COUNTING, MIXING AND EQUIDISTRIBUTION OF HOROSPHERES IN GEOMETRICALLY FINITE RANK ONE LOCALLY SYMMETRIC MANIFOLDS

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Abstract. In this paper we study the equidistribution of expanding horospheres in infinite volume geometrically finite rank one locally symmetric manifolds and apply it to the orbital counting problem in Apollonian sphere packing.

1. Introduction

In this paper, we study the equidistribution of expanding horospheres in infinite volume geometrically finite locally symmetric rank one manifolds with respect to Burger-Roblin measure. As an application we apply it to the orbital counting of geometrically finite groups.

A priori it is not clear how to count the growth of the number of orbit points in $\mathbb{F}^{n+1}$ under the infinite co-volume group $\Gamma \subset O_F(n,1)$ where $O_F(n,1) = SO(n,1), SU(n,1), Sp(n,1)$ depending on $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. If $Q_F(v_0) = 0$ for a signature $(n,1)$ quadratic form $Q_F$, we give a quantitative estimate of the asymptotic growth

$$\# \{ v \in v_0\Gamma \mid ||v|| < K \}$$

for any norm $||\cdot||$ on $\mathbb{F}^{n+1}$ with the control of the error term. Controlling the error term is crucial to our application to the counting of prime curvature spheres in Apollonian sphere packing. This orbital counting theorem follows from the equidistribution of expanding closed horospheres in geometrically finite groups. First we show that for any $\psi \in C_c^\infty(\Gamma \backslash G)^K$, the average of this function on a horosphere of height $y$ can be explicitly estimated in terms of $L^2$-product of $\psi$ and $\phi_0$, and $y^{D-\delta}$ where $\phi_0$ is a unique eigenfunction of the Laplace operator with $L^2$-norm 1 of eigenvalue $\delta(D - \delta)$, and $\delta$ is the critical exponent of $\Gamma$, $D$ is the Hausdorff dimension of the ideal boundary of the associated symmetric space. See Theorem 9.2.

The techniques involve the unitary representation theory, measure theoretic approach in algebraic Lie groups, Patterson-Sullivan measure on limit
sets and some geometrical insights in rank one space. We carry out the
computation in explicit coordinates, so-called horospherical coordinates in
rank one spaces. See sections 2 and 3. We outline necessary backgrounds as
the proof evolves in coming sections.

Let \( X \) be a real, complex or quaternionic hyperbolic space with curvature
between \(-4\) and \(-1\), and \( \Gamma \subset Iso(X) = G \) a geometrically finite group whose
critical exponent is \( \delta > D/2 \) where \( D \) is a Hausdorff dimension of \( \partial X \). Let
\( G = KAN \) be a fixed Iwasawa decomposition introduced in section 2. In
other words, \( N \) is a (generalized) Heisenberg group, \( A \) is a one-parameter
subgroup stabilizing a chosen geodesic \((0,0,y)\) in horospherical coordinates
(see section 2, 3), and \( K \) is a maximal compact subgroup stabilizing \((0,0,1)\).

The main theorem is:

**Theorem 1.1.** Let \( \Gamma \) be a geometrically finite discrete subgroup in
\( SO(n,1), SU(n,1), Sp(n,1) \) with the critical exponent \( \delta > D/2 \) where \( D \) is the Hausdorff
dimension of the boundary of the associated symmetric spaces of the sectional
curvature between \(-4\) and \(-1\). Suppose \( v_0 \) is in the light cone such that
\( v_0 \Gamma \) is discrete, and the stabilizer of \( v_0 \) in \( g_0^{-1}\Gamma g_0 \) is in \( NM \). Then for any norm
\( \| \cdot \| \) on \( \mathbb{R}^{n,1} \)

\[
\# \{ v \in v_0 \Gamma : ||v|| < T \} \sim c_{\phi_0} \delta^{-1} T^\delta \int_K \|v_0(g_0^{-1}\kappa g_0)\|^{-\delta} dk.
\]

If \( \| \cdot \| \) is \( g_0^{-1}Kg_0 \)-invariant, then

\[
\# \{ v \in v_0 \Gamma : ||v|| < T \} = c_{\phi_0} \delta^{-1} T^\delta ||v_0||^{-\delta} (1 + O(T^{-\delta'})).
\]

Here \( \delta' \) depends only on the spectral gap.

This theorem is a generalization of [23] to a general rank one case.

**Notation:** \( f(x) = O(x) \) implies that \( \lim_{x \to \infty} \frac{f(x)}{x} < \infty \).

**Outline of a proof:** One introduces a continuous counting function
\( F_T(g) = \sum_{\gamma \in (\Gamma \cap NM)\backslash \Gamma} \chi_{B_T}(v_0\gamma g) \), where \( B_T = \{ v \in v_0 G : ||v|| < T \} \).
Specially \( F_T(e) \) is the orbital counting function of \( \Gamma \). Let \( \phi_\epsilon \in C_c(\Gamma) \) be a
nonnegative function supported on a small neighborhood \( U_\epsilon \) of the identity
element with \( \int_G \phi_\epsilon = 1 \). Define a function defined on \( \Gamma \backslash G \) (i.e., invariant
under left \( \Gamma \)-action) by

\[
\Phi_\epsilon(\Gamma g) = \sum_{\gamma \in \Gamma} \phi_\epsilon(\gamma g).
\]

One is interested to estimate

\[
\langle F_T, \Phi_\epsilon \rangle = \int_G \chi_{B_T}(v_0 g) \Phi_\epsilon(g) dg
\]

\[
= \int_{M \cap K} \int_{y > T^{-1} ||v_0 k||} \int_{x \in (\Gamma \cap NM) \backslash NM/M} \psi_k(n_x a_y) dny^{-D-1} d\kappa dk
\]

where \( \psi_k \) is an average of \( \Phi_\epsilon \) over a compact group \( M \subset K \) in Langland
decomposition and \( (\Gamma \cap NM) \backslash NM/M \) denotes the projection into \( \Gamma \backslash G/M \).
Geometrically \((\Gamma \cap NM) \setminus NM/M\) denotes the quotient of a horosphere based at \(\infty\) by the action of \(\Gamma \cap NM\). See Sections 2 and 10 for details.

So it is important to estimate the average of a function over a horosphere at height \(0 < y < 1\),

\[
\int_{n_x \in (\Gamma \cap NM) \setminus NM/M} \psi(n_x a_y) dn \sim y^{D-\delta}
\]

for \(\psi \in L^2(\Gamma \setminus G)^K\), from which one deduces that \(F_T(e) \sim T^\delta\). The large part of the paper is devoted to justify this.

First using a spectral gap, one can decompose \(L^2(\Gamma \setminus G)^K\) into \(V_\delta\) where the bottom eigenfunction \(\phi_0\) is a unique \(K\)-invariant function, and the rest \(V\) where \(V\) does not contain any complementary series \(V_s, s \geq s_\Gamma\). So any \(\phi \in C^\infty(\Gamma \setminus G)^K \cap L^2(\Gamma \setminus G/M)\) can be written as

\[
\phi = \langle \phi, \phi_0 \rangle \phi_0 + \phi^\perp
\]

and hence \(\langle a_y \phi_1, \phi_2 \rangle = \langle \phi_1, \phi_0 \rangle \langle a_y \phi_0, \phi_2 \rangle + O(y^{D-\delta} S_{2m}(\phi_1) S_{2m}(\phi_2))\). See Corollary 8.2.

Secondly, we fix a positive \(\eta \in C^\infty_c((\Gamma \cap NM) \setminus NM/M)\) with \(\eta = 1\) on a bounded open set \(B\) of \((\Gamma \cap NM) \setminus NM/M\) and vanishes outside a small neighborhood of \(B\). Also for each \(\epsilon < \epsilon_0\), let \(r_\epsilon\) be a nonnegative smooth function on \(AN^-M\) whose support is contained in

\[
W_\epsilon = (U_\epsilon \cap A)(U_{\epsilon_0} \cap N^-)(U_{\epsilon_0} \cap M)
\]

and \(\int_{W_\epsilon} r_\epsilon d\nu = 1\). Finally define \(\rho_{\eta,\epsilon}\) on \(\Gamma \setminus G\) which vanishes outside \(\text{supp}(\eta)U_{\epsilon_0}\) and for \(g = n_x a_n^{-m} \in \text{supp}(\eta)W_\epsilon\),

\[
\rho_{\eta,\epsilon}(g) = \eta(n_x) r_\epsilon(a_n^{-m}).
\]

This \(\rho_{\eta,\epsilon}\) is introduced as a cut-off function to estimate the average of any function \(\psi\) in \(C^\infty_c(\Gamma \setminus G)^K\) over a horosphere at height \(y\) in terms of \(\epsilon\) and an \(L^2\) inner product of \(a_y \psi\) and \(\rho_{\eta,\epsilon}\). See Propositions 6.3 and 6.4. Finally in Theorem 9.2, one iterates the process to obtain the dominant term as the average of \(\phi_0\) over the horosphere at height \(y\), which is \(y^{D-\delta}\).

As an application to an Apollonian sphere packing in \(\mathbb{R}^3\), we have

**Corollary 1.2.** Given a bounded Apollonian sphere packing \(\mathcal{P}\) in \(\mathbb{R}^3\), the number of spheres whose curvatures are less than \(T\) grows asymptotically \(\sim c T^\alpha\) for some positive \(\alpha\) which is the Hausdorff dimension of the residual set.

Specially for asymptotic growth of the number of \(k\)-mutually tangent spheres in \(\mathbb{R}^3\) with prime curvatures, we obtain:

**Theorem 1.3.** Given a bounded primitive integral Apollonian sphere packing \(\mathcal{P}\) in \(\mathbb{R}^3\), if \(\pi_k^\mathcal{P}(T)\) denotes the number of \(k\)-mutually tangent spheres
whose curvatures are prime numbers less than $T$, then

$$\pi_k^P(T) \ll \frac{T^\alpha}{(\log T)^k}$$

for $k \leq 5$.

Some of the lower bound seems to be known by experts like Sarnak [33, 34] for Apollonian circle packing. One can attempt a lower bound for $r$-almost prime curvature spheres as follows. The following corollary is a counting of 5 spheres kissing each other with certain properties, not the counting of individual sphere with prime curvature. See the last section.

**Corollary 1.4.** Given a bounded primitive integral Apollonian sphere packing $\mathcal{P}$ in $\mathbb{R}^3$, let $\pi_k^P(T)_r$ denote the number of 5 spheres kissing each other (i.e. the number of orbits) among whose at least $k$ curvatures are $r$-almost primes and all of whose curvatures are less than $T$ where $r$ is a fixed positive integer depending only on Apollonian packing. For any $k \leq 5$,

$$\frac{T^{\delta_r}}{(\log T)^k} \ll \pi_k^P(T)^r.$$  

For abundant literatures of this subject, see [16] for example.

**Organization of the paper:** In sections 2 and 3, we give preliminary backgrounds for rank one symmetric space and specially introduce horospherical coordinates to calculate the Buseman function explicitly. In section 6 we deal with the bottom eigenfunction and its average over horospheres. This part is essential for our orbital counting problem. In section 7 we recall unitary representations of rank one semisimple Lie groups and generalize Shalom’s result on matrix coefficients of spherical unitary representations of rank one group. In section 8 we give some consequences of unitary representations using spectral gap theorem due to Hamenst"adt. After section 9 we closely follow the proofs of [23]. We record them for the reader’s convenience. In the final section, we are concerned with an integral Apollonian sphere packing, and derive some asymptotic growth on the number of prime curvature spheres using uniform spectral gap theorem due to Bourgain-Gamburd-Sarnak, Varju-Salehi Gosefidy, Breuillard-Green-Tao and Pyber-Szabo.

## 2. Preliminaries

The symmetric spaces of $\mathbb{R}$-rank one of non-compact type are the hyperbolic spaces $H^n_F$, where $F$ is either the real numbers $\mathbb{R}$, or the complex numbers $\mathbb{C}$, or the quaternions $\mathbb{H}$, or the Cayley numbers $\mathbb{O}$; in the last case $n = 2$. They are respectively called as real, complex, quaternionic and octonionic hyperbolic spaces (the latter one $H^2_O$ is also known as the Cayley hyperbolic plane). Algebraically these spaces can be described as the corresponding quotients: $SO(n,1)/SO(n)$, $SU(n,1)/SU(n)$, $Sp(n,1)/Sp(n)$ and
$F_{4}^{-20}/\text{Spin}(9)$ where the latter group $F_{4}^{-20}$ of automorphisms of the Cayley plane $H_{\mathbb{O}}^{2}$ is the real form of $F_{4}$ of rank one.

Following Mostow [28] and using the standard involution (conjugation) in $\mathbb{F}$, $z \to \bar{z}$, one can define projective models of the hyperbolic spaces $H_{\mathbb{F}}^{n}$ as the set of negative lines in the Hermitian vector space $\mathbb{F}^{n,1}$, with Hermitian structure given by the indefinite $(n,1)$-form

$$(z,w) = z_{1}w_{1} + \cdots + z_{n}w_{n} - z_{n+1}w_{n+1}.$$ 

However, it does not work for the Cayley plane since $\mathbb{O}$ is non-associative, and one should use a Jordan algebra of $3 \times 3$ Hermitian matrices with entries from $\mathbb{O}$ whose group of automorphisms is $F_{4}$, see [28].

Another models of $H_{\mathbb{F}}^{n}$ use the so called horospherical coordinates based on foliations of $H_{\mathbb{F}}^{n}$ by horospheres centered at a fixed point $\infty$ at infinity $\partial H_{\mathbb{F}}^{n}$ which is homeomorphic to $(n \dim_{\mathbb{R}} \mathbb{F} - 1)$-dimensional sphere. Such a horosphere can be identified with the nilpotent group $N$ in the Iwasawa decomposition $KAN$ of the automorphism group of $H_{\mathbb{F}}^{n}$. The nilpotent group $N$ can be identified with the product $\mathbb{F}^{n-1} \times \text{Im} \mathbb{F}$ (see [28]) equipped with the operations:

$$(\xi, v) \cdot (\xi', v') = (\xi + \xi', v + v' - 2 \text{Im} \langle \xi, \xi' \rangle) \quad \text{and} \quad (\xi, v)^{-1} = (-\xi, -v),$$

where $\langle \cdot, \cdot \rangle$ is the standard Hermitian product in $\mathbb{F}^{n-1}$, $\langle z, w \rangle = \sum z_{i}w_{i}$. The group $N$ is a 2-step nilpotent Carnot group with center $\{0\} \times \text{Im} \mathbb{F} \subset \mathbb{F}^{n-1} \times \text{Im} \mathbb{F}$, and acts on itself by the right translations $T_{h}(g) = g \cdot h, h, g \in N$.

Now we may identify

$$H_{\mathbb{F}}^{n} \cup \partial H_{\mathbb{F}}^{n} \setminus \{\infty\} \longrightarrow N \times [0, \infty) = \mathbb{F}^{n-1} \times \text{Im} \mathbb{F} \times [0, \infty),$$

and call this identification the “upper half-space model” for $H_{\mathbb{F}}^{n}$ with the natural horospherical coordinates $(\xi, v, u)$. In these coordinates, the above right action of $N$ on itself extends to an isometric action (Carnot translations) on the $\mathbb{F}$-hyperbolic space in the following form:

$$T_{(\xi_{0}, v_{0})} : (\xi, v, u) \longmapsto (\xi_{0} + \xi, v_{0} + v - 2 \text{Im} \langle \xi, \xi_{0} \rangle, u),$$

where $(\xi, v, u) \in \mathbb{F}^{n-1} \times \text{Im} \mathbb{F} \times [0, \infty)$. Hence rank one space $H_{\mathbb{F}}^{1}$ can be written in horospherical coordinates as

$$\mathcal{H}_{\mathbb{F}}^{1} = \{(z, t, y) | z \in \mathbb{F}^{n-1}, t \in \text{Im} \mathbb{F}, 0 < y \in \mathbb{R}\}.$$ 

An hyperbolic isometry fixing 0 and $\infty$ acts as

$$(z, t, y) \rightarrow (rzO_{\mathbb{F}}, r^{2}t, r^{2}y)$$

where $O_{\mathbb{F}}$ is an element of $M$. Here the group $MA$ fixing the origin of $N$ and $\infty$ is $(U(n-1) \cdot U(1)) \times \mathbb{R} \in H_{\mathbb{F}}^{n}$, $(Sp(n-1) \cdot Sp(1)) \times \mathbb{R} \in H_{\mathbb{H}}^{n}$, and $\text{Spin}(7) \times \mathbb{R} \in H_{\mathbb{R}}^{3}$. Note that once the height $y$ is fixed, which is a horosphere of $H_{\mathbb{F}}^{n}$, $NM$ leaves invariant the horosphere, i.e., $NM$ acts on each orbit of $N$ by

$$(O_{\mathbb{F}}, (\chi, v, 0)) : (z, t, y) \rightarrow ((\chi + z)O_{\mathbb{F}}, v + t - 2 \text{Im} \langle t, \chi \rangle, y).$$
Later when we use the notation \((\Gamma \cap NM) \backslash NM/M\), it denotes the projection into \(\Gamma \backslash G/M\). Since \(G/M\) is the unit tangent bundle \(T^1(G/K)\), \((\Gamma \cap NM) \backslash NM/M\) is the quotient of a horosphere under the action of \(\Gamma \cap NM\).

It is not difficult to show that \((dt - 2 \text{Im} \langle z, dz \rangle)^2 + \langle dz, dz \rangle\) is invariant under \(NM\) where \(\langle , \rangle\) is the standard positive definite Hermitian product and \(ANM\) is a Borel subgroup fixing \(\infty\). Since a metric on \(H^n_{\mathbb{F}}\) can be written as \(g_y \oplus dy^2\) where \(g_y\) is defined on a horosphere along a geodesic ending \(\infty\), one can give a metric \[ ds^2 = \frac{dy^2 + (dt - 2 \text{Im} \langle z, dz \rangle)^2 + 4y \langle dz, dz \rangle}{y^2} . \]

This metric has the sectional curvature between \(-1\) and \(-\frac{1}{4}\) so that the volume entropy for real hyperbolic \(n\)-manifold is \((n - 1)/2\), the volume entropy of complex hyperbolic \(n\)-manifold is \(n\) and the volume entropy of quaternionic hyperbolic manifold is \(2n + 1\).

The volume form on \(H^n_{\mathbb{F}}\) can be written as

(1) \[ \frac{2^{n-1}}{y^{(n+1)/2}} dVol_z dy \] (real hyperbolic)

(2) \[ \frac{4^{n-1}}{y^{n+1}} dVol_z dt dy \] (complex hyperbolic),

(3) \[ \frac{16^{n-1}}{y^{2n+2}} dVol_z dt dy \] (quaternionic hyperbolic)

where \(dVol_z\) is a volume form on \(\mathbb{F}^{n-1}\). The set \(\{x = (z,t)\}\) is identified with the Nilpotent group \(N\) in Iwasawa decomposition \(ANK\) of \(\text{Iso}(H^n_{\mathbb{F}})\) and its right action is

\[ (z,t,y)(w,s) = (z + w, t + s - 2 \text{Im} \langle z, w \rangle, y) . \]

The ideal boundary at infinity of \(H^n_{\mathbb{F}}\) is \(N \cup \infty\). As usual \(A\) will be an one-dimensional group translating along \(\{(0,0,y)|y > 0\}\) and its right action is given by \((z,t,y)a_r = (rz, r^2t, r^2y)\) so that any point \((z,t,y) \in H^n_{\mathbb{F}}\) is

\[ (0,0,1)(\frac{1}{\sqrt{y}}z, \frac{1}{y}t)a_{\sqrt{y}} . \]

In this way we will identify a point \((z,t,y)\) in \(H^n_{\mathbb{F}}\) with \([((\frac{1}{\sqrt{y}}z, \frac{1}{y}t), a_{\sqrt{y}})] \in N \times A\) in a fixed Iwasawa decomposition \(KNA\).

Note that the metric \(g_y \oplus dy^2\) is such that \(g_y = e^{2y}g_0 \oplus e^{4y}g_0\) on \(\mathbb{F}^{n-1}\) part and \(\text{Im}\mathbb{F}\) part, respectively, \(e^{-2y}g_y\) converges to nonriemannian metric, Carnot-Carathéodory metric on \(N\). It’s distance is given by

\[ d_N((x,t),(w,s)) = |(x,t)(w,s)^{-1}| = |x - w|^4 + (t - s + 2 \text{Im} \langle x, w \rangle) |^2 \]

One can even define \(N\)-invariant metric on horospherical model \(\mathcal{H}^n\) by

\[ d((x,t,y),(w,s,z)) = |x - w|^4 + (t - s + 2 \text{Im} \langle x, w \rangle) |^2 + |y - z|^2 \]
3. Busemann function and Horosphere

In this section we normalize the metric so that the sectional curvature is between $-1$ and $-\frac{1}{4}$ and we fix a reference point $o = (0,0,1)$. $H^p_0$ can be realized a unit ball $B^n$ in $\mathbb{F}^n$. Two points $(0',-1)$ and $(0',1)$ will play a special role. There is a natural map from $B^n$ to $\mathbb{P}(\mathbb{F}^n,1)$, where $\mathbb{F}^n,1$ is equipped with $\langle z,w \rangle = z_1w_1 + \cdots + z_nw_n - z_{n+1}w_{n+1}$.

defined as $(w',w_n) \mapsto (w',w_n,1)$.

From $B^n$ to the horospherical model $H^n$, one define the coordinates change as $(z',z_n) \mapsto (\frac{z'}{1+z_n}, \frac{2\text{Im}z_n}{|1+z_n|^2}, \frac{1-|z_n|^2-|z'|^2}{|1+z_n|^2})$.

It's inverse from the horospherical model $H^n$ to $\mathbb{P}(\mathbb{F}^n,1)$ is given by $(\xi,v,u) = [(\xi, 1 - |\xi|^2 - u + v, 1 + |\xi|^2 + u - v)\frac{2}{2}]$, where $v$ is pure imaginary, i.e., $iv$ in complex case, and $iv_i + jv_2 + kv_3$ in quaternionic case. According to this coordinate change, $(0',-1) = [(0',-1,1)]$ to $\infty$, and $(0',0)$ to $(0,0,1)$.

In the rest of the section, we carry out the calculations only in complex hyperbolic space but it goes through the quaternionic case. If $x,y \in H^p_0$ and $X,Y \in \mathbb{C}^{n,1}$ correspond to $x,y$, the distance between them is

$$\cosh^2 \left( \frac{d(x,y)}{2} \right) = \frac{\langle X,Y \rangle \langle Y,X \rangle}{\langle X,X \rangle \langle Y,Y \rangle}.$$ 

where $\langle X,Y \rangle = \sum_{i=1}^{n} x_i\bar{y}_i - x_{n+i}\bar{y}_{n+1}$. A Busemann function based at $\xi$ is defined as

$$B_\xi(z) = \lim_{t \to \infty} (d(z,\gamma_t) - t)$$

where $\gamma_t$ is a unit speed geodesic starting from $o$ and ending $\xi \in \partial H^p_0$. First we calculate the Busemann function based at $\infty = (0',-1,1)$. $\gamma_t$ can be chosen as

$$\gamma(t) = (0',-\tanh(t/2),1) \subset \mathbb{C}^{n,1}.$$ 

Then a straightforward calculation shows that

$$d(z,\gamma_t) \to t + \log \frac{|z_n + 1|^2}{1 - \langle z,z \rangle},$$

where $z = (z',z_n) \in B^n$ and $\langle \langle z,w \rangle \rangle = \sum z_i\bar{w}_i$ is the standard positive definite Hermitian product. We denote it as a double bracket whereas $\langle \ , \ \rangle$ is $(n,1)$ Hermitian product on $\mathbb{C}^{n,1}$. Then

$$B_\infty(z) = \log \frac{|z_n + 1|^2}{1 - \langle z,z \rangle}.$$
If we denote $Q = (0', -1, 1)$ and $Z = (z, 1)$ in $\mathbb{C}^{n,1}$, it is easy to show that
\begin{equation}
 e^{-B_Q(z)} = \frac{-\langle Z, Z \rangle}{\langle Z, Q \rangle \langle Q, Z \rangle}.
\end{equation}

Using elliptic isometry fixing $o = (0, 0, 1) \in \mathcal{H}^n$, one can send $Q$ to any other point in the ideal boundary. Since it preserves Busemann function and Hermitian inner product, above formula holds for any $Z, Q$. Using this formula, it is easy to check that $e^{-B_\infty(z)} = y$ for $z = (\xi, v, y)$ in horospherical coordinates. This explains the last height coordinate in horospherical coordinates.

We are interested in $e^{-B_Q(z)}$ for $Q \neq \infty$. Let $Q = (\xi, v)$ and $z = (x, t, y) \in \mathcal{H}^n$. These will correspond in $\mathbb{C}^{n,1}$ to $(x, \frac{1 - |x|^2 - y + it}{2}, \frac{1 + |x|^2 + y - it}{2})$ and $(\xi, \frac{1 - |\xi|^2 + iv}{2}, \frac{1 + |\xi|^2 - iv}{2})$ respectively. A direct calculation using equation (4) gives
\begin{equation}
 e^{-B_Q(z)} = \frac{4y}{|2\langle x, \xi \rangle - |x|^2 - y + i(t - v)|^2}.
\end{equation}

Using the relation $\langle x - \xi, x - \xi \rangle = |x - \xi|^2 = |x|^2 + |\xi|^2 - 2\text{Re}(\langle x, \xi \rangle)$, above formula becomes
\begin{equation}
 e^{-B_Q(z)} = \frac{4y}{(|x - \xi|^2 + y)^2 + (t - v + 2\text{Im}(\langle x, \xi \rangle))^2}.
\end{equation}

Note that the denominator is comparable with the distance defined in section 2

4. Opposite Nilpotent group

We fixed an Iwasawa decomposition $ANK$ so that $N$ is a 2-step nilpotent group which is a Heisenberg group. To describe an opposite Nilpotent group $N^-$ for later use, we introduce another coordinates changes. To define a Hermitian form of signature $(n, 1)$ one can equally use
\[
\begin{pmatrix}
0 & 0 & 1 \\
0 & I_n & 0 \\
1 & 0 & 0
\end{pmatrix}
\] for the product. Somehow this matrix simplifies the calculations the most. In this context, a coordinate change $\phi$ from the Horospherical model $\mathcal{H}^n$ to $\mathbb{P}(\mathbb{F}^{n,1})$ is given by
$$
\phi(\zeta, v, u) = \left[(-|\zeta|^2 - u + v)/2, \zeta, 1\right]
$$
see [21].

One can easily show that the right translation by $(\tau, t)$, i.e., the map $(z, s) \rightarrow (z, s)(\tau, t)$ corresponds to matrix multiplication
\[
\begin{pmatrix}
-\frac{|z|^2 + s}{2} & z & 1 \\
-\frac{|\tau|^2 + t}{2} & \tau & 1
\end{pmatrix} \begin{pmatrix}
1 & 0 & 0 \\
-\tau^* & I & 0 \\
-\tau^* & 0 & \tau
\end{pmatrix}.
\]
Since we consider right actions instead of left actions, we can define the opposite Nilpotent group $N^-$ as

$$N^- = \{ \begin{bmatrix} 1 & 0 & 0 \\ -\tau^* & I & 0 \\ -r^2 + t & \tau & 1 \end{bmatrix} \}$$

whereas $N$ consists of upper triangular matrices. Then using the multiplication rules in section 2 one can show that

$$n_xa_y = a_y n_x/y, n^-x a_y = a_y n^-x.$$  

This will be used in section 6. In these coordinates, the origin of the Heisenberg group corresponds to $(0, 0, 1)$ in $\mathbb{P}^n$ and it is the stabilized by the right action of $NM$. This fact will be used in section 10.

5. Digression to a general Riemannian geometry

Let $M$ be a Riemannian manifold, $u$ a function such that $|\nabla u| = 1$. Then

**Proposition 5.1.**

$$\Delta u(x) = \text{mean curvature at } x \text{ of } u^{-1}(u(x)).$$

If $u$ is a Busemann function on the rank one symmetric space $X$, $u^{-1}(u(x))$ is a horosphere and $\Delta u(x) = D$ where $D$ is the Hausdorff dimension of $\partial X$.

**Proof.**

$$\Delta u = \text{div} \nabla u = \text{div} \xi, \xi = \nabla u.$$  

Since $|\xi| = 1$, $\langle \nabla_v \xi, \xi \rangle = 0$ for any $v \in TM$. Also since $\nabla \xi \in \text{End}(TM)$ is symmetric (the antisymmetric part of $\nabla \xi$ is $d(du)$),

$$\langle \nabla_v \xi, w \rangle = \langle \nabla_w \xi, v \rangle,$$

hence $\langle \nabla_v \xi, w \rangle = \langle \nabla_u \xi, \xi \rangle = 0$ for any $w \in TM$. So $\nabla u \xi = 0$.

Let $e_i$ be orthonormal basis of $T_xM$ with $e_1 = \xi$. Then

$$\text{div} \xi = -\sum_{i=1}^n \langle \nabla e_i, e_i \rangle = -\sum_{i=2}^n \langle \nabla e_i, e_i \rangle = -\sum_{i=2}^n II(e_i, e_i)$$

where $II$ is a second fundamental form on $u^{-1}(u(x))$. The second claim is well-known [1] (page 638-639).

Note that for $f : \mathbb{R} \to \mathbb{R}$ and $u : M \to \mathbb{R}$, using $\text{div}(gV) = \langle \nabla g, V \rangle + g \text{div} V$ for a smooth function $g$ and a vector field $V$ on $M$,

$$\Delta(f(u)) = f''(u) |\nabla u|^2 + f'(u) \Delta u,$$

and if $|\nabla u| = 1$, $\Delta(f(u)) = f''(u) + f'(u) \Delta u$.

For example if $X$ is a symmetric space of rank one and $u$ is a distance to a fixed hyperplane, $\Delta u = \phi(u)$ where $\phi$ is a solution to

$$\phi(u) = \text{Tr}(II(u))$$
where $II: \mathbb{R} \to (n-1) \times (n-1)$ symmetric matrices satisfying $II' + II^2 + Riem = 0$. Here $TX = \mathbb{F} \nabla u \oplus (\mathbb{F} \nabla u)^\perp$ and $(\nabla u)^\perp = (\text{Im}\mathbb{F}) \nabla u \oplus (\mathbb{F} \nabla u)^\perp$. $Riem$ preserves this decomposition and

$$Riem = -4\text{Id} \text{ on } (\text{Im}\mathbb{F}) \nabla u,$$

$$= \text{Id} \text{ on } (\mathbb{F} \nabla u)^\perp.$$ 

The solution of $II' + II^2 + Riem = 0$ is

$$\phi_1(u)\text{Id}(\text{Im}\mathbb{F}) \nabla u + \phi_2(u)\text{Id}(\mathbb{F} \nabla u)^\perp$$

where $\phi_1' + \phi_1^2 - 4 = 0$ and $\phi_2' + \phi_2^2 - 1 = 0$. Then $\phi_1(u) = \frac{2}{\tanh(2u)}, \phi_2(u) = \tanh(u)$.

In conclusion

$$\triangle u = \phi(u) = Tr(II(u)) = (\text{dim}\mathbb{F} - 1)\frac{2}{\tanh(2u)} + (\text{dim}\mathbb{F})(n-1)\tanh(u).$$

For real hyperbolic 3-dimensional case, if $u$ is a distance to $H^2_{\mathbb{R}}, \triangle u = 2\tanh(u)$. So if $\phi$ is an average of the bottom eigenfunction over $H^2_{\mathbb{R}}$ at the distance $u$,

$$\triangle \phi = \phi''(u) + \phi'(u)2\tanh(u)$$

and since

$$\triangle \phi = -\delta(2-\delta)\phi$$

$\phi(u)$ grows asymptotically

$$ce^{-(2-\delta)u} + de^{-\delta u}.$$ 

If $u$ is a Busemann function on the symmetric space $X$ of rank one, then by equation (6) and $\triangle u = D$

$$\triangle(f(u)) = f''(u) + Df'(u),$$

where $D$ is a Hausdorff dimension of $\partial X$. Note that the solution space of the above differential equation is generated by $e^{-cu}$ and $e^{-(D-c)u}$ for some $c$. Hence if $\phi_0^N$ is an average of the bottom eigenfunction $\phi_0$ on a geometrically finite manifold $\Gamma \backslash X$ over a horosphere at the distance $u$,

$$\phi_0^N(u) = c_{\phi_0}e^{-(D-\delta)u} + d_{\phi_0}e^{-\delta u}.$$ 

But in later sections, we will use $-\triangle$ as the Laplace operator so that

$$\phi_0^N(u) = c_{\phi_0}e^{(D-\delta)u} + d_{\phi_0}e^{\delta u}.$$ 

In section 6 we show that $c_{\phi_0} \neq 0$. If $\Gamma_0 \subset \Gamma$ is a subgroup of index $n$, then $1/\sqrt{n}\tilde{\phi}_0$ is a unit $L^2$ norm eigenfunction of $-\triangle$ with eigenvalue $\delta(D-\delta)$ where $\tilde{\phi}_0$ denote the lift to $\Gamma_0 \backslash X$ of $\phi_0$ on $\Gamma \backslash X$. Then the average of $1/\sqrt{n}\tilde{\phi}_0$ over the horosphere at the distance $u$ is

$$\frac{1}{\sqrt{n}}\tilde{\phi}_0^N(u).$$
Specially the constant $c_{\phi_0}$ is equal to $\frac{1}{\sqrt{|\Gamma \setminus \Gamma_0|}}c_{\phi_0}$. This will be used in Section 11.

6. Bottom eigenfunction $\phi_0$

Let $X$ be a rank one symmetric space and fix a origin $(0, 0, 1) = o \in X$. Let $\Gamma \setminus X$ be geometrically finite with a critical exponent $\delta > D/2$ where $D$ is the Hausdorff dimension of $\partial X$. Let $B_\zeta$ be a Busemann function based at $\zeta$ normalized that $B_\zeta(o) = 0$. The Patterson-Sullivan measures $\theta_x$, $x \in X$ satisfies:

The measures $\theta_x$ and $\theta_y$ are mutually absolutely continuous and

$$\frac{d\theta_x}{d\theta_y}(\zeta) = e^{-\delta(B_\zeta(x) - B_\zeta(y))}$$

and for any $\gamma \in \Gamma$

$$\gamma_\ast \theta_x = \theta_{\gamma x}.$$

The function defined by

$$\phi_0(x, t, y) = \int_{\Lambda \setminus \Gamma} e^{-\delta B_\theta(x, t, y)}d\theta$$

where $d\theta = d\theta_\phi$ is a fixed Patterson-Sullivan measure associated to a fixed reference point $o \in X$, descends to a positive $L^2$-function on $\Gamma \setminus X$ whose eigenvalue with respect to the Laplace operator $-\triangle$ is $\delta(D - \delta)$ where $D$ is the Hausdorff dimension of $\partial X$. We always normalize it that its $L^2$-norm is 1. The function $\phi_0$ is given by in horospherical coordinates according to equation (5)

$$\phi_0(x, t, y) = \int_{\Lambda \setminus \Gamma} \frac{4y}{(|x - \zeta|^2 + y)^2 + (t - v + 2\text{Im}(\langle x, \zeta \rangle))^2}^\delta d\theta(\zeta, v)$$

Note that the formula for real hyperbolic case seems a bit different from this one but it is due to the fact that the $H^n_\mathbb{R} \subset H^n_\mathbb{H}$ sits as a Klein model [13](not Poincaré model) and that the curvature is $-1/4$ in this section. After we normalize the metric back to between $-4$ and $-1$, all the formulas will turn out right. See section 7.

Let

$$\phi_0^N(y) = \int_{(\Gamma \setminus \Gamma \setminus \Gamma) \setminus \Gamma \setminus \Gamma} \phi_0(x, t, y)dn$$

be the average over a horosphere where $dn = dtdx$. This is independent of the choice of a fundamental domain since $dn$ is $NM$-invariant and $\phi_0$ is $\Gamma$-invariant.

We will be interested in horospheres whose images are closed in $\Gamma \setminus \Gamma_\mathbb{H}$. Specially we will consider horospheres whose base points are either parabolic fixed points or points outside the limit set of $\Gamma$, see Lemma [10.1].
Proposition 6.1. We identify $N$ with a horosphere based at $\infty$ (in fact, the orbit of $N$) and suppose $(\Gamma \cap NM) \setminus NM/M$, which is a quotient of the horosphere by $\Gamma \cap NM$, has a closed image in $\Gamma \setminus X$. Then $\phi_N^\alpha(y) \gg y^{D-\delta}$ for all $0 < y \ll 1$ where $D$ is the Hausdorff dimension of $\partial X$, i.e., $(n-1)/2, n, 2n + 1$ for real, complex and quaternionic hyperbolic space respectively.

Proof. We carry out the calculation in complex hyperbolic case and indicate the difference in the other two cases. 

CASE I) If $\infty \notin \Lambda \Gamma$, then $NM \cap \Gamma$ is trivial and 

$$\phi_N^\alpha(y) = \int_N 4^\delta y^{-\delta} \int_{\Lambda \Gamma} \left( \frac{|x - \zeta^2 + y}{y} \right)^{\delta} d\theta(\zeta, v) dn.$$  

Change the variables to 

$$z = \frac{x}{\sqrt{y}}, s = \frac{t}{y}, x \in \mathbb{C}^{n-1}, t \in \mathbb{R},$$

to get 

$$dt dx = y^n ds dz$$

and 

$$\int_{\Lambda \Gamma} \left( \frac{|z - \zeta'|^2 + 1}{s - v' + 2 \text{Im}(\langle x, \zeta' \rangle)} \right)^{\delta} d\theta(\zeta, v) dn.$$  

(9)  

(10) 

Change $(w, t) = (z, s)(\zeta', v')^{-1}$. Then since $dn$ is $N$-invariant 

$$\phi_N^\alpha(y) = 4^\delta y^{n-\delta} \int_N \int_{\Lambda \Gamma} \left( \frac{|w|^4 + |t|^2 + 2|w|^2 + 1}{|w|^4 + |t|^2 + 2|w|^2 + 1} \right)^{\delta} d\theta$$

$$= 4^\delta y^{n-\delta} \int_N \int_{\Lambda \Gamma} \frac{dtdw}{|w|^4 + |t|^2 + 2|w|^2 + 1}.$$  

The equality holds since $d\theta$ is a probability measure. One can show that the integral converges for $\delta > \frac{n}{2}$ to a nonzero number. In this calculation, note that for real hyperbolic case, there is no $t$ factor, so from $z = x/\sqrt{y}$, 

$$dx = (\sqrt{y})^{n-1} dz,$$

this gives $y^{n-\delta}$ in front. In quaternionic case, 

$$dx = (\sqrt{y})^{4n-4} dt = y^3 ds,$$

to get $y^{2n+1-\delta}$ in front. CASE II below is similar.

CASE II) $\infty \in \Lambda \Gamma$

By the theorem of [6], $\infty$ is a bounded parabolic fixed point, so $\Gamma \setminus (\Lambda \Gamma - \infty)$ is a compact set. Hence it is bounded. Let $F_\Lambda = (\Gamma \cap NM) \setminus \Lambda \subset F_N = (\Gamma \cap NM) \setminus NM/M$ be fundamental sets under the left action of $\Gamma \cap NM$. Note that for a fixed $F_\Lambda \subset F_N$ and $X \in H^a, \zeta \in F_\Lambda$, by the property [8] of Patterson-Sullivan measure, 

$$(n_x^{-1} \theta_X)(\zeta) = \theta_{x^{-1}}(\zeta) = \theta_X(n\zeta)$$
for any $n \in NM \cap \Gamma$. Also by the property (7) of Patterson-Sullivan measure
\[
\frac{d\theta_X(n\zeta)}{d\theta_0} = e^{-\delta B_n\zeta(X)},
\]
\[
\frac{d\theta_{n^{-1}X}}{d\theta_0}(\zeta) = e^{-\delta B_c(n^{-1}X)}.
\]
This implies that for $\zeta \in F_{\Lambda}$
\[
\int_{nF_{\Lambda}} \int_{F_N} e^{-\delta B_n\zeta(x,t,y)} d\theta d\nu(n\zeta) = \int_{F_{\Lambda}} \int_{n^{-1}F_N} e^{-\delta B_c(n^{-1}(x,t,y))} d\theta d\nu(\zeta).
\]
Hence,
\[
\phi_0^N(y) = \int_{F_N} \int_{\Lambda_{\Gamma}} e^{-\delta B_0(x,t,y)} d\theta d\nu = \sum_{n \in \Gamma \cap NM} \int_{nF_{\Lambda}} \int_{F_N} e^{-\delta B_0(x,t,y)} d\theta d\nu
\]
\[
= \sum_{n \in \Gamma \cap NM} \int_{F_{\Lambda}} \int_{n^{-1}F_N} e^{-\delta B_0(x,t,y)} d\theta d\nu
\]
In terms of the coordinates
(11) \[
\phi_0^N(y) = 4^\delta y^{n-\delta}
\]
(12) \[
\int_{N} \int_{(\Gamma \cap NM) \setminus \Lambda_{\Gamma}} \frac{d\nu d\zeta}{(|z - \zeta'|^2 + 1)^2 + |s - v' + 2\text{Im} \langle z, \zeta' \rangle|^2} d\theta,
\]
where $(\zeta', v') \in a \frac{1}{\sqrt{\pi}} F_{\Lambda}$. A similar estimates holds to conclude that
\[
\phi_0^N(y) \gg y^{n-\delta}.
\]

Fix generators $v_1, \ldots, v_k$ in $NM/M$ corresponding to the axes of screw motions of $NM \cap \Gamma$ so that $v_1, \ldots, v_k$ together with $v_{k+1}, \ldots, v_{2n-1}$ are basis of $NM/M$. Denote $N^\perp$ the subspace generated by $v_{k+1}, \ldots, v_{2n-1}$. Let $F_{\Lambda} \subset B \subset F_N$ be an open set such that
(1) if $\infty \notin \Lambda_{\Gamma}$ then $\epsilon_0(B) = \inf_{u \in F_{\Lambda}, x \in B^c} |x - u|_N > 0$
(2) if $\infty$ a bounded parabolic fixed point then $\epsilon_0(B) = \inf_{u \in F_{\Lambda}, x \in B^c} |x - u|_{N^\perp} > 0$ where $B^c = F_N \setminus B$.
For such an open set $B$

**Proposition 6.2.** If $\delta > \frac{D}{2}, \phi_0^N(y) = \int_B \phi_0(x,t,y) d\nu + O_{\epsilon_0(B)}(y^\delta)$ and $\phi_0^N(y) = O(y^{D-\delta})$.

**Proof.** We give a proof in complex hyperbolic case but as in the previous Proposition, the other cases are similar. When $\infty \notin \Lambda_{\Gamma}$, using equation (9), we want to estimate
\[
4^\delta y^{n-\delta} \int_{B^c} \int_{\Lambda_{\Gamma}} \frac{d\nu d\zeta}{(|z - \zeta'|^2 + 1)^2 + |s - v' + 2\text{Im} \langle z, \zeta' \rangle|^2} d\theta.
\]
Since
\[
\frac{1}{(|z - \zeta'|^2 + 1)^2 + |s - v' + 2\text{Im} \langle z, \zeta' \rangle|^2} \ll \frac{1}{((|z - \zeta'|^2 + 1)^2 + |s - v' + 2\text{Im} \langle z, \zeta' \rangle|^2)^\delta},
\]

\[
\frac{1}{|z - \zeta'|^4 + |s - v' + 2\text{Im}(z, z')|^2} = \frac{1}{[d_N((z, s), (\zeta', v'))^4]}.
\]
Since \(dn\) is invariant under \(N\), the above integral is
\[
\leq 4^\delta y^{n-\delta} \int_{|w| \leq \epsilon_0 / \sqrt{\gamma}} \frac{dn}{(|w|_N^2)^\delta}
\]
where \(w = (z, s)(\zeta', v')^{-1}\) in Heisenberg group. Let \(w = (x, t)\) and write \(dt dx = r^{2n-3}dr dt dS\) where \(dS\) is a volume form on unit sphere in \(\mathbb{R}^{2n-2}\). Then
\[
4^\delta y^{n-\delta} \int_{|w| \leq \epsilon_0 / \sqrt{\gamma}} \frac{dn}{(|w|_N^2)^\delta} \leq 4^\delta y^{n-\delta} \int_{(r + t^2)^{1/4} \leq \epsilon_0 / \sqrt{\gamma}} \frac{r^{2n-3} dr dt dS}{(r + t^2)^\delta}.
\]
Letting \(t = \tan \theta r^2\), and for \(\delta > \frac{n}{2}\) it becomes
\[
y^{n-\delta} \int_{r \geq \epsilon_0 / \sqrt{\gamma}} \frac{r^{2n-1} dr}{(r^4 + t^2)^\delta} \int_{\theta = \tan \theta / \cos^2 \theta(1 + \tan^2 \theta)^\delta} \frac{d\theta}{\delta} \ll y^\delta.
\]
As before we have
\[
\phi_0^N(y) = 4^\delta y^{n-\delta} \int_{r, t \geq 0} \frac{r^{2n-3} dr dt dS}{(r^4 + t^2 + 1)^\delta}.
\]
Hence if \(4\delta - 2 - 2n + 3 > 1\), (i.e. \(\delta > \frac{n}{2}\)) then by letting \(t = \tan \theta \sqrt{(r^4 + 2r^2 + 1)}\) the integrals converge to a nonzero number to conclude that
\[
\phi_0^N(y) = O(y^{n-\delta}).
\]
When \(\infty\) is a bounded parabolic fixed point, the similar estimates hold.
Using
\[
\int_{B^c} \int_{\Lambda_N} e^{-\delta B_\theta(x, t, y)} d\theta dn = \sum_{n \in \Gamma \cap N M} \int_{n F_A} \int_{B^c} e^{-\delta B_\theta(x, t, y)} dnd\theta
\]
and following the equation (11), we want to estimate
\[
4^\delta y^{n-\delta} \int_{\cup_{n \in \Gamma \cap N M} B^c} \int_{(\Gamma \cap N M) \setminus \Lambda_N} \frac{d\theta dz}{(|z - \zeta'|^2 + 1)^2 + |s - v' + 2\text{Im}(z, \zeta')|^2} d\theta,
\]
where \((\zeta', v') \in a \frac{1}{y} F_A\). The same estimation as in \(\infty \notin \Lambda_N\) gives
\[
\int_{B^c} \int_{\Lambda_N} e^{-\delta B_\theta(x, t, y)} d\theta d\theta \ll y^\delta,
\]
to get
\[
\phi^N_0(y) = \int_B \int_{\Lambda} e^{-\delta B_0(x,t,y)} d\theta dn + \int_B \int_{\Lambda} e^{-\delta B_0(x,t,y)} d\theta dn
\]
\[
\ll \int_B \int_{\Lambda} e^{-\delta B_0(x,t,y)} d\theta dn + y^\delta.
\]
Also in this case the similar estimates give that for \( \delta > \frac{\varrho}{2} \)
\[
\phi^N_0(y) = O(y^{n-\delta}).
\]

We fixed Iwasawa decomposition \( ANK \) so that \( N \) is a Heisenberg group. Let \( N^- \) be the opposite Nilpotent group to \( N \) so that
\[
N \times A \times N^- \times M \to G
\]
is a diffeomorphism around a neighborhood of \( e \) and \( d\nu \) is a smooth measure on \( AN^-M \) so that \( dn \otimes d\nu \) is a Haar measure \( d\mu \) on \( G \). Fix a left invariant metric \( dG \) on \( G \) and \( U_\epsilon \) is an \( \epsilon \)-neighborhood of \( e \) in \( G \). Since \( A \times N \times K \to G \) is a diffeomorphism and hence a bi-Lipschitz map around the neighborhood of \( e \), there exists \( l > 0 \) such that \( U_\epsilon \) is contained in \( A_{\epsilon}N_{\epsilon}K_{\epsilon} \) once we fix some \( \epsilon_0 \) and take \( \epsilon \leq \epsilon_0 \). We fix a positive \( \eta \in \mathcal{C}^\infty_c((NM \cap \Gamma) \setminus NM/M) \) with \( \eta = 1 \) on a bounded open set \( B \) of \( F_N \) so that \( \epsilon_0(B) > 0 \) as in the previous Proposition 6.2 and vanishes outside a small neighborhood of \( B \) so that
\[
\phi^N_0(y) = \int_B \phi_0(n,y)\eta(n)dn + O(y^\delta).
\]
Shrinking \( \epsilon_0 \) if necessary, we further assume that \( \text{supp}(\eta) \times (U_{\epsilon_0} \cap AN^-M) \to \Gamma \setminus G \) is a bijection to its image.

For each \( \epsilon < \epsilon_0 \), let \( r_\epsilon \) be a nonnegative smooth function on \( AN^-M \) whose support is contained in
\[
W_\epsilon = (U_\epsilon \cap A)(U_{\epsilon_0} \cap N^-)(U_{\epsilon_0} \cap M)
\]
and \( \int_{W_\epsilon} r_\epsilon d\nu = 1 \). Finally define \( \rho_{\eta,\epsilon} \) on \( \Gamma \setminus G \) which vanishes outside \( \text{supp}(\eta)U_{\epsilon_0} \) and for \( g = n_xan^-m \in \text{supp}(\eta)W_\epsilon \),
\[
\rho_{\eta,\epsilon}(g) = \eta(n_x)r_\epsilon(an^-m).
\]
Then

**Proposition 6.3.** For small \( \epsilon \ll \epsilon_0 \) and for \( y < 1 \), regarding \( \phi_0 \in \mathcal{C}(\Gamma \setminus G)^K \)
\[
\phi^N_0(y) = \langle a_y \phi_0, \rho_{\eta,\epsilon} \rangle_{L^2} + O(\epsilon y^{D-\delta}) + O(y^\delta).
\]

**Proof.** As usual we give a proof only in complex hyperbolic case. Since
\[
\langle a_y \phi_0, \rho_{\eta,\epsilon} \rangle_{L^2} = \int_{W_\epsilon} r_\epsilon(h) \int_{(NM \cap \Gamma) \setminus NM/M} \phi_0(nha_y)\eta(n)dnd\nu(h)
\]
we need to estimate \( \phi_0(n_{ha_y}) \) for \( n \in F_N \) and \( h \in W_{\epsilon} \).

For \( h = a_{y_0} n_x^{-} m \in W_{\epsilon} \) so that \( |y_0 - 1| = O(\epsilon) \),
\[
n_{ha_y} = na_{y_0} n_x^{-} m.
\]

Since \( n_{y_x}^{-} = a_{y_1} n_x^{-} k_1 \in A_l \) and \( n_{y_1^{-}} K \) so that \( |y_1 - 1| = O(\epsilon) \),
\[
n_{ha_y} = nn_{x_1 y_0 y_1} a_{y_0 y_1} k_1 m.
\]

Since \( \phi_0 \) is \( K \)-invariant and \( dn \) is \( N \)-invariant,
\[
\int_{F_N} \phi_0(n_{ha_y}) \eta(n) dn = \int_{F_N} \phi_0(nn_{x_1 y_0 y_1} a_{y_0 y_1}) \eta(n) dn
\]
by letting \( n' = nn_{x_1 y_0 y_1} \) and so \( \eta(n) = \eta(n' n_{x_1 y_0 y_1}^{-1}) \) and as \( |\eta(n') - \eta(n' n_{x_1 y_0 y_1}^{-1})| = O(\epsilon) \). By Proposition 6.2,
\[
\int_{F_N} \phi_0(n_{ha_y}) \eta(n) dn = \int_{F_N} \phi_0(n a_{y_0 y_1}) \eta(n) dn + O(\epsilon \phi_0^n(a_{y_0 y_1}))
\]
\[
= \phi_0^n(y y_0 y_1) + O((y y_0 y_1)^{\delta}) + O_\eta(\epsilon \phi_0^n(a_{y_0 y_1})).
\]

Since by Proposition 6.2,
\[
\lim_{y \to 0} \frac{\phi_0^n(y y_0 y_1)}{\phi_0^n(y)} = \lim_{y \to 0} \frac{\phi_0^n(y y_0 y_1)}{(y y_0 y_1)^{D-\delta}} \frac{(y y_0 y_1)^{D-\delta}}{\phi_0^n(y)} = \lim_{y \to 0}(y y_0 y_1)^{D-\delta} = (1 + O(\epsilon)),
\]
and since \( \phi_0^n(a_{y_0 y_1}) = O((y y_0 y_1)^{D-\delta}) \) we get
\[
\int_{F_N} \phi_0(n_{ha_y}) \eta(n) dn = \phi_0^n(y) + O(y^\delta) + O_\eta(\epsilon y^{D-\delta}).
\]

Let \( \{Z_1, \cdots, Z_k\} \) denote an orthonormal basis of the Lie algebra of \( G \) and \( \Gamma \subset G \) a discrete subgroup. For \( f \in C^\infty(\Gamma \backslash G)^K \cap L^2(\Gamma \backslash G) \), one considers the Sobolev norm \( S_m(f) \):
\[
S_m(f) = \max \{|\|Z_{i_1} \cdots Z_{i_m}(f)|| : 1 \leq i_j \leq k\}.
\]
The following is standard. For \( \phi \in C_c^\infty(\Gamma \backslash G)^K \), there exists \( \phi' \in C_c^\infty(\Gamma \backslash G)^K \) so that
1. for small \( \epsilon > 0 \) and \( h \in U_{\epsilon} \),
\[
|\phi(g) - \phi(gh)| \leq \epsilon \phi'(g)
\]
for any \( g \in \Gamma \backslash G \).
2. by Sobolev embedding theorem, there exists \( q \) so that \( S_m(\phi') \ll S_q(\phi) \) for each \( m \), where the implied constant depends only on \( \supp(\phi) \).
Proposition 6.4. Let $\phi \in C^\infty(\Gamma \backslash G)^K$. Then for any $0 < y < 1$ and any small $\epsilon > 0$, 

$$|I_{\eta}(\phi)(a_y) - \langle a_y\phi, \rho_{\eta,\epsilon} \rangle| \ll (\epsilon + y)I_{\eta}(\phi')(a_y)$$

where $I_{\eta}(\phi)(a_y) = \int \phi(na_y)\eta(n)dn$ and $\eta \in C_c((NM \cap \Gamma) \backslash NM/M)$.

Proof. For $h = an^{-m} \in W_\epsilon$, $an^{-m}a_y = a_y(a_{y^{-1}}n^{-1}a_y)m$. Hence $nan^{-ma_y} = na_y(aa_{y^{-1}}n^{-1}a_ym)$ where $a_{y^{-1}}n^{-1}a_y \in U_{y\epsilon_0} \cap N^-$. Then as $\phi$ is $K$-invariant, 

$$|\phi(na_y) - \phi(nha_y)| = |\phi(na_y) - \phi(na_yh')| \ll \phi'(na_y)(\epsilon + y\epsilon_0)$$

where $h' = aa_{y^{-1}}n^{-1}a_y \in (U_\epsilon \cap A)(U_{y\epsilon_0} \cap N^-)$. Hence 

$$|\phi(na_y) - \int_{h \in W_\epsilon} \phi(nha_y)r_\epsilon(h)d\nu(h)| \ll \phi'(na_y)(\epsilon + y\epsilon_0).$$

By integrating on $(NM \cap \Gamma) \backslash NM/M$ we obtain 

$$|I_{\eta}(\phi)(a_y) - \langle a_y\phi, \rho_{\eta,\epsilon} \rangle|_{L^2(\Gamma \backslash G)} \ll (\epsilon + y\epsilon_0)I_{\eta}(\phi')(a_y).$$

Since $\epsilon_0$ is fixed, we get the desired result. \hfill $\square$

7. Unitary representations of rank one semisimple group

From this section, we normalize the metric so that its sectional curvature is between $-4$ and $-1$. Equivalently we have to multiply the metric tensors $ds^2$ by $1/4$. Then the volume form for complex hyperbolic $n$-space is multiplied by $2^{-2n}$. Then the distance from $(0,0,1)$ to $(0,0,y)$ will become $\log \sqrt{y}$. So by changing the variable $\sqrt{y} = w$ and abusing the notation by putting $w$ back to $y$, the volume form for complex hyperbolic $n$-space is

$$(13) \quad \frac{1}{2^y2^{n+1}}dVol_zdt dy$$

For real hyperbolic $n$-space,

$$(14) \quad \frac{1}{y^n}dVol_z dy$$

For quaternionic hyperbolic $n$-space

$$(15) \quad \frac{1}{8^y4^{n+3}}dVol_z dt dy$$

where $dVol_z$ is a volume form on $\mathbb{H}^{n-1}$. Also in all the formulas in section 6, $n$ should be read as $2n$ under this normalization.

Let $\eta = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ be in $so(n,1), su(n,1), sp(n,1)$ so that $e^{t\eta}$ is the 1-parameter subgroup constituting $A$ in $KAN$. A direct calculation shows that 

$$g_k = Ker(ad\eta - kId), \quad k = 0, \pm 1, \pm 2$$

are only root spaces and there are only two positive roots $\beta, 2\beta$ (in real hyperbolic case, $2\beta$ is not a root) so that $\beta(\eta) = 1$. Note that real dimension
of $g_1$ is $n - 1, 2n - 2, 4n - 4$ resp. and that of $g_2 = 0, 1, 3$ resp. Then the half sum $\rho$ of the positive roots is

$$\rho = \frac{D}{2} \beta.$$ 

Since we will work with horospheres based at $\infty$ and expanding ones as $y \to 0$, we will define a positive Weyl chamber by

$$A^+ = \{a_y|0 < y < 1\}$$

as a multiplicative group. So $-\beta, -2\beta$ will be positive roots in this paper, which will change the plus sign to the minus in all formulas in the literature.

In this section, we prove the following. The kind of estimate we look for was first established by Cowling, Haagerup and Howe for tempered representations (i.e. unitary representations weakly contained in the regular representation). A unitary representation $(V, \pi = \int_{x \in \hat{G}} \pi_x d\mu(x))$ of $G$ is called tempered if one of the following is true, see for example [9, 29].

1. For any $K$-finite unit vectors $v$ and $w$ (i.e., the dimension of the subspaces spanned by $Kv$ and $Kw$ is finite),

$$|\langle \pi(g)v, w \rangle| \leq (\dim(Kv) \dim(Kw))^{1/2} \Xi_G(g)$$

for any $g \in G$ where $\Xi_G$ is the Harish-Chandra function on $G$.

2. For almost all $x \in \hat{G}$, the irreducible representation $\pi_x$ is strongly $L^{2+\epsilon}(G/Z(G))$, i.e., for any $p > 2$, there exists a dense subset $W \subset V$ such that for any $v, w \in W$, the matrix coefficient $g \to \langle \pi_x(g)v, w \rangle$ lies in $L^p(G/Z(G))$.

3. For almost all $x \in \hat{G}$, $\pi_x$ is tempered in the sense of (1).

4. $\pi$ is weakly contained in the regular representation $L^2(G)$, i.e., any diagonal matrix coefficients of $\pi$ can be approximated, uniformly on compact sets, by convex combinations of diagonal matrix coefficients of the regular representation $L^2(G)$.

The following theorem is due to Y. Shalom, [35], Theorem 2.1 p. 125 in case $G = SO(n, 1)$ or $SU(n, 1)$. We generalize it to other rank one groups. For the notational simplicity, we fix the notations first.

Let $G$ be a $\mathbb{R}$-rank one simple Lie group. Pick an Iwasawa decomposition $G = KAN$. Let $\lambda \in \mathfrak{a}_c^*$ be a complex linear form on the Lie algebra of $A$. This gives rise to the character

$$a \mapsto a^\lambda = e^{\lambda(\log(a))}$$

on $A$. Let $M$ be the centralizer of $A$ in $K$. Let $Z^\lambda$ denote the space of $K$-finite complex valued functions on $G$ such that $f(gman) = a^{-\lambda} f(g)$ for all $a \in A, m \in M$ and $n \in N$ (notation taken from [24]). $G$ acts on $Z^\lambda$ by $(\pi_\lambda(g)u)(h) = u(g^{-1}h)$, for $g, h \in G$.

**Notation 1.** Let $\rho$ denote the half-sum of positive roots. We write $\lambda = (1 + s)\rho$ with $s \in \mathbb{C}$, $Z_s = Z^\lambda$, $\pi_s = \pi_\lambda$. 

A description of the set of $s \in \mathbb{C}$ such that $Z_s$ is irreducible and admits a $G$-invariant inner product can be found in [24], Theorem 10 page 641. It is the union of the line $\{\text{Re}(s) = 0\}$, and of the closed critical interval $CI = [-s_1(G), s_1(G)]$, a symmetric interval on the real line. It is shown in [24], that for $0 \leq \lambda \leq \rho$, $\pi_\lambda$ and $\pi_{2\rho-\lambda}$ are equivalent. The representations obtained for $\{\text{Re}(s) = 0\}$ are called spherical principal series representations of $G$. The representations obtained when $s \in CI$ are called spherical complementary series representations of $G$.

The word spherical refers to the fact that these representations contain a unique $K$-invariant line, generated by the function $v_\lambda$ which equals 1 on $K$.

According to Harish-Chandra, [19], every irreducible unitary representation of $G$ which contains a nonzero $K$-invariant vector is isomorphic to a representation from the spherical principal or complementary series (except possibly for the trivial representation).

Let $G$ be a $\mathbb{R}$-rank one simple Lie group. Let $(\pi_s, Z_s)$, $s \in [0, s_1(G)]$, be a representation belonging to the spherical complementary series of $G$. Pick a unit $K$-invariant vector $v_s$ in $Z_s$. Define the function

$$\Xi_s(g) = \langle \pi_s(g)v_s, v_s \rangle_s.$$  

When $s = 1$, $\lambda = 2\rho$, hence $\pi_{2\rho}$ is equivalent to $\pi_0$. So $\Xi_1 = \Xi_G$ Harish-Chandra function on $G$.

If $KA^+K$ is a polar decomposition of $SU(n,1)$, then the Haar measure on it is

$$(e^{-2\beta(\log a)} - e^{2\beta(\log a)})(e^{-\beta(\log a)} - e^{\beta(\log a)})^{2n-2}dkdak$$

where $\log : G \to g$ is the inverse map of the exponential map and $a(g)$ is the component of $A^+$ in the polar decomposition of $g$. Note that in this formula, $A^+$ is regarded as an additive group. Then for $y < 1$, the measure is comparable to $e^{-2n\beta(\log a)}dkdak$, that if we use log $y = a \in A^+$ to make $A^+$ a multiplicative group so that $da = \frac{dy}{y}$ it is

$$\frac{1}{y^{2n+1}}dkdydk.$$  

We hope that this switch between multiplicative and additive group does not cause any confusion to the reader. This is consistent with the volume form [13] on $H^n_C$. For real hyperbolic space, Haar measure is

$$(e^{-\beta(\log a)} - e^{\beta(\log a)})^{n-1}dkdak$$

and comparable to

$$e^{-(n-1)\beta(\log a)}dkda$$

for $y < 1$, hence after put log $y = a$, the Haar measure on $KA^+K$ is

$$\frac{1}{y^n}dkdydk,$$
which is comparable. In any case one can write the Haar measure on $KA^+K$ as

\[ \frac{1}{y^{D+1}} dk dy dk \]

for $y < 1$. For quaternionic hyperbolic case is the same.

For $\lambda \in \mathfrak{a}^*$, and for $\log a(g) \leq 0$, the principal series $\pi_\lambda$ has matrix coefficient decaying rates for $K$-invariant unit vector $v_\lambda$,

\[ |\langle \pi_\lambda(g)v_\lambda, v_\lambda \rangle| \leq \Xi(g) \leq C|1 - \beta(\log a(g))| e^{2\beta \log a(g)}. \]

See [14] section 4.6.4. For $0 \leq \lambda \leq \frac{D}{2} \beta \in \mathfrak{a}^*$, and for $\log a(g) \leq 0$, the complementary series representation has matrix coefficient decaying rates

\[ \leq C|1 - \beta(\log a(g))| e^{(-\lambda + \frac{D}{2} \beta)(\log a(g))}. \]

See [35] (equation (10) page 132) also.

**Proposition 7.1.** For all $g \in G$, there exist $c(G), C(G)$ such that

\[ c(G)\Xi_1(g)^s \leq \Xi_s(g) \leq C(G)(1 - \log \Xi_1(g))\Xi_1(g)^s. \]

**Proof.** Let $\lambda = (1 + s)\rho$, $0 \leq s \leq s_1(G)$. Then $\pi_\lambda$ is equivalent to $\pi_{2\rho - \lambda}$ where $2\rho - \lambda = \rho - sp$. In view of equation (18),

\[ \Xi_s(g) \leq C|1 - \beta(\log a(g))| e^{(-2\rho + \lambda + \rho)(\log a(g))} = C|1 - \beta(\log a(g))| e^{(sp)(\log a(g))}. \]

But it is known [14] (Theorem 4.6.5) that

\[ \Xi_1(a) = \Xi(a) \geq e^{\rho(\log a)}, \ a \in A^+. \]

Hence we get

\[ \Xi_s(g) \leq C(G)(1 - \log \Xi_1(g))\Xi_1(g)^s. \]

For the lower bound, in [14] (4.7 (4.7.13)), [35] (equation (11) page 132), it is shown that

\[ C' e^{(\rho - \lambda) \log a(g)} \leq \Xi_\lambda(g), \ 0 \leq \lambda \leq \rho. \]

Hence again using the fact $\pi_\lambda$ is equivalent to $\pi_{2\rho - \lambda}$, we get the desired lower bound. $\square$

**Theorem 7.2.** There exists a constant $C(G)$ such that for all $K$-finite vectors $u, v \in Z_s = Z^\lambda$,

\[ \langle \pi_s(g)u, v \rangle_s \leq C(G) (\dim \text{Span}(Ku) \dim \text{Span}(Kv))^{1/2} |u|_s |v|_s \Xi_s(g). \]

**Proof.** The main ingredient in the proof is Cowling, Haagerup and Howe’s temperedness criterion [9]. The following statement is a combination of their Theorems 1 and 2.

**Proposition 7.3.** Let $\pi$ be a unitary representation of a semi-simple algebraic group $G$. Then the following are equivalent.

1. $\pi$ has a dense set of vectors whose coefficients belong to $L^{2+\epsilon}(G)$ for all $\epsilon > 0$. 

\[ \langle \pi_s(g)u, v \rangle_s \leq C(G) (\dim \text{Span}(Ku) \dim \text{Span}(Kv))^{1/2} |u|_s |v|_s \Xi_s(g). \]


(2) for all $K$-finite vectors $u$ and $v$ in $\pi$,

\begin{equation}
\langle \pi(g)u, v \rangle \leq C \left( \dim \text{Span}(Ku) \dim \text{Span}(Kv) \right)^{1/2} |u|_s |v|_s \Xi_1(g)
\end{equation}

where $\Xi_1 = \Xi$ is the Harish-Chandra function on $G$.

Also, by Proposition 10.2 for all $g \in G$,

\begin{equation}
c(G)\Xi_1(g)^s \leq \Xi_s(g) \leq C(G)(1 - \log \Xi_1(g))\Xi_1(g)^s.
\end{equation}

Shalom’s trick consists in tensoring representations until they become almost square integrable. Let $s \in [0, s_1(G)]$. For $G = Sp(n, 1)$, $n \geq 2$, $s_1(G) = \frac{n^2 - 2}{2n^2 - 2} > \frac{1}{2}$. Thus there exists $t \in [0, s_1(G)]$ such that $1 - s - t \in [0, s_1(G)]$. For $G = F_{4}^{-20}$, $s_1(G) = \frac{10}{27} > \frac{1}{3}$. Thus there exists $t \in [0, s_1(G)]$ such that $1 - s - 2t \in [0, s_1(G)]$. Consider the unitary representation $\pi = \pi_s \otimes \pi_t \otimes \pi_{1-s-t}$ (resp. $\pi = \pi_s \otimes \pi_t \otimes \pi_t \otimes \pi_{1-s-2t}$).

Since all $\pi_s$ are irreducible, the translates of $v_s$ generate a dense subspace of $Z_s$. Let $u_1$, $v_1$ belong to this subspace. Then $\langle \pi_s(g)u_1, v_1 \rangle_s = O(\Xi_s(g))$. Similarly, pick $u_2$, $v_2$ (resp. $u_3$, $v_3$, $u_4$, $v_4$) in the vector space generated by translates of $v_t$ (resp. of $v_{1-s-t}$, $v_{1-s-2t}$). Let $U = u_1 \otimes u_2 \otimes u_3$, $V = v_1 \otimes v_2 \otimes v_3$ (resp. $u_1 \otimes u_2 \otimes u_3 \otimes u_4$ etc...). Then

\begin{equation}
\langle \pi(g)U, V \rangle = O(\Xi_s(g)\Xi_1(g)\Xi_{1-s-t}(g)) \quad (\text{resp. } O(\Xi_s(g)\Xi_t(g)\Xi_t(g)\Xi_{1-s-2t}(g)))
\end{equation}

Since $y^\delta \log y \to 0$ as $y \to 0$ for any $\delta > 0$, for $a_y, y < 1$ we have $(1 - \log \Xi_1(g))^{4}\Xi_1(g) \leq C y^\rho \delta$ for any $\delta > 0$. Since the Haar measure is given by

\[ \frac{1}{y^{D+1}}dkdydk, \]

for small $\delta$

\[ \int_K \int_0^1 (y^\rho - \delta)^{2+\epsilon} \frac{1}{y^{D+1}}dkdydk < \infty, \]

thus belongs to $L^{2+\epsilon}(G)$ for all $\epsilon > 0$.

Since these products generate a dense subspace of the tensor product, Proposition 10.3 applies, $\pi$ is tempered, and inequality (20) holds for all $K$-finite vectors in the tensor product. Let $u, v \in Z_s$ be $K$-finite vectors. Since

\[ \langle \pi_s(g)u \otimes v_t \otimes v_{1-s-t}, v \otimes v_t \otimes v_{1-s-t} \rangle = \langle \pi_s(g)u, v \rangle_s \Xi_t(g)\Xi_{1-s-t}(g) \]

and

\[ |u \otimes v_t \otimes v_{1-s-t}| = |u|_s, \quad |v \otimes v_t \otimes v_{1-s-t}| = |v|_s, \]

\[ \langle \pi_s(g)u, v \rangle_s \leq C \left( \dim \text{Span}(Ku) \dim \text{Span}(Kv) \right)^{1/2} |u|_s |v|_s \Xi_1(g) \Xi_t(g)\Xi_{1-s-t}(g) \]

Thanks to the lower bound (21) on spherical functions,

\[ \frac{\Xi_1(g)}{\Xi_t(g)\Xi_{1-s-t}(g)} \leq \frac{\Xi_1(g)^s}{c(G)^s} \leq \frac{\Xi_s(g)}{c(G)^s}, \]

yielding the announced inequality. □

So for \( a_y, y < 1 \), from equation (18)

\[
|\langle a_y v_\lambda, v_\lambda \rangle| \leq C|1 - \log y|y^{D-\lambda}
\]

We will change the parameter for \( \lambda \) from \( 0 \leq \lambda \leq \rho \) to \( \frac{D}{2} \beta = \rho \leq \lambda \leq D \beta = 2\rho \) so that the bound becomes

\[
C|1 - \log y|y^{D-\lambda}
\]

and when \( \lambda = D \) it represents a trivial representation and when \( \lambda = D/2 \) represents a principal series. Since \( y^\epsilon \log y \to 0 \) as \( y \to 0 \) for any \( \epsilon > 0 \), we will write

\[
|\langle a_y v_\lambda, v_\lambda \rangle| \leq Cy^{D-\lambda-\epsilon}
\]

for \( \rho \leq \lambda \leq 2\rho \).

8. Bottom eigenspectrum of Laplace operator

Henceforth, we use the notation \((V_\lambda, \pi_\lambda)\) to denote the spherical principal or complementary unitary representation of rank one group \( G \). Let \( X \) be a rank one symmetric space and fix a origin \((0, 0, 1) = o \in X \). Let \( M = \Gamma \backslash X \) be geometrically finite with a critical exponent \( \delta \). Let \( B_\theta \) be a Busemann function based at \( \theta \) normalised that \( B_\theta(o) = 0 \). The function defined as

\[
\phi_0(x, t, y) = \int_{\Lambda \Gamma} e^{-\delta B_\theta(x, t, y)} d\theta
\]

where \( d\theta \) is a fixed Patterson-Sullivan measure associated to a fixed reference point \( o \in X \), descends to a positive \( L^2 \) function on \( M \) whose eigenvalue with respect to the Laplace operator is \( \delta(D - \delta) \) where \( D \) is the Hausdorff dimension of \( \partial X \). For \( H^{m}_\mathbb{R}, H^{n}_\mathbb{C}, H^{n}_\mathbb{H} \), \( D = n - 1, 2n, 4n + 2 \) resp. under the normalization of the sectional curvature between \(-4\) and \(-1\). Note that \( L^2(\Gamma \backslash G)^K \) is naturally isomorphic to \( L^2(\Gamma \backslash H^{n}_\mathbb{H}) \) by averaging over \( K \)-orbits, and this isomorphism intertwines the action of \( C \), the Casimir operator, with that of \(-\Delta \). For geometrically finite groups, the relation between the bottom spectrum \( \lambda_0 \) and the critical exponent \( \delta(=\text{Hausdorff dimension of the limit set}) \) is

\[
\lambda_0 = \delta(D - \delta).
\]

In this paper, we will assume that \( \delta > D/2 \) and \((V_\lambda, \pi_\lambda), D/2 \leq \lambda \leq D \) to denote the spherical principal or complementary unitary representation of rank one group \( G \). Fix \( D/2 < s_\Gamma < \delta \) so that there is no eigenvalue of the Laplace operator between \( s_\Gamma(D - s_\Gamma) \) and the bottom spectrum \( \delta(D - \delta) \) in \( L^2(\Gamma \backslash G/K) \). Such \( s_\Gamma \) exists since the spectrum is discrete for small eigenvalues, see [19] for general geometrically finite manifolds with pinched negative curvature. Then

\[
L^2(\Gamma \backslash G)^K = V_\delta \oplus V
\]
where $V_δ$ is a complementary series corresponding to $δ$ and $V$ does not contain any complementary series $V_s$ for $s ≥ s_Γ$.

If $V_s$ is an irreducible factor for $V$, we want to estimate the matrix coefficient decaying rate.

For $X_i$, an orthonormal basis of the Lie algebra of $K$ with respect to an $Ad$-invariant scalar product, let $ω = 1 − \sum X_i^2$. This is a differential operator in the center of the enveloping algebra of $\text{Lie}(K)$ and acts as a scalar on each $K$-isotypic component of $V_s$.

**Proposition 8.1.** Let $(V, π)$ be a unitary representation of $G = SO(n, 1), SU(n, 1), Sp(n, 1)$ which do not weakly contain any complementary series representation $V_s$ for $s ≥ s_0$. Then for any $ε > 0$, there exists $c_ε$ such that for any smooth vectors $w_1, w_2 ∈ V$, and $y < 1$,

$$|\langle a_yw_1, w_2 \rangle| ≤ c_ε y^{D−s_0−ε}||ω^m(w_1)||||ω^m(w_2)||,$$

where $4m > \text{rank}(K) + 2\#\{\text{positive roots}\}$.

**Proof.** $π$ decomposes as $\int_\hat{G} \otimes^m ρ_2 dν$ where $\hat{G}$ is the unitary dual of $G$, $m_2$ is the multiplicity of an irreducible representation $ρ_2$, and $ν$ is a spectral measure on $\hat{G}$. Then $ρ_2$ is either tempered or isomorphic to a complementary series $V_s$ for $D/2 ≤ s < s_0 < n − 1$, $D/2 ≤ s < s_0 < 2n$, $D/2 ≤ s < s_0 < 4n$.

Note that by Kostant, $s_0$ cannot exceed $4n$ for quaternionic case, [24].

Then for $K$-finite unit vectors $w_1$ and $w_2$ of $π$, if we write

$$w_i = \oplus v_1^λ, \ v_1^λ ∈ V_λ,$$

then by Theorem [72]

$$\langle a_yv_1^λ, v_2^λ \rangle ≤ C(G)||v_1^λ||||v_2^λ||((\text{dim Span}(Kv_1^λ)) \text{dim Span}(Kv_2^λ))^{1/2} ξ(\lambda)(a_y).$$

By equation (23), using Cauchy-Schwarz inequality and dim Span$Kw_i ≥ \text{dim Span}(Kv_1^λ)$, finally we get

$$|\langle a_yw_1, w_2 \rangle| ≤ C(G)y^{D−s_0−ε}Π(\text{dim Span}(Kw_i)).$$

From $K$-finite vector to smooth vectors, it is standard, see [25], that

$$|\langle a_yw_1, w_2 \rangle| ≤ C(G)y^{D−s_0−ε}||ω^m(w_1)||||ω^m(w_2)||,$$

where $4m > \text{rank}(K) + 2\#\Sigma^+$ and $Σ^+$ is a set of positive roots of $K$. In real hyperbolic $n$ space case ($K = SO(n)$),

$$m = 0, 1, 2, 4$$

for $n = 2, 3, 4, 6$ and for $n ≥ 5$ (odd), $m = (n−1)/4$ and for $n ≥ 8$ (even), $m = n^2/4$.

From this we get

**Corollary 8.2.** Let $Γ$ be a geometrically finite discrete subgroup of $G$ with $δ$ as in the standing assumption. Then for any $φ_1 ∈ C^∞(Γ\backslash G)^K ∩ L^2(Γ\backslash G/M)$, $φ_2 ∈ C^∞_c(Γ\backslash G/M)$ and $0 < y < 1$,

$$\langle a_yφ_1, φ_2 \rangle = \langle φ_1, φ_0 \rangle \langle a_yφ_0, φ_2 \rangle + O(y^{D−s_0}S_{2m}(φ_1)S_{2m}(φ_2)).$$
Proof. Note first that
\[ L^2(\Gamma \setminus G)^K = V_\delta \oplus V \]
where \( V_\delta \) is a complementary series corresponding to \( \delta \) and \( V \) does not contain any complementary series \( V_s \) for \( s \geq s_\Gamma \). Put \( \phi_1 = \langle \phi_1, \phi_0 \rangle \phi_0 + \phi_1^\perp \).
Since \( \phi_0 \) is \( K \)-invariant, \( \phi_1^\perp \) is also \( K \)-invariant. Then
\[
\langle a_y \phi_1, \phi_2 \rangle = \langle \phi_1, \phi_0 \rangle \langle a_y \phi_0, \phi_2 \rangle + \langle a_y \phi_1^\perp, \phi_2 \rangle
= \langle \phi_1, \phi_0 \rangle \langle a_y \phi_0, \phi_2 \rangle + O(y^{D-s}S_{2m}(\phi_1)S_{2m}(\phi_2))
\]
since \( S_{2m}(\phi_1^\perp) \ll S_{2m}(\phi_1) \) and by Proposition 8.1. \( \Box \)

9. Equidistribution

Suppose the image of a horosphere is closed in \( \Gamma \setminus H^n_\Gamma \) a geometrically finite manifold. Let \( N(J) = \{ [n] \in (NM \cap \Gamma) \setminus NM : \Gamma \setminus \Gamma_nA \cap J \neq \emptyset \} \) where \( J \subset \Gamma \setminus G \) a compact set, i.e., the set of elements on \( F_N \) whose orbit under the \( A \)-flow intersects \( J \). We use the notations of section 2 and section 6.

Lemma 9.1. \( N(J) \) is bounded if \( J \) is compact.

Proof. Let \( C_\Gamma \) be the convex hull of the limit set and \( \Gamma \setminus C_\Gamma \) the convex core whose volume is finite due to the geometrical finiteness of \( \Gamma \). We claim that there exist \( L \gg 1 \) and a fundamental domain \( F \) of \( \Gamma \) such that
\[
\{(z, t, v) : (z, t) \in F_N, |(z, t)| > L \text{ or } v > L \} \subset F.
\]

If the base point \( \infty \) of the horosphere is not in the limit set of \( \Gamma \), since the domain of discontinuity is open, there exists a horoball based at \( \infty \) which embeds into the manifold \( \Gamma \setminus H^n_\Gamma \), hence embeds into a fundamental domain \( F \) of \( \Gamma \).

If the base point is a bounded parabolic fixed point and \( F_N \) is unbounded, since \( \Gamma \) is geometrically finite, the intersection \( C_\Gamma \cap (F_N \times [0, \infty)) \subset \{(z, t) \in F_N : |(z, t)| < K \} \times [0, \infty) \) for some \( K \). Otherwise since a horoball \( H = \{(z, t, v) : v > R \} \) is stabilized by \( \Gamma \cap NM \) for some large \( R \) and quotient down to a cusp of \( \Gamma \setminus H^n_\Gamma \), the quotient of \( H \cap C_\Gamma \) would have a infinite volume, hence the volume of the convex core would be infinite. Take a \( \Gamma \)-invariant small \( \varepsilon \)-neighborhood \( \mathcal{N} \) of \( C_\Gamma \). Then \( \mathcal{N} \cap (F_N \times [0, \infty)) \subset \{(z, t) \in F_N : |(z, t)| < K' \} \times [0, \infty) \) for some \( K' \). The nearest point retraction \( r : (H^n_\Gamma \setminus \mathcal{N}) \to \partial \mathcal{N} \) commutes with the action of \( \Gamma \). Since \( \partial \mathcal{N} \) is invariant under \( \Gamma \) and \( F_N \times [0, \infty) \setminus \mathcal{N} \subset r^{-1}(\partial \mathcal{N} \cap (F_N \times [0, \infty))) \) due to the convexity of \( \mathcal{N} \), if we denote \( \mathcal{F} \) a fundamental domain of \( \Gamma \) on \( \partial \mathcal{N} \), and \( \mathcal{F}_C \) a fundamental domain of \( \Gamma \) on \( C_\Gamma \),
\[
r^{-1}(\mathcal{F}) \cup \mathcal{F}_C = F,
\]
and so for some large \( L \gg 1 \)
\[
\{(z, t) \in F_N : |(z, t)| > L \} \times [0, \infty) \subset F.
\]
When \( F_N \) is bounded, there is nothing to prove.

Now suppose \( N(J) \) is unbounded. Then there exist sequences \( n_j \in F_N \to \infty, a_j \in A, \gamma_j \in \Gamma \) and \( w_j \in J \) such that \( n_ja_j = \gamma_jw_j \). As \( J \) is bounded, we
may assume that \( w_j \to w \in G \). Take \( \gamma_0 \in \Gamma \) such that \( \gamma_0 w(0, 0, 1) \in \text{int} F \) by choosing the base point different from \((0, 0, 1)\) if necessary. Then for large \( j \), \( \gamma_0 w_j(0, 0, 1) \in \text{int} F \). Since \( n_j \to \infty \), by (21), \( n_j \gamma_j a_j(0, 0, 1) \in F \) for large \( j \). Therefore for large \( j \)

\[
\gamma_0 \gamma_j^{-1} n_j a_j(0, 0, 1) = \gamma_0 w_j(0, 0, 1) \in \text{int} F \cap \gamma_0 \gamma_j^{-1}(F).
\]

Since \( F \) is a fundamental domain of \( \Gamma \), \( \gamma_0 = \gamma_j \) for all large \( j \). Since \( w_j \to w \), \( n_j a_j = \gamma_j w_j \) is bounded, a contradiction. This shows that \( N(J) \) is bounded.

So for \( \phi \in C_0(\Gamma \setminus G/M) \) and any function \( \eta \in C_0((NM \cap \Gamma) \setminus NM/M) \) with \( \eta|_{N(\text{supp} \phi)} = 1 \), we have

\[
\int_{(NM \cap \Gamma) \setminus NM/M} \phi(na_y) \eta(n) \, dn = \int_{N(\text{supp} \phi)} \phi(na_y) \, dn
\]

since \( \phi(na_y) = 0 \) for \( n \) outside \( N(\text{supp} \phi) \). Hence we may assume that

\[
\int_{(NM \cap \Gamma) \setminus NM/M} \phi(na_y) \, dn = I_\eta(\phi)(a_y).
\]

Since \( \rho_\epsilon \) is an \( \epsilon \)-approximation in \( A \)-direction \( S_{2m}(\rho_\epsilon) = O_\eta(\epsilon^{-p}) \) for some \( p > 0 \), where \( \rho_\epsilon = \rho_{\eta, \epsilon} \). Fix \( l \) an integer so that

\[
l > \frac{(D - \delta)(p + 1)}{(\delta - s_\Gamma)}.
\]

Let \( \psi_0 = \phi \) and define

\[
\psi_i = \psi_{i-1}', ~ 1 \leq i \leq l
\]

inductively.

Then a similar technique in (23) shows that

**Theorem 9.2.** For any \( \phi \in C_0^\infty(\Gamma \setminus G)^K \),

\[
\int_{F_N} \phi(n y) \, dn = \langle \phi, \phi_0 \rangle c_{\phi_0} y^{D-\delta}(1 + O(y^{\delta'}))
\]

for some constants \( c_{\phi_0} \) and \( \delta' \). Here \( \delta' \) depends only on the spectral gap and the implied constant depends only on Sobolev norm \( S_\nu(\phi) \) and \( \eta \), hence on \( F_{\Lambda_\epsilon} \).

**Proof.** Now for each \( 0 \leq i \leq l - 1 \)

\[
I_\eta(\psi_i)(a_y) = \langle a_y \psi_i, \rho_\epsilon \rangle + O((\epsilon + y) I_\eta(\psi_{i+1})(a_y))
\]

by Proposition 6.4 and

\[
I_\eta(\psi_i(a_y)) = \langle a_y \psi_i, \rho_\epsilon \rangle + O((\epsilon + y) I_\eta(\psi'_{i})(a_y)).
\]

Here since \( I_\eta(\psi'_{i})(a_y) = \int_{(NM \cap \Gamma) \setminus NM/M} \psi_i'(na_y) \eta(n) \, dn \leq C \int_{(NM \cap \Gamma) \setminus NM/M} \eta(n) \, dn \) where \( C = \max \psi_i'(na_y) \) on \( \text{supp} \eta \), we map write

\[
I_\eta(\psi_i(a_y)) = \langle a_y \psi_i, \rho_\epsilon \rangle + O((\epsilon + y))
\]

where the implied constant in \( O_\eta \) depends on \( \int_{(NM \cap \Gamma) \setminus NM/M} \eta(n) \, dn \).
By Corollary \[5.2\] for \(1 \leq i \leq l\), using \(S_k\psi_i \ll S_q\psi_{i-1}\),

\[
\langle a_y\psi_i, \rho_c \rangle = \langle \psi_i, \phi_0 \rangle \langle a_y\phi_0, \rho_c \rangle + O(y^{D-s\nu} S_m(\psi_i) S_m(\rho_c)) \\
= O(\langle a_y\phi_0, \rho_c \rangle \|\psi_i\|_2) + O(y^{D-s\nu} \epsilon^{-p} S_q(\phi)) \\
= O(\langle a_y\phi_0, \rho_c \rangle S_{2m+l}(\phi)) + O(y^{D-s\nu} \epsilon^{-p} S_q(\phi)).
\]

Hence for \(y < \epsilon\),

\[
I_\eta(\phi)(a_y) = \langle a_y\phi, \rho_c \rangle + \sum_{k=1}^{l-1} O(\langle a_y\psi_k, \rho_c \rangle (\epsilon + y)^k) + O_{\eta}(I_\eta(\psi_i)(a_y)(\epsilon + y)^l)
\]

\[
= \langle \phi, \phi_0 \rangle \langle a_y\phi_0, \rho_c \rangle + O(\langle a_y\phi_0, \rho_c \rangle \epsilon S_q(\phi)) + O(\epsilon y^{D-s\nu} \epsilon^{-p} S_q(\phi)) + O_{\eta}(\epsilon^l)
\]

\[
= \langle \phi, \phi_0 \rangle \phi_0^N(a_y) + O(\epsilon^l) + O(\epsilon y^{D-\delta}) + O(y^{D-s\nu} \epsilon^{-p} S_q(\phi)) + O_{\eta}(\epsilon^l)
\]

by Proposition \[6.3\]. All the implied constants depend on \(S_q(\phi)\) and \(\int \eta d\nu\).

Setting \(\epsilon = y^{(\delta-s\nu)/(p+1)}\), we have \(\epsilon^l < y^{D-\delta}\) and

\[
\int_{(NM\cap \Gamma) \setminus NM/M} \phi(na_y) dn = I_\eta(\phi)(a_y) = \langle \phi, \phi_0 \rangle \phi_0^N(a_y)(1 + O(\epsilon^l) y^{(\delta-s\nu)/(p+1)})
\]

using the fact that \(\phi_0^N(a_y) = c_{\phi_0} y^{D-\delta} + d_{\phi_0} y^\delta\) and the fact \(\delta > D/2 > \frac{\delta-s\nu}{p+1} + D - \delta\). The claim follows by setting \(\delta' = \frac{\delta-s\nu}{p+1}\).

There exists a Burger-Roblin measure \(\hat{\mu}\) on \(T^1 X = G/M\) which descends to \(T^1(\Gamma\setminus X)\) and satisfies

\[
\hat{\mu}(\psi) = \langle \psi, \phi_0 \rangle
\]

for \(K\)-invariant functions \(\psi \in C_c(G)\). Specially \(\hat{\mu}(\phi_0) = 1\). Indeed one can write down \(\hat{\mu}\) explicitly as follows. A fixed Patterson-Sullivan measure \(\nu_0\) on \(\partial X = K/M\) can be regarded as a measure on \(K\) via the projection \(K \to K/M\): for \(f \in C(K)\),

\[
\nu_0(f) = \int_{K/M} \int_M f(km) dm d\nu_0(k)
\]

for the probability invariant measure \(dm\) on \(M\). Using this, if \(dg = y^{-D-1} dy dndk\)

is a Haar measure on \(G = NAK\), for \(\psi \in C_c(G/M)\),

\[
(25) \hat{\mu}(\psi) = \int_K \int_{AN} \psi(g) y^\delta y^{-D-1} dy dnd\nu_0.
\]

This measure is left \(\Gamma\)-invariant and descends to the measure on \(T^1(\Gamma\setminus X) = \Gamma\setminus G/M\).

Then

**Theorem 9.3.** For \(\psi \in C_c(T^1(\Gamma\setminus X))\),

\[
\int_{n \in (NM\cap \Gamma) \setminus NM/M} \psi(na_y) dn \sim c_{\phi_0} \hat{\mu}(\psi) y^{D-\delta}
\]

as \(y \to 0\).
Proof. See [23]. The argument given there works for general rank one space. □

10. Orbital counting on a light cone

In this section we give a quantitative estimate of the orbital counting on $\mathbb{F}^{n+1}$ using the results in previous sections.

Let $\Gamma \subset G$ be a geometrically finite torsion-free subgroup with $\delta > D/2$. Let $\langle v_0, v_0 \rangle = 0$ where $\langle \cdot , \cdot \rangle$ is the standard Hermitian form of signature $(n, 1)$ on $\mathbb{F}^{n+1}$. Let $L$ be the light cone of $\mathbb{F}^{n+1}$ given by $\langle v, v \rangle = 0$. Then $G$ acts transitively on $L$ and so there is $g_0 \in G$ such that the stabilizer of $v_0 g_0$ is $N M$. By conjugating $\Gamma$ by $g_0$, we may assume that the stabilizer of $v_0$ in $G$ is $N M$.

In the language of Section 4 then $v_0$ is proportional to $(0, 0, 1)$ and the action of $a_y$ on it is

$$
(0 \ 0 \ 1) \begin{bmatrix} y & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 1/y \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1/y \end{bmatrix},
$$

i.e.,

$$
v_0 a_y = \frac{1}{y} v_0.
$$

Traditionally we prefer $G$ to act on the right and $\Gamma$ on the left. So vectors in $\mathbb{F}^{n+1}$ will be row vectors and the matrices act on the right. In right action notation, $v_0 N M = v_0$, so the stabilizer of the row vector $v_0$ is $N M$, and $v_0 a_y = \frac{1}{y} v_0$.

**Lemma 10.1.** If $v_0 \Gamma$ is discrete, then the image of the horosphere $H_0$ corresponding to $v_0$ is closed in $\Gamma \backslash H^n_F$. Furthermore the image of the horosphere $H_0$ is closed in $\Gamma \backslash H^n_F$ if and only if $H_0$ is based either at a bounded parabolic fixed point or at the point which is not in the limit set.

**Proof.** Any horosphere is obtained by projectivizing $\Pi$ intersection with the hyperboloid where $\Pi$ is an affine hyperplane $\Pi_X = \{ Y \in \mathbb{F}^{n+1} : \langle (Z, X) \rangle = 1 \}$ with $X \in L \setminus 0$. Here $\langle \cdot , \cdot \rangle$ is a standard positive definite inner product of signature $n+1$ on $\mathbb{F}^{n+1}$ and $X$ is uniquely determined up to an element in $U(1)$ or $Sp(1)$. This gives a homeomorphism between the set of horospheres and the set of $(L \setminus 0)/U(1)$ (resp. $Sp(1)$).

Now suppose the orbit $v_0 \Gamma$ is discrete, and $H_0$ to be the horosphere corresponding to $v_0$. If $\Gamma \backslash H_0$ is not closed in $\Gamma \backslash H^n_F$, there are points $p_i \in H_0$ so that $\pi(p_i)$ accumulates to a point $\bar{p} \notin \Gamma \backslash H_0$ where $\pi : H^n_F \rightarrow \Gamma \backslash H^n_F$. This means that there exist $\gamma_i \in \Gamma$ so that $p_i \gamma_i$ accumulates to $p \in H^n_F$. Hence $H_0 \gamma_i$ accumulate, which is forbidden by discreteness of $H_0 \Gamma$. The second claim is proved by Dal’Bo [11] for the Hadamard manifold whose sectional curvature is bounded above by $-1$. She showed that the projection of the horosphere based at $\infty$ is closed if and only if either $\infty$ is a bounded parabolic fixed point or $\infty$ does not belong to the limit set provided that the
length spectrum of $\Gamma$ is non-discrete, which is shown in [22] for rank one symmetric space.

Define a function on $\Gamma \backslash G$ by

$$F_T(g) = \sum_{\gamma \in (\Gamma \cap NM) \backslash \Gamma} \chi_{B_T(v_0 \gamma g)}$$

where $B_T = \{ v \in v_0 G : ||v|| < T \}$. Specially $F_T(e)$ is the orbital counting function of $\Gamma$. Note that for $\psi \in C_c(\Gamma \backslash G)$

$$\langle F_T, \psi \rangle = \int_{\Gamma \backslash G} \sum_{\gamma \in (\Gamma \cap NM) \backslash \Gamma} \chi_{B_T(v_0 \gamma g)} \psi(g) dg$$

$$= \int_{(\Gamma \cap NM) \backslash G} \chi_{B_T(v_0 g)} \psi(g) dg \quad (\psi \text{ is } \Gamma \text{ invariant})$$

Write $G$ as $G = (NM/M)AM(M \backslash K)$ and accordingly $g = [n]a_ymk$. Then the above integral is

$$= \int_{M \backslash K} \int_0^T \int_{[n_x] \in (\Gamma \cap NM) \backslash NM/M} \psi([n_x]a_ymk) y^{-D-1} d[n] dmdydk$$

$$= \int_{M \backslash K} \int_{y > T^{-1}[v_0 k]} \left( \int_{[n_x] \in (\Gamma \cap NM) \backslash NM/M} \int_{m \in M} \psi([n_x]a_y mk) dmd[n] \right) y^{-D-1} dydk$$

$$= \int_{M \backslash K} \int_{y > T^{-1}[v_0 k]} \psi_k([n_x]a_y) d[n] y^{-D-1} dydk$$

(26)

$$= \int_{M \backslash K} \int_{y > T^{-1}[v_0 k]} \psi_k^N(a_y) y^{-D-1} dydk,$$

where $\psi_k(g) = \int_{m \in M} \psi(gmk) dm$ and $\psi^N(a_y) = \int_{(\Gamma \cap NM) \backslash NM/M} \psi([n]a_y) d[n]$ as before.

**Lemma 10.2.** For any $\psi \in C_c(\Gamma \backslash G)$, as $T \to \infty$

$$\langle F_T, \psi \rangle \sim \delta^{-1} c_{\psi_0} T^\delta \int_{M \backslash K} \hat{\mu}(\psi_k)[v_0 k]^{-\delta} dk.$$
Proof. The function \( \psi_k \) is \( M \)-invariant, hence it is defined on \( T^1(\Gamma \setminus X) \), and by Theorem 9.3,

\[
\psi_k^N(a_y) = \int_{n \in (\Gamma \cap NM) \setminus NM/M} \psi_k([n]a_y)d[n] \sim c_{\phi_0} \hat{\mu}(\psi_k)y^{D-\delta}
\]
as \( y \to 0 \). By the equation (26)

\[
\langle F_T, \psi \rangle \sim \int_{M \setminus K} \int_{y > T^{-1}||v_0||} c_{\phi_0} \hat{\mu}(\psi_k)y^{D-\delta}y^{-D-1}dydk
\]

\[
= \delta^{-1}c_{\phi_0}T^{\delta} \int_{M \setminus K} \hat{\mu}(\psi_k)||v_0||^{-\delta}dk.
\]

If \( \psi \) and \( || \cdot || \) are \( K \)-invariant, since \( \psi_k = \psi \), by Theorem 9.2,

\[
\psi_k^N(a_y) = \langle \psi, \phi_0 \rangle c_{\phi_0}y^{D-\delta}(1 + O(y^\delta)).
\]

Hence

\[
\langle F_T, \psi \rangle = \int_{M \setminus K} \int_{y > T^{-1}||v_0||} \langle \psi, \phi_0 \rangle c_{\phi_0}y^{D-\delta}(1 + O(y^\delta))y^{-D-1}dydk
\]

\[
= dk(M \setminus K) \int_{y > T^{-1}||v_0||} \langle \psi, \phi_0 \rangle c_{\phi_0}y^{D-\delta}(1 + O(y^\delta))y^{-D-1}dy
\]

\[
= dk(M \setminus K) \langle \psi, \phi_0 \rangle c_{\phi_0} \delta^{-1}T^{\delta}||v_0||^{-\delta}(1 + O(T^{-\delta})).
\]

\( \square \)

**Theorem 10.3.** As \( T \to \infty \),

\[
F_T(e) \sim c_{\phi_0} \delta^{-1}T^{\delta} \int_K ||v_0||^{-\delta}dk.
\]

If \( || \cdot || \) is \( K \)-invariant,

\[
F_T(e) = \delta^{-1} \phi_0(0) c_{\phi_0}||v_0||^{-\delta}T^{\delta}(1 + O(T^{-\delta})).
\]

**Proof.** For small \( \epsilon \), choose a symmetric \( \epsilon \)-neighborhood \( U_\epsilon \) of \( e \) in \( G \), which injects to \( \Gamma \setminus G \), such that for all \( T \gg 1 \) and \( 0 < \epsilon \ll 1 \),

\[
B_TU_\epsilon \subset B((1+\epsilon)T) \text{ and } B((1-\epsilon)T) \subset \cap_{u \in U_\epsilon} B_Tu.
\]

Let \( \phi_\epsilon \in C_c^\infty(G) \) be a nonnegative function supported on \( U_\epsilon \) with \( \int_G \phi_\epsilon = 1 \).

Define \( \Phi_\epsilon \) on \( \Gamma \setminus G \) by averaging over \( \Gamma \)

\[
\Phi_\epsilon(\Gamma g) = \sum_{\gamma \in \Gamma} \phi_\epsilon(\gamma g).
\]

By definition, \( \Phi_\epsilon([g]) \neq 0 \) only if \( [g] \in \Gamma \setminus \Gamma U_\epsilon \). Also for \( g \in U_\epsilon \), if \( ||v_0\gamma|| < T \) then \( ||v_0\gamma g|| < (1+\epsilon)T \).

So

\[
\langle F_{(1+\epsilon)T}, \Phi_\epsilon \rangle
\]

\[
= \int_{\Gamma \setminus G} \sum_{\gamma \in (\Gamma \cap NM) \setminus \Gamma} \chi_{B((1+\epsilon)T)}(v_0\gamma g)\Phi_\epsilon(g)dg = \int_{U_\epsilon} \sum_{\gamma \in (\Gamma \cap NM) \setminus \Gamma} \chi_{B((1+\epsilon)T)}(v_0\gamma g)\phi_\epsilon(g)dg
\]
In equation (27), let $\epsilon$ be small.

By Lemma 10.2, \[\langle F_{(1+\epsilon)T}, \Phi_\epsilon \rangle \geq F_T(\epsilon).\]

Similarly
\[\langle F_{(1-\epsilon)T}, \Phi_\epsilon \rangle \leq F_T(\epsilon).\]

Since $||v_0\gamma\text{Supp}(\phi_\epsilon)|| < (1 - \epsilon)T$, \(\langle F_{(1-\epsilon)T}, \Phi_\epsilon \rangle \leq F_T(\epsilon).\)
So we obtain
\[\langle F_{(1\pm \epsilon)T}, \Phi_\epsilon \rangle \sim \delta^{-1}c_{\phi_0}(T(1 \pm \epsilon))^{\delta} \int_{M\setminus K} \hat{\mu}(\Phi_k) ||v_0k||^{-\delta} dk.\]

But
\[\hat{\mu}(\Phi_k) = \int_{\Gamma \setminus G} \int_M \Phi_k(gmk)dmd\hat{\mu}(g).\]

Hence
\[\langle F_{(1\pm \epsilon)T}, \Phi_\epsilon \rangle \sim \delta^{-1}c_{\phi_0}(T(1 \pm \epsilon))^{\delta} \int_{M\setminus K} \hat{\mu}(\Phi_k) ||v_0k||^{-\delta} dk = \delta^{-1}c_{\phi_0}(T(1 \pm \epsilon))^{\delta} \int_{\Gamma \setminus G} \int_{K} \Phi_k(gk) ||v_0k||^{-\delta} dk d\hat{\mu}(g).\]

For a fixed $k_0$, using Equation (25)
\[\int_{\Gamma \setminus G} \Phi_k(gk_0)d\hat{\mu}(g) = \int_{\Gamma \setminus G} \Phi_k(gk_0)y^{\delta} \frac{dg}{dk} dv_0 = \int_{\Gamma \setminus G} \Phi_k(g) \frac{y^{\delta} dg}{dk} (k_0)^{-\delta} dv_0.\]

But if $\epsilon$ is small, $y$ runs, say from $1 - \epsilon$ to $1 + \epsilon$. Since $\int \phi_\epsilon dg = 1$, $\int_{\Gamma \setminus G} \phi_\epsilon(g) y^{\delta} \frac{dg}{dk} (k_0)^{-\delta} dv_0$ is in the order of $\int_{1-\epsilon}^{1+\epsilon} y^{\delta} = O(\epsilon)$.

Hence the above equality becomes
\[\langle F_{(1\pm \epsilon)T}, \Phi_\epsilon \rangle \sim \delta^{-1}c_{\phi_0}(T(1 \pm \epsilon))^{\delta} \int_{K} ||v_0k||^{-\delta} dk + O(\epsilon).\]

In equation (27), let $\epsilon \to 0$ to obtain
\[F_T(\epsilon) \sim \delta^{-1}c_{\phi_0} T^{\delta} \int_{K} ||v_0k||^{-\delta}.\]

If $|| \cdot ||$ is $K$-invariant, take $U_\epsilon$ and $\phi_\epsilon K$-invariant, and we obtain
\[\langle F_{(1\pm \epsilon)T}, \Phi_\epsilon \rangle = \langle \Phi_\epsilon, \phi_0 \rangle c_{\phi_0} \delta^{-1}(T(1 \pm \epsilon))^{\delta} ||v_0||^{-\delta}(1 + O(T^{-\delta})).\]
Since $U_\epsilon$ must contain $K$ and $dm$ is a probability measure on $M \subset K$, we can take $\phi_\epsilon(e) = 1$ to have $\int \phi_\epsilon dg = 1$. Hence $\langle \Phi_\epsilon, \phi_0 \rangle = \int_{g \in U_\epsilon} \phi_\epsilon(g) \phi_0(g) dg = \phi_0(e) + R_\epsilon$ and $R_\epsilon \to 0$ as $\epsilon \to 0$. So we obtain

$$F_T(e) = \phi_0(e) c_{\phi_0} \delta^{-1} (T) \|v_0\|^{-\delta} (1 + O(T^{-\delta'})).$$

□

Finally the orbital counting theorem can be stated as follows:

**Theorem 10.4.** Let $\Gamma \subset G = KAN$ be a geometrically finite group with critical exponent $\delta > \frac{D}{2}$ for real, complex and quaternionic hyperbolic space, where $KAN$ is a fixed Iwasawa decomposition introduced in section 2. Suppose $v_0$ is in the light cone such that $v_0 \Gamma$ is discrete, and the stabilizer of $v_0$ in $g_0^{-1} \Gamma g_0$ is in $NM$. Then for any norm $\| \cdot \|$ on $\mathbb{F}_n$,

$$\# \{ v \in v_0 \Gamma : \| v \| < T \} \sim c_{\phi_0} \delta^{-1} T \delta \int_K \|v_0(g_0^{-1} kg_0)\|^{-\delta} dk.$$

If $\| \cdot \|$ is $g_0^{-1} K g_0$-invariant, then

$$\# \{ v \in v_0 \Gamma : \| v \| < T \} = c_{\phi_0} \delta^{-1} T \delta \|v_0\|^{-\delta} (1 + O(T^{-\delta'})).$$

11. **APPLICATION TO APOLLONIAN PACKING**

In euclidean $n$-space, the maximum number of mutually tangent $n-1$-spheres is $n+2$, if one demands that no three spheres have a common point. An Apollonian sphere packing of $n$-dimensional euclidean space is for a given $n+1$ mutually tangent spheres, one adds a sphere which is tangent to all the $n+1$ previous spheres. This procedure defines a sphere packing, called Apollonian sphere packing.

The formula for curvatures for such an $n+2$ mutually tangent spheres is \[10, 3]\]

$$n \sum_{i=1}^{n+2} \kappa_i^2 = \left( \sum_{i=1}^{n+1} \kappa_i \right)^2.$$

For a given $n+1$ mutually tangent spheres, there are exactly two spheres satisfying the above equation. The sum of curvatures of these two last spheres is

$$\kappa_{n+2} + \kappa'_{n+2} = \frac{2}{n-1} \sum_{i=1}^{n+1} \kappa_i.$$

Hence for $n \leq 3$, if the first $n+2$ generating spheres have integral curvatures, then the curvature $\kappa'_{n+2}$ of the next sphere is also integral. So we restrict our attention to integral Apollonian packings.

Let

$$Q(\kappa_1, \cdots, \kappa_{n+2}) = n \sum_{i=1}^{n+2} \kappa_i^2 - \left( \sum_{i=1}^{n+2} \kappa_i \right)^2$$

\[10\]
be a quadratic form on \( \mathbb{R}^{n+2} \). This has signature \((n + 1, 1)\). Hence the orthogonal group \( O_Q \) can be identified with the group of hyperbolic isometry in \( H_R^{n+1} \), \( O(n + 1, 1) \).

From now on we focus on \( n = 3 \), hence

\[
\kappa_5 + \kappa'_5 = \sum_{i=1}^{4} \kappa_i.
\]

There is an integral group, called Apollonian group, defined as follows. Changing the curvature \((\kappa_1, \cdots, \kappa_4, \kappa_5)\) to \((\kappa_1, \cdots, \kappa_4, \kappa'_5)\) corresponds to the integral matrix

\[
S_5 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & -1
\end{pmatrix}
\]

Similarly there exist corresponding matrices \( S_1, S_2, S_3, S_4 \). Denote \( \mathcal{A} = \langle S_1, S_2, S_3, S_4, S_5 \rangle \subset O_Q(\mathbb{Z}) \), the Apollonian group. This group acts on apollonius sphere packing \( \mathcal{P} \) generated by the initial spheres with curvature \( \kappa_1, \cdots, \kappa_5 \). We will see shortly that this Kleinian group is geometrically finite.

If \( S^1, \cdots, S^5 \) are initial reflections, then next sphere \( S^{5'} \) is obtained by a reflection along the sphere \( T^5 \) which passes through the intersection points of \( S^1, S^2, S^3, S^4 \). So the element \( S_5 \) corresponds to a reflection \( R_5 \) along \( T^5 \). It is similar for other elements \( S_i \). This way \( \mathcal{A} \) corresponds to a group \( \Gamma \) generated by reflections along \( D_i \) with a fundamental domain bounded by \( D_i \). Since the fundamental domain has finite sides, it is a geometrically finite group with cusps.

Note that \( \Gamma \) acts on \( \mathcal{P} \) by permuting spheres in \( \mathcal{P} \), so it leaves invariant the set of tangent points between spheres. But it is easy to see that any point on some sphere can be approximated by tangent points, hence the limit set of \( \Gamma \) is equal to the union of spheres in \( \mathcal{P} \). Since a sphere has Hausdorff dimension 2, the Hausdorff dimension of the limit set \( \Lambda \Gamma > 2 > 3/2 \). Hence our assumption that the critical exponent is greater than \( D/2 \) is satisfied to apply previous results to this section.

11.1. **Orbital counting for \( \mathcal{A} \).** We use a \( K \)-invariant norm \( || \cdot || \) in this section, in fact a maximum norm, and \( G = SO^0(4, 1) \). As before \( B_T = \{ v \in \mathbb{R}^3 : Q(v) = 0, ||v|| < T \} \), i.e., the set of curvatures of maximal mutually tangent spheres in \( \mathbb{R}^3 \) whose norm is less than \( T \). Fix an initial curvatures \( \xi \) so that \( Q(\xi) = 0 \) and the corresponding spheres generate an Apollonian sphere packing in \( \mathbb{R}^3 \). By conjugation if necessary, we will assume that the
stabilizer of $\xi$ is equal to $NM$. One has to recall the definitions and notations from section 10. Let $\phi_\epsilon \in C^\infty_c(H_\mathbb{R}^4) = C^\infty_c(SO^0(4, 1))^K$ be a nonnegative function supported on a $K$-invariant $U_\epsilon$ with $\int_{H_\mathbb{R}^4} \phi_\epsilon = 1$. Define a function defined on $M = \Gamma \backslash H_\mathbb{R}^4 = \Gamma \backslash G/K$ by

$$\Phi_\epsilon(\Gamma g) = \sum_{\gamma \in \Gamma} \phi_\epsilon(\gamma g).$$

Fix a primitive integral polynomial $f$ in 5 variables. One defines a positive sequence $A(T) = \{a_n(T)\}$ where

$$a_n(T) = \sum_{\gamma \in \text{Stab}_\Gamma(\xi) \backslash \Gamma} w_T(\gamma)$$

where $w_T(\gamma) = \int_{G/K} \chi_B T(\xi_\gamma g) d\mu(g)$ for each $\gamma \in \Gamma$. A subsequence $A_{d}(T) = \{a_n(T) | n = 0(\text{mod } d)\}$ and define $|A(T)| = \sum a_n(T)$, $|A_d(T)| = \sum_{n=0(\text{mod } d)} a_n(T)$.

For any subgroup $\Gamma_0 \subset \Gamma$ with $\text{Stab}_\Gamma(\xi) = \text{Stab}_{\Gamma_0}(\xi)$, define

$$F^\Gamma_0(g) = \sum_{\gamma \in \text{Stab}_\Gamma(\xi) \backslash \Gamma_0} \chi_B T(\xi_\gamma g),$$

and

$$\Phi^\Gamma_0(\epsilon_1 \gamma_1)(g) = \sum_{\gamma \in \Gamma_0} \phi_\epsilon(\gamma^{-1}_1 \gamma g), \quad \gamma_1 \in \Gamma,$$

which is an $\epsilon$-approximation to the identity around $[\gamma_1^{-1}]$ in $\Gamma_0 \backslash H_\mathbb{R}^4$. Finally let

$$\Gamma(\xi)(d) = \{\gamma \in \Gamma : \xi \gamma = \xi(\text{mod } d)\},$$

and its finite index subgroup

$$\Gamma(d) = \{\gamma \in \Gamma : \gamma = I(\text{mod } d)\}.$$

Then $\text{Stab}_\Gamma(\xi) = \text{Stab}_{\Gamma_0}(\xi)$, The following is due to Bourgain-Gamburd-Sarnak [5], Salehi Golsefidy-Varjú [32], Breuillard-Green-Tao [7] and Pyber-Szabo [31].

**Theorem 11.1.** For any square free integer $d$, the $L^2$-spectrum of $\Gamma(d) \backslash H_\mathbb{R}^4$ has a uniform spectral gap $2 \leq \theta < \delta$ so that it does not have any spectrum in $[\theta(3 - \theta), \delta \Gamma(3 - \delta \Gamma)]$.

**Proof.** (Step I: Family of Expanders) For general Apollonian sphere packing in $\mathbb{R}^n$,

$$\Gamma = \mathcal{A} = \langle S_1, \cdots, S_{n+2} \rangle \subset G = O_4(\mathbb{Z}[\frac{1}{n-1}]) \subset GL(n + 2, \mathbb{Z}[\frac{1}{n-1}]).$$

Note that $\Gamma$ is not discrete if $n \geq 4$. Then the following is proved by A. Salehi Golsefidy and P. Varjú [32].
Theorem A Let $\Gamma \subset GL(d, (\mathbb{Z}[\frac{1}{q_0}])$ be a group generated by a symmetric set $S$. Then Cayley($\Gamma / \Gamma(q); S \mod q$) form a family of expanders when $q$ ranges over square-free integers coprime to $q_0$ if and only if the connected component of the Zariski-closure of $\Gamma$ is perfect.

(Step II: Uniform spectral gap) This is a corollary of the above theorem. For reader’s convenience, following closely the argument in [5], we give the details of its extension to higher dimensional real hyperbolic spaces. Suppose $\Gamma$ is a geometrically finite discrete group in $G$ the details of its extension to higher dimensional real hyperbolic spaces. For reader’s convenience, following closely the argument in [5], we give the details of its extension to higher dimensional real hyperbolic spaces.

Fix a fundamental domain $\mathcal{F} \subset H^{n+1}_R$ of $\Gamma$. For any map $f$ defined on $\Gamma(q) \backslash H^{n+1}_R$, let $\tilde{f}$ be a lift to $H^{n+1}_R$. Then $f$ can be regarded as a vector valued function $F$ defined on $\Gamma \backslash H^{n+1}_R$ by setting

$$F(z) = (\tilde{f}(\gamma z))_{\gamma \in \Gamma(q)}$$

where $z \in \mathcal{F}$ by identifying $\Gamma \backslash H^{n+1}_R$ with $\mathcal{F}$. It satisfies the equivariant condition

$$F(\gamma z) = R(\gamma)F(z)$$

for $\gamma \in G(\mathbb{F}_q)$, where $R(\gamma)$ denotes the right regular representation of $G(\mathbb{F}_q)$.

More formally, every $L^2$ function on $\Gamma(q) \backslash H^{n+1}_R$ can be viewed as an $L^2$ section of a flat bundle $E$ of rank $|G(\mathbb{F}_q)|$ on $\Gamma \backslash H^{n+1}_R$ as follows: Given $f \in L^2(\Gamma(q) \backslash H^{n+1}_R, \mathbb{R})$, define $F : H^{n+1}_R \to L^2(G(\mathbb{F}_q))$ by

$$F(x, g) = \tilde{f}(gx)$$

for $x \in \mathcal{F}, g \in \Gamma$. Then, for $h \in \Gamma$,

$$F(hx, g) = \tilde{f}(ghx) = (R(h \mod \Gamma(q))F)(x, g),$$

so $F$ can be viewed as a section of the flat vector bundle $E$ over $\Gamma \backslash H^{n+1}_R$ associated to the $\Gamma$-principal bundle $H^{n+1}_R \to \Gamma \backslash H^{n+1}_R$ via the right regular representation of $G(\mathbb{F}_q)$ composed with $\Gamma \to \Gamma/\Gamma(q) = G(\mathbb{F}_q)$. Or, equivalently, $E$ is associated to the $G(\mathbb{F}_q)$-principal bundle $\Gamma(q) \backslash H^{n+1}_R \to \Gamma \backslash H^{n+1}_R$ via the right regular representation $R$ of $G(\mathbb{F}_q)$.

Then $E = E_0 + E_1$ according to the splitting of the right regular representation into constant functions, and functions on $G(\mathbb{F}_q)$ which are orthogonal to constant functions. This splitting is orthogonal and parallel (compatible with the flat connection). The splitting of $F = F_0 + F_1$ corresponds to $f = f_0 + f_1$ where $f_0$ is $G(\mathbb{F}_q)$-invariant.

If $f$ is orthogonal to the $\lambda_0$ eigenspace of $\Gamma(q) \backslash H^{n+1}_R$, which is generated by a $G(\mathbb{F}_q)$-invariant function, then each factor of $f$ is orthogonal to the $\lambda_0$ eigenspace of $\Gamma \backslash H^{n+1}_R$. Integrating each factor over $\Gamma \backslash H^{n+1}_R$ and taking summation, we get $||\nabla f||_2^2 \geq \lambda_1(\Gamma \backslash H^{n+1}_R)||f||_2^2$. Here $|| \cdot ||_2$ denotes the $L^2$-norm over $\Gamma(q) \backslash H^{n+1}_R$.

Hence from now on, we can assume that $f = f_1, F = F_1$ takes values in $E_1$. 
Pick a generating system $S$ of $\Gamma$. Construct corresponding right Cayley graph. By definition of the spectral gap for the discrete Laplacian, for each $z \in \Gamma \setminus H_{n+1}^\mathbb{R}$,
\[
\frac{1}{|S|} \sum_{s \in S} \|F(z) - R(s)F(z)\|^2 \geq \lambda_1(G(\mathbb{F}_q))\|F(z)\|^2.
\]
Integrate over $\Gamma \setminus H_{n+1}^\mathbb{R}$ and get
\[
\frac{1}{|S|} \sum_{s \in S} \|F - R(s)F\|_2^2 \geq \lambda_1(G(\mathbb{F}_q))\|F\|_2^2.
\]
Let $H_0$ be the subspace of such functions orthogonal to the bottom eigenfunction $\phi_0$. We need to show that there exists $\epsilon > 0$ independent of $q$ such that for any $F$ in $H_0$
\[
\frac{\int_F \|\nabla F\|^2 d\mu}{\int_F \|F\|^2 d\mu} \geq \lambda_0 + \epsilon.
\]
Above discussion about expanders implies that for $z \in \mathcal{F}$, and for any $F \in H_0$, there exists $\gamma \in S$ such that
\begin{align}
(28) \quad & \|F(\gamma z) - F(z)\|^2 \geq \lambda_1(G(\mathbb{F}_q))\|F(z)\|^2, \text{ or} \nonumber \\
(29) \quad & ||R(\gamma)F - F||_2^2 \geq \lambda_1(G(\mathbb{F}_q))\|F\|_2^2. 
\end{align}
Set $f = \|F\| = a\phi_0(z) + b(z)$ where $b$ is orthogonal to $\phi_0$, the $\lambda_0$ eigenfunction normalized that $\int_F \phi_0^2 = 1$, and $L^2$ norm of $F$, $a^2 + \int_F b(z)^2 d\mu$ is 1. Then one can show that
\[
\frac{\int_F \|\nabla F\|^2 d\mu}{\int_F \|F\|^2 d\mu} \geq \lambda_0 + (\lambda_1 - \lambda_0) \int_F b^2 d\mu.
\]
If $\int_F b^2 d\mu$ is bounded below for all $F \in H_0$ we are done. Hence suppose there is no lower bound. Then after taking a weak limit, we may assume that there is $F \in H_0$ with $\int_F b^2 d\mu = 0$ and $a = 1$. One can write
\[
F = (F_i), \quad \frac{F_i}{\phi_0} = u_i
\]
so that $\sum u_i^2 = 1$, which implies that $\sum u_j \frac{\partial u_i}{\partial x_j} = 0$. Set $u = (u_i)$. Then Equation (28) becomes
\[
(30) \quad \|R(\gamma)u(z) - u(z)\|^2 > \lambda_1(G(\mathbb{F}_q))
\]
for any $z \in \mathcal{F}$ and some $\gamma \in S$. By Step I, $\lambda_1(G(\mathbb{F}_q))$ is uniformly bounded below independent of $q$. A direct calculation as in [5] shows that
\[
\|\nabla (\phi_0 u)\|^2 = \sum_j |\nabla (\phi_0 u_j)|^2 = \sum_{j,i} \left( \frac{\partial (\phi_0 u)}{\partial x_i} \right)^2 = \sum_{j,i} u_j \frac{\partial \phi_0}{\partial x_i} + \phi_0 \frac{\partial u_j}{\partial x_i} = \phi_0^2 \|\nabla u\|^2 + ||\nabla \phi_0||^2,
\]
which implies that
\[
\frac{\int_{\mathcal{F}} ||\nabla F||^2 d\mu}{\int_{\mathcal{F}} ||F||^2 d\mu} = \frac{\int_{\mathcal{F}} ||\nabla \phi_0||^2 + \phi_0^2 ||\nabla u||^2 d\mu}{\int_{\mathcal{F}} ||\phi_0||^2 d\mu} \geq \lambda_0 + \frac{\int_{\mathcal{F}} \phi_0^2 ||\nabla u||^2 d\mu}{\int_{\mathcal{F}} \phi_0^2 d\mu}.
\]

If \(\int_{\mathcal{F}} \phi_0^2 ||\nabla u||^2 d\mu\) is bounded below, we are done again. Remember that we normalize \(\phi_0\) so that \(\int_{\mathcal{F}} \phi_0^2 = 1\).

Now \(u = (u_1, \ldots, u_k)\) for \(k = |G(\mathbb{F}_q)|\) and hence \(\nabla u = (\nabla u_1, \ldots, \nabla u_k)\).

Fix \(z_0 \in \mathcal{F}\). Take a unit speed geodesic \(\alpha(t)\) connecting \(z_0\) and \(\gamma_{z_0}\). Then
\[
u(z_0) - u(z_0) = \int_{\alpha} \frac{du(\alpha(t))}{dt} dt = \int_{\alpha} \nabla u(\alpha(t))\alpha'(t) dt.
\]

Hence
\[
||u(z_0) - u(z_0)||^2 = \left(\int_{\alpha} \frac{du(\alpha(t))}{dt} dt\right)^2 \leq d(z_0, \gamma_{z_0}) \int_{\alpha} ||\nabla u(\alpha(t)) \cdot \alpha'(t)||^2 dt
\]
\[
\leq d(z_0, \gamma_{z_0}) \int_{\alpha} ||\nabla u(\alpha(t))||^2 dt
\]

Take a Fermi coordinate along \(\alpha\) so that metric tensor is
\[
ds^2 = d\rho^2 + \sinh^2 \rho d\phi^2 + \cosh^2 \rho dt^2
\]
where \(\rho\) is a distance from \(\alpha\) so that the volume form \(d\mu = \cosh \rho \sinh^{n-1} \rho dtd\rho d\phi\).

Take a small cross section \(B\) orthogonal to \(\alpha\) passing through \(z_0\) so that the induced volume form on \(B\) can be written as \(dB = \sinh^{n-1} \rho d\rho d\phi\). Then \(N = B \times \alpha\) will be a small tubular neighborhood around \(\alpha\) so that
\[
L = \max\{d(z, \gamma z) : z \in B\} < \infty, \quad m = \min_N \phi_0^2 > 0.
\]

Since \(\phi_0\) is a \(\Gamma\)-invariant positive function, \(m > 0\), see section 8. By integrating above inequality over \(B\),
\[
\int_B ||u(z) - u(z)||^2 dB \leq L \int_B ||\nabla u||^2 dtdB \leq L \int_N ||\nabla u||^2 \cosh \rho dtdB
\]
\[
= L \int_N ||\nabla u||^2 d\mu \leq \frac{L}{m} \int_N \phi_0^2 ||\nabla u||^2 d\mu \leq \frac{L}{m} \int_{\mathcal{F}} \phi_0^2 ||\nabla u||^2 d\mu.
\]

By equation (30), \(\int_{\mathcal{F}} \phi_0^2 ||\nabla u||^2 d\mu\) has a uniform lower bound. \(\square\)

Since \(\Gamma(d)\) is a finite index subgroup of \(\Gamma_\xi(d)\), such a spectral gap theorem holds for \(\Gamma_\xi(d)\) as well. From this and Lemma 10.2, we obtain

**Proposition 11.2.** There exists \(\epsilon_0 = \delta'\) uniform over all square free integer \(d\) so that for any \(\gamma_1 \in \Gamma\) and for any \(\Gamma_\xi(d)\), we have
\[
\langle P_{\Gamma_\xi(d)} \phi_0 \phi_0 \phi_0 d\mu \rangle_{L^2(\Gamma_\xi(d)\backslash \mathbb{H}_q^2)} = \frac{c_{\phi_0} d\mu}{c_{\phi_0} \delta_\Gamma} ||\xi||^{-\delta_{\Gamma}(1 + O(T^{-\epsilon_0}))}.
\]
Proof. By Lemma 10.2 for $\Phi_{\epsilon, \gamma_1}^\Gamma \in C_c^\infty (\Gamma_\xi(d) \backslash G)^K$ and $K$-invariant $\| \cdot \|$, since $\Gamma_\xi(d)$ is finite index in $\Gamma$, $\delta_{\Gamma_\xi(d)} = \delta_{\Gamma}$ and

$$\langle F_{T, \epsilon} \Phi_{\epsilon, \gamma_1}^\Gamma, \Phi_{\epsilon, \gamma_1}^\Gamma \rangle = \langle \Phi_{\epsilon, \gamma_1}^\Gamma, \phi_0 \rangle_{\delta_{\Gamma_\xi(d)}^{-1} c_{\phi_0}} \xi \delta_{\Gamma}||\xi||^{-\delta_{\Gamma}} (1 + O(T^{-\delta_{\Gamma}})).$$

Here since $\delta'$ depends only on the spectral gap and since it is uniform over all square free integers $d$, it is uniform. Let $\phi_0$ be a lift to $M(d) = \Gamma_\xi(d) \backslash H^4_\R$ of the bottom unit eigenfunction $\phi_0$ on $M = \Gamma \backslash H^4_\R$. Then the unit bottom eigenfunction on $M(d) = \Gamma_\xi(d) \backslash H^4_\R$ is

$$\phi_{\Gamma_\xi(d)} = \frac{1}{\sqrt{|\Gamma : \Gamma_\xi(d)|}} \phi_0.$$

Since $\Phi_{\epsilon, \gamma_1}^\Gamma$ is an $\epsilon$-approximation to the identity around $[\gamma^{-1}]$ in $M(d)$ and $\tilde{\phi}_0$ is invariant under $\Gamma$,

$$\langle \Phi_{\epsilon, \gamma_1}^\Gamma, \phi_0 \rangle_{L^2(M(d))} = \frac{1}{\sqrt{|\Gamma : \Gamma_\xi(d)|}} \phi_0 \rangle_{L^2(M(d))} = \frac{1}{\sqrt{|\Gamma : \Gamma_\xi(d)|}} d_\epsilon.$$

Also note that $c_{\phi_0} = \frac{1}{\sqrt{|\Gamma : \Gamma_\xi(d)|}} c_{\phi_0}$ from Section 3. Here once we fix $\phi_\epsilon$, the implied constant depends only on $F_{\epsilon \Gamma(d)}$. But since $\Gamma_\xi(d)$ is of finite index of $\Gamma$, it is constant. Hence the claim follows.

Corollary 11.3. There exists $\epsilon_0$ uniform over all square free integer $d$ such that

$$|A_{d}(T)| = \frac{O_f^0(d)}{|\Gamma : \Gamma_\xi(d)|} (\chi + O(T^{\delta_{\Gamma} - \epsilon_0})),
$$

where $O_f^0(d) = \sum_{\gamma_1 \in \Gamma \backslash \Gamma_\xi(d) \Gamma, f(\gamma_1) = 0(d)} 1$ and $\chi = \delta_{\Gamma}^{-1} c_{\phi_0} d_\epsilon ||\xi||^{-\delta_{\Gamma} T^{\delta_{\Gamma}}}$.

Proof.

$$|A_{d}(T)| = \sum_{n=0(d)} a_n(T) = \sum_{\gamma \in \text{Stab}(\xi) \Gamma} w_T(\gamma) = \sum_{\gamma_1 \in \Gamma_\xi(d) \backslash \Gamma, f(\gamma_1) = 0(d)} \sum_{\gamma \in \text{Stab}(\xi) \Gamma_\xi(d)} w_T(\gamma \gamma_1)$$

$$= \sum_{\gamma_1 \in \Gamma_\xi(d) \backslash \Gamma, f(\gamma_1) = 0(d)} \sum_{\gamma \in \text{Stab}(\xi) \Gamma_\xi(d)} \int_{G/K} \chi_B(\xi \gamma \gamma_1 \gamma_1^{-1} g) \phi_\epsilon(\gamma_1^{-1} g) \mu(g)$$

$$= \sum_{\gamma_1 \in \Gamma_\xi(d) \backslash \Gamma, f(\gamma_1) = 0(d)} \int_{G/K} F_{T, \epsilon}^{\Gamma_\xi(d)}(g) \phi_\epsilon(\gamma_1^{-1} g) \mu(g)$$

$$= \sum_{\gamma_1 \in \Gamma_\xi(d) \backslash \Gamma, f(\gamma_1) = 0(d)} \int_{\Gamma_\xi(d) \backslash G/K} F_{T}^{\Gamma_\xi(d)}(g) \Phi_{\epsilon, \gamma_1}^\Gamma(g) \mu(g)$$

$$= \sum_{\gamma_1 \in \Gamma_\xi(d) \backslash \Gamma, f(\gamma_1) = 0(d)} \langle F_{T}^{\Gamma_\xi(d)}, \Phi_{\epsilon, \gamma_1}^\Gamma \rangle_{L^2(\Gamma_\xi(d) \backslash H^4_\R)}.$$
By Proposition 11.2,

\[ |A_d(T)| = \sum_{\gamma_1 \in \Gamma_{\xi}(d) \setminus \Gamma, \ f(\xi_{\gamma_1}) = 0(d)} \frac{c_{\phi_0}d_{\varepsilon}}{\delta_{\Gamma} \left[ \Gamma : \Gamma_{\xi}(d) \right]} \left| |\xi||^{-\delta_{\Gamma} T^{\delta_{\Gamma}} (1 + O(T^{-\epsilon_0}))} \right| \]

\[ = \frac{\mathcal{O}^0_f(d)}{[\Gamma : \Gamma_{\xi}(d)]} \left( \chi + O(T^{\delta_{\Gamma} - \epsilon_0}) \right), \]

where \( \mathcal{O}^0_f(d) = \sum_{\gamma_1 \in \Gamma_{\xi}(d) \setminus \Gamma, \ f(\xi_{\gamma_1}) = 0(d)} \) and \( \chi = \delta_{\Gamma}^{-1} c_{\phi_0}d_{\varepsilon}||\xi||^{-\delta_{\Gamma} T^{\delta_{\Gamma}}}, \) hence the claim follows. \( \square \)

11.2. Digression to an algebraic group. Let \( G \) be a semisimple algebraic group defined over \( \mathbb{Q} \). Denote \( G(\mathbb{F}_p) \) a reduction of \( G \) to an algebraic group defined over \( \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z} \). In our case \( G = \text{Spin}(4,1) \), a double cover of \( SO^0(4,1) \). Note that a stabilizer of a point \( \xi \in V = \{v = (x_1, \cdots, x_5)|Q(v) = 0\} \setminus \{0\} \) is a connected \( \mathbb{Q} \)-group \( H = NM \), so that \( V = G/H \). By Proposition 3.22, the reduction of \( V \) to \( V(\mathbb{F}_p) = \{v = (v_1, \cdots, v_5) \in \mathbb{F}_p^5 : Q(v) = 0(p)\} \) is \( G(\mathbb{F}_p)/H(\mathbb{F}_p) \). By 20, since \( \Gamma \) is Zariski dense, the reduction map

\[ \Gamma \to G(\mathbb{F}_p) \]

is surjective. Note that \( \Gamma_{\xi}(p) \) is a stabilizer of \( \bar{\xi} \) in \( V(\mathbb{F}_p) \) where \( \bar{\xi} \) is a \( p \)-reduced image of \( \xi \) in \( V(\mathbb{F}_p) \). Since the reduction of \( \Gamma \) is \( G(\mathbb{F}_p) \),

\[ [\Gamma, \Gamma_{\xi}(p)] = \#V(\mathbb{F}_p) \sim p^4 \]

by 2.

Let \( f_1(x_1, \cdots, x_5) = x_1 \), \( f_2(x_1, \cdots, x_5) = x_1 x_2 \). To estimate \( \mathcal{O}^0_{f_1(p)} = \sum_{\gamma_1 \in \Gamma_{\xi}(p) \setminus \Gamma, \ f_1(\xi_{\gamma_1}) = 0(p)} 1 \), if \( Q'(x_2, \cdots, x_5) = Q(0, x_2, \cdots, x_5) \) is a quadratic form whose zero set is \( W = \{(0, x_2, \cdots, x_5) \in V\} \), then

\[ \mathcal{O}^0_{f_1(p)} = \#W(\mathbb{F}_p) \sim p^3 \]

by 2. Similarly

\[ \mathcal{O}^0_{f_2(p)} = \#W(\mathbb{F}_p) \sim 2p^3. \]

Then

\[ g_1(p) = \frac{\mathcal{O}^0_{f_1(p)}}{[\Gamma : \Gamma_{\xi}(p)]} \sim p^{-1}, \quad g_2(p) = \frac{\mathcal{O}^0_{f_2(p)}}{[\Gamma : \Gamma_{\xi}(p)]} \sim 2p^{-1}. \]

In general, if \( f_k(x_1, \cdots, x_5) = x_1 x_2 \cdots x_k \), then \( g_k(p) \sim kp^{-1} \) for \( k = 1, \cdots, 5 \).

11.3. Asymptotic growth of number of spheres with prime curvatures. Let \( f_1(x_1, \cdots, x_5) = x_1 \), \( f_2(x_1, \cdots, x_5) = x_1 x_2 \), \( f_k(x_1, \cdots, x_5) = x_1 x_2 \cdots x_k \) as before and for square free integer \( d \), let

\[ g_i(d) = \frac{\mathcal{O}^0_{f_i(d)}}{[\Gamma : \Gamma_{\xi}(d)]}. \]

Proposition 11.4. There exists a finite set \( S \) of primes such that
(1) for any square free integer $d = d_1d_2$ with no prime factors in $S$ and for each $i = 1, 2$, $g_i(d_1)g_i(d_2)$
(2) for any prime $p$ outside $S$, $g_1(p) \in (0, 1)$ and $g_1(p) = p^{-1} + O(p^{-q})$, $g_2(p) = 2p^{-1} + O(p^{-q})$ for some $q \geq 0$.

Proof. The second claim is already shown. By [26], for a large prime $p$, the reduction of $\Gamma$ is $G(\mathbb{F}_p)$. So let $S$ be the set of primes which are less than such a prime $p$. Then for $d = p_1 \cdots p_k$ square free with $p_i \notin S$, the diagonal reduction

$$\Gamma \to G(\mathbb{Z}/d\mathbb{Z}) \to G(\mathbb{F}_{p_1}) \times \cdots \times G(\mathbb{F}_{p_k})$$

is surjective and it follows from Goursat’s lemma that $\Gamma$ surjects onto $G(\mathbb{Z}/d_1\mathbb{Z}) \times G(\mathbb{Z}/d_2\mathbb{Z})$ for any square free $d = d_1d_2$ without any prime factor in $S$. Hence for $d = d_1d_2$ square free without prime factors in $S$, the orbit of $x \mod d$ under $\Gamma$ is equal to the orbit of $G(\mathbb{Z}/d_1\mathbb{Z}) \times G(\mathbb{Z}/d_2\mathbb{Z})$ in $(\mathbb{Z}/d_1\mathbb{Z})^5 \times (\mathbb{Z}/d_1\mathbb{Z})^5$. The same thing is true for the orbit satisfying the equation $f(\xi_\gamma) = 0(d_1)$. Therefore $g(d) = g(d_1)g(d_2)$. See [6][23].

Note that $a_n(T) = \sum_{\gamma \in \text{Stab}_F(\xi) \backslash \Gamma, f_1(\xi) = n} w_T(\gamma)$ is a smooth counting function for the number of vectors $(x_1, \cdots, x_5)$ in the orbit $\xi A^I$ of maximum norm bounded above by $T$ and $x_1 = n$. We want to obtain an asymptotic growth of the number $\pi^P(T)$ of spheres in an Apollonian packing $P$ whose curvatures are prime and less than $T$. The initial spheres with curvature $\xi = (\xi_1, \cdots, \xi_5)$ is chosen so that $\xi_1 < 0$ which corresponds to the largest sphere bounding all the other spheres, and $\xi_2, \xi_3, \xi_4, \xi_5$ the smallest curvatures (largest spheres) in the packing. Upon iteration, one obtains a smaller sphere, hence larger curvature.

Since any sphere in the packing is obtained by the initial five spheres corresponding to $\xi$ under the group $A$, with the maximum norm on $\mathbb{R}^5$,

$$\pi^P(T) \leq 4 + \# \{ \gamma \in A : ||\xi \gamma||_{\text{max}} \text{ is prime } < T \}$$

$$\ll \sum_{i=1}^5 \# \{ v \in \xi A^I : ||v||_{\text{max}} < T, \ v_i \text{ is prime} \}.$$ 

Similarly the number $\pi_2^P(T)$ of twin prime curvatures less than $T$ is

$$\pi_2^P(T) \ll \# \{ \gamma \in A : ||\xi \gamma||_{\text{max}} \text{ is prime } < T, \text{ one more entry of } \xi \gamma \text{ is prime} \}$$

$$\ll \sum_{i=1}^5 \sum_{j \neq i} \# \{ v = (v_1, \cdots, v_5) \in \xi A^I : ||v||_{\text{max}} < T, \ v_i, v_j \text{ primes} \}.$$ 

This counting problem uses so-called a Selberg’s sieve; using $S$ in Proposition 11.3 and $g$ as one of $g_i$, a multiplicative function $h$ on square free integers outside $S$ that $h(p) = \frac{g(p)}{1-g(p)}$ for a prime $p$ outside $S$,

**Theorem 11.5.** ([20], Theorem 6.4) Let $P$ be a finite product of distinct primes outside $S$ such that for any square free $d|P$

$$|A_d(T)| = g(d)\chi + r_d(A(T)).$$
Then for any $D > 1$,
\[
S(A(T), P) = \sum_{(n,P) = 1} a_n(T) \leq \chi(\sum_{d < \sqrt{D|P}} h(d))^{-1} + \sum_{d < D,d | P} \tau_3(d)|r_d(A(T))|,
\]
where $\tau_3(d)$ denotes the number of representations of $d$ as the product of three natural numbers.

From all these we obtain

**Theorem 11.6.** Given a bounded primitive integral Apollonian sphere packing $\mathcal{P}$ in $\mathbb{R}^3$,
\[
\pi^\mathcal{P}(T) \ll \frac{T^\delta r}{\log T}
\]
and
\[
\pi_2^\mathcal{P}(T) \ll \frac{T^\delta r}{(\log T)^2}.
\]
Generally if $\pi_k^\mathcal{P}(T)$ denotes the number of $k$-mutually tangent spheres whose curvatures are prime numbers less than $T$, then
\[
\pi_k^\mathcal{P}(T) \ll \frac{T^\delta r}{(\log T)^k}
\]
for $k \leq 5$.

**Proof.** Since
\[
|A_d(T)| = \frac{O^0(d)}{[\Gamma : \Gamma\xi(d)\Gamma]}(\chi + O(T^{\delta r - \epsilon_0})) = g_1(d)\chi + r_d(A(T)),
\]
and since $\frac{O^0(d)}{[\Gamma : \Gamma\xi(d)\Gamma]} < 1$, $r_d(A(T)) \ll T^{\delta r - \epsilon_0}$. For any $\epsilon_1$, $\sum_{d < D} \tau_3(d) < 1^3 + 2^3 + \cdots + D^3 \ll D^3 + \epsilon_1$, hence
\[
\sum_{d < D,d | P} \tau_3(d)|r_d(A(T))| \ll D^3 + \epsilon_1 T^{\delta r - \epsilon_0} \ll T^{\delta r}/\log T
\]
if $D = T^{\epsilon_0/4}$. Hence take $P$ to be the product of all primes less than $T^{\epsilon_0/4}$ outside $S$. Denote $\mu(n) = 1$ if $n$ is square-free and 0 otherwise. Also let $s(m)$ denote the largest square-free number dividing $m$. Then for $h = \frac{q_1}{1 - q_1}$, one can deduce that (\cite{20}, Section 6.6)
\[
\sum_{d < \sqrt{D|P}} h(d) \sim \sum_{d < \sqrt{D|P}} \Pi_{p|d} \frac{p^{-1}}{1 - p^{-1}}
\]
\[
= \sum_{d < \sqrt{D|P}} d^{-1}\Pi_{p|d} \frac{1}{1 - p^{-1}} = \sum_{d < \sqrt{D|P}} d^{-1}\Pi_{p|d}(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots)
\]
\[
= \sum_{d = p_1 \cdots p_k < \sqrt{D|P}} \left(\frac{1}{d} + \frac{1}{dp_1} + \frac{1}{dp_1^2} + \cdots\right)(1 + \frac{1}{p_2} + \frac{1}{p_2^2} + \cdots)\cdots (1 + \frac{1}{p_k} + \frac{1}{p_k^2} + \cdots).
\]
But since the numbers appearing in the denominator is of the form \( m = p_1^{\alpha_1} \cdots p_k^{\alpha_k} \) and since \( S \) is a fixed finite set of primes less than some number, and \( P \) is the product of distinct primes less than \( D \) outside \( S \), when \( D \) tends to infinity, the above sum is

\[
\gg \sum_{d < \sqrt{D}} (\mu(d) \sum_{m,s(m)=d} \frac{1}{m}) = \sum_{m,s(m) < \sqrt{D}} \frac{1}{m}
\]

\[
\gg \sum_{m < \sqrt{D}} m^{-1} \gg D \gg \log T.
\]

Hence Theorem 11.5 gives

\[
S(A(T), P) \ll \frac{T^{\delta_T}}{\log T}.
\]

Since there are only 5 coordinates, we may assume that for any \( v = (v_1, \cdots, v_5) \in \xi A^4 \) \( v_1 \) is the largest coordinate among \( v_i \). Therefore by Equation (27) for the product \( P \) of distinct primes less than \( T^{\epsilon_0/4} \) outside \( S \),

\[
\frac{T^{\delta_T}}{\log T} \gg S(A((1 + \epsilon)T), P) + (T^{\epsilon_0/4})^{\delta_T} = \sum_{(n,P) = 1} a_n((1 + \epsilon)T) + (T^{\epsilon_0/4})^{\delta_T}
\]

\[
\gg \# \{ v = (x_1, \cdots, x_5) \in \xi A^4 : ||v||_{\max} < T, (x_1, P) = 1 \} + (T^{\epsilon_0/4})^{\delta_T}
\]

\[
\gg (T^{\epsilon_0/4})^{\delta_T} + \# \{ v = (x_1, \cdots, x_5) \in \xi A^4 : ||v||_{\max} < T, x_1 \text{ prime} > T^{\epsilon_0/4} \}
\]

\[
\gg 5 \# \{ v = (x_1, \cdots, x_5) \in \xi A^4 : ||v||_{\max} < T, x_1 \text{ prime} \}
\]

\[
\gg \# \{ v = (x_1, \cdots, x_5) \in \xi A^4 : ||v||_{\max} < T, \text{ some } v_i \text{ prime} \} \gg \pi^T(T).
\]

Let \( \omega(d) \) denote the number of distinct prime factors of \( d \) and \( D(d) \) the number of positive divisors of \( d \) including 1. Then \( D(d) = 2^{\omega(d)} \) for square free integer \( d \). For \( g_2 \), considering \( S \) is a finite set and \( d|P \) implies that \( d \) is a square free, as before

\[
\sum_{d < \sqrt{D}, d|P} h(d) \sim \sum_{d < \sqrt{D}, d|P} \Pi_{p|d} \frac{2p^{-1}}{1 - 2p^{-1}}
\]

\[
= \sum_{d < \sqrt{D}, d|P} d^{-1} 2^{\omega(d)} \Pi_{p|d} \frac{1}{1 - 2p^{-1}} \gg \sum_{d < \sqrt{D}} \mu(d) d^{-1} D(d) \Pi_{p|d} (1 + \frac{1}{p} + \frac{1}{p^2} + \cdots)
\]

\[
= \sum_{d < \sqrt{D}} \mu(d) D(d) \sum_{m,s(m)=d} \frac{1}{m}.
\]

If \( m = p_1^{\alpha_1} \cdots p_k^{\alpha_k} \) for distinct primes \( p_i \) then \( s(m) = d = p_1 \cdots p_k \). Then \( D(d) = 2^k \), \( D(m) = (\alpha_1 + 1) \cdots (\alpha_k + 1) \). Hence

\[
\frac{D(s(m))}{d} > \frac{D(m)}{m}.
\]
For each \( m \leq \sqrt{D} \), if \( s(m) = d \leq m \), then \( \frac{\mathcal{D}(d)}{m} \) appears in

\[
\sum_{d < \sqrt{D}} \mu(d) \mathcal{D}(d) \sum_{m, s(m) = d} \frac{1}{m}.
\]

Hence

\[
\sum_{d < \sqrt{D}} \mu(d) \mathcal{D}(d) \sum_{m, s(m) = d} \frac{1}{m} > \sum_{m < \sqrt{D}} \frac{\mathcal{D}(m)}{m}.
\]

Note by [27] (page 94),

\[
\sum_{n=1}^{N} \frac{\mathcal{D}(n)}{n} = \frac{1}{2} (\log N)^2 + O(\log N).
\]

From this

\[
\sum_{d < \sqrt{D}, d \mid P} h(d) \gg (\log D)^2 \gg (\log T)^2.
\]

If \( m = p_1^{\alpha_1} \cdots p_k^{\alpha_k} \) for distinct primes \( p_i \) then \( s(m) = d = p_1 \cdots p_k \). Hence

\[
\frac{k^\omega(d)}{d} > \frac{k^\omega(m)}{m}.
\]

For each \( m \leq \sqrt{D} \), if \( s(m) = d \leq m \), then \( \frac{k^\omega(d)}{d} \) appears in

\[
\sum_{d < \sqrt{D}} \mu(d) k^\omega(d) \sum_{m, s(m) = d} \frac{1}{m}.
\]

Hence

\[
\sum_{d < \sqrt{D}} \mu(d) k^\omega(d) \sum_{m, s(m) = d} \frac{1}{m} > \sum_{m < \sqrt{D}} \frac{k^\omega(m)}{m} \gg (\log D)^k,
\]
where the last inequality follows from the following theorem [36], and communicated to us by Pieter Moree.

**Theorem 11.7.**

\[
\sum_{n \leq x} k^{\omega(n)} = c_k(0)x(\log x)^k + O(x(\log x)^{k-2}),
\]

where

\[
c_k(0) = \frac{1}{(k-1)!} \prod_{p} (1 + \frac{k}{p-1})(1 - \frac{1}{p})^k.
\]

\[\square\]

Indeed, using a finer version (Theorem 11.13 of [13]) of Sieve method of a multiplicative function \(g(p) = k/p + O(p^{-\delta-1})\)

**Theorem 11.8.** There exists a real number \(\beta(k) > 0\) such that

\[
(f(s) + O((\log D)^{-1/6})A(T)\Pi_{p(T)}(1 - g(p)) + R(D) \leq S(A(T), P(z))
\]

\[
\leq (F(s) + O((\log D)^{-1/6})A(T)\Pi_{p(T)}(1 - g(p)) + R(D)
\]

where \(z = D^{1/s}\) with \(s > \beta(k)\), and \(P(z)\) is the product of distinct primes less than \(z\) outside \(S\), where \(F(s) > 0\) and \(f(s) > 0\) are certain functions of \(s \geq 0\), depending on \(k\), defined as solutions of explicit differential-difference equations, such that

\[
\lim_{s \to \infty} f(s) = \lim_{s \to \infty} F(s) = 1,
\]

and where

\[
R(D) = \sum_{d < D} |r_d(A(T))|.
\]

In both upper and lower bounds, the implied constant depends only on \(k\) and on the constants in the asymptotic of \(g(p)\).

we can try to give a lower bound for our counting problem. Note that the number \(\pi^p(T)\) of spheres whose curvature is prime less than \(T\) is equal to

\[
\# \{v = (x_1, \cdots, x_5) \in \xi A^t : ||v||_\text{max} = \text{prime} < T\}.
\]

But \(a_n(T)\) approximately counts the number of orbit vectors with maximum norm less than \(T\) and whose first coordinate is \(n\). Hence as long as the first coordinate is prime \(p\), both \((p, v_2, v_3, v_4, v_5)\) and \((p, v'_2, v'_3, v'_4, v'_5)\) will be counted even though none of \(v'_2, v'_3, v'_4, v'_5\) is prime. This means that \(a_n(T)\) with \(n\) prime, will count the same sphere as many times as it appears in the orbit \(\xi A^t\). Another difficulty is that

\[
\# \{v = (x_1, \cdots, x_5) \in \xi A^t : ||v||_\text{max} < T, T^{\epsilon_0/4} < x_1 \text{ prime} < T\}
\]

is not equivalent to

\[
\# \{v = (x_1, \cdots, x_5) \in \xi A^t : ||v||_\text{max} < T, (x_1, \Pi_{p(T^\epsilon_0 p)} = 1)\}.
\]
To make them comparable, we pose the condition that \( x_1 \) cannot be written as a product of more than \( r \) primes > \( T^{s_0/4} \) where \( r \) is chosen as the first positive integer to satisfy \( T^{\frac{2r}{3}} > T \). People call it \( r \)-almost prime.

These things complicate the problem for the lower bound. But if we allow this over-counting with extra assumptions, one can give a lower bound.

**Theorem 11.9.** Let \( \pi^P_k(T)^r \) denote the number of 5 spheres kissing each other (i.e., the number of orbits) among whose at least \( k \) curvatures are \( r \)-almost primes and all of whose curvatures are less than \( T \) where \( r \) is the first positive integer larger than \( \frac{4s_0}{e_0} \) for a fixed large \( s_0 \). For any \( k \leq 5 \),

\[
\frac{T^{\gamma_k}}{e^{\log T})^k} \ll \pi^P_k(T)^r.
\]

**Proof.** We know already that \( |A(T)| \sim T^{\delta_1} \). Note also that since \( r_d(A(T)) \ll T^{\delta_1 - \epsilon_0} \),

\[
R(D) = \sum_{d < D} |r_d(A(T))| \leq DT^{\delta_1 - \epsilon_0}.
\]

Hence if \( D = T^{s_0/4} \), \( R(D) \ll \frac{T^{\gamma_k}}{(\log T)^k} \). Now we estimate \( \prod_{p \mid P(z)} (1 - g(p)) \) using Mertens formula, which is communicated to us by Pieter Moree. Put \( A_k(p) = 1 - k/p \) if \( p > k \) and \( A_k(p) = 1 \) otherwise. Put \( B_k = \prod_{p \geq 2} A_k(p)(1 - 1/p)^{-k} \). Let \( \gamma \) denote Euler’s constant.

We claim that as \( x \) tends to infinity we have \( \prod_{p \leq x} A_k(p) \sim B_k e^{-k\gamma \log^{-k} x} \).

Write

\[
\prod_{p \leq x} A_k(p) = \prod_{p \leq x} (1 - 1/p)^k \prod_{p \leq x} A_k(p)(1 - 1/p)^{-k}.
\]

Note that \( \log A_k(p) - k \log(1 - 1/p) = O(k^2/p^2) \) as \( p \) tends to infinity. It thus follows that the latter product converges to \( B_k \) as \( x \) tends to infinity.

Noting that

\[
\sum_{p > x} |\log A_k(p) - k \log(1 - 1/p)| = O\left( \sum_{p > x} \frac{k^2}{p^2} \right) = O\left( \frac{k^2}{x} \right),
\]

we find that

\[
\prod_{p \leq x} A_k(p) = \prod_{p \leq x} (1 - 1/p)^k B_k(1 + O\left( \frac{k^2}{x} \right)).
\]

Using the Mertens theorem that

\[
\prod_{p \leq x} (1 - 1/p) \sim \frac{e^{-\gamma}}{\log x},
\]

the claim then follows.

Now take \( D = T^{s_0/4} \) and let \( s \) large enough and \( \epsilon \rightarrow 0 \) to get for \( z = T^{s_0/4} \) and the product \( P = P(z) \) of distinct primes less than \( T^{1/2} \) outside \( S \)

\[
\frac{T^{\gamma_k}}{(\frac{4s_0}{e_0})^k(\log T)^k} \ll (f(s) + O(\log D)^{-1/6})|A(T)|\prod_{p \mid P(z)} (1 - g(p)) + R(D)
\]
\[ \ll S(A((1 - \epsilon)T), P) \]
\[ = \sum_{(n,P) = 1} a_n((1-\epsilon)T) \ll \# \{ v = (x_1, \cdots, x_5) \in \xi A^t : \|v\|_{\text{max}} < T, (x_1 x_2 \cdots x_k, P) = 1 \} \]

Since there are only 5 coordinates, there are only 5Ck choices of k coordinates out of 5,

\[ \ll \# \{ v = (x_1, \cdots, x_5) \in \xi A^t : \|v\|_{\text{max}} < T^{\frac{\alpha}{4n}} \} + 5Ck \# \{ v = (x_1, \cdots, x_5) \in \xi A^t : \|v\|_{\text{max}} < T, (x_1 x_2 \cdots x_k, P) = 1, \exists 1 \leq i \leq k, x_i > T^{\frac{\alpha}{4n}} \} \]
\[ \ll \# \{ v = (x_1, \cdots, x_5) \in \xi A^t : \|v\|_{\text{max}} < T, (x_1 x_2 \cdots x_k, P) = 1, x_i > T^{\frac{\alpha}{4n}}, i = 1, \cdots, k \} \]
\[ + (T^{\frac{\alpha}{4n}})^{\delta r} \ll \pi_k^P(T)^r. \]

Note that to get an upper bound, we need to reverse the inequality

\[ \# \{ v = (x_1, \cdots, x_5) \in \xi A^t : \|v\|_{\text{max}} < T, (x_1 x_2 \cdots x_k, P) = 1, x_i > T^{\frac{\alpha}{4n}} \} + (T^{\frac{\alpha}{4n}})^{\delta r} \ll \pi_k^P(T)^r. \]

which is not obvious to us.

**Corollary 11.10.** Given a bounded primitive integral Apollonian sphere packing \( \mathcal{P} \) in \( \mathbb{R}^3 \), \( \pi_k^\mathcal{P}(T)^r \) denote the number of 5 spheres kissing each other whose curvatures are \( r \)-almost primes less than \( T \) where \( r \) is a fixed positive integer depending only on Apollonian packing. Then

\[ \frac{T^{\delta r}}{(\log T)^5} \ll \pi_k^\mathcal{P}(T)^r. \]

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