To which densities is spin-polarized neutron matter a weakly interacting Fermi gas?

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We study the properties of spin-polarized neutron matter at next-to-next-to-next-to-leading order in chiral effective field theory, including two-, three-, and four-neutron interactions. The energy of spin-polarized neutrons is remarkably close to a non-interacting system at least up to saturation density, where interaction effects provide less than 10% corrections. This shows that the physics of neutron matter is similar to a unitary gas well beyond the scattering-length regime. Implications for energy-density functionals and for a possible ferromagnetic transition in neutron stars are discussed. Our predictions can be tested with lattice QCD, and we present results for varying pion mass.

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Introduction. – Due to the large neutron-neutron scattering length, the physics of neutron matter exhibits properties similar to a unitary Fermi gas [1,3]. The energy of neutron matter is approximately 0.4 times the energy of a free Fermi gas, and neutrons form an S-wave superfluid for densities almost up to saturation density, for recent reviews see Refs. [3, 4]. These benchmark results, combined with the possibility to simulate low-density neutron matter with ultracold atoms near a Feshbach resonance [5], have lead to the inclusion of ab initio results for neutron matter into modern energy-density functionals for nuclei [6, 7] and into predictions for neutron stars [8, 9]. Neutron matter is also interesting theoretically, because all many-body forces among neutrons are predicted in chiral effective field theory (EFT) to next-to-next-to-next-to-leading order (N^3LO) [10, 11].

In this Letter, we study the properties of spin-polarized neutron matter at N^3LO in chiral EFT [12], including consistently two- (NN), three- (3N), and four-neutron (4N) interactions. For a unitary Fermi gas, the spin-polarized system is a non-interacting gas, so we ask the question to which densities spin-polarized neutrons behave like a weakly interacting Fermi gas? While the answer is simple at low densities relevant to ultracold atoms, because P-wave interactions between neutrons are weaker and many-body forces are suppressed by a power of the density, we find the surprising result (see Fig. 1 for a preview) that the energy of spin-polarized neutrons is close to a non-interacting system at least up to saturation density n_0 = 0.16 fm^{-3}, which is well beyond the large scattering-length regime n ≲ n_0/100.

The physics of spin-polarized neutron matter is interesting, because it can provide an additional anchor point for energy-density functionals. To this end, we explore how our results compare with state-of-the-art functionals. In addition, spin-polarized matter is ferromagnetic, so that its energy compared to the spin-symmetric system determines whether a ferromagnetic transition in neutron stars is possible [13, 14]. Finally, there are fewer non-trivial contractions for spin-polarized neutrons, so that the determination of this system is easier in lattice QCD than symmetric matter [15]. Therefore, we study how our results depend on the pion mass and provide predictions that can be tested and refined with lattice QCD.

Calculational details. – We employ the N^3LO NN potential of Entem and Machleidt (EM) with a cutoff 500 MeV [16], and the potentials developed by Epelbaum, Glöckle, and Meißen (EGM) with cutoffs Λ/Λ̃ = 450/500 and 450/700 MeV [17]. In Refs. [11, 18], it was found that these potentials are perturbative in neutron matter as a result of weaker tensor forces among neu-

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FIG. 1. (Color online) Energy per particle of spin-polarized neutron matter at N^3LO as a function of density for the different EM/EGM NN potentials and including 3N and 4N interactions. The bands provide an estimate of the uncertainty in 3N forces and in the many-body calculation (see text). The solid (orange) line is the energy of a free Fermi gas (FG). The inset shows the relative size of the interaction contributions.
tronics and restricted phase space due to Pauli blocking at finite densities. Spin-polarized matter is expected to converge even faster, because S-wave interactions among polarized neutrons vanish. P-wave interactions are weaker, and Pauli blocking becomes even more effective due to the larger Fermi momentum for a given density compared to spin-symmetric matter.

We include all NN contributions up to second order in many-body perturbation theory, as well as particle-particle/hole-hole diagrams to third order (see Ref. [10]). Restricting all spins to the same spin state, the second-order contribution to the energy per particle is given by

$$ E^{(2)}_{	ext{NN}} = \frac{1}{N} \sum_{i=1}^{4} \int \frac{dk_i}{(2\pi)^3} \langle \{12\} | V_{\text{NN}} | \{34\} \rangle | (2\pi)^3 \rangle \sum_{n_{k_1} n_{k_2}} (1 - n_{k_3})(1 - n_{k_4}) \delta(k_1 + k_2 - k_3 - k_4), $$

(1)

where \( n_k \) denotes the Fermi distribution function at zero temperature and we use the short-hand notation \( i \equiv k_i \) in the bra and ket states. Taking a free or a Hartree-Fock spectrum for the single-particle energies \( \epsilon_k \) changes the results only at the 10 keV level. This indicates that the many-body calculation is very well converged, and in the following results are given with a free spectrum. In order to simplify the numerical calculations, we average over the angles of initial and final relative momenta \( k \) and \( k' \):

$$ \int \frac{dk dk'}{(4\pi)^2} | \langle k S = 1 M_S = 1 | V_{\text{NN}} | k' S = 1 M_S = 1 \rangle |^2 $$

$$ = \sum_{l,l',J} 4(4\pi)^2 C_{lJ}^{J} \langle k | V_{l'lJ} | k' \rangle \langle k' | V_{l'lJ} | k \rangle, $$

(2)

where \( V_{Sll'J} \) denote the neutron-neutron partial-wave matrix elements and \( C_{lJ}^{J} \) is the sum of Clebsch-Gordan coefficients \( C^{(l)_{m_3}}_{l_1 m_1 m_2} \)

$$ C_{lJ}^{J} = \sum_{M} C_{lJM} C_{lJM} C_{lJM} C_{lJM} C_{lJM} C_{lJM}, $$

(3)

The angular-averaging approximation has been demonstrated to be reliable for spin-symmetric matter [10] and only affects the small contributions beyond Hartree-Fock.

The energy contributions from 3N and 4N forces up to N\(^3\)LO [19,22] are calculated in the Hartree-Fock approximation, following the strategy used in Refs. [11,13]. We expect this approximation to be reliable since we found only small contributions from 3N forces at second and third order in perturbation theory in spin-symmetric neutron matter [10]. In the polarized case we expect even smaller contributions due to the enhanced Pauli blocking effects. In addition to the 3N and 4N topologies that do not contribute to the neutron-matter energy (see Ref. [10,13]) for the spin-polarized system also the 3N N\(^3\)LO two-pion-exchange–contact topology vanishes, as a consequence of the Pauli principle excluding all leading-oder NN contacts \( C_S \) and \( C_T \). Further, the 4N N\(^3\)LO diagrams \( V^e \) and \( V^f \) (according to the nomenclature in Ref. [21]) do not contribute in polarized matter. The \( C_S/C_T \) dependence of the 3N N\(^3\)LO relativistic-corrections interaction is negligible and results only in energy differences at the 1 keV level at saturation density. Thus, the many-body forces essentially depend only on the low-energy couplings \( c_1 \) and \( c_3 \), which are chosen according to Ref. [23]: \( c_1 = -(0.75 - 1.13) \) GeV\(^{-1}\) and

![FIG. 2. (Color online) Interaction contributions at N\(^3\)LO to the energy per particle of spin-polarized neutron matter as a function of density. The left panel shows the NN contributions for the three NN potentials. The width of the bands is given by the difference between second- and third-order contributions in the many-body calculation. The dashed lines are the Hartree-Fock energies. The middle panel shows the contribution from N\(^3\)LO 3N forces, where the band corresponds to the range of \( c_i \) couplings used and the 3N cutoff variation \( \Lambda = 2 - 2.5 \) fm\(^{-1}\). The right panel gives the different N\(^3\)LO 3N and 4N contributions, with corresponding \( c_i \) and cutoff variations. The 4N contributions overlap with the relativistic-corrections 3N energies.](image)
in Eq. (1). In order to probe the cutoff dependence of our calculation we also vary the 3N/4N cutoff \( \Lambda = 2 - 2.5 \text{fm}^{-1} \).

Results and discussion.— Our central result, Fig. 1 shows that the energy of spin-polarized neutrons is close to a non-interacting system, with interaction effects providing less than 10% corrections at \( n_0 \) (see the inset). The largest dependence of our calculations is on the NN interaction used. The EM 500 MeV potential leads to weakly repulsive interactions with \( E/N \approx 61.5 \text{ MeV} \) at \( n_0 \), compared to 55.7 MeV for a free Fermi gas. Using the EGM 450/500 and 450/700 MeV potentials results in even weaker interactions with \( E/N \approx 59.5 \text{ MeV} \) and \( \approx 56 \text{ MeV} \), respectively. Because \( n_0 \) for polarized matter corresponds to a high Fermi momentum of 2.1 fm\(^{-1} \), these small differences are due to the range in NN scattering predictions at these higher momenta.

By comparing our results with the corresponding energy range for spin-symmetric matter, \( E/N \approx 14 - 21 \text{ MeV} \) at \( n_0 \) \cite{11,18}, it is clear that a phase transition to the ferromagnetic state is not possible for \( n \lesssim n_0 \). Further, we expect the energy of spin-polarized neutrons at higher densities to lie above the free Fermi gas due to repulsive 3N forces (see also Fig. 2). Assuming the energy of spin-polarized neutrons remains close to a free Fermi gas also for higher densities, we can use the general equation of state constraints of Ref. \cite{24} to provide constraints for the onset of a possible ferromagnetic phase transition. Taking the three representative equations of state \cite{24}, a phase transition to a ferromagnetic state may be possible for \( n/n_0 \gtrsim 6.1, 3.4, \) and 2.3 for the soft, intermediate, and stiff equations of state, respectively. Note that if more massive neutron stars are discovered, e.g., with \( 2.4M_\odot \), the soft case is ruled out \cite{24}.

Figure 2 shows the individual interaction contributions. All energies are small compared to the spin-symmetric system \cite{11,18}. The left panel shows the NN contributions for the three N\(^3\)LO potentials. The different behavior can be traced to different predictions for the scattering phase shifts. The EM 500 MeV potential gives a net repulsive contribution, with \( E/N \approx 3.1 \text{ MeV} \) at \( n_0 \) (5.6% relative to \( E_{\text{FG}} \)). Up to densities \( n \lesssim 0.1 \text{fm}^{-3} \) the EGM 450/500 and 450/700 MeV potentials are in good agreement and provide only \( E/N \approx -0.5 \text{ MeV} \) at \( n = 0.08 \text{fm}^{-3} \), and then start to differ. The middle panel of Fig. 2 shows the contributions from the leading N\(^3\)LO 3N forces. The 3N interactions are, as in the spin-symmetric case, repulsive but with much smaller energies in the range 0.8 – 1.9 MeV at \( n_0 \). In the right panel, we show all contributions from the N\(^3\)LO many-body forces. The dominant contributions are from two-pion-exchange 3N forces with energies \(-0.9 - 1.6 \text{ MeV} \) at \( n_0 \). This is almost as large as the leading contribution of the two-pion-exchange topology, and shows that one is pushing the chiral EFT expansion to the limits. However, all these 3N contributions are still small. In addition, there are repulsive contributions from pion-ring 3N forces, which contribute 1.1 – 2.1 MeV at \( n_0 \) and counteract these. Finally, there are small repulsive contributions from the two-pion–one-pion-exchange 3N topology of 0.1 – 0.2 MeV at \( n_0 \), small attractive contributions from the relativistic-corrections 3N topology, while three-pion-exchange 4N interactions contribute only \(-0.1 \text{ MeV} \) at \( n_0 \). In total, the 3N+4N contributions provide a net repulsion of \( E/N = 1 - 2.2 \text{ MeV} \) at \( n_0 \).

In Fig. 3 we compare our results with predictions based on state-of-the-art energy-density functionals, following Ref. \cite{25}. SIII \cite{26}, SGII \cite{27}, SkM* \cite{28}, SLy4 and SLy5 \cite{29}, SkO and SkO' \cite{30}, BS\(k\)9 \cite{31}, as well as SAMI \cite{32} and using the Gogny D1N interaction \cite{33}. At low densities \( n \lesssim 0.01 \text{fm}^{-3} \) all functionals agree with a free Fermi gas. However, at higher densities we find significant deviations. In best agreement with our calculations are the functionals SIII, SkO, SGII, SkM*, and SLy5, whereas the latter two reproduce the free Fermi gas and the former provide small repulsive contributions. The predictions of the functionals SLy4, SAMI, BS\(k\)9, and SkO' differ significantly from our N\(^3\)LO bands. Therefore, it will be interesting to use our results as additional neutron-matter constraint for modern functionals.

Note that in density functionals the instability to a ferromagnetic state is caused by the unphysical decrease in the energy of the polarized system, cf. SkO' in Fig. 3.

For comparison with lattice QCD simulations, we also vary the pion mass in NN, 3N, and 4N interactions. For this estimate we only take into account the explicit pion exchanges and do not vary the pion mass implicitly in
that including interactions gives a very similar dependence, but away from the physical pion mass, the energy starts to deviate more from the free Fermi gas. As in spin-symmetric matter, the interaction contributions also increase the chiral condensate, as determined from the slope in $m_\pi$. We emphasize that for a precision comparison, one also needs to include the $m_\pi$ dependence of the low-energy couplings in nuclear forces.

**Summary and outlook.** We have presented a complete $N^3$LO calculation of spin-polarized neutron matter, where the dominant uncertainty is due to the NN potential used, as well as due to the uncertainty in 3N forces. The uncertainty from the many-body calculation is very small (shown by the bands in the left panel of Fig. 2). Our results show that the energy of spin-polarized neutrons is remarkably close to a non-interacting system. This shows that the physics of neutron matter is similar to a unitary gas well beyond the scattering-length regime. Moreover, our results provide constraints for energy-density functionals of nuclei and show that a phase transition to a ferromagnetic state is not possible for $n \lesssim n_0$. Finally, our predictions can be tested and refined with lattice QCD calculations of spin-polarized neutrons in a box.

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