Signatures of spatial curvature on growth of structures

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Abstract. We write down Boltzmann equation for massive particles in a spatially curved FRW universe and solve the approximate line-of-sight solution for evolution of matter density, including the effects of spatial curvature to the first order of approximation. It is shown that memory of early time gravitational potential is affected by presence of spatial curvature. Then we revisit Boltzmann equation for photons in the general FRW background. Using it, we show that how the frequency of oscillations and damping factor (known as Silk damping) changed in presence of spatial curvature. At last, using this modified damping factor in hydrodynamic regime of cosmological perturbations, we find our analytic solution which shows the effects of spatial curvature on growing mode of matter density.

Keywords: CMBR theory, cosmological perturbation theory, physics of the early universe

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1 Introduction

According to Planck 2018 results [1] $\Omega_K = -0.044^{+0.018}_{-0.015}$ (68 % Planck TT, TE, EE + low E) and the joint constraint with lensing and BAO measurements constraint it to $\Omega_K = 0.001 \pm 0.002$. The constraint on spatial curvature assumes a specific cosmological model. This means that most of this results are cosmological model dependent. Many model independent methods to measure the spatial curvature of the universe are proposed [2–9]. These results shows that non-zero $\Omega_K$ can not be easily ruled out by current observations (e.g. look at the results of [10–13]). In addition, there are vast majority of study to constraint topology of the universe using cosmological data [14–17]. Beside this practical concerns, from the theoretical perspective it is also interesting to know how the topology of the universe can affect our cosmological phenomena. Furthermore, data indicates a positive cosmological constant, which leads to a de Sitter universe for vacuum solutions. Knowing that from all the de Sitter solutions only the Lorentzian de Sitter spacetime which is spatially closed, is maximally symmetric, maximally extended and geodesically complete, increased our theoretical interests for studying cosmology around this background [18, 19]. There is extensive studies on the subject of cosmological perturbations using Boltzmann hierarchy equations approach in non-flat universe. Study of some of the cosmological observables (like Hubble diagram, angular size and number density of galaxies) in closed universe models can be found in [20] and the references therein. In [21–23] solving Boltzmann equations is used for calculations on CMB anisotropy. But their approach for spatially curved universe leads to long integration time. The authors of [24] derived CMB angular power spectrum and matter transfer function in a closed FRW universe using a semi-analytic method. In [25] the semi-analytic is used to compute CMB tensor spectrum in closed universe. In [26], integral solutions are derived for CMB anisotropy in general FRW background. The solutions are in the form of time integral over source term and geometrical term. The geometrical term is described as eigenfunction of Laplacian operator in spatially curved background. In their paper, they give a fast and efficient method for computing those functions. This approach is generalized in [27] to arbitrary perturbation type and FRW metric. Furthermore, [28] use a numerical approach to linearized equations of $1+3$ covariant formalism to calculate CMB anisotropies.
in general FRW background. A new method for calculation of CMB anisotropy in non-flat universe is presented in [29], in which a new efficient algorithm is introduced to calculate eigenfunctions of Laplacian operators in non-flat universe. Corrections on spatial curvature due to relativistic effects on Large Scale Structure is studied in [30]. Some other new investigation of cosmological perturbations and Boltzmann hierarchies are presented in [31, 32]. It’s well-known that spatial curvature causes shifts of angular scale of acoustic peaks, change primordial spectrum and evolution of long wavelength modes through Integrated Sachs-Wolf effect. In this work we want to explore the effects of spatial curvature in two phenomena by studying relativistic Boltzmann equation. First it is shown that how spatial curvature affects the memory of early time gravitational potential. The second phenomena that is presented is about damping of acoustic peaks due to transportation of photons in non-relativistic plasma, which is known as silk damping [33]. In this regard, we follow closely the procedure of [34] to drive an analytical formula that shows how frequency and damping of acoustic oscillations are affected by spatial curvature. Especially, we try to give an analytical solution to our equations so that we can have a clear interpretation and perception about behavior of spatial curvature in our solutions. In section 2 we write relativistic Boltzmann equation for dark matter for a spatially curved background. We find an integro-differential solution for zeroth moment of it which gives evolution of matter density. In section 3 by writing relativistic Boltzmann equation for photons we find the effects of curvature on damping parameter and oscillation frequency. In section 4 we use the derived damping factor in hydrodynamic limit of cosmological perturbation to show how spatial curvature affects the growth of matter density and especially the ripples of Baryon Acoustic Oscillations. In the last section we summarize our results and give some comments.

Throughout this paper we use $\bar{+}$ $+$ $+$ signatures. Greek alphabets is set for 4 dimensional indices and Latin alphabet for 3 dimensional ones.

2 Relativistic Boltzmann equation for matter

Throughout this paper we assume a background of general curved FRW universe with the metric:

$$\bar{g}_{00} = -1, \quad \bar{g}_{0i} = 0, \quad \bar{g}_{ij} = a^2 \gamma_{ij}$$

(2.1)

where $\gamma_{ij}$ is the 3-dimensional spatial metric. In a semi-Cartesian coordinate it can be written as:

$$\gamma^{ij} = \delta^{ij} - K x^i x^j$$

(2.2)

where $K$ is the constant of curvature of the spatial metric. There is a useful relation for spatial metric in semi-Cartesian coordinates which helps us to simplify the following calculations:

$$\gamma^{ii'} \gamma^{jj'} \partial_{ki} \gamma_{ij} = -\partial_{k} \gamma^{ij}$$

(2.3)

Furthermore, the non vanishing components of the Christoffel’s connection of the metric are:

$$\bar{\Gamma}^k_{ij} = \frac{1}{2} \gamma^{kl} (\partial_j \gamma_{li} + \partial_l \gamma_{ij} - \partial_i \gamma_{lj})$$

(2.4)

$$\bar{\Gamma}^i_{0j} = H \delta^i_j$$

(2.5)

$$\bar{\Gamma}^0_{ij} = a^2 H \gamma_{ij}$$

(2.6)
We define the four-momentum of dark matter particles as:

\[ p^\mu = m \frac{dx^\mu}{d\tau} \]  

(2.7)

In this way, the particle’s four-momentum is related to particle’s velocity by \( \frac{dx^i}{dt} = \frac{v^i}{\rho} \) and zeroth component of four momentum would be \( p^0 = \sqrt{m^2 + g_{ij} p^i p^j} \). To write down the Boltzmann equation, we should define the number density of the particles as function of Cartesian coordinates \( x^i \), momenta \( p_i \) and time \( t \) which are the usual phase space parameters of a system of particles. Note that here we use lower index \( p_i \) only because the later calculations are less complicated in terms of them. With the help of geodesic equation, we get:

\[ \frac{dp_i}{dt} = \partial_i g_{mn} p^m p^n \]

(2.8)

Because the interactions of the cold dark matter particles are only restricted to the gravitational interaction, number density of cold dark matter satisfies collisionless Boltzmann equation, which is nothing but the fact that number density is conserved in phase space [35]. After applying chain rules and with the help of (2.8) we will have:

\[ \frac{d}{dt} n(x^i, p_i, t) = \partial_t n(x^i, p_i, t) + \frac{p^k}{p^0} \partial_k n(x^i, p_i, t) + \frac{p^p}{2p^0} \partial_k g_{ln} \frac{\partial n(x^i, p_i, t)}{\partial p^k} = 0 \]

(2.9)

Then, we write the metric perturbations around this background as:

\[ g_{ij} = a^2 \gamma_{ij} + \delta g_{ij}, \quad g^{ij} = a^{-2} \gamma^{ij} - a^{-4} \delta g^{ij} \]

(2.10)

where the covariant and contravariant forms of the metric perturbations in the above definitions, are related in this way:

\[ \delta g^{ij} = \gamma^{i'j'} \delta g_{i'j'} \]

(2.11)

After linearizing the zeroth and i-th components of 4-momenta of the particles with respect to the metric perturbation we get:

\[ p^0 = \sqrt{m^2 + p^2/a^2} \left( 1 - \frac{a^{-4} p_i p_j \delta g^{ij}}{2(m^2 + p^2/a^2)} \right) \]

\[ p^i = a^{-2} \gamma^{ij} p_j - a^{-4} p_j \delta g^{ij} \]

(2.12)

\[ p^{2.13} \]

in which \( p \) is defined as \( p \equiv \sqrt{\gamma^{ij} p_i p_j} \). The number density of the particles can be written as its background value plus perturbed number density where the background number density is function of \( a\sqrt{\gamma^{ij} p_i p_j} \):

\[ n = n\left( a\sqrt{\gamma^{ij} p_i p_j} \right) + \delta n(x^i, p_i, t) \]

(2.14)

The factor \( a(t) \) is included in its argument so that \( \bar{n}(p) \) becomes time independent. We assume that \( n(x^i, p_i, t_1) = \bar{n}(a(t_1)\sqrt{\gamma^{ij} p_i p_j}) \) as our initial condition. That is, at some initial time \( t_1 \) the dynamical correction \( \delta n \), induced by gravitational perturbation, is zero. After linearizing the background number density in terms of metric perturbations, we get:

\[ \tilde{n} = \bar{n}\left( a\sqrt{\gamma^{ij} p_i p_j} \right) = \bar{n}(p) - \bar{n}'(p) \frac{p_i p_j \delta g^{ij}}{2a^2 p} \]

(2.15)
After straightforward but tedious calculations it can be shown that:

\[
\frac{p^k}{p^0} \partial_k \bar{n} + \frac{p^l p^m}{2p^0} \partial_k g_{lm} \frac{\partial \bar{n}}{\partial p_k} = 0 \tag{2.16}
\]

Putting the above number density in Boltzmann equation (2.9) and using (2.16), we would have our Boltzmann equation for perturbed number density:

\[
\partial_t \delta n + \frac{a^{-2} \gamma^{ij} p_j}{\sqrt{m^2 + p^2/a^2}} \partial x^i \delta n + \frac{a^{-2} K \vec{x} \vec{p} p_k}{\sqrt{m^2 + p^2/a^2}} \partial p_k = \bar{n}' \frac{p_i p_j \partial_t (a^{-2} \delta g^{ij})}{2p} \tag{2.17}
\]

The Fourier expansion of the scalar modes in a spatially closed universe can be written in the following form:

\[
\delta n(x^i, p_i, t) = \int d^2 \hat{q} \sum q e^{iq \cdot \arccos (\hat{q} \cdot \vec{x})} \sqrt{1 - (\hat{q} \cdot \vec{x})^2} \delta n(q, \hat{q}, p_i, t) \tag{2.18}
\]

where \( q \) is non-negative integer, which is the well-known fact that wave numbers are discrete in spatially closed universe. In the same way, we can expand the metric perturbation, \( \delta g_{ij} \). Putting these modes expansion back in Boltzmann equation (2.17) and working in limit of \( x \ll 1 \), the equation (2.17) takes this form,

\[
\partial_t \delta n(q_i, p_i, t) + \frac{a^{-2}}{\sqrt{m^2 + p^2/a^2}} \mathcal{O} \delta n(q_i, p_i, t) = \frac{\bar{n}'(p)}{2p} p_i p_j \partial_t (a^{-2} \delta g^{ij}(q_i, t)) \tag{2.19}
\]

where \( q_i = q \hat{q}_i \) and we define the differential operator \( \mathcal{O} \) as:

\[
\mathcal{O} \equiv i\vec{q} \vec{p} + iK p_l \partial_{q_l} \tag{2.20}
\]

From now till the end of this section, we omit the arguments of the perturbation parameters, so that our relations looks much simpler. But we should note that these perturbations are in Fourier space. The integral solution of above differential equation has the following form:

\[
\delta n = \int_{t_1}^t dt' e^{\mathcal{O}} \left[ - \int_{t'}^t dt'' \frac{1}{a^2(t'')} \mathcal{O} \right] \frac{\bar{n}'(p)}{2p} p_i p_j \partial_t (a^{-2} \delta g^{ij}) \tag{2.21}
\]

By taking different moments of number density, we can get each components of energy-momentum tensor. Here we only use its zeroth moment which is:

\[
T^0_0 = \frac{1}{\sqrt{-g}} \int d^3 p (p_0 n) \tag{2.22}
\]

At the end, the fluid parameters can be extracted from different components of energy-momentum tensor, \( \delta T^0_0 = -\delta \rho \).

In this section, we want to choose the metric perturbations in Newtonian gauge. Because the form of our solutions will be simpler in this gauge. From Scalar-Vector-Tensor decomposition, we know that our perturbed equations do not mix different types of perturbations. Because our goal is to derive matter density, it is just enough to focus on scalar part of the metric perturbations:

\[
ds^2 = -(1 + 2\Phi)dt^2 + a^2(1 - 2\Psi)dx^i dx^j \tag{2.23}
\]
Using (2.22) and after linearizing the (2.25) in terms of $K$, $\delta \rho$ the integral solution of number density in (2.22) we will find:

$$
\delta \rho = \frac{1}{a^3} \int d^3p \left(- \frac{\bar{n}(p)\Phi}{\sqrt{m^2 + p^2/a^2}} (2m^2 + 3p^2/a^2) - \bar{n}'(p) \Phi p \sqrt{m^2 + p^2/a^2} + \sqrt{m^2 + p^2/a^2} \int_{t_1}^{t} dt' \exp \left[- \int_{t'}^{t} dt'' \frac{1}{a^2(t'') \sqrt{m^2 + p^2/a^2(t'')}} O \right] \bar{n}'(p)p \partial_t \Phi \right)
$$

(2.24)

After integrating over $p$ by parts in the second term, it will be simplified like this,

$$
\delta \rho = \frac{1}{a^3} \int d^3p \sqrt{m^2 + p^2/a^2} \times \left(- \bar{n}(p)\Phi + \int_{t_1}^{t} dt' \exp \left[- \int_{t'}^{t} dt'' \frac{1}{a^2(t'') \sqrt{m^2 + p^2/a^2(t'')}} O \right] \bar{n}'(p)p \partial_t \Phi \right)
$$

(2.25)

Using (2.22) and after linearizing the (2.25) in terms of $K$, $\delta \rho$ can be written as,

$$
\delta \rho = -\bar{\rho} \Phi + \frac{2\pi}{a^3} \int d^3p \sqrt{m^2 + p^2/a^2} \int_{t_1}^{t} dt' \exp \left[- \int_{t'}^{t} dt'' \frac{\bar{q} \partial_t \Phi}{a(t'') \sqrt{m^2 + p^2/a^2(t'')}} \right] \times \left(\bar{n}'(p)p \partial_t \Phi - iK \frac{\bar{q} \partial_t \Phi}{q} (p^2 \bar{n}'(p) + \bar{n}'(p)) \partial_t \Phi \int_{t'}^{t} dt'' \frac{1}{a(t'') \sqrt{m^2 + p^2/a(t'')}} \right)
$$

(2.26)

Here the prime over $\Phi$ indicates derivative with respect to wave number $q$. The momentum space volume element in the limit of small scales can be written as, $d^3p = p^2 dp dq d\phi$, where $z = \frac{\vec{p} \cdot \vec{q}}{qp}$, is the cosine of angle between $\vec{p}$ and $\vec{q}$, $\phi$ is azimuthal angle of $\vec{p}$ with respect to $\vec{q}$. So we can re-parametrize the integral solution in terms of these new parameters:

$$
\delta \rho = -\bar{\rho} \Phi + \frac{2\pi}{a^3} \int dp \int_{-\infty}^{\infty} d\phi \int_{-1}^{1} dz \sqrt{m^2 + p^2/a^2} \int_{t_1}^{t} dt' \exp \left[- \int_{t'}^{t} dt'' \frac{iqpz}{a(t'') \sqrt{p^2/a(t'')^2 + m^2}} \right] \times \left(\bar{n}'(p)p \partial_t \Phi - \int_{t'}^{t} dt'' \frac{iK \partial_t \Phi}{a(t'') \sqrt{p^2/a(t'')^2 + m^2}} (p\bar{n}'(p) + \bar{n}'(p)) \partial_t \Phi \right)
$$

(2.27)

This integral’s solution in general is very complicated, but it will have much more simpler form in non relativistic limit where $p^2/a^2 \ll m^2$, which is a permissible assumption for cold dark matter particles:

$$
\delta \rho = -\bar{\rho} \Phi + \frac{2\pi}{a^3} \int dp \int dz \int_{t_1}^{t} dt' \exp \left[- \frac{iqpz}{m} \int_{t'}^{t} dt'' \right] \times \left(\bar{n}'(p)p \partial_t \Phi - iK \frac{p^2 (p\bar{n}'(p) + \bar{n}'(p)) \partial_t \Phi}{a(t'')^2} \right)
$$

(2.28)
Also we should note that in the non relativistic limit, the rate of oscillation in the above exponential is much smaller than the rate of oscillation in metric perturbation, $\Phi$ \cite{36}. So we can pull back the time derivative to the whole of the argument:

$$
\exp \left[ - \frac{i q p z}{m} \int_v^t \frac{d t''}{a^2(t'')} \right] \left( m n'(p) p \partial_p \Phi - i K z p^2 (p n''(p) + \bar{n}'(p)) \partial_p \Phi' \int_v^t \frac{d t''}{a(t'')} \right)
$$

$$
\approx \partial_p \left( \exp \left[ - \frac{i q p z}{m} \int_v^t \frac{d t''}{a^2(t'')} \right] \left( m n'(p) p \Phi - i K z p^2 (p n''(p) + \bar{n}'(p)) \Phi' \int_v^t \frac{d t''}{a(t'')} \right) \right)
$$

(2.29)

Using this fact, we can simplify $\delta \rho$ in this manner:

$$
\delta \rho = -\bar{\rho} \Phi + \frac{2\pi}{a^3} \int dp p^2 dz \times \left( m n'(p) p \Phi(t) - e^{-\frac{i q p z}{m} \int_v^t \frac{d t''}{a^2(t'')}} \times \left( m n'(p) p \Phi(t_1) - i K z p^2 (n''(p) p + \bar{n}'(p)) \Phi'(t_1) \int_v^{t_1} \frac{d t''}{a^2(t'')} \right) \right)
$$

(2.30)

After integrating over $p$ by parts in the first term of above integral and using the fact that \( \frac{m N}{a^3} \equiv \frac{m}{a^3} \int 4\pi p^2 dp n = \bar{\rho} \) and after integration over $z$, we get:

$$
\delta \equiv \frac{\delta \rho}{\bar{\rho}} = -2 \Phi + S
$$

(2.31)

$$
S \equiv \frac{m \Phi(t_1)}{N \omega(t)} \int p^2 dp n'(p) \sin \left( \frac{p \omega(t)}{m} \right) \left( \cos \left( \frac{p \omega(t)}{m} \right) - \sin \left( \frac{p \omega(t)}{m} \right) \right)
$$

(2.32)

where the parameter $\omega(t)$ is defined as:

$$
\omega(t) \equiv \int_{t_1}^{t} \frac{q dt''}{a^2(t'')}
$$

(2.33)

Here it should be mentioned that this analytical solutions are applicable for intermediate modes. These modes enter the horizon before kinetic decoupling and after dark matter becomes non relativistic. In the absence of spatial curvature, the second term can be interpreted as memory of the gravitational field at early times. An effect like this find in \cite{36} for gravitational waves, where they show that the presence of dark matter affects propagation of gravitational wave by its memory at the time of emission. Here it can be seen that the spatial curvature modified this memory as the second term of (2.32). The function $S(t)$ is a transient function that goes to zero exponentially at late time, when dark matter particles travel a distance larger than mode’s wavelength, or in other words when $\frac{p \omega(t)}{m} \gg 1$. To clarify this fact, let us assume that $n(p)$ has familiar Maxwell-Boltzmann form of:

$$
\bar{n}(p) = A e^{-\frac{p^2}{2m^2}}
$$

(2.34)
and we set the normalization constant to $A = \sqrt{\frac{2}{3\pi}} \frac{N}{\pi m^3}$ so that $\int d^3p \bar{n} = N$, where $P$ is an arbitrary constant.

Then after performing by parts and doing integration over $p$ we will end up with:

$$S = \left( \Phi(t_1) \left( v^2 \omega_q(t)^2 - 3 \right) + \frac{K^2(t_1)}{q} \omega_q(t)^2 \bar{v}^2 \left( 25 - 13 \omega_q(t)^2 \bar{v}^2 + \omega_q(t)^4 \bar{v}^2 \right) \right) e^{-\frac{\omega_q(t)^2 v^2}{2}} \tag{2.35}$$

We define $\bar{v}^2 = \frac{p^2}{m^2}$, which is the mean square coordinate velocity for distribution function of $n$. As we mentioned earlier, dark matter particles are non relativistic and so $v^2/a^2(t) \ll 1$. For the particles that travel less than wavelength of the modes between $t_1$ and $t$, we can assume $v^2 \omega_q(t)^2 \ll 1$. Then we have $S = -3\Phi(t_1)$. In addition, at very late time, where $v^2 \omega_q^2 \gg 1$, $S(t)$ is exponentially small and this memory effect of the early time will be erased. In the between time, when $v^2 \omega_q^2 \sim 1$, we can see that spatial curvature, shows itself in memory as second term of (2.35).

### 3 Damping of photons due to non relativistic medium

The goal of this part is to compute the oscillation frequency and damping factor of acoustic peaks in the curved FRW background. Here we closely follow the procedure of [34] in deriving our results.

In the case of photons, we denote their distribution function by density matrix $n^{ij}$ and the upper indices are due to photon’s polarization. It is defined such that the number of photons with polarization $e_i$ in the phase space volume $\Pi dp_i dx^i$ at time $t$ will be $\bar{g}_{ik} g_{jl} e^k e^l n^{ij}(x^i, p_i, t)\Pi m dp_m dx^m$. For the detailed definition of the density matrix and derivation of the Boltzmann equation look at appendix A. This function obeys the Boltzmann equation as:

$$\partial_t n^{ij} + \frac{p^k}{p^0} \partial_k n^{ij} + \frac{p_i p^m}{2p^0} \partial_k g_{im} \frac{\partial n^{ij}}{\partial p_k} + \left( \Gamma^i_{kl} - \frac{p^i}{p^0} \Gamma^0_{kl} \right) \frac{p^l}{p^0} n^{kj} + \left( \Gamma^j_{kl} - \frac{p^j}{p^0} \Gamma^0_{kl} \right) \frac{p^l}{p^0} n^{ki} = C^{ij} \tag{3.1}$$

Here $C^{ij}$ is the collision term, which includes interactions of Thomson scattering of photons with electrons in the plasma. Then, we can write the perturbed distribution function of photon as:

$$n^{ij} = \frac{1}{2} \bar{n} \left( a p^0 \right) \left( g^{ij} - \frac{g^{ik} g^{jl} p_k p_l}{(p^0)^2} \right) + \delta n^{ij} \tag{3.2}$$

We can follow the same procedure of the last section, now for photon distribution. Considering the metric perturbation (2.10), the perturbed distribution function becomes: $\bar{n} = \bar{n}(p) - \frac{p^j p^l}{2a^2} \bar{n}^{ij}(p) \delta g^{ij}$. In addition, it will be easy to show that $\bar{n}$ satisfies the same relation as (2.16). Using this and Christoffel values of (2.4) streaming part of Boltzmann equation would be:

$$\frac{d}{dt} n^{ij} = \partial_t \delta n^{ij} + \frac{1}{a} \gamma^{kl} \bar{p}_i \partial_k \delta n^{ij} + \frac{2K}{a} \bar{p}_k p_l x^j \frac{\partial \delta n^{ij}}{\partial p_k} \quad \text{\textit{\textbf{Part (2.17)}}}$$

$$- \frac{p}{4a^2} \bar{p}_i \partial_i \left( a^{-2} \bar{g}^{ij} \gamma^{kl} \bar{p}_k \bar{p}_l \right) \left( \bar{g}^{ij} - \gamma^{ik} \gamma^{jl} \bar{p}_k \bar{p}_l \right) + \frac{2}{a} \delta n^{ij} + \frac{\hat{a}}{a} \delta n^{ij} \tag{3.3}$$
The $\hat{p}_i$ is the unit vector of $p_i$, which is defined as $\hat{p}_i \equiv \frac{p_i}{p}$. Furthermore, the collision part can be written as:

$$C^{ij} = -\omega_c \delta n^{ij} + \frac{3\omega_c}{8\pi} \int d\hat{p}_1 \{ \delta n^{ij}(x^i, p\hat{p}_1, t) - \gamma^{ik} \hat{p}_k \delta n^{lij}(x^i, p\hat{p}_1, t) - \gamma^{ik} \hat{p}_k \delta n^{il}(x^i, p\hat{p}_1, t) + \gamma^{ik} \gamma^{jl} \hat{p}_k \delta n^m(x^i, p\hat{p}_1, t) \} - \frac{\omega_c}{2\pi} p_k \delta u_k \gamma^{k'k} \bar{n}'[\gamma^{ij} - \gamma^{il} \gamma^{j'l'} \hat{p}_l \hat{p}_v]$$

(3.4)

where $\bar{p}_1$ in the above relation is arbitrary initial momentum of photon. Our goal is to derive damping of acoustic waves in a homogeneous and time-independent plasma in a spatially curved background. So we can consider a static universe $a = 1$ with spatial curvature as:

$$ds^2 = -dt^2 + \gamma_{ij} dx^i dx^j$$

(3.5)

When the collision rate is much larger than sound frequency, we can also assume that $\omega_c$ is time independent. This assumption of static universe is permissible, because the damping effect would only be noticeable for deep inside the horizon modes. So we can put our Boltzmann equation in this form:

$$\partial_t \delta n^{ij} + \gamma^{kl} \hat{p}_k \partial_t \delta n^{lj} + 2K \hat{p}_k p^i \frac{\partial \delta n^{ij}}{\partial p_k}$$

$$= -\omega_c \delta n^{ij} + \frac{3\omega_c}{8\pi} \int d\hat{p}_1 \{ \delta n^{ij}(x^i, p\hat{p}_1, t) - \gamma^{ik} \hat{p}_k \delta n^{lij}(x^i, p\hat{p}_1, t) + \gamma^{ik} \gamma^{jl} \hat{p}_k \delta n^m(x^i, p\hat{p}_1, t) \} - \frac{\omega_c}{2\pi} p_k \delta u_k \gamma^{k'k} \bar{n}'[\gamma^{ij} - \gamma^{il} \gamma^{j'l'} \hat{p}_l \hat{p}_v]$$

(3.6)

Then we can take the solutions of $\delta n^{ij}(t, \vec{x}, \vec{p})$ and $\delta u_j(t, \vec{x})$ the same as (2.18), but here we restrict the solutions to one energy mode $\omega$. This will help us to find the dispersion relation in the following of this section.

$$\delta n^{ij}(t, \vec{x}, \vec{p}) = \int d^2 \vec{q} \sum_q e^{i \vec{q} \cdot \vec{x}} \frac{e^{-i \omega t}}{(1 - (\vec{q}, \vec{x})^2)} \delta n^{ij}(q, \vec{q}, p)$$

(3.7)

$$\delta u_j(t, \vec{x}) = \int d^2 \vec{q} \sum_q e^{i \vec{q} \cdot \vec{x}} \frac{e^{-i \omega t}}{(1 - (\vec{q}, \vec{x})^2)} \delta u_j(q, \vec{q})$$

(3.8)

Because we want to consider the acoustic modes deep inside the horizon, we can use the same approximation of the last section, $x \ll 1$. After using this expansion in equation (3.6), we get:

$$\int d^3 \vec{p} (\omega_c - \omega + i \vec{q} \cdot \vec{p}) \delta n^{ij} + 2iK \hat{p}_k p_i \frac{\partial}{\partial p_k} \frac{\partial \delta n^{ij}}{\partial q_l} = -\frac{\omega_c}{2} \bar{n}' p \delta u(\gamma^{ij} - \delta n^{ij})$$

(3.9)

Then, we multiply $p_j$ with $kk$ component of equation (3.9), so that we come to:

$$\int d^3 \vec{p} (\omega_c - \omega + i \vec{q} \cdot \vec{p}) p_j \delta n^{kk} + 2iK \int d^3 \vec{p} \vec{p}_j p_l \frac{\partial}{\partial p_l} \frac{\partial \delta n^{kk}}{\partial q_r} = -\omega_c \int d^3 \vec{p} \vec{p}_j p_k \delta u_k \bar{n}'$$

(3.10)
Integrating by parts over $p_i$ gives the second term of left hand side of equation (3.10) as,

$$
2iK \int d^3p \hat{p}_i \hat{p}_j \frac{\partial}{\partial \hat{p}_i} \frac{\partial}{\partial \hat{q}_j} \delta n_{kk} = -2iK \int d^3p (\hat{p}_i \hat{p}_j + \hat{p}_j \hat{p}_i + \hat{p}_j \hat{p}_j) \frac{\partial}{\partial \hat{q}_j} \delta n_{kk}
$$

$$
= -10iK \int d^3p \hat{p}_j \frac{\delta n_{kk}}{\partial \hat{q}_j} 
$$

(3.11)

For simplifying the term on the right hand side of (3.10), we can write $\int d^3p \hat{p}_j \hat{n}'(p) = H_{ij}$. Contracting it with $\gamma_{ij}$ gives:

$$
\gamma^{ij} H_{ij} = \int d^3p \hat{n}'(p) = -4 \int d^3p \hat{n}(p) = -4 \bar{\rho}_\gamma 
$$

(3.12)

In which $\bar{\rho}_\gamma$ is photon energy density. So we have: $H_{ij} = -\frac{4}{3} \bar{\rho}_\gamma \gamma_{ij}$. Using (3.11) and knowing $H_{ij}$ we can rewrite equation (3.10):

$$
\int d^3p (\omega_c - i\omega + i\vec{q} \bar{p}) \hat{p}_j \delta n_{kk} = 10iK \int d^3p \hat{p}_j \frac{\partial}{\partial \hat{q}_j} \delta n_{kk} + \frac{4}{3} \omega_c \bar{\rho}_\gamma \delta u_j
$$

(3.13)

The stress tensor for photon distribution function is defined like this:

$$
\delta T_{\gamma}^{ik} = \gamma^{ii'} \int d^3p m^{kk} \hat{p}_{i'} \hat{p}_j, \quad \delta T_{\gamma}^{ij} = \int d^3p m^{kk} \hat{p}_j
$$

(3.14)

For Baryon we have: $\delta T_B^{ij} = 0$ and $\delta T_B^{ij} = \bar{\rho}_B \delta u_j$. Using momentum conservation, $\partial_\mu \delta T_{\mu}^\nu = 0$, we can write:

$$
\omega \bar{\rho}_B \delta u_j = - \int d^3p m^{kk} (\hat{p}_j (\hat{p}_i q_i - \omega))
$$

(3.15)

With the help of (3.15), the equation (3.13) simplifies to:

$$
\left(\omega \bar{\rho}_B + i \frac{4}{3} \omega_c \bar{\rho}_\gamma \right) \delta u_j = i \int d^3p \hat{p}_j \left( \omega_c \delta n_{kk} - i 10K \hat{p}_i \frac{\partial}{\partial \hat{q}_i} \delta n_{kk} \right)
$$

(3.16)

Now we can define form factors in this way:

$$
\int d^3p \hat{p} \delta n^{ij} = \bar{\rho}_\gamma (X \delta_{ij} + Y \hat{q}_i \hat{q}_j)
$$

(3.17)

$$
\int d^3p \hat{p}_j \delta n^{kk} = \bar{\rho}_\gamma Z \hat{q}_j
$$

(3.18)

$$
\int d^3p \delta n^{kk} \hat{p}_j = \bar{\rho}_\gamma (V \delta_{ij} + W \hat{q}_i \hat{q}_j)
$$

(3.19)

These integrals can be written in this way, because after integrating over $\vec{p}$, the only parameter that contains direction is $q_i$. Using these form factors in equation (3.16) leads to:

$$
\left(\omega \bar{\rho}_B + i \frac{4}{3} \omega_c \bar{\rho}_\gamma \right) \delta u_j = i \omega_c \bar{\rho}_\gamma \hat{q}_j Z + \frac{10K \bar{\rho}_\gamma}{q} \frac{\partial}{\partial \hat{q}_i} (V \delta_{ij} + W \hat{q}_i \hat{q}_j) = i \omega_c \bar{\rho}_\gamma Z \hat{q}_j + 4W \frac{\hat{q}_j}{q}
$$

(3.20)

$$
(1 - i \omega R t_c) \delta u_j = \frac{3}{4} \omega_c \left( Z - i \frac{40K t_c}{q} \right)
$$

(3.21)
where $R \equiv \frac{3\tilde{p}}{4\tilde{p}_i}$ and $t_c \equiv 1/\omega_c$ is the mean time between collisions. From the relation (3.21), the perturbed velocity can be written: $\delta u_{ij} = \frac{3\tilde{q}_i}{4(1-i\omega_c R)} (Z - i\frac{40Kt_c}{q})$. Putting this back to equation (3.9), gives:

\[
(\omega_c - i\omega + iq\hat{p})\delta n^{ij} + 2iK\hat{p}_i\hat{p}_j \frac{\partial}{\partial q_i} \frac{\partial}{\partial \hat{q}_j} \delta n^{ij} = \frac{3\tilde{q}_i\hat{p}_k}{4(1-i\omega_c R)} (Z - i\frac{40Kt_c}{q})
\]

(3.22)

We can now integrate right hand side of equation (3.22) over $\hat{d}p^3$:

\[
\int \hat{d}p^3 \text{RHS} = \frac{3\omega_c}{8\pi} \left( \delta_{ij} - \hat{p}_i\hat{p}_j \right) \left( X + \frac{Z - i\frac{40Kt_c}{q}}{1-i\omega_c R} \hat{q}_k\hat{p}_k \right) + Y \left( \hat{q}^i - \hat{p}^i \hat{q} \right) \left( \hat{q}^j - \hat{p}^j \hat{q} \right)
\]

(3.23)

After using $\frac{\partial}{\partial \hat{q}_i} = \hat{p}_k\frac{\partial}{\partial \hat{q}_i}$, on the left hand side of (3.22), our Boltzmann equation becomes:

\[
\left( \omega_c - i\omega + iq\hat{p} \right) \int \hat{d}p^3 \delta n^{ij} = \frac{3\omega_c}{8\pi} \left( \delta_{ij} - \hat{p}_i\hat{p}_j \right) \left( X + \frac{4}{3} \delta_{lk}\hat{p}_k \right) + Y \left( \hat{q}^i - \hat{p}^i \hat{q} \right) \left( \hat{q}^j - \hat{p}^j \hat{q} \right)
\]

(3.24)

We define the differential operator $Q$ like this:

\[
Q \equiv \omega - \hat{q} \hat{p} + 8K\hat{p}_i \frac{\partial}{\partial \hat{q}_i}
\]

(3.25)

Using this definition, the Boltzmann equation (3.24) gives us:

\[
4\pi \int \hat{d}p^3 \delta n^{ij} = \frac{3\tilde{p}}{2(1-iQ t_c)} \times \left( \delta_{ij} - \hat{p}_i\hat{p}_j \right) \left( X + \frac{Z - i\frac{40Kt_c}{q}}{1-i\omega_c R} \hat{p} \right) + Y \left( \hat{q}^i - \hat{p}^i \hat{q} \right) \left( \hat{q}^j - \hat{p}^j \hat{q} \right)
\]

(3.26)

Then, expanding this equation to second order in $t_c$ gives:

\[
4\pi \int \hat{d}p^3 \delta n^{ij} = \frac{3}{2} \tilde{p}_i \gamma \left( 1 + iQ t_c - t_c^2 Q^2 \right) \times \left( \delta_{ij} - \hat{p}_i\hat{p}_j \right) \left( X + (1 + i\omega_c R - t_c^2 \omega^2 R^2) \left( Z - i\frac{40Kt_c}{q} \right) \hat{p} \right)
\]

(3.27)

\[
+ Y \left( \hat{q}^i - \hat{p}^i \hat{q} \right) \left( \hat{q}^j - \hat{p}^j \hat{q} \right)
\]

(3.28)

We start by an ansatz that $X$ and $Z$ are of order $O(t_c^0)$ and $Y$ and $W$ are of order $O(t_c)$. At the end we can check that our ansatz was correct. So to second order of $t_c$, we have:

\[
4\pi \int \hat{d}p^3 \delta n^{ij} = \frac{3}{2} \tilde{p}_i \gamma \left( \delta_{ij} - \hat{p}_i\hat{p}_j \right) \left( X + t_c Q X - t_c^2 Q^2 X + \left( 1 + it_c \omega R - t_c^2 \omega^2 R^2 \right) Z \right.)
\]

\[
- \frac{40Kt_c}{q} \hat{p} \right) \hat{q} + it_c Z(1 + it_c \omega R) Q \hat{p} \hat{q} - t_c^2 Z Q^2 \hat{p} \hat{q}
\]

\[
+ (1 + it_c Q) Y(\hat{q}^i - \hat{q}^i \hat{q})(\hat{q}^j - \hat{p}^j \hat{q})
\]

(3.28)
Now we should integrate them over $d^2\hat{p}$. The calculation for each term of integration is written in the appendix. The final result will be:

$$4\pi \int d^3p\delta n^{ij} = 4\pi \rho_\gamma(X\delta_{ij} + Y\hat{q}_i\hat{q}_j)$$

$$= 4\pi \rho_\gamma \left( X + it_cX\omega - \frac{t_c^2}{4} \left( 2\omega^2 - 16K + \frac{q^2}{5} \right) X + \frac{8K}{q} \left( it_c - t_c^2\omega(1+R) \right) + \left( \frac{2it_cq}{5} + \frac{2t_c^2\omega(R+2)q}{5} \right) + \frac{1+i\omega t_c}{10} \right) \delta_{ij}$$

$$+ \left( \frac{-1}{10}t_c^2X + \frac{it_cq}{5}Z - \frac{3\omega RZ t_c^2}{5} + \frac{1}{10} \gamma \left( 1 \right) Y \right) \hat{q}_i\hat{q}_j$$

(3.29)

Equating the coefficients of $\gamma_{ij}$ and $\hat{q}_i\hat{q}_j$ on each side correspondingly, will lead to:

$$X \left( it_c\omega - \omega^2t_c^2 + 8Kt_c^2 - \frac{2q^2t_c^2}{5} \right) + \frac{1+i\omega t_c}{10} Y$$

$$+ \left( \frac{2q}{5} \right) \left( -it_c - t_c^2\omega R + 2\omega t_c^2 \right) + \frac{8K}{q} \left( -t_c^2\omega + it_c - t_c^2\omega R \right) Z = 0$$

(3.30)

$$\frac{t_c^2q^2}{5}X + \left( \frac{3}{10} \right) \left( \frac{7i}{10} \right) \omega t_c + \left( \frac{it_cq}{5} - \frac{2 + R - \omega kt_c^2}{5} \right) Y = 0$$

(3.31)

In addition, using equation (3.18) and after expanding it to second order in power of $t_c$ we get:

$$4\pi \int d^3\hat{p}\delta n^{ii} p_j = 3\hat{p} \int d^2\hat{p}\hat{p} \left( X + \frac{Y}{2} + \left( Z + it_c\omega RZ(1 + it_c\omega R) - \frac{it_c4KW}{q} \right) \hat{p}_i\hat{q}_j \right)$$

$$+ it_cQ \left( X + \frac{Y}{2} + it_cQ \left( Z - \frac{Y}{2} + it_c\omega RZ \right) \hat{p}_i\hat{q}_j - t_c^2Q^2X - t_c^2Q^2Z \hat{p}_i\hat{q}_j - \frac{Y}{2} \hat{p}_i\hat{q}_j^2 \right)$$

(3.32)

After calculating the integrals of each term (which is written the appendix), we will get:

$$(-it_cq + 2\omega t_c^2q)X - \frac{iqt_cq}{5}Y$$

$$+ \left( it_c\omega(1 + R) - t_c^2\omega(1 + R + R^2) - \frac{3}{5}q^2t_c^2 + 24Kt_c^2 \right) Z - \frac{i40KW}{q} = 0$$

(3.33)

At last, we can put the system of equations derived for form factors into a matrix:

$$\begin{bmatrix}
\frac{it_c\omega - \omega^2t_c^2 + 8Kt_c^2 - \frac{2q^2t_c^2}{5}}{2} & 1 + \frac{i\omega t_c}{10} & \frac{2q}{5} \left( -it_c + t_c^2\omega R + 2\omega t_c^2 \right) + \frac{8K}{q} \left( -t_c^2\omega + it_c - t_c^2\omega R \right) \\
\frac{t_c^2q^2}{5} & -\frac{3\omega RZ t_c^2}{5} & \frac{it_cq}{5} - \frac{2 + R - \omega kt_c^2}{5} \\
-\omega t_c^2q + 2\omega t_c^2 & \frac{-3\omega RZ t_c^2}{5} & \frac{it_cq}{5} - \frac{2 + R - \omega kt_c^2}{5} \\
0 & 0 & \frac{-3\omega RZ t_c^2}{5} \\
-\frac{3\omega RZ t_c^2}{5} & \frac{it_cq}{5} - \frac{2 + R - \omega kt_c^2}{5} & \frac{-3\omega RZ t_c^2}{5} \\
0 & 0 & \frac{-3\omega RZ t_c^2}{5}
\end{bmatrix} = 0.$$  

(3.34)

Setting the determinant of this matrix to zero and expanding it to first order in $t_c$, we get the dispersion relation as:

$$-5q^2 + 15(1 + R)\omega^2 + 120K - it_c\omega^3(5 + 5R - 15R^2) + 7i\omega t_cq^2 - 400iKWt_c = 0$$  

(3.35)
For solving the above dispersion relation, we split \( \omega \) to real and imaginary part, \( \omega = \Omega + i\Gamma \).

Insert it in equation (3.35) and set the real and imaginary part of the relations separately to zero, we get:

\[
- q^2 (5 + 7t_c \Gamma) + 400K (3 + 10t_c \Gamma) - 15 (1 + R) \Gamma^2 + 5 \left( -1 - R + 3R^2 \right) t_c \Gamma^3 \\
+ 15 (1 + R) \Omega^2 + 15t_c \Gamma^2 (1 + R - 3R^2) = 0 \tag{3.36}
\]

\[
-400Kt_c + 7q^2 t_c + 30(1 + R)\Gamma - 15(-1 - R + 3R^2)t_c \Gamma^2 5(1 + R - 3R^2)t_c R^2 = 0 \tag{3.37}
\]

Now we can solve (3.37) for getting \( \Omega \):

\[
\Omega = \pm \sqrt{\frac{400Kt_c - 7q^2 t_c + 15\Gamma(-2 - t_c \Gamma + 3R^2t_c \Gamma - R(2 + t_c \Gamma))}{5t_c(-1 - R + 3R^2)}} \tag{3.38}
\]

Putting the solution for \( \Omega \) back to equation (3.36) and solving it for \( \Gamma \) upto first power of \( t_c \) we get:

\[
\Gamma = - \frac{t_c q^2}{6(1 + R)} \left( \frac{16}{15} + \frac{R^2}{1 + R} \right) + \frac{4Kt_c}{1 + R} \left( 3 + \frac{R^2}{1 + R} \right) \tag{3.39}
\]

Inserting \( \Gamma \) back in equation (3.38) and expand it to first power of \( t_c \) we will get:

\[
\Omega = \pm \frac{1}{\sqrt{3(1 + R)}} \sqrt{q^2 - 72K \frac{3 + 3R + R^2}{-1 - R + 3R^2}} \tag{3.40}
\]

For checking our results, we can look at the limit of \( K \to 0 \), where we get back the results for flat universe as it is originally derived in [37]. It should be mentioned that our result is derived for short wavelengths perturbations. These are short enough that are well inside the horizon at the time of radiation-matter equality. They are applicable till the end of the last scattering where \( R \sim 1 \). In this domain of applicability, the function (3.40) remains regular.

### 4 Hydrodynamic solutions in a spatially curved universe

The 00-component of Einstein equation and conservation equations of energy and momentum are three coupled differential equations which are sufficient for getting hydrodynamic solutions of cosmological perturbations. These equations in synchronous gauge\(^1\) in a curved spacetime are correspondingly:

\[
-4\pi G \left( \delta \rho + 3\delta p + \nabla^2 \Pi^s \right) = \partial_t \left( a^2 \psi \right), \tag{4.1}
\]

\[
\delta \rho + \nabla^2 \Pi^s + \partial_t \left( (\bar{\rho} + \bar{p}) \delta u \right) + 3 \frac{\dot{a}}{a} (\bar{\rho} + \bar{p}) \delta u + 2K \Pi^s = 0, \tag{4.2}
\]

\[
\delta \dot{\rho} + \frac{3\dot{a}}{a} (\delta \rho + \delta p) + \nabla^2 \left( \frac{1}{a^2} (\bar{\rho} + \bar{p}) \delta u + \frac{\dot{a}}{a} \Pi^s \right) + (\bar{\rho} + \bar{p}) \psi = 0 \tag{4.3}
\]

where \( \psi \) is defined as \( \psi = \frac{1}{3}(3\dot{A} + \nabla^2 \dot{B}) \). It is important to note that modification due to spatial curvature only appears here in the form of \( 2K \Pi^s \). Ignoring \( \Pi^s \), we come to the same

\(^1\)Scalar metric perturbation in synchronous gauge is defined like this \( ds^2 = -dt^2 + a^2 ((1 + A)\gamma_{ij} + \partial_{ij} B) \).
equations as we had in flat background. So the known hydrodynamic solutions that was derived in [34] is applicable here. The fast mode solutions are:

$$\delta u_\gamma = \frac{a\sqrt{3}R_q}{q(1 + R)^{3/4}}e^{-\int \Gamma dt Sinv\left(\int \frac{\Omega dt}{a}\right)}$$  \hspace{1cm} (4.4)$$

$$\delta_D = 48\pi G\bar{\rho}_\gamma (2 + R)(1 + R)^{3/4}\left(\frac{a}{q}\right)^2 R_qe^{-\int \Gamma dt Cos\left(\int \Omega \frac{dt}{a}\right)}$$  \hspace{1cm} (4.5)$$

Here, we should put what we find in (3.40) and (3.39) for frequency of oscillation $\Omega$ and damping factor $\Gamma$. Ignoring photon and neutrino energy density combining equations (4.1), (4.2) and (4.3) we come to a second order differential equation:

$$\frac{d}{dt}\left(a^2 \frac{d}{dt}\delta M\right) = 4\pi Ga^2\bar{\rho}_M\delta M$$  \hspace{1cm} (4.6)$$

We can factorize the dependence of $t$ and $q$ by writing: $\delta_M = \Delta(q)F(t)$, where $\Delta$ satisfies:

$$\Delta(q) = \beta \delta_{\gamma q}(t_L) + (1 - \beta)\delta_D q(t_L) - t_L \psi_q(t_L) + \beta t_L \frac{q^2}{a^2} \delta u_\gamma(t_L)$$  \hspace{1cm} (4.7)$$

So the time evaluation of $F(t)$ is:

$$\frac{d}{dt}\left(a^2 \frac{d}{dt}F\right) = 4\pi Ga^2\bar{\rho}_M F$$  \hspace{1cm} (4.8)$$

with initial condition $F \to \frac{3}{5}(\frac{L}{17})^{2/3}$. From Friedman equation we have:

$$\frac{\dot{a}}{a} = \frac{8\pi G}{3}(\bar{\rho}_\Lambda + \bar{\rho}_M) - \frac{K}{a^2}$$  \hspace{1cm} (4.9)$$

Defining $X \equiv \frac{\bar{\rho}_\Lambda}{\bar{\rho}_M} = \frac{\Omega_\Lambda}{\Omega_M}\left(\frac{a}{a_0}\right)^3$, The Friedman equation can be written in terms of $X$:

$$\frac{\dot{a}}{a} = H_0\sqrt{\Omega_\Lambda} \sqrt{1 + \frac{\Omega_K}{\Omega_\Lambda^{1/3}\Omega_M^{2/3}} X^{1/3} + \frac{1}{X}}$$  \hspace{1cm} (4.10)$$

With the help of (4.10), the differential equation (4.8) can be written in terms of $X$. Then the solutions as function of $X$ is:

$$F \propto \sqrt{\frac{1 + \frac{\Omega_K}{\Omega_\Lambda^{1/3}\Omega_M^{2/3}} X^{1/3} + X}{X}} \int_0^x \frac{du}{u^{1/6} \left(1 + \frac{\Omega_K}{\Omega_\Lambda^{1/3}\Omega_M^{2/3}} u^{1/3} + u\right)^{3/2}}$$  \hspace{1cm} (4.11)$$

We know that $F \to \frac{3}{5} \frac{a}{a_L}$ at early times. In this way, we can set coefficient of proportionality and write:

$$F = \frac{3}{5} \frac{a(t)}{a_L} C\left(\frac{\Omega_\Lambda}{\Omega_M}\left(\frac{a}{a_0}\right)^3\right)$$  \hspace{1cm} (4.12)$$
Table 1. The values of $C(x)$ which is the effect of dark energy on growth of matter, as a function of $X \equiv \frac{\Omega_K}{\Omega_M} \left(\frac{a}{a_0}\right)^3$ for closed and flat spacetime. We use 2018 Planck data [1] for its calculation, which is $\Omega_M = 0.315$ and for closed case $\Omega_K = -0.044$.

| $X$ | $C(X)$ Closed | $C(x)$ Flat |
|-----|----------------|-------------|
| 0.1 | 1.0113         | 0.9826      |
| 0.2 | 1.0013         | 0.9667      |
| 0.3 | 0.9901         | 0.9520      |
| 0.5 | 0.9675         | 0.9256      |
| 0.7 | 0.9401         | 0.9025      |
| 1.0 | 0.9176         | 0.8725      |
| 1.5 | 0.8769         | 0.8314      |
| 2.0 | 0.8432         | 0.7981      |
| 2.5 | 0.8146         | 0.7702      |
| 3.0 | 0.7899         | 0.7462      |
| 3.5 | 0.7683         | 0.7254      |

\[ C(X) \equiv \frac{5}{6} X^{-5/6} \sqrt{1 + \frac{\Omega_K}{\Omega_X^{1/3} \Omega_M^{2/3}}} X^{1/3} + X \int_0^x \frac{du}{u^{1/6}} \left(1 + \frac{\Omega_K}{\Omega_X^{1/3} \Omega_M^{2/3}} u^{1/3} + u\right)^{3/2} \]  

(4.13)

We can calculate this integral numerically. The result for different values of $X$ is written in the table 1. As it can be seen from the table, for the flat universe, $C(X)$ just have a suppression effect on growth of matter due to dark energy. In the closed case, for small values of $X$, $C(X)$ gives enhancement and then it gives suppression.

Furthermore ignoring baryon’s effects (e.g. neglecting the terms of order $\beta$), from (4.7) we have $\Delta(q) = \delta_{Dq}(t_L) - t_L \psi_q(t_L)$, then we would get the solution in the form of $\Delta(q) = \frac{2q^2 R_q^O \tau(\kappa)}{3H_L a_L}$, the same as flat universe (where $\tau$ is transfer function, $t_L = \frac{2}{3H_L}$ and $H_L = \sqrt{\Omega_M H_0 (1 + z_L)^{3/2}}$). This will give us well-known hill-top shape power spectrum. When we consider the effects of the baryons, then what is the dominant term in (4.7) for fast mode is $\beta t_L \frac{q^2}{a_L} \delta u_\gamma(t_L)$, so using the fast mode solution of (4.4), we can write the $q$ dependent part of matter density as:

\[ \Delta(q) \approx \beta t_L \left(\frac{q}{a_L}\right)^2 \delta u_\gamma \]  

(4.14)

\[ = \frac{2q^2 R_q^O}{\sqrt{3a_L H_L (1+R_L)^{3/4}}} e^{-\int_0^{t_L} \frac{qdt}{a \sqrt{3(1+R)}}} \sin \left(\int_0^{t_L} \frac{qdt}{a \sqrt{3(1+R)}} \left(1 - \frac{72K}{q^2} - 1 - R + 3R^2\right)\right) \]

Here also we should use the modified form of $\Gamma$ as (3.39). This is the well-known acoustic oscillation forced by baryon’s effect.

5 Conclusion

In this work, first we write, Boltzmann equation for dark matter particles in a spatially curved spacetime. The integral solution to this equation shows that memory of gravitational
field at early time affects growth at later times. Specifically it is shown that spatial curvature modified this memory by the factor that is come in the last term of (2.35). Then we write Boltzmann equation for photons and using it find a dispersion relation in the presence of spatial curvature. This gives us damping factor and frequency of acoustic oscillations, as (3.39) and (3.40). We want to emphasis that although there is extensive literature on the subject of Boltzmann equation in non-flat FRW universe, but the effect of spatial curvature on the mentioned phenomena was not explored specifically. The damping effects are mostly important for the modes deep inside the horizon. Current uncertainty of the CMB signals at small scale does not allow us to test these effects. The modification to damping factor is of order of $t_c K$. It is hoped that future surveys such as CMB-S4 give us much more precise results. So that these effects can be tested in the more precise future surveys. In the last section we show that how results of section 3, affect evolution of matter density in the hydrodynamic regime. As it is shown, in this hydrodynamic regime spatial curvature does not affect $q$-dependent part of matter density, $\Delta(q)$. But for the time dependent part, in spite of the fact that dark energy always gives a factor of suppression to the growth of matter in a flat universe, but here when ratio of dark energy to matter is small it gives an enhancement and only when this ratio becomes larger, it changes to suppression. When considering baryons, effects of spatial curvature shows itself in terms of modification of frequency and damping parameters (the results of section 3) of acoustic oscillations.

## Appendix A

### A.1 Review of Boltzmann equation for photon

The phase space number density of photons should be defined as a matrix as:

$$n^{ij}(x^i, p_i, t) = \sum_r (\prod_{k=1}^3 \delta(x^k - x^k_r))(\prod_{k=1}^3 \delta(p_k - p_{k_r})) N^{ij}_r(t)$$  \hspace{1cm} (A.1)

where $r$ is the index which labels each photons trajectory and $N^{ij}_r$ is the polarization density matrix of $r$th photon. If each photon can have any one of several polarization vector $e^i_m$ with probabilities $P_m$ then the polarization density matrix is $N^{ij} = \sum_m P_m e^i_m e^j_m^\ast$. This number density matrix is written so that the number of photons with polarization $e^i$ in volume $\Pi dx dt$ at time $t$ will be $g^{ik} g^{jl} e^k e^l n^{ij}(x^i, p_j, t) \Pi dp dx$. In general reference frame, a photon polarization vector will undergo parallel transport up to a gauge transformation. The gauge transformation parameter fixed so that, $e^0 = 0$

$$\frac{de^i_r}{dt} = \left[ -\Gamma^i_{\lambda \delta} + \Gamma^0_{\lambda \delta} \right] \frac{dx^j_r}{dt}$$  \hspace{1cm} (A.2)

Using the above equation and equation (2.8) and knowing that $\frac{dx^i}{dt} = \frac{p^i}{p^0}$, It follows directly that number density matrix $n^{ij}(x^i, p_j, t)$ satisfies the Boltzmann equation as follows:

$$\partial_t n^{ij} + \frac{p^k}{p^0} \partial_k n^{ij} + \frac{p^j p^m}{2 p^0} \partial_k g_{lm} \partial_{p_k} n^{ij} + \left( \Gamma^i_{k \lambda} - \frac{p^i}{p^0} \Gamma^0_{k \lambda} \right) \frac{p^\lambda}{p^0} n^{kj} + \left( \Gamma^j_{k \lambda} - \frac{p^j}{p^0} \Gamma^0_{k \lambda} \right) \frac{p^\lambda}{p^0} n^{ki} = C^{ij}$$  \hspace{1cm} (A.3)

The term $C^{ij}$ on the right hand side of above equation is collision due to photon scattering. Because Thomson scattering unpolarized or linearly polarized photons only produce linearly polarized photons. We only consider photons with linear polarization so that polarization vector can be taken to be real and polarization matrix and number density matrix are real.
and symmetric. Let us first consider photon at rest in flat spacetime. Scattering of photons of momentum $\vec{p}$ into some other direction cause decrease in $n^{ij}$:

$$C^{ij}(\vec{x}, \vec{p}, t) = -\omega_c(t)n^{ij}(\vec{x}, \vec{p}, t)$$ \hspace{1cm} (A.4)

Here $\omega_c(t)$ is total collision rate. Also, scattering of photon with initial momentum $\vec{p}_1$ into momentum $\vec{p}$ cause increase in $n^{ij}$. If initial photon can have various polarizations $e_1^i n$ with probability $P_{e_1}$ then probability of finding photon in final state with polarization $e^i$ is $e^i e^j N^{ij}_1$ where $N^{ij}_1 = \sum_n P_n e_1^i n e_1^j$. This probability should be equal to $e^i e^j N^{ij}$, where $N^{ij}$ is the polarization density of photon in final state. This can help us to write polarization matrix of scattered photon in this way:

$$N^{ij}(\hat{p}) = \frac{1}{1 - \hat{p}_i \hat{p}_j N^{ij}_1} \left[ N^{ij}_1 - \hat{p}_i \hat{p}_k N^{kj}_1 - \hat{p}_j \hat{p}_k N^{ik}_1 + \hat{p}_i \hat{p}_j \hat{p}_k \hat{p}_l N^{kl}_1 \right]$$ \hspace{1cm} (A.5)

The scattering of photons from initial direction $\hat{p}_1$ to final direction $\hat{p}$ leads to increase in $n^{ij}$:

$$C^{ij}_+ = n_e \int d^2 \hat{p} \frac{d^2 \sigma}{d^2 \hat{p}} \frac{1}{1 - \hat{p}_i \hat{p}_j N^{ij}_1} \left[ n^{ij}(\vec{x}, |\vec{p}|\hat{p}_1, t) - \hat{p}_i \hat{p}_k n^{kj}(\vec{x}, |\vec{p}|\hat{p}_1, t) - \hat{p}_j \hat{p}_k n^{ik}(\vec{x}, |\vec{p}|\hat{p}_1, t) + \hat{p}_i \hat{p}_j \hat{p}_k \hat{p}_l n^{kl}(\vec{x}, |\vec{p}|\hat{p}_1, t) \right]$$ \hspace{1cm} (A.6)

where $n_e$ is the electron density and $\frac{d^2 \sigma}{d^2 \hat{p}}$ is differential cross section which can be written as Thomson scattering cross section as, $\frac{d^2 \sigma}{d^2 \hat{p}} = \frac{3\sigma T}{8\pi} S(\hat{p})$. The total change in $n^{ij}$ due to scattering of photons would be:

$$C^{ij}(\vec{x}, \vec{p}, t) = C^{ij}_+ (\vec{x}, \vec{p}, t) + C^{ij}_- (\vec{x}, \vec{p}, t)$$

$$=- \omega_c(t)n^{ij}(\vec{x}, \vec{p}, t) + \frac{3\omega_e(t)}{8\pi} \int d^2 \hat{p}_1 \left[ n^{ij}(\vec{x}, |\vec{p}|\hat{p}_1, t) - \hat{p}_i \hat{p}_k n^{kj}(\vec{x}, |\vec{p}|\hat{p}_1, t) - \hat{p}_j \hat{p}_k n^{ik}(\vec{x}, |\vec{p}|\hat{p}_1, t) + \hat{p}_i \hat{p}_j \hat{p}_k \hat{p}_l n^{kl}(\vec{x}, |\vec{p}|\hat{p}_1, t) \right]$$ \hspace{1cm} (A.7)

where we used $\omega_e(t) = n_e(t)\sigma_T$. For general arbitrary metric $g_{ij}$ we should rewrite the above equation so that it remains invariant under general three dimensional coordinate transformation.

$$C^{ij}(\vec{x}, \vec{p}, t) = -\omega_c(t)n^{ij}(\vec{x}, \vec{p}, t) + \frac{3\omega_e(t)}{8\pi} \int d^2 \hat{p}_1 \sqrt{-g} \frac{1}{p^0} \delta(p^0(\vec{x}, \vec{p}, t) - p^0(\vec{x}, \vec{p}_1, t))$$ \hspace{1cm} (A.8)

$$\times \left[ n^{ij}(\vec{x}, \vec{p}_1, t) - \frac{g^{ik}p_kp_l}{p^0} n^{ij}(\vec{x}, \vec{p}_1, t) - \frac{g^{ik}p_kp_l}{p^0} n^{il}(\vec{x}, \vec{p}_1, t) + \frac{g^{ik}g^{jl}p_kp_lp_m\rho_{mn}}{p^0} n^{ij}(\vec{x}, \vec{p}_1, t) \right]$$

Considering linear perturbations in metric and number density matrix as \(2.10\) and \(3.2\), we can expand straightforwardly the above equation up to first order of perturbation:

$$C^{ij} = -\omega_c(t)\delta n^{ij}(\vec{x}, \vec{p}, t) + \frac{3\omega_e(t)}{8\pi} \int d^2 \hat{p}_1 [\delta n^{ij}(\vec{x}, \vec{p}_1, t) - \gamma^{il}p_i \delta n^{kj}(\vec{x}, \vec{p}_1, t)$$

$$- \gamma^{jl}p_j \delta n^{ik}(\vec{x}, \vec{p}_1, t) + \gamma^{mn}p_m \rho_{mn} p_i \delta n^{kl}(\vec{x}, \vec{p}_1, t)]$$ \hspace{1cm} (A.9)

There is an additional effect on collision term due to plasma velocity, $\delta u$. Because $\delta u$ is first order perturbation, we should only consider metric and number density with no perturbation.
Photon energy is conserved in the rest frame of plasma, in which $v^\mu = (1, \vec 0)$, we have $(p_1 - p)_v = 0$. Evaluating this scalar in lab frame, in which $v^\mu = (\sqrt{1 - \delta v^2}, \delta v)$, up to first order in velocity we have: $|\vec p_1| = |\vec p| [1 + (p_1 - \vec p)_v]$. For FRW metric in which $v^k = au^k$, we get: $|\vec p_1| = |\vec p|[1 + (p_1 - \vec p)_v a]$. As an effect of plasma velocity now $|\vec p_1| \neq |\vec p|$ and so $C_{ij}$ would not be zero in local thermal equilibrium, where photons are unpolarized so that $\delta n_{ij} = \frac{n(\vec p)}{2\pi^2} (\gamma_{ij} - \gamma^{il} \gamma^{j'i'} \vec p_{i'} \vec p_{j'})$. So there should be a new term equal to difference of $-\omega_c n_{ij}$, and the same term with $|\vec p|$ replaced with $|\vec p_1|$ averaged over $\vec p_1$. So the final collision term up to first order in perturbation is:

$$C^{ij}_{ij} = -\omega_c \delta n^{ij} + \frac{3\omega_c}{8\pi} \int d^3\vec p_1 [\delta n^{ij}(x^i, p\vec p_1, t) - \gamma^{ik} \vec p_k \vec p_l \delta n^{lj}(x^i, p\vec p_1, t) - \gamma^{ikl} \vec p_k \vec p_l \vec p_m \vec p_n \delta n^{mn}(x^i, p\vec p_1, t)] - \frac{\omega_c}{2a^2} \delta n_{ik} \gamma^{kk'} \hat n |\gamma_{ij} - \gamma^{il} \gamma^{j'i'} \vec p_{i'} \vec p_{j'}|$$

(B.10)

**B Further computations**

The useful identities for following calculation is:

$$\int d^3\vec p = 4\pi$$

(B.1)

$$\int d^2\vec p \vec p_i \vec p_j = \frac{4\pi}{3} \delta_{ij}$$

(B.2)

$$\int d^2\vec p \hat p_i \hat p_j \vec p_k \vec p_l = \frac{4\pi}{15} (\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})$$

(B.3)

Here we write down the non-zero results of integration of each term in (3.28):

$$\int d^2\vec p (\delta_{ij} - \hat p_i \hat p_j) X = 4\pi X \left( \delta_{ij} - \frac{1}{3} \delta_{ij} \right) = \frac{8\pi}{3} X \delta_{ij}$$

(B.4)

$$it_c \int d^2\vec p (\delta_{ij} - \hat p_i \hat p_j) \mathcal{O} X = it_c \omega X \delta_{ij} \int d^2\vec p - it_c X w \int d^2\vec p \hat p_i \hat p_j = \frac{8\pi it_c X w}{3} \delta_{ij}$$

(B.5)

$$-t_c^2 \int d^2\vec p (\delta_{ij} - \hat p_i \hat p_j) \mathcal{O}^2 X = -t_c^2 \int d^2\vec p (\delta_{ij} - \hat p_i \hat p_j) (\omega^2 X - 2\omega \hat q_i \hat q_j X + \hat q_i \hat q_j \hat p_i \hat p_j X - 8KKX)$$

$$= -t_c^2 \frac{8\pi}{3} (\omega^2 X - 8KKX) \delta_{ij} - \frac{16\pi}{15} t_c^2 q^2 X \delta_{ij} + \frac{8\pi}{15} t_c^2 X q^2 \hat q_i \hat q_j$$

(B.6)

$$it_c Z (1 + it_c \omega R) \int d^2\vec p (\delta_{ij} - \hat p_i \hat p_j) \mathcal{O} \hat p \hat q = it_c Z (1 + i\omega t_c R) \int d^2\vec p (\delta_{ij} - \hat p_i \hat p_j)$$

$$\times \left( \omega \hat p \hat q - \hat p \hat q \hat p \hat q + \frac{8K}{q} \right)$$

$$= it_c Z (1 + i\omega t_c R) \left( \frac{8K}{q} \frac{\pi}{3} \delta_{ij} - \frac{16\pi}{15} q^2 \delta_{ij} + \frac{8\pi}{15} q^2 \hat q_i \hat q_j \right)$$

(B.7)

$$-t_c^2 \int d^2\vec p \hat p (\delta_{ij} - \hat p_i \hat p_j) \mathcal{O}^2 \hat p \hat q = -t_c^2 \int d^2\vec p (\delta_{ij} - \hat p_i \hat p_j) \left( -2\omega \hat q_i \hat q_i \hat p_k \hat p_l + \frac{8K\omega}{q} \right)$$

$$= -t_c^2 Z \left( -2\omega \frac{4\pi}{3} q^2 \delta_{ij} + 2\omega \frac{4\pi}{15} q^2 (\delta_{ij} + 2\hat q_i \hat q_j) + \frac{8K\omega}{q} \right)$$

$$= -t_c^2 Z \left( 32\pi \omega q^2 \delta_{ij} + 64\pi K \frac{\omega}{3q} \delta_{ij} + \frac{16\pi}{15} \omega \hat q_i \hat q_j \right)$$

(B.8)
\[ Y \int d^2 \hat{p} (q^i - \hat{p}^i \hat{k} q_k) (q^j - \hat{p}^j \hat{k} q_k) = \frac{4 \pi}{15} Y (\delta_{ij} + 7 \hat{q}_i \hat{q}_j) \] (B.9)

\[ it_c Y \int d^2 \hat{p} O \left( \hat{q}^i - \hat{p}^i \hat{k} q_k \right) (q^j - \hat{p}^j \hat{k} q_k) = it_c Y \left( \frac{28 \pi}{15} \omega \hat{q}_i \hat{q}_j + \frac{4 \pi}{15} \omega \delta_{ij} \right) \] (B.10)

The calculation of integration for each term in (3.32) is presented here:

\[
\left( Z + i \omega t_c R Z (1 + i \omega t_c R) - \frac{it_c 40 K W}{\omega} q \right) \int d^2 \hat{p} \hat{p} j \left( 1 \right) \left( \right) q \int d^2 \hat{p} \hat{p} j \left( \right) q \\
= \left( \frac{28 \pi}{15} \omega \hat{q}_i \hat{q}_j + \frac{4 \pi}{15} \omega \delta_{ij} \right) \] (B.11)

\[ -t_c^2 \int d^2 \hat{p} \hat{p} j O^2 \left( X + \frac{Y}{2} \right) = -t_c^2 \int d^2 \hat{p} \hat{p} j \] (B.12)

\[ -t_c^2 \int d^2 \hat{p} \hat{p} j O^2 \left( X + \frac{Y}{2} \right) = -t_c^2 \int d^2 \hat{p} \hat{p} j \] (B.13)

\[ it_c \int d^2 \hat{p} \hat{p} j O \left( Z + it_c \omega R Z - \frac{Y}{2} \right) \hat{q} \] (B.14)

\[ it_c \int d^2 \hat{p} \hat{p} j O \left( Z + it_c \omega R Z - \frac{Y}{2} \right) \hat{q} \] (B.15)

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