Integrability of diagonalizable matrices and a dual Schoenberg type inequality

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Abstract

The concepts of differentiation and integration for matrices were introduced for studying zeros and critical points of complex polynomials. Any matrix is differentiable, however not all matrices are integrable. The purpose of this paper is to investigate the integrability property and characterize it within the class of diagonalizable matrices. In order to do this we study the relation between the spectrum of a diagonalizable matrix and its integrability and the diagonalizability of the integral. Finally, we apply our results to obtain a dual Schoenberg type inequality relating zeros of polynomials with their critical points.

Keywords: polynomials, matrices, differentiators, integrators, non-derogatory

[MSC 2020] Primary: 30C10, 30C20, Secondary: 15A15

1 Introduction

Zeros and critical points of a given univariable polynomial are related by the celebrated Gauss-Lucas theorem stating that critical points belong to the convex hull of zeros. More detailed and thorough investigation of the relation between zeros and critical points of a polynomial is an interesting open problem which stimulated already a lot of research in analytic theory of polynomials. Far reaching progress in this area was established by the solution of the famous Schoenberg conjecture proved independently by Pereira [12] and Malamud [10, 11]. This progress was based on the notion of matrix differentiators introduced much earlier by Davis [5]. In this paper we use the techniques from matrix analysis and linear algebra to study the inverse concept of integrator of a matrix which was introduced by Bhat and Mukherjee [1]. Bhat and Mukherjee have shown that any
matrix is either freely integrable, or uniquely integrable or non-integrable and characterized the freely integrable matrices. However, the problem to characterize the latter two categories was left open. Our paper solves this problem for diagonalizable matrices in terms of characteristic polynomials. In order to do this we study the relation between the spectrum of diagonalizable matrix and its integrability and the diagonalizability of the corresponding integral. Moreover, we apply the obtained results to derive a dual Schoenberg type inequality providing an upper bound for the sum of squares of the absolute values of zeros by an expression in the critical points.

We start by introducing some notations that we need further for the main definitions of our paper. In this paper denote by $\mathbb{C}$ the field of complex numbers, and by $\mathbb{K}$ an arbitrary algebraically closed field. If it is not specifically mentioned, we assume that the characteristic $\text{char} \mathbb{K} = 0$. Let $M_{n,m}(\mathbb{K})$ denote the space of all $n \times m$ matrices with entries from $\mathbb{K}$, we write $M_n$ if $m = n$. Let $I_n$ be the unit matrix $n \times n$, $O_n$ be the zero $n \times n$ matrix. We write $I$ and $O$ if the size of the matrix is clear from the context. The transpose of $A \in M_{n,m}$ is denoted by $A^\top \in M_{m,n}$. Vectors in $\mathbb{K}^n$ are considered as row vectors and are identified with corresponding $n$-tuples. The $j$’th unit vector is denoted by $e_j$ and $e = (1, \ldots, 1)^\top$. For $X \in M_{n,m}(\mathbb{K})$, $Y \in M_{k,l}(\mathbb{K})$ we denote by $X \oplus Y \in M_{n+k,m+l}(\mathbb{K})$ the block matrix ($\begin{smallmatrix}X & 0 \\ 0 & Y\end{smallmatrix}$). In case $\mathbb{K} = \mathbb{C}$ by $||A||_F$ we denote the Frobenius norm of a matrix $A = (a_{ij})$, i.e. $||A||_F = \sqrt{\sum_{i,j} |a_{ij}|^2}$, by $||v||$ we denote the Euclidean norm of $n$ dimensional vector $v$, i.e. $||v|| := \sqrt{\sum_{i=1}^n |v_i|^2}$.

The following notion of matrix differentiability was introduced by Davis in [5] and further investigated in [1, 2, 3, 9, 12].

**Definition 1.1** ([1, Definition 1]). Let $A$ be a linear operator on a complex Hilbert space $H$ of the dimension $n$ and $P$ be an operator of an orthogonal projection on $H$ with $\dim \text{Ker}(P) = 1$. Let $B$ be an operator satisfying the condition $B = PAP|_{P(H)}$. Then $P$ is called a **differentiator** of the operator $A$, if characteristic polynomials of $A$ and $B$ satisfy the condition

$$p_B(x) = \frac{1}{n}p_A'(x).$$

In this case the operator $B$ is usually called a differential of the operator $A$.

Now without loss of generality we can assume that

$$P = \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} B & u^\top \\ v & \lambda \end{bmatrix}$$

where $u, v \in \mathbb{C}^n$. 2
Differentiators appear to be useful in studying the relation between the zeros of the polynomial and its critical points. In 2003, Pereira \cite{12} and Malamud \cite{10,11} independently proved the following conjecture using the method of differentiators of finite dimensional operators.

**Theorem 1.2 (Schoenberg’s conjecture (1986) \cite{12, Conjecture 3.1}).** Let \( p(z) \) be a degree \( n \) complex polynomial with zeros \( z_1, \ldots, z_n \) and critical points \( w_1, \ldots, w_{n-1} \). Then

\[
\sum_{i=1}^{n-1} |w_i|^2 \leq \left| \frac{1}{n} \sum_{i=1}^{n} z_i \right|^2 + \frac{n-2}{n} \sum_{i=1}^{n} |z_i|^2
\]

with the equality holds if and only if all \( z_i \) lie on a straight line.

Comparing the coefficients at \( x^{n-1} \), we obtain \( \text{tr}(B) = \frac{n}{n+1} \text{tr}(A) \), which implies that

\[
\lambda = \text{tr}(A) - \text{tr}(B) = \frac{n+1}{n} \text{tr}(B) - \text{tr}(B) = \frac{\text{tr}(B)}{n} := \tau(B)
\]

The converse operation of integration was introduced by Bhat and Mukherjee in \cite{1}.

**Definition 1.3 (\cite{1, Definition 3}).** Let \( B \in M_n(\mathbb{C}), A \in M_{n+1}(\mathbb{C}) \), then \( A \) is called an integral of \( B \), if

\[
A = \begin{bmatrix} B & u^\top \\ v & \tau(B) \end{bmatrix},
\]

and also \( p_B(x) = \frac{1}{n+1} p_A'(x) \). In this case the pair of vectors \( (u, v) \) is called an integrator of \( B \) and the element \( \det(A) \) is called a constant of integration.

For any algebraically closed field \( \mathbb{K} \) with \( \text{char} \mathbb{K} = 0 \) one can define a formal derivative and an integral of a polynomial \( p(x) = a_n x^n + \ldots + a_1 x + a_0 \) as follows. The derivative is \( p'(x) := n a_n x^{n-1} + \ldots + 2 a_2 x + a_1 \), and the integral \( P(x) := \frac{1}{n+1} a_n x^{n+1} + \ldots + \frac{1}{2} a_1 x^2 + a_0 x + C \), where \( C \in \mathbb{K} \) is a constant. Then Definitions 1.1 and 1.3 can be considered for matrices over any algebraically closed field \( \mathbb{K} \) of characteristic zero. It is straightforward to check that the results from \cite{1} are also true in this generality. In this paper we investigate how the integrability depends on the values of zeros, their multiplicities, and integrability property. Thus below we consider algebraically closed fields of zero characteristic.

**Definition 1.4 (\cite{1, Definition 5}).** The square matrix \( B \) is called integrable if there exists its integral. A matrix \( B \) is called uniquely integrable if it is integrable and there exists \( \alpha \in \mathbb{C} \) such that for any integral \( A \) of the matrix \( B \) it holds that \( \det(A) = \alpha \). A matrix \( B \) is called freely integrable if for any \( \alpha \in \mathbb{C} \) there exists an integral \( A \) of the matrix \( B \), such that \( \det(A) = \alpha \).

Below we collect several examples demonstrating different properties and features of matrix integrability.

Let us start with an example of freely integrable matrix.
Example 1.5. [1, Example 4] Fix $\lambda \in K \setminus \{1\}$. Consider $B = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}$. Observe that for any $t \in K$,

$$A_t = \begin{pmatrix} 1 & 0 \\ \frac{2^t - 3\lambda + 1}{2(1-\lambda)} & \frac{(\lambda-1)^2 + 2t - 3\lambda + 1}{2(1-\lambda)} \end{pmatrix}$$

is an integral of $B$ with the constant of integration $t$.

Indeed, $\det(xI - A_t) = x^3 - \frac{3(\lambda+1)}{2}x^2 + 3\lambda x - t$, therefore

$$p_{A_t}(x) = 3x^2 - 3(\lambda + 1)x + 3\lambda = 3(x - 1)(x - \lambda) = 3p_B(x)$$

and $\det(A_t) = t$. Therefore $B$ is freely integrable.

We now give an example of uniquely integrable matrix.

Example 1.6. Consider the case $\lambda = 1$, i.e. $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and write the integral in its general form $A = \begin{pmatrix} 1 & 0 & u_1 \\ 0 & 1 & u_2 \\ v_1 & v_2 & 1 \end{pmatrix}$. We have that

$$p_A(x) = x^3 - 3x^2 + (3 - u_1 v_1 - u_2 v_2)x + (u_1 v_1 + u_2 v_2 - 1),$$

$$p_A'(x) = 3x^2 - 6x + (3 - u_1 v_1 - u_2 v_2) = 3p_B(x) = 3x^2 - 6x + 3,$$

$$3 - u_1 v_1 - u_2 v_2 = 3.$$ 

Last equation has solutions and $\det(A) = 1 - (u_1 v_1 + u_2 v_2) = 1$ for any solution. This implies that $B$ is uniquely integrable.

Finally consider an example of non-integrable matrix.

Example 1.7. [1, Example 4] In [1, Theorem 16] it was shown that any matrix of size two or three is integrable. However if $\lambda_1 \neq \lambda_2$ then, for example, no integral exists for the following matrix of the size four: $B_{\lambda_1,\lambda_2} := \text{diag}(\lambda_1, \lambda_1, \lambda_2, \lambda_2)$, i.e., the diagonal matrix with the entries $\lambda_1, \lambda_1, \lambda_2, \lambda_2$ on the diagonal.

Indeed, assume that $A = \begin{pmatrix} \lambda_1 & 0 & 0 & u_1 \\ 0 & \lambda_1 & 0 & u_2 \\ 0 & 0 & \lambda_2 & u_3 \\ v_1 & v_2 & v_3 & v_4 \end{pmatrix}$ is an integral of $B_{\lambda_1,\lambda_2}$.

Then $p_A'(x) = 5p_{B_{\lambda_1,\lambda_2}}(x)$. Writing down the determinant of $xI - A$ we obtain

$$p_A(x) = (x - \tau(B_{\lambda_1,\lambda_2}))p_{B_{\lambda_1,\lambda_2}}(x) - (u_1 v_1 + u_2 v_2)(x - \lambda_1)(x - \lambda_2)^2 - (u_3 v_3 + u_4 v_4)(x - \lambda_1)^2(x - \lambda_2).$$

By direct substitution we have $p_A(\lambda_1) = p_A(\lambda_2) = 0$. Thus the multiplicities of $\lambda_1, \lambda_2$ as zeros of $p_A(x)$ are equal to 3, which is not possible since $\deg(p_A) = 5$. 
The following example shows that integration can break many matrix properties such as unitarity.

**Example 1.8.** Let $K = \mathbb{C}$. Consider an arbitrary unitary non-scalar operator $B$ and its integral $A$. It turns out that $A$ is not unitary. Indeed, suppose that $A$ is unitary. Denote $n = \deg(p_A(x))$. Since $p_B(x) = \frac{p_A'(x)}{n}$ then by Theorem 1.2 we obtain

$$
\sum_{i=1}^{n-1} |w_i|^2 \leq \left| \frac{1}{n} \sum_{i=1}^{n} z_i \right|^2 + \frac{n-2}{n} \sum_{i=1}^{n} |z_i|^2,
$$

where $z_1, \ldots, z_n$ are the zeros of $p_A(x)$, and $w_1, \ldots, w_{n-1}$ are the zeros of $p_B(x)$. Since the spectrum of unitary operator lies on the unit sphere we conclude

$$
|z_1| = \ldots = |z_n| = |w_1| = \ldots = |w_{n-1}| = 1.
$$

Therefore

$$
n - 1 \leq \left| \frac{1}{n} \sum_{i=1}^{n} z_i \right|^2 + n \cdot \frac{n-2}{n} \sum_{i=1}^{n} |z_i|^2.
$$

Thus $z_1 = \ldots = z_n$, hence $w_1 = \ldots = w_{n-1}$, which contradicts with $B$ being non-scalar.

**Remark 1.9.** The result of the previous example could be obtained by only using Gauss-Lucas theorem. Suppose that $A$ is unitary.

1) If $A$ is scalar then $B = PAP$, where $P$ is an operator of orthogonal projection, is also scalar.

2) If $A$ is not scalar then, since any unitary operator is diagonalizable, it has at least two distinct eigenvalues. Then $p_B(x) = \frac{p_A'(x)}{\deg(p_A(x))}$ has a zero $x_0$, such that $p_A(x_0) \neq 0$. By the Gauss-Lucas theorem $x_0$ lies in the convex hull of the zeros of $p_A(x)$. Since the spectrum of unitary operator lies on the unit sphere we obtain that $x_0$ lies in the open unit disc. Thus the spectrum of $B$ does not lie on the unit sphere, which contradicts with the unitarity of $B$.

Next we consider an example that shows us how one can construct non-integrable matrices of any even order $\geq 4$.

**Example 1.10.** Consider a diagonal matrix $B_0 = \text{diag}(\lambda_1, \ldots, \lambda_n)$ and $B = \begin{pmatrix} B_0 & 0 \\ 0 & B_0 \end{pmatrix}$. Then $B$ is integrable if and only if $B_0 = \lambda I_n$. Indeed, if the integral exists, then it has the form $A = \begin{pmatrix} B_0 & u_1 \\ 0 & B_0 \end{pmatrix}$. Denoting $X = xI_{2n} - B$, $Y = -(u_1, u_2)^T$, $Z = -(v_1, v_2)$, $W = xI_1 - \tau(B)$, by the formula for the determinant of block matrix

$$
\det \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} = \det(X) \det(W - ZX^{-1}Y)
$$

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Thus it is straightforward to see that \( A \neq 0 \), we obtain that for any \( x \) distinct from the zeros of \( p_B(x) \)

\[
p_A(x) = \det(xI_{2n+1} - A) = p_B^2(x)(x - \tau(B) - (v_1, v_2)(xI_{2n} - B)^{-1}(u_1, u_2)^\top).
\]

Since \((xI_{2n} - B)^{-1} = \text{diag}(1/(x-\lambda_1), \ldots, 1/(x-\lambda_n))\) then

\[
(v_1, v_2)(xI_{2n} - B)^{-1}(u_1, u_2)^\top = \sum_{i=1}^n \frac{t_i}{x - \lambda_i} \text{ for some } t_1, \ldots, t_n \in \mathbb{K}.
\]

Thus \( p_B(x)((v_1, v_2)(xI_{2n} - B)^{-1}(u_1, u_2)^\top) \) is a polynomial, which leads to \( p_B(x) \mid p_A(x) \). Therefore if \( A \) is an integral of \( B \) then

\[
p_A'(x) = (2n + 1)p_B^2(x).
\]

Thus any zero of \( p_A'(x) \) is a zero of \( p_A(x) \). Therefore \( p_A(x) = (x - \lambda)^{2n+1} \), \( p_B(x) = (x - \lambda)^{2n} \), \( B = \lambda I_{2n} \).

In the next example we show how an integrable non-scalar matrix with multiple eigenvalues can be produced.

**Example 1.11.** Consider the matrix \( B = \begin{pmatrix} a_{n-1} & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & \beta \end{pmatrix} \) with \( a \neq b \). Then the matrix \( A = \begin{pmatrix} a_{n-1} & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & \beta \tau(B) \end{pmatrix} \), where \( \beta = b\tau(B) - a(\tau(B) + b - a) \), is an integral of \( B \). Indeed,

\[
p_A(x) = (x-a)^{n-1}((x-b)(x-\tau(B)) - b\tau(B) + a(\tau(B) + b - a)) =
\]

\[
= (x-a)^{n-1}(x^2 - (b + \tau(B))x + a(\tau(B) + b - a)) = (x-a)^n(x - (\tau(B) + b - a)).
\]

Then \( p_A'(x) = n(x-a)^{n-1}(x - (\tau(B) + b - a)) + (x-a)^n = (x-a)^{n-1}(nx-n(\tau(B)+b-a)+x-a) = (x-a)^{n-1}((n+1)x-n\left(\frac{(n-1)a+b}{n} + b - a\right)-a) = (n+1)(x-a)^{n-1}(x-b) = (n+1)p_B(x) \).

It turns out that an integral of a diagonal matrix could be both diagonalizable or not, as the following example shows.

**Example 1.12.** Consider \( B = I_n \), then \( A_1 = I_{n+1} \) is a diagonalizable integral of \( B \), but it is straightforward to see that \( A_2 = \begin{pmatrix} I_n & 1 \\ 0 & 1 \end{pmatrix} \) is also an integral of \( B \), since \( p_{A_2}(x) = p_{A_1}(x) \), but is not diagonalizable.

In previous examples the integrability does not depend on the particular values of eigenvalues, it only depends on their multiplicities. This in not the case in general, as the following example shows.
Example 1.13. Consider $B = \text{diag} \, (a, a, b, b, c)$. Let us show that if $A = \begin{pmatrix} a & 0 & 0 & 0 & 0 & u_1 \\ 0 & a & 0 & 0 & 0 & u_2 \\ 0 & 0 & b & 0 & 0 & u_3 \\ 0 & 0 & 0 & b & 0 & u_4 \\ 0 & 0 & 0 & 0 & c & u_5 \\ v_1 & v_2 & v_3 & v_4 & v_5 & \tau(B) \end{pmatrix}$

is an integral of $B$ then $p_A(x) = (x - a)^3(x - b)^3$. Indeed,

$$p_A(x) = (x - a)^2(x - b)^2(x - c)(x - \tau(B)) - (v_1 u_1 + v_2 u_2)(x - a)(x - b)^2(x - c) - (v_3 u_3 + v_4 u_4)(x - a)^2(x - b)(x - c) - v_5 u_5(x - a)^2(x - b)^2.$$ 

Thus if $p'_A(x) = 6p_B(x)$ then

$$p_A(a) = p'_A(a) = p''_A(a) = p_A(b) = p'_A(b) = p''_A(b) = 0,$$

hence

$$v_1 u_1 + v_2 u_2 = v_3 u_3 + v_4 u_4 = 0,$$

$$p_A(x) = (x - a)^2(x - b)^2(x - c)(x - \tau(B)) - v_5 u_5(x - a)^2(x - b)^2.$$ 

Therefore

$$p'_A(a) = 2(a - b)^2((a - c)(a - \tau(B)) - v_5 u_5) = 0,$$

$$p'_A(b) = 2(a - b)^2((b - c)(b - \tau(B)) - v_5 u_5) = 0,$$

thus $(x - c)(x - \tau(B)) - v_5 u_5 = (x - a)(x - b)$ and $p_A(x) = (x - a)^3(x - b)^3$.

Thus $p'_A(x) = 6(x - a)^2(x - b)^2(x - \frac{a + b}{2})$. Hence if $c \neq \frac{a + b}{2}$ then $B$ is not integrable. On the other hand if $c = \frac{a + b}{2}$ then

$$A = \begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 & 0 \\ 0 & 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 & c & 0 \\ 0 & 0 & 0 & 0 & \tau(B) - ab & \tau(B) \end{pmatrix}$$

is an integral of $B$. Indeed,

$$p_A(x) = (x - a)^2(x - b)^2((x - c)(x - \tau(B)) + ab - c\tau(B)),$$

$$p_A(x) = (x - a)^2(x - b)^2(x^2 - (c + \tau(B))x + ab),$$

$$p_A(x) = (x - a)^2(x - b)^2(x^2 - (a + b)x + ab) = (x - a)^3(x - b)^3,$$

$$p'_A(x) = 6(x - a)^2(x - b)^2 \left( x - \frac{a + b}{2} \right) = 6p_B(x).$$

In [1, Corollary 10] it was proved that the following alternative holds: a matrix is either freely integrable, or uniquely integrable, or non-integrable. It is shown in [1, Theorem 9] that a matrix is freely integrable if and only if it is non-derogatory. However, the recognition question for integrability or non-integrability of a given matrix remained open.
In this paper we investigate integrability for any diagonalizable matrices in terms of the multiplicities of their eigenvalues. We find the conditions on multiplicities that determine if a matrix is integrable or not and show that for all other tuples there are both integrable and non-integrable matrices. Also we present a criterion for the diagonalizability of an integral of a diagonalizable matrix. Multiple integration is considered as well. As a corollary we characterize sequences of diagonalizable matrices in which each term is an integral of the previous one.

Our paper is organized as follows. Section 2 is devoted to studying the existence of a full integral of a polynomial (Theorem 2.14). Section 3 describes the relation between the integrability of a diagonalizable matrix and full integrals of its characteristic polynomial (Theorem 3.8). This section also describes the integrability for any diagonalizable matrices in terms of the multiplicities of their eigenvalues (Theorem 3.13). In Section 4 we provide a criterion for an integral to be diagonalizable and consider multiple integration (Theorem 4.1 and Corollary 4.3). In Section 5, we apply our results to obtain a dual Schoenberg type inequality for polynomials with full integrals (Theorem 5.1 and Corollary 5.2).

2 Existence of a full integral of a polynomial

In this section we investigate the relations between the zeros of a characteristic polynomial of a matrix, and its integral. It turns out that these questions can be reduced to the results on the full integrals of polynomials from the recent paper [4].

Definition 2.1. [4, Definition 1.1] We say that \( p \) is a polynomial of type \((k, m)\), where \( k \) and \( m \) are non-negative integers, if \( p \) has \( k \) different simple zeros and \( m \) different multiple zeros.

Consider some polynomials of different types.

Example 2.2. Assume that \( \lambda_1 \neq \lambda_2 \). Then \((x - \lambda_1)(x - \lambda_2)\) is the polynomial of type \((2, 0)\), \((x - \lambda_1)(x - \lambda_2)^7\) is the polynomial of type \((1, 1)\), \((x - \lambda_1)^4(x - \lambda_2)^5\) is the polynomial of type \((0, 2)\).

Definition 2.3. [4, Definition 1.3] The polynomial \( P \in \mathbb{K}[x] \) is called a full integral of the polynomial \( p \in \mathbb{K}[x] \), if \( P' = p \) and for any \( \lambda \in \mathbb{K} \) satisfying \((x - \lambda)^2\)\(|p\) we have that \((x - \lambda)\)|\(P\). In other words, any multiple zero of the polynomial \( p \) is a zero of \( P \).

Remark 2.4. The existence of full integrals splits into cases similarly as the existence of integrals of the matrices do. As was shown in [4, Lemmas 3.1, 3.2]:

1) If \( m = 0 \) then any integral of \( p \) is its full integral (which is similar to freely integrable matrix);
2) If \( m > 0 \) then either \( p \) does not have a full integral (similar to non-integrable matrix) or \( p \) has unique full integral (similar to uniquely integrable matrix).
Consider several examples.

**Example 2.5.** Let \( p(x) = x^2 \).
1. Polynomial \( P_1(x) = x^3 \) is a full integral of the polynomial \( p(x) \), since 0 is a zero of the polynomial \( P_1(x) \).
2. Polynomial \( P_2(x) = x^3 + 1 \) is an integral of the polynomial \( p(x) \), but it is not a full integral since 0 is not a zero of the polynomial \( P_2(x) \).

The full integrals of polynomials are closely related with the integrals of diagonalizable matrices. In the next section we will show that a diagonalizable matrix is integrable if and only if its characteristic polynomial has a full integral. Let us introduce the following notations and use them further.

**Notation 2.6.** By polynomial \( f \) we denote the polynomial of type \((k, m)\)
\[
f(x) = (x - a_1) \ldots (x - a_k)(x - b_1)^{\alpha_1} \ldots (x - b_m)^{\alpha_m},
\]
where \( a_1, \ldots, a_k, b_1, \ldots, b_m \in \mathbb{K} \) are pair-wise distinct, \( k, m \in \mathbb{N} \cup \{0\}, \alpha_1, \ldots, \alpha_m \in \mathbb{N} \setminus \{1\} \). We also denote
\[
q(x) := (x - b_1)^{\alpha_1} \ldots (x - b_m)^{\alpha_m},
\]
\[
Q(x) := q(x)(x - b_1) \ldots (x - b_m),
\]
\[
h(x) := (x - a_1) \ldots (x - a_k).
\]

Let \( \mathbb{K}_l[x] \) be the linear space of polynomials of the degree less than or equal to \( l \).

We denote by \( U_i \subseteq \mathbb{K}_l[x] \) the subspace of polynomials having zero value in \( b_i \), \( i = 1, \ldots, m \). \( U_0 \subset \mathbb{K}_l[x] \) denotes the subspace of polynomials of the degree strictly less than \( l \), and \( U = U_0 \cup U_1 \cup \ldots \cup U_m \).

For a fixed polynomial \( f \) we consider the map
\[
\varphi_{l,m} : \mathbb{K}_l[x] \rightarrow \mathbb{K}_{l+m-1}[x],
\]
defined by
\[
\varphi_{l,m} : g \mapsto \frac{(Qg)'}{q}.
\]

Below we provide several results concerning full integrals of polynomials proved in [4] since they appeared to be useful for matrix integrability.

**Lemma 2.7.** [4, Lemmas 2.7, 2.8] The map \( \varphi_{l,m} \) has following properties:
1. \( \varphi_{l,m} \) is a linear map;
2. the kernel \( \text{Ker} \varphi_{l,m} = 0 \);
3. if \( m > 1 \), then \( (\text{Im} \varphi_{l,m} \cup U) \subseteq \mathbb{K}_{l+m-1}[x] \), and this inclusion is strict;
4. if \( m = 1 \), then \( \varphi_{l,m} \) is invertible;
5. the image \( \text{Im} \varphi_{l,m} \notin U \).
Lemma 2.8. [4, Theorem 3.7] Let $m > k + 1$. Then $f$ does not have a full integral.

Lemma 2.9. [4, Theorem 3.8] Let $m = 1$. Then $f$ has a full integral.

Lemma 2.10. [4, Lemma 2.18] Let $f, g \in \mathbb{K}[x]$ be polynomials without common multiple zeros. Then the set $T := \{t \in \mathbb{K} \mid f + tg \in \mathbb{K}[x] \text{ has a multiple zero} \}$ is finite.

Lemma 2.11. Let $f$ has a full integral. Then $f \in q \cdot \text{Im} \varphi_{k-m+1,m}.$

Proof. Let $F \in \mathbb{K}[x]$ be a full integral of $f$. Since all multiple zeros of $f$ are $b_1, \ldots, b_k$, and they are zeros of $F$, it follows that $F = Qg$, for some $g \in \mathbb{K}[x]$. Thus $qh = f = F' = (Qg)'$, i.e. $h = \frac{(Qg)'}{q} = \varphi_{k-m+1,m}(g)$. Therefore $h \in \text{Im} \varphi_{k-m+1,m}$, and thus $f \in q \cdot \text{Im} \varphi_{k-m+1,m}$. □

Lemma 2.12. Let $m > 1$. Then there exist $a_i, b_j$ such that $f$ does not possess a full integral.

Proof. If $k + 1 < m$, then by Lemma 2.8 any polynomial $f$ does not possess a full integral.

Thus further we can assume that $k + 1 \geq m$.

Denote $\varphi := \varphi_{k-m+1,m}$ and consider $b_1, \ldots, b_m$ being fixed.

By Item 3 of Lemma 2.7 we can find the polynomial $g \in \mathbb{K}_k[x] \setminus (\text{Im} \varphi \cup U)$. Now consider the family of polynomials $H := \{g + c \mid c \in \mathbb{K} \}$. If two different polynomials $g + c_1, g + c_2 \in \text{Im} \varphi$ then

$$c_2(g + c_1) - c_1(g + c_2) = (c_2 - c_1)g \in \text{Im} \varphi,$$

which contradicts to the choice of $g$. Hence, $|\text{Im} \varphi \cap H| \leq 1$.

Similarly, for any $i = 0, \ldots, m$ if two different polynomials $g + c_1, g + c_2 \in U_i$, then

$$c_2(g + c_1) - c_1(g + c_2) = (c_2 - c_1)g \in U_i,$$

thus $|U_i \cap H| \leq 1$ for all $i = 0, \ldots, m$.

Therefore, we obtain that the set $H_0 := H \cap (\text{Im} \varphi \cup U)$ is finite.

Moreover from Lemma 2.10 we obtain that the set $H_1 := \{r \in H \mid r \text{ has multiple zero} \}$ is also finite.

Since the set $H$ is infinite then the set $H \setminus (H_0 \cup H_1)$ is non-empty. Choose an arbitrary polynomial $h \in H \setminus (H_0 \cup H_1)$ and observe that the polynomial $f = qh$ satisfies the conditions of the lemma. Indeed,

1) The polynomial $f$ has the form $f = (x - a_1)\ldots(x - a_k)(x - b_1)^{\alpha_1}\ldots(x - b_m)^{\alpha_m}$ since $h(b_i) \neq 0, i = 1, \ldots, m$ and $h$ has no multiple zeros and $\deg(h) = k$.

2) Since $h \notin \text{Im} \varphi$ then $f \notin q \cdot \text{Im} \varphi$.

Therefore from the Lemma 2.11 we obtain that the polynomial $f$ does not possess a full integral. □
Lemma 2.13. Let \( m > 1 \) and \( k + 1 \geq m \). Then there exist \( a_i, b_j \) such that \( f \) possesses a full integral.

Proof. Consider pair-wise different \( b_1, \ldots, b_m \in \mathbb{K} \) such that the polynomial \( Q(x) := (x - b_1)^{a_1 + 1} \cdots (x - b_m)^{a_m + 1} \) is a full integral of its derivative. Such \( b_1, \ldots, b_m \) exist by [4, Theorem 3.3].

Consider the map \( \varphi := \varphi_{k-m+1,m} \) from Definition 2.6. By Lemma 2.7, Item 5, there exists a polynomial \( h_1 \in \text{Im} \varphi \setminus (\text{Im} \varphi \cap U) \).

Case 1. Let \( k + 1 = m \). Set \( h_2 := h_1 \). By its definition \( h_1 = c \frac{Q'}{q} \) for some \( c \in \mathbb{K} \). Then the polynomials \( h_1, h_2 \) do not have multiple zeros. Indeed, \( Q \) is a full integral of \( Q' \) and therefore any multiple zero of \( Q' \) is a zero of \( Q \). Then any multiple zero of \( Q' \) is equal to \( b_i \) for some \( i = 1, \ldots, m \). However, \( \frac{Q'}{q} (b_i) \neq 0 \) for any \( i = 1, \ldots, m \). Thus \( \frac{Q'}{q}, h_1, \) and \( h_2 \) do not have multiple zeros.

Case 2. Let \( k + 1 > m \). Denote by \( x_1, \ldots, x_k \) the zeros of the polynomial \( h_1 \). From the definition of \( h_1 \) we have that \( x_i \neq b_j \) for all \( i = 1, \ldots, k, j = 1, \ldots, m \). Therefore, \( Q(x_i) \neq 0 \), \( q(x_i) \neq 0 \), \( i = 1, \ldots, k \).

Denote \( W_i = \{ r(x) \in \mathbb{K}[x] | r(x_i) = 0 \} \), \( i = 1, \ldots, k \). Let us show that \( \text{Im} \varphi \nsubseteq W := W_1 \cup \ldots \cup W_k \cup U \). Indeed, we consider \( \varphi(x + c) \), where \( c \in \mathbb{K} \).

\[
\varphi(x + c) = \frac{(x + c)Q'}{q} = \frac{Q + (x + c)Q'}{q}.
\]

If \( Q'(x_i) \neq 0 \), then for \( c = -\frac{Q(x_i) + x_iQ'(x_i) - q(x_i)}{Q'(x_i)} \), we obtain that \( \varphi(x + c)(x_i) = 0 \neq 1 \).

If \( Q'(x_i) = 0 \), then since \( Q(x_i) \neq 0 \) we have that \( \varphi(x + c)(x_i) = \frac{Q(x_i)}{q(x_i)} \neq 0 \).

Therefore \( \text{Im} \varphi \cap W_i \), \( i = 1, \ldots, k \), are proper subspaces of \( \text{Im} \varphi \). Since \( \text{Im} \varphi \nsubseteq U \), then \( \text{Im} \varphi \cap U \) is a proper subspace of \( \text{Im} \varphi \). Thus due to [8, Theorem 1.2] \( \text{Im} \varphi \nsubseteq W_1 \cup \ldots \cup W_k \cup U \). Hence \( \text{Im} \varphi \nsubseteq W \).

Let us consider the polynomial \( h_2 \in \text{Im} \varphi \setminus (\text{Im} \varphi \cap W) \). Since \( h_2 \notin W \), the polynomials \( h_1 \) and \( h_2 \) have no common zeros. In particular, they do not have common multiple zeros.

So, in both cases above we constructed two polynomials \( h_1, h_2 \in \text{Im} \varphi \setminus U \) that do not have common multiple zeros. Denote by

\[
H := \{ h_1 + th_2 \mid t \in \mathbb{K} \} \subset \text{Im} \varphi,
\]

\[
H_0 := \{ h_1 + th_2 \mid t \in \mathbb{K}, h_1 + th_2 \in \text{Im} \varphi \cap U \} \subset H,
\]

\[
H_1 := \{ h_1 + th_2 \mid t \in \mathbb{K}, h_0 + h_1 \text{ has multiple zeros} \} \subset H
\]

From Lemma 2.10 we have that the set \( H_1 \) is finite.

Let us show that the cardinality \( |H_0| \) is at most 1.

Assume that \( h_1 + t_1 h_2, h_1 + t_2 h_2 \in H_0, t_1 \neq t_2 \). Then \( h_1 + t_1 h_2, h_1 + t_2 h_2 \in U \). Therefore, \( (h_1 + t_1 h_2) - (h_1 + t_2 h_2) = (t_1 - t_2) h_2 \in U \). Thus \( h_2 \in U \) which contradicts to the definition of \( h_2 \).
Therefore since the set \( H \subset \text{Im} \varphi \) is infinite and the sets \( H_0 \) and \( H_1 \) are finite we can choose a certain polynomial \( h \in H \setminus (H_0 \cup H_1) \).

It remains to show that the polynomial \( f := qh \) satisfies the conditions of the lemma. Indeed,

1) \( f \) has the form \( f = (x - a_1) \ldots (x - a_k)(x - b_1)^{\alpha_1} \ldots (x - b_m)^{\alpha_m} \), since \( h(b_i) \neq 0 \), by its construction \( h \) does not have multiple zeros, and we have \( \deg(h) = k \).

2) The polynomial \( Q\varphi^{-1}(h) \) is a full integral of the polynomial \( f \) since an arbitrary multiple zero of \( f \) is a zero of \( q \). Therefore, the same holds for the zeros of polynomial \( Q \) and, moreover, for the polynomial \( Q\varphi^{-1}(h) \). Also

\[
f = qh = q\varphi(\varphi^{-1}(h)) = q\frac{(Q\varphi^{-1}(h))'}{q} = (Q\varphi^{-1}(h))'.
\]

\[\square\]

**Theorem 2.14.** Let \( f \in \mathbb{K}[x] \) be a polynomial of the type \((k, m)\), \( k \geq 0, m \geq 0 \). Then the following alternative is true:

1) If \( m \leq 1 \) then the polynomial \( f \) has a full integral.

2) If \( m > k + 1 \) then the polynomial \( f \) does not have a full integral.

3) For any pair \((k, m)\) which does not satisfy 1) and 2) and any sequence \( \alpha_1, \ldots, \alpha_m \) such that \( \alpha_i > 1 \), there are both possibilities:

   a) there exists a polynomial \( f_1 \) of the type \((k, m)\) with the multiplicities \( \alpha_1, \ldots, \alpha_m \) of multiple zeros, such that there exists a full integral of \( f_1 \), and

   b) there exists a polynomial \( f_2 \) of the type \((k, m)\) with the multiplicities \( \alpha_1, \ldots, \alpha_m \) of multiple zeros, such that there is no full integral of \( f_2 \).

**Proof.** The first item is a direct application of Lemma 2.9. The second one is proved in Lemma 2.8. Lemma 2.13 implies Condition 3a) and Lemma 2.12 implies Condition 3b).

\[\square\]

### 3 Matrix integrability and full integrability of polynomials

The following result is proved in [1] for matrices over the field of complex numbers, however, its proof holds for an arbitrary field \( \mathbb{K} \).

**Lemma 3.1.** [1, Lemma 7] If \( A \in M_{n+1}(\mathbb{C}) \) is an integral of \( B \in M_n(\mathbb{C}) \) with corresponding integrator \((v, u)\), here \( v, u \in \mathbb{C}^n \), then for any \( X \in GL_n(\mathbb{C}) \) it holds that \( \begin{pmatrix} X & 0 \\ 0 & 1 \end{pmatrix} A \begin{pmatrix} X^{-1} & 0 \\ 0 & 1 \end{pmatrix} \) is an integral of \( XBX^{-1} \) with corresponding integrator \((vX^{-1}, uX^T)\).

This lemma allows one to reduce different questions concerning diagonalizable matrices to the case of diagonal ones. Therefore below we restrict ourselves to the diagonal matrices.
Notation 3.2. Denote $\mathcal{B} = \text{diag}(b_1, \ldots, b_1, \ldots, b_m, \ldots, b_m, a_1, \ldots, a_k) \in M_n(\mathbb{K})$ with the characteristic polynomial $p_\mathcal{B} = f$, from Notation 2.6, i.e.

$$p_\mathcal{B} = (x - a_1) \ldots (x - a_k)(x - b_1)^{\alpha_1} \ldots (x - b_m)^{\alpha_m}.$$ 

We denote $\mathcal{A} = \begin{pmatrix} \mathcal{B} & u^\top \\ v & \tau(\mathcal{B}) \end{pmatrix} \in M_{n+1}(\mathbb{K})$, where $v = (v_1, \ldots, v_n), u = (u_1, \ldots, u_n) \in \mathbb{K}^n$. We also define $C_0 = 0$, $C_i = C_{i-1} + \alpha_i, i = 1, \ldots, m$ and $d_j = C_m + j$, here $j = 1, \ldots, k$.

Below we always assume that $\mathcal{A}$ and $\mathcal{B}$ are as in Notation 3.2.

Lemma 3.3. The following decomposition holds:

$$p_\mathcal{A}(x) = (x - \tau(\mathcal{B}))p_\mathcal{B}(x) - \sum_{i=1}^n u_i v_i \frac{p_\mathcal{B}(x)}{x - \lambda_i},$$

(3)

where $\lambda_i$ is the element of $\mathcal{B}$ located at position $(i, i)$.

Proof. Follows from the Laplace decomposition of $p_\mathcal{A}(x) = \det(xI - \mathcal{A})$ by the last column since $\mathcal{B}$ is diagonal.

Corollary 3.4. Let $\lambda \in \mathbb{K}$ be an eigenvalue of $\mathcal{B}$ of the multiplicity $l > 1$. If $\mathcal{A}$ is an integral of $\mathcal{B}$ then $\lambda$ is an eigenvalue of $\mathcal{A}$ of the multiplicity $l + 1$ and $\sum_{j: \lambda_j = \lambda} u_j v_j = 0$, where $\lambda_j$ is the element of $\mathcal{B}$ located at position $(j, j)$.

Proof. We compute $p_\mathcal{A}(\lambda)$ by the formula (3). Each summand is zero since $\lambda$ is a multiple zero of $p_\mathcal{B} = (x - \lambda_1) \ldots (x - \lambda_n)$. Hence $p_\mathcal{A}(\lambda) = 0$. Since $p'_\mathcal{A}(x) = (n + 1)p_\mathcal{B}$, it follows that $\lambda$ is a zero of $p_\mathcal{A}$ of the multiplicity $l + 1$.

Thus $(x - \lambda)^{l+1} | p_\mathcal{A}(x)$. In particular $(x - \lambda)^l | p_\mathcal{A}(x)$.

Since $(x - \lambda)^l | p_\mathcal{B}(x)$ it follows that $(x - \lambda)^l | (x - \tau(\mathcal{B}))p_\mathcal{B}(x)$ and for $\lambda_i \neq \lambda$ it holds that $(x - \lambda)^l | \frac{p_\mathcal{B}(x)}{x - \lambda_i}$.

Therefore by (3) we get $(x - \lambda)^l | \sum_{j: \lambda_j = \lambda} u_j v_j \frac{p_\mathcal{B}(x)}{x - \lambda_j}$. Then from $(x - \lambda)^l | \frac{p_\mathcal{B}(x)}{x - \lambda}$ it follows that

$$\sum_{j: \lambda_j = \lambda} u_j v_j = 0.$$

Lemma 3.5. Two coefficients at the two highest degrees of $p_\mathcal{A}(x)$ do not depend on the choice of the vectors $v$ and $u$.

Proof. In the formula (3) the degrees of all summands except the first one do not exceed $\deg(p_\mathcal{A}) - 2$. 

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Lemma 3.6. Let $A$ be an integral of $B \in M_n(\mathbb{K})$. Then

$$p_A(x) = (x - \tau(B))p_B(x) - \sum_{i=1}^{k} u_i v_i \frac{p_B(x)}{x - a_i}. \quad (4)$$

Proof. Separating the summands in the formula (3) into the two sums corresponding to multiple and simple zeros, we have by Lemma 3.3 that

$$p_A(x) = (x - \tau(B))p_B(x) + \sum_{i=1}^{k} y_i \frac{p_B(x)}{x - a_i} + \sum_{i=1}^{m} z_i \frac{p_B(x)}{x - b_i},$$

where $y_i = -u_i v_i$, $i = 1, \ldots, k$ and $z_i = \sum_{j=C_{i-1}+1}^{C_i} -u_j v_j$, $i = 1, \ldots, m$.

By Corollary 3.4 we obtain that $z_i = 0$, $i = 1, \ldots, m$. Therefore

$$p_A(x) = (x - \tau(B))p_B(x) + \sum_{i=1}^{k} y_i \frac{p_B(x)}{x - a_i}.$$ 

□

Corollary 3.7. Let $\mathbb{K} = \mathbb{C}$ and $A$ be an integral of $B$ and let

$$A' = \begin{pmatrix} B & u' \tau(B) \\ v' & \tau(B) \end{pmatrix}, \text{ where } u'_i = v'_i = \begin{cases} \sqrt{u_i v_i}, & i = d_1, \ldots, n, \\ 0, & i = 1, \ldots, C_m. \end{cases}$$

Then

1. $A'$ is also an integral of $B$.
2. $||A'||_F^2 = ||B||_F^2 + ||\tau(B)||_F^2 + 2 \sum_{i=1}^{k} |u_i v_i|$.
3. For any integral $A'' = \begin{pmatrix} B & u'' \tau(B) \\ v'' & \tau(B) \end{pmatrix}$ of $B$ with $p_{A''} = p_{A'}$ it holds that $||A'||_F^2 \leq ||A''||_F^2$.

Proof. 1. Applying the formula (4) we get

$$p_{A'}(x) = (x - \tau(B))p_B(x) - \sum_{i=1}^{k} u_i' v_i' \frac{p_B(x)}{x - a_i} = (x - \tau(B))p_B(x) - \sum_{i=1}^{k} u_i v_i \frac{p_B(x)}{x - a_i} = p_A(x).$$

Therefore $A'$ is an integral of $B$.

2. Let us compute the Frobenius norm of $A'$ by the definition of a norm and taking into account that $B$ is a submatrix of $A'$ and definition of $u'_i, v'_i$

$$||A'||_F^2 = ||B||_F^2 + ||\tau(B)||_F^2 + ||u'||^2 + ||v'||^2 = ||B||_F^2 + ||\tau(B)||_F^2 + 2 \sum_{i=1}^{k} |u_i v_i|.$$
3. Since $A''$ is an integral of $B$, the equality (4) implies

$$p_{A''}(x) = (x - \tau(B))p_B(x) - \sum_{i=1}^{k} u_i''v_i'' \frac{p_B(x)}{x - a_i}. \tag{5}$$

By the conditions $p_A = p_{A''}$. It follows that

$$u_i''v_i'' \frac{p_B}{x - a_i}(a_i) = -p_{A''}(a_i) = -p_A(a_i) = u_i v_i \frac{p_B}{x - a_i}(a_i), \quad i = 1, \ldots, k.$$ 

Therefore $u_i''v_i'' = u_i v_i, \quad i = 1, \ldots, k$. Observe that

$$||A''||^2_F = ||B||^2_F + ||\tau(B)||^2 + ||u''||^2 + ||v''||^2 \geq ||B||^2_F + ||\tau(B)||^2 + \sum_{i=1}^{k} |u_i''|^2 + \sum_{i=1}^{k} |v_i''|^2.$$ 

Combining with the item 2 we obtain that to prove $||A'||^2_F \leq ||A''||^2_F$, it is sufficient to show $|u_i''|^2 + |v_i''|^2 \geq 2|u_i v_i|$, which holds because

$$0 \leq (|u_i''| - |v_i''|)^2 = |u_i''|^2 + |v_i''|^2 - 2|u_i''v_i''| = |u_i''|^2 + |v_i''|^2 - 2|u_i v_i|. \tag{5}$$

The following statement summarizes our previous study.

**Theorem 3.8.** $B$ is integrable if and only if $p_B(x)$ has a full integral.

**Proof.** Let us prove the necessity. Let $B$ be integrable and $A$ be an integral of $B$. Then by definition $p_A' = (n + 1)p_B$ and by Lemma 3.6

$$p_A = (x - \tau(B))p_B + \sum_{i=1}^{k} w_i \frac{p_B}{x - a_i}, \quad w_i \in K. \tag{5}$$

Substituting $x = b_i$ to the formula (5) one has that $p_A(b_i) = 0$ for all $i = 1, \ldots, m$. Thus $F := \frac{1}{n+1}p_A$ is a full integral of $p_B$.

Let us prove the sufficiency. Assume now that there exists a full integral $F$ of $p_B$. Let us show that there exist $v_1, \ldots, v_d \in K$ such that $A$ is an integral of $B$, where the couple of vectors $v = (0, \ldots, 0, v_{d_1}, \ldots, v_{d_k})$ and $u = (1, \ldots, 1)$ is the corresponding integrator.

From the formula for $p_A$ and the definition for $F$ we have that $q$ divides $p_A$ and $F$. Recalling $h, q$ from Notation 2.6 we denote

$$\tilde{p}_A := \frac{p_A}{q}, \quad \tilde{F} := \frac{(n + 1)F}{q}, \quad h_i := \frac{h}{x - a_i}, \quad g := p_A - (n + 1)F, \quad i = 1, \ldots, k.$$
Consider the equation \( p_A = (n + 1)F \). If we take \( v = (0, \ldots, 0, v_{d_1}, \ldots, v_{d_k}) \) and \( u = (1, \ldots, 1) \) then this becomes an equation with \( k \) variables \( v_{d_1}, \ldots, v_{d_k} \). We now show that \( v_{d_i} = \frac{(n+1)F(a_i)}{h_{a_i}(a_i)}, \ i = 1, \ldots, k \) is the solution for this equation. By the direct substitution of \( a_i \) and the chosen values of \( u \) and \( v \) into the formula for \( p_A \) we obtain that \( p_A(a_i) = v_{d_i} \cdot h_{a_i}(a_i), \ i = 1, \ldots, k \). So
\[
g(a_i) = p_A(a_i) - (n + 1)F(a_i) = v_{d_i} \cdot h_{a_i}(a_i) - (n + 1)F(a_i) = 0, \ i = 1, \ldots, k.
\]
By Lemma 3.5 and the definition of \( F \) we obtain that the coefficients at monomials \( x^{n+1} \) and \( x^n \) in polynomials \( p_A \) and \( (n + 1)F \) are equal. Therefore \( \deg(g) \leq n - 1 \). Since \( q(a_i) \neq 0, \ i = 1, \ldots, k \), it follows from
\[
0 = g(a_i) = q(a_i)(\tilde{p}_A(a_i) - \tilde{F}(a_i)), \ i = 1, \ldots, k,
\]
that \( \tilde{p}_A(a_i) = \tilde{F}(a_i), \ i = 1, \ldots, k \), and since \( \deg(\tilde{p}_A - \tilde{F}) \leq n - 1 - \deg(q) = k - 1 \), then \( \tilde{p}_A = \tilde{F} \). Thus \( p_A = q\tilde{p}_A = q\tilde{F} = (n + 1)F \). Hence for \( v_{d_i} = \frac{(n+1)F(a_i)}{h_{a_i}(a_i)}, \ i = 1, \ldots, k \) we get
\[
p'_A(x) = (n + 1)F' = (n + 1)p_B.
\]

**Corollary 3.9.** Let \( A \) be an integral of \( B \). Then \( \frac{1}{n+1}p_A \) is a full integral of \( p_B \).

**Proof.** Directly shown at the end of the proof of sufficiency of Theorem 3.8.

**Corollary 3.10.** If \( p_B(x) \) has a full integral \( F(x) \) then an integrator of \( B \) can be chosen as follows \( u_i = 1, \ i = 1, \ldots, n, \ v_1 = \ldots = v_{C_m} = 0, \ v_{d_i} = \frac{(n+1)F(a_i)}{h_{a_i}(a_i)}, \ i = 1, \ldots, k \). In this case \( p_A(x) = (n + 1)F(x) \).

**Proof.** Directly shown in the proof of Theorem 3.8.

**Remark 3.11.** Corollary 3.10 does not describe all possible integrators and corresponding integrals. For example, if \( (u, v) \) is an integrator of \( B \), then for any \( s \in \mathbb{K} \setminus \{0\} \) the pair of vectors \((su, s^{-1}v)\) is also an integrator of \( B \), which is not described by Corollary 3.10. Indeed, as it is shown in Lemma 3.3 the characteristic polynomial depends only on the products of the coordinates of the vectors \( u \) and \( v \) with the equal indices. The integral \( A \) is determined by the choice of the integrators \( u, v \).

**Corollary 3.12.** Let \( p_B(x) \) has a full integral \( F(x) \). Then the formula
\[
u_i = \begin{cases} 0, & i = 1, \ldots, C_m, \\ \sqrt{\frac{(n+1)F(a_i)}{h_{a_i}(a_i)}}, & i = C_m + 1, \ldots, n. \end{cases}
\]
for integrators determines the integral \( A \) of \( B \) with \( p_A(x) = (n + 1)F(x) \) such that its Frobenius norm is the least possible. In this case \( ||A||_F^2 = ||B||_F^2 + ||\tau(B)||^2 + 2 \sum_{i=1}^{k} \frac{(n+1)F(a_i)}{h_{a_i}(a_i)} \).
Theorem 3.13. Let $\mathcal{B}$ be a diagonal matrix introduced in Notation 3.2. Then
1) if $m \leq 1$ then the matrix $\mathcal{B}$ has an integrator,
2) if $m > k + 1$ then the matrix $\mathcal{B}$ does not have an integrator,
3) in the other cases the existence of integrators depends on the values of the eigenvalues of $\mathcal{B}$, i.e. for any sequence of multiplicities there are eigenvalues for which an integrator exists and there are eigenvalues for which integrator does not exist.

Proof. By Theorem 3.8 the integrability of a matrix is equivalent to the full integrability of its characteristic polynomial. Then Theorem 2.14 is applicable and concludes the proof.

Remark 3.14. The subset of integrable matrices is dense in $M_n(\mathbb{C})$ and the subset of non-integrable matrices is sparse.

Proof. Indeed, the subset of non-derogatory diagonalizable matrices is dense, by the first item of Theorem 3.13 such matrices are integrable. The complement to the subset of non-derogatory matrices is sparse, therefore the subset of non-integrable matrices is sparse.

Lemma 3.15. Let $m > 1$ and $q(x) \in \mathbb{C}[x]$ be fixed. Denote by $S \subseteq M_n(\mathbb{C})$ the subset of matrices such that $q$ is a factor of their characteristic polynomials. Then the subset of non-integrable matrices $S_1 \subseteq S$ is dense in $S$ and the subset of integrable matrices $S_2 \subseteq S$ is sparse in $S$.

Proof. By Lemma 2.11 if $f \notin q \cdot \text{Im } \varphi_{k-m+1,m}$ then $f$ does not possess a full integral.

Since $$\dim \text{Im } \varphi_{k-m+1,m} = k - m + 2 < k + 1 = \dim \mathbb{C}_k[x],$$

one has $\text{Im } \varphi_{k-m+1,m}$ is sparse in $\mathbb{C}_k[x]$ and $q \cdot \text{Im } \varphi_{k-m+1,m}$ is sparse in $q \cdot \mathbb{C}_k[x]$.

Consider the map $$\varphi : S \rightarrow q \cdot \mathbb{C}_k[x],$$ $$\varphi(M) = p_M(x).$$

Since $\varphi$ is continuous then $\varphi^{-1}(q \cdot \text{Im } \varphi_{k-m+1,m})$ is sparse in $S$ as a preimage of sparse subset under the action of the surjective continuous map $\varphi$. Hence $S_2 \subseteq \varphi^{-1}(q \cdot \text{Im } \varphi_{k-m+1,m})$ is sparse in $S$.

Since $S = S_1 \cup S_2$ then $S_1$ is dense in $S$.

4 Diagonalizability of the integral

Theorem 4.1. Let $\mathcal{A}$ be an integral of $\mathcal{B}$. Then $\mathcal{A}$ is diagonalizable if and only if the following two conditions are satisfied simultaneously for the integrators $u, v$:
the characteristic polynomial \( \lambda \) is 2. Thus \( \lambda^2 \).

\[
\begin{align*}
\text{Proof.} \quad & \text{By Corollary 3.9 the characteristic polynomial } p_A(x) \text{ is a full integral of } \frac{1}{n+1} p_B(x). \\
& \text{Hence, } p_A(x) = Q(x)H(x) \text{ for some } H(x) \in \mathbb{K}[x], \text{ here } Q(x) \text{ is defined by Notation 2.6.}
\end{align*}
\]

1. The multiplicities of the zeros of \( H(x) \) are less than or equal to 2. Indeed, the zeros of \( H(x) \) of the multiplicity greater than 2 are the multiple zeros of \( p_B(x) \). But all the multiple zeros of \( p_B(x) \) are included into \( Q(x) \), and thus cannot be the zeros of \( H(x) \).

2. Let us prove the necessity. Assume that the conditions 1 and 2 are satisfied. To show that \( A \) is diagonalizable we calculate \( \dim \ker(A - \lambda I) \) for all multiple eigenvalues \( \lambda \) of \( A \) in order to show that the geometric multiplicity of each eigenvalue coincides with its algebraic multiplicity. The general situation splits into the following two cases since the only multiple zeros of \( p_A(x) \) are the zeros of \( p_B(x) \).

2.1. \( \lambda = b_i, \ i = 1, \ldots, m \). Without loss of generality we assume that \( i = 1 \). As the multiplicity of \( \lambda \) in \( A \) is \( \alpha_1 + 1 \), we need to show that \( \dim \ker(A - \lambda I) = \alpha_1 + 1 \). For any \( j = 1, \ldots, \alpha_1 \) the vector \( e_j \in \ker(A - \lambda I) \), since

\[
(A - \lambda I)e_j = \begin{pmatrix}
O_{\alpha_1} \oplus \bigoplus_{i=2}^m (b_i - b_1)I_{\alpha_i} \oplus \\
0 \cdots 0 \cdots u_{d_1} \\
\vdots \ddots \ddots \ddots \ddots \\
0 \cdots 0 a_k - b_1 u_n \\
v_{d_1} \cdots v_n \tau(B)
\end{pmatrix}
\]

Therefore \( \dim \ker(A - \lambda I) \geq \alpha_1 \).

Consider the submatrix \( A' \in M_{k+1}(\mathbb{K}) \) of \( A \) such that \( A = \text{diag} \{ b_1, \ldots, b_j, \ldots, b_m, \ldots, b_m \} \)

\[\oplus A'.\]

Then \( p_A(x) = q(x)p_{A'}(x) \). Since \( (x - b_1)^{\alpha_1 + 1} \mid p_A(x) \), it follows that \( p_{A'}(b_1) = 0. \) Hence \( \det(A' - \lambda I) = 0 \) and there exists a vector \( w = (w_1, \ldots, w_{k+1}) \) such that

\[
(A' - b_1 I)w^\top = 0.
\]

This provides the \( (\alpha_1 + 1) \)-st vector in \( \ker(A - \lambda I) \), i.e.:

\[
(A - \lambda I)(0, \ldots, 0, w_1, \ldots, w_{k+1})^\top = 0.
\]

It is straightforward to see that \( \{ e_1, \ldots, e_{\alpha_1}, w \} \) is a linearly independent system of vectors.

Thus \( \dim \ker(A - \lambda I) = \alpha_1 + 1 \) equals to the algebraic multiplicity of \( \lambda \). Hence the Jordan block corresponding to \( \lambda \) is diagonal.

2.2. \( \lambda \notin \{b_1, \ldots, b_m\} \). It is shown in the item 1 that in this case the multiplicity of \( \lambda \) is 2. Thus \( \lambda = a_j \) for some \( j = 1, \ldots, k \). Without loss of generality we assume \( j = 1 \). Then by the assumptions we have \( v_{d_1} = w_{d_1} = 0. \) Thus

\[
(A - \lambda I)(0, \ldots, 0, 1, 0, \ldots, 0)^\top = 0. \quad (6)
\]

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To construct the other vector we consider the submatrix \( A'' \in M_k(\mathbb{K}) \) of \( A \) such that
\[
A = \text{diag} \left\{ b_1, \ldots, b_1, \ldots, b_m, \ldots, b_m, a_1 \right\} \oplus A''.
\]

Then \( p_A(x) = q(x)(x-a_1)p_{A''}(x) \). Since \((x-a_1)^2 \mid p_A(x)\) then \( p_{A''}(a_1) = 0 \). Hence there exists a vector \( z = (z_1, \ldots, z_k) \) such that \((A''-\lambda I)z^T = 0\). Then \((A-\lambda I)(0, \ldots, 0, z_1, \ldots, z_k)^T = 0\). Together with the equality \((6)\) this implies \( \dim \ker(A-\lambda I) = 2 \).

3. Let us prove the sufficiency. Assume that some coordinate of \( u \) and \( v \) corresponding to the common eigenvalues of \( A \) and \( B \) is nonzero. We can consider two cases:

3.1. The common eigenvalue of \( A \) and \( B \) is a multiple eigenvalue of \( B \). Since all the multiple zeros of \( p_B \) are the zeros of \( p_A \), it follows that \( m > 0 \). Without loss of generality \( u_1 \neq 0 \). Consider the vector \( w' = \mu c_{1} + r c_{1} + 1 + \mu c_{1} + 2 r c_{1} + 2 + \cdots + \mu n r n + \mu r 1 \), where \( r_i \) is the \( i \)-th row of \((A - b_1 I)\). Then for its coordinates we have
\[
\begin{align*}
 w'_{1} &= w'_{2} = \ldots = w'_{C_{1}} = 0, \\
 w'_{C_{1}+i} &= \mu c_{1} + i (b_{2} - b_{1}), \ i = 1, \ldots, \alpha_{1}, \\
 w'_{C_{2}+i} &= \mu c_{2} + i (b_{3} - b_{1}), \ i = 1, \ldots, \alpha_{2}, \\
 \vdots & \\
 w'_{C_{m-1}+i} &= \mu c_{m-1} + i (b_{m} - b_{1}), \ i = 1, \ldots, \alpha_{m}, \\
 w'_{d_{i}} &= \mu d_{i} (a_{i} - b_{1}), \ i = 1, \ldots, k.
\end{align*}
\]

If \( w' = 0 \) then \( \mu_i = 0 \), \( i = C_{1}, \ldots, n \). In this case \( w'_{n+1} = \mu_1 u_1 \), so \( \mu_1 = 0 \). This means that \( w' = 0 \) if and only if \( \mu_1 = \ldots = \mu_n = 0 \). Therefore the first row of \((A - b_1 I)\) and the rows with the indices \( \alpha_1 + 1, \ldots, n \) form a linearly independent set. Hence the rank of \((A - b_1 I)\) is at least \( n - \alpha_1 + 1 \). Therefore \( \dim \ker(A - b_1 I) = n + 1 - rk(A) \leq \alpha_1 < \alpha_1 + 1 \).

Thus \( A \) is not diagonalizable.

3.2. The common eigenvalue of \( A \) and \( B \) is a simple eigenvalue of \( B \). Then \( k > 0 \). Without loss of generality \( u_{d_{i}} \neq 0 \), \( p_A(a_1) = 0 \). Consider the vector \( w'' = \mu_1 r_1 + \mu_2 r_2 + \cdots + \mu n r n \), where \( r_i \) is the \( i \)-th row of \((A - a_1 I)\). Then for its coordinates we have
\[
\begin{align*}
 w''_{d_{i}} &= 0, \\
 w''_{d_{i}} &= \mu d_{i} (a_{i} - a_{1}), \ i = 2, \ldots, k, \\
 w''_{c_{0}+i} &= \mu c_{0} + i (b_{1} - a_{1}), \ i = 1, \ldots, \alpha_{1}, \\
 w''_{c_{1}+i} &= \mu c_{1} + i (b_{2} - a_{1}), \ i = 1, \ldots, \alpha_{2}, \\
 \vdots & \\
 w''_{c_{m-1}+i} &= \mu c_{m-1} + i (b_{m} - a_{1}), \ i = 1, \ldots, \alpha_{m}.
\end{align*}
\]

If \( w'' = 0 \) then \( \mu_i = 0 \), \( i = 1, \ldots, n \), \( i \neq d_{i} \). In this case \( w''_{n+1} = \mu d_{i} u_{d_{i}} \), thus \( \mu d_{i} = 0 \). This means that \( w = 0 \) if and only if \( \mu_1 = \ldots = \mu_n = 0 \). Therefore the first \( n \) rows of
\((A - a_1I)\) form linearly independent set. So the rank of \((A - a_1I)\) is at least \(n\). Hence \(\dim \ker(A - a_1I) = n + 1 - rk(A) < 2\). Thus \(A\) is not diagonalizable. \(\square\)

**Corollary 4.2.** Let \(B\) be an integrable diagonalizable matrix. Then among the integrals of \(B\) there are both non-diagonalizable matrices and diagonalizable matrices.

**Proof.** Let \(A = \begin{pmatrix} B & u^\top \\ v & \tau(B) \end{pmatrix}\) be an integral of \(B\).

1) Assume that \(A\) has at least one common eigenvalue with \(B\). For given vectors \(v, u\) consider the vectors

\[
\begin{align*}
\nu' &= (1, \ldots, 1), \\
u &= (u_1', \ldots, u_n'), \text{ where } u_i' &= \begin{cases} u_i/v_i, & v_i \neq 0; \\
0, & v_i = 0. \end{cases}
\end{align*}
\]

From Lemma 3.3 we obtain that \(A' = \begin{pmatrix} B & u'^\top \\ v' & \tau(B) \end{pmatrix}\) is an integral of \(B\), since \(p_A(x) = p_{A'}(x)\). By Theorem 4.1 \(A'\) is not diagonalizable.

1.1) If \(B\) has a multiple eigenvalue, then any integral of \(B\) has a common eigenvalue with \(B\) due to Corollary 3.4. Thus by the item 1 one can construct a non-diagonalizable integral of \(B\).

1.2) Otherwise \(B\) is non-derogatory, so by [1, Theorem 9] for any \(t \in \mathbb{K}\) there exists such integral \(A_t\) of \(B\) that \(p_{A_t}(x) = p_A(x) - t\). Denote by \(\lambda\) some eigenvalue of \(B\). Then \(\lambda\) is an eigenvalue of \(A_{p_A(\lambda)}\). Thus \(A_{p_A(\lambda)}\) is an integral of \(B\) with a common eigenvalue with \(B\), therefore by the item 1 one can construct a non-diagonalizable integral of \(B\).

2) For given \(v, u\) consider vectors \(v'' = (v''_1', \ldots, v''_n'), u'' = (u''_1', \ldots, u''_n')\), where

\[
\begin{align*}
u''_i &= \begin{cases} 0, & v_iu_i = 0; \\
u_i, & u_i \text{ corresponds to a multiple eigenvalue;}
\end{cases} \\
u''_i &= \begin{cases} 0, & u_i, \text{ otherwise;} \\
u_i, & v_i \text{ corresponds to a multiple eigenvalue;}
\end{cases}
\end{align*}
\]

From Lemma 3.3 and Lemma 3.4 we obtain that \(A'' = \begin{pmatrix} B & u'^\top \\ v'' & \tau(B) \end{pmatrix}\) is an integral of \(B\), since \(p_A(x) = p_{A''}(x)\). Let \(\lambda\) be a simple eigenvalue of \(B\), which is also an eigenvalue of \(A\). Then by Lemma 3.3 we obtain

\[
0 = p_A(\lambda) = -u_i v_i \frac{p_B}{(x - \lambda)}(\lambda), \text{ for some } i.
\]

Since \(\frac{p_B}{(x - \lambda)}(\lambda) \neq 0\), then \(u_i v_i = 0\), so \(u''_i = v''_i = 0\). Thus any coordinate of \(v'', u''\) that corresponds to a common eigenvalue of \(A''\) and \(B\) is equal to 0. Hence by Theorem 4.1 \(A''\) is diagonalizable. \(\square\)
Corollary 4.3. There exists a sequence of diagonalizable matrices \( B = B_1, B_2, \ldots \), such that \( B_{i+1} \) is an integral of \( B_i \), \( i \in \mathbb{N} \) if and only if there exists a sequence of polynomials \( p_B(x) = p_1(x), p_2(x), \ldots \), such that \( p_{i+1}(x) \) is a full integral of \( p_i(x), i \in \mathbb{N} \).

Proof. 1. Let us prove the necessity. If the sequence \( B = B_1, B_2, \ldots \), such that \( B_{i+1} \) is an integral of \( B_i \), \( i \in \mathbb{N} \), exists then by Corollary 3.9 the sequence \( p_B(x), \frac{1}{\deg p_B(x)} p_B(x), \ldots \) has the desired property.

2. Let us prove the sufficiency. If the sequence \( p_B(x) = p_1(x), p_2(x), \ldots \), such that \( p_{i+1}(x) \) is a full integral of \( p_i(x), i \in \mathbb{N} \), exists then by taking an diagonalizable integral we obtain the sequence \( B_1, B_2, \ldots \) with the desired property. \( \square \)

Let us remind that \( k \) is the number of the simple zeros of a polynomial \( p \), and \( m \) is the number of its different multiple roots, in accordance with Definition 2.1.

Lemma 4.4. Let \( m = 0 \). Then there exists a sequence of diagonalizable matrices \( B = B_1, B_2, \ldots \), where \( B_{i+1} \) is an integral of \( B_i \), \( i \in \mathbb{N} \).

Proof. If \( m = 0 \) then \( B \) is integrable by Theorem 3.13. Let \( A \) be an integral of \( B \). Since \( B \) is non-derogatory then by [1, Theorem 9] for any \( t \in \mathbb{K} \) there exists an integral \( A_t \) of \( B \) such that \( p_{A_t}(x) = p_A(x) - t \). Taking \( t \) different from \( p_A(\lambda), \lambda \in \text{spec}(B) \) we obtain that \( p_A(\lambda) \neq 0 \), thus \( p_{A_t}(x) \) has no multiple zeros and \( A_t \) is non-derogatory. Therefore for any non-derogatory diagonalizable matrix there exists a non-derogatory diagonalizable integral. Thus we can construct the desired sequence. \( \square \)

Lemma 4.5. Consider a sequence of diagonalizable matrices \( B = B_1, B_2, \ldots, B_l \), where \( B_{i+1} \) is an integral of \( B_i \) for each \( i = 1, \ldots, l-1 \). Then \( m \leq 1 + \frac{k}{l-1} \).

Proof. Let \( F(x) \) be a full integral of \( p_B(x) = (x-a_1) \cdots (x-a_k)(x-b_1)^{\alpha_1} \cdots (x-b_m)^{\alpha_m} \), where \( a_1, \ldots, a_k, b_1, \ldots, b_m \in \mathbb{K} \) are pair-wise distinct and \( \alpha_1, \ldots, \alpha_m \in \mathbb{N} \setminus \{1\} \), i.e., \( p_B(x) \) has \( k \) simple zeros and \( m \) multiple zeros. Then \((x-b_1)^{\alpha_1+1} \cdots (x-b_m)^{\alpha_m+1}\) \( F(x) \). Thus \( F(x) \) has at least \( m \) multiple zeros. Since \( \deg F(x) = \deg p_B(x) + 1 \) then \( F(x) \) has at most \( k + 1 - m \) zeros different from \( b_1, \ldots, b_m \). Hence \( F(x) \) has at most \( k + 1 - m \) simple roots. If \( m > 1 \) then \( k + 1 - m < k \). Hence \( F(x) \) has less simple zeros than \( p_B(x) \). Therefore the number of simple zeros of \( p_{B_i}(x) \) is at most \( k - (m-1)(l-1) \) and the number of multiple zeros is at least \( m \). Therefore by Theorem 3.13 we have

\[
k - (m-1)(l-1) \geq m - 1, \]

\[
k \geq (l-1)(m-1), \]

\[
m \leq 1 + \frac{k}{l-1}. \]

\( \square \)
Lemma 4.6. Let \( m = 1 \) and \( k < 2 \). Then there exists a sequence of diagonalizable matrices \( \mathcal{B} = B_1, B_2, \ldots \), where \( B_{i+1} \) is an integral of \( B_i \), \( i \in \mathbb{N} \).

Proof. 1. If \( k = 0 \) then \( \mathcal{B} = \lambda I_n \) and the sequence \( \lambda I_n, \lambda I_{n+1}, \ldots \) satisfies the conditions of the lemma.

2. If \( k = 1 \) then \( p_B = (x - a)(x - b)^{n-1} \). It is straightforward to see that \( F_1 = \frac{1}{n+1} (x - \lambda_1)(x - b)^n \), where \( \lambda_1 = a + \frac{a - b}{n} \), is a full integral of \( p_B \). Similarly, \( F_2 = \frac{1}{(n+1)(n+2)} (x - \lambda_2)(x - b)^{n+1} \), where \( \lambda_2 = \lambda_1 + \frac{\lambda_1 - b}{n+1} \). Thus we obtain the sequence of polynomials \( F_0 = p_B, F_1, F_2, \ldots \), where \( F_i \) is a full integral of \( F_{i-1} \), \( i \in \mathbb{N} \). By Corollary 4.3 we obtain the required sequence \( \mathcal{B} = B_1, B_2, \ldots \).

Remark 4.7. If \( m = 1 \) and \( k \geq 2 \) then the integral \( \mathcal{A} \) of \( \mathcal{B} \) can be non-integrable. For example, if \( p_{B}(x) = x^2(x - 3)(x - 5) \) then it is straightforward to check that \( F(x) = \frac{1}{5} x^3(x - 5)^2 \) is the only full integral of \( p_{B}(x) \). Thus \( p_{A}(x) = 5F(x) \) by Corollary 3.9. Hence by Theorem 3.13 \( \mathcal{A} \) is not integrable.

5 Applications to dual Schoenberg type inequality

Sendov’s conjecture for polynomials was first formulated in 1958. It was then mentioned in Hayman’s famous research problems book [6].

Sendov conjecture (1958): Let \( p \) be a polynomial of degree \( n \geq 2 \) with zeros \( z_1, \ldots, z_n \) and critical points \( w_1, \ldots, w_{n-1} \). Then,

\[
\max_{1 \leq k \leq n} \min_{1 \leq i \leq n-1} |w_i - z_k| \leq \max_{1 \leq k \leq n} |z_k|.
\]

The conjecture remains unsolved although attempts to verify this conjecture have led to many interesting research results. The readers are referred to the survey papers [15], [17] as well as the two excellent books on the analytic theory of polynomials, [14] and [18]. Another conjecture relating the zeros and critical points of a polynomial is the Schoenberg’s conjecture [16]. Let \( z_1, z_2, \ldots, z_n \) be the zeros of a polynomial \( p = c_n x^n + \ldots + c_0 \) of degree \( n \), \( w_1, w_2, \ldots, w_{n-1} \) be the critical points of \( p \), and let \( G = (1/n) \sum_{i=1}^{n} z_i \) be the arithmetical mean of the zeros of a polynomial \( p \). It can be readily seen that this value is equal to the arithmetical mean of the critical points of \( p \), \( G = (1/(n - 1)) \sum_{i=1}^{n-1} w_i \).

Indeed, by Vieta’s formulas applied for \( p \) we obtain \( \sum_{i=1}^{n} z_i = -\frac{c_{n-1}}{c_n} \). If we now apply Vieta’s formulas for \( p' = nc_n x^{n-1} + (n - 1)c_{n-1} x^{n-2} + \ldots + c_1 \) we find that \( \frac{1}{n-1} \sum_{i=1}^{n-1} w_i = \)
we know that if 

\[ ||B|| \]

In this notation the Schoenberg’s conjecture can be written as

\[ \sum_{i=1}^{n-1} |w_i|^2 \leq |G|^2 + \frac{n-2}{n} \sum_{i=1}^{n} |z_i|^2. \]

It is natural to ask if one can bound \( \sum_{i=1}^{n} |z_i|^2 \) by some expressions in \( w_i \) similar to those in Schoenberg’s conjecture. Our results on matrix integrability and full integrability of polynomials (Theorem 3.8) are then applied to prove the dual version of the Schoenberg inequality. Namely, this inequality provides a bound for the sum of squares of the absolute values of zeros by an expression in the critical points.

**Theorem 5.1.** Following the Notation 2.6, consider a degree \( n \) polynomial of type \((k, m)\) given by

\[ f(x) = (x - a_1) \ldots (x - a_k)(x - b_1)^{\alpha_1} \ldots (x - b_m)^{\alpha_m}, \]

where \( a_1, \ldots, a_k, b_1, \ldots, b_m \in \mathbb{C} \) are pair-wise distinct, \( k, m \in \mathbb{N} \cup \{0\} \), \( \alpha_1, \ldots, \alpha_m \in \mathbb{N} \setminus \{1\} \). Let \( h(x) := (x - a_1) \ldots (x - a_k) \) and \( h_{a_i}(x) := \frac{h(x)}{x-a_i} \).

Suppose \( f \) has a full integral \( F \) and \( z_1, \ldots, z_{n+1} \) are the zeros of \( F \) and denote \( G = \frac{1}{n+1} \sum_{i=1}^{n+1} z_i = \frac{1}{n} \left( \sum_{i=1}^{k} \alpha_i + \sum_{i=1}^{m} \alpha_i b_i \right) \) then

\[ \sum_{i=1}^{n+1} |z_i|^2 \leq \sum_{i=1}^{k} |a_i|^2 + \sum_{i=1}^{m} \alpha_i |b_i|^2 + |G|^2 + 2(n+1) \sum_{i=1}^{k} \frac{|F(a_i)|}{h_{a_i}(a_i)} \]

with equality holds if and only if \( \frac{F(a_i)}{h_{a_i}(a_i)}(a_i - \tau(B)) \), \( i = 1, \ldots, k \) are real.

**Proof.** We shall make use of the Schur inequality [14, p. 56] which says that if \( \lambda_i(A) \) are eigenvalues of a square matrix \( A \) of order \( n \), then

\[ \sum_{i=1}^{n} |\lambda_i(A)|^2 \leq ||A||_F^2, \]

and the equality holds if and only if \( A \) is normal.

From Corollary 3.12 we know that if \( f \) has a full integral \( F \), then \( A = \begin{pmatrix} B & u^\top \vspace{10pt} \tau(\mathcal{B}) \end{pmatrix} \) is an integral of \( \mathcal{B} \) with \( p_A = (n + 1)F \) possessing the smallest Frobenius norm \( ||A||_F^2 = ||B||_F^2 + ||\tau(\mathcal{B})||_F^2 + 2 \sum_{i=1}^{k} \frac{|(n+1)F(a_i)|}{h_{a_i}(a_i)} \)

\[ \text{where } \tau(\mathcal{B}) := \frac{\tau(\mathcal{B})_n}{n} = G, v_1 = \ldots = v_{C_m} = 0, v_{d_i} = \]
\[
\sqrt{\frac{(n+1)F_{\tau}(a_i)}{h_{\tau}(a_i)}}, \quad i = 1, \ldots, k \text{ and } u = v. \text{ Since } (n+1)F \text{ is a characteristic polynomial of } A, \\
\text{by the Schur inequality}
\]
\[
\sum_{i=1}^{n+1} |z_i|^2 = \sum_{i=1}^{n+1} |\lambda_i(A)|^2 \leq ||A||_F^2 = \sum_{i=1}^{k} |a_i|^2 + \sum_{i=1}^{m} |b_i|^2 + |G|^2 + 2(n+1) \sum_{i=1}^{k} \left| \frac{F(a_i)}{h_{\tau}(a_i)} \right|.
\]

The equality in the Schur inequality holds if and only if \( A \) is normal, i.e. \( AA^* = A^*A \). Direct computations show
\[
A^*A = \left( \begin{array}{cc}
\mathcal{B}B + \tau^T v & \mathcal{B}u^T + \tau(B)\tau^T \\
v\mathcal{B} + \tau(B)v & \tau(B)\tau^T \end{array} \right), \quad AA^* = \left( \begin{array}{cc}
\mathcal{B}B + u^T \tau & \mathcal{B}\tau^T + \tau(B)u^T \\
v\mathcal{B} + \tau(B)v & \tau(B)\tau^T \end{array} \right).
\]

Since \( \mathcal{B}\mathcal{B} = \mathcal{B}\mathcal{B} \) then
\[
AA^* - A^*A = \left( \begin{array}{cc}
O & (B - \tau(B))\tau^T - (\mathcal{B} - \tau(B)v)^T \\
(v(B - \tau(B)) - \tau(B) - \tau(B)v)^T & O \end{array} \right).
\]

Thus \( A \) is normal if and only if this matrix is 0. Since \( B \) is diagonal, it is equivalent to \( v_B(a_i - \tau(B)) = \tau(v_B(a_i - \tau(B))), \quad i = 1, \ldots, k \). Substituting the values \( v_B \), we equivalently obtain
\[
\frac{F(a_i)}{h_{\tau}(a_i)}(a_i - \tau(B)) = \left( \frac{F(a_i)}{h_{\tau}(a_i)} \right)(a_i - \tau(B)), \quad i = 1, \ldots, k.
\]

If all the critical points of a polynomial \( p \) are distinct, then \( p \) is a full integral of \( p' \). It then follows from the case 1 of Theorem 3.13 that we have the following dual Schoenberg type inequality.

**Corollary 5.2.** Let \( p \) be a polynomial of degree \( n \) with the zeros \( z_1, \ldots, z_n \) and the critical points \( w_1, \ldots, w_{n-1} \). Let \( G = \frac{1}{n} \sum_{i=1}^{n} w_i = \frac{1}{n} \sum_{i=1}^{n} z_i \). If all the critical points of \( p \) are distinct, then
\[
\sum_{i=1}^{n} |z_i|^2 \leq |G|^2 + \sum_{i=1}^{n-1} |w_i|^2 + 2n \sum_{i=1}^{n-1} \left| \frac{p(w_i)}{p''(w_i)} \right|
\]
with equality holds if and only if all elements \( \frac{p(w_i)}{p''(w_i)} (w_i - G) , \quad i = 1, \ldots, n - 1 \) are real.

**Proof.** Without loss of generality we assume that the coefficient at the highest degree of \( p'(x) \) is 1. Since \( w_1, \ldots, w_{n-1} \) are distinct, then in notations of previous theorem \( h(x) = p'(x), \quad F(x) = p(x) \). Thus \( p''(x) = \sum_{i=1}^{n-1} p'(x) - w_i \) and \( p''(w_i) = \frac{p'(x) - w_i}{x - w_i}(w_i) \). Therefore \( \frac{F(w_i)}{h_{w_i}(w_i)} = \frac{p(w_i)}{p''(w_i)} \) and we obtain the desired formula. \( \square \)
Remark 5.3. Note that the extra term $\sum_{i=1}^{n} \frac{|p(w_i)|}{p'(w_i)}$ is indeed necessary to bound $\sum_{i=1}^{n} |z_i|^2$. To show this we consider the polynomial $p(z) = z^n - z$, $n > 1$. Then $p'(z) = nz^{n-1} - 1$, $p''(z) = (n-1)nz^{n-2}$. Thus $\sum_{i=1}^{n} |z_i|^2 = n - 1$ and $\sum_{i=1}^{n} w_i = \begin{cases} \frac{1}{n}, & \text{if } n = 2, \\ 0, & \text{if } n > 2. \end{cases}$ Moreover $\sum_{i=1}^{n} |w_i|^2 = (n-1)n^{2-n}$.

It is obvious that Sendov conjecture is equivalent to saying that all the zeros $z_i$ of a polynomial of the degree $n$ lie in the union $G = \bigcup_{i=1}^{n-1} G_i$ of the disks with the center at the critical point $w_i$:

$$G_i = \{ z \in \mathbb{C} : |z - w_i| \leq \max_{1 \leq k \leq n} |z_k| \}, \quad i = 1, \ldots, n-1.$$  

If we consider polynomials as characteristic polynomials of certain matrices it is tempting to combine the Gerschgorin’s theorem on the location of the eigenvalues together with the integration technique for matrices with simple eigenvalues to study Sendov conjecture. To state Gerschgorin’s theorem, for any square matrix $A = (a_{ij})$ of order $n \geq 2$, we shall use the following notation:

$$R_i(A) = \sum_{j \neq i}^{n} |a_{ij}|, \quad i = 1, \ldots, n.$$  

Theorem 5.4. (Gerschgorin’s theorem) ([7, p.344]). The eigenvalues of any square matrix $A = (a_{ij})$ of order $n \geq 2$ lie in the union $\bigcup_{i=1}^{n} D_i$ of the Gerschgorin disks

$$D_i = \{ z \in \mathbb{C} : |z - a_{ii}| \leq R_i(A) \}, \quad i = 1, \ldots, n.$$  

Combining the Gerschgorin’s theorem with the integration we obtain the following theorem.

Theorem 5.5. Let $p$ be a polynomial of degree $n$ with zeros $z_1, z_2, \ldots, z_n$ and distinct critical points $w_1, w_2, \ldots, w_{n-1}$. Then each zero is lies in the union $\bigcup_{i=1}^{n} D_i$ of $n$ disks

$$D_i = \{ z \in \mathbb{C} : |z - w_i| \leq \max_{1 \leq j \leq n} |z_j| \} \quad i = 1, \ldots, n-1 \text{ and } D_n = \{ z \in \mathbb{C} : \left| z - \frac{1}{n} \sum_{i=1}^{n-1} w_i \right| \leq \max_{1 \leq j \leq n} |z_j| \}.$$  

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Proof. Consider \( B = \text{diag} \left( w_1, \ldots, w_{n-1} \right) \). Since \( w_1, \ldots, w_{n-1} \) are distinct then \( p \) is a full integral of \( p_B(x) \). Consider an integral \( A \) of \( B \) given by formula from Corollary 3.10. For any \( s \neq 0 \), \( A \) is similar to \( A_0 = \left( \begin{array}{cc} B & su^\top \\ s^{-1}v & \tau(B) \end{array} \right) \) because

\[
\left( \begin{array}{cc} B & su^\top \\ s^{-1}v & \tau(B) \end{array} \right) = \left( \begin{array}{cc} sI_n & O \\ O & 1 \end{array} \right) \left( \begin{array}{cc} B & u^\top \\ v & \tau(B) \end{array} \right) \left( \begin{array}{cc} s^{-1}I_n & O \\ O & 1 \end{array} \right)
\]

Now for \( s = \max_{1 \leq j \leq n} |z_j| > 0 \) we have \( R_i(A_0) = s|u_i| = \max_{1 \leq j \leq n} |z_j|, \ 1 \leq i \leq n-1 \) and \( R_n(A_0) = s^{-1} \sum_{i=1}^{n-1} |v_i| = \frac{1}{s} \max_{1 \leq j \leq n} \sum_{i=1}^{n-1} \left| \frac{p(w_i)}{p''(w_i)} \right| \). Hence by the Gerschgorin’s theorem we obtain that

\[
z_l \in \left( \bigcup_{i=1}^{n-1} \{ z \in \mathbb{C} : |z - w_i| \leq \max_{1 \leq j \leq n} |z_j| \} \right) \bigcup \{ z \in \mathbb{C} : |z - \tau(B)| \leq \frac{n}{s} \max_{1 \leq j \leq n} \sum_{i=1}^{n-1} \left| \frac{p(w_i)}{p''(w_i)} \right| \},
\]

\( l = 1, \ldots, n \). Finally, the equality \( \tau(B) := \frac{1}{n} tr(B) = \frac{1}{n} \sum_{i=1}^{n-1} w_i \) yields the statement of the theorem. \( \Box \)

Remark 5.6. The size of the disk \( D_n \) can be quite big so that all the zeros \( z_i \) are lying inside it and in this case one cannot obtain information about the relative position between the \( z_i \) and \( w_j \).

Acknowledgments

Investigations of integrability for diagonalizable matrices (Theorem 3.13) are supported by the Ministry of Science and Higher Education of the Russian Federation (Goszadaniye No. 075-00337-20-03, project No. 0714-2020-0005). Necessary and sufficient conditions for a matrix integral to be diagonalizable (Theorem 4.1) are obtained under the financial support of the Russian Federation Government (Grant number 075-15-2019-1926).

References

[1] B.V.R.Bhat, M.Mukherjee, Integrators of matrices, Linear Algebra Appl. 426:1 (2007), 71-82.

[2] W.S. Cheung, T.W. Ng, A companion matrix approach to study of zeros and critical points of a polynomial, J. Math. Anal. Appl. 319 (2006) 690-707.
[3] W.S. Cheung, T.W. Ng, Relationship between zeros of two polynomials, Linear Algebra Appl. 432 (2010) 107-115.

[4] S. Danielyan, A. Guterman, On integral of polynomial with multiple roots, Zapiski Nauchnich Seminarov POMI, 482, 2019, 28-44; English translation: J. Math. Sci. (N. Y.) 249:2 (2020) 128-138.

[5] C. Davis, Eigenvalues of compressions, Bull. Math. Soc. Sci. Math. Phys. RPR 51 (1959) 3–5.

[6] W.K. Hayman, Research problems in function theory. University of London, London 1967. The Fiftieth Anniversary Edition, Springer Nature, 2019.

[7] R.A. Horn and C.R. Johnson. Matrix analysis. Cambridge University Press, Cambridge, 1990.

[8] A. Khare, Vector spaces as unions of proper subspaces, Linear Algebra Appl. 431:9 (2009) 1681–1686.

[9] O. Kushel, M. Tyaglov, Circulants and critical points of polynomials, J. Math. Anal. Appl. 439 (2016) 634-650.

[10] S.M. Malamud, An Analog of the Poincare Separation Theorem for Normal Matrices and the Gauss-Lucas Theorem. Functional Analysis and Its Applications 37 (2003), no.3, 232-235.

[11] S.M. Malamud, Inverse spectral problem for normal matrices and the Gauss-Lucas theorem, Trans. Amer. Math. Soc. 357 (2004), no. 10, 4043–4064.

[12] R. Pereira, Differentiators and the geometry of polynomials, J. Math. Anal. Appl. 285:1 (2003), 336-348.

[13] R. Pereira, Matrix-theoretical derivations of some results of Borcea–Shapiro on hyperbolic polynomials, Comptes Rendus Mathematique 341:11 (2005), 651-653.

[14] Q.I. Rahman and G. Schmeisser, Analytic theory of polynomials. London Mathematical Society Monographs. New Series, 26. The Clarendon Press, Oxford University Press, Oxford, 2002.

[15] G. Schmeisser, The conjectures of Sendov and Smale. Approximation theory: A volume dedicated to Blagovest Sendov), 353–369, DARBA, Sofia, 2002.

[16] I.J. Schoenberg, A conjectured analogue of Rolle’s theorem for polynomials with real or complex coefficients. Amer. Math. Monthly 93 (1986), no. 1, 8–13.
[17] Bl. Sendov, Hausdorff geometry of polynomials. East J. on Approximations 7 (2001), no. 2, 123–178.

[18] T. Sheil-Small, Complex polynomials. Cambridge Studies in Advanced Mathematics, 75. Cambridge University Press, Cambridge, 2002.