Gap probabilities in the Laguerre unitary ensemble and discrete Painlevé equations

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Received 31 December 2019, revised 7 June 2020
Accepted for publication 23 June 2020
Published 12 August 2020

Abstract

In this paper we study a certain recurrence relation, that can be used to generate ladder operators for the Laguerre unitary ensemble, from the point of view of Sakai’s geometric theory of Painlevé equations. On one hand, this gives us one more detailed example of the appearance of discrete Painlevé equations in the theory of orthogonal polynomials. On the other hand, it serves as a good illustration of the effectiveness of a recently proposed procedure on how to reduce such recurrences to some canonical discrete Painlevé equations.

Keywords: orthogonal polynomials, Painlevé equations, difference equations, birational transformations

(Some figures may appear in colour only in the online journal)

1. Introduction

By now it is clear that there are many fundamental connections between the theory of random matrices, orthogonal polynomials, and Painlevé equations, both differential and discrete. Some conceptual understanding of this fact has been given in a series of papers by Alexei Borodin and his collaborators [AB06–Bor03] and especially [BB03], see also a recent monograph of Walter Van Assche [VA18]. In a way, the geometric setting of Hidetaka Sakai’s theory of Painlevé equations [Sak01] seems to provide the natural framework for questions involving the study of various orthogonal polynomial ensembles, and so it is not surprising that various objects of interest, such as the gap probabilities, coefficients for three-term recurrence relations, or

\textsuperscript{*}In memory of Jon Nimmo.
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ladder operators, can be described in terms of solution of either differential or discrete Painlevé equations.

The purpose of the present paper is to study, from the geometric perspective of Sakai’s theory, an example of a recurrence relation obtained by Shulin Lyu and Yang Chen in their study of the largest eigenvalue distribution [LC17] for the Laguerre unitary ensemble, focusing on the reduction of this recurrence to a canonical form following step-by-step procedure recently proposed in [DFS19].

Thus, we let the weight function be \( w(x; \alpha) = x^\alpha \exp(-x) \), where \( x > 0 \) and \( \alpha > -1 \) is a parameter and consider a family of monic polynomials

\[
P_j(x, t) = x^j + p_{j-1}(t)x^{j-1} + \cdots + p_0(t)
\]

that are orthogonal with respect to the weight \( w(x; \alpha) \) on the interval \([0, t]\), \( 0 < t \leq \infty \), i.e.,

\[
\int_0^t P_n(x, t)P_m(x, t)x^\alpha \exp(-x) \, dx = \delta_{nm} h_n(t),
\]

where \( h_n(t) \) is the square of the \( L^2([0, t]; x^\alpha \exp(-x) \, dx) \) norm of \( P_n(x, t) \). This unitary ensemble is called the Laguerre unitary ensemble (or LUE for short) since when \( t = \infty \), the family \( \{P_n(x) = P_n(x, \infty)\} \) is the well-known family of monic Laguerre polynomials orthogonal w.r.t. the weight \( x^\alpha \exp(-x) \).

As usual, the orthogonality condition immediately implies the three term recurrence relations

\[
xP_n(x, t) = P_{n+1}(x, t) + \alpha_n(t)P_n(x, t) + \beta_n(t)P_{n-1}(x, t), \quad n \geq 0,
\]

with initial conditions \( P_{-1}(x, t) = 0, P_0(x, t) = 1 \).

The paper [LC17] is concerned with the study of the probability \( \mathbb{P}(n, t) \) that the largest eigenvalue in LUE on \([0, \infty)\) is not larger than \( t \), where \( n \) is the size of the corresponding random matrix. This probability can be computed as

\[
\mathbb{P}(n, t) = \frac{D_n(t)}{D_n(\infty)},
\]

where

\[
D_n(t) := \det \left[ \int_0^t x^{i+k} x^{-\alpha} \exp(-x) \, dx \right]_{0 \leq i, k \leq n-1}
\]

is the \( n \times n \) Hankel determinant, a fundamental object in the theory of orthogonal polynomials [Sze67], that can be evaluated as \( D_n(t) = \prod_{j=0}^{n-1} h_j(t) \), and

\[
D_n(\infty) = \frac{G(n + 1)G(n + \alpha + 1)}{\Gamma(\alpha + 1)}, \quad \text{where } G(.) \text{ is the Barnes } G\text{-function.}
\]

One way to study (and generate) the family of orthogonal polynomials \( \{P_n(x, t)\} \) is to use the lowering and raising ladder operators,

\[
\left( \frac{d}{dz} + B_n(z, t) \right) P_n(z, t) = \beta_n(t)A_n(z, t)P_{n-1}(z, t),
\]

\[
\left( \frac{d}{dz} - B_n(z, t) - v(z) \right) P_{n-1}(z, t) = -A_{n-1}(z, t)P_n(z, t),
\]

where \( A_n(z, t) \) and \( B_n(z, t) \) can be parameterized by the functions \( R_n(t) \) and \( r_n(t) \),
shown that it is the $\sigma$ of discrete Painlevé equations is given by

$$A_n(z, t) = \frac{R_n(t)}{z - t} + \frac{1 - R_n(t)}{z}, \quad B_n(z, t) = \frac{r_n(t)}{z - t} - \frac{r_n(t) + n}{z},$$

and where

$$R_n(t) := -\frac{P_n^2(t, t)}{h_n(t)} \rho^t e^{-t}, \quad r_n(t) := -\frac{P_n(t, t)P_{n-1}(t, t)}{h_{n-1}(t)} \rho^t e^{-t}, \quad \text{and} \quad P_j(t, t) := P_j(z, t)|_{z=t}.$$  

Let us now introduce a different parameterization $x_n(t), y_n(t)$ via

$$x_n(t) = 1 - \frac{1}{R_{n-1}(t)}, \quad y_n(t) = -r_n(t).$$

Then Lyu and Chen [LC17, remark 2.3] showed that these variables satisfy the following recurrence relations in $n$:

$$
\begin{align*}
x_n x_{n+1} &= \frac{(y_n - n)(y_n - (n + \alpha))}{y_n^2}, \\
x_n + y_{n-1} &= \frac{(-t + 2n - 1 + \alpha)x_n - (2n - 1 + \alpha)}{(x_n - 1)^2}.
\end{align*}
$$

This is the recurrence that we are interested in studying. We show, following the step-by-step procedure of [DFS19], that this recurrence is a discrete Painlevé equation that is equivalent to one of the standard examples in the d-$A^{(1)}_2$ / $D^{(1)}_4$ family. Our main result is the following theorem.

**Theorem 1.** The recurrence (1.3) is equivalent to the standard discrete Painlevé equation (A.17) written in [KNY17]. This equivalence is achieved via the following change of variables:

$$x(q, p) = \frac{q(p + t) + a_2}{qp + a_2}, \quad y(q, p) = \frac{(p + t)(qp + a_2)}{t}.$$  

The inverse change of variables is given by

$$q(x, y) = \frac{(x - 1)(x - 1)y + n}{tx}, \quad p(x, y) = \frac{y(q - n)}{(x - 1)y + n}.$$  

The relationship between the Laguerre weight recurrence parameters and the root variables of discrete Painlevé equations is given by

$$a_0 = n + \alpha, \quad a_1 = -n, \quad a_2 = n, \quad a_3 = 1 - n - \alpha.$$  

**Remark 2.** Note that for our recurrence the root variables are constrained by the condition $a_1 + a_2 = 0$ (or, equivalently, $a_0 + a_3 = 1$).

These recurrences then are particular combinations of elementary mappings that can be thought of as Bäcklund transformations of a differential $P_V$ equation that is associated with the same geometry. This is not surprising, since, if we put $\sigma_n(t) := t e^{P_n(t, t)}$ then it can be shown that it is the $\sigma$ function of a particular Painlevé $V$ equation. Estelle Basor and Yang Chen [BC09] gave an alternate derivation of this result without relying on the Christoffel–Darboux kernel (or the reproducing kernel). Note also that the quantity $S_n(t) = 1 - 1/R_n(t)$, satisfies
below, it is convenient to label them as mapping vanish, other points are found in other charts). We resolve base point singularities using the blowup procedure, see, e.g., $Y = \frac{q}{n}$ (index $n$). In this class of examples, our recurrence relation defines two natural mappings, the $\psi$ for in the mapping notation. The map $\psi$ is replaced by the $\psi^{-1}$ of discrete Painlevé equations, see, e.g., [Sha13]. That is, for each base point $q_i(x_i, y_i)$ we construct two new local charts $(u_i, v_i)$ and $(U_i, V_i)$ given by $x = x_i + u_i = x_i + U_iV_i$ and $y = y_i + u_iv_i = y_i + V_i$. The coordinates $v_i = 1/U_i$ represent all possible slopes of lines passing through the point $q_i$, and so this variable change ‘separates’ all curves passing through $q_i$ based on their slopes. This change of variables is a bijection away from $q_i$, but the point $q_i$ is replaced by the $\P^1$ line of all possible slopes, called the central fiber or the exceptional divisor of the blowup. We denote this central fiber by $F_i$, it is given in the blowup charts by local equations $u_i = 0$ and $V_i = 0$. We then extend the mapping to these new charts via the above coordinate substitution, find and resolve new base points (those would only appear on

$$S_n'(t) = \frac{3S_n(t) - 1}{2(2S_n(t) - S_n(t))} \left( S_n'(t) \right)^2 - \frac{S_n'(t)}{t} - \frac{\alpha^2(S_n(t) - 1)^2}{2S_n(t)t^2} + \frac{(2n + 1 + \alpha)S_n(t)}{t}$$

$$- \frac{1}{2} \frac{S_n(t)(S_n(t) + 1)}{S_n(t) - 1},$$

which is a $P_V$ with parameters

$$\alpha_5 = 0, \ \beta_5 = -\frac{\alpha^2}{2}, \ c_5 = 2n + 1 + \alpha, \ d_5 = -1/2.$$
the exceptional divisors $u_i = V_i = 0$) and continue this process until it terminates (it should, in the discrete Painlevé case). We summarize the result in the following lemma.

**Lemma 3.** The base points of the mapping (2.1) are

$q_1(x = 0, y = n), \quad q_2(x = 0, y = n + \alpha),

q_3 \left( x = 1, Y = \frac{1}{y} = 0 \right) \leftarrow q_4 \left( \alpha_3 = x - 1 = 0, v_3 = \frac{1}{y(x - 1)} = 0 \right)

\leftarrow q_5 \left( U_4 = y(1 - x)^2 = t, V_4 = \frac{1}{y(x - 1)} = 0 \right)

\leftarrow q_6 \left( U_5 = (x - 1)y((x - 1)^2y - t) = t(1 - 2n + t - \alpha), \right.

V_5 = \frac{1}{y(x - 1)} = 0 \right)

\leftarrow q_7 \left( X = \frac{1}{x} = 0, y = 0 \right) \leftarrow q_8 \left( U_7 = \frac{1}{xy} = 0, V_7 = y = 0 \right).

Considering the inverse mapping does not add any new base points.

Resolving these base points lifts our birational mapping $\psi : \mathbb{P}^1 \times \mathbb{P}^1 \dasharrow \mathbb{P}^1 \times \mathbb{P}^1$ to the isomorphism, also denoted by $\psi$, between the corresponding algebraic surfaces, $\psi : X_b \rightarrow X_0.$

The subscript $b$ indicates that the coordinates of the base points (and hence the resulting surface) depend on the parameters of the mapping, $b = \{\alpha, t, n\}$. These parameters can (and do) change under the mapping and so $\overline{b}$ denotes the evolved set of parameters. Sometimes we drop the parameters subscript and use the notation $\overline{X}$ for the range of the mapping.

2.2. The induced mapping on $\text{Pic}(X)$

The next step in the identification procedure is to compute the induced mapping on the Picard lattice. Recall that for a regular algebraic variety $X$, its Picard group (or Picard lattice) is the quotient of the divisor group $\text{Div}(X) = \text{Span}_\mathbb{Z}(D)$ that is a free Abelian group generated by closed irreducible subvarieties $D$ of codimension 1, by the subgroup $\mathcal{P}(X)$ of principal divisors (i.e., by the relation of linear equivalence).

$$\text{Pic}(X) \simeq \text{Cl}(X) = \text{Div}(X)/\mathcal{P}(X) = \text{Div}(X)/\sim,$$

see [SKKT00] or [Sha13]. In our case, it is enough to know that $\text{Pic}(\mathbb{P}^1 \times \mathbb{P}^1) = \text{Span}_{\mathbb{Z}}\{H_x, H_y\}$, where $H_x = [H_{x=0}]$ is the class of a vertical and $H_y = [H_{y=b}]$ is the class of a horizontal line on $\mathbb{P}^1 \times \mathbb{P}^1$. Each blowup procedure at a point $q_i$ adds the class $F_i = [F_i]$ of the exceptional divisor (i.e., the central fiber) of the blowup, so $\text{Pic}(X) = \text{Span}_\mathbb{Z}\{H_x, H_y, F_1, \ldots, F_b\}$. Further, the Picard lattice is equipped with the symmetric bilinear intersection form given by

$$H_x \cdot H_x = H_y \cdot H_y = H_x \cdot H_y \cdot F_i = H_x \cdot F_j = 0, \quad H_x \cdot H_y = 1, \quad F_i \cdot F_j = -\delta_{ij} \quad \text{(2.3)}$$

on the generators, and then extended by the linearity.

The mapping $\psi$ induces a linear mapping $\psi_* : \text{Pic}(X) \rightarrow \text{Pic}(\overline{X}).$ Note that $\text{Pic}(X)$ and $\text{Pic}(\overline{X})$ are canonically isomorphic, so we sometimes just use the notation $\text{Pic}(X)$. We also use $F_i$ to denote the divisor of the central fiber of the blowup at the point $\overline{q}_i = \psi(q_i)$, and similarly for the backwards mapping and for the classes; notation $F_{i..j}$ stands for $F_i + \ldots + F_j$. 

Lemma 4. The action of the mapping $\psi : \text{Pic}(X) \to \text{Pic}(\tilde{X})$ is given by

\begin{align*}
\mathcal{H}_x &\mapsto 5\mathcal{H}_x + 2\mathcal{H}_y - \mathcal{F}_{12}^2 - 2\mathcal{F}_{3456} - \mathcal{F}_{78}, \\
\mathcal{F}_1 &\mapsto 2\mathcal{H}_x + \mathcal{H}_y - \mathcal{F}_{23456}, \\
\mathcal{F}_2 &\mapsto 2\mathcal{H}_x + \mathcal{H}_y - \mathcal{F}_{13456}, \\
\mathcal{F}_3 &\mapsto \mathcal{H}_x - \mathcal{F}_6, \\
\mathcal{F}_4 &\mapsto \mathcal{H}_x - \mathcal{F}_5, \\
\mathcal{F}_5 &\mapsto \mathcal{F}_1 - \mathcal{F}_2, \\
\mathcal{F}_6 &\mapsto \mathcal{F}_4 - \mathcal{F}_5, \\
\mathcal{F}_7 &\mapsto \mathcal{F}_3 - \mathcal{F}_4, \\
\mathcal{F}_8 &\mapsto \mathcal{F}_5 - \mathcal{F}_6.
\end{align*}

The evolution of parameters (and hence, the base points) is given by $b = \{\alpha, t, n\} \mapsto b = \{\alpha, t, n+1\}$.

Proof. The proof of this lemma is a standard direct computation and is omitted, see [DFS19] or [DT18] for similar examples worked out in detail. □

2.3. The surface type

Given that our mapping is completely regularized by eight blowups, we know that it should fit into the discrete Painlevé equations framework. To determine the type of the resulting algebraic surface, we need to find the configuration of the irreducible components of (the proper transform of) a bi-degree $(2, 2)$ (or bi-quadratic) curve $\Gamma$ on which these points lie. Since the proper transform of $\Gamma$ for a generic choice of parameters is the unique anti-canonical divisor (i.e., the polar divisor of a symplectic form $\omega$), we denote it by $-K_X$. We also denote by $\eta$ the projection mapping back to $\mathbb{P}^1 \times \mathbb{P}^1$, $\eta : X_b = \text{Bl}_{q_1 \cdots q_8}(\mathbb{P}^1 \times \mathbb{P}^1) \to \mathbb{P}^1 \times \mathbb{P}^1$.

Lemma 5. Base points $q_1, \ldots, q_8$ of the mapping (2.1) lie on the bi-quadratic curve $\Gamma$ given in the affine chart by the equation $x = 0$ (the homogeneous equation of $\Gamma$ is $x^2y^1y^1y^1 = 0$, where $x = x^2/x^1$ and $y = y^1/y^1$, so $\Gamma$ is indeed bi-quadratic); note that some points come in infinitely-close degeneration cascades. The irreducible components $d_i$ of the proper transform $-K_X$ of $\Gamma$,

$$-K_X = 2\mathcal{H}_x + 2\mathcal{H}_y - \mathcal{F}_1 - \cdots - \mathcal{F}_8 = d_0 + d_1 + 2d_2 + 2d_3 + d_4 + d_5,$$

are given by

$$d_0 = \mathcal{H}_x - \mathcal{F}_1 - \mathcal{F}_2, \quad d_1 = \mathcal{H}_x - \mathcal{F}_7 - \mathcal{F}_8, \quad d_2 = \mathcal{H}_y - \mathcal{F}_3 - \mathcal{F}_4, \quad d_3 = \mathcal{F}_4 - \mathcal{F}_5, \quad d_4 = \mathcal{F}_3 - \mathcal{F}_4, \quad d_5 = \mathcal{F}_5 - \mathcal{F}_6;$$

(2.4)

they define the surface root basis $\delta_1, \ldots, \delta_5$ of $-2$-classes in $\text{Pic}(X)$ whose configuration is described by the Dynkin diagram of type $D_5^{(1)}$:

$$\delta_0 = \mathcal{H}_x - \mathcal{F}_1 - \mathcal{F}_2, \quad \delta_3 = \mathcal{F}_4 - \mathcal{F}_5,$$

$$\delta_1 = \mathcal{H}_x - \mathcal{F}_7 - \mathcal{F}_8, \quad \delta_4 = \mathcal{F}_3 - \mathcal{F}_4,$$

$$\delta_2 = \mathcal{H}_y - \mathcal{F}_3 - \mathcal{F}_4, \quad \delta_5 = \mathcal{F}_5 - \mathcal{F}_6.$$  

(2.5)

Figure 1. The surface root basis for the Laguerre weight recurrence.
Figure 2. The Sakai surface for the Laguerre weight recurrence \((F_i = F_1 + \cdots + F_i)\).

We show some intermediate stages of the blowup process and the resulting \(D^{(1)}_5\) surface on figure 2. Thus our recurrence belongs to the \(d\)-\(A^{(1)}_3/D^{(1)}_5\) family with the symmetry group \(\tilde{W}(A^{(1)}_3)\). We describe the choice of the standard \(d\)-\(A^{(1)}_3/D^{(1)}_5\) point configuration, choices of the root bases for the surface and the symmetry sub-lattices, and other data, in the appendix A; we follow [KNY17] in our conventions.

**Remark 6.** Looking at figure 2, we notice another \(-2\)-curve \(F_7 - F_8\) that is disjoint from the irreducible components of the anti-canonical divisor. Such curves form the class \(\Delta_{\text{nod}}\); see [Sak01, section 3.3].

### 2.4. Initial geometry identification

The next step in the identification process is to find some change of basis in \(\text{Pic}(X)\) from the basis \(\{H_x, H_y, F_i\}\) to the basis \(\{H_q, H_p, E_j\}\) that correspond to the standard geometry configuration that identifies the surface root bases; we refer to this step as matching the geometry. At this point there are many possible choices of such basis change, we later may have to adjust it to match the dynamics.

**Lemma 7.** The following change of basis of \(\text{Pic}(X)\) identifies the root bases between the standard \(D^{(1)}_5\) surface and the surface that we obtained for the Laguerre weight recurrence:

\[
\begin{align*}
H_x &= H_p, & H_q &= 2H_x + H_y - F_1 - F_3 - F_4 - F_7, \\
H_y &= H_q + 2H_p - E_1 - E_3 - E_5 - E_6, & H_p &= H_x,
\end{align*}
\]
\[ F_1 = H_p - E_1, \quad E_1 = H_x - F_1. \]
\[ F_2 = E_2, \quad E_2 = F_2. \]
\[ F_3 = H_p - E_6, \quad E_3 = H_x - F_7. \]
\[ F_4 = H_p - E_5, \quad E_4 = F_8. \]
\[ F_5 = E_5, \quad E_5 = H_x - F_4. \]
\[ F_6 = E_8, \quad E_6 = H_x - F_3. \]
\[ F_7 = H_p - E_3, \quad E_7 = F_5. \]
\[ F_8 = E_4, \quad E_8 = F_6. \]

**Proof.** This is a direct computation based on comparing the surface root bases on figures 1 and A1.

2.5. The symmetry roots and the translations

We are now in a position to compare the dynamics. Note that there are two non-equivalent model examples of discrete Painlevé equations, that we label as [**TITI**] and [**T001**], on the \( D_4^{(1)} \)-surface that are described in section A.4 in the appendix A. It is interesting that the mapping (1.3) has the multiplicative–additive from that looks very similar to the mapping (A.21), but instead it is equivalent to the mapping (A.17) that has the purely additive form. To show that, we start with the standard choice of the symmetry root basis (A.3) and use the change of basis in lemma 7 to get the symmetry roots for the applied problem shown on figure 3. From the action of \( \psi_* \) on \( \text{Pic}(X) \) given in lemma 4 we can now obtain the corresponding translation on the root lattice, decompose it in terms of the generators of the extended affine Weyl symmetry group, and compare the results with the standard mappings \( \varphi \) and \( \phi \) given in section A.4. We get

\[ \alpha_0 = F_1 - F_2, \quad \alpha_2 = F_7 - F_8, \]
\[ \alpha_1 = H_x - F_4, \quad \alpha_3 = 2H_x - F_1 - F_4 - F_5 - F_6 - F_7. \]

\[ \delta = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3. \]

**Figure 3.** The symmetry root basis for the Laguerre weight recurrence (preliminary choice).

From here we immediately see that \( \psi = w_1 \circ \varphi \circ w_1^{-1} \) (note that \( w_1 \sigma_3 \sigma_2 = \sigma_3 \sigma_2 w_1 \)) and that \( w_1 \) is an involution, \( w_1^{-1} = w_1 \). Thus, our dynamic is equivalent to the standard equation (A.17) written in [**KNY17**] but is different from equation (A.21) written in [**Sak01**].
(i.e., $[0\ 1\ 0\ 1]\ =\ [1\ 1\ 1\ 0]$). To find the change of variables matching the two equations we first need to adjust our change of basis in $\text{Pic}(\mathcal{X})$ by acting on it by $w_1$, so that we match not only the geometry, but also the dynamics. We do it in the next section.

$$
\begin{align*}
\alpha_0 &= \mathcal{H}_y - \mathcal{F}_2 - \mathcal{F}_7, & \alpha_2 &= \mathcal{H}_y - \mathcal{F}_1 - \mathcal{F}_8, \\
\alpha_1 &= \mathcal{F}_1 + \mathcal{F}_7 - \mathcal{H}_y, & \alpha_3 &= 2\mathcal{H}_x + \mathcal{H}_y - \mathcal{F}_3 - \mathcal{F}_4 - \mathcal{F}_5 - \mathcal{F}_6 - \mathcal{F}_7. \\
\delta &= \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3.
\end{align*}
$$

(2.7)

Figure 4. The symmetry root basis for the Laguerre weight recurrence (final choice).

2.6. Final geometry identification

**Lemma 8.** After the change of basis of $\text{Pic}(\mathcal{X})$ given by

$$
\begin{align*}
\mathcal{H}_x &= \mathcal{H}_y + \mathcal{H}_p - \mathcal{E}_5 - \mathcal{E}_6, & \mathcal{H}_q &= 2\mathcal{H}_x + \mathcal{H}_y - \mathcal{F}_1 - \mathcal{F}_3 - \mathcal{F}_4 - \mathcal{F}_7, \\
\mathcal{H}_y &= \mathcal{H}_q + 2\mathcal{H}_p - \mathcal{E}_1 - \mathcal{E}_3 - \mathcal{E}_5 - \mathcal{E}_6, & \mathcal{H}_p &= \mathcal{H}_x + \mathcal{H}_y - \mathcal{F}_1 - \mathcal{F}_7, \\
\mathcal{F}_1 &= \mathcal{H}_q + \mathcal{H}_p - \mathcal{E}_1 - \mathcal{E}_5 - \mathcal{E}_6, & \mathcal{E}_1 &= \mathcal{H}_x - \mathcal{F}_1, \\
\mathcal{F}_2 &= \mathcal{E}_2, & \mathcal{E}_2 &= \mathcal{F}_2, \\
\mathcal{F}_3 &= \mathcal{H}_p - \mathcal{E}_6, & \mathcal{E}_3 &= \mathcal{H}_x - \mathcal{F}_7, \\
\mathcal{F}_4 &= \mathcal{H}_p - \mathcal{E}_5, & \mathcal{E}_4 &= \mathcal{F}_8, \\
\mathcal{F}_5 &= \mathcal{E}_7, & \mathcal{E}_5 &= \mathcal{H}_x + \mathcal{H}_y - \mathcal{F}_1 - \mathcal{F}_3 - \mathcal{F}_4 - \mathcal{F}_7, \\
\mathcal{F}_6 &= \mathcal{E}_8, & \mathcal{E}_6 &= \mathcal{H}_x + \mathcal{H}_y - \mathcal{F}_1 - \mathcal{F}_3 - \mathcal{F}_7, \\
\mathcal{F}_7 &= \mathcal{H}_q + \mathcal{H}_p - \mathcal{E}_3 - \mathcal{E}_5 - \mathcal{E}_6, & \mathcal{E}_7 &= \mathcal{F}_3, \\
\mathcal{F}_8 &= \mathcal{E}_4, & \mathcal{E}_8 &= \mathcal{F}_6.
\end{align*}
$$

the recurrence relations (1.3) for variables $x_n$ and $y_n$ coincides with the discrete Painlevé equation given by (A.17). The resulting identification of the symmetry root bases (the surface root bases do not change) is shown in figure 4.

Next we need to realize this change of basis on $\text{Pic}(\mathcal{X})$ by an explicit change of coordinates. For that, it is convenient to first match the parameters between the applied problem and the reference example. This is done with the help of the period map.

2.7. The period map and the identification of parameters

For the root variable parameterization, let us consider a generic point configuration corresponding to the geometry of figure 2. Using the action of the $\text{PGL}_2(\mathbb{C}) \times \text{PGL}_2(\mathbb{C})$ gauge group we can put $H_x - F_1 - F_2 = V(x), H_x - F_3 - F_8 = V(x), H_y - F_1 - F_3 = V(y),$ and $q_7(\infty, 0).$ This leaves the scale freedom on the coordinates $x$ and $y$; we use the scaling in the $x$-coordinate to put $q_3(1, \infty).$ Then our point configuration can be described in terms of generic parameters $c_i$ as

$$
\begin{align*}
q_1(0, c_1), & \ q_2(0, c_2), \ q_3(1, \infty) \leftarrow q_4(u_3 = 0, v_3 = 0) \leftarrow q_5(c_5, 0) \leftarrow q_6(c_6, 0), \\
q_7(\infty, 0) & \leftarrow q_8(U_8 = 0, V_8 = 0).
\end{align*}
$$
with the remaining scaling gauge action in the y-coordinate given by
\[
\left( \begin{array}{c} c_1 \\ c_2 \cdot x \\ c_3 \cdot y \\ c_4 \cdot y \\ c_5 \cdot y \\ c_6 \cdot y \end{array} \right) \sim \left( \lambda c_1 \\ \lambda c_2 \\ \lambda^2 c_3 \\ \lambda^2 c_4 \\ \lambda^2 c_5 \\ \lambda^2 c_6 \right), \quad \lambda \neq 0.
\]

It is immediate that the points \( q_i \) lie on the polar divisor of a symplectic form given in the affine \((x, y)\) chart by \( \omega = k \frac{dx \wedge dy}{x} \). We then have the following lemma.

**Lemma 9.**

(a) The residues of the symplectic form \( \omega = k \frac{dx \wedge dy}{x} \) along the irreducible components of the polar divisor are given by
\[
\begin{align*}
\text{res}_{d_0} \omega &= k \, dy, \\
\text{res}_{d_2} \omega &= 0, \\
\text{res}_{d_4} \omega &= -k \frac{dy}{c_3}, \\
\text{res}_{d_1} \omega &= -k \, dy, \\
\text{res}_{d_3} \omega &= -3k \, dU_4, \\
\text{res}_{d_5} \omega &= k \frac{3 \, dU_4}{c_6}.
\end{align*}
\]

(b) The root variables are given by
\[
a_0 = -kc_2, \quad a_1 = kc_1, \quad a_2 = -kc_1, \quad a_3 = k \left( -c_1 + c_5 + \frac{c_6}{c_5} \right),
\]
and so the root variables are constrained by \( a_1 + a_2 = 0 \). Without loss of generality we can put \( k = -1 \) and then use the \( \lambda \) gauge scaling to ensure the standard normalization condition \( a_0 + a_1 + a_2 + a_3 = 1 \). Then we get
\[
c_2 = a_0, \quad c_1 = -a_1 = a_2, \quad c_5 = c_3(1 - c_1 - c_2 + c_5),
\]
which shows that the application parameters are in fact generic for this point configuration; putting \( n = a_2 \) and \( \alpha = a_0 - a_2 \), as well as denoting \( c_6 \) by \( t \), establishes this equivalence. Note that the parameter evolution is now consistent between the root variables and the application parameters; \( \pi = n + 1, \overline{\pi} = \alpha \), and \( \overline{\pi} = t \).

2.8. The change of coordinates

We are now ready to prove theorem 1. Note that at this point we have not shown that the parameter \( t \) in (A.17) is the same as in (1.3), so we continue working with generic parameters \( c_i \) from the previous section.

**Proof.** (Theorem 5). The proof is standard, and so we only outline the key steps. From the linear change of basis on Pic(\( X \)) given in lemma 8, we see that \( x \) is a projective coordinate on a pencil of \((1, 1)\) curves in the \((q, p)\)-plane passing through the points \( p_3 \) and \( p_6 \), and \( y \) is a projective coordinate on a pencil of \((1, 2)\) curves in the \((q, p)\)-plane passing through the points \( p_1, p_3, p_5, \) and \( p_6 \). The bases for these pencils are given by the curves with affine defining polynomials \( \{ q, qp - a_1 \} \) and \( \{ 1, p(qp + t) - a_1 \} \), i.e.,
\[
\begin{align*}
x &= \frac{Aq + B(qp - a_1)}{Cq + D(qp - a_1)}, \\
y &= \frac{K + Lp(qp + t) - a_1}{M + Np(qp + t) - a_1}.
\end{align*}
\]
Using the correspondence between the exceptional divisor classes for \( F_i, \ i = 1, 2, 3, 4, 7, 8 \) allows us to fix the values of the coefficients \( A, \ldots, N \) to get \( x = \frac{2(p + t) + C}{4(p + t)} \) and \( y = \frac{2(p + t) + D}{4(p + t)} \). Moreover, the correspondence \( E_7 - E_8 = H_4 + H_p - E_3 - E_4 - E_5 - E_7 \) imposes the
\(a_1 + a_2 = 0\) constraint, and the condition that \(F_3 - F_6 = F_7 - F_8\) shows that \(c_5 = t\), as expected. The inverse change of variables is obtained along the same lines. □

**Acknowledgments**

Yang Chen and Hu Jie were supported by the Macau Science and Technology Development Fund under Grant Numbers FDCT 130/2014/A3, FDCT 023/2017/A1 and by the University of Macau under Grant Numbers MYRG 2014-00011-FST, MYRG 2014-00004-FST. Part of this work was done when Anton Dzhament visited Shanghai University and the University of Macau and he would like to thank both of these Universities for their support and hospitality. We also thank Alfred Ramani, Tomoyuki Takenawa, and Ralph Willox for helpful comments and discussions.

**Appendix A. Discrete Painlevé equations in the d-P\((A_3^{(1)}/D_5^{(1)})\) family**

To make this paper self-contained, we collect in this appendix A some of the basic facts about the geometry of the \(D_5^{(1)}\)-family of Sakai surfaces and some standard discrete Painlevé equations associated with this surface family. The computations here are standard (see [DFS19], [DT18], [KNY17]) and are mostly omitted. We use \((q, p)\)-coordinates for the standard example and follow the standard reference [KNY17] for the choice of the standard point configuration and the root bases.

**A.1. The point configuration**

We start with the root basis of the surface sub-lattice that is given by the classes \(\delta_i\) of the irreducible components of the anti-canonical divisor

\[
\delta = -K_X = 2\mathcal{H}_f + 2\mathcal{H}_g - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3 - \mathcal{E}_4 - \mathcal{E}_5 - \mathcal{E}_6 - \mathcal{E}_7 - \mathcal{E}_8 \\
= \delta_0 + \delta_1 + 2\delta_2 + 2\delta_3 + \delta_4 + \delta_5.
\]

The intersection configuration of those roots is given by the Dynkin diagram of type \(D_5^{(1)}\), as shown on figure A1.

\[
\begin{align*}
\delta_0 &= \mathcal{E}_1 - \mathcal{E}_2, & \delta_3 &= \mathcal{H}_p - \mathcal{E}_6 - \mathcal{E}_7, \\
\delta_1 &= \mathcal{E}_3 - \mathcal{E}_4, & \delta_4 &= \mathcal{E}_5 - \mathcal{E}_6, \\
\delta_2 &= \mathcal{H}_q - \mathcal{E}_4 - \mathcal{E}_3, & \delta_5 &= \mathcal{E}_7 - \mathcal{E}_8.
\end{align*}
\]

**Figure A1.** The surface root basis for the standard d-\(D_5^{(1)}\) point configuration.

Using the action of the \(\text{PGL}_2(\mathbb{C}) \times \text{PGL}_2(\mathbb{C})\) gauge group we can put divisors \(d_2\) and \(d_3\), with \(\delta_i = [d_i]\), to be

\[d_2 = V(Q) = \{q = \infty\}, \quad d_3 = V(P) = \{p = \infty\}.\]
This reduces the gauge group action to that of a four-parameter subgroup, \((q, p) \mapsto (\lambda q + \mu, \zeta p + \xi)\). The corresponding point configuration and the Sakai surface are shown on figure A2.

This point configuration can be parameterized by eight parameters \(b_1, \ldots, b_8\) as follows:

\[
\begin{align*}
    &p_1(\infty, b_1) \leftarrow p_2(\infty, b_1; q(p-b_1) = b_2), \\
    &p_3(\infty, b_3) \leftarrow p_4(\infty, b_3; q(p-b_3) = b_4), \\
    &p_5(b_5, \infty) \leftarrow p_6(b_5, \infty; (q-b_5)p = b_6), \\
    &p_7(b_7, \infty) \leftarrow p_8(b_7, \infty; (q-b_7)p = b_8).
\end{align*}
\]

The four-parameter gauge group above acts on these configurations via

\[
\begin{pmatrix} b_1 & b_2 & b_3 & b_4 \q^p \\ b_5 & b_6 & b_7 & b_8' \end{pmatrix} \sim \begin{pmatrix} \zeta b_1 + \xi & \lambda \zeta b_2 & \zeta b_3 + \xi & \lambda \zeta b_4 + \lambda q + \mu \\ \lambda b_5 + \mu & \lambda b_6 & \lambda b_7 + \mu & \lambda \zeta b_8' + \zeta g + \xi \end{pmatrix}, \quad \lambda, \zeta \neq 0,
\]

and so the true number of parameters is four. The correct gauge-invariant parameterization is given by the root variables that we now describe.

### A.2. The period map and the root variables

To define the root variables we begin by choosing a root basis in the symmetry sub-lattice \(Q = \Pi(R^+) \triangleleft \text{Pic}(\Lambda)\) and defining the symplectic form \(\omega\) whose polar divisor \(-K_X\) is the configuration of \(-2\)-curves shown on figure A2. For the symmetry root basis we take the same basis as in [KNY17], see figure A3.

\[
\begin{align*}
    \alpha_0 &= \mathcal{H}_p - \varepsilon_1 - \varepsilon_2, & \alpha_2 &= \mathcal{H}_p - \varepsilon_3 - \varepsilon_4, \\
    \alpha_1 &= \mathcal{H}_q - \varepsilon_5 - \varepsilon_6, & \alpha_3 &= \mathcal{H}_q - \varepsilon_7 - \varepsilon_8. \\
    \delta &= \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3.
\end{align*}
\]

\[
\text{Figure A3. The standard root basis for the } d\text{-}P \left( A^{(1)}_1 \right) \text{ symmetry sub-lattice.}
\]

A symplectic form \(\omega \in -K_X\) such that \([\omega] = \delta_0 + \delta_1 + 2\delta_2 + 2\delta_3 + \delta_4 + \delta_5\) can be given in local coordinate charts as
\[ \omega = kdq \wedge dp = -k \frac{dQ \wedge dp}{Q^2} = -k \frac{dq \wedge dp}{p^2} = k \frac{dQ \wedge dP}{Q^2 p^2} = -k \frac{du_i \wedge dv_j}{u_i} = -k \frac{dU_j \wedge dV_j}{V_j}, \]

(A.4)

where, as usual, \( Q = 1/q, P = 1/p \) are the coordinates centered at infinity, the blowup coordinates \( u_i, v_j \) at the points \( p_i, i = 1, 3, \) are given by \( Q = u_i, p = b_i + u_i v_i, \) and the blowup coordinates \( (U_j, V_j) \) at the points \( p_j, j = 5, 7, \) are given by \( q = b_j + U_j V_j \) and \( P = V_j; k \) is some non-zero proportionality constant that we normalize later. Then we have the following lemma.

**Lemma 10.**

(a) The residue of the symplectic form \( \omega \) along the irreducible components of the polar divisor is given by

\[ \text{res}_{\alpha_i} \omega = -k \, dv_1, \quad \text{res}_{\alpha_2} \omega = -k \, dv_3, \quad \text{res}_{\alpha_3} \omega = \text{res}_{\beta_1} \omega = 0, \quad \text{res}_{\beta_2} \omega = k \, dU_5, \quad \text{res}_{\beta_3} \omega = k \, dU_7. \]

(A.5)

(b) The root variables \( a_i \) are given by

\[ a_0 = kb_2, \quad a_1 = -kb_6, \quad a_2 = kb_4, \quad a_3 = -kb_8. \]

(A.6)

It is convenient to take \( k = -1. \) We can then use the gauge action (A.2) to normalize \( b_3 = b_5 = 0, b_7 = 1, \) and \( \chi(\delta) = a_0 + a_1 + a_2 + a_3 = 1. \) In view of the relation of this example to differential Painlevé equations, it is also convenient to denote \( b_1 \) by \( -t. \) Then we get the following parameterization of this point configuration in terms of root variables:

\[ b_1 = -t, \quad b_2 = -a_0, \quad b_3 = 0, \quad b_4 = -a_2, \quad b_5 = 0, \quad b_6 = a_1, \quad b_7 = 1, \quad b_8 = a_3. \]

(A.7)

Note that if we use the notation

\[ p_{12} \left( \frac{1}{\varepsilon}, -t - \varepsilon a_0 \right), \quad p_{14} \left( \frac{1}{\varepsilon}, -\varepsilon a_2 \right), \quad p_{56} \left( a_1 \varepsilon, \frac{1}{\varepsilon} \right), \quad p_{78} \left( 1 + a_3 \varepsilon, \frac{1}{\varepsilon} \right), \]

and impose the normalization \( a_0 + a_1 + a_2 + a_3 = 1, \) we get exactly the parameterization of the point configuration in section 8.2.18 of [KNY17].

**A.3. The extended affine Weyl symmetry group**

We now describe the birational representation of the extended affine Weyl symmetry group \( \hat{W} \left( A_3^{(1)} \right) = \text{Aut} \left( A_3^{(1)} \right) \ltimes W \left( A_3^{(1)} \right), \) which is a semi-direct product of the usual affine Weyl group \( W \left( A_3^{(1)} \right) \) and the group of Dynkin diagram automorphisms \( \text{Aut} \left( A_3^{(1)} \right) \cong \mathbb{Z}_4. \)

The abstract affine Weyl group \( W \left( A_3^{(1)} \right) \) is defined in terms of generators \( w_i = w_{\alpha_i} \) and relations that are encoded by the affine Dynkin diagram \( A_3^{(1)}, \)

\[ W \left( A_3^{(1)} \right) = W \left( \begin{array}{ccc}
\alpha_0 & & \\
& \alpha_1 & \\
& & \alpha_3
\end{array} \right) = \left\langle w_0, \ldots, w_8 \left| w_i^2 = 1, \quad w_i w_j = w_j w_i \quad \text{when} \quad \begin{array}{c}
\varepsilon \alpha_i = 0, \\
\varepsilon \alpha_j = 0,
\end{array} \right. \right\rangle. \]
The natural action of this group on $\text{Pic}(\mathcal{X})$ is given by reflections in the roots $\alpha_i$,

$$w_i(C) = w_{\alpha_i}(C) = C - 2\frac{C \cdot \alpha_i}{\alpha_i \cdot \alpha_i} = C + (C \cdot \alpha_i)\alpha_i, \quad C \in \text{Pic}(\mathcal{X}), \quad (A.8)$$

which can be extended to an action on point configurations by elementary birational maps (which lifts to isomorphisms $w_i: \mathcal{X}_b \rightarrow \mathcal{X}_b$ on the family of Sakai’s surfaces), this is known as a birational representation of $W(A^{(1)}_3)$.

**Theorem 11.** Reflections $w_i$ on $\text{Pic}(\mathcal{X})$ are induced by the elementary birational mappings given below, and also denoted by $w_i$ on the family $\mathcal{X}_b$. To ensure the group structure, we require that each mapping preserves our normalization, and so it is enough to describe the mappings in terms of the root variables (note that the parameter $t$ can also change when we consider the Dynkin diagram automorphisms, so it is convenient to include it among the root variables):

$$w_0: \begin{pmatrix} a_0 & a_1 & a_2 \\ a_2 & a_3 & t; p \end{pmatrix} \mapsto \begin{pmatrix} -a_0 & a_0 + a_1 & a_2 \\ a_2 & a_0 + a_3 & t; q + \frac{a_0}{p + t} \end{pmatrix}, \quad (A.9)$$

$$w_1: \begin{pmatrix} a_0 & a_1 & a_2 \\ a_2 & a_3 & t; q \end{pmatrix} \mapsto \begin{pmatrix} a_0 + a_1 & a_2 & -a_1 \\ a_1 + a_2 & a_3 & t; p - \frac{a_1}{q} \end{pmatrix}, \quad (A.10)$$

$$w_2: \begin{pmatrix} a_0 & a_1 & a_2 \\ a_2 & a_3 & t; q \end{pmatrix} \mapsto \begin{pmatrix} a_0 + a_1 & a_2 & a_3 \\ -a_2 & a_2 + a_3 & t; q + \frac{a_2}{p} \end{pmatrix}, \quad (A.11)$$

$$w_3: \begin{pmatrix} a_0 & a_1 & a_2 \\ a_2 & a_3 & t; q \end{pmatrix} \mapsto \begin{pmatrix} a_0 + a_3 & a_1 & a_2 \\ a_2 + a_3 & -a_1 & t; p - \frac{a_3}{q - 1} \end{pmatrix}. \quad (A.12)$$

It is clear that the group of Dynkin diagram automorphisms $\text{Aut}(A^{(1)}_3) \cong D_4$, so we only describe two generators $\sigma_1, \sigma_2$, as well as one more automorphism $\sigma_3$ that we need.

**Theorem 12.** Consider the automorphisms $\sigma_1, \ldots, \sigma_3$ of $\text{Aut}(A^{(1)}_3)$ that act on the symmetry and the surface root bases as follows (here we use the standard cycle notations for permutations):

$$\sigma_1 = (\alpha_0\alpha_3)(\alpha_1\alpha_2) = (\delta_0\delta_3)(\delta_1\delta_4)(\delta_2\delta_5), \quad \sigma_2 = (\alpha_0\alpha_2) = (\delta_0\delta_1), \quad \sigma_3 = (\alpha_1\alpha_3) = (\delta_4\delta_5). \quad (A.13)$$

Then $\sigma_i$ act on the Picard lattice as

$$\sigma_1 = (E_1E_i)(E_2E_8)(E_3E_5)(E_4E_6)w_p, \quad \sigma_2 = (E_1E_3)(E_2E_4), \quad \sigma_3 = (E_5E_7)(E_6E_8),$$

where $w_p$ is a reflection (A.8) in the root $\rho = H_q - H_p$ (note also that a transposition $(E_iE_j)$ is induced by a reflection in the root $E_i - E_j$). The induced elementary birational mappings are then given by the following expressions:

$$\sigma_1: \begin{pmatrix} a_0 & a_1 & a_2 \\ a_2 & a_3 & t; p \end{pmatrix} \mapsto \begin{pmatrix} a_3 & a_2 & a_1 \\ t; -\frac{p}{q} \end{pmatrix}, \quad (A.14)$$
\[ \sigma_2: \left( \begin{array}{ccc} a_0 & a_1 & t \\ a_2 & a_3 & p \end{array} \right) \mapsto \left( \begin{array}{ccc} a_2 & a_1 & -t \\ a_0 & a_3 & p \end{array} \right) - \left( \begin{array}{ccc} q & 0 & 0 \\ 0 & 1 & 0 \end{array} \right) \cdot \left( \begin{array}{ccc} a_0 & a_1 & t \\ a_2 & a_3 & p \end{array} \right) \] (A.15)

\[ \sigma_3: \left( \begin{array}{ccc} a_0 & a_1 & t \\ a_2 & a_3 & p \end{array} \right) \mapsto \left( \begin{array}{ccc} a_0 & a_1 & t \\ a_2 & a_3 & p \end{array} \right) - \left( \begin{array}{ccc} 1 - q & 0 & 0 \\ 0 & 1 & 0 \end{array} \right) \cdot \left( \begin{array}{ccc} a_0 & a_1 & t \\ a_2 & a_3 & p \end{array} \right) \] (A.16)

Finally, the semi-direct product structure is defined by the action of \( \sigma \in \text{Aut}(A^{(1)}_3) \) on \( W(A^{(1)}_3) \) via \( w_{\sigma a}(t) = \sigma w_a \sigma^{-1} \).

### A.4. Some standard discrete d-P\((A^{(1)}_3) / D^{(1)}_5\) equations

There are infinitely many different discrete Painlevé equations of the same type corresponding to the non-conjugate translations in the affine symmetry sub-lattice \( Q \). Of those, we are interested in two particular equations that correspond to short translation vectors. One is equation (8.23) in \cite{KNY17}, section 8.1.17, the other is the so-called d-P\(\text{IV} \) equation in \cite{Sak01}, which also appears in a slightly different form (2.33) and (2.34) in \cite{Sak07}. We label these equations by \cite{HTT1} and \cite{T001} respectively, based on the induced action of the dynamics on the symmetry roots (see below), which is unambiguous. In the above references these equations are presented in a geometric way as mappings, similar to our approach. However, both classes of equations were obtained earlier by Basil Grammaticos, Alfred Ramani, and their collaborators using the singularity confinement approach; in their papers these equations are presented as recurrences with particular coefficient evolution. Equation \cite{HTT1} first appeared in \cite{GNP+94} (where it was shown that this equation actually has d-P\(\text{III} \) and not d-P\(\text{IV} \) as a continuous limit) and equation \cite{T001} first appeared in \cite{GORS98}; see also \cite{TGR02} where equations (3.1) and (3.2) is essentially the mapping (A.17) and equations (3.24a) and (3.24b) is essentially the mapping (A.21)\(^4\).

Note that equations \cite{HTT1} and \cite{T001} are not equivalent—this can be seen, for example, from the length of the corresponding words in the extended affine Weyl group, or from the lengths of the corresponding translations, or, probably in the simplest possible way, by computing the Jordan form of the matrix description of the evolution on \( \text{Pic}(\mathcal{X}) \).

#### A.4.1. The [HTT1] discrete Painlevé equation on the \( D^{(1)}_5 \) surface

In \cite{KNY17}, the standard example of a discrete Painlevé equation on the \( D^{(1)}_5 \)-surface is given in section 8.1.17 equation (8.23), and it has the following additive form, when written in coordinates \( (q, p) \):

\[ \varphi + q = 1 - \frac{a_2}{p} - \frac{a_0}{p + t}, \quad p + p = -t + \frac{a_1}{q} + \frac{a_3}{q - 1} \] (A.17)

with the root variable evolution and normalization given by

\[ a_0 = a_0 + 1, \quad a_1 = a_1 - 1, \quad a_2 = a_2 + 1, \quad a_3 = a_3 - 1, \quad a_0 + a_1 + a_2 + a_3 = 1. \] (A.18)

For this equation, the geometry of the corresponding point configuration is shown on figure A2, with the parameterization by the root variables is given by (A.7). From the root variable evolution (A.18) we immediately see that the corresponding translation on the root lattice is

\[ \varphi_\alpha: \alpha = (a_0, a_1, a_2, a_3) \mapsto \varphi_\alpha(\alpha) = \alpha + (-1, 1, -1, 1) \delta, \] (A.19)

\(^4\)We thank A Ramani for his help with historical references.
which explains our labeling for this equation (we use $\overline{T}$ instead of $-1$ for compactness).
Using the standard techniques, see [DT18] for a detailed example, we get the following decomposition of $\varphi$ in terms of the generators of $\tilde{W}(\tilde{A}_3^{(1)})$:

$$\varphi = \sigma_3 \sigma_2 w_3 w_1 w_2 w_0.$$

(A.20)

Note that equations (A.17) naturally define two half-maps, $\varphi_1 : (q, p) \to (q, -p)$ and $\varphi_2 : (q, p) \to (q, p)$ (the additional negative sign here is related to the Möbius group gauge action as explained in [DFS19, section 2.9]), and the mapping $\varphi$ that we are interested in is $\varphi = \varphi_1^{−1} \circ \varphi_1$. These individual mappings decompose as $\varphi_1 = \sigma_3 w_2 w_0$ and $\varphi_3 = \sigma_2 w_3 w_1$.

A.4.2. The $[1001]$ discrete Painlevé equation on the $D_{15}^{(1)}$ surface. In [Sak01], the following mapping $\phi : (f, g) \to (\overline{f}, \overline{g})$, written in the multiplicative–additive form, is called a d-$P_{IV}$ equation on the $D_{15}^{(1)}$ surface:

$$\overline{f} = \frac{s\overline{g}}{(\overline{g} + a_3 + \lambda)(\overline{g} + a_0 + \lambda)}, \quad \overline{g} + g = \frac{s}{f} + \frac{a_1 + a_0}{1 - f} - \lambda + a_3 - a_0,$$

(A.21)

where $\lambda = a_0 + a_1 + a_2 + a_3$ (without loss of generality it can be normalized to $\lambda = 1$), and the root variable evolution is given by $\alpha_0 = a_0 + \lambda$ and $\alpha_3 = a_3 - \lambda$. From the root variable evolution we see that the corresponding translation on the root lattice is

$$\phi_* : \alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3) \mapsto \varphi_*(\alpha) = \alpha + (-1, 0, 0, 1)\delta.$$

(A.22)

This map can be written in terms of generators as

$$\phi = \sigma_3 \sigma_1 w_2 w_1 w_0 = w_3 w_2 w_1 \sigma_3 \sigma_1,$$

(A.23)

which is the same as given in Sakai’s paper. However, the geometry of that example is slightly different from our reference model on figure A2 and is given on figure A4. This geometry can be matched to the standard one with the change of basis on $\text{Pic}(\mathcal{X})$ given by

$$\mathcal{H}_f = \mathcal{H}_q + \mathcal{H}_p - \mathcal{E}_3 - \mathcal{E}_2, \quad \mathcal{H}_g = \mathcal{H}_q + \mathcal{H}_p - \mathcal{K}_6 - \mathcal{K}_8,$$
$$\mathcal{H}_g = \mathcal{H}_q + \mathcal{H}_p - \mathcal{E}_3 - \mathcal{E}_7, \quad \mathcal{H}_p = \mathcal{H}_f + \mathcal{H}_g - \mathcal{K}_1 - \mathcal{K}_6,$$
$$\mathcal{K}_1 = \mathcal{H}_q - \mathcal{E}_1, \quad \mathcal{E}_1 = \mathcal{H}_f + \mathcal{H}_g - \mathcal{K}_1 - \mathcal{K}_6 - \mathcal{K}_8,$$
$$\mathcal{K}_2 = \mathcal{E}_3, \quad \mathcal{E}_2 = \mathcal{H}_g - \mathcal{K}_6,$$
$$\mathcal{K}_3 = \mathcal{E}_4, \quad \mathcal{E}_3 = \mathcal{K}_2.$$
\[ K_4 = \mathcal{E}_5, \quad \mathcal{E}_4 = K_3, \]
\[ K_5 = \mathcal{E}_6, \quad \mathcal{E}_5 = K_4, \]
\[ K_6 = \mathcal{H}_q + \mathcal{H}_p - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_7, \quad \mathcal{E}_6 = K_5, \]
\[ K_7 = \mathcal{E}_8, \quad \mathcal{E}_7 = \mathcal{H}_f - K_6, \]
\[ K_8 = \mathcal{H}_p - \mathcal{E}_1, \quad \mathcal{E}_8 = K_7. \]

Note that this change of basis is chosen in such a way as to match the root variables between the two examples, however the parameters \( s \) and \( t \) differ by a sign, \( s = -t \). The corresponding change of variables is given by

\[
\begin{align*}
    f(q, p) &= -\frac{p + t}{(q - 1)(p + t) + a_0}, \\
    g(q, p) &= (q - 1)(p + t),
\end{align*}
\]

\[
\begin{align*}
    q(f, g) &= 1 - \frac{g}{f(g + a_0)}, \\
    p(f, g) &= s - f(g + a_0).
\end{align*}
\]

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