Discussion on the energy content of the galactic dark matter Bose-Einstein condensate halo in the Thomas-Fermi approximation

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Received January 24, 2014
Revised February 12, 2014
Accepted March 4, 2014
Published March 7, 2014

Abstract. We show that the galactic dark matter halo, considered composed of an axion-like particles Bose-Einstein condensate [6] trapped by a self-gravitating potential [5], may be stable in the Thomas-Fermi approximation since appropriate choices for the dark matter particle mass and scattering length are made. The demonstration is performed by means of the calculation of the potential, kinetic and self-interaction energy terms of a galactic halo described by a Boehmer-Harko density profile. We discuss the validity of the Thomas-Fermi approximation for the halo system, and show that the kinetic energy contribution is indeed negligible.

Keywords: dark matter theory, gravity

ArXiv ePrint: 1401.6142
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## 1 Introduction

In the search for a description of the dark matter that is responsible for most of the matter density in galaxies, many kinds of particles have been proposed. Among the most popular, we can cite WIMP’s, *Weakly Interacting Massive Particles* \cite{1}, hypothetical particles with large masses that are being sought by many experiments. The mass proposed for this kind of particle lies in the range $10^{-100} \text{ GeV}$.

The axion, a spin-0 particle with sub-eV mass proposed in the context of the Peccei-Quinn mechanism \cite{2, 3}, is also considered a candidate for the galactic dark matter, in the form of a Bose-Einstein condensate (BEC) \cite{4} at low temperatures.

Recently, it has been shown that a halo composed of an axionic Bose-Einstein condensate may present rotation curves that are a good fit for several galaxies \cite{5}. This required the proposal of a specific density profile that we shall call Boehmer-Harko (BH) density profile.

In \cite{6}, the radius of a halo composed of a condensate of spin-0 and spin-1 particles has been derived, and a statistical analysis has been performed to constrain the range of scattering lengths related to a particle in the axion mass range ($10^{-6}–10^{-4} \text{ eV}$). This study has utilised the Boehmer-Harko density profile.

One question arises in the application of such density profile to real galaxies, that of the halo stability. It has been claimed \cite{7} that the Boehmer-Harko density profile, obtained in the framework of the so-called Thomas-Fermi (TF) approximation, leads to the formation of a halo with positive total energy. Consequently, the halo is necessarily unstable. This conclusion takes into account that the particle is ultralight, with a mass $m = 10^{-23} \text{ eV}$ and the self-interaction is very weak, i.e, the scattering length is $a = 10^{-80} \text{ m}$.

In the present work we argue that the Thomas-Fermi approximation is inescapable, due to the large number of particles in the halo. We also show, by estimating the kinetic energy contribution, that the total energy may be negative. Hence, the halo may be stable for the Boehmer-Harko density profile, given that the mass and scattering length of the axionic particle are appropriate.

The plan of this paper is as follows. The section 2 recapitulates the theoretical framework of Bose-Einstein condensation in the galactic case. The section 3 presents a calculation of the total energy for a halo with a Boehmer-Harko density profile, and also gives a justification of the use of the Thomas-Fermi approximation. In section 4 the kinetic energy is calculated and compared to the interaction energy, along with a discussion on the halo stability. Finally, the section 5 presents our conclusions and final remarks.
2 Bose-Einstein condensate halo

We recall here the theoretical description of the galactic Bose-Einstein condensate composed of axionlike particles.

We consider that each axionlike particle is represented by the field destruction and creation operators, \( \hat{\psi}(\mathbf{r}, t) \) and \( \hat{\psi}^\dagger(\mathbf{r}, t) \). These field operators satisfy simple commutation relation, \( [\hat{\psi}(\mathbf{r}, t), \hat{\psi}^\dagger(\mathbf{r}', t)] = \delta(\mathbf{r} - \mathbf{r}') \). The particles are nonrelativistic, confined by the self-gravitating potential, and only two-body collisions with small momentum transfers play an important role. Thus, the Hamiltonian operator is

\[
\hat{H} = \int d^3\mathbf{r} \left[ \hat{\psi}^\dagger(\mathbf{r}, t) \left( -\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{r}) \right) \hat{\psi}(\mathbf{r}, t) + \frac{1}{2} \frac{4\pi\hbar^2a}{m} \hat{\psi}^\dagger(\mathbf{r}, t) \hat{\psi}^\dagger(\mathbf{r}, t) \hat{\psi}(\mathbf{r}, t) \hat{\psi}(\mathbf{r}, t) \right],
\]

(2.1)

where \( m \) is the mass of the particle, \( a \) is the \( s \)-wave scattering length which characterizes the collision and the trapping potential, \( V(\mathbf{r}) \), is determined by the Poisson’s equation,

\[
\nabla^2 V = 4\pi G m \rho_{DM},
\]

(2.2)

where \( \rho_{DM} \) is the mass density of the dark matter halo.

At a sufficiently low temperature, there is a macroscopic occupation of \( N_0 \) particles in the lowest energy mode. Although the number operator \( \hat{N} = \int d^3\mathbf{r} \hat{\psi}^\dagger(\mathbf{r}, t) \hat{\psi}(\mathbf{r}, t) \) commutes with the hamiltonian (2.1), the \( U(1) \) symmetry of particle number conservation is broken due to the ground state possessing a coherent state |\( \psi \rangle \) normalized to \( N_0 \) known as BEC wavefunction. We introduce this symmetry breaking through the Bogoliubov replacement where the field operators are shifted by the BEC wavefunction,

\[
\hat{\psi}(\mathbf{r}, t) = e^{-i\mu t/\hbar} (\psi(\mathbf{r}) + \hat{\delta}(\mathbf{r})),
\]

(2.3)

where \( \mu \) is the chemical potential.

The total number of particles \( N \) is determined by the condition, \( N = N_0 + \int d^3\mathbf{r} \langle \hat{\delta}^\dagger(\mathbf{r}) \hat{\delta}(\mathbf{r}) \rangle \), where the brackets denote the ground state expectation values. For a dilute bosonic gas, it is reasonable to consider \( N - N_0 \ll N \) and the Hamiltonian (2.1) can be truncated to first order of \( \hat{\delta}(\mathbf{r}) \) and \( \hat{\delta}^\dagger(\mathbf{r}) \).

At zero temperature, the dynamics of the field destruction operator \( \hat{\psi}(\mathbf{r}, t) \) in the Heisenberg picture, \( -i\hbar \partial_t \hat{\psi}(\mathbf{r}, t) = [\hat{H}, \hat{\psi}(\mathbf{r}, t)] \), yields the time-independent Gross-Pitaevskii equation (GPE) for the BEC wavefunction \( \psi(\mathbf{r}) \)

\[
\mu \psi(\mathbf{r}) = -\frac{\hbar^2}{2m} \nabla^2 \psi + V(\mathbf{r}) \psi(\mathbf{r}) + \frac{4\pi\hbar^2a}{m} |\psi(\mathbf{r})|^2 \psi(\mathbf{r}).
\]

(2.4)

It has been demonstrated [5, 6] that (2.4) has the solution

\[
\psi_{BH}(r) = \begin{cases} \sqrt{\rho_0 \sin kr} \, \frac{kr}{k} & \text{for } r \leq R \\ 0 & \text{for } r > R \end{cases},
\]

(2.5)

with \( k = \sqrt{Gm^3/\hbar^2a} \) and \( R = \pi/k \). \( \rho_0 \) is the central particle number density of the condensate. This is the Boehmer-Harko solution in the Thomas-Fermi approximation. It results in a halo radius given by

\[
R = \pi \sqrt{\frac{\hbar^2 a}{Gm^3}}.
\]

(2.6)
Using this relation and considering the dark matter particle mass range $10^{-6} - 10^{-4}$ eV, the lower bound of the scattering length has been constrained to $10^{-29} \text{m}$ [6].

The central mass density $\rho_{\text{DM}} = m \rho_0$ will be assumed throughout this paper to be of the same order of magnitude of the local dark matter density in the Milky Way, $\rho = 0.47 \text{ GeV/cm}^3$ [8].

### 3 The total energy of a halo with Boehmer-Harko density profile

The zero-temperature mean field energy of a weakly interacting BEC confined in a self-gravitating potential, $V$, is given by [6]

$$E = \langle \hat{H} \rangle = K + W + I ,$$

where the kinetic, potential and self-interaction energies, respectively, are

$$K = \int d^3r \, \psi^* \left( -\frac{\hbar^2}{2m} \nabla^2 \right) \psi , \quad (3.2)$$

$$W = \int d^3r \, \psi^* V \psi , \quad (3.3)$$

$$I = \frac{2\pi \hbar^2 a}{m} \int d^3r \, |\psi|^4 . \quad (3.4)$$

The expression for the kinetic energy above is also called quantum pressure, and it can be shown to be negligible for a large total number of particles. In fact, this is essential for the so-called Thomas-Fermi approximation, which is the large $N$ limit of the solution for the density of the condensate. It is useful to compare the gravitationally bounded system we are investigating with a trapped condensate that can be obtained in laboratory.

For a Bose-Einstein condensate confined by a harmonic oscillator trap with a potential $V(r) = \frac{m \omega_{ho}^2}{2} r^2$, where $\omega_{ho}$ is the trap frequency, it is possible to show that the TF approximation is valid for $\frac{Na}{a_{ho}} \gg 1$ [9], where

$$a_{ho} = \left( \frac{\hbar}{m \omega_{ho}} \right)^{1/2}$$

is the harmonic oscillator length, $N$ is the total particle number and $a$ is the scattering length implementing the interaction between the atoms in the condensate. The radius of the condensate in the limit in which TF approximation is valid is

$$R = a_{ho}^{4/5} (15Na)^{1/5} . \quad (3.6)$$

Hence, in terms of the TF radius, we have

$$\frac{15(Na)^5}{R^5} \gg 1 . \quad (3.7)$$

This limit is easily achievable in laboratory since the condensate is composed of $N \sim 10^6$ atoms.

Now, making an analogy with the laboratory case, the same condition for the TF approximation must hold, i.e., $\frac{Na}{R} \gg 1$. Of course, the parameters refer now to the galactic...
Figure 1. Wavefunction $\psi_{\text{BH}}$ (in units of $\sqrt{(\rho_0/\pi)}$) and density profile $\rho_{\text{BH}}$ (in units of $(\rho_0/\pi)$) corresponding to the Boehmer-Harko solution in the Thomas-Fermi approximation.

features. Considering the typical values $N = 10^{82}$, $a = 10^{-29}$ m and $R = 10^{20}$ m [6], we have $\frac{Na}{R} = 10^{33} \gg 1$. Therefore, the TF approximation is always valid in the galactic condensate case.

An important consequence of this approximation is that we can neglect the quantum pressure term in the energy expression, because the relative contribution of the kinetic energy of the particles becomes smaller in comparison to the interaction energy when the number of particles increases. The total energy of the condensate is represented now by $E = W + I$.

Using the solution given by [5] for the density profile in terms of $|\psi_{\text{BH}}(r)|^2$ in the TF approximation, we have

$$\rho_{\text{BH}} = |\psi_{\text{BH}}(r)|^2 = \begin{cases} 
\rho_0 \frac{\sin kr}{kr} & \text{for } r \leq R \\
0 & \text{for } r > R 
\end{cases}$$

(3.8)

Both the wavefunction and the density profile for the condensate are depicted in figure 1.

With this solution, we can calculate the interaction energy as

$$I = \frac{2\pi \hbar^2 a}{m} \int d^3r \, |\psi|^4 = 4 \frac{\hbar^2 a}{m} \rho_0^2 R^3.$$  

(3.9)

We can see that this quantity is always positive, as long as the interaction is repulsive ($a > 0$).

For a radial homogeneous distribution the mass inside a radius $r$ can be computed as

$$M(r) = 4\pi \int_0^r m|\psi|^2 r^2 dr.$$  

(3.10)

Considering the BH density profile, we can write

$$M(r) = \frac{4\pi G m \rho_0 r}{k^2} \left(\frac{\sin kr}{kr} - \cos kr\right).$$  

(3.11)

The total mass of the halo is $M_T = \frac{4}{3} m \rho_0 R^3$.

The gravitational field for such radial mass distribution is given by

$$g(r) = \begin{cases} 
-\frac{GM(r)}{r^2} = -\frac{4\pi G m \rho_0}{k^2} \left(\frac{\sin kr}{kr} - \cos kr\right) & \text{for } r < R \\
-\frac{GM_T}{r^2} = -\frac{4\pi G m \rho_0 R^3}{r^2} & \text{for } r \geq R 
\end{cases}.$$  

(3.12)
Figure 2. The kinetic energy (in units of $(\pi \hbar^2 \rho_0 R)/(4m)$) as a function of the dimensionless radius $r/R$. There is a logarithmic divergence as $r/R \to 1$. The plot is made for the interval $[0, (1 - 10^{-7}) R]$, avoiding this divergence.

Hence, we find that the gravitational potential $V(r)$ is

$$ V(r) = m \int_r^\infty g(r')dr' = -\frac{4 \pi m^2 G}{k^2} |\psi|^2 - \frac{4}{\pi} m^2 G \rho_0 R^2. \quad (3.13) $$

Now we can proceed to calculate the gravitational potential energy of the condensate

$$ W = \int_0^R V(r)|\psi(r)|^2 d^3r = -4 \left( \frac{\hbar a}{m \rho_0^2 R^3} \right) = -4I. \quad (3.14) $$

The total energy becomes $E = W + I = -3I$.

The validity of this approach is shown in the next section.

4 Kinetic energy and halo stability

Computing the kinetic energy in the TF approximation, inside a halo with radius $R$, we find the expression

$$ K = \frac{\pi \hbar^2 \rho_0}{4m} \left[ \int_0^R |\Psi|^2 \left( 2k^2 + k^2 \cot^2(kr) - \frac{2k \cot(kr)}{r} + \frac{1}{r^2} \right) r^2 dr \right], \quad (4.1) $$

which is non-convergent in the interval $[0, R]$. We will refer to the integral inside the brackets in (4.1) as $I$.

It can be shown that this quantity can be estimated as

$$ K = \frac{\pi \hbar^2 \rho_0 R}{4m} \left[ \text{Si}(\pi) - \pi + \lim_{x \to \pi} \left( x \ln \left( \tan \left( \frac{x}{2} \right) \right) \right) \right], \quad (4.2) $$

where $x = \frac{\pi r}{R}$.
The last term inside the brackets in (4.2) shows a logarithmic divergence (see figure 2). This term cannot be exactly calculated up to $R$, but up to a value that is close enough to represent the halo interior it results in a small numerical value. The issue of the calculation of the kinetic energy as a border effect beyond the TF approximation has been treated in the context of atomic condensates [10–12], and it is beyond the scope of the present work. Nevertheless, it is worthy to mention that these investigations have found that the kinetic energy contribution in a region very close to the border of the condensate is negligible, even for systems with as few as $10^3$ particles.

In fact, numerical integration of (4.1) (up to $r = (1 - 10^{-16})R$) results in $I \approx 38$. Hence, the order of magnitude of the kinetic energy is mainly given by the multiplying factor $\frac{\pi \hbar^2 \rho_0 R^4}{4m}$.

In order to establish the relative importance of the kinetic energy term in relation to the total condensate energy, we can calculate the ratio $\eta$ of the kinetic energy to the interaction energy

$$\eta = \frac{K}{3I} \sim \frac{1}{a \rho_0 R^2}.$$  

(4.3)

If $\eta \leq 1$, the total energy is negative, and if $\eta > 1$ the kinetic energy should be large enough to make the total energy positive.

For the cases shown in [7], with the values $m = 10^{-22} \text{eV}$, $a \approx 10^{-80} \text{m}$ and $R \approx 10^{20} \text{m}$, this ratio results in

$$\eta \approx 10^4.$$  

(4.4)

This seems to be the reason why the authors in [7] obtained a positive total energy for the condensate in the TF approximation, and concluded that the halo described by the BF density profile is necessarily unstable. Also, such a small scattering length implies an almost null interaction parameter, justifying the use by the authors of the Gaussian approximation for the density profile instead of the TF approximation.

We want to emphasize that this result is strongly dependent on the values chosen for the quantities $m$ and $a$, and different masses and scattering lengths of the dark matter particle can lead to different conclusions about the halo stability. For instance, choosing $m = 10^{-6} \text{eV}$ and $a = 10^{-29} \text{m}$ [6], we obtain

$$\eta \approx 10^{-31},$$  

(4.5)

i.e., the kinetic energy is really negligible, and the total energy is negative, showing that the halo endowed with a Boehmer-Harko density profile can be stable in this specific case.

The figure 3 shows the parameter space for the condensate model with a BH profile. We can see that most of it allows for negative energy. The choice of mass range $10^{-6} \text{eV} < m < 10^{-4} \text{eV}$, which has been determined in [6] for axionlike particles, and resulting in galaxies’ radii ranging from $\sim 0.1 \text{kpc}$ to $\sim 10 \text{kpc}$, is shown as the blue area in this plot.

A few words must be said about the criticism on the BH solution inability to yield different halo radii. The halo radius derived from this solution is a function of the fundamental dark matter quantities $m$ and $a$, and, as a consequence, it represents a prototypical fundamental dark matter halo. Once a particle mass has been chosen, the only free parameter allowing to obtain different radii is the scattering length. As we can see from the plot 3, in order to reflect observed galactic radii this choice is constrained in the parameter space.

The main difference between the BH profile and other profiles used to study galactic dynamics is that BH is obtained from first principles of Quantum Physics, while the other ones are usually phenomenologically fit functions. This fact may be important in the investigation of microscopic properties of the dark matter particles.
Figure 3. Parameter space ($m, a$) for the condensate model. The shaded area is the region in which the total energy is positive. In the white area the energy is negative. The straight line is $\eta = 1$ for $R = 10^{20} \, m$. The blue area between the dashed lines is the region where the values of the parameters $m$ and $a$ result in the galactic radii range $0.1–10 \, kpc$ [6] (color online).

In principle, the results presented here are applicable to the galaxy cluster range, at least as a phenomenological approximation. However, we believe that the microscopic description, involving Gross-Pitaevskii equation, for instance, should require a more precise relativistic treatment for the cluster case.

We remark that, in the cluster scale, cosmological parameters regarding the Universe’s expansion are relevant for the study of the system’s dynamics and the determination of its energy content. This can be implemented, for example, by the use of the Layzer-Irvine equation for the evolution of the energy of cold dark matter in an expanding environment [13]. The relativistic approach for the problem is not the aim of the present work.

5 Conclusions

We have given expressions for the kinetic ($K$), potential ($W$) and self-interaction ($I$) energy components of the Bose-Einstein condensate dark matter halo composed of axionlike particles and described by the Boehmer-Harko density profile. These quantities have been defined by the fundamental parameters of the condensate (the particle mass $m$ and the scattering length $a$ of the interaction) and the halo parameters (central density $\rho_0$ and radius $R$). We have found that the total energy $E = W + K + I$ may be written $E = W + I$ in the Thomas-Fermi approximation. Moreover, we have found that $W = -4I$, rendering the total energy negative. By comparing the conditions for validity of this approximation in atomic condensates created in laboratory with the axionlike halo system, we have found that $\frac{Na}{\sqrt{N}} \gg 1$, where $N$ is the total particle number. This condition is always fulfilled in the galactic case since $N \sim 10^{82}$ and $10^{-6} \, eV < m < 10^{-4} \, eV$. Hence, the TF approximation should be valid for the BH density profile.

In order to stress this fact and to show the strong dependence of the energy terms expressions on the mass and scattering length of the dark matter particle, we have performed a semi-analytical calculation of the kinetic energy term, and showed that the ratio $\eta = K/3I$ indicates the sign of the total energy. For the case of the axionlike particle with mass $m = 10^{-6} \, eV$ [6], we have $\eta \approx 10^{-31}$. Therefore, the total energy in this case is negative and the system should be stable.
On the other hand, for \( m = 10^{-23} \text{eV} \) and \( a = 10^{-80} \text{m} \) \cite{7}, \( \eta \approx 10^4 \) and the system is unstable. We point out that the choice of such a small scattering length makes the particle interaction and the potential energy negligible.

Since the total energy is so sensitive to the values of mass and scattering length we claim that its magnitude alone should not be used to ascertain the stability of the system described by the BH density profile nor to rule out the validity of the TF approximation. Rather, we believe that it is necessary to find methods to obtain the values of \( m \) and \( a \) independently and then resort to procedures as phase space analysis and numerical solutions of the differential equations involved. This search will be the subject of future work.

We want to emphasize that we use Bose-Einstein condensation as an analog for a dark matter halo, taking advantage of the possibility of obtaining experimental information in a controlled manner. Nevertheless, the fact that the BH profile is capable of satisfactorily fitting rotation curves for a number of galaxies may be an indication of its adequacy in describing the underlying galactic dynamics.

Acknowledgments

J.C.C.S. thanks CAPES (Coordenação de Aperfeiçoamento de Pessoal de Nível Superior) for financial support. The authors wish to thank an anonymous referee for useful suggestions.

References

[1] Particle Data Group collaboration, J. Beringer et al., Review of Particle Physics (RPP), Phys. Rev. D 86 (2012) 010001 [arXiv:1112.4832v3 [hep-ex]].
[2] R.D. Peccei and H.R. Quinn, CP Conservation in the Presence of Instantons, Phys. Rev. Lett. 38 (1977) 1440 [arXiv:hep-th/9905221].
[3] R.D. Peccei and H.R. Quinn, Constraints Imposed by CP Conservation in the Presence of Instantons, Phys. Rev. D 16 (1977) 1791 [arXiv:hep-th/9905221].
[4] P. Sikivie and Q. Yang, Bose-Einstein Condensation of Dark Matter Axions, Phys. Rev. Lett. 103 (2009) 111301 [arXiv:0901.1106 [hep-ph]].
[5] C.G. Boehmer and T. Harko, Can dark matter be a Bose-Einstein condensate?, JCAP 06 (2007) 025 [arXiv:0705.4158 [hep-th]].
[6] M.O.C. Pires and J.C.C. de Souza, Galactic cold dark matter as a Bose-Einstein condensate of WISPs, JCAP 11 (2012) 024 [Erratum ibid. 1311 (2013) E01] [arXiv:1208.0301 [astro-ph]].
[7] F.S. Guzmán, F.D. Lora-Clavijo, J.J. González-Avilés and F.J. Rivera-Paleo, Stability of BEC galactic dark matter halos, JCAP 09 (2013) 034 [arXiv:1308.4925 [astro-ph]].
[8] F. Nesti and P. Salucci, The Dark Matter halo of the Milky Way, AD 2013, JCAP 07 (2013) 016 [arXiv:1304.5127 [astro-ph]].
[9] F. Dalfovo, S. Giorgini, L.P. Pitaevskii and S. Stringari, Theory of Bose-Einstein condensation in trapped gases, Rev. Mod. Phys. 71 (1999) 463 [arXiv:cond-mat/9704173].
[10] F. Dalfovo, L.P. Pitaevskii and S. Stringari, Order parameter at the boundary of a trapped Bose gas, Phys. Rev. A 54 (1996) 4213 [arXiv:cond-mat/9704173].
[11] E. Lundh, C.J. Pethick and H. Smith, Zero-temperature properties of a trapped Bose-condensed gas: Beyond the Thomas-Fermi approximation, Phys. Rev. A 55 (1997) 2126.
[12] A.L. Fetter and D.L. Feder, Beyond the Thomas-Fermi approximation for a trapped condensed Bose-Einstein gas, Phys. Rev. A 58 (1998) 3185 [cond-mat/9704173].
[13] M. Fukugita and P.J.E. Peebles, The Cosmic energy inventory, Astrophys. J. 616 (2004) 643 [astro-ph/0406095].