Quantum capacity of bosonic dephasing channel

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We study the quantum capacity of continuous variable dephasing channel, which is a notable example of non-Gaussian quantum channel. We prove that a single letter formula applies. We then consider input energy restriction and show that by increasing it, the capacity saturates to a finite value. The optimal input state is found to be diagonal in the Fock basis and with a distribution that is a discrete version of a Gaussian. Relations between its mean/variance and dephasing rate/input energy are put forward. We also show that quantum capacity decays exponentially for large values of dephasing rates.

I. INTRODUCTION

Any physical process can be regarded as a quantum channel, i.e. a stochastic map on the space of state that causes a state change. As such it can be characterized by its ability in conveying information. A notable example is provided by the quantum capacity of a channel, that shows its ability to transfer unaltered the entanglement of input system with a reference system [1].

Finding the quantum capacity of a channel is challenging because not only optimization of an entropic functional (coherent information) over input states is required, but also its regularization [2]. This task becomes even harder when dealing with infinite dimensional (so called continuous variable) systems. That is why till now in this framework the attention has confined to Gaussian channels, i.e. maps that transform Gaussian states into Gaussian states [3]. For instance, the coherent information of the lossy channel (a special case of Gaussian channels) is known to be additive and hence its quantum capacity is computed [4] (see [5] for the general formalism of energy-constrained quantum capacity). For more general Gaussian channels, a lower bound of quantum capacity can be obtained by evaluating one-shot coherent information of the channel [6].

Nevertheless there is an increasing pressure to go beyond the Gaussian channels paradigm [7, 8]. Heading in this direction we investigate here the quantum capacity of one of the most physically relevant non-Gaussian channel, namely the dephasing channel (see e.g. [9]). It causes the reduction of the off diagonal terms in the Fock basis, thus washing out coherence properties of the state. This happens for instance with uncertainty path length in optical fibers [10].

Here we prove that for dephasing channel the single letter formula applies for quantum capacity. We then consider input energy restriction and show that by increasing it, the capacity saturates to a finite value which depends on the noise parameter of the channel. The optimal input state is found to be a non-Gaussian state, which is diagonal in the Fock basis and with a distribution that is a discrete version of a continuous Gaussian distribution.

We show the relation between the mean/variance of the optimal distribution and dephasing rate /input energy. Finally we show that for large value of dephasing rate quantum capacity decays exponentially with dephasing rate.

The structure of the paper is as follows: In section II we shortly review quantum dephasing channel and its different representations that we are going to use in proceeding sections. Section III is devoted to quantum capacity of dephasing channel, containing analytical results for proving that single letter formula applies and showing the structure of optimal input state. In section IV we introduce our approach for using replica method to numerically evaluate quantum capacity. Its asymptotic behaviour is then discussed in section V. Finally, section VI concludes with summary and discussion of the results.

II. QUANTUM DEPHASING CHANNEL

The continuous variable quantum dephasing effect (see e.g. [9]) provides a notable example of non-Gaussian channel. Such a channel is a completely positive and trace preserving map \( N_\gamma : B(H_S) \rightarrow B(H_S) \) defined on the set of bounded operators over the infinite dimensional (separable) Hilbert space \( H_S \) of the input system as

\[
\rho \mapsto N_\gamma(\rho) = \sum_{j=0}^{\infty} K_j \rho K_j^\dagger,
\]

where the Kraus operators are given by [11]

\[
K_j = e^{-\frac{1}{2} \gamma (a^\dagger a)^2} \frac{(-i \sqrt{\gamma} a^\dagger a)^j}{\sqrt{j}}.
\]

Here \( a, a^\dagger \) are bosonic ladder operators on \( H_S \), and \( \gamma \in [0, +\infty) \) is a parameter that determines the dephasing rate. From Eq. (2) it is easy to see that the set of dephasing channel maps \( \{ N_\gamma \} \) forms a semigroup under composition, given that \( N_\gamma \circ N_{\gamma'} = N_{\gamma + \gamma'} \).

The channel can be dilated into a single mode environ-
with the following unitary
\[ U = e^{-i\sqrt{N}(a^\dagger a)(b^\dagger b)} = e^{-i\sqrt{N}(a^\dagger a)(b^\dagger b)}e^{-\frac{i}{2}(a^\dagger a)^2}. \] (3)

Here, \( b, b^\dagger \) are bosonic ladder operators on the environment space \( \mathcal{H}_E \) (isomorphic to \( \mathcal{H}_S \)). We have
\[ \rho \mapsto \mathcal{N}_\gamma(\rho) = \text{Tr}_E \left[ U \rho U^\dagger \right]. \] (4)

If we expand the input in the Fock basis \( \rho = \sum_{m,n=0}^{\infty} \rho_{m,n} |m\rangle \langle n| \) the effect of \( \mathcal{N}_\gamma \) reads
\[ \rho \mapsto \mathcal{N}_\gamma(\rho) = \sum_{m,n} e^{-\frac{1}{2}(m-n)^2} \rho_{m,n} |m\rangle \langle n|, \] (5)

which clearly shows that the diagonal elements of the input are preserved, while the off-diagonal ones tend to be washed out.

The channel action can also be written as
\[ \rho \mapsto \mathcal{N}_\gamma(\rho) = \int_{-\infty}^{+\infty} e^{-ia\phi} \rho e^{ia\phi} p(\phi) d\phi, \] (6)

with
\[ p(\phi) = \sqrt{\frac{\gamma}{2\pi}} e^{-\frac{1}{2} \gamma \phi^2}. \] (7)

This means a randomization of the phase \( \phi \) according to the probability distribution \( \gamma \). Note that \( \phi \) as random variable must be defined on the sample space \( \mathbb{R} \), not \([0, 2\pi]\).

### III. QUANTUM CAPACITY

Quantum capacity of a channel \( \mathcal{N}_\gamma \) is the highest rate of quantum information transmission via many uses of the channel that is given by
\[ Q = \lim_{n\to\infty} \frac{1}{n} \max_{\rho^{(n)}} J\left(\rho^{(n)}, \mathcal{N}_\gamma^{\otimes n}\right), \] (8)

where maximization is over all density operators on \( \mathcal{H}_S^{\otimes n} \) and the state for which the maximum is achieved is called optimal input state. Furthermore, \( J(\rho, \mathcal{N}) \) denotes the coherent information
\[ J(\rho, \mathcal{N}) \equiv S(\mathcal{N}(\rho)) - S\left(\mathcal{N}(\rho)\right), \] (9)

being \( \mathcal{N} : \mathcal{B}(\mathcal{H}_S) \to \mathcal{B}(\mathcal{H}_E) \) the complementary channel of \( \mathcal{N} \) and \( S(\rho) = -\text{Tr}(\rho \log \rho) \) the von Neumann entropy of \( \rho \) (throughout the paper we use logarithm to base 2).

It was shown that for degradable channels the coherent information is additive and the quantum capacity \( Q \) can be simplified into a single letter expression \( \frac{\gamma}{2} \). Actually, if the complementary channel is entanglement breaking, then we show that this is the case for the quantum dephasing channel, hence to compute its capacity we can restrict our attention to single letter formula. First we derive the explicit form of the complementary channel \( \mathcal{N}_\gamma : \mathcal{B}(\mathcal{H}_S) \to \mathcal{B}(\mathcal{H}_E) \) from \( \rho \)
\[ \rho \mapsto \mathcal{N}_\gamma(\rho) = \text{Tr}_S \left[ U (\rho \otimes |0\rangle \langle 0|) U^\dagger \right] \]
\[ = \text{Tr}_S \left[ \sum_{m,n} \rho_{m,n} |m\rangle \langle n| \otimes |\sqrt{\gamma} m\rangle \langle \sqrt{\gamma} m| \right] \]
\[ = \sum_{m} \rho_{m,m} |\sqrt{\gamma} m\rangle \langle \sqrt{\gamma} m|, \] (10)

where \( |\sqrt{\gamma} m\rangle \) is a coherent state of real amplitude \( \sqrt{\gamma} m \), i.e.
\[ |\sqrt{\gamma} m\rangle = e^{-\gamma m^2/2} \sum_{k=0}^{\infty} (\sqrt{\gamma} m)^k |k\rangle. \] (11)

Then, to prove that \( \mathcal{N}_\gamma \) is entanglement breaking we consider a two-mode squeezed vacuum state
\[ |\Psi\rangle_{SR} = \sum_{n=0}^{\infty} \lambda^n |n\rangle_S |n\rangle_R, \quad 0 \leq \lambda \leq 1, \] (12)

being \( R \) a reference system isomorphic to \( S \) (as well as to \( E \)). Using Eq. (10), we immediately arrive to
\[ (\mathcal{N}_\gamma \otimes \text{id}_R) |\Psi\rangle_{SR} = \sum_{m} \lambda^{2m} |\sqrt{\gamma} m\rangle \langle \sqrt{\gamma} m| \otimes |m\rangle \langle m|, \] (13)

which is a separable state for any value of \( \lambda \). Hence \( \mathcal{N}_\gamma \) is entanglement breaking and for the dephasing channel we can write
\[ Q(\mathcal{N}_\gamma) = \max_{\rho} J\left(\rho, \mathcal{N}_\gamma\right). \] (14)

In other words, computing the quantum capacity of \( \mathcal{N}_\gamma \) has been simplified to single letter formula. Next we use the covariant properties of \( \mathcal{N}_\gamma \) and the concavity of coherent information, to restrict the set of density operators over which the maximization in Eq. (14) should be performed.

**Proposition 1.** The optimal input state to \( \mathcal{N}_\gamma \) for the quantum capacity \( Q \) is diagonal in the Fock basis.

**Proof.** From Eq. (6) it follows that the quantum dephasing channel is covariant, that is for \( U_\theta = e^{-ia^\dagger a\theta} \) with \( \theta \in [0, 2\pi] \) we have
\[ \mathcal{N}_\gamma(U_\theta \rho U_\theta^\dagger) = U_\theta \mathcal{N}_\gamma(\rho) U_\theta^\dagger. \] (15)

Similarly, from Eq. (10), we conclude that also the complementary channel \( \mathcal{N}_\gamma \) is covariant:
\[ \mathcal{N}_\gamma(U_\theta \rho U_\theta^\dagger) = U_\theta \mathcal{N}_\gamma(\rho) U_\theta^\dagger. \] (16)
As the von-Neumann entropy is invariant under unitary conjugate, from Eqs. (15) and (16) we conclude that
\[ J(\rho_\theta, \mathcal{N}_\gamma) = J(\rho, \mathcal{N}_\gamma), \] (17)
with \( \rho_\theta \equiv U_\theta \rho U_\theta^\dagger \). On the other hand, single letter coherent information is concave, therefore
\[ \int_0^{2\pi} J(\rho_\theta, \mathcal{N}_\gamma) p(\theta) d\theta \leq J \left( \int_0^{2\pi} \rho_\theta p(\theta) d\theta, \mathcal{N}_\gamma \right), \] (18)
for any probability distribution \( p(\theta) \). Thus from Eq. (17) and (18) it is straightforward to see that
\[ J(\rho, \mathcal{N}_\gamma) \leq J \left( \int_0^{2\pi} \rho_\theta p(\theta) d\theta, \mathcal{N}_\gamma \right). \] (19)

Then, choosing \( p(\theta) \) as flat distribution, we find
\[ \int_0^{2\pi} \rho_\theta p(\theta) d\theta = \sum_{m,n} \int_0^{2\pi} \rho_{m,n} |m, \gamma_m \rangle \langle \gamma_m | e^{i \gamma (m-n)} \frac{d\theta}{2\pi} = \sum_n \rho_{n,n} |n, \gamma_n \rangle. \] (20)
Finally, inserting this into the r.h.s. of (19) gives
\[ J(\rho, \mathcal{N}_\gamma) \leq J \left( \sum_{n=0}^\infty \rho_{n,n} |n, \gamma_n \rangle \langle \gamma_n |, \mathcal{N}_\gamma \right), \] (21)
i.e. the desired result.

As a consequence of Proposition 1, the maximization in Eq. (14) reduces to the maximization over classical probability distribution:
\[ Q(\mathcal{N}_\gamma) = \max_{p_m} \left[ S \left( \sum_{m=0}^\infty p_m |m, \gamma_m \rangle \langle \gamma_m | \right) - S \left( \sum_{m=0}^\infty p_m |\sqrt{\gamma}m, \sqrt{\gamma}m \rangle \langle \sqrt{\gamma}m | \right) \right], \] (22)

A lower bound to (22) can be found by considering an input state to be diagonal in the Fock basis and containing only two elements with equal weight, i.e.
\[ \Omega_j = \frac{1}{2} |n-j, n+j \rangle \langle n+j, n+j |, \] (23)
where \( n, j \) are arbitrary non-negative integers. In such a case it is easy to see that \( \sum_m p_m |\sqrt{\gamma}m, \sqrt{\gamma}m \rangle \langle \sqrt{\gamma}m | \) is diagonalized in the following form:
\[ \frac{1}{\sqrt{2+2e^{-\gamma j^2/2}}} \left( |\sqrt{\gamma}n + \sqrt{\gamma}(n+j) \rangle \langle \sqrt{\gamma}(n+j) | \right), \] (24)
\[ \frac{1}{\sqrt{2-2e^{-\gamma j^2/2}}} \left( |\sqrt{\gamma}n - \sqrt{\gamma}(n+j) \rangle \langle \sqrt{\gamma}(n+j) | \right), \] (25)
with eigenvalues
\[ q_\pm (j) \equiv \frac{1}{2} \left( 1 \pm e^{-\gamma j^2/2} \right). \] (26)
Thus we have
\[ J(\Omega_j, \mathcal{N}) = 1 - H_2(q_+(j), q_-(j)), \] (27)
with \( H_2 \) the binary entropy.

We note that the eigenvalues of \( \tilde{\mathcal{N}}_\gamma(\Omega_j) \) in Eq. (26) do not depend on \( n \). Furthermore, by increasing \( j \), the distance between \( q_+(j) \) and \( q_-(j) \) decreases and as a consequence \( H_2(q_+(j), q_-(j)) \) decreases too. Therefore, \( J(\Omega_j, \mathcal{N}) \) in Eq. (27) is maximized for \( j = 1 \) and a lower bound for quantum capacity is given by \( J(\Omega_1, \mathcal{N}_\gamma) \) which is obtained for input state \( \Omega_1 \) with arbitrary \( n \).

In order to obtain the quantum capacity, it is necessary to go beyond the input state (23) considering more terms in the sum and non-trivial probability distributions. The task is complicate because computing the second term of Eq. (22) requires the diagonalization of mixture of infinite number of coherent states. Hence in the next section we will use numerical tools.

IV. NUMERICAL ANALYSIS

In this section we resort to numerical techniques to evaluate the quantum capacity. First we truncate the space \( \mathcal{H}_S \) to dimension \( N + 1 \), which somehow corresponds to constraint the (average) input energy. Then, we find the following maximum numerically
\[ Q_{N+1}(\mathcal{N}_\gamma) = \max_{p_m} \left[ S \left( \sum_{m=0}^N p_m |m, \gamma_m \rangle \langle \gamma_m | \right) - S \left( \sum_{m=0}^N p_m |\sqrt{\gamma}m, \sqrt{\gamma}m \rangle \langle \sqrt{\gamma}m | \right) \right], \] (28)
and by analyzing its behaviour by increasing \( N \), we obtain the quantum capacity in Eq. (22).

For \( N = 1 \), maximizing the right hand side of Eq. (28), yields the optimal probability distribution to be uniform, that is \( p_0 = p_1 = \frac{1}{2} \), as shown in Fig. 1 together with \( Q_2 \). This implies that for \( N = 1 \) the probability distribution in Eq. (23) is optimal. It is worth noting that even by truncating the sum in equation Eq. (22), the numerical analysis is lengthy. The root of that goes back to the fact that by increasing \( N \) not only the number of involved coherent states increases, but also their amplitudes increase. In fact, by increasing \( m \), the number of required terms at the r.h.s. of (11) to be considered increases, which is equivalent to longer time for the numerical task. In the next section, we explain an algorithm which mitigates this problem.

A. Replica method

We now explain our approach for numerical calculation of \( Q_{N+1} \) in Eq. (28). Obviously, computing the first term
is straightforward. For computing the second term, we will make use of the replica method [14].

It is known that the von-Neumann entropy of a density matrix $\Omega$ can be written as

$$ S(\Omega) = -\text{Tr}(\Omega \log \Omega) = -\partial_n \text{Tr}(\Omega^n)|_{n=1}. $$

(29)

Therefore, instead of diagonalizing $\Omega$, one can compute the entropy through the trace of $\Omega^n$. For our purpose, when the input state is a mixture of number states, we denote the output of complementary channel by

$$ \Omega \equiv \sum_{m=0}^{N} p_m |m\sqrt{\gamma}\rangle \langle m\sqrt{\gamma}|, $$

(30)

and for arbitrary $n$, express $\Omega^n$ in terms of coherent states as

$$ \Omega^n = \sum_{i,j=1}^{N} C^{(n)}_{ij} |\sqrt{n}i\rangle \langle \sqrt{n}j|, $$

(31)

with $C^{(1)}_{ij} = p_i \delta_{i,j}$. It then follows that

$$ \text{Tr} (\Omega^n) = \sum_{i,j=1}^{N} C^{(n)}_{ij} e^{-2(i-j)^2}. $$

(32)

By considering that $\Omega^n = \Omega^{n-1} \Omega$ and taking into account Eqs. (30) and (31), the following recurrence relation can be derived:

$$ C^{(n)} = C^{(n-1)} A, $$

(33)

with

$$ A_{ij} = e^{-2(i-j)^2} p_j, \quad i, j = 1, \ldots, N. $$

(34)

Thus using Eq. (33) in Eq. (32) we conclude that

$$ \text{Tr} (\Omega^n) = \text{Tr} (A^n) = \sum_{i=1}^{N} a_i^n, $$

(35)

with $\{a_i\}$ the eigenvalues of the matrix $A$. Finally, from Eqs. (29) and (35), we have

$$ S(\Omega) = -\partial_n \text{Tr}(A^n)|_{n=1} = -\sum_{i=1}^{N} a_i \log a_i, $$

(36)

which implies that the numerical computation of $S(\Omega)$ can be done through the $N \times N$ matrix $A$.

By using Eq. (36) we compute the second term of Eq. (28) numerically, and optimize the whole expression over the probability distribution, $p_m$s. We find optimal values of $p_m$, as shown in Fig. 2 for $N = 2, 3, 4, 5$. Proceeding up to $N = 8$, we observed the following relation between the optimal values of $p_m$s:

$$ p_m < p_{m+1}, \quad \text{for} \quad 0 \leq m \leq \lfloor \frac{N}{2} \rfloor, $$

(37)

$$ p_m = p_{N-m}, \quad \text{for} \quad \lfloor \frac{N}{2} \rfloor < m \leq N. $$

(38)

For the obtained optimal probability distributions, $Q_{N+1}(N\gamma)$ is shown in Fig. 3 versus $\gamma$ for $N = 1, \ldots, 8$. As expected $Q_{N+1}(N\gamma)$ monotonically decreases versus the noise parameter $\gamma$.

For probability distribution with the pattern given in (37) and (38) it is straightforward to see that the mean energy of the optimal input state is $\frac{N}{2}$ which is linearly increasing by $N$.

In the next subsection we try to figure out the probability distribution that fits well with properties in Eqs. (37) and (38).

### B. Optimal probability distribution

We discuss here the actual form of optimal probability distribution. From Eqs. (37) and (38), it is concluded that the optimal probability distribution can not have more that one peak, hence bimodal probability distributions are not optimal distributions. Furthermore, Eq. (37) and (38) imply that the optimal probability distribution is symmetric around its peak at $m = \lfloor \frac{N}{2} \rfloor$. 

![FIG. 1: Top: optimal probability distribution for $N=1$ versus $\gamma$. Bottom: $Q_{N+1}(N\gamma)$ versus $\gamma$.](image-url)
FIG. 2: Optimal value of $p_m$ for $m = 0, 1, \cdots, N$ versus $\gamma$. From top to bottom $N = 2, 3, 4, 5$. 

FIG. 3: $Q_{N+1}(N_\gamma)$, as defined in Eq. (28), versus $\gamma$ for $N = 1, \ldots, 8$.

Therefore, a non-symmetric unimodal probability distribution, such as the thermal distribution, is not an acceptable candidate for optimal probability distribution in computing $Q_{N+1}$.

A candidate for discrete probability distributions satisfying these properties is the discrete Gaussian probability distribution

$$p_m(\mu, \sigma(N, \gamma)) = \frac{1}{M(\mu, \sigma)} e^{-\frac{(m-\mu)^2}{2\sigma^2(N, \gamma)}},$$

with $m \in \{0, 1, \cdots, N\}$. It is centered around $\mu = \frac{N}{2}$ and has a width controlled by $\sigma$. Furthermore, $M(\mu, \sigma)$ is the normalization factor

$$M(\mu, \sigma(N, \gamma)) = \sum_{m=0}^{N} e^{-\frac{(m-\mu)^2}{2\sigma^2(N, \gamma)}}.$$

From Eqs. (37) and (38) we know that for all values of $\gamma$, $p_m$ attains the maximum value for $m = \lfloor \frac{N}{2} \rfloor$. Therefore, we set $\mu = \frac{N}{2}$ and vary $\sigma$ to find the best fit to the optimal probability distribution obtained numerically in Sec. IV A. It is worth mentioning that for odd $N$, the maximum value of probability distribution does not pass any $p_m$, but still $p_m$ with $m = \lfloor \frac{N}{2} \rfloor$ and $m = \lfloor \frac{N}{2} \rfloor + 1$ are equal and have maximum values.

By varying $\sigma$ we can fit discrete Gaussian distribution to the optimal probability distribution obtained in Sec. IV A for $N = 1, \ldots, 5$. We observe that $\sigma$ is linear in $N$:

$$\sigma(\gamma, N) \approx a(\gamma)N + b(\gamma),$$

and for $\gamma > 0.2$ the coefficients $a(\gamma)$ and $b(\gamma)$ are almost constant, that is $\sigma \approx 0.2N + 0.6$.

By taking $p_m$s in Eq. (28) from discrete Gaussian probability distribution as in Eq. (39) with $\mu = N/2$ and numerically maximizing it over $\sigma$, we calculate $Q_{N+1}(N_\gamma)$. The obtained quantities for $N = 1, \ldots, 5$ exactly coincide...
Fig. 4 shows the behaviour of Eq. (22) versus noise parameter close approximation to the quantum capacity. This implies that for large values of $\gamma$ become closer and closer, especially at large values of $\gamma$. The matrix $A$ up to order $O(\epsilon)$ is given by

$$A_{i,j} = p_j \delta_{i,j} + \epsilon p_j \left( \delta_{i,j+1} + \delta_{i+1,j} \right) + O(\epsilon^2).$$

Therefore, by straightforward calculation, we obtain

$$\text{Tr}(A^n) = \sum_{m=0}^{N} p_m^n + O(\epsilon^2),$$

which by considering the first equality in Eq. (36) leads to $S(\Omega) = -\sum_{m=0}^{N} p_m \log p_m$ and therefore $Q_{N+1}(\mathcal{N}_\gamma)$ as defined in Eq. (28) vanishes if we keep terms up to order $\epsilon$, because

$$Q_{N+1}(\mathcal{N}_\gamma) \approx O(\epsilon^2).$$

Hence to see the asymptotic behaviour of $Q_{N+1}(\mathcal{N}_\gamma)$ for large values of $\gamma$, we write the matrix $A$ up to order $O(\epsilon^2)$:

$$A_{i,j} = p_j \delta_{i,j} + \epsilon p_j \left( \delta_{i,j+1} + \delta_{i+1,j} \right) + O(\epsilon^2).$$

Straightforward calculations give

$$\text{Tr}(A^n) = \sum_{m=0}^{N} p_m^n + n \epsilon^2 \sum_{m=0}^{N-1} \frac{p_m p_{m+1} - p_m p_m + 1}{p_m - p_m + 1},$$

which, using Eq. (36) and replacing $\epsilon^2$ by $e^{-\gamma}$, leads to

$$Q_{N+1}(\mathcal{N}_\gamma) = e^{-\gamma} \sum_{m=0}^{N-1} \frac{p_m p_{m+1} - p_m}{p_m - p_m + 1} \log \left( \frac{p_m}{p_m - p_{m+1}} \right) + O(\epsilon^3).$$

As discussed in Sec. IV.B for large values of $\gamma$, the mean value and variance of optimal probability distribution in Eq. (39) do not depend on $\gamma$. Thus, the summation in Eq. (47) does not depend on $\gamma$. Therefore Eq. (47) implies that, for large values of $\gamma$, $Q_{N+1}(\mathcal{N}_\gamma)$ and hence $Q(\mathcal{N}_\gamma)$ approach zero exponentially.

V. ASYMPTOTIC BEHAVIOUR OF QUANTUM CAPACITY

In this section we discuss the asymptotic behaviour of quantum capacity of dephasing channel in terms of the dephasing rate, or noise parameter. As seen in Sec. III, the dephasing channel is degradable, hence its quantum capacity is equal to its private classical capacity. On the other hand, the private classical capacity is always non-negative. Therefore, $Q(\mathcal{N}_\gamma)$ is always non-negative. However, the decreasing behaviour of $Q_{N+1}(\mathcal{N}_\gamma)$ suggests that $Q_{N+1}(\mathcal{N}_\gamma)$ and hence $Q(\mathcal{N}_\gamma)$ asymptotically approaches zero from above for $\gamma \to \infty$. Actually in what follows we show that for large values of $\gamma$, $Q_{N+1}(\mathcal{N}_\gamma)$ and hence the quantum capacity decrease exponentially.

While so far we have used replica method to ease the numerical analysis of the second term of $Q_{N+1}(\mathcal{N}_\gamma)$ in Eq. (28), here we use this technique to derive the behaviour of $Q_{N+1}(\mathcal{N}_\gamma)$ and of quantum capacity $Q(\mathcal{N}_\gamma)$ for large values of $\gamma$. Elements of matrix $A$ as defined in Eq. (34) are all non-zero. Define $\epsilon \equiv e^{-\gamma}$ which is small for large values of $\gamma$. The matrix $A$ up to order $O(\epsilon)$ is given by

$$A_{i,j} = p_j \delta_{i,j} + \epsilon p_j \left( \delta_{i,j+1} + \delta_{i+1,j} \right) + O(\epsilon^2).$$

VI. CONCLUSION

Summarizing, we have studied the capability for dephasing channel of transmitting quantum information. We have analytically proved that for such a channel coherent information is additive and optimal input state is diagonal in the Fock basis, which is invariant under the noise action. Then, by using replica method which makes numerical analysis technically feasible, we have determined the optimal distribution of number states to be of the Gaussian family. Interestingly, this distribution is almost independent on the noise parameter $\gamma$, but quantum capacity varies with $\gamma$, as the output of the complementary channel depends on the noise parameter. We
found useful to truncate the dimension of Hilbert space, which is equivalent to restrict the input energy, and define $Q_{N+1}(N)$ as the maximum of coherent information in truncated space. Then, we numerically evaluated the quantum capacity by finding the asymptotic behaviour of $Q_{N+1}(N)$ when enlarging the dimension of truncated Hilbert space, as it saturates to a finite value (see Figs. 3 and 4). Our results show that the optimal input state for transmitting quantum information through a continuous variable quantum dephasing channel, is a mixture of number states with discrete Gaussian distribution, which is clearly not a Gaussian state. We also discussed that quantum capacity approaches zero from above when the noise parameter increases. For large values of dephasing rate, this decay is exponential.

We are confident that this work can pave the way for studying quantum communication with continuous variable quantum channels beyond the usual restriction of Gaussianity and this might be particularly relevant to optical communications.

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