On Factorizing Correlation Functions in String Theory Using Loop Variables

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Abstract

Factorization of string amplitudes is one way of constructing string interaction vertices. We show that correlation functions in string theory can be conveniently factorized using loop variables representing delta functionals. We illustrate this construction with some examples where one particle is off-shell. These vertices are “correct” in the sense that they are guaranteed, by construction, to reproduce S-matrix elements when combined with propagators in a well defined way.
1 Introduction

The problem of obtaining a space-time picture for string theory seems to be an outstanding one. Except in unrealistic situations it seems difficult to find exact solutions even of the tree level equations in string theory. For general backgrounds, the only convenient method seems to be the sigma-model renormalization group method \[1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\]. These methods break down as soon as one probes the high energy behaviour or ask questions about massive modes \[12\]. Recent developments in the “duality” symmetries that string theory is expected to possess \[13, 14\] make it very clear that one has to deal with all the modes of the string, more or less simultaneously, if one is to have any hope of understanding the underlying picture. These results are making it even more obvious than before that a string is more than an infinite collection of particles of different masses. This fact was already obvious from the old “duality”, namely s-channel - t-channel duality and also from related properties like modular invariance.

In order to deal with all the modes on an equal footing one has to go off-shell. One has to understand the gauge invariance associated with these (massive) modes. Furthermore it is important to be able to do this without losing contact with the low energy \(\sigma\)-model description - since it is only there that one sees important space-time properties like general covariance. An approach that we have been pursuing is an extension of the renormalization group approach - the ‘proper - time’ approach \[15\]. We showed in \[16, 17\] how one can construct off-shell vertices in this approach by keeping a finite cutoff. One of the constraints used there was gauge invariance. In fact we observed an interesting fact - namely that requiring electromagnetic gauge invariance in the presence of a finite cutoff requires the presence of massive fields as backgrounds. Another constraint one should impose of course is that these off-shell vertices be “correct” in the sense that they can be used (with appropriate Feynman rules) to construct higher N-point scattering amplitudes.

In this paper we would like to report on some progress in this second aspect. We show how one can insert delta functionals to factorize correlation functions. The delta functionals are conveniently represented as loop variables of the kind introduced in \[18\]. Thus a four tachyon scattering amplitude can be written as a sum of terms of the form: vertex \(\times\) propagator \(\times\) vertex. One can also extract vertices where one of the particles is off-shell. The idea
of factorization is very old [19, 20], however, our implementation is different. Although in this paper we will only discuss the case of four tachyon scattering, the method is easily generalized to more complicated situations. Of course, off-shell vertices in open string theory have been discussed in the context of string field theory [21, 22, 23, 24]. These vertices have nice geometric interpretations and they also reproduce scattering amplitudes. Nevertheless they do not have the simplicity of the first quantized Polyakov formalism when it comes to doing calculations. Our aim is to find something that is simpler to work with. It is therefore probably worth exploring objects that are directly obtained from scattering amplitudes, rather than the other way around.

In section II we discuss the harmonic oscillator to illustrate the basic ideas. In section III we discuss the string and derive the main result. In section IV we give some simple examples. We conclude in section V with some comments and open questions.
2 The Harmonic Oscillator

We want to understand the physical significance of certain kinds of correlators, specifically ones that involve delta functions. We will develop the basic idea by considering the example of a harmonic oscillator.

The Lagrangian for a harmonic oscillator is

$$\mathcal{L} = \frac{1}{2}(\dot{X}^2 - \omega^2 X^2)$$  \hspace{1cm} (2.1)

where we have set the mass to unity. Define, in the usual way, the creation and annihilation operators:

$$a = \sqrt{\omega} X + \frac{i \dot{p}}{\sqrt{\omega}}$$ \hspace{1cm} (2.2)

$$a^+ = \sqrt{\omega} X - \frac{i \dot{p}}{\sqrt{\omega}}$$

$$[a, a^+] = 1; [X, p] = i$$

The Hamiltonian is

$$H = \frac{1}{2} \omega (aa^+ + a^+a)$$ \hspace{1cm} (2.3)

and the ground state |0⟩, defined by a |0⟩ = 0 is given by the wave function:

$$\Psi_0(X) = \langle X |0⟩ = \left(\frac{\omega}{\pi}\right)^{\frac{1}{4}} e^{-\frac{1}{2} \omega X^2}$$ \hspace{1cm} (2.4)

Now consider the following 'one-point' function:

$$O_1 \equiv \int \mathcal{D}X \delta(X(t) - X^I) e^{-\frac{i}{\hbar} S}$$ \hspace{1cm} (2.5)

where $S = \int \mathcal{L} dt$ as in (2.1).

In operator language we can represent it as (in the Schrodinger picture)

$$O_1 = \langle +\infty, 0 | U(+\infty, t) \delta(X_S(t) - X^I) U(t, -\infty) | 0, -\infty >_S$$ \hspace{1cm} (2.6)

(where the subscript 'S' stands for "Schrodinger" and indicates that $X$ is not a function of $t$). We can also represent it as

$$O_1 = \int \frac{dk}{2\pi} < +\infty, 0 | U(+\infty, t) e^{ik(X_S(t) - X^I)} U(t, -\infty) | 0, -\infty >_S$$ \hspace{1cm} (2.7)
\[
\frac{dk}{2\pi} < 0 \mid e^{ikX_H(t)} \mid 0 >_H e^{-ikX^I} \tag{2.8}
\]
where the subscript ‘\(H\)’ stands for “Heisenberg”. Note that \(0 >_H = |0, 0>_S\) (i.e. the vacua in the two pictures coincide at \(t = 0\)) and

\[
X_H(t) = \frac{1}{\sqrt{2\omega}}(ae^{-i\omega t} + a^+ e^{i\omega t}) \tag{2.9}
\]

Due to the delta function in (2.6) we can rewrite \(O_1\) as

\[
O_1 = < +\infty, 0 \mid U(+\infty, t) \mid X^I, t > < t, X^I \mid U(t, -\infty) \mid 0, -\infty >_S \tag{2.10}
\]

\[
= e^{i\omega t/2} < 0 \mid X^I > X^I \mid 0 >_H e^{-i\omega t/2}
\]

\[
= \sqrt{\frac{\omega}{\pi}} e^{-\omega(X^I)^2}
\]

We can confirm this explicitly by calculating (2.8) using

\[
e^{ikX(t)} = e^{ikX(t)} e^{-k^2/2\omega}\tag{2.11}
\]

and

\[
< 0 \mid e^{ikX(t)} \mid 0 >= 1 \tag{2.12}
\]

Thus \(O_1\) is the square of the ground state wave function. Let us turn to the correlator of two delta functions: Thus consider the following two objects: Thus consider the following two objects:

\[
I = \int_{X(0)=X_i}^{X(T)=X_f} DX(t) e^{-\frac{S}{\hbar}} \tag{2.13}
\]

namely the Feynman path integral for a harmonic oscillator starting at \(X_i\) at time 0 and ending at \(X_f\) at time \(T\), and:

\[
II = \int DX(t) \delta(X(T) - X_f) \delta(X(0) - X_i) e^{-\frac{S}{\hbar}} \tag{2.14}
\]

the correlator of two delta functions, which also measures, roughly speaking, the amplitude to go from one point to another. Using arguments similar to that used for the one point function one can show that the two are related as follows:

\[
II = (\frac{\omega}{\pi})^{1/2} e^{i\omega T/2} e^{-\frac{S}{\hbar}(X_i^2 + X_f^2)} I \tag{2.15}
\]
This result is easily verified by explicit calculation: \( I \) is a quantity that can be found in textbooks. \( II \) is easily calculated using the vertex operator representation of the delta function and the correlation between vertex operators:

\[
<0|:e^{ikX(t_2)}::e^{ipX(t_1)}:|0>
\]

\[
e^{-kp}\langle X(t_2)X(t_1) \rangle
\]

\[
e^{-kp}e^{-i\omega(t_2-t_1)}
\]

This result is useful because objects like \( I \) are what we have in string field theory -“propagator”- whereas objects like \( II \) are what are easy to calculate in conformal field theory. We now proceed to apply these ideas to strings.
3 Factorizing String Amplitudes

Consider the following four-point function:

\[
< 0 | : V_1(z) :: V_2(u) :: V_3(v) :: V_4(w) : | 0 > \tag{3.1}
\]

Here \( V_i \) are normal ordered vertex operators used in string scattering amplitude calculation. The usual Veneziano amplitude is obtained by fixing \( z = \infty, u = 1, w = 0 \), multiplying by \( z^2 \) and integrating over \( v \) from 0 to 1.

In the proper-time equation [15], one would integrate \( u \) and \( v \). In either case, one has to first factorize (3.1). We begin by inserting two delta functionals of the form:

\[
\prod_{n=0}^{\infty} \delta (X_n(\tau_1) - X_n') \prod_{n=0}^{\infty} \delta (X_n(\tau_2) - X_n'') \tag{3.2}
\]

Here \( X_n \) is the harmonic oscillator coordinates of the \( n \)th mode of the string.

To be definite we consider an open string with a mode expansion:

\[
X(t, \sigma) = X_0(t) \sqrt{2} + \sum_{n>0} \frac{1}{\sqrt{2}} X_n e^{i n\omega t} + \sum_{n>0} \frac{1}{\sqrt{2}} X_n e^{-i n\omega t} \cos n\sigma \tag{3.3}
\]

We can rewrite this as

\[
X(z, \bar{z}) = \frac{x_0 + pt}{\sqrt{2}} \left( \sum_{n>0} \frac{1}{\sqrt{2}} a_n^+ e^{i n\omega t} + a_n e^{-i n\omega t} \right) \cos n\sigma \tag{3.4}
\]

where \( z = e^{i(\omega t+\sigma)} \) and \( \bar{z} = e^{i(\omega t-\sigma)} \). If we analytically continue \( t \to -i\tau \) to the Euclidean world sheet, then \( \bar{z} = \bar{z} \). We can then write

\[
X(z, \bar{z}) = \frac{1}{2} (X(z) + X(\bar{z})) \tag{3.5}
\]

where

\[
X(z) = \frac{x_0}{\sqrt{2}} - \frac{i\rho n z}{\sqrt{2}} + \sum_{n>0} \frac{1}{\sqrt{2n\omega}} (a_n^+ z^n + a_n z^{-n}) \tag{3.6}
\]
We will work with just $X(z)$, with the understanding that for the open string one has to add the complex conjugate. Of course, most of the time this has no effect because the vertex operators are at the boundary of the world sheet where $\sigma = 0, \pi$ and $z = \bar{z}$. After inserting the delta functionals we get:

$$
\int [dX^I_n][dX^{II}_n] < 0 | : V_1(z) :: V_2(u) : \prod_{n=0}^{\infty} \delta(X_n(\tau_1) - X^I_n) 
$$

$$
\prod_{n=0}^{\infty} \delta(X_n(\tau_2) - X^{II}_n) : V_3(v) :: V_4(w) :| 0 >
$$

$$
= \int [dX^I_n][dX^{II}_n] < 0 | : V_1(z) :: V_2(u) : \left[ X^I_n, \tau_1 \right] > 
$$

$$
\left\langle \left[ X^I_n, \tau_1 \right] | U(\tau_1, \tau_2) | \left[ X^{II}_n, \tau_2 \right] > 
$$

$$
\left\langle \left[ X^{II}_n, \tau_2 \right] | V_3(v) :: V_4(w) :| 0 >
$$

(3.7)

Consider I: It can be rewritten as:

$$
< 0 | : V_1(z) :: V_2(u) : \left[ X^I_n, \tau_1 \right] > \left\langle \left[ X^I_n, \tau_1 \right] | U(\tau_1, 0) | 0 > \mathcal{N}_1^{-1}
$$

(3.9)

where

$$
\mathcal{N}_1 = \prod_{n} \left( \frac{n\omega_1}{\pi} \right)^{1/4} e^{-\frac{n\omega(X^I_n)^2}{2}} e^{-\frac{in\omega\tau_1}{2}}
$$

(3.10)

Thus,

$$
I =< 0 | : V_1(z) :: V_2(u) : \prod_{n=0}^{\infty} \delta(X_n(\tau_1) - X^I_n) | 0 > \mathcal{N}_1^{-1}
$$

(3.11)

$$
= \prod_{n=0}^{\infty} \int \frac{dk_n}{2\pi} < 0 | : V_1(z) :: V_2(u) : e^{ik_n X_n(\tau_1)} | 0 > e^{-ik_n X^I_n} \mathcal{N}_1^{-1}
$$

(3.12)

$$
= \prod_{n=0}^{\infty} \int \frac{dk_n}{2\pi} < 0 | : V_1(z) :: V_2(u) : e^{ik_n X_n(\tau_1)} | 0 > e^{-ik_n X^I_n} e^{-\sum_{m>0} \frac{k^2_m}{4m} \mathcal{N}_1^{-1}}
$$

(3.13)
This is the final form. One can also rewrite

\[ e^{ik_nX_n(\tau_1)} = e^{i\int dt k(t)X(at)} \]  

(3.14)

where \( a = e^{i\omega \tau_1}, k(t) = \sum_n k_n t^{-n-1} \) and \( k_n = k_{-n} \). In this form one recognizes the loop variable introduced in [18]. The normal ordering of these variables was described in [25], and for the case where \( k_n = k_{-n} \) it can be seen to be the same as in (3.13). One can also write (3.14) as

\[ e^{i\sum_n a^nk_n \partial^nX(0)} \]

(3.15)

However this is only valid provided there is no singularity as \( a \to 0 \), which will be the case if there is no other vertex operator at \( z = 0 \). Performing similar manipulations on the other two factors one gets:

\[
\int \prod_n dX_n' dX_n'' \frac{dk_n dp_n dq_n dl_n}{2\pi^2} e^{-ik_nX_n'_{-i\omega_n}-i\omega_nX_n''_{-i\omega_n}-i\omega_nX_n''_{-i\omega_n}} e^{-\sum_{m>0} \left( \frac{\omega^2}{4\omega_m^2} + \frac{\omega^2}{4\omega_m^2} + \frac{\omega^2}{4\omega_m^2} \right)}
\]

\[
< 0 \mid V_1(z) \; \vdots \; V_2(u) \; e^{ik_nX_n(\tau_1)} \mid 0 > < 0 \mid e^{i\omega_nX_n(\tau_1)} \; \vdots \; e^{i\omega_nX_n(\tau_2)} \mid 0 > \]

\[
< 0 \mid e^{i\omega_nX_n(\tau_2)} V_3(v) \; \vdots \; V_4(w) \mid 0 > \mathcal{N}_1^{-1} \mathcal{N}_1^{* -1} \mathcal{M}_1^{-1} \mathcal{M}_1^{* -1} \]

(3.16)

Here, \( \mathcal{M} \) is given by (3.10) with \( X^I, \tau_1 \) replaced by \( X^{II}, \tau_2 \) and * refers to complex conjugation. One can check by explicit calculation, for particular choices of the \( V_i \) that (3.16) reproduces the right result. While this is guaranteed by construction, it is illuminating to see how it happens. In fact it gives a proof of the formula for the operator product of two vertex operators [24]. We will not do it here, though. Modulo factors required to amputate external legs (when they are off-shell) (3.10) can be recognized to be in the form \( \times \) propagator \( \times \) vertex as we will see below. (The propagator emerges after integrating over \( v \).) The normal ordering of the zero modes requires some care. Vertex operators in string theory are normal ordered - which for the zero modes usually means that \( p \) is to the right of \( x \). \[ \text{In fact the two three point vertices in (3.16) are asymmetric with respect to the} \]

\[ 1 \text{Note, however, that in (3.16), the factors } e^{i\omega_nX_n(\tau_1)} \text{ obtained as a Fourier transform of the delta functions, are not normal ordered (remember that we are only talking about the zero modes).} \]
zero modes precisely because of this ordering. This asymmetry disappears when the external particles are on shell even if the internal one is not. It is presumably possible to adopt ordering conventions that treat both vertices in a symmetric manner.
4 Examples

We will consider the case where the external particles are on-shell tachyons so that we are calculating an S-matrix element. Thus, $V_1 = e^{ikX}$, $V_2 = e^{ipX}$, $V_3 = e^{iqX}$, $V_4 = e^{i\lambda X}$. This will give us vertices with one off-shell and two on-shell particles. We will choose $e^{i\omega \tau_1} = u$ and $e^{i\omega \tau_2} = v$. This will remove as much of the $v$-dependence as possible from the vertex. In that case we get

$$V = (1 - \frac{u}{z}) \frac{\bar{p} \bar{q}}{u} e^{-\frac{k^2}{2\omega} u - \frac{p^2 + k^2}{4\omega} e^{-(\frac{kn l^2}{2\omega})}} \quad (4.1)$$

In the same way $W$ becomes

$$W = (1 - \frac{w}{v}) \frac{\bar{q} \bar{l}}{v} e^{-\frac{p^2 + q^2}{2\omega} e^{-(\frac{ln l^2}{2\omega})}} \quad (4.2)$$

The vertices do not look quite the same. However when the external particles are on shell (but not the internal one) one can choose $z = \infty$, $u = 1$ and $w = 0$. This simplifies the vertex enormously. Note that we have to multiply $V$ by $z^2$. Furthermore we can multiply $W$ by $v^2$ and compensate by dividing $P$ by $v^2$. At the end we are left with a remarkably simple form for the vertex:

$$\prod_n e^{-\frac{kn \bar{p}}{2\omega}} \quad (4.3)$$

The two point function $P$ (divided by $v^2$) takes the form

$$\prod_n \frac{1}{v^2} e^{-\frac{pn qn}{2\omega} (\frac{u}{v})^n - \frac{p^2}{2\omega}} \quad (4.4)$$

One can put the factors $V, W, P$ together and do the integrals in (3.16). The integrals are fairly straightforward. The noteworthy feature being that the normal ordering factors cancel the wave function factors. One ends up with delta functions (in $p_n, q_n$) and their derivatives multiplied by polynomials in $p_n, q_n$ from expanding the exponent in (1.4).

4.1 Tachyon-tachyon-tachyon

In the case where the intermediate state is a tachyon, on sets all $k_n = 0 = l_n$. This gives 1 for the vertex, and the propagator is just

$$\int_0^1 dvu \frac{\bar{p}^2}{4\omega - 2} = \frac{1}{\frac{p_0^2}{4\omega} - 1} \quad (4.5)$$

(where $p_0 = \bar{p} + \bar{k}$) as expected.
4.2 Tachyon-tachyon-vector

We choose one factor of $k_1, l_1$ from the vertex. This gives:

\[
\int \frac{dp_1 dq_1}{4\pi^2} \bar{p}^\mu \left[ \frac{d}{dp_1^\nu} \frac{d}{dq_1^\nu} \delta(p_1) \delta(q_1) \right] p_1 q_1 \frac{1}{4\omega} \bar{q}^\nu
\]

\[
= \frac{\bar{p} \bar{q}}{4\pi^2 (p+k)^2} 
\]

as expected. Note that the vector-tachyon-tachyon vertex has the asymmetric form $\bar{p}.k_1$ and not $(\bar{p} - \bar{k}).k_1$. This is not surprising given the asymmetric form of (3.14). However, by construction, it is guaranteed to reproduce the Veneziano amplitude.

Proceeding as above it is easy to get interaction vertices for other particles, both internal and external. Note that our factorization is in a particular channel. Of course in string theory it is sufficient to sum over only one channel, so this is consistent.
5 Conclusions

We have constructed a class of vertices by factorizing tree amplitudes and by the nature of the construction they are guaranteed to give the right answer for scattering amplitudes. The vertices have one particle off-shell and two on shell. This is because we started with an on-shell S-matrix element. By considering scattering amplitudes with more than four particles one can construct vertices where two particles are off-shell. The vertices obtained here, are not very symmetric looking, nevertheless they have the advantage of being very simple and do not require much effort to work out.

There are many open questions that arise. The most important is to extend it to the case where all the particles are off-shell. One would like to compare this with the finite cutoff vertices\[17\]. Perhaps if the calculation is done on a disc rather than the half-plane the results will look more symmetrical. Finally, it would be interesting to see if it is possible to implement gauge invariance along the lines of \[18\].
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