Projective tensor products and $A^q_p$ spaces

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Introduction

Let $G$ be a group, let $H$ and $K$ be two subgroups of $G$, and let $\pi$ and $\gamma$ be representations of $H$ and $K$, respectively. If $G$ is finite, Mackey’s results assert that the intertwining number of the two induced representations $U_\pi$ and $U_\gamma$ of $G$ can be expressed as a sum of intertwining numbers of the representations $\pi^x$ and $\gamma^y$ of the subgroups $H^x \cap K^y, x, y \in G$. In the case of an infinite group, if the subgroups are open and closed, a similar characterization is possible especially when $\pi$ and $\gamma$ are one-dimensional. If the subgroups are closed, Mackey showed that the above criteria for computing the intertwining number holds for the space of those operators which are in the Hilbert-Schmidt class.

Among other developments that are important for us, the first is the work of Rieffel on Banach G-modules and their products. He proved, in particular, that

$$(V \otimes_S W)^* \cong \text{Hom}_S(V, W^*),$$

where $S$ is a set, $V$ and $W$ are two $S$-modules, $\otimes_S$ denotes the projective tensor product of $V$ and $W$ and $\text{Hom}_S(V, W^*)$ is the space of intertwining operators of the Banach G-modules. Applying this to $L_p(G)$ spaces ($1 \leq p \leq \infty$) of complex-valued functions defined on a group $G$, Rieffel obtained the result that, under certain conditions, the corresponding intertwining operators (multipliers) form the dual space of the space of functions $A^q_p$: a subset of an $L_r$ space (where $r$ is related to $p$ and $q$ as described in Prop.3.18) consisting of those functions which can be written as a sum of convolution of functions from $L_p$ and $L_q$. This is the context in which we shall set our study of intertwining operators, that is, regarding the space of such operators as the dual of a tensor product space.

Herz studied the predual of the space of intertwining operators of the regular representations of $G$ on $L_p$ and $L_q$ and was able to show, in particular, that the tensor product space is an algebra of functions on $G$ and, in some sense, a natural analogue of the space of absolutely convergent Fourier Series. Our aim is to extend the Herz-Rieffel results from regular representations which may be seen as induced representation from the trivial subgroup to arbitrary induced representations.

In order to complete this analysis we shall need to go beyond spaces of functions on $G$ to sections of Banach (semi-)bundles on $G$. The concept of a Banach bundle was developed by Fell in 1977 and we shall use it as the appropriate device for the study of the tensor product spaces. Unfortunately, in the most general case, our semi-bundle will fail to be a bundle in the complete sense, but will be more akin to the objects studied by Dauns and Hofmann.
1 Preliminaries

1.1 \( \lambda \)-functions

We shall assume throughout that all the topological spaces under consideration are second countable.

Let \( G \) be a locally compact topological group. We denote the right-invariant Haar measure on \( G \) by \( \nu_G \). \( e \) denotes the identity element of the group. For a subgroup \( H \) of \( G \), the canonical mapping from \( G \) to the set of right-cosets \( G/H \) is denoted by \( p_H \).

A real-valued function \( \rho_H \) defined on \( G \) which satisfies

\[
\rho_H(hx) = \left( \Delta_H(h)/\Delta_G(h) \right) \rho_H(x),
\]

where \( x \in G \) and \( h \in H \), is called a \( \rho \)-function. The existence of a strictly positive continuous \( \rho \)-function has been established in a number of places in the literature (see Mackey[31], Gaal[19]). In particular, it is known that for every closed subgroup \( H \) in \( G \) there exists a function \( \beta \) on \( G \) with

\[
\int_H \beta(hx)d\nu_H(h) = 1 \quad \text{for all} \quad x \in G.
\]

The details of such a \( \beta \) function are given in the following Lemma.

**Lemma 1.1** For every closed subgroup \( H \) of a locally compact group \( G \), there exists a function \( \beta \) on \( G \) with the following properties:

(a) if \( K \) is any compact set in \( G \), then \( \beta \) coincides on the strip \( HK \) with a function in \( C^+ \_0 \_0(G) \);

(b) \( \int_H \beta(hx)d\nu_H(h) = 1 \) for all \( x \in G \).

Proof: See Reiter[35], Chapter 8, section 1.9.

A function \( \beta \) on \( G \) satisfying the properties stated in Lemma 1.1 is called a **Bruhat function** for \( H \).

Given a Bruhat function \( \beta \) for a closed subgroup \( H \), a \( \rho \)-function can be obtained by letting

\[
\rho_H(x) = \int_H \beta(hx)\Delta_G(h)\Delta_H(h^{-1})d\nu_H(h).
\]

Then \( \rho_H \) is continuous (cf. (a) and [35], Chapter 3, section 3.2, Remark) and strictly positive for all \( x \in G \).

For a given \( \rho \)-function \( \rho(sy)/\rho(s) \) is a Borel function of \( s \) and \( y \) which is constant on the right \( H \times G \) cosets in \( G \times G \). Since there is a natural homeomorphism from this coset space to \( (G/H) \times G \), these \( \rho \)-functions give rise to a unique Borel function \( \lambda_\rho \) on \( (G/H) \times G \) such that

\[
\lambda_\rho(p_H(s), y) = \frac{\rho(sy)}{\rho(s)}
\]

for all \( s \) and \( y \) in \( G \). This function \( \lambda_\rho \) has the following properties:

(a) for all \( x \in (G/H) \) and \( s, t \in G \), \( \lambda_\rho(x, st) = \lambda_\rho(x, s)\lambda_\rho(x, t) \);

(b) for all \( h \in H \), \( \lambda_\rho(p_H(e), h) = \Delta_H(h)/\Delta_G(h) \).
(c) \( \lambda_p(p_H(e), t) \) is bounded on compact sets as a function of \( t \).

(See, for example, Gaal\[19\], p.263, Lemma 10.) For a given measure \( \mu \) on \( G/H \) and \( y \in G/H \), let \( \mu_y \) denote the translated measure on \( G/H \) defined by \( \mu_y(E) = \mu([E]y) \). It is well known that for a given arbitrary \( \rho \)-function on \( G \) there exists a quasi-invariant measure \( \mu \) in the right coset space \( G/H \) such that for all \( y \in G \), the corresponding \( \lambda \)-function \( \lambda_p \) has the property that \( \lambda_p(\cdot, y) \) is a Radon-Nikodym derivative of the measure \( \mu_y \) with respect to the measure \( \mu \). Any two have the same null sets and hence are mutually absolutely continuous. A Borel set \( E \) in \( G/H \) is a null set if and only if \( p_H^{-1}(E) \) has Haar measure zero. Let us write \( \mu \asymp \lambda \) to mean that for all \( y \in G \), \( \lambda(\cdot, y) \) is a Radon-Nikodym derivative of the measure \( \mu_y \) with respect to \( \mu \). The relations \( \mu \asymp \lambda \) and \( \lambda = \lambda_\rho \) between quasi-invariant measures, \( \lambda \)-functions and \( \rho \)-functions have the following properties:

(i) Every \( \lambda \)-function is of the form \( \lambda_\rho \); \( \lambda_{\rho_1} = \lambda_{\rho_2} \) if and only if \( \rho_1/\rho_2 \) is a constant.

(ii) If \( \mu_1 \asymp \lambda \) and \( \mu_2 \asymp \lambda \) then \( \mu_1 \) is a constant multiple of \( \mu_2 \).

(iii) If \( \mu \asymp \lambda_1 \) and \( \mu \asymp \lambda_2 \) then for all \( t \), \( \lambda_1(\cdot, t) = \lambda_2(\cdot, t) \) almost everywhere in \( G/H \).

(See Mackey\[31\] for a detailed study on \( \rho \)-functions and related \( \lambda \)-functions).

The quasi-invariant measure on the homogeneous space \( G/H \) of a subgroup \( H \) of a group \( G \) will be denoted by \( \mu_H \) and the Radon-Nikodym derivative of the measure \( E \mapsto \mu_H([E]y) \) with respect to the measure \( \mu_H \) is denoted by \( \lambda_H(\cdot, y) \).

For simplicity of notation, \( \lambda_H(p_H(x), y) \) will be written as \( \lambda_H(x, y) \), or by \( \lambda(x, y) \) if the subgroup \( H \) is clearly understood.

The following result, which appears in several places in the literature, is of fundamental importance for our purposes.

**Corollary 1.2** For \( x \in G \) let \( \bar{x} = p_H(x) \). If \( \mu \) denotes the quasi-invariant measure corresponding to the function \( \rho \) then

\[
\int_G f(x)\rho(x)d\nu_G(x) = \int_H \int_{\bar{x}} f(hx)d\nu_H(h)d\mu(\bar{x}), \quad f \in C_0(G).
\]

Proof: See, for example, Gaal\[19\], p.263, Corollary to Theorem 9.

\[\Box\]

### 1.2 Banach Bundles

Here we recall definitions and a few results in terms of Banach bundles. (see \[15\], Chapter 2 and \[16\] for further details.)

A bundle \( \mathcal{B} \) over a Hausdorff space \( X \) is a pair \((\mathcal{B}, \theta)\) such that \( \mathcal{B} \) is a Hausdorff space called the **bundle space** of \( \mathcal{B} \) and \( \theta : \mathcal{B} \to X \) is a continuous open surjection called the **bundle projection** of \( \mathcal{B} \). \( X \) is called the **base space** of \( \mathcal{B} \), and for \( x \in X \), \( \theta^{-1}(x) = \{\xi : \theta(\xi) = x, \xi \in \mathcal{B}\} \) is called the **fibre** over \( X \) and is denoted by \( \mathcal{B}_x \).

A bundle \( \mathcal{B} = (\mathcal{B}, \theta) \) over \( X \) is a **Banach semi-bundle** over \( X \) if we can define a norm making each fibre \( \mathcal{B}_x \) into a Banach space satisfying the following conditions:
(a) $\xi \mapsto \|\xi\|$ is upper semi-continuous on $B$ to $R$.
(b) The operation $+$ is continuous on the set $\{(\xi, \eta) \in B \times B : \theta(\xi) = \theta(\eta)\}$ to $B$.
(c) For each $\lambda$ in $C$, the map $\xi \mapsto \lambda \xi$ is continuous on $B$ to $B$.
(d) If $x \in X$ and $\{\xi_i\}$ is a net of elements of $B$ such that $\|\xi_i\| \to 0$ and $\theta(\xi_i) \to x$, then $\xi_i \to \theta(x)$, where $\theta(x)$ denotes the zero element of the Banach space $B_x$.

A bundle $\mathcal{B} = (B, \theta)$ over $X$ is called a Banach bundle if it satisfies (b), (c) and (d) above together with the condition that

(a) $\xi \mapsto \|\xi\|$ is continuous on $B$ to $R$.

Given a Banach space $A$ and a Hausdorff space $X$, it is easy to construct a Banach bundle by letting $B = A \times X$ and $\theta(\xi, x) = \xi$. Then $(B, \theta)$ is a bundle over $X$ and if we equip each fibre $A \times \{x\}$ with the Banach structure making $\xi \mapsto (\xi, x)$ an isometric isomorphism, then it becomes a Banach bundle. The Banach bundle $(B, \theta)$ so constructed is called a trivial Banach bundle.

Let $X$ and $Y$ be any two Hausdorff spaces and $\phi : Y \to X$ be a continuous map. Suppose $\mathcal{B} = (B, \theta)$ is a Banach (semi-)bundle over $X$. Let $\mathcal{B}^\#$ be the topological subspace $\{(y, \xi) : y \in Y, \xi \in B, \phi(y) = \theta(\xi)\}$ of $Y \times B$; and define $\theta^\# : \mathcal{B}^\# \to Y$ by $\theta^\#(y, \xi) = y$. Then $\theta^\#$ is a continuous open surjection since $\theta$ is open. Hence $(\mathcal{B}^\#, \theta^\#)$ is a bundle over $Y$. For $y \in Y$, we make $B^\#_y = \theta^{-1}(y) \times B$ into a Banach space in such a way that the bijection $\xi \mapsto (y, \xi)$ of $B_\phi(y)$ onto $B^\#_y$ becomes a linear isometry. Then $(\mathcal{B}^\#, \theta^\#)$, denoted by $\mathcal{B}^\#$, becomes a Banach (semi-)bundle which is called the Banach (semi-)bundle retraction of $\mathcal{B}$ by $\phi$.

Let $i^\# : B^\# \to B$ be the surjection given by $i^\#(y, \xi) = \xi$. Then, we have the following diagram:

$\begin{array}{ccc}
B^\# & \xrightarrow{i^\#} & B \\
\downarrow \theta^\# & & \downarrow \theta \\
Y & \xrightarrow{\phi} & X
\end{array}$

Since $\theta(i^\#(y, \xi)) = \theta(\xi) = \phi(y) = \phi(\theta^\#(y, \xi))$, we have $\theta i^\# = \phi \theta^#$, and the diagram commutes.

Suppose $\mathcal{B} = (B, \theta)$ and $\mathcal{D} = (D, \vartheta)$ are Banach (semi-)bundles over the same base space $X$. Let $u : B \to D$ be a map for which the diagram

$\begin{array}{ccc}
B & \xrightarrow{u} & D \\
\downarrow \theta & & \downarrow \vartheta \\
X & & 
\end{array}$

commutes, so that $\theta(\xi) = \vartheta(u(\xi))$ for $\xi \in B$. Let $Y$ be another Hausdorff space and $\phi : Y \to X$ be a continuous map. Let $B^\#$ and $D^\#$ be the retractions of $B$ and $D$ by $\phi$ respectively. Define the map $j^\#(u) : B^\# \to D^\#$ by

$j^\#(u)(y, \xi) = (y, u(\xi)).$

Then $\vartheta^\#(j^\#(u)(y, \xi)) = \vartheta^\#((y, u(\xi))) = y = \theta^\#(y, \xi),$

for $(y, \xi) \in B^\#$, so that the diagram;

$\begin{array}{ccc}
B^\# & \xrightarrow{j^\#(u)} & D^\#
\end{array}$
\[ \theta^# \smallfrown \sqrt{\theta^#} \]

commutes.

Suppose \( u : B \to D \) is a continuous and open map. It is clear that the map \( j^#(u) \) is the restriction of the map \( (j, u) : Y \times B \to Y \times D \), where \( j \) is the identity map from \( Y \) to itself and \( (j, u)(y, \xi) = (y, u(\xi)) \). Clearly, \( (j, u) \) is a continuous, open map. Let \( U \subset B^# \) be an open set. Then there exists an open set \( U \subseteq Y \times B \) such that \( U = U \cap B^# \). Let \( j^#(u)U = V \) and \( (j, u)(U) = V \). Now \( V \) is an open set in \( Y \times D \) and \( V \subseteq V \cap D^# \). Note that if \( (y, \xi) \notin B^# \), then \( \phi(y) \neq \theta(\xi) \), and therefore \( \theta(u(\xi)) = \theta(\xi) \neq \phi(y) \), which implies that \( (y, u(\xi)) \notin D^# \). Therefore, if \( x \in V \cap D^# \) is the image of \( z \in U \), then \( z \) cannot be outside of \( B^# \). This implies that \( V = V \cap D^# \), which shows that \( V \) is an open set in \( D^# \). Hence \( j^#(u) \) is an open map.

Now we turn to the construction of a particular type of Banach (semi-)bundle. Let the Banach (semi-)bundle \( B = (B, \theta) \) over \( X \) with \( B = \mathcal{H} \times X \) be such that \( \mathcal{H} \) is a Banach space, \( X \) is a Hausdorff space and \( \theta(\xi, x) = x \). Suppose that there is an equivalence relation \( R \) on \( X \). Let \( r \) be the canonical mapping from \( X \) to \( X/R \). For \( x \in X \), let \( r(x) \in X/R \) be the equivalence class to which \( x \) belongs. Define \( B^R = (B^R, \theta^R) \) over \( X/R \) by letting \( B^R = \mathcal{H} \times X/R \) and \( \theta^R(\xi, r(x)) = r(x) \). Clearly, both bundles \( B \) and \( B^R \) are trivial bundles with constant fibre \( \mathcal{H} \).

**Proposition 1.3** The Banach bundle retraction

\[ B^{R^#} = (B^{R^#}, \theta^{R^#}) \]

of \( B^R \) by \( r \) is topologically equivalent to \( B = (B, \theta) \).

Proof: The two Banach bundles \( B^{R^#} \) and \( B \) have the same base space \( X \).

\[ B^{R^#} = \{(x', (\xi, r(x))): \theta(\xi, r(x)) = r(x'), x, x' \in X, \xi \in \mathcal{H}\} \]

\[ = \{(x', (\xi, r(x))): x' \in r(x), x', x \in X, \xi \in \mathcal{H}\}, \]

and for \( x \in X, B_x = \{(\xi, x): \xi \in \mathcal{H}\} \), while \( B^{R^#}_x = \{(x, (\xi, r(x))): \xi \in \mathcal{H}\} \). Clearly, the mapping \( (\xi, x) \mapsto (x, (\xi, r(x))) \) is a homeomorphism.

\[ \diamond \]

A cross-section of \( B \) is a function \( f : X \to B \) such that \( f(x) \in B_x \) for each \( x \in X \). The linear space of all continuous cross-sections of \( B \) is denoted by \( C(B) \) and the subspace of \( C(B) \) consisting of those cross-sections which vanish outside some compact set is denoted by \( C_0(B) \). The set of all bounded cross-sections is denoted by \( B(B) \).

We say that \( B \) has enough continuous cross-sections if for every \( \xi \in B \) there exists a continuous cross-section \( f : X \to B \) for which \( f(\theta(\xi)) = \xi \).

An unpublished result by A. Douady and L. Dal Soglio-Héralou about the existence of enough continuous cross-sections states that if \( X \) is either paracompact or locally compact, then every Banach bundle over \( X \) has enough continuous cross-sections (see Fell[15], p.324).

Let \( 1 \leq p < \infty \). A cross-section of \( B \) is said to be \( p^B \)-power summable if it is locally \( \mu \)-measurable and

\[ \|f\|_p = \left( \int_X \|f(x)\|^p d\mu(x) \right)^{1/p} < \infty. \]
The space of all \( p^\text{th} \)-power summable cross-sections is denoted by \( L_p(\mathcal{E}; \mu) \).

\( L_p(\mathcal{E}; \mu) \) is a Banach space under the norm \( \| \cdot \|_p \) defined above.

The space \( L_\infty(\mathcal{E}; \mu) \) is defined to be the space of all \( \mu \)-essentially bounded cross-sections of \( \mathcal{E} \).

\( L_\infty(\mathcal{E}; \mu) \) is a Banach space under the norm \( \| f \|_\infty = \mu \text{-ess sup}_{x \in X} |f(x)| \).

Let \( Y \) be another locally compact Hausdorff space with a regular Borel measure \( \nu \). Let \( \kappa : X \times Y \mapsto X \) be the surjection \((x, y) \mapsto x\). Then the Banach (semi-)bundle retraction \( \mathcal{E} = (\mathcal{E}, \rho) \) by \( \kappa \) is a bundle over \( X \times Y \) whose bundle space \( \mathcal{E} \) can be identified with \( \mathcal{B} \times Y \). The bundle projection is given by \( \rho : (\xi, y) \mapsto (\theta(\xi), y) \). For each \( x \in X \), \( \mathcal{E}_{(x)} \times Y \) is the trivial bundle with constant fibre \( B_x \). Therefore, for a given \( x \in C_0(\mathcal{E}) \) and for each \( x \) in \( X \), the Bochner integral \( \int_Y h(x, y) d\nu(y) \) exists and will belong to \( B_x \).

The following result has been proved by Fell\[1\] for Banach bundles. The proof is similar in the context of Banach semi-bundles.

**Lemma 1.4** For each \( h \in C_0(\mathcal{E}) \) the map \( \ell(x) = \int_Y h(x, y) d\nu(y) \) is a continuous cross-section of the Banach semi-bundle \( \mathcal{E} \).

### 1.3 The \( p \)-induced representations of locally compact groups and \( L_p(\pi) \) spaces

Let \( G \) be a locally compact group and let \( H \) be a closed subgroup of \( G \). Suppose that \( \pi \) is a representation of \( H \) on a Banach space \( \mathcal{H}(\pi) \). Let \( \mu \) be any quasi-invariant measure, in the homogeneous space \( X = G/H \) of right cosets, which belongs to a continuous \( \rho \)-function. For \( 1 \leq p < \infty \), let us denote by \( L_p(\pi, \mu) \) the set of all functions \( f \) from \( G \) to a Banach Space \( \mathcal{H}(\pi) \) such that

1. \( (f(x), v) \) is a Borel function of \( x \) for all \( v \in \mathcal{H}(\pi)^* \);
2. \( f \) satisfies the covariance condition \( f(hx) = \pi_h f(x) \) for all \( h \in H \) and \( x \in G \); and
3. \( \| f \|_p = \left( \int \| f(x) \|^p d\mu(x) \right)^{\frac{1}{p}} < \infty \).

Note that the integrand in the above integral is constant on each right coset \( Hx \) and hence defines a function on \( X \). When functions equal almost everywhere are identified, \( L_p(\pi, \mu) \) becomes a Banach space under the norm defined by (3)(for which we use the same symbol \( L_p(\pi, \mu) \)).

For each \( x, y \in G \) and \( f \in L_p(\pi, \mu) \), let us define a mapping \( "U^\pi_y \) on \( L_p(\pi, \mu) \) by

\[
(\mu U^\pi_y f)(x) := \lambda(x, y)^{\frac{1}{p}} f(xy), \tag{2}
\]

where \( \lambda(\cdot, y) \) is the Radon-Nikodym derivative of the measure \( \mu_y \) with respect to the measure \( \mu \). Then, it can be easily seen that \( "U^\pi \) is a representation of the group \( G \) on the Banach space \( L_p(\pi, \mu) \). Also, given two quasi-invariant measures \( \mu \) and \( \mu' \) on \( X \), there exists an isometry \( W \) from \( L_p(\pi, \mu) \) onto \( L_p(\pi, \mu') \) such that \( W(\mu U^\pi_y) = (\mu' U^\pi_y) W \) for all \( y \in G \). In other words, the two representations \( "U^\pi_y \) and \( \mu' U^\pi_y \) are equivalent. (cf. Mackey\[2\].)

The equivalence class of \( "U^\pi \) (denoted by \( U^\pi \) ) is called the representation of \( G \) induced by the representation \( \pi \) of \( H \). The corresponding Banach space of (equivalence classes) of functions is denoted by \( L_p(\pi) \). (The most appropriate notation for the \( p \)-induced representation (induced by \( \pi \)) would be \( U^p_\pi \); but for simplicity of notation we use \( U^\pi \) unless the former is necessary to avoid confusion.)
Let \( \pi \) and \( \gamma \) be representations of the locally compact group \( G \). A bounded linear operator \( T \) from \( \mathcal{H}(\pi) \) to \( \mathcal{H}(\gamma) \) is called an intertwining operator for \( \pi \) and \( \gamma \) if \( \pi(x)T = T\gamma(x) \) for all \( x \in G \). The vector space of all intertwining operators is denoted by \( \text{Int}_G(\pi, \gamma) \) and the dimension (possibly infinite) of this space, called the intertwining number, is denoted by \( \partial(\pi, \gamma) \).

Let \( (\Omega, \Sigma, \mu) \) be a measure space. A Banach space \( X \) is said to have the Radon-Nikodym property with respect to \( (\Omega, \Sigma, \mu) \) if for each \( \mu \)-continuous vector measure \( F : \Sigma \to X \) of bounded variation there exists \( g \in L_1(X, \mu) \) such that \( F(E) = \int_E gd\mu \) for all \( E \in \Sigma \). A Banach space \( X \) has the Radon-Nikodym property if \( X \) has the Radon-Nikodym property with respect to every finite measure space (see Gretsky and Uhl\[21\], Chapter III).

Let \( (\Omega, \Sigma, \mu) \) be a \( \sigma \)-finite measure space, \( 1 \leq p < \infty \), and let \( X \) be a Banach space. It is well known that \( L_p(\Omega, X, \mu)^* = L_{p'}(\Omega, X^{*}, \mu) \), where \( 1/p + 1/p' = 1 \), if and only if \( X^* \) has the Radon-Nikodym property with respect to \( \mu \). Also, if \( (\Omega, \Sigma, \mu) \) is a nonatomic finite measure space, then it can be seen that \( L_p(\Omega, X, \mu) \) has the Radon-Nikodym property if and only if \( 1 < p < \infty \) and \( X \) has the Radon-Nikodym property.

Throughout our work, we assume that the Banach space \( \mathcal{H}(\pi) \) of a representation \( \pi \) of a subgroup \( H \) of a group \( G \) stays within the class of spaces satisfying the Radon-Nikodym property.

Let \( \pi \) be a representation of a group \( G \) on a Banach space \( \mathcal{H}(\pi) \). We define the map \( \pi^* : G \to U((\mathcal{H}(\pi))^*) \) by letting \( \pi^*(x) = (\pi(x^{-1}))^* \). It can be easily seen that that \( \pi^* \) is a representation of \( G \) on the Banach space \( \mathcal{H}(\pi^*) = (\mathcal{H}(\pi))^* \), when \( \mathcal{H}(\pi) \) is reflexive. Assume now that the Banach space \( \mathcal{H}(\pi) \) is reflexive. Let us consider the Banach space \( L_p(\pi^*) \) and the induced representation \( U_p^{\pi^*} \) of \( G \). The dual pairing between \( L_p(\pi) \) and \( L_{p'}(\pi^*) \) is given by

\[
(f, g) = \int_{\mathcal{H}} \langle f(x), g(x) \rangle d\mu(x), \text{ for } f \in L_p(\pi) \text{ and } g \in L_{p'}(\pi^*).
\]

The above integral is well defined since, for any \( h \in H \) and \( x \in G \),

\[
\langle f(hx), g(hx) \rangle = \langle \pi(h)f(x), \pi^*(h)g(x) \rangle = \langle \pi(h)f(x), (\pi(h^{-1}))^*g(x) \rangle = \langle f(x), g(x) \rangle.
\]

Also, for any \( y \in G \),

\[
\langle U_p^{\pi^*}(y)f, U_p^{\pi^*}(y)g \rangle = \int_{\mathcal{H}} \langle \lambda(x, y)^{1/p} f(xy), \lambda(x, y)^{1/p^*} g(xy) \rangle d\mu(x) = \int_{\mathcal{H}} \lambda(x, y)\langle f(xy), g(xy) \rangle d\mu(x) = \langle f, g \rangle,
\]

the last equality of which was obtained by changing variables \( x \mapsto xy \). This implies that

\[
U_p^{\pi^*}(y) = (U_p^{y^{-1}})^* = (U_p^y)^*(y), \text{ for all } y \in G.
\]

Let \( 1 \leq p < \infty \). Let us define a convolution \( g \ast f \) for \( g \in L_p(\pi) \) and \( f \in L_1(G) \), by

\[
(g \ast f)(x) := \int_G \langle \lambda_H(x, y^{-1})^{1/p} g(xy^{-1})f(y) \rangle d\mathcal{H}(y).
\]

It is not difficult to prove that \( g \ast f \) belongs to \( L_p(\pi) \), \( g \ast (h \ast f) = (g \ast h) \ast f \) for all \( g \in L_p(\pi) \) and \( h, f \in L_1(G) \) and that \( L_p(\pi) \) is an \( L_1(G) \)-module.

Let \( \text{Hom}_G(L_p(\pi), L_q(\gamma)) \) denote the Banach space of all continuous \( G \)-module homomorphisms from \( L_p(\pi) \) to \( L_q(\gamma) \) (Rieffel\[37\]).
Proposition 1.5

\[ \text{Hom}_G(L_p(\pi), L_q(\gamma)) = \text{Int}_G(U^\pi_p, U^\gamma_q). \]  

Proof: Let \( T \) be any bounded linear operator from \( L_p(\pi) \) to \( L_q(\gamma) \) and \( T^* \) be its adjoint operator. For any \( g \in L_p(\pi), f \in L_1(G) \) and \( k \in L_q(\gamma^*), \)

\[
(T(g \ast f), k) = \langle g \ast f, T^* k \rangle,
\]

\[
= \int_G \langle \int_G (\lambda(x, y^{-1})) g(xy^{-1}) f(y) d\nu(y), T^*(k(x)) d\mu_H(x),
\]

\[
= \int_G f(y) \int_G (U^\pi_p(x)) g(x) T^*(k(x)) d\mu_H(x) d\nu(y),
\]

\[
= \int_G f(y)(U^\pi_p(y^{-1}) g, T^* k) d\nu(y).
\]

Hence,

\[
\langle T(g \ast f), k \rangle = \int_G f(y)(TU^\pi_p(y^{-1}) g, k) d\nu(y).
\]  

(5)

On the other hand,

\[
\langle T(g) \ast f, k \rangle = \int_G \langle (T(g) \ast f)(x), k(x) \rangle d\mu_K(x),
\]

\[
= \int_G \langle \int_G (\lambda(x, y^{-1})) T(g)(xy^{-1}) f(y), k(x) \rangle d\nu(y),
\]

\[
= \int_G f(y) \int_G (U^\gamma_q(y^{-1}) Tg(y), k(x)) d\mu_K(x) d\nu(y).
\]

Therefore,

\[
\langle T(g) \ast f, k \rangle = \int_G f(y)(U^\gamma_q(y^{-1}) Tg, k) d\nu(y).
\]  

(6)

If \( T \in \text{Hom}_G(L_p(\pi), L_q(\gamma)) \), we see, by (5) and (6), that

\[
TU^\pi_p(y) g = U^\gamma_q(y) Tg.
\]  

(7)

for almost all \( y \in G \). By continuity, (7) is true for all \( y \in G \). Hence \( T \in \text{Int}_G(U^\pi_p, U^\gamma_q) \). Conversely, \( T \in \text{Int}_G(U^\pi_p, U^\gamma_q) \) implies \( T \in \text{Hom}_G(L_p(\pi), L_q(\gamma)) \), by (5) and (6). Hence, (4) follows.

\[ \diamond \]

2 Some important results on \( \lambda \)-functions

First, we intend to prove an integral formula which involves integration on coset spaces. Secondly, the notion of disintegration of measures (which has been discussed in a number of places in the literature (see, for example, Mackey\[31\], Halmos\[23\])) is dealt with. Here, we derive an identity among \( \lambda \)-functions of a particular set of subgroups of a given group.
Lemma 2.1  Let $G$ be a locally compact group. Let $H$ and $K$ be subgroups of $G$ with $K \subseteq H$. Then there exist positive quasi-invariant measures $\mu_K$ on $G/K$, $\mu_H$ on $G/H$ and $\tilde{\mu}$ on $H/K$ such that, for $F \in C_0(G/K)$,

$$
\int_{G/K} F(z) d\mu_K(z) = \int_H \left( \int_{K} \frac{\lambda_K(y,t)}{\lambda_H(y,t)} F(yt) d\tilde{\mu}(y) \right) d\mu_H(t),
$$

whenever the integrals exist.

Proof: As discussed in Corollary 1.2 (see also Reiter\cite{35}, p.158, Mackey\cite{31}), there exists a continuous, strictly positive function $\rho_K$ on $G$ and a positive measure $\mu_K$ on $G/K$ such that

$$
\int_G f(u) d\nu_G(u) = \int_K \left( \int_K \frac{1}{\rho_K(sz)} f(sz) d\nu_K(s) \right) d\mu_K(z),
$$

for $f \in C_0(G)$.

Also, by the same reasoning, there exists a continuous, strictly positive function $\rho_H$ on $G$ and a positive measure $\mu_H$ on $G/H$ such that

$$
\int_G f(u) d\nu_G(u) = \int_H \left( \int_H \frac{1}{\rho_H(ht)} f(ht) d\nu_H(h) \right) d\mu_H(t).
$$

Let $\tilde{\rho} = \rho_K/\rho_H$. We see that

$$
\tilde{\rho}(sx) = \rho_K(sx)/\rho_H(sz) = (\Delta_K(s)/\Delta_H(s))\tilde{\rho}(x),
$$

for $s \in K$ and $x \in G$. Thus $\tilde{\rho}$, restricted to $H$, is a $\rho$-function for the homogeneous space $H/K$. If we let $\tilde{\mu}$ be a quasi-invariant measure associated with this $\rho$-function, we have

$$
\int_G f(u) d\nu_G(u) = \int_H \int_K \left( \int_K \frac{\rho_H(sy)}{\rho_K(sz)} f(syt) d\nu_K(s) \right) d\tilde{\mu}(y) d\mu_H(t).
$$

By Reiter\cite{35}, p.165, for a given $F \in C(G/K)$, there exists a function $f \in C(G)$ such that

$$
F(\check{z}) = \int_K \frac{1}{\rho_K(sz)} f(sz) d\nu_K(s),
$$

where $\check{z} = p_K(z)$. Comparing equations (9) and (10), and using (11), we see that

$$
\int_{G/K} F(z) d\mu_K(z) = \int_H \left( \int_H \frac{\rho_K(yt)}{\rho_K(y)\rho_H(yt)} F(yt) d\tilde{\mu}(y) \right) d\mu_H(t)
$$

$$
= \int_H \left( \int_K \frac{\lambda_K(y,t)}{\lambda_H(y,t)} F(yt) d\tilde{\mu}(y) \right) d\mu_H(t),
$$

for any $F \in C_0(G/K)$, and (8) is proved.

\hfill \Box
Let $\mu_H$ be a given quasi-invariant measure on $G/H$ with the corresponding $\lambda$-function $\lambda_H$. Consider the homeomorphism $\phi_x : G/H^x \rightarrow G/H$ given by $\phi_x(u) = xu$. Define a measure $\mu_{H^x}$ on $G/H^x$ by $\mu_{H^x}(E) = \mu_H(\phi_x(E))$ whenever $E$ is such that $x.E$ is measurable. Clearly, $\mu_{H^x}$ is quasi-invariant if and only if $\mu_H$ is. The corresponding $\lambda$-function of $\mu_{H^x}$ is denoted by $\lambda_{H^x}$.

Then, it can be easily seen that, for $x, t \in G$ and for almost all $v \in G/H$,

$$\lambda_{H^x}(x^{-1}v, t) = \lambda_H(v, t).$$

(12)

which states the relationship between $\lambda_H$ and $\lambda_{H^x}$.

Let $\Delta = \{(x, x) : x \in G\}$ be the diagonal subgroup of $G \times G$. Consider the right action of $\Delta$ on the coset space $(G \times G)/(H \times K)$. The stabilizer of the coset $(H x, K y)$ is $(H \times K)^{(x, y)} \cap \Delta$ and the orbit is the double coset $(H \times K)(x, y)\Delta$. Let $\Upsilon$ be the set of all double cosets $(H \times K) : \Delta$ of $G \times G$; that is, the set of all orbits. For each $(x, y) \in G \times G$, let $k(x, y)$ denote the $(H \times K) : \Delta$ double coset to which $(x, y)$ belongs. If $\nu_0$ is any finite measure in $G \times G$ with the same null sets as Haar measure we define a measure $\mu_{(H, K)}$ on $\Upsilon$ by $\mu_{(H, K)}(F) = \nu_0(k^{-1}(F))$ whenever $F$ is such that $k^{-1}(F)$ is measurable. Using Mackey’s terminology, we call such a measure an admissible measure in $\Upsilon$. We obtain the following result as a consequence of Lemma 11.1, Mackey[31].

**Lemma 2.2** Suppose that $H$ and $K$ are regularly related (see Mackey[31]). Let $\Delta$ be the diagonal subgroup of $G \times G$ and $\Upsilon$ denote the set of all $(H \times K) : \Delta$ double cosets in $G \times G$. Then for each double coset $D(x, y) = H \times K(x, y)\Delta$ there exists a quasi-invariant measure $\mu_{x,y}$ on $G/(H^x \cap K^y)$, $x, y \in G$, and $\lambda_{H^x \cap K^y}$ with $\mu_{x,y} \succ \lambda_{H^x \cap K^y}$ such that

$$\lambda_H(xs^{-1}, s)\lambda_K(yts^{-1}, s)\lambda_{H^x \cap K^y}(F, \Delta) = 1,$n

(13)

for all $s, t \in G$, and for almost all $(x, y) \in (G \times G)/(H \times K)$. Moreover, $\lambda_{H^x \cap K^y}(t, s)$ is defined everywhere and continuous in $(G/(H^x \cap K^y)) \times G$.

Proof: Choose two quasi-invariant measures $\mu_H$ and $\mu_K$ on $G/H$ and $G/K$ respectively, which correspond to two continuous $\rho$-functions. Define a measure $\mu_{H \times K}$ in $(G \times G)/(H \times K)$ by $\mu_{H \times K} = \mu_H \times \mu_K$ (see, for example, Halmos[22], p.144). Obviously, $\mu_{H \times K}$ is quasi-invariant to the action of $\Delta$. Let $\nu_0$ be the measure in $(G \times G)$ defined by $\nu_0(p_{H \times K}^{-1}(F)) = \mu_{H \times K}(F)$. Let $\mu_{H, K}$ be an admissible measure in $\Upsilon$ corresponding to $\nu_0$.

Let $f$ be a function defined on $(G/H) \times (G/K)$. Suppose

$$\int_{\Upsilon} \int_{G/H} f(x, y) d\mu_H(x) d\mu_K(y)$$

is integrable. Changing the variables $x \mapsto xs$ and $y \mapsto ys$, we get

$$\int_{G/H} \int_{G/K} f(x, y) d\mu_H(x) d\mu_K(y)$$

$$= \int_{G/H} \int_{G/K} \lambda_H(x, s)\lambda_K(y, s) f(xs, ys) d\mu_H(x) d\mu_K(y)$$

$$= \int_{G/H} \int_{G/K} \lambda_H(x, s)\lambda_K(y, s) f(xs, ys) d\mu_{H \times K}(x, y).$$

For each $(x, y)$ in $(G \times G)/(H \times K)$ let $r(x, y) = k(p_{H \times K}^{-1}(x, y))$. If $H$ and $K$ are regularly related then $r$ defines a measurable equivalence relation (see Mackey[31]). Then, by Lemma 11.1, Mackey[31], $\mu_{H \times K}$ is an integral of measures $\mu_{x,y}$, where $D(x, y) \in \Upsilon$, with respect to the measure $\mu_{H \times K}$ in $\Upsilon$. By Lemma 11.5, Mackey[31], each $\mu_{x,y}$ is a quasi-invariant measure on the orbit $r^{-1}(D(x, y))$. Using this disintegration, we have

$$\int_{G \times G} \lambda_H(x, s)\lambda_K(y, s) f(xs, ys) d\mu_{H \times K}(x, y)$$

$$= \int_{D \in \Upsilon} \int_{G \times G} \lambda_H(x, s)\lambda_K(y, s) f(xs, ys) d\mu_{x,y}(D) d\mu_{H, K}(D),$$

10
where \((x, y)\) is the coset representative of the coset \(D(x, y)\). Identifying the space \(\Delta/(H \times K)^{(x, y)} \cap \Delta\) with \(G/(H^x \cap K^y)\) we can regard \(\mu_{x,y}\) as a measure on \(G/(H^x \cap K^y)\). Then we have

\[
\int_{G \times G} \frac{\lambda_H(x, s)\lambda_K(y, s)f(xs, ys)d\mu_{H\times K}(x, y)}{\mu_H(D)} = \int_{D \in T} \int_{t \in \frac{G}{H \times K}} \frac{\lambda_H(xt, ts)f(xs, ys)d\mu_{x,y}(t)d\mu_{H,K}(D)}{\mu_H(D)},
\]

Changing variables \(t \mapsto ts^{-1}\), in the integral on the right-hand side, we get

\[
\int_{G \times G} \frac{\lambda_H(x, s)\lambda_K(y, s)f(xs, ys)d\mu_{H\times K}(x, y)}{\mu_H(D)} = \int_{D \in T} \int_{t \in \frac{G}{H \times K}} \frac{\lambda_H(xts^{-1}, s)\lambda_K(yts^{-1}, s)f(xt, yt)}{\mu_{H\cap K^y}(t, s^{-1})d\mu_{x,y}(t)d\mu_{H,K}(D)}. \tag{14}
\]

On the other hand, if we start with \(\int \int_{G \times G} f(x, y)d\mu_{H\times K}(x, y)\) and use Lemma 11.1, Mackey [31], we have

\[
\int \int_{G \times G} f(x, y)d\mu_{H\times K}(x, y) = \int_{D \in T} \int_{t \in \frac{G}{H \times K}} f(xt, yt)d\mu_{x,y}(t)d\mu_{H,K}(D),
\]

Hence from (14) and (15) we have

\[
\lambda_H(xts^{-1}, s)\lambda_K(yts^{-1}, s)\lambda_{H^x\cap K^y}(t, s^{-1}) = 1,
\]

for all \(s \in G\), for almost all \(t \in G/(H^x \cap K^y)\) and for almost all \((x, y) \in (G \times G)/(H \times K)\). For each such \((x_0, y_0) \in (G \times G)/(H \times K)\),

\[
\lambda_H(x_0ts^{-1}, s)\lambda_K(y_0ts^{-1}, s)\lambda_{H^{x_0}\cap K^{y_0}}(t, s^{-1}) = 1. \tag{16}
\]

By continuity of \(\lambda_H\) and \(\lambda_K\), we see that (16) is true for all \(t \in G/(H^{x_0} \cap K^{y_0})\). Furthermore, (16) implies that \(\lambda_{H^x\cap K^y}(t, s)\) is defined everywhere and continuous on \((G/(H^x \cap K^y)) \times G\), which proves the Lemma.

\diamond

The following result is a consequence of Lemma 2.2.

**Corollary 2.3** Let \((x, y) \in G \times G\) such that the identity (13) holds. Then for \(s \in H^x \cap K^y\),

\[
\frac{\Delta_H(h)\Delta_K(k)}{\Delta_G(s)\Delta_{H^x\cap K^y}(s)} = 1, \tag{17}
\]

where \(h = xsx^{-1}\) and \(k = ysy^{-1}\).
Proof: Let $t = s$ in the identity (13). Then we have
\[
\lambda_H(x, s)\lambda_K(y, s)\lambda_{H \cap K^s}(s, s^{-1}) = 1.
\]
(18)

By property (a) of $\lambda$-functions given in Section 1.1, page 3, this simplifies to
\[
\lambda_H(x, s)\lambda_K(y, s) = \lambda_{H \cap K^s}(e, s).
\]
(19)

Consider $s \in H^x \cap K^y$. Then $s = x^{-1}hx = y^{-1}ky$ for some $h \in H$ and $k \in K$. For such an $s$, we have by properties (a) and (b) of $\lambda$-functions, page 3,
\[
\lambda_H(x, s) = \lambda_H(x, x^{-1}hx) = \frac{\Delta_H(h)}{\Delta_G(h)}.
\]
(20)

Similarly,
\[
\lambda_K(y, s) = \frac{\Delta_K(k)}{\Delta_G(k)}; \text{ and } \lambda_{H \cap K^s}(e, s) = \frac{\Delta_{K^s}(s)}{\Delta_{G^s}(s)}.
\]
(21)

Using (19),(20) and (21), we obtain
\[
\frac{\Delta_H(h) \Delta_K(k)}{\Delta_G(h) \Delta_G(k)} = \frac{\Delta_{H \cap K^s}(s)}{\Delta_{G^s}(s)}.
\]
(22)

But $\Delta_G(h) = \Delta_G(x^{-1}hx) = \Delta_G(s) = \Delta_G(y^{-1}ky) = \Delta_G(k)$, hence (22) simplifies to
\[
\frac{\Delta_H(h) \Delta_K(k)}{\Delta_G(s) \Delta_{H \cap K^s}(s)} = 1,
\]
(23)
as required.

3 Projective tensor products and $A^q_p$ spaces

3.1 Construction of the convolution formula

Let $G$ be a second countable locally compact group, with closed subgroups $H$ and $K$. Thus, the corresponding homogeneous spaces are Hausdorff and second countable, which in turn implies that any Borel measure on such spaces is regular. In addition, we will assume that $H$ and $K$ are regularly related (32). $\mu_H$ and $\mu_K$ will denote fixed quasi-invariant measures on $G/H$ and $G/K$, respectively. We choose a family of quasi-invariant measures $\{\mu_{x,y} : x \in G/H, y \in G/K\}$, where $\mu_{x,y}$ is a measure on $G/(H^x \cap K^y)$, in such a manner that for a function $f$ defined and integrable on $(G/H) \times (G/K)$, we have
\[
\int_{G/H} \int_{G/K} f(x,y) d\mu_H(x)d\mu_K(y) = \int_{D(x,y) \in \mathcal{Y}} \int_{\mathcal{H} \subseteq G} f(xt, yt)d\mu_{x,y}(t)d\mu_{H,K}(D),
\]
by disintegration of measures discussed in Lemma 2.2. For a given $\mu_{x,y}$, $\rho_{H \cap K^y}$ and $\lambda_{H \cap K^y}$ will denote the corresponding $\rho$-function and the $\lambda$-function respectively. For any $x \in G$, the quasi-invariant measure $\mu_{H^x}$ on $G/H^x$ will always considered to be $\mu_{H^x} = \mu_H \circ \phi_x$, where $\phi_x : G/H^x \rightarrow G/H$ is the homeomorphism given by $\phi_x(u) = xu$. By $\rho_{H^x}$ we mean the corresponding $\rho$-function of the above $\mu_{H^x}$.

$\pi$ and $\gamma$ will denote representations of $H$ and $K$ on Banach spaces $\mathcal{H}(\pi)$ and $\mathcal{H}(\gamma)$, respectively.
Let \( L_p(\pi) \otimes^\sigma L_q(\gamma^*) \) denote the projective tensor product \((22)\) of \( L_p(\pi) \) and \( L_q(\gamma^*) \) as Banach spaces so that \( \sigma \) is the greatest cross-norm. Let \( L \) be the closed linear subspace of \( L_p(\pi) \otimes^\sigma L_q(\gamma^*) \) which is spanned by all the elements of the form

\[
U^\pi_p(s) f \otimes g - f \otimes (U^\gamma_q)^*(s) g, \quad s \in G, f \in L_p(\pi), g \in L_q.
\]

The quotient Banach space \((L_p(\pi) \otimes^\sigma L_q(\gamma^*))/L\) is called the \( G \)-module tensor product, and is denoted by \( L_p(\pi) \otimes^\sigma_G L_q(\gamma^*) \). Then we have a natural isometric isomorphism

\[
Int_G(U^\pi_p, U^\gamma_q) \cong (L_p(\pi) \otimes^\sigma_G L_q(\gamma^*))^*
\]

(see \([37]\), 2.12 and 2.13), and the ultraweak*-topology on \( Int_G(U^\pi_p, U^\gamma_q)\) corresponds to the weak*-topology on \((L_p(\pi) \otimes^\sigma_G L_q(\gamma^*))^*\) \((30)\), Theorem 1.4).

Recall that the space \( A^q_p \) in the classical case consists of convolutions of complex-valued functions of \( L_p(G) \) and \( L_q(G) \) (see, for example, Rieffel\([\)],\(]\)). Our aim is to construct \( A^q_p \) spaces using spaces of induced representations, \( L_p(\pi) \) and \( L_q(\gamma^*) \), which are spaces of vector-valued functions. Therefore, our task is to construct a formula (Definition 4.7) for a convolution of functions in \( L_p(\pi) \) and \( L_q(\gamma^*) \). The case where \( G/H \) and \( G/K \) are not compact is similar to that in the classical case (see Hörmander\([\])\) in the sense that the non-triviality of the tensor product \( L_p(\pi) \otimes^\sigma_G L_q(\gamma^*) \) depends on the value of \( 1/p + 1/q' \) as the following theorem states.

**Theorem 3.1** Let \( 1/p + 1/q' < 1, 1 < p, q' < \infty \). Suppose that for any given compact set \( F \) in \( G \), there exists \( x \in G \) such that \( HF x \cap HF = \emptyset \) and \( KF x \cap KF = \emptyset \). Then

\[
L_p(\pi) \otimes^\sigma_G L_q(\gamma^*) = \{0\}.
\]

We do not know whether Theorem 5.1 is true in the absence of the condition that there exists an element \( x \in G \) such that \( HF x \cap HF = \emptyset \) and \( KF x \cap KF = \emptyset \) for a given compact set \( F \) in \( G \).

Let us turn to the construction of the convolution formula. The following proposition states a result that equips us with the necessary ground work.

**Proposition 3.2** Let \( 1 \leq p, q' < \infty \). For \( \sum_{i=1}^{\infty} f_i \otimes g_i \) in \( L_p(\pi) \otimes^\sigma L_q(\gamma^*) \) and for almost all \( x \in G/H \) and \( y \in G/K \),

\[
\sum_{i=1}^{\infty} f_i(x) \otimes g_i(y) \in H(\pi) \otimes^\sigma H(\gamma^*).
\]

Our objective is to define a mapping on \( L_p(\pi) \otimes^\sigma L_q(\gamma^*) \) so that its image space is a generalisation of the space of convolutions as in the classical case. Let us consider the integral

\[
\int_G f(xt) \otimes^\sigma g(yt) d\nu_G(t), \quad (25)
\]

where \( f \in L_p(\pi), g \in L_q(\gamma^*) \) and \((x, y) \in G \times G\). It is easy to see that the norm of the integrand is constant on the subgroup \( H^2 \cap K^\nu \) of \( G \); for, if \( t = x^{-1}hx = x^{-1}ky \) for some \( h \in H \) and \( k \in K \), then \( f(xt) \otimes g(yt) = \pi(h)f(x) \otimes \gamma^*(k)g(y) \). This implies that the space over which we integrate must reduce to \( G/(H^2 \cap K^\nu) \), in order to avoid the integrand becoming too large. The integrand is constant over a given coset of \( G/(H^2 \cap K^\nu) \) if

\[
f(xst) \otimes g(yst) = f(xt) \otimes g(yt),
\]

13
for all $s \in H^x \cap K^y$. But
\[
f(xst) \otimes g(yst) = \pi^x(s)f(xt) \otimes \gamma^y(s)g(yt).
\]
This suggests that the integrand must have its value at $(x, y)$ in the quotient space of $H(\pi) \otimes^\sigma H(\gamma^*)$, in which we have the equality
\[
\pi^x(s)f(xt) \otimes \gamma^y(s)g(yt) = f(xt) \otimes g(yt).
\]
This calls for the following definition.

**Definition 3.3** For any $x, y \in G$, the subspace $H_{x,y}$ of $H(\pi) \otimes^\sigma H(\gamma^*)$ is defined to be the closed linear span of elements of the form
\[
\pi^x(b)\xi \otimes \eta - \xi \otimes (\gamma^y(b))^*\eta,
\]
where $b \in H^x \cap K^y, \xi \in H(\pi)$ and $\eta \in H(\gamma^*)$. The quotient Banach space $H(\pi) \otimes^\sigma H(\gamma^*)/H_{x,y}$ is denoted by $A_{x,y}$.

Note that, using the notation in Rieffel[37], $A_{x,y}$ can be written as $H(\pi) \otimes^\sigma_{H^x \cap K^y} H(\gamma^*)$.

**Proposition 3.4** The spaces $\{H_{x,y} : x, y \in G\}$, and hence the spaces $\{A_{x,y} : x, y \in G\}$, satisfy the property that
\[
H_{x,sy} = H_{x,y} \quad \text{and} \quad A_{x,sy} = A_{x,y},
\]
for any $s \in G$.

Proof: For $s \in G$, the space $H_{x,sy}$ is the closed linear span of elements of the form
\[
\pi^{xs}(b)\xi \otimes \eta - \xi \otimes (\gamma^ys(b))^*\eta,
\]
where $b \in H^{xs} \cap K^{ys}, \xi \in H(\pi)$ and $\eta \in H(\gamma)$ with
\[
\pi^{xs}(b) = \pi(xsbs^{-1}x^{-1}) = \pi^x(sbs^{-1}).
\]
Since $b \in H^{xs} \cap K^{ys}$, there exist $h \in H$ and $k \in K$ such that
\[
b = s^{-1}x^{-1}hx = s^{-1}y^{-1}ky.
\]
Hence $sbs^{-1} = x^{-1}hx = y^{-1}ky$, showing that $sbs^{-1} \in H^x \cap K^y$. Therefore,
\[
\pi^{xs}(b)\xi \otimes \eta - \xi \otimes (\gamma^ys(b))^*\eta = \pi^x(sbs^{-1})\xi \otimes \eta - \xi \otimes (\gamma^y(sbs^{-1}))^*\eta,
\]
with $sbs^{-1} \in H^x \cap K^y, \xi \in H(\pi)$ and $\eta \in H(\gamma)$. This implies that $H_{x,sy} \subseteq H_{x,y}$, which in turn gives us that $H_{x,y} = H_{x,sy}^{-1} \subseteq H_{x,sy}$, for all $s \in G$. Hence (28) follows.

\[\diamond\]

For $u \otimes v \in H(\pi) \otimes^\sigma H(\gamma^*)$, we use the notation $u \otimes_{x,y} v$ to denote the element of $A_{x,y}$ to which $u \otimes v$ belongs. Then the integral (27) must be written in the form
\[
\int_{H^{x,y}/H_{x,y}} f(xt) \otimes_{x,y} g(yt)d\mu_{x,y}(t), \quad (27)
\]
for a suitably chosen quasi-invariant measure \( \mu_{x,y} \) on the homogeneous space \( G/(H^z \cap K^y) \). For each \( x, y \in G \), the value of the integral belongs to the quotient Banach space \( A_{x,y} \). The next obvious step in this construction is to check whether the integral is finite and, to this end, we see that a further modification of the integrand is necessary. Propositions (4.5) and (4.6) state the conditions under which this modified integral is well defined and finite, respectively.

Note that if we define a function \( \rho_{H_{x,y}} \) on \( G \) by \( \rho_{H_{x,y}} := \rho_{H^z \cap K^y}/\rho_{H^z} \), we have
\[
\rho_{H_{x,y}}(sz) = \rho_{H^z \cap K^y}(sz)/\rho_{H^z}(sz) = \Delta_{H^z \cap K^y}(s)/\Delta_{H^z}(s)\rho_{H_{x,y}}(z),
\]
for \( s \in H^z \cap K^y \) and \( z \in G \). Thus \( \rho_{H_{x,y}} \), restricted to \( H^z \), is a \( \rho \)-function for the homogeneous space \( H^z/(H^z \cap K^y) \). We let \( \mu_{H_{x,y}} \) be a quasi-invariant measure associated with this \( \rho \)-function and \( \lambda_{H_{x,y}} \) be the corresponding \( \lambda \)-function. Similarly, we can define a \( \rho \)-function \( \rho_{K_{x,y}} \) for the homogeneous space \( K^y/(H^z \cap K^y) \) and the corresponding \( \lambda \)-function will be denoted by \( \lambda_{K_{x,y}} \).

**Proposition 3.5** Let \( p, q \) and \( m \) be positive real numbers with \( 1 \leq p, q' < \infty \). Then, for \( \sum_{i=1}^{\infty} f_i \otimes g_i \in L_p(\pi) \otimes^\pi L_q(\gamma^*) \) and \( x, y \in G \),
\[
t \mapsto \sum_{i=1}^{\infty} \frac{1}{\lambda_{H^z \cap K^y}(e,t)} \lambda_H(x,t)^{\frac{1}{p}} f_i(xt) \otimes_{x,y} \lambda_K(y,t)^{\frac{1}{q'}} g_i(yt) \quad (28)
\]
is a mapping on the coset space \( G/(H^z \cap K^y) \) in each of the following cases:

(a) \( p = m \) and \( G/K \) having invariant measure (or \( q' = m \) and \( G/H \) having invariant measure);
(b) \( G/K \) and \( G/H \) both having invariant measures;
(c) \( p = q' = m \);
(d) \( H^z/(H^z \cap K^y) \) and \( K^y/(H^z \cap K^y) \) having invariant measures.

**Proof:** First let us consider the expression
\[
\frac{\lambda_H(x,t)^{\frac{1}{p}} \lambda_K(y,t)^{\frac{1}{q'}}}{\lambda_{H^z \cap K^y}(e,t)^{\frac{1}{p}}},
\]
under the cases (a), (b) and (c).

Consider (a). Assuming \( p = m \) and using the identity (13), we have,
\[
\left( \frac{\lambda_H(x,t)}{\lambda_{H^z \cap K^y}(e,t)} \right)^{\frac{1}{p}} = \lambda_K(y,t)^{\frac{1}{q'}}.
\]
If the measure on \( G/K \) is invariant, then \( \lambda_K(y,t) = 1 \); hence
\[
\frac{\lambda_H(x,t)^{\frac{1}{p}} \lambda_K(y,t)^{\frac{1}{q'}}}{\lambda_{H^z \cap K^y}(e,t)^{\frac{1}{p}}} = \lambda_K(y,t)^{\frac{1}{q'}} = 1.
\]
A similar argument holds in the case where \( q' = m \) and \( G/H \) possesses an invariant measure. In the case of (b), \( \lambda_H(x,t) = \lambda_K(y,t) = 1 \), and then by identity (13), \( \lambda_{H^z \cap K^y}(e,t) = 1 \), giving
\[
\frac{\lambda_H(x,t)^{\frac{1}{p}} \lambda_K(y,t)^{\frac{1}{q'}}}{\lambda_{H^z \cap K^y}(e,t)^{\frac{1}{p}}} = 1.
\]
Clearly, under condition (c),
\[
\frac{\lambda_H(x,t)^\dagger \lambda_K(y,t)^\dagger}{\lambda_{H\cap K^v}(e,t)^\dagger} = \left( \frac{\lambda_H(x,t)\lambda_K(y,t)}{\lambda_{H\cap K^v}(e,t)} \right)^\dagger = 1,
\]
using the identity (13).

Therefore, under the conditions (a), (b) or (c), (30) can be simplified to
\[
t \mapsto \sum_{i=1}^{\infty} f_i(xt) \otimes_{x,y} g_i(yt),
\]
which is constant on each coset of \(H^x \cap K^v\) in \(G\). Hence it is a mapping on the coset space \(G/(H^x \cap K^v)\).

For the case (d), it only remains to show that
\[
\frac{1}{\lambda_{H\cap K^v}(e,st)^\dagger} \lambda_H(x,st)^\dagger \lambda_K(y,st)^\dagger = \frac{1}{\lambda_{H\cap K^v}(e,t)^\dagger} \lambda_H(x,t)^\dagger \lambda_K(y,t)^\dagger,
\]
for \(s \in H^x \cap K^v\). Letting \(s = x^{-1}hx = y^{-1}ky\), for \(h \in H\) and \(k \in K\), we have
\[
\frac{1}{\lambda_{H\cap K^v}(e,st)^\dagger} \lambda_H(x,st)^\dagger \lambda_K(y,st)^\dagger = \left( \frac{\Delta_G(s)}{\Delta_{H\cap K^v}(s)} \right)^\dagger \left( \frac{\Delta_H(h)}{\Delta_G(h)} \right)^\dagger \left( \frac{\Delta_K(k)}{\Delta_G(k)} \right)^\dagger \frac{1}{\lambda(e,t)^\dagger} \lambda_H(x,t)^\dagger \lambda_K(y,t)^\dagger.
\]
(29)

Since we assume that \(H^x/(H^x \cap K^v)\) has invariant measure, we have (see Reiter\[35\] p.159),
\[
\lambda_{H_{x,y}}(e,s) = \frac{\rho_{H_{x,y}}(s)}{\rho_{H_{x,y}}(e)} = \Delta_{H\cap K^v}(s) / \Delta_H(s) = 1.
\]

Now \(H\) and \(H^x\) are closed conjugate subgroups of \(G\) under an inner automorphism \(\tau : G \mapsto G\) given by \(\tau(y) = x^{-1}yx\). Since \(\tau\) is a topological isomorphism of \(H\) onto \(H^x\) we have \(\Delta_H = \Delta_{H^x}\). This implies that \(\Delta_{H^x}(h^x) = \Delta_H(h)\). Hence we have
\[
\frac{\Delta_H(h)}{\Delta_{H\cap K^v}(s)} = \frac{\Delta_K(k)}{\Delta_{H\cap K^v}(s)} = 1,
\]
(30)
f\(s \in H^x \cap K^v\) with \(s = x^{-1}hx = y^{-1}ky\). Considering the identity (17) and using the fact that \(H^x/(H^x \cap K^v)\) and \(K^v/(H^x \cap K^v)\) possess invariant measures, we have
\[
\left( \frac{\Delta_G(s)}{\Delta_{H\cap K^v}(s)} \right)^\dagger \left( \frac{\Delta_H(h)}{\Delta_G(h)} \right)^\dagger \left( \frac{\Delta_K(k)}{\Delta_G(k)} \right)^\dagger = \left( \frac{\Delta_G(s)}{\Delta_{H^x\cap K^v}(s)} \right)^\dagger \frac{1}{\lambda^\dagger} = 1.
\]
(31)
Thus, (31) simplifies to
\[
\frac{\lambda_H(x,st)^\dagger \lambda_K(y,st)^\dagger}{\lambda_{H\cap K^v}(e,st)^\dagger} = \frac{\lambda_H(x,t)^\dagger \lambda_K(y,t)^\dagger}{\lambda_{H\cap K^v}(e,t)^\dagger},
\]
(32)
for \(s \in H^x \cap K^v\) and therefore, (30) is a well defined mapping in case (d) as well, completing the proof of the Proposition.

\[\diamondsuit\]
Recall, from the discussion preceding Lemma 2.2, that $\Upsilon$ denotes the set of all double cosets $H \times K : \Delta$ of $G \times G$. For $x, y \in G$, let

$$M_{x,y}(\frac{\lambda}{\mu}) \ = \ \int_{H/K} \lambda_{h,y}(e, \alpha) d\mu_{h,y}(\alpha) \quad \text{and} \quad N_{x,y}(\frac{\lambda}{\mu}) = \int_{H/K} \lambda_{k,y}(e, \xi) d\mu_{k,y}(\xi).$$

**Proposition 3.6** For $\sum_{i=1}^{\infty} f_i \otimes g_i \in L_p(\pi) \otimes^\gamma L_q(\gamma^*)$ the integral

$$\int_{\frac{\lambda}{\mu} \in \Upsilon} \sum_{i=1}^{\infty} \frac{1}{\lambda_{h \cap K^\prime}(e, t)} \lambda_{h}(x,t)^{\frac{q}{q-1}} f_i(x) \otimes_x y \lambda_{K}(y,t)^{\frac{q}{q-1}} g_i(y) d\mu_{x,y}(t)$$

(33)

is finite for almost all $D(x,y) \in \Upsilon$ in each of the following cases:

(a) $p = 1$ and $G/K$ having finite invariant measure (or $q' = 1$ and $G/H$ having finite invariant measure);

(b) $G/K$ and $G/H$ both having finite invariant measures;

(c) $p = q' = 1$;

(d) $1 < p, q' < \infty$ with $1/p + 1/q' > 1$ and, $H^x/(H^x \cap K^\prime)$ and $K^\prime/(H^x \cap K^\prime)$ being compact for almost all $x, y \in G$ with $(x,y) \mapsto M_{x,y}N_{x,y}$ being a bounded function from $\Upsilon$ to $\mathbb{R}$.

**Proof:** First let us consider the cases (a), (b) and (c). Using the disintegration of measures in the spaces involved (as discussed in the proof of Lemma 2.2 ), we get

$$\sum_{i=1}^{\infty} \int_{D(x,y) \in \Upsilon} \int_{\frac{\lambda}{\mu} \in \Upsilon} \|f_i(x)\| \|g_i(y)\| d\mu_{D}(\Upsilon) d\mu_{H,K}(D)$$

$$= \sum_{i=1}^{\infty} \int_{\Upsilon} \int_{\frac{\lambda}{\mu} \in \Upsilon} \|f_i(x)\| \|g_i(y)\| d\mu_{H}(x) d\mu_{K}(y),$$

$$= \sum_{i=1}^{\infty} \|f_i\|_1 \|g_i\|_1.$$ 

Now in the case of (a), (b) or (c), we know that $\sum_{i=1}^{\infty} \|f_i\|_1 \|g_i\|_1 \leq \sum_{i=1}^{\infty} \|f_i\|_p \|g_i\|_{q'}$. Hence we have the desired result since $\sum_{i=1}^{\infty} f_i \otimes g_i \in L_p(\pi) \otimes^\gamma L_q(\gamma^*)$.

Now let us consider the case (d). In the remainder of the proof, $\lambda_{h \cap K^\prime}(e, \cdot)$ will be written as $\lambda(\cdot, \cdot)$, for simplicity of notation. Using the identity (13) we see that

$$\frac{\lambda_{h}(x,t)^{\frac{q}{q-1}} \lambda_{K}(y,t)^{\frac{q}{q-1}}}{\lambda(e,t)} = \left( \frac{\lambda_{K}(y,t)}{\lambda(e,t)} \right)^{\frac{q}{q-1}} \left( \frac{\lambda_{h}(x,t)}{\lambda(e,t)} \right)^{\frac{q}{q-1}}.$$ (34)

Let $1/p + 1/q' - 1 = 1/r$. Then $1/p' = 1 - 1/p = 1/q' - 1/r = 1/q'(1 - q'/r)$. Similarly, $1/q = 1/p(1 - p/r)$. Therefore,

$$\frac{\lambda_{h}(x,t)^{\frac{q}{q-1}} \lambda_{K}(y,t)^{\frac{q}{q-1}}}{\lambda(e,t)} = \left( \frac{\lambda_{K}(y,t)}{\lambda(e,t)} \right)^{\frac{q}{q-1}(1 - q'/r)} \left( \frac{\lambda_{h}(x,t)}{\lambda(e,t)} \right)^{\frac{q}{q-1}(1 - q'/r)}.$$ (35)

Hence we have

$$L_i(x,y) = \int_{\frac{\lambda}{\mu} \in \Upsilon} \frac{\lambda_{h}(x,t)^{\frac{q}{q-1}} \lambda_{K}(y,t)^{\frac{q}{q-1}}}{\lambda_{h \cap K^\prime}(e, t)} \|f_i(x)\| \|g_i(y)\| d\mu_{x,y}(t)$$
\[
\alpha \in K
\text{ where the three integrals are over the coset space } G/H
\]
\[
\text{Using Corollary 12.5 of Hewitt and Ross}[26], \text{ the above can be simplified to obtain}
\]
\[
I_i(x, y) \leq \left( \int \|f_i(xt)\|^p \|g_i(yt)\|^q' \, d\mu_{x,y}(t) \right)^\frac{1}{p} \times
\left( \int \frac{\lambda_H(x, t)}{\lambda(e, t)} \|f_i(xt)\|^p \, d\mu_{x,y}(t) \right)^\frac{1}{p-1} \times
\left( \int \frac{\lambda_K(y, t)}{\lambda(e, t)} \|g_i(yt)\|^q' \, d\mu_{x,y}(t) \right)^\frac{1}{q'}.
\]
\[
\quad \text{(37)}
\]
where the three integrals are over the coset space \( G/H \) via \( K^\alpha \). Let us consider \( \int_{\mathbb{R}^+} \) \( \frac{\lambda_H(x, t)}{\lambda(e, t)} \|f_i(xt)\|^p \, d\mu_{x,y}(t) \). By Lemma 2.1, there exists a quasi-invariant measure \( \mu_{x,y} \) on \( H^\alpha/(H^\alpha \cap K^\alpha) \) such that
\[
\int_{\mathbb{R}^+} \frac{\lambda_H(x, t)}{\lambda(e, t)} \|f_i(xt)\|^p \, d\mu_{x,y}(t) = \int_{\mathbb{R}^+} \lambda_{H,\alpha}(t) \|f_i(x\alpha t)\|^p \, d\mu_{x,y}(\alpha) \, d\lambda_{H^\alpha}(t).
\]
For \( \alpha = x^{-1}hx \) with \( h \in H \), we get
\[
\lambda_{H,\alpha}(\alpha, t) \frac{\lambda_H(x, t)}{\lambda(e, t)} = \lambda_{H,\alpha}(e, \alpha) = \lambda_{H,\alpha}(e, \alpha),
\]
\( \lambda_{H,\alpha} \) being a \( \lambda \)-function for \( H^\alpha/(H^\alpha \cap K^\alpha) \) corresponding to the measure \( \mu_{x,y} \). Using the assumption that \( H^\alpha/(H^\alpha \cap K^\alpha) \) is compact and the fact that \( \lambda_{H,\alpha}(e, \alpha) \) is bounded on compact sets (see property (c) on \( \lambda \)-functions, page 2), we have
\[
\int_{\mathbb{R}^+} \lambda_{H,\alpha}(e, \alpha) \, d\mu_{x,y}(\alpha) = M_{x,y}^{\frac{1}{q'}} < \infty.
\]
Thus
\[
\int_{\mathbb{R}^+} \frac{\lambda_H(x, t)}{\lambda(e, t)} \|f_i(xt)\|^p \, d\mu_{x,y}(t) \leq M_{x,y}^{\frac{1}{q'}} \int_{\mathbb{R}^+} \|f_i(xt)\|^p \, d\mu_{H^\alpha}(t)
\]
\[
= M_{x,y}^{\frac{1}{q'}} \int_{\mathbb{R}^+} \|f_i(t)\|^p \, d\mu_{H^\alpha}(t) = M_{x,y}^{\frac{1}{q'}} \|f_i\|_p^p.
\]
\[
\quad \text{(38)}
\]
Similarly, if \( K^\alpha/(H^\alpha \cap K^\alpha) \) is compact,
\[
\int_{\mathbb{R}^+} \frac{\lambda_K(y, t)}{\lambda(e, t)} \|g_i(yt)\|^q' \, d\mu_{x,y}(t) \leq N_{x,y}^{\frac{1}{q'}} \|g_i\|_{q'}^q.
\]
\[
\quad \text{(39)}
\]
The inequalities (39), (40) and (41) imply that
\[
I_i(x, y) \leq \left( \int_{\mathbb{R}^+} \|f_i(xt)\|^p \|g_i(yt)\|^q' \, d\mu_{x,y}(t) \right)^\frac{1}{p} \times
M_{x,y} \|f\|_p^{\frac{1}{p}} N_{x,y} \|g\|_{q'}^{\frac{1}{q'}}.
\]
\[
\quad \text{(40)}
\]
Note that
\[
\left( \int_{D(x,y) \in \mathcal{Y}} \left( \int_{\mathcal{Y}} \sum_{i=1}^{\infty} \frac{\lambda_H(x, t)^{\frac{1}{p}}}{{\lambda_E}^r} \right)^{\frac{1}{p}} \frac{\lambda_K(y, t)^{\frac{1}{p}}}{{\lambda_E}^r} \| f_i(x,t) \| \| g_i(y,t) \| \, d\mu_{x,y}(t) \right)^r \, d\mu_{H,K}(D) \right)_{\frac{1}{p}} = \left( \int_{D(x,y) \in \mathcal{Y}} \left( \sum_{i=1}^{\infty} I_i(x,y) \right)^r \, d\mu_{H,K}(D) \right)^{\frac{1}{p}}.
\]

Using generalised Minkowski’s inequality (see Dunford and Schwartz[13], p.529) we see that
\[
\left( \int_{D(x,y) \in \mathcal{Y}} \left( \sum_{i=1}^{\infty} I_i(x,y) \right)^r \, d\mu_{H,K}(D) \right)^{\frac{1}{p}} \leq \sum_{i=1}^{\infty} \left( \int_{D(x,y) \in \mathcal{Y}} I_i(x,y) \, d\mu_{H,K}(D) \right)^{\frac{1}{p}}
\]

(41)

Let ess sup_{D(x,y)} \{ (M_{x,y}N_{x,y})^r \} = S^r. Then, by (42) and (43) we have
\[
\left( \int_{D(x,y) \in \mathcal{Y}} \left( \sum_{i=1}^{\infty} I_i(x,y) \right)^r \, d\mu_{H,K}(D) \right)^{\frac{1}{p}} \leq \sum_{i=1}^{\infty} \left( \int_{D(x,y) \in \mathcal{Y}} I_i(x,y) \, d\mu_{H,K}(D) \right)^{\frac{1}{p}}
\]

(41)

where (44) is obtained using disintegration of measures (see proof of Lemma 2.2). Since \( p/r + p(q' - 1)/q' = p(1/r + 1 - 1/q') = p(1/p) = 1 \), and similarly \( q'/r + q'(p-1)/p = 1 \), we obtain
\[
\left( \int_{D(x,y) \in \mathcal{Y}} \left( \sum_{i=1}^{\infty} I_i(x,y) \right)^r \, d\mu_{H,K}(D) \right)^{\frac{1}{p}} \leq S \sum_{i=1}^{\infty} \| f_i \|_r \| g_i \|_{q'}.
\]

(44)

This proves the finiteness of the integral (35) for almost all \( D(x,y) \in \mathcal{Y} \) under condition (d) together with \( 1/p + 1/q' > 1 \).

Now let us consider the case (d) together with \( 1/p + 1/q' = 1 \).

Using Hölder’s inequality, we get
\[
\int_{\mathcal{X}} \sum_{i=1}^{\infty} \left( \frac{1}{\lambda(e, t)} \lambda_H(x, t)^{\frac{1}{p}} \lambda_K(y, t)^{\frac{1}{p}} \| f_i(x,t) \| \| g_i(y,t) \| \, d\mu_{x,y}(t) \right)
\]
\[
\leq \sum_{i=1}^{\infty} \left( \int_{\mathcal{X}} \left( \frac{\lambda_H(x, t)^{\frac{1}{p}}}{\lambda(e, t)^{\frac{1}{p}}} \| f_i(x,t) \| \right)^p \, d\mu_{x,y}(t) \right)^{\frac{1}{p}}
\]

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real numbers. The map \( \Psi \) on the homogeneous space \( G/H \) whenever one of the following conditions holds:

Let \( \text{Definition 3.7} \)

\[
\left( \int_{\mathcal{O} \cap K} \left( \frac{\lambda_K(y,t)}{\lambda(e,t)} \parallel g_i(yt) \parallel' \right) d\mu_{x,y}(t) \right)^{1 \over p'} \times \left( \int_{\mathcal{O} \cap K} \left( \frac{\lambda_K(y,t)}{\lambda(e,t)} \parallel f_i(xt) \parallel^p \right) d\mu_{x,y}(t) \right)^{1 \over p},
\]

\[
= \sum_{i=1}^{\infty} \left( \int_{\mathcal{O} \cap K} \left( \frac{\lambda_H(x,t)}{\lambda(e,t)} \parallel f_i(xt) \parallel^p \right) d\mu_{x,y}(t) \right)^{1 \over p} \times \left( \int_{\mathcal{O} \cap K} \left( \frac{\lambda_K(y,t)}{\lambda(e,t)} \parallel g_i(yt) \parallel' \right) d\mu_{x,y}(t) \right)^{1 \over p'}.
\]

By (40) and (41) we have

\[
\int_{\mathcal{O} \cap K} \sum_{i=1}^{\infty} \frac{\lambda_H(x,t)}{\lambda(e,t)} \frac{\lambda_K(y,t)}{\lambda(e,t)} \parallel f_i(xt) \parallel^p \parallel g_i(yt) \parallel' d\mu_{x,y}(t) \leq M_{x,y}N_{x,y} \sum_{i=1}^{\infty} \parallel f_i \parallel_p \parallel g_i \parallel_{p'}.
\]

Again, since \( \sum_{i=1}^{\infty} f_i \otimes g_i \in L_p(\pi) \otimes_G L_{q'}(\gamma^*) \), we see that \( \sum_{i=1}^{\infty} \parallel f \parallel_p \parallel g \parallel_{q'} < \infty \). Hence the result follows.

\[\Box\]

In view of Propositions 4.5 and 4.6, we can formally define the convolution of functions in \( L_p(\pi) \) and \( L_{q'}(\gamma^*) \).

**Definition 3.7** Let \( H \) and \( K \) be regularly related. For each \( x, y \in G \) let \( \mu_{x,y} \) be a quasi-invariant measure on the homogeneous space \( G/(H^x \cap K^y) \) so that the identity (9) holds. Let \( p, q \) be positive real numbers. The map \( \Psi \) on \( L_p(\pi) \otimes_G L_{q'}(\gamma^*) \) is defined by

\[
(\Psi(\Sigma_{i=1}^{\infty} f_i \otimes g_i))(x,y) := \int_{\mathcal{O} \cap K} \sum_{i=1}^{\infty} \frac{1}{\lambda_{H^x \cap K^y}(e,t)} \lambda_H(x,t)^{1 \over p} f_i(xt) \otimes_{x,y} \lambda_K(y,t)^{1 \over p'} g_i(yt) d\mu_{x,y}(t)
\]

whenever one of the following conditions holds:

(a) \( p = 1 \) and \( G/K \) has finite invariant measure (or \( q' = 1 \) and \( G/H \) has finite invariant measure);

(b) \( G/K \) and \( G/H \) both have finite invariant measures;

(c) \( p = q' = 1 \);

(d) \( 1 < p, q' < \infty \) with \( 1/p + 1/q' \geq 1 \), \( H^x/(H^x \cap K^y) \) and \( K^y/(H^x \cap K^y) \) are compact and possess invariant measures for almost all \( x, y \in G \) and the map \( (x,y) \mapsto M_{x,y}N_{x,y} \) is bounded from \( \mathcal{Y} \) to \( \mathcal{R} \).

It is clear that for each \( (x,y) \in G \times G \), the value of \( (\Psi(\Sigma_{i=1}^{\infty} f_i \otimes g_i))(x,y) \) belongs to the quotient space \( \mathcal{A}_{x,y} \). We investigate the properties of the image space of \( \Psi \) in the following section.
3.2 The space $A^g_p$

First we shall show that the image space of $\Psi$ consists of mappings which are constant on the right cosets $(G \times G) / \Delta$ under the conditions (a), (b) and (c) of Definition 4.7.

**Proposition 3.8** Let $\alpha$ be an element of the image space of $\Psi$. For any $h_0 \in H, k_0 \in K, x, y \in G$ and $s \in G / (H^x \cap K^y)$

$$\alpha(h_0xs, k_0ys) = \pi(h_0) \otimes \gamma^*(k_0) \alpha(x, y)$$

under the conditions (a), (b) and (c) of Definition 4.7.

**Proof:** By Proposition 4.4 we have

$$\mathcal{H}_{xs, ys} = \mathcal{H}_{x, y}$$

for all $x, y, s \in G$.

Now any element $\omega \otimes x, y \rho$ of $A_{x, y}$ is of the form

$$\omega \otimes x, y \rho = \mathcal{H}_{x, y} + \omega \otimes \rho$$

$$= \{\pi^\alpha(b) \xi \otimes \eta \in \mathcal{G}, b \in H^x \cap K^y, \xi \in \mathcal{H}(\eta) \}\} + \omega \otimes \rho.$$

If this element is translated by $\pi(h_0) \otimes \gamma^*(k_0)$ from the left, we get

$$\pi(h_0) \otimes \gamma^*(k_0)(\omega \otimes x, y \rho)$$

$$= \{\pi(h_0)\pi^\alpha(b) \xi \otimes \eta \in \mathcal{G}, b \in H_{h_0} \cap K_{k_0} \} + \pi(h_0)\omega \otimes \gamma^*(k_0) \rho.$$

But

$$\pi(h_0)\pi^\alpha(b) \xi = \pi_{h_0}^\alpha(b) \pi(h_0) \xi \text{ and } \gamma^*(k_0)(\gamma^\alpha(b)) \eta = (\gamma_{k_0}^\alpha(b))^* \gamma^*(k_0) \eta,$$

hence

$$\pi(h_0) \otimes \gamma^*(k_0)(\omega \otimes x, y \rho)$$

$$= \{\pi_{h_0}^\alpha(b) \pi(h_0) \xi \otimes \eta \in \mathcal{G}, b \in H_{h_0} \cap K_{k_0} \}$$

$$+ \pi(h_0)\omega \otimes \gamma^*(k_0) \rho,$$

$$= \mathcal{H}_{h_0x, k_0y} + \pi(h_0)\omega \otimes \gamma^*(k_0) \rho,$$

$$= \pi(h_0)\omega \otimes \mathcal{H}_{h_0x, k_0y} \gamma^*(k_0) \rho. \quad(46)$$

Any $\alpha$ in the image space of $\Psi$ can be expressed as $\Psi(\sum_{i=1}^{\infty} f_i \otimes g_i)$. Without loss of generality, we consider an element of the form $\Psi(f \otimes g)$; the argument is then valid for any $\alpha$ by linearity. Consider the homeomorphism $\phi_s : G / (H^x \cap K^y) \to G / (H^x \cap K^y)^\ast$ given by $\phi_s(x) = s^{-1}v$, and use the fact that $\mu_{xs, ys} = \mu_{x, y} \circ \phi_s$ to get

$$(\Psi(f \otimes g))(h_0xs, k_0ys)$$

$$= \int_{h_0xs \in G / (H^x \cap K^y)} \frac{1}{\lambda_{H \cap K^y}(e, t)} \lambda_H(xs, t)^\frac{1}{2} f(xs) \otimes xs, ys \lambda_K(ys, t)^\frac{1}{2} g(ys) d\mu_{xs, ys}(t)$$

$$= \int_{h_0xs \in G / (H^x \cap K^y)} \frac{1}{\lambda_{H \cap K^y}(e, t)} \lambda_H(xs, t) \lambda_K(ys, t)^\frac{1}{2} f(xs) \otimes xs, ys$$

$$\lambda_{H \cap K^y}(e, t)^\frac{1}{2} g(ys) d\mu_{xs, ys}(t),$$

$$= \int_{h_0xs \in G / (H^x \cap K^y)} \frac{1}{\lambda_{H \cap K^y}(e, t)} \lambda_H(xs, t) \lambda_K(ys, t)^\frac{1}{2} f(xs) \otimes xs, ys \lambda_{H \cap K^y}(e, t)^\frac{1}{2} g(ys) d\mu_{xs, ys}(t),$$

$$= \frac{\lambda_{H \cap K^y}(e, t)}{\lambda_H(xs, t) \lambda_K(ys, t)^\frac{1}{2}} \lambda_{H \cap K^y}(e, t)^\frac{1}{2} g(ys) d\mu_{xs, ys}(t),$$

$$= \frac{\lambda_{H \cap K^y}(e, t)}{\lambda_H(xs, t) \lambda_K(ys, t)^\frac{1}{2}} \lambda_{H \cap K^y}(e, t)^\frac{1}{2} g(ys) d\mu_{xs, ys}(t),$$

$$= \frac{\lambda_{H \cap K^y}(e, t)}{\lambda_H(xs, t) \lambda_K(ys, t)^\frac{1}{2}} \lambda_{H \cap K^y}(e, t)^\frac{1}{2} g(ys) d\mu_{xs, ys}(t).$$
Hence we see that

\[ \alpha(h_0xs, k_0ys) = \pi(h_0) \otimes \gamma^*(k_0) \alpha(x, y) \]

only if

\[ \frac{\lambda_{H \cap K}(e, s)}{\lambda_H(x, s)} \lambda_K(y, s) = 1 \]

for all \( s \in G/(H^r \cap K^\gamma) \). It is clear that this last condition is true in the cases (a), (b) and (c) given in Definition 4.7.

\[ \diamond \]

The structure of the image space of \( \Psi \)

The image space of \( \Psi \) is contained in a space of mappings acting on \( G \times G \), whose values at

\( (x, y) \in G \times G \) belong to a collection of Banach spaces

\( \{A_{x, y} : (x, y) \in G \times G\} \). This suggests that the image space has the structure of the space of cross-sections of a Banach bundle or a Banach semi-bundle where the bundle space is a union of quotient spaces of a given Banach space.

Let

\[
B_0 = \mathcal{H}(\pi) \otimes^{\mathcal{H}} \mathcal{H}(\gamma^*) \times G \times G,
\]

\[
B_0^\Delta = \mathcal{H}(\pi) \otimes^{\mathcal{H}} \mathcal{H}(\gamma^*) \times (G \times G)/\Delta,
\]

\[
B_1 = \cup_{(x, y) \in G \times G} \{H_{x, y} \times \{x\}\},
\]

\[
B_1^\Delta = \cup_{(x, y) \in G \times G} \{H_{x, y} \times \{x, y\}\},
\]

\[
B_2 = \cup_{(x, y) \in G \times G} \{A_{x, y} \times \{x, y\}\},
\]

\[
B_2^\Delta = \cup_{(x, y) \in G \times G} \{A_{x, y} \times \{x, y\}\}.
\]

It is clear that \( B_1 \) is a subspace of \( B_0 \), and \( B_1^\Delta \) is a subspace of \( B_0^\Delta \).

For \((x, y) \in G \times G\), let \( r(x, y) \in (G \times G)/\Delta \) be the right coset to which \((x, y)\) belongs. With \( j \) denoting any one of \( \{0, 1, 2\} \), let \( \theta_j : B_j \rightarrow G \times G \) be defined by \( \theta_j((z, (x, y))) = (x, y) \), and let \( \theta_j^\Delta : B_j^\Delta \rightarrow (G \times G)/\Delta \) be defined by \( \theta_j^\Delta((z, (x, y))\Delta) = (x, y)\Delta \), where \( \Delta \) belongs to the corresponding Banach space. Let \( q : B_0 \rightarrow B_2 \) be the quotient map given by \( q(h, x) = ((\mathcal{H}_z + h), x) \). Similarly, the quotient map \( q_\Delta : B_0^\Delta \rightarrow B_2^\Delta \) is given by \( q_\Delta(h, r(x)) = ((\mathcal{H}_z + h), r(x)) \). \( B_0 \) has the product topology, and we topologize \( B_2^\Delta \) so that the map \( p_\Delta \) is continuous and open.

Define \( B_j := (B_j, \theta_j) \) and \( B_j^\Delta := (B_j^\Delta, \theta_j^\Delta) \), for \( j \in \{0, 1, 2\} \).

The space \((G \times G)/\Delta \) is Hausdorff since \( \Delta \) is a closed subgroup of \( G \times G \). It is easy to see that \( B_j \) and \( B_j^\Delta \), for \( j \in \{0, 2\} \), are bundles over \( G \times G \) and \( \overline{G \times G}_\Delta \), respectively. Moreover, \( B_0 \) and \( B_0^\Delta \) are trivial bundles with constant fiber \( \mathcal{H}(\pi) \otimes^{\mathcal{H}} \mathcal{H}(\gamma^*) \). Although \( B_2 \) and \( B_2^\Delta \) fail to be Banach bundles in general, we see that they possess necessary properties to become Banach semi-bundles. For each \( z = \Delta(x, y) \in (G \times G)/\Delta \), the fibre of \( B_2^\Delta \) over \( z \) is \( B_2^\Delta_z = \{A_{x, y} \times \{z\}\} \). We see that \( B_2^\Delta_z \) is a Banach space with the norm \( \|\eta, z\|_{B_2^\Delta_z} \) defined by \( \|\eta, z\|_{B_2^\Delta_z} = \|\eta\| \) where \( \|\eta\| \) means the norm in \( A_{x, y} \). The operations + and . in \( B_2^\Delta_z \) are defined, in an obvious manner, using + and . in \( A_{x, y} \). We can define and topologize the fibres \( B_{2, z} \) in \( B_2 \) and define the operations + and . in a similar manner.

**Lemma 3.9** \((\eta, z) \mapsto \|\eta, z\|_{B_2^\Delta_z} \) is upper semi-continuous on \( B_2^\Delta_z \) to \( R \). A similar result holds in the case of \( B_2 \).
Proof: Let \( \{(\eta_i, z_i) : i \in I\} \) be a net of elements in \( B^{\Delta} \) with \( (\eta_i, z_i) \to (\eta, z) \). Then there exist a sequence \( \{(\varphi_i, u_i)\} \) and an element \((\varphi, u)\) in \( B^{\Delta}_0 \) such that
\[ q_\Delta((\varphi_i, u_i)) = (\eta_i, z_i) \quad \text{for all} \quad i \in I, \]
\[ q_\Delta((\varphi, u)) = (\eta, z) \quad \text{and} \quad (\varphi_i, u_i) \to (\varphi, u). \]
Now since \( \|\eta\| = \inf_{h \in H} \|\varphi + h\| \), without loss of generality we can choose \( \varphi \) such that, for a given \( \epsilon > 0 \), we have
\[ \|\varphi\| < \|\eta\| + \epsilon. \quad (47) \]
Also,
\[ \|\eta_i\| \leq \|\varphi_i\| \quad (48) \]
for all \( i \in I \). Since \( \|\varphi_i\| \to \|\varphi\| \), then from (49) and (50) we have
\[ \|\eta_i\| \leq \|\eta\| + \epsilon, \]
for \( i \) sufficiently large.

The proof is similar in the case of \( B^2 \).

\[ \square \]

**Lemma 3.10** The operation \( + \) is continuous on \( B^{\Delta}_{2,z} \times B^{\Delta}_{2,z} \) to \( B^{\Delta}_{2,z} \), and for each \( \lambda \) in \( C \), the map \( b \mapsto \lambda b \) is continuous on \( B^{\Delta}_{2,z} \) to \( B^{\Delta}_{2,z} \). A similar result holds in \( B^2 \).

Proof: Since the topology induced from \( B^{\Delta}_{2,z} \) on its fibres is just the Banach space topology, the operations \( + \) and \( . \) are continuous. Similarly for \( B^2 \).

\[ \square \]

**Lemma 3.11** If \( z \in (G \times G)/\Delta \) and \( \{b_i : i \in I\} \) is any net of elements in \( B^{\Delta}_{2,z} \) such that \( \|b_i\| \to 0 \) and \( \theta^\Delta_{2,z}(b_i) \to z \), then \( b_i \to 0_z \) where \( 0_z \) is the zero element in \( B^{\Delta}_{2,z} \). A similar result holds in \( B^2 \).

Proof: Any element \( b_i \in B^{\Delta}_{2,z} \) is of the form \( b_i = (\omega_i + H_{x_i, y_i}) \Delta \) where \( x_i, y_i \in G \). Since \( \|b_i\| = \inf\{\|\omega_i + h\| : h \in H_{x_i, y_i}\} \), with \( \omega_i \in H \), there exists an \( h_i \in H_{x_i, y_i} \) such that
\[ \|\omega_i + h_i\| < \|b_i\| + 1/2^i \]
for all \( i \in I \). This implies that the net of elements \( \omega_i + h_i \in H(\pi) \otimes^\pi H(\gamma^\ast) \) has the property that \( \omega_i + h_i \to 0, 0 \) being the zero element in \( H(\pi) \otimes^\pi H(\gamma^\ast) \). If \( \theta^\Delta_{2,z}(b_i) = (x_i, y_i) \Delta \to z \), this means that \( b_i \to 0_z \).

\[ \square \]

**Lemma 3.12** \( B^{\Delta} \) and \( B^2 \) are Banach semi-bundles over \((G \times G)/\Delta\) and \((G \times G)\) respectively.

Proof: The result follows from Lemmas 4.9, 4.10 and 4.11.
Proposition 3.13  The Banach semi-bundle retraction

\[ \mathcal{B}_2^\Delta = (\mathcal{B}_2^\Delta, \theta^\Delta_2) \]

of $\mathcal{B}_2^\Delta$ by $r$ is topologically equivalent to $\mathcal{B}_2$.

Proof: Consider the diagram

\[ \mathcal{B}_0^\Delta \xrightarrow{i^\#} \mathcal{B}_2^\Delta \]

\[ \mathcal{B}_0 \xrightarrow{\theta_0} \mathcal{B}_2 \]

\[ \mathcal{G} \times \mathcal{G} \xrightarrow{r} (\mathcal{G} \times \mathcal{G})/\Delta \]

where $q^\# = j^#(q_\Delta)$ (see Sec.1.2, p.4) and $j$ is defined so that $q^\# \circ i = j \circ q$, $i$ being the homeomorphism stated in Proposition 1.3. It is clear that $q^\#$ is the quotient map. Hence, (also by the discussion on pages 4 and 5,) $q^\#$ is continuous and open. Obviously, $j$ defines a bijection from $\mathcal{B}_2$ onto $\mathcal{B}_2^\Delta$. We need to show that $j$ and its inverse are continuous. Now $q_\Delta$ is open by the definition of the topology of $\mathcal{B}_2^\Delta$ and the right hand side of the above diagram commutes. Since the maps $i$, $q_\Delta^\#$ and $q$ are continuous and open it is clear that $j$ is continuous and open, as required.

⋄

Proposition 3.14  Let $f : \mathcal{G} \times \mathcal{G} \mapsto \mathcal{B}_2$ be a continuous cross-section which is constant on equivalence classes. Then the function $g$ defined by $g(p(x, y)) = i^#(f(x, y))$, where $(x, y) \in \mathcal{G} \times \mathcal{G}$, is a continuous cross-section from $(\mathcal{G} \times \mathcal{G})/\Delta$ to $\mathcal{B}_2^\Delta$.

Proof: By Proposition 4.13, a continuous cross-section $f$ of $\mathcal{B}_2$ can be regarded as a cross-section of $\mathcal{B}_2^\Delta$. Define $g' : (\mathcal{G} \times \mathcal{G}) \mapsto \mathcal{B}_2^\Delta$ so that

\[ g'(x, y) = i^#(f(x, y)) \]

for all $(x, y) \in \mathcal{G} \times \mathcal{G}$.

(See the diagram below.)

\[ \mathcal{B}_2 = \mathcal{B}_2^\Delta \xrightarrow{i^\#} \mathcal{B}_2^\Delta \]

Consider the function $g : (\mathcal{G} \times \mathcal{G})/\Delta \mapsto \mathcal{B}_2^\Delta$ which factors through the diagram

\[ \mathcal{B}_2^\Delta \]

\[ \mathcal{G} \times \mathcal{G} \xrightarrow{r} (\mathcal{G} \times \mathcal{G})/\Delta \]

It is clear that $g$ is well defined since $f$ is constant on the equivalence classes. Also, $g(p(x, y)) = g'(x, y)$ for any $(x, y) \in \mathcal{G} \times \mathcal{G}$ and we see that $g(z) \in \mathcal{B}_2^\Delta$ for any $z \in (\mathcal{G} \times \mathcal{G})/\Delta$. Hence $g$ is a cross-section of $\mathcal{B}_2^\Delta$. Moreover, it is continuous since $p$ is open.
Lemma 3.15 Consider the conditions (a), (b), (c) and (d) of Definition 4.7. For \( \sum_{i=1}^{\infty} f_i \otimes g_i \in L_p(\pi) \otimes L_q(\gamma) \), the element \( \Psi(\sum_{i=1}^{\infty} f_i \otimes g_i) \) is a cross-section of \( \mathcal{B}_{2}^\Delta \), if the integral (47) is constructed under one of the conditions (a), (b) or (c). It is a cross-section of \( \mathcal{B}_{2} \) if it is constructed under the condition (d).

Proof: This is an immediate consequence of Proposition 4.8.

Definition 3.16 The space \( \mathcal{A}_p^\gamma \) is defined to be the range of \( \Psi \) with the quotient norm.

In other words, \( \mathcal{A}_p^\gamma \) is contained in the space of cross-sections of the Banach semi-bundle \( \mathcal{B}_{2}^\Delta \) in the cases (a), (b) and (c) of Definition 4.7. In the case (d), it is contained in the space of cross-sections of the Banach semi-bundle \( \mathcal{B}_{2} \).

By a continuous family of functions we mean a family of functions \( \{\beta_x : x \in G\} \) such that \( (x,t) \mapsto \beta_x(t) \) is a continuous map from \( G \times G \) to \( \mathcal{R} \).

Proposition 3.17 Suppose that the spaces \( G/H, G/K \) and the numbers \( p,q \) satisfy one of the conditions (a), (b), (c) or (d) as described in Definition 4.7. Suppose further that there exists a continuous family \( \{\beta_{x,y} : (x,y) \in G \times G\} \) of functions where \( \beta_{x,y} \) is a Bruhat function for \( H^x \cap K^y \). Let \( f \) and \( g \) be functions with compact support from \( L_p(\pi) \) and \( L_q(\gamma^\ast) \) respectively. Then,

\[
(x,y) \mapsto \int_{x,y \in G} \frac{1}{\lambda_{H \cap K}(e,t)} \lambda_H(x,t)^{\frac{1}{p}} f(xt) \otimes x,y \lambda_K(x,t)^{\frac{1}{q}} g(yt) du(x,y)(t)
\]

(49)

is a continuous cross-section of the corresponding Banach semi-bundle.

Proof: It can be easily seen that for any \( x \in G \) and \( f \in L_p(\pi) \), the function \( xf(t) = f(xt) \) is a function in \( L_p(\pi^\ast) \). Similarly, a function \( g \in L_q(\gamma^\ast) \) gives rise to a function \( yg \) in \( L_q(\gamma^\ast) \).

Now suppose \( f \) and \( g \) are continuous with compact support. Then there exist compact sets \( G_1 \) and \( G_2 \) of \( G \) such that \( H^xG_1 \) and \( K^yG_2 \) are the supports of \( xf \) and \( yg \) respectively.

Suppose that the integral in (51) is constructed under one of the conditions (a), (b), (c) or (d) of Definition 4.7. Consider the map

\[
(x,y,t) \mapsto \beta_{x,y}(t) \lambda_H(x,t)^{\frac{1}{p}} f(xt) \otimes \lambda_K(y,t)^{\frac{1}{q}} g(yt)
\]

from \( (G \times G \times G) \) to \( \mathcal{B}_0 \). This is a cross-section of the Banach bundle retraction of \( \mathcal{B}_{2} \), by \( p : G \times G \times G \rightarrow G \times G \). It is a continuous cross-section since, under the assumptions, \( \{\beta_{x,y} : (x,y) \in G \times G\} \) is a continuous family of Bruhat functions. Therefore, we can form the integral

\[
\tilde{\Gamma}(x,y) := \int_{G} \beta_{x,y}(t) \lambda_H(x,t)^{\frac{1}{p}} f(xt) \otimes \lambda_K(y,t)^{\frac{1}{q}} g(yt) du_G(t)
\]
and by Lemma 1.4, we see that $\tilde{\Gamma}$ is a continuous cross-section of $\mathcal{B}_0$.

Considering the diagram

$$
\begin{array}{ccc}
\mathcal{B}_0 & \xrightarrow{\delta} & \mathcal{B}_2 \\
\tilde{\Gamma} & \uparrow & \nearrow \Gamma \\
G \times G
\end{array}
$$

where $q(\xi, x) = (\{H_x + \xi\}, x)$, we find that

$$
\Gamma(x, y) := \int_G \beta_{x,y}(t) f(xt) \otimes_{x,y} \lambda_H(x, t)^\frac{1}{\delta} g(yt) d\mu_G(t)
$$

is a continuous cross-section of $\mathcal{B}_2$. Note that the property (i) of $\lambda$-functions on page 2 implies that we can assume $\rho_{H_x \cap K_y}(e) = 1$, for all $x, y \in G$. Using Corollary 1.2, we get

$$
\Gamma(x, y) = \int_G \lambda_{H \cap K}(e, st) \beta_{x,y}(s) \lambda_H(x, t)^\frac{1}{\delta} \lambda_K(y, t)^\frac{1}{\delta} g(yt) d\mu_{x,y}(t). \quad (50)
$$

But under the conditions (a), (b), (c) or (d) in Definition 4.7, (34) implies that, for $s \in H^x \cap K^y$,

$$
\frac{\lambda_H(x, s)^\frac{1}{\delta} \lambda_K(y, s)^\frac{1}{\delta}}{\lambda_{H \cap K}(e, s)} = \frac{\lambda_H(x, t)^\frac{1}{\delta} \lambda_K(y, t)^\frac{1}{\delta}}{\lambda_{H \cap K}(e, t)}.
$$

Therefore the integral (52) can be simplified to give

$$
\Gamma(x, y) = \int_G \frac{1}{\lambda_{H \cap K}(e, t)} \lambda_H(x, t)^\frac{1}{\delta} f(xt) \otimes_{x,y} \lambda_K(x, t)^\frac{1}{\delta} g(yt) d\mu_{x,y}(t) \times
$$

$$
\left( \int_{H \cap K} \beta_{x,y}(s) d\mu_{H \cap K}(s) \right) d\mu_{x,y}(t),
$$

$$
= \int_G \frac{1}{\lambda_{H \cap K}(e, t)} \lambda_H(x, t)^\frac{1}{\delta} f(xt) \otimes_{x,y} \lambda_K(x, t)^\frac{1}{\delta} g(yt) d\mu_{x,y}(t).
$$

Hence the mapping given by (51) is continuous in the Banach semi-bundle $\mathcal{B}_2$. In the case of (a), (b) or (c) in Definition 4.7, we can consider the mapping (51) as a cross-section of the Banach semi-bundle retraction $\mathcal{B}_2^\Delta$ of $\mathcal{B}_2^\Delta$ by the canonical mapping $r : G \times G \to (G \times G)/\Delta$. By Proposition 4.14, this cross-section gives rise to the continuous cross-section in $\mathcal{B}_2^\Delta$ given by (51), as required. In the case (d), the mapping given by (51) is continuous in the Banach semi-bundle $\mathcal{B}_2$, as required.

\[\diamond\]

**Proposition 3.18**

1. If $A^q_p$ is constructed under the conditions (a), (b) or (c) of Definition 4.7, then $A^q_p \subseteq L_1(\mathcal{B}_2; \mu_{H,K})$. In particular, if $G/H$ and $G/K$ possess finite invariant measure and $1/p + 1/q' > 1$, then $A^q_p \subseteq L_r(\mathcal{B}_2^\Delta; \mu_{H,K})$ where $1/r = 1/p + 1/q' - 1$. 

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(2) If $A_p^q$ is constructed under the condition (d) and if $1/p + 1/q' > 1$, then
$A_p^q \subseteq L_r(E^\Delta_2, \mu_{H,K})$ where $1/r = 1/p + 1/q' - 1$.

(3) If $A_p^q$ is constructed under the condition (d) and if $1/p + 1/q' = 1$, then
$A_p^q \subseteq L_\infty(E^\Delta_2, \mu_{H \times K})$.

Proof: (1) Consider the space $A_p^q$ under any of the conditions (a) to (c) given in Definition 4.7. According to the calculations in Proposition 4.6, we see that

$\lambda$ finite and invariant (see also the proof of Proposition 4.6). Simplifying, $x, y$ and $K_y$ are finite invariant measures on $y$.

Note that

If $G/H$ and $G/K$ have finite invariant measure $\mu_H$ and $\mu_K$ respectively, we see that $\mu_{H^z}$ and $\mu_{K^w}$ are finite invariant measures on $G/H^z$ and $G/K^w$ for $x, y \in G$. Hence $\lambda_{H^z}(z, t) = 1 = \lambda_{K^w}(w, t)$ for $z \in G/H^z, w \in G/K^w$ and $t \in G$. Using the identity (13), we see that $\lambda_{H \cap K^w} = 1$, for almost all $(x, y) \in (G \times G)/(H \times K)$. Therefore, for $f \in L_p(\pi)$,

$$\int_{\pi^{-1}(H \times K)} \|f(xt)\|^p d\mu_{x,y}(t) = \int_{\pi^{-1}(H \times K)} \frac{\lambda_{H \cap K^w}(\alpha, t)}{\lambda_{H^z}(\alpha, t)} \|f(xt)\|^p d\mu_{H \times \alpha}(\alpha) \mu_{H^z}(t),$$

where $\mu_{H \times \alpha}$ is the measure on the coset space $H^z/(H^z \cap K^w)$ as defined in Lemma 2.1 which is finite and invariant (see also the proof of Proposition 4.6). Simplifying,

$$\int_{\pi^{-1}(H \times K)} \|f(xt)\|^p d\mu_{x,y}(t) = \int_{\pi^{-1}(H \times K)} \|f(xt)\|^p d\mu_{x,y}(\alpha) \mu_{H^z}(t) \leq \int_{\pi^{-1}(H \times K)} \|f(xt)\|^p d\mu_{x,y}(t) = \|f\|^p_p,$$

Similarly,

$$\int_{\pi^{-1}(H \times K)} \|g(\gamma t)\|^q d\mu_{x,y}(t) = \|g\|^q_q,$$

for $g \in L_q$. Therefore, using Corollary 12.5 of Hewitt and Ross [26] we obtain

$$\int_{\pi^{-1}(H \times K)} \|f(xt)\| \|g(\gamma t)\| d\mu_{x,y}(t) \leq \left( \int_{\pi^{-1}(H \times K)} \|f(xt)\|^p d\mu_{x,y}(t) \right)^{\frac{1}{p}} \left( \int_{\pi^{-1}(H \times K)} \|g(\gamma t)\|^q d\mu_{x,y}(t) \right)^{\frac{1}{q}},$$

which is similar to the right hand side of (42) (Proposition 4.6). Note that

$$\left( \int_{D(x,y) \in \Gamma} \||\Psi(\sum_{i=1}^{\infty} f_i \otimes g_i)(x,y)||^r d\mu_{H,K}(x,y) \right)^{\frac{1}{r}}$$

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Using the same notation as in (42), by generalised Minkowski’s inequality (see Dunford and Schwartz[10] p.529) we obtain

\[
\left( \int_{D(x,y) \in Y} \left( \int_{(t)} \sum_{i=1}^{\infty} \| f_i(x,t) \otimes g_i(y) \| d\mu_{x,y}(t) \right)^r d\mu_{H,K}(D) \right)^{\frac{1}{r}}
\]

\[
\leq \left( \int_{D(x,y) \in Y} \left( \sum_{i=1}^{\infty} \int_{(t)} \| f_i(x,t) \|\| g_i(y) \| d\mu_{x,y}(t) \right)^r d\mu_{H,K}(D) \right)^{\frac{1}{r}}
\]

Hence using the same calculations which follow inequality (43), we achieve the required result

\[
\| \Psi(\sum_{i=1}^{\infty} f_i \otimes g_i) \|_\infty \leq \sum_{i=1}^{\infty} \| f_i \|_p \| g_i \|_{p'}.
\]

(2) This is evident from (46) (Proposition 4.6).

(3). Suppose that \( H^x/(H^x \cap K^y) \) and \( K^y/(H^x \cap K^y) \) are compact for almost all \((x, y) \in G \times G\), and \( p = q \). Consider the supremum norm on \( B(G) \). For any \( \sum_{i=1}^{\infty} f_i \otimes g_i \in L_p(\pi) \otimes L_{p'}(\gamma^*) \),

\[
\| \Psi(\sum_{i=1}^{\infty} f_i \otimes g_i) \|_\infty = \text{ess sup}_{(x, y) \in G \times G} \| \sum_{i=1}^{\infty} \int_{(t)} f_i(x,t) \frac{1}{\lambda(x,t)} \lambda_H(x,t) \frac{1}{\lambda(x,t)} f_i(x,t) \otimes_{x,y} \lambda_K(y,t) \frac{1}{\lambda(x,t)} g_i(y) d\mu_{x,y}(t) \|.
\]

Now, following the argument in Proposition 4.6, we see that

\[
\| \Psi(\sum_{i=1}^{\infty} f_i \otimes g_i) \|_\infty \leq \text{ess sup}_{(x, y)} \sum_{i=1}^{\infty} \| f_i \|_p \| g_i \|_{p'} \leq S \sum_{i=1}^{\infty} \| f_i \|_p \| g_i \|_{p'},
\]

where \( S = \text{ess sup}_{(x, y)} M_{x,y}N_{x,y} \) is a constant, as required.

\[\diamond\]

### 4 Induced representations, Integral Intertwining Operators and \( A^q_p \) spaces

#### 4.1 Induced representations and Integral Intertwining Operators

In this section we shall investigate the possibility of generalising Rieffel’s result (see Rieffel[30] Theorem 5.5) on classical \( A^q_p \) spaces which asserts that such a space is the predual of the space of intertwining operators if and only if those operators can be approximated, in the ultraweak*-operator topology, by integral operators. To begin, we shall give the definition of an integral operator from \( L_p(\pi) \) to \( L_q(\gamma) \), and discuss some of its properties.
Definition 4.1 Let $T$ be a bounded linear operator from $L_p(\pi)$ into $L_q(\gamma)$. $T$ is called an integral operator if there exists a $\mu_H \times \mu_K$ measurable function $\Phi$, called the kernel of $T$, from $G/H \times G/K$ to $\mathcal{L}(H(\pi), H(\gamma))$ such that for a given $f \in L_p(\pi)$,

1) the function $x \mapsto \Phi(y, x)f(x)$ is integrable for almost all $y \in G/K$,
2) $y \mapsto \int_H \Phi(y, x)f(x)d\mu_H(x)$ belongs to $L_q(\gamma)$ and
3) $(Tf)(y) = \int_H \Phi(y, x)f(x)d\mu_H(x)$, for almost all $y \in G/K$.

The next result describes the properties of the kernel of an intertwining integral operator. The existence of such operators will be discussed in Proposition 5.3.

Proposition 4.2 Let $\Phi$ be the kernel of a given integral intertwining operator for induced representations $U^\pi_\gamma$ and $U^\gamma_\pi$. Then $\Phi$ satisfies the following properties.

1) For almost all $x \in G/H$, $y \in G/K$ and for all $s \in G$,
   \[ \lambda_H(x, s^{-1})^\frac{1}{|s|} \Phi(y, xs^{-1}) = \lambda_K(y, s)^\frac{1}{|s|} \Phi(y, x). \] (51)
2) For all $h \in H, k \in K$, and for almost all $x \in G/H, y \in G/K$,
   \[ \Phi(ky, hx) = \gamma_h \Phi(y, x). \] (52)
3) Under the conditions given in Definition 4.7, $\Phi(y, x)$ is an intertwining operator of the representations $\pi^\gamma$ and $\gamma^\pi$ of the subgroup $H^\gamma \cap K^\pi$ of $G$ for almost all $x \in G/H$ and $y \in G/K$.

Proof: (1) Suppose that $T$ is an integral operator from $L_p(\pi)$ to $L_q(\gamma)$ with the kernel $\Phi$. Then for $f \in L_p(\pi)$ and $y \in G$,
   \[ (Tf)(y) = \int_H \Phi(y, x)f(x)d\mu_H(x). \]
In addition, if $T \in \text{Hom}_G(L_p(\pi), L_q(\gamma))$ then
   \[ (TU^\pi_\gamma f)(y) = (U^\gamma_\pi Tf)(y) \] for almost all $y \in G/K$ and for $s \in G$.

Now
   \[ (TU^\pi_\gamma f)(y) = \int_H \Phi(y, x)\lambda_H(x, s)^\frac{1}{|s|} f(xs)d\mu_H(x). \]
Changing variables $xs \mapsto x$, we find
   \[ (TU^\pi_\gamma f)(y) = \int_H \Phi(y, xs^{-1})\lambda_H(x, s^{-1})\lambda_H(xs^{-1}, s)^\frac{1}{|s|} f(x)d\mu_H(x). \]
Since $\lambda_H(x, s^{-1})\lambda_H(xs^{-1}, s) = 1$, the above integral simplifies to
   \[ (TU^\pi_\gamma f)(y) = \int_H \Phi(y, xs^{-1})\lambda_H(x, s^{-1})^\frac{1}{|s|} f(x)d\mu_H(x). \] (53)
On the other hand,
   \[ (U^\gamma_\pi Tf)(y) = \lambda_K(y, s)^\frac{1}{|s|} \int_H \Phi(y, x)f(x)d\mu_H(x). \] (54)
Therefore, by (55) and (56), property (1) follows.

(2) For \( k \in K \) and \( y \in G \),
\[
\gamma_k(Tf)(y) = (Tf)(ky) = \int_G \Phi(ky, hx) \pi_h f(x) d\mu_H(x),
\]
for \( h \in H \). On the other hand,
\[
\gamma_k(Tf)(y) = \gamma_k \int_G \Phi(y, x) f(x) d\mu_H(x).
\]
It is clear that property (2) follows from (57) and (58).

(3) We want to show that
\[
\gamma_y b \Phi(y, x) = \Phi(y, x) \pi_x b,
\]
for all \( b \in H^x \cap K^y \) and for almost all \( x \in G/H \) and \( y \in G/K \). For any \( b \in H^x \cap K^y \) we have \( b = y^{-1}k y = x^{-1}h x \) for some \( h \in H \) and \( k \in K \). Using (54),
\[
\gamma_{yby^{-1}} \Phi(y, x) = \Phi(yby^{-1}y, xbx^{-1}x) \pi_{xbx^{-1}},
\]
which implies
\[
\gamma^x_b \Phi(y, x) = \Phi(yb, xb) \pi_b^x = \frac{\lambda_H(xb, b^{-1})}{\lambda_K(y, b)^p} \Phi(y, x) \pi_b^x, \quad \text{by (48)},\]
\[
= \frac{1}{\lambda_H(x, b)^p} \frac{\lambda_K(y, b)}{\lambda_K(e, b)} \Phi(y, x) \pi_b^x, \quad \text{(58)}.
\]
Under conditions (a), (b) or (c) of Def.4.7, (60) simplifies to (59), as required.

Now suppose that the condition given in (d) of Definition 4.7 applies. Consider the right hand side of (60). We see that
\[
\frac{1}{\lambda_H(x, b)^p} \frac{\lambda_K(y, b)}{\lambda_K(e, b)} = \left( \frac{\lambda_{H^x \cap K^y}(e, b)}{\lambda_K(e, b)} \right)^p \left( \frac{\lambda_{H \cap K}(e, b)}{\lambda_H(e, b)} \right)^q, \quad \text{by (9)},
\]
Under the condition that \( H^x/(H^x \cap K^y) \) and \( K^y/(H^x \cap K^y) \) have invariant measure, we have
\[
\frac{\lambda_{H^x \cap K^y}(e, b)}{\lambda_{H^x}(e, b)} = \frac{\lambda_{H \cap K}(e, b)}{\lambda_K(e, b)} = 1
\]
(see (32) in the proof of Proposition 4.5). Therefore,
\[
\gamma^y_b \Phi(y, x) = \Phi(y, x) \pi_b^y, \quad \text{(59)}
\]
for all \( b \in H^x \cap K^y \) and for almost all \( x \in G/H \) and \( y \in G/K \). Hence the result.

Following an argument similar to that of Moore[34], we shall obtain a result for intertwining operators between \( L_1(\pi) \) and \( L_q(\gamma), q > 1 \).
Proposition 4.3 Let $U_q^\pi$ and $U_q^\gamma$ be induced representations of the locally compact group $G$ with the corresponding Banach spaces of functions $L_q(\pi)$ and $L_q(\gamma)$ ($q > 1$), respectively. Then, if the Banach space $\mathcal{H}(\pi)$ is separable, the intertwining operators $T$ for these representations are integral operators with the corresponding kernel $\Phi$ satisfying

$$\text{ess sup}_{x \in \mathcal{H}} \left( \int_{\mathcal{H}} \| \Phi(y, x) \| q d\mu_K(y) \right) \leq \| T \|.$$ 

Proof: The proof is in two parts:

(1). Let $S$ and $R$ be fixed Borel cross-sections of $H$ and $K$ in $G$. Then $G/H \simeq S$, $G/K \simeq R$ and we regard $\mu_H$ and $\mu_K$ as measures on $S$ and $R$. Let $C$ be a continuous linear map of $L_1(S, H(\pi), \mu_H)$ into $L_q(R, H(\gamma), \mu_K)$. Firstly we prove that $C$ is an integral operator. For $u \in H(\pi)$ define $C_u : L_1(S, \mu_H) \to L_q(R, H(\gamma), \mu_K)$ by

$$C_u(g) = C(gu),$$

for $g \in L_1(S, \mu_H)$. $C_u(g)$ is bounded since $\|Cu\| \leq \|C\||u\|$. Then by Dunford and Schwartz\cite{10}, Theorem 10, p.507, there exists a $\mu_H$-essentially unique bounded measurable function $\chi_u(\cdot)$ on $S$ to a weakly compact subset of $L_q(R, H(\gamma), \mu_K)$ such that

$$C_u(g)(t) = \int_S \chi_u(s)g(s)d\mu_H(s)$$

and $\|Cu\| = \text{ess sup} \|\chi_u(s)\|$. Let $K_u(\cdot, s) = \chi_u(s)$ so that $K_u : R \times S \to H(\gamma)$. Then $K_u$ is $\mu_H \times \mu_K$ measurable (see Dunford and Schwartz\cite{10}, Theorem 17 p.198), and we have

$$(C_u(g))(t) = \int_S g(s)K_u(t, s)d\mu(s)$$

with $\text{ess sup}_{s \in S} \left( \int_R \| K_u(t, s) \| q d\mu_K(t) \right) \leq \| C \| \| u \|$. Following the same argument as in Moore\cite{33} we can define a map $K$ on $R \times S$ in to the space of bounded linear operators from $\mathcal{H}(\pi)$ to $\mathcal{H}(\gamma)$ such that $K(t, s) = 0$ for $(t, s)$ in a suitably chosen null set $N$ and $K(t, s)u = K_u(t, s)$ otherwise, for each $u \in H(\pi)$, with $\| K(s, t) \| \leq C$. Then, by (62) and (63), we have

$$C(gu)(t) = \int_S K(t, s)g(s)d\mu(s)$$

and $\text{ess sup}_{s \in S} \left( \int_R \| K(t, s) \| q d\mu_K(t) \right) \leq \| C \| \| u \|$ for any $u \in H(\pi)$, which implies that $\text{ess sup}_{s \in S} \left( \int_R \| K(t, s) \| q d\mu_K(t) \right) \leq \| C \|$. Hence for $g \in L_1(S, H(\pi), \mu_H)$ we have

$$(Cg)(t) = \int_S K(t, s)g(s)d\mu_H(s).$$

(2). Secondly, we prove that the intertwining operators $T$ from $L_1(\pi)$ to $L_q(\gamma)$ are integral operators.

Observing that $G \simeq H \times S$, for a given continuous function $f' \in L_1(S, H(\pi), \mu_H)$ we can define a function $(\Phi_1 f') \in L_1(\pi)$ by

$$(\Phi_1 f')(y) = \pi(h)f'(s),$$

where $y \in G$ with $y = hs$ for $h \in H$ and $s \in S$. Similarly, since $G \simeq K \times R$, for a given continuous function $g \in L_q(\gamma)$, we define the function $(\Phi_g g) \in L_q(R, H(\gamma), \mu_K)$ by

$$(\Phi_g g)(r) = g(r)$$

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for \( r \in R \). Clearly, \( \|f'\|_1 = \|(\Phi_1 f')\|_1 \) and \( \|g\|_q = \|(\Phi_q g)\|_q \).

For a given intertwining operator \( T \) from \( L_1(\pi) \) to \( L_q(\gamma) \) we define an operator \( \tilde{T} \) on the space of continuous functions in \( L_1(S, H(\pi), \mu_H) \) to \( L_q(R, H(\gamma), \mu_K) \) by

\[
\tilde{T} := \Phi_q T \Phi_1.
\]

Since the space of continuous functions in \( L_1(S, H(\pi), \mu_H) \) is dense in \( L_1(S, H(\pi), \mu_H) \), we have the following commutative diagram:

\[
L_1(S, H(\pi), \mu_H) \xrightarrow{T} L_q(R, H(\gamma), \mu_K) \\
\Phi_1 \downarrow \quad \uparrow \Phi_q \\
L_1(\pi) \xrightarrow{\tilde{T}} L_q(\gamma)
\]

with \( \tilde{T}(f') = \Phi_q T \Phi_1(f') \) for \( f' \in L_1(S, H(\pi), \mu_H) \). Clearly, \( \|\tilde{T}\| = \|T\| \). Using the result in part (1), we see that there exists a map \( K \) from \( S \times R \) to the set of bounded linear maps from \( H(\pi) \) to \( H(\gamma) \) such that

\[
\tilde{T}(f')(t) = \int_S K(t, s)f'(s)d\mu_H(s),
\]

for \( f' \in L^1(S, H(\pi), \mu_H) \) and \( t \in R \). Using the Borel isomorphism \( G \simeq K \times R \) any \( y \in G \) can be written as \( y = k(e, y)\ell(e, y) \) where \( k(e, y) \in K \) and \( \ell(e, y) \in R \). Both \( k \) and \( \ell \) are Borel functions on \( R \times G \). Then, for \( f \in L_1(\pi) \)

\[
(Tf)(y) = (\Phi_q^{-1}\tilde{T}\Phi_1^{-1}(f))(y) = \gamma(k(e, y))(\tilde{T}\Phi_1^{-1}(f))(\ell(e, y)),
\]

\[
= \gamma(k(e, y)) \int_S K(\ell(e, y), s)((\Phi^{-1}_1(f))(s))d\mu_H(s).
\]

But since \( (\Phi_1^{-1}f)(s) = f(s) \), we have

\[
(Tf)(y) = \gamma(k(e, y)) \int_S K(\ell(e, y), s)f(s)d\mu_H(s).
\]

Now the Borel isomorphism \( G \simeq H \times S \), allows us to express any \( x \in G \) in the form \( x = h(e, x)m(e, x) \), where \( h \) and \( m \) are Borel functions on \( H \times S \). If we define \( \Phi(y, x) = \gamma(k(e, y))K(\ell(e, y), m(e, x))\pi(h(e, x))^T \), then we have \( \|\Phi(y, x)\| = \|K(\ell(e, y), m(e, x))\| \) and

\[
(Tf)(y) = \int_S \Phi(y, s)f(s)ds = \int_{G/K} \Phi(y, x)f(x)d\mu_K(x), \tag{63}
\]

with \( \text{ess sup}_{x \in G/K} (\int_S \|\Phi(y, x)\|^q d\mu_K(y))^\frac{1}{q} \leq \|T\| \). Therefore \( T \) is an integral operator.

\[\diamondsuit\]

### 4.2 The space \( A^p_g \) as the predual of the space of intertwining operators

We are now in a position to state the main result of this section, which is a generalisation of Rieffel’s result (40 Theorem 5.5) on classical \( A^p_g \) spaces.
Theorem 4.4 Suppose that the space $A_p^q, (q' > 1)$ is constructed under one of the conditions given in Definition 4.7. Then the following statements are equivalent.

(a) $L_p(\pi) \otimes^{\sigma} L_q(\gamma^*) \simeq A_p^q$.
(b) Every element of $\text{Int}_G(U_p^\pi, U_q^\gamma)$ can be approximated in the ultraweak*-operator topology by integral operators.

Proof: (b)⇒(a) Suppose that every element of $\text{Int}_G(U_p^\pi, U_q^\gamma)$ can be approximated in the ultraweak*-operator topology by integral operators. First we show that the kernel of $\Psi$ contains the subspace $L$ of $L_p(\pi) \otimes L_q(\gamma^*)$; that is,

$$\Psi(\sum_{i=1}^{\infty} U^\pi(s) f_i \otimes g_i) = \Psi(\sum_{i=1}^{\infty} f_i \otimes (U^\gamma)^*(s) g_i)$$

for $s \in G$. In the following we write $\lambda(\cdot, \cdot)$ for $\lambda_{H \ast \gamma K \ast s}(\cdot, \cdot)$. Now

$$\Psi(\sum_{i=1}^{\infty} U^\pi(s) f_i \otimes g_i)(x, y)$$

$$= \int_{\pi} \sum_{i=1}^{\infty} \frac{1}{\lambda(e, t)} \lambda_H(x, t) \frac{1}{\lambda_H(x, s) \frac{1}{\lambda_H(y, t)} f_i(xts) \otimes_{x,y} \lambda_K(y, t) \frac{1}{\gamma} g_i(yt)d\mu_{x,y}(t),$$

on changing variables $ts \mapsto t$. Since $\lambda(t, s^{-1})/\lambda(e, t) = 1/\lambda(e, t)$, and

$$\lambda_K(y, ts^{-1}) = \lambda_K(yt, s^{-1})\lambda_K(y, t),$$

$$\Psi(\sum_{i=1}^{\infty} U^\pi(s) f_i \otimes g_i)(x, y)$$

$$= \int_{\pi} \sum_{i=1}^{\infty} \frac{1}{\lambda(e, t)} \lambda_H(x, t) \frac{1}{\lambda_H(x, s) \frac{1}{\lambda_K(y, t) \frac{1}{\gamma} g_i(yts^{-1})d\mu_{x,y}(t)},$$

$$= \int_{\pi} \sum_{i=1}^{\infty} \frac{1}{\lambda(e, t)} \lambda_H(x, t) \frac{1}{\lambda_H(x, s) \frac{1}{\lambda_K(y, t) \frac{1}{\gamma} g_i(yts^{-1})d\mu_{x,y}(t)},$$

$$= \Psi(\sum_{i=1}^{\infty} f_i \otimes (U^\gamma)^*(s) g_i).$$

Now it only requires to prove that the kernel of $\Psi$ is contained in $L$. To achieve this, it suffices to show that any bounded linear functional $F$ on $L_p(\pi) \otimes^{\sigma} L_q(\gamma^*)$ which annihilates $L$ also annihilates the kernel of $\Psi$. Since $F$ annihilates $L$, there exists $T \in \text{Int}_G(U_p^\pi, U_q^\gamma)$ such that

$$\langle r, F \rangle = \sum_{i=1}^{\infty} \langle g_i, Tf_i \rangle,$$

for any $r \in L_p(\pi) \otimes^{\sigma} L_q(\gamma^*)$ with the expansion

$$r = \sum_{i=1}^{\infty} f_i \otimes g_i.$$
Suppose now that \( r \) is in the kernel of \( \Psi \). Then,

\[
\sum_{i=1}^{\infty} \int_{\frac{G}{H} \times \mathbb{R}} \frac{1}{\lambda(e,t)} \lambda_H(x,t) g_i(xt) \otimes_{x,y} \lambda_K(y,t) g_i(yt) \mu_{x,y}(t) = 0. \tag{65}
\]

By (66), it suffices to show that

\[
\sum_{i=1}^{\infty} \langle g_i, T f_i \rangle = 0.
\]

Under the assumption that the operator \( T \) can be approximated by the integral operators \( \{T_j : j \in I\} \) in the ultraweak*-operator topology, we have

\[
\sum_{i=1}^{\infty} \langle g_i, T_j f_i \rangle \rightarrow \sum_{i=1}^{\infty} \langle g_i, T f_i \rangle.
\]

Hence in order to prove \( \sum_{i=1}^{\infty} \langle g_i, T f_i \rangle = 0 \), it is sufficient to prove

\[
\sum_{i=1}^{\infty} \langle g_i, T_j f_i \rangle = 0,
\]

for each \( T_j \). Since \( T_j \) is an integral operator, we have

\[(T_j f_i)(y) = \int_{\frac{G}{H}} \Phi_j(y,x) f_i(x) d\mu_H(x),\]

where \( \Phi_j \) is the kernel of \( T_j \) as described in Definition 5.1. Thus,

\[
\sum_{i=1}^{\infty} \langle g_i, T_j f_i \rangle
= \sum_{i=1}^{\infty} \int_{\frac{G}{H}} \int_{\frac{G}{H}} \int_{\mathbb{R}} \langle g_i(y), \Phi_j(y,x) f_i(x) \rangle d\mu_H(x) d\mu_K(y) d\lambda_H(x,t),
= \sum_{i=1}^{\infty} \int_{\frac{G}{H} \times \mathbb{R} \times \mathbb{R}} \langle g_i(y), \Phi_j(y,x) f_i(x) \rangle d\mu_{x \times K}(x,y),
= \sum_{i=1}^{\infty} \int_{D \in T} \int_{\frac{G}{H} \times \mathbb{R} \times \mathbb{R}} \langle g_i(yt), \Phi_j(yt, xt) f_i(xt) \rangle d\mu_{x,y}(t) d\mu_{(H,K)}(D),
\]

using disintegration of measures as explained in Lemma 2.2. (Also, see the discussion preceding the Lemma). By Proposition 5.2 (1), \( \lambda_H(x,t^{-1}) \Phi_j(y,x) = \lambda_K(y,t) \Phi_j(yt, xt) \) for almost all \( x \in G/H \).

Therefore,

\[
\sum_{i=1}^{\infty} \langle g_i, T_j f_i \rangle
= \sum_{i=1}^{\infty} \int_{D \in T} \int_{\frac{G}{H} \times \mathbb{R} \times \mathbb{R}} \langle g_i(yt), \lambda_H(x,t^{-1}) \Phi_j(yt, xt) \rangle d\mu_{x,y}(t) d\mu_{(H,K)}(D).
\]
From the identity (13), we see that
\[ \frac{\lambda_H(xt, t^{-1})^{\star}}{\lambda_K(y, t)^{\star}} = \frac{1}{\lambda_H(x, t)^{\star}} \lambda_K(y, t)^{\star} = \frac{1}{\lambda(e, t)} \lambda_H(x, t)^{\star} \lambda_K(y, t)^{\star}. \]

Consequently,
\[ \sum_{i=1}^{\infty} \langle g_i, T_j f_i \rangle = \sum_{i=1}^{\infty} \int_{D \in \mathcal{Y}} \int_{(\Lambda \times \Lambda) \cap \Delta} \frac{1}{\lambda(e, t)} \lambda_H(x, t)^{\star} f_i(xt) \langle \lambda_K(y, t)^{\star} g_i(yt), \Phi_j(y, x) \lambda_H(x, t)^{\star} f_i(xt) \rangle \, d\mu_{x, y}(t) \, d\mu_{\Lambda}(D). \quad (66) \]

By Proposition 5.2 (3), \( \Phi_j(y, x) \in \text{Int}_{H \ast \cap K^\prime} (H(\pi^{\ast}), H(\gamma^{\ast})) \) under the conditions given in Definition 4.7. Hence there exists \( \Theta_j(y, x) \in (H(\pi^{\ast}) \otimes_{H \ast \cap K^\prime} H((\gamma^{\ast})^{\ast}))^{\ast} \) such that
\[ \sum_{i=1}^{\infty} \langle \lambda_K(y, t)^{\star} g_i(yt), \Phi_j(y, x) \lambda_H(x, t)^{\star} f_i(xt) \rangle = \sum_{i=1}^{\infty} \langle \lambda_H(x, t)^{\star} f_i(xt) \otimes_{x, y} \lambda_K(y, t)^{\star} g_i(yt), \Theta_j(x, y) \rangle, \]
(see Rieffel [37]). Therefore we have,
\[ \sum_{i=1}^{\infty} \langle g_i, T_j f_i \rangle = \sum_{i=1}^{\infty} \int_{D \in \mathcal{Y}} \int_{(H \times K)^{\ast \ast} \cap \Delta} \frac{1}{\lambda(e, t)} \lambda_H(x, t)^{\star} f_i(xt) \otimes_{x, y} \lambda_K(y, t)^{\star} g_i(yt), \Theta_j(x, y) \rangle \, d\mu_{x, y}(t) \, d\mu_{\Lambda}(D). \quad (67) \]

Hence, by (67),
\[ \sum_{i=1}^{\infty} \langle g_i, T_j f_i \rangle = 0, \]
as required.

(a) \( \Rightarrow \) (b) Now suppose that the kernel of \( \Psi \) is \( L \). We want to show that the integral operators of the form \( T_{\phi, f}(y) = \int_{G/H} \phi(y, x) f(x) \, d\mu_H(x) \) form a dense set in \( \text{Hom}_{\cal C}(L_p(\pi), L_q(\gamma)) \) in the ultraweak* operator topology; or equivalently, the corresponding linear functionals are dense in \( (L_p(\pi) \otimes_{\sigma} L_q(\gamma^{\ast}))^{\ast} \) in the weak* topology. Hence, we only need to show that the annihilator of these functionals, regarded as functionals on \( (L_p(\pi) \otimes_{\sigma} L_q(\gamma^{\ast}))^{\ast} \), is \( L \). But by (69) we see that the annihilator of these linear functional is the kernel of \( \Psi \) which is equal to \( L \) under our assumption. This concludes the proof of the Theorem.

\[ \Box \]

**Corollary 4.5** Suppose that every element of \( \text{Int}_{\cal C}(U_p^{\ast}, U_q^{\ast}) \) can be approximated in the ultraweak* operator topology by integral operators. Then the intertwining number \( \partial(U_p^{\ast}, U_q^{\ast}) \) is equal to the
dimension of the space of all functions $\Phi$ given in Definition 5.1. Moreover, if $H$ and $K$ are discretely related,

$$\partial(U_p^+, U_q^+) = \sum_{\vartheta \in D} d_{\vartheta},$$

where $d_{\vartheta}$ is the dimension of the set of all functions $\Phi$ which vanish outside the double coset $\vartheta$.

Proof: Let $T \in \text{Int}G(U_p^+, U_q^+)$. By (69) we have

$$\sum_{i=1}^{\infty} \langle g_i, Tf_i \rangle = \int_{D \in \Upsilon} \langle \Psi(x, y), \Theta(x, y) \rangle d\mu_{(H, K)}(D) = \langle \Psi, \Theta \rangle.$$

(68)

Now using Proposition 1.5 and Theorem 5.4,

$$(A_p^q)^* \simeq \text{Hom}_G(L_p(\pi), L_q(\gamma)).$$

By (70), the intertwining number $\partial(U_p^+, U_q^+)$ is equal to the dimension of the space of all functions $\Theta$ which, in turn is equal to the dimension of the space of all functions $\Phi$.

If $H$ and $K$ are discretely related, $G$ is a union of a null set and a countable collection of double cosets. By Proposition 5.2 (2), the value of $\Phi$ on $\vartheta$ is uniquely determined by its value $\Phi(x_0, y_0)$ at $(x_0, y_0)$ where $(x_0, y_0) \in \vartheta$.

Hence

$$\partial(U_p^+, U_q^+) = \sum_{\vartheta \in D} d_{\vartheta}.$$

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