THETA HYPERGEOMETRIC SERIES

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Abstract. We formulate general principles of building hypergeometric type series from the Jacobi theta functions that generalize the plain and basic hypergeometric series. Single and multivariable elliptic hypergeometric series are considered in detail. A characterization theorem for a single variable totally elliptic hypergeometric series is proved.

Contents

1. Introduction 1
2. Theta hypergeometric series \( E_s \) and \( G_s \) 3
3. Elliptic hypergeometric series 6
4. Multiple elliptic hypergeometric series 14
References 18

1. Introduction

This note is a mostly conceptual work reflecting partially the content of a lecture on special functions of hypergeometric type associated with elliptic beta integrals presented by the author at the NATO Advanced Study Institute “Asymptotic Combinatorics with Applications to Mathematical Physics” (St. Petersburg, July 9-22, 2001). It precedes a forthcoming extended and technically more elaborate review [S4]. Here we describe general principles of building hypergeometric type series associated with the Jacobi theta functions [MS]. Some other essential results of [S4] were briefly presented in [S2, S3]. We discuss only single variable and multivariable series built out of the Jacobi theta functions though some generalizations based upon the multidimensional Riemann theta functions are possible. Moreover, the main attention will be paid to such series obeying certain ellipticity conditions, i.e. to elliptic hypergeometric series. We start from a description of the Jacobi theta functions properties [AS, WW].

Let us take two complex variables \( p \) and \( q \) lying inside the unit disk, i.e. \(|p|, |q| < 1\). The modular parameters \( \sigma \), \( \text{Im}(\sigma) > 0 \), and \( \tau \), \( \text{Im}(\tau) > 0 \), are introduced through an exponential representation

\[
p = e^{2\pi i \tau}, \quad q = e^{2\pi i \sigma}.
\]
Define the $p$-shifted factorials\footnote{GR} 
\[(a; p)_\infty = \prod_{n=0}^{\infty} (1 - ap^n), \quad (a; p)_n = \frac{(a; p)_\infty}{(ap^n; p)_\infty}.\]

For a positive integer $n$ one has 
\[(a; p)_n = (1 - a)(1 - ap) \cdots (1 - ap^{n-1})\]
and 
\[\theta(1; p; -n) = \frac{1}{(ap^{-n}; p)_n}.\]

It is convenient to use the following shorthand notations 
\[(a_1, \ldots, a_k; p)_\infty \equiv (a_1; p)_\infty \cdots (a_k; p)_\infty.\]

Let us introduce a Jacobi-type theta function\footnote{WWW} 
\[\theta(z; p) = (z, pz, -1; p)_\infty.\] (2)

It obeys the following simple transformation properties 
\[\theta(pz; p) = \theta(z; p) = -z^{-1}\theta(z; p).\] (3)

One has also $\theta(p^{-1}z; p) = -p^{-1}z\theta(z; p)$. Evidently, $\theta(z; p) = 0$ for $z = p^{-M}$, $M \in \mathbb{Z}$, and $\theta(z; 0) = 1 - z$.

The standard Jacobi's $\theta_1$-function\footnote{WWW} is expressed through $\theta(z; p)$ as follows 
\[\theta_1(u; \sigma, \tau) = -i \sum_{n=-\infty}^{\infty} (-1)^n p^{(2n+1)^2/8} q^{(n+1/2)u} \]
\[= 2 \sum_{n=0}^{\infty} (-1)^n e^{\pi i r(n+1/2)^2} \sin \pi(2n + 1)\sigma u \]
\[= 2p^{1/8} \sin \pi \sigma u (p, pe^{2\pi i \sigma u}, pe^{-2\pi i \sigma u}; p)_\infty \]
\[= p^{1/8} i q^{-u/2} (p; p)_\infty \theta(q^u; p), \quad u \in \mathbb{C}.\] (4)

We have introduced artificially the second modular parameter $\sigma$ into the definition of $\theta_1$-function—the variable $u$ will often take integer values and it is convenient to make an appropriate rescaling from the very beginning. Note that other Jacobi theta functions $\theta_{2,3,4}(u)$ can be obtained from $\theta_1(u)$ by a simple shift of the variable $u$\footnote{WWW}, i.e. these functions structurally do not differ much from $\theta_1(u)$.

In the following considerations we shall be employing convenient notations by replacing $\theta_1$-symbol in favor of the elliptic numbers $[u]$ used in\footnote{DD-O}: 
\[[u] \equiv \theta_1(u) \quad \text{or} \quad [u; \sigma, \tau] \equiv \theta_1(u; \sigma, \tau), \]
\[\{u_0, \ldots, u_k\} = \prod_{m=0}^{k} [u_m].\]

Dependence on $\sigma$ and $\tau$ will be indicated explicitly only if it is necessary. The function $[u]$ is entire, odd $[-u] = -[u]$, and doubly quasiperiodic 
\[[u + \sigma^{-1}] = -[u], \quad [u + \tau \sigma^{-1}] = e^{-\pi i \tau^{-1}}[u].\] (5)

It is well-known that the theta function $[u]$ can be derived uniquely (up to a constant factor) from the transformation properties\footnote{WWW} and the demand of entireness.
Modular transformations are described by the following $SL(2, \mathbb{Z})$ group action upon the modular parameters $\sigma$ and $\tau$

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d}, \quad \sigma \rightarrow \frac{\sigma}{c\tau + d}, \quad (6)$$

where $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$. This group is generated by two simple transformations: $\tau \rightarrow \tau + 1$, $\sigma \rightarrow \sigma$, and $\tau \rightarrow -\tau^{-1}$, $\sigma \rightarrow \sigma\tau^{-1}$. In these two cases one has

$$[u; \sigma, \tau + 1] = e^{\pi i/4}[u; \sigma, \tau], \quad (7)$$
$$[u; \sigma/\tau, -1/\tau] = i(-i\tau)^{1/2}e^{\pi i\sigma^2u^2/\tau}[u; \sigma, \tau], \quad (8)$$

where the square root sign of $(-i\tau)^{1/2}$ is fixed from the condition that the real part of this expression is positive.

2. Theta hypergeometric series $E_s$ and $G_s$

Now we are going to introduce formal power series $E_s$ and $G_s$ built out from the Jacobi theta functions (we will not consider their convergence properties here). They generalize the plain hypergeometric series $F_s$ and basic hypergeometric series $\Phi_s$ together with their bilateral partners $H_s$ and $\Psi_s$. The definitions given below follow the general spirit of qualitative (but constructive) definitions of the plain and basic hypergeometric series going back to Pochhammer and Horn [AAR, GR, GGR]. In order to define theta hypergeometric series we use as a key property the quasiperiodicity of Jacobi theta functions [5].

**Definition.** The series $\sum_{n \in \mathbb{N}} c_n$ and $\sum_{n \in \mathbb{Z}} c_n$ are called theta hypergeometric series of elliptic type if the function $h(n) = c_{n+1}/c_n$ is a meromorphic doubly quasiperiodic function of $n$ considered as a complex variable. More precisely, for $x \in \mathbb{C}$ the function $h(x)$ should obey the following properties:

$$h(x + \sigma^{-1}) = ah(x), \quad h(x + \tau\sigma^{-1}) = be^{2\pi i\gamma x}h(x), \quad (9)$$

where $\sigma^{-1}, \tau\sigma^{-1}$ are quasiperiods of the theta function $[u]$ [10] and $a, b, \gamma$ are some complex numbers.

**Theorem 1.** Let a meromorphic function $h(n)$ satisfies the properties [10]. Then it has the following general form in terms of the Jacobi theta functions

$$h(n) = \frac{[n + u_1, \ldots, n + u_r]}{[n + v_1, \ldots, n + v_s]}q^{r's}y, \quad (10)$$

where $r, s$ are arbitrary non-negative integers and $u_1, \ldots, u_r, v_1, \ldots, v_s, \beta, y$ are arbitrary complex parameters restricted by the condition of non-singularity of $h(n)$ and related to the quasiperiodicity multipliers $a, b, \gamma$ as follows:

$$a = (-1)^{r-s}e^{2\pi i\beta}, \quad b = (-1)^{s-r}e^{\pi i(r-s+2\beta)}e^{2\pi i\sigma(\sum_{m=1}^{r} v_m - \sum_{m=1}^{s} u_m)}. \quad (11)$$

**Proof.** Let us tile the complex plane of $n$ by parallelograms whose edges are formed by the theta function quasiperiods $\sigma^{-1}$ and $\tau\sigma^{-1}$. Because the multipliers in the quasiperiodicity conditions [5] are entire functions of $n$ without zeros, the meromorphic function $h(n)$ has the same finite number of zeros and poles in each parallelogram on the plane. Let us denote as $-u_1, \ldots, -u_r$ the zeros of $h(n)$ in one of such parallelograms and as $-v_1, \ldots, -v_s$ its poles. For simplicity we assume that
these zeros and poles are simple—for creating multiple zeros or poles it is sufficient to set some of the numbers \( u_m \) or \( v_m \) to be equal to each other.

Let us represent the ratio of theta hypergeometric series coefficients as follows: 
\[
c_{n+1}/c_n = h(n)g(n), \quad \text{where } h(n) \text{ has the form } (10) \text{ with some unfixed parameter } \beta. \]
Since all the zeros and poles of \( c_{n+1}/c_n \) are sitting in \( h(n) \), the function \( g(n) \) must be an entire function without zeros satisfying the constraints \( g(n + \sigma^{-1}) = a'g(n), \quad g(n + \tau\sigma^{-1}) = b'e^{2\pi i \sigma' n}g(n) \) for some complex numbers \( a', b', \gamma' \). However, the only function satisfying such demands is the exponential \( q^{\gamma n} \), where \( \beta' \) is a free parameter. Since a factor of such type is already present in (10) we may set \( g(n) = 1 \) and this proves that the most general \( c_{n+1}/c_n \) has the form (11) (note that \( y \) is just an arbitrary proportionality constant). Direct application of the properties (6) yields the connection between multipliers \( a, b, \gamma \) and the parameters \( u_m, \quad m = 1, \ldots, r, \quad v_k, \quad k = 1, \ldots, s, \quad \beta \) as stated in (11). \( \square \)

Resolving the first order recurrence relation for the series coefficients \( c_n \) and normalizing \( c_0 = 1 \) we get the following explicit “additive” expression for the general theta hypergeometric series of elliptic type
\[
\sum_{n \in \mathbb{N} \cup \mathbb{Z}} [u_1, \ldots, u_r]_n q^{\beta(n-1)/2}y^n,
\]
where we used the elliptic shifted factorials defined for \( n \in \mathbb{N} \) as follows:
\[
[u_1, \ldots, u_k]_{\pm n} = \prod_{m=1}^k [u_m]_{\pm n},
\]
\[
[u]_n = [u][u + 1] \cdots [u + n - 1], \quad \quad [u]_{-n} = \frac{1}{[u - n]_n}.
\]

In order to simplify the trigonometric degeneration limit \( \text{Im}(\tau) \to +\infty \) (or \( p \to 0 \)) in the series (12) we renormalize \( y \) and introduce as a main series argument another variable \( z \):
\[
y \equiv (i p^{1/8}) e^{-\tau (u_1 + \cdots + u_r - v_1 - \cdots - v_s)/2} z.
\]
Let us replace the parameter \( \beta \) by another parameter \( \alpha \) as well through the relation \( \beta \equiv \alpha + (r - s)/2 \). Then, we can rewrite the function (11) in the following “multiplicative” form using the functions \( \theta(tq^n; p) \):
\[
h(n) = \frac{\theta(t_1q^n, \ldots, t_rq^n; p)}{\theta(w_1q^n, \ldots, w_sq^n; p)} \cdot q^{\alpha n} z, \quad (14)
\]
where \( t_m = q^{u_m}, \quad m = 1, \ldots, r, \quad w_k = q^{v_k}, \quad k = 1, \ldots, s, \) and the following shorthand notations are employed:
\[
\theta(t_1, \ldots, t_k; p) = \prod_{m=1}^k \theta(t_m; p).
\]

Now we are in a position to introduce the unilateral theta hypergeometric series \( rE_s \). In its definition we follow the standard plain and basic hypergeometric series conventions. Namely, in the expression (13) we replace \( s \) by \( s + 1 \) and set \( u_{s+1} = u_0 \) and \( v_{s+1} = 1 \). This does not restrict generality of consideration since one can remove such a constraint by fixing one of the numerator parameters \( u_m \) to be equal
to 1. Then we fix

$$r E_s \left( t_0, \ldots, t_{r-1}; q, p; \alpha, z \right)$$

$$= \sum_{n=0}^{\infty} \frac{\theta(t_0, t_1, \ldots, t_{r-1}; q; p)_n}{\theta(q, w_1, \ldots, w_s; p; q)_n} q^{\alpha n(n-1)/2} z^n, \quad (15)$$

where we have introduced new notations for the elliptic shifted factorials

$$\theta(t_0, \ldots, t_r; p; q)_n = \prod_{m=0}^{n} \theta(t_m q^m; p)$$

and

$$\theta(t_0, \ldots, t_k; p; q)_n = \prod_{m=0}^{k} \theta(t_m; p).$$

We draw attention to the particular ordering of $q, p$ used in the notations for theta hypergeometric series (in the previous papers we were ordering $q$ after $p$ which does not match with the ordering in terms of modular parameters $\sigma, \tau$ in $[u; \sigma, \tau]$).

An important fact is that theta hypergeometric series do not admit confluence limits. Indeed, because of the quasiperiodicity of theta functions the limits of parameters $t_m, w_m \to 0$ or $t_m, w_m \to \infty$ are not well defined and it is not possible to pass in this way from $r E_s$-series to similar series with smaller values of indices $r$ and $s$.

For the bilateral theta hypergeometric series we introduce different notations:

$$r G_s \left( t_1, \ldots, t_r; q, p; \alpha, z \right)$$

$$= \sum_{n=-\infty}^{\infty} \frac{\theta(t_1, \ldots, t_r; p; q)_n}{\theta(w_1, \ldots, w_s; p; q)_n} q^{\alpha n(n-1)/2} z^n. \quad (16)$$

This expression was derived with the help of (14) without any changes. The elliptic shifted factorials for negative indices are defined in the following way:

$$\theta(q; p; q)_n = 1 \quad (n > 0), \quad n \in \mathbb{N}.$$ 

Due to the property $\theta(q; p; q)_n = 0$ (or $[1]_n = 0$) for $n > 0$, the choice $t_{s+1} = q$ (or $v_{s+1} = 1$) in the $r G_{s+1}$ series leads to its termination from one side. After denoting $t_r \equiv t_0$ (or $u_r \equiv u_0$) one gets in this way the general $r E_s$-series. Since the bilateral series are more general than the unilateral ones, it is sufficient to prove key properties of theta hypergeometric series in the bilateral case without further specification to the unilateral one.

Consider the limit $\text{Im}(\tau) \to +\infty$ or $p \to 0$. In a straightforward manner one gets

$$\lim_{p \to 0} r E_s = r \Phi_s \left( t_0, t_1, \ldots, t_{r-1}; q; \alpha, z \right)$$

$$= \sum_{n=0}^{\infty} \frac{(t_0, t_1, \ldots, t_{r-1}; q)_n}{(q, w_1, \ldots, w_s; q)_n} q^{\alpha n(n-1)/2} z^n. \quad (17)$$

This basic hypergeometric series is different from the standard one by the presence of an additional parameter $\alpha$. The definition of $r \Phi_s$ series suggested in [SI] uses
\( \alpha = 0 \). The definition given in [GR] looks as follows

\[
\Phi_s = \sum_{n=0}^{\infty} \frac{(t_0, \ldots, t_{r-1}; q)_n}{(q, w_1, \ldots, w_s; q)_n} (-1)^n q^{n(n-1)/2} z^n,
\]

which matches with (17) for \( \alpha = s+1-r \) after the replacement of \( z \) by \((-1)^{s+1-r} z\).

Actually, the \( \alpha = 0 \) and \( \alpha = s+1-r \) choices are related to each other through the inversion transformation \( q \to q^{-1} \) with the subsequent redefinition of parameters \( t_m, w_m, z \). One of the characterizations of the basic hypergeometric series \( \sum c_n \) consists in the demand for \( c_{n+1}/c_n \) to be a general rational function of \( q^n \) which is satisfied by (17) only for integer \( \alpha \). In order to get in the limit \( p \to 0 \) the standard \( q \)-hypergeometric series we fix \( \alpha = 0 \). It is not clear at the moment whether this choice is the most “natural” one or it does not play a fundamental role—this question can be answered only after the discovery of good applications for the series \( rE_s \) in pure mathematical or mathematical physics problems. From the point of view of elliptic beta integrals [S2, S4] this is the most natural choice indeed.

In the bilateral case we fix \( \alpha = 0 \) as well, so that in the \( p \to 0 \) limit the \( rG_s \) series are reduced to the general \( r\Psi_s \)-series:

\[
\begin{align*}
\Psi_s \left( t_1, \ldots, t_r; \omega_1, \ldots, \omega_s; z \right) &= \sum_{n=-\infty}^{\infty} \frac{(t_1, \ldots, t_r; q)_n}{(\omega_1, \ldots, \omega_s; q)_n} z^n. \\
\end{align*}
\]

**Definition.** The series \( r+1E_r \) and \( rG_r \) are called **balanced** if their parameters satisfy the constraints, in the additive form,

\[
u_0 + \ldots + u_r = 1 + v_1 + \ldots + v_r
\]

and

\[
u_1 + \ldots + u_r = v_1 + \ldots + v_r
\]

respectively. In the multiplicative form these restrictions look as follows: \( \prod_{m=0}^{r} t_{m} = q \prod_{k=1}^{r} w_{k} \) and \( \prod_{m=1}^{r} t_{m} = \prod_{k=1}^{r} w_{k} \) respectively.

**Remark 1.** In the limit \( p \to 0 \) the series \( r+1E_r \) goes to \( r+1\Phi_r \) provided the parameters \( u_m \) (or \( t_m \)), \( m = 0, \ldots, r \), and \( v_k \) (or \( w_k \)), \( k = 1, \ldots, r \), remain fixed. Then our condition of balancing does not coincide with the one given in [GR], where \( r+1\Phi_r \) is called balanced provided \( \prod_{m=0}^{r} t_{m} = \prod_{k=1}^{r} w_{k} \) (simultaneously one usually assumes also that \( z = q \), but we drop this requirement). A discrepancy in these definitions will be resolved after imposing some additional constraints upon the series parameters (see the very-well-poisedness condition below).

### 3. Elliptic hypergeometric series

From the author’s point of view the following definition plays a fundamental role for the whole theory of hypergeometric type series since it explains origins of some known peculiarities of the plain and basic hypergeometric series.

**Definition.** The series \( \sum_{n \in \mathbb{N}} c_n \) and \( \sum_{n \in \mathbb{Z}} c_n \) are called **elliptic hypergeometric series** if \( h(n) = c_{n+1}/c_n \) is an elliptic function of the argument \( n \) which is considered as a complex variable, i.e. \( h(x) \) is a meromorphic double periodic function of \( x \in \mathbb{C} \).
Theorem 2. Let $\sigma^{-1}$ and $\tau \sigma^{-1}$ be two periods of the elliptic function $h(x)$, i.e. $h(x + \sigma^{-1}) = h(x)$ and $h(x + \tau \sigma^{-1}) = h(x)$. Let $r + 1$ be the order of the elliptic function $h(x)$, i.e. the number of its poles (or zeros) in the parallelogram of periods. Then the unilateral (or bilateral) elliptic hypergeometric series coincides with the balanced theta hypergeometric series $r + 1 E_r$ (or $r + 1 G_{r+1}$).

Proof. It is well known that any elliptic function $h(x)$, $x \in \mathbb{C}$, of the order $r + 1$ with the periods $\sigma^{-1}$ and $\tau \sigma^{-1}$ can be written as a ratio of $\theta_1$-functions as follows [WW]:

$$h(x) = z \prod_{m=0}^{r} \frac{[x + \alpha_m; \sigma, \tau]}{[x + \beta_m; \sigma, \tau]},$$

(22)

where the zeros $\alpha_0, \ldots, \alpha_r$ and the poles $\beta_0, \ldots, \beta_r$ satisfy the following constraint:

$$\sum_{m=0}^{r} \alpha_m = \sum_{m=0}^{r} \beta_m.$$  

(23)

Now the identification of the unilateral elliptic hypergeometric series with the balanced $r + 1 E_r$-series is evident. One just has to shift $x \to x - \beta_0 + 1$, set $x \in \mathbb{N}$, denote $u_m = \alpha_m - \beta_0 + 1$, $v_m = \beta_m - \beta_0 + 1$, and resolve the recurrence relation $c_{n+1} = h(n)c_n$. After this, the condition (23) becomes the balancing condition for the $r + 1 E_r$ series. A similar situation takes place, evidently, in the bilateral series case $r + 1 G_{r+1}$, when $\alpha_m$ and $\beta_m$ just coincide with $u_m$ and $v_m$ respectively.

Note that because of the balancing condition (23) the function (22) can be rewritten as a simple ratio of $\theta(t;p)$-functions:

$$h(x) = z \prod_{m=0}^{r} \frac{\theta(t_m q^2; p)}{\theta(w_m q^2; p)},$$

where $t_m = q^{\alpha_m}$ and $w_m = q^{\beta_m}$. □

Definition. Theta hypergeometric series of elliptic type are called modular hypergeometric series if they are invariant with respect to the $SL(2, \mathbb{Z})$ group action [5].

Consider what kind of constraints upon the parameters of $r E_s$ and $r G_s$ series one has to impose in order to get the modular hypergeometric series. Evidently, it is sufficient to establish modularity of the function $h(n) = c_{n+1}/c_n$. From its explicit form [10] and the transformation laws [7] and [8] it is easy to see that in the unilateral case one must have

$$\sum_{m=0}^{r-1} (x + u_m)^2 = (x + 1)^2 + \sum_{m=0}^{s} (x + v_m)^2,$$

which is possible only if a) $s = r - 1$, b) the parameters satisfy the balancing condition (20), and c) the following constraint is valid:

$$u_0^2 + \ldots + u_{r-1}^2 = 1 + v_1^2 + \ldots + v_{r-1}^2.$$  

(24)

Under these conditions the $r E_{r-1}$-series becomes modular invariant. Let us note that modularity of theta hypergeometric series assumes their ellipticity. The opposite is not correct, but a more strong demand of ellipticity, to be formulated below, automatically leads to modular invariance. Modular hypergeometric series
represent particular examples of Jacobi modular functions in the sense of Eichler and Zagier [EZ].

In the bilateral case one must have \( r = s \), the balancing condition \( (21) \), and the constraint
\[
u_1^2 + \ldots + u_r^2 = v_1^2 + \ldots + v_r^2
\]
for the \( G_r \)-series to be modular invariant.

**Definition.** The theta hypergeometric series \( r+1E_r \) is called *well-poised* if its parameters satisfy the following constraints
\[
u_0 + 1 = u_1 + v_1 = \ldots = u_r + v_r
\]
in the additive form or
\[
qt_0 = t_1 w_1 = \ldots = t_r w_r
\]
in the multiplicative form. Similarly, the series \( rG_r \) is called well-poised if \( u_1 + v_1 = \ldots = u_r + v_r \) or \( t_1 w_1 = \ldots = t_r w_r \).

This definition of well-poised series matches with the one used in the theory of plain and basic hypergeometric series [GR]. Note that it does not imply the balancing condition.

**Definition.** The series \( r+1E_r \) is called *very-well-poised* if, in addition to the constraints \( (26) \) or \( (27) \), one imposes the restrictions
\[
u_{r-3} = \frac{1}{2}u_0 + 1, \quad u_{r-2} = \frac{1}{2}u_0 + 1 - \frac{1}{2\sigma},
\]
\[
u_{r-1} = \frac{1}{2}u_0 + 1 - \frac{\tau}{2\sigma}, \quad u_r = \frac{1}{2}u_0 + 1 + \frac{1 + \tau}{2\sigma},
\]
or, in the multiplicative form,
\[
t_{r-3} = t_0^{1/2}, \quad t_{r-2} = -t_0^{1/2}, \quad t_{r-1} = t_0^{1/2} q^{-1/2}, \quad t_r = -t_0^{1/2} q p^{1/2}.
\]

Let us derive a simplified form of the very-well-poised series. First, we notice that
\[
\theta(z p^{-1/2}; p) = -z p^{-1/2} \theta(z p^{1/2}; p)
\]
and
\[
\theta(z, -z, z p^{1/2}, -z p^{1/2}; p) = \theta(z^2; p).
\]
After application of these relations, one can find that
\[
\frac{\theta(t_{r-3}, \ldots, t_r; p; q)_n}{\theta(qt_0/t_{r-3}, \ldots, qt_0/t_r; p; q)_n} = \frac{\theta(t_0 q^{2n}; p)}{\theta(t_0; p)} (-q)^n.
\]
As a result, one gets
\[
r+1E_r \left( t_0, t_1, \ldots, t_{r-4}, qt_0^{1/2}, -qt_0^{1/2}, q p^{-1/2} t_0^{1/2}, -q p^{1/2} t_0^{1/2}, qt_0^{1/2} t_0^{1/2}, -t_0^{1/2}, p^{-1/2} t_0^{1/2}, -p^{-1/2} t_0^{1/2}; q; p^2 \right)
\]
\[
= \sum_{n=0}^{\infty} \frac{\theta(t_0 q^{2n}; p)}{\theta(t_0; p)} \prod_{m=0}^{r-4} \frac{\theta(t_m; p; q)_n}{\theta(qt_0/t_m; p; q)_n} (-q z)^n.
\]

For convenience we introduce separate notations for the very-well-poised series, since they contain an essentially smaller number of parameters than the general
theta hypergeometric series $r+1E_r$. For this we replace $z$ by $-z$ and all the parameters $t_m, m = 0, \ldots, r-4$, by $t_0t_m$ (in particular, this replaces $t_0$ by $t_0^2$). Then we write

$$r+1E_r(t_0; t_1, \ldots, t_{r-4}; q, p; z) = \sum_{n=0}^{\infty} \frac{\theta(t_0^2q^{2n}; p)}{\theta(t_0^2; p)} \prod_{m=0}^{n-4} \frac{\theta(t_0t_m; q_n)}{\theta(qt_0t_m; q_n)} (qz)^n. \tag{31}$$

In terms of the elliptic numbers this series takes the form:

$$r+1E_r(u_0; u_1, \ldots, u_{r-4}; \sigma, \tau; z) = \sum_{n=0}^{\infty} \frac{[2u_0 + 2n]}{[2u_0]} \prod_{m=0}^{n-4} \frac{[u_0 + u_m]_n}{[u_0 + 1 - u_m]_n} z^n q^{n(\sum_{m=0}^{n-4} u_m - (r-7)/2)}. \tag{32}$$

We use the same symbol $r+1E_r$ in (31) and (32) since these series can be easily distinguished from the general $r+1E_r$ series by the number of free parameters.

For $p = 0$ theta hypergeometric series (31) are reduced to the very-well-poised basic hypergeometric series:

$$r-1\Phi_{r-2}(t_0; t_1, \ldots, t_{r-4}; q; qz) = \sum_{n=0}^{\infty} \frac{1 - t_0^2q^{2n}}{1 - t_0^2} \prod_{m=0}^{n-4} \frac{(qt_0t_m; q)_n}{(qt_0t_m^{-1}; q)_n} (qz)^n,$$

which are different from the corresponding partners in [GR] by the replacement of $z$ by $qz$ and $t_m$ by $t_0t_m$ (a standard notation for these series would be $r-1W_{r-2}$ but we are not using it here).

Remind that the balancing condition is not involved into the definition of the very-well-poised theta hypergeometric series. Imposing the corresponding constraint

$$\sum_{m=0}^{r} (u_0 + u_m) = 1 + \sum_{m=1}^{r} (u_0 + 1 - u_m)$$

upon (32) we get

$$\sum_{m=0}^{r-4} u_m = \frac{r - 7}{2}.$$

In the multiplicative form this condition takes the form

$$\prod_{m=0}^{r-4} t_m = q^{(r-7)/2}.$$

But this is precisely the balancing condition for the very-well-poised series appearing in the theory of basic hypergeometric series [GR]. Thus for the very-well-poised series there is no discrepancy in the definitions of balancing condition given in [GR] and in this paper. This happens because the constraints (22) taken separately are not well defined in the limit $\text{Im}(\tau) \to +\infty$. Note that for the balanced series an extra factor standing in (22) to the right of $z^n$ disappears. Summarizing this consideration we conclude that a very natural condition of ellipticity of the function $h(n) = c_{n+1}/c_n$ in the theta hypergeometric series provides a substantial meaning to the (innatural) balancing condition for the standard basic hypergeometric series.

If we impose balancing condition in the multiplicative form then there appears an ambiguity. Indeed, substituting into the condition $\prod_{m=0}^{r-4} (t_0t_m) = q \prod_{m=1}^{r-4} (qt_0/t_m)$
the constraints \( t_{r-3} = q, t_{r-2} = -q, t_{r-1} = q p^{-1/2}, t_r = -q p^{1/2} \) (these are the restrictions) after the shift \( t_m \to t_0 t_m \) we get \( \prod_{m=0}^{r-4} t_m^2 = q^{-7} \) which yields \( \prod_{m=0}^{r-4} t_m = \pm q^{(r-7)/2} \) and it is known that only the plus sign corresponds to the correct balancing condition for odd \( r \) (the even \( r \) cases remain ambiguous, but even \( r \) do not appear in known examples of summation formulæ of basic hypergeometric series).

In the same way, the bilateral theta hypergeometric series \( rG_r \) are called very-well-poised if the constraints \( 2n \) or \( 2m \) are satisfied, where \( u_0 \) or \( t_0 = q^{u_0} \) is a free parameter. Following the unilateral series case, we replace \( z \) by \(-z\), shift the parameters \( t_m \to t_0 t_m, m = 0, \ldots, r - 4 \), and introduce the following shorthand notations for the simplified form of these series:

\[
rG_r(t_0; t_1, \ldots, t_{r-4}; q, p; z)
= \sum_{n=-\infty}^{\infty} \theta(t^2 q^{2n}; p) \prod_{m=1}^{r-4} \theta(t_0 t_m; p; q) (qz)^n \tag{33}
\]
or in terms of the elliptic numbers

\[
rG_r(u_0; u_1, \ldots, u_{r-4}; \sigma, \tau; z)
= \sum_{n=-\infty}^{\infty} \frac{[2u_0 + 2n]}{[2u_0]} \prod_{m=1}^{r-4} \frac{[u_0 + u_m]_n}{[u_0 + 1 - u_m]_n} z^n q^n \prod_{m=1}^{r-4} u_m = \frac{r - 8}{2}, \quad \text{or} \quad \prod_{m=1}^{r-4} t_m = q^{(r-8)/2}.
\tag{34}
\]

Repeating considerations for the bilateral series we find the following compact form for the balancing condition for the \( rG_r \)-series:

\[
\sum_{m=1}^{r-4} u_m = \frac{r - 8}{2}, \quad \text{or} \quad \prod_{m=1}^{r-4} t_m = q^{(r-8)/2}.
\]

Under the constraint \( u_{r-3} = u_0 \) (or \( t_{r-3} = t_0 \)) the \( r+1 \)-series is converted into the \( r+1 \).g-series. A general connection between the very-well-poised series of \( E \) and \( G \) types looks as follows:

\[
rG_r(t_0; t_1, \ldots, t_{r-4}; q, p; z) = r+1E_{r+1}(t_0; t_1, \ldots, t_{r-4}, q t_0^{-1}; q, p; z)
+ \frac{q^{-7}}{z \prod_{m=1}^{r-4} t_m^2} \prod_{m=1}^{r-4} \frac{[t_0 t_m^{-1}; q]_m}{[t_0^{-2}; q]_m} \tag{35}
\times r+2E_{r+1} \left( \frac{q}{t_0}; t_1, \ldots, t_{r-4}, t_0; q, p; \frac{q^8}{z \prod_{m=1}^{r-4} t_m^2} \right).
\]

Remark 2. Within our classification, an elliptic generalization of basic hypergeometric series \( r+1 \)F, introduced by Frenkel and Turaev in \( \mathbb{F} \) coincides with the very-well-poised balanced theta hypergeometric series \( r+1 \)E of the unit argument \( z = 1 \). Such series have their origins in elliptic solutions of the Yang-Baxter equation \( \mathbb{B} \) and \( \mathbb{A} \) and biorthogonal rational functions with self-similar spectral properties \( \mathbb{S} \).

Definition. The series \( \sum_{n \in \mathbb{C}} c_n \) and \( \sum_{n \in \mathbb{Z}} c_n \) are called totally elliptic hypergeometric series if \( h(n) = c_{n+1}/c_n \) is an elliptic function of all free parameters entering it (except of the parameter \( z \) by which one can always multiply \( h(n) \)) with equal periods of double periodicity.
Theorem 3. The most general (in the sense of a maximal number of independent free parameters) totally elliptic theta hypergeometric series coincide with the well-poised balanced theta hypergeometric series $\psi E_{r-1}$ (in the unilateral case) and $\psi G_r$ (in the bilateral case) for $r > 2$. Totally elliptic series are automatically modular invariant.

Proof. It is sufficient to prove this theorem for the bilateral series since the unilateral series can be obtained afterwards by a simple reduction. Ellipticity in $n$ leads to $h(n)$ of the form

$$h(n) = \frac{[n + u_1, \ldots, n + u_r]}{[n + v_1, \ldots, n + v_r]} z,$$

with the free parameters $u_1, \ldots, u_r$ and $v_1, \ldots, v_r$ satisfying the balancing condition $u_1 + \ldots + u_r = v_1 + \ldots + v_r$. From such a representation it is evident that there is a freedom in the shift of parameters by an arbitrary constant: $u_m \rightarrow u_m + u_0$, $v_m \rightarrow v_m + v_0$, $m = 1, \ldots, r$, which does not spoil the balancing condition.

Let us determine now the maximal possible number of independent variables in the totally elliptic hypergeometric series. In general one can denote as $a_l$, $l = 1, \ldots, L$, a set of free parameters of the elliptic hypergeometric series in which the series is doubly periodic with some periods. Then $u_m = \sum_{k=1}^{L} \alpha_{mk} a_k + \beta_m$ and $v_m = \sum_{k=1}^{L} \gamma_{mk} a_k + \delta_m$ are some linear combinations of $a_l$ with integer coefficients $\alpha_{mk}, \gamma_{mk}$. However, because of the possibilities to change variables we can take a number of $u_m$ starting, say, from $u_1$ and $v_m$ as $a_1, \ldots, a_L$ and demand the double periodicity in these parameters themselves. Since the minimal order of elliptic function is equal to 2, the function $h(n)$ should have at least two zeros and two poles (or one double zero or pole) in $u_1$. Double zeros and poles ask for additional constraints, i.e. to a reduction of the number of free parameters, and we discard such a possibility.

Let us assume that $u_r$ depends linearly on $u_1$ and suppose that $u_1, \ldots, u_{r-1}$ are independent variables. Then it is evident that all denominator parameters $v_m$, $m = 1, \ldots, r$, cannot contain additional independent variables. Indeed, if it would be so, then, inevitably, this parameter should show up at least in one $\theta$-function in the numerator, which cannot happen by the assumption.

So, $L = r - 1$ is the maximal possible number of independent variables in the totally elliptic hypergeometric series and $u_r$ together with $v_m$, $m = 1, \ldots, r$, depend linearly on $u_k$, $k = 1, \ldots, r - 1$. Because of the permutational invariance in the latter variables one must have $u_r = \alpha \sum_{k=1}^{r-1} u_k + \beta$, where $\alpha, \beta$ are some numerical coefficients to be determined (evidently, $\alpha$ must be an integer). As to the choice of $v_m$ the unique option guaranteeing permutational invariance of the product $\prod_{m=1}^{r}[x + v_m]$ in $u_1, \ldots, u_{r-1}$ is the following one

$$v_m = \gamma \sum_{k=1}^{r-1} u_k + \delta u_m + \rho, \quad m = 1, \ldots, r - 1,$$

and $v_r = \mu \sum_{k=1}^{r-1} u_k + \nu$, where $\gamma, \delta, \rho, \mu, \nu$ are some numerical parameters ($\gamma, \delta, \mu$ must be integers). This is the most general choice of $v_m$ since all other permutationally invariant combinations of $u_m$ require products of more than $r$ theta functions. Substitution of the taken ansatz into the balancing condition yields $1 + \alpha = (r - 1) \gamma + \delta + \mu$ and $\beta = (r - 1) \rho + \nu$, which guarantees invariance of $h(x)$ under the shift $u_k \rightarrow u_k + \sigma^{-1}$ and cancels the sign factor emerging from the shift.
$u_k \rightarrow u_k + \tau \sigma^{-1}$. A bit cumbersome but technically straightforward analysis of the condition of cancellation of the factors of the form $e^{-2\pi i \sigma u}$ yields the equations

$$\delta^2 = 1, \quad \alpha^2 = (r - 1)\gamma^2 + 2\gamma \delta + \mu^2$$

and

$$\alpha \beta = (r - 1)\gamma \rho + \rho \delta + \mu \nu.$$  

The constraint generated by the cancellation of the factors of the form $e^{-\pi i \tau}$ appears to be irrelevant and it will not be indicated.

Let $\delta = 1$. Then two equations upon the coefficients $\alpha, \gamma, \mu$ yield that either $\gamma = 0$ or $\gamma(r - 1)(r - 2)/2 + \mu(r - 1) = 1$. Since $\gamma$ and $\mu$ are integers, the second case cannot be valid (the integers on the left hand side are proportional to $r - 1$ whereas on the right hand such proportionality does not take place for $r > 2$). The choice $\gamma = 0$ leads to $\alpha = \mu$ and, from other two equations, one gets $\rho = 0$ and $\beta = \nu$. As a result, $h(n) = 1$, i.e. we get a trivial solution which is discarded.

Let $\delta = -1$. Solution of the taken equations gives uniquely

$$\alpha = \frac{\gamma r}{2}, \quad \beta = \frac{\rho r}{2}, \quad \mu = 1 - \gamma(r - 2), \quad \nu = -\frac{\rho(r - 2)}{2},$$

where $\gamma$ is an integer (for odd $r$ it must be an even number) and $\rho$ is an arbitrary parameter. Arbitrariness of $\rho$ seems to contradict the statement that there are no new independent parameters in the variable $u_r$. This paradox is resolved in the following way. Let us first make the shift $\rho \rightarrow \rho - \gamma \sum_{k=1}^{r-1} u_k$. It is easy to see that this leads to the removal of $\gamma$ from $h(n)$, i.e. it is a fake parameter. Denote now $\rho = 2u_0$ and make the shifts $u_m \rightarrow u_m + u_0$, $m = 1, \ldots, r - 1$. As a result, one gets

$$h(n) = \prod_{m=1}^{r-1} \frac{n + u_0 + u_m}{n + u_0 - u_m} \frac{n + u_0 - \sum_{k=1}^{r-1} u_k}{n + u_0 + \sum_{k=1}^{r-1} u_k} z,$$

i.e. the parameter $u_0$ plays the same role as $n$ and ellipticity in it is evident. It is not difficult to recognize in (36) the most general expression for $c_{n+1}/c_n$ of well-poised and balanced theta hypergeometric series. Thus we have proved that ellipticity in all free parameters in $h(n)$ leads uniquely to the balancing and well-poisedness conditions.

Let us prove now that the totally elliptic hypergeometric series are automatically modular invariant. For this it is sufficient to check that the sum of squares of the parameters $u$ entering elliptic numbers $[n + u]$ in the numerator of $h(n)$ and denominator coincide. The numerator parameters generate the sum

$$\sum_{k=1}^{r-1} u_k^2 + \left( -\sum_{k=1}^{r-1} u_k \right)^2$$

which is trivially equal to the sum appearing from the denominator:

$$\sum_{k=1}^{r-1} (-u_k)^2 + \left( \sum_{k=1}^{r-1} u_k \right)^2,$$

i.e. modular invariance is automatic. (Note that well-poised theta hypergeometric series without balancing condition are not modular invariant).

All the considerations given above were designed for the bilateral $rG_r$ series case, but a passage to the unilateral well-poised and balanced $rE_{r-1}$-series is done by a simple specification of one of the parameters. One has to set $u_{r-1} = u_0 - 1$ and
then shift \( u_0 \rightarrow u_0 + 1/2, \ u_m \rightarrow u_m - 1/2, \ m = 1, \ldots, r - 2 \). This brings \( h(n) \) to the form \( h(n) = z \prod_{m=0}^{r-1} [n + u_0 + u_m]/[n + 1 + u_0 - u_m] \), where we introduced anew the \( r - 1 \) parameter through the relation \( \sum_{m=0}^{r-1} u_m = r/2 \).

**Remark 3.** We have replaced \( t_0 \) by \( t_0^2 \) in the definition of very-well-poised elliptic hypergeometric series \( r_{+1}E_r \) in order to have \( p \)-shift invariance in the variable \( t_0 \) (or ellipticity in \( u_0 \)). Otherwise there would be \( p \)-shift invariance in \( t_0^{1/2} \).

Thus we have found interesting origins of the balancing and well-poisedness conditions for the plain and basic hypergeometric series calling for a revision of these notions. However, origins of the very-well-poisedness condition remain unknown. Probably elliptic functions \( h(n) \) obeying such a constraint have some particular arithmetic properties. An indication on this is given by the transformation and summation formulas for some of the very-well-poised balanced theta hypergeometric series of the unit argument.

The theta hypergeometric series \( rE_s \) and \( rG_s \) are defined as formal infinite series. However, because of the quasiperiodicity of the theta functions it is not a simple task to determine their convergence and this problem will not be considered here. It can be shown that for some choice of parameters the radius of convergence \( R \) of the balanced \( r_{+1}E_r \)-series is equal to 1. If \( R \) is the radius of convergence of the very-well-poised \( r_{+2}E_{r+1} \)-series without balancing condition, i.e. if these infinite series are well defined for \( |z| < R \), then from the representation \(^{[15]}\) it follows that the \( rE_r \)-series converge for \( |q^{-8}/R \prod_{m=1}^{r-1} t_m^2| < |z| < R \). A rigorous meaning to the \( rE_s \)-series can be given by imposing some truncation conditions. The theta hypergeometric series truncate if for some \( m \),

\[
\begin{align*}
    u_m &= -N - K \sigma^{-1} - M \tau \sigma^{-1}, \quad N \in \mathbb{N}, \quad K, M \in \mathbb{Z}, \\
    t_m &= q^{-N} p^{-M}, \quad N \in \mathbb{N}, \quad M \in \mathbb{Z}.
\end{align*}
\]

The well-poised elliptic hypergeometric series are double periodic in their parameters with the periods \( \sigma^{-1} \) and \( \tau \sigma^{-1} \). Therefore, these truncated series do not depend on the integers \( K, M \).

The top-level identity in the theory of basic hypergeometric series is the four term Bailey identity for non-terminating \( 10 \Phi_9 \) very-well-poised balanced series of the unit argument \(^{[8]}\). In the terminating case there remains only two terms. In \(^{[13]}\) Frenkel and Turaev have proved an elliptic generalization of the Bailey identity in the terminating case. In our notations it looks as follows

\[
12E_{11}(t_0; t_1, \ldots, t_7; q, p; 1) = \frac{\vartheta(q t_0^2, q s_0/s_4, q s_0/s_5, q/t_5^2; p; q)_N}{\vartheta(q s_0, q t_5, q t_5^2, q/s_5; p; q)_N} \times 12E_{11}(s_0; s_1, \ldots, s_7; q, p; 1),
\]

where it is assumed that \( \prod_{m=0}^{7} t_m = q^2, \ t_0 t_6 = q^{-N}, \ N \in \mathbb{N} \), and

\[
\begin{align*}
    s_0 &= q t_0, \quad s_1 = s_0 t_1, \quad s_2 = s_0 t_2, \quad s_3 = s_0 t_3, \\
    s_4 &= t_6, \quad s_5 = s_0 t_5, \quad s_6 = t_6, \quad s_7 = t_7.
\end{align*}
\]

If one sets \( t_3 t_3 = q \), then the left-hand side of \(^{[13]}\) becomes a terminating \( 10E_9 \)-series, and in the series on the right-hand side one gets \( s_1 = 1 \), i.e. only its
first term is different from zero. This gives the Frenkel-Turaev sum—an elliptic generalization of the Jackson’s sum for terminating very-well-poised balanced $_8\Phi_7$-series [GR]. After diminishing the indices of $t_{4,5,6,7}$ by two it takes the following form:

$$
t_{0}E_{9}(t_0; t_1, \ldots, t_5; q, p; 1) = \frac{\theta(qt^2_0; p; q)\prod_{1 \leq r < s \leq 3} \theta(q/t_0 t_s; p; q) N \prod_{i=1}^{5} \theta(q/t_0 t_i; p; q) N,}
$$

where the parameters $t_r$ are assumed to satisfy the balancing condition $\prod_{r=0}^{5} t_r = q$ and the truncation condition $t_{0}t_{4} = q^{-N}$, $N \in \mathbb{N}$.

Remark 4. Due to the clarification of the relation of very-well-poisedness condition with the general structure of theta hypergeometric series, starting from this paper we change notations for the elliptic hypergeometric series in the generalizations of Bailey and Jackson identities. The symbols $t_{0}E_{9}$ and $qE_{7}$ in the papers [SZ1] [SZ2] [SZ3], [S1] [S2] [DS1] [DS2] [DS3] [DS4] read in the current notations as $t_{12}E_{11}$ and $t_{10}E_{9}$ respectively.

Remark 5. Despite of the double periodicity, infinite totally elliptic hypergeometric functions are not elliptic functions of $u_r$ since they have infinitely many poles in the parallelogram of periods. Indeed, some of the poles in $t_s$, $s = 1, \ldots, r - 4$, are located at $t_s = t_0q^{n+1}p^m$, where $n \in \mathbb{N}$ and $m \in \mathbb{Z}$. If $q^k \neq p^l$ for any $k, l \in \mathbb{N}$, then, evidently, there are infinitely many integers $n$ and $m$ such that $t_s$ stays, say, within the bounds $|p| < |t_s| < 1$. This means that there are infinitely many poles in the parameters $t_s$ in this annulus.

Remark 6. The symbols $rE_s$ and $rG_s$ were chosen for denoting the one-sided and bilateral theta hypergeometric series in order to make them as close as possible to the standard notations $rF_s$ and $rH_s$ used for one-sided and bilateral plain hypergeometric series respectively. The letter “$E$” refers also to the word “elliptic”. To the author’s taste greek symbols $r\Phi_s$ and $r\Psi_s$ used for denoting basic hypergeometric series fit well enough into the aesthetics created by the sequence of letters $E, \Phi, F, G, \Psi, H$.

4. MULTIPLE ELLIPTIC HYPERGEOMETRIC SERIES

Following the definition of theta hypergeometric series of a single variable one can consider formal multiple sums of quasiperiodic combinations of Jacobi $\theta_1$-functions depending of more than one summation index. However, we shall limit ourselves only to the multiple elliptic hypergeometric series.

Definition. The formal series

$$
\sum_{\lambda_1, \ldots, \lambda_n = 0}^{\infty} c(\lambda_1, \ldots, \lambda_n) \text{ or } \sum_{\lambda_1, \ldots, \lambda_n = -\infty}^{\infty} c(\lambda_1, \ldots, \lambda_n),
$$

and

$$
\sum_{\lambda_1, \ldots, \lambda_n = 0}^{\infty} c(\lambda_1, \ldots, \lambda_n) \text{ or } \sum_{\lambda_1, \ldots, \lambda_n = -\infty}^{\infty} c(\lambda_1, \ldots, \lambda_n)
$$

are called multiple elliptic hypergeometric series if: a) the coefficients $c(\lambda)$ are symmetric with respect to an action of permutation group $S_n$ upon the summation
variables \(\lambda_1, \ldots, \lambda_n\) and the free parameters entering \(c(\lambda)\); b) for all \(k = 1, \ldots, n\) the functions
\[
h_k(\lambda) = \frac{c(\lambda_1, \ldots, \lambda_k + 1, \ldots, \lambda_n)}{c(\lambda_1, \ldots, \lambda_n)},
\]
are elliptic in \(\lambda_k, k = 1, \ldots, n\), considered as complex variables. These series are called totally elliptic if, in addition, the functions \(h_k(\lambda)\) are elliptic in all free parameters except of the free multiplication factors.

Suppose that \(h_k(\lambda)\) is symmetric in \(\lambda_1, \ldots, \lambda_{k-1}, \lambda_{k+1}, \ldots, \lambda_n\). Then, using the results of the one variable analysis, it is not difficult to see that the most general expression for the coefficients \(c(\lambda)\) is:
\[
c(\lambda) = \prod_{k=1}^{n} \left( \prod_{1 \leq i_1 < \ldots < i_k \leq n} \prod_{m=1}^{r_k} \frac{[u_{km}]_{\lambda_{i_1} + \ldots + \lambda_{i_k}}}{[v_{km}]_{\lambda_{i_1} + \ldots + \lambda_{i_k}}} \right) z_1^{\lambda_1} \cdots z_n^{\lambda_n}, \tag{41}
\]
where
\[
\sum_{k=1}^{n} C_{n-1}^{k-1} \sum_{m=1}^{r_k} (u_{km} - v_{km}) = 0.
\]
However, if the action of \(S_k\) permutes \(\lambda_1, \ldots, \lambda_n\) and simultaneously free parameters entering \(c(\lambda)\) other than \(z_1, \ldots, z_n\), then the situation is richer, e.g. more general combinations of \(\lambda_k\) are allowed than it is indicated in (41). We shall not go into further analysis of general situation but pass to some particular examples.

Currently there are two known examples of multiple elliptic hypergeometric series leading to some constructive identities (multivariable analogues of the Frenkel-Turaev \(E_9\)-summation formula). The first one corresponds to an elliptic extension of the Aomoto-Ito-Macdonald type of series [A, I, Mac]. Its structure is read off from the following multivariable generalization of the Frenkel-Turaev summation formula considered in [Wa, DS1, Ro]. Let \(N \in \mathbb{N}\) and the parameters \(t, t_r \in \mathbb{C}, r = 0, \ldots, 5\), are constrained by the balancing condition \(t^{2n-2} \prod_{r=0}^{5} t_r = q\) and the truncation condition \(t^{n-1} t_0 t_4 = q^{-N}\). Then one has the following theta-functions identity
\[
\sum_{0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_N \leq N} q^{\sum_{j=1}^{n} \lambda_j} t^{2 \sum_{j=1}^{n} (n-j) \lambda_j} \prod_{1 \leq j < k \leq n} \frac{\theta(\tau_k \tau_j q^{\lambda_k + \lambda_j} \tau_k \tau_j^{-1} q^{\lambda_k - \lambda_j}; p)}{\theta(\tau_k \tau_j; p) \theta(\tau_k \tau_j^{-1}; p)} \times \frac{\theta(t \tau_k \tau_j; p; q)_{\lambda_k + \lambda_j} \theta(q^{-1} t \tau_k \tau_j^{-1}; p; q)_{\lambda_k - \lambda_j}}{\theta(q^{-1} t \tau_k \tau_j; p; q)_{\lambda_k + \lambda_j} \theta(\tau_k \tau_j^{-1}; p; q)_{\lambda_k - \lambda_j}} \times \prod_{j=1}^{n} \frac{\theta(\tau_j^2 q^{-2 \lambda_j}; p)}{\theta(\tau_j^2; p)} \prod_{r=0}^{5} \frac{\theta(t \tau_r \tau_j; p; q)_{\lambda_j}}{\theta(q^{-1} t \tau_r \tau_j; p; q)_{\lambda_j}} = \prod_{r=0}^{5} \frac{\theta(q^{t^{n+j-2} t^2}; p; q) N \prod_{1 \leq r < s \leq 5} \theta(q^{-1-j} t^{r-1} t_s^{-1}; p; q) \theta(q^{t^3} t^{-1} t_0 t_4^{-1}; p; q)}{\theta(q^{t^{2n-j}} \prod_{r=0}^{5} t_r^{-1}; p; q) N \prod_{r=1}^{5} \theta(q^{t^{-1}} t_0 t_4^{-1}; p; q)} \tag{42}
\]
Here the parameters \(\tau_j\) are related to \(t_0\) and \(t\) as follows: \(\tau_j = t_0 t_j^{-1}, j = 1, \ldots, n\). Note that the series coefficients \(c(\lambda)\) are symmetric with respect to the simultaneous permutation of the variables \(\lambda_j\) and \(\lambda_k\) and the parameters \(\tau_j\) and \(\tau_k\) for arbitrary \(j \neq k\) (for this one has to assume that \(\tau_j\) are independent variables).
Theorem 4. The series standing on the left hand side of (42) is a totally elliptic multiple hypergeometric series.

Proof. Ratios of the series coefficients yield

\[
\begin{align*}
    h_1(\lambda) &= \prod_{j=1}^{l-1} \frac{\theta(t_j \tau_j q^{\lambda_j+\lambda_l+1}, \tau_j^{-1} q^{\lambda_l+1-\lambda_j}, t \tau_j \tau_j q^{\lambda_j+\lambda_l}, t \tau_j^{-1} \tau_j q^{\lambda_l+1-\lambda_j}; p)}{\theta(t_j \tau_j q^{\lambda_j+\lambda_l}, \tau_j^{-1} q^{\lambda_l+1-\lambda_j}, t^{-1} \tau_j \tau_j q^{\lambda_j+\lambda_l+1}, t^{-1} \tau_j^{-1} \tau_j q^{\lambda_l+1-\lambda_j}; p)} \\
    &\times \prod_{k=l+1}^{n} \frac{\theta(t_k \tau_k q^{\lambda_k+\lambda_l+1}, \tau_k^{-1} q^{\lambda_k+1-\lambda_l}, t \tau_k \tau_k q^{\lambda_k+\lambda_l}, t \tau_k^{-1} \tau_k q^{\lambda_k+1-\lambda_l}; p)}{\theta(t_k \tau_k q^{\lambda_k+\lambda_l}, \tau_k^{-1} q^{\lambda_k+1-\lambda_l}, t^{-1} \tau_k \tau_k q^{\lambda_k+\lambda_l+1}, t^{-1} \tau_k^{-1} \tau_k q^{\lambda_k+1-\lambda_l}; p)} \\
    &\times q^{t_2(n-l)} \frac{\theta(\tau_l^{2l+2}; p)}{\theta(\tau_l^{2}; p)} \prod_{m=0}^{5} \frac{\theta(t_m \tau_l q^{\lambda_l}; p)}{\theta(t_m \tau_l q^{\lambda_l+1}; p)}. \tag{43}
\end{align*}
\]

Using the equalities \(\theta\) one can easily check the ellipticity of this \(h_1(\lambda)\) in \(\lambda_i\) for \(i < l\) and \(i > l\) (for this it is simply necessary to see that \(h_1\) does not change after the replacement of \(q^{\lambda_l}\) by \(pq^{\lambda_l}\)). For the ellipticity in \(\lambda_l\) itself one has an essentially longer computation. The replacement of \(q^{\lambda_l}\) by \(pq^{\lambda_l}\) in the product \(\prod_{k=l+1}^{n}\) yields a multiplier \(t^{-4(l-1)}\). The product \(\prod_{k=l+1}^{n}\) yields the multiplier \(t^{-4(n-l)}\). The remaining part of \(h_1\) generates the factor \(q^{4 l} \prod_{m=0}^{5} q t_m^{-2}\). The product of all these three factors takes the form \(q^{2 l} t^{-4(n-l)} \prod_{m=0}^{5} t_m^{-2}\) and it is equal to 1 due to the balancing condition.

So, we have found that the taken series is indeed a multiple elliptic hypergeometric series. Let us prove now its total ellipticity or p-shift invariance in the parameters \(t_m, m = 0, \ldots, 4\) and \(t\). The \(p\)-shift invariance in the parameters \(t_1, \ldots, t_4\) follows from the balancing condition in the same way as in the single variable series case. Consider the \(l_0 \rightarrow pt_0\) shift. The product \(\prod_{j=1}^{l-1}\) yields a multiplier \(t^{-4(l-1)}\) and the product \(\prod_{k=l+1}^{n}\) yields the factor \(t^{-4(n-l)}\). The remaining part of \(h_1\) generates the factor \(q^{2 l} \prod_{m=0}^{5} t_m^{-2}\). The product of all these three multipliers is equal to 1.

Finally, the shift \(t \rightarrow pt\) calls for the most complicated computation. The product \(\prod_{j=1}^{l-1}\) yields a complicated multiplier \((q^{2 l_1+1} t^{2(l-1)} \tau_1^{2} p^{2l-3})^{2(l-1)}\). The product \(\prod_{k=l+1}^{n}\) generates no less complicated expression \((q^{2 l_1+1} t^{2(l-1)} \tau_1^{2} p^{2(l-1)})^{2(l-n)}\). The remaining part of \(h_1\) leads to the following factor (after the use of the balancing condition); \((q^{2 l_1+1} t^{2(l-1)} \tau_1^{2})^{2(n-1)} p^{(4n-6)(l-1)}\). The product of all these three multipliers yields 1. Thus we have proved the total ellipticity of the taken type of series. □

The second example of multiple series corresponds to an elliptic generalization of the Milne-Gustafson type multiple basic hypergeometric series \([Mi]\) \([Mi, G, DG]\), which are, in turn, \(q\)-analogues of the Hollman, Biedenharn, and Louck plain multiple hypergeometric series \([HBL]\). Its structure is read off from the following summation formula suggested in \([DS3]\). Let \(q^n \neq p^m\) for \(n, m \in \mathbb{N}\). Then for the parameters \(t_0, \ldots, t_{2n+3}\) subject to the balancing condition \(q^{-1} \prod_{r=0}^{2n+3} t_r = 1\) and the truncation
conditions $q^N t_j t_{n+j} = 1$, $j = 1, \ldots, n$, where $N_j \in \mathbb{N}$, one has the identity

$$
\sum_{0 \leq j \leq N_j} \prod_{1 \leq j < k \leq n} \frac{\theta(t_j t_k q^{\lambda_j + \lambda_k}, t_j t_k^{-1} q^{\lambda_j - \lambda_k}; p)}{\theta(t_j t_k, t_j t_k^{-1}; p)}
\times \prod_{1 \leq j \leq n} \frac{\theta(t_j q^{2\lambda_j}; p)}{\theta(t_j; p)}
\equiv \prod_{0 \leq r \leq 2n+3} \frac{\theta(q t_j t_k^{-1}; p; q)}{\theta(q; p) \lambda_j}
$$

(44)

where $a \equiv t_{2n+1}$, $b \equiv t_{2n+2}$, $c \equiv t_{2b+3}$. Note that this series coefficients $c(\lambda)$ are symmetric with respect to simultaneous permutation of the variables $\lambda_j$ and $\lambda_k$ together with the parameters $t_j$ and $t_k$ for arbitrary $j, k = 1, \ldots, n, j \neq k$.

**Theorem 5.** The series standing on the left hand side of (44) is a totally elliptic hypergeometric series.

**Proof.** Ratios of the successive series coefficients yield

$$
h_l(\lambda) = \prod_{j=1}^{l-1} \frac{\theta(t_j q^{\lambda_j + \lambda_j + 1}, t_j t_l^{-1} q^{\lambda_j - \lambda_j - 1}; p)}{\theta(t_j t_l q^{\lambda_j + \lambda_j}, t_j t_l^{-1} q^{\lambda_j - \lambda_j}; p)}
\times \prod_{\lambda = l+1}^{n} \frac{\theta(t_l q^{\lambda_l + \lambda_k}, t_l t_k^{-1} q^{\lambda_l + 1 - \lambda_k}; p)}{\theta(t_l t_k q^{\lambda_l + \lambda_k}, t_l t_k^{-1} q^{\lambda_l - \lambda_k}; p)}
\times q^{l \left(\frac{1}{2}q^{2\lambda_l+2} - \frac{1}{2}q^{2\lambda_l}\right)} \prod_{m=0}^{2n+3} \frac{\theta(q^{l+1}; p)}{\theta(q^{l+1}; p)}
$$

(45)

It is easy to check the ellipticity of this expression in $\lambda_j$ for $j < l$ and $j > l$. Ellipticity in $\lambda_l$ itself follows from a more complicated computation. Namely, the change of $q^{\lambda_j}$ to $q^{\lambda_l}$ leads to additional multipliers in $h_l(\lambda)$: $q^{2(l+1)}$—from the product $\prod_{\lambda=l+1}^{n}$, $q^{2(n-l)}$—from the product $\prod_{\lambda=l+1}^{n}$, and $q^{-2l} \prod_{m=0}^{2n+3} t_m^{-2}$—from the rest of $h_l(\lambda)$. Multiplication of these three expressions gives 1 due to the balancing condition.

Ellipticity in the parameters $t_m$, $m = 0, \ldots, 2n+3$, is checked separately for $m < l, m > l$ and $m = l$. The first two cases are easy enough and do not worth of special consideration. The replacement $t_l \rightarrow pt_l$ leads to the following multipliers: $q^{2(1-l)}$—from the product $\prod_{\lambda=l+1}^{n}$, $q^{2(l-n)}$—from the product $\prod_{\lambda=l+1}^{n}$, and $q^{-2n} \prod_{m=0}^{2n+3} t_m^{2n}$—from the rest of $h_l(\lambda)$. The balancing condition guarantees again that the total multiplier is equal to 1. Thus we have proved total ellipticity of this series as well.

**Remark 7.** Modular invariance of the series (12) and (14) has been established in [DS1] and [DS3] respectively. We conjecture that all totally elliptic multiple hypergeometric series are automatically modular invariant similar to the one-variable
series situation. One can introduce general notions of well-poised and very-well-poised multiple theta hypergeometric series, with the given above examples being counted as very-well-poised series, but we shall not discuss this topic in the present paper.

The author is indebted to J.F. van Diejen and A.S. Zhedanov for a collaboration in the work on elliptic hypergeometric series and for useful discussions of this paper. Valuable comments and encouragement from G.E. Andrews, R. Askey, and A. Berkovich are highly appreciated as well.

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