THE RICCI FLOW ON SURFACES WITH BOUNDARY

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ABSTRACT. We show for a non homogeneous boundary value problem for the Ricci flow on a surface with boundary that when the initial metric has positive curvature and the boundary is convex then the initial metric is deformed, via the normalized flow and along sequences of times, to a metric of constant curvature and totally geodesic boundary.

1. Introduction

Very little is known about the behavior of the Ricci flow on manifolds with boundary. One of the main difficulties arises from the fact that even trying to impose meaningful boundary conditions seems to be a challenging task. For the reader to get acquainted with the difficulty of the problem, we recommend the papers of Y. Shen ([She]) and S. Brendle ([Br]), and the interesting work of Pulemotov [Pu]. In the case of the boundary conditions imposed by Shen in [She], satisfactory convergence results have given for manifolds of positive Ricci curvature and totally geodesic boundary, and when the boundary is convex and metric is rotationally symmetric. In the case of surfaces, the Ricci flow is parabolic, and imposing meaningful boundary conditions is not difficult: one can for instance control the geodesic curvature of the boundary. In this case, Brendle in [Br] has shown that when the boundary is totally geodesic, then the behavior is completely analogous to the behavior of the Ricci flow in closed surfaces. In this case, also for non totally geodesic boundary, the author has proved, under the hypothesis of rotational symmetry of the metrics involved, convergence results in the case of positive curvature and convex boundary and for certain families of metrics with non convex boundary ([C1]).

In this paper we prove a new result concerning the asymptotic behavior of the Ricci flow in surfaces with boundary. To be more precise we will study the equation

\[
\begin{aligned}
& \frac{\partial g}{\partial t} = -R_g \quad \text{in} \quad M \times (0, T) \\
& k_g(\cdot, t) = \psi(\cdot) \quad \text{on} \quad \partial M \times (0, T) \\
& g(\cdot, 0) = g_0(\cdot) \quad \text{in} \quad M.
\end{aligned}
\]

We assume \( \psi = k_{g_0} \) for all time, i.e., the geodesic curvature of the boundary remains the same throughout the deformation.

The solution to \( (1) \) can be normalized to keep the area of the surface constant, as follows. Choose \( \phi(t) \) such that \( \phi A_g(t) = 2\pi \), where \( A_g(t) \) is the area of the surface at time \( t \) with respect to the metric \( g \). Then define

\[
\tilde{t}(t) = \int_0^t \phi(s) \, ds \quad \text{and} \quad \tilde{g} = \phi g.
\]
If the family of metrics $g(t)$ satisfies (1), then the family of metrics $\tilde{g}(\tilde{t})$ satisfies the evolution equation

$$\begin{cases}
\frac{\partial \tilde{g}}{\partial \tilde{t}} = (\tilde{r} - \tilde{R}) \tilde{g} & \text{in} \ M \times (0, \tilde{T}) \\
k_{\tilde{g}}(\cdot, t) = \tilde{\psi}(\cdot, \tilde{t}) & \text{on} \ \partial M \times (0, \tilde{T}) \\
\tilde{g}(\cdot, 0) = g_0(\cdot) & \text{on} \ M
\end{cases}$$

where $\tilde{R}$ is the scalar curvature of the metric $\tilde{g}$, and

$$\tilde{r} = \frac{\int_M \tilde{R} dA_{\tilde{g}}}{\int_M dA_{\tilde{g}}}.$$ 

Here $dA_{\tilde{g}}$ denotes the area element of $M$ with respect to the metric $\tilde{g}$. We refer to (2) as the normalized Ricci flow.

The existence theory of equation (1) is well understood. Indeed, since the deformation given by (1) is conformal, if we write $g(t) = e^{2u(t)}g_0$, it is equivalent to a nonlinear parabolic equation with Robin boundary conditions. Hence, it can be shown that (1) has a unique solution for a short time, and that this solution is $C^{2,\gamma}$, $0 < \gamma < 1$, on $M \times [0, T)$ and smooth on $M \times (0, T)$.

In this paper we prove

**Theorem 1.1.** Let $(M^2, g_0)$ be a surface with boundary with positive Gaussian curvature scalar curvature ($R_{g_0} > 0$) and such that the geodesic curvature of its boundary is nonnegative ($k_{g_0} \geq 0$) ($M^2$ is then homeomorphic to a disk). If $g(t)$ satisfies (1) with initial condition $g_0$, then its normalization $\tilde{g}(\tilde{t})$ exists for all time, and for any sequence $\tilde{t}_n \to \infty$, there is a subsequence $\tilde{t}_n \to \infty$ such that the metrics $\tilde{g}(\tilde{t}_n)$ converge smoothly to a metric of constant curvature and totally geodesic boundary.

This paper is a follow-up of [C1], where the rotationally symmetric case is treated. We give an outline of the proof: first of all, given an initial metric $g_0$ of positive scalar curvature and convex boundary, it can be shown that the curvature of $g(t)$ blows up in finite time, say $T < \infty$; the idea then is to take a blow-up limit and to show that the only possibility is for this blow-up limit to be a round hemisphere. To be able to produce this blow-up limit we will have to show that we can estimate the injectivity radius of the manifold, and since it has boundary it will require from us to show that there no geodesics hitting the boundary orthogonally that are too short with respect to the inverse of the square root of the maximum of the curvature (this is the basic new ingredient in the proof). We prove then that if such a geodesic does exist, then there exists a sequence of times $t_k \to T$ and balls of radius $r_k \to 0$ at time $t_k$ whose area is $o(r_k^2)$, which then together with a monotonicity formula given by Theorem 3.2 The monotonicity formula (Theorem 3.2) also precludes blow up limits different from the hemisphere, as it does not allow collapsing. From this Theorem 1.1 follows easily.

The layout of this paper is as follows. In section 2 we prove the basic evolution equation for the scalar curvature, when the metric evolves under (1); in section 3 we prove a monotonicity formula for Perelman’s functionals on manifolds with boundary; in section 4 it is shown that it is possible to take blow up limits for solutions to (1), by proving that we can control the injectivity radius of the manifold in terms of the curvature; finally, in section 5 we put all the ingredients together to give a proof of Theorem 1.1. This paper contains also two appendices: in appendix...
we produce a useful extension procedure for surfaces with boundary that we use in our arguments; and in appendix [B] we show how to bound derivatives of the curvature in terms of bounds on the curvature.

2. Evolution equations

Let us compute the evolution of the curvature of a metric $g$ when it is evolving under (1); of special interest to us is the computation of the (outward) normal derivative of the curvature.

Proposition 2.1. The scalar curvature satisfies the evolution equation

$$\begin{cases}
\frac{\partial R}{\partial t} = \Delta_g R + R^2 & \text{in } M \times (0,T) \\
\frac{\partial R}{\partial \eta_g} = k_g R - 2k'_g & \text{on } \partial M \times (0,T)
\end{cases}$$

where $\eta_g$ is the outward pointing unit normal with respect to the metric $g$, and the prime (‘) represents differentiation with respect to time.

Proof. Choose local coordinates so that $\{\partial_1, \partial_2\}$ is an orthogonal frame at $t = t_0$ (the instant when we want to compute the normal derivative) and which is normal at the point $P \in \partial M$ where we are computing with $\partial_2$ the exterior normal to the boundary. Since the deformation is conformal, $\partial_2$ remains normal to the boundary. Therefore the geodesic curvature is given (as long as the flow is defined for $t \geq t_0$) by

$$k_g g_{11} = -\frac{\Gamma^2_{11}}{(g_{22})^\frac{3}{2}} = -(g_{22})^\frac{3}{2} \Gamma^2_{11}$$

Now we compute at $t = t_0$ (here we use $g_{ii} = 1$)

$$(k_g g_{11})' = -\frac{1}{2(g_{22})^\frac{3}{2}} (g_{22}') \Gamma^2_{11} - (g_{22})^\frac{3}{2} (\Gamma^2_{11})'$$

$$= \frac{1}{2} R (g_{22})^\frac{3}{2} \Gamma^2_{11} - (g_{22})^\frac{3}{2} (\Gamma^2_{11})'$$

Now let’s compute (recall $\nabla$ denotes covariant derivative, and recall $g_{12} = 0$)

$$(\Gamma^2_{11})' = \frac{1}{2} g^{2j} (\nabla_1 g_{1j} + \nabla_1 g_{1j} - \nabla_j g_{11})$$

$$= \frac{1}{2} g^{2j} (-2 \nabla_1 (R g_{12}) + \nabla_2 (R g_{11}))$$

$$= \frac{1}{2} g^{22} (\partial_2 R) g_{11} = \frac{1}{2} g^{22} (\partial_2 R)$$

Therefore

$$k'_g g_{11} - k_g R g_{11} = \frac{1}{2} k_g R - \frac{1}{2(g_{22})^\frac{3}{2}} \partial_2 R$$

$$= \frac{1}{2} k_g R - \frac{1}{2} \frac{\partial R}{\partial \eta_g}$$

And the result follows. \qed

As a consequence from the Maximum Principle, since $k'_g = 0$ in the case we are considering, we obtain

Proposition 2.2. If $R \geq 0$ at time $t = 0$, it remains so along the Ricci flow. Also $R$ blows up in finite time.
3. Monotonicity of Perelman’s Functional in Manifolds with Boundary.

The purpose of this section is to show a monotonicity formula for Perelman’s celebrated \( F \) and \( W \) functionals. To begin, consider the following functional

\[
F(g_{ij}, f) = \int_M \left( R + |\nabla f|^2 \right) \exp(-f) \, dV.
\]

Here \( dV \) represents the volume (in the case of a surface, area) element of the manifold \( M \). We compute the first variation of this functional on a manifold with boundary.

**Proposition 3.1.** Let \( \delta g_{ij} = v_{ij}, \delta f = h, g^{ij}v_{ij} = v \). Then we have,

\[
\delta F = \int_M \exp(-f) \left[ -v_{ij} (R_{ij} + \nabla_i \nabla_j f) + \left( \frac{\psi}{2} - h \right) \left( 2\Delta f - |\nabla f|^2 + R \right) \right] \, dV
\]

- \( \int_{\partial M} \left[ \frac{\partial}{\partial n} + (v - 2h) \frac{\partial f}{\partial n} \right] \exp(-f) \, d\sigma + \)
- \( \int_{\partial M} \exp(-f) \nabla_i v_{ij} \eta^i \, d\sigma - \int_{\partial M} \nabla_j \exp(-f) v_{ij} \eta^i \, d\sigma. \)

Here, \( \frac{\partial}{\partial n} = \{ \eta^i \} \) is the outward unit normal to \( \partial M \) with respect to \( g \), \( \nabla \) represents covariant differentiation with respect to the metric \( g \), and \( d\sigma \) represents the volume element of \( \partial M \).

**Proof.** As in [KL], we have that

\[
\delta F(v_{ij}, h) = \int_M e^{-f} \left[ -\Delta v + \nabla_i \nabla_j v_{ij} - R_{ij} v_{ij}\right.
\]

\[-v_{ij} \nabla_i f \nabla_j f + 2g(\nabla f, \nabla h) + \left( R + |\nabla f|^2 \right) \left( \frac{\psi}{2} - h \right) \right] \, dV
\]

We must compute the integrals on the right-hand side of the previous identity.

We start by calculating

\[
\int_M e^{-f} (-\Delta v) \, dV = -\int_M \Delta e^{-f} v \, dV + \int_{\partial M} v \frac{\partial e^{-f}}{\partial \eta} \, d\sigma - \int_{\partial M} e^{-f} \frac{\partial v}{\partial \eta} \, d\sigma
\]

\[-\int_M \Delta e^{-f} v \, dV - \int_{\partial M} \left( \frac{\partial v}{\partial \eta} + v \frac{\partial e^{-f}}{\partial \eta} \right) \exp(-f) \, d\sigma.
\]

Now we compute

\[
\int_M e^{-f} \nabla_i \nabla_j v_{ij} \, dV = -\int_M \nabla_i e^{-f} \nabla_j v_{ij} \, dV + \int_{\partial M} e^{-f} \nabla_j v_{ij} \eta^i \, d\sigma
\]

\[+ \int_{\partial M} e^{-f} \nabla_j v_{ij} \eta^i \, d\sigma.
\]

And finally

\[
2 \int_M e^{-f} \langle \nabla f, \nabla h \rangle \, dV = -2 \int_M \langle \nabla e^{-f}, \nabla h \rangle \, dV
\]

\[= 2 \int_M (\Delta e^{-f}) h \, dV - \int_{\partial M} h \frac{\partial e^{-f}}{\partial \eta} \, d\sigma.
\]

 Putting all these calculations together proves the result. \( \square \)

Consider the evolution equations on a surface with boundary

\[
\begin{cases}
(g_{ij})_t = -2R_{ij} & \text{in } M \times (0, T) \\
k_j = \psi & \text{on } \partial M \times (0, T) \\
f_t = -\Delta f + |\nabla f|^2 - R & \text{in } M \times (0, T) \\
\frac{\partial}{\partial \eta} f = 0 & \text{on } \partial M \times (0, T)
\end{cases}
\]

The fundamental monotonicity property of \( F \) is given by the following result.


Theorem 3.1. Under (5) the functional $F$ satisfies

$$F_t = 2 \int_M |R_{ij} + \nabla_i \nabla_j f|^2 \exp (-f) \, dA_g + 2 \int_{\partial M} k_g R \exp (-f) \, ds_g + 2 \int_{\partial M} k_g |\nabla^\theta f|^2 \exp (-f) \, ds_g,$$

and here $\nabla^\theta f$ represents the component of $\nabla f$ tangent to $\partial M$, $dA_g$ the area element of the surface, and $ds_g$ the length element of the boundary.

Proof. For notational purposes and since parts of these computations apply to manifolds of higher dimensions, in this proof we will fix coordinates $x^1, x^2, \ldots, x^{n-1}, x^n$. We will assume that $\partial/\partial x^n$ represents the outward unit normal when these coordinates are used at a boundary point, and we will denote by a subscript $n$ quantities that are evaluated with respect to the outward unit normal. By a greek letter we will represent indices running from $1, 2, 3, \ldots, n-1$, and at a boundary point we assume the vector fields $\partial/\partial x^\alpha$ to be tangent to the boundary. Covariant differentiation will be denoted by a semicolon ($;$). If we transform the evolution equations given by (5) using the one-parameter family of diffeomorphisms $\varphi_t$ generated by $-\nabla f$, we must take,

$$\delta v_{ij} = -2 (R_{ij} + \nabla_i \nabla_j f), \quad \delta f = -\Delta f - R.$$

Observe that under this new evolution, i.e., with respect to the metric $g = (\varphi_t)_* g$ (forgive the abuse of notation), we still have $\partial R/\partial \eta = k_g R$. We will compute each of the boundary integrals in the first variation of Perelman’s functional. We change the notation from the previous proposition as follows: $dV = dA_g$ and $d\sigma = ds_g$. We start with

$$\int_{\partial M} \left[ \frac{\partial v}{\partial \eta} + (v - 2h) \frac{\partial f}{\partial \eta} \right] \exp (-f) \, ds_g, \quad \int_{\partial M} \exp (-f) \nabla_i v_{ij} \eta^j \, dA_g.$$

To compute these integrals, let us calculate $v_{in;i} = g^{ij} v_{in;j}$. First, we have

$$v_{in;i} = -2 R_{in;i} - 2 \nabla_i \nabla_i \nabla_n f = -\nabla_n R - 2 \Delta \nabla_n f \quad \text{(by the contracted Bianchi identity)}.$$

The Ricci identity says

$$\Delta \nabla_n f = \nabla_n \Delta f + g^{jk} R_{nj} \nabla_k f = \nabla_n \Delta f,$$

and therefore,

$$v_{in;i} = -\nabla_n R - 2 \nabla_n \Delta f.$$

Using the evolution equation $f_t = -\Delta f - R$, we get at the boundary,

$$\nabla_n \Delta f = -\nabla_n R.$$

This last identity has two consequences. On the one hand, it implies that $\partial v/\partial \eta = 0$, and since clearly $v - 2h = 0$, we obtain

$$\int_{\partial M} \left[ \frac{\partial v}{\partial \eta} + (v - 2h) \frac{\partial f}{\partial \eta} \right] \exp (-f) \, ds_g = 0.$$

On the other hand, it implies that

$$v_{in;i} = -\nabla_n R + 2 \nabla_n R = \nabla_n R = k_g R.$$

which shows that

$$\int_{\partial M} \exp (-f) \nabla_i v_{ij} \eta^j \, dA = \int_{\partial M} k_g R \exp (-f) \, dA_g.$$
Let us now attack the integral

$$II = -\int_{\partial M} \nabla_j \exp (-f) v_{ij} \eta^i \, dA_g.$$ 

Notice that

$$II = -\int_{\partial M} \nabla_j \exp (-f) v_{jn} \, dA_g,$$

so, under the previous conventions,

$$II = \int_{\partial M} \nabla_\alpha f \exp (-f) v_{\alpha n} \, dA_g + \int_{\partial M} \nabla_n f \exp (-f) v_{nn} \, dA_g.$$ 

We know that

$$\nabla_n \nabla_\alpha f = -g^{\alpha \sigma} h_{\alpha \sigma} \partial f,$$

where $h$ denotes the second fundamental form of the boundary. Hence, using the fact that $\nabla_n f = 0$, we get

$$II = 2 \int_{\partial M} \exp (-f) h (\nabla^{\partial} f, \nabla^{\partial} f) \, dA = 2 \int_{\partial M} \exp (-f) k_g |\nabla^{\partial} f|^2 \, dA,$$

and the formula is proved.

Consider the functional,

$$W(g, f, \tau) = \int_M \left[ \tau \left( |\nabla f|^2 + R \right) + f - \frac{2}{4 \tau} \right] (4 \pi \tau)^{-1} \exp (-f) \, dA_g.$$

Under the unnormalized Ricci flow, and the evolution

$$\begin{cases}
\frac{\partial f}{\partial t} = -\Delta f + |\nabla f|^2 - R + \frac{1}{T} & \text{in } M \times (0, T) \\
\frac{\partial \tau}{\partial t} = -1 & \text{in } (0, T) \\
\frac{\partial f}{\partial \eta} = 0 & \text{on } \partial M \times (0, T),
\end{cases}$$

we have the following monotonicity formula for the functional $W$. We omit the proof, as it is similar to the proof of theorem 3.1.

**Theorem 3.2.**

$$\frac{dW}{dt} = \int_M 2\tau \left[ R_{ij} + \nabla_i \nabla_j f - \frac{1}{2\tau} g_{ij} \right]^2 (4 \pi \tau)^{-1} \exp (-f) \, dA_g + 2\tau \left( \int_{\partial M} k_g \left( R \exp (-f) + |\nabla^{\partial} f|^2 \right) \exp (-f) \, ds_g \right).$$

### 4. Non-collapsing

In order to produce a blow-up limit we need control over the injectivity radius of the surface with boundary $M$. To define the injectivity radius of a manifold with boundary, we must define a few quantities first. Let $\nu$ be the normal bundle of $\partial M$, and let $\nu^- \nu$ be the half space of the normal space consisting of inwardly pointing vectors. Finally, define

$$\nu^- (r) = \left\{ v \in \nu^\perp : \|v\| < r \right\}.$$

Then we have

$$\iota_\partial = \sup \left\{ r > 0 : \exp : \nu^- (r) \rightarrow M \text{ is a diffeomorphism onto its image} \right\}.$$
Let \( \iota_{\text{int}} (p) \) be the supremum among all \( r > 0 \) such that for any geodesic \( \gamma : [0, t_\gamma] \to M \) starting at \( p \) it is minimizing up to time \( t = \min (t_\gamma, r) \). Define
\[
\iota_{\text{int}} = \sup_{p \in M} \iota_{\text{int}} (p).
\]
The injectivity radius, \( \iota_M \), of the manifold \( M \) is defined as
\[
\iota_M = \min \{ \iota_{\text{int}}, \iota_\partial \}.
\]
It is well known that,
\[
\iota_\partial \geq \left\{ \text{Foc} (\partial M), \frac{1}{2} l \right\}
\]
where \( l \) is the length of the shortest geodesic meeting \( \partial M \) at its two endpoints at a right angle, and \( \text{Foc} \) is the focal radius of the (boundary of) the manifold. Since for the focal radius of the manifold
\[
\text{Foc} (\partial M) \geq \frac{1}{\sqrt{K_+}} \arctan \left( \frac{\sqrt{K_+}}{\lambda_+} \right),
\]
where \( K_+ \) and \( \lambda_+ \) are upper bounds for the curvature of \( M \) and the geodesic curvature of \( \partial M \) respectively, we obtain
\[
\iota_{\partial} \geq \left\{ \frac{c}{\sqrt{K_+}}, \frac{1}{2} l \right\}
\]
On the other hand, for points \( p \in M \) at distance at least \( \frac{1}{2} \iota_{\partial} \), it is easy to show that
\[
\iota_{\text{int}} (p) \geq \left\{ \frac{1}{2} \iota_{\partial}, \frac{1}{2} L, \frac{\pi}{\sqrt{K_+}} \right\}
\]
where \( L \) is the length of the shortest closed geodesic in \( M \). By the extension results from appendix A we may assume that \( M \) is embedded in a closed surface of positive curvature, and its curvature is bounded above by \( \leq CK_+ \), where \( K_+ \) is a bound on the curvature of \( M \) and \( C \) is a universal constant as long as the minimum of the geodesic curvature of \( \partial M \) is strictly positive; hence from Klingenberg arguments, it follows that
\[
\iota_M (p) \geq \left\{ \frac{1}{2} \iota_{\partial} (M), \frac{C \pi}{\sqrt{K_+}} \right\}
\]
Then, in order to produce blow-up limits, we must show that along the Ricci flow (1) there is a constant independent of time such that at any time \( t \),
\[
\iota (M, g(t)) \geq \sup_{s \in [0, t]} \frac{c}{\sqrt{R_{\text{max}} (s)}}
\]
This is implied by the following non-collapsing result:

**Proposition 4.1.** Let \( l \) be the length of the shortest geodesic both of whose endpoints are orthogonal to the boundary. There is a \( \kappa > 0 \) which depends only on the initial metric such that
\[
l > \frac{\kappa}{\sqrt{R_{\text{max}} (s)}}.
\]

*Proof.* Here we will make use of the monotonicity formula proved in the previous section. Assume that the statement of the proposition is false. Then, there is a sequence of times \( t_k \) such that \( l_k = o \left( \frac{1}{\sqrt{R_{\text{max}} (t_k)}} \right) \). Take a geodesic ball of radius
\[ r_k \sim \frac{1}{\sqrt{R_{\text{max}}(t_k)}} \] centered at the midpoint \( p_k \) of the geodesic. As it will be shown in section 4.1, the volume of the ball is of order \( l_k r_k \), which goes to zero faster than \( r_k^2 \).

An argument similar to Perelman’s shows that this together with the monotonicity of the functional \( W \) produces a contradiction (see Section 4.2).

This shows, by the results in [Ko], reviewed in [C3] for the case of the Ricci flow, that one can take a blow up limit, as soon as we have control over the derivatives of the curvature in terms of the curvature itself, and how this can be done is shown in appendix B (see also the last section of [C2]).

**Remark 1.** A proof that limits can be taken from a sequence of pointed Ricci flows with controlled geometries (bounded curvature, bounded second fundamental forms and injectivity radius controlled from below) can be given along the same lines to the one given by Hamilton in [Ham]. To take care of the boundary one can extract from the original sequence a subsequence for which the boundaries form a convergent subsequence of pointed submanifolds by taking as a marked point in the boundary one that is close to the marked point of the manifold (unless the marked point of the manifold is getting further and further away from the boundary: in this case, there is no need to worry about the boundaries). The boundaries also have controlled geometries since we have control on the curvature, the second fundamental forms, and the injectivity radius of the manifolds (to see how to control the injectivity radius of the boundary from the mentioned quantities see [AB]; notice that in the case of the boundary being a curve, this is not a problem at all). Once we have done this, the boundary can be treated as a generalized point and Hamilton’s arguments go through.

4.1. We still have a claim to justify from the proof of Proposition 4.1. In this section we assume that the boundary of \( M \) has strictly positive geodesic curvature. Otherwise, it can be shown that by changing the original metric conformally, we can find a metric as close as the original one as wanted (in any \( C^k \) norm) so that with the new metric \( M \) has strictly positive curvature, and \( \partial M \) strictly positive geodesic curvature. Indeed, choose \( h \) so that it satisfies

\[
\begin{align*}
\Delta h &= a > 0 \quad \text{in } M \\
h &= 0 \quad \text{on } \partial M
\end{align*}
\]

Let \( \omega > 0 \) be any small number. By the deformation formula for the curvature we see that by taking \( \omega \) very small, if \((M, g)\) has strictly positive curvature then \((M, e^{2\omega h} g)\) has positive curvature and, as a consequence of Hopf’s lemma, and the deformation formula

\[ k = e^{-\varphi} \left( k_0 + \frac{\partial \varphi}{\partial \eta} \right) \]

if \( \partial M \) has positive geodesic curvature with respect to \( g \) then \( \partial M \) has strictly positive geodesic curvature with respect to \( e^{2\omega h} g \); also \( e^{\omega h} g \) is as close to \( g \) as wished. Then we can reason as it is done in what follows to justify our claims in Proposition 4.1.

So let \( M \) be a compact surface of positive Gaussian curvature and strictly convex boundary. Assume that there is a geodesic \( \gamma \) of length \( l \) parametrized by arclength

\[ \gamma : \left[ -\frac{l}{2}, \frac{l}{2} \right] \longrightarrow \overline{M} \]
that hits the boundary orthogonally at its two endpoints. Again, we may assume that \( M \) is embedded in a closed surface \( \tilde{M} \) of strictly positive curvature which is \( \leq CK_+ \). \( B \subset \tilde{M} \) be the ball centered at the midpoint of the geodesic, i.e. centered at \( \gamma(0) \), of radius \( r \sim \frac{1}{\sqrt{K}_+} \), half the convexity radius of \( \tilde{M} \).

To simplify our considerations we will consider the following region: let \( \beta_1 \) the connected component of the boundary contained in \( B \) that contains \( \gamma(l_2) \). \( \beta_1 \) divides \( B \setminus \beta_1 \) into two open sets; let \( B_1 \) the piece that contains \( \gamma(0) \). Let \( \beta_2 \) be the connected component of the boundary contained in \( B \) that contains \( \gamma(-l_2) \). \( \beta_2 \) divides \( B \setminus \beta_2 \) into two open sets. Let \( B_2 \) the piece that contains \( \gamma(0) \).

Let \( D = B_1 \cap B_2 \). It is not difficult to show that \( D \) is contained in a ball of radius \( r \) of \( M \) and that it is locally convex. Recall that a subset \( C \) of a manifold \( M \) is locally convex if for any \( p \in C \) there is a number \( \epsilon(p) > 0 \) such that \( C \cap B_{\epsilon(p)}(p) \) is strongly convex. (see Section 2 in [BCGS]; the definition we use for local convexity corresponds to the definition of convexity given in [ChG] and [Kr]).

Now, since we will estimate a volume, we may assume without lost of generality that the region considered, i.e. \( D \), is symmetric by reflection across the aforementioned geodesic (in principle the resulting metric is \( C^2 \); however, by smoothing up a little bit we may assume the reflected metric as regular as wished without changing it and its curvature too much); notice that the diameter of \( D \) once it has been symmetrized along the geodesic \( \gamma \) will change at most by \( O(l) \), so we still have that the diameter of \( D \) is \( O(r) \). Also, we may assume that by changing \( D \) a little bit, without changing its volume too much (actually as little as wished) that \( \partial D \) is smooth and \( D \) still locally convex. In fact, let us give a short argument for this:

If \( D \) is locally convex, then \( f(p) = d(p, \partial D) \) is a (geodesically) convex function (see Theorem 1.10 in [ChG]). Consider the function \( h = (d(\gamma(0), p))^2 \). \( h \) is strictly convex in a neighborhood of \( \overline{D} \); therefore, for \( \delta > 0 \) very small, \( f + \delta h \) is a strictly convex function. By a theorem of Greene and Wu (Theorem 2 of [GW]), \( f + \delta h \) can be approximated by a smooth strictly convex function; since strictly convex functions are Morse, from this our claim follows without much difficulty.

Then we have the following estimate on the area of \( D \):

**Lemma 4.1.**

\[
A_g(D) \sim lr.
\]

To show the lemma, we will conformally deform de region \( D \) to be flat, and then show that the volume of the region with the original metric is comparable with the volume of the region with the flat metric. So let \( \varphi \) be a function such that

\[
\begin{align*}
\Delta \varphi &= R \quad \text{in} \quad D, \\
\varphi &= 0 \quad \text{on} \quad \partial D
\end{align*}
\]

By our construction, the domain \( D \) is quite well behaved, so this equation has a solution that belongs to \( C^2(D) \cap C^0(\overline{D}) \), and that this solution is actually smooth in \( \overline{D} \). Consider the metric in \( D \) given by

\[
g_E = e^{2\varphi} g.
\]

It is clear that \( g_E \) has zero Gaussian curvature. By symmetry, \( \gamma \) remains a geodesic in this new metric, and still hits \( \beta_1 \cup \beta_2 \) orthogonally at both of its endpoints. Also, in this new metric \( \partial D \) has positive geodesic curvature. Indeed, since \( R \geq 0 \), \( \varphi \) is superharmonic, and hence it is negative in the interior of \( D \). Hopf’s Lemma then
implies that $\partial D$ it holds that $\frac{\partial \phi}{\partial \eta} > 0$, where $\eta$ is the outward unit normal. From this, and the deformation formula (7) our claim follows.

To proceed, we will need to show that $\phi$ is appropriately bounded. So we have the following lemma:

**Lemma 4.2.** The exists a constant $c > 0$ independent of $K_+$ such that

$$|\phi| < c.$$ 

**Proof.** Consider the function

$$f = e^{d^2} - e^{\rho^2},$$

where $d$ is the diameter of $D$ and $\rho$ is the distance function from $\gamma(0)$. Now we compute

$$\Delta f = - \left( \Delta \rho^2 \right) e^{\rho^2} - \rho^2 e^{\rho^2}$$

and it is clear that there exists a constant $A > 0$ such that

$$\Delta f < -A.$$ 

Consider the function

$$h = \frac{1}{A} f \sup \left| \frac{1}{2} R \right|. $$

Computing again,

$$\Delta (\phi - h) \geq 0,$$

and also $\phi - h \leq 0$ on $\partial B$. The Maximum Principle shows then that

$$\phi < h \leq \frac{1}{2} \left( e^{d^2} - 1 \right) K.$$ 

Using the fact that $d \sim \frac{1}{\sqrt{K_+}}$, then we obtain the conclusion of the Lemma.

\[ \square \]

The previous Lemma shows that $A_g(D) = O(A_{g_0}(D))$, so to show our claim all we need to do is to do our estimations in the Euclidean space. So consider the following situation. Let $D$ be a convex subset of the plane such that its diameter is at most $r$ and such that there exists a segment joining two points of the boundary and such that it hits the boundary orthogonally at both endpoints, and whose length is at most $l$. A simple geometric argument then shows that its volume is of order $lr$.

4.2. **Perelman’s argument for surfaces with boundary.** To finish the proof of Proposition 4.1 we will show then that the existence of a short geodesic that hits the boundary orthogonally at its endpoints contradicts the monotonicity formula given in theorem 3.2. To achieve this, consider the functional

$$\mathcal{R}(g, \Phi, \tau) = \frac{1}{4\pi \tau} \int_M \left[ 4\tau |\nabla \Phi|^2 + (\tau R - 2\log \Phi - 2) \Phi^2 \right] dA_g.$$ 

The first observation is that given a Riemannian manifold with boundary $(M, g)$

$$\mu(g, \tau) := \inf_{\Phi \in H^1(M), \|\Phi\|_{L^2(M)} = 1} \mathcal{R}(g, \Phi, \tau)$$
does exists. Then methods from Rothaus in [Rot] apply verbatim to this problem. Hence the existence of a smooth positive minimizer can be shown, and this minimizer satisfies the Euler-Lagrange equations

$$
\begin{align*}
\tau (-4\Delta \Phi + R\Phi) - 2\Phi \log \Phi &= (\mu (g, \tau) + 2) \Phi \quad \text{in } M \\
\frac{\partial}{\partial \eta} \Phi &= 0 \quad \text{on } \partial M
\end{align*}
$$

The relation between the functional $R$ and Perelman’s $W$ functional is made apparent by the use of the substitution

$$
\Phi = e^{-\frac{f}{2}},
$$

and, therefore, the existence of smooth minimizer for $W$ is guaranteed.

So in order to reach our contradiction, we must show that the existence of the aforementioned short geodesic implies that $W$ is unbounded from below; hence we must construct appropriate functions to test $W$.

Observe now that the distance function in the manifold $M$ is Lipschitz continuous. Hence it is differentiable almost everywhere, and it is possible to prove that wherever it does exist the gradient of the distance function has norm at most 1. So we can proceed as in [Pe] to show that if there is a sequence of times $t_k \to T$, of points $p_k$ and of radii $0 < r_k < \infty$ such that $0 < R < r_k^{-2}$ in the ball $B_k = B_{r_k}(p_k)$ such that

$$
r_k^{-2} A_g(t_k)(B_k) \to 0
$$

would contradict the monotonicity formula (recall that by $A_g(D)$ we denote the area of $D$ with respect to the metric $g$).

Indeed, in this case we use the same test function as in Section 4 in [Pe]: given a smooth function $\phi$ equal to 1 on $[0, \frac{1}{2}]$, decreasing on $[\frac{1}{2}, 1]$ and equal to $\epsilon_k > 0$ very small on $[1, \infty)$, define

$$
f_k(x) = -\log \left( \phi \left( d_{t_k}(p_k, x) r_k^{-1} \right) \right) + c_k,
$$

where $c_k$ is taken so that

$$
\frac{1}{4\pi r_k^2} \int_M e^{-f_k} dA_{g(t_k)} = 1.
$$

Notice that even if we make $\epsilon \to 0$ in the definition of $f_k$, $c_k$ remains bounded. Also, the choice of $f_k$ implies that there is a constant $C$ independent of $k$ such that

$$
\int_{B_k} e^{-c_k} dA_{g(t_k)} < Cr_k^2.
$$

If we test $W$ with $f_k$, we obtain the following.

$$
\frac{1}{4\pi r_k^2} \int_M 4r_k^2 |\nabla f_k|^2 e^{-f_k} dA_{g(t_k)} = \frac{1}{\pi} \int_M \left| \nabla e^{-\frac{f_k}{2}} \right|^2 dA_{g(t_k)}
$$

is bounded. Indeed, by the choice of $f_k$,

$$
\left| \nabla e^{-\frac{f_k}{2}} \right|^2 = \begin{cases} 
0 & \text{in } B_{r_k}(p_k) \\
\frac{e^{-c_k}}{r_k^2} & \text{on } B_k \setminus B_{\frac{r_k}{2}}(p_k) \\
0 & \text{outside } B_k,
\end{cases}
$$

so we have

$$
\frac{1}{\pi} \int_M \left| \nabla e^{-\frac{f_k}{2}} \right|^2 dA_{g(t_k)} \leq \frac{1}{\pi} \int_{B_k \setminus B_{\frac{r_k}{2}}(p_k)} \frac{e^{-c_k}}{r_k^2} dA_{g(t_k)},
$$
and our claim is proved.

The integral

\[ \frac{1}{4\pi r_k^2} \int_M r_k^2 R e^{-f_k} \, dA_{g(t_k)} \leq r_k^2 \sup_{t_k} R(t_k) \]

is also bounded, since we have \( R < r_k^{-2} \).

Finally we analyze the behavior of

\[ \frac{1}{4\pi r_k^2} \int_M f_k e^{-f_k} \, dA_{g(t_k)} = \frac{1}{4\pi r_k^2} \int_M (f_k - c_k) e^{-f_k} \, dA_{g(t_k)} + \frac{1}{4\pi r_k^2} \int_M c_k e^{-f_k} \, dA_{g(t_k)} \]

The first integral in the previous equality is bounded. This follows from two facts: first of all \( s \log s \) is bounded on \([0, 1]\) and also \( s \log s \to 0 \) as \( s \to 0^+ \), so by taking \( \epsilon > 0 \) very small in the definition of \( \phi \), for any \( \eta > 0 \) we have

\[ \frac{1}{4\pi r_k^2} \int_M (f_k - c_k) e^{-f_k} e^{-c_k} dA_{g(t_k)} \leq C + \frac{\eta}{4\pi r_k^2} \int_{M \setminus B_k} e^{-c_k} dA_{g(t_k)} \]

so by choosing \( \epsilon_k > 0 \) appropriately we can make \( \eta \sim r_k \), which shows that this integral is bounded.

On the other hand, the integral

\[ \frac{1}{4\pi r_k^2} \int_M c_k e^{-f_k} \, dA_{g(t_k)} \]

behaves like \( c_k \), which under assumption (8) goes to \(-\infty\). This implies, by Theorem 3.2, that \( \mu(g_0, t_k + r_k^2) \to -\infty \) as \( k \to \infty \), which is impossible since \( t_k + r_k^2 \to T < \infty \).

The final touch of our argument is then given by what is shown in Section 4.1: the existence of a sequence of times \( t_k \to T \) and a sequence of geodesics \( \gamma_k \) in the metric \( g(t_k) \) that hit the boundary at right angles at both of its endpoints and such that is length \( l_k \) satisfies

\[ \limsup_{k \to \infty} l_k \sqrt{\sup_{M} R(t_k)} = 0 \]

implies the existence of balls \( B_k \) satisfying (8).

5. Smooth convergence along sequences

Now that we can control the injectivity radius of a surface of positive curvature and convex boundary that is evolving under (1), and that bounds on the derivatives of the curvature can be produced from bounds on the curvature (see section 3), then we can form blow up limits. Recall that a blow-up limit is constructed as follows: if \((0, T)\) is the maximum interval of existence for a solution to (1), we pick a sequence of times \( t_j \to T \) and a sequence of points such that

\[ \lambda_j := R(p_j, t_j) = \max_{M \times [0, t_j]} R(x, t) \]

and then we define the dilations

\[ g_j(t) := \lambda_j g\left( t_j + \frac{t - t_j}{\lambda_j} \right), \quad -\lambda_j t_j < t < \lambda_j (T - t_j) \]

and then from the injectivity and curvature bounds from this sequence of dilated metrics we can extract a convergent subsequence towards a solution to the Ricci flow. In our case, we can classify the possible blow up limits we obtain. We recall a result proved in [C1].
Proposition 5.1. There are two possible blow-up limits for a solution of (1) with positive scalar curvature (R > 0). If the blow-up limit is compact, then it is a homotetically shrinking round hemisphere with totally geodesic boundary. If the blow-up limit is non-compact then it is (or its double is) a cigar soliton.

Proposition 5.1 has as a consequence that along a sequence of times t_k → T,
\[ \lim_{k \to \infty} \frac{R_{\max}(t_k)}{R_{\min}(t_k)} = 1. \]
where, obviously,
\[ R_{\max}(t) = \max_{p \in M} R(p, t) \quad \text{and} \quad R_{\min}(p, t) = \min_{p \in M} R(p, t). \]
Indeed, the non-collapsing results preclude the cigar as a blow-up limit.

The following interesting estimate on the evolution of the area A_g(t) of M under the Ricci flow can be given.

Proposition 5.2. There exists constants c_1, c_2 > 0 such that
\[ c_1 (T - t) \leq A_g(t) \leq c_2 (T - t). \]

Proof. Since the blow-up limit is compact, we must have
\[ \lim_{t \to T} A_g(t) = 0. \]
Since R > 0, and \( \int_{\partial M} k_g \) is decreasing, by the Gauss-Bonnet Theorem we have the inequalities
\[ -2\pi \leq \frac{dA_g}{dt} \leq -c, \]
and the result follows by integration. \( \Box \)

As a consequence of the previous proposition we can immediately conclude that:

Corollary 5.1. The normalized flow exists for all time.

Also, an estimate for the maximum of the scalar curvature can be deduced:

Corollary 5.2. There are constants c_1, c_2 > 0 such that
\[ \frac{c_1}{T - t} \leq R_{\max}(t) \leq \frac{c_2}{T - t}. \]

Proof. By the Gauss-Bonnet theorem
\[ \int_M R_{\max}(t) \, dA_g \geq C, \]
and from Corollary 5.2 there is a c' > 0 such that
\[ c'R_{\max}(t) (T - t) \geq C, \]
and the left inequality follows.

To show the other inequality we proceed by contradiction. Assume that there is no constant c_2 > 0 for which
\[ R_{\max}(t) \leq \frac{c_2}{T - t} \]
holds. Then we can find a sequence of times t_j → T such that
\[ R_{\max}(t_j) (T - t_j) \to \infty \]
and hence along this sequence the blow-up limit would not be compact (it would have infinite area by Proposition 5.2), contradicting Proposition 5.1. \( \Box \)
Corollary 5.2 shows that along any sequence of times we can take a blow-up limit since for any sequence of times, the curvature is blowing at maximal rate (i.e. \(\sim \frac{1}{T-t}\)). By Proposition 5.1 this blow-up limit is a round homotetically shrinking sphere. This proves that,

**Theorem 5.1.** Under the unnormalized flow we have that

\[
\lim_{t \to T} \frac{R_{\text{max}}(t)}{R_{\text{min}}(t)} = 1.
\]

As a consequence, under the normalized flow we have that

\[
\tilde{R}_{\text{max}}(\tilde{t}) - \tilde{R}_{\text{min}}(\tilde{t}) \to 0 \quad \text{as} \quad \tilde{t} \to \infty.
\]

Since one can produce bounds on the derivatives of the curvature from bounds on the curvature, and this remains bounded along the normalized flow, we have that along any sequence of times \(\tilde{g}(\tilde{t})\) is converging to a metric of constant curvature 1 (this metrics may be different according to the sequence considered). To be able to conclude that these limit metrics are isometric to that of a hemisphere, we must show that the geodesic curvature of the boundary approaches 0. Indeed, as it is stated in [C1], we can prove a little more. Summarizing we have the following theorem, which as stated in the introduction as the main result of this paper.

**Theorem 5.2.** For the normalized flow, given any sequence of times \(\tilde{t}_n \to \infty\) there exists a subsequence such that \(\tilde{g}(\tilde{t}_n)\) converges smoothly to a metric of constant curvature 1 and totally geodesic boundary. Furthermore, the geodesic curvature of the boundary \(k_{\tilde{g}} \to 0\) exponentially fast as \(\tilde{t} \to \infty\).

**Proof.** Notice that

\[
\frac{c}{T-t} \leq \phi(t) \leq \frac{C}{T-t},
\]

and hence

\[
T-t \leq T e^{-c\tilde{t}}
\]

\[
k_{\tilde{g}} \leq C \sqrt{T-t} \psi \leq c e^{-c\psi}.
\]

\[\Box\]

**Appendix A. An extension procedure**

The purpose of this section is to show an extension procedure for surfaces with boundary. It has been used in the arguments of section [3] (compare with the results in [Kr]).

**Theorem A.1.** Let \((M, g)\) be a surface with boundary. Assume that its curvature is strictly positive and the geodesic curvature of its boundary is strictly positive. Then there exists a closed \(C^2\) surface \((\hat{M}, \hat{g})\) such that \(M\) is isometrically embedded in \(\hat{M}\), and the curvature \(\hat{K}\) of \(\hat{M}\) satisfies

\[
0 < \hat{K} \leq 3K_+,
\]

where \(K_+\) is the maximum of the curvature of \(M\).
Let $\theta \in S^1 \sim \partial M$. Given $K(\theta) \geq 0$ the curvature function of $M$ restricted to $\partial M$, define the following family of functions. First for $\rho < 0$

$$K_\rho(\theta, \zeta) = \begin{cases} K(\theta) + \frac{\rho K(\theta)}{1-\rho} \zeta & \text{if } 0 \leq \zeta < \frac{z_0}{1-\rho} \\ 1 - \frac{\rho}{z_0} K(\theta) & \text{if } \frac{z_0}{1-\rho} \leq \zeta \leq z_0 \end{cases}$$

and for $\rho \geq 0$

$$K_\rho(\theta, \zeta) = K(\theta) + \rho \zeta \quad 0 \leq \zeta \leq z_0.$$}

Observe that for a given $\alpha > 0$ there exists exactly one member of the previously defined family, say $K_{\rho(\alpha)}$ such that

$$\alpha = \int_0^{z_0} K_{\rho(\alpha)}(\zeta) \, d\zeta.$$}

We are ready to extend the metric from a convex surface with boundary to a compact closed surface, keeping control over the maximum of the curvature. Define the warping function

$$f(\theta, z) = 1 + \alpha(\theta) z - \int_0^z \int_0^\zeta K_{\rho(\alpha(\theta))}(\theta, \xi) \, d\xi.$$}

where $\alpha(\theta)$ is the geodesic curvature of $\partial M$ at the point $\theta \in S^1 \sim \partial M$.

Notice that $z_0 > 0$ can be chosen arbitrarily. For our purposes, we will take $z_0 = \frac{\alpha}{K_+}$. If $g_{S^1}$ is the metric of $M$ restricted to its boundary, we define a metric $\hat{g}$ on $N = \partial M \times [0, z_0]$ by

$$\hat{g} = dz^2 + f^2 g_{S^1}.$$}

This metric defines an extension of the metric on the surface $M$ to the surface $\hat{M}_0 = M \cup N$ where $\partial M \subset M$ is identified with $\partial M \times \{0\} \subset N$. It is clear that $\partial N = M \times \{z_0\}$ and that it is totally geodesic.

Let us now estimate the maximum of the curvature in our extension. In a worst case scenario, $K(\theta) = K_+$, and $\alpha(\theta) = \alpha_+$. Then,

$$-f''(z) \leq K_+ + 2 \frac{(K_+)^2}{\alpha_+} z \leq 3K_+.$$}

Since $f' \geq 0$, it follows that $f \geq 1$, and so the curvature is at most $3K_+$.

The produced extension $\hat{M}_0$ is a $C^2$ surface that has totally geodesic boundary, so we can double the extended manifold to obtain a closed $C^2$ manifold of positive curvature, with curvature bounded above by $3K_+$.

**Appendix B. Derivative estimates**

In this section we will show how to produce bounds on the derivatives of the curvature in terms of bounds on the curvature. We will show how to do it in the case of first and second order derivatives, the case of higher derivatives being similar. The ideas we use are quite standard, as we produce certain quantities involving derivatives of $R$ (clearly inspired by the quantities used in the case of closed manifolds), and then we compute some differential inequalities; we will have to make computations on the boundary of $M$ in order to apply the Maximum Principle to these differential inequalities, and even though a bit tedious, these computations are certainly straightforward.
We now fix some notation. Let \( \rho ( P, t ) \) be the distance function to the boundary of \( M \) (which of course changes with time, as the metrics is \( M \) change with time). We define the set
\[
M [0, \epsilon] = \rho^{-1} ([0, \epsilon], 0),
\]
and we will refer to it as the collar of the boundary, or simply as the collar. All the quantities below depend on the time varying metric \( g \), however we will not use any subindex to indicate such dependence. Without much further ado, let us start with our estimates.

**B.1. First order derivative estimates.** We will show the following

**Theorem B.1.** Let \( \epsilon > 0 \) be such that for all \( t = t^* \),
\[
\exp : \nu^- (\epsilon) \rightarrow M
\]
is a diffeomorphism. Assume that there is a bound \( |R| < K \) and that \( 0 \leq k_g \leq \alpha \) on \([t^*, T]\). Then there is a \( \theta := \theta (K, \alpha) \) so that we can estimate,
\[
|\nabla R|^2 \leq C (\alpha) K^2 \left( \frac{1}{\epsilon^2} + \frac{1}{t - t^*} + K \right),
\]
for \( t \in (t^*, t^* + \theta] \).

**Proof.** To simplify matters a little bit, let us fix \( t^* = 0 \). Define on \( M [0, \epsilon] \) and \( t \in [0, T] \) the function
\[
F = te^{\alpha_0} |\nabla^0 R|^2 + AR^2 + 2\alpha_0 K^2,
\]
where \( \nabla^0 \) is the component of the gradient which is tangent to the level surfaces of \( \rho (\cdot, 0) \). Notice that the norm of \( \nabla^0 \) is measured with respect to the time varying metric. \( F \) satisfies a differential inequality in \( M \), namely
\[
\frac{\partial F}{\partial t} \leq \Delta F + \left( te^{\alpha_0} R + 2\alpha_0^2 + e^{\alpha_0^2} + t e^{\alpha_0^2} + t^2 \Delta e^{\alpha_0^2} - A \right) |\nabla R|^2,
\]
\[
-2\alpha_0 \nabla \rho \nabla F + 2\alpha_0 \nabla \rho \left( AR \nabla R + 2AM^2 \nabla \rho \right)
\]
\[
+2AR^3 + \frac{\partial \rho}{\partial t} AM^2 - (\Delta \rho) AM^2.
\]
We need to control a few quantities from the previous expression. For instance, we need to control \( \Delta \rho \). But this is not so difficult, as it is well known that
\[
\Delta \rho (P) = k_g (P)
\]
where \( k_g (P) \) is the geodesic curvature of the level surface of \( \rho (\cdot, 0) \) that passes through \( P \). On the other hand this geodesic curvature is easy to control in terms of the curvature, a bound on the geodesic curvature of \( \partial M \), and \( \epsilon > 0 \) since we have an equation
\[
\frac{\partial k_g}{\partial \rho} = R + k_g^2.
\]
Also, it is not difficult to estimate \( |\frac{\partial \rho}{\partial t}| \leq K \rho \). So if we have \( t \leq \theta (\alpha, K) \), and we choose \( A \) large enough, we obtain,
\[
\frac{\partial F}{\partial t} \leq \Delta F + 2\alpha_0 \nabla \rho \nabla F + C (A, \alpha) K^3.
\]
On the other hand, on \( \partial M \), \( F \) satisfies the inequality
\[
\frac{\partial F}{\partial \eta} = -t \alpha e^{\alpha \rho} |\nabla^g R|^2 + \alpha e^{\alpha \rho} k_g |\nabla^g R|^2 + 2Ak_g R^2 - 2AK^2 \leq 0,
\]
where we have used the fact
\[
\frac{\partial}{\partial \eta} |\nabla^g R|^2 = k_g |\nabla^g R|^2.
\]
In the part of the boundary of the collar that lies in the interior of the manifold, by Shi’s interior estimates, we have,
\[
F \leq C (A, \alpha) K^3 + te^{\alpha \rho} K^2 \left( \frac{1}{e^2} + \frac{1}{t} + K \right).
\]
Applying the maximum principle, we obtain
\[
F \leq \max_{t \in [0, \tau]} F + tC (A, \alpha) K^3 + C (A, \alpha) K^3 + te^{\alpha \rho} K^2 \left( \frac{1}{e^2} + \frac{1}{t} + K \right),
\]
from which we obtain
\[
|\nabla^g R|^2 \leq c \left[ K^3 + tK^2 \left( \frac{1}{e^2} + \frac{1}{t} + K \right) \right].
\]
Now, let \( \eta \) be a unit vector field orthogonal, with respect to the time varying metric, to the level curves of \( \rho (\cdot, 0) \), and which coincides with the outward unit normal on \( \partial M \) (so \( \eta \) also depends on the time \( t \)). We could take for instance \( \eta = -c \frac{\partial}{\partial \rho} \) where \( \frac{\partial}{\partial \rho} \) is the unit vector field orthogonal to the level curves of \( \rho (\cdot, 0) \) at time \( t = 0 \) which coincides with the outward unit normal to \( \partial M \) at time \( t = 0 \), and \( c \) is a constant taken so that \( g(\eta, \eta) = 1 \) (the reader must have present here that the Ricci flow in dimension 2 preserves the conformal structure). Define
\[
G = \left( \frac{\partial R}{\partial \eta} \right)^2 + AR^2.
\]
Again we have that \( G \) satisfies a differential inequality
\[
\frac{\partial G}{\partial t} \leq \Delta G + 2AR^2;
\]
on the other hand using the expression for \( \frac{\partial R}{\partial \eta} \) on \( \partial M \) and Shi’s interior derivative estimates, in both components of the boundary of the collar we have an estimate
\[
G \leq tk_g^2 K^2 + 2AK^2 + C tK^2 \left( \frac{1}{e^2} + \frac{1}{t} + K \right).
\]
This gives an estimate for \( G \) completely analogous to the estimate obtained for \( F \). This proves the theorem. \( \square \)

**B.2. Second order derivative estimates.** Here we sketch how to bound second order derivatives on an interval of time \([0, t]\) where we have assumed a bound \( K \) on absolute value of the curvature; higher derivatives estimates can be produced following a similar strategy. We will divide our estimates in two classes.

First, we let
\[
G = \left( (\nabla^g)^2 R \right)^2 + \left( \frac{\partial R}{\partial t} \right)^2.
\]
where \((\nabla^\partial)^2\) denotes two covariant derivatives taken in the direction of vectors tangent to the level curves of \(\rho(\cdot,0)\). We see \((\nabla^\partial)^2\) as a 2-tensor, that when any of its entries is a vector perpendicular (with respect to the time varying metric) to the level curves of \(\rho(\cdot,0)\) then its value is 0. We then compute its norm \(\left|\left((\nabla^\partial)^2\right) R\right|\)
with respect to the time varying metric \(g(t)\) accordingly. Define

\[F = te^{\beta\rho}G + A \left( |\nabla R|^2 + R^2 \right) + \rho N^2,\]

where \(\beta\), \(A\) and \(N\) are constants to be defined.

A tedious, but straightforward computation, shows that there exists a \(\theta = \theta(K,\alpha)\) such that on \([0,\theta]\) \(F\) satisfies a differential inequality

\[
\frac{\partial F}{\partial t} \leq \Delta F + 2\beta \nabla F \nabla \rho + B \left( |R|^2 + |\nabla R|^2 + |\nabla^\partial R|^2 \right) + \left( \frac{\partial \rho}{\partial t} - \Delta \rho \right) N^2
\]

and \(B\) is a new constant obtained from \(A\), and we also have a bound, let us call it \(C\), on \(\frac{\partial \rho}{\partial t} - \Delta \rho\) in terms of bounds on \(R\), \(k_g\) and \(\epsilon\). In \(\partial M\) we can compute

\[
\frac{\partial}{\partial \eta} \left( (\nabla^\partial)^2 R \right)^2 = (2k_g + 1) \left| (\nabla^\partial)^2 R \right|^2 + P (k_g, \nabla k_g, \nabla^2 k_g, R, \nabla R),
\]

and \(\nabla\) denotes covariant differentiation. On the other hand, denoting by \(R_\eta = \nabla_\eta R\),
we can estimate

\[
\left| \frac{\partial R_\eta^2}{\partial \eta} \right| = 2 |R_\eta R_\eta| \\
\leq 2 \left| \left( R_t - (\nabla^\partial)^2 R \right) R_\eta \right| \\
= 2 \left| \left( R_t - (\nabla^\partial)^2 R \right) \cdot k_g R \right| \\
\leq te^{\beta\rho}G + \frac{e^{-\beta\rho}}{t} k_g^2 R^2.
\]

Also, we can compute,

\[
\frac{\partial R_t^2}{\partial \eta} = 2 R_t \left( (R_t)_\eta + RR_\eta \right) \\
= 2k_g R_t^2 + k_g R^2
\]

So if we take \(\alpha \geq 2k_g + 2\) and \(N^2\) such that

\[N^2 \geq P + 2Ak_g |R| \left( |\nabla^\partial R|^2 + R \right) + \frac{e^{-\beta\rho}}{t} k_g^2 R^2.\]

on the interval \([0,\theta]\), we can guarantee that on \(\partial M\)

\[
\frac{\partial F}{\partial \eta} \leq 0.
\]

On the boundary of the collar that lies in the interior of the manifold we can estimate \(F\) by using the interior derivative estimates. From this we can conclude that an estimate

\[
\left| (\nabla^\partial)^2 R \right|^2 + \left( \frac{\partial R_t}{\partial t} \right)^2 \leq \frac{A}{t} \left( K' + CN^2 \right) + BK'\]
where $K'$ and $K''$ are bounds on $|\nabla R|^2 + R^2$ and $|R|^5 + |R||\nabla R|^2 + |\nabla R|^2$ respectively. A bound for $N$ is easily obtained from bounds $R$, $|\nabla R|$ and $k_g$ and its derivatives.

Notice that we obtain a bound on $R_{\eta\eta}$, since if $T$ is a unit vector field tangent to the level curves of $\rho(\cdot,0)$, then we have

$$\nabla^2 R = \Delta R - \nabla^2_{TT} R = \frac{\partial R}{\partial t} - R^2 - \nabla^2_{TT} R,$$

and since we just produced an estimate on $\nabla^2_{TT}$, we easily obtain a bound on $\nabla^2_{\eta\eta}$.

Now we try to estimate a second covariant derivative of the form $\nabla^2_{\eta\eta}$, where $\eta$ is a unit vector field orthogonal to the level curves of $\rho(\cdot,0)$ and which coincides with the outward unit normal on $\partial M$; as before, $\nabla^\theta$ represents a covariant derivative taken with respect to a vector field tangent to the level curves of $\rho(\cdot,0)$. Again we define

$$F = t \left| \nabla^\theta \nabla_\eta R \right|^2 + A \left( |\nabla R|^2 + R^2 \right),$$

and again, there is a $\theta = \theta(K,\alpha)$ such that $F$ satisfies a differential inequality

$$\frac{\partial F}{\partial t} \leq \Delta F + C \left( R^2 + |R||\nabla R|^2 \right).$$

on $[0,\theta]$. In $\partial M$, we have

$$F \leq B \left( R^2 + |\nabla^\theta R|^2 \right) + A |\nabla R|^2,$$

and $B$ depends on bounds on $k_g$ and its derivatives. Since we have bounds on the gradient of $R$, using the fact that in the interior boundary of the collar we also have bounds for $F$ due, once again, by the interior derivative estimates, we obtain a bound

$$\left| \nabla^\theta \nabla_\eta R \right|^2 \leq C' \frac{t}{t} \left( |\nabla R|^2 + R^2 + C'' t \right),$$

where $C''$ is a bound on $R^2 + |R||\nabla R|^2$ on $[0,\theta]$, and hence we can bound second covariant derivatives in terms of bounds on the curvature.

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THE RICCI FLOW ON SURFACES WITH BOUNDARY

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Abstract. We study a boundary value problem for the Ricci flow on a surface with boundary, where the geodesic curvature of the boundary is prescribed.

Key words: Ricci flow; surfaces with boundary.

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1. Introduction

The Ricci flow on surfaces, compact and noncompact, have been an intense subject of study since the appearance of Hamilton’s seminal work [13], where the asymptotic behavior of the flow is studied on closed surfaces, and it is used as a tool towards giving a proof of the Uniformization Theorem via parabolic methods. In addition to its obvious geometric appeal, it is of note that the study of the Ricci flow on surfaces is related to the study of the logarithmic diffusion equation, and hence, the interest on this problem goes beyond its geometric applications (see [16]).

However, not much is known about the behavior of the Ricci flow on manifolds with boundary. One of the main difficulties in studying this problem arises from the fact that even trying to impose meaningful boundary conditions for the Ricci flow, for which existence and uniqueness results can be proved so interesting geometric applications can be hoped for, seems to be a challenging task. For the reader to get an idea of the difficulty of the problem, we recommend the interesting works of Y. Shen [23], S. Brendle [3], A. Pulemotov [22] and P. Gianniotis [12]. In the case of the boundary conditions imposed by Shen [23], satisfactory convergence results have given for manifolds of positive Ricci curvature and totally geodesic boundary, and when the boundary is convex and the metric is rotationally symmetric. In the case of surfaces, the Ricci flow is parabolic, and imposing natural geometric boundary conditions is not difficult: one can for instance control the geodesic curvature of the boundary. In this case, Brendle [3] has shown that when the boundary is totally geodesic, then the behavior is completely analogous to the behavior of the Ricci flow in closed surfaces (15, 6). In this case, also for non totally geodesic boundary, the first author has proved, under the hypothesis of rotational symmetry of the metrics involved, results on the asymptotic behavior of the Ricci flow in the case of positive curvature and convex boundary, and for certain families of metrics with non convex boundary (9).

The purpose of this paper is to contribute towards the understanding of the behavior of the Ricci flow on surfaces with boundary. To be more precise, let \( M \) be a compact surface with boundary (\( \partial M \neq \emptyset \)), endowed with a smooth metric \( g_0 \);
we will study the equation

\begin{equation}
\begin{cases}
\frac{\partial g}{\partial t} = -R_g g & \text{in } M \times (0,T) \\
k_g (\cdot, t) = \psi (\cdot, t) & \text{on } \partial M \times (0,T) \\
g (\cdot, 0) = g_0 (\cdot) & \text{in } M,
\end{cases}
\end{equation}

where $R_g$ represents the scalar curvature of $M$ and $k_g$ the geodesic curvature of $\partial M$, both with respect to the time evolving metric $g$, and $\psi$ is a smooth real valued function defined on $\partial M \times [0,\infty)$, and which satisfies the compatibility condition $\psi (\cdot, 0) = k_{g_0}$.

The existence theory of equation (1) is well understood. Indeed, since the deformation given by (1) is conformal, if we write $g(p, t) = e^{u(p, t)} g_0$, problem (1) is equivalent to a nonlinear parabolic equation with Robin boundary conditions, and initial condition $u(p, 0) = 1$. Hence, via the Inverse Function Theorem and standard methods from the theory of parabolic equations [20], it can be shown that (1) has a unique solution for a short time, and that this solution is $C^{2+\alpha, 1+\frac{\alpha}{2}}$, $0 < \alpha < 1$, on $M \times [0,T)$, and smooth away from the corner.

Before we state the results we intend to prove in this paper, we must introduce a normalization of (1). As it is well known, the solution to (1) can be normalized to keep the area of the surface constant as follows. Let us assume without loss of generality assume that the area of $M$ with respect to $g_0$ is $2\pi$, and choose $\phi (t)$ such that $\phi (t) A_g (t) = 2\pi$, where $A_g (t)$ is the area of the surface at time $t$ with respect to the metric $g$. Then define

\begin{equation}
\tilde{t} (t) = \int_0^t \phi (\tau) \, d\tau \quad \text{and} \quad \tilde{g} = \phi g.
\end{equation}

If the family of metrics $g (t)$ satisfies (1), then the family of metrics $\tilde{g} (\tilde{t})$ satisfies the evolution equation

\begin{equation}
\begin{cases}
\frac{\partial \tilde{g}}{\partial \tilde{t}} = (\tilde{R}_g - \tilde{R}_\tilde{g}) \tilde{g} & \text{in } M \times (0, \tilde{T}) \\
k_{\tilde{g}} (\cdot, \tilde{t}) = \tilde{\psi} (\cdot, \tilde{t}) & \text{on } \partial M \times (0, \tilde{T}) \\
\tilde{g} (\cdot, 0) = g_0 (\cdot) & \text{on } M,
\end{cases}
\end{equation}

where $\tilde{\psi}$ is the normalization of the function $\psi$, $\tilde{R}_g$ is the scalar curvature of the metric $\tilde{g}$, and

$$\tilde{R}_\tilde{g} = \frac{\int_M \tilde{R}_{\tilde{g}} \, dA_{\tilde{g}}}{\int_M dA_{\tilde{g}}} = \frac{1}{2\pi} \int_M \tilde{R}_{\tilde{g}} \, dA_{\tilde{g}}.$$ 

Here $dA_{\tilde{g}}$ denotes the area element of $M$ with respect to the metric $\tilde{g}$. We refer to (2) as the normalized Ricci flow.

We can now state our first result.

\textbf{Theorem 1.1.} Let $(M^2, g_0)$ be a compact surface with boundary with positive scalar curvature ($R_{g_0} > 0$), and such that the geodesic curvature of its boundary is nonnegative ($k_{g_0} \geq 0$), and assume that $\psi$, as defined above, is nonnegative and also satisfies that $\frac{1}{\tilde{t}_0} \psi \leq 0$. Let $g (t)$ be the solution to (1) with initial condition $g_0$. Then the solution to the normalized flow, $\tilde{g} (\tilde{t})$, exists for all time, and for any sequence $\tilde{t}_n \to \infty$, there is a subsequence $\tilde{t}_{n_k} \to \infty$ such that the metrics $\tilde{g} (\tilde{t}_{n_k})$ converge smoothly to a metric of constant curvature and totally geodesic boundary.
Theorem 1.1 partially extends results on the asymptotic behavior of solutions to the Ricci flow known for $S^2$ (Hamilton [14] and Chow [6]), and in the case of surfaces with totally geodesic boundaries (Brendle [3]) and with rotational symmetry ([9]).

Before going any further, let us give an outline of the proof of Theorem 1.1. First of all, given an initial metric $g_0$ of positive scalar curvature and convex boundary, it can be shown that the curvature of $g(t)$ blows up in finite time, say $T < \infty$; the idea then is to take a blow up limit of the solution $(M, g(t))$ as $t \to T$, and to show that the only possibility for this blow up limit is to be a round hemisphere: this would, essentially, give a proof of Theorem 1.1. It remains then to remove one technical difficulty: we must be able to produce this blow up limit, and hence we will have to show that we can estimate the injectivity radius of the surface, and, because it has boundary, we are required to show that there are no geodesics hitting the boundary orthogonally that are too short with respect to the inverse of the square root of the maximum of the curvature: this is the basic new ingredient in the proof of Theorem 1.1, and to prove that estimate we have introduced an extension procedure for surfaces with boundary that allows some control over the maximum curvature and size of the extension.

Our second result is concerned with the behavior of the Ricci flow when the geodesic curvature of the boundary is nonpositive. Again, using blow up analysis techniques, we prove the following theorem, which generalizes similar results from [9] (notice that we make no requirements on the sign of $R$).

**Theorem 1.2.** Let $g_0$ be a rotationally symmetric metric on the two-ball. Assume that $k_{g_0} \leq 0$, and that the boundary data is given by $\psi = k_{g_0}$. Then the normalized flow corresponding to the solution to (1) with initial data $g_0$ and boundary data $\psi$ exists for all time.

The layout of this paper is as follows. In Section 2 we prove the basic evolution equation for the scalar curvature when the metric evolves under (1), and show that under certain conditions the curvature $R$ blows up in finite time; in Section 3 we prove a monotonicity formula for Perelman’s functionals on surfaces with boundary; in Section 4 it is shown that it is possible to take blow up limits for solutions to (1), by proving that we can control the injectivity radius of the surface in terms of the scalar curvature and the geodesic curvature of the boundary; in Section 5 we use the results from the previous sections to give a proof of Theorem 1.1. Finally, in Section 6 we give a proof of Theorem 1.2. This paper is complemented by an appendix where we show a procedure, completely analogous to the method used in the boundaryless case, to bound covariant derivatives of the curvature of solutions to (1) - and hence to (3) - in terms of bounds on the curvature and the boundary data (and its derivatives).

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## 2. Evolution equations

In the following proposition, which is stated in [9] without proof, we compute the evolution of the curvature of a metric $g$ when it is evolving under (1).
Proposition 2.1. Let \((M, g(t))\) be a solution to \((\mathcal{I})\). The scalar curvature satisfies the evolution equation
\[
\begin{cases}
\frac{\partial R_g}{\partial t} = \Delta_g R_g + R_g^2 & \text{in } M \times (0, T) \\
\frac{\partial R_g}{\partial \eta_g} = k_g R_g - 2k'_g = \psi R_g - 2\psi' & \text{on } \partial M \times (0, T)
\end{cases}
\]
where \(\eta_g\) is the outward pointing unit normal with respect to the metric \(g\), and the prime (’) represents differentiation with respect to time.

Proof. Since the evolution equation satisfied by \(R\) in the interior of \(M\) is known \((\mathcal{I}_4)\), we will just compute its normal derivative, with respect to the outward normal, at the boundary. To do so, we choose local coordinates \((x^1, x^2)\) at \(p \in \partial M\) such that \(x^2 = 0\) is a defining function for \(\partial M\), so that the corresponding coordinate frame \(\{\partial_1, \partial_2\}\) is orthonormal at \(p \in \partial M\) and time \(t = t_0\) (i.e., the point and instant when we want to compute the normal derivative), and so that \(\partial_2\) coincides with the outward unit normal to the boundary in the whole coordinate patch (this also at time \(t = t_0\)). Since the deformation is conformal, \(\partial_2\) remains normal to the boundary. Therefore the geodesic curvature is given (as long as the flow is defined for \(t \geq t_0\) by
\[
k_g g_{11} = -\frac{\Gamma^2_{11}}{(g^{22})^{\frac{3}{2}}} = -(g^{22})^{\frac{1}{2}} \Gamma^2_{11}.
\]
Computing the time derivative of the previous identity yields
\[
(k_g g_{11})' = -\frac{1}{2(g^{22})^{\frac{3}{2}}} (g^{22})' \Gamma^2_{11} - (g^{22})^{\frac{1}{2}} (\Gamma^2_{11})' = \frac{1}{2} R_g (g^{22})^{\frac{1}{2}} \Gamma^2_{11} - (g^{22})^{\frac{1}{2}} (\Gamma^2_{11})'.
\]
Let us calculate \((\Gamma^2_{11})'\) (as is customary \(\nabla_j\) denotes covariant differentiation with respect to \(\partial_j\), and recall that \(g_{12} = 0\) and \(g_{ii} = 1\))
\[
(\Gamma^2_{11})' = \frac{1}{2} g^{2j} (\nabla_1 g'_{1j} + \nabla_1 g'_{j1} - \nabla_j g'_{11}) = \frac{1}{2} g^{22} (-2 \nabla_1 (R_g g_{12}) + \nabla_2 (R_g g_{11})) = \frac{1}{2} g^{22} \partial_2 R_g g_{11}. \]
Therefore
\[
k'_g g_{11} - k_g R g_{11} = -\frac{1}{2} k_g R g_{11} = -\frac{1}{2} k_g R g_{11} - \frac{1}{2} \frac{\partial R_g}{\partial \eta_g},
\]
and the result follows. \(\square\)

As a consequence from the Maximum Principle, since \(k'_g = \psi' \leq 0\) in the case we are considering, we obtain the following result.

Proposition 2.2. Let \((M, g(t))\), \(M\) compact, be a solution to \((\mathcal{I})\). Assume that \(\psi\), the boundary data, satisfies \(\psi' \leq 0\). Then, if \(R_g \geq 0\) at time \(t = 0\), it remains so long as the solution exists. Furthermore, if the initial data has positive scalar curvature and the boundary data \(\psi\) is nonnegative, then \(R_g\) remains strictly positive, and \(R_g\) blows up in finite time.
2.1. In view of Proposition 2.2, this seems a good place to discuss the following fact. Let \((0, T)\) be the maximal interval of existence of a solution to (1), with \(0 < T < \infty\), then
\[
\limsup_{t \to T} \left( \sup_{p \in M} R_g(p, t) \right) = \infty.
\]
First of all if \(g_0\) is the initial metric, then as the Ricci flow preserves conformal structure, we have that the evolving metric can be represented as \(g = e^u g_0\). Hence, if \(R_{g_0}\) is the scalar curvature of the initial metric, at a fixed (but arbitrary) time, we have that \(u\) satisfies the elliptic boundary value problem
\[
\begin{aligned}
\Delta_{g_0} u + R_{g_0} &= R_{g_0} e^u \quad \text{in } M \\
\frac{\partial}{\partial \nu_{g_0}} u + 2k_{g_0} e^u &= 2k_{g_0} e^{\frac{u}{2}} \quad \text{on } \partial M.
\end{aligned}
\]
To reach a contradiction assume that \(R_g\) remains uniformly bounded on \((0, T)\). A consequence of this assumption is that \(e^u\) remains bounded away from 0 and uniformly bounded above on \((0, T)\). This follows from the equation
\[
\frac{\partial}{\partial t} e^u = -e^u R_g, \quad e^u(0) = 1.
\]
Multiplying the first equation in (4) by \(u\) and integrating by parts, and then by \(e^{2u}\) and integrating by parts, it follows that \(u \in H^1(M)\) and also \(e^u \in H^1(M)\). Now, since from bounds on the curvature and also on the geodesic curvature of the boundary (i.e., on \(\psi\)) and its derivatives, we can obtain bounds on the derivatives of \(R\), the methods in [5] guarantee that \(u\) and its derivatives (including those with respect to \(t\)) are uniformly bounded on \((0, T)\), and consequently they converge as \(t \to T\) in \(C^\infty\) to a smooth function, say \(\hat{u}\). If we start the Ricci flow at \(t = T\) with initial data \(e^\hat{u} g_0\) and the same boundary data, then we would be able to continue the original solution past \(T\), which contradicts the hypothesis. Therefore, if the Ricci flow (1) cannot be extended past \(T < \infty\), the curvature blows up. We invite the reader to consult the recent work of Gianniotis [12], where this property of the Ricci flow on manifolds with boundary is discussed in a more general context.

2.2. There are other interesting cases when solutions to (1) blow-up. We have for instance the following proposition.

**Proposition 2.3.** Assume that \(\int_M R_{g_0} \, dA_{g_0} + \int_{\partial M} k_{g_0} \, ds_{g_0} > 0\), and assume that \(\psi \leq 0\). Then the solution to (1) with initial condition \(g_0\) and boundary data \(\psi\), blows up in finite time.

**Proof.** Let \(g(t)\) be the solution to (1) with initial data \(g_0\) and boundary data \(\psi\). If \(A(t)\) represents the area of \(M\) with respect to \(g(t)\), we can calculate
\[
\frac{dA}{dt} = -\int_M R_g \, dA_g = -4\pi \chi(M) + 2 \int_{\partial M} k_g \, ds_g \leq -4\pi \chi(M).
\]
Therefore, the area cannot be positive for all time. Since we also have that \(\frac{dA}{dt} \geq -R_{\max}(t) A\), where \(R_{\max}(t)\) is the maximum of the curvature at time \(t\), a singularity must occur in finite time. \(\square\)
3. Monotonicity of Perelman’s Functionals on surfaces with boundary.

The purpose of this section is to show a monotonicity formula for Perelman’s celebrated $F$ and $W$ functionals (see [21]) in the case of surfaces with boundary. The results in this section are stated more or less in the same way in [9], although with no carefully crafted proofs. As usual, all curvature quantities, scalar products and operators depend on the time-varying metric $g$ (some of them will not bear a subindex to show that dependence). We will use the Einstein summation convention freely, and the raising and lowering of indices is done, by means of the metric $g$, in the usual way.

In order to proceed, recall the definition of Perelman’s $F$-functional:

$$F(g_{ij},f) = \int_M \left( R_g + |\nabla f|^2 \right) \exp(-f) \, dV_g,$$

where $dV_g$ represents the volume (in the case of a surface, area) element of the manifold $M$ with respect to the metric $g$. Let us compute the first variation of this functional on a manifold with boundary.

**Proposition 3.1.** Let $\delta g_{ij} = v_{ij}, \delta f = h, g^{ij}v_{ij} = v$. Then we have,

$$\delta F = \int_M \exp(-f) \left[ -v^{ij}(R_{ij} + \nabla_i \nabla_j f) + \left( v^2 - h \right) \left( 2\Delta_g f - |\nabla f|^2 + R_g \right) \right] dV_g$$

$$- \int_{\partial M} \left[ \frac{\partial v}{\partial \eta} + (v - 2h) \frac{\partial f}{\partial \eta} \right] \exp(-f) \, d\sigma_g +$$

$$\int_{\partial M} \exp(-f) \nabla_i v_{ij} \eta^i \, d\sigma_g - \int_{\partial M} \nabla_j \exp(-f) v_{ij} \eta^i \, d\sigma_g.$$

Here, $R_{ij}$ represents the Ricci tensor of the metric $g$, $\frac{\partial}{\partial \eta}$ (= $\eta^i \partial_i$ in local coordinates) is the outward unit normal to $\partial M$ with respect to $g$, $\nabla$ represents covariant differentiation with respect to the metric $g$, and $d\sigma_g$ represents the volume element of $\partial M$.

**Proof.** As in [17], we have that

$$\delta F(v_{ij}, h) = \int_M e^{-f} \left[ -\Delta_g v + \nabla_i \nabla_j v^{ij} - R_{ij}v^{ij} \right.$$

$$\left. - v^{ij} \nabla_i f \nabla_j f + 2g(\nabla f, \nabla h) + \left( R_g + |\nabla f|^2 \right) \left( v^2 - h \right) \right] \, dV_g.$$

We must compute the integrals on the righthand side of the previous identity, our main tool being integration by parts. We start by calculating

$$\int_M e^{-f} (-\Delta_g v) \, dV_g = - \int_M \Delta_g e^{-f} v \, dV_g + \int_{\partial M} v \frac{\partial e^{-f}}{\partial \eta} \, d\sigma_g - \int_{\partial M} e^{-f} \frac{\partial v}{\partial \eta} \, d\sigma_g$$

$$= - \int_M \Delta_g e^{-f} v \, dV_g - \int_{\partial M} \left( \frac{\partial v}{\partial \eta} + v \frac{\partial f}{\partial \eta} \right) \exp(-f) \, d\sigma_g.$$
Now we compute
\[ \int_M e^{-f} \nabla_i \nabla_j v^{ij} \, dV_g = - \int_M \nabla_i e^{-f} \nabla_j v^{ij} \, dV_g + \int_{\partial M} e^{-f} \nabla_i v^{ij} \eta^i \, d\sigma_g \]
\[ = \int_M \nabla_i \nabla_j e^{-f} v^{ij} \, dV_g - \int_{\partial M} \nabla_i e^{-f} v^{ij} \eta_j \, d\sigma_g \]
\[ + \int_{\partial M} e^{-f} \nabla_i v^{ij} \eta^i \, d\sigma_g. \]

Finally,
\[ 2 \int_M e^{-f} g(\nabla f, \nabla h) \, dV_g = -2 \int_M g(\nabla e^{-f}, \nabla h) \, dV_g \]
\[ = 2 \int_M (\Delta_g e^{-f}) \, h \, dV_g - \int_{\partial M} h \frac{\partial e^{-f}}{\partial \eta_g} \, d\sigma_g. \]

Putting all these calculations together proves the result. \( \Box \)

Consider the evolution equations on a surface with boundary given by
(5)
\[ \begin{align*}
\frac{\partial}{\partial t} g_{ij} &= -R_g g_{ij} - 2R_{ij} \quad \text{in } M \times (0, T) \\
\frac{\partial}{\partial t} f &= -\Delta_g f + |\nabla f|^2 - R_g \quad \text{in } M \times (0, T) \\
\frac{\partial}{\partial \eta} f &= 0 \quad \text{on } \partial M \times (0, T).
\end{align*} \]

A fundamental formula for \( \mathcal{F} \) is given by the following result.

**Theorem 3.1.** Under (5) the functional \( \mathcal{F} \) satisfies
\[ \frac{d}{dt} \mathcal{F} = 2 \int_M \left( R_{ij} + \nabla_i \nabla_j f \right)^2 \exp(-f) \, dA_g \]
\[ + \int_{\partial M} \left( k_g R_g - 2k'_g \right) \exp(-f) \, ds_g + 2 \int_{\partial M} k_g |\nabla^T f|^2 \exp(-f) \, ds_g, \]

and here \( \nabla^T f \) represents the component of \( \nabla f \) tangent to \( \partial M \), \( dA_g \) the area element of the surface, and \( ds_g \) the length element of the boundary.

**Proof.** Let us first introduce some notation and conventions. Since parts of these computations apply to manifolds of higher dimensions, in this proof we will fix coordinates \( x^1, x^2, \ldots, x^{n-1}, x^n \) at a boundary point and at fixed (but arbitrary) time \( t \), so that \( x^n = 0 \) is a defining function for \( \partial M \). We will assume that on \( \partial M \),
\[ \frac{\partial}{\partial x^n} \]
represents the outward unit normal, and hence we will denote by a subscript or superscript \( n \) quantities that are evaluated, at a boundary point, with respect to the outward unit normal. By a greek letter we will represent indices running from \( 1, 2, 3, \ldots, n-1 \), and therefore at a boundary point the vector fields \( \frac{\partial}{\partial x^i} \) are tangent to the boundary. Let us transform the evolution equations given by (5) using the one-parameter family of diffeomorphisms \( \varphi_t \) generated by \( -\nabla f \); notice that the boundary is sent to itself via this family of diffeomorphisms due to the fact that \( \frac{\partial \varphi}{\partial \eta} = 0 \). Now, by defining \( f(\cdot, t) = f(\varphi_t(\cdot), t) \) and \( g = (\varphi_t)_* g \) (forgive the abuse of notation), instead of (5) we must take the variations given by
\[ v_{ij} = \delta g_{ij} = -2(R_{ij} + \nabla_i \nabla_j f), \quad h = \delta f = -\Delta_g f - R.
For a moment let us denote with a subindex \((\varphi_t)_* g\) the quantities that depend on the pullback metric. Observe that then we have
\[
\frac{\partial}{\partial \eta} R_{(\varphi_t)_* g} (\cdot, t) = (k_g R_g - 2k'_g) (\varphi_t (\cdot), t),
\]
so keeping on with the abuse of notation, we will write, for the metric \(g = (\varphi_t)_* g\),
\[
\frac{\partial R}{\partial \eta} = k_g R_g - 2k'_g,
\]
where the prime ('') now means that we differentiate \(k_g\) (or \(\psi\)) with respect to its second variable \((t)\) and then it is evaluated at \((\varphi_t (\cdot), t)\). Notice that for the pullback metric we still have \(\frac{\partial f}{\partial \eta} = 0\).

We will now compute each of the boundary integrals in the first variation of Perelman’s functional given by Proposition 3.1 which will prove the theorem, since the computations for the integrals over \(M\) are known from the work of Perelman. We change the notation from the previous proposition as follows: \(dV_g = dA_g\) and \(d\sigma_g = ds_g\). We start with
\[
\int_{\partial M} \left[ \frac{\partial v}{\partial \eta} + (v - 2h) \frac{\partial f}{\partial \eta} \right] \exp(-f) \, ds_g, \quad \int_{\partial M} \exp(-f) \nabla_i v^i \eta^j \, ds_g.
\]
To compute these integrals, let us first calculate \(\nabla_i v^i \eta_j\). We have
\[
\nabla_i v^i_n = -2\nabla_i R^i_n - 2\nabla_i \nabla^i \nabla_n f = -\nabla_n R_g - 2\Delta_g \nabla_n f \quad \text{(by the contracted Bianchi identity)}.
\]
By the Ricci identity, using the fact that at the boundary \(\nabla_n f = 0\) and also \(R_{\alpha n} = 0\), we obtain
\[
\Delta_g \nabla_n f = \nabla_n \Delta_g f + R_n^k \nabla_k f = \nabla_n \Delta_g f,
\]
and therefore,
\[
\nabla_i v^i_n = -\nabla_n R_g - 2\nabla_n \Delta_g f.
\]
Using the evolution equation \(f_t = -\Delta_g f - R\), we get, at the boundary,
\[
\nabla_n \Delta_g f = -\nabla_n R_g.
\]
This last identity has two consequences. On the one hand, it implies that
\[
\nabla_i v^i_n = -\nabla_n R_g + 2\nabla_n R = \nabla_n R_g = k_g R_g - 2k'_g,
\]
which shows that
\[
\int_{\partial M} \exp(-f) \nabla_i v^i \eta_j \, ds_g = \int_{\partial M} (k_g R_g - 2k'_g) \exp(-f) \, ds_g.
\]
On the other hand, it implies that \(\frac{\partial v}{\partial \eta} = 0\), so we obtain
\[
\int_{\partial M} \left[ \frac{\partial v}{\partial \eta} + (v - 2h) \frac{\partial f}{\partial \eta} \right] \exp(-f) \, ds_g = 0.
\]
Let us now compute the integral
\[
II = -\int_{\partial M} \nabla_i \exp(-f) v^i \eta_j \, ds_g = -\int_{\partial M} \nabla_i \exp(-f) v^i \, ds_g.
\]
Under the previous conventions,

\[ II = \int_{\partial M} \nabla_n f \exp(-f) v_n^a \, ds_g + \int_{\partial M} \nabla_n f \exp(-f) v_n^a \, ds_g. \]

Using the fact that \( \nabla_n f = \partial_n f = 0 \) on \( \partial M \), we can compute

\[ \nabla^\alpha \nabla_n f \nabla_\alpha f = -H (\nabla^\top f, \nabla^\top f), \]

where \( H \) denotes the second fundamental form of the boundary. Hence, using the definition of \( v_\alpha n \), we get

\[ II = \int_{\partial M} 2H (\nabla^\top f, \nabla^\top f) \exp(-f) \, ds_g = \int_{\partial M} 2k_g |\nabla^\top f|^2 \exp(-f) \, ds_g, \]

and the formula is proved. \( \square \)

Next, we consider Perelman’s \( W \)-functional, namely,

\[ W(g,f,\tau) = \int_M [\tau (|\nabla f|^2 + R_g) + f - 2] (4\pi \tau)^{-1} \exp(-f) \, dA_g. \]

Under the unnormalized Ricci flow (1), and the evolution equations

\[
\begin{align*}
\frac{\partial f}{\partial t} &= -\Delta_g f + |\nabla f|^2 - R_g + \frac{1}{\tau} \quad \text{in} \quad M \times (0,T) \\
\frac{\partial \tau}{\partial t} &= -1 \quad \text{in} \quad (0,T) \\
\frac{\partial f}{\partial \eta} &= 0 \quad \text{on} \quad \partial M \times (0,T),
\end{align*}
\]

we have the following formula, which shows a monotonicity property as long as \( k_g \geq 0 \) and \( k'_g = \psi' \leq 0 \), for the functional \( W \).

**Theorem 3.2.**

\[
\frac{d}{dt} W = \int_M 2\tau \left| R_{ij} + \nabla_i \nabla_j f - \frac{1}{2\tau} g_{ij} \right|^2 (4\pi \tau)^{-1} \exp(-f) \, dA_g + \frac{1}{4\pi^2} \left( \int_{\partial M} (k_g R_g - 2k'_g + 2k_g |\nabla^\top f|^2) \exp(-f) \, ds_g \right). 
\]

**Proof.** Using the fact that

\[ 0 = \int_{\partial M} \frac{\partial e^{-f}}{\partial \eta_g} \, ds_g = \int_M \Delta_g e^{-f} \, dA_g = \int_M \left( |\nabla f|^2 - \Delta_g f \right) e^{-f} \, dA_g, \]

and recalling that under (6), \( \delta \left( \frac{1}{4\pi \tau} e^{-f} dA_g \right) = 0 \) ([17, Eq. 12.3]), we can compute the contribution to the formula due to the variation of the term

\[ \frac{1}{4\pi \tau} \int_M (f - 2) \exp(-f) \, dA_g. \]

From this, and the computations in the proof of Theorem 3.1, the theorem easily follows. \( \square \)

4. Controlling the injectivity radius of a surface with boundary

4.1. An extension procedure. Here we show an extension procedure for surfaces with boundary of positive scalar curvature and convex boundary that allows us to control easily the maximum of the curvature of the extension (compare with the results in [19]).
Theorem 4.1. Let \((M, g)\) be a compact surface with boundary, and assume that its Gaussian curvature is strictly positive and the geodesic curvature of its boundary is strictly positive. Let \(z_0 > 0\) be arbitrary. Then there exists a closed surface \((\hat{M}, \hat{g})\), a \(C^2\) metric, such that \(M\) is isometrically embedded in \(\hat{M}\), and the Gaussian curvature \(K\) of \(\hat{M}\) is strictly positive and satisfies

\[
0 < \hat{K} \leq K_+ + \frac{2\alpha_+}{z_0},
\]

where \(K_+\) is the maximum of the Gaussian curvature of \(M\), and \(\alpha_+\) is the maximum of the geodesic curvature of \(\partial M\).

Proof. Let \(\theta \in \partial M\). Given \(K(\theta) > 0\) the (Gaussian) curvature function of \(M\) restricted to \(\partial M\), define the following family of functions. First for \(y < 0\)

\[
K_y(\theta, \zeta) = \begin{cases} 
K(\theta) + \frac{yK(\theta)}{z_0} & \text{if } 0 \leq \zeta < \frac{z_0}{1-y} \\
\frac{\alpha}{1-y} & \text{if } \frac{z_0}{1-y} \leq \zeta \leq z_0,
\end{cases}
\]

and for \(y \geq 0\)

\[
K_y(\theta, \zeta) = K(\theta) + y\zeta, \quad 0 \leq \zeta \leq z_0.
\]

Observe that for a given \(\alpha > 0\) there exists exactly one member of the previously defined family, say \(K_y(\alpha)\), such that

\[
\alpha = \int_0^{z_0} K_y(\alpha)(\zeta) \, d\zeta.
\]

We are ready to extend the metric from a convex surface with boundary to a compact closed surface, keeping control over the maximum of the curvature. Define the warping function

\[
f(\theta, z) = 1 + \alpha(\theta) z - \int_0^z \int_0^{\zeta} K_y(\alpha(\theta))(\theta, \xi) \, d\xi \, d\zeta,
\]

where \(\alpha(\theta)\) is the geodesic curvature of \(\partial M\) at the point \(\theta \in \partial M\). Notice that \(z_0 > 0\) can be chosen arbitrarily, and also that \(\frac{\partial f}{\partial z} \geq 0\) on \(0 \leq z \leq z_0\) and hence \(f \geq 1\) on the same interval.

If \(g_{\partial M}\) is the metric of \(M\) restricted to its boundary, we define a metric \(\hat{g}\) on \(N = \partial M \times [0, z_0]\) by

\[
\hat{g} = dz^2 + f^2 g_{\partial M}.
\]

This metric defines an extension of the metric on the surface \(M\) to the surface \(\hat{M}_0 = M \cup N\) where \(\partial M \subset \hat{M}\) is identified with \(\partial M \times \{0\} \subset N\). It is clear that this metric is \(C^2\), that \(\partial \hat{M}_0 = M \times \{z_0\}\) and that it is totally geodesic.

Let us now estimate the maximum of the curvature in our extension. Looking for a convenient value of \(y\), assuming bounds \(K_+\) and \(\alpha_+\) on the Gaussian curvature of \(M\) and the geodesic curvature of \(\partial M\) respectively, such that the corresponding function \(K_y\) is pointwise an upper bound for the family of functions \(K_y(\alpha)(\theta, \zeta), \theta \in \partial M\), we have that

\[
K_y(\alpha(\theta))(\theta, \zeta) \leq K(\theta) + \frac{2\alpha_+}{z_0} \zeta,
\]

and hence, by taking \(z_0 = \frac{\alpha}{2C\sqrt{K_+}}\), we obtain

\[
-\frac{\partial^2}{\partial z^2} f(z) \leq K_+ + \frac{2\alpha_+}{z_0}.
\]
Since \( f \geq 1 \), it follows that the Gaussian curvature of \( \hat{M}_0 \) is at most \( K_+ + \frac{2\alpha_+}{z_0} \).

Given the fact that the produced extension \( \hat{M}_0 \) is surface with a \( C^2 \) metric and with a totally geodesic boundary, so we can double it to obtain a closed surface endowed with a \( C^2 \) metric of positive Gaussian curvature, which is bounded above by \( K_+ + \frac{2\alpha_+}{z_0} \).

From the proof of the previous theorem we can extract the following useful corollary.

**Corollary 4.1.** If there is a geodesic in \( M \) that hits the boundary orthogonally at its both endpoints of length \( l \), then there is a closed geodesic (which is \( C^3 \)) in the extension \( \hat{M} \) of length \( 2l + 2z_0 \).

### 4.2

Let \((M, g)\) be a compact surface with boundary. We will assume that its scalar curvature is positive as well as the geodesic curvature of its boundary. We will assume that the bounds \( 0 < R \leq 2K_+ \) and \( 0 \leq k_g \leq \alpha_+ \) hold, and also, without loss of generality, that \( K_+ \geq 1 \).

From the definition of the injectivity radius for a surface with boundary (we refer to [13, §1] for the relevant definitions), and the Klingenberg-type estimates for the injectivity radius of a compact surface of positive curvature, one can conclude that in the case of a surface with boundary, we have an estimate from below for the injectivity radius \( \iota_M \) of a surface with boundary is given by

\[
\iota_M \geq \min \left\{ \text{Foc}(\partial M) \frac{1}{2}, \frac{1}{\sqrt{K_+}}, \frac{c}{\sqrt{K_+}} \right\},
\]

where \( \text{Foc}(\partial M) \) is the focal distance of \( \partial M \), \( l \) is the length of the shortest geodesic meeting \( \partial M \) at its two endpoints at a right angle, and \( c > 0 \) is a universal constant.

Since it is also well known from comparison geometry that

\[
\text{Foc}(\partial M) \geq \frac{1}{\sqrt{K_+}} \arctan \left( \frac{\sqrt{K_+}}{\alpha_+} \right) \left( \geq \frac{\pi}{2\sqrt{K_+}} \text{ if } \alpha_+ = 0 \right),
\]

our estimate on the injectivity radius reduces to

\[
\iota_M \geq \min \left\{ \frac{1}{2l}, \frac{c}{\sqrt{K_+}} \right\},
\]

for a new constant \( c \).

Our wish now is to show that along the Ricci flow (1), with initial and boundary data satisfying the requirements of Theorem [1,1] on any finite interval \((0, T)\) of time where it is defined there is a constant \( \kappa > 0 \), which may depend on \( T \) but which is otherwise independent of time, such that at any time \( t \in (0, T) \)

\[
\iota(\hat{M}, g(t)) \geq \frac{\kappa}{R_{\text{max}}(t)} \quad \text{where} \quad R_{\text{max}}(t) = \max_{p \in \hat{M}} R_g(p, t).
\]

The desired estimate is then a consequence of the following estimate, which is an analogous of Klingenberg’s Lemma for surfaces of positive scalar curvature and convex boundary.

**Proposition 4.1.** Let \((M, g)\) be a compact surface with boundary. Assume that the scalar curvature of \( M \) satisfies \( 0 < R \leq 2K_+ \), \( K_+ \geq 1 \), and the geodesic curvature of the boundary satisfies \( 0 \leq k_g \leq \alpha_+ \). Let \( l \) be the length of the shortest geodesic...
in $M$ both of whose endpoints are orthogonal to the boundary. There is a constant 
\[ \kappa := \kappa (\alpha_+) > 0 \] such that 
\[ l \geq \frac{\kappa}{\sqrt{K_+}}. \]

**Proof.** For this proof we may assume $k_g > 0$, as we can deal with the case $k_g \geq 0$ by considering, instead of $M$, 
\[ M (\epsilon) = \{ p \in M : \rho (p) \geq \epsilon \}, \]
where $\rho$ is the distance function to $\partial M$; for any $\epsilon > 0$ small enough, since $R > 0$ and $k_g \geq 0$, it is not difficult to show that $M (\epsilon)$ has boundary with strictly positive geodesic curvature, so the arguments and estimates that follow apply. The proposition will be proved in this case then, by noticing that a geodesic hitting the boundary of $M (\epsilon)$ orthogonally at both of its endpoints is at least $2 \epsilon$ shorter than a geodesic with the same property in $M$.

Now, let $l$ be the length of the shortest geodesic that hits the boundary of $M$ orthogonally. Let $z_0 = \frac{\alpha_+}{2c' \sqrt{K_+}}$, $C > 0$ a constant to be chosen, in Theorem 4.1 and Corollary 4.1. By Corollary 4.1 and Klingenberg’s injectivity radius estimate applied to $\hat{M}$, the extension of $M$ given by Theorem 4.1, we have that 
\[ 2l + \frac{\alpha_+}{C \sqrt{K_+}} \geq \frac{c'}{\sqrt{K_+ + C \sqrt{K_+}}}, \]
where $c'$ is a universal constant. Hence we have an estimate for $l$: 
\[ l \geq \frac{c'}{2 \sqrt{K_+ + C \sqrt{K_+}}} - \frac{\alpha_+}{2c' \sqrt{K_+}} \geq \frac{c'}{2 \sqrt{1 + C \sqrt{K_+}}} - \frac{\alpha_+}{2c' \sqrt{K_+}}, \]
and to get the last inequality we have used that $K_+ \geq 1$. If $\alpha_+ \leq \frac{c'}{4}$, we can choose $C = 1$. If $\alpha_+ > \frac{c'}{4}$, choose $C > 0$ so that 
\[ \frac{c'}{2 \sqrt{1 + C \sqrt{K_+}}} - \frac{\alpha_+}{2c' \sqrt{K_+}} \geq \frac{\alpha_+}{2c' \sqrt{K_+}}, \]
by taking, for instance, $C = \frac{4 \alpha_+^2 + \sqrt{16 \alpha_+^4 + 16 (c' \alpha_+)^2}}{2 (c')^2}$. This shows the proposition. \[ \square \]

**Remark.** A proof that limits can be taken from a sequence of pointed Ricci flows with uniformly bounded geometries (by bounded geometry we mean bounded curvature and uniformly bounded covariant derivatives of the curvature, bounded second fundamental form and covariant derivatives of the second fundamental form, and injectivity radius controlled from below by the inverse of the square root of the maximum of the absolute value of the sectional curvatures) can be given along the same lines to the one given by Hamilton in [14] in the case of manifolds without boundary (see also [2, 11, 18]). To take care of the boundary one can extract from the original sequence a subsequence for which the boundaries form a convergent subsequence of pointed submanifolds by taking as a marked point in the boundary one that is close to the marked point of the manifold (unless the marked point of the manifold is getting farther and farther away from the boundary: in this case, there is no need to worry about the boundaries). This can be done because the boundaries also have uniformly bounded geometries since we have control on the
curvature, the second fundamental forms, and the injectivity radius of the manifolds (to see how to control the injectivity radius of the boundary from the mentioned quantities see [1, Theorem 1.3]). Once we have done this, the boundaries can be treated as a generalized point, their generalized neighborhoods being collars, which can be constructed as we already have control over the boundary injectivity radii, and Hamilton’s arguments from [14] can be readily applied.

5. Proof of Theorem 1.1

Given \((M, g(t))\) a solution to (1) in a maximal time interval \(0 < t < T < \infty\) for which we have control over the injectivity radius as given by (7) (as is the case when the solution has positive curvature and convex boundary), and for which bounds on the derivatives of the curvature can be produced from bounds on the curvature (as when the boundary data and its derivatives are bounded, see Appendix A), then we can produce blow up limits. Recall that a blow up limit is constructed as follows: if \((0, T), 0 < T < \infty\), is the maximal interval of existence for a solution to (1), we pick a sequence of times \(t_j \to T\) and a sequence of points such that

\[
\lambda_j := R_g(p_j, t_j) = \max_{M \times [0, t_j]} R_g(x, t),
\]

and then we define the dilations

\[
g_j(t) := \lambda_j g\left(t_j + \frac{t}{\lambda_j}\right), \quad -\lambda_j t_j < t < \lambda_j (T - t_j),
\]

and then, given the appropriate injectivity radius estimates (given by (7)) and curvature bounds, from this sequence of dilated metrics and pointed manifolds \((M, g_j(t), p_j)\) we can extract a subsequence which converges towards a solution of the Ricci flow. In our case, we can classify the possible blow up limits we may obtain. We have the following result which, with minor modifications, is essentially proved in [9].

**Proposition 5.1.** Let \((M, g(t))\), \(M\) a compact surface with boundary, be a solution to (1). Let \((0, T), T < \infty\), be the maximal interval of existence of \(g(t)\). Assume that there is an \(\epsilon > 0\) such that for all \(0 < t < T, R_g > -\epsilon\), and that \(k_g\) is bounded. There are two possible blow up limits for \((M, g(t))\) as \(t \to T\). If the blow up limit is compact, then it is a homotetically shrinking round hemisphere with totally geodesic boundary. If the blow up limit is non compact then it is (or its double is) a cigar soliton.

**Proof.** Just notice that any blow up limit of \((M, g(t))\) as \(t \to T\), will have non-negative scalar curvature, which is strictly positive at one point, a totally geodesic boundary, or no boundary at all, and will be defined in an interval of time \((-\infty, \Omega)\) (and \(\Omega\) could be \(\infty\)). Then, by doubling the manifold if needed, everything reduces to the boundaryless case and [15, Thm. 26.1 and Thm. 26.3] can be applied to give the proposition. \(\square\)

Now we proceed with the proof of Theorem 1.1. In what follows, we let \((M, g_0)\) be a compact surface with boundary of positive scalar curvature, and such that \(\phi M\) has nonnegative geodesic curvature, \(\psi\) be as described in the statement of Theorem 1.1, and we let \((M, g(t))\) be the solution to (1) associated to the initial data \(g_0\) and the boundary data \(\psi\). By Proposition 2.2, we have \(R > 0\), so we have control over the injectivity radius of \((M, g(t))\), by Corollary 4.1, and derivatives of the
scalar curvature can be bounded in terms of bounds on the scalar curvature (see appendix), so we are allowed to take blow up limits of \((M, g(t))\) as \(t \to T\). Hence, from Proposition 5.1 and the formulae in §3, it follows that along a sequence of times \(t_k \to T\), \(T < \infty\) being the maximum time of existence of \(g(t)\), it holds that

\[
\lim_{k \to \infty} \frac{R_{\text{max}}(t_k)}{R_{\text{min}}(t_k)} = 1,
\]

where, obviously,

\[
R_{\text{max}}(t) = \max_{p \in M} R_g(p, t) \quad \text{and} \quad R_{\text{min}}(p, t) = \min_{p \in M} R_g(p, t).
\]

Indeed, as is the case with closed surfaces, the monotonicity formula provided by Theorem 3.2 (as long as \(\psi \geq 0\) and \(\psi' \leq 0\)) precludes the cigar as a blow up limit (see [9, §§ 7.1] and [8, Corollary D.48]), and hence the only possible blow up limit is the round hemisphere with totally geodesic boundary.

The following interesting estimate on the evolution of the area \(A(t) := A_g(t)(M)\) of \(M\) under the Ricci flow (1), with initial condition \(g_0\), can now be proved.

**Proposition 5.2.** There exists constants \(c_1, c_2 > 0\) such that

\[
c_1 (T - t) \leq A(t) \leq c_2 (T - t).
\]

**Proof.** Since in our case any blow up limit is compact, we must have

\[
\lim_{t \to T} A(t) = 0.
\]

Since \(R > 0\), and \(\int_{\partial M} k_g ds_g\) is nonincreasing, by the Gauss-Bonnet theorem we have the inequalities

\[
-2\pi \leq \frac{dA}{dt} \leq -c,
\]

and the result follows by integration. \(\square\)

As a consequence of the previous proposition and from the normalization (2) we can immediately conclude the following.

**Corollary 5.1.** The normalized flow exists for all time.

**Proof.** The normalized flow exists up to time

\[
\lim_{t \to T} \int_0^t \frac{1}{A(\tau)} d\tau = \infty,
\]

by Proposition 5.2. \(\square\)

Also, an estimate for the maximum of the scalar curvature can be deduced.

**Proposition 5.3.** There are constants \(c_1, c_2 > 0\) such that

\[
\frac{c_1}{T - t} \leq R_{\text{max}}(t) \leq \frac{c_2}{T - t}.
\]

**Proof.** By the Gauss-Bonnet theorem and the fact that, under the hypothesis of Theorem 1.1, \(\int_{\partial M} k_g ds_g\) is nonincreasing, we have that

\[
\int_M R_{\text{max}}(t) dA_g \geq C,
\]
and from Proposition 5.2 there is a $c' > 0$ such that
\[ c'R_{\max}(t) (T - t) \geq C, \]
so the left inequality follows.

To show the other inequality we proceed by contradiction. Assume that there is no constant $c_2 > 0$ for which
\[ R_{\max}(t) \leq \frac{c_2}{T - t} \]
holds. Then we can find a sequence of times $t_j \to T$ such that
\[ R_{\max}(t_j) (T - t_j) \to \infty, \]
and hence along this sequence the blow up limit would not be compact, as it would have infinite area due to Proposition 5.2 and this would contradict the arguments following Proposition 5.1 □

Corollary 5.3 shows that along any sequence of times we can take a blow up limit since for any sequence of times, the curvature is blowing up at maximal rate (i.e. \( \sim \frac{1}{T - t} \)). By the arguments following Proposition 5.1 this blow up limit is a round homotetically shrinking sphere. This proves the following theorem.

**Theorem 5.1.** Let $(M, g_0)$ be a compact surface with boundary, of positive scalar curvature and such that the geodesic curvature of $\partial M$ is nonnegative, and let $\psi$ be as in the statement of Theorem 1.1. Then, the solution to the Ricci flow (1) with initial condition $g_0$ blows up in finite time $T$, and for the scalar curvature $R$ we have that
\[ \lim_{t \to T} R_{\max}(t) = 1. \]
As a consequence, under the corresponding normalized flow we obtain that
\[ R_{\max}(\tilde{t}) - R_{\min}(\tilde{t}) \to 0 \quad \text{as} \quad \tilde{t} \to \infty. \]

Since one can produce bounds on the derivatives of the curvature from bounds on the curvature, and the curvature remains bounded along the normalized flow, we have that along any sequence of times $t_n \to \infty$, there is a subsequence of times $t_{n_k} \to \infty$ such that $\tilde{g}(t_{n_k})$ is converging to a metric of constant curvature (this metrics may be different according to the sequence considered). To be able to conclude that these limit metrics are isometric to that of a standard hemisphere, we must show that the geodesic curvature of the boundary approaches 0. Let us now conclude the proof of Theorem 1.1

**Finishing the proof of Theorem 1.1** The only part of the statement that has not been proved in the previous discussion is that regarding the behavior of the geodesic curvature. Notice that
\[ \frac{c}{T - t} \leq \phi(t) \leq \frac{C}{T - t}, \]
where $\phi$ is the normalizing factor defined in the introduction, and hence
\[ T - t \leq T e^{-c\tilde{t}}, \]
which shows that
\[ k_{\tilde{g}} \leq C \sqrt{T - \tilde{t}} \psi \leq ce^{-c\tilde{t}} \psi. \]
Since $\psi \geq 0$ and $\psi' \leq 0$, it remains bounded, and the theorem follows. □
6. Proof of Theorem 1.2

In this section, we let \( g(t) \) be the solution to (1) in the two-ball \( D \), with initial and boundary data as described in the hypotheses of Theorem 1.2. It is clear, from uniqueness, that this solution is also rotationally symmetric. Let \( A(t) := A_{g(t)}(D) \) be the area of \( D \) with respect to \( g(t) \). Given the fact that

\[-c \leq \frac{d}{dt} A \leq -C < 0\]

to show that the normalized flow does exist for all time, all we must prove is that \( A(t) \to 0 \) as \( t \to T \).

First notice that the scalar curvature of the solution to (1), with initial data \( g_0 \) and boundary data \( \psi \) as in Theorem 1.2, blows up in finite time by Proposition 2.3. Let us denote by \( R(t) \) the radius of \( D \) with respect to \( g(t) \); by comparison geometry we have that at any time \( R \geq \frac{\pi}{2\sqrt{R_{\text{max}}(t)}} \), and using Hamilton’s arguments (see [6, § 5]), it can be shown that for points at distance at least \( \frac{1}{4}R \) from the boundary, the injectivity radius is conveniently bounded from below. Hence we can take a blow up limit of \((M, g(t))\) as \( t \to T \). This blow up limit might be compact, and in this case it is a round hemisphere, and as we did before, it can be shown that then \( A(t) \to 0 \) as \( t \to T \). If this blow up limit is non compact it must be the cigar. In this case, it is not difficult to prove that we can take as an origin for the blow up limit the center of \( D \). To study this case we define the following quantities which depend on \( g(t) \):

\[ I(r,t) = \frac{L_{g}(\partial D_{r})}{A_{g}(D_{r})}, \quad T(t) = \inf_{0 \leq r \leq R} I(r,t), \]

where \( D_{r} \subset D \) is the geodesic ball of radius \( r \) centered at the center of \( D \), \( L_{g}(\partial D_{r}) \) is the length of \( \partial D_{r} \) and \( A_{g}(D_{r}) \) is the area of \( D_{r} \) both with respect to the metric \( g(t) \). There are two cases to be considered. The first case is when there is a \( \delta > 0 \) such that this infimum is attained at \( r(t) \in [0, R] \) on the time interval \( (T - \delta, T) \).

If \( r = 0 \), then \( I = 4\pi \); if \( r > 0 \), we have the following formula.

**Lemma 6.1.** If \( T \) at time \( t \) is attained at \( r \in (0, R) \) then \( I \) satisfies an evolution equation

\[ \frac{\partial}{\partial t} \log I(r,t) = \frac{\partial^{2} I(r,t)}{\partial r^{2}} - \frac{1}{A_{g}(D_{r})}(4\pi - I(r,t)). \]

**Proof.** We let \( L = L_{g}(\partial D_{r}) \), \( A = A_{g}(D_{r}) \), \( k \) be the geodesic curvature of \( \partial D_{r} \) and \( K \) be the Gaussian curvature of \( D \). We have the following set of formulas

\[ \frac{\partial L}{\partial r} = \int_{\partial D_{r}} k \, ds = kL, \quad \frac{\partial^{2} L}{\partial r^{2}} = -\int_{\partial D_{r}} K \, ds = \frac{\partial L}{\partial t}, \]

and

\[ \frac{\partial A}{\partial r} = L, \quad \frac{\partial^{2} A}{\partial r^{2}} = \int_{\partial D_{r}} k \, ds, \quad \frac{\partial A}{\partial t} = -4\pi + 2 \int_{\partial D_{r}} k \, ds. \]

Notice that at the value of \( r \) where the infimum is attained we have

\[ 0 = \frac{\partial}{\partial r} \log I = \frac{2 \partial L}{L} - \frac{1}{A} \frac{\partial A}{A} \frac{\partial A}{\partial r}, \]

so we have, \( \frac{2 \partial L}{L} - \frac{1}{A} \frac{\partial A}{\partial r} = \frac{\partial A}{A} \frac{\partial A}{\partial r} \). Formula (8) now follows from all these identities by a straightforward calculation. \(\square\)
Clearly Lemma 6.1 precludes the fact that $\mathcal{T} \to 0$, since it does imply that $\mathcal{T}$ increases if $\mathcal{T} < 4\pi$. Hence, the blow up limit in this case cannot be the cigar, hence it is a round hemisphere, and we would be done. We are left with one more possibility: the infimum is achieved at $r = R$ for a sequence of times of times $t_k \to T$; if this is so, then since for the cigar $\mathcal{T} = 0$, if $A(t) \neq 0$ as $t \to T$, we must have $L(\partial D) \to 0$ as $t \to T$. Under the hypotheses of Theorem 1.2, using the Maximum Principle, it is not difficult to show that the scalar curvature remains uniformly bounded from below on $(0, T)$, and therefore $R$ is uniformly bounded above in $0 < t < T < \infty$. But then we have the following lemma (see [9]).

Lemma 6.2. Let $g_k$ be a sequence of rotationally symmetric metrics on the two-ball $D$. Assume that there is a constant $\epsilon > 0$ such that $R_{g_k} \geq -\epsilon$ and $k_{g_k} \geq -\epsilon$, and that the radius of $D$ with respect with this sequence of metrics is uniformly bounded from above by $\rho > 0$. Then if the length of the boundary of $D$, $L_{g_k}(\partial D)$, goes to 0 as $k \to \infty$, then $A_{g_k}(D) \to 0$.

Proof. By rescaling we may assume that $\epsilon = 1$. Hence a comparison argument shows that,

$$A_{g_k}(D) \leq 2\pi \int_0^\rho L_{g_k}(\partial D) e^r \, dr \leq 2\pi e\rho L_{g_k}(\partial D),$$

and the conclusion of the lemma follows. \qed

The previous lemma shows that then we must have $A(t) \to 0$ as $t \to T$. This proves that we have, in any case, $A(t) \to 0$ as $t \to T$ and the theorem follows.

**Appendix A. Derivative estimates**

Let $(M, g(t))$ be a solution to the Ricci flow (1). In this appendix we will show how to produce bounds on the derivatives of the curvature of $g(t)$ in terms of bounds on the curvature and on the geodesic curvature of the boundary and its derivatives. The ideas we use are quite standard, as we produce certain quantities involving derivatives of $R$ (clearly inspired by the quantities used in the case of closed manifolds), and then we compute some differential inequalities; we will have to make computations on the boundary of $M$ in order to apply the Maximum Principle to these differential inequalities, and even though a bit tedious, these computations are certainly straightforward.

We now fix some notation. Let $\rho(P, t)$ be the distance function to the boundary of $M$ with respect to the metric $g(t)$. We define the set

$$M[0, \epsilon] = \rho^{-1}([0, \epsilon], 0),$$

and we will refer to it as the collar of the boundary, or simply as the collar. We also will use the notation,

$$M_\epsilon = \rho^{-1}(\{\epsilon\}, 0)$$

for the level curves of $\rho(\cdot, 0)$. All the quantities below depend on the time varying metric $g$, however we will not use any subindex to indicate such dependence. Also, we must point out that similar estimates can be obtained for solutions to the normalized Ricci flow (3), either by working with it directly or using the normalization to pass back to the unnormalized flow, but we will leave that to the reader. Without much further ado, let us start with our estimates.
A.1. First order derivative estimates. Our purpose now is to show an estimate on the first derivative of the curvature near the boundary of $M$. We shall assume for simplicity that $k'_g = 0$, but it will be clear from the proof that if we have bounds on $k'_g$ and its derivatives on any interval of time, with a judicious modification, the following results are still valid. Notice the similarity with the analogous local interior estimate (see, for instance, [13, Theorem 13.1] and [8, Theorem 14.14]).

**Theorem A.1.** Let $(M, g(t))$, $M$ compact, be a solution to the Ricci flow (1) on $[0, T^*)$. Let $\epsilon > 0$ be such that at $t = 0$,

$$\exp : \nu^{-} (\epsilon) \longrightarrow M \left[0, \epsilon\right]$$

is a diffeomorphism. Let $K, \alpha > 0$ be such that $|\mathcal{R}| \leq K$ on $M \times [0, T^*)$ and $|k_g| \leq \alpha$ on $\partial M \times [0, T^*)$. Then there is a $\tau := \tau (K, \alpha)$ so that we can estimate

$$|\nabla \mathcal{R}|^2 \leq \frac{C(K, \alpha, \epsilon, T^*)}{t}, \quad \text{on} \quad M \left[0, \epsilon\right] \times (0, \tau],$$

where $C(K, \alpha, \epsilon, T^*)$ is a constant that depends on the given parameters.

**Proof.** Define on $M \left[0, \epsilon\right]$ for $t \in [0, T^*)$ the function

$$F = t e^{\alpha \rho} |\nabla \mathcal{R}|^2 + A \mathcal{R}^2 + B \rho \mathcal{K}^2,$$

where $\rho$ is the (time dependent) distance to the boundary function, $\nabla \mathcal{R}$ is the component of the gradient of $\mathcal{R}$ with respect to $g(t)$ which is tangent to the level surfaces of $\rho(\cdot, 0)$ (at time $t$ with respect to $g(t)$), and $A$ and $B$ are positive constants. $F$ satisfies a differential inequality in $M$, namely

$$\frac{\partial F}{\partial t} \leq \Delta_g F + \left( e^{\alpha \rho} \mathcal{R} + 2 \alpha^2 t e^{\alpha \rho} + e^{\alpha \rho} + t \left| \frac{\partial e^{\alpha \rho}}{\partial t} \right| + t |\Delta_g e^{\alpha \rho}| - 2A |\nabla \mathcal{R}|^2 - 2 \alpha \nabla \rho \nabla F + 2 \alpha \nabla \rho \left( 2A \mathcal{R} \nabla \mathcal{R} + 2AK^2 \nabla \rho \right) + 2AR^3 + \left| \frac{\partial \rho}{\partial t} \right| BK^2 + |\Delta_g \rho| BK^2.$$

where $c$ is a constant that can be computed, but whose actual value is irrelevant. Now we need to control a few quantities from the previous expression. The term $A \mathcal{R} \nabla \mathcal{R}$ can be dealt with via the inequality

$$|A \mathcal{R} \nabla \mathcal{R}| \leq \frac{1}{2} \left( A^2 \mathcal{R}^2 + |\nabla \mathcal{R}|^2 \right).$$

To control the term $\Delta_g \rho$ we use the identity

$$\Delta_g \rho (P) = -k_g (P),$$

where $k_g (P)$ is the geodesic curvature of the level surface of $\rho$ that passes through $P$. On the other hand the geodesic curvature of the level curves of $\rho$ can be controlled in terms of the curvature, a bound on the geodesic curvature of $\partial M$, and $\epsilon > 0$, since we have an equation

$$\frac{\partial k_g}{\partial \rho} = \frac{R}{2} + k_g^2.$$

Also, it is not difficult to estimate $\left| \frac{\partial \rho}{\partial t} \right| \leq K \rho$, and we have $|\nabla \rho| = 1$. Now, by a convenient choice of $\tau (K, \alpha) > 0$ and $A > 0$, we obtain a differential inequality for $F$, valid on $0 < t \leq \tau (\alpha, K)$, namely

$$\frac{\partial F}{\partial t} \leq \Delta_g F - 2 \alpha \nabla \rho \nabla F + C(A, B, \alpha) K^3.$$
From now on, we will use the convention that $C$ is a constant (that may change from estimate to estimate) that depends on, or any subset of $A, B, \epsilon, K, \alpha, T^*$.

On the other hand, on $\partial M$, we have the identity
\[
\frac{\partial F}{\partial \eta} g = -t\alpha e^{\alpha \rho} \left| \nabla^\top R \right|^2 + te^{\alpha \rho} k_g \left| \nabla^\top R \right|^2 + 2A k_g R^2 - BK^2,
\]
where we have used the fact that $\frac{\partial}{\partial \eta} g \left| \nabla^\top R \right|^2 = 2k_g \left| \nabla^\top R \right|^2$. So by taking $B \geq 2A \alpha$ we have that $\frac{\partial F}{\partial \eta} g \leq 0$.

In the part of the boundary of the collar that lies in the interior of the manifold, i.e. $M_\epsilon$, by Shi’s interior estimates ([15, Theorem 13.1]), we have,
\[
F \leq C(A, B, \alpha) K^3 + Cte^{C(K, \alpha, \epsilon)} K^2 \left( \frac{1}{\epsilon^2} + \frac{1}{t} + K \right)
\leq C(A, B, \alpha) K^3 + C\epsilon^{C(K, \alpha, \epsilon)} K^2 \left( \frac{T^*}{\epsilon^2} + 1 + KT^* \right).
\]
Applying the Maximum Principle yields
\[
F \leq \max_{t=0} F + tC(A, B, \alpha) K^3 + C(A, B, \alpha) K^3 + C\epsilon^{C(K, \alpha, \epsilon)} K^2 \left( \frac{T^*}{\epsilon^2} + 1 + KT^* \right),
\]
from which we obtain
\[
\left| \nabla^\top R \right|^2 \leq \frac{C}{t} \left[ K^3 + K^2 \left( \frac{T^*}{\epsilon^2} + 1 + K \right) \right].
\]

Now, let $\nabla^\bot R$ be the part of the gradient of $R$ perpendicular at time $t$ to the level curves of $\rho(\cdot, 0)$. Define
\[
G = t \left| \nabla^\bot R \right|^2 + AR^2.
\]
Again we have that $G$ satisfies a differential inequality
\[
\frac{\partial G}{\partial t} \leq \Delta_g G + 2AR^3;
\]
on the other hand using the expression for $\frac{\partial R}{\partial \eta}$ on $\partial M$ and Shi’s interior derivative estimates on $M_\epsilon$, in both components of the boundary of the collar we have an estimate
\[
G \leq T^* \alpha^2 K^2 + AK^2 + Ck^2 \left( \frac{T^*}{\epsilon^2} + 1 + K \right) + 2AK^3.
\]
This gives an estimate for $G$ completely analogous to the estimate obtained for $F$. This proves the theorem.

A.2. Second order derivative estimates. Here we sketch how to bound second order derivatives in a collar $M [0, \epsilon]$ of the boundary, on an interval of time $(0, T^*)$, where we have assumed a bound $K$ on the absolute value of the curvature and its gradient (this just to simplify matters, as we know how to produce bounds for it in any compact subinterval of $(0, T^*)$), and on the geodesic curvature of the boundary and its covariant derivatives (at the boundary) on $M [0, \epsilon] \times [0, T^*]$. Before we start, let us give a couple of definitions. For a tensor time dependent tensor $T$, at a fixed (but arbitrary) time $t$ we define
\[
\nabla^\top T(X, \ldots) = \begin{cases} \nabla_X T(\ldots) & \text{if } X \text{ is tangent to } M_\epsilon \text{ for } c \geq 0, \\ 0 & \text{if } X \text{ is orthogonal to } M_\epsilon \text{ for } c \geq 0, \end{cases}
\]
\[ \nabla^\perp T (X, \ldots) = \begin{cases} \nabla_X T (\ldots) & \text{if } X \text{ is perpendicular to } M_c \text{ for a } c \geq 0, \\ 0 & \text{if } X \text{ is tangent to } M_c \text{ for a } c \geq 0, \end{cases} \]

where the covariant derivative \( \nabla \) is taken with respect to the metric \( g(t) \), and the orthogonality of \( X \) to \( M_c \) is also with respect to \( g(t) \).

**Theorem A.2.** Under the hypotheses of Theorem A.1 and assume also a bound \( |\nabla R| \leq K \) on \( M \times [0, T^*] \), and bounds \( |\nabla k_g| \leq \alpha \) and \( |\nabla^2 k_g| \leq \alpha \) on \( \partial M \times [0, T^*] \), where \( \nabla \) denotes covariant differentiation on \( \partial M \) induced by \( g(t) \). Then there is a \( \tau(K, \alpha) > 0 \) so that on \( M \times [0, \epsilon] \times (0, \tau] \) we can bound

\[ |\nabla^2 R|^2 \leq \frac{C(K, \alpha, \epsilon, T^*)}{t^2}, \]

where \( C(K, \alpha, \epsilon, T^*) \) is a constant that depends on the given parameters.

To begin our estimations, we let

\[ G = \left| \left( \nabla^\top \right)^2 R \right|^2 + \left( \frac{\partial R}{\partial t} \right)^2 \]

Define

\[ F = t^2 e^{\beta \rho} G + At |\nabla R|^2 + \rho N^2, \]

where \( \beta, A \) and \( N \) are constants to be defined.

A tedious, but straightforward computation, and an analogous reasoning to what is done in the case of manifolds without boundary (see for instance the computations from [7, Chapter 7 \$2]), shows that given a choice of \( \beta \) (that only depends on a bound on \( k_g \), which we have called \( \alpha \), see below) and a good choice of \( \tau = \tau(K, \beta) \) (or \( \tau = \tau(K, \alpha) \) as \( \beta \) depends on \( \alpha \)), we can choose \( A \) such that on \( (0, \tau] \), \( F \) satisfies a differential inequality

\[ \frac{\partial F}{\partial t} \leq \Delta_g F - 2\beta \nabla F \nabla \rho + (t + 1) Q \left( |R|, |\nabla R| \right) + E N^2 \]

where \( Q \) is a polynomial whose coefficients depend only on the constants \( A, \beta \) and \( \epsilon \), and \( E \) is a constant that depends on a bound on \( \left| \frac{\partial \rho}{\partial t} - \Delta_g \rho \right| \) which can be given in terms of bounds on \( R, k_g \) and \( \epsilon \).

First, we can estimate

\[ \frac{\partial}{\partial \eta_l} |\nabla R|^2 = \frac{\partial}{\partial \eta_l} \left( \nabla_{e_2} R \right)^2 \leq 2 \left| \nabla_{e_2} \nabla_{e_2} R \right| |\nabla_{e_2} R| \]

\[ = \left( \left| \left( R_l - \nabla_{e_1} \nabla R - R^2 \right) \right| k_g R \right) \leq t e^{\beta \rho} G + \frac{e^{-\beta \rho}}{t} k_g^2 R^2 + 2 |k_g| |R|^3. \]

Notice that the previous computation gives us estimates on \( \frac{\partial}{\partial \eta_l} |\nabla R|^2 \), since

\[ |\nabla R|^2 = |\nabla^\top R|^2 + |\nabla^\perp R|^2 \text{ and } \frac{\partial}{\partial \eta_l} |\nabla^\top R|^2 \leq 2 k_g |\nabla^\top R|^2 + 2 |k_g| |\nabla^\top R|. \]
On the other hand we have
\[
\frac{\partial}{\partial \eta} \left| (\nabla^\top)^2 R \right|^2 \leq 2k_g \left| (\nabla^\top)^2 R \right|^2 + 2 |R| \left| (\nabla^\top)^2 R \right|
\]
\[
+ 2 \left( \left| \nabla^\top k_g \right| \left| \nabla^\top R \right| + \left| (\nabla^\top)^2 k_g \right| \left| R \right| \right) \left| (\nabla^\top)^2 R \cdot R \right|
\]
\[
\leq (2k_g + 2) \left| (\nabla^\top)^2 R \right|^2 + P \left( \left| \nabla^\top k_g \right|, \left| (\nabla^\top)^2 k_g \right|, \left| R \right|, \left| \nabla R \right| \right),
\]
where \( P \) is a polynomial whose coefficients are universal constants. Let \( e_1, e_2 \) be a frame, so that \( e_1 \) is tangent to the level curves of \( \rho \), and which is orthonormal at the point in the boundary where we are doing our calculations. Then it can be shown that at the boundary
\[
\nabla_{e_1} \nabla_{e_1} k_g = \nabla_{e_1} \nabla_{e_1} k_g + k_g \nabla_{e_2} k_g = \nabla_{e_1} \nabla_{e_1} k_g - k_g \left( \frac{R}{2} + k_g^2 \right),
\]
so \( \left| (\nabla^\top)^2 k_g \right|^2 \) can be estimated in terms of \( K \) and \( \alpha \), and since \( \left| \nabla^\top k_g \right| = \left| \nabla k_g \right| \), \( P \) can be bounded from above in terms of \( K \) and \( \alpha \). Also, we can compute,
\[
\frac{\partial R^2_t}{\partial \eta} = 2R_t \left( \left( \frac{\partial R}{\partial \eta} \right)_t + R \frac{\partial R}{\partial \eta} \right) = 2k_g R^2_t + k_g R^2.
\]
Using the previous computations and the definition of \( F \), we have
\[
\frac{\partial F}{\partial \eta} \leq -t^2 e^{2\beta \rho} + t^2 e^{2\beta \rho} \left\{ (2k_g + 2) \left| (\nabla^\top)^2 R \right|^2 + P (\ldots) + 2k_g R^2_t + k_g R^2 \right\}
\]
\[
+ 2Atk_g \left| (\nabla^\top) R \right|^2 + At^2 e^{2\beta \rho} G + A e^{-\beta \rho} k_g^2 R^2 - N^2.
\]
It follows that if we take
\[
\beta = 2\alpha + 3 \geq 2k_g + 3 \quad \text{on} \quad [0, T^*],
\]
and \( N^2 \) such that
\[
N^2 \geq t^2 e^{2\beta \rho} P (\ldots) + t^2 e^{2\beta \rho} k_g R^2 + A e^{-\beta \rho} k_g^2 R^2 + 2Atk_g \left| (\nabla^\top) R \right|^2,
\]
on the interval \([0, T^*] \), we can guarantee that on \( \partial M, \frac{\partial F}{\partial \eta} \leq 0 \).

It must be now clear to the reader that we choose the constants to define \( F \) in the following order: first we choose \( \beta \), then \( \tau \), then \( A \), and finally \( N \). In what follows we will use the convention that \( C \) represents a constant that may change from line to line that depends on (or a subset of) \( K, \epsilon, T^* \) and \( \beta \) (or \( \alpha \)).

On the boundary of the collar that lies in the interior of the manifold we can estimate \( F \) by using the interior derivative estimates on a ball whose radius is \( \frac{1}{2} \) at time \( t = 0 \). Indeed, by the interior derivative estimates we have
\[
F \leq t^2 e^{2\beta \rho} \frac{C (K, \epsilon, T^*)}{t^2} + \frac{A t C (K, \epsilon, T^*)}{t} + AtK + \rho N^2
\]
\[
\leq C (K, \beta, \epsilon, T^*) + C (K, \epsilon, T^*) N^2
\]
Using the Maximum Principle, we obtain an estimate
\[
F \leq \max_{t=0} F + C (K, \beta, \epsilon, T^*) + t \left( B_1 + C (K, \beta, \epsilon, T^*) N^2 \right)
\]
where $B_1$ is a bound, which depends on $T^*$, $K$, $\epsilon$ and $\beta$, on $(t + 1) Q (|R|, |\nabla R|)$ on $[0, T^*]$. From this we obtain an estimate

$$
\left| (\nabla^\top)^2 R \right|^2 + \left( \frac{\partial R}{\partial t} \right)^2 \leq \frac{C (K, \beta, \epsilon)}{t^2} \left( C (K, \beta, \epsilon, T^*) + \epsilon N^2 + t \left( B_1 + CN^2 \right) \right) \\
\leq \frac{C (K, \alpha, \epsilon, T^*)}{t^2}.
$$

To obtain a bound on $\left| (\nabla^\bot)^2 R \right|$, given a frame $e_1, e_2$, so that at the point where we are calculating is orthonormal, and $e_1$ is tangent to the corresponding level surface of $\rho (\cdot, 0)$, we use that

$$
\nabla e_2 \nabla e_2 R = \Delta_g R - \nabla e_1 \nabla e_1 R = \frac{\partial R}{\partial t} - R^2 - \nabla e_1 \nabla e_1 R,
$$

and that we already have an estimate on $\frac{\partial R}{\partial t}$ and on $\nabla e_1 \nabla e_1 R$.

We now estimate second covariant derivatives of the form $\nabla^\top \nabla^\bot R$. Define

$$
F = t^2 \left| (\nabla^\top \nabla^\bot R) \right|^2 + At |\nabla R|^2.
$$

Again, there is a $\tau = \tau (K, \alpha)$ such that $F$ satisfies a differential inequality

$$
\frac{\partial F}{\partial t} \leq \Delta_g F + (t + 1) S (|R|, |\nabla R|),
$$

on $[0, \tau]$, where $S$ is a polynomial. On $\partial M$, using the formula for $\nabla e_2 R = k_g R$ (Proposition 2.1), we have

$$
F \leq B' \left\{ t^2 \left( \left| \nabla^\top R \right|^2 + R^2 \right) \right\} + At |\nabla R|^2 \\
\leq 2B' T^* K^2 + AT^* K^2,
$$

where $B'$ is a constant that depends on bounds on $k_g$ and its derivatives, and in our estimation we have used the assumed bound $K$ on the gradient of $R$. In $M$, the boundary of the collar that lies in the interior of the manifold, using the interior derivative estimates for first and second derivatives of the curvature, we can also bound $F$ by a constant $C (K, \epsilon, T^*)$. Hence, proceeding as before, via the Maximum Principle, we obtain a bound

$$
\left| \nabla^\top \nabla^\bot R \right|^2 \leq \frac{C (K, \alpha, \epsilon, T^*)}{t^2},
$$

on $M [0, \epsilon] \times (0, \tau]$. This shows the theorem.

A.3. higher derivative estimates.

**Theorem A.3.** Under the hypotheses of Theorem [A.1] and assuming bounds

$$
|\nabla R|, \ldots, |\nabla^{m-1} R| \leq K \quad \text{on} \quad M \times [0, T^*],
$$

and bounds $|\nabla k_g|, \ldots, |\nabla^m k_g| \leq \alpha$ on $\partial M \times [0, T^*]$, where $\nabla$ denotes covariant differentiation on $\partial M$ with respect to the metric induced by $g (t)$, there is a $\tau (K, \alpha) > 0$ so that for $M [0, \epsilon] \times (0, \tau]$ we can bound

$$
|\nabla^m R|^2 \leq \frac{C (K, \alpha, \epsilon, T^*)}{t^m},
$$

where $C (K, \alpha, \epsilon, T^*)$ is a constant that depends on the given parameters.
Proof. We estimate covariant derivatives of the form

\[
\text{case 1: } (\nabla^\top)^{m_1} (\nabla_t)^{m_2} R \quad \text{with } m_1 + 2m_2 = m,
\]

and

\[
\text{case 2: } (\nabla^\top)^{m_1} (\nabla_t)^{m_2} \nabla^\bot R \quad \text{with } m_1 + 2m_2 + 1 = m,
\]

where \(\nabla_t := \frac{\partial}{\partial t}\), and it is considered as a covariant derivative of order 2. Every other covariant derivative can be transformed into one of this form, up to lower order terms. Indeed, given \(e_1, e_2\) orthonormal and so that \(e_1\) is tangent to a level curve of \(\rho\), one has

\[
\nabla_{e_2} \nabla_{e_2} R = \frac{\partial R}{\partial t} - \nabla_{e_1} \nabla_{e_1} R - R^2,
\]

also the exchange of two (spatial) covariant derivatives produces only lower order terms. And regarding the exchange of a time derivative (which is regarded as a covariant derivative of order 2) with a (spatial) covariant derivative, it only produces lower order terms since, for the Christoffel symbols we have

\[
\frac{\partial}{\partial t} \Gamma_{jk}^l = \frac{1}{2} g^{lq} (\nabla_j R_{qk} + \nabla_k R_{qj} - \nabla_q R_{jk}).
\]

To measure the norm of this covariant derivatives we extend the metric in the obvious way: if \(0\) represents the index for the time direction, we define \(g_{0j} = \delta_{0j}\), where \(\delta_{ij}\) is Kronecker’s delta. We will use once again the convention that \(C\) is a constant that may change from line to line, but that only depends on (or subsets of) \(K, \alpha, \epsilon, T^*, \beta\) (\(\beta\) is to be defined below, but the careful reader will notice that it will depend on \(\alpha\)) and \(A\) (a constant that is defined appropriately).

Now, to estimate the first class of covariant derivatives, we set

\[
G = \sum_{m_1+2m_2=m} \left| (\nabla^\top)^{m_1} (\nabla_t)^{m_2} R \right|^2
\]

\[
H = t^m e^{\beta t} G + A t^{m-1} \left| \nabla^{m-1} R \right|^2 + \rho N^2.
\]

In the interior of the collar, the usual computations, by a judicious choice of \(\tau(K, \beta) > 0\) and \(A\), give the differential inequality

\[
\frac{\partial H}{\partial \tau} \leq \Delta H - 2\beta \nabla \rho \nabla H + Q (|R|, \ldots, |\nabla^{m-1} R|) + CN^2.
\]

on the interval of time \((0, \tau]\). Now comes the interesting part. We need to find differential inequalities at the boundary of the collar, so that we can apply the Maximum Principle to \(H\). In this case, we start by computing at \(\partial M\). We will need to compute \(\frac{\partial}{\partial \eta} |\nabla^{m-1} R|^2\). Remember that every covariant derivative of \(R\) is equivalent to case 1 or 2, except for terms of order \(m - 2\) or lower, so when differentiated give terms of order \(m - 1\) which by assumption we know how to estimate. If we are in case 1, here with \(n_1 + 2n_2 = m - 1\), we have

\[
\nabla_{e_2} (\nabla^\top)^{n_1} (\nabla_t)^{n_2} R = (\nabla^\top)^{n_1} (\nabla_t)^{n_2} \nabla_{e_2} R + \mathcal{L}_{m-2},
\]

where \(\mathcal{L}_{m-2}\) denotes (and will denote from now on) a term that depends on covariant derivatives of \(R\) of order at most \(m - 2\). Using the fact that \(\nabla_{e_2} R = k_g R\), we obtain

\[
\nabla_{e_2} (\nabla^\top)^{n_1} (\nabla_t)^{n_2} R = k_g (\nabla^\top)^{n_1} (\nabla_t)^{n_2} R + \mathcal{L}_{m-1},
\]

where \(\mathcal{L}_{m-1}\) denotes (and will denote) a term that depends on covariant derivatives \((\nabla^\top)^{m'} k_g\) of the geodesic curvature with \(m' \leq m - 1\), and covariant derivatives of
the curvature of order at most \( m - 2 \). If we are in case 2, with \( n_1 + 2n_2 + 1 = m - 1 \), we have

\[
\nabla_{e_2} (\nabla^T)^{n_1} (\nabla_t)^{n_2} \nabla_{e_2} R = (\nabla^T)^{n_1} (\nabla_t)^{n_2} \nabla_{e_2} \nabla_{e_2} R + \mathcal{L}_{m-2}. 
\]

Hence, by the previous discussion we end up with

\[
\nabla_{e_2} (\nabla^T)^{n_1} (\nabla_t)^{n_2} \nabla_{e_2} R = (\nabla^T)^{n_1} (\nabla_t)^{n_2} R + (\nabla^T)^{n_1} (\nabla_t)^{n_2+1} R + \mathcal{L}_{m-2},
\]

i.e., a term of order \( m \), up to lower order terms, of the form described in case 1. Using this, just as we did in the case of second order derivative estimates, yields

\[
\frac{\partial}{\partial \eta} |\nabla^{m-1} R|^2 = 2 \left\langle c_{m_1,m_2} (\nabla^T)^{m_1} (\nabla_t)^{m_2} R, \nabla^{m-1} R \right\rangle + \langle \mathcal{L}_{m-2} + \mathcal{L}_{m-1}', \nabla^{m-1} R \rangle \leq C t e^{\beta \rho} G + \frac{1}{t} |\nabla^{m-1} R|^2 + B_{m-1} |\nabla^{m-1} R|,
\]

with \( m_1 + 2m_2 = m \); where \( c_{m_1,m_2} \) are functions which depend on \( k_g \), and therefore \( C \) is a constant that only depends on \( \alpha \), and \( B_{m-1} \) denotes (and will denote from now on) a bound on terms that contains covariant derivatives (with respect to the induced metric on \( \partial M \)) of the geodesic curvature of order at most \( m - 1 \), and covariant derivatives of the curvature of order at most \( m - 2 \); here we have used the fact that \( (\nabla^T)^m k_g, m' \leq m - 1 \), can be estimated in terms of estimates on covariant derivatives of the form \( \nabla^{m'} k_g \) with \( m'' \leq m - 1 \) and covariant derivatives of the curvature of order at most \( m - 2 \). Using similar arguments, we arrive at

\[
\frac{\partial}{\partial \eta} G \leq 2k_g G + B_m,
\]

where \( B_m \) is as defined above with \( m \) instead of \( m - 1 \). We can estimate now

\[
\frac{\partial}{\partial \eta} H \leq -\beta t^n e^{\beta \rho} G + C t^m e^{\beta \rho} G + t^{m-2} |\nabla^{m-1} R|^2 + B_m - N^2.
\]

This shows that by carefully choosing \( \beta \), which only depends on \( \alpha \), and \( N \), we have \( \frac{\partial}{\partial \eta} H \leq 0 \). On the other hand, in the boundary of the collar lying in the interior of the manifold, we can use the interior estimates to bound

\[
H \leq e^{\beta \rho} t^m G + A t^{m-1} |\nabla^{m-1} R|^2 + \rho N \leq e^{C(K,\beta,\epsilon)} t^m \frac{C}{t^{m-1}} + A t^{m-1} \frac{C}{t^{m-1}} + C(K,\epsilon) N \leq C + C(K,\epsilon) N.
\]

Now we are in position to apply the Maximum Principle. If we have that \( Q(\ldots) \leq N \), then

\[
H \leq C(K,\epsilon) N + T^* N + C(K,\epsilon) T^* N,
\]

which implies the theorem in this case.

If we are in case 2, again we define

\[
H = t^m e^{\beta \rho} \sum_{m_1 + 2m_2 = m-1} \left| (\nabla^T)^{m_1} (\nabla_t)^{m_2} \nabla R \right|^2 + A t^{m-1} |\nabla^{m-1} R|^2 + \rho N^2.
\]

However, this time is easier to deal with as we do not have to compute normal derivatives of this expression on \( \partial M \), since we have

\[
(\nabla^T)^{m_1} (\nabla_t)^{m_2} \nabla_{e_2} R = (\nabla^T)^{m_1} (\nabla_t)^{m_2} (k_g R).
\]
and the expression on the right hand side is of order $m - 1$ up to lower order terms in derivatives of the curvature, which by hypothesis we know how to bound. In $M_\varepsilon$ we use the interior estimates again, and via the Maximum Principle, we obtain the conclusion of the theorem.

□

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