Hölder-Zygmund Estimates for Degenerate Parabolic Systems

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Abstract
We consider energy solutions of the inhomogeneous parabolic $p$-Laplacien system
\[ \partial_t u - \text{div}(|\nabla u|^{p-2}\nabla u) = -\text{div}g. \]
We show in the case $p \geq 2$ that if the right hand side $g$ is locally in $L^\infty(BMO)$, then $u$ is locally in $L^\infty(C^1)$, where $C^1$ is the 1-Hölder–Zygmund space. This is the borderline case of the Calderón-Zygmund theory. We provide local quantitative estimates. We also show that finer properties of $g$ are conserved by $\nabla u$, e.g. Hölder continuity. Moreover, we prove a new decay for gradients of $p$-caloric solutions for all $\frac{2n}{n+2} < p < \infty$.

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1. Introduction
We study local behavior of solutions $u : Q_T \rightarrow \mathbb{R}^N$ to the inhomogeneous parabolic $p$-Laplace system.
\[ \partial_t u - \Delta_p u = \partial_t u - \text{div}(|\nabla u|^{p-2}\nabla u) = -\text{div}g. \] (1.1)
If $g \in L^{p'}(Q_T)$ this problem is well-posed and local solutions exist; here $Q_T$ is a space time cylinder. Solutions with this type of term on the right hand side are called energy solutions. It is the aim of the non-linear Calderón-Zygmund theory to transfer information from $g$ to $\nabla u$, the gradient of the solution. The theory started with the important paper of Iwaniec [9]. In this article the elliptic $p$-Laplace is considered. It states that if $g \in L^{p'}$ for $1 \leq q < \infty$, then $\nabla u \in L^{pq}$. In [1] the same was proved for the parabolic $p$-Laplace including local estimates. On the other hand Misawa [14] proved that if $g$ is Hölder continuous, then $\nabla u$ is Hölder continuous for conveniently small Hölder exponents. Later this result was refined and extended by Kuusi and Mingione [11] (see also [13]). It is the concern of this article to close the
gap between higher integrability and H"older continuity, especially the limit case $q = \infty$. Even in the linear elliptic setting (i.e. Poisson’s equation) we know that $g \in L^\infty$ does not imply $\nabla u \in L^\infty$. As in this case where $f \mapsto \nabla u$ is a singular integral operator, the right limit space is the space of bounded mean oscillation (BMO). In the case of the (non-linear) elliptic $p$-Laplacian the right limit space is the same. Indeed, it was shown in \cite{4} and \cite{6} that $g \in \text{BMO}$ implies $|\nabla u|^{p-2} \nabla u \in \text{BMO}$ (locally). The task to find a satisfactory limit space in the parabolic setting turns out to be difficult. We will introduce this matter by looking at the inhomogeneous heat equation.

For the linear theory we have the natural space of parabolic bounded mean oscillation. We say that $f \in \text{BMO}_{\text{par}}(\Omega)$, if $f \in L^1(\Omega)$ and

$$
\|f\|_{\text{BMO}_{\text{par}}(\Omega)} := \sup_{Q_{\ell^2}, \ell^2 \subseteq \Omega} \int_{Q_{\ell^2}} |f - \langle f \rangle_{Q_{\ell^2}}| \, dz < \infty.
$$

If $p = 2$, then we find that $g \in \text{BMO}_{\text{par}}(Q_T)$ implies $\nabla u \in \text{BMO}_{\text{par}}(Q_T)$.

The non-linear version of this result is the boundedness over mean oscillation of the so called natural scaled cylinders: $Q_{\lambda^2} \subseteq Q_T$ defines

$$
\lambda^p \geq \int_{Q_{\lambda^2}} |\nabla u|^p \, dz. \quad (1.2)
$$

We carefully construct cubes of the above type and are able to bound the mean oscillations of $\nabla u$ over these natural scaled cylinders for $p \geq 2$: see Proposition \ref{prop:oscillation-estimate}. However, these oscillation estimates are not very satisfactory. They depend very strongly on the solution itself. We will overcome this by proving some Bochner estimates. To motivate this result, we want to mention a result on which we worked simultaneously to this paper. There we prove $|g|^p \in L^\infty(I, L^q(B))$ implies $|\nabla u|^p \in L^\infty(I, L^q(B))$ (locally). If one let $q \to \infty$ on this quantity we realize that the right borderline space should be a Bochner space of type $L^\infty(I, X)$. The first guess is of course $X = \text{BMO}(B)$. It turns out that this space is too small. Instead we obtained the following main theorem.

**Theorem 1.1.** Let $u$ be a solution on $I \times B$, for $p \geq 2$. If $g \in L^\infty(I, \text{BMO}(B))$, then $u \in L^\infty_{\text{loc}}(I, C^1_{\text{loc}}(B))$. Moreover, for every parabolic cylinder $Q_{2r} \subseteq I \times B$

$$
\|u\|_{L^\infty(I \times B, C^1(B))} \leq c \|g\|_{L^\infty(I, \text{BMO}(B))}^{\frac{1}{p-1}} + c \|\nabla u\|_{L^p(Q_{2r})} + c,
$$

where the constant $c$ only depends on $n, N, p$.

Here $C^1$ is the $1$-H"older-Zygmund space (see \cite{17} and Section \ref{sec:definition} for the exact definition). It is a known substitute for $C^1$ in the setting of PDE’s.
To fortify this we mention the following order of spaces on a bounded set $B \subset \mathbb{R}^n$

$$C^1(B) \subset W^{1,\text{BMO}}(B) \subset C^1(B) \subset \bigcap_{1 \leq q < \infty} W^{1,q}(B).$$

All estimates can be found in Triebel’s book \cite{16}. The difference between these spaces and details will be discussed in Section 2 and interpolation estimates, that follow from our estimates can be found in Remark 4.9.

Theorem 1.1 is the limit case which has not been proven before. To the authors knowledge these estimates are new even for the linear case $p = 2$. Our estimates are general enough so that we can go beyond. Indeed, all our estimates can be stated in the form of weighted $\text{BMO}_\omega$ (see Section 2 for details). These imply, for example, that Hölder continuity can be transferred from $g$ to $\nabla u$ (see Proposition 4.10). This was already proven for all $\frac{2n}{n+2} \leq p$ in \cite{14} and more recently in \cite{11} and \cite{15}. However, for the model case \cite{11} and $p \geq 2$ considered here, all such estimates are regained by our technique. Moreover, we can weaken the condition on $g$. Indeed, if $g \in L^\infty(I, C^{\gamma(p-1)}(B))$, this already implies that $\nabla u \in C^\gamma_{\text{par}}(I \times B)$ locally for small $\gamma$; see Proposition 4.10 at the end of the paper.

The sub-quadratic case requires more difficult analysis. This can be seen in the elliptic case, where the sub-quadratic case was much more problematic to treat (see \cite{6} for details on that matter). Also in the parabolic case it is not straightforward extension, but needs other sophisticated tools. We hope to present these in a future work. Some advances for the $\frac{2n}{n+2} < p < 2$ are achieved in this paper. The first important step to gain BMO estimates is a decay estimate for homogeneous solutions (called $p$-caloric). In Theorem 3.2 we prove a decay in the spirit of Giaquinta and Modica \cite{8} for $p$-caloric solutions. This decay is a distinctively stronger estimate on the Hölder behavior for the gradients of $p$-caloric solutions than known before. It tightens the famous result of DiBenedetto and Friedman \cite{3} and is therefore of independent interest.

Let us mention some results if the right hand side of (1.1) can be characterized by Radon measures. In case of systems little is known. In the case where $u$ is scalar valued, Kuusi and Mingione provided pointwise estimates, which allow a direct control of $\nabla u$ by the right hand side, such that many regularity properties can be carried over. See \cite{12}, \cite{13}.

Finally we want to give another motivation. In \cite{7} it was possible to extend the techniques of \cite{6} to stationary power law fluids. We hope to gain some generalizations of the estimates given in this article to instationary power law fluids in the future.

The structure of the paper is as follows: first we prove the decay for $p$-caloric solutions (for all $\frac{2n}{n+2} < p < \infty$). This is done in Section 3. In Section 4.3 we derive a comparison estimate on so-called intrinsic cylinders (see Lemma 4.5). This leads to the boundedness of the intrinsic mean oscil-
lations, which implies the Hölder-Zygmund estimate.

2. Preliminaries

Through the paper we will denote by \( I \) a (time) interval and \( B \) to be a ball (in space). By \( I_r, B_r \) we mean a time interval or ball in space with radius \( r \). A time space cylinder with “center point” \((t, x)\) \( Q_{s,r}(t, x) := (t-s) \times B_r(x) \) and its parabolic boundary as \( \partial_{par}Q_{s,r}(t, x) := [t, t-s] \times \partial B_r(x) \cup (t-s) \times B_r(x) \). As the “center point” is mostly of no importance, it will be often omitted. We will use the notation \( \langle f \rangle_E := \frac{1}{|E|} \int_E f \, dx \).

We have to introduce a few function spaces. Let \( \omega : \mathbb{R}_+ \to \mathbb{R}_+ \) almost increasing. This means, that there is a \( c > 0 \) fixed, such that \( \omega(r) \leq c \omega(\rho) \) for all \( r < \rho \). We say that \( f \in \text{BMO}^\omega_{\text{par}}(Q) \) the weighted space of mean oscillations, if

\[
\|f\|_{\text{BMO}^\omega_{\text{par}}(Q)} = \sup_{Q_{s,r} \subset Q} \frac{1}{\omega(r)} \int_{Q_{s,r}} |f - \langle f \rangle_{Q_{s,r}}| \, dx \, dt < \infty.
\]

For \( \omega(r) = 1 \), we get the space of parabolic bounded mean oscillation: \( \text{BMO}^\omega_{\text{par}}(Q) \). By the Campanato characterization, of Hölder spaces we find for \( \beta \in (0, 1) \) and \( \omega(r) = r^\beta \) the space of Hölder continuous function in the parabolic metric.

We will now look at the Bochner spaces of refined BMO. Let \( \omega : \mathbb{R}^2_+ \to \mathbb{R}_+ \). We say that \( f \in \text{BMO}_\omega(I \times B) \) if

\[
\|f\|_{\text{BMO}_\omega(I \times B)} := \sup_{I_s \times B_r \subset Q} \frac{1}{\omega(s, r)} \int_{I_s \times B_r} |f - \langle f(t) \rangle_{B_r}| \, dx \, dt < \infty.
\]

if \( \omega \equiv 1 \), then we have the space \( L^\infty(I, \text{BMO}(B)) \). More general, if \( \omega \) only depends on \( r \), then we have the \( L^\infty(I, \text{BMO}_\omega(B)) \) spaces.

Through the paper we will need the following typical estimate for mean oscillations, which we will refer to as best constant property. For \( f \in L^p(Q) \), \( p \in [1, \infty) \) we have that

\[
\int_Q |f - \langle f \rangle_Q|^p \, dx \leq 2^p \int |f - c|^p \, dx \text{ for all } c \in \mathbb{R}.
\]

We will also need the famous John-Nirenberg estimate \[10\]

\[
\int_B |f - \langle f \rangle_B|^q \, dx \leq c_q \|f\|_{\text{BMO}(B)}^q
\]
for \( 1 \leq q < \infty \). Let us introduce the Hölder–Zygmund spaces. We say that \( f \in C^r(\Omega) \) if
\[
\|f\|_{C^r(\Omega)} := \sup_{x \in \Omega} \sup_{|x+2h| \leq \Omega} \frac{|f(x+2h) - 2f(x+h) + f(x)|}{|h|^r} + \|f\|_{\infty} < \infty.
\]
This is a Banach space. By [16, Sec. 1.2.2] we find that \( C^r(\Omega) = C^r(\Omega) \) if \( \gamma \not\in \mathbb{N} \) but \( C^1(\Omega) \subseteq C^1(\Omega) \).

We find in [16, Section 1.7.2], that \( C^1 \) has a Campanato space like interpretation. Analogous to the spaces of BMO\( _\omega \) we define the space of weighted bounded linear oscillation BLO\( _\omega \) by the semi-norm
\[
\|f\|_{\text{BLO}^q(\Omega)} := \sup_{B_r \subseteq \Omega} \inf_{\ell \in P^1(B_r)} \left( \int_{B_r} \frac{f - \ell}{r} \frac{q}{|x-y|^q} dx \right)^{\frac{1}{q}}, 1 < q < \infty.
\]
Here \( P^1 \) is the set of all polynomials with degree 1. For \( q = 2 \) we define \( \ell_r(f) \) as the best linear approximation of \( f \) on \( B_r \) in with respect to \( \|\cdot\|_2 \), which is well defined for all \( r > 0 \) and \( f \in L^2_{B_r} \). We find by [16, Section: 1.7.2] that \( \text{BLO}(\Omega) := \text{BLO}^1(\Omega) \equiv \text{BLO}^q(\Omega) \equiv C^1(\Omega) \) for all \( 1 \leq q < \infty \); more general, for \( \gamma \in (0,1) \) and \( \omega(r) = r^\gamma \) the space \( \text{BLO}^q(\Omega) = C^{1+\gamma}(\Omega) \) for \( 1 \leq q < \infty \). We define that \( f \) is in the space of vanishing linear oscillations VLO if \( \|f\|_{\text{BLO}(B_r(x))} \to 0 \) for \( r \to 0 \) uniform in \( x \). Please note
\[
\frac{1}{\omega(r)} \|f\|_{\text{BMO}^q(B_r)} \leq c \|f\|_{\text{BLO}^q(B_r)} \text{ or } \frac{1}{\omega(r)} \|f\|_{\text{BLO}^q(B_r)} \leq \|f\|_{\text{BLO}^q(B_r)},
\]
because \( \omega \) is almost increasing. We will use this in this work without further reference.

We denote by
\[
\text{osc}_E(f) := \sup_{x,y \in E} |f(x) - f(y)|
\]
the oscillations of \( f \) on \( E \).

We define the following natural quantity: for \( Q \in \mathbb{R}^{N \times n} \) we have \( V(Q) := |Q| = \sqrt{Q^T Q} \). If \( \nabla u \in L^p \), then \( V(\nabla u) \in L^2 \), therefore \( V(\nabla u) \) can be seen as a linear substitute. First remark that we will use without further mentioning that for any set \( E \subset \mathbb{R}^n \) and \( f, h \in L^p(E, \mathbb{R}^{N \times n}) \)
\[
\langle |f|^p \rangle_E \leq c \int_E \langle V(f) - V(h) \rangle^2 dx + \langle |h|^p \rangle_E.
\]
We will need [3, Lemma 3]. It quantifies the ellipticity of (1.1) in terms of \( V \). In our case it states for \( P, Q \in \mathbb{R}^{N \times n} \) and \( 1 < p < \infty \)
\[
\begin{align*}
|Q|^p - |P|^p &\sim |Q - P|^p\langle (Q - P) \rangle, \\
|Q|^p - |P|^p &\sim |Q - P|^p\langle |Q - P| \rangle, \\
|Q| &\sim |P|\langle |Q^p - P^p| \rangle.
\end{align*}
\] (2.1)
This implies for \( p \geq 2 \)
\[
|P - Q|^p \leq c|V(Q) - V(P)|^2. \tag{2.2}
\]
We also need some estimate which makes use of so called shifted N–functions \([5, \text{Lemma 32}]\) and \([6, (2.5)]\) we gain for \( P, Q, G_1, G_0 \in \mathbb{R}^{N \times n} \) and \( \delta > 0 \)
\[
|G_1 - G_0||P - Q|
\leq c(|Q| + |G_1 - G_0|^{p-2}|G_1 - G_0|^2 + \delta|V(Q) - V(P)|^2). \tag{2.3}
\]
Here \( c \) only depends on \( p, n, N \) and \( \delta \). We use \( p' := \frac{p}{p-1} \) as the dual exponent to \( p \).

Finally we introduce the \( \lambda \)-scaled cylinders \( Q_{\lambda}^r(t, x) := (t, t - \lambda r^2) \times B_r(x) \), where \( \lambda, r \) is the exponent of (1.1). For \( \theta \in \mathbb{R}^+ \) we define \( \theta Q_{\lambda}^r(t, x) := (t, t - \lambda^{2-p}r^2) \times B_r(x) \). If \( \lambda = 1 \), then we have a standard parabolic cylinder and we write \( Q_r^1(t, x) =: Q_r(t, x) \). As solutions are translation invariant and our estimates are local, the center \( (t, x) \) of the cube is mostly of no importance and will often be omitted, to shorten notation. Finally, we call a cylinder \( K \)-intrinsic with respect to \( f \), when
\[
\frac{\lambda}{K} \leq \langle |Df|^p \rangle_{Q_{\lambda}^r} \leq K \lambda \text{ and } K \text{-sub-intrinsic w.r.t } f, \text{ when (2.4)}
\]
\[
\langle |Df|^p \rangle_{Q_{\lambda}^r} \leq K \lambda.
\]
We say (sub-)intrinsic if \( K = 1 \).

3. Decay for \( p \)-Caloric Functions

In this section we consider \( h : Q_T \rightarrow \mathbb{R}^N \) to be locally \( p \)-caloric on a space time domain \( Q_T \). I.e. \( h \) is a solution to the following system
\[
\partial_t h - \text{div}(|\nabla h|^{p-2} \nabla h) = 0
\]
locally in \( Q_T \). In this section we provide a decay for the natural quantity \( V(\nabla h) = |\nabla h|^{\frac{p}{p-2}} \nabla h \). It is an extension to the known result of DiBenedetto and Friedmann \([3]\) providing finer estimates for the continuity behavior. Our results are very much in the spirit of Giaquinta and Modica \([8, \text{Proposition 3.1-3.3}]\). We will prove a parabolic version of their decay for the \( p \)-caloric setting.

The first theorem we will need is the well-known weak Harnack inequality first proved by DiBenedetto and Friedmann \([3, \text{Proposition 3.1-3.3}]\). We will use the K-sub-intrinsic version of \([1, \text{Lemma 1+2}]\).

**Theorem 3.1.** Let \( p > \frac{2n}{n+2} \) and \( h \) be \( p \)-caloric on \( Q_T \). If for \( Q_{\lambda}^r \subset Q_T \)
\[
\int_{Q_{\lambda}^r} |\nabla h|^p \, dz \leq K \lambda^p,
\]

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then
\[ \sup_{Q^\lambda_r} |\nabla h| \leq c\lambda. \]

The constant only depends on $K, p$ and the dimensions.

**Proof.** If $p \geq 2$ it is the same statement as in [1, Lemma 1]. But also in the case of $\frac{2n}{n+2} < p < 2$ the statement holds. In [1, Lemma 2] it is proved that if
\[ \int_{Q^\lambda_{r^2s^{2}}} |\nabla h|^p \, dz \leq K\lambda^p, \]
it follows
\[ \sup_{Q^\lambda_{r^2s^{2}}} |\nabla h| \leq c\lambda. \]
Now we define $r = \lambda^{\frac{n-2}{2}} s$ which implies, that $s^2 = \lambda^{2-p} r^2$. Therefore the estimate holds for all $\frac{2n}{n+2} < p < \infty$. □

The main theorem of this section is the following.

**Theorem 3.2.** Let $\partial_t h - \text{div}(|\nabla h|^{p-2} \nabla h) = 0$ on $Q^\lambda_\rho$, such that
\[ \frac{\lambda}{K} \leq \left( \int_{Q^\lambda_\rho} |\nabla h|^p \, dz \right)^{\frac{1}{p}} \leq K\lambda, \]
then there exists a $c > 0$ and $\alpha, \tau \in (0, 1)$ depending only on $n, N, p, K$, such that for every $\theta \in (0, \tau]$
\[ \sup_{z, w \in \theta Q^\lambda_\rho} |V(\nabla h(w)) - V(\nabla h(z))|^2 \leq c\theta^\alpha \int_{Q^\lambda_\rho} |V(\nabla h) - \langle V(\nabla h) \rangle_{Q^\lambda_\rho}|^2 \, dz. \]

We start with a $K$-intrinsic cube $Q^\lambda_\rho \subset Q_T$ fixed. To be able to state the result neatly we define for $r < \rho$
\begin{align*}
M(r) &:= \sup_{Q^\lambda_r} |Dh| \quad (3.1) \\
\Phi(r) &:= \left( \int_{Q^\lambda_r} \left| V(Dh) - \langle V(\nabla h) \rangle_{Q^\lambda_r} \right|^2 \, dz \right)^{\frac{1}{2}}. \quad (3.2)
\end{align*}

The classic elliptic result of Giaquinta and Modica [8] was that there is a uniform constant $c$ and an $\alpha \in (0, 1)$, such that $\Phi(\theta \rho) \leq c\theta^\alpha \Phi(\rho)$. It is then a standard procedure to gain the estimate of the oscillations. It actually follows by Lemma [Appendix A.2] which can be found in the appendix.
Theorem 3.3. Let \( h \) be \( p \)-caloric on \( Q^\lambda_r \), such that
\[
\left( \int_{Q^\lambda_r} |\nabla h|^p \, dz \right)^{\frac{1}{p}} \leq K\lambda,
\]
then there exists an \( \alpha, c > 0 \) depending only on \( n, N, p, K \), such that for every \( \theta \in (0, \frac{1}{4}] \)
\[
\sup_{z,w \in \theta Q^\lambda_r} |V(\nabla h(w)) - V(\nabla h(z))|^2 \leq c\theta^\alpha \lambda^p.
\]

The theorem is a consequence of [2, IX, Prop 1.1,1.2], resp. [11, Prop. 3.1-3.3]. We combine these statements in the following proposition, as we will use them.

Proposition 3.4. Let \( h \) be \( p \)-caloric. Let
\[
M(\rho) \leq K\lambda.
\]
Then one of the two alternatives hold:

Case 1, non degenerate: There exist \( \beta, \delta_0 \in (0, 1) \) depending only on \( n, N, p, K \) such that
\[
\frac{\lambda}{4} \leq \inf_{2\delta_0 Q^\lambda_p} |\nabla h| \leq \sup_{2\delta_0 Q^\lambda_p} |\nabla h| \leq K\lambda
\]
and \( \text{osc}_{Q^\lambda_p} (V(\nabla h))^{\frac{1}{2}} \leq c\delta^\beta \Phi(\rho) \) for all \( \delta \in (0, \delta_0) \).

Case 2, degenerate: There exist \( \sigma, \eta \in (0, 1) \) depending only on \( n, N, p, K \) such that
\[
M(\sigma \rho) \leq \eta K\lambda.
\]

Proof. We only have to show that in Case 1, \( \text{osc}_{Q^\lambda_p} (V(\nabla h))^{\frac{1}{2}} \leq c\delta^\beta \Phi(\rho) \) for \( \delta \in (0, \delta_0) \). Anything else can be found in [11, Proposition 3.1-3.3].

By [11, Proposition 3.1] we know, that if Case 1 does not hold, there exists \( \delta_1 \in (0, 1) \) such that for every sub cube \( Q^\lambda_r(z) \subset \delta_1 Q^\lambda_p \) we have
\[
\frac{\lambda}{4} \leq \inf_{Q^\lambda_r(z)} |\nabla h| \leq \sup_{Q^\lambda_r(z)} |\nabla h| \leq K\lambda.
\]

Therefore we have by [11, Proposition 3.2] for all these sub cubes
\[
\int_{\theta Q^\lambda_r(z)} |V(\nabla h) - \langle V(\nabla h) \rangle_{\theta Q^\lambda_r(z)}|^2 \leq c\theta^{2\beta} \int_{Q^\lambda_r(z)} |V(\nabla h) - \langle V(\nabla h) \rangle_{Q^\lambda_r(z)}|^2.
\]
this implies the result by Lemma [Appendix A.2] with \( \delta_0 = \frac{\delta_1}{2} \). \( \square \)
Proof of Theorem 3.2. Before we can prove the decay we have to do some preliminary work. If for $Q^\lambda_\rho$ Case 1 of Proposition 3.4 holds, we have the desired decay.

If Case 2 holds, we shall iterate. In this case the degenerate alternative of Proposition 3.4 holds for $Q^\lambda_\rho$. We will now construct another smaller cube on which we can apply Proposition 3.4 again.

We find for $\lambda_1 = \eta \lambda$,

$$Q^{\lambda_1}_{\sigma \eta^{\frac{2-p}{p}} \rho} \subset Q^{\lambda_1}_{\sigma \rho} \subset Q^{\lambda_1}_{\sigma \eta^{\frac{p-2}{p}} \rho} \subset Q^{\lambda_1}_{\sigma \rho} \text{ if } p < 2,$$

and

$$Q^{\lambda_1}_{\sigma \rho} \subset Q^{\lambda_1}_{\sigma \eta^{\frac{p-2}{p}} \rho} \subset Q^{\lambda_1}_{\sigma \rho} \text{ if } p \geq 2.$$

We define

$$\rho_1 = a \rho \text{ where } a = \sigma \text{ for } p < 2 \text{ and } a = \sigma \eta^{\frac{p-2}{p}} \text{ for } p \geq 2$$

and $r_1 = b \rho$ with $b = \eta^{\frac{2-p}{p}} \sigma$ if $p < 2$ and $b = \sigma$ if $p \geq 2$.

We find

$$M(r_1) \leq \sup_{Q^{\lambda_1}_{\sigma \rho}} |\nabla h| \leq M(\sigma \rho) \leq K \eta \lambda = K \lambda_1.$$

Thus $Q^{\lambda_1}_{\rho \rho}$ satisfies the assumption of Proposition 3.4 If Case 2 holds for this cube we can iterate further with

$$\lambda_i = \eta^i \lambda; \rho_i = a \rho_{i-1} \text{ and } r_i = b r_{i-1}, \quad (3.3)$$

and $a, b$ defined above. If Case 2 holds also for $Q^{\lambda_j}_{\rho_j}$ and $1 \leq j \leq i - 1$, then we find

$$Q^{\lambda_i}_{\rho_i} \subset Q^{\lambda_{i-1}}_{\rho_{i-1}} \text{ and } \sup_{Q^{\lambda_i}_{\rho_i}} |\nabla h| \leq \sup_{Q^{\lambda_{i-1}}_{\rho_{i-1}}} |\nabla h| \leq K \eta \lambda_{i-1} = K \eta^i \lambda.$$

Let us fix $m \in \mathbb{N}$, such that $\eta^m K^2 \leq \frac{1}{2}$. This implies that if the degenerate alternative holds for all $i \leq m$, then

$$\sup_{Q^{\lambda_m}_{\rho_m}} |\nabla h| \leq \sup_{Q^{\lambda_m}_{\rho_m}} |\nabla h| \leq K \eta^m \lambda \leq \frac{1}{2} \langle |\nabla h|^p \rangle_{Q^\lambda_\rho} \quad (3.4)$$

by the assumption that $Q^\lambda_{\rho}$ is intrinsic.

Now we are able to prove the decay. Let us first assume, that for one $i \in \{0, \ldots, m\}$ the non-degenerate Case 1 of Proposition 3.4 holds. This implies for $\delta \in (0, \tau)$, where $\tau = \frac{\delta_0}{\rho^m}$, that

$$\text{osc}_{\delta \rho^m} (V(\nabla h))^{\frac{1}{2}} \leq \text{osc}_{\delta \rho^m} (V(\nabla h))^{\frac{1}{2}} \leq c \delta^\beta \Phi^\lambda(\rho_i) \leq c \delta^\beta \Phi^\lambda(\rho),$$

as

$$Q^{\lambda_i}_{\rho_i} \subset Q^\lambda_{\rho} \text{ and } \frac{|Q^{\lambda_i}_{\rho_i}|}{|Q^\lambda_{\rho}|} \leq \frac{|Q^\lambda_{\rho}|}{|Q^\lambda_{\rho}|} \leq c \text{ depending only on } n, N, p, K.$$
This leaves the case, when for all \( i \in \{0, \ldots, m \} \) the degenerate alternative (Case 2) holds. In this case we know by (3.4)

\[
\sup_{Q_{\lambda m}^m} |\nabla h| \leq K \eta^m \lambda \leq \frac{1}{2} (|\nabla h|^p)_{Q_{\rho}^\lambda}^{\frac{1}{p}}.
\]

This implies that

\[
|\langle V(\nabla h) \rangle_{Q_{\rho}^\lambda}| \leq \frac{1}{2^p} (|V(\nabla h)|^2)_{Q_{\rho}^\lambda}^{\frac{1}{2}}.
\]

Therefore we gain by Lemma [Appendix A.1] 4.1. Finding a scaled sequence of cubes

\[
\lambda^p \leq K^p (|\nabla h|^p)_{Q_{\rho}^\lambda} \leq c \int_{Q_{\rho}^\lambda} |V(\nabla h) - \langle V(\nabla h) \rangle_{Q_{\rho}^\lambda}|^2 \, dz,
\]

again, as

\[
Q_{\rho_i}^\lambda \subset Q_{\rho}^\lambda \quad \text{and} \quad \frac{|Q_{\rho}^\lambda|}{|Q_{\rho_i}^\lambda|} \leq \frac{|Q_{\rho_i}^\lambda|}{|Q_{\rho_i}^\lambda|} \leq c \text{ depending only on } n, N, p, K.
\]

Finally, the last estimate combined with Theorem [3.3] implies the decay also in this case.

\[
\square
\]

4. A BMO result for \( p \geq 2 \)

Theorem [1.1] is a consequence of a more general result. From this we will conclude also other Campano like estimates.

Before proving the main result we will have to prove some intermediate results. The key ingredient is to carefully choose a family of intrinsic cylinders.

4.1. Finding a scaled sequence of cubes

To treat the scaling behavior in a way to gain a BMO result for (1.1) is quite delicate. Our estimates are based on comparison principles: Whenever one knows that \( \|g\|_{L^\infty(I, BMO(B_r))} \) is small, then \( u \) is “close” to a \( p \)-caloric comparison solution.

In the following we will construct sub-intrinsic cubes with properties convenient for our needs.

**Lemma 4.1.** Let \( p \geq 2 \). Let \( Q_{S,R}(t, x) \subset Q_T \) and \( b \in (0, 2) \). For every \( 0 < r \leq R \) there exists \( s(r) \), \( \lambda_r \) and \( Q_{s(r), r}(t, x) \) with the following properties. Let \( r, \rho \in (0, R] \) and \( r \leq \rho \), then

(a) \( 0 \leq s(r) \leq S \) and \( s(r) = \lambda_r^2 - r^2 \). Especially \( Q_{s(r), r}(t, x) = Q_{\lambda r} \subset Q_T \).
(b) \( s(r) \leq \left( \frac{r}{S} \right)^b s(\rho) \), the function \( s \) is continuous and strictly increasing on \([0, R]\). Especially \( Q_{\lambda r}^\lambda \subset Q_{\rho \lambda r}^\rho \).

(c) \( \int_{Q_{\lambda r}^\lambda} |\nabla u|^p \, dz \leq \lambda^p \), i.e. \( Q_{\lambda r}^\lambda \) is sub-intrinsic.

(d) if \( s(r) < \left( \frac{r}{S} \right)^b s(\rho) \), then there exists \( r_1 \in [r, \rho) \) such that \( Q_{\lambda r_1}^\lambda \) is intrinsic.

(e) if for all \( r \in (r_1, \rho) \), \( Q_{\lambda r}^\lambda \) is strictly sub-intrinsic, then \( \lambda_r \leq \left( \frac{r}{S} \right)^\beta \lambda_\rho \) for all \( r \in [r_1, \rho) \) and \( \beta = \frac{2-p}{p-2} \in (0, \frac{2}{p-2}) \).

(f) for \( \theta \in (0, 1) \), \( \theta^\beta \lambda_r \leq \lambda_{\theta r} \leq c \theta^\frac{2-p}{p-2} \).

(g) for \( \theta \in (0, 1) \), \( |Q_{\lambda r}^\lambda|^{-1} \leq c\theta^{-(n+2)(1+\frac{p-2}{2})}|Q_{\rho \lambda r}^\rho|^{-1} \).

(h) for \( \theta \in (0, 1) \), we find \( Q_{\lambda r}^\lambda \subset \theta Q_{\lambda r}^\lambda \) for \( \sigma = \theta^\frac{2}{p} \).

The constant only depends on the dimensions and \( p \).

Proof. Let \( Q_{S,R}(t, x) \subset Q_T \). In the following we often omit the point \((t, x)\). We start, for every \( \theta \in (0, 1) \),

\[
\tilde{s}(r) = \max \left\{ s \leq S \left| \left( \int_{t-s B_{\rho}(x)} |\nabla u|^p \, dz \right)^{\frac{p-2}{2}} \leq r^{2p} |B_{r}|^{p-2} \right. \right\}. \tag{4.1}
\]

The function \( \tilde{s}(r) \) is well defined and strictly positive for \( r > 0 \). We define \( \tilde{\lambda}_r \) by the equation \( r^2 \tilde{\lambda}_r^{2-p} = \tilde{s}(r) \). We will first show, that \( Q_{\lambda r}^\lambda := Q_{\tilde{s}(r),r}^\lambda \) holds \([c]\) By construction we find, that

\[
\left( \int_{Q_{\lambda r}^\lambda} |\nabla u|^p \, dz \right)^{\frac{p-2}{2}} \tilde{s}(r)^2 \leq r^{2p} |B_{r}|^{p-2}. \tag{4.2}
\]

This implies that

\[
\left( \int_{Q_{\lambda r}^\lambda} |\nabla u|^p \, dz \right)^{\frac{p-2}{2}} \tilde{s}(r)^p \leq r^{2p} = (\tilde{\lambda}_r^{(p-2)} \tilde{s}(r))^p
\]

which implies

\[
\int_{Q_{\lambda r}^\lambda} |\nabla u|^p \, dz \leq \tilde{\lambda}_r^{p}, \text{ and if } \int_{Q_{\tilde{s}(r),r}} |\nabla u|^p \, dz \leq \lambda_r^{p}, \text{ then } \tilde{s}(r) = S. \tag{4.3}
\]
Next we will show, that \( \tilde{s}(r) \) is continuous for \( r \in (0, R] \). For \( \varepsilon \leq \tilde{s}(r) \leq S - \varepsilon \) and \( r_0 > 0 \), we find that \( \left( \int_{t - \tilde{s}(r)}^{t} \int_{B_r} |\nabla u|^p \, dz \right)^{p-2} s^2 \) is growing of order 2. Because the growth rate is explicitly bounded by

\[
\frac{|B_r|^{p-2} R^{2p}}{\varepsilon^2} \geq \left( \int_{t - \tilde{s}(r)}^{t} \int_{B_{r_1}(x)} |\nabla u|^p \, dz \right)^{p-2} \geq \frac{r_0^{2p} |B_{r_0}|^{p-2}}{S^2},
\]

for \( r \in [r_0, R] \). This implies that there exists a \( \delta_{\varepsilon, r_0} > 0 \), such that for all \( r, r_1 \in [r_0, R] \) with \( |r - r_1| < \delta_{\varepsilon, r_0} \),

\[
\left( \int_{t - \tilde{s}(r)}^{t} \int_{B_{r_1}(x)} |\nabla u|^p \, dz \right)^{p-2} \tilde{s}(r) - \varepsilon^2 < r_1^{2p} |B_{r_1}|^{p-2}
\]

as \( \left( \int_{t - \tilde{s}(r)}^{t} \int_{B_{r_1}(x)} |\nabla u|^p \, dz \right)^{p-2} \) and \( r^{2p} |B_r|^{p-2} \) are both uniformly continuous in \( r \). Now we gain immediately

\[
\left( \int_{t - \tilde{s}(r) + \varepsilon}^{t} \int_{B_{r_1}(x)} |\nabla u|^p \, dz \right)^{p-2} \tilde{s}(r) - \varepsilon^2 < r_1^{2p} |B_{r_1}|^{p-2}
\]

which implies that \( |\tilde{s}(r) - \tilde{s}(r_1)| < 2\varepsilon \).

Let us define \( s_{\varepsilon}(r) = \max \{ \varepsilon, \min \{ \tilde{s}(r), S - \varepsilon \} \} \). By the previous calculations we find that \( s_{\varepsilon} \) is uniformly continuous, especially \( |s_{\varepsilon}(r) - s_{\varepsilon}(r_1)| \leq 2\varepsilon \) for \( r, r_1 \in [r_0, R] \) with \( |r - r_1| < \delta_{\varepsilon, r_0} \). Therefore

\[
|\tilde{s}(r_1) - \tilde{s}(r)| \leq |\tilde{s}(r_1) - s_{\varepsilon}(r_1)| + |s_{\varepsilon}(r_1) - s_{\varepsilon}(r)| + |s_{\varepsilon}(r) - \tilde{s}(r)| \leq 4\varepsilon.
\]

As \( r_0 \) was arbitrary we find that \( \tilde{s}(r) \) is continuous on \( (0, R] \).

Now it might happen, that \( r < \rho \) and \( \tilde{s}(r) > \tilde{s}(r) \). To avoid that we define for \( b \in (0, 2) \)

\[
s(r) = \min_{R \geq a \geq r} \left( \frac{r}{a} \right)^b \tilde{s}(a).
\]

The minimum exists, as \( \left( \frac{r}{a} \right)^b \tilde{s}(a) \) is continuous in \( a \). As for \( \rho \in (r, R] \)

\[
s(r) = \min \left\{ \min_{R \geq a \geq r} \left( \frac{r}{a} \right)^b \tilde{s}(a), \left( \frac{r}{\rho} \right)^b \tilde{s}(\rho) \right\}
\]

(4.4)

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we find that \( s(r) < s(\rho) \). Now we define \( \lambda_r := \left( \frac{r^2}{\pi(r)} \right)^{\frac{1}{4-n}} \geq \tilde{\lambda}_r \) and \( Q_r^{\lambda_r} := Q_{s(r),r} \). By this definition we find \([a]\) and \([b]\) as \( \lim_{r \to 0} s(r) \leq \lim_{r \to 0} \left( \frac{r}{\pi} \right)^b S(R) = 0 \).

We show \([c]\) by \((4.2)\)

\[
\int_{Q_{s(r),r}} |\nabla u|^p \leq \frac{\tilde{s}(r)}{s(r)} \int_{Q_{s(r),r}} |\nabla u|^p = \left( \frac{\lambda_r}{R} \right)^{p-2} \int_{Q_{s(r),r}} |\nabla u|^p \leq \tilde{\lambda}_r^{p-2} \leq \lambda_r^p.
\]

(4.5)

To prove \([d]\) we assume that \( s(r) < \left( \frac{a}{\rho} \right)^b s(\rho) \). Then there exist a \( r_1 \in [r, \rho) \), such that

\[
\left( \frac{r}{r_1} \right)^b \tilde{s}(r_1) = s(r) = \min_{R \geq a \geq r_1} \left( \frac{r}{r_1} \right)^b \tilde{s}(a) = \left( \frac{r}{r_1} \right)^b \min_{R \geq a \geq r_1} \left( \frac{r_1}{a} \right)^b \tilde{s}(a) = \left( \frac{r}{r_1} \right)^b s(r_1).
\]

Now because \( \tilde{s}(r_1) \geq s(r_1) \) we find \( \tilde{s}r_1 = s(r_1) \). Since also \( s(r) < \left( \frac{r_1}{\rho} \right)^b s(\rho) \) we find by \((4.2)\) that \( Q_{s(r),r_1} = Q_r^{\lambda_{r_1}} \) is intrinsic. This implies \([d]\).

To prove \([e]\) we gain by \([d]\) that if \( Q_{a}^{\lambda_a} \) is strictly sub-intrinsic for all \( a \in (r, \rho) \), then \( s(a) = \left( \frac{a}{\rho} \right)^b s(\rho) \) for all \( a \in (r, \rho) \). Now we calculate

\[
\lambda_{a-2} = \frac{a^2}{s(a)} = \frac{a^2}{\left( \frac{a}{\rho} \right)^b s(\rho)} = \left( \frac{a}{\rho} \right)^{2-b} \lambda_{\rho-2}.
\]

this proves \([e]\) with \( \beta = \frac{2-b}{2} \).

To prove \([f]\) we take \( \theta \in (0, 1) \). If \( s(\theta r) = \theta^b s(r) \) we are finished. If \( s(\theta r) < \theta^b s(r) \), we find by \([d]\) that there is a \( \sigma \in [\theta, 1) \) with \( s(\theta r) = \left( \frac{\sigma}{\theta} \right)^b s(\sigma r) \) and \( Q_{\sigma r}^{\lambda_{\sigma r}} \) is intrinsic. This implies using also \([e]\)

\[
\lambda_{\sigma r}^2 = \frac{c}{(\sigma r)^{n+2}} \int_{Q_{s(\sigma r),\sigma r}} |\nabla u|^p \, dz \leq \frac{c s(r)}{r^{2\sigma n+2}} \int_{s(r),r} |\nabla u|^p \, dz \leq \frac{c \lambda_r^2}{\theta^{2 \sigma n+2}}.
\]

By the definition of \( \lambda_r \) we find for \( \beta = \frac{2-b}{\rho^2} \) and the previous that

\[
\lambda_r \leq \theta^{-\beta} \lambda_\theta r \text{ and } \lambda_{\theta r} \leq \frac{c}{\theta^{n+2}} \lambda_r,
\]

which implies \([f]\) and \([g]\). To prove \([h]\) we take \( \theta Q_{s(r),r} = Q_{\theta^2 s(r),\theta r} \) we define \( \sigma < \theta \), such that \( \sigma^b = \theta^2 \). Now we find by \((4.4)\), that \( s(\sigma r) \leq \sigma^b s(r) = \theta^2 s(r) \).

\[
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\]
4.2. Comparison

In this section we will derive a comparison estimate which will allow us to gain BMO estimates. Let \( u \) be a solution to (1.1) on \( I \times B \). As we want to use Theorem 3.2, we will have to start with an intrinsic cylinder. We therefore take any intrinsic cylinder \( Q^\lambda_R(z) \subset I \times B \), i.e.

\[
\int_{Q^\lambda_R(z)} |\nabla u|^p = \lambda^p_0.
\]

In this section we define \( Q^\lambda_r \) as the sub-intrinsic cylinders all sharing the same center, which are constructed by Lemma 4.1. By comparison we mean the local comparison to a \( p \)-caloric function. I.e. for \( r \in (0, R) \) we will compare \( u \) to solutions of

\[
\partial_t h - \text{div}(|\nabla h|^{p-2} \nabla h) = 0 \quad \text{on } Q^\lambda_r \quad h = u \quad \text{on } \partial_{\text{par}} Q^\lambda_r.
\]

(4.6)

Lemma 4.2. Let \( p \geq 2 \), \((t, t - \lambda^2 - p) \times B_r(x) =: Q^\lambda_r \subset I \times B \) and \( g \in L^\infty(I, \text{BMO}(B)) \). For \( h \) the solution of (4.6) and \( u \) the solution of (1.1) we have

\[
\lambda^{p-2} \int_{B_r(x)} \frac{|u - h|^2(t)}{r^2} \, dy + \int_{Q^\lambda_r} |V(\nabla u) - V(\nabla h)|^2 \, dz \leq c \|g\|^{p'}_{L^\infty(I, \text{BMO}(B_r(x)))}. 
\]

Proof. We take \( u - h \) as a test function for both systems (1.1) and (4.6). We take the difference and find

\[
\int_{Q^\lambda_r} \frac{|u - h|^2}{2} \, dz + \int_{Q^\lambda_r} (|\nabla u|^{p-2} \nabla u - |\nabla h|^{p-2} \nabla h) \cdot \nabla (u - h) \, dz \\
= \int_{Q^\lambda_r} g \cdot \nabla (u - h) \, dy \, d\tau + \int_{Q^\lambda_r} (g - \langle g(\tau) \rangle_{B_r}) \cdot \nabla (u - h) \, dy \, d\tau.
\]
We find by (2.1), (2.3) and as $p' \leq 2$

$$\lambda^{p-2} \int_{B_r} \frac{|u - h(t)|^2}{r^2} \, dy + \int_{Q^{2r}_\lambda} |V(\nabla u) - V(\nabla h)|^2 \, dz$$

$$\leq c \int_{Q^{2r}_\lambda} \left( |\nabla u| + |g - \langle g(\tau) \rangle_{B_r(x)}| \right)^{p'-2} |g - \langle g(\tau) \rangle_{B_r(x)}|^2 \, dy \, d\tau$$

$$+ \delta \int_{Q^{2r}_\lambda} |V(\nabla u) - V(\nabla h)|^2 \, dz$$

$$\leq c \int_{t - \lambda^2 - r^2} \int_{B_r(x)} |g - \langle g(\tau) \rangle_{B_r(x)}|^{p'} \, dy \, d\tau + \delta \int_{Q^{2r}_\lambda} |V(\nabla u) - V(\nabla h)|^2 \, dz.$$

We absorb and use John-Nirenberg to find that

$$\int_{B_r(x)} |g - \langle g(\tau) \rangle_{B_r(x)}|^{p'} \, dx \leq c\|g(\tau)\|_{BMO(B_r(x))}^{p'},$$

which leads to the result.

\[ \square \]

**Proposition 4.3.** Let $Q^{\lambda_0}_R$ be intrinsic and $r \in (0, R)$ and $g \in L^\infty(I, BMO(B))$. Let $\beta \leq \frac{\alpha}{1 + \alpha - \delta}$, such that $\beta < \frac{2}{p - 2}$, where $\alpha$ is defined by Theorem 3.3.

Then there exist $K, c > 1$ depending only on $n, N, p, \beta$, such that one of the following two alternatives holds:

**Case 1:** $\lambda^{p}_r \leq K\|g\|_{L^\infty(I, BMO(B_r))}^{p'}$

**Case 2:** For the $p$-caloric comparison function $h$ of \(4.6\) there exist a $\rho \in [r, R]$ such that

$$\text{osc}_{Q^{2r}_\lambda}(V(\nabla h))^2 \leq c\left(\frac{\sigma \tau}{\rho}\right)^{\beta} \int_{Q^{2r}_\lambda} |V(\nabla u) - \langle V(\nabla u) \rangle_{Q^{2r}_\lambda}|^2 \, dz$$

$$+ c\sigma \beta \|g\|_{L^\infty(I, BMO(B_r))}^{p'}$$

for every $\sigma \in (0, \delta]$ and $Q^{\lambda_0}_{p'}$ defined by Lemma 4.1. The constant $\delta \in (0, 1)$ only depends on $n, N, p$.

**Proof.** Suppose Case 1 does not hold. We find for $\varepsilon = \frac{1}{K}$

$$\|g\|_{L^\infty(I, BMO(B_r))}^{p'} \leq \varepsilon \lambda^{p}_r.$$  \(4.7\)

Now let $h$ be the solution of \(4.6\) on $Q^{\lambda_0}_{p'}$, then Lemma 4.2 implies

$$\int_{Q^{2r}_\lambda} |\nabla h|^p \, dz \leq 2^p \int_{Q^{2r}_\lambda} |\nabla u|^p \, dz + c\|g\|_{L^\infty(I, BMO(B_r))}^{p'} \leq c\lambda^{p}_r.$$  \(4.8\)

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We therefore can apply Theorem 3.3 and find for \( \theta < \frac{1}{4} \)

\[
\text{osc}_{\theta Q_{\lambda^2}} (V(\nabla h))^2 \leq c \theta^\alpha \lambda^p.
\]  

(4.9)

We define

\[
\rho := \min \{ a \geq r | Q^\lambda_a \text{ is intrinsic} \}.
\]

(4.10)

By construction \( \rho \leq R \) exists as \( Q^\lambda_R \) is intrinsic. Moreover, (see Lemma 4.1 (e)), we find that \( \lambda_a \leq (\frac{a}{\rho})^\beta \lambda_\rho \) for every \( r \leq a \leq \rho \).

If \( \frac{\rho}{2} > r \), we find

\[
\langle |\nabla u|^p \rangle_{Q^\lambda_{\rho/2}} \leq \lambda_{\rho/2}^p \leq \frac{1}{2\beta} \langle |\nabla u|^p \rangle_{Q^\lambda_{\rho}}.
\]

Therefore Lemma Appendix A.1 implies

\[
\lambda_{\rho}^p \leq c \left( \frac{r}{\rho} \right)^\beta \lambda_{\rho}^p \leq c \left( \frac{r}{\rho} \right)^\beta \int_{Q^\lambda_{\rho}} |V(\nabla u) - \langle V(\nabla u) \rangle_{Q^\lambda_{\rho}}|^2 \, dz. \tag{4.11}
\]

If \( \frac{\rho}{2} \leq r \leq \rho \), we either find that

\[
\langle |\nabla u|^p \rangle_{Q^\lambda_{\rho}} \leq \lambda_{\rho}^p \leq c \left( \frac{r}{\rho} \right)^\beta \int_{Q^\lambda_{\rho}} |V(\nabla u) - \langle V(\nabla u) \rangle_{Q^\lambda_{\rho}}|^2 \, dz.
\]

We find by Lemma 4.2 and as Case 1 does not hold

\[
\lambda_{\rho}^p \sim \int_{Q^\lambda_{\rho}} |\nabla h|^p \, dz. \tag{4.13}
\]

Now we can apply Theorem 3.2. This implies together with Lemma 4.2 for \( \theta \in (0, \tau) \)

\[
\text{osc}_{\theta Q_{\lambda^2}} (V(\nabla h))^2 \leq c \theta^\alpha \int_{Q^\lambda_{\rho}} |V(\nabla u) - \langle V(\nabla u) \rangle_{Q^\lambda_{\rho}}|^2 \, dz + c \theta^\alpha \|g\|^p_{L^\infty(I, \text{BMO}(B_r))}.
\]
Combining the last estimate with (4.9) and (4.11) we find
\[
\text{osc}_{\theta Q^\lambda_r}(V(\nabla h))^2 \leq c\theta^\alpha \left( \frac{r}{R} \right)^\beta \int_{Q^\lambda_r} |V(\nabla u) - \langle V(\nabla u) \rangle_{Q^\lambda_r}|^2 \, dz + c\theta^\alpha \|g\|_{L^\infty(I,\text{BMO}(B_r))}.
\]

To conclude the proof we use Lemma 4.1, (h): For \( \sigma^\frac{b}{p} = \theta \) we have \( Q^\lambda_{\sigma r} \subset \theta Q^\lambda_r \), therefore
\[
\text{osc}_{Q^\lambda_{\sigma r}}(V(\nabla h))^2 \leq c\sigma^\alpha \left( \frac{r}{R} \right)^\beta \int_{Q^\lambda_{\sigma r}} |V(\nabla u) - \langle V(\nabla u) \rangle_{Q^\lambda_{\sigma r}}|^2 \, dz + c\sigma^\alpha \|g\|_{L^\infty(I,\text{BMO}(B_r))}.
\]

by the choice of \( \beta = \frac{2-b}{p-2} \leq \frac{b}{2} \) by our assumptions on \( \beta \).

4.3. An intrinsic BMO result

The next proposition gives an intrinsic BMO estimate. We will prove it for the refined spaces \( \text{BMO}_\omega \). In the following let \( \omega : [0, \infty) \to [0, \infty) \) be almost increasing. Moreover,

\[
\frac{\omega(r)}{\omega(\sigma r)} \leq c_1 \sigma^{-\gamma} \quad \text{for } \sigma \in (0, 1) \text{ where } \gamma < \min \left\{ \frac{\alpha}{1 + \alpha \frac{p-2}{2}}, \frac{2}{p-2} \right\}, \quad (4.14)
\]

**Lemma 4.4.** Let \( Q^\lambda_R \) be intrinsic, \( \omega \) hold (4.14) and \( g \in L^\infty(I, \text{BMO}_\omega(B)) \), with \( \omega' \equiv \omega^{p-1} \). Then there exist constants \( c, \beta \) depending on \( \gamma, c_1, n, N, p \) such that

\[
\sup_{0<r<R} \frac{1}{\omega^p(r)} \int_{Q^\lambda_r} |V(\nabla u) - \langle V(\nabla u) \rangle_{Q^\lambda_r}|^2 \leq c\|g\|_{L^\infty(I,\text{BMO}_\omega(B))}^p + \frac{c}{\omega^p(R)} \int_{Q^\lambda_R} |V(\nabla u) - \langle V(\nabla u) \rangle_{Q^\lambda_R}|^2,
\]

where \( Q^\lambda_r \) is defined by Lemma 4.1 for a \( \beta > \gamma \) fixed.

**Proof.** We fix \( \gamma < \beta < \min \left\{ \frac{\alpha}{1 + \alpha \frac{p-2}{2}}, \frac{2}{p-2} \right\} \). Now we take \( \sigma \in (0, 1) \). We will define the size of \( \sigma \) in the end of the proof. If \( r \geq \sigma R \), we find by Lemma 4.1
Now we will prove the estimate for \( \sigma r \in (0, \sigma R] \). We apply Proposition 4.3 on the cylinder \( Q^{\lambda r}_r \). If Case 1 holds, we find as \( Q^{\lambda r}_{\sigma r} \) is sub-intrinsic

\[
\frac{1}{\omega^p(r)} \int_{Q^{\lambda r}_{\sigma r}} |V(\nabla u) - \langle V(\nabla u) \rangle_{Q^{\lambda r}_{\sigma r}}|^2 \leq \frac{c(\sigma)}{\omega^p(R)} \int_{Q^{\lambda 0}_R} |V(\nabla u) - \langle V(\nabla u) \rangle_{Q^{\lambda 0}_R}|^2.
\]

(4.15)

where we used that \( \omega \) is almost increasing and that \( \omega' \equiv \frac{\omega}{p} \).

If Case 2 of Proposition 4.3 holds, we find using the best constant property, Lemma 4.2, (4.14) and Lemma 4.1 (g)

\[
\frac{1}{\omega^p(\sigma r)} \int_{Q^{\lambda r}_{\sigma r}} |V(\nabla u) - \langle V(\nabla u) \rangle_{Q^{\lambda r}_{\sigma r}}|^2 \leq \frac{cK^p}{\omega^p(r)} \|g\|_{L^\infty(I, \text{BMO}^\prime(B_r))} \leq c \|g\|_{L^\infty(I, \text{BMO}^\prime(B_r))}.
\]

(4.16)

By Proposition 4.3 and (4.14) we find for \( \sigma \in (0, \delta) \) and \( \rho \geq r \)

\[
\frac{1}{\omega^p(\sigma r)} \int_{Q^{\lambda r}_{\sigma r}} (V(\nabla h))^2 + \frac{c}{\omega^p(r)} \int_{Q^{\lambda r}_{\sigma r}} |V(\nabla u) - V(\nabla h)|^2 \leq \sigma^{\beta - \gamma} \frac{c}{\omega^p(\rho)} \int_{Q^{\lambda \rho}_\rho} |V(\nabla u) - \langle V(\nabla u) \rangle_{Q^{\lambda \rho}_\rho}|^2 \]

\[
+ \sigma^{\beta - \gamma} \frac{c}{\omega^p(r)} \|g\|_{L^\infty(I, \text{BMO}(B_r))}^2.
\]

(4.17)

By Proposition 4.3 and (4.14) we find for \( \sigma \in (0, \delta) \) and \( \rho \geq r \)

\[
\frac{1}{\omega^p(\sigma r)} \int_{Q^{\lambda r}_{\sigma r}} (V(\nabla h))^2 + \frac{c}{\omega^p(r)} \int_{Q^{\lambda r}_{\sigma r}} |V(\nabla u) - V(\nabla h)|^2 \leq \sigma^{\beta - \gamma} \frac{c}{\omega^p(\rho)} \int_{Q^{\lambda \rho}_\rho} |V(\nabla u) - \langle V(\nabla u) \rangle_{Q^{\lambda \rho}_\rho}|^2 \]

\[
+ \sigma^{\beta - \gamma} \frac{c}{\omega^p(r)} \|g\|_{L^\infty(I, \text{BMO}(B_r))}^2.
\]

(4.17)
Combining the last estimate with (4.15), (4.16) and (4.17) leads to

\[
\sup_{a<r<R} \frac{1}{\omega(r)} \int_{Q^r_{\lambda^0}} |V(\nabla u) - \langle V(\nabla u) \rangle_{Q^r_{\lambda^0}}|^2 \\
\leq c\|g\|_{L^\infty(I,\text{BMO}_{\omega_{(B)})}} + \frac{c}{\omega(R)} \int_{Q^R_{\lambda^0}} |V(\nabla u) - \langle V(\nabla u) \rangle_{Q^R_{\lambda^0}}|^2 \\
+ c\sigma^{\beta-\gamma} \sup_{a<r<R} \frac{1}{\omega(r)} \int_{Q^r_{\lambda^0}} |V(\nabla u) - \langle V(\nabla u) \rangle_{Q^r_{\lambda^0}}|^2.
\]

(4.18)

Now fix \(\sigma\) conveniently, such that we can absorb the last term. The result follows by \(a \to 0\). \(\square\)

In Proposition 4.6 we show the intrinsic BMO estimate. Before we need another lemma on cylinders.

**Lemma 4.5.** Let \(Q^\lambda_{R} \) be sub-intrinsic. For every \(z \in Q^\lambda_{R/2} \) there exist a sub-intrinsic cube \(Q^{\lambda R/2}_R(z) \subset Q^\lambda_{R} \) and \(\lambda_{R/2} \sim \lambda_0 \).

Let \(Q_R = (t, t-R^2) \times B_R(x) \). Then for every \(z \in Q_{R/2} \) there exists a sub-intrinsic cube \(Q^{\lambda R/2}_R(z) \subset Q_R \) and \(\lambda_{R/2} \sim \max \{ \left( \int_{Q_R} |\nabla u|^p \right)^\frac{1}{p}, 1 \} \).

**Proof.** We start with the first statement. Since \(Q^\lambda_{R} \) is sub-intrinsic we find for fixed \(z \in Q^\lambda_{R/2} \)

\[
\frac{1}{|Q^\lambda_{R/2}_R(z)|} \int_{Q^\lambda_{R/2}_R(z)} |\nabla u|^p \leq \lambda^p_0.
\]

Hence, for \(2^{\frac{n+2}{p-2}} = \lambda_{R/2} \geq \lambda_0 \) we find

\[
\left( \int_{Q^{\lambda R/2}_R(z)} |\nabla u|^p \right)^\frac{1}{p} \leq \lambda_{R/2} \leq 2^{\frac{n+2}{p-2}} \lambda_0.
\]

To prove the second statement we define \(\lambda_0 \) by \(\int_{Q_R} |\nabla u|^p = \lambda^2_0 \). If \(\lambda_0 \leq 1 \), then \(\int_{Q_R} |\nabla u|^p \leq 1^p \), in this case we define \(\lambda_0 = 1 \). If \(\lambda_0 \geq 1 \) (and \(\lambda_0^{2-p} \leq 1 \)), we define \(\lambda_0 = \lambda_0 \) and find for any \(Q^\lambda_{R}(t) := (t, t-\lambda_0^{2-p} R^2 ) \times B_R \subset Q_R \) that \(\int_{Q^\lambda_{R}(t)} |\nabla u|^p \leq \lambda^p_0 \). Now we gain the result by proceeding as before. \(\square\)
Proof. We fix $\beta > \gamma$ by Lemma 4.1 for $r$ fixed. Then there exist a constant $c, \beta$ such that

$$
\sup_{z \in Q_R^{\lambda \nu}(z)} \sup_{r < \frac{R}{2}} \frac{1}{\omega(r)} \left( \int_{Q_R^{\lambda \nu}(z)} |V(\nabla u) - \langle V(\nabla u) \rangle_{Q_R^{\lambda \nu}(z)}|^2 \right)^{\frac{1}{p}} 
\leq c \|g\|_{L^\infty(I, \text{BMO}_\nu(B))} + \frac{c\lambda_0}{\omega(R)}
$$

where $Q_R^{\lambda R/2}(z)$ is defined by Lemma 4.5 and $Q_R^{\lambda \nu}(z) \subset Q_R^{\lambda R/2}(z)$ is defined by Lemma 4.1 for $\beta > \gamma$ fixed.

Proof. We fix $\rho := \sup \{a < \frac{R}{2} | Q_a^{\lambda \nu}(z) \text{ is intrinsic} \}$. By (e) of Lemma 4.1 (4.14) and Lemma 4.5 we find for $\rho \leq r \leq \frac{R}{2}$

$$
\frac{1}{\omega^p(r)} \int_{Q_R^{\lambda \nu}(z)} |V(\nabla u) - \langle V(\nabla u) \rangle_{Q_R^{\lambda \nu}(z)}|^2 \leq \frac{c\lambda_0^p}{\omega^p(r)} \leq c(\sigma) \frac{\lambda_0^{R/2}}{\omega^p(R)} \leq \frac{c\lambda_0^p}{\omega^p(r)}.
$$

For $r \leq \rho$ we can apply Lemma 4.4 and find by the previous that

$$
\frac{1}{\omega^p(r)} \int_{Q_R^{\lambda \nu}(z)} |V(\nabla u) - \langle V(\nabla u) \rangle_{Q_R^{\lambda \nu}(z)}|^2 
\leq c \|g\|_{L^\infty(I, \text{BMO}_\nu(B))} \frac{\rho_0}{\omega(r)} + \frac{c\lambda_0^p}{\omega^p(r)} \int_{Q_R^{\lambda \nu}(z)} |V(\nabla u) - \langle V(\nabla u) \rangle_{Q_R^{\lambda \nu}(z)}|^2
$$

This finishes the proof. \qed

We can generalize this result by the following purely intrinsic result

**Corollary 4.7.** Let $Q_R^{\lambda \nu}$ be sub-intrinsic, $\omega$ hold (4.14) and for every cube $Q_R^{\lambda \nu}(z)$ constructed as in Proposition 4.6

$$
\sup_{z \in Q_R^{\lambda \nu}(z)} \sup_{r < \frac{R}{2}} \frac{1}{\omega^p(r)} \int_{Q_R^{\lambda \nu}(z)} |g - \langle g \rangle_{Q_R^{\lambda \nu}(z)}|^p =: \|g\|_{p'} < \infty,
$$

then there exist a constant $c, \beta$ depending on $\gamma, c_1, n, p$ such that

$$
\sup_{z \in Q_R^{\lambda \nu}(z)} \sup_{r < \frac{R}{2}} \frac{1}{\omega^p(r)} \int_{Q_R^{\lambda \nu}(z)} |V(\nabla u) - \langle V(\nabla u) \rangle_{Q_R^{\lambda \nu}(z)}|^2 
\leq c \|g\|_{L^\infty(I, \text{BMO}_\nu(B))} \omega^p(R) + \frac{c\lambda_0^p}{\omega^p(R)}.
$$

Proof. One simply replaces $\|g\|_{L^\infty(I, \text{BMO}_\nu(B))}$ by $\|g\|$ in Lemma 4.2, Lemma 4.3 and Proposition 4.6. Anything else follows analogously. \qed

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4.4. Main Results

We are now able to prove the main theorem on weighted BLO spaces.

**Theorem 4.8.** Let $p \geq 2$ and the weight $\omega : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be almost increasing and satisfy (4.11). Let $u$ be a solution to (4.11) on $I \times B$ and $g \in L^\infty(I, \text{BMO}_{\omega'}(B))$, with $\omega' \equiv \omega^{p-1}$, then $u \in L^\infty(I, \text{BLO}_{\omega}(B))$ locally. Moreover, there exists $c, \delta$ depending on $n, N, p, \gamma, c_1$ such that for every sub-intrinsic cylinder $Q^\lambda_{R} \subset I \times B$

$$\|u\|_{L^\infty(I, \text{BLO}_{\omega}(B))} \leq c \|g\|_{L^\infty(I, \text{BMO}_{\omega'}(B))} + \frac{c \lambda_0}{\omega(R)}.$$  

**Proof.** We fix $(t, x) \in Q_{R/2}^{\lambda_0}$ and construct $Q_{R/2}^{\lambda_R/2}(t, x)$ by Lemma 4.5. Then we define $Q_{r}^{\lambda_R} := (t - \lambda_R^{2-p}r^2) \times B_r(x) \subset Q_{R/2}^{\lambda_0}$ for $R > 0$ by Lemma 4.1 with respect to $Q_{R/2}^{\lambda_R/2}(t, x)$ for a convenient $\beta$. In the following all balls in space are centered in $(t, x)$. Our aim is to estimate

$$N_\omega(u)(t, x) := \frac{1}{\omega_2^2(r)} \int_{B_r(x)} \frac{|u(y, t) - \ell_r(u)(t)|^2}{r} \, dy.$$  

Here $\ell_r(u)(t)$ is the best linear approximation of $u$ on $\{t\} \times B_r(x)$. We show the result for $bR \in (0, \frac{\delta R}{2})$. The constant $\delta$ is fixed by Proposition 4.3. We will divide the proof in the two cases of Proposition 4.3.

Case 1: $\lambda_r^p \leq K\|g\|_{L^\infty(I, \text{BMO}(B_r))}^p$.

Case 2: $\|g\|_{L^\infty(I, \text{BMO}(B_r))} \leq \frac{1}{\mu} \lambda_r^p$.

If Case 1 holds, we find $\langle |\nabla u|^p \rangle_{Q_r^{\lambda_r}} \leq \lambda_r^p \leq \mu_r^p : = K\|g\|_{L^\infty(I, \text{BMO}(B_r))}^p$. As $\lambda_r \leq \mu_r$ we find that $Q_r^{\mu_r} \subset Q_r^{\lambda_r}$ and by (4.5) that $\langle |\nabla u|^p \rangle_{Q_r^{\mu_r}} \leq \mu_r^p$. We take $h$ to be the solution of (4.15) on $Q_r^{\mu_r}$. Now Lemma 4.2 gives

$$\mu_r^{p-2} \int_{\{t\} \times B_r} \frac{|u - h|^2}{r} \, dx + \int_{Q_r^{\mu_r}} |V(\nabla u) - V(\nabla h)|^2 \, dz \leq c \mu_r^p. \quad (4.19)$$  

We use that $\ell_{bR}(u)$ is the best linear approximation of $u$ on $B_{bR} := \{t\} \times
$B_{\delta r}(x)$ and Poincaré's inequality to gain

$$N^\omega_{\delta r}(u) \leq \frac{1}{\omega^2(\delta r)} \int_{B_{\delta r}} \left| \frac{u - \ell_{\delta r}(h)}{r} \right|^2 dx \leq \frac{c}{\omega^2(\delta r)} \int_{B_{\delta r}} \left| \frac{u - h}{r} \right|^2 + \left| \frac{h - \ell_{\delta r}(h)}{r} \right|^2 dx$$

$$\leq \frac{c(\delta)}{\omega^2(r)} \int_{B_{r}} \left| \frac{u - h}{r} \right|^2 dx + \frac{c}{\omega^2(\delta r)} \sup_{B_{\delta r}} |\nabla h|^2 = I + II.$$ \hspace{1cm} (4.20)

For $I$ we find by \[4.19\]

$$I \leq \frac{c(\delta)}{\omega^2(r)} \frac{\mu_r^2}{\mu_r} \leq c \|g\|_{L^2(I,BMO_{\omega}(B_r))}.$$ To estimate $II$ we find by \[4.19\]

$$\langle |\nabla h|^p \rangle_{Q_{\delta r}^r} \leq \langle |\nabla u|^p \rangle_{Q_{\delta r}^r} + \int_{Q_{\delta r}^r} |V(\nabla u) - V(\nabla h)|^2 dz \leq c \mu_r^p.$$ Now Theorem \[3.1\] implies

$$\frac{c}{\omega^2(\delta r)} \sup_{\{t\} \times B_{\delta r}(x)} |\nabla h|^2 \leq \frac{c}{\omega^2(r)} \sup_{Q_{\delta r}^r} |\nabla h|^2 \leq \frac{c}{\omega^2(\delta r)} \frac{\mu_r^2}{\mu_r} \leq c \|g\|_{L^\infty(I,BMO_{\omega}(B_r))}^2.$$ This closes Case 1.

In the following Case 2 holds. Remember, that $\delta r \in (0,\delta)^2$. We start similar to \[4.20\], we take $h$ to be the solution of \[4.6\] on $Q_{\delta r}^r$. Now Lemma \[1.2\] gives

$$\lambda_r^{p-2} \int_{\{t\} \times B_{r}} \left| \frac{u - h}{r} \right|^2 dx + \int_{Q_{\delta r}^r} |V(\nabla u) - V(\nabla h)|^2 dz \leq c \|g\|_{L^\infty(I,BMO(B_r))}.$$ \hspace{1cm} (4.21)

Similar to Case 1 we find

$$N^\omega_{\delta r}(u) \leq \frac{1}{\omega^2(\delta r)} \int_{B_{\delta r}} \left| \frac{u - \ell_{\delta r}(h)}{r} \right|^2 dx \leq \frac{c}{\omega^2(\delta r)} \int_{B_{\delta r}} \left| \frac{u - h}{r} \right|^2 + \left| \frac{h - \ell_{\delta r}(h)}{r} \right|^2 dx$$

$$\leq \frac{c(\delta)}{\omega^2(r)} \int_{B_{r}} \left| \frac{u - h}{r} \right|^2 dx + \frac{c}{\omega^2(\delta r)} \text{osc}_{B_{\delta r}}(\nabla h)^2 = I + II.$$ where we used the Poincaré's inequality. We estimate $I$ by Lemma \[1.2\] as Case 2 holds we deduce from \[4.21\]

$$\int_{B_{r}} \left| \frac{u - h}{r} \right|^2 dx \leq \lambda_r^{2-p} \|g\|_{L^\infty(I,BMO(B_r))} \leq \|g\|_{L^\infty(I,BMO(B_r))}^2.$$
and consequently
\[
I \leq c\|g\|_{L^\infty(I, BMO_{\omega}(B_r))}^{\frac{3}{p-1}}. \quad (4.22)
\]

We estimate II by using \(p \geq 2\) and Proposition 4.3. As in the proof of Proposition 4.6 we fix \(\rho := \sup\{a < \frac{R}{4} \mid \lambda_0(t, x) \text{ is intrinsic}\}\). If \(r \leq \rho\) Proposition 4.3 provides an \(\lambda_1 \leq \rho\) such that
\[
\text{osc}_{B_{2r}}(V(\nabla h))^2 \leq c\left(\frac{\delta r}{r_1}\right)^{2(\beta - \gamma)} \int_{Q_{r_1}^{\lambda_1}} |V(\nabla u) - (V(\nabla u))_{Q_{r_1}^{\lambda_1}}|^2 \, dz + c\delta^2 \|g\|_{L^\infty(I, BMO(B_r))}^{p'}.
\]

This implies using (2.2)
\[
II = \frac{c}{\omega^2(\delta r)} \text{osc}_{B_{2r}}(\nabla h)^2 \leq \frac{c}{\omega^2(\delta r)} \text{osc}_{B_{2r}}(V(\nabla h))^{\frac{2}{p'}}
\]
\[
\leq c\left(\frac{\delta r}{r_1}\right)^{2(\beta - \gamma)} \int_{Q_{r_1}^{\lambda_1}} |V(\nabla u) - (V(\nabla u))_{Q_{r_1}^{\lambda_1}}|^2 \, dz + \|g\|_{L^\infty(I, BMO(B_r))}^{p'}
\]
\[
\leq c\left(\frac{\delta r}{r_1}\right)^{2(\beta - \gamma)} \left(\frac{1}{\omega^p(r_1)}\int_{Q_{r_1}^{\lambda_1}} |V(\nabla u) - (V(\nabla u))_{Q_{r_1}^{\lambda_1}}|^2 \, dz\right)^{\frac{2}{p}}
\]
\[
+c\|g\|_{L^\infty(I, BMO_{\omega}(B_r))}^{\frac{2}{p-1}}.
\]

as \(\omega\) holds (4.14). On this we can apply Proposition 4.3 and find as \(\gamma < \beta\)
\[
II \leq c\|g\|_{L^\infty(I, BMO_{\omega}(B_r))}^{\frac{2}{p-1}} + \frac{c\lambda_0^2}{\omega^2(R)}. \quad (4.23)
\]

If \(\rho < r < \frac{R}{4}\) we have by \([e]\) of Lemma 4.1 and the construction of \(Q_{R/2}^{\lambda_2}(t, x)\), that \(\lambda_r \leq \left(\frac{r}{R/2}\right)^{\beta - \gamma} \lambda_0\) and therefore we find by (4.21) and its consequences (4.8) and (4.9)
\[
II \leq c\|g\|_{L^\infty(I, BMO_{\omega}(B_r))}^{\frac{2}{p-1}} + \frac{c\lambda_0^2}{\omega^2(R)} \left(\frac{r}{R}\right)^{\beta - \gamma} \lambda_0^2.
\]

Combining the last estimate with (4.22) and (4.23) closes case 2. As all estimates are independent of \((t, x) \in Q_{R/2}^{\lambda_0}\), the result is proved. \(\square\)

**Proof of Theorem 1.1.** One fixes \(\omega(r) \equiv 1\) and combines Lemma 4.5 with Theorem 4.8. Then the result follows by the Campanato characterization of \(C^1(B_{R/2}(x))\). \(\square\)
Remark 4.9. In [16, Section: 1.7.2] we find that \( \text{BLO} = C^1 = F^1_{\infty, \infty} \), here \( F^1_{\infty, \infty} \) is the Triebel-Lizorkin space. The space \( W^{1, \text{BMO}} = F^1_{\infty, 2} \). We therefore find by our estimates that, if \( g \) is in \( L^\infty(2I, \text{BMO}(B)) \), then we have \( u \in L^p(I, W^{1,p}(B)) \cap L^\infty(I, \text{BLO}(B)) = L^p(I, P^1_{p,2}(B)) \cap L^\infty(I, F^1_{\infty, \infty}(B)) \). By interpolation \( u \in L^q(I, W^{1,q}(B)) \) for every \( 1 \leq q \leq \infty, 1 \leq r < \infty \) (see [16, Section: 1.6.2]); natural local estimates are available.

Proposition 4.10. Let \( \gamma p < \min \left\{ \frac{\alpha}{1 + \frac{\alpha}{p}}, \frac{3}{p-2} \right\} \). If \( g \in L^\infty(I, C^\gamma(p^{-1})(B)) \), then \( \nabla u \in C^\gamma_{\text{par}}(I \times B) \). Moreover, for every sub-intrinsic cylinder \( Q_{R/4}^{\lambda_0} \) we find

\[
\|\nabla u\|_{C^\gamma_{\text{par}}(Q_{R/4}^{\lambda_0})} \leq c \left( \frac{1}{R^\gamma} + \frac{1}{(K^{2-p}R^2)^{\frac{\gamma}{p}}} \right),
\]

where \( K = c\lambda_0 + cR^\gamma \|g\|_{L^\infty(I, C^\gamma(p^{-1})(B))} \) and \( c \) depends on \( \gamma, n, p, K \).

Proof. We start by showing Hölder continuity in space. By Theorem \ref{holder continuity in space} and \( \omega(r) = r^\gamma \) we gain by the Campanato characterization that

\[
\|\nabla u\|_{L^\infty(I, c^{\gamma(p^{-1})}(B))} \leq c \|g\|_{L^\infty(I, C^\gamma(p^{-1})(B))} + \frac{c\lambda_0}{R^\gamma} \tag{4.24}
\]

This implies that \( \nabla u \) is Hölder continuous in space. It implies also, that \( \nabla u \) is bounded in \( Q_{R/2}^{\lambda_0} \). Moreover, the previous implies

\[
\max_{Q_{R/2}^{\lambda_0}} |\nabla u| \leq K < \infty
\]

for \( K = c\lambda_0 + cR^\gamma \|g\|_{L^\infty(I, C^\gamma(p^{-1})(B))} \).

In the following we prove Hölder continuity in time. I.e. we show for \( (t, x) \in Q_{R/4}^{\lambda_0} \),

\[
\left( \int_{t-s}^t |V(\nabla u)(\tau, x) - (V(\nabla u)(\tau, x))_{(t,t-s)}|^2 \, d\tau \right)^{\frac{1}{p}} \leq K \left( \frac{8}{S} \right)^{\frac{\gamma}{2}}. \tag{4.25}
\]

for all \( s \in (0, S), S := K^{p-2}R^2 \). From this estimate the Hölder continuity in time follows by (2.2) and the Campanato characterization of Hölder spaces.

In the following we prove (4.25). We take \( (t, x) \in Q_{R/4}^{\lambda_0}, \) fix \( S(R) = K^{2-p}R \) and take \( Q_{R/2}^{\lambda_0}(t, x) \subset Q_{R/4}^{\lambda_0} \) as starting cylinder. Then for all \( r < \frac{R}{4} \) we take \( Q_r^{\lambda_0}(t, x) \) constructed by Lemma \ref{maximal functions}. We have that \( \lambda_r \leq K \), (as
\( \lambda \leq K \) by (4.3). Therefore Proposition 4.6 provides for all \( r \in (0, \frac{R}{4}] \)

\[
\int_{t-s}^{t} \int_{B_r(x)} |V(\nabla u) - \langle V(\nabla u) \rangle_{Q^\lambda_{r}(z)}|^2 d\tau d\sigma \leq c \left( \frac{r}{R} \right)^{p \gamma} K^p = c \left( \frac{\lambda r}{K} \right)^{p \gamma} \left( \frac{s(\tau)}{S(R)} \right)^{p \gamma} K^p \leq c \left( \frac{s(\tau)}{S(R)} \right)^{p \gamma} K^p,
\]

as \( \lambda r \leq K \). Now we find by Lemma 4.1, (b), that \( s(\tau) = \lambda^{2-p(r)} - p(r)^2 \) is continuous and \( s(0) = 0 \) and \( s(R) = S(R) \). Therefore we can choose an \( r(s) \) for every \( 0 < s \leq S \) such that \( s = s(r) = \lambda^{2-p(r)} - p(r)^2 \). We estimate for \( x \in B_{R/4} \) and \( s \) fixed

\[
\int_{t-s}^{t} |(V(\nabla u)(\tau, x) - \langle V(\nabla u)(\tau, x) \rangle_{(t, t-s)}|^2 d\tau \leq c \int_{t-s}^{t} |V(\nabla u)(\tau, x) - \langle V(\nabla u)(\tau) \rangle_{B_r(s)}|^2 d\tau
\]

\[
+ c \int_{t-s}^{t} \langle V(\nabla u)(\tau) \rangle_{B_r(s)} - \langle V(\nabla u) \rangle_{Q^\lambda_{r}(s)}|^2 d\tau = I + II.
\]

\( I \) can be estimated by the \( L^\infty(I_{R^2/4}, C^1, \gamma(B_{R/2})) \) estimate

\[
I \leq K^p \left( \frac{r(s)}{R(S)} \right)^{p \gamma} \leq K^p \left( \frac{s}{S} \right)^{p \gamma}.
\]

\( II \) can be estimated by (4.26)

\[
II \leq \int_{t-s}^{t} \int_{B_r(s)} |V(\nabla u) - \langle V(\nabla u) \rangle_{Q^\lambda_{r}(s)}|^2 d\tau \leq c \left( \frac{s}{S} \right)^{p \gamma} K^p,
\]

where we used that \( Q^\lambda_{r}(s) = (t, t-s(r)) \times B_r(s)(x) \). This finishes the proof of (4.25). \( \square \)

**Remark 4.11.** The last result can be weakened. As long as the modulus of continuity is strong enough to imply the boundedness of \( |\nabla u| \) we find the same natural estimates as in Proposition 4.10. We expect that the sharp bound would be the Dini continuity. I.e. \( f \) is Dini continuous on \( B_R \) if it’s
modulus of continuity $\omega$ holds $\sum_{i=1}^{\infty} \omega(2^{-i}R) < \infty$. We conjecture that in this case $BLO_\omega \equiv C^1_\omega$. If this would be true, then the Dini result of [11] could be gained similar to Proposition 4.10 with a weaker condition on $g$, i.e. $g \in L^\infty(I, C_{\omega(p-1)}(B))$, but restricted to (1.1) and $p \geq 2$.

If we follow the estimates of [2, Corr. 5.4], we find directly, that $g \in L^\infty(I, VMO(B))$ implies that locally $u \in L^\infty(I, VLO(B))$.

Appendix A.

For $Q_1 \subset Q_2$ and $q \in [1, \infty)$ we find that
\[
|\langle f \rangle_{Q_1} - \langle f \rangle_{Q_2}| \leq \left( \frac{1}{Q_2} \int_{Q_2} |f - \langle f \rangle_{Q_2}|^q \right)^{\frac{1}{q}} \leq \left( \frac{|Q_2|}{|Q_1|} \int_{Q_1} |f - \langle f \rangle_{Q_2}|^q \right)^{\frac{1}{q}}. \tag{A.1}
\]

This estimate can be iterated for $i = \{0 \ldots k\}$ and $Q_i \subset Q_{i-1}$ with $\frac{|Q_{i-1}|}{|Q_i|} \leq c$
\[
|\langle f \rangle_{Q_k} - \langle f \rangle_{Q_1}| \leq \sum_{i=1}^{k} |\langle f \rangle_{Q_i} - \langle f \rangle_{Q_{i-1}}| \leq c \sum_{i=1}^{k} \left( \frac{1}{Q_{i-1}} \int_{Q_{i-1}} |f - \langle f \rangle_{Q_{i-1}}|^q \right)^{\frac{1}{q}}. \tag{A.2}
\]

Lemma Appendix A.1. Let $Q_1 \subset Q$ be two Cylinders and $f \in L^q(Q)$ for $q \in [1, \infty)$. For $\varepsilon \in (0,1)$ we find:
If $|\langle f \rangle_{Q_1}| \leq \varepsilon |\langle |f|^q \rangle_{Q}^{\frac{1}{q}}$, then
\[
|\langle f \rangle_{Q_1}| \leq \varepsilon |\langle |f|^q \rangle_{Q}^{\frac{1}{q}} \leq \frac{\varepsilon}{1 - \varepsilon} (1 + \left( \frac{|Q|}{|Q_1|} \right)^{\frac{1}{q}} \left( \int_Q |f - \langle f \rangle_Q|^q \, dx \right)^{\frac{1}{q}}).
\]

Proof. We find
\[
|\langle |f|^q \rangle_{Q}^{\frac{1}{q}} \leq \left( \int_Q |f - \langle f \rangle_{Q_1}|^q \, dx \right)^{\frac{1}{q}} + |\langle f \rangle_{Q_1}|
\leq \left( \int_Q |f - \langle f \rangle_Q|^q \, dx \right)^{\frac{1}{q}} + |\langle f \rangle_Q - \langle f \rangle_{Q_1}| + \varepsilon |\langle |f|^q \rangle_{Q}^{\frac{1}{q}}
\]
This implies that
\[
|\langle |f|^q \rangle_{Q}^{\frac{1}{q}} \leq \frac{1}{1 - \varepsilon} \left( \int_Q |f - \langle f \rangle_Q|^q \, dx \right)^{\frac{1}{q}} + \frac{1}{1 - \varepsilon} |\langle f \rangle_{Q_1} - \langle f \rangle_Q|
\]

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We estimate the second integral by
\[
|\langle f \rangle_{Q_1} - \langle f \rangle_Q| \leq \int_{Q_1} |f - \langle f \rangle_Q| \, dx \leq \left( \int_{Q_1} |f - \langle f \rangle_Q|^q \, dx \right)^\frac{1}{q}.
\]
\[
\leq \left( \frac{|Q_1|}{|Q|} \int_{Q} |f - \langle f \rangle_Q|^q \, dx \right)^\frac{1}{q}.
\]

\[\square\]

Lemma Appendix A.2. Let \( f \in L^q(Q_R) \) with \( q \in [1, \infty) \). Suppose that \( \omega : \mathbb{R}^+ \to \mathbb{R}^+ \) is increasing and holds the following Dini condition: \( \sum_i^\infty \omega(2^{-i} R) \leq K \) (e.g. \( \omega(r) = r^\alpha \)). If
\[
\left( \int_{\theta B} |f - \langle f \rangle_B|^q \right)^\frac{1}{q} \leq c_1 \omega(\theta) \left( \int_{B} |f - \langle f \rangle_B|^q \right)^\frac{1}{q},
\]
then
\[
\text{osc}_{Q_\rho}(f) \leq cK \omega(\theta) \left( \int_{Q_\rho} |f - \langle f \rangle_{Q_\rho}|^q \right)^\frac{1}{q},
\]
for all \( \theta \in (0, \frac{1}{2}) \), \( \rho \leq R \) and \( c \) depending only on \( q, n, c_1 \).

Proof. We only proof the first statement. For \( k \in \mathbb{N} \) we define for \( z \in \frac{1}{2}\theta Q_\rho \) we define \( Q_i(z) := 2^{-i} \frac{1}{2} Q_{\theta \rho}(z) \) for \( i = 1, \ldots, k \) and \( Q_0(z) = \theta Q_\rho \). We estimate by (A.2)
\[
|\langle f \rangle_{Q_k(z)} - \langle f \rangle_{Q_\rho}| \leq \sum_{i=0}^{k-1} \left( \int_{Q_i(z)} |f - \langle f \rangle_{Q_i(z)}|^q \right)^\frac{1}{q},
\]
this can be estimated by assumption by and because \( \omega \) is increasing
\[
|\langle f \rangle_{Q_k(z)} - \langle f \rangle_{Q_\rho}| \leq c \sum_{i=1}^{k-1} \omega(2^{-i} \theta \rho) \left( \int_{\theta Q_\rho} |f - \langle f \rangle_{Q_\rho}|^q \right)^\frac{1}{q}
\]
\[
\leq cK \omega(\theta) \left( \int_{Q_\rho} |f - \langle f \rangle_{Q_\rho}|^q \right)^\frac{1}{q};
\]
the constant is independent of \( k \); this implies that
\[
|f(z) - \langle f \rangle_{Q_\rho}| \leq cK \omega(\theta) \left( \int_{Q_\rho} |f - \langle f \rangle_{Q_\rho}|^q \right)^\frac{1}{q}.
\]
Consequently, we find for $z, w \in \frac{1}{2} \theta Q_\rho$ we have

$$|f(z) - f(w)| \leq cK\omega(\theta) \left( \frac{1}{Q_\rho} \int_{Q_\rho} |f - \langle f \rangle_{Q_\rho}|^q \right)^{\frac{1}{q}}.$$

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