Asymptotes of Space Curves

Angel Blasco and Sonia Pérez-Díaz
Departamento de Física y Matemáticas
Universidad de Alcalá
E-28871 Madrid, Spain
angel.blasco@uah.es, sonia.perez@uah.es

Abstract

In this paper, we generalize the results presented in [4] for the case of real algebraic space curves. More precisely, given an algebraic space curve \( C \) implicitly defined, we show how to compute the generalized asymptotes. In addition, we show how to deal with this problem for the case of a given curve \( C \) parametrically defined. The approaches are based on the notion of approaching curves introduced in [3].

Keywords: Algebraic Space Curve; Parametric Representation; Implicit Representation; Convergent Branches; Infinity Branches; Asymptotes; Perfect Curves

1 Introduction

In the first part of the paper (Sections 2, 3 and 4), we consider \( C \) an irreducible real algebraic space curve over the field of complex numbers \( C \) implicitly defined by two irreducible polynomials \( f_1(x_1, x_2, x_3), f_2(x_1, x_2, x_3) \in \mathbb{R}[x_1, x_2, x_3] \). That is, we work over \( C \), but \( C \) has infinitely many points in the affine plane over the field of real numbers \( \mathbb{R} \). Since every irreducible real curve has a real defining polynomial, we assume that \( C \) is defined by irreducible polynomials in \( \mathbb{R}[x_1, x_2, x_3] \) (see Chapter 7 in [12]).
In the second part of the paper (Section 5), we are given an irreducible real algebraic space curve $C$ defined by a parametrization of the form $P(s) = (p_1(s), p_2(s), p_3(s))$, where $p_i(s) = p_{i1}(s)/p(s)$, $i = 1, 2, 3$. Similarly as above, since every real curve can be parametrized over $\mathbb{R}$, we assume that $P(s) \in \mathbb{R}(s)^3$ (see Chapter 7 in [12]).

In both cases, the assumption of reality is included because of the nature of the problem, but the theory can be similarly developed for the case of complex non-real curves.

Under these conditions, we deal with the problem of computing the asymptotes of the infinity branches of $C$. Intuitively speaking, the asymptotes of some branch of an algebraic curve reflect the status of this branch at the points with sufficiently large coordinates. In analytic geometry, an asymptote of a curve is a line such that the distance between the curve and the line approaches zero as they tend to infinity. In some contexts, such as algebraic geometry, an asymptote is defined as a line which is tangent to a curve at infinity. Thus, the problem of computing the asymptotes is very important in the study of real algebraic curves since asymptotes contain much of the information about the behavior of the curves in the large.

Determining the linear asymptotes of an algebraic curve is a topic considered in many text-books on analysis (see e.g. [10]). In [8], it is presented a simple method for obtaining the linear asymptotes of a curve defined by an irreducible polynomial, with emphasis on second order polynomials. In [15], an algorithm for computing all the linear asymptotes of a real plane algebraic curve implicitly defined, is obtained. In [11], it is briefly studied the linear asymptotes of space curves. In particular, it is proved how the tangents at the simple points at infinity of the curve (i.e. non-singular points at infinity) are related with the asymptotes.

However, an algebraic curve may have more general curves than lines that describe the status of a branch at the points with sufficiently large coordinates. This motivates the interest in analyzing and computing these generalized asymptotes. Intuitively speaking, a curve $\tilde{C}$ is a generalized asymptote (or g-asymptote) of another curve $C$ at some infinity branch $B \subset C$ if the distance between $\tilde{C}$ and $B$ tends to zero as they tend to infinity, and $C$ can not be approached by a new curve of lower degree at $B$ (see [11]).

A deeply elaborated theory in this sense is developed by the authors
In that paper, a method for computing all the g-asymptotes of a real plane algebraic curve $C$ implicitly defined by an irreducible polynomial $f(x_1, x_2) \in \mathbb{R}[x_1, x_2]$ is presented. The approach is based on the notion of approaching curves introduced in [5].

In this paper, we generalize these results, and we present an algorithm for computing g-asymptotes of a real algebraic space curve $C$ implicitly defined by two irreducible polynomials $f_1(x_1, x_2, x_3), f_2(x_1, x_2, x_3) \in \mathbb{R}[x_1, x_2, x_3]$. In addition, we also show how to compute the g-asymptotes if the given curve is defined parametrically. This parametric approach can be easily generalized for parametric plane curves and in general, for a rational parametrization of a curve in the $n$-dimensional space.

The structure of the paper is as follows. In Section 2, we present the notation and we generalize some previous results developed in [5]. In particular, we characterize whether two implicit algebraic space curves approach each other at the infinity, and we present a method to compare the asymptotic behavior of two space curves (i.e., the behavior at the infinity). In Section 3, we show the relation between infinity branches of plane curves and infinity branches of space curves. More precisely, we obtain the infinity branches of a given space curve $C$ from the infinity branches of a certain plane curve obtained by projecting $C$ along some “valid projection direction”.

The study of approaching curves and convergent branches leads to the notions of perfect curve (a curve of degree $d$ that cannot be approached by any curve of degree less than $d$) and g-asymptote (a perfect curve that approaches another curve at an infinity branch). These concepts are introduced in Section 4. In this section, we obtain an algorithm that computes a g-asymptote for each infinity branch of a given curve. Section 5 is devoted to the computation of g-asymptotes for a given parametric space curve. We remark that the method presented in this section is easily applicable to parametric plane curves and in general, for rational parametrizations of curves in the $n$-dimensional space.

2 Notation and terminology

In this section, we present some notions and terminology that will be used throughout the paper. In particular, we need some previous results concerning local parametrizations and Puiseux series. For further details see [1], [3].
We denote by $\mathbb{C}[[t]]$ the domain of formal power series in the indeterminate $t$ with coefficients in the field $\mathbb{C}$, i.e. the set of all sums of the form $\sum_{i=0}^{\infty} a_i t^i$, $a_i \in \mathbb{C}$. The quotient field of $\mathbb{C}[[t]]$ is called the field of formal Laurent series, and it is denoted by $\mathbb{C}((t))$. It is well known that every non-zero formal Laurent series $A \in \mathbb{C}((t))$ can be written in the form $A(t) = t^k \cdot (a_0 + a_1 t + a_2 t^2 + \cdots)$, where $a_0 \neq 0$ and $k \in \mathbb{Z}$. In addition, the field $\mathbb{C} \ll t \gg := \bigcup_{n=1}^{\infty} \mathbb{C}((t^{1/n}))$ is called the field of formal Puiseux series. Note that Puiseux series are power series of the form $\varphi(t) = m + a_1 t^{N_1/N} + a_2 t^{N_2/N} + a_3 t^{N_3/N} + \cdots \in \mathbb{C} \ll t \gg$, $a_i \neq 0$, $\forall i \in \mathbb{N}$, where $N, N_i \in \mathbb{N}$, $i \geq 1$, and $0 < N_1 < N_2 < \cdots$. The natural number $N$ is known as the ramification index of the series. We denote it as $\nu(\varphi)$ (see [6]).

The order of a non-zero (Puiseux or Laurent) series $\varphi$ is the smallest exponent of a term with non-vanishing coefficient in $\varphi$. We denote it by ord($\varphi$). We let the order of 0 be $\infty$.

The most important property of Puiseux series is given by Puiseux’s Theorem, which states that if $K$ is an algebraically closed field, then the field $K \ll x \gg$ is algebraically closed (see Theorems 2.77 and 2.78 in [12]). A proof of Puiseux’s Theorem can be given constructively by the Newton Polygon Method (see e.g. Section 2.5 in [12]).

Let $\mathcal{C} \in \mathbb{C}^3$ be an irreducible space curve defined by two polynomials $f_1(x_1, x_2, x_3), f_2(x_1, x_2, x_3) \in \mathbb{R}[x_1, x_2, x_3]$. We assume that $\mathcal{C}$ is not planar (for planar space curves, one may apply the results in [4]).

We note that we work over $\mathbb{C}$, but we assume that the curve has infinitely many points in the affine plane over $\mathbb{R}$ and then, $\mathcal{C}$ has a real defining polynomial (see Chapter 7 in [12]). We recall that the assumption of reality is included because of the nature of the problem, but the theory developed in this paper can be applied for the case of complex non-real curves.

Let $\mathcal{C}^*$ be the corresponding projective curve defined by the homogeneous polynomials $F_1(x_1, x_2, x_3, x_4), F_2(x_1, x_2, x_3, x_4) \in \mathbb{R}[x_1, x_2, x_3, x_4]$. Furthermore, let $P = (1 : m_2 : m_3 : 0)$, $m_2, m_3 \in \mathbb{C}$, be an infinity point of $\mathcal{C}^*$.

In addition, we consider the curve defined implicitly by the polynomials $g_i(x_2, x_3, x_4) := F_i(1, x_2, x_3, x_4) \in \mathbb{R}[x_2, x_3, x_4]$, for $i = 1, 2$. Observe that $g_i(p) = 0$, where $p = (m_2, m_3, 0)$. Let $I \in \mathbb{R}(x_4)[x_2, x_3]$ be the ideal generated
by \(g_i(x_2, x_3, x_4), \ i = 1, 2\) in the ring \(\mathbb{R}(x_4)[x_2, x_3]\). Since \(C\) is not contained in some hyperplane \(x_4 = c, c \in \mathbb{C}\), we have that \(x_4\) is not algebraic over \(\mathbb{R}\). Under this assumption, the ideal \(I\) (i.e. the system of equations \(g_1 = g_2 = 0\)) has only finitely many solutions in the 3-dimensional affine space over the algebraic closure of \(\mathbb{R}(x_4)\) (which is contained in \(\mathbb{C} \ll x_4 \gg\)). Then, there are finitely many pairs of Puiseux series \((\varphi_2(t), \varphi_3(t))\) \(\in \mathbb{C} \ll t \gg^2\) such that \(g_i(\varphi_2(t), \varphi_3(t), t) = 0, \ i = 1, 2\). Each of the pairs \((\varphi_2(t), \varphi_3(t))\) is a solution of the system, and \(\varphi_2(t)\) and \(\varphi_3(t)\) converge in a neighborhood of \(t = 0\).

It is important to remark that if \(\varphi(t) := (\varphi_2(t), \varphi_3(t))\) is a solution of the system, then \(\sigma_\epsilon(\varphi(t)) := (\sigma_\epsilon(\varphi_2(t)), \sigma_\epsilon(\varphi_3(t)))\) is another solution of the system, where

\[
\sigma_\epsilon(\varphi_k(t)) = \sum_{i \geq 0} a_{i,k}^\epsilon t^{N_{i,k}/N_k}, \ N_k, N_{i,k} \in \mathbb{N}, \ 0 < N_{1,k} < N_{2,k} < \cdots ,
\]

\(N := \text{lcm}(N_2, N_3)\), \(\lambda_{i,k} := N_{i,k} N/N_k \in \mathbb{N}\), and \(\epsilon^N = 1\) (see [1]). We refer to these solutions as the *conjugates* of \(\varphi\). The set of all (distinct) conjugates of \(\varphi\) is called the *conjugacy class* of \(\varphi\), and the number of different conjugates of \(r\) is \(N\). We denote the natural number \(N\) as \(\nu(\varphi)\).

Under these conditions and reasoning as in [5], we get that there exists \(M \in \mathbb{R}^+\) such that for \(i \in \{1, 2\}\),

\[
F_i(1 : \varphi_2(t) : \varphi_3(t) : t) = g_i(\varphi_2(t), \varphi_3(t), t) = 0, \ \text{for} \ t \in \mathbb{C} \text{ and } |t| < M,
\]

where

\[
\varphi_k(t) = \sum_{i \geq 0} a_{i,k} t^{N_{i,k}/N_k}, \ N_k, N_{i,k} \in \mathbb{N}, \ 0 < N_{1,k} < N_{2,k} < \cdots .
\]

This implies that

\[
F_i(t^{-1} : t^{-1} \varphi_2(t) : t^{-1} \varphi_3(t) : 1) = f_i(t^{-1}, t^{-1} \varphi_2(t), t^{-1} \varphi_3(t)) = 0,
\]

for \(t \in \mathbb{C}\) and \(0 < |t| < M\).

Now, we set \(t^{-1} = z\), and we obtain that for \(i \in \{1, 2\}\),

\[
f_i(z, r_2(z), r_3(z)) = 0, \ \text{z} \in \mathbb{C} \text{ and } |z| > M^{-1}, \ \text{where}
\]

\[
r_k(z) = z \varphi_k(z^{-1}) = m_k z + a_{1,k} z^{1-N_{1,k}/N_k} + a_{2,k} z^{1-N_{2,k}/N_k} + a_{3,k} z^{1-N_{3,k}/N_k} + \cdots ,
\]

5
a_{j,k} \neq 0, N_k, N_{j,k} \in \mathbb{N}, j = 1, \ldots, \text{ and } 0 < N_{1,k} < N_{2,k} < \cdots.

Since \nu(\varphi) = N, we get that there are N different series in its conjugacy class. Let \varphi_{j,k}, j = 1, \ldots, N be these series, and \quad r_{j,k}(z) = z\varphi_{j,k}(z^{-1}) = m_kz + a_{1,k}c_j^{\lambda_{1,k}}z^{1-N_{1,k}/N_k} + a_{2,k}c_j^{\lambda_{2,k}}z^{1-N_{2,k}/N_k} + a_{3,k}c_j^{\lambda_{3,k}}z^{1-N_{3,k}/N_k} + \cdots \quad (1)

where \quad N := \text{lcm}(N_2, N_3), \quad \lambda_{i,k} := N_{i,k}N/N_k \in \mathbb{N}, \text{ and } c_1, \ldots, c_N \text{ are the } N \text{ complex roots of } x^N = 1. \text{ Now we are ready to introduce the notion of infinity branch. The following definitions and results generalize those presented in \cite{5} for algebraic plane curves.}

**Definition 2.1.** An infinity branch of a space curve \mathcal{C} associated to the infinity point \mathcal{P} = (1 : m_2 : m_3 : 0), m_2, m_3 \in \mathbb{C}, is a set \mathcal{B} = \bigcup_{j=1}^{N} L_j, where \quad L_j = \{(z, r_{j,2}(z), r_{j,3}(z)) \in \mathbb{C}^3 : z \in \mathbb{C}, |z| > M\}, \quad M \in \mathbb{R}^+, \text{ and the series } r_{j,2} \text{ and } r_{j,3} \text{ are given by } (1). \text{ The subsets } L_1, \ldots, L_N \text{ are called the leaves of the infinity branch } \mathcal{B}.

**Remark 2.2.** An infinity branch is uniquely determined from one leaf, up to conjugation. That is, if \quad B = \bigcup_{j=1}^{N} L_i, \text{ where } L_i = \{(z, r_{i,2}(z), r_{i,3}(z)) \in \mathbb{C}^3 : z \in \mathbb{C}, |z| > M\}, \text{ and } r_{i,k}(z) = z\varphi_{i,k}(z^{-1}) = m_kz + a_{1,k}c_j^{\lambda_{1,k}}z^{1-N_{1,k}/N_k} + a_{2,k}c_j^{\lambda_{2,k}}z^{1-N_{2,k}/N_k} + a_{3,k}c_j^{\lambda_{3,k}}z^{1-N_{3,k}/N_k} + \cdots

then \quad r_{j,k} = r_{i,k}, j = 1, \ldots, N, \text{ up to conjugation; i.e. } r_{j,k}(z) = z\varphi_{j,k}(z^{-1}) = m_kz + a_{1,k}c_j^{\lambda_{1,k}}z^{1-N_{1,k}/N_k} + a_{2,k}c_j^{\lambda_{2,k}}z^{1-N_{2,k}/N_k} + a_{3,k}c_j^{\lambda_{3,k}}z^{1-N_{3,k}/N_k} + \cdots

N, N_{i,k} \in \mathbb{N}, \lambda_{i,k} := N_{i,k}N/N_k \in \mathbb{N}, \text{ and } c_j^{N} = 1, j = 1, \ldots, N.

**Remark 2.3.** Observe that the above approach and Definition 2.1 is presented for infinity points of the form (1 : m_2 : m_3 : 0). For the infinity points (0 : m_2 : m_3 : 0), with m_2 \neq 0 or m_3 \neq 0, we reason similarly but we dehomogenize w.r.t \ x_2 (if m_2 \neq 0) or \ x_3 (if m_3 \neq 0). More precisely, we distinguish two different cases:

1. If \quad (0 : m_2 : m_3 : 0), m_2 \neq 0 \text{ is an infinity point of the given space curve } \mathcal{C}, \text{ we consider the curve defined by the polynomials } g_i(x_1, x_3, x_4) := F_i(x_1, 1, x_3, x_4) \in \mathbb{R}[x_1, x_3, x_4], \text{ } i = 1, 2, \text{ and we reason as above. We get that an infinity branch of } \mathcal{C} \text{ associated to the infinity point } \mathcal{P} = (0 : m_2 : m_3 : 0), m_2 \neq 0, \text{ is a set } \mathcal{B} = \bigcup_{j=1}^{N} L_j, \text{ where } L_j = \{(r_{j,1}(z), z, r_{j,3}(z)) \in \mathbb{C}^3 : z \in \mathbb{C}, |z| > M\}, \quad M \in \mathbb{R}^+.\quad \text{ 6}
2. If \((0 : m_2 : m_3 : 0), m_3 \neq 0\) is an infinity point of the given space curve \(C\), we consider the curve defined by the polynomials \(g_i(x_1, x_2, x_4) := F_i(x_1, x_2, 1, x_4) \in \mathbb{R}[x_1, x_2, x_4] , \ i = 1, 2\), and we reason as above. We get that an infinity branch of \(C\) associated to the infinity point \(P = (0 : m_2 : m_3 : 0)\), \(m_3 \neq 0\), is a set \(B = \bigcup_{j=1}^{N} L_j\), where \(L_j = \{(r_{j,1}(z), r_{j,2}(z), z) \in \mathbb{C}^3 : z \in \mathbb{C}, |z| > M\}\), \(M \in \mathbb{R}^+\).

Additionally, instead of working with this type of branches, if the space curve \(C\) has infinity points of the form \((0 : m_2 : m_3 : 0)\), one may consider a linear change of coordinates. Thus, in the following, we may assume w.l.o.g that the given algebraic space curve \(C\) only has infinity points of the form \((1 : m_2 : m_3 : 0)\). More details on this type of branches are given in [5].

In the following, we introduce the notions of convergent branches and approaching curves. Intuitively speaking, two infinity branches converge if they get closer as they tend to infinity. This concept will allow us to analyze whether two space curves approach each other and it generalizes the notion introduced for the plane case (see [3]).

**Definition 2.4.** Two infinity branches, \(B\) and \(\overline{B}\), are convergent if there exist two leaves \(L = \{(z, r_2(z), r_3(z)) \in \mathbb{C}^3 : z \in \mathbb{C}, |z| > M\} \subset B\) and \(\overline{L} = \{(z, \overline{r}_2(z), \overline{r}_3(z)) \in \mathbb{C}^3 : z \in \mathbb{C}, |z| > M\} \subset \overline{B}\) such that

\[
\lim_{z \to \infty} d((r_2(z), r_3(z)), (\overline{r}_2(z), \overline{r}_3(z))) = 0.
\]

In this case, we say that the leaves \(L\) and \(\overline{L}\) converge.

**Remark 2.5.**

1. In Definition [2.4], we consider any distance \(d(u, v)\), \(u, v \in \mathbb{C}^2\). Taking into account that all norms are equivalent in \(\mathbb{C}^2\), we easily get that \(\lim_{z \to \infty} d((r_2(z), r_3(z)), (\overline{r}_2(z), \overline{r}_3(z))) = 0\) if and only if \(\lim_{z \to \infty}(r_i(z) - \overline{r}_i(z)) = 0\), \(i = 2, 3\).

2. Two convergent infinity branches are associated to the same infinity point (see Remark 4.5 in [3]).

In the following lemma, we characterize the convergence of two given infinity branches. This result is obtained similarly as in the case of plane curves and thus, we omit the proof (see Lemma 4.2, and Proposition 4.6 in [3]).
Lemma 2.6. The following statements hold:

- Two leaves $L = \{(z, r_2(z), r_3(z)) \in \mathbb{C}^3 : z \in \mathbb{C}, |z| > M\}$ and $\overline{L} = \{(z, \overline{r}_2(z), \overline{r}_3(z)) \in \mathbb{C}^3 : z \in \mathbb{C}, |z| > \overline{M}\}$ are convergent if and only if the terms with non-negative exponent in the series $r_i(z)$ and $\overline{r}_i(z)$ are the same, for $i = 2, 3$.

- Two infinity branches $B$ and $\overline{B}$ are convergent if and only if for each leaf $L \subset B$ there exists a leaf $\overline{L} \subset \overline{B}$ convergent with $L$, and reciprocally.

In Definition 2.7, we introduce the notion of approaching curves that is, curves that approach each other. For this purpose, we recall that given an algebraic space curve $C$ over $\mathbb{C}$ and a point $p \in \mathbb{C}^3$, the distance from $p$ to $C$ is defined as $d(p, C) = \min\{d(p, q) : q \in C\}$.

Definition 2.7. Let $C$ be an algebraic space curve over $\mathbb{C}$ with an infinity branch $B$. We say that a curve $\overline{C}$ approaches $C$ at its infinity branch $B$ if there exists one leaf $L = \{(z, r_2(z), r_3(z)) \in \mathbb{C}^3 : z \in \mathbb{C}, |z| > M\} \subset B$ such that $\lim_{z \to \infty} d((z, r_2(z), r_3(z)), \overline{C}) = 0$.

In the following, we state some important results concerning two curves that approach each other. These results can be proved similarly as in the case of plane curves (see Lemma 3.6, Theorem 4.11, Remark 4.12 and Corollary 4.13 in [5]).

Theorem 2.8. Let $C$ be a space algebraic curve over $\mathbb{C}$ with an infinity branch $B$. A space algebraic curve $\overline{C}$ approaches $C$ at $B$ if and only if $\overline{C}$ has an infinity branch, $\overline{B}$, such that $B$ and $\overline{B}$ are convergent.

Remark 2.9. 1. Note that $\overline{C}$ approaches $C$ at some infinity branch $B$ if and only if $C$ approaches $\overline{C}$ at some infinity branch $\overline{B}$. In the following, we say that $C$ and $\overline{C}$ approach each other or that they are approaching curves.

2. Two approaching curves have a common infinity point.

3. $\overline{C}$ approaches $C$ at an infinity branch $B$ if and only if for every leaf $L = \{(z, r_2(z), r_3(z)) \in \mathbb{C}^3 : z \in \mathbb{C}, |z| > M\} \subset B$, it holds that $\lim_{z \to \infty} d((z, r_2(z), r_3(z)), \overline{C}) = 0$. 

8
Corollary 2.10. Let \( C \) be an algebraic space curve with an infinity branch \( B \). Let \( \overline{C}_1 \) and \( \overline{C}_2 \) be two different curves that approach \( C \) at \( B \). Then:

1. \( \overline{C}_i \) has an infinity branch \( \overline{B}_i \) that converges with \( B \), for \( i = 1, 2 \).
2. \( \overline{B}_1 \) and \( \overline{B}_2 \) are convergent. Then, \( \overline{C}_1 \) and \( \overline{C}_2 \) approach each other.

For the sake of simplicity, and taking into account that an infinity branch \( B \) is uniquely determined from one leaf, up to conjugation (see statement 1 in Remark 2.2), we identify an infinity branch by just one of its leaves. Hence, in the following

\[
B = \{(z, r_2(z), r_3(z)) \in \mathbb{C}^3 : z \in \mathbb{C}, |z| > M\}, \quad M \in \mathbb{R}^+
\]

will stand for the infinity branch whose leaves are obtained by conjugation on

\[
r_k(z) = m_k z + a_{1,k} z^{1-N_{1,k}/N_k} + a_{2,k} z^{1-N_{2,k}/N_k} + a_{3,k} z^{1-N_{3,k}/N_k} + \cdots,
\]

\( a_{i,k} \neq 0, \forall i \in \mathbb{N}, i \geq 1, N_k, N_{i,k} \in \mathbb{N}, k = 2, 3, \) and \( 0 < N_{1,k} < N_{2,k} < \cdots \) for \( k = 2, 3 \). Observe that the results stated above hold for any leaf of \( B \). In addition, we also will show that the results obtained in the following sections hold for any leaf (see statement 3 in Remark 4.1).

3 Computation of infinity branches

Let \( C \) be an irreducible algebraic space curve defined by the polynomials \( f_1(x_1, x_2, x_3), f_2(x_1, x_2, x_3) \in \mathbb{R}[x_1, x_2, x_3] \). In [2], it is proved that there exists a plane curve, say \( C^p \), which is birationally related with \( C \). That is, there exists a birational correspondence between the points of \( C^p \) and the points of \( C \). Furthermore, it is shown that \( C^p \) can always be obtained by projecting \( C \) along some “valid projection direction”.

In the following we assume that the \( x_3 \)-axis is a valid projection direction (otherwise, we apply a linear change of coordinates). Let \( C^p \) be the projection of \( C \) along the \( x_3 \)-axis, and let \( f^p(x_1, x_2) \in \mathbb{R}[x_1, x_2] \) be the implicit polynomial defining \( C^p \). In [2], it is shown how to construct a birational mapping \( h(x_1, x_2) = h_1(x_1, x_2)/h_2(x_1, x_2) \) such that \( (x_1, x_2, x_3) \in C \) if and only if \( (x_1, x_2) \in C^p \) and \( x_3 = h(x_1, x_2) \). We refer to \( h(x_1, x_2) \) as the lift function,
since we can obtain the points of the space curve $C$ by applying $h$ to the points of the plane projected curve $C^p$. In addition, note that $x_3 = h(x_1, x_2)$ if and only if $h_1(x_1, x_2) - h_2(x_1, x_2)x_3 = 0$. Thus, $C$ can be implicitly defined by the polynomials $f^p(x_1, x_2)$ and $f_3(x_1, x_2, x_3) = h_1(x_1, x_2) - h_2(x_1, x_2)x_3$.

In Theorem 3.1 we study the relation between the infinity branches of $C$ and $C^p$. The idea is to use the lift function $h$ to obtain the infinity branches of the space curve $C$ from the infinity branches of the plane curve $C^p$. An efficient method to compute the infinity branches of a plane curve is presented in [5].

**Theorem 3.1.** $B^p = \{(z, r_2(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| > M^p\}$ is an infinity branch of $C^p$ for some $M^p \in \mathbb{R}^+$ if there exists a series $r_3(z) = z\varphi_3(1/z)$, $\varphi_3(z) \in \mathbb{C} \ll z \gg$, such that $B = \{(z, r_2(z), r_3(z)) \in \mathbb{C}^3 : z \in \mathbb{C}, |z| > M\}$ is an infinity branch of $C$ for some $M \in \mathbb{R}^+$.

**Proof:** Clearly, if $B$ is an infinity branch of $C$, then $B^p$ is an infinity branch of $C^p$. Conversely, let $B^p = \{(z, r_2(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| > M^p\}$ be an infinity branch of $C^p$, and we look for a series $r_3(z) = z\varphi_3(1/z)$, $\varphi_3(z) \in \mathbb{C} \ll z \gg$, such that $B = \{(z, r_2(z), r_3(z)) \in \mathbb{C}^3 : z \in \mathbb{C}, |z| > M\}$ is an infinity branch of $C$. Note that, from the discussion above, we can get it as $r_3(z) = h(z, r_2(z))$. However, we need to prove that $r_3(z) = z\varphi_3(1/z)$ for some Puiseux series $\varphi_3(z)$.

As we stated above, given $(a_1, a_2, a_3) \in C$, it holds that $f_3(a_1, a_2, a_3) = h_1(a_1, a_2) - h_2(a_1, a_2)a_3 = 0$. Thus, in particular, $(z, r_2(z), r_3(z)) \in B \subset C$ verifies that $f_3(z, r_2(z), r_3(z)) = 0$. Hence, $F_3(z, r_2(z), r_3(z), 1) = 0$, where $F_3(x_1, x_2, x_3, x_4)$ is the homogeneous polynomial of $f_3(x_1, x_2, x_3)$.

Taking into account the results in [5], we have that $r_2(z) = z\varphi_2(1/z)$, where $\varphi_2(z) \in \mathbb{C} \ll z \gg$. Now, we look for $\varphi_3(z) \in \mathbb{C} \ll z \gg$ such that $r_3(z) = z\varphi_3(1/z)$. This series must verify that (see statement above)

$$F_3(z, z\varphi_2(1/z), z\varphi_3(1/z), 1) = 0 \quad \text{for } |z| > M.$$ 

We set $z = t^{-1}$, and we get that $F_3(t^{-1}, t^{-1}\varphi_2(t), t^{-1}\varphi_3(t), 1) = 0$ or equivalently

$$F_3(1, \varphi_2(t), \varphi_3(t), t) = 0. \quad (I)$$ 

Note that equality (I) holds for $|t| < 1/M$. That is, equality (I) must be satisfied in a neighborhood of the infinity point $(1, \varphi_2(0), \varphi_3(0), 0)$.
At this point, we observe that $F_3$ has the form

$$F_3(x_1, x_2, x_3, x_4) = x_4^{n_1}H_1(x_1, x_2, x_4) - x_4^{n_2}H_2(x_1, x_2, x_4)x_3$$

where $H_i(x_1, x_2, x_4)$ is the homogeneous polynomial of $h_i(x_1, x_2)$, $i = 1, 2$, and $n_1, n_2 \in \mathbb{N}$. Then, we have that

$$F_3(1, \varphi_2(t), \varphi_3(t), t) = t^{n_1}H_1(1, \varphi_2(t), t) - t^{n_2}H_2(1, \varphi_2(t), t)\varphi_3(t)$$

and since (I) must hold, we obtain that

$$\varphi_3(t) = t^{n_1-n_2}\frac{H_1(1, \varphi_2(t), t)}{H_2(1, \varphi_2(t), t)}$$

Obviously, $\varphi_3(t)$ can be expressed as a Puiseux series since $\mathbb{C} \ll t \gg$ is a field. Therefore, we conclude that $B = \{(z, r_2(z), r_3(z)) : z \in \mathbb{C}, |z| > M\}$, where $r_3(z) = z\varphi_3(1/z)$, is an infinity branch of $C$. □

In the following, we illustrate the above theorem with an example.

**Example 3.2.** Let $C$ be the irreducible space curve defined over $\mathbb{C}$ by the polynomials

$$f_1(x_1, x_2, x_3) = -x_2^2-2x_1x_3+2x_2x_3-x_1+3, \quad \text{and} \quad f_2(x_1, x_2, x_3) = x_3+x_1x_2-x_2^2.$$

The projection along the $x_3$-axis, $C^p$ is given by the polynomial

$$f^p(x_1, x_2) = x_2^2 + x_1 - 3 - 2x_2x_1^2 + 4x_1x_2^2 - 2x_2^3$$

(this polynomial can be obtained by computing resultant $x_3(f_1, f_2)$; see [12]).

By applying the method described in [5], we compute the infinity branches of $C^p$. We obtain the branch $B_1^p = \{(z, r_{12}(z)) : |z| > M_{12}^p\}$, where

$$r_{12}(z) = \frac{z^{-1}}{2} - \frac{3z^{-2}}{2} + \frac{z^{-3}}{2} - \frac{23z^{-4}}{8} + \frac{37z^{-5}}{8} - \frac{25z^{-6}}{4} + \cdots,$$

that is associated to the infinity point $P_1 = (1 : 0 : 0)$, and the branch $B_2^p = \{(z, r_{22}(z)) : |z| > M_{22}^p\}$, where

$$r_{22}(z) = z + \frac{\sqrt{2}z^{1/2}}{2} + \frac{1}{4} + \frac{9\sqrt{2}z^{-1/2}}{32} - \frac{z^{-1}}{4} - \frac{785\sqrt{2}z^{-3/2}}{1024} + \cdots.$$
that is associated to the infinity point \( P_2 = (1 : 1 : 0) \). Note that \( B_2^p \) has ramification index 2, so it has two leaves.

Once we have obtained the infinity branches of the projected curve \( C_p \), we compute the infinity branches of the space curve \( C \). We use the lift function \( h(x_1, x_2) = -x_1x_2 + x_2^2 \) to get the third component of these branches (we apply the results in [2] to compute \( h \)). Thus, the infinity branches of the space curve are \( B_1 = \{ (z, r_{12}(z), r_{13}(z)) : |z| > M_1 \} \), where
\[
r_{13}(z) = h(z, r_{12}(z)) = -\frac{1}{2} - \frac{3z^{-1}}{2} - \frac{z^{-2}}{4} + \frac{11z^{-3}}{8} - \frac{15z^{-4}}{8} + \frac{15z^{-5}}{8} + \ldots
\]
and \( B_2 = \{ (z, r_{22}(z), r_{23}(z)) : |z| > M_2 \} \), where
\[
r_{23}(z) = h(z, r_{22}(z)) = \frac{\sqrt{2}z^{3/2}}{2} + \frac{3z}{4} + \frac{17\sqrt{2}z^{1/2}}{32} + \frac{3}{8} - \frac{897\sqrt{2}z^{-1/2}}{1024} + \ldots.
\]
In Figure 1, we plot the curve \( C \) and some points of the infinity branches \( B_1 \) and \( B_2 \).

Figure 1: Curve \( C \) and infinity branches \( B_1 \) (left) and \( B_2 \) (right).

4 Computation of an asymptote of a given infinity branch

In [4], we show how some algebraic plane curves can be approached at infinity by other curves of less degree. A well-known example is the case of hyperbolas that are curves of degree 2 approached at infinity by two lines (their
asymptotes). Similar situations may also arise when we deal with curves of higher degree.

For instance, let \( C \) be the plane curve defined by the equation 
\[
-xy - y^2 - x^3 + 2x^2y + x^2 - 2y = 0.
\]
The curve \( C \) has degree 3 but it can be approached at infinity by the parabola 
\[
y - 2x^2 + 3/2x + 15/8 = 0
\]
(see Figure 2). This example leads us to introduce the notions of perfect curve and \( g \)-asymptote. Some important properties on these concepts are presented for a given plane curve in Sections 3 and 4 in [4]. Most of these results can be easily generalized for a given algebraic space curve.

![Figure 2: Curve \( C \) (left) approached by a parabola and a line (right).](image)

**Definition 4.1.** A curve of degree \( d \) is a perfect curve if it cannot be approached by any curve of degree less than \( d \).

A curve that is not perfect can be approached by other curves of less degree. If these curves are perfect, we call them \( g \)-asymptotes. More precisely, we have the following definition.

**Definition 4.2.** Let \( C \) be a curve with an infinity branch \( B \). A \( g \)-asymptote (generalized asymptote) of \( C \) at \( B \) is a perfect curve that approaches \( C \) at \( B \).

The notion of \( g \)-asymptote is similar to the classical concept of asymptote. The difference is that a \( g \)-asymptote does not have to be a line, but a perfect curve. Actually, it is a generalization, since every line is a perfect curve (this remark follows from Definition 4.1). Throughout the paper we refer to \( g \)-asymptote simply as asymptote.
Remark 4.3. The degree of an asymptote is less or equal than the degree of the curve it approaches. In fact, an asymptote of a curve $C$ at a branch $B$ has minimal degree among all the curves that approach $C$ at $B$ (see Remark 3 in [3]).

In the following, we prove that every infinity branch of a given algebraic space curve has, at least, one asymptote and we show how to obtain it (see Theorem 4.10). Most of the results introduced below to the space case generalize the results presented in [4] for the plane case.

Let $C$ be an irreducible space curve implicitly defined by the polynomials $f_1(x_1, x_2, x_3), f_2(x_1, x_2, x_3) \in \mathbb{R}[x_1, x_2, x_3]$, and let $B = \{(z, r_2(z), r_3(z)) : z \in \mathbb{C}, |z| > M\}$ be an infinity branch of $C$ associated to the infinity point $P = (1 : m_2 : m_3 : 0)$. We know that $r_2$ and $r_3$ are given as

$$r_2(z) = m_2z + a_{1,2}z^{-N_2/N_2+1} + a_{2,2}z^{-N_2/N_2+1} + a_{3,2}z^{-N_3/N_2+1} + \ldots$$

$$r_3(z) = m_3z + a_{1,3}z^{-N_1/N_3+1} + a_{2,3}z^{-N_1/N_3+1} + a_{3,3}z^{-N_2/N_3+1} + \ldots$$

where $a_{i,2} \neq 0$, $N_2, N_i, 2 \in \mathbb{N}$, $i \geq 1$, $0 < N_1 < N_2 < \ldots$, and $a_{i,3} \neq 0$, $N_3, N_i, 3 \in \mathbb{N}$, $i \geq 1$, and $0 < N_2 < N_3 < \ldots$. Let $N := \text{lcm}(N_2, N_3)$, and note that $\nu(B) = N$.

Lemma 4.4. It holds that $\deg(C) \geq N$.

Proof: In Section 2, we show that there exist $N := \text{lcm}(N_2, N_3)$ conjugate tuples, $(\varphi_2(z), \varphi_3(z))$, which are solutions of the system $g_i(x_1, x_2, x_3, x_4) = 0$, $i = 1, 2$. Hence, the tuples $(z, r_{j,2}(z), r_{j,3}(z))$ with $r_{j,2}(z) = z\varphi_{j,2}(z^{-1})$ and $r_{j,3}(z) = z\varphi_{j,3}(z^{-1})$ for $j = 1, \ldots, N$, are solutions of the system $f_i(x_1, x_2, x_3) = 0$, $i = 1, 2$. That is, they are points of the curve $C$.

Then, given $z_0$ such that $|z_0| > M$, we have $N$ intersections between the curve $C$ and the plane defined by the equation $x_1 - z_0 = 0$ (these points are $(z_0, r_{j,2}(z_0), r_{j,3}(z_0))$, $j = 1, \ldots, N$). Thus, by definition of degree of a space curve (see e.g. [3] or [7]), we get that $\deg(C) \geq N$. \hfill $\square$

In the following, we write

$$r_2(z) = m_2z + a_{1,2}z^{-n_2/N_2+1} + \ldots + a_{2,2}z^{-n_2/N_2+1} + a_{3,2}z^{-n_3/N_2+1} + \ldots$$

$$r_3(z) = m_3z + a_{1,3}z^{-n_3/N_3+1} + \ldots + a_{2,3}z^{-n_3/N_3+1} + a_{3,3}z^{-n_3/N_3+1} + \ldots$$

(2)
From Theorem 2.8, we get that

**Proof:**

Let 

$$\begin{align*}
\{z, r_2(z), r_3(z)\} &\subset \mathbb{C}^3 : z \in \mathbb{C}, |z| > M \}
\end{align*}$$

an infinity branch associated to 

$$\begin{align*}
P = (1 : m_2 : m_3 : 0), \quad m_j \in \mathbb{C}, \ j = 1, 2.
\end{align*}$$

We say that 

$$\begin{align*}
\text{gcd}(n_2, n_3) = \deg(B) \leq N = \text{lcm}(N_2, N_3), \quad \text{and from Lemma 4.4 we get that } \deg(C) \geq \deg(B).
\end{align*}$$

**Remark 4.6.** Note that 

$$\begin{align*}
n_i \leq N_i, \quad i = 1, 2.
\end{align*}$$

Thus, 

$$\begin{align*}
n = \text{lcm}(n_2, n_3) = \deg(B) \leq N = \text{lcm}(N_2, N_3), \quad \text{and from Lemma 4.4 we get that } \deg(C) \geq \deg(B).
\end{align*}$$

**Proposition 4.7.** Let 

$$\begin{align*}
\overline{C}
\end{align*}$$

be a curve that approaches 

$$\begin{align*}
C
\end{align*}$$

at its infinity branch 

$$\begin{align*}
B.
\end{align*}$$

It holds that 

$$\begin{align*}
\deg(C) \geq \deg(B).
\end{align*}$$

**Proof:** From Theorem 2.8 we get that 

$$\begin{align*}
\overline{C}
\end{align*}$$

has an infinity branch 

$$\begin{align*}
\overline{B} = \{\{z, \overline{r}_2(z), \overline{r}_3(z)\} \subset \mathbb{C}^3 : z \in \mathbb{C}, |z| > M\}
\end{align*}$$

convergent with the branch 

$$\begin{align*}
B = \{(z, r_2(z), r_3(z)) \subset \mathbb{C}^3 : z \in \mathbb{C}, |z| > M\}
\end{align*}$$

From Lemma 2.6 we deduce that the terms with non negative exponent in the series 

$$\begin{align*}
r_i(z) \quad \text{and} \quad \overline{r}_i(z), \quad i = 2, 3,
\end{align*}$$

are the same, and hence 

$$\begin{align*}
\overline{B}
\end{align*}$$

is a branch of degree 

$$\begin{align*}
n
\end{align*}$$

of the form given in (2). Now, the result follows taking into account Remark 4.6.

### 4.1 Construction of asymptotes

Let 

$$\begin{align*}
C
\end{align*}$$

be a space curve with an infinity branch 

$$\begin{align*}
B = \{(z, r_2(z), r_3(z)) \subset \mathbb{C}^3 : z \in \mathbb{C}, |z| > M\}
\end{align*}$$

Taking into account the results presented above, we have that any curve 

$$\begin{align*}
\overline{C}
\end{align*}$$

approaching 

$$\begin{align*}
C
\end{align*}$$

at 

$$\begin{align*}
B
\end{align*}$$

an infinity branch 

$$\begin{align*}
\overline{B} = \{(z, \overline{r}_2(z), \overline{r}_3(z)) \subset \mathbb{C}^3 : z \in \mathbb{C}, |z| > M\}
\end{align*}$$

such that the terms with non negative exponent in 

$$\begin{align*}
r_i(z) \quad \text{and} \quad \overline{r}_i(z), \quad (i = 2, 3),
\end{align*}$$

We consider the series 

$$\begin{align*}
\overline{r}_2(z) \quad \text{and} \quad \overline{r}_3(z),
\end{align*}$$

obtained from 

$$\begin{align*}
r_2(z) \quad \text{and} \quad r_3(z)
\end{align*}$$

by removing the terms with negative exponent (see equation (2)). Then, we have that

$$\begin{align*}
\overline{r}_2(z) &= m_2 z + a_{12,2} z^{-n_{12,2}/n_2} + \ldots + a_{l_2,2} z^{-n_{l_2,2}/n_2} \quad (3) \\
\overline{r}_3(z) &= m_3 z + a_{13,3} z^{-n_{13,3}/n_3} + \ldots + a_{l_3,3} z^{-n_{l_3,3}/n_3}
\end{align*}$$

where 

$$\begin{align*}
a_{i,j} \in \mathbb{C} \setminus \{0\}, \quad m_k \in \mathbb{C}, \quad n_k, n_{j,k} \in \mathbb{N}, \quad \text{gcd}(n_k, n_{1,k}, \ldots, n_{l,k}) = 1, \quad \text{and } 0 < n_{1,k} < n_{2,k} < \ldots.
\end{align*}$$

That is, 

$$\begin{align*}
\overline{r}_k
\end{align*}$$

has the same terms with non negative
exponent that $r_k$, and $\tilde{r}_k$ does not have terms with negative exponent.

Let $\tilde{C}$ be the space curve containing the branch $\tilde{B} = \{(z, \tilde{r}_2(z), \tilde{r}_3(z)) \in \mathbb{C}^3 : z \in \mathbb{C}, |z| > \tilde{M}\}$. Observe that

$$\tilde{Q}(t) = \left( t^n, m_2 t^n + a_{1,2} t^{r_2(n_2-n_1,2)} + \cdots + a_{\ell_2,2} t^{r_2(n_2-n_{\ell_2,2})}, \right.$$  
$$m_3 t^n + a_{1,3} t^{r_3(n_3-n_1,3)} + \cdots + a_{\ell_3,3} t^{r_3(n_3-n_{\ell_3,3})}\right) \in \mathbb{C}[t]^3,$$  

where $n = \text{lcm}(n_2, n_3)$, $r_k = n/n_k$, $n_k, n_{1,k}, \ldots, n_{\ell_k,k} \in \mathbb{N}$, $0 < n_{1,k} < \cdots < n_{\ell_k,k}$, and $\gcd(n_k, n_{1,k}, \ldots, n_{\ell_k,k}) = 1$, $k = 2, 3$, is a polynomial parametrization of $\tilde{C}$. In addition, in Lemma 4.8, we prove that $\tilde{Q}$ is proper (i.e. invertible).

**Lemma 4.8.** The parametrization $\tilde{Q}$ given in (4) is proper.

**Proof:** Let us assume that $\tilde{Q}$ is not proper. Then, there exists $R(t) \in \mathbb{C}[t]$, with $\text{deg}(R) = r > 1$, and $\tilde{Q}(t) = (q_1(t), q_2(t), q_3(t)) \in \mathbb{C}[t]^3$, such that $\tilde{Q}(R) = \tilde{Q}$ (see [9]). In particular, we get that $q_1(R(t)) = t^n$, which implies that $q_1(t) = (t - R(0))^k$, and $R(t) = t^r + R(0)$, $r_k = n$.

Let us consider $R^*(t) = R(t) - R(0) = t^r \in \mathbb{C}[t]$, and

$$\tilde{Q}^*(t) = \tilde{Q}(t + R(0)) = \left( t^k, q_2^*(t), q_3^*(t) \right) = \left( t^k, c_0 + c_1 t + c_2 t^2 + \cdots + c_u t^u, d_0 + d_1 t + d_2 t^2 + \cdots + d_v t^v \right) \in \mathbb{C}[t]^3.$$

Then, $\tilde{Q}^*(R^*) = \tilde{Q}(R) = \tilde{Q}$ and, in particular,

$$q_2^*(R^*) = q_2^*(t^r) = m_2 t^n + a_{1,2} t^{r_2(n_2-n_1,2)} + \cdots + a_{\ell_2,2} t^{r_2(n_2-n_{\ell_2,2})}$$

That is,

$$c_0 + c_1 t^r + c_2 t^{2r} + \cdots + c_u t^{ur} = m_2 t^n + a_{1,2} t^{r_2(n_2-n_1,2)} + \cdots + a_{\ell_2,2} t^{r_2(n_2-n_{\ell_2,2})}.$$  

From this equality, and taking into account that $r_2 = n/n_2 = r{k/n_2}$, we deduce that $k/n_2(n_2 - n_{i,2}) \in \mathbb{Z}$, and thus $kn_{i,2}/n_2 \in \mathbb{Z}$ for $i = 1, \ldots, \ell_2$. This implies that $n_2$ divides $k$ since, otherwise, $n_2$ should divide $n_{i,2}$ for $i = 1, \ldots, \ell_2$, which contradicts the assumption that $\gcd(n_2, n_{1,2}, \ldots, n_{\ell_2,2}) = 1$ (see equation [9]).
On the other hand, reasoning similarly with the third component, we have that 
$q^*_3(R^*) = q^*_3(t^*) = \bar{q}_3(t)$ and we get that $n_3$ also divides $k$. Therefore, 
k is a common multiple of $n_2$ and $n_3$, which is impossible since $k < n$ (note 
that $rk = n$, $r > 1$) and $n = \text{lcm}(n_2,n_3)$.

From Lemma 4.8 and using the definition of degree for an implicitly algebraic space curve (see e.g. [3] or [7]), we obtain the following lemma.

**Lemma 4.9.** Let \( \tilde{C} \) be the plane curve containing the infinity branch given in (3). It holds that \( \deg(\tilde{C}) = \deg(B) \).

**Proof:** The intersection of \( \tilde{C} \) with a generic plane provides \( n \) points since \( \tilde{C} \) is parametrized by the proper parametrization \( \tilde{Q} \) that has degree \( n \) (see Lemma 4.8). In addition, we remark that \( n = \deg(B) \) (see Definition 4.5).

In the following theorem, we prove that for any infinity branch \( B \) of a space curve \( C \), there always exists an asymptote that approaches \( C \) at \( B \). Furthermore, we provide a method to obtain it (see algorithm Space Asymptotes Construction). The proof of this theorem is obtained from Lemmas 4.4 and 4.9 and Proposition 4.7. This proof is similar to the proof of Theorem 2 in [4], but for the sake of completeness, we include it.

**Theorem 4.10.** The curve \( \tilde{C} \) is an asymptote of \( C \) at \( B \).

**Proof:** From the construction of \( \tilde{C} \), we have that \( \tilde{C} \) approaches \( C \) at \( B \). Thus, we need to show that \( \tilde{C} \) cannot be approached by any curve with degree less than \( \deg(\tilde{C}) \) (that is, \( \tilde{C} \) is perfect).

For this purpose, we first note that \( \tilde{C} \) has a polynomial parametrization given by the form in (4). Hence, the unique infinity branch of \( \tilde{C} \) is \( \tilde{B} \) (see [3]). In addition, we observe that by construction, \( \tilde{B} \) and \( B \) are convergent.

Under these conditions, we consider a plane curve \( \overline{C} \), that approaches \( \tilde{C} \) at \( \tilde{B} \). Then, \( \overline{C} \) has an infinity branch \( \overline{B} \) convergent with \( \tilde{B} \) (see Theorem 2.8). Since \( \tilde{B} \) and \( B \) are convergent, we deduce that \( \overline{B} \) and \( B \) are convergent (see Corollary 2.10) which implies that \( \overline{C} \) approaches \( C \) at \( B \). Finally, from Proposition 4.7 and Lemma 4.9 we deduce that \( \deg(\overline{C}) \geq \deg(\tilde{C}) \) and thus, we conclude that \( \tilde{C} \) is perfect.

From these results, in the following we present an algorithm that computes an asymptote for each infinity branch of a given space curve.
We assume that we have prepared the input curve \( C \), by means of a suitable linear change of coordinates if necessary, such that \( (0 : a : b : 0) \) \((a \neq 0 \text{ or } b \neq 0)\) is not an infinity point of \( C \) (see Remark 2.3). In addition, we assume that there exists a birational correspondence between the points of \( C^p \) and the points of \( C \), where \( C^p \) is the plane curve obtained by projecting \( C \) along the \( x_3 \)-axis (see Section 3).

**Algorithm Space Asymptotes Construction.**

Given an irreducible real algebraic space curve \( C \) implicitly defined by two polynomials \( f_1(x_1, x_2, x_3), f_2(x_1, x_2, x_3) \in \mathbb{R}[x_1, x_2, x_3] \), the algorithm outputs an asymptote for each of its infinity branches.

1. Compute the projection of \( C \) along the \( x_3 \)-axis. Let \( C_p \) be this projection and \( f^p(x_1, x_2) \) the implicit polynomial defining \( C_p \).

2. Determine the lift function \( h(x_1, x_2) \) (see [2]).

3. Compute the infinity branches of \( C_p \) by applying Algorithm Asymptotes Construction in [4].

4. For each branch \( B^p_i = \{(z, r_{i,2}(z)) \in \mathbb{C}^2 : z \in \mathbb{C}, |z| > M^p_{i,2}\} \), \( i = 1, \ldots, s \), do:

   4.1. Compute the corresponding infinity branch of \( C \):

   \[
   B_i = \{(z, r_{i,2}(z), r_{i,3}(z)) \in \mathbb{C}^3 : z \in \mathbb{C}, |z| > M_i\}
   \]

   where \( r_{i,3}(z) = h(z, r_{i,2}(z)) \) is given as a Puiseux series.

   4.2. Consider the series \( \tilde{r}_{i,2}(z) \) and \( \tilde{r}_{i,3}(z) \) obtained by eliminating the terms with negative exponent in \( r_{i,2}(z) \) and \( r_{i,3}(z) \), respectively. Note that, for \( j = 2, 3 \), the series \( \tilde{r}_{i,j} \) has the same terms with non negative exponent that \( r_{i,j} \), and \( \tilde{r}_{i,j} \) does not have terms with negative exponent.

   4.3. Return the asymptote \( \tilde{C}_i \) defined by the proper parametrization (see Lemma [4,8]), \( \tilde{Q}_i(t) = (t^{n_i}, \tilde{r}_{i,2}(t^{n_i}), \tilde{r}_{i,3}(t^{n_i})) \in \mathbb{C}[t]^3 \), where \( n_i = \deg(B_i) \) (see Definition [4,5]).
Remark 4.11. 1. The implicit polynomial \( f^p(x_1, x_2) \) defining \( C_p \) (see step 1) can be computed as \( f^p(x_1, x_2) = \text{resultant}_{x_3}(f_1, f_2) \) (see Section 4.5 in [12]).

2. Since we have assumed that the given algebraic space curve \( C \) only has infinity points of the form \((1 : m_2 : m_3 : 0)\) (see Remark 2.3), we have that \((0 : m : 0)\) is not an infinity point of the plane curve \( C_p \) and thus, Algorithm Asymptotes Construction in [4] (see step 3) can be applied.

3. Reasoning as in the correctness of the algorithm Asymptotes Construction in [4], one may prove that the algorithm Space Asymptotes Construction outputs an asymptote \( \tilde{C} \) that is independent of the leaf chosen to define the branch \( B \) (see Section 2).

In the following example, we illustrate algorithm Space Asymptotes Construction.

Example 4.12. Let \( C \) be the algebraic space curve over \( \mathbb{C} \) introduced in Example 3.2. The curve \( C \) is defined by the polynomials

\[
f_1(x_1, x_2, x_3) = -x_2^2 - 2x_1x_3 + 2x_2x_3 - x_1 + 3, \quad \text{and} \quad f_2(x_1, x_2, x_3) = x_3 + x_1x_2 - x_2^2.
\]

In Example 3.2, we show that \( C \) has two infinity branches given by:

\[
B_1 = \{(r_{11}(z), r_{12}(z), r_{13}(z)) : |z| > M_1\}, \quad \text{where}
\]

\[
r_{11}(z) = z, \\
r_{12}(z) = \frac{z^{-1}}{2} - \frac{3z^{-2}}{2} + \frac{z^{-3}}{8} - \frac{23z^{-4}}{8} + \frac{37z^{-5}}{8} - \frac{25z^{-6}}{8} + \cdots, \\
r_{13}(z) = \frac{1}{2} - \frac{3z^{-1}}{4} - \frac{z^{-2}}{8} + \frac{11z^{-3}}{8} - \frac{15z^{-4}}{8} + \frac{15z^{-5}}{8} + \cdots,
\]

and

\[
B_2 = \{(r_{21}(z), r_{22}(z), r_{23}(z)) : |z| > M_2\}, \quad \text{where}
\]

\[
r_{21}(z) = z, \\
r_{22}(z) = z + \frac{\sqrt{2}z^{1/2}}{2} + \frac{1}{4} + \frac{9\sqrt{2}z^{-1/2}}{32} - \frac{z^{-1}}{4} - \frac{785\sqrt{2}z^{-3/2}}{1024} + \cdots, \\
r_{23}(z) = \frac{\sqrt{2}z^{3/2}}{2} + \frac{3z}{4} + \frac{17\sqrt{2}z^{1/2}}{32} + \frac{3}{8} - \frac{897\sqrt{2}z^{-1/2}}{1024} + \cdots.
\]
These branches were obtained by applying steps 1, 2, 3, and 4.1 of Algorithm Space Asymptotes Construction. Now we apply step 4.2, and we compute the series $\tilde{r}_{i,j}(z)$ by removing the terms with negative exponent from the series $r_{i,j}(z)$, $i = 1, 2, j = 1, 2, 3$. We get:

$$
\begin{align*}
\tilde{r}_{11}(z) &= z, & \tilde{r}_{21}(z) &= z, \\
\tilde{r}_{12}(z) &= 0, & \tilde{r}_{22}(z) &= z + \frac{\sqrt{2}z^{1/2}}{2} + \frac{1}{4}, \\
\tilde{r}_{13}(z) &= -\frac{1}{2}, & \tilde{r}_{23}(z) &= \frac{\sqrt{2}z^{3/2}}{2} + \frac{3z}{4} + \frac{17\sqrt{2}z^{1/2}}{32} + \frac{3}{8}.
\end{align*}
$$

Figure 3: Curve $C$ approached by asymptotes $\tilde{C}_1$ (left) and $\tilde{C}_2$ (right).

Thus, in step 4.3, we obtain:

$$
\tilde{Q}_1(t) = (t, \tilde{r}_{1,2}(t), \tilde{r}_{1,3}(t)) = (t, 0, -1/2), \quad \text{and}
$$

$$
\tilde{Q}_2(t) = (t^2, \tilde{r}_{2,2}(t^2), \tilde{r}_{2,3}(t^2)) = \left(t^2, t^2 + \frac{\sqrt{2}t}{2} + \frac{1}{4}, \frac{\sqrt{2}t^3}{2} + \frac{3t^2}{4} + \frac{17\sqrt{2}t}{32} + \frac{3}{8}\right).
$$

$\tilde{Q}_1$ and $\tilde{Q}_2$ are proper parametrizations (see Lemma 4.8) of the asymptotes $\tilde{C}_1$ and $\tilde{C}_2$, which approach $C$ at its infinity branches $B_1$ and $B_2$, respectively. In Figure 3, we plot the curve $C$ and its asymptotes $\tilde{C}_1$ and $\tilde{C}_2$. 

20
5 Asymptotes of a parametric curve

Throughout this paper, we have dealt with real algebraic space curves defined implicitly by two polynomials. In this section, we present a method to compute infinity branches and asymptotes of rational curves from their parametric representation (without implicitizing).

Thus, in the following, we deal with real space curves defined parametrically. However, the method described can be trivially applied to the case of parametric real plane curves and in general, for a rational parametrization of a curve in the $n$-dimensional space. Similarly as in the previous sections, we work over $\mathbb{C}$, but we assume that the curve has infinitely many points in the affine plane over $\mathbb{R}$ and then, the curve has a real parametrization (see Chapter 7 in [12]).

Under these conditions, in the following, we consider a real space curve $C$ defined by the parametrization

$$P(s) = (P_1(s), P_2(s), P_3(s)) \in \mathbb{R}(s)^3 \setminus \mathbb{R}^3, \quad p_i(s) = p_{i1}(s)/p(s), \quad i = 1, 2, 3.$$ 

We assume that we have prepared the input curve $C$, by means of a suitable linear change of coordinates (if necessary) such that $(0 : a : b : 0)$ ($a \neq 0$ or $b \neq 0$) is not an infinity point (see Remark 2.3). Note that this implies that $\deg(p_1) \geq 1$.

Observe that if $C^*$ represents the projective curve associated to $C$, we have that a parametrization of $C^*$ is given by $P^*(s) = (p_{11}(s) : p_{21}(s) : p_{31}(s) : p(s))$ or, equivalently,

$$P^*(s) = \left(1 : \frac{p_{21}(s)}{p_{11}(s)} : \frac{p_{31}(s)}{p_{11}(s)} : \frac{p(s)}{p_{11}(s)}\right).$$

A method to construct the asymptotes of $C$.

In order to compute the asymptotes of $C$, first we need to determine the infinity branches of $C$. That is, the sets $B = \{(z : r_2(z) : r_3(z)) : z \in \mathbb{C}, |z| > M\}$, where $r_j(z) = z \varphi_j(z^{-1}), j = 2, 3$. For this purpose, we note that from Section 2 we have that $F_i(1 : \varphi_2(t) : \varphi_3(t) : t) = 0$ around $t = 0$, where $F_i, i = 1, 2$ are the polynomials defining implicitly $C^*$. Observe that in this section, we are given the parametrization $P^*$ of


\( \mathcal{C}^* \) and then, \( F_i(\mathcal{P}^*(s)) = F_i (1 : \frac{p_{2i}(s)}{p_{11}(s)} : \frac{p_{3i}(s)}{p_{11}(s)} : \frac{p(s)}{p_{11}(s)}) = 0. \) Thus, intuitively speaking, in order to compute the infinity branches of \( \mathcal{C} \), and in particular the series \( \varphi_j, j = 2, 3 \), one needs to rewrite the parametrization \( \mathcal{P}^*(s) = \left(1 : \frac{p_{2i}(s)}{p_{11}(s)} : \frac{p_{3i}(s)}{p_{11}(s)} : \frac{p(s)}{p_{11}(s)}\right) \) in the form \((1 : \varphi_2(t) : \varphi_3(t) : t)\) around \( t = 0 \). For this purpose, the idea is to look for a value of the parameter \( s \), say \( \ell(t) \in \mathbb{C} \ll t \gg \), such that \( \mathcal{P}^*(\ell(t)) = (1 : \varphi_2(t) : \varphi_3(t) : t) \) around \( t = 0 \).

Hence, from the above reasoning, we deduce that first, we have to consider the equation \( p(s)/p_{11}(s) = t \) (or equivalently, \( p(s) - tp_{11}(s) = 0 \)), and we solve it in the variable \( s \) around \( t = 0 \) (note that \( \deg(p_i) \geq 1 \)). From Puiseux’s Theorem, there exist solutions \( \ell_1(t), \ell_2(t), \ldots, \ell_k(t) \in \mathbb{C} \ll t \gg \) such that, \( p(\ell_i(t)) - tp_{11}(\ell_i(t)) = 0, i = 1, \ldots, k \), in a neighborhood of \( t = 0 \).

Thus, for each \( i = 1, \ldots, k \), there exists \( M_i \in \mathbb{R}^+ \) such that the points \((1 : \varphi_{i,2}(t) : \varphi_{i,3}(t) : t)\) or equivalently, the points \((t^{-1} : t^{-1} \varphi_{i,2}(t) : t^{-1} \varphi_{i,3}(t) : 1)\), where

\[ \varphi_{i,j}(t) = \frac{p_{1j}(\ell_i(t))}{p_{11}(\ell_i(t))}, \quad j = 2, 3, \]

are in \( \mathcal{C}^* \) for \(|t| < M_i \) (note that \( \mathcal{P}^*(\ell(t)) \in \mathcal{C}^* \) since \( \mathcal{P}^* \) is a parametrization of \( \mathcal{C}^* \)). Observe that \( \varphi_{i,j}(t), j = 2, 3 \) are Puiseux series, since \( p_{j1}(\ell_i(t)), j = 2, 3 \) and \( p_{11}(\ell_i(t)) \) can be written as Puiseux series and \( \mathbb{C} \ll t \gg \) is a field.

Finally, we set \( z = t^{-1} \). Then, we have that the points \((z : r_{i,2}(z) : r_{i,3}(z))\), where \( r_{i,j}(z) = z\varphi_{i,j}(z^{-1}), j = 2, 3 \), are in \( \mathcal{C} \) for \(|z| > M_i^{-1} \). Hence, the infinity branches of \( \mathcal{C} \) are the sets

\[ B_i = \{(z : r_{i,2}(z) : r_{i,3}(z)) : z \in \mathbb{C}, |z| > M_i^{-1}\}, \quad i = 1, \ldots, k. \]

**Remark 5.1.** Note that the series \( \ell_i(t) \) satisfies that \( p(\ell_i(t))/p_{11}(\ell_i(t)) = t, \) for \( i = 1, \ldots, k. \) Then, from equality (5), we have that for \( j = 2, 3 \)

\[ \varphi_{i,j}(t) = \frac{p_{j1}(\ell_i(t))}{p(\ell_i(t))} t = p_j(\ell_i(t)) t, \quad \text{and} \quad r_{i,j}(z) = z\varphi_{i,j}(z^{-1}) = p_j(\ell_i(z^{-1})). \]

Once we have the infinity branches, we can compute an asymptote for each of them by simply removing the terms with negative exponent from \( r_{i,2} \) and \( r_{i,3} \) (see Subsection 4.1).
The following algorithm computes the infinity branches of a given parametric space curve and provides an asymptote for each of them. We remind that the input curve $C$ is prepared such that $(0 : a : b : 0)\ (a \neq 0 \ or \ b \neq 0)$ is not an infinity point of $C^*$ (see Remark 2.3).

**Algorithm** Space Asymptotes Construction-Parametric Case.

Given a rational irreducible real algebraic space curve $C$ defined by a parametrization $P(s) = (p_1(s), p_2(s), p_3(s)) \in \mathbb{R}(s)^3, p_j(s) = p_{j1}(s)/p(s), j = 1, 2, 3$, the algorithm outputs one asymptote for each of its infinity branches.

1. Compute the Puiseux solutions of $p(s) - tp_{11}(s) = 0$ around $s = 0$. Let them be $\ell_1(t), \ell_2(t), \ldots, \ell_k(t) \subset C \ll t \gg$.

2. For each $\ell_i(t) \subset C \ll t \gg, i = 1, \ldots, k$, do:
   2.1. Compute the corresponding infinity branch of $C$:
   
   $$B_i = \{(z, r_{i,2}(z), r_{i,3}(z)) \subset \mathbb{C}^3 : z \subset \mathbb{C}, |z| > M_i\},$$
   
   where $r_{i,j}(z) = p_j(\ell_i(z^{-1})), j = 2, 3$ is given as Puiseux series (see Remark 5.1).

   2.2. Consider the series $\tilde{r}_{i,2}(z)$ and $\tilde{r}_{i,3}(z)$ obtained by eliminating the terms with negative exponent in $r_{i,2}(z)$ and $r_{i,3}(z)$, respectively. Note that, for $j = 2, 3$, the series $\tilde{r}_{i,j}$ has the same terms with non negative exponent that $r_{i,j}$, and $\tilde{r}_{i,j}$ does not have terms with negative exponent.

   2.3. Return the asymptote $\tilde{C}_i$ defined by the proper parametrization (see Lemma 4.8), $\tilde{Q}_i(t) = (t^{n_i}, \tilde{r}_{i,2}(t^{n_i}), \tilde{r}_{i,3}(t^{n_i})) \subset \mathbb{C}[t]^3$, where $n_i = \deg(B_i)$ (see Definition 4.5).

**Remark 5.2.** We note that:

1. In step 1 of the algorithm, some of the solutions $\ell_1(t), \ell_2(t), \ldots, \ell_k(t) \subset C \ll t \gg$ might belong to the same conjugation class. Thus, we only consider one solution for each of these classes.
2. Reasoning as in statement 3 in Remark 4.14, one also gets that the algorithm Space Asymptotes Construction-Parametric Case outputs an asymptote $\tilde{C}$ that is independent of the solutions $\ell_1(t), \ell_2(t), \ldots, \ell_k(t) \in \mathbb{C} \ll t \gg$ chosen in step 1 (see statement 1 above), and of the leaf chosen to define the branch $B$.

In the following example, we study a parametric space curve with only one infinity branch. We use algorithm Space Asymptotes Construction-Parametric Case to obtain the branch and compute an asymptote for it.

**Example 5.3.** Let $C$ be the space curve defined by the parametrization

$$P(s) = \left(\frac{-1+s^2}{s^3}, \frac{-1+s^2}{s^2}, \frac{1}{s}\right) \in \mathbb{R}(s)^3.$$

**Step 1:** We compute the solutions of the equation

$$p(s) - tp_{11}(s) = s^3 - t(-1 + s^2) = s^3 - ts^2 + t = 0$$

around $t = 0$. There is only one solution that is given by the Puiseux series (see Proposition 5.4)

$$\ell(t) = (-t)^{1/3} + 1/3t + 1/9(-t)^{5/3} - 2/81(-t)^{7/3} + 2/729(-t)^{11/3} + \cdots$$

(note that $\ell(t)$ represents a conjugation class composed by three conjugated series; one of them is real and the other two are complex).

**Step 2:**

**Step 2.1:** We compute (see Proposition 5.4)

$$r_2(z) = p_2(\ell(z^{-1})) = -z^{2/3} + 1/3 - 1/9z^{-2/3} + 2/81z^{-4/3} - 2/729z^{-8/3} + \cdots$$

$$r_3(z) = p_3(\ell(z^{-1})) = -z^{1/3} - 1/3z^{-1/3} + 1/81z^{-5/3} - 1/243z^{-7/3} + \cdots.$$  

The curve has only one infinity branch given by

$$B = \{(z, r_2(z), r_3(z)) : z \in \mathbb{C}, |z| > M\}$$

for some $M \in \mathbb{R}^+$ (note that this branch has three leaves; one of them is real and the other two are complex).
Step 2.2: We obtain \( \tilde{r}_2(z) \) and \( \tilde{r}_3(z) \) by eliminating the terms with negative exponent in \( r_2(z) \) and \( r_3(z) \) respectively:

\[
\tilde{r}_2(z) = -z^{2/3} + 1/3 \quad \text{and} \quad \tilde{r}_3(z) = -z^{1/3}.
\]

Step 2.3: The input curve \( C \) has an asymptote \( \tilde{C} \) at \( B \) that can be polynomially parametrized by:

\[
\tilde{Q}(t) = (t^3, \tilde{r}_2(t^3), \tilde{r}_3(t^3)) = (t^3, -t^2 + 1/3, -t).
\]

In Figure 4, we plot the curve \( C \), the infinity branch \( B \), and the asymptote \( \tilde{C} \).

![Figure 4: Curve C (left), infinity branch B (center) and asymptote \( \tilde{C} \) (right)](image)

**Correctness.**

The application of the algorithm Space Asymptotes Construction-Parametric Case presents some technical difficulties since infinite series are involved. In particular, when we compute the series \( \ell_i \) in step 1, we cannot handle its infinite terms so it must be truncated, which may distort the computation of the series \( r_{i,j} \) in step 2. However, this distortion may not affect to all the terms in \( r_{i,j} \). In fact, the number of affected terms depends on the number of terms considered in \( \ell_i \). Nevertheless, note that we do not need to know the full expression of \( r_{i,j} \) but only the terms with non negative exponent. Proposition 5.4 states that the terms with non negative exponent in \( r_{i,j} \) can be obtained from a finite number of terms considered in \( \ell_i \). In fact, it provides a lower bound for the number of terms needed in \( \ell_i \).
Proposition 5.4. Let $\ell(z) \in \mathbb{C} \ll z \gg$ be a solution obtained in step 1 of the algorithm Space Asymptotes Construction-Parametric Case. Let $B = \{(z, r_2(z), r_3(z)) \in \mathbb{C}^3 : z \in \mathbb{C}, |z| > M\}$, $r_j(z) = p_j(\ell(z^{-1}))$, $j = 2, 3$, be the infinity branch of $C$ obtained in step 2.1 of the algorithm Space Asymptotes Construction-Parametric Case. It holds that the terms with non negative exponent in $r_2$ and $r_3$ can be obtained from the computation of $2\deg(p_1) + 1$ terms of $\ell$.

Proof. We prove the proposition for $r_2$ (similarly, one gets the result for $r_3$). For this purpose, we write $\ell(z)$ as

$$\ell(z) := b_0 + b_1z^{-1/N} + \cdots + b_kz^{-k/N} + B(z), \quad B(z) = \sum_{j=1}^{\infty} a_jz^j/N, \quad N \in \mathbb{N}^+,$$

$a_i, b_i \in \mathbb{C}$, and we consider $\ell^*(z) := \ell(z^N) = \nu/z^k$ where

$$\nu := b_0z^k + b_1z^{k-1} + \cdots + b_{k-1}z + b_k + z^kB(z^N), \quad B(z^N) = \sum_{j=1}^{\infty} a_jz^{jN}.$$

Note that the terms with non negative exponent in $r_2(z)$ are the terms with non positive exponent in $r_2(1/z)$. In addition, these terms are the terms with non positive exponent in $r_2(1/z^N)$. On the other hand, $r_2(z) = p_2(\ell(z^{-1}))$ so $r_2(1/z^N) = p_2(\ell^*(z))$. Therefore, we need to determine the terms with non positive exponent in $p_2(\ell^*(z))$.

Now, we distinguish two different cases:

1. Let us assume that $\ell(z)$ has terms with negative exponent and thus, we assume w.l.o.g. that $b_k \neq 0$, $k > 0$. Thus,

$$p_2(\ell^*(z)) = \frac{p_{2,1}(\nu/z^k)}{p(\nu/z^k)} = \frac{\tilde{p}_{2,1}(z)}{z^{k(m-n)}\tilde{p}(z)}, \quad m := \deg(p_{2,1}), n := \deg(p),$$

$$\tilde{p}_{2,1}(z) = c_m\nu^m + c_{m-1}z^{k}\nu^{m-1} + c_{m-2}z^{2k}\nu^{m-2} + \cdots + c_0z^{km}, \quad c_m \neq 0$$

$$\tilde{p}(z) = d_n\nu^n + d_{n-1}z^{k}\nu^{n-1} + d_{n-2}z^{2k}\nu^{n-2} + \cdots + d_0z^{kn}, \quad d_n \neq 0.$$

Under these conditions, the generalized series expansion of $p_2(\ell^*(z))$ around $z = 0$ is given by $\frac{\tilde{p}_{2,1}(z)}{z^{k(m-n)}} G(z)$, where $G(z)$ is the Taylor series.
of $1/\bar{p}(z)$ at $z = 0$. Observe that $G(z)$ exists since all the derivatives of $1/\bar{p}(z)$ at $z = 0$ exist (note that the denominator of all the derivatives is a power of the polynomial $\bar{p}(z)$, and $\bar{p}(0) = d_n\nu(0)^n = d_nb_k^n \neq 0$). In addition, taking into account that

$$\nu^j(0) = b_{k-j}, \quad 0 \leq j \leq k,$$
and $$\nu^j(0) = a_{j-k}, \quad j \geq k + 1,$$
and that $\left. \frac{\partial^i(1/\bar{p}(z))}{\partial z^i} \right|_{z=0}$ is obtained from $\nu^i(0)$, $0 \leq i \leq j$, we get that

$$G(z) = \frac{1}{\bar{p}(0)} + z \left. \frac{\partial(1/\bar{p}(z))}{\partial z} \right|_{z=0} + \cdots = h_0(b_k) + \cdots + z^k h_k(b_k, \ldots, b_0) + z^{k+1} h_{k+1}(b_k, \ldots, b_0, a_1) + \cdots + z^{k+u} h_{k+u}(b_k, \ldots, b_0, a_1, \ldots, a_u) + \cdots,$$

where $h_j(b_k, \ldots, b_0, a_1, \ldots, a_{j-k}), j \geq 0$, denotes a rational function depending on $b_k, \ldots, b_0, a_1, \ldots, a_{k}$. As we stated above, we need to determine the terms with non positive exponent in

$$p_2(\ell^*(z)) = \frac{\bar{p}_{2,1}(z)}{z^{k(m-n)}} G(z).$$

In the following, we prove that they can be obtained by just computing $b_k, \ldots, b_0, a_1, \ldots, a_{k m}$. Indeed:

1.1. Let $m = n$. Then, we need to compute the terms with non positive exponent in

$$\bar{p}_{2,1}(z)G(z) = (c_m \nu^m + c_{m-1} z^k \nu^{m-1} + c_{m-2} z^{2k} \nu^{m-2} + \cdots + c_0 z^m)$$

$$(h_0(b_k) + \cdots + z^k h_k(b_k, \ldots, b_0) + z^{k+1} h_{k+1}(b_k, \ldots, b_0, a_1) + \cdots).$$

Thus, we only need the independent term $c_m b_k^m h_0(b_k)$.

1.2. Let $m < n$. In this case, we need to determine the terms with non positive exponent in $z^{k(n-m)} \bar{p}_{2,1}(z)G(z)$. However, since $n - m > 0$, we conclude that there are no such terms.

1.3. Let $m > n$. Then, we need to compute the terms with non positive exponent in $\bar{p}_{2,1}G/ z^{k(m-n)}$ which implies that we need to determine the terms having degree less or equal to $k(m-n)$ in the product $\bar{p}_{2,1}(z)G(z)$. Those terms are included in the product
Let us assume that \( b \) is the maximum exponent of \( k \) we have to compute \( k \) that has a negative exponent in \( \ell \). Hence, \((z^m(b_0z^k + b_1z^{k-1} + \cdots + b_{k-1}z + b_k)^m + c_{m-1}z^k(b_0z^k + b_1z^{k-1} + \cdots + b_{k-1}z + b_k)^{m-1} + \cdots + c_0z^km) \cdot (h_0(b_k) + \cdots + z^kh_k(b_k, \ldots, b_0) + z^{k+1}h_{k+1}(b_k, \ldots, b_0, a_1) + \cdots + z^{k(m-n)}h_{k(m-n)}(b_k, \ldots, b_0, a_1, \ldots, a_{k(m-n)})\)

(we do not include the term \( z^kB(z^N) \) in this product since after multiplying, it only provides terms of degree greater than \( km \)). Therefore, at most we have to compute \( \ell(z) \) till the terms \( b_k, \ldots, b_0, a_1, \ldots, a_{k(m-n)} \) appear. That is, \( k + 1 + k(m-n) \) terms are needed.

Taking into account the cases 1.1, 1.2, and 1.3, we deduce that at most we have to compute \( k + 1 + k(m-n) \) terms in \( \ell(z) \). Finally, we prove that \( k + 1 + k(m-n) \leq 2\deg(p_1) + 1 \). For this purpose, let \( d(r_2) \) denote the maximum exponent of \( z \) in \( r_2(z) \). We observe that \( d(r_2) \leq 1 \); otherwise, since

\[
F(z : r_2(z) : r_3(z) : 1) = F(z/r_2(z) : 1 : r_3(z)/r_2(z) : 1/r_2(z)) = 0
\]

(for \( |z| > M \)) by continuity, we get

\[
\lim_{z \to \infty} F(z/r_2(z) : 1 : r_3(z)/r_2(z) : 1/r_2(z)) = F(0 : 1 : C : 0) = 0
\]

where \( C := \lim_{z \to \infty} r_3(z)/r_2(z) \). If \( C \in \mathbb{C} \), we get that \( (0 : 1 : C : 0) \) is an infinity point of the input curve which is impossible since we have assumed that the input curve does not have infinity points of the form \((0 : a : b : 0)\). If \( C = \infty \), we reason as above but we divide by \( r_3(z) \).

In this case, we get the infinity point \((0 : 0 : 1 : 0)\) which is again impossible.

On the other hand, since \( r_2(z) = p_2(\ell(z^{-1})) = \frac{p_{21}(\ell(z^{-1}))}{\ell(z^{-1})} \), we get that \( d(r_2) = (m-n)k/N \), where \( m = \deg(p_{21}) \) and \( n = \deg(p) \) (see Chapter 4 in [14]). Hence, \((m-n)k/N \leq 1 \) which implies that \((m-n)k \leq N \). In addition, since \( N \leq \deg_s(p(s) - tp_{11}(s)) = \deg(p_1) \) (see Remark 4 in [4]), we get that \( k + 1 + k(m-n) \leq 2k(m-n) + 1 \leq 2\deg(p_1) + 1 \).

2. Let us assume that \( b_k = 0 \) for \( k > 0 \). That is, there are no terms with negative exponent in \( \ell(z) \). Then, we write \( \ell(z) := b_0 + B(z) \), where

\[
B(z) = \sum_{j=1}^{\infty} a_jz^{q_j/N}, \quad N \in \mathbb{N}^+, \quad q_j \in \mathbb{N}^+, \quad 0 < q_1 < q_2 < \cdots, \quad a_j \in \mathbb{C}\setminus\{0\},
\]

28
and
\[ \ell^*(z) := \ell(z^N) = b_0 + B(z^N) = b_0 + z^{q_1} \left( a_1 + \sum_{j=2}^{\infty} a_j z^{q_j - q_1} \right), \quad B(z^N) = \sum_{j=1}^{\infty} a_j z^{q_j}. \]

In this case, we denote \( \nu := b_0 + z^{q_1} (a_1 + \sum_{j=2}^{\infty} a_j z^{q_j - q_1}) \). In addition, we write
\[ p(t) = p^*(t)(t - b_0)^r, \quad \gcd(p^*(t), t - b_0) = 1 \quad \text{for some } r \in \mathbb{N}. \]

Under these conditions, we get that \( p_2(\ell^*(z)) = 0 \) for some \( r \in \mathbb{N} \).

Under these conditions, we get that \( p_2(\ell^*(z)) = 0 \) for some \( r \in \mathbb{N} \).

Under these conditions, we get that \( p_2(\ell^*(z)) = 0 \) for some \( r \in \mathbb{N} \).

Under these conditions, we get that \( p_2(\ell^*(z)) = 0 \) for some \( r \in \mathbb{N} \).

The generalized series expansion of \( p_2(\ell^*(z)) \) around \( z = 0 \) is given by \( \frac{\bar{p}_2(0)}{z^{q_1}} G(z) \), where \( G(z) \) is the Taylor series of \( 1/\bar{p}(z) \) at \( z = 0 \). Observe that \( G(z) \) exists since all the derivatives of \( 1/\bar{p}(z) \) at \( z = 0 \) exist (note that the denominator of all the derivatives is a power of the polynomial \( \bar{p}(z) \)), and \( \bar{p}(0) = p^*(0) a_1 = p^*(b_0) a_1 \neq 0 \). Reasoning as in case 1, one may check that \( G(z) = \frac{1}{\bar{p}(0)} + \frac{\bar{p}^*(1/\bar{p}(z))}{\partial z} |_{z=0} + \cdots = \)
\[ = h_0(b_0, a_1) + z h_1(b_0, a_1, a_2) + \cdots + z^k h_k(b_0, a_1, \ldots, a_{k+1}) + \cdots, \]
where \( h_j(b_0, a_1, \ldots, a_{j+1}), j \geq 0 \) is a rational function depending on \( b_0, a_1, \ldots, a_{j+1} \).

Since we need to compute the terms with non positive exponent in
\[ p_2(\ell^*(z)) = \frac{\bar{p}_2(0)}{z^{q_1}} G(z), \]
we reason as in case 1.1 (if \( r = 0 \)), or case 1.3 (if \( r > 0 \)), and we conclude that at most, we have to determine \( \ell(z) \) till the terms \( b_0, a_1, \ldots, a_{rq_1+1} \) appear. That is, in this case, at most \( rq_1 + 2 \) terms are needed. Finally, we prove that \( rq_1 + 2 \leq 2\deg(p_1) + 1 \). For this purpose, we reason as above and since

\[
    r_2(z) = \frac{p_{21}(\ell(z^{-1}))}{p(\ell(z^{-1}))} = \frac{p_{21}(\ell(z^{-1}))}{(\sum_{j=1}^{\infty} a_j z^{-q_j}/N) r^* p^*(\ell(z^{-1}))},
\]

and \( \lim_{z \to \infty} p_{21}(\ell(z^{-1}))/p^*(\ell(z^{-1})) = p_{21}(b_0)/p^*(b_0) \in \mathbb{C} \) (and thus, \( d(p_{21}(\ell(z^{-1}))) = d(p^*(\ell(z^{-1})) \)), we get that \( d(r_2) = rq_1/N \) (see Chapter 4 in [14]). Since \( d(r_2) \leq 1 \), we deduce that \( rq_1 \leq N \). In addition, since \( N \leq \deg(p_1) \) (see Remark 4 in [4]), we get that \( rq_1 \leq \deg(p_1) \), and thus \( rq_1 + 2 \leq 2\deg(p_1) + 1 \). \( \square \)

References

[1] Alonso, M.E., Mora, T., Niesi, G., Raimondo, M. (1992). Local Parametrization of Space Curves at Singular Points. Computer Graphics and Mathematics. Focus on Computer Graphics. pp: 61-90.

[2] Abhyankar, S.S., Bajaj, C. (1987). Automatic Parametrization of Rational Curves and Surfaces IV: Algebraic Curve Spaces. Computer Science Technical Reports. Paper 608.

[3] Abhyankar, S.S., Chandrasekar, S., Chandru, V. (1991). Intersection of algebraic space curves. Discrete Applied Mathematics. Vol. 31. pp: 81-96.

[4] Blasco, A., Pérez-Díaz, S. (2014). Asymptotes and Perfect Curves. Computer Aided Geometric Design. Vol. 31, Issue 2. pp: 81-96.

[5] Blasco, A., Pérez-Díaz, S. (2014b). Asymptotic Behavior of an Implicit Algebraic Plane Curve. Computer Aided Geometric Design (to appear). arxiv.org/abs/1302.2522v2.

[6] Duval, D. (1989). Rational Puiseux Expansion. Compositio Mathematica. Vol. 70. pp: 119–154.
[7] Farouki, R., (2008). Pythagorean-Hodograph Curves: Algebra and Geometry Inseparable. Series: Geometry and Computing. Vol. 1. Springer.

[8] Kečkić, J. D. (2000). A Method for Obtaining Asymptotes of Some Curves. The Teaching of Mathematics. Vol. III, 1. pp: 53-59.

[9] Manocha, D., Canny, J.F. (1991). Rational Curves with Polynomial Parametrizations. Computer Aided Design. Vol. 23/9. pp: 645-652.

[10] Maxwell, E. A. (1962). An Analytical Calculus. Vol. 3. Cambridge.

[11] Rueda, S., Sendra, J.R., Sendra, J. (2013). An Algorithm to Parametrize Approximately Space Curves. Journal of Symbolic Computation. Vol. 56. pp: 80-106

[12] Sendra, J.R., Winkler, F., Perez-Diaz, S. (2007). Rational Algebraic Curves: A Computer Algebra Approach. Series: Algorithms and Computation in Mathematics. Vol. 22. Springer Verlag.

[13] Stadelmeyer, P. (2000). On the Computational Complexity of Resolving Curve Singularities and Related Problems. Ph.D. thesis, RISC-Linz, J. Kepler Univ. Linz, Austria, Techn. Rep. RISC 00-31.

[14] Walker, R.J. (1950). Algebraic Curves. Princeton University Press.

[15] Zeng, G. (2007). Computing the Asymptotes for a Real Plane Algebraic Curve. Journal of Algebra. Vol. 316. pp: 680705.