Counting Primes Rationally And Irrationally

N. A. Carella

Abstract: The recent technique for estimating lower bounds of the prime counting function \( \pi(x) = \# \{ p \leq x : p \text{ prime} \} \) by means of the irrationality measures \( \mu(\zeta(s)) \geq 2 \) of special values of the zeta function claims that \( \pi(x) \gg \log \log x/\log \log \log x \). This note improves the lower bound to \( \pi(x) \gg \log x \), and extends the analysis to the irrationality measures \( \mu(\zeta(s)) \geq 1 \) for rational ratios of zeta functions.

1 Introduction

Let \( x \geq 1 \) be a large number and define the prime counting function by

\[
\pi(x) = \# \{ p \leq x : p \text{ is prime} \}. \tag{1}
\]

Dozens of proofs are known for estimating a lower bound of the prime counting function \( \pi(x) = \# \{ p \leq x : p \text{ is prime} \} \), see \[11, \text{pp. 3-11}, \[9\]. The oldest Euclidean technique has dozens of versions and refinements, and remains a research topic today, see \[1\] and Section 7 for some information.

The recent techniques introduced in \[6\] and \[8\] for estimating lower bounds of the prime counting function \( \pi(x) = \# \{ p \leq x : p \text{ prime} \} \) by means of the irrationality measures \( \mu(\zeta(s)) \geq 2 \) of special values of the zeta function claims that \( \pi(x) \gg \log \log x/\log \log \log x \). This note improves the lower bound to \( \pi(x) \gg \log x \), and extends the analysis to the irrationality measures \( \mu(\zeta(s)) \geq 1 \) for rational ratio of zeta functions. The proofs are independent of the irrationality measures \( \mu(\alpha) \geq 1 \) of the real numbers \( \alpha \in \mathbb{R} \).

2 For Irrational Values \( \zeta(s) \)

Theorem 2.1. Let \( x \geq 2 \) be a large number, and let \( \pi(x) = \# \{ p \leq x : p \text{ is prime} \} \). Then

\[
\pi(x) \geq c_1 \log x + c_0, \tag{2}
\]

where \( c_0 \) and \( c_1 > 0 \) are constants.

Proof. Fix an even number \( s > 1 \). Let \( x \gg 1 \) be a large number, and let

\[
\frac{1}{\zeta(s)} = \prod_{p \geq 2} \left( 1 - \frac{1}{p^s} \right) \quad \text{and} \quad \frac{p_x}{q_x} = \prod_{p \leq x} \left( 1 - \frac{1}{p^s} \right), \tag{3}
\]

where \( \gcd(p_x, q_x) = 1 \). Then

\[
\left| \frac{1}{\zeta(s)} - \frac{p_x}{q_x} \right| = \frac{1}{\zeta(s)} \left( 1 - \prod_{p \leq f(x)} \left( 1 - \frac{1}{p^s} \right)^{-1} \right), \tag{4}
\]

where \( f(x) = x \). By Lemma 5.1, the difference has the upper bound

March 18, 2022
AMS MSC: Primary 11N05, Secondary 11A41, 11M26.
Keywords: Distribution of prime; Prime counting function.
\[
\left| \frac{1}{\zeta(s)} - \frac{p_x}{q_x} \right| = \frac{1}{\zeta(s)} \left( 1 - \prod_{p > f(x)} \left( 1 - \frac{1}{p^s} \right)^{-1} \right)
\leq \frac{1}{(s-1)\zeta(s) f(x)^{\alpha-1}}.
\]

Let \(\mu = \mu(\zeta(s)) \geq 2\) be the irrationality measure of the number \(\zeta(s)\), see Definition 8.1. Then, applying Lemma 6.1 yields

\[
\frac{1}{q_x^2} \leq \frac{1}{2(\pi(x)-s-1)(\mu+\varepsilon)} \leq \frac{1}{\zeta(s)} - \frac{p_x}{q_x},
\]

where \(\varepsilon > 0\) is an arbitrary small constant. Comparing (5) and (6) yield

\[
\frac{1}{2(\pi(x)-s-1)(\mu+\varepsilon)} \leq \frac{1}{(s-1)\zeta(s) f(x)^{\alpha-1}}.
\]

Solving for \(\pi(x)\) yields

\[
\frac{s-1}{(\mu+\varepsilon) \log 2} \log f(x) + s + 1 + \frac{\log(s-1)\zeta(s)}{(\mu+\varepsilon) \log 2} = c_1 \log f(x) + c_0 \leq \pi(x)
\]

where \(c_0\), and \(c_1 = (s-1)/(\mu+\varepsilon) \log 2 > 0\) are constants.

Significant improvement of the lower bound \(\pi(x) \gg \log x\) to \(\pi(x) \gg x/\log x\) can be achieved by setting \(f(x) = 2^{c x/\log x + o(x/\log x)}\), for some constant \(c > 0\). In fact, this can be viewed as a near proof of the Prime Number Theorem by elementary methods. The earlier related analysis appear almost simultaneously in [3], and [8].

3 For Rational Values \(\zeta(s)/\zeta(t)\)

**Theorem 3.1.** Let \(x \geq 2\) be a large number, and let \(\pi(x) = \#\{p \leq x : p \text{ is prime}\}\). Then

\[
\pi(x) \geq c_3 \log x + c_2,
\]

where \(c_2\) and \(c_3 > 0\) are constants.

**Proof.** Let \(x \gg 1\) be a large number, and let

\[
\frac{2}{5} = \prod_{p \geq 2} \frac{p^2 - 1}{p^2 + 1} \quad \text{and} \quad \frac{p_x}{q_x} = \prod_{p \leq x} \frac{p^2 - 1}{p^2 + 1},
\]

where \(\gcd(p_x, q_x) = 1\). Then

\[
\left| \frac{2}{5} - \frac{p_x}{q_x} \right| = \frac{2}{5} \left( 1 - \prod_{p > f(x)} \left( \frac{p^2 - 1}{p^2 + 1} \right)^{-1} \right),
\]

where \(f(x) = x\). By the Euclidean Prime Number Theorem, there are infinitely many primes, so the product

\[
\prod_{p > f(x)} \left( \frac{p^2 - 1}{p^2 + 1} \right)^{-1} \neq 0.
\]

Applying Lemma 3.1 yields

\[
\left| \frac{2}{5} - \frac{p_x}{q_x} \right| = \frac{2}{5} \left( 1 - \prod_{p > f(x)} \left( \frac{p^2 - 1}{p^2 + 1} \right)^{-1} \right) \leq \frac{c_4}{f(x)},
\]

where $c_4 > 0$ is a constant. Let $\mu = \mu(5/2) = 1$ be the irrationality measure of the number $5/2$, see Definition 5.1. Then, applying Lemma 6.2 yields

$$
\frac{c_5}{q^\mu \varepsilon} \leq \frac{1}{2^{|\pi(x) - s - 1|}(\mu + \varepsilon)} \leq \left| \frac{2}{5} - \frac{p_x}{q_x} \right|,
$$

where $c_5 > 0$ is a constant. Comparing (13) and (14) yield

$$
\frac{1}{2^{|\pi(x) - s - 1|}(\mu + \varepsilon)} \leq \frac{c_4}{f(x)},
$$

Solving for $\pi(x)$ yields

$$
\frac{1}{(\mu + \varepsilon) \log 2} \log f(x) + s + 1 - \frac{\log c_4}{(1 + \varepsilon) \log 2} = c_3 \log f(x) + c_2 \leq \pi(x)
$$

where $c_2$ and $c_3 = 1/(\mu + \varepsilon) \log 2 > 0$ are constants.

\section{4 Infinite Products}

The evaluations of some infinite prime products have rational values. The best known cases are generated by some ratios of zeta functions, and by some ratios of $L$-functions. The zeta function and the $L$-function are defined by

$$
\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} = \prod_{p \geq 2} \left( 1 - \frac{1}{p^s} \right)^{-1},
$$

and

$$
L(s, \chi) = \sum_{n \geq 1} \frac{\chi(n)}{n^s} = \prod_{p \geq 2} \left( 1 - \frac{\chi(p)}{p^s} \right)^{-1},
$$

where $s \in \mathbb{C}$ is a complex variable, and $\chi$ is a character modulo $q \geq 1$, respectively.

\textbf{Lemma 4.1.} For any integer $n \geq 1$, the following primes products are rational numbers.

(i) For any integer $n \geq 1$,

$$
\frac{\zeta(2n)^2}{\zeta(4n)} = \prod_{p \geq 2} \left( \frac{1 - p^{-2n}}{1 - p^{-4n}} \right)^{-1} = \prod_{p \geq 2} \left( 1 + \frac{2}{p^{2n} - 1} \right).
$$

(ii) For any odd integers $m, n \geq 1$, and a character $\chi$ modulo $q$, $\mod q$,

$$
\frac{L(m, \chi)^n}{L(mm, \chi)} = \prod_{p \geq 2} \left( \frac{1 - \chi(p)p^{-mn}}{1 - \chi(p)p^{-mn}} \right)^{-n}.
$$

Proof. (i) Let $B_{2n}$ be the $2n$th Bernoulli number. Then, zeta ratio has a rational value

$$
\frac{\zeta(2n)^2}{\zeta(4n)} = \left( \frac{(-1)^{n-1}(2\pi)^{2n}B_{2n}}{2(2n)!} \right)^2 \left( \frac{(-1)^{2n-1}2(4n)!}{(2\pi)^{4n}B_{4n}} \right)
$$

$$
= \frac{(4n)!B_{2n}^2}{2((2n)!)^2B_{4n}}
$$

since any $B_k$ is rational. On the other hand, the infinite product has the expression

$$
\frac{\zeta(2n)^2}{\zeta(4n)} = \prod_{p \geq 2} \frac{(1 - p^{-2n})^{-2}}{(1 - p^{-4n})^{-1}}
$$

$$
= \prod_{p \geq 2} \left( 1 + \frac{2}{p^{2n} - 1} \right).
$$

as claimed. (ii) The proof of this case has similar calculations but lengthier.

\textbf{■}
Example 4.1. A pair of rational primes products.

(1) The simplest case occurs for \( n = 2 \). The zeta values are \( \zeta(2) = \pi^2/6 \) and \( \zeta(4) = \pi^4/90 \). Thus,

\[
\frac{5}{2} = \frac{\zeta(2)^2}{\zeta(4)} = \prod_{p \geq 2} \frac{(1 - p^{-2})^{-2}}{(1 - p^{-4})^{-1}} \prod_{p \geq 2} \frac{p^2 + 1}{p^2 - 1}
\]

(2) The simplest case occurs for \( m = 1, n = 3 \), and the quadratic character \( \chi \mod q = 4 \). The \( L \)-function values are \( L(1, \chi) = \pi/4 \) and \( L(3, \chi) = \pi^3/32 \). Thus,

\[
\frac{1}{2} = \frac{L(1, \chi)^3}{L(3, \chi)} = \prod_{p \geq 2} \frac{(1 - \chi(p)p^{-3})^{-1}}{(1 - \chi(p)p^{-3})^{-1}} = \prod_{p \equiv 1 \mod 4} p^3 - 1 \prod_{p \equiv 3 \mod 4} (p + 1)^3.
\]

5 Partial Products

Lemma 5.1. Fix a real number \( s \geq 1 \). Let \( x \gg 1 \) be a large number. Then,

(i)

\[
1 - \prod_{p > x} \left(1 - \frac{1}{p^s}\right) \leq \sum_{n > x} \frac{1}{n^s} = \frac{1}{s} - \frac{1}{s^s - 1},
\]

(ii)

\[
\zeta(s) - \sum_{n \leq x} \frac{1}{n^s} = O \left( \frac{1}{x^{s-1}} \right).
\]

Proof. These are standard results, see [12, Theorem 4.11].

Lemma 5.2. Fix a real number \( s \geq 1 \). Let \( x \gg 1 \) be a large number. Then,

\[
1 - \prod_{p > x} \left(\frac{p^{2s} - 1}{p^{2s} + 1}\right)^{-1} \leq \sum_{n > x} \frac{1}{n^{2s}} = \frac{2}{2s} - \frac{1}{x^{2s-1}} + O \left( \frac{1}{x^{s-2}} \right).
\]

Proof. Routine calculations yield

\[
1 - \prod_{p > x} \left(\frac{p^{2s} - 1}{p^{2s} + 1}\right)^{-1} = 1 - \prod_{p > x} \left(1 + \frac{2}{p^{2s} - 1}\right)
\]

\[
\leq \sum_{n > x} \frac{1}{n^{2s}} = \frac{2}{2s} - \frac{1}{x^{2s-1}} + O \left( \frac{1}{x^{2s-2}} \right).
\]

6 Rational Approximations

The denominator of the Euler approximation \( \frac{p_x}{q_x} \) has a trivial bound \( q_x \leq (x!)^2 \). Sharper and more effective descriptions of the rational approximations generated by some Euler products are provided here.

Lemma 6.1. Let \( s \geq 1 \) be a fixed integer, and let \( x \gg 1 \) be a large number. Then, the product

\[
\frac{p_x}{q_x} = \prod_{p \leq x} \left(1 - \frac{1}{p^s}\right),
\]

where \( \gcd(p_x, q_x) = 1 \), satisfies the followings properties.
(i) The integer \( p_x \geq 2^{\pi(x) - s - 1} \) has exponential growth.

(ii) The integer \( q_x \geq 2^{\pi(x) - s - 1} \) has exponential growth.

**Proof.** For a large number \( x \gg 1 \), consider

\[
A_x = \prod_{p \leq x} (p^s - 1) \quad \text{and} \quad B_x = \prod_{p \leq x} p^s. \tag{28}
\]

Here, the integer \( A_x \) is divisible by an increasing high power of 2 as \( x \to \infty \), but the integer \( B_x \) is divisible by a small fixed power of 2:

\[
2^{\pi(x) - 2} | A_x \quad \text{and} \quad 2^s || B_x. \tag{29}
\]

Thus, the even part of the product can be precisely factored as

\[
\prod_{p \leq x} \left( \frac{p^s - 1}{p^s} \right) = \frac{A_x}{B_x} = \frac{2^{\pi(x) - s - 1} \times A}{2^s B} = \frac{p_x}{q_x}, \tag{30}
\]

where \( A > 1 \) and \( B > 1 \) are integers such that \( \gcd(2, B) = 1 \), and \( \gcd(p_x, q_x) = 1 \).

(i) To verify this statement, observe that in (30), the integer \( B \) is odd and that these integers are nearly relatively prime, \( 1 \leq \gcd(A, B) \leq A \). Hence,

\[
p_x = \frac{2^{\pi(x) - s - 1} \times A}{\gcd(A, B)} \geq 2^{\pi(x) - s - 1}. \tag{31}
\]

(ii) To verify this statement, observe that \( 1/2 \leq p_x/q_x \leq 1 \). Equivalently,

\[
q_x/2 \leq p_x \leq q_x. \tag{32}
\]

Hence,

\[
q_x \geq p_x \geq 2^{\pi(x) - s - 1}. \tag{33}
\]

These complete the verifications of (i) and (ii). \( \blacksquare \)

**Lemma 6.2.** Let \( s \geq 1 \) be a fixed integer, and let \( x \gg 1 \) be a large number. Then, the product

\[
\frac{p_x}{q_x} = \prod_{p \leq x} \frac{p^2 - 1}{p^2 + 1}, \tag{34}
\]

where \( \gcd(p_x, q_x) = 1 \), satisfies the following properties.

(i) The integer \( p_x \geq 2^{\pi(x) - s - 1} \) has exponential growth.

(ii) The integer \( q_x \geq 2^{\pi(x) - s - 1} \) has exponential growth.

**Proof.** For a large number \( x \gg 1 \), consider

\[
A_x = \prod_{p \leq x} (p^{2s} - 1) = \prod_{p \leq x} (p^s - 1)(p^s + 1) \quad \text{and} \quad B_x = \prod_{p \leq x} (p^{2s} + 1). \tag{35}
\]

Here, the integer \( A_x \) is divisible by an increasing double high power of 2 as \( x \to \infty \), but the integer \( B_x \) is divisible by a high power of 2:

\[
2^{2\pi(x) - 2} | A_x \quad \text{and} \quad 2^{\pi(x) - 1} || B_x. \tag{36}
\]

The last expression in (36) follows from \( p^{2s} + 1 \equiv 2 \mod 4 \) for \( s \in \mathbb{N} \). Thus, the even part of the product can be precisely factored as

\[
\prod_{p \leq x} \left( \frac{(p^s - 1)(p^s + 1)}{p^{2s} + 1} \right) = \frac{A_x}{B_x} = \frac{2^{2\pi(x) - 2} \times A}{2^{\pi(x) - 1} B} = \frac{2^{\pi(x) - s - 1} \times A}{B} = \frac{p_x}{q_x}. \tag{37}
\]
where $A > 1$ and $B > 1$ are integers such that $\gcd(2, B) = 1$, and $\gcd(p_x, q_x) = 1$.

(i) To verify this statement, observe that in (37), the integer $B$ is odd, (follows from $p^2 + 1 \equiv 2 \mod 4$), and the these integers are nearly relatively prime, $1 \leq \gcd(A, B) \leq A$. Hence,

$$p_x = \frac{2^{\pi(x) - s - 1}A}{\gcd(A, B)} \geq 2^{\pi(x) - s - 1}.$$  

(38)

(ii) To verify this statement, observe that $1/4 \leq p_x/q_x \leq 1$. Equivalently,

$$\frac{q_x}{4} \leq p_x \leq q_x.$$  

(39)

Hence,

$$q_x \geq p_x \geq 2^{\pi(x) - s - 1}.$$  

(40)

These complete the verifications of (i) and (ii).

7  Euclidean Sequences

The Euclidean sequence $q_n = \max\{p \mid n! + 1\}$ established the existence of infinitely many primes around 23 centuries ago. The first few terms of the sequence are these:

2, 3, 7, 5, 11, 103, 71, 61, 661, 19, 269, 329891, 39916801, 13, 83, ...

(41)

The primes are generated in a chaotic manner. Many variations of this sequence are studied in the literature, see [1] and similar references.

The Hermite sequence $q_n = \min\{q \mid (p - 1)! + 1\}$, where $q \geq 2$ ranges over the primes, generates all the prime numbers, and the primes are generated in increasing order. These nice properties follow from Wilson theorem $(p - 1)! + 1 \equiv 0 \mod p$, see [4, p. 303] for more details. The first few terms of the sequence are these:

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, ...

(42)

However, the opposite Hermite sequence $q_n = \max\{q \mid (p - 1)! + 1\}$ has more complex properties, and generates primes in a chaotic manner.

About 2 centuries ago Euler introduced a new primes counting method based on the prime harmonic sum

$$\sum_{p \leq x} \frac{1}{p} \to \log \log x$$  

as $x \to \infty$, see [7]. More recently, about a century ago, Hadamard and delaVallee Poussin independently proved using different methods, that the sequence of increasing prime numbers up to a fixed number $x \geq 2$ has

$$\pi(x) = \#\{p \leq x\} = \frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right)$$  

(44)

primes, confer the literature for additional details.

8  Irrationality Measures

The irrationality measure $\mu(\alpha)$ of a real number $\alpha \in \mathbb{R}$ is the infimum of the subset of real numbers $\mu(\alpha) \geq 1$ for which the Diophantine inequality

$$|\alpha - \frac{p}{q}| \ll \frac{1}{q^{\mu(\alpha)}}$$  

(45)

has finitely many rational solutions $p$ and $q$. 
Definition 8.1. A measure of irrationality $\mu(\alpha) \geq 2$ of an irrational real number $\alpha \in \mathbb{R}^*$ is a map $\psi : \mathbb{N} \rightarrow \mathbb{R}$ such that for any $p, q \in \mathbb{N}$ with $q \geq q_0$,

$$|\alpha - \frac{p}{q}| \geq \frac{1}{\psi(q)}.$$  \hspace{1cm} (46)

Furthermore, any measure of irrationality of an irrational real number satisfies $\psi(q) \geq \sqrt{5}q^{\mu(\alpha)} \geq \sqrt{5}q^2$.

The concept of measures of irrationality of real numbers is discussed in [13, p. 556], [2, Chapter 11], et alii.

Lemma 8.1. ([3, Theorem 2]) The map $\mu : \mathbb{R} \rightarrow [2, \infty) \cup \{1\}$ is surjective. Any number in the set $[2, \infty) \cup \{1\}$ is the irrationality measure of some number.

More precisely,

1. A rational number has an irrationality measure of $\mu(\alpha) = 1$, see [5, Theorem 186].
2. An algebraic irrational number has an irrationality measure of $\mu(\alpha) = 2$, an introduction to the earlier proofs of Roth Theorem appears in [10, p. 147].
3. Any irrational number has an irrationality measure of $\mu(\alpha) \geq 2$.
4. A Mahler number $\psi_b = \sum_{n \geq 1} b^{-|\tau|^n}$ in base $b \geq 3$ has an irrationality measure of $\mu(\psi_b) = \tau$, for any real number $\tau \geq 2$, see [3, Theorem 2].

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