Leskelä, Lasse

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Ross’s second conjecture and supermodular stochastic ordering

Lasse Leskelä

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1 Introduction

In 1978, about eight years before the first issue of Queueing Systems: Theory and Applications, Sheldon M. Ross presented in [13] two famous conjectures about the classical single server queue subject to a randomly time-varying arrival rate \( \lambda(t) \). Intuitively, more autocorrelated arrival rate processes lead to more bursty arrival patterns and therefore to larger workloads and longer waiting times. This question is naturally approached by modelling the arrival times using a Cox process (doubly stochastic Poisson process) in which the instantaneous arrival rate \( \lambda(t) \) is a stationary and ergodic random process.

Ross’s first conjecture in [13] states that the mean stationary waiting time decreases if the Cox arrival process is replaced by a homogeneous Poisson process with constant rate \( \mathbb{E}\lambda(0) \). In 1981, Tomasz Rolski [11] published an elegant proof of this conjecture. After five years, he proved a complementary upper bound [12], showing that the mean stationary waiting time increases if the intensity of the Cox process is replaced by a random constant \( \lambda(0) \). These bounds display two extreme dependence structures:
- The lower bound corresponds to a fully averaged random environment and leads to a completely independent (Lévy) dependence structure.
- The upper bound corresponds to a frozen or quenched random environment and leads to a maximally autocorrelated dependence structure.

Ross’s second conjecture [13] concerns an arrival intensity \( \lambda_c(t) = \lambda(ct) \) parameterised by a modulation rate \( c \in (0, \infty) \) and a stationary and ergodic baseline intensity \( \lambda(t) \). Here \( c \to 0 \) corresponds to freezing the environment to a (random) constant, and \( c \to \infty \) corresponds to observing the environment at its ergodic average value. Intuitively, a smaller modulation rate should correspond to more bursty arrival point patterns and longer waiting times. Therefore, Ross conjectured that the mean stationary waiting time in the queue should be a decreasing function of the modulation rate \( c \).
2 Problem statement

It appears that, after more than 40 years, Ross’s second conjecture still remains not fully solved. This motivates highlighting the following problem. Consider a single server queue with a Cox arrival process and IID service times, where the arrival rate $\lambda_c(t) = \lambda(ct)$ is defined in terms of a modulation rate $c \in (0, \infty)$ and a stationary and ergodic random baseline intensity $\lambda(t)$ for which the queueing system is stable. Denote by $w(c)$ the mean stationary workload in the system.

Problem 1 Is it always true that
$$c_1 \leq c_2 \implies w(c_1) \geq w(c_2),$$
and if not, what is a simple necessary and sufficient condition for $\lambda(t)$ that is needed for (1) to be valid?

3 Discussion

During past decades, the study of the above question has inspired the development of novel methods, e.g. stochastic dependence orders [10, 14], for analysing spatial and temporal autocorrelations in depth far greater than pairwise correlation coefficients. Key milestones partially answering the above question include:

- In the original article [13], Ross proved (1) for a case in which the baseline intensity is a two-state continuous-time Markov chain (CTMC) with one state being zero.
- In 1991, Chang, Chao, and Pinedo [3] proved (1) in case where the baseline intensity is a finite-state CTMC with transition rates of form $q_{ij} = \alpha_i > 0$ for all $i \neq j$.
- In 1998, Bäuerle and Rolski [1] established (1) in case where the baseline intensity is a finite-state CTMC which is doubly stochastically monotone in the sense that the transition rate matrix $Q$ and its time-reversal $Q^*$ with entries $Q^*_{ij} = \frac{\pi_j}{\pi_i} Q_{ji}$ are stochastically monotone with respect to the strong stochastic order on the real line.

As a consequence of the latter milestone, we know that (1) holds whenever the baseline intensity $\lambda(t)$ is a reversible and stochastically monotone CTMC, such as a two-state CTMC, or a birth-and-death process. However, monotonicity in the strong stochastic order is a property of a dynamic system preserving the ordering of states [5–7] and as such is not directly related to autocorrelation properties. If stochastic monotonicity precisely characterises CTMCs satisfying (1), this would be a fundamental, and perhaps surprising, discovery. A possible step in analysing this question, pointed to me by an anonymous referee, is studying the following problem.

Problem 2 Does (1) hold when the baseline intensity is of the form $\lambda(t) = f(X(t))$ in which $f$ is a Boolean or more general deterministic function defined on the state space of a doubly stochastically monotone CTMC $X(t)$?

Bäuerle and Rolski [1] also showed that a sufficient condition for (1) is that $c \mapsto \lambda_c = (\lambda_c(t))$ is decreasing in the supermodular stochastic ordering, in the sense that
$$c_1 \leq c_2 \implies \lambda_{c_1} \geq_{sm} \lambda_{c_2},$$
where we recall that for stochastic processes $X \leq_{sm} Y$ means that $E\phi(X_{t_1}, \ldots, X_{t_n}) \leq E\phi(Y_{t_1}, \ldots, Y_{t_n})$ for all integers $n \geq 1$, all time instants $t_k$, and all supermodular functions $\phi : \mathbb{R}^n \to \mathbb{R}$ for which the expectations exist [10, 14]. Furthermore, any doubly
stochastically monotone CTMC studied in [1] satisfies (2). These discoveries have inspired impressive analytical works analysing supermodular regularity of Markov chains [4, 8, 9] and more recently also spatial random measures and point patterns [2]. Nevertheless, the following problems apparently are still unanswered in full generality.

**Problem 3** Does there exist a stationary and ergodic CTMC \( \lambda = (\lambda(t)) \) which is not stochastically monotone but satisfies (2)?

**Problem 4** Describe a simple necessary and sufficient condition for a stationary and ergodic CTMC \( \lambda = (\lambda(t)) \) under which (2) holds.

Problems 3 and 4 are appealing because, outside the domain of stochastically monotone CTMCs, not much seems to be known about them even for state spaces of size 3. Furthermore, the above problems may also be formulated for discrete-time Markov chains, either by uniformising the CTMC, or by analysing chains generated by dampened transition probability matrices \((1-c)I + cP\) with \(c \in (0, 1]\) and \(P\) representing a baseline Markov chain. To the best of my knowledge, all of the above problems remain equally open also in the discrete-time setting.

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