Hecke symmetries associated with the polynomial algebra in 3 commuting indeterminates

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All Hecke symmetries corresponding to quantum groups of the GL(3) family can be found by solving a large set of equations in the coordinates of two cubic tensors associated with a pair of Artin-Schelter regular graded algebras of global dimension 3. This method worked out by Ewen and Ogievetsky [3] requires a case by case analysis. Each case is distinguished by the choice of one cubic tensor and a linear operator whose action should fix that tensor.

The alternating cubic tensors correspond to the free commutative algebras with 3 generators. This case appears to be somewhat troublesome in the approach of Ewen and Ogievetsky as it involves no restriction on the linear operators which extend to automorphisms of the algebra. In fact, the necessary equations with this choice of a cubic tensor were not investigated in [3]. Instead it was proposed to interchange the roles of two tensors and look at all solutions where the second tensor was found to be alternating.

In the present paper we pursue quite a different approach which enables us to derive a general formula for all Hecke symmetries $R$ with the $R$-symmetric algebra $S(V, R)$ being the polynomial algebra in 3 commuting indeterminates. It will be shown that such a Hecke symmetry is determined by a bivector and a symmetric bilinear form on a 3-dimensional vector space $V$. Precise statements of results are given in section 5 of the paper.

Rather than trying to fix all possibilities for the second cubic tensor we reformulate the braid equation for $R$ in terms of a certain collection of linear functions $\ell_{xy} \in V^*$ indexed by pairs of vectors $x, y \in V$. In this form equations are easily manageable. Especially, it will be immediately clear that all $\ell_{xx}, x \in V$, are scalar multiples of one function. This property will further lead to the determination of $\ell_{xy}$ for arbitrary $x, y$ and the recovery of the Hecke symmetry itself. At the end we will still need some checks to ensure that the braid equation holds in full generality.

Our solution exploits invariance of the ordinary symmetric algebra $S(V)$ and some peculiarities of the 3-dimensional space. Unfortunately, this method does not generalize to higher dimensions.

As a byproduct we will see that each Hecke symmetry with the prescribed algebra $S(V, R)$ is a deformation of the flip operator $R_0$ that sends $x \otimes y$ to $y \otimes x$ for all $x, y \in V$. This was by no means clear a priori. The already mentioned preprint [3] gives examples of $R$-matrices not obtained by such a deformation.

In section 6 we determine the equivalence classes of Hecke symmetries with the prescribed algebra $S(V, R)$. There are two families with an arbitrary parameter $q \neq 1$ of the quadratic Hecke relation and six equivalence classes with $q = 1$. The latter can be obtained as specializations of the operators in the 1-parameter families, but we list all cases with $q = 1$ as separate types. We provide explicit formulas for the Hecke symmetry $R$ in each type and for the corresponding classical $r$-matrix.
As a matter of comparison, Ewen and Ogievetsky mention a solution with an alternating cubic tensor only once [3, p. 18]. This does not tell the full story as the twists used there distort the tensor and the corresponding algebra. One can search for the other solutions by twisting back.

1. Notation

Let $V$ be a vector space of dimension 3 over a field $k$. In the tensor algebra $T(V)$ we have

$$xy = x \otimes y, \quad xyz = x \otimes y \otimes z$$

for $x, y, z \in V$. This will be understood in all formulas. For each $k \geq 0$ denote by $\text{Alt}_k$ the subspace of alternating tensors in $V^\otimes k$. There is a multiplication $\bowtie$ which makes the direct sum of spaces $\text{Alt}_k$ an associative algebra isomorphic to the exterior algebra $\bigwedge V$. Explicitly,

$$x \wedge y = xy - yx, \quad x \wedge y \wedge z = xyz + yzx + zyx - zyx - xyz - yxz$$

for $x, y, z \in V$. Essentially, we may identify $x \wedge y$ and $x \wedge y \wedge z$ with the bivector $x \wedge y$ and the trivector $x \wedge y \wedge z$.

Fix a nonzero alternating trilinear form $\omega : V \times V \times V \rightarrow k$. Since the space $\text{Alt}_3$ has dimension 1, there is a linear bijection $\tilde{\omega} : \text{Alt}_3 \rightarrow k$ such that

$$\tilde{\omega}(x \wedge y \wedge z) = \omega(x, y, z) \quad \text{for } x, y, z \in V.$$

Let $\omega_{xy} \in V^*$ be the linear forms defined by the rule

$$\omega_{xy}(v) = \omega(x, y, v), \quad x, y, v \in V.$$

Note that $\omega_{xy} \neq 0$ if and only if $x$ and $y$ are linearly independent, and in this case $\ker \omega_{xy}$ is spanned by $x$ and $y$. If $l \in V^*$ is any nonzero linear form, then $l = \omega_{ab}$ for a suitably normalized basis $a, b$ of the 2-dimensional subspace $\ker l \subset V$. Thus $V^* = \{ \omega_{xy} \mid x, y \in V \}$.

2. Reformulation of the braid equation

Let $0 \neq q \in k$. According to Gurevich [5] a Hecke symmetry with parameter $q$ on $V$ is any linear operator $R : V \otimes V \rightarrow V \otimes V$ satisfying the braid equation

$$(R \otimes \text{Id}_V)(\text{Id}_V \otimes R)(R \otimes \text{Id}_V) = (\text{Id}_V \otimes R)(R \otimes \text{Id}_V)(\text{Id}_V \otimes R)$$

and the quadratic Hecke relation

$$(R - q \cdot \text{Id}_V \otimes V)(R + \text{Id}_V \otimes V) = 0.$$

The $R$-symmetric algebra $S(V, R)$ is the factor algebra of the tensor algebra $T(V)$ by the graded ideal generated by the subspace $\text{Im} Y \subset V^\otimes 2$ where

$$Y = q \cdot \text{Id}_V \otimes V - R \quad (2.1)$$

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is the $R$-skewsymmetrizer.

Our aim is to describe all Hecke symmetries $R \in GL(V \otimes V)$ such that $S(V, R)$ is the ordinary symmetric algebra of $V$, i.e., $S(V, R) = T(V)/I$ where $I$ is the ideal of $T(V)$ generated by the 3-dimensional subspace $Alt_2$ of alternating tensors in $V \otimes V$.

So let $R$ be such a Hecke symmetry with parameter $q$. Then $\text{Im} \ Y = Alt_2$. The Hecke relation for $R$ implies that $Y^2 = (q + 1)Y$. Hence

$$Yw = (q + 1)w \quad \text{for all} \ w \in Alt_2. \quad (2.2)$$

The braid equation for $R$ is equivalent to the following equation for $Y$:

$$(\text{Id}_V \otimes Y)(Y \otimes \text{Id}_V)(Y \otimes Y) - q(\text{Id}_V \otimes Y)$$

$$= (Y \otimes \text{Id}_V)(Y \otimes Y)(Y \otimes \text{Id}_V) - q(Y \otimes Y). \quad (2.3)$$

Here the linear operators $\text{Id} \otimes Y$ and $Y \otimes \text{Id}$ acting on $V \otimes V$ have images, respectively, $V \otimes Alt_2$ and $Alt_2 \otimes V$. Therefore the two equal operators in the above equality have images in the 1-dimensional subspace $Alt_3 = (V \otimes Alt_2) \cap (Alt_2 \otimes V)$ of alternating tensors in $V \otimes V$. So it follows that

$$(\text{Id}_V \otimes Y)(Y \otimes \text{Id}_V)w - qw \in Alt_3 \quad \text{for all} \ w \in V \otimes Alt_2. \quad (2.4)$$

We will eventually find a general form of such operators $Y$. First we are going to reformulate the requested condition in a different form starting with its coordinate representation as an intermediate step. Let $e_1, e_2, e_3$ be a linear basis for $V$, and let $Y_{ij}^{kl} \in \mathbb{k}$ be the coefficients in the expressions

$$Y(e_i e_j) = \sum Y_{ij}^{kl} e_k e_l, \quad Y_{ij}^{kl} = -Y_{ij}^{lk}, \quad Y_{ij}^{kk} = 0.$$  

Here and later we assume that the summation is over the indices repeated as subscripts and superscripts. Then

$$(\text{Id} \otimes Y)(Y \otimes \text{Id})(e_i e_j e_k - e_i e_k e_j) = \sum (Y_{ij}^{r lj} Y_{lk}^{rt} - Y_{lk}^{r lj} Y_{lj}^{rt}) e_r e_t e_s.$$ 

Since $e_i e_j e_k - e_i e_k e_j \in V \otimes Alt_2$, we must have

$$\sum (Y_{ij}^{r lj} Y_{lk}^{rt} - Y_{lk}^{r lj} Y_{lj}^{rt}) e_r e_s e_t = q(e_i e_j e_k - e_i e_k e_j) \pmod{Alt_3}. \quad (\text{mod} \ Alt_3).$$

If $s = r$, then the basis monomial $e_r e_s e_t$ has zero coefficient in the elements of $Alt_3$.

Comparison of coefficients yields

$$\sum (Y_{ij}^{r lj} Y_{lk}^{rt} - Y_{lk}^{r lj} Y_{lj}^{rt}) = \begin{cases} 
0 & \text{if } i \neq r, \text{ or } t = r, \text{ or } \{j, k\} \neq \{r, t\}, \\
q & \text{if } i = j = r \text{ and } k = t \neq r, \\
-q & \text{if } i = k = r \text{ and } j = t \neq r.
\end{cases} \quad (2.5)$$

where the summation is over $l$. We will not need to look at the basis monomials $e_r e_s e_t$ with $s \neq r$ due to the following observation:
**Lemma 2.1.** Let \( Y : V^\otimes 2 \rightarrow V^\otimes 2 \) be a linear operator such that \( \text{Im} \ Y \subset \text{Alt}_2 \). For the containments (2.4) to hold it is necessary and sufficient that in every basis of \( V \) the matrix components of \( Y \) satisfy (2.5).

**Proof.** Equalities (2.5) mean that the image of the linear operator

\[
Z = (\left( \text{Id}_V \otimes Y \right)(Y \otimes \text{Id}_V) - q \cdot \text{Id}_{V^\otimes 3})|_{V^\otimes \text{Alt}_2}
\]

is contained in the 6-dimensional linear subspace \( W_{e_1,e_2,e_3} \) of \( V^\otimes 3 \) spanned by the basis monomials \( e_re_se_t \) with \( r, s, t \) pairwise distinct. If these equalities hold in every basis of \( V \), then \( \text{Im} \ Z \subset W \) where \( W \) is the intersection of those subspaces \( W_{e_1,e_2,e_3} \) taken for all bases of \( V \). Since \( W \) is a \( \text{GL}(V) \)-submodule of \( V^\otimes 3 \), it is easy to see that \( W = \text{Alt}_3 \). Thus \( \text{Im} \ Z \subset \text{Alt}_3 \). \( \square \)

Next we will find a coordinate-free interpretation of equalities (2.5). The multiplication \( \wedge \) produces the elements \( x \wedge Y(yz) \in \text{Alt}_3 \) for \( x, y, z \in V \). So there are linear forms \( \ell_{xy} \in V^* \) defined by the rule

\[
\ell_{xy}(z) = \tilde{\omega}(x \wedge Y(yz)), \quad x, y, z \in V.
\] (2.6)

Note that the expression \( \ell_{xy}(z) \) is a trilinear function of \((x, y, z)\). For \( f, g \in V^* \) we identify \( f \wedge g \in \wedge^2 V^* \) with an alternating bilinear form on \( V \) setting

\[
(f \wedge g)(u, v) = f(u)g(v) - f(v)g(u), \quad u, v \in V.
\]

**Lemma 2.2.** Let \( Y : V^\otimes 2 \rightarrow V^\otimes 2 \) be a linear operator such that \( \text{Im} \ Y \subset \text{Alt}_2 \). For the containments (2.4) to hold it is necessary and sufficient that

\[
(\ell_{xy} \wedge \ell_{xz} - \ell_{xx} \wedge \ell_{yz})(u, v) = q \omega(x, y, z) \omega(x, u, v)
\] (2.7)

for all \( x, y, z, u, v \in V \).

**Proof.** The expressions in the left and right hand sides of (2.7) are linear in \( y, z, u, v \). If either \( x = 0 \) or \( y = x \), then both sides are equal to 0. So we may assume that \( x \) and \( y \) are linearly independent. Let \( e_1, e_2, e_3 \) be any basis of \( V \) such that \( x = e_2 \) and \( y = e_3 \). Put \( \alpha = \omega(e_1, e_2, e_3) \). So \( 0 \neq \alpha \in k \). By linearity it suffices to check (2.7) for all triples \( z, u, v \) consisting of basis elements. Let \( z = e_i \). Since

\[
e_2 \wedge Y(e_ie_j) = Y_{ij}^{13}e_2 \wedge e_1 \wedge e_3, \quad e_3 \wedge Y(e_ie_j) = Y_{ij}^{12}e_1 \wedge e_2 \wedge e_3,
\]

we get

\[
\ell_{xz}(e_j) = \tilde{\omega}(e_2 \wedge Y(e_ie_j)) = -\alpha Y_{ij}^{13}, \quad \ell_{yz}(e_j) = \tilde{\omega}(e_3 \wedge Y(e_ie_j)) = \alpha Y_{ij}^{12}
\]

for each \( j = 1, 2, 3 \). With \( i = 2 \) and \( i = 3 \) the first of the above equalities gives

\[
\ell_{xz}(e_j) = -\alpha Y_{2j}^{13}, \quad \ell_{xy}(e_j) = -\alpha Y_{3j}^{13}.
\]

Hence
\[(\ell_{xy} \wedge \ell_{xz} - \ell_{xx} \wedge \ell_{yz})(e_k, e_j) = \alpha^2 (Y_{ij}^{13} Y_{k3}^{13} - Y_{ik}^{13} Y_{3j}^{13}) + \alpha^2 (Y_{ij}^{12} Y_{2k}^{13} - Y_{ik}^{12} Y_{2j}^{13}) \]

\[= \alpha^2 \sum (Y_{ij}^1 Y_{ik}^{13} - Y_{ik}^1 Y_{ij}^{13}) \]

(the \(l = 1\) term in the sum vanishes since \(Y_{ij}^{11} = Y_{ik}^{11} = 0\)). On the other hand,

\[\omega(e_2, e_3, e_i) \omega(e_2, e_k, e_j) = \begin{cases} 
0 & \text{if either } i \neq 1 \text{ or } \{j, k\} \neq \{1, 3\}, \\
\alpha^2 & \text{if } i = j = 1 \text{ and } k = 3, \\
-\alpha^2 & \text{if } i = k = 1 \text{ and } j = 3.
\end{cases} \]

We see that (2.7) holds for all \(x, y, z, u, v \in V\) if and only if (2.5) holds in every basis of \(V\) for \(r = 1, t = 3\) and arbitrary \(i, j, k\). For any other pair of indices \(r \neq t\) equalities (2.5) reduce to the case \(r = 1, t = 3\) by renumbering the basis elements. If \(r = t\), then the sum in the left hand side of (2.5) vanishes since \(Y_{ij}^{rr} = 0\) for all \(l, j\). Thus Lemma 2.2 is just a reformulation of Lemma 2.1. \(\square\)

3. Solution

Now comes the main step. Assume that \(\text{char } k \neq 2\).

**Proposition 3.1.** Let \(Y : V \otimes^2 \rightarrow V \otimes^2\) be a linear operator such that \(\text{Im } Y \subset \text{Alt}_2\). Then \(Y\) satisfies conditions (2.2) and (2.4) if and only if there exist two vectors \(a, b \in V\) and a symmetric bilinear form \(g : V \times V \rightarrow k\) such that

\[(q - 1)^2 = -4\Delta \quad \text{where } \Delta = g(a, a) g(b, b) - g(a, b)^2 \quad (3.1)\]

and \(Y\) is given by the formula

\[Y(xy) = g(x, y) a \wedge b + x \wedge T y + y \wedge T x + \frac{q + 1}{2} x \wedge y, \quad x, y \in V, \quad (3.2)\]

where \(T : V \rightarrow V\) is the linear operator defined by the rule

\[Tv = g(b, v) a - g(a, v) b, \quad v \in V. \quad (3.3)\]

Moreover, such an operator \(Y\) also satisfies (2.3).

**Proof.** Since the assignment \((x, t) \mapsto \tilde{\omega}(x \wedge t)\) defines a nondegenerate bilinear pairing between the vector spaces \(V\) and \(\text{Alt}_2\), it is seen from (2.6) that the operator \(Y\) is uniquely determined by the collection of linear forms \(\ell_{xy}\). We have

\[\tilde{\omega}(x \wedge Y(yz - zy)) = \ell_{xy}(z) - \ell_{xz}(y), \quad x, y, z \in V. \]

Since the space \(\text{Alt}_2\) is spanned by the tensors \(y \wedge z = yz - zy\), it follows that condition (2.2) amounts to the identity

\[\ell_{xy}(z) - \ell_{xz}(y) = (q + 1) \omega(x, y, z). \quad (3.4)\]

By Lemma 2.2 condition (2.4) is expressed by means of identity (2.7). We will determine the forms \(\ell_{xy}\) satisfying (2.7) and (3.4), and then derive a formula for \(Y\).

We first use a simple consequence of (2.7). Substituting \(z = y\) in (2.7) we get
\[ \ell_{xx} \land \ell_{yy} = 0 \quad \text{for all } x, y \in V \]

since \( \ell_{xy} \land \ell_{yx} = 0 \) and \( \omega(x, y, z) = 0 \). This means that all nonzero functions in the set \( \{ \ell_{xx} \mid x \in V \} \) are scalar multiples of each other. Note also that the assignment \( x \mapsto \ell_{xx} \) gives a homogeneous polynomial map of degree 2. Hence there is a pair \( l, \varphi \) consisting of a linear form and a quadratic form on \( V \) such that

\[ \ell_{xx} = \varphi(x) l = g(x, x) l \quad \text{for all } x \in V \quad (3.5) \]

where \( g \) is the unique symmetric bilinear form such that \( \varphi(x) = g(x, x) \) for all \( x \).

If \( \ell_{xx} = 0 \) for all \( x \), then we may use \( l = 0 \) and \( g = 0 \). Let \( a, b \in V \) be such that \( l = \omega_{ab} \). So

\[ l(v) = \omega(a, b, v) \quad \text{for all } v \in V. \quad (3.6) \]

Formula (3.4) with \( y = x \) gives \( \ell_{xx}(x) = \ell_{xx}(z) \). Hence

\[ \ell_{xy}(y) = \ell_{yx}(y) = g(y, y) l(x). \]

Since \( \ell_{xy} + \ell_{yx} = 2g(x, y) l \) by linearization of (3.5), we get

\[ \ell_{xy}(y) = 2g(x, y) l(y) - \ell_{yx}(y) = 2g(x, y) l(y) - g(y, y) l(x) \]

and

\[ \ell_{xy}(z) + \ell_{xx}(y) = 2g(x, y) l(z) + 2g(x, z) l(y) - 2g(y, z) l(x). \]

by linearization. Combined with (3.4) the last identity yields

\[ \ell_{xy}(z) = g(x, y) l(z) + g(x, z) l(y) - g(y, z) l(x) + \frac{q + 1}{2} \omega(x, y, z). \quad (3.7) \]

It is readily seen that for any \( l \) and \( g \) the linear forms \( \ell_{xy} \) defined by (3.7) satisfy (3.4). We will show that (2.7) is satisfied if and only if (3.1) holds. This step is somewhat harder.

Since \( \wedge^4 V = 0 \), every alternating multilinear form of 4 arguments in \( V \) vanishes. It follows that for any \( \xi \in V^* \) and \( v_1, v_2, v_3, v_4 \in V \) there is an equality

\[ \xi(v_1) \omega(v_2, v_3, v_4) - \xi(v_2) \omega(v_1, v_3, v_4) + \xi(v_3) \omega(v_1, v_2, v_4) - \xi(v_4) \omega(v_1, v_2, v_3) = 0. \quad (3.8) \]

Taking \( \xi \) such that \( \xi(v) = g(v, z) \) and using (3.6), we get

\[ g(x, z) l(y) - g(y, z) l(x) = g(b, z) \omega(a, x, y) - g(a, z) \omega(b, x, y) \]

\[ = \omega(Tz, x, y) = \omega(x, y, Tz). \quad (3.9) \]

Thus

\[ \ell_{xy}(z) = g(x, y) l(z) + \omega(x, y, Tz) + \frac{q + 1}{2} \omega(x, y, z) \quad (3.10) \]

for all \( x, y, z \), i.e.,
\[
\ell_{xy} = g(x, y)l + \omega_{xy}T + \frac{q+1}{2}\omega_{xy} = g(x, y)l + \omega_{xy}T'
\]

where we put \( T' = T + \frac{q+1}{2} \text{Id} \). Now

\[
\ell_{xy} \wedge \ell_{xz} - \ell_{xx} \wedge \ell_{yz} = l \wedge hT' + \omega_{xy}T' \wedge \omega_{xz}T'
\]

where \( h = g(x, y)\omega_{xz} - g(x, z)\omega_{xy} - g(x, x)\omega_{yz} \). As a special case of (3.8) we have

\[
h(u) = -g(x, u)\omega(x, y, z)
\]

for all \( u \in V \).

Hence

\[
(l \wedge hT')(u, v) = (g(x, T'u) l(v) - g(x, T'v) l(u)) \omega(x, y, z).
\]

As another special case of (3.8) we take \( \xi = \omega_{xy} \) and deduce that

\[
\xi(u) \omega(x, z, v) - \xi(v) \omega(x, z, u) = \xi(z) \omega(x, u, v)
\]

since \( \xi(x) = 0 \). In other words,

\[
(\omega_{xy} \wedge \omega_{xz})(u, v) = \omega(x, y, z) \omega(x, u, v).
\]

It follows that

\[
(\omega_{xy}T' \wedge \omega_{xz}T')(u, v) = (\omega_{xy} \wedge \omega_{xz})(T'u, T'v) = \omega(x, y, z) \omega(x, T'u, T'v).
\]

Thus the verification of (2.7) reduces to the identity

\[
g(x, T'u) l(v) - g(x, T'v) l(u) + \omega(x, T'u, T'v) = q \omega(x, u, v). \tag{3.11}
\]

Let us rewrite the left hand side of (3.11) applying several known identities. By (3.9)

\[
g(x, u) l(v) - g(x, v) l(u) = \omega(Tx, u, v).
\]

Since \( g(x, Tu) = g(u, b)g(x, a) - g(u, a)g(x, b) = -g(Tx, u) \), we also have

\[
g(x, Tu) l(v) - g(x, TV) l(u) = -\omega(T^2x, u, v).
\]

Hence

\[
\begin{align*}
g(x, T'u) l(v) - g(x, T'v) l(u) + \omega(x, T'u, T'v) \\
= \frac{q+1}{2} (\omega(Tx, u, v) + \omega(x, Tu, v) + \omega(x, u, Tv) + (q+1)^2) \omega(x, u, v) \\
- \omega(T^2x, u, v) + \omega(x, Tu, Tv)
\end{align*}
\]

Since \( \text{tr} T = 0 \), the first 3 terms sum up to 0, and also

\[
-\omega(T^2x, u, v) + \omega(x, Tu, Tv) = \omega(Tx, u, Tv) + \omega(Tx, Tu, v) + \omega(x, Tu, Tv)
\]

\[
= c_2(T) \omega(x, u, v)
\]
where \( c_2(T) \) is the second coefficient of the characteristic polynomial of \( T \).

If \( a \wedge b = 0 \), then \( T = 0 \) and \( c_2(T) = 0 \). Otherwise the image of \( T \) is contained in the 2-dimensional subspace \( \langle a, b \rangle \) of \( V \) spanned by \( a \) and \( b \). The restriction of \( T \) to this subspace has the matrix

\[
\begin{pmatrix}
g(b, a) & g(b, b) \\
-g(a, a) & -g(a, b)
\end{pmatrix},
\]

and so \( c_2(T) = \det T|_{\langle a,b \rangle} = g(a, a)g(b, b) - g(a, b)^2 \). We see that \( c_2(T) = \Delta \) in any case. As a consequence, identity (3.11), and therefore also (2.7), are satisfied if and only if

\[
\frac{(q + 1)^2}{4} + \Delta = q,
\]

which can be rewritten as (3.1).

Finally we come to the operator \( Y \). To derive (3.2) we use (3.9) to obtain the expression

\[
g(x, y) l(z) = g(y, z) l(x) + \omega(Ty, x, z).
\]

Now (3.10) can be rewritten as

\[
\ell_{xy}(z) = g(y, z) l(x) + \omega(x, z, Ty) + \omega(x, y, Tz) + \frac{q+1}{2} \omega(x, y, z) = \tilde{\omega}(x \wedge w)
\]

where \( w = g(y, z) a \wedge b + y \wedge Tz + z \wedge Ty + \frac{q+1}{2} y \wedge z \).

Hence \( Y(yz) = w \) as \( Y(yz) \) is characterized as the unique element \( w \in \text{Alt}_2 \) such that \( \tilde{\omega}(x \wedge w) = \ell_{xy}(z) \) for all \( x \in V \).

It remains to establish (2.3). This will be done separately in the next section. \( \square \)

4. Verification of the braid equation

We will check that any linear operator \( Y : V \otimes V \otimes V \rightarrow V \otimes V \otimes V \) given by formula (3.2) satisfies (2.3). This equation for \( Y \) is equivalent to the braid equation for the operator \( R = q \cdot \text{Id} - Y \). It would be quite tiresome to compute separately the left and right hand sides of (2.3). We will find an expression for the difference between the two sides making use of additional symmetry properties.

Put \( Y_1 = Y \otimes \text{Id}_V \) and \( Y_2 = \text{Id}_V \otimes Y \) for short. So we have to check that

\[
Y_1 Y_2 Y_1 w - q Y_1 w = Y_2 Y_1 Y_2 w - q Y_2 w
\]

for all \( w \in V \otimes V \).

By the construction in the previous section \( Y \) satisfies (2.2) and (2.4). The operator \( Y_1 \) gives by restriction a linear map \( V \otimes \text{Alt}_2 \rightarrow \text{Alt}_2 \otimes V \), and \( Y_2 \) gives a linear map in the opposite direction. Both \( Y_1 \) and \( Y_2 \) act on alternating tensors in \( V \otimes V \otimes V \) as the multiplication by \( q + 1 \). Hence there are induced linear maps

\[
(V \otimes \text{Alt}_2)/\text{Alt}_3 \rightarrow (\text{Alt}_2 \otimes V)/\text{Alt}_3 \rightarrow (V \otimes \text{Alt}_2)/\text{Alt}_3
\]

whose composition is \( q \cdot \text{Id} \) in view of (2.4). Since the two vector spaces involved here have equal dimension, those maps are invertible. Hence the composition of the
two maps in reversed order is also $q \cdot \text{Id}$, i.e.,

$$Y_1 Y_2 w - qw \in \text{Alt}_3 \quad \text{for all } w \in \text{Alt}_2 \otimes V. \quad (4.2)$$

If $w \in V \otimes \text{Alt}_2$, then $Y_2 w = (q + 1)w$ by (2.2) and $Y_2 Y_1 w - qw \in \text{Alt}_3$ by (2.4). In this case both sides of (4.1) are equal to $(q + 1)(Y_2 Y_1 w - qw)$. Similarly, both sides of (4.1) are equal to $(q + 1)(Y_1 Y_2 w - qw)$ when $w \in \text{Alt}_2 \otimes V$.

Thus (4.1) holds for all $w \in I_3$ where $I_3 = \text{Alt}_2 \otimes V + V \otimes \text{Alt}_2$ is the degree 3 homogeneous component of the ideal $I \subset T(V)$ defining the symmetric algebra $S(V)$. So full generality of (4.1) will follow once we check this equality for some set of tensors $w \in V^{\otimes 3}$ whose images in $S(V)$ span the vector space $S_3(V) = V^{\otimes 3}/I_3$.

If $\text{char } k \neq 2, 3$ then $S_3(V)$ is spanned by the monomials $x^3$ with $x \in V$. This is no longer true when $\text{char } k = 3$, but we note that it suffices to check (4.1) in characteristic 0 only. Indeed, taking a commutative local domain $K$ with residue field $k$ and the field of fractions $Q(K)$ of characteristic 0, one can lift the vectors $a, b$ and the bilinear form $g$ to the rank 3 free $K$-module $K^3$. The same formula (3.2) defines then a $K$-linear endomorphism of $K^3 \otimes_K K^3$, and the verification that it satisfies (4.1) can be done over $Q(K)$.

The proof of (4.1) is thus reduced to the case where $w$ is the tensor $x^3 \in V^{\otimes 3}$ for some $x \in V$, i.e., we have to show that

$$Y_1 Y_2(tx) - qt x = Y_2 Y_1(xt) - qxt \quad \text{when } t = Y(x^2) \in \text{Alt}_2. \quad (4.3)$$

Let $\sigma : V^{\otimes 3} \to V^{\otimes 3}$ be the linear operator defined by the formula

$$\sigma(xyz) = yzx, \quad x, y, z \in V.$$

It maps $V \otimes \text{Alt}_2$ onto $\text{Alt}_2 \otimes V$ and acts as the identity operator on $\text{Alt}_3$. Since both sides of (4.3) lie in $\text{Alt}_3$ by (2.4) and (4.2), equality (4.3) can be rewritten as

$$Y_1 Y_2(tx) - qt x = \sigma(Y_2 Y_1(xt) - qxt).$$

Since $\sigma(xt) = tx$, it is equivalent to the equality

$$Y_1 Y_2(tx) = \sigma Y_2 Y_1(xt). \quad (4.4)$$

We will compare the left and right hand sides above by means of the following

**Lemma 4.1.** \( Y_1 Y_2(tx) - \sigma Y_2 Y_1(xt) = 2Tx \wedge t \) for all \( x \in V \) and \( t \in \text{Alt}_2 \).

**Proof.** We may assume that \( t = y \wedge z = yz - zy \) for some \( y, z \in V \). Then

$$Y_1(xt) + \sigma Y_2(tx) = Y(xy)z - Y(xz)y + \sigma(yY(zx) - zY(yx))$$

$$= Y(xy - yx)z - Y(xz - zx)y$$

$$= (q + 1)(xyz - yxz - xzy + zyx) = (q + 1)(x \wedge t - tx).$$

Here \( x \wedge t = x \wedge y \wedge z \in \text{Alt}_3 \). Let $\alpha : \text{Alt}_2 \otimes V \to \text{Alt}_3$ be the linear map such that $\alpha(tx) = t \wedge x = x \wedge t$ for $x \in V$ and $t \in \text{Alt}_2$. We get the identity
Each Hecke symmetry \( \sigma \), as for \( w = tx \) it amounts to the equality written previously. For \( w = Y_1(xt) \) it gives

\[
Y_1(\sigma^{-1}w) = -\sigma Y_2w + (q + 1)(aw - w), \quad w \in \text{Alt}_2 \otimes V,
\]
as \( w = tx \) it amounts to the equality written previously. For \( w = Y_1(xt) \) it gives

\[
Y_1(\sigma^{-1}Y_1(xt)) = -\sigma Y_2Y_1(xt) + (q + 1)(Y(xy) \wedge z - Y(xz) \wedge y - Y_1(xt)).
\]

On the other hand, applying \( \sigma^{-1} \), we find

\[
Y_2(tx) = -\sigma^{-1}Y_1(xt) + (q + 1)(x \wedge t - xt),
\]
and it follows that

\[
Y_1Y_2(tx) = -Y_1(\sigma^{-1}Y_1(xt)) + (q + 1)^2x \wedge t - (q + 1)Y_1(xt)
= \sigma Y_2Y_1(xt) - (q + 1)Y(xy) \wedge z + (q + 1)Y(xz) \wedge y + (q + 1)^2x \wedge t.
\]

Plugging in the explicit expression (3.2), this gives

\[
Y_1Y_2(tx) - \sigma Y_2Y_1(xt) = (q + 1)(-g(x, y) a \wedge b \wedge z + g(x, z) a \wedge b \wedge y
- x \wedge T y \wedge z + x \wedge T z \wedge y - y \wedge T x \wedge z + z \wedge T x \wedge y).
\]

Note that

\[
g(x, z) a \wedge b \wedge y = g(x, y) a \wedge b \wedge z
= g(x, a) b \wedge y \wedge z - g(x, b) a \wedge y \wedge z = -T x \wedge y \wedge z
\]
since \( \wedge^4 V = 0 \) and \(-x \wedge T y \wedge z - x \wedge y \wedge T z = T x \wedge y \wedge z \) since \( \text{tr} T = 0 \). This yields

\[
Y_1Y_2(tx) - \sigma Y_2Y_1(xt) = 2Tx \wedge y \wedge z,
\]
completing the proof. \( \square \)

It is immediately clear from Lemma 4.1 that (4.4) is satisfied for \( t = x \wedge Tx \). This equality is satisfied also for \( t = a \wedge b \) since \( Tx \) lies in the subspace of \( V \) spanned by \( a \) and \( b \). Since \( Y(x^2) = g(x, x) a \wedge b + 2 x \wedge Tx \), equality (4.4) holds for \( t = Y(x^2) \) as well, and we are done.

5. Final results
We assume that \( V \) is a vector space of dimension 3 over an arbitrary field \( k \) of characteristic \( \neq 2 \). Denote by \( \text{HeckeSym}(V) \) the set of all Hecke symmetries on \( V \) and by \( \text{HeckeSym}_0(V) \) its subset consisting of those Hecke symmetries for which \( S(V, R) = S(V) \), the ordinary symmetric algebra of \( V \).

**Theorem 5.1.** Each Hecke symmetry \( R \in \text{HeckeSym}_0(V) \) with parameter \( q \) of the Hecke relation is given by the formula

\[
R(xy) = \frac{q - 1}{2}xy + \frac{q + 1}{2}yx - g(x, y) a \wedge b - x \wedge Ty - y \wedge Tx, \quad x, y \in V,
\]
where \( a, b \in V \) are two vectors, \( g : V \times V \rightarrow k \) a symmetric bilinear form satisfying (3.1) and \( T : V \rightarrow V \) the linear operator defined in terms of \( a, b \) and \( g \) by (3.3).

**Proof.** The \( R \)-skewsymmetrizer \( Y \) is determined in Proposition 3.1, and \( R \) is found by the formula \( R = q \cdot \text{Id} - Y \) according to (2.1). \( \square \)

The vector space \( V^{\otimes 2} \) is a \( \text{GL}(V) \)-module in a natural way. Hence there is also an action of \( \text{GL}(V) \) on the algebra \( \text{End}_k V^{\otimes 2} \). It is clear that the set \( \text{HeckeSym}(V) \) and its subset \( \text{HeckeSym}_0(V) \) are stable under this action of \( \text{GL}(V) \).
Theorem 5.2. There is a \( \text{GL}(V) \)-equivariant bijection between the set of Hecke symmetries \( \text{HeckeSym}_0(V) \) and the set consisting of all pairs \( (q, F) \) where \( 0 \neq q \in \mathbb{k} \) and \( F : V^\otimes 2 \to V^\otimes 2 \) is a linear operator satisfying the following conditions:

(i) \( F(xy) = F(yx) \) for all \( x, y \in V \),

(ii) \( \text{Im } F \subset \text{Alt}_2 \),

(iii) \( \text{rank } F \leq 1 \),

(iv) \( (q - 1)^2 = -4 \Delta(F) \)

where \( \Delta(F) \) is defined by the formula \( \Delta(F) = g(a, a) g(b, b) - g(a, b)^2 \) with \( a, b \in V \) and a symmetric bilinear form \( g : V \times V \to \mathbb{k} \) taken so that \( F(xy) = g(x, y) a \bowtie b \) for all \( x, y \in V \).

Proof. Each Hecke symmetry \( R \in \text{HeckeSym}_0(V) \) is determined by a symmetric bilinear form \( g : V \times V \to \mathbb{k} \) and a bivector \( t = a \wedge b \in \bigwedge^2 V \). On the other hand, the pair \( (g, t) \) gives rise to a linear operator \( F \) defined by formula (5.1). Clearly \( F \) satisfies (i)–(iii). We can describe a relationship between \( R \) and \( F \) in invariant terms. Let \( Y \) be the \( R \)-skewsymmetrizer (2.1). Then

\[
x \bowtie Y(xz) = g(x, x) a \bowtie b \bowtie z \quad \text{for all } x, z \in V
\]

since the bijection \( \bar{\omega} : \text{Alt}_3 \to \mathbb{k} \) transforms the above equality into the equality

\[
\ell_{xx}(z) = g(x, x) \omega(a, b, z)
\]

which holds in view of (3.5) and (3.6). Hence \( F(xx) \bowtie z = x \bowtie Y(xz) \), and linearizing this identity we get

\[
2F(xy) \bowtie z = x \bowtie Y(yz) + y \bowtie Y(xz), \quad x, y, z \in V.
\]

(5.2)

It follows that \( F \) is uniquely determined by \( R \). Since the multiplication \( \bowtie \) is \( \text{GL}(V) \)-invariant, the map given by the assignment \( R \mapsto F \) is \( \text{GL}(V) \)-equivariant.

We put into correspondence to \( R \) the pair \( (q, F) \) where \( q \) is the parameter of the Hecke relation satisfied by \( R \) and \( F \) is found from (5.2). Then (iv) is a consequence of (3.1). By means of the bijection \( \bar{\omega} \) formula (3.7) translates into the identity

\[
x \bowtie Y(yz) = F(xy) \bowtie z + F(xz) \bowtie y - F(yz) \bowtie x + \frac{q + 1}{2} x \bowtie y \bowtie z.
\]

Hence \( Y \), and therefore also \( R \), are uniquely determined by the pair \( (q, F) \). \( \square \)

Corollary 5.3. Given a Hecke symmetry \( R \in \text{HeckeSym}_0(V) \) with parameter \( q \), put

\[
R_\lambda = R_0 + \lambda(R - R_0)
\]

where \( R_0 : V^\otimes 2 \to V^\otimes 2 \) is the flip of tensorands \( xy \mapsto yx \). Then \( R_\lambda \in \text{HeckeSym}_0(V) \) for all \( \lambda \in \mathbb{k} \) such that \( \lambda(q - 1) \neq -1 \). Thus \( R \) is a deformation of \( R_0 \).
Proof. We use the bijective correspondence described in Theorem 5.2. Let \((q, F)\) be the pair corresponding to \(R\). Then \(R_\lambda\) is the Hecke symmetry corresponding to the pair \((q_\lambda, \lambda F)\) where \(q_\lambda = 1 + \lambda(q-1)\). Note that
\[(q_\lambda - 1)^2 = \lambda^2(q-1)^2 = -4\Delta(\lambda F)\]
since \(\Delta(\lambda F) = \lambda^2\Delta(F)\). If \(\lambda\) is such that \(\lambda(q-1) = -1\), then \(q_\lambda = 0\), and the corresponding operator \(R_\lambda\) is singular. This value of \(\lambda\) should be excluded. \(\square\)

**Corollary 5.4.** If \(R \in \text{HeckeSym}_0(V)\), then the operator \(r = R_0 R - \text{Id}\), regarded as an element of \(\mathfrak{gl}(V) \otimes \mathfrak{gl}(V)\), is a solution to the classical Yang-Baxter equation. Moreover,
\[r + r_{21} = (q-1)(R_0 + \text{Id})\]
where \(r_{21} = R_0 r R_0 = RR_0 - \text{Id}\). Thus \(r\) is skewsymmetric precisely when \(q = 1\).

Proof. Since \(R_\lambda\) in Corollary 5.3 satisfies the braid equation, the operator
\[R_0 R_\lambda = \text{Id} + \lambda r\]
satisfies the constant quantum Yang-Baxter equation, for all \(\lambda \in k\). This entails the classical Yang-Baxter equation for \(r\) (see [1]). We have
\[R_\lambda^2 = \text{Id} + (\lambda - \lambda^2)(r + r_{21}) + \lambda^2(R^2 - \text{Id}).\]
Note that \(R^2 - \text{Id} = (q-1)(R + \text{Id})\). Hence
\[R_\lambda^2 - \lambda(q-1)(R_\lambda + \text{Id}) - \text{Id} = (\lambda - \lambda^2)(r + r_{21} - (q-1)(R_0 + \text{Id})),\]
which is 0 since \(R_\lambda\) satisfies the Hecke relation with parameter \(q_\lambda = 1 + \lambda(q-1)\). This proves the desired formula for \(r + r_{21}\). \(\square\)

**6. Determination up to equivalence**

Suppose that the base field \(k\) is algebraically closed of characteristic \(\neq 2\). There is a choice of a basis \(x_1, x_2, x_3\) for the vector space \(V\) which brings any Hecke symmetry \(R\) described in Theorem 5.1 to one of 8 types. If \(a \wedge b = 0\), then \(R = R_0\) is the flip of tensorands. Suppose that \(a \wedge b \neq 0\). The first two basis vectors \(x_1, x_2\) will be chosen so that \(x_1 \wedge x_2 = a \wedge b\). Changing \(a\) and \(b\), we may assume that \(x_1 = a\) and \(x_2 = b\). Clearly, the rank of the bilinear form \(g\) and the rank of its restriction to the 2-dimensional subspace \(\langle a, b \rangle\) are \(\text{GL}(V)\)-invariants of \(R\). It is seen from (3.1) that \(q = 1\) if and only if \(g|_{\langle a, b \rangle}\) is degenerate.

If \(g|_{\langle a, b \rangle}\) is nondegenerate, then \(\langle a, b \rangle\) has a basis consisting of isotropic vectors. So we may assume in this case that \(g(a, a) = g(b, b) = 0\), and then \(4g(a, b)^2 = (q-1)^2\) by (3.1). Since \(g(-b, a) = -g(a, b)\), we can achieve \(g(a, b) = (q-1)/2\) by replacing the pair \(a, b\) with \(-b, a\), if necessary. The third basis vector \(x_3\) can be chosen in the orthogonal complement of \(\langle a, b \rangle\). We thus obtain two types of Hecke symmetries with \(q \neq 1\) distinguished by whether \(g\) is nondegenerate or not.

If \(g|_{\langle a, b \rangle}\) is degenerate, then it either has rank 1 or is identically zero, while
\[ \text{rank } g \big|_{(a,b)} \leq \text{rank } g \leq 2 + \text{rank } g \big|_{(a,b)}. \]

If \( \text{rank } g \big|_{(a,b)} = 1 \), then \( x_2 \) can be taken in the radical of \( g \big|_{(a,b)} \), and \( x_3 \) orthogonal to \( x_1 \). If \( g \big|_{(a,b)} = 0 \), then \( x_1 \) can be taken in the radical of \( g \). So \( g(x_1, x_3) = 0 \) in both cases. If \( g(x_2, x_3) \neq 0 \), then \( x_3 \) can be chosen isotropic. Altogether there are 6 types of Hecke symmetries with \( q = 1 \).

The matrices of the bilinear forms \( g \) in suitably chosen bases of \( V \) are listed below for all 8 types of Hecke symmetries \( R \) with the \( R \)-symmetric algebra \( S(V) \):

Types 1, 2 (\( q \neq 1 \)):
\[
M(g) = \begin{pmatrix}
0 & \frac{q-1}{2} & 0 \\
\frac{q-1}{2} & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

Types 3–8 (\( q = 1 \)):
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}, \quad
\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}, \quad
\begin{pmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}, \quad
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

It is seen from the above description that equivalence classes of Hecke symmetries in the set \( \text{HeckeSym}_0(V) \) are distinguished by the parameter \( q \) and the ranks of \( g \) and \( g \big|_{(a,b)} \).

The values of \( R \) at the basis elements \( x_i x_j \) of the space \( V \otimes 2 \) can be computed readily by means of the explicit formula in Theorem 5.1. In Type 1, \( x_1, x_2, x_3 \) are eigenvectors of \( T \) with respective eigenvalues \( (q - 1)/2, -(q - 1)/2, 0 \), and we get the formulas:

\[
\begin{align*}
R(x_1^2) &= qx_1^2 & R(x_1 x_2) &= (q - 1)x_1 x_2 + x_2 x_1 & R(x_1 x_3) &= (q - 1)x_1 x_3 + x_3 x_1 \\
R(x_2 x_1) &= qx_1 x_2 & R(x_2^2) &= qx_2^2 & R(x_2 x_3) &= qx_3 x_2 \\
R(x_3 x_1) &= qx_1 x_3 & R(x_3 x_2) &= (q - 1)x_3 x_2 + x_2 x_3 & R(x_3^2) &= qx_3^2 - x_1 x_2 + x_2 x_1
\end{align*}
\]

Let \( \{ E_{ij} \mid 1 \leq i, j \leq 3 \} \) be the basis of \( \mathfrak{gl}(V) \) consisting of the matrix units with respect to the chosen basis of \( V \). Then the classical \( r \)-matrix \( r = R_0 R - \text{Id} \) described in Corollary 5.4 is written as
\[
r = (q - 1) (E_{11} \otimes E_{11} + E_{22} \otimes E_{22} + E_{33} \otimes E_{33} + E_{22} \otimes E_{11} + E_{33} \otimes E_{22} + E_{11} \otimes E_{33} + E_{11} \otimes E_{11})
\]
\[
+ (q - 1) (E_{21} \otimes E_{12} + E_{23} \otimes E_{32} + E_{31} \otimes E_{13}) + E_{13} \otimes E_{23} - E_{23} \otimes E_{13}.
\]

In Type 2 the operator \( R \) has the same values at the monomials \( x_i x_j \) except that \( R(x_2^2) = qx_2^2 \), and the \( r \)-matrix is as displayed above, but without the last two summands \( E_{13} \otimes E_{23} \) and \( E_{23} \otimes E_{13} \). Specializing the parameter \( q \) to 1 gives the formulas for Types 7 and 8.

In Type 2 we have \( T x_1 = -x_2, T x_2 = 0, T x_3 = x_1 \), which gives
\[
\begin{align*}
R(x_1^2) &= x_1^2 + x_1 x_2 - x_2 x_1 & R(x_1 x_2) &= x_2 x_1 & R(x_1 x_3) &= x_3 x_1 - x_2 x_3 + x_3 x_2 \\
R(x_2 x_1) &= x_1 x_2 & R(x_2^2) &= x_2^2 & R(x_2 x_3) &= x_3 x_2 \\
R(x_3 x_1) &= x_1 x_3 - x_2 x_3 + x_3 x_2 & R(x_3 x_2) &= x_2 x_3 & R(x_3^2) &= x_3^2 + 2(x_1 x_3 - x_3 x_1)
\end{align*}
\]

and \( r = E_{21} \wedge (E_{11} + E_{33}) + E_{23} \wedge E_{31} + 2 E_{33} \wedge E_{13} \).

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The formulas for Types 4, 5, 6 are almost the same as for Type 3 with only a few changes. In Type 4 the differences are in the formulas

\[ T x_3 = 0, \quad R(\alpha(x_2^3)) = x_2^3 - x_1 x_2 + x_2 x_1 \quad \text{and} \quad r = E_{21} \land (E_{11} + E_{33}) + E_{23} \land (E_{31} - E_{13}). \]

In Type 5 we have

\[ T x_3 = 0, \quad R(\alpha(x_2^3)) = x_2^3 \quad \text{and} \quad r = E_{21} \land (E_{11} + E_{33}) + E_{23} \land E_{31}, \]

and in Type 6 we have

\[ T x_1 = 0, \quad R(x_1 x_3) = x_3 x_1, \quad R(x_3 x_1) = x_1 x_3, \quad r = 2 E_{33} \land E_{13}. \]

If \( \text{char} \neq 3 \), then the Lie algebra \( \mathfrak{g}l(3) \) is the direct sum of \( \mathfrak{sl}(3) \) and the onedimensional center. In this case the projection \( \mathfrak{g}l(3) \to \mathfrak{sl}(3) \) transforms each classical \( r \)-matrix in \( \mathfrak{g}l(3) \otimes \mathfrak{g}l(3) \) to one in \( \mathfrak{sl}(3) \otimes \mathfrak{sl}(3) \). All classical \( r \)-matrices in \( \mathfrak{sl}(3) \otimes \mathfrak{sl}(3) \) were described by Gerstenhaber and Giaquinto, and for each a quantization was found [4]. Certainly, a large part of those \( r \)-matrices do not correspond to Hecke symmetries with \( S(V, R) = S(V) \).

Recall that a finite-dimensional Lie algebra \( L \) is said to be quasi-Frobenius if there exists a nondegenerate alternating bilinear form on \( L \) which is a 2-cocycle, and if such a form is a 2-coboundary, then \( L \) is Frobenius. It is a general fact that the carriers of skewsymmetric classical \( r \)-matrices are quasi-Frobenius Lie algebras (see [2] and [1, Prop. 2.2.6]). In our present study the carriers of the \( r \)-matrices corresponding to Hecke symmetries of Types 3-6 are the following four Frobenius subalgebras of the Lie algebra \( \mathfrak{g}l(3) \):

\[ \langle E_{11}, E_{13}, E_{21}, E_{23}, E_{31}, E_{33} \rangle, \quad \langle E_{11} + E_{33}, E_{13} - E_{31}, E_{21}, E_{23} \rangle, \]
\[ \langle E_{11} + E_{33}, E_{21}, E_{23}, E_{31} \rangle, \quad \langle E_{13}, E_{33} \rangle. \]

The carrier in Type 7 is the abelian subalgebra \( (E_{13}, E_{23}) \), and in Type 8 the carrier is 0. One can see that these six Lie algebras are pairwise nonisomorphic. The conjugacy classes of all quasi-Frobenius subalgebras in the Lie algebra \( \mathfrak{sl}(3) \) were determined by Stolin [6].

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