Quantum Group and Magnetic Translations.  
Bethe-Ansatz for Asbel-Hofstadter Problem

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We present a new approach to the problem of Bloch electrons in magnetic (sometimes called Azbel-Hofstadter problem) field, by making explicit a natural relation between the group of magnetic translations and the quantum group $U_q(sl_2)$. The approach allows us to express the "mid" band spectrum of the model and the Bloch wave function as solutions of the Bethe-Ansatz equations typical for completely integrable quantum systems. The zero mode wave functions are found explicitly in terms of $q$-deformed classical orthogonal polynomials. In this paper we present solution for the isotropic problem. We also present a class of solvable quasiperiodic equations related to $U_q(sl_2)$.

I. INTRODUCTION

The peculiar problem of Bloch electrons in magnetic field [1], [2], [3], [4], [5] often emerges in various branches physics. Every time presenting a new face to describe another physical application. The Hamiltonian of a particle on a two dimensional square lattice in magnetic field is

$$H = \sum_{<n,m>} t_{n,m} e^{iA_{n,m}} c_n^\dagger c_m,$$

(1)

$$\prod_{\text{plaquette}} e^{iA_{\bar{n},\bar{m}}} = e^{i\Phi}$$

(2)

where $\Phi = 2\pi \frac{P}{Q}$ is a flux per plaquette, $P$ and $Q$ are mutual prime integers and $t_{n,m}$ is a hoping amplitude between the nearest neighbors. In the most conventional Landau gauge $A_x = A_{\bar{n},\bar{n}+1_x} = 0$, $A_y = \Phi n_x$ the Bloch wave function is

$$\psi(\bar{n}) = e^{i\vec{k}\bar{n}} \psi_{n_x}(\vec{k}), \quad \psi_n = \psi_{n+Q}$$

(3)
where \( n_x \equiv n = 1, \ldots, Q \) is a coordinate in the magnetic cell. With these substitution the Schrodinger equation turns into a famous one-dimensional quasiperiodic difference equation ("Harper’s" equation):

\[
t_x (e^{ik_x \psi_{n+1}} + e^{-ik_x \psi_{n-1}}) + 2t_y \cos(k_y + n\Phi) \psi_n = E \psi_n
\]  \hspace{1cm} (4)

The spectrum of this equation has \( Q \) bands and feels the difference between rational and irrational numbers [2]. The beauty and complexity of this problem is the hierarchy of the spectrum. If the flux is irrational, the spectrum is singular continuum - uncountable but measure zero set of points (Cantor set) [4]. The multifractal behaviour (see [3] for a review) of the spectrum is a long standing theoretical challenge.

The Harper’s equation (i.e the Schrodinger equation for the Bloch particle in the Landau gauge) formally describes one dimensional electrons in quasiperiodic potential and also one dimensional quasicrystal. Depending of the strength of the potential \( t_x/t_y \) (it corresponds to anisotropy in hopping amplitude of a particle in magnetic field) the Harper equation describes localization- delocalization transition in one dimensional incommensurate potential.

The list of others applications may be continued.

Recently it has been conjectured that the symmetry of magnetic group may appear dynamically in strongly correlated electronic systems [11], [12].

In this paper we made explicit a long time anticipated connection of the noncommutative geometry in magnetic field and the structure of the Quantum Group \( U_q(sl_2) \) and therefore with Quantum Integrable Systems [13]. We present the Bethe-Ansatz algebraic equations for the spectrum of isotropic problem. Although we do not solve the Bethe-Ansatz equations here, we hope that they provide a basis for analytical studying the fractal properties of the spectrum. Although, the Bethe Ansatz solution is available for an anisotropic case, in this paper we present only isotropic solution \( t_x = t_y = 1 \). Anisotropic case will be considered elsewhere.

The spectrum of our problem is complex but not chaotic. Instead it is governed by the Quantum integrability. It may happened that the Bethe Ansatz technique will be a powerfool tool to study general properties of multifractality. In fact multifractality has been already observed in quantum integrable models [4], [10]. However, this aspect of integrability has never been seriously developed.

Let us stress that we apply the Bethe Ansatz technique to Quantum Mechanics - not to a Quantum Field Theory. Therefore we found convinient to use the theory of representation of the Quantum Group, instead of traditional Bethe Ansatz technique.

The Harper’s equation is a member (perhaps the most interesting but at least the most famous) of class of solvable quasiperiodic equations related to \( U_q(sl_2) \). It will be briefly described below (for a more extended discussion of algebraization of quasiperiodic equations see Ref. [13]).

To ease the reference we state the main result now:

It is known that due to the gauge invariance, the energy depends on a single parameter \( \Lambda = \cos(Qk_x) + \cos(Qk_y) \) [4]. We find that the spectrum at \( \Lambda = 0 \) ("mid" band spectrum) is given by the sum of roots \( z_l \)

\[
E = iq^Q(q - q^{-1}) \sum_{l=1}^{Q-1} z_l,
\]  \hspace{1cm} (5)
of the Bethe-Ansatz equations for the Quantum Group $U_q(sl_2)$ with

$$\frac{z_i^2 + q}{qz_i^2 + 1} = -q^Q \prod_{m=1,m \neq i}^{Q-1} \frac{qz_i - z_m}{z_i - qz_m}, \quad l = 1, ..., Q - 1.$$  \tag{6}$$

$$q = e^{i\Phi}, \tag{7}$$

Another version of the Bethe-Ansatz equations is presented in Sect.7. An ambitious problem of extending the Bethe Ansatz solution for arbitrary momenta $k_x, k_y$ is beyond the scope of this paper. The paper is organized as follows. In Sect.2 we present the Schrödinger equation in two other gauges in which the quantum group structure of the model is more transparent. In Sect.3 we review some necessary facts about the quantum group $U_q(sl_2)$ and its representations. The relation between the quantum group and magnetic translations is revealed in Sect.4. The Hamiltonian can be rewritten entirely via the quantum group generators. We give two equivalent (dual) representations of the Hamiltonian as a linear or quadratic form in the quantum group generators. In Supplement A we consider a general quadratic forms in $U_q(sl_2)$ generators and introduce an ”integrable” class of the second order ”quasiperiodic” equations related to the quantum group. In Supplement B we show that a general quadratic form in quantum group generators is a trace of monodromy matrix of some integrable models with nonperiodic boundary conditions. In Sect.5 we apply the functional Bethe Ansatz to obtain the ”mid” band spectrum of the model. A promising connection between the zero mode wave functions and $q$-deformations of certain classical orthogonal polynomials is discussed in Sect.6.

II. MAGNETIC TRANSLATIONS AND GAUAGES INVARIANCE

The wave functions of a particle in a magnetic field form a representation of the group of magnetic translations: let generators of the translations be

$$T_{\vec{\mu}}(\vec{i}) = e^{iA_{\vec{i}}\cdot\vec{\mu}} | \vec{i} \rangle < \vec{i} + \vec{\mu} |$$ \tag{8}

They form the algebra

$$T_{\vec{\mu}} = T_{-\vec{\mu}}^{-1}, \quad T_{\vec{n}}T_{\vec{m}} = q^{-\vec{n} \times \vec{m}}T_{\vec{n} + \vec{m}},$$

$$T_yT_x = q^2gT_y, \quad T_xT_{-x} = q^{-2}T_{-x}T_y$$ \tag{9}

with $q = \exp i\pi \frac{P}{\tilde{Q}}$. The Hamiltonian therefore is

$$H = T_x + T_{-x} + T_y + T_{-y}.$$ \tag{10}

Integrability of the problem can be seen from the first glance. It has been observed (see e.g. [1]) that the spectrum depends only on one parameter $\Lambda$. That means that there is a parametric family of Hamiltonians

$$H(u) = uT_x + u^{-1}T_{-x} + vT_y + v^{-1}T_{-y}.$$ \tag{11}
with the same spectra at
\[ u^Q e^{ik_x} + u^{-Q} e^{-ik_x} + v^Q e^{ik_y} + v^{-Q} e^{-ik_y} = \text{const} \]

. The mid band spectrum \( \Lambda = 0 \) is a very special point where the Bethe Ansatz is especially simple. We shall consider only this point below. The useful way to explore this idea is representation theory of quantum groups.

We found two specific gauges for which the quantum group structure is more transparent. In these gauges (we call them "regular") the wave function has the form
\[ \Psi_n = \prod_{m=1}^{Q-1} (q_0^n - z_m) \]  
(12)
where \( q_0 = q \) or \( q_0 = q^2 \) and \( z_m \)'s are the roots of the Eq.(8) and are independent on \( n \). The Landau gauge of the Introduction is not "regular" in this sense and is not convenient for revealing the quantum group structure of the model.

1. Modified Landau gauge
Consider the gauge
\[ A_x = -A_y = -\Phi n_x \]  
(13)
which we refer as a modified Landau gauge. Now the Bloch wave function is
\[ \phi(\vec{n}) = e^{i\vec{k}' \cdot \vec{n}} \phi_{n_x}(\vec{k'}), \quad \phi_n = \phi_{n+Q} \]  
(14)
where \( \vec{k}' \) is the wave vector (different from that in (3)). The Schrodinger equation turns in to
\[ e^{ik'_x - i\Phi n} \phi_{n+1} + e^{-ik'_x + i\Phi(n-1)} \phi_{n-1} + 2 \cos(k'_y + n\Phi)\phi_n = E\phi_n \]  
(15)
(we drop the argument in \( \phi_n \)).

The "mid" band spectrum (\( \Lambda = 0 \)) corresponds to the values
\[ \vec{k} = \left( \frac{1}{2} \Phi(Q - 1), \pi \right) \]  
(16)
in the Landau gauge and is given by
\[ \vec{k}' = (0, \pi) \]  
(17)
in the Modified Landau gauge.

At the "mid" band all the values of \( \vec{k}' \) on the line \( \exp i(k'_x - k'_y) = -1 \) are physically equivalent i.e. connected by a gauge transformation. We have chosen \( \exp i(k'_x + k'_y) = -1 \) in (17).

2. Chiral gauge
Consider the chiral gauge defined by:
\[ A_x = -\frac{\Phi}{2} (n_x + n_y), \quad A_y = \frac{\Phi}{2} (n_x + n_y + 1), \]  
(18)
in which the Bloch wave function takes the form
\[ \chi(\vec{n}) = e^{i\vec{p}\cdot\vec{n}}\chi_n(\vec{p}) \] (19)
where \(\vec{p} = (p_x, p_y)\), \(n = n_x + n_y\), \(p_{\pm} = (p_x \pm p_y)/2\) are light cone coordinates and momenta.

This wave function is defined in two magnetic cells because \(n\) runs now from 1 to 2\(Q\). Accordingly, the light cone momenta are confined to the half of the Brillouin zone \([0, \pi/Q]\).

The equivalent form of the Harper’s equation (4) is
\[
2e^{+\Phi+ip_x} \cos\left(\frac{1}{2}\Phi n + \frac{1}{4}\Phi - p_-(n+1)\right) + \\
2e^{-\Phi-\frac{1}{2}ip_y} \cos\left(\frac{1}{2}\Phi n - \frac{1}{4}\Phi - p_-(n-1)\right) = E\chi_n
\] (20)
The wave function \(\chi_n\) is 2\(Q\)-periodic. This doubling of period in comparison with (15) is, of course, artificial. Although the coefficients in (20) have period 2\(Q\) there exists a simple transformation of \(\chi_n\) which makes them \(Q\)-periodic:
\[ \chi_n = \exp\left(\frac{i\Phi}{4}n(n-2)\right)\xi_n \] (21)

This new wave function \(\xi_n\) is \(Q\)-periodic and satisfies the equation
\[
(e^{-\frac{1}{4}\Phi+ip_x} + e^{i\Phi n+\frac{1}{4}\Phi+ip_y})\xi_{n+1} + \\
(e^{\frac{1}{4}\Phi-\frac{1}{2}ip_y} + e^{-i\Phi n+\frac{3}{4}\Phi-\frac{1}{2}ip_y})\xi_{n-1} = E\xi_n
\] (22)
The ”mid” band spectrum corresponds to \(\vec{p} = (\frac{1}{2}\pi, \frac{1}{2}\pi + \frac{1}{4}\Phi)\) in the chiral gauge (compare with (17)). Again all the values \(\vec{p} = (\frac{1}{2}\pi, \delta)\) for any real \(\delta\) are physically equivalent. We choose \(\vec{p}\) at the ”mid” band to be
\[ \vec{p} = (\frac{\pi}{2}, \frac{\pi}{2}) \] (23)
The relation between the three gauges are given in the Appendix A.

III. QUANTUM GROUP

The algebra \(U_q(sl_2)\) (a \(q\)-deformation of the universal enveloping of the \(sl_2\)) is generated by the elements \(A, B, C, D\), with the commutation relations
\[ AB = qBA, \quad BD = qDB, \]
\[ DC = qCD, \quad CA = qAC, \]
\[ AD = 1, \quad [B, C] = \frac{A^2 - D^2}{q - q^{-1}} \] (24)
We shall take the deformation parameter \(q\) as a root of \(\pm 1\) of degree \(Q\): \(q = exp(i\pi P/Q)\) where \(P\) and \(Q\) are mutually prime integers. The central element of this algebra (for arbitrary \(q\)) is a \(q\)–analog of the Casimir operator
\[
c_0 = \left( \frac{q^{-\frac{1}{2}}A - q^{\frac{1}{2}}D}{q - q^{-1}} \right)^2 + BC
\]  

(25)

When \( q \) is a root of unity some additional central elements appear.

In the classical limit \( q \to 1 + \frac{i}{2} \Phi \), the quantum group turns to the \( sl_2 \) algebra: \( (A - D)/(q - q^{-1}) \to S_3 \), \( B \to S_+ \), \( C \to S_- \), \( c \to \hat{S}^2 + 1/4 \).

The commutation relations (24) are simply another way to write the intertwining relation for the \( L \) - operator:

\[
R(u/v)(L(u) \otimes 1)(1 \otimes L(v)) = (1 \otimes L(v))(L(u) \otimes 1)R(u/v)
\]

(26)

with the trigonometric \( R \)–matrix

\[
R(u) = \frac{1}{2}(q + 1)(u - q^{-1}u^{-1}) + \frac{1}{2}(q - 1)(u + q^{-1}u^{-1})\sigma_3 \otimes \sigma_3 + (q - q^{-1})(\sigma_+ \otimes \sigma_- + \sigma_- \otimes \sigma_+)
\]

(27)

satisfying the Yang-Baxter relation (\( \sigma_j \) are Pauli matrices; \( \sigma_{\pm} = (\sigma_1 \pm i\sigma_2)/2 \)). Generators \( A, B, C, D \) are matrix elements of the \( L \)-operator

\[
L(u) = \begin{bmatrix}
\frac{ukA - u^{-1}k^{-1}D}{q - q^{-1}} & C \\
B & \frac{ukD - u^{-1}k^{-1}A}{q - q^{-1}}
\end{bmatrix}
\]

(28)

Here \( u \) is the spectral parameter and \( k \) is an additional parameter (rapidity at the site). Note that the \( R \)-matrix is the \( L \)-operator in the spin \( 1/2 \) - representation. It is given by the same matrix (28) for \( k = q^{1/2} \) with elements: \( A = q^{j\pi_3}, \; D = q^{-j\pi_3}, \; B = \sigma_+, \; C = \sigma_- \).

Irreducible finite \( 2j + 1 \) dimensional representations can be expressed in the weight basis where \( A \) and \( D \) are diagonal matrices: \( A = \text{diag}(q^j, \ldots, q^{-j}) \). An integer or half-integer \( j \) is the spin of

the representation. The value of the Casimir operator (25) in this representation is given by the \( q \)-analog of \( (j + 1/2)^2 \).

\[
c_0 = \left( \frac{q^{j+1/2} - q^{-j-1/2}}{q - q^{-1}} \right)^2 = [j + 1/2]^2_q
\]

(29)

The representation can be realized by difference operators acting in the space of polynomials \( \Psi(z) \) of degree \( 2j \).

\[
A\Psi(z) = q^{-j}\Psi(qz), \; D\Psi(z) = q^{j}\Psi(q^{-1}z), \\
B\Psi(z) = z(q - q^{-1})^{-1} \left(q^{2j}\Psi(q^{-1}z) - q^{-2j}\Psi(qz)\right) \\
C\Psi(z) = -z^{-1}(q - q^{-1})^{-1} \left(q^{2j}\Psi(q^{-1}z) - q^{-2j}\Psi(qz)\right)
\]

(30)

Then \( \Psi_0(z) = 1 \) is the lowest weight vector whereas \( \Psi_{2j}(z) = z^{2j} \) is the highest weight vector, i.e. \( C\Psi_0(z) = 0, \; B\Psi_{2j}(z) = 0 \).

We call this series of representations “regular”. They are a smooth deformation of the representation of the \( sl_2 \) algebra by difference operators:
\[ S_3 = z \frac{d}{dz} - j, \quad S_+ = z(2j - z \frac{d}{dz}), \quad S_- = \frac{d}{dz} \] (31)

In addition, in the special dimension \( Q \) there is three parametric family of representations having, in general, no lowest and no highest weight. Sometimes they are called cyclic or unrestricted representations. These representations have no classical finite dimensional limit. They can be written in the Weyl basis \[29], [20], [19]. Let \( X \) and \( Y \) satisfy

\[ qXY = YX \] (32)

Then the representation of \( U_q(sl_2) \) is

\[
A = X, \quad D = X^{-1}, \\
B = (bX + \bar{b}X^{-1})Y^{-1}, \\
C = Y(cX + \bar{c}X^{-1}),
\]

and is characterized by parameters \( b, c, \bar{b}, \bar{c} \) obeying the conditions \( qbc = q^{-1}\bar{b}\bar{c} = -(q - q^{-1})^{-2} \). The value of the Casimir operator (29) depends on the parameters and is equal to \( \bar{c}b + \bar{c}b - 2(q - q^{-1})^{-2} \). Comparing it with (29) we find that at \( q^2b/\bar{b} = \bar{c}/c = \mp q \) (for \( P \)-odd, even) cyclic representations belong to the “regular” series with \( q^{2j+1} = \mp 1 \) for \( P \)-odd (even) and the Casimir operator (29) is

\[
c_0 = -4(q - q^{-1})^{-2}, \text{ for } P - odd \\
c_0 = 0, \text{ for } P - even \] (34)

**IV. MAGNETIC TRANSLATIONS AS A SPECIAL REPRESENTATION OF THE QUANTUM GROUP**

For a given value of the Bloch wave vector \( \vec{k} \) the dimension of physical Hilbert space of our problem is \( Q \). A representation of dimension \( 2j + 1 = Q \) of the quantum group \( U_q(sl_2) \) for \( q = \exp(i\pi \frac{\vec{P}}{Q}) \) naturally acts in the space of Bloch states in the magnetic field. Therefore magnetic translations and the Hamiltonian (1),(5) can be expressed through generators of the quantum group. Let us set

\[
T_y = e^{ip_y}YX^{-1}, \\
T_x = e^{ip_x}XY,
\] (35)

Then, the Hamiltonian (11) could be expressed as a linear form in quantum group generators. At the middle of the band (an integrable point)

\[ e^{i(p_x - p_y)} = -q^{-1} \] (36)

we may choose

\[
b = e^{\frac{i}{2}(p_y - p_x)}(q - q^{-1})^{-1}, \\
\bar{b} = q^2(q - q^{-1})^{-1}, \\
\bar{c} = (1 - q^2)^{-1}, \\
c = e^{\frac{i}{2}(p_y - p_x)}(1 - q^2)^{-1},
\] (37)
Then $B$ and $C$ given by (33) form a representation of the quantum group with the value of the Casimir operator Eq.(34) which corresponds to the regular representation i.e. with the highest and the lowest weight. Using (33) and (35) we may identify the quantum group generators and magnetic translations

\[
\begin{align*}
T_{-x} + T_{-y} &= (q - q^{-1})e^{-\frac{i}{2}(p_y + p_x)} B, \\
T_x + T_y &= \mp(q - q^{-1})e^{\frac{i}{2}(p_y + p_x)} C, \\
T_{-y}T_x &= \pm q^{-1}A^2, \\
T_{-x}T_y &= \pm qD^2
\end{align*}
\]

The Hamiltonian now acquires the form

\[
H = (q - q^{-1})(\mp e^{\frac{i}{2}(p_y + p_x)} C + e^{-\frac{i}{2}(p_y + p_x)} B)
\]

In fact at the integrable point (36) the physical states and their energies (midband states) do not depend on the value of $\exp\frac{i}{2}(p_y + p_x)$. It simply can be gauged away. As a result for the midband states in the isotropic case the relation between magnetic translations and the quantum group is as follows

\[
H = i(q - q^{-1})(C \pm B)
\]

The Hamiltonian (40) can be written as the trace of a modified $L$-operator (28) $\tilde{L}(u)$. Say at $P$ odd

\[
H = Tr\tilde{L}(u) = TrL(u)\sigma_1
\]

Modified $L$-operator also obeys the intertwining relation (26). Realization of the quantum group in terms of magnetic translations is not unique. There are different realizations which may be used for different physical applications. Below we present, another "dual" realization. Say for an odd $P$ we have:

\[
\begin{align*}
T_{-x} + T_{-y} &= -i(q - q^{-1})q^{-\frac{1}{2}}BD, \\
T_x + T_y &= -i(q - q^{-1})q^{-\frac{1}{2}}CA, \\
T_{-y}T_x &= q^{-1}A^2, T_{-x}T_y &= qD^2
\end{align*}
\]

that also forms a representation of $U_q(sl_2)$ with the same value of the Casimir operator (29) and with the same dimension $Q$. Now the Hamiltonian turns into quadratic form in terms of the $U_q(sl_2)$ generators:

\[
H = -i(q - q^{-1})q^{-\frac{1}{2}}(CA + BD)
\]

Two different forms of the Hamiltonian are in fact gauge equivalent: they correspond to the two choices of gauge discussed in Sect.2. To stress ambiguity of representation of the

\[1^*\]

$L$-operator of such kind is used in the sine-Gordon model [23]. We are grateful to L.Faddeev and A.Volkov for pointing our attention to the Ref. [22].

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quantum group by magnetic translations we notice that the Hamiltonian of our problem can be also represented as a quadratic form of another quantum group $U_{q^2}(sl_2)$

$$H = CA - BD + q^2 BA - q^2 CD$$

Diagonalization of this form leads to the same Bethe Ansatz equation as the form (43).

In the Appendix B we show that the Hamiltonian (43) represented as a quadratic form of the quantum group generators can also be understood as a trace of a quantum monodromy matrix, though of a more complicated nature - the monodromy matrix for a system with non-periodic boundary conditions.

Since the main ingredients of the quantum inverse scattering method have appeared in the problem there is a strong evidence that the Hofstadter Hamiltonian can be actually diagonalized for arbitrary $Q$, i.e. its energy spectrum may be expressed by solutions of the Bethe equations. The ambitious goal of solving the problem for arbitrary parameters $p_x, p_y$ and $t_x/t_y$ is beyond the scope of this paper. Instead we shall present a solution for the fixed values of the parameters when representation of the quantum group (expressed through magnetic translations) belong to the ”regular” series and, therefore, has the lowest and the highest weight. This point corresponds to the middle of band and isotropic hoping. In this case the so-called functional Bethe Ansatz is applicable.

V. FUNCTIONAL BETHE ANSATZ

The Hamiltonian of the Bloch particle in a magnetic field in the midband point of the spectrum $\Lambda = 0$ can be expressed as a polynomial form in the quantum group operators in a regular representation. Let us use this advantage and replace generators by their functional realizations.

Let us first consider the representation in which the Hamiltonian is a linear form (40). Substituting the functional realization (30) in the Hamiltonian (40) we obtain a difference equation for a meromorphic function $\Psi(z)$

$$i(z^{-1} + qz)\Psi(qz) - i(z^{-1} + q^{-1}z)\Psi(q^{-1}z) = E\Psi(z)$$

This equation can be easily obtained directly from the original Harper equation in the chiral gauge (20) without a reference to the quantum group. Indeed, Eq.(45) is an extension of the (20) at $\vec{p} = (\pi/2, \pi/2)$ to the whole complex plane provided

$$\chi_n = \Psi(q^n)$$

However the miracle which would be very hard to see without the quantum group is that the energy in the extended equation (45) does not depends on $z$.

Now it is easy to get a family of Hamiltonians which have the same spectrum. Let us set $z = q^n u$ and $\Psi(q^n u) = \chi_n(u)$. Then, all Hamiltonians

$$H(u) = i(q^{-n}u^{-1} + q^{n+1}u)\chi_{n+1}(u) - i(q^{-n}u^{-1} + q^{n-1}u)\chi_{n-1}(u) = E\chi_n(u)$$

have the same spectrum and correspond to $\Lambda = 0$. The original Harper’s Hamiltonian (1) is $H(1)$.
Another miracle is that we know in advance that the function $\Psi(z)$ is a polynomial of degree $Q - 1$:

$$\Psi(z) = \prod_{m=1}^{Q-1} (z - z_m) \quad (47)$$

Note that last fact is not valid in an arbitrary gauge, in particular in the Landau gauge.

The similar arguments work in case of the modified Landau gauge. The functional realization of the quantum group Hamiltonian (43) gives the difference equation

$$z^{-1}\Psi(q^2 z) + q^{-2}z \Psi(q^{-2}z) - (z + z^{-1})\Psi(z) = E\Psi(z) \quad (48)$$

with a polynomial solution. It is an extension of the Harper equation in the modified Landau gauge for $\vec{k}' = (0, \pi)$ can be extended to the whole complex plane with

$$\phi_n = \Psi(q^{2n}) \quad (49)$$

The most suitable method to solve the spectral problems (45,48) is the so called functional Bethe-Ansatz.

Let us plug (47) in (45,48) and divide both sides by $\Psi(z)$.

$$i(z^{-1} + qz) \prod_{m=1, m \neq l}^{Q-1} \frac{qz - z_m}{z - z_m}$$

$$-i(q^{-1} + q^{-1}z) \prod_{m=1, m \neq l}^{Q-1} \frac{q^{-1}z - z_m}{z - z_m} = E \quad (50)$$

The l.h.s. of this equation is a meromorphic function, whereas the r.h.s. is a constant. To make them equal we must null all residues of the l.h.s.. They appear at $z = 0$, at $z = \infty$ and at $z = z_m$. The residue at $z = 0$ vanishes automatically.

The residue at $z = \infty$ is $-iq^{Q} + iq^{Q^{-1}}$. Its null determines the degree of the polynom.

Comparing the coefficients of $z^{Q-1}$ in the both sides of Eq.(50), we obtain the energy given by Eq.(5) advertised in the Introduction.

Finally, annihilation of poles at $z = z_m$ gives the Bethe-Ansatz equations (6) for roots of the polynomial (47). Here we write them in a more conventional form. Let $z_l = \exp (2\varphi_l)$, then

$$\frac{\cosh(2\varphi_l - i\frac{\varphi_m}{2})}{\cosh(2\varphi_l + i\frac{\varphi_m}{2})} = \pm \prod_{m=1, m \neq l}^{Q-1} \frac{\sinh(\varphi_l - \varphi_m + i\frac{\varphi_m}{2})}{\sinh(\varphi_l - \varphi_m - i\frac{\varphi_m}{2})} \quad (51)$$

The Bethe equations for the quadratic realization of the Hamiltonian (13), can be obtained via similar method. They are

$$z_l^2 = q^{Q} \prod_{m=1, m \neq l}^{Q-1} z_l - \frac{z_m}{z_l - q^{2}z_m}, \quad l = 1, ..., Q - 1 \quad (52)$$

The energy spectrum is again proportional to the sum of roots:

$$E = -q(q - q^{-1}) \sum_{l=1}^{Q-1} z_l \quad (53)$$

Inspite of the apparent difference eqs. (6) and (4) must be equivalent to the eqs. (52) and (53). The former is useful to describe the middle of the spectrum, whether the latter is good for the bottom of the spectrum.
VI. Q-ANALOG OF ORTHOGONAL POLYNOMIALS AS EXACT ZERO MODE
WAVE FUNCTIONS

There is an intriguing connection between the wave function of the zero energy state
and q-generalization of the classical orthogonal polynomials [34]. The q-analogy of orthogonal polynomials (so called Askey-Wilson polynomials) are the most general polynomials orthogonal on a discrete support. They depend on four parameters (except $q$) and satisfy a q-analogy of the differential hypergeometric equation. It is the difference equation

$$A(z)P_n(q^2z) + A(z^{-1})P_n(q^{-2}z) - (A(z) + A(z^{-1}))P_n(z) = (q^{-2n} - 1)(1 - abcdq^{2n-2})P_n(z) \quad (54)$$

where

$$A(z) = \frac{(1 - az)(1 - bz)(1 - cz)(1 - dz)}{(1 - z^2)(1 - q^2z^2)} \quad (55)$$

and $a, b, c, d$ are parameters and $n$ is the degree of the polynomial. Note, that $P_n$ is a Laurent polynomial in $z$ and usual polynomial in $z + z^{-1}$ of degree $n$. Recently, the Askey-Wilson polynomials appeared in connection with representation of the quantum group. It has been found that they are closely related to $6 - j$-symbol of $U_q(sl_2)$ [24]. They also appeared in "q-harmonic analysis" to be spherical functions on the quantum $SL(2)$ group [35]. Some of them are closely related to our problem.

Choosing $c = -d = q$, $a = b = 0$ we arrive at the equation for the continuous $q$-Hermite polynomials [32] $H_n^{(q)}$

$$H_n^{(q)}(q^2z) - z^2H_n^{(q)}(q^{-2}z) = q^{-2n}(1 - z^2)H_n^{(q)}(z).$$

(They are called continuous because of their orthogonality on the unit circle with a continuous measure).

It is clear from (48) that for an odd $Q$ $q$-Hermite polynomial of the order $n = (Q - 1)/2$ yields zero energy solution of our problem:

$$\Psi^{(E=0)}(iz) = z^{(Q-1)/2}H_{(Q-1)/2}^{(q)}(z). \quad (56)$$

The explicit form of the $q$-Hermite polynomials is

$$H_n^{(q)}(z) = \sum_{m=0}^{n} \frac{(q^2; q^2)_n}{(q^2; q^2)_m(q^2; q^2)_{n-m}} z^{2m-n} \quad (57)$$

where the standard notation

$$(a; q)_n = \prod_{l=0}^{n-1} (1 - aq^l)$$

is used.

Another choice is $c = -d = q$, $a = -b = q$ and then the replacement of $q$ by $q^{1/2}$ gives the $q$-Legendre equation

$$\frac{1 - qz^2}{1 - z^2} P_n^{(q)}(qz) + \frac{q - z^2}{1 - z^2} P_n^{(q)}(q^{-1}z) = (q^{-n} + q^{n+1})P_n^{(q)}(z) \quad (58)$$
Comparing with the Eq.(45) we conclude that the zero mode solution is given by the continuous $q$-Legendre polynomial

$$
\Psi^{(E=0)}(iz) = z^{(Q-1)/2}P_{(Q-1)/2}^{(q)}(z)
$$

Their explicit form is

$$
P_n^{(q)}(z) = \sum_{m=0}^{n} \frac{(q;q)_n(q^{1/2};q)_m(q^{1/2};q)_{n-m}}{(q;q)_m(q;q)_{n-m}}z^{2m-n}
$$

(59)

VII. CONCLUSION

We have showed that motion of a particle in a periodic potential in a magnetic field has posses a structure of the quantum group. We found the Bethe Ansatz equation for the midband spectrum and the wave function for a particle on a square lattice. We also presented a class of quasiperiodic equations related to $U_q(sl_2)$ which can be solved in a similar way. The most interesting feature of that type of equation is the multifractality of their spectrum at $P,Q \to \infty$, i.e when flux $\Phi/2\pi$ is irrational. The spectrum is complex but not chaotic. Quite opposite, it is determined by the quantum integrability.

As we already mentioned the Bethe Ansatz solution is available (but not presented here) for an arbitrary strength of the potential of the Harper equation. It provides the basis to study Anderson localization-delocalization transition of electrons in quasiperiodic potential.

As the best of our knowledge even elementary questions regarding the Hofstadter problem at irrational flux remained unclear. For example the low temperature thermodynamics, dispersion of excitations, conductivity etc. are not known. In all previous examples of quantum integrable models all these quantities have been found by solving Bethe Ansatz equations at large $Q$. In this limit dispersion of bands becomes negligible. Therefore, the lack of solution away from the midband is therefore is not an obstacle for study multifractality and other physical properties. The Bethe Ansatz solution does not give an advantage at finite $P$ and $Q$. Just contrary, up to $Q \sim 6000$ direct diagonalization of the Hofstadter Hamiltonian is more effective. However, as usual the Bethe Ansatz solution becomes a powerful tool at $Q \to \infty$. Perhaps the most interesting task is to solve the Bethe Ansatz equations in this limit. The strategy of solving the Bethe equations is well known: at large $Q$ the roots $z_l$ form dense groups (“strings”) and can be described by their distributions. The algebraic equations then replaced by the system of integral equations for the distribution functions of strings. This program is in progress.

Another ambitious problem is to generalize the Bethe ansatz solution to the whole Hilbert space of the model.

VIII. SUPPLEMENT A:A CLASS OF DIFFERENCE AND DISCRETE EQUATIONS RELATED TO THE QUANTUM GROUP

A general quadratic form in generators of the quantum group provide a class of difference and discrete operators of the second order which allows complete or partial algebraization.
A problem of the Bloch particle in magnetic field is a particular case (the most physically valuable, perhaps) of the general class.

The idea of algebraization is simple. Consider a regular representation of the quantum group with the lowest and the highest weight. The space of this representation can be can realized by the ring of polynomials of degree 2j. Any form in generators A, B, C, D preserves the space of polynomials. Therefore eigenfunctions of a form, which by means of the representation (30) is a difference operator, are polynomials as well. Vice versa is probably also true - one may look on the quantum group as an algebra of difference operators which preserve the ring of polynomials.

Quadratic forms of the "classical" $sl_2$ as a class of second-order differential equations solvable in polynomials (so called "quasi exactly solvable" models of quantum mechanics ) have been extensively studied [25] [26] (see also [32] for a review) in the last few years. They include second order differential equations for all classical orthogonal polynomials and a set of Schrodinger operators with solvable potentials. Attempt of q-generalization for difference functional equations (in fact very closed to our approach) has been made in Ref. [33].

Consider a general quadratic form (in the Appendix B we show that it can be consider as a trace of monodromy matrix of some integrable model)

$$G = aA^2 + dD^2 + (q - q^{-1})(c_2CA + b_2BD + b_3BA + c_3CD) + (q - q^{-1})^2(b_1B^2 + c_1C^2)$$  (60)

with a set of parameters $a, d, c_i, b_i$ ($i = 1, 2, 3$).

Under the representation (30) the quadratic form is a difference operator. Let us consider its spectral problem.

$$G\Psi(z) = a(z)\Psi(q^2z) + d(z)\Psi(q^{-2}z) - v(z)\Psi(z) = E\Psi(z)$$  (61)

where

$$a(z) = b_1q^{-4j+1}z^2 - b_3q^{-3j}z + aq^{-2j} + c_2q^{-j}z^{-1} + c_1q^{-1}z^{-2}$$  (62)

$$d(z) = b_1q^{4j-1}z^2 + b_2q^{3j}z + dq^{2j} - c_3q^jz^{-1} + c_1qz^{-2}$$  (63)

$$v(z) = (q + q^{-1})(b_1z^2 + c_1z^{-2}) + (c_2q^{-j} - c_3q^j)z^{-1} + (b_2q^{-j} - b_3q^j)z$$  (64)

If all $b_{1,2,3} = 0$ the difference equation (61) is known as q-hypergeometric equation (see e.g. [34]). Their solutions are certain q-Jacobi polynomials - a degenerate case of Askey Wilson polynomials [34] [35]. At the "classical" limit $q \rightarrow 1$ they reproduce classical orthogonal polynomials. The difference equation (61) in its general form in the classical limit leads to Mathieu and Lame equations. One may call the equation (61) q- analog of the Mathieu (Lame) equations.

From the theory of representation of the quantum group, we know that this equation has $(2j + 1)$ polynomial solutions of degree no greater than $2j$.

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2* We are indebted to A.Turbiner who informed us about this work

13
\[ \Psi(z) = \prod_{m=1}^{N} (z - z_m) \]  

(65)

The class of "integrable" equations can be extended if we apply a "gauge" transformation
\[ \Psi(z) \rightarrow f(z)\Psi(z), \quad a(z) \rightarrow a(z)f(z)/f(q^2z), \quad d(z) \rightarrow d(z)f(z)/f(q^{-2}z), \quad v(z) \rightarrow v(z) \].

A class of "integrable" discrete equations can be obtained from (61) by setting
\[ z = q^{2n}, \quad a(q^{2n}) = a_n, \quad d(q^{2n}) = d_n, \quad v(q^{2n}) = v_n, \quad \Psi(q^{2n}) = \Psi_n. \]

\[ a_n \Psi_{n+1} + d_n \Psi_{n+1} - v_n \Psi_n = E \Psi_n, \quad n = 1, \ldots, Q \]  

(66)

Solutions of the discrete equations are given by discrete polynomials
\[ \Psi_n = \prod_{m=1}^{N} (q^{2n} - z_m) \]  

(67)

with the same roots.

The functional Bethe-Ansatz described above determines algebraic equations for roots of the solution (63) of eqs. (61) or (66). We already applied this method for the Harper Equation. For complete reference we repeat it for a general case below. First of all, let us plug (67) in (61) and divide both sides by \( \Psi(z) \). We get
\[ a(z) \prod_{m=1}^{N} \frac{q^2z - z_m}{z - z_m} + d(z) \prod_{m=1}^{N} \frac{q^{-2}z - z_m}{z - z_m} - v(z) = E \]  

(68)

If at least one of the coefficients \( c_1, c_2, c_3 \) is nonzero, there are two different cases:
(i) at least one of \( b_1, b_2, b_3 \) is nonzero,
(ii) All \( b ' s \) are zero - the quadratic form (60) includes only \( A, D \) and \( C \) (generators of the Borel subalgebra of \( U_q(sl_2) \)). That type of difference equation (61) is known as \( q \)-hypergeometric equation. Their solutions are certain \( q \)-Jacobi polynomials - a class of polynomials orthogonal with a discrete measure (see e.g. [34]).

Let us consider the case (i) first. The l.h.s. of (68) is a meromorphic function, whereas the r.h.s. is a constant. To make them equal we must cancel all the singularities of the l.h.s. They appear at singular points of \( a(z), d(z) \) and \( v(z) \) (double and simple poles at \( z = 0 \) and \( z = \infty \)) and at \( z = z_m \).

The singular part at \( z = 0 \) vanishes automatically.
Vanishing of the singular part at \( z = \infty \)
\[ b_1(q^{2N-4j+1} + q^{-2N+4j-1} - q - q^{-1})z^2 + b_2(q^{-2N+3j} - q^{-j})z + b_3(q^j - q^{2N-3j})z + z(q - q^{-1})b_1(q^{2N-4j} - q^{-2N+4j}) \sum_{m=1}^{N} z_m \]  

(69)

*** We note that this method is most close to the approach of algebraization of quantum mechanical problem suggested by Ushveridze 26
determines the degree of the polynomial: $N = 2j$.

Comparing the constant terms in the both sides of (68) we find the energy spectrum:

$$E = b_1(q - q^{-1})(q^2 - q^{-2}) \sum_{n<m} z_n z_m - (q - q^{-1})(b_2q^{-j+1} + b_3q^{j-1}) \sum_{m=1}^{N} z_m + a q^{2j} + dq^{-2j}$$

(70)

Finally, annihilation of poles at $z = z_m$ gives the following Bethe-Ansatz equations

$$\frac{d(z_l)}{a(z_l)} = q^{4j} \prod_{m=1, m \neq l}^{2j} \frac{q^2 z_l - z_m}{z_l - q^2 z_m}, \ l = 1, ..., 2j.$$  

(71)

The Bethe equations is a system of $2j$ algebraic equations. It must have exactly $2j + 1$ solutions corresponding to different eigenfunctions. In the case (i) all of them are polynomials of one and the same degree $2j$.

Similar arguments are applicable in the case (ii). The difference is that the l.h.s. of (68) is now regular at $z = \infty$ from the very beginning and the condition of vanishing of (69) does not bring a restriction on degree of the polynomials. The Bethe equations are valid for any $N < 2j + 1$

$$\frac{d(z_l)}{a(z_l)} = q^{2N} \prod_{m=1, m \neq l}^{N} \frac{q^2 z_l - z_m}{z_l - q^2 z_m}, \ l = 1, ..., N.$$  

(72)

Apparently, for each degree there is exactly one such polynomial. This means that for each $N < 2j + 1$ the Bethe equations (72) must have exactly one solution. This fact is far from obvious when we look at (72). Anyway, we derived equations (72) for zeros of $q$-orthogonal polynomials. It seems to be interesting to review the theory of orthogonal polynomials from this point of view.

The main property of the $q$-hypergeometric equation (the case (ii)) that makes it ”solvable” is triangularity of the matrix connecting the original basis $z^m (m = 0, 1, ..., 2j)$ with the basis formed by eigenfunctions of the operator $G$. In particular, this leads to a very simple structure of the spectrum of $G$:

$$E_N = a q^{2N-2j} + dq^{2j-2N}, \ N = 0, ..., 2j.$$  

(73)

IX. SUPPLEMENT B: QUANTUM INTEGRABLE MODELS WITH NON-PERIODIC BOUNDARY CONDITIONS AND HOFSTADTER PROBLEM

A general quadratic form in quantum group generators is related to a quantum magnetic chain with one site and non-periodic boundary conditions.

To begin with, let us give a brief summary of the formalism treating the integrable systems with boundaries. The boundary conditions of an integrable model are determined
by c-number $2 \times 2$ matrices $K_+(u)$ and $K_-(u)$ depending on spectral parameter and satisfying the "reflection equations"\ [27].

$$R(u/v)(K_-(u) \otimes 1)R(uv^{-1})(1 \otimes K_-(v)) = (1 \otimes K_-(v))R(uv^{-1})(K_-(u) \otimes 1)R(u/v)$$

$$R(v/u)(K^t_+(u) \otimes 1)R((uvq)^{-1})(1 \otimes K^t_+(v)) = (1 \otimes K^t_+(v))R((uvq)^{-1})(K^t_+(v) \otimes 1)R(v/u) \quad (74)$$

($t$ means the transposition) with the $R$-matrix\ [27]. Each solution of the "reflection equations" specify a boundary condition consistent with integrability. Solutions for $K_+$ and for $K_-$ are related

$$K_+(u) = K^t_-(u^{-1}) \quad (75)$$

The monodromy matrix for an integrable model with non-periodic boundary conditions is a quadratic form in the monodromy matrix $L(u)$ of a model with periodic boundary conditions\ [28].

$$T(u) = L(u)K_-(u)\sigma_2 L^t(u^{-1})\sigma_2 \quad (76)$$

The trace of the monodromy matrix

$$\tau(u) = Tr(K_+(u)T(u)) \quad (77)$$

forms a commutative family $[\tau(u), \tau(v)] = 0$.

It is known that for the models with the trigonometric $R$-matrix there is a 3-parametric family of boundary $K$-matrices\ [30, 31]

$$K_-(u) = \begin{pmatrix} \alpha(q^{-1}s^{-1}u - qsu^{-1}) & \beta(q^{-1}u^2 - qu^{-2}) \\ \gamma(q^{-1}u^2 - qu^{-2}) & -\alpha(su - s^{-1}u^{-1}) \end{pmatrix} \quad (78)$$

$$K_+ = \begin{pmatrix} \lambda(qu - q^{-1}t^{-1}u^{-1}) & \mu(qu^2 - q^{-1}u^{-2}) \\ \nu(qu^2 - q^{-1}u^{-2}) & -\lambda(t^{-1}u - tu^{-1}) \end{pmatrix} \quad (79)$$

where $\alpha, \beta, \gamma, s$ and $\lambda, \mu, \nu, t$ are arbitrary parameters characterized the boundary conditions. Substituting the $L$-operator\ [28] into\ (77) we find matrix elements of the transfer matrix

$$T_{11}(u) = \frac{\alpha(q^{-1}s^{-1}u - qsu^{-1})}{(q - q^{-1})^2}(k^2 + k^{-2} - u^2A^2 - u^{-2}D^2) + \frac{q^{-1}u^2 - qu^{-2}}{q - q^{-1}}(\gamma kuCD - \gamma k^{-1}uCA - \beta kuAB + \beta k^{-1}u^{-1}DB) + \alpha(su - s^{-1}u^{-1})CB \quad (80)$$

$$T_{12}(u) = \frac{q^{-1}u^2 - qu^{-2}}{q - q^{-1}} \left( \frac{\beta(k^2A^2 + k^{-2}D^2 - u^2 - u^{-2})}{q - q^{-1}} - \gamma(q - q^{-1})C^2 + \alpha(sk^{-1}DC - s^{-1}kAC) \right) \quad (81)$$

$$T_{21}(u) = \frac{q^{-1}u^2 - qu^{-2}}{q - q^{-1}} \left( \frac{\gamma(k^2D^2 + k^{-2}A^2 - u^2 - u^{-2})}{q - q^{-1}} - \beta(q - q^{-1})B^2 + \alpha(skBD - s^{-1}k^{-1}BA) \right) \quad (82)$$
\[ T_{22}(u) = \frac{\alpha(s^{-1}u^{-1} - su)}{(q - q^{-1})^2} (k^2 + k^{-2} - u^2D^2 - u^{-2}A^2) + \]
\[
\frac{q^{-1}u^2 - qu^{-2}}{q - q^{-1}} (\beta ku^{-1}BA - \beta k^{-1}uBD - \gamma kuDC + \gamma k^{-1}u^{-1}AC) - \alpha(q^{-1}s^{-1}u - qu^{-1})BC \quad (83)
\]

Summing diagonal elements (80) and (83) we find the trace of the transfer matrix \( \tau(u) \)
\[
\tau(u) = \left( \frac{u^4 + u^{-4} - q^2 - q^{-2}}{(q - q^{-1})^2} \right) [(\mu \gamma k^2 + \nu \beta k^2 - \alpha \lambda t s^{-1})A^2 + (\mu \gamma k^2 + \nu \beta k^{-2} - \alpha \lambda t^{-1}s)D^2 + (q - q^{-1})(-tk^{-1}\lambda \gamma + q^{-1}s^{-1}k\alpha \nu)CA + (q^{-1}t^{-1}k^{-1}\lambda \beta + sk\alpha \mu)BD + (t^{-1}k\lambda \gamma + qsk^{-1}\alpha \nu)CD - (gtk\lambda \beta + s^{-1}k^{-1}\alpha \mu)BA) - (q - q^{-1})^2(\mu \beta B^2 + \nu \gamma C^2)] + c - number\ term \quad (84)
\]

(here we replace the Casimir element (23) by it \( c \)-number value)

The trace of the transfer matrix \( \tau(u) \) is a general quadratic form in \( A, B, C, D \) since it depends on 7 parameters - 3 in each \( K \)-matrix and rapidity \( k \). Indeed, the total number of coefficients of a general quadratic form is 10 but one of them is a common multiplier and another two contribute to the \( c - number\ term \) in (84) due to the two central elements \( (AD = 1 \text{ and the Casimir operator}) \).

As we showed previously the Hamiltonian of the Bloch particle in magnetic field is a particular quadratic form of the quantum group generators and therefore can be considered as an integrable model.

**APPENDIX A: A**

Gauges 1. The connection between Landau gauge and the modified Landau gauge (13) is very simple
\[
\phi_n(\vec{k}') = \exp\left(\frac{i\Phi}{2} n(n - Q)\right)\psi_n(\vec{k}) \quad (A1)
\]
\[
k'_x = k_x + \frac{i\pi P(Q - 1)}{Q}, \quad k'_y = k_y \quad (A2)
\]
Both \( k'_x \) and \( k_x \) are defined modulo \( \Phi \).

2. The relationship between Landau gauge and the chiral gauge is a bit sophisticated

We are indebted to A.Abanov who found this connection. They are connected by the Fourier transformation
\[
\tilde{\phi}_m = \sum_{n=0}^{Q-1} e^{i\Phi nm} \phi_n \quad (A3)
\]
where $\phi_n$ is the wave function in the modified Landau gauge. The new function $\tilde{\phi}_n$ obeys the equation

\[
(e^{ik_y'} + e^{i\Phi_n - ik_x'})\tilde{\phi}_{n+1} + (e^{-ik_y'} + e^{-i\Phi_n + i\Phi + ik_x'})\tilde{\phi}_{n-1} = E\tilde{\phi}_n
\]  

(A4)

Comparing with (22) we can identify the momenta as follows:

\[
k'_x = -p_y - \frac{1}{4}\Phi, \quad k'_y = p_x - \frac{1}{4}\Phi
\]  

(A5)

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Postscriptum: when this paper has been finished we have learned that L.Faddev and R.Kashaev succeeded in extending the Bethe Ansatz solution for an arbitrary momentum and an arbitrary anisotropy. We indebted to L.Faddeev and R.Kashaev for informing us of their results.
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