ABELIAN LANDAU–GINZBURG ORBITOLDS AND MIRROR SYMMETRY

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ABSTRACT

We construct a class of Heterotic String vacua described by Landau–Ginzburg theories and consider orbifolds of these models with respect to abelian symmetries. For LG–vacua described by potentials in which at most three scaling fields are coupled we explicitly construct the chiral ring and discuss its diagonalization with respect to its most general abelian symmetry. For theories with couplings between at most two fields we present results of an explicit construction of the LG–potentials and their orbifolds. The emerging space of (2,2)–theories shows a remarkable mirror symmetry. It also contains a number of new three–generation models.

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1 Introduction

The question of the structure of the configuration space of string theory is an important one for theoretical as well as practical reasons. Unfortunately there are few well developed tools available for a general analysis of this space from first principles. Instead much of the insight gained over the last years stems from explicit constructions of string vacua. These use a variety of methods ranging from lattice techniques and exactly solvable models to mean field theory and algebraic geometry.

Recently techniques from Landau–Ginzburg mean field theory have been utilized to construct a set of several thousand consistent Heterotic vacua [1]. This construction extends the number of known vacua by an order of magnitude and provides a large enough slice of the moduli space to expose an important property of this space, its mirror symmetry. Considering that these models have been constructed as completely independent LG theories this provides strong evidence that the space of left–right symmetric Heterotic String vacua indeed features mirror symmetry.

An a priori independent technique of constructing Heterotic vacua was pursued in refs. [2][3][4]. The starting point of those papers is the set of exactly solvable N=2 superconformal tensor models [5] constructed explicitly in [3][6]. These models always have discrete symmetries and hence it is possible to consider orbifolds of any of these tensor models by modding out any of the subgroups of their symmetries. It was observed in refs. [4][3][2] that in some cases this orbifolding procedure produces mirror pairs.

It turns out that these two modes of construction are not completely independent. It was shown in ref. [3] that certain classes of mirror pairs of vacua can be related via a process involving two steps: first a LG–vacuum is orbifolded and then the order parameters of the LG–potential are transformed into new fields with a nonlinear transformation involving fractional powers. This technique can be applied not only to mean field theories associated to the exactly solvable models but also to the much more general class constructed in [1]. Its application is not restricted to mirror pairs but is completely general, depending only on the type of symmetry considered. Hence this result suggests that a general relation might exist between Landau–Ginzburg potentials and their orbifolds. To investigate this question further it is clearly useful to consider orbifolds of the tensor models and, more generally, LG theories in a systematic way.

Our results indeed show that there is substantial overlap between the Landau–Ginzburg theories constructed in [1] and our orbifolds. Figure 1 shows a plot of the difference between the number of generations and antigenerations versus the sum of these numbers for the class of LG theories we constructed and all their orbifolds with respect to phase symmetries. Similar to the results in [1] the

1Here we consider those vacua as distinct that have a different number of generations or antigenerations; this is a very rough measure since it does not take into account the Yukawa couplings between these fields.

2An analysis of mirror orbifolds of Calabi–Yau manifolds of Fermat type in weighted IP4 has been performed in [7].
diagram shows a remarkable symmetry with respect to the exchange \( n_g \leftrightarrow \bar{n}_g \). Even though our implementation is not complete already 94\% of the Hodge pairs have mirror partners.

A second motivation for our work is the fact that it is surprisingly difficult to find ground states of the Heterotic String theory that accommodate the Standard Model in a painless manner. Despite all interest in the general structure of the configuration space the search for realistic models remains an important challenge. Knowledge of the general structure is, after all, aimed toward a mechanism to lift the degeneracy of the groundstates and hence a much more ambitious goal.

In the present paper we construct a class of Landau–Ginzburg theories which contains the class of minimal tensor models as a small subset. We work out the formulae needed for the evaluation of the number of 27 and \( \overline{27} \) \( E_6 \)-representations for any abelian orbifold of a large set of LG–vacua. Then we proceed to orbifold them with respect to abelian symmetries with determinant 1. These theories correspond to supersymmetric orbifolds of Calabi–Yau spaces.

This paper is organised as follows: In section 2 we briefly review the results of Vafa and Intriligator [9] [10] on the construction of LG orbifolds. In section 3 we determine the local algebra, which corresponds to the chiral ring of the SCFT, for a class of quasihomogeneous singularities. In section 4 we define the class of models which we considered in our explicit constructions. In section 5 the diagonalization of the chiral ring with respect to its most general discrete abelian symmetry and the eigenvalues and dimensions of the eigenspaces are computed. Section 6 contains some general considerations about the symmetries we have implemented. In section 7 we present our results for phase symmetries and in section 8 for cyclic symmetries. Finally we present our conclusions.

## 2 Landau – Ginzburg orbifolds

The Landau–Ginzburg description of an \( N = 2 \) SCFT [11] is determined by an action of the form

\[
\int d^2z d^4\theta K(\Phi_i, \bar{\Phi}_i) + \left( \int d^2z d^2\theta W(\Phi_i) + \text{c.c.} \right),
\]

where the superpotential \( W \) is quasihomogeneous of degree \( d \) in the chiral superfields \( \Phi_i(z, \bar{z}, \theta^\pm, \bar{\theta}^\pm) \) of weight \( k_i \)

\[
W(\lambda^k \Phi_i) = \lambda^d W(\Phi_i)
\]

with an isolated singularity at \( \Phi_i = 0 \). The central charge of the superconformal theory is given by the highest weight

\[
\hat{c} = \frac{c}{3} = \sum_i (1 - 2q_i)
\]

with \( q_i = \frac{k_i}{d} \). Its chiral ring is isomorphic to what mathematicians call the local algebra of the non-degenerate quasihomogeneous function \( W(z_i) \), defined as the ring of all polynomials in some complex
variables $z_i$ modulo the ideal generated by $dW/dz_i$ [12]. In the present context the $z_i$ denote the constant values of the lowest components of the chiral superfields. The zero locus of $W$ defines a complex variety in some complex space $\mathbb{C}^n$. We will use a short hand and denote the space of such polynomials by

$$\mathbb{C}(k_1,k_2,\ldots,k_n)[d]$$

and call it a configuration. For the computation of the spectrum of the LG theory it is not important to know the precise form of the polynomial; only the set of weights $k_i$ is important as well as the fact that the configuration does have a member with an isolated singularity. The nondegeneracy of $W$ implies that the local algebra (and therefore the chiral ring) is finite dimensional. The Poincaré polynomial $P(t)$ is defined as the generating function for the number of basis monomials of the local algebra of a specific degree of quasihomogeneity, i.e. the number of states of a given conformal weight. It can be computed with the formula

$$P(t) = \prod \frac{(1 - t^{1-q_i})}{(1 - t^{q_i})}.$$  

Note that this expression is not a polynomial in $t$ but rather in $t^{1/d}$. For convenience, however, we will refer to $P(t)$ and not to $P(t^d)$ as the Poincaré polynomial.

If we want to use an $N=2$ superconformal theory for constructing a string vacuum with $N=1$ space-time supersymmetry, we require $\hat{c} = 3$ and integral $U(1)$ charges. By orbifolding an LG theory it is possible to obtain a theory with integral charges. One way to achieve this is to orbifoldize the theory with respect to the $U(1)$ symmetry of the $N = 2$ superconformal algebra. Since we are considering only rational theories all the fields have rational charges and hence this $U(1)$ projection translates, in the mean field description of the superconformal theory, into an orbifolding with respect to the $\mathbb{Z}_d$ symmetry of the superpotential $W(\Phi_i)$. In this case the numbers of states with charges $(q_L,q_R)$ are given [9] by the coefficients of $t^{q_L}\bar{t}^{q_R}$ in

$$P(t,\bar{t}) = \text{tr } t^0\bar{t}^0 \sum_{0 \leq i < d, \tilde{\theta}_i \in \mathbb{Z}} \prod_{\tilde{\theta}_i \in \mathbb{Z}} \frac{1 - (tt)^{1-q_i}}{1 - (tt)^{q_i}} \prod_{\tilde{\theta}_i \notin \mathbb{Z}} (tt)^{\tilde{\theta}_i - \frac{1}{2}} \bigg|_{\text{int}},$$

where $\tilde{\theta}_i = \theta_i - [\theta_i]$ and $\theta_i = lq_i$ is the non-integer part of $lq_i$. The subscript int means that only integral powers of $t$ and $\bar{t}$ are kept in this expression. If we have a Calabi–Yau interpretation of our theory, these coefficients correspond to the Hodge numbers of the CY–manifold. For $\hat{c} = 3$

$$P(t,\bar{t}) = (1 + t^3)(1 + \bar{t}^3) + n_g(tt + t^2\bar{t}^2) + \bar{n}_g(t\bar{t}^2 + t^2\bar{t}),$$

where $n_g$ and $\bar{n}_g$ denote the numbers of 27 and $\bar{27}$ representations of $E_6$ occurring in the construction of the Heterotic String vacua. If it is possible to give masses to all $(27,\bar{27})$ pairs, then the Euler number $\chi = 2(n_g - \bar{n}_g)$ is twice the net number of fermion generations.

Of course, if our potential $W$ has more symmetries we are free to orbifoldize with respect to any of them, and if certain constraints are imposed then a consistent vacuum is obtained [10]. The formula
given above has to be modified because, unfortunately, the Poincaré polynomial does not contain the information on the transformation properties of the states under general symmetries. So for genuine orbifolds we need to rely on an explicit basis of the chiral ring. The expression for the left and right charges of a state $\prod X_i^{\lambda_i}|0\rangle_{NS}$ in some twisted sector, however, remains valid,

$$q_\pm = \sum_{\tilde{\theta}_i > 0} \left(\frac{1}{2} - q_i \pm (\tilde{\theta}_i - \frac{1}{2})\right) + \sum_{\tilde{\theta}_i = 0} \lambda_i q_i$$

with $\theta_i$ now being the phase of the $i^{th}$ field in a diagonal basis under the action of the group element defining the twist. Note that in general the twisted vacua have nontrivial transformation properties under all symmetries (see [10]).

3 Local algebra of quasihomogeneous functions

In this section we work out the local algebra for all nondegenerate quasihomogeneous functions of three or less complex variables. Since our explicit construction of potentials with $c = 9$ and the implemention of symmetries discussed in later sections covers only models with couplings of up to two superfields the reader who is more interested in the results for the emerging string vacua rather than the general theory may wish to skip some of the details in the present section.

The nondegenerate quasihomogeneous functions of three or less complex variables have been classified in the mathematical literature [12]. They are sums of functions of the form

(I) $z_1^{a_1} z_2 + \cdots + z_n^{a_n} z_n + z_n^{a_n}$,

(II) $z_1^{a_1} z_2 + \cdots + z_n^{a_n} z_n + z_n^{a_n} z_1$,

(III) $z_1^a z_2 + z_2^b + z_2 c + \epsilon z_1^p z_3^q$,

(IV) $z_1^a z_2 + z_2^b z_3 + z_2 c + \epsilon z_1^p z_3^q$,

where the first two types represent nondegenerate quasihomogeneous functions for any $n \geq 1$ or 2.

The variables in a Type (I) function have degrees of quasihomogeneity

$$q_i = \frac{1}{a_i} - \frac{1}{a_i a_{i+1}} + \cdots + (-1)^{n-i} \frac{1}{a_i a_{i+1} \cdots a_n}. \quad (13)$$

Using the abbreviations $x_i := t^{q_i}$, the Poincaré polynomial fulfills the recursion relation

$$P_n(t; a_1, \cdots, a_n) = \frac{x_1^{q_1} - 1}{x_1 - 1} \frac{x_2^{q_2} - 1}{x_2 - 1} \cdots \frac{x_n^{q_n} - 1}{x_n - 1} + x_1^{q_1} P_{n-2}(t; a_3, \cdots, a_n) \quad (14)$$

with $P_1(t; a) = (x^a - 1)/(x - 1)$ and $P_0(t) = 1$. The local algebra is determined by the equations

$$a_1 z_1^{a_1} z_2 = z_1^{a_1} + a_2 z_2^{a_2} z_3 = \cdots = z_n^{a_n-1} + a_n z_n^{a_n-1} = 0. \quad (15)$$
The monomials $\prod z_i^{\alpha_i}$ with $\alpha_1 \leq a_1 - 2$ and $\alpha_i \leq a_i - 1$ for all other $i$’s are nonvanishing and independent, and all other monomials with $\alpha_1 \leq a_1 - 2$ or $\alpha_1 \geq a_1$ can be written as linear combinations of them. Monomials with $\alpha_1 = a_1 - 1$ can only be nonvanishing if $\alpha_2 = 0$. They are independent if and only if the other exponents correspond to a chiral ring of the same type for $z_3, \cdots, z_n$, as the form of the Poincarè polynomial suggests. The highest weight is $\hat{c} = n - 2 \sum q_i$ with

$$\sum_{i=1}^{n} q_i = \sum_{i=1}^{n} \frac{1}{a_i} - \sum_{i=1}^{n-1} \frac{1}{a_i a_{i+1}} + \sum_{i=1}^{n-2} \frac{1}{a_i a_{i+1} a_{i+2}} + \cdots + (-1)^{n-1} \frac{1}{a_1 a_2 \cdots a_n}. \quad (16)$$

For **Type (II)** the degrees are given by

$$q_i (1 + (-1)^{n-1} a_1 \cdots a_n) = 1 - a_{i-1} + a_{i-1} a_{i-2} - a_{i-1} a_{i-2} a_{i-3} + \cdots + (-1)^{n-1} a_{i-1} a_{i-2} \cdots a_{i+1}, \quad (17)$$

where the indices are to be understood modulo $n$. With the same abbreviations as before, the Poincarè polynomial is

$$P(t) = \frac{x_1^{a_1} - 1 x_2^{a_2} - 1}{x_1 - 1} \frac{x_3^{a_3} - 1}{x_2 - 1} \cdots \frac{x_n^{a_n} - 1}{x_{n-1} - 1}. \quad (18)$$

The local algebra is determined by the equations

$$z_i^{a_i-1} + a_i z_i^{a_i} z_{i+1} = 0. \quad (19)$$

The monomials $\prod z_i^{\alpha_i}$ with $\alpha_i \leq a_i - 1$ form a basis.

The analysis for the types (III) and (IV), for which not all coefficients can be normalized to 1, is more complicated.

The degrees in **Type (III)** are

$$q_1 = \frac{1}{a} \left( 1 - \frac{1}{b} \right), \; q_2 = \frac{1}{b}, \; q_3 = \frac{1}{c} \left( 1 - \frac{1}{b} \right), \quad (20)$$

yielding

$$\sum 1 - 2q_i = 3 - 2 \frac{ac + (b-1)(a+c)}{abc} = (b-1) \frac{2ac + (p-2)c + (q-2)a}{abc}. \quad (21)$$

for the highest weight. Quasihomogeneity of $z_1^p z_3^q$ implies

$$\left( \frac{p}{a} + \frac{q}{c} \right) \left( 1 - \frac{1}{b} \right) = 1, \; \frac{p}{a} + \frac{q}{c} = 1 + \frac{1}{b-1}. \quad (22)$$

Therefore, in order to allow a suitable last term for given values of $a$ and $c$, $b-1$ must be a divisor of the least common multiple of $a$ and $c$. This is exactly the condition for the expression (3) for the Poincarè polynomial to be a polynomial. With $x_1 = t^{c(b-1)}$, $x_2 = t^{ac}$, $x_3 = t^{a(b-1)}$ and $T = x_1^a = x_3^c = t^{ac(b-1)}$, one can show that the Poincarè polynomial is determined by

$$P(t) \simeq \frac{x_1^{a-1} - 1 x_2^{b-1} - 1 x_3^{c-1} - 1}{x_1 - 1} + \frac{x_1^{a-1} - 1}{x_2 - 1} \frac{1}{x_3 - 1} (1 + T + T^2) x_3^{c-1} + \frac{x_1^{a-1} - 1 x_2^{b-1} - 1}{x_1 - 1} \frac{1}{x_3 - 1} (1 + T + T^2) x_3^{c-1} + \frac{x_1^{a-1} - 1 x_2^{b-1} - 1}{x_1 - 1} \frac{1}{x_3 - 1} (2T + T^2 + T^3 - T x_1^{p-1} x_3^{q-1}), \quad (23)$$

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where $\simeq$ stands for “equality up to the highest weight”, i.e. on the r.h.s. we neglect powers of $t$ that are higher than in the highest weight term $T^2 x_1^{p-2} x_3^{q-2}$. The local algebra is determined by the equations

$$a z_1^{a-1} z_2 + e p z_1^{p-1} z_3^q = z_1^a + bz_2^{b-1} + z_3^c = cz_2 z_3^{c-1} + eq z_1^{p-1} z_3^{q-1} = 0. \quad (24)$$

We construct a basis of the local algebra in the following way: The first step is “$z_2$ elimination”: While possible, we use the second and then the first and third equation in order to substitute monomials by polynomials which contain smaller powers of $z_2$. The remaining $z_2$-dependent monomials are of the form $z_1^a z_2^b z_3^c$ with $0 \leq \alpha \leq a - 2, 1 \leq \beta \leq b - 2$ and $0 \leq \gamma \leq c - 2$. They are obviously represented by the first term in our expression for the Poincaré polynomial. All other basis monomials can be chosen of the form $z_1^a z_3^c$. Eliminating $z_2$ in a combination of the first and third equation, we obtain

$$e p z_1^{p-1} z_3^{c+q-1} = a q z_1^{a+p-1} z_3^{q-1}. \quad (25)$$

This restriction on the independence of different monomials is related to the last term in eq. (23). Further equations are found by combining the first or last two equations:

$$z_1^{2a-1} + z_1^{a-1} z_3^c = a^{-1} e p z_1^{p-1} z_2^{b-2} z_3^q, \quad (26)$$

$$z_1^a z_3^{c-1} + z_3^{2c-1} = c^{-1} e b q z_2^{p-1} z_3^{q-1}, \quad (27)$$

where the r.h.s.’s are still subject to “$z_2$ elimination”. The arbitrariness of $\epsilon$ permits us to neglect terms proportional to $\epsilon$ in the explicit construction of some basis of the algebra, once eq. (23) is derived. One thus shows that a completion of the basis is given by $z_1^a z_3^c$ with $(\alpha, \gamma) \in [0, a - 2] \times [0, 2c - 2] \cup [a - 1, a + p - 2] \times [0, c - 1] \cup [a + p - 1, 2a + p - 2] \times [0, q - 2]$ for $p \geq a$ or $q \geq c$, and with $(\alpha, \gamma) \in [0, p - 2] \times [0, 2c - 2] \cup [p - 1, a + p - 2] \times [0, c + q - 2] \cup [a + p - 1, 2a - 2] \times [0, c - 2]$ otherwise. This is true for all $\epsilon \in \mathbb{C}$ except for a finite number of values for which the singularity is degenerate. Note that equations (26) and (27) are not completely independent. Multiplying the first of them with $z_3^{c-1}$ yields, after $z_2$ elimination, the same as multiplying the second one with $z_1^{a-1}$. Even taking this into account, when we go to high degrees, we have more restrictions than there are monomials. If these restrictions were not redundant, the $x_2$-independent part of the Poincaré polynomial would be (exactly)

$$\frac{1}{1-x_1} \frac{1}{1-x_3} (1 - T x_1^{a-1} - T x_3^{c-1} + T x_1^{a-1} x_3^{c-1} - T x_1^{p-1} x_3^{q-1}), \quad (28)$$

with $x_1$ and $x_3$ interpreted as counting $z_1$ and $z_3$, respectively. This expression is equal to the one given above except for terms exceeding the highest weight. We conclude that the interpretation of the Poincaré polynomial in the form (23) or (28) is unique up to $T = x_1^a = x_3^c = t^{ac(b-1)}$; $x_1$ is interpreted as the formal variable for counting $z_1$, $x_3$ is interpreted as counting $z_3$.

The analysis of Type (IV) is similar to the previous case. The degrees are

$$q_1 = \frac{c b - 1}{a b c - 1}, \quad q_2 = \frac{c - 1}{b c - 1}, \quad q_3 = \frac{b - 1}{b c - 1}. \quad (29)$$
yielding the highest weight
\[
\sum 1 - 2q_i = 3 - 2\frac{ab + ac + bc - 2a - c}{a(bc - 1)} = (b - 1)\frac{2ac + (p - 2)c + (q - 2)a}{a(bc - 1)}.
\]
(30)

Quasihomogeneity of \(z_1^px_3^q\) implies
\[
\left(\frac{pc}{a} + q\right)\frac{b - 1}{bc - 1} = 1, \quad \frac{p}{a} + \frac{q}{c} = 1 + \frac{c - 1}{c(b - 1)}.
\]
(31)

Thus for given \(a\) and \(bc - 1\) has to be a divisor of \(\text{lcm}(a, c) \cdot (c - 1)/c\). With \(x_1 = t^{c(b - 1)}\), \(x_2 = t^{a(c - 1)}\), \(x_3 = t^{a(b - 1)}\) and \(T = x_1^a = x_3^c = t^{ac(b - 1)}\) the Poincarè polynomial is given by
\[
P(t) \simeq \frac{x_1^{a - 1} - 1}{x_1 - 1}x_2^{b - 1} - 1 \frac{x_3^c - 1}{x_3 - 1} + \frac{x_2^{b - 1}x_1^{a - 1} - 1}{x_1 - 1} + \frac{x_1^{a - 1} - 1}{x_1 - 1}(T + T^2)x_3^{c - 1}
\]
\[
+ \frac{x_3^{c - 1} - 1}{x_3 - 1}(1 + T + T^2)x_1^{a - 1} + (1 + T)x_1^{a - 1}x_3^{c - 1}
\]
\[
+ \frac{x_1^{a - 1} - 1}{x_1 - 1} \frac{x_3^{c - 1} - 1}{x_3 - 1}(2T + T^2 + T^3 - Tx_1^{a - 1}x_3^{q - 1}).
\]
(32)

The local algebra is determined by the equations
\[
a^\alpha z_1^{a - 1}z_2 + b^\beta z_2^{b - 1}z_3^c + c^\gamma z_3^{c - 1} + \epsilon dqz_1^{p - 1}z_3^{q - 1} = 0.
\]
(33)

Again the first step in the construction of a basis of the local algebra is “\(z_2\) elimination”. We end up with \(z_2\)-dependent monomials of the form \(z_1^\alpha z_2^\beta z_3^c\) with \(0 \leq \alpha \leq a - 2, 1 \leq \beta \leq b - 2\) and \(0 \leq \gamma \leq c - 1\) or with \(0 \leq \alpha \leq a - 2, \beta = b - 1\) and \(\gamma = 0\). They are represented by the first two terms in our expression for the Poincarè polynomial. With arguments similar to the ones used before, one can show that the part containing only \(z_1\) and \(z_3\) is the same as in the previous model.

4 Fermats, loops, and tadpoles

As already mentioned we have in our explicit construction of LG-potentials and their orbifolds focused on models described by superpositions of potentials in which at most two fields interact. In this section we thus specialize the above considerations to this case and describe how we obtained our list of 7579 different LG potentials.

We call a contribution \(\Phi^a\) to the potential Fermat type, whereas the simplest possible couplings look like
\[
W^{(L)} = \Phi^e \Psi + \Phi^f, \quad q_\Phi = \frac{f - 1}{ef - 1}, \quad q_\Psi = \frac{e - 1}{ef - 1}, \quad \hat{c}^{(L)} = 2\frac{(e - 1)(f - 1)}{ef - 1},
\]
(34)
\[
W^{(T)} = \Phi^e + \Phi^f, \quad q_\Phi = \frac{1}{e}, \quad q_\Psi = \frac{e - 1}{ef}, \quad \hat{c}^{(T)} = 2\frac{(e - 1)(f - 1)}{ef}.
\]
(35)
and are called loops and tadpoles, respectively.

The corresponding chiral rings are represented by linear combinations of monomials of the form $\Phi^i$, $0 \leq i \leq a - 2$ for Fermat type. A basis for loops and tadpoles is given by

\[
L : \{\Phi^i \Psi^j, \quad 0 \leq i \leq e - 1, \quad 0 \leq j \leq f - 1\}, \quad (36)
\]

\[
T : \{\Phi^i \Psi^j, \quad 0 \leq i \leq e - 1, \quad 0 \leq j \leq f - 2\} \cup \{\Psi^{f-1}\}. \quad (37)
\]

These formulae are of course contained in the results for type (I) and (II) in the previous section.

It is now straightforward to construct all possible combinations of these potentials which add up to a total central charge of $c \equiv 3\hat{c} = 9$. First we note that the accumulation points of possible contributions $\hat{c}$ are $2\frac{i-1}{i}$ with $2 \leq i \leq \infty$. If we define the finite set

\[
M_\epsilon = \{\hat{c}\} - (1 - \epsilon, 1) \cup (\frac{4}{3} - \epsilon, \frac{4}{3}) \cup (\frac{2}{3} - \epsilon, \frac{2}{3}) \cup (\frac{2}{3} - \epsilon, \frac{2}{3}) \cup (\frac{2}{3} - \epsilon, \frac{2}{3}) \cup (\frac{2}{3} - \epsilon, \frac{2}{3}), \quad (38)
\]

one can show that for $\epsilon = \frac{1}{12}$ at most one of the contributions to $\sum \hat{c}_i = 3$ can lie in the excluded open intervals ($M_{\frac{1}{12}}$ has 502 elements). Thus a computer program can produce all 830 solutions where members of $M_\epsilon$ add up to 3 or to 3 minus a possible value of $\hat{c}$ which lies in the excluded range. Most of these combinations of $\hat{c}$'s can originate from several (up to 90) different potentials of tadpole and Fermat type. The complete number of inequivalent models of this type (i.e. $a, e^{(T)} \geq 3, f^{(T)} \geq 2$ and $e^{(L)} \geq f^{(L)} \geq 2$) turns out to be 7579. $d$ in (3) is a multiple of all $a$'s, $(\frac{ef-1}{\text{gcd}(e-1,f-1)})^{(L)}$'s, and $(\frac{ef}{\text{gcd}(e-1,f)})^{(T)}$'s.

All these models are distinct from the orbifold point of view. They only correspond to 3112 different combinations of weights, however, which in turn give rise to some 1200 different Hodge pairs. Considering a specific set of weights, different points in the configuration space (3) can have different symmetries, leading to different possibilities for orbifoldizing. Even taking this into account, our analysis given here is complete for all potentials in which at most two fields are coupled: One can show that the potentials (36,37) represent the points of maximal symmetry in the respective moduli spaces.

## 5 Abelian symmetries of LG potentials

In this section we first determine pure phase symmetries in the canonical basis (9-12). Then we discuss the diagonalization of states for abelian combinations of phase symmetries and cyclic permutations and work out the formulae for the dimensions and eigenvalues of the eigenspaces for general group actions. The final ingredients for the calculation of the chiral ring of the orbifold are the quantum numbers of the vacuum and the LG description of the twisted sectors.

We want to construct all abelian symmetries which respect quasihomogeneity. These consist of combinations of cyclic permutations and phase transformations. We need not consider permutations
within a coupled sector, because due to the resulting restrictions on the phases we would not get any new models.

Let us first discuss all possible phase groups for the models (I)–(IV) of the previous section, generated by transformations \( \mathcal{P} : \Phi_i \rightarrow \rho_i \Phi_i \) with \( \ln \rho_i = 2 \pi i \varphi_i \).

In the first two cases we have \( \rho_{i+1} = \rho_i^{-a_i} \) for \( i \geq 1 \) with
\[
\varphi_1^{(I)} = \frac{1}{\prod a_i}, \quad \varphi_1^{(II)} = \frac{1}{\prod a_i - (-1)^n} \tag{39}
\]
for type (I) and type (II), respectively.

Type three is more complicated. In general there will be two generators, because there is no particular \( \rho_i \) which determines all others. Invariance of \( W \) implies \( \rho_2 = \rho_1^{-a} = \rho_3^{-c} \) and \( \rho_2^b = \rho_1^b \rho_3^b = 1 \), from which we conclude \( \varphi_1 = \frac{m}{ab}, \varphi_3 = \frac{n}{bc} \) with \( n = m + bj \) and \( m \) and \( j \) chosen in such a way that \( l = p \varphi_1 + q \varphi_3 \in \mathbb{Z} \).

The minimal value of \( m = \frac{ab(lc - jq)}{pc + aq} \) is given by
\[
m_0 = \frac{ab(c \cap q)}{abq \cap (pc + aq)} = \frac{b - 1}{(b - 1) \cap \frac{c}{q^c}}, \tag{40}
\]
where “\( \cap \)” means “greatest common divisor” (The equality of these two expressions can be seen with the help of eq. (22). We define our first generator \( \mathcal{P}_1 \) of the maximal phase symmetry as the group element determined by \( m_0 \) and \( n_0 = m_0 + jb \) with any \( j \) fulfilling \( \frac{2j}{b} + \frac{m_0}{b-1} \in \mathbb{Z} \). Taking any group element, repeated application of \( \mathcal{P}_1 \) will produce an element with \( m = 0 \). The remaining freedom can be described by \( \mathcal{P}_2 : \rho_1 = \rho_2 = 1, \varphi_3 = \frac{1}{c \cap q} \).

Type (IV) is similar to the first two cases: There is only one generator \( \mathcal{P} \) determined by \( \rho_2 = \rho_1^{-a} \) and \( \rho_3 = \rho_2^{-b} \) with
\[
\varphi_1 = \frac{1}{(a(bc - 1)) \cap (p + abq)}. \tag{41}
\]

We now consider additional cyclic symmetries. If we have \( n \) copies of the same Fermat type model in \( W = \sum_{i=1}^n \Phi_i^n + \ldots \), then the maximal abelian group which mixes all \( n \) fields can have 2 generators. By a linear change of variables these act as
\[
\mathcal{C} : \Phi_n \rightarrow \sigma \Phi_1, \quad \Phi_i \rightarrow \Phi_{i+1} \quad i < n, \quad \ln \sigma = 2 \pi i \frac{s}{a} \tag{42}
\]
\[
\mathcal{P} : \Phi_i \rightarrow \rho \Phi_i, \quad \ln \rho = 2 \pi i \frac{r}{a} \tag{43}
\]
with
\[
\ln \det \mathcal{C} = 2 \pi i \left( \frac{n - 1}{2} + \frac{s}{a} \right), \quad \ln \det \mathcal{P} = 2 \pi i \frac{nr}{a}. \tag{44}
\]
Upon diagonalization of \( \mathcal{C} \) we find that its eigenvalues are equidistant on the unit circle:
\[
\mathcal{C} \Phi_j = \rho_j \Phi_j, \quad \varphi_j = \frac{s}{na} + \frac{j}{n}, \quad 1 \leq j \leq n. \tag{45}
\]
It would be complicated to work in the diagonal basis $\tilde{\Phi}_j = \frac{1}{n} \sum_{i=1}^{n} \rho_j^{-i} \Phi_i$, but fortunately it is not difficult to calculate the dimensions of the eigenspaces, which is all we need: The number of states for a given degree of homogeneity $h$ in $\Phi_i$ is

$$\mathcal{A}_a(n, h) = \sum_{j=0}^{\frac{n}{a}} (-1)^j \binom{n}{j} \left( \frac{h + n - 1 - j(a - 1)}{n - 1} \right).$$

(46)

$h$ is an integer between 0 and $n(a - 2)$. For all states $\Phi_1^{\lambda_1} \ldots \Phi_n^{\lambda_n}$ $C^n$ is diagonal. The states in the orbit of $C$ in general yield $n$ diagonalized states with eigenvalues

$$\exp 2\pi i \left( \frac{hs}{na} + \frac{j}{n'} \right),$$

(47)

$n' = n$. If the $\lambda_i$'s have a cyclic symmetry $\lambda_i = \lambda_{i+n'}$, then $C^{n'}$ is diagonal for a divisor $n'$ of $n$, with the above formula for the eigenvalues still being valid. We thus need to calculate the number of states with definite $h = \sum_{i=1}^{n} \lambda_i = g \sum_{i=1}^{n'} \lambda_i$ and a cyclic symmetry of order $g = n/n'$ in the exponents $\lambda_i$. Including a factor $1/n'$ for the number of eigenspaces into which these states decompose, this number $\bar{\mathcal{A}}(\frac{n}{n'}, \frac{h}{n'})$ can be calculated recursively by

$$n \bar{\mathcal{A}}_a(n, h) = \mathcal{A}_a(n, h) - \sum_{1 < m | \gcd(n, h)} \frac{n}{m} \bar{\mathcal{A}}_a \left( \frac{n}{m}, \frac{h}{m} \right),$$

(48)

where $m$ runs over the common divisors of $n$ and $h$. For a definite $h$ the dimension of an eigenspace with phase $\theta = \frac{hs}{na} + \frac{j}{n}$ of the respective eigenvalue of $C$ is

$$\bar{\mathcal{A}}_a(n, h, g) = \sum_{\gamma | g} \bar{\mathcal{A}}_a \left( \frac{n}{\gamma}, \frac{h}{\gamma} \right),$$

(49)

where $\gamma$ runs over all divisors (including 1) of $g = \gcd(j, n, h)$ and $j = n\theta - \frac{hs}{a}$ has to be integer. The phase of the eigenvalue of $P$ is, of course, $\frac{hs}{a}$ modulo 1.

Producing an orbifold by modding out with respect to the symmetries considered above, we have to pay special attention to twisted states. In a particular twisted sector, where the string closes (on the considered fields) up to a group transformation $C^I P^J$, only invariant fields contribute to the ground states in the Ramond sector, and thus to the chiral rings in the Neveu Schwarz sector. Let $t = \gcd(I, n)$. Then there exist exactly $t$ invariant fields if the equation $I(\frac{s}{am} + \frac{j}{n}) + J \in \mathbb{Z}$ has at least one integer solution $j$, which is the case if and only if

$$\frac{Is + Jrn}{ta} \in \mathbb{Z}.$$

(50)

The chiral states can be calculated from an “effective” Landau Ginzburg model with $n_{\text{eff}} = t$ fields and with

$$s_{\text{eff}} = t(s + j_0a)/n \mod a,$$

(51)
where \((I(s + j_0a) + Jnr)/(na)\) is integer (we may choose \(0 \leq j_0 < \frac{n}{a}\)). Using \(\Phi_i = \sum_{j=1}^{n} \rho^i_j \tilde{\Phi}_j\),

\[
W_{\text{eff}} = \sum_{i=1}^{n} \left( \sum_{k=1}^{t} \rho^i_{j_0+ka} \tilde{\Phi}_{j_0+ka} \right)^a = n \sum_{i=1}^{t} \Phi_{i \text{eff}}^a
\]

(52)

with \(\Phi_{i \text{eff}} = \frac{1}{t} \sum_{k=0}^{t-1} \exp(-2\pi is_{\text{eff}}k/n) \Phi_{i+tk}\). According to eq. (53) the contribution of the non-invariant fields to the charges, i.e. the left/right charge of the twisted vacuum, is

\[
\Delta q_{\pm} = \sum_{\tilde{\theta}_i \notin \mathbb{Z}} \left( \frac{1}{2} - q_i \pm \left( \tilde{\theta}_i - \frac{1}{2} \right) \right) = (n - n_{\text{eff}}) \left( \frac{1}{2} - q_\Phi \right) \pm (t - n_{\text{eff}}) \left( \left[ \frac{Ia + Jnr}{ta} \right] - \frac{1}{2} \right)
\]

(53)

where \(n_{\text{eff}} = t = \gcd(n, I)\) if (54) is satisfied and \(n_{\text{eff}} = 0\) otherwise.

The same analysis can be done for the phase symmetries we found at the beginning of this chapter, combined with a cyclic symmetry which now permutes complete coupled sectors. For type (I) and (II) models with \(n = 2\) the results are listed in the remainder of this section.

For \(W^{(L)} = \sum_{i=1}^{n} (\Phi_i^e \Psi_i + \Psi_i^f \Phi_i)\) the number of monomials of degree \(k\) in \(\Phi\) and of degree \(l\) in \(\Psi\) is

\[
\mathcal{L}_{\text{ef}}(n, k, l) = \mathcal{A}_{e+1}(n, k) \mathcal{A}_{f+1}(n, l),
\]

(54)

whereas for \(W^{(T)} = \sum_{i=1}^{n} (\Phi_i^e + \Psi_i^f \Phi_i)\) there are

\[
\mathcal{T}_{\text{ef}}(n, k, l) = \sum_{j \leq n} \left( \begin{array}{c} n \\ j \end{array} \right) \mathcal{A}_{e+1}(j, k) \mathcal{A}_{f}(j, l - (n - j)(f - 1))
\]

(55)

monomials of degree \(k\) in \(\Phi\) and of degree \(l\) in \(\Psi\). As before, it is useful to define

\[
\mathcal{L}_{\text{ef}}(n, k, l) = \sum_{m \mid \gcd(n, k, l)} \frac{n}{m} \tilde{\mathcal{L}}_{\text{ef}}(\frac{n}{m}, \frac{k}{m}, \frac{l}{m})
\]

(56)

\[
\tilde{\mathcal{L}}_{\text{ef}}(n, k, l, g) = \sum_{\gamma \mid g} \tilde{\mathcal{L}}_{\text{ef}}(\frac{n}{\gamma}, \frac{k}{\gamma}, \frac{l}{\gamma})
\]

(57)

and the analogous quantities for \(\mathcal{T}\). \(\mathcal{L}(n, k, l, g)\) and \(\mathcal{T}(n, k, l, g)\) are the dimensions of the eigenspaces with eigenvalue \(\exp(2\pi i \theta)\) of \(\mathcal{C}\), where \(j = n\theta - \frac{sI - f}{O}\) has to be integer, \(g = \gcd(n, k, l, j)\) and \(O = ef - 1\) are \(O = ef\) the orders of the respective maximal phase symmetries. The phases of the eigenvalues of \(\mathcal{P}\) are \(r(l - f)/O\) and the determinants are given by

\[
\ln \det \mathcal{C} = 2\pi i \frac{s(1 - f)}{O}, \quad \ln \det \mathcal{P} = 2\pi i \frac{nr(1 - f)}{O}.
\]

(58)

These formulae are valid for both loops and tadpoles.

Now we consider the twisted sector for a group element \(\mathcal{C}^I \mathcal{P}^J\). \(t = \gcd(n, I)\) pairs of fields \((\Phi, \Psi)\) contribute to the chiral ring in this sector iff

\[
\frac{sI + nJr}{tO} \in \mathbb{Z},
\]

(59)
implying charges
\[ \Delta q_{\pm} = (n - n_{\text{eff}})(1 - q_{\Phi} - q_{\Psi}) \pm t \left( \left[ \frac{Is + Jnr}{tO} \right] - \left[ \frac{f(Is + Jnr)}{tO} \right] \right) \] (60)
of the twisted vacuum (in this case \( n_{\text{eff}} = t \)). If eq. (59) is not fulfilled, the chiral ring only consists of the twisted vacuum and (60) is still valid, but now \( n_{\text{eff}} = 0 \). The effective LG theory describing the twisted sectors has
\[ s_{\text{eff}} = t(s + j_0 O) / n \text{ mod } O, \] (61)
where \( j_0 \) is determined by \( (I(s + j_0 O) + Jnr)/(nO) \in \mathbb{Z} \).

A pecularity of the tadpole type is that even if \( \Psi \) is not invariant, \( \Phi \) still can be invariant. Thus, if equation (59) is not satisfied, but
\[ \frac{sI + nJr}{et} \in \mathbb{Z}, \] (62)
then the effective LG-theory is of Fermat type with \( n_{\text{eff}} = t, a_{\text{eff}} = e, r_{\text{eff}} = -r \text{ mod } e \) and
\[ s_{\text{eff}} = t(-s + j_0 e) / n \text{ mod } e \] (63)
with \( (j_0 e - Is - Jnr)/(ne) \in \mathbb{Z} \) and
\[ \Delta q_{\pm} = n(1 - q_{\Phi} - q_{\Psi}) + t(q_{\Phi} - \frac{1}{2}) \pm t \left( \left[ \frac{Is + Jnr}{tef} \right] - \frac{1}{2} \right). \] (64)

6 Actions of Symmetries: General Considerations

In this section we will discuss some general aspects that are important for group actions on Landau–Ginzburg theories that have been orbifolded with respect to the U(1)–symmetry in order to describe string vacua with \( N = 1 \) spacetime supersymmetry.

An obvious question when considering orbifolds is whether there is any a priori insight into what spectra are possible for the orbifolds of a given model with respect to a particular set of symmetries. This question is of particular interest if the goal is to produce orbifolds with prescribed spectra, say models with a small number of fields where the difference between the number of generations and antigenerations is three.

Even though it is possible to formulate constraints on the orbifold spectrum for particular types of actions, we know of no constraints that hold in full generality, or even for arbitrary cyclic actions. One very simple class of symmetries are those without fixed points. For such actions there are no twisted sectors and hence there exists a simple formula expressing the Euler number \( \chi_{\text{orb}} \) of the orbifold in terms of the Euler characteristic \( \chi \) of the covering space and the order \( |G| \) of the group
\[ \chi_{\text{orb}} = \frac{\chi}{|G|}. \] (65)
The vast majority of actions however do have fixed points and hence the result above does not apply very often.

For orbifolds with respect to cyclic groups of prime order there exists a generalization of this result. For such group actions it was shown in [13] that

\[ \bar{n}_{\text{orb}}^g - n_{\text{orb}}^g = (|G| + 1) (\bar{n}_{\text{inv}}^g - n_{\text{inv}}^g) - (\bar{n}^g - n^g), \] (66)

where \( n_{\text{orb}}^g, n_{\text{inv}}^g, n^g \) are the numbers of generations of the orbifold theory, the invariant sector and the original LG theory, respectively.

Consider then the problem of constructing an orbifold with a prescribed Euler number \( \chi_{\text{orb}} \) from a given theory. Only for fixed point free actions will the order of the group be completely specified as \( |G| = \chi/\chi_{\text{orb}} \). It is important to realize that in general the order of the group by which a theory is orbifolded does not determine its spectrum – the precise form of the action of the symmetry is important.

Nevertheless we can derive some constraints on the order of the action that we are looking for. Even though we don’t know a priori what the invariant sector of the orbifold will be we do know that its associated Euler number must be an integer

\[ \chi_{\text{inv}} = \frac{\chi + \chi_{\text{orb}}}{|G| + 1} \in \mathbb{N}. \] (67)

This simple condition does lead to restrictions for the order of the group. Suppose, e.g., that we wish to check whether the quintic threefold admits a three–generation orbifold: For the deformation class of the quintic

\[ \mathcal{C}(1,1,1,1)[5] : \chi = -200 \] (68)

the order of the discrete group in question must satisfy the constraint \(-206/(|G| + 1) \in \mathbb{Z}\), implying \( |G| = 102 \). Hence there exists no three–generation orbifold of the quintic with respect to a discrete group with prime order. A counterexample for nonprime orders is furnished by the following theory

\[ \mathcal{C}(2,2,2,3,3,3)[9] : (\bar{n}^g, n^g, \chi) = (8, 35, -54), \] (69)

which corresponds to a CY theory embedded in a product of two projective spaces by two polynomials of bidegree \((0, 3)\) and \((3, 1)\) \[[4]\]. Suppose we are searching for three–generation orbifolds of this space with \( \chi_{\text{orb}} = \pm 6 \). If \( \chi = -6 \) the constraint is not very restrictive and allows a number of possible groups \( |G| \in \{2, 3, 5, 11, 19, 29\} \). Even though it is not known whether any of these groups lead to a three–generation model it is known that at a particular point in the configuration space of (69) described by the superpotential

\[ W = \sum_{i=1}^{3} (\Phi_i^3 + \Phi_i \Psi_i^3) + \Phi_4^3 \] (70)

\(^3\text{i.e. the Calabi–Yau manifold of this model is embedded in an ambient space consisting of a product of two projective spaces } \mathbb{P}_2 \times \mathbb{P}_3. \)
a symmetry of order nine exists that leads to a three–generation model [14].

Our interest however is not restricted to models with particular spectra for reasons explicated in the introduction. Hence we wish to implement general types of actions regardless of their fixed point structure and order. A general analysis of symmetries for an arbitrary Landau–Ginzburg potential is beyond the scope of this paper; instead we restrict our attention to the types of potentials that we have constructed explicitly. Before we discuss these types we should remark upon a number of aspects concerning actions on string vacua defined by LG–theories.

It is important to note that depending on the weights (or charges) of the original LG theory it can and does happen that actions that take rather different forms when considered as actions on the LG theory actually are isomorphic when viewed as action of the string vacuum proper because of the U(1) projection. It is easiest to explain this with an example. Consider the superpotential

$$W = \Phi_1^{18} + \Phi_2^{18} + \Phi_3^3 + \Phi_4^3 + \Phi_5^3$$

(71)

which belongs to the configuration $C_{(1,1,6,6,4)}[18]^9_{-204}$ (here the superscript denotes the number of anti-generations and the subscript denotes the Euler number of the configuration). At this particular point in moduli space we can, e.g., consider the orbifolds with respect to the actions

$$\mathbb{Z}_3 : [0 \ 0 \ 1 \ 0 \ 2], \ (13, 79, -132)$$

$$\mathbb{Z}_3 : [1 \ 1 \ 1 \ 0 \ 0], \ (13, 79, -132)$$

$$\mathbb{Z}_3 : [1 \ 0 \ 1 \ 0 \ 1], \ (14, 44, -60),$$

(72)

where the notation $\mathbb{Z}_a : [p_1 \ldots p_n]$ indicates that the fields $\Phi_i$ transform with phases $(2\pi ip_i/a)$ under the generator of the $\mathbb{Z}_a$ symmetry. It is clear from the last action in (72) that the order of a group is, in general, not sufficient to determine the resulting orbifold spectrum but that the specific form of the way the symmetry acts is essential.

Since the first two actions lead to the same spectrum we are led to ask whether the two resulting orbifolds are equivalent. Theories with the same number of light fields need, of course, not be equivalent and to show whether they are is, in general a rather involved analysis, entailing the transformation behaviour of the fields and the computation of the Yukawa couplings.

In the case at hand it is, however, very easy to check this question. The first two actions only differ by the $6^{th}$ power of the canonical $\mathbb{Z}_{18}$ which is given by $\mathbb{Z}_3 : [1 \ 1 \ 0 \ 0 \ 1]$. Since the orbifolding with respect to this group is always present in the construction of an LG vacuum the fist two orbifolds in eq. (72) are trivially equivalent.

Another important point is the role of trivial factors in the LG theories. Given a superpotential $W_0$ with the correct central charge to define a Heterotic String vacuum we always have the freedom to add

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trivial factors to it

\[ W = W_0 + \sum_i \Phi_i^2, \quad (73) \]

since neither the central charge nor the chiral ring are changed by this operation. As we restrict our attention to symmetries with unit determinant, we gain, however, the possibility to cancel a negative sign of the determinant by giving some \( \Phi_i \) a nontrivial transformation property under a \( \mathbb{Z}_{2n} \). Adding a trivial factor hence changes the symmetry properties of the LG–potential with regards to this class of symmetries. If we wish to relate the vacuum described by the potential to a Calabi–Yau manifold, consideration of trivial factors becomes essential. Consider e.g. the LG–potential

\[ W_0 = \Phi_1^{12} + \Phi_2^{12} + \Phi_3^6 + \Phi_4^6 \quad (74) \]

which has \( c = 9 \) and charges \( (\frac{1}{12}, \frac{1}{12}, \frac{1}{6}, \frac{1}{6}) \) and hence is a member of the configuration \( C_{(1,1,2,2)}^{[12]} \).

Only after adding the necessary trivial factor this theory can be orbifolded with an action defined by \( \mathbb{Z}_2 : [1 \ 0 \ 0 \ 0 \ 1] \) acting on the Fermat polynomial in \( C_{(1,1,2,2,6)}^{[12]} \); this action leads to the orbifold spectrum \( (4, 94, -180) \) and is not equivalent to any symmetry that acts only on the first four variables with determinant 1. Neglecting the addition of the quadratic term to the LG potential \( W_0 \) would have meant missing the above spectrum as one of the possible orbifold results.

Finally it should be noted that obviously we have to make some choice about which points in moduli space we wish to consider. Different members of a moduli space have, in general, drastically different symmetry properties. An example is the well known quintic theory which we already mentioned. The most symmetric point in the 101 dimensional space of complex deformations of the quintic is described by the Fermat polynomial

\[ W = \sum_i \Phi_i^5, \quad (75) \]

which has a discrete symmetry group of order \( 5! \cdot 5^4 \). Any deformation breaks most of these symmetries. It turns out that for the quintic the different points in moduli space that we have considered do not lead to actions which provide additional spectra. In other examples it does happen that the consideration of additional points leads to new result. The configuration \( C_{(15,3,2,20,20)}^{[60]} \) e.g. admits Fermat type polynomials as well as tadpole type polynomials, and even though none of the Fermat actions we have implemented leads to a three–generation model the following tadpole–Fermat type potential

\[ W = \Phi_1^4 + \Phi_2^{20} + \Phi_3^{30} + \Phi_4^3 + \Phi_4 \Phi_5^2 \quad (76) \]

leads to a model with spectrum \( (36, 39, -6) \) when orbifolded with the symmetry \( \mathbb{Z}_2 : [1 \ 0 \ 0 \ 0 \ 1] \).

\(^4\)In LG theories the determinant restriction is necessary for modular invariance and can be avoided by introducing discrete torsion \([10]\).
7 Phase Actions: Implementation and Results

Consider then a potential $W$ with $n$ order parameters normalized such that the degree $d$ takes the lowest value such that all order parameters have integer weight. In the following we discuss potentials of the type

$$W = \sum_i \Phi_i^{a_i} + \sum_j \left( \Phi_j^{f_j} + \Phi_j^{\psi_j} \right) + \sum_k \left( \Phi_k^{e_k} \psi_k + \Phi_k^{\psi_k} \right)$$

(77)

which consist of Fermat parts, tadpole parts and loop parts.

**Fermat Potentials**: Clearly the potential $W = \sum_i \Phi_i^{a_i}$ is invariant under $\prod_i Z_{a_i}$, i.e. the phases of the individual fields, acting like

$$\Phi_i \rightarrow e^{2\pi i \frac{m_i}{a_i}} \Phi_i.$$  

(78)

For some divisor $a$ of lcm($a_1, \ldots, a_n$) and $\frac{m_i}{a_i} = \frac{p_i}{a}$ we denote such an action by

$$Z_a : [p_1, p_2, \ldots, p_n], \quad 0 \leq p_i \leq a - 1.$$  

(79)

and require that $a$ divides $\sum p_i$ in order to have determinant 1.

We have implemented such symmetries in the form

$$Z_a : [ (a - \sum_i i_l) \ i_1 \ \cdots \ i_p \ (a - \sum_j j_m) \ j_1 \ \cdots \ j_q \ \cdots ]$$  

(80)

with the obvious divisibility conditions. For small $p$ and $q$ these symmetries can act on a large number of spaces and therefore lead to many different orbifolds, but as $p, q$ get larger the number of resulting orbifolds decreases rapidly. We have stopped implementation of more complicated actions when the number of results for the different orbifold Hodge pairs was of the order of a few tens. As already mentioned above, the precise form of the action is very important when considering symmetries with fixed points since the order itself is not sufficient to determine the orbifold spectrum.

More complicated symmetries can be constructed via multiple actions by multiplying single actions of the type described above

$$\prod_c Z_{a_c} : [ (a_c - \sum_l i_{c,l}) \ i_{c,1} \ \cdots \ i_{c,p} \ (a_c - \sum_m j_{c,m}) \ j_{c,1} \ \cdots \ j_{c,q} \ \cdots ].$$  

(81)

We have considered (an incomplete set of) actions of this type with up to six twists (i.e. six $Z_a$ factors). Again the precise form of the action is rather important.

**Tadpole and Loop Polynomials**: The action of the generator of the maximal phase symmetry within a tadpole or loop sector is

$$Z_\sigma : [-f \ 1].$$  

(82)
where $O = ef$ or $ef - 1$, respectively. If we want unit determinant within one sector, we must take our generator to the $n^{th}$ power with some $n$ fulfilling $n(f - 1)/O \in \mathbb{Z}$. With $\omega = \gcd(f - 1, O)$ the action of the resulting subgroup can be chosen to be

$$\mathbb{Z}_\omega : \begin{bmatrix} (\omega - 1) & 1 \end{bmatrix}. \quad (83)$$

Other types of actions that we have considered for superpotentials consisting of Fermat parts and tadpole/loop parts involve phases acting both on the tadpole/loop part as well as on a number of Fermat monomials. As was the case with pure Fermat polynomials we have also implemented multiple actions of the type considered above.

We have implemented some forty different actions of the types described in the previous paragraphs. These symmetries lead to a large number of orbifolds not all of which are distinct however for reasons explained in the previous section. Our computations have concentrated on the number of generations and anti–generations of these models and we have found some 2000 distinct Hodge pairs \footnote{This number is very close to the number of spectra found in [1] for the number of distinct Hodge pairs in a large class of LG–theories equivalent to CY manifolds embedded in weighted $\mathbb{P}_4$}. In Fig. 1 we have plotted the difference of the number of generations and antigenerations versus their sum for all the Hodge pairs.

It is obvious from this plot that there is a large overlap between the results of \footnote{This number is very close to the number of spectra found in [1] for the number of distinct Hodge pairs in a large class of LG–theories equivalent to CY manifolds embedded in weighted $\mathbb{P}_4$} and the orbifolds constructed here. This might indicate that the relation established in [8] between orbifolds of Landau–Ginzburg theories and other Landau–Ginzburg theories is a general phenomenon and not restricted to the particular classes of actions which were analysed in [8].

Models with a low number of fields are clearly of particular interest. There are two aspects to this question, as mentioned in the introduction – low numbers for the difference of generations and anti–generations (more precisely one wants the number 3 here) and low values for the total number of generations and anti–generations. As far as the latter are concerned the following ‘low–points’ are the ‘highlights’ among the results for phase symmetry orbifolds.

The lowest models have $\chi = 0$, more precisely the spectra (9,9,0) and (11,11,0). These spectra appear many times in different orbifolds of Fermat type; an example for the first one being

$$\mathcal{C}_{(1,...,1)[9]} / \mathbb{Z}_3^2 : \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \ 0 \ 0 \end{bmatrix}. \quad (84)$$

or, even simpler,

$$\mathcal{C}_{(4,4,4,4,4,3,3)} / \mathbb{Z}_3 : [1 1 1 0 0 0 0 0]. \quad (85)$$

The second one can be constructed e.g. as

$$\mathcal{C}_{(4,3,3,3,3,3)} / \mathbb{Z}_4^2 : \begin{bmatrix} 0 & 1 & 1 & 2 & 0 & 0 \ 0 & 0 & 2 & 1 & 1 \end{bmatrix}. \quad (86)$$
Other examples with a total of 22 generations and anti–generations are the following orbifolds of the Fermat quintic:

\[
\mathbb{Z}_5 : [0 \ 1 \ 2 \ 3 \ 4], \quad (1, 21, -40) \tag{87}
\]

and

\[
\mathbb{Z}_5^2 : \begin{bmatrix} 3 & 1 & 1 & 0 & 0 \\ 0 & 3 & 1 & 1 & 0 \end{bmatrix}, \quad (21, 1, 40) \tag{88}
\]

Of particular interest, of course, are three–generation models. In the list of 3112 models there are, aside from the known [1] three–generation models embedded in weighted \( \mathbb{P}_4 \), no new three generation models. For completeness we list these models in Table 1.

**Table 1.** Three–generation LG models.

| Configuration Potential | Action Spectrum |
|-------------------------|-----------------|
| \( C_{(21,21,3,4)}[63]^{32}_{-6} \) | \( \Phi^3 + \Phi_2^3 + \Phi_2 \Phi_3^5 + \Phi_4^{21} + \Phi_4 \Phi_5^{15} \) |
| \( C_{(15,5,8,3,14)}[45]^{20}_{-6} \) | \( \Phi^3 + \Phi_2^9 + \Phi_2 \Phi_3^5 + \Phi_4^{15} + \Phi_4 \Phi_5^3 \) |
| \( C_{(17,6,9,17,2)}[51]^{34}_{-6} \) | \( \Phi^3 + \Phi_2 \Phi_3 + \Phi_2 \Phi_3^5 + \Phi_4^3 + \Phi_4 \Phi_5^{17} \) |

Via orbifolding we find a number of such models which all however have a fairly large number of generations and antigenerations. We list those in Table 2.

**Table 2.** Three–generation orbifold models: we do not list models separately which are equivalent up to the U(1) projection.

| # | Configuration Potential | Action | Spectrum |
|----|-------------------------|--------|----------|
| 1  | \( C_{(9,2,5,9,2)}[27]^{16}_{-66} \) \( \Phi_1^3 + \Phi_2^{11} \Phi_3 + \Phi_2 \Phi_3^5 + \Phi_3^3 + \Phi_4 \Phi_5^9 \) | \( \mathbb{Z}_3 : [1 \ 0 \ 0 \ 2] \) | \( (18, 21, -6) \) |
| 2  | \( C_{(17,6,9,3,16)}[51]^{15}_{-102} \) \( \Phi_1^3 + \Phi_2^7 \Phi_3 + \Phi_2 \Phi_3^5 + \Phi_4^{17} + \Phi_4 \Phi_5^3 \) | \( \mathbb{Z}_2 : [0 \ 1 \ 1 \ 0 \ 0] \) | \( (31, 34, -6) \) |
| 3  | \( C_{(9,2,5,3,8)}[27]^{10}_{-54} \) \( \Phi_1^4 + \Phi_2^{11} \Phi_3 + \Phi_2 \Phi_3^5 + \Phi_3^9 + \Phi_4 \Phi_5^3 \) | \( \mathbb{Z}_2 : [0 \ 1 \ 1 \ 0 \ 2] \) | \( (21, 18, 6) \) |
| 4  | \( C_{(15,15,2,9,4)}[45]^{22}_{-30} \) \( \Phi_1^3 + \Phi_2^3 + \Phi_2 \Phi_3^5 + \Phi_3^3 + \Phi_4 \Phi_5^9 \) | \( \mathbb{Z}_3 : [1 \ 0 \ 2 \ 0 \ 0] \) | \( (23, 20, 6) \) |
| 5  | \( C_{(15,15,10,3,2)}[45]^{22}_{-34} \) \( \Phi_1^3 + \Phi_2^3 + \Phi_2 \Phi_3^5 + \Phi_4^{15} + \Phi_4 \Phi_5^{21} \) | \( \mathbb{Z}_3 : [1 \ 0 \ 2 \ 0 \ 0] \) | \( (35, 32, 6) \) |

By using the relation established in [8] between LG/CY–theories via fractional transformations it can be shown that the orbifold \#1 in Table 2,

\[
C_{(2,5,9,2,9)}[27]^{16}_{-66}/\mathbb{Z}_3 : [0 \ 0 \ 0 \ 2 \ 1], \tag{89}
\]
for which the covering model is described by the polynomial
\[ W = \Phi_1^{11} \Phi_2 + \Phi_1 \Phi_2^5 + \Phi_3^3 + \Phi_3 \Phi_4^6 + \Phi_5^3, \]
(90)
is isomorphic to the orbifold
\[ \mathfrak{C}_{(2,5,9,3,8)[27]}^{10}/\mathbb{Z}_2 : [0 0 0 1 1] \]
(91)
where the covering theory is described by the polynomial
\[ W = \Phi_1^{11} \Phi_2 + \Phi_1 \Phi_2^5 + \Phi_3^3 + \Phi_4 \Phi_5^3. \]
(92)
The latter is a theory involving a subtheory with couplings among three scaling fields and hence goes beyond the types of potentials we have implemented. This example indicates that more complicated examples than the ones investigated here are likely to yield more (perhaps more realistic) three generation models.

The covering spaces of all the three generation models are described by either tadpole or loop type polynomials, and with our actions none of the Fermat type polynomials leads to a three generation model. It should be noted that these orbifolds exist only at particular points in moduli space. In some cases the tadpole polynomial defining the covering space configuration admits a Fermat representation, but it turns out that this is not the point in moduli space that leads to a three generation model.

8 Cyclic Permutations

Consider a cyclic permutation of order \( r = 2n \) generated by
\[ (\Phi_1, \Phi_2, \ldots, \Phi_r) \mapsto (\Phi_2, \ldots, \Phi_r, \Phi_1). \]
(93)
Such an action is not allowed since its determinant is \(-1\). The direct sum of an even number of such permutations is, of course, a good symmetry.

Since the total number of fields is at most nine, there is only a small number of possible pure cyclic permutations. The implementation of all these cyclic permutations leads to several hundred orbifolds which however lead only to some 100 different Hodge pairs which we plot in Figure 2.

It should be noted that the model with the smallest number of particles among all our orbifolds is in this set; it has the spectrum \((0, 12, -24)\) and comes from the cyclic permutation orbifold
\[ \mathfrak{C}_{(1,\ldots,1)[9]}/\mathbb{Z}_{9,cyclic} : (0, 12, -24). \]
(94)
The model with the next smallest number of fields is the \(\mathbb{Z}_3\) permutation orbifold of the theory described by the potential
\[ W = \Phi^3 + \sum_{i=1}^{3}(\Phi_i^3 + \Phi_i \Psi_i^2) \]
(95)
in the configuration $\mathcal{C}_{(3,3,3,2,2,2)}[9]^{8}_{-54}$ mentioned already. The cyclic symmetry permutes the pairs of fields $(\Phi_i, \Psi_i)$ for $i = 1, 2, 3$ and leads to a theory with the spectrum $(4, 13, -18)$. Another model with a comparatively low number of fields is the Fermat orbifold

$$\mathcal{C}_{(1,\ldots,1)}[9]/\mathbb{Z}_{7,\text{cyclic}} : (6, 18, -24).$$

\section{9 Conclusions}

Using the methods we have worked out for the computation of the spectra of a large class of abelian orbifolds, we have performed a computer search for all potentials consisting of a superposition of polynomials in which at most two fields are coupled.

For pure phase symmetries we observe a large overlap with canonically orbifolded LG vacua, but we also find additional models, in particular with a small total number of fields. New three generation orbifolds are found only for tadpole and loop type theories. There are also some models with a relatively low number of fields.

It is clear from our results that the most promising avenue to produce phenomenologically interesting models is to consider orbifolds with mixed actions of phase symmetries and permutations since these models likely lead to spectra that will populate the lower part of the plot. Generalizing the implementation of cyclic permutations to arbitrary permutations is also of interest for the breaking of the $E_6$ gauge symmetry present in this class of theories. Nonabelian symmetries allow to recover the gauge symmetries of the Standard Model.

As mentioned previously our construction of potentials and implementation of symmetries is not complete. A very rough measure of the completeness of our implementation can be gained by considering the orbifold ‘descendents’ of particular models. Consider e.g. the Fermat potential

$$W = \sum_{i=1}^{3} \Phi_i^{10} + \Phi_i^{5} + \Phi_i^{2}$$

in $\mathcal{C}_{(1,1,1,2,5)}[10]^{1}_{-288}$. For this theory our code produced a completely mirror symmetric space of orbifold ‘descendents’ which we list in Table 3\footnote{the pair $(7, 67, -120)$ and $(67, 7, 120)$ was missed in ref. \cite{2}.}.
Table 3. Orbifolds of the Fermat theory in $C_{(1,1,1,2,5)}[10]^{1-288}$.

| Group     | Action       | Spectrum    |
|-----------|--------------|-------------|
| $\mathbb{Z}_2$ | [1 1 0 0 0]   | (3, 99, -192) |
| $\mathbb{Z}_{10}^2$ | [8 1 1 0 0] [0 9 1 0 0] | (99, 3, 192) |
| $\mathbb{Z}_2^3$ | [1 1 0 0 0] [0 1 1 0 0] | (7, 67, -120) |
| $\mathbb{Z}_5^2$ | [4 1 0 0 0] [0 0 4 1 0] | (67, 7, 120) |
| $\mathbb{Z}_5$ | [0 0 4 1 0] | (11, 47, -72) |
| $\mathbb{Z}_{10} \times \mathbb{Z}_2$ | [9 1 0 0 0] [0 1 1 0 0] | (47, 11, 72) |
| $\mathbb{Z}_5$ | [4 1 0 0 0] | (13, 37, -48) |
| $\mathbb{Z}_{10} \times \mathbb{Z}_2$ | [8 1 1 0 0] [0 0 1 0 1] | (37, 13, 48) |
| $\mathbb{Z}_{10}$ | [9 1 0 0 0] | (15, 39, -48) |
| $\mathbb{Z}_{10}$ | [8 1 1 0 0] | (39, 15, 48) |
| $\mathbb{Z}_{10}$ | [7 2 1 0 0] | (17, 29, -24) |
| $\mathbb{Z}_{10}$ | [5 4 1 0 0] | (29, 17, 24) |
| $\mathbb{Z}_{10}^2$ | [9 1 0 0 0] [0 9 1 0 0] | (145, 1, 288) |

There are, however, in our list a fair number of LG–potentials where we have not yet found a completely mirror symmetric ‘descendant’ space of orbifolds.
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Fig. 1 A plot of Euler numbers against $\bar{n}_g + n_g$ for the 1898 spectra of all the LG potentials and phase orbifolds constructed.
Fig. 2 A plot of Euler numbers against the total number of particles for the orbifolds with respect to cyclic permutations.