TATE CLASSES ON SIEGEL 3-FOLDS

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Abstract. This article proves that Tate classes on Siegel modular 3-folds are spanned by the images of Hilbert modular surfaces at degree 2 and by the images of Shimura curves at degree 4. The proof involves a careful study of the period pairing between degree-4 rapidly decreasing cohomological forms and Hilbert modular surfaces.

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[An earlier version of this paper has circulated as a preprint for some years. Due to a few recent requests of the paper made to the author, the current version is prepared with some minor notations changed.]

Let \( F \) be a number field, \( X \) be a \( n \)-dimensional smooth projective \( F \)-variety and set \( X_\mathbb{Q} = X \otimes_F \mathbb{Q} \). With respect to the \( \ell \)-adic cycle map

\[
\text{cl}_{et} : CH^i(X_\mathbb{Q}) \rightarrow H^{2i}(X_\mathbb{Q}, \mathbb{Z}_\ell(i)),
\]

(0.1)

Tate Conjecture predicts that for any finite extension \( E/F \), the space \( H^{2i}(X_\mathbb{Q}, \mathbb{Q}_\ell(i))^{\text{Gal}(\overline{\mathbb{Q}}/E)} \) is spanned by the images of codimension-\( i \) \( E \)-cycles. An accompanying question is to construct sufficient concrete \( E \)-cycles so that their images span \( H^{2i}(X_\mathbb{Q}, \mathbb{Q}_\ell(i))^{\text{Gal}(\overline{\mathbb{Q}}/E)} \).

When \( X \) is a Shimura variety associated to a reductive \( \mathbb{Q} \)-group \( G \) with the reflex field \( F \), \( X \) is defined over \( F \). Each reductive \( \mathbb{Q} \)-subgroup of \( G \) yields a Shimura subvariety, whose connected components are defined over finite abelian extensions of \( F \). These connected components and their Hecke translates are called special cycles in \( X \). Ramakrishnan raised a general question in [16]: are special cycles enough to generate the \( \text{Gal}(\overline{\mathbb{Q}}/F^{ab}) \)-invariant subspace of the intersection cohomology \( IH^{2i}(X_\mathbb{Q}, \mathbb{Q}_\ell(i)) \)?

The original example motivating Ramakrishnan’s question is Hilbert modular surfaces \( S \), for which the reflex field is \( F = \mathbb{Q} \). Harder, Langlands and Rapport [7] proved the Tate conjecture for \( H^2(S_\mathbb{Q}, \mathbb{Q}_\ell(1)) \). They also discovered that modular curves are enough to generate the \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}^{ab}) \)-invariant subspace of the interior cohomology \( IH^2(S_\mathbb{Q}, \mathbb{Q}_\ell(1)) \).

This paper provides Siegel 3-folds as a second affirmative example in this direction. Let \( \mathbb{A} := \mathbb{A}_\mathbb{Q} \) be the ring of adels over \( \mathbb{Q} \) and \( \mathbb{A}_f \) be the subring of finite adels. For a neat compact open subgroup \( K_f \) of \( \text{GSp}_4(\mathbb{A}_f) \), let \( M_{K_f} \) denote the Siegel 3-fold of level \( K_f \). \( M_{K_f} \) is defined over \( \mathbb{Q} \) and we let \( \tilde{M}_{K_f} \) be a smooth toroidal compactification which is defined over \( \mathbb{Q} \) and whose boundary divisors have normal crossings. Weissauer proved the Tate conjecture for \( H^2(\tilde{M}_{K_f} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathbb{Q}_\ell(1)) \) [21, Thm. 9.4] and showed that \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}^{ab}) \) acts trivially on \( CH^1(M_{K_f} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}) \otimes_{\mathbb{Z}} \mathbb{Q} \).
Based on his work, we prove that Tate classes of degree 2 and 4 are spanned by the images of special cycles and cycles on the boundary.

Let \( SC^i(M_{K_f}) \) denote the subgroup of \( CH^i(M_{K_f} \otimes \overline{\mathbb{Q}}) \) consisting of special cycles of dimension \( i \) (see Sect. 1.4). When \( i = 1 \), special cycles are Hecke translates of Hilbert modular surfaces in \( M_{K_f} \); when \( i = 2 \), special cycles are Hecke translates of Shimura curves in \( M_{K_f} \). For a cycle \( Z \) in \( M_{K_f} \otimes \overline{\mathbb{Q}} \), let \( \overline{Z} \) denote its Zariski closure in \( \overline{M}_{K_f} \otimes \overline{\mathbb{Q}} \).

With respect to the cycle map \( (\overline{\mathbb{Q}}) \), let \( CH^i(X_{\overline{\mathbb{Q}}}) \) denote the kernel and \( \overline{CH^i(X_{\overline{\mathbb{Q}}})} \) be the quotient group \( CH^i(X_{\overline{\mathbb{Q}}})/CH^i_0(X_{\overline{\mathbb{Q}}}) \). Let \( Ta^i(X_{\overline{\mathbb{Q}}}) \) be the union of \( H^{2i}(X_{\overline{\mathbb{Q}}}, Q_f(i))^{Gal(\overline{\mathbb{Q}}/E)} \) for all number fields \( E \supset F \); it is the space of all degree-2i Tate classes on \( X_{\overline{\mathbb{Q}}} \).

Our main theorem is below.

**Theorem 1.** Let \( \{B_i\}_{i=1}^m \) be the boundary divisors of \( \overline{M}_{K_f} \otimes \overline{\mathbb{Q}} \).

1. \( CH^1(M_{K_f} \otimes \overline{\mathbb{Q}}) \otimes \mathbb{Z} = SC^1(M_{K_f}) \otimes \mathbb{Z} \).
2. \( Ta^1(\overline{M}_{K_f}) \) is spanned by the images of \( [Z] \) and \( [B_i] \), with \( [Z] \in SC^1(M_{K_f}) \) and \( 1 \leq i \leq m \).
3. \( Ta^2(\overline{M}_{K_f}) \) is spanned by the images of \( [Z] \), \( [B_i \cdot B_j] \) and \( [Z \cdot B_i] \), with \( [Z] \in SC^2(M_{K_f}) \) and \( 1 \leq i, j \leq m \).
4. \( \overline{CH^2(M_{K_f} \otimes \overline{\mathbb{Q}})} \otimes \mathbb{Z} \) is spanned by the image of \( SC^2(M_{K_f}) \).

**Remark 1.** When the variety \( X \) in \( (0.1) \) is smooth, let \( H^*(X, R) \) denote the singular cohomology of the complex manifold \( X(\mathbb{C}) \) with coefficients in an abelian group \( R \). There is a cycle map

\[
cl : CH^i(X_{\overline{\mathbb{Q}}}) \to H^{2i}(X, \mathbb{Z})
\]

The Artin comparison theorem of etale cohomology and singular cohomology gives a canonical isomorphism \( H^i(X_{\overline{\mathbb{Q}}}, \mathbb{Z}) \cong H^i(X, \mathbb{Z}) \otimes \mathbb{Z} \) that is compatible with \( cl_{\mathbb{Z}} \) and \( cl \), whence \( CH^i(X_{\overline{\mathbb{Q}}}) \) is also the kernel of \( cl \).

**Remark 2.** \( CH^1(M_{K_f} \otimes \overline{\mathbb{Q}}) \otimes \mathbb{Q} \) vanishes because of the well-known fact \( H^1(M_{K_f}, \mathbb{C}) = 0 \) (See Lemma 1). \( CH^2(M_{K_f} \otimes \overline{\mathbb{Q}}) \) is more difficult to study and is related to the intermediate Jacobian. It is a very interesting problem to characterize \( CH^2(M_{K_f} \otimes \overline{\mathbb{Q}}) \cap SC^2(M_{K_f}) \).

In Theorem 1 there is \( (1) \implies (2) \implies (3) \implies (4) \). By Remark 2 the following cycle map with coefficients in \( \overline{\mathbb{Q}} \) is injective,

\[
cl_{M_{K_f}} : CH^i(M_{K_f} \otimes \overline{\mathbb{Q}}) \otimes \mathbb{Z} \to H^{2i}(M_{K_f}, \overline{\mathbb{Q}}).
\]

To prove (1), it suffices that the images of \( CH^1(M_{K_f} \otimes \overline{\mathbb{Q}}) \) and \( SC^1(M_{K_f}) \) span the same subspace in \( H^2(M_{K_f}, \mathbb{Q}) \), or equivalently, in \( H^2(M_{K_f}, \mathbb{C}) \).

We choose to handle all \( K_f \) simultaneously. Write \( M := \lim_{\leftarrow K_f} M_{K_f}, SC^i(M) := \lim_{\leftarrow K_f} SC^i(M_{K_f}), CH^i(M \otimes \overline{\mathbb{Q}}) := \lim_{\rightarrow K_f} CH^i(M_{K_f} \otimes \overline{\mathbb{Q}}) \), and \( H^{2i}(M, \mathbb{C}) := \lim_{\rightarrow K_f} H^{2i}(M_{K_f}, \mathbb{C}) \). The according cycle map

\[
cl_M : CH^i(M \otimes \overline{\mathbb{Q}}) \otimes \mathbb{Z} \to H^{2i}(M, \mathbb{C}).
\]

is injective, \( GSp_4(\mathbb{A}_f) \)-equivariant and factors through \( H^{1,1}(M, \mathbb{C}) \).

**Theorem 2.** There is an isomorphism

\[
cl_M|_{SC^1(M)} : SC^1(M) \otimes \mathbb{C} \to H^{1,1}(M, \mathbb{C}).
\]

Theorem 2 implies \( SC^1(M) \otimes \mathbb{Z} = CH^1(M \otimes \overline{\mathbb{Q}}) \otimes \mathbb{Z} \) and is sufficient to deduce Part (1) of Theorem 1. The main body of this paper is to prove Theorem 2 by using the period pairing between cycles and cohomological differential forms. We sketch the main steps of the proof.

Step 1. \( H^2(M, \mathbb{C}) \) is isomorphic to the \( (\mathfrak{g}, K_{\infty}) \)-cohomology of the discrete spectrum \( L^2_{disc}(GSp_4(\mathfrak{g}) \Gamma_{\mathbb{C}}^{\mathbb{C}}, \mathfrak{g}) \) and hence is completely reducible as an admissible \( GSp_4(\mathbb{A}_f) \)-module. (See Sect. 1.4 and 1.3 for the definition of \( \mathfrak{g} \) and \( K_{\infty} \).) For an irreducible admissible unitary representation \( \pi_f \) of \( GSp_4(\mathbb{A}_f) \), let \( SC^1(\pi_f), H^2(\pi_f) \) and \( H^{1,1}(\pi_f) \) be the \( \pi_f \)-isotypic component of \( SC^1(M) \otimes \mathbb{C}, H^2(M, \mathbb{C}) \) and \( H^{1,1}(M, \mathbb{C}) \) respectively. It
reduces to check that the cycle map \( cl(\pi_f) : SC^1(\pi_f) \rightarrow H^{1,1}(\pi_f) \) on each isotypic component is an isomorphism. We show that when \( H^{1,1}(\pi_f) \) is nonzero, \( H^{1,1}(\pi_f) = H^2(\pi_f) \) is irreducible. So it suffices that \( SC^1(\pi_f) \) is nonzero when \( \pi_f \) occurs in \( H^{1,1}(M, \mathbb{C}) \).

Step 2. The map \( cl \) is defined with respect to the Poincaré duality,

\[
H^2(M, \mathbb{C}) \times H^2(M, \mathbb{C}) \rightarrow \mathbb{C}.
\]  

(0.3)

Here the singular cohomology \( H^*(M, \mathbb{C}) \) and \( H^*_c(M, \mathbb{C}) \) can be identified with the de Rham cohomology and de Rham cohomology with compact support. Also, the pairing in (0.3) respects the action of \( GSp_4(\mathbb{A}_f) \) and hence is a direct sum of the perfect pairings.

\[
H^2(\pi_f) \times H^4(\pi_f) \rightarrow \mathbb{C}.
\]

To show \( SC^1(\pi_f) \neq 0 \), one needs a cycle class \( [Z] \in SC^1(\pi_f) \) and a closed form \( \Omega \in H^2(\pi_f) \) satisfying \( \int_Z \Omega \neq 0 \). However, compactly supported closed forms are hard to construct. As a substitute, rapidly decreasing differential forms are easier to construct and define cohomology groups that are isomorphic to the de Rham cohomology with compact support (see [3]). We propose the following Proposition (See Prop. 2 in Sect. 4) and show that it is sufficient for \( SC^1(\pi_f) \neq 0 \).

**Proposition.** When \( H^{1,1}(\pi_f) \) is nonzero, there exists a special divisor \( Z \) and a \( \mathbb{K}_f \)-finite rapidly decreasing closed form \( \Omega \) on \( M \) such that (i) \( < H^{1,1}(\pi_f), \Omega >= 0 \) for all \( \pi_f \neq \pi_f \), (ii) \( \int_Z \Omega \neq 0 \).

Step 3. To verify the above proposition, we utilize the automorphic description of \( H^{1,1}(M, \mathbb{C}) \) in [20]. When \( H^{1,1}(\pi_f) \neq 0 \), there exists a unique irreducible unitary \( GSp_4(\mathbb{R}) \)-representation \( \pi_\infty \) such that \( \pi = \pi_\infty \times \pi_f \) occurs in the discrete spectrum. All such \( \pi \) are the character twist of three types of basic representations (See Thm. 4). It suffices to treat the basic representations and, briefly speaking, they are

(I) a Siegel-type CAP representation of \( PGSp_4(\mathbb{A}) \cong SO(3,2)(\mathbb{A}) \),

(II) a residue representation of Siegel-type Eisenstein series,

(III) the trivial representation 1.

We prove the Proposition for these three types in Section 5, 6 and 7.

When \( \pi \) is of type I, \( \pi_\infty \) is a cohomological parameter \( \pi^{2+} \) (see Sect. 1.6) and \( \pi \) is the global theta lift of an irreducible cuspidal metaplectic \( SL_2(\mathbb{A}) \)-representation. These two facts select out certain non-split \( SO_4 \subset SO(3,2) \) so that the automorphic periods

\[
P(\varphi) := \int_{SO_4(\mathbb{Q}) \backslash SO_4(\mathbb{A})} \varphi(h) dh, \quad \varphi \in \pi
\]

can be nonzero. We let \( Z \) be the Hilbert modular surface associated to this \( SO_4 \) and consider forms \( \Omega \in H^{2,2}(g, K_\infty, \pi) \). Condition (i) is satisfied automatically. The cohomological periods \( \int_Z \Omega \) are related to the values of \( P \) on a subspace \( v_0 \times \pi_f \subset \pi \), where \( v_0 \) is a special vector in \( \pi_\infty \). We use the Vogan-Zuckerman description of \( \pi^{2+} \) to deduce that when \( P \) is nonzero on \( \pi \), it must be nonzero on \( v_0 \otimes \pi_f \), whence certain cohomological period is nonzero. This subtle phenomenon happens partly because the \( K_\infty \)-type containing \( v_0 \) is minimal in \( \pi^{2+} \).

When \( \pi \) is of type II or III, the candidate for the cycle \( Z \) is a product of modular curves and associated to a subgroup \( SO(2,2) \subset SO(3,2) \). Let \( P \) be the Siegel parabolic subgroup in case II and the Borel subgroup in case III. We specify certain type of closed degree-4 form \( \eta \) on the space \( P(\mathbb{Q}) \backslash GSp_4(\mathbb{A}) / K_\infty \). Let \( L \) denote the left translation on \( GSp_4(\mathbb{R}) \). We follow Harder’s method of Eisenstein cohomology and construct

\[
E(\eta) = \sum_{\gamma \in P(\mathbb{Q}) \backslash GSp_4(\mathbb{Q})} L^*_\gamma \eta.
\]

When \( \eta \) is carefully chosen, the form \( E(\eta) \) would be closed and meet Condition (i) and (ii).
The results of this paper were obtained in 2007 and its writing was completed in 2008 when the author visited University of Wisconsin at Madison.

We later heard that He and Hoffman [8] independently proved a result similar to Theorem 2 in the classical setting using a different method. As a comparison, [8] proves the result by applying the Kudla-Millson theory [11, 12, 13]; we do not rely on the work of Kudla-Millson but take the direct approach of constructing rapidly decreasing cohomological forms and pairing them with concrete cycles. Such an explicit study of period pairing is sufficient to show that Tate classes on Siegel 3-folds are from special cycles; additionally, the explicit construction of representatives of the Eisenstein cohomology could be useful for other purposes.

1. Notations

Let $\mathbb{A}$ be the ring of adeles over $\mathbb{Q}$ and $\mathbb{A}_f$ be the subring of finite adeles. Let $\mathbb{A}^\times$ be the multiplicative group of invertible elements in $\mathbb{A}$ and $\mathbb{A}_f^\times$ denote the subgroup consisting of norm 1 elements. For $a \in \mathbb{Q}^\times$ (resp. $\mathbb{Q}_p^\times$), let $\chi_a = \langle a, \cdot \rangle$ be the associated quadratic character of $\mathbb{Q}^\times \backslash \mathbb{A}^\times$ (resp. $\mathbb{Q}_p^\times$), where $\langle, \rangle$ is the Hilbert symbol. For a character $\psi$ of $\mathbb{Q} \backslash \mathbb{A}$ (resp. $\mathbb{Q}_p^\times$), $\psi_a$ denotes the $a$-twist $\psi_a(x) = \psi(ax)$.

For a reductive group $H$ over $\mathbb{A}$, let $H(\mathbb{R})^+$ denote the identity component of $H(\mathbb{R})$ with respect to the real topology. Set $\mathbb{R}_+ := \{ t \in \mathbb{R} : t > 0 \}$ and embed it into $\mathbb{A}^\times$ by sending $t$ to $(t, 1_{A_f})$. Put $c = Vol(\mathbb{Q}^\times \mathbb{R}^+ \backslash \mathbb{A}^\times) = Vol(\mathbb{Q}^\times \backslash \mathbb{A}_f^\times)$.

1.1. The group $\text{GSp}_4$. Set $J_n = \left( \begin{smallmatrix} -I_n & I_n \\ I_n & -I_n \end{smallmatrix} \right)$ and define $\text{GSp}_4 = \{ g \in \text{GL}_4 : \text{tr}(J_n g) = \nu(g) J_2, \nu(g) \in \text{GL}_1 \}$. One calls $\nu(g)$ the similitude character. Let $Z$ denote the center of $\text{GSp}_4$. Set $\text{Sp}_4 = \text{Ker} \nu$ and $\text{PGSp}_4 = \text{GSp}_4/Z$.

There are three types of parabolic subgroups in $\text{GSp}_4$: the Borel subgroup $B$, the Siegel parabolic subgroup $P$, and the Klingen parabolic subgroup $Q$. We fix a choice of them as below,

$$B = \left\{ \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix} \right\}, \quad P = \left\{ \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix} \right\}, \quad Q = \left\{ \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix} \right\}. $$

We also fix a maximal compact subgroup $K = \prod_v K_p$ of $\text{GSp}_4(\mathbb{A})$, with $K_\infty = K_\mathbb{R} \cdot \left\{ \begin{pmatrix} I_2 & \pm I_2 \\ \pm I_2 & I_2 \end{pmatrix} \right\}$ and $K_p = \text{GSp}_4(\mathbb{Z}_p)$ when $p$ is finite. Here $K_\mathbb{R}$ is specified by its Lie algebra as in Section 1.2.

Write $K_f = \prod_{p < \infty} K_p$.

There is an exceptional isomorphism $\text{GSp}_4 \cong \text{GSpin}(V)$, where $V$ is a 5-dimensional quadratic space over $\mathbb{Q}$ of Witt index 2 and determinant 1. To make the isomorphism explicit, set $w = \left( \begin{smallmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{smallmatrix} \right)$ and

$$V = \{ Y = \begin{pmatrix} X & w^t \\ -w & X \end{pmatrix} | x', x'' \in \mathbb{Q}, X \in M_{2 \times 2}(\mathbb{Q}), \text{Tr}(X) = 0 \},$$

$$q(Y) = \frac{1}{4} \text{Tr}(Y^2).$$

$\text{GSp}_4$ acts on $V$ by $g \circ Y = g Y g^{-1}$ and this action induces an isomorphism $\text{PGSp}_4 \sim \rightarrow \text{SO}(V)$, which is then lifted to an isomorphism $\text{PGSp}_4 \sim \rightarrow \text{GSpin}(V)$.

1.2. The Lie algebra of $\text{GSp}_4(\mathbb{R})$. Put $\mathfrak{g} = \text{Lie}(G(\mathbb{R})) \otimes \mathbb{C}$, $\mathfrak{g}_0 = \text{Lie}(\text{Sp}_4(\mathbb{R})) \otimes \mathbb{C}$, and $\mathfrak{z} = \text{Lie}(Z(\mathbb{R})) \otimes \mathbb{C}$. There is $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{z}$. We describe the structure of $\mathfrak{g}_0$ as a complex semi-simple Lie algebra:

(i) The Cartan involution $\theta : X \rightarrow -X$ leads to a Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k} + \mathfrak{p}$, with $\theta$ acting on $\mathfrak{k}$ and $\mathfrak{p}$ by 1 and $-1$ respectively.

$$\mathfrak{k} = \{ X | \theta(X) = X \} = \left\{ \begin{pmatrix} s_0 & \ell_1 \\ -\ell_1 & s_0 \end{pmatrix} | s_0 \in \mathbb{R}, \ell \in \text{Sym}_{2 \times 2}(\mathbb{C}) \right\},$$

$$\mathfrak{p} = \{ X | \theta(X) = -X \} = \left\{ \begin{pmatrix} s_1 & s_2 \\ s_2 & -s_1 \end{pmatrix} | s_1, s_2 \in \text{Sym}_{2 \times 2}(\mathbb{C}) \right\}. $$
Lemma 1. A linear functional $\gamma : t \to \mathbb{C}$ is the highest weight of a finite-dimensional irreducible complex representation of $\mathfrak{t}$ if and only if it is of the form $\gamma = n_1 \alpha + n_2 \beta$, $n_1 \in \frac{1}{2}\mathbb{Z}_{\geq 0}$, $n_2 \in \frac{1}{2}\mathbb{Z}$.

1.3. Siegel 3-folds. We follow Deligne \cite{Deligne} to define the Siegel 3-folds associated to the $\mathbb{Q}$-group $G := \text{GSp}_4$. Fix a $\mathbb{R}$-group homomorphism

$$h : \text{Res}_{\mathbb{R}}^\mathbb{C} \mathbb{C}^* \to G(\mathbb{R})$$

$$x + iy \to \left( \frac{x I_2}{y I_2} \right).$$

The centralizer of $h$ in $G(\mathbb{R})$ is $K_\infty = Z(\mathbb{R})K_\mathbb{R}$. For a compact open subgroup $K_f$ of $G(\mathbb{A}_f)$, the Siegel 3-fold $M_{K_f}$ of level $K_f$ is a 3-dimensional quasi-projective variety over $\mathbb{Q}$ whose set of complex points are

$$M_{K_f}(\mathbb{C}) = G(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty K_f.$$

The family $\{M_{K_f}\}$ forms an inverse system of $\mathbb{Q}$-varieties and its inverse limit $M := \varprojlim M_{K_f}$ is a $\mathbb{Q}$-scheme (not of finite type).

1.4. Special cycles. We follow \cite{Beilinson} to define special cycles on Siegel 3-folds. Recall the isomorphism $G \cong \text{GSpin}(V)$. For a positive-definite subspace $V_0$ of $V$, regard $G_{V_0} := \text{GSpin}(V_0^\perp)$ as a $\mathbb{Q}$-subgroup of $G$.

(i) Choose an element $g_\infty \in G(\mathbb{R})$ such that

$$L_\infty := g_\infty K_\infty g_\infty^{-1} \cap G_{V_0}(\mathbb{R})$$

contains a maximal connected compact subgroup of $G_{V_0}(\mathbb{R})$. All such choices then form a double coset $G_{V_0}(\mathbb{R})g_\infty K_\infty$.

(ii) For a compact open subgroup $K_f \subset G(\mathbb{A}_f)$ and an element $g_f \in G_{V_0}(\mathbb{A}_f)$, put $L_f = g_f K_f g_f^{-1} \cap G_{V_0}(\mathbb{A}_f)$.

(iii) Let $Z_{V_0,g_f,K_f}$ be the Shimura variety associated to $G_{V_0}$ of level $L_f$. It is quasi-projective and defined over $\mathbb{Q}$, with $Z_{U,g_f,K_f}(\mathbb{C}) = G_{V_0}(\mathbb{Q})\backslash G_{V_0}(\mathbb{A})/L_\infty L_f$ and $\dim Z_{V_0,g_f,K_f} = \dim V_0$. 

There is a natural morphism \( i_{g_f,K_f} : Z_{\mathbb{Q},g_f,K_f} \to M_{K_f} \); over complex points, it is given by \( Z_{\mathbb{Q},g_f,K_f}(\mathbb{C}) \to M_{K_f}(\mathbb{C}), (x_\infty, x_f) \to (x_\infty g_\infty, x_f g_f) \). We write \( Z_{\mathbb{Q},g_f,K_f} \) for the \( \mathbb{Q} \)-cycle \( i_{g_f,K_f}(Z_{\mathbb{Q},g_f,K_f}) \).

**Definition 2.** \( i = 1, 2, 3 \). \( SC^i(M_{K_f}) \) is the subgroup of the Chow group \( CH^i(M_{K_f} \otimes_{\mathbb{Q}} \mathbb{Q}) \) generated by connected components of \( Z_{\mathbb{Q},g_f,K_f} \otimes_{\mathbb{Q}} \mathbb{Q} \), with \( g_f \) running over \( G(\mathbb{A}_f) \) and \( V_0 \) running over positive-definite subspaces of dimension \( i \).

The family \( \{SC^i(M_{K_f})\} \) forms a direct system: for two compact open subgroups \( K_f \subset K'_f \), the projection \( M_{K_f} \to M_{K'_f} \) is flat and induces a pull-back homomorphism \( SC^i(M_{K'_f}) \to SC^i(M_{K_f}) \).

The direct limit
\[
SC^i(M) := \varinjlim K_f SC^i(M_{K_f})
\]
is a \( G(\mathbb{A}) \)-module. For \( g_f \in GSpin(V)_K \), the translation map \( \rho(g_f)_{K_f} : M_{K_f} \to M_{g_f^{-1}K_fg_f} \), \((x_\infty, x_f) \to (x_\infty, x_f g_f)\) induces an isomorphism \( \rho(g_f)_{K_f}^* : SC^i(M_{g_f^{-1}K_fg_f}) \to SC^i(M_{K_f}) \), then \( g_f \) acts on \( SC^i(M) \) by \( \rho(g_f)_{K_f}^* = \varinjlim K_f \rho(g_f)_{K_f}^* \). When \( K_f \) is neat, \( SC^i(M)_{K_f} = SC^i(M_{K_f}) \).

We comment that the group \( G_{V_0} \) has a realization other than \( GSpin \).

(i) \( V_0 = \mathbb{Q}^v \), then \( G_{V_0} \cong GL_2(F) \), with \( F = \mathbb{Q}(\sqrt{q(v)}) \) and
\[
GL_2(F) := \{ g \in GL_2(F) | \det g \in \mathbb{Q}^x \}.
\]
Specifically, when \( q(v) \in \mathbb{Q} \times^2 \), there are \( F = \mathbb{Q} \oplus \mathbb{Q} \) and \( GL_2(F) = \{(g_1, g_2) \in GL_2 \times GL_2 : \det g_1 = \det g_2\} \); the connected components of \( Z_{Q,v,g_f,K_f} \) are essentially the product of two modular curves associated to \( GL_2 \).

(ii) \( \dim V_0 = 2 \). There exists an indefinite quaternion algebra \( D \) over \( \mathbb{Q} \) such that \( G_{V_0} \cong D^x \). Here indefinite means \( D(\mathbb{R}) \cong M_{2 \times 2}(\mathbb{R}) \).

**Remark 3.** Let \( SC^1(M) \) (resp. \( SC^1_{ns}(M) \)) denote the subspace of \( SC^1(M) \) spanned by connected components of \( Z_{Q,v,g_f,K_f} \) with \( q(v) \in \mathbb{Q} \times^2 \) (resp. \( q(v) \in \mathbb{Q}_+ \cup \mathbb{Q}^x \times^2 \)). We call cycles in \( SC^1_{ns}(M) \) split divisors and cycles in \( SC^1_{ns}(M) \) non-split divisors.

1.5. **CAP representations.** Let \( \tau \) be an irreducible cuspidal automorphic representation of \( GL_2(\mathbb{A}) \) and \( V_\tau \subset A_{cusp}(PGL_2) \) be the underlying space of \( \tau \). Let \( \chi \) be a character of \( \mathbb{A}^\times / \mathbb{Q}^\times \) and \( z \in \mathbb{C} \). Let \( II(\tau \boxtimes \chi, z) \) denote the induced representation of \( G(\mathbb{A}) \) consisting of smooth \( K \)-finite functions \( f : G(\mathbb{A}) \to V_\tau \) that satisfy
\[
f \left( \left( \begin{array}{cc} A & u \\ x & A^{-1} \end{array} \right) \right) g = |\det A|_x^{1/2} \chi(x) \tau(A) f(g)
\]
for \( u \in \text{Sym}_2(\mathbb{A}) \), \( A \in GL_2(\mathbb{A}) \), and \( x \in \mathbb{A}^\times \).

**Definition 2.** An irreducible cuspidal automorphic representation \( \pi \) of \( GSp_4(\mathbb{A}) \) is called CAP of Siegel type (\( \tau \boxtimes \chi, z \)) if there exists an irreducible constituent \( II(\tau \boxtimes \chi, z) \) such that \( \pi_p \cong \Pi_p \) for almost all \( p \).

To describe CAP representations of Siegel type, it is necessary to use the theta correspondence between \( PGL_2(\mathbb{A}) \) and the metaplectic group \( \tilde{SL}_2 \). We refer to Section 1.7 for notions of theta correspondence.

Choose a non-trivial character \( \psi \) of \( \mathbb{Q} \backslash \mathbb{A} \). Let \( \mathcal{A}_{00}(\tilde{SL}_2) \) be the space of cuspidal automorphic forms on \( \tilde{SL}_2(\mathbb{A}) \) that are orthogonal to elementary theta series. Waldspurger \cite{18} \cite{19} proved a packet decomposition
\[
\mathcal{A}_{00}(\tilde{SL}_2) = \bigsqcup_{\tau \subset A_{cusp}(PGL_2)} Wd_\psi(\tau),
\]
where \( \tau \) runs over irreducible cuspidal automorphic representation of \( PGL_2(\mathbb{A}) \) and the Waldspurger packet \( Wd_\psi(\tau) \) is the collection of all non-zero global theta lifts \( \Theta_{\tilde{SL}_2 \times PGL_2}(\tau \boxtimes \chi_\alpha, \psi_\alpha) \), \( \alpha \in \mathbb{Q}^\times \). Define the \( \ell_\psi \)-Whittaker functional on \( \mathcal{A}(\tilde{SL}_2) \) by
\[
\ell_\psi(\varphi) := \int_{\mathbb{Q} \backslash \mathbb{A}} \varphi(\left( \begin{array}{cc} 1 & n \\ 0 & 1 \end{array} \right)) \psi(-n) dn.
\]
For $\sigma \in \text{Wd}_\psi(\tau)$ and $a \in \mathbb{Q}^\times$, there is
\[ \Theta_{\text{SL}_2 \times \text{PGL}_2}(\sigma, \psi_a) = \begin{cases} \tau \otimes \chi_a, & \ell_{\psi_a}|_{\sigma} \neq 0, \\ 0, & \ell_{\psi_a}|_{\sigma} = 0. \end{cases} \]

**Theorem 3.** [15] $\pi$ is CAP of Siegel type $(\tau \boxtimes \chi, z)$ if and only if

(i) $z = \pm \frac{1}{2}$ and $\tau$ has a trivial central character,
(ii) $\pi = \Theta_{\text{SL}_2 \times \text{PGL}_2}(\sigma, \psi) \otimes \chi$, where $\sigma$ is an irreducible cuspidal automorphic representation of $\text{SL}_2(\mathbb{A})$ belonging to the Waldspurger packet $\text{Wd}_\psi(\tau)$ and satisfying $\Theta_{\text{SL}_2 \times \text{PGL}_2}(\sigma, \psi) = 0$.

CAP representation of Siegel type all occur with multiplicity one in the discrete spectrum $L^2_{\text{disc}}(G(\mathbb{Q})Z(\mathbb{R})^+\backslash G(\mathbb{A}))$.

### 1.6. Cohomological parameters

A cohomological parameter of $G$ is an irreducible unitary $(\mathfrak{g}, K_{\infty})$-module that has non-trivial $(\mathfrak{g}, K_{\infty})$-cohomology. By the Vogan-Zuckerman theory [17], one can determine all seven cohomological parameters of $G$. Five of them contribute to $H^2$ and, among these five, four contribute to $H^{1,1}$. The four parameters are $\{1, \text{sgn} \circ \nu, \pi^{2+}, \pi^{2-}\}$ and their nonzero $(\mathfrak{g}, K_{\infty})$-cohomology groups are

- $H^{1,1}(\mathfrak{g}, K_{\infty}, \pi^{2,\pm}) = H^{2,2}(\mathfrak{g}, K_{\infty}, \pi^{2,\pm}) = \mathbb{C}$,
- $H^{1,0}(\mathfrak{g}, K_{\infty}, 1) = \mathbb{C} (i = 0, 1, 2, 3)$,
- $H^{1,1}(\mathfrak{g}, K_{\infty}, \text{sgn}) = \mathbb{C} (i = 0, 1, 2, 3)$.

#### 1.6.1. The Langlands parameter of $\pi^{2,\pm}$

Let $\tau_\infty$ be an irreducible tempered unitary representation of $\text{GL}_2(\mathbb{R})$ on the space $V_{\tau_\infty}$. Let $\chi_\infty$ be a character of $\mathbb{R}^\times$ and $z \in \mathbb{C}$. Let $I_{P, \tau_\infty \boxtimes \chi_\infty, z}$ be the smooth induced representation of $G(\mathbb{R})$ consisting of smooth functions $f : G(\mathbb{R}) \to V_{\tau_\infty}$ that satisfy

\[ f_\infty \left( \begin{pmatrix} I_2 & u \\ 0 & I_2 \end{pmatrix} \right) \left( A \right) = \left| \frac{\det A}{x} \right|^{z+\frac{1}{2}} \chi_\infty(x) \tau_\infty(A) f_\infty(g) \]

for $A \in \text{GL}_2(\mathbb{R}), u \in \text{Sym}_2(\mathbb{R})$ and $x \in \mathbb{R}^\times$. Let $I_{P, \tau_\infty \boxtimes \chi_\infty, z}$ be the underlying $(\mathfrak{g}, K_{\infty})$-module of $I_{P, \tau_\infty \boxtimes \chi_\infty, z}$ consisting of $K_{\infty}$-finite functions. When $\text{Re}(z) > 0$, $I_{P, \tau_\infty \boxtimes \chi_\infty, z}$ is irreducible for almost all $z$ and, when it is reducible, has a unique quotient. We set $J_{P, \sigma \boxtimes \chi_\infty}$ to be $I_{P, \sigma \boxtimes \chi_\infty}$ in the former case and to be the unique quotient in the latter case.

The calculation in [17] allows one to identify $\pi^{2,\pm}$ as a Langlands quotient of the above type. Let $\mathcal{D}_{2n}$ $(n \geq 1)$ be the discrete series representation of $\text{GL}_2(\mathbb{R})$ with trivial central character and of weight $2n$. In other words, $\mathcal{D}_{2n}$ is the unique subrepresentation of the GL representation induced from the quasi-character $(a_1, a_2) \rightarrow |a_2|^{n-\frac{1}{2}}$. There is $\{\pi^{2+, \pi^{-}}\} = \{J_{P, \mathcal{D}_{4} \boxtimes \text{sgn}, \frac{1}{2}}, J_{P, \mathcal{D}_{4} \boxtimes \text{sgn}, \frac{3}{2}}\}$ and we set

\[ \pi^{2+} = J_{P, \mathcal{D}_{4} \boxtimes \text{sgn}, \frac{1}{2}}, \quad \pi^{2-} = J_{P, \mathcal{D}_{4} \boxtimes \text{sgn}, \frac{3}{2}}. \]

#### 1.6.2. The structure of $H^2(\mathfrak{g}, K_{\infty}, \pi^{2+})$

We keep the notations as in Section 1.2. For a vector space $a$ over $\mathbb{C}$, let $a^*$ denote its dual $\text{Hom}(a, \mathbb{C})$. For $j \geq 0$, there is a canonical isomorphism $(\wedge^j a)^* \cong \wedge^j a^*$ and we identify $(\wedge^j a)^*$ with $\wedge^j a^*$.

Now we compute the $(\mathfrak{g}, K_{\infty})$-cohomology of $\pi^{2+}$. Set $B_0 = B \cap \text{Sp}_4$, $b_0 = \text{Lie}(B_0(\mathbb{R})) \otimes \mathbb{C}$ and $\mathfrak{t} := \text{Lie}(K_{\infty}) = \mathfrak{t} \oplus \mathfrak{z}$. There is $\mathfrak{g} = b_0 \oplus \mathfrak{t}$, whence $b_0 \cong \mathfrak{g} / \mathfrak{t} \cong \mathfrak{p}$ and $b_0^* \cong (\mathfrak{g} / \mathfrak{t})^* \cong \mathfrak{p}^*$. $K_{\infty}$ and $\mathfrak{t}$ act on $b_0$ and $b_0^*$ in an according way. There is

\[ H^2(\mathfrak{g}, K_{\infty}, \pi^{2+}) = \text{Hom}_{K_{\infty}}(\wedge^2(\mathfrak{g} / \mathfrak{t}), \pi^{2+}) \]

\[ = \text{Hom}_t(\wedge^2 b_0, \pi^{2+}) \]

\[ = ((\wedge^2 b_0)^* \otimes \pi^{2+})^\mathfrak{t} \]

\[ = (\wedge^2 b_0^* \otimes \pi^{2+})^\mathfrak{t} \]

The $\mathfrak{t}$-module structure of $\wedge^2 b_0^*$ can be determined as below.
(i) Chose the following explicit basis of $\mathfrak{b}_0$:

$$a = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 2 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 2 \end{pmatrix}, \quad n_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$n_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad n_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad n_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

The weight space decomposition of $\mathfrak{b}_0$ with respect to the action of $t$ is $\mathfrak{b}_0 = \mathbb{C}e_{-\alpha-\beta} \oplus \mathbb{C}e_{\alpha-\beta} \oplus \mathbb{C}e_{\alpha+\beta} \oplus \mathbb{C}e_{\beta} \oplus \mathbb{C}e_{\alpha+\beta}$, where the subscripts denote the weights and

$$e_{-\alpha-\beta} = \frac{1}{2} h + n_0 i + 2 n_2 - n_3,$$

$$e_{\alpha-\beta} = \frac{1}{2} h - n_0 i - 2 n_2 - n_3,$$

$$e_{\alpha+\beta} = \frac{1}{2} h - n_0 i - 2 n_2 - n_3,$$

$$e_{\beta} = \frac{1}{2} a - n_1 i,$$

$$e_{\alpha+\beta} = \frac{1}{2} a + n_1 i.$$

(ii) $\wedge^2 \mathfrak{b}_0, \wedge^4 \mathfrak{b}_0, \wedge^2 \mathfrak{b}_0^*, \wedge^4 \mathfrak{b}_0^*$ are isomorphic as $\mathfrak{t}$-modules and each is the direct sum of five irreducible $\mathfrak{t}$-submodules with highest weights $0, \alpha - 2 \beta, \alpha, \alpha + 2 \beta, 2 \alpha$ respectively.

(iii) Let $\{ e_{*\alpha-\beta}, e_{*\alpha-\beta}, e_{*\alpha-\beta}, e_{*\alpha-\beta}, e_{*\alpha-\beta} \}$ be the basis of $\mathfrak{b}_0^*$ dual to the basis $\{ e_{-\alpha-\beta}, e_{-\alpha-\beta}, e_{-\alpha-\beta}, e_{-\alpha-\beta}, e_{-\alpha-\beta} \}$ of $\mathfrak{b}_0$. The irreducible $\mathfrak{t}$-submodule of $\wedge^2 \mathfrak{b}_0^*$ with highest weight $2\alpha$ is of dimension 5 and spanned by

$$\eta_2 = e_{*\alpha-\beta} \wedge e_{*\alpha-\beta},$$

$$\eta_1 = e_{*\alpha-\beta} \wedge e_{*\alpha-\beta} + e_{*\alpha+\beta} \wedge e_{*\alpha+\beta},$$

$$\eta_0 = e_{*\alpha-\beta} \wedge e_{*\alpha+\beta} + 2 e_{*\alpha+\beta} \wedge e_{*\alpha+\beta} + e_{*\alpha-\beta} \wedge e_{*\alpha+\beta},$$

$$\eta_{-1} = e_{*\alpha-\beta} \wedge e_{*\alpha+\beta} + e_{*\alpha+\beta} \wedge e_{*\alpha+\beta},$$

$$\eta_{-2} = e_{*\alpha-\beta} \wedge e_{*\alpha+\beta}.$$

The vector $\eta_j (j = 2, \ldots, 4)$ is of weight $j\alpha$.

Lemma 2. (i) The irreducible $\mathfrak{t}$-submodules of $\pi^{2+}$ are of highest weights $m_1 \alpha + m_2 \beta$, with $m_1 \in \mathbb{Z}_{\geq 2}, m_2 \in \mathbb{Z}$ and $m_1 \equiv m_2 \mod 2$. Specifically, the irreducible $\mathfrak{t}$-submodule of $\pi^{2+}$ with highest weight $2\alpha$ occurs with multiplicity one in $\pi^{2+}$.

(ii) Let $\pi^{2+}(\delta \alpha)$ be the irreducible $\mathfrak{t}$-submodule of $\pi^{2+}$ with highest weight $2\alpha$. There is a basis $\{ v_j \mid -2 \leq j \leq 2 \}$ of $\pi^{2+}(\delta \alpha)$ such that $v_j$ is of weight $j\alpha$ and $H^2(\mathfrak{g}, K, \pi^{2+}) = \bigoplus_{j=-2}^{2} v_j \otimes \eta_j$.

Proof. Part (i) is a direct application of the Vogan-Zuckerman theory to $G$. For (ii), recall that $H^2(\mathfrak{g}, K, \pi^{2+}) = \text{Hom}_t(\wedge^2 \mathfrak{b}_0, \pi^{2+})$: by (i) and the structure of $\wedge^2 \mathfrak{b}_0$, this space is 1-dimensional and an $\mathfrak{t}$-invariant homomorphism from $\wedge^2 \mathfrak{b}_0$ to $\pi^{2+}$ can only happen between their $\mathfrak{t}$-submodules with highest weight $2\alpha$. Choose a basis $\{ v_j \mid -2 \leq j \leq 2 \}$ of $\delta$ such that $v_j$ is of weight $j\alpha$, then such a homomorphism is of the form $\sum_{j=-2}^{2} c_j v_j \otimes \eta_j$, where $c_j$ are nonzero numbers. Replacing $v_j$ by $c_{-j} v_j$, one gets the desired basis of $\delta$. \( \square \)

1.7. Theta correspondence. We prepare here the relevant notions of theta lifting for the discussion of CAP representations in Section 1.3.

1.7.1. Local theta correspondence. Let $k$ be a local field of characteristic zero and $\psi$ a non-trivial character of $k$. For a reductive group $G$ over $k$, let $\text{Irr}(G)$ denote the set of irreducible admissible representations of $G$ when $k$ is non-archimedean and the set of irreducible admissible $(\text{Lie}(G) \otimes_{\mathbb{R}} \mathbb{C}, K)$-modules when $k$ is archimedean. $(K$ denotes a maximal compact subgroup of $G$ in case of $k$ being archimedean.)

Let $\tilde{\text{SL}}_2(k)$ be the 2-fold metaplectic cover of $\text{SL}_2(k)$; it is $\text{SL}_2(k) \times \mathbb{Z}_2$ as a set and the upper triangular unipotent group in $\text{SL}_2(k)$ has a lift to $\tilde{\text{SL}}_2(k)$. Let $(U, q)$ be a quadratic space
over $k$ of odd dimension and $S(U)$ be the space of Bruhat-Schwartz functions on $U$. The Weil representation $\omega := \omega_{\psi,U}$ of $\widetilde{SL}_2(k) \times O(U)$ on $S(U)$ is given by:

$$\omega(h)\phi(X) = \phi(h^{-1}X), \quad h \in O(U),$$

$$\omega((1, \eta), e)\phi(X) = e\psi(nq(X))\phi(X), \quad n \in k, \quad e \in \mathbb{Z}_2,$$

$$\omega((\sigma, a^{-1}), e)\phi(X) = e\chi_{\phi,U}(a)|a|^\frac{\dim U}{2}\phi(aX), \quad a \in k^\times,$$

$$\omega((-1, 1), e)\phi(X) = e\gamma(\psi,U)\int_U \phi(Y)\phi(q(X,Y))dY.$$

Here $q(X,Y) = q(X+Y) - q(X) - q(Y)$ is the associated quadratic pairing, $\gamma(\psi,U)$ is a constant of norm 1, and $\chi_{\phi,U} : k^\times \to S^1$ is a function satisfying $\chi_{\phi,U}(a_1a_2) = \chi_{\phi,U}(a_1)\chi_{\phi,U}(a_2) < a_1, a_2 >$.

Let $\pi \in \Irr(O(U))$ and $\sigma \in \Irr(\widetilde{SL}_2)$ satisfying $\sigma((I_2, e)) = e$. The maximal $\pi$-isotypic quotient of $\omega$ is of the form $\pi \boxtimes \Theta_0(\pi)$ and the maximal $\sigma$-isotypic quotient of $\omega$ is of the form $\theta_0(\pi) \boxtimes \sigma$, where $\theta_0(\pi)$ and $\Theta_0(\pi)$ are finite-length admissible representations of $\widetilde{SL}_2(k)$ and $O(U)$ respectively. Let $\theta(\pi)$ and $\sigma(\pi)$ be the maximal semi-simple quotient of $\theta_0(\pi)$ and $\Theta_0(\pi)$ respectively. The Howe Duality Conjecture asserts that $\theta(\pi)$ and $\sigma(\pi)$ are irreducible. The Howe Duality Conjecture has been proved for general reductive dual pairs when $k$ is not a dyadic field; when $k$ is dyadic, it is also known for the current pair $(\widetilde{SL}_2, O(U))$ when $\dim U = 1, 3$ and 5.

When $\theta(\pi) = \pi$ and $\sigma(\pi) = \pi$, we say that $\pi$ and $\sigma$ are in local theta correspondence with respect to $\psi$ and call $\sigma$ (resp. $\pi$) the local theta lift of $\pi$ (resp. $\sigma$). We use the notations $\theta(\pi, \psi)$ and $\sigma(\pi, \psi)$ when the role of $\psi$ needs to be emphasized.

Because $\dim U$ is assumed to be odd, each $\pi \in \Irr(SO(U))$ has two extensions to $O(U)$. When $\dim U \geq 3$, at most one of the extensions occurs in the local theta correspondence with $\widetilde{SL}_2(k)$ and one actually considers local theta correspondence between $\Irr(SO(U))$ and $\Irr(\widetilde{SL}_2)$.

1.7.2. Global theta lifting. Let $F$ be a number field, $\mathbb{A}_F$ be the ring of adeles over $F$, and $\psi$ be a non-trivial character of $\mathbb{A}_F/F$. For a semisimple group $G$ over $F$, $[G]$ denotes the quotient space $G(F) \backslash G(\mathbb{A}_F)$.

Let $\widetilde{SL}_2(\mathbb{A}_F)$ be the two-fold metaplectic cover of $SL_2(\mathbb{A}_F)$. Let $(U, q)$ be a quadratic space over $F$ of odd dimension and $S(U(\mathbb{A}_F))$ be the space of Bruhat-Schwartz functions on $U(\mathbb{A}_F)$. Let $\omega := \omega_{\psi,U} = \otimes_v \omega_{\psi_v,U(F_v)}$ be the global Weil representation of $\widetilde{SL}_2(\mathbb{A}_F) \times O(U)_{\mathbb{A}_F}$ on $S(U(\mathbb{A}_F))$ with respect to $\psi$. For $\phi \in S(U(\mathbb{A}_F))$, the associated theta kernel function

$$\theta_\phi(h, g) = \sum_{\xi \in U(F)} \omega(h, g)\phi(\xi)$$

is a slowly increasing function on $(O(V)_{\mathbb{F}_q}) \times (\text{Sp}_{n}(F) \backslash \mathbb{A}_F)$. Specifically, when $\dim U = 1$, the functions $\theta_\phi(g)$ are called elementary theta series on $\widetilde{SL}_2(\mathbb{A})$.

Suppose that $\dim U \geq 3$. Let $\sigma$ be an irreducible cuspidal representation of $\widetilde{SL}_2(\mathbb{A}_F)$ and $\pi$ be an irreducible cuspidal representation of $SO(U)_{\mathbb{A}_F}$. We define the global theta lift of $\sigma$ to $SO(U)_{\mathbb{A}_F}$ and the global theta lift of $\pi$ to $\widetilde{SL}_2(\mathbb{A}_F)$ as

$$\Theta(\sigma, \psi) := \{\Theta(\phi, \varphi) : \phi \in S(U(\mathbb{A}_F)), \varphi \in \sigma\},$$

$$\Theta(\pi, \psi) := \{\Theta(\phi, f) : \phi \in S(U(\mathbb{A}_F)), f \in \pi\},$$

where

$$\Theta(\phi, \varphi) := \int_{[SL_2]} \overline{\varphi(g)}\Theta_\phi(h, g), \quad \Theta(\phi, f) := \int_{[SO(U)]} \overline{f(h)}\Theta_\phi(h, g).$$

1.7.3. Theta correspondence between $\widetilde{SL}_2$, $PGL_2$ and $PGSp_4$. There is an isomorphism $PGL_2 \cong SO(V')$, where $V' = \{X \in M_{2 \times 2}(\mathbb{Q}) : \text{Tr}(X) = 0\}, \quad q'(X) = -\det X,$
Recall that $\text{PGSp}_4 \cong \text{SO}(V)$. We refer the theta correspondence between $\widetilde{\text{SL}}_2$ and $\text{PGL}_2$ (resp. $\text{PGSp}_4$) to the theta correspondence between $\text{SL}_2$ and $\text{SO}(V)$ (resp. $\text{SO}(V)$).

Choose a non-trivial character $\psi_\infty$ of $\mathbb{R}$. The following lemma gives a description of $\pi^{2+}$ in terms of local theta correspondence.

**Lemma 3.** Set $\sigma_\infty = \theta_{\widetilde{\text{SL}}_2 \times \text{PGL}_2} (\mathfrak{D}_{2n}, \psi_\infty)$, then $\theta_{\widetilde{\text{SL}}_2 \times \text{PGSp}_4} (\sigma_\infty, \psi_\infty) = J_{P, 2n} \otimes 1, \frac{1}{2}$ and $\theta_{\widetilde{\text{SL}}_2 \times \text{PGSp}_4} (\sigma_\infty, \psi_\infty^{-1})$ is a discrete series.

**Proof.** This lemma is a special case of Proposition 5.5 in [5].

### 2. Cycle maps and Proof of Theorem

2.1. **The maps** $\text{cl}_{et}$ and $\text{cl}$. Let $X$ be an algebraic variety over a number field $F$. For $0 \leq i \leq 2 \dim X$, let $Z^i(X_{\overline{Q}})$ be the free abelian group generated by codimension-$i$ irreducible closed subvarieties of $X_{\overline{Q}}$ and $\text{CH}^i(X_{\overline{Q}})$ be the quotient of $Z^i(X_{\overline{Q}})$ modulo the relation of rational equivalence. For a cycle $Z \in Z^i(X_{\overline{Q}})$, $[Z]$ refers to its class in $\text{CH}^i(X_{\overline{Q}})$.

Let $H^*(X_{\overline{Q}}, \mathbb{Z})$ and $H^*_c(X_{\overline{Q}}, \mathbb{Z})$ be the $\ell$-adic cohomology and $\ell$-adic cohomology with compact support. For a $Z_\ell$-module $R$, set $H^*_c(X_{\overline{Q}}, Z_\ell) = H^*_c(X_{\overline{Q}}, \mathbb{Z}) \otimes R$. The image of $H^*_c$ in $H^*$ is denoted by $H^*_c$ and called the interior cohomology. Let $\mu_n$ be the multiplicative group of the $n$-th roots of unity in $\overline{Q}$ and set $Z_\ell(1) := \varprojlim_n \mu_n$; $Z_\ell(1)$ is isomorphic to $Z_\ell$ as a $Z_\ell$-module but $\text{Gal}(\overline{Q}/Q)$ acts on it by the cyclotomic character. Let $Z_\ell(i)$ be the $i$-fold tensor of $Z_\ell$ and set $Q_\ell(i) = Z_\ell(i) \otimes Q_\ell$. There is a $\text{Gal}(\overline{Q}/F)$-equivariant $\ell$-adic cycle map

$$\text{cl}_{et} : \text{CH}^i(X_{\overline{Q}}) \to H^{2i}(X_{\overline{Q}}, Z_\ell(i)).$$

For a finite field extension $E \supset F$ contained in $\overline{Q}$, let $H^i(X_E)$ (resp. $H^*_c(X_E)$) be the subgroup of $\text{CH}^i(X_{\overline{Q}})$ generated by irreducible closed subvarieties defined over $E$. There is $\text{CH}^i(X_E) = \text{CH}^i(X_{\overline{Q}})^{\text{Gal}(\overline{Q}/E)}$. Elements in $\text{Ta}_{2i}^E(X_{\overline{Q}}) := H^{2i}(X_{\overline{Q}}, Q_\ell(i))^{\text{Gal}(\overline{Q}/E)}$ are called degree-$2i$ Tate classes over $E$. The union $\text{Ta}^{2i}(X_{\overline{Q}}) = \cup_E \text{Ta}^{2i}_E(X_{\overline{Q}})$ for all finite extensions $E \supset F$ is the space of all degree-$2i$ Tate classes. Tate’s conjecture asserts that $\text{Ta}^E(X_{\overline{Q}})$ is generated by the image of $\text{CH}^i(X_E)$.

When $X$ is smooth, for an abelian group $R$, let $H^*(X, R)$ (resp. $H^*_c(X, R)$) denote the singular cohomology (resp. the singular cohomology with compact support) of $X(\mathbb{C})$ with respect to its complex topology. When $R$ is $\mathbb{R}$ and $\mathbb{C}$, $H^*(X, R)$ (resp. $H^*_c(X, R)$) is isomorphic to the de Rham cohomology (resp. de Rham cohomology with compact support) defined with the cochain of differential forms (resp. differential forms with compact support). There is a cycle map

$$\text{cl} : \text{CH}^i(X_{\overline{Q}}) \to H^{2i}(X, \mathbb{Z}).$$

defined with regard to the Poincaré duality between $H^*_c$ and $H^*$.

**Remark 4.** When $X$ is smooth, by [1], Exposé XVI], there are canonical isomorphisms between the etale cohomology with constant sheaf $\mathbb{Z}/n\mathbb{Z}$ and the singular cohomology with coefficients in $\mathbb{Z}/n\mathbb{Z}$.

$$H^i(X_{\overline{Q}}, \mathbb{Z}/n\mathbb{Z}) \cong H^i(X, \mathbb{Z}/n\mathbb{Z}), \quad H^*_c(X_{\overline{Q}}, \mathbb{Z}/n\mathbb{Z}) \cong H^j(X, \mathbb{Z}/n\mathbb{Z}).$$

The comparison map is functorial and respect Poincaré duality, whence it is compatible with the cycle map $\text{cl}_{et}$ into $H^{2i}(X_{\overline{Q}}, \mathbb{Z}/n\mathbb{Z}(i))$ and the cycle map $\text{cl}$ into $H^{2i}(X, \mathbb{Z}/n\mathbb{Z})$. By passing to the inverse limit, one gets that $\text{cl}_{et} : \text{CH}^i(X_{\overline{Q}}) \to H^{2i}(X_{\overline{Q}}, Z_\ell(i))$ and $\text{cl} : \text{CH}^i(X_{\overline{Q}}) \to H^{2i}(X, \mathbb{Z})$ are compatible with the canonical isomorphism

$$H^{2i}(X_{\overline{Q}}, Z_\ell(i)) \cong H^{2i}(X, \mathbb{Z}) = H^{2i}(X, \mathbb{Z}) \otimes_{\mathbb{Z}} Z_\ell$$
2.2. The cycle map on $M_K$ and $M$. Let $K_f$ be a neat compact open subgroup of $G(\mathbb{A}_f)$. Let $\tilde{M}_K$ be a smooth projective toroidal compactification of $M_K$, which is defined over $\mathbb{Q}$ and whose boundary is the union of divisors $B_i$ ($1 \leq i \leq m$) with normal crossings. Consider the cycle maps on $M_K$ and $\tilde{M}_K$, respectively,

$$\text{cl}_{M_K} : CH^i(M_K \otimes \mathbb{Q}) \to H^{2i}(M_K, \mathbb{Z}),$$

$$\text{cl}_{\tilde{M}_K} : CH^i(\tilde{M}_K \otimes \mathbb{Q}) \to H^{2i}(\tilde{M}_K, \mathbb{Z}).$$

For a codimension-$i$ irreducible closed subvariety of $M_K \otimes \bar{\mathbb{Q}}$, its Zariski closure $\bar{Z}$ in $\tilde{M}_K \otimes \bar{\mathbb{Q}}$ is also irreducible. Extend the map $Z \to \bar{Z}$ to a homomorphism $\bar{Z}(M_K) \to \bar{Z}(\tilde{M}_K)$ and let $j : H^*(\tilde{M}_K, \mathbb{Z}) \to H^*(M_K, \mathbb{Z})$ denote the restriction map, then

$$\text{cl}_{M_K}([Z]) = j \circ \text{cl}_{\tilde{M}_K}(\bar{Z}).$$

There is a short exact sequence (see [21]),

$$0 \to \bigoplus_{i=1}^m C[B_i] \xrightarrow{\text{cl}_{\tilde{M}_K}} H^2(\tilde{M}_K, \mathbb{C}) \xrightarrow{j} H^2(M_K, \mathbb{C}) \to 0,$$

(2.2)

Also, it is well-known that $H^1(M_K, \mathbb{C}) = 0$.

**Lemma 4.**

(i) $CH^0_1(M_K \otimes \mathbb{Q}) \otimes \mathbb{Z} \otimes \mathbb{Q} = 0$

(ii) Suppose $Z_1, Z_2 \in \bar{Z}(1)(M_K \otimes \mathbb{Q})$. If $\sum Z_1 = [Z_2]$ in $CH^1(M_K \otimes \mathbb{Q}) \otimes \mathbb{Q}$, then

$$\sum Z_1 \equiv [Z_2] + \bigoplus_i Q \cdot \text{cl}_{\tilde{M}_K}([B_i]).$$

*Proof.* Because $H^1(M_K, \mathbb{C}) = 0$, there is $H^1(\tilde{M}_K, \mathbb{C}) = 0$. So the Picard variety is trivial and therefore $CH^0_1(\tilde{M}_K \otimes \mathbb{Q}) = 0$.

We first verify (i). Suppose $[Z] \in CH^0_1(M_K \otimes \mathbb{Q})$, then $j \circ \text{cl}_{\tilde{M}_K}(\bar{Z}) = 0$ by (2.1). By [22], $\text{cl}_{\tilde{M}_K}(\bar{Z})$, considered as an element in $H^2(\tilde{M}_K, \mathbb{Q})$, belongs to $\bigoplus_i \bigoplus \text{cl}_{\tilde{M}_K}([B_i])$. Hence there exists a nonzero integer $n$ and integers $n_i$ such that $n \cdot \text{cl}_{\tilde{M}_K}([Z]) = \sum n_i \cdot \text{cl}_{\tilde{M}_K}([B_i])$ is equal to zero in $H^2(\tilde{M}_K, \mathbb{Z})$. Because $CH^0_1(\tilde{M}_K \otimes \mathbb{Q})$ vanishes, the cycle $n\bar{Z} - \sum n_i B_i$ is rationally equivalent to zero and of the form $\text{Div}(F)$ for certain rational function $F$ on $\tilde{M}_K \otimes \mathbb{Q}$. Therefore $\text{Div}(F)|_{M_K \otimes \mathbb{Q}} = n\bar{Z}$ and $[Z]$ is equal to 0 in $CH^0_1(\tilde{M}_K \otimes \mathbb{Q}) \otimes \mathbb{Q}$.

Now consider (ii). By the hypothesis, $[Z_1 - Z_2]$ is a torsion element in $CH^1(M_K \otimes \mathbb{Q})$, whence there exists $n \in \mathbb{N}$ so that $nZ_1 - nZ_2$ is rationally equivalent to zero. Write $nZ_1 - nZ_2 = \text{Div}(f)$ for certain rational function $f$ on $M_K \otimes \mathbb{Q}$. The divisor of $f$ on $M_K \otimes \mathbb{Q}$ is of the shape $n\bar{Z}_1 - n\bar{Z}_2 + \sum n_i B_i$ with $n_i \in \mathbb{Z}$. So (ii) holds.

We now pass to the direct limits $CH^* = \varprojlim CH^*(M_K \otimes \mathbb{Q})$, $H^*(M, \mathbb{C}) = \varprojlim H^*(M_K, \mathbb{C})$ and consider the according cycle map

$$\text{cl}_M : CH^1(M \otimes \mathbb{Q}) \otimes \mathbb{Q} \otimes \mathbb{C} \to H^2(M, \mathbb{C}).$$

**Proposition 1.** $\text{cl}_M$ is $G(\mathbb{A}_f)$-equivariant and injective. $H^2(M, \mathbb{C})$ is a completely reducible admissible $G(\mathbb{A}_f)$-module.

*Proof.* The $G(\mathbb{A}_f)$-equivariance is obvious. Because of Lemma[1](i), the map $\text{cl}_{M} : CH^1(M_K \otimes \mathbb{Q}) \otimes \mathbb{Q} \to H^2(M_K, \mathbb{Q})$ is injective for neat $K_f$, whence $\text{cl}_M$ is injective. By Section 13 and specifically Lemma 12 in [20], $H^2(M, \mathbb{C})$ is isomorphic to the $(\mathfrak{g}, K_{\infty})$-cohomology of the discrete spectrum $L^2_{\text{disc}}(G(\mathbb{Q}) \mathbb{Z} \mathbb{R}^+) \setminus G(\mathbb{A})$, whence $H^2(M, \mathbb{C})$ is a completely reducible admissible $G(\mathbb{A}_f)$-module.

We now prove Theorem 1 by assuming Theorem 2
Proof of Theorem. (1) Theorem 3 implies
\[ \text{CH}^1(M \otimes \mathbb{Q}) \otimes \mathbb{C} = \text{SC}^1(M) \otimes \mathbb{C}. \]
Hence \( \text{CH}^1(M \otimes \mathbb{Q}) \otimes \mathbb{C} = \text{SC}^1(M) \otimes \mathbb{C} \). Taking the \( K_f \)-invariant subspaces on both sides, we get \( \text{CH}^1(M_{K_f} \otimes \mathbb{Q}) \otimes \mathbb{C} = \text{SC}^1(M_{K_f}) \otimes \mathbb{C} \).

(1) \implies (2) By Theorem 9.4 in [21], the space of Tate classes \( \text{Ta}(\tilde{M}_{K_f}) \) is spanned by the image of \( \text{CH}^1(\tilde{M}_{K_f} \otimes \mathbb{Q}) \). By (1) and Lemma 8 (ii), \( \text{CH}^1(\tilde{M}_{K_f} \otimes \mathbb{Q}) \otimes \mathbb{Q} \) is spanned by \([B_i] \) (for \( 1 \leq i \leq m \) and \([Z] \) with \( |Z| \in \text{SC}^1(M_{K_f} \otimes \mathbb{Q}) \)). The assertion in (2) then follows.

(2) \implies (3) Choose a \( \mathbb{Q} \)-embedding \( \tilde{M}_{K_f} \to \mathbb{P}^N \) for certain \( N \in \mathbb{N} \). Let \( L_0 \) be a \( \mathbb{Q} \)-hyperplane in \( \mathbb{P}^N \) that has a non-trivial intersection with \( \tilde{M}_{K_f} \). Put \( \mathcal{L} = L_0 \cap \tilde{M}_{K_f} \), then the class \( \text{cl}_{et}(\mathcal{L}) \in H^2(\tilde{M}_{K_f} \otimes \mathbb{Q}, \mathcal{Q}_\ell(1)) \) is invariant by \( \text{Gal}(\mathbb{Q}/\mathbb{Q}) \). By Hard Lefschetz Theorem, the map
\[ \mathcal{L} : H^2(\tilde{M}_{K_f} \otimes \mathbb{Q}, \mathcal{Q}_\ell(1)) \to H^4(\tilde{M}_{K_f} \otimes \mathbb{Q}, \mathcal{Q}_\ell(2)) \]
\[ t \to t \cup \text{cl}_{et}(\mathcal{L}). \]
is an isomorphism. It respects the action of \( \text{Gal}(\mathbb{Q}/\mathbb{Q}) \) and hence, for every number field \( E \), induces an isomorphism
\[ H^4(\tilde{M}_{K_f} \otimes \mathbb{Q}, \mathcal{Q}_\ell(2))^{\text{Gal}(\mathbb{Q}/E)} \cong \mathcal{L} \left( H^2(\tilde{M}_{K_f} \otimes \mathbb{Q}, \mathcal{Q}_\ell(1))^{\text{Gal}(\mathbb{Q}/E)} \right). \]
Thus, \( \text{Ta}^2(\tilde{M}_{K_f}) = \mathcal{L}(\text{Ta}^1(\tilde{M}_{K_f})). \)

By (2), \( \text{Ta}(\tilde{M}_{K_f}) \) is spanned by the images of \([B_i] \) and \([Z] \) for \( 1 \leq i \leq m \) and \([Z] \in \text{SC}^1(M_{K_f}). \)

Specifically, \( \text{cl}_{et}(\mathcal{L}) \) is a linear combination of the images of these \([B_i \) and special divisors. Therefore, \( \text{Ta}^2(\tilde{M}_{K_f}) = \mathcal{L}(\text{Ta}^1(\tilde{M}_{K_f})) \) is spanned by the listed elements in (3). (The map \( \mathcal{L} \) may depend on the embedding into \( \mathbb{P}^N \).

(3) \implies (4) Suppose \( Z \in H^2(\tilde{M}_{K_f} \otimes \mathbb{Q}) \). By (3), there is a cycle class \([Z'] \) in \( \text{CH}^1(\tilde{M}_{K_f} \otimes \mathbb{Q}) \otimes \mathbb{Q} \) that satisfies \( \text{cl}_{\tilde{M}_{K_f}, et}(\mathcal{L}) = \text{cl}_{\tilde{M}_{K_f}, et}(\mathcal{L}) \) and is of the shape \( \sum_i q_i[Z_i] + \sum_{i'j} q_{i'j} [Z_{i'} \cdot B_j] + \sum_{j_1, j_2} [B_{j_1} \cdot B_{j_2}] \) with \([Z_i], [Z_{i'}] \in SC^2(M_{K_f} \otimes \mathbb{Q}) \) and the coefficients in \( \mathbb{Q} \). Because the cycle maps are compatible with the comparison maps between etale cohomology and singular cohomology, there is \( \text{cl}_{\tilde{M}_{K_f}}(\mathcal{L}) = \text{cl}_{M_{K_f}}(\mathcal{L}) \) in \( H^4(\tilde{M}_{K_f}, \mathbb{Q}) \). Thus
\[ \text{cl}_{M_{K_f}}([Z]) = j \circ \text{cl}_{\tilde{M}_{K_f}}([Z]) = j \circ \text{cl}_{\tilde{M}_{K_f}}([Z']) = \sum_i q_i \text{cl}_{M_{K_f}}([Z_i]). \]
Here \( j \circ \text{cl}_{\tilde{M}_{K_f}}([Z_{i'} \cdot B_j]) \) and \( j \circ \text{cl}_{\tilde{M}_{K_f}}([B_{j_1} \cdot B_{j_2}]) \) vanish because the cycles are not supported on \( M_{K_f}(\mathbb{C}) \). So \([Z] = \sum_i q_i[Z_i] \) in \( \text{CH}^2(M_{K_f} \otimes \mathbb{Q}) \) and the assertion in (4) follows. \( \square \)

3. The decomposition of \( H^{1,1}(M, \mathbb{C}) \)

For an irreducible unitary automorphic representation \( \pi \) of \( G(\mathbb{A}) \), let \( m(\pi) \) denote its multiplicity in \( L^2(\mathbb{Z}(\mathbb{R})^+; G(\mathbb{A})) \). For an irreducible admissible unitary representation \( \pi_f \) of \( G(\mathbb{A}_f) \), let \( H^{1,1}(\pi_f) \) be the \( \pi_f \)-isotypic component of \( H^{1,1}(M, \mathbb{C}) \).

There is a decomposition
\[ H^{1,1}(M, \mathbb{C}) = \bigoplus_{\pi} m(\pi) H^{1,1}(g, K_\infty, \pi) = \bigoplus_{\pi_f} H^{1,1}(\pi_f). \] (3.1)
Here \( H^{1,1}(g, K_\infty, \pi) = H^{1,1}(g, K_\infty, \pi_\infty) \otimes \pi_f \). Recall that \( H^{1,1}(g, K_\infty, \pi_\infty) \) is \( \mathbb{C} \) when \( \pi_\infty \in \{ \pi^2, \pi^2, 1, \text{sgn} \circ \nu \} \) and zero otherwise. Similarly, one writes \( H^2(M, \mathbb{C}) = H^2(\pi_f) \) as the direct sum of isotypic components.

Weissauer determined that all \( \pi \) occurring in (3.1) are the twists of three types of basic representations.

Theorem 4. [20] Thm. 4, Lemma 6) If \( H^{1,1}(g, K_\infty, \pi) \neq 0 \), then \( \pi = \pi' \otimes \chi \), where \( \chi \) is a character with \( \chi_\infty \in \{ 1, \text{sgn} \circ \nu \} \) and \( \pi' \) is one of the following types:
(I) a CAP representation of $\PGSp_4(\mathbb{A})$ of Siegel type $(\tau \boxtimes 1, \frac{1}{2})$ with $\tau \subset \mathcal{A}_{\text{cusp}}(\PGL_2))$, $\pi_{\infty} = \pi^{2+}$ and $\tau_{\infty} = \mathcal{D}_4$. 

(II) $J(P, \tau, \frac{1}{2})$, with $\tau \subset \mathcal{A}_{\text{cusp}}(\PGL_2)$, $\tau_{\infty} = \mathcal{D}_4$, and $L(\frac{1}{2}, \tau) \neq 0$. $J(P, \tau, \frac{1}{2})$ denotes the unique irreducible quotient of $\Pi(\tau \boxtimes 1, \frac{1}{2})$ and is realized as the residue representation of the Eisenstein series associated to $\Pi(\tau \boxtimes 1, z)$ at $z = \frac{1}{2}$.

(III) the trivial representation $\rho$. 

Because representations of type I, II and III all have multiplicity one in the discrete spectrum, there is $m(\pi) = 1$ when $H^{1,1}(g, K_{\infty}, \pi) \neq 0$.

**Lemma 5.** If $H^{1,1}(\pi_f)$ is nonzero, then there exists a unique $\pi_{\infty}$ such that $\pi = \pi_{\infty} \times \pi_f$ occurs in $L^2_{\text{disc}}(G(\mathbb{Q})\mathbb{Z}(\mathbb{R})^+ \backslash G(\mathbb{A}))$. As a consequence, $H^2(\pi_f) = H^{1,1}(\pi_f)$ is an irreducible admissible $G(\mathbb{A})$-module.

**Proof.** Since $H^{1,1}(\pi_f)$ is nonzero, there is $\pi_{\infty} \in \{\pi^{2+}, \pi^{2-}, 1, \text{sgn} \nu\}$ such that $\pi = \pi_{\infty} \times \pi_f$ occurs in the discrete spectrum. We show that if $\pi' = \pi_{\infty} \times \pi_f$ occurs in the discrete spectrum, then $\pi' = \pi$. By Theorem 1 one may assume that $\pi$ is one of the basic types.

(i) $\pi$ is of type I or II with respect to $\tau \subset \mathcal{A}_{\text{cusp}}(\PGL_2)$. In this situation, there is an irreducible cuspidal $\overline{\SL}_2(\mathbb{A})$-representation $\sigma \in \text{Wd}_\psi(\tau)$ such that $\pi = \Theta_{\overline{\SL}_2 \times \PGSp_4}(\sigma, \psi)$. When $\pi$ is of type I, $\Theta_{\overline{\SL}_2 \times \PGSp_4}(\sigma, \psi) = 0$ (See Thm. 3); when $\pi$ is of type II, $\Theta_{\overline{\SL}_2 \times \PGSp_4}(\sigma, \psi) = \tau$ (See Lemma 10 in [20]).

Note that for almost all finite $p$, $\pi'_p = \pi_p$ is equal to $J(P, \tau_p, \frac{1}{2})$, the $p$-component of $J(P, \tau, \frac{1}{2})$. $\pi'$ is cuspidal, then it is CAP of Siegel type $(\tau \boxtimes 1, \frac{1}{2})$. $\pi'$ is in the discrete residue spectrum, then there must be $L(\frac{1}{2}, \tau) \neq 0$ and $\pi' = J(P, \tau, \frac{1}{2})$; this is because the residue spectrum of $\PGSp_4(\mathbb{A})$ have been known explicitly and the fact $\pi'_p = J(P, \tau_p, \frac{1}{2})$ for almost all $p$ selects out only one possible choice $J(P, \tau, \frac{1}{2})$. In either case, there is $\pi' = \Theta_{\overline{\SL}_2 \times \PGSp_4}(\sigma', \psi)$ for certain $\sigma' \in \text{Wd}_\psi(\tau)$.

Because $\pi'_f = \pi_f$, there is $\sigma'_f = \sigma_f$. However, for two representations $\sigma$ and $\sigma'$ in the same Waldspurger packet, their local components can differ only at a even number of places. (See [5] (1.8), (1.9) or [13] Coro. 1, 2]). It forces that $\sigma'_{\infty} = \sigma_{\infty}$, whence $\sigma' = \sigma$ and $\pi' = \pi$.

(ii) $\pi = 1$ is of type III. So $\pi'_f = 1$ and this forces $\pi'_f = 1$. Actually, $G(\mathbb{Q})\mathbb{Z}(\mathbb{R})^+ \backslash G(\mathbb{A})/G(\mathbb{A}) \neq \mathbb{Q}(\mathbb{Q})\mathbb{Z}(\mathbb{R})^+ \backslash G(\mathbb{R})$ and any continuous function on it must be a constant function.

Since $m(\pi) = 1$, $H^2(\pi_f) = H^2(g, K_{\infty}, \pi_{\infty}) \times \pi_f = H^{1,1}(g, K_{\infty}, \pi_{\infty}) \times \pi_f = H^{1,1}(\pi_f)$ is irreducible.

**Corollary 1.** The map $\text{cl}_M : \text{SC}^1(M) \otimes_{\mathbb{Z}} \mathbb{C} \rightarrow H^{1,1}(M, \mathbb{C})$ is an isomorphism if and only if $\text{SC}^1(\pi_f)$ is nonzero when $H^{1,1}(\pi_f)$ is nonzero.

**Proof.** Because $\text{cl}_M : \text{SC}^1(M, \mathbb{C}) \rightarrow H^{1,1}(M, \mathbb{C})$ is injective and $G(A_f)$-equivariant, it is an isomorphism if and only if $\text{cl}(\pi_f) : \text{SC}^1(\pi_f) \rightarrow H^{1,1}(\pi_f)$ is an isomorphism for each $\pi_f$ occurring in $H^{1,1}(M, \mathbb{C})$. Since $H^{1,1}(\pi_f)$ is irreducible by Lemma 5 the injective homomorphism $\text{cl}(\pi_f)$ is an isomorphism if and only if $\text{SC}^1(\pi_f) \neq 0$.

### 4. The period pairing

A cycle $[Z] \in \text{SC}^1(M, \mathbb{C})$ can be written as

$$[Z] = \sum_{\pi_f} [Z]_{\pi_f},$$

where $[Z]_{\pi_f} \in \text{SC}^1(\pi_f)$ is called the $\pi_f$-component of $[Z]$. (Given $[Z]$, $[Z]_{\pi_f}$ is nonzero for only finitely many $\pi_f$ because $[Z]$ is invariant under certain compact open subgroup $K_f \subset G(A_f)$ and only finitely many $\pi_f$ occurring in $H^{1,1}(M, \mathbb{C})$ have a nontrivial $K_f$-invariant subspace.)

The natural way to test the image of a cycle in cohomology is to use the period pairing between cycles and cohomology with compact support. It is difficult to construct closed differential forms with compact support. So we consider the alternative of rapidly decreasing closed differential forms. There are two basic facts:
(i) The pairing between $H^2(M, \mathbb{C})$ and $H^4_\mathbb{C}(M, \mathbb{C})$ given by $< [\omega_1], [\omega_2] > := \int_M \omega_1 \wedge \omega_2$ is perfect and $G(\mathbb{A}_f)$-equivariant. The restricted pairing on the isotypic components

$$H^2(\pi_f) \times H^4_\mathbb{C}(\pi_f') \rightarrow \mathbb{C}$$

is zero when $\pi_f' \neq \pi_f''$ and perfect when $\pi_f' = \pi_f''$. Here $\pi_f''$ denotes the dual representation.

(ii) $H^*_\mathbb{C}(M, \mathbb{C})$ is defined using the cochain $\Omega^*_\mathbb{C}(M, \mathbb{C})$ consisting of compactly supported differential forms on $M$. Let $H^*_\mathbb{r}(M, \mathbb{C})$ be the cohomology groups defined using the cochain $\Omega^*_\mathbb{r}(M, \mathbb{C})$ consisting of rapidly decreasing differential forms on $M$. Borel \[3\] proved that the inclusion map $\Omega^*_\mathbb{C}(M, \mathbb{C}) \hookrightarrow \Omega^*_\mathbb{r}(M, \mathbb{C})$ induces an isomorphism $\Lambda : H^*_\mathbb{C}(M, \mathbb{C}) \xrightarrow{\sim} H^*_\mathbb{r}(M, \mathbb{C})$. 

We make a simple but very useful observation below.

**Lemma 6.** Suppose $H^{1,1}(\pi_f) \neq 0$. If there exists $[Z] \in SC^1(M)$ and a $\mathbb{K}_f$-finite rapidly decreasing closed form $\Omega$ on $M$ such that (i) $< H^{1,1}(\pi_f^'), \Omega >= 0$ for $\pi_f' \neq \pi_f$ and (ii) $\int_Z \Omega \neq 0$, then $[Z]_{\pi_f}$ is nonzero and as a consequence $SC^1(\pi_f)$ is nonzero. 

**Proof.** Let $[\Omega]$ be the cohomology class of $\Omega$ in $H^4_\mathbb{r}(M, \mathbb{C})$ and $\omega$ be a compactly supported closed form on $M$ that represents the class $\Lambda([\Omega])$ in $H^4(M, \mathbb{C})$, then $\Omega - \omega$ is the boundary of a rapidly decreasing form. Hence

$$\int_Z \Omega = \int_Z \Omega - \omega + \int_Z \omega = 0 + < \text{cl}_M([Z]), [\omega] > .$$

Write $[Z] = \sum_{\pi_f'} [Z]_{\pi_f'}$ and $[\omega] = \sum_{\pi_f''} [\omega]_{\pi_f''}$, where $[Z]_{\pi_f'}$ is the $\pi_f'$-component of $[Z]$ and $[\omega]_{\pi_f''}$ is the $\pi_f''$-component of $[\omega]_{\pi_f'}$. These are finite sums because $[Z]$ and $[\Omega]$ (and hence $[\omega]$) are $\mathbb{K}_f$-finite. Note that $\text{cl}_M([Z]_{\pi_f'}) \in H^{1,1}(\pi_f^{'})$.

By Condition (i), one has $< H^{1,1}(\pi_f'), [\omega] > = 0$ for $\pi_f' \neq \pi_f$, whence

$$< \text{cl}_M([Z]), [\omega] > = \sum_{\pi_f'} < \text{cl}_M([Z]_{\pi_f'}), [\omega] > = < \text{cl}_M([Z]_{\pi_f}), [\omega] > .$$

Since the pairing on $H^2(\pi_f^{'}) \times H^4(\pi_f^{'})$ is zero for $\pi_f'' \neq \pi_f'$, there is

$$< \text{cl}_M([Z]_{\pi_f}), [\omega] > = \sum_{\pi_f'} < \text{cl}_M([Z]_{\pi_f'}), [\omega]_{\pi_f'} > = < \text{cl}_M([Z]_{\pi_f}), [\omega]_{\pi_f''} > .$$

So $\int_Z \Omega = < \text{cl}_M([Z]_{\pi_f}), [\omega]_{\pi_f''} >$. With Condition (ii), one sees that both $\text{cl}_M([Z]_{\pi_f})$ and $[\omega]_{\pi_f''}$ are nonzero. Specifically, $[Z]_{\pi_f}$ is a nonzero element in $SC^1(\pi_f)$. \hfill $\square$

So we propose the following proposition.

**Proposition 2.** If $H^{1,1}(\pi_f)$ is nonzero, then there exist $[Z] \in SC^1(M)$ and a $\mathbb{K}_f$-finite rapidly decreasing closed form $\Omega$ on $M$ such that (i) $< H^{1,1}(\pi_f^{'}) , \Omega >= 0$ for $\pi_f' \neq \pi_f$, (ii) $\int_Z \Omega \neq 0$.

**Lemma 7.** Suppose that $H^{1,1}(\pi_f)$ is nonzero. Let $\pi_\infty$ be the unique member of $\{ \pi^+, \pi^-, 1, \text{sgn } \nu \}$ such that $\pi_\infty \times \pi_f$ occurs in the discrete spectrum. Let $\chi$ be a character of $\mathbb{Q}^* \backslash \mathbb{Q}^*$ with $\chi_\infty \in \{ 1, \text{sgn} \}$. If Proposition 2 holds for $\pi_f$, then it also holds for $\pi_f \otimes \chi_f$.

**Proof.** Suppose that Proposition 2 holds for $\pi_f$, then there exist a $\mathbb{K}_f$-finite rapidly decreasing closed form $\Omega$ on $M$ and $[Z] \in SC^1(M)$ such that $< H^{1,1}(\pi_f'), \Omega >= 0$ for $\pi_f' \neq \pi_f$ and $\int_Z \Omega \neq 0$.

Choose a neat compact open subgroup $K_f$ sufficiently small so that (i) $\chi_f \circ \nu$ is invariant by $K_f$, (ii) $[Z] = \sum_i \zeta_i [Z_i]$, where $Z_i$ are irreducible special divisors on $M_{K_f} \otimes \mathbb{Q}$. For each $i$, select an element $g_i \in G(\mathbb{A})$ that represents a point on $Z_i \subset M_{K_f}$. Define the $\chi$-twist of $Z$ by $Z_\chi := \sum_i \chi \nu^{-1}(\nu(g_i)) Z_i$. Then $\Omega_\chi := \chi(\nu(g)) \Omega'$ and $Z_\chi$ satisfy the requirement of Proposition 2 for $\pi_f \otimes \chi_f$. \hfill $\square$
By Corollary 4 and Lemma 5, Theorem 2 is a consequence of Proposition 2. Furthermore, by Theorem 4 and Lemma 7 it is sufficient to verify Proposition 2 when \( \pi = \pi_\infty \times \pi_f \) is of basic type I, II, and III in Theorem 3. In next three section, we prove Proposition 2 when \( \pi \) is one of the three basic types I, II, and III. This would complete the proof of Theorem 2. Note that \( \pi \) is self-dual in these cases.

5. Nonvanishing Periods I

We verify Proposition 2 when \( \pi \) is of type I. The candidates for \([\mathcal{Z}]\) are non-split cycles in \( \text{SC}^1_{ns}(M) \) and the candidates for \( \Omega \) are forms in \( H^{2,2}(g, K_\infty, \pi) \), which are cuspidal and hence rapidly decreasing.

**Proposition 3.** Let \( \pi \) be a CAP representation of \( \text{PGSp}_4(\mathbb{A}) \) of Siegel type \( (\tau, \frac{1}{2}) \) with \( \pi_\infty = \pi^{2+} \). There exist \([\mathcal{Z}] \in \text{SC}^1_{ns}(M) \) and \( \Omega \in H^{2,2}(g, K, \pi) \) such that (i) \( <\Theta^{(1, \frac{1}{2})}(\pi_f), \Omega> = 0 \) for all \( \pi_f \neq \pi_f \), (ii) \( \int_Z \Omega \neq 0 \).

Proposition 3 has an immediate corollary below. We prove Proposition 3 at the end of this section, after preparing relevant lemmas.

**Corollary 2.** Let \( H^{1,1}_{\text{cusp}}(M, \mathbb{C}) \) be the subspace of \( H^{1,1}(M, \mathbb{C}) \) spanned by cuspidal closed differential forms, then \( H^{1,1}_{\text{cusp}}(M, \mathbb{C}) \) is spanned by the images of certain cycle classes in \( \text{SC}^1_{ns}(M) \).

5.1. Nonvanishing of automorphic periods. Recall the identification \( \text{GSp}_4 \cong \text{GSpin}(V) \) in Section 1. For a nonisotropic vector \( v \in V \). Let \( \pi \) be an irreducible cuspidal unitary representation in \( L^2_{\text{disc}}(G(\mathbb{Q})Z(\mathbb{R})^\pm + G(\mathbb{A})) \). For a nonisotropic vector \( v \in V \), define a period functional \( \mathcal{P}_v \) on the space of smooth vectors in \( \pi \),

\[
\mathcal{P}_v(\varphi) = \int_{[\text{SO}(v^\perp)]} \varphi(h)dh.
\]

**Lemma 8.** Let \( \pi \) be a CAP representations of \( \text{PGSp}_4(\mathbb{A}) \cong \text{SO}(V_\mathbb{A}) \) of Siegel type \( (\tau, \frac{1}{2}) \) with \( \pi_\infty = \pi^{2+} \). There exists a vector \( v \in V \) with \( q(v) \in \mathbb{Q}_+\backslash \mathbb{Q}^\times \) such that \( \mathcal{P}_v(\cdot) \) is nonzero.

**Proof.** Choose a non-trivial character \( \psi \) of \( \mathbb{A}/\mathbb{Q} \). By Theorem 3 \( \pi = \Theta_{\text{SL}_2 \times \text{PGSp}_4}(\sigma, \psi) \) with \( \sigma \in \text{Wd}_v(\tau) \) satisfying \( \Theta_{\text{SL}_2 \times \text{PGSp}_2}(\sigma, \psi) = 0 \). There exists \( a \in \mathbb{Q} \) such that the \( \psi_a \)-Whittaker functional \( \ell_{\psi_a} \) is nonzero on \( \sigma \). This implies \( \theta_{\text{SL}_2 \times \text{PGSp}_2}(\sigma, \psi_a) = \tau \otimes \chi_a \), whence \( \sigma \notin \mathbb{Q}^\times \).

Because \( \pi_\infty = \pi^{2+} \), we must have \( a \in \mathbb{Q}_+ \). Otherwise, \( \pi_\infty = \theta_{\text{SL}_2 \times \text{PGSp}_2}(\Omega_4 \otimes \chi_a, \psi_{a_\infty}) = \theta_{\text{SL}_2 \times \text{PGSp}_2}(\Omega_4, \psi_{a_\infty}) \) and, by Lemma 3 \( \pi^{2+} = \theta_{\text{SL}_2 \times \text{PGSp}_4}(\pi_\infty, \pi_\infty) \) is a discrete series, which is a contradiction.

We have \( \sigma = \theta_{\text{SL}_2 \times \text{PGSp}_4}(\pi, \psi) \). For a form \( \varphi = \Theta(\phi, f) \) with \( f \in \pi \) and \( \phi \in \text{S}(V(\mathbb{A})) \), there is

\[
\ell_{\psi_a}(\varphi) = \int_{[\text{SO}(V)]} \left[ \int_{[\text{SO}(V)]} \frac{f(h)}{\xi E} \sum_{\xi \in \mathbb{V}} \omega((1, \frac{1}{2}), h)\phi(\xi)dh \right] \psi(a n)dn = \int_{[\text{SO}(V)]} \frac{f(h)}{\xi E} \sum_{\xi \in \mathbb{V}} \omega(h)\phi(\xi) \int_{[\text{SO}(V)]} \psi((q(\xi) - a)n)dn.
\]

The integral \( \int_{\mathbb{A}/\mathbb{Q}} \psi((q(\xi) - a)n)dn \) is zero when \( q(\xi) \neq a \) and 1 when \( q(\xi) = a \). Let \( v \) be a vector in \( V \) with \( q(v) = a \), then vectors \( \xi \) with \( q(\xi) = a \) can be expressed as \( \gamma \cdot v \), with \( \gamma \in \text{SO}(v^\perp)Q_{\text{SO}(V)} \). So,

\[
\ell_{\psi_a}(\varphi) = \int_{[\text{SO}(V)]} \sum_{\gamma \in \text{SO}(v^\perp)Q_{\text{SO}(V)}} \frac{f(h)}{\xi E} \phi(h^{-1} \gamma^{-1} v)dh = \int_{\text{SO}(v^\perp)Q_{\text{SO}(V)}} \frac{f(h)}{\xi E} \phi(h^{-1} \gamma^{-1} v)dh = \int_{\text{SO}(v^\perp)Q_{\text{SO}(V)}} \left( \int_{[\text{SO}(v^\perp)]} f(hh')dh \right) \phi(h'^{-1} v)dh'.
\]
Because \( \ell_{\psi_f} \) is nonzero on \( \sigma \), the function \( \int_{[SO(v^+)]} f(hh')dh = P_v(\pi(h')f) \) is not identically zero. Therefore, \( P_v \) is nonzero.

5.2. Periods of cohomological forms. We describe the periods of cohomological forms in \( H^4(g, K_\infty, \pi) \) on \( Z_{Qv,g_0,K_f} \). Note the following:

(i) The homomorphism \( GSp_4(\mathbb{R}) \twoheadrightarrow SO(V)(\mathbb{R}) \) induces an isomorphism \( \text{Lie}(Sp_4(\mathbb{R})) \xrightarrow{\sim} \text{Lie}(SO(V)_{\mathbb{R}}) \). We take the Cartan decomposition \( g_0 = p \oplus t \) in Section 1.6.2 as the Cartan decomposition of \( \text{Lie}(SO(V)_{\mathbb{R}}) \otimes \mathbb{C} \).

(ii) Recall the compact torus \( T \) whose Lie algebra is \( t_\mathbb{R} \). There exists \( g_\infty \in G(\mathbb{R}) \) such that \( g_\infty T g_\infty^{-1} \) is a maximal connected compact subgroup of \( GSpin(v^+)_{\mathbb{R}} \). With this choice of \( g_\infty \), \( p' := p \cap \text{Ad}_{g_\infty}^{-1} \left( \text{Lie}(SO(V)_{\mathbb{R}}) \otimes \mathbb{C} \right) \) is 4-dimensional and \( t \)-invariant. Thus \( \text{Ad}_{g_\infty} p' \oplus \text{Ad}_{g_\infty} t \) is a Cartan decomposition of \( \text{Lie}(SO(V)_{\mathbb{R}}) \otimes \mathbb{C} \).

Recall that the \( t \)-submodule \( \pi^{2+}(\delta_{2\alpha}) \) of \( \pi^{2+} \) has a basis \( \{v_j, -2 \leq j \leq 2\} \) consisting of weight vectors (see Sect. 1.6.2). The vector \( v_0 \) of weight 0 is particularly important, as shown by the lemma below.

**Lemma 9.** Let \( \pi \) be a CAP representations of \( GSpin(4,\mathbb{A}) \) of Siegel type \((\tau, \frac{1}{2})\) with \( \pi_\infty = \pi^{2+} \). Let \( v \) be as in Lemma 3 and choose \( g_\infty \in G(\mathbb{R}) \) such that \( g_\infty T g_\infty^{-1} \) is a maximal connected compact subgroup of \( GSpin(v^+)_{\mathbb{R}} \). For the special divisor \( Z_{Qv,g_0,K_f} \), there is

\[
\left\{ \int_{Z_{Qv,g_0,K_f}} \omega : \omega \in H^4(g, K_\infty, \pi) \right\} = \{ P_v(\pi(g_\infty)\varphi)| \varphi \in v_0 \otimes \pi_f \}.
\]

**Proof.** Choose a basis \( \{X_i\}_{i=1}^6 \) of \( p' \) consisting of weight vectors with respect to \( t \) and add two other weight vectors \( X_5, X_6 \) to form a basis of \( p \). Let \( \{\omega_i\}_{i=1}^6 \) be the dual basis in \( p^* \). For a subset \( I = \{i_1 < \cdots < i_q\} \) of \( \{1, \ldots, 6\} \), set \( \omega_I = \omega_{i_1} \wedge \cdots \wedge \omega_{i_q} \). By the discussion in Section 1.6.2 about the \( t \)-module structure of \( \pi^{2+} \) and \( \wedge^4 p \cong \wedge^4 b_0 \), there is

\[
H^4(g, K_\infty, \pi^{2+}) = \text{Hom}_{K_\infty}(\wedge^4 p, \pi^{2+}) = C \cdot (\sum_{|I|=4} v_I \cdot \omega_I),
\]

where \( v_I \in \pi^{2+}(\delta_{2\alpha}) \) and its weight is the negative of the weight of \( \omega_I \).

Thus, a form in \( H^4(g, K_\infty, \pi) \) is of the shape \( \omega = \sum_{|I|=4} \varphi_I \omega_I \), with \( \varphi_I = v_I \otimes v_f \in \pi \) for certain \( v_f \in \pi_f \). Let \( R \) denote the right translation action on \( G(\mathbb{A}) \). There is

\[
\int_{Z_{Qv,g_0,K_f}} \omega = \int_{GSpin(v^+)_{\mathbb{Q}} \setminus GSpin(v^+)_{\mathbb{A}}/L_{\infty}} R^*_g \omega = c \int_{SO(v^+)_{\mathbb{Q}} \setminus SO(v^+)_{\mathbb{A}}/L_{\infty}} \varphi_I(g g_\infty) \text{Ad}_{g_\infty}^*(\omega_I).
\]

Here \( L_{\infty} = g_\infty K_\infty g_\infty^{-1}, T_{\infty} = L_{\infty}/Z(\mathbb{R})^+, \) and \( c \) is as in Section 1.

One needs to determine the restriction of \( \text{Ad}^*_{g_\infty}^1(\omega_I) \) to \( SO(v^+)_{\mathbb{R}}/T_{\infty} \). Because \( \text{Ad}_{g_\infty} p' \oplus \text{Ad}_{g_\infty} t \) is a Cartan decomposition of \( \text{Lie}(SO(v^+)_{\mathbb{R}}) \otimes \mathbb{C} \) and \( \text{Ad}_{g_\infty} t = \text{Lie}(T_{\infty}) \otimes \mathbb{C} \), left invariant vector fields on \( SO(v^+)_{\mathbb{R}}/T_{\infty} \) are identified with elements in \( \text{Ad}_{g_\infty} p' \). Hence the restriction of \( \text{Ad}^*_{g_\infty}^1 \omega_i \) to \( SO(v^+)_{\mathbb{R}}/T_{\infty} \) is nonzero if and only if \( i = 1, 2, 3, 4 \). Therefore, among all \( I \) with \( |I| = 4 \), only \( \text{Ad}^*_{g_\infty}^1(\omega_{1234}) \) has nonzero restriction. Its restriction, when combined with the volume form of \( T_{\infty} \), gives the volume form of \( SO(v^+)_{\mathbb{R}} \). So

\[
\int_{Z_{Qv,g_0,K_f}} \omega = c \int_{SO(v^+)_{\mathbb{Q}} \setminus SO(v^+)_{\mathbb{A}}} \varphi_{1234}(g g_\infty) dg = c P_v(\pi(g_\infty)\varphi_{1234}).
\]

Note that \( \varphi_{1234} \) belongs to \( v_0 \otimes \pi_f \) because it is of weight 0.

On the other hand, given \( \varphi = v_0 \otimes v_f \in v_0 \otimes \pi_f \), set \( \omega = \sum_{|I|=4} \varphi_I \omega_I \) with \( \varphi_I = v_I \otimes v_f \), then \( P_v(\pi(g_\infty)\varphi) = c^{-1} \int_{Z_{Qv,g_0,K_f}} \omega \). So we obtain the equality of sets in the Lemma. \qed
5.3. Period relation. We show that $P_v$ is nonzero on $\pi$ if and only if it is nonzero on the subspace $v_0 \otimes \pi_f$. We prove this assertion by arguing that the value of $P_v$ on a general vector is a scalar multiple of its value on a vector in $v_0 \times \pi_f$. Such a relation exists partly because the $K_\mathbb{R}$-type to which $v_0$ belongs is minimal in $\pi^{2+}$.

Lemma 10. Let $\pi$ be an irreducible cuspidal automorphic representation of $SO(V)(\mathbb{A})$ and $v$ be a nonisotropic vector of $V$. For a smooth vector $\varphi \in \pi$ and $X \in \text{Lie}(SO(V)_\mathbb{R})$, there is $P_v(X \circ \varphi) = 0$.

Proof. Write $F(h, t) = \frac{\varphi(h^t \gamma) - \varphi(h)}{t}$, then $X \circ \varphi(h) = \lim_{t \to 0} F(h, t)$ and

$$P_v(X \circ \varphi) = \int_{[SO(v^\perp)]} \lim_{t \to 0} F(h, t) dh.$$ 

By Mean Value Theorem, for each couple $(h, t)$, there exists some number $\xi_{h, t}$ between $0$ and $t$ such that $F(h, t) = X \circ \varphi(h \xi_{h, t}^t \gamma)$. Because $\pi$ is cuspidal, both $\varphi$ and $X \circ \varphi$ are rapidly decreasing. So there exists an integrable function $F(h)$ on $SO(V)_\mathbb{Q} \setminus SO(v^\perp)$ such that $|F(h, t)| \leq F(h)$ when $|t|$ is small. By the dominated convergence theorem, there is

$$P_v(X \circ \varphi) = \lim_{t \to 0} \int_{[SO(v^\perp)]} F(h, t) dh = \lim_{t \to 0} \frac{1}{t} [P_v(\pi(e^{tX}) \varphi) - P_v(\varphi)].$$

Because $X \in \text{Lie}(SO(v^\perp)_\mathbb{R})$, there is $P_v(\pi(e^{tX}) \varphi) = P_v(\varphi)$, whence $P_v(X \circ \varphi) = 0$. \hfill $\Box$

Lemma 11. Let $\pi$ be a CAP representations of $PGSp_4(\mathbb{A})$ of Siegel type $(\alpha, \frac{1}{2})$ with $\pi_\infty = \pi^{2+}$. Let $v$ and $g_\infty$ be as in Lemma 8 and 9. For a $K_\infty$-finite smooth vector $\varphi = v_\infty \otimes v_f \in \pi$, put $\varphi_0 = v_0 \otimes v_f$, then $P_v(\pi(g_\infty) \varphi) = CP_v(\pi(g_\infty) \varphi_0)$ for a number $C$ depending on $v_\infty$.

Proof. Let $\mathcal{U}$ denote the universal enveloping algebra of $g_\mathbb{0}$. Because $v_\infty$ is $K_\infty$-finite, $v_\infty = R \cdot v_0$ for certain $R \in \mathcal{U}$. Put $\ell := P_v \circ \pi(g_\infty)$ and observe the following facts:

(i) By the choice of $g_\infty$, there is $\text{Lie}(SO(v^\perp)_\mathbb{R}) \otimes \mathbb{C} = \text{Ad}_{g_\infty} p' \oplus \text{Ad}_{g_\infty} t$ with $p' \subset p$. Because $p' \oplus t$ is closed under Lie brackets, there is

$$p' = V_{\alpha+\beta} \oplus V_{-\alpha-\beta} \oplus V_{-\alpha+\beta}.$$ 

Set $\mathfrak{h} := t \oplus V_{\alpha+\beta} \oplus V_{-\alpha-\beta} \oplus V_{-\alpha+\beta} \oplus V_{-\alpha-\beta}. \quad \text{By Lemma 11, } \ell \circ X = 0 \text{ for } X \in \mathfrak{h}.$

(ii) The Casimir element $\Omega$ in $\mathcal{U}$ acts by a scalar, say $\lambda$, on the smooth vectors of $\pi^{2+}$. Choose a nonzero vector $E_\gamma \in V_\gamma$ for each positive root $\gamma$ of $g_\mathbb{0}$. By choosing a suitable nonzero vector $E_{-\gamma} \in V_{-\gamma}$ and setting $H_\alpha = [E_\alpha, E_{-\alpha}], H_\beta = [E_\beta, E_{-\beta}]$, one can write $\Omega$ as

$$\Omega = F(H_\alpha, H_\beta) + E_\alpha E_{-\alpha} + E_\beta E_{-\beta} + E_{\alpha+\beta} E_{-\alpha-\beta} + E_{\alpha-\beta} E_{-\alpha+\beta},$$

where $F(\cdot, \cdot)$ is a certain degree-2 polynomial. By the observation made in (i), there is

$$\ell \circ E_\beta E_{-\beta} = \lambda \ell - \ell \circ E_\alpha E_{-\alpha}.$$ 

(iii) By Poincaré-Birkhoff-Witt theorem, we may write $R$ as

$$R = R' + \sum_{0 \leq i_1, i_2, j_1, j_2 \leq n} c_{i_1, i_2, j_1, j_2} E_{-\beta}^{i_1} E_{-\alpha}^{j_1} E_{-\beta}^{i_2} E_{-\alpha}^{j_2}$$

where $R' \in \mathfrak{h} \cdot \mathcal{U}$ and $n \in \mathbb{N}$.

(iv) $\ell$ vanishes on vectors of nonzero weight. Suppose that $\varphi'$ is of weight $\gamma \neq 0$; choose $X \in t$ satisfying $\gamma(X) \neq 0$, then $0 = \ell(X \varphi') = \ell(\gamma(X) \varphi') = \gamma(X) \ell(\varphi')$, whence $\ell(\varphi') = 0$.

Now for $\varphi_0$, by (i) and (iii), there is

$$\ell(R \varphi_0) = \sum_{0 \leq i_1, i_2, j_1, j_2 \leq n} c_{i_1, i_2, j_1, j_2} \ell(E_{-\beta}^{i_1} E_{-\alpha}^{j_1} E_{-\beta}^{i_2} E_{-\alpha}^{j_2} \varphi_0).$$
Because \( \varphi_0 \) is of weight zero, the vector \( E_{\beta}^i E_{-\beta}^j E_{\alpha}^k \varphi_0 \) is of weight \( (i_1 - i_2)\beta + (j_1 - j_2)\alpha \).

Applying (iv), we get that
\[
\ell(R \varphi_0) = \sum_{0 \leq i, j, l \leq n} c_{i, j, l} \ell([E_{\beta}^i E_{-\beta}^j E_{\alpha}^k \varphi_0]).
\]

Because \( E_\alpha \) and \( E_{-\alpha} \in \mathfrak{f} \), the vector \( E_{\alpha}^i E_{-\alpha}^j \varphi_0 \) is a scalar multiple of \( v_0 \). So there are number \( c_i (0 \leq i \leq n) \) such that
\[
\ell(R \varphi_0) = \sum_{0 \leq i \leq n} c_i \ell(E_{\alpha}^i \varphi_0).
\]

Furthermore, the relation \( E_{\beta} E_{-\beta} = E_{-\beta} E_{\beta} + H_\beta \) implies \( E_{\beta}^j H_\beta = (H_\beta - j\beta(H_\beta)) E_{\beta}^j \) and
\[
E_{\beta}^i E_{-\beta} = E_{-\beta} E_{\beta}^i - \frac{i(i-1)\beta(H_\beta)}{2} E_{\beta}^{i-1} E_{-\beta}.
\]

Applying (i) and (ii), one gets
\[
\ell(E_{\beta}^i E_{-\beta} \varphi_0) = \ell(E_{\beta} E_{-\beta} E_{\beta}^{i-1} E_{-\beta} \varphi_0) - \frac{i(i-1)\beta(H_\beta)}{2} \ell(E_{\beta} E_{-\beta} E_{\beta}^{i-1} E_{-\beta} \varphi_0).
\]

Note that the second term is a multiple of \( \ell(E_{\beta}^j E_{-\beta} \varphi_0) \) and that, by (i) and PBW theorem, the third term is a linear combination of \( \ell([E_{\beta}^j E_{-\beta} \varphi_0]) \) (\( 0 \leq j \leq i - 1 \)). By induction, we have \( \ell(E_{\beta}^j E_{\beta}^i \varphi_0) = C_i \ell(\varphi_0) \). (It obviously holds when \( i = 0 \).) Therefore, \( \ell(R \varphi_0) = C \ell(\varphi_0) \) for \( C = \sum_{0 \leq i \leq n} c_i C_i \).

**Proof of Proposition 3** We first apply Lemma 8 to get a vector \( v \in V \) with \( q(v) \in \mathbb{Q}^+\backslash \mathbb{Q}^{\times 2} \) such that \( \mathcal{P}_v \) is nonzero on the space of smooth vectors of \( \pi \). Choose a \( K_\infty \)-finite smooth vector \( \varphi = v_\infty \otimes v_f \) of \( \pi \) such that \( \mathcal{P}_v(\varphi) \neq 0 \). (This is possible because \( K_\infty \)-finite smooth decomposable vectors span a dense subspace in the space of smooth vectors and \( \mathcal{P}_v \) is continuous.) By Lemma 11, \( \mathcal{P}_v(\varphi_0) \neq 0 \) for \( \varphi_0 = v_0 \otimes v_f \in v_0 \times \pi_f \). By Lemma 9, there exists \( \Omega \in H^2(\mathfrak{g}, K_\infty, \pi) = H^2(\pi_f) \) such that \( \int_{Z_{G_0, K_\infty} \pi_\infty} \Omega \neq 0 \). Condition (i) in Proposition 3 obviously holds: for \( \langle H^1(\pi_f), H^2(\pi_f) \rangle \neq 0 \), it is necessary that \( \pi_f = \pi_f^\vee = \pi_f \).

6. **Nonvanishing Periods II**

We verify Proposition 2 when \( \pi \) is of type II. The candidates for \([Z] \) are cycles in \( SC^1_s(M) \) (see Sect. [4]) and the form \( \Omega \) will be constructed using Harder’s method of Eisenstein cohomology.

It is more convenient to describe \( SC^1_s(M) \) in terms of the group
\[
H := \{(g_1, g_2) : g_1, g_2 \in GL_2, \det g_1 = \det g_2 \},
\]
and the embedding \( H \hookrightarrow GSp_4 \) given by
\[
\left( \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \right) \rightarrow \begin{pmatrix} a_1 & b_1 & b_2 \\ c_1 & c_2 & d_1 & d_2 \end{pmatrix}.
\]
6.1. Eisenstein cohomology. We construct $\Omega$ by the method of Eisenstein cohomology, which was first used by Harder in [10]. The key observation is that $H^0(\mathfrak{g}, \mathfrak{k}_\infty, \Pi(\tau, \frac{1}{2}))$ is nonzero when $\tau_\infty = \mathcal{D}_4$. We wedge a form in this space with a degree-1 form on $P(\mathbb{Q}) \backslash G(\mathfrak{a})$. The wedge product is a closed form on $P(\mathbb{Q}) \backslash G(\mathfrak{a}) / K_\infty$ and we then sum its translates by $P(\mathbb{Q}) \backslash G(\mathbb{Q})$ to obtain a form on $G(\mathbb{Q}) \backslash G(\mathfrak{a}) / K_\infty$.

Fix a Levi decomposition $P = UM$ as below,

$$U := \left\{ u = \begin{pmatrix} l_2 & n \end{pmatrix} : n \in \text{Sym}_2 \right\}, \quad M := \left\{ m = \begin{pmatrix} A & x' A^{-1} \end{pmatrix} : A \in \text{GL}_2, x \in \text{GL}_1 \right\}.$$  

Set $m = \text{Lie}(M(\mathbb{R})) \otimes \mathbb{C}$, $u = \text{Lie}(U(\mathbb{R})) \otimes \mathbb{C}$, $K_{M, \infty} = K_\infty \cap M(\mathbb{R})$ and $\tilde{t}_M = \text{Lie}(K_{M, \infty}) \otimes \mathbb{R} \otimes \mathbb{C}$. Define a quasi-character $\lambda$ on $P(\mathbb{Q}) \backslash P(\mathfrak{a})$,

$$\lambda(p) = \left| \frac{\det A}{x} \right|, \quad p = \begin{pmatrix} A & x' A^{-1} \end{pmatrix}.$$  

Set $P_H = H \cap P$, $M_H = H \cap M$, $U_H = H \cap U$.

6.1.1. The $(\mathfrak{g}, K_\infty)$-cohomology. By \cite[Sect. 3.3, 3.4]{[2]}, there is

$$H^3(\mathfrak{g}, K_\infty, I_P \mathcal{D}_4 \otimes M_4 \mathbb{C}) = H^3(\mathfrak{p}, K_{M, \infty}, \mathcal{D}_4 \otimes \lambda^2)
= H^1(m, K_{M, \infty}, \mathcal{D}_4 \otimes H^2(u, \mathbb{C}) \otimes \lambda^2)
= \text{Hom}_{K_{M, \infty}}(\mathfrak{m} / \tilde{t}_M, \mathcal{D}_4 \otimes H^2(u, \mathbb{C}) \otimes \lambda^2).$$  

Note that $t \cap \tilde{t}_M = \mathbb{C}H$ and that $\mathfrak{m} / \tilde{t}_M = \mathbb{C}h \oplus \mathbb{C} n_0$ with respect to the identification $\mathfrak{g} / \tilde{t} = b_0$. Observe the following:

(i) $(\mathfrak{m} / \tilde{t}_M)^* = \mathbb{C}\eta^+ \oplus \mathbb{C}\eta^-$, where $\eta^+ = h^* \pm \frac{1}{2} n_0^*$ are of weights $\pm \alpha$ with respect to $t \cap \tilde{t}_M$.
(ii) $H^2(u, \mathbb{C}) = \mathbb{C}\eta^+ \oplus \mathbb{C}\eta^-$, where $\eta_{\pm} = n_{1}^* \wedge n_{2}^* \pm i n_{1}^* \wedge n_{2}^*$ are of weights $\pm \alpha$ with respect to $t \cap \tilde{t}$.
(iii) $\mathcal{D}_4 = \oplus_{n \in \mathbb{Z}, |n| \geq 2} \mathcal{D}_4(n\alpha)$, where $\mathcal{D}_4(n\alpha)$ refers to the weight space of $\mathcal{D}_4$ with weight $n\alpha$ and is 1-dimensional.
(iv) $K_{M, \infty} \cong Z(\mathbb{R})^+ \times O(2)$. An element $(\alpha, A)$ on the RHS corresponds to $(\alpha A_{\alpha A^{-1}})$ on the LHS. The element $(1_{-1}) \in O(2)_{\mathbb{R}}$ sends $\eta^+ \rightarrow -\tilde{\eta}^+$, $\eta_{\pm} \rightarrow -\eta_{\mp}$, and $\mathcal{D}_4(n\alpha)$ to $\mathcal{D}_4(-n\alpha)$.

By choosing $v_+ \in \mathcal{D}_4(2\alpha)$ and setting $v_- = -\left( 1_{-1} \right) \circ v_+$, we can write

$$\text{Hom}_{K_{M, \infty}}(\mathfrak{m} / \tilde{t}_M, \mathcal{D}_4 \otimes H^2(u, \mathbb{C}) \otimes \lambda^2).
= \mathbb{C} \cdot (v_+ \eta^- \wedge \eta_- + 1 + v_- \eta^+ \wedge \eta_+ \otimes 1).$$  

(6.2)
6.1.2. The global induction. By (6.2), there is a global isomorphism
\[ H^3(g, K_\infty, \Pi(\tau, \frac{1}{2})) \cong H^1(m, K_M, \mathcal{O}_4 \otimes H^2(u, \mathbb{C}) \otimes \lambda^2) \otimes \Pi(\tau, \frac{1}{2}). \] (6.3)
Here \( \Pi(\tau, \frac{1}{2}) \) refers to the abstract induced representation of \( G(\mathbb{A}) \) consisting of \( \mathbb{K}_f \)-finite functions \( \phi_f : G(\mathbb{A}) \to \tau_f \) that satisfy
\[ \phi_f \left( \left( \begin{array}{c} A \\ x \end{array} \right) \right) = |\det A|^2 \tau_f(\phi(g)). \]
We define a map from the RHS of (6.3) to the LHS.
(i) Let \( v_\pm \) be as in (6.2). To \( \phi_f \in \Pi(\tau, \frac{1}{2}) \), we associate two functions \( F_\pm(g) \) on \( P(\mathbb{R}) \times G(\mathbb{A}) \):
\[ F_\pm(g) = \lambda^2(p) \cdot [v_\pm \otimes \phi_f(g)](A). \]
Here \( [v_\pm \otimes \phi_f(g)] \) refer to the cuspidal automorphic forms in the space of \( \tau \) that correspond to \( v_\pm \otimes \phi_f(g) \) under the isomorphism \( \tau = \mathcal{O}_4 \otimes \tau_f \).
(ii) Accordingly, define a differential form \( \omega \) on \( P(\mathbb{R}) \times G(\mathbb{A}) \):
\[ \omega := F_+(g)\eta_+ \wedge \eta_- + F_-(g)\eta^+ \wedge \eta_+. \]
Because of (6.2), \( \omega \) is right invariant by \( K_M, \mathbb{K} \) and descends to a form on \( P(\mathbb{R}) \times G(\mathbb{A}) / K_\infty \).
By the identification \( P(\mathbb{R}) / K_M, \mathbb{K} = G(\mathbb{R}) / K_\infty \), it is regarded as a form on \( G(\mathbb{A}) / K_\infty \). Its pullback to \( G(\mathbb{A}) \) is
\[ \omega(p_\infty k_\infty, g_f) = R^i_{k_\infty} \left( \omega(p_\infty, g_f) \right), \quad p_\infty \in P(\mathbb{R}), k_\infty \in K_\infty, g_f \in G(\mathbb{A}). \]
By (6.3), the form \( \omega \) is closed and belongs to \( H^3(\mathbb{L}, K_\infty, \Pi(\tau, \frac{1}{2})) \).

6.1.3. Eisenstein series operation. Regard \( \lambda \) as a function on \( G(\mathbb{A}) \) by setting \( \lambda(pk) = \lambda(p) \) for \( p \in P(\mathbb{A}), k \in \mathbb{K} \). Define \( \eta_0 := \lambda^*(\frac{d\tau}{\tau}) \). It is a closed degree-1 form on \( P(\mathbb{Q}) \setminus G(\mathbb{A}) / K_\infty \) and \( \eta_0(p) = 2a^*_f \) for \( p \in P(\mathbb{A}) \).
To \( \kappa \in C^\infty_\mathbb{C}(\mathbb{R}^+) \) and \( \omega \in H^3(\mathbb{L}, K_\infty, \Pi(\tau, \frac{1}{2})) \), we associate a differential form \( \eta^\kappa \omega \) on \( P(\mathbb{Q}) \setminus G(\mathbb{A}) / K_\infty \):
\[ \eta^\kappa \omega := \kappa(\lambda)(g) \omega \wedge \eta_0. \]
Because \( \omega \) and \( \eta_0 \) are closed and \( d(\kappa(\lambda(g))) = \lambda(g)\kappa'(\lambda(g))\eta_0 \), there is \( d(\eta^\kappa \omega) = d(\kappa(\lambda(g))) \wedge \omega \wedge \eta_0 = 0 \), whence \( \eta^\kappa \omega \) is a closed form.

Imitating the construction of Eisenstein series, we define
\[ E(\eta^\kappa \omega) = \sum_{\gamma \in P(\mathbb{Q}) \setminus G(\mathbb{Q})} L^*_\gamma(\eta^\kappa \omega), \]
It is a \( \mathbb{K}_f \)-finite differential form on \( G(\mathbb{Q}) \setminus G(\mathbb{A}) / K_\infty \).

Remark 5. Choose a basis \( \{\omega_i\} \) of \( p^* \) and write \( \eta_0 = \sum c_i(g)\omega_i \), then \( c_i(g) \) are left \( P(\mathbb{A}) \)-invariant and right \( \mathbb{K}_f \)-invariant because \( \eta_0 \) is so. Write \( \omega = \sum F_1(g)\omega_I \) with \( F_1(g) \in P(\tau, \frac{1}{2}) \) and \( \omega_I \in \wedge^3 p^* \), then
\[ E(\eta^\kappa \omega) = \sum_{i, I} E(\kappa \circ \lambda \cdot c_i F_1)\omega_I \wedge \omega_i. \] (6.4)
Here \( c_i F_1 \in P(\tau, \frac{1}{2}) \) and \( E(\kappa \circ \lambda \cdot c_i F_1) = \sum_{\gamma \in P(\mathbb{Q}) \setminus G(\mathbb{Q})} [\kappa \circ \lambda \cdot c_i F_1](\gamma g) \) are pseudo-Eisenstein series of type \( (P, \tau) \). \( E(\eta^\kappa \omega) \) is closed by Lemma 12 and we may call it a pseudo-Eisenstein cohomological form.

Lemma 12. Suppose \( \tau \subset \mathcal{A}_{\text{cusp}}(\text{PGL}_2) \) with \( \tau_\infty = \mathcal{O}_4 \). \( E(\eta^\kappa \omega) \) is a \( \mathbb{K}_f \)-finite rapidly decreasing closed form on \( G(\mathbb{Q}) \setminus G(\mathbb{A}) / K_\infty \).
Proof. By Proposition II.1.10 in [13], the pseudo-Eisenstein series \( E(\kappa \circ \lambda \cdot c, F_I) \) in \( \mathcal{O}_A \) are rapidly decreasing. So is \( E^{\kappa, \omega} \).

For closedness, as observed by Harder in [6], it suffices to show that for each \( \mathbb{Q} \)-parabolic subgroup \( P \) of \( G \), the constant term of \( E(\eta^{\kappa, \omega}) \) along the unipotent radical of \( P \) is a closed form on \( P(\mathbb{Q}) \backslash G(\mathbb{A})/K_{\infty} \). One may suppose that \( P \) is one of \( B, P \) and \( Q \). The constant terms along \( B \) and \( Q \) are zero because of the cuspidality of \( \tau \).

We now compute the constant term along \( P \). Set

\[
w_1 = 1, \ w_2 = \left( -1, 1 \right) - \left( 1, 1 \right), \ w_3 = \left( 1, 1 \right), \ w_4 = \left( 1, 1 \right).
\]

There is \( G(\mathbb{Q}) = \bigcup_{j=1}^d P(\mathbb{Q})w_j P(\mathbb{Q}) \). Set \( U_j = w_j^{-1}Pw_j \cap U \) and \( M_j = w_j^{-1}Pw_j \cap M \), then \( P(\mathbb{Q})\omega_j P(\mathbb{Q}) \) is the disjoint union of \( P(\mathbb{Q})\omega_j \gamma_j \gamma_j' \) with \( \gamma_j \in U_j(\mathbb{Q}) \backslash U(\mathbb{Q}) \) and \( \gamma_j' \in M_j(\mathbb{Q}) \backslash M(\mathbb{Q}) \). So

\[
E_P(\eta^{\kappa, \omega}) = \int [U] L_u^* E(\eta^{\kappa, \omega}) du = \sum_j \sum_{\gamma_j' \gamma_j} \int [U] L_{\omega_j \gamma_j \gamma_j'}^* \eta^{\kappa, \omega} du.
\]

We make a change of variable \( u \to \gamma_j^{-1}u \gamma_j' \) and then combine the summation over \( \gamma_j \) with the integration over \( [U] \). It yields

\[
E_P(\eta^{\kappa, \omega}) = \sum_j \sum_{\gamma_j} \int_{U_j(\mathbb{Q}) \backslash U_j(\mathbb{A})} L_{\omega_j \gamma_j}^* \eta^{\kappa, \omega}.
\]  

(6.5)

When \( j = 2, 3 \), there exists a 1-dimensional subgroup \( U_j' \subset U_j \) satisfying \( w_j U_j' w_j^{-1} \subset M \), whence \( \int_{U_j(\mathbb{Q}) \backslash U_j(\mathbb{A})} L_{\omega_j \gamma_j}^* \eta^{\kappa, \omega} = 0 \) by the cuspidality of \( \tau \). So only the terms for \( j = 1, 4 \) remain in (6.5).

\[
E_P(\eta^{\kappa, \omega}) = \eta^{\kappa, \omega} + \int_{U(\mathbb{A})} L_{w_4}^* \eta^{\kappa, \omega} du
\]

We now show \( \eta' = \int_{U(\mathbb{A})} L_{w_4}^* \eta^{\kappa, \omega} du \) is closed. Write \( \omega = \sum_l F_l \omega_l \) with \( F_l \in \Pi(\tau, \frac{1}{2}) \) and \( \omega_l \in \wedge^3 \mathbb{P}^* \). Put \( F_{l, \kappa} = F_l(g)\kappa(\lambda(g)) \), then

\[
\eta^{\kappa, \omega} = \sum_l F_{l, \kappa}(g) \omega_l \land \eta_0,
\]

\[
\eta' = \sum_l \omega_l \land \left[ \int_{U(\mathbb{A})} L_{w_4}^* (F_{l, \kappa} \eta_0) du \right].
\]

Observe that

\[
d \left[ \int_{U(\mathbb{A})} L_{w_4}^* (F_{l, \kappa} \eta_0) du \right] = \int_{U(\mathbb{A})} d \left[ L_{w_4}^* (F_{l, \kappa} \eta_0) \right] du.
\]

Here one can change the order of differentiation and integration because \( \kappa \) is compactly supported and \( F_l \) is cuspidal on \( M(\mathbb{A}) \). So \( d\eta' \) equals

\[
\sum_l d \omega_l \land \left[ \int_{U(\mathbb{A})} L_{w_4}^* (F_{l, \kappa} \eta_0) du \right] - \sum_l \omega_l \land d \left[ \int_{U(\mathbb{A})} L_{w_4}^* (F_{l, \kappa} \eta_0) du \right]
\]

\[
= \sum_l \int_{U(\mathbb{A})} d \omega_l \land L_{w_4}^* (F_{l, \kappa} \eta_0) - \omega_l \land d \left[ L_{w_4}^* (F_{l, \kappa} \eta_0) \right]
\]

\[
= \sum_l \int_{U(\mathbb{A})} d \left[ L_{w_4}^* (\eta^{\kappa, \omega}) \right] du.
\]

Because \( \eta^{\kappa, \omega} \) is closed, there is \( d \left[ L_{w_4}^* (\eta^{\kappa, \omega}) \right] = L_{w_4}^* (d\eta^{\kappa, \omega}) = 0 \), whence \( d\eta' = 0 \) and \( \eta' \) is closed. Thus, \( E(\eta^{\kappa, \omega}) \) is closed. □
6.2. Properties of $E(\eta^{\kappa, \omega})$.

Lemma 13. $< H^{1,1}(\pi'_j), E(\eta^{\kappa, \omega}) > = 0$ for $\pi'_j \neq \pi_j$.

Proof. By Section 3 nonzero $H^{1,1}(\pi'_j)$ are of the form

$$H^{1,1}(\pi'_j) = \chi'(\nu(g)) \cdot H^{1,1}(g, K_\infty, \pi'')$$

where $\chi'$ is a character with $\chi'_\infty \in \{1, \text{sgn}\}$ and $\pi''$ is of type I, II or III. Hence a form $\omega'$ in $H^{1,1}(\pi'_j)$ is of the shape

$$\omega' = \chi'(\nu(g)) \sum_j \varphi_j \omega_j, \quad \varphi_j \in \pi'', \omega_j \in \wedge^2 p^*.$$ 

Write $E(\eta^{\kappa, \omega}) = \sum_i \int_{A(\mathbb{R})} E(\kappa \cdot c_i F_i) \varphi_i \omega_i \wedge \omega_i$ as in (6.31) of Remark 5 with $\omega_i \in \wedge^3 p^*$, $\omega_i \in p^*$, $F_i(g) \in \Pi(\tau, \frac{1}{2})$, and $c_i(g)$ left invariant by $P(\mathbb{A})$. It follows that

$$< \omega', E(\eta^{\kappa, \omega}) > = \sum_{i,j} \int_M \chi'(\nu(g)) \varphi_j(g) E(\kappa \cdot \lambda \cdot F_j) \varphi_i(g) \omega_j \wedge \omega_j \wedge \omega_i.$$ 

The form $\omega_j \wedge \omega_j \wedge \omega_i$ is either zero or a scalar multiple of the volume form on $M$. So $< \omega', E(\eta^{\kappa, \omega}) >$ is a finite sum of integrals of the following type,

$$\int_G (Q_{(\mathbb{R})}) \chi'(\nu(g)) \varphi(g) E(\kappa \cdot \lambda \cdot F) dg$$

where $\varphi \in \pi''$, $F \in \Pi(\tau, \frac{1}{2})$ and $\varphi P(g) := \int_M \varphi(ug) du$ refers to the constant term of $\varphi$ along $P$. (Note that $c_i(g) F_i(g) \in \Pi(\tau, \frac{1}{2})$.)

(i) When $\pi''$ is of type I, $\varphi_P = 0$ and the above integral vanishes. When $\pi''$ is of type III, $\varphi$ is constant and the above integral vanishes because $F$ is cuspidal when restricted to $M(\mathbb{A})$. So $< \omega', E(\eta^{\kappa, \omega}) > = 0$.

(ii) $\pi''$ is of type II. So $\pi'' = J(P, \tau'', \frac{1}{2})$ with $\tau'' \subset A_{\text{cusp}}(\text{PGL}_2)$ and $L(\frac{1}{2}, \tau'') \neq 0$. For $\varphi \in \pi''$, the constant term $\varphi_P$ belongs to the space $\Pi(\tau'', -\frac{1}{2})$. Writing elements in $G(\mathbb{A})$ as $g = (\begin{vmatrix} A & x \end{vmatrix} \cdot k \in k, \mathbb{K})$, we can rewrite the integral in (6.6) as

$$\int_K \int_{\mathcal{M}(Q_{(\mathbb{R})})} \chi'(x) \kappa(| \det A x |) f_k''(A) f_k(A) d^x A d^x x,$$

with

$$f_k''(A) = | \det A |^{-1} \varphi_P \left( \begin{vmatrix} A & x \end{vmatrix} \cdot k \in \tau'', f_k(A) = | \det A |^{-2} F \left( \begin{vmatrix} A & x \end{vmatrix} \cdot k \in \tau. \right.$$ 

Setting $x = x \det A$, we can turn the inner integral in (6.7) into

$$\int_{Q_x \setminus A^x} \chi'(x) \kappa(| x' |^{-1}) d^x x \cdot \int_{Q_x \mathcal{K} \setminus A^x} \chi'(z) d^x z \int_{[\text{PGL}_2]} f_k''(A) f_k(A) d^x A.$$ 

For the above expression to be nonzero, it is necessary that $\chi' = 1$ and $\tau'' = \tau' = \tau$. Thus, for (6.7) and (6.6) to be nonzero, it is necessary that $\pi' = \pi$. Therefore, $< \omega', E(\eta^{\kappa, \omega}) > = 0$ for $\pi'_j \neq \pi_j$. □

Lemma 14. Suppose $\tau \in A_{\text{cusp}}(\text{PGL}_2)$ with $\tau_\infty = \mathcal{D}_4$ and $L(\frac{1}{2}, \tau) \neq 0$. There exists $\kappa \in C^\infty_c(\mathbb{R}_+)$, $\omega \in H^3(\mathbb{R}, K_\infty, \Pi(\tau, \frac{1}{2}))$ and a sufficiently small $K_f$ such that $\int_{Z_{H,K_f}} E(\eta^{\kappa, \omega}) \neq 0$.

Proof. The map $\phi_f \in \Pi(\tau, \frac{1}{2}) \to \omega \in H^3(\mathbb{R}, K_\infty, \Pi(\tau, \frac{1}{2}))$ in Section 6.1.2 guides the choice of $\omega$. To a decomposable vector $v_f = \otimes v_p \in \tau_f$, we associate a specific section $\phi(v_f) = \otimes \phi_p \in \Pi(\tau_f, \frac{1}{2})$: let $S(v_f)$ be the set of finite places of $\mathbb{Q}$ such that $v_p$ is spherical; for $p \in S(v_f)$, choose the $\phi_p$
with \( \phi_p|_{K_p} = v_p \); for \( p \not\in S(v_f) \), set \( \overline{U} = J_{2}^{-1}U J_2 \) and let \( \phi_p \) be the one supported in \( P(\mathbb{Q}_p) \overline{U}(\mathbb{Q}_p) \) with

\[
\phi_p \left( \begin{array}{c} l_2 \\ n \\ l_2 \end{array} \right) = \begin{cases} v_p, & n \in \text{Sym}_{2 \times 2}(\mathbb{Z}_p), \\ 0, & n \not\in \text{Sym}_{2 \times 2}(\mathbb{Z}_p). \end{cases}
\]

(i) Choose a decomposable \( v_f \in \tau_f \), set \( \phi_f = \phi_{v_f} \) and let \( \omega \) be the form associated to \( \phi_f \).

Choose \( K_f \) sufficiently small so that \( \phi_f \) is \( K_f \)-invariant. Because \( H(\mathbb{Q}) \) acts transitively on \( P(\mathbb{Q}) \setminus G(\mathbb{Q}) \), there is

\[
\int_{Z_H K_f} E(\eta^\kappa \omega) = \int_{H(\mathbb{Q}) \setminus H(\mathbb{Q})/K_{H, \infty}} \sum_{\gamma \in P(\mathbb{Q}) \setminus G(\mathbb{Q})} L^* \eta^\kappa \omega dg_f = \int_{P_H(\mathbb{Q}) \setminus H(\mathbb{Q})/K_{H, \infty}} \eta^\kappa \omega df_f. \tag{6.8}
\]

Note that \( H(\mathbb{R})/K_{H, \infty} = P_H(\mathbb{R})/(K_{H, \infty} \cap P_H(\mathbb{R})) \) and that \( \eta^\kappa \omega \) is right-invariant by \( K_{H, \infty} \cap P_H(\mathbb{R}) = Z(\mathbb{R})^+ \{ I_4, \text{diag} (1, -1, 1, -1) \} \).

So

\[
\int_{P_H(\mathbb{Q}) \setminus H(\mathbb{Q})/K_{H, \infty}} \eta^\kappa \omega df_f = 2 \int_{P_H(\mathbb{Q}) \setminus H(\mathbb{Q})/K_{H, \infty}} \eta^\kappa \omega df_f \tag{6.9}
\]

(ii) Recall that \( \eta^\kappa \omega = \kappa(\lambda(g)) \omega \wedge \eta_\circ \). Also recall from Section 6.1.2 that \( \omega = F_+(g) \eta^- \wedge \eta_0 \). The right translation by \( \text{diag}(1, -1, 1, -1) \) sends \( F_+(g) \eta^- \wedge \eta_0 \) to \( F_-(g) \eta^+ \wedge \eta_0 \), and vice versa. So the RHS of (6.9) is equal to

\[
4 \int_{P_H(\mathbb{Q}) \setminus H(\mathbb{Q})/K_{H, \infty}} \kappa(\lambda(h)) F_+(h) \eta^- \wedge \eta_0 \cdot dh_f \tag{6.10}
\]

The form \( (\eta^- \wedge \eta_0)|_{P_H(\mathbb{R})} = 2i a^* \wedge h^* \wedge n_1^* \wedge n_2^* \) represents a left Haar measure on \( P_H(\mathbb{R}) \). Write it as \( c_H dp_\infty^\kappa \), then (6.10) is equal to

\[
4 c_{H, \infty} \int_{P_H(\mathbb{Q}) \setminus H(\mathbb{Q})/K_{H, \infty}} \kappa(\lambda(h)) F_+(p'_\infty, h_f) dp'_\infty dh_f. \tag{6.11}
\]

(iii) Set \( \overline{U}_H := \overline{U} \cap H \) and \( H_{v_f} := P_H(H_k) \cdot H_{v_f} \), with

\[
H_{v_f} = \left( \prod_{p \not\in S(v_f)} \overline{U}_H(\mathbb{Z}_p) \right) \times \left( \prod_{p \in S(v_f)} \mathbb{K}_H(p) \right).
\]

By the choice of \( \phi_f \), the function \( F_+(p'_\infty, h_f) \) is supported in \( P_H(\mathbb{R}) \times H_{v_f} \) and is right invariant by \( H_{v_f}' \). So (6.11) is equal to

\[
4 c_{H, \infty} \text{Vol}(H_{v_f}) \int_{P_H(\mathbb{Q}) \setminus H(\mathbb{Q})/K_{H, \infty}} \kappa(\lambda(p')) F_+(p') dp' \tag{6.12}
\]

Write \( m' = \text{diag}(a_1, a_2, a_0^{-1} a_1, a_0^{-1} a_2) \) and \( f_+ = [v_f \otimes v_f] \in \tau_f \), then

\[
\kappa(\lambda(m')) = \kappa(\frac{a_1 a_2}{a_0}), \quad F_+(m') = \lambda(m')^2 f_+ (a_1, a_2).
\]

By changing variables, one can simplify (6.12) and turn it into

\[
4 c_{H, \infty} \text{Vol}(H_{v_f}) \text{Vol}(Q^\times R_+ \setminus A^\times)^2 \int_{R_+} \kappa(t) dt \int_{Q^\times \setminus A^\times} f_+ (a_1) d^\times a_1.
\]
Thus, setting $C = 4c_{H,∞} \text{Vol}(H_f')c^2$, we have

$$\int_{Z_{H,K_f}} E(\eta^{κ,ω}) = C \int_{\mathbb{R}_+} \kappa(t) d^2 t \cdot \int_{Q^x \setminus A^x} f_+^{(a_1)} d^x a_1.$$  

(iv) When $L(\frac{1}{2}, \tau) \neq 0$, the integral $\int_{Q^x \setminus A^x} f_+^{(a_1)} d^x a_1$ is nonvanishing on $\nu \otimes T_f$. (This is well-known and follows from the Jacquet-Langlands theory [9] for $L$-functions of $GL(2)$-representations.) Choose $\nu_f$ such that this integral is nonzero and choose $κ$ such that $\int_{\mathbb{R}_+} \kappa(t) d^2 t \neq 0$, then the period $\int_{Z_{H,K_f}} E(\eta^{κ,ω})$ is nonzero. □

7. Nonvanishing Periods III

We verify Proposition [2] for $π = 1$. As in Section [6], we consider the split divisor $Z_{H,K_f}$ associated to the group $H \subset G$ and construct a form $Ω$ by using Eisenstein cohomology. Note that $H^2(\mathfrak{g}, K_∞, 1) = \mathbb{C}ω_0$ with

$$ω_0 = h^* \wedge n_2 + \frac{1}{2} n_0^* \wedge n_3^* + a^* \wedge n_1.$$  

Note that $ω_0 \in ∧^2 b_0^* \cong ∧^2 p^*$ is $K_∞$-invariant.

**Proposition 5.** There exists a $\mathbb{K}_f$-finite rapidly decreasing closed form $Ω$ on $M$ such that (i) $\langle H^1, 1(π_f') \rangle$, $Ω > 0$ for $π'_f \neq 1$, (ii) $\int_{Z_{H,K_f}} Ω \neq 0$.

We prove Proposition [4] at the end of this section. Here is an immediate corollary.

**Corollary 4.** Let $χ$ be a character of $Q^x R_+ \setminus A^x$ and $π = \chi \circ ν$, then $H^{1,1}(π_f)$ is spanned by the image of certain split special divisor.

7.1. Eisenstein cohomology. Let $N$ be the unipotent radical of $B$ and $A$ be the diagonal subgroup of $B$. Set $K_{A,∞} = K_∞ \cap A(\mathbb{R})$.

To $τ ∈ C^∞(\mathbb{R}_+ \times \mathbb{R}_+)$, we associate a function $f^τ$ on $B(\mathbb{Q}) \setminus G(\mathbb{A}) / K_∞$:

$$f^τ(g) = τ(\frac{a_1}{a_2}, |\frac{a_1}{a_2}|),$$  

for $g = nak$ with $n ∈ N(\mathbb{A})$, $a = \text{diag}(a_1, a_2, a_0, a_0) ∈ A(\mathbb{A})$, and $k ∈ \mathbb{K}$.

To $τ_1, τ_2 ∈ C^∞(\mathbb{R}_+ × \mathbb{R}_+)$, we associate a differential form $η^{τ_1, τ_2}$ on $B(\mathbb{Q}) \setminus B(\mathbb{R}) \times G(\mathbb{A}_f) / K_{A,∞}$:

$$η^{τ_1, τ_2}(g) = f^{τ_1}(g) a^* \wedge h^* \wedge n_1^* \wedge n_2^* + f^{τ_2}(g) a^* \wedge n_0^* \wedge (n_1^* - n_2^*) \wedge n_3^*.$$  

Because $B(\mathbb{R}) / K_{A,∞} = G(\mathbb{R}) / K_∞$, the form $η^{τ_1, τ_2}$ can be regarded as a differential form on $B(\mathbb{Q}) \setminus B(\mathbb{R}) \times G(\mathbb{A}_f) / K_∞$. Its pullback to $G(\mathbb{A})$ is

$$η^{τ_1, τ_2}(b_∞, g_f) = R_{K∞}(η^{τ_1, τ_2}(b_∞, g_f)),$$  

for $b_∞ ∈ B(\mathbb{R})$, $k_∞ ∈ K_∞$, and $g_f ∈ G(\mathbb{A}_f)$.

Imitating the construction of Eisenstein series, we define

$$E(η^{τ_1, τ_2}) = \sum_{γ ∈ B(\mathbb{Q}) \setminus G(\mathbb{Q})} L^*_γ(η^{τ_1, τ_2}).$$  

**Remark 6.** $η^1 := a^* \wedge h^* \wedge n_1^* \wedge n_2^*$ and $η^2 := h^* \wedge n_0^* \wedge (n_1^* - n_2^*) \wedge n_3^*$ are left-invariant differential forms on $B(\mathbb{Q}) \setminus B(\mathbb{R}) \times G(\mathbb{A}_f) / K_{A,∞}$. We use the same notations to denote their pullback to $G(\mathbb{A})$. Choose a basis $\{ω_I\}$ of $∧^2 \mathbb{P}^*$ and write $η^i = ∑_I C_{i,I}(g)ω_I$, then $C_{i,I}(g)$ are left $B(\mathbb{A})$-invariant and right $\mathbb{K}_f$-invariant because $η^i$ are so. There is

$$E(η^{τ_1, τ_2}) = ∑_I E(C_{i,I} f^τ)(ω_I).$$  

(7.1)

Here $E(C_{i,I} f^τ) := ∑_{γ ∈ B(\mathbb{Q}) \setminus G(\mathbb{A})} C_{i,I}(γ g) f^τ(γ g)$ are Borel-type pseudo-Eisenstein series and are rapidly decreasing. So $E(η^{τ_1, τ_2})$ is rapidly decreasing.

**Lemma 15.** $η^{τ_1, τ_2}$ is closed if and only if $τ_1 = 2τ_2 - 2τ_1 \frac{∂_1}{∂_2} + 2τ_2 \frac{∂_2}{∂_2}$. 

Proof. One needs to calculate \( d(\eta^{1,2}) \). Recall that for a differential form \( \Omega \) of degree \( m \), \( d\Omega \) is defined by the expression

\[
d\Omega(X_0, \cdots, X_m) = \sum_{i=0}^{m} (-1)^i X_i \left( \Omega(X_0, \cdots, \hat{X}_i, \cdots, X_m) \right)
+ \sum_{0 \leq i < j \leq m} (-1)^{i+j} \Omega([X_i, X_j], X_0, \cdots, \hat{X}_i, \cdots, \hat{X}_j, \cdots, X_m),
\]

where \( X_0, \cdots, X_m \) are smooth vector fields.

\( \{a^*, h^*, n_0^*, n_1^*, n_2^*, n_3^*\} \) is a frame of the cotangent bundle on \( G(\mathbb{R})/K_\infty = B(\mathbb{R})/K_{A,\infty} \). Direct calculation shows that

\[
d(a^*) = d(h^*) = 0, \quad d(n_0^*) = -2h^* \wedge n_0^*,
\]

\[
d(n_1^*) = -2a^* \wedge n_1^* - 2h^* \wedge n_2^* - n_0^* \wedge n_3^*,
\]

\[
d(n_2^*) = -2a^* \wedge n_2^* - 2h^* \wedge n_1^* - n_0^* \wedge n_3^*,
\]

\[
d(n_3^*) = -2a^* \wedge n_3^* - n_0^* \wedge (n_1^* + n_2^*),
\]

\[
d(f^\tau) = f^{2t_1} \frac{\partial}{\partial t_1} a^* + f^{2t_2} \frac{\partial}{\partial t_2} h^*.
\]

It follows that

\[
d(n_1^* \wedge n_2^*) = 4a^* \wedge n_1^* \wedge n_2^* - n_0^* \wedge (n_1^* - n_2^*) \wedge n_3^*;
\]

\[
d(n_0^* \wedge (n_1^* - n_2^* \wedge n_3^*) = -4a^* \wedge n_0^* \wedge (n_1^* - n_2^*) \wedge n_3^*.
\]

Hence

\[
d(\eta^{1,2}) = (-f^{\tau_1} - f^{2t_1} \frac{\partial}{\partial t_1} - 2t_2 \frac{\partial}{\partial t_2} + 2f^{\tau_2}) a^* \wedge h^* \wedge n_0^* \wedge (n_1^* - n_2^*) \wedge n_3^*
\]

\[
= -f^{\tau_1 - 2t_1} \frac{\partial}{\partial t_1} + 2t_2 \frac{\partial}{\partial t_2} + 2f^{\tau_2} a^* \wedge h^* \wedge n_0^* \wedge (n_1^* - n_2^*) \wedge n_3^*.
\]

Therefore, \( d(\eta^{1,2}) = 0 \) if and only \( \tau_1 = 2\tau_2 - 2t_1 \frac{\partial}{\partial t_1} + 2t_2 \frac{\partial}{\partial t_2} \).

\[\square\]

Lemma 16. \( E(\eta^{1,2}) \) is closed when \( \tau_1 = 2\tau_2 - 2t_1 \frac{\partial}{\partial t_1} + 2t_2 \frac{\partial}{\partial t_2} \).

Proof. Express \( E(\eta^{1,2}) \) using equation (1.1). We first use reduction theory to show that the summation defining \( E(C_{1,f}^{\tau_1}) \) is locally finite. Write \( G(\mathbb{A})^1 = \{ g \in G(\mathbb{A}) : |\nu(g)| = 1 \} \), then \( G(\mathbb{A}) = Z(\mathbb{R})^+ G(\mathbb{A})^1 \).

For \( c > 0 \), let \( \mathcal{S}(c) \) be the set of \( g \in G(\mathbb{A})^1 \) of the form \( n_k \mathfrak{a}(\mathbb{A}), a \in \mathfrak{a}(\mathbb{A}), k \in \mathbb{K} \) and \( \frac{\nu_a}{|a|}, \frac{\nu_b}{|b|} \geq c \), then

(i) \( G(\mathbb{A})^1 = G(\mathbb{Q}) \mathcal{S}(c) \) when \( c \) is small enough.

(ii) the set \( \{ \gamma \in G(\mathbb{Q}) : \gamma \mathcal{S}(c) \cap \mathcal{S}(c') \neq \emptyset \} \) for given \( c, c' \) is a finite union of cosets in \( B(\mathbb{Q}) \) \( G(\mathbb{Q}) \).

Given \( \tau_1, \tau_2 \), there exists \( c' \) such that \( f^{\tau_i}(g) = 0 \) \( (i = 1, 2) \) when \( g \notin Z(\mathbb{R})^+ \mathcal{S}(c') \). Choose a small \( c \) so that \( G(\mathbb{A})^1 = G(\mathbb{Q}) \mathcal{S}(c) \). By (ii), there are only finitely many \( \gamma_j \in B(\mathbb{Q}) \setminus G(\mathbb{Q}) \) such that \( \gamma_j \mathcal{S}(c) \cap \mathcal{S}(c') \neq \emptyset \), whence \( E(C_{1,f}^{\tau_1})(g) = \sum_j \gamma_j f^{\tau_1} \gamma_j g \) is a finite summation for \( g \in \mathcal{S}(c) \). It follows that \( E(\eta^{1,2}) = \sum_j L_{\gamma_j}(\eta^{1,2}) \) is a finite summation for \( g \in \mathcal{S}(c) \).

When \( \tau_1 = 2\tau_2 - 2t_1 \frac{\partial}{\partial t_1} + 2t_2 \frac{\partial}{\partial t_2} \), Lemma 14 tells that \( \eta^{1,2} \) is a closed form on \( G(\mathbb{A})/K_\infty \), whence each \( L_{\gamma_j}(\eta^{1,2}) \) is closed. So \( E(\eta^{1,2}) \) is a closed form on \( \mathcal{S}(c)^o \), the interior of \( \mathcal{S}(c) \). Since \( G(\mathbb{A})^1 = G(\mathbb{Q}) \mathcal{S}(c)^o \) when \( c \) is small enough, \( E(\eta^{1,2}) \) is closed on \( G(\mathbb{A})/K_\infty \).

\[\square\]

7.2. Properties of \( E(\eta^{1,2}) \). We suppose \( \tau_1 = 2\tau_2 - 2t_1 \frac{\partial}{\partial t_1} + 2t_2 \frac{\partial}{\partial t_2} \).

Lemma 17. \( \langle H^{1,1}(\pi'_f), E(\eta^{1,2}) \rangle = 0 \) when \( \pi'_f \neq 1 \).

Proof. By Section 3 a nonzero \( H^{1,1}(\pi'_f) \) is of the form

\[
H^{1,1}(\pi'_f) = \chi'(\nu(g)) \cdot H^{1,1}(g, K_\infty, \pi''),
\]

where \( \chi' \) is a character with \( \chi'_\infty \in \{ 1, \text{sgn} \} \) and \( \pi'' \) is of type I, II or III.
Lemma 18. 

(i) If \( \pi'' = 1 \) is of type III, then \( H^{1,1}(\pi'') = \mathbb{C} \cdot \chi'(\nu(g))\omega_0 \). Since \( \pi'_f \neq 1 \), there is \( \chi'_f \neq 1 \). By unfolding the integral, we have

\[
< \chi'(\nu(g))\omega_0, E(\eta^{\tau_1, \tau_2}) > = \int_{B(\mathbb{Q}) \setminus G(\mathbb{A})/K_{A,\infty}} \chi'(\nu(g))\left( \frac{1}{2}f^{\tau_1}(g) + f^{\tau_2}(g) \right)\omega_p,
\]

where \( \omega_p := a^* \wedge h^* \wedge n^*_0 \wedge n^*_1 \wedge n^*_2 \wedge n^*_3 \) is the volume form on \( B(\mathbb{R})/K_{A,\infty} \). Noticing \( |K_{A,\infty}/Z(\mathbb{R})^+| = 4 \), we rewrite the RHS as

\[
4 \int_{B(\mathbb{Q})Z(\mathbb{R})\setminus G(\mathbb{A})} \chi'(\nu(g))\left( \frac{1}{2}f^{\tau_1}(g) + f^{\tau_2}(g) \right) dg
\]

(7.2)

Because \( \chi'_f \neq 1 \) and \( \chi'_\infty \in \{1, \text{sgn}\} \), there is \( \chi'|_{A^1_1} \neq 1 \). However, \( f^{\tau_i} \) are left-invariant under \( \text{diag}(a_1, a_2, \frac{a_1}{1}, \frac{a_2}{1}) \) when \( a_i \in A^1_1 \). Hence, \( (7.2) \) vanishes because \( \int_{\mathbb{Q}^1 \setminus A^1_1} \chi'(z)d^x z = 0 \). So \( < H^{1,1}(\pi'_f), E(\eta^{\tau_1, \tau_2}) > = 0 \).

(ii) If \( \pi'' \) is of type I or II, then \( \pi''_{\infty} = \pi^{2+} \). By the description of \( H^{1,1}(g, K_{\infty}, \pi^{2+}) \) in Lemma 18, forms in \( H^{1,1}(\pi'_f) \) are of the shape

\[
\omega' = \chi'(\nu(g)) \sum_{j=-2}^2 \varphi_j \eta_{-j}, \quad \varphi_j \in \pi'', \eta_j \in \mathbb{A}^2_1.
\]

(7.3)

Recalling the expression for \( E(\eta^{\tau_1, \tau_2}) \) in (7.1), we have

\[
\omega' \wedge E(\eta^{\tau_1, \tau_2}) = \sum_{i,j,t} \chi'(\nu(g))\varphi_j(g)E(C_{i,t}f^{\tau_t})(\eta_{-j} \wedge \omega_I).
\]

The form \( \eta_{-j} \wedge \omega_I \), being left-invariant and of degree 6, is either zero or a scalar multiple of the volume form \( \omega_I \) on \( M \). Thus, \( < \omega', E(\eta^{\tau_1, \tau_2}) >= \int_M \omega' \wedge E(\eta^{\tau_1, \tau_2}) \) is a finite sum of integrals of the following type: \( \varphi \in \pi'' \),

\[
\int_{G(\mathbb{Q})Z(\mathbb{R})\setminus G(\mathbb{A})} \chi'(\nu(g))\varphi(g)E(C_{i,t}f^{\tau_t})(g)dg
\]

\[
= \int_{B(\mathbb{Q})N(\mathbb{A})Z(\mathbb{R})\setminus G(\mathbb{A})} \chi'(\nu(g))\varphi_B(g)C_{i,t}(g)f^{\tau_t}(g)dg.
\]

Here \( \varphi_B(g) := \int_{\mathcal{N}(\mathbb{Q})\setminus \mathbb{A}(\mathbb{A})} \varphi(\nu g)dn \) is the constant term of \( \varphi \) along \( B \). When \( \pi'' \) is of type I or II, \( \varphi_B \) is zero and the above integral vanishes. Therefore, \( < H^{1,1}(\pi'_f), E(\eta^{\tau_1, \tau_2}) > = 0 \).

Lemma 18. Suppose \( \tau_i(t_1, t_2) = 0 \) when \( t_1 \in (0, 1) \) \( (i = 1, 2) \), then

\[
\int_{Z_{H, K_f}} E(\eta^{\tau_1, \tau_2}) = 8^3 \int_{R_+ \times R_+} \frac{\tau_1(t_1, t_2)}{|t_1t_2|^3} dt_1 dt_2.
\]

Proof. Because \( H(\mathbb{Q}) \) acts transitively on \( P(\mathbb{Q}) \setminus G(\mathbb{Q}) \), there is

\[
\int_{Z_{H, K_f}} E(\eta^{\tau_1, \tau_2}) = \int_{P_H(\mathbb{Q}) \setminus H(\mathbb{A})/K_{H, \infty}} \sum_{\gamma \in B(\mathbb{Q}) \setminus P(\mathbb{Q})} L_\gamma^{*}[\eta^{\tau_1, \tau_2}] d\gamma
\]

\[
= 4 \int_{P_H(\mathbb{Q})Z(\mathbb{R})\setminus P_H(\mathbb{A})} \sum_{\gamma \in B(\mathbb{Q}) \setminus P(\mathbb{Q})} L_\gamma^{*}[\eta^{\tau_1, \tau_2}] d\gamma.
\]

(7.4)

(i) The coset space \( B(\mathbb{Q}) \setminus P(\mathbb{Q}) \) is parameterized by 1 and \( \gamma_\delta \) \( (\delta \in \mathbb{Q}) \), with \( \gamma_\delta = \left( \frac{\beta_\delta}{\beta_{\delta-1}} \right) \) and \( \beta_\delta = \left( \begin{smallmatrix} 1 & 1 \\ -1 & 0 \end{smallmatrix} \right) \). One needs to restrict

\[
L_\gamma^{*}[\eta^{\tau_1, \tau_2}] = f^{\tau_1}(\gamma g)L_\gamma^{*}\eta^{\gamma 1} + f^{\tau_2}(\gamma g)L_\gamma^{*}\eta^{\gamma 2}
\]

to \( P_H(\mathbb{A}) \), that is, to restrict \( L_\gamma^{*}\eta^{\gamma} \) to \( P_H(\mathbb{R}) \). By definition,

\[
(L_\gamma^{*}\eta^{\gamma})|_{p_{\infty}^{'}} = L_\gamma^{*}(\eta^{\gamma}(p_{\infty}')) \quad p_{\infty}' \in P_H(\mathbb{R}).
\]

(7.5)
Write \( \gamma p'_{\infty} = p_{\infty}k_{\infty} \) with \( p_{\infty} \in B(\mathbb{R}) \) and \( k_{\infty} \in K_{\infty} \), then
\[
\eta^i(p_{\infty}k_{\infty}) = R^i_{k_{\infty}^{-1}}\eta^i(p_{\infty}). \tag{7.6}
\]
For \( \eta = \sum_{I} \omega_I \) with \( \omega_I \in \Lambda^4 p^* \) on \( B(\mathbb{R}) \), there is \( R^*_i \eta = \sum_{I} \text{Ad}^*_{k_{\infty}} \omega_I \).

(i) When \( \gamma = 1 \), there are \( \eta_1|_{P_H(\mathbb{R})} = \eta_1 \) and \( \eta_2|_{P_H(\mathbb{R})} = 0 \) because \( n^*_0 \) and \( n^*_3 \) restrict to zero on \( P_H(\mathbb{R}) \).

(ii) Consider \( \gamma = \gamma_\delta \). Set \( k_\delta := \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \) and \( k(\theta) := \begin{pmatrix} k_\delta & 0 \\ \omega & 1 \end{pmatrix} \in K_{\mathbb{R}} \) for \( \theta \in \mathbb{R} \). For
\[
p'_{\infty} = \left[ \begin{pmatrix} \tau_1 & 0 \\ \rho \tau_1^{-1} \end{pmatrix}, \begin{pmatrix} \tau_2 & 0 \\ \rho \tau_2^{-1} \end{pmatrix} \right] \in P_H(\mathbb{R}) \subset H(\mathbb{R}),
\]
there is \( \gamma_\delta p'_{\infty} \in B(\mathbb{R})k(\theta) \) with \( \theta = -\tan^{-1} \frac{\tau_2}{\tau_1} \), whence
\[
(L_{\gamma_\delta}^* \eta^i)|_{p'_{\infty}} = \text{Ad}_{k(\theta)}^* \eta^i.
\]

(iii) The action of \( \text{Ad}_{k(\theta)} \) on \( b_0 \cong p \) is given by
\[
a \to a, \quad n_0 \to n_0, \\
h \to (\cos 2\theta)h - (\sin 2\theta)n_0, \quad n_1 \to n_1, \\
n_2 \to (\cos 2\theta)n_2 - (\sin 2\theta)n_3, \quad n_3 \to -(\sin 2\theta)n_2 + (\cos 2\theta)n_3.
\]

Hence \( \text{Ad}_{k(\theta)}^* \) acts on \( b_0 \) by
\[
a^* \to a^*, \quad n^*_0 \to n^*_0 - (\sin 2\theta)h^*, \\
h^* \to (\cos 2\theta)h^*, \quad n^*_1 \to n^*_1, \\
n^*_2 \to (\cos 2\theta)n^*_2 - (\sin 2\theta)n^*_3, \quad n^*_3 \to -(\sin 2\theta)n^*_2 + (\cos 2\theta)n^*_3.
\]

Therefore \( \text{Ad}_{k(\theta)}^* \eta^i = (\cos 2\theta)a^* \wedge h^* \wedge n^*_1 \wedge (\cos 2\theta \cdot n^*_2 - \sin 2\theta \cdot n^*_3) \). Note that \( n^*_3 \) restricts to zero on \( P_H(\mathbb{R}) \subset B(\mathbb{R}) \), so
\[
(L_{\gamma_\delta}^* \eta^i)|_{p'_{\infty}} = (\cos 2\theta)^2 a^* \wedge h^* \wedge n^*_1 \wedge n^*_2 = \frac{(\tau_2^2 - \tau_1^2 \delta^2)^2}{(\tau_1^2 + \tau_2^2 \delta^2)^2} \cdot \eta^i.
\]

Similarly, \( L_{\gamma_\delta}^* \eta^2|_{p'_{\infty}} = (\sin 2\theta)^2 a^* \wedge h^* \wedge n^*_1 \wedge n^*_2 = \frac{(2r_1 r_2 \delta)^2}{(r_1^2 + r_2^2 \delta^2)^2} \cdot \eta^1 \).

To summarize, we have \( L_{\gamma_\delta}^* \eta^i|_{p'} = f_{i,\delta}(p') \eta^1 \), where \( f_{i,\delta} \) are functions on \( P_H(\mathbb{A}) \) given by
\[
f_{1,\delta}(p') = \frac{(r_2^2 - r_1^2 \delta^2)^2}{(r_1^2 + r_2^2 \delta^2)^2}, \quad f_{2,\delta}(p') = \frac{(2r_1 r_2 \delta)^2}{(r_1^2 + r_2^2 \delta^2)^2},
\]
where \( p' \in P_H(\mathbb{A}) \) with \( p'_{\infty} = \left[ \begin{pmatrix} \tau_1 & 0 \\ \rho \tau_1^{-1} \end{pmatrix}, \begin{pmatrix} \tau_2 & 0 \\ \rho \tau_2^{-1} \end{pmatrix} \right] \).

Noticing that \( f_{1,0}(p') = 1 \) and \( f_{2,0}(p') = 0 \), we have
\[
\sum_{\gamma \in B(\mathbb{Q}) \setminus P(\mathbb{Q})} L_{\gamma}^i|_{\gamma_\delta} \eta_{\gamma_\delta}(p') = \left[ 2f^{\tau_1} + \sum_{\delta \in Q^\times} f_{1,\delta} f^{\tau_1}(\gamma_\delta \cdot) + f_{2,\delta} f^{\tau_2}(\gamma_\delta \cdot) \right](p') \eta^1.
\]

(ii) Now we compute the integral in \( (7.4) \). Note that \( \eta^1 \) is the volume form on \( Z(\mathbb{R})^+ \setminus P_H(\mathbb{R}) \).

By doing integration on \( U_H(\mathbb{Q}) \setminus U_H(\mathbb{A}) \) first, we turn the integral in \( (7.4) \) to
\[
\int_{M_H(\mathbb{Q})Z(\mathbb{R})^+ \setminus M_H(\mathbb{A})} \lambda(m')^{-2} \left[ 2f^{\tau_1} + \sum_{\delta \in Q^\times} f_{1,\delta} f^{\tau_1}(\gamma_\delta \cdot) + f_{2,\delta} f^{\tau_2}(\gamma_\delta \cdot) \right] (m')dm'. \tag{7.7}
\]

We analyze the integrands \( f^{\tau_1} \) and \( f_{1,\delta} f^{\tau_1}(\gamma_\delta \cdot) + f_{2,\delta} f^{\tau_2}(\gamma_\delta \cdot) \) separately.

(iia) By changing variables, one computes quickly that
\[
\int_{M_H(\mathbb{Q})Z(\mathbb{R})^+ \setminus M_H(\mathbb{A})} \frac{f^{\tau_1}(m')}{\lambda(m')^2} dm' = c^3 \int_{\mathbb{R}_+ \times \mathbb{R}_+} \frac{\tau_1(t_1, t_2)}{|t_1 t_2|^2} d^x t_1 d^x t_2.
\]
Remark 7. In the above computation, we only care about nonvanishing result and hence take \( \text{Vol}(K_f) = 1 \) for simplicity. In general, one needs to carefully choose the measure in order to get a precise equality.

Proof of Proposition 5. Set \( \tau_1(t_1, t_2) = t_1^2 \tilde{\tau}_1(t_1, t_2), \) then the condition \( \tau_1 = 2\tau_2 - 2t_1 \frac{\partial \tau_1}{\partial t_1} + 2t_2 \frac{\partial \tau_1}{\partial t_2} \) becomes \( \tilde{\tau}_1 = 2\tilde{\tau}_2 - 2t_1 \frac{\partial \tilde{\tau}_1}{\partial t_1} + 2t_2 \frac{\partial \tilde{\tau}_1}{\partial t_2}. \) When \( \tilde{\tau}_1 \) vanishes for \( t_1 \in (0, 1), \) Lemma 18 tells that

\[
\int_{Z_{H, K_f}} E(\eta^{\tau_1, \tau_2}) = 8c^3 \int_{R_+} \frac{\tau_1(t_1, t_2)}{|t_1|^{3/2}} dt_1 dt_2 = 8c^3 \int_{R_+} \frac{\tilde{\tau}_2(t_1, t_2)}{t_1 t_2} dt_1 dt_2.
\]

So we choose \( \bar{\tau} \in C^c_c((1, \infty)^2) \) with \( \int_{R_+} \bar{\tau} = 0 \) and set \( \tau_1 = t_1^2 \bar{\tau}(2\bar{\tau} - 2t_1 \frac{\partial \bar{\tau}}{\partial t_1} + 2t_2 \frac{\partial \bar{\tau}}{\partial t_2}), \) \( \tau_2 = t_1^2 \frac{\partial \bar{\tau}}{\partial t_2}. \) The form \( E(\eta^{\tau_1, \tau_2}) \) then meets the requirement of Proposition 5. \( \square \)

REFERENCES

[1] Théorie des topos et cohomologie étale des schémas. Tome 3. Lecture Notes in Mathematics, Vol. 305. Springer-Verlag, Berlin, 1973. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck et J. L. Verdier. Avec la collaboration de P. Deligne et B. Saint-Donat.

[2] A. Borel and N. Wallach. Continuous cohomology, discrete subgroups, and representations of reductive groups, volume 67 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, second edition, 2000.

[3] Armand Borel. Stable real cohomology of arithmetic groups. II. In Manifolds and Lie groups (Notre Dame, Ind., 1980), volume 14 of Progr. Math., pages 21–55. Birkhäuser Boston, Mass., 1981.
[4] Pierre Deligne. Travaux de Shimura. In Séminaire Bourbaki, 23ème année (1970/71), Exp. No. 389, pages 123–165. Lecture Notes in Math., Vol. 244. Springer, Berlin, 1971.

[5] Wee Teck Gan. The Saito-Kurokawa space of $PGSp_4$ and its transfer to inner forms. In Eisenstein series and applications, volume 258 of Progr. Math., pages 87–123. Birkhäuser Boston, Boston, MA, 2008.

[6] G. Harder. On the cohomology of discrete arithmetically defined groups. In Discrete subgroups of Lie groups and applications to moduli (Internat. Colloq., Bombay, 1973), pages 129–160. Oxford Univ. Press, Bombay, 1975.

[7] G. Harder, R. P. Langlands, and M. Rapoport. Algebraische Zyklen auf Hilbert-Blumenthal-Flächen. J. Reine Angew. Math., 366:53–120, 1986.

[8] Hongyu He and Jerome William Hoffman. Picard groups of Siegel modular 3-folds and $\theta$-liftings. J. Lie Theory, 22(3):769–801, 2012.

[9] H. Jacquet and R. P. Langlands. Automorphic forms on $GL(2)$. Lecture Notes in Mathematics, Vol. 114. Springer-Verlag, Berlin, 1970.

[10] Stephen S. Kudla. Algebraic cycles on Shimura varieties of orthogonal type. Duke Math. J., 86(1):39–78, 1997.

[11] Stephen S. Kudla and John J. Millson. The theta correspondence and harmonic forms. I. Math. Ann., 274(3):353–378, 1986.

[12] Stephen S. Kudla and John J. Millson. The theta correspondence and harmonic forms. II. Math. Ann., 277(2):267–314, 1987.

[13] Stephen S. Kudla and John J. Millson. Tubes, cohomology with growth conditions and an application to the theta correspondence. Canad. J. Math., 40(1):1–37, 1988.

[14] C. Mœglin and J.-L. Waldspurger. Spectral decomposition and Eisenstein series, volume 113 of Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge, 1995. Une paraphrase de l’Écriture [A paraphrase of Scripture].

[15] I. I. Piatetski-Shapiro. On the Saito-Kurokawa lifting. Invent. Math., 71(2):309–338, 1983.

[16] Dinakar Ramakrishnan. Problems arising from the Tate and Beilinson conjectures in the context of Shimura varieties. In Automorphic forms, Shimura varieties, and $L$-functions, Vol. II (Ann Arbor, MI, 1988), volume 11 of Perspect. Math., pages 227–252. Academic Press, Boston, MA, 1990.

[17] David A. Vogan, Jr. and Gregg J. Zuckerman. Unitary representations with nonzero cohomology. Compositio Math., 53(1):51–90, 1984.

[18] J.-L. Waldspurger. Correspondance de Shimura. J. Math. Pures Appl. (9), 59(1):1–132, 1980.

[19] Jean-Loup Waldspurger. Correspondances de Shimura et quaternions. Forum Math., 3(3):219–307, 1991.

[20] R. Weissauer. The Picard group of Siegel modular threefolds. J. Reine Angew. Math., 430:179–211, 1992. With an erratum: “Differential forms attached to subgroups of the Siegel modular group of degree two” [J. Reine Angew. Math. 391 (1988), 100–156; MR0961166 (89i:32074)] by the author.

[21] Rainer Weissauer. Differentialformen zu Untergruppen der Siegelschen Modulgruppe zweiten Grades. J. Reine Angew. Math., 391:100–156, 1988.

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