Phase shift experiments identifying Kramers doublets in a chaotic superconducting microwave billiard of threefold symmetry

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The spectral properties of a two-dimensional microwave billiard showing threefold symmetry have been studied with a new experimental technique. This method is based on the behavior of the eigenmodes under variation of a phase shift between two input channels, which strongly depends on the symmetries of the eigenfunctions. Thereby a complete set of 108 Kramers doublets has been identified by a simple and purely experimental method. This set clearly shows Gaussian unitary ensemble statistics, although the system is time-reversal invariant.

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In a recent experiment with a superconducting microwave resonator, which possesses threefold symmetry $C_3$ symmetry), first signatures for spectral properties according to Gaussian unitary ensemble (GUE) statistics have been observed for a time-reversal invariant system $[1]$. Discrepancies between experiment and theory remained, although these results confirmed the theoretical prediction $[2]$, that time-reversal invariant systems may show properties, characteristic of violation of time-reversal invariance (TRI). A comparison with numerical results allowed to understand the discrepancies, which were due to a less than perfect identification of Kramers doublets that are supposed to display GUE statistics. The only criterion used to identify these doublets, which are split due to $C_3$ symmetry breaking perturbations, was the spacing between consecutive resonances. The results of $[3]$ were even worse and demanded for an explanation by ambitious numerical studies. In the present paper we report on a new experimental technique, which identifies the doublets properly and uniquely, and agreement between experiment and theory is obtained within statistical uncertainty.

Billiards are model systems, which allow to study the properties of classically chaotic dynamics: a point-like particle moving without losses inside a closed boundary, where it is reflected specularly at impact, possesses non-integrable equations of motion, if the boundary is suitably shaped (see, e.g., $[4]$). These systems have a quantum mechanical analogue, where the chaotic behavior manifests itself in the properties of the eigenfunctions and spectra $[5]$. It is common knowledge that the statistical fluctuations of the eigenvalues of a quantum system with a classically chaotic counterpart can be described by random matrix theory $[6, 7]$. For chaotic quantum systems one usually finds fluctuations similar to those of the Gaussian orthogonal ensemble (GOE), which reflects TRI. It has been recognized, however, that certain systems with TRI show spectral fluctuations which follow the GUE and thus mimic a violation of TRI $[2]$. A simple example for a time-reversal invariant system with GUE statistics is a two-dimensional chaotic billiard, the highest symmetry of which is $C_3$. Such a billiard has been used in $[1]$ and in the present work (see fig. 1). As was pointed out in $[1, 2]$ the Hamiltonian of this triangular billiard possesses three classes of eigenfunctions $\Psi(l)$ with different symmetry properties ($l = -1, 0, +1$). A rotation $R$ of $120^\circ$ transforms these eigenfunctions according to $R\Psi(l) = \exp(2\pi il/3)\Psi(l)$. Two of the classes of wave functions are degenerate due to TRI and one finds a spectrum composed of singlet ($l = 0$) and doublet ($l = \pm 1$) modes. The doublets were predicted to show GUE statistics, while the singlets should follow GOE statistics. In a complex representation time-reversal will transform one eigenfunction of a doublet into its complex conjugate and thus interchanges the two eigenfunctions, i.e. $\Psi(-1) = T\Psi(1) = (\Psi(1))^\ast$. This is an example for Kramers degeneracy: whenever TRI holds, i.e. $[H, T] = 0$, the time-reversed quantity $T\Psi$ of an eigenfunction $\Psi$ is also an eigenfunction of $H$. Thus, if $\Psi \neq T\Psi$, TRI forces a degeneracy $[8]$. This is known as Kramers theorem and its consequences play an important role in studies of systems with half-integer spin $[9]$. Two-dimensional quantum billiards can be studied experimentally with the help of macroscopic analog systems $[10, 11]$: in flat cylindric resonators only transverse magnetic modes exist below a critical frequency. These modes have electric field vectors perpendicular to the bottom and lid of the resonator, while the magnetic field lines lie parallel to them. The electric field $E_n(\vec{r}) = \phi_n(x, y)e_z$ as well as the magnetic field is fully described by the solutions $\phi_n(x, y)$ of the scalar Helmholtz equation

$$(\Delta + k_n^2)\phi_n = 0 \quad (1)$$

with Dirichlet boundary conditions. There is a complete analogy between the two-dimensional Helmholtz equation for a flat microwave cavity and the Schrödinger equation describing a quantum billiard of the same shape. The wave numbers $k_n$ of the electromagnetic resonator
correspond to the energy eigenvalues $\varepsilon_n$ of the Hamiltonian of the quantum billiard, i.e. $k_n^2 \propto \varepsilon_n$.

The usual way to determine the eigenfrequencies of a microwave cavity $f_n = k_n c / (2\pi)$, where $c$ denotes the speed of light, is to measure the power transmitted from one antenna emitting microwaves into the cavity to another antenna receiving the microwaves. As a result one gets a resonance line (see fig. 2), where the positions of the maxima correspond to the eigenfrequencies $f_n$, while the widths of the resonances $\Delta f_n$ reflect the damping, characterized by the quality factor $Q = f_n / \Delta f_n$. The widths of the resonances define the experimental limit of resolution. Thus, a superconducting cavity of high $Q$ is a prerequisite for resolving small splittings of degenerate eigenvalues, which stem from mechanical imperfections, and for identifying Kramers doublets properly. In our experiment a resonator made from lead plated copper was used. The measurements were carried out in a cryostat with liquid helium at a temperature of 4.2 K. The resonator becomes superconducting at temperatures below approximately 7.2 K and possesses a quality factor of the order of $10^4$. The microwave signals are processed by a HP-8510C network analyzer, which also controls the signal generator providing the microwaves. This technique has been used in several experiments [10, 11], including the first measurements with the cavity under consideration [1], and now was modified to improve the results in the following way (see fig. 1).

![FIG. 1: Schematic view of the experimental setup. The chaotic microwave cavity (left side) possesses threefold symmetry. The microwaves provided by port 1 (P1) of the network analyzer are split into two signals before they are coupled into the cavity by two different channels. The phase shift $\Delta \Phi$ between these two channels can be varied by a phase shifter. Power is coupled out by a third antenna and led back to the network analyzer, where the transmission to port 2 (P2) is measured.](image)

An additional antenna is used for feeding microwave power into the resonator. Thus, power is coupled into the system through two different channels at the same time, while the output power is coupled out through a third channel. The positions of the three antennas were chosen in a way, which preserves the threefold symmetry of the billiard. A Midwest Microwave PWD-2533-02-SMA-79 power divider splits the microwave power provided by the signal generator into two signals. These two input signals have a certain phase shift $\Delta \Phi_{\text{total}} = \Delta \Phi + \Delta \Phi_0$, (2)

where $\Delta \Phi$ can be varied by an ARRA 9428A phase shifter, while $\Delta \Phi_0$ is an offset, which is constant for a given frequency and is due to the different lengths of the two signal paths. The phase shifter and the power divider operate up to a frequency of 18 GHz. For frequencies below 6 GHz the maximum achievable $\Delta \Phi$ is less than 360°. For the frequency range from 8 GHz up to 18 GHz 50 different transmission spectra were measured, covering phase shifts from $\Delta \Phi \approx 0^\circ$ up to $\Delta \Phi \approx 350^\circ$ in steps of 7°. Data points were taken in steps of 100 kHz, i.e. a resolution of the order of $10^{-5}$, which has to be compared with $1/Q \approx 10^{-4}$. Furthermore, conventional transmission spectra - as described above - have been recorded in the frequency range from 0 to 18 GHz with a spectral resolution of 50 kHz (see fig. 2). This was done for two different antenna combinations, with a fourth antenna not used for the phase shift experiment. These transmission spectra served to identify the resonance frequencies, while the results of the phase shift measurements were utilized to identify the eigenmodes as singlets or doublets.

As we had conjectured, singlets and doublets typically show a very different behavior, when the phase shifter setting $\Delta \Phi$ is varied. For a typical singlet mode (fig. 3) the shape of the resonance line does not change, although

![FIG. 2: Part of a transmission spectrum measured at $T = 4.2 \, \text{K}$. The maxima of the resonance line mark the resonance frequencies. One clearly observes a signal-to-noise ratio of about four orders of magnitude (40 dB). Singlets and doublets are marked by S and D, resp.](image)
there is a change in the amplitude: the transmitted power at resonance depends on $\Delta \Phi$ and shows a minimum and a maximum, corresponding to destructive or constructive interferences. Because of the special symmetries of the antennas and of the singlet wave functions these should occur at a distance of $180^\circ$. However, the symmetries are not perfect and thus the minimum amplitude is non zero, but for some singlets the intensities at the maximum and the minimum differ by two orders of magnitude. Furthermore minimum and maximum do not always have an exact phase difference of $180^\circ$ as expected, but $179^\circ \pm 33^\circ$. There is a correlation between the deviation from $180^\circ$ and the minimum intensity. Large shifts only appear, when the minimum and the maximum intensity differ only marginally. The shape of the resonance, however, does not change due to those effects.

In contrast to the singlet levels, typical doublets show a strong variation of the shape of the resonance curve (fig. 4). The relative and the absolute height of each of the two peaks within a doublet depends on the phase difference and sometimes an almost complete vanishing of one of the peaks is achieved. In some cases a doublet can only be identified for certain values of the phase shift. However, the behavior of the doublet amplitudes depending on $\Delta \Phi$ is much harder to understand, as one has to deal with two eigenmodes being excited simultaneously with different amplitudes, the ratio of which depends on the properties of the two individual states involved and is uncontrollable in our setup. Nevertheless, a strong influence of the induced phase shift on the shape of a resonance line composed of a doublet is obvious (see fig. 4), which provides an important additional criterion for distinguishing doublets from singlets.

With the help of the method described above, 102 singlets and 108 doublets have been identified unambiguously in the frequency range up to 18 GHz. The sets of eigenfrequencies $f_i$ were unfolded using Weyl’s law \[ f_i : = \frac{N_{Weyl}}{2\pi} \left( f_i^2 \right) = v_1 f_i^2 + v_2 f_i + v_3 \] (3)

This procedure leads to spectra $x_i$ with normalized mean level spacings of 0.988 and 1.006 for singlets and doublets, respectively. As the number of levels in the two spectra is only about 100, we chose to analyze the long-range correlations of the spectra. Evaluating the Dyson-Mehta ($\Delta_3$) statistics \(^{13}\) yields perfect GUE behavior for the doublets (see fig. 5), including saturation \(^{13,14}\), i.e. $\Delta_3(L)$ stops increasing for $L > L_{max}$. The value of $L_{max} \approx 10$ is consistent with an estimate of the length of the shortest periodic orbit in the billiard. All this indicates that the corresponding spectrum is indeed complete. Otherwise deviations would occur, as observed in our previous work \(^1\). For the singlets we still observe deviations from GOE for small spacings (see fig. 5). As there are fewer singlets than doublets, this has to be due to missing levels. A check with a newly developed method based upon the assumptions that resonances below a certain critical strength are not detected and that the positions of the missing levels are distributed randomly indeed indicates that we lost about 5 percent of the singlets, while we have
all of the doublets [14]. It is most likely that some of the singlet modes cannot be detected with our antennas, because their positions show the same symmetry properties ($C_3$) as the singlet wave functions. We cannot circumvent this problem without destroying the $C_3$ symmetry, e.g. when inducing a perturbing body. Thus, if one antenna lies on a nodal line, all antennas do so at frequencies corresponding to singlets, while this is not the case for the doublets. These considerations are also supported by results we obtain for the Weyl coefficients $v_1$, $v_2$, and $v_3$, which were determined by a fit of eq. (3) to the experimental data. The coefficients $v_1$ and $v_2$ are connected to area $A$ and perimeter $U$, resp., of the billiard. We found $A = 0.03 \text{ m}^2$ and $U = 1.16 \text{ m}$ for the singlet spectrum and $A = 0.03 \text{ m}^2$ and $U = 0.81 \text{ m}$ for the doublets. Therefore, the Weyl parameters of the doublet spectrum are much closer to the physical dimensions of the cavity, which are approximately $A = 0.03 \text{ m}^2$ and $U = 0.82 \text{ m}$ at room temperature. Note, that for Dirichlet boundary conditions a larger perimeter $U$ corresponds to a smaller number of eigenfrequencies [12].

We can be sure, however, not to have spurious singlet levels. This was the case in [1], where we did not have a reliable method of separating the subspectra, which caused misassignments, i.e. lost and extra eigenvalues in the resulting spectra of 196 singlets and 213 doublets up to 25 GHz. Thus, these spectra showed deviations from GOE- and GUE-like statistics, respectively [1]. Also there was no saturation behavior of the Dyson-Mehta statistics (see fig. 5). Now saturation is obtained both for singlets and doublets. Although the modified setup restricts us to a smaller frequency range, it nevertheless allows a much more reliable identification and classification of states as singlets or doublets.

To conclude, the spectrum of a microwave cavity with $C_3$ symmetry shows degenerate and non-degenerate modes. Under variation of a phase shift between two feeding antennas these classes of states behave differently. Therefore, the experimental technique presented allows us for the first time to classify singlets and doublets as such. For the doublet spectrum fluctuation properties of the eigenvalues perfectly matching GUE statistics have been observed for a system with TRI – with 108 Kramers doublets only. This constitutes a clear improvement upon earlier results, where additional numerical information was needed to fully identify doublets [1] or to explain the effects of level splitting and level loss [4].

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**FIG. 5:** Long range correlations for the singlets and doublets: in the upper half the Dyson-Mehta statistics obtained in the framework of our previous work is shown. As one can see there are distinct deviations from GOE for the singlets. Also there is no saturation behavior for singlets and doublets. In the lower part the Dyson-Mehta statistics for the results of the present work are plotted. The statistics show for both singlets and doublets saturation, although the singlets still deviate slightly from GOE for small interval lengths $L$.

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