Abstract

In this paper we derive time-decay and Strichartz estimates for the generalized Benjamin-Bona-Mahony equation on the framework of modulation spaces $M^{s}_{p,q}$. We use these results to analyze the existence of local and global solutions of the corresponding Cauchy problem with rough data in modulation spaces. The results improve known results in Sobolev spaces in some sense.

Key words. Benjamin-Bona-Mahony equation, modulation spaces, Strichartz estimates, well-posedness.

AMS subject classifications. 35Q53; 35A01; 35Q35; 35C15

1 Introduction

In this paper we study the Cauchy problem for the generalized Benjamin-Bona-Mahony equation, (gBBM)

\[
\begin{cases}
  u_t + u_x - uu_{xx} + u^\lambda u_x = 0, & x \in \mathbb{R}, \quad t \in \mathbb{R}, \\
  u(x, 0) = u_0(x), & x \in \mathbb{R},
\end{cases}
\]

(1.1)

where $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a real-valued function, $u_0 : \mathbb{R} \to \mathbb{R}$ is the initial data and $\lambda \geq 1$ is an integer. The case $\lambda = 1$ corresponds to the Benjamin-Bona-Mahony equation (BBM), which has been derived as a model to describe the gravity water waves in the long-wave regime, see Benjamin, Bona and Mahony [4], and Peregrine [23, 24]. Also, BBM equation is well suited for modeling wave propagation on star graphs, which gives some interesting applications as shown in Bona and Cascaval [6]. For $\lambda = 2$ the equation (1.1) is known as the modified BBM equation, which describes wave propagation in one dimensional nonlinear lattice (cf. Wadati [29, 30]; thus, the generalization considered in (1.1) is not only of mathematical interest.

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The BBM equation is a good substitute for the famous Korteweg-de Vries equation (KdV)

\[ u_t + u_x - u_{xxx} + u_x u = 0, \quad x, t \in \mathbb{R}, \]

in the case of shallow waters in a channel (see Whitham [36], Bona, Pritchard and Scott [9]). Furthermore, the solutions of the KdV and the BBM stay “close” to each other over relatively long time intervals, see [9] for more details. As the KdV equation, the gBBM possesses solitary and periodic wave solutions, which are particular solutions very important for applications in physics. The existence, orbital, asymptotic and spectral stability or instability of the solitary or periodic traveling waves, associated to the gBBM equation, have been studied by several researchers, see for instance [1, 2, 12, 16, 21, 27, 35, 37].

Moreover, the solutions of the KdV and the BBM stay “close” to each other over relatively long time in the case of shallow waters in a channel (see Whitham [36], Bona, Pritchard and Scott [9]).

Let \( \lambda \) be the Schwartz space on \( \mathbb{R} \), and \( H^s(\mathbb{R}) \) with \( \lambda \) in Sobolev spaces \( H^s(\mathbb{R}) \), \( s \geq 0 \), was obtained by Bona and Tzvetkov in [8], and Carvajal and Panthee [10] in the periodic case on \( H^s_{\text{per}}([-L, L]) \) with \( s \geq 0 \). On the other hand, the initial value problem \( \text{(1.1)} \) with \( \lambda = 1 \) is ill-posed in \( H^s(\mathbb{R}) \) or \( H^s_{\text{per}}([-L, L]) \), for \( s < 0 \) (cf. [8, 22]). Previous results have been obtained in finite energy spaces \( H^s \). Results of well-posedness for a generalization on the space dimension of the BBM equation, in Sobolev spaces \( W^{k,p} \), have been obtained by Goldstein and Wichnoski [15], and Avrin and Goldstein [3]. In [15] the authors analyzed the local well-posedness in \( W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega) \) for \( p > n \), with \( \Omega \) a bounded domain of \( \mathbb{R}^n \), and in [3] the authors studied the existence of local weak solutions in \( W^{1,p}(\mathbb{R}^n) \), \( p \geq 1 \). Recently, Wang [33] established an interesting result of global well-posedness for the BBM equation in Bessel potential spaces \( H^s_p(\mathbb{R}) \), with \( s \geq \max\{0, \frac{1}{2} - \frac{1}{p}\} \) and \( 1 \leq p < \infty \). The results of [33] are sharp in the sense that equation \( \text{(1.1)} \) with \( \lambda = 1 \) is ill-posed in \( H^s_p(\mathbb{R}) \) for \( s < \max\{0, \frac{1}{2} - \frac{1}{p}\} \), \( 1 \leq p < \infty \). More recently, Bona and Dai in [7] obtained an ill-posedness result for the BBM equation on the periodic homogeneous Sobolev spaces \( H^s_{\text{per}} \), with \( r < 0 \). So far, to the best of our knowledge, the larger initial data classes for BBM equation are those of [33].

Our interest in this paper is to analyze the well-posedness in some spaces of low regularity than \( H^s \) and \( H^s_p \) for large \( s \), namely, modulation spaces \( M^s_{p,q} \). Modulation spaces are decomposition spaces that emerge from a uniform covering of the underlying frequency space; they were introduced by Feichtinger in [14], prompted by the idea of measuring the smoothness classes of functions or distributions. Since their introduction, modulation spaces have become canonical for both time-frequency and phase-space analysis, see Chaichenets et al. [11]. Wang and Hudzik [31] gave an equivalent definition of modulation spaces by using the frequency-uniform-decomposition operators. In the same work, the existence of global solutions for nonlinear Schrödinger and Klein-Gordon equations in modulation spaces were analyzed. After them, several studies on nonlinear PDEs in the framework of modulation spaces have been addressed (cf. [11, 17, 18, 20, 25, 34, 38] and references therein). In this context, the contribution of this paper is to analyze the existence of solutions for the gBBM equation with initial data in modulation spaces. To get this aim, first we establish a careful harmonic analysis in order to derive some time-decay estimates of the one parameter group given by the corresponding linear equation, as well as some Strichartz estimates and nonlinear estimates on modulation spaces which allow us to control the nonlinearity in the gBBM equation (cf. Section 2 and 3). In particular, we prove some Strichartz estimates in \( M^s_{p,q} \) for a general dispersive semigroup \( U(t) = \mathcal{F}^{-1}e^{itP(\xi)}\mathcal{F} \), with \( P : \mathbb{R}^n \to \mathbb{R} \) a real-valued function, complementing the ones established in Wang and Hudzik [31], which can be used to analyze the well-posedness of another dispersive models.

Before stating our main results, we recall some preliminary definitions and notations related to the modulation spaces \( M^s_{p,q}(\mathbb{R}^n) \) (for more details see for instance Wang and Hudzik [31] and Kato [19]). Let \( \mathcal{S} \) be the Schwartz space on \( \mathbb{R}^n \) and \( \mathcal{S}' \) its dual space. Let \( Q_0 = \{ \xi : \xi \in [1/2, 1/2] \}, i = 1, \ldots, n \} \) and \( Q_k = k + Q_0, k \in \mathbb{Z}^n \). Thus, \( \{Q_k\}_{k \in \mathbb{Z}^n} \) constitutes a decomposition of \( \mathbb{R}^n \), that is, \( Q_i \cap Q_j = \emptyset \)
and \( \bigcup_{k \in \mathbb{Z}^n} Q_k = \mathbb{R}^n \). Let \( \rho : \mathbb{R}^n \to [0,1] \) be a smooth function satisfying \( \rho(\xi) = 1 \) for \( |\xi| \leq \frac{\sqrt{n}}{2} \) and \( \rho(\xi) = 0 \) for \( |\xi| \geq \frac{\sqrt{n}}{2} \). Let \( \rho_k(\xi) = \rho(\xi - k) \), \( k \in \mathbb{Z}^n \), a translation of \( \rho \). It holds that \( \rho_k(\xi) = 1 \) in \( Q_k \), and thus, \( \sum_{k \in \mathbb{Z}^n} \rho_k(\xi) \geq 1 \) for all \( \xi \in \mathbb{R}^n \). Let

\[
\sigma_k(\xi) = \rho_k(\xi) \left( \sum_{l \in \mathbb{Z}^n} \rho_l(\xi) \right)^{-1}, \quad k \in \mathbb{Z}^n.
\]

Then, the sequence \( \{\sigma_k(\xi)\}_{k \in \mathbb{Z}^n} \) verifies the following properties:

\[
|\sigma_k(\xi)| \geq C, \quad \forall \xi \in Q_k,
\]

\[
\text{supp}(\sigma_k) \subset \{ \xi : |\xi - k|_\infty \leq \sqrt{n} \},
\]

\[
\sum_{k \in \mathbb{Z}^n} \sigma_k(\xi) = 1, \quad \forall \xi \in \mathbb{R}^n,
\]

\[
|D^\alpha \sigma_k(\xi)| \leq C_m, \quad \forall \xi \in \mathbb{R}^n, \quad |\alpha| \leq m.
\]

Modulation spaces \( M_{p,q}^s = M_{p,q}^s(\mathbb{R}^n) \) are Banach spaces constituted by frequency uniform decomposition \( \sigma_k, \ k \in \mathbb{Z}^n \). Explicitly, we consider the frequency-uniform decomposition operators \( \Box_k := \mathcal{F}^{-1}[\sigma_k \mathcal{F}] \), \( k \in \mathbb{Z}^n \). Then, for \( s \in \mathbb{R}, 1 \leq p, q \leq \infty \), modulations spaces \( M_{p,q}^s \) are defined as (cf. [14, 19, 31]):

\[
M_{p,q}^s := \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{M_{p,q}^s} < \infty \right\},
\]

where

\[
\|f\|_{M_{p,q}^s} = \begin{cases} 1 \left( \sum_{k \in \mathbb{Z}^n} (1 + |k|)^s \|\Box_k f\|_p^q \right)^{1/q}, & \text{for } 1 \leq q < \infty, \\ \sup_{k \in \mathbb{Z}^n} (1 + |k|)^s \|\Box_k f\|_p, & \text{for } q = \infty. \end{cases}
\]

For simplicity, we will write \( M_{p,q}^0(\mathbb{R}) = M_{p,q}(\mathbb{R}) \). Many of their properties, including embeddings in other known function spaces, can be found in Wang and Hudzik [31] (see also Kato [19]). In particular, the following properties hold:

i) If \( \Omega \) is a compact subset of \( \mathbb{R}^n \), then \( \mathcal{S}\Omega = \{ f : f \in \mathcal{S} \text{ and supp} \hat{f} \subset \Omega \} \) is dense in \( M_{p,q}^s, s \in \mathbb{R}, 0 < p, q < \infty \).

ii) \( M_{p_1,q_1}^{s_1} \subset M_{p_2,q_2}^{s_2}, \) if \( s_1 \geq s_2, 0 < p_1 \leq p_2, 0 < q_1 \leq q_2 \).

iii) \( M_{p_1,q_1}^{s_1} \subset M_{p_2,q_2}^{s_2}, \) if \( q_1 > q_2, s_1 > s_2, s_1 - s_2 > n/q_2 - n/q_1. \)

iv) \( M_{p,1}^1 \subset L^\infty \cap L^p, \) for \( 1 < p \leq \infty \).

v) \( B_{p,q}^{s+n/q} \subset M_{p,q}^s, \) for \( 0 < p, q \leq \infty \) and \( s \in \mathbb{R} \).

vi) \( B_{p,q}^{s_1} \subset M_{p,q}^{s_2} \) if and only if \( s_1 \geq s_2 + \nu_1(p,q) \).

vii) \( \nu(p,q) = \begin{cases} 0, & \text{if } (\frac{1}{p}, \frac{1}{q}) \in \left\{ (\frac{1}{p}, \frac{1}{q}) \in [0,\infty)^2 : \frac{1}{q} \leq \frac{1}{p} \text{ and } \frac{1}{q} \leq 1 - \frac{1}{p} \right\}, \\ \frac{1}{p} + \frac{1}{q} - 1, & \text{if } (\frac{1}{p}, \frac{1}{q}) \in \left\{ (\frac{1}{p}, \frac{1}{q}) \in [0,\infty)^2 : \frac{1}{p} \geq \frac{1}{2} \text{ and } \frac{1}{q} \geq 1 - \frac{1}{p} \right\}, \\ -\frac{1}{p} + \frac{1}{q}, & \text{if } (\frac{1}{p}, \frac{1}{q}) \in \left\{ (\frac{1}{p}, \frac{1}{q}) \in [0,\infty)^2 : \frac{1}{q} \leq \frac{1}{2} \text{ and } \frac{1}{q} \geq \frac{1}{p} \right\}. \end{cases} \)
In order to establish the main results, we need to consider the integral formulation associated to the Cauchy problem (1.1). Applying the operator $(1 - \partial_x)^{-1}$ on both sides of (1.1) it holds that

\[
\begin{align*}
  \begin{cases}
    u_t + i \varphi(D)u - \frac{\lambda}{\lambda + 2} \varphi\left[D^{\lambda+1}\right], & x \in \mathbb{R}, \ t \in \mathbb{R}, \\
    u(x, 0) = u_0(x), & x \in \mathbb{R},
  \end{cases}
\end{align*}
\]

(1.2)

where $\varphi(D)$ is defined as the Fourier multiplier with symbol $\varphi(\xi) = \frac{\xi}{1 + \xi^2}$. Let $S(t)$ be the unitary group on $L^2$ generated by $-i\varphi(D)$, namely, $S(t)u_0 = e^{-it\varphi(D)}u_0$ with $\varphi(D)u_0(\xi) = \varphi(\xi)\hat{u}_0(\xi)$. Then, by the Duhamel principle (1.2) is equivalent to the following integral equation

\[
u(x, t) = S(t)u_0(x) - \frac{i}{\lambda + 1} \int_0^t S(t - \tau)\varphi(D)[u^{\lambda+1}(x, \tau)]d\tau.
\]

(1.3)

Now we are in position to establish the main results of this paper. From now on, we consider the function $\beta_{\sigma} : (-\infty, -1) \rightarrow [-\frac{1}{q}, 0)$ defined by:

\[
\beta_{\sigma} := \beta(\sigma) = \begin{cases}
  \frac{\sigma + 1}{\lambda - 2\sigma} & \text{if } -4 \leq \sigma < -1, \\
  \frac{-3}{1 - 2\sigma} & \text{if } -\infty < \sigma \leq -4.
\end{cases}
\]

(1.4)

Function $\beta_{\sigma}$ controls the time-decay of the $L^p - H^s$ estimate of the group $S(t)$, which plays a key role in the analysis of the existence of global solutions for (1.3).

**Theorem 1.1** Consider $\lambda \geq 1$ an integer, $p = \lambda + 2$, $s \geq 0$, and $0 < \theta \leq -\frac{1}{\sigma}$ with $\sigma < -1$ such that $0 < -\frac{\lambda \theta \beta_{\sigma}}{\lambda + 2} < 1$ and define $r = -\frac{\lambda(\lambda + 2)}{\lambda + 2 + \lambda \theta \beta_{\sigma}}$. Then, there exists $\epsilon > 0$ such that if $\|S(t)u_0\|_{L^r(\mathbb{R}; M^s_{p,q})} < \epsilon$, equation (1.3) has a unique global solution $u \in L^r(\mathbb{R}; M^s_{p,q})$.

The restriction $q = 1$ in Theorem 1.1 comes from the use of the product estimate in modulation spaces (cf. Lemma 2.11 below). Considering a different product estimate given in Iwabuchi [18] we can include a global existence result with values in $M^s_{p,q}$, for $1 \leq q < 2$ by penalizing the regularity coefficient $s$, imposing that $1 - \frac{1}{q} \leq s < \frac{1}{q}$. This is the content of next theorem.

**Theorem 1.2** Consider $\lambda \geq 1$ an integer, $p = \lambda + 2$, $1 \leq q < \infty$ and $1 - \frac{1}{q} \leq s < \frac{1}{q}$. Also let $0 < \theta \leq -\frac{1}{\sigma}$, with $\sigma < -1$ such that $0 < -\frac{\lambda \theta \beta_{\sigma}}{\lambda + 2} < 1$ and define $r = -\frac{\lambda(\lambda + 2)}{\lambda + 2 + \lambda \theta \beta_{\sigma}}$. Then, there exists $\epsilon > 0$ such that if $\|S(t)u_0\|_{L^r(\mathbb{R}; M^s_{p,q})} < \epsilon$, equation (1.3) has a unique global solution $u \in L^r(\mathbb{R}; M^s_{p,q})$.

In Theorems 1.1 and 1.2 the condition $0 < -\frac{\lambda \theta \beta_{\sigma}}{\lambda + 2} < 1$ and the value $r = -\frac{\lambda(\lambda + 2)}{\lambda + 2 + \lambda \theta \beta_{\sigma}}$ come from the application of the Hardy-Littlewood-Sobolev’s inequality, which allow us to consider the range $\lambda \geq 1$. However, we are forced to assume a smallness condition on the norm $\|S(t)u_0\|_{L^r(\mathbb{R}; M^s_{p,q})}$ in place of a weaker smallness condition on the initial data $u_0$ directly. By using Strichartz estimates we can control the nonlinearity and establish the existence of global solution by assuming $\|u_0\|_{M^s_{p,q}}$ small enough; in this case, we need to impose the condition $\lambda \geq 6$. This is the content of next theorem.
Theorem 1.3 Consider \( \lambda \geq 6 \) an integer, \( 0 < \theta \leq -\frac{1}{\sigma} \), with \( \sigma = \frac{\lambda + 2}{4 - \lambda} \) or \( \sigma = \frac{2 - 3\lambda}{4} \). Take \( p = \lambda + 2 \), \( q \in [\gamma', \gamma] \) with \( \gamma = -\frac{2(\lambda + 2)}{\lambda \theta} \) and \( 1 - \frac{1}{q} \leq s < \frac{1}{q} \). Then, there exists \( \epsilon > 0 \) such that if \( \|u_0\|_{M^{\lambda - \sigma q}_{\rho,s}} < \epsilon \), equation (1.3) has a unique global solution \( u \) in \( C(\mathbb{R}; M^{\lambda}_{s,q}) \cap L'(\mathbb{R}; M^{\lambda}_{p,q}) \).

By using only the time-decay estimates for the group \( S(t) \) on Modulation spaces, we are able to obtain the existence of global solutions in the time-weighted based space \( X_{p,q}^{\rho,s} \) based on the modulation spaces:

\[
X_{p,q}^{\rho,s} := \left\{ u \in L^\infty_{\text{loc}}(\mathbb{R}; M^{s}_{p,q}) : \sup_{-\infty < t < \infty} (1 + |t|)^\rho \|u(t)\|_{M^{\rho,s}_{p,q}} < \infty \right\}, \quad \rho = -2\theta \left( \frac{1}{2} - \frac{1}{p} \right) \beta_\sigma > 0.
\]

Theorem 1.4 Consider \( \lambda \geq 3 \) an integer, \( 0 < \theta \leq -\frac{1}{\sigma} \), with \( \sigma < -1 \), \( p = \lambda + 2 \), \( s \geq 0 \). Then, there exists \( \epsilon > 0 \) such that if \( \|u_0\|_{M^{\lambda - \sigma p}_{s,1}} < \epsilon \), equation (1.3) has a unique global solution \( u \) in \( X_{p,q}^{\rho,s} \).

Theorem 1.5 Consider \( \lambda \geq 3 \) an integer, \( 0 < \theta \leq -\frac{1}{\sigma} \), with \( \sigma < -1 \), \( p = \lambda + 2 \), \( 1 \leq q < \infty \), and \( 1 - \frac{1}{q} \leq s < \frac{1}{q} \). Then, there exists \( \epsilon > 0 \) such that if \( \|u_0\|_{M^{\lambda - \sigma q}_{\rho,s}} < \epsilon \), equation (1.3) has a unique global solution \( u \) in \( X_{p,q}^{\rho,s} \).

In Theorems 1.3 and 1.5, the condition \( \lambda \geq 3 \) comes from the integrability of the function \( f(\tau) = (1 + |t - \tau|)^{-\rho}(1 + |\tau|)^{-\sigma(\lambda + 1)} \), on the real line. Next theorems provide local existence results in \( C([-T,T]; M^{s}_{p,q}) \) for the general case \( \lambda \geq 1 \).

Theorem 1.6 Consider \( \lambda \geq 1 \) an integer, \( 2 \leq p < \infty \), \( s \geq 0 \), and assume \( u_0 \in M^{s}_{p,q} \). Then there exists \( T > 0 \) and a unique solution \( u \) in \( C([0,T]; M^{s}_{p,q}) \) solution of equation (1.3).

Theorem 1.7 Consider \( \lambda \geq 1 \) an integer, \( 2 \leq p < \infty \), \( 1 \leq q < \infty \), \( 1 - \frac{1}{q} \leq s < \frac{1}{q} \), and assume \( u_0 \in M^{s}_{p,q} \). Then there exists \( T > 0 \) and a unique solution \( u \) in \( C([0,T]; M^{s}_{p,q}) \) solution of equation (1.3).

Remark 1.8 i) Theorems 1.3, 1.4, 1.5, 1.6 and 1.7 continue true if we replace the time interval \( \mathbb{R} \) by the compact interval \([-T, T]\) throughout their statements. Notice that for \( s_1 > s + \nu_1(p,q) \), \( p = \lambda + 2 \), \( H^{r}_{p,q} \subset M^{r}_{p,q} \), and therefore for \( 1 < r < \infty \) it holds

\[
\|S(t)u_0\|_{L^{r}([-T,T]; M^{s}_{p,q})} \leq C\|S(t)u_0\|_{L^{r}([-T,T]; H^{s}_{p,q})} \leq CT\|u_0\|_{H^{s}_{p,q}}.
\]

In particular, observing the statement of Theorem 1.3 for instance, if \( u_0 \in H^{s}_{p,q} \) and \( 0 < T \leq \epsilon C^{-1}\|u_0\|^{-1}_{H^{s}_{p,q}} \), where \( \epsilon \) is as in Theorem 1.3 it follows that \( \|S(t)u_0\|_{L^{r}([-T,T]; M^{s}_{p,q})} \leq \epsilon \). Thus, without assume any smallness condition on the initial data, the local in time version of Theorem 1.3 gives a solution \( u \in L^{r}([-T,T]; M^{s}_{p,q}) \), for initial data in \( H^{s}_{p,q} \). Therefore, if \( \bar{u} \in C([-T_0, T_0]; H^{s}_{p,q}) \) is the local solution obtained in Wang [33], for some \( T_0 > 0 \), by uniqueness \( u = \bar{u} \).

ii) The initial data class in Theorems 1.3, 1.4 and 1.5 is larger than the \( H^{s}_{p,q} \) of Wang [33] provided \( s_1 \) be large enough. This is consequence of the embedding \( H^{s_1}_{p,q} \subset M^{s_1}_{p,q} \) if \( s_1 > s_2 + \nu_1(p,q) \).

iii) In Theorems 1.3, 1.4, 1.5 and 1.7, the condition \( 1 - \frac{1}{q} \leq s < \frac{1}{q} \) and the integer nature of \( \lambda \) come from Lemma 2.10 below, to use the estimate \( \|u^p\|_{M^{s}_{p,q}} \leq C\|u\|_{M^{s}_{p,q}}^{p} \). In fact, it is an open problem to see if \( \|u^p\|_{M^{s}_{p,q}} \leq C\|u\|_{M^{s}_{p,q}}^{p} \) holds for any positive real constant \( p \).

This paper is organized as follows. In Section 2, we state time-decay and the Strichartz estimates in \( M^{s}_{p,q} \) for the group \( S(t) \), as well as some nonlinear estimates to deal with the nonlinear term in gBBM equation. In Section 3, we prove some Strichartz estimates in \( M^{s}_{p,q} \) for a general dispersive semigroup \( U(t) = e^{-itP(\xi)}F \), with \( P : \mathbb{R}^{n} \to \mathbb{R} \) a real-valued function, which, applied to the particular case \( S(t) \), allow us to obtain some existence results. Finally, we prove Theorems 1.3, 1.4, 1.5, 1.6, 1.7 in Section 4.
2 Dispersive and nonlinear estimates

The first aim of this section is to derive some decay estimate of the group $S(t)$ on Modulation spaces $M^s_{p,q}$. For that, a useful tool is the van der Corput’s Lemma, whose proof can be found in Stein [28], see also Linares and Ponce [13].

Lemma 2.1 (van der Corput) Let $h$ be either convex or concave twice differentiable function and $F$ be continuously differentiable function on $[a, b]$, with $-\infty < a < b < \infty$. Then

$$\left| \int_a^b F(r)e^{ih(r)}dr \right| \leq 4 \left\{ \min_{r \in [a,b]}|h''(r)| \right\}^{-1/2} \left[ |F(b)| + \int_a^b |F'(r)|dr \right],$$

(2.1)

for $h'' \neq 0$ on $[a, b]$.

Using Lemma 2.1 and a meticulous analysis of the Fourier symbol of the $S(t)$ we can obtain the following $L^1 - L^{\infty}$ estimate.

Lemma 2.2 Let $\sigma < 0$, $0 < \epsilon < \frac{1}{5}$ and $N > 2$. Define $K(t) = S(t)J^\sigma$, where $J^\sigma = (I - \Delta)^{\sigma/2}$. Then, there exists $C > 0$ such that

$$\|S(t)f\|_{L^\infty} \leq C \left\{ \epsilon + (1 - \sigma)|t|^{-1/2}\epsilon^{-1/2} + |t|^{-1/2} \max\{N^{3/2}, \epsilon^{-1/2}\} \left[ N^\sigma + (\sqrt{3} + \epsilon)^\sigma \right] - \frac{N^{\sigma+1}}{\sigma + 1} \right\} \|f\|_{L^1},$$

for all $f \in \mathcal{S}(\mathbb{R})$ and $t \neq 0$.

Proof: Define $h_{x,t}(\xi) = x\xi - t\varphi(\xi)$ and $F(\xi) = (1 + \xi^2)^{\sigma/2}$. Then

$$K(t)f(x) = \int_{-\infty}^{\infty} F(\xi)e^{ih_{x,t}(\xi)}\hat{f}(\xi)d\xi$$

$$= \int_{|\xi| \leq \epsilon} + \int_{\epsilon < |\xi| < \sqrt{3} - \epsilon} + \int_{\sqrt{3} - \epsilon \leq |\xi| \leq \sqrt{3} + \epsilon} + \int_{\sqrt{3} + \epsilon < |\xi| < N} + \int_{|\xi| \geq N}$$

$$=: I_1 + I_2 + I_3 + I_4 + I_5.$$

From the choice of $\sigma$, we have that $|F(\xi)| \leq 1$ for all $\xi$. Then using the Riemann-Lebesgue’s Lemma, we obtain

$$|I_1| \leq \| \hat{f} \|_\infty \int_{-\epsilon}^{\epsilon} |F(\xi)|d\xi \leq 2\epsilon \|f\|_{L^1}. \quad (2.2)$$

In a similar way we get

$$|I_5| \leq 2\epsilon \|f\|_{L^1}. \quad (2.3)$$

Next, to estimate $I_2$ note that $h_{x,t}$ is concave in $[\epsilon, \sqrt{3} - \epsilon]$; furthermore,

$$h_{x,t}'(\xi) = x - \frac{t(1 - \xi^2)}{(1 + \xi^2)^2} \quad \text{and} \quad h_{x,t}''(\xi) = \frac{2t(3 - \xi^2)}{(1 + \xi^2)^3}.$$

Then, easily we can see that

$$|h_{x,t}''(\xi)| \geq \frac{2t(|\sqrt{3} - \epsilon|)(3 - (\sqrt{3} - \epsilon)^2)}{(1 + (\sqrt{3} - \epsilon)^2)^3} \geq |t|\epsilon, \quad (2.4)$$

for all $\xi \in [\epsilon, \sqrt{3} - \epsilon]$. On the other hand, since $\sigma < 0$, we get

$$|F(\sqrt{3} - \epsilon)| \leq (1 + (\sqrt{3} - \epsilon)^2)^{\sigma/2} \leq 1. \quad (2.5)$$

follows.
Since \( \sigma < 0 \), for all \( \xi \in [\epsilon, \sqrt{3} - \epsilon] \), we have
\[
|F'(\xi)| = -\sigma|\xi|(1 + \xi^2)^{\sigma/2-1} \leq -\sigma(1 + \xi^2)^{\sigma/2-1/2} \leq -\sigma.
\] (2.6)

Now, applying Lemma 2.1 from (2.4), (2.5) and (2.6) we arrive at
\[
\left| \int_{\epsilon}^{\sqrt{3}-\epsilon} F(\xi)e^{ih_{x,t}(\xi)}d\xi \right| \leq (1 - \sigma)|t|^{-1/2}e^{-1/2}.
\]
Therefore,
\[
|I_1| \leq (2 - \sigma)|t|^{-1/2}e^{-1/2}\|f\|_{L^1}.
\] (2.7)

Next, in order to estimate \( I_4 \), note that \( h_{x,t} \) is concave in \([\sqrt{3} + \epsilon, N]\). Then,
\[
|h''_{x,t}(\xi)| \geq \min \left\{ \frac{2|t|N(3 - N^2)}{(1 + N^2)^3}, \frac{2|t|((\sqrt{3} + \epsilon)3 - (\sqrt{3} + \epsilon)^2)}{(1 + (\sqrt{3} + \epsilon)^2)^3} \right\} \geq |t|\min\{N^{-3}, \epsilon\},
\] (2.8)
for all \( \xi \in [\sqrt{3} + \epsilon, N] \). On the other hand,
\[
|F(N)| \leq (1 + N^2)^{\sigma/2} \leq N^{\sigma},
\] (2.9)
and since \( \sigma \) is negative, for all \( \xi \in [\sqrt{3} + \epsilon, N] \) we obtain
\[
|F'(\xi)| = (1 + \xi^2)^{\sigma/2-1}|\sigma| \leq -\sigma(1 + \xi^2)^{\sigma/2-1/2} \leq -\sigma \xi^{\sigma-1}.
\] (2.10)

Now, applying Lemma 2.1 from (2.8), (2.9) and (2.10) we arrive at
\[
\left| \int_{\sqrt{3}+\epsilon}^{N} F(\xi)e^{ih_{x,t}(\xi)}d\xi \right| \leq |t|^{-1/2} \max\{N^{3/2}, \epsilon^{-1/2}\} \left[ N^\sigma - \sigma \int_{\sqrt{3}+\epsilon}^{N} \xi^{\sigma-1}d\xi \right]
\leq |t|^{-1/2} \max\{N^{3/2}, \epsilon^{-1/2}\} \left[ N^\sigma + (\sqrt{3} + \epsilon)^{\sigma} - N^{\sigma} \right]
\leq |t|^{-1/2} \max\{N^{3/2}, \epsilon^{-1/2}\} \left[ N^{\sigma} + (\sqrt{3} + \epsilon)^{\sigma} \right].
\] (2.11)
Therefore,
\[
|I_4| \leq \|f\|_{L^1}|t|^{-1/2} \max\{N^{3/2}, \epsilon^{-1/2}\} \left[ N^{\sigma} + (\sqrt{3} + \epsilon)^{\sigma} \right].
\] (2.12)

To estimate \( I_5 \), since \( \sigma < 0 \), we arrive at
\[
|I_5| \leq \|\hat{f}\|_{L^\infty} \int_{N}^{\infty} |F(\xi)|d\xi \leq \|f\|_{L^1} \int_{N}^{\infty} (1 + \xi^2)^{\sigma}d\xi \leq \|f\|_{L^1} \int_{N}^{\infty} \xi^{\sigma}d\xi.
\]

Thus,
\[
|I_5| \leq -\frac{N^{\sigma+1}}{\sigma + 1}\|f\|_{L^1}.
\] (2.13)

From (2.2), (2.3), (2.7), (2.12) and (2.13), we obtain the desired result. \( \square \)

In a similar way as in Lemma 2.2 we also obtain the following result.

**Lemma 2.3** Let \( \sigma < -1 \), \( 2 \leq p \leq \infty \) and \( 1/p + 1/p' = 1 \). There is \( C_\sigma > 0 \) such that
\[
\|S(t)f\|_{H^s_p} \leq C_\sigma|t|^{2(\frac{1}{2} - \frac{1}{p})}\beta_s\|f\|_{L^{p'}}.
\]
for all \( f \in \mathcal{S}(\mathbb{R}) \) and \( t \neq 0 \). Here \( \beta_s \) is given by (1.4).
Lemma 2.4

On the other hand, from the Hölder and Young's inequalities, we obtain

\[ \|K(t)f\|_{L^\infty} \leq \left\{ N^{-3} + (1 - \sigma)|t|^{-1/2}N^{3/2} + |t|^{-1/2}N^{3/2} \left[ N^\sigma + (\sqrt{3N^{-3}})^\sigma \right] - \frac{N^{3+1}}{\sigma + 1} \right\} \|f\|_{L^1} \]

\[ \leq \left\{ N^{-3} + (1 - \sigma)|t|^{-1/2}N^{3/2} + |t|^{-1/2}N^{3/2} \left[ N^\sigma + 1 \right] - \frac{N^{3+1}}{\sigma + 1} \right\} \|f\|_{L^1} \]

\[ \leq \left\{ N^{-3} + (1 - \sigma)|t|^{-1/2}N^{3/2} + |t|^{-1/2}N^{3/2} + |t|^{-1/2}N^{3/2} - \frac{N^{3+1}}{\sigma + 1} \right\} \|f\|_{L^1} \]

\[ \leq \left\{ N^{-3} + (2 - \sigma)|t|^{-1/2}N^{3/2} + |t|^{-1/2}N^{3/2} - \frac{N^{3+1}}{\sigma + 1} \right\} \|f\|_{L^1}. \]

Proof: From Lemma 2.3, taking \( N^{3/2} = \epsilon^{-1/2} \), we obtain

\[ \|K(t)f\|_{L^\infty} \leq \left\{ t^{-3\theta} + (2 - \sigma)t^{\frac{3\sigma}{2} - \frac{1}{2} + \theta(\sigma + \frac{1}{2})^{-\frac{1}{2}} - \frac{1}{\sigma + 1}} \right\} \|f\|_{L^1}. \]

Taking \( \theta = \frac{1}{1 - 2\sigma} \), with \( \sigma < -1 \), we obtain

\[ \|K(t)f\|_{L^\infty} \leq C_\sigma t^{\beta_\sigma} \|f\|_{L^1}. \]

Note that \( C_\sigma \to +\infty \) as \( \sigma \to -1^- \) or \( \sigma \to -\infty \). If \( 0 < t \leq 1 \) and \( \sigma < -1 \), one easily get

\[ |K(t)f(x)| = \left| \int_{-\infty}^{\infty} e^{ix\xi - it\varphi(\xi)} (1 + \xi^2)^{\sigma/2} \hat{f}(\xi) d\xi \right| \leq \|f\|_{L^1} \int_{-\infty}^{\infty} (1 + \xi^2)^{\frac{\sigma}{2}} d\xi \]

\[ \leq t^{\beta_\sigma} \|f\|_{L^1}. \]

On the other hand, it is clear that \( K(t) : L^2 \to L^2 \) is continuous; thus an interpolation argument and recalling that \( S(t) = K(t)J^\sigma \), permit us to finishes the proof of the lemma. \( \Box \)

Next lemma gives a time-decay estimate of the group \( S(t) \) on modulation spaces \( M^\sigma_{p,q} \).

Lemma 2.4 Let \( s \in \mathbb{R}, \sigma < -1, 2 \leq p < \infty, \frac{1}{p} + \frac{1}{p'} = 1, 0 < q < \infty, \theta \in [0,1], \beta_\sigma \) as in (2.14). Then we have

\[ \|S(t)f\|_{M^\sigma_{p,q}} \leq (1 + |t|)^{2\theta(\frac{1}{2} - \frac{1}{p})} \|f\|_{M^{-\sigma}_{p',q}}. \] (2.14)

Proof: From Lemma 2.3 and taking into account that \( S(t) \) and \( \Box_k \) commute, we obtain

\[ \|\Box_k S(t)f\|_{H^p_q} \leq |t|^{2\left(\frac{1}{2} - \frac{1}{p}\right)} \|\Box_k f\|_{L^p}. \] (2.15)

Using the Berstein's multiplier estimate (cf. Wang [34]), we have

\[ \|\Box_k (I - \Delta)^{\delta/2} g\|_{L^p} \leq (1 + |k|)^{\delta} \|g\|_{L^p}. \] (2.16)

Then, from (2.15) and (2.16), we arrive at

\[ \|\Box_k S(t)f\|_{L^p} \leq (1 + |k|)^{-\sigma} \sum_{l \in \Lambda} \|\Box_{k+l} S(t)f\|_{H^p_q} \leq (1 + |k|)^{-\sigma} |t|^{2\left(\frac{1}{2} - \frac{1}{p}\right)} \|\Box_k f\|_{L^p}. \] (2.17)

On the other hand, from the Hölder and Young's inequalities, we obtain

\[ \|\Box_k S(t)f\|_{L^p} \leq \|\sigma_k e^{-it\varphi(\xi)} \varphi(\xi) \hat{f}\|_{L^p} \leq \sum_{l \in \Lambda} \|\sigma_k e^{-it\varphi(\xi)} \varphi(\xi) \Box_{k+l} f\|_{L^p} \]

\[ \leq \sum_{l \in \Lambda} \|\Box \Box_{k+l} f\|_{L^p} \leq \sum_{l \in \Lambda} \|\Box_{k+l} f\|_{L^p}. \] (2.18)
From (2.17) and (2.18) and an interpolation argument we get
\[ \|\Box k S(t)f\|_{L^p} \leq (1 + |k|)^{-\sigma \theta} |t|^{20(\frac{1}{2} - \frac{1}{p}) \beta_\sigma} \sum_{l \in \Lambda} \|\Box k + t f\|_{L^{p'}}. \tag{2.19} \]
for any \( \theta \in [0, 1] \). Since \( -\sigma \geq 0 \), from (2.18) we have
\[ \|\Box k S(t)f\|_{L^p} \leq (1 + |k|)^{-\sigma \theta} \sum_{l \in \Lambda} \|\Box k + t f\|_{L^{p'}}. \tag{2.20} \]
Combining (2.19) and (2.20), we arrive at
\[ \|\Box k S(t)f\|_{L^p} \leq (1 + |k|)^{-\sigma \theta} (1 + |t|)^{20(\frac{1}{2} - \frac{1}{p}) \beta_\sigma} \sum_{l \in \Lambda} \|\Box k + t f\|_{L^{p'}}. \tag{2.21} \]
Finally, multiplying (2.21) by \((1 + |k|)^s\) and then taking the \( l^p \)-norm, we obtain the desired result. \( \Box \)

**Proposition 2.5** Let \( s \in \mathbb{R}, \sigma < -1, 2 \leq p < \infty, 0 < q < \infty, 1 \leq r \leq \infty, \) and \( \theta \in [0, 1] \). Then
\[ \|S(t)u_0\|_{L^r(I, M_{p,q}^s)} \lesssim \|u_0\|_{M_{p',q}^{2\sigma}}, \tag{2.22} \]
where \( I \) is a compact subset of \( \mathbb{R} \) containing zero.

**Proof:** From Lemma 4.1 and since \( t \to (1 + |t|)^{20(\frac{1}{2} - \frac{1}{p}) \beta_\sigma} \) is a continuous function on \( I \), we obtain
\[ \|S(t)u_0\|_{L^r(I, M_{p,q}^s)} \lesssim \left\| (1 + |t|)^{20(\frac{1}{2} - \frac{1}{p}) \beta_\sigma} \|u_0\|_{M_{p',q}^{2\sigma}} \right\|_{L^1(I)} \]
\[ = \|u_0\|_{M_{p',q}^{2\sigma}} \left\| (1 + |t|)^{20(\frac{1}{2} - \frac{1}{p}) \beta_\sigma} \right\|_{L^1(I)} \lesssim \|u_0\|_{M_{p',q}^{2\sigma}}, \]
which concludes the result. \( \Box \)

**Lemma 2.6** Let \( s \in \mathbb{R}, 2 \leq p < \infty, 0 < q < \infty, \) \( \theta \in [0, 1] \). Then, we have
\[ \|S(t)\varphi(D)f\|_{M_{p,q}^s} \lesssim \|t\|^{2(\frac{1}{2} - \frac{1}{p})} \|f\|_{M_{p,q}^s}. \tag{2.23} \]
Here, the symbol \( \langle t \rangle \) denotes \((1 + |t|^2)^{1/2}\).

**Proof:** From Lemma 3.1 of Wang \( [23] \) it holds
\[ \|S(t)\varphi(D)f\|_{L^p} \lesssim \|t\|^{2(\frac{1}{2} - \frac{1}{p})} \|\varphi(D)f\|_{L^p}. \tag{2.24} \]
Notice that since \( p \geq 2 \) we can write \( \partial_x (1 - \partial_x)^{-1}(f) = \partial_x J^s J^{-\frac{s}{2} - \frac{1}{p}}(f) \), where \( J^s \) is the fractional differential operator defined by the symbol \( (1 + |\xi|^2)^{s/2} \). Since the symbol of the operator \( \partial_x J^s \frac{s}{2} - \frac{1}{p} \) is \( i\xi \frac{s}{2} - \frac{1}{p} \), it is an \( L^p \)-multiplier (cf. Theorem 2.1 in \( [23] \)). Furthermore, taking into account that the mapping \( J^{-\frac{s}{2}} : L^{p/2} \to L^p \) is bounded, it holds that \( \|\varphi(D)f\|_{L^p} \leq \|f\|_{L^p} \). Consequently, from (2.24) we get
\[ \|S(t)\varphi(D)f\|_{L^p} \lesssim \|t\|^{2(\frac{1}{2} - \frac{1}{p})} \|f\|_{L^p}. \tag{2.25} \]
From (2.25) and since \( \Box k \) and \( S(t)\varphi(D) \) commute, we have
\[ \|\Box k S(t)\varphi(D)f\|_{L^p} \lesssim \|t\|^{2(\frac{1}{2} - \frac{1}{p})} \|\Box k f\|_{L^p}. \tag{2.26} \]
Multiplying (2.20) by \((1 + |k|)^s\) and taking the \( l^q \)-norm in both sides of (2.20), we obtain the desired result. \( \Box \)
Lemma 2.7 Define $\varphi_k(\xi) = \varphi(\xi - k)$, where $\varphi(\xi) = \frac{\xi}{1 + \xi^2}$. Then,

$$\left| \frac{\partial^2 \varphi_k(\xi)}{\partial \xi^2} \right| \leq \langle k \rangle^{-3},$$

for all $|\xi| \leq C$ and $k \in \mathbb{Z}$.

**Proof:** Notice that

$$\frac{\partial^2 \varphi_k(\xi)}{\partial \xi^2} = \frac{2(\xi + k)((\xi + k)^2 - 3)}{(1 + (\xi + k)^2)^3}.$$

Therefore,

$$\left| \frac{\partial^2 \varphi_k(\xi)}{\partial \xi^2} \right| \leq \frac{6(1 + |\xi + k|^2)^{\frac{3}{2}}(1 + |\xi + k|^2)}{(1 + |\xi + k|^2)^3} = 6(1 + |\xi + k|^2)^{-\frac{3}{2}}.$$

Since $|\xi| \leq C$, we obtain the desired result. □

**Proposition 2.8** Let $1 < p \leq \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$. Then,

$$\|\varphi(D)g\|_{M^s_{p,q}} \leq \|g\|_{M^s_{p,q}},$$

for all $g \in M^s_{p,q}$.

**Proof:** The proof is inspired in the proof of Proposition A.1 in Kato [19]. First, we choose an auxiliary smooth function $\kappa \in \mathcal{S}$ satisfying

$$\kappa(\xi) = \begin{cases} 1, & \text{if } |\xi| \leq 1, \\ 0, & \text{if } |\xi| \geq 2. \end{cases}$$

We also define

$$\kappa_k(\xi) := \kappa \left( \frac{\xi - k}{C} \right).$$

Then $\kappa_k = 1$ on the support of $\sigma_k$. Here, the constant $C > 1$ is that one taking from the support of $\sigma_k$ in Section [1]. By the Young’s inequality and the change of variables $\xi - k \mapsto \xi$ we have

$$\left\| \mathcal{F}^{-1}[\kappa_k \cdot \mathcal{F}(\varphi(D)g)] \right\|_{L^p} = \left\| \mathcal{F}^{-1}[\kappa_k \varphi \cdot \sigma_k \mathcal{F} g] \right\|_{L^p} \leq \left\| \mathcal{F}^{-1}[\kappa_k \varphi] \right\|_{L^1} \left\| \mathcal{F}^{-1}[\sigma_k \mathcal{F} g] \right\|_{L^p} = \left\| \int_{\mathbb{R}} e^{ix\xi} \kappa \left( \frac{\xi - k}{C} \right) \varphi(\xi) d\xi \right\|_{L^1} \left\| \mathcal{F}^{-1}[\sigma_k \mathcal{F} g] \right\|_{L^p} = \left\| \int_{\mathbb{R}} e^{ix\xi} \kappa \left( \frac{\xi}{C} \right) \varphi(\xi - k) d\xi \right\|_{L^1} \left\| \Box g \right\|_{L^p}.$$

Now, since $|\varphi(\xi - k)| \leq \langle k \rangle^{-1}$ for all $|\xi| \leq C$ and for all $k \in \mathbb{Z}$, we obtain

$$\int_{|x| \leq 1} \left| \int_{\mathbb{R}} e^{ix\xi} \kappa \left( \frac{\xi}{C} \right) \varphi(\xi - k) d\xi \right| dx \leq \int_{|x| \leq 1} \langle k \rangle^{-1} dx \leq \langle k \rangle^{-1}.$$

Notice that

$$\frac{1}{x} \frac{\partial e^{ix\xi}}{\partial \xi} = ie^{ix\xi}, \quad \text{for } x \neq 0.$$
Proof: Thus, integrating by parts twice, from Lemma 2.7 we easily see that
\begin{align*}
\int_{|x| \geq 1} \left| \int \frac{e^{i \xi \cdot x}}{C} \varphi(\xi - k) \, d\xi \right| \, dx & \leq \int \frac{1}{x^2} (k)^{-1} \, dx \\
& \leq (k)^{-1}.
\end{align*}
Thus
\[ \left\| \mathcal{F}^{-1} [\varphi_k \cdot \mathcal{F}(\varphi(D))g] \right\|_{L^p} \leq (k)^{-1} \left\| k \right\|_{L^p}. \]
Now, with the usual modifications in the case \( q = \infty \), we have
\[ \| \varphi(D)g \|_{M^s_{p,q}} = \left( \sum_{k \in \mathbb{Z}} (k)^q \| \mathcal{F}(\varphi(D))g \|^2_{L^p} \right)^{1/q} \leq \left( \sum_{k \in \mathbb{Z}} (k)^{(q-1)q} \| \mathcal{F}(g) \|^2_{L^p} \right)^{1/q} \leq \| g \|_{M^s_{p,q}^1}, \]
as desired. \qed

With the aim of making the reading easier, we present three lemmas which allow us to deal with the nonlinearity \( f(u) = u^{\lambda+1} \). The proof of the first two can be found in Iwabuchi [18] (Proposition 2.7 (ii) and Corollary 2.9 (ii)). For the proof of the third one we refer to Bényi and Okoudjou [3].

Lemma 2.9 Let \( 1 \leq p, p_1, p_2 \leq \infty \), \( 1 < \sigma, \sigma_1, \sigma_2 < \infty \). If \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \frac{1}{\sigma} = \frac{1}{\sigma_1} + \frac{1}{\sigma_2} + 1 \leq \frac{1}{n} < \frac{1}{\sigma} \), there exists \( C > 0 \), such that for any \( u \in M^s_{p_1, \sigma_1} (\mathbb{R}^n) \) and \( v \in M^s_{p_2, \sigma_2} (\mathbb{R}^n) \), it holds
\[ \| uv \|_{M^s_{p, \sigma}} \leq C \| u \|_{M^s_{p_1, \sigma_1}} \| v \|_{M^s_{p_2, \sigma_2}}. \]

Lemma 2.10 Let \( 1 \leq q \leq \infty \), \( p \in \mathbb{N} \), \( 0 < s < n/\nu \), and \( 1 \leq \mu, \nu < \infty \) satisfying
\[ \frac{1}{\nu} = \frac{(p-1)s}{n} \leq \frac{p}{\mu} - p + 1, \quad 1 \leq \nu \leq \mu. \]
Then, there exists \( C > 0 \) such that for any \( u \in M^s_{p, \mu} (\mathbb{R}^n) \), we have
\[ \| u^p \|_{M^{s, \mu}_p} \leq C \| u \|_{M^{s, \mu}_p}^p. \]

Lemma 2.11 (Product estimate) Let \( m \) a positive integer and \( s \geq 0 \). Assume that \( \frac{1}{p_1} + \cdots + \frac{1}{p_m} = \frac{1}{p_0} \), \( \frac{1}{q_1} + \cdots + \frac{1}{q_m} = m - 1 + \frac{1}{q_0} \), with \( 0 < p_i \leq \infty \), \( 1 \leq q_i \leq \infty \) for \( 1 \leq i \leq m \). Then it holds that
\[ \left\| \prod_{i=1}^m u_i \right\|_{M^s_{p_0, q_0}} \leq C \left\| u_i \right\|_{M^s_{p_i, q_i}}^p, \]
where \( C \) is independent of \( u_i \), and \( u_i \in M^s_{p_i, q_i}, i = 1, \ldots, m. \)

Proposition 2.12 Let \( \lambda \geq 1 \) an integer, \( p = \lambda + 2 \), \( 1 \leq q < \infty \), \( 1 - \frac{1}{q} \leq s < \frac{1}{q} \), \( r = \frac{\lambda(\lambda+2)}{\lambda+2+\theta \lambda \beta s} \), and \( f(u) = u^{\lambda+1} \). Also assume that \( 0 < \theta < -\frac{1}{\sigma} \) and \( \sigma < -1 \) such that \( 0 < -\frac{-\theta \lambda^2}{\lambda+2} < 1 \). Then
\[ \left\| \int_0^t S(t-\tau) \varphi(D)f(u(\tau)) \, d\tau \right\|_{L^r(\mathbb{R}, M^s_{p,q})} \leq \| u \|_{M^{\lambda+1, \theta}_{p,q}}^{\lambda+1}. \quad (2.27) \]
Proof: From Lemma 2.4 and Proposition 2.8 we obtain
\begin{align*}
\left\| \int_0^t S(t-\tau) \varphi(D)f(u(\tau)) \, d\tau \right\|_{L^r(\mathbb{R}, M^s_{p,q})} & \leq \left\| \int_0^t \| S(t-\tau) \varphi(D)f(u(\tau)) \|_{M^s_{p,q}} \, d\tau \right\|_{L^r} \\
& \leq \left\| \int_0^t (1 + |t-\tau|)^{2\beta(\frac{1}{2} - \frac{1}{p})} \| \varphi(D)f(u(\tau)) \|_{M^s_{p,q}} \, d\tau \right\|_{L^r} \\
& \leq \left\| \int_0^t (1 + |t-\tau|)^{2\beta(\frac{1}{2} - \frac{1}{p})} \| f(u(\tau)) \|_{M^{\alpha-\sigma-1}_{p,q}} \, d\tau \right\|_{L^r}. \end{align*}
Since $\sigma < -1$, we have $-\frac{1}{\sigma} < 1$. Therefore, we can choose $0 < \theta \leq -\frac{1}{\sigma}$, and apply the embedding $M^s_{\theta,q} \subset M^{s-\theta-1}_{\theta,q}$ and Lemma 2.10 in last inequality to arrive at

$$\left\| \int_0^t S(t-\tau)\varphi(D)f(u(\tau))d\tau \right\|_{L^r(\mathbb{R},M^s_{\theta,q})} \leq \left\| \int_0^t (1 + |t-\tau|)^{2p(\frac{1}{2} - \frac{1}{p})} \|f(u(\tau))\|_{M^s_{\theta,q}} d\tau \right\|_{L^r_t} \leq \left\| \int_0^t (1 + |t-\tau|)^{2p(\frac{1}{2} - \frac{1}{p})} \|u(\tau)\|^\lambda_{M^s_{\theta,q}} d\tau \right\|_{L^r_t}. \quad (2.28)$$

From the choice of $\theta, \sigma$ and $r$, we have that

$$0 < 1 + 2\theta \left( \frac{1}{2} - \frac{1}{p} \right) \beta_\sigma < 1 \quad \text{and} \quad \frac{1}{r} = \frac{\lambda + 1}{r} - \left( 1 + 2\theta \left( \frac{1}{2} - \frac{1}{p} \right) \beta_\sigma \right) .$$

Therefore, we can apply the Hardy-Littlewood-Sobolev’s inequality in (2.28) in order to obtain the desired result. \(\square\)

**Proposition 2.13** Let $\lambda \geq 1$ an integer, $p = \lambda + 2$, $s \geq 0$, $r = \frac{\lambda(\lambda+2)}{\lambda+2+\theta\beta_\sigma}$, and $f(u) = u^{\lambda+1}$. Also assume that $0 < \theta \leq -\frac{1}{\sigma}$ and $\sigma < -1$ such that $0 < \frac{-\theta\beta_\sigma}{\lambda+2} < 1$. Then

$$\left\| \int_0^t S(t-\tau)\varphi(D)f(u(\tau))d\tau \right\|_{L^r(\mathbb{R},M^s_{\theta,q})} \leq \|u\|^\lambda_{L^r(\mathbb{R},M^s_{\theta,q})}. \quad (2.29)$$

**Proof:** The proof of Proposition 2.13 is analogous to the proof of Proposition 2.12 by using Lemma 2.11 in place of Lemma 2.10. \(\square\)

### 3 General Strichartz estimates in $M^s_{p,q}$

The aim of this section is to derive some Strichartz estimates in modulation spaces $M^s_{p,q}$ for a general dispersive semigroup

$$U(t) = \mathcal{F}^{-1} e^{itP(\xi)} \mathcal{F}, \quad (3.1)$$

where $P : \mathbb{R}^n \rightarrow \mathbb{R}$ is a real-valued function. In Section 4, we will use these estimates in the particular case of $U(t) = S(t)$ in order to analyze the existence of global solutions for the gBBM equation. We assume that $U(t)$ is a semigroup which satisfies the next estimate

$$\|U(t)f\|_{M^s_{p,q}} \leq (1 + |t|^{-\delta})\|f\|_{M^{s+\delta}_{p,q}}, \quad (3.2)$$

where $2 \leq p < \infty$, $1 \leq q < \infty$, $\alpha \in \mathbb{R}$, $\mu = \mu(p)$, with $0 < \mu < 1$ and $\alpha, \delta, \mu$ are independent of $t \in \mathbb{R}$. Taking into account Lemma 2.13 it holds that an example of a group $U(t)$ verifying (3.1) and (3.2) is given by the BBM group $S(t)$. Next, we derive some time-space estimates of $U(t)$ satisfying (3.1) and (3.2). Here is worthwhile to remark that the results of this section can be applied to analyze existence results for other dispersive models in the framework of modulation spaces. These estimates complement those proved by Wang and Hudzik 31. We need to establish some additional notations: Given the Banach space $X$, for instance $X = L^\gamma(\mathbb{R}; L^p(\mathbb{R}^n))$, $1 \leq p, \gamma \leq \infty$, we consider the function spaces $L^{\gamma,q}_\square(X)$, $s \in \mathbb{R}, 1 \leq q < \infty$, introduced in 31, which are defined as follows:

$$L^{\gamma,q}_\square(X) = \left\{ f \in \mathcal{S}'(\mathbb{R}^{n+1}) : \|u\|_{L^{\gamma,q}_\square(X)} := \left( \sum_{k \in \mathbb{Z}^n} (1 + |k|)^{\alpha q} \|\Box_k f\|_X^q \right)^{1/q} < \infty \right\}.$$

In the definition of $L^{\gamma,q}_\square(X)$, if $s = 0$, then we write $L^{\gamma,0}_\square(X)$. The following duality results is known (cf. Wang and Hudzik 31).
Theorem 3.1 (Dual space) \([31]\) Let \(s \in \mathbb{R}\) and \(1 \leq p, \gamma < \infty\). We have

\[
\left(c_c^\infty(L^\gamma(\mathbb{R}, L^p))\right)^* = l_{\infty}^{s,1}(L^\gamma(\mathbb{R}, L^{p'})).
\]

Proposition 3.2 Let \(U(t)\) satisfying \([31]\) and \([32]\). Then, for all \(s \in \mathbb{R}\) and \(\gamma = \frac{2}{\mu}\), we have

\[
\|U(t)f\|_{L^\gamma(\mathbb{R}, L^p)} \leq \|f\|_{M_{2,q}^{s+\frac{1}{2}}}. \tag{3.3}
\]

In addition, if \(\gamma \geq q\), we have

\[
\|U(t)f\|_{L^\gamma(\mathbb{R}, M_{p,q}^s)} \leq \|f\|_{M_{2,q}^{s+\frac{1}{2}}}. \tag{3.4}
\]

**Proof:** The proof is based on a duality argument. Without loss of generality take \(s = -\delta/2\). First, we consider the case \(1 < q < \infty\). We show that

\[
\int_{\mathbb{R}} (U(t)f, \phi(t)) dt \leq \|f\|_{M_{2,q}^s} \|\phi\|_{\mathcal{H}^s(\mathbb{R}, L^{p'})} \tag{3.5}
\]

holds for all \(f \in \mathcal{S}(\mathbb{R}^n)\) and \(\phi \in C_0^\infty(\mathbb{R}; \mathcal{S}(\mathbb{R}^n))\). Since \(\mathcal{S}(\mathbb{R}^n)\) and \(C_0^\infty(\mathbb{R}; \mathcal{S}(\mathbb{R}^n))\) are dense in \(M_{2,q}^s\) and \(l_{\infty}^{s,1}(L^\gamma(\mathbb{R}, L^{p'}))\), respectively (cf. \([31]\)), we see that \((3.5)\) implies \((3.3)\). By duality we obtain that

\[
\int_{\mathbb{R}} (U(t)f, \phi(t)) dt \leq \|f\|_{M_{2,q}^s} \left\| \int_{\mathbb{R}} U(-t)\phi(t) dt \right\|_{M_{2,q}^s}. \tag{3.6}
\]

Now, for \(k \in \mathbb{Z}^n\) we get

\[
\left\| \Box_k \int_{\mathbb{R}} U(-t)\phi(t) dt \right\|_{L^2} \leq \|\Box_k \phi\|_{L^\gamma(\mathbb{R}; L^{p'})} \left\| \Box_k \int_{\mathbb{R}} U(t-\tau)\phi(\tau) d\tau \right\|_{L^\gamma(\mathbb{R}; L^{p'})}. \tag{3.7}
\]

Since the sequence \(\{\Box_k\}_{k \in \mathbb{Z}^n}\) is almost orthogonal, using \([32]\) with \(\alpha = -\delta\), the definition of the norm of \(M_{\mu,q}^\gamma\) and the Bernstein’s multiplier estimate (cf. Wang and Huang \([31]\)) we arrive at

\[
\|\Box_k U(t)f\|_{L^p} \leq (1 + |k|\delta) \|U(t)f\|_{M_{\mu,q}^\gamma} \leq (1 + |t|)^{-\mu(p)}(1 + |k|)^\delta \sum_{l \in \Lambda} \|\Box_k \Box_{k+l} f\|_{M_{\mu,q}^\gamma}
\]

\[
\leq (1 + |t|)^{-\mu(p)}(1 + |k|)^\delta \|k f\|_{L^{p'}}, \tag{3.8}
\]

where \(\Lambda = \{l \in \mathbb{Z}^n : B(0, \sqrt{n}) \cup B(l, \sqrt{n}) \neq \emptyset\}\). Since \(0 < \mu < 1\) and \(\gamma = \frac{2}{\mu}\), we can use the Hardy-Littlewood-Sobolev’s inequality to obtain

\[
\left\| \Box_k \int_{\mathbb{R}} U(t-\tau)\phi(\tau) d\tau \right\|_{L^\gamma(\mathbb{R}; L^{p'})} \leq (1 + |k|)^\delta \|\Box_k \phi\|_{L^\gamma(\mathbb{R}; L^{p'})}. \tag{3.9}
\]

So, in view of \((3.7)\) and \((3.9)\), we have

\[
\left\| \Box_k \int_{\mathbb{R}} U(-t)\phi(t) dt \right\|_{L^2} \leq (1 + |k|)^{\delta/2} \|\Box_k \phi\|_{L^\gamma(\mathbb{R}; L^{p'})}. \tag{3.10}
\]

Taking the \(l^{p'}\) norm in both sides of the last inequality, we get

\[
\left\| \int_{\mathbb{R}} U(-t)\phi(t) dt \right\|_{M_{2,q}^s} \leq \|\phi\|_{l_{\infty}^{s,1}(L^{p'}(\mathbb{R}; L^{p'}))}. \tag{3.11}
\]

From \((3.10)\) and \((3.11)\), we arrived at \((3.3)\), as desired.
If $\gamma \geq q$, using the Minkowski’s inequality, we obtain the left-hand of (3.3) is controlled by the left-hand of (3.3).

Next, we consider the case $q = 1$. From Theorem 3.1 it enough to show that

$$
\int_{\mathbb{R}} (U(t) f, \phi(t)) dt \leq \|f\|_{L_{q}(\mathbb{R}, L^2)} \|\phi\|_{L^{2}_p(\mathbb{R}; (L^p, L^q)'))}
$$

holds for all $f \in \mathcal{S}(\mathbb{R}^n)$ and $\phi \in C_0^\infty(\mathbb{R}; \mathcal{S}(\mathbb{R}^n))$. Repeating the above procedure, we can obtain the desire result. Finally, from the Minkowski’s inequality we obtain (3.4) from (3.3).

Next, we estimate the nonlinear part. We denote by

$$
(\mathcal{N}f)(t, \cdot) = \int_0^t U(t - \tau) f(\tau, \cdot) d\tau.
$$

**Proposition 3.3** Let $U(t)$ satisfying (3.1) and (3.2). Then, for all $s \in \mathbb{R}$ and $\gamma = \frac{2}{p}$ we have

$$
\|\mathcal{N}f\|_{L^q(\mathbb{R}, L^2)} \leq \|f\|_{L^{2}_p(\mathbb{R}; (L^p, L^q)'))}.
$$

In addition, if $\gamma' \leq q$, we have

$$
\|\mathcal{N}f\|_{L^q(\mathbb{R}, L^2)} \leq \|f\|_{L^{2}_p(\mathbb{R}; (L^p, L^q)')}.
$$

**Proof:** From the definition of the norm of the space $L^{2}_p(\mathbb{R}; (L^p, L^q)')$ and using the same ideas as in Proposition 3.2 the crucial inequality

$$
\|\square_k \mathcal{N}f\|_{L^q}^2 \leq \langle k \rangle^\delta \|\square_k f\|_{L^{2}_p(\mathbb{R}; L^q')}^2,
$$

implies (3.11). From the Minkowski’s inequality we obtain (3.12) from (3.11).

**Proposition 3.4** Let $U(t)$ satisfying (3.1) and (3.2). Then, for all $s \in \mathbb{R}$ and $\gamma = \frac{2}{p}$, we have

$$
\|\mathcal{N}f\|_{L^q(\mathbb{R}, L^2)} \leq \|f\|_{L^{2}_p(\mathbb{R}; (L^p, L^q)')}.
$$

In addition, if $\gamma \geq q$, we have

$$
\|\mathcal{N}f\|_{L^q(\mathbb{R}, L^2)} \leq \|f\|_{L^{2}_p(\mathbb{R}; (L^p, L^q)')}.
$$

**Proof:** Let $f, \phi \in C_0^\infty(\mathbb{R}, \mathcal{S}(\mathbb{R}))$. Following Proposition 3.2 we have

$$
\left| \int_{\mathbb{R}} \left( \int_0^t U(t - \tau) f(\tau, \cdot) d\tau \right) dt \right| \leq \|f\|_{L^1(\mathbb{R}, L_{2,q})} \left\| \int_{\mathbb{R}} U(\cdot - t) \phi(t) dt \right\|_{L^{\infty}(\mathbb{R}; L_{2,q'})} \leq \|f\|_{L^1(\mathbb{R}, L_{2,q})} \|\phi\|_{L^{2}_p(\mathbb{R}; (L^p, L^q)')}.
$$

Since $\phi \in C_0^\infty(\mathbb{R}, \mathcal{S}(\mathbb{R}))$ is dense in $L^{2}_p(\mathbb{R}; (L^p, L^q')$ and in $L^{2}_p(\mathbb{R}; (L^p, L^q))$, by duality we get (3.13). Finally, from the Minkowski’s inequality we obtain (3.14) from (3.13). □
Proposition 3.5 Let \( U(t) \) satisfying (3.1) and (3.2). Then, for all \( s \in \mathbb{R} \) and \( q = \frac{2}{\mu} \), we have
\[
\|Nf\|_{l^q_s(L^\gamma(\mathbb{R}, L^r))} \lesssim \|f\|_{l^{2\gamma/\mu}_s(L^{2\gamma}(\mathbb{R}, L^{2\gamma'}))}.
\]

In addition, if \( q \in [\gamma', \gamma] \), we have
\[
\|Nf\|_{L^\gamma(\mathbb{R}, M^r_{p,q})} \lesssim \|f\|_{L^{\gamma'}(\mathbb{R}, M^r_{p,q})}.
\]

Proof: The proof of (3.15) is also based on a duality argument, so we only sketch the proof of (3.16). From (3.2) we obtain
\[
\|Nf\|_{M^r_{p,q}} \leq \int_0^t (t - \tau)^{-\mu} \|f(\tau)\|_{M^r_{p,q}} d\tau.
\]
Taking the \( L^\gamma \)-norm and using the Hardy-Littlewood-Sobolev inequality we obtain the desired result.

\[ \Box \]

4 Existence results

4.1 Proof of Theorems 1.1 and 1.2

We first focus on the proof of Theorem 1.2, which is based on a fixed point argument by applying the time-decay and Strichartz estimates, as well as the nonlinear estimates obtained in Section 2 and 3. For that, let us consider the closed ball \( B_{2\varepsilon} = \{ u : \|u\|_{L^\gamma(\mathbb{R}, M^r_{p,q})} \leq 2\varepsilon \} \), with \( \varepsilon > 0 \), and define the map \( \Gamma \) on the metric space \( B_{2\varepsilon} \):
\[
(\Gamma u)(x, t) = S(t)u_0(x) - \frac{i}{\lambda + 1} \int_0^t S(t - \tau)\varphi(D)|u|^{\lambda+1}(x, \tau)|d\tau.
\]

We want to choose a \( \varepsilon > 0 \) such that \( \Gamma : B_{2\varepsilon} \rightarrow B_{2\varepsilon} \) is a contraction. From Proposition 2.12 and the smallness assumption on \( \|S(t)u_0\|_{L^\gamma(\mathbb{R}, M^r_{p,q})} \) we have that, if \( u \in B_{2\varepsilon} \), then
\[
\|\Gamma u\|_{L^\gamma(\mathbb{R}, M^r_{p,q})} \leq \|S(t)u_0(x)\|_{L^\gamma(\mathbb{R}, M^r_{p,q})} + C\|u\|_{L^{\gamma'}(\mathbb{R}, M^r_{p,q})}^{\lambda+1} \leq \varepsilon + C(2\varepsilon)^{\lambda+1} = \varepsilon + C2^{\lambda+1}\varepsilon^\lambda.
\]

Taking \( \varepsilon > 0 \) such that \( C2^{\lambda+1}\varepsilon^\lambda < 1 \), we get that \( \Gamma : B_{2\varepsilon} \rightarrow B_{2\varepsilon} \). From Lemma 2.4 and Proposition 2.8 (see the proof of Proposition 2.12) and taking into account Lemmas 2.9 and 2.10 we get
\[
\|\Gamma u - \Gamma v\|_{L^\gamma(\mathbb{R}, M^r_{p,q})} \leq \left\| \int_0^t (1 + |t - \tau|)^{2\theta(\frac{\lambda}{2} + \frac{1}{\beta})} \|u^{\lambda+1} - v^{\lambda+1}\|_{M^r_{p',q'}} d\tau \right\|_{L_\gamma^1}
\]
\[
\leq \left\| \int_0^t (1 + |t - \tau|)^{2\theta(\frac{\lambda}{2} + \frac{1}{\beta})} \|u - v\| \left( \sum_{k=1}^{\lambda+1} u^{\lambda+1-k} v^{k-1} \right) \|_{M^r_{p',q'}} d\tau \right\|_{L_\gamma^1}
\]
\[
\leq \left\| \int_0^t (1 + |t - \tau|)^{2\theta(\frac{\lambda}{2} + \frac{1}{\beta})} \|u - v\|_{M^r_{p,q}} \left( \sum_{k=1}^{\lambda+1} u^{\lambda+1-k} \|M^r_{p',q'}\|_{L_\gamma^1} v^{k-1} \right) d\tau \right\|_{L_\gamma^1}
\]
\[
\leq \left\| \int_0^t (1 + |t - \tau|)^{2\theta(\frac{\lambda}{2} + \frac{1}{\beta})} \|u - v\|_{M^r_{p,q}} \left( \sum_{k=1}^{\lambda+1} u^{\lambda+1-k} \|v^{k-1}\|_{M^r_{p',q'}} \right) d\tau \right\|_{L_\gamma^1}.
\]
Considering the assumption $0 < -\frac{\lambda \theta \beta}{\lambda^2 + 2} < 1$ and $r = \frac{\lambda(\lambda + 2)}{\lambda^2 + 2 + \beta \theta \sigma}$, we can apply the Hölder and Hardy-Littlewood-Sobolev's inequalities in last inequality in order to obtain

$$\|\Gamma u - \Gamma v\|_{L^r(\mathbb{R}; M^\alpha_{q,q})} \leq \|u - v\|_{L^r(\mathbb{R}; M^\alpha_{q,q})}\left(\|u\|_{L^r(\mathbb{R}; M^\beta_{p,q})} + \|v\|_{L^r(\mathbb{R}; M^\beta_{p,q})}\right) \leq 2^{\lambda + 1}e^C\|u - v\|_{L^r(\mathbb{R}; M^\beta_{p,q})}.$$  \hspace{1cm} (4.2)

From (4.1) and (4.2), it holds that $\Gamma : B_{2\epsilon} \rightarrow B_{2\epsilon}$ is a contraction, which implies the existence of a unique fixed point, as desired. The proof of Theorem 1.1 is analogous to the proof of Theorem 1.2 by using Proposition 2.13 in place of Proposition 2.12. \hfill \Box

### 4.2 Proof of Theorem 1.3

The proof is based on a fixed point argument by applying the time-decay and Strichartz estimates, as well as the nonlinear estimates obtained in Section 2 and Section 3. For that, let us consider the closed ball $B_{2\epsilon} = \{ u : \|u\|_{L^\infty(\mathbb{R}; M^\alpha_{q,q})} + \|u\|_{L^\gamma(\mathbb{R}; M^\beta_{p,q})} \leq 2\epsilon \}$, with $\epsilon > 0$, and define the map $\Gamma$ on the metric space $B_{2\epsilon}$:

$$(\Gamma u)(x,t) = S(t)u_0(x) - \frac{i}{\lambda + 1} \int_0^t S(t - \tau)\varphi(D)[u^{\lambda + 1}(x,\tau)]d\tau.$$  \hspace{1cm}

From Proposition 3.2 (Inequality (3.4)) and Proposition 3.5 (Inequality (3.10)), we have

$$\|\Gamma u\|_{L^\gamma(\mathbb{R}; M^\alpha_{p,q})} \lesssim \|u_0\|_{M^\alpha_{2,q}} + \|\varphi(D)(u^{\lambda + 1})\|_{L^\gamma(\mathbb{R}; M^\beta_{p,q})}.$$  \hspace{1cm}

In view of Proposition 2.8, the choice of $\theta$, Lemma 2.10 and the values of $p$ and $\gamma$, we arrive at

$$\|\Gamma u\|_{L^\gamma(\mathbb{R}; M^\alpha_{p,q})} \lesssim \|u_0\|_{M^\alpha_{2,q}} + \|u^{\lambda + 1}\|_{L^\gamma(\mathbb{R}; M^\beta_{p,q})}.$$  \hspace{1cm} (4.3)

On the other hand, since $\|\Box \Delta S(t)u_0\|_2 \leq \|\Box u_0\|_2$ and Proposition 3.3 (Inequality (3.12)), we have

$$\|\Gamma u\|_{L^\infty(\mathbb{R}; M^\alpha_{q,q})} \lesssim \|u_0\|_{M^\alpha_{2,q}} + \|\varphi(D)(u^{\lambda + 1})\|_{L^\gamma(\mathbb{R}; M^\beta_{p,q})}.$$  \hspace{1cm}

In view Proposition 2.8, the choice of $\theta$, Lemma 2.10 and the values of $p$ and $\gamma$, we obtain

$$\|\Gamma u\|_{L^\infty(\mathbb{R}; M^\alpha_{q,q})} \lesssim \|u_0\|_{M^\alpha_{2,q}} + \|u^{\lambda + 1}\|_{L^\gamma(\mathbb{R}; M^\beta_{p,q})}.$$  \hspace{1cm} (4.4)

From (4.3) and (4.4), if $\|u_0\|_{M^\alpha_{2,q}} \leq \epsilon$ and $u \in B_{2\epsilon}$, we conclude that $\Gamma : B_{2\epsilon} \rightarrow B_{2\epsilon}$. Also, in the same spirit of the proof of Theorem 1.2 together with (4.3) and (4.4), we get that $\Gamma : B_{2\epsilon} \rightarrow B_{2\epsilon}$ is a
contraction on \(B_{2\epsilon}\), which implies the existence of a unique fixed point, as desired. Now we prove that the fixed point \(u \in C(\mathbb{R}; M_{2,\varrho}^{\beta})\), that is, that \(\|u(t_n) - u(t)\|_{M_{2,\varrho}^{\beta}} \to 0\), as \(t_n \to t\). For that, notice that

\[
\|u(t_n) - u(t)\|_{M_{2,\varrho}^{\beta}} \leq \|S(t_n)u_0 - S(t)u_0\|_{M_{2,\varrho}^{\beta}} + \frac{1}{\lambda + 1} \left| \int_0^{t_n} S(t_n - \tau)\varphi(D)[u^{\lambda+1}(x, \tau)]d\tau - \int_0^{t} S(t - \tau)\varphi(D)[u^{\lambda+1}(x, \tau)]d\tau \right|_{M_{2,\varrho}^{\beta}}
\]

\[
:= I_1 + \frac{1}{\lambda + 1} I_2.
\]

(4.5)

Observe that for \(v \in \mathcal{S}_t^{\Omega} = \{f : f \in \mathcal{S} \text{ and } \text{supp } f \subset \Omega\} \) (see Section 1) it holds

\[
\|\Box_k(S(t_n)v - S(t)v)\|_2 \leq \|\sigma_k(e^{-it_n \xi} - e^{-it \xi})\hat{\varphi}(\xi)\|_2 \leq \|e^{-it_n \xi} - e^{-it \xi}\hat{\varphi}(\xi)\|_2.
\]

If \(v \in \mathcal{S}\), then from the Lebesgue’s dominated convergence Theorem we get

\[
\left\|\left(e^{-it_n \xi} - e^{-it \xi}\right)\hat{u}_0(\xi)\right\|_2 \to 0, \text{ as } t_n \to t.
\]

Since \(v \in \mathcal{S}_t^{\Omega}\), then \(\Box_k(S(t_n)v - S(t)v) \neq 0\) only for a finite number of \(k\). Therefore we can conclude that

\[
I_1 = \|S(t_n)u_0 - S(t)u_0\|_{M_{2,\varrho}^{\beta}} \to 0, \text{ as } t_n \to t.
\]

Now we deal with the term \(I_2\) in (4.5). For that, notice that

\[
I_2 = \left\|(S(t_n) - S(t)) \int_0^{t} S(-\tau)\varphi(D)[u^{\lambda+1}(x, \tau)]d\tau + \int_t^{t_n} S(t_n - \tau)\varphi(D)[u^{\lambda+1}(x, \tau)]d\tau \right\|_{M_{2,\varrho}^{\beta}}
\]

\[
:= I_2^1 + I_2^2.
\]

Thus, arguing as in (4.3) we get

\[
\left\|\int_0^{t} S(-\tau)\varphi(D)[u^{\lambda+1}(x, \tau)]d\tau \right\|_{M_{2,\varrho}^{\beta}} \leq \|u\|_{L^\infty(\mathbb{R}; M_{2,\varrho}^{\beta})} < \infty,
\]

which allow us to conclude that \(I_2^1 \to 0\), as \(t_n \to t\). In the same way,

\[
\left\|\int_t^{t_n} S(t_n - \tau)\varphi(D)[u^{\lambda+1}(x, \tau)]d\tau \right\|_{M_{2,\varrho}^{\beta}} \leq \|u\|_{L^\infty(\mathbb{R}; M_{2,\varrho}^{\beta})} < \infty,
\]

which implies that \(I_2^2 \to 0\), as \(t_n \to t\). Thus, \(I_2 \to 0\), as \(t_n \to t\), and then \(u \in C(\mathbb{R}; M_{2,\varrho}^{\beta})\). \(\square\)

### 4.3 Proof of Theorems 1.4 and 1.5

We first present the proof of Theorem 1.5. This proof is also based on a fixed point argument and the time-decay and product estimates established in Section 2. Let us consider the closed ball

\[
B_{2\epsilon} = \left\{ u : \sup_{-\infty < t < \infty} (1 + |t|^\rho)\|u(t)\|_{M_{2,\varrho}^{\beta}} \leq 2\epsilon \right\}, \epsilon > 0,
\]

where \(\rho = -2\theta(\frac{1}{2} - \frac{1}{p})\beta > 0\), and define the map \(\Gamma\) on the metric space \(B_{2\epsilon}\) :

\[
(\Gamma u)(x, t) = S(t)u_0(x) - \frac{i}{\lambda + 1} \int_0^{t} S(t - \tau)\varphi(D)[u^{\lambda+1}(x, \tau)]d\tau.
\]
From Lemma 2.4 Proposition 2.3, the embedding $M_{p',q}^s \subset M_{p',q}^{s-\sigma \theta - 1}$, and Lemma 2.10 we have

$$
\|\Gamma u\|_{M_{p',q}^s} \leq (1 + |t|)^{-\rho} \|u_0\|_{M_{p',q}^{s-\sigma \theta}} + \int_0^t (1 + |t - \tau|)^{-\rho} \|\varphi(D) u^{\lambda + 1}\|_{M_{p',q}^{s-\sigma \theta - 1}}
$$

$$
\leq (1 + |t|)^{-\rho} \|u_0\|_{M_{p',q}^{s-\sigma \theta}} + \int_0^t (1 + |t - \tau|)^{-\rho} \|u^{\lambda + 1}\|_{M_{p',q}^{s-\sigma \theta - 1}}
$$

$$
\leq (1 + |t|)^{-\rho} \|u_0\|_{M_{p',q}^{s-\sigma \theta}} + \int_0^t (1 + |t - \tau|)^{-\rho} \|u\|_{M_{p',q}^{s-\sigma \theta}}
$$

$$
\leq (1 + |t|)^{-\rho} \|u_0\|_{M_{p',q}^{s-\sigma \theta}} + \left(\sup_{t > 0} (1 + |t|)^\rho \|u(t)\|_{M_{p',q}^s}\right)^{\lambda + 1}
$$

$$
\times \int_0^t (1 + |t - \tau|)^{-\rho} (1 + |\tau|)^{-\rho(\lambda + 1)} d\tau. \quad (4.6)
$$

Since $\lambda \geq 3$, then $\rho(\lambda + 1) > 1$. Thus, it holds that

$$
\int_0^{t/2} (1 + t - \tau)^{-\rho} (1 + \tau)^{-\rho(\lambda + 1)} d\tau \leq (1 + t)^{-\rho} \int_0^{t/2} (1 + \tau)^{-\rho(\lambda + 1)} d\tau \leq (1 + t)^{-\rho}. \quad (4.7)
$$

Also, it is straightforward to get

$$
\int_{t/2}^t (1 + t - \tau)^{-\rho} (1 + \tau)^{-\rho(\lambda + 1)} d\tau = \int_0^{t/2} (1 + \tau)^{-\rho} (1 + t - \tau)^{-\rho(\lambda + 1)} d\tau
$$

$$
\leq (1 + t)^{-\rho(\lambda + 1)} \int_0^{t/2} (1 + \tau)^{-\rho} d\tau \leq (1 + t)^{-\rho}. \quad (4.8)
$$

Therefore, from (4.6)-(4.8) we obtain

$$
\sup_{t > 0} (1 + |t|)^\rho \|\Gamma u\|_{M_{p',q}^s} \leq \|u_0\|_{M_{p',q}^{s-\sigma \theta}} + \left(\sup_{t > 0} (1 + |t|)^\rho \|u(t)\|_{M_{p',q}^s}\right)^{\lambda + 1}. \quad (4.9)
$$

From (4.9), if $\|u_0\|_{M_{p',q}^{s-\sigma \theta}} \leq \epsilon$ and $u \in B_{2\epsilon}$, we conclude that $\Gamma : B_{2\epsilon} \to B_{2\epsilon}$. Also, following the proof of (4.6) we get that $\Gamma : B_{2\epsilon} \to B_{2\epsilon}$ is a contraction on $B_{2\epsilon}$, which implies the existence of a unique fixed point, as desired. The proof of Theorem 1.3 follows analogously to the proof of Theorem 1.5 by using Lemma 2.11 in place of Lemma 2.10.

\[\square\]

4.4 Proof of Theorems 1.6 and 1.7

We first prove Theorem 1.7. The proof of Theorem 1.7 is also based on a fixed point argument. Let us consider the closed ball

$$
B_{2\epsilon} = \left\{ u \in C([0,T]; M_{p',q}^s) : \sup_{0 < t < T} \|u(t)\|_{M_{p',q}^s} \leq 2\epsilon \right\}, \quad \epsilon > 0,
$$

and define the map $\Gamma$ on the metric space $B_{2\epsilon}$

$$
(\Gamma u)(x, t) = S(t)u_0(x) - \frac{i}{\lambda + 1} \int_0^t S(t-\tau)\varphi(D)[u^{\lambda+1}(x, \tau)]d\tau.
$$
From Lemma [2.4] with θ = 0, and Lemma [2.6] the embedding $M^s_{\frac{3}{2}+T} \subset M^s_{p,q}$ and Lemma [2.10] we get

$$\|\Gamma u\|_{M^s_{p,q}} \leq (1 + |t|)^{2(\frac{3}{2} - \frac{1}{p})\beta_e} \|u_0\|_{M^s_{p,q}} + \int_0^t \|S(t-\tau)\varphi(D)u^{\lambda+1}(\tau)\|_{M^s_{p,q}} d\tau$$

$$\leq (1 + |t|)^{2(\frac{3}{2} - \frac{1}{p})\beta_e} \|u_0\|_{M^s_{p,q}} + \int_0^t (t-\tau)^{2(\frac{3}{2} - \frac{1}{p})}\|u^{\lambda+1}(\tau)\|_{M^s_{p,q}} d\tau$$

$$\leq (1 + |t|)^{2(\frac{3}{2} - \frac{1}{p})\beta_e} \|u_0\|_{M^s_{p,q}} + \int_0^t (t-\tau)^{2(\frac{3}{2} - \frac{1}{p})}\|u(\tau)\|_{M^s_{p,q}}^{\lambda+1} d\tau. \quad (4.10)$$

Taking θ = 0 in (4.10) we get

$$\|\Gamma u\|_{M^s_{p,q}} \leq C\|u_0\|_{M^s_{p,q}} + C\|T\|^{2(\frac{3}{2} - \frac{1}{p})}T\|u\|_{M^s_{p,q}}^{\lambda+1}. \quad (4.11)$$

Let $\epsilon = C\|u_0\|_{M^s_{p,q}}$ and consider $T > 0$ such that $C\|T\|^{2(\frac{3}{2} - \frac{1}{p})}T^{\lambda+1}\lambda^{\epsilon} < 1$. Then, if we assume that

$$\sup_{0 < t < T} \|u(t)\|_{M^s_{p,q}} \leq 2\epsilon,$$

from (4.11) we get

$$\|\Gamma u\|_{M^s_{p,q}} \leq \epsilon + C\|T\|^{2(\frac{3}{2} - \frac{1}{p})}T(2\epsilon)^{\lambda+1} \leq 2\epsilon. \quad (4.12)$$

Therefore, $\Gamma : B_{2\epsilon} \rightarrow B_{2\epsilon}$. In an analogous way we get

$$\|\Gamma u - \Gamma v\|_{M^s_{p,q}} \leq C \int_0^t (t-\tau)^{2(\frac{3}{2} - \frac{1}{p})}\|u^{\lambda+1}(\tau) - v^{\lambda+1}(\tau)\|_{M^s_{p,q}} d\tau$$

$$\leq C \sup_{0 < t < T} \|u - v\|_{M^s_{p,q}} \|T\|^{2(\frac{3}{2} - \frac{1}{p})}T^{\lambda+1}\lambda^{\epsilon}, \quad (4.13)$$

which implies that $\Gamma$ is a contraction on $B_{2\epsilon}$, and thus, the integral equation (1.3) has a unique local solution $u \in B_{2\epsilon}$. The proof of the time-continuity follows in the same way to the one of Theorem [1.3] and therefore we omit it. The proof of Theorem [1.6] follows analogously to the proof of Theorem [1.7] by using Lemma [2.11] in place of Lemma [2.10].

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