilon_n}\right) + o_n(1), \quad \text{strongly in } D^s(\mathbb{R}^{N+1}_+),
\]

where \( U \) is the extremal function for \( S \) introduced in (10), \( D^s(\mathbb{R}^{N+1}_+) \) is defined as the completion of \( \mathcal{C}_0^\infty(\mathbb{R}^{N+1}_+) \) with respect to the norm

\[
||U||_{\mathbb{R}^{N+1}_+} = \left( \int_{\mathbb{R}^{N+1}_+} y^{1-2\alpha} |\nabla U|^2 \right)^{\frac{1}{2}}.
\]

It is well known that \( \varepsilon^{2\alpha-N} U(\frac{z}{\varepsilon}) \) is a solution of the following equation

\[
\begin{cases}
(-\Delta)^s w = w^{2^*_s-1}, & x \in \mathbb{R}^N, \\
w \in D^s(\mathbb{R}^{N+1}_+).
\end{cases}
\] (30)

Since \( \Omega \) is a bounded domain and \( \{z_n\} \subset \Omega \), then \( \varepsilon_n \to 0 \). Suppose the subsequence \( z_n \to z_0 \in \Omega \) as \( n \to \infty \), we claim that \( z_0 \in K \). By the Lebesgue dominated convergence theorem, then

\[
\mathcal{S}^{\alpha s} = \int_{\Omega \times \{y=0\}} g(x)|w|^{2^*_s}dx + a_0(1)
\]

\[
= \int_{\Omega} g(x)(\frac{1}{s_n} \varepsilon_n^{-\frac{N-2\alpha}{2}} U(\frac{z - x_n}{\varepsilon_n}))^{2^*_s}dz + o_n(1)
\]

\[
= g(z_0)\mathcal{S}^{\alpha s}, \quad \text{that is } z_0 \in K.
\]

Let \( \Omega_n = \{x|\varepsilon_n x + z_n \in \Omega\} \). Since \( \varepsilon_n \to 0 \), \( z_n \to z_0 \in \Omega \) and \( \frac{1}{\varepsilon_n} \text{dist}(z_n, \partial\Omega) \to \infty \) as \( n \to \infty \), then \( \Omega_n \to \mathbb{R}^N \) as \( n \to \infty \). By the Lebesgue dominated convergence theorem again, we have

\[
Q(w_n) = \frac{\int_{\Omega} \chi(z)|U(\varepsilon_n x + z_n)|^{2^*_s}dz}{\int_{\Omega} |U(\varepsilon_n x + z_n)|^{2^*_s}dz}
\]

\[
= \frac{\int_{\Omega} \chi(\varepsilon_n x + z_n)|U|^{2^*_s}dz}{\int_{\Omega} |U|^{2^*_s}dz} \to z_0 \in K_{2^*_s} \quad \text{as } n \to \infty,
\]

which is a contradiction. We complete the proof. \( \Box \)

**Lemma 4.7.** There exists \( \Lambda^* \in (0, \Lambda) \) such that if \( \lambda \in (0, \Lambda^*) \) and \( w \in \mathcal{M}_\lambda^- \) with

\[
J_\lambda(w) \leq \frac{\alpha s}{2\alpha s} \mathcal{S}^{\alpha s} + \frac{a_0}{2},
\]

then \( Q(w) \in K_{2^*_s} \).

**Proof.** Similar to Lemma 3.4, there is a unique positive number

\[
s^w = \left( \frac{||w||_{X_0^0(\mathcal{C}_1)}}{\int_{\Omega \times \{y=0\}} g(x)|w|^{2^*_s}dx} \right)^{\frac{1}{2^*_s-1}},
\]

such that \( s^w w \in \mathcal{M}_0 \).
We want to show that \( s^w < c \) for some constant \( c > 0 \) (where \( c \) is independent of \( w \) and \( \lambda \in (0, \Lambda_0) \)). First, by (18), if \( w \in \mathcal{M}_{\Lambda_0} \), then
\[
\|w\|^2_{X_0^\Lambda(c_0)} \geq (S_{\mathsf{h}}^{\frac{2}{2s} - \frac{2}{2s} - q} - \frac{2 - q}{2s - q})^\frac{2}{2s} S_{\mathsf{h}}^\infty = c_1.
\]
For \( \lambda \in (0, \Lambda_0) \), since \( J_{\Lambda}(w) \) is coercive on \( \mathcal{M}_{\Lambda} \) and \( J_{\Lambda}(w) \leq \frac{\lambda}{N} S_{\mathsf{h}}^{\infty} + \frac{\delta_0}{2} \), then by (12), we have that \( \|w\|^2_{X_0^\Lambda(c_0)} < c_2 \) for some constant \( c_2 \).

Next, we claim that \( \|w(x, 0)\|_{L^{2s}} > c_3 > 0 \). If \( w \in \mathcal{M}_{\Lambda_0} \), by (13),
\[
\frac{2 - q}{2s - q} < \int_{\Omega \times \{y = 0\}} g(x)|s^w w|^{2s} dx - \frac{\lambda}{q} \int_{\Omega \times \{y = 0\}} f(x)|s^w w|^q dx < \int_{\Omega \times \{y = 0\}} f(x)|s^w w|^q dx.
\]
Then \( s^w < c \) for some constant \( c > 0 \). Now, we get
\[
\frac{s}{N} S_{\mathsf{h}}^{\infty} + \frac{\delta_0}{2} \geq J_{\lambda}(w) = \sup_{t \geq 0} J_{\lambda}(tw) \geq J_{0}(s^w w)
\]
\[
= \frac{1}{2} \|s^w w\|^2_{X_0^\Lambda(c_0)} - \frac{1}{2s} \int_{\Omega \times \{y = 0\}} g(x)|s^w w|^{2s} dx - \frac{\lambda}{q} \int_{\Omega \times \{y = 0\}} f(x)|s^w w|^q dx
\]
\[
\geq J_{0}(s^w w) - \frac{\lambda}{q} \int_{\Omega \times \{y = 0\}} f(x)|s^w w|^q dx.
\]
From the above inequality, we deduce
\[
J_{0}(s^w w) \leq \frac{s}{N} S_{\mathsf{h}}^{\infty} + \frac{\delta_0}{2} + \frac{\lambda}{q} \int_{\Omega \times \{y = 0\}} f(x)|s^w w|^q dx
\]
\[
\leq \frac{s}{N} S_{\mathsf{h}}^{\infty} + \frac{\delta_0}{2} + \frac{\lambda}{q} \int_{\Omega \times \{y = 0\}} f(x)|s^w w|^q dx < \frac{s}{N} S_{\mathsf{h}}^{\infty} + \frac{\delta_0}{2} + \frac{\lambda}{q} c_1 \int_{\Omega \times \{y = 0\}} f(x)|s^w w|^q dx,
\]
where \( \lambda = \varepsilon^* \).

Hence, there exists \( 0 < \varepsilon^* \leq \min\{\varepsilon_0, \varepsilon^0\} \) such that \( \Lambda^* = (\varepsilon^*)^\gamma \) and if for \( 0 < \lambda < \Lambda \),
\[
J_{0}(s^w w) \leq \frac{s}{N} S_{\mathsf{h}}^{\infty} + \delta_0, \text{ where } s^w w \in \mathcal{M}_0.
\]
By Lemma 4.6, we obtain,
\[
Q(s^w w) = \frac{\int_{\Omega} \chi(x)|s^w w(x, 0)|^{2s} dx}{\int_{\Omega} |s^w w(x, 0)|^{2s} dx} \in K_{\mathsf{h}}^{\infty} \text{ for } 0 < \lambda < \Lambda^*.
\]
or \( Q(w) \in K_{\mathsf{h}}^{\infty} \) for any \( 0 < \lambda < \Lambda^* \).

Applying the above Lemma 4.7, we get that
\[
\tilde{\beta}_{\lambda} > \frac{s}{N} S_{\mathsf{h}}^{\infty} \text{ for any } 0 < \lambda < \Lambda^*.
\]
(31)
By Lemma 4.3, then
\[
\beta_{\lambda} \leq \alpha_{\lambda} < \frac{s}{N} S_{\mathsf{h}}^{\infty} - C_0 \lambda^{\frac{2s}{2s - q}}, \text{ for any } 0 < \lambda < \Lambda^*.
\]
(32)
Lemma 4.8. Given \( w \in O_{\lambda}^1 \), then there exist \( \eta > 0 \) and a differentiable functional \( l : B(0, \eta) \subset X_0^1(\Omega) \rightarrow \mathbb{R}^+ \) such that \( l(0) = 1 \), \( l(v)(w - v) \in O_{\lambda}^1 \) for any \( v \in B(0, \eta) \) and

\[
\langle l'(0), \phi \rangle = \frac{\langle \psi_{\lambda}(w), \phi \rangle}{\psi_{\lambda}(w, w)}, \quad \text{for any } \phi \in C^\infty(\Omega) \tag{33}
\]

where \( \psi_{\lambda}(w) = \langle J_{\lambda}(w), w \rangle \).

Proof. We define

\[
F(t, v) = t||w - v||^2_{X_0^1(\Omega)} - t^{p-1} \int_{\Omega \times \{y=0\}} f(x)||w - v|^p \, dx
\]

\[
- t^{2^*_s-1} \int_{\Omega \times \{y=0\}} g(x)||w - v|^{2^*_s} \, dx.
\]

Since \( F(1,0) = 0 \) and

\[
F_t(1,0) = ||w||^2_{X_0^1(\Omega)} - (p-1) \int_{\Omega \times \{y=0\}} f(x)||w|^p \, dx - (2^*_s-1) \int_{\Omega \times \{y=0\}} g(x)||w|^{2^*_s} \, dx.
\]

We can apply the implicit theorem at the point \((1,0)\) and get the results similar to Cao and Zhou [9]. \( \square \)

Lemma 4.9. For each 1 \( \leq i \leq k \), there is a \((PS)_{\beta_{\lambda}^i}^s\) sequence \( \{w_n^i\} \subset O_{\lambda}^1 \) in \( X_0^1(\Omega) \) for \( J_{\lambda} \).

Proof. For each 1 \( \leq i \leq k \), by (31) and (32), we get that

\[
\beta_{\lambda}^i < \beta_{\lambda}^i, \quad \text{for any } 0 < \varepsilon < \varepsilon^*. \tag{36}
\]

Then

\[
\beta_{\lambda}^i = \inf_{w \in O_{\lambda}^1 \cup \partial O_{\lambda}^1} J_{\lambda}(w), \quad \text{for any } 0 < \varepsilon < \varepsilon^*. \tag{37}
\]

Let \( \{w_n^i\} \subset O_{\lambda}^1 \cup \partial O_{\lambda}^1 \) be a minimizing sequence for \( \beta_{\lambda}^i \). Applying Ekeland’s variational principle, there exists a subsequence \( \{w_n^i\} \) such that \( J_{\lambda}(w_n^i) = \beta_{\lambda}^i + \frac{1}{n} \) and

\[
J_{\lambda}(w_n^i) \leq J_{\lambda}(w) + \frac{||w_n^i - w||_{X_0^1(\Omega)}}{n} \quad \text{for all } w \in O_{\lambda}^1 \cup \partial O_{\lambda}^1. \tag{38}
\]

Using (36), we may assume that \( w_n^i \in O_{\lambda}^1 \) for sufficiently large \( n \). By Lemma 4.8, then there exists an \( \eta_n^i > 0 \) and a differentiable functional \( l_n^i : B(0, \eta_n^i) \subset X_0^1(\Omega) \rightarrow \mathbb{R}^+ \) such that \( l_n^i(0) = 1 \) and \( l_n^i(v)(w_n^i - v) \in O_{\lambda}^1 \) for \( v \in B(0, \eta_n^i) \).

Let \( v_\sigma = \sigma v \) with \( ||v||_{X_0^1(\Omega)} = 1 \) and 0 \( < \sigma < \eta_n^i \). Then \( v_\sigma \in B(0, \eta_n^i) \) and \( w^i_\sigma = l_n^i(v_\sigma)(w_n^i - v_\sigma) \in O_{\lambda}^1 \). From (38) and by the mean value theorem, we get that as \( \sigma \rightarrow 0 \),

\[
\frac{||w_n^i - w^i_\sigma||_{X_0^1(\Omega)}}{n} \geq J_{\lambda}(w_n^i) - J_{\lambda}(w_n^i_\sigma)
\]

\[
= \langle J'_{\lambda}(t_0 w_n^i + (1 - t_0) w_n^i, w_n^i - w_n^i_\sigma) \rangle
\]

\[
= \langle J'_{\lambda}(w_n^i), w_n^i - w_n^i_\sigma \rangle + o(||w_n^i - w_n^i_\sigma||_{X_0^1(\Omega)})
\]

\[
= \sigma l_n^i(v_\sigma) \langle J_{\lambda}(w_n^i), v \rangle + (1 - l_n^i(v_\sigma))(J_{\lambda}(w_n^i), w_n^i) + o(||w_n^i, w_n^i_\sigma||_{X_0^1(\Omega)})
\]

\[
= \sigma l_n^i(v_\sigma) \langle J_{\lambda}(w_n^i), v \rangle + o(||w_n^i - w_n^i_\sigma||_{X_0^1(\Omega)}),
\]
where \( o(\|w_n^i - w_{n,\sigma}\|_{X_0^s(C_\Omega)})/\|w_n^i - w_{n,\sigma}\|_{X_0^s(C_\Omega)} \to 0 \) as \( \sigma \to 0 \). Hence,

\[
|J'_\lambda(w_n^i, v)| \leq \frac{\|w_n^i - w_{n,\sigma}\|_{X_0^s(C_\Omega)}(\frac{1}{n} + |o(1)|)}{\sigma |l_n^i(\sigma v)|} \\
\leq \frac{\|w_n^i\|_{X_0^s(C_\Omega)}(|l_n^i(\sigma v) - l_n^i(0)|) + \sigma |v||l_n^i(\sigma v)|}{\sigma |l_n^i(\sigma v)|} \left( \frac{1}{n} + |o(1)| \right) \\
\leq C(1 + |(l_n^i)'(0)|)(\frac{1}{n} + |o(1)|),
\]

where \( o(1) \to 0 \) as \( \sigma \to 0 \). Since we can deduce that \( |(l_n^i)'(0)| \leq c \) for all \( n \) and \( i \) from (33), then \( J'_\lambda(w_n^i) = o_n(1) \) strongly in \( (X_0^s(C_\Omega))^{-1} \) as \( n \to \infty \).

**Proof of Theorem 1.2.** From Lemma 4.9, there is a \((PS)_{\beta^*}\)-sequence \( \{w_n\} \subset O_\lambda \) in \( X_0^s(C_\Omega) \) for \( J_\lambda \) for each \( 1 \leq i \leq k \). Since \( J_\lambda \) satisfies the \((PS)_{\beta}\) condition for \( \beta \in (-\infty, \frac{\Lambda^*}{8nC_0} - C_0\lambda^{\frac{1}{n-1}}) \), by (37), then \( J_\lambda \) has at least \( k \) critical points in \( M_\lambda \) for \( 0 < \lambda < \Lambda^{**} = (c^*)^7 \). It follows that problem (2) has \( k \) nonnegative solutions in \( \Omega \). Applying the maximum principle and Theorem 1.1, problem has \( k+1 \) positive solutions.

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