On the Number of Chiral Generations in $Z_2 \times Z_2$ Orbifolds

Ron Donagi$^1$ and Alon E. Faraggi$^2$

1 Department of Mathematics, University of Pennsylvania, Philadelphia, PA, 19104–6395, USA
2 Theoretical Physics Department, University of Oxford, Oxford, OX1 3NP, UK and
School of Natural Sciences, Institute for Advanced Study, Princeton, NJ 08540, USA

Abstract

The data from collider experiments and cosmic observatories indicates the existence of three light matter generations. In some classes of string compactifications the number of generations is related to a topological quantity, the Euler characteristic. However, these do not explain the existence of three generations. In a class of free fermionic string models, related to the $Z_2 \times Z_2$ orbifold compactification, the existence of three generations is correlated with the existence of three twisted sectors in this class of compactifications. However, the three generation models are constructed in the free fermionic formulation and their geometrical correspondence is not readily available. In this paper we classify quotients of the $Z_2 \times Z_2$ orbifold by additional symmetric shifts on the three complex tori. We show that three generation vacua are not obtained in this manner, indicating that the geometrical structures underlying the free fermionic models are more esoteric.

$^*$donagi@math.upenn.edu
$^†$faraggi@thphys.ox.ac.uk
1 Introduction

One of the important clues in the quest for the unification of the elementary matter and interactions is the triple replication of the Standard Model fermion states. While the possibility exists that there are additional families, contemporary data suggests the existence of only three chiral generations. The precision electroweak data obtained at LEP and SLC show that the width of the $Z$–boson can only accommodate three light left–handed neutrinos [1]. The constraints from observations of light element abundances also constrain the number of relativistic degrees of freedom during primordial nucleosynthesis to correspond to three light left–handed neutrinos [2]. Similarly, existence of three quark generations is compatible with the constraints arising from unitarity of the Cabbibo–Kobayashi–Maskawa mixing matrix. In the context of grand unification, gauge coupling unification and the mass ratio $m_b/m_{\tau}$ are only compatible with the low energy data in the presence of three chiral generations [3]. Understanding the origin of the number of flavors and of their mass and mixing spectrum is therefore one of the vital issues in the phenomenology of the Standard Model and unification.

In the context of point quantum field theories the number of generations and the flavor variables are mere parameters that fit the data. It is plausible that understanding of the origin of these fundamental constants can only be obtained in the framework of quantum gravity, i.e. their origin is of a geometrical characteristic. It is then encouraging that in the context of heterotic string theories [4] compactified on Calabi–Yau manifolds [5] the number of generation in the low energy spectrum is dictated by a topological quantity, the Euler number $\chi$. However, the Euler number of a random Calabi–Yau manifold can take many values and therefore does not yet provide a compelling explanation for the existence of three chiral generations.

In this context it has been suggested that a particular class of Calabi–Yau manifolds may provide a plausible insight to the existence of three chiral generations [6]. The relevant manifolds are those related to the $Z_2 \times Z_2$ orbifolds [9] of six dimensional toroidal spaces [7, 8], that have been studied most extensively in the free fermionic formulation of the heterotic string in four dimensions [10, 11]. The origin of the number of three generations in the $Z_2 \times Z_2$ orbifold is associated with the existence of three twisted sectors and the fact that each of the three $Z_2$ twists leaves one torus fixed. The enumeration of the number of generations then corresponds to the number of fixed points on the two twisted tori. The realization of the three generations in the free fermionic models then corresponds to reducing the number of generations to one generation from each of the twisted sectors of the $Z_2 \times Z_2$ orbifold. Thus, this class of string compactifications correlates the existence of three generations with the structure of the underlying $Z_2 \times Z_2$ compactified manifold [6, 9].

The free fermionic models, however, are a particular realization of the $Z_2 \times Z_2$ orbifold at a fixed point in the moduli space. Furthermore, the geometrical correspondence is well established only in specific cases and is lacking in the case of the
three generation models. For many of the issues pertaining to the phenomenology of these models, it will be beneficial to abandon the fermionic realization and to resort to the bosonic, or geometric description. It is therefore particularly important to understand the precise geometrical structure of the three generation models.

The aim of this paper is therefore to study the question of the number of generations in $Z_2 \times Z_2$ orbifolds with symmetric shifts. The goal is to examine whether such constructions can reproduce the free fermionic picture of obtaining one chiral generation from each twisted sector. While string theory, in general, and its free fermionic formulation, in particular, allows more general operations, i.e. those that are asymmetric between the left– and right–moving coordinates on the world–sheet, the restriction to symmetric shifts may be viewed as what is allowed by “classical geometry”. In this respect our conclusion will be in fact negative. Namely, we will prove that it is not possible to produce the three generation manifolds solely by utilizing symmetric shifts. This, in our view, is a substantial outcome with important possible consequences. First, it indicates that the geometry underlying the three generation free fermionic models is not “classical geometry” as it necessarily involves operations that are not symmetric between the left– and the right–moving coordinates. The relevant geometrical structures may therefore be of intrinsic “quantum” or “stringy” character. Second, the fact that the three generation models necessarily employ asymmetric operations may prove to be important for the question of moduli stabilization.

Our paper is organized as follows. In section 2 we discuss how three generations are obtained in the free fermionic models, which serves as our motivation for the ensuing analysis. In section 3 we discuss the free fermion–orbifold correspondence and set the ground for the subsequent analysis. In sections 4–8 we present the complete analysis of the $Z_2 \times Z_2$ with symmetric shifts. We identify the geometric condition for producing chiral matter, and present the proof that $Z_2 \times Z_2$ orbifold with solely symmetric shifts cannot yield a three generation vacuum. Section 11 concludes the paper.

2 Three generations in the free fermion models

In the free fermionic formulation of the heterotic string in four dimensions all the world–sheet degrees of freedom required to cancel the conformal anomaly are represented in terms of free fermions. For the left–movers one has the usual space–time fields $X^\mu$, $\psi^\mu$, $(\mu = 0, 1, 2, 3)$, and in addition the following eighteen real free fermion fields: $\chi^I, y^I, \omega^I$ $(I = 1, \cdots, 6)$, transforming as the adjoint representation of $SU(2)^6$. A model in this construction is defined by a set of boundary condition basis vectors, and the one–loop GSO projection coefficients. The basis vectors generate a finite additive group $\Xi$. The physical states in the Hilbert space, of a given sector $\alpha \in \Xi$, are obtained by acting on the vacuum with bosonic and fermionic operators. For a periodic complex fermion $f$, there are two degenerate vacua $|+\rangle, |-\rangle$, annihilated
by the zero modes \( f_0 \) and \( f_0^* \) and with fermion numbers \( F(f) = 0, -1 \), respectively. The physical spectrum is obtained by applying the generalized GSO projections.

The free fermion three generation models are constructed in two stages. The first corresponds to the NAHE set of boundary basis vectors \( \{1, S, b_1, b_2, b_3\} \) \([6, 11]\). The second consists of adding to the NAHE set three additional boundary condition basis vectors, typically denoted \( \{\alpha, \beta, \gamma\} \). The sector \( S \) generates \( N = 4 \) space–time supersymmetry, which is broken to \( N = 2 \) and \( N = 1 \) space–time supersymmetry by \( b_1 \) and \( b_2 \), respectively. The gauge group after the NAHE set is \( SO(10) \times E_8 \times SO(6)^3 \), which is broken to \( SO(4)^3 \times U(1)^3 \times SO(10) \times SO(16) \) by the vector \( 2\gamma \). At the level of the NAHE set, each sector \( b_1, b_2 \) and \( b_3 \) give rise to 16 spinorial 16 of \( SO(10) \). The Neveu-Schwarz (NS) sector produces some massless states that transform as \( (5 \oplus 5) \) of \( SO(10) \) and some others that are singlets of \( SO(10) \). All the states from the NS sector are singlets of the hidden \( E_8 \).

The NAHE set divides the internal world–sheet fermions into several groups. The internal 44 right–moving fermionic states are divided in the following way: \( \bar{\psi}^{1, \cdots, 5} \) are complex and produce the observable \( SO(10) \) symmetry; \( \bar{\phi}^{1, \cdots, 8} \) are complex and produce the hidden \( E_8 \) gauge group; \( \{\bar{\eta}^1, \bar{\eta}^{3, \cdots, 6}\}, \{\bar{\eta}^2, \bar{\eta}^{1, 2, \bar{\omega}^{5, 6}}\}, \{\bar{\eta}^3, \bar{\omega}^{1, \cdots, 4}\} \) give rise to the three horizontal \( SO(6) \) symmetries. The left–moving \( \{\eta, \omega\} \) states are divided to, \( \{y^{3, \cdots, 6}\}, \{y^{1, 2, \omega^{5, 6}}\}, \{\omega^{1, \cdots, 4}\} \). The left–moving \( \chi^{12}, \chi^{34}, \chi^{56} \) states carry the supersymmetry charges.

An important consequence of the NAHE set is observed by extending the \( SO(10) \) symmetry to \( E_6 \). Adding to the NAHE set a vector \( \xi_2 \) with periodic boundary conditions for the set \( \{\bar{\psi}^{1, \cdots, 5}, \bar{\eta}^{1, 2, 3}\} \), extends the gauge symmetry to \( E_6 \times U(1)^2 \times SO(4)^3 \). Each spinorial 16 of \( SO(10) \), produced by one of the three sectors \( b_j \), combines with a 10+1 of \( SO(10) \), produced by the sector \( b_j + \xi_2 \), to give a 27 of \( E_6 \). The sectors \( (b_j; b_j + \xi_2), \ (j = 1, 2, 3) \) each give eight 27 of \( E_6 \). The untwisted \( (NS; NS + \xi_2) \) sector gives, in addition to the vector bosons and spin two states, three copies of scalar representations in 27 + 27 of \( E_6 \). Alternatively, we can start with an extended NAHE set \( \{1, S, \xi_1, \xi_2, b_1, b_2\} \), with \( \xi_1 = 1 + b_1 + b_2 + b_3 \). The set \( \{1, S, \xi_1, \xi_2\} \) produces a toroidal Narain model with \( SO(12) \times E_8 \times E_8 \) or \( SO(12) \times SO(16) \times SO(16) \) gauge group depending on the GSO phase \( c^{(\xi_1)} \). The basis vectors \( b_1 \) and \( b_2 \) then break \( SO(12) \rightarrow SO(4)^2 \), and either \( E_8 \times E_8 \rightarrow E_6 \times U(1)^2 \times E_8 \) or \( SO(16) \times SO(16) \rightarrow SO(10) \times U(1)^3 \times SO(16) \). The vectors \( b_1 \) and \( b_2 \) correspond to \( Z_2 \times Z_2 \) orbifold modding. The three sectors \( b_1, b_2 \) and \( b_3 \) correspond to the three twisted sectors of the \( Z_2 \times Z_2 \) orbifold, with each producing eight generations in the 27, or 16, representations of \( E_6 \), or \( SO(10) \), respectively. In the case of \( E_6 \) the untwisted sector produces an additional \( 3 \times (27 + 27) \), whereas in the \( SO(10) \) model it produces \( 3 \times (10 + \overline{10}) \). Therefore, the Calabi–Yau manifold that corresponds to the \( Z_2 \times Z_2 \) orbifold at the free fermionic point in the Narain moduli space has \( (h_{11}, h_{21}) = (27, 3) \).

In this model the fermionic states which count the multiplets of \( E_6 \) are the internal fermions \( \{y, w|\bar{\eta}, \bar{\omega}\} \). The vacuum of the sectors \( b_j \) contains twelve periodic fermions.
Each periodic fermion gives rise to a two dimensional degenerate vacuum $|+\rangle$ and $|−\rangle$ with fermion numbers 0 and −1, respectively. After applying the GSO projections, we can write the vacuum of the sector $b_1$ in combinatorial form

$$
\left[\left(\binom{4}{0} + \binom{4}{2} + \binom{4}{4}\right)\{0\} \binom{5}{0} + \binom{5}{2} + \binom{5}{4}\right]\{1\} + \binom{2}{2} \left[\binom{5}{1} + \binom{5}{3} + \binom{5}{5}\right]\{1\}\right) \right) \right)
$$

where $4 = \{y^3y^4, y^5y^6, \bar{y}^3\bar{y}^4, \bar{y}^5\bar{y}^6\}$, $2 = \{\psi^\mu, \chi^{12}\}$, $5 = \{\bar{\psi}^{1\cdots 5}\}$ and $1 = \{\bar{\eta}^1\}$. The combinatorial factor counts the number of $|−\rangle$ in a given state. The two terms in the curly brackets correspond to the two components of a Weyl spinor. The $10 + 1$ in the $27$ of $E_6$ are obtained from the sector $b_j + \xi_1$. The states which count the multiplicities of $E_6$ are the internal fermionic states $\{y^{3\cdots 6}|\bar{y}^{3\cdots 6}\}$. A similar result is obtained for the sectors $b_2$ and $b_3$ with $\{y^{1,2}, \omega^{5,6}|\bar{y}^{1,2}, \bar{\omega}^{5,6}\}$ and $\{\omega^{1\cdots 4}|\bar{\omega}^{1\cdots 4}\}$, respectively, which suggests that these twelve states correspond to a six dimensional compactified orbifold with Euler characteristic equal to 48.

The construction of the free fermionic models beyond the NAHE–set entails the construction of additional boundary condition basis vectors and the associated one–loop GSO phases. Their function is to reduce the number of generations and at the same time break the four dimensional gauge group. In terms of the former, the reduction is primarily by the action on the set of internal world–sheet fermions $\{y, \omega|\bar{y}, \bar{\omega}\}$. As elaborated in the next section this set corresponds to the internal compactified manifold and the action of the additional boundary condition basis vectors on this set also breaks the gauge symmetries from the internal lattice enhancement. The latter is obtained by the action on the gauge degrees of freedom which correspond to the world–sheet fermions $\{\bar{\psi}^{1\cdots 5}, \bar{\eta}^{1\cdots 3}, \bar{\phi}^{1\cdots 8}\}$. In the bosonic formulation this would correspond to Wilson–line breaking of the gauge symmetries, hence for the purpose of the reduction of the number of generations we can focus on the assignment to the internal world–sheet fermions $\{y, \omega|\bar{y}, \bar{\omega}\}$.

We can therefore examine basis vectors that do not break the gauge symmetries further, i.e. basis vectors of the form $b_j$, with $\{\psi^\mu_{1,2}\chi_{j,j+1}, (y, \omega|\bar{y}, \bar{\omega}), \bar{\psi}^{1\cdots 5}, \bar{\eta}_j\} = 1$ for some selection of $(y, \omega|\bar{y}, \bar{\omega}) = 1$ assignments such that the additional vectors $b_j$ produce massless $SO(10)$ spinorials. We will refer to such vectors as spinorial vectors. The additional basis vectors $b_j$ can then produce chiral, or non–chiral, spectrum. The condition that the spectrum from a given such sector $b_j$ be chiral is that there exist another spinorial vector, $b_i$, in the additive group $\Xi$, such that the overlap between the periodic fermions of the internal set $(y, \omega|\bar{y}, \bar{\omega})$ is empty, i.e.

$$
\{b_j(y, \omega|\bar{y}, \bar{\omega})\} \cap \{b_i(y, \omega|\bar{y}, \bar{\omega})\} \equiv \emptyset .
$$

If there exists such a vector $b_i$ in the additive group then it will induce a GSO projection that will select the chiral states from the sector $b_j$. Interchangeably, if
such a vector does not exist, states from the sector $b_j$ will be non–chiral, i.e. there will be an equal number of 16 and 16 or 27 and 27. For example, we note that for the NAHE–set basis vectors the condition (2.2) is satisfied. In section (3) we will discuss the geometrical correspondence of this condition. The reduction to three generations in a specific model is illustrated in table 2.3.

$$\begin{array}{ccccccccc}
\alpha & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
\beta & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
\gamma & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\
\end{array}$$

(2.3)

In the realistic free fermionic models the vector $X$ is replaced by the vector $2\gamma$ in which $\{\bar{\psi}_1, \ldots, \bar{\psi}_5, \bar{\eta}_1, \bar{\eta}_2, \bar{\eta}_3, \bar{\phi}_1, \ldots, \bar{\phi}_4\}$ are periodic. This reflects the fact that these models have (2,0) rather than (2,2) world-sheet supersymmetry. At the level of the NAHE set we have 48 generations. One half of the generations is projected because of the vector $2\gamma$. Each of the three vectors in table 2.3 acts nontrivially on the degenerate vacuum of the fermionic states $\{y, \omega|\bar{y}, \bar{\omega}\}$ that are periodic in the sectors $b_1, b_2$ and $b_3$ and reduces the combinatorial factor of Eq. (2.1) by a half. Thus, we obtain one generation from each sector $b_1, b_2$ and $b_3$. By replacing the basis vectors $\alpha, \beta, \gamma$ with $\alpha + \beta, \alpha + \gamma$ and $\alpha + \beta + \gamma$, it is seen that the action on the internal coordinates of two of the basis vectors beyond the NAHE–set correspond to symmetric shifts, whereas the third corresponds to a fully asymmetric shift [13]. In sections (4–8) we will classify all the possible symmetric shifts on $Z_2 \times Z_2$ orbifolds.

**3 The $Z_2 \times Z_2$ correspondence**

In this section we elaborate on the correspondence between the free fermion models and $Z_2 \times Z_2$ orbifold. The aim is to set the stage for the analysis of the $Z_2 \times Z_2$ orbifold beyond the NAHE–set correspondence. In this respect we remark that the NAHE–set is a particular realization of the $Z_2 \times Z_2$ orbifold at a fixed point in the moduli space. However, its crucial property is precisely its correspondence with a $Z_2 \times Z_2$ orbifold. For many issues pertaining to the phenomenology of the relevant string vacua, it will prove beneficial to abandon the free fermionic realization and to resort to the bosonic, or geometrical description. In this respect the NAHE based free fermionic models merely indicate that the relevant geometrical structure for this class of models is that of the $Z_2 \times Z_2$ orbifold.
To translate the fermionic boundary conditions to twists and shifts in the bosonic formulation we bosonize the real fermionic degrees of freedom, \{y, \omega|\bar{y}, \bar{\omega}\}. Defining, 
\[ \xi_i = \sqrt{2}(y_i + i\omega_i) = -ie^{iX_i}, \eta_i = \sqrt{2}(y_i - i\omega_i) = -ie^{-iX_i} \]
with similar definitions for the right movers \{\bar{y}, \bar{\omega}\} and \[X^I(z, \bar{z}) = X^I(z) + X^I_\bar{h}(\bar{z})\]. With these definitions the world-sheet supercurrents in the bosonic and fermionic formulations are equivalent, \[T^a_{int} = \sum_i \chi_i y_i \omega_i = i \sum_i \chi_i \eta_i = \sum_i \chi_i \partial X_i. \]
The momenta \(P^I\) of the compactified scalars in the bosonic formulation are identical with the \(U(1)\) charges \(Q(f)\) of the unbroken Cartan generators of the four dimensional gauge group,
\[ Q(f) = \frac{1}{2} \alpha(f) + F(f) \]
where \(\alpha(f)\) are the boundary conditions of complex fermions \(f\), reduced to the interval \((-1, 1]\) and \(F(f)\) is a fermion number operator.

The extended NAHE–set model is generated in the orbifold language by modding out an \(SO(12)\) lattice by a \(Z_2 \times Z_2\) discrete symmetry with standard embedding [9]. The \(SO(12)\) lattice is obtained for special values of the metric and antisymmetric tensor and at a fixed point in compactification space. The metric is the Cartan matrix of \(SO(12)\) and the antisymmetric tensor is given by \(b_{ij} = g_{ij}\) for \(i > j\). The boundary condition vectors \(b_1\) and \(b_2\) translate into \(Z_2 \times Z_2\) twists on the bosons \(X_i\) and fermions \(\chi_i\) and to shifts on the gauge degrees of freedom. The massless spectrum of the resulting orbifold model consist of the untwisted sector and three twisted sectors, \(b_1, b_2\) and \(b_3\). Starting from the Narain model with \(SO(12) \times E_8 \times E_8\) symmetry [7], and applying the \(Z_2 \times Z_2\) twisting on the internal coordinates, we then obtain the orbifold model with \(SO(4)^3 \times E_6 \times U(1)^2 \times E_8\) gauge symmetry. There are eight fixed points in each twisted sector, yielding the 24 generations from the three twisted sectors. The three additional pairs of 27 and \(\bar{27}\) are obtained from the untwisted sector. This orbifold model exactly corresponds to the free-fermion model with the six-dimensional basis set \{1, S, \xi_1, \xi_2, b_1, b_2\}. The Euler characteristic of this model is 48 with \(h_{11} = 27\) and \(h_{21} = 3\). We refer to this model as \(X_2\).

This \(Z_2 \times Z_2\) orbifold, corresponding to the extended NAHE set, at the core of the realistic free fermionic models, differs from the one at a generic point in the moduli space. In that \(Z_2 \times Z_2\) orbifold model the Euler characteristic is equal to 96, or 48 generations, and \(h_{11} = 51\), \(h_{21} = 3\). We refer to this model as \(X_1\).

For the purpose of the analysis in section 8 it is instructive to discuss the connection between the \(X_1\) and \(X_2\) models. We consider here only symmetric shifts on the toroidal coordinates. We start by constructing the \(Z_2 \times Z_2\) at a generic point in the moduli space. The compactified \(T^2 \times T^2 \times T^3\) torus is parameterized by three complex coordinates \(z_1, z_2\) and \(z_3\), with the identification
\[ z_i = z_i + 1 \quad ; \quad z_i = z_i + \tau_i, \]
where \(\tau\) is the complex parameter of each torus \(T^2\). We consider \(Z_2\) twists and
possible shifts of order two:

\[ z_i \rightarrow (-1)^{\epsilon_i} z_i + \frac{1}{2} \delta_i, \]  

(3.2)

subject to the condition that \( \Pi_i (-1)^{\epsilon_i} = 1 \). This condition insures that the holomorphic three–form \( \omega = dz_1 \wedge dz_2 \wedge dz_3 \) is invariant under the \( Z_2 \) twist. Under the identification \( z_i \rightarrow -z_i \), a single torus has four fixed points at

\[ z_i = \{0, 1/2, \tau/2, (1 + \tau)/2\}. \]  

(3.3)

The first model that we consider is produced by the two \( Z_2 \) twists

\[
\alpha: (z_1, z_2, z_3) \rightarrow (-z_1, -z_2, z_3) \\
\beta: (z_1, z_2, z_3) \rightarrow (z_1, -z_2, -z_3).
\]  

(3.4)

There are three twisted sectors in this model, \( \alpha, \beta \) and \( \alpha \beta = \alpha \cdot \beta \), each producing 16 fixed tori, for a total of 48. The untwisted sector adds three additional Kähler and complex deformation parameters producing in total a manifold with \( (h_{11}, h_{21}) = (51, 3) \).

Next we consider the model generated by the \( Z_2 \times Z_2 \) twists in (3.4), with the additional shift

\[
\gamma: (z_1, z_2, z_3) \rightarrow (z_1 + \frac{1}{2}, z_2 + \frac{1}{2}, z_3 + \frac{1}{2}).
\]  

(3.5)

This model again has fixed tori from the three twisted sectors \( \alpha, \beta \) and \( \alpha \beta \). The product of the \( \gamma \) shift in (3.5) with any of the twisted sectors does not produce any additional fixed tori. Therefore, this shift acts freely. Under the action of the \( \gamma \) shift, half the fixed tori from each twisted sector are paired. Therefore, the action of this shift is to reduce the total number of fixed tori from the twisted sectors by a factor of \( 1/2 \). Consequently, in this model \( (h_{11}, h_{21}) = (27, 3) \). This model therefore reproduces the data of the \( Z_2 \times Z_2 \) orbifold at the free-fermion point in the Narain moduli space.

To facilitate the discussion of the subsequent examples, we briefly describe the calculation of the cohomology for this orbifold: a more complete discussion can be found in [14]. Consider first the untwisted sector. The Hodge diamond for a single untwisted torus is given by

\[
\begin{pmatrix}
1 & 1 \\
1 & 1
\end{pmatrix}
\]  

(3.6)

which displays the dimensions of the \( H^{p,q}(T_i) \), with \( H^{0,0}, H^{0,1}, H^{1,0} \) and \( H^{1,1} \) being generated by the differential forms \( 1, dz_i, dz_i \) and \( dz_i \wedge d\bar{z}_i \), respectively. Under the \( Z_2 \) transformation \( z \rightarrow -z \), \( H^{0,0} \) and \( H^{1,1} \) are invariant, whereas \( H^{1,0} \) and \( H^{0,1} \) change sign.

The untwisted sector of the manifold produced by the product of the three tori \( T_1 \times T_2 \times T_3 \) is then given by the product of differential forms which are invariant
under the $Z_2 \times Z_2$ twists $\alpha \times \beta$. The invariant terms are summarized by the Hodge diamond
\[
\begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & 3 & 3 & 0 \\
0 & 3 & 3 & 0 \\
1 & 0 & 0 & 1 \\
\end{pmatrix}
\] (3.7)

For example, $H^{1,1}$ is generated by $dz_i \wedge \bar{z}_i$ for $i = 1, 2, 3$, and $H^{2,1}$ is produced by $dz_1 \wedge z_2 \wedge \bar{z}_3, dz_2 \wedge z_3 \wedge \bar{z}_1, dz_3 \wedge z_1 \wedge \bar{z}_2$, etc.. We next turn to the twisted sectors, of which there are three, produced by $\alpha$, $\beta$ and $\alpha \beta$, respectively. In each sector, two of the $z_i$ are identified under $z_i \to -z_i$, and one torus is left invariant. We need then consider only one of the twisted sectors, say $\alpha$, and the others will contribute similarly. The sector $\alpha$ has 16 fixed points from the action of the twist on the first and second tori. Since the action is trivial on the third torus, we get 16 fixed tori. The cohomology is given by sixteen copies of the cohomology of $T_3$, where each $H^{p,q}$ of $T_3$ contributes $H^{p+1,q+1}$ to that of the orbifold theory [14]. The Hodge diamond from each twisted sector then has the form
\[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 16 & 16 & 0 \\
0 & 16 & 16 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}
\] (3.8)

It remains to find the forms from the $\alpha$ twisted sector which are invariant under the action of the $\beta$ twist. Since $z_3 \to -z_3$ under $\beta$, it follows that 1 and $dz_3 \wedge d\bar{z}_3$ are invariant, whereas $dz_3$ and $d\bar{z}_3$ are not. Consequently, only the contributions of $H^{1,1}$ and $H^{2,2}$ in (3.8) are invariant under the $\beta$ twist. Therefore, we see that the invariant contribution from each twisted sector is only along the diagonal of (3.8), and that the total Hodge diamond of the $Z_2 \times Z_2$ orbifold is
\[
\begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & 51 & 3 & 0 \\
0 & 3 & 51 & 0 \\
1 & 0 & 0 & 1 \\
\end{pmatrix}
\] (3.9)

Next we consider the model generated by the $Z_2 \times Z_2$ twists in (3.4), with the additional shift Eq. (3.5). This model again has fixed tori from the three twisted sectors $\alpha$, $\beta$ and $\alpha \beta$. The product of the $\gamma$ shift in (3.5) with any of the twisted sectors does not produce any additional fixed tori. Therefore, this shift acts freely. Under the action of the $\gamma$ shift, half the fixed tori from each twisted sector are paired. Therefore, the action of this shift is to reduce the total number of fixed tori from the twisted sectors by a factor of 1/2. Consequently, the Hodge diamond for this model is
\[
\begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & 27 & 3 & 0 \\
0 & 3 & 27 & 0 \\
1 & 0 & 0 & 1 \\
\end{pmatrix}
\] (3.10)
with \((h_{11}, h_{21}) = (27, 3)\). This model therefore reproduces the data of the \(Z_2 \times Z_2\) orbifold at the free-fermion point in the Narain moduli space.

Finally, let us consider the model generated by the twists (3.4) with the additional shift given by

\[
\gamma : (z_1, z_2, z_3) \rightarrow (z_1 + \frac{1}{2}, z_2 + \frac{1}{2}, z_3)
\]  

(3.11)

This model, denoted by \(X_3\), again has three twisted sectors \(\alpha\), \(\beta\) and \(\alpha\beta\). Under the action of the \(\gamma\) shift, half of the fixed tori from these twisted sectors are identified. These twisted sectors therefore contribute to the Hodge diamond as in the previous model. However, the \(\gamma\) shift in (3.11) does not act freely, as its combination with \(\alpha\) produces additional fixed tori, since, under the action of the product \(\alpha \cdot \gamma\), we have

\[
\alpha \gamma : (z_1, z_2, z_3) \rightarrow (-z_1 + \frac{1}{2}, -z_2 + \frac{1}{2}, z_3)
\]  

(3.12)

This sector therefore has 16 additional fixed tori. Repeating the analysis as in the previous cases, we see that, under the identification imposed by the \(\alpha\) and \(\beta\) twists, the invariant states from this sector give rise to four additional (1,1) forms and four additional (2,1) forms. The Hodge diamond for this model therefore has the form

\[
\begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & 31 & 7 & 0 \\
0 & 7 & 31 & 0 \\
1 & 0 & 0 & 1
\end{pmatrix}
\]  

(3.13)

with \((h_{11}, h_{21}) = (31, 7)\).

String theory allows more complicated operations that involve shifts of momentum, winding, or both [12]. Whereas the first is symmetric between left and right movers, the last is asymmetric. In fact, it was shown in [9] that the \(SO(12)\) lattice at the free fermionic point is reproduced by such an asymmetric shift, that differs from (3.5). These different freely acting shifts induce the same projection on the massless spectrum, but the partition function and the massive spectrum differ. We can regard all the resulting quotient manifolds as existing in the same moduli space, that may be connected by continuous extrapolations. In this paper we restrict the analysis to symmetric shifts, in the spirit that the free fermionic models merely reveal the central role of the \(Z_2 \times Z_2\) orbifold. Thus, our aim is to promote the understanding of the geometry of the \(Z_2 \times Z_2\) orbifolds, and in particular with respect to the phenomenological features exhibited by the free fermionic models. Namely, in respect to the fashion in which the three massless chiral generations arise in the free fermionic models.

In this regard, in the orbifold picture we start with the \((51,3) Z_2 \times Z_2\) orbifold. We can regard each twisted sector as generated by a two coordinate base and one coordinate fiber, with the twists acting on the coordinates of the base. Above each of the 16 fixed points we then have the untwisted fiber which is an unfixed torus. We
can then imagine that in this picture the reduction to three generations entails the
reduction of the number of independent fixed point, by identifying points on the base
by shifts. Indeed, the action of (3.5) is precisely such a reduction of the number of
twisted fixed points from 48 to 24. Thus, we can imagine imposing additional shift
operations that will reduce the number of fixed points further. From the analysis
of (3.5) and (3.11) we note that the additional shifts can be freely acting or non–
freely acting. In the following we perform a complete classifications of all possible
symmetric shifts and prove that a reduction to three fixed points is not possible
solely with symmetric shifts. This result concurs with the free fermionic analysis that
indicates that at least one asymmetric operation is required to reduce the number of
families to three generations [13].

4 Classification of low rank orbifolds

We are interested in orbifolds of the form

$$X = (E_1 \times E_2 \times E_3)/G$$

(4.1)

Here the three $E_i$ are elliptic curves, or topologically they are tori $T^2$. The group $G$
contains the $(Z_2)^2$ of twists, as well as a subgroup $\mathcal{G}$, which acts on $E_1 \times E_2 \times E_3$ by
translations. In this work we consider only translations of order 2. This means that
our $\mathcal{G}$ is a subgroup of the group

$$E_1[2] \times E_2[2] \times E_3[2] \approx (Z_2)^6$$

(4.2)

of points of order 2 on the elliptic curves. Here $E_i[2]$ is the group $(Z_2)^2$ of points of
order 2 in $E_i$. We denote its four elements by: 0, 1, $\tau$, $1 + \tau$ (they should be more
accurately labeled as $1/2$, $\tau/2$, $(1 + \tau)/2$, but the notation would then get out of
hand). The full group $G$ is an extension

$$0 \to (Z_2)^2 \to G \to \mathcal{G} \to 0,$$

(4.3)

i.e. it contains $(Z_2)^2$ as a subgroup and the quotient $G/(Z_2)^2$ is $\mathcal{G}$. In general, this
extension can be non–trivial. However, in our case of translations of order 2, the
entire $G$ is commutative, so the extension is really a product,

$$G \approx (Z_2)^2 \times \mathcal{G}.$$  

(4.4)

The obvious invariant of a group $\mathcal{G}$ is its rank $r$, i.e. the number of its generators
(or its dimension as a vector space over $Z_2$). The set of all subgroups $\mathcal{G} \subset (Z_2)^6$ of a
given rank $r$ is called a (finite) Grassmannian $Gr_{Z_2}(r, 6)$. The number of subgroups
of rank $r$ is given by:

$$\prod_{i=1}^{r} \frac{64 - 2(i-1)}{2r - 2(i-1)}.$$  

(4.5)
This is seen by noting that the first generator of $G$ can be any of the 63 non–zero elements. Having chosen it, the second generator cannot be a multiple, so has only $62 = 64 - 2$ options. The third generator needs to avoid the four elements, which are combinations of the first two, hence $60 = 64 - 4$ options, and so on. But each subgroup has now been counted as many times as we can choose bases for it. The choice of a basis amounts to $2^r - 1$ possibilities for the first vector, $2^r - 2$ for the second, and so on.

Explicitly, these numbers are

| $r$ | #  |
|-----|----|
| 0   | 1  |
| 1   | 63 |
| 2   | 651|
| 3   | 1395|
| 4   | 651|
| 5   | 63 |
| 6   | 1  |

Fortunately, we do not need to consider that many cases. The permutation group $S_3$ acts on the 3 non–zero elements of each $E_i[2]$. This gives a total symmetry group of $(S_3)^3$. In addition, one more copy of $S_3$ acts on the product $E_1[2] \times E_2[2] \times E_3[2]$ by permuting the 3 tori. The full symmetry group $S$ is therefore an extension of $S_3$ by $(S_3)^3$:

$$0 \rightarrow (S_3)^3 \rightarrow S \rightarrow S_3 \rightarrow 0.$$  \hfill (4.7)

A group of rank 1 is uniquely specified by a single non–zero element $(a_1, a_2, a_3) \in E_1[2] \times E_2[2] \times E_3[2]$. The 63 original possibilities are reduced by $S$ to just 3 equivalence classes, namely those of,

$$\begin{align*}
(1,1,1); \\
(1,1,0); \\
(1,0,0).
\end{align*} \hfill (4.8)$$

Indeed, we use the overall $S_3$ to move the non–zero $a_i$ to the left and the zeroes to the right. Then we use the individual symmetries $S_3^i$ of the $E_i$ to change each non–zero entry to a 1, resulting in just the above three classes. Note that the first of these groups corresponds to the $(27,3)$ model, denoted $X_2$ in section 3. The $(51,3)$ model, denoted $X_1$ in section 3, corresponds to the unique $G$ of rank 0. The Hodge numbers of the other two groups listed above can be computed according to rules explained in section 6 below. They are $(31,7)$ for $(1,1,0)$ and $(51,3)$ for $(1,0,0)$.

With a bit of patience, one can similarly work out the complete list of rank 2
Next to each group we itemized the number of zero entries in each of its three non–zero elements. There is only one pair (the 6th and 7th groups) with the same distribution of zeroes, but these two are easily seen to be non–equivalent anyway (e.g. because the zeroes are always in the third entry for group # 6, but in varying entries for # 7). In the last column we listed the Hodge numbers \( h^{11}, h^{21} \) for each orbifold \( T^6/G \). The rules for calculating these Hodge numbers are explained in section 6 below.

## 5 Classification of rank 3 orbifolds

A subgroup \( G \) of rank 3 contains 7 non–zero elements and 7 rank 2 subgroups. Each rank 2 subgroup contains 3 non–zero elements, and each non–zero element belongs to 3 rank 2 subgroups. We can display the situation by the following diagram:

The vertices represent non–zero elements while the 6 edges and the circle represent rank 2 subgroups. To each rank 3 group \( G \) we assign a decorated diagram, e.g. for the group \( G : (1, 1, 1), (1, 0, 0), (\tau, 0, 0) \) the decorated diagram is:

![Decorated Diagram](image-url)
The 0 on the top vertex gives the number of zero entries in the first generator \((1, 1, 1)\). The bottom left and right vertices correspond to those in \((1, 0, 0)\) and \((\tau, 0, 0)\). The entries on the edges correspond similarly to the other group elements.

The quotients of the \((27, 3)\) model \(X_2\) correspond to subgroups \(\tilde{G}\) which contain the element \((1, 1, 1)\). We can work out the complete list, which is displayed in figure 1.

| (1,1,1) (1,0,0) (τ,0,0) (31,7) | (1,1,1) (1,0,0) (0,0,1) (51,3) |
|-------------------------------|-------------------------------|
| 1 1 0                          | 1 0 2                         |
| 0 0 2                          | 2 1 0                         |

| (1,1,1) (1,0,0) (0,0,τ) (31,7) | (1,1,1) (τ,0,0) (0,0,τ) (27,3) |
|-------------------------------|-------------------------------|
| 0 1 1                          | 1 0 0                         |
| 2 1 0                          | 0 2 1                         |

| (1,1,1) (1,0,0) (τ,1,0) (27,3) | (1,1,1) (1,0,0) (τ,τ,0) (19,7) |
|-------------------------------|-------------------------------|
| 0 1 1                          | 1 0 0                         |
| 1 1 0                          | 0 1 0                         |

| (1,1,1) (1,0,0) (0,τ,τ) (21,9) | (1,1,1) (τ,τ,τ) (17,5) |
|-------------------------------|-------------------------------|
| 0 1 1                          | 1 0 0                         |
| 1 1 0                          | 0 1 0                         |

| (1,1,1) (τ,1,0) (1,τ,0) (17,5) | (1,1,1) (τ,1,0) (0,τ,τ) (12,6) |
|-------------------------------|-------------------------------|
| 0 1 1                          | 1 0 0                         |
| 1 1 1                          | 0 1 0                         |

| (1,1,1) (0,τ,τ) (τ,0,τ) (15,3) |
|-------------------------------|
| 0 0 0                          |
| 1 1 0                          |

Figure 1: rank 3 classification
Clearly no two of these eleven subgroups can be equivalent, as they all have distinct decorated diagrams. This classification can be extended to cover the additional subgroups which do not contain \((1, 1, 1)\). A cleaner way to see this will emerge shortly.

\section{Rules for orbifold cohomology}

Having worked out some examples of groups \(G\), we need to compute the orbifold cohomology of the quotient \(T^6/G\). The general rule is:

\[ H^*(T^6/G) = \oplus_{g \in G} H^*(T^g)^G \]  

(6.1)

Here \(T^g\) is the fixed locus of \(g \in G\) acting on \(T^6\), \(H^*(T^g)\) is its cohomology, and the superscript \(G\) denotes cohomology classes on \(T^g\) which are invariant under the action of \(G\) on \(T^g\).

We describe an element \(g \in G\) by the data \((\epsilon_1, \epsilon_2, \epsilon_3)(a_1, a_2, a_3)\). Here each \(\epsilon_i \in \pm 1\), \(\Pi_{i=1}^3 \epsilon_i = +1\), and each \(a_i\) is in \(E_3[2]\). First let us describe the fixed locus \(T^g\):

- An element \(g = (+, +, +)(a_1, a_2, a_3) \in G\) acts on \(T^6\) by translations. Therefore \(T^g = T^6\) if \(g = 0\), and \(T^g = \emptyset\) otherwise.
- An element \(g = (−, −, +)(a_1, a_2, a_3)\) sends \((x_1, x_2, x_3) \in T^6 = E_1 \times E_2 \times E_3\) to \((a_1 − x_1, a_2 − x_2, a_3 + x_3)\). So the fixed points are of the form \((x_1 = a_1/2, x_2 = a_3/2, x_3\) arbitrary), and the fixed locus consists of \(4 \times 4 = 16\) copies of \(T^2\), if \(a_3 = 0\); otherwise, \(T^g = \emptyset\).

Next, we need the action of each element \(h \in G\) on \(T^g\), in those cases where \(T^g \neq \emptyset\).

- For \(g = 0\), we already saw that the \((Z_2)^2\) invariants in \(H^*(T^6)\) contribute a 3-dimensional space to each of the Hodge groups \(H^{11}, H^{21}, H^{12}, H^{22}\). The action of any translation \(g \in G\) is trivial on these spaces of invariants. We abbreviate this by saying that the \(g = 0\) sector contributes \((3, 3)\) to the orbifold cohomology.
- We need the contribution to orbifold cohomology of \(g = (−, −, +)(a_1, a_2, 0)\).

Consider the subgroup

\[ \hat{G}_3 = \overline{G} \cap (0, 0, E_3[2]) \]  

(6.2)

consisting of elements of \(\overline{G}\) for which \(a_1 = a_2 = 0\). Let \(\rho_3\) denote the rank of \(\hat{G}_3\); it equals 0, 1 or 2. Each of the \(2^r\) translations \(h \in \overline{G}\) permutes the \(16 = 2^4\) components of \(T^g\). This permutation is trivial if and only if \(h \in \hat{G}_3\). Therefore, the quotient of \(T^g\) by the action of \(\overline{G}\) consists of \(2^{4−r+\rho}\) tori.

We still have to impose invariance under \((Z_2)^2\). Now the action of the twist \((−, −, +)\) on \(T^g\) coincides with the action of the translation \((a_1, a_2, 0)\) for which we have already accounted. So we are left solely with invariance under the action of the \((−, +, −)\) twist. This action sends a component labeled \((a_1/2, a_3/2)\) to the component labeled \((a_1/2 + a, a_3/2)\), \(i.e.\) it shifts the labels by \((a_1 − 1, 0)\). There are therefore two possibilities:
• If the projection of $\overline{G}$ to the first two factors $E_1[2] \times E_2[2]$ contains the element $(a_1, 0)$, then the action of $(-, +, -)$ relates two tori which had already been glued previously by the action of the translation group $\overline{G}$. So it acts as the $\pm$ involution on each of these $2^{4-r+p_2}$ components. The result is $2^{4-r+p_2}$ copies of the quotient $T^2/(\pm 1) = P^1$. So the contribution to orbifold cohomology is $(2^{4-r+p_2}, 0)$.

• Otherwise, the action of $(-, +, -)$ relates pairs of tori which had previously been unrelated, resulting in $2^{3-r+p_3}$ tori $T^2$. The contribution to orbifold cohomology is therefore $(2^{3-r+p_3}, 2^{3-r+p_3})$.

To summarize, the contributions of the various sectors to orbifold cohomology are:

$$
g = 0 \quad (3, 3)$$

$$
g \in \overline{G}, \ g \neq 0 \quad (0, 0)$$

$$
g = (-, -, +)(a_1, a_2, a_3) \Rightarrow$$

if $a_3 \neq 0$ \hspace{2cm} (6.3)

if $a_3 = 0$ and $(a_1, 0, b) \in \overline{G}$ for some $b$ \hspace{2cm} $(2^{4-r+p_2}, 0)$

if $a_3 = 0$ but $(a_1, 0, b) \notin \overline{G}$ for any $b$ \hspace{2cm} $(2^{3-r+p_3}, 2^{3-r+p_3})$

Finally, it is convenient to add the contributions of the four elements $g$ corresponding to each $\bar{g} \in \overline{G}$ combined with the possible twists:

$$
al a_i \neq 0 \quad (0, 0)$$

$$a_1 \neq 0, a_2 \neq 0, a_3 = 0$$

if $(a_1, 0, b) \in \overline{G}$ for some $b$ \hspace{2cm} $(2^{4-r+p_2}, 0)$

otherwise \hspace{2cm} $(2^{3-r+p_3}, 2^{3-r+p_3})$ (6.4)

$$a_1 \neq 0, a_2 = a_3 = 0$$

$(2^{3-r}(2^{p_2} + 2^{p_3}), 0)$

$$a_1 = a_2 = a_3 = 0$$

$(3 + 2^{4-r}(2^{p_1} + 2^{p_2} + 2^{p_3}), 3)$

It is routine to apply these rules to each of the groups $G$ encountered so far. The resulting Hodge numbers were tabulated in sections 4 and 5.

7 A reduction

In principle, we could list all subgroups of ranks 4, 5, 6 as we did for lower ranks, and we could compute their Hodge numbers according to the rules in section 6. However, there is a shortcut: If our translation subgroup $\overline{G}$ contains an element $g = (a_1, a_2, a_3)$ where exactly one of the $a_i$’s (say $a_1$) is non–zero then there is another subgroup $G'$ of lower rank such that the orbifolds $T^6/G$ and $T^6/G'$, where $G'$ is the corresponding extension of $\overline{G}$, are topologically equivalent, and in fact live in the same moduli space. This reduces the calculation of the Hodge numbers for $G$ to those for the smaller group $G'$. 

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The reason for this is that the action of \( (Z_2)^2 \) commutes with the translation \( g \):

Note that when more than one of the \( a_i \) is non-zero, we still have a commutative diagram, but the quotient \( (E_1 \times E_2 \times E_3)/g \) cannot be naturally identified as a product of three \( T^2 \) factors. (In algebro geometric language, the quotient is isogenous, but not isomorphic, to the product of 3 elliptic curves.)

We conclude that our \( X = (E_1 \times E_2 \times E_3)/G \) can also be described as \( (X_1/g)/G' \), where \( G' \) is the image of \( G \) which acts on \( X_1/g \). It is a subgroup of \( (E_1/a_1)[2] \times E_2[2] \times E_3[2] \) of rank \( r - 1 \). Since the Hodge numbers of \( X = X_G \) depend only on the group \( G' \) (and not on the particular elliptic curves used), we see that \( H^*(X_G) = H^*(X_G') \). Inductively we can therefore assume that our group \( G' \) contains no elements with a single non-zero entry.

8 Rank \( \geq 4 \)

We are therefore led to study the groups of translations \( G \approx Z_2^3 \) contained in \( Z_2^6 = H^1(T^6, Z_2) \), with the property that: \( \forall g \in G, g \neq 0, g = (a_1, a_2, a_3), a_i \in E_i[2] \), at most one \( a_i = 0 \).

We claim that there exists such a group, and that it is in fact unique up to natural equivalences. We then take the \((51,3)\) model modulo \( G' \), which is a \( Z_2^3 \) quotient of the \((27,3)\) model. Our group \( G' \) has 9 elements with fixed points, each has a fixed point locus \( \approx Z_2 \times \text{elliptic} \) (e.g. \( g = 1/2(0, \tau, 1) \) sends \( (x, y, z) \leftrightarrow (x, y + \tau/2, z + 1/2) \), so the fixed points are \( (x, \tau/4, 1/4) \), which gives \( 4 \times 4 = 16 \) fixed tori mod 8 identifications). We still need to mod out by the \( Z_2^3 \) translations. One of these acts trivially on its own fixed points. The remaining \( Z_2^3 \)'s act as follows: the first interchanges the two tori, and the other acts as \( z \to -z \), so the fixed torus degenerates into \( P^1 \). The resulting Hodge numbers are \((15,3)\).

We proceed to classify all such subgroups of \( G \) which are generated by 4 vectors, one of which is \((3.5)\). In each \( E_i \), the projection of \( G \) must be all of \( E_i[2] \). Otherwise \( G \) contains 8 elements projecting to 0 in \( E_i \). These 8 elements then form a hyperplane in \( (Z_2)^4 \), given as perpendicular of some vector in \( (Z_2)^4 \), say \((a, b, c, d)\), but then \((0, 0, d, c)\) or \((b, a, 0, 0)\) is a non-zero vector in \( Z_2^3 \) with 0 in another \( E_j \). We conclude that we can take the third and fourth basis vectors of \( G \) to project to 0 in \( E_1 \), and the second vector projects to \( \tau \)

\[
G = 1/2\{(1,1,1); \}
\]
\((\tau, x_3, a);\) \\
\((0, x_1, b);\) \\
\((0, x_2, c)\) \quad (8.1)

Now, \(x_1\) and \(x_2\) can be taken to be \(\tau\) and 1. Then by subtracting multiples of the third and fourth vectors from the second, it follows that we can take \(x_3 = 0\). It remains to impose the conditions on \(a\), \(b\) and \(c\). These are

\[a \neq 0; b \neq 0; c \neq 0; c \neq b; c \neq a + 1; c \neq 1\] \quad (8.2)

If these hold, no vector of \(Z_4^2\) has 0’s in \(E\) and another factor, since the vectors with 0 in \(E\) are combinations of last two vectors and in columns 2,3 these each have 2 distinct non zero entries. Last case to exclude is zeroes in column 2 + 3. The vectors with zeroes in column 2 are \(v_2, v_1 + v_4\) and \(v_1 + v_2 + v_4\). The third entries are then \(a, 1 + c, 1 + a + c\), which impose the condition in eq. (8.2).

We now proceed to compute the full set of solutions for \(a\), \(b\) and \(c\).

| \(a\) | \(b\) | \(c\) |
|------|------|------|
| 1    | 1    | \(\tau\) |
| 1    | 1    | \(\tau + 1\) |
| 1    | \(\tau\) | \(\tau + 1\) |
| 1    | \(\tau + 1\) | \(\tau\) |
| \(\tau\) | 1    | \(\tau\) |
| \(\tau\) | \(\tau + 1\) | \(\tau\) |
| \(\tau + 1\) | 1    | \(\tau + 1\) |
| \(\tau + 1\) | \(\tau\) | \(\tau + 1\) |

These solutions are invariant under two operations. One involves the interchange of columns \(E_1\) and \(E_2\) with \(a \leftrightarrow b\) and \(c \leftrightarrow c + 1\). The second operation interchanges \(\tau \leftrightarrow \tau + 1\) and replaces \(a \leftrightarrow a + c + 1\), but only in column \(E_1\). These two operations can be seen to mix all eight solutions and therefore the solution is unique up to equivalences.

Since the rank 4 group is unique, it follows that every group \(\overline{G}\) of higher rank can be reduced to rank 4 or less.

## 9 The complete list

Starting with the \((51,3)\) model and analyzing the complete set of models that are obtained by identifying fixed points on the three complex tori by shifts, we analyze the complete set of models that are obtained from the \(Z_2 \times Z_2\) orbifold on a product of three complex tori. The complete set of models is given in table (9.1).
Subgroups of $E[2]^3$ free of single-entry elements

We note that the model with $|h_{11} - h_{21}| = 3$, that would correspond to the three generation case, does not arise in this classification. We conclude that the $Z_2 \times Z_2$ orbifold cannot produce three generations solely with symmetric order 2 shift identifications on the three internal complex tori.

10 The chirality condition

We now discuss a geometric picture of the chirality condition (2.2) that was discussed in section 2. We examine the fixed points of an element $(a, b, 0)$, with $a$ and $b$ of order 2, i.e. $(\epsilon + a/2, \delta + b/2, z)$. Under the action of the two twists the torus with parameters $(\epsilon, \delta)$ is shifted under $(-, +, -)$ by $(a, 0)$ and under $(+, -, -)$ by $(0, b)$. The chirality question is whether one of these is a chirality projection of a group element in $G$. Geometrically, we are trying to change the difference between the Hodge numbers, $h^{1,1} - h^{2,1}$. This can happen only when an involution acts on the $T^2$ above some fixed point as $-1$, so in the quotient this $T^2$ is replaced by a $\mathbb{P}^1$; this preserves $h^{1,1}$ but reduces $h^{2,1}$ by 1.

For an element of the form $(a, 0, 0)$, the fixed points in $T^6/(Z_2)^2$ are $(a/2 + \epsilon, z, \delta)$ and $(a/2 + \epsilon, \delta, z)$. In the resolution, these curves no longer intersect. The actions of the three non-trivial elements are:

$(-, +, -): (a, 0, 0)$ on either curve, with twist $z \rightarrow -z$ on second curve.
$(-, -, +): (a, 0, 0)$ on either curve, with twist $z \rightarrow -z$ on first curve.
$(+,−,−)$: sends each curve to itself with $z \to -z$.

We see that in this case the action is always chiral. We conclude that the quotient is non–chiral iff: 1) no element has two zeroes. 2) If an element has a zero, e.g. $(a, b, 0)$, then $(a, 0, c)$ and $(0, b, c)$ are not in the group for any $c$. This is therefore the translation of the chirality condition to the geometric language.

11 Conclusion

We have demonstrated in this paper that quotients of the $Z_2 \times Z_2$ orbifold of a product of three $T^2$ tori by additional identifications by shifts of order 2 on the three complex tori cannot reduce the net number of generations to three. The motivation for this analysis stems from the relation of these geometries to the free fermionic heterotic string models and the manner in which three generations are obtained in these string models. Namely, there each one of the twisted sectors produces a net number of one generation and it is therefore of interest to explore the geometrical correspondence of this picture. A complementary analysis, performed by using the free fermionic techniques [13], reaches the same conclusion. Namely, the three generation cases cannot be obtained by utilizing solely left–right symmetric shifts on the complex tori, but necessarily involve an asymmetric projection. This observation may have far reaching implication on the issue of moduli stabilization and vacuum selection. The reason is that the asymmetric operation cannot be performed at an arbitrary point in the moduli space, but has to be performed at special points. Extending the analysis of the $Z_2 \times Z_2$ orbifold class models to the non–perturbative regime, along the lines of ref. [15], will be facilitated by starting from the $X_1$ manifold as the internal Calabi–Yau and solving the anomaly constraints in the case of non–standard embedding and in the presence of five branes. This will elucidate how and whether the $Z_2 \times Z_2$ reasoning for the origin of the three generations is modified in the nonperturbative regime. Similarly understanding the implication of the asymmetric operation in the context of strong–weak transformation is of further interest. We should note that the type of geometries that correspond to the left–right asymmetric action are not yet readily understood.

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