Perturbations of Functional Inequalities for Lévy Type Dirichlet Forms

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Abstract

Perturbations of super Poincaré and weak Poincaré inequalities for Lévy type Dirichlet forms are studied. When the range of jumps is finite our results are natural extensions to the corresponding ones derived earlier for diffusion processes; and we show that the study for the situation with infinite range of jumps is essentially different. Some examples are presented to illustrate the optimality of our results.

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1 Introduction

Functional inequalities of Dirichlet forms are powerful tools in the study of Markov semigroups and spectral theory of Dirichlet operators, see [1, 6, 8, 14] for accounts on functional inequalities and applications. To establish a functional inequality, one often needs
to verify some conditions on the generator, for instances the Bakry-Emery curvature condition in the diffusion setting or the Lyapunov condition in a general setting, see e.g. [8, 9, 14]. Since these conditions exclude generators with less regular coefficients, to establish functional inequalities in a more general setting one treats the singularity part as a perturbation. So, it is important to investigate perturbations of functional inequalities.

In [2], sharp growth conditions have been presented for perturbations of super Poincaré and weak Poincaré inequalities in the diffusion setting (i.e. the underlying Dirichlet form is local). Note that these two kinds of functional inequalities are general enough to cover all Poincaré/Sobolev/Nash type inequalities, and thus, have a broad range of applications. Recently, explicit sufficient conditions were derived in [4, 5, 10, 15] for functional inequalities of stable-like Dirichlet forms. The aim of this paper is to extend perturbation results derived in [2] to the non-local setting, so that combining with the existing sufficient conditions we are able to establish functional inequalities for more general Dirichlet forms. Due to the lack of the chain rule, the study of the non-local setting is usually more complicated. Nevertheless, we are able to present some relatively clean perturbation results, which are sharp as illustrated by some examples latter on, and when the range of jumps is finite, are natural extensions to the corresponding results derived earlier for diffusion processes.

Let \((E, d)\) be a Polish space equipped with the Borel \(\sigma\)-field \(\mathcal{F}\) and a probability measure \(\mu\). Let \(\mathcal{B}(E)\) be the set of all measurable functions on \(E\), and let \(\mathcal{B}_b(E)\) be the set of all bounded elements in \(\mathcal{B}(E)\). Let \(q \in \mathcal{B}(E \times E)\) be non-negative with \(q(x, x) = 0, x \in E\), such that

\[
\lambda := \sup_{x \in E} \int_{E} (1 \wedge d(x, y)^2)q(x, y)\mu(\text{dy}) < \infty.
\]

Then

\[
\Gamma(f, g)(x) := \int_{E} (f(x) - f(y))(g(x) - g(y))q(x, y)\mu(\text{dy}), \quad \forall x \in E,
\]

\[
f, g \in \mathcal{A} := \{f \in \mathcal{B}_b(E) : \Gamma(f, f) \in \mathcal{B}_b(E)\}
\]

gives rise to a non-negatively definite bilinear map from \(\mathcal{A} \times \mathcal{A}\) to \(\mathcal{B}_b(E)\). We assume that \(\mathcal{A}\) is dense in \(L^2(\mu)\). Then it is standard that the form

\[
\mathcal{E}(f, g) := \mu(\Gamma(f, g))
\]

\[
= \int_{E \times E} (f(x) - f(y))(g(x) - g(y))q(x, y)\mu(\text{dy})\mu(\text{dx}), \quad f, g \in \mathcal{A}
\]

is closable in \(L^2(\mu)\) and its closure \((\mathcal{E}, \mathcal{D}(\mathcal{E}))\) is a symmetric conservative Dirichlet form, see e.g. [7, Example 1.2.6]. A typical example of the framework is the \(\alpha\)-stable-like Dirichlet form, where \(E = \mathbb{R}^m\) with \(d(x, y) = |x - y|\), and

\[
q(x, y) = \frac{\tilde{q}(x, y)}{u(y)|x - y|^{m+\alpha}}, \quad \mu(\text{dx}) = u(x)dx
\]
for some $\alpha \in (0, 2)$, non-negative $\tilde{q} \in \mathcal{B}_b(\mathbb{R}^m \times \mathbb{R}^m)$, and positive $u \in \mathcal{B}(\mathbb{R}^m)$ such that $\mu(dx)$ is a probability measure. In this case we have $\mathcal{A} \supset C_0^2(\mathbb{R}^m)$ which is dense in $L^2(\mu)$ for any probability measure $\mu$ on $\mathbb{R}^m$.

To investigate perturbations of functional inequalities using growth conditions as in [2], we fix a point $o \in E$ and denote $\rho(x) = d(o, x), x \in E$. Now, for a $\rho$-locally bounded measurable function $V$ on $E$ (i.e. $V$ is bounded on the set $\{x \in E : \rho(x) \leq r\}$ for all $r > 0$) such that $\mu(e^V) = 1$, let $\mu_V(dx) = e^{V(x)}\mu(dx)$. Since for every $f \in \mathcal{A}$, $\Gamma(f, f)$ given by (1.2) is a bounded measurable function on $E$, we have

$$\mathcal{E}_V(f, f) := \int_E \Gamma(f, f)(x) \mu_V(dx) < \infty.$$  

Again by the argument in [7, Example 1.2.6], the form

$$\mathcal{E}_V(f, g) := \mu_V(\Gamma(f, g)) = \int_{E \times E} (f(x) - f(y))(g(x) - g(y))q(x, y)\mu(dy)\mu_V(dx)$$

defined for $f, g \in \mathcal{A}$ is closable in $L^2(\mu_V)$ and its closure $(\mathcal{E}_V, \mathcal{D}(\mathcal{E}_V))$ is a symmetric conservative Dirichlet form.

In this paper, we shall assume that $\mathcal{E}$ satisfies a functional inequality and then search for conditions on $V$ such that $\mathcal{E}_V$ satisfies the same type of functional inequality. In the following two sections, we study perturbations of the super Poincaré inequality and the weak Poincaré inequality respectively. Each section includes some typical examples to illustrate the main results. Throughout the paper, we simply denote that $\Gamma(f) = \Gamma(f, f)$, $\mathcal{E}(f) = \mathcal{E}(f, f)$ and $\mathcal{E}_V(f) = \mathcal{E}_V(f, f)$.

## 2 Perturbations of the super Poincaré inequality

In Subsection 2.1 we state two results for perturbations of the super Poincaré inequality using growth conditions and present some examples to illustrate the results. Subsection 2.2 includes proofs of these results. Finally, in Subsection 2.3 we prove that the super Poincaré inequality is stable for perturbations under a variation condition on the support of $q$, which extends a known result for diffusion processes.

### 2.1 Main results and examples

We consider the following super Poincaré inequality introduced in [12, 13]:

$$\mu(f^2) \leq r\mathcal{E}(f) + \beta(r)\mu(|f|^2), \quad r > 0, f \in \mathcal{D}(\mathcal{E}),$$  

(2.1)

where $\beta : (0, \infty) \to (0, \infty)$ is a decreasing function. Note that if (2.1) holds for some non-decreasing function $\beta$, it holds also for the decreasing function $\tilde{\beta}(r) := \inf_{s \in [0, r]} \beta(s)$ in place of $\beta(r)$. 

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To establish a super Poincaré inequality for $\mathcal{E}_V$, we need the following quantities. For any $n \geq 1$ and $k \geq 1$, let

\[ K_{n,k}(V) = \sup_{\rho(x) \leq n+2} V(x) - \inf_{\rho(x) \leq n+k+2} V(x), \]
\[ J_{n,k}(V) = \sup_{\rho(x) \leq n+1} V(x) - 2 \inf_{\rho(x) \leq n+k+2} V(x), \]
\[ \varepsilon_{n,k}(V) = \sup_{m \geq n} \left\{ \beta^{-1} \left( 1/[2\mu(\rho > m - 1)] \right) e^{K_{m,k}(V)} \right\}, \]

where $\beta^{-1}(s) := \inf\{ r > 0 : \beta(r) \leq s \}$ for $s > 0$, with $\inf \emptyset := \infty$ by convention.

When the jump is of finite range, i.e. there exists $k_0 > 0$ such that $q(x,y) = 0$ for $d(x,y) > k_0$, we have the following result similar to [2, Theorem 3.1] for local Dirichlet forms.

**Theorem 2.1.** Assume that (2.1) holds and there exists $k_0 \geq 1$ such that $q(x,y) = 0$ for $d(x,y) > k_0$.

1. If $\inf_{n \geq 1, k \geq k_0} \varepsilon_{n,k}(V) = 0$, then the super Poincaré inequality (2.5) holds with

\[ \beta_V(r) := \inf \left\{ (1 + 8\lambda r')e^{J_{n,k}(V)} \beta(s) : s > 0, r' \in (0,r], n \geq 1, k \geq k_0 \right\} \]

such that $8\varepsilon_{n,k}(V) + s e^{K_{n,k}(V)} \leq \frac{r'}{2 + 16\lambda r'}$.

2. If $\inf_{n \geq 1, k \geq k_0} \varepsilon_{n,k}(V) < \infty$, then the defective Poincaré inequality

\[ \mu_V(f^2) \leq C_1 \mathcal{E}_V(f) + C_2 \mu_V(|f|^2), \ f \in \mathcal{D}(\mathcal{E}_V) \]

holds for some constants $C_1, C_2 > 0$.

We note that according to [11, Proposition 1.3] (see also [14, Proposition 4.1.2]), if $\mathcal{E}_V$ satisfies the weak Poincaré inequality (see Section 4 below) then the defective Poincaré inequality (2.2) implies the Poincaré inequality

\[ \mu_V(f^2) \leq C \mathcal{E}_V(f), \ f \in \mathcal{D}(\mathcal{E}_V), \mu_V(f) = 0 \]

for some constant $C > 0$.

When the jump is of infinite range, we will need additional notation and assumptions to control the uniform norm appearing in the perturbed functional inequalities (see Lemmas 2.6 and 2.7 below), which is an essentially different feature from the diffusion setting. For any $n, k \geq 1$ and $\delta > 1$, let

\[ Z_n(V) = \sup_{\rho(x) \leq n+1} V(x), \]
\[ \zeta_n(V) = \sup_{m \geq n} \left\{ \beta^{-1} \left( 1/[2\mu(\rho > m - 1)] \right) e^{Z_{m+1}(V)} \right\}, \]
\[ t_{i,n,k}(\delta, V) := \beta^{-1} \left( \frac{1}{4} \delta^i e^{-J_{n,k}(V)} \right). \]
Moreover, let
\[ \gamma_{n,k} = \int \int_{\{d(x,y) > k, \rho(y) \geq n-1\}} q(x,y) \mu(dy) \mu(dx), \]
\[ \eta_{n,k} = \int \int_{\{\rho(x) > n+k+2, \rho(y) \leq n+1\}} q(x,y) \mu(dy) \mu(dx), \]

By (1.1), we see that \( \gamma_{n,k} + \eta_{n,k} \downarrow 0 \) as \( n \uparrow \infty \) holds for any \( k \geq 1 \).

We assume

(A) There exist \( \delta > 1 \) and sequences \( \{(n_i, k_i)\}_{i \geq 1} \subset \mathbb{N}^2 \) such that \( n_i \uparrow \infty \) and

(A1) \( \lim_{i \to \infty} (\varepsilon_{n_i, k_i}(V) + t_{i, n_i, k_i}(\delta, V)e^{K_{n_i, k_i}(V)}) = 0; \)

(A2) \( \sum_{i=1}^{\infty} \{\zeta_{n_i}(V) \gamma_{n_i, k_i} + t_{i, n_i, k_i}(\delta, V)e^{Z_{n_i}(V)} \eta_{n_i, k_i}\} < \infty. \)

We shall let \( I_\delta \) denote the set of all sequences \( \{(n_i, k_i)\} \subset \mathbb{N}^2 \) such that \( n_i \uparrow \infty \) and (A1)-(A2) hold. Moreover, for any \( r > 0 \) and \( \{n_i, k_i\} \in I_\delta \), let \( D(r, \{(n_i, k_i)\}) \) be the set of \( j \in \mathbb{N} \) such that

(2.3) \( \sup_{i \geq j} (8\varepsilon_{n_i, k_i}(V) + t_{i, n_i, k_i}(\delta, V)e^{K_{n_i, k_i}(V)}) \leq \frac{1}{64} \wedge \frac{c(\delta)r}{16}, \quad i \geq j, \)

where \( c(\delta) := (\frac{2^{\delta} - 1}{\delta})^2 \), and such that

(2.4) \( \sum_{i=j}^{\infty} \left(6\zeta_{n_i}(V) \gamma_{n_i, k_i} + t_{i, n_i, k_i}(\delta, V)e^{Z_{n_i}(V)} \eta_{n_i, k_i}\right) \delta^{i+2} \leq \frac{1}{256}. \)

By (A), we see that for any \( r > 0 \) and \( \{(n_i, k_i)\} \in I_\delta \), the set \( D(r, \{(n_i, k_i)\}) \) is non-empty.

**Theorem 2.2.** Assume that (2.1) holds.

1. If (A) is satisfied, then the super Poincaré inequality

(2.5) \( \mu_V(f^2) \leq r \delta_V(f) + \beta_V(r) \mu_V(|f|^2), \quad r > 0, \quad f \in \mathcal{D}(\delta_V) \)

holds with

\( \beta_V(r) := \inf \left\{2\delta^j : \{(n_i, k_i)\} \in I_\delta, \quad j \in D(r, \{(n_i, k_i)\})\right\} < \infty, \quad r > 0. \)

2. If (A2) is satisfied and (A1) is replaced by the following weaker assumption

(A1') \( \limsup_{i \to \infty} (\varepsilon_{n, k_i}(V) + t_{i, n, k_i}(\delta, V)) < \infty, \)

then the defective Poincaré inequality (2.2) holds for some \( C_1, C_2 > 0. \)
The following example shows that Theorem 2.2 is sharp in some specific situations.

**Example 2.3.** Let $E = \mathbb{R}^m$ with $d(x,y) = |x - y|$, and let

$$q(x,y) = \frac{(1 + |y|)^{m+\alpha} \log(1 + |y|)}{|x - y|^{m+\alpha}}, \quad \mu(dx) = \frac{c_{m,\alpha} dx}{(1 + |x|)^{m+\alpha} \log(1 + |x|)},$$

where $\alpha \in (0, 2)$ and $c_{m,\alpha} > 0$ is the normalizing constant such that $\mu$ is a probability measure. It is easy to see that all the assumptions in the introduction for $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ are satisfied. We consider $V$ satisfying

$$- s\varepsilon \log \log(e + |x|) - K \leq V(x) \leq (1 - s)\varepsilon \log \log(e + |x|) + K, \quad x \in \mathbb{R}^m$$

for some constants $\varepsilon \in (0, 1]$, $s \in [0, 1]$ and $K \in \mathbb{R}$ such that $\mu(e^V) = 1$.

1. If $\varepsilon < 1$ then (2.5) holds with

$$\beta_V(r) = \exp(C_1(1 + r^{-1/(1-\varepsilon)})},$$

for some constant $C_1 > 0$.

2. $\beta_V$ in (1) cannot be replaced by any essentially smaller functions, i.e. when $V(x) = \varepsilon \log \log(e + |x|) + K_0$ for some constant $K_0 \in \mathbb{R}$ such that $\mu(e^V) = 1$, the estimate (2.7) is sharp in the sense that the super Poincaré inequality (2.5) does not hold if

$$\lim_{r \to 0} r^{1/(1-\varepsilon)} \log \beta_V(r) = 0.$$

3. In (1) the constant $K$ cannot be replaced by any unbounded positive function, i.e. for any increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(r) \uparrow \infty$ as $r \uparrow \infty$, there exists $V$ such that

$$\varepsilon \log \log(e + |x|) \leq V(x) \leq \varepsilon \log \log(e + |x|) + \phi(|x|)$$

with $\mu(e^V) < \infty$, but the super Poincaré inequality (2.5) with $\beta_V$ given by (2.7) does not hold for $\mu_V(dx) := \frac{e^{V(x)} dx}{\mu(e^V)}$.

4. If $\varepsilon = 1$, then $(\mathcal{E}_V, \mathcal{D}(\mathcal{E}_V))$ satisfies the Poincaré inequality

$$\mu_V(f^2) \leq C\mathcal{E}_V(f,f) + \mu_V(f)^2, \quad f \in \mathcal{D}(\mathcal{E}_V)$$

with some constant $C > 0$.

**Proof.** As (2) is included in [15 Corollary 1.3], we only prove (1), (3) and (4).

(a) According to [15 Corollary 1.3(3)], we know the logarithmic Sobolev inequality holds for $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$, i.e. the super Poincaré inequality (2.1) holds for $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ with the rate function $\beta(r) = \exp(c_1(1 + r^{-1}))$ for some constant $c_1 > 0$. 

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Next, by (2.6), there exists a constant $c_2 > 0$ such that for $n$ large enough
\[
K_{n,n}(V) \leq \varepsilon \log \log n + c_2, \quad J_{n,n}(V) \leq (1 + s)\varepsilon \log \log n + c_2,
\]
\[
Z_n(V) \leq (1 - s)\varepsilon \log \log n + c_2, \quad \zeta_n(V) \leq \frac{c_2}{\log^{1-(1-s)\varepsilon} n}, \quad \varepsilon_{n,n}(V) \leq \frac{c_2}{\log^{1-\varepsilon} n},
\]
\[
\gamma_{n,n} \leq \frac{c_2}{n^\alpha}, \quad \eta_{n,n} \leq \frac{c_2}{n^\alpha \log(1 + n)}.
\]

For any $\delta > 1$, taking $\{n_i\} = \{\delta^i\}_{i=1}^\infty$, and $k_i = n_i$, we get that $t_{i,n_i,k_i}(\delta, V) \leq c_3i^{-1}$ for $i$ large enough and some constant $c_3 > 0$. So, assumption (A) is fulfilled.

Moreover, by (2.3) and (2.4), it is easy to check that for $r > 0$ small enough we have
\[
\left\{ j \geq 1 : j \geq c_4r^{-\frac{1}{1-r}} \right\} \subseteq D(r, \{(n_i, k_i)\}).
\]

Then, the assertion in (1) follows from Theorem 2.2(1) and (2.10).

(b) Let $\psi(r) = (1 + \log(1 + r)) \land e^{\phi(r)}$ for $r \geq 0$. Then $1 \leq \psi \leq e^\phi$, $\psi(r) \uparrow \infty$ as $r \uparrow \infty$ and $\psi(r) \leq 2 \log r$ for large $r$. Let
\[
V(x) = \varepsilon \log \log(e + |x|) + \log \psi(|x|).
\]

Then $V$ satisfies (2.8) and $\mu(e^V) < \infty$. Up to a normalization constant we may simply assume that $\mu_V(dx) = e^{V(x)}dx$.

Now, suppose that (2.5) holds with $\beta_V$ given by (2.7). For any $n \geq 1$, let $f_n \in C^\infty(\mathbb{R}^d)$ satisfy that
\[
f_n(x) = \begin{cases} 
0, & |x| \leq n, \\
\in [0, 1], & n \leq |x| \leq 2n, \\
1, & |x| \geq 2n,
\end{cases}
\]
and $|\nabla f_n| \leq \frac{2}{n}$. Then, there exists a constant $c_5 > 0$ independent of $n$ such that
\[
\Gamma(f_n, f_n)(x) = c_{m,\alpha} \int \frac{(f_n(y) - f_n(x))^2}{|y - x|^{m+\alpha}} dy
\]
\[
\leq \frac{4c_{m,\alpha}}{n^2} \int_{|y-x| \leq n} \frac{dy}{|y - x|^{m+\alpha-2}} + c_{m,\alpha} \int_{|y-x| > n} \frac{dy}{|y - x|^{m+\alpha}}
\]
\[
\leq \frac{c_5}{n^\alpha}, \quad n \geq 1.
\]

According to the definition of $f_n$ and the increasing property of $\psi$, there exists $c_6 > 0$ such that
\[
\mu_V(f_n^2) \geq \frac{c_6\psi(n)}{n^\alpha \log^{1-\varepsilon} n}, \quad n \geq 2.
\]

On the other hand, since $\psi(r) \leq 2 \log r$ for large $r$, we have
\[
\mu_V(|\cdot| > n) \leq c_7 \int_n^\infty \frac{\psi(r) dr}{r^{1+\alpha} \log^{1-\varepsilon} r} \leq \frac{c_8 \log^\varepsilon n}{n^\alpha}, \quad n \geq 2.
\]
Then
\[ \mu_V(f_n)^2 \leq \mu_V(f_n^2) \mu_V(\{| \cdot | > n\}) \leq \frac{c_8 \log^\varepsilon n}{n^\alpha} \mu_V(f_n^2), \ n \geq 2. \]

Combining this with (2.5), (2.11) and (2.12), we obtain
\[ (2.13) \ 1 - \frac{c_8 \log^\varepsilon n}{n^\alpha} \beta_V(r) \leq \frac{c_9 r}{n^\alpha \mu_V(f_n^2)} \leq \frac{c_9 r \log^{1-\varepsilon} n}{\psi(n)}, \ r > 0, n \geq 2 \]

for some constant \( c_9 > 0 \). Since \( \psi(n) \leq 2 \log n \) for large \( n \), it is easy to see that
\[ r_n := \beta_V^{-1}\left(\frac{n^\alpha}{2c_8 \log^\varepsilon n}\right) \leq \frac{c_{10}}{\log^{1-\varepsilon} n} \]

holds for some constant \( c_{10} > 0 \) and large \( n \). Therefore, it follows from (2.13) that
\[ \frac{1}{2} \leq \lim_{n \to \infty} \frac{c_9 r_n \log^{1-\varepsilon} n}{\psi(n)} = 0, \]

which is a contradiction.

(c) If \( \varepsilon = 1 \) in (2.6), the estimate (2.10) is still true. According to Theorem 2.2 (2), the defective Poincaré inequality (2.2) holds for \((E_V, \mathcal{D}(E_V))\). On the other hand, by [15, Theorem 1.1(3)], the weak Poincaré inequality holds for \((E_V, \mathcal{D}(E_V))\). Therefore, the required Poincaré inequality follows from [11, Proposition 1.3] or [14, Proposition 4.1.2].

Finally, we consider an example for finite range of jumps to illustrate Theorem 2.1.

**Example 2.4.** Let \( E = \mathbb{R}^m \) with \( d(x, y) = |x - y| \) and let
\[ q(x, y) = \frac{e^{\beta |y|^\kappa}}{|x - y|^{m+\alpha} 1_{\{|x-y| \leq 1\}}}, \ \mu(dx) = c_{m, \kappa} e^{-|x|^\alpha} dx \]

for some constants \( 0 < \alpha < 2, \ \kappa > 1 \) and \( c_{m, \kappa} \geq 1 \) such that \( \mu \) is a probability measure. It is easy to see that all the assumptions in the introduction for \((E, \mathcal{D}(E))\) is satisfied. Consider \( V \) satisfying
\[ (2.14) \ - C_1 (1 + |x|^{\theta-1}) - K \leq V(x) \leq C_1 (1 + |x|^{\theta-1}) + K, \ x \in \mathbb{R}^m \]

for some constants \( \theta \in (1, \kappa], \ C_1 > 0 \) and \( K \in \mathbb{R} \) such that \( \mu(e^V) = 1 \).

(1) If \( \theta < \kappa \), then the super Poincaré inequality holds for \((E_V, \mathcal{D}(E_V))\) with
\[ \beta_V(r) = C_2 \exp \left[ C_2 \log^{\frac{\kappa}{\theta - 1}}(1 + r^{-1}) \right] \]

for some positive constants \( C_2 > 0 \).

(2) Let \( \theta = \kappa \). Then if \( C_1 > 0 \) is small enough, then the Poincaré inequality (2.9) holds for some constants \( C > 0 \).
Proof. According to [5, Example 1.2(2)], we know the super Poincaré inequality (2.1) holds for \((E, D(E))\) with

\[
\beta(r) = c_1 \exp \left( c_1 \log \frac{r}{\kappa - 1} (1 + r^{-1}) \right)
\]

for some constant \(c_1 > 0\).

(1) By (2.14) and the fact that \(\theta < \kappa\), one can find some constants \(c_2, c_3 > 0\) such that for \(n\) large enough

\[
K_{n,n}(V) \leq c_2 n^{\theta - 1}, \quad J_{n,n}(V) \leq c_2 n^{\theta - 1}, \quad \varepsilon_{n,n}(V) \leq e^{-c_3 n^{\kappa - 1}}.
\]

Therefore, for every \(r\) small enough, taking \(n = c_4 \log \frac{1}{\kappa - 1} (1 + r^{-1})\) and \(s = \exp \left(-c_5 \log(1 + r^{-1})\right)\) for some constants \(c_5 >> c_4 >> 1\), we obtain

\[
8 \varepsilon_{n,n}(V) + s e^{K_{n,n}(V)} \leq \frac{r}{2(1 + 8\lambda r)}.
\]

The first required assertion for super Poincaré inequality of \((E_V, D(E_V))\) follows from Theorem 2.1(1) and all the estimates above.

(2) Let \(\theta = \kappa\). By (2.14), (2.15) and the definition of \(\mu\), for \(n\) large enough we have

\[
K_{n,n}(V) \leq 3C_1 n^{\kappa - 1}, \quad \beta^{-1}(\mu(|x| > n - 1)) \leq e^{-c_6 n^{\kappa - 1}},
\]

for some constant \(c_6 > 0\) depending only on \(\beta\) and \(\mu\). So, if \(C_1 < c_6 / 3\), then \(\inf_{n \geq 1} \varepsilon_{n,n}(V) < \infty\). Therefore, by Theorem 2.1(2), the defective Poincaré inequality holds for \((E_V, D(E_V))\). Finally, according to [5, Proposition 2.6], the weak Poincaré inequality holds for \((E_V, D(E_V))\). Therefore, the Poincaré inequality holds for \((E_V, D(E_V))\).

To show that in Example 2.4(2) it is essential to assume that \(C_1 > 0\) is small, we present below a counterexample inspired by [2, Proposition 5.1].

**Proposition 2.5.** In the situation of Example 2.4 let \(\theta = \kappa, \alpha \in (0, 1)\) and \(m = 1\). Let

\[
V(x) = K_0 + L \sum_{n=1}^{\infty} 1_{[nH,(n+1)H]}(x)(n + 1)^{\kappa - 1} \left(2n + 1 - \frac{2x}{H}\right),
\]

where \(H > 4, L > \frac{\kappa H^\kappa}{H - 2}\) and \(K_0 \in \mathbb{R}\) are constants such that \(\mu(e^V) = 1\). Then (2.14) holds for some constant \(C_1 > 0\) and \(K \in \mathbb{R}\); however, for any \(C > 0\), the Poincaré inequality (2.9) does not hold.
Proof. It suffices to disprove the Poincaré inequality, for which we are going to construct a sequence of functions \( \{f_n\} \subset A \) such that

\[
\lim_{n \to \infty} \frac{\mathcal{E}_V(f_n)}{\text{Var}_{\mu_V}(f_n)} = 0.
\]

For \( n \geq 1 \), let

\[
f_n(x) = \begin{cases} 
\frac{f_{H(n+1)}^2 \exp(y^\kappa - V(y))dy}{f_{H(n+1)}^2 \exp(y^\kappa - V(y))dy}, & x \in (Hn + 1, H(n + 1) - 1), \\
1, & x \in [H(n + 1) - 1, +\infty), \\
0, & \text{otherwise},
\end{cases}
\]

which is a bounded Lipschitz continuous function on \( \mathbb{R} \), so that \( f_n \in A \). In the following calculations \( C \) stands for a constant which varies from line to line but is independent of \( n \) (may depend on \( H \) or \( L \)). We simply denote \( K_n = L(n+1)^{\kappa-1} \) for \( n \geq 1 \), so that

\[
V(x) = K_0 + \sum_{n=1}^{\infty} 1_{[nH,(n+1)H]}(x)K_n(2n + 1 - \frac{2x}{H}).
\]

(a) Estimate on \( \text{Var}_{\mu_V}(f_n) \). For \( n \) large enough, since \( z \mapsto (z^\kappa - K_{n+1}(2n + 3 - \frac{2z}{H})) \) is increasing, we have

\[
\mu_V(f_n^2) \geq \mu_V(H(n + 1) \leq x \leq H(n + 2)) \\
\geq C \int_{H(n+1)}^{H(n+2)} \exp \left( \frac{K_{n+1}(2n + 3 - \frac{2z}{H}) - z^\kappa}{K_{n+1}^{\kappa-1} + \frac{2K_{n+1}}{H}} \right) d\left( z^\kappa - K_{n+1}(2n + 3 - \frac{2z}{H}) \right) \\
\geq C \exp \left( - (H(n + 1))^{\kappa} + K_{n+1} \right) \frac{\exp\left( - (H(n + 2))^{\kappa} - K_{n+1} \right)}{(n + 2)^{\kappa-1}} \\
\geq C \frac{\exp\left( - (H(n + 1))^{\kappa} + K_{n+1} \right)}{(n + 2)^{\kappa-1}}.
\]

Noting that for \( n \) large enough,

\[
\mu_V(f_n) = \mu_V(f_n 1_{\{x \geq n\}})^2 \leq \mu_V(f_n^2) \mu_V(x \geq n) \leq \frac{\mu_V(f_n^2)}{2},
\]

we arrive at

\[
(2.16) \quad \text{Var}_{\mu_V}(f_n) \geq C \exp \left( - (H(n + 1))^{\kappa} + K_{n+1} \right) \frac{\exp\left( - (H(n + 2))^{\kappa} - K_{n+1} \right)}{(n + 2)^{\kappa-1}}.
\]

(b) Estimate on \( \mathcal{E}_V(f_n) \). Let

\[
g_n(x) = \left( \int_{H(n+1)}^{H(n+1)-1} \exp(y^\kappa - V(y))dy \right) f_n(x).
\]
Noting that $g_n'(x) \neq 0$ only when $x \in (Hn + 1, H(n + 1) - 1)$, we have

$$E_V(g_n) \leq C \int_{Hn}^{H(n+1)} \left( \int_{|x-y| \leq 1} \frac{|\int_x^y g_n'(z)dz|^2}{|x-y|^{1+\alpha}}dy \right) \exp(-x^\kappa + V(x)) dx$$

$$= C \int_{Hn}^{H(n+1)} \left( \int_{|x-y| \leq 1, x \leq y} \frac{|\int_x^y g_n'(z)dz|^2}{|x-y|^{1+\alpha}}dy \right) \exp(-x^\kappa + V(x)) dx$$

$$+ C \int_{Hn}^{H(n+1)} \left( \int_{|x-y| \leq 1, x > y} \frac{|\int_x^y g_n'(z)dz|^2}{|x-y|^{1+\alpha}}dy \right) \exp(-x^\kappa + V(x)) dx$$

$$=: C(I_1 + I_2).$$

Since $g_n'(x) = \exp(x^\kappa - V(x))$ for $x \in (Hn + 1, H(n + 1) - 1)$, and since for large $n$ the function $z \mapsto z^\kappa - V(z)$ is increasing on $(Hn, H(n+1))$, by the Cauchy-Schwarz inequality we have, for $x \in (Hn, H(n + 1) - 1)$ and $x \leq y \leq x + 1$,

$$\frac{|\int_x^y g_n'(z)dz|^2}{|x-y|^{1+\alpha}}$$

$$= \frac{|\int_x^{y\wedge(H(n+1)-1)} g_n'(z)dz|^2}{|x-y|^{1+\alpha}}$$

$$\leq \frac{|\int_x^{y\wedge(H(n+1)-1)} \exp(z^\kappa - V(z)) dz|^2}{|x-y|^{1+\alpha}} \int_x^{y\wedge(H(n+1)-1)} |g_n'|^2(z) e^{-z^\kappa + V(z)} dz$$

$$\leq \exp((x+1)^\kappa - V(x+1)) \left( \int_{H(n+1)-1}^{H(n+1)} |g_n'|^2(z) e^{-z^\kappa + V(z)} dz \right).$$

Thus, since $\alpha \in (0, 1)$,

$$I_1 \leq C \left( \int_{Hn+1}^{H(n+1)-1} |g_n'|^2(z) \exp(-z^\kappa + V(z)) dz \right)$$

$$\times \left[ \int_{Hn}^{H(n+1)-1} \left( \int_{|x-y| \leq 1} \frac{1}{|x-y|^{1+\alpha}}dy \right) \exp((x+1)^\kappa - x^\kappa - V(x+1) + V(x)) dx \right]$$

$$\leq C \left( \int_{Hn+1}^{H(n+1)-1} \exp(z^\kappa - V(z)) dz \right) \exp(\kappa(H(n+1))^{\kappa-1} + \frac{2K_n}{H}).$$

Similarly, we have

$$I_2 \leq C \int_{Hn+1}^{H(n+1)-1} \exp(z^\kappa - V(z)) dz.$$

So, for $n$ large enough,

$$E_V(g_n) \leq C \left( \int_{Hn+1}^{H(n+1)-1} \exp(z^\kappa - V(z)) dz \right) \exp(\kappa(H(n+1))^{\kappa-1} + \frac{2K_n}{H}).$$
Moreover, for large $n$,
\[
\int_{H^{n+1}}^{H^{(n+1)-1}} \exp \left( z^\kappa - V(z) \right) dz \\
= \int_{H^{n+1}}^{H^{(n+1)-1}} \exp \left( z^\kappa - \frac{2K_n(Hn-z)}{H} - K_n \right) \frac{\kappa z^{\kappa-1} + \frac{2K_n}{H}}{\kappa z^{\kappa-1}} d\left( z^\kappa - \frac{2K_n(Hn-z)}{H} - K_n \right) \\
\geq C \exp \left( \left( H(n+1) - 1 \right)^\kappa + \frac{(H-2)Kn}{H} \right) - \exp \left( \left( H(n+1) - 1 \right)^\kappa - \frac{(H-2)Kn}{H} \right) \\
\geq \frac{C \exp \left( \left( H(n+1) - 1 \right)^\kappa + \frac{(H-2)Kn}{H} \right)}{(n+1)^{\kappa-1}},
\]

Therefore, for large $n$,
\[
E_V(f_n) = \frac{E_V(g_n)}{(\int_{H^{n+1}}^{H^{(n+1)-1}} \exp \left( z^\kappa - V(z) \right) dz)^2} \\
\leq \frac{C \exp \left( \kappa (H(n+1))^{\kappa-1} + \frac{2Kn}{H} \right)}{\int_{H^{n+1}}^{H^{(n+1)-1}} \exp \left( z^\kappa - V(z) \right) dz} \\
\leq C(n+1)^{\kappa-1} \exp \left( \kappa (H(n+1))^{\kappa-1} - (H(n+1) - 1)^\kappa - \frac{(H-4)Kn}{H} \right).
\]

(c) Combining (2.16) with (2.17) and noting that
\[
(H(n+1))^{\kappa} - (H(n+1) - 1)^\kappa \leq \kappa (H(n+1))^{\kappa-1},
\]
we obtain
\[
\lim_{n \to \infty} \frac{E_V(f_n, f_n)}{\operatorname{Var}_{\mu_V}(f_n)} \\
\leq C \lim_{n \to \infty} (n+2)^{2(\kappa-1)} \exp \left( 2\kappa (H(n+1))^{\kappa-1} - \frac{2(H-2)Kn}{H} \right) = 0
\]
since
\[
L > \frac{\kappa H^\kappa}{H-2}.
\]

### 2.2 Proofs of Theorems 2.1 and 2.2

As in [2, Theorem 3.1], we shall adopt a split argument by estimating $\mu(f^{2\{\rho\geq n\}})$ and $\mu(f^{2\{\rho\leq n\}})$ respectively. Unlike in the local setting where the chain rule is available, in the present situation the uniform norm $\|f\|_\infty$ will appear in our estimates when the range of jumps is infinite. Below we simply denote $K_{n,k} = K_{n,k}(V)$, $J_{n,k} = J_{n,k}(V)$, $Z_n = Z_n(V)$, $\varepsilon_{n,k} = \varepsilon_{n,k}(V)$, $\zeta_n = \zeta_n(V)$ and $t_{i,n,k} = t_{i,n,k}(\delta, V)$. 

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Lemma 2.6. For any $n \geq 1$, $k \geq 1$ and $f \in \mathcal{A}$,
\[
\mu_V(f^21_{\{\rho \geq n\}}) \leq 12\varepsilon_{n,k}\mathcal{E}_V(f) + 128\lambda\varepsilon_{n,k}\mu_V(f^2) + 96\zeta_n\gamma_{n,k}\|f\|_{\infty}^2.
\]
Proof. If $f|_{\{\rho \leq n-1\}} = 0$, by the Cauchy-Schwartz inequality we have
\[
\mu(|f|)^2 = \mu(|f|1_{\{ho > n-1\}})^2 \leq \mu(\rho > n-1)\mu(f^2).
\]
Substituting this into (2.18) with $r = \beta^{-1}\left(\frac{1}{2\mu(\rho > n-1)}\right)$ we obtain
\[
(2.18) \quad \mu(f^2) \leq 2\beta^{-1}\left(1/[2\mu(\rho > n-1)]\right)\mathcal{E}(f), \quad f \in \mathcal{A}, \quad f|_{\{\rho \leq n-1\}} = 0.
\]
To apply (2.18) for general $f \in \mathcal{A}$, we consider $fl_n$ instead of $f$, where $l_n := h_n \circ \rho$ for some function $h_n \in C^\infty([0, \infty); [0, \infty))$ satisfying
\[
h_n(s) \begin{cases} 
1, & n \leq s \leq n + 1, \\
\in [0, 1], & n - 1 \leq s < n \text{ or } n + 1 < s \leq n + 2, \\
0, & s < n - 1 \text{ or } s > n + 2,
\end{cases}
\]
and
\[
\sup_{s \in [0, \infty)} |h'_n(s)| \leq 2.
\]
It is easy to see that
\[
|l_n(x) - l_n(y)| \leq 2(1 \wedge d(x, y))\{1_{\{\rho(x) \in (n-2, n+3)\}} + 1_{\{\rho(y) \in (n-1, n+2), \rho(x) \notin (n-2, n+3)\}}\}.
\]
This implies
\[
\Gamma(fl_n(x)) \leq 2 \int_E \{l_n(y)^2(f(x) - f(y))^2 + f(x)^2(l_n(x) - l_n(y))^2\}q(x, y)\mu(dy)
\]
\[
\leq 2 \int_{\{\rho(y) \in (n-1, n+2)\}} (f(x) - f(y))^2q(x, y)\mu(dy) + 8\lambda f(x)^21_{\{\rho(x) \in (n-2, n+3)\}}
\]
\[
+ 8f^2(x)1_{\{\rho(x) \notin (n-2, n+3)\}} \int_{\{\rho(y) \in (n-1, n+2)\}} q(x, y)\mu(dy).
\]
(2.19)
Since
\[
1_{\{\rho(x) \notin (n-2, n+3)\}} \int_{\{\rho(y) \in (n-1, n+2)\}} q(x, y)\mu(dy) \leq 1_{\{\rho(x) \notin (n-2, n+3)\}} \int_{\{d(x, y) > 1\}} q(x, y)\mu(dy) \leq \lambda,
\]
we have $\Gamma(fl_n) \in \mathcal{B}_b(E)$, and $fl_n \in \mathcal{A}$.
Let
\[
\delta_{n,k} = e^{K_{n,k}\beta^{-1}\left(1/[2\mu(\rho > n-1)]\right)}, \quad \theta_n := e^{\sup_{\rho \in \mathbb{N}^2}V\beta^{-1}\left(1/[2\mu(\rho > n-1)]\right)}.
\]
Combining \((2.18)\) with \((2.19)\), and noting that \(\rho(y) \in (n-1, n+2)\) and \(\rho(x) \geq n+k+2\) imply \(d(x, y) > k\), and

\[
\begin{align*}
&\int \int_{\{\rho(x) > n+k+2, \rho(y) \in (n-1, n+2)\}} (f(x) - f(y))^2 q(x, y) \mu(dy) \mu(dx) \\
&\leq 4\|f\|_2^2 \int \int_{\{\rho(x) > n+k+2, \rho(y) \in (n-1, n+2), d(x, y) > k\}} q(x, y) \mu(dy) \mu(dx),
\end{align*}
\]

we obtain

\[
\begin{align*}
\mu_V(f^2 1_{\{\rho \geq n\}}) &\leq e^{\sup_{\rho \leq n+2} V} \mu(f^2 1_{\{\rho \geq n\}}) \\
&\leq 2e^{\sup_{\rho \leq n+2} V} \beta^{-1}(1/2\mu(\rho > n - 1)) \mathcal{E}(f_n) \\
&\leq 2e^{\sup_{\rho \leq n+2} V} \beta^{-1}(1/2\mu(\rho > n - 1)) \mu(1_{\{\rho \leq n+k+2\}} \Gamma(f_n)) \\
&\quad + 2e^{\sup_{\rho \leq n+2} V} \beta^{-1}(1/2\mu(\rho > n - 1)) \mu(1_{\{\rho \geq n+k+2\}} \Gamma(f_n)) \\
&\leq 4\delta_{n,k} \int \int_{\{\rho(x) \leq n+k+2, \rho(y) \in (n-1, n+2)\}} (f(x) - f(y))^2 q(x, y) \mu(dy) \mu_V(dx) \\
&\quad + 16\delta_{n,k} \lambda \mu_V(1_{\{\rho \in (n-2, n+3)\}} f^2) \\
&\quad + 16\delta_{n,k} \int \int_{\{\rho(x) \leq n+k+2, \rho(y) \notin (n-2, n+3), \rho(y) \in (n-1, n+2)\}} f(x)^2 q(x, y) \mu(dy) \mu_V(dx) \\
&\quad + 32\theta_n \|f\|_2^2 \int \int_{\{\rho(y) \in (n-1, n+2), d(x, y) > k\}} q(x, y) \mu(dy) \mu(dx).
\end{align*}
\]

Noting that \(\varepsilon_{n,k} = \sup_{m \geq n} \delta_{m,k}, \ z_n = \sup_{m \geq n} \theta_{m}, \sum_{m=1}^{\infty} 1_{\{\rho(y) \in (m-1, m+2)\}} \leq 3\) and

\[
1_{\{\rho(x) \leq n+k+2, \rho(y) \notin (n-2, n+3), \rho(y) \in (n-1, n+2)\}} \leq 1_{\{d(x, y) > 1, \rho(y) \in (n-1, n+2)\}},
\]

and taking summations in \((2.20)\) from \(n\), we arrive at

\[
\begin{align*}
\mu_V(f^2 1_{\{\rho \geq n\}}) &\leq \sum_{m=n}^{\infty} \mu_V(f^2 1_{\{\rho \geq m\}}) \\
&\leq 12\varepsilon_{n,k} \int \int_{E \times E} (f(x) - f(y))^2 q(x, y) \mu(dy) \mu_V(dx) + 80\varepsilon_{n,k} \lambda \mu_V(f^2) \\
&\quad + 48\varepsilon_{n,k} \int \int_{\{d(x, y) > 1\}} f(x)^2 q(x, y) \mu(dy) \mu_V(dx) \\
&\quad + 96\varepsilon_n \|f\|_2^2 \int \int_{\{\rho(y) \geq n-1, d(x, y) > k\}} q(x, y) \mu(dy) \mu(dx) \\
&\leq 12\varepsilon_{n,k} \mathcal{E}_V(f) + 128\lambda \varepsilon_{n,k} \mu_V(f^2) + 96\varepsilon_n \gamma_{n,k} \|f\|_\infty^2.
\end{align*}
\]

\(\Box\)

**Lemma 2.7.** For any \(n, k \geq 1\), \(s > 0\) and \(f \in \mathcal{A}\),

\[
\mu_V(f^2 1_{\{\rho \leq n\}}) \leq 2s e^{K_{n,k}} \mathcal{E}_V(f) + 16s e^{K_{n,k}} \mu_V(f^2) + 16se^{\eta_{n,k}} \|f\|_\infty^2 + \beta(s) e^{J_{n,k}} \mu_V(|f|)^2.
\]

\(14\)

Proof. Let \( \phi_n : [0, \infty) \to [0, \infty) \) be a smooth function such that
\[
\phi_n(s) = \begin{cases} 
1, & s \leq n, \\
0, & s > n+1,
\end{cases}
\]
and
\[
\sup_{s \in [0,\infty)} |\phi_n'(s)| \leq 2.
\]
Set \( g_n = \phi_n \circ \rho \). Then \( g_n \in \mathcal{A} \) and
\[
|g_n(x) - g_n(y)| \leq 2(1 + d(x, y)) \left( 1_{\{\rho(x) \leq n+1\}} + 1_{\{\rho(x) > n+1, \rho(y) \leq n+1\}} \right).
\]
So, similarly to (2.19) we have
\[
\Gamma(fg_n)(x) \leq 2 \int_{\{\rho(y) \leq n+1\}} (f(x) - f(y))^2 q(x, y) \mu(\mathrm{d}y)
\]
\[
+ 8 \lambda f(x)^2 1_{\{\rho(x) \leq n+k+2\}} + 8 f^2(x) 1_{\{\rho(x) > n+k+2\}} \int_{\{\rho(y) \leq n+1\}} q(x, y) \mu(\mathrm{d}y).
\]
Note that \( \rho(x) > n + k + 2 \) and \( \rho(y) \leq n + 1 \) imply that \( d(x, y) > k \), and
\[
\int_{\{\rho(x) > n+k+2, \rho(y) \leq n+1\}} (f(x) - f(y))^2 q(x, y) \mu(\mathrm{d}y) \mu(\mathrm{d}x)
\]
\[
\leq 4 \|f\|^2_\infty \int_{\{\rho(x) > n+k+2, \rho(y) \leq n+1, d(x, y) > k\}} q(x, y) \mu(\mathrm{d}y) \mu(\mathrm{d}x).
\]
Combining all the estimates above with (2.1), we obtain
\[
\mu_V(f^2 1_{\{\rho \leq n+1\}}) \leq \mu_V(f^2 g_n^2) \leq e^{\sup_{\rho \leq n+1} V} \mu(f^2 g_n^2)
\]
\[
\leq e^{\sup_{\rho \leq n+1} V} \left\{ s \mu(\Gamma(f g_n)) + \beta(s) \mu(\|f g_n\|)^2 \right\}
\]
\[
= e^{\sup_{\rho \leq n+1} V} \left\{ s \mu(1_{\{\rho \leq n+k+2\}} \Gamma(f g_n)) + s \mu(1_{\{\rho > n+k+2\}} \Gamma(f g_n)) + \beta(s) \mu(\|f g_n\|)^2 \right\}
\]
\[
\leq s e^{K_{n,k}} \mu_V(1_{\{\rho \leq n+k+2\}} \Gamma(f g_n)) + s e^{\sup_{\rho \leq n+1} V} \mu(1_{\{\rho > n+k+2\}} \Gamma(f g_n)) + \beta(s) e^{J_{n,k}} \mu_V(\|f\|)^2
\]
\[
\leq 2 s e^{K_{n,k}} \delta_V(f) + 16 \lambda s e^{K_{n,k}} \mu_V(f^2) + 16 s e^{Z_{n,k}} \|f\|_\infty^2 + \beta(s) e^{J_{n,k}} \mu_V(\|f\|)^2.
\]
Now, we are in a position to prove Theorem 2.1.

Proof of Theorem 2.1 It suffices to prove for \( f \in \mathcal{A} \). 

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(1) Since \( q(x, y) = 0 \) for \( d(x, y) > k_0 \), we have \( \gamma_{n,k} = \eta_{n,k} = 0 \) for all \( k \geq k_0 \). So, by Lemmas 2.6 and 2.7,
\[
\mu_V(f^2) \leq 2(6\varepsilon_{n,k} + s\varepsilon_{n,k})\delta_V(f) + 16\lambda(8\varepsilon_{n,k} + s\varepsilon_{n,k})\mu_V(f^2) + \beta(s)e^{n,k}\mu_V(|f|^2), \quad s > 0, n \geq 1, k \geq k_0. 
\]
(2.21)
If \( \inf_{n \geq 1, k \geq k_0} \varepsilon_{n,k} = 0 \), then for any \( r' \in (0, r] \) there exist \( s > 0, n \geq 1 \) and \( k \geq k_0 \) such that
\[
8\varepsilon_{n,k} + s\varepsilon_{n,k} \leq \frac{r'}{2 + 16\lambda r'}. 
\]
Combining this with (2.21) we obtain
\[
\mu_V(f^2) \leq r'\delta_V(f) + (1 + 8\lambda r')\beta(s)e^{n,k}\mu_V(|f|^2) \leq r\delta_V(f) + (1 + 8\lambda r')\beta(s)e^{n,k}\mu_V(|f|^2). 
\]
This implies the super Poincaré inequality for the desired \( \beta_V \).

(2) If
\[
\inf_{n \geq 1, k \geq k_0} \varepsilon_{n,k} < \frac{1}{128\lambda}, 
\]
then there exist \( n \geq 1, k \geq k_0 \) and \( s > 0 \) such that
\[
16\lambda(8\varepsilon_{n,k} + s\varepsilon_{n,k}) < 1. 
\]
Therefore, the defective Poincaré inequality follows from (2.21).

To prove (2.22) without condition (2.22), we follow the approach in the proof of [2, Theorem 3.1(2)] by making bounded perturbations of \( V \). For any \( N \geq 1 \), write
\[
V = V_N + (V \wedge N) \vee (-N), \quad V_N := (V - N)1_{\{V \geq N\}} + (V + N)1_{\{V \leq -N\}}. 
\]
Since \( (V \wedge N) \vee (-N) \) is bounded and the defective Poincaré inequality is stable under bounded perturbations of \( V \), we only need to prove that when \( V \) is unbounded we may find \( N \geq 1 \) such that the defective Poincaré inequality holds for \( V_N \) in place of \( V \). It is easy to check that for \( n \) large enough
\[
\sup_{\rho \leq n} V_N \leq \begin{cases} 
(\sup_{\rho \leq n} V) - N, & \text{if } \sup_{E} V = +\infty, \\
\sup_{\rho \leq n} V, & \text{if } \sup_{E} V < +\infty,
\end{cases}
\]
and
\[
\inf_{\rho \leq n} V_N \geq \begin{cases} 
(\inf_{\rho \leq n} V) + N, & \text{if } \inf_{E} V = -\infty, \\
\inf_{\rho \leq n} V, & \text{if } \inf_{E} V > -\infty.
\end{cases}
\]
Thus, for large \( n \) we have \( K_{n,k}(V_N) \leq K_{n,k}(V) - N \), so that
\[
\inf_{n \geq 1, k \geq k_0} \varepsilon_{n,k}(V_N) \leq e^{-N} \inf_{n \geq 1, k \geq k_0} \varepsilon_{n,k}(V). 
\]
Since \( \inf_{n \geq 1, k \geq k_0} \varepsilon_{n,k}(V) < \infty \), we see that (2.22) holds for \( V_N \) in place of \( V \) when \( N \) is large enough. Therefore, the defective Poincaré inequality holds for \( V_N \) in place of \( V \) as observed above. \( \square \)
Next, we turn to the proof of Theorem 2.2. To get rid of the uniform norm included in Lemmas 2.6 and 2.7, we adopt a cut-off argument as in the proof of [12, Theorem 3.2] or [14, Theorem 3.3.3]. More precisely, for $\delta > 1$ in assumption (A) and a non-negative function $f$, let

$$f_{\delta,i} = (f - \delta^\frac{i}{2})^+ \wedge (\delta^\frac{i+1}{2} - \delta^\frac{i}{2}), \quad i \geq 0.$$  

(2.24)

According to [14, Lemma 3.3.2], for $f \in \mathcal{D}(|f|)$ and $i, j \geq 0$, we have $f_{\delta,i}^+ \wedge f_{\delta,j}^+$ and $f \wedge f_{\delta,i}^+ \in \mathcal{D}(\mathcal{E}_V)$. Moreover,

$$\sum_{i=j}^\infty \mathcal{E}_V(f_{\delta,i}) \leq \mathcal{E}_V((f - \delta^\frac{i}{2})^+), \quad \mathcal{E}_V((f - \delta^\frac{i}{2})^+) + \mathcal{E}_V(f \wedge \delta^\frac{i}{2}) \leq \mathcal{E}_V(f).$$

(2.25)

We also have the following lemma.

**Lemma 2.8.** For any non-negative function $f$ and $k \in \mathbb{Z}_+ := \{0, 1, 2, \cdots \}$,

$$\sum_{i=k}^\infty f_{\delta,i}^2 \geq c(\delta)((f - \delta^\frac{i}{2})^+)^2.$$

(2.26)

**Proof.** We shall simply use $f$ to denote its value at a fixed point. If $f \leq 1$, then both sides in (2.26) are equal to zero. Assume that $f \in (\delta^\frac{l}{2}, \delta^\frac{l+1}{2}]$ for some $l \in \mathbb{Z}_+$. If $l \leq k$ then

$$\sum_{i=k}^\infty f_{\delta,i}^2 = ((f - \delta^\frac{l}{2})^+)^2 \geq c(\delta)((f - \delta^\frac{l}{2})^+)^2.$$

Next, if $l > k$ then

$$\sum_{i=k}^\infty f_{\delta,i}^2 \geq (\delta^\frac{l}{2} - \delta^\frac{l-1}{2})^2 = c(\delta)\delta^{l+1} \geq c(\delta)f^2.$$

In conclusion, (2.26) holds. $\square$

**Proof of Theorem 2.2.** Since $\mathcal{E}_V(|f|, |f|) \leq \mathcal{E}_V(f, f)$ for every $f \in \mathcal{A}$, without loss of generality, we may and do assume that $f \in \mathcal{A}$ with $f \geq 0$ and $\mu_V(f^2) = 1$.

(1) By Lemmas 2.6 and 2.7 we have

$$\mu_V(f^2) \leq 2(6\varepsilon_{n,k} + se^{K_{n,k}})\mathcal{E}_V(f) + 16\lambda(8\varepsilon_{n,k} + se^{K_{n,k}})\mu_V(f^2) + \beta(s)e^{J_{n,k}}\mu_V(\|f\|) + 16(6\varepsilon_{n,k} + se^{Z_{n,k}})\|f\|_\infty^{2s}, \quad s, k \geq 1, n \geq 1.$$

(2.27)

Next, let $r > 0$, $\{(n_i, k_i)\} \in I_\delta$, $j \in D(r, \{(n_i, k_i)\})$ be fixed, and let $f_{\delta,i}$ be defined by (2.24). Since $\|f_{\delta,i}\|_\infty \leq c(\delta)\delta^{i+2}$ and by the Cauchy-Schwarz inequality

$$\mu_V(\|f_{\delta,i}\|_\infty^2) \leq \mu_V(\|f_{\delta,i}\|_\infty^2) \leq \mu_V(f_{\delta,i}^2) \mu_V(f^2) \geq \delta^i \leq \mu_V(f_{\delta,i}^2)\delta^{-i},$$

$$\mu_V(f_{\delta,i})$$
it follows from (2.3) and (2.27) with \( n = n_i \) and \( s = t_{i,n_i,k_i} \) that
\[
\mu_V(f_{\delta,i}^2) \leq \frac{c(\delta)r}{8} \mathcal{E}_V(f_{\delta,i}) + \frac{1}{2} \mu_V(f_{\delta,i}^2) + 16c(\delta)(6\zeta n_i \gamma n_i, k_i + t_{i,n_i,k_i} e^{Z_{n_i} \eta_{n_i,k_i}}) \delta^{i+2}, \quad i \geq j.
\]
That is,
\[
\mu_V(f_{\delta,i}^2) \leq \frac{c(\delta)r}{4} \mathcal{E}_V(f_{\delta,i}) + 32c(\delta)(6\zeta n_i \gamma n_i, k_i + t_{i,n_i,k_i} e^{Z_{n_i} \eta_{n_i,k_i}}) \delta^{i+2}, \quad i \geq j.
\]
Taking summation over \( i \geq j \) and using (2.25), (2.26) and (2.4), we obtain
\[
(2.28) \quad \mu_V((f - \delta \hat{z})^2) \leq \frac{r}{4} \mathcal{E}_V((f - \delta \hat{z})^2) + \frac{1}{8}.
\]
On the other hand, noting that \( c(\delta) \in (0, 1) \) and \( \delta > 1 \), applying (2.27) with \( n = n_j \) and \( s = t_{j,n_j,k_j} \) to \( f \wedge \delta \hat{z} \), and combining with (2.3), (2.4), we obtain
\[
\mu_V(f^2 \wedge \delta^i) \leq \frac{c(\delta)r}{8} \mathcal{E}_V(f \wedge \delta \hat{z}) + \frac{1}{4} \mu_V(f^2 \wedge \delta^i) + \frac{\delta^i}{4} \mu_V(|f|)^2
\]
\[+ 16(6\varepsilon n_j, k_j \gamma n_j, k_j + t_{j,n_j,k_j} e^{Z_{n_j} \eta_{n_j,k_j}}) \delta^i \varepsilon_{n_j,k_j} \eta_{n_j,k_j} \delta_i^2 \]
\[\leq \frac{r}{8} \mathcal{E}_V(f \wedge \delta \hat{z}) + \frac{1}{2} \mu_V(f^2 \wedge \delta^i) + \frac{\delta^i}{4} \mu_V(|f|)^2 + \frac{1}{16}.
\]
Thus,
\[
\mu_V(f^2 \wedge \delta^i) \leq \frac{r}{4} \mathcal{E}_V(f \wedge \delta \hat{z}) + \frac{\delta^i}{2} \mu_V(|f|)^2 + \frac{1}{8}.
\]
Combining this with (2.28) and using (2.25), we arrive at
\[
1 = \mu_V(f^2) \leq \mu_V\{(f^2 - \delta \hat{z})^{+} + (f \wedge \delta \hat{z})^2\}
\]
\[\leq 2 \mu_V\{(f^2 - \delta \hat{z})^{+})^2\} + 2 \mu_V(f^2 \wedge \delta \hat{z}) \]
\[\leq \frac{r}{2} \left( \mathcal{E}_V((f - \delta \hat{z})^+) + \mathcal{E}_V(f \wedge \delta \hat{z}) \right) + \frac{1}{2} + \delta^i \mu_V(|f|)^2 \]
\[\leq \frac{r}{2} \mathcal{E}_V(f) + \frac{1}{2} + \delta^i \mu_V(|f|)^2.
\]
Therefore,
\[
(2.29) \quad \mu_V(f^2) \leq r \mathcal{E}_V(f) + 2 \delta^i \mu_V(|f|)^2
\]
holds for all \( \{(n_i, k_i)\} \in I_\delta \) and \( j \in D(r, \{(n_i, k_i)\}) \). This proves (2.5) for the desired \( \beta_V \).

(2) If
\[
(2.30) \quad \limsup_{i \to \infty} (8\varepsilon_{n_i, k_i} + t_{i,n_i,k_i} e^{K_{n_i,k_i}}) \leq \frac{1}{64\lambda},
\]

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then there exist a constant \( r > 0 \) and \( j \geq 1 \) such that (2.3) and (2.4) hold, i.e. \( j \in D(r, \{(n_i, k_i)\}) \). So, the arguments in (1) ensure (2.29), and so the defective Poincaré inequality for \( C_1 = r \) and \( C_2 = 2\delta^j \). Let \( V_N \) be in the proof of Theorem 2.1(2). It follows from the proof of Theorem 2.1(2) that for \( n \) large enough \( K_{n,k}(V_N) \leq K_{n,k}(V) - N \). Since \( J_{n,k}(V_N) \leq J_{n,k}(V) \) and \( \beta(r) \) is decreasing, we have

\[
\limsup_{i \to \infty} t_{i,n_i,k_i}(\delta, V_N) e^{K_{n_i,k_i}(V_N)} \leq e^{-N} \limsup_{i \to \infty} t_{i,n_i,k_i}(\delta, V) e^{K_{n_i,k_i}(V)},
\]

which combined with (2.23) yields that

\[
\limsup_{i \to \infty} \left( \varepsilon_{n_i,k_i}(V_N) + t_{i,n_i,k_i}(\delta, V_N) e^{K_{n_i,k_i}(V_N)} \right) \leq e^{-N} \limsup_{i \to \infty} \left( \varepsilon_{n_i,k_i}(V) + t_{i,n_i,k_i}(\delta, V) e^{K_{n_i,k_i}(V)} \right).
\]

Then the remainder of the proof is similar to that of Theorem 2.1(2) by using (2.30) instead of (2.22).

2.3 Perturbations for the super Poincaré inequality under a variation condition

It is well known that in the diffusion case the super Poincaré inequality is stable under Lipschitz perturbations (see [12, Proposition 2.6]). The aim of this section is to extend this result to the non-local setting using a variation condition on \( \text{supp} \ q := \{(x,y) : q(x,y) > 0\} \).

**Theorem 2.9.** Assume that (2.1) holds. If there exists a constant \( \kappa_1 > 0 \) such that

\[
\kappa_2 := \mu(e^{-2V}) < \infty \text{ and } (2.31) \quad |V(x) - V(y)| \leq \kappa_1(1 \land d(x,y)), \quad (x,y) \in \text{supp} \ q.
\]

Then (2.5) holds for

\[
\beta_V(s) := \inf \left\{ 16\kappa_2 \beta(r)^3(4 + \lambda \kappa_1^2 s') : s' \in (0,s], 0 < r \leq \frac{s' e^{-\kappa_1}}{4 + \lambda \kappa_1^2 s'}, s > 0 \right\}.
\]

**Proof.** To prove (2.1) for the desired \( \beta_V \), we may and do assume that \( V \) is bounded. Indeed, for any \( n \geq 1 \) let

\[
V_n = (V \land n) \lor (-n) - \log \mu(e^{(V \land n) \lor (-n)}).
\]

Then \( \mu(e^{V_n}) = 1 \), (2.31) holds for \( V_n \) in place of \( V \), and

\[
\lim_{n \to \infty} \mu(e^{-2V_n}) = \kappa_2.
\]

Thus, applying the assertion to the bounded \( V_n \) and letting \( n \to \infty \), we complete the proof.
Now, let $V$ be bounded and let $f \in \mathcal{A}$ with $\mu_V(|f|) = 1$. Take $\tilde{f} = f e^\frac{y}{2}$. By (2.31) we have for every $x, y \in \text{supp}\,q$,

$$(1 - e^{\frac{V(x) - V(y)}{2}})^2 \leq \frac{\kappa_1}{4} (1 \wedge d^2(x, y)), $$

hence

$$\Gamma(\tilde{f})(x) \leq 2 \int_E \left\{ e^{V(y)}(f(x) - f(y))^2 + f^2(x) (e^{\frac{V(y)}{2}} - e^{\frac{V(x)}{2}})^2 \right\} q(x, y) \mu(dy)$$

$$\leq 2e^{V(x) + \kappa_1} \Gamma(f)(x) + \frac{1}{2} \kappa_1^2 \lambda e^{V(x) + \kappa_1} f^2(x).$$

Since $V$ is bounded, this implies $\tilde{f} \in \mathcal{A}$. Moreover, combining this with (2.1), we obtain

$$\mu_V(f^2) = \mu(\tilde{f}^2) \leq r \mu(\Gamma(\tilde{f})) + \beta(r) \mu(|\tilde{f}|)^2$$

$$\leq 2r e^{\kappa_1} \mathcal{E}_V(f) + r \frac{\kappa_1^2}{2} \lambda e^{\kappa_1} \mu_V(f^2) + \beta(r) \mu_V(|f| e^{-\frac{V}{2}})^2.$$ 

(2.32)

Since $\mu_V(|f|) = 1$, for any $R > 0$ we have $\mu_V(|f| > R) \leq \frac{1}{R}$, and hence,

$$\mu_V(|f| e^{-\frac{V}{2}})^2 \leq 2\mu_V(|f| e^{-\frac{V}{2}} 1_{\{|f| \leq R\}})^2 + 2\mu_V(|f| e^{-\frac{V}{2}} 1_{\{|f| > R\}})^2$$

$$\leq 2R \mu_V(|f| e^{-\frac{V}{2}})^2 + 2\mu_V(f^2) \mu_V(e^{-V} 1_{\{|f| > R\}})$$

$$\leq 2R \mu_V(|f|) \mu_V(e^{-V}) + 2\mu_V(f^2) \sqrt{\mu_V(|f| > R) \mu_V(e^{-2V})}$$

$$\leq 2R + 2\mu_V(f^2) \frac{\sqrt{\kappa_2}}{\sqrt{R}}.$$

Taking $R = 16\kappa_2 \beta(r)^2$, we get

$$\beta(r) \mu_V(|f| e^{-\frac{V}{2}})^2 \leq 32 \kappa_2 \beta(r)^3 + \frac{1}{2} \mu_V(f^2).$$

Substituting this into (2.32), we arrive at

$$\mu_V(f^2) \leq 4r e^{\kappa_1} \mathcal{E}_V(f) + r \kappa_1^2 \lambda e^{\kappa_1} \mu_V(f^2) + 64 \kappa_2 \beta(r)^3.$$ 

Therefore, for any $s' \in (0, s]$ such that

$$r \leq \frac{s' e^{-\kappa_1}}{4 + \lambda \kappa_1^2 s'},$$

we have

$$\mu_V(f^2) \leq s' \mathcal{E}_V(f) + 16 \kappa_2 \beta(r)^3 (4 + \lambda \kappa_1^2 s') \leq s \mathcal{E}_V(f) + 16 \kappa_2 \beta(r)^3 (4 + \lambda \kappa_1^2 s').$$

This implies the desired super Poincaré inequality. 

\square
Let $E = \mathbb{R}^m$ and $d(x, y) = |x - y|$. If the jump has a finite range, i.e. there is a constant $k \geq 1$ such that $q(x, y) = 0$ for $|x - y| > k$, then (2.31) holds for any Lipschitz function $V$. Therefore, the above theorem implies that the super Poincaré inequality is stable for all Lipschitz perturbations as is known in the diffusion case. In particular, since the defective log-Sobolev inequality
\[
\mu(f^2 \log f^2) \leq C_1 \mathcal{E}(f) + C_2, \quad f \in \mathcal{D}(\mathcal{E}), \mu(f^2) = 1
\]
holds for some $C_1, C_2 > 0$ if and only if the super Poincaré inequality (2.1) holds for $\beta(r) = e^{c(1+r^{-1})}$ for some $c > 0$, see [13, Corollary 1.1] for $\delta = 1$, we conclude from Theorem 2.2 that the defective log-Sobolev inequality is stable under perturbations of Lipschitz functions $V$. See [5, Example 1.2] for examples of $\mu$ and $q$ having finite range of jumps such that the log-Sobolev inequality holds.

### 3 Perturbations for the weak Poincaré inequality

Suppose that the weak Poincaré inequality
\[
(3.1) \quad \mu(f^2) \leq \beta(r) \mathcal{E}(f, f) + r \|f\|_{2,\infty}^2, \quad r > 0, f \in \mathcal{D}(\mathcal{E}), \mu(f) = 0,
\]
holds for some decreasing function $\beta : (0, \infty) \to (0, \infty)$. To derive the weak Poincaré inequality for $\mathcal{E}_V$ using growth conditions on $V$, for any $n, k \geq 1$ let
\[
\tilde{K}_{n,k}(V) = \sup_{\rho \leq n} V - \inf_{\rho \leq n, k+1} V, \quad \tilde{Z}_n(V) = \sup_{\rho \leq n} V, \quad \tilde{\eta}_{n,k} = \int \int_{\{d(x,y)>n,k+1, \rho(y)\leq n\}} q(x,y)\mu(dy)\mu(dx), \quad \tilde{\gamma}_k = \int \int_{\{d(x,y)>k\}} q(x,y)\mu(dy)\mu(dx).
\]
It is clear that $\tilde{\eta}_{n,k} \leq \tilde{\gamma}_k$. By (1.1) we have $\tilde{\eta}_{n,k} \downarrow 0$ as $n \uparrow \infty$ or $k \uparrow \infty$.

#### Theorem 3.1

Assume that the weak Poincaré inequality (3.1) holds. If for any $\varepsilon > 0$
\[
(3.2) \quad \inf_{n,k \geq 1} e^{\tilde{Z}_n(V)}\beta(\varepsilon e^{-\tilde{Z}_n(V)})\left(\tilde{\eta}_{n,k} + \tilde{\gamma}_k + \mu(\rho > n-k)\right) = 0,
\]
then
\[
(3.3) \quad \mu_V(f^2) \leq \beta_V(r) \mathcal{E}_V(f, f) + r \|f\|_{2,\infty}^2, \quad r > 0, f \in \mathcal{D}(\mathcal{E}_V), \mu_V(f) = 0
\]
holds for
\[
\beta_V(r) := \inf \left\{ 2\beta\left(\frac{r}{8}\tilde{Z}_n(V)^2\right)e^{\tilde{K}_{n,k}(V)} : \, 6\mu_V(\rho > n) + 2e^{\tilde{Z}_n(V)}\beta\left(\frac{r}{8}\tilde{Z}_n(V)^2\right)\left(4\tilde{\eta}_{n,k} + \tilde{\gamma}_k + 4\lambda\mu(\rho > n-k)\right) \leq \frac{r}{2} \right\} < \infty, \quad r > 0.
\]
Proof. It is easy to see that \((3.2)\) implies \(\beta_V(r) < \infty\) for \(r > 0\). Let \(g_n\) be in the proof of Lemma 2.7. Then for any \(f \in \mathcal{A}\) with \(\mu_V(f) = 0\), we have

\[\mu_V(f g_n)^2 = \mu_V(f(1 - g_n))^2 \leq \|f\|_\infty^2 \mu_V(\rho > n)^2.\]

Moreover,

\[
\text{Var}_{\mu_V}(f g_n) \leq \mu_V((f g_n - \mu(f g_n))^2) \\
\leq e^{\sup_{\rho \leq n} V} \mu(1_{\{\rho \leq n\}}(f g_n - \mu(f g_n))^2) + \mu_V(1_{\{\rho > n\}}(f g_n - \mu(f g_n))^2) \\
\leq e^{\sup_{\rho \leq n} V} \text{Var}_\mu(f g_n) + 4\|f\|_\infty^2 \mu_V(\rho > n).
\]

Then

\[
\mu_V(f^2) \leq \text{Var}_{\mu_V}(f g_n) + \mu_V(f g_n)^2 + \mu_V(f^2 1_{\{\rho > n\}}) \\
\leq e^{\sup_{\rho \leq n} V} \text{Var}_\mu(f g_n) + 6\|f\|_\infty^2 \mu_V(\rho > n).
\]

On the other hand, we have

\[
e^{\sup_{\rho \leq n} V} \mathcal{E}(f g_n) \leq 2e^{\sup_{\rho \leq n} V} \int_{E \times E} g_n^2(y)(f(x) - f(y))^2 q(x, y) \mu(dy) \mu(dx) \\
+ 2e^{\sup_{\rho \leq n} V} \int_{E \times E} f^2(x)(g_n(x) - g_n(y))^2 q(x, y) \mu(dy) \mu(dx) \\
\leq 2e^{\tilde{K}_{n,v}(V)} \mathcal{E}_V(f) + 2e^{\tilde{z}_n(V)}(4\tilde{n}_{n,k} + \tilde{\gamma}_k + 4\lambda \mu(\rho > n - k))\|f\|_\infty^2,
\]

where in the last inequality we made the first integral on the sets \(\{\rho(x) \leq n + k + 1\}\) and \(\{\rho(x) > n + k + 1\}\), and the second integral on the sets \(\{\rho(x) \leq n - k\}\) and \(\{\rho(x) > n - k\}\), and also used the facts that

\[
\int_{\{\rho(x) > n + k + 1\}} g_n^2(y)(f(x) - f(y))^2 q(x, y) \mu(dy) \mu(dx) \leq 4\eta_n,k\|f\|_\infty^2, \\
\int_{\{\rho(x) \leq n - k\}} f^2(x)(g_n(x) - g_n(y))^2 q(x, y) \mu(dy) \mu(dx) \leq \tilde{\gamma}_k\|f\|_\infty^2, \\
\int_{\{\rho(x) > n - k\}} f^2(x)(g_n(x) - g_n(y))^2 q(x, y) \mu(dy) \mu(dx) \leq 4\lambda \mu(\rho > n - k)\|f\|_\infty^2.
\]

Combining this with \((3.4)\) and \((3.1)\), we arrive at

\[
\mu_V(f^2) \leq 2\beta(s)e^{\tilde{K}_{n,v}(V)} \mathcal{E}_V(f) \\
+ \|f\|_\infty^2 \left\{6\mu_V(\rho > n) + 4s\tilde{z}_n(V) + 2\beta(s)e^{\tilde{z}_n(V)}(4\tilde{n}_{n,k} + \tilde{\gamma}_k + 4\lambda \mu(\rho > n - k))\right\}.
\]

So, for any \(r > 0\), let \(s = \frac{r}{8}e^{-\tilde{z}_n(V)}\). If for some \(n, k \geq 1\) one has

\[
6\mu_V(\rho > n) + 2e^{\tilde{z}_n(V)}\beta\left(\frac{r}{8}e^{-\tilde{z}_n(V)}(4\tilde{n}_{n,k} + \tilde{\gamma}_k + 4\lambda \mu(\rho > n - k))\right) \leq \frac{r}{2},
\]

then

\[
\mu_V(f^2) \leq 2\beta(s)e^{\tilde{K}_{n,v}(V)} \mathcal{E}_V(f) + r\|f\|_\infty^2.
\]

Therefore, the proof is finished.
To conclude this section, we present an example where \((E, \mathcal{D}(E))\) satisfies the Poincaré inequality, i.e. the weak Poincaré inequality \((3.1)\) holds for a constant function \(\beta\).

Example 3.2. Let \(E = \mathbb{R}^m\) with \(d(x, y) = |x - y|\). Let

\[
q(x, y) = \frac{(1 + |y|)^{m+\alpha}}{|x - y|^{(m+\alpha)}}, \quad \mu(dx) = \frac{c_{m,\alpha}dx}{(1 + |x|)^{m+\alpha}}
\]

for some constant \(0 < \alpha < 2\), where \(c_{m,\alpha}\) is a normalizing constant such that \(\mu\) is a probability measure. Then \(\mathcal{A} \supset C_0(\mathbb{R}^m)\), and according to \([15, \text{Corollary 1.2}(1)]\), \((3.1)\) holds for a constant rate function \(\beta(r) \equiv \beta > 0\).

Now, let \(V\) be measurable satisfying

\[
(3.5) \quad -s\varepsilon \log(1 + |x|) - K \leq V(x) \leq (1 - s)\varepsilon \log(1 + |x|) + K, \quad x \in \mathbb{R}^m
\]

for some constants \(\varepsilon \in [0, \alpha), s \in [0, 1]\) and \(K \in \mathbb{R}\) such that \(\mu(e^V) = 1\). Then the weak Poincaré inequality \((3.3)\) holds with

\[
(3.6) \quad \beta_V(r) = C \left(1 + r^{-\varepsilon/(\alpha-(1-s)\varepsilon)}\right)
\]

for some constant \(C > 0\).

Moreover, the assertion is sharp in the following two cases with \(s = 0\).

(i) \(\beta_V\) in \((3.6)\) is sharp, i.e. \(\beta_V\) can not be replaced by any essentially smaller functions: if

\[
\lim_{r \to 0} r^{\varepsilon/(\alpha-\varepsilon)} \beta_V(r) = 0,
\]

then the weak Poincaré inequality \((3.3)\) does not hold.

(ii) The constant \(K\) can not be replaced by any unbounded functions: for

\[
(3.7) \quad V(x) = \varepsilon \log(1 + |x|) + \phi(|x|) + K_0,
\]

where \(\varepsilon \in [0, \alpha), \phi : [0, +\infty) \to [0, +\infty)\) is an increasing function with \(\phi(r) \uparrow +\infty\) as \(r \uparrow +\infty\) such that \(\mu(e^{\varepsilon \log(1+|.|) + \phi(|.|)}) < \infty\), and \(K_0 \in \mathbb{R}^d\) is such that \(\mu(e^V) = 1\), the weak Poincaré inequality \((3.3)\) with the rate function \(\beta_V\) given by \((3.6)\) does not hold.

Proof. Take \(k = \frac{\alpha}{2}\). Then there is a constant \(c_1 > 0\) such that for \(n\) large enough

\[
\tilde{K}_{n, \frac{\alpha}{2}}(V) \leq \varepsilon \log(1 + n) + c_1, \quad \tilde{Z}_n(V) \leq \varepsilon(1 - s) \log(1 + n) + c_1,
\]

\[
\tilde{\eta}_{n, \frac{\alpha}{2}} + \tilde{\gamma}_{n, \frac{\alpha}{2}} + \mu(\rho > \frac{n}{2}) \leq \frac{c_1}{n^{\alpha}}, \quad \mu(\rho > n) \leq \frac{c_1}{n^{\alpha-(1-s)\varepsilon}}.
\]

Since \(\varepsilon \in [0, \alpha)\), we see that \((3.2)\) holds and there exists \(c_2 > 0\) such that

\[
6\mu_V(\rho > n) + 2\beta e^{\tilde{Z}_n(V)}(4\tilde{\eta}_{n, \frac{\alpha}{2}} + \tilde{\gamma}_{n, \frac{\alpha}{2}} + 4\lambda \mu(\rho > \frac{n}{2})) \leq \frac{c_2}{2n^{\alpha-(1-s)\varepsilon}}, \quad n \geq 1.
\]
Thus, in the definition of $\beta_V$ for small $r > 0$ we may take $n = \left(\frac{c_2}{r}\right)^{\frac{1}{\alpha - (1-s)\varepsilon}}$ to get

$$\beta_V(r) \leq 2\beta e^{\tilde{K}_n\Phi(V)} \leq c_3 r^{-\varepsilon/(\alpha - (1-s)\varepsilon)}$$

for some constant $c_3 > 0$. Therefore, there exists $C > 0$ such that the weak Poincaré inequality (3.3) holds for $\beta_V$ given in (3.6).

It remains to verify (i) and (ii), where the assertion (i) has been included in [15, Corollary 1.2(3)]. So, it suffices to consider (ii). Now, assume that the weak Poincaré inequality holds for $(\mathcal{E}_V, \mathcal{D}(\mathcal{E}_V))$ with the rate function

$$\beta_V(r) = c_4 (1 + r^{-\frac{\varepsilon}{\alpha - \varepsilon}})$$

for some constant $c_4 > 0$. For any $n \geq 1$, let $f_n \in C^\infty(\mathbb{R}^d)$ be in part (b) of the proof of Example 2.3. By (2.11) and noting that $\mu_V(f_n)^2 \leq \frac{1}{2} \mu_V(f_n^2)$ for large $n$, there exist constants $c_5, c_6 > 0$ such that for $n$ large enough,

$$\mathcal{E}_V(f_n) \leq c_5 n^{-\alpha}, \quad \mu_V(f_n^2) - \mu_V(f_n)^2 \geq \frac{c_6 e^{\phi(n)}}{n^{\alpha - \varepsilon}}.$$

Combining these with (3.3), we obtain that there exists $c_7 > 0$ such that for all $r > 0$ and for $n$ large enough,

$$\frac{c_7 e^{\phi(n)}}{n^{\alpha - \varepsilon}} \leq \frac{\beta_V(r)}{n^{\alpha}} + r.$$

Taking $r = \frac{c_7 e^{\phi(n)}}{2n^{\alpha - \varepsilon}}$ in the inequality above, we arrive at

$$(3.8) \quad \beta_V\left(\frac{c_7 e^{\phi(n)}}{2n^{\alpha - \varepsilon}}\right) \geq \frac{c_7}{2} n^\varepsilon e^{\phi(n)}.$$  

Since there is $c_8 > 0$ such that for $n$ large enough

$$\frac{e^{\phi(n)}}{n^{\alpha - \varepsilon}} \leq c_8 \int_{\{|x| \geq n\}} \frac{e^{\phi(|x|)}}{(1 + |x|)^{m + \alpha - \varepsilon}} \, dx,$$

it holds that

$$\lim_{n \to \infty} \frac{e^{\phi(n)}}{n^{\alpha - \varepsilon}} = 0,$$

which, along with the definition of $\beta_V$, yields that there is a constant $c_9 > 0$ such that for $n$ large enough

$$\beta_V\left(\frac{c_7 e^{\phi(n)}}{2n^{\alpha - \varepsilon}}\right) \leq c_9 n^\varepsilon e^{-\varepsilon \phi(n)/(\alpha - \varepsilon)},$$

which is a contradiction to (3.8) since $\lim_{r \to \infty} \phi(r) = \infty$. Therefore, the weak Poincaré inequality does not hold with the rate function (3.6). □
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