From the work of Morgan [M], we know that the fundamental group of a complex algebraic variety carries a mixed Hodge structure, which really means that a certain linearization of it does. This linearization, called the Malčev or pro-unipotent completion destroys the group completely in some cases; for example if it were perfect. So a natural question is whether can one give a Hodge structure on a larger chunk of the fundamental group. There have been a couple of approaches to this. Work of Simpson [Si] continued by Katzarkov, Pantev, Toen [KPT1] has shown that one has a weak Hodge-like structure (essentially an action by \( \mathbb{C}^* \) viewed as a discrete group) on the entire pro-algebraic completion of the fundamental group when the variety is smooth projective. Hain [H2] refining his earlier work [H1], has shown that a Hodge structure of a more conventional sort exists on the so called relative Malčev completions (under appropriate hypotheses). In this paper, I want to propose a third alternative. I define a quotient of the pro-algebraic completion called the Hodge theoretic fundamental group \( \pi_{\text{hodge}}^1(X,x) \) as the Tannaka dual to the category local systems underlying admissible variations of mixed Hodge structures on \( X \), or in more prosaic terms the inverse limit of Zariski closures of their associated monodromy representations. This carries a nonabelian mixed Hodge structure in a sense that will be explained below. The group \( \pi_{\text{hodge}}^1(X,x) \) dominates the Malčev completion and the Hodge structures are compatible, and I expect a similar statement for the relative completions.

The basic model here comes from arithmetic. Suppose that \( X \) is a variety over a field \( k \) with separable closure \( \bar{k} \). Let \( \bar{X} = X \times_{\text{Spec}k} \text{Spec} \bar{k} \). Suppose that \( x \in X(k) \) is a rational point, and that \( \bar{x} \in X(\bar{k}) \) a geometric point lying over it. Then there is an exact sequence of étale fundamental groups

\[
1 \to \pi_1^{et}(\bar{X}, \bar{x}) \to \pi_1^{et}(X, \bar{x}) \to \text{Gal}(\bar{k}/k) \to 1
\]

The point \( x \) gives a splitting, and so an action of \( \text{Gal}(\bar{k}/k) \) on \( \pi_1^{et}(\bar{X}, \bar{x}) \). This action will pass to the Galois cohomology of \( \pi_1^{et}(\bar{X}, \bar{x}) \). In the translation into Hodge theory, \( \pi_1^{et}(\bar{X}, \bar{x}) \) is replaced by \( \pi_{\text{hodge}}^1(X, x) \) and the Galois group by a certain universal Mumford-Tate group \( MT \). The action of \( MT \) on \( \pi_{\text{hodge}}^1(X, x) \) is precisely what I mean by a nonabelian mixed Hodge structure. The cohomology \( H^*(\pi_{\text{hodge}}^1(X, x), V) \) will carry induced mixed Hodge structures for admissible variations \( V \). In fact there is a canonical morphism \( H^*(\pi_{\text{hodge}}^1(X, x), V) \to H^*(X, V) \). I call \( X \) a Hodge theoretic \( K(\pi, 1) \) if this is an isomorphism for all \( V \). Basic examples of such spaces are abelian varieties, and smooth affine curves (modulo [HMPT]). I want to add that part of my motivation for this paper is to test out some ideas.

Partially supported by NSF.
which could be applied to motivic sheaves. So consequently certain constructions are phrased in more generality than is strictly necessary for the present purposes.

My thanks to Roy Joshua for the invitation to the conference. I would also like to thank Dick Hain, Tony Pantev, Jon Pridham, and the referee for their comments, and also Hain for informing me of his recent work with Matsumoto, Pearlstein and Terasoma.

1. Review of Tannakian categories

In this section, I will summarize standard material from [DM, D2, D3]. Let \( k \) be a field of characteristic zero. Given an affine group scheme \( G \) over \( k \), we can express its coordinate ring as a directed union of finitely generated sub Hopf algebras \( \mathcal{O}(G) = \lim_{\to} A_i \). Thus we can, and will, identify \( G \) with the pro-algebraic group \( \lim_{\leftarrow} \text{Spec} A_i \) and conversely (see [DM, cor. 2.7] for justification).

By a tensor category \( \mathcal{T} \) over \( k \), I will mean a \( k \)-linear abelian category with bilinear tensor product making it into a symmetric monoidal (also called an ACU) category; we also require that the unit object \( 1 \) satisfies \( \text{End}(1) = k \). \( \mathcal{T} \) is rigid if it has duals. The category \( \text{Vect}_k \) of finite dimensional \( k \)-vector spaces is the key example of a rigid tensor category. A neutral Tannakian category \( \mathcal{T} \) over \( k \) is a \( k \)-linear rigid tensor category, which possesses a faithful functor \( F : \mathcal{T} \to \text{Vect}_k \), called a fibre functor, preserving all the structure. The Tannaka dual \( \Pi(\mathcal{T}, F) \) of such a category (with specified fibre functor \( F \)) is the group of tensor preserving automorphisms of \( F \). In more concrete language, an element \( g \in \Pi(\mathcal{T}, F) \) consists of a collection \( g_V \in GL(F(V)) \), for each \( V \in \text{Ob}\mathcal{T} \), satisfying the following compatibilities:

\begin{align*}
&C1 \quad g_V \otimes U = g_V \otimes g_U, \\
&C2 \quad g_V \otimes U = g_V \oplus g_U, \\
&C3 \quad \text{the diagram}
\begin{array}{ccc}
F(V) & \xrightarrow{g_V} & F(V) \\
\downarrow & & \downarrow \\
F(U) & \xrightarrow{g_U} & F(U)
\end{array}
\end{align*}

commutes for every morphism \( V \to U \).

\( \Pi = \Pi(\mathcal{T}, F) \) is (the group of \( k \)-points of) an affine group scheme. Suppose that \( \mathcal{T} \) is generated, as a tensor category, by an object \( V \) i.e. every object of \( \mathcal{T} \) is finite sum of subquotients of tensor powers

\[ T_{m,n}(V) = V^ \otimes m \otimes V^* \otimes n. \]

Observe that for all \( m, n \geq 0 \)

\[ \text{Hom}_k(F(1), F(T_{m,n}(V))) = T_{m,n}(F(V)) \]

**Lemma 1.1.** Suppose that \( \mathcal{T} \) is generated as a tensor category by \( V \), then

1. \( \Pi \) can be identified with the largest subgroup of \( GL(F(V)) \) that fixes all tensors in the subspaces

\[ \text{Hom}_\mathcal{T}(1, T_{m,n}(V)) \subset T_{m,n}(F(V)) \]

2. \( \Pi \) leaves invariant each subspace of \( T_{m,n}(F(V)) \) that corresponds to a sub-object of \( T_{m,n}(V) \).
In particular, $\Pi$ is an algebraic group.

Proof. This is standard cf. \cite{DM}, \hfill \square

In general,
$$\Pi(T, F) = \lim_{\substack{T' \subset T \text{ finitely generated}}} \text{Aut}(F|_{T'})$$

exhibits it as a pro-algebraic group and hence a group scheme.

The following key example will explain our choice of notation.

Example 1.2. Let $\text{Loc}(X)$ be the category of local systems (i.e. locally constant sheaves) of finite dimensional $\mathbb{Q}$-vector spaces over a connected topological space $X$. This is a neutral Tannakian category over $\mathbb{Q}$. For each $x \in X$, $F_x(L) = L_x$ gives a fibre functor. The Tannaka dual $\Pi(\text{Loc}(X), F_x)$ is isomorphic to the rational pro-algebraic completion
$$\pi_1(X, x)^{\text{alg}} = \lim_{\rho: \pi_1(X)^{\text{alg}} \to \text{GL}_n(\mathbb{Q})} \rho(\pi_1(X, x))$$
of $\pi_1(X, x)$.

Given an affine group scheme $G$, let $\text{Rep}_\infty(G)$ (respectively $\text{Rep}(G)$) be the category of (respectively finite dimensional) $k$-vector spaces on which $G$ acts algebraically. When $T$ is a neutral Tannakian category with a fibre functor $F$, the basic theorem of Tannaka-Grothendieck is that $T$ is equivalent to $\text{Rep}(\Pi(T, F))$. The role of a fibre functor is similar to the role of base points for the fundamental group of a connected space. We can compare the groups at two base points by choosing a path between them. In the case of pair of fiber functors $F$ and $F'$, a “path” is given by the tensor isomorphism $p \in \text{Isom}(F, F')$ between these functors. An element $p \in \text{Isom}(F, F')$ determines an isomorphism $\Pi(T, F) \cong \Pi(T, F')$ by $g \mapsto p gp^{-1}$. More canonically, one can define $\Pi(T)$ as an “affine group scheme in $T$” independent of any choice of $F$ \cite[D2 §6]. For our purposes, we can view $\Pi(T)$ as the Hopf algebra object which maps to the coordinate ring $\mathcal{O}(\Pi(T, F))$ for each $F$.

Note that $\Pi$ is contravariant. That is, given a faithful exact tensor functor $E : T' \to T$ between Tannakian categories, we have an induced homomorphism $\Pi(T, F) \to \Pi(T', F \circ E)$.

2. Enriched local systems

Before giving the definition of the Hodge theoretic fundamental group, it is convenient to start with some generalities. A theory of enriched local systems $E$ on the category smooth complex varieties consist of

1. an assignment of a neutral $\mathbb{Q}$-linear Tannakian category $E(X)$ to every smooth variety $X$,
2. a contravariant exact tensor pseudo-functor on the category of smooth varieties, i.e., a functor $f^* : E(Y) \to E(X)$ for each morphism $f : X \to Y$ together with natural isomorphisms for compositions,
3. faithful exact tensor functors $\phi : E(X) \to \text{Loc}(X)$ compatible with base change, i.e., a natural transformation of pseudo-functors $E \to \text{Loc}$,
(E4) a δ-functor \( h^\bullet : E(X) \to E(pt) \) with natural isomorphisms \( \phi(h^i(L)) \cong H^i(X, \phi(L)) \). We also require there to be a canonical map \( p^*h^0(L) \to L \) corresponding the adjunction map on local systems, where \( p : X \to pt \) is the projection.

By a weak theory of enriched local systems, we mean something satisfying (E1)-(E3). We have the following key examples.

**Example 2.1.** Choosing \( E = \text{Loc} \) and \( h^i = H^i \) gives the tautological example of a theory enriched of local systems.

**Example 2.2.** Let \( E(X) = \text{MHS}(X) \) be the category of admissible variations of mixed Hodge structure \([K, SZ]\) on \( X \). This carries a forgetful functor \( \text{MHS}(X) \to \text{Loc}(X) \). Let \( h^i(L) = H^i(X, L) \) equipped with the mixed Hodge structure constructed by Saito \([Sa1, Sa2]\). Set \( \pi^E_{\text{loc}}(X, x) = \Pi(\phi(E(X))), F_x \), where \( F_x \) is the fibre functor associated to a base point \( x \). More explicitly, this is the inverse limit of the Zariski closures of monodromy representations of objects of \( E(X) \). It follows that the isomorphism class of \( \pi^E_{\text{loc}}(X, x) \) as a group scheme is independent of \( x \). However, certain additional structure will depend on it.

Given a theory of enriched local systems \((E, \phi, h^\bullet)\), let \( \phi(E(X)) \) denote the Tannakian subcategory of \( \text{Loc}(X) \) generated by the image of \( E(X) \). So \( \phi(E(X)) \) is the full subcategory whose objects are sums of subquotients of objects in the image of \( \phi \). Set \( \pi^E_1(X, x) = \Pi(\phi(E(X))), F_x \), where \( F_x \) is the fibre functor associated to a base point \( x \). More explicitly, this is the inverse limit of the Zariski closures of monodromy representations of objects of \( E(X) \). It follows that the isomorphism class of \( \pi^E_1(X, x) \) as a group scheme is independent of \( x \). However, certain additional structure will depend on it.

Let \( \kappa : E(pt) \to E(X) \) and \( \psi : E(X) \to E(pt) \) be given by \( p^* \) and \( i^* \) respectively, where \( p : X \to pt \) and \( i : pt \to X \) are the projection and inclusion of \( x \). We also have \( \phi : E(X) \to \phi(E(X)) \). These functors yield a diagram

\[
\begin{array}{ccc}
\pi^E_1(X, x) & \to & \Pi(E(X), x) \\
\downarrow & & \downarrow \\
\pi^E_1(Y, y) & \to & \Pi(E(Y), y)
\end{array}
\]

where \( \Pi(E(X), x) = \Pi(E(X), F_x) \) and \( \Pi(E(pt)) = \Pi(E(pt), pt) \). The diagram is clearly canonical in the sense that a morphism \( f : (X, x) \to (Y, y) \) of pointed varieties gives rise to a larger commutative diagram

\[
\begin{array}{ccc}
\pi^E_1(X, x) & \to & \Pi(E(X), x) \\
\downarrow & & \downarrow \\
\pi^E_1(Y, y) & \to & \Pi(E(Y), y)
\end{array}
\]

By a weak theory of enriched local systems, we mean something satisfying (E1)-(E3). We have the following key examples.
Theorem 2.6. The sequence

$$1 \to \pi_1^E(X, x) \to \Pi(E(X), x) \odot \Pi(E(pt)) \to 1$$

is split exact. Therefore there is a canonical isomorphism $\Pi(E(X), x) \cong \Pi(E(pt)) \odot \pi_1^E(X, x)$.

Proof. Since $\psi \circ \kappa = id$, the induced homomorphisms $\Pi(E(pt)) \to \Pi(E(X), x) \to \Pi(E(pt))$ compose to the identity. The injectivity of $\pi_1^E(X, x) \to \Pi(E(X), x)$ follows from [DX] 2.21.

Therefore it remains to check exactness in the middle. An element of $im[\pi_1^E(X, x) \to \Pi(E(X), x)]$ is given by a collection of elements $g_v \in GL(\phi(V)_x)$, with $V \in ObE(X)$, satisfying (C1)-(C3) such that

$$g_v = \alpha_x \circ g_v \circ \alpha_x^{-1} \quad (1)$$

holds for any isomorphism $\alpha : \phi(V) \cong \phi(U)$. While an element of $ker[\Pi(E(X), x) \to \Pi(E(pt))]$ is given by a collection $\{g_v\}$ such that $g_v = I$ for any object in the image of $\kappa$. If $V$ is in the image of $\kappa$, the underlying local system is trivial. This implies that $g_v = \alpha_x \circ \alpha_x^{-1} = I$, if (1) holds. Thus, we have

$$im[\pi_1^E(X, x) \to \Pi(E(X), x)] \subseteq ker[\Pi(E(X), x) \to \Pi(E(pt))]$$

Conversely, suppose that $\{g_v\}$ is an element of the kernel on the right. Given an isomorphism $\alpha : \phi(V) \to \phi(U)$, we have to verify (1). Let $H = \kappa(h^0(V^* \otimes U))$, then $g_H = 1$ by assumption. $H$ gives a subobject of $V^* \otimes U$ which maps to the invariant part $\phi(V^* \otimes U)^{\pi_1(X)}$ of the local system $\phi(V^* \otimes U)$. This follows from our axiom (E4). Therefore, $\alpha$ gives a section of $\phi(H)$. The evaluation morphism $ev : (V^* \otimes U) \otimes V \to U$ restricts to give a morphism $H \otimes V \to U$. We claim that the diagram

$$\begin{array}{ccc}
\phi(V)_x & \xrightarrow{\alpha} & \phi(H)_x \otimes \phi(V)_x \\
\alpha \otimes I \downarrow & & \phi(U)_x \\
\phi(V)_x \otimes \phi(H)_x \otimes \phi(V)_x & \xrightarrow{ev} & \phi(U)_x \\
\end{array}$$

commutes. The commutativity of the square on the left is clear. For the commutativity on right, apply (C1) and (C3) and the fact that $g_H = I$. Equation (1) is now proven. \[\square\]

Since any representation of an affine group scheme is locally finite, it follows that $Rep_{\infty}(E(pt))$ can be identified with the category of ind-objects $Ind-E(pt)$. We will often refer to an object of this category as an $E$-structure. To simplify notation, we generally will not distinguish between $V$ and $\phi(V)$. By a nonabelian $E$-structure, we will mean an affine group scheme $G$ over $\mathbb{Q}$ with an algebraic action of $\Pi(E(pt))$. Equivalently, $O(G)$ possesses an $E$-structure compatible with the Hopf algebra operations. A morphism of nonabelian $E$-structures is a homomorphism of group schemes commuting with the $\Pi(E(pt))$-actions. The previous theorem yields a nonabelian $E$-structure on $\pi_1^E(X, x)$ which is functorial in the category of pointed varieties. When $X$ is connected, for any two base points $x_1, x_2$, $\pi_1^E(X, x_1)$ and $\pi_1^E(X, x_2)$ are isomorphic as group schemes although the $E$-structures need not be the same.
Lemma 2.7. An algebraic group $G$ carries a nonabelian $E$-structure if and only if there exists a finite dimensional $E$-structure $V$ and an embedding $G \subseteq GL(V)$, such that $G$ is normalized by the action of $\Pi(E(pt))$. A nonabelian $E$-structure is an inverse limit of algebraic groups with $E$-structures.

Proof. Suppose that $G$ is an algebraic group with an $E$-structure. Then $\Pi = \Pi(E(pt))$ acts on $G$; denote this left action by $m^g$. We also have a left action of $\Pi$ on $\mathcal{O}(G)$, written in the usual way, such that

$$m(gf) = (m^g)(mf)$$

(2)

for $m \in \Pi$, $g \in G$ and $f \in \mathcal{O}(G)$. This gives an action of $\Pi \ltimes G$ on $\mathcal{O}(G)$. Let $V' \subseteq \mathcal{O}(G)$ be a subspace spanned by a finite set of algebra generators. Let $V \subseteq \mathcal{O}(G)$ be the smallest $\Pi \ltimes G$-submodule containing $V'$. By standard arguments, $V$ is finite dimensional and faithful as a $G$-module. So we get $G \subseteq GL(V)$. Equation (2) implies that $G$ is normalized by $\Pi$. This proves one direction. The converse is clear.

For the second statement write $\mathcal{O}(G)$ as direct limit $\mathcal{O}(G) = \lim V_i$ of finite dimensional $E$-structures. Let $A_i \subseteq \mathcal{O}(G)$ be the smallest Hopf subalgebra containing $V_i$. This is an $E$-substructure. So we have $G = \lim A_i$ which gives the desired conclusion. \qed

An $E$-representation of a nonabelian $E$-structure $G$ is a representation on an $E$-structure $V$ such that (2) holds. The adjoint representation gives a canonical example of an $E$-representation. The above lemma says that every finite dimensional nonabelian $E$-structure has a faithful $E$-representation. The following is straightforward and was used implicitly already.

Lemma 2.8. There is an equivalence between the tensor category of $E$-representations of a nonabelian $E$-structure $G$ and the category of representations of the semidirect product $\Pi(E(pt)) \ltimes G$.

Corollary 2.9. In the notation of theorem 2.7, $E(X)$ is equivalent to the category of finite dimensional $E$-representations of $\pi_1^E(X, x)$.

3. Nonabelian Hodge structures

The category $MHS = MHS(pt)$ (respectively $HS = HS(pt)$) is just the category of rational graded polarizable rational (respectively pure) Hodge structures \cite{Die}; its Tannaka dual will be called the universal (pure) Mumford-Tate group and will be denoted by $MT$ (PMT). The Tannaka dual of the category of real Hodge structures $\mathbb{R}HS$ is just Deligne’s torus $S = Res_{\mathbb{C}/\mathbb{R}}\mathbb{C}^*$. So the obvious functor $\mathbb{R}HS \to HS \otimes \mathbb{R}$ yields an embedding $S(\mathbb{R}) \hookrightarrow PMT(\mathbb{R})$. In more concrete terms, $(z_1, z_2) \in S(\mathbb{C}) = \mathbb{C}^* \times \mathbb{C}^*$ acts by multiplication by $z_1^p z_2^q$ on the $(p, q)$ part of a pure Hodge structure. Since $HS$ is semisimple, $PMT$ is pro-reductive. The inclusion $HS \subseteq MHS$ gives a homomorphism $MT \to PMT$. We have a section $PMT \to MT$ induced by the functor $V \mapsto Gr^W(V) = \oplus Gr_i^W(V)$. Thus $MT$ is a semidirect product of $PMT$ with $\ker[MT \to PMT]$. The kernel is pro-unipotent since it acts trivially on $W^* \mathbb{R}$ for any $V \in MHS$.

$Rep_{\infty}(MT)$ is the category $Ind-MHS$ of direct limits of mixed Hodge structures. Given an object $V = \lim V_i$ in this category, we can extend the Hodge and weight filtrations by $F^pV = \lim F^pV_i$ and $W_kV = \lim W_kV_i$. 

Set $\pi_1^{\text{hodge}} = \pi_1^E$ for $E = \text{MHS}$. So this is the inverse limit of Zariski closures of monodromy representations of variations of mixed Hodge structures. The key definition is:

(NH1) A nonabelian mixed (respectively pure) Hodge structure, or simply an \textit{NMHS} (or an \textit{NHS}), is an affine group scheme $G$ over $\mathbb{Q}$ with an algebraic action of $MT$ (respectively $PMT$).

A morphism of these objects is a homomorphism of group schemes commuting with the $MT$-actions. A Hodge representation is an $MT$-equivariant representation; it is the same thing as an $E$-representation for $E = \text{MHS}$. The coordinate ring of an \textit{NMHS} is a Hopf algebra in $\text{Ind-MHS}$. Let us recapitulate the results of the previous section in the present setting.

- $\pi_1^{\text{hodge}}(X, x)$ has an \textit{NMHS} which is functorial in the category of smooth pointed varieties (and this structure will usually depend on the choice of $x$).
- An algebraic group $G$ admits an \textit{NMHS} if and only if it has a faithful representation to the general linear group of a mixed Hodge structure for which $MT$ normalizes $G$.
- Admissible variations of MHS correspond to Hodge representations of $\pi_1^{\text{hodge}}(X, x)$.

The notion of a nonabelian mixed Hodge structure is fairly weak, although sufficient for some of the main results of this paper. At this point it is not clear what the optimal set of axioms should be. We would like to spell out some further conditions which will hold in our basic example $\pi_1^{\text{hodge}}$.

(NH2) An \textit{NMHS} $G$ satisfies (NH2) or has nonpositive weights if $W_{-1} \mathcal{O}(G) = 0$.

\textbf{Remark 3.1.} To see why this is “nonpositive”, observe that if $V$ is an MHS with $W_1 V = 0$, then $W_{-1} \mathcal{O}(V) = W_{-1} \text{Sym}^*(V^*) = 0$. (My thanks to the referee for pointing out that the weights gets flipped.)

The significance of this condition is explained by the following:

\textbf{Lemma 3.2.} An \textit{NMHS} $G$ has nonpositive weights if and only if the left and right actions of $G$ on $\mathcal{O}(G)$ preserves the weight filtration.

\textbf{Proof.} Note that the left and right $G$-actions preserves the weight filtration if and only if

- $\mu^*(W_i \mathcal{O}(G)) \subseteq \mathcal{O}(G) \otimes W_i \mathcal{O}(G)$
- $\mu^*(W_i \mathcal{O}(G)) \subseteq W_i \mathcal{O}(G) \otimes \mathcal{O}(G)$

Comultiplication $\mu^* : \mathcal{O}(G) \rightarrow \mathcal{O}(G) \otimes \mathcal{O}(G)$ is a morphism of $\text{Ind-MHS}$. If $G$ has nonpositive weights, then

$$\mu^*(W_i \mathcal{O}(G)) \subseteq \sum_{j \geq 0} W_j \mathcal{O}(G) \otimes W_{i-j} \mathcal{O}(G) \subseteq \mathcal{O}(G) \otimes W_i \mathcal{O}(G)$$

(3) follows by symmetry.

Suppose $W_{-1} \mathcal{O}(G) \neq 0$ and that $3$ and $4$ hold. Let $f \in W_{-1} \mathcal{O}(G)$ be a nonzero element, and let $n > 0$ be the largest integer such that $f \in W_{-n}$. Suppose that $\mu^*(f) = \sum g_i \otimes h_i$. By (4),

$$\sum g_i \otimes h_i \equiv 0 \mod \mathcal{O}(G)/W_{-n} \otimes \mathcal{O}(G)$$
Therefore all \( g_i \in W_{-n} \). By a similar argument \( h_i \in W_{-n} \). Therefore \( \mu^*(f) \in W_{-2n}(O(G) \otimes O(G)) \). Since morphisms of MHS (and therefore Ind-MHS) strictly preserve weight filtrations \( \text{(D1)} \), \( \mu^*(f) = \mu^*(f') \) for some \( f' \in W_{-2n} \). But \( \mu^* \) is injective because \( \mu \) is dominant. Therefore \( f = f' \in W_{-2n} \) which is a contradiction. \( \square \)

**Lemma 3.3.** If \( G \) satisfies (NH2), then there is a unique maximal pure quotient \( G^\text{pure} \).

**Proof.** Since \( G \) satisfies (NH2) it acts on the pure Ind-MHS \( Gr^W O(G) \) on the left. As a group we take \( G^\text{pure} \) to be the image of \( G \) in \( \text{Aut}(Gr^W O(G)) \). To get the finer structure, we apply lemma 2.7 to write \( G = \lim G_i \) with \( G_i \subset \text{GL}(V_i) \) where \( V_i \subset O(G) \) are mixed Hodge structures such that \( G_i \) is normalized by \( MT \). \( G \) preserves \( W_i, V_i \) by assumption. Then

\[
G^\text{pure} = \lim \text{im}[G_i \to \text{GL}(Gr^W V_i)]
\]

This group carries a nonabelian pure Hodge structure since each group of the limit does. By construction, there is a surjective morphism \( G \to G^\text{pure} \).

Suppose that \( G \to H \) is another pure quotient. Then \( G \) will act on \( O(H) \) through this map. The image of \( G \) in \( \text{Aut}(O(H)) \) is precisely \( H \). We have a \( G \)-equivariant morphism of Ind-MHS \( O(H) \subset O(G) \). By purity \( O(H) = Gr^W O(H) \subset Gr^W O(G) \), which shows that the \( G \)-action on \( O(H) \) factors through \( G^\text{pure} \). Therefore the homomorphism \( G \to H \) also factors through this. \( \square \)

Before explaining the next condition, we recall that some standard definitions. Fix a real algebraic group \( G \). We have a conjugation \( g \mapsto \bar{g} \) on the group of complex points \( G(\mathbb{C}) \), whose fixed points are exactly \( G(\mathbb{R}) \). A **Cartan involution** \( C \) of \( G \) is an algebraic involution of \( G(\mathbb{C}) \) defined over \( \mathbb{R} \), such that the group of fixed points of \( g \mapsto C(g) = \overline{C(g)} \) is compact for the classical topology and has a point in every component. An involution of a pro-algebraic group is Cartan if it descends to a Cartan involution in the usual sense on a cofinal system of finite dimensional quotients. Recall that Deligne’s torus \( S = \text{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{C}^* \) embeds into \( PMT \) in such a way that \( (z_1, z_2) \in S(\mathbb{C}) = \mathbb{C}^* \times \mathbb{C}^* \) acts by multiplication by \( z_1^p z_2^q \) on the \( (p, q) \) part of a pure Hodge structure. It will be convenient to say that a group is reductive (or pro-reductive) when its connected component of the identity is.

(NH3) A NMHS \( G \) satisfies (NH3) or is \( S \)-polarizable if it has nonpositive weights, and the action \( C \) of \( (i, i) \in S(\mathbb{C}) \) on \( G^\text{pure} \) gives a Cartan involution.

The “\( S \)” stands for Simpson, since this condition is related to (and very much inspired by) the notion of a pure nonabelian Hodge structure introduced by him \[84\] p. 61]. Simpson has shown that the pro-reductive completion of the fundamental group of a smooth projective variety carries a pure nonabelian Hodge structure in his sense. These ideas have been developed further in \[KPT1, KT2\]. Although there is no direct relation between these notions of nonabelian Hodge structure, there are a number of close parallels (e.g. lemma 3.7 below holds for both). The meaning of (NH3) is explained by the following:

**Lemma 3.4.** Let \( G \) be an algebraic group with an NMHS. Then \( G \) is \( S \)-polarizable if and only if there exists a pure Hodge structure \( V \) and an embedding \( G^\text{pure} \subseteq \text{GL}(V) \), such that \( G^\text{pure} \) is normalized by the action of \( PMT \) and such that \( G^\text{pure} \) preserves a polarization on \( V \). Under these conditions, \( G^\text{pure} \) is reductive.
Proof. After replacing $G$ by $G^{\text{pure}}$, we may assume $G = G^{\text{pure}}$ is already pure. Given an embedding $G \subseteq GL(V)$ as above, choose a $G$-invariant polarization $(\cdot, \cdot)$ on $V$. The image $W$ of $(i, i) \in S(C)$ in $GL(V)$ is nothing but the Weil operator for the Hodge structure on $V$. Therefore $(u, v) = (u, Wv)$ is positive definite Hermitian. The group of fixed points under $\sigma(g) = W^{-1}gW$ is easily seen to preserve $(\cdot, \cdot)$, so it is compact. In other words, $Cg = W^{-1}gW$ is a Cartan involution. Conversely, if $C$ is a Cartan involution then a $G$-invariant polarization can be constructed by averaging an existing polarization over the Zariski dense compact group of $\sigma$-fixed points. When these conditions are satisfied the reductivity of $G$ follows from the existence of a Zariski dense compact subgroup. \hfill $\square$

**Corollary 3.5.** The underlying group of a pure $S$-polarizable nonabelian Hodge structure is pro-reductive.

**Corollary 3.6.** If $G$ is $S$-polarizable, then $G^{\text{pure}} = G^{\text{red}}$ is the maximal pro-reductive quotient of $G$. In particular, if $G$ is pro-reductive then $G = G^{\text{pure}}$ is pure.

**Proof.** We have an exact sequence of pro-algebraic groups

$$1 \to U \to G \to G^{\text{pure}} \to 1$$

where $U$ is simply taken to be the kernel. This can also be described as above by

$$U = \lim \ker[G_i \to GL(Gr^W V_i)]$$

We can see from this that $U$ is pro-unipotent. By the previous corollary, $G^{\text{pure}}$ is pro-reductive. Thus $U$ is the pro-unipotent radical and $G^{\text{pure}} = G^{\text{red}}$ is the maximal pro-reductive quotient. This forces $G = G^{\text{pure}}$ if $G$ is pro-reductive. \hfill $\square$

Recall that Simpson [Si, p 46] defined a real algebraic group $G$ to be of **Hodge type** if $\mathbb{C}^*$ acts on $G(\mathbb{C})$ such that $U(1)$ preserves the real form and $-1 \in U(1)$ acts as a Cartan involution. Such groups are reductive, and also subject to a number of other restrictions [loc. cit.]. For example, $SL_n(\mathbb{R})$ is not of Hodge type when $n \geq 3$.

**Lemma 3.7.** If an algebraic group admits a pure $S$-polarizable nonabelian Hodge structure then it is of Hodge type.

**Proof.** Choose an embedding $G \subseteq GL(V)$ as in lemma 3.3. The Hodge structure on $V$ determines a representation $S(\mathbb{C}) \to GL(V_C)$. The group $G(\mathbb{C})$ is stable under conjugation by elements of $PMT(\mathbb{C}) \supset S(\mathbb{C})$. Embed $\mathbb{C}^* \subset S(\mathbb{C})$ by the diagonal. Then $-1 \in \mathbb{C}^*$ acts trivially on $G(\mathbb{C})$. Therefore the $\mathbb{C}^*$ action factors through $\mathbb{C}^*/\{\pm 1\} \cong \mathbb{C}^*$. To see that $U(1)/\{\pm 1\}$ preserves $G(\mathbb{R}) \subseteq GL(V_{\mathbb{R}})$, it suffices to note that the image of $e^{i\theta} \in U(1)$ in $GL(V_C)$, which acts on $V^{\mathbb{R}q}$ by multiplication by $e^{i(p-q)\theta}$, is a real operator. Under the isomorphism $U(1) \cong U(1)/\{\pm 1\}$, $-1$ on the left corresponds to the image of $i$ on the right. Thus $-1$ acts by a Cartan involution on $G(\mathbb{C})$. \hfill $\square$

**Theorem 3.8.** Given a smooth variety $X$, $\pi_1^{\text{hodge}}(X, x)$ carries an $S$-polarizable nonabelian Hodge mixed structure. The category of Hodge representations of $\pi_1^{\text{hodge}}(X, x)^{\text{red}}$ is equivalent to the category of pure variations of Hodge structure on $X$. 
whose fibres are
variation
Spec
variation by the
symbol
called
inner
where
Any object
Example 4.1.
usual sense, and this is an NMHS. Basic examples are given as follows.

For

\[ \pi_1^{\text{hodge}}(\langle V \rangle, x) \]

is normalized by \( MT(V) \). We can also see this directly. The group \( \pi_1^{\text{hodge}}(\langle V \rangle, x) \)
is characterized as the group of automorphisms that fix all monodromy invariant
tensors \( T^{m,n}(V_x)_{\pi_1(X,x)} \). While \( MT(V) \) leaves all sub MHS of \( T^{m,n}(V_x) \) invariant
by lemma 4.1. Let \( g \in MT(V) \) and let \( \gamma \in \pi_1^{\text{hodge}}(\langle V \rangle, x) \), then it is enough to
see that \( g^{-1} \gamma g \) fixes every tensor in \( T^{m,n}(V_x)_{\pi_1(X,x)} \). This space is a sub MHS of
\( T^{m,n}(V_x) \) by [SZ 4.19]. Therefore \( T^{m,n}(V_x)_{\pi_1(X,x)} \) is invariant under \( g \) (although
it need not fix elements pointwise). This shows that \( g^{-1} \gamma g \) fixes the elements of
this space as claimed. Therefore \( \pi_1^{\text{hodge}}(\langle V \rangle, x) \) carries a NMHS.

Since the weight filtration of \( V \) is a filtration by local systems, \( \pi_1^{\text{hodge}}(\langle V \rangle, x) \)
preserves \( W_1 V_x \). So it satisfies (NH2) by lemma 3.2. Let \( \pi_1^{\text{hodge}}(V) \) be the image
of \( \pi_1^{\text{hodge}}(\langle V \rangle, x) \) in \( GL(G^W V_x) \). This can be identified with the Zariski closure
of the monodromy representation of the pure variation of Hodge structure \( G^W V \).
This pure variation is polarizable by definition of admissibility [SZ K]. Therefore
\( G^W V_x \) possesses a \( \pi_1^{\text{hodge}}(V) \) invariant polarization. Consequently \( \pi_1^{\text{hodge}}(\langle V \rangle, x) \)
satisfies (NH3) by lemma 3.4. Moreover \( \pi_1^{\text{hodge}}(V) = \pi_1^{\text{hodge}}(\langle V \rangle, x)^{\text{red}} \) is the Tan-
naka dual to the subcategory of \( \text{Loc}(X) \) generated by the local system \( G^W V \).
Putting this all together, we see that

\[
\pi_1^{\text{hodge}}(X, x) = \lim_{V} \pi_1^{\text{hodge}}(\langle V \rangle, x)
\]
satisfies (NH3), and that the Tannaka dual to \( \text{HS}(X) \) is \( \text{PMT} \cong \pi_1^{\text{hodge}}(X, x)^{\text{red}} \).
Lemma 2.8 implies that \( \text{HS}(X) \) is equivalent to the category of Hodge representations of \( \pi_1^{\text{hodge}}(X, x)^{\text{red}} \).

4. Nonabelian variations

The goal of this section is to give a characterization of \( \pi_1^{\text{hodge}}(X, x) \). To this
end, we introduce the category of nonabelian variation of mixed Hodge structures
over \( X \), by which we mean the opposite of the category of Hopf algebras in Ind-
\( \text{MHS}(X) \). Given such a Hopf algebra \( A \), we denote the corresponding nonabelian
variation by the symbol \( \text{Spec} A \). For any \( x \in X \), \( \text{Spec} A_x \) can be understood in the
usual sense, and this is an NMHS. Basic examples are given as follows.

Example 4.1. Any object \( V \in \text{MHS}(X) \) can be identified with the nonabelian variation \( \text{Spec}(\text{Sym}^*(V^*)) \).

By applying the forgetful functor \( \text{MHS}(X) \to \text{Loc}(X) \), we can see that any
nonabelian variation \( \text{Spec} A \) carries a monodromy action of \( \pi_1(X, x) \to \text{Aut}(G_x) \),
where \( G_x = \text{Spec} A_x \). A nonabelian variation of mixed Hodge structures will be
called inner if the monodromy action lifts to a homomorphism \( \pi_1(X, x) \to G_x \). The
examples of \( 4.1 \) are rarely inner. However, an ample supply of such examples is
given by the following.

Example 4.2. For \( V \in \text{MHS}(X) \), we have a nonabelian variation \( \pi_1^{\text{hodge}}(\langle V \rangle) \)
whose fibres are \( \pi_1^{\text{hodge}}(\langle V \rangle, x) \) by [D2 §6]. To construct this directly, note that we
can realize the coordinate ring of $\pi_1^{\text{hodge}}((V), x)$ as a quotient of $\mathcal{O}(GL(V_x))$ by a Hopf ideal $\sum f_k \mathcal{O}(GL(V_x))$. The generators $f_k$ can be regarded as sections of

$$\mathcal{R} = \bigoplus_{i,j \geq 0} \text{Sym}^i(V^*) \otimes \text{det}(V^*)^{-j}$$

Thus we can define $\pi_1^{\text{hodge}}((V))$ as Spec of the Ind-MHS$(X)$ Hopf algebra $\mathcal{R}/(\sum_k f_k \mathcal{R})$. This is inner since the monodromy is given by homomorphism $\pi_1(X, x) \to \pi_1^{\text{hodge}}((V), x)$.

Let $\pi_1^{\text{hodge}}(X)$ be the inverse limit of $\pi_1^{\text{hodge}}((V))$ over $V \in MHS(X)$. The fibres are $\pi_1^{\text{hodge}}(X, x)$. This is the universal inner nonabelian variation in the following sense:

**Proposition 4.3.** If $G$ is an inner nonabelian variation of mixed Hodge structure over $X$, with monodromy given by a homomorphism $\rho : \pi_1(X, x) \to G_x$. Then $\rho$ extends to a morphism $\pi_1^{\text{hodge}}(X) \to G$ of nonabelian variations.

**Proof.** Let $G = \text{Spec} A$. Since $A \in \text{Ind-MHS}(X)$, $\pi_1^{\text{hodge}}(X, x)$ will act on it. By an argument similar to the proof of lemma [2.7] we can write $A$ as a direct limit of finitely generated Hopf algebras with $\pi_1^{\text{hodge}}(X, x)$-action. So that $G_x$ becomes an inverse limit of algebraic groups carrying inner nonabelian variations. Thus we can assume that $G_x$ is an algebraic group. By standard techniques we can find a finite dimensional faithful (left) $G$-submodule $V \subset \mathcal{O}(G)$. After replacing this with the span of the $\pi_1^{\text{hodge}}(X, x)$-orbit, we can assume that $V$ is stable under $\pi_1^{\text{hodge}}(X, x)$, Therefore $V$ corresponds to a variation of mixed Hodge structure. By assumption, the image of $\pi_1(X, x)$ in $GL(V_x)$ lies in $G$. This implies that $G$ contains $\pi_1^{\text{hodge}}((V), x)$ and that $\rho$ factors through it. \hfill \Box

## 5. Unipotent and Relative Completion

Morgan [M] and Hain [H1] have shown that the pro-unipotent completion of the fundamental group of a smooth variety carries a mixed Hodge structure. We want to compare this with our nonabelian Hodge structure. We start by recalling some standard facts from group theory (c.f. [HZ], [Q] appendix A). Fix a finitely generated group $\pi$. Then

(a) $\mathbb{Q}[\pi]$, and its quotients by powers of the augmentation ideal $J$, carry Hopf algebra structures with comultiplication $\Delta(g) = g \otimes g \mod J^r$.

(b) A finite dimensional $\mathbb{Q}[\pi]$-module is unipotent if and only if it factors through some power of $J$. (The smallest power will be called the index of unipotency).

(c) The set of group-like elements

$$G_r(\pi) = \{ f \in \mathbb{Q}[\pi]/J^{r+1} \mid \Delta(f) = f \otimes f, f \equiv 1 \mod J \}$$

forms a group under multiplication. This is a unipotent algebraic group.

(d) The Lie algebra of $G_r(\pi)$ can be identified with the Lie algebra of primitive elements

$$\mathfrak{g}_r(\pi) = \{ f \in \mathbb{Q}[\pi]/J^{r+1} \mid \Delta(f) = f \otimes 1 + 1 \otimes f \}$$

with bracket given by commutator.

(e) The exponential map gives a bijection of sets $\mathfrak{g}_r(\pi) \cong G_r(\pi)$. The Lie algebra and group structures determine each other via the Baker-Campbell-Hausdorff formula.
(f) $\mathbb{Q}[\pi]/J^{r+1}$ is isomorphic to a quotient of the universal enveloping algebra of $G_r(\pi)$ by a power of its augmentation ideal.

Let $ULoc(X) (U_r Loc(X))$ denote the category of local systems with unipotent monodromy (with index of unipotency at most $r$). The category $U_r Loc(X)$ can be identified with the category of $\mathbb{Q}[\pi_1(X, x)]/J^{r+1}$-modules. We note that this category has a tensor product: $\mathbb{Q}[\pi_1(X, x)]/J^{r+1}$ acts on the usual tensor product of representations $U \otimes_{\mathbb{Q}} V$ through $\Delta$. With this structure, the category of $\mathbb{Q}[\pi_1(X, x)]/J^{r+1}$-modules is Tannakian. Its Tannaka dual $\pi_1(X, x)^{un_r}$ is isomorphic to the group $G_r(\pi_1(X, x))$ above, and the Tannaka dual $\pi_1(X, x)^{un}$ of $ULoc(X)$, is the inverse limit of these groups.

We need to impose Hodge structures on these objects.

**Lemma 5.1.** There is a bijection between

1. The set of nonabelian mixed Hodge structures on $G_r(\pi)$.
2. The set of mixed Hodge structures on $G_r(\pi)$ compatible with Lie bracket.
3. The set of mixed Hodge structures on $\mathbb{Q}[\pi]/J^{r+1}$ compatible with the Hopf algebra structure.

**Proof.** To go from (1) to (2), observe that a nonabelian mixed Hodge structure always induces a mixed Hodge structure on its Lie algebra compatible with bracket. A Lie compatible mixed Hodge structure on $G_r(\pi)$ induces an ind-MHS on its universal enveloping algebra, compatible with the Hopf algebra structure. This descends to $\mathbb{Q}[\pi]/J^{r+1}$ by (f) above. A Hopf compatible mixed Hodge structure on $\mathbb{Q}[\pi]/J^{r+1}$ induces one on $G_r(\pi)$ by restriction. \qed

Hain [HI] constructed a mixed Hodge structure on the Hopf algebra $\mathbb{Q}[\pi_1(X, x)]/J^{r+1}$ which is equivalent (in the sense of the lemma) to the one constructed by Morgan on $G_r(\pi_1(X, x))$. These fit together to form an inverse system as $r$ increases. In brief outline, Chen had shown that $\mathbb{C} \otimes \lim_i \mathbb{Q}[\pi_1(X, x)]/J^{r+1}$ can be realized as the zeroth cohomology $H^0(B(x, \mathcal{E}^\bullet(X), x))$ of a complex built from the $C^\infty$ de Rham complex via the bar construction:

$$B^\bullet(x, \mathcal{E}^\bullet(X), x) = (\mathcal{E}^\bullet(X))^\otimes -$$

$$\pm d_B(\alpha_1 \otimes \ldots \otimes \alpha_n) = \ i_x(\alpha_1)\alpha_2 \otimes \ldots \otimes \alpha_n$$

$$+ \sum (-1)^i \alpha_1 \otimes \ldots \otimes \alpha_i \wedge \alpha_{i+1} \otimes \ldots \otimes \alpha_n$$

$$+ (-1)^n i_x(\alpha_n)\alpha_1 \otimes \ldots \otimes \alpha_{n-1}$$

$$\pm d\alpha_1 \otimes \alpha_2 \otimes \ldots \otimes \alpha_n$$

$$+ \ldots$$

where $i_x : \mathcal{E}^\bullet(X) \to \mathbb{C}$ is the augmentation given by evaluation at $x$. Hain showed how to extend $B(x, \mathcal{E}^\bullet(X), x)$ to a cohomological mixed Hodge complex (or more precisely a direct limit of such), and was thus able to deduce the corresponding structure on cohomology. One thing that is more readily apparent in Hain’s approach is the dependence on base points. As $x$ varies, $\mathbb{Q}[\pi_1(X, x)]/J^{r+1}$ forms part of an admissible variation of mixed Hodge structure over $X$ called the tautological variation. This is nontrivial since the monodromy representation is the natural conjugation homomorphism $\pi_1(X, x) \to Aut(\mathbb{Q}[\pi_1(X, x)]/J^{r+1})$.

Wojtkowiak [W] gave a more algebro-geometric interpretation of Hain’s construction, which will be briefly described. Bousfield and Kan defined the total
space functor \( \text{Tot} \) [BK, chap X §3], which is a kind of geometric realization, from the category of cosimplicial spaces to the category of spaces. The image of the map of cosimplicial schemes

\[
\begin{array}{c}
X^{\Delta[1]} = X \times X \xrightarrow{\pi} X \times X \times X \quad \cdots \\
\downarrow \quad \downarrow \quad \downarrow \\
X^{0\Delta[1]} = X \times X \xrightarrow{\pi} X \times X \quad \cdots
\end{array}
\]

under this functor is the path space fibration \( X^{[0,1]} \rightarrow X^{(0,1)} = X \times X \). The horizontal maps on the top are diagonals (from left to right) or projections (from right to left); on the bottom they are all identities. The total space of the fibre \( \pi^{-1}(x,x) \) is the space of loops of \( X \) based at \( x \), which is an \( H \)-space. Therefore \( H^0(\text{Tot}(\pi^{-1}(x,x)), \mathbb{Q}) \) is naturally a Hopf algebra. This Hopf algebra, which is described more precisely in [W], can be identified with the coordinate ring of \( \lim G_r(\pi_1(X,x)) \), or \( G_r(\pi_1(X,x)) \) if we truncate the cosimplicial space at the \( r \)th stage. This follows from the fact \( H^0(\text{Tot}(\pi^{-1}(x,x)), \mathbb{C}) \) can be computed using the total complex of the de Rham complex of the cosimplicial fibre, which is none other than \( B(x, \mathcal{E}^s(X), x) \). Under this identification the filtration by truncations induced on \( H^0 \) coincides with the filtration by length of tensors on \( B \). The mixed Hodge structure on \( H^0(\text{Tot}(\pi^{-1}(x,x)), \mathbb{Q}) \) can now be constructed using standard machinery: take compatible multiplicative mixed Hodge complexes on each component of the cosimplicial space and then form the total complex [W §5]. Furthermore the tautological variations are given by the 0th total direct images of \( \mathbb{Q} \) under \( \pi|_X \) under the diagonal embedding \( X \subset X \times X \). A useful consequence of this point of view is that the MHS on \( \mathbb{Q}[\pi_1(X,x)]/J^{r+1} \) can be seen to come from a motive in Nori’s sense [K].

Let \( \text{UMHS}(X) \) (\( U_r \text{MHS}(X) \)) denote the subcategory of unipotent admissible variations of mixed Hodge structure (with index of unipotency at most \( r \)). The tautological variation associated to \( \mathbb{Q}[\pi_1(X,x)]/J^{r+1} \) lies in \( U_r \text{MHS}(X) \). Given an object \( V \) in \( U_r \text{MHS}(X) \), the monodromy representation extends to an algebra homomorphism

\[
\mathbb{Q}[\pi_1(X,x)]/J^{r+1} \rightarrow \text{End}(V_x)
\]

which is compatible with mixed Hodge structures.

**Theorem 5.2 (Hain-Zucker [HZ]).** The above map gives an equivalence between \( U_r \text{MHS}(X) \) and the category of Hodge representations of \( \mathbb{Q}[\pi_1(X,x)]/J^{r+1} \).

The above equivalence respects the tensor structure.

We note that every object of \( U_r \text{Loc}(X) \) is a sum of subquotients of the local system associated to the tautological representation \( \pi_1(X,x) \rightarrow \text{Aut}(\mathbb{Q}[\pi_1(X,x)]/J^{r+1}) \). Therefore \( \phi(U_r \text{MHS}(X)) = U_r \text{Loc}(X) \), where \( \phi : U_r \text{MHS}(X) \rightarrow \text{Loc}(X) \) is the forgetful functor. Consequently, we get a split exact sequence

\[
1 \longrightarrow \pi_1(X,x)^{unr} \longrightarrow \Pi(U_r \text{MHS}(X)) \xrightarrow{MT} MT \longrightarrow 1
\]

by theorem 2.6. In particular, \( \pi_1(X,x)^{unr} \) carries an NMHS which is a quotient of the one on \( \pi_1^{\text{hodge}}(X,x) \).

**Proposition 5.3.** The above NMHS on \( \pi_1(X,x)^{unr} \) is equivalent to the Morgan-Hain structure on \( \mathbb{Q}[\pi_1(X,x)]/J^{r+1} \).
Proof. By lemma 5.1, the Morgan-Hain structure on $\mathbb{Q}[\pi_1(X,x)]/J^{r+1}$ induces an NMHS on $\pi_1(X,x)^{unr}$. Let $MH$ denote the semidirect product of $MT$ with this Hodge structure on $\pi_1(X,x)^{unr}$. By theorem 5.2 we have a commutative diagram

$$
\begin{array}{ccc}
U_r \text{Loc}(X) & \xleftarrow{\phi} & U_r \text{MHS}(X) \\
\downarrow & & \downarrow \cong \\
\text{Rep}(\pi_1(X)^{unr}) & \xrightarrow{\kappa} & \text{Rep}(MH) \\
\end{array}
\quad \begin{array}{ccc}
\text{MHS} & \xrightarrow{\psi} & \text{Rep}(MT)
\end{array}
$$


where the functors $\psi$ and $\kappa$ are given by the fibre and the pullback along the constant map. Therefore $MH$ is isomorphic to $\Pi(U_r \text{MHS}(X))$ as a semidirect product.

\[ \square \]

**Corollary 5.4.** The Morgan-Hain NMHS on $\pi_1(X,x)^{un}$ is a quotient of $\pi_1^{hodge}(X,x)$.

Hain has extended the above construction in [H2]. Given a representation $\rho : \pi_1(X,x) \to S$ to a reductive algebraic group, the relative Mal’cev completion is the universal extension

$$1 \to U \to \mathfrak{g} \to S \to 1$$

of $S$ by a prounipotent group with a homomorphism $\pi_1(X,x) \to \mathfrak{g}$ such that

$$
\begin{array}{ccc}
\pi_1(X,x) & \to & \mathfrak{g} \\
\rho & \downarrow & \downarrow \\
& S & 
\end{array}
$$

commutes. When $\rho : \pi_1(X,x) \to S = \text{Aut}(\mathcal{V}_x,\langle , \rangle)$ is the monodromy representation of a variation of Hodge structure with Zariski dense image, Hain [H2] has shown that the relative Mal’cev completion carries a NMHS.

**Conjecture 5.5.** The relative completion should carry an NMHS in general with $S$ equal to the Zariski closure of $\pi_1(X,x) \to \text{Aut}(\mathcal{V}_x,\langle , \rangle)$. This should be a quotient of $\pi_1^{hodge}(X,x)$.

I am quite confident about this. The essential point would construct an inner nonabelian variation of mixed Hodge structure on the family of $\mathfrak{g}$ as the base point varies. Then proposition 4.3 would give a homomorphism $\pi_1^{hodge}(X,x) \to \mathfrak{g}$. The main step would be to establish an appropriate refinement of [H2] cor 13.11, and I understand that Hain, Matsumoto, Pearlstein, and Terasoma [HMPT] have done this. J. Pridham has pointed to me that his preprint [P] may also have some bearing on this conjecture.

**Remark 5.6.** For the applications given later in section 7, only this weaker statement on the existence of a morphism $\pi_1^{hodge}(X,x) \to \mathfrak{g}$ extending $\rho$ is needed. There is one notable case in which this can be deduced immediately. If $\pi_1(X,x)$ is abelian, then $\mathfrak{g}$ is necessarily abelian, so it splits into a product $U \times S$. Morphisms $\pi_1^{hodge}(X,x) \to U$ and $\pi_1^{hodge}(X,x) \to S$ can be constructed directly from propositions 4.3 and 4.5.
6. Cohomology

Fix a theory of enriched local systems \( E \). Let \( G \) be a nonabelian \( E \)-structure. The category of representations of \( \Pi(E(pt)) \times G \) is equivalent to the category of \( E \)-representations of \( G \). Given such a representation \( V \), let \( H^0(G, V) = V^G \) and \( H^0(\Pi(E(pt)) \times G, V) = V^{\Pi(E(pt)) \times G} \) be the subspaces of invariants. The action of \( \Pi(E(pt)) \) on \( V \) descends to an action on \( H^0(G, V) \). These are left exact functors on the category of representations of \( \text{Rep}_{\infty}(\Pi(E(pt)) \times G) \). Since this category has enough injectives (c.f. [J I 3.9]), we can define higher derived functors \( H^i(G, V) \) and \( H^i(\Pi(E(pt)) \times G, V) \). Note that \( H^i(G, V) \) is an \( E \)-structure since it is derived from a functor from \( \text{Rep}_{\infty}(\Pi(E(pt)) \times G) \rightarrow \text{Rep}_{\infty}(\Pi(E(pt))) \). Alternatively, \( H^i(G, V) \) can be computed as the cohomology of a bar or Hochschild complex \( C^\bullet(G, V) \) [J I 4.14], which is a complex in \( \text{Rep}_{\infty}(\Pi(E(pt))) \). We can define cohomology of \( \Pi(E(pt)) \) by taking \( G \) trivial. We observe that these functors are covariant in \( V \) and contravariant in \( G \).

There are products

\[
H^i(G, V) \otimes H^j(G, V') \rightarrow H^{i+j}(G, V \otimes V')
\]

compatible with \( E \)-structures. These can be constructed by either using standard formulas for products on the complexes \( C^\bullet(G, -) \), or by identifying \( H^i(G, V) = \text{Ext}^i(Q, V) \) and using the Yoneda pairing

\[
\text{Ext}^i(Q, V) \otimes \text{Ext}^j(Q, V') \rightarrow \text{Ext}^i(Q, V) \otimes \text{Ext}^j(V, V' \otimes V') \rightarrow \text{Ext}^{i+j}(Q, V \otimes V')
\]

**Lemma 6.1.** Given an \( E \)-representation \( V \) of \( G \), \( H^i(G, V) = \lim H^i(G/H_j, V^{H_j}) \) where \( H_j \) runs over all normal subgroups stable under \( \Pi(E(pt)) \) such that \( G/H_j \) is finite dimensional.

**Proof.** By lemma 3.4 \( G = \lim G/H_j \) is an inverse limit of algebraic groups with nonabelian \( E \)-structures. Clearly \( V = \lim V^{H_j} \) and so

\[
H^0(G, V) = \lim H^0(G/H_j, V^{H_j})
\]

The lemma follows from exactness of direct limits. \( \square \)

**Lemma 6.2.** Fix a Hodge representation \( V \) of an NMHS \( G \), and an Ind-MHS \( U \).

Then

1. \( H^i(MT, U) = 0 \) for \( i > 1 \).
2. We have an exact sequence

\[
0 \rightarrow H^1(MT, H^{i-1}(G, V)) \rightarrow H^i(MT \ltimes G, V) \rightarrow H^0(MT, H^i(G, V)) \rightarrow 0
\]

**Proof.** Write \( U \) as a direct limit of finite dimensional Hodge structures \( U_j \), then

\[
H^i(MT, U) = \lim H^i(MT, U_j) = \lim \text{Ext}^i_{\text{MHS}}(Q, U_j)
\]

Beilinson [B] shows that the higher \( \text{Ext} \)'s vanish, which implies the first statement.

For the second statement, note that we can construct a Hochschild-Serre spectral sequence

\[
E_2^{pq} = H^p(MT, H^q(G, V)) \Rightarrow H^{p+q}(MT \ltimes G, V)
\]

in the usual way. This reduces to the given exact sequence thanks to (1). \( \square \)
Lemma 6.3. Let $G$ be a nonabelian $E$-structure and $V$ an $E$-representation. If $G$ is pro-reductive then $H^i(G, V) = 0$ for $i > 0$. In general

$$H^i(G, V) = H^0(G^{\text{red}}, H^i(U, V))$$

where $U$ is the pro-unipotent radical and $G^{\text{red}} = G/U$.

Proof. When $G$ is pro-reductive, its category of representations is semisimple. Therefore $H^0(G, -)$ is exact. So higher cohomology must vanish.

In general, the Hochschild-Serre spectral sequence

$$E_2^{pq} = H^p(G^{\text{red}}, H^q(U, V)) \Rightarrow H^{p+q}(G, V)$$

will collapse to yield the above isomorphism. □

Proposition 6.4. Let $X$ be a smooth variety. Then for each object $V \in E(X)$:

1. $H^i(\pi_1^F(X, x), V)$ carries a canonical $E$-structure.
2. There is natural morphism $H^i(\pi_1^F(X, x), V) \to H^i(X, V)$ of $E$-structures.
3. There are $\mathbb{Q}$-linear maps

$$H^i(\pi_1^F(X, x), V) \to H^i(\pi_1(X, x), V) \to H^i(X, V)$$

whose composition is the map given in (2).

Proof. $H^i(\pi_1^F(X, x), V)$ carries an $E$-structure by the discussion preceding lemma 6.1. Note that $H^0(\pi_1^F(X, x), V) = V_{\pi_1^F(X)}$ is nothing but the monodromy invariant part of $V$, and $H^i(\pi_1^F(X, x), V)$ is the universal $\delta$-functor extending it, in the sense of [G]. By axiom (E4), $H^i(X, V)$ with its $E$-structure also forms a $\delta$-functor, and there is an isomorphism

$$H^0(\pi_1^F(X, x), V) \cong H^0(X, V)$$

of $E$-structures. Therefore (2) is a consequence of universality.

After disregarding $E$-structure, we can view $H^i(\pi_1^F(X, x), V)$ as a universal $\delta$-functor from $E(X) \to \text{Vect}_{\mathbb{Q}}$. Therefore, from the isomorphisms

$$H^0(\pi_1^F(X, x), V) \cong H^0(\pi_1(X, x), V) \cong H^0(X, V),$$

we deduce $\mathbb{Q}$-linear maps

$$H^i(X, V) \leftarrow H^i(\pi_1^F(X, x), V) \to H^i(\pi_1(X, x), V)$$

The leftmost map was the one constructed in the previous paragraph. Since group cohomology is also a universal $\delta$-functor from the category of $\mathbb{Q}[\pi_1(X, x)]$-modules to $\text{Vect}_{\mathbb{Q}}$, we can complete this to a commutative triangle

$$\begin{array}{c}
H^i(\pi_1^F, V) \longrightarrow H^i(\pi, V) \\
\downarrow \\
H^i(X, V)
\end{array}$$

□
The map
\[
(*) 
H^i(\pi_1^E(X,x),V) \to H^i(X,V)
\]
constructed in the previous section is trivially an isomorphism for \(i = 0\), but usually not in general. For instance if \(i = 2, V = \mathbb{Q}\) and \(X = \mathbb{P}^1\), the cohomology groups are 0 on the left and \(\mathbb{Q}\) on the right.

Let us say that \(X\) is a \(K(\pi^E, 1)\) (or a Hodge theoretic \(K(\pi, 1)\) when \(E = \text{MHS}\)) if (**) is an isomorphism for every \(V \in E(X)\). When \(X\) is a \(K(\pi, 1)\) in the usual sense, then it is a \(K(\pi^E, 1)\) if and only if
\[
(**) 
H^i(\pi_1^E(X,x), V) \to H^i(\pi_1(X,x), V)
\]
is an isomorphism for all \(V \in E(X)\). The following is a straightforward modification of [Sc, p 13, ex 1].

**Lemma 7.1.** The following are equivalent

1. (**) is an isomorphism for all \(i \leq n\) and injective for \(i = n + 1\) for all \(V \in \text{Ind-E}(X)\).
2. (**) is an isomorphism for all \(i \leq n\) and injective for \(i = n + 1\) for all \(V \in E(X)\).
3. (**) is surjective for all \(i \leq n\) and all \(V \in E(X)\).
4. (**) is surjective for all \(i \leq n\) and all \(V \in \text{Ind-E}(X)\).
5. For all \(V \in \text{Ind-E}(X), 1 \leq i \leq n\), and \(\alpha \in H^i(X,V)\), there exists \(V' \in \text{Ind-E}(X)\) containing \(V\) such that the image of \(\alpha\) in \(H^i(X,V')\) vanishes.

In particular, \(X\) is a \(K(\pi^E, 1)\) if (**) is surjective for all \(i\) and \(V \in E(X)\).

**Proof.** The implications (1)\(\Rightarrow\)(2) and (2)\(\Rightarrow\)(3) are clear. For (3)\(\Rightarrow\)(4), we can use the fact that cohomology commutes with filtered direct limits. The implication (4)\(\Rightarrow\)(5) follows because \(\text{Ind-E}(X)\) contains enough injectives [3]. Any injective object \(V' \supset V\) will satisfy the conditions of (5) assuming (4).

Finally, we prove (5)\(\Rightarrow\)(1). This is the only nontrivial step. An exact sequence
\[
0 \to V \to V' \to V'\big/ V \to 0
\]
yields a diagram
\[
\begin{array}{cccccc}
H^{i-1}(\pi_1^E,V') & \longrightarrow & H^{i-1}(\pi_1^E,V'/V) & \longrightarrow & H^i(\pi_1^E,V) & \longrightarrow & H^i(\pi_1^E,V') \\
\downarrow f & & \downarrow g & & \downarrow & & \downarrow \\
H^{i-1}(X,V') & \longrightarrow & H^{i-1}(X,V'/V) & \longrightarrow & H^i(X,V) & \longrightarrow & H^i(X,V')
\end{array}
\]
We first prove surjectivity of (1) for \(i \leq n\) by induction. This is trivially true when \(i = 0\), so we may assume \(i > 0\). Given \(\alpha \in H^i(X,V)\), we may choose \(V'\) so that \(\alpha\) has trivial image in \(H^i(X,V')\). Then \(\alpha\) lifts to \(H^{i-1}(X,V'/V)\) and hence to some \(\beta \in H^{i-1}(\pi_1^E,V'/V)\). The image of \(\beta\) in \(H^i(\pi_1^E,V)\) will map to \(\alpha\) as required.

Now we prove injectivity of (1) for \(i \leq n + 1\) by induction. We may assume that \(i > 0\). Let \(V'\) be injective in \(\text{Ind-E}(X)\). Suppose that
\[
\alpha \in \ker[H^i(\pi_1^E(X,x),V) \to H^i(X,V)]
\]
Then \(\alpha\) can be lifted to \(\beta \in H^{i-1}(\pi_1^E,V'/V)\). Since the maps labelled \(f\) and \(g\) are isomorphisms, a simple diagram chase shows that \(\beta\) lies in the image of \(H^{i-1}(\pi_1^E,V')\). Therefore \(\alpha = 0\).
\[\square\]
Proposition 7.2. When $E = MHS$, for all $V \in MHS(X)$ the map $(\ref{eq:7.2})$ is an isomorphism for $i \leq 1$ and injective for $i = 2$ assuming conjecture 5.5 holds. This is true unconditionally if $\pi_1(X)$ is abelian.

Proof. As is well known, for any $V$ the map

$$H^i(\pi_1(X,x),V) \to H^i(X,V)$$

is an isomorphism for $i = 1$ and injective for $i = 2$. Thus, by this remark and the previous lemma, it is enough to prove that the map $(\ref{eq:7.2})$ to group cohomology is surjective for $i = 1$. By the 5-lemma and induction on the length of the weight filtration, it is sufficient to the prove this when $V$ is pure.

Let $V$ be a variation of pure Hodge structure, and let $1 \to U \to G \to S \to 1$ be the associated relative Malčev completion. By lemma 6.3,

$$H^1(G,V) \cong H^0(S,H^1(U,V)) \cong \text{Hom}_S(U/[U,U],V)$$

By [H2, prop 10.3]

$$H^1(\pi_1(X,x),V) \cong \text{Hom}_S(U/[U,U],V)$$

Therefore the natural map $H^1(G,V) \cong H^0(S,H^1(U,V)) \cong \text{Hom}_S(U/[U,U],V)$.

Hence (**) to group cohomology is surjective for $i = 1$. By the 5-lemma and induction on the length of the weight filtration, it is sufficient to the prove this when $V$ is pure.

Theorem 7.3. A (not necessarily affine) connected commutative algebraic group is a Hodge theoretic $K(\pi,1)$. Assuming conjecture 5.5, a smooth affine curve is a Hodge theoretic $K(\pi,1)$.

Proof. Suppose that $X$ is a commutative algebraic group. Then it is homotopy equivalent to a torus. In particular, it is a $K(\pi,1)$. So it suffices to check surjectivity of $(\ref{eq:7.2})$. The group $\pi_1(X)$ is abelian and finitely generated, which implies that the pro-algebraic completion of $\pi_1(X)$ is a commutative algebraic group. Therefore the same is true for $\pi_1^{\text{hodge}}(X)$. After extending scalars, it follows that the group $\pi_1^{\text{hodge}}(X) \otimes \bar{\mathbb{Q}}$ is a product of $\mathbb{G}_a$’s, $\mathbb{G}_m$’s and a finite abelian group. Consequently, any irreducible representation $V$ of $\pi_1^{\text{hodge}}(X) \otimes \bar{\mathbb{Q}}$ is one dimensional. For such a module, the Künneth formula implies that

$$H^i(\pi_1(X),V) = \begin{cases} \wedge^i H^1(\pi_1(X),V) & \text{if } V \text{ is trivial} \\ 0 & \text{otherwise} \end{cases}$$

Therefore $(\ref{eq:7.2})$ is surjective in this case. By applying (an appropriate modification of) lemma 7.1 to the category of semisimple $\pi_1^{\text{hodge}}(X) \otimes \bar{\mathbb{Q}}$ representations, we can see that $(\ref{eq:7.2})$ is an isomorphism. We note that

$$H^i(\pi_1^{\text{hodge}}(X),V \otimes \bar{\mathbb{Q}}) \cong H^i(\pi_1^{\text{hodge}}(X),V) \otimes \bar{\mathbb{Q}}$$

and likewise for the map between them. Thus we may extend scalars in order to test bijectivity in $(\ref{eq:7.2})$. After doing so, we see that $(\ref{eq:7.2})$ is an isomorphism by an induction on the length of a Jordan-Holder series.

Let $X$ be a smooth affine curve. This is a $K(\pi,1)$ with a free fundamental group. Since a free group has cohomological dimension one, $(\ref{eq:7.2})$ is surjective in all degrees assuming 5.5 by the previous proposition.

□
I expect that Artin neighbourhoods are also Hodge theoretic $K(\pi, 1)$’s. This would give a large supply of such spaces. Katzarkov, Pantev and Toen have established an analogous result in their setting [KPT2, rmk 4.17]. Although, their proofs do not translate directly into the present framework, I suspect that an appropriate modification may.

**References**

[A] D. Arapura, *A category of motivic sheaves*, arXiv:0801.0261

[B] A. Beilinson, *Notes on absolute Hodge cohomology*, Applications of algebraic K-theory to algebraic geometry and number theory, AMS (1987)

[BK] A. Bousfield, D. Kan, *Homotopy limits, completions and localizations*, LNM 304, Springer-Verlag (1972)

[C] M. Cushman, *Morphisms of curves and the fundamental group*, Contemp Trends in Alg. Geom. and Alg. Top., World Scientific (2002)

[DM] P. Deligne, J. Milne, *Tannakian categories*, in LNM 900, Springer-Verlag (1982)

[D1] P. Deligne, *Theorie de Hodge II*, Inst. Hautes Études Sci. Publ. Math 40 (1971)

[D2] P. Deligne, *Le groupes fondemental de la droit projective moin trois points*, Galois Groups over $\mathbb{Q}$, Springer-Verlag (1989)

[D3] P. Deligne, *Catégories Tannakiennes*, Grothendieck Festschrift, Birkhauser (1990)

[G] A. Grothendieck, *Sur quelques points d algèbre homologiques*, Tohoku Math. J. 9 (1957)

[H1] R. Hain, *The de Rham homotopy theory of complex algebraic varieties. I*, K-theory 3 (1987)

[H2] R. Hain, *The Hodge de Rham theory of relative Malcev completion*, Ann. Sci. cole Norm. Sup. 31 (1998), 47–92.

[HZ] R. Hain, S. Zucker, *Unipotent variations of mixed Hodge structure*, Inv. Math 88 (1987)

[HMPT] R. Hain, M. Matsumoto, G. Pearlstein, T. Terasoma, *Tannakian Fundamental Groups of Categories of Variations of Mixed Hodge Structure*, in preparation

[I] J. Jantzen, *Representations of algebraic groups*, 2nd ed., AMS (2003)

[K] M Kashiwara, *A study of a variation of mixed Hodge structure*, Publ. Res. Inst. Math. Sci. 22 (1986)

[KPT1] L. Katzarkov, T. Pantev, B. Toen, *Schematic homotopy types and non-abelian Hodge theory*, Compositio Math (2008)

[KPT2] L. Katzarkov, T. Pantev, B. Toen, *Algebraic and topological aspects of the schematization functor*, arXiv:math.AG/0503418

[Ko] J. Kollár, *Shafarevich maps and automorphic forms*, Princeton U. Press (1995)

[M] J. Morgan, *The algebraic topology of smooth algebraic varieties*, Inst. Hautes Études Sci. Publ. Math 48 (1978)

[P] J. Pridham, *Formality and splitting of real non-abelian mixed Hodge structures*, arXiv:0902.0770

[Q] D. Quillen, *Rational homotopy theory*, Ann. Math 90 (1969)

[Sa1] M. Saito, *Mixed Hodge modules and admissible variations*, CR Acad. Sci. Paris 309 (1989)

[Sa2] M. Saito, *Mixed Hodge modules*, Publ. Res. Inst. Math. Sci. 26 (1990)

[Se] J.P. Serre, *Cohomologie Galois*, 5th ed., LNM 5, Springer-Verlag (1994)

[Si] C. Simpson, *Higgs bundles and local systems*, Inst. Hautes Études Sci. Publ. Math 75 (1992)

[SZ] J. Steenbrink, S. Zucker, *Variation of mixed Hodge structures I*, Inv. Math. 80 (1985)

[W] Z. Wojtkowiak, *Cospicuous objects in algebraic geometry*, Alg. K-theory and Alg. Top., Kluwer (1993)

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