Two-sided inequalities
for the density function’s maximum
of weighted sum of chi-square variables

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Abstract. Two–sided bounds are constructed for a probability density
function of a weighted sum of chi-square variables. Both cases of cen-
tral and non-central chi-square variables are considered. The upper and
lower bounds have the same dependence on the parameters of the sum
and differ only in absolute constants. The estimates obtained will be use-
ful, in particular, when comparing two Gaussian random elements in a
Hilbert space and in multidimensional central limit theorems, including
the infinite-dimensional case.

Keywords: two–sided bounds, weighted sum, chi-square variable, Gaus-
sian element

1 Introduction

In many statistical and probabilistic applications, we have to solve the problem
of Gaussian comparison, that is, one has to evaluate how the probability of a ball
under a Gaussian measure is affected, if the mean and the covariance operators
of this Gaussian measure are slightly changed. In \cite{1} we present particular exam-
pies motivating the results when such “large ball probability” problem naturally
arises, including bootstrap validation, Bayesian inference and high-dimensional
CLT, see also \cite{2} and \cite{3}. The tight non-asymptotic bounds for the Kolmogorov
distance between the probabilities of two Gaussian elements to hit a ball in a
Hilbert space have been derived in \cite{1} and \cite{4}. The key property of these bounds
is that they are dimension-free and depend on the nuclear (Schatten-one) norm
of the difference between the covariance operators of the elements and on the
norm of the mean shift. The obtained bounds significantly improve the bound
based on Pinsker’s inequality via the Kullback–Leibler divergence. It was also
established an anti-concentration bound for a squared norm $\|Z-a\|^2$, $a \in H$, of a shifted Gaussian element $Z$ with zero mean in a Hilbert space $H$. The decisive role in proving the results was played by the upper estimates for the maximum of the probability density function $g(x,a)$ of $\|Z-a\|^2$, see Theorem 2.6 in [1]:

$$\sup_{x \geq 0} g(x,a) \leq c (A_1 A_2)^{-1/4}, \quad (1)$$

where $c$ is an absolute constant and

$$A_1 = \sum_{k=1}^{\infty} \lambda_k^2, \quad A_2 = \sum_{k=2}^{\infty} \lambda_k^2$$

with $\lambda_1 \geq \lambda_2 \geq \ldots$ are the eigenvalues of a covariance operator $\Sigma$ of $Z$.

It is well known that $g(x,a)$ can be considered as a density function of a weighted sum of non-central $\chi^2$ distributions. An explicit but cumbersome representation for $g(x,a)$ in finite dimensional space $H$ is available (see, e.g., Section 18 in Johnson, Kotz and Balakrishnan [5]). However, it involves some special characteristics of the related Gaussian measure which makes it hard to use in specific situations. Our result (1) is much more transparent and provide sharp uniform upper bounds. Indeed, in the case $H = \mathbb{R}^d$, $a = 0$, $\Sigma$ is the unit matrix, one has that the distribution of $\|Z\|^2$ is the standard $\chi^2$ with $d$ degrees of freedom and the maximum of its probability density function is proportional to $d^{-1/2}$. This is the same as what we get in (1).

At the same time, it was noted in [1] that obtaining lower estimates for $\sup_x g(x,a)$ remains an open problem. The latter problem was partially solved in [6], Theorem 1. However, it was done under additional conditions and we took into account the multiplicity of the largest eigenvalue.

In the present paper we get two–sided bounds for $\sup_x g(x,0)$ in the finite-dimensional case $H = \mathbb{R}^d$, see Theorem 2 below. The bounds are dimension-free, that is they do not depend on $d$. Thus, for the upper bounds (1), we obtain a new proof, which is of independent interest. And new lower bounds show the optimality of (1), since the upper and lower bounds differ only in absolute constants. Moreover, new two-sided bounds are constructed for $\sup_x g(x,a)$ with $a \neq 0$ in the finite-dimensional case $H = \mathbb{R}^d$, see Theorem 3 below. Here we consider a typical situation, where $\lambda_1$ does not dominate the other coefficients.

## 2 Main results

For independent standard normal random variables $Z_k \sim N(0,1)$, consider the weighted sum

$$W_0 = \lambda_1 Z_1^2 + \ldots + \lambda_n Z_n^2, \quad \lambda_1 \geq \ldots \geq \lambda_n > 0.$$  

It has a continuous probability density function $p(x)$ on the positive half-axis. Define the functional

$$M(W_0) = \sup_{x} p(x).$$
Theorem 1. Up to some absolute constants \( c_0 \) and \( c_1 \), we have
\[
c_0(A_1A_2)^{-1/4} \leq M(W_0) \leq c_1(A_1A_2)^{-1/4},
\] (2)
where
\[
A_1 = \sum_{k=1}^{n} \lambda_k^2, \quad A_2 = \sum_{k=2}^{n} \lambda_k^2
\]
and
\[
c_0 = \frac{1}{4e^2\sqrt{2\pi}} > 0.013, \quad c_1 = \frac{2}{\sqrt{\pi}} < 1.129.
\]

Theorem 1 can be extended to more general weighted sums:
\[
W_a = \lambda_1(Z_1 - a_1)^2 + \ldots + \lambda_n(Z_n - a_n)^2
\] (3)
with parameters \( \lambda_1 \geq \ldots \geq \lambda_n > 0 \) and \( a = (a_1, \ldots, a_n) \in \mathbb{R}^n \).

It has a continuous probability density function \( p(x,a) \) on the positive half-axis \( x > 0 \). Define the functional
\[
M(W_a) = \sup_{x} p(x,a).
\]

Remark. It is known that for any non-centred Gaussian element \( Y \) in a Hilbert space, the random variable \( ||Y||^2 \) is distributed as \( \sum_{i=1}^{\infty} \lambda_i(Z_i - a_i)^2 \) with some real \( a_i \) and \( \lambda_i \) such that
\[
\lambda_1 \geq \lambda_2 \geq \ldots \geq 0 \quad \text{and} \quad \sum_{i=1}^{\infty} \lambda_i < \infty.
\]
Therefore, the upper bounds for \( M(W_a) \) immediately imply the upper bounds for the probability density function of \( ||Y||^2 \).

Theorem 2. If \( \lambda_1^2 \leq A_1/3 \), then one has a two-sided bounds
\[
\frac{1}{4\sqrt{3}} A_1 \leq M(W_a) \leq \frac{2}{\sqrt{A_1 + B_1}},
\]
where
\[
A_1 = \sum_{k=1}^{n} \lambda_k^2, \quad B_1 = \sum_{k=1}^{n} \lambda_k^2 a_k^2.
\]

Moreover, the left inequality holds without any assumptions on \( \lambda_1^2 \).

Remark. In Theorem 2 we only consider a typical situation, where \( \lambda_1 \) does not dominate the other coefficients. Moreover, the condition \( \lambda_1^2 \leq A_1/3 \) necessarily implies that \( n \geq 3 \). If this condition is violated, the behaviour of \( M(W_a) \) should be studied separately.
3 Auxiliary results

For the lower bounds in the theorems, one may apply the following lemma, which goes back to the work by Statulyavichus [7], see also Proposition 2.1 in [8].

**Lemma 1.** Let \( \eta \) be a random variable with \( M(\eta) \) denoting the maximum of its probability density function. Then one has

\[
M^2(\eta) \text{Var}(\eta) \geq \frac{1}{12}.
\]

(4)

Moreover, the equality in (4) is attained for the uniform distribution on any finite interval.

**Remark.** There are multidimensional extensions of (4), see e.g. [9], [10] and Section III in [11].

**Proof.** Without loss of generality we may assume that \( M(\eta) = 1 \).

Put \( H(x) = P(|\eta - E\eta| \geq x), \quad x \geq 0 \).

Then, \( H(0) = 1 \) and \( H'(x) \geq -2 \), which gives \( H(x) \geq 1 - 2x \), so

\[
\text{Var}(\eta) = 2 \int_0^\infty xH(x) \, dx \geq 2 \int_0^{1/2} xH(x) \, dx \geq 2 \int_0^{1/2} x(1 - 2x) \, dx = \frac{1}{12}.
\]

Lemma is proved.

The following lemma will give the lower bound in Theorem 2.

**Lemma 2.** For the random variable \( W_a \) defined in (3), the maximum \( M(W_a) \) of its probability density function satisfies

\[
M(W_a) \geq \frac{1}{4\sqrt{3}} \frac{1}{\sqrt{A_1 + B_1}},
\]

(5)

where

\[
A_1 = \sum_{k=1}^n \lambda_k^2, \quad B_1 = \sum_{k=1}^n \lambda_k^2 a_k^2.
\]

**Proof.** Given \( Z \sim N(0, 1) \) and \( b \in \mathbb{R} \), we have

\[
E(Z - b)^2 = 1 + b^2, \quad E(Z - b)^4 = 3 + 6b^2 + b^4,
\]

so that \( \text{Var}((Z - b)^2) = 2 + 4b^2 \). It follows that

\[
\text{Var}(W_a) = \sum_{k=1}^n \lambda_k^2 (2 + 4a_k^2) = 2A_1 + 4B_1 \leq 4(A_1 + B_1).
\]

Applying (4) with \( \eta = W_a \), we arrive at (5).

Lemma is proved.

The proofs of the upper bounds in the theorems are based on the following lemma.
Lemma 3. Let 
\[ \alpha_1^2 + \ldots + \alpha_n^2 = 1. \]
If \( \alpha_k^2 \leq 1/m \) for \( m = 1, 2, \ldots \), then the characteristic function \( f(t) \) of the random variable 
\[ W = \alpha_1 Z_1^2 + \ldots + \alpha_n Z_n^2 \]
satisfies 
\[ |f(t)| \leq \frac{1}{(1 + 4t^2/m)^{m/4}}. \] (6)
In particular, in the cases \( m = 4 \) and \( m = 3 \), \( W \) has a bounded density with 
\( M(W) \leq 1/2 \) and \( M(W) < 0.723 \) respectively.

Proof. Necessarily \( n \geq m \). The characteristic function has the form 
\[ f(t) = \prod_{k=1}^{n} (1 - 2\alpha_k t)^{-1/2}, \]
so
\[ -\log |f(t)| = \frac{1}{4} \sum_{k=1}^{n} \log(1 + 4\alpha_k^2 t^2). \]

First, let us describe the argument in the simplest case \( m = 1 \).
For a fixed \( t \), consider the concave function 
\[ V(b_1, \ldots, b_n) = \sum_{k=1}^{n} \log(1 + 4b_k t^2) \]
on the simplex 
\[ Q_1 = \{ (b_1, \ldots, b_n) : b_k \geq 0, \ b_1 + \ldots + b_n = 1 \}. \]
It has \( n \) extreme points \( b^k = (0, \ldots, 0, 1, 0, \ldots, 0) \). Hence 
\[ \min_{b \in Q_1} V(b) = V(b^k) = \log(1 + 4t^2), \]
that is, \( |f(t)| \leq (1 + 4t^2)^{-1/4} \), which corresponds to (6) for \( m = 1 \).

If \( m = 2 \), we consider the same function \( V \) on the convex set 
\[ Q_2 = \{ (b_1, \ldots, b_n) : 0 \leq b_k \leq \frac{1}{2}, \ b_1 + \ldots + b_n = 1 \}, \]
which is just the intersection of the cube \([0, \frac{1}{2}]^n\) with the hyperplane. It has \( n(n-1)/2 \) extreme points 
\[ b^{kj}, \quad 1 \leq k < j \leq n, \]
with coordinates \( 1/2 \) on the \( j \)-th and \( k \)-th places and with zero elsewhere. Indeed, suppose that a point 
\[ b = (b_1, \ldots, b_n) \in Q_2 \]
has at least two non-zero coordinates $0 < b_k, b_j < 1/2$ for some $k < j$. Let $x$ be the point with coordinates

$$x_l = b_l \quad \text{for} \quad l \neq k, j, \quad x_k = b_k + \varepsilon, \quad \text{and} \quad x_j = b_j - \varepsilon,$$

and similarly, let $y$ be the point such that

$$y_l = b_l \quad \text{for} \quad l \neq k, j, \quad y_k = b_k - \varepsilon, \quad \text{and} \quad y_j = b_j + \varepsilon.$$

If $\varepsilon > 0$ is small enough, then both $x$ and $y$ lie in $Q_2$, while

$$b = (x + y)/2, \quad x \neq y.$$

Hence such $b$ cannot be an extreme point. Equivalently, any extreme point $b$ of $Q_2$ is of the form

$$b^{kj}, \quad 1 \leq k < j \leq n.$$

Therefore, we conclude that

$$\min_{b \in Q_2} V(b) = V(b^{kj}) = 2 \log(1 + 2t^2),$$

which is the first desired claim.

In the general case, consider the function $V$ on the convex set

$$Q_m = \{(b_1, \ldots, b_n) : 0 \leq b_k \leq \frac{1}{m}, \quad b_1 + \ldots + b_n = 1\}.$$

By a similar argument, any extreme point $b$ of $Q_m$ has zero for all coordinates except for $m$ places where the coordinates are equal to $1/m$. Therefore,

$$\min_{b \in Q_m} V(b) = V\left(\frac{1}{m}, \ldots, \frac{1}{m}, 0, \ldots, 0\right) = m \log(1 + 4t^2/m),$$

and we are done.

In case $m = 4$, using the inversion formula, we get

$$M(W) \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} |f(t)| \, dt \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{1 + t^2} \, dt = \frac{1}{2}.$$

Similarly, in the case $m = 3$,

$$M(W) \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{(1 + \frac{4}{3}t^2)^{3/4}} \, dt < 0.723.$$

Lemma is proved.
4 Proofs of main results

Proof of Theorem 1. In the following we shall write $W$ instead of $W_0$.

If $n = 1$, then the distribution function and the probability density function of $W = \lambda_1 Z_1^2$ are given by

\[ F(x) = 2 \Phi \left( \frac{\sqrt{x}}{\lambda_1} \right) - 1, \quad p(x) = \frac{1}{\sqrt{2\pi\lambda_1}} e^{-x/(2\lambda_1)} \quad (x > 0), \]

respectively. Therefore, $p$ is unbounded near zero, so that $M(W) = \infty$. This is consistent with (2), in which case $A_1 = \lambda_2^2$ and $A_2 = 0$.

If $n = 2$, the density $p(x)$ is described as the convolution

\[ p(x) = \frac{1}{2\pi\sqrt{\lambda_1\lambda_2}} \int_0^1 \frac{1}{\sqrt{(1-t)t}} \exp \left\{ -\frac{x}{2} \left[ \frac{1-t}{\lambda_1} + \frac{t}{\lambda_2} \right] \right\} dt \quad (x > 0). \quad (7) \]

Hence, $p$ is decreasing and attains maximum at $x = 0$:

\[ M(W) = \frac{1}{2\pi\sqrt{\lambda_1\lambda_2}} \int_0^1 \frac{1}{\sqrt{(1-t)t}} dt = \frac{1}{2\sqrt{\lambda_1\lambda_2}}. \]

Since $A_1 = \lambda_1^2 + \lambda_2^2$ and $A_2 = \lambda_2^2$, we conclude, using the assumption $\lambda_1 \geq \lambda_2$, that

\[ \frac{1}{2} (A_1 A_2)^{-1/4} \leq M(W) \leq \frac{1}{2^{3/4}} (A_1 A_2)^{-1/4}. \]

As for the case $n \geq 3$, the density $p$ is vanishing at zero and attains maximum at some point $x > 0$.

The further proof of Theorem 1 is based on the following observations and Lemma 3.

By homogeneity of (2), we may assume that $A_1 = 1$.

If $\lambda_1 \leq 1/2$, then all $\lambda_k^2 \leq 1/4$, so that $M(W) \leq 1/2$, by Lemma 3.

Hence, the inequality of the form

\[ M(W) \leq \frac{1}{2} (A_1 A_2)^{-1/4} \]

holds true.

Now, let $\lambda_1 \geq 1/2$, so that $A_2 \leq 3/4$. Write

\[ W = \lambda_1 Z_1^2 + \sqrt{A_2} \xi, \quad \xi = \sum_{k=2}^n \alpha_k Z_k^2, \quad \alpha_k = \frac{\lambda_k}{\sqrt{A_2}}. \]

By construction, $\alpha_2^2 + \ldots + \alpha_n^2 = 1$.

Case 1: $\lambda_2 \geq \sqrt{A_2}/2$. Since the function $M(W)$ may only decrease when adding an independent random variable to $W$, we get using (7) that

\[ M(W) \leq M(\lambda_1 Z_1^2 + \lambda_2 Z_2^2) = \frac{1}{2\sqrt{\lambda_1\lambda_2}} \leq c (A_1 A_2)^{-1/4}, \]
where the last inequality holds with \( c = 1 \). This gives the upper bound in (2) with constant 1.

Case 2: \( \lambda_2 \leq \sqrt{A_2}/2 \). It implies that \( n \geq 5 \) and all \( \alpha_k^2 \leq 1/4 \) for \( k > 1 \).

By Lemma 3 with \( m = 4 \), the random variable \( \xi \) has the probability density function \( q \) bounded by \( 1/2 \). The distribution function of \( W \) may be written as

\[
P\{W \leq x\} = \int_{0}^{x/\sqrt{A_2}} P\{|Z_1| \leq \frac{1}{\sqrt{\lambda_1}} (x - y \sqrt{A_2})^{1/2}\} q(y) \, dy, \quad x > 0,
\]

and its density has the form

\[
p(x) = \frac{1}{\sqrt{2\pi \lambda_1}} \int_{0}^{x/\sqrt{A_2}} \frac{1}{\sqrt{x - y \sqrt{A_2}}} e^{-(x - y \sqrt{A_2})/(2\lambda_1)} q(y) \, dy.
\]

Equivalently,

\[
p(x \sqrt{A_2}) = \frac{1}{\sqrt{2\pi \lambda_1}} A_2^{-1/4} \int_{0}^{x} \frac{1}{\sqrt{x - y}} e^{-(x - y \sqrt{A_2})/(2\lambda_1)} q(y) \, dy. \tag{8}
\]

Since \( \lambda_1 \geq 1/2 \), we immediately obtain that

\[
M(W) \leq A_2^{-1/4} \frac{1}{\sqrt{\pi}} \sup_{x > 0} \int_{0}^{x} \frac{1}{\sqrt{x - y}} q(y) \, dy.
\]

But, using \( q \leq 1/2 \), we get

\[
\int_{0}^{x} \frac{1}{\sqrt{x - y}} q(y) \, dy = \int_{0 < y < x, \, x - y < 1} \frac{1}{\sqrt{x - y}} q(y) \, dy + \int_{0 < y < x, \, x - y > 1} \frac{1}{\sqrt{x - y}} q(y) \, dy \\
\leq \frac{1}{2} \int_{0}^{1} \frac{1}{\sqrt{z}} \, dz + 1 = 2.
\]

Thus,

\[
M(W) \leq 2A_2^{-1/4} \frac{1}{\sqrt{\pi}}.
\]

Combining the obtained upper bounds for \( M(W) \) in all cases we get the upper bound in (2).

For the lower bound, one may apply the inequality (4) in Lemma 1. Thus, we obtain that

\[
M(W) \geq \frac{1}{2\sqrt{6}}
\]

due to the assumption \( A_1 = 1 \) and the property \( \text{Var}(Z_1^2) = 2 \).

If \( \lambda^2 \leq 1/2 \), we have \( A_2 \geq 1/2 \). Hence,

\[
M(W) \geq \frac{1}{2\sqrt{6}} \geq c_0 (A_1 A_2)^{-1/4}, \tag{9}
\]
where the last inequality holds true with

\[ c_0 = \frac{1}{2^{5/4}\sqrt{6}} \geq 0.171. \]

In case \( \lambda_1^2 \geq \frac{1}{3} \), we have \( A_2 \leq 1/2 \). Returning to the formula (S), let us choose \( x = E\xi + 2 \) and restrict the integration to the interval

\[ \Delta : \max(E\xi - 2, 0) < y < E\xi + 2. \]

On this interval necessarily \( x - y \leq 4 \).

Therefore, (S) yields

\[ M(W) \geq \frac{A_2^{-1/4}}{2\sqrt{2\pi}\lambda_1} \cdot e^{-2\sqrt{A_2}/\lambda_1} P\{\xi \in \Delta\}. \]

Here,

\[ \frac{A_2}{\lambda_1^2} = \frac{1}{\lambda_1^2} - 1 \leq 1, \]

and we get

\[ M(W) \geq \frac{A_2^{-1/4}}{2\sqrt{2\pi}} \cdot e^{-2} P\{\xi \in \Delta\}. \]

Now, recall that \( \xi \geq 0 \) and \( \text{Var}(\xi) = 2 (\alpha_2^2 + \ldots + \alpha_n^2) = 2 \). Hence, by Chebyshev’s inequality,

\[ P\{|\xi - E\xi| \geq 2\} \leq \frac{1}{4} \text{Var}(\xi) = \frac{1}{2}. \]

That is, \( P\{\xi \in \Delta\} \geq 1/2 \), and thus

\[ M(W) \geq \frac{(A_1A_2)^{-1/4}}{4\sqrt{2\pi}} e^{-2} \geq 0.013 \cdot (A_1A_2)^{-1/4}. \]

Theorem 1 is proved.

**Proof of Theorem 2** In the following we shall write \( W \) instead of \( W_a \).

The lower bound in Theorem 2 immediately follows from (5) in Lemma 2 without any assumption on \( \lambda_1^2 \).

Our next aim is to reverse this bound up to a numerical factor under suitable natural assumptions.

Without loss of generality, let \( A_1 = 1 \). Our basic condition will be that \( \lambda_1^2 \leq 1/3 \), similarly to the first part of the proof of Theorem 1. Note that if \( \lambda_1^2 \leq 1/3 \) then necessarily \( n \geq 3 \).

As easy to check, for \( Z \sim N(0, 1) \) and \( a \in \mathbb{R} \),

\[ E e^{it(Z-a)^2} = \frac{1}{\sqrt{1-2it}} \exp \left\{ a^2 \frac{it}{1-it} \right\}, \quad t \in \mathbb{R}, \]
so that
\[
\left| E e^{it(Z-a)²} \right| = \frac{1}{(1+4t²)^{1/4}} \exp \left\{ -2a² \frac{t²}{1+4t²} \right\}.
\]
Hence, the characteristic function \( f(t) \) of \( W \) satisfies
\[
-\log |f(t)| = \frac{1}{4} \sum_{k=1}^{n} \log(1 + 4\lambda_k^2 t²) + 2 \sum_{k=1}^{n} \alpha_k^2 \frac{\lambda_k^2 t²}{1 + 4\lambda_k^2 t²}.
\]

Since \( \lambda_k^2 \leq \frac{1}{3} \), by the monotonicity, all \( \lambda_k^2 \leq \frac{1}{3} \) as well. But, as we have already observed, under the conditions
\[
0 \leq b_k \leq \frac{1}{3}, \quad b_1 + \ldots + b_k = 1,
\]
and for any fixed value \( t \in \mathbb{R} \), the function
\[
\psi(b_1, \ldots, b_n) = \sum_{k=1}^{n} \log(1 + 4b_k t²)
\]
is minimized for the vector with coordinates
\[
b_1 = b_2 = b_3 = \frac{1}{3} \quad \text{and} \quad b_k = 0 \quad \text{for} \quad k > 3.
\]
Hence,
\[
\psi(b_1, \ldots, b_n) \geq 3 \log(1 + 4t²/3) \geq 3 \log(1 + t²).
\]
Therefore, one may conclude that
\[
|f(t)| \leq \frac{1}{(1 + t²)^{3/4}} \exp \left\{ -2 \sum_{k=1}^{n} \alpha_k^2 \frac{\lambda_k^2 t²}{1 + 4\lambda_k^2 t²} \right\}. \tag{10}
\]
It is time to involve the inversion formula which yields the upper bound
\[
M(W) \leq \frac{1}{\pi} \int_{0}^{\infty} |f(t)| \, dt. \tag{11}
\]
In the interval
\[
0 < t < T = \frac{1}{2\lambda_1},
\]
we have \( \lambda_k^2 t² \leq 1/4 \) for all \( k \), and the bound (8) is simplified to
\[
|f(t)| \leq \frac{1}{(1 + t²)^{3/4}} e^{-B_1 t²}.
\]
This gives
\[
\int_{0}^{T} |f(t)| \, dt \leq I(B_1) \equiv \int_{0}^{\infty} \frac{1}{(1 + t²)^{3/4}} e^{-B_1 t²} \, dt.
\]
If $B_1 \leq 1$, 
\[ I(B_1) \leq \int_0^{\infty} \frac{1}{(1 + t^2)^{3/4}} \, dt < 3, \]
while for $B_1 \geq 1$, 
\[ I(B_1) \leq \int_0^{\infty} e^{-B_1 t^2} \, dt = \frac{\sqrt{\pi}}{2\sqrt{B_1}} < \frac{1}{\sqrt{B_1}}. \]
The two estimates can be united by 
\[ I(B_1) \leq 3\sqrt{2} \sqrt{1 + B_1}. \]

To perform the integration over the half-axis $t \geq T$, a different argument is needed. Put $p_k = a_k^2 \lambda_k^2 / B_1$, so that $p_k \geq 0$ and $p_1 + \ldots + p_k = 1$. By Jensen’s inequality applied to the convex function $V(x) = 1/(1 + x)$ for $x \geq 0$ with points $x_k = 4\lambda_k^2 t^2$, we have 
\[ \sum_{k=1}^{n} a_k^2 \frac{\lambda_k^2 t^2}{1 + 4\lambda_k^2 t^2} = B_1 t^2 \sum_{k=1}^{n} p_k V(x_k) \geq B_1 t^2 V(p_1 x_1 + \ldots + p_n x_n) = \frac{B_1 t^2}{1 + \frac{1}{4} t^2}, \]
where we used the property $\lambda_k^2 \leq 1/3$. Moreover, since 
\[ t^2 \geq \frac{1}{(2\lambda_1)^2} \geq \frac{3}{4}, \]
necessarily 
\[ \frac{t^2}{1 + \frac{3}{4} t^2} \geq \frac{3}{8}. \]
Hence, from (10) we get 
\[ |f(t)| \leq \frac{1}{(1 + t^2)^{3/4}} e^{-3B_1/4}, \quad t \geq T, \]
and 
\[ \int_T^{\infty} |f(t)| \, dt \leq e^{-3B_1/4} \int_{\sqrt{3}/2}^{\infty} \frac{1}{(1 + t^2)^{3/4}} \, dt < 1.68 e^{-3B_1/4} < \frac{1.85}{\sqrt{1 + B_1}}. \]

Combining the two estimates together for different regions of integration with $(3\sqrt{2} + 1.85)/\pi < 1.94$, the bound (11) leads to 
\[ M(W) < \frac{2}{\sqrt{A_1 + B_1}}. \]
Thus, this inequality, together with Lemma 2, completes the proof of the theorem.
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