Spatial analyticity and exponential decay of Fourier modes for the stochastic Navier-Stokes equation

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Abstract. We construct a local in time spatially real-analytic solution to the 2D and 3D stochastic Navier–Stokes equation driven by a spatially real-analytic multiplicative and transport noise but emanating from an initial condition that is only required to have bounded enstrophy. Under the condition that the solution is global in time, we also establish the exponential decay of the finite-dimensional Galerkin approximation, with respect to its maximum wavenumber, to the strong pathwise solution of the stochastic Navier–Stokes equation. This decay is uniform in time, uniform with respect to the initial enstropy, and uniform in the noise coefficients.

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1. Introduction

The Navier–Stokes equation is a widely used model to describe the evolution of incompressible viscous fluids. It represents a balance law between the forces present in the fluid and the acceleration of the fluid. Classically, the main forces taken into account will consist of the deterministic conservative pressure, non-conservative viscous term, and perhaps, all other external deterministic forces lumped up together. However, to take uncertainties in the fluid into account, it is desirable to incorporate noise into the system. This noise may be understood as a random force that is a function of the underlying deterministic unknowns. Another form of noise is the transport noise of Stratonovich-type, that has the desirable property of preserving some of the physical laws satisfied by the deterministic system.

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In this work, we are interested in the regularity of solutions to the 2D and 3D stochastic Navier–Stokes equation for incompressible fluids. In particular, we explore a specific Gevrey class regularity that renders the solution spatially real-analytic. The analysis of Gevrey class regularity for fluid-dynamic systems has a long history spanning many decades. The fundamental work by Foias and Temam [14], in particular, bridged the gap between the functional analysis of general differential equations and the study of regularity for solutions to fluid dynamic equations. Since the work done by Foias and Temam, the treatment of Gevrey class regularity for deterministic fluid dynamic systems has gained considerable attention. We cannot give an exhaustive review but some fundamental works include [1, 6, 11, 15, 21, 22, 30]. Unfortunately, very few works exist on the Gevrey class regularity for stochastic systems. From an analytic point of view, the only work we have found is by Mattingly [25] who analysed the Navier–Stokes equation with a white-in-time stochastic forcing. An analytic solution is derived from analytic data and discussions on stationary measures and ergodicity are also presented. From the numerical analysis point of view, a few more results are present [20, 24]. In [20], the authors show the convergence of the Galerkin approximation to the strong solution of the 2D Navier–Stokes equation defined on a bounded domain with Dirichlet boundary condition and that is driven by a multiplicative noise. However, the rate of convergence is not given and the convergence holds for the expectation of a logarithmic function of the norm of the difference of the two solutions, rather than for the norm itself. In [24], however, the authors give an inverse polynomial rate of convergence for general (viscous) stochastic PDEs driven by additive noise whose coefficients are analytic.

In this work, we analyze a generalized Navier–Stokes equation on a 2D and 3D torus that accounts for all the deterministic and random forces discussed above. We explore its Gevrey class regularity that results in the unknowns becoming spatially real-analytic. In particular, for data consisting of just a bounded initial enstrophy and spatially real-analytic forcing terms, we construct a local in time spatially real-analytic solution to the Navier–Stokes equation. This is a regularization result in the sense that we obtain an analytic solution from an initial condition that is far less regular as observed in many deterministic dissipative systems [3, 6, 11, 14, 30]. However, a caveat is that we need more regularity for the forced data of stochastic transport type than the anticipated regularity of the solution to be constructed. As such, although the solution becomes regularized with respect to its initial condition, the trade-off is that we need more regular transport noise coefficients, in particular we assume they are constant vectors in the time variable. However, the assumptions are optimal for noise with linear growth and Lipschitz coefficients (like additive and multiplicative noise) where we only require the same regularity for the noise coefficient as that of the anticipated solution. To our knowledge, this regularity result is a first for the stochastic Navier–Stokes equation.
Our second main result involves the analysis of the rate of decay of the finite-dimensional solution used in the approximation of the constructed solution. Here, we assume that the constructed solution is global in time. We measure the decay rate in the space of enstrophy functionals and show that the rate of decay is exponential as a function of its Fourier modes. We refer to Theorem 3.13 for the exact statement. As a consequence, in particular, we can infer that irrespective of the addition of noise, each Fourier coefficient of our solution decays faster than exponentially (see the first estimate in (6.3)), uniformly in time, uniform with respect to the initial enstrophy and uniformly in the noise coefficients, with the same rate as we will expect of the corresponding deterministic system [9, 15]. Also, see [10] for a related result for the generalized Ginzburg-Landau equation. Subsequently, the difference between the continuum and discrete velocities in the spatial $H^1$-norm decays exactly exponentially fast to zero.

1.1. Plan

In order to improve readability, we begin with some preliminaries covered in Section 2 where we introduce the various notations and functional set-up needed to present and prove our main results. We then state the precise equation we wish to study in Section 3 and give the assumptions under which the equation is analysed. We also make precise in this section the various concepts of a solution and finally, state our main results. In order to avoid long computations in our proofs, we present in Section 4 some key results that will be used in the proof of our main results. Next, we devote the entirety of Section 5 to the proof of our first main result concerning the construction of a unique spatially real-analytic solution of the stochastic Navier-Stokes equation. Here, the main tool in our construction is the application of the extension of the Cauchy method [16, Lemma 5.1] to the Gevrey classes taking into account the additional transport noise and its Stratonovich-to-Itô corrector. Finally, we use Section 6 to study the asymptotic behaviour of the finite-dimensional Galerkin approximations which is the subject of our second main result, Theorem 3.13.

2. Preliminaries

2.1. Notation

We consider a spacetime cylinder consisting of spatial points $\mathbf{x} = (x_1, \ldots, x_d)$ on the torus $\mathbb{T}^d = (\mathbb{R}/2\pi\mathbb{Z})^d$, $d = 2, 3$ with periodic boundary condition and a time variable $t \in [0, T]$ where $T > 0$ is fixed and arbitrary. For functions $F$ and $G$, we write $F \lesssim G$ if there exists a generic constant $c > 0$ such that $F \leq cG$. We also write $F \lesssim_p G$ if the constant $c(p) > 0$ depends on a variable $p$. If $F \lesssim G$ and $G \lesssim F$ both hold (respectively, $F \lesssim_p G$ and $G \lesssim_p F$), we use the notation $F \sim G$ (respectively, $F \sim_p G$). The symbol $|\cdot|$ may be used in four different context. For a scalar function $f \in \mathbb{R}$, $|f|$ denotes the absolute value of $f$. For a vector $\mathbf{f} \in \mathbb{R}^d$, $|\mathbf{f}|$ denotes the Euclidean norm.
of $f$. For a square matrix $F \in \mathbb{R}^{d \times d}$, $|F|$ shall denote the Frobenius norm $\sqrt{\text{trace}(F^T F)}$. Lastly, if $S \subseteq \mathbb{R}^d$ is a measurable subset, then $|S|$ is the $d$-dimensional Lebesgue measure of $S$.

Throughout this paper, we fix a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ with the complete right-continuous filtration $(\mathcal{F}_t)_{t \geq 0}$. In addition, $\mathbb{E}(\cdot) = \int_{\Omega} (\cdot) \, d\mathbb{P}$ is the expectation of the argument denoted by the dot. Let $(W^k)_{k \geq 1}$ be a family of one-dimensional $(\mathcal{F}_t)$-adapted Wiener/Brownian processes. A stopping time $\tau$ is a random variable $\tau : \Omega \to [0, T]$ such that $\{\omega : \tau(\omega) \leq t\} \in \mathcal{F}_t$ for each $t \in [0, T]$. We denote by $L^p_{\mathcal{F}_t}(\Omega; H)$, where $p \in [1, \infty)$, the space of equivalence class of random variables $f : \Omega \to H$ that are $\mathcal{F}_t$-measurable and for which $\|f\|_H$ has finite $p$-moments.

### 2.2. The functional setting

Let $C^\infty_{\text{div}}(\mathbb{T}^d)$ denote the space of all divergence-free smooth periodic functions. For $r \geq 0$, let the Hilbert space $(H^r(\mathbb{T}^d), \|\cdot\|_{H^r})$ represent the $L^2$-homogeneous Sobolev space of mean-free functions defined as

$$H^r(\mathbb{T}^d) = \left\{ f(x) = \sum_{k \in \mathbb{Z}^d} \hat{f}_k e^{i k \cdot x} \in L^2(\mathbb{T}^d) \left| \hat{f}_{k} = \langle f(x), e^{i k \cdot x} \rangle_{L^2} \right. \right\}.$$

$$= \int_{\mathbb{T}^d} e^{-i k \cdot x} f(x) \, dx \in \mathbb{C}^d, \quad \hat{f}_k = \hat{f}_{-k}, \quad \hat{f}_0 = 0, \quad (2.1)$$

$$\|f\|_{H^r} = \left( \sum_{k \in \mathbb{Z}^d} |k|^{2r} |\hat{f}_k|^2 \right)^{\frac{1}{2}} < \infty.$$

Here, the $\hat{f}_k$s are the Fourier coefficients of the function $f \in L^2(\mathbb{T}^d)$.

**Remark 2.1.** Note the $\hat{f}_0 = 0$ in (2.1) automatically renders elements in these spaces mean-free, i.e., $\int_{\mathbb{T}^d} f(x) \, dx = 0$ for any $f \in H^r(\mathbb{T}^d)$. Also, note that the underlining $L^2$-inner product is given by $\langle f, g \rangle = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x) \cdot \overline{g}(x) \, dx$.

With (2.1) in hand, we now define $(\mathbb{H}^r(\mathbb{T}^d), \|\cdot\|_{H^r})$ as the subclass of $(H^r(\mathbb{T}^d), \|\cdot\|_{H^r})$ satisfying

$$\mathbb{H}^r(\mathbb{T}^d) = \left\{ f(x) = \sum_{k \in \mathbb{Z}^d} \hat{f}_k e^{i k \cdot x} \in H^r(\mathbb{T}^d) \left| \hat{f}_k \cdot k = 0 \right. \right\}.$$

**Remark 2.2.** The condition $\hat{f}_k \cdot k = 0$ corresponds to the divergence-free condition $\text{div} f = 0$ in frequency space.

The space $H^2(\mathbb{T}^d)$ is the domain of the positive, self-adjoint Stokes operator

$$A = P(-\Delta), \quad A : D(A) \to \mathbb{H}^0(\mathbb{T}^d) =: L^2(\mathbb{T}^d);$$

the domain $D(A)$ is compactly embedded in $L^2(\mathbb{T}^d)$ and where $P = I - \nabla \Delta^{-1} \text{div}$ is the Leray projector. It is well-known that $P$ commutes with the derivative operators and it can be restricted to a bounded linear operator from $H^r$ to $\mathbb{H}^r$, see [23, 31, 32]. Also, $A$ has a compact inverse $A^{-1}$ and
possesses a nondecreasing sequence \( \{\lambda_j\}_{j \in \mathbb{N}} \) of strictly positive eigenvalues approaching infinity as \( j \to \infty \) with associated orthonormal basis \( \{e_j\}_{j \in \mathbb{N}} \) in \( L^2(\mathbb{T}^d) \). By using the properties \( \bar{f}_k = \hat{f}_{-k} \) and \( \hat{f}_k \cdot k = 0 \), explicitly, we obtain for each \( k \in \mathbb{Z}^d \),

\[
\lambda_k = |k|^2, \quad e_k = a_k e^{ik \cdot x} + \bar{a}_k e^{-ik \cdot x}
\]

where for each \( k, a_k \) is a complex vector in \( \mathbb{C}^d \) with the property that \( a_k \cdot k = 0 \) and \( \bar{a}_k = a_{-k} \). The pair of sequences \( \{(\lambda_j, e_j)\}_{j \in \mathbb{N}} \) then corresponds to the rearrangement of \( \{(\lambda_k, e_k)\}_{k \in \mathbb{Z}^d} \) in nondecreasing order of \( \lambda_k \).

Next, by using the definition of the Stokes operator \( A \), for any \( r \in \mathbb{R} \), we can define its fractional power \( A^r \) as the mapping

\[
f(x) = \sum_{k \in \mathbb{Z}^d} \hat{f}_k e^{ik \cdot x} \mapsto A^r f = \sum_{k \in \mathbb{Z}^d} |k|^{2r} \hat{f}_k e^{ik \cdot x} = \sum_{k \in \mathbb{Z}^d} \lambda_k^r \langle f(x), e^{ik \cdot x} \rangle_{L^2} e^{ik \cdot x}
\]

with \( A^0 = I \) so that

\[
\|A^r f\|^2_{L^2} = \sum_{k \in \mathbb{Z}^d} |k|^{4r} |\hat{f}_k|^2 = \sum_{k \in \mathbb{Z}^d} \lambda_k^{2r} |\hat{f}_k|^2 = \|f\|^2_{H^{2r}}.
\]

We let \( P^N : L^2(\mathbb{T}^d) \to H_N \) be the \( L^2 \)-orthogonal projection onto \( H_N = \text{span}\{e^{ik \cdot x} \mid |k| \leq N\} \). Note that by using the definition of \( P^N \), we directly obtain these inequalities for trading Fourier modes for regularity

\[
\|P^N f\|_{H^s}^2 \leq N^{2(s-r)} \|P^N f\|_{H^r}^2, \quad \|(I - P^N) f\|_{H^r}^2 \leq N^{2(r-s)} \|(I - P^N) f\|_{H^s}^2
\]

for \( 0 \leq r \leq s \) and for all \( N \in \mathbb{N} \). Also, the continuity property

\[
\|P^N f\|_{H^r}^2 \lesssim \|f\|_{H^r}^2, \quad \|(I - P^N) f\|_{H^r}^2 \lesssim \|f\|_{H^r}^2
\]

holds uniformly in \( N \in \mathbb{N} \) for all \( r \in \mathbb{R} \).

With (2.2) in hand, we can recall the so-called Gevrey class spaces. For \( s > 0 \) and \( r \geq 0 \), we say that \( f \) is of Gevrey class \( s \) (i.e. \( f \in G^s(\mathbb{T}^d) \)) if and only if there exists \( \varphi > 0 \) such that

\[
f \in D(e^{\varphi A^{1/2s}} : \mathbb{H}^r(\mathbb{T}^d)) \equiv \{f(x) \in \mathbb{H}^r(\mathbb{T}^d) : \|e^{\varphi A^{1/2s}} f(x)\|_{H^r} < \infty\}.
\]

see [22]. In the Gevrey spaces, we refer to \( r \) as the Sobolev corrector and \( \varphi \) is the radius of analyticity or width of analyticity. To account for variations in the radius of analyticity, we consider the parameter \( \varphi \) as a function \( \varphi = \varphi(t) \) of time, see [15] Page 71.

**Remark 2.3.** When we don’t need to use divergence-free vector fields, we can define the Gevrey space in terms of (2.1) rather than (2.2) and still retain the same properties above.
3. The main results

With our preparation in hand, we can now present the system understudy. Here, we are interested in a velocity field $u : \Omega \times [0, T] \times \mathbb{T}^d \to \mathbb{R}^d$ and a pressure $p : \Omega \times [0, T] \times \mathbb{T}^d \to \mathbb{R}$ that satisfy

$$
\begin{align*}
\frac{d}{dt}u + [(u \cdot \nabla)u - \nu \Delta u] dt + d\nabla p &= \sum_{k \geq 1} [g_k(u) - (\xi_k \cdot \nabla)u] \circ dW^k_t, \\
\text{div} u &= 0, \\
u > 0 \text{ is the viscosity coefficient and the symbol } \circ \text{ means that the stochastic integral is understood in the Stratonovich sense. Details on the body forces } (g_k)_{k \geq 1} \text{ and the coefficients of the noise advection } (\xi_k)_{k \geq 1} \text{ will be given in the next section.}
\end{align*}
$$

(3.1)

The system is a generalisation of the deterministic Navier–Stokes equation that takes into account several random phenomena in the fluids. In particular, this may cover additive noise, multiplicative noise and transport noise of Stratonovich type, and a first rigorous analysis can be found in [28]. When $\xi_k \equiv 0$, the resulting stochastic partial differential equation have been studied intensively by many authors including [2, 5, 7, 13].

On the other hand, when $g_k(u) \equiv 0$, the system (3.1) becomes a so-called SALT (Stochastic Advection by Lie Transport) model introduced by Holm [19] but adjusted for viscous effects as studied in [12] in vorticity form. The (inviscid) SALT models are derived from a stochastic variational principle and their solutions follow the flow of a stochastic vector field. The viscous SALT variant above are, however, closely related to the LU (Location Uncertainty) models introduced by Mémin [26] that relies on a decomposition of the velocity fields into a random non-differentiable part and a differentiable deterministic component. In the specific case of the Stochastic Navier–Stokes under LU, one considers a random vector fields $\sigma : \Omega \times [0, T] \times \mathbb{T}^d \to \mathbb{R}^d$ and define an elliptic variance tensor $A = (A_{ij})_{i,j=1}^d \in \mathbb{R}^{d \times d}$ as $A(x, t)\delta(t-s) dt = \mathbb{E}[(\sigma(x, t) dW_t)(\sigma(x, s) dW_s)^T]$ where $W_t = (W^k_t)_{k \geq 1}$. The system of equation is then given by

$$
\begin{align*}
\partial_t u + (u \cdot \nabla)u - \frac{1}{2} (\text{div} A \cdot \nabla)u - \frac{1}{2} \text{div}(A \nabla u) &= \nu \Delta u - \nabla p, \\
\nabla \cdot \nabla u &= 0, \\
div (\sigma dW_t) &= 0, \\
u > 0 \text{ is the viscosity coefficient and the symbol } \circ \text{ means that the stochastic integral is understood in the Stratonovich sense. Details on the body forces } (g_k)_{k \geq 1} \text{ and the coefficients of the noise advection } (\xi_k)_{k \geq 1} \text{ will be given in the next section.}
\end{align*}
$$

(3.2)

where $p(x, t)$ denotes the large-scale pressure contribution and $\hat{p}_t$ is a zero-mean turbulent pressure related to the small-scale velocity component. From the analytic point of view, (3.3) is strongly related to (3.1) (with $g_k(u) \equiv 0$) in the sense that both system preserves the deterministic energy since the noise terms and the variance tensor terms cancels out when deriving an identity for $\|u\|_{L^2}^2$, assuming that $u$ is smooth enough.
Returning to (3.1), it is convenient to eliminate the pressure in (3.1) and only ask to find the velocity field. This is done by applying Leray’s projection $P = I - \nabla \Delta^{-1} \text{div}$ to the first equation in (3.1). Doing so and converting the Stratonovich integral to an Itô integral results in

$$
\begin{align*}
\text{d}u + [P((u \cdot \nabla)u) - \nu \Delta u] \, \text{d}t &= \frac{1}{2} \sum_{k \geq 1} P((\xi_k \cdot \nabla)P(\xi_k \cdot \nabla)u) \, \text{d}t \\
&\quad + \sum_{k \geq 1} P[g_k(u) - ((\xi_k \cdot \nabla)u)] \, \text{d}W_t^k,
\end{align*}
\tag{3.3}
$$

$$
\text{div} u = 0,
$$

$$
u u(x,0) = u_0(x).
$$

Note that by using the incompressibility condition $\text{div} u = 0$, one can recover the pressure from (3.1) by solving

$$
\text{d}p = (-\Delta)^{-1} \text{div}((u \cdot \nabla)u) \, \text{d}t - \sum_{k \geq 1} (-\Delta)^{-1} \text{div} [g_k(u) - (\xi_k \cdot \nabla)u] \, \text{d}W_t^k.
$$

Also note that if we consider the orthogonal complement $Q$ of $P$, then for incompressible vector fields $u$ and $\xi_k$, we have that

$$
\langle P((\xi_k \cdot \nabla)P(\xi_k \cdot \nabla)u), u \rangle = \langle ((\xi_k \cdot \nabla)P(\xi_k \cdot \nabla)u), u \rangle \\
= -\langle P(\xi_k \cdot \nabla)u, (\xi_k \cdot \nabla)u \rangle \\
= -\langle P(\xi_k \cdot \nabla)u, P(\xi_k \cdot \nabla)u \rangle \\
- \langle P(\xi_k \cdot \nabla)u, Q(\xi_k \cdot \nabla)u \rangle \\
= -\|P(\xi_k \cdot \nabla)u\|_{L^2}^2.
$$

This gives a useful cancellation property between the Stratonovich to Itô corrector and the quadratic variation term that appears when Itô’s formula is used to obtain an energy estimate from (3.3).

3.1. Assumptions

Let $\sigma_1 \geq 0$, $s > 0$ and $r \geq 0$ be parameters that will be chosen later. For the family of mappings $(g_k)_{k \geq 1} : [0,T] \times \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}^d$, the following estimates

$$
\sum_{k \geq 1} \|e^{\sigma_1 A^{1/2s}}g_k(t,x,v)\|_{H^r} \lesssim 1 + \|e^{\sigma_1 A^{1/2s}}v\|_{H^r},
\tag{Growth} (3.4)
$$

$$
\sum_{k \geq 1} \|e^{\sigma_1 A^{1/2s}}[g_k(t,x,v) - g_k(t,x,w)]\|_{H^r} \lesssim \|e^{\sigma_1 A^{1/2s}}[v - w]\|_{H^r}
\tag{Lipschitz} (3.5)
$$

holds a.s. for any $v, w \in D(e^{\sigma_1 A^{1/2s}}: \mathbb{H}^r(\mathbb{T}^d))$, where the constants in (3.4)–(3.5) are independent of $t \in [0,T]$. Additionally, we assume that

$$
\int_{\mathbb{T}^d} g_k(t,x,v) \, \text{d}x = 0 \text{ for each } k \geq 1,
\tag{Mean – free} (3.6)
$$

$$
\hat{T}d g_k(t,x,v) \, \text{d}x = 0 \text{ for each } k \geq 1.
\tag{Mean – free} (3.6)
$$
Remark 3.1. The assumptions imposed on the noise coefficients \((g_k)_{k \geq 1}\) cover but are not limited to the cases of additive and linear multiplicative noise.

With regards to the transport noise, we require that the sum of the coefficients \((\xi_k)_{k \geq 1}\) is spatially analytic and each vector is solenoidal. For example, \((\xi_k)_{k \geq 1}\) encodes data from satellites, drifters, and floats in large-scale ocean dynamics \cite{18}. More precisely, we assume that for any \(r \geq 0, s > 0\) and \(\sigma_2 \geq 0\),

\[
\sum_{k \geq 1} \|e^{\sigma_2 A^{1/2s}} \xi_k\|_{H^r} \leq K < \infty, \quad \text{div} \xi_k = 0 \quad \text{(3.7)}
\]

holds for a constant \(K > 0\).

Finally, in order to treat the multiplication noise and the transport noise separately, we impose an orthogonality constraint between them. More precisely, we assume that

\[
\langle A^r e^{\sigma_1 A^{1/2s}} g_k(v), A^r e^{\sigma_1 A^{1/2s}} ((\xi_k \cdot \nabla) w) \rangle_{L^2} = 0 \quad \text{(3.8)}
\]

holds for any \(k \geq 1\) and any \(v, w \in D(e^{\sigma_1 A^{1/2s}} : \mathbb{H}^r(\mathbb{T}^d))\).

3.2. Concepts of solution

We now make precise the various notion of a solution that we shall refer to throughout this work.

**Definition 3.2 (Local strong pathwise solution).** Let \(\tau\) be an a.s. strictly positive stopping time and let \(u : [0, \tau) \to \mathbb{H}^r(\mathbb{T}^d)\) be a stochastic process for some \(r \geq 1\). We call the pair \((u, \tau)\) a local strong pathwise solution of (3.3) if there exists an increasing sequence of strictly positive stopping times \((\tau_l)_{l \geq 1}\) such that:

- \(\tau_l < \tau\) and \(\lim_{l \to \infty} \tau_l = \tau\) a.s.;
- \(u\) is \((\mathcal{F}_t)\)-progressively measurable and for each \(l \geq 1\),
  
  \[
  u(\cdot \wedge \tau_l) \in C([0, T]; \mathbb{H}^r(\mathbb{T}^d)) \cap L^2((0, T); \mathbb{H}^{r+1}(\mathbb{T}^d)) \quad \mathbb{P}\text{-a.s.};
  \]
- For each \(l \geq 1\), the equation

  \[
  u(t \wedge \tau_l) = u_0 - \int_0^{t \wedge \tau_l} P((u, \nabla) u) \, d\sigma + \nu \int_0^{t \wedge \tau_l} \Delta u \, d\sigma + \frac{1}{2} \sum_{k \geq 1} \int_0^{t \wedge \tau_l} P((\xi_k \cdot \nabla) P(\xi_k \cdot \nabla) u) \, d\sigma
  \]

  \[
  + \sum_{k \geq 1} \int_0^{t \wedge \tau_l} Pg_k(u) \, dW^k_\sigma - \sum_{k \geq 1} \int_0^{t \wedge \tau_l} P((\xi_k \cdot \nabla) u) \, dW^k_\sigma \quad \text{(3.9)}
  \]

holds a.s. for all \(t \in [0, T]\).
Definition 3.3 (Maximal strong pathwise solution). A local strong pathwise solution \((u, \tau)\) of \((3.3)\) is called a maximal strong pathwise solution of \((3.3)\) if for any other local strong pathwise solution \((u', \tau')\), we have
\[
\tau' \leq \tau \quad \text{a.s. and} \quad u = u' \quad \text{a.e. on} \quad \Omega \times [0, \tau') \times \mathbb{T}^d. \tag{3.10}
\]

Remark 3.4. Note that by definition, maximal strong pathwise solutions are essentially unique in the class of local strong pathwise solutions in the following sense. For any two maximal strong pathwise solutions \((u_1, \tau_1)\) and \((u_2, \tau_2)\), it follows from \((3.10)\) that
\[
\tau_1 = \tau_2 = \tau \quad \text{a.s. and} \quad u_1 = u_2 \quad \text{a.e. on} \quad \Omega \times [0, \tau) \times \mathbb{T}^d.
\]

But one could say even more. Using continuity of the trajectories of \(u_1\) and \(u_2\) - as \(\mathbb{H}^r(\mathbb{T}^d)\)-valued random variables - one could replace the condition \(u_1 = u_2 \text{ a.e. on } \Omega \times [0, \tau) \times \mathbb{T}^d\) with the stronger statement that there exists a full measure set \(\tilde{\Omega} \subseteq \Omega\) such that, for every \(\omega \in \tilde{\Omega}\) it holds \(u_1(\omega, t) = u_2(\omega, t)\) (as an identity in \(\mathbb{H}^r(\mathbb{T}^d)\)) for every \(t < \tau(\omega)\). In particular, the two processes \(u_1\) and \(u_2\) are indistinguishable.

In the following, we state a result from \(\cite{16}\) on the existence of an essentially unique local (global when \(d = 2\)) strong pathwise solution of \((3.3)\) without the transport noise term. An extension to the full system \((3.3)\) applies and coincides with the construction that is to be performed in Section 5 when the radius of analyticity is \(\varphi = 0\).

Theorem 3.5 (An essentially unique local strong pathwise solution). Let \((g_k)_{k \geq 1}\) satisfy \((3.4)\) - \((3.6)\) with \(r = 1\), \(s > 0\) and \(\sigma_1 = 0\) and let \((\xi_k)_{k \geq 1}\) satisfy \((3.7)\) with \(r > 4\), \(s > 0\) and \(\sigma_2 = 0\). For \(u_0 \in L^p_{\mathbb{F}_0}(\Omega; \mathbb{H}^1(\mathbb{T}^d))\) with \(p \in (1, \infty)\), there exists an essentially unique local strong pathwise solution \((u, \tau)\) of \((3.3)\) such that
\[
\mathbb{E}\left[\sup_{t \in [0, T \wedge \tau]} \|u\|_{H^s}^p + \int_0^{T \wedge \tau} \|u\|_{H^{s-2}}^2 \|u\|_{H^s}^{p-2} dt\right] \leq c \tag{3.11}
\]
holds with a constant \(c = c(u_0, K)\) where \(K\) is the upper bound appearing in \((3.7)\).

We can now make precise, what we mean by a solution of \((3.3)\) being of Gevrey class \(s\).

Definition 3.6 (Gevrey class \(s\) solution). Let \((g_k)_{k \geq 1}\) and \((\xi_k)_{k \geq 1}\) satisfy the assumptions in Section 3.1 for some \(r \geq 0\), \(\sigma_1, \sigma_2 > 0\) and \(s > 0\). Let \(\tau\) be an a.s. strictly positive stopping time and let \(u : [0, \tau) \to \mathbb{H}^r(\mathbb{T}^d)\) be a stochastic process with \(r \geq 1\). We call the pair \((u, \tau)\) a Gevrey class \(s\) solution of \((3.3)\) if:

1. \((u, \tau)\) is a local strong pathwise solution of \((3.3)\);
2. \(u(\cdot \wedge \tau) \in C((0, \tau); \mathbb{D}(e^{F(\cdot)}, A_1^{1/2s}) : \mathbb{H}^r(\mathbb{T}^d))\) a.s.

Remark 3.7. Henceforth, we will set \(r = s = 1\) and \(\varphi(t) = t\) and concentrate on the Gevrey class 1 solution. Please see the preliminary section of \(\cite{22}\) for...
further discussions on the relationship between this Gevrey class and the space of analytic functions.

Remark 3.8. Here, \( C((0, \tau); D(e^{tA^{1/2s}} : \mathbb{H}^r(T^d))) \) denotes the set of functions \( u : (0, \tau) \to D(e^{tA^{1/2s}} : \mathbb{H}^r(T^d)) \) being continuous with respect to the norm topology, i.e.,

\[
e^{t_kA^{1/2s}}u(t_k) \to e^{tA^{1/2s}}u(t) \quad \text{in} \quad \mathbb{H}^r(T^d)
\]

for any sequence \((t_k)_{k \in \mathbb{N}} \subset (0, \tau)\) with \( t_k \to t \).

### 3.3. Main results

Our first main theorem is the following.

**Theorem 3.9 (An essentially unique Gevrey class 1 solution).** Let \((g_k)_{k \geq 1}\) satisfy (3.4)–(3.6) with \( r = s = 1 \) and \( \sigma_1 = t \) and let \((\xi_k)_{k \geq 1}\) satisfy (3.7) with \( r > 4, s = 1 \) and \( \sigma_2 = T \). Now suppose that there exists a deterministic \( K_0 > 0 \) such that

\[
\|u_0\|_{H^1}^2 \leq K_0 \quad \text{a.s.} \quad (3.12)
\]

Then there exists an essentially unique Gevrey class 1 (real-analytic) solution \((u, \tau)\) of (3.3).

**Remark 3.10.** Note that due to Theorem 3.5, the proof of Theorem 3.9 is done once we show the second item in Definition 3.6.

**Remark 3.11.** It is possible to generalize the assumption (3.12) to \( u_0 \in L^p_F(\Omega; \mathbb{H}^1(T^d)) \) with \( p \in (1, \infty) \) by employing a truncating argument in the spirit of [16]. That is, for \( u_0 \in L^p_F(\Omega; \mathbb{H}^1(T^d)) \), we define

\[
u^k_0 := u_0 1\{k \leq \|u_0\|_{H^1} < k+1\}
\]

for \( k \geq 0 \) so that the boundedness assumption (3.12) holds for \( u^k_0 \). Subsequently, as we shall soon see, this will generate a Gevrey class 1 solution \((u^k, \tau^k)\) of (3.3). We then define the pair

\[
\begin{align*}
u := \sum_{k \geq 0} u^k 1\{k \leq \|u_0\|_{H^1} < k+1\}, \\
\tau := \sum_{k \geq 0} \tau^k 1\{k \leq \|u_0\|_{H^1} < k+1\},
\end{align*}
\]

and find that \((\nu, \tau)\) is indeed a Gevrey class 1 solution of (3.3) with the general initial condition \( u_0 \in L^p_F(\Omega; \mathbb{H}^1(T^d)) \). However, for clarity of exposition, we impose the bounded initial condition (3.12) throughout the paper.

By using the fact that the Gevrey class 1 functions are contained in the space of smooth functions, an immediate corollary of Theorem 3.9 is the following smoothing result. See [22, Lemma 3] for more details.

**Corollary 3.12 (Instantaneous smoothing).** Let assumptions of Theorem 3.9 hold. Any Gevrey class 1 solution \((\nu, \tau)\) of (3.3) satisfies \( \nu(\cdot \wedge \tau) \in C((0, \tau); C^\infty_{\text{div}}(T^d)) \) a.s.
Finally, by using the spatial discretization in the construction of the above Gevrey class regularity, we give a quantitative error estimate for the difference between the original continuum solution and the solution to the equation solved by the truncated finite-dimensional approximation. This estimate is uniform over the whole time interval $[0, T]$, uniform with respect to the initial enstrophy, and uniform with respect to the noise coefficients. In particular, uniformly with respect to the aforementioned parameters, we can infer that the Fourier coefficient $\hat{u}_k(t)$ decay exponentially with respect to its Fourier modes or wavenumber $|k|$. The exact statement of the result is given in Theorem 3.13 below. To state this theorem, we let $u^N$ be a unique solution to a finite-dimensional Galerkin approximation of (3.3). Further details on this approximation will be given in Section 5 before we give the proof of Theorem 3.13.

**Theorem 3.13 (Rate of decay).** Let the assumptions in Theorem 3.9 hold so that for $u_0$ satisfying (3.12), $(u, \tau)$ is the corresponding essentially unique Gevrey class solution of (3.3) obtained as the limit of the Galerkin approximation $u^N$. Assume that $\tau = \infty$ $\mathbb{P}$-a.s. and for each $R > 0$, let

$$
\tau_R := \inf \left\{ t \in (0, T) : \int_0^t \left( \|u^N(\sigma)\|_{H^2}^2 + \|u(\sigma)\|_{H^2}^2 \right) \, d\sigma \geq R \right\}
$$

be a stopping time. Then there exist a constant $c = c(\nu, K_0, K)e^{c(\nu, K)(T+R)}$ such that for any $t \in (0, T]$, we have that

$$
\mathbb{E}\|\left(u - u^N\right)(t \wedge \tau_R)\|_{H^1}^2 \leq c e^{-2\delta N}
$$

for some deterministic time $\delta > 0$.

### 4. Preparatory results

Before proving our main results, we collect in this section, various essential results that will be needed in the sequel. To begin with, we give an estimate for the convective term in the Navier-Stokes equation that is useful in showing the required Gevrey class regularity. The proof can be found in [14, Lemma 2.1].

**Remark 4.1.** Henceforth, we drop the $L^2$ in the inner product $\langle \cdot, \cdot \rangle_{L^2}$.

**Lemma 4.2.** Let $u, v, w \in D(e^{\varphi A^{1/2}} : \mathbb{H}^2(\mathbb{T}^d))$ with $\varphi \geq 0$. We have

$$
P((u \cdot \nabla)v) \in D(e^{\varphi A^{1/2}} : \mathbb{L}^2(\mathbb{T}^d))$$

and the inequality

$$
\left| \left\langle A^{1/2} e^{\varphi A^{1/2}} P((u \cdot \nabla)v), A^{1/2} e^{\varphi A^{1/2}} w \right\rangle \right| \lesssim \left\| A^{1/2} e^{\varphi A^{1/2}} u \right\|_{L^2}^2 \left\| A e^{\varphi A^{1/2}} u \right\|_{L^2}^{1/2} \left\| A e^{\varphi A^{1/2}} v \right\|_{L^2} \left\| A e^{\varphi A^{1/2}} w \right\|_{L^2}^{1/2}
$$

holds.

The next lemma, whose proof can be found in the appendix, is a further useful estimate for the deterministic nonlinear term in the Navier–Stokes equation.
Lemma 4.3. For \( u, v \in D\left(e^{\varphi A^{1/2}} : \mathbb{H}^1(\mathbb{T}^d)\right) \) with \( \varphi \geq 0 \), it holds that

\[
\|A^{1/2} e^{\varphi A^{1/2}} (u \cdot v)\|_{L^2} \lesssim \|e^{\varphi A^{1/2}} u\|_{H^1} \|e^{\varphi A^{1/2}} v\|_{H^1} + \|e^{\varphi A^{1/2}} u\|_{H^1} \|e^{\varphi A^{1/2}} v\|_{L^2}.
\]

Our next goal is to give a cancellation property for the sum of the quadratic variation term and the Stratonovich-to-Itô’s correction term when we apply Itô’s formula to the mapping \( t \mapsto \frac{1}{2} \|A^{1/2} e^{\varphi A^{1/2}} u(t)\|_{L^2}^2 \). Its proof can also be found in the appendix, Section 7.

Lemma 4.4. Let \( r \geq 0, s > 0 \) and \( \varphi \geq 0 \). For any \( \xi \in D\left(e^{\varphi A^{1/2}} : \mathbb{H}^r(\mathbb{T}^d)\right) \) and \( u \in D\left(e^{\varphi A^{1/2}} : \mathbb{H}^{r+2}(\mathbb{T}^d)\right) \), we have that

\[
\langle A^r e^{\varphi A^{1/2}} (\xi \cdot \nabla)(\xi \cdot \nabla) u \rangle + \|A^r e^{\varphi A^{1/2}} (\xi \cdot \nabla) u\|_{L^2}^2 = 0
\]

for any \( k \geq 1 \)

Our final preparatory result is an adaptation of [16, Lemma 5.1] to Gevrey spaces. Compare with [17, Lemma 7.1] and [4, Lemma 2.1] and see the appendix below for the proof. In the following, we fix \( r \geq 0, s > 0, \nu \geq 0 \) and let \( \varphi \geq 0 \) be an arbitrary time-dependent function of bounded variation. Now define \( \mathcal{E}(T) \) as

\[
\mathcal{E}(T) := C([0, T]; D(e^{\varphi(t)A^{1/2}} : \mathbb{H}^{r+1}(\mathbb{T}^d))) \cap L^2((0, T); D(e^{\varphi(t)A^{1/2}} : \mathbb{H}^{r+1}(\mathbb{T}^d)))
\]

with the norm

\[
\|u^N\|_{\mathcal{E}(T)}^2 := \sup_{\sigma \in [0, T]} \|A^{r/2} e^{\varphi(\sigma)A^{1/2}} u^N(\sigma)\|_{L^2}^2 + \nu \int_0^T \|A^{(r+1)/2} e^{\varphi(\sigma)A^{1/2}} u^N\|_{L^2}^2 \ d\sigma.
\]

Lemma 4.5. Let \( s > 0 \) and \( r, \varphi \geq 0 \). Let \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}) \) be a stochastic basis with the complete right-continuous filtration \( (\mathcal{F}_t)_{t \geq 0} \). Let \( u^N \) be an \( (\mathcal{F}_t) \)-adapted continuous stochastic process valued in \( D(e^{\varphi(t)A^{1/2}} : \mathbb{H}^{r+1}(\mathbb{T}^d)) \). For a deterministic \( M > 1 \) and \( T > 0 \), define the stopping times \( T^M_N \) as

\[
T^M_N := \inf \left\{ \tau \in [0, T] : \|u^N\|_{\mathcal{E}(\tau)} > \|e^{\varphi(0)A^{1/2}} u^N_0\|_{H^r} + M \right\} \wedge T
\]

and set \( T^M_{Nn} := T^M_N \wedge T^M_n \). If

\[
\lim_{t \to 0} \sup_{N \in \mathbb{N}} \mathbb{P}\left(\|u^N\|_{\mathcal{E}(t \wedge T^M_N)} > \|e^{\varphi(0)A^{1/2}} u^N_0\|_{H^r} + (M - 1)\right) = 0 \tag{4.1}
\]

and

\[
\lim_{n \to \infty} \sup_{N \geq n} \mathbb{E}\|u^N - u^N\|_{\mathcal{E}(T^M_{Nn})} = 0 \tag{4.2}
\]

hold, then

- there exists a stopping time \( \tau \) with

\[
\mathbb{P}(0 < \tau \leq T) = 1 \tag{4.3}
\]
• there exists a process $u(\cdot) = u(\cdot \wedge \tau) \in E(\tau)$ such that up to taking a subsequence (not relabelled),
  $$\|u^N - u\|_{E(\tau)} \to 0 \quad a.s.;$$  \hfill (4.4)
• also,
  $$\|u\|_{E(\tau)} \leq M + \sup_N \|e^{\nu(0)A^{1/2}}u_0^N\|_{H^r} \quad a.s. \quad (4.5)$$

5. Gevrey regularity: the construction

5.1. Estimates
With an initial condition $u_0 \in L^\infty(\Omega; H^1(T^d))$ and a dataset $(g_k, \xi_k)_{k \geq 1}$ satisfying the assumptions in Section 3.1, our goal now is to construct a solution $u$ of (3.3) that lives in the Bochner space $C([0, T); D(e^{tA^{1/2}} : H^1(T^d)))$ a.s. We will achieve this result by using a Galerkin approximation. In particular, we let $P^N : L^2(T^d) \to H_N$ be the $L^2$-orthogonal projection onto $H_N = \text{span}\{e^{ik \cdot x} \mid |k| \leq N\}$, and consider the finite-dimensional stochastic differential equation (SDE)

$$d u^N + [P^N P((u^N \cdot \nabla)u^N) - \nu \Delta u^N] \, dt$$
$$= \frac{1}{2} \sum_{k \geq 1} P^N P((\xi_k \cdot \nabla)P(\xi_k \cdot \nabla)u^N) \, dt$$
$$+ \sum_{k \geq 1} P^N P[g_k(u^N) - ((\xi_k \cdot \nabla)u^N)] \, dW^k_t,$$
$$u_0^N(x) = P^N u_0(x).$$

Since (5.1) is an $N$-dimensional system of SDEs, for $u_0^N \in L^\infty(\Omega; H_N)$ we can construct a solution $u^N$ in $L^\infty(0, T; H_N)$ that solves (5.1) a.e in space-time. See for example, [8, 13]. Furthermore, we can show the following.

Lemma 5.1. Suppose that there exists a deterministic $K_0 > 0$ such that

$$\|u_0^N\|_{H^1}^2 \leq K_0 \quad a.s.$$  \hfill (5.2)

For $u_0^N \in H_N$, let $u^N \in C([0, T]; H_N)$ be the corresponding Galerkin solution of (5.1). Now for any deterministic constant $M > 1$, define the stopping times $T_N^M$ as

$$T_N^M := \inf \left\{ \tau \in [0, T] : \sup_{\sigma \in [0, \tau]} \|e^{\sigma A^{1/2}}u^N(\sigma)\|_{H^1}^2 + \nu \int_0^\tau \|e^{\sigma A^{1/2}}u^N\|_{H^2}^2 d\sigma > \|u_0^N\|_{H^1}^2 + M \right\} \wedge T$$
and set $T_{Nn}^M := T_{N}^M \cap T_{n}^M$. Then for any $\tilde{M} \in (0, M)$,

$$
\lim_{t \to 0} \sup_{N \in \mathbb{N}} \mathbb{P} \left( \sup_{\sigma \in [0, t \wedge T_{Nn}^M]} \|e^{\sigma A^{1/2}} u^N(\sigma)\|_{H^1}^2 + \nu \int_0^{t \wedge T_{Nn}^M} \|e^{\sigma A^{1/2}} u^N\|_{H^2}^2 d\sigma > \|u_0^N\|_{H^1}^2 + \tilde{M} \right) = 0 \quad (5.3)
$$

and

$$
\lim_{n \to \infty} \sup_{N \geq n} \mathbb{E} \left( \sup_{\sigma \in [0, T_{N}^M]} \|e^{\sigma A^{1/2}} [u^N(\sigma) - u^n(\sigma)]\|_{H^1}^2 + \nu \int_0^{T_{N}^M} \|e^{\sigma A^{1/2}} [u^N - u^n]\|_{H^2}^2 d\sigma \right) = 0. \quad (5.4)
$$

**Proof.** By applying Itô’s formula to the mapping $t \mapsto \frac{1}{2}\|e^{\sigma A^{1/2}} u^N(t)\|_{H^1}^2$, it follows from (5.1) that

$$
\frac{1}{2}\|e^{\sigma A^{1/2}} u^N(t)\|_{H^1}^2 + \nu \int_0^t \|e^{\sigma A^{1/2}} u^N\|_{H^2}^2 d\sigma = \frac{1}{2}\|u_0^N\|_{H^1}^2 - \int_0^t \langle A^{1/2} e^{\sigma A^{1/2}} P_N P(u^N \cdot \nabla) u^N, A^{1/2} e^{\sigma A^{1/2}} u^N \rangle d\sigma
$$

$$
+ \int_0^t \langle A e^{\sigma A^{1/2}} u^N, A^{1/2} e^{\sigma A^{1/2}} u^N \rangle d\sigma
$$

$$
+ \frac{1}{2} \int_0^t \sum_{k \geq 1} \|e^{\sigma A^{1/2}} P_N P[g_k(u^N) - ((\xi_k \cdot \nabla)u^N)]\|_{H^1}^2 d\sigma
$$

$$
+ \frac{1}{2} \int_0^t \sum_{k \geq 1} \langle A^{1/2} e^{\sigma A^{1/2}} P_N P[(\xi_k \cdot \nabla)P(\xi_k \cdot \nabla)u^N), A^{1/2} e^{\sigma A^{1/2}} u^N \rangle d\sigma
$$

$$
+ \int_0^t \sum_{k \geq 1} \langle A^{1/2} e^{\sigma A^{1/2}} P_N P[g_k(u^N), A^{1/2} e^{\sigma A^{1/2}} u^N \rangle dW^k_\sigma
$$

$$
- \int_0^t \sum_{k \geq 1} \langle A^{1/2} e^{\sigma A^{1/2}} P_N P((\xi_k \cdot \nabla)u^N), A^{1/2} e^{\sigma A^{1/2}} u^N \rangle dW^k_\sigma. \quad (5.5)
$$

We now estimate the various deterministic integrals to the right of the equation above. First of all, note that by using Lemma 4.2 we have

$$
\int_0^{t \wedge T_{Nn}^M} \|e^{\sigma A^{1/2}} u^N\|_{H^2}^2 d\sigma + c(\nu) \int_0^{t \wedge T_{Nn}^M} \|e^{\sigma A^{1/2}} u^N\|_{H^1}^6 d\sigma. \quad (5.6)
$$
On the other hand, we have that
\[
\left| \int_0^{t \land T_N^M} \left< Ae^{\sigma A^{1/2}} u^N, A^{1/2} e^{\sigma A^{1/2}} u^N \right> d\sigma \right| 
\leq \frac{\nu}{4} \int_0^{t \land T_N^M} \| e^{\sigma A^{1/2}} u^N \|_{H^2}^2 d\sigma + c(\nu) \int_0^{t \land T_N^M} \| e^{\sigma A^{1/2}} u^N \|_{H^1}^2 d\sigma.

(5.7)
\]

Now use the Polarization Identity and (3.8) to write
\[
\| e^{\sigma A^{1/2}} P^N P[\mathbf{g}_k(u^N) - ((\xi_k \cdot \nabla) u^N)] \|_{H^1}^2 = \| e^{\sigma A^{1/2}} P^N P\mathbf{g}_k(u^N) \|_{H^1}^2 + \| e^{\sigma A^{1/2}} P^N P((\xi_k \cdot \nabla) u^N) \|_{H^1}^2.

(5.8)
\]

By using the assumption on \((\mathbf{g}_k)_{k \geq 1}\) in Section 3.1 and the continuity of \(P^N P\), we can estimate the first term in (5.8) as follows
\[
\frac{1}{2} \int_0^{t \land T_N^M} \sum_{k \geq 1} \| e^{\sigma A^{1/2}} P^N P\mathbf{g}_k(u^N) \|_{H^1}^2 d\sigma
\leq \int_0^{t \land T_N^M} (1 + \| e^{\sigma A^{1/2}} u^N \|_{H^1}^2) d\sigma.

(5.9)
\]

Next, we combine the second term in (5.8) with the fifth right-hand term in (5.5) and estimate both by using Lemma 4.4. This yields
\[
\left| \int_0^{t \land T_N^M} \sum_{k \geq 1} \left< A^{1/2} e^{\sigma A^{1/2}} P^N P[(\xi_k \cdot \nabla) P((\xi_k \cdot \nabla) u^N), A^{1/2} e^{\sigma A^{1/2}} u^N] \right> d\sigma
\right.
\leq \sum_{k \geq 1} \| e^{\sigma A^{1/2}} P^N P((\xi_k \cdot \nabla) u^N) \|_{H^1}^2 d\sigma
\leq \int_0^{t \land T_N^M} \| e^{\sigma A^{1/2}} u^N \|_{H^1}^2 d\sigma.

(5.10)
\]

By collecting the various estimates above, we obtain
\[
\sup_{\sigma \in [0,t \land T_N^M]} \| e^{\sigma A^{1/2}} u^N(\sigma) \|_{H^1}^2 + \nu \int_0^{t \land T_N^M} \| e^{\sigma A^{1/2}} u^N \|_{H^2}^2 d\sigma \leq \| u^N_0 \|_{H^1}^2
\]
\[+ c(\nu) \int_0^{t \land T_N^M} (1 + \| e^{\sigma A^{1/2}} u^N \|_{H^1}^6) d\sigma
\]
\[+ c(\nu) \sup_{\sigma \in [0,t \land T_N^M]} \left| \int_0^\sigma \sum_{k \geq 1} \left< A^{1/2} e^{\sigma A^{1/2}} P^N P\mathbf{g}_k(u^N), A^{1/2} e^{\sigma A^{1/2}} u^N \right> dW^k_s \right|
\]
\[+ c(\nu) \sup_{\sigma \in [0,t \land T_N^M]} \left| \int_0^\sigma \sum_{k \geq 1} \left< A^{1/2} e^{\sigma A^{1/2}} P^N P((\xi_k \cdot \nabla) u^N), A^{1/2} e^{\sigma A^{1/2}} u^N \right> dW^k_s \right|.

(5.11)
It therefore follow that for every $\tilde{M} \in (0, M)$,

\[
\mathbb{P}\left( \sup_{\sigma \in [0, t \wedge T_N^M]} \|e^{\sigma A^{1/2}} u^N(\sigma)\|_{H^1}^2 + \nu \int_0^{t \wedge T_N^M} \|e^{\sigma A^{1/2}} u^N\|_{H^2}^2 \, d\sigma > \|u^0\|_{H^2}^2 + \tilde{M} \right)
\leq \mathbb{P}\left( c(\nu) \int_0^{t \wedge T_N^M} \left( 1 + \|e^{\sigma A^{1/2}} u^N\|_{H^1}^6 \right) \, d\sigma > \frac{2M}{3} \right)
\]

\[+ \mathbb{P}\left( c(\nu) \sup_{\sigma \in [0, t \wedge T_N^M]} \left| \int_0^\sigma \sum_{k \geq 1} \langle A^{1/2} e^{s A^{1/2}} P^N P g_k(u^N) \rangle \, dW_s^k \right| > \frac{2M}{3} \right) \quad (5.12)\]

\[+ \mathbb{P}\left( c(\nu) \sup_{\sigma \in [0, t \wedge T_N^M]} \left| \int_0^\sigma \sum_{k \geq 1} \langle A^{1/2} e^{s A^{1/2}} P^N P \left( (\xi_k \cdot \nabla) u^N \right) \rangle \, dW_s^k \right| > \frac{2M}{3} \right). \]

To estimate the first term on the right, we use Chebyshev’s inequality just as in [16] which yields

\[
\mathbb{P}\left( c(\nu) \int_0^{t \wedge T_N^M} \left( 1 + \|e^{\sigma A^{1/2}} u^N\|_{H^1}^6 \right) \, d\sigma > \frac{2M}{3} \right)
\leq c_{\nu, \tilde{M}} \mathbb{E} \int_0^{t \wedge T_N^M} \left( 1 + \|e^{\sigma A^{1/2}} u^N\|_{H^1}^6 \right) \, d\sigma \quad (5.13)
\]

\[\leq c_{\nu, \tilde{M}, M} \int_0^t \, d\sigma \quad \leq c_{\nu, \tilde{M}, M}. \]

On the other hand, by Doob’s inequality and the assumption on the noise in Section 3.1,

\[
\mathbb{P}\left( c(\nu) \sup_{\sigma \in [0, t \wedge T_N^M]} \left| \int_0^\sigma \sum_{k \geq 1} \langle A^{1/2} e^{s A^{1/2}} P^N P g_k(u^N) \rangle \, dW_s^k \right| > \frac{2M}{3} \right)
\leq c_{\nu, \tilde{M}} \mathbb{E} \int_0^{t \wedge T_N^M} \|e^{\sigma A^{1/2}} u^N\|_{H^2}^2 \sum_{k \geq 1} \|e^{\sigma A^{1/2}} P^N P g_k(u^N)\|_{H^1}^2 \, d\sigma \quad (5.14)
\]

\[\leq c_{\nu, \tilde{M}, M}. \]
Similarly, we obtain by using the first estimate of Lemma 4.2 and the assumption on the transport noise in Section 3.1 that
\[
P\left( c(\nu) \sup_{\sigma \in [0, t \wedge T_N^M]} \left| \int_0^\sigma A^{1/2} e^{s A^{1/2}} P^N P((\xi_k \cdot \nabla) u^N), A^{1/2} e^{s A^{1/2}} u^N \right) dW_s^k \right| > \frac{\bar{M}^2}{3} \right)
\]
\[
\leq c_{\nu, \bar{M}, M} t.
\]
We therefore conclude that
\[
P\left( \sup_{\sigma \in [0, t \wedge T_N^M]} \| e^{\sigma A^{1/2}} u^N(\sigma) \|_{H^1}^2 + \nu \int_0^{t \wedge T_N^M} \| e^{\sigma A^{1/2}} u^N \|_{H^2}^2 \, d\sigma \right)
\]
\[
> \| u_0^N \|_{H^1}^2 + \bar{M} \right) \leq c_{\nu, \bar{M}, M} t
\]
an thus, (5.3) immediately follows.

Our next goal is to show (5.4). For this, we take $N, n \in \mathbb{N}$ with $N \geq n$. We now let $u^N$ and $u^n$ be the two Galerkin solutions of (5.1) with corresponding data $P^N u_0$ and $P^n u_0$ respectively. Set $u^{Nn} = u^N - u^n$ so that $u^{Nn}$ solves
\[
du^{Nn} + [P^N P((u^N \cdot \nabla) u^N) - P^n P((u^n \cdot \nabla) u^n) - \nu \Delta u^{Nn}] dt
\]
\[
= \frac{1}{2} \sum_{k \geq 1} \left[ P^N P((\xi_k \cdot \nabla) P(\xi_k \cdot \nabla) u^N) - P^n P((\xi_k \cdot \nabla) P(\xi_k \cdot \nabla) u^n) \right] dt
\]
\[
+ \sum_{k \geq 1} [P^N P g_k(u^N) - P^n P g_k(u^n)] dW_t^k
\]
\[
- \sum_{k \geq 1} [P^N P((\xi_k \cdot \nabla) u^N) - P^n P((\xi_k \cdot \nabla) u^n)] dW_t^k,
\]
\[
u_0^N = (P^N - P^n) u_0.
\]
We have by the product rule,
\[
d(A^{1/2} e^{t A^{1/2}} u^{Nn}) = A e^{t A^{1/2}} u^{Nn} dt
\]
\[
- A^{1/2} e^{t A^{1/2}} [P^N P(u^N \cdot \nabla) u^N - P^n P(u^n \cdot \nabla) u^n + \nu A u^{Nn}] dt
\]
\[
+ \frac{1}{2} \sum_{k \geq 1} A^{1/2} e^{t A^{1/2}} [P^N P((\xi_k \cdot \nabla) P(\xi_k \cdot \nabla) u^N)
\]
\[
- P^n P((\xi_k \cdot \nabla) P(\xi_k \cdot \nabla) u^n)] dt
\]
\[
+ \sum_{k \geq 1} A^{1/2} e^{t A^{1/2}} [P^N P g_k(u^N) - P^n P g_k(u^n)] dW_t^k
\]
\[
- \sum_{k \geq 1} A^{1/2} e^{t A^{1/2}} [P^N P((\xi_k \cdot \nabla) u^N) - P^n P((\xi_k \cdot \nabla) u^n)] dW_t^k
\]
and thus, it follows that

\[
\begin{align*}
\frac{1}{2}\|e^{tA^{1/2}}u_{Nn}(t)\|_{H^{1}}^2 + \nu \int_{0}^{t}\|e^{\sigma A^{1/2}}u_{Nn}\|_{H^{2}}^2 \, d\sigma &= \frac{1}{2}\|u_{Nn}^{0}\|_{H^{1}}^2 \\
+ \int_{0}^{t} \left\langle Ae^{\sigma A^{1/2}}u_{Nn}, A^{1/2}e^{\sigma A^{1/2}}u_{Nn} \right\rangle \, d\sigma \\
- \int_{0}^{t} \left\langle A^{1/2}e^{\sigma A^{1/2}}[P_{N}P(u^{N} \cdot \nabla)u^{N} - P_{n}P(u^{N} \cdot \nabla)u^{n}], A^{1/2}e^{\sigma A^{1/2}}u_{Nn} \right\rangle \, d\sigma \\
+ C_{1}(g_{k}, \xi_{k}, u^{n}, u^{N}, u_{Nn})(t) + C_{2}(g_{k}, \xi_{k}, u_{n}, u^{N}, u_{Nn})(t) \\
+ \int_{0}^{t} \sum_{k \geq 1} \left\langle A^{1/2}e^{\sigma A^{1/2}}[P_{N}Pg_{k}(u^{N}) - P_{n}Pg_{k}(u^{n})], A^{1/2}e^{\sigma A^{1/2}}u_{Nn} \right\rangle \, dW_{\sigma}^{k} \\
- \int_{0}^{t} \sum_{k \geq 1} \left\langle A^{1/2}e^{\sigma A^{1/2}}[P_{N}P((\xi_{k} \cdot \nabla)u^{N}) - P_{n}P((\xi_{k} \cdot \nabla)u^{n})], A^{1/2}e^{\sigma A^{1/2}}u_{Nn} \right\rangle \, dW_{\sigma}^{k}
\end{align*}
\]  

where

\[
C_{1}(g_{k}, \xi_{k}, u^{n}, u^{N}, u_{Nn})(t) \\
:= \frac{1}{2} \int_{0}^{t} \sum_{k \geq 1} \|e^{\sigma A^{1/2}}[P_{N}Pg_{k}(u^{N}) - P_{n}Pg_{k}(u^{n})] \\
- e^{\sigma A^{1/2}}[P_{N}P((\xi_{k} \cdot \nabla)u^{N}) - P_{n}P((\xi_{k} \cdot \nabla)u^{n})]\|_{H^{1}}^2 \, d\sigma
\]

is the Itô correction term and

\[
C_{2}(g_{k}, \xi_{k}, u^{n}, u^{N}, u_{Nn})(t) \\
:= \frac{1}{2} \int_{0}^{t} \sum_{k \geq 1} \left\langle A^{1/2}e^{\sigma A^{1/2}}P_{N}P[(\xi_{k} \cdot \nabla)P(\xi_{k} \cdot \nabla)u^{N}], A^{1/2}e^{\sigma A^{1/2}}u_{Nn} \right\rangle \, d\sigma \\
- \frac{1}{2} \int_{0}^{t} \sum_{k \geq 1} \left\langle A^{1/2}e^{\sigma A^{1/2}}P_{n}P[(\xi_{k} \cdot \nabla)P(\xi_{k} \cdot \nabla)u^{n}], A^{1/2}e^{\sigma A^{1/2}}u_{Nn} \right\rangle \, d\sigma
\]

is the Stratonovich corrector. For \(T_{NN}^{M} := T_{N}^{M} \wedge T_{n}^{M}\), we have that

\[
\left| \int_{0}^{T_{N}^{M}} \left\langle Ae^{tA^{1/2}}u^{Nn}, A^{1/2}e^{tA^{1/2}}u^{Nn} \right\rangle \, dt \right| \\
\leq \frac{\nu}{8} \int_{0}^{T_{N}^{M}} \|e^{tA^{1/2}}u^{Nn}\|_{H^{2}}^2 \, dt + c(\nu) \int_{0}^{T_{N}^{M}} \|e^{tA^{1/2}}u^{Nn}\|_{H^{1}}^2 \, dt.
\]
In order to estimate the nonlinear convective term, we rewrite it as follows
\[
\langle A^{1/2}e^{tA^{1/2}}[P^N P(u^N \cdot \nabla)u^N - P^n P(u^n \cdot \nabla)u^n], A^{1/2}e^{tA^{1/2}} u^{Nn} \rangle = I_1 + I_2 + I_3
\]
(5.21)
where
\[
I_1 := \langle A^{1/2}e^{tA^{1/2}}[P^N P(u^{Nn} \cdot \nabla)u^N], A^{1/2}e^{tA^{1/2}} u^{Nn} \rangle,
\]
\[
I_2 := \langle A^{1/2}e^{tA^{1/2}}[P^N P(u^n \cdot \nabla)u^{Nn}], A^{1/2}e^{tA^{1/2}} u^{Nn} \rangle,
\]
\[
I_3 := \langle A^{1/2}e^{tA^{1/2}}[(P^N - P^n)P(u^n \cdot \nabla)u^n], A^{1/2}e^{tA^{1/2}} u^{Nn} \rangle.
\]

By using Lemma 4.2, we obtain
\[
|I_1| \lesssim \|A^{1/2}e^{tA^{1/2}} u^{Nn}\|_{L^2}^2 \|Ae^{tA^{1/2}} u^{Nn}\|_{L^2}^2 \|A^{1/2}e^{tA^{1/2}} u^{N}\|_{L^2}^2 \|Ae^{tA^{1/2}} u^{N}\|_{L^2}^2 \leq \frac{\nu}{8} \|e^{tA^{1/2}} u^{Nn}\|_{L^2}^2 + c(\nu) \|e^{tA^{1/2}} u^{Nn}\|_{H^2}^2 \|e^{tA^{1/2}} u^{N}\|_{H^1}^2.
\]
(5.22)

Next,
\[
|I_2| \lesssim \|A^{1/2}e^{tA^{1/2}} u^n\|_{L^2}^2 \|Ae^{tA^{1/2}} u^n\|_{L^2}^2 \|A^{1/2}e^{tA^{1/2}} u^{Nn}\|_{L^2}^2 \|Ae^{tA^{1/2}} u^{Nn}\|_{L^2}^2 \leq \frac{\nu}{8} \|e^{tA^{1/2}} u^{Nn}\|_{L^2}^2 + c(\nu) \|e^{tA^{1/2}} u^n\|_{H^2}^2 \|e^{tA^{1/2}} u^{Nn}\|_{H^2}^2.
\]
(5.23)

Finally, if we set \(Q^n = I - P^n\) as the projection onto the modes higher than \(n\), then we can use the identity \(P^N - P^n = Q^n P^N\) which holds for \(N \geq n\) to obtain
\[
|I_3| = |\langle e^{tA^{1/2}} Q^n P^N P(u^n \cdot \nabla)u^n, A e^{tA^{1/2}} u^{Nn} \rangle| \leq \frac{\nu}{8} \|e^{tA^{1/2}} Q^n P^N e^{tA^{1/2}} P(u^n \cdot \nabla)u^n\|_{L^2}^2.
\]
(5.24)

However, by employing the second inequality in (2.3), the continuity of \(P^N\) shown in (2.4) and Lemma 4.3, we obtain
\[
\|Q^n P^N e^{tA^{1/2}} P(u^n \cdot \nabla)u^n\|_{L^2}^2 \leq \frac{2}{n^2} \|e^{tA^{1/2}} P(u^n \cdot \nabla)u^n\|_{H^1}^2 \leq \frac{1}{n^2} \left( \|e^{tA^{1/2}} u^n\|_{L^2}^2 \|e^{tA^{1/2}} u^n\|_{H^2}^2 + \|e^{tA^{1/2}} u^n\|_{H^1} \|e^{tA^{1/2}} u^n\|_{H^2} \right) \leq \frac{1}{n^2} \|e^{tA^{1/2}} u^n\|_{H^1}^2 \|e^{tA^{1/2}} u^n\|_{H^2}^2
\]
(5.25)

where the last inequality follows from the first estimate in (2.3). Thus, it follows that
\[
|I_3| \leq \frac{\nu}{8} \|e^{tA^{1/2}} u^{Nn}\|_{H^2}^2 + \frac{c(\nu)}{n} \|e^{tA^{1/2}} u^n\|_{H^1} \|e^{tA^{1/2}} u^n\|_{H^2}^2 \]
(5.26)
and as such,
\[
\left| \int_0^{T_N^M} \left( A^{1/2} e^{tA^{1/2}} [P^N P(u^N \cdot \nabla)u^N - P^n P(u^n \cdot \nabla)u^n], A^{1/2} e^{tA^{1/2}} u^{Nn} \right) dt \right|
\]
\[
\leq \frac{3\nu}{8} \int_0^{T_N^M} \| e^{tA^{1/2}} u^{Nn} \|_{H^2}^2 dt + c(\nu) \int_0^{T_N^M} \frac{1}{n} \| e^{tA^{1/2}} u^n \|_{H^1}^2 \| e^{tA^{1/2}} u^n \|_{H^2}^2 dt
\]
\[
+ c(\nu) \int_0^{T_N^M} \| e^{tA^{1/2}} u^{Nn} \|_{H^1}^4 \left( \| e^{tA^{1/2}} u^n \|_{H^1}^4 + \| e^{tA^{1/2}} u^n \|_{H^2}^2 \right) dt.
\]

Next, we use the Polarization Identity and (3.8) to obtain
\[
\| \epsilon^{tA^{1/2}} [P^N P\hat{g}_k(u^N) - P^n P\hat{g}_k(u^n)]
\]
\[
- \epsilon^{tA^{1/2}} [P^N P((\xi_k \cdot \nabla)u^N) - P^n P((\xi_k \cdot \nabla)u^n)] \|_{H^1}^2
\]
\[
= \| \epsilon^{tA^{1/2}} [P^N P\hat{g}_k(u^N) - P^n P\hat{g}_k(u^n)] \|_{H^1}^2
\]
\[
+ \| \epsilon^{tA^{1/2}} [P^N P((\xi_k \cdot \nabla)u^N) - P^n P((\xi_k \cdot \nabla)u^n)] \|_{H^1}^2.
\]

To estimate the first term on the right of (5.28), we use the triangle inequality again to obtain
\[
\| \epsilon^{tA^{1/2}} [P^N P\hat{g}_k(u^N) - P^n P\hat{g}_k(u^n)] \|_{H^1}^2
\]
\[
\leq \| P^N P \epsilon^{tA^{1/2}} [\hat{g}_k(u^N) - \hat{g}_k(u^n)] \|_{H^1}^2 + \| Q^n P^N P \epsilon^{tA^{1/2}} \hat{g}_k(u^n) \|_{H^1}^2
\]
where \( Q^n = I - P^n \). Thus, we obtain by using the assumptions on \( (\hat{g}_k)_{k \geq 1} \) in Section 3.1 the second inequality in (2.3) and the continuity of \( P^N P \) that,
\[
\left| \int_0^{T_N^M} \sum_{k \geq 1} \| \epsilon^{tA^{1/2}} [P^N P\hat{g}_k(u^N) - P^n P\hat{g}_k(u^n)] \|_{H^1}^2 dt \right|
\]
\[
\leq \int_0^{T_N^M} \left( \| \epsilon^{tA^{1/2}} u^{Nn} \|_{H^1}^2 + \frac{1}{n^2} \left( 1 + \| \epsilon^{tA^{1/2}} u^n \|_{H^2}^2 \right) \right) dt.
\]

We now combine the treatment of the last term in (5.28) with the \( C_2 \)-term in (5.19). In particular, we note from (7.12) and (2.3) that
\[
J_1^k := \| \epsilon^{tA^{1/2}} [P^N P((\xi_k \cdot \nabla)u^N) - P^n P((\xi_k \cdot \nabla)u^n)] \|_{H^1}^2
\]
\[
= \| \epsilon^{tA^{1/2}} Q^n P^N P(\xi_k \cdot \nabla u^N) \|_{H^1}^2 + \| \epsilon^{tA^{1/2}} P^n P(\xi_k \cdot \nabla u^n) \|_{H^1}^2
\]
\[
\leq \frac{1}{n^2} \| \epsilon^{tA^{1/2}} P^N P(\xi_k \cdot \nabla u^N) \|_{H^2}^2 + \| \epsilon^{tA^{1/2}} P^n P(\xi_k \cdot \nabla u^n) \|_{H^1}^2
\]
and
\[
J_2^k := \langle A^{1/2} \epsilon^{tA^{1/2}} P^N P[(\xi_k \cdot \nabla)P(\xi_k \cdot \nabla)u^N], A^{1/2} \epsilon^{tA^{1/2}} u^{Nn} \rangle,
\]
\[
J_3^k := -\langle A^{1/2} \epsilon^{tA^{1/2}} P^n P[(\xi_k \cdot \nabla)P(\xi_k \cdot \nabla)u^n], A^{1/2} \epsilon^{tA^{1/2}} u^{Nn} \rangle
\]
are such that

\[
J_k^2 + J_k^3 := \langle A^{1/2} e^{tA^{1/2}} Q^n P^N P[\langle \xi_k \cdot \nabla \rangle P(\xi_k \cdot \nabla) u^N], A^{1/2} e^{tA^{1/2}} u^N \rangle \\
+ \langle A^{1/2} e^{tA^{1/2}} P^n P[(\xi_k \cdot \nabla) P(\xi_k \cdot \nabla) u^{Nn}], A^{1/2} e^{tA^{1/2}} u^{Nn} \rangle \\
- \langle A^{1/2} e^{tA^{1/2}} Q^n P^N P[(\xi_k \cdot \nabla) P(\xi_k \cdot \nabla) u^N], A^{1/2} e^{tA^{1/2}} u^n \rangle.
\]

(5.33)

Now note that since \( u^n = P^n u^n \), the last term above is

\[
\langle Q^n A^{1/2} e^{tA^{1/2}} P^N P[(\xi_k \cdot \nabla) P(\xi_k \cdot \nabla) u^N], A^{1/2} e^{tA^{1/2}} u^n \rangle = 0. \tag{5.34}
\]

On the other hand, since \( Q^n = Q^n Q^n \) is a projection, it follows from the second estimate in (2.3) that

\[
\langle A^{1/2} e^{tA^{1/2}} Q^n P^N P[(\xi_k \cdot \nabla) P(\xi_k \cdot \nabla) u^N], A^{1/2} e^{tA^{1/2}} u^N \rangle \leq \frac{1}{n^2} \langle A e^{tA^{1/2}} P^N P[(\xi_k \cdot \nabla) P(\xi_k \cdot \nabla) u^N], A e^{tA^{1/2}} u^N \rangle. \tag{5.35}
\]

Therefore, by pairing each of the two terms estimating \( J_k^k \) with the first two terms of the summand \( J_k^2 + J_k^3 \), respectively, and using Lemma 4.4, keeping in mind the continuity property of \( P^N \), we obtain

\[
|J_k^1 + J_k^2 + J_k^3| \\
\lesssim \left\| e^{tA^{1/2}} P^n P[(\xi_k \cdot \nabla) u^{Nn}] \right\|^2_{H^1} \\
+ \left\| e^{tA^{1/2}} P^n P[(\xi_k \cdot \nabla) P(\xi_k \cdot \nabla) u^{Nn}], A^{1/2} e^{tA^{1/2}} u^{Nn} \right\| \\
+ \frac{1}{n^2} \left\| e^{tA^{1/2}} P^N P[(\xi_k \cdot \nabla) u^N] \right\|^2_{H^2} \\
+ \left\| A e^{tA^{1/2}} P^N P[(\xi_k \cdot \nabla) P(\xi_k \cdot \nabla) u^N], A e^{tA^{1/2}} u^N \right\| \\
\lesssim \left\| e^{tA^{1/2}} \xi_k \right\|^2_{H^r} \left( \left\| e^{tA^{1/2}} u^n \right\|^2_{H^1} + \frac{1}{n^2} \left\| e^{tA^{1/2}} u^N \right\|^2_{H^2} \right). \tag{5.36}
\]
with \( r > 4 \). By the Burkholder–Davis–Gundy inequality and the same argument used in (5.30), we obtain

\[
\mathbb{E} \sup_{t \in [0, T_N^M]} \left| \int_0^t \sum_{k \geq 1} \left\langle A^{1/2} e^{tA^{1/2}} [P^N P g_k(u^N) - P^n P g_k(u^n)] \right\rangle \, dW^k_\sigma \right|
\]

\[
\lesssim \mathbb{E} \left( \int_0^{T_N^M} \sum_{k \geq 1} \left\langle A^{1/2} e^{tA^{1/2}} [P^N P g_k(u^N) - P^n P g_k(u^n)] \right\rangle \, dt \right)^{1/2}
\]

\[
\lesssim \mathbb{E} \left( \int_0^{T_N^M} \sum_{k \geq 1} \left\| e^{tA^{1/2}} [P^N P g_k(u^N) - P^n P g_k(u^n)] \right\|_{H^1}^2 \, dt \right)^{1/2}
\]

\[
\lesssim \frac{1}{4} \mathbb{E} \sup_{t \in [0, T_N^M]} \left\| e^{tA^{1/2}} u^{N_n}(t) \right\|_{H^1}^2 + c \mathbb{E} \int_0^{T_N^M} \left( \| e^{tA^{1/2}} u^{N_n} \|_{H^1}^2 + \frac{1}{n^2} \left( \| e^{tA^{1/2}} u^n \|_{H^2}^2 \right) \right) dt.
\]

It now remains to estimate the last term in (5.19). For this, we need the following preliminary estimate. Similar to (5.21), we rewrite

\[
\left\langle A^{1/2} e^{tA^{1/2}} [P^N P((\xi_k \cdot \nabla)u^N) - P^n P((\xi_k \cdot \nabla)u^n)], A^{1/2} e^{tA^{1/2}} u^{N_n} \right\rangle = K_1^k + K_2^k
\]

where

\[
K_1^k := \left\langle A^{1/2} e^{tA^{1/2}} [P^N P((\xi_k \cdot \nabla)u^N)], A^{1/2} e^{tA^{1/2}} u^{N_n} \right\rangle,
\]

\[
K_2^k := \left\langle A^{1/2} e^{tA^{1/2}} [(P^N - P^n)P((\xi_k \cdot \nabla)u^n)], A^{1/2} e^{tA^{1/2}} u^{N_n} \right\rangle.
\]

Since \((\xi)_{k \geq 1}\) satisfies (3.7) by assumption, it follows by using Lemma 4.3 that

\[
\sum_{k \geq 1} \left| K_1^k \right| \lesssim \| e^{tA^{1/2}} u^{N_n} \|_{H^1} \| e^{tA^{1/2}} u^{N_n} \|_{H^2}
\]

and thus, by the Burkholder–Davis–Gundy inequality,

\[
\mathbb{E} \sup_{t \in [0, T_N^M]} \left| \int_0^t \sum_{k \geq 1} K_1^k \, dW_\sigma^k \right| \lesssim \mathbb{E} \left( \int_0^{T_N^M} \sum_{k \geq 1} \left| K_1^k \right|^2 \, dt \right)^{1/2}
\]

\[
\lesssim \frac{c}{2} \mathbb{E} \sup_{t \in [0, T_N^M]} \left\| e^{tA^{1/2}} u^{N_n}(t) \right\|_{H^1}^2 + \frac{c}{2} \mathbb{E} \int_0^{T_N^M} \left\| e^{tA^{1/2}} u^{N_n} \right\|_{H^2}^2 \, dt.
\]
for some $\epsilon > 0$. On the other hand, since $P^N - P^n = Q^n P^N$ is a continuous operator \((2.4)\), it follows from Lemma 4.3 and (3.7) that

$$\sum_{k \geq 1} |K_k| \lesssim \|e^{tA^{1/2}} u^N\|_{H^1} \|e^{tA^{1/2}} u^n\|_{H^2}.$$  

Thus, by the Burkholder–Davis–Gundy inequality,

$$\mathbb{E} \sup_{t \in [0, T_{N_n}^M]} \left| \int_0^t \sum_{k \geq 1} K_k \, dW_k^\sigma \right| \lesssim \mathbb{E} \int_0^{T_{N_n}^M} \|e^{tA^{1/2}} u^N\|_{H^1}^2 \left(1 + \|e^{tA^{1/2}} u^n\|_{H^2}^2\right) dt.$$

From (5.39) and (5.40), it follows that

$$\mathbb{E} \sup_{t \in [0, T_{N_n}^M]} \left| \int_0^t \sum_{k \geq 1} (A^{1/2} e^{\sigma A^{1/2}} [P^N P((\xi_k \cdot \nabla) u^n) - P^n P((\xi_k \cdot \nabla) u^n)] A^{1/2} e^{\sigma A^{1/2}} u^N) \, dW_k^\sigma \right| \leq \frac{c}{2} \mathbb{E} \sup_{t \in [0, T_{N_n}^M]} \|e^{tA^{1/2}} u^N(t)\|_{H^1}^2 + \frac{c}{2} \mathbb{E} \int_0^{T_{N_n}^M} \|e^{tA^{1/2}} u^n\|_{H^2}^2 dt + c(\nu) \mathbb{E} \int_0^{T_{N_n}^M} \|e^{tA^{1/2}} u^n\|_{H^1}^2 \left(1 + \|e^{tA^{1/2}} u^n\|_{H^2}^2\right) dt.$$

By collecting the various estimates of (5.19), we obtain

$$\kappa_1 \mathbb{E} \sup_{t \in [0, T_{N_n}^M]} \|e^{tA^{1/2}} u^N(t)\|_{H^1}^2 + \kappa_2 \mathbb{E} \int_0^{T_{N_n}^M} \|e^{tA^{1/2}} u^n\|_{H^2}^2 dt \leq \frac{1}{2} \mathbb{E} \|u_0^N\|_{H^1}^2,$$

$$+ c(\nu) \mathbb{E} \int_0^{T_{N_n}^M} \|e^{tA^{1/2}} u^N\|_{H^1}^2 \left(1 + \|e^{tA^{1/2}} u^n\|_{H^1}^4 + \|e^{tA^{1/2}} u^n\|_{H^2}^2\right) dt$$

$$+ c(\nu) \mathbb{E} \int_0^{T_{N_n}^M} \frac{1}{n} \left(1 + \|e^{tA^{1/2}} u^n\|_{H^2}^2 + \|e^{tA^{1/2}} u^n\|_{H^2}^2\right) dt$$

where $\kappa_1 := \frac{1}{4} - \frac{c \epsilon}{2}$ and $\kappa_2 = \frac{\nu}{2} - \frac{c \epsilon}{2}$. We can now choose $\epsilon = \frac{\nu}{2c}$ (recall that $c = c(\nu)$) appropriately so that both $\kappa_1$ and $\kappa_2$ are strictly positive. Since all constants are independent of $n$ or $N$, our required convergence (5.4) follows from Grönwall’s lemma [16, Lemma 5.3].

With Lemma 5.1 in hand, our next goal is to apply Lemma 4.5 to get our final result that leads to the proof of Theorem 3.9.

**Proposition 5.2.** Let $(g_k)_{k \geq 1}$ satisfy (3.4)–(3.6) with $r = s = 1$ and $\sigma_1 = t$ and let $(\xi_k)_{k \geq 1}$ satisfy (3.7) with $r > 4$, $s = 1$ and $\sigma_2 = T$. If there exists a deterministic $K_0 > 0$ such that (3.12) holds, then $u(t \wedge \tau) \in C((0, \tau); D(e^{tA^{1/2}} : H^r(\mathbb{T}^d)))$ a.s. for all $t \in (0, T)$. 
Proof. We note that the projection of the initial condition satisfying \( \|u_0\|_{H^1} \leq K_0 \) a.s. will also satisfies \( \|u_0^N\|_{H^1} \leq K_0 \) a.s. Now take the associated family \( \{u^N\}_{N \in \mathbb{N}} \) family of Galerkin solution. Due to Lemma \ref{lem5.1} we see that the assumptions of Lemma \ref{lem4.5} are satisfied and as such, there exists a pair \((u, \tau)\) with \( \tau > 0 \) a.s. and \( u(\cdot) = u(\cdot \wedge \tau) \in C([0, \tau); D(e^{-A^{1/2}t} : H^1(\mathbb{T}^d)) \cap L^2((0, \tau); D(e^{-A^{1/2}t} : H^2(\mathbb{T}^d)))) \) such that up to taking a subsequence (not relabelled) 

\[
\sup_{\sigma \in [0, \tau]} \|e^{\sigma A^{1/2}}[u^N(\sigma) - u(\sigma)]\|_{H^1}^2 + \nu \int_0^\tau \| e^{\sigma A^{1/2}}[u^N - u]\|_{H^2}^2 \, d\sigma \to 0 \quad \text{a.s.}
\]

5.2. Identification of the limit

Note that in the definition of a Gevrey class solution, Definition \ref{def3.6}, identifiaction of the limit system has nothing to do with the Gevrey class regularity. Indeed, the limit equation is identified at the level of construction of a maximal strong pathwise solution \((u, \tau_m)\). We refer to \cite{16,27,28} for this procedure. Due to Theorem \ref{thm3.5}, it follows that \( \tau \) from Proposition \ref{prop5.2} is such that \( \tau \leq \tau_m \) a.s. since the latter is maximal. Nevertheless, similar to \cite[Section 4.2]{16}, we can use a contradiction argument to extend \( \tau \) to a maximal stopping time \( \tau_{\max} \leq \tau_m \) whose announcing times are given by an increasing sequence of stopping times \( (\tau_l)_{l \geq 1} \). However, it is an open question if \( \tau_{\max} = \infty \) a.s. in 2D (but certainly not expected in 3D) and one main drawback to proving this result lies in the fact that when working in the Gevrey class, 

\[
\langle A^{1/2}e^{tA^{1/2}}P(u \cdot \nabla)u, A^{1/2}e^{tA^{1/2}}u \rangle \neq 0
\]

unlike the cancellation property \cite[Lemma 3.1]{33} 

\[
\langle A^{1/2}P(u \cdot \nabla)u, A^{1/2}u \rangle = 0.
\]

enjoyed when working in the usual Sobolev space defined on \( \mathbb{T}^2 \).

6. Decay Rate

In the section, we show Theorem \ref{thm3.13} which gives the decay rate of the Fourier modes of the velocity field solving the Navier–Stokes equation under the assumption that \( \tau = \infty \) a.s. To begin with, we recall \( u \) and its finite-dimensional approximation \( u^N \) that solves

\[
du + [P((u \cdot \nabla)u) + \nu A u] \, dt = \frac{1}{2} \sum_{k \geq 1} P[(\xi_k \cdot \nabla) P(\xi_k \cdot \nabla)u] \, dt + \sum_{k \geq 1} P[g_k(u) - ((\xi_k \cdot \nabla)u)] \, dW_t^k, \tag{6.1}
\]

\[
u(x, 0) = u_0(x)
\]
and
\[ \text{d}u^N + [P^N P((u^N \cdot \nabla)u^N) + \nu Au^N] \, \text{d}t = \frac{1}{2} \sum_{k \geq 1} P^N P[(\xi_k \cdot \nabla)P(\xi_k \cdot \nabla)u^N] \, \text{d}t + \sum_{k \geq 1} P^N P[g_k(u^N) - ((\xi_k \cdot \nabla)u^N)] \, \text{d}W_t^k, \]

respectively. Having shown the Gevrey class 1 regularity of \( u \) and with the assumption \( \tau = \infty \) in hand, we wish to use this information to obtain the following exponential decay rate for the difference of \( u \) and \( u^N \) in the \( H^1 \)-norm (i.e. in the class of strong solutions). Before we begin, we note that from Theorem \ref{thm:3.9}, we have that the estimate
\[ \|u(t)\|_{C^1}^2 = \sum_{k \in \mathbb{Z}^d} |k|^2 e^{2\delta_1|k|}|\hat{u}_k(t)|^2 \lesssim 1 \]
holds a.s. for all \( t \in (0, T] \) with a constant depending only on \( \nu, u_0 \) and \( K \) from \ref{eq:3.7}. Thus, there exists \( \delta_1 \in (0, T) \) such that
\[ |\hat{u}_k(t)|^2 \lesssim \frac{1}{|k|^2} e^{-2\delta_1|k|}, \quad |k|^2 |\hat{u}_k(t)|^2 \lesssim e^{-2\delta_1|k|} \] 
holds a.s. for \( t \in [\delta_1, T] \). The first bound means that each Fourier coefficient \( \hat{u}_k \) decays better-than-exponential, with respect to its wavenumber \( \frac{1}{|k|} \), uniformly with respect to \( t \in [\delta_1, T] \), uniformly with respect to the initial enstrophy \( \|u_0\|_{H^1}^2 \) and uniformly with respect to the noise coefficients \( (g_k, \xi_k)_{k \geq 1} \). With this preliminary information in hand, we can now show Theorem \ref{thm:3.13}.

First of all, note that
\[ \|(u^N - u)(t)\|_{H^1} \leq \|Q^N u(t)\|_{H^1} + \|(P^N u - u^N)(t)\|_{H^1} \] 
where
\[ \|Q^N u(t)\|_{H^1}^2 = \|(I - P^N)u(t)\|_{H^1}^2 = \sum_{|k| > N} |k|^2 |\hat{u}_k(t)|^2 \]
and so, the estimate
\[ \|Q^N u(t)\|_{H^1}^2 \lesssim e^{-2\delta_1 N} \] 
holds a.s. for \( t \in [\delta_1, T] \). In order to obtain our desired decay, therefore, it remains to estimate \( \|P^N u - u^N\|_{H^1}^2 \). For this, we first note that
\[ \frac{1}{2} \|P^N u - u^N\|_{H^1}^2 = \frac{1}{2} \|u^N\|_{H^1}^2 + \frac{1}{2} \|P^N u\|_{H^1}^2 - \langle A^{1/2} P^N u, A^{1/2} u^N \rangle \] 
and as such, we can compute the right-hand side using Itô’s formula. Firstly, by applying Itô’s formula to the mapping \( t \mapsto \frac{1}{2} \|A^{1/2} u^N(t)\|_{L^2}^2 \), we obtain
from (6.2),

\[
\frac{1}{2} \|u^N(t)\|^2_{H^1} + \nu \int_0^t \|Au^N\|^2_{L^2} \, d\sigma \\
= \frac{1}{2} \|u^N_0\|^2_{H^1} - \int_0^t \langle A^{1/2} P^N [(u^N \cdot \nabla)u^N], A^{1/2} u^N \rangle \, d\sigma \\
+ \frac{1}{2} \int_0^t \sum_{k \geq 1} \langle A^{1/2} P^N [(\xi_k \cdot \nabla)P(\xi_k \cdot \nabla)u^N], A^{1/2} u^N \rangle \, d\sigma \\
+ \frac{1}{2} \int_0^t \sum_{k \geq 1} \|P^N [g_k(u^N) - ((\xi_k \cdot \nabla)u^N)]\|^2_{H^1} \, d\sigma \\
+ \int_0^t \sum_{k \geq 1} \langle A^{1/2} P^N [g_k(u^N) - ((\xi_k \cdot \nabla)u^N)], A^{1/2} u^N \rangle \, dW^k. 
\]

(6.7)

Similarly, by applying Itô’s formula to the mapping \( t \mapsto \frac{1}{2} \|A^{1/2} P^N u(t)\|^2_{L^2} \), we obtain from (6.1)

\[
\frac{1}{2} \|P^N u(t)\|^2_{H^1} + \nu \int_0^t \|AP^N u\|^2_{L^2} \, d\sigma \\
= \frac{1}{2} \|P^N u_0\|^2_{H^1} - \int_0^t \langle A^{1/2} P^N [(u \cdot \nabla)u], A^{1/2} P^N u \rangle \, d\sigma \\
+ \frac{1}{2} \int_0^t \sum_{k \geq 1} \langle A^{1/2} P^N [(\xi_k \cdot \nabla)P(\xi_k \cdot \nabla)u], A^{1/2} P^N u \rangle \, d\sigma \\
+ \frac{1}{2} \int_0^t \sum_{k \geq 1} \|P^N [g_k(u) - ((\xi_k \cdot \nabla)u)]\|^2_{H^1} \, d\sigma \\
+ \int_0^t \sum_{k \geq 1} \langle A^{1/2} P^N [g_k(u) - ((\xi_k \cdot \nabla)u)], A^{1/2} P^N u \rangle \, dW^k. 
\]

(6.8)
Next, we use Itô’s product rule to obtain
\[
\langle A^{1/2}P^N u, A^{1/2}u^N \rangle \\
= \langle A^{1/2}P^N u_0, A^{1/2}u_0^N \rangle - 2\nu \int_0^t \langle Au^N, AP^N u \rangle \, d\sigma \\
- \int_0^t \langle A^{1/2}P^N P[(u \cdot \nabla)u], A^{1/2}u^N \rangle \, d\sigma \\
- \int_0^t \langle A^{1/2}P^N P[(u^N \cdot \nabla)u^N], A^{1/2}P^N u \rangle \, d\sigma \\
+ C(g_k, \xi_k, u, u^N)(t) \\
+ \int_0^t \sum_{k \geq 1} \langle A^{1/2}P^N P[g_k(u^N) - ((\xi_k \cdot \nabla)u^N)], A^{1/2}P^N u \rangle \, dW^k_\sigma \\
+ \int_0^t \sum_{k \geq 1} \langle A^{1/2}P^N P[g_k(u) - ((\xi_k \cdot \nabla)u)], A^{1/2}u^N \rangle \, dW^k_\sigma.
\]
and for $Q^N = I - P^N$,
\[
|I_2| \leq \|A(P^N u - u^N)\|_{L^2} \|Q^N u\|_{L^4} \|\nabla u^N\|_{L^4} \\
\leq \frac{\nu}{6} \|A(P^N u - u^N)\|_{L^2}^2 + c(\nu) \|Q^N u\|_{H^1} \|u^N\|_{H^2}^2
\] (6.11)
and
\[
|I_3| \leq \|A(P^N u - u^N)\|_{L^2} \|u\|_{L^\infty} \|\nabla (u^N - u)\|_{L^2} \\
\leq \frac{\nu}{6} \|A(P^N u - u^N)\|_{L^2}^2 + c(\nu) \|P^N u - u^N\|_{H^1} \|u\|_{H^2}^2
\] (6.12)

Next, we also note that
\[
\frac{1}{2} \left\{ \langle A^{1/2}P^N P[(\xi_k \cdot \nabla)(\xi_k \cdot \nabla)u^N], A^{1/2}u^N \rangle \\
+ \langle A^{1/2}P^N P[(\xi_k \cdot \nabla)P(\xi_k \cdot \nabla)u], A^{1/2}P^N u \rangle \\
- \langle A^{1/2}P^N P[(\xi_k \cdot \nabla)P(\xi_k \cdot \nabla)u], A^{1/2}u^N \rangle \\
- \langle A^{1/2}P^N P[(\xi_k \cdot \nabla)P(\xi_k \cdot \nabla)u^N], A^{1/2}P^N u \rangle \right\}
\]
\[
= \frac{1}{2} \left\{ \langle A^{1/2}P^N P[(\xi_k \cdot \nabla)(\xi_k \cdot \nabla)(u^N - u)], A^{1/2}(u^N - u) \rangle \\
- \langle A^{1/2}P^N P[(\xi_k \cdot \nabla)(\xi_k \cdot \nabla)(u^N - u)], A^{1/2}(P^N u - u) \rangle \right\}
\]
\[
= \frac{1}{2} \langle A^{1/2}P^N P[(\xi_k \cdot \nabla)(\xi_k \cdot \nabla)(u^N - u)], A^{1/2}(u^N - u) \rangle
\] (6.13)

where we have used in the last equation, the identity $\langle P^N x, Q^N y \rangle = 0$ which holds for the orthogonal projection $P^N$ and its complement $Q^N = I - P^N$. Moving on, we estimate the following three terms as follows
\[
\frac{1}{2} \|P^N P[g_k(u^N) - ((\xi_k \cdot \nabla)(\xi_k \cdot \nabla)u^N)]\|_{H^1}^2 + \frac{1}{2} \|P^N P[g_k(u) - ((\xi_k \cdot \nabla)u)]\|_{H^1}^2
\]
\[- \langle A^{1/2}P^N P[g_k(u) - ((\xi_k \cdot \nabla)u)], A^{1/2}P^N P[g_k(u^N) - ((\xi_k \cdot \nabla)u^N)] \rangle \]
\[
= \frac{1}{2} \|P^N P[g_k(u^N) - ((\xi_k \cdot \nabla)u^N)] - P^N P[g_k(u) - ((\xi_k \cdot \nabla)u)]\|_{H^1}^2
\]
\[
\leq \|P^N P[g_k(u^N) - g_k(u)]\|_{H^1}^2 + \|P^N P[(\xi_k \cdot \nabla)(u^N - u)]\|_{H^1}^2
\] (6.14)

where by the continuity of $P^N P$ and the Lipschitz property of $(g_k)_{k \geq 1}$, it follows that
\[
\sum_{k \geq 1} \|P^N P[g_k(u^N) - g_k(u)]\|_{H^1}^2 \leq \|u^N - u\|_{H^1}^2
\]
\[
\leq \|P^N u - u^N\|_{H^1}^2 + \|Q^N u\|_{H^1}^2
\]
holds uniformly in $N \in \mathbb{N}$. On the other hand, by using Lemma 4.4 and the assumption on the transport noise in Section 3.1, we can combine (6.13) with
the second right-hand term in (6.14) to obtain
\[
\left| \sum_{k \geq 1} \left\{ \| P^N P[(\xi_k \cdot \nabla)(u^N - u)] \|^2_{H^1} + \langle A^{1/2} P^N P[(\xi_k \cdot \nabla)P(\xi_k \cdot \nabla)(u^N - u)], A^{1/2}(u^N - u) \rangle \right\} \right| \leq \frac{1}{2}\| P^N u^N \|_{L^2}^2 - \frac{1}{2}\| P^N u_0 - u^N \|_{H^1}^2 + \int_0^t \mathcal{M}(P^N u | u^N) \, dW^k_{\sigma} + c(\nu) \int_0^t \mathcal{R}(P^N u | u^N) \, d\sigma
\] (6.16)

where
\[
\mathcal{M}(P^N u | u^N) := \sum_{k \geq 1} \langle A^{1/2} P^N P[g_k(u) - ((\xi_k \cdot \nabla)u)], A^{1/2}(P^N u - u^N) \rangle
\]
\[
= \sum_{k \geq 1} \langle A^{1/2} P((\xi_k \cdot \nabla)(u^N - u)), A^{1/2}(P^N u - u^N) \rangle
\]
\[
- \sum_{k \geq 1} \langle A^{1/2} P[g_k(u^N) - g_k(u)], A^{1/2}(P^N u - u^N) \rangle
\] (6.17)
is such that \( \int_0^t \mathcal{M}(\cdot) \, dW^k_{\sigma} \) is a family of real-valued centred martingales and
\[
\mathcal{R}(P^N u | u^N) := \left( \| P^N u - u^N \|^2_{H^1} + \| Q^N u \|^2_{H^1} \right) \times \left( 1 + \| u^N \|^2_{H^2} + \| u \|^2_{H^2} \right)
\] (6.18)
is the remainder term. Note from (6.1) and (6.2) that
\[
\| P^N u_0 - u^N_0 \|^2_{H^1} = 0.
\] (6.19)

In order to get rid of the martingale term in (6.16), we are going to take the expectation of (6.16). However, in order to also handle the nonlinear interactions due to the fluid’s advection, we also introduce the a.s. strictly positive stopping time \( \tau_R \) defined as
\[
\tau_R := \inf \left\{ t \in (0, T) : \int_0^t \left( \| u^N(\sigma) \|^2_{H^2} + \| u(\sigma) \|^2_{H^2} \right) \, d\sigma \geq R \right\}
\]
for some $R > 0$. Note that $\mathbb{P}[\tau_R \leq 0] = 0$. Since the martingale term is centred, it follows that
\[
\mathbb{E} \int_0^{t \wedge \tau_R} \mathcal{M}(P^N u \mid u^N) \, dW^k_\sigma = 0. \tag{6.20}
\]
By collecting the various estimates above and adding the resulting inequality to (6.5), we obtain
\[
\mathbb{E} \left( \| (P^N u - u^N) (t \wedge \tau_R) \|_{H^1}^2 + \| (Q^N u) (t \wedge \tau_R) \|_{H^1}^2 \right) \\
+ \nu \int_0^{t \wedge \tau_R} \| A(P^N u - u^N) \|^2_{L^2} \, d\sigma \\
\leq c(\nu) \mathbb{E} \int_0^{t \wedge \tau_R} \left( \| P^N u - u^N \|_{H^1}^2 + \| Q^N u \|_{H^1}^2 \right) \\
\times \left( 1 + \| u^N \|_{H^2}^2 + \| u \|_{H^2}^2 \right) \, d\sigma \\
+c(\nu, u_0, K) \mathbb{E} e^{-2tN} \tag{6.21}
\]
for $t = \delta_1 \wedge \tau_R$. Now if $t = \delta_1$ as given in (6.5), then $\mathbb{E} e^{-2tN} = e^{-2\delta_1 N}$. On the other hand, if $t = \tau_R$, then we can use the strict positivity of $\tau_R$ to infer the existence of a deterministic constant $\delta_2 > 0$ such that $\mathbb{P}[\tau_R \geq \delta_2] > 0$. Consequently, it also follows that
\[
\mathbb{E} e^{-2tN} \leq \mathbb{E} e^{-2\delta_2 N} \mathbb{1}_{\tau_R \geq \delta_2} \leq \mathbb{E} e^{-2\delta_2 N} \mathbb{P}[\tau_R \geq \delta_2] \leq Ce^{-2\delta_2 N}
\]
when $t = \tau_R$. If we now use the definition of the stopping time $\tau_R$, then it follows from Grönwall’s lemma that
\[
\mathbb{E} \left( \| (P^N u - u^N) (t \wedge \tau_R) \|_{H^1}^2 + \| (Q^N u) (t \wedge \tau_R) \|_{H^1}^2 \right) \\
\leq e^{c(\nu) (T + R) c(\nu, u_0, K)} e^{-2\delta N}, \tag{6.22}
\]
where $\delta = \min\{\delta_1, \delta_2\}$. Our desired result thus follow due to (6.4).

7. Appendix

In this section, we give the proofs of the results stated in Section 4.

Proof of Lemma 4.3 The proof can be directly deduced from the proof of [29, Theorem 5.3] where the Gevrey space $D(e^{\varphi_A^{1/2}} : \mathbb{H}^r(T^d))$ is shown to be a Banach algebra when $r > d/2$. The condition $r > d/2$ is only used to obtain [29] (5.22). Our desired result is however obtained from [29] (5.21)]. See also, the original proof in [11, Lemma 1]. For completeness, however, we
reproduce the proof below.

\[
\|A^{1/2}e^{\varphi A^{1/2}}(u \cdot v)\|^2_{L^2} = \sum_{\ell} \left| \sum_{j+k=\ell} \hat{u}_j \cdot \hat{v}_k \right|^2 |\ell|^2 e^{2\varphi|\ell|} \\
\leq \sum_{\ell} \left( \sum_{j+k=\ell} \left| \hat{u}_j \right| \left| \hat{v}_k \right| |\ell| e^{\varphi|\ell|} \right)^2 \\
\lesssim \sum_{\ell} \left( \sum_{j+k=\ell} \left| \hat{u}_j \right| |j| e^{\varphi|j|} |\hat{v}_k| e^{\varphi|k|} \right)^2 \\
+ \sum_{\ell} \left( \sum_{j+k=\ell} \left| \hat{u}_j \right| e^{\varphi|j|} |\hat{v}_k| |k| e^{\varphi|k|} \right)^2.
\] (7.1)

If we now use Young’s convolution inequality followed by Jensen’s inequality, we obtain,

\[
\|A^{1/2}e^{\varphi A^{1/2}}(u \cdot v)\|^2_{L^2} \lesssim \sum_j \left| \hat{u}_j \right|^2 |j|^2 e^{2\varphi|j|} \left( \sum_k \left| \hat{v}_k \right| e^{\varphi|k|} \right)^2 \\
+ \sum_k \left| \hat{v}_k \right|^2 |k|^2 e^{2\varphi|k|} \left( \sum_j \left| \hat{u}_j \right| e^{\varphi|j|} \right)^2 \\
\lesssim \|A^{1/2}e^{\varphi A^{1/2}}u\|^2_{L^2} \|e^{\varphi A^{1/2}}v\|^2_{L^2} \\
+ \|A^{1/2}e^{\varphi A^{1/2}}v\|^2_{L^2} \|e^{\varphi A^{1/2}}u\|^2_{L^2}.
\] (7.2)

\[\square\]

**Proof of Lemma 4.4.** To simplify notations, for \(j, k \in \mathbb{Z}^d\), we let \(j := |j|\) and \(k := |k|\) and note that for \(n \in \mathbb{Z}^d\),

\[
\frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} e^{i n \cdot x} \, dx = \begin{cases} 
0 & \text{if } n \neq 0, \\
1 & \text{if } n = 0.
\end{cases}
\]

Combining this information with \(\langle f, g \rangle = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x) \overline{g(x)} \, dx\), we obtain for

\[
\xi_k(x) = \sum_{j \in \mathbb{Z}^d} \hat{\xi}_{kj} e^{ij \cdot x} \quad \text{and} \quad u(x) = \sum_{k \in \mathbb{Z}^d} \hat{u}_k e^{ik \cdot x}
\]
that
\begin{align*}
\langle A^r e^{\varphi A^{1/2s}} (\xi_k \cdot \nabla (\xi_k \cdot \nabla) u), A^r e^{\varphi A^{1/2s}} u \rangle \\
= \frac{1}{(2\pi)^d} \langle \sum_j \xi_{kj} e^{ijx} \cdot \sum_k i \xi_{kj} e^{ijx} \cdot \sum_k d \xi_k e^{ikx} \cdot \sum_k k^4 r e^{2\varphi k^{1/s}} \hat{u}_k e^{ikx} \rangle \\
= \sum_n \sum_{2(j+k)=n} (\xi_{kj} \cdot i \xi_{kj} \cdot i k) k^4 r e^{2\varphi k^{1/s}} |\hat{u}_k|^2 \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} e^{im \cdot x} \, dx \\
= \sum_{2(j+k)=0} k^4 r e^{2\varphi k^{1/s}} |\hat{\xi}_{j-k} \cdot i k|^2 |\hat{u}_k|^2 \\
= -\| A^r e^{\varphi A^{1/2s}} ((\xi_k \cdot \nabla) u) \|_{L^2}^2
\end{align*}

Next, we present the proof of Lemma 4.5 for completeness. The proof essentially follow the same argument as [16, Lemma 5.1] with the obvious modification with respect to the exponential weight.

Proof of Lemma 4.5. Given that (4.2) holds, we can find an increasing family $(N_l)_{l \in \mathbb{N}_0}$ such that

\[ \mathbb{E} \| u_{N_l+1} - u_{N_l} \|_{\mathcal{E}(\mathcal{T}^{M}_{N_l+1})} \leq 2^{-2l}. \]

With this, we define the stopping time

\[ \tau_l := \inf \left\{ t \in [0, T] : \| u_{N_l} \|_{\mathcal{E}(t)} > \| e^{\varphi(0) A^{1/2s}} u_0 \|_{H^s} + M - 1 + 2^{-l} \right\} \wedge T \]

so that $\tau_l \wedge \tau_{l+1} \in \mathcal{T}^{M}_{N_{l+1}}$ and thus, by Chebyshev’s inequality,

\[ \mathbb{P} (\| u_{N_l+1} - u_{N_l} \|_{\mathcal{E}(\tau_l \wedge \tau_{l+1})} \geq 2^{-(l+2)}) \leq 2^{l+2} \mathbb{E} \| u_{N_l+1} - u_{N_l} \|_{\mathcal{E}(\tau_l \wedge \tau_{l+1})} \]

\[ \leq 2^{l+2} 2^{-2l} = 2^{2-l}. \]

Given the finiteness of this probability, by setting

\[ \Omega_p := \bigcap_{l=p}^{\infty} \{ \| u_{N_l+1} - u_{N_l} \|_{\mathcal{E}(\tau_l \wedge \tau_{l+1})} < 2^{-(l+2)} \}, \]

(7.3)
we can use Borel–Cantelli lemma to conclude that
\[
0 = \mathbb{P} \left( \bigcap_{p=1}^{\infty} \bigcup_{l=p}^{\infty} \left\{ \|u^{N_{l+1}} - u^{N_{l}}\|_{\mathcal{C}(\tau_{l} \wedge \tau_{l+1})} \geq 2^{-(l+2)} \right\} \right)
= \mathbb{P} \left( \bigcap_{p=1}^{\infty} \Omega_{p}^{c} \right)
= 1 - \mathbb{P} \left( \bigcup_{p=1}^{\infty} \Omega_{p} \right)
\tag{7.4}
\]
and thus, the set \( \tilde{\Omega} = \bigcup_{p=1}^{\infty} \Omega_{p} \) has full measure.

Now consider the set \( \{ \tau_{l} < \tau_{l+1} \} \cap \Omega_{p} \) where \( l \geq p \), so that \( \tau_{l} < T \). By using the continuity of \( \|u^{N_{l}}\|_{\mathcal{C}(t)} \) in \( t \) (in the sense of Remark 3.8), it follows that
\[
\|u^{N_{l}}\|_{\mathcal{C}(\tau_{l})} = \|e^{\phi(0)}A^{1/2}u_{0}\|_{H^{r}} + M - 1 + 2^{-l}. \tag{7.5}
\]
On the other hand, we obtain on \( \Omega_{p} \) (c.f. (7.3)) that
\[
\|u^{N_{l}}\|_{\mathcal{C}(\tau_{l} \wedge \tau_{l+1})} - \|u^{N_{l+1}}\|_{\mathcal{C}(\tau_{l} \wedge \tau_{l+1})} < 2^{-(l+2)}, \tag{7.6}
\]
\[
\|e^{\phi(0)}A^{1/2}u_{0}\|_{H^{r}} - \|e^{\phi(0)}A^{1/2}u_{0}\|_{H^{r}} < 2^{-(l+2)}. \tag{7.7}
\]
If we now combine (7.5)–(7.7), then we obtain on the set \( \{ \tau_{l} < \tau_{l+1} \} \cap \Omega_{p} \) that
\[
\|u^{N_{l+1}}\|_{\mathcal{C}(\tau_{l} \wedge \tau_{l+1})} > \|u^{N_{l}}\|_{\mathcal{C}(\tau_{l} \wedge \tau_{l+1})} - 2^{-(l+2)}
= \|u^{N_{l}}\|_{\mathcal{C}(\tau_{l})} - 2^{-(l+2)}
= \|e^{\phi(0)}A^{1/2}u_{0}\|_{H^{r}} + M - 1 + 2^{-l} - 2^{-(l+2)}
> \|e^{\phi(0)}A^{1/2}u_{0}\|_{H^{r}} + M - 1 + 2^{-(l+1)}. \tag{7.8}
\]
Similarly, on \( \{ \tau_{l} < \tau_{l+1} \} \cap \Omega_{p} \), we can also show that
\[
\|u^{N_{l+1}}\|_{\mathcal{C}(\tau_{l} \wedge \tau_{l+1})} \leq \|e^{\phi(0)}A^{1/2}u_{0}\|_{H^{r}} + M - 1 + 2^{-(l+1)} \tag{7.9}
\]
by noticing that the reverse inequalities
\[
\|u^{N_{l+1}}\|_{\mathcal{C}(\tau_{l} \wedge \tau_{l+1})} - \|u^{N_{l}}\|_{\mathcal{C}(\tau_{l} \wedge \tau_{l+1})} < 2^{-(l+2)},
\]
\[
\|e^{\phi(0)}A^{1/2}u_{0}\|_{H^{r}} - \|e^{\phi(0)}A^{1/2}u_{0}\|_{H^{r}} < 2^{-(l+2)}.
\]
also holds on \( \Omega_{p} \). The two estimates (7.8) and (7.9) implies that \( \{ \tau_{l} < \tau_{l+1} \} \cap \Omega_{p} = \emptyset \) which further implies that for every \( l \geq p \) and \( \omega \in \Omega_{p} \), we have that
\[
\tau_{l+1}(\omega) \leq \tau_{l}(\omega). \tag{7.10}
\]
Thus, by combining (7.4) with the decreasing sequence (7.10), we obtain that
\[
\tau_{l} \to \tau \quad \text{a.s.}
\]
with
\[
\tau \leq T \quad \text{a.s.}
\]
following due to the definition of $\tau_l$. To show (4.3), it remains to show that $\tau > 0$ a.s. or equivalently that $\mathbb{P}(\tau = 0) = 0$. For this, we fix $\varepsilon > 0$ with $T > \varepsilon > 0$ so that

$$
\lim_{l \to \infty} \inf \{ \tau_l < \varepsilon \} = \{ \tau < \varepsilon \}
$$

and

$$
\subset \{ \| u^{N_l} \|_{\mathcal{E}(\tau_l \wedge \varepsilon)} = \| e^{\varphi(0)}A^{1/2} u^{N_l}_0 \|_{H^r} + M - 1 + 2^{-l} \}
$$

and

$$
\subset \{ \| u^{N_l} \|_{\mathcal{E}(\tau_l \wedge \varepsilon)} > \| e^{\varphi(0)}A^{1/2} u^{N_l}_0 \|_{H^r} + M - 1 \}.
$$

Therefore,

$$
\mathbb{P}(\tau = 0) = \lim_{\varepsilon \to 0} \mathbb{P}(\tau < \varepsilon) = \lim_{\varepsilon \to 0} \mathbb{P}\left( \liminf_{l \to \infty} \{ \tau_l < \varepsilon \} \right) \leq \liminf_{\varepsilon \to 0} \mathbb{P}(\tau_l < \varepsilon) \leq \limsup_{\varepsilon \to 0} \mathbb{P}(\tau_l < \varepsilon) \leq \lim_{\varepsilon \to 0} \sup_{l \in \mathbb{N}_0} \mathbb{P}\left( \| u^{N_l} \|_{\mathcal{E}(\tau_l \wedge \varepsilon)} > \| e^{\varphi(0)}A^{1/2} u^{N_l}_0 \|_{H^r} + M - 1 \right)
$$

= 0.

This completes the proof of (4.3).

Now, to show (4.4), we observe from (7.10) that for any $\omega \in \tilde{\Omega}$, we can choose $p = p(\omega)$ so that whenever $l \geq p$, we have that $\omega \in \Omega_p$ and

$$
\tau(\omega) \leq \tau_{l+1}(\omega) \leq \tau_l(\omega).
$$

(7.11)

Thus, it follows from (7.3) that

$$
\| u^{N_{l+1}} - u^{N_l} \|_{\mathcal{E}(\tau(\omega))} \leq \| u^{N_{l+1}} - u^{N_l} \|_{\mathcal{E}(\tau_l(\omega) \wedge \tau_{l+1}(\omega))} < 2^{-(l+2)}.
$$

We can, therefore, find a process $u(\cdot) = u(\cdot \wedge \tau) \in \mathcal{E}(\tau)$ such that (4.4) holds. Finally, to show (4.5), we use (7.5), (7.9) and (7.11) and obtain

$$
1_{\Omega_{l+1}} \| u^{N_l} \|_{\mathcal{E}(\tau)} \leq 1_{\Omega_{l+1}} \| u^{N_{l+1}} \|_{\mathcal{E}(\tau)} + 2^{-(l+2)}
$$

$$
\leq \| e^{\varphi(0)}A^{1/2} u^{N_{l+1}}_0 \|_{H^r} + M - 1 + 2^{-(l+1)} + 2^{-(l+2)}
$$

$$
\leq \| e^{\varphi(0)}A^{1/2} u^{N_{l+1}}_0 \|_{H^r} + M
$$

$$
\leq \sup_n \| e^{\varphi(0)}A^{1/2} u^{N_l}_0 \|_{H^r} + M
$$

almost surely. \(\square\)

Finally, we present a useful identify for final dimensional orthogonal projection. In particular, for a Hilbert space $V$, we note that since the identity

$$
\langle (P^N - P^n)f^N, P^n(f^N - f^n) \rangle_V = \langle P^n(P^N - P^n)f^N, P^n(f^N - f^n) \rangle_V = 0
$$
holds for all $f^N, f^n \in V$ and $N \geq n$, it follows that
\[
\|P_N f^N - P_n f^n\|_V^2 = \|(P_N - P_n)f^N + P_n(f^N - f^n)\|_V^2 \\
= \|(P_N - P_n)f^N\|_V^2 + \|P_n(f^N - f^n)\|_V^2 \\
= \|Q^n P_N f^N\|_V^2 + \|P_n(f^N - f^n)\|_V^2.
\] (7.12)

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7.2. Author Contribution
All authors wrote and reviewed the manuscript.

7.3. Conflict of Interest
The authors declare that they have no conflict of interest.

7.4. Data Availability Statement
Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

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