Contradiction-Tolerant Process Algebra with Propositional Signals

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Abstract. In a previous paper, an ACP-style process algebra was proposed in which propositions are used as the visible part of the state of processes and as state conditions under which processes may proceed. This process algebra, called ACPps, is built on classical propositional logic. In this paper, we present a version of ACPps built on a paraconsistent propositional logic which is essentially the same as CLuNs. There are many systems that would have to deal with self-contradictory states if no special measures were taken. For a number of these systems, it is conceivable that accepting self-contradictory states and dealing with them in a way based on a paraconsistent logic is an alternative to taking special measures. The presented version of ACPps can be suited for the description and analysis of systems that deal with self-contradictory states in a way based on the above-mentioned paraconsistent logic.

Keywords: process algebra, propositional signal, propositional condition, paraconsistent logic.

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1 Introduction

Algebraic theories of processes such as ACP [11], CCS [27], and CSP [25], as well as most algebraic theories of processes in the style of these ones, are concerned with the behaviour of processes only. That is, the state of processes is kept invisible. In [8], an ACP-style process algebra, called ACPps, was proposed in which processes have their state to some extent visible. The visible part of the state of a process, called the signal emitted by the process, is a proposition of classical propositional logic. Propositions are not only used as signals emitted by processes, but also as conditions under which processes may proceed. The intuition is that the signal emitted by a process is a proposition that holds at its start and the condition under which processes may proceed is a proposition that must hold at its start. Thus, by the introduction of signal emitting processes, an answer is given to the question what determines whether a condition under which a process may proceed is met.

If the signals emitted by two processes are contradictory, then the signal emitted by the parallel composition of these processes is self-contradictory. For
example, if the signals emitted by the two processes, being propositions, are each others negation, then they are contradictory and their conjunction, which is the signal emitted by the parallel composition of these processes, is self-contradictory. Intuitively, a process emitting a self-contradictory signal is an impossibility. Therefore, a special process has been introduced in ACPps to deal with it. In practice, there are many systems that would have to deal with self-contradictory states if no special prevention measures or special detection and resolution measures were taken. Some typical examples are web-service-oriented applications and autonomous robotic agents (see e.g. [23, 29, 30]). At least for a number of these systems, it is conceivable that accepting self-contradictory states and dealing with them in a way based on a suitable paraconsistent logic is an alternative to taking special measures. It may even be the only workable alternative because a system may have to cope with inconsistencies occurring on a large scale.

What exactly does it mean to deal with self-contradictory states in a way based on a paraconsistent logic? The systems referred to above are systems whose behaviour is made up of discrete steps where, upon each step performed, the way in which the behaviour proceeds is conditional on the current state of the system concerned. If the propositions by which the visible part of the possible states of a system can be characterized are used as conditions, then it can be established in accordance with a paraconsistent propositional logic whether a condition is met in a state. This is what is meant by dealing with self-contradictory states in a way based on a paraconsistent logic. We think that a version of the process algebra ACPps that is built on an appropriate paraconsistent propositional logic instead of classical propositional logic can be suited for the description and analysis of systems that deal with self-contradictory states in a way based on a paraconsistent logic. The important point here is that, in such a logic, it is generally not possible to deduce an arbitrary formula from two contradictory formulas.

The question remains: what is an appropriate paraconsistent propositional logic? The ones that have been proposed differ in many ways and whether one of them is more appropriate than another is fairly difficult to make out. A paraconsistent propositional logic is a logic that does not have the property that every proposition is a logical consequence of every set of hypotheses that contains contradictory propositions. A paraconsistent propositional logic with the property that every proposition is a logical consequence of every set of hypotheses that contains contradictory propositions but one is far from appropriate. Such a logic is a minimal paraconsistent logic. Maximal paraconsistency, i.e. a logical consequence relation that cannot be extended without loosing paraconsistency, is generally considered an important property. There are various other properties that have been proposed as characteristic of reasonable paraconsistent propositional logics, but their importance remains to some extent open to question.

The properties that have been proposed as characteristic of reasonable paraconsistent propositional logics do not include all properties that are required of an appropriate one to build a version of ACPps on. These properties include,
among other things, properties needed to retain the basic axioms of ACP-style process algebras. In this paper, we present a version of ACP\textsubscript{ps} built on the paraconsistent propositional logic for which the name LP\textsuperscript{⊃,F} was coined in [26]. This logic, which is essentially the same as J3 [20], CLuNs [12], and LF1I [19], has virtually all properties that have been proposed as characteristic of reasonable paraconsistent propositional logics as well as all properties that are required of an appropriate one to build a version of ACP\textsubscript{ps} on. LP\textsuperscript{⊃,F} can be replaced by any paraconsistent propositional logics with the latter properties, but among the paraconsistent propositional logics with the former properties, LP\textsuperscript{⊃,F} is the only one with the latter properties.

The structure of this paper is as follows. First, we give a survey of the paraconsistent propositional logic LP\textsuperscript{⊃,F} (Section 2). Next, we present BPA\textsubscript{ct}\textsubscript{ps}, the subtheory of the version of ACP\textsubscript{ps} built on LP\textsuperscript{⊃,F} that does not support parallelism and communication (Sections 3 and 4). After that, we present ACP\textsubscript{ct}\textsubscript{ps}, the version of ACP\textsubscript{ps} built on LP\textsuperscript{⊃,F}, as an extension of BPA\textsubscript{ct}\textsubscript{ps} (Sections 5 and 6). Following this, we introduce a useful additional feature, namely a generalization of the state operators from [6] (Section 7). Then, we treat the addition of guarded recursion to ACP\textsubscript{ct}\textsubscript{ps} (Section 8). Finally, we make some concluding remarks (Section 9).

## 2 The Paraconsistent Logic LP\textsuperscript{⊃,F}

A set of propositions \( \Gamma \) is contradictory if there exists a proposition \( A \) such that both \( A \) and \( \neg A \) can be deduced from \( \Gamma \). A proposition \( A \) is called self-contradictory if \( \{ A \} \) is contradictory. In classical propositional logic, every proposition can be deduced from a contradictory set of propositions. A paraconsistent propositional logic is a propositional logic in which not every proposition can be deduced from each contradictory set of propositions.

In [28], Priest proposed the paraconsistent propositional logic LP (Logic of Paradox). The logic introduced in this section is LP enriched with an implication connective for which the standard deduction theorem holds and a falsity constant. This logic, called LP\textsuperscript{⊃,F}, is in fact the propositional fragment of CLuNs [12] without bi-implications.

LP\textsuperscript{⊃,F} has the following logical constants and connectives: a falsity constant \( F \), a unary negation connective \( \neg \), a binary conjunction connective \( \wedge \), a binary disjunction connective \( \vee \), and a binary implication connective \( \supset \). Truth and bi-implication are defined as abbreviations: \( T \) stands for \( \neg F \) and \( A \equiv B \) stands for \( (A \supset B) \wedge (B \supset A) \).

A Hilbert-style formulation of LP\textsuperscript{⊃,F} is given in Table I. In this formulation, which is taken from [4], \( A, B, \) and \( C \) are used as meta-variables ranging over all formulas of LP\textsuperscript{⊃,F}. The axiom schemas on the left-hand side of Table I and the single inference rule (modus ponens) constitute a Hilbert-style formulation of the positive fragment of classical propositional logic. The first four axiom schemas on the right-hand side of Table I allow for the negation connective to be moved inward. The fifth axiom schema on the right-hand side of Table I is the law of
The classical truth-conditions and falsehood-conditions for the logical connectives are retained. Except for implications, a formula is classified as both-true-
and-false exactly when when it cannot be classified as true or false by the classical truth-conditions and falsehood-conditions. The definition of a valuation given above shows that the logical connectives of LP_{3, F} are (three-valued) truth-functional, which means that each n-ary connective represents a function from \{t, f, b\}^n to \{t, f, b\}.

For LP_{3, F}, the semantic logical consequence relation, denoted by \vdash, is based on the idea that a valuation \nu satisfies a formula \varphi if \nu(\varphi) \in \{t, b\}. It is defined as follows: \Gamma \vdash \varphi if for every valuation \nu, either \nu(\varphi') = f for some \varphi' \in \Gamma or \nu(\varphi) \in \{t, b\}. We have that the Hilbert-style formulation of LP_{3, F} is strongly complete with respect to its semantics, i.e. \Gamma \vdash \varphi if \Gamma \models \varphi (see e.g. [12]).

For all formulas \varphi of LP_{3, F}, the logical equivalence relation \equiv is defined as for classical propositional logic: \varphi \equiv \varphi' if for every valuation \nu, \nu(\varphi) = \nu(\varphi'). Unlike in classical propositional logic, we do not have that \varphi \equiv \bot \iff \varphi \equiv \top. For LP_{3, F}, the consistency property is defined as to be expected: \varphi is consistent if for every valuation \nu, \nu(\varphi) \neq b.

The following are some important properties of LP_{3, F}:

(a) containment in classical logic: \vdash \subseteq \vdash_{\text{CL}}
(b) proper basic connectives: for all sets \Gamma of formulas of LP_{3, F} and all formulas \varphi, \psi, and \chi of LP_{3, F}:
   (b_1) \Gamma \cup \{ \varphi \} \vdash \psi if \Gamma \vdash \varphi \vdash \psi,
   (b_2) \Gamma \vdash \varphi \land \psi if \Gamma \vdash \varphi and \Gamma \vdash \psi,
   (b_3) \Gamma \cup \{ \varphi \lor \psi \} \vdash \chi if \Gamma \vdash \chi and \Gamma \cup \{ \varphi \} \vdash \psi\lor \chi;
(c) weakly maximal paraconsistency relative to classical logic: for all formulas \varphi of LP_{3, F} with \not \vdash \varphi and \vdash_{\text{CL}} \varphi, for the minimal consequence relation \vdash' such that \vdash \subseteq \vdash' and \vdash' \varphi, for all formulas \psi of LP_{3, F}, \vdash' \psi \iff \vdash_{\text{CL}} \psi;
(d) strongly maximal absolute paraconsistency: for all logics \mathcal{L} with the same logical constants and connectives as LP_{3, F} and a consequence relation \vdash' such that \vdash \subseteq \vdash', \mathcal{L} is not paraconsistent;
(e) internalized notion of consistency: \varphi is consistent if \vdash (\varphi \land F) \lor (\neg F) ;
(f) internalized notion of logical equivalence: \varphi \equiv \psi if \vdash (\varphi \equiv \psi) \land (\neg \varphi \equiv \neg \psi);
(g) the laws given in Table 2 hold for the logical equivalence relation of LP_{3, F}.

Properties (a)–(f) have been mentioned relatively often in the literature (see e.g. [12][13][14][15]). Properties (a), (b_1), (c), and (d) make LP_{3, F} an ideal paraconsistent logic in the sense made precise in [2]. By property (e), LP_{3, F} is a logic of formal inconsistency in the sense made precise in [19]. Properties (a)–(c) indicate that LP_{3, F} retains much of classical propositional logic. Actually, property (c) can be strengthened to the following property: for all formulas \varphi of LP_{3, F}, \vdash \varphi if \vdash_{\text{CL}} \varphi.
From Theorem 4.42 in [1], we know that there are exactly 8192 different three-valued paraconsistent propositional logics with properties (a) and (b). From Theorem 2 in [2], we know that properties (c) and (d) are common properties of all three-valued paraconsistent propositional logics with properties (a) and (b). From Fact 103 in [19], we know that property (f) is a common property of all three-valued paraconsistent propositional logics with properties (a), (b) and (e). Moreover, it is easy to see that that property (e) is a common property of all three-valued paraconsistent propositional logics with properties (a) and (b). Hence, each three-valued paraconsistent propositional logic with properties (a) and (b) has properties (c)–(f) as well.

Property (g) is not a common property of all three-valued paraconsistent propositional logics with properties (a) and (b). To our knowledge, properties like property (g) are not mentioned in the literature. However, like property (f), property (g) is essential for the process algebra presented in this paper. Among the 8192 three-valued paraconsistent propositional logics with properties (a)–(e), which are considered desirable properties, LP$^{\supset F}$ is one out of four with the essential properties (f) and (g).

**Proposition 1 (Almost Uniqueness).** There are exactly four three-valued paraconsistent propositional logics with the logical constants and connectives of LP$^{\supset F}$ that have the properties (a)–(g) mentioned above.

**Proof.** Because property (f) is a common property of all 8192 three-valued paraconsistent propositional logics with properties (a)–(e), it is sufficient to prove that, among these 8192 logics, there exists only one that has property (g). Because 'non-deterministic truth tables' that uniquely characterize the 8192 logics are given in [2], the theorem can be proved by showing that, for each of the connectives, only one of the ordinary truth tables represented by the non-deterministic truth table for that connective is compatible with the laws given in Table 2. It can be shown by short routine case analyses that only one of the 8 ordinary truth tables represented by the non-deterministic truth tables for conjunction is compatible with laws (1), (3), (5), and (7) and only one of the 32 ordinary truth tables represented by the non-deterministic truth tables for

| (1) $A \land F \Leftrightarrow F$ | (2) $A \lor T \Leftrightarrow T$ |
| (3) $A \land T \Leftrightarrow A$ | (4) $A \lor F \Leftrightarrow A$ |
| (5) $A \lor A \Leftrightarrow A$ | (6) $A \lor A \Leftrightarrow A$ |
| (7) $A \land B \Leftrightarrow B \land A$ | (8) $A \lor B \Leftrightarrow B \lor A$ |
| (9) $(A \land B) \land C \Leftrightarrow A \land (B \land C)$ | (10) $(A \lor B) \lor C \Leftrightarrow A \lor (B \lor C)$ |
| (11) $A \land (B \lor C) \Leftrightarrow (A \land B) \lor (A \land C)$ | (12) $A \lor (B \land C) \Leftrightarrow (A \lor B) \land (A \lor C)$ |
| (13) $(A \supset B) \land (A \supset C) \Leftrightarrow A \supset (B \land C)$ | (14) $(A \supset C) \land (B \supset C) \Leftrightarrow (A \lor B) \supset C$ |
| (15) $(A \lor \neg A) \supset B \Leftrightarrow B$ | (16) $A \lor (B \supset C) \Leftrightarrow (A \land B) \supset C$ |
disjunction is compatible with laws (2), (4), (6), and (8). The truth tables concerned are compatible with laws (9)–(12) as well. Given the ordinary truth table for conjunction and disjunction so obtained, it can be shown by slightly longer routine case analyses that exactly four of the 16 ordinary truth tables represented by the non-deterministic truth table for implication are compatible with laws (13)–(15). The four truth tables concerned are compatible with law (16) as well.

The next corollary follows from the proof of Proposition 1.

**Corollary 1 (Uniqueness).** $LP^\supset F$ is the only three-valued paraconsistent propositional logic with the logical constants and connectives of $LP^\supset F$ that has the properties (a)–(g) mentioned above and moreover the property that the law $\neg\neg A \iff A$ holds for its logical equivalence relation.

Corollary 1 may be of independent importance to the area of paraconsistent logics.

From now on, we will use the following abbreviations: $A \leftrightarrow B$ stands for $(A \equiv B) \land (\neg A \equiv \neg B)$ and $\circ A$ stands for $(A \supset F) \lor (\neg A \supset F)$.

In Section 3, where we will use formulas of $LP^\supset F$ as terms, equality of formulas will be interpreted as logical equivalence. This means that equality of formulas can be formally proved using the fact that $A \leftrightarrow B$ iff $\vdash A \leftrightarrow B$. This fact also suggests that $LP^\supset F$ may be Blok-Pigozzi algebraizable [18]. It is shown in [19] that actually all 8192 three-valued paraconsistent propositional logics referred to above are Blok-Pigozzi algebraizable. Although there must exist one, a conditional-equational axiomatization of the algebras concerned in the case of $LP^\supset F$ has not yet been devised. Owing to this, the equations derivable in the version of ACPs built on $LP^\supset F$ presented in this paper cannot always be derived by equational reasoning only.

### 3 Contradiction-Tolerant BPA with Propositional Signals

BPAps is a subtheory of ACPps that does not support parallelism and communication. In this section, we present the contradiction-tolerant version of BPAps. In this version, which is called BPApsct, processes have their state to some extent visible. The visible part of the state of a process, called the signal emitted by the process, is a proposition of $LP^\supset F$. These propositions are not only used as signals emitted by processes, but also as conditions under which processes may proceed. The intuition is that the signal emitted by a process is a proposition that holds at its start and the condition under which processes may proceed is a proposition that must hold at its start.

In BPApsct, just as in BPAps, it is assumed that a fixed but arbitrary finite set $A$ of actions, with $\delta \notin A$, and a fixed but arbitrary finite set $B_{at}$ of atomic propositions have been given. We write $A_\delta$ for $A \cup \{\delta\}$.

The algebraic theory BPApsct has two sorts:

- the sort $P$ of processes;
- the sort $B$ of propositions.
The algebraic theory $\text{BPA}_{\text{ct}}$ has the following constants and operators to build terms of sort $\mathbf{B}$:

- for each $P \in \mathbf{B}_{\text{at}}$, the atomic proposition constant $P : \mathbf{B}$;
- the falsity constant $\mathbf{F} : \mathbf{B}$;
- the unary negation operator $\neg : \mathbf{B} \rightarrow \mathbf{B}$;
- the binary conjunction operator $\land : \mathbf{B} \times \mathbf{B} \rightarrow \mathbf{B}$;
- the binary disjunction operator $\lor : \mathbf{B} \times \mathbf{B} \rightarrow \mathbf{B}$;
- the binary implication operator $\supset : \mathbf{B} \times \mathbf{B} \rightarrow \mathbf{B}$.

The algebraic theory $\text{BPA}_{\text{ps}}$ has the following constants and operators to build terms of sort $\mathbf{P}$:

- the deadlock constant $\delta : \mathbf{P}$;
- for each $a \in \mathbf{A}$, the action constant $a : \mathbf{P}$;
- the inaccessible process constant $\bot : \mathbf{P}$;
- the binary alternative composition operator $+: \mathbf{P} \times \mathbf{P} \rightarrow \mathbf{P}$;
- the binary sequential composition operator $\cdot : \mathbf{P} \times \mathbf{P} \rightarrow \mathbf{P}$;
- the binary guarded command operator $\Rightarrow : \mathbf{B} \times \mathbf{P} \rightarrow \mathbf{P}$;
- the binary signal emission operator $\land \uparrow : \mathbf{B} \times \mathbf{P} \rightarrow \mathbf{P}$.

It is assumed that there are infinitely many variables of sort $\mathbf{P}$, including $x, y, \text{ and } z$.

We use infix notation for the binary operators. The following precedence conventions are used to reduce the need for parentheses. The operators to build terms of sort $\mathbf{B}$ bind stronger than the operators to build terms of sort $\mathbf{P}$. The operator $\cdot$ binds stronger than all other binary operators to build terms of sort $\mathbf{P}$ and the operator $+$ binds weaker than all other binary operators to build terms of sort $\mathbf{P}$.

Let $p$ and $q$ be closed terms of sort $\mathbf{P}$ and $\phi$ be a closed term of sort $\mathbf{B}$. Intuitively, the constants and operators to build terms of sort $\mathbf{P}$ can be explained as follows:

- $\delta$ is not capable of doing anything, the proposition that holds at the start of $\delta$ is $\top$;
- $a$ is only capable of performing action $a$ unconditionally and next terminating successfully, the proposition that holds at the start of $a$ is $\top$;
- $\bot$ is not capable of doing anything; there is an inconsistency at the start of $\bot$;
- $p + q$ behaves either as $p$ or as $q$ but not both, the proposition that holds at the start of $p + q$ is the conjunction of the propositions that hold at the start of $p$ and $q$;
- $p \cdot q$ first behaves as $p$ and on successful termination of $p$ it next behaves as $q$, the proposition that holds at the start of $p \cdot q$ is the proposition that holds at the start of $p$;
- $\phi : \Rightarrow p$ behaves as $p$ under condition $\phi$, the proposition that holds at the start of $\phi : \Rightarrow p$ is the implication with $\phi$ as antecedent and the proposition that holds at the start of $p$ as consequent;
The axioms of BPA$_{pa}^{ct}$ are the axioms given in Table 3. In this table, $a$ stands for an arbitrary constant from $A \cup \{\delta\}$, $\phi$ and $\psi$ stand for arbitrary closed terms of sort B, and $\vdash$ is the logical consequence relation of LP$_{\supset:F}$. A1–A7 are the axioms of BPA$_3$, the subtheory of ACP that does not support parallelism and communication (see e.g. [11]). NE1–NE3, GC1–GC7, and SE1–SE8 have been taken from [8], using a different numbering. By IMP, the axioms of BPA$_{pa}^{ct}$ include all equations $\phi = \psi$ for which $\phi \leftrightarrow \psi$ is a theorem of LP$_{\supset:F}$. This is harmless because the connective $\leftrightarrow$, which is the internalization of the logical equivalence relation $\Leftrightarrow$ of LP$_{\supset:F}$, is a congruence.

The following generalizations of axioms SE4 and SE7 are among the equations derivable from the axioms of BPA$_{pa}^{ct}$:

\[
\phi \ast x + \psi \ast y = (\phi \land \psi) \ast (x + y), \\
(\phi \land \psi) \ast (\phi \rightarrow x) = (\phi \land \psi) \ast x, \\
\phi \ast ((\phi \land \psi) \rightarrow x) = \phi \ast (\psi \rightarrow x);
\]

$\phi \ast p$ behaves as $p$ if the proposition that holds at its start does not equal $\bot$ and as $\bot$ otherwise, in the former case, the proposition that holds at the start of $\phi \ast p$ is the conjunction of $\phi$ and the proposition that holds at the start of $p$.

| $x + y = y + x$ | A1 | $x + \bot = \bot$ | NE1 |
| $(x + y) + z = x + (y + z)$ | A2 | $\bot \cdot x = \bot$ | NE2 |
| $x + x = x$ | A3 | $a \cdot \bot = \delta$ | NE3 |
| $(x + y) \cdot z = x \cdot z + y \cdot z$ | A4 |
| $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ | A5 |
| $x + \delta = x$ | A6 |
| $\delta \cdot x = \delta$ | A7 | $\phi = \psi$ if $\vdash \phi \leftrightarrow \psi$ | IMP |

$\top :\rightarrow x = x$

$\bot :\rightarrow x = \delta$

$\phi :\rightarrow \delta = \delta$

$\phi :\rightarrow (x + y) = \phi :\rightarrow x + \phi :\rightarrow y$

$\phi :\rightarrow x \cdot y = (\phi :\rightarrow x) \cdot y$

$\phi :\rightarrow (\psi :\rightarrow x) = (\phi \land \psi) :\rightarrow x$

$\phi \lor \psi :\rightarrow x = \phi :\rightarrow x + \psi :\rightarrow x$

$\phi :\rightarrow (\psi \ast x) = (\phi \lor \psi) \ast x$

$\phi :\rightarrow (\psi \ast x) = (\phi \lor \psi) \ast (\phi \rightarrow x)$

Table 3. Axioms of BPA$_{pa}^{ct}$

3 The axioms of BPA$_{pa}^{ct}$ are not independent: A3, A6, and A7 are derivable from GC1–GC7 and IMP, NE1 and NE2 are derivable from SE1–SE8, and SE3 is derivable from SE6 and IMP.
the following specialization of axiom SE4 is among the equations derivable from the axioms of $\text{BPA}_{ps}^{ct}$.

$$\phi \cdot \delta + x = \phi \cdot x ;$$

and the following equations concerning the inaccessible process are among the equations derivable from the axioms of $\text{BPA}_{ps}^{ct}$.

$$\phi \cdot \perp = \perp ,$$

$$\phi :\rightarrow \perp = (\phi \triangleright F) \cdot \delta .$$

The derivable equations mentioned above are derivable from the axioms of $\text{BPA}_{ps}$ as well. The equation $\phi :\rightarrow \perp = \neg \phi \cdot \delta$, which is derivable from the axioms of $\text{BPA}_{ps}$, is not derivable from the axioms of $\text{BPA}_{ps}^{ct}$.

Let $\phi$ be a closed term of sort $B$ such that not $\vdash \phi \leftrightarrow F$ and not $\vdash \neg \phi \leftrightarrow F$. Then, because not $\vdash \phi \land \neg \phi \leftrightarrow F$, we have that $a \cdot (\phi \cdot x + \neg \phi \cdot y) = a \cdot (F \cdot (x + y)) = \delta$, which is derivable from the axioms of $\text{BPA}_{ps}$, is not derivable from the axioms of $\text{BPA}_{ps}^{ct}$. This shows the main difference between $\text{BPA}_{ps}^{ct}$ and $\text{BPA}_{ps}$: the alternative composition of two processes of which the propositions that hold at the start of them are contradictory does not lead to an inconsistency in $\text{BPA}_{ps}^{ct}$, whereas it does lead to an inconsistency in $\text{BPA}_{ps}$. This is why $\text{BPA}_{ps}^{ct}$ is called the contradiction-tolerant version of $\text{BPA}_{ps}$.

Let $\phi$ be a closed term of sort $B$ such that not $\vdash \phi \leftrightarrow F$ and not $\vdash \neg \phi \leftrightarrow F$. We can derive $a \cdot (\phi \cdot b + \neg \phi \cdot c) = a \cdot ((\phi \land \neg \phi) \cdot (b + c)) = \delta$ from the axioms of $\text{BPA}_{ps}$ because, in the case of $\text{BPA}_{ps}$, $a \cdot (\phi \cdot b + \neg \phi \cdot c)$ is not capable of doing anything. We can only derive $a \cdot (\phi \cdot b + \neg \phi \cdot c) = a \cdot ((\phi \land \neg \phi) \cdot (b + c))$ from the axioms of $\text{BPA}_{ps}^{ct}$ because, in the case of $\text{BPA}_{ps}^{ct}$, $a \cdot (\phi \cdot b + \neg \phi \cdot c)$ is capable of first performing $a$ and next either performing $b$ and after that terminating successfully or performing $c$ and after that terminating successfully — although the proposition that holds at the start of the process that remains after performing $a$ is the contradiction $\phi \land \neg \phi$.

Let $\phi$ be a closed term of sort $B$ such that not $\vdash \phi \leftrightarrow F$ and not $\vdash \neg \phi \leftrightarrow F$. Then, because $\vdash \circ \phi \land \phi \land \neg \phi \leftrightarrow F$, we have that $a \cdot (\circ \phi \cdot (\phi \cdot x + \neg \phi \cdot y)) = a \cdot (F \cdot (x + y)) = \delta$ is derivable from the axioms of $\text{BPA}_{ps}^{ct}$. This shows that it can be enforced by means of a consistency proposition $(\circ \phi)$ that the alternative composition of two processes of which the propositions that hold at the start of them are contradictory leads to an inconsistency in $\text{BPA}_{ps}^{ct}$.

Hereafter, we will write $[\phi]$ for the equivalence class of $\phi$ modulo $\leftrightarrow$. That is, $[\phi] = \{ \psi \mid \phi \leftrightarrow \psi \}$. Hence, $[\phi] = \{ \psi \mid \vdash \phi \leftrightarrow \psi \}$.

All processes that can be described by a closed term of $\text{BPA}_{ps}^{ct}$, can be described by a basic term. The set $B$ of basic terms is inductively defined by the following rules:

- $\perp \in B$;
- if $\phi \notin [F]$, then $\phi \cdot \delta \in B$;
- if $\phi \notin [F]$ and $a \in A$, then $\phi :\rightarrow a \in B$;
- if $\phi \notin [F]$, $a \in A$, and $p \in B$, then $\phi :\rightarrow a \cdot p \in B$;
- if $p, q \in B$, then $p + q \in B$.  

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Each basic term can be written as \( \bot \) or in the form
\[
\chi \cdot \delta + \sum_{i \in \{1,...,n\}} \phi_i :\to a_i \cdot p_i + \sum_{j \in \{1,...,m\}} \psi_j :\to b_j,
\]
where \( n, m \in \mathbb{N} \), where \( \chi \notin [F] \), where \( \phi_i \notin [F] \), \( a_i \in A \), and \( p_i \in B \) for all \( i \in \{1,...,n\} \), and where \( \psi_j \notin [F] \) and \( b_j \in A \) for all \( j \in \{1,...,m\} \). The subterm \( \chi \) is called the root signal of the basic term and the subterms \( \phi_i :\to a_i \cdot p_i \) and \( \psi_j :\to b_j \) are called the summands of the basic term.

All closed BPA\textsubscript{ps} terms of sort \( P \) can be reduced to a basic term.

**Proposition 2 (Elimination).** For all closed BPA\textsubscript{ps} terms \( p \) of sort \( P \), there exists a \( q \in B \) such that \( p = q \) is derivable from the axioms of BPA\textsubscript{ps}.

**Proof.** The proof is straightforward by induction on the structure of closed term \( p \). If \( p \) is of the form \( \bot \), \( a \), \( p' + p'' \) or \( \phi \cdot p' \), then it is trivial to show that there exists a \( q \in B \) such that \( p = q \) is derivable from the axioms of BPA\textsubscript{ps}. If \( p \) is of the form \( p' \cdot p'' \) or \( \phi :\to p' \), then it follows immediately from the induction hypothesis and the following claims:

- for all \( p, p' \in B \), there exists a \( p'' \in B \) such that \( p \cdot p' = p'' \) is derivable from the axioms of BPA\textsubscript{ps};
- for all \( \phi \notin [F] \) and \( p \in B \), there exists a \( p' \in B \) such that \( \phi :\to p = p' \) is derivable from the axioms of BPA\textsubscript{ps}.

Both claims are easily proved by induction on the structure of basic term \( p \). \( \square \)

### 4 Semantics of BPA\textsubscript{ps}

In this section, we present a structural operational semantics of BPA\textsubscript{ps}, define a notion of bisimulation equivalence based on this semantics, and show that the axioms of BPA\textsubscript{ps} are sound and complete with respect to this bisimulation equivalence.

We start with the presentation of the structural operational semantics of BPA\textsubscript{ps}. The following transition relations on closed terms of sort \( P \) are used:

- for each \( \ell \in C \times A \), a binary action step relation \( \xrightarrow{\ell} \);
- for each \( \ell \in C \times A \), a unary action termination relation \( \xrightarrow{\ell} \sqrt{\cdot} \);
- for each \( \phi \in C \), a unary signal emission relation \( s^{\phi} \);

where \( C \) is the set of all closed terms \( \phi \) of sort \( B \) such that \( \phi \notin [F] \). We write \( p \xrightarrow{\phi} a \rightarrow q \) instead of \( (p,q) \in (\phi,a) \), \( p \xrightarrow{\phi} a \rightarrow \sqrt{\cdot} \) instead of \( p \in (\phi,a) \rightarrow \sqrt{\cdot} \), and \( s(p) = \phi \) instead of \( p \in s^{\phi} \). These relations can be explained as follows:

- \( p \xrightarrow{\phi} a \rightarrow \sqrt{\cdot} \): \( p \) is capable of performing action \( a \) under condition \( \phi \) and then terminating successfully.
Table 4. Transition rules for BPA$_{ps}^{ct}$

\[
\begin{array}{ll}
(a \triangleright) & a \\
\hline
x \overset{(\phi)}{\to} y, s(x + y) = \psi & \psi \not\in \mathbb{F} \\
\hline
x \overset{(\phi)}{\to} y, s(x + y) = \psi & \psi \not\in \mathbb{F} \\
\hline
x \overset{(\phi)}{\to} x', s(x + y) = \psi & \psi \not\in \mathbb{F} \\
\hline
x \overset{(\phi)}{\to} y, s(y) = \psi & \psi \not\in \mathbb{F} \\
\hline
x \overset{(\phi)}{\to} y, s(y) = \chi & \chi \not\in \mathbb{F} \\
\hline
\end{array}
\]

\[s(x) = \phi, s(y) = \psi \quad s(x) = \phi \quad s(x) = \phi \quad s(x) = \phi \]

\[-p \overset{(\phi)}{\to} q: p \text{ is capable of performing action } a \text{ under condition } \phi \text{ and then proceeding as } q;\]

\[-s(p) = \phi: \text{the proposition that holds at the start of } p \text{ is } \phi.\]

The structural operational semantics of BPA$_{ps}^{ct}$ is described by the transition rules given in Table 4. In this table, $a$ stands for an arbitrary constant from $A \cup \{\delta\}$ and $\phi, \psi,$ and $\chi$ stand for arbitrary closed terms of sort $B$.

A bisimulation is a binary relation $R$ on closed BPA$_{ps}^{ct}$ terms of sort $P$ such that, for all closed BPA$_{ps}^{ct}$ terms $p, q$ of sort $P$ with $(p, q) \in R$, the following conditions hold:

- if $p \overset{(\phi)}{\to} p'$, then, for all valuations $\nu$ with $\nu(s(p)) \neq f$ and $\nu(\phi) \neq f$, there exists a closed term $\psi$ of sort $B$ and a closed term $q'$ of sort $P$ such that $\nu(\phi) = \nu(\psi), q \overset{(\psi)}{\to} q'$, and $(p', q') \in R$;
- if $q \overset{(\psi)}{\to} q'$, then, for all valuations $\nu$ with $\nu(s(q)) \neq f$ and $\nu(\psi) \neq f$, there exists a closed term $\phi$ of sort $B$ and a closed term $p'$ of sort $P$ such that $\nu(\psi) = \nu(\phi), p \overset{(\phi)}{\to} p'$, and $(p', q') \in R$;
- if $p \overset{(\phi)}{\to} \sqrt{ },$ then, for all valuations $\nu$ with $\nu(s(p)) \neq f$ and $\nu(\phi) \neq f$, there exists a closed term $\psi$ of sort $B$ such that $\nu(\phi) = \nu(\psi)$ and $q \overset{(\psi)}{\to} \sqrt{ }$; and
- if $q \overset{(\psi)}{\to} \sqrt{ },$ then, for all valuations $\nu$ with $\nu(s(q)) \neq f$ and $\nu(\psi) \neq f$, there exists a closed term $\phi$ of sort $B$ such that $\nu(\psi) = \nu(\phi)$. 

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– if \( s(p) = \phi \), then there exists a closed term \( \psi \) of sort \( B \) such that \( s(q) = \psi \) and \( \phi \Leftrightarrow \psi \);
– if \( s(q) = \psi \), then there exists a closed term \( \phi \) of sort \( B \) such that \( s(p) = \phi \) and \( \psi \Leftrightarrow \phi \).

Two closed \( \text{BPA}_{\text{ct}} \) terms \( p, q \) of sort \( P \) are bisimulation equivalent, written \( p \leftrightarrow q \), if there exists a bisimulation \( R \) such that \( (p, q) \in R \). Let \( R \) be a bisimulation such that \( (p, q) \in R \). Then we say that \( R \) is a bisimulation witnessing \( p \leftrightarrow q \).

Henceforth, we will loosely say that a relation contains all closed substitution instances of an equation if it contains all pairs \((t, t')\) such that \( t = t' \) is a closed substitution instance of the equation.

Because a transition on one side may be simulated by a set of transitions on the other side, a bisimulation as defined above is called a splitting bisimulation in \([15]\).

Bisimulation equivalence is a congruence with respect to the operators of \( \text{BPA}_{\text{ct}} \).

**Proposition 3 (Congruence).** For all closed \( \text{BPA}_{\text{ct}} \) terms \( p, q, p', q' \) of sort \( P \) and closed \( \text{BPA}_{\text{ct}} \) terms \( \phi \) of sort \( B \), \( p \leftrightarrow q \) and \( p' \leftrightarrow q' \) imply \( p + p' \leftrightarrow q + q' \), \( p \cdot p' \leftrightarrow q \cdot q' \), \( \phi \rightarrow \phi \leftrightarrow q \rightarrow q' \), and \( \phi \leftrightarrow \phi \rightarrow q \leftrightarrow q' \).

**Proof.** We can reformulate the transition rules such that:

– bisimulation equivalence based on the reformulated transition rules according to the standard definition of bisimulation equivalence coincides with bisimulation equivalence based on the original transition rules according to the definition of bisimulation equivalence given above;
– the reformulated transition rules make up a complete transition system specification in panth format.

The reformulation goes like the one for the transition rules for \( \text{BPA}_{\text{ps}} \) outlined in \([8]\). The proposition follows now immediately from the well-known result that bisimulation equivalence according to the standard definition of bisimulation equivalence is a congruence if the transition rules concerned make up a complete transition system specification in panth format (see e.g. \([21]\)). \( \square \)

The underlying idea of the reformulation referred to above is that we replace each transition \( p \xrightarrow{\{\phi\}_a} p' \) by a transition \( p \xrightarrow{\{\nu\}_a} p' \) for each valuation \( \nu \) such that \( \nu(\phi) \neq f \), and likewise \( p \xrightarrow{\{\phi\}_a} \sqrt{.} \) and \( s(p) = \phi \). Thus, in a bisimulation, a transition on one side must be simulated by a single transition on the other side. We did not present the reformulated structural operational semantics in this paper because it is, in our opinion, intuitively less appealing.

\( \text{BPA}_{\text{ct}} \) is sound with respect to \( \leftrightarrow \) for equations between closed terms.

**Theorem 1 (Soundness).** For all closed \( \text{BPA}_{\text{ct}} \) terms \( p, q \) of sort \( P \), \( p = q \) is derivable from the axioms of \( \text{BPA}_{\text{ct}} \) only if \( p \leftrightarrow q \).
Proof. Because of Proposition 3, it is sufficient to prove the theorem for all closed substitution instances of each axiom of $\text{BPA}^{ct}_{ps}$.

For each axiom, we can construct a bisimulation $R$ witnessing $p \leftrightarrow q$ for all closed substitution instances $p = q$ of the axiom as follows:

- in the case of A1–A4 and A6, we take the relation $R$ that consists of all closed substitution instances of the axiom concerned and the equation $x = x$;
- in the case of A5, we take the relation $R$ that consists of all closed substitution instances of A5, SE5, and the equation $x = x$;
- in the case of A7, NE1–NE3, GC2–GC3, and SE2–SE3, we take the relation $R$ that consists of all closed substitution instances of the axiom concerned;
- in the case of GC1, GC4–GC7, SE1, and SE4–SE8, we take the relation $R$ that consists of all closed substitution instances of the axiom concerned and the equation $x = x$.

The laws from property (8) of $\text{LP}^{\supset F}$ mentioned in Section 2 are needed to check that these relations are witnessing ones.

The proof of Theorem 1 goes along the same line as the soundness proof for $\text{BPA}^{ct}_{ps}$ outlined in [8]. The laws from property (8) of $\text{LP}^{\supset F}$ mentioned in Section 2 are laws that $\text{LP}^{\supset F}$ has in common with classical propositional logic. They are needed in the soundness proof for $\text{BPA}^{ct}_{ps}$ as well, but their use is left implicit in the proof outline given in [8].

$\text{BPA}^{ct}_{ps}$ is complete with respect to $\leftrightarrow$ for equations between closed terms.

**Theorem 2 (Completeness).** For all closed $\text{BPA}^{ct}_{ps}$ terms $p, q$ of sort $P$, $p = q$ is derivable from the axioms of $\text{BPA}^{ct}_{ps}$ if $p \leftrightarrow q$.

Proof. By Proposition 2 and Theorem 1 it is sufficient to prove the theorem for basic terms $p$ and $q$.

For $p, p' \in B$, $p'$ is called a basic subterm of $p$ if $p' \equiv p$ or there exists an $a \in A$ such that $a \cdot p'$ is a subterm of $p$.

We introduce a reduction relation $\Rightarrow$ on $B$. The one-step reduction relation $\Rightarrow$ on $B$ is inductively defined as follows:

- if $p'$ is a basic subterm of $p$ and $q'$ occurs twice as summand in $p'$, then $p \Rightarrow r$ where $r$ is $p$ with one occurrence of $q'$ removed;
- if $p'$ is a basic subterm of $p$ and both $\phi : \rightarrow a \cdot q'$ and $\psi : \rightarrow a \cdot q'$ occur as summand in $p'$, then $p \Rightarrow r$ where $r$ is $p$ with the occurrence of $\phi : \rightarrow a \cdot q'$ replaced by $\phi \lor \psi : \rightarrow a \cdot q'$ and the occurrence of $\psi : \rightarrow a \cdot q'$ removed;
- if $p'$ is a basic subterm of $p$ and both $\phi : \rightarrow a$ and $\psi : \rightarrow a$ occur as summand in $p'$, then $p \Rightarrow r$ where $r$ is $p$ with the occurrence of $\phi : \rightarrow a$ replaced by $\phi \lor \psi : \rightarrow a$ and the occurrence of $\psi : \rightarrow a$ removed.

The one-step reductions correspond to sharing of double states and joining of transitions as in [16]. The reduction relation $\Rightarrow$ is the reflexive and transitive closure of $\Rightarrow$, and the conversion relation $\rightarrow$ is the reflexive and transitive closure of $\Rightarrow \cup \Rightarrow^{-1}$.

The following are important properties of $\Rightarrow$:
(1) \(\leftrightarrow\) is strongly normalizing;
(2) for all \(p, q \in B\), \(p \leftrightarrow q\) only if \(p \sqsubseteq q\);
(3) for all \(p, q \in B\) that are in normal form, \(p \sqsubseteq q\) only if \(p = q\) is derivable from axioms A1 and A2;
(4) for all \(p, q \in B\), \(p \rightarrow q\) only if \(p = q\) is derivable from the axioms of BPA

Verifying properties (1), (2), and (4) is trivial. Property (3) can be verified by proving it, simultaneously with the property
for all \(p \in B\) that are in normal form, any bisimulation between \(p\) and itself is the identity relation,
by induction on the number of occurrences of a constant from \(A\) in \(p\) and \(q\). The proof is similar to the proof of Theorem 2.12 from [13], but easier.

From properties (1), (2) and (3), it follows immediately that, for all \(p, q \in B\), \(p \sqsubseteq q\) iff \(p \rightarrow q\). From this and property (4), it follows immediately that, for all \(p, q \in B\), \(p \sqsubseteq q\) only if \(p = q\) is derivable from the axioms of BPA

\[\square\]

5 Contradiction-Tolerant ACP with Propositional Signals

In this section, we present the contradiction-tolerant version of ACPPs. This version, which is called ACPCt, is an extension of BPA\(^{ct}\) that supports parallelism and communication.

In ACPCt, just as in BPA\(^{ct}\), it is assumed that a fixed but arbitrary finite set \(A\) of actions, with \(\delta \not\in A\), and a fixed but arbitrary finite set \(B_{at}\) of atomic propositions have been given. In ACPCt, it is further assumed that a fixed but arbitrary commutative and associative communication function \(\mid: A_{\delta} \times A_{\delta} \rightarrow A_{\delta}\), such that \(\delta \mid a = \delta\) for all \(a \in A_{\delta}\), has been given. The function \(\mid\) is regarded to give the result of synchronously performing any two actions for which this is possible, and to be \(\delta\) otherwise.

The algebraic theory ACPCt has the sorts, constants and operators of BPA\(^{ct}\) and in addition the following operators:

- the binary parallel composition operator \(\parallel: P \times P \rightarrow P\);
- the binary left merge operator \(\lfloor\lfloor: P \times P \rightarrow P\);
- the binary communication merge operator \(\mid: P \times P \rightarrow P\);
- for each \(H \subseteq \delta\), the unary encapsulation operator \(\partial_H: P \rightarrow P\).

We use infix notation for the additional binary operators as well.

The constants and operators of ACPCt to build terms of sort \(P\) are the constants and operators of ACP and additionally the guarded command operator and the signal emission operator.

Let \(p\) and \(q\) be closed terms of sort \(P\). Intuitively, the additional operators can be explained as follows:

- \(p \parallel q\) behaves as the process that proceeds with \(p\) and \(q\) in parallel, the proposition that holds at the start of \(p \parallel q\) is the conjunction of the propositions that hold at the start of \(p\) and \(q\);
Table 5. Additional axioms for $\text{ACP}_{ps}^\text{ct}$

| Equation                                                                 | Axiom |
|--------------------------------------------------------------------------|-------|
| $x \parallel y = x \parallel y + y \parallel x + y$                    | CM1   |
| $a \parallel x = a \cdot x + \partial_\Delta(x)$                       | CM2S  |
| $a \cdot x \parallel y = a \cdot (x \parallel y) + \partial_\Delta(y)$ | CM3S  |
| $(x + y) \parallel z = x \parallel z + y \parallel z$                  | CM4   |
| $a \cdot x \parallel b = (a \parallel b) \cdot x$                     | CM5   |
| $(\phi \mapsto x) \parallel y = \phi \mapsto (x \parallel y) + \partial_\Delta(y)$ | GC8S  |
| $(\phi \mapsto x) \parallel y = \phi \mapsto (x \parallel y) + \partial_\Delta(y)$ | GC9S  |
| $(\phi \mapsto x) \parallel y = \phi \mapsto (x \parallel y)$         | GC10S |
| $\partial_H(\phi \mapsto x) = \phi \mapsto \partial_H(x)$             | GC11  |
| $\partial_H(x \parallel y) = \partial_H(x) + \partial_H(y)$           | D3    |
| $\partial_H(x \parallel y) = \partial_H(x) \cdot \partial_H(y)$       | D4    |
| $\phi \parallel q$ behaves the same as $p \parallel q$, except that it starts with performing an action of $p$, the proposition that holds at the start of $p \parallel q$ is the conjunction of the propositions that hold at the start of $p$ and $q$; $- p \parallel q$ behaves the same as $p \parallel q$, except that it starts with performing an action of $p$ and an action of $q$ synchronously, the proposition that holds at the start of $p \parallel q$ is the conjunction of the propositions that hold at the start of $p$ and $q$; $- \partial_H(p)$ behaves the same as $p$, except that the actions in $H$ are blocked, the proposition that holds at the start of $\partial_H(p)$ is the proposition that holds at the start of $p$. |

The axioms of $\text{ACP}_{ps}^\text{ct}$ are the axioms of $\text{BPA}_{ps}^\text{ct}$ and the additional axioms given in Table 5. In this table, $a, b, c$ stand for arbitrary constants from $A \cup \{\delta\}$ and $\phi$ stands for an arbitrary closed term of sort $B$. A1–A7, CM1–CM9 with CM1S and CM2S replaced by $a \parallel x = a \cdot x$ and $a \cdot x \parallel y = a \cdot (x \parallel y)$, C1–C3, and D1–D4 are the axioms of $\text{ACP}$ (see e.g. [11]). GC11 and SE9–SE12 have been taken from [8] and GC9S and GC10S have been taken from [8] with subterms of the form $s(x) \ast \delta$ replaced by $\partial_\Delta(x)$. CM2S, CM3S and GC8S differ really from the corresponding axioms in [8] due to the choice of having as the proposition that holds at the start of the merge of two processes, as in the case of the communication merge, always the conjunction of the propositions that hold at the start of the two processes.

The following equations are among the equations derivable from the axioms of $\text{ACP}_{ps}^\text{ct}$:

$$(\phi \ast x) \parallel (\psi \ast y) = (\phi \land \psi) \ast (x \parallel y),$$

$x \parallel \bot = \bot, \quad \bot \parallel x = \bot.$
Let $\phi$ be a closed term of sort $B$ such that $\not\vdash \phi \leftrightarrow F$ and $\not\vdash \neg \phi \leftrightarrow F$. Then, because $\not\vdash \phi \land \neg \phi \leftrightarrow F$, we have that $a \cdot (\phi \land \neg \phi) = a \cdot (F \land (x \parallel y)) = \delta$, which is derivable from the axioms of $ACP^{\text{pt}}$. This shows the main difference between $ACP^{\text{pt}}$ and $ACP$: the parallel composition of two processes of which the propositions that hold at the start of them are contradictory does not lead to an inconsistency in $ACP^{\text{pt}}$, whereas it does lead to an inconsistency in $ACP$. This is why $ACP^{\text{pt}}$ is called the contradiction-tolerant version of $ACP$.

Let $\phi$ be a closed term of sort $B$ such that $\not\vdash \phi \leftrightarrow F$ and $\not\vdash \neg \phi \leftrightarrow F$. Assume that $b \parallel c = d$. Then, we can derive $a \cdot (\phi \land b \parallel c) = a \cdot ((\phi \land b) \land c) = a \cdot (\phi \land (b \parallel c)) = a \cdot (\phi \land \neg \phi) = \delta$ from the axioms of $BPA$. We can only derive $a \cdot (\phi \land b \parallel c) = a \cdot (\phi \land b) \land c$ from the axioms of $BPA^{\text{pt}}$ because, in the case of $BPA^{\text{pt}}$, $a \cdot (\phi \land \neg b \parallel c)$ is not capable of doing anything. We can only derive $a \cdot (\phi \land b \parallel c) = a \cdot ((\phi \land \neg b) \land c)$ from the axioms of $BPA^{\text{pt}}$, because, in the case of $BPA^{\text{pt}}$, $a \cdot (\phi \land b) \land c$ is capable of first performing $a$ and next either performing $b$ and $c$ in either order and after that terminating successfully or performing $d$ and after that terminating successfully — although the proposition that holds at the start of that remains after performing $a$ is the contradiction $\phi \land \neg \phi$.

Let $\phi$ be a closed term of sort $B$ such that $\not\vdash \phi \leftrightarrow F$ and $\not\vdash \neg \phi \leftrightarrow F$. Then, because $\vdash \phi \land \neg \phi \leftrightarrow F$, we have that $a \cdot (\phi \land \neg \phi) = a \cdot (F \land (x \parallel y)) = \delta$ is derivable from the axioms of $ACP^{\text{pt}}$. This shows that it can be enforced by means of a consistency proposition $(\phi \land \neg \phi)$ that the parallel composition of two processes of which the propositions that hold at the start of them are contradictory leads to an inconsistency in $ACP^{\text{pt}}$.

All closed $ACP^{\text{pt}}$ terms of sort $P$ can be reduced to a basic term.

**Proposition 4 (Elimination).** For all closed $ACP^{\text{pt}}$ terms $p$ of sort $P$, there exists a $q \in B$ such that $p = q$ is derivable from the axioms of $ACP^{\text{pt}}$.

**Proof.** The proof is straightforward by induction on the structure of closed term $p$. If $p$ is of the form $\perp$, $a$, $p' + p''$, $p' \cdot p''$, $\phi \rightarrow p'$ or $\phi \land p'$, then it follows immediately from the induction hypothesis and Proposition 3 that there exists a $q \in B$ such that $p = q$ is derivable from the axioms of $ACP^{\text{pt}}$. If $p$ is of the form $p' \parallel p''$, $p' \parallel p''$, $p' \parallel p''$ or $\partial_H(p')$, then it follows immediately from the induction hypothesis and claims similar to the ones from the proof of Proposition 3. The claims concerning $\parallel$, $\parallel$, and $\parallel$ are easily proved simultaneously by structural induction. The claim concerning $\partial_H$ is easily proved by structural induction. □

### 6 Semantics of $ACP^{\text{pt}}$

In this section, we present a structural operational semantics of $ACP^{\text{pt}}$ and show that the axioms of $ACP^{\text{pt}}$ are sound and complete with respect to this bisimulation equivalence.

We start with the presentation of the structural operational semantics of $ACP^{\text{pt}}$. The structural operational semantics of $ACP^{\text{pt}}$ is described by the transition rules for $BPA^{\text{pt}}$ and the additional transition rules given in Table 6. In
### Table 6. Additional transition rules for $\text{ACP}^\text{ct}$

| Rule | Description |
|------|-------------|
| $x \xrightarrow{(\phi)\alpha} \sqrt{\,\,\,} \downarrow s(x \parallel y) = \psi, s(y) = \chi \psi, \chi \notin [F]$ |  |
| $y \xrightarrow{(\phi)\alpha} \sqrt{\,\,\,} \downarrow s(x \parallel y) = \psi, s(x) = \chi \psi, \chi \notin [F]$ |  |
| $x \xrightarrow{(\phi)\alpha} \sqrt{\,\,\,} \downarrow s(x \parallel y) = \psi, s(x') \parallel y) = \chi \psi, \chi \notin [F]$ |  |
| $y \xrightarrow{(\phi)\alpha} \sqrt{\,\,\,} \downarrow s(x \parallel y) = \psi, s(x') \parallel y) = \chi \psi, \chi \notin [F]$ |  |
| $x \xrightarrow{(\phi)\alpha} \sqrt{\,\,\,} \downarrow s(x \parallel y) = \psi, s(x') \parallel y) = \chi \psi, \chi \notin [F]$ |  |
| $y \xrightarrow{(\phi)\alpha} \sqrt{\,\,\,} \downarrow s(x \parallel y) = \psi, s(x') \parallel y) = \chi \psi, \chi \notin [F]$ |  |
| $x \xrightarrow{(\phi)\alpha} \sqrt{\,\,\,} \downarrow s(x \parallel y) = \psi, s(x) = \chi \psi, \chi \notin [F]$ |  |
| $y \xrightarrow{(\phi)\alpha} \sqrt{\,\,\,} \downarrow s(x \parallel y) = \psi, s(x') \parallel y) = \chi \psi, \chi \notin [F]$ |  |
| $x \xrightarrow{(\phi)\alpha} \sqrt{\,\,\,} \downarrow s(x \parallel y) = \psi, s(x') \parallel y) = \chi \psi, \chi \notin [F]$ |  |
| $y \xrightarrow{(\phi)\alpha} \sqrt{\,\,\,} \downarrow s(x \parallel y) = \psi, s(x') \parallel y) = \chi \psi, \chi \notin [F]$ |  |
| $x \xrightarrow{(\phi)\alpha} \sqrt{\,\,\,} \downarrow s(x \parallel y) = \psi, s(x) = \chi \psi, \chi \notin [F]$ |  |
| $y \xrightarrow{(\phi)\alpha} \sqrt{\,\,\,} \downarrow s(x \parallel y) = \psi, s(x') \parallel y) = \chi \psi, \chi \notin [F]$ |  |
these tables, \(a\), \(b\), and \(c\) stand for arbitrary constants from \(\mathcal{A} \cup \{\delta\}\) and \(\phi\), \(\psi\), \(\chi\), and \(\chi'\) stand for arbitrary closed terms of sort \(\mathcal{B}\).

In Sections 3 and 5, we have touched upon the main difference between \(\text{ACP}^{\text{ct}}\) and \(\text{ACP}^{\text{ps}}\); the alternative and parallel composition of two processes of which the propositions that hold at the start of them are contradictory does not lead to an inconsistency in \(\text{ACP}^{\text{ct}}\) whereas it does lead to an inconsistency in \(\text{ACP}^{\text{ps}}\). However, the transition rules for \(\text{ACP}^{\text{ct}}\) and \(\text{ACP}^{\text{ps}}\) seem to be the same. The difference is fully accounted for by the fact that \(\mathcal{F}\), the equivalence class of \(\mathcal{F}\) modulo logical equivalence, contains in the case of \(\text{LP} \supset F\) only propositions of the form \(\phi \land \neg \phi\) with \(\phi\) such that either \(\phi \iff F\) or \(\neg \phi \iff F\), whereas it contains in the case of classical propositional logic all propositions of the form \(\phi \land \neg \phi\).

By this fact, in the case of \(\text{ACP}^{\text{ct}}\), \(a \cdot (\phi \land \Box b \parallel \neg \phi \land \Box c)\) from the example preceding Proposition 4 is capable of first performing \(a\) and next either performing \(b\) and \(c\) in either order and after that terminating successfully or performing \(d\) and after that terminating successfully — although the proposition that holds at the start of the process that remains after performing \(a\) is the contradiction \(\phi \land \neg \phi\) — and, in the case of \(\text{ACP}^{\text{ps}}\), it is not capable of doing anything.

Bisimulation equivalence is a congruence with respect to the operators of \(\text{ACP}^{\text{ct}}\).

**Proposition 5 (Congruence).** For all closed \(\text{ACP}^{\text{ct}}\) terms \(p,q,p',q'\) of sort \(\mathcal{P}\) and closed \(\text{ACP}^{\text{ct}}\) terms \(\phi\) of sort \(\mathcal{B}\), \(p \leftrightarrow q\) and \(p' \leftrightarrow q'\) imply \(p + p' \leftrightarrow q + q'\), \(p \cdot p' \leftrightarrow q \cdot q'\), \(p \parallel p' \leftrightarrow q \parallel q'\), \(p \parallel\parallel p' \leftrightarrow q \parallel\parallel q'\), \(p \parallel\parallel\parallel p' \leftrightarrow q \parallel\parallel\parallel q'\), and \(\partial H(p) \leftrightarrow \partial H(q)\).

**Proof.** The proof goes along the same line as the proof of Proposition 3. \(\square\)

\(\text{ACP}^{\text{ct}}\) is sound with respect to \(\leftrightarrow\) for equations between closed terms.

**Theorem 3 (Soundness).** For all closed \(\text{ACP}^{\text{ct}}\) terms \(p,q\) of sort \(\mathcal{P}\), \(p = q\) is derivable from the axioms of \(\text{ACP}^{\text{ct}}\) only if \(p \leftrightarrow q\).

**Proof.** Because of Proposition 5 it is sufficient to prove the theorem for all closed substitution instances of each axiom of \(\text{ACP}^{\text{ps}}\).

For each axiom, we can construct a bisimulation \(R\) witnessing \(p \leftrightarrow q\) for all closed substitution instances \(p = q\) of the axiom as follows:

- in the case of the axioms of \(\text{BPA}^{\text{ct}}\), we take the same relation as in the proof of Theorem 1;
- in the case of CM1, we take the relation \(R\) that consists of all closed substitution instances of CM1, the equation \(x \parallel y = y \parallel x\), and the equation \(x = x\);
- in the case of CM2–CM9, we take the relation \(R\) that consists of all closed substitution instances of the axiom concerned and the equation \(x = x\);
- in the case of CI–C3 and DI–D2, we take the relation \(R\) that consists of all closed substitution instances of the axiom concerned;

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– in the case of D3–D4, GC8S–GC11, and SE9–SE12, we take the relation \( R \) that consists of all closed substitution instances of the axiom concerned and the equation \( x = x \).

The laws from property (8) of LP\(^{\geq f}\) mentioned in Section 2 are needed to check that these relations are witnessing ones.

ACP\(_{ps}^{ct}\) is complete with respect to \( \equiv \) for equations between closed terms.

**Theorem 4 (Completeness).** For all closed ACP\(_{ps}^{ct}\) terms \( p, q \) of sort \( P \), \( p = q \) is derivable from the axioms of ACP\(_{ps}^{ct}\) if \( p \equiv q \).

**Proof.** We have that the axioms of BPA\(_{ps}^{ct}\) are complete with respect to \( \equiv \) (Theorem 2), the axioms of ACP\(_{ps}^{ct}\) are sound with respect to \( \equiv \) (Theorem 3), and for each closed ACP\(_{ps}^{ct}\) term \( p \) of sort \( P \), there exists a closed BPA\(_{ps}^{ct}\) term \( q \) such that \( p = q \) is derivable from the axioms of ACP\(_{ps}^{ct}\) (Proposition 4). By Theorem 3.14 from [31], the result immediately follows from this and the claim that the set of transition rules for ACP\(_{ps}^{ct}\) is an operational conservative extension of the set of transition rules for BPA\(_{ps}^{ct}\).

This claim can easily be proved if we reformulate the transition rules for ACP\(_{ps}^{ct}\) in the same way as the transition rules for BPA\(_{ps}^{ct}\) have been reformulated to prove Proposition 3. The operational conservativity can then easily be proved by verifying that the reformulated transition rules for ACP\(_{ps}^{ct}\) make up a complete transition system specification, the reformulated transition rules for BPA\(_{ps}^{ct}\)— which are included in the reformulated transition rules for ACP\(_{ps}^{ct}\)— are source-dependent, and the additional transition rules have fresh sources (see e.g. [22]).

7 State Operators

In this section, we extend ACP\(_{ps}^{ct}\) with state operators. The resulting theory is called ACP\(_{ps}^{ct} + SO\). The state operators introduced here generalize the state operators added to ACP in [6].

The state operators from [6] were introduced to make it easy to represent the execution of a process in a state. The basic idea was that the execution of an action in a state has effect on the state, i.e. it causes a change of state. Moreover, there is an action left when an action is executed in a state. The main difference between the original state operators and the state operators introduced here is that, in the case of the latter, the state in which a process is executed determines the proposition that holds at its start. Thus, one application of a state operator may replace many applications of the signal emission operator.

It is assumed that a fixed but arbitrary set \( S \) of states has been given, together with functions \( \text{act} : A \times S \to A \), \( \text{eff} : A \times S \to S \), and \( \text{sig} : S \to B \), where \( B \) is the set of all closed terms \( \phi \) of sort \( B \).

For each \( s \in S \), we add a unary state operator \( \lambda_s : P \to P \) to the operators of ACP\(_{ps}^{ct}\).
The state operator $\lambda_s$ allows, given the above-mentioned functions, processes to be executed in a state. Let $p$ be a closed term of sort $P$. Then $\lambda_s(p)$ is the process $p$ executed in state $s$. The function $\text{act}$ gives, for each action $a$ and state $s$, the action that results from executing $a$ in state $s$. The function $\text{eff}$ gives, for each action $a$ and state $s$, the state that results from executing $a$ in state $s$. The function $\text{sig}$ gives, for each state $s$, the proposition that holds at the start of any process executed in state $s$.

The additional axioms for $\lambda_s$, where $s \in S$, are given in Table 7. In this table, $a$ stands for an arbitrary constant from $A \cup \{\delta\}$ and $\phi$ stands for an arbitrary closed term of sort $B$. SO1–SO5 have been taken from [8].

The following equations are among the equations derivable from the axioms of $\text{ACP}_{\text{ps}}^+\text{SO}$:

$$\lambda_s(\bot) = \bot, \quad \lambda_s(\delta) = \text{sig}(s) \cdot \delta.$$ 

All closed $\text{ACP}_{\text{ps}}^+\text{SO}$ terms of sort $P$ can be reduced to a basic term.

**Proposition 6 (Elimination).** For all $\text{ACP}_{\text{ps}}^+\text{SO}$ closed terms $p$ of sort $P$, there exists an $q \in B$ such that $p = q$ is derivable from the axioms of $\text{ACP}_{\text{ps}}^+\text{SO}$. 

**Proof.** The proof goes along the same line as the proof of Proposition 2. $\square$

The additional transition rules for the state operators are given in Table 8. In this table, $a$ stands for an arbitrary constant from $A \cup \{\delta\}$ and $\phi$ stands for an arbitrary closed term of sort $B$. Bisimulation equivalence is a congruence with respect to the operators of $\text{ACP}_{\text{ps}}^+\text{SO}$.
Proposition 7 (Congruence). For all closed $\text{ACP}_{\text{ps}}^{\text{ct}}+\text{SO}$ terms $p, q, p', q'$ of sort $P$ and closed $\text{ACP}_{\text{ps}}^{\text{ct}}+\text{SO}$ terms $\phi$ of sort $B$, $p \leftrightarrow q$ and $p' \leftrightarrow q'$ imply $p + p' \leftrightarrow q + q'$, $p \cdot p' \leftrightarrow q \cdot q'$, $\phi : \rightarrow p \leftrightarrow \phi : \rightarrow q$, $\phi \cdot p \leftrightarrow \phi \cdot q$, $p \parallel p' \leftrightarrow q \parallel q'$, $p \parallel p' \leftrightarrow q \parallel q'$, $\partial H(p) \leftrightarrow \partial H(q)$, and $\lambda s(p) \leftrightarrow \lambda s(q)$.

Proof. The proof goes along the same line as the proof of Proposition 3. \hfill \Box

$\text{ACP}_{\text{ps}}^{\text{ct}}+\text{SO}$ is sound with respect to $\leftrightarrow$ for equations between closed terms.

Theorem 5 (Soundness). For all closed $\text{ACP}_{\text{ps}}^{\text{ct}}+\text{SO}$ terms $p, q$ of sort $P$, $p = q$ is derivable from the axioms of $\text{ACP}_{\text{ps}}^{\text{ct}}+\text{SO}$ only if $p \leftrightarrow q$.

Proof. The proof goes along the same line as the proof of Theorem 3. \hfill \Box

$\text{ACP}_{\text{ps}}^{\text{ct}}+\text{SO}$ is complete with respect to $\leftrightarrow$ for equations between closed terms.

Theorem 6 (Completeness). For all closed $\text{ACP}_{\text{ps}}^{\text{ct}}+\text{SO}$ terms $p, q$ of sort $P$, $p = q$ is derivable from the axioms of $\text{ACP}_{\text{ps}}^{\text{ct}}+\text{SO}$ if $p \leftrightarrow q$.

Proof. The proof goes along the same line as the proof of Theorem 4. \hfill \Box

8 Guarded Recursion

In order to allow for the description of processes without a finite upper bound to the number of actions that it can perform, we add in this section guarded recursion to $\text{ACP}_{\text{ps}}^{\text{ct}}$ and $\text{ACP}_{\text{ps}}^{\text{ct}}+\text{SO}$. The resulting theories are called $\text{ACP}_{\text{ps}}^{\text{ct}}+\text{REC}$ and $\text{ACP}_{\text{ps}}^{\text{ct}}+\text{SO}+\text{REC}$, respectively.

A recursive specification over $\text{ACP}_{\text{ps}}^{\text{ct}}$ is a set of recursion equations $E = \{X = t_X \mid X \in V\}$ where $V$ is a set of variables of sort $P$ and each $t_X$ is a term of sort $P$ that only contains variables from $V$. We write $V(E)$ for the set of all variables that occur on the left-hand side of an equation in $E$. A solution of a recursive specification $E$ is a set of processes (in some model of $\text{ACP}_{\text{ps}}^{\text{ct}}$) $\{P_X \mid X \in V(E)\}$ such that the equations of $E$ hold if, for all $X \in V(E)$, $X$ stands for $P_X$.

Let $t$ be a $\text{ACP}_{\text{ps}}^{\text{ct}}$ term of sort $P$ containing a variable $X$. We call an occurrence of $X$ in $t$ guarded if $t$ has a subterm of the form $a \cdot t'$, where $a \in A$, with $t'$ containing this occurrence of $X$. A recursive specification $E$ over $\text{ACP}_{\text{ps}}^{\text{ct}}$ is called a guarded recursive specification if all occurrences of variables in the right-hand sides of its equations are guarded or it can be rewritten to such a recursive specification using the axioms of $\text{ACP}_{\text{ps}}^{\text{ct}}$ in either direction and/or the equations in $E$ from left to right. We are only interested in a model of $\text{ACP}_{\text{ps}}^{\text{ct}}$ in which guarded recursive specifications have unique solutions.

For each guarded recursive specification $E$ over $\text{ACP}_{\text{ps}}^{\text{ct}}$ and each variable $X \in V(E)$, we add a constant of sort $P$, standing for the unique solution of $E$ for $X$, to the constants of $\text{ACP}_{\text{ps}}^{\text{ct}}$. This constant is denoted by $(X | E)$. 

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### Table 9. Axioms for guarded recursion

| Equation | Description |
|----------|-------------|
| \( \langle X | E \rangle = \langle t_X | E \rangle \) if \( X = t_X \in E \) | RDP |
| \( E \rightarrow X = \langle X | E \rangle \) if \( X \in V(E) \) | RSP |

### Table 10. Transition rules for guarded recursion

| Rule | Description |
|------|-------------|
| \( \langle t_X | E \rangle \rightarrow X = \langle X | E \rangle \) if \( X = t_X \in E \) | RDP |
| \( \langle t_X | E \rangle \rightarrow x' = \langle X | E \rangle \) if \( X = t_X \in E \) | RSP |

We will use the following notation. Let \( t \) be a ACP ct term of sort \( P \) and \( E \) be a guarded recursive specification over ACP ct. Then we write \( \langle t | E \rangle \) for \( t \) with, for all \( X \in V(E) \), all occurrences of \( X \) in \( t \) replaced by \( \langle X | E \rangle \).

The additional axioms for guarded recursion are the equations given in Table 9. In this table, \( X \), \( t_X \), and \( E \) stand for an arbitrary variable of sort \( P \), an arbitrary ACP ct term, and an arbitrary guarded recursive specification over ACP ct, respectively. Side conditions are added to restrict the variables, terms and guarded recursive specifications for which \( X \), \( t_X \) and \( E \) stand. The additional axioms for guarded recursion are known as the recursive definition principle (RDP) and the recursive specification principle (RSP). The equations \( \langle X | E \rangle = \langle t_X | E \rangle \) for a fixed \( E \) express that the constants \( \langle X | E \rangle \) make up a solution of \( E \). The conditional equations \( E \rightarrow X = \langle X | E \rangle \) express that this solution is the only one.

The additional transition rules for the constants \( \langle X | E \rangle \) are given in Table 10. In this table, \( X \), \( t_X \) and \( E \) stand for an arbitrary variable of sort \( P \), an arbitrary ACP ct term and an arbitrary guarded recursive specification over ACP ct, respectively.

Bisimulation equivalence is a congruence with respect to the operators of ACP ct + REC.

**Proposition 8 (Congruence).** For all closed ACP ct + REC terms \( p, q, p', q' \) of sort \( P \) and closed ACP ct + REC terms \( \phi \) of sort \( B \), \( p \equiv q \) and \( p' \equiv q' \) imply \( p + p' \equiv q + q' \), \( p \cdot p' \equiv q \cdot q' \), \( \phi \rightarrow p \equiv \phi \rightarrow q \), \( \phi \wedge p \equiv \phi \wedge q \), \( p \parallel p' \equiv q \parallel q' \), \( \partial_H(p) \equiv \partial_H(q) \).

**Proof.** The proof goes along the same line as the proof of Proposition 3.

ACP ct + REC is sound with respect to \( \equiv \) for equations between closed terms.

**Theorem 7 (Soundness).** For all closed ACP ct + REC terms \( p, q \) of sort \( P \), \( p = q \) is derivable from the axioms of ACP ct + REC only if \( p \equiv q \).
**Proof**. Because of Proposition 8 it is sufficient to prove the theorem for all closed $\text{ACP}_{\text{ps}}^{\text{ct}}+\text{REC}$ terms $p$ and $q$ for which $p = q$ is a closed substitution instance of an axiom of $\text{ACP}_{\text{ps}}^{\text{ct}}+\text{REC}$. With the exception of the closed substitution instances of RSP, the proof goes along the same line as the proof of Theorem 9. The proof of the validity of RSP is rather involved. We confine ourselves to a very brief outline of the proof. The transition rules for $\text{ACP}_{\text{ps}}^{\text{ct}}+\text{REC}$ determines a transition system for each process that can be denoted by a closed $\text{ACP}_{\text{ps}}^{\text{ct}}+\text{REC}$ term of sort $P$. A model of $\text{ACP}_{\text{ps}}^{\text{ct}}+\text{REC}$ based on these transition systems can be constructed along the same line as the models of a generalization of ACPps constructed in [12]. An equation $p = q$ between closed $\text{ACP}_{\text{ps}}^{\text{ct}}+\text{REC}$ terms holds in this model iff $p \models q$. Based on this model, the validity of RSP can be proved along the same line as in the proof of Theorem 10 from [12]. The underlying ideas of that proof originate largely from [9].

Guarded recursion can be added to $\text{ACP}_{\text{ps}}^{\text{ct}}+\text{SO}$ in the same way as it is added to $\text{ACP}_{\text{ps}}^{\text{ct}}$ above, resulting in $\text{ACP}_{\text{ps}}^{\text{ct}}+\text{SO}+\text{REC}$. It is easy to see that the above results, i.e. Proposition 8 and Theorem 7, go through for $\text{ACP}_{\text{ps}}^{\text{ct}}+\text{SO}+\text{REC}$. $\text{ACP}_{\text{ps}}^{\text{ct}}+\text{REC}$ and $\text{ACP}_{\text{ps}}^{\text{ct}}+\text{SO}+\text{REC}$ are not complete with respect to $\models$ for equations between closed terms. Completeness can be obtained by restriction to the guarded recursive specifications that are linear, i.e. the ones of which the right-hand sides of the recursion equations can be written in the form $\chi \land \delta + \sum_{i \in \{1, \ldots, n\}} \phi_i \rightarrow a_i \cdot X_i + \sum_{j \in \{1, \ldots, m\}} \psi_j \rightarrow b_j$, where $n, m \in \mathbb{N}$, where $\chi / \notin [\mathcal{F}]$, where $\phi_i / \notin [\mathcal{F}]$, $a_i \in A$, and $X_i$ is variable of sort $P$ for all $i \in \{1, \ldots, n\}$, and where $\psi_j / \notin [\mathcal{F}]$ and $b_j \in A$ for all $j \in \{1, \ldots, m\}$.

### 9 Concluding Remarks

We have presented $\text{ACP}_{\text{ps}}^{\text{ct}}$, a version of ACPps built on a paraconsistent propositional logic called $\text{LP}_{\supset \lor \land \land}$. $\text{ACP}_{\text{ps}}^{\text{ct}}$ deals with processes with possibly self-contradictory states by means of this paraconsistent logic. To our knowledge, processes with possibly self-contradictory states have not been dealt with in any theory or model of processes. This leaves nothing to be said about related work. However, it is worth mentioning that the need for a theory or model of processes with possibly self-contradictory states was already expressed in [24].

In order to streamline the presentation of $\text{ACP}_{\text{ps}}^{\text{ct}}$, we have left out the terminal signal emission operator, the global signal emission operator, and the root signal operator of ACPps and also the additional operators introduced in [8] other than the state operators. To our knowledge, these are exactly the operators that have not been used in any work based on ACPps. The root signal operator is an auxiliary operator which can be dispensed with and the global signal emission operator is an auxiliary operator which can be dispensed with in the absence of the terminal signal emission operator. The terminal signal emission operator makes it possible to express that a proposition holds at the termination of a process.
ACP\textsuperscript{ct}ps is a contradiction-tolerant version of ACPps \cite{8}. ACPps itself can be viewed as a simplification and specialization of ACPS \cite{7}. The simplification consists of the use of conditions instead of special actions to observe signals. The specialization consists of the use of the set of all propositions with propositional variables from a given set instead of an arbitrary free Boolean algebra over a given set of generators. Later, the generalization of ACPps to arbitrary such Boolean algebras has been treated in \cite{15}. Moreover, a timed version of ACPps has been used in \cite{14} as the basis of a process algebra for hybrid systems and a timed version of ACPps has been used in \cite{17} to give a semantics to a specification language that was widely used in telecommunications at the time.

Timed versions of ACP\textsuperscript{ct}ps may be useful in various applications. We believe that they can be obtained by combining ACP\textsuperscript{ct}ps with a timed version of ACP, such as ACP\textsuperscript{drt} or ACP\textsuperscript{srt} from \cite{10}, in much the same way as timed versions of ACPps have been obtained in \cite{13,17}. Because idling of processes is taken into account, two forms of the guarded command operator can be distinguished in these timed versions, namely a non-waiting form and a waiting form (see e.g. \cite{17}). A version of ACP\textsuperscript{ct}ps with abstraction features like in ACP\textsuperscript{t} (see e.g. \cite{11}) may be useful in various applications as well. Working out a timed version of ACP\textsuperscript{ct}ps and working out a version of ACP\textsuperscript{ct}ps with abstraction features are options for further work. It is very important that case studies are carried out in conjunction with the theoretical work just mentioned to assess the degree of usefulness in practical applications.

LP\textsuperscript{3,5} is Blok-Pigozzi algebraizable. However, although there must exist one, a conditional-equational axiomatization of the algebras concerned has not yet been devised. Owing to this, the equations derivable in ACP\textsuperscript{ct}ps cannot always be derived by equational reasoning only. Another option for further work is devising the axiomatization referred to.

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