ON CONTINGENT CLAIMS PRICING IN INCOMPLETE MARKETS: A RISK SHARING APPROACH

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Abstract. In an incomplete market setting, we consider two financial agents, who wish to price and trade a non-replicable contingent claim. Assuming that the agents are utility maximizers, we propose a transaction price which is a result of the minimization of a convex combination of their utility differences. We call this price the risk sharing price, we prove its existence for a large family of utility functions and we state some of its properties. As an example, we analyze extensively the case where both agents report exponential utility.

Keywords: incomplete markets, utility function, exponential utility, indifference price, risk sharing price.

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1. Introduction

Realistic financial markets are incomplete, as evidenced by empirical studies and manifested by the difficulty or inability to replicate any contingent claim as a portfolio of traded assets. Thus, the problems of contingent claim pricing and hedging in incomplete markets are of paramount importance in mathematical finance. Seminal studies in the field of incomplete markets (see [32], [14] for overviews in relevant topics) have led to an ever increasing research activity in this area, that by now has allowed for a deeper understanding of the way that financial markets function. It is well known for instance, that in an incomplete market setup there is no longer a unique pricing kernel and this may at best point out a whole band of non-arbitrage prices for a contingent claim.

There is an extensive and very interesting literature focusing on the determination of the upper and lower hedging prices (see e.g., [13], [11], [46] etc.). On the other hand though, further criteria seem to be needed for the determination of the single price, out of the whole band of non-arbitrage prices, at which the contingent claim will eventually be traded. The majority of such criteria, as they have been proposed in the relevant literature, are based on, or related to, the minimization of entropy-like functions (see e.g., [19], [18] etc.) and lead to subjective non-linear pricing rules. However a complete theory on the procedure of price selection in incomplete markets is still missing. The aim of this paper is to address the question of price selection in incomplete markets and provide new ideas and insights, by proposing and elaborating on a “risk sharing” scenario involving two financial agents who negotiate the price of a single indivisible non-replicable contingent claim.

We consider two agents, named “buyer” and “seller”, who are risk averse von Neumann-Morgenstern expected utility maximizers and have access to some financial market that is liquid, incomplete and arbitrage free. In our context, the agents’ utility functions are to be understood as benchmarks that guide them to informed decision making; however, their final decisions are subject to further “judgment” by the agents, so that departures from these benchmarks are meaningful and can be considered. Furthermore, it should be pointed out that the use of such utilities is not restrictive for the purpose of this paper; as discussed in Section 5 our results can be generalized, using convex or coherent risk measures (see [17] and [2] for the exact definitions of such measures) as decision making rules instead of utility functions.

The utility maximization problem has captured an important part of the mathematical finance literature (see e.g., [30], [9], [38], [43] etc). In these related papers, there is a main distinction of utility functions into two types on the basis of their domain (see [33], Chapter 1). The first type refers to those utility functions that are defined on wealth that may take values from the whole real axis, while the utility functions of the second type are defined only for values of wealth on the positive part of the real axis. In this work, we do not impose restrictions on the agents’ type of utility function. In fact, the pricing scheme that we suggest below can be applied to both types of
utility. However, for presentation purposes and in order to facilitate the reading of the paper, we consider the case of utilities of the first type in the main body of the paper, while the technicalities of utility functions of the second type are treated in the Appendix. We emphasize though that, under the appropriate assumptions, whenever a utility function is considered in this paper, it can be taken to be of either type.

At time 0 the two agents are determined to conclude a transaction on a contingent claim, by agreeing on a price $P$ immediately payable by the “buyer” to the “seller” in exchange of a non-replicable contingent payoff $B$ that is due from the seller to the buyer at the end of their common and mutually beforehand agreed investment horizon $T$.

The incompleteness of the market dictates that there is an infinity of prices $P$ that are consistent with the absence of arbitrage. The main question that we address is the following:

“Out of this infinity of prices, which one will actually be realized?”

The incompleteness of the market means that the agents cannot fully hedge their positions in the contingent claim $B$ by trading in the market. Therefore, each of our agents resorts to (expected) utility maximization criteria to produce a reference price that makes her indifferent between an optimally invested portfolio that contains the position on the contingent claim and an optimally invested portfolio without the position on the claim. Most probably however, the indifference price of the “buyer” and the indifference price of the “seller” will not coincide (see e.g., [47], [3]). This means that in order to agree on a common transaction price for the contingent claim, they should depart somehow from the indifference prices that are dictated by their benchmark utility functions, i.e., some “gain” or “loss” of indirect utility, which we call hereafter “risk” (see Remark 3.1 for the justification of the choice of this term), should take place. Clearly, if the indifference price of the seller is lower than the indifference price of the buyer, then by agreeing at any price in between, they both experience some gain of utility, so they both undertake “negative risk”. In the case however that the indifference price of the seller is higher than the indifference price of the buyer, then by agreeing at any price in between, they both experience some loss in utility, i.e., they undertake some “positive risk” with regard to their benchmark utility functions. In general, there is an infinity of “risk” allocations that will allow for the transaction to take place. It is reasonable therefore to ask which is the optimal one and through the resolution of this problem propose a unique pricing mechanism for the claim.

As a result of undertaking some positive risk the buyer quotes a higher reservation price than her original indifference price, while the opposite effect happens for the seller, who quotes a lower reservation price than her original indifference price when undertaking positive risk. Should the new reservation prices be such that the buyer’s reservation price is higher then the seller’s reservation price, the transaction may take place.
In the scenario we propose, we assume that each of the two agents has firm beliefs about the future states of the world but deliberately undertakes some risk so that the transaction will be made possible. The optimal risk allocation is defined by the solution of the optimization problem, in which a convex combination of the risks undertaken by each of the two agents is minimized, under the constraint that the transaction is made possible, i.e., under the constraint that the buyer’s price is greater or equal than the seller’s price. From the unique optimal risk allocation one is led to a unique common price for the contingent claim in question.

For the purposes of this paper we adopt the view that utility is cardinal and allows for interpersonal and intrapersonal comparison as is usually done in many branches of economic theory such as social welfare theory (see e.g., [34], [44], [21] but also [27] for an opposite view), the axiomatic theory of bargaining (see e.g., [36], [26]), the theory of coalitional bargaining ([45]) etc. Without entering the long standing debate concerning the ordinal or cardinal nature of utilities and the problem of strength of preferences [5], [6], [16], [37], we will not eschew from exploring the implications that such an assumption has in contingent claim pricing theory in incomplete markets.

The structure of the paper is as follows: In Section 2, we fix ideas and notation, we present the general setup of the market and of the agents’ risk preferences, we introduce the concept of “loss” or “gain” of indirect utility and we extend the concept of utility indifference pricing in order to accommodate situations of loss or gain of indirect utility. In Section 3, we deal with our main problem of price selection in incomplete markets by imposing our risk sharing criterion which leads to our main result, a pricing scheme, represented by an optimization problem, which we prove offers a unique solution to the price selection problem. In Section 4, we examine in detail the example of agents who report exponential utility, in which case we are able to provide a closed form solution to the price selection problem and discuss further its properties. In Section 5, we summarize and conclude, after discussing possible variations or extensions of our framework (convex risk measures, optimal trading time, other forms of total risk) which hint at some possible directions for future research. Finally, in the Appendix we treat the technicalities involved with utility functions that are defined only for wealths that take values on the positive real axis.

2. The Market and the Agents

2.1. The Market. We consider an incomplete market setting over a fixed finite time horizon \( T \), based on a filtered probability space \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})\), where the filtration \( \mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]} \) satisfies the usual conditions of right continuity and completeness. We assume that the market consists of \( d + 1 \) tradable assets, the discounted price process of which is denoted by \( S = (S_t)_{t \in [0,T]} = \)
The first of these assets is considered to be riskless (the bond) with discounted price process $S_t^0$ equal to 1 at any time $t \in [0,T]$, i.e., it plays the rôle of the numéraire. The other assets are risky (the stocks) with discounted price process modeled by a $\mathbb{R}^d$-valued (locally bounded) semimartingale $\left(S_t^1, S_t^2, ..., S_t^d\right)_{t \in [0,T]}$ on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$.

Furthermore, we make the following assumption which ensures that our market is incomplete and free of arbitrage opportunities (see [13]).

Assumption 2.1. The set $\mathcal{M}_e := \{Q \sim \mathbb{P} : S \text{ is local martingale under } Q\}$ is non empty and not a singleton.

According to F. Delbaen and W. Schachermayer, [13], given a contingent claim $B$, the set of non-arbitrage prices for this claim is the interval $\left(\inf_{Q \in \mathcal{M}_e} \{\mathbb{E}_Q[B]\}, \sup_{Q \in \mathcal{M}_e} \{\mathbb{E}_Q[B]\}\right)$. Clearly, if the market were complete, i.e., if $\mathcal{M}_e$ were a singleton, this interval would degenerate to a single point, thus leading to the unique price of the claim.

2.2. The Utility Functions. The incompleteness of the market implies the existence of contingent claims that can not be replicated by the market assets $S$. Thus, an agent who includes such a non-replicable claim in her portfolio will always face unhedgeable risk. Therefore, in this market setting, the agent’s risk preferences play a crucial rôle with regard to her investment decisions. We assume that the risk preferences of an agent are modeled by some utility function on her terminal wealth at time horizon $T$. Then, the agent faces the problem of maximizing the expected utility of her terminal wealth by employing an appropriate portfolio strategy for trading in the market $S$ (see problems (4) and (5)).

So let us suppose that an agent decides on the basis of some utility function $U : \mathbb{R} \rightarrow \mathbb{R}$ that is strictly increasing, strictly concave, continuously differentiable and satisfies the Inada conditions:

$$\lim_{x \rightarrow -\infty} U'(x) =: U'(-\infty) = +\infty \text{ and } \lim_{x \rightarrow +\infty} U'(x) =: U'(+\infty) = 0. \tag{1}$$

Moreover, we assume that $U$ has reasonable asymptotic elasticity, i.e.,

$$\liminf_{x \rightarrow -\infty} \frac{xU'(x)}{U(x)} > 1 \text{ and } \limsup_{x \rightarrow +\infty} \frac{xU'(x)}{U(x)} < 1. \tag{2}$$

As proved in [42] (see also [30]), this assumption is required for the well posedness of the utility maximization problem of the agent.

Furthermore, it is convenient to assume that $\lim_{x \rightarrow -\infty} U(x) = -\infty$. The notation $\lim_{x \rightarrow +\infty} U(x) = U(+\infty)$ is also used.

2.3. The Agents and the Admissible Strategies. Consider now an agent with initial wealth $x \in \mathbb{R}$ (measured in numéraire units) and risk preferences that are modeled by a utility function
U. The agent invests her initial wealth in the market assets by creating self-financing portfolios (investment strategies), with the goal to maximize her expected utility of terminal wealth.

A self-financing portfolio for this agent is a \(d + 1\)-dimensional stochastic process \(\vartheta = (\vartheta_t)_{t \in [0,T]}\), that is predictable and \(S\)-integrable, specifying the number of units of each asset he holds in the portfolio at each time \(t \in [0,T]\). Then, the wealth process \(X = (X_t)_{t \in [0,T]}\) that such a portfolio produces is given as the stochastic integral \(X_t = (\vartheta \cdot S)_t = \int_0^t \vartheta_s dS_s\). The portfolio \(\vartheta\) is called admissible if the wealth process \(X_t = (\vartheta \cdot S)_t\) that it produces is uniformly bounded from below by some constant.

Let \(\Theta\) denote the set of all admissible portfolios that can be formed by this agent.

In the sequel, we use the following notation: \(\mathbb{L}^p = \mathbb{L}^p(\Omega, \mathcal{F}_T, \mathbb{P}), 1 \leq p < \infty\), denotes the set of equivalence classes of \(\mathcal{F}_T\)-measurable random variables \(X\) such that \(\mathbb{E}[|X|^p] < \infty\), where \(\mathbb{E}[\cdot]\) denotes the expectation under the probability measure \(\mathbb{P}\); the case where \(p = \infty\) corresponds to the essentially bounded random variables. By \(\mathbb{L}^0\) we denote the set of \(\mathcal{F}_T\)-measurable random variables.

Assume now that the agent has a (possibly non-replicable) claim \(B \in \mathbb{L}^\infty\) that matures at the fixed time horizon \(T\). Then we define the set \(\mathcal{F}_{U,B}(x) := \{F \in \mathbb{L}^0 : U(F) \in \mathbb{L}^1\} \cup \{x + (\vartheta \cdot S)_T + B | \vartheta \in \Theta\}\) and according to M. Owen, [38], the set of attainable wealths for this agent, which describes the wealths that she can attain at time \(T\) by employing some admissible portfolio strategy \(\vartheta\), is given as

\[
\mathcal{X}_{U,B}(x) := \{X \in \mathbb{L}^0 : U(X + B) \in \{U(F) : F \in \mathcal{F}_{U,B}(x)\}^1\}. \quad (3)
\]

Then, the problem of utility maximization that the agent faces is

\[
u(x; B) := \sup_{X \in \mathcal{X}_{U,B}(x)} \mathbb{E}\{U(X + B)\} \quad (4)
\]

In the special case that \(B = 0\), the corresponding problem becomes

\[
u(x) := \sup_{X \in \mathcal{X}_{U}(x)} \mathbb{E}\{U(X)\} \quad (5)
\]

where, by \(\mathcal{X}_{U}(x)\) we denote the attainable wealths of the agent in the special case that \(B = 0\), i.e., when there is no liability at the time horizon \(T\).

The functions \(\nu(x)\) and \(\nu(x; B)\) are usually called the “indirect” utilities (the latter under the liability \(B\)) or simply the value functions and have been studied by many authors (see e.g., [42], [38], [39] etc.).

The next theorem due to [38] (see also [42]) states some useful properties of the indirect utility.

**Theorem 2.1.** (W. Schachermayer, 2001, M. Owen, 2002)

Assume that:

(i) \(B \in \mathbb{L}^\infty\).

(ii) Assumption 2.1 holds.
(iii) There exists \( x \in \mathbb{R} \) such that \( u(x;B) < U(\infty) \).

Then, the value function \( u(\cdot;B) \) is finitely valued, strictly increasing, strictly concave and continuously differentiable on \( \mathbb{R} \) with

\[
\lim_{x \to +\infty} u'(x;B) = 0 \quad \text{and} \quad \lim_{x \to -\infty} u'(x;B) = +\infty
\]

(6)

Furthermore, there exists a solution to problem (4), i.e., the supremum in (4) is attained.

Remark 2.1. Since, \( u(x;B) \) is a strictly concave \( C^1(\mathbb{R}) \) function of \( x \), we obtain that \( \frac{du(x;B)}{dx} \) is strictly decreasing and its range is \((0, +\infty)\). We also observe that \( \lim_{x \to -\infty} u(x;B) = \lim_{x \to -\infty} U(x) = -\infty \) and \( \lim_{x \to +\infty} u(x;B) = U(+\infty) \).

2.4. The Agents and the Indifference Price. Having defined the indirect utility functions, we recall the notion of the seller’s and buyer’s indifference price for the given claim \( B \in \mathbb{L}^\infty \). The seller’s indifference price, \( v^s(B) \), is given as the unique solution of

\[
u(x) = u \left( x + v^s(B); -B \right)
\]

(7)

while the corresponding buyer’s indifference price, \( v^b(B) \), solves

\[
u(x) = u \left( x - v^b(B); B \right)
\]

(8)

Indifference pricing has been examined by several authors (see e.g., [23], [10], [40], [35], [28] etc.). It can be shown that both indifference prices are non-arbitrage prices, i.e., they belong to the interval \( \left( \inf_{Q \in \mathbb{M}_e} \{E_Q[B]\}, \sup_{Q \in \mathbb{M}_e} \{E_Q[B]\} \right) \) (see among others [39], Proposition 7.2) and when \( B \) is replicable, they are both equal to the unique non-arbitrage price \( E_Q[B] \), for some \( Q \in \mathbb{M}_e \).

The indifference pricing rule provides the agents with subjective reference prices of the claim \( B \in \mathbb{L}^\infty \). An agent who reports this pricing rule is willing to sell the claim \( B \) at any price \( P \) such that \( v^s(B) \leq P \). In particular, if the agent sells the claim \( B \) at a price \( P \) strictly greater than \( v^s(B) \), then the seller has “gained” some indirect utility. Similarly, the agent is willing to buy the claim \( B \) at any price \( P \) such that \( v^b(B) \geq P \), while again a strict inequality transaction results in a “gain” of indirect utility to the buyer. On the other hand, selling claim \( B \) for a price \( P < v^s(B) \) or buying claim \( B \) for a price \( P > v^b(B) \) results in a “loss” of indirect utility.

Although a transaction that does not cause any “loss” of indirect utility to any of the transacting agents sounds natural, it seems that the existence of transactions that may result in some “loss” of indirect utility may be in need of some defense. The following motivating examples try to serve this purpose, although we should emphasize that the acceptance of loss of indirect utility is used in the sequel as a means for pricing and could better be interpreted as departure from the benchmark utility function.
Example 2.1. A company $C$ owns two subsidiaries $S_1$ and $S_2$. For some reason (e.g., internal politics, book-keeping, tax purposes) the top management of $C$ is convinced that it would be beneficial for $C$ if $S_1$ and $S_2$ perform a transaction on a structured claim $B$ with $S_1$ acting as the buyer and $S_2$ acting as the seller of the claim. The mother company $C$ is not interested in the exact price at which the transaction will take place, as long as it is a non-arbitrage price, and so it “orders” the management of $S_1$ and $S_2$ to perform the transaction at a price convenient to them. However, the managers of the subsidiaries are naturally concerned about the price of the transaction. Let $v^{(b)}(B)$, $v^{(s)}(B)$ be the indifference prices of $S_1$ and $S_2$ respectively. If $v^{(b)}(B) < v^{(s)}(B)$ then, since $S_1$ and $S_2$ are forced to perform the transaction, at least one of them will lose some indirect utility (most probably they will both lose some indirect utility as the bargaining procedure may eventually lead to a price $P$ such that $v^{(b)}(B) < P < v^{(s)}(B)$).

Example 2.2. Suppose that two investment companies $I_1$ and $I_2$ are interested in transacting on a claim $B$, the one acting as the buyer of the claim and the other acting as the seller of the claim. The investment company $I_1$ has an investment committee that is to decide on the pricing of the claim $B$. The committee members have utility functions $U_1, ..., U_k$. After extensive discussion, they decide to use for pricing purposes the maximin or Rawlsian (see [41]) utility function $U = \min(U_1, ..., U_k)$, so that all members will be happy. Let $v^{(b)}(B)$ the resulting indifference price. In a similar manner the investment committee of the seller has reached to an indifference price $v^{(s)}(B)$. If $v^{(b)}(B) < v^{(s)}(B)$, the transaction cannot take place unless some risk (utility-wise) is undertaken. Notice that the notion of risk to be undertaken refers to departure from the indifference prices, which however were calculated according to the “conservative” minimum utility function. It is probable that a majority of the investment committee, consisting of members that are indifferent with less favorable prices, will insist on some departure from the benchmark.

Example 2.3. An employer $E$ offers a bonus scheme to an employee $M$, which is agreed to have a fixed value $C$ at time zero. The bonus consists of two parts: the first one is a contingent non-replicable claim $B$ maturing at time $T$. The second part of the bonus is just the lump cash amount that remains after subtracting the time zero value of $B$ from the total value $C$. Therefore, the value $C$ of the bonus is the sum of the time zero value of $B$ and whatever remains in cash; however, the time zero value of $B$ is clearly negotiable. In this sense, the time zero value of $B$ is the price at which the employer “sells” the claim to the employee, who in turn “buys” it from the employer. It is clear that the price of $B$ is important and the two counter-parties may find that it is mutually beneficial to reach an agreement on the price of $B$ even if some loss of indirect utility is needed.

The above examples indicate that in certain cases we need to extend the notion of the utility indifference pricing to accommodate situations of loss or gain of indirect utility. Let us fix a contingent claim $\hat{B} \in \mathbb{L}^\infty$ and following [47] consider an agent with utility function $U$ that faces...
where \( \lim \) and is compensated at time 0 by the certain amount \( \hat{P} \in \mathbb{R} \). We define her loss of indirect utility by the quantity:

\[
\varepsilon := u(x) - u(x + \hat{P}; B).
\]

Clearly, when \( \hat{B} = -B \), \( \hat{P} =: P \) corresponds to the “reservation price” stated by the seller of the claim \( B \), when a suboptimal decision leading to indirect utility differing by \( \varepsilon \) from the optimal level is taken. On the other hand, when \( \hat{B} = B \), \( \hat{P} =: -P \) corresponds to the buyer’s side.

Alternatively, one can define the function \( P^{(s)}(\varepsilon) \), which gives the exact price under which the loss of indirect utility for the seller is \( \varepsilon \). In other words, \( P^{(s)}(\varepsilon) \) is given implicitly as the solution of the equation

\[
\varepsilon = u(x) - u\left(x + P^{(s)}(\varepsilon); -B\right).
\]  

Similarly, for the buyer’s side, \( P^{(b)}(\varepsilon) \) is given as the solution of the equation

\[
\varepsilon = u(x) - u\left(x - P^{(b)}(\varepsilon); B\right).
\]

Notice that \( P^{(b)}(0) = v^{(b)}(B) \) and \( P^{(s)}(0) = v^{(s)}(B) \) and in general both \( P^{(s)}(\varepsilon) \) and \( P^{(b)}(\varepsilon) \) depend on the initial wealth \( x \).

2.5. The Functions \( P^{(s)}(\cdot) \) and \( P^{(b)}(\cdot) \). To study the properties of \( P^{(s)}(\cdot) \) and \( P^{(b)}(\cdot) \), we need to introduce the following notation: For every utility, \( U \), and for every initial wealth, \( x \in \mathbb{R} \), we define

\[
A_{U,x} := (u(x) - u(+\infty), +\infty) = (u(x) - U(+\infty), +\infty)
\]

where \( u(x) \) is the corresponding indirect utility given in (5). We also define the function \( \varphi(\cdot;B) : (-\infty, U(+\infty)) \rightarrow \mathbb{R} \) by

\[
\varphi(y; B) := u^{-1}(y; B), \quad \text{for } y \in (-\infty, U(+\infty)).
\]

\( \varphi(y; B) \) is well-defined on account of Theorem 2.1.

Using the above definitions, the solutions of equations (10) and (11) are given by

\[
P^{(s)}(\varepsilon) = \begin{cases} 
\varphi(u(x) - \varepsilon; -B) - x, & \text{if } \varepsilon \in A_{U,x}, \\
+\infty, & \text{if } \varepsilon \leq u(x) - U(+\infty)
\end{cases}
\]

and

\[
P^{(b)}(\varepsilon) = \begin{cases} 
x - \varphi(u(x) - \varepsilon; B), & \text{if } \varepsilon \in A_{U,x}, \\
-\infty, & \text{if } \varepsilon \leq u(x) - U(+\infty)
\end{cases}
\]

where \( \lim_{\varepsilon \rightarrow +\infty} P^{(s)}(\varepsilon) = -\infty \) and \( \lim_{\varepsilon \rightarrow -\infty} P^{(b)}(\varepsilon) = +\infty \).

We should point out that although the indifference prices (i.e., \( P^{(s)}(0) \) and \( P^{(b)}(0) \)) are non-arbitrage prices, the values \( P^{(s)}(\varepsilon) \) and \( P^{(b)}(\varepsilon) \) may lie outside the interval of non-arbitrage prices \( \left( \inf_{Q \in M_{\varepsilon}} \{ E_{Q}(B) \}, \sup_{Q \in M_{\varepsilon}} \{ E_{Q}(B) \} \right) \) for certain values of \( \varepsilon \). In fact, \( P^{(s)}(A_{U,x}) = P^{(b)}(A_{U,x}) = \mathbb{R} \) for
any pair of initial wealths \( x \in \mathbb{R} \) and utility functions \( U \). By (14) and (15), \( P^{(s)}(\varepsilon) \) is within the interval of non-arbitrage prices for those \( \varepsilon \)'s that belong to the interval

\[
\left( u(x) - u\left( \sup_{Q \in \mathcal{M}_e} \{ E_Q(B) \} + x; -B \right), u(x) - u\left( \inf_{Q \in \mathcal{M}_e} \{ E_Q(B) \} + x; -B \right) \right)
\]

(16)

and \( P^{(b)}(\varepsilon) \) is within the interval of non-arbitrage prices for those \( \varepsilon \)'s that belong to the interval

\[
\left( u(x) - u(x - \inf_{Q \in \mathcal{M}_e} \{ E_Q(B) \}; B), u(x) - u(x - \sup_{Q \in \mathcal{M}_e} \{ E_Q(B) \}; B) \right)
\]

(17)

The following proposition establishes some useful properties of the functions \( P^{(s)} \) and \( P^{(b)} \).

**Proposition 2.1.** For a given claim \( B \in L^\infty \) and initial wealth \( x \in \mathbb{R} \), \( P^{(s)} \) (\( P^{(b)} \)) is a continuously differentiable, strictly decreasing (increasing) and strictly convex (concave) function of \( \varepsilon \) on \( A_{U,x} \).

**Proof.** We restrict our attention to \( P^{(s)} \), since the proof for \( P^{(b)} \) follows along the same lines. The fact that \( P^{(s)} \in C^1(A_{U,x}) \) follows directly from the definition of \( P^{(s)} \) and Theorem 2.1. Let us now consider \( \varepsilon_1, \varepsilon_2 \in A_{U,x} \) such that \( \varepsilon_1 < \varepsilon_2 \). Then,

\[
\varepsilon_1 = u(x) - u\left( x + P^{(s)}(\varepsilon_1); -B \right) < \varepsilon_2 = u(x) - u\left( x + P^{(s)}(\varepsilon_2); -B \right)
\]

hence \( P^{(s)}(\varepsilon_1) > P^{(s)}(\varepsilon_2) \), because \( u(x; -B) \) is strictly increasing for \( x \in \mathbb{R} \).

For the convexity, let us take any \( \lambda \in (0,1) \) and some \( \varepsilon_1, \varepsilon_2 \in A_{U,x} \), with \( \varepsilon_1 \neq \varepsilon_2 \). Then,

\[
\lambda \varepsilon_1 + (1-\lambda) \varepsilon_2 = u(x) - u\left( x + P^{(s)}(\lambda \varepsilon_1) + (1-\lambda) \varepsilon_2; -B \right),
\]

but also

\[
\lambda \varepsilon_1 + (1-\lambda) \varepsilon_2 = u(x) - \lambda u\left( x + P^{(s)}(\varepsilon_1); -B \right) - (1-\lambda) u\left( x + P^{(s)}(\varepsilon_2); -B \right).
\]

Therefore,

\[
u\left( x + P^{(s)}(\lambda \varepsilon_1) + (1-\lambda) \varepsilon_2; -B \right) = \lambda u\left( x + P^{(s)}(\varepsilon_1); -B \right) - (1-\lambda) u\left( x + P^{(s)}(\varepsilon_2); -B \right)
\]

and by the strict convexity of \( u(x; -B) \) we have

\[
u\left( x + P^{(s)}(\lambda \varepsilon_1 + (1-\lambda) \varepsilon_2); -B \right) < u\left( x + \lambda P^{(s)}(\varepsilon_1) + (1-\lambda) P^{(s)}(\varepsilon_2); -B \right)
\]

which gives that

\[
P^{(s)}(\lambda \varepsilon_1 + (1-\lambda) \varepsilon_2) < \lambda P^{(s)}(\varepsilon_1) + (1-\lambda) P^{(s)}(\varepsilon_2).
\]

This completes the proof. \( \square \)
3. The Risk Sharing Pricing Scheme

In this Section, we deal with the main problem of this paper. Consider a fixed claim \( B \in \mathbb{L}^\infty \) and two financial agents whose risk preferences are modeled by utility functions that satisfy the properties listed in subsection 2.2 (or the ones in subsection A.1). One of them is supposed to be the seller of \( B \), while the other is the buyer. In other words, we suppose that at time \( T \), the seller will face the liability \(-B\) payable to the buyer, as a result of receiving some certain amount \( P \in \mathbb{R} \) at time 0 by the buyer. On the other hand, the buyer will receive the payoff \( B \) at time \( T \), as a result of paying \( P \) to the seller at time 0. Given that the transaction has to take place,

\[ \text{How should price } P \text{ be determined?} \]

Before we give our proposed answer, we need some further notation. Let \( U_s \) and \( U_b \) denote the utility functions of the seller and the buyer respectively, \( u^{(s)}(.; -B), u^{(b)}(.; B) \) their indirect utilities, \( x_s, x_b \in \mathbb{R} \) their initial wealths and \( P^{(s)}(.), P^{(b)}(.) \) their respective pricing functions. Furthermore, we set \( A_s := A_{U_s, x_s} \) and \( A_b := A_{U_b, x_b} \). Since both agents are considered to be utility maximizers (i.e., they both encounter problems of the form [4] and [5]), it is reasonable to expect that they give value to non-replicable contingent claims using the indifference pricing rule. Then, two cases are possible, namely, \( v^{(s)}(B) \leq v^{(b)}(B) \) and \( v^{(s)}(B) > v^{(b)}(B) \). In the former case, the indifference pricing rule leads to an agreement price (see [3] for the exact definition of agreement and [24] for a relevant discussion), in the sense that there is a price \( P \in [v^{(s)}(B), v^{(b)}(B)] \), the transaction at which implies gain in terms of indirect utility for at least one of the agents. However, determination of the exact price \( P \) at which the transaction will take place needs an extra criterion. In the latter situation, the indifference pricing rule implies no agreement between the agents, in other words, transaction at any price leads to loss of indirect utility for at least one of the agents. Hence even in this case, provided though that the agents have to proceed to the transaction, a criterion is needed in order to determine the exact price at which the transaction will take place.

A desired criterion (common in both cases) can be given through the functions \( P^{(s)}(.) \) and \( P^{(b)}(.) \). For this, we consider the following minimization problem

\[
\min_{(\varepsilon_s, \varepsilon_b) \in A_s \times A_b} \{ \lambda \varepsilon_s + (1 - \lambda) \varepsilon_b \} \tag{18}
\]

subject to \( P^{(s)}(\varepsilon_s) \leq P^{(b)}(\varepsilon_b) \)

where, \( \lambda \in (0, 1) \) and \( \varepsilon_s, \varepsilon_b \) are the loss (gain) of indirect utilities of the seller and the buyer respectively. Assuming the existence of a solution \( (\varepsilon^*_s, \varepsilon^*_b) \) of (18), we are then able to define a commonly agreed price, \( P = P^* \), for the claim \( B \) by

\[
P^* := P^{(s)}(\varepsilon^*_s) = P^{(b)}(\varepsilon^*_b)
\]
In words, according to the above scenario, the agents will proceed to the transaction on claim $B$ when a convex combination (a weighted average) of their loss of indirect utilities is minimized. This convex combination corresponds to an allocation, between the two agents, of the total risk needed for the completion of the transaction. As Theorem 3.1 states, the uniqueness of the solution provides a unique price $P^*$, which will be called the risk sharing price of the claim $B$ (see Definition 3.1 below) of these two agents.

The following theorem provides the existence and the uniqueness of the solution to the minimization problem (18).

**Theorem 3.1.** Under Assumption 2.1, for any choice of utilities $U_s$ and $U_b$ and for every weight $\lambda \in (0,1)$, the minimization problem (18) has a unique solution $(\varepsilon^*_s, \varepsilon^*_b) \in A_s \times A_b$, which provides a unique price $P^* = P(s) (\varepsilon^*_s) = P(b) (\varepsilon^*_b)$.

**Proof.** First, notice that in (18), the inequality in the constraint can be replaced by the equality $P(s) (\varepsilon_s) = P(b) (\varepsilon_b)$. Indeed, if $(\varepsilon_s, \varepsilon_b) \in A_s \times A_b$ minimizes $\lambda \varepsilon_s + (1-\lambda) \varepsilon_b$ and $P(s) (\varepsilon_s) < P(b) (\varepsilon_b)$, there exists $\delta > 0$ such that $\varepsilon_s - \delta \in A_s$ and $P(s) (\varepsilon_s - \delta) = P(b) (\varepsilon_b) \in \mathbb{R}$ (since $P(s)$ is strictly decreasing and $\sup \{P(s) (\varepsilon) : \varepsilon \in A_s \} = +\infty$). But then $\lambda (\varepsilon_s - \delta) + (1-\lambda) \varepsilon_b < \lambda \varepsilon_s + (1-\lambda) \varepsilon_b$, which is a contradiction.

Now, since $P(s)$ ($P(b)$) is in $C^1(A_s)$ ($C^1(A_b)$), we apply the Lagrange multiplier method in order to solve (18). To this end, we define the Lagrangian $L : A_s \times A_b \times \mathbb{R} \rightarrow \mathbb{R}$

$$L(\varepsilon_s, \varepsilon_b, m) = \lambda \varepsilon_s + (1-\lambda) \varepsilon_b + m \left( P(s) (\varepsilon_s) - P(b) (\varepsilon_b) \right)$$

and we are looking for the triples $(\varepsilon_s, \varepsilon_b, m)$ that solve the following system

$$\frac{\partial L}{\partial \varepsilon_s} = \frac{\partial L}{\partial \varepsilon_b} = \frac{\partial L}{\partial m} = 0$$

or equivalently

$$(P(s))' (\varepsilon_s) = -\frac{\lambda}{m} \quad (19)$$

$$(P(b))' (\varepsilon_b) = \frac{(1-\lambda)}{m} \quad (20)$$

$$P(s) (\varepsilon_s) = P(b) (\varepsilon_b) \quad (21)$$

Since $P(s)$ is strictly decreasing and $P(b)$ is strictly increasing, we can restrict ourselves to $m \in \mathbb{R}^+_*$.

Thanks to the representations (13) and (15) of the functions $P(s)$ and $P(b)$ with respect to the indirect utilities, equations (19) and (20) can be written as

$$u_s' \left( x_s + P(s) (\varepsilon_s) ; -B \right) = \frac{m}{\lambda} \quad (22)$$

$$u_b' \left( x_b - P(b) (\varepsilon_b) ; B \right) = \frac{m}{(1-\lambda)} \quad (23)$$
and therefore the required inequalities (25) and (26) hold for any choice of utilities and hence there exists a unique \( \epsilon^* (m^*) \in A_s \) (24).

By Theorem 2.1 (see also Remark 2.1), we have that for every \( m \in \mathbb{R}_+^* \) and \( \lambda \in (0, 1) \), there exists a unique \( k (m) \in \mathbb{R} \) such that

\[
\lim_{m \to 0} P^{(s)} (\epsilon^*_s (m)) > \lim_{m \to 0} P^{(b)} (\epsilon^*_b (m))
\]

and

\[
\lim_{m \to +\infty} P^{(s)} (\epsilon^*_s (m)) < \lim_{m \to +\infty} P^{(b)} (\epsilon^*_b (m)) .
\]

Since, \( \epsilon^*_s (m) \) and \( \epsilon^*_b (m) \) are strictly increasing and continuous (because \( u_s, u_b \) are), \( P^{(s)} (\epsilon^*_s (m)) \) is strictly decreasing and continuous and \( P^{(b)} (\epsilon^*_b (m)) \) is strictly increasing and continuous as functions of \( m \). Now, to establish that (24) has a unique solution, it is enough to ensure that

\[
\lim_{m \to 0} P^{(s)} (\epsilon^*_s (m)) > \lim_{m \to 0} P^{(b)} (\epsilon^*_b (m))
\]

and

\[
\lim_{m \to +\infty} P^{(s)} (\epsilon^*_s (m)) < \lim_{m \to +\infty} P^{(b)} (\epsilon^*_b (m)) .
\]

Indeed, we have

\[
\lim_{m \to 0} P^{(s)} (\epsilon^*_s (m)) = \lim_{m \to +\infty} P^{(b)} (\epsilon^*_b (m)) = +\infty
\]

and

\[
\lim_{m \to 0} P^{(b)} (\epsilon^*_b (m)) = \lim_{m \to +\infty} P^{(s)} (\epsilon^*_s (m)) = -\infty
\]

and therefore the required inequalities (25) and (26) hold for any choice of utilities and hence there exists a unique \( m^* \in \mathbb{R}_+^* \), which solves (24). Given \( m^* \), \( \epsilon^*_s = \epsilon^*_s (m^*) \) and \( \epsilon^*_b = \epsilon^*_b (m^*) \) solve (19) and (20) respectively. The fact that \( (\epsilon^*_s, \epsilon^*_b) \) is then indeed the unique minimizer of (18) is a direct consequence of the strict convexity of \( L (\epsilon_s, \epsilon_b, m) \) on \( (\epsilon_s, \epsilon_b) \), for any given \( m \). \( \square \)

Thanks to the uniqueness of the solution of (18), we are able to define the risk sharing price as follows.

**Definition 3.1.** The price \( P^* = P^{(s)} (\epsilon^*_s) = P^{(b)} (\epsilon^*_b) \), where \( (\epsilon^*_s, \epsilon^*_b) \) is the solution of problem (18) is called the risk sharing price of claim \( B \) for the two agents.
Remark 3.1. Using the notation $X^{(s)}(P^{(s)}(\varepsilon)) := \text{argmax}_{X \in \mathcal{X}_{s,-B(x_s+P^{(s)}(\varepsilon))}} \mathbb{E}[U_s(X - B)]$, equation (10) can be written as

$$
\mathbb{E}\left[U_s\left(X^{(s)}(P^{(s)}(\varepsilon)) - B\right)\right] = u_s(x_s) - \varepsilon.
$$

Then, we define the seller’s residual risk of claim $B$ at price $P$ (see also [35]) as

$$(29)$$

Similarly, the buyer’s residual risk of claim $B$ at $P$ is defined as the difference $X^{(b)}(P) - P$. Then problem (18) is equivalent to maximization of

$$
\lambda \mathbb{E}[U_s(R^{(s)}(P^{(s)}(\varepsilon_s)))] + (1 - \lambda)\mathbb{E}[U_b(R^{(b)}(P^{(b)}(\varepsilon_b)))]
$$

Equation (30) describes the total utility in terms of residual risks. The price $P^*$ is the maximizer of (30) and corresponds to the optimal residual risk allocation according to the sharing rule $\lambda$. This provides a clarification of the terminology that we used in Definition 3.1.

Remark 3.2. The risk sharing parameter $\lambda$ decides how the risk is going to be distributed between the two agents. The role of parameter $\lambda$ is twofold: On the one hand, it reflects the relative bargaining power between the two agents, while on the other hand, it may serve as an adjustment parameter allowing for the interpersonal comparison of utilities (if that has not already been accounted for, by the cardinality properties of the utility functions). In this paper, we consider $\lambda$ as exogenously given (e.g., by some authorities or by policy makers or even by some mutual agreement). It would be interesting to examine the possibility of determining $\lambda$ in an endogenous manner, for example as the equilibrium of a properly designed game; however, this is clearly beyond the scope of this work.

The following proposition discusses the effect of parameter $\lambda$ on the risk sharing price.

Proposition 3.1. The utility sharing price $P^*$ is strictly decreasing and continuous function of $\lambda \in (0,1)$.

Proof. We first show that $P^*$ is strictly decreasing with respect to $\lambda$. Consider two strictly positive numbers $\lambda_1 < \lambda_2 < 1$. Since $u'_s(\cdot; - B)$ and $P^{(s)}(\cdot)$ are strictly decreasing functions (see Theorem 2.1), equation (22) implies that $\varepsilon_{s,\lambda_1}^*(m) > \varepsilon_{s,\lambda_2}^*(m)$, for every $m \in \mathbb{R}_+$, where $\varepsilon_{s,\lambda}^*(m)$ denotes the solution the equation (22). Similar arguments imply $\varepsilon_{b,\lambda_1}^*(m) < \varepsilon_{b,\lambda_2}^*(m), \forall m \in \mathbb{R}_+$, where $\varepsilon_{b,\lambda}^*(m)$ denotes the solution of equation (23). Therefore,

$$
P^{(s)}(\varepsilon_{s,\lambda_1}^*(m)) < P^{(s)}(\varepsilon_{s,\lambda_2}^*(m)) \quad \text{and} \quad P^{(b)}(\varepsilon_{b,\lambda_1}^*(m)) < P^{(b)}(\varepsilon_{b,\lambda_2}^*(m))
$$

for every $m$, hence by (24), we obtain the desired monotonicity property of $P^*$ with respect to $\lambda$. 
The continuity with respect to $\lambda$ is an easy consequence of Berge’s maximum theorem; Indeed Berge’s theorem (see e.g., [1], Theorem 16.31) guarantees the upper semicontinuity of the correspondence $P^*_\lambda$. Since, the solution of the problem is unique, this correspondence is single-valued, hence continuous. □

Since, the functions $P(s)$ and $P(b)$ have unbounded ranges (see (14) and (15)), one can observe that for values of $\lambda$ close enough to 0 or 1, it may be the case that $P^*$ lies outside the interval of non-arbitrage prices for $B$ (see for example (40) for $\gamma_s = \gamma_b$). This can be seen as an inconsistency, the resolution of which requires that lower and upper bounds are imposed on $\lambda$, which intuitively means that “extreme” loss (or gain) of utilities is not allowed.

In the light of Proposition 3.1, there exists a continuous function $b(.)$ (which in fact is the inverse of $P^*$ as a function of $\lambda$) such that

$$g < P^* < G \iff b(g) < \lambda < b(G)$$ \hspace{1cm} (31)

In particular, the risk sharing price $P^*$ is a non-arbitrage if and only if the weight of the pricing scheme $\lambda$ belongs in the interval $(b(\inf_{Q \in M_e} \{E_Q[B]\}), b(\sup_{Q \in M_e} \{E_Q[B]\}))$.

Remark 3.3. Notice the formal similarity of problem (18) with Hicks’ expenditure minimization problem ([34], Chapter 3). In our setting, risk plays the rôle of expenditure, $\lambda$ and $1-\lambda$ play the rôle of the prices, while the price functions $P(s)$ and $P(b)$ play the rôle of utilities. Through this similarity, we may clarify the effect of $\lambda$ on the risk sharing price $P^*$. Furthermore, one can consider the dual problem, namely the utility maximization, which in our framework corresponds to the maximization of the difference of the reservation prices of the two agents, given the total quantity of the risk undertaken.

Remark 3.4. The risk sharing pricing scheme as provided by the solution of (18), gives a price for $B$ traded at $t = 0$; it can be extended to include the case where the claim can be traded at any time $t \in [0, T]$.

The utility maximization problems (4) and (5) can be naturally expressed in a dynamic way, i.e., for any time $t \in [0, T]$, we consider the problem

$$u(x, t; B) := \sup_{X \in \mathcal{X}_{U,B}(x,t)} \mathbb{E}[U(X + B) | \mathcal{F}_t]$$ \hspace{1cm} (32)

where $\mathcal{X}_{U,B}(x,t)$ is defined similarly as in (3), taking into account the market conditions from time $t$ onwards and $x$ is the agent’s (random) wealth at time $t$. Using the notation $u(x, t) = u(x, t; 0)$, we can define the loss of indirect utility at time $t$ of the writer and the buyer of the given claim $B$, by

$$\varepsilon_{s,t} := u(x, t) - u(x + P, t; -B) \quad \text{and} \quad \varepsilon_{b,t} := u(x, t) - u(x - P, t; B),$$ \hspace{1cm} (33)
so that the reservation prices for the seller and the buyer at time \( t \) given that “risk” \( \varepsilon \) is undertaken is given by the solutions of the equations

\[
\varepsilon = u(x, t) - u\left(x + P_t^s(\varepsilon), t; -B\right) \quad \text{and} \quad \varepsilon = u(x, t) - u\left(x - P_t^b(\varepsilon), t; B\right).
\]

Hence, we can consider the family of minimization problems

\[
\min_{(\varepsilon_s, \varepsilon_b) \in A_{s,t} \times A_{b,t}} \{ \lambda \varepsilon_s + (1 - \lambda) \varepsilon_b \}
\]

subject to

\[
P_t^s(\varepsilon_s) \leq P_t^b(\varepsilon_b)
\]

where, \( A_{s,t} := (u_s(x_s, t) - U^+(\infty), +\infty) \) and \( A_{b,t} := (u_b(x_b, t) - U^+(\infty), +\infty) \).

By similar assumptions and arguments as in problem (18), one can conclude the existence of (unique) solutions \( P_t^\ast \) for the family of problems (34).

Instead of providing the proof for the above arguments, which follows closely that of Theorem 3.1, we prefer to give a representative example of how problem (34) can be used to determine the price of claims that can be traded at any time before maturity in Section 4.

4. The Case of the Exponential Utility

The exponential utility, defined as \( U(x) = -e^{-\gamma x} \), where \( \gamma > 0 \) is the risk aversion coefficient, is a widely used utility function in the literature, because it offers closed form solutions to a variety of utility maximization problems. In this Section, we show that in the case where the agents report exponential utilities, problem (18) can be solved explicitly, thus leading to a closed form expression for the risk sharing price \( P^\ast \). This allows us to perform a detailed analysis of the properties of \( P^\ast \).

In the special case of the exponential utility, the problem of choosing the set of admissible strategies \( \Theta \) simplifies considerably to

\[
\Theta_{exp} := \{ \vartheta : (\vartheta \cdot S)_{t \in [0, T]} \text{ is a true-martingale under any } Q \in \mathcal{M}_{e,f} \}
\]

where \( \mathcal{M}_{e,f} := \{ Q \in \mathcal{M}_e : \mathcal{H}(Q|P) < +\infty \} \) (assumed to be non-empty) and \( \mathcal{H}(Q|P) \) is the relative entropy with respect to the probability measure \( P \), which is defined as follows

\[
\mathcal{H}(Q|P) = \begin{cases} 
E\left[ \frac{dQ}{dP} \ln \left( \frac{dQ}{dP} \right) \right], & Q \ll P, \\
+\infty, & \text{otherwise}.
\end{cases}
\]

(see e.g., [12], [25], [33]). The occurrence of the relative entropy in the definition of the set of the admissible strategies is not coincidental, since as shown in [19] and [12], the entropy minimization problem is the dual of the exponential utility maximization problem.

For every claim \( B \in L^\infty \), the utility maximization (4) is well-defined, the supremum is attained, the indirect utility, \( u(x;B) \) is of the form

\[
u(x;B) = e^{-\gamma x} u(0;B), \quad (35)\]
is finite for every initial wealth $x \in \mathbb{R}$, and both the optimal strategy and the indifference prices are independent of $x$.

4.1. On Price $P^*$ and its Properties. Let $\gamma_s$ and $\gamma_b > 0$ be the seller’s and buyer’s risk aversion coefficients respectively. Using (35), it follows that (14) and (15) simplify to

$$
P^{(s)}(\varepsilon) = \frac{1}{\gamma_s} \ln \left( \frac{u_s(x_s; -B)}{u_s(x_s) - \varepsilon} \right) = v^{(s)}(B) - \frac{1}{\gamma_s} \ln \left( 1 - \frac{\varepsilon}{u_s(x_s)} \right), \quad \text{for } \varepsilon \in A_s = (u_s(x_s), +\infty)
$$

and

$$
P^{(b)}(\varepsilon) = \frac{1}{\gamma_b} \ln \left( \frac{u_b(x_b; -B)}{u_b(x_b) - \varepsilon} \right) = v^{(b)}(B) + \frac{1}{\gamma_b} \ln \left( 1 - \frac{\varepsilon}{u_b(x_b)} \right), \quad \text{for } \varepsilon \in A_b = (u_b(x_b), +\infty),
$$

where $v^{(s)}(B)$ and $v^{(b)}(B)$ denote the seller’s and the buyer’s indifference prices and $u_s(x_s), u_b(x_b)$ their respective indirect utilities when no position on the claim $B$ is undertaken.

Remark 4.1. Using equations (36) and (37), and straightforward algebra, we observe that

$$
(P^{(s)})'(\varepsilon) = \frac{1}{\gamma_s} \frac{1}{u_s(x_s) - \varepsilon} \quad \text{and} \quad (P^{(b)})'(\varepsilon) = -\frac{1}{\gamma_b} \frac{1}{u_b(x_b) - \varepsilon},
$$

while,

$$
\frac{\partial P^{(s)}}{\partial x_s} = \frac{\varepsilon}{u_s(x_s) - \varepsilon} \quad \text{and} \quad \frac{\partial P^{(b)}}{\partial x_b} = \frac{\varepsilon}{\varepsilon - u_b(x_b)},
$$

i.e., the sensitivity of the reservation prices with respect to the risk undertaken and the initial wealth are independent of the nature of the claim $B$. The above calculations indicate that, given a loss of utility ($\varepsilon > 0$), a seller with greater initial wealth asks for lower prices, while given a gain of utility ($\varepsilon < 0$), she asks for higher prices. Similar results can be drawn for the buyer.

For the case of exponential utility, we can solve the problem (18) explicitly.

**Proposition 4.1.** The risk sharing price is given by

$$
P^* = P^{(s)}(\varepsilon_s^*) = P^{(b)}(\varepsilon_b^*) = \frac{\gamma_s v^{(s)}(B) + \gamma_b v^{(b)}(B)}{\gamma_s + \gamma_b} + \frac{1}{\gamma_s + \gamma_b} \ln \left( \frac{u_s(x_s) \gamma_s \lambda}{u_b(x_b) \gamma_b (1 - \lambda)} \right)
$$

**Proof.** Consider again the Lagrangian

$$
L(\varepsilon_s, \varepsilon_b, m) = \lambda \varepsilon_s + (1 - \lambda) \varepsilon_b + m \left( P^{(s)}(\varepsilon_s) - P^{(b)}(\varepsilon_b) \right)
$$

where, $(\varepsilon_s, \varepsilon_b) \in A_s \times A_b$ and $m \in \mathbb{R}_+$. Using (38), the first order conditions yield

$$
\varepsilon_s^*(m) = u_s(x_s) + \frac{m}{\lambda \gamma_s} \quad \text{and} \quad \varepsilon_b^*(m) = u_b(x_b) + \frac{m}{(1 - \lambda) \gamma_b}.
$$

The value of the Lagrange multiplier $m^*$ such that the constraint is satisfied is given by the solution of the algebraic equation

$$
P^{(s)}(\varepsilon_s^*(m^*)) = P^{(b)}(\varepsilon_b^*(m^*))$$
which using \( \text{(35)} \) and \( \text{(37)} \) readily gives

\[
m^* = (-u_s(x_s)\lambda \gamma_s \gamma_b)^{\gamma_b/(\gamma_s+\gamma_b)} (-u_b(x_b)(1-\lambda)\gamma_b)^{\gamma_s/(\gamma_s+\gamma_b)} \exp \left( \frac{\gamma_s \gamma_b}{\gamma_s + \gamma_b} (v^{(s)}(B) - v^{(b)}(B)) \right). \]

Therefore, the risk sharing price is

\[
P^* = P^{(s)}(\varepsilon_s^*) = P^{(b)}(\varepsilon_b^*) = \frac{\gamma_s v^{(s)}(B) + \gamma_b v^{(b)}(B)}{\gamma_s + \gamma_b} + \frac{1}{\gamma_s + \gamma_b} \ln \left( \frac{u_s(x_s) \gamma_s \lambda}{u_b(x_b) \gamma_b (1-\lambda)} \right) \tag{40} \]

Since the indirect utilities have the necessary smoothness, we can conclude that the critical point is a minimum if the determinant of the matrix

\[
H = \begin{pmatrix}
\frac{\partial^2 L}{\partial \varepsilon_s \partial \varepsilon_b} (\varepsilon_s^*, \varepsilon_b^*) & \frac{\partial^2 L}{\partial \varepsilon_s \partial \varepsilon_b} (\varepsilon_s^*, \varepsilon_b^*) & \frac{\partial \phi}{\partial \varepsilon_s} (\varepsilon_s^*, \varepsilon_b^*) \\
\frac{\partial^2 L}{\partial \varepsilon_s \partial \varepsilon_b} (\varepsilon_s^*, \varepsilon_b^*) & \frac{\partial^2 L}{\partial \varepsilon_s \partial \varepsilon_b} (\varepsilon_s^*, \varepsilon_b^*) & \frac{\partial \phi}{\partial \varepsilon_b} (\varepsilon_s^*, \varepsilon_b^*) \\
\frac{\partial \phi}{\partial \varepsilon_s} (\varepsilon_s^*, \varepsilon_b^*) & \frac{\partial \phi}{\partial \varepsilon_b} (\varepsilon_s^*, \varepsilon_b^*) & 0
\end{pmatrix}
\]

where \( \phi(\varepsilon_s, \varepsilon_b) := P^{(s)}(\varepsilon_s) - P^{(b)}(\varepsilon_b) \) is negative. Indeed,

\[
\|H\| = -P^{(s)}''(\varepsilon_s^*) \left( \frac{1-\lambda}{m^*} \right)^2 + \frac{\lambda}{m^*} P^{(b)}''(\varepsilon_b^*) < 0,
\]

since \( P^{(s)} \) is strictly convex and \( P^{(b)} \) is strictly concave.

\[\square\]

**Remark 4.2.** In the special case where the agents have the same characteristics \( x_s = x_b = x \) and \( \gamma_s = \gamma_b = \gamma \), the risk sharing price simplifies to

\[
P^* = \frac{v^{(s)}(B) + v^{(b)}(B)}{2} + \frac{1}{2\gamma} \ln \left( \frac{\lambda}{1-\lambda} \right). \tag{41}
\]

Therefore, the proposed price is the midpoint between the two agents' indifference prices plus a correction term reflecting the risk aversion and the risk sharing rule \( \lambda \). This correction becomes zero if and only if \( \lambda = 1/2 \), i.e., if the risk is equally shared among the agents.

The difference between the losses of indirect utilities that lead to the price \( P^* \) is given by

\[
\varepsilon_s^* - \varepsilon_b^* = \sqrt{u(x;B)u(x;-B)} \left( \frac{1-2\lambda}{\sqrt{\lambda}(1-\lambda)} \right)
\]

This difference is positive for \( \lambda < 1/2 \), negative for \( \lambda > 1/2 \) and zero if \( \lambda = 1/2 \).

**Remark 4.3.** Formula \( \text{(40)} \), can easily reveal the sensitivity of the risk sharing price on the risk aversion coefficients of the agents. Namely, \( P^* \) is an increasing function of \( \gamma_s \) and a decreasing function of \( \gamma_b \), reflecting the fact that as the seller becomes less risk averse, she is willing to ask for lower prices while the opposite happens for the bid prices of the buyer. It is also clear from \( \text{(40)} \) that \( P^* \) decreases as \( x_s \) increases and \( P^* \) increases as \( x_b \) increases.

In the special case where they have the same risk aversion coefficient, i.e., \( \gamma_s = \gamma_b = \gamma \), \( P^* \) is an increasing function of \( \gamma \) when \( \lambda < 1/2 \) and decreasing when \( \lambda > 1/2 \). For \( \lambda = 1/2 \) and for \( \gamma_s = \gamma_b = \gamma \) we have that \( P^* = \frac{v^{(s)}(B)+v^{(b)}(B)}{2} + \frac{x_s-x_b}{2} \), which generally is not a monotonic function.
of \( \gamma \). An interesting limit is the limit as \( \gamma \to 0 \): 
\[
\lim_{\gamma \to 0} P^* = \mathbb{E}_{Q(0)}[B] + \frac{x_s - x_b}{2},
\]
where \( Q(0) \) is the measure that minimizes the relative entropy \( H(Q|P) \) (see [12] on the behavior of \( v(s)(B) \) and \( v(b)(B) \) as \( \gamma \) goes to zero).

**Remark 4.4.** Explicit bounds on \( \lambda \) can be provided such that the risk sharing price is a non-arbitrage price; Using equation (40), one can see through straightforward calculations that 
\[
b(\inf_{Q \in \mathcal{M}} \{\mathbb{E}_Q[B]\}) < \lambda < b(\sup_{Q \in \mathcal{M}} \{\mathbb{E}_Q[B]\}),
\]
where 
\[
b(a) = \frac{K(a)}{K(a) + 1},
\]
with 
\[
K(a) = e^{a(\gamma_s + \gamma_b)} MB(x_s, x_b, \gamma_s, \gamma_b) \quad \text{and} \quad MB(x_s, x_b, \gamma_s, \gamma_b) = e^{(\gamma_s v(s)(B) - x_s) - \gamma_b (v(b)(B) + x_b)} \frac{\gamma_s u_s(0)}{\gamma_s u_s(0)}.
\]
These explicit bounds allow us to draw some interesting conclusions. For example, by keeping the rest of the parameters constant, we get that as \( \gamma_s \) (\( \gamma_b \)) goes to zero, the function \( \frac{K(a)}{K(a) + 1} \) goes to one (zero), for every \( a \in \mathbb{R} \), i.e., as an agent reduces her aversion level, her loss of utility weight approaches one.

### 4.2. Example: A European Claim on a Non-traded Asset.

In this Section, we build on the model proposed in [35], concerning the utility pricing of a European claim written on a non-traded asset. This model considers two traded assets, a riskless one (the numéraire) and a risky one, whose discounted price \( S_t \) follows the dynamics
\[
dS_t = \mu S_t dt + \sigma S_t dW_{t}^{(1)}
\]
where \( W_{t}^{(1)} \) is a standard Wiener process, \( \mu \in \mathbb{R} \) and \( \sigma \in \mathbb{R}_+^* \). It further assumes the existence of a non-traded asset, whose discounted price dynamics follow the stochastic differential equation
\[
dY_t = b(Y_t, t) dt + a(Y_t, t) \left( \rho dW_{t}^{(1)} + \rho' dW_{t}^{(2)} \right)
\]
where \( W_{t}^{(2)} \) is another standard Wiener process (independent of \( W_{t}^{(1)} \)). The constant \( \rho \in (-1, 1) \) is the correlation coefficient between the factors driving the dynamics of the prices of the traded and the non-traded asset.

Consider now a European contingent claim \( B \), whose payoff depends on the value \( Y_T \) of the non-traded asset at time \( T \), through \( g(Y_T) \), where \( g : \mathbb{R} \to \mathbb{R} \) is a bounded Borel function, so that \( B \in \mathbb{L}^\infty \). In [35], expressions for indirect utilities and the bid and ask indifference prices in the case of exponential utilities are stated. In general, these prices do not lead to an agreement concerning a commonly acceptable price at which the option can be traded. We now propose an alternative approach to this problem using the definition of the risk sharing price as extended in Remark 3.4. The following theorem, which quotes the relevant results of [35] concerning the indirect utilities achieved, needed in this work, is included to enhance the readability of the present paper.
Theorem 4.1. (M. Musiela and T. Zariphopoulou, 2004)
Let $\mathbb{Q}^{(0)}$ be the minimal relative entropy martingale probability measure and

$$
\tilde{\mathcal{X}}_{\exp}(x,t) := \{ x + \int_t^T \vartheta_u dS_u : \text{for } (\vartheta_u)_{u \in [t,T]} \text{ such that } \mathbb{E} \left[ \int_t^T \vartheta_u^2 d\mathbb{Q}_u \right] < \infty \}.
$$

Then,

(i) The indirect utility of an agent with no position on the contingent claim is

$$
u(x,t) = \sup_{\mathcal{X} \in \tilde{\mathcal{X}}_{\exp}(x,t)} \mathbb{E} \left[ -e^{-\gamma X} \bigg| \mathcal{F}_t \right] = -e^{-\gamma x} e^{-\frac{g(T-t)}{2\sigma^2}}$$

(ii) The indirect utility for the seller, $u(x,y,t; -g(Y_T))$, is given by

$$u(x,y,t; -g(Y_T)) = -e^{-\gamma x} \left( \mathbb{E}_{\mathbb{Q}^{(0)}} \left[ e^{\gamma x (1-\rho^2) g(Y_T)} e^{-\frac{(1-\rho^2)\sigma^2 (T-t)}{2\sigma^2}} \big| Y_t = y, \mathcal{F}_t \right] \right)^{\frac{1}{1-\rho^2}}.$$ 

(iii) The indirect utility for the buyer, $u(x,y,t; g(Y_T))$, is given by

$$u(x,y,t; g(Y_T)) = -e^{-\gamma x} \left( \mathbb{E}_{\mathbb{Q}^{(0)}} \left[ e^{-\gamma b (1-\rho^2) g(Y_T)} e^{-\frac{(1-\rho^2)\sigma^2 (T-t)}{2\sigma^2}} \big| Y_t = y, \mathcal{F}_t \right] \right)^{\frac{1}{1-\rho^2}}.$$ 

For details on the measure $\mathbb{Q}^{(0)}$, we refer the interested reader to Theorem 2 of [35].

We are now able to characterize the price functions $P^{(s)}$ and $P^{(b)}$ and the risk sharing price at time $t$.

Proposition 4.2.

(i) The seller’s price $P_t^{(s)}(\varepsilon)$ is given by

$$P_t^{(s)}(\varepsilon) = v_t^{(s)}(B) - \frac{1}{\gamma_s} \ln \left( 1 + \varepsilon \delta_t^{(s)} \right)$$

where

$$v_t^{(s)}(B) = \frac{1}{(1-\rho^2)\gamma_s} \ln \left( \mathbb{E}_{\mathbb{Q}^{(0)}} \left[ e^{\gamma_s (1-\rho^2) g(Y_T)} \big| Y_t = y, \mathcal{F}_t \right] \right)$$

is the seller’s indifference price at time $t$ and $\delta_t^{(s)} := e^{\gamma_s x_s + \sigma^2 (T-t)}$.

(ii) The buyer’s price $P_t^{(b)}(\varepsilon)$ is given by

$$P_t^{(b)}(\varepsilon) = v_t^{(b)}(B) + \frac{1}{\gamma_b} \ln \left( 1 + \varepsilon \delta_t^{(b)} \right)$$

where

$$v_t^{(b)}(B) = -\frac{1}{(1-\rho^2)\gamma_b} \ln \left( \mathbb{E}_{\mathbb{Q}^{(0)}} \left[ e^{-\gamma_b (1-\rho^2) g(Y_T)} \big| Y_t = y, \mathcal{F}_t \right] \right)$$

is the buyer’s indifference price at time $t$ and $\delta_t^{(b)} := e^{-\gamma_b x_b + \sigma^2 (T-t)}$.

(iii) The risk sharing price $P_t^*$ for any time $t \in [0,T]$, is given by

$$P_t^* = \frac{\gamma_s v_t^{(s)}(B) + \gamma_b v_t^{(b)}(B)}{\gamma_s + \gamma_b} + \frac{1}{\gamma_s + \gamma_b} \ln \left( \frac{\gamma_s \lambda \delta_t^{(b)}}{\gamma_b (1-\lambda) \delta_t^{(b)}} \right) \tag{42}$$
Proof. Using (14) and (15) and Theorem 4.1, the proof of (i) and (ii) is straight-forward. Taking into account Theorem 4.1 and the formula (40), we obtain the solution of the dynamic version of problem (34) as stated in (iii). □

Remark 4.5. The function $P_t^s(\varepsilon)$ can be considered as a function of time $t$ and the market condition at this time, $y = Y_t$, and as such it can be shown to be the solution of a deterministic quasilinear PDE. For fixed $\varepsilon$, define

$$P_t^s(\varepsilon) := P^s(y, t) := \frac{1}{\gamma_s (1 - \rho^2)} \ln \Phi^s(y, t).$$

Using the Feynman-Kac representation one may show by extension of the arguments in [35], that $\Phi^s(y, t)$ is the solution of the linear backward Cauchy problem

$$\frac{\partial \Phi^s(y, t)}{\partial t} + \frac{1}{2} a(y, t)^2 \frac{\partial^2 \Phi^s(y, t)}{\partial y^2} + \left( b(y, t) - \frac{\rho \mu}{\sigma} a(y, t) \right) \frac{\partial \Phi^s(y, t)}{\partial y} + \frac{R^s(t)}{R^s}\Phi^s = 0$$

where

$$\Phi^s(y, T) := \frac{e^{\gamma_s (1 - \rho^2) y}}{(1 + \varepsilon e^{\gamma_s x_s}) (1 - \rho^2)},$$

and

$$R^s(t) := (1 + \varepsilon \delta_t^s)^{(1 - \rho^2)}.$$ 

By straight-forward algebraic manipulation, one may show that the price is the solution of a quasilinear deterministic PDE of the form

$$\frac{\partial P^s(y, t)}{\partial t} + \frac{1}{2} a(y, t)^2 \frac{\partial^2 P^s(y, t)}{\partial y^2} + \left( b(y, t) - \frac{\rho \mu}{\sigma} a(y, t) \right) \frac{\partial P^s(y, t)}{\partial y} + \frac{1}{2} \gamma_s (1 - \rho^2) a(y, t)^2 \left( \frac{\partial P^s(y, t)}{\partial y} \right)^2 + \Lambda^s(t) = 0$$

with final condition

$$P^s(y, T) = g(y) - \frac{1}{\gamma_s} \ln (1 + \varepsilon e^{\gamma_s x_s})$$

where

$$\Lambda^s(t) := \frac{1}{\gamma_s} \ln (1 + \varepsilon \delta_t^s).$$

The indifference prices are recovered when setting $R^s(t) = 0$ and $\Lambda^s(t) = 0$ for all $t \in [0, T]$ in the above PDEs. Similar arguments give a quasilinear PDE for the evolution of the buyer’s price after undertaking some risk. However, even though the risk sharing price is a linear combination of the indifference prices plus the addition of a known factor depending only on time, the quasilinear nature of the PDEs that the indifference prices satisfy do not allow us to write down a single PDE that the risk sharing price as a function of $Y_t = y$ and $t$, will satisfy.
5. Extensions, Directions for future research and Conclusion

In this Section, we discuss some possible extensions of the pricing scheme proposed in this paper and conclude.

5.1. Risk Sharing Price Using Risk Measures. Consider that the agents model their risk preferences using convex risk measures $\rho_s$ and $\rho_b$ respectively, rather than expected utility functions. Since they have access to a liquid market, their investment goal is to minimize their risk, as quantified by the risk measures employed, by trading into this market. More precisely, for a contingent claim $B$, the agents’ marketed risk measures (as defined in [18], [48], [20]) are given by

$$\hat{\rho}_s(x_s; B) := \inf_{X \in \mathcal{X}(x_s)} \rho_s(X + B)$$

and

$$\hat{\rho}_b(x_b; B) := \inf_{X \in \mathcal{X}(x_b)} \rho_b(X + B).$$

Along the lines of equations (10) and (11), we can define the analogous of the reservation prices $P(s)$ and $P(b)$, when the risk is represented by $\hat{\rho}_s$ and $\hat{\rho}_b$, as the solutions of the following equations

$$\epsilon_s = -\hat{\rho}_s(x_s; 0) + \hat{\rho}_s(x_s + P(s)(\epsilon_s); -B)$$

and

$$\epsilon_b = -\hat{\rho}_b(x_b; 0) + \hat{\rho}_b(x_b - P(b)(\epsilon_b); B).$$

Using these definitions and similar arguments as in Section 3, we define a pricing scheme as dictated by the solution of the minimization problem (18). Clearly, the convexity of the risk measures guarantees that problem (18) is well-posed. Notice however, that if the marketed risk measures are cash invariant, the above general arguments may break down since then problem (18) is equivalent to the minimization of

$$\lambda \epsilon_s + (1 - \lambda) \epsilon_b = c(B) + P(s)(\epsilon_s)(2\lambda - 1)$$

under the constraint $P(s)(\epsilon_s) = P(b)(\epsilon_b)$, where $c(B) := -\lambda(\hat{\rho}_s(0; 0) - \hat{\rho}_s(0; -B)) - (1 - \lambda)(\hat{\rho}_b(0; 0) - \hat{\rho}_b(0; B))$ which is ill-posed.

However, this is not a problem, since even though the risk measures $\rho_s$ and $\rho_b$ are by definition cash invariant, the corresponding property for the marketed risk measures requires that $\mathcal{X}(x + y) = \mathcal{X}(x) + y$, for every possible initial wealths $x$ and $y$. This clearly does not hold in most cases of some practical interest, as for instance when borrowing constraints are imposed. Therefore in general, minimization of $\lambda \epsilon_s + (1 - \lambda) \epsilon_b$ is a well-defined problem leading to meaningful risk sharing prices.

5.2. Optimal Trading Time. The family of risk sharing prices for the asset at time $t$ as given by equation (42) is clearly a stochastic process, on account of the stochastic nature of $Y_t$. One may at the cost of considerable algebraic manipulations, apply Itô’s rule and determine the evolution of the process $P^*_t$ as a stochastic differential equation. For every time $t \in [0, T]$ the price $P^*_t$, as given by equation (42), is the price corresponding to the minimal total risk that the two agents have to undertake, at any time $t$, so that the transaction may take place. However, this total risk is a stochastic process itself since it depends on the market conditions at time $t$. We may then consider the stochastic process $\epsilon^*_t := \lambda \epsilon^*_{s,t} + (1 - \lambda) \epsilon^*_{b,t} = \mathcal{E}(t, Y_t)$ which is the minimal total risk allowing
the transaction at time $t$. The function $E$ is a deterministic function the form of which is known explicitly (see Proposition 4.1). In principle, a straightforward application of Itô’s lemma gives us the stochastic differential equation this process satisfies. One may now consider the following problem: Suppose that we can find a stopping time $\tau \in [0, T]$ such that the expectation of the total risk $\varepsilon^\tau$ is minimized, over all such stopping times. When choosing to trade the contingent claim at this time, clearly both agents undertake the minimum possible total expected risk. We may then define this time $\tau$ as the *optimal trading time* for the contract, and the price $P^\tau$ is then the optimal trading price. The resolution of this problem can be based on optimal stopping techniques, using variational inequalities, the solution of which will help us to determine the optimal stopping rule.

5.3. Different Forms of the Total Risk. The expression of the total risk undertaken by the agents, can be generalized to $\lambda \psi_s(\varepsilon_s) + (1 - \lambda) \psi_b(\varepsilon_b)$, where $\lambda \in (0, 1)$ and $\psi_s(\cdot), \psi_b(\cdot)$ are strictly convex increasing functions. We then may consider the problem

$$\min_{(\varepsilon_s, \varepsilon_b) \in A_s \times A_b} \{ \lambda \psi_s(\varepsilon_s) + (1 - \lambda) \psi_b(\varepsilon_b) \}$$

subject to $P^{(s)}(\varepsilon_s) \leq P^{(b)}(\varepsilon_b)$  \hspace{1cm} (44)

In the special case where $\psi_s(\varepsilon) = \psi_b(\varepsilon) = \varepsilon$, for every $\varepsilon$, problem (44) reduces to problem (18).

Through the solution of problem (44), we can define a risk sharing price. The solvability of (44) follows along the same lines of the proof of Theorem 3.1.

5.4. Conclusion. In this work, we considered two agents, a seller and a buyer, who are interested in trading a given, non-divisible, non-replicable contingent claim. Assuming that they are determined to complete the transaction, they need to agree on a commonly accepted price, out of the infinity of possible non-arbitrage prices, at which the transaction will take place. The risk preferences of the agents indicate that some risk may have to be undertaken if a common price is to be reached. We proposed a pricing mechanism according to which the unique common price is determined by the minimization of the total risk undertaken according to a fixed sharing rule, under the constraint that the transaction is feasible. We proved that such a problem is well-posed and we illustrated in detail this pricing mechanism in the special case of the exponential utility.

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Appendix A. Utility Functions on the Positive Real Line

In this Appendix, we provide the necessary technical details that guarantee the validity of our results in the case where one or both agents has utility function defined on the positive real line.

A.1. Assumptions on the Utility Functions. The second type of utilities includes the functions that are defined only for positive wealth, i.e.,

$$ U : \mathbb{R}_+ \rightarrow \mathbb{R} $$

where we set $U(x) = -\infty$ for every $x < 0$. $U$ is taken to be strictly increasing, strictly concave and continuously differentiable on $\mathbb{R}_+$ and to satisfy the Inada conditions:

$$ \lim_{x \rightarrow 0} U'(x) := U'(0) = +\infty \quad \text{and} \quad \lim_{x \rightarrow +\infty} U'(x) := U'(+\infty) = 0. \quad (45) $$

Moreover, we assume reasonable asymptotic elasticity for $U$, meaning that

$$ \limsup_{x \rightarrow +\infty} \frac{xU'(x)}{U(x)} < 1 \quad (46) $$

(see [30], [22] and [9]). Again, $U(+\infty)$ stands for $\lim_{x \rightarrow +\infty} U(x)$.

A.2. Admissible Strategies and Initial Weights. For utilities that are defined only for positive wealths, the properties of indirect utility have been given in the seminal works of D. Kramkov and W. Schachermayer, [30], for the case of no random endowment (i.e., no random liability $B$) and J. Cvitanić, W. Schachermayer and H. Wang, [9], under the presence of random endowment. In this case, for every initial wealth $x \in \mathbb{R}_+$, the set of admissible wealths $X(x)$ is

$$ X(x) = \{ X \in L^0(\Omega, \mathcal{F}_T, \mathbb{P}) : X \leq (\vartheta \cdot S)_T + x \text{ for some } \vartheta \in \Theta \} \quad (47) $$

where $\Theta$ is the set of admissible strategies given in Section 2. We also need to define the set

$$ D := \{ Q \in (L^\infty)^\ast : \|Q\| = 1 \text{ and } \langle Q, X \rangle \leq 0, \forall X \in C \} \quad (48) $$

where $C = \Theta \cap L^\infty$ and $\langle ., . \rangle$ is the duality pairing of $L^\infty$ and the space of finitely additive measures, $(L^\infty)^\ast$. Finally, given any claim $B \in L^\infty$, we set $x_0(B) := \sup_{Q \in D} \langle Q, B \rangle$ (notice that in [13], Theorem 5.6, it is stated that $\mathcal{M}_e \subseteq D$).

The following theorem, quoted from [9] (Theorem 3.1 and Lemma 4.3) and [30] (Theorem 2.2) gives the properties which are needed in the present work.

**Theorem A.1.** \textit{(K. Kramkov and W. Schachermayer, 1999, J. Cvitanić, W. Schachermayer and H. Wang, 2001)}

Assume that:

(i) $B \in L^\infty$.

(ii) Assumption 2.1 holds (see Section 2).
(iii) There exists \( y \in \mathbb{R}_+ \) such that \( u(y; B) < U(\infty) \) (where \( u(\cdot, B) \) is defined \([5]\)).

Then, there exists a unique optimal solution in the problem \([6]\) for every \( x > x_0(-B) \). Furthermore, the value function \( u(\cdot; B) \) is finitely valued, strictly increasing and strictly concave and continuously differentiable on \((x_0(-B), +\infty)\), \( u(x; B) = -\infty \) for every \( x < x_0(-B) \) with \( \lim_{x \to x_0(-B)} u(x; B) = -\infty \).

Finally,

\[
\lim_{x \to +\infty} u'(x; B) = 0 \quad \text{and} \quad \lim_{x \to x_0(-B)} u'(x; B) = \infty. \tag{49}
\]

**Remark A.1.** Since, \( u(x; B) \) is a strictly concave \( C^1 \) \((x_0(-B), +\infty)\) function of \( x \), we obtain that

\[
\frac{du(x; B)}{dx} \quad \text{is strictly decreasing for every} \quad x > x_0(-B) \quad \text{and its range is} \quad (0, +\infty).
\]

We also observe that \( \lim_{x \to +\infty} u(x; B) = U(+\infty) \).

**A.3. Utilities of Both Types and the Risk Sharing Price.** Taking into account Theorem \([A.1]\) and the intervals \([16]\) and \([17]\), in the case of utility functions of this type, we impose the following assumption on the agents’ initial wealths, in order to exclude infinite valued indirect utilities and to guarantee that the intersections of the images of the pricing rules \( P(s) \) and \( P(b) \) and the non-arbitrage prices for the fixed claim \( B \) are not empty.

**Assumption A.1.** If the seller has utility of the second type, we assume that her initial wealth \( x_s \in \mathbb{R}_+ \) satisfies the following inequality

\[
x_s > \max\{x_0(B), x_0(B) - \sup_{Q \in \mathcal{M}_e} \{\mathbb{E}_Q[B]\}\} \tag{50}
\]

Similarly, if the buyer has utility of the second type, we assume that her initial wealth \( x_b \in \mathbb{R}_+ \) satisfies the following inequality

\[
x_b > \max\{x_0(-B), x_0(-B) + \inf_{Q \in \mathcal{M}_e} \{\mathbb{E}_Q[B]\}\} \tag{51}
\]

Under the above assumptions, the pricing scheme induced by the problem \([18]\) is valid even in the case where one or both of the agents have utility function of the second type. More precisely, we define the functions \( P(s) \) and \( P(b) \) in exactly the same way as in \([14]\) and \([15]\) with the difference that

\[
P(s)(A_{x_s}) = (x_0(B) - x_s, +\infty) \quad \text{and} \quad P(b)(A_{x_b}) = (-\infty, x_b - x_0(-B))
\]

and also that \( \lim_{\varepsilon \to +\infty} P(s)(\varepsilon) = x_0(B) - x_s \) and \( \lim_{\varepsilon \to +\infty} P(b)(\varepsilon) = x_b - x_0(-B) \).

Without repeating the proof of Theorem \([3.1]\) we state its modification in this slightly generalized case.

**Theorem A.2.** Let Assumptions \([2.1]\) and \([A.1]\) hold. For any choice of utilities \( U_s, U_b \) (of either type) and for any weight \( \lambda \in (0, 1) \), the minimization problem \([18]\) has a unique solution \((\varepsilon^*_s, \varepsilon^*_b) \in A_s \times A_b\). This provides a unique price \( P^* = P(s)(\varepsilon^*_s) = P(b)(\varepsilon^*_b) \).
Remark A.2. It should be mentioned that the only difference between the proof of Theorem 3.1 and Theorem A.2 is that in the case that agents choose utility function of the second type the limits that guarantee the existence of the Lagrange multiplier $m^*$ have to be modified to
\[
\lim_{m \to 0} P^s(\varepsilon^*_s(m)) = + \infty \quad \text{and} \quad \lim_{m \to 0} P^b(\varepsilon^*_b(m)) = - \infty
\]
\[
\lim_{m \to +\infty} P^s(\varepsilon^*_s(m)) = x_0(B) - x_s \quad \text{and} \quad \lim_{m \to +\infty} P^b(\varepsilon^*_b(m)) = x_b - x_0(-B)
\]
(compare to limits given in [27]).

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