Abstract

Following [6, 12], we study coupled map networks over arbitrary finite graphs. An estimate from below for a topological entropy of a perturbed coupled map network via a topological entropy of an unperturbed network by making use of the covering relations for coupled map networks is obtained. The result is quite general, particularly no assumptions on hyperbolicity of a local dynamics or linearity of coupling are made.

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1 Introduction

A coupled map network is characterized by local dynamics operating at each node of a graph and their interaction along the edges of the graph. Coupled map networks are useful in applications: they appear naturally as electronic circuits in engineering, as chemical reactions in physics, as neuronal networks in the biological sciences, and as various agent-based models in the social sciences, etc. These networks are usually finite in size, but can be very large. The couplings of them can be linear as well as nonlinear. Much attention has been paid to linear coupling and simple dynamical behaviors such as existence of a global attractor in [1, 2] and synchronization in [3, 5]; see also [4] and references therein, while more complex phenomena are known to occur but not as well recognized and understood. Recently in [6], a linear coupling of expanding circle maps was studied. It was found there that an increasing of coupling strength leads to a cascade of bifurcations in which unstable subspaces in the coupled map systematically become stable.

We study here a topological dynamics in coupled map networks without assuming hyperbolicity of local maps and linearity of interactions. Consider a coupled map network with local dynamics having covering relations and coupling having linear model. We give sufficient conditions for existence of periodic points and for existence of a positive topological entropy, in the coupled map network. Both conditions allow for a weak as well as for a strong coupling. Moreover, both results are also valid for small perturbations of the coupled map network, of which coupling might be nonlinear.

Our approach is based on the concept of covering relations, introduced by [13]. The covering relation is a topological technique which does not require hyperbolicity (see e.g. [7, 8, 9]). Assuming that a coupling is locally topologically conjugate to a linear coupling, we show that the unperturbed coupled map network and its small perturbations both have covering relations of local dynamics as well as existence of periodic points. To implement a topological chaos from local dynamics to perturbed coupled map networks, we introduce a notion of unified sets to guarantee the conjugacy relation between the coupling and its linear model.
The paper is organized as follows. In Section 2, we define coupled map networks (here and in other places) of two types and state main results for each of them. In Section 3, we begin with formulation of known results needed in what follows and present the proofs of our results. In Appendix, the definition of covering relations determined by a transition matrix is briefly recalled.

## 2 Definitions and statements of the main results

We start with the definition of a general class of coupled map networks which will be studied.

**Definition 1.** A coupled map network is a triple \((G, \{T_k\}, A)\) where

1. \(G\) is a connected directed graph specified by a finite set \(\Omega\) of nodes and a collection of edges \(E \subset \Omega \times \Omega\);
2. to each node \(k \in \Omega\) there corresponds a local space \(X_k\) and a local map \(T_k : X_k \to X_k\);
3. network dynamics is defined by the iteration of \(\Phi : X \to X\), where \(X = \prod_{k \in \Omega} X_k\) is the product space and \(\Phi = A \circ T\), where \(T = \prod_{k \in \Omega} T_k\) is the (independent) application of local maps and \(A : X \to X\) is the spatial interaction or coupling; for \(x = (x_k)_{k \in \Omega} \in X\), the \(k\)th coordinate of \(A(x)\) depends only on \(x_k\) and those \(x_j\) for which \((j, k) \in E\).

We consider perturbations of coupled map networks in the following sense.

**Definition 2.** Let \((G, \{T_k\}, A)\) and \((\tilde{G}, \{\tilde{T}_k\}, \tilde{A})\) be two coupled map networks with the same set of nodes (might with distinct edges) and local spaces. For \(\varepsilon > 0\), we say that \((G, \{T_k\}, A)\) is \(\varepsilon\)-close to \((G, \{T_k\}, A)\) if \(|\tilde{A}(z) - A(z)| < \varepsilon\) for all \(z \in X\) and \(|\tilde{T}_k(x_k) - T_k(x_k)| < \varepsilon\) for all \(x_k \in X_k\) for all nodes \(k\), where \(|\cdot|\) denotes norms on the product space and on the local spaces without ambiguity.

Given a coupled map network \((G, \{T_k\}, A)\), if \((\tilde{G}, \{\tilde{T}_k\}, \tilde{A})\) is a family of coupled map networks of the same nodes and local spaces, with a real parameter \(\varepsilon\), such that \(\tilde{T}_k(x_k) = T_k(x_k) + \varepsilon \alpha(x_k)\) and \(\tilde{A}(x) = A(x) + \varepsilon \beta(x)\), where both \(\alpha\) and \(\beta\) are bounded and continuous functions, then \((G, \{T_k\}, A)\) is approaching \((G, \{T_k\}, A)\) as \(\varepsilon\) tends to zero.

For a positive integer \(m\), let \(\mathbb{R}^m\) denote the space of all \(m\)-tuples of real numbers. Let \(|\cdot|\) be a given norm on \(\mathbb{R}^m\), and let \(||\cdot||\) denote the operator-norm on the space of linear maps on \(\mathbb{R}^m\) induced by \(|\cdot|\). For \(x \in \mathbb{R}^m\) and \(r > 0\), we denote \(B^m(x, r) = \{z \in \mathbb{R}^m : |z - x| < r\}\); for the particular case when \(x = 0\) and \(r = 1\), we write \(B^m = B^m(0, 1)\), that is, the open unit ball in \(\mathbb{R}^m\). Furthermore, for any subset \(S\) of \(\mathbb{R}^m\), let \(\overline{S}\), \(\text{int}(S)\) and \(\partial S\) denote the closure, interior and boundary of \(S\), respectively. For the definition of covering relations determined by a transition matrix, see Appendix.

**Definition 3.** Let \(F\) be a continuous map on \(\mathbb{R}^m\). Define the maximum stretch \(||F||_{\text{max}} = \max\{|F(x)| : x \in \overline{B^m}\}\) and the minimum stretch \(||F||_{\text{min}} = \min\{|F(x)| : x \in \partial B^m\}\).

The maximum and minimum stretches are the radii of the smallest ball with center at origin that contains \(F(\overline{B^m})\) and of the largest open ball with center at origin not intersecting \(F(\partial B^m)\). If \(F\) is a linear map, the maximum and minimum stretches are the norm and conorm of \(F\).

From now on, we consider a coupled map network \((G, \{T_k\}, A)\) such that \(G\) is a connected directed graph with nodes \(\Omega = \{1, \ldots, d\}\) and edges \(E \subset \Omega \times \Omega\), and for \(1 \leq k \leq d\), \(T_k\) is a continuous local map on \(\mathbb{R}^{m_+}\) having covering relations on \(h\)-sets \(\{M_{ki}\}_{i=1}^{d_k}\) determined by a transition matrix \(W_k = [w_{kij}]_{1 \leq i,j \leq d_k}\) such that \(u(M_{ki}) = u\) and \(s(M_{ki}) = s\).

We say that the coupled map network \((G, \{T_k\}, A)\) is of type \(I\) with locally linear coupling, if for each nonzero entry \(\prod_{k=1}^{d} w_{kij} k_{ij} \neq 0\) of the Kronecker product \(\bigotimes_{k=1}^{d} W_k\), there exists a \(d \times d\) invertible real matrix \([a_{lm}]\) satisfying \((m, l) \notin E\) implies \(a_{lm} = 0\), and the following conditions hold:

- For \(1 \leq k \leq d\) and \(1 \leq i', j' \leq d_k\) with \(i' \neq i\) and \(j' \neq j\),
  \[T_k(M_{kij}) \cap (M_{kij'} \cup T(M_{kij})) = \emptyset.\] \hspace{1cm} \(\text{(1)}\)
For $1 \leq k \leq d$ and $x \in \overline{B}^u$, $y \in \overline{B}^s$,
\[ c_{M_{k,jk}} \circ T_k \circ c^{-1}_{M_{k,jk}}(x,y) = (U_{kis,jk}(x),V_{kis,jk}(y)), \]
(2)

where $U_{kis,jk}$ and $V_{kis,jk}$ are continuous maps on $\mathbb{R}^u$ and $\mathbb{R}^s$, respectively, such that
\[ ||U_{kis,jk}||_{\text{min}} > 1, \deg(U_{kis,jk}, B^u, 0) \neq 0, \text{ and } ||V_{kis,jk}||_{\text{max}} < 1. \]
(3)

- For $z \in (\prod_{k=1}^{d} c_{M_{k,jk}}) \circ T(\prod_{k=1}^{d} M_{k,jk}),$
\[ (\prod_{k=1}^{d} c_{M_{k,jk}}) \circ A \circ (\prod_{k=1}^{d} c_{M_{k,jk}})^{-1}(z) = ([a_{lm}] \otimes I)z, \]
(4)

where $I$ is the $(u+s) \times (u+s)$ identity matrix.

Notice that (2) and (3) are only to specify the covering relation $M_{k,jk} \rightarrow M_{k,jk}$. From (1) it follows that (4) is well defined and it says that the restriction of $A$ to the set $T(\prod_{k=1}^{d} M_{k,jk})$ is topologically conjugate to the linear map $A_c$, by the homeomorphism $\prod_{k=1}^{d} c_{M_{k,jk}}$. In general, the map on the left-hand side of (4) is not well defined.

Now, we state the first result about covering relations and existence of periodic points for perturbations of coupled map networks, under permutation transition matrices.

**Theorem 1.** Let $(G, \{T_k\}, A)$ be a coupled map network of type I with locally linear coupling as in (7) such that each of the transition matrices $W_k$, $1 \leq k \leq d$, is a permutation. Suppose that for each nonzero entry $\prod_{k=1}^{d} w_{kis,jk}$ of the Kronecker product $\bigotimes_{k=1}^{d} W_k$, there exists a permutation $\tau$ on $\{1, 2, \ldots, d\}$ such that for $1 \leq k \leq d$,
\[ ||a_{k\tau(k)}U_{\tau(k)\iota_{k,\tau(k)}}||_{\text{min}} - \sum_{l=1, l \neq \tau(k)}^{d} ||a_{kl}U_{l,jk}||_{\text{max}} > 1 \text{ and } \sum_{l=1}^{d} ||a_{kl}V_{l,jk}||_{\text{max}} < 1. \]
(5)

Then any coupled map network $(\tilde{G}, \{\tilde{T}_k\}, \tilde{A})$ sufficiently close to $(G, \{T_k\}, A)$ has covering relations determined by $\bigotimes_{k=1}^{d} W_k$ and has a periodic point of period $1 \text{lcm}(\dim W_1, \ldots, \dim W_d)$, where $1 \text{lcm}$ stands for the least common multiple.

Before further investigating topological chaos for perturbations of coupled map networks, we introduce a notion of unified h-sets.

**Definition 4.** A $d$-tuple $(M_1, \ldots, M_d)$ of disjoint h-sets in $\mathbb{R}^m$ with $u(M_i) = u$ is said to be unified by a subset $N$ of $\mathbb{R}^m$ if $\bigcup_{i=1}^{d} M_i \subset N$, and there exists a homeomorphism $\hat{c}_N : \mathbb{R}^m \rightarrow \mathbb{R}^m = \mathbb{R}^u \times \mathbb{R}^{m-u}$ such that for $1 \leq i \leq d$,
\[ \hat{c}_N(N) = \overline{B}^u(q_i^u, (3d-1)/2) \times \overline{B}^s(0, 1) \text{ and } \hat{c}_N(M_i) = \overline{B}^u(q_i^u, 1) \times \overline{B}^s(q_i^s, r_i), \]
(6)

where $q_i^u = ((3d-3)/2, 0, \ldots, 0)$ and $q_i^u = (3(i-1), 0, \ldots, 0)$ belong to $\mathbb{R}^u$, and $q_i^s = (q_i^s, 0, \ldots, 0)$ belongs to $\mathbb{R}^s$ for some real numbers $|q_i^s| < 1$ and $0 < r_i \leq 1$. Here, we call $q_i^u$, $q_i^s$, and $r_i$, the $i$th unstable center, stable center, and radius of stability of $N$ respectively. In particular, any h-set is unified by itself.

Being unified is a topological aspect: a unified tuple means the union of elements in the tuple is an enlarged h-set such that under the change of coordinates as (6), the union looks like a product of unstable and stable balls, while the choice of centers and radii is flexible.

Let $A_c$ be the following Kronecker product
\[ A_c = [a_{lm}] \otimes I, \]
(7)

where $[a_{lm}]$ is a $d \times d$ invertible real matrix such that if $(l, m) \notin E$ then $a_{lm} = 0$, and $I$ is the $(u+s) \times (u+s)$ identity matrix. We say that the coupled map network $(G, \{T_k\}, A)$ is of type II with the linear coupling model $A_c$, if the following conditions hold:
• For $1 \leq k \leq d$, there exists a set $N_k$ such that the tuple $(M_{k1}, \ldots, M_{kd})$ is unified by $N_k$ with the $j$-th unstable center at $p_{kj}^u$, stable center at $p_{kj}^s$, and stable radius $r_{kj}$ for all $1 \leq j \leq d_k$.

• For $1 \leq k \leq d$, if $w_{kj} = 1$, then for $x \in \mathbb{B}^u, y \in \mathbb{B}^s$,

$$c_{N_k} \circ T_k \circ c_{M_{k1}}^{-1}(x, y) = (U_{ki}(x), V_{ki}(y)),$$

where

$$g_{ki}(x, y) = (x - p_{kj}^u, (y - p_{kj}^s)/r_{kj})$$

and $U_{ki}, V_{ki}$ are continuous maps on $\mathbb{R}^u, \mathbb{R}^s$, respectively, such that

$$||U_{ki} - p_{kj}^u||_{\min} > 1, \deg(U_{ki}, B^u, p_{kj}^u) \neq 0, \text{ and } ||V_{ki} - p_{kj}^s||_{\max} < r_{kj}.$$

• For $z \in (\prod_{k=1}^d c_{N_k}) \circ T(\prod_{k=1}^d M_{ki}),$

$$d \prod_{k=1}^d c_{N_k} \circ A \circ (\prod_{k=1}^d c_{N_k})^{-1}(z) = A_{c}z.$$  

Notice that (8) and (10) are only to specify the covering relation $M_{ki} \xrightarrow{T_{ki}} M_{kj}$ under the unified structure. With a help of (9), each quadruple $(M_{ki}, \hat{c}_{M_{ki}}, u, s)$ now is an $h$-sets in $\mathbb{R}^{u+s}$. Moreover, since $(\prod_{k=1}^d c_{N_k})$ is independent of $i$ and $j$, (11) is always well defined and it says that the restriction of $A$ to the set $T(\prod_{k=1}^d M_{ki})$ is topologically conjugate to the linear map $A_c$, by the homeomorphism $(\prod_{k=1}^d c_{N_k})$. We give examples of coupled map networks of type II: one for $A = A_c$ and the other for $A \neq A_c$.

**Example 5.** Define local dynamics by

$$T_1(x) = \begin{cases} 3.5x + 1.5, & \text{if } x \leq 3/2, \\ 2x - 6, & \text{if } x > 3/2, \end{cases} \text{ and } T_2(x) = \begin{cases} 2x + 3, & \text{if } x \leq 3/2, \\ 3.5x - 9, & \text{if } x > 3/2. \end{cases}$$

Then $T_1$ has covering relations determined by $W_1 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ on the $h$-set tuple $(M_{11} = [-1, 1], M_{12} = [2, 4])$, with $c_{M_{11}}(x) = x, c_{M_{12}}(x) = x - 3$ and $u = 1, s = 0$, which is unified by $N_1 = [-1, 4]$ with $\hat{c}_{N_1}(x) = x$ and unstable centers at $p_{11}^u = 0$ and $p_{12}^u = 3$. Also, $T_2$ has covering relations determined by $W_2 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ on the $h$-set tuple $(M_{21} = [-1, 1], M_{22} = [2, 4])$, with $c_{M_{21}}(x) = x, c_{M_{22}}(x) = x - 3$ and $u = 1, s = 0$, which is unified by $N_2 = [-1, 4]$ with $\hat{c}_{N_2}(x) = x$ and unstable centers at $p_{21}^u = 0$ and $p_{22}^u = 3$. For $-1 \leq x \leq 1$, let $U_{11}(x) = 3.5x + 1.5, U_{12}(x) = 2x, U_{21}(x) = 2x + 3,$ and $U_{22}(x) = 3.5x + 1.5$. Then (8) and (10) hold. Let $G$ be a complete graph with two nodes and define a coupling by $A(x, y) = (a_{11}x + a_{12}y, a_{21}x + a_{22}y)$. Then, $(G, \{T_k\}, A)$ is of type II with the linear coupling model $A_c = A$.

**Example 6.** Define local dynamics by

$$T_1(x) = \begin{cases} 3.5x - 1, & \text{if } x \leq 5/2, \\ 2x - 7, & \text{if } x > 5/2, \end{cases} \text{ and } T_2(x) = \begin{cases} 2x + 1, & \text{if } x \leq 7/2, \\ 3.5x - 14, & \text{if } x > 7/2. \end{cases}$$

Then $T_1$ has covering relations determined by $W_1 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ on the $h$-set tuple $(M_{11} = [0, 2], M_{12} = [3, 5])$, with $c_{M_{11}}(x) = x - 1, c_{M_{12}}(x) = x - 4$ and $u = 1, s = 0$, which is unified by $N_1 = [0, 5]$ with $\hat{c}_{N_1}(x) = x - 1$ and unstable centers at $p_{11}^u = 0$ and $p_{12}^u = 3$. Also, $T_2$ has covering relations determined by $W_2 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ on the $h$-set tuple $(M_{21} = [1, 3], M_{22} = [4, 6])$, with $c_{M_{21}}(x) = x - 2, c_{M_{22}}(x) = x - 5$ and $u = 1, s = 0$, which is unified by $N_2 = [1, 6]$ with $\hat{c}_{N_2}(x) = x - 2$ and unstable centers at $p_{21}^u = 0$ and $p_{22}^u = 3$. For $-1 \leq x \leq 1$, let $U_{11}(x) = 3.5x + 1.5, U_{12}(x) = 2x, U_{21}(x) = 2x + 3,$ and $U_{22}(x) = 3.5x + 1.5$. Then (8) and (10) hold. Let $G$ be a complete graph with two nodes and define a coupling by $A(x, y) = (a_{11}(x - 1) + a_{12}(y - 2) + 1, a_{21}(x - 1) + a_{22}(y - 2) + 2)$. Then, $(G, \{T_k\}, A)$ is of type II with the linear coupling model $A_c(x, y) = (a_{11}x + a_{12}y, a_{21}x + a_{22}y)$. 


Now, we state our result on covering relations and topological entropy of perturbed coupled map networks.

**Theorem 2.** Let \((G, \{T_k\}, A)\) be a coupled map network of type II with the linear coupling model \(A\) as in \([7]\). Suppose that for each nonzero entry \(\prod_{k=1}^{d} w_{ki}j_k\) of the Kronecker product \(\bigotimes_{k=1}^{d} W_k\), there exists a permutation \(\tau\) on \(\{1, \ldots, d\}\) such that for \(1 \leq k \leq d\), \(p^u_{kjk} \in a_{k\tau(k)}U_{\tau(k)\tau^r(k)}(B^u)\), and

\[
\|a_{k\tau(k)}U_{\tau(k)\tau^r(k)}-p^u_{kjk}\|_{\min} - \sum_{l=1, l \neq \tau(k)}^{d} \|a_{kl}U_{li}\|_{\max} > 1, \quad \text{and} \quad \sum_{l=1}^{d} \|a_{kl}V_{li} - p^u_{kjk}\|_{\max} < r_{kjk}. \quad (12)
\]

Then any coupled map network \((\tilde{G}, \{\tilde{T}_k\}, \tilde{A})\) sufficiently close to \((G, \{T_k\}, A)\) has covering relations determined by \(\bigotimes_{k=1}^{d} W_k\) and has topological entropy bounded below by \(\log(\prod_{k=1}^{d} \rho(W_k))\).

**Remark 7.** Let \(\{A^\varepsilon\}\) and \(\{T^\varepsilon\}\) be one-parameter families of maps on \(\mathbb{R}^{(u+s)d}\), where \(\varepsilon \in \mathbb{R}\) is a parameter, such that \(A^0 = A, T^0 = \prod_{k \in B} T_k\), and \(A^\varepsilon(z)\) and \(T^\varepsilon(z)\) are both continuous jointly in \(\varepsilon\) and \(z\), then Theorem 2 holds for \(A^\varepsilon \circ T^\varepsilon\) if \(\varepsilon\) is sufficiently small.

## 3 Proofs of Theorems 1 and 2

First, we list some known results \([8, 13]\) which will be needed in the proofs. The following one ensures persistence of covering relations for \(C^0\) perturbations.

**Proposition 8.** \([8]\) Proposition 14] Let \(M\) and \(N\) be \(h\)-sets in \(\mathbb{R}^m\) with \(u(M) = u(N) = u\) and \(s(M) = s(N) = s\) and let \(f, g : M \to \mathbb{R}^m\) be continuous. Assume that \(M \xrightarrow{f} N\). Then there exists \(\delta > 0\) such that if \(|f(x) - g(x)| < \delta\) for all \(x \in M\) then \(M \xrightarrow{g} N\).

The next statement says that a closed loop of covering relations implies existence of a periodic point.

**Proposition 9.** \([13]\) Theorem 9] Let \(\{f_i\}_{i=1}^{k}\) be a collection of continuous maps on \(\mathbb{R}^m\) and \(\{M_i\}_{i=1}^{k}\) be a collection of \(h\)-sets in \(\mathbb{R}^m\) such that \(M_{k+1} = M_1\) and \(M_i \xrightarrow{f_i} M_{i+1}\) for \(1 \leq i \leq k\). Then there exists a point \(x \in \text{int}(M_1)\) such that

\[
f_i \circ f_{i-1} \circ \cdots \circ f_1(x) \in \text{int}(M_{i+1}) \text{ for } i = 1, \ldots, k, \text{ and } \quad f_k \circ f_{k-1} \circ \cdots \circ f_1(x) = x.
\]

It is known that a continuous map having covering relations determined by a transition matrix is topologically semi-conjugate to a one-sided subshift of finite type.

**Proposition 10.** \([8]\) Proposition 15] Let \(f : \mathbb{R}^m \to \mathbb{R}^m\) be a continuous map which has covering relations determined by a transition matrix \(W\). Then there exists a compact subset \(\Lambda\) of \(\mathbb{R}^m\) such that \(\Lambda\) is maximal positively invariant for \(f\) in the union of the \(h\)-sets (with respect to \(W\)) and \(f|\Lambda\) is topologically semi-conjugate to \(\sigma_W\).

Finally, we summarize basic properties of the local Brouwer degree; refer to \([11]\) Chapter III] for the proof.

**Proposition 11.** Let \(S\) be an open and bounded subset of \(\mathbb{R}^m\) with \(m \geq 1\), and let \(\varphi : \tilde{S} \to \mathbb{R}^m\) be continuous and \(q \in \mathbb{R}^m\) such that \(q \notin \varphi(\partial S)\). Then the following holds:

1. If \(\varphi\) is \(C^1\) and for each \(x \in \varphi^{-1}(q) \cap S\) the Jacobian matrix of \(\varphi\) at \(x\), denoted by \(D\varphi_x\), is nonsingular, then

\[
\deg(\varphi, S, q) = \sum_{x \in \varphi^{-1}(q) \cap S} \text{sgn}(|\det D\varphi_x|),
\]

where \(\text{sgn}\) is the sign function.
2. Let $\psi : \mathbb{R}^m \to \mathbb{R}^m$ be a $C^1$ map and $p \in \mathbb{R}^m$ such that $\psi^{-1}(p)$ consists of a single point and lies in a bounded connected component $\Delta$ of $\mathbb{R}^m \setminus \varphi(DS)$, and $D\psi|_{\psi^{-1}(p)}$ is nonsingular. Then
\[
\deg(\psi \circ \varphi, S, p) = \text{sgn}(\det D\psi|_{\psi^{-1}(p)}) \deg(\varphi, S, v),
\]
for any $v \in \Delta$.

3. Let $S'$ be an open and bounded subset of $\mathbb{R}^n$ with $n \geq 1$, and let $\psi : \tilde{S'} \to \mathbb{R}^n$ be continuous and $q' \in \mathbb{R}^n$ such that $q' \notin \psi(DS')$. Define a map $(\varphi, \psi) : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m \times \mathbb{R}^n$ by $(\varphi, \psi)(x, y) = (\varphi(x), \psi(y))$ for $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$. Then
\[
\deg((\varphi, \psi), S \times S', (q, q')) = \deg(\varphi, S, q) \deg(\psi, S', q').
\]

Now, we are in a position to prove our main results.

**Proof of Theorem** Let $\prod_{k=1}^d w_{k_{ik},j_k}$ be a nonzero entry of $\bigotimes_{k=1}^d W_k$. We shall prove that the following covering relation holds
\[
\prod_{k=1}^d M_{k_{ik},j_k} \xrightarrow{A \circ T} \prod_{k=1}^d M_{k_{jk},j_k}.
\]

In the sequel, we use the following notations: $x_k \in \mathbb{R}^a$ and $y_k \in \mathbb{R}^r$ for $1 \leq k \leq d$, $x = \prod_{k=1}^d x_k \in (\mathbb{R}^a)^d = \mathbb{R}^{ad}$, $y = \prod_{k=1}^d y_k \in (\mathbb{R}^r)^d = \mathbb{R}^{rd}$. $\prod_{k=1}^d (x_k, y_k) \in (\mathbb{R}^{a+r})^d = \mathbb{R}^{(a+r)d}$, and $(x, y) \in \mathbb{R}^{(a+r)d}$.

First, we check conditions on h-sets. Let $M = \prod_{k=1}^d M_{k_{ik},j_k}$ and $N = \prod_{k=1}^d M_{k_{jk},j_k}$. Then $M$ and $N$ are h-sets, with constants $u(M) = u(N) = ud$ and $s(M) = s(N) = sd$, and homeomorphisms $c_M, c_N : \mathbb{R}^{(a+r)d} \to \mathbb{R}^{ud} \times \mathbb{R}^{rd}$, defined as follows
\[
c_M(\prod_{k=1}^d (x_k, y_k)) = c \circ \prod_{k=1}^d c_{M_{k_{ik},j_k}} (x_k, y_k), \quad \text{and} \quad c_N(\prod_{k=1}^d (x_k, y_k)) = c \circ \prod_{k=1}^d c_{M_{k_{jk},j_k}} (x_k, y_k),
\]
where $c : \mathbb{R}^{(a+r)d} \to \mathbb{R}^{(a+r)d}$ is defined by $c(\prod_{k=1}^d (x_k, y_k)) = (x, y)$.

Second, we construct a homotopy such that (15) holds. Define a homotopy $H : [0, 1] \times \mathbb{B}^{ad} \times \mathbb{B}^{rd} \to \mathbb{R}^{(a+r)d}$ as
\[
H(t, x, y) = (1 - t)c_N \circ A \circ T \circ \tilde{\varphi}_{-1}(x, y) + t \pi_1 \circ c_N \circ A \circ T \circ \tilde{c}_{-1}(x, y)
\]
where $\pi_1 : \mathbb{R}^{(a+r)d} \to \mathbb{R}^{(a+r)d}$ is defined by $\pi_1(x, y) = (x, 0)$ for all $x \in \mathbb{R}^{ad}$ and $y \in \mathbb{R}^{rd}$. Clearly, (15) holds.

Before checking (16) and (17) (see Appendix), we derive a new form for the homotopy. Define $\tilde{c}_M = \prod_{k=1}^d c_{M_{k_{ik},j_k}}$ and $\tilde{c}_N = \prod_{k=1}^d c_{M_{k_{jk},j_k}}$. Then $c_M = c \circ \tilde{c}_M$ and $c_N = c \circ \tilde{c}_N$. Let $(x, y) \in \mathbb{B}^{ad} \times \mathbb{B}^{rd} = M_c$. Then $\tilde{c}_N \circ T \circ \tilde{c}_M^{-1}(\prod_{k=1}^d (x_k, y_k)) \in \tilde{c}_N(T(M))$ and $\tilde{c}_N \circ T \circ \tilde{c}_M^{-1}(\prod_{k=1}^d (x_k, y_k)) = \prod_{k=1}^d (U_{k_{ik,j_k}}(x_k), V_{k_{ik,j_k}}(y_k))$. Moreover, by the definition of $A_c$, we obtain that
\[
c_N \circ A \circ T \circ \tilde{c}_M^{-1}(x, y) = c \circ \tilde{c}_N \circ A \circ T \circ \tilde{c}_M^{-1} \circ c \circ \tilde{c}_N \circ T \circ \tilde{c}_M^{-1}(\prod_{k=1}^d (x_k, y_k))
\]
\[
= \circ A_c \circ \tilde{c}_N \circ T \circ \tilde{c}_M^{-1}(\prod_{k=1}^d (x_k, y_k)) = \circ A_c \circ \prod_{k=1}^d (U_{k_{ij,j_k}}(x_k), V_{k_{ik,j_k}}(y_k))
\]
\[
= \circ (\prod_{k=1}^d a_k U_{k_{ik,j_k}}(x_k), V_{k_{ik,j_k}}(y_k)) = \circ \prod_{k=1}^d \left( \sum_{l=1}^d a_k U_{k_{ik,j_k}}(x_k), \sum_{l=1}^d a_k V_{k_{ik,j_k}}(y_k) \right)
\]
\[
= \prod_{k=1}^d \left( \sum_{l=1}^d a_k U_{k_{ik,j_k}}(x_k), \prod_{l=1}^d a_k V_{k_{ik,j_k}}(y_k) \right).
\]
Therefore, 
\[ H(t, x, y) = \left( \prod_{l=1}^{d} \sum_{k=1}^{d} a_{lk} U_{ki_{lk}j_{lk}}(x_k) \right) \left( \prod_{l=1}^{d} \left( (1 - t) \sum_{k=1}^{d} a_{lk} V_{ki_{lk}j_{lk}}(y_k) \right) \right). \]

To prove (16), consider \((x, y) \in M_\varepsilon^c\). Then there exists \(1 \leq \beta \leq d\) such that \(|x_\beta| = 1\). Since \(\tau\) is a permutation one can find \(\gamma, 1 \leq \gamma \leq d\) such that \(\tau(\gamma) = \beta\). By (5), we have that 
\[ ||a_{\gamma\beta} U_{\beta i_{\beta} j_{\beta}}||_{\min} - \sum_{k=1, k \neq \beta}^{d} ||a_{\gamma k} U_{ki_{\gamma} j_{\gamma}}||_{\max} > 1. \]

Hence, 
\[ \left| \sum_{k=1}^{d} a_{\gamma k} U_{ki_{\gamma} j_{\gamma}}(x_k) \right| \geq ||a_{\gamma \beta} U_{\beta i_{\beta} j_{\beta}}(x)\beta|| - \sum_{k=1, k \neq \beta}^{d} ||a_{\gamma k} U_{ki_{\gamma} j_{\gamma}}(x_k)|| \geq ||a_{\gamma \beta} U_{\beta i_{\beta} j_{\beta}}||_{\min} - \sum_{k=1, k \neq \beta}^{d} ||a_{\gamma k} U_{ki_{\gamma} j_{\gamma}}||_{\max} > 1. \]  
(13)
It implies that \(H(t, x, y) \notin N_\varepsilon\) and thus (16) holds. 

For checking (17), consider \((x, y) \in M_\varepsilon^c\) Then we get that 
\[ |(1 - t) \sum_{k=1}^{d} a_{lk} V_{ki_{lk}j_{lk}}(y_k)| \leq |\sum_{k=1}^{d} a_{lk} V_{ki_{lk}j_{lk}}(y_k)| \leq \sum_{k=1}^{d} ||a_{lk} V_{ki_{lk}j_{lk}}||_{\max} < 1, \]
where the last inequality follows from (5). Therefore \(H(t, x, y) \notin N_\varepsilon\) and hence (17) is true.

Next, we check the item 2 in Definition 4. Consider a map \(\varphi: \mathbb{R}^{ud} \rightarrow \mathbb{R}^{ud}\), where 
\[ \varphi(x) = \prod_{l=1}^{d} \sum_{k=1}^{d} a_{lk} U_{ki_{lk}j_{lk}}(x_k). \]
Then \(H(1, x, y) = (\prod_{l=1}^{d} \sum_{k=1}^{d} a_{lk} U_{ki_{lk}j_{lk}}(x_k)), 0) = (\varphi(x), 0)\). By (13), we have \(\varphi(\partial B^{ud}) \subset \mathbb{R}^{ud} \setminus B^{ud}\).

Finally, we show that the local Brouwer degree \(\deg(\varphi, B^{ud}, 0)\) is nonzero. Observe that we can rewrite 
\[ \varphi(x) = ([a_{lk}] \otimes I_u) \circ \prod_{k=1}^{d} U_{ki_{lk}j_{lk}}, \]
where \(I_u\) is the \(u \times u\) identity matrix. Since the matrix \([a_{lk}]\) is invertible, \([a_{lk}] \otimes I_u\) is also invertible. Since \(||U_{ki_{lk}j_{lk}}||_{\min} > 1\) and \(\deg(U_{ki_{lk}j_{lk}}, B^u, 0) \neq 0\) for all \(1 \leq k \leq d\), we have \(0 \in U_{ki_{lk}j_{lk}}(B^u)\), and hence \(0 = ([a_{lk}] \otimes I_u)^{-1}(0)\) lies in a bounded connected component of \(\mathbb{R}^{ud} \setminus (\prod_{k=1}^{d} U_{ki_{lk}j_{lk}})(\partial B^{ud})\).

By Proposition 11 we obtain that 
\[ \deg(\varphi, B^{ud}, 0) = \text{sgn}(\det([a_{lk}] \otimes I_u)) \prod_{k=1}^{d} \deg(U_{ki_{lk}j_{lk}}, B^u, 0) \]
\[ = \text{sgn}(\det([a_{lk}] \otimes I_u)) \prod_{k=1}^{d} \deg(U_{ki_{lk}j_{lk}}, B^u, 0) \neq 0. \]
We have proved that the needed covering relation holds. If \( \tilde{T}_k \) and \( \tilde{A} \) are both \( C^0 \) close enough to \( T_k \) and \( A \) respectively. Then by Proposition 9, the following covering relation holds, for all nonzero entries \( \prod_{k=1}^{d} w_{k_{i_k}j_k} \) of \( \bigotimes_{k=1}^{d} W_k \),

\[
\prod_{k=1}^{d} M_{k_{i_k}j_k} \xrightarrow{\tilde{A} \circ \tilde{T}} \prod_{k=1}^{d} M_{k_{i_k}j_k}.
\]

Therefore, \( (\tilde{G}, \{\tilde{T}_k\}, \tilde{A}) \) has covering relations determined by \( \bigotimes_{k=1}^{d} W_k \). Since each \( W_k \) is a permutation, there exists a closed loop of covering relations for \( \tilde{A} \circ \tilde{T} \) with loop length \( 1 \varepsilon k(\dim W_1, \ldots, \dim W_d) \). By Proposition 9, \( \tilde{A} \circ \tilde{T} \) has a periodic point of period \( 1 \varepsilon \varepsilon M(\dim W_1, \ldots, \dim W_d) \). 

Next, we prove the second main result.

**Proof of Theorem** Let \( \prod_{k=1}^{d} w_{k_{i_k}j_k} \) be a nonzero entry of \( \bigotimes_{k=1}^{d} W_k \). We shall prove that the following covering relation holds

\[
\prod_{k=1}^{d} M_{k_{i_k}j_k} \xrightarrow{\tilde{A} \circ \tilde{T}} \prod_{k=1}^{d} M_{k_{i_k}j_k}.
\]

We shall keep the use of the following notations: \( x_k \in \mathbb{R}^u \) and \( y_k \in \mathbb{R}^s \) for \( 1 \leq k \leq d \), \( x = \prod_{k=1}^{d} x_k \in (\mathbb{R}^u)^d = \mathbb{R}^{ud} \), \( y = \prod_{k=1}^{d} y_k \in (\mathbb{R}^s)^d = \mathbb{R}^{sd} \), \( \prod_{k=1}^{d} (x_k, y_k) \in (\mathbb{R}^{u+s})^d = \mathbb{R}^{(u+s)d} \), and \( (x, y) \in \mathbb{R}^{ud} \times \mathbb{R}^{sd} = \mathbb{R}^{(u+s)d} \).

First, we check conditions on h-sets. For convenience, we denote \( M = \prod_{k=1}^{d} M_{k_{i_k}} \) and \( M' = \prod_{k=1}^{d} M_{k_{j_k}} \). Then \( M \) and \( M' \) are h-sets, together with constants \( u(M) = u(M') = ud \) and \( s(M) = s(M') = sd \), and homeomorphisms \( c_M, c_{M'} : \mathbb{R}^{(u+s)d} \to \mathbb{R}^{ud} \times \mathbb{R}^{sd} \), defined as follows, for all \( x_k \in \mathbb{R}^u \) and \( y_k \in \mathbb{R}^s \), \( 1 \leq k \leq d \),

\[
c_M(\prod_{k=1}^{d} x_k, y_k) = \varsigma \circ \prod_{k=1}^{d} \tilde{c}_{M_{i_k}}(x_k, y_k), \quad \text{and} \quad c_{M'}(\prod_{k=1}^{d} x_k, y_k) = \varsigma \circ \prod_{k=1}^{d} \tilde{c}_{M_{j_k}}(x_k, y_k),
\]

where \( \varsigma : \mathbb{R}^{(u+s)d} \to \mathbb{R}^{(u+s)d} \) is defined by \( \varsigma(\prod_{k=1}^{d} x_k, y_k) = (x, y) \).

Second, we construct a homotopy such that equations \( 16 \) and \( 17 \) holds. Define a homotopy \( H : [0, 1] \times \overline{B}^{ud} \times \overline{B}^{sd} \to \mathbb{R}^{(u+s)d} \) by, if \( 0 \leq t \leq 1/2 \), then

\[
H(t, x, y) = (1 - 2t)c_{M'} \circ A \circ T \circ c_{M}^{-1}(x, y) + 2t\pi_1 \circ c_{M'} \circ A \circ T \circ c_{M}^{-1}(x, y),
\]

and if \( 1/2 < t \leq 1 \), then

\[
H(t, x, y) = (2 - 2t)\pi_1 \circ c_{M'} \circ A \circ T \circ c_{M}^{-1}(x, y) + (2t - 1)\left( \prod_{k=1}^{d} (a_{k_{\tau(k)}} U_{\tau(k)j_{\tau(i_k)}} (x_{\tau(k)}) - p^u_{j_{k_k}}), 0 \right),
\]

where \( \pi_1 : \mathbb{R}^{(u+s)d} \to \mathbb{R}^{(u+s)d} \) is defined by \( \pi_1(x, y) = (x, 0) \) for all \( x \in \mathbb{R}^{ud} \) and \( y \in \mathbb{R}^{sd} \). Clearly, \( 16 \) holds.

Before checking \( 16 \) and \( 17 \), we derive a new form for the homotopy. Define \( \tilde{c}_N = (\prod_{k=1}^{d} \tilde{c}_{N_{i_k}}) \). Let \((x, y) \in \overline{B}^{ud} \times \overline{B}^{sd} = M_e \). Then by the definitions of \( \tilde{c}_{N_{i_k}} \), we get that

\[
\tilde{c}_N \circ T \circ \prod_{k=1}^{d} \tilde{c}_{N_{i_k}}^{-1}(x_k, y_k) = \prod_{k=1}^{d} (U_{k_{i_k}}(x_k), V_{k_{i_k}}(y_k)) \in c_N(T(M)).
\]
Thus, 
\[ c_{M'} \circ A \circ T \circ c_{M}^{-1}(x, y) = \varsigma \circ \left( \prod_{l=1}^{d} \hat{c}_{N(l)} \right) \circ A \circ T \circ \left( \prod_{k=1}^{d} \tilde{c}_{M(k)} \right) \circ \varsigma^{-1}(x, y) \]
\[ = \varsigma \circ \left( \prod_{l=1}^{d} \hat{c}_{N(l)} \right) \circ A \circ \tilde{c}_{N}^{-1} \circ \tilde{c}_{N} \circ T \circ \left( \prod_{k=1}^{d} \tilde{c}_{M(k)} \right) \circ (x, y) \]
\[ = \varsigma \circ \left( \prod_{l=1}^{d} g_{lji} \right) \circ A \circ \tilde{c}_{N}^{-1} \circ \left( U_{ki}(x), V_{ki}(y) \right) \]
\[ = \varsigma \circ \left( \prod_{l=1}^{d} g_{lji} \right) \circ \tilde{c}_{N} \circ A \circ \tilde{c}_{N}^{-1} \circ \left( U_{ki}(x), V_{ki}(y) \right). \]

Moreover, by the definitions of \( A_c \) and \( g_{lji} \), we obtain that 
\[ c_{M'} \circ A \circ T \circ c_{M}^{-1}(x, y) = \varsigma \circ \left( \prod_{l=1}^{d} g_{lji} \right) \circ A \circ \left( \prod_{k=1}^{d} U_{ki}(x), V_{ki}(y) \right) \]
\[ = \varsigma \circ \left( \prod_{l=1}^{d} g_{lji} \right) \circ \left( \prod_{k=1}^{d} \left( U_{ki}(x), V_{ki}(y) \right) \right) \]
\[ = \varsigma \circ \left( \prod_{l=1}^{d} \sum_{k=1}^{d} a_{lk} U_{ki}(x) - p_{lj}^{u} \right) - \frac{\sum_{k=1}^{d} a_{lk} V_{ki}(y) - p_{lj}^{v}}{r_{lj}} \right) \]
\[ \left( \prod_{l=1}^{d} \sum_{k=1}^{d} a_{lk} U_{ki}(x) - p_{lj}^{u} \right), \prod_{l=1}^{d} \left( \sum_{k=1}^{d} a_{lk} V_{ki}(y) - p_{lj}^{v} \right)) \right). \]

Therefore, for \( 0 \leq t \leq 1/2 \),
\[ H(t, x, y) = \prod_{l=1}^{d} \sum_{k=1}^{d} a_{lk} U_{ki}(x) - p_{lj}^{u}, (1 - 2t) \prod_{l=1}^{d} \left( \sum_{k=1}^{d} a_{lk} V_{ki}(y) - p_{lj}^{v} \right)), \]
and, for \( 1/2 < t \leq 1 \),
\[ H(t, x, y) = \prod_{l=1}^{d} \left( a_{l\tau(l)} U_{\tau(l)\tau^{-1}}(x) - p_{lj}^{u} \right) + (2 - 2t) \sum_{k=1, k \neq \tau(l)}^{d} a_{lk} U_{ki}(x), 0) \right). \]

For checking (10), consider \( (x, y) \in M_{c}^{-} \). Then there exists \( 1 \leq \beta \leq d \) such that \( |x_{\beta}| = 1 \). Since \( \tau \) is a permutation, there exists \( 1 \leq \gamma \leq d \) such that \( \tau(\gamma) = \beta \). By (12), we have that 
\[ ||a_{\gamma\beta}U_{\beta^i} - p_{\gamma j}^{w}||_{min} - \sum_{k=1, k \neq \beta}^{d} ||a_{\gamma k} U_{ki}||_{max} > 1. \]

It implies that 
\[ \left| \sum_{k=1}^{d} a_{\gamma k} U_{ki}(x) - p_{\gamma j}^{w} \right| \geq \left| a_{\gamma\beta}U_{\beta^i}(x) - p_{\gamma j}^{w} \right| - \sum_{k=1, k \neq \beta}^{d} \left| a_{\gamma k} U_{ki}(x) \right| \]
\[ \geq ||a_{\gamma\beta}U_{\beta^i} - p_{\gamma j}^{w}||_{min} - \sum_{k=1, k \neq \beta}^{d} ||a_{\gamma k} U_{ki}||_{max} > 1. \] (14)
and, for $1/2 < t \leq 1$,

$$|a_{\gamma \beta}U_{\beta i}(x_{\beta}) - p^u_{\gamma j} + (2 - 2t) \sum_{k=1, k \neq \beta}^d a_{\gamma k}U_{ki}(x_k)|$$

$$\geq |a_{\gamma \beta}U_{\beta i}(x_{\beta}) - p^u_{\gamma j}| - (2 - 2t) \sum_{k=1, k \neq \beta}^d |a_{\gamma k}U_{ki}(x_k)|$$

$$\geq \|a_{\gamma \beta}U_{\beta i} - p^u_{\gamma j}\|_{\min} - \sum_{k=1, k \neq \beta}^d \|a_{\gamma k}U_{ki}\|_{\max}$$

$$> 1.$$

Thus $H(t, x, y) \notin M$, and hence (10) holds.

For checking (17), consider $(x, y) \in M_c$. Then, for $0 \leq t \leq 1/2$, we have that

$$|(1 - 2t) \sum_{k=1}^d \frac{a_{lk}V_{ki}(y_k) - p^s_{lj}}{r_{lj}}| \leq \frac{1}{r_{lj}} \sum_{k=1}^d a_{lk}V_{ki}(y_k) - p^s_{lj}$$

$$\leq \frac{1}{r_{lj}} \sum_{k=1}^d \|a_{lk}V_{ki} - p^s_{lj}\|_{\max} < 1.$$

where the last inequality follows from (11). Thus, $H(t, x, y) \notin N^+_c$ and hence (17) holds.

Next, we check item 2 in Definition 13. Define a map $\varphi : \mathbb{R}^d \to \mathbb{R}^q$ by,

$$\varphi(x) = \prod_{l=1}^d (a_{l\tau(l)}U_{\tau(l)\tau(l)}(x_{\tau(l)}) - p^u_{lj}).$$

Then $H(1, x, y) = (\prod_{l=1}^d (a_{l\tau(l)}U_{\tau(l)\tau(l)}(x_{\tau(l)}) - p^u_{lj}), 0) = (\varphi(x), 0)$. By equation (14), we have $\varphi(\partial B^{u^d}) \subset \mathbb{R}^q \setminus \overline{B^{u^d}}$.

Finally, we prove that the local Brouwer degree $\deg(\varphi, B^{u^d}, 0)$ is nonzero. Define a function $g(x) = \prod_{l=1}^d (x_l - p^u_{lj})$ for $x \in \mathbb{R}^d$, and a $d \times d$ matrix $[b_{lk}]$ such that $b_{l\tau(l)} = a_{l\tau(l)}$ and $b_{lk} = 0$ for all $k \neq \tau(l)$. Then

$$\varphi(x) = g \circ ([b_{lk}] \otimes I_u) \circ \prod_{k=1}^d U_{ki}(x_k),$$

where $I_u$ is the $u \times u$ identity matrix. In order to apply Proposition 11 for $\deg(\varphi, B^{u^d}, 0)$, we need to check conditions on the affine maps $g$ and $[b_{lk}] \otimes I_u$. By the definition of $g$, we get that $g^{-1}(0)$ consists of a single point $p \equiv \prod_{l=1}^d p^u_{lj}$. By the definition of $[b_{lk}]$ and $b_{l\tau(l)} = a_{l\tau(l)}$, the hypothesis $p^u_{lj} \in a_{l\tau(l)}U_{\tau(l)\tau(l)}(B^u)$, together with 12 implies that the matrix $[b_{lk}]$ is invertible; otherwise, $|a_{l\tau(l)}U_{\tau(l)\tau(l)} - p^u_{lj}|_{\min} = 0$ leads a contradiction. Hence, $([b_{lk}] \otimes I_u)^{-1}(p)$ lies in a bounded connected component of $\mathbb{R}^d \setminus (\prod_{l=1}^d U_{ki}(\partial B^{u^d})$. Since $\deg(U_{li}, B_u, p^u_{lj}) \neq 0$, we have $p^u_{lj} \in U_{li}(B^u)$ and hence $p \in \prod_{l=1}^d U_{li}(B^u)$. By applying Proposition 11 since $\deg(U_{li}, B_u, p^u_{lj}) \neq 0$ for all $1 \leq l \leq d$, we obtain that

$$\deg(\varphi, B^{u^d}, 0) = \text{sgn}(\det(Dg_p))\text{sgn}(\det([b_{lk}] \otimes I_u)) \deg(\prod_{l=1}^d U_{li}, B^{u^d}, p)$$

$$= \text{sgn}(\det(Dg_p))\text{sgn}(\det([b_{lk}] \otimes I_u)) \prod_{l=1}^d \deg(U_{li}, B^u, p^u_{lj})$$

$$\neq 0.$$
This concludes the proof of the needed covering relation. If $\tilde{T}_k$ and $\tilde{A}$ are both $C^0$ close enough to $T_k$ and $A$ respectively, then by Proposition we have the following covering relation holds, for all nonzero entries $\prod_{l=1}^d w_{li(j)}$ of $W = \bigotimes_{l=1}^d W_l$,

$$\prod_{l=1}^d M_{li} \overset{\tilde{A} \circ \tilde{T}}{\longrightarrow} \prod_{l=1}^d M_{ij_l}.$$ 

Therefore, $(\tilde{G}, \{\tilde{T}_k\}, \tilde{A})$ has covering relations determined by $W$. By Proposition there exists a compact subset $\Lambda$ of $\mathbb{R}^{(w+s)d}$ such that $\Lambda$ is a maximal positively invariant for $A \circ \tilde{T}$ in the union of the h-sets (with respect to $W$) and $A \circ \tilde{T}|\Lambda$ is topologically semi-conjugate to $\sigma^+_W$. Therefore,

$$h_{\text{top}}(A \circ \tilde{T}) \geq h_{\text{top}}(\sigma^+_W) = \log(\rho(W) = \log(\prod_{l=1}^d \rho(W_l)).$$

\[\square\]

**Appendix**

First, we briefly recall some definitions from concerning covering relations.

**Definition 12.** [13, Definition 6] An h-set in $\mathbb{R}^m$ is a quadruple consisting of the following data:

- a nonempty compact subset $M$ of $\mathbb{R}^m$,
- a pair of numbers $u(M), s(M) \in \{0, 1, \ldots, m\}$ with $u(M) + s(M) = m$,
- a homeomorphism $c_M : \mathbb{R}^m \rightarrow \mathbb{R}^{u(M)} \times \mathbb{R}^{s(M)}$ with $c_M(M) = \overline{B^u(M)} \times \overline{B^s(M)}$, where $S \times T$ is the Cartesian product of sets $S$ and $T$.

For simplicity, we will denote such an h-set by $M$ and call $c_M$ the coordinate chart of $M$; furthermore, we use the following notations:

$$M_c = \overline{B^u(M)} \times \overline{B^s(M)}, \quad M_c^- = \partial B^u(M) \times \overline{B^s(M)}, \quad M_c^+ = \overline{B^u(M)} \times \partial B^s(M),$$

$$M^- = c_M^{-1}(M_c^-), \text{ and } M^+ = c_M^{-1}(M_c^+).$$

A covering relation between two h-sets is defined as follows.

**Definition 13.** [13, Definition 7] Let $M, N$ be h-sets in $\mathbb{R}^m$ with $u(M) = u(N) = u$ and $s(M) = s(N) = s$, $f : M \rightarrow \mathbb{R}^m$ be a continuous map, and $f_c = c_N \circ f \circ c_M^{-1} : M_c \rightarrow \mathbb{R}^u \times \mathbb{R}^s$. We say $M$ $f$-covers $N$, and write

$$M \overset{f}{\Rightarrow} N,$$

if the following conditions are satisfied:

1. there exists a homotopy $h : [0, 1] \times M_c \rightarrow \mathbb{R}^u \times \mathbb{R}^s$ such that
   
   $$h(0, x) = f_c(x) \text{ for } x \in M_c,$$
   
   $$h([0, 1], M_c) \cap N_c = \emptyset,$$
   
   $$h([0, 1], M_c) \cap N_c^+ = \emptyset;$$

2. there exists a map $\varphi : \mathbb{R}^u \rightarrow \mathbb{R}^u$ such that
   
   $$h(1, p, q) = (\varphi(p), 0) \text{ for any } p \in \overline{B^u} \text{ and } q \in \overline{B^s},$$
   
   $$\varphi(\partial B^u) \subset \mathbb{R}^u \setminus \overline{B^u}; \text{ and}$$
3. There exists a nonzero integer \( w \) such that the local Brouwer degree \( \deg(\varphi, B^u, 0) \) of \( \varphi \) at 0 in \( B^u \) is \( w \); refer to [13, Appendix] for its properties.

A transition matrix is a square matrix which satisfies the following conditions:

(i) all entries are either zero or one,

(ii) all row sums and column sums are greater than or equal to one.

For a transition matrix \( W \), let \( \rho(W) \) denote the spectral radius of \( W \). Then \( \rho(W) \geq 1 \) and moreover, if \( W \) is irreducible and not a permutation, then \( \rho(W) > 1 \). Let \( \Sigma_W^+ \) (resp. \( \Sigma_W^- \)) be the space of all allowable one-sided (resp. two sided) sequences generated by the transition matrix \( W \) with a usual metric, and let \( \sigma_W^+ : \Sigma_W^+ \to \Sigma_W^+ \) (resp. \( \sigma_W^- : \Sigma_W^- \to \Sigma_W^- \)) be the one-sided (resp. two sided) subshift of finite type for \( W \). Then \( h_{\text{top}}(\sigma_W^+) = h_{\text{top}}(\sigma_W^-) = \log(\rho(W)) \) (Refer to [10] for more background).

**Definition 14.** Let \( W = [w_{ij}]_{1 \leq i,j \leq \gamma} \) be a transition matrix and \( f \) be a continuous map on \( \mathbb{R}^m \). We say that \( f \) has covering relations determined by \( W \) if the following conditions are satisfied:

1. there are \( \gamma \) pairwisely disjoint \( h \)-sets \( \{M_i\}_{i=1}^\gamma \) in \( \mathbb{R}^m \);

2. if \( w_{ij} = 1 \) then the covering relation \( M_i \overset{f}{\longrightarrow} M_j \) holds;

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