From Sylvester’s determinant identity to Cramer’s rule

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Abstract

The object of this paper is to introduce a new and fascinating method of solving large linear equations, based on Cramer’s rule or Gaussian elimination but employing Sylvester’s determinant identity in its computation process. In addition, a scheme suitable for parallel computing is presented for this kind of generalized Chiò’s determinant condensation processes, which makes this new method have a property of natural parallelism. Finally, some numerical experiments also confirm our theoretical analysis.

Keywords: Sylvester’s determinant identity; Cramer’s rule; Chiò’s method; Parallel process.

1 Introduction

As is well-known, how to solve effectively linear systems is a very important problem in scientific and engineering fields. Many of linear solvers have been researched such as Gaussian elimination [9, 15], relaxation methods [14], row-action iteration schemes [6, 13] and (block) Krylov subspace [5, 15].

Recently, a low communication condensation-based linear system solver utilizing Cramer’s Rule is presented in [12]. As the authors stated that unique combination between Cramer’s rule and matrix condensation techniques yields an elegant parallel
computing architectures, by constructing a binary, tree-based data flow in which the
algorithm mirrors the matrix at critical points during the condensation process. More-
over, the accuracy and computational complexity of the proposed algorithm are similar
to LU-decomposition [9].

In this paper, we will continue research this kind of parallel algorithms and give
some theoretical analysis and a generalized Chio’s determinant condensation process,
which perfect the corresponding conclusions.

This paper is organized as follows. In Section 2, we will review some more general
determinant condensation algorithms—Sylvester’s determinant identity, and then give
theoretical basis on the above parallel computing architectures [12], which shows the
negation in mirroring process is not necessary to arrive at the correct answer. Moreover,
a more general scheme utilizing Cramer’s Rule and matrix condensation techniques is
also given. In addition, the scheme suitable for parallel computing on the sylvester’s
identity is proposed in Section 3. Finally, a simple example is used to illustrate this
new algorithm in Section 4.

\section{Sylvester’s determinant condensation algorithms}

Throughout this section, we mainly consider an \( n \times n \) matrix \( A = (a_{ij}) \) \((i, j = 1, 2, \ldots, n)\) with elements \( a_{ij} \) and determinant \( |A| \), also written \( \det A \). Recently, a Chio condensation method [7] is applied to solve large linear systems in [12]. In fact, the
prototype of this method may be traced back to the following Sylvester’s determinant
identity for calculating a determinant of arbitrary order in 1851.

\textbf{Theorem 2.1.} \textit{(Sylvester’s identity, [11, 5, 4, 7]).} Let \( A = (a_{ij}) \) be an \( n \times n \) matrix
over a commutative ring. For a submatrix \( A_0 = (a_{ij}) \), \( i, j = 1, \ldots, k \) of \( A \), set

\[
\hat{c}_{pq} = \det \begin{bmatrix} a_{1q} & \cdots & a_{pk} \\ \vdots & \ddots & \vdots \\ a_{1q} & \cdots & a_{pk} \end{bmatrix},
\]

(2.1)

Let \( \hat{C} = (\hat{c}_{pq}) \), \( p, q = k + 1, \ldots, n \). Then

\[
(\det A_0)^{n-k-1} \det A = \det \hat{C}.
\]

\textit{Specially when \( A_0 \) is an invertible matrix, we have that}

\[
\det A = \frac{\det \hat{C}}{(\det A_0)^{n-k-1}}.
\]

(2.2)
Corollary 2.2. (Chiò’s method, [8, 12]). For an $n \times n$ matrix $A = (a_{ij})$ with $a_{nn} \neq 0$, let $E = (e_{ij})$ be the $(n-1) \times (n-1)$ matrix defined by

$$e_{ij} = \begin{vmatrix} a_{ij} & a_{in} \\ a_{nj} & a_{nn} \end{vmatrix} = a_{ij}a_{nn} - a_{in}a_{nj}, \; i, j = 1, \ldots, n-1. \quad (2.3)$$

Then

$$\det A = \frac{1}{a_{nn}^{n-2}} \det E. \quad (2.4)$$

Obviously, the above Theorem 2.1 reduces a matrix of order $n$ to order $n-k$ to evaluate its determinant. Repeating the procedure numerous times can reduce a large matrix to a small one, which is convenient for the calculation. This process is called by condensation method [7, 8]. As an example of Chiò’s condensation, the paper [12] considers the following $3 \times 3$ matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \text{and its condensed form} \quad \begin{bmatrix} \times & \times & \times \\ \times & a_{11}a_{22} - a_{21}a_{12} & a_{11}a_{23} - a_{21}a_{13} \\ \times & a_{11}a_{32} - a_{31}a_{12} & a_{11}a_{33} - a_{31}a_{13} \end{bmatrix}.$$

In fact, the above condensation processes are not only used to evaluate determinants but also can be used to solve linear systems. For example, one can derive the following equivalence relation on the solution formula of linear systems.

Theorem 2.3. (Equivalence relation). The linear system in the form $Ax = b$ (where $A = (a_{ij})$ is an $n \times n$ invertible coefficient matrix) has the same corresponding solution as the linear system $\hat{C}x^{(k)} = b^{(k)}$, where $\hat{C}$ is defined as in Theorem 2.1, $x^{(k)} = [x_{k+1}, x_{k+2}, \ldots, x_n]^T$ and $b^{(k)} = [b'_1, b'_{k+2}, \ldots, b'_n]^T$. Here

$$b'_j = \det \begin{bmatrix} A_0 & b_1 \\ \vdots & \vdots \\ a_{j1} & \cdots & a_{jk} & b_j \end{bmatrix}, \; j = k+1, \ldots, n.$$

Proof. According to Theorem 2.1 or Eq. (2.2), we know that there exists a constant $(\det A_0)^{n-k-1}$ between the determinant of $A$ and the determinant of $\hat{C}$, which is only dependent on the given submatrix $A_0$. Therefore, for any given submatrix $A_0$, there also exists the same constant $(\det A_0)^{n-k-1}$ between the determinant of $A_j(b)$ ($j = k+1, \ldots, n$), the matrix $A$ with its $j$th column replaced by $b$, and the determinant of $\hat{C}_j(b^{(k)})$. Thus, by Cramer’s rule, we have that

$$x_j = \frac{\det(A_j(b))}{\det A} = \frac{\det(A_0)^{n-k-1} \det(\hat{C}_j(b^{(k)}))}{\det A_0^{n-k-1} \det \hat{C}} = \frac{\det(\hat{C}_j(b^{(k)}))}{\det \hat{C}}, \; j = k+1, \ldots, n.$$

The conclusion holds.
Obviously, when the submatrix $A_0$ is singular, the solution of linear systems cannot be evaluated by this method. Since interchanging the $r$th and $n$th rows and the $s$th and $n$th columns of linear systems has only an effect on the order of the unknowns $x_i$, which has no effect on the whole solution $x$. Therefore, we may obtain the following more general conclusion.

For convenience, we firstly define the ordered index list $N_n = (1, 2, \ldots, n)$ for any positive integer $n$. For two ordered index (i.e., for any $\alpha < \beta, i_\alpha < i_\beta$) lists $I = (i_1, \ldots, i_t) \subset N_n$ and $J = (j_1, \ldots, j_t) \subset N_n$, we denote the corresponding complementary ordered index lists by $I'$ and $J'$, respectively. That is, $I \cup I' = J \cup J' = N_n$.

**Corollary 2.4.** Let $A = (a_{ij})$ be an $n \times n$ matrix and $k$ be a fixed integer $0 \leq k \leq n-1$. $I = (i_1, \ldots, i_k) \subset N_n$ and $J = (j_1, \ldots, j_k) \subset N_n$ are two ordered index lists. We denote the corresponding submatrix, extracted from $A$, as

$$A[I \ J] = A[i_1, \ldots, i_k \ j_1, \ldots, j_k] \triangleq \begin{bmatrix} a_{i_1j_1} & \cdots & a_{i_1j_k} \\ \vdots & \ddots & \vdots \\ a_{i_kj_1} & \cdots & a_{i_kj_k} \end{bmatrix}.$$ 

Suppose that the invertible submatrix $A_0 = A[I \ J]$ in the Theorem 2.1, then the linear system $Ax = b$ has the same corresponding solutions as the linear system $\hat{C}x^{(k)} = \hat{b}^{(k)}$, where $\square_{J'}$ is defined as the subset of $\square$ with the index coming from $J'$.

According to the above theorem, one can easily see that though the condensation process removes information associated with discarded columns, we may obtain certain variables values by controlling the elements in the set $J'$, see Example 4.1.

In addition, the matrix mirroring and the negation of matrix mirroring process in [12] are also not necessary to arrive at the correct answer, since we may obtain the similar parallel computing process by condensing the index set $J'$ from both sides (left and right), see Figure 1.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \rightarrow \text{mirrored} \rightarrow \begin{vmatrix} a_{13} & a_{12} & -a_{11} \\ a_{23} & a_{22} & -a_{21} \\ a_{33} & a_{32} & -a_{31} \end{vmatrix}.$$ 

Similar to [12], copying occurs with the initial matrix and then each time a matrix is reduced in half. An $N \times N$ matrix is copied when it reaches the size of $N \setminus 2 \times N \setminus 2$. Once the matrix is copied, there is double the work. In other words, two $N \setminus 2 \times N \setminus 2$ matrices each require a condensation. Obviously, the amount of work for two matrices of half the size is much lower than that of one $N \times N$ matrix, which avoids the $O(N^4)$
Figure 1: A process flow depicting the proposed framework.
growth pattern in computations. This is due to the \( O(N^3) \) nature of the condensation process (see [12]).

Similarly, one may consider a scenario in which the algorithm creates more than two matrices during each copying step, according to Corollary 2.4. On its computational complexity and more details, see [12].

3 A scheme suitable for parallel computing on the Sylvester’s identity

The Sylvester’s identity (2.1) reduces a matrix of order \( n \) to order \( n - k \) when evaluating its determinant. Since when \( k = 1 \), it is just the Chiò’s method. Therefore, for convenience, we call the Sylvester’s identity a K-Chiò’s method from now on.

As have been shown above, repeating the procedure numerous times can reduce a large matrix to a size convenient for the computations. However, in order to condense a matrix from \( N \times N \) to \( (N - M) \times (N - M) \), the core calculation is repeated \( (N - M)^2 \) times. Obviously, this is very expensive. In fact, we may parallel computing each row of the matrix \( \hat{C} \) in (2.1), since \( A_0 \) or \( A \left[ \begin{array}{cccc} i_1, \ldots, i_t \\ j_1, \ldots, j_t \end{array} \right] \) is common to each element of the row and may be calculated but once for each row via expanding by the last column. For example, for the \( p \)-th row \( \alpha_p \) of \( \hat{C} \), we may write

\[
\alpha_p = [\hat{c}_{p,k+1}, \hat{c}_{p,k+2}, \ldots, \hat{c}_{p,k+n}]
\]

\[
= \left[ \begin{array}{cccc} A_{p,1}^{0} & A_{p,2}^{0} & \ldots & A_{k+1}^{0} \end{array} \right] \left[ \begin{array}{cccc} a_{1,k+1} & a_{1,k+2} & \cdots & a_{1,n} \\ a_{2,k+1} & a_{2,k+2} & \cdots & a_{2,n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{p,k+1} & a_{p,k+2} & \cdots & a_{p,n} \end{array} \right], \quad (3.1)
\]

where

\[
A_{0}^{p,j} = \left[ \begin{array}{cccc} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \cdots & \cdots & \cdots & \cdots \\ a_{p1} & a_{p2} & \cdots & a_{pk} \\ \cdots & \cdots & \cdots & \cdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{array} \right]_{j}.
\]

Therefore, only \( k \) determents \( A_{0}^{p,j} (j = 1, \ldots, k) \) and a common \( k \times k \) determinant \( \det A_0 \) are needed for each row of the matrix \( \hat{C} \). Therefore, the matrix \( \hat{C} \) is essentially suitable for parallel computations since the each row of matrix \( \hat{C} \) may be independently computed by (3.1). See Example 3.1 below.
Example 3.1. Consider the following four order determinant

\[
|A| = \begin{vmatrix}
1 & -2 & 3 & 1 \\
4 & 2 & -1 & 0 \\
0 & 2 & 1 & 5 \\
-3 & 3 & 1 & 2
\end{vmatrix}.
\]

Let \(A_0 = \begin{bmatrix} 1 & -2 \\ 4 & 2 \end{bmatrix}\), then \(|A_0| = 10\), and

\[
\alpha_3 = \begin{vmatrix} 0 & 2 \\ 4 & 2 \end{vmatrix}, -\begin{vmatrix} 1 & -2 \\ 0 & 2 \end{vmatrix}, 10 \begin{vmatrix} 3 & 1 \\ -1 & 0 \\ 1 & 5 \end{vmatrix} = \begin{bmatrix} 36 \\ 58 \end{bmatrix};
\]

\[
\alpha_4 = \begin{vmatrix} -3 & 3 \\ 4 & 2 \end{vmatrix}, -\begin{vmatrix} 1 & -2 \\ -3 & 3 \end{vmatrix}, 10 \begin{vmatrix} 3 & 1 \\ -1 & 0 \\ 1 & 2 \end{vmatrix} = \begin{bmatrix} 61 \\ 38 \end{bmatrix}.
\]

Therefore,

\[
|A| = \frac{1}{10^{1.3}} \begin{vmatrix} 36 & 58 \\ 61 & 38 \end{vmatrix} = -217.
\]

From here, we note that only six \(2 \times 2\) determinants is needed. However, Chiò’s method will require fourteen \(2 \times 2\) determinants to be computed. In addition, comparing with the Gaussian elimination, our method increases only two multiplications. But Gaussian elimination method is not too suitable for parallel computing. Thus, the whole computational amount on the matrix \(\hat{C}\) will be much less than that involved in the old process of computation [12]. Concretely speaking, if we denote the total of multiplications/diversions on the \(k\)-order determinant \(|A_0|\) by \(m\), then the total of multiplications/diversions by using the K-Chiò’s method (2.2) is about

\[
(k + 1) \left[ k^2 + (2k)^2 + \ldots + (n - k)^2 \right] + km \left[ k + 2k + \ldots + (n - k) \right] + \frac{n-k}{k} (m + 1) + m
\approx O \left( \frac{1}{3}(1 + \frac{1}{k})n^3 \right).
\]

Similarly, the computational complexity of other algorithms is also described as follows, see Table 1.

From Table 1, we note that additions/subtractions on these algorithms are almost the same. However, multiplications/diversions mainly depend on the parameter \(k\) for K-Chiò’s condensation method. But this does not show that the total computational complexity on K-Chiò’s method (2.2) is tending to decrease with the \(k\) increasing, since the core loop of the K-Chiò’s condensation method involves the calculation of \(k \times k\) determinants for each element of the matrix during condensation. Normally,
Table 1: Comparisons of the computational complexity for different algorithms on determinant calculations in the nonparallel setting.

| Algorithms                     | Multiplications/divisions       | Additions/subtractions       |
|--------------------------------|--------------------------------|-----------------------------|
| Gaussian Elimination ([10])    | $O\left(\frac{1}{3}n^3\right)$ | $O\left(\frac{1}{3}n^3\right)$ |
| Chiò’s condensation method ([3]) | $O\left(\frac{2}{3}n^3\right)$ | $O\left(\frac{1}{3}n^3\right)$ |
| K-Chiò’s condensation method([2]) | $O\left(\frac{1}{3}(1 + \frac{1}{k})n^3\right)$ | $O\left(\frac{1}{3}n^3\right)$ |

This would necessitate the standard computational workload to calculate the $k$-order determinant, i.e., $\frac{1}{3}k^3$ multiplications/divisions and $\frac{1}{3}k^3$ additions/subtractions, using a method such as Gaussian elimination [12]. Therefore, the parameter $k$ is not the better for the bigger number, see the following experimental results Figure 1 and 2 on the 5000-order and 20000-order determinants, respectively. The small subgraphs in Fig. 1 and 2 show the optimal parameter $k$ value ranges. For example, the optimal parameter $k$ is approximately ten for a 20000-order determinant. In addition, for matrices of different dimensions, we specifically compute the optimal parameters $k$, we find the optimal parameter values increasing as the matrix dimension increases. But this increase is still relatively slow, see Fig. 3.

Since the optimal parameter $k$ is usually small, by (3.1), we may normalize the each row of matrix $\hat{C}$ by dividing the determinant of $A_0$, which will further reduce the computational complexity of K-Chiò’s condensation method, see Example 4.1.

## 4 An application in the Cramer’s rule

As is well-known, the classical Cramer’s rule states that the components of the solution to a linear system in the form $Ax = b$ (where $A = (a_{ij})$ is an $n \times n$ invertible coefficient matrix) are given by

$$x_i = \frac{\det(A_i(b))}{\det(A)}, \quad i = 1, 2, \ldots, n,$$

where $x_i$ is the $i$th unknown.

In [12], an algorithm based on Chiò’s condensation and Cramer’s rule for solving large-scale linear systems is achieved by constructing a binary, tree-based data flow in which the algorithm mirrors the matrix at critical points during the condensation process. However, according to the above corollary 2.4 one may obtain certain unknowns values by freely controlling the elements in the set $J'$ without matrix mirroring, see Example 4.1. This also makes it more easily for more CPUs to be used in computing process and even without any communication. At the same time, the scheme (3.1) also reduce the memory space.
Figure 2: The relationship between the parameter $k$ and the number of multiplication/diversion for a 5000-order determinant on K-Chiø's condensation method.
Figure 3: The relationship between the parameter $k$ and the number of multiplication/diversion for a 20000-order determinant on K-Chio's condensation method.
Figure 4: The relationship between the optimal parameter $k$ and the dimension of matrices, Left: only consider multiplications; Right: consider all computational complexity.
Example 4.1. Solve the following six-order linear system

\[
\begin{bmatrix}
1 & 3 & 5 & 7 & 9 & 11 \\
2 & 0 & 0 & 0 & 9 \\
3 & 0 & 5 & 7 & 0 & 7 \\
4 & 0 & 6 & 8 & 0 & 5 \\
5 & 0 & 0 & 0 & 0 & 3 \\
6 & 5 & 4 & 3 & 2 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6
\end{bmatrix}
= \begin{bmatrix}
1 \\
-1 \\
1 \\
-1 \\
1 \\
-1
\end{bmatrix}. \tag{4.2}
\]

Let \( I_1 = (1, 2, 3) \) and \( J_1 = (1, 3, 5) \), then \( J'_1 = (2, 4, 6) \). Denote

\[
|A_0| = A \begin{bmatrix} I_1 \\ J_1 \end{bmatrix} = \begin{bmatrix} 1 & 5 & 9 \\ 2 & 0 & 0 \\ 3 & 5 & 0 \end{bmatrix}.
\]

Then, \( |A_0| = 90 \) and

\[
\alpha_4 = \begin{bmatrix}
4 & 6 & 0 \\
-2 & 0 & 0 \\
3 & 5 & 0
\end{bmatrix}, \quad
\begin{bmatrix}
1 & 5 & 9 \\
-2 & 0 & 0 \\
3 & 5 & 0
\end{bmatrix}, \quad
\begin{bmatrix}
1 & 5 & 9 \\
\end{bmatrix}, \quad
\begin{bmatrix}
3 & 7 & 11 \\
0 & 0 & 9 \\
0 & 7 & 7
\end{bmatrix}, \quad
\begin{bmatrix}
1 \\
-1 \\
1
\end{bmatrix}, \quad
\begin{bmatrix}
0 & 8 & 5 \\
0 & 7 & 7 \\
0 & 3 & 1
\end{bmatrix}
\end{bmatrix} = \begin{bmatrix} 0, -36, -468, -180 \end{bmatrix};
\]

\[
\alpha_5 = \begin{bmatrix}
5 & 0 & 0 \\
-2 & 0 & 0 \\
3 & 5 & 0
\end{bmatrix}, \quad
\begin{bmatrix}
1 & 5 & 9 \\
-2 & 0 & 0 \\
3 & 5 & 0
\end{bmatrix}, \quad
\begin{bmatrix}
1 & 5 & 9 \\
\end{bmatrix}, \quad
\begin{bmatrix}
3 & 7 & 11 \\
0 & 0 & 9 \\
0 & 7 & 7
\end{bmatrix}, \quad
\begin{bmatrix}
1 \\
-1 \\
1
\end{bmatrix}, \quad
\begin{bmatrix}
0 & 0 & 3 \\
0 & 7 & 7 \\
0 & 3 & 1
\end{bmatrix}
\end{bmatrix} = \begin{bmatrix} 0, 0, -1755, 315 \end{bmatrix};
\]

\[
\alpha_6 = \begin{bmatrix}
6 & 4 & 2 \\
-2 & 0 & 0 \\
3 & 5 & 0
\end{bmatrix}, \quad
\begin{bmatrix}
1 & 5 & 9 \\
-2 & 0 & 0 \\
3 & 5 & 0
\end{bmatrix}, \quad
\begin{bmatrix}
1 & 5 & 9 \\
\end{bmatrix}, \quad
\begin{bmatrix}
3 & 7 & 11 \\
0 & 0 & 9 \\
0 & 7 & 7
\end{bmatrix}, \quad
\begin{bmatrix}
1 \\
-1 \\
1
\end{bmatrix}, \quad
\begin{bmatrix}
5 & 3 & 1 \\
0 & 7 & 7 \\
0 & 3 & 1
\end{bmatrix}
\end{bmatrix} = \begin{bmatrix} 390, -234, -2132, 20 \end{bmatrix}.
\]

Therefore, we need only solve the condensed linear system \( \hat{C} \hat{x}^{(k)}_j = b^{(k)}_j \), i.e.,

\[
\begin{bmatrix}
0 & -36 & -468 \\
0 & 0 & -1755 \\
390 & -234 & -2132
\end{bmatrix}
\begin{bmatrix}
x_2 \\
x_4 \\
x_6
\end{bmatrix}
= \begin{bmatrix}
-180 \\
315 \\
20
\end{bmatrix}. \tag{4.3}
\]
By Cramer’s rule or Gaussian elimination, the solution of above sub-linear system (4.3) is

\[ x_2 = \frac{406}{117}, \quad x_4 = \frac{22}{3}, \quad x_6 = -\frac{7}{39}, \]

which is also the corresponding solution of original linear system (4.2). Similarly, let \( I_2 = (1, 2, 3) \) and \( J_2 = (2, 4, 6) \), then we may also obtain the solution of the unknown \( x_1, x_2 \) and \( x_3 \):

\[ x_1 = \frac{4}{13}, \quad x_3 = -10, \quad x_5 = -\frac{118}{117}. \]

In addition, we may continue condense the above \( \alpha_4, \alpha_5 \) and \( \alpha_6 \). For example, we condense them from right side for \( J'_1 = (2, 4, 6) \). Without loss of generality, we may let \( I_3 = (6), \quad J_3 = (6) \), then \( J'_3 = (2, 4) \) and we have

\[
\begin{align*}
\alpha'_4 &= \begin{bmatrix} 468 & -2132 \end{bmatrix} \begin{bmatrix} 390 & -234 & 20 \\ 0 & -36 & -180 \end{bmatrix} = \begin{bmatrix} 182520 & -32760, 393120 \end{bmatrix}; \\
\alpha'_5 &= \begin{bmatrix} 1755 & -2132 \end{bmatrix} \begin{bmatrix} 390 & -234 & 20 \\ 0 & 0 & 315 \end{bmatrix} = \begin{bmatrix} 684450 & -410670, -636480 \end{bmatrix}.
\end{align*}
\]

(4.4)

By Gaussian elimination, we obtain the solution of the above linear system (4.4):

\[ x_2 = \frac{406}{117}, \quad x_4 = \frac{22}{3}. \]

Moreover, to further reduce the computational complexity of K-Chiò’s condensation method, we may normalize the each row of matrix \( \hat{C} \) by dividing the determinant of \( A_0 \). For example, the above \( \alpha'_4 \) and \( \alpha'_5 \) may be written as

\[
\begin{align*}
\alpha'_4 &= \begin{bmatrix} 468/ -2132, 1 \end{bmatrix} \begin{bmatrix} 390 & -234 & 20 \\ 0 & -36 & -180 \end{bmatrix} = \begin{bmatrix} -3510/41, 630/41, -7560/41 \end{bmatrix}; \\
\alpha'_5 &= \begin{bmatrix} 1755/ -2132, 1 \end{bmatrix} \begin{bmatrix} 390 & -234 & 20 \\ 0 & 0 & 315 \end{bmatrix} = \begin{bmatrix} -26325/82, 15795/82, 12240/41 \end{bmatrix}.
\end{align*}
\]

From the above example, we know that applying Gaussian elimination method instead of Cramer’s rule to solve the small sub-linear system \( \hat{C}x^{(k)}_{j'} = b^{(k)}_{j'} \) is also very convenient.

5 Concluding remarks

From the above discussion, one can see that unique utilization of matrix condensation techniques yields an elegant process that has promise for parallel computing architectures. Moreover, as was also mentioned in [12], these condensation methods become
extremely interesting, since they still retain an $O(n^3)$ complexity with pragmatic forward and backward stability properties when they are applied to solve large-scale linear systems by the Cramer’s rule or Gaussian elimination.

In this paper, some condensation methods are introduced and some existing problems on these techniques are also discussed. Though the condensation process removes information associated with discarded columns, this makes the computation of linear systems become feasible by more freely parallel process.

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References

[1] Francine F. Abeles. Dodgson condensation: the historical and mathematical development of an experimental method. *Linear Algebra Appl.*, 429(2-3): 429-438, 2008.

[2] Francine F. Abeles. Chioś and dodgson’s determinantal identities. *Linear Algebra Appl.*, 454(1): 130-137, 2014.

[3] A.C. Aitken. *Determinants and Matrices*. Interscience Publishers, third edition, 1944.

[4] A.G. Akritas, E.K. Akritas, and G.I. Malaschonok. Various proofs of Sylvester’s (determinant) identity. *Math. Comput. Simulat.*, 42(4-6):585-593, 1996.

[5] J.C.R. Bloch and S. Heybrock. A nested Krylov subspace method to compute the sign function of large complex matrices. *Comput. Phys. Commun.*, 182(4):878-889, 2011.

[6] R. Bramley and A. Sameh. Row projection methods for large nonsymmetric linear systems. *SIAM J. Sci. Stat. Comput.*, 13(1): 168-193, 1992.

[7] F. Chioś. Memoire sur les fonctions connues sous le nom de resultantes ou de determinants, Turin: E. Pons, 1853.

[8] L. E. Fuller and J. D. Logan. On the Evaluation of Determinants by Chioś Method. *The Two-Year College Mathematics Journal*, 6(1):8-10, 1975.
[9] N. Galoppo, N.K. Govindaraju, M. Henson, and D. Manocha. LU-GPU: Efficient algorithms for solving dense linear systems on graphics hardware. Proceedings of the ACM/IEEE SC2005 Conference on High Performance Networking and Computing, University of North Carolina at Chapel Hill, 2005.

[10] G.H. Golub and C.F. Van Loan. Matrix computations. Johns Hopkins, Baltimore, third edition, 1996.

[11] Israel Gelfand, Sergei Gelfand, Vladimir Retakh, and Robert Lee Wilson. Quasideeterminants. Adv. Math., 193(1):56-141, 2005.

[12] Ken Habgood and Itamar Arel. A condensation-based application of Cramer’s rule for solving large-scale linear systems. J. Discrete Algorithms, 10:98-109, 2012.

[13] A.S. Householder and F.L. Bauer. On certain iterative methods for solving linear systems. Numer. Math., 2:55-59, 1960.

[14] Young D. M. Iterative Solution of Large Linear Systems. Academic Press, NewYork-London, second edition, 1971.

[15] Saad Y. Iterative methods for sparse linear systems. SIAM, Baltimore, second edition, 2003.