EXTENDING WEAKLY POLYNOMIAL FUNCTIONS FROM HIGH RANK VARIETIES

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Abstract. Let $k$ be a field, $V$ a $k$-vector space $Y \subset V$. A function $f : Y \to k$ is weakly polynomial of degree $\leq a$, if the restriction of $f$ on any affine subspace $L \subset Y$ is a polynomial of degree $\leq a$. In this paper we consider the case when $Y = \mathcal{Y}(k)$ where $\mathcal{Y}$ is a hypersurface defined by a high rank polynomial of degree $d$, $\text{char}(k) > d$ or $\text{char}(k) = 0$ and either $k$ is algebraically closed, or $k$ is finite and $|k| > a$. We show that under these assumptions any $k$-valued weakly polynomial function of degree $\leq a$ on $Y$ is a restriction of a polynomial of degree $\leq a$ on $V$. The same results hold for complete intersections $Y \subset V$ of bounded degree and codimension.

Our proof is based on a result on the uniformity of fibers of maps $P : \mathbb{F}_q^n \to \mathbb{F}_q^m$ of high rank which has an independent interest.

1. Introduction

Let $k$ be a field. We denote $k$-algebraic varieties by bold letters such as $\mathbb{X}$ and the sets of $k$-points of $\mathbb{X}$ by $X$ or $\mathbb{X}(k)$. We denote by $\mathbb{A}^n$ the $n$-dimensional vector space. So $\mathbb{A}^n(k) = k^n$. We fix $d, a \geq 1$ and $L \geq 1$. We always assume that $|k| > a$.

Definition 1.1. Let $V$ be a $k$-vector space and $X \subset V$. We say that a function $f : X \to k$ is weakly polynomial of degree $\leq a$ if the restriction $f|_L$ to any affine subspace $L \subset X$ is a polynomial of degree $\leq a$.

Remark 1.2. Since we assume $|k| > a$ it suffices to check this on 2-dimensional subspaces (see [12]), namely a function is weakly polynomial of degree $\leq a$ if the restriction $f|_L$ to 2-dimensional affine subspace $L \subset X$ is a polynomial of degree $\leq a$.

One of goals of this paper is to construct classes of hypersurfaces $\mathbb{X} \subset \mathbb{V}$ such that any weakly polynomial function $f$ on $X$ of degree $\leq a$ is a restriction of a polynomial $F$ of degree $\leq a$ on $V$. The main difficulty is in the case when $a \geq d$ since in this case an extension $F$ of $f$ to $V$ is not unique.

Remark 1.3. When $d = 1$ then $X$ is vector space, and the result is easy. The case $a < d$ was studied in [15]. The case $a = d = 2$ was studied in [13], and a bilinear version of it was studied in [6], where it was applied as part of

\begin{footnote}
The second author is supported by ERC grant ErgComNum 682150.
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a quantitative proof for the inverse theorem for the $U_4$-norms over finite fields. We expect the results in this paper to have similar applications to a quantitative proof for the inverse theorem for the higher Gowers uniformity norms, for which at the moment only a non quantitative proof using ergodic theoretic methods exists \cite{2,17,18}.

To state our result properly we introduce some definitions:

**Definition 1.4.** Let $X \subset V$ be an algebraic $k$-subvariety.

1. $X \subset V$ satisfies $\star_k$ if any weakly polynomial function of degree $\leq a$ on $X$ is a restriction of a polynomial function of degree $\leq a$ on $V$.
2. A $k$-subvariety $X \subset V$ satisfies $\star_a$ if $X(l)$ satisfies $\star_a^l$ for any finite extension $l/k$.
3. For any polynomial $P$ on $V$ we define $X_P := \{v \in V | P(v) = 0\}$.

The following example demonstrates the existence of cubic surfaces $X \subset \mathbb{A}^2$ which do not have the property $\star_k^a$ for any field $k$.

**Example 1.5.** Let $V = k^2$, $Q = xy(x + y)$. Then $X = X_0 \cup X_1 \cup X_2$ where $X_0 = \{v \in V | x = 0\}$, $X_1 = \{v \in V | y = 0\}$, $X_2 = \{v \in V | x = y\}$. The function $f : X \to k$ such that $f(x, 0) = f(0, y) = 0$, $f(x, x) = x$ is weakly linear but one can not extend it to a linear function on $V$.

Our goal is to prove that high rank hypersurfaces have the property $\star_k^a$.

**Definition 1.6 (Algebraic rank).** Let $P : V \to k$ be a polynomial. We define the $k$-rank $r_k(P)$ as the minimal number $r$ such that $P = \sum_{i=1}^r Q_i R_i$, with $Q_i, R_i$ of $k$-polynomials of degrees strictly smaller than the degree of $P$. When $k$ is fixed we write $r(P)$ instead of $r_k(P)$.

**Question 1.7.** Does there exists a function $R(d, r) \to \infty$ for $r \to \infty$ such that $r_k(P) \geq R(d, r)$ for any polynomial $P$ of degree $d$ and $k$-rank $r$, where $\bar{k}$ is the algebraic closure of $k$?

Let $a, d > 0$. Call a finite field $k$ admissible if $\text{char}(k) > d$ (and $|k| > a$). Call an algebraically closed field admissible if $\text{char}(k) > d$ or $\text{char}(k) = 0$.

Our main result is that high rank hypersurfaces over admissible fields satisfy $\star_k^a$.

**Theorem 1.8.** There exists $r = r(a, d)$ such that for any admissible field $k$, any $k$-vector space $V$, and any hypersurface $X \subset V$ of degree $d$ and rank $\geq r$ satisfies $\star_k^a$.

The first step in our proof of Theorem 1.8 is to construct an explicit collection of hypersurfaces $X_n$ satisfying $\star_k^a$ for an arbitrary fields $k, |k| > a$. Let $V_n = (\mathbb{A}^d)^n$. We write $v = \{x_i^j\}, 1 \leq i \leq d, 1 \leq j \leq n$. Let $P_n : V_n \to \mathbb{A}$ be a polynomial by given $P_n(v) := \sum_{j=1}^n \prod_{i=1}^d x_i^j$ and $X_n \subset V_n$ be the hypersurface defined by $P_n$.\footnote{Also known as the Schmidt $h$-invariant.}
Theorem 1.9. For any field $k$, $|k| > a$, the hypersurface $X_n \subset V_n$ has the property $\star_a$.

The second step in our proof of Theorem 1.8 is of an independent interest. To formulate the statement we introduce some definitions.

Definition 1.10. Let $V$ be $k$-vector space.

1. We denote by $P_d(V)$ the $k$-variety of polynomials of degree $\leq d$ on $V$.
2. For any $k$-vector space $W$ we denote by $\text{Hom}_{af}(W, V)$ the $k$-variety of affine maps $\phi : W \to V$.
3. For any $P \in P_d(V)$ we denote by $\kappa_P : \text{Hom}_{af}(W, V) \to P_d(V)$ the map given by $\phi \mapsto \phi^*(P) := P \circ \phi$.
4. A field $k$ has the property $c_d$ if for any $m \geq 2$, there exists $\rho = \rho(m, d)$ such that for $k$-vector spaces $V, W$, $\dim(W) \leq m$, any polynomial $P : V \to A$ of degree $d$ and rank $\geq \rho$ the map $\kappa_P(k) : \text{Hom}_{af}(W, V) \to P_d(W)(k)$ is onto.

Theorem 1.11. Finite fields of characteristic $> d$ and algebraically closed fields of characteristic $> d$ or characteristic 0 the property $c_d$.

Remark 1.12. Theorem 1.11 is first proven for finite fields. We show in Appendix D how to derive the validity Theorem 1.11 for algebraically closed fields from the corresponding statement for finite fields.

The results extend without difficulty to complete intersections $X \subset V$ of bounded degree and codimension, and high rank (see Definition C.1).

Theorem 1.13. For any $L > 0$, there exists $r = r(a, d, L)$ such that for any admissible field $k$, any $k$-vector space $V$, and any subvariety $X \subset V$ codimension $L$, degree $d$ and rank $\geq r$ the subset $X \subset V$ satisfies $\star_a$.

Conjecture 1.14. (1) For any $m, d \geq 0$, there exists $\rho = \rho(m, d)$ such that if the rank of a polynomial $P$ is $\geq \rho(m, d)$, then the map $\kappa_P$ is flat.

(2) Non-archimedian local fields have the property $c$.

(3) The bound on $r$ depends polynomially on $L$. This will follow from Conjecture A.4 which is currently known for $d = 2, 3$.

Remark 1.15. From now on the claims in the main body of the paper are stated for the case that $X$ a hypersurface ($L = 1$). The general case is completely analogous, and explained in Appendix C.

The third step in the proof of Theorem 1.8 consists of the following two results. Let $X \subset V$ be a hypersurface of degree $d$, $l : V \to k$ a non-constant affine function, $X_b := l^{-1}(b) \cap X$. We denote by $\mathcal{L}$ the variety of planes $L \subset X_1$ and by $\mathcal{L}^0 \subset \mathcal{L}$ the constructible subset of $\mathcal{L}$ for which there exists an affine cube $M \subset X$ containing $L$ such that $M \cap X_0 \neq \emptyset$. Let $Y := \mathcal{L} - \mathcal{L}^0$, and let $r(X)$ be the rank of the polynomial defining the hypersurface $X_0 \subset V^0_n := l^{-1}(0)$. 
The first result states that $\mathcal{Y}$ is small:

**Claim 1.16.** For any $s > 0$, there exists $r = r(d, s)$ such that if $r(X) > r$ then

1. If $k$ is finite and $\text{char}(k) > d$ then $|Y(k)|/|L(k)| \leq |k|^{-s}$.
2. $\dim(\mathcal{Y}) < \dim(\mathcal{L})$.

**Remark 1.17.** We first prove the part (1) using techniques from Additive Combinatorics. The part (2) is then an easy corollary.

The second result demonstrates that one can extend a weakly polynomial function vanishing on $X_0$ to a polynomial function on $X$:

**Proposition 1.18.** There exists $r = r(d, a)$ such that if $r(X) > r$ then

1. If $k$ is a finite field, $|k| > a$ then any weakly polynomial function on $X$ of degree $\leq a$ vanishing on $X_0$ is a restriction of a polynomial on $V$ of degree $\leq a$.

2. If $k$ is an algebraically closed field then any weakly polynomial function on $X$ of degree $\leq a$ vanishing on $X_0$ is a restriction of a polynomial on $V$ of degree $\leq a$.

As an immediate corollary we obtain:

**Corollary 1.19.** There exists $r = r(d, a)$ such that if $r(X) > r$ then for any hypersurface $X = \{v \in V | P(v) = 0\} \subset V$ of degree $\leq d$, an affine subspace $W \subset V$ such that rank of $P|_W \geq r$ and any weakly polynomial function $f : X \to k$ of degree $\leq a$ there exists a polynomial $F : V \to k$ of degree $\leq a$ such that $f = F|_X$.

**Proof.** Choose a flag $F = \{W_0 = W \subset W_1 \cdots \subset W_{\dim(V) - \dim(W)} = V\}, \dim(W_i) = \dim(W) + i,$

and extend $f$ by induction in $i, 1 \leq i \leq \dim(V) - \dim(W)$ to a polynomial $F$ on $V$. \hfill $\Box$

**Remark 1.20.** The choice of $F$ depends on a choice of flag $F$ and on choices involved in the inductive arguments.

Theorem 1.8 is an almost immediate corollary of the results stated above:

**Proof of Theorem 1.8 assuming Theorems 1.9 and 1.11, and Corollary 1.19.** The hypersurfaces $X_n$ are of high rank (Lemma 2.10) for large $n$. Let $\tilde{r}$ be from Corollary 1.19 and let $n$ be sufficiently large so that $X_n \subset W$ is of rank $\geq \tilde{r}$. Let $r = r(a, d) := \rho(\dim(W), d)$ from Theorem 1.11. Let $X \subset V$ be a hypersurface of rank $\geq r$. By Theorem 1.11 there exists a linear map $\phi : W \to V$ such that $X_n = \{w \in W | \phi(w) \in X\}$. Since $X_n$ satisfies $\ast_a^k$, Corollary 1.19 implies that $X$ satisfies $\ast_a^k$. \hfill $\Box$

**Remark 1.21.** We believe that the restrictions on the characteristic is not necessary in Theorem 1.8.
We finish the introduction with a different application of Theorem 1.11. We show how to derive the following strengthening of the main Theorem from [4].

Let $V$, $W$, $\dim(W) \leq m$ be vector spaces over an algebraically closed field $k$, and let $P : V \to k$ be a polynomial of degree $d$ and rank $\geq \rho(m,d)$ such that the map $\kappa_P(k) : \text{Hom}_{af}(W,V)(k) \to \mathcal{P}_d(V)(k)$ is onto. Let $Gr$ be the Grassmanian of $m$-dimensional affine imbeddings $\phi : \mathbb{A}^m \to V$ and the map $a : Gr \to \mathcal{P}_d(\mathbb{A}^m)$ be given by $\phi \to \phi^*(P)$. Theorem 1.11 shows the surjectivity of the map $a$. The proof also shows that all fibers $a^{-1}(y)$, $y \in Y$ are of dimension $\dim(Gr) - \dim(\mathcal{P}_d(\mathbb{A}^m))$. The proof is in Appendix E.

We can derive from Theorem 1.11 the following strengthening of the main Theorem from [4]:

**Lemma 1.22.** Let $k$ be an algebraically closed field, $C$ the category of finite-dimensional affine $k$-vector spaces with morphisms being affine maps, let $\mathcal{F}_d$ be the contravariant endofunctor on $C$ given by

$$\mathcal{F}_d(V) = \{\text{Polynomials on } V \text{ of degree } \leq d\},$$

and let $G \subset \mathcal{F}$ be a proper subfunctor. Then there exists $r$ such that $r(P) \leq r$ for any finite-dimensional $k$-vector space $V$ and $P \in G(V)$.

**Proof.** Let $G$ be a subfunctor of $\mathcal{F}_d$ such that $r(P), P \in G(W)$ is not bounded above. We want to show that $G(W) = \mathcal{F}_d(W)$ for any finite-dimensional $k$-vector space $W$.

Let $m = \dim(W)$ and choose a polynomial $P \in G(V)$ where $V$ where $V$ is a $k$-vector space $V$ such that $r(P) \geq r(m,d)$ where $r(m,d)$ is as in Theorem 1.11. Then for any polynomial $Q$ on $W$ of degree $d$ there exist an affine map $\phi : W \to V$ such that $Q = \phi^*(P)$. We see that $G(W) = \mathcal{F}_d(W)$. \hfill $\square$

**Remark 1.23.** The paper [4] assumed that $G(V) \subset \mathcal{P}_d(V)$ are Zariski closed subsets.

### 2. Proof of Theorem 1.9

In this section we prove that varieties $X_n$ have the property $\star_a$.

**Lemma 2.1.** Let $V \subset V$ be a variety which has the property $\star_c$, for $c \leq a$ and let $f$ be a weakly polynomial function on $Y \times k$ of degree $\leq a$ which is polynomial of degree $b \leq |k|$. Then $f$ is a polynomial of degree $a$.

**Proof.** Let $P$ a polynomial on $V \times k$ of degree $b$ such that $f = P|_X$. We can write $P$ in the form $P(v,t) = \sum_{i=0}^b P_i(v)t^i$ where $\deg(P_i) \leq b - i$. Let $f_i := P_i|_Y$.

**Claim 2.2.** $f_i$ is weakly polynomial function on $Y$ of degree $\leq a - i$.

**Proof.** Suppose that the function $f_i$ is not a weakly polynomial function on $Y$ of degree $\leq a - i$ then there exists a plane $\phi : k^2 \to \mathbb{C}$, $(s, u) \to \phi(s, u) \in Y$.
such that the function $\tilde{f}_i(s, u) := f_i \circ \phi(s, u)$ is of degree $> a - i$. Consider the polynomial $P_3(t, s, u) := P(t, \phi(s, u)) = \sum t^j \tilde{f}_j(s, u)$. Since $\deg(\tilde{f}_i) > a - i$ there exists a plane $L \subset k^3$ such that the restriction of $P_3$ on $L$ has degree $> a - i$. But this contradicts the assumption that $f$ is a weakly polynomial function on $X$ of degree $\leq a$. □

Since $Y$ has the property $*_{a-i}$ there exists a polynomial $Q_i$ on $V$ of degree $a-i$ such that $f_i = Q_i|Y$. □

**Corollary 2.3.** Let $Y \subset V$ be a variety which has the property $*_{c}$ for $c \leq a$.

1. $Y \times A$ has the property $*_{c}$ for $c \leq a$.
2. Any weakly polynomial function on $Y \times A^n$ of degree $\leq a$ which is polynomial of degree $b \leq |k|$ is a polynomial of degree $a$.

**Definition 2.4.**

1. For any sequences $\bar{c} = \{c^j\}$, $1 \leq c^j \leq d$ we define $V_\bar{c} = \prod_{j=1}^n A^{c^j}$.
2. We write elements $v \in V$ in the form $v = (x^j_i)(v)$, $1 \leq j \leq n$, $1 \leq i \leq c^j$.
3. $c_+ (\bar{c}) := \max_j c^j, c_- (\bar{c}) := \min_j c^j$.
4. $j_+ (\bar{c})$ is the minimal $j$, $1 \leq j \leq n$ such that $c_j = c_+ (\bar{c})$.
5. $j_- (\bar{c})$ be the minimal $j$, $1 \leq j \leq n$ such that $c_j = c_- (\bar{c})$.
6. We always assume that $c_+ (\bar{c}) - c_- (\bar{c}) \leq 1$ and that $c_- (\bar{c}) \geq 1$.
7. We denote by and $\bar{\alpha} = \{\alpha^j\} \in k^*$ elements of $(k^*)^n$.
8. We define $\bar{c}' := \{(c^j')\}$ by $(c^j') := c_j - \delta_{j, j_+ (\bar{c})}$.
9. We define $\bar{c}^-$ by omitting $c^{j_+ (\bar{c})}$ from $\bar{c}$ (so $\bar{c}^-$ is of length $n - 1$) and analogously define $\bar{\alpha}^-$ by omitting $\alpha^{j_+ (\bar{c})}$ from $\bar{\alpha}$.
10. For $\bar{\alpha} = \{\alpha^j\} \in (k^*)^n$ we denote by $P_{\bar{c}, \bar{\alpha}}$ the polynomial on $V_\bar{c}$ given by
    $$P_{\bar{c}, \bar{\alpha}}(v) = \sum_{j=1}^n \prod_{i=1}^{c^j} \alpha^j x^j_i.$$  

11. We define $X_{\bar{c}, \bar{\alpha}} = \{v \in V_\bar{c} = |P_{\bar{c}, \bar{\alpha}}(v) = 0\}.$
12. We denote by $l_\bar{c} : V_\bar{c} \to A$ the projection $l_\bar{c} (v) = x_i^{j_+ (\bar{c})} (v)$. If there is no confusion we write $l$ instead of $l_\bar{c}$.
13. For $b \in k$ we define $X_{\bar{c}, \bar{\alpha}}^b = \{x \in X_{\bar{c}, \bar{\alpha}} | l(x) = b\}$

**Claim 2.5.**

1. If $b \neq 0$ then $X_{\bar{c}, \bar{\alpha}}^b \sim X_{\bar{c}', \bar{\alpha}'}$ where $\alpha' = \{\alpha'^j\}$ is given by $\alpha'^j = \alpha^j, j \neq j_+ (\bar{c})$ and $\alpha'^{j_+ (\bar{c})} = b \alpha^{j_+ (\bar{c})}$.
2. $X_{\bar{c}, \bar{\alpha}}^0 = A^{c_+ (\bar{c})-1} \times X_{\bar{c}, \bar{\alpha}^-}$.

**Theorem 2.6.** Let $k$ be a field with $|k| > a$. The subvarieties $X_{\bar{c}, \bar{\alpha}} \subset V_\bar{c}$ have property $*_{a}$. 

**Proof.** Observe that in the case when \( c^j = 1 \) for all \( j \), \( 1 \leq j \leq n \), or \( a = 0 \) the Claim is obviously true. So we may assume that \( c^j \geq 2 \) for some \( j \), \( 1 \leq j \leq n \).

The proof is by induction in \(( \sum_{j=1}^n c^j - n ) \) and \( a \).

Fix \( n \geq 1 \) and \( \bar{c} \) and \( \bar{a} \) as in Definition 2.4. Let \( l : V_{\bar{c}} \to k \) be the linear function \( l = x_1^{j+\bar{c}} \). As follows from Claim 2.5 the varieties \( X_{c,\alpha}^b \) have property \(*_a\) for all \( b \in k \).

Let \( f \) be a weakly polynomial function on \( X_{\bar{c},\bar{\alpha}} \). Since \( X_{\bar{c},\bar{\alpha}}^0 = X_{\bar{c},\bar{\alpha}} \times k^{\bar{c}+\bar{\alpha}-1} \) it follows from the inductive assumptions and Corollary 2.3 that there exists a polynomial \( P \) on \( V_{\bar{c}} \) of degree \( \leq a \) such that \( f|_{X_{\bar{c},\bar{\alpha}}^0} = P|_{X_{\bar{c},\bar{\alpha}}^0} \). So subtracting \( P \) we can assume that \( f|_{X_{\bar{c},\bar{\alpha}}^0} \equiv 0 \).

**Lemma 2.7.** Any plane \( L \subset X_{\bar{c},\bar{\alpha}}^b \) is contained in a cube \( M \subset X_{\bar{c},\bar{\alpha}}^b \) that intersects \( X_{\bar{c},\bar{\alpha}}^0 \).

**Proof.** A parametrized plane \( L \hookrightarrow V \) is a family of affine functions \( \phi(t) = \{ x_i^j(t, r), 1 \leq j \leq n, 1 \leq i \leq c^j \}. \) It belongs to \( X_{\bar{c},\bar{\alpha}} \) if \( \sum_{j=1}^n \prod_{i=1}^{c^j} x_i^j(t, r) \equiv 0 \) and it belongs to \( X_{\bar{c},\bar{\alpha}}^b \) if \( x_1^{j+\bar{c}}(t, r) \equiv b \). We can define parametrized cube \( M \hookrightarrow V \) is a family of affine functions \( \phi_2(t, s) = \{ x_i^j(t, r, s), 1 \leq i \leq d, 1 \leq j \leq n \} \) where \( x_i^j(x, r, s) = (1-s)x_i^j(t, r), 1 \leq j \leq n \) and \( x_i^j(x, r, s) = x_i^j(t, r), 2 \leq i \leq c^j, 1 \leq j \leq n \). Since \( \phi_2(t, r, 0) \equiv \phi \) the plane \( M \) contains \( L \). Since \( P_{\bar{c},\bar{\alpha}}(\phi_2(t, r, s) = (1-s)P_{\bar{c},\bar{\alpha}}(\phi(t, r)) = 0, \)

we see that \( M \subset X_{\bar{c},\bar{\alpha}} \).

For any \( S \subset k \), denote \( X_{\bar{c},\bar{\alpha}}^S = \bigcup_{s \in S} X_{\bar{c},\bar{\alpha}}^s \).

**Lemma 2.8.** For any finite subset \( S \subset k \), any a weakly polynomial function \( f \) of degree \( a \) on \( X \) such that \( f|_{X_{\bar{c},\bar{\alpha}}^S} \equiv 0 \), and any \( b \in k \) there exists a polynomial \( Q \) of degree \( \leq a \) on \( V \) such that \( Q|_{X_{\bar{c},\bar{\alpha}}^S} \equiv 0 \) and \( (Q-f)|_{X_{\bar{c},\bar{\alpha}}^S} \equiv 0 \).

**Proof.** We start with the following result.

**Claim 2.9.** Under the assumptions of Lemma 2.8 the restriction \( f|_{X_{\bar{c},\bar{\alpha}}^S} \) is a weakly polynomial function of degree \( \leq a - |S| \).

**Proof.** Since \(|k| > a \) it suffices to show that for any plane \( L \subset X_{\bar{c},\bar{\alpha}}^S \) the restriction \( f|_L \) is a polynomial of degree \( \leq a - |S| \). As follows from the Lemma 2.4 for any affine plane \( L \subset X_{\bar{c},\bar{\alpha}}^S \) there exists an affine cube \( M \subset X_{\bar{c},\bar{\alpha}}^S \) containing \( L \) and such that \( M \nsubseteq X_{\bar{c},\bar{\alpha}}^S \). Then \( M \cap X_{\bar{c},\bar{\alpha}}^t \neq \emptyset \) for any \( t \in k \). Since \( f \) is a weakly polynomial function of degree \( a \) its restriction to \( M \) is a polynomial \( R \) of degree \( \leq a \). Since the restriction of \( R \) to \( l^{-1}(S) \cap M \equiv 0 \) we see that \( R = R' \prod_{s \in S}(l - s) \). Since \( l|L \equiv b \) we see that the restriction \( f|_L \) is equal to \( R' \) which is a is a polynomial of degree \( \leq a - |S| \).

Now we show that this Claim implies Lemma 1.1. Indeed, assume that \( f|_{X_{\bar{c},\bar{\alpha}}^S} \) is a weakly polynomial function of degree \( \leq a - |S| \). It follows from the inductive
assumption on $a$ that there exists a polynomial $Q'$ of degree $\leq a - |S|$ on $V$ such that $f|_{X_{\alpha}^b} = Q'|_{X_{\alpha}^b}$. Let $Q := \frac{Q'}{\prod_{l \in S} (l - s)}$. Then $(f - Q)|_{X_{\alpha}^{S \cup \{b\}}} \equiv 0$. □

This completes the proof of Theorem 2.6. □

We conclude this section by showing the varieties $X_n$ are of high rank for large $n$:

**Lemma 2.10.** Let $k$ be an admissible finite field. $r(P_n) \geq C_d \cdot n$, for some constant $C_d$ depending on $d$.

**Proof.** We start with the following result.

**Claim 2.11.** For $d > 1$ we have

$$\mathbb{E}_{w^1, \ldots, w^d \in k} e_q(w^1, \ldots, w^d) = (1 - 1/q^{d-1})/(q - 1).$$

**Proof.** We prove that for $t \neq 0$,

$$\sum_{w^1, \ldots, w^d \in k} e_q(tw^1 \ldots w^d) = q^{d-1} + \ldots + q.$$

For $d = 2$,

$$\sum_{w^1, w^2 \in k} e_q(tw^1 w^2) = \sum_{w^1 \in k, w^1 \neq 0} \sum_{w^2 \in k} e_q(tw^1 w^2) + q = q.$$

Now proceed by induction. □

By Proposition A.6 we obtain

$$q^{-r(P_n)} \leq \mathbb{E}_{w^1, \ldots, w_n \in k^d} e_q(P_n(w_1, \ldots, w_n)) = [\mathbb{E}_{w^1, \ldots, w^d \in k} e_q(w^1 \ldots w^d)]^n.$$

□

3. **Proof of Theorem 1.11.**

In this section we assume that $k$ is a finite field. Theorem 1.11 follows from the following Proposition:

**Proposition 3.1.** For any $m \geq 0$ there exists $r = r(m, d)$ such that for any finite field $k$ of characteristic $>d$, any polynomial $P \in k[V_n]$ of degree $d$ and of $d$-rank $\geq r$ and any polynomial $R \in k[x_1, \ldots, x_m]$ of degree $\leq d$ there exists a linear map $\phi : k^m \to V_n$ such that $R = \phi^*(P)$.

**Proof.** Let $P : V = k^n \to k$ be a polynomial. Denote by $W$ the space linear maps $w : k^m \to k^n$. Consider the polynomial $Q$ on $W \times k^m$ ded by

$$Q(w, x) = P(w(x)) = \sum_{\lambda \in \Lambda} c_\lambda(w)x^\lambda,$$

where $\Lambda$ is the set of ordered tuples $(j_1, \ldots, j_m)$ with $j_i \geq 0$ and $\sum_{i=1}^m j_i \leq d$.

**Lemma 3.2.** If $p > d$ and $\text{rank}(P) > r$ then $\{c_\lambda(w)\}_{\lambda \in \Lambda}$ is of rank $\geq r$. 

Proof. We begin with the argument for the case $d = 2$. We are given $P(t) = \sum_{1 \leq i \leq j \leq n} a_{ij} t_i t_j$ of rank $r$. Note that for any linear form $l(t) = \sum_{i=1}^n c_i t_i$ we have that $P(t) + l(t)$ is of rank $\geq r$.

We can write

$$P(w(x)) = \sum_{1 \leq i \leq j \leq n} a_{ij} w^i(x) w^j(x) = \sum_{1 \leq i \leq j \leq n} a_{ij} \sum_{k,l=1}^m w^i_k w^j_l x_k x_l$$

Which we can write as

$$\sum_{1 \leq k < l \leq m} \sum_{1 \leq i \leq j \leq n} a_{ij} (w^i_k w^j_l + w^i_l w^j_k) x_k x_l + \sum_{1 \leq i \leq j \leq n} \sum_{1 \leq l \leq m} a_{ij} w^i_l w^j_l x_l^2$$

We want to show that the collection

$$\{ \sum_{1 \leq i \leq j \leq n} a_{ij} (w^i_k w^j_l + w^i_l w^j_k) \}_{1 \leq k < l \leq m} \bigcup \{ \sum_{1 \leq i \leq j \leq n} a_{ij} w^i_l w^j_l \}_{1 \leq l \leq m}$$

is of rank $\geq r$. Namely we need to show that if $B = (b_{kl})$ is not 0 then

$$\sum_{1 \leq k < l \leq m} \sum_{1 \leq i \leq j \leq n} a_{ij} (w^i_k w^j_l + w^i_l w^j_k) + \sum_{1 \leq l \leq m} b_{ll} \sum_{1 \leq i \leq j \leq n} a_{ij} w^i_l w^j_l$$

is or rank $\geq r$. Suppose $b_{11} \neq 0$. Then we can write the above as

$$b_{11} P(w_1) + l_{w_2, \ldots, w_m}(w_1)$$

where $w_j = (w^1_j, \ldots, w^n_j)$, and $l_{w_2, \ldots, w_m}$ is linear in $w_1$, so as a polynomial in $w_1$ this is of rank $\geq r$ and thus also of rank $\geq r$ as a polynomial in $w$. Similarly in the case where $b_{ll} \neq 0$, for some $1 \leq l \leq m$.

Suppose $b_{12} \neq 0$. We can write the above as

$$(*) \quad b_{12} Q(w_1, w_2) + l_{w_3, \ldots, w_m}(w_1, w_2)$$

where $Q : V^2 \to k$, and $Q(t, t) = 2P(t)$, and $l_{w_3, \ldots, w_m} : V^2 \to k$ is linear. Thus restricted to the subspace in $W$ where $w_1 = w_2$ we get that $(*)$ is of rank $\geq r$ and thus of rank $\geq r$ on $W$. Similarly if $b_{kl} \neq 0$ for some $k < l$.

For $d > 2$ the argument is similar: We are given $P(t) = \sum_{I \in \mathcal{I}} a_I t_I$ of rank $r$, where $\mathcal{I}$ is the set of ordered tuples $(i_1, \ldots, i_d)$ with $1 \leq i_1 \leq \ldots \leq i_d \leq n$, and $t_I = t_{i_1} \ldots t_{i_d}$.

Note that for any polynomial $R(t)$ of degree $< d$ we have that $P(t) + R(t)$ is also of rank $> r$.

We can write

$$P(w(x)) = \sum_{I \in \mathcal{I}} a_I w^I(x) = \sum_{I \in \mathcal{I}} a_I \sum_{l_1, \ldots, l_d=1}^m w^{i_1}_{l_1} \ldots w^{i_d}_{l_d} x_{l_1} \ldots x_{l_d}$$
For $1 \leq l_1 \leq \ldots \leq l_d \leq m$ the term $x_{l_1} \ldots x_{l_d}$ has as coefficient
\[ \sum_{I \in \mathcal{I}_{l_1,\ldots,l_d}} a_I w_{l_1}^{i_1} \ldots w_{l_d}^{i_d} \]
where $\mathcal{I}_{l_1,\ldots,l_d}$ is the set of permutations of $l_1, \ldots, l_d$. We wish to show that the set of the collection
\[ \{ \sum_{I \in \mathcal{I}_{l_1,\ldots,l_d}} a_I w_{l_1}^{i_1} \ldots w_{l_d}^{i_d} \} \]
is of rank $\geq r$. Namely we need to show that if $B = (b_{\mathcal{I}_{l_1,\ldots,l_d}})$ is not $0$ then
\[ \sum_{\mathcal{I}_{l_1,\ldots,l_d}} b_{\mathcal{I}_{l_1,\ldots,l_d}} \sum_{I \in \mathcal{I}_{l_1,\ldots,l_d}} a_I w_{l_1}^{i_1} \ldots w_{l_d}^{i_d} \]
is of rank $\geq r$. Suppose $b_{\mathcal{I}_{l_1,\ldots,l_d}} \neq 0$. Then restricted to the subspace $w_{l_1} = \ldots = w_{l_d}$ we can write the above as
\[ b_{\mathcal{I}_{l_1,\ldots,l_d}} |_{\mathcal{I}_{l_1,\ldots,l_d}} |P(w_{l_1}) + R(w) \]
where $w_j = (w_{l_j}^1, \ldots, w_{l_j}^{n_j})$, and $R(w)$ is of lower degree in $w_{l_1}$, so as a polynomial in $w_{l_1}$ this is of rank $\geq r$ and thus also of rank $\geq r$ as a polynomial in $w$. □

**Lemma 3.3.** There exists $r = r(d,m)$ such that for $P : k^n \to k$ is of rank $> r$ the map $f : W \to k^{\left| \Lambda \right|}$ given by $w \to \{ c_\lambda(w) \}_{\lambda \in \Lambda}$ is surjective.

**Proof.** We need to solve the system of equations $c_\lambda(w) = b_\lambda$ for any $b \in k^\Lambda$. The number of solutions is given by
\[ q^{-\left| \Lambda \right|} \sum_{\alpha \in k^\Lambda} e_q \left( \sum_{\lambda \in \Lambda} a_\lambda (c_\lambda(w) - b_\lambda) \right). \]

Let $s = 2\left| \Lambda \right| \leq d^m$. By Proposition A.3 there exists $r = r(d,m)$ such that each term other that the term corresponding to $\alpha = 0$ is at most $q^{-s}$, so that the total contribution from these terms is at most $q^{-\left| \Lambda \right|}$. □

This completes the proof of Proposition 3.1. □

**4. PROOF OF Proposition 1.18**

A key tool in our proof of this Proposition is a testing result from [15] which roughly says that any weakly polynomial function of degree $a$ that is “almost” weakly polynomial of degree $< a$, namely it is a polynomial of degree $< a$ on almost all affine subspaces, is weakly polynomial of degree $< a$. This does not require high rank. We use high rank to show that almost any isotropic line is contained in an isotropic plane that not contained in $l^{-1}\{0\}$.

**Proof of Claim 1.18** Let $V$ be a vector space and $l : V \to k$ be a non-constant affine function. For any subset $I$ of $k$ we denote $\mathbb{W}_I = \{ v \in V | l(v) \in I \}$ so that $\mathbb{W}_b = \mathbb{W}_{\{b\}}$, for $b \in k$. For a hypersurface $X \in \mathbb{V}$ we write $X_I = X \cap \mathbb{W}_I$. 


Lemma 4.1. For any finite subset $S \subset k$, any a weakly polynomial function $f$ of degree $a$ on $X$ such that $f|_{X_S} \equiv 0$, and any $b \in k$ there exists a polynomial $Q$ of degree $\leq a$ on $V$ such that $Q|_{X_S} \equiv 0$ and $(Q-f)|_{X_S} \equiv 0$.

Proof. We start with the following result.

Claim 4.2. Under the assumptions of Lemma 4.1 the restriction $f|_{X_b}$ is a weakly polynomial function of degree $\leq a - |S|$. 

Proof. Since $|k| > a$ it suffices to show that for any plane $L \subset X_b$ the restriction $f|_L$ is a polynomial of degree $\leq a - |S|$. 

As follows from Proposition B.3 there is a constant $A = A(d,a)$ such that it suffices to check the restriction $f|_L$ on $q^{-A}$-almost any affine plane $L \subset X_b$ is a polynomial of degree $\leq a - 1$.

As follows from Proposition B.3 for any $s > 0$ there is an $r = r(d,s)$ such that if $X$ is of rank $> r$ then for $q^{-s}$-almost any affine plane $L \subset X_b$ there exists an affine cube $M \subset X$ containing $L$ and such that $M \cap W_0 \neq \emptyset$. Then $M \cap X_t \neq \emptyset$ for any $t \in k$. Since $f$ is a weakly polynomial function of degree $a$ its restriction to $M$ is a polynomial $R$ of degree $\leq a$. Since the restriction of $R$ to $l^{-1}(S) \cap M \equiv 0$ we see that $R = R' \prod_{l \in S}(l-s)$. Since $l|L \equiv b$ we see that the restriction $f|_L$ is equal to $R'$ which is a is a polynomial of degree $\leq a - |S|$. 

Now we show that this Claim implies Lemma 4.1. Indeed, assume that $f|_{X_b}$ is a weakly polynomial function of degree $\leq a - |S|$. It follows from the inductive assumption on $a$ that there exists a polynomial $Q'$ of degree $\leq a - |S|$ on $V$ such that $f|_{X_b} = Q'|_{X_b}$. Let $Q = \frac{Q'}{\prod_{l \in S}(l-s)}$. Then $(f - Q)|_{X_{S \cup \{b\}}} \equiv 0$. 

This completes the proof the Claim 1.18.

To formulate a useful variant of Corollary 1.19 we introduce some notations.

Definition 4.3. Let $k[X]$ be the space of $k$-valued functions on $X$.

1. We denote by $\mathcal{P}^a_w(X) \subset k[X]$ the subspace of weakly polynomial functions of degree $\leq a$.
2. We denote by $\mathcal{P}_a(X) \subset \mathcal{P}^a_w(X)$ the subspace of functions $f : X \rightarrow k$ which are restrictions of polynomial functions on $V$ of degree $\leq a$.
3. We denote by $\overline{\mathcal{P}}_a(X)$ the quotient $\mathcal{P}_a(X)/\mathcal{P}_a(X)$.
4. For an imbedding $\phi : V' \rightarrow V$ we define $X_\phi = \{v' \in V' : \phi(v') \in X\}$. The map $\phi$ defines an imbedding $X_\phi \rightarrow X$ which we also denote by $\phi$.
5. We denote by $\phi^* : k[X] \rightarrow k[X_\phi]$ the map $f \rightarrow f \circ \phi$.
6. We denote by $\overline{\phi} : \overline{\mathcal{P}}_a(X) \rightarrow \overline{\mathcal{P}}_a(X_\phi)$ the linear map induced by $\phi^*$.
Corollary 4.4. Let $a > 0$, and let $k$ be a field with $a < q$. There exists $\rho = \rho(d, a)$ such that for any hypersurface $X = \{ v \in V | P(v) = 0 \} \subset V$ of degree $\leq d$, any affine imbedding $\phi : W \hookrightarrow V$ such that rank of $P|_W \geq \rho$ the restriction map $\bar{\phi} : \overline{P}(X) \rightarrow \overline{P}(X \cap W)$ is an imbedding.

Appendix A. Counting tools

The main property of high rank varieties is that it is easy to estimate the number of points on various important varieties. The main counting tool comes from the relation between the bias of exponential sums and algebraic rank.

Let $k$ be a finite field, $\text{char}(k) = p$, $|k| = q$. Let $V$ a vector space over $k$. We denote $e_q(x) = e^{2\pi i \psi(x)/p}$ where $\psi : k \rightarrow \mathbb{F}_p$ is the trace function. Let $P : V \rightarrow k$ be a polynomial of degree $d$. We denote by $(h_1, \ldots, h_d)_P$ the multilinear form $(h_1, \ldots, h_d)_P = \sum_{\omega \in \{0, 1\}^d} -1^{\omega/d} P(x + \omega \cdot \bar{h})$; $|\omega| = \sum_{i=1}^d \omega_i$.

We denote by $\mathbb{E}_{x \in S} f(x)$ the average $|S|^{-1} \sum_{x \in S} f(x)$.

Definition A.1 (Gowers norms [5]). For a function $g : V \rightarrow \mathbb{C}$ we define the norm $\|g\|_{U_d}$ by

$$\|g\|_{U_d}^2 = \mathbb{E}_{x, v_1, \ldots, v_d \in V} \prod_{\omega \in \{0, 1\}^d} g^{\omega}(x + \omega \cdot \bar{v}),$$

where $g^{\omega} = g$ if $|\omega|$ is even and $g^{\omega} = \bar{g}$ otherwise.

Definition A.2 (Analytic rank). The analytic rank of a polynomial $P : V \rightarrow k$ of degree $d$ is defined by $\text{arank}(P) = - \log q \|e_q(P)\|_{U_d}$.

The following Proposition relating bias and rank was proved in increasing generality in [7, 11, 3]. The most general version can be found at the survey [8] (Theorem 8.0.1):

Proposition A.3 (Bias-rank). Let $s, d > 0$. There exists $r = r(s, k, d)$ such that for any finite field $k$ of size $q = p^l$, any vector space $V$ over $k$, any polynomial $P : V \rightarrow k$ of degree $d$. If $P$ is of rank $\geq r$ then $|\mathbb{E}_{v \in V} e_q(P(v))| < q^{-s}$.

In the case when $p > d$, the bound on $r$ is uniform in $k$.

Conjecture A.4. For $p > d$ we have $r = s^{-O_d(1)}$. The conjecture is known for $d = 2, 3$ ([9]).

Remark A.5. When $p > d$ then one can recover $P$ from $(h_1, \ldots, h_d)_P$ so that if the rank of $(h_1, \ldots, h_d)_P$ is $< r$ of then so is the rank of $P$.

For multilinear functions; in particular for $(h_1, \ldots, h_d)_P$, the converse is also true:
Proposition A.6 ([16]). Let \( r, d > 0 \). There exists \( s = C_d \cdot r \) such that for any finite field \( k \) of size \( q = p^l \), \( p > d \), any vector space \( V \) over \( k \), any polynomial \( P : V \to k \) of degree \( d \), if
\[
\| e_q(P) \|^2_{U_d} = |E_{h_1, \ldots, h_d} e_q(h_1, \ldots, h_d)P| < q^{-s}
\]
for some polynomial, then \( P \) is of rank \( > r \).

Remark A.7. If \( P \) is of degree \( d \) and \( p > d \) then \( (h_1, \ldots, h_d)P = P(h)/d! \) so that if \( P \) is of rank \( > r \) then also \( (h_1, \ldots, h_d)P \) is of rank \( > r \) as a polynomial on \( V^d \).

Lemma A.8. For any \( R \) of degree \( < d \) we have
\[
|E_{h_1, \ldots, h_d} e_q((h_1, \ldots, h_d)P + R(h_1, \ldots, h_d))| \leq |E_{h_1, \ldots, h_d} e_q((h_1, \ldots, h_d)P)|
\]

Lemma A.9. Let \( P : V \to k \) be a polynomial of degree \( d \) and rank \( R \), and let \( W \subset V \) be a subspace of codimension \( s \). Then the rank of \( P|_W \) is \( \geq R - s \).

Lemma A.10. Let \( s > 0, \bar{d} = (d_1, \ldots, d_c), k \) a finite field. There exists \( r = r(\bar{d}, s, k) \) such that for any \( \bar{P} = \{P_1, \ldots, P_c\}, P_i : V \to k \) with \( \deg(P_i) \leq d_i \), \( |X_{\bar{P}}| = q^{c \dim(V) - c}(1 + q^{-s}) \). In the case when \( p > \max_i d_i \), the bound on \( r \) is uniform in \( k \).

Proof. The number of points on \( X \) is given by
\[
q^{-c} \sum_{\bar{a} \in k^c} \sum_{x \in V} e_q(\sum_{i=1}^c a_i P_i(x)).
\]
By Proposition A.3 for any \( s > 0 \) we can choose \( r \) so that for any \( \bar{a} \neq 0 \) we have
\[
|\sum_{x \in V} e_q(\sum_{i=1}^c a_i P_i(x))| < q^{-s}|V|.
\]

\[\square\]

Appendix B. Almost-sure results

In [12] (Theorem 1) the following description of degree \( < m \) polynomials is given:

Proposition B.1. Let \( P : V \to k \). Then \( P \) is a polynomial of degree \( \leq a \) if and only if the restriction of \( P \) to any affine subspace of dimension \( l = \left\lceil \frac{a+1}{q-q/p} \right\rceil \) is a polynomial of degree \( < m \).

Note that when \( a < q \) then \( l \leq 2 \).

In [12] the above criterion is used for polynomial testing over general finite fields. In [15] (Corollary 1.14) it is shown how the arguments in [12] can be adapted to polynomial testing within a subvariety variety \( X \subset V \) (high rank is not required).
**Theorem B.2** (Subspace splining on $X$). For any $a,d,L > 0$ there exists an $A = A(d, L, a) > 0$ such that the following holds. Let $X \subset V(k)$ be a complete intersection of degree $d$, codimension $L$. Then any weakly polynomial function $f$ of degree $a$ such that the restriction of $f$ to $q^{-A}$-a.e $l$-dimensional affine subspace, $l = \lceil \frac{a}{q^{-A}} \rceil$ is a polynomial of degree $< a$ is weakly polynomial of degree $< a$.

Let the notation be as in Section 4.

**Proposition B.3.** (1) For any $s > 0$ there exists $r = r(d)$ such that for any finite field $k$ with char$(k) > d$, any hypersurface $X(k)$ of rank $> r$, for $q^{-s}$-almost any affine line $L \subset X_b$ there exists an affine plane $L \subset M \subset X$ such that $M \cap X_0 \neq \emptyset$.

(2) For any $s > 0$ there exists $r = r(d)$ such that for any finite field $k$ with char$(k) > d$, any hypersurface $X(k)$ of rank $> r$, for $q^{-s}$-almost any affine plane $L \subset M \subset X$ there exists an affine cube $L \subset M \subset X$ such that $M \cap X_0 \neq \emptyset$.

*Proof.* We will prove (1); the proof of (2) is similar.

Let $k = \mathbb{F}_q$. We fix $d$ and define $d' = \min(d + 1, q)$. Let $M_0 = \{a_0, \ldots, a_d\} \subset k$ be a subset of $d'$ distinct points.

**Claim B.4.** Let $Q(x)$ be a polynomial of degree $\leq d$ such that $Q|_{M_0} \equiv 0$. Then $Q(a) = 0$ for all $a \in k$.

*Proof.* If $q \geq d + 1$ the Claim follows from the formula for the Vandermonde determinant. On the other hand if $d \geq q$ then there is nothing to prove. \hfill $\Box$

From now on we assume that $q \geq d + 1$. For any $0 \leq i \leq d$ we define a subset $S_i$ of $k^2$ by

$$S_i = \{(a_i, a_j), 0 \leq j \leq i\}$$

Let $T = \bigcup_{0 \leq i \leq d} S_i$.

**Claim B.5.** Let $Q(x, y)$ be a polynomial of degree $\leq d$ such that $Q|_T \equiv 0$. Then $Q = 0$.

*Proof.* We prove the claim by induction in $d$. Let $Q = \sum_{a,b} q_{a,b} x^a y^b, a + b \leq d$. The restriction of $Q$ on the line $\{x = 0\}$ is equal to $Q^0(y) = \sum_{b \leq d} q_{0,b} y^b$. Since $Q^0|_{X_d} \equiv 0$ we see that $Q^0 = 0$. So $Q(x, y) = xQ'(x, y)$. By the inductive assumption we have $Q' = 0$ \hfill $\square$

We will assume from now on that $a_0 = 0$. Denote $I(d) = \{(i, j) : 1 \leq i \leq d, 1 \leq j \leq i\}$, and $m = |I(d)|$.

An affine line in $X_b$ is parametrized as $x + ty, t \in k$, with

$$(*) \quad Q(x + ty) = 0, l(x) = b, l(y) = 0.$$  

Let $Y$ be the set of $(x, y)$ satisfying $(*)$.  

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We need to show that almost every \((x, y) \in Y\) we can find \(z\) with

\[
(**) \quad Q(x + ty + sz) = 0, \quad l(z) = -b, \quad s, t \in k
\]
or alternatively

\[
Q(x + ty + sz) = 0, \quad l(x + ty + sz) = (1 - s)b, \quad s, t \in k
\]

We can reduce this system to

\[
Q(x + a_iy + a_jz) = 0, \quad l(x + a_iy + a_jz) = (1 - a_j)b, \quad (i, j) \in I(d)
\]

Fix \((x, y) \in Y\) and estimate the number of solutions:

\[
(*) \quad q^{-2m} \sum_z \sum_{a, c \in k^n} e_q(\sum_{a_j} a_j Q(x + a_iy + a_jz) + c_{ij}(l(x + a_iy + a_jz) + (a_j - 1)b))
\]

Suppose \(a = (a_{ij}) = 0\), but \(c = (c_{ij}) \neq 0\), and recall that \(l(x) = b, l(y) = 0\). We have

\[
\sum_z e_q(\sum_{a_j} c_{ij}(l(x + a_iy + a_jz) + (a_j - 1)b)) = \sum_z e_q(\sum_{a_j} c_{ij}(a_j l(z) + a_j b))
\]

Now if \(\sum c_{ij} a_j l(z) \neq 0\) then the sum is 0. Otherwise also \(\sum c_{ij} a_j b = 0\) so that the sum is \(|V|\).

Now suppose \(a \neq 0\). Say \(a_{i_0 j_0} \neq 0\). We estimate

\[
(1) \quad \mathbb{E}_{x, y \in V} \mathbb{E}_z e_q(\sum a_{ij} Q(x + a_iy + a_jz) + c_{ij}(l(x + a_iy + a_jz) + (a_j - 1)b))^2
\]

**Lemma B.6.**

\[
\mathbb{E}_{x, y, z, z'} \prod_{(i, j) \in I(d)} f_{i, j}(x + a_iy + a_jz) f_{i, j}(x + a_iy + a_jz + a_j z') \leq \|f_{i_0, j_0}\|_{U_d}
\]

**Proof.** Without loss of generality \(a_1 = 1\) (make a change of variable \(y \rightarrow a_1^{-1}y, z \rightarrow a_1^{-1}z\)). We prove this by induction on \(d\). When \(d = 1\) we have \(x + y + z, x + y + z'\), and the claim is obvious. Assume \(d > 1\). We can write the average as

\[
\mathbb{E}_{x, y, z, z'} \prod_{(i, j) \in I(d-1)} f_{i, j}(x + a_iy + a_jz) f_{i, j}(x + a_iy + a_jz + a_j z')
\]

\[
\cdot \prod_{1 \leq j \leq d} f_{d, j}(x + a_dy + a_jz) f_{d, j}(x + a_dy + a_jz + a_j z').
\]

Shifting \(x\) by \(a_d y\) we get

\[
\mathbb{E}_{x, y, z, z'} \prod_{(i, j) \in I(d-1)} f_{i, j}(x + (a_i - a_d)y + a_jz) f_{i, j}(x + (a_i - a_d)y + a_jz + a_j z')
\]

\[
\cdot \prod_{1 \leq j \leq d} f_{d, j}(x + a_jz) f_{d, j}(x + a_jz + a_j z').
\]
Applying the Cauchi-Schwartz inequality we get

\[
\left[ \mathbb{E}_{x,y,z,z'} \prod_{(i,j) \in I(d-1)} f_{i,j}(x + (a_i - a_d)y + a_j z + a_j z') \right. \\
\left. \prod_{(i,j) \in I(d-1)} f_{i,j}(x + (a_i - a_d)y + a_j z) \right]^{1/2}
\]

Shifting \( x \) by \( a_d y \) and rearranging we get

\[
\left[ \mathbb{E}_{x,y',z,z'} \prod_{(i,j) \in I(d-1)} f_{i,j}(x + a_i y + a_j z) \tilde{f}_{i,j}(x + a_i y + (a_i - a_d)y' + a_j z) \right. \\
\left. \prod_{(i,j) \in I(d-1)} \tilde{f}_{i,j}(x + a_i y + a_j z + a_j z') f_{i,j}(x + a_i y + (a_i - a_d)y' + a_j z + a_j z') \right]^{1/2}
\]

Now if we denote

\[ g_{i,j,y'}(x) = f_{i,j}(x)\tilde{f}_{i,j}(x + y' + (a_i - a_d)y''). \]

then by the induction hypothesis we get that the above is bounded by

\[
\mathbb{E}_{y'} \| f_{i,j}(x)(x + y' + (a_i - a_d)y'')\|_{u_{d-1}}^{1/2} \leq \| f_{i,j}(x)\|_{u_d},
\]

for any \((i,j) \in I(d-1)\).

We do a similar computation for \((i,j) \in I(d) \setminus \{I(d-1), (d, 1)\}\), splitting

\[
\mathbb{E}_{x,y,z,z'} \prod_{(i,j) \in I(d-1)} f_{i+1,j+1}(x + a_{i+1}y + a_{j+1}z) \bar{f}_{i+1,j+1}(x + a_{i+1}y + a_{j+1}z + a_{j+1}z') \\
\prod_{1 \leq j \leq d} f_{j,1}(x + a_j z) \bar{f}_{j,1}(x + a_j z + a_j z')
\]

The only term left uncovered is \( f_{d,1} \), so we split

\[
\mathbb{E}_{x,y,z,z'} \prod_{(i,j) \in I(d-1)} f_{i+1,j}(x + a_{i+1}y + a_j z) \bar{f}_{i+1,j}(x + a_{i+1}y + a_j z + a_j z') \\
\prod_{1 \leq i \leq d} f_{i,i}(x + a_i y + a_i z) \bar{f}_{i,i}(x + a_i y + a_i z + a_i z').
\]

We make the change of variable \( z \to z - y \) to get

\[
\mathbb{E}_{x,y,z,z'} \prod_{(i,j) \in I(d-1)} f_{i+1,j}(x + a_{i+1}y + a_j (z - y)) \bar{f}_{i+1,j}(x + a_{i+1}y + a_j (z - y) + a_j z') \\
\prod_{1 \leq i \leq d} f_{i,i}(x + a_i z) \bar{f}_{i,i}(x + a_i z + a_i z').
\]

\( \square \)
By the Lemma [A.6] we obtain that $(\vec{1})$ is bounded by $\|e_q(\sum a_{i,j}Q + c_{i,j}f)\|_{U_d}$. By Propositions [A.3, A.8] for there exists $r = r(s)$ such that if $Q$ is of rank $> r$ then $\|e_q(\sum a_{i,j}Q + c_{i,j}f)\|_{U_d} < q^{-s}$. It follows that we can choose $r$ so that for $q^{-s}$ almost all $x, y \in Y$ the contribution to $(\ast)$ from all $(a, c)$ with $a \neq 0$ is bounded by $|V|q^{-4m}$.

**Proposition B.7.** For any $s > 0$ there exists $r = r(d, q)$ such that for any hypersurface $X(k)$ of rank $> r$, for $q^{-s}$-almost any affine plane $L \subset X_b$ there exists an affine cube $L \subset M \subset X$ such that $M \cap X_0 \neq \emptyset$.

**APPENDIX C. COMPLETE INTERSECTIONS OF BOUNDED CODIMENSION**

The arguments in the paper are written in the case when $X$ is hypersurface. Most of our results extend easily to the case when $X$ is a complete intersections of bounded codimension. Here is an example.

Fix

$$\vec{d} = \{c \geq 1, \; d_s \geq 2, \; 1 \leq s \leq c\}.$$ 

For a $k$-vector space $V$ we denote by $P_d(V)$ the space of polynomials $\vec{P} = \{P_s\}$,

$P_s \colon V \to k$, $1 \leq s \leq c$ of degree $d_s$.

**Definition C.1** (Algebraic rank of a variety). (1) Given a collection $\vec{P} \in P_d(V)$ of polynomials we define $X_\vec{P} = \{v \in V | P_s(v) = 0, 1 \leq s \leq c\}$.

(2) A collection $\vec{P} = \{P_i\}$ of polynomials is *admissible* if $X_\vec{P} \subset V$ is a complete intersection (that is $\dim(X_\vec{P}) = \dim(V) - c$).

(3) In the case when all polynomials $P_i$ are of the same degree we define $r(\vec{P})$ as the minimal rank of a non trivial $k$-linear combination of $P_i$.

(4) If $\vec{P} = \bigcup_j \vec{P}_j$ where $\vec{P}_j$ is a collection of polynomials of degree $d_j$ we say that $r(X_\vec{P}) > r$ if for all $j$, $r(\vec{P}_j) > r$.

**Remark C.2.** One can show that any collection $\vec{P}$ of a high $\vec{k}$-rank is admissible (see Section [E]).

The following is the counterpart of Proposition [3.1]. We can consider $P_d$ as a contravariant autofunctor on the category $\text{Vect}_k$ of finite-dimensional $k$-affine vector spaces where to an affine map $f : W \to V$ we associate the map

$$f^* : P_d(V) \to P_d(W), \quad f^*(\{P_s\}) = \{P_s \circ f\}.$$

**Proposition C.3.** For any $m \geq 1$ there exists $r = r(d, m)$ such that for any $\vec{P} \in P_d(V)$, $\vec{Q} \in P_d(U)$, with $r(\vec{P}) \geq r$, and $\dim(U) \leq m$ there exists $f \in \text{Hom}_{\text{Vect}_k}(U, V)$ such that $\vec{Q} = f^*(\vec{P})$.

**Proof.** Consider the polynomials $R_s, 1 \leq s \leq c$ on $W \times k^m$ defined by

$$R_s(w, x) = P_s(w(x)) = \sum_{\lambda \in \Lambda_s} c^s_\lambda(w)x^\lambda,$$

where $\Lambda_s$ is the set of ordered tuples $(j_1, \ldots, j_m)$ with $j_i \geq 0$ and $\sum_{i=1}^m j_i \leq d_s$. 

Lemma C.4. If \( p > d \) and \( \text{rank}(\bar{P}) > r \) then \( \{c^i_{\lambda}(w)\}_{1 \leq s \leq L, \lambda \in \Lambda} \) is of rank \( > r \).

Proof. We are given \( P_s(t) = \sum_{l \in I_s} a^s_l t^l, \ 1 \leq s \leq c, \) of rank \( r \), where \( I_s \) is the set of ordered tuples \((i_1, \ldots, i_{d_s})\) with \( 1 \leq i_1 \leq \ldots \leq i_{d_s} \leq n \), and \( t_I = t_{i_1} \ldots t_{i_{d_s}} \).

Note that for any polynomials \( l_s(t) \) of degrees \( < d_s \) we have that \( \{P_s(t) + l_s(t)\} \) is also of rank \( > r \).

We can write

\[ P_s(w(x)) = \sum_{l \in I_s} a^s_l w^l(x) = \sum_{l \in I_s} a^s_l \sum_{i_1, \ldots, i_{d_s} = 1}^m w_{i_1}^1 \ldots w_{i_{d_s}}^{d_s} x_{i_1} \ldots x_{i_{d_s}} \]

For \( 1 \leq l_1 \leq \ldots \leq l_{d_s} \leq m \) the term \( x_{i_1} \ldots x_{i_{d_s}} \) has as coefficient

\[ \sum_{l \in I_{l_1, \ldots, l_{d_s}}} a^s_l w_{i_1}^1 \ldots w_{i_{d_s}}^{d_s}, \]

where \( I_{l_1, \ldots, l_{d_s}} \) is the set of permutations of \( l_1, \ldots, l_{d_s} \). We wish to show that the collection

\[ \{ \sum_{l \in I_{l_1, \ldots, l_{d_s}}} a^s_l w_{i_1}^1 \ldots w_{i_{d_s}}^{d_s} \}_{1 \leq s \leq c, l_1, \ldots, l_{d_s}} \]

is of rank \( > r \). Write \([1, c] = \bigcup_{f=2}^d C_f \) where \( C_f = \{ s : d_s = f \} \).

We need to show that for any \( f = 2, \ldots, c \) if \( B = (b_{I_{l_1, \ldots, l_{d_s}}})_{s \in C_f, l_1, \ldots, l_{d_s}} \) is not \( 0 \), then

\[ \sum_{s \in C_f} \sum_{I_{l_1, \ldots, l_{d_s}}} b_{I_{l_1, \ldots, l_{d_s}}} \sum_{l \in I_{l_1, \ldots, l_{d_s}}} a^s_l w_{i_1}^1 \ldots w_{i_{d_s}}^{d_s} \]

is of rank \( > r \). Suppose \((b_{I_{l_1, \ldots, l_{d_s}}})_{s \in C_f} \neq \bar{0} \). Then restricted to the subspace \( w_{i_1} = \ldots = w_{i_{d_s}} \), we can write the above as

\[ \sum_{s \in C_f} b_{I_{l_1, \ldots, l_{d_s}}} |I_{l_1, \ldots, l_{d_s}}| P_s(w_{i_1}) + R(w) \]

where \( w_j = (w^1_j, \ldots, w^n_j) \), and \( R(w) \) is of lower degree in \( w_{i_1} \), so as a polynomial in \( w_{i_1} \) this is of rank \( > r \) and thus also of rank \( > r \) as a polynomial in \( w \). \( \square \)

Lemma C.5. There exists \( r = r(d, c, m) \) such that for \( \bar{P} : k^n \to k^c \) is of rank \( > r \) the map \( f : W \to \prod_s k^{|\Lambda_s|} \) given by \( w \to \{c^i_{\lambda}(w)\}_{s, \lambda \in \Lambda_s} \) is surjective.

Proof. Same proof as the corresponding Lemma C.3 for hypersurfaces. \( \square \)

This completes the proof of Proposition C.3 for algebraically closed fields.

Appendix D. The case of algebraically close fields

We start with a proof of Theorem 1.11 for algebraically closed fields.
Proof. Let $k$ be an algebraically closed field either of characteristic 0 or of characteristic $> d$, $\mathcal{W}$ a $k$-vector space of dimension $m$, $\mathcal{V}$ a $k$-vector space and $\mathcal{Y} \subset \mathcal{P}_d(\mathcal{V})$ the subvariety of polynomials of rank $\geq r(m, d)$. Then $\mathcal{Y}$ is a constructible subset of $\mathcal{P}_d(\mathcal{V})$ defined over $\mathcal{Z}$. Let $\kappa : \mathcal{Y} \times \text{Hom}_f(\mathcal{W}, \mathcal{V}) \to \mathcal{Y} \times \mathcal{P}_d(\mathcal{W})$ be given by $(P, \phi) \mapsto (P, \phi^*(P))$. For a proof of Theorem 1.11 it is sufficient to show the surjectivity of the map $\kappa$.

Let $\mathcal{Z} = \mathcal{Y} \times \mathcal{P}_d(\mathcal{W}) - \text{Im}(\kappa)$. Then $\mathcal{Z}$ is a constructible set and by Theorem 1.11 we have $\mathcal{Z}(\mathcal{F}_q) = \emptyset$ for sufficiently large $q$ of characteristic $> d$. It follows now from Claim 3.1 in 14 that $\mathcal{Z}(\mathcal{k}) = \emptyset$. □

The same arguments as in the proof of Theorem 1.11 imply the validity of the following result.

Claim D.1. Proposition B.3 implies the validity of the part (2) of Claim 1.18.

Theorem B.2 is a precise form of the part (1) of Claim 1.18. Now the arguments used in the derivation of Theorem B.2 from Proposition B.3 show that the part (1) of Claim 1.18 follows from Claim D.1.

Appendix E. Reduction to the case of complete intersection

Lemma E.1. Fix $\tilde{d} = \{d_i, 1 \leq i \leq c\}$ and $l$ be an algebraically closed field of characteristic $> \max d_i$, $1 \leq i \leq c$. Then $\mathbb{X}_\mathcal{P}$ is irreducible and $\text{dim}(\mathbb{X}_\mathcal{P}) = \text{dim}(\mathcal{V}) - c$ for any $l$-vector space $\mathcal{V}$ and a system $\tilde{P} = \{P_i\}$, $P_i : \mathcal{V} \to \mathbb{A}_l$, $i = 1, \ldots, c$ of polynomials of degrees $d_i$ and of rank $\geq r(\tilde{d})$ (see Lemma A.10).

Proof. We first consider the case $l = \bar{k}$, $k = \mathbb{F}_q$. We can assume (after we replace $q$ by $q^n$) that $P_i \in k[x_1, \ldots, x_n]$ and that all irreducible components $\mathcal{Y}$ of $\mathbb{X}_\mathcal{P}$ are defined over $k$. Let $k_N := \mathbb{F}_{q^N}$. Since $r_{k_N}(\tilde{P}) \geq r_l(\tilde{P}) \geq r(\tilde{d})$ we see that $|X_{\mathcal{P}}(\mathcal{k}_N)| \sim q^{N(\text{dim}(\mathcal{V})) - c}$. Therefore the Weil’s estimates imply that $\text{dim}(\mathcal{Y}) \leq \text{dim}(\mathcal{V}) - c$ for all irreducible components $\mathcal{Y}$ of $\mathbb{X}_\mathcal{P}$ and there is exactly one such component of dimension $\text{dim}(\mathcal{V}) - c$. On the other hand as well know $\text{dim}(\mathcal{Y}) \geq \text{dim}(\mathcal{V}) - c$ for any irreducible component $\mathcal{Y}$ of $\mathbb{X}_\mathcal{P}$. We see that $\mathbb{X}_\mathcal{P}$ is irreducible and of dimension $\text{dim}(\mathcal{V}) - c$.

The same arguments as in Appendix D allow the extension Lemma to the case of arbitrary algebraically closed field. □

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