[0,1] IS NOT A MINIMALITY DETECTOR FOR [0, 1]²

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Abstract. This paper shows that there exists a non-minimal sequence \( \bar{x} \in \{0, 1\}^\mathbb{N} \) such that for any continuous function \( f : [0, 1]^2 \to [0, 1] \), the sequence obtained by mapping terms of \( \bar{x} \) by \( f \) is minimal.

Let \( (X, d) \) be a compact metric space. \( X^\mathbb{N} \), the set of infinite sequences with terms taken from \( X \), is also a compact metric space under the metric \( d(\bar{x}, \bar{y}) = \sum_{i=1}^{\infty} \frac{d(x_i, y_i)}{2^i} \).

\( X^\mathbb{N} \) has a natural dynamical system, the (left) shift:

\[ S : X^\mathbb{N} \to X^\mathbb{N} \quad S((x_1, x_2, x_3, \ldots)) = (x_2, x_3, \ldots). \]

Definition 1. \( \bar{z} \in X^\mathbb{N} \) is called minimal if the closure of \( \{ \bar{z}, S(\bar{z}), S^2(\bar{z}), \ldots \} \) is minimal as a dynamical system under the action of \( S \).

An equivalent formulation is that a sequence is minimal iff for any finite block, \((z_i, \ldots, z_{i+r})\), and \( \epsilon > 0 \) there exists an \( N \in \mathbb{N} \) such that any block of \( \bar{z} \) of length \( N \) has a sub-block of length \( r+1 \) which is within \( \epsilon \) of \((z_i, \ldots, z_{i+r})\). The distance between \( n \)-blocks is given by \( d((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \sum_{i=1}^{\infty} \frac{d(x_i, y_i)}{2^i} \). (For an introduction to minimality see [1].)

Definition 2. Let \( X, Y \) be compact metric spaces. \( C(X, Y) \) denotes all continuous maps from \( X \) into \( Y \). We say \( Y \) is a minimality detector (MD) for \( X \) if for every \( \bar{x} = (x_1, x_2, \ldots) \in X^\mathbb{N} \), which is not minimal, there exists \( f \in C(X, Y) \) such that \((f(x_1), f(x_2), \ldots) \in Y^\mathbb{N} \) is not minimal.

Remark 1. The continuous image of a minimal sequence is minimal.

Definition 3. If \( Y \) is a minimality detector (MD) for all compact metric spaces \( X \), then \( Y \) is called a universal minimality detector (UMD).

The main result of this note is showing that \([0, 1]\) is not an MD (minimality detector). The following is an earlier result motivating this.

Theorem 1. (Boshernitzan) \([0, 1]^2\) is a UMD (universal minimality detector).

Proof: Let \( (X, d) \) be a compact metric and \( \bar{x} \in (X)^\mathbb{N} \) be not minimal. By definition, there exists an \( \epsilon > 0 \) and \((x_1, \ldots, x_{i+r})\), a sub-block of \( \bar{x} \), such that there are arbitrarily long blocks of \( \bar{x} \) having no sub-blocks of length \( r+1 \) within \( \epsilon \) of \((x_1, \ldots, x_{i+r})\). Fix \( \delta > 0 \) such that any \( x_{i+j} \neq x_{i+k}, 0 \leq j, k \leq r \), has \( d(x_{i+j}, x_{i+k}) > 2 \delta \). Fix \( a_j \in [0, 1], 0 \leq j \leq r \) such that \( a_j = a_k \) iff \( x_{i+j} = x_{i+k} \). Let \( f : X \to [0, 1] \) be a continuous function that \( f(y) = a_j \) if \( y \in B(x_{i+j}, \delta) \). (By our choice of \( \delta \) and Tietze extension theorem ([2] Theorem 35.1) such a continuous function exists).

Let \( g : X \to [0, 1] \) by:

\[ g(y) = \min_{0 \leq j \leq r} \frac{d(x_{i+j}, y)}{\sup_{x_1, x_2 \in X} d(x_1, x_2)}. \]
The sequence \( \tilde{e} = ((f(x_1), g(x_1)), (f(x_2), g(x_2)), \ldots) \in (\{0, 1\}^2)^\mathbb{N} \) is not minimal because there exists \( \epsilon' \) such that arbitrarily long blocks of \( \tilde{e} \) have no sub-blocks within \( \epsilon' \) of \( ((f(x_1), g(x_1)), (f(x_{i+1}), g(x_{i+1})), \ldots, (f(x_{i+r}), g(x_{i+r}))) \). This is because in a sub-block of length \( r + 1 \) in a long block of \( \tilde{e} \) which avoids \( (x_1, \ldots, x_{i+r}) \) either some term is far away from an \( x_{i+s} \) or it is close, but appears in the wrong order. In the first case \( g \) notices the difference, while \( f \) notices the difference in the second case.

**Remark 2.** This argument produces a residual set of functions in \( C(X, [0, 1]^2) \), each of which detects the non-minimality of \( \tilde{e} \). As a result, the non-minimality of any particular countable collection of non-minimal sequences in \( X^\mathbb{N} \) can be detected by a residual set of functions.

**Remark 3.** It is an observation of Professor B. Weiss [4] that an infinite fan (say \( \{(r \cos(\theta), r \sin(\theta)) : r \in [0, 1], \theta \in \{2\pi, \frac{3\pi}{2}, \ldots\}\} \)) is a UMD. The proof is similar. This stands in contrast to the situation in which the range is \([0, 1]\) as the next theorem shows.

**Theorem 2.** \([0, 1]\) is not a UMD. In particular, \([0, 1]\) is not an MD for \([0, 1]^2\).

**Definition 4.** Let \( B = \{b_1, b_2, \ldots\} \) be an infinite sequence. Its 2-Toeplitz sequence is \( \bar{a} = (a_1, a_2, a_3, \ldots) \) where \( a_j = b_i \) \( \forall j \equiv 2^i - 1 \bmod 2^i \).

**Remark 4.** Equivalently, \( a_j = b_p \) where \( 2^{p-1} | j \) and \( 2^p \not| j \).

**Remark 5.** This terminology is not standard. Compare with the sequence A001511 from [3].

The 2-Toeplitz sequence \( \bar{a} \) begins:
\[
(b_1, b_2, b_1, b_3, b_1, b_2, b_1, b_1, b_2, b_1, b_3, b_1, b_2, b_1, b_1, \ldots).
\]
Observe that every block in \( \bar{a} \) appears in \( \bar{a} \) with bounded gaps. In fact, if \( b_i \) is the largest term in a block then that block will appear in any block of \( 2^{i+1} \) consecutive terms. This gives the following lemma:

**Lemma 1.** Let \( B = \{b_1, b_2, \ldots\} \) be a sequence whose elements lie in a compact metric space. Then its 2-Toeplitz sequence is minimal.

The following example provides a toy model for the proof of the main result.

**Proposition 1.** Let \( \bar{a} = (a_1, a_2, \ldots) \) be the 2-Toeplitz sequence associated with an enumeration of \( \mathbb{Q} \cap [0, 1] \). Then \( (c, a_1, a_2, \ldots) \) is minimal for any \( c \in [0, 1] \).

**Proof:** Choose \( c \). For any \( \epsilon > 0 \) there exists infinitely many \( a_r \) such that \( |a_r - c| < \epsilon \). Choose \( r \) such that \( \frac{1}{2^r} < \epsilon \). \( d(S^{2^r-1}(\bar{a}), (c, a_1, a_2, \ldots)) \leq \epsilon + \frac{1}{2^{2^r}} < 2\epsilon \). The first inequality is because their first coordinates are at most \( \epsilon \) apart and then they agree for the next \( 2^r - 1 \) coordinates. By the previous discussion, this block will appear at least every \( 2^{r+1} \) terms in \( \bar{a} \).

One obviously can not guarantee minimality for any pair \( c, d \) placed at the start of \( \bar{a} \). However, one can modify this construction so that this is the case by enumerating \( \mathbb{Q}^2 \cap [0, 1]^2 \) and examining the 2-Toeplitz sequence associated with this enumeration (thought of as a sequence in \([0, 1]\]).
1. Building the sequence for Theorem 2

**Definition 5.** Given a finite sequence \( \bar{x} = (x_1, ..., x_t) \) and a (finite or infinite) sequence \( \bar{y} \), we define \( \bar{x} * \bar{y} \) to be the word formed by concatenating them.

That is, \( (\bar{x} * \bar{y})_i = x_i \) if \( i \leq t \) and \( y_{i-t} \) otherwise.

Our sequence will begin \( ((0, 0), (1, 0), (0, 1)) \). Before we continue the sequence, it is necessary to define some sets.

Let \( A_1 \) be a countable dense set on the line from \((1, 0)\) to \((0, 1)\).

Let \( A_2 \) be a countable dense set on the line from \((0, 0)\) to \((1, 0)\).

Let \( A_3 \) be a countable dense set on the line from \((0, 0)\) to \((0, 1)\).

Finally, let \( V = \{v_1, v_2, \ldots \} \) be a sequence of all the terms in:
\[
(A_1 \times \{(1, 0)\} \times \{(0, 1)\}) \cup (\{(0, 0)\} \times A_2 \times \{(0, 1)\}) \cup (\{(0, 0)\} \times \{(1, 0)\} \times A_3) \subset \mathbb{R}^6.
\]

Let \( \bar{b} \) be the 2-Toeplitz sequence associated with \( V \). Our sequence is
\[
\bar{x} = ((0, 0), (1, 0), (0, 1)) \ast \bar{b}
\]
thought of as a sequence in \([0, 1]^2\) (so each “letter” in \( \bar{b} \) gives us 3 “letters” in \( \bar{x} \)).

**Lemma 2.** \( \bar{x} \) is not minimal.

Proof: It suffices to show that no block of 3 consecutive letters after the first 3 gets close to the block of the first 3. To be more precise, 
\[
d((0, 0), (1, 0), (0, 1)), (x_i, x_{i+1}, x_{i+2})) \geq \frac{1}{8} \text{ for } i > 1.
\]
If \( x_i = (0, 0) \) then either
\[
x_{i+1} \in A_2 \text{ or } x_{i+2} \in A_3 \text{ and } d((0, 0), (1, 0), (0, 1)), (x_i, x_{i+1}, x_{i+2})) \geq \frac{1}{4}.
\]
If
\[
0 < d((0, 0), x_i) < \frac{1}{\sqrt{2}} \text{ then } x_i \in A_2 \cup A_3.
\]
If \( x_i \in A_2 \) then \( x_{i+1} = (0, 1) \) and
\[
d((0, 0), (1, 0), (0, 1)), (x_i, x_{i+1}, x_{i+2})) \geq \frac{1}{4}\sqrt{2}.
\]
It suffices to consider \( x_i \in A_3 \). Then \( x_{i+1} = (0, 0) \) or \( x_{i+1} \in A_1 \). If \( x_{i+1} = (0, 0) \) then
\[
d((0, 0), (1, 0), (0, 1)), (x_i, x_{i+1}, x_{i+2})) \geq \frac{1}{4}.
\]
If \( x_{i+1} \in A_1 \) then \( x_{i+2} = (1, 0) \) and
\[
d((0, 0), (1, 0), (0, 1)), (x_i, x_{i+1}, x_{i+2})) \geq \frac{1}{4}\sqrt{2}.
\]

**Lemma 3.** If \( f : [0, 1]^2 \to [0, 1] \) continuously, then \( f(x_1), f(x_2), \ldots \) is minimal.

Proof: Without loss of generality assume \( f((0, 0)) \leq f((1, 0)) \leq f((0, 1)) \). By the intermediate value theorem, for any \( \epsilon > 0 \) there exists \( a \in A_2 \) such that
\[
|f(a) - f((1, 0))| < \epsilon.
\]
By the construction of \( V \), \( (0, 0) \ast a \ast (0, 1) = v_i \) for some \( i \). By the construction of \( \bar{x} \),
\[
d(f(x_{3(2^i-1)+1}), f(x_{3(2^i-1)+2}), \ldots), (f(x_1), f(x_2), \ldots)) \leq \frac{1}{4} + \epsilon + 0 + \frac{1}{2^{2^i-1} \text{min}_{-1}}.
\]
In fact, by the construction of \( \bar{x} \), this is achieved with bounded gaps of \( 3 \cdot 2^i \). This shows that the image is minimal.

**Remark 6.** This sequence lives on the triangle. It shows that the interval is not an MD for the triangle.

**Remark 7.** This proof can be modified to show that no finite graph is an MD for \([0,1]^2\). In fact, with the previous remark, one can show the complete graph on \( n+1 \) vertices has a non-minimal sequence \( \bar{x} \) such that \( (f(x_1), f(x_2), \ldots) \) is minimal for any continuous \( f \) from the complete graph on \( n+1 \) vertices to a graph on \( n \) vertices. Compare this to remark 3. In fact, no finite graph is an MD for an infinite fan.

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