HIGHER-ORDER WEIERSTRASS WEIGHTS OF BRANCH POINTS ON SUPERELLIPITIC CURVES

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Abstract. In this paper we consider the problem of calculating the higher-order Weierstrass weight of the branch points of a superelliptic curve $C$. For any $q > 1$, we give an exact formula for the $q$-weight of an affine branch point. We also find a formula for the $q$-weight of a point at infinity in the case where $n$ and $d$ are relatively prime. With these formulas, for any fixed $n$, we obtain an asymptotic formula for the ratio of the $q$-weight of the branch points, denoted $BW_q$, to the total $q$-weight of points on the curve:

$$\liminf_{d \to \infty} \frac{BW_q}{g(g-1)^{2q}(2q-1)^2} \geq \frac{n+1}{3(n-1)^2(2q-1)^2},$$

with equality when the limit is taken such that $\gcd(n,d) = 1$.

1. Introduction

Let $q \in \mathbb{N}$. A $q$-Weierstrass point (or higher-order Weierstrass point) is a point $P$ on a curve for which there exist holomorphic $q$-differentials that have higher than expected orders of vanishing at $P$. Each $q$-Weierstrass point has an associated $q$-weight, denoted $w^{(q)}(P)$, which measures how much higher than expected those orders of vanishing are. A curve of genus $g \geq 2$ has finitely many $q$-Weierstrass points.

Importantly, the $q$-weight of a point is invariant under automorphism. Thus, higher-order Weierstrass points are important in the study of automorphisms of algebraic curves. For instance, Lewittes showed in [6] that if an automorphism has at least five fixed points, then all of its fixed points are 1-Weierstrass points. Further, Mumford, in [8], has suggested that $q$-Weierstrass points on an algebraic curve are analogous to $q$-torsion points on an elliptic curve. For more on the history of Weierstrass points, we refer the reader to [2]. For background material of Weierstrass points specifically on superelliptic curves, see [12].

Let $C$ be a curve of genus $g \geq 2$ of the form $y^n = f(x)$ where $f(x)$ is a separable polynomial of degree $d > n \geq 2$. Such a curve is said to be superelliptic. In the cover $C \to \mathbb{P}^1$, the points above the roots of $f(x)$ are branch points. If $n \nmid d$, the point (or points) above $\infty$ in the nonsingular model of $C$ is also a branch point. One can show that each branch point is a $q$-Weierstrass point for all $q$; in the case where $n = 2$, the branch points are exactly the 1-Weierstrass points. Let $B$ be an affine branch point on $C$ and, if $n \nmid d$, $P^\infty$ a nonsingular branch point at infinity. In [16, Theorem 8], Towse calculated the 1-weight of the branch points (affine and
at infinity) as a function of \( n \) and \( d \). In the case of \( \gcd(n, d) = 1 \), he found that

\[
w^{(1)}(B) = \frac{g(n + 1)(d - 7)}{12} + (d - 1) \sum_{j=1}^{n-1} \left\{ -\frac{dj}{n} \right\}_j,
\]

where \( \{x\} \) denotes the fractional part of \( x \), and

\[
w^{(1)}(P^\infty) = \frac{(n^2 - 1)(d^2 - 1)}{24} - g.
\]

Given that the total 1-weight of points on a curve of genus \( g \) is \( g^3 - g \), he was able to calculate the fraction that the branch points’ 1-weight (denoted \( BW \)) accounted for as

\[
\lim_{d \to \infty} \frac{BW}{g^3 - g} \geq \frac{n + 1}{3(n - 1)^2},
\]

with equality when the limit is taken over integers \( d \) such that \( \gcd(n, d) = 1 \).

The goal of this paper is to extend Towse’s results to higher-order Weierstrass weights of branch points on a superelliptic curve. To achieve this, we first produce a basis for the space of holomorphic \( q \)-differentials on a superelliptic curve \( C \). A common approach to calculate the \( q \)-weight is to work with the Wronskian of this basis, a method first described by Hurwitz in [5]. However, we take a different approach, instead using results from numerical semigroups and non-representable numbers. In this way we obtain a formula for the \( q \)-weight of an affine branch point as a function of \( n \), \( d \), and \( q \). The main result is Theorem 2. In particular, when \( \gcd(n, d) = 1 \) we find for \( q \geq 2 \)

\[
w^{(q)}(B) = \frac{g(n + 1)(d - 7)}{12} + g + (d - 1) \sum_{j=1}^{n-1} \left\{ -\frac{(d + 1)q + dj}{n} \right\}_j.
\]

As for the points at infinity, with two examples, we show that if \( \gcd(n, d) > 1 \), one cannot get a formula for \( w^{(q)}(P^\infty) \) based only on \( n \), \( d \), and \( q \). However, if \( \gcd(n, d) = 1 \) and \( q \geq 2 \), then in Theorem 3 we have

\[
w^{(q)}(P^\infty) = \frac{(n^2 - 1)(d^2 - 1)}{24}.
\]

Since the total \( q \)-weight of points on a curve of genus \( g \) is \( g(g - 1)^2(2q - 1)^2 \) for \( q \geq 2 \), we get an similar asymptotic result in Proposition 6 for the proportion of branch points’ \( q \)-weight (denoted \( BW_q \)). We find

\[
\lim_{d \to \infty} \frac{BW_q}{g(g - 1)^2(2q - 1)^2} \geq \frac{n + 1}{3(n - 1)^2(2q - 1)^2},
\]

with equality when the limit is taken over integers \( d \) such that \( \gcd(n, d) = 1 \).

The paper is organized as follows. In Section 2 we provide some background material on calculating the \( q \)-Weierstrass weight of a point. We also include some notation and results for non-representable integers in numerical semigroups with two generators. In Section 3 we find a basis for the space of holomorphic \( q \)-differentials on a curve \( C \) by presenting a set of linearly independent holomorphic differentials and then counting them to make sure there are as many as the Riemann-Roch theorem predicts. In Section 4 we have our main results. We find a formula for the \( q \)-weight of an affine branch point, and we use that to derive a few corollaries for specific cases of \( n \) and \( d \). We also note that, for given \( n \) and \( d \), the \( q \)-weight of a branch point depends only on the value of \( q \) modulo \( n \). We also give examples to
show that if \( \gcd(n, d) > 1 \) the \( q \)-weight of a point at infinity cannot necessarily be determined just by knowing \( n \) and \( d \). If \( \gcd(n, d) = 1 \), however, we can calculate the weight. In both cases, we obtain some asymptotic results about the proportion of \( q \)-weight that the branch points contain.

2. Preliminaries and notation

2.1. \( q \)-Weierstrass points. In this paper, we will follow the approach given in [13, Section 2]. We describe the notation and major results for calculating weights of \( q \)-Weierstrass points here.

Let \( k \) be an algebraically closed field, \( C \) be a non-singular projective curve over \( k \) of genus \( g \geq 2 \), and \( k(C) \) its function field. For any \( f \in k(C) \), let \( \text{div}(f) \) denote the divisor associated to \( f \). For any divisor \( D = \sum P \cdot P \) and any point \( P \), let \( \nu_P(D) = n \), and let \( \text{ord}_P(f) = \nu_P(\text{div}(f)) \).

For any \( q \in \mathbb{N} \), let \( H^0(C, (\Omega^1)^q) \) be the \( C \)-vector space of holomorphic \( q \)-differentials on \( C \), a space of dimension

\[
d_q = \begin{cases} 
g & \text{if } q = 1, \\
(g-1)(2q-1) & \text{if } q \geq 2. 
\end{cases}
\]

For \( P \) a degree 1 point on \( C \), consider a basis \( \{\psi_1, \ldots, \psi_{d_q}\} \) of \( H^0(C, (\Omega^1)^q) \) where

\[
\text{ord}_P(\psi_1) < \text{ord}_P(\psi_2) < \cdots < \text{ord}_P(\psi_{d_q}).
\]

The \( q \)-weight of \( P \) is

\[
w^{(q)}(P) = \sum_{i=1}^{d_q} \text{ord}_P(\psi_i) - \sum_{j=0}^{d_q-1} j.
\]

We call the point \( P \) a \( q \)-Weierstrass point if \( w^{(q)}(P) > 0 \).

**Proposition 1.** [4, III.5.10] Let \( C \) be a curve of genus \( g \geq 2 \) and let \( q \geq 1 \). Then the total \( q \)-weight of points on \( C \) is

\[
\sum_{P \in C} w^{(q)}(P) = (g-1)d_q(2q-1 + d_q) = \begin{cases} 
g^3 - g & \text{if } q = 1, \\
g(g-1)^2(2q-1)^2 & \text{if } q \geq 2. 
\end{cases}
\]

2.2. Non-representable numbers. For notation, let \( \mathbb{N}_0 \) be the set of non-negative integers. Let \( a, b \in \mathbb{N} \) and consider the set

\[
R(a, b) = \{ax + by : x, y \in \mathbb{N}_0 \}.
\]

Elements of \( R(a, b) \) are called \((a, b)\)-representable numbers. The complement of \( R(a, b) \) in \( \mathbb{N}_0 \), denoted \( NR(a, b) \), is the set of \((a, b)\)-representable numbers. When there is no confusion, we will omit the \((a, b)\) and simply refer to these numbers as representable or non-representable.

The problem of calculating the cardinality of \( NR(a, b) \) dates to the late 19th century in [15]. Clearly, if \( \gcd(a, b) > 1 \) then \( NR(a, b) \) is an infinite set. It is straightforward to show that the converse is true too. For example, see [10, Theorem 1.0.1] for two proofs of the following result.

**Lemma 1.** For \( a, b \in \mathbb{N} \), if \( \gcd(a, b) = 1 \) then \( NR(a, b) \) is a finite set.

For the rest of this section, we will assume \( \gcd(a, b) = 1 \). Thus, \( NR(a, b) \) is finite and so we can compute its cardinality.
Proposition 2. [15, Page 134] For $a, b \in \mathbb{N}$ with $\gcd(a, b) = 1$,
$$|NR(a, b)| = \frac{(a - 1)(b - 1)}{2}.$$  

This result is important in the theory of algebraic curves. Suppose a plane curve
$C$ is given by the affine equation
$$\alpha_{a,0}x^a + \alpha_{0,b}y^b + \sum_{i,j} \alpha_{i,j}x^iy^j = 0$$
for constants $\alpha_{a,0}, \alpha_{0,b} \neq 0$ and where the summation is over non-negative
$i, j$ such that $aj + bi < ab$. Such a curve is called a $C_{a,b}$ curve. These curves can
be seen as a generalization of elliptic and hyperelliptic curves in Weierstrass form.

With the Riemann-Roch Theorem, if the affine part of the curve is non-singular
then one can show that the genus of such a curve is exactly $(a - 1)(b - 1)/2$, the
 cardinality of $NR(a, b)$. For details, see [7] or [14].

For the purposes of this paper, we will also need to know the sum of the elements
of $NR(a, b)$. This problem was solved in [1] using generating functions.

Proposition 3. For $a, b \in \mathbb{N}$ with $\gcd(a, b) = 1$,
$$\sum_{n \in NR(a, b)} n = \frac{(a - 1)(b - 1)(2ab - a - b - 1)}{12}.$$  

This result was generalized to a formula for the sum of the $m$th powers of ele-
ments of $NR(a, b)$. See [11] or [17] for details.

3. A basis of holomorphic $q$-differentials

In this section, we give a basis for the space of holomorphic $q$-differentials on a
superelliptic curve $C$. The main result of this section is as follows:

Theorem. Let $C$ be a curve of genus $g \geq 2$ given in affine coordinates by $y^n = f(x)$,
for $f(x)$ a separable polynomial of degree $d > n \geq 2$. For $q \geq 1$, let $H^0(C, (\Omega^1)^q)$
be the space of holomorphic $q$-differentials on $C$. Let
$$\mathfrak{B}_{n,d,q} = \left\{ x^iy^j \left( \frac{dx}{y^n} \right)^q : 0 \leq i, 0 \leq j < n, ni + dj \leq (2g - 2)q \right\}.$$  

Then $\mathfrak{B}_{n,d,q}$ is a basis for $H^0(C, (\Omega^1)^q)$.

In order to prove this, we need the following results (based on the work in [16])
and a useful lemma.

Let $G = \gcd(n, d)$. For $g$ the genus of $C$, we have $2g - 2 = nd - n - d - G$. Let
$\{\alpha_1, \ldots, \alpha_d\}$ denote the $d$ distinct roots of $f(x)$ and let $B_i = (\alpha_i, 0)$ for $i = 1, \ldots, d$.
For each non-root $\omega$ of $f(x)$, let $P_1^\omega, \ldots, P_n^\omega$ denote the $n$ points on $C$ over $x = \omega$.
And let $P_1^\infty, \ldots, P_G^\infty$ denote the $G$ points over $\infty$ in the non-singular model of $C$.

One then has the following principal divisors.

- $\text{div}(y) = \sum_{j=1}^{d} B_j - \frac{d}{G} \sum_{m=1}^{G} P_m^\infty$,
- $\text{div}(x - \alpha_i) = nB_i - \frac{n}{G} \sum_{m=1}^{G} P_m^\infty$,
- $\text{div}(x - \omega) = \sum_{j=1}^{n} P_j^\omega - \frac{n}{G} \sum_{m=1}^{G} P_m^\infty$.  

\textbf{Theorem 1.} Let \( C \) be a curve of genus \( g \geq 2 \) given in affine coordinates by \( y^n = f(x) \), for \( f(x) \) a separable polynomial of degree \( d > n \). For \( g \geq 1 \), let

\[ \text{div}(dx) = (n - 1) \sum_{j=1}^{d} B_j - \left( \frac{n}{G} + 1 \right) \sum_{m=1}^{G} P_m^\infty, \]

From these, we see that \( \text{div}((dx/y^{n-1})^q) = \frac{2(n-2)q}{G} \sum_{m=1}^{G} P_m^\infty. \)

For integers \( i, j \), let \( f_{i,j} = x^i y^j (dx/y^{n-1})^q \). We want to find conditions on \( i \) and \( j \) such that \( f_{i,j} \) is a holomorphic \( q \)-differential. Note that \( f_{i,j} \) can have poles only at the points above \( \infty \) if \( i, j \geq 0 \). In that situation, we find

\[ \text{ord}_{i,j}^m(f_{i,j}) = \left( \frac{2g - 2 - (ni + dj)}{G} \right) \]

for each pair \((i, j)\) and each \( m \). Hence \( f_{i,j} \) is a holomorphic \( q \)-differential as long as \( i \geq 0, j \geq 0, \) and \( ni + dj \leq (2g - 2)q \).

\textbf{Lemma 2.} Let \( n, d, q \in \mathbb{Z} \) with \( 2 \leq n < d \) and \( q \geq 2 \). As above, let \( G = \gcd(n, d) \) and \( 2g - 2 = nd - n - d - G \). For all but finitely many triples \((n, d, q)\), one has

\[ (2g - 2)q - d(n - 1) \geq 0. \]

The exceptional cases are \((n, d, q) \in \{ (2, 5, 2), (2, 6, 2) \} \).

\textit{Proof.} First, note that \((2g - 2)q - d(n - 1) = (nd - n - d - G)q - d(n - 1) = d(n - 1)(q - 1) - q(n + G) \). Thus, to show our desired inequality, it is equivalent to show

\[ d \geq \left( \frac{q}{q-1} \right) \left( \frac{n+G}{n-1} \right). \]

For notation, let \( h(q, n) = \left( \frac{q}{q-1} \right) \left( \frac{n+G}{n-1} \right) \). We aim to show \( d \geq h(q, n) \).

For \( q \geq 2 \), the maximum value of \( q/(q - 1) \), which occurs when \( q = 2 \), is 2. For \( n \geq 2 \), the maximum value of \((n + G)/(n - 1) \) occurs when \( G \) is largest, so when \( G = n \), and when \( n = 2 \). The maximum value is 4. Thus \( h(q, n) \leq 2 \cdot 4 = 8 \) so \( d \geq h(q, n) \) for all \( d \geq 8 \).

Now we consider cases of \( n \). If \( n \geq 4 \), then \( G = n \) or \( G < n \). If \( G = n \), then \( n|d \), so \( d \geq 2n \geq 8 \), so \( d \geq h(q, n) \). If \( G < n \), then \( G \leq n/2 \), so \( h(q, n) \leq 2(n + n/2)/(n - 1) = 3 + 3/(n - 1) \leq 4 \). Since \( d > n \), we have \( d > n \geq 4 \geq h(q, n) \), as desired.

If \( n = 3 \), then \( G = 1 \) or \( G = 3 \). If \( G = 1 \), then \( h(q, 3) \leq 2 \cdot 2 = 4 \). In this case, since \( d > n = 3 \), we have \( d \geq 4 \), so \( d \geq h(q, 3) \). If \( G = 3 \), then \( h(q, 3) \leq 2 \cdot 3 = 6 \). In this case, \( d \geq 2n = 6 \geq h(q, 3) \), as desired.

If \( n = 2 \), then \( G = 1 \) or \( G = 2 \). Note that to have \( g \geq 2 \), we only consider \( d \geq 5 \). If \( G = 1 \) and \( q = 2 \), then \( h(2, 2) = 2 \cdot 3 = 6 \). Since \( G = 1 \), \( d \) is odd, so \( d \geq h(2, 2) \) for all \( d \) except for \( d = 5 \). If \( G = 1 \) and \( q \geq 3 \), then \( h(q, 2) \leq (3/2) \cdot 3 = 9/2 \), so \( d \geq h(q, 2) \) for all \( d \geq 5 \). If \( G = 2 \) and \( q = 2 \), then \( h(2, 2) = 2 \cdot 4 = 8 \). Since \( G = 2 \), \( d \) is even, so \( d \geq h(2, 2) \) for all \( d \) except for \( d = 6 \). If \( G = 2 \) and \( q \geq 3 \), then \( h(q, 2) \leq (3/2) \cdot 4 \), so \( d \geq h(q, 2) \) for all \( d \geq 6 \).

Thus, the only exceptional cases are \((n, d, q) = (2, 5, 2) \) or \((2, 6, 2) \). For all other triples, we find that \( d \geq h(q, n) \), or, equivalently, that

\[ (2g - 2)q - d(n - 1) \geq 0, \]

as desired. \( \square \)

\textbf{Theorem 1.} Let \( C \) be a curve of genus \( g \geq 2 \) given in affine coordinates by \( y^n = f(x) \), for \( f(x) \) a separable polynomial of degree \( d > n \). For \( g \geq 1 \), let
$H^0(C, (\Omega^1)^q)$ be the space of holomorphic $q$-differentials on $C$. Let
\[ \mathfrak{B}_{n,d,q} = \left\{ x^i y^j \left( \frac{dx}{y^{n-1}} \right)^q : 0 \leq i, 0 \leq j < n, ni + dj \leq (2g - 2)q \right\}. \]

Then $\mathfrak{B}_{n,d,q}$ is a basis for $H^0(C, (\Omega^1)^q)$.

**Proof.** The $q = 1$ case, in a slightly different form, is proved in [10]. For completeness, we will first prove the $q \geq 2$ case here and then adapt our argument to cover the $q = 1$ case.

Suppose $q \geq 2$. With the restriction that $0 \leq j < n$, we see that these holomorphic $q$-differentials are linearly independent. We therefore need to show that $|\mathfrak{B}_{n,d,q}| = d_q = (2q - 1)(g - 1)$.

We first consider the case where $(n,d,q) \notin \{(2,5,2),(2,6,2)\}$ and let $\mathfrak{B} = \mathfrak{B}_{n,d,q}$. Note that we require $i \geq 0$ and $ni + dj \leq (2g - 2)q$, so
\[
0 \leq i \leq \left\lfloor \frac{(2g - 2)q - dj}{n} \right\rfloor.
\]

For each $j = 0, \ldots, n - 1$, we have $(2g - 2)q - dj \geq (2g - 2)q - d(n - 1)$, which is non-negative by Lemma [2]. (This is why we handle the $(2,5,2)$ and $(2,6,2)$ cases separately.) Thus, to calculate the number of pairs $(i,j)$ in $\mathfrak{B}$, we will let $j$ go from $0$ to $n - 1$ and count the number of indices $i$ that correspond to each $j$ value. I.e.,
\[
|\mathfrak{B}| = \sum_{j=0}^{n-1} \left( 1 + \left\lfloor \frac{(2g - 2)q - dj}{n} \right\rfloor \right).
\]

Since $|x| = x - \{x\}$, we simplify the sum to get
\[
|\mathfrak{B}| = n + (2g - 2)q - \frac{d(n - 1)}{2} - \sum_{j=0}^{n-1} \left\{ \frac{(-d - G)q - dj}{n} \right\}.
\]

Now, we consider cases of $G$. If $G = n$, then $n|d$, so $n|(-d - G)q - dj)$, so each term in the summation is 0. Note that $n - d(n - 1)/2 = -(nd - d - 2n)/2 = -(nd - n - d - G)/2 = -(g - 1)$. Then $|\mathfrak{B}| = (2g - 2)q - (g - 1) = (2q - 1)(g - 1) = d_q$, as desired.

Next, suppose $G \neq n$. Let $n' = n/G$ and $d' = d/G$. Dividing the numerator and denominator by $G$, the summation equals $\sum_{j=0}^{n-1} \left\{ \frac{(-d')q - dj}{n'} \right\}$. Since $\gcd(n',d') = 1$, as $j$ goes from 0 to $n' - 1$ modulo $n'$, the numerators are distinct modulo $n'$ and therefore in every congruence class exactly once modulo $n'$. Since $n/n' = G$, this summation equals $G\sum_{k=0}^{n'-1} \frac{1}{n'} = G(n' - 1)/2$. All together,
\[
|\mathfrak{B}| = n + (2g - 2)q - d(n - 1)/2 - (n - G)/2.
\]

I.e. $|\mathfrak{B}| = n + 2q(g - 1) - ((1/2)(nd - d - n - G) = 2q(g - 1) - (1/2)(nd - d - n - G) = (2q - 1)(g - 1) = d_q$, as desired.

To complete the proof for $q \geq 2$, we consider the exceptional cases. Suppose $(n,d,q) = (2,5,2)$, so $g = 2$ and $d_2 = 3$. Then
\[
\mathfrak{B}_{2,5,2} = \left\{ x^i y^j (dx/y)^2 : i \geq 0, 0 \leq j < 2, 2i + 5j \leq 4 \right\}.
\]

Thus, $\mathfrak{B}_{2,5,2} = \{ (dx/y)^2, x(dx/y)^2, x^2(dx/y)^2 \}$, so $|\mathfrak{B}_{2,5,2}| = 3 = d_2$, as desired.

Suppose $(n,d,q) = (2,6,2)$, so $g = 2$ and $d_2 = 3$. Then
\[
\mathfrak{B}_{2,6,2} = \left\{ x^i y^j (dx/y)^2 : i \geq 0, 0 \leq j < 2, 2i + 6j \leq 4 \right\}.
\]

I.e. $|\mathfrak{B}_{2,6,2}| = 3 = d_2$, as desired.
Thus, $\mathfrak{B}_{2,6,2} = \{(dx/y)^2, x(dx/y)^2, x^2(dx/y)^2\}$, so $|\mathfrak{B}_{2,6,2}| = 3 = d_2$, as desired.

Now, suppose $q = 1$. Following the approach above, given $j$ we need integers $i$ such that

$$0 \leq i \leq \left\lfloor \frac{(2g-2) - dj}{n} \right\rfloor.$$

If $j \leq n - 2$, then $(2g-2) - dj \geq (2g-2) - d(n-2) = d - n - G \geq 0$ since $d \geq n$. If $j = n - 1$, then $(2g-2) - d(n-1) = -n - G < 0$, so there are no such $i$. Thus, our summation for the summation above, we can subtract the $j = n - 1$ term out front to get

$$|\mathfrak{B}| = -\left(1 + \left\lfloor \frac{-n - G}{n} \right\rfloor\right) + \sum_{j=0}^{n-1} \left(1 + \left\lfloor \frac{(2g-2) - dj}{n} \right\rfloor\right).$$

Since $q = 1$ the summation equals $g - 1$, so $|\mathfrak{B}| = -(1 - 2) + (g - 1) = g = d_1$, as desired. □

4. Weights of branch points

In this section, we use the bases we found in the previous section to calculate the $q$-weight of the affine branch points and, in the case that $\gcd(n, d) = 1$, the point at infinity.

4.1. Weights of affine branch points. Suppose $q \geq 2$. For $C$ given by $y^n = f(x)$ with $f(x)$ separable of degree $d$, let $\alpha$ be a root of $f(x)$. Then $B = (\alpha, 0)$ is an affine branch point of $C$. Note that we can replace $x$ by $(x - \alpha)$ in our basis $\mathfrak{B}_{n,d,q}$ to produce a new basis $\mathfrak{B}_{n,d,q,\alpha}$. That is,

$$\mathfrak{B}_{n,d,q,\alpha} = \{(x - \alpha)^i y^j (dx/y^{n-1})^q : i \geq 0, 0 \leq j < n, ni + dj \leq (2g-2)q\}$$

is a basis for $H^q(C, (\Omega^1)^q)$.

Let $f_{i,j,\alpha} = (x - \alpha)^i y^j (dx/y^{n-1})^q \in \mathfrak{B}_{n,d,q,\alpha}$. Then

$$\nu_B(f_{i,j,\alpha}) = ni + j.$$

Since $0 \leq j < n$, these valuations are all different, and thus

$$w^{(q)}(B) = \sum_{(i,j) \in S} (ni + j) - \sum_{k=0}^{d_q-1} k,$$

where $S = \{(i,j) \in \mathbb{Z}^2 : i \geq 0, 0 \leq j < n, ni + dj \leq (2g-2)q\}$. We rewrite this as

$$w^{(q)}(B) = W_1 - W_2 - W_3$$

where

$$(1) \quad W_1 = \sum_{(i,j) \in S} (ni + dj), \quad W_2 = (d-1) \sum_{(i,j) \in S} j, \quad W_3 = \sum_{k=0}^{d_q-1} k.$$

We have

$$W_3 = \frac{(d_q-1)d_q}{2} = \frac{1}{2} ((2g-2)^2 q^2 + (2g-2)(1-2g)q + g(g-1)).$$

We will evaluate $W_1$ and $W_2$ with the following propositions.

**Proposition 4.** Let $n, d, q \in \mathbb{N}$ such that $n < d$ and $q \geq 2$. Then

$$W_1 = 2(g-1)^2 q^2 + (g-1)Gq + \frac{G^2 - 1 - (n-1)(d-1)(2nd - n - d - 1)}{12}.$$
We will first sketch the proof in the situation where \( \gcd(n, d) = 1 \). Afterward, we will prove the theorem for any \( \gcd \).

When \( \gcd(n, d) = G = 1 \), for \((i, j) \in S\), the terms \( ni + dj \) are distinct integers from 0 to \((2g - 2)q\). From Proposition 2, we have \((2g - 2)q \geq nd - n - d\) by Lemma 3 below, all of the \((n - 1)(d - 1)/2\) \((n, d)\)-non-representable integers are in that interval. The sum of the non-representable integers, as is given in Proposition 3, is \((n - 1)(d - 1)(2nd - n - d - 1)/12\). Thus, if \( \gcd(n, d) = 1 \), we add up all of the integers from 0 to \( (2g - 2)q \) and subtract off the non-representable integers to get

\[
W_1 = (2g - 2)q((2g - 2)q + 1)/2 - (n - 1)(d - 1)(2nd - n - d - 1)/12.
\]

If \( \gcd(n, d) > 1 \), then the terms \( ni + dj \) are no longer distinct, so we need to evaluate the sum more carefully.

**Proof.** Let \( G = \gcd(n, d) \). First, we observe that \( W_1 = G \sum_{(i, j) \in S} (n'i + d'j) \) for \( n' = n/G \) and \( d' = d/G \). Note that \( \gcd(n', d') = 1 \). For \( k \) from 0 to \( G - 1 \), let

\[
S_k = \{(i, j) : i \geq 0, \text{ } kn' \leq j < (k + 1)n', \text{ } ni + dj \leq (2g - 2)q\}.
\]

In particular, \( S \) is the disjoint union of the sets \( S_k \). Let \( W_{1,k} = \sum_{(i,j) \in S_k} (n'i + d'j) \). Then

\[
W_1 = G \sum_{k=0}^{G-1} W_{1,k}.
\]

Letting \( j' = j - kn' \), we rewrite \( S_k \) as

\[
S_k = \{(i, j' + kn') : i \geq 0, 0 \leq j' < n', ni + dj' \leq (2g - 2)q - n'dk\}
\]

and dividing the last inequality through by \( G \) we obtain

\[
S_k = \{(i, j' + kn') : i \geq 0, 0 \leq j' < n', ni + d'j' \leq \frac{(2g - 2)q - n'dk}{G}\}.
\]

Let \( m_k = \frac{(2g - 2)q - n'dk}{G} \), the upper bound in \( S_k \). The following lemma will allow us to conclude that all of the \((n', d')\)-non-representable integers are less than \( m_k \).

**Lemma 3.** Let \( m_k = \frac{(2g - 2)q - n'dk}{G} \). Then \( m_k \geq n'd' - n' - d' \) for all \( n, d, q \in \mathbb{N} \) with \( 0 \leq k \leq G - 1 \), \( n < d' \), \( g \geq 2 \), and \( q \geq 2 \).

**Proof.** First, note that for \( 0 \leq k \leq G - 1 \), \( m_k = \frac{(2g - 2)q - n'dk}{G} \geq \frac{(2g - 2)q - n'dk}{G} - n'd' = (2g - 2)q - G(n - 1)d' \leq (2g - 2)q - n'd' = nd - n - d \).

We compute the exceptional cases separately. For \((n, d, q) = (2, 5, 2)\), then \( g = 2 \) and \((2g - 2)q = 4 = \text{nd} - n - d \). If \((n, d, q) = (2, 6, 2)\), then \( g = 2 \) and \((2g - 2)q = 4 = \text{nd} - n - d \). Thus, the bound holds for the exceptional cases as well.

For \( S_k \), since we are considering \( i \geq 0 \) and \( 0 \leq j' < n' \), and since \( m_k \geq n'd' - n' - d' \) for all \( k \), our ordered pairs \((i, j' + kn')\) are in one-to-one correspondence with the \((n', d')\)-representable numbers in the interval \([0, m_k] \). And since \( m_k \geq n'd' - n' - d' \), all of the \((n' - 1)(d' - 1)/2\) \((n', d')\)-non-representable numbers are in this interval as well. Thus \( S_k \) contains \(|S_k| = m_k + 1 - (n' - 1)(d' - 1)/2\) ordered pairs.
Then
\[ W_{1,k} = \sum_{(i,j) \in S_k} (n'i + d'j) \]
\[ = \sum_{(i,j'+kn') \in S_k} (n'i + d'j' + n'd'k) \]
\[ = n'd'k \cdot |S_k| + \sum_{(i,j'+kn') \in S_k} (n'i + d'j'). \]

The summation is the sum of the \((n',d')\)-representable numbers from 0 to \(m_k\).
We calculate this by summing all of the integers from 0 to \(m_k\) and subtracting the \((n',d')\)-non-representable integers, which all lie in this interval. Using Proposition 3, the summation is
\[ m_k(m_k + 1)/2 - (n' - 1)(d' - 1)(2n'd' - n' - d' - 1)/12. \]
Thus,
\[ W_{1,k} = n'd'k \left( m_k + 1 - \frac{(n' - 1)(d' - 1)}{2} \right) + \frac{m_k(m_k + 1)}{2} - \frac{(n' - 1)(d' - 1)(2n'd' - n' - d' - 1)}{12}, \]

so
\[ W_1 = G \sum_{k=0}^{G-1} \left[ n'd'k \left( m_k + 1 - \frac{(n' - 1)(d' - 1)}{2} \right) + \frac{m_k(m_k + 1)}{2} - \frac{(n' - 1)(d' - 1)(2n'd' - n' - d' - 1)}{12} \right]. \]

To evaluate this sum, we need the following calculations which are straightforward to compute.

- \( \sum_{k=0}^{G-1} m_k = (2g - 2)q - \frac{n'd'G(G-1)}{2} \).
- \( \sum_{k=0}^{G-1} m_k^2 = \frac{(2g-2)^2}{6}q^2 - (2g - 2)(G - 1)d'n'q + \frac{d'^2n'^2(G-1)G(2G-1)}{6} \).
- \( \sum_{k=0}^{G-1} km_k = (g - 1)(G-1)q - \frac{d'n'(G-1)G(2G-1)}{6} \).

Simplifying the resulting expression, we find
\[ W_1 = \frac{(2g-2)^2}{2}q^2 + (g - 1)Gq + \frac{G^2 - 1 - (n - 1)(d - 1)(2nd - n - d - 1)}{12}, \]
which completes the proof of Proposition 3.

**Proposition 5.** Let \( n, d, q \in \mathbb{N} \) such that \( n < d \) and \( q \geq 2 \). Let
\[ D(a, b, c) = \sum_{j=0}^{c-1} \left\{ \frac{a + bj}{c} \right\}. \]
Then
\[ W_2 = (d - 1) \left( n - 1 \right) \left( (g - 1)q + \frac{-2nd + 3n + d}{6} \right) - D(-(d + G)q, -d, n) \].
First, we consider the case where $\gcd(n, d) = 1$. For $W_2 = (d-1)\sum_{(i, j) \in S} I_j$, we have

$$W_2 = (d-1)\sum_{j=0}^{n-1} I_j,$$

for $I_j = \left\lfloor \frac{(2q-2)d - dj}{n} \right\rfloor$. By Lemma 2, $I_j \geq 0$ so $W_2 = (d-1)\sum_{j=0}^{n-1} (I_j + 1)j$. Since $[x] = x - \{x\}$,

$$W_2 = (d-1)\sum_{j=0}^{n-1} \left( \frac{(2q-2)d - dj}{n} - \left\lfloor \frac{(2q-2)d - dj}{n} \right\rfloor + 1 \right) j.$$

Note that $\left\lfloor \frac{(2q-2)d - dj}{n} \right\rfloor = \left\lfloor \frac{(nd - dG)q - dj}{n} \right\rfloor = \left\lfloor \frac{(n - G)q - dj}{n} \right\rfloor$.

Expanding out, we get

$$W_2 = (g-1)(d-1)(n-1)q + \frac{(d-1)(n-1)}{6} (3n - d(2n-1))$$

$$- (d-1)\sum_{j=0}^{n-1} \left\{ \frac{(d + G)q + dj}{n} \right\} j,$$

which can be rearranged to give the desired result.

Finally, if $(n, d, q) \in \{(2, 5, 2), (2, 6, 2)\}$, then $S = \{(0, 0), (1, 0), (2, 0)\}$, and so $W_2 = \sum_{(i, j) \in S} I_j = 0$. We get the same value if we plug each these $(n, d, q)$ triples into the above formula for $W_2$.

**Remark.** The summation $D(a,b,c)$ is related to a Dedekind sum. There is no closed form for such sums, though there is a reciprocity law. For a general reference, see [1].

Finally, we can combine and simplify $W_1 - W_2 - W_3$. Note that the $q^2$ and $q$ terms (other than in the summation) cancel. With further manipulation, we have our main result.

**Theorem 2.** Let $C$ be given in affine coordinates by $y^n = f(x)$ for $f(x)$ a separable polynomial of degree $d > n$. Let $G = \gcd(n, d)$, and let $q \in \mathbb{Z}$ with $q \geq 2$. For any root $\alpha$ of $f(x)$, let $B = (\alpha, 0)$ be a branch point.

The $q$-weight of $B$ is $w_q(B) = \frac{1}{24} \left( (n-1)(d-1)(n+1)(d-7) + 12g(G+1) + 5(G^2-1) \right) + (d-1) \cdot D(-(d+G)q, -d, n)$

Note that, for given values of $n$ and $d$, the $q$-weight of $B$ depends only on the value of $q$ modulo $n$.

We will give results for some combinations of $n$ and $d$ in the corollaries below. First, we consider the case where $\gcd(n, d) = 1$.

**Corollary 1.** If $\gcd(n, d) = 1$,

$$w_q(B) = \frac{g}{12} (n+1)(d-7) + g + (d-1) \cdot D(-(d+1)q, -d, n).$$
Fix \( n \) and \( d \) (with any gcd). If one varies \( q \), then one sees the value of \( w^{(q)}(B) \) depends only on the congruence class of \( q \) modulo \( n/G \). Further, if \( d \equiv -G \pmod{n} \), then the summation term simplifies to \( \sum_{j=0}^{n-1} \left\{ \frac{Gj}{n} \right\} j \), for which there is a closed form.

**Corollary 2.** If \( d \equiv -G \pmod{n} \), then \( w^{(q)}(B) \) doesn’t depend on \( q \). In particular,

\[
w^{(q)}(B) = \frac{1}{24} \left( (n-1)(d-1)(n+1)(d-7) + 12g(G+1) + 5G^2 - 1 \right) + 2(d-1)(n-G)(3n+n'-2).
\]

**Proof.** The summation term is \( \sum_{j=0}^{n-1} \left\{ \frac{Gj}{n} \right\} j \). Each \( j \) can be written uniquely as \( j = j' + k n' \) for \( 0 \leq k < G \) and \( 0 \leq j' < n' \). Thus, the summation is \( \sum_{k=0}^{G-1} \sum_{j'=0}^{n'-1} \frac{j'}{n} j = \sum_{k=0}^{G-1} \sum_{j'=0}^{n'-1} \left( \frac{j'}{n} + j'k \right) \), which simplifies to \( (n-G)(3n+n'-2)/12 \).

Combining the two corollaries above, we obtain the following.

**Corollary 3.** If \( d \equiv -1 \pmod{n} \), then

\[
w^{(q)}(B) = \frac{g(n+1)(d+1)}{12} = \frac{(n^2-1)(d^2-1)}{24}
\]

for all \( q \geq 2 \).

**Corollary 4.** If \( n \mid d \), then

\[
w^{(q)}(B) = \frac{(n^2-1)(d^2-2d)}{24}.
\]

**Proof.** If \( n \mid d \), then \( G = n \) and \( n \mid ((d+G)q + dj) \) for all \( j \), so the summation is zero. Since \( 2g - 2 = nd - n - d - n \), we have \( g = \frac{4(d-2)(n-1)}{4} \). Plugging in, the result follows. \( \square \)

### 4.2. Weights of points at infinity

If \( n \mid d \), then there are \( n \) points at infinity in the smooth model of \( C \), so these points are not branch points. However, we can still investigate their \( q \)-weights. If \( \gcd(n, d) > 1 \), then we need to know more about \( f(x) \) to determine \( w^{(q)}(P^\infty_m) \). We give a few examples to illustrate this.

In [3], the authors consider curves of the form \( y^2 = f(x) = x^6 + ax^4 + bx^2 + 1 \), where \( a, b \) are parameters and \( f(x) \) is separable. In the non-singular models of these curves, there are \( G = \gcd(n, d) = 2 \) points at infinity \( P^\infty_1 \) and \( P^\infty_2 \). If \( 4b = a^2 \), then \( w^{(3)}(P^\infty_1) = w^{(3)}(P^\infty_2) = 2 \). If \( 4b \neq a^2 \), then \( w^{(3)}(P^\infty_1) = w^{(3)}(P^\infty_2) = 0 \).

In [3], Lemma 4 and Proposition 3], the authors consider hyperelliptic curves of genus 3 of the form \( y^2 = f(x) \) where \( \deg(f) = 8 \). In the non-singular models of these curves, there are \( G = \gcd(n, d) = 2 \) points at infinity \( P^\infty_1 \) and \( P^\infty_2 \). If \( C \) is given by \( y^2 = x^8 + x^6 + 16x^4 + x^2 + 1 \), then \( w^{(2)}(P^\infty_1) = w^{(2)}(P^\infty_2) = 1 \). If \( C \) is given by \( y^2 = x^8 + x^4 + 1 \), then \( w^{(2)}(P^\infty_1) = w^{(2)}(P^\infty_2) = 3 \).

Thus, simply knowing \( n \) and \( d \) is not enough to calculate the \( q \)-weight of the points at infinity. However, there are some cases where we can get a result.

First, if \( d = n+1 \), then the lone point at infinity is a nonsingular branch point, so it will have the same \( q \)-weight as the affine branch points. By Corollary 3, since \( d \equiv -1 \pmod{n} \), \( w^{(q)}(B) = \frac{(n^2-1)(d^2-1)}{24} \) for \( q \geq 2 \), so we will have \( w^{(q)}(P^\infty) = \frac{(n^2-1)(d^2-1)}{24} \) for \( q \geq 2 \).
Suppose $\text{BW}$ denotes the branch points.\footnote{The branch weight.} Let $P_1^\infty$ be the lone point at infinity in the non-singular model of $C$. Then

$$w^{(q)}(P_1^\infty) = \begin{cases} \frac{g(n+1)(d+1)}{12} - g = \frac{(n^2 - 1)(d^2 - 1)}{24} - g & \text{if } q = 1, \\ \frac{g(n+1)(d+1)}{12} = \frac{(n^2 - 1)(d^2 - 1)}{24} & \text{if } q \geq 2. \end{cases}$$

\textbf{Theorem 3.} Suppose $C$ is a curve of genus $g \geq 2$ given by the affine equation $y^n = f(x)$ for $f(x)$ a separable polynomial of degree $d$ where $n < d$ and $\gcd(n, d) = 1$. Then

$$w^{(q)}(P_1^\infty) = \sum_{i,j \in S} \text{ord}_{P_1^\infty}(f_{i,j}) - \sum_{k=0}^{d_q-1} k.$$ 

Proof. For $q = 1$, the formula is given at the end of the proof of Theorem 8. For $q \geq 2$ and $G = 1$, let $\mathcal{B}_{n,d,q}$ be as in Section 4 and again let $S = \{(i, j) \in \mathbb{Z}^2 : i \geq 0, 0 \leq j < n, ni + dj \leq (2g - 2)q\}$. Then $f_{i,j} \in \mathcal{B}_{n,d,q}$ if and only if $(i, j) \in S$. Recall that $\text{ord}_{P_1^\infty}(f_{i,j}) = (2g - 2)q - (ni + dj)$. These orders of vanishing are unique, so

$$w^{(q)}(P_1^\infty) = d_q(2g - 2)q - \left( \sum_{i,j \in S} (ni + dj) \right) - \frac{(d_q - 1)d_q}{2}.$$

The summation, which we called $W_1$ in Equation 11 is evaluated in Proposition 4. Plugging this and $d_q$ in, the expression simplifies to $w^{(q)}(P_1^\infty) = \frac{(n^2 - 1)(d^2 - 1)}{24}$. \qed

4.3. Branch weight. In the case where $\gcd(n, d) = 1$, we can calculate the total $q$-weight of the branch points (both affine and at infinity) for $q \geq 2$, which we denote $\text{BW}_q$.

\textbf{Corollary 5.} Suppose $\gcd(n, d) = 1$, so $g = \frac{(n-1)(d-1)}{2}$. Then the total branch $q$-weight is given by $\text{BW}_q = d \cdot w^{(q)}(B) + w^{(q)}(P_1^\infty)$.$$

Rewritten in terms of $g$, we get

$$\text{BW}_q = \frac{n + 1}{3(n - 1)^2} \left( g^3 - 2g^2(n - 1) - g(n - 1)^2 + d(d - 1) \cdot D(-(d + 1)q, -d, n) \right) + g \frac{(n + 1)(d + 1)}{12}.$$ 

From Proposition 4, we know the total weight of the $q$-Weierstrass points, for $q \geq 2$, is $g(g - 1)^2(2q - 1)^2$. We can now calculate the proportion of $q$-weight of the branch points.

\textbf{Proposition 6.} Fix $n$ and let $q \geq 2$. Then

$$\liminf_{d \to \infty} \frac{\text{BW}_q}{g(g - 1)^2(2q - 1)^2} \geq \frac{n + 1}{3(n - 1)^2(2q - 1)^2}.$$ 

If we restrict to values of $d$ that are relatively prime to $n$ then

$$\lim_{d \to \infty, (n,d)=1} \frac{\text{BW}_q}{g(g - 1)^2(2q - 1)^2} = \frac{n + 1}{3(n - 1)^2(2q - 1)^2}.$$
Proof. For general \( n \) and \( d \), since we do not have an exact formula for the \( q \)-weight of the points at infinity, we can only say \( BW_q \geq d \cdot w^{(q)}(B) \). Using the result from Theorem 2 since

\[
\sum_{j=0}^{n-1} \left\{ \frac{(d + G)q + dj}{n} \right\} \leq \sum_{j=0}^{n-1} \frac{n - 1}{n} j = \frac{(n - 1)^2}{2},
\]

in terms of \( d \), the dominant term of \( d \cdot w^{(q)}(B) \) is \( d^3 \frac{(n-1)(n+1)}{24} \). Since \( q \) is on the order of \( d(n-1)/2 \), the dominant term of the denominator is \( d^3 \frac{(n-1)^3(2n-1)^2}{8} \). The result follows.

For \( \gcd(n,d) = 1 \), the lone point at infinity has weight \( \frac{(d^2-1)(n^2-1)}{24} \). Thus, the dominant term of \( BW_q \) is precisely \( d^3 \frac{(n-1)(n+1)}{24} \), and we thus have an equality if we take a limit involving integers \( d \) such that \( \gcd(n,d) = 1 \). \( \square \)

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