Information-causality and extremal tripartite correlations

Tzyh Haur Yang\textsuperscript{1,3}, Daniel Cavalcanti\textsuperscript{1}, Mafalda L Almeida\textsuperscript{1}, Colin Teo\textsuperscript{1} and Valerio Scarani\textsuperscript{1,2}

\textsuperscript{1} Centre for Quantum Technologies, National University of Singapore, 3 Science Drive 2, Singapore 117543, Singapore
\textsuperscript{2} Department of Physics, National University of Singapore, 2 Science Drive 3, Singapore 117542, Singapore
E-mail: hermitian.yang@gmail.com

\textit{New Journal of Physics} 14 (2012) 013061 (11pp)
Received 22 August 2011
Published 30 January 2012
Online at http://www.njp.org/
doi:10.1088/1367-2630/14/1/013061

Abstract. We study the principle of information-causality (IC) in the presence of extremal no-signaling correlations on a tripartite scenario. We prove that all, except one, of the non-local correlations lead to violation of IC. The remaining non-quantum correlation is shown to satisfy any bipartite physical principle.

Contents

1. Introduction 2
2. Review of the tools 3
  2.1. No-signaling, quantum and local tripartite probability distributions 3
  2.2. Sufficient criteria to violate the information-causality (IC) principle 4
3. Tripartite correlations that violate IC 4
  3.1. Approach 4
  3.2. Results and examples 6
4. Class 4: extremal no-signaling correlations satisfying any bipartite criterion 8
  4.1. Description of the correlations 8
  4.2. No violation by any bipartite criterion 8
5. Perspectives and conclusions 10
Acknowledgments 10
Appendix. Class 4 is non-quantum 10
References 11

\textsuperscript{3} Author to whom any correspondence should be addressed.
1. Introduction

Quantum theory was developed as a set of mathematical rules from which predictions on physical phenomena could successfully be obtained. Naturally, much effort has been put into finding intuitive physical principles on which to lay the foundations of quantum theory. In the case of quantum correlations, several such principles have already been proposed. For instance, it has been proven that physical correlations satisfy the no-signaling principle \[1\] (instantaneous transmission of information between two locations is impossible) and macroscopic locality \[2\] (a macroscopic coarse-graining of quantum correlations can be explained classically). However, these principles are not exclusive to quantum theory: several other theories satisfy them but still allow for non-quantum correlations.

Is there a physical principle that completely identifies the set of quantum correlations? Information-causality (IC) has been proposed as a solution to this question \[3\]. The principle of IC imposes a limit on the amount of information-gain an observer (Bob) can reach when another observer (Alice), in a different location, sends him an amount of information. If \(m\) bits are sent, Bob cannot learn more than \(m\) bits from Alice’s system. This principle is satisfied in the presence of any classical and quantum correlations, but it is not proven to exclude all possible post-quantum correlations \[4, 5\]. Even in the simplest scenario, where correlations are established by two distant observers measuring a pair of two-valued observables on their physical systems, IC might not be sufficient to rule out all non-quantum correlations. So far, it has proven strong enough to exclude a significant portion of supra-quantum theories: for instance, any correlation stronger than the strongest quantum correlation violates IC. More precisely, IC is violated by all correlations violating the Clauser–Horn–Shimony–Holt (CHSH) Bell inequality \[6\] by a value superior to that allowed by quantum theory \[3\].

Until now, the search for physical principles defining quantum correlations has focused essentially on the bipartite scenario. However, recent results show that the structure of multipartite correlations is much richer than its bipartite counterpart \[7\]. For instance, non-trivial non-local games without quantum advantage over classical theory were found \[8\] and the generalization of Gleason’s theorem \[9\] to the multipartite case is known to be problematic \[10\]. It is thus pertinent to ask if IC is a good principle in a multipartite scenario.

In the present paper, we apply the principle of IC to correlations arising in the simplest multipartite scenario. It consists of three observers (Alice, Bob and Charlie), each performing two space-like separated local measurements \((x, y, z = 0, 1)\), with two possible outcomes \((a, b, c = 0, 1)\), on their local systems. The obtained correlations between the parties’ outcomes, conditioned on the choice of observables, are described by the joint probability distribution \(P(abc|xyz)\). Here we assume that the no-signaling principle holds and we consider the set of no-signaling extremal correlations (those from which all other no-signaling correlations can be obtained by statistical mixing). We then test the principle of IC in the presence of these correlations. We conclude that IC rules out all extremal no-signaling non-local tripartite correlations, except one. Using the criterion derived in \[11\] and similarly to an example pointed out there, we show that this singular correlation cannot be ruled out by any bipartite principle.

This paper is organized as follows. In section 2, we introduce our working tools: we briefly overview the properties of the sets of tripartite probability distributions and present sufficient conditions for the violation of the IC principle. In section 3, we start by explaining our method. We show how we derive bipartite correlations, on which we can test IC, from tripartite correlations. For that, we introduce simple versions of a general processing called
wiring. We then present our results; namely, we show how we are able to exclude all extremal non-local tripartite correlations using IC, except one. In section 4, we analyze this correlation in detail and prove that it satisfies any bipartite criterion. Finally, in section 5 we discuss the perspectives arising from the present results, especially on the need for a genuine multipartite physical principle.

2. Review of the tools

2.1. No-signaling, quantum and local tripartite probability distributions

The set of probability distributions for tripartite systems, with binary input and output for each party, is a convex set in a 26-dimensional space, defined by the conditions of positivity \( P(abc|xyz) \geq 0 \) and normalization \( \sum_{abc} P(abc|xyz) = 1, \forall x, y, z \). The set of no-signaling distributions is obtained by imposing that the choice of measurement by one of the parties cannot affect the distribution of the remaining, that is,

\[
P(ab|xyz) = \sum_{c} P(abc|xyz) = P(ab|xy), \forall a, b, x, y, z,
\]

and any permutation of the parties. This defines the no-signaling polytope, which has been completely characterized in [7]. It is extremely complex, especially taking into account that it considers just the simplest non-trivial tripartite scenario. Indeed, the no-signaling polytope has 53,856 extremal points, belonging to 46 different classes: 45 of them comprise non-local points, while the remaining class contains all deterministic local points. These consist of the extreme points of the local polytope, i.e. the set of correlations obtained by a local (classical) model,

\[
P(abc|xyz) = \sum_{\lambda} p_{\lambda} P(a|x, \lambda) P(b|y, \lambda) P(c|z, \lambda),
\]

where \( \lambda \) is some random variable, with distribution \( p_{\lambda} \), shared by the parties. Curiously, the local polytope is defined by exactly 46 classes of facets, forming one class of trivial constraints and 45 inequivalent Bell-type inequalities [12]. Unfortunately, there is no obvious correspondence between local facets and extremal no-signaling points.

The set of quantum correlations is the convex body defined by Born’s law, \( P(abc|xyz) = \text{tr}(\rho M^a_x \otimes M^b_y \otimes M^c_z) \), where \( \rho \) is a quantum state and \( M \) define quantum measurements. The quantum set is clearly not a polytope and little is known about its boundaries. A convergent hierarchy of semi-definite programs is known [14], but there is no guarantee that the quantum set is reached after a finite number of steps. Also, no analytical bounds have been derived and the techniques used in the bipartite case [13] would probably not be very tight. Fortunately, for the non-local correlations we analyze here, the problem turns out to be simpler and we will always be able to decide on their quantumness.

Our goal is then to determine the ability of the IC principle to exclude post-quantum tripartite correlations, in particular the set of extremal points of the no-signaling polytope\(^4\). Following the classification from [7], class 1 corresponds to local deterministic probability distributions, which arise in classical systems and necessarily satisfy IC. We are then interested in the other 45 non-local classes, which all turn out to be non-quantum. We will show that 44 of \( 4^\text{th} \) It is enough to discuss the violation of IC for one representative of each class, since all correlations in the same class are equivalent under relabeling of parties, inputs or outputs.
them violate IC (note that the violation of IC by boxes of class 46, and their natural multipartite
generalization, has been studied in detail in [15]). The remaining class (class 4) accounts for
126 extremal points and will be analyzed in more detail in section 4. There we show that these
correlations not only satisfy IC, but any other bipartite information principle aimed at defining
the set of quantum correlations.

2.2. Sufficient criteria to violate the information-causality (IC) principle

The principle of IC states that Bob cannot learn more than $m$ bits from a distant Alice, after
she sends him $m$ bits. This principle has been formulated based on a specific communication
protocol [3]. It starts with Alice receiving a string of $n$ random and uncorrelated bits,
($a_1, \ldots, a_n$), and Bob a number $b$, between 1 and $n$. The goal is for Bob to guess the value
of Alice’s bit $a_b$. For that, the parties are allowed to share physical resources and operate
locally on these, and Alice can send $m$ classical bits to Bob. Using this scheme, in [3, 4]
it has been proven that IC is violated whenever the parties share correlations that either:
(i) achieve a larger-than-quantum violation of the CHSH inequality, i.e.

$$\text{CHSH} = E_{00} + E_{01} + E_{10} - E_{11} > 2\sqrt{2},$$

where $E_{xy} = P(a = b|x,y) - P(a \neq b|x,y)$; or (ii) violate the quadratic inequality proposed by
Uffink [16], i.e.

$$(E_{00} + E_{10})^2 + (E_{01} - E_{11})^2 > 4.$$  

3. Tripartite correlations that violate IC

3.1. Approach

We have seen that IC is a principle formulated for bipartite systems, and the simple criteria for
its violation (equations (3) and (4)) are based on correlations between single input/output bits
per party. Therefore our tripartite distributions must be transformed into appropriate bipartite
ones before being tested. Our method is the following. First, we bipartite the system by
grouping two of the parties together, from which we obtain bipartitions $A|BC$, $AB|C$ or
$B|AC$. As such, the two-party block receives two bits as inputs and outputs two bits. We
transform these into effective single bits by a process called wiring$^5$, where the parties inside
the new partitions perform some form of processing (possibly using communication). Clearly,
communication is forbidden outside the partitions, in order to guarantee that the non-local
correlations across the bipartition remain the same as in the original tripartite distribution. More
precisely, for each tripartite box $P(a, b, c|x, y, z)$, a bipartition and a wiring define an effective
box $P_{\text{eff}}(a', b'|x', y')$, where all the inputs and outputs$^6$ are considered to be in \{0, 1\}. The goal
is to find, for each $P$, a bipartition and a wiring such that $P_{\text{eff}}$ violates IC, according to the
sufficient criteria in section 2.2.

5 The term has been introduced in [17]; the notion is a rather natural one and had been used previously in a
non-systematic way.

6 Note that our outputs $a, b, c$ were written $\hat{a}, \hat{b}, \hat{c}$ in [7], where the letters $a, b, c$ were used to denote outcomes
belonging to \{-1, +1\}.

New Journal of Physics 14 (2012) 013061 (http://www.njp.org/)
Table 1. Violation of bipartite IC as detected by the CHSH inequality (3) or the Uffink inequality (4). The table follows the conventions of table 2 in [7]: both the settings $x, y, z$ and the outcomes $a, b, c$ take the value 0 or 1. All the sums are to be taken modulo 2. The bipartitions are implied by the outputs $a', b'$; for instance, if $b' = b + c$, clearly the bipartition must be $A|BC$. Note that the inequality which is violated may not necessarily be (3) or (4), but one of their equivalent forms under relabeling of the parties and/or the inputs and/or the outputs.

| No. | Wiring | $a'$ | $b'$ | CHSH | Uffink |
|-----|--------|------|------|------|--------|
| 1   | –      | –    | –    | –    | –      |
| 2   | –      | $b$  | $c$  | 4    | 8      |
| 3   | $x = z + zc$ | $a$ | $b$  | 3    | 5      |
| 4   | –      | –    | –    | –    | –      |
| 5   | $x = z + zc$ | $a$ | $b$  | 3    | 5      |
| 6   | $x = 1$ | $a + b$ | $c$  | 4    | 8      |
| 7   | $y = 1 + x$ | $a + b$ | $c$  | 4    | 8      |
| 8   | $z = ax$ | $b$  | $c$  | 3    | 5      |
| 9   | $y = 1 + x$ | $a + b$ | $c$  | 4    | 8      |
| 10  | $x = 0$ | $a + b$ | $c$  | 4    | 8      |
| 11  | $z = ax$ | $a + c$ | $b$  | 3    | 5      |
| 12  | $z = ax$ | $a + c$ | $b$  | 3    | 5      |
| 13  | $y = 1 + z$ | $a$  | $b + c$ | –   | 40/9   |
| 14  | $y = 1 + x$ | $a + b$ | $c$  | 10/3 | 52/9   |
| 15  | $x = 0$ | $a + b$ | $c$  | 4    | 8      |
| 16  | $y = 1$ | $a + b$ | $c$  | –    | 40/9   |
| 17  | $x = 0$ | $a + b$ | $c$  | 4    | 8      |
| 18  | $z = 0$ | $a$  | $b + c$ | 3   | 5      |
| 19  | $z = 1$ | $a$  | $b + c$ | 3   | 9/2    |
| 20  | $z = 0$ | $a$  | $b + c$ | 16/5| 26/5   |
| 21  | $z = 1$ | $a$  | $b + c$ | 3   | 9/2    |
| 22  | $z = 1 + a + ax$ | $a + c$ | $b$  | –    | 40/9   |
| 23  | $y = 1 + x$ | $a + b$ | $c$  | 4    | 8      |
| 24  | $x = 1$ | $a + b$ | $c$  | 4    | 8      |
| 25  | $z = 1$ | $a$  | $b + c$ | 10/3| 52/9   |
| 26  | $z = 0$ | $a$  | $b + c$ | –   | 40/9   |
| 27  | $z = 1$ | $a$  | $b + c$ | 3   | 5      |
| 28  | $x = 1$ | $a + b$ | $c$  | 4    | 8      |
| 29  | $z = 1$ | $a$  | $b + c$ | 10/3| 52/9   |
| 30  | $z = 0$ | $a$  | $b + c$ | 18/5| 114/25 |
| 31  | $y = 1$ | $a + b$ | $c$  | 14/5 | 4      |
| 32  | $z = 0$ | $a$  | $b + c$ | 18/5| 114/25 |
| 33  | $y = 1$ | $a + b$ | $c$  | 14/5 | 116/25 |
| 34  | $z = 0$ | $a$  | $b + c$ | 10/3| 50/9   |
| 35  | $z = 0$ | $a$  | $b + c$ | 10/3| 50/9   |
| 36  | $z = 1$ | $a$  | $b + c$ | 7/2  | 49/8   |
| 37  | $z = 0$ | $a$  | $b + c$ | 7/2  | 25/4   |
| 38  | $z = 0$ | $a$  | $b + c$ | 10/3| 52/9   |
We basically use two simple types of wirings, which we will illustrate with examples ahead. Interestingly, like in any other processing, wirings may result in loss of correlations. However, for the purpose of ruling out these extremal correlations, they will prove powerful enough.

3.2. Results and examples

The results are summarized in table 1. Only the local deterministic correlations comprising class 1 and non-local correlations in class 4 are not found to violate IC. For all the others, a bipartition and a wiring can be found such that either (3) or (4), or both, happen. We now illustrate the two distinct types of wiring with examples taken from table 1.

1. **Wiring type I.** Consider a bipartition \( A | BC \). In the first type of wiring, the inputs \( yz \) of the original tripartite box will only be a function of the effective input \( y' \) given to the block BC. Take the wiring on box 44, characterized by the relation \( a \oplus b \oplus c = x \cdot y \cdot z \) as an example (see figure 1). The inputs of the tripartite box are defined by \( x = x' \), \( y = y' \) and \( z = 1 \). Therefore, inside the BC partition, the input \( y' \) is given to B and the input of C is fixed. The outputs are chosen to be \( a' = a \) and \( b' = b \oplus c \), which realizes \( a' \oplus b' = x' \cdot y' \). In other words, using box 44 we are actually able to obtain an effective bipartite Popescu–Rohrlich

![Diagram](http://www.njp.org/)

**Figure 1.** *Wiring type I.* Bipartition and wiring that lead to the violation of IC by the extremal points of class 44. Here, \( A \) has as an input \( x = x' \) and outputs \( a = a' \). On the other partition however, the input of \( C \) is always \( z = 1 \), while the input of \( B \) is \( y = y' \); the final output is \( b' = b \oplus c \).
Figure 2. Wiring type II. Bipartition and wiring that lead to the violation of IC by the extremal points of class 3. Here, $B$ has as an input $y = y'$ and outputs $b = b'$. On the other partition, first the input $z = x'$ is used for $C$, and then the corresponding output $c$ is used to define the input $x$ of $A$, according to $x = x' \cdot (c \oplus 1)$; the final output of this block is $a' = a$.

Box [1], known to violate maximally IC [3]. Consequently, correlations from box 44 are also forbidden by the principle of IC.

2. Wiring type II. In the second type of wiring, one of the inputs $z$ or $y$ additionally depends on the output of the other party in the block. This necessarily imposes a time order in the use of the tripartite box: if, for example, the input $z$ depends on the output $b$, Charlie can only use his part of the box after receiving the information from Bob’s outcome. A few of the classes of extremal points require this type of wiring to violate IC; its study is slightly more complicated. Take as an example the points of the class 3. The explicit form of one of its representatives can be derived from table 2 in [7]:

$$P(a, b, c|x, y, z) = \frac{1}{8}[1 + (-1)^{a+b} \delta_{x,0} + (-1)^{a+c} \delta_{x,1} \delta_{z,0} + (-1)^{a+b+c} \delta_{x,1}(\delta_{y,0} - \delta_{y,1})\delta_{z,1}].$$

(5)

The bipartition (now $B|AC$) and wiring are sketched in figure 2. In Bob’s part, $y = y'$ and $b = b'$. In the AC partition, the input $x'$ is first used as the input for $C$, $z = x'$. The output of this box, $c$, will then be used for the input of $A$, according to $x = z \cdot (c \oplus 1)$. The corresponding outcome $a$ is used as the final output $a' = a$.

In order to work out this example, note that the wiring relation $x = z \cdot (c \oplus 1)$ explicitly reads: if $c = 0$, then $x = z$; if $c = 1$, then $x = 0$ independently of $z$. So:

$$P_{\text{eff}}(a', b'|x', y') = P(a, b, c = 0|x = z, y, z) + P(a, b, c = 1|x = 0, y, z)$$

$$= \frac{1}{8}[1 + (-1)^{a+b} \delta_{x,0} + (-1)^{a+c} \delta_{x,1} \delta_{z,0} + (-1)^{a+b+c} \delta_{x,1}(\delta_{y,0} - \delta_{y,1})\delta_{z,1}] + \frac{1}{8}[1 + (-1)^{a+b}]$$

$$= \frac{1}{4} \left[1 + (-1)^{a+b} \delta_{x,0} + (\delta_{y,0} - \delta_{y,1})\delta_{z,1} \right] \equiv \frac{1}{2}[1 + (-1)^{a+b} E_{x'y'}].$$

From this last expression, one finds that $E_{00} = E_{01} = E_{10} = 1$ and $E_{11} = 0$, whence $\text{CHSH} = 3$.

New Journal of Physics 14 (2012) 013061 (http://www.njp.org/)
4. Class 4: extremal no-signaling correlations satisfying any bipartite criterion

We turn now to the detailed study of correlations in class 4. We have seen that using our simple techniques we were unable to observe a violation of IC in the presence of these correlations. Since they are supra-quantum (see proof in the appendix), it is pertinent to ask if different wirings or sharper violation criteria would lead to such a violation. Here, using the results derived in [11], we will see that this is not the case: non-quantum class 4 cannot be excluded by the IC principle. Actually, it cannot be excluded by any bipartite information-theoretical principle aimed at defining the set of quantum (or even local) correlations. But before turning to this, let us describe the points in this class and some of their properties.

4.1. Description of the correlations

The correlations in class 4 have a very precise structure. As its representative, we choose the one that is obtained from the version in table 2 in [7] by changing $y \rightarrow 1 - y$. In this formulation, class 4 is defined by the following deterministic correlations:

$$
\begin{align*}
    a_0 \oplus b_1 &= 0, \\
    b_0 \oplus c_1 &= 0, \\
    c_0 \oplus a_1 &= 0, \\
    a_0 \oplus b_0 \oplus c_0 &= 0, \\
    a_1 \oplus b_1 \oplus c_1 &= 1,
\end{align*}
$$

and random statistics for any other combination. Here, $a_x$ refers to the output of $A$ when its input is $x$ and similarly for $B$ and $C$. It is clear from these equations that any cyclic permutations of $(A, B, C)$ will have the same statistics.

Class 4 is clearly non-local, because the sum of the first four correlations would imply $a_1 \oplus b_1 \oplus c_1 = 0$, which contradicts the last relation. It also cannot be realized within quantum theory; see the appendix for details. Another remarkable property is that these correlations cannot be created by distributing arbitrary many-bipartite PR-boxes between the parties. What we prove in the following subsection yields as a corollary that the converse is also true: these correlations cannot be used to create a bipartite PR-box.

4.2. No violation by any bipartite criterion

We now show that extremal points in class 4 satisfy IC and any other bipartite information principles aimed at singling out quantum correlations. The main idea is to prove that the probability distributions belonging to this class are local for any bipartition, even after any local wirings. A sufficient criterion was provided in [11]: if $P(a, b, c|x, y, z)$ belongs to the set of time-ordered bi-local (TOBL) probability distributions, all possible bipartite distributions derived from $P(a, b, c|x, y, z)$ are local. A probability distribution belongs to TOBL if it can be written as

$$
P(a, b, c|x, y, z) = \sum_\lambda p_\lambda \ P(a|x, \lambda) \ P_B(b|y, \lambda) \ P_C(c|b, y, z, \lambda)
$$

$$
= \sum_\lambda p_\lambda \ P(a|x, \lambda) \ P'_B(b|c, y, z, \lambda) \ P'_C(c|z, \lambda),
$$

where $P'_B$ and $P'_C$ denote the probability distributions after any local wirings. The proof (S Pironio 2010 private communication) is similar to the one used for a five-partite distribution in [18].

New Journal of Physics 14 (2012) 013061 (http://www.njp.org/)
for bipartition $A|BC$, and analogously for $B|AC$ and $C|AB$. Here $p_x$ is the probability distribution over the shared random variable $\lambda$. We see that the model allows signaling from Bob to Charlie (first line) or from Charlie to Bob (second line), but never both at the same time. This accounts for the unknown time-ordered events. Also, Alice is unable to infer the direction of signaling inside the bipartition $BC$.

We show that correlations in class 4 belong to TOBL by constructing a model (7) that generates its statistics. Let the hidden variable $\lambda$ be a vector of two bits $\lambda = (\lambda_0, \lambda_1)$, distributed along the uniform probability distribution $P(\lambda) = \frac{1}{4}$. To start with, consider the splitting $A|BC$. Alice always outputs according to the strategy

$$a = \lambda_0 \oplus (\lambda_0 \oplus \lambda_1) \cdot x.$$  \hfill (8)

Inside the partition $BC$, the instructions change according to the direction of signaling. If Bob receives his input before Charlie, it must be independent of $z$ and $c$; therefore only $B \rightarrow C$ signaling is possible. In this case, the strategy is

$$b = \lambda_0 \oplus \lambda_1 \oplus \lambda_1 \cdot y,$$

$$c = \lambda_1 \oplus (\lambda_0 \oplus y) \cdot z.$$  \hfill (9)

If Charlie receives his input before Bob ($C \rightarrow B$), they follow the instructions

$$b = \lambda_0 \oplus (\lambda_1 \oplus z) \cdot (y \oplus 1)$$

$$c = \lambda_1 \oplus (\lambda_0 \oplus 1) \cdot z.$$  \hfill (10)

Since correlations in class 4 are invariant under cyclic permutation of the parties ($A$, $B$, $C$), similar models are valid for the bipartitions $C|AB$ and $B|CA$. We can easily check that this completely specifies a TOBL model (7) that reproduces the correlations of class 4 (6). We observe that these have local (classical) statistics for any bipartition we consider, even after wirings, so they will always respect any bipartite information-theoretic principle aimed to single out quantum (or even local) correlations.

A non-extremal correlation with the same property has already been found in another region of the polytope [11], specifically above the guess-your-neighbor-input (GYNI) tripartite inequality [8],

$$P(000|000) + P(110|011) + P(011|101) + P(101|110) \leq 1.$$  \hfill (11)

Note that this conclusion extends at least to all points which are convex combinations of those examples and some local deterministic points. Then, the non-quantum correlations that cannot be ruled out with bipartite criteria form sets of non-zero measure.

To finish, we show that our class of extremal points does not violate any GYNI inequality, which proves that our example and that in [11] exhibit intrinsically different kinds of non-locality. First, we recall that the most general form of tripartite GYNI inequalities [8] is

$$\sum_{x_1, x_2, x_3} q(x_1, x_2, x_3) P(x_2 x_3 x_1 | x_1 x_2 x_3) \leq \max_{x_1, x_2, x_3} (q_{x_1, x_2, x_3} + q_{\bar{x}_1, \bar{x}_2, \bar{x}_3}),$$  \hfill (12)

where $\sum_{x_1, x_2, x_3} q(x_1, x_2, x_3) = 1$ and the overbars denote negation. For the points in class 4, the tripartite probabilities are never larger than $1/4$: $P(a_1, a_2, a_3 | x_1, x_2, x_3) \leq \frac{1}{4}$. So, in the best case, these correlations provide the value

$$\frac{1}{4} \left( \sum_{x_1 \oplus x_2 \oplus x_3 = 0} (q_{x_1, x_2, x_3} + q_{\bar{x}_1, \bar{x}_2, \bar{x}_3}) \right)$$  \hfill (13)
to the left-hand side of equation (12). Clearly, it never exceeds the local bound 
\[
\max_{x_1, x_2, x_3} (q_{x_1, x_2, x_3} + q_{\bar{x}_1, \bar{x}_2, \bar{x}_3}) 
\] 
and non-local boxes in class 4 are unable to violate GYNI inequalities.

5. Perspectives and conclusions

We set out to apply the principle of IC to multipartite correlations. We took the only existing 
form to test IC, which involves a bipartite task, and checked its violation on the extremal points 
of the simplest tripartite no-signaling polytope. The IC principle excludes 44 out of 45 classes of 
non-local extremal points (which consequently are non-quantum); the remaining class, which 
can also be proved not to be quantum by other means, satisfies this and any other bipartite 
principle.

IC remains a powerful criterion to rule out no-signaling correlations which cannot 
be achieved within quantum physics. But it started out with a more ambitious conjecture, 
namely, as a physical principle that might identify the set of quantum correlations exactly. 
Generalizations of IC beyond the basic CHSH scenario cannot be said to have brought 
clarification: instead, we are discovering extremely complex structures. These are not devoid 
of interest and deserve further studies; certainly, for instance, it will be very instructive to find 
a meaningful multipartite task that detects the non-quantumness of our example and of that 
reported in [11]. However, ultimately, completely different approaches will probably have to be 
found in order to prove the main conjecture.

Acknowledgments

We acknowledge illuminating discussions with Antonio Acín, Nicolas Brunner, Rodrigo 
Gallego, Miguel Navascués and Lars Würflinger. We thank one of the anonymous referees for 
pointing out the elegant argument used in the proof in the appendix. This work was supported 
by the National Research Foundation and the Ministry of Education, Singapore.

Appendix. Class 4 is non-quantum

Here we show that probability distributions of class 4 cannot be obtained within quantum 
physics. For that, we will see that they violate an inequality satisfied by any local and quantum 
correlations. This inequality reads as follows:

\[
k = \frac{15}{2} + \frac{1}{2} (A_1 B_1 C_1) - 2 ((A_0 B_0 C_0) + (A_0 B_1) + (B_0 C_1) + (A_1 C_0)) \geq 0, \tag{A.1}
\]

where \(A_i, B_i, C_i\) are local observables with possible values \(\pm 1\). To prove that it holds true 
for any quantum observables, we define the operator \(K\) as

\[
K \equiv \left( \frac{\alpha \beta + \gamma \delta}{2} - 1 \right) \left( \frac{\alpha \beta + \gamma \delta}{2} - 1 \right)^\dagger + 2 \left( \frac{\alpha + \beta}{2} - 1 \right)^2 + 2 \left( \frac{\gamma + \delta}{2} - 1 \right)^2, \tag{A.2}
\]

where \(1\) is the identity operator, while \(\alpha, \beta, \gamma\) and \(\delta\) are any arbitrary operators. One can 
easily check by inspection that \(K\) is positive semi-definite and \(\langle K \rangle \geq 0\). Now, performing the
substitution
\begin{align*}
\alpha &= A_1 C_0, \\
\beta &= A_0 B_1, \\
\gamma &= A_0 B_0 C_0, \\
\delta &= B_0 C_1
\end{align*}
\hspace{1cm} (A.3)

and some simple algebra, we observe that \( \langle K \rangle \) defines exactly expression (A.1). Hence, inequality (A.1) is valid for any theory where local observables are defined by operators on a Hilbert space: in particular, quantum theory.

Consider now the correlations that define class 4. In the case where local observables have possible outcomes \( \pm 1 \), relations (6) can be transformed into
\begin{align*}
\langle A_0 B_1 \rangle &= 1, \\
\langle C_0 A_1 \rangle &= 1, \\
\langle B_0 C_1 \rangle &= 1, \\
\langle A_0 B_0 C_0 \rangle &= 1, \\
\langle A_1 B_1 C_1 \rangle &= -1.
\end{align*}
\hspace{1cm} (A.4)

Simple substitution in (A.1) gives \( \langle K \rangle = -1 \), from which we conclude that class 4 cannot be quantum.

References

[1] Popescu S and Rohrlich D 1994 *Found. Phys.* **24** 379
[2] Navascués M and Wunderlich H 2009 *Proc. R. Soc.* A **466** 881–90
[3] Pawlowski M, Paterek T, Kaszlikowski D, Scarani V, Winter A and Zukowski M 2009 *Nature* **461** 1101
[4] Allcock J, Brunner N, Pawlowski M and Scarani V *Phys. Rev.* A 2009 **80** 040103
[5] Cavalcanti D, Salles A and Scarani V 2010 *Nat. Commun.* **1** 136
[6] Clauser J F, Horne M, Shimony A and Holt R A 1969 *Phys. Rev. Lett.* **23** 880
[7] Pironio S, Bancal J-D and Scarani V 2011 *J. Phys. A: Math. Theor.* **44** 065303
[8] Almeida M L et al 2010 *Phys. Rev. Lett.* **104** 230404
[9] Gleason A 1957 *J. Math. Mech.* **6** 885
[10] Acín A et al 2010 *Phys. Rev. Lett.* **104** 140404
[11] Gallego R, Würflinger L, Acín A and Navascués M 2011 arXiv:1107.3738
[12] Śliwa C 2003 *Phys. Lett.* A **317** 165–8
[13] Yang T H, Navascués M, Sheridan L and Scarani V 2011 *Phys. Rev. A* **83** 022105
[14] Navascués M, Pironio S and Acín A 2007 *Phys. Rev. Lett.* **98** 010401

Navascués M, Pironio S and Acín A 2008 *New J. Phys.* **10** 073013

Pironio S, Navascués M and Acín A 2010 *SIAM J. Optim.* **20** 2157
[15] Xiang Y and Ren W 2011 arXiv:1101.2971
[16] Uffink J 2002 *Phys. Rev. Lett.* **88** 230406
[17] Allcock J, Brunner N, Linden N, Popescu S, Skrzypczyk P and Vertesi T 2009 *Phys. Rev. A* **80** 062107
[18] Barrett J and Pironio S 2005 *Phys. Rev. Lett.* **95** 140401