A KÄHLER STRUCTURE ON THE PUNCTURED COTANGENT BUNDLE OF THE CAYLEY PROJECTIVE PLANE

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Dedicated to the memory of Jean Leray (1906-1998)

Abstract. We construct a Kähler structure on the punctured cotangent bundle of the Cayley projective plane whose Kähler form coincides with the natural symplectic form on the cotangent bundle and we show that the geodesic flow action is holomorphic and is expressed in a quite explicit form. We also give an embedding of the punctured cotangent bundle of the Cayley projective plane into the space of $8 \times 8$ complex matrices.

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Introduction

In the paper [Ra1] a Kähler structure on the punctured cotangent bundle $T_0^*S^n = T^*S^n \setminus S^n$ of the sphere $S^n$ is constructed through the mapping $\tau_S$

\begin{equation}
\tau_S : T_0^*S^n \longrightarrow \mathbb{C}^{n+1}
\end{equation}

\begin{equation}
(x, y) \mapsto \|y\|x + \sqrt{-1}y.
\end{equation}

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It is shown that the natural symplectic form $\omega = \omega_S$ on the cotangent bundle coincides with the Kähler form $\sqrt{-2} \partial \bar{\partial} ||z||$ (see also [So]). Moreover the geodesic flow action is holomorphic.

In the paper [FT] we constructed a Kähler structure on the punctured cotangent bundle of complex and quaternion projective spaces with similar properties as for the sphere cases (see also [Ii2]). This Kähler structure is just a positive complex polarization on the cotangent bundle as a symplectic manifold and is applied to construct a quantization operator by the method of pairing of polarizations. The operator quantizes geodesic flows of such manifolds. In other words such an operator gives a correspondence between the geodesic flow and the one parameter group of Fourier integral operators generated by the square root of the Laplacian ([FY], [Ra2]).

In this paper we construct a Kähler structure on the punctured cotangent bundle of the Cayley projective plane whose Kähler form coincides with the natural symplectic form on the cotangent bundle and is invariant under the action of the geodesic flow (Theorem 2.1). In general it will not be easy to find such Riemannian manifolds whose (punctured)cotangent bundle has a Kähler structure where the symplectic form coincides with the Kähler form and is invariant under the action of the geodesic flow (see [Sz1] and [Sz2]). Such a Kähler structure for complex and quaternion projective spaces is constructed by making use of the Hopf fibration and the map above for the sphere. Although the Cayley projective plane has no fiber bundle like the Hopf fiber bundle, we prove here that a map similar to the cases of complex and quaternion projective spaces gives an embedding of the punctured cotangent bundle of the Cayley projective plane into the space of a complexified exceptional Jordan algebra. It is well-known that the Cayley projective plane is one of the compact symmetric spaces of rank one and that the exceptional Lie group $F_4$ acts on it two-point homogeneously. Some properties which we prove in Theorem 2.1 could be shown quite easily if we used this property of minimal rank for symmetric spaces, but we prove our main theorem through elementary calculi in a Jordan algebra where the Cayley projective plane is realized as a subset consisting of primitive idempotents. Further we give an embedding of this image in the complexified exceptional Jordan algebra into the space of $8 \times 8$ complex matrices by composing with a map given by [Y].
In §1 we describe the Cayley projective plane as a subset consisting of primitive idempotents in an exceptional Jordan algebra. In §2 we state our main theorem and a corollary. In §3 we recall some basic facts about the Jordan algebra including, so called, the Freudenthal product and the determinant on the Jordan algebra. In §4 we prove our main theorem and in §5 we describe an embedding of the punctured cotangent bundle of the Cayley projective plane into the space of $8 \times 8$ complex matrices.

1. CAYLEY PROJECTIVE PLANE

In this section we describe the Cayley projective plane as a subset in the exceptional Jordan algebra over the real number field $\mathbb{R}$ (see [M] and [Be]).

Let $\mathbb{H}$ be the quaternion number field, that is, $\mathbb{H}$ is an algebra over $\mathbb{R}$ generated by $\{e_i\}_{i=0}^3$ with the relations:

\begin{align*}
(1.1) & \quad e_0 e_i = e_i e_0 \quad (i = 0, 1, 2, 3) \\
(1.2) & \quad e_i^2 = -e_0 \quad (i = 1, 2, 3) \\
(1.3) & \quad e_i e_j = -e_j e_i = e_k \quad (\text{mod } 3).
\end{align*}

The Cayley number field $\mathcal{O}$ is a division algebra over $\mathbb{R}$ generated by $\{e_i\}_{i=0}^7$ with $e_i e_j$ given by the table

\begin{table}[h]
\begin{tabular}{cccccccc}
  & $e_0$ & $e_1$ & $e_2$ & $e_3$ & $e_4$ & $e_5$ & $e_6$ & $e_7$ \\
$e_0$ & $e_0$ & $e_1$ & $e_2$ & $e_3$ & $e_4$ & $e_5$ & $e_6$ & $e_7$ \\
$e_1$ & $e_1$ & $e_0$ & $-e_0$ & $e_2$ & $e_3$ & $e_4$ & $-e_5$ & $e_6$ \\
$e_2$ & $e_2$ & $-e_0$ & $-e_0$ & $e_1$ & $e_2$ & $e_3$ & $e_4$ & $e_5$ \\
$e_3$ & $e_3$ & $e_2$ & $-e_1$ & $-e_0$ & $e_3$ & $e_4$ & $e_5$ & $-e_2$ \\
$e_4$ & $e_4$ & $e_5$ & $-e_6$ & $-e_7$ & $-e_0$ & $e_1$ & $e_2$ & $e_3$ \\
$e_5$ & $e_5$ & $e_6$ & $e_7$ & $e_6$ & $-e_1$ & $-e_0$ & $e_3$ & $e_2$ \\
$e_6$ & $e_6$ & $e_7$ & $-e_5$ & $-e_2$ & $e_3$ & $e_0$ & $-e_1$ & $e_0$ \\
$e_7$ & $e_7$ & $-e_6$ & $e_5$ & $e_4$ & $-e_3$ & $-e_2$ & $e_1$ & $-e_0$
\end{tabular}
\end{table}

Especially,

\begin{equation}
(1.5) \quad e_1 e_4 = e_5, \quad e_2 e_4 = e_6 \quad \text{and} \quad e_3 e_4 = e_7.
\end{equation}

Hence $\mathcal{O}$ is identified with

\begin{equation}
(1.6) \quad \mathbb{H} \oplus \mathbb{H} e_4
\end{equation}
and the multiplication between $x = a + be_4$ and $y = h + ke_4 \in \mathbb{H} \oplus \mathbb{H}e_4$
is given by

$$x \cdot y = ah - \theta(k)b + \{ka + b\theta(h)\}e_4,$$

where $h = \sum_{i=0}^{3} h_i e_i$ ($h_i \in \mathbb{R}$) and $\theta(h) = h_0 e_0 - h_1 e_1 - h_2 e_2 - h_3 e_3,$
and so on. We assume that the basis $\{e_i\}_{i=0}^{7}$ are orthonormal and we will sometimes omit
$e_0 (= identity element)$ and identify $\mathbb{R} = \mathbb{R}e_0 \subset \mathbb{H} \subset \mathcal{O}.$

For $h = \sum_{i=0}^{7} h_i e_i \in \mathcal{O},$ we denote

$$\theta(h) = h_0 e_0 - \sum_{i=1}^{7} h_i e_i,$$

as for $h \in \mathbb{H}.$

Let $M(3, \mathcal{O})$ be the space of $3 \times 3$ matrices with entries in $\mathcal{O}.$ By identifying $M(3, \mathcal{O}) \cong M(3, \mathbb{R}) \otimes_{\mathbb{R}} \mathcal{O},$ we denote for $X \in M(3, \mathcal{O})$

$$\theta(X) = X_0 \otimes e_0 - \sum_{i=1}^{7} X_i \otimes e_i,$$

$$\iota X = \sum_{i=0}^{7} \iota X_i \otimes e_i$$
and

$$\text{tr} X = \sum_{i=0}^{7} (\text{tr} X_i) e_i,$$

where $X = \sum_{i=0}^{7} X_i \otimes e_i,$ $X_i \in M(3, \mathbb{R}).$

Now consider the subspace $\mathfrak{J}$ in $M(3, \mathcal{O})$ defined by

$$\mathfrak{J} = \{X \in M(3, \mathcal{O}) \mid \theta(\iota X) = X\},$$
then $\dim_\mathbb{R} \mathfrak{J} = 27$ and any $X \in \mathfrak{J}$ has the form

$$X = \begin{pmatrix}
\xi_1 e_0 & x_3 & \theta(x_2) \\
\theta(x_3) & \xi_2 e_0 & x_1 \\
x_2 & \theta(x_1) & \xi_3 e_0
\end{pmatrix},$$

where $\xi_i \in \mathbb{R},$ $x_i \in \mathcal{O}.$

The space $\mathfrak{J}$ is called an exceptional Jordan algebra with the Jordan product

$$X \circ Y = \frac{1}{2}(XY + YX) \in \mathfrak{J},$$
$X, Y \in J$.

$J$ has an inner product given by

\begin{equation}
\text{tr}(X \circ Y) = (X, Y)e_0. \tag{1.14}
\end{equation}

In fact, for $X \in J$, we have $\text{tr} X \in \mathbb{R}e_0$, and

\begin{equation}
(X, Y) = \sum_{i=1}^{3} (\xi_i \eta_i + 2 \langle x_i, y_i \rangle) \tag{1.15}
\end{equation}

where

$$X = \begin{pmatrix}
\xi_1 e_0 & x_3 & \theta(x_2) \\
\theta(x_3) & \xi_2 e_0 & x_1 \\
x_2 & \theta(x_1) & \xi_3 e_0
\end{pmatrix}$$

$$Y = \begin{pmatrix}
\eta_1 e_0 & y_3 & \theta(y_2) \\
\theta(y_3) & \eta_2 e_0 & y_1 \\
y_2 & \theta(y_1) & \eta_3 e_0
\end{pmatrix}$$

and $\langle x_i, y_i \rangle$ denote the inner product on $O$.

Now the Cayley projective plane $P^2O$ is defined as

**Definition 1.1.** $P^2O = \{X \in J \mid X \circ X = X, \text{tr} X = 1\}$.

The exceptional Lie group $F_4$ is defined as a group of algebra automorphisms of $J$, and acts on $P^2O$ two-point homogeneously. So we have $P^2O \cong F_4/\text{Spin}(9)$, where Spin(9) is realized as a subgroup of $F_4$ consisting of those elements which leave $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in J$ invariant.

**2. A Kähler structure**

In this section we describe a Kähler structure on the punctured cotangent bundle $T_0^*P^2O = T^*P^2O \setminus P^2O$ and state our main Theorem 2.1.

The tangent bundle $TP^2O$ is identified with the subset in $J \times J$ as

\begin{equation}
TP^2O = \left\{(X, Y) \in J \times J \mid X \circ X = X, \text{tr} X = 1, X \circ Y = \frac{1}{2} Y \right\}. \tag{2.1}
\end{equation}
We introduce the Riemannian metric on \( P^2 \mathcal{O} \) such that

\[
(Y_1, Y_2)_P = \frac{1}{2} \operatorname{tr}(Y_1 \circ Y_2) = \frac{1}{2} (Y_1, Y_2),
\]

where \((X, Y_1), (X, Y_2) \in TP^2 \mathcal{O} \).

Then from the inclusions \( \mathbb{C} \subset \mathbb{H} \subset \mathcal{O} \), the complex projective plane \( P^2 \mathbb{C} \) and the quaternion projective plane \( P^2 \mathbb{H} \) are embedded isometrically into \( P^2 \mathcal{O} \) as totally geodesic submanifolds (\( \text{Be} \)).

In the following we identify the tangent bundle \( TP^2 \mathcal{O} \) and the cotangent bundle \( T^*P^2 \mathcal{O} \) through the metric above. Under this identification the symplectic form \( \omega_\mathcal{O} \) on \( T^*P^2 \mathcal{O} \) is given by

\[
\omega_\mathcal{O} = -\frac{1}{2} (dX, dY),
\]

where we should interpret the inner product \((dX, dY)\) as a two-form in such a way that

\[
-(dX, dY) = \sum_{i=1}^{3} d\eta_i \wedge d\xi_i + 2 \sum_{i=1}^{3} \sum_{\alpha=0}^{7} dy^i_\alpha \wedge dx^i_\alpha
\]

restricted to \( TP^2 \mathcal{O} \), that is, we notice that \( \sum \eta_i \xi_i \) is replaced by \( \sum d\eta_i \wedge d\xi_i \) and so on in the definition of the inner product on \( \mathfrak{J} \).

We can extend \( \theta : \mathfrak{J} \rightarrow \mathfrak{J}, \, {}^t : \mathfrak{J} \rightarrow \mathfrak{J}, \, \text{tr} : \mathfrak{J} \rightarrow \mathcal{O} \), and the inner product \( (\cdot, \cdot) \) to the complexification \( \mathfrak{J} \otimes \mathbb{R} \mathbb{C} = \mathfrak{J}^\mathbb{C} \) in a natural way. So the Hermite inner product \( \langle \cdot, \cdot \rangle \) on \( \mathfrak{J}^\mathbb{C} \) is given by

\[
\langle X, Y \rangle = (X, Y),
\]

where \( X = \sum_{\alpha=0}^{7} X_\alpha \otimes e_\alpha \), \( X_\alpha \in M(3, \mathbb{C}) \) and \( X_\alpha \) is the complex conjugate of \( X_\alpha \). The norm of these elements in \( \mathfrak{J} \) and \( \mathfrak{J}^\mathbb{C} \) is always written as \( ||\cdot|| \), and we write the norm of the tangent vector \( Y \in T_X(P^2 \mathcal{O}) \) by \( ||Y||_P \).

Now consider the map \( \tau_\mathcal{O} : T_0^*P^2 \mathcal{O}(\cong T_0P^2 \mathcal{O}) \rightarrow \mathfrak{J}^\mathbb{C} \) defined by

\[
\tau_\mathcal{O}(X, Y)
= (||Y||^2 X - Y \circ Y) \otimes 1 + \frac{1}{\sqrt{2}} ||Y|| Y \otimes \sqrt{-1}
= (2||Y||^2 P X - Y \circ Y) \otimes 1 + ||Y||_P Y \otimes \sqrt{-1},
\]

then we have
**Theorem 2.1.** The map $\tau_O$ gives an isomorphism between $T_0^*P^2O$ and $E = \{A \in J^C \mid A \circ A = 0, A \neq 0\}$. Moreover

\[
\tau^*_O(\sqrt{-1} \partial\bar{\partial}\|A\|^2) = \frac{1}{\sqrt{2}}\omega_O.
\]

The two-form $\sqrt{-2} \partial\bar{\partial}\|A\|^2$ is itself a Kähler form on $J^C \setminus \{0\}$, so that we can regard $J^C \setminus \{0\}$ is a symplectic manifold. On this symplectic manifold the flow $\{\phi_t\}_{t \in \mathbb{R}}$ defined by

\[
\phi_t : A \mapsto \phi_t(A) = e^{-2\sqrt{-1}t} \cdot A
\]

is a Hamilton flow. The Hamiltonian of this flow is given by the function $f : A \mapsto \frac{1}{\sqrt{2}}\|A\|^2$. Since $E$ is holomorphic and the flow $\{\phi_t\}$ leaves $E$ invariant, the Hamiltonian of this flow on $E$ is just the restriction of $f$ to $E$, that is, the Hamiltonian is the square root of the metric function. So the flow $\{\phi_t\}$ is the bicharacteristic flow of the square root of the Laplacian on $P^2O$. Especially the flow restricted to the unit sphere $= \{(X, Y) \in TP^2O : \|Y\|_E = 1\}$ coincides with the geodesic flow. So we have

**Corollary 2.2.** The geodesics $\gamma(t)$ on $P^2O$ through a point $X$ with the direction $Y$ ($\|Y\|_E = 1$ and $X \circ Y = \frac{1}{2}Y$) is given by

\[
\gamma(t) = \cos 2t \cdot (X - \frac{1}{2}Y \circ Y) + \frac{1}{2} \sin 2t \cdot Y + \frac{1}{2}Y \circ Y.
\]

### 3. Freudenthal product and determinant

In this section we recall several formulas in the Jordan algebra $\mathfrak{J}$ for later use (see [11]).

Let $X, Y \in \mathfrak{J}$, then the “Freudenthal product” $X \times Y \in \mathfrak{J}$ is defined by the formula

\[
X \times Y = \frac{1}{2} \{2X \circ Y - (\text{tr} X)Y - (\text{tr} Y)X + (\text{tr} X \cdot \text{tr} Y - \text{tr}(X \circ Y))E\}
\]

where $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and the determinant, “det $X$”, for $X \in \mathfrak{J}$ is defined by

\[
det X = \frac{1}{3} \text{tr}(X \circ (X \times X)).
\]
Then we have

**Proposition 3.1.**

(i) \((X \circ Y, Z) = (X, Y \circ Z)\), for \(\forall X, Y, Z \in \mathcal{J}\)

(ii) \(X \circ (X \times X) = \det X \cdot E (\text{Cayley-Hamilton})\)

(iii) \((X \times X) \times (X \times X) = \det X \cdot X\).

As we explained above

\[(3.3) \quad F_4 = \{g \in GL(\mathcal{J}) \mid g(X \circ Y) = g(X) \circ g(Y), \forall X, Y \in \mathcal{J}\}.\]

Now \(F_4\) is also given in the following ways:

\[
F_4 = \{g \in GL(\mathcal{J}) \mid \text{for any } X, Y \in \mathcal{J}, \det(gX) = \det X, (gX, gY) = (X, Y)\}
\]

\[
= \{g \in GL(\mathcal{J}) \mid \text{for any } X, Y \in \mathcal{J}, \det(gX) = \det X, g(E) = E\}
\]

\[
= \{g \in GL(\mathcal{J}) \mid \text{for any } X, Y \in \mathcal{J}, g(X \times Y) = g(X) \times g(Y)\}.
\]

We also have for \(g \in F_4\)

\[(3.4) \quad \text{tr} gX = \text{tr} X.\]

The “Freudenthal product” and “\(\det\)” on \(\mathcal{J}\) are extended naturally to the complexification \(\mathcal{J}^C\), and we denote them with the same notations. Then the complexification of \(F_4\) is defined in the same way:

**Definition 3.2.** The complex simple Lie group \(F_4^C\) is

\[
F_4^C = \{g \in GL(\mathcal{J}^C) \mid g(X \circ Y) = g(X) \circ g(Y)\}
\]

\[
= \{g \in GL(\mathcal{J}^C) \mid \det(gX) = \det X, (gX, gY) = (X, Y)\}
\]

\[
= \{g \in GL(\mathcal{J}^C) \mid g(X \times Y) = g(X) \times g(Y)\}.
\]

The two-point homogeneity of \(F_4\) on \(P^2O\) is equivalent to

**Proposition 3.3.** Let \(S(P^2O) = \{(X, Y) \mid (X, Y) \in TP^2O \subset \mathcal{J} \times \mathcal{J}, \|Y\| = 1\}\), then \(F_4\) acts on \(S(P^2O)\) transitively and we have

\[(3.5) \quad S(P^2O) = F_4/\text{Spin}(7),\]

where the stationary subgroup at the point

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad \begin{pmatrix}
0 & \frac{1}{\sqrt{2}} & 0 \\
\frac{1}{\sqrt{2}} & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

\(\in S(P^2O)\) is identified with \(\text{Spin}(7)\).

**Remark 3.4.** The representation of \(F_4^C\) on \(\mathcal{J}_0^C(=\{A \in \mathcal{J}^C \mid \text{tr} A = 0\})\) is irreducible, and the subspace \(E\) is the orbit of the highest weight vector. According to the theorem by Lichtenstein ([Li]), then such an orbit is
characterized as the null set of a certain system of quadric equations. Thus our equation $A \circ A = 0$ (also this is equivalent to $A \times A = 0$ and $\text{tr} \ A = 0$) is nothing else than an example of this theorem in [Li], however we need the map $\tau_0$ to put a Kähler structure on $T^*_0 P^2 O$.

4. Proof of the theorem

We give a proof of Theorem 2.1 in a series of lemmas and proposition. First we prove

Lemma 4.1. Let $(X, Y) \in TP^2 O$, then

(i) $\det X = 0$,

(ii) $\text{tr} Y = 0$,

(iii) $\det Y = 0$.

Proof. Since

$$X \times X = \frac{1}{2} \{ 2X \circ X - 2 \text{tr} X \cdot X + ((\text{tr} X)^2 - (X, X)) E \} = 0,$$

we have

$$\det X = 0$$

by the definition of the determinant

$$X \circ (X \times X) = \det X \cdot E = 0.$$

From the equality

$$(X \circ Y, Z) = (X, Y \circ Z) \quad (X, Y, Z \in \mathcal{J})$$

we have

$$(X \circ Y, X) = \frac{1}{2} (Y, X)$$

$$= (Y, X \circ X) = (Y, X).$$

Hence $\text{tr} Y = \frac{1}{2} \text{tr}(X \circ Y) = (X, Y) = 0$.

Since $\text{tr} Y = 0$, the Freudenthal product $Y \times Y$ is expressed as

$$Y \times Y = Y \circ Y - \frac{1}{2} \|Y\|^2 E,$$

and we have

$$(4.1) \quad Y \circ (Y \times Y) = Y \circ (Y \circ Y) - \frac{1}{2} \|Y\|^2 Y = \det Y \cdot E.$$
Then by taking the trace we have
\[ \text{tr}(Y \circ (Y \circ Y)) = 3 \det Y. \]
Now
\[
\begin{align*}
\text{tr}(Y \circ (Y \circ Y)) &= \left( Y, Y \times Y + \frac{1}{2} \|Y\|^2 E \right) \\
&= 2 \left( X \circ Y, Y \times Y + \frac{1}{2} \|Y\|^2 E \right) \\
&= 2 \left( X, Y \circ \left( Y \times Y + \frac{1}{2} \|Y\|^2 E \right) \right) \\
&= 2(X, Y \circ (Y \times Y)) + (X, \|Y\|^2 Y) \\
&= 2(X, \det Y \cdot E) = 2 \det Y.
\end{align*}
\]
Hence we have
\[ (4.2) \quad \det Y = 0. \]
\[ \square \]

**Lemma 4.2.** For \((X, Y) \in T_0 P^2 \mathcal{O}, \tau_{\mathcal{O}}(X, Y) \in \mathbb{E},\) that is, \(\tau_{\mathcal{O}}(X, Y) \circ \tau_{\mathcal{O}}(X, Y) = 0.\)

**Proof.** Let \(X, Y \in T_0 P^2 \mathcal{O},\) then
\[
\begin{align*}
\tau_{\mathcal{O}}(X, Y) \circ \tau_{\mathcal{O}}(X, Y)
&= \left( \|Y\|^2 X - Y \circ Y \right)^2 - \frac{1}{2} \|Y\|^2 Y \circ Y \right) \otimes 1
\&+ \frac{2}{\sqrt{2}} \|Y\| Y \circ \left( \|Y\|^2 X - Y \circ Y \right) \otimes \sqrt{-1} \\
&= \left( \|Y\|^4 X - 2 \|Y\|^2 X \circ (Y \circ Y) + (Y \circ Y) \circ (Y \circ Y) - \frac{1}{2} \|Y\|^2 Y \circ Y \right) \otimes 1
\&+ \sqrt{2} \left( \frac{1}{2} \|Y\|^3 Y - \|Y\| Y \circ (Y \circ Y) \right) \otimes \sqrt{-1}.
\end{align*}
\]
Here we notice the following formulas: let \((X, Y) \in T(P^2 \mathcal{O}),\) then
\[ \begin{align*}
&\text{(i) } Y \circ (Y \circ Y) = \frac{1}{2} \|Y\|^2 Y, \\
&\text{(ii) } (X + Y) \times (X + Y) = (X - Y) \times (X - Y) = Y \circ Y - \frac{1}{2} \|Y\|^2 E, \\
&\text{(iii) } \det(X \pm Y) = 0, \\
&\text{(iv) } X \circ (Y \circ Y) = \frac{1}{2} \|Y\|^2 X.
\end{align*} \]
(i) is obtained by the Cayley-Hamilton

\[ Y \circ (Y \times Y) = \det Y \cdot E = 0. \]

and (ii) is easily shown. (iii) and (iv) are proved by first calculating

\[
(X \pm Y) \circ ((X \pm Y) \times (X \pm Y))
\]
\[
= (X \pm Y) \circ \left( Y^2 - \frac{1}{2} \|Y\|^2 E \right)
\]
\[
= X \circ (Y \circ Y) - \frac{1}{2} \|Y\|^2 X
\]
\[
= \det(X \pm Y) \cdot E,
\]

and by taking trace of both sides, we know \( \det(X \pm Y) = 0 \). Hence we have also (iv).

Next, from the formula \((Y \times Y) \times (Y \times Y) = (\det Y)Y = 0 \) we have

\[
0 = \left( Y \circ Y - \frac{1}{2} \|Y\|^2 E \right) \times \left( Y \circ Y - \frac{1}{2} \|Y\|^2 E \right)
\]

and so we have

\[
\|Y \circ Y\|^2 = \frac{1}{2} \|Y\|^4
\]

and

\[
(Y \circ Y) \circ (Y \circ Y) = \frac{1}{2} \|Y\|^2 Y \circ Y.
\]

Finally by making use of these formulas we can prove

\[
= \tau_C(X, Y) \circ \tau_C(X, Y)
\]
\[
= \left( \|Y\|^4 X - 2 \|Y\|^2 \cdot \frac{1}{2} \|Y\|^2 X + \frac{1}{2} \|Y\|^2 Y \circ Y - \frac{1}{2} \|Y\|^2 Y \circ Y \right) \otimes 1
\]
\[
+ \sqrt{2} \left( \|Y\|^3 \cdot \frac{1}{2} Y - \|Y\| \cdot \frac{1}{2} \|Y\|^2 Y \right) \otimes \sqrt{-1}
\]
\[
= 0.
\]

□

Let a map \( \sigma : \mathbb{E} \rightarrow \mathfrak{J} \times \mathfrak{J} \) be

\[
(4.5) \quad \sigma : A \mapsto (X, Y),
\]

where \( X \) and \( Y \) are given by the following formulas:
(4.6) \[ X = \frac{1}{2} \frac{1}{\|A\|} (A + \overline{A}) + \frac{A \circ \overline{A}}{\|A\|^2}, \]

(4.7) \[ Y = -\frac{\sqrt{-1}}{\sqrt{2}} \|A\|^{-\frac{1}{2}} (A - \overline{A}). \]

**Proposition 4.3.** Let \( X, Y \) be defined as above for \( A \in \mathbb{E} \), then

(i) \( X \circ X = X, \quad t^\theta(X) = X, \quad \text{tr} X = 1 \)

(ii) \( Y \circ X = \frac{1}{2} Y, \quad \text{tr} Y = 0. \)

**Proof.** By the definition of the Freudenthal product we have

\[
A \times A = \frac{1}{2} \{ 2A \circ A - 2 \text{tr} A \cdot A + ((\text{tr} A)^2 - (A, A)) E \}
\]

\[
= \frac{1}{2} \{ -2 \text{tr} A \cdot A + (\text{tr} A)^2 E \},
\]

and

\[
A \circ (A \times A) = \det A \cdot E
\]

\[
= A \circ \left( - \text{tr} A \cdot A + \frac{1}{2}(\text{tr} A)^2 E \right) = \frac{1}{2}(\text{tr} A)^2 A.
\]

So we have

(4.8) \[ \det A \cdot A = \frac{1}{2}(\text{tr} A)^2 A \circ A = 0 \]

and then we have

(4.9) \[ \det A = 0, \quad \text{tr} A = 0. \]

It follows easily that \( t^\theta(X) = X \) because of \( t^\theta(A) = A \) and \( t^\theta(\overline{A}) = \overline{A} \).

Put \( A = a \otimes 1 + b \otimes \sqrt{-1} \in \mathbb{E} \), where \( a, b \in \mathbb{J} \). Now we can assume \( \|A\| = 1 \), because of the homogeneity of the map \( \sigma \). Then we have \( a \circ a = b \circ b, a \circ b = 0, \|a\|^2 = \|b\|^2 = \frac{1}{2}, \) and \( \text{tr} a = \text{tr} b = 0. \) Also we show

(4.10) \[ \det a = \det b = 0, \quad \det(a \pm b) = 0. \]

The last equalities are proved by the following argument: from the equalities \( a \times a = a \circ a - \frac{1}{4} E \) and \( b \times b = b \circ b - \frac{1}{4} E \), we have

\[ a \circ (a \times a) = a \circ (a \circ a) - \frac{1}{4} a = \det a \cdot E, \]
\[ b \circ (b \times b) = b \circ (b \circ b) - \frac{1}{4} b = \det b \cdot E, \]
\[ (a \pm b) \times (a \pm b) = 2a \circ a - \frac{1}{2} E = 2b \circ b - \frac{1}{2} E. \]

Then
\[ (a \pm b) \circ ((a \pm b) \times (a \pm b)) = (a \pm b) \circ \left( 2a \circ a - \frac{1}{2} E \right) \]
\[ = (a \pm b) \circ \left( 2b \circ b - \frac{1}{2} E \right) \]
\[ = 2a \circ (a \circ a - \frac{1}{4} E) \pm 2b \circ (b \circ b - \frac{1}{4} E) \]
\[ = \det(a \pm b)E. \]

Hence we have
\[ 2(\det a \pm \det b) = \det(a \pm b). \]

On the other hand
\[ ((a \pm b) \times (a \pm b)) \times ((a \pm b) \times (a \pm b)) = \det(a \pm b)(a \pm b) \]
\[ = \left( 2a \circ a - \frac{1}{2} E \right) \times \left( 2a \circ a - \frac{1}{2} E \right). \]

Note that the last equality shows that (4.11) does not depend on the sign. Hence
\[ \det(a + b) \cdot (a + b) = \det(a - b) \cdot (a - b). \]

So we have
\[ \det(a + b) = \det(a - b), \]

since \( \text{tr}(a \circ a) = \frac{1}{2} \) and
\[ \det a = \det b = 0. \]

Then these finally imply
\[ \det(a \pm b) = 0. \]

By making use of these formulas we prove \( X \circ X = X \). Again we may assume \( \|A\| = 1 \), then \( X \) is written as
\[ X = a \otimes 1 + (a \otimes 1 + b \otimes \sqrt{-1}) \circ (a \otimes 1 - b \otimes \sqrt{-1}) \]
\[ = a + 2a \circ a. \]
Hence

\[ X \circ X = (a + 2a \circ a) \circ (a + 2a \circ a) \]
\[ = a \circ a + 4a \circ (a \circ a) + 4(a \circ a) \circ (a \circ a) \]
\[ = a \circ a + 4 \cdot \frac{a}{4} + a \circ a = a + 2a \circ a = X, \]

where we used the equality

\[ (a \circ a) \circ (a \circ a) = \frac{1}{4} a \circ a. \]

Also we have \( \text{tr} X = 1 = 2 \text{tr}(a \circ a). \)

Next we show

\[ X \circ Y = \frac{1}{2} Y. \]

Since

\[ Y = -\sqrt{-1} \frac{1}{\sqrt{2}} (a \otimes 1 + b \otimes \sqrt{-1} - (a \otimes 1 - b \otimes \sqrt{-1})) = \sqrt{2} b, \]

\[ X \circ Y = \sqrt{2} (a + 2a \circ a) \circ b \]
\[ = \sqrt{2} (a \circ b + 2b \circ (a \circ a)) \]
\[ = 2\sqrt{2} b \circ \left( a \times a + \frac{1}{4} E \right) \]
\[ = 2\sqrt{2} b \circ \left( b \times b + \frac{1}{4} E \right) \]
\[ = 2\sqrt{2} \left( \text{det} b \cdot E + \frac{1}{4} b \right) \]
\[ = \frac{1}{2} \cdot (\sqrt{2} b) = \frac{1}{2} Y. \]

From these we have proved that \( \tau_O \) is a bijection between \( T^*_0 P^2 \mathcal{O} \) in \( \mathfrak{J} \times \mathfrak{J} \) and \( \mathbb{E} \) in \( M(3, \mathcal{O}) \otimes_{\mathbb{R}} \mathbb{C} \) and that \( \sigma \) is the inverse map. \( \square \)

Next we prove \( \tau_{\mathcal{O}}(\sqrt{-2} \bar{\partial} \partial \sqrt{\|A\|}) = \omega_{\mathcal{O}}, \) that is, the Kähler form \( \sqrt{-2} \bar{\partial} \partial \sqrt{\|A\|} \) coincides with the symplectic form \( \omega_{\mathcal{O}} \) on \( T^*_0 P^2 \mathcal{O}. \)
First we have
\[
\tau^*_O(\sqrt{-1} \partial \partial \|A\|^2)
= \tau^*_O(\sqrt{-1} \partial \partial (A, A)^2)
= \frac{\sqrt{-1}}{4} d(\tau^*_O(A, A)^2 (dA, A))
= \frac{\sqrt{-1}}{4} d(\|Y\|^2 (d\tau^*_O A, \tau^*_O(A))).
\]

Here we should consider \( A \in J^C \) to be the section
\[
(4.16) \quad A : \quad E \longrightarrow E \times J^C
A \longmapsto (A, A)
\]
of the trivial bundle \( E \times J^C \) on \( E \), and \( dA \) the section of \( J^C \otimes T^*E \). Note that the inner product \((\cdot, \cdot)\) defines the pairing \( J^C \otimes J^C \otimes T^*E \rightarrow T^*E \). In the calculations below we will use this pairing with the same notation \((\cdot, \cdot)\). Also we note \( \|A\|^2 = \|Y\|^4 \) under the mapping \( \tau_O \).

We write
\[
(4.17) \quad \tau^*_O(A) = (\|Y\|^2 X - Y \circ Y) \otimes 1 + \frac{1}{\sqrt{2}} \|Y\| Y \otimes \sqrt{-1}
\]
\[
(4.18) \quad = a \otimes 1 + b \otimes \sqrt{-1},
\]
where \( a = a(X, Y), b = b(X, Y) \). Then
\[
\tau^*_O(dA, A)
= (d\tau^*_O A, \tau^*_O A)
= (da \otimes 1 + d(b \otimes \sqrt{-1}), a \otimes 1 - b \otimes \sqrt{-1})
= (da, a) + (db, b) + ((db, a) - (da, b)) \sqrt{-1}.
\]

Now from
\[
(a, a) = \frac{1}{2} \|Y\|^4 = (b, b)
\]
we have
\[
(da, a) = (Y, Y)(dY, Y)
(db, b) = (Y, Y)(dY, Y).
\]

So the real part of
\[
\tau^*_O((dA, A))
\]
is a closed form, since \(d((Y, Y)(dY, Y)) = 2(dY, Y) \wedge (dY, Y) = 0\). From

\[
(a, b) = \left(\|Y\|^2 X - Y \circ Y, \frac{1}{\sqrt{2}} Y \circ Y\right) = 0,
\]

\[
(db, a) - (b, da) = 2(db, a) = \frac{1}{\sqrt{2}} \left( Y \otimes \frac{(dY, Y)}{\|Y\|} + \|Y\|dY, \|Y\|^2 X - Y \circ Y \right)
\]

\[
= \frac{2}{\sqrt{2}} \left\{ \|Y\|^3 (dY, X) - \|Y\|((dY, Y \circ Y)) \right\}.
\]

Hence

\[
\frac{(-1)^2}{4} \cdot \frac{2}{\sqrt{2}} \left[ d\left(\|Y\|^{-3} \cdot (\|Y\|^3 (dY, X) - \|Y\|((dY, Y \circ Y)))\right) \right]
\]

\[
= -\frac{1}{2\sqrt{2}} \left\{ d(dY, X) - d\left(\frac{(dY, Y \circ Y)}{\|Y\|^2}\right) \right\}.
\]

By the equality \((X, Y \circ Z) = (X \circ Y, Z)\) we have

\[
(dY, Y \circ Y) = (Y, dY \circ Y)
\]

and by \((Y, Y \circ Y) = 0\), we have

\[
(dY, Y \circ Y) = 0.
\]

Hence we finally proved

\[
(4.19) \quad \tau_\mathcal{O}^\ast (-\partial \partial \sqrt{\|A\|}) = \frac{1}{2\sqrt{2}} (dY, dX) = \frac{1}{\sqrt{2}} \omega_\mathcal{O}.
\]

**Remark 4.4.** The map \(\tau_\mathcal{O}\) commutes with the actions of \(F_4\) on \(T_0 P^2 \mathcal{O}\) and on \(\mathfrak{J}^C\) (as a subgroup in \(F_4^C\)). Of course all the elements in \(F_4\) preserve the symplectic form \(\omega_\mathcal{O}\). Then it will be true that the subgroup of \(F_4^C\) consisting of those elements that preserve the symplectic form \(\omega_\mathcal{O}\) is compact. Hence it will coincide with \(F_4\).

### 5. An Embedding into \(M(8, \mathbb{C})\)

In this section we describe an embedding of the space \(E \subset \mathfrak{J}^C\) into the space of \(8 \times 8\) complex matrices \(M(8, \mathbb{C})\).

By identifying \(\mathcal{O} \cong \mathbb{H} \oplus \mathbb{H} e_4\), we define \(\gamma : \mathcal{O} \rightarrow \mathcal{O}\) by

\[
(5.1) \quad \gamma(h + ke_4) = h - ke_4.
\]
The map $\gamma$ is naturally extended to the complexification $O \otimes \mathbb{C}$, where we regard $O \otimes \mathbb{C} \cong H \otimes \mathbb{C} \oplus H \otimes \mathbb{C} e_4$. It is easily verified that $\gamma$ is an algebra isomorphism of $O$ (and of $O \otimes \mathbb{C}$), that is, $\gamma \in G_2(\cong$ the group of algebra isomorphisms of $O$), $\gamma^2 = \text{Id}$, and $\theta \circ \gamma = \gamma \circ \theta$.

Let $X \in \mathcal{J}^C$ be

$$X = \begin{pmatrix} \xi_1 & x_3 & \theta(x_2) \\ \theta(x_3) & \xi_2 & x_1 \\ x_2 & \theta(x_1) & \xi_3 \end{pmatrix},$$

$\xi_i \in \mathbb{C}, x_i \in O \otimes \mathbb{C}, x_i = \sum_{\alpha=0}^{7} x_i^\alpha e_\alpha \ (x_i^\alpha \in \mathbb{C})$, and we denote by

$$\gamma(X) = \begin{pmatrix} \xi_1 & \gamma(x_3) & \theta(\gamma(x_2)) \\ \theta(\gamma(x_3)) & \xi_2 & \gamma(x_1) \\ \gamma(x_2) & \theta(\gamma(x_1)) & \xi_3 \end{pmatrix},$$

then

$$\gamma : \mathcal{J}^C \longrightarrow \mathcal{J}^C$$

is also an algebra isomorphism.

We decompose elements $X \in \mathcal{J}^C$ as

$$X = \begin{pmatrix} \xi_1 & x_3 & \theta(x_2) \\ \theta(x_3) & \xi_2 & x_1 \\ x_2 & \theta(x_1) & \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 & m_3 & \theta(m_2) \\ \theta(m_3) & \xi_2 & m_1 \\ m_2 & \theta(m_1) & \xi_3 \end{pmatrix} + \begin{pmatrix} 0 & a_3 & -a_2 \\ -a_3 & 0 & a_1 \\ a_2 & -a_1 & 0 \end{pmatrix} e_4 = M + A e_4$$

where $x_i = m_i + a_i e_4 \ (m_i, a_i \in \mathbb{H} \otimes \mathbb{C})$. Then we can regard

$$\mathcal{J} \otimes \mathbb{C} = \mathcal{J}(3, \mathbb{H}) \otimes \mathbb{C} \oplus \mathbb{H}^3 \otimes \mathbb{C}.$$ 

Here we denote by $\mathcal{J}(3, \mathbb{H})$ the Jordan algebra of $3 \times 3$ Hermitian matrices with entries in $\mathbb{H}$.
Now we define a map $g$ following (5.4):

$$g : J \otimes \mathbb{C} \longrightarrow J(4, \mathbb{H})_0 \otimes \mathbb{C}$$

where we denote by $J(4, \mathbb{H})_0$ the subspace in the Jordan algebra $J(4, \mathbb{H})$ consisting of elements whose trace is zero.

The map $g$ satisfies

(i) $g(X) \circ g(Y) = g(\gamma(X \times Y)) + \frac{1}{4}(\gamma(X), Y) \cdot E$

(ii) $(g(X), g(Y)) = (\gamma(X), Y) = \text{tr}(\gamma(X) \circ Y)$

$$= \text{tr}(g(X) \circ g(Y))$$

We remark that $\text{tr}(A \circ B)$ for $A, B \in J(4, \mathbb{H})$ defines a Euclidean inner product on $J(4, \mathbb{H})$. We extend it to the complexification $J(4, \mathbb{H}) \otimes \mathbb{C}$.

If $A \in J^\mathbb{C}$, $A \circ A = 0$, then from (i) above we have at once

$$g(A) \circ g(A) = \frac{1}{4}(\gamma(A), A) \cdot E.$$

Let $\rho : \mathbb{H} \otimes \mathbb{C} \to M(2, \mathbb{C})$ be the isomorphism given by

$$\rho \left( \sum_{i=0}^{3} z_i e_i \right) \longmapsto \begin{pmatrix} z_0 + z_1 i & z_2 + z_3 i \\ -z_2 + z_3 i & z_0 - z_1 i \end{pmatrix},$$

and we denote with the same notation $\rho$ the map

$$\rho : J(4, \mathbb{H}) \otimes \mathbb{C} \longrightarrow M(8, \mathbb{C})$$

$$\psi \Longmapsto (\rho(h_{ij}))$$

where $h_{ij} \in \mathbb{H} \otimes \mathbb{C}$.

Proposition 5.1.

$$\rho(J(4, \mathbb{H})) = \{A \in M(8, \mathbb{C}) \mid \mathbb{J}A = 'A\mathbb{J}\},$$

where

$$\mathbb{J} = \begin{pmatrix} J & 0 & 0 & 0 \\ 0 & J & 0 & 0 \\ 0 & 0 & J & 0 \\ 0 & 0 & 0 & J \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$
Let $A \in \mathfrak{J} \otimes \mathbb{C}$ and $A \circ A = 0$, then as we know $\rho(g(A))^2$ is a scalar matrix and $\text{tr} \rho(g(A)) = 0$. Conversely we have

**Proposition 5.2.** Let $A \in M(8, \mathbb{C})$, $A \neq 0$, $\mathbb{J}A = ^tA\mathbb{J}$, and $A^2 = \lambda E$, $\lambda \in \mathbb{C}$, $\text{tr} A = 0$, then there exists an element $X \in \mathfrak{J}\mathbb{C}$ such that

$$X \circ X = 0$$

$$\rho(g(X)) = A$$

under the condition that $A$ is of the form

\[
A = \begin{pmatrix}
0 & 0 & \xi_1 & 0 \\
0 & 0 & 0 & \xi_1 \\
\xi_2 & 0 & 0 & \xi_2 \\
0 & \xi_3 & 0 & \xi_3
\end{pmatrix}, \quad \sum_{i=1}^{3} \xi_i = 0.
\]

**Remark 5.3.** The canonical line bundle of this complex structure on $T^*_0 \mathbb{P}^2 \mathcal{O}$ will be holomorphically trivial and this realization of $T^*_0 \mathbb{P}^2 \mathcal{O}$ in the space $M(8, \mathbb{C})$ given in the above proposition will be useful to construct an explicit holomorphic trivialization of the canonical line bundle.

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