Quantum and Classical Dissipative Effects on Tunnelling in Quantum Hall Bilayers

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We discuss the interplay between transport and dissipation in quantum Hall bilayers. We show that quantum effects are relevant in the pseudospin picture of these systems, leading either to direct tunnelling currents or to quantum dissipative processes that damp oscillations around the ground state. These quantum effects have their origins in resonances of the classical spin system.

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Over the past ten years, a series of elegant experiments on quantum Hall bilayer systems have allowed investigation of the correlated many-body quantum state at total filling fraction v = 1. The bilayers consist of two closely-spaced parallel two-dimensional electron layers in a double quantum well. If the Landau level fillings are \( \nu_1 = \nu_2 = 1/2 \) then separate layers will not show the quantum Hall effect. However, Coulomb interactions between the layers drive a transition into a state in which the bilayer as a whole exhibits quantised Hall conductance. Predictions that these states would undergo spontaneous breaking of a U(1) symmetry have been supported by experiments.

The ground state of the system may be viewed as an easy-plane ferromagnet. There also are analogies with Josephson junctions and excitonic superfluidity. The transport properties of the bilayers are rich and often surprising. They provide experimental evidence for the Goldstone mode associated with the broken U(1) symmetry and, more recently, ‘excitonic superfluidity’.

The interlayer tunnelling properties of the bilayer remain a subject of considerable theoretical study. In a previous paper, we introduced a simple thought experiment in which we investigated the dissipation and interlayer tunnelling properties of a clean bilayer system at zero temperature, and gave a brief comparison of our theoretical predictions with the experimental results. A key point is the existence of a crossover in behaviour at a very small bias. As mentioned in Ref. 17, we predict a tunnelling current that varies as \( 1/V \) in that region. This negative differential conductance is in qualitative agreement with experiments.

In this paper, we extend the discussion of Ref. 17, including details of the behaviour for \( V < V_0 \), and a discussion of the relationship between quantum dissipative processes and instabilities of a classical spin system. Section I contains a discussion of the model and of our thought experiment. In section II we discuss the results both for \( V > V_0 \), and for \( V < V_0 \). In the latter regime, the theoretical treatment is made rather complicated by the presence of an out of equilibrium Bose condensate in the low energy bosonic theory of the bilayer: we show how these issues may be resolved. In section III we show how the quantum dissipative mechanisms in the system arise from instabilities of an underlying classical spin system. Finally we draw conclusions in section IV, and identify outstanding issues in our treatment.

I. THE MODEL

In this section, we define our model, and describe the thought experiment that we will use to investigate the link between dissipation and tunnelling. The bilayer system may be cast in a pseudospin formulation. The layer index of each electron defines a two dimensional Hilbert space: it resembles the usual spin-(1/2) degree of freedom of electrons. We assume throughout that physical electron spin is completely polarised by the applied magnetic field. In the quantum Hall state, we may work in a basis of localised orthogonal states in the lowest Landau level. Configurations in which two electrons occupy the same localised state represent quasiparticles, and have a large energy gap. Therefore, a theory of discrete spins \( S = 1/2 \) on a lattice is appropriate at low energies. The spins interact by an exchange interaction with an easy plane, and experience a field in the easy plane that arises from tunnelling between the layers.

In our calculations, we generalise the model to \( S > 1/2 \): this generalisation may be treated as a coarse-graining procedure. The limit of large \( S \) is also the classical limit for the spin system. After coarse-graining, we replace the easy-plane exchange interaction by a combination of isotropic exchange and an on-site anisotropy term. The resulting Hamiltonian is:

\[
\frac{H}{2S} = -\frac{\rho_E}{2} \sum_{(ij)} \vec{m}_i \cdot \vec{m}_j + \frac{D}{4} \sum_i (m_i^z)^2 - \frac{\Delta_{\text{SAS}}}{2} \sum_i m_i^z (1)
\]

where \( \vec{S}_i = \vec{S} \vec{m}_i = S(m_x^i, m_y^i, m_z^i) \) is the spin operator on site \( i \) of a square lattice with spacing \( c_0 = \sqrt{2\pi l_B} \), where \( l_B = (\hbar c/eB)^{1/2} \) is the magnetic length. The interlayer exchange \( \rho_E \) and the strength of the on-site repulsion \( D \)
were derived from microscopic considerations by Moon et al. \((D = 8\pi\beta l_B^2\) in the notation of that paper). The tunnelling between the layers enter the problem through \(\Delta_{\text{SAS}}\): the splitting of the “bonding” and “anti-bonding” single-particle states in the double well. We use a Hamiltonian with isotropic exchange purely for simplicity: the interlayer and intralayer exchange constants will differ in general. However, if we generalise to an exchange interaction that is anisotropic (in the spin space), then only the interlayer part is relevant at the large lengthscales and small charge imbalances that are relevant to our calculations (we require \(m^2 \ll 1\), and \(\rho E q^2 \ll D\) for all relevant wavevectors, \(q\)). Since these conditions are satisfied for the experimental comparisons that we make, we set both the interlayer and intralayer exchange constants to their interlayer value, \(\rho E\). In our thought experiment, a gate is used to control the charge imbalance on the bilayer: this adds a term \(HV = -SV \sum_i m_i^z\) to the Hamiltonian.

Typical values for the model parameters in physical bilayer systems are \(l_B \simeq 20\text{nm}, \Delta_{\text{SAS}} \simeq 90\mu\text{K}, \rho E \simeq 0.5\text{K}, D \simeq 30\text{K}\).

Hamiltonians of the form (1) permit both ferromagnetic and quantum disordered phases at zero temperature. At large \(D\) (or small \(S\)), the uncertainty in \(S^2\) associated with ferromagnetic order means that the paramagnetic phase with \(S^2 \rightarrow 0\) becomes energetically favoured. We interpret the observation of a linearly dispersing peak in the tunnelling conductance as evidence that the experimentally accessible states have ferromagnetic order — the peak arises from the Goldstone mode of the system. In other words, we assume that the system is connected adiabatically to the large-\(S\) limit of our model, ignoring the possibility of the quantum disordered phase that exists for \(D/\rho ES^2 \gg 1\).

We seek a minimal theory of the bilayer. We therefore work in a clean system at \(T = 0\). This allows us to calculate dissipation rates starting from purely microscopic considerations: this is distinct from the more phenomenological approach of Ref. 12 in which dissipation is added to the model in the form of damping terms in the classical equations of motion. Simulations\(^1\) indicate that disorder may be relevant to the tunnelling at small bias voltages. While these effects are beyond the scope of this paper, we argue below that our framework may be suitable for further investigation of these effects. Possible effects of topological defects such as merons\(^2\) are also neglected in our treatment, as are inhomogeneities in the order parameter associated with the spontaneous symmetry breaking\(^3\).

A feature of our framework is that we work on transport in an isolated bilayer. This means that our calculations are well-controlled. Coupling the bilayer to leads, as in experiments, complicates the picture. We believe that the bulk effects that we study should dominate over boundary effects associated with the leads, such as those studied by Park\(^4\), especially in the (experimentally relevant) geometry where the current source and drain are separated by a distance of the order of the sample size.

![FIG. 1: Sketch of the constant energy trajectories on the spin sphere.](image)

Having reduced the system to a minimal model, we now proceed to discuss the behaviour of the spin system of equation (1).

### A. Classical limit of the spin system

In the ferromagnetic phase, we expect the qualitative behaviour of the system to be accessible through an expansion about the classical limit. In this limit, an initially uniform configuration of the spins remains uniform forever. Each spin precesses along one of the trajectories of constant energy shown in figure (1). There are two qualitatively different types of trajectory in that figure: those which wind around the \(x\)-axis and those which wind around the \(z\)-axis. This distinction underlies the crossover in the tunnelling \(IV\) relation mentioned above. We now discuss the form of the trajectories and the classical properties of the spin system.

Trajectories with low energies are localised near the ground state, in which the spins are aligned along the \(x\) axis. The spins precess around that axis with a frequency \(\Delta_{sw}/\hbar\) where

\[
\Delta_{sw} = [\Delta_{\text{SAS}}(\Delta_{\text{SAS}} + D)]^{1/2}
\]

is the energy gap for spinwave excitations. These excitations describe density waves of charge imbalance across the bilayer. For the case \(\Delta_{\text{SAS}} = 0\) these excitations form the Goldstone mode that appears when the symmetry around the easy plane is spontaneously broken. In that case they disperse linearly with a velocity \(v = l_B(2\pi D\rho E)^{1/2}\). Introducing the tunnelling \(\Delta_{\text{SAS}}\) breaks the symmetry explicitly, and the gap opens up. As the mean magnetisation precesses around the \(x\)-axis, the charge imbalance on the bilayer oscillates around zero.

For trajectories with much larger energies, the mean spin precesses around the maximal energy state, which is close to the \(z\)-axis of the spin sphere. This yields a Josephson-like alternating current \(I \simeq e\Delta_{\text{SAS}} \cos(eVt/\hbar)\) where \(V\) is the voltage across the bilayer due to capacitive charging. We stress that this is valid only for large charge imbalance (large \(V\)).

A saddle point on the negative \(x\)-axis of the spin sphere marks the boundary between oscillations around the ground state and oscillations around the maximal energy
state. The saddle point trajectory crosses the $xz$-plane at the saddle point, and at $(x, z) = (S \cos \theta_0, S \sin \theta_0)$ where $\theta_0$ is the angle shown in Fig. 1. It is given by $\cos \theta_0 = 1 - 2(\Delta_{SAS}/D)$.

When the gate is used to induce a charge imbalance on the bilayer, the mean spin is tilted out of the easy plane by an angle $\theta$, which satisfies:

$$eV = D \sin \theta + \Delta_{SAS} \tan \theta$$  \hspace{1cm} (3)

where $V$ is the gate voltage. The saddle point then corresponds to a voltage difference of

$$V_0 = (2\Delta_{sw}/e)[1 + \mathcal{O}(\Delta_{SAS}/D)]$$  \hspace{1cm} (4)

across the layers. (We write $V_0$ in this form since the tunnelling energy scale set by $\Delta_{SAS}$, is much smaller than the charging energy scale set by $D$.) Since the dynamics of this system depends strongly on whether the applied bias is above or below this threshold value, the calculations we describe below will treat these two regimes separately.

### B. A thought experiment

We now turn to our thought experiment which will form the basic strategy behind our calculations. We imagine using a gate to induce a uniform charge imbalance on the bilayer. In the spin picture, the magnetisation is tilted out of the easy $(xy)$ plane with

$$m^z = \sin \theta \approx eV/D$$  \hspace{1cm} (5)

where $V$ is the voltage bias applied across the bilayer and the second approximate equality is valid to leading order in $(\Delta_{SAS}/D)$ and $(eV/D)$, both of which will be small (recall equation (3)). The magnetisation remains in the $xz$-plane due to the tunnelling field. The bias is then instantaneously removed and the bilayer finds itself in a highly excited state. In this paper, we will calculate the quantum and classical dynamics of this highly excited system.

From the discussion of the classical dynamics, we expect the thought experiment to give different results according to whether the initial gate voltage $V$ is above or below the saddle point value $V_0$. For small initial energy, $V < V_0$, the charge imbalance oscillates around zero: the effect of any dissipation will be to damp this oscillation (lower trace in figure 2). Conversely, if the initial charge is larger, the oscillation is about a finite charge imbalance: dissipation leads to a decay of that charge imbalance, and therefore a direct current (upper trace in figure 2). The link between dissipation and transport is clear in this case, since dissipative processes lead directly to a direct current.

In our calculations, we will treat these two regimes in separate perturbation theories. Common to both calculations, we will expand around the classical limit using spinwave theory in a $1/S$ expansion. In a quantum theory of the spin system, the coherent oscillations of the charge imbalance do not remain coherent forever. This is different from the classical case and is consistent with the mixing between modes with different wavevectors observed in the exact diagonalisations of Nakajima. Dissipation is possible because the tunnelling term breaks the global spin rotation symmetry so that the long-wavelength modes are no longer protected from decay by Goldstone’s theorem. We will find that the leading terms in the expansion about the classical limit result in the leading terms in the relations between the dissipation rate and the initial gate voltage, $V$.

## II. DISSIPATION RATES IN THE QUANTUM SYSTEM

In this section, we calculate the dissipation rates out of the coherent classical oscillation of the spin system. As shown in figure 2, the effect of the dissipation is different, depending on the initial charge on the bilayer. However, in both cases, the dissipation rate, $\Gamma$ is given by

$$\Gamma = \partial_t \langle i | e^{iHt} \sum_{q\neq 0} E_q e^{-iHt} | i \rangle$$  \hspace{1cm} (6)

where $E_q$ measures the energy in the bosonic mode with momentum $q$ and $| i \rangle$ is an initial state for the system. The procedure used to calculate the dissipation rate is different in each case.

The smallest energy scale in the physical bilayer systems is $\Delta_{SAS}$, so it seems natural to treat this parameter perturbatively. However, the tunnelling term breaks a continuous symmetry of the Hamiltonian, so care must be taken in the limit $\Delta_{SAS} \rightarrow 0$. It is clear from figure 1 that the trajectories with energies above the saddle point energy are perturbatively connected to the $\Delta_{SAS} = 0$ limit. Conversely, trajectories with very low energies are qualitatively different to those in the limit of zero tunnelling: in this case perturbation theory in $\Delta_{SAS}$ is not appropriate. Instead, we treat the energy of the trajectory (or equivalently the initial gate voltage) perturbatively.
We now calculate the extent to which quantum dissipative processes affect charge relaxation on the bilayer, or damping of charge oscillations (recall figure 2).

A. Charge relaxation: $V > V_0$

We begin by considering the situation for initial energies above the saddle point ($V > V_0$). This calculation was outlined in Ref. 17. We review the method here since the calculation in the following subsection is a refinement of the same approach. Evaluation of $\Gamma$ requires the choice of an initial state, and calculation of matrix elements between that state and final states with different values of $E_q$. We use a bosonic representation of the spin algebra, which defines the basis for initial and final states. We then identify the initial state and the dominant process that leads to dissipation; this allows calculation of $\Gamma$.

To begin, we define our bosonic representation of the Hamiltonian, by using the Holstein-Primakov representation of the spin algebra $S_j^z = (2S - a_j^\dagger a_j)^{1/2}a_j$, $S_j^z = S - a_j^\dagger a_j$. We treat tunnelling perturbatively, so we begin by setting $\Delta_{\text{SAS}} = 0$ and expanding around $m^2 = (eV/D) = \sin \theta$. The quadratic part of the Hamiltonian is then easily diagonalised in the Fourier basis to give

$$H_{\theta > \theta_0}^{(0)} = (D \sin \theta)\delta n_q = 0 + \sum_{q \neq 0} \varepsilon_q a_q^\dagger a_q$$

(7)

where the $a_q$ are bosonic operators ($[a_q, a_q^\dagger] = \delta_{q,q'}$) describing the spinwave modes. The spinwave dispersion is given by

$$\varepsilon_q = [\rho E\gamma(q)(D + \rho E\gamma(q))]^{1/2}$$

$$\gamma(q) = 4 - 2 \cos(q_x c_0) - 2 \cos(q_y c_0).$$

As discussed above, the tunnelling is being treated perturbatively. Therefore, this dispersion relation is that of a model with spin rotational symmetry in the $xy$-plane, and $\varepsilon_q$ is gapless linear in $q$ at long wavelengths. The absence of an energy gap in this approximation does not affect our calculation for $V > V_0 = 2\Delta_{\text{SAS}}/\varepsilon$. Observe that the $q = 0$ mode has been singled out in the Hamiltonian, and its energy is not given by the long wavelength limit of $\varepsilon_q$. The quanta of this mode carry $S^z = 1$ while spin waves with finite wavevector have $S^z = 0$. Thus the eigenspectrum of the system consists of multiple branches: each branch has an associated $S^z$, and consists of a continuum of linearly dispersing collective modes. We use a basis $|Q, q, q', \ldots\rangle$ where $Q$ is the number of quanta in the $q = 0$ mode (which labels the branch), and the set of wavevectors indicate indicate the presence of collective modes with finite momenta.

The next step is to express the initial state of the system in this basis. The thought experiment stipulates an initial state with the expectation of the spin equal to $(S \cos \theta, 0, S \sin \theta)$: this is a coherent state. However, a choice of an eigenstate of $S^z$ with the same expectation of the energy leads to the same average rate. Thus we use $|i\rangle_{V > V_0} = |(eV/D)(SL^2), \{\}\rangle$.

The final step in the calculation is to identify the dominant process contributing to $\Gamma$. The dissipation arises from the destruction of one quantum in the $q = 0$ mode, and the generation of multiple spinwaves during tunnelling across the bilayer. To leading order in $1/S$, a pair of spin waves is excited, each with energy $eV/2$, but with opposite momenta (Fig. 3b). The relevant vertex in the interaction Hamiltonian is:

$$H_{\theta > \theta_0}^{(1)} = -\frac{\Delta_{\text{SAS}}}{8} \sum_q \gamma_2 a_q^\dagger a_q^\dagger + \text{h.c.}$$

(8)

in which $\tilde{\Delta}_{\text{SAS}} = \Delta_{\text{SAS}} \exp(-S^{-1} \sqrt{D/\rho E})$ is a renormalised tunnelling amplitude, and the vertex factor is given by $\gamma_2 = \cos\theta (u_q + v_q)^2 + 2 \sin \theta \sec^2 \theta - (u_q - v_q)^2 \sec^4 \theta$ where $u_q$ and $v_q$ are coherence factors: $(u_q + v_q)^2 = (D + \rho E\gamma(q))/\varepsilon_q$ with $u_q^2 - v_q^2 = 1$. The $\theta$-dependence arises because the $x$-component of the pseudospin depends on the angle $\theta$ of the spin with the easy plane, as well as the azimuthal angle $\phi$ around the plane. This dependence is weak for $\theta \ll 1$ when the charge imbalance is small compared to the Landau level filling.

Hence we calculate the power dissipation $\Gamma$ for a given initial voltage $V$, using $E_q = \varepsilon_q a_q^\dagger a_q$. The calculation proceeds according to Fermi’s Golden Rule. The steady-state tunnelling current density at a bias $V$ can then be computed from this dissipation by $I = \Gamma/V L^2 S$. We find:

$$\frac{I_{\theta > \theta_0}}{L^2 S} = I_{\theta > \theta_0} V = \frac{D\Delta_{\text{SAS}}^2}{32\pi\rho E L^2 S} \left[1 + X(\theta)\right]^2$$

(9)

where $X(\theta) = (\sec^2 \theta - 1)(\sec^2 \theta - 2 \sec \theta - 1)$ is small for a small charge imbalance ($\theta \ll 1$). The dissipation saturates at small initial gate voltages, leading rise to an increasing current as the gate voltage is reduced.

As discussed in Ref. 17, this form for the current is qualitatively consistent with the experiments of Ref. 2 in that the region of negative differential conductance is observed for $V > V_0 \sim 6uV$. From that experimental data it seems that the dissipation rate does increase with increased applied voltage rather than remaining constant: we attribute this increase to dissipative channels not captured within our approach. We also note that the current choice of an eigenstate of $S^z$ with the same expectation of the energy leads to the same average rate.
that flows in our thought experiment is the same current defined in Ref. 10, but that the calculations in that paper neglect the $O(1/S)$ contribution to the current that we calculate.

Quantitative comparison between theory and experiment is complicated by the fact that the area of the sample over which tunnelling takes place remains an open question. It may depend on the experimental geometry and on spatial fluctuations in the tunnelling amplitude. Our calculation gives a current density per unit area of bilayer. There is a momentum scale associated with the Aharonov-Bohm flux is introduced in the plane of the bilayer. There is a momentum scale associated with the new flux, given by $Q = (eB_yd/hc)^{1/2}$ where $d$ is the spacing between the two quantum wells, and takes a value around 30nm. With an appropriate gauge choice, the effect of the flux is to introduce a spatial dependence to the tunnelling term in the Hamiltonian: $\Delta_{\text{SAS}} \rightarrow \Delta_{\text{SAS}}(m^x \cos Qx + m^y \sin Qx)$.

The result is that a new dissipative process appears in which a single quantum of the $q = 0$ mode decays into a single spinwave with momentum $Q$. This leads to a contribution to the current at order $S^0$. In other words, even the purely classical spin system has a decaying oscillation in this case. Evaluating the leading terms in the $(1/S)$ expansion, we arrive at

$$I_{\theta>\theta_0, Q} = \frac{D \Delta_{\text{SAS}}^2}{16\pi\hbar B} \times \left[ 2\pi V^{-1} \delta(eV - vQ) + \frac{1}{2\rho eS} \frac{e}{\sqrt{(eV)^2 - (vQ)^2}} \Theta(eV - vQ) \right]$$

where $\Theta(x)$ is the step function. The second term is the two spinwave contribution that reduces to 0 at $Q = 0$.

The $\delta$-function in the first term indicates a resonance in the classical spin system when the capacitative energy released on transferring an electron across the bilayer matches the energy of the collective mode with momentum $Q$. This infinitely narrow form of the peak is clearly unphysical. This is because we are calculating a transition rate to a discrete level, by specifying both the momentum and the energy of the final one-spinwave state. The final state can itself decay at higher order in $1/S$ and so the delta function will be broadened.

Moreover, the delta function can be broadened by disorder or by finite temperature. In figure 4 we show the $IV$ characteristic of 10, where an uncertainty in the momenta of the collective modes has been introduced to mimic the effect of weak disorder. Quantitative calculation of the broadening introduced by these effects is beyond the scope of this paper: the degree broadening also controls the peak height, which again frustrates a detailed comparison with experiment. However, we emphasised in Ref. 17 that the asymmetric lineshape of this peak arises from the quantum corrections that we have calculated: these are multi-spinwave decay channels.

**B. Damping of oscillations: $V < V_0$**

We now turn to initial charging voltages $V < V_0$, in which the effect of dissipation is to damp oscillations in the charge imbalance on the bilayer. This behaviour was shown in the lower trace of figure 2. We note that this case is different from $V > V_0$ in that our thought experiment does not result in a state with a steady direct current. For that reason we cannot directly address the $IV$ relation at these small biases: the strong feature in the differential conductance observed experimentally at small $V$ is beyond the scope of this paper. However, the coherent oscillations of charge between the layers that are predicted by the classical limit of the spin system are not observed in experiments. The intrinsic dissipation rate that we calculate sets a minimal decay rate for these oscillations. Adding disorder or inelastic scattering to the model might increase this decay rate; we consider only the clean system in this paper, but the framework that we now describe could be extended to consider this case.

The calculation of $\Gamma$ is slightly more complex than for $V < V_0$, but proceeds along the same lines. The result is a slow (power law) decay of the oscillations predicted by the classical limit of the pseudospin system.

In order to obtain a bosonic theory, we again use the Holstein-Primakov representation. However, we may choose either the $x$ axis or $z$ axis as the principal axis for our bosonic theory. Choosing the $x$ axis is natural since that is the direction of the mean magnetisation. On the other hand, choosing the $z$ axis has the advantage that no interactions arise from the anisotropy term in the Hamiltonian. We have checked that the results we present below are independent of this choice.

Let us consider first the spinwave spectrum $\omega_q$. The non-interacting part of the Hamiltonian for the spinwaves takes the form

$$H_0 = \sum_q \omega_q \hat{a}_q^\dagger \hat{a}_q$$

$$V < V_0$$

**FIG. 4: Contributions to current at finite $Q$**
where $\beta_q^\dagger$ and $\beta_q$ are the creation and annihilation operators for a spinwave with momentum $q$ and

$$\omega_q = [(\Delta_{\text{SAS}} + \rho E\gamma(q))(D + \Delta_{\text{SAS}} + \rho E\gamma(q))]^{1/2}. \quad (12)$$

where $\gamma(q)$ was defined in section [IA] above. Since we are treating the tunnelling non-perturbatively, the spinwaves now have an energy gap of $\Delta_{\text{sw}} = [\Delta_{\text{SAS}}(D + \Delta_{\text{SAS}})]^{1/2}$.

The non-interacting eigenspectrum has multiple branches, labelled by the total number of spinwave quanta. This should be contrasted with the high-bias branches, labelled by the total number of spinwave $\Delta_w$ waves now have an energy gap of $\Delta_w$.

We can now discuss the Hamiltonian for our calculation in this low-bias regime. Our perturbative parameter $V < V_0$ describes the interactions within the condensate:

$$\langle N_0 - 3|H_{\text{int}}|N_0 \rangle = (L^2 S)\gamma_{4c}(n_0/S)^2 c_0^2 + \ldots \quad (18)$$

This seems to suggest that the condensate with $N_0 q = 0$ modes decays immediately without the generation of finite-$q$ modes. This divergence of the Fermi’s Golden Rule amplitude originates from the macroscopic occupation of the $q = 0$ mode.

In fact, these condensate interactions do not cause decay within the condensate. Instead, they contribute to a coherent non-linear evolution of all the condensed bosons: this effect is beyond the perturbative framework of Fermi’s Golden Rule. We will see in the next section that this evolution is part of the semiclassical spin dynamics. To avoid divergences, we need to modify the interaction-picture operators to take into account the condensate interactions: $O(t) = \exp[iHt/\hbar] \exp[-i\Delta t]$ where $H = H_0 + H_{\text{int}}$ contains the non-quadratic interaction terms in $\beta_q$.

We observe that the occupancy of the bosonic mode with $q = 0$ is proportional both to the area of the system and to $S$: the bosons in this mode are Bose-condensed with areal density $n_0$. We will see below that interactions within the condensate must be treated carefully.

We can now discuss the Hamiltonian for our calculation in this low-bias regime. Our perturbative parameter for $V < V_0$ is $(n_0/S)$. It is independent of $S$ and does not vanish in the classical limit. We will calculate the dissipation rate $\Gamma$ to leading order in this parameter. We give the full details of the Hamiltonian in the Appendix. It can be written in the form

$$H = H_0 + H_{\text{int}} + H_{\text{diss}} \quad (15)$$

$H_{\text{int}}$ describes the interactions within the condensate:

$$H_{\text{int}} = \frac{c_0^2}{L^2 S} \left[ \gamma_{4d}\beta_q^\dagger\beta_q^\dagger + \gamma_{4c}\beta_0^\dagger + \text{h.c.} \right] \quad (16)$$

and $H_{\text{diss}}$ contains dissipative processes involving finite-momentum spinwaves

$$H_{\text{diss}} = \frac{c_0^2}{L^2 S} \sum_{q \neq 0} \left[ \gamma_{4a} \beta_q^\dagger \beta_q^\dagger - \gamma_{4b} \beta_0^\dagger \beta_0 \right] + \gamma_{4c} \left( \beta_q^\dagger \beta_q + \beta_0^\dagger \beta_0 + \text{h.c.} \right) \quad (17)$$

where we have included only terms that affect the dissipation rate, $\Gamma$, to leading order in $(1/S)$. The vertex factors, $\gamma$, are functions of $q$ which depend on the choice of principal axis in the Holstein-Primakoff transformation. They are independent of $L$ and are finite in the large-$S$ limit.

Before discussing the dissipative processes, we note that we have isolated the interactions within the condensate in $H_{\text{int}}$ because these processes do not directly contribute to the dissipation of energy from the $q = 0$ modes into finite-momentum spinwaves. However, at first sight, they do appear to contribute to the decay of our initial state. Moreover, the rate for the process is apparently divergent in both the thermodynamic (large $L$) and classical (large $S$) limits! For instance, for the process $\gamma_{4c}$, Fermi’s Golden Rule would give a transition amplitude of

$$\langle N_0 - 3|H_{\text{int}}|N_0 \rangle = (L^2 S)\gamma_{4c}(n_0/S)^2 c_0^2 + \ldots \quad (18)$$

This seems to suggest that the condensate with $N_0 q = 0$ modes decays immediately without the generation of finite-$q$ modes. This divergence of the Fermi’s Golden Rule amplitude originates from the macroscopic occupation of the $q = 0$ mode.

The effect of the non-linear evolution of the $q = 0$ mode is illustrated by the annihilation operator $\beta_q = 0(t)$ for the $q = 0$ mode. For example, we have:

$$\langle N_0 - 3|\beta_q = 0(t)|N_0 \rangle = \frac{c_0^2}{L^2 S} \gamma_{4d} (N_0 - 3|\beta_q = 0|N_0 \rangle \times (e^{-i\Delta_{\text{sw}}t/\hbar} - e^{-i\Delta_{\text{sw}}t/\hbar}) (19)$$

Thus $\beta_q = 0(t)$ has a finite amplitude for destroying more than one particle. (There is an analogous amplitude for particle creation as well.) As a result of this non-trivial time dependence, the $\gamma_{4a}$ term in $H_{\text{diss}}$ may now contribute to the decay of the initial state, giving rise...
FIG. 5: Schematic decay process for $V < V_0$. Energy conservation requires $2\omega_q = 4\Delta_{sw}$.

to a contribution to the dissipation rate proportional to $\gamma_{4a}\gamma_{4d}$.

We now have all the ingredients required to calculate the dissipation rate. We note that the dissipative part, $H_{\text{diss}}$, contains only terms of even order in the bosonic operators. Together with energy-momentum conservation, we can see that the simplest transition in which energy is transferred out of the $q = 0$ mode involves the annihilation of four quanta of that mode, combined with the creation of a pair of spin waves with equal and opposite momenta:

$$ |N_0\rangle \rightarrow |N_0 - 4, \{q, -q\}\rangle $$

(20)

where the momentum $q$ has to satisfy energy conservation: $4\Delta_{sw} = 2\epsilon_q$. This kinematic constraint leads to an intrinsic dissipation which depends on the fourth power of the density of spin waves. At low bias, this density is small and so the intrinsic dissipation is rather weak.

As expected from kinematics, the dominant processes contributing to dissipation are of the form shown in figure 5. The modified interaction picture yields an effective matrix element for the transition, given by

$$ \gamma_{fi} = \Delta_{sw}^{-1}(n_0g_0^2/S)^2(\gamma_{4b}\gamma_{4e} + \gamma_{4a}\gamma_{4c} - \gamma_{4a}\gamma_{4d} - \Delta_{sw}\gamma_6) $$

(21)

where the coefficients ($\gamma_{4a}, \gamma_{4b}, \ldots$) were defined in equation (17) as the vertex factors for various processes involving spin wave quanta. The different terms in $\gamma_{fi}$ correspond to different coherent routes for destroying 4 $q = 0$ spin wave quanta and generating a pair of finite-$q$ spin waves. This can be achieved with a direct vertex ($\gamma_6$), or through an intermediate virtual state $|N_0 - 2, \{q, -q\}\rangle$ ($\gamma_{4a}$ followed by $\gamma_{4c}$). The $\gamma_{4a}$ term in $H_{\text{diss}}$ also contributes to the decay through the non-linear component of $\beta_q=0$ from (19), giving the term $\gamma_{4a}\gamma_{4d}$. The $\gamma_{4b}\gamma_{4e}$ term arises in a similar way.

Substituting the relevant coefficients into (21), and denoting the (small) number $(\Delta_{SAS}/D)$ by $x$, the result is

$$ \gamma_{fi} = \Delta_{sw}(n_0g_0^2/S)^2 \frac{3 - 4x - 2x\sqrt{1 + 16x + 16x^2} - 8x^2}{128(1 + x)^3} $$

(22)

We have checked that this quantity is independent of the choice of quantisation axis. The density of states for spin wave pairs, in which the spin waves have equal and opposite momentum, and each has energy $\epsilon$ is

$$ g(\epsilon) = \frac{\epsilon}{4\pi\rho_F g_0^2} \frac{1}{\sqrt{D^2 + 4\epsilon^2}} $$

(23)

The dissipation rate is now simply given by $\Gamma = 8\pi L^2\Delta_{sw}^2\gamma_{fi}^2g(2\Delta_{sw})/\hbar$, and we arrive at our result for the dissipation rate in the low-bias regime ($V < V_0$):

$$ \Gamma_{V < V_0} = L^2 \left( \frac{V}{V_0} \right)^8 \frac{9\hbar \Delta_{sw}^2 \Delta_{SAS}^3}{512\pi^3 \rho_F D^2} \cdot 1 + O(\Delta_{SAS}/D) $$

(24)

We can express this as the rate at which the number of $q = 0$ modes is decreasing: $dN_0/dt = -\Gamma(V)/4\Delta_{sw}$. However, $N_0$ is related to the instantaneous bias $V$ by $|\sas|_0$. The instantaneous bias is in turn related to the $z$-magnetisation by $|\sas|_0$: $V \propto m_z \sim \theta$ at low bias. In terms of the tilt angle, we see that $dN_0/dt \propto -\theta^8$. Hence the amplitude, $A$, of the oscillating charge imbalance on the bilayer decays slowly in time, according to

$$ A(t) \sim 1/t^{1/6} $$

(25)

This is weak dissipation, but we believe that it is the first microscopic calculation of an intrinsic dissipation rate in these systems. The existence of such mechanisms is an important issue because coherent oscillations of the charge are not observed in physical bilayer systems. Introducing disorder or inelastic scattering from topological excitations might relax the kinematic constraint that lead to the weak dissipation. Simulations indicate that these effects are important at small energies: the framework of the modified interaction picture may be a useful framework in which to investigate these effects.

Equations (9), (10) and (24) represent the main results of this section. We have calculated dissipation rates in the three regimes of our thought experiment that are most relevant to experimental systems. As we commented in Ref. 17, the results agree qualitatively with experiment; more quantitative agreement may require the taking into account both disorder, and the effects of finite temperature.

III. CLASSICAL ORIGIN OF THE DISSIPATION

In this section we show that the dissipation mechanisms in the bilayer have their origins in dynamical instabilities of the classical spin system with a Hamiltonian given by (1). We consider the case $V < V_0$, and emphasise the strong links between the calculation of the strength of the classical resonances and the quantum calculation above. We emphasised in the previous section that an initially coherent oscillation remains coherent forever in the classical system. However, we show here that there are instabilities of the classical dynamics whereby an initially very small spatial modulation of the spin field will grow resonantly, leading to dissipation out of the coherent oscillation. In the quantum system, zero point fluctuations are sufficient to seed this instability: this is the cause of the dissipation rates calculated in the previous section.
The classical equations of motion of the spin system are described by the Landau-Lifshitz equation:

$$\hbar \frac{d}{dt} S_i^a = \epsilon^{abc} \frac{\partial H}{\partial S_i^b} S_i^c \quad (26)$$

where $S_i = (S_i^1, S_i^2, S_i^3)$ is the spin on site $i$ and $\epsilon^{abc}$ is the completely antisymmetric tensor. We parametrise the magnetisation in terms of a mean value and a weak spatial modulation: $\vec{m}_i = \bar{m}_0 + (\vec{m}_q e^{-i\omega t} \cdot \vec{r}_i \cdot c.c.)$. Assuming that the spatial modulation is small compared to the mean value, we may linearise the equation of motion for $\bar{m}_0$, arriving at

$$\frac{d}{dt} \bar{m}_0 = \bar{m}_0 \times [\Delta_{SAS} \vec{e}_x - D \vec{e}_z \bar{m}_0] \quad (27)$$

where $\vec{e}_{x,y,z}$ are the unit vectors in the $x,y,z$-directions in spin space.

Let us consider first the time evolution of the uniform mode $\bar{m}_0(t)$. Suppose we have an initial state with a small component in the $S_z$-direction: $\vec{m}_0(t = 0) = (\sqrt{1 - z_0^2}, 0, z_0)$ with $z_0 \ll 1$. We will use $z_0$ as our perturbative parameter. (Compare with the quantum calculation of section 11B where the perturbative parameter was the energy in the $q = 0$ mode.)

At very small amplitudes, the driving mode is harmonic: the spin precesses along elliptical trajectories with frequency $\Delta_{sw}/\hbar$. The perturbation expansion over $z_0$ is straightforward, and is equivalent to solving for the motion of the operator $\beta_{q=0}(t)$ in the modified interaction picture (see appendix A). The time dependence of the $z$-component of the uniform mode $\bar{m}_0$ is

$$\bar{m}_0(t) = z_0 \cos(\Delta_{sw} t/\hbar) + \frac{Dz_0^3}{64\Delta_{sw}^3} \times \left[ \cos(3\Delta_{sw} t/\hbar) - \cos(\Delta_{sw} t/\hbar) \right] + \mathcal{O}(z_0^5) \quad (29)$$

where $\Delta_{sw} = \Delta_{sw}[1 - (Dz_0^2/4\Delta_{sw}^2)^2 + \mathcal{O}(z_0^4)]$. The other components of $\vec{m}_0$ are easily obtained from $\Delta_{SAS} \vec{m}_0 = \hbar S)(\vec{m}_0/dt)$ and $(\vec{m}_0^2)^2 = 1 - (m_0^2)^2 - (m_0^2)^2$.

Let us now turn to the spatial modulations $\vec{m}_q(t)$. It is simple to verify that $|\vec{m}_0\rangle$ and $(\vec{m}_0 \cdot \vec{m}_q)$ are constants of the motion. To make this explicit, we write

$$\vec{m}_q = p_{2q}(\vec{m}_0 \times \vec{e}_z) + p_{1q}(\vec{m}_0 \times (\vec{m}_0 \times \vec{e}_z)) \quad (30)$$

This is the equivalent of casting the spin system (with three components per site) in terms of bosons (one complex or two real degrees of freedom) via the Holstein-Primakov representation. In this representation, the linearised equation of motion for the spatial modulation is:

$$\hbar \frac{d}{dt} \begin{pmatrix} p_{1q} \\ p_{2q} \end{pmatrix} = \begin{pmatrix} M_{dq} & M_{aq} \\ -M_{aq} & M_{dq} \end{pmatrix} \begin{pmatrix} p_{1q} \\ p_{2q} \end{pmatrix} \quad (31)$$

where $M_{dq} = \Delta_{SAS} m_0^6 / (1 - (m_0^2)^2)$, $M_{aq} = \Delta_{SAS} m_0^6 + \rho E \gamma(q)$ and $M_{aq} = \Delta_{SAS} m_0^6 / (1 - (m_0^2)^2) + \rho E \gamma(q) + D(1 - (m_0^2)^2)$.

Consider first the limit of small amplitude $z_0 \to 0$. Then, $M_d = 0$, $M_a = \Delta_{SAS} + \rho \gamma(q)$ and $M_b = \rho \gamma(q) + D$. We can see that each $q$-mode is harmonic with a natural frequency $\omega_q/\hbar = (M_a M_q)^{1/2}/\hbar$ which is the same as the dispersion relation given in [12]. These are the classical spinwaves of the system.

However, at larger amplitudes $z_0$, we cannot ignore the driving of these spinwaves by the uniform mode. This will provide the energy to amplify any small spatial modulations. Substituting $\bar{m}_0(t)$ into (31), we can see that these matrix elements are periodic in time. They take the form

$$M_{aq}^{ab}(t) = \sum_n M_{aq,nn}^{ab} e^{(i/h)n\Delta_{sw}t} \quad (32)$$

We may therefore find solutions of the Bloch form

$$\begin{pmatrix} p_{1q} \\ p_{2q} \end{pmatrix} = e^{i\vec{q} \cdot \vec{r}} \begin{pmatrix} u_{1q}(t) \\ u_{2q}(t) \end{pmatrix} \quad (33)$$

where $u_1$ and $u_2$ are periodic in $t$ with period $(2\pi/\Delta_{sw})$. The relation between the wavenumber of the spatial modulation, $\vec{q}$, and the Bloch wavenumber, $k$, is a complex band structure, in which complex wavenumbers represent resonantly growing or exponentially decaying solutions. The relation $k(q)$ is obtained by solving, for any given spatial mode $\vec{q}$, an equation of the form

$$\det_{ab,mn} \left\{ [k(q) + 2n\Delta_{sw}] \delta_{m-n} \delta^{ab} + iM_{aq,nn}^{ab} \right\} = 0 \quad (34)$$

where the determinant is of an (infinite dimensional) matrix operating in the product space of Fourier components $(mn)$ and components $(ab)$ of the vector $\vec{p}$.

This calculation closely resembles the calculation of the quantum matrix element between initial and final states: see figure [6]. The non-interacting $(z_0 = 0)$ problem defines energy levels which we may identify as initial and final states. At finite $z_0$, we treat the interactions between the states perturbatively: the resonance condition occurs when initial and final states become close in energy (as in the quantum case).

The resulting complex band structure for spatial wavevectors around $\omega_q = 2\Delta_{sw}$ is shown in figure [7].

FIG. 6: ‘Energy level’ picture of the classical calculation. Resonance occurs when initial (left) and final (right) states become degenerate.
There is a resonance at $2\omega_q = 4\Delta_{sw}$. At the centre of the resonance, the imaginary part of the Bloch wavenumber, $\lambda$, coincides exactly with the matrix element for the quantum decay process. To be precise,

$$
\lambda = (\Delta_{sw}/\hbar)(z_0/S)^4(\Delta_{sw}/2\Delta_{SAS})^2 \times \frac{3 - 4x - 2x\sqrt{1 + 16x + 16x^2} - 8x^2}{128(1 + x)^3} \tag{35}
$$

where we have again abbreviated $(\Delta_{SAS}/D) = x$. Identifying $\theta$ with $(z_0/S)$, and using equation (22), we see that

$$
\lambda = \gamma_{fi}/\hbar \tag{36}
$$

Equation (36) is strong evidence that the quantum decay process is linked to the instability of the classical system at the same wavenumber. (Note that even the non-trivial dependence on $\Delta_{SAS}/D$ contained in the factor $\gamma_{fi}$ is identical in both cases).

To make this correspondence closer, we demonstrate that with an appropriate choice of boundary conditions, the rate at which the classical oscillation decays is approximately equal to the rate given in (22). An outline of this calculation is given in appendix B. The boundary conditions of the classical calculation have been chosen so as to mimic the quantum system. To set the boundary conditions, the initial energy in the driven mode is chosen such that the energy can never drop below its zero point value.

The significance of the result (36) is that the dissipation rate in the classical system takes the universal form:

$$
\Gamma_{cl} = 4\pi E(0)g_c(2\Delta_{sw})\lambda^2 C(\lambda t) \tag{37}
$$

where $E(0)$ is the initial energy in the spinwave modes with frequency $(2\Delta_{sw}/\hbar)$, $g_c$ is the classical density of states at that frequency, and the form of $C(t)$ is shown in figure S.

Figure S shows the similarities and differences between the classical and quantum dissipation rates. For the classical trace, we used $E(0) = 2\Delta_{sw}$, which is of the order of the zero point energy in the relevant spinwave mode. At short times ($\lambda t \ll 1$), the classical system has a rapid transfer of energy into modes far away from the resonance. Once this transient behaviour is over, the quantum and classical rates are both approximately constant in time at intermediate times. The two results diverge again for long times $\lambda t \gg 1$. However, the quantum calculation is non-degenerate perturbation theory, and is therefore valid only up to times $\lambda t \sim 1$ and so we do not expect the agreement to continue beyond this time scale. We therefore emphasise that the form of the dissipation rate calculated in appendix B and plotted in figure S is a universal result for dissipation out of a periodic oscillation into a bath of “driven” oscillators (as long as the equations of motion of the driven oscillators may be linearised). This may be regarded as the classical analogue of Fermi’s Golden Rule.

IV. CONCLUSION

We have used a thought experiment on an isolated, clean system at zero temperature to investigate the link between tunnelling transport and dissipation in quantum Hall bilayers. The thought experiment allows us to identify a crossover in the behaviour of the system that may be underlying the small bias crossover in the tunnelling IV characteristic. Our theory predicts an IV characteristic that is in qualitative agreement with experiment at large bias.

In the very low bias regime we have identified an intrinsic source of dissipation in the system, and showed that its origin is in a dynamic instability of a classical spin system.

In the light of these results, two main questions remain. Firstly, we were unable to address the $IV$ characteristic at small biases, which is the position of a prominent feature in the conductance. An extension to the method so that a steady-state current can be forced through the bilayer would be required if this issue is to be resolved. Secondly, the effects of disorder on the model have not
been considered, although this thought experiment and theoretical framework does seem appropriate for investigating that problem.

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APPENDIX A: MODIFIED INTERACTION PICTURE

In this appendix, we show how modifying the usual interaction picture allows us to avoid divergences associated with interactions between quanta of the macroscopically occupied $q = 0$ mode in the low-bias regime (section II B). We begin by describing our modifications, before commenting briefly on their physical interpretation.

For our calculation of the dissipation rate in the low-bias regime in section II B, we wish to evaluate matrix elements of the form

\[
\mathcal{M} = \langle 0, \{q, -q\} \mid \beta_0^N - m e^{i Ht} E_q e^{-i Ht} (\beta_0^\dagger)^N \mid 0, \{\} \rangle \quad (A1)
\]

where the operator $E_q$ measures the energy in the spin-wave mode with finite momentum $q$ and $H$ is the Hamiltonian from (15). $N$ is a macroscopically large number and $m$ is a number of order unity. For convenience, we use the notation $\beta_0 = \beta_{q=0}$ for the annihilation operator of the $q = 0$ mode.

In the usual interaction picture, we would use time-dependent operators $O(t) = e^{i H_0 t} O e^{-i H_0 t}$ and the wavefunctions have a time dependence of $|\psi(t)\rangle = e^{i H_0 t} e^{-i Ht} |\psi(0)\rangle$. We would write the matrix element in (A1) as:

\[
\langle 0, \{q, -q\} \mid \beta_0^N - m e^{i Ht} e^{-i H_0 t} e^{i H_0 t} e^{-i Ht} (\beta_0^\dagger)^N \mid 0, \{\} \rangle \quad (A2)
\]

where $H_0$ is the quadratic Hamiltonian (14) that commutes with $E_q$. The product $\hat{U}(t) = e^{i H_0 t} e^{-i Ht}$ may be written as a time-ordered exponential: $\hat{U}(t) = T \exp[-i \int_0^t dt' H_1(t')]$ which has a perturbative expansion in terms of time integrals of the operator $H_1(t) = e^{i H_0 t} (H - H_0) e^{-i H_0 t}$.

Our modification to the interaction picture replaces the time evolution $e^{-i H_0 t}$ in the definition of the interaction picture with $e^{-i Ht}$ where $\hat{H}$ includes the non-quadratic terms from $H_{\text{int}}$ (15), but maintaining the commutation relation $[E_q, \hat{H}] = 0$ for all modes except $q = 0$. Thus $\hat{H}$ includes all terms that are functions only of $(\beta_0, \beta_0^\dagger)$: we have

\[
\hat{H} = \sum_q \omega_q \beta_q^\dagger \beta_q + H_{\text{int}}^c + \ldots
\]

\[
H_{\text{int}}^c = \frac{1}{L^2 S} \left[ \gamma_{4d}(\beta_0^2 \beta_0^\dagger + \text{h.c.}) + \gamma_{4e}(\beta_0^4 + \text{h.c.}) \right] (A3)
\]

where the omitted terms have coefficients that are $O[(L^2 S)^{-2}]$. The operator $H_1(t)$ must now be evaluated in a perturbative series before the expansion of the exponential in $\hat{U}(t)$. This procedure requires the (non-trivial) time dependence of the operator $\beta_0$ which is given (for $m \neq 0$) by

\[
\langle N - m | e^{i Ht} \beta_0 e^{-i Ht} | N \rangle = \sqrt{N} \delta_{m-1} e^{-i \Delta_{\omega t}} + \left( e^{-i m \Delta_{\omega t}} - e^{-i \Delta_{\omega t}} \right) \frac{\langle N - m | [\beta_0, H_{\text{int}}] | N \rangle}{m \Delta_{\omega}}
\]

\[(A4)\]

These matrix elements converge even in the classical (or thermodynamic) limit, in which $N \to \infty$ but $NS^{-1}$ and $NL^{-2}$ are constants.

The purpose of this calculation is to evaluate a decay rate into a continuum of final states. To interpret the modified interaction picture in this light, observe that we can write the matrix element, $\mathcal{M}$ in the form

\[
\mathcal{M} = \sum_{|f\rangle} \langle f | E_q | f \rangle |\langle f | e^{-i Ht} | i \rangle|^2
\]

\[(A5)\]

where $|i\rangle$ and $|f\rangle$ are initial and final states. Then, writing $|f\rangle = e^{i Ht} |q\rangle$, the fact that $\hat{H}$ and $E_q$ commute means that

\[
\mathcal{M} = \sum_{|g\rangle} \langle g | E_q | g \rangle |\langle g | \hat{U}(t) | i \rangle|^2
\]

\[(A6)\]

Note that the state $|g\rangle = e^{-i Ht} |f\rangle$ is obtained from $|f\rangle$ by evolving the state $|f\rangle$ in parallel with the initial state, taking into account the evolution of the condensate itself, but ignoring interactions that involve modes with finite wavevectors. This does not affect our results because the quantity of interest, $E_q$, is independent of the occupancy of the condensate: formally $\langle g | E_q | g \rangle = \langle f | E_q | f \rangle$. In other words, this is a unitary change of basis for our subspace of final states of the form $|N, \{q, q', \ldots\}\rangle$.

The advantage of using the $|g\rangle$ basis is that it allows us to use conventional perturbation theory for the evolution operator $\hat{U}(t)$. Otherwise, we find divergences when evaluating matrix elements to individual final states in the $|f\rangle$ basis. Physically, these divergences arise from the fact that the particles in the condensate have evolved in time and so the initial condensate $|N_0\rangle$ has become strongly admixed with states with different numbers of $q = 0$ modes: $|N_0 + m, \{q, -q\}\rangle$. The $|g\rangle$ basis tracks this evolution and is therefore the natural basis for the calculation.

Equations (A3) and (A6) define a convergent perturbative expansion for the dissipation rate $\Gamma$, defined in (6): this leads to equation (24).

APPENDIX B: DISSIPATION RATES IN CLASSICAL SYSTEMS

In this appendix we discuss the extent to which a broad class of quantum problems with solutions based
on Fermi’s Golden Rule may be addressed within a classical framework. We investigate the decay of a coherent oscillation by dissipation into an environment modelled by a ‘bath’ of harmonic oscillators. The only quantum ingredient will be the zero-point fluctuations of this bath.

To illustrate the point we use a simple model system with a Hamiltonian of the form

\[ H = \frac{1}{2}(p^2 + \Omega^2 y^2) + \sum_n \left[ \frac{1}{2}(p_{xn}^2 + \omega_n^2 x_n^2) + 4\lambda_n \Omega x_n y_n^2 \right] \]

where the coordinate \( y \) describes the coherent (driving) oscillation and the coordinates \( x_n \) describe the bath of driven oscillators. We work perturbatively around \( \lambda = 0 \) which is the limit of independent oscillators in which the coherent oscillation remains coherent for ever. The energy in the driven oscillator is initially much greater than all other energy scales, so \( y \sim \cos \Omega t \), and the \( x_n \) obey Mathieu’s equation \( \ddot{z} + \omega_n^2 + \lambda \sin 2\Omega t \z \approx 0 \) (B2)

For each \( n \), solutions can be written in the Bloch form \( x_n(t) = e^{ikvt}x_n(t) \) where \( x_n(t) \) has a period of \( 2\pi/\Omega \).

The interesting behaviour occurs when \( (\omega_n - \Omega) \) is very close to an integer. The Bloch wavenumber \( k \) becomes complex and gaps open up. Near \( \omega_n = \Omega \), it can be shown that

\[ (k + \Omega)^2 = (\omega_n - \Omega)^2 - \lambda^2 \]

Thus there is a gap at the edge of the Brillouin zone: \( \text{Re}(k) = -\Omega \). For \( |\omega_n - \Omega| > \lambda_n \), \( k \) is real and the solution for a given \( n \)-mode is

\[ x_n(t) = \text{Re} \left[ R e^{i\phi} \left[ (k + \omega_n - 2\Omega n)e^{(\Omega - k)t} - \lambda_n e^{i(k - \Omega)t} \right] \right] \]

where \( R \) and \( \phi \) are determined by boundary conditions. For \( |\omega_n - \Omega| < \lambda_n \), \( k \) is complex. Writing \( \kappa = i(k - \Omega) \) (which is real) and \( 2\theta = \text{arg}(i\kappa - \omega_n + \Omega) \), the solution is

\[ x_n(t) = Ae^{\kappa t} \cos(\Omega t + \theta) - Be^{-\kappa t} \cos(\Omega t - \theta) \]

where \( A \) and \( B \) are determined by initial conditions.

In our quantum calculation based on Fermi’s Golden Rule, we calculated the energy transferred to the driven oscillators, using for the energy of each mode the energy given by the non-interaction Hamiltonian. The analogous quantity here would be the dissipation rate

\[ \Gamma_{\text{cl}} = \dot{x}_n \sum_n (1/2)(p_{xn}^2 + \omega_n^2 x_n^2). \]

Using the solutions above, the energy in each driven mode with real \( k \) is

\[ E^{(r)}_n = E(0) \frac{w_n - \lambda_n \cos(2(k + \Omega)t + 2\phi')}{w_n - \lambda_n \cos 2\phi'} \]

with \( w_n = (\lambda_n^2 + |\Omega|^2)^{1/2} \). The angle \( \phi' \) is equal to \( \phi \) for \( \omega > \Omega \); if \( \omega < \Omega \) then \( \phi' = \phi + (\pi/2) \). Similarly, if \( k \) is complex then we have

\[ E^{(i)}_n = E(0) \frac{\lambda_n (\cos 2\kappa + \sin 2\kappa \sin \phi'') - w_n \cos \phi''}{\lambda_n - w_n \cos \phi''} \]  

where the phase angle \( \phi'' \) depends on the ratio of \( A \) and \( B \) in equation (B5).

For correspondence between the classical and quantum calculations, the initial energy in the driven oscillators should be of the order of the zero point value of the quantum calculation. Further, we choose the relative phase between driven and driving oscillators such that the energy in the classical oscillators can never be reduced below the same zero point value (since that would violate the uncertainty principle).

Finally the result is that

\[ \Gamma_{\text{cl}} = E(0)\lambda^2 g_c(\Omega)C(\lambda t) \]

where \( g_c(\omega) \) is the classical density of states:

\[ \sum_n \rightarrow \int d\omega g_c(\omega) \text{ in the limit in the frequencies in the oscillator bath form a continuum.} \]

The universal function \( C(\lambda t) \) was plotted in figure 8. It takes the form

\[ C(x) \approx \begin{cases} (1/2\pi x) - (1/4) & x \ll 1 \\ 1 + O(x) & x \gg 1 \end{cases} \]

The diverging rate at small \( x \) is cut off by the finite width of the oscillator bath. At small times there is a rapid transfer of a small amount of energy into modes far from the resonance.

The rate calculated in (B9) is compared with the dissipation rate for a quantum calculation on the same system in figure 8. The agreement breaks down at long times since the classical calculation corresponds to degenerate perturbation theory for states near the resonance, and is able to capture the resonant growth of the energy. At small times there is a transient effect that is unique to the classical system. However, at intermediate times then choosing \( E_0 = \hbar \omega_n \) gives good agreement between quantum and classical rates. This is in accordance with the correspondence principle, as expected for systems with linear equations of motion.

The applicability of this treatment to more complicated models than that of equation (B11) is not obvious at first sight. However, the forms for the energy in the driven modes, (B11) and (B15), are in fact universal for problems in which the equation of motion of the driven modes is linearisable (that is, problems in which the periodic forcing multiplies a coordinate as in (B2), so that the forcing vanishes as the energy in the driven oscillator gets small). In this general problem the parameter \( A \) in (B9) is equal to the maximal value of the imaginary part of the Bloch wavenumber. Neglecting the effect of the initial transient on the upper cutoff, (B12) is then a universal function of \( \lambda t \), so the dissipation rate associated with resonance is completely characterised by the \( \lambda \) and
the density of driven oscillators \( g_c(\omega) \). This is the general result applicable to the spin system of equation (1), and shows that the quantum decay channels have corresponding instabilities in the classical dynamics.

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