Multigrid for propagators of staggered fermions in four-dimensional $SU(2)$ gauge fields

T. Kalkreuter

*a) Institut für Theoretische Physik der Universität Hamburg
Luruper Chaussee 149, W-2000 Hamburg 50, Germany

Multigrid (MG) methods for the computation of propagators of staggered fermions in non-Abelian gauge fields are discussed. MG could work in principle in arbitrarily disordered systems. The practical variational MG methods tested so far with a “Laplacian choice” for the restriction operator are not competitive with the conjugate gradient algorithm on lattices up to $18^d$. Numerical results are presented for propagators in $SU(2)$ gauge fields.

1. INTRODUCTION

In Monte Carlo simulations of lattice gauge theories with fermions the most time-consuming part is the computation of the gauge field dependent fermion propagators. Great hopes to compute propagators without any critical slowing down (CSD) are attached to multigrid (MG) methods [1–10]. The MG methodology was reviewed by Brandt last year [11].

We wish to solve an equation

$$(-\nabla^2 + m^2)\chi = f \tag{1}$$

by MG, where $\nabla$ is the gauge covariant staggered Dirac operator, and $m$ is a small quark mass.

The following MG notations will be used. The fundamental lattice of lattice spacing $a_0$ is denoted by $\Lambda^0$. The first block lattice $\Lambda^1$ has lattice spacing $a_1 = L_b a_0$. Restriction and interpolation operators $C$ and $A$, respectively, are given by kernels $C(x, z)$ and $A(z, x)$ with $z \in \Lambda^0$, $x \in \Lambda^1$. If $z$ is a site in block $x$, we write $z \in x$. Advantage is taken of the fact that we work in $d = 4$ dimensions and that $L_b = 3$ will be chosen, so that only a two-grid algorithm was implemented. The residual equation on the coarse grid was solved exactly by the conjugate gradient (CG) algorithm.

2. BLOCKED STAGGERED FERMIONS

We use a blocking procedure for staggered fermions which is consistent with the lattice symmetries of free fermions [8]. This forces us to choose $L_b = 3$. Even $L_b$ are not allowed.

Fig. 1 illustrates our choice of blocks. The different fermionic degrees of freedom are called “pseudoflavor” [8]. Different pseudoflavors are distinguished by different symbols in Fig. 1. Block centers $\hat{x}$ are encircled. The boundaries of seven blocks are marked. The averaging kernel $C(x, z)$ is only nonvanishing if site $z$ has the same pseudoflavor (symbol) as the block center $\hat{x}$. Therefore the seemingly overlapping blocks have actually no sites in common.

3. GROUND-STATE PROJECTION MG

The averaging kernel $C$ is chosen according to the ground-state projection (GSP) definition. The idea behind this definition is that the lowest mode of the fermion matrix, which is responsible for CSD in traditional algorithms, should be represented as well as possible on the block lattice. GSP was introduced in Refs. [12,13,2]. An alternative MG approach to the inversion of the fermion matrix is the “parallel-transported MG” of Ben-Av et al. [1,10].

In GSP the kernel $C^*$ of the adjoint of $C$ fulfills a gauge covariant eigenvalue equation

$$(-\Delta_{N,x} C^*)(z, x) = \lambda_0(x) C^*(z, x) \tag{2}$$

$\lambda_0(x)$ is the lowest (gauge invariant) eigenvalue of the positive (semi-)definite operator $-\Delta_{N,x}$.

$C^*$ depends on the gauge field although this is not indicated explicitly.

*Work supported by Deutsche Forschungsgemeinschaft.
by imposing the normalization condition

\[ CC = 1 \]  

as usual. The solution of Eq. (2) is made unique by a positive Hermitean matrix.) The covariance condition (4) ensures that the kernel \( C(x, z) \) is the solution of Eq. (2) 

\[ (\Delta N, x C^*)(z, x) = \sum_{z', n.n. \ z' \in x} [U(z, z') C^*(z', x) - C^*(z, x)] \]  

for \( z \in x \), \( U(z, z') \) is the gauge field on the link \( (z, z') \). As a supplement we enforce Dirichlet boundary conditions in such a way that \( C(x, z) = 0 \) unless \( z \in x \).

### 3.2. Staggered fermions

For fermions \( \Delta \) in (2) was chosen as the gauge covariant fermionic “2-link Laplacian” which is defined through \(-\nabla^2 = -\Delta + F_{\mu\nu} \). With this “Laplacian choice” \( C(x, z) \) is only nonvanishing when \( z \) and \( x \) carry the same pseudoflavor.

Other proposals were made in Ref. [8], but have not been implemented yet.

There exists an efficient algorithm for the solution of Eq. (2) [9]. Therefore it is not necessary to sacrifice the gauge covariance of GSP by wasting computer time for gauge fixing.

### 3.3. Idealized MG algorithm

Given the averaging kernel \( C \) there exists an associated idealized interpolation kernel \( A \). By means of this \( A \) a (nearly) critical system on \( \Lambda^0 \) is mapped onto a noncritical system on the MG. The construction of the optimal \( A \) was described in Mack’s Cargèse lectures [15]. Its origin are works on constructive quantum field theory.

The gauge covariant optimal \( A \) is the solution of the equation

\[ ([D + \kappa C^* C] A)(z, x) = \kappa C^*(z, x) \]  

for large \( \kappa \). \( D = -\Delta + m^2 \) for bosons, and \( D = -\nabla^2 + m^2 \) for fermions.

An averaging kernel \( C \) is said to be “good” (or to define a “good” block spin) if the associated \( A(z, x) \) decays exponentially in \(|z - x|\). GSP defines a good \( C \) in arbitrarily disordered gauge fields, both for bosons and for staggered fermions. Fig. [2] visualizes an example. \( A \) decays exponentially over distance \( L_b \) (nearly) as fast in the presence of gauge fields than in their absence.

By means of the idealized \( A \) it was shown in Ref. [8] that MG computations of bosonic propagators without CSD are possible in arbitrarily
Figure 2. An optimal fermionic interpolation kernel $\mathcal{A}(z,0)$ in a quenched $SU(2)$ gauge field at $\beta = 2.7$ on an $18^4$ lattice. Shown is a two-dimensional cut through the block center $x = 0$; $z_3$ and $z_4$ are fixed. The vertical axis gives the trace norm of $\mathcal{A}(z,x)$.

Disordered gauge fields (including the case $\beta = 0$). These computations showed for the first time that the MG method could work in principle in arbitrarily strong disorder. Unfortunately, the idealized MG algorithm is not practical for production runs because of computational complexity and storage space requirements, but it was important to answer questions of principle.

3.4. Variational MG

A practical MG method is variational coarsening. There $\mathcal{A} = C^*$, and the coarse grid operator is $CDC^*$. This method reduces to piecewise constant interpolation in the absence of gauge fields, where it is known that CSD is eliminated.

However, in nontrivial gauge fields variational MG with the Laplacian choice of $C$ is not competitive with CG on lattices up to $18^4$; see Fig. 3. The non-optimized MG program requires a factor of 4.5 more arithmetic operations than CG.

A further numerical finding is the following. Relaxation times $\tau$ of conventional one-grid relaxation and of variational MG with the Laplacian choice of $C$ follow a scaling relation

$$\tau = \frac{\text{const.}}{\Delta m^2} \quad \text{with} \quad \Delta m^2 = m^2 - m_{cr}^2,$$

where $m_{cr}^2$ is the lowest eigenvalue of $-\nabla^2$, and const. is independent of the lattice size. $m_{cr}^2$ is small and is usually neglected, but on relatively small lattices this neglect is not justified. From 3 we conclude that variational MG with the Laplacian choice of $C$ will not be able to eliminate CSD on large lattices.

4. UPDATING ON AN MG LAYER CONSISTING OF A SINGLE SITE

Because of 3, (MG) relaxation continues to have CSD even on a lattice of only $2^d$ sites. Therefore it seems necessary to update on a “last site” in order to eliminate the appearance of CSD.

Given an approximate propagator $\chi^{(n)}$, this updating (in the unigrid point of view) amounts to globally rescaling $\chi^{(n)}$ by a gauge covariant $N_c \times N_c$ matrix $\Omega$ (in case of bosons or Wilson

\footnote{In case of one-grid relaxation for bosons Eq. 3 is known analytically.}
fermions)

\( \chi^{(n)}(z) \mapsto \chi^{(n)}(z) \Omega \).

The matrix \( \Omega \) is chosen such that the energy functional

\[ K[\chi] = \frac{1}{2} \langle \chi, D\chi \rangle - \langle \chi, f \rangle \]

of the rescaled approximation \( \chi^{(n)} \Omega \) is minimized.

In case of staggered fermions we have to consider that there are \( 2^d \) different pseudoflavors, and we replace \( \Omega \) in (8) by \( \Omega(H(z)) \) where \( H(z) \) denotes the pseudoflavor of \( z \). The \( 2^d N_c \times N_c \) matrices \( \Omega(H) \) are again chosen automatically by the algorithm in such a way that \( K[\chi^{(n)}\Omega] \) gets minimized.

We note that \( \Omega \) and the \( \Omega(H) \)'s practically equal \( \mathbb{I} \) as soon as errors decay exponentially.

4.1. Results for bosons

In a fixed volume the asymptotic fall-off properties do not depend on \( \triangle m^2 \) when (8) is included in algorithms. When the lattice size is increased, the asymptotic fall-off rate does not change, there is only a mild volume effect with respect to how long it takes until errors decay exponentially. This volume effect is much milder than the one in (unpreconditioned) CG, and variational MG plus (8) begins to become superior in computer time for lattices \( \gtrsim 18^4 \).

4.2. Results for staggered fermions

In a fixed volume the asymptotic \( 1/\triangle m^2 \) divergence is eliminated when the analog of (8) is included in algorithms. However, it takes a relatively large number of iterations until fall-offs are exponential. Also, in contrast to the bosonic case, asymptotic decay rates increase at fixed \( \triangle m^2 \) when the lattice size is increased. Therefore CSD seems not to be eliminated, and it is questionable whether the method pays in practice on lattices of realizable sizes. For details see Refs. (1)-(4).

5. CONCLUSIONS

The variational MG methods tested so far for staggered fermions are not competitive with CG. We expect better results when blocks are used which overlap in a nontrivial way. The question, whether it is sufficient to average only over sites with the same pseudoflavor in nontrivial gauge fields, also has to be investigated. The Diracian proposal for \( C \) of Ref. (3) should be tested.

ACKNOWLEDGMENTS

I am indebted to G. Mack for many stimulating discussions. Financial support by Deutsche Forschungsgemeinschaft is gratefully acknowledged. For providing resources and help I wish to thank HLRZ Jülich and its staff.

REFERENCES

1. R. Ben-Av, A. Brandt, and S. Solomon, Nucl. Phys. B329 (1990) 193; R. Ben-Av, A. Brandt, M. Harmatz, E. Katznelson, P.G. Lauwers, S. Solomon, and K. Wolowesky, Phys. Lett. B253 (1991) 185; Nucl. Phys. B (Proc. Suppl.) 20 (1991) 102; R. Ben-Av, P.G. Lauwers, and S. Solomon, Nucl. Phys. B 374 (1992) 249; P.G. Lauwers and S. Solomon, Int. J. Mod. Phys. C3 (1992) 149.
2. R.C. Brower, C. Rebbi, and E. Vicari, Phys. Rev. D43 (1991) 1965.
3. R.C. Brower, C. Rebbi, and E. Vicari, Phys. Rev. Lett. 66 (1991) 1263; R.C. Brower, K.J.M. Moriarty, C. Rebbi, and E. Vicari, Nucl. Phys. B (Proc. Suppl.) 20 (1991) 89; Phys. Rev. D43 (1991) 1974.
4. A. Hulsebos, J. Smit, and J.C. Vink, Nucl. Phys. B (Proc. Suppl.) 20 (1991) 94; Int. J. Mod. Phys. C3 (1992) 161; Nucl. Phys. B368 (1992) 379.
5. R.C. Brower, R.G. Edwards, C. Rebbi, and E. Vicari, Nucl. Phys. B366 (1991) 689.
6. T. Kalkreuter, G. Mack, and M. Speh, Int. J. Mod. Phys. C3 (1992) 121.
7. J.C. Vink, Phys. Lett. B272 (1991) 81; Nucl. Phys. B (Proc. Suppl.) 26 (1992) 607.
8. T. Kalkreuter, Phys. Lett. B276 (1992) 485.
9. T. Kalkreuter, preprint DESY 92–108 (July 1992), to appear in Int. J. Mod. Phys. C.
10. P.G. Lauwers and T. Wittlich, preprint BONN 92–29 (September 1992); these proceedings.
11. A. Brandt, Nucl. Phys. B (Proc. Suppl.) 26 (1992) 137.
12. G. Mack, unpublished.
13. A. Hulsebos, J. Smit, and J.C. Vink, Nucl. Phys. B (Proc. Suppl.) 9 (1989) 512.
14. T. Kalkreuter, Nucl. Phys. B376 (1992) 637.
15. G. Mack, in: Nonperturbative quantum field theory, eds. G. ’t Hooft et al. (Plenum Press, New York, 1988).
16. T. Kalkreuter, *Ph.D. thesis* (fall of 1992).