SEMiclassal Resolvent Bound for Compactly Supported $L^\infty$ Potentials

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Abstract. We give an elementary proof of a weighted resolvent estimate for semiclassical Schrödinger operators in dimension $n \geq 1$. We require the potential to belong to $L^\infty(\mathbb{R}^n)$ and have compact support, but do not require that it have distributional derivatives in $L^\infty(\mathbb{R}^n)$. The weighted resolvent norm is bounded by $e^{C h^{-4/3} \log(h^{-1})}$, where $h$ is the semiclassical parameter.

1. Introduction

Let $\Delta \leq 0$ be the Laplacian on $\mathbb{R}^n$, $n \geq 1$. We consider semiclassical Schrödinger operators of the form

$$P = P(h) := -h^2 \Delta + V : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n), \quad h > 0,$$

where the potential $V \in L^\infty(\mathbb{R}^n)$ is real-valued and compactly supported. By the Kato-Rellich Theorem, the operator $P$ is self-adjoint with respect to the domain $H^2(\mathbb{R}^n)$. Therefore, the resolvent $(P - z)^{-1}$ is bounded $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ for all $z \in \mathbb{C} \setminus \mathbb{R}$. We establish an $h$-dependent bound on a weighted resolvent norm that is uniform up to the positive real spectrum.

Theorem. Let $n \geq 1$, $V \in L^\infty_{\text{comp}}(\mathbb{R}^n)$, and $[E_{\min}, E_{\max}] \subseteq (0, \infty)$. For any $s > 1/2$, there exist $C, h_0 > 0$ such that

$$\| (1 + |x|)^{-s} (P(h) - i \varepsilon)^{-1} (1 + |x|)^{-s} \|_{L^2(\mathbb{R}^n) \to H^2(\mathbb{R}^n)} \leq e^{C h^{-4/3} \log(h^{-1})},$$

for all $E \in [E_{\min}, E_{\max}]$, $0 < \varepsilon < 1$, and $h \in (0, h_0]$.

Exponential resolvent bounds are known to hold under a wide range of geometric, regularity, and decay assumptions. In [Bu98], Burq showed the resolvent is $O(e^{C h^{-1}})$ for smooth, compactly supported perturbations of the Laplacian outside an obstacle. He later established the same bound for smooth, long-range perturbations [Bu02]. Cardoso and Vodev [CaVo02] extended Burq’s estimate in [Bu02] to infinite volume Riemannian manifolds which may contain cusps.

In lower regularity, Datchev [Da14] and the author [Sh16] proved the weighted resolvent norm in (1.1) is still $O(e^{C h^{-1}})$, provided $V$ and $\nabla V$ belong to $L^\infty$ and have long-range decay. Vodev [Vo14] showed an $O(e^{C h^{-1}})$ bound, $0 < \ell < 1$, for potentials that are Hölder continuous, $h$-dependent, and have decay depending on $\ell$.

Since the completion of the first draft of this paper, the author has learned about the independent and parallel work of Klopp and Vogel [KiVo18]. They use a different Carleman estimate to show that, if the support of $V$ is contained in the ball $B(0, R) := \{x \in \mathbb{R}^n : |x| < R\}$, and $\chi$ is a smooth cutoff supported near $B(0, R)$, then for any compact interval $I \subseteq \mathbb{R} \setminus \{0\}$, there exist constants $C > 0$ and $h_0 \in (0, 1]$ such that

$$\|(h^2 \Delta + V - \lambda^2)^{-1} v\|_{H^1(B(0, R))} \leq e^{C h^{-4/3} \log(h^{-1})} \|v\|_{L^2(B(0, R))},$$

for all $h \in (0, h_0]$, $v \in L^2_{\text{comp}}(B(0, R))$, and $\lambda \in I$.

The novelty of (1.1) and (1.2) is that they are the first explicit $h$-dependent weighted resolvent bounds in such low regularity. The only previous result for general $L^\infty$ potentials of which the author is aware is by Rodnianski and Tao [RoTa15, Theorem 1.7]. They consider short-range, $L^\infty$ potentials on asymptotically conic manifolds of dimension $n \geq 3$, and prove a non-semiclassical version of (1.1) in which the right side is replaced by an unspecified function of $h$.

A related result for compactly supported $L^\infty$ potentials in dimension one is [DyZw, Theorem 2.29]. It says that, given $V \in L^\infty_{\text{comp}}(\mathbb{R})$ and $[E_{\min}, E_{\max}] \subseteq (0, \infty)$, there exists a constant $c > 0$ such that the meromorphic continuation of the cutoff resolvent

$$\chi(h^2 \partial_x + V - z)^{-1} \chi, \quad \chi \in C^\infty_0(\mathbb{R}),$$

for all $h \in (0, h_0]$, $v \in L^2_{\text{comp}}(B(0, R))$, and $\lambda \in I$. 


from $\text{Im } z > 0$, $\text{Re } z > 0$ to $\text{Im } z \leq 0$, $\text{Re } z > 0$ has the property

$$z \text{ is a pole, or } \text{resonance, of the continuation, } \text{Re } z \in [E_{\text{min}}, E_{\text{max}}] \implies \text{Im } z \geq -e^{-c/h}.$$ (1.3)

See section [DyZw, Section 2.8] for further details. Resonance free regions are closely related to resolvent bounds, see, for instance, [Vo14, Theorem 1.5], and [DyZw, Theorem 2.8], [KlVo18, Theorem 3], so (1.3) strongly suggests that, in dimension one, the right side of (1.1) can be improved to $e^{\sqrt{c/h}}$.

The $O(e^{\sqrt{c/h}})$ bound appearing in [Bu98, Bu02, Da14, Sh16, CaVo02] is well-known to be optimal. See for instance, [DDZ15] and the references cited there. However, it is still an open problem to determine the optimal resolvent bound for $V \in L^\infty$.

Resolvent bounds such as (1.1) imply local energy decay for the wave equation

$$\begin{cases}
(\partial_t^2 - c^2(x)\Delta)u(x,t) = 0, & (x,t) \in (\mathbb{R}^n \setminus \Omega) \times (0,\infty), \\
u(x,0) = u_0, & \\
\partial_t u(x,0) = u_1(x), & (x,t) \in \partial \Omega \times (0,\infty), \\
u(t,x) = 0, & (x,t) \in \partial \Omega \times (0,\infty),
\end{cases}$$ (1.4)

where $\Omega$ is a compact (possibly empty) obstacle with smooth boundary.

Burq used his $O(e^{\sqrt{c/h}})$ resolvent bounds to show that, if $c$ is smooth and decaying sufficiently quickly to unity at infinity, then for any compact $K \subseteq \mathbb{R}^n \setminus \Omega$ and any compactly supported initial data, the local energy $E_K(t)$ of the solution to (1.4) decays like

$$E_K(t) \leq C_{K,u_0,u_1} \frac{\log(2 + t)}{(2 + t)^{\ell}}, \quad t \geq 0.$$ (1.5)

See [Bu98, Theorem 1] and [Bu02, Theorem 2].

Logarithmic decays also hold in lower regularity. In [Sh17], the author showed that (1.5) holds if $\Omega = \emptyset$, $n \geq 2$, and $c$ is Lipschitz perturbation of unity.

Adapting the methods from these articles, one expects to find that if $c$ is a sufficiently decaying $L^\infty$-perturbation of unity, then

$$O(e^{\sqrt{c/h}}) \text{ resolvent bound, } \ell \geq 1, \text{ implies }$$

$$E_K(t) \leq C_{K,u_0,u_1} \frac{\log^{\ell/2}(2 + t)}{(2 + t)^{\ell}}, \quad t \geq 0.$$ (1.6)

In fact, the author has been informed by Georgi Vodev that (1.6) also follows by adapting the methods of [CaVo04] in a straightforward manner.

Stronger resolvent bounds are known when $V$ is more regular and additional assumptions are made about the Hamilton flow $\Phi(t) = \text{exp}(2\xi \cdot \nabla x - \nabla_x V \cdot \nabla \xi)$. For example, if $V \in C_0^\infty(\mathbb{R}^n)$ and is nontrapping at the energy $E$, then it is well-known that (1.1) improves to

$$\| (1 + |x|)^{-s} (P(h) - E - i\varepsilon)^{-1} (1 + |x|)^{-s} \|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)} \leq C/h.$$ 

Nontrapping resolvent estimates have application to Strichartz and local smoothing estimates [BT07, MMT08], resonance counting [Ch15], and integrated local energy decay [RoTa15]. For more about resolvent bounds under various dynamical assumptions, see chapter 6 from [DyZw], and the references therein. Note that, in our case, $\Phi(t)$ may be undefined, since $V$ may not be differentiable.

Let $1_{\leq 1}$ be the characteristic function of $\{x \in \mathbb{R}^n : |x| \leq 1\}$, and define $1_{\geq 1}$ similarly. The key to proving the Theorem is to establish the following global Carleman estimate.

**Lemma 1** (Carleman estimate). Let $R_0 > 3$ so that $\text{supp } V \subseteq B(0,R_0/2)$. There exist $K, C > 0$, $h_0 \in (0,1]$, and $\varphi = \varphi_h \in C^2(0,\infty)$ depending on $E_{\text{min}}, E_{\text{max}}, \|V\|_{L^\infty}, R_0, n$, and $s$ such that

$$\max \varphi = K \log(h^{-1}), \quad h \in (0,h_0),$$ (1.7)

and

$$\| (1_{\leq 1} |x|^{1/2} + 1_{\geq 1} (1 + |x|)^{-s}) e^{\varphi/h^{\delta/3}} v \|_{L^2(\mathbb{R}^n)}^2 \leq$$

$$\frac{C}{h^{10/3}} \| (1 + |x|)^s e^{\varphi/h^{\delta/3}} (P(h) - i\varepsilon)v \|_{L^2(\mathbb{R}^n)}^2 + \frac{C_\varepsilon}{h^{10/3}} \| e^{\varphi/h^{\delta/3}} v \|_{L^2(\mathbb{R}^n)}^2,$$ (1.8)
for all \( E \in [E_{\text{min}}, E_{\text{max}}] \), \( \varepsilon > 0 \), \( h \in (0, h_0) \), and \( v \in C_0^\infty(\mathbb{R}^n) \).

The key properties of the Carleman weight \( \varphi = \varphi_h \) are that \( \partial_r \varphi \) is large on \( \text{supp} \, V \) and that \( \max \varphi = K \log(h^{-1}) \), where \( K > 0 \) depends on \( E_{\text{min}} \), \( \|V\|_\infty \), and \( \text{supp} \, V \), but not on \( h \). We construct \( \varphi \) to have these properties in Lemma 3.

To prove Lemma 1, we adopt the strategy appearing in [CaVo02, Da14, RoTa15, Sh16]. The common starting point is a certain spherical energy functional \( F : (0, \infty) \to \mathbb{R} \) that includes \( \varphi \), see (3.3). Typically, \( F \) also includes \( V \). However, we intend to differentiate the product \( wF \), where \( w : (0, \infty) \to \mathbb{R} \) is a second weight function defined by (2.5). Since we cannot differentiate \( V \) in our case, we initially leave \( V \) out of \( F \), but add it back after differentiation. By doing so, we recover the terms needed to prove (1.8), at the cost of introducing a remainder term that may be large on the support of \( V \), which we must control. We control the remainder with two innovations that go beyond the techniques used in [CaVo02, Da14, RoTa15, Sh16]. First, we increase the \( h \)-dependence of the exponent in (1.8) to \( h^{-4/3} \). This differs from the Carleman estimates in the previous works, which use a factor of the form \( e^{c h^{-1}} \). Second, we require that \( \partial_r \varphi \geq c \) on \( \text{supp} \, V \), where \( c \) is chosen large enough to satisfy (2.1) and (2.2).

The outline of the paper is as follows. In Section 2, we construct the weights \( w \) and \( \varphi \) and prove their key properties. In Section 3, we prove the Carleman estimate. In Section 4, we first glue two versions of the Carleman estimate together to remove the loss at the origin. Then we prove the Theorem via a density argument. The density argument is straightforward and closely follows proofs in [Da14, Sh16, DyZw], but we recall it for the reader’s convenience.

Acknowledgements: A previous version of this paper asserted only that \( \max \varphi \leq Kh^{-1/3} \), resulting in an larger \( h^{-5/3} \) exponent on the right side of (1.1). Not until seeing the estimate (1.2) of Klopp and Vogel did the author realize that, without changing the construction, \( \varphi' \) could be estimated more sharply outside the support of \( V \) (see (2.23)). As a result, the author was able to improve the exponent in (1.1) from \( h^{-5/3} \) to \( h^{-4/3} \log(h^{-1}) \) where it currently stands. The author is grateful to Klopp and Vogel for helping to bring about this improvement.

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2. Notation and construction of the Carleman weight

In this section, we establish notation, construct the weight functions \( w \) and \( \varphi \), and prove elementary estimates needed for the proof of Lemma 1.

Throughout the paper, we use prime notation to denote differentiation with respect to the radial variable \( r := |x|, x \in \mathbb{R}^n \). For instance, \( u' := \partial_r u \).

Put
\[
\delta := 2s - 1.
\]
Without loss of generality, we assume \( 0 < \delta < 1 \). Fix \( R_0 > 3 \) large enough so that
\[
\text{supp} \, V \subseteq B(0, R_0/2).
\]
Next, choose \( c > 1 \) large enough so that
\[
c > \|V\|_\infty R_0/4, \quad \sqrt{c} \tanh(\sqrt{c}/2) > \max\{\|V\|_\infty/4, 1\}. \quad (2.1)
\]
Set
\[
\psi = \psi_h(r) := \begin{cases} 
c, & 0 < r \leq R_0, \\
\frac{h^{2/3} E_{\text{min}}}{4} - \frac{h^{2/3} E_{\text{min}}}{4}, & R_0 < r < R_1, \\
0, & r \geq R_1,
\end{cases} \quad (2.3)
\]
Figure 1. The graph of $\psi$. 

Figure 2. The graph of $w$. 

where we put

$$B = B(h) := \left( c + \frac{h^{2/3}E_{\text{min}}}{4} \right) R_0^2,$$

$$R_1^2 = R_1^2(h) := \frac{4B}{h^{2/3}E_{\text{min}}} = \left( 1 + \frac{4c}{h^{2/3}E_{\text{min}}} \right) R_0^2,$$

so that $\psi$ is continuous. In Lemma 3, we will construct the Carleman weight $\varphi$ so that $(\varphi')^2$ is approximately equal to $\psi$ for $h$ small. From this relationship, we will deduce the properties of $\varphi$ needed to prove the Carleman estimate.

To continue, define

$$w = w_{h, \delta}(r) := \begin{cases} r^2 & 0 < r < R_1, \\ R_1^2 + 1 - (1 + (r - R_1))^{-\delta} & r \geq R_1. \end{cases}$$

According to (2.3), $\psi$ and $w$ satisfy the inequality

$$h^{-2/3}(w\psi)' \geq -\frac{E_{\text{min}}}{4} w', \quad r > 0, \ r \neq R_0, \ r \neq R_1.$$

We use (2.6) in the proof of the Carleman estimate to ensure that a group of remainder terms is not too negative, see (3.6).
The next lemma proves elementary estimates involving \( w \) and \( w' \). We use them in the proof of Lemma 1 to bring intermediate steps closer to (1.8), note in particular (3.10).

**Lemma 2.** Suppose \( h \in (0,1] \). There exists \( C > 1 \) depending on \( E_{\text{min}}, R_0, c, \) and \( \delta \) so that for each \( r \neq R_1 \), it holds that

\[
2wr^{-1} - w' \geq 0, \tag{2.7}
\]

\[
w(r) \leq C h^{-2/3}, \tag{2.8}
\]

\[
w^2(r)/w'(r) \leq Ch^{-4/3}(1 + r)^{1+\delta}, \tag{2.9}
\]

\[
w'(r) \geq C^{-1} (1 \leq r + 1 \geq 1 (1 + r)^{-1-\delta}). \tag{2.10}
\]

**Proof.** When \( r < R_1 \), \( 2wr^{-1} - w' = 0 \). If \( r > R_1 \), then

\[
2wr^{-1} - w' = 2r^{-1} (R_1^2 + 1 - (1 + (r - R_1))^{-\delta}) - \delta(1 + (r - R_1))^{-1-\delta}.
\]

So to finish proving (2.7), it is enough to show,

\[
2R_1^2 \geq 2(1 + r - R_1)^{-\delta} + \delta r(1 + r - R_1)^{-1-\delta}, \quad r > R_1.
\]

Using \( 0 < \delta < 1 \) and \( R_1 > R_0 > 3 \), we estimate,

\[
2(1 + (r - R_1))^{-\delta} + \delta r (1 + r - R_1)^{-1-\delta} \leq 2 + \delta + \delta R_1
\]

\[
\leq 2R_1^2.
\]

To show (2.8), simply note that

\[
w(r) \leq R_1^2 + 1
\]

\[
\leq 2R_1^2
\]

\[
= 8h^{-2/3}BE_{\text{min}}^{-1}.
\]

For (2.9), when \( 0 < r \leq R_1 \),

\[
w^2(r)/ (w'(r)(1 + r)^{1+\delta}) \leq 2^{-1}r^2
\]

\[
\leq 2^{-1}R_1^2
\]

\[
= 2h^{-2/3}BE_{\text{min}}^{-1}.
\]

If \( r \geq R_1 \), then we use the bound \( w(r) \leq 2R_1^2 \),

\[
w^2(r)/(w'(r)(1 + r)^{1+\delta}) = \delta^{-1}w^2(r) \left( \frac{1 + r - R_1}{1 + r} \right)^{1+\delta}
\]

\[
\leq 4\delta^{-1}R_1^2
\]

\[
= 64h^{-4/3}\delta^{-1}BE_{\text{min}}^{-2}.
\]

As for (2.10), observe that when \( 1 < r \leq R_1 \),

\[
w'(r)(1 + r)^{1+\delta} \geq 2r(1 + r)^{1+\delta}
\]

\[
\geq 2,
\]

and when \( r > R_1 \),

\[
w'(r)(1 + r)^{1+\delta} = \delta \left( \frac{1 + r}{1 + r - R_1} \right)^{1+\delta}
\]

\[
\geq \delta.
\]

We now construct the Carleman weight \( \varphi \in C^2(0, \infty) \) as a solution to an ODE with right hand side equal to \( \psi \). The argument is modeled after Proposition 3.1 [DdeH16].
Lemma 3. Let $h \in (0, 1]$. There exists $\varphi = \varphi_h \in C^2(0, \infty)$ with the properties that

\[
\begin{align*}
(\varphi')^2 - h^{4/3} \varphi'' &= \psi, & r > 0, \\
0 &\leq \varphi'(r) \leq \sqrt{c}, & r > 0, \\
0 &\leq \varphi'(r) \leq Kr^{-1}, & R_0 < r < R_1 \\
1 &\leq \max \varphi = K \log(h^{-1}), \\
\varphi'(r) &\geq \sqrt{c} \tanh(\sqrt{c}/2), & 0 < r < R_0/2,
\end{align*}
\]

where $K > 0$ depends on $\|V\|_{\infty}$, $R_0$ and $E_{\min}$ but not on $h$.

Once we construct $\varphi$ according to (2.11), it holds that $\varphi' \approx \sqrt{\psi}$ for $h$ small, and so (2.12) through (2.15) follow naturally from the definition of $\psi$.

Proof. To begin, consider the solution to the initial value problem

\[
y' = h^{-4/3}(y^2 - \psi), \quad y(R_1) = 0.
\]

According to Theorem 1.2 in Chapter 1 of [CoLe], there exists an open interval $I$ containing $R_1$ and a solution $y \in C^1(I)$ to (2.16). In fact, this solution is unique on $I$. For if $y_1$, $y_2$ are two solutions to (2.16), then $\bar{y} := y_1 - y_2$ solves $\bar{y}' = (y_1 + y_2)\bar{y}$, $\bar{y}(R_1) = 0$, and hence is identically zero.

We take

\[
\varphi(r) := \int_0^r y(s)ds.
\]

Hence $\varphi$ satisfies (2.11). We now analyze $y$ to establish (2.12), (2.14) and (2.15).

First, we show that $y(r) = 0$ for $r \geq R_1$, $r \in I$, and therefore $y$ extends to be identically zero on $[R_1, \infty)$. Because $y(R_1) = 0$, there exists $\varepsilon \in (0, h^{4/3})$ so that $[R_1, R_1 + \varepsilon] \subseteq I$ and $|y(r)| \leq 1/2$ on $[R_1, R_1 + \varepsilon]$. Therefore, using (2.16), we see that $|y'(r)| = h^{-4/3}|y(r)|^2 \leq (4h^{4/3})^{-1}$ on $[R_1, R_1 + \varepsilon]$.

\[
|y(r)| \leq \int_{R_1}^r |y'(s)|ds \leq \frac{\varepsilon}{4h^{4/3}}\leq \frac{1}{4}, \quad r \in [R_1, R_1 + \varepsilon).
\]

Applying $|y'(r)| = h^{-4/3}|y(r)|^2$ on $[R_1, R_1 + \varepsilon]$ another time, we then get $|y'(r)| \leq (16h^{4/3})^{-1}$ and use it to show that $|y(r)| \leq 16^{-1}$, $r \in [R_1, R_1 + \varepsilon)$. Continuing in this fashion, we see that $y(r) = 0$ for $r \in [R_1, R_1 + \varepsilon)$. Therefore $y$ extends to be identically zero on $[R_1, \infty)$.

Moving on, we now show that

\[
0 \leq y \leq \sqrt{\psi(R_0)} = \sqrt{c}
\]

where it is defined on $(0, R_1)$. To show $y \geq 0$, assume for contradiction that there exists $0 < r_0 < R_1$ with $y(r_0) < 0$. Then, because $y' = h^{-4/3}(y^2 - \psi) \leq h^{-4/3}y^2$, we have $y'(r)/y(r)^2 \leq h^{-4/3}$, for $r$ near $r_0$. This implies

\[
y(r_0)^{-1} - y(r)^{-1} = \int_{r_0}^r \frac{y'(s)}{y(s)^2}ds \leq \frac{r - r_0}{h^{4/3}}, \quad r \geq r_0, \ r \ near \ r_0.
\]

As $r$ approaches $\inf\{r \in [r_0, \infty) : y(r) = 0\} \leq R_1$, (2.19) must hold. But this is a contradiction because the left side becomes arbitrarily large, while the right side remains bounded. So $y(r) \geq 0$ where it is defined on $(0, R_1)$.

To show $y \leq \sqrt{\psi(R_0)}$, we compare $y$ to the solution of the initial value problem

\[
z' = h^{-4/3}(z^2 - \psi(R_0)) = h^{-4/3}(z^2 - c), \quad z(R_1) = 0,
\]
This solution exists for all \( r > 0 \) and is given by
\[
z(r) = \sqrt{c} \frac{1 - \exp\left(-2h^{-4/3}\sqrt{c}(R_1 - r)\right)}{1 + \exp\left(-2h^{-4/3}\sqrt{c}(R_1 - r)\right)} = \sqrt{c} \tanh\left(h^{-4/3}\sqrt{c}(R_1 - r)\right).
\]
Suppose for contradiction that there exists \( r_0 < R_1 \) such that \( y(r_0) > z(r_0) \). Set \( \zeta := y - z \). Then \( \zeta' \geq h^{-4/3}(y + z)\zeta, \zeta(r_0) > 0 \), and \( \zeta(R_1) = 0 \).

Put \( r_1 := \inf\{r \in (r_0, R_1] : \zeta(r) = 0\} \). By the mean value theorem, there exists \( \tilde{r} \in (r_0, r_1) \) so that
\[
\zeta'(\tilde{r}) = \frac{\zeta(r_1) - \zeta(r_0)}{r_1 - r_0} = \frac{-\zeta(r_0)}{r_1 - r_0} < 0.
\]
In addition, \( \zeta(\tilde{r}) > 0 \) by the definition of \( r_1 \). But this contradicts \( \zeta'(\tilde{r}) \geq h^{-4/3}\zeta(\tilde{r})(y(\tilde{r}) + z(\tilde{r})) \) since \( y + z \geq 0 \) where \( y \) is defined on \((0, R_1)\).

So we have shown that \( 0 \leq y \leq z \leq \sqrt{c} \) where it is defined on \((0, R_1)\). It then follows by Theorem 1.3 in Chapter 2 of [CoLe] that \( y \) extends to all of \((0, R_1)\), where it obeys the same bounds.

We omit the proof of (2.13). However, we remark that one can show
\[
y \leq \xi(r) := \tilde{B}/r \quad \text{on} \quad (R_0, R_1),
\]
where
\[
\tilde{B} := \left(\sqrt{4B + h^{8/3} - h^{4/3}}\right)/2,
\]
by first noting that \( \xi \) solves
\[
\xi' = h^{-4/3}(\xi^2 - (B/r^2)), \quad \xi(R_1) = \tilde{B}/r,
\]
and then comparing \( y \) and \( \xi \) by the same method as in the preceding paragraph.

Lastly, we show that
\[
y(r) \geq \sqrt{c}\tanh\left(\sqrt{c}/2\right), \quad r \in (0, R_0/2].
\]
To see this, let \( \tilde{\zeta} \) solve the initial value problem
\[
\tilde{\zeta}' = h^{-4/3}(\tilde{\zeta}^2 - \psi), \quad \tilde{\zeta}(R_0) = 0.
\]
Then \( \tilde{\zeta} \) is given by
\[
\tilde{\zeta}(r) = \sqrt{c}\tanh\left(h^{-4/3}\sqrt{c}(R_0 - r)\right).
\]
Set \( \tilde{\xi} := y - \tilde{\zeta} \). To show (2.22), it is enough to see that \( \tilde{\xi} \geq 0 \) on \((0, R_0)\), and we give an argument similar to the one in the preceding paragraph. For contradiction, suppose there exists \( 0 < r_2 \leq R_0 \) such that \( \tilde{\xi}(r_2) < 0 \). Put \( r_3 := \inf\{r \in (r_2, R_0) : \tilde{\xi}(r) = 0\} \). Such an \( r_3 \) exists because \( \tilde{\xi}(R_0) = y(R_0) \geq 0 \). By the mean value theorem, there is some \( r^* \in (r_2, r_3) \) so that \( \tilde{\xi}'(r^*) = -(r_3 - r_2)^{-1}\tilde{\xi}(r_2) > 0 \), and furthermore \( \tilde{\xi}'(r^*) < 0 \) by the definition of \( r_3 \). But also \( \tilde{\xi}'(r^*) = h^{-4/3}\tilde{\xi}(r^*)(y(r^*) + \tilde{\zeta}(r^*)) \leq 0 \), and so we have contradiction.

We now have enough properties of \( y \) to finish the proof. With \( \varphi \) defined by (2.17), we observe that (2.12) follows from (2.18), and (2.15) from (2.22).

Lastly, we use (2.4), (2.2), (2.20), \( R_0 > 3 \), and \( h \in (0, 1] \) to see
\[
\max\varphi \geq \sqrt{c} \int_0^{R_0/2} \tanh\left(\sqrt{c}/2\right) ds \geq 1,
\]
\[
\max\varphi \leq \int_0^{R_0} \sqrt{c} ds + \int_{R_0}^{R_1} \tilde{B}/s ds \leq \sqrt{c} R_0 + \tilde{B} \log(R_1/R_0) \leq K \log(h^{-1}),
\]
where \( K > 0 \) depends on \( \|V\|_{\infty}, R_0 \) and \( E_{\min} \) but not on \( h \). This shows (2.14) and completes the proof. \( \square \)
3. Proof of the Carleman Estimate

In this section, we use the weight functions \( w \) and \( \varphi \) constructed in the previous section to prove Lemma 1. We make integral estimates using polar coordinates \((r, \theta) \in (0, \infty) \times S^{n-1}\) on \( \mathbb{R}^n \). As in the previous chapter, the starting point is a conveniently chosen conjugation

\[
P_\varphi := e^{2/h^{4/3}} r^{(n-1)/2} (P(h) - E - i \varepsilon) r^{-(n-1)/2} e^{-\varphi/h^{4/3}}
\]

\[
= -h^2 \partial_r^2 + 2h^{2/3} \varphi' \partial_r + \Lambda + \rho + V - h^{-2/3} \psi - E - i \varepsilon,
\]

where

\[
0 \leq \Lambda = \Lambda_h(r) := -h^2 r^{-2} \Delta_{S^{n-1}}, \quad \rho = \rho_h(r) := h^2 (2r)^{-2} (n-1)(n-3).
\]

To prove the Carleman estimate, we need another simple estimate, this time involving involving \( w, w' \) and \( \rho \).

**Lemma 4.** There exists \( h_0 \in (0, 1) \) depending on \( E_{\text{min}} \) and \( n \) so that

\[
(2w(r)r^{-1} - w'(r)) \rho(r) \geq -\frac{E_{\text{min}}}{4} w'(r), \tag{3.1}
\]

for all \( E \geq E_{\text{min}}, r \neq R_1 \), and \( h \in (0, h_0] \).

**Proof.** If \( r < R_1 \), then \( 2wr^{-1} - w' = 0 \) and (3.1) follows immediately. On the other hand, if \( r > R_1 \), we use \( R_1 > 3 \) to see that

\[
(2w(r)r^{-1} - w'(r)) \rho(r) \geq -h^2 (2r)^{-2} |n-1||n-3|w'(r)
\]

\[
\geq -h^2 |n-1||n-3|w'(r)/36.
\]

So we obtain (3.1) for \( r > R_1 \) by taking \( h_0 \) sufficiently small.

\[\square\]

**Proof of Lemma 1.** Let \( \int_{r, \theta} \) denote the integral over \((0, \infty) \times S^{n-1}\) with respect to \( drd\theta \), where \( d\theta \) is the usual surface measure on \( S^{n-1} \).

To show (1.8), it suffices to prove that

\[
\int_{r, \theta} (1 \leq r + 1 \geq (1 + r)^{-1-\delta})|u|^2 \leq \frac{C}{h^{10/3}} \left( \int_{r, \theta} (1 + r)^{1+\delta}|P_\varphi u|^2 + \varepsilon \int_{r, \theta} |u|^2 \right), \quad u \in r^{(n-1)/2} e^{\varepsilon/h^{4/3}} C_0^\infty(\mathbb{R}^n). \tag{3.2}
\]

Without loss of generality, we may assume \( \varepsilon \leq h^{10/3} \). To show (3.2), we proceed in the spirit of the previous chapter and of [CaVo02, Da14, RoTa15] and define the functional \( F \) by

\[
F(r) := \|hu'\|_S^2 - \langle (\Lambda + \rho - h^{-2/3} \psi - E)u, u \rangle_S, \quad r > 0, \tag{3.3}
\]

where \( \| \cdot \|_S \) and \( \langle \cdot, \cdot \rangle_S \) denote the norm and inner product on \( S^{n-1} \), respectively.

We compute the derivative of \( F \), which exists for all \( r \neq R_0, r \neq R_1 \),

\[
F'(r) = 2 \text{Re} \langle h^2 u'', u' \rangle_S - 2 \text{Re} \langle (\Lambda + \rho - h^{-2/3} \psi - E)u, u' \rangle_S
\]

\[
+ 2w^{-1} \langle (\Lambda + \rho)u, u \rangle_S + \langle h^{-2/3} \varphi' u, u \rangle_S.
\]

Next, we calculate, for \( r \neq R_0, r \neq R_1 \),

\[
wF' + w'F = 2w \text{Re} \langle h^2 u'', u' \rangle_S - 2w \text{Re} \langle (\Lambda + \rho - h^{-2/3} \psi - E)u, u' \rangle_S
\]

\[
+ 2wr^{-1} \langle (\Lambda + \rho)u, u \rangle_S + h^{-2/3} \varphi' \|u\|_S^2
\]

\[
+ w'\|hu'\|_S^2 - w''\langle (\Lambda + \rho)u, u \rangle_S + w''\langle h^{-2/3} \psi + E \rangle u, u \rangle_S
\]

\[
= -2w \text{Re} \langle P_\varphi u, u' \rangle_S + 2w \varepsilon \text{Im} \langle u, u' \rangle_S
\]

\[
+ h^2 w' \|u'\|_S^2 + (2wr^{-1} - w')\langle (\Lambda + \rho)u, u \rangle_S
\]

\[
+ Ew'\|u\|_S^2 + 4h^{2/3} w \varphi' \|u'\|_S^2 + h^{-2/3} \langle w' \psi' \rangle u, u \rangle_S + 2w \text{Re} \langle Vu, u' \rangle_S.
\]
Note that we have have added and subtracted $2w \text{Re}\langle Vu, u' \rangle_S$, $4\hbar^{2/3}w \varphi' ||u||_S^2$, and $2w \text{Im}\langle u, u' \rangle_S$ in order to recover $P_\varphi u$ in line four. Using $w' > 0$, $2w^{-1} - w' \geq 0$, $\Lambda \geq 0$ and $-2 \text{Re}(a,b) + ||b||^2 \geq -||a||^2$, we estimate, for $r \neq R_0$, $r \neq R_1$,

$$wF' + w' F \geq \frac{u^2}{h^2 w'} ||P_\varphi u||_S^2 + 2w \text{Im}\langle u, u' \rangle_S$$

$$+ Eu' ||u||_S^2 + (2w r^{-1} - w') ||u||_S^2$$

$$+ 4h^{2/3}w \varphi' ||u||_S^2 + h^{-2/3}(w \psi)' ||u||_S^2 + 2w \text{Re}(Vu, u')_S.$$  

(3.4)

To continue, let $1_{B(0,R_0/2)}$ denote the characteristic function of $B(0,R_0/2)$. We bound $2w \text{Re}\langle Vu, u' \rangle_S$ from below by

$$2w \text{Re}\langle Vu, u' \rangle_S \geq -2w(r) \int_0^r |V(r,\theta)u(r,\theta)u'(r,\theta)|d\theta$$

$$\geq -\gamma ||V||_\infty 1_{B(0,R_0/2)}(r)w(r)||u'(r,\theta)||^2_S$$

$$- \gamma^{-1} ||V||_\infty 1_{B(0,R_0/2)}(r)w(r)||u(r,\theta)||^2_S, \quad \gamma > 0.$$  

Plugging this lower bound into (3.4), we get for $r \neq R_0, r \neq R_1$,

$$wF' + w' F \geq \frac{u^2}{h^2 w'} ||P_\varphi u||_S^2 + 2w \text{Im}\langle u, u' \rangle_S$$

$$+ \left(4h^{2/3}w' - \gamma ||V||_\infty 1_{B(0,R_0/2)} \right) w' ||u||_S^2$$

$$+ \left(\frac{E w' + (2w r^{-1} - w') ||u||_S^2 + h^{-2/3}(w \psi)' - \gamma^{-1} ||V||_\infty 1_{B(0,R_0/2)} w' ||u||_S^2. \right.$$  

(3.5)

Now, fix $\gamma = h^{2/3}$ (the author is grateful to Jeff Galkowski for the suggestion to use an $h$-dependent $\gamma$). Then, use $\psi = c$ on $(0,R_0)$ along with (2.1) to get

$$(w \psi)' - ||V||_\infty 1_{B(0,R_0/2)} w \geq r (2c - ||V||_\infty R_0/2)$$

$$\geq 0, \quad r \in (0,R_0/2].$$

Combining this with (2.6) and (3.1), we have

$$\left(E w' + (2w r^{-1} - w') ||u||_S^2 + h^{-2/3}(w \psi)' - \gamma^{-1} ||V||_\infty 1_{B(0,R_0/2)} w \right) ||u||_S^2 \geq \frac{E_{\min}}{2} ||u||_S^2.$$  

(3.6)

for all $r > 0, r \neq R_0, r \neq R_1$, and all $h \in (0,h_0)$, where $h_0$ is as given in Lemma 4.

On the other hand, according to (2.2), (2.12), and (2.15), we have

$$4\varphi' - ||V||_\infty 1_{B(0,R_0/2)} \geq 0, \quad r > 0.$$  

Updating (3.5) with these lower bounds, we get

$$wF' + w' F \geq \frac{u^2}{h^2 w'} ||P_\varphi u||_S^2 + 2w \text{Im}\langle u, u' \rangle_S$$

$$+ \frac{E_{\min}}{2} w' ||u||_S^2, \quad r \neq R_0, R_1.$$  

(3.7)

Next, we apply Fatou's lemma, along with the fundamental theorem of calculus to get

$$\int_0^\infty (w(r) F(r))' \leq -\liminf_{r \to 0} w(r) F(r) = 0.$$  

(3.8)

Integrating (3.7) with respect to $dr$ and using (3.8), we arrive at

$$\frac{E_{\min}}{2} \int_{r,\theta} w' ||u||^2 \leq \frac{1}{h^2} \int_{r,\theta} \frac{u^2}{w'} ||P_\varphi u||^2 + 2\varepsilon \int_{r,\theta} w ||uu'||.$$  

(3.9)

Combining (3.9) with, (2.9) and (2.10) gives for $h \in (0,h_0)$

$$\int_{r,\theta} (1 \leq 1 + (1 + r)^{-1-\delta}) ||u||^2 \leq \frac{C}{h^{10/3}} \int_{r,\theta} (1 + r)^{1+\delta} ||P_\varphi u||^2 + 2\varepsilon \int_{r,\theta} w ||uu'||,$$  

(3.10)

where $C > 1$ is a constant that depends on $E_{\min}, R_0, n, c$ and $\delta$, but is independent of $h$ and $u$. We will reuse $C$ is the ensuing estimates, but its precise value will change from line to line.
We focus on the last term in (3.10). Our goal is to show
\[ 2 \int_{r,\theta} w|u'| \leq \frac{C}{h^2} \left( \int_{r,\theta} w^2|P_{\varphi} u|^2 + \int_{r,\theta} (1 + w^2 + \rho w^2)|u|^2 \right), \quad h \in (0, h_0]. \tag{3.11} \]
If we have shown (3.11), we can substitute it into (3.10) and use (2.8) along with
\[ |\rho w^2| \leq C h^{2/3}, \quad r > 0 \]
to get
\[ \int_{r,\theta} (1 \leq_1 r + 1 \leq_1 (1 + r)^{-1-\delta}) |u|^2 \leq \frac{C}{h^{10/3}} \int_{r,\theta} (1 + r)^{1+\delta}|P_{\varphi} u|^2 + \frac{C\varepsilon}{h^{10/3}} \int_{r,\theta} |P_{\varphi} u|^2 \]
\[ + \frac{C\varepsilon}{h^{10/3}} \int_{r,\theta} |u|^2, \quad h \in (0, h_0], \]
which will complete the proof of the Lemma. To show (3.11), we first integrate by parts,
\[ \int_{r,\theta} w^2|h'\varphi|^2 \leq C \int_{r,\theta} w^2|P_{\varphi} u|^2 + \frac{C}{h^{2/3}} \int_{r,\theta} (w^2 + |\rho w^2|)|u|^2, \quad h \in (0, h_0], \tag{3.13} \]
which will complete the proof of the Lemma. To show (3.13), we first integrate by parts,
\[ \int_{r,\theta} w^2|h'\varphi|^2 = \text{Re} \left( \int_{r,\theta} w^2\bar{u}(-h^2u'') - 2h^2ww'\bar{u}' \right), \]
and then estimate,
\[ \text{Re} \int_{r,\theta} -2h^2ww'\bar{u}' \leq \frac{h^2}{\eta_1} \int_{r,\theta} (w')^2|u|^2 + \eta_1 \int_{r,\theta} w^2|h'\varphi|^2, \quad \eta_1 > 0, \tag{3.14} \]
\[ \text{Re} \int_{r,\theta} w^2\bar{u}(-h^2u'') \]
\[ = \text{Re} \int_{r,\theta} w^2\bar{u}(P_{\varphi} - 2h^{2/3}\varphi\partial_r - \Lambda - \rho - V + h^{-2/3}\psi + E + i\varepsilon)u \]
\[ \leq \int_{r,\theta} w^2|P_{\varphi} u||u| + 2 \int_{r,\theta} w^2\varphi|h^{2/3}\psi||u| \]
\[ + \frac{1}{2} \int_{r,\theta} w^2|E - \rho - V + h^{-2/3}\psi||u|^2 \]
\[ \leq \frac{1}{2} \int_{r,\theta} w^2|P_{\varphi} u|^2 + \eta_2 \int_{r,\theta} w^2|h'\varphi|^2 + \int_{r,\theta} |\rho w^2||u|^2 \]
\[ + \left( \frac{\sqrt{\varepsilon}}{h^{2/3}n_2} + E_{\text{max}} + \|V\|_{\infty} + \frac{c}{h^{2/3}} + \frac{1}{2} \right) \int_{r,\theta} w^2|u|^2, \quad \eta_2 > 0. \tag{3.15} \]
Now, take \( \eta_1 = 1/4, \eta_2 = 1/(4\sqrt{\varepsilon}) \), and bound \( \int_{r,\theta} w^2|h'\varphi|^2 \) from above in (3.12) using (3.14) and (3.15).
We get, for \( h \in (0, h_0] \),
\[ \int_{r,\theta} w^2|h'\varphi|^2 \leq C \int_{r,\theta} w^2|P_{\varphi} u|^2 + \frac{C}{h^{2/3}} \int_{r,\theta} (w^2 + \rho w^2)|u|^2 + \frac{1}{2} \int_{r,\theta} w^2|h'\varphi|^2. \]
Subtracting the last term to the left side and multiplying through by 2, we arrive at (3.13). □
4. Proof of the theorem

In this final section, we use Lemma 1 to prove the Theorem. We condense notation by setting $L^2 = L^2(\mathbb{R}^n)$, $H^1 = H^1(\mathbb{R}^n)$, $H^2 = H^2(\mathbb{R}^n)$, $C_0^\infty = C_0^\infty(\mathbb{R}^n)$, and by renaming the weight appearing on the left side of (1.8),

$$b(r) := 1_{1 \leq r^{1/2}} + 1_{2 \leq (1 + r)^{-s}}.$$  

We also employ of a smooth version of the weight $(1 + r)^s$, which we denote by $m$,

$$m = m_0(r) := (1 + r^2)^{(1 + \delta)/4}.$$  

Before giving the main argument, we make two reductions. First, since $(1 + r)^s/\sqrt{2} \leq m(r) \leq (1 + r)^s$, $r > 0$,

to prove the Theorem it suffices to show (1.1) holds except with each instance of $(1 + |x|)^{-s}$ replaced by $m^{-1}$. Second, to obtain the desired $L^2 \to H^2$ bound, we merely need to show

$$\|m^{-1}(P(h) - E - i\varepsilon)^{-1}m^{-1}f\|_{L^2} \leq e^{Ch^{-4/3} \log(h^{-1})},$$  

$$E \in [E_{\min}, E_{\max}], \quad 0 < \varepsilon < 1, \quad h \in (0, h_0].$$  

The argument for making this reduction is standard, but we give it now for the sake of completeness.

Throughout the followings estimates, and later in the proof of the Theorem, $C > 1$ denotes a constant that is independent of $h$, but may depend on $E_{\min}$, $E_{\max}$, supp$V$, $\|V\|_{\infty}$, $n$, and $s$. It’s precise value will change from the line to line.

For each $f \in H^2$, it holds that

$$\|f\|_{H^2} \leq C (\|f\|_{L^2} + \|\Delta f\|_{L^2}).$$  

Therefore to show (1.1), we only need that

$$\|\Delta m^{-1}(P(h) - E - i\varepsilon)^{-1}m^{-1}f\|_{L^2} \leq e^{Ch^{-4/3} \log(h^{-1})}\|f\|_{L^2},$$  

$$E \in [E_{\min}, E_{\max}], \quad 0 < \varepsilon < 1, \quad h \in (0, h_0], \quad f \in L^2.$$

(4.1)

If $[\Delta, m^{-1}]$ denotes the commutator of $\Delta$ and $m^{-1}$, then a simple calculation shows

$$[\Delta, m^{-1}]m f = m^{-1}(\Delta m)f + 2m^{-1}\nabla m \cdot \nabla f,$$  

which is a bounded map $H^1 \to L^2$. Using (4.1) along with

$$\|\nabla f\|_{L^2} \leq C_{\gamma}\|f\|_{L^2} + \gamma^{-1}\|\Delta f\|_{L^2}, \quad \gamma > 0, \quad f \in L^2,$$

and

$$\Delta(P(h) - E - i\varepsilon)^{-1} = h^{-2}(V - E - i\varepsilon)(P(h) - E - i\varepsilon)^{-1} - h^{-2},$$

we have for $E \in [E_{\min}, E_{\max}]$ and $h$ small enough,

$$\|\Delta m^{-1}(P(h) - E - i\varepsilon)^{-1}m^{-1}f\|_{L^2} \leq \|[\Delta, m^{-1}]m^{-1}(P(h) - E - i\varepsilon)^{-1}m^{-1}f\|_{L^2}$$

$$\quad + \|m^{-1}\Delta(P(h) - E - i\varepsilon)^{-1}m^{-1}f\|_{L^2}$$

$$\quad \leq C\|m^{-1}(P(h) - E - i\varepsilon)^{-1}m^{-1}f\|_{H^1}$$

$$\quad + Ch^{-2}e^{Ch^{-4/3} \log(h^{-1})}\|f\|_{L^2}$$

$$\quad \leq C(1 + \gamma)e^{Ch^{-4/3} \log(h^{-1})}\|f\|_{L^2}$$

$$\quad + C\gamma^{-1}\|\Delta m^{-1}(P(h) - E - i\varepsilon)^{-1}m^{-1}f\|_{L^2}$$

$$\quad + e^{Ch^{-4/3} \log(h^{-1})}\|f\|_{L^2}.$$  

If we set $\gamma = 2C$, we can absorb the term in line six on the right side into the left side, and then multiply through by 2. This establishes (4.2).

Proof of the Theorem. Let $\tilde{R_0} > 3$ be large enough so that supp$V \subseteq B(0, \tilde{R_0}/4)$. Pick $x_0 \in \mathbb{R}^n$ with $1/2 < |x_0| < 3/4$, which implies

$$\text{supp} V_0(\cdot + x_0) \subseteq B(0, \tilde{R_0}/2).$$
We shift coordinates, apply (1.8) to the operator $P_0 = P_0(h) := -h^2 \Delta + V(\cdot + x_0) - E$ in place of $P$, and then shift back:

$$\left\| b(|\cdot - x_0|) e^{\varphi(|\cdot - x_0|)h^{-4/3}} v \right\|_{L^2}^2 = \left\| b e^{\varphi h^{-4/3}} v(\cdot + x_0) \right\|_{L^2}^2 \leq \frac{C}{h^{10/3}} \left\| m e^{\varphi h^{-4/3}} (P_0 - i\varepsilon) v(\cdot + x_0) \right\|_{L^2}^2 + \frac{C\varepsilon}{h^{10/3}} \left\| e^{\varphi h^{-4/3}} v(\cdot + x_0) \right\|_{L^2}^2$$

$$+ \frac{C\varepsilon}{h^{10/3}} \left\| e^{\varphi h^{-4/3}} v \right\|_{L^2}^2$$

Summarizing in a single inequality, we have

$$\left\| b(|\cdot - x_0|) e^{\varphi(|\cdot - x_0|)h^{-4/3}} v \right\|_{L^2}^2 \leq \frac{C}{h^{10/3}} \left\| m(|\cdot - x_0|) e^{\varphi(|\cdot - x_0|)h^{-4/3}} (P - i\varepsilon) v \right\|_{L^2}^2$$

$$+ \frac{C\varepsilon}{h^{10/3}} \left\| e^{\varphi(|\cdot - x_0|)h^{-4/3}} v \right\|_{L^2}^2, \quad h \in (0, h_0].$$

Set $C_{\varphi} = C_{\varphi}(h) := 2 \max \varphi$. Recall that by (2.14),

$$1 \leq C_{\varphi} \leq K \log(h^{-1}),$$

for $K > 0$ depending on $R_0$, $\|V\|_{\infty}$, and $E_{\min}$, but not on $h$. Multiply (1.8) and (4.3) through by $e^{-C_{\varphi}h^{-4/3}}$ to obtain for $h \in (0, h_0]$,

$$e^{-C_{\varphi}h^{-4/3}} \left\| b v \right\|_{L^2}^2 \leq \frac{C}{h^{10/3}} \left\| m(P - i\varepsilon) v \right\|_{L^2}^2 + \frac{C\varepsilon}{h^{10/3}} \left\| v \right\|_{L^2}^2,$$

$$e^{-C_{\varphi}h^{-4/3}} \left\| b(|\cdot - x_0|) v \right\|_{L^2}^2 \leq \frac{C}{h^{10/3}} \left\| m(|\cdot - x_0|)(P - i\varepsilon) v \right\|_{L^2}^2 + \frac{C\varepsilon}{h^{10/3}} \left\| v \right\|_{L^2}^2.$$

It is straightforward to show that

$$4^{-1} m^{-2} \leq b^2 + b^2(|\cdot - x_0|), \quad m^2 + m^2(|\cdot - x_0|) \leq 17m^2,$$

We add (4.6) and (4.5) and apply (4.7) to arrive at

$$e^{-C_{\varphi}h^{-4/3}} \left\| m^{-1} v \right\|_{L^2}^2 \leq \frac{C}{h^{10/3}} \left\| m(P - i\varepsilon) v \right\|_{L^2}^2 + \frac{C\varepsilon}{h^{10/3}} \left\| v \right\|_{L^2}^2.$$

For any $\eta > 0$,

$$2\varepsilon \left\| v \right\|_{L^2}^2 = -2 \text{Im}(\langle P - i\varepsilon \rangle v, v)_{L^2} \leq \eta^{-2} \left\| m(P - i\varepsilon) v \right\|_{L^2}^2 + \eta \left\| m^{-1} v \right\|_{L^2}^2.$$

Setting $\eta = h^{10/3} (2C)^{-1} e^{-C_{\varphi}h^{-4/3}}$ and applying (4.4), we estimate $\varepsilon \left\| v \right\|_{L^2}^2$ from above and find that

$$\left\| m^{-1} v \right\|_{L^2}^2 \leq e^{2C_{\varphi}h^{10/3} \log(h^{-1})} \left\| m(P - i\varepsilon) v \right\|_{L^2}^2, \quad h \in (0, h_0].$$

The final task is to use (4.8) to show that for any $f \in L^2$,

$$\left\| m^{-1}(P - i\varepsilon)^{-1} m^{-1} f \right\|_{L^2}^2 \leq e^{C_{\varphi}h^{-4/3} \log(h^{-1})} \left\| f \right\|_{L^2}^2, \quad h \in (0, h_0].$$

from which (1.1) follows. To establish (4.9), we prove a simple Sobolev space estimate and then apply a density argument which relies on (4.8).

In what follows, we use $a \lesssim b$ to denote $a \leq C_{a,b} h$ for $C_{a,b}$ depending on $\varepsilon$ and $h$, but not on $v \in H^2$. The commutator $[P, m] = -h^2 \Delta m + 2h^2 \nabla m \cdot \nabla : H^2 \to L^2$ is bounded. So for $v \in H^2$ such that $mv \in H^2$, we have

$$\left\| m(P - i\varepsilon) v \right\|_{L^2} \lesssim \left\| (P - i\varepsilon) mv \right\|_{L^2} + \left\| [P, m] v \right\|_{L^2} \lesssim \left\| mv \right\|_{H^2} + \| v \|_{H^2} \lesssim \left\| mv \right\|_{H^2}.$$
Thus we have shown
\[ \|m(P - i\varepsilon)v\|_{L^2} \leq C_{\varepsilon, h}\|mv\|_{H^2}, \quad v \in H^2 \text{ such that } mv \in H^2. \quad (4.10) \]

For fixed \( f \in L^2 \), the function \( m(P - i\varepsilon)^{-1}m^{-1}f \in H^2 \) because
\[ m(P - i\varepsilon)^{-1}m^{-1}f = (P - i\varepsilon)^{-1}f + [m, (P - i\varepsilon)^{-1}]m^{-1}f \]
\[ = (P - i\varepsilon)^{-1}f + (P - i\varepsilon)^{-1}[P, m](P - i\varepsilon)^{-1}m^{-1}f. \]
Now, choose a sequence \( v_k \in C_0^\infty \) such that \( v_k \to m(P - i\varepsilon)^{-1}m^{-1}f \) in \( H^2 \). Define \( \tilde{v}_k := m^{-1}v_k \). Then, as \( k \to \infty \)
\[ \|m^{-1}\tilde{v}_k - m^{-1}(P - i\varepsilon)^{-1}m^{-1}f\|_{L^2} \leq \|v_k - m(P - i\varepsilon)^{-1}m^{-1}f\|_{H^2} \to 0. \]
Also, applying (4.10)
\[ \|m(P - i\varepsilon)\tilde{v}_k - f\|_{L^2} \lesssim \|v_k - m(P - i\varepsilon)^{-1}m^{-1}f\|_{H^2} \to 0. \]
We then achieve (4.9) by replacing \( v \) by \( \tilde{v}_k \) in (4.8) and sending \( k \to \infty \).

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