Anarchy in the neutrino sector?

J.R. Espinosa

IMAFF, CSIC, Serrano 113 bis, 28006 Madrid, Spain
IFT, C-XVI, Univ. Autónoma de Madrid, 28049 Madrid, Spain
(December 3, 2003)

If the connection between the fundamental theory and neutrino mass matrices $M_\nu$ is sufficiently complicated all traces of underlying symmetries might be erased and an ‘anarchical’ $M_\nu$, with random entries, can arise. It has been claimed that such random matrices generically prefer large mixing making unnecessary to invoke flavour symmetries to explain the observed pattern of neutrino mixing angles. A critical analysis of this idea (paying particular attention to the role of unphysical phases and coordinate dependence) shows that basis-independent, random $M_\nu$’s are not biased towards large values of the neutrino mixing angles, contrary to previous claims. As a result, the case for anarchy in the neutrino sector is considerably weakened.

1. Introduction. In a series of three interesting papers [1–3] the idea that the measured neutrino parameters [4] could arise from a random mass matrix has been proposed and studied. A neutrino mass matrix with random entries could be the result of a fundamental theory that is itself complicated or that requires a convoluted mechanism (e.g. with several sources) for generating neutrino masses.

In ref. [1], a random scan of $10^6$ neutrino mass matrices (of Dirac, Majorana or see-saw type) is performed and the resulting distributions for mixing angles and for the ratio $R = \Delta m^2_{sol}/\Delta m^2_{atm}$ are studied. The distributions for the atmospheric and solar angles peak at maximal mixing (which is welcome) while $U_{e3}$ is in general not particularly small (which can be a problem, or a promise for its measurement in the future). Finally, it is easy to get a small $R$ ($\sim 0.1$), especially in the see-saw case.

Ref. [2] sheds light on the previous results by realizing that the distributions of mixing angles are in fact dictated by the requirement of basis independence of the random scan. If $M_\nu$ is diagonalized by a mixing matrix $U$ that belongs to a certain Lie group [e.g. $U(3)$ for a complex $3 \times 3$ Majorana $M_\nu$], then the probability distribution for the mixing angles that parametrize $U$ follow the Haar measure of that Lie group. This nice property allows an analytic understanding of the predictions of anarchy.

Finally, ref. [3] tests the hypothesis of anarchy in the neutrino sector by performing a Kolmogorov-Smirnov (KS) test taking as data the observed neutrino mixing angles (choosing the best fit points for the solar and atmospheric mixing angles and leaving the CHOOZ angle free, but below its bound). The test should measure how well the anarchy hypothesis describes the data or, alternatively, see if the data are able to rule out this hypothesis (with some degree of C.L.). It is found that the KS probability in favor of the anarchy hypothesis is $\sim 12\%$.

In this letter, the predictions of the anarchy hypothesis for different types of neutrino mass matrices are carefully re-examined in section 2. This analysis shows that anarchy has no preference for certain ranges of mixing angles. In particular, there is no preference for maximal mixing. A critical analysis of statistical tests of the anarchy hypothesis, performed in section 3, confirms this result.

2. Mixing angles predicted by Anarchy. One of the main ingredients of the hypothesis of anarchy in the neutrino sector is the requirement of basis-independence in flavour space: i.e. the entries of $M_\nu$ are random numbers distributed in some specified range according to a flat probability density, that remains flat in any basis. As an example, take a $2 \times 2$ real and symmetric $M_\nu$. Factoring out the absolute neutrino mass scale $m$, take the entries of $\hat{M}_\nu = M_\nu/m$ as random numbers in the interval $[-1, 1]$ with uniform probability density. The differential probability density for each entry is simply $d\hat{M}_{ij}$ and in the 3-D space $\{M_{11}, M_{12}, M_{22}\}$ that parametrizes $M_\nu$, the differential probability density is given by the $3$-form $dP = dM_{11} \wedge dM_{12} \wedge dM_{22}$. It is straightforward to prove the invariance of $dP$ under rotations in flavour space. This result can be extended to more general matrices: for complex entries one simply uses $d\hat{M}_{ij} \equiv d\Re e \, \hat{M}_{ij} \wedge d\Im m \, \hat{M}_{ij}$. A uniform probability density for the entries of $M_\nu$ is not enough to guarantee basis-independence. The range in which these entries can vary must also be a basis-independent domain. In the previous example, this domain was a cube, which is not rotation-invariant, but one can simply take $\hat{M}_{ij}$ inside a sphere of some specified radius: $\text{Tr} \, \hat{M}_{ij}^\dagger \hat{M}_{ij} < r^2$.

Once the basis-independence of the random matrices is enforced, the mixing matrices $U$ that diagonalize them are distributed in a well determined way: if the $U$’s are matrices of a given Lie group, the differential probability density for $U$, $d\Omega$, is simply the invariant Haar measure of that Lie group (see [2]). In terms of the invariant Maurer-Cartan 1-forms $\omega^a = -i\text{Tr} \, T^a U^{-1} dU$, where $T^a$ are the generators of the Lie group, then $d\Omega = \epsilon_{a_1 a_2 \ldots a_N} \omega^{a_1} \wedge \ldots \wedge \omega^{a_N}$, with $N$ the dimension of the group.

Let us now examine in turn several examples of Majorana mass matrices (the results can be extended to Dirac or see-saw matrices).
2.1 Real 2×2 $M_\nu$

These matrices are diagonalized by rotations, $U = R(\theta)$, so that the corresponding Lie group is $U(1)$ and $d\Omega = d\theta$, which is explicitly rotation invariant. The distribution of the mixing angle $\theta$ is therefore flat and no mixing angle is preferred. The conclusions of ref. [2] are different: by writing $d\theta = d\sin^22\theta/(2\sin4\theta)$ it is shown that the distribution of $\sin^22\theta$ peaks at zero and maximal angles, which are claimed to be preferred. Such peaks, however, do not imply that angles near zero or maximal mixing are favored: these peaks are simply the result of using a particular coordinate in the circle, $\sin^22\theta$, which distorts the uniform distribution in angles. To see this even more clearly, notice that we have not specified in what basis we are measuring $\theta$. We would find peaks for $\sin^22\theta$ in any basis, but if they were to imply a preference for maximal and zero mixing, one arrives at the paradoxical conclusion that different physical angles are preferred depending on the basis used.

Given the flat distribution in $\theta$, all one can say is that large mixing angles are not unlikely. The relative likelihood of finding $\theta$ in a region near zero mixing or in a region near maximal mixing is simply given by the relative size of the two regions. In section 3 we discuss the statistical tests that might be applied to data in order to see how well they support the anarchy hypothesis.

2.2 Complex 2×2 $M_\nu$

These Majorana matrices are diagonalized by $U(2)$ matrices that can be written as $U = \exp[i(\eta I_2 + \omega \sigma_3)]R(\theta)\exp[i\phi \sigma_3]$, where $\eta$ and $\omega$ are unphysical phases, $\theta$ is the rotation angle and $\phi$ a CP violating phase. The distributions of these quantities in the anarchy scenario follow from the $U(2)$ Haar measure

$$d\Omega = ds_\theta^2 \wedge d\eta \wedge d\omega \wedge d\phi,$$  \hspace{1cm} (1)

(with $s_x \equiv \sin x$). Rewriting $ds_\theta^2 = s_2d\theta$, ref. [2] concludes that, in this case, maximal mixing is preferred. This conclusion, which seems now inescapable, leaves still the paradox that the preferred angle seems to depend on the basis chosen to measure $\theta$ (in spite of the fact that $d\Omega$ is indeed basis independent). One might argue that the measured $\theta$ is defined in the flavour eigenstate basis and therefore such basis should be used. However, one should be allowed to determine the probability distributions and the most probable angle in a rotated basis, provided one rotates back later to the flavour eigenstate basis. The paradox is that, doing so, the final result for the expected $\theta$ depends on the intermediate rotated basis used.

In order to understand why, according to (1), $\theta = \pi/4$ seems to be preferred while $\theta = 0$ is disfavoured, it is useful to inspect the explicit expressions for the invariant Maurer-Cartan forms:

$$\omega^1 = d\eta,$$
$$\omega^3 = s_2\theta,$$
$$\omega^4 = s_2\theta d\omega + c_2\theta d\phi,$$  \hspace{1cm} (2)

with $d\Omega = \omega^1 \wedge \omega^2 \wedge \omega^3 \wedge \omega^4$. We see that, for $\theta \rightarrow 0$ one has $\omega^2 \wedge \omega^3 \rightarrow 0$, while this is not the case for finite values of $\theta$ due to the presence of $d\omega$ terms. In other words, the subspace of parameter space corresponding to $\theta = 0$ has one dimension less than that for $\theta = \theta_0 \neq 0$ and the extra volume associated with the latter case opens up in the direction of the unphysical parameter $\omega$. However, all matrices related to each other by changes in an unphysical parameter should be considered as physically indistinguishable and therefore the preference for maximal mixings does not stand on firm ground.

Keeping the unphysical phases $\eta, \omega$ helps dealing with basis independence but introduces a bias in the distributions of physical parameters. How do we get rid of this problem? We simply fix $\eta = \omega = 0$, which we are free to do for any $M_\nu$. The crucial point is that this condition is rotation invariant, as can be readily checked. After fixing this ‘gauge’, $\theta$ transforms under a rotation by angle $\alpha$ simply by $\theta \rightarrow \theta + \alpha$ while the phase $\phi$ does not change (notice that in the presence of non-zero unphysical phases the transformation properties under rotations are more complicated). Therefore, after ‘gauge-fixing’, the only volume element in the space $\{\theta, \phi\}$ that is rotation invariant must be of the form

$$d\Omega_{gf} = \mathcal{F}(\phi)d\theta \wedge d\phi.$$  \hspace{1cm} (3)

Because invariance under rotations is not enough to fix $\mathcal{F}(\phi)$ in (3), further assumptions besides anarchy are necessary to make any prediction about the distribution of the phase $\phi$. This situation is similar to that of the distribution of mass eigenvalues [2] which are unknown functions of mass differences. In analogy with that case, one might choose these functions in such a way as to get a volume element in parameter space that is simple when expressed in terms of mass matrix elements. Such choice is useful to generate samples of random matrices like in [1] but now in a ‘fixed gauge’; however, its theoretical motivation is not clear. A simple choice for a rotation invariant volume element (in terms of matrix entries) is

$$dV_{gf} = d\Re M_{11}d\Re M_{22}d\Re M_{12}/d\log 3|m|_{M_{12}},$$

which leads to

$$dV_{gf} = -2(m_1 - m_2)dm_1 \wedge dm_2 \wedge \hat{c}_3^2d\phi \wedge d\theta.$$  \hspace{1cm} (4)

This privileges small values of $\phi$ but, as explained, other choices of invariant volume are possible. The uncertainty in the distribution of $\phi$ does not affect the distribution of $\theta$, which is again flat: all mixing angles are equally probable and the comments at the end of section 2.1 apply also to this case.

2.3 Real 3×3 $M_\nu$

A symmetric real $M_\nu$ can be diagonalized by an orthogonal matrix $U$ belonging to $SO(3)$. The parametrization $U = R_{23}(\theta_1)R_{31}(\theta_2)R_{12}(\theta_3)$, with $R_{ij}(\theta)$ a rotation in the plane $ij$ by an angle $\theta$, is quite convenient to discuss neutrino oscillations. The angle $\theta_3$ is the relevant angle for the oscillations of solar neutrinos, $\theta_2$ is the small mixing angle bounded by CHOOZ experiment and $\theta_1$ is the relevant angle for the oscillations of atmospheric neutrinos [4]. The differential probability distributions for the
mixing angles $\theta_i$ derived from anarchy is then dictated by the SO(3) invariant Haar measure

$$d\Omega = d\theta_1 \wedge \cos \theta_2 d\theta_2 \wedge d\theta_3.$$  \tag{5}

In order to understand better what are the implications of (5) concerning the likelihood of different values for the mixing angles, it is convenient to switch to a different parametrization of $U$. Let $\vec{n}$ be the axis of rotation in flavour space [parametrized in terms of polar angles $\alpha, \beta$ as $\vec{n} = (s_\beta c_\alpha, s_\beta s_\alpha, c_\beta)$], and $\theta$ the rotation angle. We can then write $U = R_{12}(\alpha) R_{31}(\beta) R_{12}(\theta) R_{31}(-\beta) R_{12}(-\alpha)$. In terms of these parameters one obtains

$$d\Omega = d\alpha \wedge \sin \beta d\beta \wedge \sin^2(\theta/2) d\theta.$$  \tag{6}

For a fixed value of $\theta$ this measure is just that on the sphere $(d\alpha \wedge \sin \beta d\beta)$ with polar coordinates $\alpha, \beta$. Although this measure is zero for $\sin \beta = 0$, this does not mean that rotations along the $z$ axis are disfavoured but rather that the distribution of the rotation axis $\vec{n}$ is uniform in flavour space, as it should be the case if one starts with a rotation-invariant mass matrix. A useful analogy is to consider a uniform distribution of points over the surface of the Earth. In terms of longitude ($\alpha$) and latitude ($\beta$) the probability density has the same form as above and the expected number of points inside a certain region is simply proportional to its area no matter where this region is on the surface of the Earth.

The previous discussion shows that in order to make meaningful statements about what angles are preferred, one should take into account the area (or volume) of the regions of parameter space that are compared. What anarchy implies, as encoded in (5), is that all regions of parameter space are equally likely. The ‘preference’ for small $\theta_2$ in (5) is due to the fact that the volume of parameter space with small $\theta_2$ is larger [in the same sense that $\beta \sim 0, \pi/2$ is disfavored in (6) because the area of the poles is smaller than the rest of the Earth’s area].

2.4 Complex 3×3 $\nu$

A complex symmetric $M_\nu$ is diagonalized by a $U(3)$ matrix that may be written in this way: $U = \text{diag}(e^{i\alpha_1}, e^{i\alpha_2}, e^{i\alpha_3}) R_{31}(\theta_1) e^{-i\delta} R_{12}(\theta_2) e^{i\delta} R_{13}(\theta_3) \text{diag}(e^{-i(\xi_1 + \xi_2)}, e^{i\xi_1}, e^{i\xi_2})$ with $\lambda' = \text{diag}(1, 0, -1)/2$. In this notation $\alpha_{1,2,3}$ are unphysical phases. The distribution of angles and phases according to the original idea of anarchy would then follow the distribution

$$d\Omega = ds_3^2 \wedge dc_2^2 \wedge ds_2^2 \wedge d\delta \wedge d\xi_1 \wedge d\xi_2 \wedge d\alpha_1 \wedge d\alpha_2 \wedge d\alpha_3$$

which corresponds to the $U(3)$ Haar measure. This results from combining 9 invariant Maurer-Cartan forms (straightforward to compute but lengthy).

As was discussed for the complex 2×2 case, one cannot use $d\Omega$ above to extract physical implications without getting rid of the unphysical degrees of freedom (which otherwise affect the distribution of physical parameters). This problem is solved by ‘choosing a gauge’ like $\alpha_1 = \alpha_2 + \delta$ and $\alpha_2 = \alpha_3 = -\xi_2$, which is invariant under rotations of basis. The invariant volume element after this gauge-fixing is of the form [6]

$$d\Omega_{gf} = \mathcal{F}(\delta - 3\xi_2) G(s_3 s_2 c_3)(c_2 d\theta_1 \wedge d\theta_2 \wedge d\theta_3) \wedge (d\xi_1 \wedge d\xi_2 \wedge d\delta/s_3).$$  \tag{7}

This volume element is composed of three parts: $\mathcal{F}$ and $G$ are unknown functions of their invariant arguments; the angular element is the volume element for real 3×3 matrices (5), which corresponds to a homogeneous distribution of mixing angles; and finally there is a phase volume element. The predictions of anarchy for the angles $\theta_1$ and $\theta_2$ are similar to those of the real 3×3 case: all values of $\theta_1$ are equally likely; small $\theta_2$ is preferred due to volume effects. Without further assumptions on the form of $\mathcal{F}$ and $G$, anarchy does not make more detailed predictions for $\theta_3$ and the physical phases (except for the prediction of a flat distribution for $\xi_1$). If $G(x) = x$, $d\Omega_{gf}$ gives a preference for large solar mixing.

3. Statistical test of the hypothesis of anarchy

In order to assess how likely it is that the observed neutrino angles arise from a distribution dictated by the hypothesis of anarchy, ref. [3] performs an statistical test as follows. Consider for simplicity the solar angle $\theta_3$ and assume that the probability density is flat in $x = \sin^2 \theta_3$ (as was the prediction for 3×3 complex Majorana matrices in ref. [3], revised in this paper) and take $x$ as a random variable in the interval $[0, 1]$ (the same analysis applies to a flat distribution in $d\theta_3$ with $x = 2\theta/\pi$). Experimentally, the best fit point for solar neutrino data corresponds to $x_e = 0.3$. Based on this, ref. [3] tests the anarchy hypothesis using standard methods, by comparing the distribution function predicted by anarchy, $F(x) = \int_0^x f(x') dx' = x$, with the ‘empirical’ distribution function, $F_e(x)$, obtained from the best guess probability density, $f_e(x') = \delta(x' - x_e)$, that leads to $F_e(x) = \int_0^x f_e(x') dx' = \Theta(x - x_e)$. The mismatch between the two distribution functions is measured by the (two-sided) Kolmogorov-Smirnov statistic

$$D = \max_x |F_e(x) - F(x)|.$$  \tag{8}

The higher $D$ is, the worst $F(x)$ accomodates the observed data. More precisely, the probability of obtaining a worse value of $D$ from a different observed $x$ is

$$P(D \geq D_e) = \begin{cases} 2x_e, & \text{if } x_e \leq 1/2 \\ 2(1 - x_e), & \text{if } x_e \geq 1/2 \end{cases}$$  \tag{9}

where $D_e$ is the KS statistic for $x_e$. It is clear that, according to this procedure, the value $x_e = 1/2$ (i.e., maximal mixing) is the best possible result in favor of the anarchy hypothesis.

This is a puzzling result. How can a single data point be able to provide evidence in favor of a flat distribution at all? Why should data values close to $x_e = 1/2$ be the best support for a flat probability distribution for $x$? While this seems reasonable for $x = \sin^2 \theta_3$ it is certainly
not for $x = 2\theta/\pi$, and ref. [3] uses the above test, without modification, also for distributions flat in $x = \cos^2 \theta$. There are three problems with the previous analysis.

The first is that $x$ is an angular variable ($0 \leq \theta < \pi/2$ is mapped into the interval $[0, 1]$ with $x = 0$ and $x = 1$ corresponding to the same physical point) and the test used is suited for linear variables. In fact the statistical analysis of angular or directional data requires some departure from the usual linear statistical methods and there is a well developed corpus of knowledge on this topic (for an introduction to this subject see e.g. [7]) which is relevant in many areas of science, from the analysis of directional patterns in bird migration to magnetic pole wandering, coincidence of planetary orbital planes or distribution of earthquakes in the globe, among others. The version of the Kolmogorov statistic that is best suited to angular random variables is Kuiper’s test [7], defined as [8]

$$D = \max_\theta[F_e(\theta) - F(\theta)] + \max_\theta[F(\theta) - F_e(\theta)].$$

(10)

This statistic has the desirable property of being independent of the choice of origin for $\theta$ (see Appendix for a simple proof), which is not the case for the linear Kolmogorov statistic, Eq. (8).

However, the previous improvement is not sufficient in view of the second problem: the smallness of the sample. One can easily see that the Kuiper statistic for a single random draw, $\theta_e$, is $D_e = 1$, which is independent of $x_e$. Although this seems to be a reasonable result (it implies that a single data point cannot give evidence in favor or against the hypothesis of a flat probability distribution) it cannot be taken seriously because the Kuiper test is always $D_e = 1$ for a single data point irrespective of the distribution being tested! One possible way out would be to improve the test by taking into account the difference between the distribution functions over the whole interval (‘integral’ variants of the Kuiper test exist [9]) but, as we show below, these improvements would also be insufficient.

The third problem is that the K-S test is invariant under reparametrizations of $x$, e.g. it cannot tell apart distributions that are flat in $x = \sin^2 \theta$ from those that are flat in $x = 2\theta/\pi$, and this is inappropriate.

An alternative approach (for $x = 2\theta/\pi$) is to enlarge the data sample by considering the origin, $\theta_0 = 0$, defined by the flavour basis, as another data point. In that case the probability of getting a fit worse than the observed one, according to the Kuiper test, is given again by Eq. (9) with $x_e = 2(\theta_e - \theta_0)/\pi$, which is explicitly rotation invariant. From this result one would conclude that $\theta_e - \theta_0 \sim \pi/4$ would give the strongest support to a flat distribution and $\theta_e - \theta_0 \sim 0, \pi/2$ the weakest, which agrees with the conclusions of Ref. [3]. However, this result cannot be trusted. The reason is that even two points are too few to give evidence in favor of a flat distribution function: any set of two data points has the same probability of any other (this situation changes only for three or more data points).

We therefore abandon the attempt at performing a meaningful statistical test of the anarchy hypothesis, at least for uniformly distributed angles (in principle a partial test could be devised for angles which are not uniformly distributed, like $\theta_2$) and prefer to limit ourselves to the following estimate, similar to the analysis performed in Ref. [1]. Assuming a large number of random neutrino mass matrices, we can estimate now what fraction of them will have angles in some neighbourhood of the observed values (say in a subspace $v$ of the total volume $V$). For $3 \times 3$ real Majorana matrices [or for complex ones with $\mathcal{G}(x) = 1$ in Eq. (7)], that fraction will be given simply by the ratio $\int_{v} d\Omega/\int_{v} d\Omega$. Using ranges similar to those in Ref. [1] ($\sin^2 2\theta_2 < 0.15, \sin^2 2\theta_{1,3} > 0.5$) the interesting area is about a 4.8% of the total. This number is similar to the one obtained in Ref. [1] for the real case (in which case we both agree on the angle distributions) before applying the cut in mass splittings. If $\mathcal{G}(x) = x$ is chosen in (7), then the preference for maximal solar mixing raises the previous percentage to 6.8%. Still, we regard this as a very small basis upon which to build the case in favour of anarchy. In contrast, attempts based on flavour symmetries fare considerably better [10,11] and are physically much more appealing: they imply that what we learn at low-energy from measurements of the parameters in the neutrino sector will be a window (with a view) to fundamental physics at much higher energies.

**Appendix** To see that the Kuiper’s test is invariant under rotations, note first that, due to the ‘periodicity’ property of angular distribution functions, $[F(\theta + 2\pi) = F(\theta) + 1]$, any angular interval $[\alpha, \alpha + 2\pi]$ is suitable to compute $D$ in Eq.(10). If we now change from the random variable $\theta$ to the rotated $\theta' = \theta + \alpha$, the new distribution functions are $F'(\theta') = F(\theta + \alpha) - F(\theta)$ and

$$D' = \max_{\theta'}[F_e(\theta') - F'(\theta')] + \max_{\theta'}[F'(\theta') - F_e(\theta')]$$

$$= \max_{\theta}[F_e(\theta + \alpha) - F(\theta + \alpha)] - F_e(\alpha) + F(\alpha)$$

$$+ \max_{\theta}[F(\theta + \alpha) - F_e(\theta + \alpha)] + F_e(\alpha) - F(\alpha)$$

$$= \max_{\theta}[F_e(\theta) - F(\theta)] + \max_{\theta}[F(\theta) - F_e(\theta)],$$

so that $D' = D$. A less straightforward proof of this property can be found in [7].

**Acknowledgments:** Useful discussions with Guido Altarelli, Alberto Casas, Concha González-García, Alejandro Ibarra and Raúl Rabadán are gratefully acknowledged. I also thank CERN, where most of this work was carried out, for hospitality and partial financial support.

[1] L. J. Hall, H. Murayama and N. Weiner, Phys. Rev. Lett. 84 (2000) 2572 [hep-ph/9911341].

[2] N. Haba and H. Murayama, Phys. Rev. D 63 (2001) 053010 [hep-ph/0009174].

[3] A. de Gouvêa and H. Murayama, [hep-ph/0301050].

[4] For a review of the present status see, M. C. González-García, [hep-ph/0211054].
[5] The same result can be obtained setting $d\eta = d\omega = 0$ in the Maurer-Cartan forms (2). One gets the three invariant forms $s_2 d\theta, c_2 d\theta$ and $d\phi$. Invariance of the first two forms implies invariance of $\phi$ and $d\theta$ separately, and the combination of these forms leads to Eq. (3).

[6] This can be checked either by direct calculation or by setting $d\alpha_1 = d\alpha_2 + d\delta$ and $d\alpha_2 = d\alpha_3 = -d\xi_2$ in the $U(3)$ Maurer-Cartan forms to identify invariant zero- and one-forms out of which $d\Omega_{gf}$ is constructed.

[7] K. V. Mardia, *Statistics of Directional data*, Academic Press, London and New York, 1972.

[8] Distribution functions for an angular variable $\theta$ are simply defined by $F(\theta) = \int_0^\theta f(\theta')d\theta'$ for $0 < \theta \leq 2\pi$ and extended to all values of $\theta$ by $F(\theta + 2\pi) = F(\theta) + 1$.

[9] G.S. Watson, Biometrika 48 (1961) 109.

[10] M. Hirsch, [hep-ph/0102102].

[11] G. Altarelli, F. Feruglio and I. Masina, JHEP 0301 (2003) 035 [hep-ph/0210342].