On Quantum Channels and Operations Preserving Finiteness of the von Neumann Entropy

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Abstract—We describe the class (semigroup) of quantum channels mapping states with finite entropy into states with finite entropy. We show, in particular, that this class is naturally decomposed into three convex subclasses, two of them are closed under concatenations and tensor products. We obtain asymptotically tight universal continuity bounds for the output entropy of two types of quantum channels: channels with finite output entropy and energy-constrained channels preserving finiteness of the entropy.

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1. INTRODUCTION

The output entropy $H_\Phi(\rho) = H(\Phi(\rho))$ of a quantum channel $\Phi$ is an important characteristic of this channel essentially used in analysis of its information abilities [1, 2]. For an arbitrary channel $\Phi$ between quantum systems $A$ and $B$ the output entropy $H_\Phi$ is a lower semicontinuous concave function on the set $\mathcal{S}(\mathcal{H}_A)$ of all states of the input system $A$ taking values in $[0, +\infty]$. If the function $H_\Phi$ is finite for all input states then it is bounded and continuous on the set $\mathcal{S}(\mathcal{H}_A)$ [3, Theorem 1].

In contrast to many other characteristics of quantum channels (such as the mutual and coherent informations, the constrained Holevo capacity, etc., see [4]) the output entropy $H_\Phi(\rho)$ may take infinite values at states $\rho$ with finite entropy (the simplest example is the channel transforming all input states into a single state with infinite entropy). Nevertheless, there is a large class (semigroup) of quantum channels mapping states with finite entropy into states with finite entropy, i.e. such channels $\Phi$ that

$$H(\rho) < +\infty \Rightarrow H_\Phi(\rho) < +\infty.$$  

(1)

It is shown in [5] that any channel $\Phi$ possessing property (1) preserves local continuity of the entropy: if a sequence $\{\rho_n\}$ converges to a state $\rho_0$ then

$$\lim_{n \to +\infty} H(\rho_n) = H(\rho_0) < +\infty \Rightarrow \lim_{n \to +\infty} H_\Phi(\rho_n) = H_\Phi(\rho_0) < +\infty.$$  

(2)

In this paper we describe the structure of the semigroup of all quantum channels possessing the equivalent properties (1) and (2). We show, in particular, that this semigroup is naturally decomposed into three classes $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$. Classes $\mathcal{A}$ and $\mathcal{B}$ are convex and closed under concatenations and tensor products. The class $\mathcal{C}$ contains channels of the form $\lambda \Phi + (1 - \lambda) \Psi$, $\lambda \in (0, 1)$, where $\Phi$ and $\Psi$ are channels from the classes $\mathcal{A}$ and $\mathcal{B}$ correspondingly. The class $\mathcal{C}$ is convex but not closed under concatenations.

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concatenations and tensor products. Channels from this class are characterised by existence of input state at which both the output entropy and the entropy exchange are infinite.

It is well known that the von Neumann entropy is continuous on the set
\[ \mathcal{E}_{H_A,E} = \{ \rho \in \mathcal{S}(H_A) \mid \text{Tr} H_A \rho \leq E \} \]
of states with mean energy not exceeding \( E \) if (and only if) the Hamiltonian \( H_A \) of system \( A \) satisfies the Gibbs condition \( \text{Tr} e^{-\lambda H_A} < +\infty \) for all \( \lambda > 0 \). Under this condition the set \( \mathcal{E}_{H_A,E} \) is compact [6]. So, in this case the equivalence of (1) and (2) implies that the output entropy of any channel \( \Phi \) satisfying (1) is uniformly continuous on the set \( \mathcal{E}_{H_A,E} \) for any \( E \). In this paper we obtain uniform continuity bound for the function \( \rho \mapsto H_\Phi(\rho) \) on the set \( \mathcal{E}_{H_A,E} \) in explicit form. This will be done by using the technique developed in [7] under the condition
\[ \lim_{\lambda \to 0^+} \left[ \text{Tr} e^{-\lambda H_A} \right]^{\lambda} = 1, \]
which is a slightly stronger than the Gibbs condition but holds for the Hamiltonians of many real quantum systems (see Section 2.2).

2. PRELIMINARIES

2.1. Basic Notation

Let \( \mathcal{H} \) be a separable infinite-dimensional Hilbert space, \( \mathcal{B}(\mathcal{H}) \) the algebra of all bounded operators on \( \mathcal{H} \) with the operator norm \( \| \cdot \| \) and \( \mathfrak{T}(\mathcal{H}) \) the Banach space of all trace-class operators on \( \mathcal{H} \) with the trace norm \( \| \cdot \|_\text{tr} \). Let \( \mathcal{S}(\mathcal{H}) \) be the set of quantum states (positive operators in \( \mathfrak{T}(\mathcal{H}) \) with unit trace) [1, 2, 8].

Denote by \( I_\mathcal{H} \) the unit operator on a Hilbert space \( \mathcal{H} \) and by \( \text{Id}_\mathcal{H} \) the identity transformation of the Banach space \( \mathfrak{T}(\mathcal{H}) \).

The von Neumann entropy of a quantum state \( \rho \in \mathcal{S}(\mathcal{H}) \) is defined by the formula \( H(\rho) = \text{Tr} \eta(\rho) \), where \( \eta(x) = -x \ln x \) for \( x > 0 \) and \( \eta(0) = 0 \). It is a concave lower semicontinuous function on the set \( \mathcal{S}(\mathcal{H}) \) taking values in \([0, +\infty]\) [1, 2, 8–10]. The homogeneous concave extension of the von Neumann entropy to the cone \( \mathcal{S}_+(\mathcal{H}) \) is given by the formula
\[ H(\rho) = \left[ \text{Tr} \rho \right] H\left( \frac{\rho}{\text{Tr} \rho} \right) = \text{Tr} \eta(\rho) - \eta(\text{Tr} \rho). \]

By using Theorem 11.10 in [8] it is easy to obtain the inequality
\[ H\left( \sum_k p_k \rho_k \right) \leq \sum_k p_k H(\rho_k) + S\left( \{ p_k \} \right), \]
valid for any finite or countable collection \( \{ \rho_k \} \) of positive operators in the unit ball of \( \mathfrak{T}(\mathcal{H}) \) and any probability distribution \( \{ p_k \} \), where \( S(\{ p_k \}) \) is the Shannon entropy. In particular, for arbitrary positive operators \( \rho \) and \( \sigma \) in the unit ball of \( \mathfrak{T}(\mathcal{H}) \) and any \( p \in (0, 1) \) the following inequality holds
\[ H(pp + (1 - p)\sigma) \leq p H(\rho) + (1 - p) H(\sigma) + h_2(p), \]
where \( h_2(p) = \eta(p) + \eta(1 - p) \) is the binary entropy. By using inequality (4) it is easy to show that
\[ H\left( \sum_k \rho_k \right) \leq \sum_k H(\rho_k) + S\left( \{ \text{Tr} \rho_k \} \right), \]
for any finite or countable collection \( \{ \rho_k \} \) of positive operators in the unit ball of \( \mathfrak{T}(\mathcal{H}) \) such that \( \sum_k \text{Tr} \rho_k < +\infty \), where \( S(\{ x_k \}) \) is the extended Shannon entropy of the vector \( \{ x_k \} \) from the positive cone of \( \ell_1 \) defined as \(^{1}\)
\[ S(\{ x_k \}) = \sum_k \eta(x_k) - \eta\left( \sum_k x_k \right). \]

\(^{1}\)It is easy to see that \( S \) is the homogeneous extension of the “classical” Shannon entropy defined on the set of probability distributions to the positive cone of \( \ell_1 \).
Note that an equality holds in (6) if and only if the supports of all the operators \( \rho_k \) are mutually orthogonal.\(^2\)

Uniform continuity bound for the von Neumann entropy in a finite dimensional quantum system was obtained by Fannes [11]. An optimized version of Fannes’s continuity bound obtained by Audenaert in [12] states that

\[
|H(\rho) - H(\sigma)| \leq \varepsilon \ln(d - 1) + h_2(\varepsilon), \quad \varepsilon = \frac{1}{2} ||\rho - \sigma||_1, \quad (8)
\]

for any states \( \rho \) and \( \sigma \) in a \( d \)-dimensional Hilbert space provided that \( \varepsilon \leq 1 - 1/d \).

2.2. The Set of Quantum States with Bounded Energy

One of the main results of this paper is the uniform continuity bound for the output entropy of any positive linear map possessing property (1) under the input energy constraint (Theorem 2 in Section 5.2). In this subsection we describe some preliminary results that are necessary for deriving this continuity bound.

Let \( H_A \) be a positive (semi-definite) densely defined operator on a Hilbert space \( \mathcal{H}_A \). We will assume that \( \text{Tr} H_A \rho = \sup_n \text{Tr} P_n H_A \rho \) for any positive operator \( \rho \in \mathfrak{S}(\mathcal{H}_A) \), where \( P_n \) is the spectral projector of \( H_A \) corresponding to the interval \([0, n]\). Let \( E_0 \) be the infimum of the spectrum of \( H_A \) and \( E \geq E_0 \). Then

\[
\mathfrak{C}_{H_A,E} = \{ \rho \in \mathfrak{S}(\mathcal{H}_A) | \text{Tr} H_A \rho \leq E \}
\]

is a closed convex subset of \( \mathfrak{S}(\mathcal{H}_A) \). If \( H_A \) is treated as Hamiltonian of a quantum system \( A \) then \( \mathfrak{C}_{H_A,E} \) is the set of states with the mean energy not exceeding \( E \).

It is well known that the von Neumann entropy is continuous on the set \( \mathfrak{C}_{H_A,E} \) for any \( E > E_0 \) if (and only if) the Hamiltonian \( H_A \) satisfies the condition

\[
\text{Tr} e^{-\lambda H_A} < +\infty \quad \text{for all} \quad \lambda > 0 \quad (9)
\]

and that the maximal value of the entropy on this set is achieved at the Gibbs state \( \gamma_A(E) \overset{\text{def}}{=} e^{-\lambda(E)H_A} / \text{Tr} e^{-\lambda(E)H_A} \), where the parameter \( \lambda(E) \) is determined by the equality \( \text{Tr} H_A e^{-\lambda(E)H_A} = E \text{Tr} e^{-\lambda(E)H_A} \) [10]. Condition (9) implies that \( H_A \) is an unbounded operator having discrete spectrum of finite multiplicity. It can be represented as follows

\[
H_A = \sum_{k=0}^{+\infty} E_k \langle \tau_k | \tau_k \rangle,
\]

where \( \{\tau_k\}_{k=0}^{+\infty} \) is the orthonormal basis of eigenvectors of \( H_A \) corresponding to the nondecreasing sequence \( \{E_k\}_{k=0}^{+\infty} \) of eigenvalues tending to \( +\infty \).

Consider the function

\[
F_{H_A}(E) = \sup_{\rho \in \mathfrak{C}_{H_A,E}} H(\rho) = H(\gamma_A(E)).
\]

It is easy to show that \( F_{H_A} \) is a strictly increasing concave function on \([E_0, +\infty)\) such that \( F_{H_A}(E_0) = \ln m(E_0) \), where \( m(E_0) \) is the multiplicity of \( E_0 \) [13].

In this paper we will assume that the Hamiltonian \( H_A \) satisfies the condition

\[
\lim_{\lambda \to 0^+} \left[ \text{Tr} e^{-\lambda H_A} \right]^{1/\lambda} = 1, \quad (10)
\]

which is slightly stronger than condition (9).\(^3\) Condition (10) holds if and only if

\[
F_{H_A}(E) = o(\sqrt{E}) \quad \text{as} \quad E \to +\infty, \quad (11)
\]

\(^2\)The support \( \text{supp}\rho \) of a positive trace class operator \( \rho \) is the closed subspace spanned by the eigenvectors of \( \rho \) corresponding to its positive eigenvalues.

\(^3\)In terms of the sequence \( \{E_k\} \) of eigenvalues of \( H_A \) condition (9) means that \( \lim_{k \to \infty} E_k / \ln k = +\infty \), while (10) is valid if \( \lim \inf_{k \to \infty} E_k / \ln^q k > 0 \) for some \( q > 2 \) [7, Section 2.2].
while condition \( (9) \) is equivalent to \( F_{H_A}(E) = o(E) \) as \( E \to +\infty \) [7, Section 2.2]. It is essential that condition \( (10) \) holds for the Hamiltonians of many real quantum systems [14].4)

The function
\[
\hat{F}_{H_A}(E) = F_{H_A}(E + E_0) = H(\gamma_A(E + E_0))
\]
is concave and nondecreasing on \([0, +\infty)\). Let \( \hat{F}_{H_A} \) be a continuous function on \([0, +\infty)\) such that
\[
\hat{F}_{H_A}(E) \geq \hat{F}_{H_A}(E) \quad \forall E > 0, \quad \hat{F}_{H_A}(E) = o(\sqrt{E}) \quad \text{as} \quad E \to +\infty
\]
and
\[
\hat{F}_{H_A}(E_1) < \hat{F}_{H_A}(E_2), \quad \hat{F}_{H_A}(E_1)/\sqrt{E_1} \geq \hat{F}_{H_A}(E_2)/\sqrt{E_2}
\]
for any \( E_2 > E_1 > 0 \). Sometimes we will additionally assume that
\[
\hat{F}_{H_A}(E) = \hat{F}_{H_A}(E)(1 + o(1)) \quad \text{as} \quad E \to +\infty.
\]
By property \( (11) \) the role of \( \hat{F}_{H_A} \) can be played by the function \( \hat{F}_{H_A} \) provided that the function \( E \mapsto \hat{F}_{H_A}(E)/\sqrt{E} \) is nonincreasing. In general case the existence of a function \( \hat{F}_{H_A} \) with the required properties is established in the following proposition proved in [7].

**Proposition 1.** A) If the Hamiltonian \( H_A \) satisfies condition \( (10) \) then
\[
\hat{F}_{H_A}^*(E) = \sqrt{E} \sup_{E' \geq E} \hat{F}_{H_A}(E')/\sqrt{E'}
\]
is the minimal function satisfying all the conditions in \( (12) \) and \( (13) \).

B) Let
\[
N_\uparrow[H_A](E) = \sum_{k,j:E_k + E_j \leq E} E_k^2 \quad \text{and} \quad N_\downarrow[H_A](E) = \sum_{k,j:E_k + E_j \leq E} E_k E_j
\]
for any \( E > E_0 \). If
\[
\exists \lim_{E \to +\infty} N_\uparrow[H_A](E)/N_\downarrow[H_A](E) = a > 1
\]
then
- there is \( E_* \) such that the function \( E \mapsto \hat{F}_{H_A}(E)/\sqrt{E} \) is nonincreasing for all \( E \geq E_* \) and hence \( \hat{F}_{H_A}^*(E) = \hat{F}_{H_A}(E) \) for all \( E \geq E_* \);
- \( \hat{F}_{H_A}^*(E) = (a - 1)^{-1}(\ln E)(1 + o(1)) \) as \( E \to +\infty \).

In [14] it is shown that condition \( (15) \) is valid for the Hamiltonians of many real quantum systems.

Practically, it is convenient to use functions \( \hat{F}_{H_A} \) defined by simple formulae. The example of such function \( \hat{F}_{H_A} \) satisfying all the conditions in \( (12), (13) \) and \( (14) \) in the case when \( A \) is a multimode quantum oscillator is considered in Section 5.2.

3. ON POSITIVE LINEAR MAPS PRESERVING FINITENESS OF THE ENTROPY

Many results concerning quantum channels preserving finiteness of the entropy are valid for arbitrary positive linear maps possessing this property, i.e. property \( (1) \), where the entropy \( H(\Phi(\rho)) \) of a positive operator \( \Phi(\rho) \) is defined according to formula \( (3) \).

We will call such maps (channels, operations) PFE-maps (channels, operations) for brevity.5)

4) Theorem 3 in [14] shows that \( F_{H_A}(E) = O(\ln E) \) as \( E \to +\infty \) provided that condition \( (15) \) holds.

5) In [3, 5] these maps were called PCE-maps, since they also preserve local continuity of the entropy by Theorem 1 below.
3.1. Characterization of PFE-Maps

The following theorem proved in [5] gives characterisations of the class (semigroup) of all PFE-maps.

**Theorem 1.** Let $\Phi$ be a positive linear map from $\mathcal{T}(\mathcal{H}_A)$ into $\mathcal{T}(\mathcal{H}_B)$. The following properties are equivalent:

(i) $\Phi$ preserves finiteness of the entropy, i.e. property (1) holds;
(ii) $\Phi$ preserves continuity of the entropy, i.e. property (2) holds;
(iii) the output entropy $H_\Phi(\rho)$ is bounded on the set $\text{ext}\mathcal{E}(\mathcal{H}_A)$ of pure states.\(^6\)

Theorem 1 implies, in particular, that property (1) holds if and only if

$$H^p_{\text{max}}(\Phi) = \sup_{\rho \in \text{ext}\mathcal{E}(\mathcal{H}_A)} H_\Phi(\rho) < +\infty. \quad (16)$$

The parameter $H^p_{\text{max}}(\Phi)$ will be used below in quantitative continuity analysis of the function $H_\Phi$. It can be estimated by using a concrete expression for the map $\Phi$. Below, in Section 3.2, we obtain an upper bound on $H^p_{\text{max}}(\Phi)$ by using the Kraus representation for $\Phi$.

By using inequality (4) and the spectral decomposition of an arbitrary state $\rho$ in $\mathcal{E}(\mathcal{H}_A)$ it is easy to show that

$$H_\Phi(\rho) \leq H(\rho) + H^p_{\text{max}}(\Phi). \quad (17)$$

The simplest PFE-maps are maps with finite-dimensional output space and unitary transformations, i.e. maps of the form $\Phi(\rho) = U\rho U^*$, where $U$ is an isometry from $\mathcal{H}_A$ into $\mathcal{H}_B$. More interesting examples are considered in the next subsections.

3.2. Completely Positive Maps

In this subsection we apply the criteria in Theorem 1 to the class of quantum channels and operations—completely positive trace-preserving and trace-non-increasing linear maps [1, 2, 8].

It is well known that any quantum operation (correspondingly, channel) $\Phi$ has the Kraus representation

$$\Phi(\rho) = \sum_k V_k \rho V_k^*, \quad (18)$$

where $\{V_k\}$ is a collection of linear operators from $\mathcal{H}_A$ to $\mathcal{H}_B$ such that $\sum_k V_k^* V_k \leq I_{\mathcal{H}_A}$ (correspondingly, $\sum_k V_k^* V_k = I_{\mathcal{H}_A}$). The minimal number of nonzero summands in representation (18) is called Choi rank of the operation $\Phi$ [1, 2, 8].

If a quantum operation $\Phi$ has finite Choi rank $m$ then it follows from (17) that $H_\Phi(\rho) \leq \ln m$ for any pure state $\rho$. Hence, Theorem 1 shows that $\Phi$ is a PFE-operation.

Inequality (6) implies that for any unit vector $\varphi$ in $\mathcal{H}_A$ the following inequality holds

$$H(\Phi(|\varphi\rangle\langle\varphi|)) = H\left(\sum_k V_k |\varphi\rangle\langle\varphi| V_k^*\right) \leq S\left(\{||V_k\varphi||^2\}_k\right), \quad (19)$$

where $S(\cdot)$ is the extended Shannon entropy defined in (7). Inequality (19) shows that

$$H^p_{\text{max}}(\Phi) \leq \sup_{\varphi \in \mathcal{H}_A^1} S\left(\{||V_k\varphi||^2\}_k\right) \leq S\left(\{||V_k||^2\}_k\right), \quad (20)$$

where $\mathcal{H}_A^1$ is the unit sphere of $\mathcal{H}_A$ and it is assumed that $S\left(\{||V_k||^2\}_k\right) = +\infty$ if $\sum_k ||V_k||^2 = +\infty$.

Theorem 1 implies the following conditions of the PFE-property (1) for quantum operations with infinite Choi rank.

**Corollary 1.** A quantum operation $\Phi$ having representation (18) possesses the PFE-property (1) if one of the following conditions holds:

\(^6\)Pure states are rank-one projectors—extreme points of the convex set $\mathcal{E}(\mathcal{H}_A)$.
a) the function $\varphi \mapsto S\left(\{\|V_k\varphi\|^2\}_k\right)$ is bounded on the unit sphere of $\mathcal{H}_A$;

b) $\sum_k \|V_k\|^2$ and $S\left(\{\|V_k\|^2\}_k\right)$ are finite;

c) there exists a sequence $\{h_k\}$ of nonnegative numbers such that

$$\left\|\sum_k h_k V_k^* V_k\right\| < +\infty \quad \text{and} \quad \sum_k e^{-h_k} < +\infty.$$ 

If $\text{Ran}V_k \perp \text{Ran}V_j$ for all $k \neq j$ then a) is a necessary and sufficient condition of property (1) for the operation $\Phi$.

**Proof.** By Theorem 1 the sufficiency of conditions a) and b) follows from inequality (20). The necessity of condition a) in the case $\text{Ran}V_k \perp \text{Ran}V_j$ follows from the remark after inequality (6).

To prove the implication c) $\Rightarrow$ a) it suffices to note that the extended Shannon entropy is bounded on the subset of the positive part of the unit ball in $\ell_1$ consisting of vectors $\{p_k\}$ such that $\sum_k h_k p_k \leq C$ for any $C > 0$.

**Remark 1.** Condition b) in Corollary 1 is the most easily verified but is too rough because it depends only on the norms of the Kraus operators. Condition c) is more subtle, since it takes “geometry” of the sequence $\{V_k\}$ into account. This is confirmed by the following example.

**Example 1.** Let $\{P_k\}_{k=1}^\infty$ be any sequence of mutually orthogonal projectors in $\mathfrak{B}(\mathcal{H}_A)$ and $\alpha \in [0, \ln 2]$. Consider the quantum channel $\Phi_\alpha(\rho) = \sum_{k \geq 1} c_k P_k \rho P_k$, where $c_1 = 1$, $c_k = \alpha / \ln k$ for $k > 1$ and $P_1 = \sqrt{I_{\mathcal{H}_A} - \sum_{k=1}^\infty c_k P_k}$. Condition c) in Corollary 1 shows that $\Phi_\alpha$ is a PFE-channel, while condition b) is not valid in this case.

Sometimes it is possible to prove PFE-property (1) of a quantum channel (operation) without using its Kraus representation.

**Example 2.** Let $\mathcal{H}_a$ be the Hilbert space $L_2([-a, +a])$, where $a < +\infty$, and $\{U_t\}_{t \in \mathbb{R}}$ be the group of unitary operators in $\mathcal{H}_a$ defined as follows

$$(U_t \varphi)(x) = e^{-itx} \varphi(x), \quad \forall \varphi \in \mathcal{H}_a.$$ 

For given probability density function $p(t)$ consider the quantum channel

$$\mathfrak{I}(\mathcal{H}_a) \ni \rho \mapsto \Phi^a_p(\rho) = \int_{-\infty}^{+\infty} U_t \rho U_t^* p(t) dt \in \mathfrak{I}(\mathcal{H}_a).$$

One can show that the function $H_{\Phi^a_p}$ is bounded (and continuous) on the set $\text{ext} \mathfrak{G}(\mathcal{H}_a)$ provided that the differential entropy of the distribution $p(t)$ is finite and that the function $p(t)$ is bounded and monotone on $(-\infty, -b)$ and on $[+b, +\infty)$ for sufficiently large $b$ [3, Example 3]. So, in this case $\Phi^a_p$ is a PFE-channel with infinite Choi rank.

### 3.3. Types of PFE-Channels and Tensor Products

It follows from the definition that the class of PFE-channels is closed under composition: if $\Phi : A \to B$ and $\Psi : B \to C$ are PFE-channels then $\Psi \circ \Phi$ is a PFE-channel. Inequality (5) shows that any convex mixture of PFE-channels between given quantum systems is a PFE-channel. But the class of PFE-channels is not closed under tensor products: the tensor product of the identity channel and the completely depolarising channel with a pure output state is not a PFE-channel. Moreover, it is easy to see that the tensor square of the PFE-channel $\rho \mapsto (1 - p) \rho + ps$, where $\sigma$ is a given pure state, is not a PFE-channel. On the other hand, there are nontrivial PFE-channels $\Phi$ and $\Psi$ such that $\Phi \otimes \Psi$ is a PFE-channel (see below). Note also that the PFE-property (1) of a channel $\Phi \otimes \Psi$ obviously implies the same property of the channels $\Phi$ and $\Psi$.

To analyse the PFE-property (1) of tensor products we will introduce a classification of PFE-channels based on the notion of a complementary channel.
For any quantum channel $\Phi : A \to B$ the Stinespring theorem implies existence of a Hilbert space $\mathcal{H}_{E}$ and of an isometry $V : \mathcal{H}_{A} \to \mathcal{H}_{B} \otimes \mathcal{H}_{E}$ such that $\Phi(\rho) = \text{Tr}_{E}V\rho V^{*}$, $\rho \in \mathcal{S}(\mathcal{H}_{A})$. The quantum channel
\[
\mathcal{I}(\mathcal{H}_{A}) \ni \rho \mapsto \hat{\Phi}(\rho) = \text{Tr}_{B}V\rho V^{*} \in \mathcal{I}(\mathcal{H}_{E})
\]
is called complementary to the channel $\Phi$ [1, Ch. 6].

Since the functions $H_{\Phi}$ and $H_{\hat{\Phi}}$ coincide on the sets of pure states in $\mathcal{S}(\mathcal{H}_{A})$, Theorem 1 implies the following

**Corollary 2.** $\Phi$ is a PFE-channel if and only if $\hat{\Phi}$ is a PFE-channel.

The function $H_{\hat{\Phi}}$ is an important entropic characteristic of a channel $\Phi$ called the entropy exchange of $\Phi$ [1, 2].

**Proposition 2.** Let $A$ be an infinite-dimensional quantum system. Any PFE-channel $\Phi : A \to B$ belongs to one of the classes $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ characterized, respectively, by the conditions:

a) $H_{\Phi}(\rho) < +\infty$ for any state $\rho \in \mathcal{S}(\mathcal{H}_{A})$;

b) $H_{\hat{\Phi}}(\rho) < +\infty$ for any state $\rho \in \mathcal{S}(\mathcal{H}_{A})$;

c) $\sup_{\rho \in \text{ext}\mathcal{S}(\mathcal{H}_{A})} H_{\Phi}(\rho) < +\infty$, but $H_{\Phi}(\rho) = H_{\hat{\Phi}}(\rho) = +\infty$ for some state $\rho \in \mathcal{S}(\mathcal{H}_{A})$.

The classes $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ are convex. Convex mixtures of channels from different classes belong to the class $\mathcal{C}$. The classes $\mathcal{A}$ and $\mathcal{B}$ are closed under composition, while the class $\mathcal{C}$ is not.

**Proof.** By Theorem 1 in [3] condition a) (correspondingly, b)) implies boundedness and continuity of the function $H_{\Phi}$ (correspondingly, $H_{\hat{\Phi}}$) on the set $\mathcal{S}(\mathcal{H}_{A})$. So, any of these conditions implies the PFE-property (1) of the channel $\Phi$. The inequality
\[
H(\rho) \leq H_{\Phi}(\rho) + H_{\hat{\Phi}}(\rho), \quad \rho \in \mathcal{S}(\mathcal{H}_{A})
\]
(which follows from subadditivity of the entropy) shows that the classes $\mathcal{A}$ and $\mathcal{B}$ are disjoint. The convexity of all the classes can be established by using basic properties of the entropy and the relation
\[
H_{\hat{\Phi}}(\rho) = H_{\Phi \otimes \text{Id}_{R}}(\hat{\rho}), \quad \rho \in \mathcal{S}(\mathcal{H}_{A}),
\]
where $\hat{\rho}$ is a purification of $\rho$ in $\mathcal{S}(\mathcal{H}_{AR})$ [1].

The closedness of the class $\mathcal{A}$ under composition is obvious. To prove the same property of the class $\mathcal{B}$ we will use Proposition 3 below (proved independently).

Assume that $\Phi : A \to B$ and $\Psi : B \to C$ are PFE-channels of the class $\mathcal{B}$. Let $R$ be an infinite-dimensional quantum system. By Proposition 3A $\Phi \otimes \text{Id}_{R}$ and $\Psi \otimes \text{Id}_{R}$ are PFE-channels. So, $[\Psi \otimes \text{Id}_{R}] \circ [\Phi \otimes \text{Id}_{R}] = [\Psi \circ \Phi] \otimes \text{Id}_{R}$ is a PFE-channel. By Proposition 3B $\Psi \circ \Phi$ is a PFE-channel from the class $\mathcal{B}$.

To show that the class $\mathcal{C}$ is not closed under composition consider the PFE-channels
\[
\Phi(\rho) = P_{\rho}P \oplus [\text{Tr}\hat{P}\rho]\sigma \quad \text{and} \quad \Psi(\rho) = [\text{Tr}\rho]\varsigma \oplus P_{\rho}\hat{P},
\]
where $P$ and $\hat{P} = I_{\mathcal{H}} - P$ are infinite rank projectors, $\sigma$ and $\varsigma$ are pure states such that $\hat{P}\sigma\hat{P} = \sigma$ and $P\varsigma P = \varsigma$. These channels belong to the class $\mathcal{C}$. The simplest way to show this is to note that $\Phi \otimes 2$ and $\Psi \otimes 2$ are not PFE-channels and to use Proposition 3A below. It is easy to see that $\Psi \circ \Phi$ is a completely depolarizing channel belonging to the class $\mathcal{A}$.

The class $\mathcal{A}$ consists of channels with continuous output entropy. The simplest example of such channels is the completely depolarizing channel $\rho \mapsto [\text{Tr}\rho]\sigma$, where $\sigma$ is a given state with finite entropy. More interesting channels from the class $\mathcal{A}$ are considered in [3, Section 3].

The class $\mathcal{B}$ contains the identity channel and all channels with finite Choi rank. The channel from the class $\mathcal{B}$ having infinite Choi rank is presented in the above Example 2. The finiteness of the function

\[
7) \text{in the sense described at the begin of this subsection.}
\]
output entropy plays an important role [1, 15]. The function \( \Phi \) is defined as the maximal closed (lower semicontinuous) convex function on \( \mathcal{S}(\mathcal{H}_A) \) majorized by the function \( H_{\Phi} \). In finite dimensions \( \mathcal{S}(\mathcal{H}_\Phi) \) coincides with the convex hull \( \text{co} H_{\Phi} \) of \( H_{\Phi} \), i.e., the maximal convex function on \( \mathcal{S}(\mathcal{H}_A) \) majorized by the function \( H_{\Phi} \) which is given by the formula

\[
\text{co} H_{\Phi}(\rho) = \inf_{\sum_i p_i = 1} \sum_i p_i H_{\Phi}(\rho_i),
\]

where the infimum is over all finite ensembles \( \{p_i, \rho_i\} \) of input states with the average state \( \rho \).

In infinite dimensions the function \( \mathcal{S}(\mathcal{H}_\Phi) \) coincides with \( \text{co} H_{\Phi} \) only for positive maps (channels) with finite output entropy, but one can assume that it coincides with the \( \sigma \)-convex hull \( \sigma \text{-co} H_{\Phi} \) of \( H_{\Phi} \) defined by formula (21) in which the infimum is over all countable ensembles \( \{p_i, \rho_i\} \) of input states with the average state \( \rho \).

On the other hand, the compactness criterion for families of probability measures on \( \mathcal{S}(\mathcal{H}) \) makes it possible to show that

\[
\mathcal{S}(\mathcal{H}_\Phi) = \inf_{\rho(\mu) = \rho} \int H_{\Phi}(\mu) d\mu,
\]

where the infimum is over all Borel probability measures on the set \( \mathcal{S}(\mathcal{H}_A) \) with the barycenter of \( \rho \) [15]. So, to prove the conjecture \( \sigma \text{-co} H_{\Phi} = \mathcal{S}(\mathcal{H}_\Phi) \) it suffices to show that the infimum in (22) can be taken

4. THE CONVEX CLOSURE OF THE OUTPUT ENTROPY

4.1. General Results

In analysis of information properties of a quantum channel \( \Phi \) the convex closure \( \mathcal{S}(\mathcal{H}_\Phi) \) of its output entropy plays an important role [1, 15]. The function \( \mathcal{S}(\mathcal{H}_\Phi) \) is defined as the maximal closed (lower semicontinuous) convex function on \( \mathcal{S}(\mathcal{H}_A) \) majorized by the function \( H_{\Phi} \). In finite dimensions \( \mathcal{S}(\mathcal{H}_\Phi) \) coincides with the convex hull \( \text{co} H_{\Phi} \) of \( H_{\Phi} \), i.e., the maximal convex function on \( \mathcal{S}(\mathcal{H}_A) \) majorized by the function \( H_{\Phi} \) which is given by the formula

\[
\text{co} H_{\Phi}(\rho) = \inf_{\sum_i p_i = 1} \sum_i p_i H_{\Phi}(\rho_i),
\]

where the infimum is over all finite ensembles \( \{p_i, \rho_i\} \) of input states with the average state \( \rho \).

In infinite dimensions the function \( \mathcal{S}(\mathcal{H}_\Phi) \) coincides with \( \text{co} H_{\Phi} \) only for positive maps (channels) with finite output entropy, but one can assume that it coincides with the \( \sigma \)-convex hull \( \sigma \text{-co} H_{\Phi} \) of \( H_{\Phi} \) defined by formula (21) in which the infimum is over all countable ensembles \( \{p_i, \rho_i\} \) of input states with the average state \( \rho \).

On the other hand, the compactness criterion for families of probability measures on \( \mathcal{S}(\mathcal{H}) \) makes it possible to show that

\[
\mathcal{S}(\mathcal{H}_\Phi) = \inf_{\rho(\mu) = \rho} \int H_{\Phi}(\mu) d\mu,
\]

where the infimum is over all Borel probability measures on the set \( \mathcal{S}(\mathcal{H}_A) \) with the barycenter of \( \rho \) [15]. So, to prove the conjecture \( \sigma \text{-co} H_{\Phi} = \mathcal{S}(\mathcal{H}_\Phi) \) it suffices to show that the infimum in (22) can be taken.
only over all discrete probability measures. We don’t know how to prove (or disprove) this conjecture in general, but it seems reasonable to mention that it is true for PFE-channels.

**Corollary 4.** Let $\Phi : \mathcal{S}(\mathcal{H}_A) \to \mathcal{S}(\mathcal{H}_B)$ be a positive map possessing PFE-property (1). Then

A) $\sigma$-co$\Phi(\rho) = \varpi\Phi(\rho)$ for any state $\rho \in \mathcal{S}(\mathcal{H}_A)$;

B) the function $\sigma$-co$\Phi$ is continuous and bounded on $\mathcal{S}(\mathcal{H}_A)$.

**Proof.** Since the assumption implies, by Theorem 1, continuity and boundedness of the function $\Phi$ on the set $\text{ext}\mathcal{S}(\mathcal{H}_A)$, both assertions of the corollary can be derived from Corollary 2 in [15] by using Remark 3 below.

**Remark 3.** By using concavity and lower semicontinuity of the function $\Phi$ one can show that $\sigma$-co$\Phi = \hat{H}_\Phi^\sharp$ and $\varpi\Phi = \hat{H}_\Phi^\dagger$, where $\hat{H}_\Phi^\sharp$ and $\hat{H}_\Phi^\dagger$ are discrete and continuous convex roof extensions of the function $H_\Phi$ defined, respectively, by the right hand sides of (21) and (22) in which the infima are over all ensembles (measures) consisting of (supported by) pure states [15, Section 2.3]. Thus, the assertion of Corollary 4 can be reformulated in terms of the functions $\hat{H}_\Phi^\sharp$ and $\hat{H}_\Phi^\dagger$ (instead of $\sigma$-co$\Phi$ and $\varpi\Phi$).

#### 4.2. Applications to the Entanglement Theory

Important task of the entanglement theory consists in finding appropriate characteristics of entanglement of composite states and in exploring their properties [16, 17].

If $\rho_{AB}$ is a pure state of a bipartite system $AB$ of any dimension then its entanglement is characterized by the von Neumann entropy of partial states: $E(\rho_{AB}) = H(\rho_A) = H(\rho_B)$. Entanglement of mixed states of a bipartite system $AB$ is characterized by different entanglement measures [18–20]. One of the most important of them is the Entanglement of Formation (EoF). In the case of finite-dimensional bipartite system $AB$ the EoF is defined as the convex roof extension to the set $\mathcal{S}(\mathcal{H}_{AB})$ of the function $\rho_{AB} \mapsto H(\rho_A)$ on the set $\text{ext}\mathcal{S}(\mathcal{H}_{AB})$ of pure states, i.e.

$$E_F(\rho_{AB}) = \inf_{\sum_i p_i \rho_{iAB} = \rho_{AB}} \sum_i p_i H(\rho_A^i),$$

where the infimum is over all ensembles $\{p_i, \rho_{iAB}\}$ of pure states with the average state $\rho_{AB}$.

In infinite dimensions there are two versions $E_F^d$ and $E_F^c$ of the EoF defined, respectively, by using discrete and continuous convex roof extensions, i.e.

$$E_F^d(\rho_{AB}) = \inf_{\sum_i p_i \rho_{iAB} = \rho_{AB}} \sum_i p_i H(\rho_A^i), \quad E_F^c(\rho_{AB}) = \inf_{b(\mu) = \rho_{AB}} \int H(\varrho_A) \mu(d\varrho_{AB}),$$

where the first infimum is over all countable convex decompositions of the state $\rho_{AB}$ into pure states and the second one is over all Borel probability measures on the set $\text{ext}\mathcal{S}(\mathcal{H}_{AB})$ with the barycenter $\rho_{AB}$ [15, Section 5].

The continuous version $E_F^c$ is a lower semicontinuous function on the set $\mathcal{S}(\mathcal{H}_{AB})$ of all states of infinite-dimensional bipartite system possessing basic properties of entanglement measures (including monotonicity under generalized selective measurements) [15]. The discrete version $E_F^d$ seems more preferable from the physical point of view but the assumption $E_F^d \neq E_F^c$ leads to several problems with this version, in particular, it is not clear how to prove its vanishing for countably non-decomposable separable states.\(^{10}\)

In [15] it is shown that $E_F^d(\rho_{AB}) = E_F^c(\rho_{AB})$ for any state $\rho_{AB}$ such that

$$\min\{H(\rho_A), H(\rho_B), H(\rho_{AB})\} < +\infty,$$

\(^{10}\)It is shown in [15] that $\sigma$-co$\neq \varpi$ for a particular lower semicontinuous concave nonnegative unitarily invariant function $f$ on $\mathcal{S}(\mathcal{H})$, so the above conjecture can not be proved by using only general entropy-type properties of the function $H_\Phi$.

\(^{9}\)The convex roof extension is widely used for construction of different characteristics of states in finite-dimensional quantum systems [17, 19].

\(^{10}\)In general, the discrete convex roof construction applied to an entropy type function (in the role of $H$) may give a function which is not equal to zero at countably non-decomposable separable states [15, Remark 6].
but the coincidence of $E_F^d$ and $E_F^c$ on the whole set $\mathcal{G}(H_{AB})$ is not proved yet (as far as we know). It is equivalent to the lower semicontinuity of $E_F^d$ on $\mathcal{G}(H_{AB})$ (since $E_F^c$ coincides with the convex closure of the entropy of a partial trace).

By applying Corollary 4 and Remark 3 to the channel $\Phi(\rho_{AB}) = \rho_A$ we obtain the following

**Proposition 4.** Let $\mathcal{K}$ be a subspace of $\mathcal{H}_{AB}$ such that all unit vectors in $\mathcal{K}$ have bounded entanglement, i.e. $\sup_{\varphi \in \mathcal{K}, ||\varphi|| = 1} E(|\varphi\rangle\langle\varphi|) < +\infty$. Then

A) $E_F^c(\rho) = E_F^d(\rho)$ for any state $\rho$ in $\mathcal{G}(\mathcal{K})$;

B) the function $E_F^c = E_F^d$ is continuous on the set $\mathcal{G}(\mathcal{K})$.

The existence of nontrivial subspaces satisfying the condition of Proposition 4 follows, by the Stinespring representation, from the existence of PFE-channels of the class $\mathfrak{B}$ with infinite Choi rank and of PFE-channels of the class $\mathcal{C}$ (see Section 3).

5. **UNIFORM CONTINUITY BOUNDS FOR THE OUTPUT ENTROPY**

5.1. **Positive Linear Maps with Finite Output Entropy**

Assume that $\Phi$ is a positive trace-non-increasing linear map from $\mathcal{S}(H_A)$ to $\mathcal{S}(H_B)$ such that

$$H_\Phi(\rho) = H(\Phi(\rho)) < +\infty$$

for any state $\rho$ in $\mathcal{S}(H_A)$. (23)

The simplest example of such a map is a map with finite-dimensional output system, i.e. when $\dim H_B < +\infty$. Nontrivial examples of quantum channels satisfying condition (23) are considered in [3].

By Theorem 1 in [3] for any positive trace-non-increasing linear map $\Phi$ possessing property (23) the function $H_\Phi$ is continuous and bounded on the whole set $\mathcal{S}(H_A)$ of input states. The concavity of the entropy and inequality (5) imply that

$$0 \leq H_\Phi(p\rho + (1-p)\sigma) - (pH_\Phi(\rho) + (1-p)H_\Phi(\sigma)) \leq h_2(p), \quad p \in [0,1],$$

for any states $\rho$ and $\sigma$ in $\mathcal{S}(H_A)$, where $h_2(p)$ is the binary entropy.

Note also that for any states $\rho$ and $\sigma$ in $\mathcal{S}(H_A)$ we have

$$|H_\Phi(\rho) - H_\Phi(\sigma)| \leq H_{\max}(\Phi) - H_{\min}(\Phi),$$

(25)

where

$$H_{\min}(\Phi) = \inf_{\rho \in \mathcal{S}(H_A)} H_\Phi(\rho) \geq 0 \quad \text{and} \quad H_{\max}(\Phi) = \sup_{\rho \in \mathcal{S}(H_A)} H_\Phi(\rho) < +\infty.$$

Inequalities (24) and (25) allow to apply the Alicki–Fannes–Winter method (proposed in the optimal form in [13] and described in a full generality in the proof of Proposition 1 in [22]) to the function $H_\Phi(\rho)$. As a result we obtain

**Proposition 5.** Let $\Phi : \mathcal{S}(H_A) \to \mathcal{S}(H_B)$ be a positive trace-non-increasing linear map possessing property (23). Then

$$|H_\Phi(\rho) - H_\Phi(\sigma)| \leq \varepsilon[H_{\max}(\Phi) - H_{\min}(\Phi)] + g(\varepsilon)$$

(26)

for any states $\rho$ and $\sigma$ in $\mathcal{S}(H_A)$, where $\varepsilon = \frac{1}{2} ||\rho - \sigma||_1$ and $g(\varepsilon) = (1 + \varepsilon)h_2\left(\frac{\varepsilon}{1+\varepsilon}\right)$.

**Remark 4.** The l.h.s. of (26) can be made arbitrarily close to $H_{\max}(\Phi) - H_{\min}(\Phi)$ by appropriate choice of $\rho$ and $\sigma$. Since $\frac{1}{2} ||\rho - \sigma||_1 \leq 1$ for any $\rho$ and $\sigma$, this shows that continuity bound (26) is close to tight when $H_{\max}(\Phi) - H_{\min}(\Phi) \gg g(1) = 2\ln 2$.

If $d_B = \dim H_B < +\infty$ then the quantity $H_{\max}(\Phi)$ in (26) can be replaced by $k \log d_B$, where $k = \sup_{\rho \in \mathcal{S}(H_A)} \text{Tr} \Phi(\rho)$. In this case continuity bound for the function $H_\Phi$ can be also obtained by using Audenaert’s continuity bound (8). But if the quantity $H_{\max}(\Phi) - H_{\min}(\Phi)$ is less then $\log(d_B - 1)$ then continuity bound (26) is sharper than the continuity bound for $H_\Phi$ obtained via (8), since the functions $h_2(\varepsilon)$ and $g(\varepsilon)$ are equivalent for small $\varepsilon$.

**Remark 5.** For many quantum channels $\Phi$ the quantity $H_{\max}(\Phi) - H_{\min}(\Phi)$ coincides with the Holevo capacity of $\Phi$ that gives the ultimate rate of transmission of classical information through this channel when non-entangled input encoding is used (in many cases it coincides with the classical capacity of $\Phi$) [1, 2]. This holds, in particular, for quantum channels covariant w.r.t. irreducible representation of a unitary group [23, 24].
5.2. PFE-Maps on the Set of States with Bounded Energy

By Theorem 1 the output entropy $H_\Phi$ of any positive linear map $\Phi : \mathfrak{X}(\mathcal{H}_A) \to \mathfrak{X}(\mathcal{H}_B)$ preserving finiteness of the entropy is continuous on any subset of $\mathfrak{S}(\mathcal{H}_A)$, where the von Neumann entropy is continuous.

Assume that $H_A$ is the Hamiltonian of a system $A$ with the minimal energy $E_0$ satisfying condition (9). Then the von Neumann entropy is continuous on the set $\mathfrak{C}_{H_A,E}$ of states $\rho$ with mean energy $\text{Tr} \ H_A \rho$ non exceeding $E$ (see Section 2.2). Hence the output entropy $H_\Phi$ of any positive PFE-map is continuous on $\mathfrak{C}_{H_A,E}$. Moreover, it is uniformly continuous on $\mathfrak{C}_{H_A,E}$, since this set is compact [6].

We will obtain uniform continuity bound for the output entropy of any trace-non-increasing PFE-map $\Phi : \mathfrak{X}(\mathcal{H}_A) \to \mathfrak{X}(\mathcal{H}_B)$ on the set $\mathfrak{C}_{H_A,E}$ assuming that the Hamiltonian $H_A$ satisfies the condition (10), which is slightly stronger than condition (9), but holds for many real quantum systems.

In the following theorem we use the notation introduced in Section 2.2. We assume that $\hat{F}_{H_A}$ is any continuous function on $\mathbb{R}_+$ satisfying conditions (12) and (13), $d_0$ is the minimal natural number such that $\text{ln} \ d_0 > \hat{F}_{H_A}(0)$ and $\gamma(d) = \hat{F}_{H_A}(\ln d)$ for any $d \geq d_0$.

**Theorem 2.** Let $\Phi : \mathfrak{X}(\mathcal{H}_A) \to \mathfrak{X}(\mathcal{H}_B)$ be a positive trace non-increasing linear map satisfying condition (1) and $\Delta = H_{\max}^p(\Phi) + 1/d_0 + \ln 2$, where $H_{\max}^p(\Phi)$ is the parameter defined in (16). Let $\bar{E} = E - E_0 > 0$ and $\epsilon > 0$. Then

$$|H_\Phi(\rho) - H_\Phi(\sigma)| \leq \epsilon(1 + 4t) \left[ \hat{F}_{H_A} \left( \frac{\bar{E}}{(\epsilon t)^2} \right) + \Delta \right] + 2g(\epsilon t) + g(\epsilon(1 + 2t))$$

(27)

for any states $\rho$ and $\sigma$ in $\mathfrak{S}(\mathcal{H}_A)$ such that $\text{Tr} \ H_A \rho, \text{Tr} \ H_A \sigma \leq E$ and $\frac{1}{2}|\rho - \sigma|_1 \leq \epsilon$ and any $t \in (0, T]$, where $T = (1/\epsilon) \min \{1, \sqrt{E/\gamma(d_0)}\}$ and $g(x) = (1 + x)h_2 \left( \frac{x}{1+x} \right)$.

If conditions (14) and (15) hold $^{11}$ then the r.h.s. of (27) can be written as

$$\epsilon(1 + 4t) \left( \text{ln} \left[ \frac{\bar{E}}{(\epsilon t)^2} \right] \frac{1 + o(1)}{a - 1} + \Delta \right) + 2g(\epsilon t) + g(\epsilon(1 + 2t)), \quad \epsilon \to 0^+.$$

(28)

and continuity bound (27) with optimal $t$ is asymptotically tight for large $E$. $^{12}$

*Proof.* We will use the technique from the proof of Theorem 1 in [7].

Note first that inequality (17) implies that

$$H_\Phi(\rho) \leq \hat{F}_{H_A}(E(\rho) - E_0) + H_{\max}^p(\Phi)$$

(29)

for any state $\rho$ in $\mathfrak{S}(\mathcal{H}_A)$ with finite $E(\rho) \equiv \text{Tr} \ H_A \rho$.

Let $\rho$ and $\sigma$ be states in $\mathfrak{S}(\mathcal{H}_A)$ such that $\text{Tr} \ H_A \rho, \text{Tr} \ H_A \sigma \leq E$ and $\frac{1}{2}|\rho - \sigma|_1 \leq \epsilon$. By Lemma 1 in [7] (with trivial system $B$) for any $d > d_0$ such that $\bar{E} \leq \gamma(d)$ there exist states $\varrho, \varsigma, \alpha_k, \beta_k, k = 1, 2$, in $\mathfrak{S}(\mathcal{H}_A)$ and numbers $p, q \leq \sqrt{E/\gamma(d)}$ such that rank $\varrho$, rank $\varsigma \leq d$, $\text{Tr} H_A \varrho, \text{Tr} H_A \varsigma \leq E$, $\frac{1}{2}|\rho - \varrho|_1 \leq p, \frac{1}{2}|\sigma - \varsigma|_1 \leq q$, $\text{Tr} H_A \alpha_k \leq E/p^2$, $\text{Tr} H_A \beta_k \leq E/q^2$, $k = 1, 2$, and

$$\begin{align*}
(1 - p') \rho + p' \alpha_2 &= (1 - p') \rho + p' \alpha_2, \\
(1 - q') \sigma + q' \beta_2 &= (1 - q') \sigma + q' \beta_2,
\end{align*}$$

(30)

where $\bar{H}_A = H_A - E_0 I_A$, $p' = \frac{1}{1+p}$ and $q' = \frac{1}{1+q}$. If rank $\rho \leq d$ we assume that $\varrho = \rho$ and do not introduce the states $\alpha_k$. Similar assumption holds if rank $\varsigma \leq d$.

All the states $\varrho, \varsigma, \alpha_1, \alpha_2, \beta_1$ and $\beta_2$ have finite entropy. Hence the function $H_\Phi$ is finite at these states. By using the first relation in (30) and inequality (24) it is easy to show that

$$H_\Phi(\rho) - H_\Phi(\varrho) - H_\Phi(\varsigma) \leq p'(H_\Phi(\alpha_2) - H_\Phi(\alpha_1)) + h_2(p')$$

By Proposition 1 in Section 2.2 this holds, in particular, if $\hat{F}_{H_A} = \hat{F}_{H_A}$. $^{11}$

$^{11}$A continuity bound $\sup_{x,y \in X_a} |f(x) - f(y)| \leq B_a(x,y)$ depending on a parameter $a$ is called asymptotically tight for large $a$ if $\lim_{a \to +\infty} \sup_{x,y \in X_a} |f(x) - f(y)| = 1$. $^{12}$
and 
\[(1 - p')(H_\Phi(q) - H_\Phi(\rho)) \leq p'(H_\Phi(\alpha_1) - H_\Phi(\alpha_2)) + h_2(p').\]

These inequalities imply that 
\[|H_\Phi(q) - H_\Phi(\rho)| \leq p|H_\Phi(\alpha_2) - H_\Phi(\alpha_1)| + g(p).\]  

Similarly, by using the second relation in (30) and inequality (24) we obtain 
\[|H_\Phi(\varsigma) - H_\Phi(\sigma)| \leq q|H_\Phi(\beta_2) - H_\Phi(\beta_1)| + g(q).\]  

Since \(\text{Tr} \tilde{H}_A \alpha_k \leq \tilde{E}/p^2\) and \(\text{Tr} \tilde{H}_A \beta_k \leq \tilde{E}/q^2, k = 1, 2\), it follows from (29) that 
\[|H_\Phi(\alpha_2) - H_\Phi(\alpha_1)| \leq \tilde{F}_{H_A} \left(\tilde{E}/p^2\right) + H_{\max}^p(\Phi)\]  
and 
\[|H_\Phi(\beta_2) - H_\Phi(\beta_1)| \leq \tilde{F}_{H_A} \left(\tilde{E}/q^2\right) + H_{\max}^p(\Phi).\]  

Since \(p, q \leq y = \sqrt{\tilde{E}/\gamma(d)}\) and the function \(E \mapsto \tilde{F}_{H_A}(E)/\sqrt{E}\) is non-increasing, we have 
\[x \tilde{F}_{H_A} \left(\tilde{E}/x^2\right) \leq y \tilde{F}_{H_A} \left(\tilde{E}/y^2\right) = \sqrt{\tilde{E}/\gamma(d)} \tilde{F}_{H_A}(\gamma(d)) = \sqrt{\tilde{E}/\gamma(d)} \ln d,\]  
x = \(p, q\), where the last equality follows from the definition of \(\gamma(d)\). Thus, it follows from (31)–(34) and the monotonicity of the function \(g(x)\) that 
\[|H_\Phi(q) - H_\Phi(\rho)|, |H_\Phi(\varsigma) - H_\Phi(\sigma)| \leq \sqrt{\tilde{E}/\gamma(d)} \left(\ln d + H_{\max}^p(\Phi)\right) + g \left(\sqrt{\tilde{E}/\gamma(d)}\right).\]  

Since rank \(\rho \leq d\) and rank \(\varsigma \leq d\), the supports of both states \(\rho\) and \(\varsigma\) are contained in some 2\(d\)-dimensional subspace of \(H_A\). By the triangle inequality we have 
\[||\rho - \varsigma||_1 \leq ||\rho - \rho||_1 + ||\varsigma - \sigma||_1 + ||\rho - \sigma||_1 \leq 2\varepsilon + 4\sqrt{\tilde{E}/\gamma(d)}.\]  

So, by using the Alički–Fannes–Winter method (mentioned in Section 5.1) and inequality (17) one can show that 
\[|H_\Phi(\rho) - H_\Phi(\varsigma)| \leq \left(2\sqrt{\tilde{E}/\gamma(d)} + \varepsilon\right) \left(\ln(2d) + H_{\max}^p(\Phi)\right) + g \left(2\sqrt{\tilde{E}/\gamma(d)} + \varepsilon\right).\]  

It follows from (35) and (36) that 
\[|H_\Phi(\rho) - H_\Phi(\sigma)| \leq \left(4\sqrt{\tilde{E}/\gamma(d)} + \varepsilon\right) \left(\ln d + H_{\max}^p(\Phi)\right) + \left(2\sqrt{\tilde{E}/\gamma(d)} + \varepsilon\right) \ln 2 + g \left(2\sqrt{\tilde{E}/\gamma(d)} + \varepsilon\right) + 2g \left(\sqrt{\tilde{E}/\gamma(d)}\right).\]  

If \(t \in (0, T]\) then, since the sequence \(\gamma(d)\) is increasing, there is a natural number \(d_* > d_0\) such that \(\gamma(d_*) > \tilde{E}/(\varepsilon t)^2 \geq \tilde{E}\) but \(\gamma(d_* - 1) \leq \tilde{E}/(\varepsilon t)^2\). It follows that 
\[\sqrt{\tilde{E}/\gamma(d_*)} \leq \varepsilon t \leq 1 \quad \text{and} \quad \ln(d_* - 1) = \tilde{F}_{H_A}(\gamma(d_* - 1)) \leq \tilde{F}_{H_A}(\tilde{E}/(\varepsilon t)^2),\]  
where the first condition in (13) was used. Since \(\ln d_* \leq \ln(d_* - 1) + 1/(d_* - 1) \leq \ln(d_* - 1) + 1/d_0\), inequality (37) with \(d = d_*\) implies continuity bound (27). 

If conditions (14) and (15) hold then Proposition 1B shows that 
\[\tilde{F}_{H_A}(E) = (a - 1)^{-1} \ln(E)(1 + o(1)) \quad \text{as} \quad E \to +\infty.\]  
This implies the asymptotic representation (28) of the r.h.s. of (27). The asymptotic tightness of continuity bound (27) in this case follows from the asymptotic tightness of the continuity bound for the von Neumann entropy presented in [7, Example 2], since the right hand sides of these continuity bounds coincide provided that \(\Phi = \text{Id}_A\) (in this case \(H_{\max}^p(\Phi) = 0\)).
Remark 6. Since the function $\hat{F}_{H_A}$ satisfies condition (12) and (13), the r.h.s. of (27) is a nondecreasing function of $\varepsilon$ and $\tilde{E}$ tending to zero as $\varepsilon \to 0^+$ for any given $\tilde{E}$ and $t \in (0, T]$.

Remark 7. The “free” parameter $t$ can be used to optimize continuity bound (27) for given values of $E$ and $\varepsilon$.

Assume now that the input system $A$ is the $\ell$-mode quantum oscillator with the frequencies $\omega_1, \ldots, \omega_\ell$. The Hamiltonian of this system has the form

$$H_A = \sum_{i=1}^\ell h\omega_i a_i^* a_i + E_0 I_A, \quad E_0 = \frac{1}{2} \sum_{i=1}^\ell h\omega_i,$$

where $a_i$ and $a_i^*$ are the annihilation and creation operators of the $i$-th mode [1]. Note that this Hamiltonian satisfies condition (15) with $a = 1 + 1/\ell$ [14]. In this case the function

$$\hat{F}_{\ell, \omega}(E) = \ell \ln \frac{E + 2E_0}{\ell E^*_s} + \ell, \quad E^*_s = \left[ \prod_{i=1}^\ell h\omega_i \right]^{1/\ell},$$

is an upper bound on the function $\hat{F}_{H_A}(E) \doteq F_{H_A}(E + E_0)$ satisfying all the conditions in (12), (13) and (14) [7]. By using the function $\hat{F}_{\ell, \omega}$ in the role of function $\hat{F}_{H_A}$ in Theorem 2 we obtain the following

Corollary 5. Let $\Phi : \mathcal{S}(H_A) \to \mathcal{S}(H_B)$ be a positive trace non-increasing linear map satisfying condition (1), where $A$ is the $\ell$-mode quantum oscillator with the frequencies $\omega_1, \ldots, \omega_\ell$. Let $\tilde{E} = E - E_0 > 0$, $\varepsilon > 0$ and $T^*_s = (1/\varepsilon) \min \{1, \sqrt{E/E_0}\}$. Then

$$|H_\Phi(\rho) - H_\Phi(\sigma)| \leq \varepsilon (1 + 4t) \left( \frac{\ell \ln (E / (\varepsilon t))^2 + 2E_0}{\ell E_s} + \ell + \Delta^* \right) + 2g(\varepsilon t) + g(\varepsilon (1 + 2t))$$

(38)

for any states $\rho$ and $\sigma$ in $\mathcal{S}(H_A)$ such that $\text{Tr} H_A \rho \leq E$, $\text{Tr} H_A \sigma \leq E$ and $\frac{1}{2} ||\rho - \sigma||_1 \leq \varepsilon$ and any $t \in (0, T^*_s]$, where $\Delta^* = H^0_{\max}(\Phi) + e^{-t} + \ln 2$ (the parameter $H^0_{\max}(\Phi)$ is defined in (16)).

Continuity bound (38) with optimal $t$ is asymptotically tight for large $E$.

Proof. All the assertions of the corollary directly follow from Theorem 2. It suffices to note that in this case $d_0$ is the minimal natural number not less than $x^\ell$, where $x = 2E_0e / (\ell E_s) \geq e$, and hence

$$\gamma(d_0) = \hat{F}_{\ell, \omega}^{-1}(\ln d_0) = (\ell/e)E_s\sqrt{d_0} - 2E_0 \leq (\ell/e)E_s\sqrt{1 + e^{-t} - 2E_0} \leq E_0.$$ 

In the case $\Phi = \text{Id}_A$ continuity bounds (27) and (38) coincide with the continuity bounds for the von Neumann entropy under the energy constraint obtained in [7].

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