ON FUNCTIONS OF FINITE BAIRE INDEX

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Abstract. It is proved that every function of finite Baire index on a separable metric space \( K \) is a \( D \)-function, i.e., a difference of bounded semi-continuous functions on \( K \). In fact it is a strong \( D \)-function, meaning it can be approximated arbitrarily closely in \( D \)-norm, by simple \( D \)-functions. It is shown that if the \( n \)th derived set of \( K \) is non-empty for all finite \( n \), there exist \( D \)-functions on \( K \) which are not strong \( D \)-functions. Further structural results for the classes of finite index functions and strong \( D \)-functions are also given.

1. Introduction

Throughout, let \( K \) be a separable metric space. A function \( f : K \to \mathbb{R} \) is called a difference of bounded semi-continuous functions if there exist bounded lower semi-continuous functions \( u \) and \( v \) on \( K \) with \( f = u - v \). We denote the class of all such functions by \( DBSC(K) \). We shall also refer to members of \( DBSC(K) \) as \( D \)-functions. A classical theorem of Baire (cf. [H, p.274]) yields that \( f \in DBSC(K) \) if and only if there exists a sequence \( (\varphi_j) \) of continuous functions on \( K \) so that

\[
\sup_{k \in K} \sum |\varphi_j(k)| < \infty \quad \text{and} \quad f = \sum \varphi_j \text{ point-wise.}
\]

(1)

Now defining \( \|f\|_D = \inf \{ \sup_{k \in K} \sum |\varphi_j|(k) : (\varphi_j) \text{ is a sequence of continuous functions on } K \text{ satisfying } (1) \} \), it easily follows that \( DBSC(K) \) is a Banach algebra; and of course \( DBSC(K) \subset B_1(K) \) where \( B_1(K) \) denotes the (bounded) first Baire class of functions on \( K \); i.e., the space of all bounded functions on \( K \) which are the limit of a point-wise convergent sequence of continuous functions on \( K \).

\( DBSC(K) \) appears as a natural object in functional analysis. For example, if \( X \) is a separable Banach space and \( K \) is the unit ball of \( X^* \) in the weak*-topology, then \( X \) contains a subspace isomorphic to \( c_0 \) if and only if there is an \( f \) in \( X^{**} \sim X \)

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with $f | K$ in $DBSC(K)$ (cf. [HOR], [R1]). Natural invariants for $DBSC(K)$ are used in a fundamental way in [R1], to prove that $c_0$ embeds in $X$ provided $X$ is non-reflexive and $Y^*$ is weakly sequentially complete for all subspaces $Y$ of $X$.

We investigate here a special subclass of $DBSC(K)$, which we term $SD(K)$, and show that all functions of finite Baire index belong to this class.

To motivate the definitions of these objects we first recall the following class of functions. Define $B_{1/2}(K)$ to be the set of all uniform limits of functions in $DBSC(K)$. (The terminology follows that in [HOR].) Functions in $B_{1/2}(K)$ may be characterized in terms of an intrinsic oscillation behavior, which we now give.

For $f : K \to \mathbb{R}$ a given bounded function, let $Uf$ denote the upper semi-continuous envelope of $f$; $Uf(x) = \liminf_{y \to x} f(y)$ for all $x \in K$. (We use non-exclusive lim sups; thus equivalently, $Uf(x) = \inf_U \sup_{y \in U} f(y)$, the inf over all open neighborhoods of $x$.) Now we define $\text{osc} f$, the lower oscillation of $f$, by

$$\text{osc} f(x) = \liminf_{y \to x} |f(y) - f(x)| \quad \text{for all } x \in K.$$  

Finally, we define $\text{osc} f$, the oscillation of $f$, by

$$\text{osc} f = U\text{osc} f.$$  

Now let $\varepsilon > 0$. We define the (finite) oscillation sets of $f$, $\text{os}_j(f, \varepsilon)$, as follows. Set $\text{os}_0(f, \varepsilon) = K$. Suppose $j \geq 0$ and $\text{os}_j(f, \varepsilon)$ has been defined. Let $\text{os}_{j+1}(f, \varepsilon) = \{x \in L : \text{osc} f | L(x) \geq \varepsilon\}$, where $L = \text{os}_j(f, \varepsilon)$.

We recall the following fact ([HOR]).

**Proposition 1.1.** Let $f : K \to \mathbb{R}$ be a given function. The following are equivalent:

1. $f \in B_{1/2}(K)$.
2. For all $\varepsilon > 0$, there is an $n$ with $\text{os}_n(f, \varepsilon) = \emptyset$.

(The proof given in [HOR] for compact metric spaces works for arbitrary separable ones; cf. also [R2].)

**Remark.** Actually, the sets defined in [HOR] use what we term here the upper oscillation of $f$, defined by $\overline{\text{osc}} f(x) = \limsup_{y, z \to x} |f(y) - f(z)|$. It is easily seen that $\overline{\text{osc}} f$ is upper semi-continuous and

$$\frac{1}{2} \overline{\text{osc}} f \leq \text{osc} f \leq \overline{\text{osc}} f.$$  

Now define \( K_j(f, \varepsilon) \) inductively by

\[
K_0(f, \varepsilon) = K \quad \text{and} \quad K_{j+1}(f, \varepsilon) = \{ x \in K_j : \overline{\text{osc}} f | K_j(x) \geq \varepsilon \}.
\]

We then have by (4) that

\[
K_j(f, 2\varepsilon) \subset \text{os}_j(f, \varepsilon) \subset K_j(f, \varepsilon) \quad \text{for all} \quad j.
\]

Thus \( f \) satisfies 2 of 1.1 if and only if for all \( \varepsilon > 0 \), there is an \( n \) with \( K_n(f, \varepsilon) = \emptyset \).

Proposition 1.1 suggests the following quantitative notion.

**Definition 1.** Let \( f : K \to \mathbb{R} \) be a given bounded function and \( \varepsilon > 0 \). We define \( i(f, \varepsilon) \), the \( \varepsilon \)-oscillation index of \( f \), to be \( \sup \{ n : \text{os}_n(f, \varepsilon) \neq \emptyset \} \).

Thus Proposition 1.1 says that \( f \in B_{1/2}(K) \) if and only if \( i(f, \varepsilon) < \infty \) for all \( \varepsilon > 0 \).

**Definition 2.** A bounded function \( f : K \to \mathbb{R} \) is said to be of finite Baire index if there is an \( n \) with \( \text{os}_n(f, \varepsilon) = \emptyset \) for all \( \varepsilon > 0 \). We then define \( i(f) \), the oscillation index of \( f \), by

\[
i(f) = \max_{\varepsilon > 0} i(f, \varepsilon).
\]

Evidently \( f \) is continuous if and only if \( i(f) = 0 \).

**Remark.** In [HOR], an index \( \beta(f) \) is defined as \( \beta(f) = \sup_{\varepsilon > 0} \min \{ j : K_j(f, \varepsilon) = \emptyset \} \). It follows from the remark following Proposition 1.1 that \( f \) is of finite index if and only if \( \beta(f) < \infty \), and then in fact \( \beta(f) = i(f) + 1 \).

In [HOR], it is proved that finite index functions belong to \( B_{1/4}(K) \), a class properly containing the \( D \)-functions. We obtain here that every function of finite Baire index belongs to \( DBSC(K) \). In fact, we show that it belongs to the following subclass:

**Definition 3.** A function \( f : K \to \mathbb{R} \) is said to be a strong \( D \)-function if there exists a sequence \( (\varphi_n) \) of simple \( D \)-functions with \( \| f - \varphi_n \|_D \to 0 \). We denote the class of all strong \( D \)-functions by \( SD(K) \).

We may thus formulate one of our main results as follows:
Theorem 1.2. Let $f : K \to \mathbb{R}$ be a function of finite Baire index. Then $f$ belongs to $SD(K)$.

As we show below it is easily seen that every simple $D$-function has finite Baire index. Thus Theorem 1.2 yields that $SD(K)$ equals the closure, in $D$-norm, of the functions of finite index on $K$. Our proof essentially proceeds from first principles. An alternate argument, using transfinite oscillations, is given in [R2].

An interesting special case of 1.2: Let $f : [0, 1] \to \mathbb{R}$ be bounded such that $\lim_{y \uparrow x} f(y), \lim_{y \downarrow x} f(y)$ exist for all $x$. Then $f$ is in $SD[0, 1]$. The fact that such functions are in $DBSC[0, 1]$ was initially proved jointly by the first and third named authors, and precedes the work given here [C]. (It is a standard elementary result that if $f$ has these properties, then $os_1(f, \varepsilon)$ is finite for all $\varepsilon > 0$, hence $i(f) = 1$.)

It is evident that the simple $D$-functions form an algebra, hence $SD(K)$ is a Banach algebra. It is proved in [R2] that $SD(K)$ is a lattice, i.e., $|f| \in SD(K)$ if $f \in SD(K)$. We prove here that the functions of finite index form an algebra and a lattice. This follows immediately from the following result.

Theorem 1.3. Let $f, g$ be bounded real-valued functions on $K$, of finite index. Let $h$ be any of the functions $f + g, f \cdot g, \max\{f, g\}, \min\{f, g\}$. Then

$$i(h) \leq i(f) + i(g).$$

It is evident that if $f$ is of finite index, then for any non-zero scalar $\lambda$, $i(\lambda f) = i(f)$; also it is easy to show that $i(|f|) \leq i(f)$. However the assertions of Theorem 1.3 appear to lie below the surface. The quantitative result which does the job (Theorem 2.8 below), is then applied to yield a necessary condition for a function to be in $SD(K)$, which is also sufficient in the case of upper semi-continuous functions.

Theorem 1.4. Let $f : K \to \mathbb{R}$ be a given bounded function.

(a) If $f \in SD(K)$, then

$$\lim_{\varepsilon \to 0} \varepsilon i(f, \varepsilon) = 0$$

(b) If $f$ is semi-continuous and satisfies (7), then $f \in SD(K)$.

It is proved in [R2] that every $SD$-function is a difference of strong $D$-semi-continuous functions. Evidently Theorem 1.4 yields an effective criterion for distinguishing the class of strong $D$-semi-continuous functions. However, one may
construct functions, e.g., on $K = \omega^\omega + 1$, which are not $D$-functions but satisfy (7), or which are $D$-functions but not $SD$-functions, and still satisfy (7). An effective intrinsic criterion involving the \( \omega \text{th oscillation} \), which does distinguish $SD$-functions from $D$-functions, is given in [R2].

We conclude the article by applying Theorem 1.4(a) to show that $DBSC(K) \sim SD(K)$ is non-empty for all interesting $K$.

**Proposition 1.5.** Assume that $K^{(j)}$, the $j$th derived set of $K$, is non-empty for all $j = 1, 2, \ldots$. There exists a function $f$ on $K$ which is in $DBSC(K)$ but not in $SD(K)$.

(An alternate proof of 1.5, using transfinite oscillations, is given in [R2].)

Recall that $K^{(j)}$ is defined inductively: For $M$ a topological Hausdorff space, let $M'$ denote the set of cluster points of $M$. Let $K^{(0)} = K$ and $K^{(j+1)} = (K^{(j)})'$ for all $j$. Now if $K$ fails the hypotheses of 1.5 there is an integer $n$ with $K^{(n+1)} = \emptyset$. Then every bounded function on $K$ is of index at most $n$, hence belongs to $SD(K)$.

It can also be shown that if $K$ satisfies the hypotheses of 1.5, there exists an $f \in B_{1/2}(K) \sim DBSC(K)$, and also an $f \in B_1(K) \sim B_{1/2}(K)$.

**Section 2.**

We begin with some preliminary results.

**Lemma 2.1.** Let $f$ be a bounded non-negative lower semi-continuous function on $K$. Then $f \in DBSC(K)$ and $\|f\|_D = \|f\|_\infty$. Hence if $f$ is bounded semi-continuous, $\|f\|_D \leq 3\|f\|_\infty$.

**Proof.** By a classical result of Baire (cf. [H]), there exists a sequence $(\varphi_j)$ of continuous functions on $K$ with $0 \leq \varphi_1 \leq \varphi_2 \leq \cdots$ and $\varphi_j \to f$ pointwise. Setting $u_1 = \varphi_1$, $u_j = \varphi_j - \varphi_{j-1}$ for $j > 1$, we have that $u_j \geq 0$ for all $j$ and $\sum u_j = f$ point-wise. Thus $\|f\|_D \geq \|f\|_\infty$; the reverse inequality is trivial.

To see the last statement, let e.g., $f$ be bounded upper semi-continuous, $\lambda = \|f\|_\infty$, and note that $\lambda - f$ is non-negative lower semi-continuous. Thus $\|\lambda - f\|_D = \|\lambda - f\|_\infty \leq 2\lambda$, so $\|f\|_D \leq \lambda + \|\lambda - f\|_D \leq 3\lambda$. \( \square \)

**Remark.** It thus follows that if $f$ is a $D$-function, then $\|f\|_D = \inf \{\|u + v\|_\infty : u, v \geq 0 \text{ are bounded lower semi-continuous with } f = u - v\}$. 

Of course it follows immediately from Lemma 2.1 that if $U$ is an open non-empty subset of $K$, then $\|\chi_U\|_D = 1$, for $\chi_U$ is lower semi-continuous. In this case, the sequence $(\varphi_j)$ mentioned above can be easily chosen, using Urysohn’s lemma. Indeed, if $U$ is closed, this is trivial. Otherwise, let $\varepsilon_0 > 0$ be such that $\dist(x_0, \partial U) > \varepsilon_0$ for some $x_0 \in U$; set $F_n = \{x \in U : \dist(x, \partial U) \geq \frac{\varepsilon_0}{n}\}$. Then $U = \bigcup_{j=1}^{\infty} F_j$ and for all $j$, $F_j$ is closed, $F_j \subset \text{Int} F_{j+1}$. Now choose $[0, 1]$-valued continuous functions $(\varphi_j)$ on $K$ so that for all $j$, $\varphi_j = 1$ on $F_j$ and $\{x : \varphi_j(x) \neq 0\} \subset \text{Int} F_{j+1}$. Then $\varphi_j \to \chi_U$ pointwise.

Evidently it follows that if $W$ is a closed subset of $K$, then $\|\chi_W\|_D \leq 2$. In fact, if $W$ is a difference of closed sets; i.e., $W = W_1 \sim W_2$, with $W_i$ closed for $i = 1, 2$, we again have that $\|\chi_W\|_D \leq 2$, for $\|\chi_W\|_D \leq \|\chi_{W_1}\|_D \|\chi_{W_2}\|_D \leq 2 \cdot 1 = 2$.

The following result shows that the simple $D$-functions are precisely those functions built up from the differences of closed sets.

**Proposition 2.2.** Let $f$ be a simple real-valued function on $K$. The following are equivalent:

1) $f \in B_{1/2}(K)$;
2) $f$ is of finite Baire index;
3) $f \in \text{DBSC}(K)$;
4) There exist disjoint differences of closed sets $W_1, \ldots, W_m$ and scalars $c_1, \ldots, c_m$ with

$$f = \sum_{i=1}^{m} c_i \chi_{W_i}.$$ 

**Proof.** Let us suppose $f$ is non constant, let $r_1, \ldots, r_k$ be the distinct values of $f$, and set $\varepsilon = \min\{|r_i - r_j| : i \neq j, 1 \leq i, j \leq k\}$. Now if $W$ is a non-empty subset of $K$, $w \in W$, and $\text{osc} f \mid W(w) < \varepsilon$, then $f \mid W$ is continuous at $w$; in fact there is an open neighborhood $U$ of $w$ with $f(x) = f(w)$ for all $x \in U \cap W$.

Now suppose 1) holds, and let $n = i(f, \varepsilon)$. By Proposition 1.1, $n < \infty$. We then obtain that defining $K_0 = K$ and $K_{j+1} = \{x \in K_j : f|K_j$ is discontinuous at $x\}$, for $1 \leq j \leq n + 1$, then $K_{n+1} = \emptyset$ and if $0 < \varepsilon' \leq \varepsilon$, $\text{osc}_j(f, \varepsilon') = K_j$ for all $1 \leq j \leq n$. Hence in fact $i(f) = i(f, \varepsilon) = n$, so 2) is proved. Of course 2) implies 1) by Proposition 1.1.

It remains only to show that 1) $\Rightarrow$ 4), for evidently 4) $\Rightarrow$ 3) $\Rightarrow$ 1). Now fixing $0 \leq i \leq n$, we have that $f$ is continuous on $K_{i-1} \cap K_i$. Let then $\ell = \ell(i)$ and
Let $r_j^1, \ldots, r_j^{\ell}$ be the distinct values of $f$ on $K_j \sim K_{j+1}$; let $W_i^j = \{x \in K_j \sim K_{j+1} : f(x) = r_i^j\}$. Then $W_i^j$ is a clopen subset of $K_j \sim K_{j+1}$; it follows easily that in fact $W_i^j$ is then again a difference of closed sets in $K$, for all $i$, $1 \leq i \leq \ell$, and thus

$$f = \sum_{j=0}^{n} \sum_{i=1}^{\ell(j)} r_i^j \chi_{W_i^j},$$

proving 4). \qed

**Remark.** The above proof yields that moreover if $W \subset K$, and $\chi_W$ is a $D$-function, then $W$ is a (disjoint) finite union of differences of closed sets; the converse is again immediate. This condition is incidentally equivalent to the condition that $W$ belongs to the algebra $D$ of sets generated by the closed subsets of $K$.

We give some more preliminary results, before passing to the proof of Theorem 1.2. For $f : K \to \mathbb{R}$, we set $\text{supp} f = \{k \in K : f(k) \neq 0\}$. If $W \subset K$, we say that $f$ is supported on $W$ if $\text{supp} f \subset W$.

**Lemma 2.3.** Let $U$ be a non-empty open subset of $K$, and $f$ a bounded function on $K$, supported and continuous on $U$. Then $f \in SD(K)$ and $\|f\|_D = \|f\|_\infty$.

**Proof.** Let us first show the norm identity. Note that since $f$ is bounded, if $u$ is a continuous function on $K$ with $u(x) = 0$ for all $x \notin U$, then $f \cdot u$ is continuous on $K$. Now choose $u_1, u_2, \ldots$ continuous non-negative functions on $K$ with $\chi_U = \sum u_j$ point-wise. But then $f = \sum f \cdot u_j$ point-wise, $f \cdot u_j$ is continuous on $K$ for all $j$, and $\sum |fu_j| \leq \|f\|_\infty \sum u_j \leq \|f\|_\infty$, so $\|f\|_D \leq \|\sum |fu_j|\|_\infty \leq \|f\|_\infty$; the reverse inequality is trivial.

To see that $f$ is a strong $D$-function, assume without loss of generality that $\|f\|_\infty = 1$. Now fix $n$ a positive integer, and for each $j$, $-n \leq j \leq n$, define $K_j^n$ by

$$(8) \quad K_j^n = \left\{x \in U : \frac{j}{n} \leq f(x) < \frac{j+1}{n} \right\}.$$  

Finally, define $\varphi_n$ by

$$\varphi_n = \sum_{j=-n}^{n} \frac{j}{n} \chi_{K_j^n}. \tag{9}$$

Then evidently by the continuity of $f$, $K_j^n$ is a difference of closed sets in $U$, and hence in $K$, for all $j$, so $\varphi_n$ is a simple $D$-function; moreover we have

$$0 \leq f - \varphi_n \leq \frac{1}{n}. \tag{10}$$
Thus to show that \( \|f - \varphi_n\|_D \to 0 \) as \( n \to \infty \), we need only show that \( f - \varphi_n \) is lower semi-continuous; for then \( \|f - \varphi_n\|_D \leq \frac{1}{n} \) by (10) and Lemma 2.1.

Let \( \psi = f - \varphi_n \), and suppose it were false that \( \psi \) is lower semi-continuous. We may then choose \( x \in K \) and \((x_m)\) a sequence in \( K \) with \( x_m \to x \) so that \( \psi(x_m) \) converges and

\[
\lim_{m \to \infty} \psi(x_m) < \psi(x). \tag{11}
\]

Evidently then \( x \in U \), since \( x \notin U \) implies \( \psi(x) = 0 \leq \psi(x_m) \) for all \( m \). By passing to a subsequence, we may then assume without loss of generality that there is a \( j \), \(-n \leq j \leq n \), with \( x_m \in K_j^n \) for all \( m \). But since \( f \) is continuous on \( U \), \( \lim_{m \to \infty} f(x_m) = f(x) \); if also \( x \in K_j^n \), then since \( \psi(x_m) = f(x_m) - \frac{j}{n} \) for all \( m \), we have that \( \lim_{m \to \infty} \psi(x_n) = f(x) - \frac{j}{n} = \psi(x) \), a contradiction. If \( x \notin K_j^n \), by continuity of \( f \) we must have that \( f(x) = \frac{j+1}{n} \). But then \( x \in K_{j+1}^n \), so \( \psi(x) = 0 < \frac{j+1}{n} - \frac{j}{n} = \lim_{m \to \infty} \psi(x_m) \) again contradicting (11). \( \Box \)

Our next preliminary result deals with extension issues. (For \( W \subset K \) and \( f : W \to \mathbb{R} \), \( f \cdot \chi_W \) denotes the function which is zero off \( W \) and agrees with \( f \) on \( W \).)

**Lemma 2.4.** Let \( W \subset K \) be a difference of closed sets and \( f \) in DBSC(\( W \)). Then \( f \cdot \chi_W \) is in DBSC(\( K \)) and

\[
\|f \cdot \chi_W\|_{D(K)} \leq 2\|f\|_{D(W)} ; \tag{12}
\]

if \( W \) is an open set, then

\[
\|f \cdot \chi_W\|_{D(K)} = \|f\|_{D(W)} . \tag{13}
\]

Moreover if \( f \in SD(W) \), then \( f\chi_W \in SD(K) \).

**Proof.** Suppose first that \( W \) is open, and let \((\varphi_j)\) in \( C(K) \) be such that the \( \varphi_j \)'s are non-negative and \( \sum \varphi_j = \chi_W \) point-wise. Let \( \varepsilon > 0 \) and choose \((\psi_j)\) in \( C(W) \) with \( \sum |\psi_j| < \|f\|_{D(W)} + \varepsilon \) and \( f = \sum \psi_j \) point-wise on \( W \). Now identifying \( \psi_j \) with \( \psi_j \cdot \chi_W \), \( \psi_j \cdot \varphi_i \) is continuous on \( K \) for all \( i \) and \( j \), and we have that \( \sum_{i,j} \psi_j \varphi_i \leq \|f\|_{D(W)} + \varepsilon \), with \( \sum_{i,j} \psi_j \varphi_i = f\chi_W \). Thus \( \|f\chi_W\|_{D(K)} \leq \|f\|_{D(W)} + \varepsilon \) for all \( \varepsilon > 0 \); so \( \|f\chi_W\|_{D(K)} \leq \|f\|_{D(W)} \). The reverse inequality is trivial, so (13) is established.
Next, suppose that $W$ is closed, and again let $\varepsilon > 0$. As noted following Lemma 2.1, we may choose $u, v$ non-negative lower semi-continuous on $W$ with

$$f = u - v \quad \text{and} \quad \|u + v\|_\infty < \|f\|_{D(W)} + \varepsilon.$$ 

Now let $\lambda = \|u + v\|_\infty$ and let $\tilde{u} = \lambda \chi_{\sim W} + u \chi_W$, $\tilde{v} = \lambda \chi_{\sim W} + v \chi_W$. It follows easily that $\tilde{u}$ and $\tilde{v}$ are both non-negative lower semi-continuous on $K$ and of course

$$f \chi_W = \tilde{u} - \tilde{v}, \quad \|\tilde{u} + \tilde{v}\|_\infty = 2\lambda.$$ 

Thus by the observation following Lemma 2.1, $\|f \chi_W\|_D \leq 2\lambda < 2\|f\|_{D(W)} + 2\varepsilon$.

Since $\varepsilon > 0$ is arbitrary, (12) is proved for closed $W$.

Now suppose $W$ is a difference of closed sets. Choose $U$ open, $L$ closed with $W = U \cap L$. Then $W$ is a relatively closed subset of $U$, so we have that $f \cdot \chi_L \mid U$ belongs to $DBSC(U)$ with $\|f \cdot \chi_L \mid U\|_{D(U)} \leq 2\|f\|_{D(W)}$. But then by (13), $f \cdot \chi_W = (f \cdot \chi_L) \mid U \cdot \chi_U$ belongs to $DBSC(K)$ and $\|f \cdot \chi_W\| \leq \|f \cdot \chi_L \mid U\|_{D(W)} \leq 2\|f\|_{D(W)}$, proving (12).

Finally, suppose $f \in SD(W)$. Then given $\varepsilon > 0$, choose $g$ a simple $D$-function on $W$ with

$$\|g - f\|_{D(W)} < \varepsilon.$$ 

By Proposition 2.2, there are disjoint differences of closed sets in $W, W_1, \ldots, W_k$, and scalars $c_1, \ldots, c_k$ with $g = \sum_{i=1}^k c_i \chi_{W_i}$ on $W$. But then for all $i$, $W_i$ is actually a difference of closed sets in $K$, and thus $g \cdot \chi_{W_i}$ is a simple $D$-function on $K$. Then by (12),

$$\|(g - f) \chi_W\| = \|g \chi_W - f \chi_W\| < 2\varepsilon.$$ 

Thus the final assertion of the Lemma is established. □

Remark. Using the comment following Proposition 2.2, we obtain that if $W \subset K$ is in $\mathcal{D}$ (i.e., $\chi_W$ is a $D$-function), then for $f : W \to \mathbb{R}$ a bounded function, $f$ is a $D$-function on $W$ if and only if $f \chi_W$ is a $D$-function on $K$; moreover $f \in SD(W)$ if and only if $f \chi_W \in SD(K)$.

Before giving the proof of Theorem 1.2, we recall the following standard result.
**Lemma 2.5.** Let \( \varepsilon > 0 \), and suppose \( f : K \to \mathbb{R} \) is such that \( \text{osc} \, f \leq \varepsilon \) on \( K \). There exists \( \varphi : K \to \mathbb{R} \) continuous with \( |f - \varphi| \leq \varepsilon \) on \( K \).

**Proof.** Let \( Lf \) be the lower semi-continuous envelope of \( f \); \( Lf(x) = \lim_{y \to x} f(y) \) for all \( x \in X \). Then we have that

\[
\text{osc} \, f = Uf - Lf.
\]

Since \( \text{osc} \, f \leq 2 \text{osc} \, f \), \( \text{osc} \, f \leq 2\varepsilon \) on \( K \). Thus we have by assumption that

\[
Uf - \varepsilon \leq Lf + \varepsilon.
\]

By the Hahn interposition theorem (cf. [H], p.276), there exists \( \varphi \) continuous with

\[
Uf - \varepsilon \leq \varphi \leq Lf + \varepsilon.
\]

Since \( f \leq Uf \) and \( Lf \leq f \), \( \varphi \) satisfies the conclusion of the Lemma. \( \Box \)

We now treat the proof of Theorem 1.2. It is convenient to consider a larger class; for \( n \geq 0 \), let \( G_n \) denote the family of all bounded functions \( f : K \to \mathbb{R} \) so that there exists an open set \( U \) with \( f \) supported on \( U \) and \( i(f \mid U) \leq n \). The following quantitative result yields Theorem 1.2 immediately.

**Theorem 2.6.** Let \( n \geq 0 \) and \( f \in G_n \). Then \( f \in SD(K) \) and

\[
\|f\|_D \leq (2^{n+1} - 1)\|f\|_\infty.
\]

**Remark.** Of course it follows a-posteriori that if we prove the result just for functions \( f \) of index \( n \), then it holds immediately for functions in \( G_n \), by Lemma 2.4. The class \( G_n \) is needed for our proof, however. We also note that the argument given in [R2], using transfinite oscillations, gives the optimal estimate: if \( i(f) \leq n \), then \( \|f\|_D \leq (2n + 1)\|f\|_\infty \).

We prove 2.6 by induction on \( n \). The case \( n = 0 \) follows immediately from Lemma 2.3. Now let \( n > 0 \) and suppose 2.6 proved for \( "n" = n - 1 \).

**Lemma 2.7.** Let \( f \in G_n \) and \( \varepsilon > 0 \). There exist functions \( g \) and \( h \) with \( f = g + h \), \( g \in G_n \), \( h \in SD(K) \), and

\[
\|h\|_D \leq (2^{n+1} - 1)\|f\|_\infty, \quad \|g\|_\infty \leq \varepsilon.
\]
Proof. Let \( \lambda_j = 2^{i+1} - 1 \) for \( j = 0, 1, 2, \ldots \). Let \( U \) be chosen with \( f \) supported in \( U \) and \( i(f \mid U) \leq n \). Let \( W = \{ x \in U : \text{osc} f(x) \geq \epsilon \} \). It follows that \( W \) is a relatively closed subset of \( U \) and

\[
(22) \quad \lambda(f \mid W) \leq n - 1.
\]

Thus by induction hypothesis and Lemma 2.4,

\[
(23) \quad f \cdot \chi_W \in SD(K) \quad \text{and} \quad \| f \cdot \chi_W \|_D \leq 2\lambda_{n-1} \| f \|_\infty.
\]

Now by Lemma 2.5, we may choose \( \varphi : U \sim W \rightarrow \mathbb{R} \), \( \varphi \) continuous on \( U \sim W \), with

\[
(24) \quad \| \varphi \|_\infty \leq \| f \|_\infty \quad \text{and} \quad | \varphi(x) - f(x) | \leq \epsilon \quad \text{for all} \quad x \in U \sim W,
\]

Indeed, 2.5 gives \( \tilde{\varphi} \) with \( \tilde{\varphi} \) continuous and \( | \tilde{\varphi} - f | \leq \epsilon \) on \( U \sim W \). But simply define \( \varphi(x) = \tilde{\varphi}(x) \) if \( | \tilde{\varphi}(x) | \leq \| f \|_\infty \), and \( \varphi(x) = \| f \|_\infty \text{sgn} f(x) \) otherwise.

Let \( g \) and \( h \) be defined by

\[
(25) \quad g = (f - \varphi)\chi_{U \sim W}, \quad h = f \cdot \chi_W + \varphi \cdot \chi_{U \sim W}.
\]

Now evidently \( \text{supp} g \subset U \sim W \); since \( \varphi \) is continuous on \( U \sim W \), it follows that \( i((f - \varphi) \mid U \sim W) \leq i(f \mid U) \leq n \); hence \( g \in \mathcal{G}_n \), and by (24), \( \| g \|_\infty \leq \epsilon \).

Evidently, \( f = g + h \); finally, by (23) and Lemma 2.3, \( h \in SD(K) \) and

\[
\| h \|_D \leq (2\lambda_{n-1} + 1) \| f \|_\infty = \lambda_n \| f \|_\infty.
\]

□

Proof of Theorem 2.6 for \( n \). Fix \( \epsilon > 0 \). We may choose by induction sequences \( (h_j) \) and \( (g_j) \) so that for all \( j \),

\[
(26i) \quad f = h_1 + \cdots + h_j + g_j
\]

\[
(26ii) \quad h_j \in SD(K), \quad g_j \in \mathcal{G}_n
\]

\[
(26iii) \quad \| h_1 \|_D \leq \lambda_n \| f \|_\infty, \quad \| h_j \|_D \leq \frac{\epsilon}{2^{j-1}} \quad \text{for} \quad j > 1
\]

\[
(26iv) \quad \| g_j \|_\infty \leq \frac{\epsilon}{\lambda_n 2^j}.
\]

Indeed, by Lemma 2.7, we may choose \( h_1 \in SD(K) \) and \( g_1 \in \mathcal{G}_n \) with \( f = h_1 + g_1 \),

\[
\| h_1 \|_D \leq \lambda_n \| f \|_\infty, \quad \| g_1 \|_\infty \leq \frac{\epsilon}{\lambda_n 2^1}.
\]
Now suppose \( j \geq 1 \) and \( h_1, \ldots, h_j, g_j \) chosen satisfying (26i)–(26iv). Since \( g_j \in G_n \), by Lemma 2.7 we may choose \( h_{j+1} \in SD(K) \) and \( g_{j+1} \in G_n \) with \( g_j = h_{j+1} + g_{j+1} \).

\[
\| h_{j+1} \|_D \leq \lambda_n \| g_j \|_\infty \quad \text{and} \quad \| g_{j+1} \|_\infty \leq \frac{\varepsilon}{\lambda_n 2^{j+1}}.
\]

Then (26i)–(26iv) hold at \( j + 1 \).

Since the \( D \)-norm is trivially larger than the sup-norm and \( \| g_j \|_\infty \to 0 \), it follows from (26i) and (26iii) that \( \sum h_i \) converges uniformly to \( f \). Since \( DBSC(K) \) is a Banach space, \( \sum \| h_j \|_D < \infty \), and \( h_j \in SD(K) \) for all \( j \), it follows that \( f \in SD(K) \).

Finally, we have by (26iii) that

\[
\| f \|_D \leq \lambda_n \| f \|_\infty + \sum_{j=2}^{\infty} \frac{\varepsilon}{2^{j-1}} = \lambda_n \| f \|_\infty + \varepsilon.
\]

Since \( \varepsilon > 0 \) is arbitrary, Theorem 2.6 is proved. \( \square \)

We turn now to Theorem 1.3. This follows immediately from the following result.

**Theorem 2.8.** Let \( f, g \in B_{1/2}(K) \), and \( \varepsilon > 0 \). Then the following hold.

(a) \( i( f + g, \varepsilon ) \leq i( f, \frac{\varepsilon}{2} ) + i( g, \frac{\varepsilon}{2} ) \).

(b) \( i( f \cdot g, \varepsilon ) \leq i( f, \frac{\varepsilon}{2F} ) + i( g, \frac{\varepsilon}{2G} ) \) where \( F = \| f \|_\infty, G = \| g \|_\infty \), and it is assumed that \( F, G > 0 \).

(c) \( i( h, \varepsilon ) \leq i( f, \varepsilon ) + i( g, \varepsilon ) \) where \( h = f \lor g \) or \( h = f \land g \).

We give the detailed proof of (a) (which is also needed later), and then indicate how (b), (c) follow by the same method.

We first note the following fact.

**Lemma 2.9.** Let \( W_1, \ldots, W_n \) be closed non-empty sets with \( K = \bigcup_{i=1}^{n} W_i \) and \( f : K \to \mathbb{R} \) a bounded function. Then

\[
\text{osc} f = \max_{1 \leq i \leq n} \left( \text{osc} f \mid W_i \right) \chi_{W_i}.
\]

**Proof.** We first note that

\[
\text{osc} f = \max_{1 \leq i \leq n} \left( \text{osc} f \mid W_i \right) \chi_{W_i}.
\]

For let \( x \in K \) and choose \( (x_m) \) in \( K \) with \( x_m \to x \) and \( \text{osc} f(x) = \lim_{n \to \infty} |f(x_n) - f(x)| \). We may choose \( i \) and \( m_j < m_{j+1} \) with \( x_m \in W_i \) for all \( j \). But then
If \( x \in W_i \) and so \( \text{osc} f(x) \leq \text{osc} f \ | \ W_i(x) \leq \max_{i} \text{osc} f \ | \ W_i(x) \chi_{W_i}(x) \). The reverse inequality is trivial, so (30) follows.

Now again let \( x \in K \) and choose \( (x_m) \) in \( K \) with \( x_m \to x \) and \( \text{osc} f(x) = \lim_{n \to \infty} \text{osc} f(x_m) \). By (30), we may again choose \( m_1 < m_2 < \cdots \) and \( i \) with \( \text{osc} f(x_{m_j}) = \text{osc} f \ | \ W_i \chi_{W_i}(x_{m_j}) \) for all \( j \). Now if \( f(x) = 0 \), (29) is trivial. Otherwise, without loss of generality, \( \text{osc} f(x_{m_j}) > 0 \) for all \( j \); hence then \( x_{m_j} \in W_i \) and so \( x \in W_i \), whence \( \text{osc} f(x) \leq \text{osc} f \ | \ W_i(x) \leq \max_{i} \text{osc} f \ | \ W_i \chi_{W_i}(x) \). Again the reverse inequality is trivial, so (29) holds. \( \square \)

Now let \( f, g \) be as in Theorem 2.8, and \( \varepsilon > 0 \) be given. For each \( n = 1, 2, \ldots \) and \( \theta = (\theta_1, \ldots, \theta_n) \) with \( \theta_i = 0 \) or 1 for all \( 1 \leq i \leq n \), we define closed subsets \( L(\theta) \) of \( K \) as follows:

\[
(31) \quad L(0) = \left\{ x \in K : \text{osc} f(x) \geq \frac{\varepsilon}{2} \right\} ; \quad L(1) = \left\{ x \in K : \text{osc} g(x) \geq \frac{\varepsilon}{2} \right\}.
\]

If \( n \geq 1 \) and \( L(\theta) = L(\theta_1, \ldots, \theta_n) \) is defined, let

\[
(32) \quad \begin{cases} 
L(\theta_1, \ldots, \theta_{n+1}) = \left\{ x \in L(\theta) : \text{osc} f \ | \ L(\theta) \geq \frac{\varepsilon}{2} \right\} & \text{if } \theta_{n+1} = 0 \\
L(\theta_1, \ldots, \theta_{n+1}) = \left\{ x \in L(\theta) : \text{osc} g \ | \ L(\theta) \geq \frac{\varepsilon}{2} \right\} & \text{if } \theta_{n+1} = 1.
\end{cases}
\]

These sets are closed, since \( \text{osc} f, \text{osc} g \) are upper semi-continuous functions. We then have for all \( n \) that

\[
(33) \quad \text{os}_n(f + g, \varepsilon) \subset \bigcup_{\theta \in \{0,1\}^n} L(\theta).
\]

We prove this by induction on \( n \). Now for \( n = 1 \), since it is easily seen that \( \text{osc}(f + g) \leq \text{osc} f + \text{osc} g \), we then have that \( \text{osc}(f + g)(x) \geq \varepsilon \) implies \( \text{osc} f(x) \geq \frac{\varepsilon}{2} \) or \( \text{osc} g(x) \geq \frac{\varepsilon}{2} \); this gives \( \text{os}_1(f + g, \varepsilon) \subset L(0) \cup L(1) \). Suppose (33) is proved for \( n \), and suppose \( K_n = \text{os}_n(f + g, \varepsilon) \) and \( x \in \text{os}_{n+1}(f + g, \varepsilon) \). Thus \( \text{osc}(f + g) \ | \ K_n(x) \geq \varepsilon \). By the preceding lemma and (33), we may then choose \( \theta \in \{0,1\}^n \) with \( x \in K_n \cap L(\theta) \) and

\[
\text{osc}(f + g) \ | \ K_n(x) = \text{osc}(f + g) \ | \ K_n \cap L(\theta)(x) \\
\leq \text{osc}(f + g) \ | \ L(\theta)(x) \\
\leq \text{osc} f \ | \ L(\theta)(x) + \text{osc} g \ | \ L(\theta)(x).
\]

It follows immediately that \( x \in L(\theta_1, \ldots, \theta_n, 0) \cup L(\theta_1, \ldots, \theta_n, 1) \); thus (32) holds at \( n + 1 \).
Next, fix \( n \) and \( \theta \in \{0, 1\}^n \). Let
\[
j = j(\theta) = \# \{1 \leq i \leq n : \theta_i = 0\}, \quad k = k(\theta) = \# \{1 \leq i \leq n : \theta_i = 1\}.
\]

Then we claim
\[
L(\theta) \subset \text{os}_j\left(f, \frac{\varepsilon}{2}\right) \cap \text{os}_k\left(g, \frac{\varepsilon}{2}\right).
\]

Again we prove this by induction on \( n \). The case \( n = 1 \) is trivial, by the definitions of \( L(0) \) and \( L(1) \). Now suppose (35) is proved for \( n \), and \((\theta_1, \ldots, \theta_{n+1})\) is given; let \( j = j(\theta_1, \ldots, \theta_n) \) and \( k = k(\theta_1, \ldots, \theta_n) \). Now if \( \theta_{n+1} = 0 \), then \( j(\theta_1, \ldots, \theta_{n+1}) = j + 1 \) and \( k(\theta_1, \ldots, \theta_{n+1}) = k \); then by (35), \( L(\theta_1, \ldots, \theta_{n+1}) \subset L(\theta_1, \ldots, \theta_n) \subset \text{os}_k(g, \frac{\varepsilon}{2}) \) and by definition and (35),
\[
L(\theta_1, \ldots, \theta_n) \subset \left\{x \in \text{os}_j\left(f, \frac{\varepsilon}{2}\right) : \text{osc} f | \text{os}_j\left(f, \frac{\varepsilon}{2}\right)(x) \geq \frac{\varepsilon}{2}\right\}
= \text{os}_{j+1}\left(f, \frac{\varepsilon}{2}\right).
\]

Of course if \( \theta_{n+1} = 1 \), we obtain by the same reasoning that \( L(\theta_1, \ldots, \theta_{n+1}) \subset \text{os}_j(f, \frac{\varepsilon}{2}) \cap \text{os}_{k+1}(g, \frac{\varepsilon}{2}) \) and \( j = j(\theta_1, \ldots, \theta_{n+1}), \ k + 1 = k(\theta_1, \ldots, \theta_{n+1}) \); thus (35) is proved for \( n + 1 \), and so established for all \( n \) by induction.

Now suppose, for a given \( n \), that \( \text{os}_n(f + g, \varepsilon) \neq \emptyset \). Then by (33), there is a \( \theta \in \{0, 1\}^n \) with \( L(\theta) \neq \emptyset \). Thus letting \( j \) and \( k \) be as in (34), we have by (35) that \( \text{os}_j(f, \frac{\varepsilon}{2}) \neq \emptyset \) and \( \text{os}_k(g, \frac{\varepsilon}{2}) \neq \emptyset \). But then \( n = j + k \leq i(f, \frac{\varepsilon}{2}) + i(g, \frac{\varepsilon}{2}) \).

Theorem 2.8(a) is thus established.

To see 2.8(b), note for any \( y \) and \( x \in K \) that
\[
|f(y)g(y) - f(x)g(x)| \leq G|f(y) - f(x)| + F|g(y) - g(x)|.
\]

Hence we have that fixing \( x \in K \), then \( \text{osc} f g(x) \leq G \text{osc} f(x) + F \text{osc} g(x) \), whence
\[
\text{osc} f g(x) \leq G \text{osc} f(x) + F \text{osc} g(x).
\]

Thus \( \text{osc} f g(x) \geq \varepsilon \) implies \( \text{osc} f(x) \geq \frac{\varepsilon}{2G} \) or \( \text{osc} g(x) \geq \frac{\varepsilon}{2F} \). We now prove (b) by defining the sets \( L(\theta) \) by \( L(0) = \text{os}_1(f, \frac{\varepsilon}{2G}), L(1) = \text{os}_1(g, \frac{\varepsilon}{2F}) \), and for \( \theta = (\theta_1, \ldots, \theta_{n+1}) \), \( L(\theta_1, \ldots, \theta_{n+1}) = \{x \in L(\theta) : \text{osc} f | L(\theta) \geq \frac{\varepsilon}{2G} \} \) if \( \theta_{n+1} = 0 \), and \( L(\theta_1, \ldots, \theta_{n+1}) = \{x \in L(\theta) : \text{osc} g | L(\theta) \geq \frac{\varepsilon}{2F} \} \) if \( \theta_{n+1} = 1 \). Then we proceed...
exactly as in case (a). Finally, for case (c), we note that if $h$ is as in (c) and $x \in K$, then

\begin{equation}
\text{osc } h(x) \geq \varepsilon \text{ implies osc } f(x) \geq \varepsilon \text{ or osc } g \geq \varepsilon .
\end{equation}

Suppose this were false. Then we can choose $0 < \varepsilon' < \varepsilon$ and $U$ an open neighborhood of $x$ with

\begin{equation}
\text{osc } f(u) < \varepsilon' \text{ and osc } g(u) < \varepsilon' \text{ for all } u \in U.
\end{equation}

Now fix $u \in U$; we can then choose $V$ an open neighborhood of $u$ with $V \subset U$ and

\begin{equation}
|f(v) - f(u)| < \varepsilon' \text{ and } |g(v) - g(u)| < \varepsilon' \text{ for all } v \in V.
\end{equation}

Suppose e.g., $h = f \lor g$ and $v \in V$ with $(f \lor g)(v) = f(v)$, $(f \lor g)(u) = g(u)$. But then by (40) and the above,

\begin{equation}
f(v) \geq g(v) > g(u) - \varepsilon' \text{ so } f(v) - g(u) > -\varepsilon'
\end{equation}

and

\begin{equation}
f(v) < f(u) + \varepsilon' \leq g(u) + \varepsilon' \text{ so } f(v) - g(u) < \varepsilon'.
\end{equation}

It thus follows from (40)–(42) that

\begin{equation}
|h(v) - h(u)| < \varepsilon'.
\end{equation}

If e.g., $f \lor g(v) = f(v)$ and $f \lor g(u) = f(u)$, (43) follows immediately from (40), so (43) holds for all $v \in V$. Thus we obtain \(\text{osc } h(u) \leq \varepsilon'\); but since $u \in U$ is arbitrary, we also have \(\text{osc } h(x) \leq \varepsilon',\) a contradiction. The proof for $h = f \land g$ is the same.

Evidently (38) yields that $\text{os}_1(h, \varepsilon) \subset \text{os}_1(f, \varepsilon) \cup \text{os}_1(g, \varepsilon)$; we then proceed as in case (a), except that the sets $L(\theta_1, \ldots, \theta_n)$ are defined by replacing “$\varepsilon$” by \(\frac{\varepsilon}{2}\)” in (31), (32). □

We next treat Theorem 1.4. We first recall the following fact.

**Lemma 2.9.** Let $f \in D(K)$. Then $\varepsilon i(f, \varepsilon) \leq 4\|f\|_D$.

This follows immediately from the definitions, the fact that $\text{os}_j(f, \varepsilon) \subset K_j(f, \varepsilon)$ for all $j$, and Lemma 2.4 of [HOR]. (A direct proof of 2.9 is given in [R2] yielding the refinement that $\varepsilon i(f, \varepsilon) \leq \|f\|_D$.)
Proof of Theorem 1.4. Suppose first that \( f \in SD(K), \eta > 0 \), and choose \( g \) a simple \( D \)-function with \( \| f - g \|_D \leq \eta \). It then follows by Lemma 2.9 that

\[
\varepsilon_i(f - g, \varepsilon) \leq 4 \eta \quad \text{for all } \varepsilon > 0 \tag{44}
\]

Now since \( g \) is a simple \( D \)-function, \( g \) has finite index (by Proposition 2.2); say \( N = i(g) \). Then by Theorem 2.8(a) and (44), for any \( \varepsilon > 0 \),

\[
\varepsilon_i(f, \varepsilon) \leq \varepsilon_i(f - g, \varepsilon) + \varepsilon_i(g, \varepsilon) \leq 8 \eta + \varepsilon N .
\]

Hence \( \lim_{\varepsilon \to 0} \varepsilon_i(f, \varepsilon) \leq 8 \eta \). Since \( \eta > 0 \) is arbitrary, (7) is proved.

Finally, to prove (b) of Theorem 1.4, suppose without loss of generality that \( f \) is upper semi-continuous and satisfies (7), let \( \eta > 0 \), and choose \( 0 < \varepsilon < \eta \) with

\[
\varepsilon_i(f, \varepsilon) < \eta \tag{45}
\]

Let then \( n = i(f, \varepsilon) \) and set \( K^j = \text{os}_j(f, \varepsilon) \) for all \( j \). Thus \( K^n \neq \emptyset \), \( K^{n+1} = \emptyset \), and for \( 0 \leq j \leq n \), \( \text{osc}(f \mid K^j \sim K^{j+1}) < \varepsilon \). Thus for all \( j \), we may choose by Lemma 2.5 a continuous function \( \varphi_j \) on \( K^j \sim K^j+1 \) with

\[
|\varphi_j - f| \leq \varepsilon \quad \text{on } K^j \sim K^j+1 . \tag{46}
\]

Now set \( g = \sum_{j=0}^n \varphi_j \chi_{K^j \sim K^{j+1}} \). By Lemmas 2.3 and 2.4, \( g \in SD(K) \). Now fixing \( j \) and letting \( W = K^j \sim K^{j+1} \), then \( (f - g) \mid W \) is upper semi-continuous, hence by Lemma 2.1 and (46),

\[
\| f - g \|_{D(W)} \leq 3\| f - g \|_\infty \leq 3 \varepsilon . \tag{47}
\]

Then by Lemma 2.4,

\[
\|(f - g)\chi_W\|_{D(K)} \leq 6 \varepsilon . \tag{48}
\]

Hence

\[
\| f - g \|_D = \sum_{j=0}^n \|(f - g)\chi_{K^j \sim K^{j+1}}\|_D \\
\leq \sum_{j=0}^n \|(f - g)\chi_{K^j \sim K^{j+1}}\|_D \\
\leq 6n\varepsilon + 6 \varepsilon \\
\leq 7 \varepsilon , \text{ by (45)}.
\]
Since \( \eta > 0 \) is arbitrary and \( SD(K) \) is closed in \( DBSC(K) \), we obtain that \( f \in SD(K) \), thus completing the proof of Theorem 1.4. \( \square \)

**Remark.** Define \( B_{1/2}^0(K) \) to be the family of all bounded functions \( f : K \to \mathbb{R} \) which satisfy (7). Evidently we have (by the preceding result) that \( SD(K) \subset B_{1/2}^0(K) \subset B_{1/2}(K) \). We have moreover that \( B_{1/2}^0(K) \) is an algebra and a lattice, by Theorem 2.8. As noted in the introduction, it can be shown that there are non-\( D \)-functions in \( B_{1/2}^0(K) \), and also \( (DBSC(K) \sim SD(K)) \cap B_{1/2}^0(K) \neq \emptyset \) (for suitable \( K \)). It can be seen that \( B_{1/2}^0(K) \) is a complete linear topological space under the quasi-norm \( \|f\| = \sup_{\varepsilon > 0} \varepsilon i(f, \varepsilon) + \|f\|_{\infty} \).

We finally consider Proposition 1.5. The construction uses some preliminary results.

**Lemma 2.10.** Let \( n \geq 1 \) and \( K = K_0 \supset K_1 \supset \cdots \supset K_n \) be closed non-empty sets with \( K_i \) nowhere dense relative to \( K_{i-1} \) for all \( 1 \leq i \leq n \). Also let \( K_{n+1} = \emptyset \). Let \( E = \bigcup_{0 \leq i \leq [n/2]} K_{2i} \sim K_{2i+1} \). Then

\[
i(\chi_E) = i(\chi_E, \varepsilon) = n \quad \text{for all} \quad 0 < \varepsilon \leq 1.
\]

Moreover \( \|\chi_E\|_D \leq n + 1 \).

**Proof.** Fix \( 0 < \varepsilon \leq 1 \). We prove by induction on \( j \) that

\[
\text{osc}_j(\chi_E, \varepsilon) = K_j \quad \text{for all} \quad 0 \leq j \leq n.
\]

Then since \( \chi_E \) is constant on \( K_n \), \( \text{osc}_{n+1}(\chi_E, \varepsilon) = \emptyset \), yielding (49).

Now \( \chi_E \) is constant on \( K_0 \sim K_1 \), an open set; since \( K_1 \) is nowhere dense in \( K \), given \( x \in K_1 \), there exists a sequence \( (x_m) \) in \( K_0 \sim K_1 \) with \( x_m \to x \). But then

\[
\text{osc}_E(x) = \lim_{m \to \infty} (\chi_E(x_m) - \chi_E(x)) = 1,
\]

hence (50) is proved for \( j = 0 \).

Suppose now (50) is proved for \( 0 \leq j < n \). Again if \( x \in K_{j+1} \), since \( K_{j+1} \) is nowhere dense in \( K_j \), choose a sequence \( (x_m) \) in \( K_j \) with \( x_m \to x \). Now by definition of \( E \), \( |\chi_E(x_m) - \chi_E(x)| = 1 \) for all \( m \). Thus \( \text{osc}_E | K_j(x) \geq 1 \), which proves that \( K_{j+1} \subset \text{osc}_{j+1}(\chi_E, \varepsilon) \). But \( \chi_E \) is constant on \( K_j \sim K_{j+1} \), whence \( K_{j+1} \supset \text{osc}_{j+1}(\chi_E, \varepsilon) \). Thus (50) holds.

To see the final inequality in 2.10, we have that \( \|\chi_E\|_\infty = 1 \) and...
∥χ_{K_{2i} \sim K_{2i+1}}∥_D ≤ 2 for all 1 ≤ i ≤ \lfloor n/2 \rfloor (by Lemma 2.4); hence
\[
\|\chi_E\|_D ≤ \sum_{i=0}^{\lfloor n/2 \rfloor} \|\chi_{K_{2i} \sim K_{2i+1}}\|_D ≤ 1 + 2\lfloor n/2 \rfloor ≤ n + 1 \quad \square
\]

Remark. Actually the final inequality in 2.10 follows from (49). In fact it is proved in [R2] that if \( E \subset K \) is such that \( i(\chi_E) = n \), then \( \|\chi_E\|_D = n \) or \( n + 1 \) (and both possibilities can occur).

Lemma 2.11. (a) Let \( n \geq 1 \) and suppose \( K^{(n)} \neq \emptyset \). There exist non-empty closed sets \( K_1, \ldots, K_n \) satisfying the hypotheses of Lemma 2.10.

(b) Suppose \( K^{(n)} \neq \emptyset \) for all \( n = 1, 2, \ldots \). There exist disjoint open subsets \( U_1, U_2, \ldots \) of \( K \) with \( U_n^{(n)} \neq \emptyset \) for all \( n \).

Proof.
(a) If \( K \) is perfect, it can be seen that there exists a closed perfect nowhere dense subset \( L \) of \( K \); we then easily obtain the desired sets \( (K_j) \) with \( K_j \) a perfect nowhere dense result of \( K_{j-1} \). Evidently the same reasoning holds if \( K \) has a perfect non-empty subset. Otherwise, simply let \( K_j = K^{(j)}, 1 ≤ j ≤ n \). Alternatively, we may just observe that the hypotheses imply \( K \) has a closed subset homeomorphic to \( \omega^n + 1 \).

(b) First note that if \( x \in K^{(n)} \), then
\[
(51) \quad x \in U^{(n)} \quad \text{for all open neighborhoods } U \quad \text{of} \quad x .
\]

Next, note that the hypotheses imply that \( K^{(n)} \) is infinite for all \( n \). We may thus choose distinct points \( x_1, x_2, \ldots \), with \( x_n \in K^{(n)} \) for all \( n \). Now it follows that if \( U \) is an open set containing infinitely many of the \( x_j \)'s, there exists an \( n \) and an open neighborhood \( V \) of \( x_n \) with \( \bar{V} \subset U \) so that \( U \sim \bar{V} \) contains infinitely many of the \( x_j \)'s. We may then choose \( k_1 < k_2 < \cdots \) and \( U_1, U_2, \ldots \) open sets with \( \bar{U}_i \cap \bar{U}_j = \emptyset \) for all \( i \neq j \) and \( x_{k_n} \in U_n \) for all \( n \). (51) then yields that (b) holds. \( \square \)

We finally observe the following simple “localization” property for \( D \)-functions.

Lemma 2.12. Let \( U_1, U_2, \ldots \) be disjoint non-empty open subsets of \( K \), \( U = \bigcup_{j=1}^{\infty} U_j, \lambda < \infty \), and \( f : K \to \mathbb{R} \) a function supported on \( U \) with \( \|f | U_j\|_D ≤ \lambda \) for all \( j \). Then \( f \in DBSC(K) \) and \( \|f\|_D \leq \lambda \).
Proof. Let $\varepsilon > 0$. For each $j$, choose a sequence of continuous functions on $K$, 
$(\varphi^j_i)_{i=1}^{\infty}$, with $0 \leq \varphi^j_i \leq 1$ for all $i$ and $\chi_{U_j} = \sum_{i=1}^{\infty} \varphi^j_i$ pointwise. Also, choose 
$(h^j_i)_{i=1}^{\infty}$ continuous functions on $U_j$, with $\sum |h^j_i| \leq \lambda + \varepsilon$ and $f \mid U_j = \sum h^j_i$ pointwise. Now let 

$$f_{jk\ell} = \varphi^j_k h^j_\ell \chi_{U_j} \text{ for all } j, k, \ell.$$ 

Then $f_{jk\ell}$ is continuous on $K$ since $h^j_\ell$ is bounded continuous on $K$ and supported on $U_j$, and 

$$\sum_{j,k,\ell} |\varphi^j_k h^j_\ell \chi_{U_j}| = \sum_j \sum_{\ell} |h^j_\ell| \chi_{U_j} \leq \lambda + \varepsilon,$$

$$\sum_j \sum_{\ell} \sum_k \varphi^j_k h^j_\ell \chi_{U_j} = \sum_j \sum_{\ell} h^j_\ell \chi_{U_j} = \sum_j f \chi_{U_j} = f.$$

Thus $\|f\|_D \leq \lambda + \varepsilon$; since $\varepsilon > 0$ is arbitrary, the result follows. □

We are now prepared for the

Proof of Proposition 1.5.

By Lemmas 2.10 and 2.11, we may choose disjoint non-empty open subsets $U_1, U_2, \ldots$ of $K$, and for each $n$ a subset $E_n$ of $U_n$ so that 

$$i(\chi_{E_n}) = n = i(\chi_{E_n}, \varepsilon) \text{ for all } 0 < \varepsilon \leq 1,$$

and 

$$\|\chi_{E_n}\|_{D(U_n)} \leq n + 1.$$ 

Now let $f = \sum_{n=1}^{\infty} \chi_{E_n} / n$ pointwise. Thus by Lemma 2.12 and (54), $f \in \text{DBSC}(K)$ (with $\|f\|_D \leq 2$). However fixing $n$ and letting $\varepsilon = \frac{1}{n}$, then by (53), 

$$i(\chi_{E_n}, 1) = n \left(= i\left(\frac{1}{n} \chi_{E_n}, \frac{1}{n}\right)\right) \text{ and so}$

$$\varepsilon i(f, \varepsilon) \geq \frac{1}{n} i\left(f \mid U_n, \frac{1}{n}\right) = 1.$$ 

Thus $f$ fails (7), so $f \notin \text{SD}(K)$ by Theorem 1.4. □

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