GEOMETRIC RESOLUTION OF SINGULAR RIEMANNIAN FOLIATIONS

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Abstract. We prove that an isometric action of a Lie group on a Riemannian manifold admits a resolution preserving the transverse geometry if and only if the action is infinitesimally polar. We provide applications concerning topological simplicity of several classes of isometric actions, including polar and variationally complete ones. All results are proven in the more general case of singular Riemannian foliations.

1. Introduction

For an isometric action of a Lie group $G$ on a Riemannian manifold $M$ the presence of singular orbits is the main source of difficulties to understand the geometric and topological properties of the action. It seems natural to look for some procedure resolving the singularities, i.e., some way to pass from $M$ to some other $G$-manifold $\hat{M}$ with only regular orbits, related to $M$ in some canonical way. For the choice of the procedure it is crucial, what kind of information one would like to preserve by this resolution. If one only would like to let the regular part of the action unchanged, then there is a canonical procedure resolving an arbitrary action. One starts with the most singular stratum, replaces it by the projectivized normal bundle and proceeds inductively. The reader is referred, for instance, to [Was97] or to [Mol84] for this topological approach. The disadvantage of this method is that many crucial geometric and topological properties of the action are “concentrated” in the singular locus and in the transverse geometry and cannot be traced by this procedure.

In geometry it seems natural to consider the quotient $M/G$ with the induced metric as the essence of the action. Thinking of the action as of a (singular) foliation, one considers the transverse geometry as the

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most important object. Therefore it seems natural to look only for such resolutions $\hat{M}$ with a $G$-equivariant surjective map $f : \hat{M} \to M$ such that the induced map $f : \hat{M}/G \to M/G$ is an isometry (some partial resolutions of this type have already been considered, for instance in [GS00]). The main technical result of this paper (Theorem 1.1) states that such a resolution exists if and only if all isotropy representations of the action are polar. Many natural classes of actions, for instance polar ones, variationally complete ones or actions of cohomogeneity at most two satisfy this property of being infinitesimally polar. Moreover, if the action is infinitesimally polar there is a canonical resolution that inherits many properties of the original action. This provides a way to reduce the study of some topological and geometric properties of actions to the case of regular actions, where they can be easily established, see the subsequent results in the introduction.

It turns out that the action itself does not play a role in our considerations, but only the decomposition of the manifold into orbits, i.e., a singular Riemannian foliation. We refer the reader to [Mol88b] or to the preliminaries in Section 2 for basics about singular Riemannian foliations. Readers only interested in the special case of group actions may just consider all singular Riemannian foliations as orbit decompositions of an isometric group action. We also would like to mention [Wie08], where the ideas of this paper are elaborated and simplified in the case of isometric group actions.

**Definition 1.1.** Let $\mathcal{F}$ be a singular Riemannian foliation on a Riemannian manifold $M$. A geometric resolution of $(M, \mathcal{F})$ is a smooth surjective map $F : \hat{M} \to M$ from a smooth Riemannian manifold $\hat{M}$ with a regular Riemannian foliation $\hat{\mathcal{F}}$ such that the following holds true. For all smooth curves $\gamma$ in $\hat{M}$ the transverse lengths of $\gamma$ with respect to $\hat{\mathcal{F}}$ and of $F(\gamma)$ with respect to $\mathcal{F}$ coincide.

Here the transverse length is defined as usual in the theory of foliations as the length of the projection to local quotients (Subsection 2.6). The last requirement in the definition above means that $F$ sends leaves of $\hat{\mathcal{F}}$ to leaves of $\mathcal{F}$ and induces a length-preserving map between the quotients $F : \hat{M}/\hat{\mathcal{F}} \to M/\mathcal{F}$, see Section 3. Considering the quotient space $M/\mathcal{F}$ with its local metric structure as the essence of the singular Riemannian foliation $(M, \mathcal{F})$, the above definition becomes the most natural one.

Our main result reads as follows:
Theorem 1.1. Let $M$ be a Riemannian manifold and let $\mathcal{F}$ be a singular Riemannian foliation on $M$. Then $(M, \mathcal{F})$ has a geometric resolution if and only if $\mathcal{F}$ is infinitesimally polar. If $\mathcal{F}$ is infinitesimally polar then there is a canonical resolution $\hat{F} : \hat{M} \to M$ with the following properties. The resolution $\hat{M}$ is of the same dimension as $M$ and the map $\hat{F}$ induces a bijection between the spaces of leaves. Moreover, $\hat{F}$ is a diffeomorphism, when restricted to the preimage of the set of regular points of $(M, \mathcal{F})$. The map $\hat{F}$ is proper and 1-Lipschitz. In particular, the resolution $\hat{M}$ is compact or complete if $M$ has the corresponding property. The isometry group $\Gamma$ of $(M, \mathcal{F})$ acts by isometries on $(\hat{M}, \hat{\mathcal{F}})$ and the map $\hat{F} : \hat{M} \to M$ is $\Gamma$-equivariant. If $\mathcal{F}$ is given by the orbits of a group $G$ of isometries of $M$ then $G$ acts by isometries on $\hat{M}$, and $\hat{\mathcal{F}}$ is given by the orbits of $G$. If $M$ is complete then the singular Riemannian foliation $\mathcal{F}$ has no horizontal conjugate points if and only if $\hat{\mathcal{F}}$ has no horizontal conjugate points. If $M$ is complete then the singular Riemannian foliation $\mathcal{F}$ is polar if and only if $\hat{\mathcal{F}}$ is polar.

The infinitesimal polarity of $\mathcal{F}$ means that locally the singular Riemannian foliation $\mathcal{F}$ is diffeomorphic to an isoparametric singular Riemannian foliation on a Euclidean space (Subsection 2.5). For polar singular Riemannian foliations we refer to Subsection 2.3 (cf. [Ter85], [Bou95], [Ale04], [Ale06]) and for singular Riemannian foliations without horizontal conjugate points we refer to [BS58], [LT07b], [LT07a].

Before we are going to comment on this theorem and related results, we state some consequences that motivated our study of geometric resolutions. Recall that a (regular) Riemannian foliation $\mathcal{F}$ on a Riemannian manifold $M$ is called simple if it is given by the fibers of a Riemannian submersion. If $M$ is complete (or, more generally, if $\mathcal{F}$ is full, see Section 5) then $\mathcal{F}$ is simple if and only if all leaves of $\mathcal{F}$ are closed and have no holonomy ([Her60]). The next result generalizes [BH83] and [Heb86], Theorem 2 to the realm of singular Riemannian foliations.

Theorem 1.2. Let $M$ be a complete, simply connected Riemannian manifold, and let $\mathcal{F}$ be a singular Riemannian foliation on $M$. If $\mathcal{F}$ is polar, or if $\mathcal{F}$ has no horizontal conjugate points then the leaves of $\mathcal{F}$ are closed. Moreover, the restriction of $\mathcal{F}$ to the regular part of $M$ is a simple foliation.

In [Ter85] it is shown that isoparametric foliations on simply connected spaces of constant curvature have closed leaves and that there
are no exceptional leaves, i.e., that all regular leaves have trivial holonomy. In \cite{Tol06} it is shown that if \( \mathcal{F} \) is a polar singular Riemannian foliation on a simply connected symmetric space \( M \) then properness of all leaves implies vanishing of holonomy of regular leaves. Finally, in \cite{AT06} the same result was shown for an arbitrary complete, simply connected space \( M \). Thus, in the case of polar singular Riemannian foliations only the closedness of \( \mathcal{F} \) is new.

Since a connected group of isometries of a Riemannian manifold is closed if and only its orbits are closed, Theorem 1.2 reads in the case of group actions as follows:

**Corollary 1.3.** Let \( M \) be a complete, simply connected manifold and let a connected group \( G \) act by isometries of \( M \). If the action is polar or variationally complete then the image of \( G \) in the isometry group of \( M \) is closed and there are no exceptional orbits of the action.

From Theorem 1.2 and \cite{LT07a}, Theorem 1.7 we immediately get a complete description of singular Riemannian foliations without horizontal conjugate points in terms of their quotient spaces. Since singular Riemannian foliations without horizontal conjugate points generalize the concept of variationally complete actions introduced in \cite{Bot56} and \cite{BS58} and investigated in \cite{Con72}, \cite{GT02}, \cite{DO01} and \cite{LT07b}, the next result also gives a description of variationally complete actions in terms of the quotient spaces. Since complete non-negatively curved Riemannian orbifolds without conjugate points are flat, the next result generalizes the main results of \cite{DO01}, \cite{GT02} and \cite{LT07b}.

**Corollary 1.4.** Let \( M \) be a complete Riemannian manifold and let \( \mathcal{F} \) be a singular Riemannian foliation. Then \( \mathcal{F} \) does not have horizontal conjugate points if and only if the lift \( \tilde{\mathcal{F}} \) of \( \mathcal{F} \) to the universal covering \( \tilde{M} \) of \( M \) is closed and the quotient \( \tilde{M}/\tilde{\mathcal{F}} \) is a Riemannian orbifold without conjugate points.

To deduce Theorem 1.2 from Theorem 1.1 we proceed as follows. If \( \mathcal{F} \) is polar then \( \mathcal{F} \) is also infinitesimally polar. If \( \mathcal{F} \) has no horizontal conjugate points then it is infinitesimally polar as well, due to \cite{LT07a}, Theorem 1.7. Thus we may apply Theorem 1.1 and obtain a regular Riemannian foliation \( \hat{\mathcal{F}} \) on a complete Riemannian manifold \( \hat{M} \) that is polar or has no horizontal conjugate points. In the first case we apply \cite{BH83} and deduce that the lift of \( \hat{\mathcal{F}} \) to the universal covering of \( \hat{M} \) is a simple Riemannian foliation. In the second case, the leaves of the regular Riemannian foliation \( \hat{\mathcal{F}} \) on the complete Riemannian manifold \( \hat{M} \) have no focal points and the proof of \cite{Heb86}, Theorem 2 shows that the lift of \( \hat{\mathcal{F}} \) to the universal covering of \( \hat{M} \) is again a simple Riemannian foliation.
foliation. But \((\hat{M}, \hat{\mathcal{F}})\) coincides with \((M, \mathcal{F})\) on the regular part \(M_0\) of \(M\). Therefore, the restriction of \(\mathcal{F}\) to \(M_0\) becomes simple, when lifted to the universal covering \(\hat{M}_0\) of \(M_0\). Thus, Theorem 1.2 follows from the next general topological observation whose proof will be given in Section 5. The proof of this result is implicitly contained in \([\text{Mol88b}],\) p.213-214 (see also \([\text{Mol88a}]\)).

**Theorem 1.5.** Let \(M\) be a complete, simply connected Riemannian manifold and let \(\mathcal{F}\) be a singular Riemannian foliation on \(M\). If the restriction of \(\mathcal{F}\) to the regular part \(M_0\) becomes simple, when lifted to the universal covering \(\hat{M}_0\) of \(M_0\), then the restriction of \(\mathcal{F}\) to \(M_0\) is a simple foliation.

In Section 5 we will discuss a more general version of the theorem above. We also will derive some further consequences of Theorem 1.5 concerning the general structure of infinitesimally polar foliations with closed leaves on simply connected manifolds. These results are independent of our main Theorem 1.1 but are related to Theorem 1.2 and therefore included here. To state these results we will need some notations.

For a Riemannian orbifold \(B\), we denote by \(\partial B\) the union of all closures of all singular strata of \(B\) that have codimension 1 in \(B\) and call it the *boundary* of \(B\) (This coincides with the boundary in the sense of Alexandrov geometry).

**Theorem 1.6.** Let \(\mathcal{F}\) be a closed infinitesimally polar singular Riemannian foliation on a complete manifold \(M\) with quotient orbifold \(B\). Then all singular leaves of \(\mathcal{F}\) are contained in the boundary \(\partial B\). If \(M\) is simply connected then the converse is also true, i.e., \(\partial B\) is the set of all singular leaves. In particular, for simply connected \(M\), the quotient \(B\) has no boundary if and only if \(\mathcal{F}\) is a regular foliation.

For foliations of codimension 2 we will deduce from the last theorem a result generalizing a known statement about compact transformation groups (Theorem 8.6 in Chapter IV of \([\text{Bre72}]\)):

**Corollary 1.7.** Let \(M\) be a complete simply connected Riemannian manifold and let \(\mathcal{F}\) be a closed singular Riemannian foliation with a quotient \(B = M/\mathcal{F}\) of dimension 2. Then either the foliation is regular or there are no exceptional leaves.

For further investigations of exceptional orbits we need another definition. A *Coxeter orbifold* (cf. \([\text{AKLM07}]\)) is Riemannian orbifold locally diffeomorphic to Weyl chambers, i.e., to quotients of the Euclidean space by finite Euclidean Coxeter groups. Note that in a Coxeter orbifold each non-manifold point is contained in the boundary. In
dimension 2 the converse holds as well, i.e., a two-dimensional orbifold is a Coxeter orbifold if it does not have isolated singularities. In particular, a Coxeter orbifold does not have to be a good orbifold, as it was claimed in [AKLM07] and cited in the previous version of this paper (a disc with an additional conical singularity on the boundary is a counterexample, cf. Remark [1.4]).

Now we can state:

**Theorem 1.8.** Let \( M \) be a complete, simply connected Riemannian manifold and let \( \mathcal{F} \) be a closed infinitesimally polar singular Riemannian foliation on \( M \) with quotient \( B = M/\mathcal{F} \). Then the following are equivalent:

1. There are no exceptional leaves;
2. The regular part \( B_0 := M_0/\mathcal{F} \) is a good orbifold;
3. The quotient \( B \) is a Coxeter orbifold;
4. All non-manifold points of the orbifold \( B \) are contained in the boundary \( \partial B \).

**Example 1.1.** Closed singular Riemannian foliations that are polar or have no horizontal conjugate points have good Riemannian orbifolds as quotients (thus \( B_0 \) is good as well). In the case of closed polar singular Riemannian foliations on simply connected manifolds it was shown in [AT06], that the quotients are Coxeter orbifolds.

**Example 1.2.** If the singular Riemannian foliation \( \mathcal{F} \) is induced by the action of a connected group \( K \) of isometries, the equivalent conditions of Theorem 1.8 are also equivalent to the following one: For all \( x \) in \( M \) the action of the isotropy group \( K_x \) on the horizontal space \( H_x \) has connected fibers. In fact, the sufficiency is clear. Assume on the other hand that there are no exceptional orbits. Then the finite group \( K_x/K_x^0 \) acts on the quotient \( H_x/K_x^0 \) which is a Weyl chamber. The set of its regular point is contractible, thus if the action of \( K_x/K_x^0 \) is non-trivial there are elements of \( K_x \) that fix some but not all points in \( H_x/K_x^0 \). But such points correspond to exceptional orbits. See also the proof of Theorem 1.8, where the same argument is used.

In view of Theorem 1.8 it seems natural to ask the following

**Question 1.3.** What simply connected Coxeter orbifolds \( B \) can be represented as quotient spaces \( B = M/\mathcal{F} \) for some singular Riemannian foliation \( \mathcal{F} \) on some simply connected Riemannian manifold \( M \).

**Remark 1.4.** Note, that if under the assumptions of Theorem 1.8 the quotient \( B \) is a good orbifold, then \( B_0 \) is a good orbifold as well, thus \( B \) is a Coxeter orbifold. On the other hand, using [HQ84] and the
arguments of [AKLM07], it is not difficult to deduce, that a Coxeter orbifold, that is simply connected as a topological space, is a good orbifold if and only if the following two conditions are fulfilled: A wall (a stratum of codimension 1) intersects a small tube around any stratum of codimension 2 in a connected set. If the closures of two walls intersect at different connected components, then the intersection angles at these components do not depend on the component. Thus, it is not to difficult to decide, when the quotient $B$ as in Theorem 1.8 is a good orbifold.

The proof of Theorem 1.1 is provided in Section 4 and Section 3 along the following lines. For an infinitesimally polar $\mathcal{F}$ on a Riemannian manifold $M$ one uses the ideas of [Bou95] and [To06] and defines the resolution $\hat{M}$ to be the subset of the Grassmannian bundle $Gr_k(M)$ consisting of all infinitesimal horizontal sections of $\mathcal{F}$. In the polar case the result is contained in [Bou95] and [To06]. In the general case one follows an idea from [LT07a] and uses transformation relating horizontal geometry of different Riemannian metrics adapted to a given foliation to reduce the question to the polar case.

**Remark 1.5.** The proof shows (and is based on) the fact that the resolution $(\hat{M}, \hat{\mathcal{F}})$ considered as a foliation on a manifold (disregarding the Riemannian metric on $\hat{M}$) does not depend on the Riemannian metric adapted to the singular Riemannian foliation $\mathcal{F}$ on $M$.

To see that a singular Riemannian foliation $\mathcal{F}$ with a metric resolution $\hat{\mathcal{F}}$ is infinitesimally polar one observes that in a regular Riemannian foliation transversal sectional curvatures remain bounded on compact subsets. Now, one uses the transverse equivalence of $\mathcal{F}$ and $\hat{\mathcal{F}}$ and deduces from [LT07a], Theorem 1.4 that this property characterize infinitesimally polar singular Riemannian foliations. This already proves the claim in the case of a compact resolution $\hat{M}$. In the general case one needs to be more careful and to extend some results from [LT07a] slightly (Lemma 3.4).

We would like to mention that Sections 3, 4 and 5 do not depend on each other. Thus, reader only interested in Theorem 1.5 and subsequent results may directly proceed to Section 5 and reader only interested in the (more important) if part of Theorem 1.1 may skip Section 3.

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2. Preliminaries

2.1. Singular Riemannian foliations. A transnormal system $\mathcal{F}$ on a Riemannian manifold $M$ is a decomposition of $M$ into smooth injectively immersed connected submanifolds, called leaves, such that geodesics emanating perpendicularly to one leaf stay perpendicularly to all leaves. A transnormal system $\mathcal{F}$ is called a singular Riemannian foliation if there are smooth vector fields $X_i$ on $M$ such that for each point $p \in M$ the tangent space $T_pL(p)$ of the leaf $L(p)$ through $p$ is given as the span of the vectors $X_i(p) \in T_pM$. We refer to [Mol88b] and [Wil07] for more on singular Riemannian foliations. Examples of singular Riemannian foliations are (regular) Riemannian foliations and the orbit decomposition of an isometric group action.

2.2. Stratification. Let $\mathcal{F}$ be a singular Riemannian foliation on the Riemannian manifold $M$. The dimension of $\mathcal{F}$, $\dim(\mathcal{F})$, is the maximal dimension of its leaves. The codimension of $\mathcal{F}$, $\text{codim}(\mathcal{F}, M)$, is defined by $\dim(M) - \dim(\mathcal{F})$. For $s \leq \dim(\mathcal{F})$, denote by $\Sigma_s$ the subset of all points $x \in M$ with $\dim(L(x)) = s$. Then $\Sigma_s$ is an embedded submanifold of $M$ and the restriction of $\mathcal{F}$ to $\Sigma_s$ is a Riemannian foliation. For a point $x \in M$, we denote by $\Sigma^x$ the connected component of $\Sigma_s$ through $x$, where $s = \dim(L(x))$. We call the decomposition of $M$ into the manifolds $\Sigma^x$ the canonical stratification of $M$.

The subset $\Sigma_{\dim(\mathcal{F})}$ is open, dense and connected in $M$. It is the regular stratum $M$. It will be denoted by $M_0$ and will also be called the set or regular points of $M$. All other strata $\Sigma^x$, called singular strata, have codimension at least 2 in $M$. For any singular stratum $\Sigma$, we have $\text{codim}(\mathcal{F}, \Sigma) < \text{codim}(\mathcal{F}, M)$.

2.3. Infinitesimal singular Riemannian foliations. Let $M$ be a Riemannian manifold and let $\mathcal{F}$ be a singular Riemannian foliation on $M$. Let $x \in M$ be a point. Then there is a well defined singular Riemannian foliation $T_x\mathcal{F}$ on the Euclidean space $(T_xM, g_x)$ with the following properties:

1. There is a neighborhood $O$ of $x$ and a diffeomorphic embedding $\phi: O \to T_xM$, with $D_x\phi = Id$ and $\phi^*(T_x\mathcal{F}) = \mathcal{F}|_O$.
2. $T_x\mathcal{F}$ is homogeneous, i.e., for each non-zero real number $\lambda$, the multiplication by $\lambda$ on $T_xM$ preserves $T_x\mathcal{F}$.
3. The singular foliation $T_x\mathcal{F}$ on the tangent space $T_xM$ does not depend on the Riemannian metric adapted to $\mathcal{F}$.

The singular Riemannian foliation $T_x\mathcal{F}$ on the tangent space $T_xM$ will be called the infinitesimal singular Riemannian foliation of $\mathcal{F}$ at the point $x$. 

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2.4. **Horizontal sections.** We refer to [Bou95, Ale04, Ale06] for more on polar singular Riemannian foliations. Let $\mathcal{F}$ be a singular Riemannian foliation on a Riemannian manifold $M$. A global (local) horizontal section through $x$ is a smooth immersed submanifold $x \in N \subset M$ that intersects all leaves of $\mathcal{F}$ (all leaves in a neighborhood of $x$), such that all intersections are orthogonal. $\mathcal{F}$ is called polar (locally polar) if there are (local) global horizontal sections through every point $x \in M$. Each local section $N$ of a singular Riemannian foliation is totally geodesic. Moreover, for each $x \in N$, $T_x N \subset T_x M$ is a horizontal section of the infinitesimal singular Riemannian foliation $T_x \mathcal{F}$. On the other hand, if $\mathcal{F}$ is locally polar then each horizontal section $V \subset T_x M$ of the infinitesimal singular Riemannian foliation $T_x \mathcal{F}$ is the tangent space to a local horizontal section of $\mathcal{F}$.

Recall, that a singular Riemannian foliation $\mathcal{F}$ is locally polar if and only if the restriction of $\mathcal{F}$ to the regular part $M_0$ has integrable horizontal distribution ([Ale06]). Moreover, a locally polar singular Riemannian foliation on a complete Riemannian manifold is polar.

2.5. **Infinitesimal polarity.** The singular Riemannian foliation $\mathcal{F}$ is called infinitesimally polar at the point $x \in M$ if the infinitesimal singular Riemannian foliation $T_x \mathcal{F}$ is polar. We say that $\mathcal{F}$ is infinitesimally polar if it is infinitesimally polar at all points. In [LT07a] it is shown that $\mathcal{F}$ is infinitesimally polar at the point $x$ if and only if for all sequences $x_i$ of regular points converging to $x$, the supremum $\bar{\kappa}(x_i)$ of the sectional curvatures at projections of $x_i$ to local quotients remain bounded away from infinity. Another equivalent condition derived in [LT07a], is that $\mathcal{F}$ is locally closed at $x$ and that local quotients at $x$ are smooth Riemannian orbifolds.

2.6. **Transverse length.** Let $M$ be a Riemannian manifold and let $\mathcal{F}$ be a singular Riemannian foliation on $M$. For $x$ in $M$, we denote by $V_x$ the tangent space to the leaf $V_x = T_x L(x)$ and call it the *vertical space at $x$*. The orthogonal complement of $V_x$ will be denoted by $H_x$ (or by $H_x(g)$, if we want to specify the Riemannian metric $g$). This subspace $H_x$ will be called the *horizontal subspace at $x$*. By $P_x : T_x \to H_x$ we denote the orthogonal projection. The spaces $H_x$ vary semi-continuously. Therefore, for each smooth curve $\gamma$ in $M$, the value $L_{\text{hor}}(\gamma) := \int |P_{\gamma(t)}(\gamma'(t))| dt$ is well defined. We call this quantity the *transversal length of $\gamma$*. If $B = M/\mathcal{F}$ is a Hausdorff metric space then $L_{\text{hor}}(\gamma)$ is the length of the projection of $\gamma$ to $B$. Note that a smooth curve has transversal length zero if and only it is completely contained in one leaf.
3. THE ONLY IF PART

We are going to prove the only if part of the first statement of Theorem 1.1 in this section. Thus, let \( \tilde{F} \) be a regular Riemannian foliation on a Riemannian manifold \( M \), let \( F \) be a singular Riemannian foliation on a Riemannian manifold \( M \) and let \( F : \hat{M} \to M \) be a geometric resolution. We are going to analyze \( F \) and to prove that \( F \) is infinitesimally polar. The proof in the case of compact \( \hat{M} \) was explained in the introduction. In the general case, we will give a proof along the same lines, but the proof becomes technically more involved.

First of all, \( F \) sends curves of zero transversal length to curves of zero transversal length, therefore \( F \) sends leaves into leaves, i.e., \( F(\tilde{L}(x)) \subset L(F(x)) \) for all \( x \in \hat{M} \).

For each open subset \( O \) of \( M \) the restriction \( F : F^{-1}(O) \to O \) is again a geometric resolution. As usual, let \( M_0 \) denote the set of regular points of \( M \) and set \( \hat{M} := F^{-1}(M_0) \). Since the restriction of \( F \) to \( M_0 \) is a regular Riemannian foliation, we deduce from continuity reasons, that for all \( x \in \hat{M} \) the map \( G_x := P_{F(x)} \circ D_x F : H_x \to H_{F(x)} \) is an isometric embedding. Here, the horizontal subspaces \( H \) and the projections \( P \) are defined as in Subsection 2.6.

On the other hand, \( F \) is smooth and surjective. By Sard’s theorem there is at least one point \( x \in \hat{M} \) such that \( D_x F : T_x \hat{M} \to T_{F(x)} M \) is surjective. Since \( D_x F \) sends \( T_x (L(x)) \) to a subspace of \( T_{F(x)} (L(F(x))) \) we deduce that the map \( G_x : H_x \to H_{F(x)} \) must be surjective at such points. Therefore, \( \dim(H_x) = \dim(H_{F(x)}) \). Hence, \( \text{codim}(M,F) = \text{codim}(\hat{M},\tilde{F}) \). Moreover, for each \( x \in \hat{M} \), the map \( G_x : H_x \to H_{F(x)} \) is an isometry.

Thus, for each point \( x \in \hat{M} \), we find a small neighborhood \( O \) of \( x \) such that \( \tilde{F} \) on \( O \) is given by a Riemannian submersion \( s_1 : O \to B_1 \), such that \( F \) on \( F(O) \) is given by a Riemannian submersion \( s_2 : F(O) \to B_2 \), and such that \( F \) induces an isometry \( \tilde{F} : B_1 \to B_2 \) between the local quotients.

This finishes the analysis of \( F \) on \( \hat{M} \). The picture over the singular points is more complicated. We start our discussion of the singular part with the following easy observation.

**Lemma 3.1.** Let \( \gamma_1 : [0,a] \to \hat{M} \) and \( \gamma_2 : [0,a] \to M \) be horizontal geodesics with \( \gamma_2((0,a)] \subset M_0 \). If \( F(\gamma_1(t)) \subset L(\gamma_2(t)) \), for all \( t \), then the sectional curvatures in local quotients at \( L(\gamma_2(t)) \), \( t \in (0,a] \), are uniformly bounded.

**Proof.** From the discussion above we know that the sectional curvatures in local quotients at \( L(\gamma_1(t)) \) and \( \tilde{L}(\gamma_2(t)) \) coincide for all \( t \in
(0, a]. Since [0, a] is compact and \( \hat{F} \) is a regular Riemannian foliation, the sectional curvatures in local quotients at \( \hat{L}(\gamma_2(t)) \) are uniformly bounded. □

The idea is now to find such curves starting at all points and to deduce infinitesimal polarity from this existence.

**Lemma 3.2.** The open subset \( \tilde{M} \) is dense in \( \hat{M} \).

*Proof.* Assume the contrary and choose an open subset \( O \) of \( \hat{M} \setminus \tilde{M} \). By making \( O \) smaller we may assume that \( F(O) \) is contained in a singular stratum \( \Sigma \) of \( M \). Now, the restriction of \( F \) to \( \Sigma \) is again a regular Riemannian foliation. Thus, for each \( x \in O \), we obtain by continuity that \( D_xF \) maps \( H_x \) injectively onto the subspace \( D_xF(H_x) \) that intersects \( T_{F(x)}L(F(x)) \) only in \{0\}. Thus we deduce

\[
\text{codim}(\hat{M}, \hat{F}) = \dim(H_x) \leq \text{codim}(\Sigma, F) < \text{codim}(M, F)
\]

since \( \Sigma \) is a singular stratum. This contradicts the previously obtained equality \( \text{codim}(\hat{M}, \hat{F}) = \text{codim}(M, F) \). □

Now we can prove:

**Lemma 3.3.** For each \( x \in M \), there are horizontal geodesics \( \gamma_1 : [0, a] \to \hat{M} \) and \( \gamma_2 : [0, a] \to M \) such that \( \gamma_2(0) = x \), \( \gamma_2((0, a] \subset M_0 \) and \( F(\gamma_1(t)) \subset L(\gamma_2(t)) \), for all \( t \).

*Proof.* Choose a distinguished tubular neighborhood \( U \) at \( x \) and a preimage \( y \) of \( x \) in \( \hat{M} \). Make the diameter \( \epsilon \) of \( U \) so small that all geodesics starting in the \( \epsilon \)-neighborhood \( O \) of \( y \) are defined at least for the time \( \epsilon \). Take a point \( z \in \hat{M} \cap O \) with \( \bar{z} = F(z) \in U \). Let \( \bar{x} \) be the projection of \( \bar{z} \) onto the leaf of \( F \) through \( x \) in \( U \). Then \( \bar{x} \) is the only possibly non-regular point on the geodesic \( \gamma_3 = \bar{z}\bar{x} \). Consider the horizontal geodesic \( \gamma_1 \) in \( \hat{M} \) starting at \( z \) in the direction \( h \) with \( G_z(h) = \gamma_3' \). From the understanding of \( F \) on \( M \), we deduce that \( F(\gamma_1(t)) \) is contained in \( L(\gamma_3(t)) \) for all \( t \in [0, d(\bar{z}, \bar{x})] \). Now, replacing \( \gamma_3 \) through a horizontal geodesic starting in a point on \( L(\bar{z}) \) and ending in \( x \), we obtain a horizontal geodesic \( \gamma_2 \) ending in \( x \) with \( F(\gamma_1(t)) \subset L(\gamma_2(t)) \). It remains to reverse the orientations of \( \gamma_1 \) and \( \gamma_2 \). □

Now the proof of the infinitesimal polarity of \( F \) is finished by combining Lemma 3.1, Lemma 3.3 and the following lemma, that we consider to be of independent interest.

**Lemma 3.4.** Let \( F \) be a singular Riemannian foliation on a Riemannian manifold \( M \). Let \( x \in M \) be a point. Let \( \gamma : [0, \epsilon] \to M \) be a horizontal geodesic starting at \( x \), such that \( \gamma((0, \epsilon]) \) is contained in the set
of regular points $M_0$. If all sectional curvatures in local quotients are uniformly bounded along $\gamma(0, \epsilon]$ then $F$ is infinitesimally polar at $x$.

**Proof.** Consider $T_xF$ as the limit of rescaled singular Riemannian foliations $(M, F)$ as in [LT07a], p.10. As in [LT07a], we deduce that $T_xF$ is a singular Riemannian foliation on the Euclidean space $T_xM$ such that at the regular point $v = \gamma'(0) \in T_xM$ all sectional curvatures vanish in local quotients. In this case, Proposition 3.5 below implies that $T_xF$ is polar. □

**Proposition 3.5.** Let $F$ be a singular Riemannian foliation on the Euclidean space $\mathbb{R}^n$. Let $L$ be a regular leaf such that in local quotients all sectional curvatures vanish at the image of this leaf. Then $F$ is polar.

**Proof.** Since $\mathbb{R}^n$ is flat, the sectional curvatures at the point $\{L\}$ in local projections vanish if and only if the O'Neill tensor $A : H_x \times H_x \to T_x(L(x))$ vanishes identically at all points $x \in L$. But this implies that each Bott-parallel normal field $H$ along $L$ is a parallel normal field. Since all these fields are equifocal (cf. [AT08]), we get that $L$ is an isoparametric submanifold of $\mathbb{R}^n$ and that $F$ coincides with the isoparametric foliation defined by the isoparametric submanifold $L$. □

4. Desingularization

4.1. Notations. First, let $T$ be a finite-dimensional real vector space with scalar products $g$ and $g^+$. Let $A : T \to T$ be the linear map defined by $g^+(A(v), w) = g(v, w)$ for all $v, w \in T$. Then, for each linear subspace $H$ of $T$, the image $A(H)$ of $H$ satisfies $H^+ = (A(H))^{g^+}$, i.e., the $g$-orthogonal complement of $H$ coincides with the $g^+$-orthogonal complement of $A(H)$. We will denote the map $A$ by $I_{g, g^+}$. By the same symbol $I_{g, g^+}$ we denote the induced map on the Grassmannians $Gr_k(T)$, i.e., on the spaces of $k$-dimensional linear subspaces of $T$. Note that $I_{g, g^+} \circ I_{g^+, g} = Id.$

If $M$ is a Riemannian manifold with Riemannian metrics $g, g^+$ then we get a bundle automorphism $I_{g, g^+} : TM \to TM$ of the tangent bundle $TM$ of $M$. For $k \geq 0$, we denote by $Gr_k = Gr_k(M)$ the Grassmannian bundle of the tangent bundle of $M$, i.e., the bundle of $k$-dimensional subspaces of tangent spaces of $M$. By the same symbol $I_{g, g^+}$ we will denote the induced bundle automorphism $I_{g, g^+} : Gr_k \to Gr_k$.

Let now $F$ be a singular foliation adapted to the Riemannian metrics $g$ and $g^+$, i.e., $F$ is a singular Riemannian foliation with respect to the Riemannian metrics $g$ and $g^+$. For any point $x \in M$, we have the
subspaces \( H_x(g) \) and \( H_x(g^+) \) of \( g \)-horizontal and of \( g^+ \)-horizontal vectors, respectively. By construction, our transformation \( I_{g,g^+} \) satisfies \( I_{g,g^+}(H_x(g)) = H_x(g^+) \), since \( H_x \) is defined as orthogonal complement of the vertical space \( V_x \) that does not depend on the adapted Riemannian metric.

4.2. **Basic construction.** Let \((M, g)\) be a Riemannian manifold and let \( \mathcal{F} \) be an infinitesimally polar singular Riemannian foliation on \( M \) of codimension \( k \).

We denote by \( \hat{M} \subset Gr_k \) the set of all \( k \)-dimensional infinitesimal sections of \( \mathcal{F} \). Thus \( p^{-1}(x) \subset \hat{M} \) is the manifold of horizontal sections of the polar Riemannian foliation \( T_x \mathcal{F} \) on \( T_x M \). In particular, for each regular point \( x \in M_0 \subset M \), the preimage \( p^{-1}(x) \) consists of only one point \( H_x \in Gr_k M \).

We are going to prove:

1. \( \hat{M} \) is a closed smooth submanifold of \( Gr_k \).
2. The decomposition of \( \hat{M} \) into preimages \( \hat{L} = p^{-1}(L) \) of the leaves of \( \mathcal{F} \) is a smooth foliation \( \hat{\mathcal{F}} \) of \( \hat{M} \).

The definition of \( \hat{M} \) and of \( \hat{\mathcal{F}} \) are local on \( M \) and so are the claims. Thus we may restrict ourselves to a small distinguished neighborhood \( U \) of a given point \( x \in M \). Pulling back the flat metric on \( T_x M \) by the diffeomorphism \( \phi \) (Subsection 2.3), we thus reduce the question to the following situation, to which we will refer later as the **standard case**. The manifold \( \hat{M} \) is an open subset of the Euclidean space \( \mathbb{R}^n \) with a flat (constant) Riemannian metric \( g^+ \); and \( \mathcal{F} \) is the restriction of an isoparametric foliation on \( \mathbb{R}^n \) to \( M \). Moreover, \( g \) is a Riemannian metric on \( M \) adapted to \( \mathcal{F} \).

Let \( \hat{M}^+ \) be the subset of the Grassmannian \( Gr_k \) of all infinitesimal horizontal sections of \( \mathcal{F} \) with respect to the Riemannian metric \( g^+ \). Moreover, by \( \hat{\mathcal{F}}^+ \) we denote the decomposition of \( \hat{M}^+ \) into preimages of leaves of \( \mathcal{F} \). Due to [Bou95], \( \hat{M}^+ \) is a closed submanifold of \( Gr_k \) and \( \hat{\mathcal{F}}^+ \) is a foliation on \( \hat{M}^+ \). (In fact, we only use the result of Boualem in the case of an isoparametric foliation on the flat \( \mathbb{R}^n \)).

We claim that the gauge \( I_{g,g^+} : Gr_k \to Gr_k \) sends \( M \) to \( \hat{M}^+ \). As soon as the claim is verified, we deduce that \( I_{g,g^+} \) sends \( \hat{\mathcal{F}} \) to \( \hat{\mathcal{F}}^+ \), because \( I_{g,g^+} \) is a bundle morphism, i.e., it commutes with the projection \( p \). Thus this claim would imply that \( \hat{M} \) is a smooth closed submanifold and that \( \hat{\mathcal{F}} \) is a foliation on \( \hat{M} \).

Thus it remains to prove the following
Lemma 4.1. Let $M$ be a manifold and let $\mathcal{F}$ be an infinitesimally polar singular Riemannian foliation with respect to Riemannian metrics $g$ and $g^+$. Then $I_{g,g^+} : Gr_k \to Gr_k$ sends $\hat{M}$ to $\hat{M}^+$.

Proof. Choose a point $x \in M$. The singular foliation $T_x\mathcal{F}$ on the tangent space $T_xM$ is defined independently of $g$ and $g^+$. The preimages of $x$ in $\hat{M}$ and in $\hat{M}^+$ are defined only in terms of $T_x\mathcal{F}$, $g_x$ and $g_x^+$, thus it is enough to prove the claim for the case $M = \mathbb{R}^n$, where $\mathcal{F}$ is a polar singular Riemannian foliation with respect to the flat metrics $g$ and $g^+$ (by replacing $\mathcal{F}$ through $T_x\mathcal{F}$). In this case $\hat{M}$ and $\hat{M}^+$ are closed submanifolds of $Gr_kM$ and the regular part $p^{-1}(M_0)$ is open and dense in both $\hat{M}$ and $\hat{M}^+$ ([Bou95]). By definition, $I_{g,g^+}$ sends $p^{-1}(M_0) \cap \hat{M}$ to $p^{-1}(M_0) \cap \hat{M}^+$.

By continuity, we deduce $I_{g,g^+}(\hat{M}) \subset \hat{M}^+$. Reversing the role of $g$ and $g^+$ and using that $I_{g,g^+} \circ I_{g^+,g} = Id$, we deduce $I_{g,g^+}(M) = \hat{M}^+$. □

4.3. Regular vectors. Before we are going to define a Riemannian structure on $\hat{M}$, we will need some observations concerning the space of horizontal vectors. Let $\mathcal{F}$ be again a singular Riemannian foliation on a Riemannian manifold $(M, g)$. As in [LT07a], we denote by $D(g)$ the space of all unit horizontal vectors on $M$. By $D^0 = D^0(g) \subset D(g)$ we denote the space of all regular horizontal vectors. Recall, that a horizontal vector $v \in H_x$ is called regular if the horizontal geodesic $\gamma^v$ starting in the direction of $v$ contains at least one regular point ([LT07a]). In this case all but discretely many points on $\gamma^v$ are regular. Equivalently, one can say that a vector $v \in H_x$ is regular, if $v \in T_xM$ is a regular point of the infinitesimal singular Riemannian foliation $T_x\mathcal{F}$. Recall finally, that $D^0$ is a smooth, injectively immersed submanifold of the unit tangent bundle $U^gM$ of $M$, that is invariant under the geodesic flow.

If $\mathcal{F}$ is infinitesimally polar then a horizontal vector $v$ is regular if and only if it is contained in only one horizontal section $S$ of the isoparametric foliation $T_x\mathcal{F}$. The assignment of the section $S$ to the regular horizontal vector $v$ defines a map $m = m(g) : D^0 \to \hat{M}$. We are going to prove that $m$ is a smooth submersion.

First, recall that for another Riemannian metric $g^+$ adapted to $\mathcal{F}$ we have an induced map $I_{g,g^+} : D(g) \to D(g^+)$ that is the restriction of the smooth map $I_{g^+,g}$ between the unit tangent bundles $I_{g^+,g} : U^gM \to U^{g^+}M$ (induced by the fiber-wise linear isomorphisms $I_{g,g^+} : TM \to TM$).
Lemma 4.2. Let $\mathcal{F}$ be an infinitesimally polar singular Riemannian foliation with respect to the Riemannian metrics $g$ and $g^+$. Then the map $I_{g,g^+} : D(g) \to D(g^+)$ sends $D^0(g)$ to $D^0(g^+)$. 

Proof. Since $I_{g,g^+}$ sends infinitesimal $g$-horizontal sections containing a $g$-horizontal vector $v$ to infinitesimal $g^+$-horizontal sections containing the $g^+$-horizontal vector $I_{g,g^+}(v)$, the result follows from the characterization of $D^0$ as the set of all horizontal vectors, contained in precisely one infinitesimal horizontal section. □

Question 4.1. Is the statement of the last lemma true for general singular Riemannian foliations, that are not infinitesimally polar?

Let $M, \mathcal{F}, g, g^+$ be as in the lemma above, and let $\hat{M}$ and $\hat{M}^+$ be the manifolds of horizontal infinitesimal sections with respect to $g$ and $g^+$ respectively. We have the diffeomorphisms $I_{g,g^+} : D(g) \to D(g^+)$ and $I_{g^+,g} : \hat{M}^+ \to \hat{M}$ and the maps $m(g) : D^0(g) \to \hat{M}$ and $m(g^+) : D^0(g^+) \to \hat{M}$. By construction, the maps commute, i.e., $m(g) = I_{g^+,g} \circ m(g^+) \circ I_{g,g^+}$. Therefore, $m(g)$ is a smooth submersion if and only if $m(g^+)$ is a smooth submersion. Now we can prove:

Lemma 4.3. Let $\mathcal{F}$ be an infinitesimal Riemannian foliation on a Riemannian manifold $(M, g)$. Then the map $m(g) : D^0(g) \to \hat{M}$ is a smooth submersion.

Proof. The objects $m(g), D^0, \hat{M}$ are defined locally on $M$. Thus it is enough to prove the statement in a neighborhood of each point $x$ in $M$. This reduces the question to the standard case. Then the observation preceding this proposition reduces the question to the case $\mathcal{F} = T_x\mathcal{F}$. Thus we may assume that $M$ is the Euclidean space $\mathbb{R}^n$ and that $\mathcal{F}$ is a polar singular Riemannian foliation on $\mathbb{R}^n$.

In this case the claim can be deduced as follows. Given a regular horizontal vector $v \in D^0$, choose a small number $\epsilon$ and a neighborhood $O$ of $v$ in $D^0$ such that $p(\phi_t(O))$ is contained in the set of regular points of $M$. Here, $p : UM \to M$ is the projection from the unit tangent bundle to $M$ and $\phi_t$ is the geodesic flow. The Grassmannian bundle of $\mathbb{R}^n$ is a trivial bundle with a canonical trivialization. With respect to this trivialization we have $m(v) = m(\phi_t(v))$ for all $v \in D^0$ and all $t$. Thus $m$ is preserved by the geodesic flow $\phi$, and the above choice of $O$ reduces the question to the regular part of $M$. However, in the regular part $M_0$ of $M$ the claim is clear. □

4.4. Normal distribution. We are going to define now, what is going to be the normal distribution of the foliation $\hat{\mathcal{F}}$ with respect to
the Riemannian metric $\hat{g}$ to be defined later. Let $F$ be an infinitesimally polar singular Riemannian foliation on a Riemannian manifold $M$. (Since we are not going to use auxiliary metrics $g^+$ anymore, we are going to suppress the Riemannian metric $g$ in the sequel). Let $\hat{M}$ be defined as in Subsection 4.2. Let $\hat{M}_0$ be the regular part of $\hat{M}$ and let $\hat{M}_0$ be the preimage $p^{-1}(M_0)$. The restriction $p : \hat{M}_0 \to M_0$ is a diffeomorphism, thus on $\hat{M}_0$ there is a smooth distribution $\hat{H}_0$ that is sent by $p$ to the horizontal distribution of the Riemannian foliation $F$ on the Riemannian manifold $M_0$. We claim:

**Lemma 4.4.** There is a unique smooth $k$-dimensional distribution $\hat{H}$ on $\hat{M}$ that extends $\hat{H}_0$.

**Proof.** The uniqueness is clear, since $\hat{M}_0$ is dense in $\hat{M}$. In order to prove the existence, it is enough to show that for each element $S \in \hat{M}$ there are $k$ linearly independent smooth vector fields $W_i$ defined on an open neighborhood $O$ of $S$ in $\hat{M}$, such that the restriction of each $W_i$ to $O \cap \hat{M}_0$ is a section of $\hat{H}_0$.

Thus, let $S \in \hat{M}$ be given and let $x = p(S) \in M$ be the foot point of $S$. Let $w \in T_xM$ be a regular unit horizontal vector contained in $S$. Since the map $m : D^0 \to \hat{M}$ is a smooth submersion, we find an open neighborhood $O$ of $S$ in $\hat{M}$ and a smooth section $n : O \to D^0$ with $m \circ n = \text{Id}$ and $n(S) = w$.

Let $I$ be a small interval around 0. Consider the map $\bar{\xi} : O \times I \to D^0$ given by $\bar{\xi}(\bar{S}, t) = \phi_t(n(\bar{S}))$, where $\phi_t$ denote the restriction of the geodesic flow to $D^0$. By construction, $\bar{\xi}$ is a smooth map. This implies smoothness of the composition $\xi : O \times I \to O$ given by $\xi = m \circ \bar{\xi}$. By construction, $\xi(S, 0) = S$ for all $\bar{S} \in O$. Therefore, the map

$$W(\bar{S}) := \frac{d}{dt}\xi(\bar{S}, t)$$

is a smooth vector field on $O$.

Now, the map $m : D^0 \to \hat{M}$ commutes with the projections to $M$, i.e., $p(m(v)) = p(v)$ for all $v \in D^0$. Thus the projection of any $\xi$-trajectory to $M$ is the projection of the corresponding $\bar{\xi}$ trajectory to $\hat{M}$. By definition, $\xi$-trajectories are flow lines of the geodesic flow. Thus the $\xi$-trajectory of a point $\bar{S} \in O$ is sent by the projection $p : \hat{M} \to M$ to the regular horizontal geodesic that starts at $p$ in the direction $n(\bar{S})$. In particular, we deduce that the restriction of $W$ to $M_0$ is a section of $\hat{H}_0$. Moreover, by construction, $p_*(W(\bar{S})) = w$.

Now, we choose a basis $w_i$ of $S$ that consists of regular vectors and applying the above construction, we get the linearly independent smooth vector fields $W_i$, we were looking for. □
4.5. **Riemannian structure.** Now we are in position to define the right Riemannian structure $\hat{g}$ on $\hat{M}$. We start with the canonical Riemannian metric $h$ on the Grassmannian bundle $Gr_k(M)$ (cf. [1506] for its definition and properties) and denote by the same letter $h$ its restriction to the submanifold $\hat{M}$. The projection $p : (Gr_k, h) \to (M, g)$ is a Riemannian submersion. In particular, the restriction $p : (\hat{M}, h) \to (M, g)$ is 1-Lipschitz.

Let $\mathcal{H}$ be the distribution of $k$-dimensional spaces on $\hat{M}$ defined in the previous subsection. In the proof of Lemma 4.4 we have seen that for each $S \in \hat{M}$ it is possible to choose a base $W_1, ..., W_n$ of $\mathcal{H}(S)$ that are mapped by the differential $p_*$ to a base of $S \subset T_{p(S)}M$. In particular, for each $S \in \hat{M}$, the restriction of $p_* : T_S \hat{M} \to T_{p(S)}M$ sends $\mathcal{H}(S)$ bijectively to $S \subset T_{p(S)}M$. Since $S$ is normal to the leaf of $\mathcal{F}$ through $p(S)$, we deduce that $\mathcal{H}$ and $\mathcal{F}$ are transversal.

Now we define the Riemannian metric $\hat{g}$ on $\hat{M}$ uniquely by the following three properties. On $\mathcal{F}$ we let $\hat{g}$ coincide with the canonical metric $h$. We require $\mathcal{F}$ and $\mathcal{H}$ to be orthogonal with respect to $\hat{g}$. Finally, on $\mathcal{H}$ we define $\hat{g}$ such that $p_*$ induces an isometry between $\mathcal{H}(S)$ and $S$, for all elements $S \in \hat{M}$. In other words, we set $\hat{g}(v, w) = g(p_*(v), p_*(w))$, for all $v, w \in \mathcal{H}(S)$.

By construction, $\hat{g}$ is a smooth Riemannian metric on $\hat{M}$. For each point $S \in \hat{M}$, the differential $p_*$ sends the orthogonal subspaces $\hat{F}(S)$ and $\mathcal{H}(S)$ to orthogonal subspaces of $T_{p(S)}M$ and the restrictions of $p_*$ to $\hat{F}(S)$ and to $\mathcal{H}(S)$ are 1-Lipschitz. Therefore, the map $p : (\hat{M}, \hat{g}) \to (M, g)$ is 1-Lipschitz.

On the regular part $\hat{M}_0$ the foliation $\hat{F}$ is a Riemannian foliation with respect to the metric $\hat{g}$. (If $\hat{M}_0$ and $M_0$ are identified via the diffeomorphism $p : \hat{M}_0 \to M_0$, the metric $\hat{g}$ arises from the metric $g$ by changing $g$ only on $\mathcal{F}$ and by leaving the metric on the normal part unchanged). Since $\hat{M}_0$ is dense in $\hat{M}$, the foliation $\hat{F}$ is a Riemannian foliation on the whole manifold $(\hat{M}, \hat{g})$.

By construction, $p_*$ sends horizontal vectors on $\hat{M}$ to horizontal vectors on $M$ of the same length; therefore, $p$ preserves transverse length of curves. Thus $p : (\hat{M}, \hat{F}) \to (M, \mathcal{F})$ is a geometric resolution.

4.6. **Proof of Theorem 1.1.** Now we can finish the proof of Theorem 1.1. If $(M, \mathcal{F})$ admits a geometric resolution, then $\mathcal{F}$ is infinitesimally polar, as was shown in Section 3.

Let now $\mathcal{F}$ be infinitesimally polar. Consider the manifold $\hat{M}$ with the foliation $\hat{F}$ defined in Subsection 4.2 and let $F : \hat{M} \to M$ be the
canonical projection \( p \). Let \( \hat{g} \) be the Riemannian metric on \( \hat{M} \) defined in Subsection 4.5. As we have seen, \( \hat{F} \) is a Riemannian foliation on the Riemannian manifold \((\hat{M}, \hat{g})\) and \( F : \hat{M} \rightarrow M \) is a geometric resolution.

We have seen in Subsection 4.5 that the map \( F \) is 1-Lipschitz. By construction, the leaves of \( \hat{F} \) are preimages of leaves of \( F \), thus \( p \) induces a bijection between spaces of leaves. Moreover, by construction, the preimage of a compact subset \( K \) on \( M \) is a closed subset of a compact subset of the Grassmannian bundle \( Gr_k(M) \). Thus the map \( F \) is proper.

If \( M \) is compact then \( \hat{M} \) is compact, since \( F \) is proper. Since \( F \) is 1-Lipschitz, a ball of radius \( r \) around a point \( S \in \hat{M} \) is contained in the preimage of the ball of radius \( r \) around \( F(S) \) in \( M \). If \( M \) is complete, the properness of \( F \) implies that all balls in \( \hat{M} \) are compact. Therefore, \( \hat{M} \) is complete in this case.

The objects \((\hat{M}, \hat{F}, \hat{g})\) are defined only in terms of \( M, F \) and \( g \). Therefore, they are invariant under isometries of \((M, F)\). This proves the statement about \( \Gamma \)-equivariance. The claim about singular Riemannian foliations \( F \) given by orbits of an isometric action of a group \( G \) is a direct consequence of the last claim.

Assume now that \( M \) and therefore \( \hat{M} \) are complete. The notion of the absence of horizontal conjugate point is a transverse notion, i.e., it can be formulated only in terms of local quotients (cf. [LT07a]). Since the transverse geometries of \((M, F)\) and of \((\hat{M}, \hat{F})\) coincide, due to the definition of a geometric resolution, the singular Riemannian foliation \( F \) has no horizontal conjugate points if and only if the Riemannian foliation \( \hat{F} \) has no horizontal conjugate points.

Identifying the regular part \( \hat{M}_0 \) with \( M_0 \) via \( F \), we see that, by construction, the horizontal distributions of \( F \) with respect to the metrics \( g \) and \( \hat{g} \) coincide. Thus, one of them is integrable if and only if the other one is integrable. The integrability of the normal distribution on the regular part is equivalent to polarity ([Alc06]). This shows that \( F \) is polar if and only if \( \hat{F} \) is polar.

This finishes the proof of Theorem 1.1.

5. Simplicity in the regular part

We are going to prove Theorem 1.5 in a slightly more general setting that we are going to describe now.

**Definition 5.1.** A singular Riemannian foliation on a Riemannian manifold \( M \) is **full** if for each leaf \( L \) there is some \( \epsilon > 0 \) such that \( \exp(\epsilon v) \) is defined for each unit vector in the normal bundle \( L \).
Each singular Riemannian foliation on a complete Riemannian manifold is full. In a full singular Riemannian foliation each pair of leaves is equidistant. If \( \mathcal{F} \) is full on \( M \) and if \( U \subset M \) is an open subset that is a union of leaves of \( \mathcal{F} \) then the restriction of \( \mathcal{F} \) to \( U \) is full again (this follows from \([LT07a]\), Proposition 4.3). Moreover, for each covering \( N \) of \( M \) the lift of \( \mathcal{F} \) to \( N \) is full on \( N \).

If \( \mathcal{F} \) is a full singular Riemannian foliation on a Riemannian manifold \( M \) with all leaves closed, then \( M/\mathcal{F} \) is a metric space, with a natural inner metric that has curvature locally bounded below in the sense of Alexandrov. Note that an isometry of such a space is uniquely determined by its restriction to an open subset. Finally, a full regular Riemannian foliation is simple, i.e., has closed leaves with trivial holonomy, if and only if the quotient \( M/\mathcal{F} \) is a Riemannian manifold.

Let now \( \mathcal{F} \) be a full singular Riemannian foliation on a connected Riemannian manifold \( M \), with \( \pi_1(M) = \Gamma \). Let \( \hat{M} \) be the universal covering of \( M \) and let \( \hat{\mathcal{F}} \) be the lifted singular Riemannian foliation on \( \hat{M} \). Assume that \( \hat{\mathcal{F}} \) has closed leaves and denote by \( B \) the quotient space \( \hat{M}/\hat{\mathcal{F}} \). The fundamental group \( \Gamma \) acts on \( (\hat{M}, \hat{\mathcal{F}}) \). Thus we get an induced action of \( \Gamma \) on the quotient \( B \). Denote by \( \Gamma_0 \) the kernel of the action of \( \Gamma \) on \( B \), i.e., the set of all elements of \( \Gamma \) that act trivially on \( B \).

**Lemma 5.1.** In the notations above let \( g \in \Gamma \) be an element. Then the following are equivalent:

1. \( g \in \Gamma_0 \);
2. Each leaf \( L \) of \( \mathcal{F} \) contains a closed curve whose free homotopy class is the conjugacy class of \( g \);
3. There is a non-empty open subset \( U \) in \( M \) such that each leaf \( L \) of \( \mathcal{F} \), which has a non-empty intersection with \( U \), contains a closed curve whose free homotopy class is the conjugacy class of \( g \).

**Proof.** Let \( \hat{L} \) be a leaf of \( \hat{\mathcal{F}} \) through a point \( y \in \hat{M} \). Then the translate \( gy \) is contained in \( \hat{L} \) if and only if \( g \) fixes the point \( \hat{L} \in B \). On the other hand, if \( gy \) is contained in \( \hat{L} \) then connecting \( y \) and \( gy \) by a curve in \( \hat{L} \) one obtains a closed curve in the image \( L \) of \( \hat{L} \) in \( M \) whose free homotopy class is in the conjugacy class of \( g \). Note that this image \( L \) is a leaf of \( \mathcal{F} \).

Let \( L \) be a leaf in \( M \) that contains a closed curve \( \gamma \) whose free homotopy class is in the conjugacy class of \( g \). Then each lifted leaf \( \hat{L} \) of \( L \) contains a lift of the curve \( \gamma \). Thus, in this case, each lift \( \hat{L} \) of the leaf \( L \) is fixed by some conjugate of \( g \).
Now the implications $1 \implies 2 \implies 3$ are clear. Assume 3. Let $\tilde{U}$ be the preimage of $U$ in $\tilde{M}$ and $V$ the projection of $\tilde{U}$ to $B$. Then $V$ is a non-empty open subset of the quotient $B$ and each point in $V$ is fixed by some conjugate of $g$. There are only countably many conjugates of $g$, each of them fixing a closed subset of $B$. By Baire’s theorem, at least one conjugate of $g$ fixes a non-empty open subset of $V$. Since $B$ is an inner metric space with curvature locally bounded from below, $g$ fixes all of $B$. Therefore, $g \in \Gamma_0$. □

The following result generalizes Theorem 1.5.

**Proposition 5.2.** Let $\mathcal{F}$ be a full singular Riemannian foliation on a simply connected Riemannian manifold $M$. Let $M_0$ denote the regular stratum of $M$ and let the Riemannian foliation $\mathcal{F}_0$ be the restriction of $\mathcal{F}$ to $M_0$. Assume that the lift $\tilde{\mathcal{F}}_0$ of $\mathcal{F}_0$ to the universal covering $\tilde{M}_0$ is closed. Then $\mathcal{F}_0$ is closed as well and the canonical projection $\tilde{M}_0/\tilde{\mathcal{F}}_0 \to M_0/\mathcal{F}_0$ is an isometry. In particular, if $\tilde{\mathcal{F}}_0$ is a simple Riemannian foliation then $\mathcal{F}_0$ is a simple Riemannian foliation on $M_0$.

**Proof.** The assumptions and conclusions do not change if one deletes from $M$ all strata of codimension $\geq 3$. Thus we may assume that such strata do not exist. Then the complement $\Sigma = M \setminus M_0$ is a disjoint union of closed submanifolds $\Sigma_i$ of codimension 2.

Choose a point $x_i$ on $\Sigma_i$, a small neighborhood $P_i$ of $x_i$ in $\Sigma_i$ and a small tubular neighborhood $U_i$ of $P_i$ in $M$. Let $q : U_i \to P_i$ be the foot point projection. The restriction of $q$ to $U_i \setminus P_i$ is a fiber bundle with circles as fibers. By construction, each of these circles is contained in a leaf of $\mathcal{F}$.

On the other hand, all these circles are in the same free homotopy class $[g_i]$ of $U_i \setminus P_i$. Since $M$ is simply connected, the fundamental group $\Gamma$ of $M_0$ is generated by conjugates of the elements $g_i$ (i.e., $\Gamma$ is normally generated by the elements $g_i$). Due to Lemma 5.1, each of these free homotopy classes acts trivially on the Riemannian orbifold $B = \tilde{M}_0/\tilde{\mathcal{F}}_0$. Thus $\Gamma = \pi_1(M_0)$ acts trivially on $B$ and we get $M_0/\mathcal{F}_0 = B$. This proves the theorem. □

For a full Riemannian foliation $\mathcal{F}$ with closed leaves one has an induced surjective homomorphism from $\pi_1(M)$ onto $\pi_1^{orb}(B)$, the orbifold fundamental group of the quotient orbifold $B$ (cf. [Sal88] or [Hae88]). Thus as a consequence of the above Proposition we deduce:

**Corollary 5.3.** Let $\mathcal{F}$ be a full singular Riemannian foliation on a simply connected Riemannian manifold $M$, with all leaves closed. Then the quotient $B_0$ of the restriction of $\mathcal{F}$ to the regular part $M_0$ is a Riemannian orbifold with $\pi_1^{orb}(B) = 1$.  

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Remark 5.1. The above Lemma 5.1 is true also in the case of non-closed $\tilde{\mathcal{F}}$, as one sees by localizing the arguments. Corollary 5.3 is also valid without the assumption that $\tilde{\mathcal{F}}_0$ is closed, in the sense, that the fundamental group of the pseudo-group of isometries $M_0/\mathcal{F}_0$ is simply connected, cf. [Sa88].

We are going to use two simple observations about orbifolds. First of all, an orbifold $B$ with $\pi_1^{\text{orb}}(B) = 1$ is orientable. Hence it does not have strata of codimension 1, i.e., $\partial B = \emptyset$. On the other hand, any non-compact 2-dimensional orbifold is a good orbifold. Thus if $B$ is a non-compact two-dimensional orbifold with $\pi_1^{\text{orb}}(B) = 1$ then $B$ is a manifold (nessesarily an open disc).

Now we are going to provide:

**Proof of Theorem 1.6.** Let $p \in B$ be a point representing a singular leaf $L$ of $\mathcal{F}$. Choose some $x \in L$. Choose a small distinguished neighborhood $U$ at the point $x$. Then the restriction of $\mathcal{F}$ to $U$ is given by a (restriction of a) non-trivial isoparametric foliation on $\mathbb{R}^n$, thus $U/\mathcal{F}$ is a Weyl chamber. The embedding $U \to M$ induces a finite-to-one projection $U/\mathcal{F} \to B$. Moreover, this projection is given by a finite isometric action of a group $\Gamma$ on $U/\mathcal{F}$ (cf. [LT07a], p.7). Since the Weyl chamber has non-empty boundary, so does its finite quotient. Hence any neighborhood of $p$ contains boundary points. Since the boundary is closed, $p \in \partial B$.

Assume now that $M$ is simply connected. Denote by the orbifold $B_0 \subset B$ the quotient of the regular part of $\mathcal{F}$. We have seen in Corollary 5.3 that $B_0$ is simply connected as orbifold. Thus it cannot contain strata of codimension 1. But $B_0$ is open thus, if it has a point in $\partial B$, then it has a point lying on a stratum of codimension 1 in $B$. Then the whole stratum is contained in $B_0$, contradiction. □

Now it is easy to obtain:

**Proof of Corollary 1.7.** Recall that $\mathcal{F}$ is infinitesimally polar, since $B$ has dimension 2 ([LT07a]). Assume that the foliation is not regular. Then the quotient has non-empty boundary, by Theorem 1.6. Therefore, the complement of the boundary $B_0 = B \setminus \partial B$ (the quotient orbifold of the regular part) is not compact. But its orbifold fundamental group is trivial by Corollary 5.3. Since it is a 2-dimensional orbifold, it must be a manifold. Thus there are no exceptional orbits. □

Now we are going to provide:
Proof of Theorem 1.8. The equivalence of (1), (2) and (4) has already been established (Theorem 1.5 and Theorem 1.6). By definition of a Coxeter orbifold, (3) implies (4).

Now assume (1). Take a point \( p \in B \). As we have seen in the proof of the first part of Theorem 1.6 there is a Weyl chamber \( W \) with a Riemannian metric (a local quotient at a point \( x \in M \) over \( p \) and an action of a finite group \( \Gamma \) on \( W \) by isometries, such that the quotient \( W/\Gamma \) is isometric to an open neighborhood of \( p \). Note that the set of regular points \( W_0 \) in \( W \) is projected to \( B_0 \), the set of regular leaves. The assumption, that there are no exceptional orbits, i.e., that \( B_0 \) and therefore \( W_0/\Gamma \) is a Riemannian manifold, implies that the action of \( \Gamma \) on the regular part of \( W \) is free. But the regular part \( W_0 \) of \( W \) is contractible! Hence the finite group \( \Gamma \) must act trivially on \( W_0 \) (since \( \Gamma \) it has infinite cohomological dimension). Thus \( \Gamma \) acts trivially on \( W \). Therefore, a neighborhood of \( p \) isometric to \( W \). Thus \( B \) is a Coxeter orbifold.

\[ \square \]

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