TAYLOR EXPANSION PROOF OF THE MATRIX TREE THEOREM - PART I

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ABSTRACT. The Matrix-Tree Theorem states that the number of spanning trees of a graph is given by the absolute value of any cofactor of the Laplacian matrix of the graph. We propose a very short proof of this result which amounts to comparing Taylor expansions.

1. INTRODUCTION

Call a matrix $L = (L_{ij})$ Laplace-like if it is symmetric and $\forall i \sum_j L_{ij} = 0$.

Let $M_{ij}(L)$ denote the submatrix of $L$ obtained by deleting the $i$'th row and $j$'th column. The $i,j$ cofactor of $L$ is $C_{ij} := (-1)^{i+j} \det M_{ij}(L)$. It is not hard to see that if $L$ is Laplace-like then all its cofactors are the same; we denote their common value by $C(L)$.

Theorem 1. If $L$ is an $n \times n$ Laplace-like matrix then

$$C(L) = (-1)^{n+1} \sum_\tau A_\tau(L)$$

where the sum is taken over all trees $\tau$ whose set of vertices is $\{1, ..., n\}$ and

$$A_\tau(L) = \prod_{i<j, (i,j) \in \text{Edges}(\tau)} L_{ij}.$$ 

A special case of the above gives the well-known Matrix-Tree Theorem.

Corollary 2. The number of spanning trees of an unoriented graph is equal to $|C(L)|$ where $L = A - D$ for $A$ the adjacency matrix of the graph and $D$ the diagonal matrix of valencies of the graph.

The original formulation and proof of this beautiful theorem is due to Kirchoff [2]. Since then, several different proofs have been proposed. Our proof of Theorem 1 is given in Section 2 below, and is shorter than any we’ve seen. Roughly speaking, it amounts to comparing the Taylor expansion of both sides of eq (1.1). In the second part of this paper, we shall use the same technique to prove a generalization of the Matrix Tree Theorem for (not necessarily symmetric) matrices whose columns sum to zero.

I would like to thank David Kazhdan, Nati Linial, Ori Parzanchevski, Ron Peled, Ron Rosenthal and Ran Tessler for their suggestions and comments. I am especially grateful to Nati Linial for suggesting reference [1].

2. PROOF OF MATRIX TREE THEOREM

We denote by $\mathcal{L}$ the linear space of all Laplace-like matrices.

Let $T(L) = \sum_\tau A_\tau(L)$. The tangent space to $\mathcal{L}$ at any point is spanned by $v_{ij} := \frac{\partial}{\partial L_{ij}} + \frac{\partial}{\partial L_{ji}} - \frac{\partial}{\partial L_{ii}} - \frac{\partial}{\partial L_{jj}}$ for $i < j$. We will prove that $v_{ij}(C(L)) = v_{ij}(T(L))$ for all $i,j$ and $L$; the theorem then follows since $\mathcal{L}$ is path connected and $C(0) = T(0) = 0$. 

arXiv:1308.2633v1  [math.CO]  9 Aug 2013
Let $L' = M_{ij}(L_\tau)$, where $L_\tau$ is obtained from $L$ by adding the $j$'th row to the $i$'th row and the $j$'th column to the $i$'th column. We will show that $v_{ij}(T(L)) = T(L')$ and $v_{ij}(C(L)) = -C(L')$. The result then follows by induction on the size of the matrix $n$.

To see that $v_{ij}(T(L)) = T(L')$, note that $v_{ij}(A_\tau(L)) = \frac{\partial}{\partial L_{ij}} A_\tau(L)$ is nonzero only if $i,j$ are connected by an edge in $\tau$, in which case we may contract the edge to produce a new spanning tree $\tau' = \text{contract}_{ij}(\tau)$ with vertices $\{1, ..., n-1\}$, see Fig. 2.1. Explicitly, we erase the vertex $j$ and the $i,j$ edge, and reconnect all the other neighbours of $j$ to the vertex $i$. We also relabel the vertices $j+1, ..., n$ by $j, ..., n-1$, respectively. Fixing some $\tau'$, it is then easy to see that

$$\sum_{\{\tau|\text{contract}_{ij}(\tau) = \tau\}} \frac{\partial}{\partial L_{ij}} A_\tau(L) = A_\tau(L'),$$

hence $v_{ij}(T(L)) = T(L')$.

Next we show $-v_{ij}(C(L)) = C(L')$. Indeed,

$$-v_{ij}(C(L)) = -v_{ij}(C_{ii}(L)) = \frac{\partial}{\partial L_{jj}} C_{ii}(L) = \frac{\partial}{\partial L_{ij}} C_{ii}(L_\tau) =$$

$$= \frac{\partial}{\partial L_{ij}} \det(M_{ii}(L_\tau)) = \det(M_{j-1,j-1}(M_{ii}(L_\tau))) = \det M_{ii}(M_{jj}(L_\tau)) =$$

$$= \det M_{ii}(L') = C(L')$$

which completes the proof. □

Remark 3. Note even if $L$ is not symmetric (but its rows and columns sum to zero) it is still true that $-v_{ij}(C(L)) = C(L')$. This has led us to suspect that a non-symmetric version of the theorem should exist. In the second part of this paper we discuss a more general, non-symmetric formulation of the Matrix Tree Theorem that we managed to prove using a very similar technique.

REFERENCES

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