Supersymmetric extension of a coupled Korteweg-de Vries system

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Abstract. We introduce a supersymmetric extension for a parametric coupled Korteweg-de Vries system. The supersymmetric system has two real Hamiltonian functionals and two associated basic Poisson structures. The basic Poisson structures allows the construction of a pencil of Poisson brackets and associated to them a Hamiltonian functional depending on the parameter of the pencil. The two basic Poisson brackets are compatible.

1. Introduction
It was shown in [1] that among the one parameter supersymmetric extensions of the Korteweg-de Vries (KdV) equation there is a special one, for a particular value of one of the coefficients on the equations, that has an infinite number of conservation laws. It was called the $N = 1$ supersymmetric extension of KdV equation. It is equivalent to the super KdV equation obtained in [2] by reduction from the super Kadomtsev-Petviashvili hierarchy. Other supersymmetric extensions of the KdV equations and interesting analysis of them can be found in [3, 4, 5, 6, 7].

The $N = 1$ supersymmetric extension of KdV equation is a system of coupled equations formulated in terms of fields $u, \xi$ valued on the even and odd part of a given Grassmann algebra respectively.

The system is
\begin{align}
    u_t &= -u''' + 6u' + 3\xi'', \\
    \xi_t &= -\xi''' + 3(\xi u)'.
\end{align}

The Grassmann algebra may have any number of generators. In particular, if we consider only one generator $e$, then
\begin{align*}
    u &= u_0 e, \\
    \xi &= \xi_0 e
\end{align*}

since $1 \circ 1 = 1, 1 \circ e = e \circ 1 = e, e \circ e = 0$.

$u_0, \xi_0$ are real fields $u_0 = u_0(x, t), \xi_0 = \xi_0(x, t)$.

In this particular case (1) reduces to ninth Hirota-Satsuma coupled KdV system [8, 9]. Other extensions are obtained by considering more generators of the Grassmann algebra. In all these cases the equation for $u_0$ is the KdV equation.
The super KdV system (1) has one local Hamiltonian structure obtained via a super Miura transformation. The associated Poisson algebra is equivalent to the super Virasoro algebra with central terms. It is consequently related to conformal field theory and string theory.

There exists also a super Gardner transformation [1] which allows to obtain in a direct way an infinite sequence of local conserved quantities. Moreover, in [10] it was found an infinite sequence of non-local conserved quantities of odd weight. In [11] this infinite sequence was derived from a non-local conserved quantity of the super Gardner system and later on, using a supersymmetric cohomology, a new infinite sequence of non-local conserved quantities of even weight was derived [12].

In the present work we consider a supersymmetric extension of a parametric coupled KdV system which we have analyzed recently [13]. For $$\lambda = 1$$ the system is equivalent to the complexification of the KdV equation. For $$\lambda = +1$$ the system is equivalent to two decoupled KdV equations. For $$\lambda = 0$$ the system corresponds to the ninth Hirota-Satsuma coupled system [9].

2. The supersymmetric extension of a coupled KdV system

The parametric coupled KdV system was formulated in terms of two real fields

$$u_t = -u''' + 6uu' + 6\lambda uv'$$
$$v_t = -v''' + 6(uv)'$$.

(2)

We found [13] a Bäcklund transformation, proved the permutability theorem and obtained the Gardner transformation, which allows to obtain directly infinite local conserved quantities for the system. We found new solutions for the system, a class of them describes multi-solitonic solutions while another class describes periodic solutions. We also found the complete Hamiltonian and Poisson structure for the system [14].

We now consider a supersymmetric extension of the system (2). We start from the supersymmetric extension [1] of KdV equation and consider the corresponding fields to be valued on a $$\mathbb{Z}_2$$ algebra [15]. If we denote the generators of the associative and commutative algebra over $$\mathbb{R}$$ by $$e_1$$ and $$e_2$$ we then have

$$U = ue_1 + ve_2$$
$$\xi = \mu e_1 + \nu e_2$$

(3)

where $$u, v$$ are even and $$\mu, \nu$$ are odd Grassmann valued fields respectively. The generators of $$\mathbb{Z}_2$$ satisfy $$e_1 \circ e_1 = e_1, e_1 \circ e_2 = e_2, e_2 \circ e_2 = \lambda e_1$$.

The resulting supersymmetric extension of the coupled KdV system (2) becomes

$$u_t = -u''' + 6uu' + 6\lambda uv' - 3\mu\mu'' - 3\lambda\nu\nu''$$
$$v_t = -v''' + 6(uv)' - 3\mu\nu'' - 3\nu\mu''$$
$$\mu_t = -\mu''' + 3(\mu u + \lambda uv)'$$
$$\nu_t = -\nu''' + 3(\nu v + v\mu)'$$

(4)

which are invariant under the supersymmetric transformation

$$\delta u = \epsilon_1 \mu' + \lambda \epsilon_2 v', \delta v = \epsilon_1 \nu' + \epsilon_2 \mu'$$
$$\delta \mu = \epsilon_1 u + \lambda \epsilon_2 v, \delta \nu = \epsilon_1 v + \epsilon_2 u$$

(5)

where $$\epsilon_1$$ and $$\epsilon_2$$ are the odd Grassmann supersymmetric parameters.

We notice that for $$\mu = \nu = 0$$ the system (4) reduces to (2). If $$\lambda = 0$$ the system becomes

$$u_t = -u''' + 6uu' - 3\mu\mu''$$
$$\mu_t = -\mu''' + 3(\mu u)'$$
$$v_t = -v''' + 6(uv)' - 3\mu\nu'' - 3\nu\mu''$$
$$\nu_t = -\nu''' + 3(\nu v + v\mu)'$$

(6)
where the first two equations define the \( N = 1 \) supersymmetric extension of KdV equation [1], the third and fourth equations couple the \( u \) and \( \mu \) fields to \( v \) and \( \nu \). This system is the \( N = 1 \) supersymmetric extension of the ninth Hirota-Satsuma coupled KdV system [8, 9]. This bosonic system has all the integrability properties of KdV systems, in particular integrable hierarchies related to polynomial Lie algebras [16].

3. The Miura transformation, Poisson structure and Gardner transformation

We consider the supersymmetric Miura transformation

\[
U = V' + V^2 - \Upsilon \Upsilon' \\
\xi = \Upsilon' + V \Upsilon,
\]

where \( V \) and \( \Upsilon \) are valued on the \( \mathcal{E}^\lambda \) algebra with coefficients on the even and odd parts of a Grassmann algebra respectively.

The hamiltonians of the system are

\[
H_1 = \int_{-\infty}^{+\infty} \left[ u^2 + \lambda v^2 + \mu' \mu + \lambda \nu' \nu \right] \, dx \\
H_2 = \int_{-\infty}^{+\infty} \left[ 2uv + \mu' \nu + \nu' \mu \right] \, dx.
\]

Associated to each hamiltonian there is a Poisson structure which can be deduced in terms of the Miura fields \( V, \Upsilon \) by following a Dirac approach on the constrained phase space obtained from two real singular lagrangians. The two Poisson structures are compatible. Moreover one may construct a pencil of Poisson brackets from these basic Poisson structures. The resulting Poisson algebra is a deformation of the super Virasoro algebra which is the Poisson algebra of the supersymmetric extension of the KdV equation.

The supersymmetric system (4) has, for any \( \lambda \), an infinite sequence of conserved quantities. They can be obtained in a direct way, from the associated super Gardner system which follows from (4) via a Gardner transformation. The few basic conserved quantities of the super Gardner system provides the infinite sequence of conserved quantities of (4).

The super Gardner system is the supersymmetric extension of the Gardner system in terms of the two real fields presented in [13].

4. Conclusions

We have introduced a supersymmetric extension of a coupled KdV system. For particular values of the parameter it corresponds to supersymmetric of interesting coupled KdV systems. The supersymmetric extension has a sequence of infinite conserved quantities and a natural extension of the hamiltonian structure of the supersymmetric KdV equation.

Acknowledgments

A. R. and A. S. are partially supported by Projects Fondecyt 1121103 and Mecesup ANT398 (Chile).

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