Fractional quantum-Hall liquid spontaneously generated by strongly correlated $t_{2g}$ electrons

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For topologically nontrivial and very narrow bands, Coulomb repulsion between electrons has been predicted to give rise to a spontaneous fractional quantum-Hall (FQH) state in absence of magnetic fields. Here we show that strongly correlated electrons in a $t_{2g}$-orbital system on a triangular lattice self-organize into a spin-chiral magnetic ordering pattern that induces precisely the required topologically nontrivial and flat bands. This behavior is very robust and does not rely on fine tuning. In order to go beyond mean field and to study the impact of longer-range interactions, we map the low-energy electronic states onto an effective one-band model. Exact diagonalization is then used to establish signatures of a spontaneous FQH state.

The Integer Quantum Hall (IQH) effect [1] is a prime example of an electronic state that cannot be classified within the traditional framework of symmetry breaking, but is instead characterized by a topological invariant [2]. It is by now theoretically well established that an external magnetic field is in principle not needed and that states within the same topological class as IQH states can be realized in lattice models, if time-reversal symmetry is broken by other mechanisms, e.g., by complex electron hoppings [3]. Related topologically nontrivial Quantum Spin-Hall (QSH) states even occur in systems where time-reversal symmetry is not broken at all [4-8], see Refs. [9, 10] for reviews. At present, many intriguing features intrinsic to topologically non-trivial states have been observed in the absence of magnetic fields, such as the metallic Dirac cones at the surface of a topological insulator [11, 12], or the QSH effect in quantum wells [13, 14].

Fractional Quantum Hall (FQH) states [15] are topological states that can be seen as composed of quasi-particles carrying an exact fraction of the elementary electronic charge [10]. Apart from the fundamental interest in observing a quasi-particle that behaves in many ways like a fraction of an electron, some FQH states also have properties relevant to fault-tolerant quantum computation [17]. Very recently [18-20], it was suggested that lattice-FQH states may similarly arise without a magnetic field, in fractionally filled topologically nontrivial bands.

In contrast to the IQH and QSH effects, which can be fully understood in terms of non-(or weakly-)interacting electrons, interactions are an essential requirement for FQH states, which places demanding restrictions on candidate systems: One needs a topologically nontrivial band that must be nearly flat – similar to the highly degenerate Landau levels – so that the electron-electron interaction can at the same time be large compared to the band width and small compared to the gap separating it from other bands [18-20]. If the requirements can be fulfilled, however, the temperature scale of the FQH state is set by the energy scale of the interaction. This can allow temperatures considerably higher than the sub-Kelvin range of the conventional FQH effect, which would be extremely desirable in view of potential quantum-computing applications. Moreover, the lattice version of FQH states [21] may have unique and different properties [22].

In most recently proposed model Hamiltonians [18-20, 23-25], the topological nature of the bands was introduced by hand and model parameters have to be carefully tuned to obtain very flat bands. As potential realizations, “purpose built” physical systems in oxide heterostructures [26] or optical lattices [19] were suggested. On the other hand, topologically nontrivial bands can in principle emerge spontaneously in interacting electron systems [27, 28], e.g., for charge-ordered systems [29, 30] or for electrons coupling to spins in a non-coplanar magnetic order [31, 32]. We demonstrate here that such a scenario indeed arises in a Hubbard model describing electrons with a $t_{2g}$ orbital degree of freedom on a triangular lattice: a ground state with topologically nontrivial and nearly flat bands is stabilized by onsite Coulomb interactions. Upon doping the flat bands, longer-range Coulomb repulsion induces FQH states.

$t_{2g}$ orbitals on the triangular lattice.— The building blocks of our system are oxygen octahedra with a transition-metal (TM) ion in the center, the most common building block in the large and versatile material class of TM oxides. Local cubic symmetry due to the oxygen ions splits the $d$-orbitals into $t_{2g}$ and $e_g$ levels, and it has been shown that orbital degrees of freedom of either kind can substantially reduce the width of topologically nontrivial bands [24]. Here, we concentrate on the $t_{2g}$ orbitals illustrated in Fig. 1(a), which are further split by a crystal-field due to the overall lattice geometry. On a triangular lattice, we find one $a_{1g}$ and two $e_{g,\pm}$ states, see Fig. 1(b), with a splitting $H_{JT} = \Delta_{JT}(n_{e_{g+}} + n_{e_{g-}} - 2n_{a_{1g}})/3$ depending on the Jahn-Teller effect and the lattice [33]. Electron hopping along nearest-neighbor (NN) bonds consists of terms $t$ via ligand oxygens and $t_{dd}$ due to direct $d$-$d$ overlap [33, 34]. hopping matrices are given in [35]. We set here $n < 3$ and

\[ \left( \begin{array}{c} a_{1g} \\ e_{g+} \\ e_{g-} \end{array} \right) \Rightarrow \left( \begin{array}{c} a_{1g} \\ e_{g+} \\ e_{g-} \end{array} \right) \]
that can support spontaneous FQH states. Forms nearly flat bands with non-trivial topological character filled, while the \( a_\sigma \) orbital (bottom). In (b), the orbitals reflecting the three-fold lattice symmetry are shown: The two \( \epsilon'_\sigma \) orbitals (bottom), which differ by their complex phases, will turn out to be half filled, while the \( a_{1g} \) orbital (pointing out of the plane, see top) forms nearly flat bands with non-trivial topological character that can support spontaneous FQH states.

**FIG. 1.** (Color online) Triangular perovskite lattice and \( t_{2g} \) orbitals. Oxygen octahedra are indicated by lines, with black lines illustrating the front facets. Thick dotted (dashed, solid) lines indicate nearest-neighbor bonds along lattice vector \( a_1 \) \((a_2, a_3)\). (a) Shows two \( d_{xy} \) orbitals (top) and one \( d_{xz} \) and \( d_{yz} \) orbital (bottom). In (b), the orbitals reflecting the three-fold lattice symmetry are shown: The two \( \epsilon'_\sigma \) orbitals (bottom), which differ by their complex phases, will turn out to be half filled, while the \( a_{1g} \) orbital (pointing out of the plane, see top) forms nearly flat bands with non-trivial topological character that can support spontaneous FQH states.

**FIG. 2.** (Color online) Spin-chiral magnetic phase with topologically nontrivial bands stabilized by onsite Coulomb interactions in \( t_{2g} \) electrons on a triangular lattice. (a) Chiral magnetic order, the sites of the unit cell are labeled by 1 to 4. (b) The spins on the four sites can be seen as pointing to the corners of a tetrahedron, i.e., the pattern is non-coplanar. (c) One-particle energies on a cylinder (periodic boundary conditions along \( z \)) in the mean-field [35] ground state of the \( t_{2g} \) multiorbital Hubbard model, which is given by the pattern shown in (a). States drawn in black (grey) have more (less) than 33% \( a_{1g} \) character, dashed and dotted lines indicate edge states with more than 33% of their weight on the top (bottom) row of sites. The arrows \( \uparrow (\downarrow) \) indicate states with electron spin mostly (anti-)parallel to the local quantization axis, which can be seen as the lower (upper) Hubbard band. The filling is 2.5 electrons per site, slightly less than half filling. Parameters used were \( t = 1, t_{dd} = 0, U/t = 12, J/t = 3, \Delta_{JT}/t = -6 \). The figure of merit \( M \), which is given by the ratio of the gap separating the two \( a_{1g} \) subbands of the lower Hubbard band and the band width of the highest subband of the lower Hubbard band, is \( M \approx 14 \).

Choose \( t > 0 \) [34] as unit of energy, but analogous results hold for \( n > 3, t < 0 \), and \( t_{dd} \) \( \to -t_{dd} \). \( \Delta_{JT} \to -\Delta_{JT} \) due to particle-hole symmetry.

In TM oxides, Coulomb interaction is substantial compared to the kinetic energy of \( t_{2g} \) orbitals and spin-orbital physics induced by correlations are known to be rich in \( t_{2g} \) systems on triangular lattices [31] [30]. We take into account the onsite interaction including Coulomb repulsion \( U \) (intra-orbital) and \( U' \) (interorbital) as well as Hund’s-rule coupling \( J \). We employ a mean-field approximation with a decoupling into expectation values of densities \( \langle \eta_{i,\alpha,\sigma} \rangle = \langle c_{i,\alpha,\sigma}^\dagger c_{i,\alpha,\sigma} \rangle \) for site \( i \), orbital \( \alpha \), and spin \( \sigma \) [37] [38]. The spin is thus reduced to its \( z \)-component \( \eta_{i,\alpha} = (\eta_{i,\alpha,\uparrow} - \eta_{i,\alpha,\downarrow})/2 \) and non-collinear magnetic patterns are treated by allowing for a site-dependent spin-quantization axis expressed by angles \( \theta_i \) and \( \phi_i \). The change in quantization axis from site to site manifests itself in a complex Berry phase for the hopping terms [39]. Numerical optimization is used to find the \( \theta_i \) and \( \phi_i \) giving the magnetic ground state, permitting arbitrary magnetic orderings with unit cells of up to four sites, including all phases considered in Ref. [40]. For simplicity, we present here results for \( J/U = 1/4 \) and the relation \( U' = U - 2J \) between the Kanamori parameters was used, but we have verified that the results presented remain robust for other choices. Details are in [39].

For wide parameter ranges (see below), the ground state is the non-coplanar spin-chiral phase illustrated in Fig. [3] (a,b). As demonstrated in the context of the Kondo-lattice [38] [40] and the Hubbard [38] [41] models, this magnetic order leads to topologically nontrivial bands, which can also be seen in the one-particle bands shown in Fig. [3] (c). The chemical potential lies within the \( a_{1g} \) states of the lower Hubbard band, where the electron spin is mostly parallel (labelled by \( \uparrow \)) to the direction defined by the spin-chiral pattern. Dashed and dotted lines decorate states living on the top and bottom edges of a cylinder, they cross the chiral gap exactly once as one left- and one right-moving edge mode, indicating the different Chern numbers associated with the two bands directly above and below the chemical potential. Such a spontaneous IQH state is already rather exotic and has recently been shown to support fractionalized excitations bound to vortices [42].

Figure [3] (c) also indicates that the upper chiral subband has a very small width, \( \sim 14 \) times smaller than the chiral gap. One can quantify the band flatness by a figure of merit \( M \) given by the ratio of the gap to the band width. Its dependence on various parameters of the Hamiltonian is shown in Fig. [4]. It peaks at \( M > 40 \), but the more striking observation is that it is above 5 or even 10 for wide ranges of \( U, \Delta_{JT} \) and \( t_{dd} \), in contrast to many other proposals that require carefully fine-tuned parameters [18] [20] [23] [25] [43]. Nearly flat chiral bands are thus very robust in this system and both their topological character and their flat dispersion emerge spontaneously with purely onsite interaction and short-range hopping, without spin-orbit coupling or any explicit breaking of time-reversal symmetry.

**Mapping to an effective model.**— For large onsite interactions and large crystal field splitting \( U, J, |\Delta_{JT}| \gg t, t_{dd} \), the three-orbital model with fillings between 2 and 3 electrons per site can be mapped onto the one-band
Kondo-lattice model (KLM). Low-energy configurations minimize onsite interactions and thus contain two or three electrons per site, with parallel spins due to Hund’s rule. In order to additionally minimize the crystal-field energy, the \( e'_g \) levels will always be half filled and form an effective spin, while any holes will be found in the \( a_{1g} \) sector. The electrons in the partially filled \( a_{1g} \) states can delocalize with an isotropic hopping \( t_{a1g} = (2t + t_{dd})/3 \), however, their spin must remain parallel to the local \( e'_g \) spin. In the low-energy limit, each site can thus be described as a spin coupled to a charge degree of freedom and we arrive at the situation described by the KLM in the limit of strong Hund’s rule coupling. Our numeric mean-field results corroborate this picture, see Fig. 2(c), where the \( e'_g \) levels are found far below the chemical potential. The KLM supports spin-chiral phases on many frustrated lattices like the triangular \cite{38, 40, 41, 45}, pyrochlore \cite{40}, and face-centered cubic \cite{47} lattices.

In addition to processes within the low-energy Hilbert space, virtual excitations involving high-energy states can be taken into account in second-order perturbation theory. This leads to (i) effective longer-range hopping of the \( a_{1g} \) electrons and (ii) an effective antiferromagnetic superexchange between the \( e'_g \) spins. The latter stabilizes the spin-chiral pattern \cite{45} and is due to excitations into the upper Hubbard/Kondo band. When it is suppressed for \( U \gtrsim 24|t| \), the ground state consequently becomes FM, as in the KLM with a large Kondo gap \cite{40, 44}. Nevertheless, the exotic spin-chiral state is remarkably stable in the present \( t_{2g} \) system considering its sensitivity to Hund’s coupling in the KLM \cite{40}.

The effective longer-range hopping of \( a_{1g} \) electrons involves processes via excitations into the upper Kondo/Hubbard band \((\propto 1/J \text{ and } \propto 1/U)\) as well as virtual excitations of \( e'_g \) electrons into \( a_{1g} \) states \((\propto 1/\Delta_{JT})\), for details see \cite{35}. Second-neighbor hopping \( \propto 1/J \) does not significantly modify the low-energy bands and drops out completely in the limit of a large Mott/Hubbard gap, but third-neighbor hopping \( t_3 \) is crucial in cancelling the dispersion coming from NN hopping \( t_1 \) for one of the bands \cite{45}. The simplest description of the effective low-energy bands around the Fermi level is thus

\[
H_{\text{eff}}(\mathbf{k}) = 2t_1 \sum_j \sigma^j \cos k a_j + 2t_3 \sum_j \sigma^0 \cos 2k a_j ,
\]

where \( a_j (j = 1, 2, 3) \) denote the unit vectors on the triangular lattice. Pauli matrices \( \sigma^j \) and unit matrix \( \sigma^0 \) refer to the two sites of the electronic unit cell in the chiral phase \cite{38}. Formally, this describes electrons moving in a constant (and very strong) magnetic field with a flux of \( \pi/2 \) threading each triangle of the lattice \cite{38}.

**FQH groundstates of an effective spinless one-band model.**— We now address the impact of NN Coulomb interaction \( V \sum_{\langle i,j \rangle} n_i n_j \) on the fractionally filled flat band. The spin-chiral state can only be expected to remain stable for densities close to 2.5 electrons per site, i.e., low doping factions \( \nu \) of the flat band \cite{40}. FQH states corresponding to such low fillings are generally separated from the rest of the spectrum by only a small gap, making their analysis on finite-size clusters difficult \cite{48}. Here, we use Lanczos exact diagonalization \cite{20, 39, 43, 48} to study a number of simple filling fractions \((1/3, 2/3, 1/5, \text{ and } 2/5)\) available on accessible lattices and consistently find \( V \) to induce signatures of a FQH state. It is thus plau-
sible that the FQH behavior discussed next persists to fractional fillings in a low doping range of the spin-chiral state.

As an example, we present here the case of 16 electrons on a $4 \times 6$-site cluster of the model Eq. (1), a filling that would correspond to 2.6 in the original three-orbital model. After a particle-hole transformation, it corresponds to 2/3 filling of the nearly flat band. Figure 1(a) shows that with increasing $V$, three low-energy states split off from the rest of the spectrum. Inserting a magnetic flux $\phi_y = 2\pi$ interchanges the three states, $\phi_y = 6\pi$ recovers the original situation, see Fig. 1(b), as reported for other systems \[20, 23–26, 43\]. However, an experimental realization appears for NN bonds along the three directions $a_1$, $a_2$, $a_3$ as illustrated in Fig. 1 of the main text. The transformation into the \{e_{g,+}, e'_{g,+}, e_{g,-}\} can be found, e.g., in Ref. \[33\].

**Conclusions.** The possibility of a spontaneous FQH effect without a magnetic field is currently hotly discussed, and various models have been suggested \[13–20, 23–26\]. However, an experimental realization appears for wide parameter ranges in strongly correlated $t_{2g}$ orbitals on a triangular lattice, and that these bands support FQH ground states. Both $t_{2g}$ systems and triangular lattices occur in various TM oxides, and signatures of the unconventional integer QH state have been reported for a triangular-lattice palladium-chromium oxide \[51\]. This harbors the prospect that a suitable material can be synthesized in this highly versatile material class. As such a material is by default strongly correlated, one also naturally expects an inter-site Coulomb repulsion that is strong enough to stabilize spontaneous FQH states in the absence of a magnetic field.

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**The appendix contains Supplemental Material:**

**One-particle terms of the multi-orbital $t_{2g}$ Hamiltonian**

The multi-orbital kinetic energy is

$$H_{\text{kin}} = \sum_{\langle ij \rangle, \alpha, \beta, \sigma} t_{ij}^{\alpha, \beta} c_{i, \alpha, \sigma}^\dagger c_{j, \beta, \sigma} + \text{H.c.},$$

where $c_{i, \alpha, \sigma}^\dagger$ ($c_{i, \alpha, \sigma}$) creates (annihilates) an electron on site $i$, in orbital $\alpha$, and with spin $\sigma$. $\langle ij \rangle$ denotes nearest-neighbor (NN) bonds, $\alpha$ and $\beta$ denote the orbital. Using as basis states the $xy$, $xz$, and $yz$ orbitals shown in Fig. 1(a) of the main text, the orbital- and direction-dependent hopping parameters $t_{ij}^{\alpha, \beta}$ are given by the matrices

$$\hat{T}_1 = \begin{pmatrix} t_{dd} & 0 & 0 \\ 0 & t_0 & 0 \\ 0 & 0 & t_0 \end{pmatrix}, \quad \hat{T}_2 = \begin{pmatrix} 0 & 0 & t \\ 0 & t_0 & 0 \\ t & 0 & 0 \end{pmatrix}, \quad \hat{T}_3 = \begin{pmatrix} 0 & t & 0 \\ t & 0 & 0 \\ 0 & 0 & t_{dd} \end{pmatrix}$$

for NN bonds along the three directions $a_1$, $a_2$, $a_3$ as illustrated in Fig. 1 of the main text. The transformation into the \{e_{g,+}, e'_{g,+}, e_{g,-}\} can be found, e.g., in Ref. \[33\].

**Mean-field approximation**

Onsite interaction is described by Kanamori parameters $U$ ($U'$) for Coulomb repulsion between electrons in the same (different) orbitals as well as ferromagnetic Hund’s-rule coupling between electrons in different orbitals. The relation $U' = U - 2J$ is used here, “pair-hopping” $J'$ is left out, because it drops out of the mean field decoupling

$$H_{\text{int}} \approx U \sum_{i, \alpha} \langle n_{i, \alpha, \uparrow} n_{i, \alpha, \downarrow} + n_{i, \alpha, \downarrow} n_{i, \alpha, \uparrow} \rangle$$

$$+ \left[ U' - J/2 \right] \sum_{i, \alpha < \beta} \langle n_{i, \alpha} n_{i, \beta} + n_{i, \beta} n_{i, \alpha} \rangle$$

$$- 2J \sum_{i, \alpha < \beta} \langle n_{i, \alpha} n_{i, \beta} + n_{i, \beta} n_{i, \alpha} \rangle$$

$$- U \sum_{i, \alpha} \langle n_{i, \alpha, \uparrow} \rangle - \left[ U' - J/2 \right] \sum_{i, \alpha < \beta} \langle n_{i, \alpha} \rangle$$

$$+ 2J \sum_{i, \alpha < \beta} \langle n_{i, \alpha} \rangle \langle n_{i, \beta} \rangle, \quad (3)$$

where $i$ labels the site, $\alpha$ and $\beta$ orbitals. $n_{i, \alpha, \sigma} = \langle c_{i, \alpha, \sigma}^\dagger c_{i, \alpha, \sigma} \rangle$ is the density operator. We keep here only expectation values for diagonal operators, i.e., only $\langle n_{i, \alpha, \sigma} \rangle = \langle c_{i, \alpha, \sigma}^\dagger c_{i, \alpha, \sigma} \rangle \[37, 38\]$ which reduces the spin to its $z$-component $m_{i, \alpha} = \langle n_{i, \alpha, \uparrow} - n_{i, \alpha, \downarrow} \rangle/2$. In order to treat non-collinear spin patterns, one has to allow for a site-dependent spin-quantization axis given by angles $\theta_i$ and $\phi_i$. The change in quantization axis from site to site manifests itself in a complex phase for the hopping terms, \[39\] which is between sites $i$ and $j$

$$\Omega_{ij}^{\alpha, \sigma} = \cos \frac{\theta_j}{2} \cos \frac{\theta_i}{2} + \sin \frac{\theta_j}{2} \sin \frac{\theta_i}{2} e^{-i\sigma(\phi_i - \phi_j)}$$

$$\Omega_{ij}^{\uparrow, \downarrow} = \frac{\theta_j}{2} \sin \theta_i e^{-i\phi_j} - \cos \frac{\theta_j}{2} \sin \theta_i e^{-i\phi_i}. \quad (4)$$

where $\Omega_{ij}^{\alpha, \sigma}$ ($\Omega_{ij}^{\uparrow, \downarrow}$) modulates the hopping of an electron with spin parallel (antiparallel) to the chosen spin-quantization axis. In the site-dependent quantization,
spin is not conserved and there are spin-mixing hoppings with $\Omega_{ij}^{\uparrow,\downarrow}$ given by the complex conjugate of $\Omega_{ji}^{\uparrow,\downarrow}$.

We use numerical optimization routines to find the spin pattern with the lowest energy among all orderings with unit cells of up to four sites, including all patterns considered in Ref. [40] of the main text. In each step, the mean-field energy is calculated self-consistently for a lattice of $16 \times 16$ (four-site unit cell) or $24 \times 16$ (three-site unit cell). (For selected points in parameter space, we also used larger lattices and did not find a significant difference.) In order to minimize the impact of our approximations on the symmetries of the orbital degrees of freedom, we perform the mean-field decoupling in the $\{a_{1g}, e_g', e_g''\}$ basis, where the symmetry between the half-filled $e_g'$ and the quarter-filled $a_{1g}$ orbitals (for the fillings discussed here) is already broken by the crystal field. We verified that decoupling directly in the $\{xy, xz, yz\}$ basis, where all three orbitals have the same electronic density, leads to qualitatively identical and quantitatively very similar results.

**Effective one-band model and exact diagonalization**

The mapping to the effective one-band model is most easily carried out in the Kondo-lattice picture, where the localized spins are assumed to consist of the $e_g'$ electrons. Without a magnetic order, the $a_{1g}$ orbital has an isotropic hopping $t_{a1g} = (2t + t_{dd})/3$, i.e., the same along all three directions on the triangular lattice, but in the spin-chiral phase, this hopping is modulated by a direction-dependent Berry phase Eq. (4). The electronic unit cell of the spin-chiral pattern has two sites $[38]$, and the Berry phases can then be expressed in terms of Pauli matrices as given in the main text. The absolute value of the NN hopping is renormalized to $t_1 = (2t + t_{dd})/3\sqrt{3}$.

Corrections to this simplest approximation can be obtained by second-order perturbation theory, which yields longer-range hopping processes mediated by virtual excitations. We are first going to discuss processes within the lower Kondo/Hubbard band, where an electron from the $e_g'$ levels, which are half filled in the low-energy Hilbert space, is excited into an empty $a_{1g}$ state in the virtual intermediate state, involving an excitation energy $\Delta_{1T}$. In a second step, an electron from a different occupied $a_{1g}$ state can take the empty place in the $e_g'$ orbital, which corresponds to an effective hopping. There are two possible hopping paths connecting pairs of either nearest or next-nearest neighbor (NNN) sites and it turns out that the corresponding effective hoppings drop out in the spin-chiral phase because the Berry phases for the two paths interfere destructively. Third-neighbor sites, on the other hand, are only connected by a single path and the combined Berry phase $\Omega_{ij}^{\uparrow,\uparrow}\Omega_{jk}^{\downarrow,\downarrow}$ only renormalizes the effective hopping by a factor of 3, because the spins at sites $i$ and $k$ are parallel in the spin-chiral phase. One thus obtains an effective hopping $t_3 = -2(t - t_{dd})^2/(27\Delta_{1T})$, which is the only second-order correction for the limit of infinite Hund’s rule. In this limit, $t_3$ flattens the lower subbands for the fillings discussed here $[24]$.

In the more realistic case of strong but finite Hund’s rule coupling, there are additional processes where the virtual excitation involves an electron in the upper Hubbard/Kondo band. The corresponding excitation energies then contain Hund’s coupling $J$ and the effective hoppings from site $i$ to $k$ involve the Berry phases $\Omega_{ij}^{\uparrow,\uparrow}\Omega_{jk}^{\downarrow,\downarrow}$, see Eq. (4). For the parameters of Fig. 2 of the main text, the chiral bands around the Fermi level are compared to this second-order treatment in Fig. 5. NNN hopping does here not drop out, and NN hopping is also slightly renormalized, however, both these processes have only a small impact on the low-energy bands. Again, we find third-neighbor hopping $t_3$ to flatten the dispersion of one of the bands. Since excitations into the upper Hubbard/Kondo band involve electrons rather than holes as before, the sign of the effective $t_3$ is reversed, and it is the upper chiral subband that is flattened.

In order to investigate the FQH groundstate, we used Lanczos exact diagonalization to study the Hamiltonian given by the kinetic energy of the effective one-band model, Eq. (1) of the main text, and NN Coulomb repul-
sion \( V \sum_{(i,j)} n_i n_j \). Hopping parameters \( t_1 = 0.27 \) and \( t_3 = -0.058 \) were used, giving a dispersion similar to Fig. 3. NN bonds \((i,j)\) are defined on the original triangular lattice and \(V\) acts both between the two sites within one unit cell and between NN sites belonging to different unit cells. As mentioned in the main text, the spin-chiral state can actually be expected to be stable for low doping of the flat band [10], which is close to half filling for the effective one-band model. FQH states corresponding to such small fillings \( \nu \) tend to have smaller gaps than those for large \( \nu \), and the low-energy manifold giving the quasi-degenerate FQH states contains more states. On the small clusters that we can study with exact diagonalization, eigenenergies always have spacings between them, as an illustration see the \( V = 0 \) energies in Fig. 4(a) of the main text, which would form a continuous band in the thermodynamic limit. It is thus far harder to reliably resolve a small gap than a larger one, and it is moreover highly desirable that we can study the system on at least two lattice sizes in order to see a gap. This severely restricts our access to very low dopings. We thus study several filling fractions corresponding to “simple” FQH states. In all cases where we find a low-energy manifold to separate from the rest of the spectrum, the states of this low-energy manifold shows signatures of FQH behavior, which is thus also a very robust feature of the doped flat band.

Inserting a flux \((\phi_x, \phi_y)\) means that electrons gain a phase \( e^{i\phi_x} (e^{i\phi_y}) \) for going once around the whole lattice in \((x,y)\) direction. This is implemented by changing the hopping \( t_{i,j} \) from site \( \mathbf{i} = i_x \mathbf{a}_1 + i_y \mathbf{a}_2 \) to site \( \mathbf{j} = j_x \mathbf{a}_1 + j_y \mathbf{a}_2 \) to

\[
t_{i,j} \to t_{i,j} e^{i \left( \frac{\phi_x}{x} \right) + i \left( \frac{\phi_y}{y} \right)},
\]

leading to a flux-dependent Hamiltonian \( H_{\text{eff}}(\phi_x, \phi_y) \). In the case of \( \nu = 1/3 \), we find three low-energy states separated from the remaining spectrum by a gap as in \( \nu = 2/3 \) case discussed in the main text, both on a \( 6 \times 6 \) and a \( 4 \times 6 \) system. For \( 6 \times 6 \) sites, however, all three low-energy states have total momentum \((0,0)\) for \((\phi_x, \phi_y) = (0,0)\). Due to finite-size effects, the states do not then cross upon flux insertion [18], but avoid crossings. For the smaller \( 4 \times 6 \) system, the three low-energy states have different total momenta, and this good quantum number allows us to clearly resolve their crossing when we insert a flux \( \phi_y \), even on a finite system.

The Chern numbers were evaluated by integrating the Berry curvature \( \Omega^{\nu}(\phi_x, \phi_y) \) over the square \( 0 \leq \phi_x, \phi_y < 6\pi \). \( \Omega^{\nu}(\phi_x, \phi_y) \) was obtained by the Kubo formula [49, 50].

\[
\Omega^{\nu}(\phi_x, \phi_y) = i L_x L_y \sum_{n' \neq n} \frac{\partial H_{\text{eff}}(\phi_x, \phi_y)}{\partial \phi_x} |n\rangle \langle n'| \frac{\partial H_{\text{eff}}(\phi_x, \phi_y)}{\partial \phi_y} |n\rangle - \langle n| \frac{\partial H_{\text{eff}}(\phi_x, \phi_y)}{\partial \phi_y} |n'| \langle n'| \frac{\partial H_{\text{eff}}(\phi_x, \phi_y)}{\partial \phi_x} |n\rangle}{(\epsilon_n - \epsilon_{n'})^2},
\]

where \( n' \) and \( n \) label eigenstates with energies \( \epsilon_{n'}/n \) and \( \partial H_{\text{eff}}(\phi_x, \phi_y)/\partial \phi_{x/y} \) are currents.

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