Nonparametric Independence Testing for Right-Censored Data using Optimal Transport

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1 Abstract

We propose a nonparametric test of independence, termed OPT-HSIC, between a covariate and a right-censored lifetime. Because the presence of censoring creates a challenge in applying the standard permutation-based testing approaches, we use optimal transport to transform the censored dataset into an uncensored one, while preserving the relevant dependencies. We then apply a permutation test using the kernel-based dependence measure as a statistic to the transformed dataset. The type 1 error is proven to be correct in the case where censoring is independent of the covariate. Experiments indicate that OPT-HSIC has power against a much wider class of alternatives than Cox proportional hazards regression and that it has the correct type 1 control even in the challenging cases where censoring strongly depends on the covariate.

2 Introduction

This paper proposes a nonparametric test of independence between a covariate and a right-censored lifetime. Although our methods may be extended to higher dimensions, we consider the case in which the covariate is real-valued. There are several existing approaches to this challenge. Firstly, one might cluster the continuous covariate into groups, and then test for equality of lifetime distributions among the groups. The results will depend on the arbitrary choice of boundaries between the groups, and the range of covariates within groups reduces power. Alternatively, one might fit a (semi-)parametric regression model, and test whether the regression coefficient corresponding to the covariate differs significantly from zero. The most commonly used such method is the Cox proportional hazards (CPH) model, which makes two assumptions (Cox (1972)): First, the hazard function must factorize into a function of time and a function of the covariate (the proportional hazards or relative risk condition); second, the effect of a covariate on the logarithm of the hazard
function must be linear. Although this is a flexible model, it can have limited power when the data generation deviates from these model assumptions.

Since distance- and kernel-based approaches have been used successfully for independence testing on uncensored data (Gretton, Fukumizu, Teo, Song, Schölkopf, and Smola (2008), Szekely and Rizzo (2009)), it is natural to investigate whether these methods can be extended to the case of right-censored lifetimes. To this end propose applying optimal transport to transform the censored dataset into an uncensored dataset in such a way that, 1) the new uncensored dataset preserves the dependencies of the original dataset, and 2) we can apply a standard permutation test to the new dataset with test statistic given by Distance Covariance (DCOV) (Szekely and Rizzo (2009)) or, equivalently, the Hilbert–Schmidt Independence Criterion (HSIC) (Gretton et al. (2008)). While our exposition can straightforwardly be applied to general covariates, and the approach can model dependencies between event times and random elements in generic domains following the framework of DCOV/HSIC, we will for clarity present the case of one-dimensional covariates.

Section 3 provides a brief introduction to survival analysis and distance- and kernel-based independence testing. Section 4 proposes an optimal transport based transformation of the data, and a testing procedure named OPT-HSIC. Although we have not yet been able to prove control of the type 1 error in full generality, we do show the type 1 error to be correct in the case where censoring is independent of the covariate. Furthermore we obtain very promising results in simulation studies, showing correct type 1 control even under censoring that depends strongly on the covariate. Section 5 explores some alternative kernel-based approaches under the additional assumption that censoring is independent of the covariate. Sections 6 compares the power and type 1 error of all tests in simulated data and compares them with CPH regression. Section 7 discusses the special case of binary covariates $X$, where testing dependence between $X$ and $T$ is equivalent to two-sample testing.

3 Background Material

3.1 Right-Censored Lifetimes

Let $T \in \mathbb{R}_{\geq 0}$ be a lifetime subject to right-censoring, so that we do not observe $T$ directly, but instead observe $Z := \min\{T, C\}$ for some censoring time $C \in \mathbb{R}_{\geq 0}$, as well as the indicator $\Delta := 1\{C > T\}$ and a covariate vector $X \in \mathbb{R}^d$, where $\mathbb{R}^d$ is equipped with the Borel sigma algebra. In total we thus observe $D = \{(x_i, z_i, \delta_i)\}_{i=1}^n$. When we say ‘the $i$–th observed event’, we refer to the $i$–th event, in order of time, for which $\delta = 1$. Denote $F_{T|X}(t|x) = P(T \leq t|X = x)$ and $F_{C|X}(t|c) = P(C \leq t|X =$
x). The $X_i$ can be treated either as random or as fixed. If they are random, we analyze the outcomes conditioned on an arbitrary realization of $X$.

Throughout this paper we will make the following assumptions.

**Assumption 1:** Conditional on $\{X_i\}_{i=1}^n$, the random variables $\{(T_i, C_i)\}_{i=1}^n$ are mutually independent.

**Assumption 2:** We assume that $C \perp 
 T | X$.

Let $S(t) = P(T > t)$ and let $f(t)$ be the density of $T$. Then we define the hazard rate of an individual with covariate $x$ to be $\lambda(t | x) = \frac{f(t | X = x)}{S(t | X = x)}$. The Cox proportional hazards model assumes that the hazard rate can be written as $\lambda(t | x) = \lambda(t) \exp(\beta x)$. This model enables estimation of $\beta$ and testing the significance of $\beta$’s difference from zero. The CPH model is by far the most commonly used regression methods in survival analysis. A last important concept is that of the at risk set. We denote the covariates at risk by the time of the $i$–th observed event by $AR_i$. Technically $AR_i$ is a multiset, as different individuals may have the same covariate. The setminus operation removes one instance of the covariate.

### 3.2 Independence testing using kernels

In recent years kernel methods have been successfully used for nonparametric independence- and two-sample testing (Gretton et al. (2008), Gretton, Borgwardt, Rasch, Schölkopf, and Smola (2012)). We now give some of the relevant background in kernel methods.

**Definition 3.1. (Reproducing Kernel Hilbert Space)** (B. Schölkopf (2001)) Let $\mathcal{X}$ be a non-empty set and $\mathcal{H}$ a Hilbert space of functions $f : \mathcal{X} \rightarrow \mathbb{R}$. Then $\mathcal{H}$ is called a reproducing kernel Hilbert (RKHS) space endowed with dot product $\langle \cdot, \cdot \rangle$ if there exists a function $k : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$ with the following properties.

1. $k$ has the reproducing property

$$\langle f, k(x, \cdot) \rangle = f(x) \text{ for all } f \in \mathcal{H}, x \in \mathcal{X}. \quad (1)$$

2. $k$ spans $\mathcal{H}$, that is, $\mathcal{H} = \text{span}\{k(x, \cdot) | x \in \mathcal{X}\}$ where the bar denotes the completion of the space.

Let $\mathcal{X}$ together with a sigma-algebra be a measurable space and let $H_{\mathcal{X}}$ be an RKHS on $\mathcal{X}$. Let $P$ be a probability measure on $\mathcal{X}$. If $E_P \sqrt{k(X, X)} < \infty$, ...
then there exists an element $\mu_P \in H_\mathcal{X}$ such that $E_P f(X) = \langle f, \mu_P \rangle$ for all $f \in H_\mathcal{X}$ (Gretton et al. (2012)), where we used the notation $E_P f(X) := \int f(x) P(dx)$. The element $\mu_P$ is called the mean embedding of $P$ in $H_\mathcal{X}$. Given a second distribution $Q$ on $\mathcal{X}$ of which a mean embedding exists, we can measure the similarity of $P$ and $Q$ by the distance between their mean embeddings in $H_\mathcal{X}$. The distance between their mean embeddings

$$\text{MMD}(P, Q) := ||\mu_P - \mu_Q||_{H_\mathcal{X}}$$

is also called the Maximum Mean Discrepancy (MMD). The name comes from the following equality (Gretton et al. (2012)),

$$||\mu_P - \mu_Q||_{H_\mathcal{X}} = \sup_{f \in H_\mathcal{X}} E_P f(X) - E_Q f(X)$$

showing that MMD is an integral probability metric. Given a sample $\{x_i\}_{i=1}^n$ and the corresponding empirical distribution, $\sum_{i=1}^n \delta_{x_i}$, the corresponding mean embedding is given by $\frac{1}{n} \sum_{i=1}^n k(x_i, \cdot)$.

Suppose now that $\mathcal{Y}$, equipped with some sigma algebra, is a second measurable space, and let $H_\mathcal{Y}$ be an RKHS on $\mathcal{Y}$ with kernel $l$. Let $X$ be a random variable in $\mathcal{X}$ with law $P_X$ and similarly let $Y$ be a random variable in $\mathcal{Y}$ with law $P_Y$. Finally let $P_{XY}$ denote the joint distribution on $\mathcal{X} \times \mathcal{Y}$ equipped with the product sigma-algebra. We let $H$ denote the RKHS on $\mathcal{X} \times \mathcal{Y}$ with kernel

$$K(((x, y), (x', y')), k(x, x') l(y, y').$$

In Gretton et al. (2008) it was proposed that the dependence of $X$ and $Y$ could be quantified by the following measure:

**Definition 3.2.** The Hilbert–Schmidt independence criterion (HSIC) of $X$ and $Y$ is defined by

$$\text{HSIC}(X, Y) := ||\mu_{P_{XY}} - \mu_{P_X P_Y}||_H^2$$

where $P_X P_Y$ denotes the product measure of $P_X$ and $P_Y$.

Using the reproducing property and the definition of mean embeddings, this can be shown to equal

$$\text{HSIC}(X, Y) = E_{XY} E_{X'Y'} k(X, X') l(Y, Y') + E_{XX'} k(X, X') E_{YY'} l(Y, Y') - 2 E_{XY} E_{XY'} k(X, X') l(Y, Y').$$
Definition 3.3. Let $(X', Y')$ be an independent copy of $(X, Y)$. Then

\[
\text{HSIC}(X, Y) = E_{XY}E_{X'Y'}k(X, X')l(Y, Y') + E_{XX'}k(X, X')E_{YY'}l(Y, Y') - 2E_{XY}E_{X'Y'}k(X, X')l(Y, Y').
\]

(2)

A biased estimator is obtained as follows:

\[
\widehat{\text{HSIC}}_b(X, Y) = \frac{1}{n^2} \sum_{i,j=1}^n k(x_i, x_j)l(y_i, y_j) + \frac{1}{n^2} \sum_{i,j=1}^n k(x_i, x_j) \frac{1}{n^2} \sum_{q,r=1}^n l(y_q, y_r)
- \frac{2}{n^3} \sum_{i,j,r=1}^n k(x_i, x_j)l(y_i, y_r).
\]

One can further show that when the RKHSs $H_X$ and $H_Y$ are rich enough function classes, $\text{HSIC}(X, Y) = 0$ if and only if $X \perp \!\!\!\!\!\!\!\!\!\!\! Y$ (Gretton et al. (2008)).

Now assume we are given a sample $D = \{(x_i, y_i)\}_{i=1}^n$. A biased estimate of HSIC can be obtained by measuring the distance between the embedding of the empirical distribution of the data and the embedding of the product of the marginal empirical distributions. That is,

\[
\text{HSIC}(D) := \left\| \frac{1}{n} \sum_{i=1}^n K((x_i, y_i), \cdot) - \frac{1}{n^2} \sum_{i,j=1}^n K((x_i, y_j), \cdot) \right\|_H^2.
\]

Using the reproducing property of the kernel and the definition of $K$ in terms of $k$ and $l$, $\text{HSIC}(D)$ can be shown to equal

\[
\text{HSIC}(D) = \frac{1}{n^2} \sum_{i,j=1}^n k(x_i, x_j)l(y_i, y_j) + \frac{1}{n^2} \sum_{i,j=1}^n k(x_i, x_j) \frac{1}{n^2} \sum_{q,r=1}^n l(y_q, y_r)
- \frac{2}{n^3} \sum_{i,j,r=1}^n k(x_i, x_j)l(y_i, y_r).
\]

The following algorithm shows how HSIC is commonly combined with a permutation test for independence testing.
Algorithm 1: Independence testing using HSIC and a permutation test

**Input**: Observed data \( D = \{(X_i, Y_i)\}_{i=1}^n \), significance level \( \alpha \)

1. Sample \( \pi_j, 1 \leq j \leq B \) distributed uniformly and independent from the symmetric group \( S_n \) of all permutations on \( n \) elements. Denote \( \pi(D) := \{(X_{\pi(i)}, Y_i)\}_{i=1}^n \).

2. Breaking ties at random, compute the rank \( R \) of HSIC \((D)\) in the vector

\[
(\text{HSIC}(D), \text{HSIC}(\pi_1(D)), \text{HSIC}(\pi_2(D)), \ldots, \text{HSIC}(\pi_B(D)))
\]

Reject if \( R \geq (1 - \alpha)(B + 1) + 1 \).

Throughout this paper we set \( \mathcal{X} = \mathcal{Y} = \mathbb{R} \) and choose the kernels to be

\[ k(x, x') = l(x, x') = (|x| + |x'| - |x - x'|), \]

(Sejdinovic, Sriperumbudur, Gretton, and Fukumizu (2012)). One can substitute this expression in the biased estimate of HSIC given above, to find that HSIC \((D)\) equals

\[
\text{HSIC}(D) = \frac{1}{n^2} \sum_{i,j=1}^n |x_i - x_j||y_i - y_j| + \frac{1}{n^2} \sum_{i,j=1}^n |x_i - x_j|\frac{1}{n} \sum_{q,r=1}^n |y_q - y_r| \\
- \frac{2}{n^3} \sum_{i,j,r=1}^n |x_i - x_j||y_i - y_r|.
\]

This quantity is also known as Distance Covariance (DCOV) (Székely and Rizzo (2009)). The equivalence of HSIC and DCOV was discussed in Sejdinovic et al. (2012).

4 OPT-HSIC

4.1 Objective of the algorithm

It is not obvious how to apply kernel based independence tests to right censored data. In this section we propose a transformation of the original, censored dataset, into a synthetic dataset consisting of \( n \) observed events. The algorithm uses optimal transport and its goal is twofold: first, it should return a dataset to which we can apply a permutation test with test-statistic HSIC, and obtain correct \( p \)-values under
the null hypothesis; second, it should return a dataset in which the dependence between $X$ and $T$ is similar to that in the original dataset. Indeed, the transformation of the data is of independent interest: we use the standard permutation test with test statistic HSIC/DCOV, but other statistics could be considered.

4.2 Optimal transport based transformation

Say we are given a fixed dataset $D = \{(x_i, z_i, \delta_i)\}_{i=1}^n$, and say there are $k$ observed ($\delta = 1$) events, with the data labelled so that $\delta_i = 1$ for $i = 1, \ldots, k$ and $z_1 \leq z_2 \leq \cdots \leq z_k$. Recall that we denote the covariates at risk at the $i$–th observed event time, $z_i$, by $AR_i$, for $i = 1, \ldots, k$. We will generate a synthetic dataset $\tilde{D}$, that will consist of the same $n$ covariates $x_1, \ldots, x_n$, each associated to some time $\tilde{t}_1, \ldots, \tilde{t}_n$. These times are related to the times in the original dataset. We define $L_i$ to be the covariates at risk in the synthetic dataset by time $z_i$, thus being analogous to $AR_i$, but for the synthetic dataset. Set $L_1 = (x_1, \ldots, x_n)$ and note that, since there is no censoring in the synthetic dataset, $L_i$ consists only of the covariates in the synthetic dataset that haven’t had an event by the $i$–th event time of the original data. The sets $L_i$ for $i > 1$ will thus be defined recursively.

The algorithm works as follows. Let $i = 1$, and let $X$ denote the random variable that chooses uniformly from $AR_i$. Let $Y$ denote the random variable that chooses uniformly from $L_i$. Now use optimal transport to define a coupling between $X$ and $Y$, so that the expected distance of $|Y - X|$ is minimal. Given the computed joint distribution of $X$ and $Y$, we sample

$$y_i \sim Y | X = x_i.$$ 

Having chosen the covariate $y_i$, we add $(y_i, z_i)$ to the synthetic dataset and set $L_{i+1} = L_i \setminus \{y_i\}$.

Having done so, we increment $i$ and repeat these steps up to and including $i = k$, at each step adding $(y_i, z_i)$ to the synthetic dataset. Once we have looped through all $k$ observed events in the original data, there are $n - k$ covariates that have not been associated to any time in the synthetic dataset. Let $z_{\text{last}}$ be the last time (of either a censored event or of an observed failure) in the original dataset. For all $n - k$ covariates $y$ left in $L$ after picking the first $k$ covariates, add $(y, z_{\text{last}})$ to $\tilde{D}$, so that the synthetic dataset also consists of $n$ covariates with an event time. Algorithm 2 below describes this in algorithmic notation. Figure 1 shows the transformation of a given dataset.
Figure 1: Above: The dataset as originally observed. Below: the synthetic dataset after applying the OPT-based transformation. The labels indicate which individual the observation corresponds to in each dataset. The lifetimes are sampled from a parabolic relationship between covariates and lifetimes.
Algorithm 2: Optimal transport based transformation of $D$ to $\tilde{D}$.

**Input**: $D = \{(x_i, z_i, \delta_i)\}_{i=1}^n$, with $z_0 \leq z_1 \leq \cdots \leq z_n$, significance level $\alpha$.

1. Set $AR = L = (x_1, \ldots, x_n)$;
2. for $i = 1, \ldots, n$ do
   3. if $\delta_i = 1$ then
      4. Let $X$ be the random variable that chooses uniformly from $AR$;
      5. Let $Y$ be the random variable that choose uniformly from $L$;
      6. Use optimal transport to define a coupling $P$ of $X$ and $Y$;
      7. Choose $y$ from $P(Y = y | X = x_i)$;
      8. Update $L = L.delete(y)$;
      9. Add $(y, z_i)$ to $\tilde{D}$
   end
   10. Update $AR = AR.delete(x_i)$
end
13. For $y$ in $L$, add $(y, z_n)$ to $\tilde{D}$.
14. Return $\tilde{D}$

4.3 Intuition behind the transformation

Before we prove properties of the proposed transformation, we briefly comment on the intuition behind the transformation. To this end, first consider a permutation test in the absence of censoring, when we simply observe $D = \{(X_i, T_i)\}_{i=1}^n$. The permuted datasets can be generated as follows: loop through the events in order of time, and to each time, associate a covariate that you have not associated to any earlier time. Comparing the original dataset with the datasets obtained in this manner, is justified because the original dataset can be generated by first sampling $T_i$ for $i = 1, \ldots, n$ i.i.d. and then looping through the times in order of time, and to each time, associate a covariate that has not been chosen before. The original dataset and the permuted datasets are thus equal in distribution. Intuitively, the permutation test checks if the dataset looks as if, at each time a covariate is picked uniformly from those not chosen before.

It is not obvious how to translate this to censored data. For this reason survival analysis often compares the $i$–th event covariate with the covariates at risk (not failed and not censored) just before the $i$–th event. Phrased in this way, independence means that the event covariate is chosen uniformly from those at risk just before the event. Since our algorithm couples a uniform pick from those at risk, to a uniform choice from those that have not yet been chosen in the synthetic dataset, in
the synthetic dataset, at each time, the covariate is chosen uniformly from those not chosen before. But our intuition behind a permutation test is exactly that: namely, we test if the dataset looks as if, at each time a covariate is picked uniformly from those not chosen before.

4.4 Applying HSIC to the transformed dataset: OPT–HSIC

We have thus far described how to transform the dataset. To perform a test, we propose to apply a permutation test with test statistic DCOV/HSIC to the transformed dataset. This approach is summarized in Algorithm 3.

Algorithm 3: OPT–HSIC

Input : Observed data \( D = \{ (x_i, z_i, \delta_i) \}_{i=1}^n \), significance level \( \alpha \).

1. Apply Algorithm 2 to \( D \) to obtain the transformed (and uncensored) dataset \( \tilde{D} \);
2. Apply the standard HSIC permutation test of Algorithm 1 to \( \tilde{D} \)

4.5 Correct type 1 control when \( C \perp \perp X \)

This section proves that, if censoring is independent of the covariate, the type 1 error of the OPT–HSIC test is correct. This is not a property of the specific choice of HSIC, but would be true for any statistic. We first prove an auxiliary result: namely, although we propose to permute the transformed dataset, this is equivalent to permuting the original dataset, and then transforming the permuted datasets.

Theorem 4.1. Let \( \pi_1, \ldots, \pi_B \) be independent uniform random permutations, and let \( T \) be the OPT–HSIC transformation.

\[
\left[ T(D), \pi_1(T(D)), \ldots, \pi_B(T(D)) \right] \overset{d}{=} \left[ T(D), T_1(\pi_1(D)), \ldots, T_B(\pi_B(D)) \right]
\]

Proof. See appendix.

The \( T_i \) are independent transformations. That is, given the input data, each transformation samples from distributions computed through optimal transport at several times. Independent transformations sample from these distributions independently. The result above enables us to show that the type 1 control is correct when \( C \perp \perp X \).
**Theorem 4.2.** (Permutation test on transformed data) Let $D = \{(X_i, Z_i, \Delta_i)\}_{i=1}^n$ be the dataset and denote $\tilde{D}$ the transformed dataset after applying OPT-HSIC. Let $\pi_1, \ldots, \pi_B$ be permutations sampled uniformly and independently from $S_n$, the symmetric group. Let $H$ be any statistic of the data. Let $R$ be the rank of the first coordinate in the vector:

$$(H(\tilde{D}), H(\pi_1(\tilde{D})), \ldots, H(\pi_B(\tilde{D})))$$

Then under the null hypothesis $H_0: X \perp \perp T$:

$$P(R \geq (1 - \alpha)(B + 1) + 1) = \frac{|\alpha(B + 1)|}{B + 1} \leq \alpha.$$

**Proof.** See appendix. \(\square\)

### 4.6 Type 1 error when $C \not\perp \perp X$

A type 1 error is made when a test rejects the null hypothesis when the null hypothesis was true. The above proves the type 1 error is controlled if $C \perp \perp X$. Extensive simulations investigated the type I error in a range of synthetic datasets. All simulations indicate a correct type 1 control, even when there is a strong dependence between censoring and covariates. The results are shown in Section 6. Theoretical guarantees of this control are a topic for future work.

### 4.7 Computational cost of OPT-HSIC

The computational cost of the transformation of the data comes from the computation of the optimal transport couplings and the required distance matrices. We implemented the algorithm in Python 3.6. To compute the distance matrices we used the ‘spatial distance’ function in the package ‘scipy’. To compute the joint distribution we used the Earth Mover’s Distance function ‘emd’ from the optimal transport package ‘ot’. We timed the transformation of a dataset consisting of 1000 individuals. We performed the timing on a Dell PC with 7 Intel i7-6700 CPUs running at 3.40 GHz. When 75% is observed, the transformation of the dataset took 22 seconds. When 25% of the data is observed, the transformation took 10 seconds. To the transformed dataset we applied a standard permutation test using DCOV/H-SIC. Using 1999 permutations, this permutation test took 40 seconds, making for a total of transformation and permutation of 62 or 50 seconds. We note that we did not utilize parallelization of the permutation test, which would further speed up the procedure. In addition, existing methods to speed up independence-testing for large sample sizes can be applied directly to the transformed dataset (Zhang, Filippi, Gretton, and Sejdinovic (2018)).
Alternative approaches when censoring is independent of the covariate

When $C$ is independent of $X$ the challenge of independence testing on right-censored data is mitigated, as it is then appropriate simply to permute the covariates in the original dataset. Since a permutation test can be applied to every choice of test-statistic, the remaining challenge is to find a test-statistic that measures dependency of a censored sample in a meaningful way. To see why we can use a standard permutation test on right censored data, we begin by stating the theorem in the uncensored case. We include the proof following Berrett and Samworth (2017) in the appendix for completeness.

**Theorem 5.1.** Let $D = \{(X_i, Y_i)\}_{i=1}^n$ be an i.i.d. sample of size $n$ with distribution $P_{XY}$ on $\mathcal{X} \times \mathcal{Y}$ equipped with some sigma algebra. Denote $\pi(D) = \{(X_{\pi(i)}, Y_i)\}_{i=1}^n$. Let $\pi_1, \ldots, \pi_B$ be permutations sampled uniformly and independently from $S^n$. Let $H$ be any statistic of the data. Let $R$ be the rank of the first coordinate in the vector:

$$(H(D), H(\pi_1(D)), \ldots, H(\pi_B(D)))$$

Then under the null hypothesis $H_0 : X \perp \!\!\!\!\perp T$:

$$P(R \geq (1 - \alpha)(B + 1) + 1) = \frac{|\alpha(B + 1)|}{B + 1} \leq \alpha.$$

**Proof.** See appendix.

In survival analysis we apply the above with $X$ being the covariate and setting $Y = (Z, \Delta)$, where $Z = \min\{T, C\}$ and $\Delta = 1\{T \leq C\}$.

**Theorem 5.2.** If $C \perp X$ and $T \perp C|X$, then a standard permutation test can be applied to right censored data. That is, write $D = \{(X_i, Z_i, \Delta_i)\}_{i=1}^n$, and let $\pi_1, \ldots, \pi_B$ be i.i.d. uniform from $S_n$. Write $\pi(D) := \{X_{\pi(i)}, Z_i, \Delta_i\}$. Let $H(D)$ be any statistic of the data. Let $R$ be the rank of the first coordinate in the vector:

$$(H(D), H(\pi_1(D)), \ldots, H(\pi_B(D)))$$

Then under the null hypothesis $H_0 : X \perp \!\!\!\!\perp T$:

$$P(R \geq (1 - \alpha)(B + 1) + 1) = \frac{|\alpha(B + 1)|}{B + 1} \leq \alpha.$$

**Proof.** See appendix.

Having shown that we can use a permutation test on any test statistic when $C \perp X$, the next two sections explore meaningful measures of dependency on right-censored data.
5.1 wHSIC

HSIC relies on estimating the mean embedding of the joint distribution $P_{X,T}$. The estimated embedding is then compared to embedding of the product of marginal distributions. In the uncensored case the estimated embedding is simply: $(1/n) \sum_{i=1}^{n} k(x_i, \cdot)l(t_i, \cdot)$ corresponding to the mean embedding of the empirical distribution $(1/n) \sum_{i=1}^{n} \delta_{(x_i,t_i)}$.

Since we do not observe the $t_i$’s we could consider replacing the empirical distribution by a weighted version $\sum_{i=1}^{n} w_i \delta_{(x_i,z_i)}$ where we try to find weights $w_i$ such that $\sum_{i=1}^{n} w_i f(x_i, z_i) \approx E f(X, T)$ for every measurable function $f$ such that the expectation exists. A natural idea is to give an observed point $(x, z, \delta)$ a weight of zero if it is censored, and a weight equal to the inverse probability of being uncensored otherwise. This can be motivated by the following lemma, that applies also if $C \not\perp X$.

First write the probability of being uncensored by time $t$ given you have covariate $x$ as

$$g(t,x) = P(C > t | X = x).$$

The following lemma proposes a weight function $W$ in terms of $g$.

**Lemma 5.1.** Let $f$ be an $(X, T)$-measurable function such that $E f(X, T) < \infty$ and define a random variable $W$ on the same probability space as $(X, Z, \Delta)$ by

$$W = \begin{cases} 
0 & \text{if } \Delta = 0 \\
1 & \text{if } g(x, z) \text{ if } \delta = 1.
\end{cases}$$

Then

$$E W f(X, Z) = E f(X, T).$$

**Proof.** See Appendix. \(\square\)

We would thus like to use weights $w_i = 0$ if $\delta_i = 0$ and $1/g(x_i, z_i)$ if $\delta = 1$. The function $g(x, t)$, however, is unknown, and estimating it is hard. However, we now show that under the assumption that $C \perp X$ we may estimate $1/P(C > t | X) = 1/P(C > t)$ using Kaplan–Meier weights, which we define now. Assume that there are no ties in the data and that $z_i < z_{i+1}$ for $i = 1, \ldots, n$. Then the Kaplan-Meier survival curve is given by:

$$\hat{S}(z_k) = \prod_{i=1}^{k} \left( \frac{n-i}{n-i+1} \right) \delta_i$$
This results in weights:

\[ w_k = \hat{P}(T = z_k) = \hat{S}(z_{k-1}) - \hat{S}(z_k) \]

\[ = \prod_{i=1}^{k-1} \left( \frac{n-i}{n-i+1} \right) \delta_i \left( 1 - \frac{n-k}{n-k+1} \right) \delta_k \]

\[ = \prod_{i=1}^{k-1} \left( \frac{n-i}{n-i+1} \right) \delta_i \left( \frac{1}{n-k+1} \right) \delta_k \]

The following lemma shows that these weights correspond, up to a constant of $1/n$, to the inverse probability of being uncensored.

**Lemma 5.2.** Let $\hat{P}(C > z_k)$ denote the Kaplan–Meier estimate of the survival probability of the censoring distribution. Then:

\[ w_k = \frac{1}{n} \frac{1}{\hat{P}(C > z_k)} \]

**Proof.** See appendix.

We now propose to measure the distance between the embedding of $\sum_{i=1}^{n} w_i \delta_{x_i, z_i}$ and the embedding of the product of the marginals, $\sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \delta_{x_i} \delta_{z_j}$. This suggests the statistic

\[ \text{wHSIC}(D) := \left\| \sum_{i=1}^{n} w_i K((x_i, z_i), \cdot) - \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j K((x_i, y_j), \cdot) \right\|_H^2, \]

which is very similar to a statistic proposed in [Matabuena (2019)] for the two-sample case (See also Section 6.1). Here $K((x, y), (x', y')) = k(x, x')l(y, y')$. The following theorem shows how to compute wHSIC efficiently.

**Theorem 5.3.** (Computation of wHSIC) Given a dataset $D = \{(x_i, y_i)\}_{i=1}^{n}$ with a weight vector $w \in \mathbb{R}^n$,

\[ \text{wHSIC}(D) = \text{tr}(H_w K H_w L), \]

where $K_{ij} = k(x_i, x_j)$ and $L_{ij} = l(y_i, y_j)$ and $H_w = D_w - w w^T$, where $D_w = \text{diag}(w)$, a diagonal matrix with $(D_w)_{ii} = w_i$.

**Proof.** See appendix.
The products $H_wK$ and $H_wL$ can be computed in $O(n^2)$ time, as $H_w$ is a diagonal matrix plus a rank one matrix. Finally, the trace of the product is equal to the sum of all the elements of the entry-wise product of $H_wK$ and $(H_wL)^T$. So in total the weighted HSIC can be computed, like the standard HSIC, in $O(n^2)$ time. To implement a permutation test effectively, note that the weights and the times they correspond to are not affected by permutation. Hence the matrices $ww^T$ and $L$ only need to be computed once, and we only need to permute the entries of the matrix $K$.

**Algorithm 4: weighted-HSIC**

**Input**: $D = \{(x_i,z_i,\delta_i)\}_{i=1}^n$, significance level $\alpha$.

1. Compute $w\text{HSIC}(D)$ as in Theorem 5.3.
2. Sample permutations $\pi_1, \ldots, \pi_B$ i.i.d. uniformly from $S_n$.
3. Breaking ties at random, compute the rank $R$ of $\text{HSIC}(D)$ in the vector
   
   \[(w\text{HSIC}(D), w\text{HSIC}(\pi_1(D)), w\text{HSIC}(\pi_2(D)), \ldots, w\text{HSIC}(\pi_B(D)))\]

   Reject if $R \geq (1 - \alpha)(B + 1) + 1$.

**5.2 zHSIC**

If $C \perp X$, then any dependence between $X$ and $Z$ must be due to dependence between $X$ and $T$. Hence we may compute $\text{HSIC}(X,Z)$ to test for independence. This test is, however, inherently flawed if the assumption $C \perp X$ fails to hold, and will lack power when a large portion of the events is censored.

**Algorithm 5: zHSIC**

**Input**: $\tilde{D} = \{(x_i,z_i)\}_{i=1}^n$, significance level $\alpha$.

1. Denote $D = \{(x_i,z_i)\}_{i=1}^n$ the sample as if there was no censoring. Sample permutations $\pi_1, \ldots, \pi_B$ i.i.d. uniformly from $S_n$. Denote $\pi(D) = \{(x_{\pi(i)},z_i)\}_{i=1}^n$.
2. Breaking ties at random, compute the rank $R$ of $\text{HSIC}(Data)$ in the vector
   
   \[(\text{HSIC}(D), \text{HSIC}(\pi_1(D)), \text{HSIC}(\pi_2(D)), \ldots, \text{HSIC}(\pi_B(D)))\]

   Reject if $R \geq (1 - \alpha)(B + 1) + 1$. 
6 Numerical evaluation of the methods

We generate data from various distributions of \(X\), \(T\) and \(C\) to compare the power and type 1 error of the following methods: OPT-HSIC, wHSIC, zHSIC and CPH. CPH stands for the Cox proportional hazards model with a test of significance of the coefficient corresponding to the covariate. In each scenario we let the \(n\) values range from \(n = 20\) to \(n = 400\) in intervals of 20. To obtain \(p\)-values in the three HSIC based methods we use a permutation test with 1999 permutations. We reject the null hypothesis if our obtained \(p\)-value is less than 0.05.

6.1 Comparison of power

We test the power of each approach to detect different dependencies between \(T\) and \(X\), each under different censoring intensities. In the first four scenarios the censoring rate is independent of the covariate. In particular we consider the power if 1) the Cox proportional hazards assumption holds, if 2) \(X\) and \(T\) are linearly related, 3) \(X\) and \(T\) are quadratically related, and 4) \(T\) satisfies a different bimodal distribution for each \(X\). The precise distributions are given in the table in Table 1. In Scenarios 5 and 6 both the lifetime distribution and the censoring distribution depend on the covariate. Since we see in Section 6.2 that wHSIC and zHSIC do not control the type 1 error under dependent censoring, we compare only CPH and OPT-HSIC in Scenarios 5 and 6. The resulting power is displayed in Figures 2, 3 and 4.

| Sc. | Event-time | \(X\) | \(T|X\) | \(C|X\) |
|-----|------------|------|--------|--------|
| 1   | CPH        | \(N(0,2)\) | \(\text{Exp}(\text{exp}(X/5))\) | \(\text{Exp}(\lambda)\) |
| 2   | Linear     | \(N(0,2)\) | \(20 + X + \text{Exp}(1/10)\) | \(17 + \text{Exp}(\lambda)\) |
| 3   | Quadratic  | \(\text{Unif}[-5,5]\) | \(X^2/2 + \text{Exp}(1/10)\) | \(\text{Exp}(\lambda)\) |
| 4   | Bimodal    | \(\text{Unif}[0,1]\) | \(10 + XY + N(0,1/4)\) | \(8.5 + \text{Unif}[0, \lambda]\) |
| 5   | Quadratic  | \(\text{Unif}[-5,5]\) | \(15 + X^2/2 + \text{Exp}(10)\) | \(\max(a + 2X, 15) + \text{Exp}(b)\) |
| 6   | CPH        | \(N(0,2)\) | \(\text{Exp}(\text{exp}(X/5))\) | \(\text{Exp}(\text{exp}(X/5) \times a)\) |

Table 1: The four scenarios in which we test the power. In scenario 4 \(Y\) takes on \(\pm 1\) with probability \(1/2\). We let the parameters in the censoring distribution vary, so that approximately 75 %, 50 and 25% is censored for each scenario. \(\text{Exp}(\lambda)\) is an exponential random variable with rate \(\lambda\), and \(\exp(x) = e^x\).
Figure 2: Left: Scenario 1: the relationship between $X$ and $T$ satisfies the Cox-proportional-hazards model. Right: the relationship between $X$ and $T$ is linear. The three plots have different censoring rates $\lambda$.  

(a) $\lambda = 1/3$: 75% of the events is observed.  
(b) $\lambda = 1$: 50% of the events is observed.  
(c) $\lambda = 3$: 25% of the events is observed.  
(d) $\lambda = 1/40$: 75% of the events is observed.  
(e) $\lambda = 1/15$: 50% of the events is observed.  
(f) $\lambda = 1/7$: 25% of the events is observed.
Figure 3: Left: Scenario 3: the relationship between $X$ and $T$ is parabolic. The three plots have different censoring rates $\lambda$. Right: Scenario 4: the relationship between $X$ and $T$ is bimodal. The three plots have uniform censoring with different widths $\lambda$. 

(a) $\lambda = 1/45$: 75% of the events is observed. 
(b) $\lambda = 1/17$: 50% of the events is observed. 
(c) $\lambda = 1/7$: 25% of the events is observed. 
(d) $\lambda = 6$: 75% of the events is observed. 
(e) $\lambda = 3$: 50% of the events is observed. 
(f) $\lambda = 1.75$: 25% of the events is observed.
Figure 4: Power under dependent censoring. Left: Scenario 5: the relationship between $X$ and $T$ is quadratic. The three plots have different censoring parameters $a, b$. Right: Scenario 6: the relationship between $X$ and $T$ satisfies the CPH assumption. The three plots have different censoring parameters $a$. 

(a) $a = 15, b = 1/35$: 75% observed. 
(b) $a = 19, b = 1/9$: 50% observed. 
(c) $a = 15, b = 1/10$: 25% observed. 
(d) $a = 1/3$: 75% of the events is observed. 
(e) $a = 1$: 50% of the events is observed. 
(f) $a = 3$: 25% of the events is observed.
6.2 Comparison of type 1 error

In this section we present the results of simulations of the type 1 error of each method for six different distributions of \((X, T, C)\) in which \(T\) is independent of \(X\). The exact distributions can be found in Table 2. In the first two scenarios censoring is independent of the covariate so that each of the three kernel methods should display the correct type 1 error. We then choose four scenarios in which \(C\) depends strongly on \(X\). In these scenarios we expect wHSIC and zHSIC to return false positives as both methods relied strongly on the assumption of censoring being independent of the covariate. Section 4 discussed why intuitively OPT-HSIC may control the type 1 error at the correct level even when censoring depends on the covariate. The results are plotted in Figure 6. Figure 1 illustrates the effect of the OPT-HSIC under dependent censoring.

![Figure 5: The OPT-based transformation of the original dataset, displayed on the left, into the synthetic dataset on the right. The original data is sampled from Scenario 5 of the scenarios used in simulation of the type 1 error with \(n = 200\).](image)

| Sc. | Censoring | \(X\) | \(T \mid X\) | \(C \mid X\) | % Observed |
|-----|-----------|-------|-------------|-------------|------------|
| 1   | Independent | \(N(0, 1)\) | \(\text{Exp}(1)\) | \(\text{Exp}(1)\) | 50 %       |
| 2   | Independent | \(\text{Chisquare}(3)\) | \(\text{Unif}[0, 1]\) | \(\text{Unif}[0, 1]\) | 50 %       |
| 3   | Linear | \(\text{Unif}[-5, 5]\) | \(\text{Chisquare}(16)\) | \(15 + 2 \times X\) | 45 %       |
| 4   | Linear+Noise | \(\text{Chisquare}(5)\) | \(\text{Chisquare}(5)\) | \(X/2 + \text{Exp}(2)\) | 45 %       |
| 5   | Quadratic | \(\text{Unif}[-5, 5]\) | \(\text{Unif}[0, 10]\) | \(X^2\) | 55 %       |
| 6   | Exponential | \(\text{Chisquare}(3)\) | \(N(10, 1)\) | \(\exp(X/6) + \text{Exp}(4)\) | 60 %       |

Table 2: The 6 scenarios in which we test the type 1 error.
Figure 6: Type 1 error ($\alpha = 0.05$) of each method in the 6 scenarios. Under independent censoring (plots a and b), each method obtains the correct rejection rate. OPT-HSIC is the only method that, even under very dependent censoring (plots c-f), achieves the correct rejection rate.
6.3 Comments on the simulations

As to power, Scenarios 3 and 4 show that OPT-HSIC has power against a wider class of alternatives than the CPH model. Scenarios 1 and 2 show that even in the cases in which CPH performs well (e.g. when the CPH assumption is actually true), the power of OPT-HSIC is close to that of the CPH method. The other kernel based tests lose much power, especially when many of the observations are censored. As expected, the type 1 error of every method is correct in the scenarios in which censoring is independent of the covariate. We note that that even in scenarios in which censoring strongly depends on the covariate, OPT-HSIC achieves the correct rejection rate. In Scenario 5 of the scenarios used to assess the type 1 error, we did an additional investigation of the distribution of the $p$-values: We took 1.2 million samples of size 100 from the given distribution. On each of the samples we performed the OPT-HSIC test, resulting in 1.2 million $p$-values. A Kolmogorov–Smirnov test of uniformity resulted in a $p$-value of 0.21, indicating that the $p$-values returned by OPT-HSIC appeared uniformly distributed, even though censoring depended strongly on the covariate.

7 Binary covariates

As a special case of independence testing we consider the case of a single binary covariate, i.e., $X \in \{0, 1\}$. If one groups the data by covariate, then testing independence of $T$ and $X$ is equivalent to testing equality of lifetime distribution between the two groups. This is known as two-sample testing on right censored data. A popular approach to this challenge is the logrank test, and various weighted logrank tests. OPT-HSIC can be applied to this problem without any adjustments, while wHSIC can be improved in this case in two ways: first, the weights can be estimated even when the censoring distribution differs between the two groups; and second, there exists an alternative permutation strategy that, experiments show, seems to control the type 1 error effectively even under dependent censoring. These adjustments are described in Section 7.1 and Section 7.2 respectively. We omit consideration of zHSIC, as it is fundamentally more limited, and because there is less need in this setting, given the larger number of available methods. We now review some alternative methods one may consider and then compare the performance. As OPT-HSIC is designed for the more challenging case of continuous covariates where there are no real alternatives, we expect that some of the specialized methods for the two-sample test may perform better, but we wish to investigate in what cases OPT-HSIC performs well.
7.1 wHSIC for two-sample testing

Let $P_0$ and $P_1$ denote the distribution of $T|X = 0$ and $T|X = 1$ respectively. Let the total sample be $D = \{(x_i, z_i, \delta_i)\}_{i=1}^n$ as before, and write $\{(z_{i,j}^0, \delta_{i}^0)\}_{i=1}^{n_0}$ and $\{(z_{i,j}^1, \delta_{i}^1)\}_{i=1}^{n_1}$ for the event times and indicators of individuals with covariate $X = 0$ and $X = 1$ respectively. We want to assess if $P_0 = P_1$. We now define $H$ to be an RKHS on $\mathbb{R}_{\geq 0}$, with kernel $k(a, b) = \min(a, b)$. If all of the $n$ times were observed ($\delta = 1$), we could measure the difference in empirical distributions between both groups by the MMD between the two distributions:

$$\left\| \frac{1}{n_0} \sum_{i=1}^{n_0} k(z_{i,j}^0, \cdot) - \frac{1}{n_1} \sum_{j=1}^{n_1} k(z_{i,j}^1, \cdot) \right\|_H.$$ 

Similar to Section 5.1, when some observations are censored, we might reweight the empirical distributions, and instead compare the weighted empirical distributions

$$\sum_{i=1}^{n_0} w_{i}^0 k(z_{i,j}^0, \cdot)$$

and

$$\sum_{i=1}^{n_0} w_{i}^1 k(z_{i,j}^1, \cdot).$$

We propose that the weights $w_i$ are computed by the Kaplan–Meier weights within each group. The test statistic thus becomes:

$$\text{wHSIC}(D) := \left\| \sum_{i=1}^{n_0} w_{i}^0 k(z_{i,j}^0, \cdot) - \sum_{i=1}^{n_0} w_{i}^1 k(z_{i,j}^1, \cdot) \right\|^2_H.$$ 

This statistic is proposed by Matabuena (2019) independently. Under the hypothesis that $C \perp \perp X$, one can obtain $p$-values using a permutation test, resulting in the following algorithm. Section 7.2 provides an alternative permutation test under dependent censoring, that was proposed by Wang, Lagakos, and Gray (2010).

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**Algorithm 6:** wHSIC for two-sample data  

**Input:** $D = \{(x_i, z_i, \delta_i)\}_{i=1}^n$  

1. Sample permutations $\pi_1, \ldots, \pi_B$ i.i.d. uniformly from $S_n$.  
2. Breaking ties at random, compute the rank $R$ of HSIC($D$) in the vector

$(\text{wHSIC}(D), \text{wHSIC}(\pi_1(D)), \text{wHSIC}(\pi_2(D)), \ldots, \text{wHSIC}(\pi_B(D)))$

where wHSIC is as defined in Section 7.1;  
3. Reject if $R \geq (1 - \alpha)(B + 1) + 1$.  

---
7.2 ipxHSIC

This subsection overviews a test we name ipxHSIC, which uses the statistic \( w_{HSIC}(D) \) defined in Section 7.1 above, but which uses a different permutation strategy that is robust against differences in the censoring distributions of both groups. The permutation strategy was proposed in [Wang et al. (2010)] to provide reliable \( p \)-values for the logrank test in the case of small or unequal sample sizes. In fact [Wang et al. (2010)] propose two permutation strategies: the first one, which they call ‘ipz’ (section 2.1.1), permutes group membership and the second, which they call ‘ipt’ (section 2.1.2), permutes survival times. The first algorithm, which permutes the covariates, is referred to in their work as ‘ipz’ since the procedure first imputes several unobserved times, and then permutes the covariate, which in their work is denoted by \( z \). We refer to it as ‘ipx’, as our covariate is denoted by \( x \). The algorithm uses the Kaplan–Meier estimator to estimate three distributions: 1) \( G^0 \), the censoring distribution in group 0, based on the data observed in group 0; 2) \( G^1 \), the censoring distribution in group 1, based on the data observed in group 1; 3) the distribution of the lifetimes \( F \) based on the pooled dataset containing both groups. With these estimates, a new dataset is constructed, consisting of \( n \) observations, each consisting of a covariate, an event time, and a two censoring times, one for each censoring distribution. This larger dataset is then permuted, and transformed back to a censored dataset. [Wang et al. (2010)] describe the algorithm in full detail. This algorithm was proposed also in [Matabuena (2019)].

7.3 Numerical comparison of methods in two-sample case

We generate data from four different distributions for each of \( X, T, \) and \( C \) to compare the power and type 1 error of the proposed methods OPT-HSIC, \( w_{HSIC} \), ipxHSIC to the power and type 1 error of the classic logrank test and a weighted logrank test proposed by [Ditzhaus and Friedrich (2018)]. The classical logrank test is known to have low power against certain alternatives, such as crossing survival curves. A weighted logrank test assigns weights to data, giving the logrank test power against different alternatives. In [Ditzhaus and Friedrich (2018)] a combination of weights is proposed, so as to achieve power against a wider class of alternatives. In particular [Ditzhaus and Friedrich (2018)] propose a combination of two sets of weights, corresponding to proportional and crossing hazards. As this section mostly serves to provide an example of our methods, we simulate fewer scenarios than in Section 6. In each scenario we let the \( n \) values range from \( n = 20 \) to \( n = 400 \) in intervals of 20. To obtain \( p \)-values in the three HSIC based methods as well as the weighted logrank test we use a permutation test with 1999 permutations. We reject the null
hypothesis if our obtained $p$-value is less than 0.05.

| Sc. | $T_0$         | $T_1$         | $C_0$         | $C_1$         | % Observed |
|-----|---------------|---------------|---------------|---------------|------------|
| 1   | Exp(1)        | Exp(1/1.6)    | Exp(1/2)      | Exp(1/2)      | 60 %       |
| 2   | Weib(1, 5)    | Weib(1, 1.5)  | exp(1/2)      | Exp(1/2)      | 60 %       |
| 3   | Exp(1)        | 0.43, 1.39+Exp(1) | 1 + Exp(1/2) | 1 + Exp(1/2) | 90 %       |
| 4   | Exp(1)        | Exp(1)        | Exp(2)        | None          | 65 %       |

Table 3: The 4 scenarios in which we perform two-sample tests. $T_1$ is 0.43 w.p. 0.75 and 1.39 + Exp(1) w.p. 1/4.

Figure 7: Power of the various two-sample tests.
7.4 Comments on two-sample simulations

The results show that the logrank test and the weighted logrank test have little power in scenario 2 and 3 and scenario 3 respectively, even though large differences between the samples are present. The logrank is designed to detect differences as in scenario 1, and the weighted logrank is designed to detect differences as in scenario 1 and 2, sacrificing power slightly compared to the logrank test in the first. Scenario 3 is designed to defeat the weighted logrank test, since we constructed an extreme version of an early crossing survival curve, and the test does not contain weights for early crossing. The kernel methods are fully non-parametric. This implies that they may sacrifice power in some scenarios, but that they are able to detect a much wider range of differences between the two samples. OPT-HSIC in particular has power comparable to the logrank test in the proportional hazards scenario, and a much higher power in the two other scenarios. We note again that a binary covariate is not optimally suited to the optimal transport approach, as one may lose too much information, given that the transformation relies on ‘similarity’ between covariates.

8 Discussion

The main contribution of this paper is the proposal of OPT-HSIC, combining a novel way of using optimal transport to transform right-censored datasets, with non-parametric independence testing by HSIC or DCOV. We have shown OPT-HSIC has power against a much wider range of alternatives than the commonly used CPH model, while forfeiting very little power when the CPH assumptions are satisfied exactly. Extensive numerical simulations suggest that the approach is well calibrated, yielding reliable $p$-values even when censoring strongly depends on the covariate. Under the assumption that censoring does not depend on the covariate, we have proven that the $p$-values of the algorithm are correct. Theoretical guarantees for the type 1 error under dependent censoring are a topic of future work. A second challenge is extending this methods to higher dimensional covariates. While the methodology is inherently applicable to fairly general covariates, by analogy with kernel-based tests on uncensored data, the extension depends on some choices that require some care in this setting, such as defining the scale of each covariate, and selecting the distance measure underlying optimal transport. Furthermore, it may be more difficult to preserve dependence in the transformed dataset, especially when the dependency arises from a low-dimensional subspace of the covariates. We furthermore proposed reweighting the original dataset, and measuring the distance between the resulting weighted mean embeddings. While these methods showed some promise, they do rely on the very strong assumption...
that $C$ and $X$ are independent, except in the case of a single binary covariate, where there is more flexibility. It may be worth investigating whether bootstrap methods or asymptotic analysis of null distributions of test statistics can provide more flexible methods of testing significance. Mittelbach, Goossens, Braams, Carlisle, Rowley, Detig, and Schrod (2004)

9 Supplementary materials and code

Supplementary materials contain proofs of the stated results and tables of the plots provided. Code to replicate the experiments and of the tests is available on www.github.com/davidrindt

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