ALGEBRAIC HYPERBOLICITY OF VERY GENERAL HYPERSURFACES IN PRODUCTS OF PROJECTIVE SPACES

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Abstract. We study the algebraic hyperbolicity of very general hypersurfaces in $\mathbb{P}^m \times \mathbb{P}^n$ by using three techniques that build on past work by Ein, Voisin, Pacienza, Coskun and Riedl, and others. As a result, we completely answer the question of whether or not a very general hypersurface of bidegree $(a, b)$ in $\mathbb{P}^m \times \mathbb{P}^n$ is algebraically hyperbolic, except in $\mathbb{P}^3 \times \mathbb{P}^1$ for the bidegrees $(a, b) = (7, 3), (6, 3)$ and $(5, b)$ with $b \geq 3$. As another application of these techniques, we improve the known result that very general hypersurfaces in $\mathbb{P}^n$ of degree at least $2n - 2$ are algebraically hyperbolic when $n \geq 6$ to $n \geq 5$, leaving $n = 4$ as the only open case.

1. Introduction

A complex projective variety is algebraically hyperbolic if for some $\varepsilon > 0$ and some ample divisor $H$, every integral curve $C$ that lies in the variety satisfies

\[(1)\quad 2g(C) - 2 \geq \varepsilon \cdot \deg_H C,\]

where $g(C)$ is the geometric genus of the curve. It follows that varieties that contain any rational or elliptic curves are not algebraically hyperbolic.

Algebraic hyperbolicity was introduced by Demailly in [De95] as an algebraic analogue to Kobayashi hyperbolicity for complex manifolds, which was shown to be equivalent to Brody hyperbolicity when the manifold is compact [Br78]. A compact complex manifold $X$ is Brody hyperbolic if all holomorphic maps $\mathbb{C} \to X$ are constant, i.e. $X$ contains no entire curves. Demailly [De95] proved that for smooth projective varieties Kobayashi hyperbolicity implies algebraic hyperbolicity, and conjectured that the converse holds.

The algebraic and Brody hyperbolicity of very general degree $d$ hypersurfaces $X_d \subseteq \mathbb{P}^n$ is well-studied. (See for instance the surveys [Co05, De18, Vo03].) For $n = 3$, Xu [Xu94] proved that very general surfaces $X_d \subseteq \mathbb{P}^3$ of degree $d \geq 6$ are algebraically hyperbolic by using a degeneration argument. Then Coskun and Riedl [CR19a] improved Xu’s bound to $d \geq 5$ by considering the degrees of surface scrolls that contain any particular curve in $X_d$. Their result completes the classification in $\mathbb{P}^3$ since surfaces of degree at most 4 contain rational curves. For larger $n$, Clemens [Cl86] and Ein [Ei88] proved that $X_d$ is algebraically hyperbolic when $d \geq 2n$ for $n \geq 4$. The bound was subsequently improved by Voisin [Vo96, Vo98] to $d \geq 2n - 1$ for $n \geq 4$, and then by Pacienza [Pa04] and Clemens and Ran [ClR04] to $d \geq 2n - 2$ for $n \geq 6$. We confirm in this article that very general degree 8 hypersurfaces in $\mathbb{P}^5$ are algebraically hyperbolic, hence improving the previously-known result to $d \geq 2n - 2$ for $n \geq 5$ (see Theorem $3.21$). When $d \leq 2n - 3$, $X_d$ contains lines and hence, is not algebraically hyperbolic. Thus, the only remaining open question in $\mathbb{P}^n$ is the case of sextic threefolds.
The hyperbolicity of very general hypersurfaces has also been studied in other ambient varieties besides $\mathbb{P}^n$. Haase and Ilten [HI19] studied very general surfaces in Gorenstein toric threefolds by building on the focal loci techniques of Chiantini and Lopez [CL99]. As an application, they almost completely classified very general surfaces in the threefolds $\mathbb{P}^2 \times \mathbb{P}^1$, $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{P}^1 \times \mathbb{P}^n$ and the blowup of $\mathbb{P}^3$ at a point. Then, Coskun and Riedl completed this classification in [CR19b] as a result of their techniques for three-dimensional complex projective varieties that admit a group action with dense orbit. More recently, Robins [Ro21] almost completely classified very general surfaces in smooth projective toric threefolds with Picard rank 2 or 3 by combining Haase and Ilten’s focal loci techniques with a study of the combinatorics of such threefolds.

In this paper, we study very general hypersurfaces in $\mathbb{P}^m \times \mathbb{P}^n$ and prove the following main theorem. We assume throughout that $m \geq n$.

**Theorem 1.1.** A very general hypersurface in $\mathbb{P}^m \times \mathbb{P}^n$ of bidegree $(a, b)$ is

1. algebraically hyperbolic if
   1. $m + n \geq 5$: $a \geq 2m + n - 2$ and $b \geq m + 2n - 2$; or
   2. $(m, n) = (2, 2)$: $a \geq 5$ and $b \geq 5$; or
   3. $(m, n) = (3, 1)$: $a \geq 6$ and $b \geq 4$, or $a \geq 8$ and $b = 3$,

2. not algebraically hyperbolic if
   1. $m + n \geq 5$: either $a < 2m + n - 2$ or $b < m + 2n - 2$; or
   2. $(m, n) = (2, 2)$: either $a < 5$ or $b < 5$; or
   3. $(m, n) = (3, 1)$: either $a < 5$ or $b < 3$.

This completely resolves algebraic hyperbolicity of very general hypersurfaces in $\mathbb{P}^m \times \mathbb{P}^n$, except in $\mathbb{P}^3 \times \mathbb{P}^1$ for bidegrees $(a, b) = (7, 3), (6, 3)$ and $(5, b)$ with $b \geq 3$.

Note that the classification of very general surfaces in $\mathbb{P}^2 \times \mathbb{P}^1$ was already done by Haase and Ilten [HI19] and Coskun and Riedl [CR19b], while hypersurfaces in $\mathbb{P}^1 \times \mathbb{P}^1$ are curves, which are easy to classify. We give a description of the remaining open cases in $\mathbb{P}^3 \times \mathbb{P}^1$ in Remark 4.3.

1.1. **Organization.** In §2 we introduce the setup developed by Ein, Voisin, Pacienza, and others. In §3, we present the three techniques that are used in §4 to prove the main theorem. As another application of these techniques, we give a streamlined proof of the algebraic hyperbolicity of very general hypersurfaces in $\mathbb{P}^n$ of degree at least $2n - 1$ when $n \geq 4$ and degree at least $2n - 2$ when $n \geq 5$.

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2. **Setup**

We use a similar setup as [CR04, CR19a, CR19b, Pa03, Pa04, Vo96]. Denote by $\pi_1$ and $\pi_2$ the projection maps from $\mathbb{P}^m \times \mathbb{P}^n$. Let $H_1 = \pi_1^* \mathcal{O}_{\mathbb{P}^m}(1)$ and $H_2 = \pi_2^* \mathcal{O}_{\mathbb{P}^n}(1)$. Then $\text{Pic}(\mathbb{P}^m \times \mathbb{P}^n) = \mathbb{Z} \cdot H_1 \oplus \mathbb{Z} \cdot H_2$. For $a, b > 0$, denote the line bundle $aH_1 + bH_2$ by $\mathcal{E}$. Suppose that a very general section $X$ of $\mathcal{E}$ contains a curve $Y$ of geometric genus $g$ and degree $e$ with respect to the ample class $H_1 + H_2$. Denote $V = H^0(\mathbb{P}^m \times \mathbb{P}^n, \mathcal{E})$, and let $\mathcal{X}_V \subseteq (\mathbb{P}^m \times \mathbb{P}^n) \times V$ be the universal hypersurface of bidegree $(a, b)$. Let $T \subseteq \text{Hilb}(\mathbb{P}^m \times \mathbb{P}^n) \times V$ be an irreducible variety such that
a general element \((Y, F)\) of \(T\) projects to a point in \(\text{Hilb}(\mathbb{P}^m \times \mathbb{P}^n)\) representing a curve \(Y\) of geometric genus \(g\) and \((H_1 + H_2)\)-degree \(e\) that is contained in the hypersurface defined by \(F\), and

2. the projection map \(T \to V\) is dominant, and

3. \(T\) is stable under the \(G := GL_{m+1} \times GL_{n+1}\)-action on \(\text{Hilb}(\mathbb{P}^m \times \mathbb{P}^n) \times V\).

Let \(\mathcal{Y}_T \subseteq \mathcal{X}_T := \mathcal{X}_V \times_V T\) denote the universal curve over \(T\). By (3), there is a \(G\)-invariant subvariety \(U \subseteq T\) such that \(U \to V\) is étale, and we restrict the universal curve to \(\mathcal{Y}_U\). Finally after taking a \(G\)-equivariant resolution of \(\mathcal{Y}_U\) and restricting \(U\) to a \(G\)-invariant open set, we obtain a smooth family \(\mathcal{Y} \to U\) whose fibers are smooth curves mapping to curves of geometric genus \(g\) and \((H_1 + H_2)\)-degree \(e\). Similarly, pull back \(\mathcal{X}_T\) to \(U\) to get a smooth family \(\mathcal{X} \to U\) of hypersurfaces. Denote the projection maps by \(p_1 : \mathcal{X} \to U\) and \(p_2 : \mathcal{X} \to \mathbb{P}^m \times \mathbb{P}^n\). Denote by \(h : \mathcal{Y} \to \mathcal{X}\) the natural map, which is generically injective. Note that

\[
\text{codim}(\mathcal{Y} \subseteq \mathcal{X}) = m + n - 2.
\]

By \(G\)-invariance, we can define the vertical tangent sheaves \(T_{\mathcal{X}/\mathbb{P}^m \times \mathbb{P}^n}\) and \(T_{\mathcal{Y}/\mathbb{P}^m \times \mathbb{P}^n}\) by the following short exact sequences:

\[
\begin{align*}
0 & \to T_{\mathcal{X}/\mathbb{P}^m \times \mathbb{P}^n} \to T_{\mathcal{X}} \to p_2^*T_{\mathbb{P}^m \times \mathbb{P}^n} \to 0 \\
0 & \to T_{\mathcal{Y}/\mathbb{P}^m \times \mathbb{P}^n} \to T_{\mathcal{Y}} \to h^*p_2^*T_{\mathbb{P}^m \times \mathbb{P}^n} \to 0
\end{align*}
\]

Let \(t \in U\) be a general element. Denote the fibers of \(h^*\mathcal{Y}\) and \(\mathcal{X}\) over \(t\) by \(Y_t\) and \(X_t\) respectively, and the restriction of \(h\) by \(h_t : Y_t \to X_t\). Let us also define the normal bundles \(N_{h/\mathcal{X}}\) and \(N_{h_t/\mathcal{X}_t}\) by the following short exact sequences:

\[
\begin{align*}
0 & \to T_{Y_t} \to h^*T_{X_t} \to N_{h/\mathcal{X}} \to 0 \\
0 & \to T_{Y_t} \to h_t^*T_{X_t} \to N_{h_t/\mathcal{X}_t} \to 0
\end{align*}
\]

Taking degrees in the second short exact sequence above, we have

\[
2g(Y_t) - 2 - K_{X_t} \cdot h_t(Y_t) = \deg N_{h_t/\mathcal{X}_t}.
\]

Hence, one way to confirm algebraic hyperbolicity is to find a suitable lower bound for \(\deg N_{h_t/\mathcal{X}_t}\).

We denote by \(M_E\) the Lazarsfeld-Mukai bundle associated to \(E\), which is defined by the short exact sequence

\[
0 \to M_E \to H^0(\mathbb{P}^m \times \mathbb{P}^n, E) \otimes \mathcal{O}_{\mathbb{P}^m \times \mathbb{P}^n}^{ev} \to E \to 0.
\]

We can define similarly the Lazarsfeld-Mukai bundles \(M_{H_1}\) and \(M_{H_2}\) associated to \(H_1\) and \(H_2\) respectively. These bundles play a central role in the proof techniques in \S3.

The following result follows immediately from Proposition 2.1 in [CR19b] and summarizes the relationship between the vertical tangent sheaves, the normal bundles, and the Lazarsfeld-Mukai bundle \(M_E\).

**Proposition 2.1** (cf Proposition 2.1 in [CR19b]).

1. \(N_{h/\mathcal{X}}|_{Y_t} \simeq N_{h_t/\mathcal{X}_t}\).
2. The quotient \((h^*T_{\mathcal{X}/\mathbb{P}^m \times \mathbb{P}^n}) / (T_{\mathcal{Y}/\mathbb{P}^m \times \mathbb{P}^n})\) of vertical tangent sheaves is isomorphic to \(N_{h/\mathcal{X}}\).
3. The vertical tangent sheaf \(T_{\mathcal{X}/\mathbb{P}^m \times \mathbb{P}^n}\) on \(\mathcal{X}\) is isomorphic to the pullback \(p_2^*M_E\) of the Lazarsfeld-Mukai bundle.
3. Proof techniques

In order to estimate the positivity of $N_{h_t/X_t}$, we start with the notion of a section-dominating collection of line bundles introduced in \cite{CR19b}.

**Definition 3.1** (cf Definition 2.3 in \cite{CR19b}). Let $E$ be a vector bundle on a smooth, complex projective variety $A$. A collection of non-trivial, globally generated line bundles $L_1, \ldots, L_u$ is called a *section-dominating collection* of line bundles for $E$ if

1. $E \otimes L_i^\gamma$ is globally generated for every $1 \leq i \leq u$, and
2. the map

$$
\bigoplus_{i=1}^u \left( H^0(L_i \otimes I_p) \otimes H^0(E \otimes L_i^\gamma) \right) \to H^0(E \otimes I_p)
$$

is surjective at every point $p \in A$.

We can view a section-dominating collection of line bundles for $E$ as a collection of simpler building blocks for $E$, in the sense of the following proposition.

**Proposition 3.2** (cf Proposition 2.6 in \cite{CR19b}). Let $E$ be a globally generated vector bundle and $M_E$ the Lazarsfeld-Mukai bundle associated to $E$. Let $L_1, \ldots, L_u$ be a section-dominating collection of line bundles for $E$. Then for some integers $s_i$, there is a surjection

$$
\bigoplus_{i=1}^u M_{L_i}^{\otimes s_i} \to M_E
$$

induced by multiplication by some choice of basis elements of $H^0(E \otimes L_i^\gamma)$ for $0 \leq i \leq u$.

Example 2.4 in \cite{CR19b} shows that on $A = \mathbb{P}^m \times \mathbb{P}^n$, the line bundles $H_1, H_2$ form a section-dominating collection for $E = aH_1 + bH_2$. By Proposition 3.2, this gives a surjection induced by multiplication by generic polynomials in $H^0(E \otimes H_i^\gamma)$ for $i = 1, 2$. We can use this surjection to obtain a first pair of lower bounds $a \geq a_0$ and $b \geq b_0$ such that a very general hypersurface of bidegree $(a, b)$ is algebraically hyperbolic.

**Lemma 3.3.** Let $h_t : Y_t \to X_t$ be a curve in a very general hypersurface $X_t \subseteq \mathbb{P}^m \times \mathbb{P}^n$ of bidegree $(a, b)$. Let $N$ be a vector bundle on the curve. Suppose that there is a surjective map

$$
\beta : M_{H_1}^{\otimes t_1} \oplus M_{H_2}^{\otimes t_2} \to N
$$

for some $t_1, t_2$. Then,

$$
deg N \geq -t_1 \cdot (Y_t \cdot H_1) - t_2 \cdot (Y_t \cdot H_2).
$$

In particular, setting $N = N_{h_t/X_t}$, we obtain

$$
2g(Y_t) - 2 \geq (a - (m + 1 + t_1)) \cdot (Y_t \cdot H_1) + (b - (n + 1 + t_2)) \cdot (Y_t \cdot H_2).
$$

**Proof.** Since $\beta$ is surjective, we have

$$
deg N \geq \deg \beta \left( M_{H_1}^{\otimes t_1} \right) + \deg \beta \left( M_{H_2}^{\otimes t_2} \right)
$$

$$
\geq t_1 \cdot (\deg H_1 |_{Y_t}) + t_2 \cdot (\deg H_2 |_{Y_t})
$$

where the second inequality follows from Proposition 2.7 in \cite{CR19b}. If $N = N_{h_t/X_t}$, then by (3), we obtain

$$
2g(Y_t) - 2
$$

$$
= (K_{X_t} \cdot Y_t) + \deg N_{h_t/X_t}
$$

$$
\geq (a - (m + 1 + t_1)) \cdot (Y_t \cdot H_1) + (b - (n + 1 + t_2)) \cdot (Y_t \cdot H_2).
$$
Proposition 3.4. Suppose \( a \geq 2m + n \) and \( b \geq m + 2n \), then a very general hypersurface \( X \subseteq \mathbb{P}^n \times \mathbb{P}^n \) of bidegree \((a, b)\) is algebraically hyperbolic.

Proof. Suppose \( X \) is parametrized by \( t \in U \) and let \( h_t : Y_t \to X_t \) be a curve in \( X \). By Propositions 2.1 and 3.2, we have a surjection
\[
\alpha : M_{H_1} \oplus M_{H_2} \to M_{\mathcal{E}} \to N_{h_t/X_t}
\]
for some integers \( s_1, s_2 \). Since rank \( N_{h_t/X_t} = m + n - 2 \), we can take \( s_1 = s_2 = m + n - 2 \). Therefore, by \([6]\), we have
\[
2g(Y_t) - 2 \geq ((a - m - 1) - (m + n - 2)) \cdot (Y_t \cdot H_1) + ((b - n - 1) - (m + n - 2)) \cdot (Y_t \cdot H_2)
\]
\[
= (a - (2m + n - 1)) \cdot (Y_t \cdot H_1) + (b - (m + 2n - 1)) \cdot (Y_t \cdot H_2)
\]
which is at least \((Y_t \cdot H_1) + (Y_t \cdot H_2)\) when \( a \geq 2m + n \) and \( b \geq m + 2n \).

Remark 3.5. The proof of Proposition 3.4 is essentially the approach taken by Ein in \([Ei88, Ei91]\) to prove that if \( X \subseteq \mathbb{P}^n \) is a generic complete intersection of type \((d_1, \ldots, d_k)\) satisfying \( \sum_i d_i \geq 2n - k + 1 \), then the desingularization of every subvariety of \( X \) of general type. In particular, such varieties \( X \) are algebraically hyperbolic. The above proof is also in the same spirit as Theorem 3.6 in \([HI19]\), which was proved using focal loci techniques.

The lower bounds in Proposition 3.4 are within 1 or 2 of the lower bounds stated in Theorem 111(a). In the rest of this section, we present two techniques that improve the bounds in Proposition 3.4.

In the proof of Proposition 3.4 one sees that it is possible for the surjection (7) to be achieved with smaller \( s_1, s_2 \), which would give us better control of the positivity of \( N_{h_t/X_t} \). This motivates the following definition.

Definition 3.6. We say that a curve \( h_t : Y_t \to X_t \) is of type \((s_1, s_2)\) if the induced map
\[
\alpha : M_{H_1}^{\oplus s_1} \oplus M_{H_2}^{\oplus s_2} \to N_{h_t/X_t}
\]
is surjective, with no summand having torsion image. In particular, \( s_1 + s_2 \leq m + n - 2 \).

Note that a curve may be of several different types, since the integers \( s_1, s_2 \) as defined above may not be unique.

Remark 3.7. (cf \([Cl03, ClR04]\)) Due to the genericity of the polynomials inducing the map \( \alpha \), for a fixed \( 0 \leq u_2 \leq s_2 \), the rank \( \gamma(i) \) of \( \alpha(M_{H_1}^{\oplus s_1} \oplus M_{H_2}^{\oplus u_2}) \) is a strictly increasing function for \( 0 \leq i \leq s_1 \), while the increase \( \gamma(i + 1) - \gamma(i) \) in rank is a non-decreasing function of \( i \) for \( 0 \leq i \leq s_1 - 1 \). The analogous statement with a fixed \( 0 \leq u_1 \leq s_1 \) and varying \( 0 \leq j \leq s_2 \) is true as well.

Now, we give an outline of the next two techniques. When \( s_1 + s_2 \) is large, i.e. \( s_1 + s_2 > (m + n - 2)/2 \), there is a summand \( M_{H_i} \) whose image in a quotient of \( N_{h_t/X_t} \), induced by \( \alpha \) has rank one. In this case, \([\text{3.1}]\) shows that the curve \( h_t : Y_t \to X_t \) lies in a special surface in \( \mathbb{P}^n \times \mathbb{P}^n \) made of “lines” (see \([\text{3.1}]\) for definition) passing through each point of the curve. This surface is described in Lemma 3.8. This method produces an improvement in the estimate
for the positivity of $N_{h_t/X_t}$, which then improves the lower bounds for $a$ and $b$ in Proposition 3.4 by 1.

When $s_1$ (or $s_2$) achieves the maximum, i.e. $s_1$ (or $s_2$) equals $m + n - 2$, $N_{h_t/X_t}$ modulo torsion is a direct sum of $m + n - 2$ rank-one images of $M_{H_1}$ (or $M_{H_2}$) via $\alpha$. In this case, 3.2 shows that the curve lies in a special locus on the hypersurface, which turns out to be a curve of general type when $a = 2m + n - 2$ (or $b = m + 2n - 2$). This method improves the lower bounds for $a$ and $b$ in Proposition 3.3 by 2 when $m + n \geq 5$. When $(m, n) = (3, 1)$, this method improves the lower bound for $b$ in Proposition 3.4 by 2.

3.1. Scroll considerations. Let us define a $\mathbb{P}^m$-line (resp. $\mathbb{P}^n$-line) in $\mathbb{P}^m \times \mathbb{P}^n$ to mean an integral curve $L$ whose numerical class is $H_1^{m-1}H_2^n$ (resp. $H_1^mH_2^{n-1}$), and a line may refer to either a $\mathbb{P}^m$-line or a $\mathbb{P}^n$-line. A surface in $\mathbb{P}^m \times \mathbb{P}^n$ is called a $\mathbb{P}^m$-scroll (resp. $\mathbb{P}^n$-scroll) if there is a $\mathbb{P}^m$-line (resp. $\mathbb{P}^n$-line) through every point of the surface.

Lemma 3.8 (cf Lemma 2.12 in [CR19b]). A rank-one quotient $Q$ of $M_{H_1}|Y_t$ induces a $\mathbb{P}^m$-scroll containing $Y_t$ of $\pi_1^*\mathcal{O}_{\mathbb{P}^m}(1)$-degree equal to $\deg Q + (Y_t \cdot H_1)$.

Proof. Recall the Euler sequence on $\mathbb{P}^m$:

$$0 \to \Omega_{\mathbb{P}^m}(1) \to \mathcal{O}_{\mathbb{P}^m}^{m+1} \to \mathcal{O}_{\mathbb{P}^m}(1) \to 0.$$ 

The pullback of $\Omega_{\mathbb{P}^m}(1)$ via $\pi_1$ is isomorphic to the Lazarsfeld-Mukai bundle $M_{H_1}$. By pulling back the above short exact sequence via $\pi_1$ and then restricting to the curve $Y_t \subseteq X_t$, we get

$$0 \to M_{H_1}|Y_t \to H^0(\mathbb{P}^m \times \mathbb{P}^n, \pi_1^*\mathcal{O}_{\mathbb{P}^m}(1)) \otimes \mathcal{O}_{Y_t} \to \pi_1^*\mathcal{O}_{\mathbb{P}^m}(1)|Y_t \to 0.$$ 

We define $S$ and $Q'$ to be the sheaves that make this commutative diagram exact:

$$
\begin{array}{c}
0 & \to & 0 \\
& \downarrow & \downarrow \\
& S & \to \\
& \downarrow & \downarrow \\
0 & \to & M_{H_1}|Y_t \\
& \downarrow & \downarrow \\
& Q & \to \mathcal{O}_{Y_t}^{m+1} \\
& \downarrow & \downarrow \\
& \to \pi_1^*\mathcal{O}_{\mathbb{P}^m}(1)|Y_t \\
& \downarrow & \downarrow \\
& Q' & \to 0 \\
& \downarrow & \downarrow \\
0 & \to 0
\end{array}
$$

The triangle (*) induces a map $\gamma : Y_t \to G(1, m)$ that sends a point $p \in Y_t$ to a line in $\mathbb{P}^m$ given by the rank-two vector bundle $Q'$, which contains $\pi_1(p)$. $\gamma$ lifts to a map that sends $p \in Y_t$ to the $\mathbb{P}^m$-line $\pi_1^{-1}(\gamma(p)) \cap \pi_2^{-1}(\pi_2(p))$ which contains $p$, giving a $\mathbb{P}^m$-scroll containing $Y_t$. The $\pi_1^*\mathcal{O}_{\mathbb{P}^m}(1)$-degree of this surface follows from taking degrees in the above diagram. □

The following is the analogous result with $H_1$ and $H_2$ switched.

Lemma 3.9. A rank-one quotient $Q$ of $M_{H_2}|Y_t$ induces a $\mathbb{P}^m$-scroll containing $Y_t$ of $\pi_2^*\mathcal{O}_{\mathbb{P}^m}(1)$-degree equal to $\deg Q + (Y_t \cdot H_2)$. 

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Lemma 3.11. Suppose that $h_t : Y_t \to X_t$ is a curve of type $(s_1, s_2) = (m + n - 2, 0)$. Then the map

$$M_{H_1} \to N_{h_t/X_t} / \alpha(M_{H_1}^{\oplus m + n - 3})$$

induced by (7) has a rank-one image $Q$, which by Lemma 3.8 produces a $\mathbb{P}^m$-scroll containing the curve whose intersection number with the class $H_1^2$ equals $\deg Q + (Y_t \cdot H_1)$. By comparing the numerical classes of the curve, the hypersurface and the $\mathbb{P}^m$-scroll, we obtain a lower bound for $\deg Q$, which in turn gives the following lower bound for $\deg N_{h_t/X_t}$ by (5):

$$\deg N_{h_t/X_t} \geq \deg Q + \deg \alpha(M_{H_1}^{\oplus m + n - 3}) \geq \deg Q - (m + n - 3) \cdot (Y_t \cdot H_1).$$

Remark 3.10. The above scroll considerations were first introduced in [CR19a] to show that a curve $C$ in a very general degree $d$ surface $X \subseteq \mathbb{P}^3$ satisfies the inequality

$$2g(C) - 2 \geq \left( d - 5 + \frac{1}{d} \right) \cdot (C \cdot H),$$

hence $X$ is algebraically hyperbolic if $d \geq 5$. Similarly, when applied to curves $C$ contained in very general degree $d$ hypersurfaces $X \subseteq \mathbb{P}^n$ of higher dimensions, the scroll method yields the inequality

$$2g(C) - 2 \geq \left( d - (2n - 1) + \frac{1}{d} \right) \cdot (C \cdot H).$$

Hence, it gives an alternate proof (see Theorem 3.21) that $X$ is algebraically hyperbolic if $d \geq 2n - 1$ and $n \geq 4$, which is a result first obtained in [Vo96, Vo98]. Prior to [CR19a], Clemens-Ran used a similar scroll construction in §4 of [ClR04] to discount the existence of curves of genus $\leq 2$ in very general sextic 3-folds $X \subseteq \mathbb{P}^4$, in the case where there is a surjection from $M_H$ onto the normal bundle of the curve in $X$.

3.2. Osculating lines. A $\mathbb{P}^m$-line (resp. $\mathbb{P}^n$-line) in $\mathbb{P}^m \times \mathbb{P}^n$ is called an osculating $\mathbb{P}^m$-line (resp. osculating $\mathbb{P}^n$-line) of $X_t$ if it intersects $X_t$ in exactly one point. We denote by $\Lambda^m_{a,t}$ (resp. $\Lambda^m_{b,t}$) the locus on $X_t$ swept out by osculating $\mathbb{P}^m$-lines (resp. osculating $\mathbb{P}^n$-lines).

First, we show the following by using modifications of Lemmas 2.10–11 in [CR19b]:

1. When $a = 2m + n - 2$ and $s_1 = m + n - 2$, the curve $h_t : Y_t \to X_t$ lies in $\Lambda^m_{a,t}$.
2. When $b = m + 2n - 2$ and $s_2 = m + n - 2$, the curve $h_t : Y_t \to X_t$ lies in $\Lambda^m_{b,t}$.

Then, we show that $\Lambda^m_{a,t}$ and $\Lambda^m_{b,t}$ are curves of general type when $a, b$ are large enough. Note that the hypothesis of the following lemma is equivalent to the condition that the curve $h_t : Y_t \to X_t$ is of type $(s_1, s_2) = (m + n - 2, 0)$ and $m + n \geq 4$.

Lemma 3.11 (cf Lemma 2.10 in [CR19b]). Suppose that at a general point $(p, t) \in \mathcal{V}$,

1. $M_{H_1} \to N_{h_t/X_t}$ has rank-one image for a generic polynomial in $H^0(\mathcal{E} \otimes H^\vee_1)$ and is not surjective, and

2. $M_{H_2} \to N_{h_t/X_t}$ has torsion image for a generic polynomial in $H^0(\mathcal{E} \otimes H^\vee_1)$.

Then there exists a $\mathbb{P}^m$-line $p \in \ell \subseteq \mathbb{P}^m \times \mathbb{P}^n$ whose ideal $H^0(\mathcal{E} \otimes \mathcal{I}_\ell)$ is contained in $T_{\mathcal{E} \otimes \mathcal{I}_\ell}(p, t)$. Moreover, there is only one such $\mathbb{P}^m$-line $\ell$, and it is the line induced by the generic rank-one image $Q$ of $M_{H_1}$ in $N_{h_t/X_t}$, as in Lemma 3.8.

Proof. Consider the map $M_{H_1} \to N_{h_t/X_t}$ induced by a generic polynomial in $H^0(\mathcal{E} \otimes H^\vee_1)$ at the point $p \in Y_t$. Since the map is not surjective, its kernel is independent of the generic
polynomial by Lemma 2.2(i) in [Ci03]. We denote its kernel by \( S \subseteq H^0(H_1 \otimes I_p) \) and its image by \( Q \subseteq N_{ht/X_t}|_p \). By Proposition 2.1, \( N_{ht/X_t}|_p \) is isomorphic to the quotient \( M_\ell|_p / T_{Y/P^m \times P^n}|_{(p,t)} \). Hence by the above assumptions, \( T_{Y/P^m \times P^n}|_{(p,t)} \) contains the image of the map

\[
(S \otimes H^0(E \otimes H_1)) \oplus \left( H^0(H_2 \otimes I_p) \otimes H^0(E \otimes H_2^\vee) \right) \to H^0(E \otimes I_p) \cong M_\ell|_p,
\]

which is the ideal of the \( \mathbb{P}^m \)-line containing \( p \) that is given by Lemma 3.8.

Uniqueness follows from a dimension count. We can view \( T_{Y/P^m \times P^n}|_{(p,t)} \subseteq T_X/P^m \times P^n|_{(p,t)} \) as spaces of polynomials. If there are two distinct \( \mathbb{P}^m \)-lines whose ideals are contained in \( T_{Y/P^m \times P^n}|_{(p,t)} \), then we would have \( T_{Y/P^m \times P^n}|_{(p,t)} = T_X/P^m \times P^n|_{(p,t)} \), a contradiction. □

The following is the analogous result with \( H_1 \) and \( H_2 \) switched.

**Lemma 3.12.** Suppose that at a general point \((p,t) \in \mathcal{Y},\)

1. \( M_{H_1} \to N_{ht/X_t} \) has torsion image for a generic polynomial in \( H^0(E \otimes H_1^\vee) \), and
2. \( M_{H_2} \to N_{ht/X_t} \) has rank-one image for a generic polynomial in \( H^0(E \otimes H_2^\vee) \) and is not surjective.

Then there exists a \( \mathbb{P}^n \)-line \( \ell \subseteq \mathbb{P}^n \times \mathbb{P}^n \) whose ideal \( H^0(E \otimes I_{\ell}) \) is contained in \( T_{Y/P^m \times P^n}|_{(p,t)} \). Moreover, there is only one such \( \mathbb{P}^n \)-line \( \ell \), and it is the line induced by the generic rank-one image \( Q \) of \( M_{H_2} \) in \( N_{ht/X_t} \), as in Lemma 3.9.

The following result is a generalization of Lemma 2.11 in [CR19b].

**Lemma 3.13** (cf Lemma 2.11 in [CR19b]). Let \((p,t) \in \mathcal{Y} \) be a general point and let \( Z := \mathcal{P}_2^{-1}(p) \). Let \( T \subseteq \mathbb{P}^n \times \mathbb{P}^n \) be a subvariety containing \( p \) whose ideal \( H^0(E \otimes I_T) \) is contained in \( T_{Y/P^m \times P^n}|_{(p,t)} \). Then \( W := h(\mathcal{Y}) \cap Z \) is a union of fibers of the restriction map \( \beta : Z \to H^0(E|_T) \). In particular, \( W \) contains the fiber containing \((p,t)\), which is the affine space \((p,t) + H^0(E \otimes I_T)\).

**Proof.** The tangent space to a fiber of the above restriction map \( \beta \) is \( H^0(E \otimes I_T) \) at every point of the fiber, so the fiber containing \((p,t)\) is the affine space \((p,t) + H^0(E \otimes I_T)\). Since \( H^0(E \otimes I_T) \subseteq T_{Y/P^m \times P^n}|_{(p,t)}, \) Lemma 2.11 in [CR19b] implies that \( W \) is a union of fibers of the restriction map \( \beta : Z \to H^0(E|_T) \). Therefore, \( W \) contains the fiber containing \((p,t)\), which is \((p,t) + H^0(E \otimes I_T)\). □

Now, we will use Lemmas 3.11 and 3.13 to prove that if \( a = 2m + n - 2 \), then a curve of type \((s_1, s_2) = (m + n - 2, 0) \) lies in the osculation locus \( \Lambda^m_a|_t \subseteq X_t \).

**Lemma 3.14** (cf §2 in [No98], §3.3 in [Pa03]). Suppose that \( a = 2m + n - 2 \), and that at a general point \((p,t) \in \mathcal{Y},\)

1. \( M_{H_1} \to N_{ht/X_t} \) has rank-one image for a generic polynomial in \( H^0(E \otimes H_1^\vee) \) and is not surjective, and
2. \( M_{H_2} \to N_{ht/X_t} \) has torsion image for a generic polynomial in \( H^0(E \otimes H_2^\vee) \).

Then the curve \( h_t : Y_t \to X_t \) lies in the locus \( \Lambda^m_a|_t \subseteq X_t \).

**Proof.** Let \( \ell \) denote the \( \mathbb{P}^m \)-line associated to \((p,t)\) obtained from Lemma 3.11. We can assume without loss of generality that

\[
p = \{X_1 = \ldots = X_m = Y_1 = \ldots = Y_n = 0\},
\]
Lemma 3.15. Suppose that \( \mathcal{Y} \) contains the elements \( G \) tangent to the \( \mathcal{X} \) by a dimension count, we have \( \mathcal{Y} \) equals \( \mathcal{M} \). Consider the restriction map

\[ \beta : T_{\mathcal{Y}/\mathbb{P}^m \times \mathbb{P}^n}|_{(p,t)} \to H^0(\mathcal{O}_\ell(a)(-p)), \]

whose kernel is \( H^0(\mathcal{E} \otimes \mathcal{I}_\ell) \). Denote by \( F \) the polynomial corresponding to \( t \).

Since \( \mathcal{Y} \) is \( G \)-invariant, \( T_{\mathcal{Y}/\mathbb{P}^m \times \mathbb{P}^n}|_{(p,t)} \) contains the elements of \( T_{(\mathbb{P}^m \times \mathbb{P}^n) \times V}|_{(p,t)} \) that are tangent to the \( G \)-orbit of \( (p,t) \) and project to 0 in \( T_{\mathbb{P}^m \times \mathbb{P}^n}|_p \). In particular, \( T_{\mathcal{Y}/\mathbb{P}^m \times \mathbb{P}^n}|_{(p,t)} \) contains the elements

\[ F, X_1 \frac{\partial F}{\partial X_0}, \ldots, X_1 \frac{\partial F}{\partial X_m}. \]

By Lemma 3.13 we may assume that \( F \) is generic in the space \( (p,t) + H^0(\mathcal{E} \otimes \mathcal{I}_\ell) \), so we can assume that the \( m-1 \) elements

\[ X_1 \frac{\partial F}{\partial X_2}|_\ell, \ldots, X_1 \frac{\partial F}{\partial X_m}|_\ell \]

in \( \text{Im} \ \beta \) are independent modulo the subspace

\[ K := \langle F|_\ell, X_1 \frac{\partial F}{\partial X_0}|_\ell, X_1 \frac{\partial F}{\partial X_1}|_\ell \rangle \subseteq H^0(\mathcal{O}_\ell(a)(-p)). \]

By a dimension count, we have

\[
\dim \text{Im} \ \beta = \dim (T_{\mathcal{Y}/\mathbb{P}^m \times \mathbb{P}^n}|_{(p,t)}) - \dim H^0(\mathcal{E} \otimes \mathcal{I}_\ell)
\]

\[
= \dim (T_{\mathcal{X}/\mathbb{P}^m \times \mathbb{P}^n}|_{(p,t)}) - \dim H^0(\mathcal{E} \otimes \mathcal{I}_\ell) - \dim \left( h^*T_{\mathcal{X}/\mathbb{P}^m \times \mathbb{P}^n}|_{(p,t)}/T_{\mathcal{Y}/\mathbb{P}^m \times \mathbb{P}^n}|_{(p,t)} \right)
\]

\[
= a - (m + n - 2)
\]

when \( a = 2m + n - 2 \). Hence, \( \dim K \leq \dim \text{Im} \ \beta - (m-1) = m - (m-1) = 1 \), which implies that \( F|_\ell = a \cdot X^a Y^b \). Therefore, the curve \( h_t : Y_t \to X_t \) lies in the locus \( \Lambda^m_a|_\ell \subseteq X_t \).

The following is the analogous result when \( b = m + 2n - 2 \) and the curve is of type \( (s_1, s_2) = (0, m + n - 2) \) instead.

**Lemma 3.15.** Suppose that \( b = m + 2n - 2 \), and that at a general point \( (p,t) \in \mathcal{Y} \),

1. \( M_{H_1} \to N_{h_t/X_t} \) has torsion image for a generic polynomial in \( H^0(\mathcal{E} \otimes H_1^\vee) \), and
2. \( M_{H_2} \to N_{h_t/X_t} \) has rank-one image for a generic polynomial in \( H^0(\mathcal{E} \otimes H_2^\vee) \) and is not surjective.

Then this curve lies in the locus \( \Lambda^m_b|_\ell \subseteq X_t \).

**Remark 3.16.** When \( n = 1 \), Lemma 3.15 follows directly from Lemmas 2.10 and 2.11 in [CR19b] by the argument used there for classifying very general surfaces in \( \mathbb{P}^2 \times \mathbb{P}^1 \). (See [CR19b], Proof of Theorem 3.4, Case 4.) This argument goes as follows. By Lemmas 3.12 and 3.13 above, \( W \) is a union of fibers of the restriction map \( \beta : Z \to H^0(\mathcal{E}|_\ell) \) where \( \ell \) is the \( \mathbb{P}^n \)-line \( \pi_1^{-1}(p) \), so \( X_t \cap \ell \) is a subscheme \( B \) of \( b \) (not necessarily distinct) points on \( \mathbb{P}^1 \). If \( B \) contained \( \geq 2 \) distinct points, then its \( GL(2) \)-orbit would have codimension \( \leq b - 2 \), which equals \( m - 2 \) when \( b = m + 2n - 2 = m \). Since \( \mathcal{Y} \) is \( G \)-invariant, \( W \) contains the fibers of \( \beta \) over \( B \), so we would have \( \text{codim}(\mathcal{Y} \subseteq \mathcal{X}) \leq m - 2 \). However, this is a contradiction since
codim(\( \mathcal{V} \subseteq \mathcal{X} \)) = m + n - 2 = m - 1 by (2). Hence, \( B \) has to be one unique point with multiplicity \( b \).

For the rest of this section, we investigate the geometry of the osculation loci \( \Lambda^{\text{pm}}_a \mid_t \) and \( \Lambda^{\text{pm}}_b \mid_t \). Let us define the incidence variety

\[
\Delta^{\text{pm}}_a = \{(L, p, t) \mid \pi_1^{-1}(L) \cap \pi_2^{-1}(\rho_2(p)) \cap X_t = a \cdot [p] \} \subseteq \mathbb{G}(1, m) \times (\mathbb{P}^m \times \mathbb{P}^n) \times V,
\]

and denote its fiber over \( t \) by \( \Delta^{\text{pm}}_a \mid_t \).

**Lemma 3.17** (cf Theorem 5.2 in [CR20]). For a very general \( t \in U \), \( \Delta^{\text{pm}}_a \mid_t \) is a smooth, irreducible variety of dimension \( 2m - 1 + n - a \). Moreover, it is of general type when \( a \geq \max\{\sqrt{2m + 1}, 3\} \) and \( b \geq n + 2 \).

**Proof.** Since \( \Delta^{\text{pm}}_a \) is a vector bundle over the space of pairs \((L, p) \in \mathbb{G}(1, m) \times (\mathbb{P}^m \times \mathbb{P}^n)\) such that \( \pi_1(p) \in L \), it is a smooth and irreducible variety.

Now we prove that the canonical bundle \( K_{\Delta^{\text{pm}}_a} \) is very ample when \( a, b \) are large enough. For convenience, let us denote \( \mathbb{G} := \mathbb{G}(1, m) \times (\mathbb{P}^m \times \mathbb{P}^n) \times V \), and let us denote the various projection maps by \( \rho_1 : \mathbb{G} \to \mathbb{G}(1, m) \), \( \rho_2 : \mathbb{G} \to \mathbb{P}^m \times \mathbb{P}^n \), \( \rho_{2,m} : \mathbb{G} \to \mathbb{P}^m \), \( \rho_{2,n} : \mathbb{G} \to \mathbb{P}^n \), and \( \rho_3 : \mathbb{G} \to V \). We will construct \( \Delta^{\text{pm}}_a \) as the vanishing locus of some vector bundle on \( \mathbb{G} \).

Let \( \sigma, H_1, H_2 \) be the pullback of the hyperplane divisors via \( \rho_1, \rho_{2,m}, \rho_{2,n} \) respectively. Then the canonical bundle of \( \mathbb{G} \) is of class

\[
K_\mathbb{G} = (\sigma - 1)H_1 + (\sigma - 1)H_2 =: (\sigma - 1, -\sigma - 1).
\]

First consider the natural map

\[
e : \rho_3^*\mathcal{O}_V(-1) \to \rho_2^*\mathcal{E}
\]

of vector bundles on \( \mathbb{G} \). Its zero scheme is the locus of points \((L, p, t)\) with \( p \in X_t \). Then consider the natural map

\[
f : \rho_{2,m}^*\mathcal{O}_{\mathbb{P}^m}(-1) \hookrightarrow \rho_1^*\mathcal{O}_{\mathbb{G}(1, m)}^{m+1} \twoheadrightarrow \rho_1^*\mathcal{Q},
\]

where the surjection comes from the following tautological sequence on \( \mathbb{G}(1, m) \)

\[
0 \to S \to \mathcal{O}_{\mathbb{G}(1, m)}^{m+1} \to \mathcal{Q} \to 0.
\]

The zero scheme of \( f \) is the locus of points \((L, p, t)\) with \( \pi_1(p) \in L \). Since \( \Delta^{\text{pm}}_1 \) is the common zero scheme of \( e, f \), we can compute the class of its canonical bundle as follows:

\[
K_{\Delta^{\text{pm}}_1} = K_\mathbb{G} + c_1(\rho_2^*\mathcal{E}) + c_1(\rho_1^*\mathcal{Q} \otimes \rho_{2,m}^*\mathcal{O}_{\mathbb{P}^m}(1))
\]

\[
= K_\mathbb{G} + c_1(\rho_2^*\mathcal{E}) + c_1(\rho_1^*\mathcal{Q}) + (m - 1) \cdot c_1(\rho_{2,m}^*\mathcal{O}_{\mathbb{P}^m}(1))
\]

\[
= (-m - 1, -m - 1, -n - 1) + (0, a, b) + (1, 0, 0) + (0, m - 1, 0)
\]

\[
= (-m, a - 2, b - n - 1).
\]

Note that on \( \Delta^{\text{pm}}_1 \), the natural map

\[
\rho_{2,m}^*\mathcal{O}_{\mathbb{P}^m}(-1) \hookrightarrow \rho_1^*\mathcal{O}_{\mathbb{G}(1, m)}^{m+1}
\]

from above factors as

\[
\rho_{2,m}^*\mathcal{O}_{\mathbb{P}^m}(-1) \quad \mu \quad \rho_1^*\mathcal{O}_{\mathbb{G}(1, m)}^{m+1}
\]

\[
\rho_1^*S
\]
Let $R^\vee$ be the cokernel of $\mu$. We have the short exact sequence

$$0 \to \rho^*_{2,m} O_{\mathbb{P}^m}(-1) \xrightarrow{\mu} \rho^*_1 S \to R^\vee \to 0,$$

on $\Delta_1^{\mathbb{P}^m}$, which induces a filtration $F^\bullet$ on $\rho^*_1 S^a \otimes S^\vee$ with

$$F^i/F^{i+1} = \rho^*_{2,m} O_{\mathbb{P}^m}(a-i) \otimes R^i.$$

Note that

$$c_1(F^i/F^{i+1}) = (0, a-i, 0) + i \cdot (1, -1, 0) = (i, a-2i, 0).$$

Since $\Delta_{\mathbb{P}^m}$ is the zero scheme of the (well-defined) natural map $O_V(-1) \to F^1/F^a$,
we can compute its canonical bundle using adjunction again as follows:

$$K_{\Delta_{\mathbb{P}^m}} = K_{\Delta_1^{\mathbb{P}^m}} + c_1(F^1/F^a)$$

$$= K_{\Delta_1^{\mathbb{P}^m}} + \sum_{i=1}^{a-1} c_1(F^i/F^{i+1})$$

$$= (-m, a-2, b-n-1) + \sum_{i=1}^{a-1} (i, a-2i, 0)$$

$$= \left(-m + \frac{a(a-1)}{2}, a-2, b-n-1\right).$$

Hence $K_{\Delta_{\mathbb{P}^m}}$ is very ample when $a \geq \max\{\sqrt{2m}+1, 3\}$ and $b \geq n+2$.

Finally we verify the dimension of $\Delta_{\mathbb{P}^m} \mid t$. Since a very general fiber of the projection map $(L, p, t) \mapsto (L, p)$ has codimension $a$ in $V$, we have

$$\dim \Delta_{\mathbb{P}^m} \mid t = \dim \mathbb{G}(1, m) + n + 1 + (\dim V - a) = 2m - 1 + n - a + \dim V,$$

and so $\dim \Delta_{\mathbb{P}^m} \mid t = 2m - 1 + n - a$. □

**Lemma 3.18** (cf Proposition 5.3 in [CR20]). For a very general $t \in U$, the projection map $\Delta_{\mathbb{P}^m} \mid t \to \Lambda_{\mathbb{P}^m} \mid t$ is an isomorphism when $a \geq 2m + n - 2$.

**Proof.** Let us define

$$D = \{(p, t) \mid p \in X_t \text{ and } \exists L_1 \neq L_2 \in \mathbb{G}(1, m) \text{ s.t. } L_1 \cap L_2 = \pi_1(p)\} \subseteq X_V.$$

From the proof of Lemma 3.17, $X_t$ does not contain $\mathbb{P}^m$-lines when $a \geq 2m + n - 2$, so it is enough to prove that codim $(D \subseteq X_V) \geq m + n$.

Define the incidence variety

$$I = \{(p, t, L_1, L_2) \mid p \in X_t, L_1 \neq L_2 \text{ and } L_1 \cap L_2 = \pi_1(p)\}.$$

The space of tuples $(p, L_1, L_2)$ such that $L_1 \cap L_2 = \pi_1(p)$ has dimension equal to

$$2(m-1) + (1 + n) + (m-1) = 3m + n - 2.$$

Since a fiber of $I$ over such a tuple has dimension $\dim V - 2a$, we have

$$\dim I = \dim V + 3m + n - 2 - 2a,$$
so \( \mathcal{D} \) being the image of \( I \) must have dimension \( \leq \dim V + 3m + n - 2 - 2a \). Therefore,

\[
\text{codim} (\mathcal{D} \subseteq \mathcal{X}_V) \geq \dim \mathcal{X}_V - (N + 3m + n - 2 - 2a)
= (\dim V + m + n - 1) - (\dim V + 3m + n - 2 - 2a)
= 2a - 2m + 1,
\]

which is \( \geq m + n \) when \( a \geq \frac{3m+n-1}{2} \).

\( \square \)

The following are the analogous results to Lemmas 3.17 and 3.18 for \( \Delta_b^{\mathbb{P}^n} |_t \) and \( \Lambda_b^{\mathbb{P}^n} |_t \).

**Lemma 3.19.** For a very general \( t \in U \), \( \Delta_b^{\mathbb{P}^n} |_t \) is a smooth, irreducible variety of dimension \( m - 1 + 2n - b \). Moreover, it is of general type when \( a \geq m + 2 \) and \( b \geq \max\{\sqrt{2n+1}, 3\} \).

**Lemma 3.20.** For a very general \( t \in U \), the projection map \( \Delta_b^{\mathbb{P}^n} |_t \to \Lambda_b^{\mathbb{P}^n} |_t \) is an isomorphism when \( b \geq m + 2n - 2 \).

One can use the analogues of the above tools in the \( \mathbb{P}^n \) setting to show that a very general hypersurface \( X_t \subseteq \mathbb{P}^n \) of degree \( d \geq 2n - 2 \) is algebraically hyperbolic when \( n \geq 5 \). Our proof is built upon the proofs of [Pa04] and [ClR04] for the \( n \geq 6 \) case. The key to our improved result is the use of Coskun and Riedl’s scroll argument from [CR19a], which also gives an alternate proof of Voisin’s result on the algebraic hyperbolicity of very general hypersurfaces of degree \( d \geq 2n - 1 \) when \( n \geq 4 \) [Vo96, Vo98].

**Theorem 3.21.** A very general hypersurface \( X \subseteq \mathbb{P}^n \) of degree \( d \) is algebraically hyperbolic when (1) \( n \geq 4 \) and \( d \geq 2n - 1 \); or (2) \( n \geq 5 \) and \( d \geq 2n - 2 \).

**Proof.** Suppose \( X \) is parametrized by \( t \in U \) and let \( h_t : Y_t \to X_t \) be a curve in \( X \). Then, there is a surjection

\[
\alpha : M_H^{\oplus s} \to N_{h_t/X_t}
\]

for some integer \( s \). Take \( s \) to be the minimum. Since \( \text{rank} N_{h_t/X_t} = n - 2 \), we have \( s \leq n - 2 \).

Suppose that \( n \geq 4 \). If \( s \leq n - 3 \), then

\[
\deg N_{h_t/X_t} \geq (n - 3) - \deg \alpha (M_H) \geq (n - 3) - \deg H|_{Y_t}.
\]

Therefore,

\[
2g(Y_t) - 2 = (K_{X_t} \cdot Y_t) + \deg N_{h_t/X_t}
\geq ((d - n - 1) - (n - 3)) \cdot (Y_t \cdot H)
= (d - 2n - 2) \cdot (Y_t \cdot H)
\]

which is at least \( (Y_t \cdot H) \) when \( d \geq 2n - 1 \). If \( s = n - 2 \), then the map

\[
M_H \to N_{h_t/X_t}/\alpha(M_H^{\oplus n-3})
\]

given by multiplication by a generic polynomial in \( H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d - 1)) \) has a rank-one image \( Q \), which induces a surface scroll \( \Sigma \subseteq \mathbb{P}^n \) of degree \( \deg Q + (Y_t \cdot H) \). Since the curve \( Y_t \) is contained in the curve \( \Sigma \cap X_t \) whose degree is \( d \cdot (\deg Q + (Y_t \cdot H)) \), we must have

\[
\deg Q \geq \left( \frac{1}{d} - 1 \right) \cdot (Y_t \cdot H).
\]
Therefore,
\[
2g(Y_t) - 2 = (K_{X_t} \cdot Y_t) + \deg N_{h_t/X_t} \\
\geq ((d - n - 1) - (n - 3)) \cdot (Y_t \cdot H) + \deg Q \\
\geq (d + \frac{1}{d} - (2n - 1)) \cdot (Y_t \cdot H)
\]
which is at least \(\frac{1}{d}(Y_t \cdot H)\) when \(d \geq 2n - 1\). This concludes the proof for \(n \geq 4\) and \(d \geq 2n - 1\).

Now suppose that \(n \geq 5\) and \(d = 2n - 2\). If \(s \leq n - 4\), then as in the case \(s \leq n - 3\) from above, we have \(\deg N_{h_t/X_t} \geq (n - 4) \cdot -\deg H|_{Y_t}\). Therefore,
\[
2g(Y_t) - 2 \geq ((d - n - 1) - (n - 4)) \cdot (Y_t \cdot H) = (d - (2n - 3)) \cdot (Y_t \cdot H) = (Y_t \cdot H).
\]
If \(s = n - 3\), then as in the case \(s = n - 2\) from above, the map
\[
M_H \to N_{h_t/X_t} / \alpha(M_H \oplus_{\mathbb{P}^n-4})
\]
has a rank-one image \(Q\), since we assume that \(n \geq 5\). Then it follows from the scroll method as applied above that \(2g(Y_t) - 2 \geq \frac{1}{d}(Y_t \cdot H)\). If \(s = n - 2\), then the map
\[
M_H \to N_{h_t/X_t}
\]
has rank-one image and is not surjective. By the analogues of Lemmas 3.11, 3.13 and 3.14 for \(\mathbb{P}^n\), the curve lies in the 1-osculating locus of \(X_t\), which is a curve of general type by \S 4.1 in [Pa03].

Remark 3.22. Consider a curve in a very general sextic threefold, i.e. when \(n = 4\) and \(d = 2n - 2\). If the curve is of type \(s = 2\), then it satisfies the inequality (1) by the same argument as the case \(n \geq 5\), \(d = 2n - 2\) and \(s = n - 2\) in Theorem 3.21. If the curve is of type \(s = 1\), i.e. the map \(M_H \to N_{h_t/X_t}\) induced by a generic degree 5 polynomial has a rank-two image, then the scroll method as above does not apply (see also Remark 3.10).

4. Proof of Theorem 1.1

4.1. Proof of Theorem 1.1(2). We verify that very general hypersurfaces of low bidegrees \((a, b)\) contain lines or elliptic curves. Hence, they are not algebraically hyperbolic.

Lemma 4.1. Suppose that \(a \leq 2m+n-3\) or \(b \leq m+2n-3\). Then a very general hypersurface \(X \subseteq \mathbb{P}^m \times \mathbb{P}^n\) of bidegree \((a, b)\) contains a line. In particular, it is not algebraically hyperbolic.

Proof. View \(X\) as a very general \(\mathbb{P}^n\)-family of degree \(a\) hypersurfaces in \(\mathbb{P}^m\) via the projection map \(\pi_2 : X \to \mathbb{P}^n\). The space of degree \(a\) hypersurfaces in \(\mathbb{P}^m\) containing a line has codimension \((a + 1) - 2(m - 1)\) in \(H^0(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(a))\), so it has non-trivial intersection with \(X\) when \(a \leq 2m + n - 3\). Therefore \(X\) contains a \(\mathbb{P}^m\)-line when \(a \leq 2m + n - 3\). The same argument shows that \(X\) contains a \(\mathbb{P}^n\)-line when \(b \leq m + 2n - 3\).

Lemma 4.2. Let \((m, n) = (2, 2)\). Suppose that \(a = 4\) or \(b = 4\). Then a very general hypersurface \(X \subseteq \mathbb{P}^2 \times \mathbb{P}^2\) of bidegree \((a, b)\) contains an elliptic curve. In particular, it is not algebraically hyperbolic.

Proof. As above, we can view \(X\) as a very general net of degree 4 curves in \(\mathbb{P}^2\). By the degree-genus formula for plane curves, a general member of this net has geometric genus 3. We prove that there exists a curve in this net that has two nodes. Such a curve would be an elliptic curve since its geometric genus is \(3 - 2 = 1\). Requiring a quartic plane curve to
contain two particular distinct nodes in \( \mathbb{P}^2 \) imposes \( 2 \times 3 = 6 \) conditions on the equation of the curve. Since the two points can vary over a \((2 \times 2 = 4)\)-dimensional family, the space of quartic plane curves containing any two distinct nodes has codimension \( 6 - 4 = 2 \) in \( H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(4)) \). Hence, this family intersects the two-dimensional family that sweeps out the very general hypersurface \( X \).

4.2. **Proof of Theorem 1.1(1).** We divide up the proof by \((m, n)\). Then, for each possible type \((s_1, s_2)\), we produce a uniform \( \varepsilon > 0 \) such that all curves \( h_t : Y_t \to X_t \) of type \((s_1, s_2)\) (see Definition 3.6) lying in a very general hypersurface \( X_t \subseteq \mathbb{P}^m \times \mathbb{P}^n \) satisfies

\[
2g(Y_t) - 2 \geq \varepsilon \cdot ((Y_t \cdot H_1) + (Y_t \cdot H_2))
\]

provided that the bidegree \((a, b)\) of \( X_t \) lies in the stated range. We assume without loss of generality that \( s_1 \geq s_2 \).

4.2.1. **Proof of Theorem 1.1(1)(a).** Let \( m + n \geq 5 \).

**Case 1:** Suppose that \( s_1, s_2 \leq m + n - 4 \). Then, by (6), we have

\[
2g(Y_t) - 2 \\
\geq ((a - m - 1) - (m + n - 4)) \cdot (Y_t \cdot H_1) + ((b - n - 1) - (m + n - 4)) \cdot (Y_t \cdot H_2) \\
= (a - (2m + n - 3)) \cdot (Y_t \cdot H_1) + (b - (m + 2n - 3)) \cdot (Y_t \cdot H_2),
\]

which is at least \((Y_t \cdot H_1) + (Y_t \cdot H_2)\) when \( a \geq 2m + n - 2 \) and \( b \geq m + 2n - 2 \).

**Case 2:** Suppose that \( s_1 = m + n - 3 \), so \( s_2 \leq 1 \). Consider the map

\[
M_{H_1} \to N_{\alpha/\alpha} \cap \alpha(M_{H_1}^{m+n-4})
\]

induced by \( \alpha \). Since \( m + n \geq 5 \), this map must have a rank-one image \( Q \), so we can apply the scroll considerations of Lemma 3.8. Let \( \Sigma \subseteq \mathbb{P}^m \times \mathbb{P}^n \) denote the \( \mathbb{P}^m \)-scroll given by \( Q \). The \( H_1 \)-degree of \( \Sigma \) is \( \Sigma \cdot H_1^2 \), which is equal to \( \deg Q + (Y_t \cdot H_1) \) by Lemma 3.8. Since \( \Sigma \) is a \( \mathbb{P}^m \)-scroll, we have \( \Sigma \cdot H_2 = (Y_t \cdot H_2) \cdot H_1^{m-1}H_2^n \). Hence, \( \Sigma \) is of numerical class

\[
(Y_t \cdot H_2) \cdot H_1^{m-1}H_2^n + (\deg Q + (Y_t \cdot H_1)) \cdot H_1^{m-2}H_2^n.
\]

\( Y_t \) is contained in the curve \( \Sigma \cap X_t \) whose class is

\[
(a \cdot (H_2 \cdot Y_t)) \cdot H_1^mH_2^{n-1} + (b \cdot (H_2 \cdot Y_t) + a \cdot (\deg Q + (H_1 \cdot Y_t))) \cdot H_1^{m-1}H_2^n.
\]

Since \( \Sigma \cap X_t - Y_t \) is effective, we have \( H_1 \cdot (\Sigma \cap X_t - Y_t) \geq 0 \). Hence, we obtain the inequality

\[
b \cdot (H_2 \cdot Y_t) + a \cdot (\deg Q + (H_1 \cdot Y_t)) - (H_1 \cdot Y_t) \geq 0,
\]

which can be rearranged into

\[
\deg Q \geq \left(\frac{1}{a} - 1\right) \cdot (H_1 \cdot Y_t) + \left(-\frac{b}{a}\right) \cdot (H_2 \cdot Y_t).
\]

Therefore, by Lemma 3.3

\[
2g(Y_t) - 2 \\
\geq ((a - m - 1) - (m + n - 4)) \cdot (Y_t \cdot H_1) + ((b - n - 1) - 1) \cdot (Y_t \cdot H_2) + \deg Q \\
\geq \left(a + \frac{1}{a} - 2m + n - 2\right) \cdot (H_1 \cdot Y_t) + \left(b - \frac{b}{a} - n - 2\right) \cdot (H_2 \cdot Y_t),
\]

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which is at least some multiple of \((Y_t \cdot H_1) + (Y_t \cdot H_2)\) when \(a \geq 2m + n - 2\) and \(b > (n+2) \cdot \left(\frac{a}{a-1}\right)\). One can check that \(b \geq m + 2n - 2 > (n+2) \cdot \left(\frac{a}{a-1}\right)\) when \(m + n \geq 5\) and \(m \geq n\).

**Case 2:** Suppose that \(s_1 = m + n - 2\), so \(s_2 = 0\). Then, the map

\[ M_{H_t} \to N_{h_t/X_t} \]

induced by \(\alpha\) has a rank-one image \(Q\), so we can apply the scroll considerations as in Case 2 to obtain

\[ \deg Q \geq \left(\frac{1}{a} - 1\right) \cdot (H_1 \cdot Y_t) + \left(-\frac{b}{a}\right) \cdot (H_2 \cdot Y_t). \]

Therefore, by Lemma 3.3,

\[ 2g(Y_t) - 2 \geq ((a - m - 1) - (m + n - 3)) \cdot (Y_t \cdot H_1) + (b - n - 1) \cdot (Y_t \cdot H_2) + \deg Q \]

\[ \geq \left(a + \frac{1}{a} - (2m + n - 1)\right) \cdot (H_1 \cdot Y_t) + \left(b - \frac{b}{a} - n - 1\right) \cdot (H_2 \cdot Y_t), \]

which is at least some multiple of \((Y_t \cdot H_1) + (Y_t \cdot H_2)\) when \(a \geq 2m + n - 1\) and \(b > (n+1) \cdot \left(\frac{a}{a-1}\right)\). One can check that \(b \geq m + 2n - 2 > (n+1) \cdot \left(\frac{a}{a-1}\right)\) when \(m + n \geq 5\) and \(m \geq n\). Hence, we focus on the case when \(a = 2m + n - 2\) and \(b \geq m + 2n - 2\).

Suppose that the hypothesis of Lemma 3.11 is satisfied. Then, by Lemma 3.14, the curve \(h_t : Y_t \to X_t\) lies in the subvariety \(\Lambda^\text{fr}_a |_{X_t}\). Lemmas 3.17 and 3.18 imply that \(\Lambda^\text{fr}_a |_{X_t}\) is a smooth, irreducible curve of general type. Therefore, there is some \(\varepsilon > 0\) such that (10) is satisfied.

\[ 4.2.2. \text{Proof of Theorem 1.1(1)(b).} \text{ Let } (m, n) = (2, 2). \]

**Case 1:** Suppose that \(s_1, s_2 \leq 1\). We proceed as in Case 1 of 4.2.1 and obtain the inequality

\[ 2g(Y_t) - 2 \geq (a - 4) \cdot (Y_t \cdot H_1) + (b - 4) \cdot (Y_t \cdot H_2), \]

which is at least \((Y_t \cdot H_1) + (Y_t \cdot H_2)\) when \(a, b \geq 5\).

**Case 2:** Suppose that \(s_1 = 2\), so \(s_2 = 0\). We proceed as in Case 2 of 4.2.1. Let \(Q\) be the rank-one image of \(\alpha : M_{H_t} \to N_{h_t/X_t}\), and let \(\Sigma \subseteq \mathbb{P}^m \times \mathbb{P}^n\) denote the \(\mathbb{P}^m\)-scroll given by \(Q\). By Lemma 3.3 and the same calculations as above, we have

\[ \deg Q \geq \left(\frac{1}{a} - 1\right) \cdot (H_1 \cdot Y_t) + \left(-\frac{b}{a}\right) \cdot (H_2 \cdot Y_t), \]

and so

\[ 2g(Y_t) - 2 \geq ((a - 3) - 1) \cdot (Y_t \cdot H_1) + (b - 3) \cdot (Y_t \cdot H_2) + \deg Q \]

\[ \geq \left(a + \frac{1}{a} - 5\right) \cdot (H_1 \cdot Y_t) + \left(b - \frac{b}{a} - 3\right) \cdot (H_2 \cdot Y_t), \]

which is at least some multiple of \((Y_t \cdot H_1) + (Y_t \cdot H_2)\) when \(a, b \geq 5\). \(\square\)
4.2.3. Proof of Theorem 4.1(1)(c). Let \((m, n) = (3, 1)\).

**Case 1:** Suppose that \((s_1, s_2) = (1, 0)\). We proceed as in Case 1 of §4.2.1 and obtain the inequality
\[
2g(Y_i) - 2 \geq (a - 5) \cdot (Y_i \cdot H_1) + (b - 2) \cdot (Y_i \cdot H_2),
\]
which is at least \((Y_i \cdot H_1) + (Y_i \cdot H_2)\) when \(a \geq 5\) and \(b \geq 3\).

**Case 2:** Suppose that \((s_1, s_2) = (1, 1)\). We proceed as in Cases 2 and 3 of §4.2.1. For \(i = 1, 2\), let \(Q_i\) be the rank-one image of \(M_{H_i} \to N_{H_i} / X_i\), and let \(\Sigma_i \subseteq \mathbb{P}^n \times \mathbb{P}^n\) denote the scroll given by \(Q_i\). By Lemma 3.8 and the same calculations as above, we have
\[
\deg Q_1 \geq \left( \frac{1}{a} - 1 \right) \cdot (H_1 \cdot Y_i) + \left( -\frac{b}{a} \right) \cdot (H_2 \cdot Y_i),
\]
\[
\deg Q_2 \geq \left( -\frac{a}{b} \right) \cdot (H_1 \cdot Y_i) + \left( \frac{1}{b} - 1 \right) \cdot (H_2 \cdot Y_i).
\]
Using (11), we have
\[
2g(Y_i) - 2 \geq (a - 4) \cdot (Y_i \cdot H_1) + ((b - 2) - 1) \cdot (Y_i \cdot H_2) + \deg Q_1
\]
\[
\geq \left( a + \frac{1}{a} - 5 \right) \cdot (H_1 \cdot Y_i) + \left( b - \frac{b}{a} - 3 \right) \cdot (H_2 \cdot Y_i),
\]
which is at least some multiple of \((Y_i \cdot H_1) + (Y_i \cdot H_2)\) when \(a \geq 5\) and \(b \geq 4\).

Using (12), we have
\[
2g(Y_i) - 2 \geq ((a - 4) - 1) \cdot (Y_i \cdot H_1) + (b - 2) \cdot (Y_i \cdot H_2) + \deg Q_2
\]
\[
\geq \left( a - \frac{a}{b} - 5 \right) \cdot (H_1 \cdot Y_i) + \left( b + \frac{1}{b} - 3 \right) \cdot (H_2 \cdot Y_i),
\]
which is at least some multiple of \((Y_i \cdot H_1) + (Y_i \cdot H_2)\) when \(a \geq 8\) and \(b \geq 3\).

**Case 3:** Suppose that \((s_1, s_2) = (2, 0)\). Then the same proof as Case 4 of §4.2.1 applies to show that there is some \(\varepsilon > 0\) such that (10) is satisfied when \(a \geq 5\) and \(b \geq 3\). \(\square\)

**Remark 4.3.** §4.2.3 shows where our proof techniques could not resolve the remaining open cases in \(\mathbb{P}^3 \times \mathbb{P}^1\). Specifically, our techniques could not produce an \(\varepsilon > 0\) such that curves of some types \((s_1, s_2)\) satisfy (10), hence we could not confirm algebraic hyperbolicity. We list such types of curves for each open case:

1. For \((a, b) = (5, 3)\), curves of types \((s_1, s_2) = (1, 0)\) or \((1, 1)\);
2. For \((a, b) = (5, b)\) with \(b \geq 4\), curves of type \((s_1, s_2) = (1, 0)\);
3. For \((a, b) = (6, 3)\) or \((7, 3)\), curves of type \((s_1, s_2) = (1, 1)\).

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