Representations of nets of $\mathbb{C}^*$-algebras over $S^1$

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Abstract

In recent times a new kind of representations has been used to describe superselection sectors of the observable net over a curved spacetime, taking into account of the effects of the fundamental group of the spacetime. Using this notion of representation, we prove that any net of $\mathbb{C}^*$-algebras over $S^1$ admits faithful representations, and when the net is covariant under Diff($S^1$), it admits representations covariant under any amenable subgroup of Diff($S^1$).

1 Introduction

Nets of $\mathbb{C}^*$-algebras are the basic objects of study in algebraic quantum field theory and, as well-known to the specialists, code the basic idea that any suitable region $Y$ of a spacetime defines an abstract $\mathbb{C}^*$-algebra $A_Y$, interpreted as the one generated by the quantum observables localized in $Y$; from this assumption it is natural to require that there are inclusions morphisms $j_{Y'Y} : A_Y \to A_{Y'}$, $\forall Y \subseteq Y'$, which, for coherence, must fulfil the equalities

$$j_{Y''Y'} \circ j_{Y'Y} = j_{Y''Y} , \quad \forall Y \subseteq Y' \subseteq Y''$$

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Open, relatively compact and simply connected subsets of the spacetime.
In the Minkowski spacetime $X$ the set of regions $Y \subset X$ is upward directed under inclusion, so the pair $(\mathcal{A}, j)$, $\mathcal{A} := \{\mathcal{A}_Y\}$, $j := \{j_{Y'Y}\}$, is indeed a net and we can construct the inductive limit $\tilde{\mathcal{A}} := \text{lim}(\mathcal{A}, j)$. In this way, families of Hilbert space representations of $\mathcal{A}_Y$, $Y \subset X$, coherent with the inclusion morphisms (that we call Hilbert space representations of the net) are obtained by considering representations of $\tilde{\mathcal{A}}$. This point is important for the applications, because crucial physical properties of the quantum system described by $(\mathcal{A}, j)$, like the charge structure of elementary particles, are encoded by certain Hilbert space representations of the net, called sectors [8, 9, 2].

Now, general relativity and conformal theory lead to consider spacetimes $X$ such that the set of regions $\{Y \subset X\}$ is not directed under inclusion anymore, and the above scenario breaks down. In this case we should say, to be precise, that $(\mathcal{A}, j)$ is a precosheaf of $C^*$-algebras, and the search for Hilbert space representations may be vain (see [16]).

In recent times a more general notion of representation has been given for ”nets” of $C^*$-algebras over generic spacetimes $X$, defined in such a way that the obstacle to get coherence is encoded by a family $\{U_{Y'Y}\}_{Y' \subset Y}$ of unitaries fulfilling the cocycle relations (see [6, 3]). These, that we simply call representations, reduce to Hilbert space representations when $X$ is simply connected and maintain the properties of charge composition, conjugation and covariance under eventual spacetime symmetries, typical of the usual sectors.

In a previous paper [16], we introduced the notion of enveloping net bundle of the given net of $C^*$-algebras $(\mathcal{A}, j)$: this is a net of $C^*$-algebras $(\overline{\mathcal{A}}, \overline{j})$ such that any $\overline{j}_{Y'Y}$ is an isomorphism, and fulfils the universal property of lifting any representation of $(\mathcal{A}, j)$. So the question of existence of representations is reduced to nondegeneracy of the canonical embedding $\epsilon : (\mathcal{A}, j) \rightarrow (\overline{\mathcal{A}}, \overline{j})$. We call injective those nets such that $\epsilon$ is faithful.

In the present work we focus on nets of $C^*$-algebras defined over $S^1$, a remarkable class due to its applications in conformal quantum field theory [13, 7]. We show that any net over $S^1$ is injective, so it has faithful representations. Moreover, when the net is covariant under the action of $\text{Diff}(S^1)$, we show the existence of covariant representations of any amenable subgroup of $\text{Diff}(S^1)$. The technique that we will use shall be the one of approximate the set of proper intervals of $S^1$ with finite subsets (roughly speaking, an analogue of the decomposition of $S^1$ as a CW-complex in the setting of partially ordered sets), then to show that the restriction of $(\mathcal{A}, j)$ on these subsets is injective, and finally to prove injectivity of the initial net performing an inductive limit. We have postponed to Appendix A some rather technical computations showing that injectivity is preserved under inductive limits, a result which plays a key rôle in the analysis of nets over $S^1$ and that, we hope, could play a similar role for other spacetimes too.

2 Some preliminaries on nets.

To make the present work self-contained in this section we recall the basic properties of the main objects of our study, namely nets of $C^*$-algebras. All the material presented here appeared in [16]; the reader may pass to the next section whenever he is already
2.1 Posets

A *poset* (partially ordered set) is a set endowed with an (antisymmetric, reflexive and transitive) order relation $\leq$. A *poset morphism* is a map $f : K \to K'$ such that $o \leq \tilde{\circ}$ implies $f(o) \leq' f(\tilde{\circ})$ for all $o, \tilde{\circ} \in K$, where $\leq'$ is the order relation of $K'$. A *disjointness relation* on $K$ is a symmetric binary relation $\perp$ such that $\tilde{\circ} \perp a \iff o \leq \tilde{\circ} \Rightarrow o \perp a$.

A group $G$ is said to be a *symmetry group* for $K$ whenever it acts by automorphisms on $K$, namely $go \leq g\tilde{\circ} \iff o \leq \tilde{\circ}$ for all $g \in G$ and $o, \tilde{\circ} \in K$, and we assume that, whenever $K$ has a disjointness relation, $o \perp a \iff go \perp ga$.

The classical covariant poset used in algebraic quantum field theory is the set of doublecones in the Minkowski spacetime, having the inclusion as order relation, the spacelike separation as the disjointness relation and the Poincaré group as the group of symmetries. We shall focus in §3 to the case of proper intervals in $S^1$, of interest in low dimensional quantum field theory.

We now give a brief description of the notion of connectedness and simply connectedness for posets and refer the reader to the paper [16] for details. A poset $K$ is *pathwise connected* if for any pair $a, \tilde{\circ} \in K$, there are two finite sequences $a \ldots a_{n+1}$ and $o \ldots o_n$ of elements of $K$, with $a_1 = a$ and $a_{n+1} = \tilde{\circ}$, satisfying the relations $a_i, a_{i+1} \leq o_i$, $i = 1, \ldots, n$.

In the sequel, we will always assume that our poset is pathwise connected. As already said, there is a notion of the first homotopy group $\pi_1^o(K)$ for $K$. The subscript $o$ denotes the base point in $K$ where the homotopy is calculated, however the isomorphism class does not depend on the choice of $o$ (for this reason, often we shall write $\pi_1(K)$ without specification of the base point). We shall say that $K$ is *simply connected* whenever $\pi_1(K)$ is trivial.

If $X$ is a space having a subbase $K$ of arcwise and simply connected open sets, and if $K$ is ordered under inclusion, then there is an isomorphism $\pi_1(X) \simeq \pi_1(K)$ ([15]).

We now deal with continuous actions of symmetry groups. Let $G$ be topological symmetry group acting on a poset $K$, and $\mathcal{O}(e)$ denote the set of open neighbourhoods of the identity of the group $G$. Then we define

$$ o \ll a \iff \exists U \in \mathcal{O}(e), \; go \leq a, \; \forall g \in U. $$

(2.1)

Now, a topological symmetry group $G$ of $K$ is said to be a *continuous symmetry group of* $K$ if

$$ \forall o \in K, \; \exists a \in K, \; o \ll a, $$

(2.2)

The standard way to introduce these topological notions makes use of a simplicial set associated to the poset. We prefer do not introduce this simplicial set since it will be not explicitly used in the present paper.
\[ o \ll a \Rightarrow \exists \tilde{a} \in K \, , \, o \ll \tilde{a} \ll a \, , \quad (2.3) \]

and
\[ o \ll a_1, a_2 \Rightarrow \exists \tilde{o} \in K \, , \, o \ll \tilde{o} \ll a_1, a_2 \, . \quad (2.4) \]

This condition is suited for posets arising as subbases of topological \( G \)-spaces and, roughly speaking, encodes the idea that the sets \( \{ g_0, g \in U \} \), \( U \in \mathcal{O}(e) \), yield a neighbourhood system for \( o \). Double cones in Minkowski spacetime and the open intervals of \( S^1 \) are examples of posets acted upon continuously (in the above sense) by the Poincaré group and the \( \text{Diff}(S^1) \) group respectively. In these cases it is easily seen that \( a \ll o \) is equivalent to the condition that the closure of \( a \) is contained in \( o \). Note, in addition, that the above conditions are always verified when the symmetry group \( G \) has the discrete topology.

**Remark 2.1.** The notion of a continuous symmetry group of a poset introduced in the present paper is different from that used in [16]. We prefer this new notion of continuity because it involves in its definition only the poset and the group. The older, instead, involved the simplicial set associated to the poset. In Appendix A.3 we shall prove that the new notion of continuity is stronger than the older one, so that all the results obtained in that paper continue to hold.

### 2.2 Nets of \( C^* \)-algebras

A **net of \( C^* \)-algebras** over the poset \( K \) is given by a family \( \mathcal{A} := \{ \mathcal{A}_o \}_{o \in K} \) of unital \( C^* \)-algebras (called the *fibres*), and a family \( j := \{ j_{\tilde{o} o} : \mathcal{A}_o \to \mathcal{A}_{\tilde{o}}, o \leq \tilde{o} \} \) of unital monomorphisms (called the *inclusion maps*) fulfilling the **net relations**
\[ j_{\tilde{o} o} \circ j_{\tilde{o} o'} = j_{\tilde{o} o'} , \quad o \leq \tilde{o} \leq o' . \]

In the sequel we shall denote a net of \( C^* \)-algebras by \( (\mathcal{A}, j)_K \). When every \( j_{\tilde{o} o} \) is an isomorphism we say that \( (\mathcal{A}, j)_K \) is a **\( C^* \)-net bundle** and, to be short, we write \( j_{\tilde{o} o} := j_{\tilde{o} o}^{-1} \), \( \forall o \leq \tilde{o} \). The **restriction of \( (\mathcal{A}, j)_K \) over** \( S \subset K \) is given by the same families restricted to elements of \( S \) and is denoted by \( (\mathcal{A}, j)_S \).

Clearly, the definition of net can be given for other categories and in particular we shall use the one of Hilbert spaces, especially the case of Hilbert net bundles (whose net structure is given by unitary operators).

In all the cases of interest \( K \) shall be a subbase for the topology of a space, in general not directed, so if we would use the correct terminology in the setting of algebraic topology we should use the term **precosheaf of \( C^* \)-algebras**; nevertheless, we prefer to maintain the usual term *net*, since it is standard in algebraic quantum field theory.

A **morphism** of nets is written
\[ (\pi, f) : (\mathcal{A}, j)_K \to (\mathcal{B}, i)_P , \]
where \( f : K \to P \) is a poset morphism and \( \pi := \{ \pi_o : \mathcal{A}_o \to \mathcal{B}_{i(o)} \} \) is a family of unital morphisms such that \( i_{(\tilde{o}), i(o)} \circ \pi_o = \pi_{\tilde{o}} \circ j_{\tilde{o} o}, \forall o \leq \tilde{o} \). When \( f \) is the identity we shall write
π instead of (π, id_K). We say that (π, f) is faithful on the fibres if π_o is a monomorphism for any o; it is an isomorphism when both π and f are isomorphisms. We say that a net is trivial if it is isomorphic to the constant net (C, id), where C_o = F for a fixed C* -algebra F and any id_o is the identity of F.

The structures that we introduce in the following lines are familiar in the setting of quantum field theory, and reflect Poincaré (Möbius) symmetry and Einstein causality respectively. If G is a continuous symmetry group of K, then we say that the net (A, j)_K is G-covariant whenever there are isomorphisms α_o^g : A_o → A_{go}, ∀o ∈ K, g ∈ G, such that α_o^g ◦ j_o = j_{go, o} ◦ α_o^g, α_o^h ◦ α_o^g = α_o^{hg}, o ≤ o, ∈ K, g, h ∈ G, and fulfilling the following continuity condition: if {g_λ} ⊂ G is a net converging to e, then for any o ∈ K there exists a ≫ o and an index λ_o such that g_λ o ≤ a, ∀λ ≥ λ_o, and

\[ \| j_{g_\lambda o} \circ α_o^g(A) − j_o(A) \| → 0, \quad A ∈ A_o. \] (2.5)

If (A, j, α)_K and (B, i, β)_K are G-covariant nets, a morphism (π, f) : (A, j, α)_K → (B, i, β)_K is said to be G-covariant whenever

\[ f(\beta) = g(\alpha), \quad \pi_o \circ α_o^g = β_{f(o)} \circ \pi_o, \quad o ∈ K, \quad g ∈ G. \]

Finally, when K has a causal disjointness relation ⊥, we say that the net (A, j)_K is causal whenever

\[ [ j_o(t), j_{o'}(s) ] = 0, \quad o ⊥ o', \quad o, o' ≤ a, \]

where t ∈ A_o and s ∈ A_o.

### 2.3 The enveloping net bundle and injectivity

The importance of net bundles in the analysis of nets resides in the following fact. Let (A, j)_K be a net bundle. Since the inclusion maps j are isomorphism, they induce, for any o ∈ K, an action, the holonomy action,

\[ j_o : π_o^K(K) → \text{aut} A_o, \] (2.6)

of the homotopy group π_o^K(K) into the fibre A_o. The C*-dynamical system (A_o, π_o^K(K), j_o) is unique up to isomorphism at varying of o in K and is a complete invariant of the net bundle since the net bundle can be reconstructed (up to isomorphism) starting from the C*-dynamical system. The net bundle (A, j)_K is trivial if and only if j_o is the trivial action. We shall refer to (A_o, π_o^K(K), j_o) as the holonomy dynamical system of the net bundle.

On these grounds it is crucial to understand whether and how a net can be embedded into a net bundle.

It turns out that any net of C* -algebras can be embedded into a C* -net bundle. To be precise, the enveloping net bundle of a net of C* -algebras (A, j)_K is a C* -net bundle by (A, j)_K, which comes equipped with a morphism ε : (A, j)_K → (A, j)_K, called
the canonical embedding, satisfying the following remarkable universal properties: given morphisms with values in $\mathbb{C}^*$-net bundles,

$$(\varphi, h), (\theta, h) : (\mathcal{A}, \mathcal{J})_K \to (\mathbb{C}, y)_P, \quad (\psi, f) : (\mathcal{A}, \mathcal{J})_K \to (\mathcal{B}, \mathcal{I})_S,$$

we have

$$\{ (\varphi, h) \circ \epsilon = (\theta, h) \circ \epsilon \Rightarrow \varphi = \theta, \quad \exists (\psi^\dagger, f) \text{ such that } (\psi^\dagger, f) \circ \epsilon = (\psi, f) \}, \quad (2.7)$$

where $(\psi^\dagger, f)$ is the pullback

$$(\psi^\dagger, f) : (\mathcal{A}, \mathcal{J})_K \to (\mathcal{B}, \mathcal{I})_S. \quad (2.8)$$

These properties characterize the enveloping net bundle, that is, it is the unique, up to isomorphism, $\mathbb{C}^*$-net bundle satisfying the above relations, and this leads to the following classification: a net of $\mathbb{C}^*$-algebras is degenerate if its enveloping net bundle vanishes, and is nondegenerate otherwise. A nondegenerate net of $\mathbb{C}^*$-algebras is injective if the canonical embedding is a monomorphism.

Remark 2.2. (1) When $K$ is simply connected the $(\mathcal{A}, \mathcal{J})_K$ is a trivial $\mathbb{C}^*$-net bundle with fibres isomorphic to the Fredenhagen universal $\mathbb{C}^*$-algebra of $(\mathcal{A}, \mathcal{J})_K$ (see [10]).

(2) If $G$ is a continuous symmetry group, then the enveloping net bundle of a $G$-covariant net is $G$-covariant as well.

We can now state the functoriality property and its relation with injectivity: for any morphism $(\rho, f) : (\mathcal{A}, \mathcal{J})_K \to (\mathcal{D}, \mathcal{K})_P$ there exists a morphism $(\overline{\rho}, \overline{f}) : (\overline{\mathcal{A}}, \overline{\mathcal{J}})_K \to (\overline{\mathcal{D}}, \overline{\mathcal{K}})_P$ which fulfils the the relation

$$(\overline{\rho}, \overline{f}) \circ \epsilon = \tilde{\epsilon} \circ (\rho, f),$$

where $\epsilon$ and $\tilde{\epsilon}$ are, respectively, the canonical embeddings of the nets $(\mathcal{A}, \mathcal{J})_K$ and $(\mathcal{D}, \mathcal{K})_P$. This makes the assignment of the enveloping net bundle a functor.

Remark 2.3. (1) Note that if $(\rho, f)$ is faithful on the fibres and $(\mathcal{D}, \mathcal{K})_P$ is injective, then $(\mathcal{A}, \mathcal{J})_K$ is injective too.

(2) The functor assigning the enveloping net bundle preserves inductive limits (Prop. A.4).

### 2.4 States and representations

A state of a net of $\mathbb{C}^*$-algebras $(\mathcal{A}, \mathcal{J})_K$ is a family of states of $\mathbb{C}^*$-algebras $\omega := \{ \omega_o : \mathcal{A}_o \to \mathbb{C}, o \in K \}$ fulfilling

$$\omega_o = \omega_a \circ j_{ao}, \quad o \leq a. \quad (2.9)$$

It turns out that the set of states of a $\mathbb{C}^*$-net bundle $(\mathcal{A}, \mathcal{J})_K$ is in one-to-one correspondence with the set of invariant states of the associated holonomy dynamical system $(\mathcal{A}_o, \pi_1^o(K), j_a)$. Since in a $\mathbb{C}^*$-dynamical system having amenable group invariant states always exist, we conclude that when the fundamental group of $K$ is amenable then any nondegenerate net has states; in fact we can compose states of the enveloping net bundle.
with the canonical embedding. If \((A,j,\alpha)_K\) is \(G\)-covariant net, then a state of the net \(\varphi\) is said to be \(G\)-invariant whenever

\[
\varphi_{go} \circ \alpha^g_o := \varphi_o , \quad \forall o \in K , \ g \in G .
\]

The next result concerns the existence of \(G\)-invariant states and is proved in \([16]\):

**Proposition 2.4 \([16]\).** Let \(G\) be amenable. Then: (i) Any \(G\)-covariant \(C^*\)-net bundle having states has \(G\)-invariant states. (ii) If \(\pi_1(K)\) is amenable, then any nondegenerate \(G\)-covariant net over \(K\) has \(G\)-invariant states.

A representation of the net \((A,j)_K\) is given by a pair \((\pi, U)\), where \(\pi := \{\pi_o : A_o \to B(H_o)\}\) is a family of representations and \(U := \{U_{\delta o} : H_o \to H_{\delta o} , o \leq \delta\}\) is a family of unitary operators fulfilling the relations

\[
U_o \in (\pi_o , \pi_{\delta o} \circ j_{\delta o}) , \quad U_{o' \delta} \circ U_{\delta o} = U_{o' \delta} , \quad \forall o \leq \delta \leq o' . \tag{2.10}
\]

We call \(U\) the family of inclusion operators. We say that \((\pi, U)\) is faithful whenever \(\pi_o\) is faithful for any \(o \in K\), and that \((\pi, U)\) is a Hilbert space representation whenever any \(U_{\delta o}\) is the identity on a fixed Hilbert space (and in this case we write \((\pi, 1)\)).

It follows from (2.10) that the pair \((\mathcal{H}, U)_K\), \(\mathcal{H} := \{H_o\}\), is a Hilbert net bundle in the sense of the previous section. Using the adjoint action we obtain the \(C^*\)-net bundle \((\mathcal{B}\mathcal{H}, \text{ad}U)_K\), so that \((\pi, U)\) can be regarded as a morphism

\[
\pi : (A,j)_K \to (\mathcal{B}\mathcal{H}, \text{ad}U)_K .
\]

Hilbert space representations correspond, in essence, to morphisms with values in trivial nets. When \(K\) is simply connected any \((\mathcal{B}\mathcal{H}, \text{ad}U)_K\) is trivial, so we have only Hilbert space representations. When \(K\) is not simply connected it is very easy to give examples of nets having faithful representations but no Hilbert space representations (see \([16]\), and this is the reason why it is convenient to use the more general definition. In algebraic quantum field theory it is customary to use Hilbert spaces representations, also because the usual background is the Minkowski spacetime that is simply connected. In curved spacetimes and in \(S^1\) it is of interest to give results stating the existence of (possibly) faithful representations, and this is the motivation of our work.

Let us focus for a moment on \(C^*\)-net bundles. There exists a one-to-one correspondence between representations \((\pi, U)\) of \((A,j)_K\) and covariant representations \((\pi_o, U_o)\) of the holonomy dynamical system \((A_o, \pi^o_1(K), j_o)\). In particular \(U_o\), which is a unitary representation of the fundamental group of \(K\), is nothing but that the holonomy of the Hilbert net bundle \((\mathcal{H}, U)_K\) (see [2.6]). Since any \(C^*\)-dynamical system has faithful covariant representations, we conclude that any \(C^*\)-net bundle has faithful representations.

We now return to the general case in which \((A,j)_K\) is a net of \(C^*\)-algebras. Using the pullback (see [2.8]), we see that any representation \((\pi, U)\) of \((A,j)_K\) extends to a representation \((\pi^+, U)\) of \((\overline{A}, \overline{j})_K\) and this yields a one-to-one correspondence between representations of \((A,j)_K\) and those of its enveloping net bundle. Thus, as \(C^*\)-net bundles are faithfully represented, we conclude that a net of \(C^*\)-algebras is injective if, and only if, it has faithful representations.
Let \( G \) be a continuous symmetry group of \( K \) and \((A, \gamma, \alpha)_K\) a \( G \)-covariant net. A \( G \)-covariant representation of \((A, \gamma, \alpha)_K\) is a representation \((\pi, U)\) of \((A, \gamma)_K\) with a strongly continuous family \( \Gamma \) of unitaries \( \Gamma^g : \mathcal{H}_o \to \mathcal{H}_{go}, g \in G, o \in K \), such that

\[
\Gamma^h \circ \Gamma^g _o = \Gamma^{hg} _o, \quad \text{ad}\Gamma^g _o \circ \pi = \pi \circ \alpha^g _o, \quad \Gamma^g _o \circ U_{go} = U_{go} \circ \Gamma^g _o,
\]

for all \( g, h \in G, o \leq \tilde{o} \in K \). With the term strongly continuous we mean the following property: if \( \{g_\lambda\} \subset G \) converges to the identity, then for any \( o \in K \) there exists \( a \gg o \) and an index \( \lambda_a \) such that \( g_\lambda o \leq a \) for any \( \lambda \geq \lambda_a \) and

\[
\|U_{a g_\lambda o} \Gamma^o \circ \Omega - \Omega\| \to 0, \quad \forall \Omega \in \mathcal{H}_a.
\]

(2.11)

We have the following result (see, as usual, [16]):

**Proposition 2.5.** Let \( K \) be a poset with amenable fundamental group and \( G \) an amenable continuous symmetry group of \( K \). Then every injective, \( G \)-covariant net of \( C^* \)-algebras over \( K \) has strongly continuous \( G \)-covariant representations.

### 3 Nets over \( S^1 \)

Let \( \mathcal{I} \) be the poset formed by the set of connected, open intervals of \( S^1 \) having closure \( \text{cl}(o) \) properly contained in \( S^1 \), ordered by inclusion; that is, \( o \leq a \) if, and only if, \( o \subseteq a \). The homotopy group of this poset is \( \mathbb{Z} \), since \( \mathcal{I} \) is a base for the topology of \( S^1 \). By a net of \( C^* \)-algebras over \( S^1 \) we mean a net of \( C^* \)-algebras over \( \mathcal{I} \).

On \( \mathcal{I} \) there is a natural causal disjointness relation: \( o \bot a \) if, and only if, \( o \cap a = \emptyset \). Important symmetries for nets over \( S^1 \) are given by \( \text{Diff}(S^1) \) or the Möbius subgroup. These groups act continuously on \( S^1 \) and, hence, on the poset \( \mathcal{I} \) as well, according to (2.2, 2.3, 2.4). These groups are non-amenable. However there are important amenable subgroups: the rotation group, the semidirect product of the translations and the dilations.

In the present section we show that any net of \( C^* \)-algebras over \( S^1 \) is injective, so it admits faithful representations. As a consequence any such a net has states and, if the net is covariant under \( \text{Diff}(S^1) \), states which are invariant under any amenable subgroup of \( \text{Diff}(S^1) \).

To prove injectivity, we shall follow the strategy of finding a family of finite, "approximating" subposets of \( \mathcal{I} \), that we call cylinders, where injectivity can be established.

#### 3.1 Idea and scheme of the proof

A strategy for proving injectivity is suggested by the analysis of inductive systems of nets (see in appendix). Let \((\mathcal{A}, J)_K\) be a net of \( C^* \)-algebras over a poset \( K \). Assume that there is a family \( \{K^\alpha\} \) of subsets of \( K \) satisfying the following conditions:

1. the family \( \{K^\alpha\} \) is upward directed under inclusion and \( K \) is the inductive limit poset of \( \{K^\alpha\} \) (each \( K^\alpha \) equipped with the order relation inherited by \( K \));
2. the net $(\mathcal{A}, j)_{K^\alpha}$, that is the restriction of $(\mathcal{A}, j)_K$ to $K^\alpha$, is injective for any $\alpha$.

Condition 1 says that the net $(\mathcal{A}, j)_K$ is the inductive limit of the system $(\mathcal{A}, j)_{K^\alpha}$. Condition 2 implies the injectivity of the inductive limit net (Theorem A.5).

Although it is a hard problem, even impossible in some cases, to find the right family of subsets of a poset where injectivity can be established, this problem, in the case of $S^1$, can be fully solved. We briefly explain how. Given a net $(\mathcal{A}, j)_I$ we will find a sequence $\{I_N\}, N \in \mathbb{N}$, of subsets of $I$ satisfying the condition 1. Concerning condition 2., first we will construct (using $I_N$) a finite poset $P_N$ and show that the net $(\mathcal{A}, j)_{I_N}$ embeds, faithfully on the fibres, into a suitable net over $P_N$. Secondly, we shall show that any net of $C^*$-algebras over $P_N$ is injective. These facts imply that the restrictions $(\mathcal{A}, j)_{I_N}$ are injective for any $N$.

The proof that any net over $P_N$ is injective relies on the isomorphism between $P_N$ and an abstract poset $C_N$, called the $N$-cylinder, and on the fact that any net over this poset is injective.

### 3.2 Nets over the $N$-Cylinder

We now introduce a class $\{C_N, N \in \mathbb{N}\}$ of finite posets. These are of interest for two reasons; the first one is that any net of $C^*$-algebras over some $C_N$ is injective, and the second one is that, as we shall explain in the following, each $C_N$ arises from a suitable simplicial approximation of the circle.

As we shall see soon $C_N$ can be seen as a lattice of $N^2$ elements on a cylinder of finite height. To deal with periodicity we shall use the equivalence relation $\text{mod } N$ with the following convention: we choose as representative elements of the classes associated with the equivalence relation $\text{mod } N$ the numbers $1, 2, \ldots, N$. So, for instance, for $N = 4$ we have, $(0)_4 = 4, (-1)_4 = 3, (5)_4 = 1$, etc. Using this convention, elements of the $N$-cylinder $C_N$ are pairs $(i, l)$ with $i, l \in \{1, \ldots, N\}$. We shall think of $C_N$ as a matrix whose rows and columns are indexed by $l$ and $i$ respectively. The order relation is defined inductively, as follows: given an element $(i, l)$ of the $l$-row, with $l < N$, it has only two majorants in the $(l + 1)$-row, given by

\[(i, l) < (i, l + 1), \quad (i, l) < ((i - 1)_N, l + 1).\]  

Finally, the relation among $(i, l)$ and that of the $(l + t)$-rows with $t > 1$ is obtained by
transitivity. For $N = 4$, the poset $C_4$ is represented by the following diagram,

\[
\begin{array}{cccccc}
(4,4) & (1,4) & (2,4) & (3,4) & (4,4) \\
(4,3) & (1,3) & (2,3) & (3,3) & (4,3) \\
(4,2) & (1,2) & (2,2) & (3,2) & (4,2) \\
(4,1) & (1,1) & (2,1) & (3,1) & (4,1) \\
\end{array}
\]

where, for simplicity, the first column is the repetition of the last. The order relation is represented by an arrow from the smaller element to the greater one. So, $C_N$ has $N$ maximal elements, those belonging to the $N$-row, and $N$ minimal elements, those belonging to the 1-row.

The rest of the section is devoted to proving that any net of $C^*$-algebras $(A_j)_{j \in C_N}$ admits a faithful representation, a property equivalent to injectivity (see §2.4). To this end we shall use an idea of Blackadar [4]. Consider the algebras associated with the maximal elements: $A_{i,(i,N)}$. Take a cardinal $\kappa$ greater than the cardinality of any such an algebra. Let $\rho_i$ denote the tensor product of the universal representation of the algebra $A_{i,(i,N)}$ and of $1_\kappa$. Then define

\[
\pi_{i,l} := \rho_i \circ j_{i,(i,N)}(i,l), \quad l = 1, 2, \ldots, N. \quad (3.2)
\]

In words, the representation of the algebras associated with elements of the $i$-column is obtained by restricting $\rho_i$ to such algebras. In particular $\pi_{i,i} = \rho_i$. In the case of $N = 4$ we will label the columns of the above diagram as follows

\[
\begin{array}{cccccc}
\rho_4 & \rho_1 & \rho_2 & \rho_3 & \rho_4 \\
(4,4) & (1,4) & (2,4) & (3,4) & (4,4) \\
(4,3) & (1,3) & (2,3) & (3,3) & (4,3) \\
(4,2) & (1,2) & (2,2) & (3,2) & (4,2) \\
(4,1) & (1,1) & (2,1) & (3,1) & (4,1) \\
\end{array}
\]

Since universal representations and the inclusion maps are faithful, any representation $\pi_{i,l}$ is faithful.
We now define the inclusion operators. We proceed by defining the inclusion operators from a \( l \)-row to \((l - 1)\)-row, starting from \( l = N \); the others will be obtained by composition. To this end we note that the maximal element \((i, N)\) has two minorants in the \((N - 1)\)-row, \((i, N - 1)\) and \((i + 1)_N, N - 1\).

According to Definition 3.2 we take the identity operator \( \mathbb{1} \) as the inclusion operator from \((i, N - 1)\) and \((i, N)\), because these two elements belong to the same column \( i \). Concerning the inclusion operator from \(((i + 1)_N, N - 1)\) to \((i, N)\), the representations \( \pi_{(i,N)} \circ J(i,N)((i+1)_N, N-1) \) and \( \pi_{((i+1)_N, N-1)} \) are unitarily equivalent, because they are unitarily equivalent to tensor product of the universal representation of the algebra \( A_{((i+1)_N, N-1)} \) and \( 1_\kappa \) (see 3.1). So, there is a unitary operator \( V_{l,(i+1)_N} \) such that

\[
V_{l,(i+1)_N} \pi_{((i+1)_N, N-1)} = \pi_{(i,N)} \circ J(i,N)((i+1)_N, N-1) V_{l,(i+1)_N}.
\]  
(3.3)

So we take \( V_{l,(i+1)_N} \) as inclusion operator from \(((i + 1)_N, N - 1)\) to \((i, N)\); the reason way it is labelled only by the column indices will become clear soon. Given an element \((i, l)\), with \( 1 < l < N \), consider the minorants \((i, l - 1)\) and \(((i + 1)_N, l - 1)\). As before we take the identity \( \mathbb{1} \) as the inclusion operator from \((i, l - 1)\) and \((i, l)\), since they belong to the same column. But the important fact is that we may take the same operator \( V_{l,(i+1)_N} \) satisfying equation (3.3) as inclusion operator from \(((i + 1)_N, l - 1)\) to \((i, l)\). In fact by using equation (3.3) and Definition 3.2 we have

\[
V_{l,(i+1)_N} \pi_{((i+1)_N, l-1)} = V_{l,(i+1)_N} \rho_{(i+1)_N} \circ J((i+1)_N, N) ((i+1)_N, l-1) = V_{l,(i+1)_N} \pi_{((i+1)_N, N-1)} \circ J((i+1)_N, N-1) ((i+1)_N, l-1) = \pi_{(i,N)} \circ J(i,N)((i+1)_N, N-1) V_{l,(i+1)}.
\]

where we have used the relations \(((i + 1)_N, N - 1) > ((i + 1)_N, l - 1)\) and \((i, N) > (i, l)\) for \( N > l > 1 \). Hence

\[
V_{l,(i+1)_N} \pi_{((i+1)_N, l-1)} = \pi_{(i,l)} \circ J(i,l) ((i+1)_N, l-1) V_{l,(i+1)}.
\]  
(3.4)
This choice, in the case of $C_4$, corresponds to the diagramme

\[
\begin{array}{cccccc}
\rho_4 & \rho_1 & \rho_2 & \rho_3 & \rho_4 \\
(4,4) & (1,4) & (2,4) & (3,4) & (4,4) \\
V_{i_1} & V_{i_2} & V_{i_3} & V_{i_4} & V_{i_1} \\
(4,3) & (1,3) & (2,3) & (3,3) & (4,3) \\
V_{i_1} & V_{i_2} & V_{i_3} & V_{i_4} & V_{i_1} \\
(4,2) & (1,2) & (2,2) & (3,2) & (4,2) \\
V_{i_1} & V_{i_2} & V_{i_3} & V_{i_4} & V_{i_1} \\
(4,1) & (1,1) & (2,1) & (3,1) & (4,1) \\
\end{array}
\]

Finally, consider a generic inclusion $(i_k, l_k) > (i_1, l_1)$. This inclusion can be obtained by a composition

\[
(i_1, l_1) < (i_2, l_2) < \cdots < (i_{k-1}, l_{k-1}) < (i_k, l_k) ,
\]

of the generators (3.1). Given such a composition we define the inclusion operator

\[
V_{(i_k, l_k)(i_1, l_1)} := V_{(i_k, l_k)(i_{k-1}, l_{k-1})} V_{(i_{k-1}, l_{k-1})(i_{k-2}, l_{k-2})} \cdots V_{(i_2, l_2)(i_1, l_1)} ,
\]

where the inclusion operators for the generators of $C_N$ are defined according to the above prescriptions. However note that for $l_k \geq l_1 + 2$ the inclusion $(i_k, l_k) > (i_1, l_1)$ can be obtained by different compositions of the generators (3.1). For instance, for $C_4$, the inclusion $(3,1) < (2,3)$ can be obtained either

\[
(3,1) < (3,2) < (2,3) ,
\]

or

\[
(3,1) < (2,2) < (2,3) .
\]

However, the definition (3.6) does not depend on the chosen composition. Because in any such composition the following ordered sequence of transitions from a column to the preceding one ($mod \ N$) must be present:

\[
i_1 \to (i_1 - 1)_N \to \cdots \to (i_k + 1)_N \to i_k ,
\]

and no other. So the difference between two compositions of inclusions leading to $(i_k, l_k) > (i_1, l_1)$ depends on how inclusions preserving the column index are inserted between the elements of the sequence. However the inclusion operator, associated to inclusions which preserve the column index, is the identity. Since the inclusion operators associated with inclusions of the form $(i, l) < ((i - 1)_N, l + 1)$ depend only on the column index, the definition (3.6) does not depend on the chosen path. Thus the pair $(\pi, V)$ is a faithful representation of $(A, \rho)_{C_N}$ (see 2.10), and in conclusion we have the following

**Proposition 3.1.** Any net of $C^*$-algebras over $C_N$ is injective.
3.3 Finite approximations of $S^1$, and injectivity

Following the strategy outlined in §3.1 we prove that any net of $C^*$-algebras $(A, j)_I$ over $S^1$ is injective.

The subsets $I_N$. Let $\{x_n\}$ be a dense sequence of points in $S^1$. Define

$$ I_N := \bigcup_{i=1}^N I_{x_i}, \quad N \in \mathbb{N}, $$

where $I_x := \{ o \in I \mid x \notin \text{cl}(o) \}$, for a point $x$ of $S^1$, and $\text{cl}(o)$ denotes the closure of the interval $o$. Note that $I_x$ is, as a subposet of $I$, upward directed. For $N \geq 2$ the poset $I_N$ is a base of neighbourhoods for the topology of $S^1$, so its homotopy group is $\mathbb{Z}$.

The family $\{I_N\}$ satisfy condition 1. outlined in §3.1. To this end we first observe that note that $I_N \subset I_{N+1}$ for any $N \in \mathbb{N}$. So given a net of $C^*$-algebras $(A, j)_I$ consider the restrictions $(A, j)_{I_N}$ for any $N$, and for any inclusion $N \leq M$ define

$$ \left\{ \begin{array}{l}
  i^{M,N}_o(o) := o, \quad o \in I_N, \\
  i^{M,N}_o(A) := A, \quad o \in I_N, \; A \in A_o,
\end{array} \right. $$

giving unital monomorphisms $(i^{M,N}, i^{M,N}) : (A, j)_{I_N} \to (A, j)_{I_M}$.

**Lemma 3.2.** Given a net of $C^*$-algebras $(A, j)_I$ over $S^1$, then

(i) $\{(A, j)_{I_N}, (i^{M,N}, i^{M,N})\}_{N}$ is an inductive system of nets of $C^*$-algebras whose linking morphisms $(i^{M,N}, i^{M,N})$ are monomorphisms.

(ii) $(A, j)_I$ is isomorphic to the inductive limit of $\{(A, j)_{I_N}, (i^{M,N}, i^{M,N})\}_{N}$.

**Proof.** (i) easily follows from the definition of inductive system [A.2]. (ii) Once we have shown that $I$ is the inductive limit poset of $(I_N, i^{M,N})_{N}$ (see [A.1], the proof follows from the definition of $(A, j)_{I_N}$ and from the universal property of inductive limits Prop.[A.3].

To this end it is enough to observe that since $\{x_n\}$ is, by assumption, dense in $S^1$, for any $o \in I$ there exist $N_o \in \mathbb{N}$ such that $o \in I_N$ for any $N \geq N_o$. In fact, by density, there exists $N_o$ such that $x_{N_o} \in S^1 \setminus \text{cl}(o)$; hence $o \in I_{N_o}$.

So what remains to be shown is that the nets $(A, j)_{I_N}$ are injective for any $N$. We first introduce some suited finite approximations of $S^1$.

**Finite approximations of $S^1$: the poset $P_N$.** Starting from the sequence $\{x_n\}$ introduced in the previous step, we construct for any $N \in \mathbb{N}$ a finite poset $P_N$ associated to the first $N$ elements $x_1, \ldots, x_N$ of the sequence.

The definition of the poset $P_N$ is notably simplified if we assume that the points $x_1, \ldots, x_N$ are ordered as

$$ x_i < x_{i+1}, \quad i = 1, \ldots, N - 1, $$

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under the clockwise orientation of $S^1$. This does not affect the generality of the proof of the injectivity of the net $(A,j)_{2N}$ since $I_N$ does not depend on the order of the points $x_1, \ldots, x_N$.

Given the ordered $N$-ple $x_1, \ldots, x_N$, the elements of the poset $P_N$ are the open intervals $(x_i, x_k)$, for $i, k = 1, \ldots, N$, having, with respect to the clockwise orientation, $x_i$ as initial extreme and $x_k$ as final extreme respectively, ordered under inclusion. In this way, each $(x_i, x_i)$ is the interval $S^1 \setminus \{x_i\}$ and hence is maximal; on the other hand, the intervals $(x_i, x_{i+1})$, $i \neq N$, and $(x_N, x_1)$, have contiguous extreme points and hence are minimal.

To handle the periodicity with respect to clockwise rotations we use classes mod $N$, in the following way: for each $n \in \mathbb{N}$ we denote its class mod $N$ by $(n)_N \in \{1, \ldots, N\}$ (here we use the convention of §3.2), and write consequently $x_{(n)} \in S^1$. Moreover, we introduce the length function assigning to each $(x_i, x_k) \in P_N$ the positive integer

$$\ell_{i,k} := \sharp\{ j \in \{1, \ldots, N\} : (x_{(j)}_N, x_{(j+1)}_N) \subseteq (x_i, x_k) \} ,$$

where $\sharp$ stands for the cardinality. In this way each minimal element of $P_N$ has length 1 and each maximal element has length $N$. Note that the length of an interval can be easily calculated from the indices,

$$\ell_{i,k} = (k-i)_N .$$

Any interval $(x_i, x_k)$ of length $\ell_{i,k} < N$ has only two majorants among the intervals of length $\ell_{i,k}+1$, namely $(x_i, x_{(k+1)}_N)$ and $(x_{(i-1)}_N, x_k)$. Finally, note that with our conventions the points of $\{x_1, \ldots, x_N\}$ belonging to $(x_i, x_k)$ are $x_{(i+1)}_N, x_{(i+2)}_N, \ldots, x_{(i+\ell_{i,k}-1)}_N$.

**Lemma 3.3.** The poset $P_N$ is isomorphic to the cylinder $C_N$.

**Proof.** We define a mapping $f : P_N \to C_N$ as follows:

$$f(x_i, x_k) := (i, \ell_{i,k}) , \quad i, k \in \{1, 2, \ldots N\} .$$

This map has inverse

$$f'(i, \ell) := (x_i, x_{(i+\ell)}_N) , \quad (i, \ell) \in C_N ,$$

and is thus bijective. To prove that $f$ is order preserving it suffices to compute, for $\ell_{i,k} < N$

$$f(x_i, x_{(k+1)}_N) = (i, \ell_{i,k}+1) > (i, \ell_{i,k}) = f(x_i, x_{(\ell_{i,k}+i)}_N) = f(x_i, x_k) ,$$

and

$$f(x_{(i-1)}_N, x_k) = ((i-1)_N, \ell_{i,k}+1) > (i, \ell_{i,k}) = f(x_i, x_{(\ell_{i,k}+i)}_N) = f(x_i, x_k) .$$

So $f$ is an isomorphism and the proof follows. \qed
Inducing nets over $P_N$. We now show that the net $(A, j)_{I}$ induces a net over $P_N$. In the next paragraph we shall see that the restrictions $(A, j)_{I_N}$ embed, faithfully on the fibres, into such a nets.

For any $i, k = 1, 2, \ldots, N$, let

$$I_N^{(i,k)} := \left\{ o \in I \mid cl(o) \subseteq (x_i, x_k) \right\} . \tag{3.9}$$

According to this definition $(x_i, x_k) \notin I_N^{(i,k)}$ and $I_N^{(i,k)} \subseteq I_N$. The poset $I_N^{(i,k)}$ is upward directed with respect to inclusion; so $(A_o, j_{ao})_{I_N^{(i,k)}}$ is an inductive system, and we define the inductive limit C*-algebra

$$\hat{A}_{(i,k)} := \lim \{A_o, j_{ao}\}_{I_N^{(i,k)}} . \tag{3.10}$$

The algebras $\hat{A}_{(i,k)}$ are associated with the interval $(x_i, x_k)$ of $P_N$. So we have defined the fibres of a net over $P_N$. We now define the inclusion maps. Let $J_o^{(i,k)} : A_o \to \hat{A}_{(i,k)}$ be the canonical embedding for $o \in I_N^{(i,k)}$. This is a unital monomorphism satisfying the relations

$$J_o^{(i,k)} \circ j_{ao} = J_a^{(i,k)} , \quad a \leq o. \tag{3.11}$$

Note that if $(x_i, x_k) \subseteq (x_j, x_s)$ then $I_N^{(i,k)} \subseteq I_N^{(j,s)}$. We then define, for $a \in I_N^{(i,k)}$,

$$\hat{j}_{(j,s),(i,k)}(J_a^{(i,k)}(A)) := J_o^{(j,s)} \circ j_{ao}(A) , \quad A \in A_o , \tag{3.12}$$

where we take some $o \in I_N^{(j,s)}$ with $a \leq o$ since $I_N^{(i,k)}$ is directed. We easily find, applying (3.11), that our definition does not depend on the choice of $o \geq a$. It turns out that $\hat{j}_{(j,s),(i,k)}$ extends to a unital monomorphism from $\hat{A}_{(i,k)}$ into $\hat{A}_{(j,s)}$; applying (3.12), we immediately find that $\hat{j}_{(j,s),(i,k)}$ fulfills the net relations

$$\hat{j}_{(j,s),(i,k)} \circ \hat{j}_{(i,k),(m,r)} = \hat{j}_{(j,s),(m,r)} , \quad (x_m, x_r) \subseteq (x_i, x_k) \subseteq (x_j, x_s) ,$$

therefore the system $(\hat{A}, \hat{j})_{P_N}$ is a net of C*-algebras.

Lemma 3.4. The net $(\hat{A}, \hat{j})_{P_N}$ is injective for any $N$.

Proof. This follows by Proposition [3.1] and Lemma [3.3]

Conclusion: the embedding of the nets $(A, j)_{I_N}$ We now are ready to prove that any net over $S^1$ is injective.

Theorem 3.5. Any net of C*-algebras $(A, j)_{I}$ over $S^1$ is injective. In particular, the net $(A, j)_{I_N}$ embeds, faithfully on the fibres, into $(\hat{A}, \hat{j})_{P_N}$ for any $N$.

Proof. Injectivity of $(A, j)_{I}$ follows from Lemma [3.2] and Theorem [A.3] once we have shown that $(A, j)_{I_N}$ is injective for any $N$. To this end, as observed in Remark [2.3], it will be enough to show that $(A, j)_{I_N}$ embeds, faithfully on the fibres, into $(\hat{A}, \hat{j})_{P_N}$.
because the latter is an injective net.

As a first step we prove that $P_N$ is a quotient of $I_N$. Define

$$f(o) := (x_i, x_k), \quad o \in I_N$$

(3.13)

if

$$\left\{ \begin{array}{l}
cl(o) \subset (x_i, x_k) \text{ and } \\
x_{(i+1)N}, x_{(i+2)N}, \ldots, x_{(i+\ell_k-1)N} \in cl(o),
\end{array} \right.$$  

(3.14)

where the points listed in the above equation are those $x_1, x_2, \ldots, x_N$ that belong to $(x_i, x_k)$. It is clear that $f$ is order preserving and surjective, i.e. it is an epimorphism.

As a second step, we define a unital morphism faithful on the fibres from $(A, \gamma)_{I_N}$ into the injective net $(\hat{A}, \hat{\gamma})_{P_N}$, and this will suffice to conclude the proof (by functoriality).

Given $o \in I_N$, define

$$\eta_o(A) := J_{f(o)}(A), \quad A \in A_o.$$  

(3.15)

So $\eta_o : A_o \to \hat{A}_{f(o)}$ is a unital monomorphism. Given $a \leq o$, by (3.15) and (3.12), we have

$$\eta_o \circ \gamma_o = J_{f(o)}(o) \circ \gamma_o = \hat{J}_{f(o)}(f(a)) \circ J_{f(o)}(a),$$

and can conclude that $(\eta, f) : (A, \gamma)_{I_N} \to (\hat{A}, \hat{\gamma})_{P_N}$ is a unital morphism faithful on the fibres, completing the proof.

Using this theorem we deduce the following result for Diff($S^1$)-covariant nets.

**Corollary 3.6.** Any Diff($S^1$)-covariant net of $C^*$-algebras over $S^1$ has $H$-invariant states and strongly continuous $H$-covariant representations for any amenable subgroup $H$ of Diff($S^1$).

**Proof.** Since any net of $C^*$-algebras over $S^1$ is injective and the homotopy group of $S^1$ is amenable, the proof follows by Prop. 2.4 and Prop. 2.5.

We stress that, by Prop. 2.5, the covariant representations of the previous corollary induce strongly continuous $H$-representations in the sense of (2.11).

4 Comments and outlook

We list some topics and questions arising from the present paper.

1. A problem which deserves a further investigation is whether any net over $S^1$ admits faithful Hilbert space representations. This is a stronger condition than injectivity and is related with the existence of a proper ideal of the fibres of the enveloping net bundle [16].

2. An interesting question is whether it is possible to construct representations of nets over $S^1$ directly from the representations of the associated nets over $N$-cylinders [3.3].

This could be interesting because $N$-cylinders have some symmetries that might be inherited by the induced representations of the nets over $S^1$. 

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3. Having shown that any net over $S^1$ is injective, it would be interesting to understand the rôle of representations, carrying a nontrivial representation of the fundamental group of $S^1$, in chiral conformal quantum field theories and, in particular, to relate them to the new superselection sectors introduced in [7], since they carry the same topological content.

4. It is hoped that the method used in the present paper to prove injectivity of nets over $S^1$ can be used for other manifolds. This point is the object of a work in progress.

A  Inductive limits

The constructions that can be made in the category of $C^*$-algebras can be also made in the category of nets of $C^*$-algebras. The direct sum of two nets, for instance, has fibres and inclusion maps given by, respectively, the direct sum of the fibres and of the inclusion maps of the two nets. On the same line the tensor product is defined. In this appendix, however, we focus on a single construction which will turns out to be very useful for analysing injectivity: the inductive limit. We show that the category of nets of $C^*$-algebras has inductive limits. The functor assigning the enveloping net bundle turns out to preserve inductive limits, and this implies that inductive limits preserve injectivity.

A.1 Basic properties

A inductive system of nets of $C^*$-algebras is given by the following data: an upward directed poset $\Lambda$ (we shall denote the elements of $\Lambda$ by Greek letters $\alpha, \beta, \text{etc...}$, and the order relation by $\preceq$); a family of nets of $C^*$-algebras $(A^\alpha, \mathcal{J}^\alpha)_{K^\alpha}$, with $\alpha \in \Lambda$, and a family of unital morphisms $(\psi^\alpha, f^\alpha) : (A^\sigma, \mathcal{J}^\sigma)_{K^\sigma} \rightarrow (A^\alpha, \mathcal{J}^\alpha)_{K^\alpha}$ for $\sigma \preceq \alpha$, with $f^\alpha$ injective, such that

\[(\psi^\alpha \circ f^\alpha) \circ (\psi^\sigma \circ f^\sigma) = (\psi^{\alpha \circ \sigma} \circ f^{\alpha \circ \sigma}), \quad \delta \preceq \sigma \preceq \alpha.\]  

We shall denote such an inductive system by

\[\{(A^\alpha, \mathcal{J}^\alpha)_{K^\alpha}, (\psi^\alpha, f^\alpha)\}_\Lambda,\]  

and call $(\psi^\alpha, f^\alpha)$ the linking morphisms of the system.

We stress that do not assume in the definition of inductive system that the linking morphisms are monomorphisms, but we only require that $f^\alpha$ is a monomorphism of posets.

Note that by the definition of morphisms of nets, for any $\alpha \preceq \sigma$,

\[\psi^\alpha_e \circ f^\alpha_e = f^\alpha_{\mathcal{I}^\alpha_e(\sigma) \mathcal{I}^\alpha_e(\alpha)} \circ \psi^\alpha_e, \quad e \preceq \alpha,\]  

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where \( \leq^\alpha \) is the order relation of \( K^\alpha \). Moreover, it is worth observing that an inductive system of posets \( \{ K^\alpha, f^{\sigma \alpha} \}_\Lambda \) is associated with our inductive system of nets. In the following we show that any inductive system of nets of \( C^* \)-algebras has an inductive limit, which turns out to be a net of \( C^* \)-algebras over the inductive limit poset.

First of all we explain the inductive limit poset of \( \{ K^\alpha, f^{\alpha \sigma} \}_\Lambda \). This is a poset \( K \), with order relation denoted by \( \leq \), such that for any \( \alpha \in \Lambda \) there is a monomorphism \( F^\alpha : K^\alpha \rightarrow K \) satisfying the following properties:

1. \( F^\alpha \circ f^{\alpha \beta} = F^\beta \) for any \( \beta \leq^\alpha \alpha \);
2. \( K = \bigcup_{\alpha \in \Lambda} F^\alpha(K^\alpha) \);
3. for any other poset \( K' \) with a family of morphisms \( H_\alpha : K^\alpha \rightarrow K' \) such that \( H^\alpha \circ f^{\alpha \beta} = H^\beta \) there exists a unique morphism \( H : K \rightarrow K' \) such that \( H \circ F^\alpha = H^\alpha \) for any \( \alpha \in \Lambda \).

Existence of the inductive limit poset can be proved as a consequence of the more general construction of colimits in the setting of small categories (see [14, §III.3]).

Now, given \( o \in K \), we consider \( \Lambda_o := \{ \alpha \in \Lambda : o \in F^\alpha(K^\alpha) \} \subseteq \Lambda \).

Clearly \( \Lambda_o \) is not empty because of property (2) of the inductive limit poset.

**Lemma A.1.** If \( K \) is the inductive limit poset of \( (K^\alpha, f^{\sigma \alpha})_\Lambda \), then the following properties hold: (i) \( \Lambda_o \) is an upper set of \( \Lambda \) for any \( o \in K \) and, as a subposet of \( \Lambda \), it is upward directed. (ii) For any \( o \in K \) there is a map \( \Lambda_o \ni \alpha \mapsto o_\alpha \in K^\alpha \) such that

\[
\begin{align*}
 f^{\sigma \alpha}(o_\alpha) &= o_\sigma, \quad \alpha \leq^\sigma \sigma; \\
 F^\alpha(o_\alpha) &= o, \quad \alpha \in \Lambda_o.
\end{align*}
\]  

(A.4)

**Proof.** (i) Given \( \alpha \in \Lambda_o \) the element \( a \in K^\alpha \) satisfying \( F^\alpha(a) = o \) is unique because \( F^\alpha \) is injective, so we denote this element by \( o_\alpha \). Now, given \( \alpha \in \Lambda_o \) and \( \beta \in S \) with \( \alpha \leq^\beta \beta \) then \( \beta \in \Lambda_o \), since \( F^\beta(f^{\beta \alpha}(o_\alpha)) = F^\alpha(o_\alpha) = o \), so \( \Lambda_o \) is an upper set of \( \Lambda \). Thus \( \Lambda_o \) is upward directed because \( \Lambda \) is. (ii) has been proved in the course of proving (i). \( \square \)

We now construct the inductive limit of \( \{ (A^\alpha, \iota^\alpha)_K, (\psi^{\sigma \alpha}, f^{\sigma \alpha}) \}_\Lambda \). Given \( o \in K \), using the properties of the function \( \Lambda_o \ni \alpha \mapsto \sigma^\alpha \) and applying (A.4), (A.3), we see that \( \{ A^\alpha_{\Lambda_o}, \psi^{\sigma \alpha}_{\Lambda_o} \}_{\Lambda_o} \) is an inductive system of \( C^* \)-algebras over \( \Lambda_o \). So we can define the \( C^* \)-inductive limit

\[
A_o := \lim_{\Lambda_o} A^\alpha_{\Lambda_o}.
\]  

(A.5)

Now, it is worth recalling some facts about inductive limits (see [5]). First of all, there is a unital morphism \( \psi^\alpha_o : A^\alpha_{\Lambda_o} \rightarrow A_o \) such that

\[
\psi^\alpha_o \circ \psi^{\sigma \alpha}_{\Lambda_o} = \psi^\sigma_o, \quad \alpha \leq^\sigma \sigma.
\]  

(A.6)

\[3\] An upper set \( P \) of \( \Lambda \) is a subset such that if \( \sigma \in \Lambda \) and there is \( \alpha \in P \) such that \( \sigma \leq^\alpha \alpha \), then \( \sigma \in P \).
Furthermore, the unital *-algebra \( \mathcal{A}_0' \) defined by
\[
\mathcal{A}_0' := \bigcup_{\alpha \in \Lambda_o} \psi_0^\alpha (\mathcal{A}_o^\alpha) \quad (A.7)
\]
is a dense *-subalgebra of \( \mathcal{A}_o \) for any \( o \in K \). Finally, the norm \( \| \cdot \|_\infty \) of the inductive limit satisfies, for any \( A \in \mathcal{A}_o^\alpha \), the relation
\[
\| \psi_0^\alpha (A) \|_\infty = \inf_{\sigma \geq \alpha} \| \psi_{\sigma, o} \| \| \psi_{\sigma, o}^\alpha (A) \| = \lim_{\sigma \geq \alpha} \| \psi_{\sigma, o}^\alpha (A) \| \quad ,
\]
(A.8)
because the norm \( \| \psi_{\sigma, o}^\alpha (A) \| \) is monotone decreasing in \( \sigma \).

The correspondence \( \mathcal{A} : K \ni o \to \mathcal{A}_o \), where \( \mathcal{A}_o \) is the unital C\(^*\)-algebra defined by equation (A.5), is the fibre of the inductive limit net over \( K \). What is yet missing are the inclusion maps. Given \( o, \tilde{o} \in K \) with \( o \leq \tilde{o} \), we first define the inclusion map \( j_{\tilde{o}o} \) on the *-algebra \( \mathcal{A}_0' \). Afterwards we shall prove that these maps can be isometrically extended to all of \( \mathcal{A}_o \). Given \( o, \tilde{o} \) as above, take \( \alpha \in \Lambda_o \) and choose \( \sigma \in \Lambda_{\tilde{o}} \) such that \( \sigma \geq \alpha \), and, for any \( A \in \mathcal{A}_o^\alpha \), define
\[
j_{\tilde{o}o} (\psi_{\sigma, o}^\alpha (A)) := \psi_{\sigma, o}^\alpha \circ j_{\tilde{o}, o} \circ \psi_{\sigma, o}^\alpha (A) \quad .
\]
(A.9)
This definition does not depend on the choice of \( \sigma \in \Lambda_{\tilde{o}} \) with \( \sigma \geq \alpha \). In fact, take \( \gamma \in \Lambda_{\tilde{o}} \) with \( \gamma \geq \sigma \). Relations (A.6), (A.3), (A.4), and (A.1) give
\[
\psi_{\sigma, o}^\alpha \circ j_{\tilde{o}, o} \circ \psi_{\sigma, o} = \psi_{\sigma, o}^\gamma \circ j_{\tilde{o}, o} \circ \psi_{\sigma, o} \circ \psi_{\sigma, o}^\alpha
\]
\[
= \psi_{\sigma, o}^\gamma \circ j_{\tilde{o}, o} \circ j_{\tilde{o}, o} \circ \psi_{\sigma, o} \circ \psi_{\sigma, o}^\alpha
\]
\[
= \psi_{\sigma, o}^\gamma \circ j_{\tilde{o}, o} \circ \psi_{\sigma, o}^\alpha \circ \psi_{\sigma, o}^\gamma
\]
By the definition of \( \mathcal{A}_0' j_{\tilde{o}o} \) maps from \( \mathcal{A}_o' \) into \( \mathcal{A}_0' \) since \( \Lambda_o \) is upward directed. We now prove that these maps satisfy all the properties of inclusion maps.

**Lemma A.2.** Given an inductive system \( \{ (\mathcal{A}^\alpha, j^\alpha)_{\Lambda}, (\psi_{\sigma, o}^\alpha, f^\alpha) \}_{\Lambda} \) of nets of C\(^*\)-algebras the following assertions hold:

(i) The triple \( (\mathcal{A}, j)_K \) is a net of C\(^*\)-algebras.
(ii) If the system is composed of C\(^*\)-net bundles, then \( (\mathcal{A}, j)_K \) is a C\(^*\)-net bundle.

**Proof.** (i) By definition \( j_{\tilde{o}o} : \mathcal{A}_0' \to \mathcal{A}_0' \) is a unital morphism such that
\[
j_{\tilde{o}o} \circ j_{\tilde{o}o} = j_{\tilde{o}o} \quad , \quad o \leq \tilde{o} \leq \tilde{o} \quad .
\]
We now prove that \( j_{\tilde{o}o} \) are isometries. To this end, given \( \alpha \in \Lambda_o \), take \( \sigma \in \Lambda_{\tilde{o}} \) with \( \sigma \geq \alpha \). By (A.3), (A.3) and (A.1) we have
\[
\| j_{\tilde{o}o} \circ \psi_{\sigma, o}^\alpha (A) \|_\infty = \inf_{\gamma \geq \sigma} \| \psi_{\sigma, o}^\gamma \circ j_{\tilde{o}, o} \circ \psi_{\sigma, o}^\alpha (A) \| = \inf_{\gamma \geq \sigma} \| j_{\tilde{o}, o} \circ \psi_{\sigma, o}^\gamma \circ \psi_{\sigma, o}^\alpha (A) \|
\]
\[
= \inf_{\gamma \geq \sigma} \| \psi_{\sigma, o}^\gamma (A) \| = \inf_{\gamma \geq \sigma} \| \psi_{\sigma, o}^\alpha (A) \|
\]
\[
= \| \psi_{\sigma, o}^\alpha (A) \|_\infty \quad ,
\]
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since the $j_{0, o}$ are isometries. So $j_{0 o}$ admits an isometric extension to a mapping from $A_0$ into $A_o$.

(ii) We study the image of $j_{0 o}$. First of all we show that the following relations hold for every $\alpha \in \Lambda_o$,

$$j_{0 o} \circ \psi^o_\alpha (A) = \psi^o_\alpha \circ j_{0 o, \alpha} (A), \quad A \in A^o_{\alpha} ; \quad \text{(A.10)}$$

in fact, given $\sigma \in \Lambda_o$ with $\sigma \geq \alpha$, by (A.6) and (A.9) we have

$$j_{0 o} \circ \psi^o_\alpha (A) = \psi^o_\sigma \circ j_{0 o, \sigma} \circ \psi^\sigma_\alpha (A)$$

$$= \psi^o_\sigma \circ \psi^\sigma_\alpha \circ j_{0 o, \alpha} (A) = \psi^o_\alpha \circ j_{0 o, \alpha} (A) .$$

Since every $j_{0 o, \alpha} : A^o_{\alpha} \rightarrow A^o_{\sigma, \alpha}$ is an isomorphism, the previous relation gives $j_{0 o} \circ \psi^o_\alpha (A_{\alpha}) = \psi^o_\sigma (A_{\alpha})$. So, $j_{0 o}(A^o_{\alpha}) = A^o_{\sigma, \alpha}$, and this, in turn, implies that $j_{0 o}$ extends to an isomorphism from $A_0$ to $A_o$, completing the proof.

Finally, we have the following result.

**Proposition A.3.** Let $\{(A^o, j^o)^{K_o}, (\psi^\sigma_\alpha, f^\sigma_\alpha)\}_{\Lambda}$ be an inductive system of nets of $C^*$-algebras. Then for each $\alpha \in \Lambda, \beta \in \Lambda$, there is a unique morphism

$$(\Psi^\alpha, F^\alpha) : (A^\alpha, j^\alpha)^{K_o} \rightarrow (A, j)^{K_o} \quad \text{(A.1)}$$

such that

(i) $(\Psi^\sigma, F^\sigma) \circ (\psi^\sigma_\alpha, f^\sigma_\alpha) = (\Psi^\alpha, F^\alpha)$ for any $\alpha \leq \sigma$ ;

(ii) The image $\Psi^\alpha_{\alpha} (A^\alpha_{\alpha})$, as $\alpha$ varies in $\Lambda_\alpha$, is dense in $A_\alpha$ for any $\alpha \in K$;

(iii) If there is a net of $C^*$-algebras $(B, y)^{K_o}$ and a collection of morphisms $(\Phi^\alpha, F^\alpha) : (A^\alpha, j^\alpha)^{K_o} \rightarrow (B, y)^{K_o}, \alpha \in \Lambda$, such that $(\Phi^\sigma, F^\sigma) \circ (\psi^\sigma_\alpha, f^\sigma_\alpha) = (\Phi^\alpha, F^\alpha), \alpha \leq \sigma$, then there exists a unique morphism $\Phi : (A, j)^{K_o} \rightarrow (B, y)^{K_o}$ such that $\Phi \circ (\Psi^\alpha, F^\alpha) = (\Phi^\alpha, F^\alpha)$ for any $\alpha \in \Lambda$.

**Proof.** Define

$$\Psi^\alpha_{\alpha} := \psi^\alpha_{F^\alpha(a)}, \quad a \in K^\alpha . \quad \text{(A.11)}$$

By relation (A.10) we have

$$\Psi^\alpha_{\alpha} \circ j^\alpha_{ae} = \psi^\alpha_{F^\alpha(a)} \circ j^\alpha_{ae} = j^{\alpha}_{F^\alpha(a)} F^\alpha_{\alpha} (e) \circ \psi^\alpha_{F^\alpha(e)} = j^{\alpha}_{F^\alpha(a)} F^\alpha_{\alpha} (e) \circ \Psi^\alpha_{\alpha} ,$$

and this proves that $(\Psi^\alpha, F^\alpha) : (A^\alpha, j^\alpha)^{K_o} \rightarrow (A, j)^{K_o}$ is a unital morphism.

In this way (i) and (ii) are easy to prove. (iii) follows from the universal property of inductive limits of $C^*$-algebras: in fact, if $\alpha \in \Lambda_\alpha$ then $\Phi^\alpha_{\alpha} : A^\alpha_{\alpha} \rightarrow B_\alpha$ is a collection of morphisms satisfying the relation

$$\Phi^\sigma_{\alpha} \circ \psi^\sigma_{\alpha} = \Phi^\alpha_{\alpha} ,$$

by the universal property of the inductive limit $A_\alpha$ there is a unique morphism $\Phi_\alpha : A_\alpha \rightarrow B_\alpha$ such that $\Phi_\alpha \circ \Psi^\alpha_{\alpha} = \Phi^\alpha_{\alpha}$. So let $\Phi$ denotes the collection $\Phi_\alpha$, with $\alpha \in K$. 

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We now prove that $\Phi$ is a morphism of nets. Take $o, \dot{o} \in K$ and $\alpha \in \Lambda_o$, $\sigma \in \Lambda_\dot{o}$ as in Definition \ref{def:embedding}. Then we have

$$
\Phi_o \circ j_{\dot{o}o} \circ \Psi_{\dot{o}\alpha} = \Phi_o \circ \Psi_{\dot{o}\alpha} \circ j_{\dot{o}o,\alpha} \circ \psi_{\dot{o}\alpha} = \Phi_o \circ \Psi_{\dot{o}\alpha} \circ j_{\dot{o}o,\alpha} \circ \psi_{\dot{o}\alpha}
$$

for any $\alpha$. Density implies that $\Phi: (A, j)_K \to (B, y)_K$ is a morphism, and it is clear that $\Phi \circ (\Psi^o, F^\sigma) = (\Phi^o, F^\sigma)$. Finally, uniqueness follows in a similar fashion. \hfill \Box

Given a net of $C^*$-algebras satisfying the properties of the previous proposition, an inductive system $\{(A^\alpha, j^\alpha)_{K^\alpha}, (\psi^{\sigma\alpha}, f^{\sigma\alpha})\}_\Lambda$, will be called the \textit{inductive limit} of the system and, from now on, will be denoted by $\lim_{\alpha}(A^\alpha, j^\alpha)_{K^\alpha}$. The property $(iii)$ of the proposition is the \textit{universal} property of inductive limits.

\section{Injectivity of inductive limits}

By functoriality, to any inductive system of nets there corresponds an inductive system of enveloping net bundles. The functor of taking the enveloping net bundle commutes with inductive limits. As a consequence, injectivity is preserved under inductive limits indicating a strategy for analyzing injectivity.

Consider an inductive system of nets of $C^*$-algebras $\{(A^\alpha, j^\alpha)_{K^\alpha}, (\psi^{\sigma\alpha}, f^{\sigma\alpha})\}_\Lambda$. Let $(\overline{A}^\alpha, \overline{j}^\alpha)_{K^\alpha}$ be the enveloping net bundle of $(A^\alpha, j^\alpha)_{K^\alpha}$, and let

$$
\epsilon^\alpha: (A^\alpha, j^\alpha)_{K^\alpha} \to (\overline{A}^\alpha, \overline{j}^\alpha)_{K^\alpha}
$$

be the canonical embedding. By the universal property of the enveloping net bundle, for any $\alpha \leq \sigma$ there is a unique morphism $(\overline{\psi}^{\sigma\alpha}, \overline{f}^{\sigma\alpha}): (\overline{A}^\alpha, \overline{j}^\alpha)_{K^\alpha} \to (\overline{A}^\sigma, \overline{j}^\sigma)_{K^\sigma}$ satisfying the relation.

$$
(\overline{\psi}^{\sigma\alpha}, \overline{f}^{\sigma\alpha}) \circ \epsilon^\alpha = \epsilon^\sigma \circ (\psi^{\sigma\alpha}, f^{\sigma\alpha}) .
$$

(A.12)

This implies

$$
(\overline{\psi}^{\sigma\alpha}, \overline{f}^{\sigma\alpha}) \circ (\overline{\psi}^{\sigma\beta}, \overline{f}^{\sigma\beta}) = (\overline{\psi}^{\sigma\beta}, \overline{f}^{\sigma\beta}) , \quad \beta \leq \alpha \leq \sigma ,
$$

(A.13)

hence $\{ (\overline{A}^\alpha, \overline{j}^\alpha)_{K^\alpha}, (\overline{\psi}^{\sigma\alpha}, \overline{f}^{\sigma\alpha})\}_\Lambda$ is an inductive system of $C^*$-net bundles; \textit{the system of the enveloping net bundles}.

Note that even if the linking morphisms of the original system are monomorphism, in general the linking morphisms of the system of enveloping net bundles may not be monomorphisms. This depends on the injectivity of the nets of the original system.

Let $\lim_{\alpha}(\overline{A}^\alpha, \overline{j}^\alpha)_{K^\alpha}$ be the $C^*$-net bundle inductive limit of the system of the enveloping net bundles, and denote by $(\overline{\Psi}^o, F^\sigma)$ be the embedding of the nets $(\overline{A}^\alpha, \overline{j}^\alpha)_{K^\alpha}$ into this limit. We now show that this limit is nothing but the enveloping net bundle of the net $\lim_{\alpha}(A^\alpha, j^\alpha)_{K^\alpha}$.
Proposition A.4. Let \( \{ (\mathcal{A}^\alpha, \mathcal{J}^\alpha)^{\Lambda}, (\psi^\alpha, f^\alpha) \} \) be an inductive system. Then the inductive limit \( \varinjlim (\mathcal{A}^\alpha, \mathcal{J}^\alpha)_{\Lambda} \) is isomorphic to the enveloping net bundle of the inductive limit \( \varinjlim (\mathcal{A}^\alpha, \mathcal{J}^\alpha)_{\Lambda} \).

Proof. Let \((\mathcal{A}, \mathcal{J})_K\) be the enveloping net bundle of \( \varinjlim (\mathcal{A}^\alpha, \mathcal{J}^\alpha)_{\Lambda} \) and \( \varepsilon : \varinjlim (\mathcal{A}^\alpha, \mathcal{J}^\alpha)_{\Lambda} \to (\mathcal{A}, \mathcal{J})_K \) denote the canonical embedding. We start by showing that \((\mathcal{A}, \mathcal{J})_K\) embeds into \( \varinjlim (\mathcal{A}^\alpha, \mathcal{J}^\alpha)_{\Lambda} \). To this end, we use the universal property of inductive limits of nets (Proposition A.3.iii): given \( \alpha \in \Lambda \), note that \((\Psi^\alpha, F^\alpha) \circ \varepsilon^\alpha : (\mathcal{A}^\alpha, \mathcal{J}^\alpha)^{\Lambda} \to \varinjlim (\mathcal{A}^\alpha, \mathcal{J}^\alpha)_{\Lambda}\) is a morphism such that

\[
(\Psi^\alpha, F^\alpha) \circ \varepsilon^\alpha = (\Psi^\alpha, F^\alpha) \circ \varepsilon^\alpha = (\Psi^\beta, F^\beta) \circ \varepsilon^\beta,
\]

so, by the universal property of the inductive limit \( \varinjlim (\mathcal{A}^\alpha, \mathcal{J}^\alpha)_{\Lambda} \), there exists a morphism \( \theta : \varinjlim (\mathcal{A}^\alpha, \mathcal{J}^\alpha)_{\Lambda} \to \varinjlim (\mathcal{A}^\alpha, \mathcal{J}^\alpha)_{\Lambda} \) such that

\[
\theta \circ (\Psi^\alpha, F^\alpha) = (\Psi^\alpha, F^\alpha) \circ \varepsilon^\alpha, \quad \alpha \in \Lambda .
\]

On the other hand, since \( \varinjlim (\mathcal{A}^\alpha, \mathcal{J}^\alpha)_{\Lambda} \) is a \( C^\ast \)-net bundle, the universal property of the enveloping net bundle \((\mathcal{A}, \mathcal{J})_K \) says that there is a unique morphism \( \Theta : (\mathcal{A}, \mathcal{J})_K \to \varinjlim (\mathcal{A}^\alpha, \mathcal{J}^\alpha)_{\Lambda} \) such that

\[
\Theta \circ \varepsilon = \theta.
\]

We now prove that there is a morphism in the opposite direction. Consider the morphisms

\[
\varepsilon \circ (\Psi^\alpha, F^\alpha) : (\mathcal{A}^\alpha, \mathcal{J}^\alpha)^{\Lambda} \to (\mathcal{A}, \mathcal{J})_K , \quad \alpha \in S .
\]

The universal property of the enveloping net bundle \((\mathcal{A}^\alpha, \mathcal{J}^\alpha)^{\Lambda} \) implies that there are morphisms \( (\chi^\alpha, F^\alpha) : (\mathcal{A}^\alpha, \mathcal{J}^\alpha)^{\Lambda} \to (\mathcal{A}, \mathcal{J})_K \), intertwining the canonical embeddings

\[
(\chi^\alpha, F^\alpha) \circ \varepsilon^\alpha = \varepsilon \circ (\Psi^\alpha, F^\alpha) ,
\]

and compatible with the inductive structures of the enveloping net bundles

\[
(\chi^\alpha, F^\alpha) \circ (\tilde{\psi}^\alpha, \tilde{f}^\alpha) = (\chi^\alpha, F^\alpha) .
\]

By the universal property of inductive the limit \( \varinjlim (\mathcal{A}^\alpha, \mathcal{J}^\alpha)_{\Lambda} \), there is a unique morphism \( \Theta' : \varinjlim (\mathcal{A}^\alpha, \mathcal{J}^\alpha)_{\Lambda} \to (\mathcal{A}, \mathcal{J})_K \) satisfying the equation

\[
\Theta' \circ (\Psi^\alpha, F^\alpha) = (\chi^\alpha, F^\alpha) , \quad \alpha \in \Lambda .
\]

We now prove that \( \Theta \) is the inverse of \( \Theta' \). First, using equations (A.16), (A.15) and (A.19), we note that for any \( \alpha \in \Lambda ,
\]

\[
\Theta' \circ \Theta \circ \varepsilon \circ (\Psi^\alpha, F^\alpha) = \Theta' \circ \theta \circ (\Psi^\alpha, F^\alpha)
\]

\[= \Theta' \circ (\Psi^\alpha, F^\alpha) \circ \varepsilon^\alpha = (\chi^\alpha, F^\alpha) \circ \varepsilon^\alpha
\]

\[= \varepsilon \circ (\Psi^\alpha, F^\alpha) ,
\]

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so $\Theta' \circ \Theta \circ \epsilon = \epsilon$. By (2.7) we conclude that $\Theta' \circ \Theta$ is the identity automorphism of $(\mathcal{A}, j)_K$.

Conversely, equations (A.19), (A.17), (A.16) and (A.15), for any $\alpha \in \Lambda$, imply

$$
\Theta \circ \Theta' \circ (\Psi^\alpha, F^\alpha) \circ \epsilon^\alpha = \Theta \circ \epsilon (\Psi^\alpha, F^\alpha) = \Theta \circ (\Psi^\alpha, F^\alpha) = (\Psi^\alpha, F^\alpha) \circ \epsilon^\alpha,
$$

thus (2.7), implies that $\Theta \circ \Theta' \circ (\Psi^\alpha, F^\alpha)$ is the identity automorphism of $\lim\nrightarrow (\mathcal{A}^\alpha, j^\alpha)_{K^\alpha}$.

We now prove the main result of the present section.

**Theorem A.5.** Let $\{(\mathcal{A}^\alpha, j^\alpha)_{K^\alpha}, (\psi^\alpha, \epsilon^\alpha)\}_{\Lambda}$ be an inductive system of nets of $C^*$-algebras. If the linking morphisms are monomorphisms and the nets $(\mathcal{A}^\alpha, j^\alpha)_{K^\alpha}$ are all injective, then the inductive limit $\lim\nrightarrow (\mathcal{A}^\alpha, j^\alpha)_{K^\alpha}$ is an injective net.

**Proof.** It is enough to prove that the morphism

$$
\theta : \lim\nrightarrow (\mathcal{A}^\alpha, j^\alpha)_{K^\alpha} \to \lim\nrightarrow (\overline{\mathcal{A}}^\alpha, \overline{j}^\alpha)_{K^\alpha},
$$

defined in the previous proposition, is a monomorphism. To this end, considering $o \in K$ and $A \in \mathcal{A}^o_{o^o}$, we find

$$
\|\theta_o \circ \psi^\alpha_{o^o} (A)\|_\infty = \|\Psi^\alpha_o \circ \epsilon^\alpha (A)\|_\infty = \inf_{\sigma \geq \alpha} \|\psi_{o^\sigma}^\alpha \circ \epsilon^\alpha_{o^\sigma} (A)\| = \inf_{\sigma \geq \alpha} \|\epsilon^\alpha_{o^\sigma} \circ \psi^\sigma_{o^\sigma} (A)\| = \|A\|,
$$

as both $\epsilon^\alpha_{o^\sigma}$ and $\psi^\sigma_{o^\sigma}$ are isometries. Since $\bigcup_\alpha \psi^\alpha_{o^o} (\mathcal{A}^o_{o^o})$ is dense in $\mathcal{A}_o$ (Proposition A.3 ii), we conclude that $\theta_o$ is an isometry for any $o \in K$; hence $\theta$ is a monomorphism.

**A.3 On the continuity condition**

We prove that the notion of a continuous symmetry group $G$ of a poset $K$ as given in the present paper implies that introduced in [16]. We shall use the simplicial set associated to the poset and the corresponding notion of homotopy equivalence of paths. For all these notions and related results we refer the reader to the cited paper.

**Lemma A.6.** Let $G$ be a continuous symmetry group of a poset $K$. Then for any path $p : o \to a$ there $\bar{o}, \bar{a} \in K$, with $o \leq \bar{o}$ and $a \leq \bar{a}$, and an open neighbourhood $U$ of the identity $e$ of $G$ such that $g \bar{a} \leq \bar{a}$ and $g \bar{o} \leq \bar{o}$ and

$$(\bar{a}a) * p * (\bar{o}o) \sim (\bar{a}g(a)) * gp * (\bar{o}g(o)), \quad g \in U.$$
Proof. We give a proof by induction. Let \( b \) a 1-simplex. By continuity of the action of \( G \) there is \( O \in K \) such that \( |b| \ll O \). Note in particular that the faces of the 1-simplex satisfy \( \partial_0 b, \partial_1 b \ll O \). Let \( V \) be the neighbourhood of identity of \( G \) associated to \( |b| \ll O \). Then

\[
(O, \partial_0 b) * b * (O, \partial_1 b) \sim (O, g(\partial_0 b)) * g(b) * (O, g(\partial_1 b)) , \quad g \in V .
\]  

(A.20)

In fact, note that all the elements of the poset involved in the above relation are smaller than \( O \) for any \( g \in V \). Then homotopy equivalence follows because any upward directed poset is simply connected.

Assume that the above relation holds for paths which are composition of \( n \) 1-simplices. Let \( p : o \to a \) be such a path and let \( b \) a 1-simplex such that \( \partial_0 b = o \). By hypothesis there are \( o \leq \hat{o} \) and \( a \leq \hat{a} \), and an open neighbourhood \( W \) of the identity \( e \) of \( G \) such that \( ga \leq \hat{a} \) and \( go \leq o \) and

\[
(\hat{a}a) * p * (oo) \sim (\hat{a}g(a)) * g(p) * (o \hat{g}(o)) , \quad g \in W .
\]  

(A.21)

Let \( O \) and \( V \) be as in the equation (A.20). Since \( \hat{o}, O \gg o \), there exists \( o' \) such that \( o \ll o' \ll o, O \). Let \( V' \) be the neighborhood of the identity of \( G \) associated to \( o \ll o' \). If \( U := V \cap W \cap V' \), then the equation (A.20) and (A.21) are verified for any \( g \in U \).

Furthermore since \( o \leq o' \leq \hat{o} \), we have \( (\hat{o}, o') * (o' \hat{o}) \sim (\hat{o}, o) \). Since homotopy equivalence is stable under composition, we have \( (oo) * (\hat{o}o') \sim (o' \hat{o}) \). This and equation (A.21) yield

\[
(\hat{a}a) * p * (o' \hat{o}) \sim (\hat{a}g(a)) * g(p) * (o' \hat{g}(o)) , \quad g \in U .
\]  

(A.21)

The same argument applied to equation (A.20) yields

\[
(o', o) * b * (O, \partial_1 b) \sim (o', g(o)) * g(b) * (O, g(\partial_1 b)) , \quad g \in U ,
\]  

(A.23)

(recall that \( o = \partial_0 b \)). The composition of the left hand sides of the equations (A.22) (A.23) gives

\[
(\hat{a}a) * p * (o' \hat{o}) * (o', o) * b * (O, \partial_1 b) \sim (\hat{a}a) * p * b * (O, \partial_1 b) ,
\]  

(A.24)

while the composition of the right hand sides gives

\[
(\hat{a}g(a)) * g(p) * (o' \hat{g}(o)) * (o', g(o)) * g(b) * (O, g(\partial_1 b)) \sim (\hat{a}g(a)) * g(p) * b * (O, g(\partial_1 b))
\]  

(A.25)

for any \( g \in U \). Finally, the equations (A.22), (A.23), (A.24) and (A.25) give

\[
(\hat{a}a) * p * b * (O, \partial_1 b) \sim (\hat{a}g(a)) * g(p) * b * (O, g(\partial_1 b)) , \quad g \in U ,
\]

completing the proof.

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