INTEGRATION OVER QUANTUM PERMUTATION GROUPS

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Abstract. We find a combinatorial formula for the Haar measure of quantum permutation groups. This leads to a dynamic formula for laws of diagonal coefficients, explaining the Poisson/free Poisson convergence result for characters.

Introduction

A remarkable fact, discovered by Wang in [14], is that the set $X_n = \{1, \ldots, n\}$ has a quantum permutation group. For $n = 1, 2, 3$ this is the usual symmetric group $S_n$. However, starting from $n = 4$ the situation is different: for instance the dual of $\mathbb{Z}_2 \ast \mathbb{Z}_2$ acts on $X_4$. In other words, “quantum permutations” do exist. They form a compact quantum group $Q_n$, satisfying the axioms of Woronowicz in [15].

There are several motivations for study of $Q_n$:

1. This quantum group is supposed to be to noncommutative theories what the usual symmetric group is to classical theories. One can expect for instance that $Q_n$ might be of help in connection with Connes’ approach ([8]).

2. The subgroups of $Q_n$ are in correspondence with subalgebras of the spin planar algebra ([2]), and several questions from Jones’ paper [11] should have formulations in terms of $Q_n$. Work here is in progress.

3. Some connections with Voiculescu’s free probability theory ([13]) are pointed out in [3, 5]. The idea is that $Q_n$ and its subgroups should provide new illustrations for the old principle “integrating = counting diagrams”.

In this paper we clarify some questions coming from free probability. The main tool is a Weingarten type formula, for Haar integration over $Q_n$:

$$\int u_{i_1 j_1} \cdots u_{i_k j_k} = \sum_{pq} \delta_{p i_1} \delta_{q j_1} W_{kn}(p, q)$$

Here $u_{ij}$ are the coefficients of the fundamental representation of $Q_n$. Their products are known to form a basis of the algebra of representative functions on $Q_n$, so the above formula gives indeed all integrals over $Q_n$.

The sum on the right is over non-crossing partitions, and $W_{kn}$ is the Weingarten matrix, obtained as inverse of the Gram matrix of partitions $G_{kn}$.

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This leads to a general formula for moments of diagonal coefficients of $u$:

$$\int (u_{11} + \ldots + u_{ss})^k = \text{Tr}(G_{kn}^{-1}G_{ks})$$

This is similar to the formula in our previous paper \[5\], for the free analogue of the orthogonal group $O(n)$. The change comes from the scalar product on the space of partitions, different from the one in \[5\]. This leads to a number of subtleties at level of applications, as well as at the conceptual level: for instance, the matrix $G_{kn}$ in this paper is no longer equal to Di Francesco’s meander matrix (\[9\]).

As an application, we solve a problem arising from \[3\]. It is pointed out there that the character of $u$ is Poisson for $S_n$ with $n \to \infty$, and free Poisson for $Q_n$ with $n \geq 4$. The fact that $S_n \to Q_n$ corresponds to a passage Poisson $\to$ free Poisson is definitely positive, and not surprising. However, there is problem with correspondence of convergences, which is asymptotic $\to$ exact.

We show here that a fully symmetric statement can be obtained by looking at laws of diagonal coefficients $u_{11} + \ldots + u_{ss}$:

| $s$ | $u_{11} + \ldots + u_{ss} \in \mathbb{C}(S_n)$ | $u_{11} + \ldots + u_{ss} \in \mathbb{C}(Q_n)$ |
|-----|-----------------------------------------------|-----------------------------------------------|
| 1   | projection $(1/n)$                             | projection $(1/n)$                             |
| $o(n)$ | $\simeq$ projection $(s/n)$                    | $\simeq$ projection $(s/n)$                    |
| $tn$ | $\simeq$ Poisson $(t)$                         | $\simeq$ free Poisson $(t)$                    |
| $n$  | $\simeq$ Poisson $(1)$                         | free Poisson $(1)$                             |

As a conclusion, this paper, together with \[5\], gathers information obtained from direct application of Weingarten philosophy to free quantum groups. This is part of a general analytic approach to several classical and quantum enumeration problems, in the spirit of the Wick formula. See \[3\], \[4\] and \[6\], \[7\] for related work.

The paper is organized as follows: 1 is a preliminary section, in 2–3 we present the integration formulae, and in 4–5 we discuss Poisson laws.

1. Quantum permutation groups

Let $A$ be a $\mathbb{C}^*$-algebra. A projection is an element $p \in A$ satisfying $p^2 = p = p^*$. Two projections $p, q$ are called orthogonal when $pq = 0$. A partition of unity is a set of mutually orthogonal projections, which sum up to 1.

**Definition 1.1.** A magic unitary is a square matrix over a $\mathbb{C}^*$-algebra, all whose rows and columns are partitions of unity with projections.

As a first example, consider the situation $G \curvearrowright X$ where a finite group acts on a finite set. The functions $f : G \to \mathbb{C}$ form a $\mathbb{C}^*$-algebra, with the sup norm and usual involution. Inside this algebra we have the characteristic functions

$$\chi_{ij} = \chi \{ \sigma \in G \mid \sigma(j) = i \}$$

which altogether form a magic unitary matrix. Indeed, when $i$ is fixed and $j$ varies, or vice versa, the corresponding sets form partitions of $G$. 
Definition 1.2. \( \chi \) is called magic unitary associated to \( G \ltimes X \).

The interest in \( \chi \) is that it encodes the structural maps of \( G \ltimes X \). These are the multiplication, unit, inverse and action map:

\[
\begin{align*}
m & : G \times G \to G \\
u & : \{\cdot\} \to G \\
i & : G \to G \\
a & : X \times G \to X
\end{align*}
\]

Consider the algebras \( A = \mathbb{C}(G) \) and \( V = \mathbb{C}(X) \). The dual structural maps of \( G \ltimes X \), called comultiplication, counit, antipode and coaction, and denoted \( \Delta, \varepsilon, S, \alpha \), are obtained as functional analytic duals of \( m, u, i, a \):

\[
\begin{align*}
\Delta & : A \to A \otimes A \\
\varepsilon & : A \to \mathbb{C} \\
S & : A \to A \\
\alpha & : V \to V \otimes A
\end{align*}
\]

We denote by \( e_i \in V \) the Dirac mass at \( i \in X \). The following result is known since Wang’s paper [14]; for the magic unitary formulation, see [1], [2].

Proposition 1.1. The dual structural maps of \( G \ltimes X \) are given by

\[
\begin{align*}
\Delta(\chi_{ij}) & = \sum \chi_{ik} \otimes \chi_{kj} \\
\varepsilon(\chi_{ij}) & = \delta_{ij} \\
S(\chi_{ij}) & = \chi_{ji} \\
\alpha(e_i) & = \sum e_j \otimes \chi_{ji}
\end{align*}
\]

where \( \chi \) is associated magic unitary matrix.

Proof. The structural maps are given by the following formulae:

\[
\begin{align*}
m(\sigma, \tau) & = \sigma \tau \\
u(\cdot) & = 1 \\
i(\sigma) & = \sigma^{-1} \\
a(i, \sigma) & = \sigma(i)
\end{align*}
\]

Thus the dual structural maps are given by the following formulae:

\[
\begin{align*}
\Delta(f) & = (\sigma, \tau) \to f(\sigma \tau) \\
\varepsilon(f) & = f(1) \\
S(f) & = \sigma \to f(\sigma^{-1}) \\
\alpha(f) & = (i, \sigma) \to f(\sigma(i))
\end{align*}
\]

This gives all equalities in the statement, after a routine computation. \( \square \)

In the particular case of the symmetric group \( S_n \) acting on \( X_n = \{1, \ldots, n\} \), we have the following presentation result, which together with Proposition 1.1 gives a purely functional analytic description of \( S_n \ltimes X_n \). See [1], [2], [14].
**Theorem 1.1.** $\mathbb{C}(S_n)$ is the universal commutative $\mathbb{C}^*$-algebra generated by $n^2$ elements $\chi_{ij}$, with relations making $\chi$ a magic unitary matrix.

*Proof.* Let $A$ be the universal algebra in the statement. We have an arrow $A \to \mathbb{C}(S_n)$, which is surjective by Stone-Weierstrass. As for injectivity, this follows from the fact that $A$ with maps as in Proposition 1.1 is a commutative Hopf algebra coacting on $X_n$, hence corresponds to a group acting on $X_n$. □

A free analogue of $\mathbb{C}(S_n)$ can be obtained by removing the commutativity relations in Theorem 1.1.

**Definition 1.3.** $A_s(n)$ is the universal $\mathbb{C}^*$-algebra generated by $n^2$ elements $u_{ij}$, with relations making $u$ a magic unitary matrix.

This algebra fits into Woronowicz’s formalism in [15]. By using the universal property of $A_s(n)$ we can define maps as follows:

\[
\Delta(u_{ij}) = \sum u_{ik} \otimes u_{kj}, \\
\varepsilon(u_{ij}) = \delta_{ij}, \\
S(u_{ij}) = u_{ji}, \\
\alpha(e_i) = \sum e_j \otimes u_{ji}.
\]

These satisfy the axioms for a comultiplication, counit, antipode and coaction. In other words, $A_s(n)$ is a Hopf algebra coacting on $X_n$.

The following fundamental result is due to Wang ([14]).

**Theorem 1.2.** $A_s(n)$ is the biggest Hopf algebra coacting on $X_n = \{1, \ldots, n\}$.

*Proof.* The key remark here is that a linear map $\alpha : V \to V \otimes A$ is a morphism of $\mathbb{C}^*$-algebras if and only if its matrix of coefficients consists of projections, and has partitions of unity on all rows. Now if $\alpha$ is a coaction, one can use the antipode to get that the same is true for columns, and this leads to the result. □

For $n = 1, 2, 3$ the canonical map $A_s(n) \to \mathbb{C}(S_n)$ is an isomorphism. This is because 1 or 4 projections which form a magic unitary mutually commute, and the same can be shown to happen for 9 projections. See [2], [14].

For $n \geq 4$ the algebra $A_s(n)$ is non commutative and infinite dimensional. This is because 16 projections which form a magic unitary don’t necessarily commute, and can generate an infinite dimensional algebra. See [14].

2. **Integration formula**

For the rest of the paper, we assume $n \geq 4$. We use the notation $V = \mathbb{C}^n$.

In this section we find a formula for the Haar functional of $A_s(n)$. This is a linear form satisfying a certain bi-invariance condition, whose existence and uniqueness are shown by Woronowicz in [15]. We denote this form as an integral:

\[
\int : A_s(n) \to \mathbb{C}
\]
The integrals of various combinations of generators $u_{ij}$ can be computed by using
Temperley-Lieb diagrams, by plugging results in [1] into the method in [5].

We present here an alternative approach, by using non-crossing partitions. The
choice of partitions vs. diagrams is due to some simplifications in integration for-mlae, to become clear later on. Let us also mention that such partitions and diagrams
are known to be in correspondence, via fatgraphs.

**Definition 2.1.** $\text{NC}(k)$ is the set of non-crossing partitions of $\{1, \ldots, k\}$.

In this definition $\{1, \ldots, k\}$ is regarded as an ordered set. The sets of the partition
are unordered, and are called blocks. The fact that $p$ is non-crossing means that we
cannot have $a < b < c < d$ with both $a, c$ and $b, d$ in the same block of $p$.

Given an index set $I$, we can plug multi-indices $i = (i_1 \ldots i_k)$ into partitions
$p \in \text{NC}(k)$ in the following way: we take the partition $p$, and we replace each
element $s \in \{1, \ldots, k\}$ of the set which is partitioned by the corresponding index $i_s$.

What we get is a collection of subsets with repetitions of the index set $I$.

The number $\delta_{pi}$ is defined to be 0 if some of these subsets contains two different
indices of $I$, and to be 1 if not. Observe that we have $\delta_{pi} = 1$ if and only if each of
the above subsets with repetitions contains a single element of $I$, repeated as many
times as the cardinality of the subset with repetitions is.

We can summarize this definition in the following way:

**Definition 2.2.** Given a partition $p \in \text{NC}(k)$ and a multi-index $i = (i_1 \ldots i_k)$, we
can plug $i$ into $p$, and we define the following number:

$$\delta_{pi} = \begin{cases} 
0 & \text{if some block of } p \text{ contains two different indices of } i \\
1 & \text{if not}
\end{cases}$$

Consider now the fundamental corepresentation $u = (u_{ij})$ of the Hopf algebra
$A_s(n)$. Its $k$-th tensor power in the corepresentation sense is the following matrix,
having as indices the multi-indices $i = (i_1 \ldots i_k)$ and $j = (j_1 \ldots j_k)$:

$$u \otimes k = (u_{i_1 j_1} \ldots u_{i_k j_k})$$

Our first task is to find the fixed vectors of $u \otimes k$. We recall that a vector $\xi$ is fixed
by a corepresentation $r$ when we have $r(\xi \otimes 1) = \xi \otimes 1$.

The following result uses [16], and we refer to [1] for missing details.

**Proposition 2.1.** The partitions in $\text{NC}(k)$ create tensors in $V \otimes k$ via the formula

$$p(\cdot) = \sum_i \delta_{pi} e_{i_1} \otimes \ldots \otimes e_{i_k}$$

and we get in this way a basis for the space of fixed vectors of $u \otimes k$.

**Proof.** The non-crossing partitions $p \in \text{NC}(l + k)$ transform tensors of $V \otimes l$ into
tensors of $V \otimes k$, according to the following formula:

$$p(e_{j_1} \otimes \ldots \otimes e_{j_l}) = \sum_i \delta_{p,ij} e_{i_1} \otimes \ldots \otimes e_{i_k}$$
Here the multi-index $ij$ is obtained by concatenating the multi-indices $i, j$. Observe that this notation extends the one in the statement, where $l = 0$.

It is routine to check that linear maps corresponding to different partitions are linearly independent, so the abstract vector space $TL(l, k)$ spanned by $NC(l + k)$ can be viewed as space of linear maps between tensor powers of $V$:

$$TL(l, k) \subset Hom(V^\otimes l, V^\otimes k)$$

Consider now the following linear spaces:

$$Hom(u^\otimes l, u^\otimes k) \subset Hom(V^\otimes l, V^\otimes k)$$

These can be interpreted as follows:

1. The spaces on the right form a tensor subcategory $C(V)$ of the tensor category $H$ of finite dimensional Hilbert spaces.
2. The spaces on the left form a tensor subcategory $C(u)$ of the tensor category $R$ of finite dimensional corepresentations of $A_s(n)$.
3. The middle embeddings give an embedding of tensor categories $C(u) \subset C(V)$, which is the restriction of the canonical embedding $R \subset H$.

Now recall that Woronowicz’s Tannakian duality in [16] shows that $A_s(n)$ can be reconstructed from $R \subset H$. Moreover, the matrix version of duality, also from [16], shows that $A_s(n)$ can be reconstructed from $C(u) \subset C(V)$. In particular the presentation relations of $A_s(n)$ should correspond to some generation property of $C(u)$, and this is worked out in [1]: the conclusion is that $C(u)$ is the tensor subcategory of $C(V)$ generated by $M$ and $U$, the multiplication and unit of $V$.

We have the following equality of subcategories of $C(V)$, where $< A >$ is the category generated by a set of arrows $A$, and $1_k \in NC(k)$ is the $k$-block partition:

$$\{Hom(u^\otimes l, u^\otimes k)\}_{lk} = C(u) = < M, U > = < 1_3, 1_1 > = \{TL(l, k)\}_{lk}$$

Indeed, the first two equalities follows from the above discussion, the third equality follows from $M = 1_3$ and $U = 1_1$, and the fourth equality follows from a routine computation. With $l = 0$ this gives the result.

**Definition 2.3.** The Gram and Weingarten matrices are given by

$$G_{kn}(p, q) = n^{[\nu\nu]}$$

$$W_{kn}(p, q) = G_{kn}^{-1}(p, q)$$

where indices $p, q$ are partitions in $NC(k)$, and $|.|$ is the number of blocks.

The matrix $G_{kn}$ is indeed a Gram matrix, as shown by the following computation in $V^\otimes k$, with respect to the canonical scalar product:

$$< p(\cdot), q(\cdot) > = \left< \sum_i \delta_{pi} e_{i_1} \otimes \ldots \otimes e_{i_k}, \sum_i \delta_{qi} e_{i_1} \otimes \ldots \otimes e_{i_k} \right>$$
\[= \sum_i \delta_{pi} \delta_{qi}\]
\[= \sum_i \delta_{p \cup q, i}\]
\[= n_{|p \cup q|}\]

As for the matrix \(W_{kn}\), this is indeed an analogue of the Weingarten matrix, as shown by the following result:

**Theorem 2.1.** We have the integration formula
\[
\int u_{i_1 j_1} \ldots u_{i_k j_k} = \sum_{pq} \delta_{pi} \delta_{qj} W_{kn}(p, q)
\]
where the sum is over all partitions \(p, q \in NC(k)\).

**Proof.** By standard results of Woronowicz in [18], the linear map
\[\pi(e_{i_1} \otimes \ldots \otimes e_{i_k}) = \sum_j e_{j_1} \otimes \ldots \otimes e_{j_k} \int u_{i_1 j_1} \ldots u_{i_k j_k}\]
is the orthogonal projection onto the fixed point space \(F = Hom(1, u^\otimes k)\). For computing \(\pi\), we use the factorisation
\[V^\otimes k \xrightarrow{\pi} V^\otimes k\]
\[\downarrow \gamma \quad \quad \cup\]
\[F \xrightarrow{\omega} F\]
where the linear map \(\gamma\) is given by the formula
\[\gamma(x) = \sum_p <x, p> \cdot p\]
and where \(\omega\) is the inverse of the restriction of \(\gamma\) to \(F\):
\[\omega = (\gamma|_F)^{-1}\]

With the notation \(e_i = e_{i_1} \otimes \ldots \otimes e_{i_k}\), this gives:
\[
\int u_{i_1 j_1} \ldots u_{i_k j_k} = < \pi(e_i), e_j >
\]
\[= < \omega \gamma(e_i), e_j >
\]
\[= \sum_p < e_i, p > \cdot < \omega(p), e_j >
\]
\[= \sum_{pq} < e_i, p > \cdot < q, e_j > \cdot < \omega(p), q >
\]
\[= \sum_{pq} \delta_{pi} \delta_{qj} < \omega(p), q >
\]
Now the restriction of $\gamma$ to $F$ being the linear map corresponding to $G_{kn}$, its inverse $\omega$ is the linear map corresponding to $W_{kn}$. This gives the result.

**Theorem 2.2.** We have the moment formula

$$\int (u_{11} + \ldots + u_{ss})^k = \text{Tr}(G_{kn}^{-1}G_{ks})$$

where $u$ is the fundamental corepresentation of $A_s(n)$.

**Proof.** We have the following computation:

$$\int (u_{11} + \ldots + u_{ss})^k = \sum_{i_1=1}^{s} \ldots \sum_{i_k=1}^{s} \int u_{i_1i_1} \ldots u_{i_ki_k}$$

$$= \sum_{i_1=1}^{s} \ldots \sum_{i_k=1}^{s} \sum_{pq} \delta_{pi} \delta_{qi} G_{kn}^{-1}(p,q)$$

$$= \sum_{pq} G_{kn}^{-1}(p,q) \sum_{i_1=1}^{s} \ldots \sum_{i_k=1}^{s} \delta_{pi} \delta_{qi}$$

Now the last term on the right is an entry of the Gram matrix:

$$\int (u_{11} + \ldots + u_{ss})^k = \sum_{pq} G_{kn}^{-1}(p,q) G_{ks}(p,q)$$

$$= \sum_{pq} G_{kn}^{-1}(p,q) G_{ks}(q,p)$$

$$= \text{Tr}(G_{ks}^{-1}G_{ks})$$

This gives the result.

\[\square\]

3. **Numeric results**

We know that the order $k$ moments of diagonal coefficients of $u$ can be explicitly computed, provided we know how to invert the Gram matrix $G_{kn}$. This matrix has integer entries, and its size is $C_k$, the $k$-th Catalan number.

The sequence of Catalan numbers is as follows:

$$1, 2, 5, 14, 42, 132, 249, \ldots$$

These numbers tell us that:

(1) For $k = 1, 2, 3$ the moments can be computed directly.

(2) For $k = 4$ we can use a computer.

(3) For $k = 5$ we need a computer implementation of $NC(k)$.

(4) For $k \geq 6$ we need a supercomputer (or a new idea).

In this section we compute the moments for $k = 1, 2, 3, 4$. The formulae below can be regarded as experimental data, illustrating the combinatorics of Gram and Wein-garten matrices. They are useful for checking validity of various general statements, and this is how most results in next sections were obtained.
Theorem 3.1. We have the moment formulae

\[
\int (u_{11} + \ldots + u_{ss}) = \frac{s}{n}
\]

\[
\int (u_{11} + \ldots + u_{ss})^2 = \frac{s}{n} \cdot \frac{n + (s - 2)}{n - 1}
\]

\[
\int (u_{11} + \ldots + u_{ss})^3 = \frac{s}{n} \cdot \frac{n^2 + 3(s - 2)n + (s^2 - 9s + 10)}{(n - 1)(n - 2)}
\]

\[
\int (u_{11} + \ldots + u_{ss})^4 = \frac{s}{n} \cdot \frac{P_s(n)}{(n - 1)(n - 2)(n^2 - 3n + 1)}
\]

where \( P_s(n) \) is the following polynomial:

\[
P_s(n) = n^4 + (6s - 12)n^3 + (6s^2 - 46s + 52)n^2 + (s^3 - 26s^2 + 104s - 88)n + (12s^2 - 38s + 28)
\]

Proof. Here are the matrices \( G_{2n} \) and \( G_{2n}^{-1} \):

\[
G_{2n} = n \begin{pmatrix} 1 & 1 \\ 1 & n \end{pmatrix}
\]

\[
G_{2n}^{-1} = \frac{1}{n(n-1)} \begin{pmatrix} n & -1 \\ -1 & 1 \end{pmatrix}
\]

Here are the matrices \( G_{3n} \) and \( G_{3n}^{-1} \):

\[
G_{3n} = n \begin{pmatrix} n & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & n & 1 \\ 1 & 1 & n & n \\ n & 1 & n & n^2 \end{pmatrix}
\]

\[
G_{3n}^{-1} = \frac{1}{n(n-1)(n-2)} \begin{pmatrix} n-1 & -n & 1 & 1 & -1 \\ -n & n^2 & -n & -n & 2 \\ 1 & -n & n-1 & 1 & -1 \\ 1 & -n & 1 & n-1 & -1 \\ -1 & 2 & -1 & -1 & 1 \end{pmatrix}
\]
And here is $G_{4n}$, whose inverse will not be given here:

$$G_{4n} = n \begin{pmatrix}
\begin{array}{ccccccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & n & n & n & n \\
n & n^2 & n & n & 1 & 1 & 1 & 1 & n & n & n & n^2 \\
n & n & 1 & 1 & 1 & 1 & 1 & 1 & n & 1 & 1 & n \\
n & n & 1 & 1 & 1 & 1 & 1 & 1 & n & 1 & 1 & n \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
n & n & 1 & 1 & n & n & 1 & n & n & n & n & n^2 \\
n & n & 1 & 1 & n & n & n & n & n & n & n^2 & n^2 \\
n & n & 1 & 1 & n & n & n & n & n & n & n^2 & n^2 \\
n & n & 1 & 1 & n & n & n & n & n & n & n^2 & n^2 \\
n & n & n & 1 & 1 & n & n & n & n & n & n & n \\
n & n & 1 & 1 & n & n & n & n & n & n & n & n^2 \\
n & n & 1 & 1 & n & n & n & n & n & n & n & n \\
n & n & n & 1 & 1 & n & n & n & n & n & n & n \\
n & n & n & 1 & 1 & n & n & n & n & n & n & n \\
n & n & n & 1 & 1 & n & n & n & n & n & n & n \\
n & n & n & 1 & 1 & n & n & n & n & n & n & n \\
n & n & n & 1 & 1 & n & n & n & n & n & n & n \\
n & n & n & 1 & 1 & n & n & n & n & n & n & n \\
\end{array}
\end{pmatrix}$$

By computing $\text{Tr}(G_{kn}^{-1}G_{ks})$ we get the formulae in the statement. □

We would like now to point out the fact that some simplifications appear for $s = 2$. This is not surprising, because the operator $u_{11} + u_{22}$ is a sum of two projections, and such sums have in general reasonably simple combinatorics.

**Proposition 3.1.** We have the following moment formulae:

$$\int (u_{11} + u_{22})^k = \frac{2}{n} \cdot \frac{n + k}{n - 1}$$

**Proof.** This follows from Theorem 3.1. □

### 4. Asymptotic laws

According to Voiculescu’s free probability theory, the free analogue of the Poisson law of parameter 1 is the following probability measure on $[0, 4]$:

$$\mu_1 = \frac{1}{2\pi} \sqrt{4x^{-1} - 1} \, dx$$

This is also known as Marchenko-Pastur law of parameter 1. See [10], [12]. For reasons that will become clear later on, we prefer the Poisson terminology.

The following result is pointed out in [3].

**Proposition 4.1.** Let $u$ be the fundamental corepresentation of $A_s(n)$.

1. $u_{11}$ is a projection of trace $1/n$. 


(2) $u_{11} + \ldots + u_{nn}$ is free Poisson of parameter 1.

Proof. The first assertion is clear. The moments of the variable in the second assertion are the Catalan numbers

$$\int (u_{11} + \ldots + u_{nn})^l = \text{Tr}(G_{kn}^{-1}G_{ks})$$

$$= \text{Tr}(1)$$

$$= \#NC(k)$$

$$= C_k$$

known to be equal to the moments of the free Poisson law. \(\square\)

The measure $\mu_1$ is part of a one-parameter family of real measures. The free Poisson law of parameter $t > 0$ is the probability measure on the set

$$\{0\} \cup [(1 - \sqrt{t})^2, (1 + \sqrt{t})^2]$$

given by the following formula, with the notation $K = \max(0, 1 - t)$:

$$\mu_t = K \delta_0 + \frac{1}{2\pi x} \sqrt{4t - (x - 1 - t)^2} dx$$

The free Poisson laws form a one-parameter semigroup with respect to free convolution, in the sense that we have $\mu_{s+t} = \mu_s \boxplus \mu_t$. See [10], [12].

Lemma 4.1. We have the estimates

$$G_{kn} = \Delta_{kn}^{1/2}(Id + O(n^{-1/2}))\Delta_{kn}^{1/2}$$

$$G_{kn}^{-1} = \Delta_{kn}^{-1/2}(Id + O(n^{-1/2}))\Delta_{kn}^{-1/2}$$

where $\Delta_{kn}(p, p) = n^{|p|}$ is the diagonal matrix formed by diagonal entries of $G_{kn}$.

Proof. We have the following formula:

$$(\Delta_{kn}^{-1/2}G_{kn}\Delta_{kn}^{-1/2})(p, q) = \Delta_{kn}^{-1/2}(p, p)G_{kn}(p, q)\Delta_{kn}^{-1/2}(q, q)$$

The $(p, q)$ coefficient of this matrix is given by:

$$\sum_p n^{|p|}n^{|p|}n^{-|p|}n^{-|q|} = n^{|p\vee q|}n^{-\frac{|p|+|q|}{2}}$$

It is standard combinatorics to check that the last exponent is negative for $p \neq q$, and zero for $p = q$. This gives the result. \(\square\)

Theorem 4.1. Let $u$ be the fundamental corepresentation of $A_s(n)$.

(1) $u_{11} + \ldots + u_{ss}$ with $s = o(n)$ is a projection of trace $s/n$.

(2) $u_{11} + \ldots + u_{ss}$ with $s = tn + o(n)$ is free Poisson of parameter $t$.

Proof. We use the following moment estimate:

$$\int (u_{11} + \ldots + u_{ss})^k = \text{Tr}(G_{kn}^{-1}G_{ks})$$

$$\simeq \text{Tr}(\Delta_{kn}^{-1}\Delta_{ks})$$

$$= \sum_p n^{-|p|}g_0^{|p|}$$
\[ = \sum_p (s/n)^{|p|} \]

(1) With \( s = o(n) \) the terms that count are those corresponding to minimal values of \(|p|\). But the minimal value is \(|p| = 1\), and this value appears only once. Thus the above sum is asymptotically equal to \( s/n \), and this gives the assertion.

(2) With \( s = tn + o(n) \) we have the following estimate:

\[
\int (u_{11} + \ldots + u_{ss})^k \approx \sum_p t^{|p|} \]

On the other hand, the term on the right is the order \( k \) moment of the free Poisson law of parameter \( t \), and we are done (see [10]).

\[ \Box \]

Corollary 4.1. Let \( u_{ij}(n) \) be the fundamental corepresentation of \( A_s(n) \). Then for any \( t \in (0, 1] \) the following limit converges

\[ \rho_t = \lim_{n \to \infty} \text{law} \left( \sum_{i=1}^{[tn]} u_{ii}(n) \right) \]

and we get a one-parameter (truncated) semigroup with respect to free convolution.

Proof. This is clear from Theorem 4.1 and from the fact that free Poisson laws form a one-parameter semigroup with respect to free convolution. \[ \Box \]

5. SYMMETRIC GROUPS

We present here classical analogues of results in previous section. These justify the table and comments in the introduction.

The following result is pointed out in [3] in the case \( s = n \):

Lemma 5.1. We have the law formula

\[
\text{law}(u_{11} + \ldots + u_{ss}) = \frac{s!}{n!} \sum_{p=0}^{s} \frac{(n-p)!}{(s-p)!} \frac{(\delta_1 - \delta_0)^p}{p!} \]

where \( u \) is the fundamental corepresentation of \( C(S_n) \).

Proof. We have the moment formula

\[
\int (u_{11} + \ldots + u_{ss})^k = \frac{1}{n!} \sum_{f=0}^{s} m_f \delta_f \]

where \( m_f \) is the number of permutations of \( \{1, \ldots, n\} \) having exactly \( f \) fixed points in the set \( \{1, \ldots, s\} \). Thus the law in the statement, say \( \nu_{sn} \), is the following average of Dirac masses:

\[
\nu_{sn} = \frac{1}{n!} \sum_{f=0}^{s} m_f \delta_f \]
Permutations contributing to $m_f$ are obtained by choosing $f$ points in the set \{1, \ldots, s\}, then by permuting the remaining $n - f$ points in \{1, \ldots, n\} in such a way that there is no fixed point in \{1, \ldots, s\}. These latter permutations are counted as follows: we start with all permutations, we subtract those having one fixed point, we add those having two fixed points, and so on. We get:

$$\nu_{sn} = \frac{1}{n!} \sum_{f=0}^{s} \binom{s}{f} \left( \sum_{k=0}^{s-f} (-1)^k \binom{s-f}{k} (n-f-k)! \right) \delta_f$$

$$= \sum_{f=0}^{s} \sum_{k=0}^{s-f} (-1)^k \frac{1}{n!} \frac{s!}{f!(s-f)!} \frac{(s-f)!((n-f-k)!}{k!(s-f-k)!} \delta_f$$

$$= \frac{s!}{n!} \sum_{f=0}^{s} \sum_{k=0}^{s-f} (-1)^k (n-f-k)! \frac{1}{f!k!(s-f-k)!} \delta_f$$

We continue the computation by using the index $p = f + k$:

$$\nu_{sn} = \frac{s!}{n!} \sum_{p=0}^{s} \sum_{k=0}^{p} (-1)^k (n-p)! \frac{1}{(p-k)!k!(s-p)!} \delta_{p-k}$$

$$= \frac{s!}{n!} \sum_{p=0}^{s} \frac{(n-p)!}{(s-p)!p!} \sum_{k=0}^{p} (-1)^k \binom{p}{k} \delta_{p-k}$$

$$= \frac{s!}{n!} \sum_{p=0}^{s} \frac{(n-p)!}{(s-p)!} \left( \frac{\delta_1 - \delta_0}{p!} \right)^p$$

Here $*$ is convolution of real measures, and the assertion follows.

The Poisson law of parameter 1 is the following real probability measure:

$$\nu_1 = \frac{1}{e} \sum_{p=0}^{\infty} \frac{1}{p!} \delta_p$$

The following result is a classical analogue of Proposition 4.1.

**Proposition 5.1.** Let $u$ be the fundamental corepresentation of $A_s(n)$.

1. $u_{11}$ is a projection of trace $1/n$.
2. $u_{11} + \ldots + u_{nn}$ with $n \to \infty$ is Poisson.

**Proof.** The first assertion is clear. For the second one, we have

$$\text{law}(u_{11} + \ldots + u_{nn}) = \sum_{p=0}^{n} \frac{(\delta_1 - \delta_0)^p}{p!}$$

and the measure on the right converges with $n \to \infty$ to the Poisson law.
The Poisson law of parameter $t \in (0, 1]$ is the following real probability measure:

$$\nu_t = e^{-t} \sum_{p=0}^{\infty} \frac{t^p}{p!} \delta_p$$

The following results are classical analogues of Theorem 5.1 and Corollary 5.1.

**Theorem 5.1.** Let $u$ be the fundamental corepresentation of $\mathbb{C}(S_n)$.

1. $u_{11} + \ldots + u_{ss}$ with $s = o(n)$ is a projection of trace $s/n$.
2. $u_{11} + \ldots + u_{ss}$ with $s = tn + o(n)$ is Poisson of parameter $t$.

**Proof.** (1) With $s$ fixed and $n \to \infty$ we have the estimate

$$\text{law}(u_{11} + \ldots + u_{ss}) = \sum_{p=0}^{s} \frac{(n-p)!}{n!} \cdot \frac{s!}{(s-p)!} \cdot \frac{\delta_1 - \delta_0)^p}{p!}$$

$$= \delta_0 + \frac{s}{n} (\delta_1 - \delta_0) + O(n^{-2})$$

and the law on the right is that of a projection of trace $s/n$.

(2) We have a law formula of the following type:

$$\text{law}(u_{11} + \ldots + u_{ss}) = \sum_{p=0}^{s} c_p \cdot \frac{\delta_1 - \delta_0)^p}{p!}$$

The coefficients $c_p$ can be estimated by using the Stirling formula:

$$c_p = \frac{(tn)!}{n!} \cdot \frac{(n-p)!}{(tn-p)!}$$

$$\approx \frac{(tn)^n}{n^n} \cdot \frac{(n-p)^{n-p}}{(tn-p)^{n-p}}$$

$$= \left( \frac{tn}{tn-p} \right)^{n-p} \left( \frac{n-p}{n} \right)^{n-p} \left( \frac{tn}{n} \right)^p$$

The last expression is estimated by using the definition of exponentials:

$$c_p \approx e^p e^{-p} t^p = t^p$$

We compute now the Fourier transform with respect to a variable $y$:

$$\mathcal{F}(\text{law}(u_{11} + \ldots + u_{ss})) \approx \sum_{p=0}^{s} t^p \cdot \frac{(e^y-1)^p}{p!}$$

The sum of the series on the right is $e^{t(e^y-1)}$, and this is known to be the Fourier transform of the Poisson law $\nu_t$. This gives the second assertion. \qed

**Corollary 5.1.** Let $u_{ij}(n)$ be the fundamental corepresentation of $\mathbb{C}(S_n)$. Then for any $t \in (0, 1]$ the following limit converges

$$\rho_t = \lim_{n \to \infty} \text{law} \left( \sum_{i=1}^{[tn]} u_{ii}(n) \right)$$
and we get a one-parameter (truncated) semigroup with respect to usual convolution.

**Proof.** This is clear from Theorem 5.1 and from the fact that Poisson laws form a one-parameter semigroup with respect to usual convolution. □

This result is to be compared to Corollary 4.1, which asserts that for $A_s(n)$ we get a one-parameter semigroup with respect to free convolution.

It follows from [5], [7] that similar statements hold for $C(O(n))$, $C(U(n))$ and for $A_o(n)$, $A_u(n)$. We believe that further work in this direction can lead to an abstract notion of free Hopf algebra, but we don’t have any other example so far.

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