Determination of Spatial-Temporal Characteristics of Sea Wind Waves

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Abstract. The nonlinearity of sea surface waves is an important property of them. Despite the fact that the nonlinearity is weak, it must be taken into account when solving many applied and fundamental problems. In this work, we study the effect of nonlinearity on the space-time structure of sea waves. The results of determining the spatial-temporal structure from the data of measurements by an array of wave sensors and from optical images of the sea surface are compared. It is shown that there is a discrepancy between the wave characteristics determined in different ways. Partially, but not completely, these discrepancies are explained by incomplete consideration of the directional spreading function.

1. Introduction

In recent years, much attention has been paid to the study of the nonlinearity of sea surface waves. First of all, this is due to the need to clarify the nature of the impact of waves on ships, offshore platforms and coastal structures [1, 2]. Deviations from the linear wave field are also a source of errors in altimetric measurements from spacecraft [3, 4], they must also be taken into account when reconstructing the spectrum of the sea surface from remote sensing data [5].

The creation of optical equipment with a high spatial resolution, which allows obtaining images with a small temporal lag, led to the emergence of a new method for reconstructing the surface current velocity. Within the framework of this method, it is assumed that nonlinear effects can be neglected and that the deviations of the estimates of the phase velocities of surface waves from the values obtained from the dispersion equation are due only to the current. The phase velocities are calculated by cross-spectral analysis from two consecutive images of the sea surface [6]. It is also assumed that in the absence of current on scales close to the scale of dominant waves, the space-time characteristics of surface waves correspond to the linear dispersion relation for gravitational waves

\[ \omega^2 = gk \]  

where \( \omega \) is the cyclic frequency, \( g \) is the gravitational acceleration; \( k \) is wave number.

The accuracy of measuring the current velocity is determined by how much the nonlinear effects in sea waves distort the dispersion relation (1). According to optical sensing data [6], the measured phase velocities are in good agreement with the theoretical values \( C_T \) following from (1)

\[ C_T = \frac{\omega}{k} = \frac{\sqrt{g/k}}{g/\omega} \]  

where \( C_T \) is the phase velocity of the surface wave.
Earlier, experiments on the remote determination of the space-time characteristics of surface waves were carried out using a digital airborne camera [7]. In these experiments, the linear dispersion relation was confirmed.

A series of in situ experiments performed using an array of spaced wave sensors showed deviations from the dispersion relation (1) [8]. The measured phase velocities exceeded their theoretical values $C_T$. This was interpreted as a manifestation of the nonlinearity of sea waves. With increasing distance between the sensors, the coherence level decreased faster than it follows from the linear wave model [9].

There are two groups of results of determining the spatial and temporal characteristics of surface waves, which contradict each other. According to the measurements of the phase velocities from images of the sea surface in the absence of a current, the phase velocities correspond to the theoretical value (2). At the same time, the results of direct in situ measurements indicate that there are noticeable deviations from the dispersion relation, which are interpreted as a manifestation of nonlinearity.

2. Spatial array measurements
Consider a spatially uniform, stationary wave field. Such a field can be described by a three-dimensional spectrum obtained by the Fourier transform of the space-time covariance function of the sea surface displacements $Z(\vec{r}, t) - \xi(\vec{x}, t_0) - \xi(\vec{x} + \vec{r}, t_0 + t)$ [10]

$$X(\vec{k}, \omega) = (2\pi)^3 \int Z(\vec{r}, t) \exp(-i\vec{k} \cdot \vec{r} - i\omega t) d\vec{r} dt$$  \hspace{1cm} (3)

where $\vec{r}$ is the vector connecting two points in space, $t$ is time; $\xi$ is wave elevation of the sea surface, $\vec{x}$ is spatial coordinate; $t_0$ is time shift; $\vec{k}$ is wave vector; $\omega$ is circular frequency.

The cross-spectrum of wave displacements at two points connected by a vector is represented as

$$\Phi_A(\vec{r}_0, \omega) = (2\pi)^3 \int Z(\vec{r}, t) \exp(i\omega t) dt$$  \hspace{1cm} (4)

Taking into account (3), we obtain

$$\Phi_A(\vec{r}_0, \omega) = \int X_A(\vec{k}, \omega) \exp(i\vec{k} \cdot \vec{r}_0) d\vec{k}$$  \hspace{1cm} (5)

Here and further, the lower indices $A$ and $I$ denote the spatio-temporal characteristics obtained from measurements by an array of wave sensors and from optical images of the sea surface, respectively.

If the waves correspond to the dispersion relation (1), then

$$X_A(\vec{k}, \omega) = \Xi_A(\vec{k}) \delta(\omega - \sqrt{gk})$$  \hspace{1cm} (6)

where $\Xi_A(\vec{k})$ is the spectrum of wave vectors, $\delta$ is the Dirac delta.

Let’s pass into the polar coordinate system. The transition procedure is described by the equation [10]

$$\Phi_A(\vec{r}_0, \omega) = \int \Psi_A(\vec{k}, \alpha) \delta(\omega - \sqrt{gk}) \exp(ikr_0 \cos \alpha) k dk d\alpha$$  \hspace{1cm} (7)

where the direction $\alpha = 0$ coincides with the direction of the vector $\vec{r}_0$. For gravitational waves satisfying (1), we obtain

$$\Phi_A(\vec{r}_0, \omega) = \int \Psi_A(\omega, \alpha) \exp\left(i \frac{\omega^2 r_0}{g} \cos \alpha \right) d\alpha$$  \hspace{1cm} (8)

The frequency-angle spectrum is usually represented as
where \( S_{\omega}(\omega) \) is the frequency spectrum; \( \Theta(\omega, \alpha) \) is the directional spreading function, satisfying the normalization condition

\[
\int_0^{2\pi} \Theta(\omega, \alpha) d\alpha = 1
\]

Substituting (9) in (8), we get

\[
\Phi_A(r_0, \alpha, \omega) = \int S_{\omega}(\omega) \Theta(\omega, \alpha) \exp\left( i \frac{\omega^2 r_0}{g} \cos \alpha \right) d\alpha .
\]

The phase spectrum is determined by the ratio of the imaginary and real parts of the cross-spectrum

\[
F_A(r_0, \alpha, \omega) = \frac{\text{Im} \left[ \int \Theta(\omega, \alpha) \sin \left( \frac{\omega^2 r_0}{g} \cos \alpha \right) d\alpha \right]}{\text{Re} \left[ \int \Theta(\omega, \alpha) \cos \left( \frac{\omega^2 r_0}{g} \cos \alpha \right) d\alpha \right]}
\]

The phase velocity is related to the phase shift by the ratio

\[
C_A(\omega) = \omega r_0 / F_A(r_0, \alpha, \omega)
\]

Thus, the phase velocity \( C_A(\omega) \) when measured with an array of wave sensors depends on the angular distribution function. The phase shift at a fixed frequency depends non-linearly on the distance and, accordingly, the calculated values of the phase velocity also depend on the distance.

Another function that determines the space-time characteristics of the wave field is the quadratic coherence function

\[
R_A^2(r_0, \alpha, \omega) = \left| \frac{\Phi_A(r_0, \alpha, \omega)}{S_{\omega}(\omega)} \right|^2 = \left| \int \Theta(\omega, \alpha) \exp\left( i \frac{\omega^2 r_0}{g} \cos \alpha \right) d\alpha \right|^2
\]

The function \( R_A^2(r_0, \alpha, \omega) \) is a measure of the stability of phase relations [11].

3. Optical measurements
As in the previous section, we will consider a linear wave field, which is described by a spectrum \( X(k, \omega) \). Let’s construct a cross-spectrum of two images obtained with an temporal lag equal \( \Delta t \)

\[
\Phi_i(k, \Delta t) = \int X(k, \omega) \exp(i\omega \Delta t) d\omega = \int \Xi_i(k) \delta(\omega) \exp(i\omega \Delta t) d\omega
\]

In the polar coordinate system, the two-dimensional spectrum \( \Psi_i(k, \alpha) \) can be represented, like (9), in the form

\[
\Psi_i(k, \alpha) = S_i(k) \Theta(k, \alpha)
\]

where \( S_i(k) \) is the one-dimensional spectrum of wave numbers, \( \Theta(k, \alpha) \) is directional spreading function corresponding to \( \Theta(\omega, \alpha) \). Taking into account

\[
\omega \Delta t = k C_i \Delta t = \frac{2\pi}{\lambda} \frac{\Delta t}{L}
\]
where $L$ is the distance that the wave crest with the wave number $k$ passes in a time interval equal to $\Delta t$, we get

$$\Phi_I(k, \Delta t) = S_A(k)\Theta(k, \alpha)\exp\left(i\frac{2\pi}{\lambda} L\right)$$  \hspace{1cm} (18)

From where follows

$$F_I(k, \Delta t) = \frac{2\pi}{\lambda} L$$  \hspace{1cm} (19)

It is important to note here that the phase spectrum varies linearly with distance. The quadratic coherence function is

$$R^2_I(k, \Delta t) = \frac{|\Phi_I(k, \Delta t)|^2}{S^2_A(\omega)} = 1$$  \hspace{1cm} (20)

4. Numerical simulation

It follows from the above analysis that the cross-spectrum of surface displacements constructed from optical images of a linear wave field does not depend on the angular distribution function of the wave energy. The phase velocity satisfies the equality $C_I(k) = C_T(k)$.

In a two-dimensional linear wave field, in which the function $\Theta(\omega, \alpha)$ cannot be described by the Dirac delta, the phase spectrum $\Phi_A(\vec{r}_0, \omega)$ obtained from the data of an array of wave sensors depends on the directional spreading function. As a consequence $C_A(\omega) \neq C_T(\omega)$. This is due to the fact that the contribution to the phase spectrum is given by waves from different directions and having different projections of the phase velocity on the direction of the vector $\vec{r}_0$. A feature of the results of calculating the phase spectrum $\Phi_A(\vec{r}_0, \omega)$ is that its values change nonlinearly with a change in the distance between the sensors, therefore, the estimates of the phase velocity also depend on $r_0$ [12].

Let’s introduce a dimensionless distance $\varepsilon = r_0/\lambda$, i.e. distance measured in wavelengths. For gravitational waves, we obtain from the dispersion relation (1) $\omega^2 r_0/g = kC\Delta t = 2\pi \varepsilon$.

The dependences of the space-time characteristics of the linear wave field on the dimensionless parameter obtained within the framework of two approaches are shown in figure 1. In the calculations, the directional spreading function was used, which takes into account the dependence of the width of the angular distribution on the stage of development of the wave field. The stage of development is usually characterized by a dimensionless parameter $\tau = C_0/U_{10}$, where $C_0$ is the phase velocity of the dominant waves, $U_{10}$ is the wind speed at an height of 10 m. The directional spreading function is based on the results of the international experiment “Joint North Sea Wave Project” (JONSWAP) [13].

It is described by the equation

$$\Theta(k, \alpha) = N \cos^2(s\epsilon) \left(\frac{\alpha - \alpha_0}{2}\right)$$  \hspace{1cm} (21)

where $N$ is the normalizing coefficient, $s$ is the dimensionless parameter that determines the width of the angular distribution, $\alpha_0$ is the main direction of wave propagation. The functions shown in Figure 1 are obtained for $\alpha_0 = 0$.

As the wave field develops, it becomes more narrowly directed. The simulation was carried out for dominant waves at $\tau = 0.83$ and $\tau = 1.5$. The smaller the value $\tau$, the later the stage of development it corresponds to. For fully developed waves $\tau = 0.83$. It follows from Figure 1 that the deviations of the phase velocity values calculated from the in situ measurements from the theoretical value are
significant even on the scale of dominant waves. At higher frequencies (for shorter wavelengths), the angular distribution of the wave energy expands and the deviations increase.

![Figure 1](image1.png)

**Figure 1.** The dependence of coherence $R^2$, phase shift $F$, and relative phase velocity on the dimensionless distance $C/C_T$. Curve 1 is calculations based on the cross-spectrum $\Phi_f$, curves 2 and 3 are calculations based on the cross-spectrum $\Phi_A$ at two stages of development of the wave field $\tau = 1.5$ and $\tau = 0.83$, respectively.

On the scales of dominant waves, the differences between the values of $C_A$ and $C_T$ caused by the angular distribution are not large. However, there remains one more experimentally established fact, which has not yet received proper attention. As shown in Figure 2, on the scale of dominant waves, the coherence decreases faster than it follows from models that take into account the directional spreading function [14]. Here the calculations were carried out according to three models, in addition to model (21), well-known models proposed in [15, 16] were used.

![Figure 2](image2.png)

**Figure 2.** Measured and model values of the quadratic coherence function $R^2$ in the direction of wave propagation, constructed as a function of the dimensionless distance $\varepsilon$. Points are experimental data, curves 1-3 are simulation results based on known directional spreading function [13, 15, 16].

5. **Conclusion**

There are two groups of results for determining the spatio-temporal characteristics of surface waves, obtained from measurements of wave sensors array and from optical images of the sea surface. These results contradict each other. In the first case, noticeable deviations from the values following from the linear dispersion relation are observed. The measured phase velocity significantly exceeds the theoretical value; the decrease in the level of coherence occurs much faster than it follows from the linear theory. In the second case, there is a fairly good correspondence of the wave characteristics to the linear theory, which allows us to solve a number of oceanographic problems, in particular, to remotely determine the speed of surface currents.
It is shown that this contradiction can be caused by the fact that, in situ measurements, the contribution to the cross-spectrum of the surface elevations at two points connected by a vector is made by waves arriving from different directions and having different projections of the phase velocity onto the direction of the vector \( \vec{r}_0 \). It is shown that this contradiction can be caused by the fact that when measuring the wave sensors array, the contribution to the cross-spectrum of surface elevations at two points connected by the vector \( \vec{r}_0 \) is given by waves that come from different directions and have different projections of the phase velocity on the direction of the vector \( \vec{r}_0 \). Calculation of space-time characteristics from optical images sea surface eliminates the effects associated with the angular distribution of wave energy.

Currently, optical images obtained from spacecraft allow us to estimate the characteristics of sea waves at scales close to the scale of dominant waves. Deviations from the linear theory at these scales are caused by effects related to the group structure of surface waves, as well as their kinematic nonlinearity. Thus, along with the study of surface currents based on optical methods, it is possible to study the nonlinearity of sea waves, which is their fundamental property.

6. References

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