SUBGROUPS OF CHEVALLEY GROUPS OF TYPES $B_l$ AND $C_l$, CONTAINING THE GROUP OVER A SUBRING, AND CORRESPONDING CARPETS

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Abstract. We continue the study of subgroups of the Chevalley group $G_P(\Phi, R)$ over a ring $R$ with root system $\Phi$ and weight lattice $P$, containing the elementary subgroup $E_P(\Phi, K)$ over a subring $K$ of $R$. A. Bak and A. V. Stepanov considered recently the case of symplectic group (simply connected group with root system $\Phi = C_l$) in characteristic 2. In the current article we extend their result for the case $\Phi = B_l$ and for the groups with other weight lattices. As well as in the Ya. N. Nuzhin’s work on the case where $R$ is an algebraic extension of a nonperfect field $K$ and $\Phi$ is not simply laced the description uses carpet subgroups parametrized by two additive subgroups. In the second part of the article we establish the Bruhat decomposition for these carpet subgroups and prove that they have a split saturated Tits system. As a corollary we obtain that they are simple as abstract groups.

Introduction

Let $G_P(\Phi, R)$ be a Chevalley–Demazure group scheme, $E_P(\Phi, K)$ its elementary subgroup, and let $K \subseteq R$ be a pair of rings (by default all rings are commutative with a unit element 1 and all ring homomorphisms preserve 1) In the current article we study the lattice $L$ of subgroups of $G_P(\Phi, R)$, containing $E_P(\Phi, K)$, assuming that the root system $\Phi$ is not simply laced, i.e. $\Phi = B_l$, $C_l$, $F_4$ or $G_2$.

For root simply laced root systems reasonable description can be obtained only if $R$ is quasi algebraic over $K$, see. [32] [33]. The standard description for an algebraic field extension was obtained in [32] (in this case $R$ is quasi algebraic over $K$ as well). For not simply laced root systems the situation is quite different. In particular, for doubly laced root systems ($\Phi = B_l$, $C_l$ or $F_4$) the standard description was obtained in [34] for an arbitrary pair of rings provided that 2 is invertible in $K$. To specify the meaning of the standard description of the lattice $L$ consider the lattice $L(D, G)$ consisting of subgroups of an abstract group $G$, containing a given subgroup $D$. A subgroup

$F \in L(D, G)$

is called $D$-full, if the normal closure $D^F$ coincides with $F$. A sandwich of the lattice $L(D, G)$ is the set of all subgroups containing a given $D$-full subgroup $F$ and lying in its normalizer $N_G(F)$. We say that the lattice $L(D, G)$ satisfies the sandwich classification theorem if it splits into the union of sandwiches (with our definition the union is disjoint as the only $D$-full subgroup in a sandwich equals $D^H$ for all $H$ in this sandwich). Now, by the standard description of the intermediate subgroup lattice we mean the sandwich classification theorem together with a description of all $D$-full subgroups.

In the cases of the standard description of the lattice $L$ mentioned above the sandwiches were parametrized by subrings $S$ of $R$, containing $K$, and $E_P(\Phi, S)$-full subgroups were the elementary subgroups $E_P(\Phi, S)$. Observe that if $S$ and $R$ are fields, then the normalizer of $E_P(\Phi, S)$ in $G_P(\Phi, R)$ in equal to the product of $G_P(\Phi, S)$ by the center of $G_P(\Phi, R)$. If the structure constants from the Chevalley commutator formula are not invertible, then there exist other $E_P(\Phi, K)$-full subgroups.

Denote by $p = p(\Phi)$ the maximal multiplicity of an edge in the Dynkin diagram of, what amounts to be the same, the maximal structure constant in the Chevalley commutator formula. In other words, $p = 2$ if $\Phi = B_l$, $C_l$ ($l \geq 2$), $F_4$, and $p = 3$ if $\Phi = G_2$. Let $\Lambda = (\Lambda_l, \Lambda_s)$ be a pair of additive subgroups of $R$, satisfying the following conditions:

AP1. $p\Lambda_s \subseteq \Lambda_l \subseteq \Lambda_s$;
AP2. $t^p\Lambda_t \subseteq \Lambda_l$ for any $t \in \Lambda_s$;
AP3. $\Lambda_s$ is a subring if $\Phi \neq B_l$;
AP4. $\Lambda_l$ is a subring if $\Phi \neq C_l$.

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(if \( \Phi = B_2 = C_2 \), then neither \( \Lambda_s \), nor \( \Lambda_l \) must be subrings). If \( \Phi = C_l \), \( n \geq 3 \) such a pair is a particular case of a form ring in the sense of A. Bak \[3\]. In general we call such a pair an admissible pair of type \( \Phi \). The notion of admissible pair is defined in the works E. Abe and K. Suzuki \[2, 1\] in a similar way.

Define a family of subgroups \( \mathfrak{A} = \{ \mathfrak{A}_\alpha \mid \alpha \in \Phi \} \) by

\[
\mathfrak{A}_\alpha = \begin{cases} 
\Lambda_s, & \text{if } \alpha \text{ is a short root,} \\
\Lambda_l, & \text{if } \alpha \text{ is a long root.}
\end{cases}
\]

It turns out that this family is an elementary carpet of type \( \Phi \) in the sense of V. M. Levchuk \[18\]. For the carpet \( \mathfrak{A} \), corresponding to an admissible pair \( \Lambda \), its elementary carpet subgroup

\[
E(\Phi, \Lambda) = E(\Phi, \mathfrak{A}) = \langle x_\alpha(\mathfrak{A}_\alpha) \mid \alpha \in \Phi \rangle
\]

belongs to the lattice \( \mathcal{L} \), if \( K \subseteq \Lambda_l \) and almost always is \( E(\Phi, K) \)-full. Thus, if the structure constants are not invertible, then there appear new sandwiches, corresponding to admissible pairs instead of rings.

The sandwich classification theorem for the lattice \( \mathcal{L} \) with noninvertible structure constants are established in the works \[22\] and \[4\]. The former considers the case of an algebraic field extension, whereas the latter deals with the group \( \text{Sp}_{2l} \) over arbitrary rings \( K \subseteq R \) with \( 2 = 0 \). It is well-known that in characteristic 2 Chevalley groups of types \( B_l \) and \( C_l \) are almost the same. In particular, over perfect fields they are isomorphic. In the current article we extend the results of \[4\] to all forms of Chevalley groups of types \( B_l \) and \( C_l \) and on the Steinberg group of type \( C_l \). This is done by the homomorphisms \( G(C_l, R) \to G(B_l, R) \) and group-theoretic arguments. Conjecturally, for the groups of type \( F_4 \) the same result is also true, but the proof needs an extra knowledge about the normalizer of the elementary carpet subgroup. Therefore, it will be addressed in a separate article.

Besides, for fields of bad characteristic we study elementary carpet subgroups corresponding to admissible pairs. In particular, we prove the Bruhat decomposition for these subgroup and establish their simplicity.

The paper is organized as follows. In \[1]\ we introduce the main notation to be used in the article. Section 2 is devoted to the group-theoretic aspects of the sandwich classification theorem. In \[3]\ we prove the existence of scheme morphisms

\[
G_{sc}(C_l, -) \xrightarrow{\sim} G_{sc}(B_l, -).
\]

Based on the results of two previous sections, in \[4]\ the sandwich classification theorem is extended from simply connected group of type \( C_l \) to all \( \Phi \) Chevalley groups of types \( B_l \) and \( C_l \) and to the Steinberg group of type \( C_l \). In the remaining part of the article we study the carpet groups over fields. For the reader’s convenience in \[4]\ we recall known statements about these groups to be used in the sequel. In section 6 we state a corollary from the work \[22\] and pose some problems on admissible pairs, which are negatively solved in \[7\]. Also in \[7\] we discuss the question why there are no admissible pairs between a principle ideal domain and its field of fractions. In \[8\] we establish the Bruhat decomposition in the carpet subgroup, corresponding to an admissible pair. The last section is to prove simplicity of this subgroup. This is done with help of the notion of \( (B, N) \)-pair.

## 1. Main notation

Some definitions and notation were already formulated in the introduction, the others are given in the current section.

Let \( G \) be a group and \( a, b, c \in G \). Denote by \( a^b = b^{-1}ab \) the conjugate to \( a \) by \( b \). The commutator \( a^{-1}b^{-1}ab \) is denoted by \( [a, b] \). Let \( F \) and \( H \) be subsets of \( G \). By \( \langle F \rangle \) we denote the subgroup, generated by \( F \). We denote by \( F^H \) the subgroup of \( G \), generated by the elements \( a^b \) over all \( a \in F \) and \( b \in H \). If \( H \) is a subgroup, \( F^H \) is the smallest subgroup in \( G \) containing \( F \) and normalized by \( H \). The mutual commutator subgroup is a subgroup of \( G \) generated by all commutators \( [a, b] \), \( a \in F \), \( b \in H \). The normalizer of a subgroup \( H \) in \( G \) is denoted by \( N_G(H) \).

For a ring \( R \) by \( R^2 \) we denote the set of all squares of elements of \( R \). In the current article the set \( R^2 \) is considered for a ring \( R \) of characteristic 2. In this case \( R^2 \) is a subring.

Let \( \Phi \) be a reduced irreducible root system and \( G_\Phi(\Phi, -) \) a Chevalley–Demazure group scheme of type \( \Phi \) with the weight lattice \( P \). If the weight lattice is not important we write simply \( G(\Phi, -) \). The simply

\[\text{Since the center of the groups of type } B_l \text{ and } C_l \text{ in characteristic 2 is trivial, it may seem as they do not depend upon the weight lattice. However, the statement about the center holds only over reduced rings, and moreover, the map from the simply connected group the adjoint one is not necessarily surjective even for reduced rings.}\]
connected scheme (i.e. where $P = P(\Phi)$) is denoted by $G_{\text{sc}}(\Phi, \varnothing)$. Let $R$ be a ring. The elementary subgroup $E(\Phi, R)$ of a Chevalley group $G(\Phi, R)$ is generated by the root subgroups

$$X_\alpha(R) = \{x_\alpha(r) \mid r \in R\},$$

over all $\alpha \in \Phi$. In case, where $R$ is a field or, more generally, a semilocal ring, $E_{\text{sc}}(\Phi, R)$ coincides with the whole Chevalley group $G_{\text{sc}}(\Phi, R)$.

For each $\alpha \in \Phi$ the scheme $X_\alpha$ is isomorphic to $\mathbb{G}_a$. The isomorphism $\mathbb{G}_a \to X_\alpha$ is denoted by $x_\alpha$. Thus for all $r, s \in R$ we have

$$x_\alpha(r)x_\alpha(s) = x_\alpha(r + s).$$

Furthermore, the Chevalley commutator formula

$$[x_\alpha(r), x_\beta(s)] = \prod_{i, j > 0, \ i \alpha + j \beta \in \Phi} x_{i \alpha + j \beta}(C_{i \alpha, j \beta} r^{i} s^{j}), \ \alpha \neq \pm \beta \in \Phi,$$

holds in the elementary subgroup. Here $C_{i \alpha, j \beta}$ are nonzero integer constants not bigger than 3 (set $C_{i \alpha, j \beta} = 0$ if $i \alpha + j \beta \notin \Phi$, this will allow to skip the condition $i \alpha + j \beta \notin \Phi$).

The group with generators $y_\alpha(r), \ \alpha \in \Phi, \ r \in R$ subject to relations \[1.1\] and \[1.2\] with all $x$ substituted by $y$ is called the Steinberg group of type $\Phi$ over a ring $R$. It is denoted by $\text{St}(\Phi, R)$. The kernel of the natural epimorphism $\pi: \text{St}(\Phi, R) \to E(\Phi, R)$ that sends $y_\alpha(r)$ to $x_\alpha(r)$ is denoted by $K_{2}(\Phi, R)$. Except certain root systems of small rank, the Steinberg group is centrally closed. Conjecturally in all these cases $K_{2}(\Phi, R)$ is central in $\text{St}(\Phi, R)$ and hence, is the Schur multiplier of the elementary subgroup. Up to now the conjecture is proved for the root systems $A_l, l \geq 3$ (W. van der Kallen [14]), $C_l, l \geq 3$ (A. Lavrenov [16]), $E_{6}, E_{7}, E_{8}$ (S. Sinchuk [20]) and $D_{l}, l \geq 4$ (A. Lavrenov, S. Sinchuk [17]). We can not include the group $\text{St}(B_{l}, R)$ in the theorem [1.1] exactly because this conjecture is not proved for the group of type $B_{l}$.

Let us call a carpet of type $\Phi$ over $R$ a family of additive subgroups

$$\mathfrak{A} = \{\mathfrak{A}_\alpha \mid \alpha \in \Phi\}$$

of $R$ satisfying the condition

$$C_{i \alpha, j \beta} \mathfrak{A}_\alpha \mathfrak{A}_\beta \subseteq \mathfrak{A}_{i \alpha + j \beta}, \ \text{при} \ \alpha \neq \pm \beta \in \Phi, \ i > 0, \ j > 0,$$

where $\mathfrak{A}_\alpha = \{a^{i} \mid a \in \mathfrak{A}_\alpha\}$, and the constants $C_{i \alpha, j \beta}$ are defined by the Chevalley commutator formula.

An elementary carpet $\mathfrak{A}$ of type $\Phi$ over $R$ defines the elementary carpet subgroup

$$E(\Phi, \mathfrak{A}) = \{x_\alpha(\mathfrak{A}_\alpha) \mid \alpha \in \Phi\}$$

defines the element of $G(\Phi, R)$. In the current article we deal only with elementary carpets and elementary carpet subgroups. Therefore, we shall skip the word “elementary” in this context. A carpet $\mathfrak{A}$ of type $\Phi$ over a ring $R$ is called closed, if its carpet subgroup does not contain extra root elements, i.e. if

$$E(\Phi, \mathfrak{A}) \cap x_\alpha(R) = x_\alpha(\mathfrak{A}_\alpha) \ \text{for all} \ \alpha \in \Phi.$$

In the scheme $G(\Phi, \varnothing)$ we fix a split maximal torus $T$ and an ordering of the root system. Denote by $N$ the scheme normalizer of the torus $T$ and let $U$ be the unipotent radical of the Borel subgroup:

$$U(R) = \{x_\alpha(R) \mid \alpha \in \Phi^{+}\}.$$

In an appropriate matrix representation we may assume that $T(R)$ is the set of diagonal matrices, $N(R)$ is the set of monomial matrices, and $U(R)$ is the set of upper unitriangular matrices inside $G(\Phi, R)$.

For a ring $R$ with a connected spectrum the quotient group $N(R)/T(R)$ is isomorphic to the Weyl group $W = W(\Phi)$ of $\Phi$. Note that over any ring $R$ there exists a preimage of a given element $w \in W$ in $N(R)$, because it exists already in $N(\mathbb{Z})$.

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\[2\] In the book by R. Steinberg [29] by a Chevalley group over a field the author means its elementary subgroup. However, in non simply connected case this group is not algebraic. For instance, the group $\text{PSL}_n(F)$ is the elementary subgroup of the adjoint Chevalley group of type $\Lambda_{n-1}$ over $F$, but cannot be defined by polynomial equations as soon as $F$ is infinite and does not contain $n$th root of at least one element.
2. Sandwich classification theorem

In this section we develop group theoretic methods of proof of the sandwich classification theorem. First, recall two facts obtained in the works [30, 31, 34], see also [35]. In the sequel the group is called perfect, if it coincides with its commutator subgroup.

Let $D$ be a perfect subgroup of an abstract group $G$. Then for a subgroup $H \in L(D, G)$ the normal closure $D^H$ coincides with the mutual commutator subgroup $[H, D]$. It is also clear that the sandwich classification theorem for the lattice $L(D, G)$ is equivalent to the equalities $[H, D, D] = [H, D]$ for all subgroup $H$ from this lattice.

Lemma 2.1 ([30] Lemma 1). Let $D$ be a perfect subgroup of a group $G$, normalizing a subgroup $H \subseteq G$. Then $[H, D, D]$ is normal in $H$ if and only if $[H, D, D] = [H, D]$.

Lemma 2.2 ([31] Proposition 1.9). Let $D \subseteq G$ and let $\varphi: G \rightarrow \overline{G}$ be a group epimorphism. The sandwich classification theorem for the lattice $L(D, G)$ implies the sandwich classification theorem for the lattice $L(\varphi(D), \overline{G})$. Moreover, $\varphi(D)$-full subgroups of $\overline{G}$ are the images of $D$-full subgroups of $G$ and their normalizers are the images of the normalizers of the corresponding $D$-full subgroups of $G$.

The next two lemmas allows to lift the sandwich classification theorem to a central extension.

Lemma 2.3. Let $D$ be a perfect subgroup of $G$, $F$ a $D$-full subgroup of $G$, and $\pi: S \rightarrow G$ an epimorphism with a central kernel.

1. There exists the smallest subgroup $\bar{D} \leq S$ such that $\pi(\bar{D}) = D$.
2. Let $\bar{F}$ be the smallest, whereas $\bar{F}$ an arbitrary preimage of $F$ under $\pi$ (since any $D$-full subgroup is perfect, $\bar{F}$ exist in accordance to the first item). For subgroups $\bar{D}$ and $\bar{F}$ the group $S$ such that $\pi(\bar{D}) = D$ and $\pi(\bar{F}) = F$ we have $[\bar{F}, \bar{F}] = [\bar{D}, \bar{F}] = \bar{F}$. In particular the group $\bar{D}$ is perfect and $\bar{F}$ is $D$-full.

Proof. Put $\bar{D} = [\pi^{-1}(D), \pi^{-1}(D)]$ and $\bar{F} = [\pi^{-1}(F), \pi^{-1}(D)]$. Since $D$ is perfect and $F$ is $D$-full, $\pi(\bar{D}) = D$ and $\pi(\bar{F}) = F$. For arbitrary preimages $\bar{D}$ and $\bar{F}$ of the groups $D$ and $F$ respectively we have $\pi^{-1}(D) = D \ker \varphi$ and $\pi^{-1}(F) = \bar{F} \ker \varphi$. Hence,

$$\bar{F} = [\pi^{-1}(F), \pi^{-1}(D)] = [\bar{F} \ker \varphi, \bar{D} \ker \varphi] = [\bar{F}, \bar{D}]$$

as $\ker \varphi$ lies in the center of $S$. For $F = D$ we get $\bar{D} \leq \bar{D}$, otherwise, for $\bar{D} = \bar{D}$ we conclude that $\bar{D}$ is perfect. In general it is clear that $\bar{F}$ contains $\bar{D}$. This observation with $\bar{D} = \bar{D}$ implies that $\bar{F} = [\bar{F}, \bar{D}] \leq \bar{F}$. Thus, $\bar{F}$ as well as $\bar{D}$ is the smallest preimage. Finally, the second assertion of the lemma is the general case of the displayed formula.

Note that $\bar{F}$ can be defined by the same formula as $\bar{D}$, i.e. $\bar{F} = [\bar{F}, \bar{F}]$.

Lemma 2.4. In the notation of the previous lemma the sandwich classification theorems for the lattices $L(D, G)$ and $L(\bar{D}, S)$ are equivalent and $\pi$ induces a bijection between the set of all $\bar{D}$-full subgroups of $S$ onto the set of of all $D$-full subgroups of $G$.

Proof. It follows from the Lemma 2.2 that the sandwich classification theorem for the lattice $L(\bar{D}, S)$ implies the sandwich classification theorem for the lattice $L(D, G)$ and the map induced by $\pi$ is surjective. Injectivity of this map follows immediately from the previous lemma as in the set of all preimages of a $D$-full subgroup is only one $\bar{D}$-full, namely the smallest one.

Now, suppose that the sandwich classification theorem holds for the lattice $L(D, G)$. Let $H \in L(\bar{D}, S)$. Then $\pi(H)$ normalizes some $D$-full subgroup $F$. Let $\bar{F}$ be the smallest preimage of $F$ in $S$. Then $\pi([H, \bar{D}]) = [\pi(H), D] = F$. By the previous lemma $[H, \bar{D}, \bar{D}] = [[H, \bar{D}], [H, \bar{D}]] = \bar{F}$. Hence, $[H, \bar{D}, \bar{D}]$ is normal in $H$ as a commutator subgroup of a normal subgroup. Now, Lemma 2.1 implies that $[H, \bar{D}] = [H, \bar{D}, \bar{D}] = \bar{F}$, which completes the proof.

Clearly, if $G \geq E \geq D$, then the sandwich classification of the lattice $L(D, G)$ implies the sandwich classification of the lattice $L(D, E)$. Moreover, $D$-full subgroups of the latter lattice are just $D$-full subgroups of the lattice $L(D, G)$ contained in $E$. The converse is not true in general, even if we assume that $D$ is perfect, $H$ normal in $G$, and $G/H$ is abelian.\(^3\)

\(^3\)Let $S$ be a nonabelian simple group, $E = S \ast S$, $D$ the normal closure of one of free factors in $E$, and $G \cong E \times \mathbb{Z}/2\mathbb{Z}$ an extension of $E$ by the automorphism that changes free summands. Then the lattice $L(D, E)$ consists of one sandwich, $D^G = E$, whereas $D^{G'} = D$, i.e. the sandwich classification theorem does not hold for $L(D, G)$.
However, under certain additional assumption this implication holds. This statement will allow us to extend the sandwich classification theorem from the elementary subgroup to the whole Chevalley group.

**Lemma 2.5.** Let $D \leq E \triangleleft G$. Assume that the sandwich classification theorem holds for $L(D,E)$. Suppose further that the following conditions are satisfied.

1. The group $D$ is generated by a union of finitely generated perfect subgroups.
2. Given a $D$-full subgroup $F \leq E$, the quotient group $N_E(F)/F$ is quasi-solvable, i.e. is the union of an ascending chain of solvable groups.

Then the lattice $L(D,G)$ satisfies the sandwich classification theorem and the sets of $D$-full subgroups in $G$ and $E$ coincide.

**Proof.** Since $D$ is generated by a union of perfect subgroups, it is perfect itself. Then any $D$-full subgroup $F$ is perfect as well as it is generated by perfect subgroups $D^f$ over all $f \in F$.

Let $H \leq G$. Since $E$ is normal in $G$, the subgroup $D^H$ belongs to the lattice $L(D,E)$ and hence, is contained in some sandwich $L(F,N)$. Let $D$ be a perfect finitely generated subgroup in $D$ and $h \in H$. Then the subgroup $D^h$ is finitely generated and perfect. By condition (2), $N$ is the union of an ascending chain of subgroups $N_i$ such that $N_i/F$ are solvable. In particular, $F$ is the largest perfect subgroup in each $N_i$. Since $D^h$ is finitely generated it is contained in $N_i$ for some index $i$. Hence, $D^h \leq F$. Since $h$ is an arbitrary element of the group $H$ and $D$ is generated by its perfect finitely generated subgroups, we obtain $D^H \leq F$. Thus, given a subgroup $H$ in $G$ the group $D^H = F$ is $D$-full, as required.

The statement about the sets of $D$-full subgroups is obvious.

Properties (1) and (2) of the previous lemma are internal properties of the group $E$. With a help of Lemma 2.5 it easy to see that these properties are preserved by epimorphisms. Therefore, the sandwich classification theorem can be extended from an epimorphic image of the group $E$ to its arbitrary extension.

**Corollary 2.6.** Let $D \leq E' \leq E$. Suppose that groups $D$ and $E$ satisfy the conditions of Lemma 2.5, the sandwich classification theorem holds for the lattice $L(D,E)$, and let $\varphi: E' \to E$ be an epimorphism. Then, given a subgroup $\overline{G} \triangleright \overline{E}$ the lattice $L(\varphi(D),\overline{G})$ satisfies the sandwich classification theorem and $\varphi(D)$-full subgroups of this lattice are the images of $D$-full subgroups of the group $E'$.

**Proof.** Clearly the lattice $L(D,E')$ satisfies the sandwich classification theorem, $D$-full subgroups of this lattice are $D$-full subgroups in $E$, and $N_{E'}(F) = N_E(F) \cap E'$. Therefore, the conditions of the previous lemma hold for groups $D$ and $E'$ as well.

By Lemma 2.5 the lattice $L(\varphi(D),\overline{G})$ satisfies the sandwich classification theorem, $\varphi(D)$-full subgroups of $\overline{G}$ are the images of $D$-full subgroups of $E'$, and their normalizers are the images of the normalizers of the corresponding $D$-full subgroups of $E'$. Obviously, the property (1) of Lemma 2.5 as well as the property of being quasi-solvable are inherited by epimorphic images. Now the result follow from Lemma 2.5. \[\square\]

### 3. Exceptional morphism

In this section for group schemes over $\mathbb{F}_2$ we construct morphisms from $Sp_{2l} = G_{sc}(C_l, -)$ to $Spin_{2l+1} = G_{sc}(B_l, -)$ and back whose composition in any order is equal to the Frobenius endomorphism. On elementary groups over fields these morphisms are mentioned in the book by R. Steinberg [26, propesfa 28]. For $l = 2$ over a finite field this morphism is a key point in a construction of the Suzuki groups.

We start with recalling the construction of the Steinberg group. Let $R$ be an algebra over the field $\mathbb{F}_2$ and $\Phi \neq G_2$. In this case the Steinberg group $St(\Phi, R)$ is generated by symbols $y_{\alpha}(r)$, where $\alpha \in \Phi$, and $r \in R$ subject to relations:

1. $y_{\alpha}(r)y_{\alpha}(s) = y_{\alpha}(r + s)$;
2. $[y_{\alpha}(r), y_\beta(s)] = 1$, if $\alpha + \beta \notin \Phi$ or $\alpha \perp \beta$;
3. $[y_{\alpha}(r), y_\beta(s)] = y_{\alpha + \beta}(rs)$, if $\alpha$ and $\beta$ are of the same length;
4. $[y_{\alpha}(r), y_\beta(s)] = y_{\alpha + \beta}(rs)y_{\alpha + 2\beta}(rs^2)$, if $\alpha$ is long, whereas $\beta$ and $\alpha + \beta$ are short roots.

To avoid a confusion let us denote the generators of the Steinberg group of type $C_l$ by $c_{\alpha}(r)$ and of the Steinberg group of type $B_l$ by $b_{\alpha}(r)$. We use the standard presentation of the root systems $B_l$ and $C_l$ in a euclidean space with an orthonormal basis $e_i$.

$$B_l = \{e_i, e_i - e_j \mid 1 \leq i, j \leq l, \ i \neq j\} \text{ and } C_l = \{2e_i, e_i - e_j \mid 1 \leq i, j \leq l, \ i \neq j\}.$$
Define maps \( \varphi \) and \( \psi \) between the generating sets of the groups \( \text{St}(C_1, R) \) and \( \text{St}(B_1, R) \) by the following formulas.

\[
\varphi(c_\alpha(r)) = \begin{cases} 
  b_{\alpha/2}(r), & \text{if } \alpha \text{ is long,} \\
  b_\alpha(r^2), & \text{if } \alpha \text{ is short;
}\end{cases}
\]

\[
\psi(b_\alpha(r)) = \begin{cases} 
  c_\alpha(r), & \text{if } \alpha \text{ is long,} \\
  c_{2\alpha}(r^2), & \text{if } \alpha \text{ is short.
}\end{cases}
\]

(3.1)

(The same letters will denote the maps between the generating sets of the groups \( E(C_1, R) \) and \( E(B_1, R) \). It is easy to verify that both \( \varphi \) and \( \psi \) take relations to relations. Hence, they can be extended to group homomorphisms \( \text{St}(C_1, R) \to \text{St}(B_1, R) \).

Recall that the group \( K_2(\Phi, R) \) is the kernel of the natural map \( \text{St}(\Phi, R) \to G_{sc}(\Phi, R) \) sending \( y_\alpha(r) \) to \( x_\alpha(r) \). If \( F \) is a field, then \( K_2(\Phi, F) \) is generated by the elements \( \{ r, s \} = h_\alpha(r)h_\alpha(s)h_\alpha(rs)^{-1} \) over all \( \alpha \in \Phi \) and invertible elements \( r, s \in F \), where \( h_\alpha(r) = w_\alpha(r)w_\alpha(-1) \) and \( w_\alpha(r) = y_\alpha(r)y_{-\alpha}(-r^{-1})y_\alpha(r) \), see e.g. [29] §6. In case \( \Phi \neq C_1 \) the elements \( \{ r, s \} \) are the Steinberg symbols, i.e. they satisfy certain relations. If \( \Phi = C_1 \) there are less relations between these elements, but this is not important for our purposes. Straightforward computation shows that the generators of \( K_2 \) goes to \( K_2 \) under the homomorphisms \( \varphi \) and \( \psi \), therefore these maps induce the group homomorphisms between \( G_{sc}(C_1, F) = E_{sc}(C_1, F) \) and \( G_{sc}(B_1, F) = E_{sc}(B_1, F) \). Denote these homomorphisms by \( \tilde{\varphi} \) and \( \tilde{\psi} \) respectively. It is easy to see that the images of these homomorphisms are equal to the elementary groups, corresponding to the admissible pair \((F, F^2)\).

Our next aim is to show that these homomorphisms are regular, i.e. that they are induced by group scheme morphisms, and hence, are defined over an arbitrary ring. Morally, this follows from the fact that a simply connected Chevalley group is the sheafification of its elementary subgroup, but formally we can not apply this argument as \( \tilde{\varphi} \) and \( \tilde{\psi} \) even are not defined over all local rings. The leading idea of the proof is that the morphism of affine schemes is uniquely defined by the image of the generic element and that the affine algebra of a Chevalley group is a domain.

The notion of generic element rarely appears in the theory of algebraic groups, therefore we recall some relevant definitions. Let \( G \) be an arbitrary affine scheme over a ring \( K \). By definition, \( G \) is a functor from the category of \( K \)-algebras to the category of sets, isomorphic to the functor \( \text{Hom}_{K-alg}(A, -) \), where \( A = K[G] \) is the affine algebra of \( G \). Thus, for any \( K \)-algebra \( R \) the element \( h \in G(R) \) corresponds to the \( K \)-algebra homomorphism \( A \to R \), which will be denoted by \( \varepsilon_h \). The generic element \( g_G \) of the affine scheme \( G \) is an element of \( G(A) \), corresponding to the identity homomorphism \( \text{id}: A \to A \). If \( G' \) is another scheme over \( K \), then the scheme morphism \( \theta: G \to G' \) is uniquely defined by the image of the generic element \( g_G \) in the set \( G'(A) \) or, what amounts to be the same, the \( K \)-algebra homomorphism \( K[G'] \to A \). In more details this point of view to affine schemes is discussed in the work by Demazure–Gabriel [9], see also the book [13].

**Lemma 3.1.** Let \( G \) and \( G' \) affine group schemes over a domain \( K \) and let \( E \leq G \) be a group subfunctor. Suppose that \( G \) is smooth and connected and that \( G(R) = E(R) \) for all local rings \( R \). Let \( \theta: E \to G' \) be a natural transformation of the restrictions of the functors \( E \) and \( G' \) on the full subcategory of \( K \)-algebras that are domains. Then there exists a unique scheme morphism \( \tilde{\theta}: G \to G' \) such that for any domain \( R \) the restriction of \( \tilde{\theta}_R: G(R) \to G'(R) \) to \( E(R) \) coincides with \( \theta_R \).

**Proof.** Let \( A = K[G] \) be the affine algebra of the scheme \( G \). Since \( G \) is smooth and connected and \( K \) is a domain, \( A \) is a domain as well. Denote by \( F \) its field of fractions. Consider the restriction of the homomorphism \( \theta_F: E(F) = G(F) \to G'(F) \) to the group \( G(A) \). We claim that the image of this restriction lies in \( G'(A) \). Let \( p \) be a prime ideal of \( A \). Then

\[
\theta_F(G(A)) \subseteq \theta_F(G(A_p)) = \theta_F(E(A_p)) = \theta_{A_p}(E(A_p)) \subseteq G'(A_p).
\]

Since \( \bigcap_{p \in \text{Spec}_A} A_p = A \), the image of \( G(A) \) under \( \theta_F \) is contained in \( G'(A) \). In particular, \( g' = \theta_F(g_G) \in G'(A) \), where \( g_G \in G(A) \) is the generic element of the scheme \( G \). Now, for a \( K \)-algebra \( R \) and an element \( h \in G(R) \) put

\[
\tilde{\theta}_R(h) = g'(\varepsilon_h)(g').
\]

As we have mentioned above, the element \( g' \) defines the scheme morphism \( G \to G' \) uniquely, which immediately implies the uniqueness statement. It remains to prove that for any domain \( R \) the restriction \( \tilde{\theta}_R: G(R) \to G'(R) \) to \( E(R) \) coincide with \( \theta_R \). Clearly, it suffices to give a proof for a field \( R \) (the fraction
field of a domain). Let \( h \in G(R) \). The kernel of the homomorphism \( \varepsilon_h : A \to R \) is a prime ideal. Denote it by \( p \). Let \( \varepsilon \) be the homomorphism \( A_p \to R \) induced by \( \varepsilon_h \). Consider the diagram

\[
\begin{array}{ccc}
G(A_p) = E(A_p) & \xrightarrow{\theta_A} & G'(A_p) \\
E(\varepsilon) \downarrow & & \downarrow G'(\varepsilon) \\
G(R) = E(R) & \xrightarrow{\theta_R} & G'(R)
\end{array}
\]

Since \( \theta \) is a natural transformation, this diagram is commutative. By definition of the homomorphism \( \varepsilon \), the image of the generic element \( g_G \) under \( \varepsilon \) equals \( h \) (we identify elements of the groups \( G(A) \) and \( G'(A) \) with their canonical images in \( G(A_p) \) and \( G'(A_p) \) respectively).

Therefore, the image of \( g_G \) in \( G'(R) \) equals \( \theta_R(h) \). Choosing another path we see that \( g_G \) goes to \( G'(\varepsilon)(g') = G'(\varepsilon_h)(g') = \tilde{\theta}_R(h) \). Thus, \( \tilde{\theta}_R(h) = \theta_R(h) \), which completes the proof. \( \square \)

**Corollary 3.2.** There exist morphisms of group schemes

\[
G_{sc}(C_l, -) \xrightarrow{\varphi} G_{sc}(B_l, -) \xrightarrow{\psi} G_{sc}(C_l, -)
\]

over the field \( \mathbb{F}_2 \) defined on the root unipotent elements by formulas \( \text{[371]} \). The composition of these morphisms in any order is equal to the Frobenius endomorphism.

When this text has been already written A. V. Smolenski in his work \( \text{[27]} \) has given explicit formulas for these morphisms.

Over a perfect field these homomorphisms are isomorphisms and over a reduced rings they are injective. However, as group scheme morphisms they are epimorphisms but not monomorphisms. Indeed, the kernel of both of them is a direct product of several copies of the scheme \( \mu_2 \), whereas the image is dense.

### 4. Distribution of Subgroups

In this section we prove the sandwich classification theorem for the lattices \( L(\text{St}(\Phi, K), G(\Phi, R)) \) and \( L(E(\Phi, K), G(\Phi, R)) \), where \( K \subseteq R \) are \( \mathbb{F}_2 \)-algebras, \( \Phi = B_l \) or \( C_l \), and \( G \) is not necessarily simply connected.

**Theorem 4.1.** Let \( K \subseteq R \) be \( \mathbb{F}_2 \)-algebras, \( \Phi = B_l \) or \( C_l \), and \( l \geq 3 \).

Then given a subgroup \( H \subseteq G(\Phi, R) \), containing \( E(\Phi, K) \), there exists a unique admissible pair \( \Lambda \) if type \( \Phi \) such that

\[
E(\Phi, \Lambda) \leq H \leq N_{G(\Phi, R)}(E(\Phi, \Lambda)).
\]

Similarly, given a subgroup \( H \subseteq \text{St}(C_l, R) \), containing \( \text{St}(C_l, K) \), there exists a unique admissible pair \( \Lambda \) such that

\[
\text{St}(C_l, \Lambda) \leq H \leq N_{\text{St}(C_l, R)}(\text{St}(C_l, \Lambda)).
\]

**Proof.** The statement of the theorem for an arbitrary subring \( K \) of \( R \) follows from the statement for \( K = \mathbb{F}_2 \).

Without loss of generality we may assume that \( K = \mathbb{F}_2 \).

In case of the simply connected Chevalley group \( \text{Sp}_{2l} \) with the root system \( C_l \) the result is obtained in the paper \( \text{[4]} \). Clearly, it implies the standard description of the lattice \( L(E_{sc}(C_l, K), E_{sc}(C_l, R)) \).

By Lavrenov’s theorem \( \text{[16]} \) the kernel of the canonical homomorphism \( \text{St}(C_l, R) \to E_{sc}(C_l, R) \) lies in the center. By Corollary 4.4 from \( \text{[28]} \) the groups \( \text{St}(C_l, K) \) and \( E_{sc}(C_l, K) \) are perfect. Therefore, Lemma \( \text{[24]} \) implies the second assertion.

Let \( S \) be an \( R \)-algebra generated by symbols \( t_r \) over all \( r \in R \) subject to relations \( t_r^2 = r \). Since \( 2 = 0 \) in \( R \), squaring is a ring homomorphism, hence, \( S^2 = R \), i.e. \( \Lambda = (S, R) \) is an admissible pair in \( S \) of type \( C_l \). The standard description of the lattice \( L(E_{sc}(C_l, K), E_{sc}(C_l, S)) \) implies the standard description of the lattice \( L(E_{sc}(C_l, K), E_{sc}(C_l, \Lambda)) \).

The function \( \varphi \) maps the generators of the group \( E_{sc}(C_l, \Lambda) \) onto the generators of the group \( E_{sc}(B_l, R) \). Therefore the function \( \varphi \) maps \( E_{sc}(C_l, \Lambda) \) onto \( E_{sc}(B_l, R) \). Since \( K^2 = K \), the image of the group \( E_{sc}(C_l, K) \) equals \( E_{sc}(B_l, K) \). For any ring \( R \), any root system \( \Phi \) and any weight lattice \( P \), the canonical map from \( E_{sc}(\Phi, R) \) to \( E_{sc}(\Phi, R) \) is surjective. Therefore, there exists an epimorphism onto \( E_{sc}(C_l, K) \) or \( E_{sc}(B_l, R) \) from a certain subgroup of the group \( E_{sc}(\Phi, R) = E_{sc}(C_l, Q) \), containing \( E_{sc}(C_l, K) \) is finite and perfect as we assume that \( K = \mathbb{F}_2 \). On the other hand, the property \( \text{[2]} \) follows immediately from Theorem 2 of \( \text{[4]} \), which completes the proof. \( \square \)
5. Preliminary results on carpet subgroups over a field

In this section we present known statements on carpet subgroups of a field that we shall use in a sequel. The following lemma appears first in [36]. It is a particular case of Theorem 3 from [18]. In a sequel by default $F$ denotes an arbitrary field and all computations are performed in a Chevalley group $G(\Phi, F)$.

**Lemma 5.1.** Suppose that a subgroup $M \leq U(F)$ is normalized by $T(K)$ for a subfield $K \subseteq F$ such that $|K| > 4$. If $x_{\alpha_1}(t_1) \ldots x_{\alpha_k}(t_k) \in M$, where $\alpha_1 < \cdots < \alpha_k \in \Phi^+$, then each factor $x_{\alpha_i}(t_i)$, $i = 1, \ldots, k$, lies in $M$.

The next statement follows from the definition of a carpet subgroup and Lemma 5.1.

**Lemma 5.2.** Suppose that a subgroup $M \leq U(F)$ is normalized by $T(K)$ for a subfield $K \subseteq F$ such that $|K| > 4$. Then the subgroup of $M$ generated by the intersections

$$M \cap x_{\alpha}(F) = x_{\alpha}(\mathfrak{A}_\alpha), \quad \alpha \in \Phi,$$

is a carpet subgroup defined by a closed carpet $\mathfrak{A} = \{\mathfrak{A}_\alpha \mid \alpha \in \Phi\}$.

It is well-known that the special linear group of degree two $SL_2(F)$ over a field $F$ is generated by the elementary transvections

$$t_{12}(u) = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, \quad t_{21}(u) = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}, \quad u \in F,$$

and there exists a homomorphism $\varphi$ of $SL_2(F)$ onto the subgroup $\langle X_\alpha, X_{-\alpha} \rangle$, $\alpha \in \Phi$ that extends the map $t_{12}(u) \rightarrow x_{\alpha}(u)$, $t_{21}(u) \rightarrow x_{-\alpha}(u)$.

**Lemma 5.3** ([21 Lemma 1]). Let $r$ be an element of a field $F$ that is algebraic over a subfield $K \subseteq F$ and does not belong to $K$. Put $M = (t_{21}(K), t_{12}(rK))$. Then one of the following holds:

1. $|K| = 2$ and $M$ is the dihedral group;
2. $|K| = 3$, $r^2 = -1$, and the image of $M$ in $PSL_2(K)$ is isomorphic to $A_5$;
3. $M \cap t_{21}(F) \neq t_{21}(K)$.

Let $K$ be a finite field of characteristic $p$. For $p > 2$ Lemma 5.3 is a particular case of well-known theorem of L. Dickson (see [10] or [11, теорема 2.8.4]). For $p = 2$ it is a particular case of the main theorem of [19] by V. M. Levchuk, which describes up to equality all periodic subgroups of $SL_2(F)$ over an arbitrary field $F$, that have nontrivial intersections with subgroups of upper and lower unitriangular matrices. If the field $K$ is infinite, E. L. Bashkirov obtained the following improvement of Lemma 5.3.

**Lemma 5.4** (see [3]). Let $r$ be an element of a field $F$ that is algebraic over a subfield $K \subseteq F$ and does not belong to $K$. Suppose that if $\text{char} F = 2$, then $r$ is separable over $K$. Then $(t_{21}(K), t_{12}(rK)) = SL_2(K(r))$.

A modification of the proof of Lemma 5.4 for the case where char $K \neq 2$, or char $K = 2$ and $K$ is a perfect field is presented in appendix by A. E. Zalesski to the paper of F. G. Timmesfeld [37].

Let us say that roots $\alpha, \beta \in \Phi$ commute, if $\alpha + \beta \notin \Phi$. The following statement is a particular case of Lemma 2 from [21].

**Lemma 5.5.** Let $\Delta$ be a nonempty subset of $\Phi^+$. Then there exist roots $\alpha \in \Phi^+$ and $\beta \in \Delta$ such that $\alpha$ commutes with all roots from $\Delta$ and the inner product $(\alpha, \beta)$ is nonzero.

Lemmas 5.1, 5.3 and 5.5 from the works [21, 22, 23] of the first author were key steps in the description of intermediate subgroups between groups of Lie type over different fields in the case where the bigger field is an algebraic extension of the smaller one. In a sequel we shall use two special analogs of Lemma 5.5 for types $B_l$ and $C_l$.

**Lemma 5.6.** Let $\Delta$ be a nonempty subset of $\Phi^+$, where $\Phi = B_l$, ($l \geq 2$). Then there exists a long root $\alpha \in \Phi^+$ and a root $\beta \in \Delta$ such that $\alpha$ commutes with all roots from $\Delta$ and $(\alpha, \beta) \neq 0$.

**Proof.** The root $\alpha$ from $\Phi^+$ of maximal height is long and commutes with all roots from $\Phi^+$. If $\Delta$ contains a root $\beta$ not orthogonal to $\alpha$, then the statement is trivial. Hence, it suffices to consider the case $\Delta \subseteq \Phi_\alpha^+ \cap \Phi^+$, where $\Phi_\alpha^+ = \{\gamma \in \Phi \mid (\gamma, \alpha) = 0\}$. For any long root $\alpha$ the set $\Phi_\alpha^+$ is an orthogonal sum of a root system of rank 1 consisting of long roots and a root system of type $B_{l-2}$. It is easy to see if the root $\alpha$ is a terminal vertex in the Coxeter graph. For an orthogonal sum the lemma follows from the statement for irreducible summands. For one dimensional set $\Delta$ the statement is obvious, therefore, the induction on the rank of $\Phi$ completes the proof. □
Lemma 6.4. Let $\Delta$ be a nonempty subset of $\Phi^+$, where $\Phi = C_l$ ($l \geq 2$). Then there exists a short root $\alpha \in \Phi^+$ and a root $\beta \in \Delta$ such that the root subgroup $X_\alpha(F)$ of a Chevalley group $G(\Phi, F)$ over a field $F$ of characteristic 2 commutes with all root subgroups $X_\gamma(F)$, $\gamma \in \Delta$, and $(\alpha, \beta) \neq 0$.

Proof. Let $\alpha$ be the short root from $\Phi^+$ of maximal height. Then the root subgroup $X_\alpha(F)$ of a Chevalley group $G(\Phi, F)$ over a field $F$ characteristic 2 commutes with all root subgroups $X_\gamma(F)$, $\gamma \in \Phi^+$. The rest of the proof is the same as for Lemma 5.5 with replacing the word “long” by “short”. $\square$

6. Intermediate subgroups between Chevalley groups of type $B_l$, $C_l$, $F_4$, or $G_2$

Theorems 3.1 and 4.1 from [22] imply the following result.

Proposition 6.1. Let $F$ be an algebraic extension of a field $K$ of characteristic $p$ and let $M$ be a group lying between Chevalley groups $G(\Phi, K)$ and $G(\Phi, F)$ of type $\Phi = B_l$, $C_l$, $F_4$, or $G_2$. Let $p = 2$ if $\Phi = B_l$, $C_l$, or $F_4$, and $p = 3$ if $\Phi = G_2$. Then $M$ is a product of the carpet subgroup $E(\Phi, A)$ and a diagonal subgroup $H_M$ normalizing $E(\Phi, A)$. The carpet $A = \{A_\alpha \mid \alpha \in \Phi\}$ is closed and is defined by the equality

$$A_\alpha = \begin{cases} P, & \text{if } \alpha \text{ is short}, \\ Q, & \text{if } \alpha \text{ is long}. \end{cases}$$

(6.1)

for some additive subgroups $P$ and $Q$ of the field $F$ such that $K \subseteq P^p \subseteq Q \subseteq P \subseteq F$.

Moreover, depending on the type of the Chevalley group $G(\Phi, F)$ the following additional conditions for $P$, $Q$ and $H_M$ hold.

(a) if $\Phi = B_l$ and $l \geq 3$, then $Q$ is a field;
(b) if $\Phi = C_l$ and $l \geq 3$, then $P$ is a field;
(c) if $\Phi = F_4$ or $G_2$, then both additive subgroups $P$ and $Q$ are fields and $H_M$ is the unit subgroup.

For groups of types $B_l$ and $C_l$, where $l \geq 3$, this proposition improves Theorem 4.1 of the current article in the case of fields. The paper [22] asserts that over fields of characteristic 2 Chevalley groups of types $B_l$ and $C_l$ are isomorphic and, therefore, it considers only type $B_l$. But actually, as we have shown in section 3 the exceptional morphism exists but is an isomorphism only over perfect fields of characteristic 2. However, the proof for type $B_l$ is valid also for type $C_l$. The only difference is that for $B_l$ ($l \geq 3$) the smallest additive subgroup $Q$ is a field, whereas for $C_l$ ($l \geq 3$) the largest subgroup $P$ is a field.

In section 7 below we give a negative answer on the following question from [22] cpr. 160.

Problem 6.2. In case of $\Phi = B_l$, $C_l$, are both additive subgroups $P$ and $Q$ from Proposition 6.1 fields?

Inclusions (6.1) follows easily from carpet conditions [152] applied for the family $A$ from proposition 6.1. However, the inverse implication holds only for types $F_4$ and $G_2$. Indeed, for $F_4$ in our situation there are two nontrivial carpet conditions: $PQ \subseteq P$ and $P^2Q \subseteq Q$, which follows from (6.1) because $P$ and $Q$ are fields. By the same reason all nontrivial carpet conditions $PQ \subseteq P$, $P^2Q \subseteq P$, $P^3Q \subseteq Q$, $P^3Q^2 \subseteq Q$, $QQ \subseteq Q$, and $2PP \subseteq P$ for a family $A$ of type $G_2$ follows from (6.1) as well. In section 7 we give examples of pairs of additive subgroups in a nonperfect field of characteristic 2 satisfying conditions (6.1) that do not define carpets of types $B_l$ and $C_l$. Namely, we give a negative answer to the following question.

Problem 6.3. Let $F$ be an algebraic extension of a nonperfect field $K$ of characteristic 2 and let $P$ and $Q$ be additive subgroups of $K$. Suppose that both $P$ and $Q$ are $K$-modules and that one of them is a field. Is it true that the inclusions $PQ \subseteq P$ and $P^2Q \subseteq Q$ follow from the inclusions $K \subseteq P^2 \subseteq Q \subseteq P \subseteq F$?

For an additive subgroup $A$ of a field $F$ define

$$A^{-1} = \{0\} \cup \{r^{-1} \mid 0 \neq r \in A\}.$$

In a sequel we need the following statement, which is established in the proof of the main theorem of [21] p. 535. It follows from the fact that $F$ is algebraic over $K$ and the carpet conditions, e.g. for type $B_2$ it follows from the inclusions $PQ \subseteq P$ and $P^2Q \subseteq Q$.

Lemma 6.4 (cm. [21]). Let $p$, $K$, $F$, $P$, $Q$ and $\Phi$ are the same as in Proposition 6.1. Then, given $r \in P \setminus \{0\}$ (respectively $r \in Q \setminus \{0\}$) the ring $K[\bar{r}]$ (respectively $K^p[\bar{r}]$) is contained in $P$ (respectively in $Q$). In particular, $P = P^{-1}$ and $Q = Q^{-1}$.

The intermediate subgroups of Chevalley groups of skew types over nonperfect fields of exceptional characteristic are described in [23].
7. Examples concerning admissible pairs

At the beginning of this section we construct counterexamples to Problems 6.2 and 6.3. For it suffices to define fields $K \subseteq F$ in the following way. Let $n$ be a positive integer greater than one and $x_1, \ldots, x_n$ independent commutative variables, i.e. transcendental elements over the field of two elements $\mathbb{F}_2$. Consider the field of rational functions $F = \mathbb{F}_2(x_1, \ldots, x_n)$ and its subfield $K = \mathbb{F}_2(x_1^2, \ldots, x_n^2)$, generated by the squares of the variables. Obviously, $F$ is an algebraic extension of $K$ of degree $2^n$ and the set of monomials

$$\{x_{i_1} \cdots x_{i_m} \mid i_1 < \cdots < i_m, \{i_1, \ldots, i_m\} \subseteq \{1, \ldots, n\}\}$$

is a basis of $F$ over $K$.

**Proposition 7.1.** Let $\Phi = B_l$, $C_l$, $l \geq 2$. There exist fields $K \subseteq F$ of characteristic 2 and an admissible pair $(P, Q)$ of type $\Phi$ such that $K \subseteq Q \subseteq P \subseteq F$ and:

1. if $\Phi = B_l$, $l \geq 3$, then $P$ is not a field;
2. if $\Phi = C_l$, $l \geq 3$, then $Q$ is not a field;
3. if $\Phi = B_2 = C_2$, then neither $P$, nor $Q$ is a field.

**Proof.**

**Type $B_l$, $l \geq 3$.** Let $Q = K$ and let $P$ be the $K$-module with the basis $\{1, x_1, x_2\}$. The following properties can be verified immediately: (1) $PQ \subseteq P$; (2) $P^2Q \subseteq Q$; (3) $\dim_K P = 3$. Since 3 does not divide $2^n$, the additive subgroup $P$ is not a field.

**Type $C_l$, $l \geq 3$.** Let $P = F$, and let $Q$ be a $K$-module with the basis $\{1, x_1, x_2\}$. Properties (1)-(3) from the previous part of the proof hold also in this case, except that in the third property one substitutes $P$ by $Q$. The rest of the proof repeated one for type $B_l$.

**Type $B_2 = C_2$.** Suppose in this case that $n \geq 4$. Define the additive subgroups $P$ and $Q$ of the field $F$ in the following way. Let $P$ be a $K$-module with the basis

$$\{1, x_1, x_2, x_3, x_4, x_1x_2, x_1x_3, x_1x_4, x_2x_3, x_2x_4, x_1x_2x_3, x_1x_2x_4\},$$

and let $Q$ be a $K$-module with the basis $\{1, x_1, x_2\}$. Again, the following properties can be verified immediately: (1) $PQ \subseteq P$; (2) $P^2Q \subseteq Q$; (3) $\dim_K P = 12$, $\dim_K Q = 3$. Since 3 and 12 do not divide $2^n$, both $P$ and $Q$ are not fields.

The carpet, corresponding to the admissible pair from the previous proposition gives the negative answer to Problem 6.3. The following statement gives the negative answer to Problem 6.2.

**Proposition 7.2.** There exist fields $K \subseteq F$ of characteristic 2 and $K$-submodules $P$ and $Q$ in $F$ such that $K \subseteq P^2 \subseteq Q \subseteq P \subseteq F$, the module $Q$ (respectively $P$) is a field, but the inclusion $PQ \subseteq P$ (respectively $P^2Q \subseteq Q$) does not hold, i.e. the pair $(P, Q)$ is not an admissible pair of type $B_l$ (respectively $C_l$).

**Proof.** Define additive subgroups $P$ and $Q$ in the field $F$ as $K$-modules with the bases $\{1, x_1, x_2\}$ and $\{1, x_1\}$ respectively. Note that $Q$ is a field. Subgroups $P$ and $Q$ satisfy conditions of the proposition, however $x_1x_2 \notin P$, and hence, $PQ \notin P$.

To prove the statement in the case where $P$ is a field we reduce the field $K$ to $K = \mathbb{F}_2(x_1^2, \ldots, x_n^2)$. Define additive subgroups $P$ and $Q$ of the field $F$ as $K$-modules in the following way. Put $P = F$ and $Q = P^2 + Kx_1 + Kx_2$. Evidently, $x_1^2 \in P^2Q$. However, $x_1^4 \notin Q$. Indeed, if $x_1^4 \in Q$, then $x_1^4 = r + sx_1 + tx_2$, where $r \in P^2$, $s, t \in K$. It follows that $0 = r + (s-x_1^2)x_1 + tx_2$ and all the coefficients $r$, $s$, $t$ belong to the field $P^2$. Elements $1, x_1, x_2$ are linearly independent over $P^2$. Therefore, $s - x_1^2 = 0$, but $x_1^4 \notin K$. The contradiction shows that $P^2Q \notin Q$.

In [25] the authors describe intermediate subgroups between Chevalley groups $G(\Phi, R)$ and $G(\Phi, F)$ of an arbitrary type $\Phi$ over the fraction field $F$ of the principle ideal domain $R$. It turns out that in this case such a subgroup coincides with $G(\Phi, P)$ for an intermediate subring $P$, $R \subseteq P \subseteq F$. Why in this case admissible pairs do not appear? The following proposition answers this question.

**Proposition 7.3.** Let $F$ be the fraction field of a principle ideal domain $R$ and $Q \subseteq P$ a pair of additive subgroups in $F$, containing $R$. If $P$ is an $R$-module and $P^mQ \subseteq Q$ for some positive integer $m$, then $P = Q$.

For an admissible pair $(P, Q)$ of an arbitrary type such that $R \subseteq Q$, we have $P = Q$ and $P$ is a subring of $F$.

**Proof.** Let $r, s \in R$ and $\frac{r}{s} \in P$. Without loss of generality we may assume that $r$ and $s$ are mutually prime. Then there exist $u, v \in R$ such that $ur + vs = 1$. Since $P$ is an $R$-module and contains $R$, we have $\frac{1}{s} = u\frac{r}{s} + v \in P$. The product $rs^{m-1}$ lies in $R \subseteq Q$ and in $P^mQ \subseteq Q$, therefore, $\frac{r}{s} = (\frac{1}{s})^m(rs^{m-1}) \in Q$. Thus, $P \subseteq Q$, and hence, $P = Q$. 
The second assertion follows from the first one and the definition of admissible pair given in the introduction.

For Chevalley groups of skew types $2A_{2n+1}, 2D_n, 2E_6, 3D_4$ over the fraction field $F$ of a PID $R$ the intermediate subgroups were described in [20] with certain restrictions on the cardinality of the multiplicative subgroup of the ring $R$.

8. Bruhat decomposition

The factorization $G(F) = U(F)N(F)U(F)$ of a Chevalley group $G(F)$ over a field $F$ is usually called the Bruhat decomposition. Such presentation of an element from $G(F)$ is not unique. However, it can be transformed to the reduced Bruhat decomposition (canonical presentation of elements of Chevalley groups, see [5, Corollary 8.4.2]), which is unique. The following theorem establishes the reduced Bruhat decomposition in the carpet subgroup corresponding to an admissible pair.

The image of $N(\mathbb{Z})$ in $N(F)$ is denoted by $N^±(F)$ or simply by $N^±$. The expressions $T(\mathfrak{A})$ and $N(\mathfrak{A})$ are used to denote the intersection of the carpet subgroup $E(\Phi, \mathfrak{A})$ with the subgroups $T(F)$ and $N(F)$ respectively. In the unipotent subgroups

$$U(F) = \langle X_\alpha(F) \mid \alpha \in \Phi^+ \rangle \quad \text{and}$$

$$V(F) = \langle X_\alpha(F) \mid \alpha \in -\Phi^+ \rangle$$

a carpet $\mathfrak{A}$ naturally defines the unipotent carpet subgroups

$$U(\mathfrak{A}) = \langle x_\alpha(\mathfrak{A}_u) \mid \alpha \in \Phi^+ \rangle \quad \text{and}$$

$$V(\mathfrak{A}) = \langle x_\alpha(\mathfrak{A}_u) \mid \alpha \in -\Phi^+ \rangle,$$

These subgroups coincide with the intersections of $E(\Phi, \mathfrak{A})$ with $U(F)$ and $V(F)$ respectively if and only if the carpet $\mathfrak{A}$ is closed. For an element $w \in W(\Phi)$ and a carpet $\mathfrak{A}$ of type $\Phi$ put

$$U_w(\mathfrak{A}) = \langle x_\alpha(\mathfrak{A}_u) \mid \alpha \in \Phi^+_w \rangle,$$

where $\Phi^+_w = \Phi^+ \cap w^{-1}(-\Phi^+)$. 

**Theorem 8.1.** Let $E(\Phi, \mathfrak{A})$ be a carpet subgroup lying between Chevalley groups $G(\Phi, K)$ and $G(\Phi, F)$ of type $\Phi = B_l, C_l, F_4, G_2$ ($l \geq 2$), where $F$ is an algebraic extension of a nonperfect field $K$ of characteristic $p = 2$ for $\Phi = B_l, C_l, F_4$, and $p = 3$ for $\Phi = G_2$. For each element $w \in W$ choose its representative $n_w$ in $N^±$. Then an element $g \in E(\Phi, \mathfrak{A})$ has a unique presentation $g = uhv n_w$, where $u \in U(\mathfrak{A}), h \in T(\mathfrak{A}), v \in W$, and $v \in U_w(\mathfrak{A})$. In particular, the carpet $\mathfrak{A}$ is closed, i.e. the carpet subgroup $E(\Phi, \mathfrak{A})$ contains no new root elements.

**Proof.** According to Proposition 6.1 the carpet $\mathfrak{A}$ is defined by a pair of additive subgroups $P$ and $Q$ of the field $F$, containing $K$.

Let $g \in E(\Phi, \mathfrak{A})$. Then $g = uhv n$ for some $u \in U(F), v \in V(F), h \in T(F), n \in N^±$, and $n^{-1} v n \in U(F)$ by the reduced Bruhat decomposition in the group $G(\Phi, F)$. Since $N^± \leq E(\Phi, \mathfrak{A})$, we have $uhv n \in E(\Phi, \mathfrak{A})$.

Each element in $u \in U(F)$ can be written in the form

$$u' a_{\alpha_k} \ldots a_{\alpha_1}, \quad \alpha_k > \cdots > \alpha_1, \quad a_{\alpha_i} \in X_{\alpha_i}(F) \setminus x_{\alpha_i}(\mathfrak{A}_u), \quad i = 1, \ldots, k,$$

where $u' \in U(\mathfrak{A}), k \geq 0$, if $k = 0$, then $u = u'$, and the ordering of roots is compatible with their heights. This can be easily deduced from the carpet conditions for the carpet subgroup $U(\mathfrak{A})$ see also Lemma 3 from [21].

Similarly, each element $v \in V(F)$ can be written in the form

$$a_{\beta_m} \ldots a_{\beta_1} v', \quad \beta_m > \cdots > \beta_1, \quad a_{\beta_i} \in X_{\beta_i}(F) \setminus x_{\beta_i}(\mathfrak{A}_u), \quad i = 1, 2, \ldots, m,$$

where $v' \in V(\mathfrak{A}), n^{-1} v' n \in U(\mathfrak{A}), m \geq 0$, and if $m = 0$, then $v = v'$. Since $u', v' \in E(\Phi, \mathfrak{A})$, then $E(\Phi, \mathfrak{A})$ contains the element

$$a_{\alpha_k} \ldots a_{\alpha_1} h a_{\beta_m} \ldots a_{\beta_1}, \quad \alpha_k > \cdots > \alpha_1 > 0 > \beta_1 > \cdots > \beta_m.$$ 

Suppose that $k + m > 0$. By Lemmas 5.5.7 there exists a root $\alpha \in -\Phi^+ \Phi^+$ such that the root subgroup $X_\alpha(F)$ commutes with root subgroups $X_{\beta_i}(F)$ for all $i = 1, \ldots, m$ and with $X_{-\alpha_i}$ for all $i = 1, \ldots, k$. Moreover, $\alpha$ is not orthogonal to one of the roots $\beta_i$ or $-\alpha_i$. Suppose that it is not orthogonal to $-\alpha_j, 1 \leq j \leq k$, if it is orthogonal to all roots $\alpha_i$ but not orthogonal to one of $\beta_i$’s is essentially the same. Taking the smallest possible $j$ we may assume that $(\alpha, \alpha_j) = 0$ for all $i < j$. This implies that $X_\alpha(F)$ commutes with $X_{\alpha_j}(F)$, $i = 1, \ldots, j$, and in characteristic 2 any two root subgroups corresponding to orthogonal roots commute.

Let $r = \alpha(h)$, so that $hx_\alpha(t)h^{-1} = x_\alpha(rt)$. Put

$$y = a_{\alpha_k} \ldots a_{\alpha_1} h, \quad z = a_{\beta_m} \ldots a_{\beta_1}.$$
Then \( yhz \in E(\Phi, \mathfrak{A}) \) and
\[
(yhz)x_\alpha(\mathfrak{A}_\alpha)(yhz)^{-1} = yx_\alpha(r\mathfrak{A}_\alpha)y^{-1} \in E(\Phi, \mathfrak{A})
\]
On the other hand the choice of \( \alpha \) implies that
\[
x_{-\alpha}(\mathfrak{A}_{-\alpha}) = yx_{-\alpha}(\mathfrak{A}_\alpha)y^{-1}
\]
Thus,
\[
y \cdot (x_\alpha(r\mathfrak{A}_\alpha), x_{-\alpha}(\mathfrak{A}_\alpha)) \cdot y^{-1} \subseteq E(\Phi, \mathfrak{A})
\]
Now the proof splits in two cases: (1) \( \Phi \neq B_2 = C_2 \); (2) \( \Phi = B_2 = C_2 \).

(1) Let \( \Phi \neq B_2 = C_2 \). Here it becomes important that by Lemmas 5.3 and 5.4 we can take long root \( \alpha \) for \( \Phi = B_1 \) and short root \( \alpha \) for \( \Phi = C_1 \). Therefore, according to items (a)–(c) of Proposition 6.4 the additive subgroup \( \mathfrak{A}_\alpha \) is a field. If \( r \notin \mathfrak{A}_\alpha \), then by Lemma 5.3 there exists \( s \notin \mathfrak{A}_\alpha = \mathfrak{A}_{-\alpha} \) such that \( x_{-\alpha}(s) = yx_{-\alpha}(s)yx^{-1} \in E(\Phi, \mathfrak{A}) \), which contradicts to the fact that the carpet \( \mathfrak{A} \) is closed. Therefore, \( r \in \mathfrak{A}_{-\alpha} = \mathfrak{A}_\alpha \). Put \( y' = a_{\alpha_k} \cdots a_{\alpha_{i+1}} \) and \( a_{\alpha_i} = x_{\alpha_i}(q) \). Note that \( [a_{\alpha_i}, h_{\alpha_i}(t)] = x_{\alpha_i}(q(t_m - 1)) \) for all \( t \in K \setminus \{0\} \), where \( m = -2(\alpha_j, \alpha)/(\alpha, \alpha) \neq 0 \). Since the field \( K \) is not perfect, it is infinite. Hence, one can choose \( t \in K \setminus \{0\} \) such that \( t_m - 1 \neq 0 \). Inclusion (8.1) implies that
\[
[y, h_{\alpha}(t)] = [y', h_{\alpha}(t)]_{\alpha_j} = \prod_{\gamma > \alpha_j} x_{\gamma}(s_{\gamma})x_{\alpha_j}(q(t_m - 1)) \in E(\Phi, \mathfrak{A})
\]
for some \( s_{\gamma} \in F \). Applying Lemma 5.1 to this element and the subgroup \( E(\Phi, \mathfrak{A}) \) we obtain the inclusion \( q(t_m - 1) \notin \mathfrak{A}_{\alpha_j} \), hence, \( q \notin \mathfrak{A}_{\alpha_j} \). This means that the root element \( a_{\alpha_j} \) lies in \( E(\Phi, \mathfrak{A}) \), which is a contradiction.

(2) Let \( \Phi = B_2 = C_2 \). In this case in view of an example from section 7 both subgroups \( P \) and \( Q \) are not necessarily fields. According to Lemmas 5.6 and 5.7 we can take long or short root \( \alpha \). Let \( \alpha = -2\gamma - \delta \) in the notation of item (1), where \( \{\gamma, \delta\} \) is the set of fundamental positive roots in \( B_2 \). Then
\[
(yhz)x_{-2\gamma - \delta}(Q)(yhz)^{-1} = yx_{-2\gamma - \delta}(rQ)y^{-1}
\]
where \( h \in -2\gamma - \delta(1)h^{-1} = x_{-2\gamma - \delta}(r) \). On the other hand, since the root subgroups \( X_{2\gamma + \delta}(F) \) and \( X_{\gamma + \delta}(F) \) are central in \( U(F) \), we have
\[
x_{2\gamma + \delta}(Q) = yx_{2\gamma + \delta}(Q)y^{-1}
\]
\[
x_{\gamma + \delta}(Q) = yx_{\gamma + \delta}(Q)y^{-1}
\]
Commuting the elements of the shapes (8.2) and (8.3), we obtain
\[
yx_{-\gamma}(r)x_{3}(r)y^{-1}
\]
Now commuting the elements of the shapes (8.3) and (8.4), we get
\[
yx_{\gamma + \delta}(r)x_{3}(r^2)y^{-1}
\]
By Lemma 5.1 equation (8.6) implies the inclusion \( x_{3}(r^2) \in E(\Phi, \mathfrak{A}) \). Therefore, if \( r^2 \notin Q \), then we arrive at the contradiction with the equation \( E(\Phi, \mathfrak{A}) \cap X_{3} = x_{3}(Q) \).

Let \( r^2 \in Q \) but \( r \notin Q \). By Lemma 5.4 \( Q^{-1} = Q \). Therefore, the last inclusion is equivalent to the equation \( qr^2 = 1 \) for some nonzero \( q \in Q \). Moreover, \( q \neq 1 \), as \( r = 1 \) when \( q = 1 \) and in this case we again obtain the inclusion \( a_{\alpha_j} \in E(\Phi, \mathfrak{A}) \), which leads to a contradiction. Further, for all \( q_1 \in Q \) the subgroup \( \langle x_{2\gamma + \delta}(Q), x_{-2\gamma - \delta}(rQ) \rangle \) contains the image of the matrix
\[
a = \begin{pmatrix}
1 & q_1 r \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
q_3 & 1
\end{pmatrix}
\begin{pmatrix}
1 & q_2 r \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
q_1 & 1
\end{pmatrix}
\]
under the homomorphism \( \varphi \) from \( SL_2(F) \) onto \( (X_{2\gamma + \delta}, X_{-2\gamma - \delta}) \), that extends the map \( t_{12}(u) \mapsto x_{-2\gamma - \delta}(u) \), \( t_{21}(u) \mapsto x_{2\gamma + \delta}(u) \). Let \( q_1 + q_3 = 1, q_2 = q_1^{-1}, q_4 = 1 \). Then
\[
a = \begin{pmatrix}
1 + (q_1 + 1)r^2 \\
1 + (q_1 + 1)r
\end{pmatrix}
\]
Obviously, \( (q_1 + 1) \neq 0, 1 \) when \( q_1 \neq 0, 1 \). Since \( qr^2 = 1 \), for \( (q_1 + 1) = q \) we have
\[
a = \begin{pmatrix}
0 & (1 + qr)^{-1} \\
1 + qr & (1 + qr)^{-1}
\end{pmatrix}
\begin{pmatrix}
0 & (1 + qr)^{-1} \\
1 + qr & (1 + qr)^{-1}
\end{pmatrix}
\]

Let $\varphi(a) = z$. Then
\[
yzx_{-2\gamma - \delta}(rQ)z^{-1}y^{-1} = yzx_{2\gamma + \delta}((1 + qr^2)rQ)y^{-1} = x_{2\gamma + \delta}((1 + qr^2)rQ) \subseteq E(\Phi, A).
\]

Recall that $qr^2 = 1$, hence
\[
(1 + qr^2)r = (1 + q^2r^2)r = (1 + qr)r = (1 + r^{-2})r = r + r^{-1} = \frac{r^2 + 1}{r} = q^{-1} + 1.
\]

Since $0, 1 \neq q \in Q$, then $0 \neq q^{-1} + 1 \in Q$ according to the equality $Q = Q^{-1}$. Therefore formula (8.7) implies that $r^{-1} \in Q$, and hence $r \in Q$, which is a contradiction. \( \square \)

Remark 8.2. In the proof of [22, Theorem 3.1] for type $B_l$ the inclusions of the factors of the product when $n \in M$ into $M$ are claimed to have the same proof as in [21]. However, in [21] all additive subgroups $\mathfrak{A}_\alpha$, which are defined by $M \cap x_\alpha(F) = x_\alpha(\mathfrak{A}_\alpha)$, $\alpha \in \Phi$, coincide with a subfield of the ground field $F$. The proof of Theorem 8.1 given above at the same time fills this gap (when at least one of additive subgroups $\mathfrak{A}_\alpha$ is not a field).

Remark 8.3. The Bruhat decomposition of a Chevalley group $G(F)$ over a field $F$ implies the Gauss decomposition $G(F) = U(F)T(F)V(F)U(F)$, which holds even for semilocal commutative rings. In 1976 Z. I. Borevich [10] established the Gauss decomposition for matrix subgroups of $GL_n$ and $SL_n$, that are defined by net of ideals of a semilocal ring. The same result was obtained by N. A. Vavilov and E. B. Plotkin [39, 40] for net subgroups of all Chevalley groups over commutative semilocal rings.

9. INTERMEDIATE SUBGROUPS AS GROUPS WITH A $(B, N)$-PAIR

Subgroups $B$ and $N$ of an arbitrary group $G$ are called a $(B, N)$-pair they satisfy the following axioms.

BN1. Subgroups $B$ and $N$ generate $G$.

BN2. $B \cap N \subseteq N$.

BN3. The quotient group $W = N/B \cap N$ is generated by involutions $w_i, i \in I$.

BN4. For any preimage $n_i \in N$ of $w_i$ under the natural homomorphism $N$ onto $W$ we have
\[
Bn_iB \cdot Bn_iB \subseteq Bn_iB \cup BnB, \quad n \in N.
\]

BN5. If $n_i$ is the element from axiom (BN4), then $n_iBn_i \neq B$.

In different terminology with $S = \{w_i | i \in I\}$ the quadruple $(G, B, N, S)$ is called a Tits system [7, p. 26]). A $(B, N)$-pair is called split if $B = U(B \cap N)$, where $U$ is a normal nilpotent subgroup of $B$, see [12, p. 149]. A $(B, N)$-pair is called saturated, if $\bigcap_{n \in N} B^n = B \cap N$, see [7, p. 58]).

It is well-known that a Chevalley group $G(\Phi, F)$ over a field $F$ admits a split saturated $(B, N)$-pair. The group $B(F) = U(F)T(F)$ can be taken as $B$, and $N(F)$ as $N$. Note that $(B(F), N(K))$ is also a $(B, N)$-pair of the group $G(\Phi, F)$ for an arbitrary subfield $K$ of $F$, but it is saturated only if $K = F$.

Theorem 9.1. Let $E(\Phi, A)$ be a carpet subgroup that lies between Chevalley groups $G(\Phi, K)$ and $G(\Phi, F)$ of type $\Phi = B_l$, $C_l$ ($l \geq 2$), $F_4$, or $G_2$, where $F$ is an algebraic extension of a nonperfect field $K$ of characteristic $p = 2$ for $\Phi = B_l$, $C_l$, $F_4$, and $p = 3$ for $\Phi = G_2$. Then the group $E(\Phi, A)$ is simple and admits a split saturated $(B, N)$-pair.

Proof. By Proposition 8.1 the subgroup $E(\Phi, A)$ is parametrized by two additive subgroups $P$ and $Q$, satisfying the conditions $K \subseteq P^p \subseteq Q \subseteq P \subseteq F$. Let $Y(P, Q) = Y(F) \cap E(\Phi, A)$, if $Y$ is $U$, $V$, $B$, $N$ or $H$.

We show that $(B(P, Q), N(P, Q))$ is a required $(B, N)$-pair. The monomial subgroup $N(K)$ by definition lies in $E(\Phi, A)$ and acts by conjugation transitively of its root subgroups $x_\alpha(\mathfrak{A}_\alpha)$, $\alpha \in \Phi$. Therefore axioms (BN1) and BN5) are satisfied. Axioms (BN2), and (BN3) as well as the facts that the pair $(B(P, Q), N(P, Q))$ is split and saturated follows easily from the definition of the groups $B(P, Q)$, $N(P, Q)$ and $T(P, Q)$. The proof of axiom (BN4) for the whole Chevalley group $G(\Phi, F)$ from [8, crp. 106] is valid in our situation with obvious changes and, therefore, is omitted.

Next we establish the simplicity of the group $E(\Phi, A)$. It is well-known (see e.g. [8, p. 170]), that a group $G$ with a $(B, N)$-pair is simple if the following conditions hold.

(a) $G = G'$,
(b) $B$ is solvable,
(c) $\bigcap_{g \in G} gBg^{-1} = 1$,
(d) the set $I$ is not a disjoint union of two nonempty elementwise commuting subsets $J$ and $J'$.
The group $E(\Phi, A)$ is generated by its root subgroups $x_\alpha(A_\alpha)$, $\alpha \in \Phi$ and

$$[x_\alpha(A_\alpha), T(P, Q)] = x_\alpha(A_\alpha).$$

Therefore the group $E(\Phi, A)$ is perfect. Clearly, the group $B(P, Q)$ is solvable. The equality

$$\bigcap_{g \in E(\Phi, A)} gB(P, Q)g^{-1} = 1$$

can be established by the same arguments as for the whole Chevalley group $G(\Phi, F)$ [8, p. 172]. Finally, the quotient group $N(P, Q)/T(P, Q)$ is isomorphic to the Weyl group of the system $\Phi$. Thus, the $(B, N)$-pair $(B(P, Q), N(P, Q))$ satisfy conditions (a)-(d), and hence, the group $E(\Phi, A)$ is simple.

The groups $E(\Phi, A)$ from Theorem 9.1 are interesting also with respect to the following problem posed by A. V. Borovik, see [15].

**Problem 9.2.** Describe infinite groups with a split saturated $(B, N)$-pair.

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