Integrability of Particle System around a Ring Source
as the Newtonian Limit of a Black Ring

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The geodesic equation in the five-dimensional singly rotating black ring is non-integrable unlike the case of the Myers-Perry black hole. In the Newtonian limit of the black ring, its geodesic equation agrees with the equation of motion of a particle in the Newtonian potential due to a homogeneous ring gravitational source. In this paper, we show that the Newtonian equation of motion allows the separation of variables in the spheroidal coordinates, providing an non-trivial constant of motion quadratic in momenta. This shows that the Newtonian limit of a black ring recovers the symmetry of its geodesic system, and the geodesic chaos is caused by relativistic effects.

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I. INTRODUCTION

The Kerr geometry in four-dimensional general relativity possesses remarkable properties, such as black hole thermodynamics and the uniqueness of stationary, axisymmetric, asymptotically flat vacuum solution of the Einstein equation with a regular event horizon (see, for example, \cite{1}). One of the most important features is the integrability of its geodesic equation. This was proven by showing the separability of the Hamilton-Jacobi equation for the geodesic system in the Boyer-Lindquist coordinates and the existence of an additional constant of motion, called the Carter constant \cite{2}. This non-trivial constant is given by a second-rank Killing tensor in the Kerr geometry \cite{3}, which is

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a generalization of Killing vectors to symmetric tensors. Consequently, the separation of variables has great advantages in studying freely falling particle motion around a Kerr black hole.

For the last few decades, many efforts are devoted to studying the higher-dimensional black hole solutions of the Einstein equation and their physical properties highly motivated by superstring theories and the scenarios with large extra dimensions (see, for example, [4]). In those works, it turned out that the stationary rotating black holes in higher dimensions do not inherit all of the properties in the four-dimensional Kerr geometry. For example, the Myers-Perry geometries [5], which describe black holes with spherical horizon topology, have no uniqueness in stationary rotating black hole solutions in higher dimensions. However, the integrability of the geodesic equation remains valid in the Myers-Perry geometry. In five dimensions, the separation of variables occurs in the Hamilton-Jacobi equation for geodesic system [6]. After these findings, the geodesic equation in the most general known vacuum black hole solution with a horizon of spherical topology is found to be integrable in all dimensions (see, for a review, [7]).

In five-dimensional spacetimes, there also exist the black hole solutions that have unusual horizon topology, called the black ring solutions [8–15]. The simplest one is the Emparan-Reall black ring geometry [8], which possesses $S^2 \times S^1$ horizon topology and rotates on its symmetric axis. Unlike the case of the Myers-Perry geometry, the Hamilton-Jacobi equation for the geodesic system in the black ring geometry is not separable at least in the ring coordinates except for special cases of geodesics [17, 18]. Although non-separability of the Hamilton-Jacobi equation is a necessary condition for the non-integrability of the geodesic equation, there exists a strong indication for the non-integrability of the geodesic equation. Indeed, the existence of chaotic behavior of geodesics in the black ring geometry was demonstrated by the Poincaré map method [24]. Recently, the absence of the Killing-Yano and conformal Killing-Yano tensors in the black ring geometry has been rigorously proved in [25]. This result indicates that the black ring geometry is less symmetric compared to the Myers-Perry geometry. However, since the existence of constants of geodesic motion is related to the Killing tensor, the mathematical proof of non-integrability of the geodesic system in black ring spacetimes still remains an open problem.

Chaos is universal phenomena in various nonlinear systems, and the appearance of geodesic chaos also is one of the important issues in general relativity. In the Majumdar-Papapetrou geometry [26, 27] that represents two-fixed extremal black holes there exist geodesic chaos [28, 33]. The results in [34] suggest that its geodesic system is chaotic in the higher-dimensional Majumdar-

\footnote{Other discussions on geodesics in the black ring geometry can be seen in [19–23].}
Papapetrou geometry as well. In the Newtonian limit of four-dimensional two-fixed black holes, the geodesic system reduces to Newtonian particle system around two-fixed centers, which is known as Euler’s three-body problem \cite{35}. This reduced system is integrable, because there exists an additional constant that is quadratic in momentum. Therefore, the geodesic chaos in the Majumdar-Papapetrou geometry is caused by relativistic effects.

Thus, a natural question arises in the study of geodesic chaos in the black ring geometry: whether or not the chaos is caused by relativistic effects. The purpose of this paper is to demonstrate that the geodesic chaos does not appear in the weak-field approximation, and thus it is generated by strong gravitational effects. In order to show this, we show the integrability of the geodesic equation in the black ring geometry in the Newtonian limit.

This paper is organized as follows. In the following section, the Newtonian limit of the geodesic equation in the singly rotating black ring geometry is demonstrated. In Sec. \textbf{III} the Newtonian potential obtained in Sec. \textbf{II} is shown to agree with the one generated by a homogeneous ring gravitational source. In Sec. \textbf{IV} the Newtonian equation of particle motion around the ring source is demonstrated to allow the separation of variables in the spheroidal coordinates, and provide an additional constant of motion that is quadratic in momentum. Finally, the implication of our results is discussed in Sec. \textbf{V}.

II. NEWTONIAN LIMIT OF THE GEODESIC EQUATION IN THE SINGLY ROTATING BLACK RING GEOMETRY

In this section, we derive a Newtonian potential from the Newtonian limit of the geodesic equation in the thin black ring geometry. The Newtonian limit of the geodesic equation requires slow motion and weak gravitational field limit. Let \( X^\mu(\tau) \) be the world line of a freely falling particle with a proper time \( \tau \). For the motion much slower than the speed of light, i.e., in the slow motion limit, the four velocity \( dX^\mu/d\tau \) is approximately one for the time component and zero for the spatial components. In the weak field limit, a metric \( g_{\mu\nu} \) is decomposed as

\[
g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{1}
\]

in global inertial coordinates, where \( \eta_{\mu\nu} \) denotes the flat metric and \( |h_{\mu\nu}| \ll 1 \). Hence the geodesic equation in the Newtonian limit becomes

\[
\frac{d^2 X^\mu}{dt^2} + \Gamma^\mu_{\nu0} = 0, \tag{2}
\]
where the second term of the Christoffel symbol in the linear approximation of $h_{\mu\nu}$ is given by

$$\Gamma^\mu_{00} = -\frac{1}{2} \eta^\mu_{\nu} \partial_\nu h_{00}. \quad (3)$$

Finally, (2) reduces to the Newtonian equation of motion of a particle subjected to the gravitational force due to the Newtonian potential that is defined as

$$\phi(r) = -\frac{1}{2} h_{00}. \quad (4)$$

Note that the time derivative of $\phi$ is neglected, because we consider a stationary spacetime.

Let us determine the explicit form of $\phi(r)$ in the case of the black ring geometry. The five-dimensional singly rotating black ring metric in the ring coordinates is of the form

$$ds^2 = -\frac{F(y)}{F(x)} \left( dt - CR \frac{1+y}{F(y)} d\psi \right)^2 + \frac{R^2 F(x)}{(x-y)^2} \left( \frac{G(y)}{F(y)} dy^2 - \frac{dy^2}{G(y)} + \frac{dx^2}{G(x)} + \frac{G(x)}{F(x)} d\phi^2 \right), \quad (5)$$

with

$$F(\chi) = 1 + \lambda \chi, \quad (6)$$
$$G(\chi) = (1 - \chi^2) (1 + \nu \chi), \quad (7)$$
$$C = \sqrt{\lambda (\lambda - \nu) \frac{1 + \lambda}{1 - \lambda}}, \quad (8)$$

where $R$, $\lambda$, and $\nu$ are constants and characterize the ring radius, rotational velocity, and thickness of the black ring horizon, respectively. From the regularity of the solutions, $\lambda$ must be

$$\lambda = \frac{2 \nu}{1 + \nu^2}, \quad (9)$$

and then the regular black ring solutions have two-free parameters $R$ and $\nu$. Note that absence of naked singularities requires the thickness parameters to be $0 < \nu < 1$.

We introduce the polar coordinates given by

$$x = \frac{R^2 - \zeta^2 - \rho^2}{\Sigma}, \quad (10)$$
$$y = -\frac{R^2 + \zeta^2 + \rho^2}{\Sigma}, \quad (11)$$

and

$$\Sigma = \sqrt{((\zeta + R)^2 + \rho^2)((\zeta - R)^2 + \rho^2)}, \quad (12)$$

where the flat limit of the metric in these coordinates is of the form

$$ds^2 = -dt^2 + d\zeta^2 + \zeta^2 d\psi^2 + d\rho^2 + \rho^2 d\phi^2. \quad (13)$$
The time-time component of the metric \([5]\) in the new coordinates is given by
\[
g_{00} = -\frac{\Sigma - \lambda (R^2 + \zeta^2 + \rho^2)}{\Sigma + \lambda (R^2 - \zeta^2 - \rho^2)}, \tag{14}\]
From \([1]\) and \([14]\), the time-time component of \(h_{\mu \nu}\) is given by
\[
h_{00} = \frac{2\lambda R^2}{\Sigma + \lambda (R^2 - \zeta^2 - \rho^2)}. \tag{15}\]
The leading order of the expansion in \(\nu\) takes the form
\[
h_{00} = \frac{2\lambda R^2}{\Sigma}. \tag{16}\]
Note that the other components of \(h_{\mu \nu}\) defined in \([1]\) must be much smaller than unity in the case \(\nu \ll 1\), because the metric \([5]\) goes to flat in the limit \(\nu \to 0\). Finally, Eqs. \([1]\) and \([16]\) lead
\[
\phi(r) = -\frac{GM}{2\Sigma}, \tag{17}\]
where
\[
M = \frac{4\nu R^2}{G}, \tag{18}\]
which is the Arnowitt-Deser-Misner mass of the singly rotating black ring geometry \([36, 37]\) linearized in \(\nu\).\(^2\) Hence, the Newtonian limit of the geodesic equation in the singly rotating thin black ring geometry reduces to the Newtonian equation of motion of a particle subjected to gravitational force due to the Newtonian potential \([17]\). Since the topology of the curvature singularity in the black ring geometry is ring shape and the horizon in this case is very thin, it is expected that the matter distribution producing \([17]\) is a ring source.

**III. NEWTONIAN POTENTIAL DUE TO A GRAVITATIONAL RING SOURCE**

In this section we provide the explicit form of the Newtonian potential due to a homogeneous ring source placed on four-dimensional flat space by integrating the basic equation of higher-

\(^2\) The choice of the gravitational coupling constant here gives the Einstein equation in the form
\[
R_{\mu \nu} - \frac{1}{2} R g_{\mu \nu} = \alpha T_{\mu \nu} \tag{19}\]
with
\[
\alpha = \frac{D - 2}{D - 3} S_{D-2} G_D, \tag{20}\]
where \(S_{D-2}\) and \(G_D\) denote the area of \((D-2)\)-sphere and gravitational constant in \(D\) dimensions, respectively, and \(D = 5\) in this case.
dimensional Newton gravity. As will be seen in the last of this section, the Newtonian potential takes the same form of the potential obtained in the previous section.

By analogy with the configuration of the curvature singularity of the black ring geometry, let us place a homogeneous ring source on a circle of radius $R$ from the coordinate origin in the two-dimensional plane with $\rho = 0$. Then the mass density is of the form

$$\sigma(r) = \frac{M}{(2\pi)^2 \zeta \rho} \delta(\zeta - R) \delta(\rho),$$

(21)

where $M$ denotes the total mass of the ring source in this section and $\delta(\cdot)$ is the delta function. The basic equation to determine the form of gravitational potential is the Newtonian gravitational field equation in higher dimensions

$$\nabla^2 \phi(r) = S_3 G \sigma(r),$$

(22)

where $\nabla^2$ denotes Laplacian of the flat space, $G$ is the gravitational constant in five dimensions. Note that this equation is consistent with the Newtonian limit of the Einstein equation given in (19) and (20) up to the numerical factor. The solution to (22) is written in the form

$$\phi(r) = -\frac{G}{2} \int_{\mathbb{R}^4} \frac{\sigma(r')}{|r - r'|^2} dV(r'),$$

(23)

where $dV$ is the volume element on $\mathbb{R}^4$. The integration of (23) with the matter distribution (21) provides the explicit form of the Newtonian potential due to the ring source as

$$\phi(r) = \frac{-GM}{2\Sigma}.$$  

(24)

Thus (24) agrees with (17) under the identification of the total mass $M$ of the homogeneous ring source as the Arnowitt-Deser-Misner mass of the black ring (13). Thus, it is shown that the gravitational field of the black ring in the Newtonian limit reduces to the Newtonian potential produced by a homogeneous ring source. The equipotential surfaces of (24), or equivalently, of (17) are shown in Fig. 1

IV. SEPARABILITY OF THE HAMILTON-JACOBI EQUATION

In this section, let us consider the Hamiltonian system of particle motion in the gravitational field due to a homogeneous ring source. In order to show the integrability of this system, the separability of the Hamilton-Jacobi equation is demonstrated in the spheroidal coordinates. As a
result, we obtain the expression for a constant of motion, which is analogous to the Carter constant in the Kerr geometry, in addition to the energy and the two angular momenta.

The Hamiltonian of the particle system in the gravitational field (24) is given by

$$H = \frac{1}{2m} \left( p_\zeta^2 + \frac{p_\psi^2}{\zeta^2} + p_\rho^2 + \frac{p_\phi^2}{\rho^2} \right) - \frac{GMm}{2 r_+ r_-},$$

(25)

where $m$ is the mass of a particle, $p_i$ is the momentum of a particle canonically conjugate to the polar coordinates, and $r_\pm$ are defined by

$$r_\pm = \sqrt{(\zeta \pm R)^2 + \rho^2}.$$

(26)

Note that the product of $r_+$ and $r_-$ agrees with (12),

$$\Sigma = r_+ r_-.$$  

(27)

The Hamiltonian shows that the Hamilton-Jacobi equation of this system does not lead to the separation of variables in the polar coordinates.

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$^3$ Particles described by (24) have stable bound orbits by a balance between centrifugal force and gravitational force due to the ring source, which is consistent to the existence of stable bound orbits at a distant region from a black ring with $\nu < 1/3$ as demonstrated in (21).
FIG. 2: Relation between the polar coordinates \((\zeta, \rho)\) and the spheroidal coordinates \((\xi, \eta)\). The solid lines show constant \(\xi\), which are confocal ellipsoids, and the dashed lines shows constant \(\eta\), which are confocal hyperbolas, where the focal point is located at \((\zeta, \rho) = (R, 0)\).

Here we introduce the spheroidal coordinates \((\xi, \eta)\) defined by

\[
\begin{align*}
\zeta &= R \xi \eta, \\
\rho &= R \sqrt{(\xi^2 - 1)(1 - \eta^2)}. 
\end{align*}
\]

The contours for \(\xi\) and \(\eta\) in \((\zeta, \rho)\)-plane are shown in Fig. 2. In the spheroidal coordinates, the four-dimensional flat metric is of the form

\[
ds^2 = R^2 (\xi^2 - \eta^2) \left( \frac{d\xi^2}{\xi^2 - 1} + \frac{d\eta^2}{1 - \eta^2} \right) + R^2 \xi^2 \eta^2 d\psi^2 + R^2 (\xi^2 - 1)(1 - \eta^2) d\phi^2. \tag{31}
\]

The Hamiltonian (25) is given in the spheroidal coordinates as

\[
H = \frac{1}{2mR^2 (\xi^2 - \eta^2)} \left[ (\xi^2 - 1) p_\xi^2 + (1 - \eta^2) p_\eta^2 \\
+ \left( \frac{1}{\eta^2} - \frac{1}{\xi^2} \right) p_\phi^2 + \left( \frac{1}{\xi^2 - 1} + \frac{1}{1 - \eta^2} \right) p_\phi^2 \right] - \frac{GMm}{2R^2(\xi^2 - \eta^2)}, \tag{32}
\]

\[4 \text{ The transformation also is written in the simple form}
\]

\[
r_\pm = R(\xi \mp \eta). \tag{28}
\]
From this form of $H$, the Hamilton-Jacobi equation is given by

\[
\frac{1}{2mR^2(\xi^2 - \eta^2)} \left[ (\xi^2 - 1) \left( \frac{\partial S}{\partial \xi} \right)^2 + (1 - \eta^2) \left( \frac{\partial S}{\partial \eta} \right)^2 \right]
+ \left( \frac{1}{\eta^2} - \frac{1}{\xi^2} \right) \left( \frac{\partial S}{\partial \psi} \right)^2 + \left( \frac{1}{\xi^2 - 1} + \frac{1}{1 - \eta^2} \right) \left( \frac{\partial S}{\partial \phi} \right)^2 - GMm^2 \right] - \frac{\partial S}{\partial t} = 0, \tag{33}
\]

where $S$ is Hamilton’s principal function. To find a complete solution to the Hamilton-Jacobi equation, $S$ is assumed to have the completely separated form

\[
S = p_\psi \psi + p_\phi \phi + S_\xi(\xi) + S_\eta(\eta) - Et, \tag{34}
\]

where $p_\psi$, $p_\phi$, and $E$ are conjugate momenta to $\psi$, $\phi$, and $t$, respectively, which are constants of motion, and $S_\xi(\xi)$ and $S_\eta(\eta)$ are the functions of $\xi$ and $\eta$, respectively, to be determined later. Substitution of (34) into (33) shows that the Hamilton-Jacobi equation allows the separation of variables such that

\[-P(\xi) = Q(\eta) = C, \tag{35}\]

where

\[
P(\xi) = (\xi^2 - 1) \left( \frac{dS_\xi}{d\xi} \right)^2 - \frac{p_\psi^2}{\xi^2} + \frac{p_\phi^2}{\xi^2 - 1} - 2mR^2\xi^2 - \frac{GMm^2}{2}, \tag{36}\]

\[
Q(\eta) = (1 - \eta^2) \left( \frac{dS_\eta}{d\eta} \right)^2 + \frac{p_\psi^2}{\eta^2} + \frac{p_\phi^2}{1 - \eta^2} + 2mR^2\eta^2 - \frac{GMm^2}{2}, \tag{37}\]

and $C$ is the constant of the separation. This additional constant of motion $C$ is analogous to the Carter constant for the geodesic system in the Kerr geometry.

As a result, the Newtonian particle system \([25]\) is separable in the spheroidal coordinates. In addition, the set of the constants of motion, $H$, $p_\psi$, $p_\phi$, and $C$, commute with each other under the Poisson bracket, which means that those constants are independent and this particle system is integrable in Liouville’s sense. Hence, it is concluded that chaotic orbits of a particle in a five-dimensional black ring is caused by relativistic effect.

For physical understanding, let us rewrite $C$ in the polar coordinates as

\[
C = L^2 + R^2 \left( p_\zeta^2 + \frac{p_\psi^2}{\zeta^2} \right) + m^2f, \tag{38}\]

where

\[
L^2 = (\zeta p_\rho - \rho p_\zeta)^2 + (\zeta^2 + \rho^2) \left( \frac{p_\psi^2}{\zeta^2} + \frac{p_\phi^2}{\rho^2} \right), \tag{39}\]

\[
f = -\frac{GM}{4} \frac{r_+^2 + r_-^2}{r_+r_-}, \tag{40}\]
Note that $L^2$ is the squared total angular momentum. Therefore, the constant $C$ is essentially interpreted as the squared total angular momentum, because in the limit of a point source, $R \to 0$, goes to central force potential, and $C$ reduces to $L^2$ except for constant shift. Note that the quadratic terms in the momenta in (38), $C^{ij}p_i p_j$, are written by the Killing tensor in the four-dimensional flat space,

$$C^{ij} = (\rho^2 + R^2) \partial_i \zeta \partial_j \zeta + \zeta^2 \partial_i \rho \partial_j \rho - 2\zeta \rho \partial_i \zeta \partial_j \rho + \frac{\zeta^2 + \rho^2 + R^2}{\zeta^2} \partial_i \psi \partial_j \psi + \frac{\zeta^2 + \rho^2}{\rho^2} \partial_i \phi \partial_j \phi. \quad (41)$$

Note that $C^{ij}$ is reducible, i.e., a linear combination of symmetric tensor product of the Killing vectors in the four-dimensional flat space, because the background is maximally symmetric, and accordingly, there exist only reducible second-rank Killing tensors in contrast to the Kerr geometry. The pair of (41) and (40) solves the Killing hierarchy equation given in [38].

V. DISCUSSION

In this paper, the integrability of the geodesic equation has been investigated in the singly rotating black ring geometry in the Newtonian limit. The geodesic equation in the thin black ring geometry reduces to the Newtonian equation of motion of a particle moving in the gravitational potential generated by a homogeneous ring source. This is consistent with the fact that the curvature singularity of the black ring has $S^1$ topology and the event horizon is thin in this case. The main result is that the Newtonian equation of particle motion allows the separation of variables in the spheroidal coordinates, and then, there exists an additional constant of motion quadratic in momentum analogous to the Carter constant in the Kerr geometry. This means that the Newtonian system of particle motion around a ring source is integrable. Hence, it is concluded that the appearance of geodesic chaos in the black ring geometry is caused by relativistic effects.

Our result implies the existence of an approximate constant for the geodesic system and an approximate Killing tensor in the thin black ring geometry. The approximate Killing tensor must be useful in the study for the physics in the black ring geometry, e.g., field dynamics and perturbation. Clarifying the condition for break down of the integrability of the geodesic system is one of the interesting issues for future work.
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