Optimization of security costs in nested purification protocol

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We analyse security costs in one segment of nested purification protocol in a large quantum cryptography network, employing the quantum switchers and repeaters. We demonstrate that exponential or even super-exponential grow of entanglement resources occurs in dependence on number of the network switchers. For this reason, an optimization in the nested strategy is suggested, preventing a stronger than the exponential grow of entanglement resources.

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Recently, quantum cryptography link using entangled states [8] has been experimentally realized [9], which is basic step to future quantum cryptographic network. To construct this network, original Ekert’s protocol has been extended by the quantum memories [4] and quantum switchers [5]. Quantum memories carefully store the entangled states before key distribution and the quantum switchers are able to swap entanglement and consequently, establish a secret communication between distant users. However, the entangled states transmitted between the users or stored in the quantum memories cannot be precisely protected against undesirable decoherence or the deliberate eavesdropper attacks. Due to these influences, the entanglement exponentially vanishes with increasing distance between the users. Fortunately, quantum cryptography based on Ekert’s protocol presents a potential advantage, since the users could regenerate the secure key by implementing a purification protocol [6]. Quantum purification is able to extract sufficiently entangled pair of photons from a number of the pairs with weak entanglement, utilizing only the local operations and classical communications. Thus, the links between users in the networks achieve a sufficient security and the almost maximally entangled pairs can be stored in the quantum memories for the next usages.

Practical implementation of the swapping/purification idea based on the quantum repeaters in connection points of network extended standard purification to nested purification protocol (NPP) [8]. We can assume a link consisting of total number \( N - 1 \) of the switchers connecting \( N \) pairs of imperfectly entangled states, having non-unit fidelity with maximally entangled state. On the first level, we implement the quantum repeaters in the check points with \( L \) switchers between two repeaters. To purify one pair with required amount of entanglement between nearest repeaters, we need a certain number \( M \) of copies that we construct in parallel fashion, as is depicted in Fig. 1. The total number of elementary pairs is \( M(L + 1) \). On the second level, we repeat this procedure considering the outcomes of first level as input pairs for second level. In summary, the total number of elementary pairs will be [8]

\[
R = N^{\log_{L+1} M+1}
\]

which shows that the resources grow polynomially with the distance \( N \). However, the number of needed entangled pairs \( M \) per segment and per one nesting level depends on number of switchers \( L \), on fidelity of pairs between the nearest users and required fidelity of outgoing pairs between the distant users. To simplify discussion, we require the same value of a long-distance fidelity as the nearest users fidelity and denote it as “working” fidelity. It has been previously shown [8] that for \( L = 2 \), the optimal “working” fidelity \( F \approx 0.95 \) requires minimal number of four pairs per segment and nesting level. However, if we want to create entanglement over arbitrary distance (large \( L \)), there is still a question, how much weakly entangled pairs are needed between two repeaters for purification up to working fidelity. Shortly, what is the relation \( M = M(L, F) \) in the nested purification protocol?

In this paper, we present an analysis of these security costs, assuming an almost perfect entanglement between the users in the neighbourhood. To distribute entanglement between the distant users in cryptographic network, an idealized swapping/purification protocol will be assumed. These procedures can be implemented probabilistically, with the help of only linear optics elements [11]. We demonstrate a new and important fact, that the number of imperfectly entangled states between two quantum repeaters grows exponentially or even super-exponentially with the number of implemented switchers. It is a cost of exponential decoherence elimination, which must be paid to achieve a secure key distribution. For this reason, we suggest an optimized nested purification protocol (ONPP), which prevents the super-exponential grow of the security costs. It will be important for a construction of the quantum cryptographic network, expected in the next future.

Now we will theoretically discuss both the entanglement swapping and distillation procedures in one segment of nested purification protocol, as is depicted in Fig. 1. We assume that the nearest users \( (i, j = i + 1) \) share two-qubit state

\[
\rho^{(i,j)} = \sum_{k,l} p_{(i,j)}^{k,l} |B_k\rangle\langle B_l|
\]
which is given in the Bell basis

\[ |B_{1,2}⟩ = \frac{1}{\sqrt{2}}(|00⟩ \pm |11⟩), \]
\[ |B_{3,4}⟩ = \frac{1}{\sqrt{2}}(|01⟩ \pm |10⟩). \] (3)

Note that diagonal elements \( B_{k,k} = ⟨ B_k | ρ | B_k⟩ \) are the fidelities \( B_k \) with particular Bell states \( B_k \). First, we discuss a propagation of entanglement in the network utilizing the standard entanglement swapping \( S \) between two links \((i, i+1)\) and \((i+1, i+2)\). Assuming Bell state analysis, which, at least conditionally, distinguishes between four Bell states \( B_k \), the diagonal elements \( B_{k,k} = B_k \) change along rule

\[

B_{i,i+2}^{1} = B_{i,i+1}^{1} B_{(i+1,i+2)}^{1} + B_{i,i+1}^{2} B_{(i+1,i+2)}^{2} + B_{i,i+1}^{3} B_{(i+1,i+2)}^{3} + B_{i,i+1}^{4} B_{(i+1,i+2)}^{4},
\]
\[

B_{i,i+2}^{2} = B_{i,i+1}^{1} B_{(i+1,i+2)}^{2} + B_{i,i+1}^{2} B_{(i+1,i+2)}^{1} + B_{i,i+1}^{3} B_{(i+1,i+2)}^{4} + B_{i,i+1}^{4} B_{(i+1,i+2)}^{3},
\]
\[

B_{i,i+2}^{3} = B_{i,i+1}^{1} B_{(i+1,i+2)}^{3} + B_{i,i+1}^{2} B_{(i+1,i+2)}^{1} + B_{i,i+1}^{3} B_{(i+1,i+2)}^{2} + B_{i,i+1}^{4} B_{(i+1,i+2)}^{4},
\]
\[

B_{i,i+2}^{4} = B_{i,i+1}^{1} B_{(i+1,i+2)}^{4} + B_{i,i+1}^{2} B_{(i+1,i+2)}^{2} + B_{i,i+1}^{3} B_{(i+1,i+2)}^{1} + B_{i,i+1}^{4} B_{(i+1,i+2)}^{3},
\] (4)

individually on the off-diagonal elements. If the swapping procedure is carried out many times with an ensemble of pairs distributed along the link, the iterative rule \( S \) describes the evolution of fidelities in the link with the switchers. However, the entanglement swapping of weakly entangled states leads to generation of lower entangled pairs at a large distance. To improve this long-distance entanglement, we can use LOCC purification procedure \( T \) requiring multiple copies of the long-distance states. Deutsch’s purification allows us to distill a state with needed amount of entanglement from imperfectly entangled pairs \( ρ^{(0,L+1)} \) having fidelity \( B_1 > 1/2 \). It is composed from three steps: (i) the local unitary operations on the sender side

\[ |0⟩ \rightarrow \frac{1}{\sqrt{2}}(|0⟩ - i|1⟩), \quad |1⟩ \rightarrow \frac{1}{\sqrt{2}}(|1⟩ - i|0⟩), \] (5)

and on the receiver side

\[ |0⟩ \rightarrow \frac{1}{\sqrt{2}}(|0⟩ + i|1⟩), \quad |1⟩ \rightarrow \frac{1}{\sqrt{2}}(|1⟩ + i|0⟩), \] (6)

followed by (ii) action of the quantum C-NOT operation on the both sides

\[ |a⟩_C |b⟩_T \rightarrow |a⟩_C |a ⊕ b⟩_T \] (7)

where \( C \) is control qubit and \( T \) is target qubit, and (iii) measurement of the target qubits on the both sides. If the measurement outcomes coincide, the control pair is kept for next round. For two different states \( ρ^{(0,L+1)} \) and \( ρ′^{(0,L+1)} \) described by fidelities \( B_i \) and \( B'_i \), the distilled pair \( ρ^{(0,i+1)} \) has the following shifted fidelities

\[

\tilde{B}_1 = \frac{1}{N}(B_1 B'_1 + B'_1 B_4), \quad \tilde{B}_2 = \frac{1}{N}(B_1 B'_4 + B'_4 B_1),
\]
\[

\tilde{B}_3 = \frac{1}{N}(B_2 B'_2 + B'_2 B_3), \quad \tilde{B}_4 = \frac{1}{N}(B_2 B'_3 + B'_3 B_2),
\] (8)

where \( N = (B_1 + B_4) (B'_1 + B'_4) + (B_2 + B_3) (B'_2 + B'_3) \), \( B_i, B'_i \) are the fidelities of input pairs and \( B_i \) are the fidelities of an output distilled pair. Particularly, if \( B_1 > 1/2 \) for all shared states, there is only one fixed point for \( B_1 = 1 \). After some steps of procedure, the outgoing state is sufficiently close to this fixed point (as it is necessary) and the sender and receiver establish a resource for key distribution with a given security. After \( m \) repetition of purification procedure, we can generate a strongly entangled pair from \( 2^m \) input lower entangled pairs. In this way, an eavesdropping attack is factorized from the sender-receiver state. On the other hand, if the purification procedure does not converge, we cannot use the resulting pairs for the key distribution. Thus, a convergence of the procedure must be simultaneously proved in the communication. Note, when there are the imperfect local operations, only a maximal non-unit fidelity is obtained, however as has been recently proved, the users may nevertheless use these pairs for secure communication \( T \).

The iterative procedures \( S \) and \( T \) represent a complete solution of the presented problem for any two-qubit states. To discuss the security costs in dependence on the number of switchers, we will analyse two interesting cases of the decoherence process. As first, we will analyse the security costs in the network with the states \( ρ^{(i,i+1)} \) after an optimal individual eavesdropping attack - quantum nondemolition (QND) measurement. We assume that state \( |B_1⟩ \) is decohered by the standard QND monitoring of basis states \( |0⟩ \) and \( |1⟩ \)

\[ |00⟩_E \rightarrow |00⟩_E, \]
\[ |11⟩_E \rightarrow |11⟩_E (R |0⟩_E + \sqrt{1 - R^2} |1⟩_E), \] (9)

where \( R \) is a measure of robustness against decoherence and \( |0⟩_E \) and \( |1⟩_E \) are basis states of the environment. After monitoring process and tracing out the environmental
approximately determine a lower bound on \(M\) robustness in form of \(\bar{\mu}\) between near-users robustness procedure. Due to this relation, we can express a relation on parameters \(r\) robustness \(\bar{\mu}\), which can be simplified for the same states utilizing the \(B\) structure of state (10) is preserved and only the fidelity for every \(B\) decreases, as is depicted in Fig. 2 for almost maximal “working” fidelities \(B_1 = 0.9925, 0.985, 0.9625\). This exponential overhead is necessary cost that must be paid to establish a security of quantum channel at a large distance. It is still open question, whether the LOCC purification procedure can be modified in such a way to obtain only a sub-exponential overhead.

As a second example, we consider Werner state between the nearest users. Note, that Werner state results from the simplest eavesdropping strategy on the state \(|B_1\rangle\): Eve steals the photon transmitted from sender station with probability \(p\) and subsequently, sends another photon with random polarization towards receiver station. Particularly, for the \(L\) switches with the \(L + 1\) Werner states (having \(B_2 = B_3 = B_4\))

\[
\rho^{(i,j)} = p|B_1\rangle\langle B_1| + \frac{1-p}{4} 1 \otimes 1, \tag{14}
\]

where \(p = (4B_1 - 1)/3\), an outgoing state is the Werner state \(|\bar{\mu}\rangle\) with \(p' = p^L\). This is a signature a \(\text{exponential}\) decrease of nonlocality and entanglement in a network with large \(L\), even if particular \(p\) approaches unity. Note, that Werner state \(|\bar{\mu}\rangle\) is entangled if the entanglement factor \(\Lambda = 1/2(1-3p)\) is negative and does not admit a local realistic explanation if the Bell factor \(B_{\text{max}} = 2\sqrt{2p}\) is larger than 2. After \(L\) swapping procedures, we obtain a long distance Werner state with the following fidelity

\[
B'_1 = \frac{3}{4} \left(\frac{4}{3} B_1 - \frac{1}{3}\right)^L + \frac{1}{4}. \tag{15}
\]

which is entangled if and only if \(B'_1 > \frac{1}{2}\). Then if is possible to use the purification procedure, i.e. numerically iterate the map \(|\bar{\mu}\rangle\). Number \(M = 2^m\) (where \(m\) is the number of purification steps) of pairs needed to achieve the same fidelity, as has been between the nearest users, is depicted in Fig. 3 in dependence on the number \(L\) of the switchers for the “working” fidelities \(B_1 = 0.9925, 0.985, 0.9625\). As can be seen, we have two distinct regions of overhead: for given \(B_1\), the number \(M\) firstly \(\text{exponentially}\) grows with \(L\) and this divergence changes to rapid \(\text{super-exponential}\) behavior. Note, that super-exponential overhead is pronounced as the fidelity \(B_1\) approaches unity. For sufficiently large \(B_1 > 0.95\), we can optimize NPP to prevent super-exponential overhead, if we restrict the number \(L\) of switchers between two repeaters

\[
L < \frac{1}{2 \left(1 - \frac{\ln(4B_1 - 1)}{\ln 3}\right)}. \tag{16}
\]

to half of the maximal value \(L_{\text{max}}\), where \(L_{\text{max}}\) is depicted in Fig. 2 by a cross symbol. In addition, it can be numerically checked that the analysed case, employing the Werner states, is the most expensive from all the cases with the same state \(|\bar{\mu}\rangle\) between the nearest users. Because we are not able to priori know what are the states distributed between nearest users, the restriction \(|\bar{\mu}\rangle\) is a general how to prevent super-exponential overhead and it represents \(\text{optimized}\) nested purification protocol (ONPP).

In summary, irrespective to very encouraging experimental results in entangled state cryptography [3], entanglement swapping procedure [3], a strong \(\text{exponential}\) decoherence arises at a large distance from “multiplication” of small deviations from perfect near-users entangled states. To avoid this decoherence, we can use the nested purification procedure [3, 1]. In this paper, we have found a general formula for the fidelity evolution in this nested purification procedure and we have demonstrated that, per nesting segment, this procedure exhibits \(\text{exponential}\) or even \(\text{super-exponential}\) overhead in the entanglement resources, in dependence on the number of the entanglement switchers. We analysed the case with a maximal overhead and optimized NPP by a restriction of maximal number of switchers between two repeaters to avoid \(\text{super-exponential}\) overhead. It is important for any construction of the sufficiently secure large cryptographic networks in the next future. However, there is
a still opened question whether it is possible to obtain a sub-exponential overhead by a modification of swapping/purification procedure.

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[1] A.K. Ekert, Phys. Rev. Lett. 67, 661 (1991).
[2] C.H. Bennett, G. Brassard and N.D. Mermin, Phys. Rev. Lett. 68, 557 (1992).
[3] D.S. Naik, C.G. Peterson, A.G. White, A.J. Berglund, and P.G. Kwiat, Phys. Rev. Lett. 84, 4733 (2000).
[4] E. Biham, B. Huttner, and T. Mor, Phys. Rev. A 54, 2651 (1996).
[5] C.H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W.K. Wootters, Phys. Rev. Lett. 70, 1895 (1993); M. Zukowski, A. Zeilinger, M.A. Horne, and A. Ekert, Phys. Rev. Lett. 71, 4287 (1993); S. Bose, V. Vedral and P.L. Knight, Phys. Rev. A 57, 822 (1998); J.-W. Pan, D. Bouwmeester, H. Weinfurther and A. Zeilinger, Phys. Rev. Lett. 80, 3891 (1998).
[6] Ch. Bennett, G. Brassard, S. Popescu, B. Schumacher, J.A. Smolin and W. K. Wootters, Phys. Rev. Lett. 76, 722 (1996).
[7] D. Deutsch, A. Ekert, R. Jozsa, Ch. Macchiavello, S. Popescu and A. Sanpera, Phys. Rev. Lett. 77, 2818 (1996);
[8] H.-J. Briegel, W. Dür, J.I. Cirac, and P. Zoller, Phys. Rev. Lett. 81, 5932 (1998);
[9] W. Dür, H.-J. Briegel, J.I. Cirac, and P. Zoller, Phys. Rev. A 59, 169 (1999);
[10] H. Aschauer and H.J. Briegel, Phys. Rev. Lett. 88, 047902-1 (2002).
[11] J.W. Pan, Ch. Simon, Č. Brukner, and A. Zeilinger, Nature 410, 1067 (2001); T. Jennewein, G. Weihs, J.W. Pan, and A. Zeilinger, Phys. Rev. Lett. 88, 017903-1 (2002).
[12] Ch. Macchiavello, Phys. Lett. A 246, 385 (1998).

FIG. 1: Segment of nested purification protocol: R- quantum repeaters, S- switchers based on entanglement swapping employed by particular users.

FIG. 2: Number of entangled pairs $M = 2^m$ needed for distillation of one entangled pair between distant users with the same security as for the nearest users, $L$ number of the switchers in the link mutually sharing the same QND states with given $R$: (a) $R=0.985$ ($B_1 = 0.9925$), (b) $R=0.97$ ($B_1 = 0.9850$), (c) $R=0.925$ ($B_1 = 0.9625$).

FIG. 3: Number of entangled pairs $M = 2^m$ needed for distillation of one entangled pair between distant users with the same security as for the nearest users, $L$ number of the switchers in the link mutually sharing the same Werner states with given $p$: (a) $p=0.99$ ($B_1 = 0.9925$), (b) $p=0.98$ ($B_1 = 0.9850$), (c) $p=0.95$ ($B_1 = 0.9625$). The crosses denote a threshold number $L_{max}$ of the switchers between two repeaters.
$M = 2^m$

Diagram showing the relationship between $M$, $L$, and $m$ with steps indicating super-exponential and exponential behavior.