ADDITIVITY OF BRIDGE NUMBERS OF KNOTS

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Abstract. We provide a new proof of the following results of H. Schubert: If
K is a satellite knot with companion J and pattern (Y, L) with index k, then
the bridge numbers satisfy the following: b(K) ≥ k · (b(J)). In addition, if K
is a composite knot with summands J and L, then b(K) = b(J) + b(L) − 1.

In “Über eine numerische Knoteninvariante” [1], Horst Schubert proved that
for a satellite knot K with companion J and pattern of index k, bridge numbers
satisfy the inequality b(K) ≥ k · (b(J)). He also proved that for a composite knot
K with summands J and L, the bridge numbers satisfy b(K) = b(J) + b(L) − 1.
His investigation was motivated by the question as to whether a knot can have only
finitely many companions. Together with the fact that the only bridge number one
knot is the unknot, his result showed that the answer to this question is yes.

Schubert’s main result may be recovered by a much shorter proof. This shorter
proof grew out of an endeavour to recast the problem within the framework of the
thin position of a knot. This framework turns out to be far more refined than
necessary. The proof here does not employ the notion of thin position. It does,
however, rely heavily on the idea of rearranging the order in which critical points
occur to suit one’s purpose, an idea fundamental to the notion of thin position
of knots and 3-manifolds. In this way it differs dramatically from Schubert’s proof. It
also differs from Schubert’s in that it relies on the consideration of Morse functions
on S^3 whose level sets are spheres (except for the maximum and minimum) and
their induced foliations. This streamlines the terminology and the complexity of the
argument. Schubert’s proofs of the results reproven here involve 25 pages containing
15 lemmas which involve a consideration of up to three cases.

I wish to thank both Ray Lickorish and Marty Scharlemann for independently
suggesting that a more modern proof of this theorem would be a welcome addition
to the existing literature and for helpful conversations. I also wish to thank the
Department of Pure Mathematics and Mathematical Statistics at the University of
Cambridge for its hospitality.

In the following K will always be a knot in S^3 and h : S^3 → R a Morse function
with exactly two critical points. This last assumption guarantees that h induces a
foliation of S^3 by spheres, along with one maximum that we denote by ∞ and one
minimum that we denote by −∞.

Definition 1. If the minima of h_K occur below all maxima of h_K, then we say
that K is in bridge position with respect to h. The bridge number of K, b(K), is the
minimal number of maxima required for h_K. (Note that this number is independent
of whether or not we require K to be in bridge position. Indeed, if h_K has n maxima,
then the maxima of $h_K$ can be raised, and the minima of $h_K$ lowered, to obtain a copy of $K$ in bridge position with $n$ maxima.)

**Definition 2.** Let $J$ be a knot in $S^3$ and denote a small closed regular neighborhood of $J$ by $V$. Let $V$ be an unknotted solid torus in $S^3$ containing a knot $L$. A map of $V$ into $V$ maps $L$ onto a knot $K$. We call $K$ a satellite knot, $J$ a companion of $K$, $V$ the companion torus of $K$ with respect to $J$ and $(V, L)$ the pattern (of $K$ with respect to $J$). The least number of times which a meridian disk of $V$ intersects $L$ is called the index of the pattern. (It is also called the wrapping number.)

In the special case in which the index of the pattern is 1, this construction yields the connected sum of $J$ and $L$, and $V$ is called a swallow-follow torus.

**Definition 3.** Suppose that $K$ is homotopically nontrivially contained in a solid torus $V$. Set $T = \partial V$. Then $V$ is taut with respect to $b(K)$, if the number of critical points of $h_T$ is minimal subject to the condition that $h_K$ has $b(K)$ maxima.

**Definition 4.** Consider the singular foliation, $\mathcal{F}_T$, of $T$ induced by $h_T$. Let $\sigma$ be a leaf corresponding to a saddle singularity. Then $\sigma$ consists of two circles, $s_1, s_2$, wedged at a point. If either $s_1$ or $s_2$ is inessential in $T$, then we call $\sigma$ an inessential saddle. Otherwise, $\sigma$ is an essential saddle.

**Lemma 1.** (The Pop Over Lemma) Let $h, K, V, \mathcal{F}_T$ be as above. If $\mathcal{F}_T$ contains inessential saddles, then, after an isotopy of $T$ that does not change $b(K)$ or the number of critical points of $h_T$, there is an inessential saddle $\sigma$ in $\mathcal{F}_T$ for which the following conditions hold:

1) $s_1$ bounds a disk $D_1 \subset T$ such that $\mathcal{F}_T$ restricted to $D_1$ contains only disjoint circles and one maximum or minimum; and

2) for $L$ the level surface of $h$ containing $\sigma$, $D_1$ cobounds a 3-ball $B$ with a disk $D_1 \subset L - T$, such that $B$ does not contain $\infty$ or $-\infty$, and such that $s_2$ does not meet $B$ (i.e., such that $s_2$ lies outside of $D_1$).

**Proof.** The first condition on $\sigma$ may be satisfied by choosing $\sigma$ to be an inessential saddle in $\mathcal{F}_T$ that is innermost in $T$. In this case $L - \partial D_1$ consists of two disks, $\hat{D}_1$ and $\hat{D}_2$. Together with $D_1$, both $\hat{D}_1$ and $\hat{D}_2$ cobound 3-balls $\hat{B}_1, \hat{B}_2$, respectively. One of these 3-balls, say $\hat{B}_2$, contains either $\infty$ or $-\infty$ and the other contains neither.

If $s_2 \subset \hat{D}_2$, we may take $B = \hat{B}_1$, so suppose $s_2 \subset \hat{D}_1$. Without loss of generality, we may assume that the critical point of $D_1$ is a maximum. In this case, consider a monotone arc $\alpha$ disjoint from $K$, beginning at the maximum of $D_1$, passing only through maxima of $T$ and ending at $\infty$. Let $a_1, \ldots, a_n$ be the points at which $\alpha$ meets $T$, with $a_n$ the highest such point.

Let $\beta$ be the subarc of $\alpha$ between $a_n$ and $\infty$ and let $C'$ be a collar neighborhood of $\beta$. After a small isotopy, $T \cap C'$ consists of a small disk $D = a_n \times \text{disk} \subset T$. Let $C''$ be a small 3-ball centered at $\infty$ that is disjoint from $T$. Set $C = C' \cup C''$ and consider $T' = (T - D) \cup (\partial C - D)$. This describes an isotopy of $T$ that replaces $\hat{B}_1$ by $\hat{B}_1 \cup C$ and replaces $\hat{B}_2$ by $\hat{B}_2 - C$. After a small tilt which turns $h_{T'}$ into a Morse function, the maximum $a_n$ of $h_T$ has turned into a maximum of $h_{T''}$ at a higher level. No critical points need have been introduced for $h_K$ and the number of critical points of $h_{T''}$ is the same as that of $h_T$. 


By induction, we may assume that $\alpha$ is disjoint from $T$ except at its initial point. Then if $s_2 \subset \hat{D}_1$, this same construction using $\beta = \alpha$ describes an isotopy of $T$ augmenting $\hat{B}_1$ to contain $\infty$ and shrinking $\hat{B}_2$ to exclude $\infty$ without introducing any critical points of $h_K$ or $h_T$. We may then choose $B$ to be the shrunk version of $\hat{B}_2$.

\[ \square \]

**Lemma 2.** (The Pop Out Lemma) Let $h, K, V, F_T$ be as above. If $V$ is taut with respect to $b(K)$, then there are no inessential saddles in $F_T$.

**Proof.** Suppose there are inessential saddles. Alter $T$ as in Lemma 1 so that there is an inessential saddle $\sigma$ satisfying the conclusions of Lemma 1. We may assume that $D_1$ contains a maximum and lies above $L$. (The other case is analogous.) Here $(K \cup T) \cap \text{int}(B)$ can be shrunk horizontally and lowered via an isotopy to lie just below $\hat{D}_1$ (and above any critical points of $h_K$ or $h_T$ below $\hat{D}_1$). This does not change the nature or number of critical points of $h_K$ or $h_T$.

Now $D_1 \subset T$ can be replaced by $\hat{D}_1$ to obtain $\hat{T}$. After a small tilt, $\hat{T}$ bounds a solid torus $\hat{V}$ containing a copy of $K$ with $b(K)$ maxima, and $\hat{T}$ is isotopic to $T$, yet $h_T$ has two fewer critical points than $h_T$. (A maximum and an inessential saddle have been cancelled). This contradicts the assumption that $V$ is taut with respect to $b(K)$.

\[ \square \]

**Remark 1.** Consider a bicollar of an essential saddle $\sigma$ in $F_T$. It has three boundary components, $c_1, c_2, c_3$, where $c_i$ is parallel to $s_i$ for $i = 1, 2$. Since $\chi(T) = 0$, it follows that $c_3$ bounds a disk. If there are no inessential saddles, then the disk bounded by $c_3$ contains exactly one singular point, a maximum or minimum. We consider this maximum or minimum, $m_\sigma$, to be the maximum or minimum corresponding to $\sigma$.

Conversely, if there are no inessential saddles in $F_T$, then every maximum or minimum corresponds to a saddle in this way, since $\chi(T) = 0$.

**Definition 5.** Let $\sigma, c_1, c_2, c_3$ be as above. We may assume that $c_1$ and $c_2$ are in the same level surface $L$ of $h$. Then since $L$ is a sphere, $c_1$ and $c_2$ cobound an annulus in $L$. If a collar of $c_1 \cup c_2$ in this annulus is contained in $V$, then $\sigma$ is a nested saddle.

**Lemma 3.** Let $h, K, V, F_T$ be as above. If $V$ is taut with respect to $b(K)$, then $F_T$ has no nested saddles.

**Proof.** Suppose that there are nested saddles in $F_T$.

Claim: There are also saddles in $F_T$ that are not nested.

Let $\sigma$ be the highest saddle in $F_T$. For $c_1, c_2, L$ as above, let $\hat{D}_1, \hat{D}_2$ be the (disjoint) disks bounded by $c_1, c_2$ in $L$. As $\sigma$ is the highest saddle in $F_T$, any curve in $T \cap \text{interior}(\hat{D}_i)$ bounds a disk lying above $L$. This implies that $\hat{D}_i$ is isotopic to a disk whose interior is disjoint from $T$, i.e., lies either entirely in $V$ or entirely in $S^3 - V$. Since $c_i$ is parallel to $s_i$, $c_i$ is essential in $T$. Furthermore, since $V$ is knotted, $T$ is incompressible, whence $c_i$ is essential in the closure of $S^3 - V$. This implies that $\hat{D}_i$ must be isotopic to a disk whose interior lies entirely in $V$ (in particular, $\hat{D}_i$ is a meridian disk). Thus $\sigma$ is not nested.
If there are both saddles that are nested and saddles that are not nested, then there must be an “adjacent” pair $\sigma_1, \sigma_2$ of essential saddles in $T$ with $\sigma_1$ nested, $\sigma_2$ not nested, where “adjacent” means that one component, say $C$, of $T - (\sigma_1 \cup \sigma_2)$ contains no critical points of $h_T$. Consider the circles $s_1^i, s_2^i$ whose wedge is $\sigma_i$. Without loss of generality, $s_1^1$ and $s_2^1$ meet $C$.

Again without loss of generality, we may assume that $\sigma_1$ lies above $\sigma_2$ and hence that the component of $T - \sigma_1$ lying above $\sigma_1$ and meeting both $s_1^1$ and $s_2^1$ is a disk $D_3^1$. Construct a disk $D$ by adding $D_3^1$ to $C$ and capping off $s_2^1$ with a level disk (a component of $h^{-1}(h(\sigma_1)) - \sigma_1$). Note that by the discussion above, this latter horizontal portion of $D$ meets $K$ and $T$.

We now proceed as in Lemma 1 and Lemma 2. Here $\partial D = s_1^2$, so $\partial D$ divides $h^{-1}(h(\sigma_2))$ into two disks, $D_1$ and $D_2$, that cobound 3-balls $\hat{B}_1$ and $\hat{B}_2$ together with $D$. By the proof of Lemma 1, we may assume that $\hat{B}_2$ contains $\infty$ and that $s_2^2 \subset \hat{B}_2$. We may thus shrink horizontally and lower $(K \cup T) \cap B$ as in the proof of Lemma 2. The difference is that here $K \cup T$ meets $D$ along its horizontal portion. As $(K \cup T) \cap B$ is shrunk horizontally, the horizontal portion of $D$ is lowered while remaining horizontal. The portion of $B$ lying above $h^{-1}(h(\sigma_1))$ is shrunk horizontally as necessary. In the end, a product neighborhood below the original horizontal portion of $D$ ends up intersecting $K \cup T$ in vertical arcs and surfaces.

As in the proof of Lemma 2, the number of critical points of $h_T$ can be reduced without altering the number of critical points of $h_K$, contradicting the fact that $V$ is taut with respect to $b(K)$.

\[ \text{fig. 3} \]

**Remark 2.** If $V$ is a knotted solid torus that is taut with respect to $b(K)$ then all saddles are essential and there are no nested saddles. It follows that if $L = h^{-1}(r)$ for some regular value $r$, then $V \cap L$ consists of disks. More specifically, let $\sigma_1, \ldots, \sigma_n$ be the saddles in $F_T$, and let $L_i = h^{-1}(h(\sigma_i))$. Recall that each saddle $\sigma$ corresponds to a maximum or minimum $m_\sigma$ of $h_T$. Between the level surfaces $h^{-1}(h(\sigma))$ and $h^{-1}(h(\sigma_i))$ lies a portion $B_\sigma$ of $V$ that is a 3-ball. Here $L_1 \cup \cdots \cup L_n$ cuts $V$ into $B_{\sigma_1}, \ldots, B_{\sigma_n}$ and vertical cylinders.

**Theorem 1.** Suppose $K$ is a satellite knot with companion $J$, companion torus $\hat{V}$, pattern $(V, L)$ and index $k$. Then $b(K) \geq k \cdot b(J)$. In addition, if $K$ is the connected sum of two knots $K_1$ and $K_2$, then $b(K) = b(K_1) + b(K_2) - 1$.

**Proof.** We may assume that $V$ is taut with respect to $b(K)$. Then $V$ is as described in Remark 2. We obtain a Morse function on $(S^3, J)$ by making $V$ very thin. So $b(J)$ is less than or equal to the number of maxima of $h_{T=0_V}$.

Consider a maximum of $T$. It corresponds to a saddle $\sigma$, where $\sigma$ is the wedge of the circles $s_1, s_2$, bounding level meridian disks $D_1, D_2$ of $V$. Here $D_1 \cup D_2$ cuts off a 3-ball $B_\sigma$ as in Remark 2. For distinct saddles $\sigma_i$ and $\sigma_j$, $B_{\sigma_i}$ and $B_{\sigma_j}$ are disjoint. Since at least $k$ strands pass through both $D_1$ and $D_2$, there are at least $k$ maxima of $K$ in $B_{\sigma}$. Whence $b(K) \geq k \cdot b(J)$.

In the special case where $K$ is the connected sum of $K_1$ and $K_2$, the satellite construction may be used with $K_1$ the companion and $(V, K_2)$ the pattern (of index 1). By renumbering, if necessary, we may assume that $b(K_1) \geq b(K_2)$. Then we still obtain a Morse function on $(S^3, K_1)$ as above. Furthermore, if, for each maximum of $T$ and $\sigma$, $B_{\sigma}, D_1, D_2$ as above, $D_i \cap K \geq 2$ for $i = 1, 2$, then $B_{\sigma}$ contains at
least two maxima of $K$. Hence $b(K) \geq 2 \cdot b(K_1) \geq b(K_1) + b(K_2) - 1$. Thus we may assume that there is a meridian disk contained in a level surface of $h$ that intersects $K$ once.

Recall from Remark 2 that $V$ is comprised of $B_{\sigma_1}, \ldots, B_{\sigma_n}$ and vertical cylinders. In a vertical cylinder, a meridian disk contained in a level surface that intersects $K$ once may be used to move all critical points upwards or downwards and out of the cylinder. Thus the intersection of $K$ with the cylinder becomes a monotone arc and the number of critical points of $h_K$ is unchanged.

In $B_{\sigma_1}$, assume that $[D_1 \cap K] = 1$. We may assume that $\sigma_1$ corresponds to a maximum of $h_T$. Let $\alpha$ be the subarc of $B_{\sigma_1} \cap K$ that connects $D_1 \cap K$ to the closest maximum of $K$ in $B_{\sigma_1}$. We may assume that this maximum is the highest maximum of $K$ in $B_{\sigma_1}$. Then consider a disk $E$ in $B_{\sigma_1}$ for which $\partial E$ consists of four subarcs: $\alpha, a_1, a_2, a_3$, where $a_1$ and $a_3$ are horizontal arcs connecting the endpoints of $\alpha$ to $\partial B_{\sigma_1} \subset T$, and $a_2$ is an arc in $\partial B_{\sigma_1}$ connecting the other endpoints of $a_1$ and $a_3$, that runs over the maximum of $\partial B_{\sigma_1}$, and has no other critical points. We further require that $E \cap T = a_2$.

Claim: After an isotopy that does not change the number of critical points of $h_K$, $E \cap K = \alpha$.

Let $p_1, \ldots, p_k$ be the points in $E \cap K - \alpha$ with $p_n$ the highest such point. A small monotone subarc $\beta$ of $K$ containing $p_n$ may be replaced by a monotone arc $\beta'$ that begins at one endpoint of $K - \beta$, travels parallel to $E$ until it almost reaches $\partial B_{\sigma_1}$, then circles around to the other side of $E$ along $\partial B_{\sigma_1}$ and travels parallel to $E$ on the other side of $E$ until it meets the other endpoint of $K - \beta$. See fig. 4. The result is isotopic to $K$ and has the same number of critical points as $K$, yet one fewer intersection with $E$. The Claim follows by induction.

Now $B_{\sigma_1} \cap K$ may be isotoped horizontally and downward, so that after the isotopy this intersection consists of one arc with exactly one critical point. In the vertical solid cylinder meeting $B_{\sigma_1}$ at $D_2$, $[D_2 \cap K] = 1$ and $D_2$ may be used to isotope $K$ so that all critical points are moved to $B_{\sigma_2}$ (after relabelling) and $K$ intersects the solid cylinder in a single monotone arc. Here $\sigma_2$ corresponds to a minimum, but an identical argument shows how proceed. After a finite number of iterations of this procedure, $B_{\sigma_i} \cap K$ consists of a single arc with one critical point for $i = 1, \ldots, n - 1$ and $K$ intersects all cylindrical portions of $V$ in monotone arcs. Then, (since $\sigma_n$ corresponds to a minimum) $\partial B_{\sigma_2}$ cuts $K$ into $K_1 - (\text{subarc containing a minimum})$ and $K_2 - (\text{subarc containing a maximum})$. This proves that $b(K_1 \# K_2) \geq b(K_1) + b(K_2) - 1$.

The other inequality follows by considering a copy of $K_2$ in bridge position realizing $b(K_2)$ lying below a copy of $K_1$ in bridge position realizing $b(K_1)$ and taking the connected sum.

References

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