Predictability of the Appearance of Anomalous Waves at Sufficiently Small Benjamin-Feir Indices

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The numerical simulation of the nonlinear dynamics of random sea waves at moderately small Benjamin-Feir indices and its comparison with the linear dynamics (at the coincidence of spatial Fourier harmonics near a spectral peak at a certain time $t_p$) indicate that the appearance of a rogue wave can be predicted in advance. If the linear approximation shows the presence of a sufficiently extensive and/or high group of waves in the near future after $t_p$, an anomalous wave is almost necessarily formed in the nonlinear model. The interval of reliable forecasting covers several hundred wave periods, which can be quite sufficient in practice for, e.g., avoiding the meeting of a ship with a giant wave.

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Anomalous waves or rogue waves (freak waves, giant waves) are of both applied and theoretical interest. They are rare individual extremely high waves among moderate waves. This subject is actively studied (see, e.g., reviews [1-3], special issues of journals [4, 5], and numerous references therein). Extremely high and steep waves with heights reaching 20-30 m at a length of 200-250 m are dangerous even for large ships and platforms [6, 7].

One of the characteristic properties of a rogue wave is its sudden appearance: “a giant wave appears out of nowhere and disappears without any trace”. For this reason, it seems hopeless to predict an anomalous wave at least two hundred wave periods before its appearance in order to take measures. This is generally the case [8]. However, there is a widespread natural regime of seas where the place and time of the appearance of a rogue wave can be simply and efficiently forecasted. This regime is studied in this work. It should be emphasized that here we simulate essentially three-dimensional flows of a liquid with a two-dimensional free surface, which cannot be analyzed with the methods and results reported in [9-15], where similar problems were studied for planar flows primarily within the focusing one-dimensional nonlinear Schrödinger equation.

Let the main energy-carrying part of the random wave spectrum be concentrated near the wave vector $k_0$. We shall speak about sea states with moderately low Benjamin-Feir indices, $I_{BF} \sim \tilde{v} k_0 A \lesssim 1$, where $\tilde{v}$ is the average number of waves in a group, $k_0$ is the characteristic wavenumber, and $A$ is the typical amplitude of a wave. This dimensionless quantity appears naturally when describing a quasimonochromatic wave within the nonlinear Schrödinger equation [16, 17] as a parameter characterizing the presence of nonlinear coherent structures. The case under consideration implies that such structures are almost completely absent. The situation where $\tilde{v} = 3\ldots 5$ and the vertical standard deviation of the (two-dimensional) free surface is $\sigma = (0.005\ldots 0.007) \lambda_0$, where $\lambda_0 = 2\pi/k_0$ is the typical wavelength, will be considered below. It can be accepted that $\tilde{A} = 2\sigma$; in this case, $I_{BF} = 0.3\ldots 0.5$. Strictly speaking, wave groups on the two-dimensional surface are characterized by additional important parameters, in particular, by the characteristic area occupied by a group. This parameter depends on the length of crests. It is assumed that the length of wave crests is $\tilde{l}_c \sim \tilde{v} \lambda_0$; i.e., motion of the liquid is assumed sufficiently three-dimensional. This wave regime is noticeably distinguished in low nonlinearity from quasi-two-dimensional (long-crested) waves [18-22] and, the more so, from plane waves [23-26], as well as from long-correlated three-dimensional wave fields with large BenjaminFeir indices [27-31]. In addition, the action of wind on the main part of the wave state, large-scale inhomogeneous flows, vorticity of the liquid, etc., are neglected.

The dominant dynamic factor in such random fields is the linear dispersion with the law $\omega_k = (gk)^{1/2}$, where $g$ is the gravitational acceleration (waves on deep water are considered). Nonlinearity generally plays a secondary role except for rare events where dispersion accidentally begins to form a sufficiently extensive and/or high group of waves with the local index $\nu k_0 A \gtrsim 1$. For example, two moderate wave groups can collide with each other. In this case, nonlinearity is strongly manifested and significantly distorts the two-dimensional plot of the envelope of such a group as compared to the linear theory, making it narrower in the direction of propagation of waves and relatively longer in the transverse direction [32-34]. As a result, this leads to formation of a single, extremely nonlinear wave with a sharp, sometimes even breaking crest (or two or three closely located anomalous waves when the orientation of the elongated envelope of the group is inclined; this configuration is a possible reason for the term “three sisters” in sea folklore).

The above scenario implies that the vertical deviation of the free surface until the appearance of a large group...
should be satisfactorily described by the simple formulas

\[ \eta(r, t) \approx \text{Re} A(r, t), \quad A(r, t) = \sum_k \alpha_k e^{i(kr - \omega_k t)}, \quad (1) \]

and the Fourier components \( \alpha_k(t) \) at \( k \) values near the spectral peak are slowly varying functions of time such that their variations hardly affect the wave dynamics during several hundred wave periods \( T_0 = 2\pi/\omega_0 \) (although dispersion noticeably transforms the picture of the wave field in this time interval). Thus, a reasonable approximation is

\[ \alpha_k(t) \approx \alpha_k(t_p), \quad \text{if} \quad |t - t_p| \lesssim T_0/k_0^2 \sigma^2. \quad (2) \]

This estimate follows from a known formula for the nonlinear frequency shift of a quasimonochromatic wave

\[ \delta \omega_{\text{nonlin}} = \omega_0 k_0^2 |A|^2/2. \]

Formulas (1) and (2) correspond to the so-called second-order theory (see reviews [1-3] and references therein), where resonant four-wave processes are neglected, whereas nonresonant three-wave interactions are taken into account in the form of the second harmonic.

It is important that Eqs. (1) and (2) allow forecasting the appearance or absence of a rogue wave at \( t > t_p \) if the spatial spectrum \( \eta_k \) of the vertical deviation of the free surface is measured at the time \( t_p \) in a sufficiently large region of the sea. A map of the sea surface can be obtained from satellites and be digitized; then, the fast Fourier transform can be performed. The forecasting procedure is as follows.

First, the complex conjugate harmonics of the real function \( \eta(r, t_p) \) near \( -k_0 \) should be rejected, and \( \alpha_k = 2\eta_k \) should be set at \( +k_0 \). It is also desirable to filter high wavenumbers, retaining only the energy-carrying vicinity of the spectral peak. This will allow using a large spatial resolution and accelerating computer calculations.

Second, it is necessary to perform a linear forecast of the sea state by Eqs. (1) and (2) with a time step of about \( 12T_0 \). Such a forecast on a PC takes about a minute. In particular, the time dependence of the maximum \( |A| \) value in the region under consideration should be plotted.

Further, the dependence \( |A|_{\text{max}}(t) \) should be analyzed. The possible appearance of a maximum on the plot at \( t_*(t_p) \) soon after \( t_p \) with the height exceeding, say, \( 6\sigma \) is a serious indication of the nucleation of an anomalous wave.

Additional information can be obtained from the map of the sea surface at \( t = t_*(t_p) \) plotted according to linear forecasting. The place of the appearance of the rogue wave, as well as the local Benjamin-Feir index, can be estimated on the map. The larger the local Benjamin-Feir index, the larger the expected nonlinear increase in the height of the wave as compared to the linear prediction.

Three numerical examples presented below demonstrate the efficiency of the above method for forecasting anomalous waves. The wave dynamics was calculated within the completely nonlinear weakly three-

FIG. 1: Anomalous waves formed in three numerical experiments.
dimensional model [35, 36] in a $2\pi \times 2\pi$ square with periodic boundary conditions and the dimensionless gravitational acceleration $\tilde{g} = 1$, where waves propagate on average along the x axis. The characteristic wavenumber was chosen to be 50. In the recalculation to a square with a side of 5 km and $g = 9.81 \text{ m/s}^2$, the dimensionless time unit corresponds to $\tau = \left[ \frac{5000}{(9.81 \cdot 2\pi)} \right]^{1/2} = 9.01 \text{ s}$. In this case, the wave-length is $\lambda_0 = 100 \text{ m}$ and the period is $T_0 \approx 8 \text{ s}$.

FIG. 2: Extremal deviations of the free surface in comparison with linear forecasts for different $t_p$.

FIG. 3: Linear forecasts indicate the formation of sufficiently high groups of waves.
In all three numerical experiments, quasirandom initial data for nonlinear simulation were chosen by means of a special procedure (which is not described here) such that an anomalous wave is formed in the system in a time of $(250-350)T_0$. The corresponding portraits of these rogue waves near the times of their highest elevation are shown in Fig. 1. All three waves undergo breather oscillations because of the difference between the phase and group velocities, particularly, the “straight” wave C (in contrast to “inclined” waves A and B, where crests and troughs seemingly move along the inclined elongated envelope from its “end” to the “beginning”; this motion can be visually observed as the lateral propagation of the anomalous wave).

Figure 2 shows the results of nonlinear calculations of the height of the highest crest and the depth of the deepest trough in comparison to linear forecasts made at different times $t_p$. As a whole, the closer the time $t_p$ to the time of rise of the rogue wave, the clearer the prediction of the formation of a high group by the linear theory. Figure 3 shows examples of such linear groups. Differences from anomalous waves in Fig. 1 are very significant. It is seen that even the longest group in case B “has been transformed” by nonlinearity in a single, very high wave in spite of a somewhat smaller average Benjamin-Feir index. It is remarkable that a large group appears owing to the collision of two wave groups (not shown in the figures), which confirms the assumption made in [34] that this mechanism is topical. In Fig. 1b, two deep troughs are located slantwise on two opposite sides of the high crest near it and two less high crest are located further. In general, such a configuration corresponds to three sisters. The height of the crest in this case is more than an order of magnitude (!) larger than the standard deviation and the crest itself is very sharp, as is seen in Fig. 4. It is clear that the second-order theory is inapplicable to the description of such really anomalous waves. Furthermore, the rogue wave in case C rises noticeably earlier in time (and closer along the trajectory of motion) than that predicted by the linear model. Such nonlinear effects should be taken into account when forecasting the time and place of the appearance of a large wave.

It is also noteworthy that, in very rare cases, a long-term forecast can predict the appearance of a large group in a time of, e.g., $(200\ldots300)T_0$ after $t_p(1)$, but an anomalous wave does not appear in reality. However, later forecasts with $t_p(2) > t_p(1)$, $t_p(3) > t_p(2)$, etc., approaching the forecasting time will show gradually smaller linear groups at the indicated time. For this reason, the method is sufficiently reliable.

According to the linear estimate of the frequency of appearance of extreme waves, a high group exceeding 6$\sigma$ appears on average once in $(4\ldots8) \times 10^4 T_0$ on an area of $(50)^2 \lambda_0^2$. This means that a giant wave at the taken parameters appears on an area of, e.g., $50 \times 50$ km$^2$ once in several hours. Obviously, less anomalous but quite high waves in the form of groups of two or three waves appear much more frequently. They correspond to smaller linear groups in the forecast (they are not shown in the figures).

To summarize, the fundamental possibility of advanced prediction of anomalous waves in sea states with sufficiently small Benjamin-Feir Indices within the linear model has been demonstrated. The problem of possible implementation and applied usefulness of such an approach requires additional studies and discussions. In particular, it is still unclear how the interaction of waves with wind and other factors disregarded in our model can change the quality of the forecast. For this reason, details of the method such as the choice of the sizes of the forecasting region and boundary conditions at the Fourier transformation, as well as the possibility of estimating the parameters of anomalous waves in the nonlinear stage by using the Gaussian variation model considered in [33] or by modifying the forecasting equation by introducing local nonlinearity such as the nonlinear Schrödinger equation and using the split-step method, were not discussed.

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