The No-Binding Regime of the Pauli-Fierz Model

Fumio Hiroshima\textsuperscript{1}, Herbert Spohn\textsuperscript{2}, and Akito Suzuki\textsuperscript{3}

\textsuperscript{1}Faculty of Mathematics, Kyushu University, Fukuoka, 819-0395, Japan
\textsuperscript{2}Zentrum Mathematik and Physik Department, TU München, D-80290, München, Germany
\textsuperscript{3}Department of Mathematics, Faculty of Engineering, Shinshu University, Nagano, 380-8553, Japan

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\textbf{Abstract}

The Pauli-Fierz model $H(\alpha)$ in nonrelativistic quantum electrodynamics is considered. The external potential $V$ is sufficiently shallow and the dipole approximation is assumed. It is proven that there exist constants $0 < \alpha_- < \alpha_+$ such that $H(\alpha)$ has no ground state for $|\alpha| < \alpha_-$, which complements an earlier result stating that there is a ground state for $|\alpha| > \alpha_+$. We develop a suitable extension of the Birman-Schwinger argument. Moreover for any given $\delta > 0$ examples of potentials $V$ are provided such that $\alpha_+ - \alpha_- < \delta$. 
1 Introduction

Let us consider a quantum particle in an external potential described by the Schrödinger operator

\[ H_p(m) = -\frac{1}{2m}\Delta + V(x) \]

acting on \( L^2(\mathbb{R}^d) \). If the potential \( V \) is short ranged and attractive and if the dimension \( d \geq 3 \), then there is a transition from unbinding to binding as the mass \( m \) is increased. More precisely, there is some critical mass, \( m_c \), such that \( H_p(m) \) has no ground state for \( 0 < m < m_c \) and a unique ground state for \( m_c < m \). In fact, the critical mass is given by

\[ \frac{1}{2m_c} = \| |V|^{1/2} (\Delta)^{-1} |V|^{1/2} \|, \]

see Lemma 3.3. We now couple \( H_p(m) \) to the quantized electromagnetic field with coupling strength \( \alpha \geq 0 \). The corresponding Hamiltonian is denoted by \( H(\alpha) \). On a heuristic level, through the dressing by photons the particle becomes effectively more heavy, which means that the critical mass \( c_0\alpha^2(\alpha) \) should be decreasing as a function of \( \alpha \) with \( m_c(0) = m_c \). In particular, if \( m < m_c \), then there should be an unbinding-binding transition as the coupling \( \alpha \) is increased. This phenomenon has been baptized enhanced binding and has been studied for a variety of models by several authors \([AK03, BV04, HVV03, HHS05, HS01, HS08]\). In case \( m > m_c \) more general techniques are available and the existence of a unique ground state for the full Hamiltonian is proven in \([AH97, BFS99, GLL01, LL03, Ger00, Spo98]\).

The heuristic picture also asserts that the full Hamiltonian has a regime of couplings with no ground state. This property is more difficult to establish and the only result we are aware of is proved by Benguria and Vougalter \([BV04]\). In essence they establish that the line \( m_c(\alpha) \) is continuous as \( \alpha \to 0 \). (In fact, they use the strength of the potential as parameter). From this it follows that the no binding regime cannot be empty. In our paper, as in \([HS01]\), we will use the dipole approximation for simplicity, but provide a fairly explicit bound on the critical mass. In the dipole approximation the effective mass \( m_{\text{eff}}(\alpha) = m + c_0\alpha^2 \) with some explicitly computable coefficient \( c_0 \), see Eq. (2.10) below. Thus the most basic guess for \( m_c(\alpha) \) would be \( m_c(\alpha) + c_0\alpha^2 = m_c \). The corresponding curve is displayed in Fig. 1. In fact the guess turns out to be a lower bound on the true \( m_c(\alpha) \). We will
complement our lower bound with an upper bound of the same qualitative form.

The unbinding for the Schrödinger operator $H_p(m)$ is proven by the Birman-Schwinger principle. Formally one has

$$H_p(m) = \frac{1}{2m}(-\Delta)^{1/2}(\mathbb{1} + 2m(-\Delta)^{-1/2}V(-\Delta)^{-1/2})(-\Delta)^{1/2}.$$  

If $m$ is sufficiently small, then $2m(-\Delta)^{-1/2}V(-\Delta)^{-1/2}$ is a strict contraction. Hence the operator $\mathbb{1} + 2m(-\Delta)^{-1/2}V(-\Delta)^{-1/2}$ has a bounded inverse and $H_p(m)$ has no eigenvalue in $(-\infty, 0]$. More precisely the Birman-Schwinger principle states that

$$\dim \mathbb{1}_{(-\infty, 0]}(H_p(m)) \geq \dim \mathbb{1}_{(2m, \infty]}(V^{1/2}(-\Delta)^{-1}V^{1/2}).$$

For small $m$ the left hand side equals 0 and thus $H_p(m)$ has no eigenvalues in $(-\infty, 0]$.

Our approach will be to generalize (1.2) to the Pauli-Fierz model of non-relativistic quantum electrodynamics. The Pauli-Fierz Hamiltonian $H(\alpha)$ is defined on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^d) \otimes \mathcal{F}$, where $\mathcal{F}$ denotes the boson Fock space. Transforming $H(\alpha)$ unitarily by $U$ one arrives at

$$U^{-1}H(\alpha)U = H_0(\alpha) + W + g$$
as the sum of the free Hamiltonian
\begin{equation}
H_0(\alpha) = -\frac{1}{2m_{\text{eff}}(\alpha)} \Delta \otimes 1 + 1 \otimes H_f,
\end{equation}
involving the effective mass of the dressed particle and the Hamiltonian $H_f$ of the free boson field, the transformed interaction
\begin{equation}
W = T^{-1}(V \otimes 1)T,
\end{equation}
and the global energy shift $g$. $m_{\text{eff}}(\alpha)$ is an increasing function of $\alpha$. We will show that (1.3) has no ground state for sufficiently small $|\alpha|$ by means of a Birman-Schwinger type argument such as (1.2). In combination with the results obtained in [HS01] we provide examples of external potentials $V$ such that for some given $\delta > 0$ there exist two constants $0 < \alpha_- < \alpha_+$ satisfying
\begin{equation}
\delta > \alpha_+ - \alpha_- > 0
\end{equation}
and $H(\alpha)$ has no ground state for $|\alpha| < \alpha_-$ but has a ground state for $|\alpha| > \alpha_+$.

Our paper is organized as follows. In Section 2 we define the Pauli-Fierz model and in Section 3 we prove the absence of ground states. Section 4 lists examples of external potentials exhibiting the unbinding-binding transition.

## 2 The Pauli-Fierz Hamiltonian

We assume a space dimension $d \geq 3$ throughout, and take the natural unit: the velocity of light $c = 1$ and the Planck constant divided $2\pi$, $\hbar = 1$. The Hilbert space $\mathcal{H}$ for the Pauli-Fierz Hamiltonian is given by
\[
\mathcal{H} = L^2(\mathbb{R}^d) \otimes \mathcal{F},
\]
where
\[
\mathcal{F} = \bigoplus_{n=0}^{\infty} \left[ \otimes^n (\oplus^{d-1} L^2(\mathbb{R}^d)) \right]
\]
denotes the boson Fock space over the $(d-1)$-fold direct sum $\oplus^{d-1} L^2(\mathbb{R}^d)$. Let $\Omega = \{1, 0, 0, \ldots\} \in \mathcal{F}$ denote the Fock vacuum. The creation operator and the annihilation operator are denoted by $a^*(f, j)$ and $a(f, j)$, $j = 1, \ldots, d-1$, $f \in L^2(\mathbb{R}^d)$, respectively, and they satisfy the canonical commutation relations
\[
[a(f, j), a^*(g, j')] = \delta_{jj'}(f, g) 1, \quad [a(f, j), a(g, j')] = 0 = [a^*(f, j), a^*(g, j')].
\]
with \((f, g)\) the scalar product on \(L^2(\mathbb{R}^d)\). We write

\[
(2.1) \quad \alpha^x(f, j) = \int \alpha^x(k, j)f(k)dk, \quad \alpha^x = a, a^*,
\]

The energy of a single photon with momentum \(k \in \mathbb{R}^d\) is

\[
(2.2) \quad \omega(k) = |k|.
\]

The free Hamiltonian on \(\mathcal{F}\) is then given by

\[
(2.3) \quad H_f = \sum_{j=1}^{d-1} \int \omega(k)a^*(k, j)a(k, j)dk.
\]

Note that \(\sigma(H_f) = [0, \infty)\), and \(\sigma_p(H_f) = \{0\}\). \(\{0\}\) is a simple eigenvalue of \(H_f\) and \(H_f \Omega = 0\).

Next we introduce the quantized radiation field. The \(d\)-dimensional polarization vectors are denoted by \(e_j(k) \in \mathbb{R}^d, j = 1, \ldots, d-1\), which satisfy \(e_i(k) \cdot e_j(k) = \delta_{ij}\) and \(e_j(k) \cdot k = 0\) almost everywhere on \(\mathbb{R}^d\). The quantized vector potential then reads

\[
(2.4) \quad A(x) = \sum_{j=1}^{d-1} \int \frac{1}{\sqrt{2\omega(k)}} e_j(k)\left(\hat{\phi}(k)a^*(k, j)e^{-ikx} + \hat{\phi}(-k)a(k, j)e^{ikx}\right)dk
\]

for \(x \in \mathbb{R}^d\) with ultraviolet cutoff \(\hat{\phi}\). Conditions imposed on \(\hat{\phi}\) will be supplied later. Assuming that \(V\) is centered, in the dipole approximation \(A(x)\) is replaced by \(A(0)\). We set \(A = A(0)\). The Pauli-Fierz Hamiltonian \(H(\alpha)\) in the dipole approximation is then given by

\[
(2.5) \quad H(\alpha) = \frac{1}{2m} \left( p \otimes \mathbb{1} - \alpha \mathbb{1} \otimes A \right)^2 + V \otimes \mathbb{1} + \mathbb{1} \otimes H_f,
\]

where \(\alpha \in \mathbb{R}\) is the coupling constant, \(V\) the external potential, and \(p = (-i\partial_1, \ldots, -i\partial_d)\) the momentum operator. For notational convenience we omit the tensor notation \(\otimes\) in what follows.

**Assumption 2.1** Suppose that \(V\) is relatively bounded with respect to \(-\frac{1}{2m} \Delta\) with a relative bound strictly smaller than one, and

\[
(2.6) \quad \hat{\varphi}/\omega \in L^2(\mathbb{R}^d), \quad \sqrt{\omega} \hat{\varphi} \in L^2(\mathbb{R}^d).
\]
By this assumption $H(\alpha)$ is self-adjoint on $D(-\Delta) \cap D(H_f)$ and bounded below for arbitrary $\alpha \in \mathbb{R}$ [Ara81, Ara83]. We need in addition some technical assumptions on $\hat{\varphi}$ which are introduced in [HS01, Definition 2.2]. We list them as

**Assumption 2.2** The ultraviolet cutoff $\hat{\varphi}$ satisfies (1)-(4) below.

1. $\hat{\varphi}/\omega^{3/2} \in L^2(\mathbb{R}^d)$;
2. $\hat{\varphi}$ is rotation invariant, i.e. $\hat{\varphi}(k) = \chi(|k|)$ with some real-valued function $\chi$ on $[0, \infty)$; and $\rho(s) = |\chi(\sqrt{s})|^2 s^{(d-2)/2} \in L^r([0, \infty), ds)$ for some $1 < r$, and there exists $0 < \beta < 1$ such that $|\rho(s + h) - \rho(s)| \leq K|h|^\beta$ for all $s$ and $0 < h \leq 1$ with some constant $K$;
3. $\|\hat{\varphi}\omega^{(d-1)/2}\|_\infty < \infty$;
4. $\hat{\varphi}(k) \neq 0$ for $k \neq 0$.

The Hamiltonian $H(\alpha)$ with $V = 0$ is quadratic and can therefore be diagonalized explicitly, which is carried out in [Ara83, HS01]. Assumption 2.2 ensures the existence of a unitary operator diagonalizing $H(\alpha)$.

Let

$$D_+(s) = m - \alpha^2 d - 1 \int \frac{|\hat{\varphi}(k)|^2}{s - \omega(k)^2 + i0} \, dk, \quad s \geq 0.$$ 

We see that $D_+(0) = m + \alpha^2 d - 1 \|\hat{\varphi}/\omega\|^2 > 0$ and the imaginary part of $D_+(s)$ is $\alpha^2 d - 1 \pi S_{d-1} \rho(s) \neq 0$ for $s \neq 0$, where $\rho$ is defined in (2) of Assumption 2.2 and $S_{d-1}$ is the volume of the $d-1$ dimensional unit sphere, and the real part of $D_+(s)$ satisfies that $\lim_{s \to \infty} \Re D_+(s) = m > 0$. These properties follows from Assumption 2.2. In particular

$$(2.7) \quad \inf_{s \geq 0} |D_+(s)| > 0.$$ 

Define

$$(2.8) \quad \Lambda^\mu_j(k) = \frac{e(\mu)(k) \hat{\varphi}(k)}{\omega^{3/2}(k) D_+(\omega^2(k))}.$$ 

Then $\|\Lambda^\mu_j\| \leq C\|\hat{\varphi}/\omega^{3/2}\|$ for some constant $C$. 


Proposition 2.3 Under the assumptions 2.1 and 2.2, for each \( \alpha \in \mathbb{R} \), there exist unitary operators \( U \) and \( T \) on \( \mathcal{H} \) such that both map \( D(-\Delta) \cap D(H_f) \) onto itself and

\[
U^{-1}H(\alpha)U = -\frac{1}{2m_{\text{eff}}(\alpha)} \Delta + H_f + T^{-1}VT + g,
\]

where \( m_{\text{eff}}(\alpha) \) and \( g \) are constants given by

\[
m_{\text{eff}}(\alpha) = m + \alpha^2 \left( \frac{d-1}{d} \right) \| \hat{\varphi}/\omega \|_2^2,
\]

\[
g = \frac{d}{2\pi} \int_{-\infty}^{\infty} \frac{t^2}{m + \alpha^2 \left( \frac{d-1}{d} \right)} \left\| \frac{\hat{\varphi}}{\sqrt{t^2 + \omega^2}} \right\|_2^2 dt.
\]

Here \( U \) is defined in (4.29) of [HS01] and \( T \) by

\[
T = \exp \left( -i \frac{\alpha}{m_{\text{eff}}(\alpha)} p \cdot \phi \right),
\]

where \( \phi = (\phi_1, ..., \phi_d) \) is the vector field

\[
\phi_\mu = \frac{1}{\sqrt{2}} \sum_{j=1}^{d-1} \int \left( \Lambda_j^\mu(k)a^*(k,j) + \Lambda_j^\mu(k)a(k,j) \right) dk.
\]

Proof: See [HS01] Appendix. \( \Box \)

3 The Birman-Schwinger principle

3.1 The case of Schrödinger operators

Let \( h_0 = -\frac{1}{2} \Delta \). We assume that \( V \in L^1_{\text{loc}}(\mathbb{R}^d) \) and \( V \) is relatively form-bounded with respect to \( h_0 \) with relative bound \( a < 1 \), i.e., \( D(|V|^{1/2}) \supset D(h_0^{1/2}) \) and

\[
||V|^{1/2}\varphi||^2 \leq a||h_0^{1/2}\varphi||^2 + b||\varphi||^2, \quad \varphi \in D(h_0^{1/2}),
\]

with some \( b > 0 \). Then the operators

\[
R_E = (h_0 - E)^{-1/2} |V|^{1/2}, \quad E < 0,
\]
are densely defined. From (3.1) it follows that \( R^*_E = |V|^{1/2}(h_0 - E)^{-1/2} \) is bounded and thus \( R_E \) is closable. We denote its closure by the same symbol. Let

\[
(3.3) \quad K_E = R^*_E R_E.
\]

Then \( K_E (E < 0) \) is a bounded, positive self-adjoint operator and it holds

\[
K_E f = |V|^{1/2} (h_0 - E)^{-1} |V|^{1/2} f, \quad f \in C_0^\infty (\mathbb{R}^d).
\]

Now let us consider the case \( E = 0 \). Let

\[
(3.4) \quad R_0 = h_0^{-1/2} |V|^{1/2}.
\]

The self-adjoint operator \( h_0^{-1/2} \) has the integral kernel

\[
h_0^{-1/2} (x,y) = \frac{a_d}{|x - y|^{d-1}}, \quad d \geq 3,
\]

where \( a_d = \sqrt{2\pi}^{(d-1)/2}/\Gamma((d-1)/2) \) and \( \Gamma(\cdot) \) the Gamma function. It holds that

\[
\left| \langle h_0^{-1/2} g, |V|^{1/2} f \rangle \right| \leq a_d \|g\|_2 \| |V|^{1/2} f \|_{2d/(d+2)}
\]

for \( f, g \in C_0^\infty (\mathbb{R}^3) \) by the Hardy-Littlewood-Sobolev inequality. Since \( f \in C_0^\infty (\mathbb{R}^3) \) and \( V \in L^1_{\text{loc}} (\mathbb{R}^3) \), one concludes \( \| |V|^{1/2} f \|_{2d/(d+2)} < \infty \). Thus \( |V|^{1/2} f \in D(h_0^{-1/2}) \) and \( R_0 \) is densely defined. Since \( V \) is relatively form-bounded with respect to \( h_0 \), \( R_0^* \) is also densely defined, and \( R_0 \) is closable. We denote the closure by the same symbol. We define

\[
(3.5) \quad K_0 = R^*_0 R_0.
\]

Next let us introduce assumptions on the external potential \( V \).

**Assumption 3.1** \( V \) satisfies that (1) \( V \leq 0 \) and (2) \( R_0 \) is compact.

**Lemma 3.2** Suppose Assumption 3.1. Then

(i) \( R_E, R^*_E \) and \( K_E (E \leq 0) \) are compact.

(ii) \( \|K_E\| \) is continuous and monotonously increasing in \( E \leq 0 \) and it holds that

\[
(3.6) \quad \lim_{E \to -\infty} \|K_E\| = 0, \quad \lim_{E \to 0} \|K_E\| = \|K_0\|.
\]
Proof: Under (2) of Assumption 3.1, \( R_0^* \) and \( K_0 \) are compact. Since

\[
(f, K_E f) \leq (f, K_0 f), \quad f \in C_c^\infty(\mathbb{R}^d),
\]

extends to \( f \in L^2(\mathbb{R}^3), K_E, R_E \) and \( R_E^* \) are also compact. Thus (i) is proven.

We will prove (ii). It is clear from (3.7) that \( K_E \) is monotonously increasing in \( E \). Since \( R_0 \) is bounded, (3.7) holds on \( L^2(\mathbb{R}^d) \) and

\[
K_E = R_0^* ((h_0 - E)^{-1} h_0) R_0, \quad E \leq 0.
\]

From (3.8) one concludes that

\[
\| K_E - K_{E'} \| \leq \| K_0 \| \frac{|E - E'|}{|E'|}
\]

for \( E, E' < 0 \). Hence \( \| K_E \| \) is continuous in \( E < 0 \). We have to prove the left continuity at \( E = 0 \). Since \( R_0 \) is bounded, \( (3.7) \) holds on \( L^2(\mathbb{R}^d) \) and

\[
K_E \to K_0 = \text{s-lim}_{E \uparrow 0} K_E \quad K_0 f = \lim_{E \uparrow 0} \| K_E f \|.
\]

Hence we have \( \| K_0 \| \leq \liminf_{E \uparrow 0} \| K_E \| \) and \( \lim_{E \uparrow 0} \| K_E \| = \| K_0 \| \). It remains to prove that \( \lim_{E \to -\infty} \| K_E \| = 0 \). Since \( R_0^* \) is compact, for any \( \epsilon > 0 \), there exists a finite rank operator \( T_\epsilon = \sum_{k=1}^n (\varphi_k, \cdot) \psi_k \) such that \( n = n(\epsilon) < \infty, \varphi_k, \psi_k \in L^2(\mathbb{R}^d) \) and \( \| R_0^* - T_\epsilon \| < \epsilon \). Then it holds that \( \| K_E \| \leq (\epsilon + \| T_\epsilon h_0(h_0 - E)^{-1} \|) \| R_0 \| \). For any \( f \in L^2(\mathbb{R}^d) \), we have

\[
\| T_\epsilon h_0(h_0 - E)^{-1} f \| \leq \left( \sum_{k=1}^n \| h_0(h_0 - E)^{-1} \varphi_k \| \psi_k \| \right) \| f \|
\]

and \( \lim_{E \to -\infty} \| T_\epsilon h_0(h_0 - E)^{-1} \| = 0 \), which completes (ii).

Let

\[
H_p(m) = -\frac{1}{2m}\Delta + V.
\]

By (ii) of Lemma 3.2 we have \( \lim_{E \to -\infty} \| |V|^{1/2}(h_0 - E)^{-1/2} \| = 0 \). Therefore \( V \) is infinitesimally form bounded with respect to \( h_0 \) and \( H_p(m) \) is the self-adjoint operator associated with the quadratic form

\[
f, g \mapsto \frac{1}{m}(h_0^{1/2} f, h_0^{1/2} g) + (|V|^{1/2} f, |V|^{1/2} g)
\]
for \( f, g \in D(h_0^{1/2}) \). Note that the domain \( D(H_0(m)) \) is independent of \( m \).

Under (2) of Assumption 3.1 the essential spectrum of \( H_0(m) \) coincides with that of \(- \frac{1}{2m} \Delta\), hence \( \sigma_{\text{ess}}(H_0(m)) = [0, \infty) \). Next we will estimate the spectrum of \( H_0(m) \) contained in \((-\infty, 0]\). Let \( \mathbb{I}_\mathcal{O}(T), \mathcal{O} \subset \mathbb{R} \), be the spectral resolution of self-adjoint operator \( T \) and set

\[
N_\mathcal{O}(T) = \dim \text{Ran} \mathbb{I}_\mathcal{O}(T).
\]

The Birman-Schwinger principle [Sim05] states that

\[
\begin{align*}
(E < 0) & \quad N_{(-\infty, \frac{1}{m}]}(H_0(m)) = N_{[\frac{1}{m}, \infty)}(K_E), \\
(E = 0) & \quad N_{(-\infty, 0]}(H_0(m)) = N_{[\frac{1}{m}, \infty)}(K_0).
\end{align*}
\]

Now let us define the constant \( m_c \) by the inverse of the operator norm of \( K_0 \),

\[
m_c = \| K_0 \|^{-1}.
\]

**Lemma 3.3** Suppose Assumption 3.1.

1. If \( m < m_c \), then \( N_{(-\infty, 0]}(H_0(m)) = 0 \).
2. If \( m > m_c \), then \( N_{(-\infty, 0]}(H_0(m)) \geq 1 \).

**Proof:** It is immediate to see (1) by the Birman-Schwinger principle (3.11). Suppose \( m > m_c \). Then, using the continuity and monotonicity of \( E \to \| K \| \), see Lemma 3.2, there exists \( \epsilon > 0 \) such that \( m_c < \| K_{-\epsilon} \|^{-1} \leq m \). Since \( K_{-\epsilon} \) is positive and compact, \( \| K_{-\epsilon} \| \in \sigma_p(K_{-\epsilon}) \) follows and hence \( N_{[\frac{1}{m}, \infty)}(K_{-\epsilon}) \geq 1 \). Therefore (2) follows again from the Birman-Schwinger principle. \( \square \)

**Remark 3.4** By Lemma 3.3 the critical mass at zero coupling \( m_c(0) = m_c \).

In the case \( m > m_c \), by the proof of Lemma 3.3 one concludes that the bottom of the spectrum of \( H_0(m) \) is strictly negative. For \( \epsilon > 0 \) we set

\[
m_\epsilon = \| K_{-\epsilon} \|^{-1}.
\]

**Corollary 3.5** Suppose Assumption 3.1 and \( m > m_\epsilon \). Then

\[
\inf \sigma(H_0(m)) \leq -\frac{\epsilon}{m}.
\]

**Proof:** The Birman-Schwinger principle states that \( 1 \leq N_{(-\infty, -\frac{\epsilon}{m}]}(H_0(m)) \), since \( 1/m < \| K_{-\epsilon} \| \), which implies the corollary. \( \square \)
3.2 The case of the Pauli-Fierz model

In this subsection we extend the Birman-Schwinger type estimate to the Pauli-Fierz Hamiltonian.

**Lemma 3.6** Suppose Assumption 3.1. If \( m < m_c \), then the zero coupling Hamiltonian \( H_p(m) + H_f \) has no ground state.

**Proof:** Since the Fock vacuum \( \Omega \) is the ground state of \( H_f \), \( H_p(m) + H_f \) has a ground state if and only if \( H_p(m) \) has a ground state. But \( H_p(m) \) has no ground state by Lemma 3.3. Therefore \( H_p(m) + H_f \) has no ground state. \( \square \)

From now on we discuss \( U^{-1}H(\alpha)U \) with \( \alpha \neq 0 \). We set

\[
U^{-1}H(\alpha)U = H_0(\alpha) + W + g,
\]

where

\[
H_0(\alpha) = -\frac{1}{2m_{\text{eff}}(\alpha)}\Delta + H_f,
\]

\[
W = T^{-1}VT.
\]

**Theorem 3.7** Suppose Assumptions 2.1, 2.2 and 3.1. If \( m_{\text{eff}}(\alpha) < m_c \), then \( H_0(\alpha) + W + g \) has no ground state.

**Proof:** Since \( g \) is a constant, we prove the absence of ground state of \( H_0(\alpha) + W \). Since \( V \) is negative, so is \( W \). Hence \( \inf \sigma(H_0(\alpha) + W) \leq \inf \sigma(H_0(\alpha)) = 0 \). Then it suffices to show that \( H_0(\alpha) + W \) has no eigenvalues in \( (-\infty, 0] \). Let \( E \in (-\infty, 0] \) and set

\[
\mathcal{K}_E = |W|^{1/2}(H_0(\alpha) - E)^{-1}|W|^{1/2},
\]

where \( |W|^{1/2} \) is defined by the functional calculus. We shall prove now that if \( H_0(\alpha) + W \) has eigenvalue \( E \in (-\infty, 0] \), then \( \mathcal{K}_E \) has eigenvalue 1. Suppose that \( (H_0(\alpha) + W - E)\varphi = 0 \) and \( \varphi \neq 0 \), then

\[
\mathcal{K}_E|W|^{1/2}\varphi = |W|^{1/2}\varphi.
\]

Moreover if \( |W|^{1/2}\varphi = 0 \), then \( W\varphi = 0 \) and hence \( (H_0(\alpha) - E)\varphi = 0 \), but \( H_0(\alpha) \) has no eigenvalue by Lemma 3.6. Then \( |W|^{1/2}\varphi \neq 0 \) is concluded and \( \mathcal{K}_E \) has eigenvalue 1. Then it is sufficient to see \( \|\mathcal{K}_E\| < 1 \) to show that
$H_0(\alpha) + W$ has no eigenvalues in $(-\infty, 0]$. Notice that $-\frac{1}{2m_{\text{eff}}(\alpha)}\Delta$ and $T$ commute, and
\[
\left\|(-\Delta)^{1/2}(H_0(\alpha) - E)^{-1}(-\Delta)^{1/2}\right\| \leq 2m_{\text{eff}}(\alpha).
\]
Then we have
\[
\|K_E\| \leq \left\||V|^{1/2} \left(-\frac{1}{2m_{\text{eff}}(\alpha)}\Delta\right)^{-1/2}\right\|^2 = m_{\text{eff}}(\alpha)\|K_0\| = \frac{m_{\text{eff}}(\alpha)}{m_c} < 1
\]
and the proof is complete. \hfill \Box

4  Absence and existence of a ground state

In this section we establish the absence, resp. existence, of a ground state of the Pauli-Fierz Hamiltonian $H_0(\alpha) + W$. Let $\kappa > 0$ be a parameter and let us define the Pauli-Fierz Hamiltonian with scaled external potential $V_\kappa(x) = V(x/\kappa)/\kappa^2$ by
\[
H_\kappa = \frac{1}{2m}(p - \kappa A)^2 + V_\kappa + H_f.
\]
We also define $K_\kappa$ by $H(\alpha)$ with $a^\sharp$ replaced by $\kappa a^\sharp$. Then
\[
K_\kappa = \frac{1}{2m}(p - \kappa \alpha A)^2 + V + \kappa^2 H_f.
\]
$H_\kappa$ and $\kappa^{-2}K_\kappa$ are unitarily equivalent,
\[
H_\kappa \simeq \kappa^{-2}K_\kappa.
\]
Let $m < m_c$ and $\epsilon > 0$. We define the function
\[
\alpha_\epsilon = \left(d - \frac{1}{d}\|\hat{\phi}/\omega\|^2\right)^{-1/2} \sqrt{m_c - m}, \quad \epsilon > 0
\]
\[
\alpha_0 = \left(d - \frac{1}{d}\|\hat{\phi}/\omega\|^2\right)^{-1/2} \sqrt{m_c - m},
\]
where we recall that $m_c = \|K_{-\epsilon}\|^{-1}$ for $\epsilon \geq 0$. Note that

1. $|\alpha| < \alpha_0$ if and only if $m_{\text{eff}}(\alpha) < m_c$;
(2) \(|\alpha| > \alpha_\epsilon\) if and only if \(m_{\text{eff}}(\alpha) > m_\epsilon\).

Note that \(\alpha_0 < \alpha_\epsilon\) because of \(m_\epsilon > m_c\). Since \(\lim_{\epsilon \to 0} m_\epsilon = m_c\), it holds that \(\lim_{\epsilon \to 0} \alpha_\epsilon = \alpha_0\). We furthermore introduce assumptions on the external potential \(V\) and ultraviolet cutoff \(\hat{\phi}\).

**Assumption 4.1** The external potential \(V\) and the ultraviolet cutoff \(\hat{\phi}\) satisfies:

1. \(V \in C^1(\mathbb{R}^d)\) and \(\nabla V \in L^\infty(\mathbb{R}^d)\);
2. \(\hat{\phi}/\omega^{5/2} \in L^2(\mathbb{R}^d)\).

We briefly comment on (1) of Assumption 4.1. We know that

\[
H_0(\alpha) + W = -\frac{1}{2m_{\text{eff}}(\alpha)} \Delta + V + H_f + V(\cdot - \frac{\alpha}{m_{\text{eff}}(\alpha)} \phi)^2 - V.
\]

The term on the right-hand side above, \(H_{\text{int}} = V(\cdot - \frac{\alpha}{m_{\text{eff}}(\alpha)} \phi)^2 - V\), is regarded as the interaction, and

\[
H_{\text{int}} \sim \frac{\alpha}{m_{\text{eff}}(\alpha)} \nabla V(\cdot) \cdot \phi.
\]

By (1) of Assumption 4.1 we have

\[
\|H_{\text{int}} \Phi\| \leq C \|(H_f + 1)^{1/2} \Phi\|
\]

with some constant \(C\) independent of \(\alpha\). This estimate follows from the fundamental inequality \(\|a^2(f) \Phi\| \leq \|f/\sqrt{\Omega}\| \|(H_f + 1)^{1/2} \Phi\|\). Then the interaction has a uniform bound with respect to the coupling constant \(\alpha\). Since the decoupled Hamiltonian \(-\frac{1}{2m_{\text{eff}}(\alpha)} \Delta + V + H_f\) has a ground state for sufficiently large \(\alpha\), it is expected that \(H_0(\alpha) + W\) also has a ground state for sufficiently large \(\alpha\). This is rigorously proven in (1) of Theorem 4.2 below. Now we are in the position to state the main theorem.

**Theorem 4.2** Suppose Assumptions 2.1, 2.2, 3.1 and 4.1. Then (1) and (2) below hold.

1. For any \(\epsilon > 0\), there exists \(\kappa_\epsilon\) such that for all \(\kappa > \kappa_\epsilon\), \(H_\kappa\) has a unique ground state for all \(\alpha\) such that \(|\alpha| > \alpha_\epsilon\).
2. \(H_\kappa\) has no ground state for all \(\kappa > 0\) and all \(\alpha\) such that \(|\alpha| < \alpha_0\).
Proof: Let $U_\kappa$ (resp. $T_\kappa$) be defined by $U$ (resp. $T$) with $\omega$ and $\hat{\phi}$ replaced by $\kappa^2 \omega$ and $\kappa \hat{\phi}$. Then

\begin{equation}
U_\kappa^{-1} K_\kappa U_\kappa = H_p(m_{\text{eff}}(\alpha)) + \kappa^2 H_f + \delta V_\kappa + g,
\end{equation}

where $\delta V_\kappa = T_\kappa^{-1} V T_\kappa - V$. Note that $g$ is independent of $\kappa$. Since $U_\kappa^{-1} K_\kappa U_\kappa$ is unitary equivalent to $\kappa^2 H_\kappa$, we prove the existence of a ground state for $U_\kappa^{-1} K_\kappa U_\kappa$. Let $N = \sum_{j=1}^{d-1} \int a^*(k,j) a(k,j) dk$ be the number operator. Since $H_p(m_{\text{eff}}(\alpha))$ has a ground state by the assumption $|\alpha| > \alpha_\epsilon$, i.e., $m_{\text{eff}}(\alpha) > m_\epsilon$, it can be shown that $U_\kappa^{-1} K_\kappa U_\kappa + \nu N$ with $\nu > 0$ also has a ground state, see [HS01, p.1168] for details. We denote the normalized ground state of $U_\kappa^{-1} K_\kappa U_\kappa + \nu N$ by $\Psi_\nu = \Psi_\nu(\kappa)$. Since the unit ball in a Hilbert space is weakly compact, there exists a subsequence of $\Psi_\nu'$ such that the weak limit $\Psi = \lim_{\nu' \to 0} \Psi_\nu'$ exists. If $\Psi \neq 0$, then $\Psi$ is a ground state [AH97]. Let $P = \mathbb{1}_{\{\Sigma \geq 0\}}(-1/2m_{\text{eff}}(\alpha) \Delta + V) \otimes \mathbb{1}_{\{0\}}(H_f)$ and $\Sigma = \inf \sigma(H_p(m_{\text{eff}}(\alpha)))$. Adopting the arguments in the proof of [HS01, Lemma 3.3], we conclude

\begin{equation}
(\Psi, P \Psi) \geq 1 - \frac{|\alpha| \|\hat{\phi}/\omega^{5/2}\|^2}{\kappa^3 m_{\text{eff}}(\alpha)} - \frac{3 D}{2\kappa} - \frac{3 D}{2\kappa},
\end{equation}

where $\varepsilon > 0$ and $D$ are constants independent of $\kappa$ and $\alpha$. Since $m_{\text{eff}}(\alpha) > m_\epsilon > m_{\epsilon}/2$,\n
\begin{equation}
\Sigma \leq \inf \sigma(H_p(m_\epsilon)) \leq -\frac{\varepsilon}{2m_\epsilon}
\end{equation}

by Corollary [35]. By (4.8) and (4.7) we have

\begin{equation}
(\Psi, P \Psi) \geq \kappa^{-3} \left( \rho(\kappa) - \varepsilon \|\hat{\phi}/\omega^{5/2}\|^2 \frac{|\alpha|}{m_{\text{eff}}(\alpha)} \right),
\end{equation}

where $\rho(\kappa) = \kappa^3 - \frac{\kappa}{\xi \kappa - 1}$ with $\xi = \frac{2\epsilon}{3m_{\ell}D}$. Then there exists $\kappa_\epsilon > 0$ such that the right-hand side of (4.9) is positive for all $\kappa > \kappa_\epsilon$ and all $\alpha \in \mathbb{R}$. Actually a sufficient condition for the positivity of the right-hand side of (4.9) is

\begin{equation}
\rho(\kappa) > \frac{\varepsilon \|\hat{\phi}/\omega^{5/2}\|^2}{2\sqrt{m} \|\hat{\phi}/\omega\|^2},
\end{equation}

since $\sup_{\alpha} \frac{|\alpha|}{m_{\text{eff}}(\alpha)} = (2\sqrt{m} \|\hat{\phi}/\omega\|)^{-1}$. Then $\Psi \neq 0$ for all $\kappa > \kappa_\epsilon$. Thus the ground state exists for all $|\alpha| > \alpha_\epsilon$ and all $\kappa > \kappa_\epsilon$ and (1) is complete.
We next show (2). Notice that

\[ U_\kappa^{-1} H_\kappa U_\kappa = -\frac{1}{2m_{\text{eff}}(\alpha)} \Delta + H_f + T^{-1} V_\kappa T + g. \]

Define the unitary operator \( u_\kappa \) by \((u_\kappa f)(x) = k^{d/2} f(x/\kappa)\). Then we infer

\[ V_\kappa = \kappa^{-2} u_\kappa V u_\kappa^{-1}, -\Delta = \kappa^{-2} u_\kappa (-\Delta) u_\kappa^{-1} \]

and

\[ |||V_\kappa|^{1/2}(-\Delta)^{-1}|V_\kappa|^{1/2}|| = \kappa^{-2}||u_\kappa|V|^{1/2}u_\kappa^{-1}(-\Delta)^{-1}u_\kappa|V|^{1/2}u_\kappa^{-1}|| = ||K_0||. \]

(2) follows from Theorem 3.7.

Corollary 4.3 Let arbitrary \( \delta > 0 \) be given. Then there exists an external potential \( \tilde{V} \) and constants \( \alpha_+ > \alpha_- \) such that

1. \( 0 < \alpha_+ - \alpha_- < \delta \);
2. \( H(\alpha) \) has a ground state for \( |\alpha| > \alpha_+ \) but no ground state for \( |\alpha| < \alpha_- \).

Proof: Suppose that \( V \) satisfies Assumption 3.1. For \( \delta > 0 \) we take \( \epsilon > 0 \) such that \( \alpha_\epsilon - \alpha_0 < \delta \). Take a sufficiently large \( \kappa \) such that (4.10) is fulfilled, and set \( \tilde{V}(x) = V(x/\kappa)/\kappa^2 \). Define \( H(\alpha) \) by the Pauli-Fierz Hamiltonian with potential \( \tilde{V} \). Then \( H(\alpha) \) satisfies (1) and (2) with \( \alpha_+ = \alpha_\epsilon \) and \( \alpha_- = \alpha_0 \).

Remark 4.4 (Upper and lower bound of \( m_c(\alpha) \)) Corollary 4.3 implies the upper and lower bounds

\[ m_-(\alpha) \leq m_c(\alpha) \leq m_+(\alpha), \]

\[ m_c(0) = m_c, \]

where

\[ m_-(\alpha) = m_0 - \alpha^2 \frac{d-1}{d} \|\tilde{\psi}/\omega\|^2, \]

\[ m_+(\alpha) = m_\epsilon - \alpha^2 \frac{d-1}{d} \|\tilde{\psi}/\omega\|^2. \]

Fix the coupling constant \( \alpha \). If \( m < m_-(\alpha) \), then there is no ground state, and if \( m > m_+(\alpha) \), then the ground state exists, compare with Fig. 1.

Remark 4.5 ( \( m_c(\alpha) \) for sufficiently large \( \alpha \) ) Let \((\frac{d-1}{d} \|\tilde{\psi}/\omega\|^2)^{-1} m_\epsilon < \alpha^2 \). Then by Remark 4.4, \( H(\alpha) \) has a ground state for arbitrary \( m > 0 \). It is an open problem to establish whether this is an artifact of the dipole approximation or in fact holds also for the Pauli-Fierz operator.
5 Examples of external potentials

In this section we give examples of potentials \( V \) satisfying Assumption 3.1. The self-adjoint operator \( h_0^{-1} \) has the integral kernel

\[
h_0^{-1}(x, y) = \frac{b_d}{|x - y|^{d-2}}, \quad d \geq 3,
\]

with \( b_d = 2\Gamma((d/2) - 1)/\pi^{(d/2)-2} \). It holds that

\[
(f, K_0 f) = \int dx \int dy f(x) K_0(x, y) f(y),
\]

where

\[
K_0(x, y) = b_d \frac{|V(x)|^{1/2}|V(y)|^{1/2}}{|x - y|^{d-2}}, \quad d \geq 3,
\]

is the integral kernel of operator \( K_0 \). We recall the Rollnik class \( \mathcal{R} \) of potentials is defined by

\[
\mathcal{R} = \left\{ V \big| \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy \frac{|V(x)V(y)|}{|x - y|^2} < \infty \right\}.
\]

By the Hardy-Littlewood-Sobolev inequality, \( \mathcal{R} \supset L^p(\mathbb{R}^3) \cap L^r(\mathbb{R}^3) \) with \( 1/p + 1/r = 4/3 \). In particular, \( L^{3/2}(\mathbb{R}^3) \subset \mathcal{R} \).

Example 5.1 (\( d = 3 \) and Rollnik class) Let \( d = 3 \). Suppose that \( V \) is negative and \( V \in \mathcal{R} \). Then \( K_0 \in L^2(\mathbb{R}^3 \times \mathbb{R}^3) \). Hence \( K_0 \) is Hilbert-Schmidt and Assumption 3.1 is satisfied.

The example can be extended to dimensions \( d \geq 3 \).

Example 5.2 (\( d \geq 3 \) and \( V \in L^{d/2}(\mathbb{R}^d) \)) Let \( L_w^p(\mathbb{R}^d) \) be the set of Lebesgue measurable function \( u \) such that sup\( \beta > 0 \) \( \left| \{ x \in \mathbb{R}^d | u(x) > \beta \} \right|_L^{1/p} < \infty \), where \( |E|_L \) denotes the Lebesgue measure of \( E \subset \mathbb{R}^d \). Let \( g \in L^p(\mathbb{R}^d) \) and \( u \in L_w^p(\mathbb{R}^d) \) for \( 2 < p < \infty \). Define the operator \( B_{u,g} \) by

\[
B_{u,g} h = (2\pi)^{-d/2} \int e^{ikx} u(k) g(x) h(x) dx.
\]

It is shown in [Cwi77, Theorem, p.97] that \( B_{u,g} \) is a compact operator on \( L^2(\mathbb{R}^d) \). It is known that \( u(k) = 2|k|^{-1} \in L_w^d(\mathbb{R}^d) \) for \( d \geq 3 \). Let \( F \) denote Fourier transform on \( L^2(\mathbb{R}^d) \), and suppose that \( V \in L^{d/2}(\mathbb{R}^d) \). Then \( B_{u,[V]^{1/2}} \) is compact on \( L^2(\mathbb{R}^d) \) and then \( R_0^* = FB_{u,V^{1/2}}F^{-1} \) is compact. Thus \( R_0 \) is also compact.
Assume that $V \in L^{d/2}(\mathbb{R}^d)$. Let us now see the critical mass of zero coupling $m_c = m_0$. By the Hardy-Littlewood-Sobolev inequality, we have
\begin{equation}
|\langle f, K_0 f \rangle| \leq D_V \|f\|_2^2,
\end{equation}
where
\begin{equation}
D_V = \sqrt{2\pi} \sqrt[4/d]{\frac{\Gamma((d/2) - 1)}{\Gamma((d/2) + 1)} \left( \frac{\Gamma(d)}{\Gamma(d/2)} \right)^{2/d} \|V\|_{d/2}^2},
\end{equation}
a constant in (5.4) is proved by Lieb \cite{Lie83}. Then
\begin{equation}
\|K_0\| \leq D_V.
\end{equation}
By (5.5) we have $m_c \geq D_V^{-1}$. In particular in the case of $d = 3$,
\begin{equation}
m_c \geq \frac{3}{\sqrt{2\pi}^{2/3} 4^{5/3} \|V\|_{3/2}^2}.
\end{equation}

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