Quantum Black Hole Entropy
from 4d Supersymmetric Cardy formula

Masazumi Honda*

Department of Applied Mathematics and Theoretical Physics,
Centre for Mathematical Sciences, Wilberforce Road, Cambridge, CB3 0WA, UK

January 2019

Abstract
We study supersymmetric index of 4d $SU(N)$ $\mathcal{N} = 4$ super Yang-Mills theory on $S^1 \times M_3$. We compute asymptotic behavior of the index in the limit of shrinking $S^1$ for arbitrary $N$ by a refinement of supersymmetric Cardy formula. The asymptotic behavior for the superconformal index case ($M_3 = S^3$) at large $N$ agrees with the Bekenstein-Hawking entropy of rotating electrically charged BPS black hole in $AdS_5$ via a Legendre transformation as recently shown in literature. We also find that the agreement formally persists for finite $N$ if we slightly modify the AdS/CFT dictionary between Newton constant and $N$. This implies an existence of non-renormalization property of the quantum black hole entropy. We also study the cases with other gauge groups and additional matters, and the orbifold $\mathcal{N} = 4$ super Yang-Mills theory. It turns out that the entropies of all the CFT examples in this paper are given by $2\pi \sqrt{Q_1 Q_2 + Q_1 Q_3 + Q_2 Q_3 - 2c(J_1 + J_2)}$ with charges $Q_{1,2,3}$, angular momenta $J_{1,2}$ and central charge $c$. The results for other $M_3$ make predictions to the gravity side.

* mh974@damtp.cam.ac.uk
1 Introduction

Since string theory is the candidate of consistent quantum gravity, string theory should give microscopic explanation of black hole entropy \[1\]. As well known, the seminal paper \[2\] by Strominger and Vafa has derived the Bekenstein-Hawking entropy of asymptotically flat black hole by counting BPS states in string theory.

In the context of AdS/CFT \[3\], this problem is mapped into whether an entropy of an asymptotically AdS black hole is explained by counting states of a dual CFT. Recently there has been great steps to understand this problem along two directions. First, the black hole entropies of static dyonic BPS black holes has been reproduced by topologically twisted indices of 3d $\mathcal{N} = 6$ superconformal theory \[4, 5\] by using supersymmetry localization \[6\]. Then there appeared agreements in various setups involving static magnetic charged black holes \[7\].

The second type of the progress has been made in the canonical AdS/CFT correspondence between the 4d $SU(N) \mathcal{N} = 4$ super Yang-Mills theory (SYM) and type IIB superstring theory on $AdS_5 \times S^5$, which is also the subject of this paper. It is known that there are rotating electrically charged black hole solutions in $AdS_5$ \[8\] which are embedded in the type IIB supergravity in $AdS_5 \times S^5$ as $1/16$-BPS solutions \[9\]. The black holes have three charges $(Q_1, Q_2, Q_3)$ associated with $U(1)^3 \subset SO(6)$ and two angular momenta $(J_1, J_2)$ associated with Cartan part of $SU(2)^2 \sim SO(4) \subset SO(4, 2)$. They are related to the black hole mass $M$ by

$$M = g (|J_1| + |J_2| + |Q_1| + |Q_2| + |Q_3|), \quad (1.1)$$

where $g$ is the gauge coupling. The Bekenstein-Hawking entropy of the black hole is \[11\]

$$S_{BH} = \frac{\text{Area}}{4G_N} = 2\pi \sqrt{Q_1Q_2 + Q_1Q_3 + Q_2Q_3 - \frac{\pi}{4G_Ng^3}(J_1 + J_2)}, \quad (1.2)$$

where the AdS/CFT dictionary between $G_Ng^3$ and $N$ is

$$\frac{\pi}{2G_Ng^3} = N^2. \quad (1.3)$$

A long-standing question is whether this black hole entropy is holographically explained by counting 1/16-BPS states in the $\mathcal{N} = 4$ SYM on $S^1 \times S^3$. Technically it is much easier to analyze the superconformal index \[12, 13\] rather than the net sum of the 1/16-BPS states:

$$I_{S^1 \times S^3} = \text{Tr} \left[ (-1)^F e^{-\beta Q} p^{J_1+\tilde{J}_1} q^{J_2+\tilde{J}_2} v_1^{q_1} v_2^{q_2} \right] = \text{Tr}_{\text{BPS}} \left[ (-1)^F p^{J_1+\tilde{J}_1} q^{J_2+\tilde{J}_2} v_1^{q_1} v_2^{q_2} \right], \quad (1.4)$$

where $r = \frac{2}{3}(Q_1 + Q_2 + Q_3)$ and $q_{1,2} = Q_{1,2} - Q_3$ taking charges of $U(1)^3 \subset SO(6)_R$ symmetry to be $Q_{1,2,3}/2$. One common worry is that the index may have huge cancellation between bosonic and fermionic states so that it does not capture the black hole entropy \[12\] (see also \[14\] for other early attempts).

\[1\] See also \[10\] for another embedding.
However, very recent papers have updated our understanding. First, the paper \cite{15} has shown that a Legendre transformation of the black hole entropy called entropy function is given by a generalization of supersymmetric Casimir energy $E_{\text{Casimir}}$ \cite{16, 17} in the large-$N$ limit which is defined as a relative factor between partition function and index\cite{3}:

$$Z_{S^1 \times S^3} = e^{-\beta E_{\text{Casimir}}} I_{S^1 \times S^3}. \quad (1.5)$$

Second, the authors of \cite{20} have analyzed the index of the $U(N)$ $\mathcal{N} = 4$ SYM in a limit of shrinking $S^1$ at large-$N$ which we refer to as Cardy limit, and identified a saddle point of holonomy integral which gives the black hole entropy function. Then they have assumed the dominance of the saddle point and derived the asymptotic behavior of the index in the Cardy limit which agrees with the black hole entropy \cite{1.2} via a Legendre transformation with respect to the chemical potentials. They have also discussed a deconfinement transition in another paper \cite{21}. Third, the authors of the paper \cite{22} have analyzed the index for $p = q$ in the large-$N$ limit by using Bethe ansatz type formula of the index \cite{23}. They have identified a saddle point which reproduces the black hole entropy function corresponding to the equal angular momenta case: $J_1 = J_2$. They have also assumed that the saddle point is most dominant. It has also been stressed in \cite{20, 21, 22} that the index with real fugacities have more cancellations than generic complex fugacities.

Aims of this paper are to provide further evidence that the index gives microscopic explanation of the black hole entropy and make predictions for the black hole physics in more general case. We mainly study supersymmetric index of the $SU(N)$ $\mathcal{N} = 4$ SYM on $S^1 \times M_3$. We compute an asymptotic behavior of the index in the limit of shrinking $S^1$ for arbitrary $N$ by using a refinement \cite{21, 25} of supersymmetric Cardy formula \cite{26}. Therefore our approach for the superconformal index case ($M_3 = S^3$) is basically the same as the one in \cite{20}. The asymptotic behavior of the superconformal index at large $N$ agrees with the Bekenstein-Hawking entropy \cite{1.2} via a Legendre transformation with respect to the chemical potentials. This agreement at large-$N$ has been already found in \cite{20} recently. We also find that the agreement formally persists for finite $N$ if we slightly modify the AdS/CFT dictionary \cite{1.3} as

$$\frac{\pi}{2G_N g^3} \bigg|_{\text{finite}N} = N^2 - 1 = 4c, \quad (1.6)$$

where $c = (N^2 - 1)/4$ is the central charge of the $SU(N)$ $\mathcal{N} = 4$ SYM. This implies an existence of non-renormalization property for the black hole entropy function in the small-$S^1$ limit at quantum level. We also study the cases with other gauge groups and additional matters in conjugate representations, and orbifold $\mathcal{N} = 4$ SYM. It turns out that the entropies of all the CFT examples in this paper are given by

$$S_{\text{QFT}}(Q, J) = 2\pi \sqrt{Q_1 Q_2 + Q_1 Q_3 + Q_2 Q_3 - 2c(J_1 + J_2)}, \quad (1.7)$$

\footnote{The entropy function of the black hole was first computed in \cite{18}. It was also argued in \cite{18} that the entropy function is formally equal to the SUSY Casimir energy. The SUSY Casimir energy of the $\mathcal{N} = 4$ SYM with the fugacities of $SO(6)_R$ was first computed in \cite{19}.}
with the central charge $c$. This formula is our prediction for the black hole entropy with full quantum corrections. The results for other $M_3$ are also regarded as predictions to the gravity side. It is also interesting to note that the authors in [11] first wrote down the black hole formula for the dual of the $SU(N) \mathcal{N} = 4$ SYM as

$$S_{BH} = 2\pi \sqrt{Q_1 Q_2 + Q_1 Q_3 + Q_2 Q_3 - 2c(J_1 + J_2)}, \quad (1.8)$$

and then substituted $c = N^2/4$ to get the formula

$$S_{BH} = 2\pi \sqrt{Q_1 Q_2 + Q_1 Q_3 + Q_2 Q_3 - \frac{N^2}{2}(J_1 + J_2)}, \quad (1.9)$$

in their derivation. Of course there is no difference in the large-$N$ limit but our result suggests that (1.8) is more accurate for finite $N$.

Our argument for the $M_3 = S^3$ case is overlapped with the one made in [20]. While the approach is the same up to technical details and the final result at large-$N$ has been already obtained in [20], there are mainly three differences. First, we mainly consider the $SU(N) \mathcal{N} = 4$ SYM rather than the $U(N)$ case while the difference is irrelevant at large-$N$ and we also finally consider the $\mathcal{N} = 4$ SYM with general gauge group as well as other theories. Second, we analyze the index for finite $N$ but we will see that the result in [20] is formally correct also for finite $N$. Finally we do not only identify a saddle point giving the black hole entropy (1.2) but also prove that the saddle point is most dominant. This amounts to justify the assumption made in [20] at large-$N$ and make sure that the most dominant contribution of the index gives the black hole entropy. Some contents discussed in [20] but not in this paper are Macdonald limit [28] and the case for $AdS_7$ black holes.

This paper is organized as follows. In sec. 2, we compute the asymptotic behavior of the SUSY index of the $SU(N) \mathcal{N} = 4$ SYM in the Cardy limit $\beta \to 0$. In sec. 3 we generalize the analysis in sec. 2 to the cases with other gauge groups and additional matters, and the orbifold $\mathcal{N} = 4$ SYM. Sec. 4 is devoted to conclusion and discussions.

## 2 Asymptotic behavior of supersymmetric index in $SU(N) \mathcal{N} = 4$ SYM

Let us consider the $SU(N) \mathcal{N} = 4$ SYM on Euclidean compact manifold of the form $S^1_\beta \times M_3$ with the radius $\beta$. We take $M_3$ to preserve a part of supersymmetry and this condition constrains $S^1_\beta \times M_3$ to be complex [29]. Different choices of $M_3$ count different quantum numbers as different $M_3$’s have different isometries. One of the most well-studied cases is the index on $S^1 \times S^3$ known as superconformal index [12, 13]:

$$I_{S^1 \times S^3} = \text{Tr}_{\text{BPS}} \left[ (-1)^F p^{J_1 + \frac{1}{2}} q^{J_2 + \frac{1}{2}} v_1 q_1 v_2 q_2 \right], \quad (2.1)$$

3 A proposal for quantum black hole entropy for the $M_3 = S^1 \times T^2$ case is written in eq. (1.82) of [27].

4 For example, an index on $T^4$ counts momenta along three “spatial” $S^1$’s as well as flavor charges.
where
\[ p = e^{2\pi i\sigma}, \quad q = e^{2\pi i\tau}, \quad v_{1,2} = e^{2\pi im_{1,2}}. \]  

We are interested in an asymptotic behavior of the partition function in the shrinking \( S^1 \) limit: \( \beta \to 0 \). In this limit, the partition function is exactly the same as the supersymmetric index since we can ignore the contribution from the SUSY Casimir energy in (1.5). Therefore we are essentially looking at the asymptotic behavior of the index. There is a general formula to describe such asymptotic behavior for general 4d \( \mathcal{N} = 1 \) SUSY theory with \( U(1)_R \) symmetry and Lagrangian description which is a refinement \([24, 25]\) of SUSY Cardy formula \([26]\).

For simplicity of explanations, we first consider the superconformal index. We will consider more general \( M_3 \) later. The superconformal index is defined through supersymmetric partition function on a space with topology of \( \mathbb{S}^3 \) for the squashed sphere \( S^3 \) valued in the maximal torus of \( G \). For example, if we take \( M_3 \) to be the squashed sphere \( S^3 \), \( \tau \) and \( \sigma \) are given by \( \tau = -\beta b/2\pi i \) and \( \sigma = -\beta b / 2\pi i \). For any choices, the Cardy limit \( \beta \to 0 \) for the superconformal index is equivalent to \( |\tau|, |\sigma| \to 0 \). The refined SUSY Cardy formula for the superconformal index is given by
\[
I_{S^3 \times S^3} \sim e^{-\frac{i(\tau + \sigma)}{12\pi^2} \text{Tr}(R)} \int d^{\text{rank}G} a \, e^{\frac{i\sigma}{2\pi^2} V_2(a) + \frac{i(\tau + \sigma)}{2\pi^2} V_1(a)},
\]
which has been derived in two ways: taking the limit in localization formula \([25]\) and effective theory consideration \([24]\)

Several definitions are in order. First, \( G \) is the gauge group and \( e^{2\pi i a_j} \) with \( j = 1, \cdots, \text{rank}G \) is holonomy around \( S^1 \) valued in the maximal torus of \( G \). Second, \( \text{Tr}(R) \) is anomaly coefficient of the \( U(1)_R \) symmetry and related to conformal anomalies by \( \text{Tr}(R) = -16(c - a) \) for superconformal case. Third, \( V_2(a) \) and \( V_1(a) \) are piecewise polynomials of \( a_j \) and flavor chemical potentials whose forms are explicitly determined if we specify representations, \( U(1)_R \)-charges and flavor charges of chiral multiplets (see app [A]). Their explicit forms for the \( SU(N) \) \( \mathcal{N} = 4 \) SYM are
\[
V_2(a) = - \sum_{1 \leq i \neq j \leq N} \left[ \kappa(a_{ij} + m_1) + \kappa(a_{ij} + m_2) + \kappa(a_{ij} - m_1 - m_2) \right] \\
- (N - 1) \left[ \kappa(m_1) + \kappa(m_2) + \kappa(-m_1 - m_2) \right],
\]
\[
V_1(a) = \frac{1}{3} \sum_{1 \leq i \neq j \leq N} \left[ 3\theta(a_{ij}) - \theta(a_{ij} + m_1) - \theta(a_{ij} + m_2) - \theta(a_{ij} - m_1 - m_2) \right] \\
- \frac{N - 1}{3} \left[ \theta(m_1) + \theta(m_2) + \theta(-m_1 - m_2) \right],
\]

---

5 See \([30]\) for earlier related works.

6 This is simply the sum of \( U(1)_R \) charges of fermions in theory under consideration.

7 For \( m_1 = 0 = m_2 \), \( V_2(a) \) and \( V_1(a) \) are zero. The leading asymptotic behavior of the index for this case is \((N - 1) \log \beta \) as shown in \([25]\).
where

\[ a_{ij} = a_i - a_j, \quad \sum_{j=1}^{N} a_j = 0, \]

\[ \kappa(x) = \{x\}(1 - \{x\})(1 - 2\{x\}), \quad \theta(x) = \{x\}(1 - \{x\}), \quad (2.5) \]

with fractional part \( \{x\} \equiv x - [x] \) (see fig. 1 for shapes of \( \kappa(x) \) and \( \theta(x) \)). \( V_2(a) \) (\( V_1(a) \)) is apparently a piecewise cubic (quadratic) polynomial but this is actually quadratic (linear) because there is a cancellation of the highest order terms physically coming from cancellation of anomalies involving the gauge symmetry.

Here we restrict ourselves to

\[ \text{Re}\left(\frac{i}{\tau \sigma}\right) < 0, \quad (2.6) \]

and mention other regime later. In this regime, the integral in the limit is dominated by saddle point configuration(s) to minimize the function \( V_2(a) \). We can easily find a dominant saddle point as follows. Noting \( \kappa(-x) = -\kappa(x) \) and \( \kappa(x + 1) = \kappa(x) \), we rewrite \( V_2(a) \) as

\[ V_2(a) = \sum_{i<j} f(a_{ij}) + \frac{N-1}{2} f(0), \quad (2.7) \]

where

\[ f(a_{ij}) = \kappa(a_{ij} - \{m_1\}) - \kappa(a_{ij} + \{m_1\}) + \kappa(a_{ij} - \{m_2\}) - \kappa(a_{ij} + \{m_2\}) + \kappa(a_{ij} + \{m_1\} + \{m_2\}) - \kappa(a_{ij} - \{m_1\} - \{m_2\}). \quad (2.8) \]

It is sufficient to minimize each \( f(a_{ij}) \) and show that we can realize a simultaneously minimizing configuration. As a result, the minimizing configuration is simply \( a_j = 0 \) for any \( j \) as illustrated in fig. 2 for specific values of \((m_1, m_2)\). To see this generally, it is convenient to first analyze the regime

\[ 0 \leq \{m_2\} \leq \{m_1\}, \quad \{m_1\} + \{m_2\} \leq \frac{1}{2}, \quad (2.9) \]

\[ \text{Physically this periodicity reflects invariance under large gauge transformation.} \]
and extend it to other regime by using the periodicity \( m_{1,2} \sim m_{1,2} + 1 \). In this regime, noting \( \kappa(x) = 2x^3 - 3x|x| + x \) for \( |x| \leq 1 \), the function \( f(x) \) in “the fundamental region” \(|x| < 1 - \{m_1\} + \{m_2\}\) is given by

\[
f(x) = \begin{cases} 
6x^2 + 12\{m_1\}\{m_2\}(\{m_1\} + \{m_2\} - 1) & \text{for } |x| \leq \{m_2\} \\
12m_2|x| + 6m_2(2m_1^2 + 2m_1m_2 - 2m_1 - m_2) & \text{for } \{m_2\} \leq |x| \leq \{m_1\} \\
-6(|x| - \{m_1\} - \{m_2\})^2 + 12\{m_1\}\{m_2\}(\{m_1\} + \{m_2\}) & \text{for } \{m_1\} \leq |x| \leq \{m_1\} + \{m_2\} \\
12\{m_1\}\{m_2\}(\{m_1\} + \{m_2\}) & \text{for } \{m_1\} + \{m_2\} \leq |x| 
\end{cases}
\]

(2.10)

which has the minimum at the origin:

\[
f(x)|_{\min} = f(0) = 12\{m_1\}\{m_2\}(\{m_1\} + \{m_2\} - 1). \quad (2.11)
\]

Therefore the minimum of \( V_2(a) \) is realized by \( a_{ij} = 0 \) for all \( i, j \) with the traceless condition \( \sum_{j=1}^N a_j = 0 \), which is nothing but \( a_j = 0 \). Thus we find the minimum of \( V_2(a) \) as

\[
V_2(a)|_{\min} = V_2(0) = 6(N^2 - 1)\{m_1\}\{m_2\}(\{m_1\} + \{m_2\} - 1). \quad (2.12)
\]

The next order \( \mathcal{O}(\beta^{-1}) \) is simply obtained by substituting the saddle point into \( V_1(a) \):

\[
V_1(a)|_{a_{ij}=0} = \frac{2(N^2 - 1)}{3} \left[ \{m_1\}^2 + \{m_2\}^2 + \{m_1\}\{m_2\} - \{m_1\} - \{m_2\} \right]. \quad (2.13)
\]

Then, noting \( c - a = 0 \) in the \( \mathcal{N} = 4 \) SYM, we find the Cardy limit of the superconformal index to be

\[
\log I_{S^1 \times S^3} \underset{|\tau|,|\sigma|\to 0}{\sim} \frac{i\pi(N^2 - 1)}{\tau \sigma} \left[ \{m_1\}\{m_2\}(\{m_1\} + \{m_2\} - 1) \right].
\]

\( \text{The saddle point of } V_2(a) \text{ also realizes the minimum of } V_1(a) \text{ as a result though this property is not necessary for our analysis. Beyond this order, we need to take into account fluctuations around the saddle point.} \)
\[
+ \frac{\tau + \sigma}{3} \left( \{m_1\}^2 + \{m_2\}^2 + \{m_1\}\{m_2\} - \{m_1\} - \{m_2\} \right). \tag{2.14}
\]

In order to directly compare this with the Bekenstein-Hawking entropy, it is convenient to rewrite the result in the following two steps. First we redefine the chemical potentials \( m_{1,2} \) as

\[
m_{1,2} = \Delta_{1,2} - \frac{\tau + \sigma}{3}, \tag{2.15}
\]

so that our index becomes

\[
\text{Tr}_{\text{BPS}} \left[ (-1)^F p^{J_1+Q_3} q^{J_2+Q_3} e^{2\pi i \Delta_1 (Q_1-Q_3)} e^{2\pi i \Delta_2 (Q_2-Q_3)} \right]. \tag{2.16}
\]

This object is the same as the grand canonical partition function

\[
\text{Tr}_{\text{BPS}} \left[ p^{J_1} q^{J_2} \prod_{a=1}^3 e^{2\pi i \Delta_a Q_a} \right], \tag{2.17}
\]

with the constraint\(^{10}\) \( \Delta_1 + \Delta_2 + \Delta_3 - \tau - \sigma - 1 \in 2\mathbb{Z} \). In this parametrization, the asymptotic behavior of the index becomes

\[
\log I_{S^1 \times S^3} \sim \frac{i\pi (N^2 - 1)\{\Delta_1\}\{\Delta_2\}\{\Delta_1\} + \{\Delta_2\} - 1 - \sigma - \tau}{\tau \sigma}. \tag{2.18}
\]

Second, we perform a Legendre transformation \(^{18}\) with respect to \((\sigma, \tau, \Delta_1, \Delta_2)\) to directly obtain entropy or equivalently degeneracy of states with fixed charges and angular momenta. We will perform this analysis in next subsection.

**Comments on other regime of \((\tau, \sigma)\)**

So far we have taken \( \text{Re} \left( \frac{1}{\tau \sigma} \right) < 0 \). If we take it oppositely i.e. \( \text{Re} \left( \frac{1}{\tau \sigma} \right) > 0 \), then we need to minimize \(-V_2(a)\) or equivalently maximize \(V_2(a)\). Then the dominant saddle points are given by the points maximizing \(f(x)\). According to (2.10), the saddle points are any configurations giving the plateau regime of \(f(x)\), namely the ones satisfying \(\{m_1\} + \{m_2\} \leq |\{a_{ij}\}| < 1 - \{m_1\} + \{m_2\}\). We immediately see that the saddle points are no longer isolated and therefore it remains integration over the saddle points which seems complicated since \(V_1(a)\) is not constant in this regime. As a result, the asymptotic behavior of the index is

\[
\log I_{S^1 \times S^3} \sim \frac{i\pi \{m_1\}\{m_2\}}{\tau \sigma} \left[ (N^2 - 1)(\{m_1\} + \{m_2\}) - (N - 1) \right] + \log \int_{\text{saddles}} d^N a \, \delta \left( \sum_{j=1}^N a_j \right) e^{\frac{i\pi (\tau + \sigma)}{2\pi \sigma} V_1(a)}. \tag{2.19}
\]

This implies that we have anti-Stokes line at \( \text{Re} \left( \frac{1}{\tau \sigma} \right) = 0 \) since the dominant saddle point changes there. The above saddle points are unstable in the regime \( \text{Re} \left( \frac{1}{\tau \sigma} \right) < 0 \) which we

\(^{10}\) We have used \((-1)^F = e^{2\pi i Q_3}\. 
have mainly considered in this paper. Relatedly Stokes phenomena have been observed in the large-$N$ analysis of the Bethe ansatz type formula [22]. It is interesting to understand the above phenomena in more detail and find their physical interpretations especially from the gravity side. This might be related to hairy black holes discussed in [31].

2.1 Comparison with Bekenstein-Hawking entropy

This subsection is essentially a review of various papers [18, 15, 20, 22] up to identifications [11]. The Legendre transformation of the black hole entropy is referred to as entropy function [33]. Suppose that we have the entropy function $S$:

$$S = 2\pi i\nu \frac{X_1 X_2 X_3}{\omega_1 \omega_2}, \quad (2.20)$$

with the constraint

$$X_1 + X_2 + X_3 - \omega_1 - \omega_2 = n. \quad (2.21)$$

These quantities in our case are

$$S = -\log I_{S_1 \times S_2}, \quad \nu = \frac{N^2 - 1}{2}, \quad \omega_1 = \sigma, \quad \omega_2 = \tau, \quad X_a = \{\Delta_a\}, \quad n = 1. \quad (2.22)$$

The entropy $S(Q, J)$ is obtained by the Legendre transformation

$$S(Q, J) = S(X_a, \omega_i) + 2\pi i \left( \sum_{a=1}^3 X_a Q_a + \sum_{l=1}^2 \omega_l J_l \right) + 2\pi i \Lambda \left( \sum_{a=1}^3 X_a - \sum_{l=1}^2 \omega_l - n \right) \bigg|_{X_a, \omega_i}, \quad (2.23)$$

where $\Lambda$ is Lagrange multiplier. The extremization conditions are

$$\frac{\partial S}{\partial X_a} = -2\pi i (Q_a + \Lambda), \quad \frac{\partial S}{\partial \omega_l} = -2\pi i (J_l - \Lambda), \quad (2.24)$$

with the constraint (2.21). Note that we do not need explicit solutions for $(X_a, \omega_l)$ to compute $S$ if we use the relation

$$S = \sum_{a=1}^3 X_a \frac{\partial S}{\partial X_a} + \sum_{l=1}^2 \omega_l \frac{\partial S}{\partial \omega_l}. \quad (2.25)$$

Then the entropy is simply given by

$$S = 2\pi i n \Lambda, \quad (2.26)$$

where $\Lambda$ satisfies

$$0 = (Q_1 + \Lambda)(Q_2 + \Lambda)(Q_3 + \Lambda) + \nu (J_1 - \Lambda)(J_2 - \Lambda) = \Lambda^3 + p_2 \Lambda^2 + p_1 \Lambda + p_0, \quad (2.27)$$

The original argument was in sec. 3 of [18]. This subsection is also a review of appendix B of [15], sec. 2.3 of [20] and sec. 6 of [22]. See also [32] for a similar argument for AdS$_7$ black holes.
with
\[ p_0 = Q_1Q_2Q_3 + \mu J_1J_2, \]
\[ p_1 = Q_1Q_2 + Q_2Q_3 + Q_3Q_1 - \nu(J_1 + J_2), \]
\[ p_2 = Q_1 + Q_2 + Q_3 + \mu J_1J_2. \]  

(2.28)

The equation for \( \Lambda \) has the three solutions \( \Lambda = \{ -p_2, \pm i\sqrt{p_1}\} \) with \( p_1, p_2 \in \mathbb{R}_{\geq 0} \). Imposing the entropy to be real positive, the physical solution among the three is \( \Lambda = -i\text{sign}(n)\sqrt{p_1} \) which leads us to the entropy
\[ S = 2\pi|n|\sqrt{p_1}. \]  

(2.29)

Under the identifications (2.22), the entropy computed by the superconformal index of the \( SU(N) \mathcal{N} = 4 \) SYM is
\[ S_{\text{QFT}}(Q, J) = 2\pi \sqrt{Q_1Q_2 + Q_1Q_3 + Q_2Q_3 - \frac{N^2 - 1}{2}(J_1 + J_2)}, \]  

(2.30)

which agrees with the Bekenstein-Hawking entropy (1.2) via the AdS/CFT dictionary (1.3) in the large-\( N \) limit. Interestingly, the agreement persists for finite \( N \) if we slightly modify the AdS/CFT dictionary for finite \( N \) as
\[ \frac{\pi}{2G_N g^3} \bigg|_{\text{finite } N} = N^2 - 1 = 4c, \]  

(2.31)

where \( c = \frac{N^2 - 1}{4} \) is the central charge. This may suggest that the black hole entropy with full quantum corrections is captured by the Bekenstein-Hawking entropy with the renormalized Newton constant (2.31) in the Cardy limit.

\[ \text{2.2 General } M_3 \]

The refined SUSY Cardy formula for the \( SU(N) \mathcal{N} = 4 \) SYM on \( S^1 \times M_3 \) is
\[ I_{S^1 \times M_3} \bigg|_{\beta \to 0} \sim \int d^N a \delta \left( \sum_{j=1}^N a_j \right) e^{-\frac{\pi^2 A_{M_3} \theta_1(a)}{6\beta^2} + \frac{\pi^2 L_{M_3} \theta_3(a)}{2\beta} - \frac{\pi^2 \theta_1(a)}{2\beta}} \]  

(2.32)

where \( \theta_1(a) \) is the contribution absent in the superconformal index:
\[ \theta_1(a) = \sum_{i \neq j} \left( \ell_i^M_3 - \ell_j^M_3 \right) \left[ \theta(a_{ij} + m_1) + \theta(a_{ij} + m_2) + \theta(a_{ij} - m_1 - m_2) + \theta(a_{ij}) \right]. \]  

(2.33)

The quantities \( A_{M_3}, L_{M_3} \) and \( \ell_i^M_3 \) are local functionals on \( M_3 \) given by bosonic fields in the 3d new minimal supergravity multiplet \( (h_{\mu \nu}, A_\mu^{(R)}, H, c_\mu) \) and 3d \( \mathcal{N} = 2 \) vector multiplet \( (A_\mu, \sigma, D) \):
\[ A_{M_3} = \frac{i}{\pi^2} \int_{M_3} d^3 x \sqrt{h} \left[ -c_\mu v_\mu + 2H \right]. \]

\[ ^{12} \text{This is both for gauge and global symmetries.} \]
\[ L_{M_3} = \frac{1}{\pi^2} \int_{M_3} d^3x \sqrt{h} \left[ -A^{(R)}_{\mu} v^\mu + v_\mu v^\mu - \frac{1}{2} H^2 + \frac{1}{4} R \right], \]
\[ \ell^i_{M_3} = \frac{1}{\pi^2} \int_{M_3} d^3x \sqrt{h} \left[ -A^i_{\mu} v^\mu + D^i \right], \] (2.34)

which come from induced Chern-Simons terms of \( U(1)_{KK} \) \( U(1)_{KK} \), \( U(1)_{KK} \) \( U(1)_R \) and \( U(1)_{KK} \) \( U(1)_{KK} \) Gauge/Flavor respectively, from the viewpoint of 3d effective theory on \( M_3 \). Technically \( A_{M_3} \) and \( L_{M_3} \) are just constants for fixed \( M_3 \) while \( l^i_{M_3} \) generally depends on (supersymmetric configurations of) the dynamical vector multiplets though it has typically a simple form because of SUSY\(^{13}\).

Here we restrict ourselves to
\[ \text{Re} \left( \frac{iA_{M_3}}{\beta^2} \right) > 0, \] (2.35)
which generalizes the condition (2.6). Then the integral in the \( \beta \to 0 \) limit is dominated by the saddle point of \( V_2(a) \) which is already found as \( a_j = 0 \). Thus, noting \( V_1(a) \bigg|_{a_j=0} = 0 \), the asymptotic behavior of the index for general \( M_3 \) is
\[ \log I_{S^1 \times M_3} \bigg|_{\beta \to 0} \approx \frac{2\pi^3}{\beta^2} \left\{ m_1 \right\} \left\{ m_2 \right\} \left\{ \left\{ m_1 \right\} + \left\{ m_2 \right\} - 1 \right\} + \frac{2}{\beta} \left[ \left\{ m_1 \right\}^2 + \left\{ m_2 \right\}^2 + \left\{ m_1 \right\} \left\{ m_2 \right\} - \left\{ m_1 \right\} - \left\{ m_2 \right\} \right]. \] (2.36)

This makes predictions to the gravity side for more general \( M_3 \). For example, the case for Lens space index is
\[ \log I_{S^1 \times S^3 / \mathbb{Z}_n} \bigg|_{|\tau|,|\sigma| \to 0} \approx \frac{i\pi}{n \tau \sigma} \left[ \left\{ m_1 \right\} \left\{ m_2 \right\} \left\{ \left\{ m_1 \right\} + \left\{ m_2 \right\} - 1 \right\} \right. \]
\[ + \frac{\tau + \sigma}{3} \left( \left\{ m_1 \right\}^2 + \left\{ m_2 \right\}^2 + \left\{ m_1 \right\} \left\{ m_2 \right\} - \left\{ m_1 \right\} - \left\{ m_2 \right\} \right) \]
\[ = \frac{\log I_{S^1 \times S^3}}{n}, \] (2.37)

which implies that the dual black hole entropy is \( 1/n \) of the one for the superconformal index.

### 3 Generalizations

#### 3.1 Other gauge groups

Generalization to other gauge groups is straightforward because we can still apply the technique in the \( SU(N) \) case. For the \( \mathcal{N} = 4 \) SYM with gauge group \( G \), the functions appearing

---

\(^{13}\) See [26, 24] for details.

\(^{14}\) For example, \( \ell_{S_3/\mathbb{Z}_n} \equiv 0 \) and \( \ell_{S_3 \times \Sigma_g} \propto \text{(magnetic charge)} \) with Riemann surface \( \Sigma_g \).
in the SUSY Cardy formula are
\[
V_2(a) = - \sum_{\alpha \in \text{root}} \left[ \kappa(\alpha \cdot a + m_1) + \kappa(\alpha \cdot a + m_2) + \kappa(\alpha \cdot a - m_1 - m_2) \right]
- \text{rank}(G) \left[ \kappa(m_1) + \kappa(m_2) + \kappa(-m_1 - m_2) \right],
\]
\[
V_1(a) = \frac{1}{3} \sum_{\alpha \in \text{root}} \left[ 3\theta(\alpha \cdot a) - \theta(\alpha \cdot a + m_1) - \theta(\alpha \cdot a + m_2) - \theta(\alpha \cdot a - m_1 - m_2) \right]
- \frac{N - 1}{3} \left[ \theta(m_1) + \theta(m_2) + \theta(-m_1 - m_2) \right],
\]
\[
\tilde{V}_1(a) = \sum_{\alpha \in \text{root}} \alpha \cdot L_{M_3} \left[ \theta(\alpha \cdot a + m_1) + \theta(\alpha \cdot a + m_2) + \theta(\alpha \cdot a - m_1 - m_2) + \theta(\alpha \cdot a) \right].
\]
(3.1)

In terms of \( f(x) \), we rewrite \( V_2(a) \) as
\[
V_2(a) = \sum_{\alpha \in \text{root}^+} f(\alpha \cdot a) + \frac{\text{rank}(G)}{2} f(0),
\]
(3.2)
which has the global minimum at \( a_j = 0 \) by the same logic\(^{15}\) as in sec. 2. Thus the index asymptotically behaves as
\[
\log I_{S^1 \times M_3} \sim \frac{-2\pi^3 i A_{M_3} \text{dim}(G)}{\beta^2} \left( m_1 \{ m_2 \} \{ m_1 \} + \{ m_2 \} - 1 \right)
+ \frac{\pi^2 L_{M_3} \text{dim}(G)}{3\beta} \left( \{ m_1 \}^2 + \{ m_2 \}^2 + \{ m_1 \} \{ m_2 \} - \{ m_1 \} - \{ m_2 \} \right).
\]
(3.3)
Especially, the superconformal index is\(^{16}\)
\[
\log I_{S^1 \times S^3} \sim \frac{i \pi \text{dim} G \{ \Delta_1 \} \{ \Delta_2 \} \{ \Delta_1 \} + \{ \Delta_2 \} - 1 - \sigma - \tau}{\tau \sigma}.
\]
(3.4)
The Legendre transformation leads us to the entropy
\[
S_{\text{QFT}}(Q, J) = 2\pi \sqrt{Q_1 Q_2 + Q_1 Q_3 + Q_2 Q_3 - \frac{\text{dim} G}{2} (J_1 + J_2)}
= 2\pi \sqrt{Q_1 Q_2 + Q_1 Q_3 + Q_2 Q_3 - 2c(J_1 + J_2)},
\]
(3.5)
where we have used \( c = \text{dim} G/4 \). This implies that the dual black hole entropy for gauge group \( G \) is captured by \(^{17,22}\) under the identification
\[
\frac{\pi}{2G_{NG}^3} \bigg|_{\text{finite } N} = 4c,
\]
(3.6)
even if \( G \) is not necessarily \( SU(N) \) or \( U(N) \).

\(^{15}\) For \( G = U(N) \), this is sufficient but not necessary due to decoupling the diagonal \( U(1) \). The same minimum is realized by any configuration satisfying \( a_1 = \cdots = a_N \) which is the same as the one obtained in \(^{20}\). This flat direction affects \( O(\log \beta) \).

\(^{16}\) For \( G = U(N) \), the result is the same as the one obtained in \(^{20}\) which takes the large-\( N \) limit. However, our result shows that the result of \(^{20}\) is formally correct also for finite \( N \). This implies that contributions which are ignored in \(^{20}\) vanish in the Cardy limit.
3.2 Adding matters in conjugate representations

Let us add pairs of chiral multiplets in conjugate representations to the $\mathcal{N} = 4$ SYM with the gauge group $G$. In general this theory may have new flavor symmetries but let us keep to turn off fugacities of the new symmetries for simplicity. For this case, the function $V_2(a)$ does not receive contributions from the additional matters essentially because of $\kappa(-x) = -\kappa(x)$. Therefore the holonomy integral of the SUSY Cardy formula is still dominated by $a_j = 0$. Furthermore, contributions from the additional matters to the $V_1(a)$ and $\tilde{V}_1(a)$ are zero at $a_j = 0$. Thus, the asymptotic behavior of the index is

$$\log I_{S^1 \times M_3} \overset{\beta \to 0}{\sim} -\frac{\pi^2 L_{M_3}}{12\beta} \text{Tr}(R) - \frac{2\pi^3 iA_{M_3} \dim(G)}{\beta^2} \{m_1\} \{m_2\} \{\{m_1\} + \{m_2\} - 1\} + \frac{\pi^2 L_{M_3} \dim(G)}{3\beta} \left[ (\{m_1\})^2 + (\{m_2\})^2 + \{m_1\} \{m_2\} - \{m_1\} - \{m_2\} \right].$$

(3.7)

Note that the difference from the $\mathcal{N} = 4$ SYM is only the first term, which is simply captured by the unrefined SUSY Cardy formula [26]. Specifying to the superconformal index case, we find

$$\log I_{S^1 \times S^1} \overset{|\tau|,|\sigma| \to 0}{\sim} \frac{i\pi \dim G\{\Delta_1\} \{\Delta_2\} \{(\Delta_1) + (\Delta_2) - 1 - \sigma - \tau\}}{\tau \sigma} - \frac{i\pi(\tau + \sigma)}{12\tau \sigma} \text{Tr}(R).$$

(3.8)

This indicates that the entropies in theories with $|\text{Tr}(R)|/N^2 \ll 1$ in the large-$N$ limit are universally captured by the one of the $\mathcal{N} = 4$ SYM. An interesting example of such theories is the $SU(N)$ $\mathcal{N} = 4$ SYM plus $N_f$ fundamental hypermultiples known as D3-D7 system.

3.3 Orbifold $\mathcal{N} = 4$ SYM

Let us consider so-called orbifold $\mathcal{N} = 4$ SYM which is the circular quiver $\mathcal{N} = 2$ gauge theory with $U(N)_1 \times \cdots U(N)_K$ gauge group and one bi-fundamental hypermultiplet of neighboring gauge group $U(N)_1 \times U(N)_{i+1}$. We turn on chemical potentials $m_1, m_2$ of flavor symmetry $U(1)_1 \times U(1)_2$ in which the $U(1)_1$ $(U(1)_2)$ symmetry assigns charge 1 to each $\mathcal{N} = 1$ (anti-)bi-fundamental chiral multiplet and charge -1 to each $\mathcal{N} = 1$ adjoint chiral multiplet in the $\mathcal{N} = 2$ vector multiplet. The function $V_2(a)$ for this theory is

$$V_2(a) = -\sum_{I=1}^{K} \sum_{1 \leq i,j \leq N} \left[ \kappa \left( a_i^{(I)} - a_j^{(I+1)} + m_1 \right) + \kappa \left( -a_i^{(I)} + a_j^{(I+1)} + m_2 \right) + \kappa \left( a_{ij}^{(I)} - m_1 - m_2 \right) \right].$$

(3.9)

It is not easy to find global minimum of this function in contrast to the $\mathcal{N} = 4$ SYM. Instead of solving this problem completely, we proceed by taking the physically motivated ansatz:

$$a_j^{(I)} = a_j^{(j)} = a_j,$$

(3.10)

$^{17}$ in the notation $U(N)_{K+1} = U(N)_1$. 

12
which reflects $Z_k$ rotation symmetry of the quiver diagram or equivalently all the gauge groups are “democratic”\footnote{This type of ansatz was taken also in large-$N$ analysis of $S^4$ partition function in the orbifold $\mathcal{N} = 4$ SYM \cite{34}.}. Under this ansatz, $V_2(a)$ becomes

$$V_2(a)\bigg|_{a^{(j)}_i = a^{(j)}_{aj}} = -K \sum_{1 \leq i, j \leq N} \left[ \kappa (a_{ij} + m_1) + \kappa (-a_{ij} + m_2) + \kappa (a_{ij} - m_1 - m_2) \right],$$

which is proportional to $V_2(a)$ of the $U(N) \mathcal{N} = 4$ SYM. Thus, the asymptotic behavior of the index is

$$\log I_{S^1 \times M_3} \underset{\beta \to 0}{\simeq} -\frac{2\pi^3 i A_{M_3} KN^2}{\beta^2} \{m_1\} \{m_2\} \{(m_1) + (m_2) - 1\} + \frac{\pi^2 L_{M_3} KN^2}{3\beta} \left[ \{m_1\}^2 + \{m_2\}^2 + \{m_1\} \{m_2\} - \{m_1\} - \{m_2\} \right].$$

This result has a nice interpretation from the viewpoint of so-called large-$N$ orbifold equivalence \cite{35} which states that a free energy of a “daughter” theory obtained by a projection of a “parent” theory by a group $\Gamma$ obeys

$$\lim_{N \to \infty} F_{\text{daughter}} \frac{N^2}{|\Gamma|} = \lim_{N \to \infty} F_{\text{parent}} \frac{N^2}{|\Gamma|},$$

where $|\Gamma|$ is the order of $\Gamma$. Since the orbifold $\mathcal{N} = 4$ SYM is obtained by a $Z_k$ projection of the $U(KN) \mathcal{N} = 4$ SYM, the above result is expected from the orbifold equivalence. The result for the superconformal index is

$$\log I_{S^1 \times S^3} \underset{|\tau|, |\sigma| \to 0}{\simeq} \frac{i \pi KN^2 \{\Delta_1\} \{\Delta_2\} \{(\Delta_1) + (\Delta_2) - 1 - \sigma - \tau\}}{\tau \sigma},$$

which gives the entropy

$$S_{\text{QFT}}(Q, J) = 2\pi \sqrt{Q_1 Q_2 + Q_1 Q_3 + Q_2 Q_3 - \frac{KN^2}{2}(J_1 + J_2)}.$$

Noting $c = KN^2/2$, we can also express this as

$$S_{\text{QFT}}(Q, J) = 2\pi \sqrt{Q_1 Q_2 + Q_1 Q_3 + Q_2 Q_3 - 2c(J_1 + J_2)}.$$

### 4 Conclusion and Discussions

In this paper we have mainly studied the supersymmetric index of the $SU(N) \mathcal{N} = 4$ super Yang-Mills theory on $S^1 \times M_3$. We have computed the asymptotic behavior of the index in the Cardy limit for arbitrary $N$ by the refined supersymmetric Cardy formula. We have seen that the asymptotic behavior of the superconformal index in the large-$N$ limit agrees with
the Bekenstein-Hawking entropy (1.2) of the rotating electrically charged BPS black hole in $AdS_5$ via the Legendre transformation as recently shown in [20]. We have also found that the agreement formally persists for finite $N$ if we slightly modify the AdS/CFT dictionary (1.3) as
$$\frac{\pi}{2G_{N}g^{3}} = 4c.$$ This implies an existence of non-renormalization property for the black hole entropy in the Cardy limit. We have also studied the cases with other gauge groups and additional matters, and the orbifold $\mathcal{N} = 4$ SYM. It has turned out that the entropies of all the CFT examples in this paper are given by (1.7).

There are several questions and interesting future directions. Perhaps the most immediate question is whether or not our results match at quantum level. The first step to test this would be to compute a logarithmic correction to the black hole entropy by one-loop analysis of the supergravity as in the case of the magnetically charged $AdS_4$ black holes [36]. Our result suggests that the logarithmic correction is absent in the Cardy limit. It is also of course illuminating to include higher derivative corrections. Another question is what are physical interpretations of the dominant saddle points in the regime $\text{Re} \left( \frac{i}{\tau \sigma} \right) > 0$, which we have not mainly considered in this paper. The dominant saddle points in this regime are not isolated and technically give the plateau in the function $f(x)$ given in (2.10) but they are not degenerate at $O(\beta^{-1})$. This question might be related to hairy black holes discussed in [31]. It is also interesting to study higher order corrections of $\beta$ to the Cardy limit in order to interpolate our result to the one in [22] which does not take the Cardy limit. The higher order corrections might be significantly different between large-$N$ and finite $N$. Another interesting direction is to extend our results for more general holographic 4d CFT such as less supersymmetric case. Perhaps there is an efficient way to compute the asymptotic behavior of the index especially for class-$S$ theories.

Acknowledgement

The author thanks Nikolay Bobev, Seyed Morteza Hosseini, Seok Kim and Leopoldo A. Pando Zayas for comments to the earlier versions of arXiv. This work has been partially supported by STFC consolidated grant ST/P000681/1.

A Explicit forms of $V_2(a)$, $V_1(a)$ and $\tilde{V}_1(a)$ for general Lagrangian 4d $\mathcal{N} = 1$ theory

Let us consider 4d $\mathcal{N} = 1$ SUSY gauge theory with gauge group $G$ coupled to chiral multiplets of representation $R_I$ having $U(1)_R$ charge $R_I$ and flavor charge $Q^I_j$ of $U(1)_j$ flavor symmetry. The refined Cardy formula takes the form [24]

$$I_{S^1 \times M_3} \approx \frac{e^{-\pi^2 \text{Tr}(R)L_{M_3}}}{\beta^3} \int \text{d}^\text{rank}G \, a \, e^{-\frac{\pi^2 L_{M_3}}{6g^2}V_2(a) + \frac{\pi^2 L_{M_3}}{2g^2}V_1(a) - \frac{1}{2g^2}V_1(a)}.$$

(A.1)
where
\[
V_2(a) = - \sum I \in \text{matters} \sum \rho_I \in R_I \kappa(\rho_I \cdot a + \sum j \in \text{flavor} Q^j m_j),
\]
\[
V_1(a) = \sum \alpha \in \text{root} \theta(\alpha \cdot a) + \sum I \in \text{matters} \sum \rho_I \in R_I (R_I - 1) \theta(\rho_I \cdot a + \sum j \in \text{flavor} Q^j m_j)
\]
\[
\tilde{V}_1(a) = \sum I \in \text{matters} \sum \rho_I \in R_I \rho_I \cdot \ell_{M_3} \theta(\rho_I \cdot a + \sum j \in \text{flavor} Q^j m_j).
\]

(A.2)

References

[1] J. D. Bekenstein, *Black holes and the second law*, Lett. Nuovo Cim. 4 (1972) 737–740; *Black holes and entropy*, Phys. Rev. D7 (1973) 2333–2346; *Generalized second law of thermodynamics in black hole physics*, Phys. Rev. D9 (1974) 3292–3300, S. W. Hawking, *Particle Creation by Black Holes*, Commun. Math. Phys. 43 (1975) 199–220. [167(1975)]. *Black hole explosions*, Nature 248 (1974) 30–31.

[2] A. Strominger and C. Vafa, *Microscopic origin of the Bekenstein-Hawking entropy*, Phys. Lett. B379 (1996) 99–104, [hep-th/9601029].

[3] J. M. Maldacena, *The Large N limit of superconformal field theories and supergravity*, Int. J. Theor. Phys. 38 (1999) 1113–1133, [hep-th/9711200]. [Adv. Theor. Math. Phys.2,231(1998)], S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, *Gauge theory correlators from noncritical string theory*, Phys. Lett. B428 (1998) 105–114, [hep-th/9802109], E. Witten, *Anti-de Sitter space and holography*, Adv. Theor. Math. Phys. 2 (1998) 253–291, [hep-th/9802150].

[4] F. Benini, K. Hristov, and A. Zaffaroni, *Black hole microstates in AdS_4 from supersymmetric localization*, JHEP 05 (2016) 054, [arXiv:1511.04085].

[5] F. Benini, K. Hristov, and A. Zaffaroni, *Exact microstate counting for dyonic black holes in AdS_4*, Phys. Lett. B771 (2017) 462–466, [arXiv:1608.07294].

[6] F. Benini and A. Zaffaroni, *A topologically twisted index for three-dimensional supersymmetric theories*, JHEP 07 (2015) 127, [arXiv:1504.03698]; *Supersymmetric partition functions on Riemann surfaces*, Proc. Symp. Pure Math. 96 (2017) 13–46, [arXiv:1605.06120], C. Closset and H. Kim, *Comments on twisted indices in 3d supersymmetric gauge theories*, JHEP 08 (2016) 059, [arXiv:1605.06531], M. Honda and Y. Yoshida, *Supersymmetric index on T^2\times S^2 and elliptic genus*, [arXiv:1504.04355].

More generally, \(\tilde{V}_1(a)\) can have \(\ell_{M_3}\) for flavor symmetry background. For example, \(\ell_{M_3}\) for \(M_3 = S^1 \times \Sigma_g\) is proportional to magnetic charge and we have to specify the background magnetic charges.
S. M. Hosseini and A. Zaffaroni, *Large N matrix models for 3d $\mathcal{N} = 2$ theories: twisted index, free energy and black holes*, JHEP 08 (2016) 064, [arXiv:1604.03122].

S. M. Hosseini, A. Nedelin, and A. Zaffaroni, *The Cardy limit of the topologically twisted index and black strings in AdS$_5$*, JHEP 04 (2017) 014, [arXiv:1611.09374].

A. Cabo-Bizet, V. I. Giraldo-Rivera, and L. A. Pando Zayas, *Microstate counting of AdS$_4$ hyperbolic black hole entropy via the topologically twisted index*, JHEP 08 (2017) 023, [arXiv:1701.07893].

F. Azzurli, N. Bobev, P. M. Crichigno, V. S. Min, and A. Zaffaroni, *A universal counting of black hole microstates in AdS$_4$*, JHEP 02 (2018) 054, [arXiv:1707.04257].

S. M. Hosseini, K. Hristov, and A. Passias, *Holographic microstate counting for AdS$_4$ black holes in massive IIA supergravity*, JHEP 10 (2017) 190, [arXiv:1707.06884].

F. Benini, H. Khachatryan, and P. Milan, *Black hole entropy in massive Type IIA*, Class. Quant. Grav. 35 (2018), no. 3 035004, [arXiv:1707.06886].

N. Halmagyi and S. Lal, *On the on-shell: the action of AdS$_4$ black holes*, JHEP 03 (2018) 146, [arXiv:1710.09580].

N. Bobev, V. S. Min, and K. Pilch, *Mass-deformed ABJM and black holes in AdS$_4$*, JHEP 03 (2018) 050, [arXiv:1801.03135].

S. M. Hosseini, I. Yaakov, and A. Zaffaroni, *Topologically twisted indices in five dimensions and holography*, JHEP 11 (2018) 119, [arXiv:1808.06626].

P. M. Crichigno, D. Jain, and B. Willett, *5d Partition Functions with A Twist*, JHEP 11 (2018) 058, [arXiv:1808.06744].

M. Suh, *Supersymmetric AdS$_5$ black holes from F(4) gauged supergravity*, JHEP 01 (2019) 035, [arXiv:1809.03517].

S. M. Hosseini, K. Hristov, A. Passias, and A. Zaffaroni, *6D attractors and black hole microstates*, [arXiv:1809.10685].

M. Suh, *D4-branes wrapped on supersymmetric four-cycles from matter coupled F(4) gauged supergravity*, [arXiv:1810.00675].

H. K. Kunduri, J. Lucietti, and H. S. Reall, *Supersymmetric multi-charge AdS(5) black holes*, JHEP 04 (2006) 036, [hep-th/0601156].

J. B. Gutowski and H. S. Reall, *Supersymmetric AdS(5) black holes*, JHEP 02 (2004) 006, [hep-th/0401042]; *General supersymmetric AdS(5) black holes*, JHEP 04 (2004) 048, [hep-th/0401129].

Z. W. Chong, M. Cvetic, H. Lu, and C. N. Pope, *Five-dimensional gauged supergravity black holes with independent rotation parameters*, Phys. Rev. D72 (2005) 041901, [hep-th/0505112]; *General non-extremal rotating black holes in minimal five-dimensional gauged supergravity*, Phys. Rev. Lett. 95 (2005) 161301, [hep-th/0506029].

H. K. Kunduri, J. Lucietti, and H. S. Reall, *Supersymmetric multi-charge AdS(5) black holes*, JHEP 04 (2006) 036, [hep-th/0601156].

M. Cvetic, M. J. Duff, P. Hoxha, J. T. Liu, H. Lu, J. X. Lu, R. Martinez-Acosta, C. N. Pope, H. Sati, and T. A. Tran, *Embedding AdS black holes in ten-dimensions and eleven-dimensions*, Nucl. Phys. B558 (1999) 96–126, [hep-th/9903214].

E. O. Colgain, M. M. Sheikh-Jabbari, J. F. Vazquez-Poritz, H. Yavartanoo and Z. Zhang, *Warped Ricci-flat reductions*, Phys. Rev. D 90, no. 4, 045013 (2014).
[arXiv:1406.6354],

[11] S. Kim and K.-M. Lee, 1/16-BPS Black Holes and Giant Gravitons in the AdS(5) × S**5 Space, JHEP 12 (2006) 077, [hep-th/0607085].

[12] J. Kinney, J. M. Maldacena, S. Minwalla, and S. Raju, An Index for 4 dimensional super conformal theories, Commun.Math.Phys. 275 (2007) 209–254, [hep-th/0510251].

[13] C. Romelsberger, Counting chiral primaries in N = 1, d=4 superconformal field theories, Nucl. Phys. B747 (2006) 329–353, [hep-th/0510060].

[14] M. Berkooz, D. Reichmann, and J. Simon, A Fermi Surface Model for Large Supersymmetric AdS(5) Black Holes, JHEP 01 (2007) 048, [hep-th/0604023], R. A. Janik and M. Trzetrzelewski, Supergravitons from one loop perturbative N=4 SYM, Phys. Rev. D77 (2008) 085024, [arXiv:0712.2714], L. Grant, P. A. Grassi, S. Kim, and S. Minwalla, Comments on 1/16 BPS Quantum States and Classical Configurations, JHEP 05 (2008) 049, [arXiv:0803.4183], M. Berkooz and D. Reichmann, Weakly Renormalized Near 1/16 SUSY Fermi Liquid Operators in N=4 SYM, JHEP 10 (2008) 084, [arXiv:0807.0559], C.-M. Chang and X. Yin, 1/16 BPS states in N = 4 super-Yang-Mills theory, Phys. Rev. D88 (2013), no. 10 106005, [arXiv:1305.6314].

[15] A. Cabo-Bizet, D. Cassani, D. Martelli, and S. Murthy, Microscopic origin of the Bekenstein-Hawking entropy of supersymmetric AdS5 black holes, [arXiv:1810.11442].

[16] B. Assel, D. Cassani, and D. Martelli, Localization on Hopf surfaces, JHEP 1408 (2014) 123, [arXiv:1405.5144].

[17] B. Assel, D. Cassani, L. Di Pietro, Z. Komargodski, J. Lorenzen, and D. Martelli, The Casimir Energy in Curved Space and its Supersymmetric Counterpart, JHEP 07 (2015) 043, [arXiv:1503.05537].

[18] S. M. Hosseini, K. Hristov, and A. Zaffaroni, An extremization principle for the entropy of rotating BPS black holes in AdS5, JHEP 07 (2017) 106, [arXiv:1705.05383].

[19] N. Bobev, M. Bullimore and H. C. Kim, Supersymmetric Casimir Energy and the Anomaly Polynomial, JHEP 1509, 142 (2015) [arXiv:1507.08553].

[20] S. Choi, J. Kim, S. Kim, and J. Nahmgoong, Large AdS black holes from QFT, [arXiv:1810.12067]

[21] S. Choi, J. Kim, S. Kim, and J. Nahmgoong, Comments on deconfinement in AdS/CFT, [arXiv:1811.08646]

[22] F. Benini and P. Milan, Black holes in 4d N = 4 Super-Yang-Mills, [arXiv:1812.09613]
[23] F. Benini and P. Milan, *A Bethe Ansatz type formula for the superconformal index*, [arXiv:1811.04107], C. Closset, H. Kim, and B. Willett, $\mathcal{N} = 1$ supersymmetric indices and the four-dimensional A-model, JHEP 08 (2017) 090, [arXiv:1707.05774].

[24] L. Di Pietro and M. Honda, *Cardy Formula for 4d SUSY Theories and Localization*, JHEP 04 (2017) 055, [arXiv:1611.00380].

[25] A. Arabi Ardehali, *High-temperature asymptotics of supersymmetric partition functions*, JHEP 07 (2016) 025, [arXiv:1512.03376].

[26] L. Di Pietro and Z. Komargodski, *Cardy formulae for SUSY theories in $d = 4$ and $d = 6$*, JHEP 12 (2014) 031, [arXiv:1407.6061].

[27] S. M. Hosseini, *Black hole microstates and supersymmetric localization*, [arXiv:1803.01863].

[28] A. Gadde, L. Rastelli, S. S. Razamat and W. Yan, *Gauge Theories and Macdonald Polynomials*, Commun. Math. Phys. 319, 147 (2013) [arXiv:1110.3740].

[29] T. T. Dumitrescu, G. Festuccia, and N. Seiberg, *Exploring Curved Superspace*, JHEP 1208 (2012) 141, [arXiv:1205.1115].

[30] O. Aharony, S. S. Razamat, N. Seiberg, and B. Willett, *3d dualities from 4d dualities*, JHEP 1307 (2013) 149, [arXiv:1305.3924]. A. Arabi Ardehali, J. T. Liu, and P. Szepietowski, *The spectrum of IIB supergravity on $AdS_5 \times S^5/Z_3$ and a $1/N^2$ test of $AdS/CFT$*, JHEP 06 (2013) 024, [arXiv:1304.1540]; *$1/N^2$ corrections to the holographic Weyl anomaly*, JHEP 01 (2014) 002, [arXiv:1310.2611], S. Golkar and D. T. Son, *Non-renormalization of the chiral vortical effect coefficient*, JHEP 02 (2015) 169, [arXiv:1207.5806]. A. A. Ardehali, J. T. Liu, and P. Szepietowski, *c - a from the $\mathcal{N} = 1$ superconformal index*, JHEP 12 (2014) 145, [arXiv:1407.6024]; *Central charges from the $\mathcal{N} = 1$ superconformal index*, Phys. Rev. Lett. 114 (2015), no. 9 091603, [arXiv:1411.5023]; *High-Temperature Expansion of Supersymmetric Partition Functions*, JHEP 07 (2015) 113, [arXiv:1502.0737], E. Shaghoulian, *Modular forms and a generalized Cardy formula in higher dimensions*, Phys. Rev. D93 (2016), no. 12 126005, [arXiv:1508.02728]; *Black hole microstates in $AdS$*, Phys. Rev. D94 (2016), no. 10 104044, [arXiv:1512.06855], M. Buican and T. Nishinaka, *On the superconformal index of Argyres-Douglas theories*, J. Phys. A49 (2016), no. 1 015401, [arXiv:1505.05884].

[31] S. Bhattacharyya, S. Minwalla, and K. Papadodimas, *Small Hairy Black Holes in $AdS_5 \times S^5$*, JHEP 11 (2011) 035, [arXiv:1005.1287], O. J. C. Dias, P. Figueras, S. Minwalla, P. Mitra, R. Monteiro, and J. E. Santos, *Hairy black holes and solitons in global $AdS_5$*, JHEP 08 (2012) 117, [arXiv:1112.4447], J. Markevicute and J. E. Santos, *Hairy black holes in $AdS_5 \times S^5$*, JHEP 06 (2016) 096, [arXiv:1602.03893]; *Evidence for
the existence of a novel class of supersymmetric black holes with $AdS_5 \times S^5$ asymptotics, Class. Quant. Grav. 36 (2019), no. 2 02LT01, [arXiv:1806.01849]. J. Markeviciute, Rotating Hairy Black Holes in $AdS_5 \times S^5$, [arXiv:1809.04084].

[32] S. M. Hosseini, K. Hristov and A. Zaffaroni, A note on the entropy of rotating BPS $AdS_7 \times S^4$ black holes, JHEP 1805, 121 (2018) arXiv:1803.07568.

[33] A. Sen, Black hole entropy function and the attractor mechanism in higher derivative gravity, JHEP 09 (2005) 038, hep-th/0506177.

[34] T. Azeyanagi, M. Hanada, M. Honda, Y. Matsuo, and S. Shiba, A new look at instantons and large-N limit, JHEP 05 (2014) 008, arXiv:1307.0809.

[35] S. Kachru and E. Silverstein, 4-D conformal theories and strings on orbifolds, Phys. Rev. Lett. 80 (1998) 4855–4858, hep-th/9802183, M. Bershadsky and A. Johansen, Large N limit of orbifold field theories, Nucl. Phys. B536 (1998) 141–148, hep-th/9803249, P. Kovtun, M. Unsal, and L. G. Yaffe, Necessary and sufficient conditions for non-perturbative equivalences of large $N(c)$ orbifold gauge theories, JHEP 07 (2005) 008, hep-th/0411177.

[36] J. T. Liu, L. A. Pando Zayas, V. Rathee and W. Zhao, One-Loop Test of Quantum Black Holes in anti-de Sitter Space, Phys. Rev. Lett. 120, no. 22, 221602 (2018) arXiv:1711.01076.