Energy Formula and Energy Extraction for a Charged Rotating Black Hole in Heterotic String Theory

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Abstract

We evaluate the surface energy ($E_{s}^{\pm}$), the rotational energy ($E_{r}^{\pm}$) and the electromagnetic energy ($E_{em}^{\pm}$) for a charged rotating black hole (BH) in heterotic string theory having the event horizon ($H^{+}$) and the Cauchy horizon ($H^{-}$). Interestingly, we find that the sum of these three energies is equal to the Arnowitt-Deser-Misner (ADM) mass parameter i.e. $E_{s}^{\pm} + E_{r}^{\pm} + E_{em}^{\pm} = M$. Moreover in terms of the scale parameter ($\eta_{\pm}$), the distortion parameter ($\beta_{\pm}$) and a new parameter ($\epsilon_{\pm}$) which corresponds to the area ($A_{\pm}$), the angular momentum ($J$) and the charge parameter ($Q$), interestingly we find that the ADM mass parameter in a compact form

$$E_{s}^{\pm} + E_{r}^{\pm} + E_{em}^{\pm} = M = \frac{\eta_{\pm}}{2} \sqrt{\frac{1 + 2\epsilon_{\pm}^{2}}{1 - \beta_{\pm}^{2}}}$$

which is valid for $H^{\pm}$. Moreover, using this ADM mass decomposition formula, we compute the exact expression of rotational energy that should be extracted from the BH via Penrose process. The maximum value of rotational energy which is extractable should occur for extremal charged rotating BH in heterotic string theory i.e.

$$E_{r}^{\pm} = \left(\frac{\sqrt{2} - 1}{2}\right) \sqrt{2M^2 - Q^2} = \left(\sqrt{2} - 1\right) \sqrt{J}$$

1 Introduction

One of the most fascinating objects in Einstein’s general theory of relativity is BH. It is fascinating in a sense that it has the entropy which is proportional to the BH surface area [1] (a two dimensional surface) and the temperature which is proportional to the surface gravity [2].

In our previous work [3], we have studied in detail the thermodynamic properties of the said BH. We computed the thermodynamic parameters such as area (or entropy) product, area (or entropy sum) and other thermodynamic parameters. We also computed the Smarr like mass formula and the Christodoulou-Ruffini mass formula. Moreover, we verified the first law as well as the second law of BH thermodynamics. Furthermore, we computed the central charges of the left moving sectors and right moving sectors of the dual conformal field theory (CFT) and showed that the central charges are equal in both sectors.

However in the previous study we have not computed the exact expressions for the surface energy ($E_{s}^{\pm}$), the rotational energy ($E_{r}^{\pm}$) and the electromagnetic energy ($E_{em}^{\pm}$). This is the first motivation to study behind this work. The other motivation of this work is to come from the crucial point that does these three energies relate to any energy extraction process i.e. the famous Penrose process? Perhaps the answer should be YES because using this energy decomposition formula we can derive the exact value of the rotational energy that should be extracted via Penrose process.

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Therefore in the present work we wish to evaluate these three energies by using the mass formula derived in [3]. By computing this energies we prove that the sum of three energies is equal to the ADM mass parameter. Also we show that in terms of the scale parameter \((\eta_{\pm})\), the distortion parameter \((\beta_{\pm})\) and a new parameter \((\epsilon_{\pm})\) which corresponds to the BH area \((A_{\pm})\), the BH angular momentum \((J)\) and the BH charge parameter \((Q)\) then the ADM mass parameter should be written in a most compact form as

\[
\mathcal{E}_s^\pm + \mathcal{E}_r^\pm + \mathcal{E}_{em}^\pm = M = \frac{\eta_{\pm}}{2} \sqrt{\frac{1 + 2\epsilon_{\pm}^2}{1 - \beta_{\pm}^2}}
\]  

which is valid for both the horizons \((\mathcal{H}^\pm)\). Moreover, using this ADM mass decomposition relation we evaluate the rotational energy that should be extracted from the BH via Penrose process. The maximum value of extracted energy should be found for maximally Kerr-Sen BH i.e.

\[
\mathcal{E}_r^+ = \left(\frac{\sqrt{2} - 1}{2}\right) \sqrt{2M^2 - Q^2} = \left(\sqrt{2} - 1\right) \sqrt{\frac{J}{2}}
\]  

In the limit \(Q = 0\) or \(J = M^2\) i.e. for extremal Kerr BH, the extractable rotational energy is \(\mathcal{E}_r^+ = (1 - \frac{1}{\sqrt{2}})M\). To the best of our knowledge this is the first time we have reported in literature such remarkable result for a charged rotating BH in heterotic string theory.

In the next section (Sec. 2) we will familiar with the Kerr-Sen metric and its thermodynamic properties. In Sec. 3, we will derive the surface energy, the rotational energy and the electromagnetic energy for Kerr-Sen BH. In Sec. 4, we will introduce the scale parameter and the distortion parameter for Kerr-Sen BH. In Sec. 5, we shall calculate the amount of rotational energy that should be extracted from the BH in heterotic string theory. Finally in Sec. 6 we have given our conclusions.

2 Kerr-Sen BH metric and its properties:

The metric of rotating charged BH solution (in the geometrized units i.e. \((c = G = 1)\) in heterotic string theory [4] is

\[
ds^2 = -\frac{\Delta - a^2 \sin^2 \theta}{\rho^2} dt^2 - \frac{4aMr \sin^2 \theta}{\rho^2} dt d\phi + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \frac{\Upsilon}{\rho^2} \sin^2 \theta d\phi^2
\]

where

\[
\rho^2 = r^2 + a^2 \cos^2 \theta + 2qr
\]

\[
\Delta = r^2 - 2(M - q)r + a^2
\]

\[
\Upsilon = (r^2 + a^2 + 2qr)^2 - \Delta a^2 \sin^2 \theta
\]

\[
q = \frac{Q^2}{2M}
\]

The above spacetime metric implies that there exists a BH solution with mass \((M)\), charge \((Q)\), angular momentum \((J)\), and the magnetic dipole moment \((\mu = aQ)\).

There exists two horizons for Kerr-Sen BH namely the event horizon \((\mathcal{H}^+)\) or outer horizon and Cauchy horizon \((\mathcal{H}^-)\) or inner horizon. Therefore the horizon radius may be determined from the following functions:

\[
\Delta \equiv \Delta(r) = r^2 - 2(M - q)r + a^2 = 0
\]
which gives

\[ r_+ = (M - q) + \sqrt{(M - q)^2 - a^2} \]  
\[ r_- = (M - q) - \sqrt{(M - q)^2 - a^2} \].

Note that \( r_+ > r_- \). Where \( r_+ \) is the radius of outer horizon while \( r_- \) is the radius of inner horizon.

From Eq. (9) one must see that the horizon disappears unless \( a \leq (M - q) \). Thus the extremal limit of the Sen BH corresponds to \( a = (M - q) \) and the horizon for extremal Sen BH is located at \( r_{ex} = r_+ = r_- = a = (M - q) \). The ergosphere for Kerr-Sen BH is located at

\[ r_{\text{ergo}}(\theta) = (M - q) + \sqrt{(M - q)^2 - a^2 \cos^2 \theta} \].

The thermodynamic quantities like the BH area is

\[ A_{\pm} = \int_0^{2\pi} \int_0^\pi \sqrt{g_{\theta \theta} g_{\phi \phi}} \, d\theta d\phi = 8\pi M r_{\pm} \].

The horizon area of the BH is a 2D surface and it is constant as pointed out by the Hawking [2]. Also it never decreases.

The other thermodynamic quantities like the surface gravity, the angular velocity and the Hawking temperature of \((\mathcal{H}^+)\) [4] and of \((\mathcal{H}^-)\) [5] are

\[ \Omega_{\pm} = \frac{a}{2M r_{\pm}} \].
\[ T_{\pm} = \frac{r_{\pm} - r_+}{8\pi M r_{\pm}}, \ T_+ > T_- .

The horizon Killing vector field may be defined for \( \mathcal{H}^\pm \)

\[ \chi_{\pm}^a = (\partial_t)^a + \Omega_{\pm} (\partial_\phi)^a \].

3 The Surface energy, The Rotational energy and The Electromagnetic energy for Kerr-Sen BH:

Larry Smarr [6, 7] first derived the mass formula for charged spinning BH i.e. Kerr-Newman BH which could be obtained by integrating term by term of the mass differential which consists of three terms interpreted as the surface energy, rotational energy, and electromagnetic energy respectively. Here we extend this computations for a rotating charged BH in heterotic string theory.

The squared ADM mass for Kerr-Sen BH derived in [3] is

\[ M^2 = \frac{A_+}{16\pi} + \frac{4\pi J^2}{A_+} + \frac{Q^2}{2} .

The mass differential for both the horizons is

\[ dM = T_+ dA_+ + \Omega_{\pm} dJ + \Phi_{\pm} dQ .\]

where

\[ T_\pm = \frac{\partial M}{\partial A_\pm} = \frac{1}{M} \left( \frac{1}{32\pi} - \frac{2\pi J^2}{A_\pm^2} \right) \]
\[ \Omega_\pm = \frac{\partial M}{\partial J} M A_\pm = \frac{a}{2M r_\pm} \]
\[ \Phi_\pm = \frac{\partial M}{\partial Q} = \frac{Q}{2M} .\]

\(^1\)For recent calculation of these energies for NUT class of BHs one must see the Ref. [12].
and
\[ T_{\pm} = \text{The effective surface tension for } H_{\pm}, \]
\[ \Omega_{\pm} = \text{The angular velocity for } H_{\pm}, \]
\[ \Phi_{\pm} = \text{The electromagnetic potentials for } H_{\pm}. \]

Remarkably, the parameters \( T_{\pm}, \Omega_{\pm} \) and \( \Phi_{\pm} \) should be defined and are constant on the horizons for any stationary axisymmetric BH. Since \( dM \) is a perfect differential hence one could choose any convenient path of integration in \((A, J, Q)\) space. In particular, one could choose a path which will be defined for a charged BH in heterotic string theory. It has three energy components: the surface energy \( E_{s}^{\pm} \) defined in [3] [Eq.(76)] for \( H_{\pm} \)

\[ E_{s}^{\pm} = \int_{0}^{A_{\pm}} T_{\pm}(\tilde{A}_{\pm}, 0, 0) d\tilde{A}_{\pm}; \quad (19) \]

the rotational energy for \( H_{\pm} \) [Eq.(77) in Ref. [3]] is

\[ E_{r}^{\pm} = \int_{0}^{J} \Omega_{\pm}(A_{\pm}, \tilde{J}, 0) d\tilde{J}, A_{\pm} \text{ fixed}; \quad (20) \]

and the electromagnetic energy [Eq.(78) in Ref. [3]] for \( H_{\pm} \) is

\[ E_{em}^{\pm} = \int_{0}^{Q} \Phi_{\pm}(A_{\pm}, J, \tilde{Q}) d\tilde{Q}, A_{\pm}, J \text{ fixed}; \quad (21) \]

The ADM mass parameter or BH mass parameter in Eq. (16) may be rewritten as

\[ M(A_{\pm}, J, Q) = \sqrt{\frac{A_{\pm}}{16\pi} + \frac{4\pi J^2}{A_{\pm}} + \frac{Q^2}{2}}. \quad (22) \]

The above energy integrals should be directly computed using the variational definitions which is defined in Eq. (18). First, we would like to compute the surface energy of \( H_{\pm} \)

\[ E_{s}^{\pm} = \sqrt{\frac{A_{\pm}}{16\pi}} \quad (23) \]

Next we evaluate the rotational energy of \( H_{\pm} \) as

\[ E_{r}^{\pm} = \sqrt{\frac{A_{\pm}}{16\pi} + \frac{4\pi J^2}{A_{\pm}}} - \sqrt{\frac{A_{\pm}}{16\pi}}, \quad (24) \]

and finally we calculate the electromagnetic energy of \( H_{\pm} \) as

\[ E_{em}^{\pm} = \sqrt{\frac{A_{\pm}}{16\pi} + \frac{4\pi J^2}{A_{\pm}}} + \frac{Q^2}{2} - \sqrt{\frac{A_{\pm}}{16\pi} + \frac{4\pi J^2}{A_{\pm}}}. \quad (25) \]

Now we compute the sum of three energies

\[ E_{s}^{\pm} + E_{r}^{\pm} + E_{em}^{\pm} = \sqrt{\frac{A_{\pm}}{16\pi} + \frac{4\pi J^2}{A_{\pm}}} + \frac{Q^2}{2}. \quad (26) \]

Using Eq. (22), we can rewrite the above equation as

\[ E_{s}^{\pm} + E_{r}^{\pm} + E_{em}^{\pm} = M(A_{\pm}, J, Q). \quad (27) \]

Remarkably, the ADM mass can be expressed as the sum of three energies, namely the surface energy of \( H_{\pm} \), the rotational energy of \( H_{\pm} \) and the electromagnetic energy of \( H_{\pm} \). This is one of the key results of the paper.
4 The Scale Parameter and Distortion Parameter for Kerr-Sen BH:

Like Kerr-Newman BH \cite{8}, it is convenient to study the geometry intrinsic to the surface of charged BH in heterotic string theory. Therefore we would like to introduce the scale parameter \((\eta_\pm)\), the distortion parameter \((\beta_\pm)\) and a new parameter \((\epsilon_\pm)\) which corresponds to the BH area \((A_\pm)\), angular momentum \((J)\) and charge \((Q)\). It is helpful to study the intrinsic geometry of both \(\mathcal{H}^\pm\). They should be defined as

\[
\eta_\pm = \sqrt{\frac{A_\pm}{4\pi}}, \quad \beta_\pm = \frac{a}{\eta_\pm}, \quad \epsilon_\pm = \frac{Q}{\eta_\pm}.
\]  

In terms of these parameters, the three energy can be written as

\[
\mathcal{E}_s^\pm = \frac{\eta_\pm}{2}
\]  

\[
\mathcal{E}_r^\pm = \frac{\eta_\pm}{2} \left[ \frac{1}{\sqrt{1 - \beta_0^2}} - 1 \right]
\]

where

\[
\beta_0 = \beta(A_\pm, J, Q = 0)
\]

and

\[
\mathcal{E}_{em}^\pm = \frac{\eta_\pm}{2} \left[ \sqrt{\frac{1 + 2 \epsilon_\pm^2}{1 - \beta_\pm^2}} - \frac{1}{\sqrt{1 - \beta_0^2}} \right].
\]

The integrated mass formula is then given by

\[
\mathcal{M} = \frac{\eta_\pm}{2} \sqrt{\frac{1 + 2 \epsilon_\pm^2}{1 - \beta_\pm^2}}.
\]

This is another key observation of this work.

Alternatively, a set of values for \(\eta_\pm\) and \(\beta_\pm\) determine only the spin parameter uniquely:

\[
a = \beta_\pm \eta_\pm.
\]

on the other hand an algebraic expression relates the mass parameter and the charge parameter:

\[
\mathcal{M} = \frac{\eta_\pm}{2} \left(1 - \beta_\pm^2\right)^{-\frac{1}{2}} \left(1 + 2 \frac{Q^2}{\eta_\pm^2}\right)^{1/2}.
\]

It should be noted that the angular momentum parameter relates the charge parameter as

\[
J = \frac{\eta_\pm^2}{2} \beta_\pm \left(1 - \beta_\pm^2\right)^{-\frac{1}{2}} \left(1 + 2 \frac{Q^2}{\eta_\pm^2}\right)^{1/2}.
\]

Eq. (33) and Eq. (34) determine the identical intrinsic surface geometry of whole rotating charged BH in heterotic string theory.
5 How much rotational energy can be extracted from Kerr-Sen BH?

In a seminal work [9] Penrose & Floyd first showed how one can extract the rotational energy from a spinning (Kerr) BH. This is possible only due to the presence of the finite region of space (ergosphere) which is located between the event horizon \((r_+)\) and stationary limit surface \((r_{\text{ergo}})\) i.e. \(r_+ < r < r_{\text{ergo}}(\theta)\). The finite region has an important effect because it allows to extract the rotational energy from the BH. Here, we will derive this rotational energy for a charged rotating BH in heterotic string theory by using Eq. (27).

We find the rotational energy for Kerr-Sen BH as

\[
\mathcal{E}_r^+ = \mathcal{M} - \mathcal{E}_s^+ - \mathcal{E}_{\text{em}}^+ .
\]  

The surface energy \(\mathcal{E}_s^+\) is often called to as irreducible mass [10] of the Kerr BH as described by Christodoulou.

Now the above calculation yields

\[
\mathcal{E}_r^+ = \sqrt{\frac{\mathcal{M}}{2}} \left[ \left( \mathcal{M} - \frac{Q^2}{2\mathcal{M}} \right) + \sqrt{\left( \mathcal{M} - \frac{Q^2}{2\mathcal{M}} \right)^2 - a^2} \right] \times
\]

\[
\sqrt{\frac{2}{\sqrt{\mathcal{M} + \sqrt{\mathcal{M}^2 - a^2}}}} - 1 .
\]  

(37)

It indicates that this is the maximum amount of rotational energy we can extract from the rotating charged BH in heterotic string theory. It also implies that the quantity \(\mathcal{E}_r^+\) depends on both the charge and spin parameter.

It should be noted that when the charge parameter goes to zero value we get the value of rotational energy for Kerr BH [11]

\[
\mathcal{E}_r^+ = \sqrt{\frac{\mathcal{M}}{2}} \left( \mathcal{M} + \sqrt{\mathcal{M}^2 - a^2} \right) \sqrt{\frac{2\mathcal{M}}{\mathcal{M} + \sqrt{\mathcal{M}^2 - a^2}} - 1} .
\]

Taking the extremal limit of Kerr BH \((a = \mathcal{M})\), we find \(\mathcal{E}_r^{\text{XKerr}} = \left( \frac{\sqrt{2} - 1}{\sqrt{2}} \right) \mathcal{M}\) i.e. we can get approximately 29 percentage of its total energy.

Now taking the extremal limit of Kerr-Sen BH, we get

\[
\mathcal{E}_r^{\text{XSen}} = \left( \frac{\sqrt{2} - 1}{2} \right) \sqrt{2\mathcal{M}^2 - Q^2} = \left( \frac{\sqrt{2} - 1}{\sqrt{2}} \right) \sqrt{J} .
\]

It follows that the energy value strictly depends on charge parameter or the angular momentum parameter.

Now we compute the ratio

\[
\frac{\mathcal{E}_r^{\text{XSen}}}{\mathcal{E}_r^{\text{XKerr}}} = \sqrt{\frac{1 - \frac{Q^2}{2\mathcal{M}^2}}{\mathcal{M}^2}} = \sqrt{\frac{J}{\mathcal{M}^2}} .
\]

Note that the ergosphere coincides with \(r_+\) only at the poles \(\theta = 0\) and \(\theta = \pi\).
In the limit $J = M^2$, the rotational energy is indeed equal for extremal Kerr BH and extremal Kerr-Sen BH. Now if we increase the spin value we have

$$\mathcal{E}_{t}^{\pm\text{Sen}} > \mathcal{E}_{t}^{\pm\text{Kerr}}$$

(38)

This is the last key result of this work. To the best of our knowledge, this is the first time we have examined the energy extraction process for a BH in heterotic string theory.

6 Discussion

In this paper we have considered the three parameter charged rotating BH solutions in the low energy limit of effective field theory in heterotic string theory. We examined four important new results:

(i) We computed the surface energy ($\mathcal{E}_s^\pm$), the rotational energy ($\mathcal{E}_r^\pm$) and the electromagnetic energy ($\mathcal{E}_{em}^\pm$) for a spinning charged BH in heterotic string theory having the event horizon and the Cauchy horizon.

(ii) Remarkably, we showed that the sum of three energies is equal to the ADM mass parameter i.e.

$$\mathcal{E}_s^\pm + \mathcal{E}_r^\pm + \mathcal{E}_{em}^\pm = M$$

(iii) Moreover, in terms of the scale parameter, the distortion parameter and a new parameter ($\epsilon^\pm$) which corresponds to the surface area of the BH, the angular momentum and the charge parameter. Remarkably we proved that the ADM mass parameter in a compact form like

$$\mathcal{E}_s^\pm + \mathcal{E}_r^\pm + \mathcal{E}_{em}^\pm = M = \frac{\eta^\pm}{2} \sqrt{\frac{1 + 2\epsilon^2_{\pm}}{1 - \beta^2_{\pm}}}$$

which is valid for $\mathcal{H}^\pm$.

(iv) Finally, using this mass decomposition formula, we computed the value of rotational energy that should be extracted from the spinning charged BH in heterotic string theory via famous Penrose process. The maximum value of extracted rotational energy for extremal Kerr-Sen BH is found to be

$$\mathcal{E}_t^+ = \left(\sqrt{2} - 1\right) \sqrt{\frac{J}{2}}$$

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