Gradient-based estimation of linear Hawkes processes with general kernels

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November 23, 2021

Abstract

Linear multivariate Hawkes processes (MHP) are a fundamental class of point processes with self-excitation. When estimating parameters for these processes, a difficulty is that the two main error functionals, the log-likelihood and the least squares error (LSE), as well as the evaluation of their gradients, have a quadratic complexity in the number of observed events. In practice, this prohibits the use of exact gradient-based algorithms for parameter estimation. We construct an adaptive stratified sampling estimator of the gradient of the LSE. This results in a fast parametric estimation method for MHP with general kernels, applicable to large datasets, which compares favourably with existing methods.

Keywords: Hawkes processes; stochastic gradient descent; point processes; Monte Carlo methods; adaptive stratified sampling.

MSC: 60G55,62M09,90C52,93E10.

Acknowledgements

Álvaro Cartea and Samuel Cohen acknowledge the support of the Oxford–Man Institute for Quantitative Finance. Samuel Cohen also acknowledges the Alan Turing Institute under the Engineering and Physical Sciences Research Council grant EP/N510129/1. Saad Labyad acknowledges the support of St John’s College, University of Oxford, under the Ioan and Rosemary Jones scholarship.

1 Introduction

Temporal point processes are widely applied as models of asynchronous streams of events. One way to specify these models is through their conditional intensity, that is, the expected infinitesimal rate of events per unit of time, conditioned on the history of the process. Linear multivariate Hawkes processes (MHP) are a specific class of point processes in which the conditional intensity takes a linear autoregressive form parameterized by a matrix of kernel functions and a vector of constant background rates; see Hawkes \cite{22}. The widespread use of MHP is mainly due to their explainability: their matrix of kernel functions accounts for self-excitation and cross-excitation between different types of events, and their cluster representation can be a proxy for causality between events.

MHP have applications in a variety of domains including finance, particularly in market microstructure (see Cartea, Jaimungal, and Ricci \cite{11}, Bacry, Mastromatteo, and Muzy \cite{1}, and Hawkes \cite{22} for an extensive review); social networks, with an emphasis on modeling information cascades such as retweets (see Zhao et al. \cite{45}, Kobayashi and Lambiotte \cite{26} and Chen and Tan \cite{12}); seismology, to study the occurrence of earthquakes and their aftershocks (see Veen and Schoenberg \cite{42}); and criminology, to examine criminal contagion mechanisms, notably in burglaries and gang violence (see Mohler et al. \cite{34}, Lewis et al. \cite{29}, and Mohler \cite{33}).
Despite their popularity, estimation of the kernel matrix and background rates of MHP remains difficult. In practice, the absence of a fast parametric estimation method prohibits the use of MHP with significant amounts of data (i.e., of order higher than $10^6$–$10^7$ jumps in an observed sample path), and with arbitrary kernels, in particular non-Markovian kernels. These limitations arise because the evaluation of the conditional intensity at each time $t$ has linear complexity in the number of jumps up to time $t$, leading to quadratic complexity overall. Therefore, objective functions based on the conditional intensity of the MHP are expensive to evaluate and to minimize (see Section 2.2.2 for a detailed analysis). A notable exception is the MHP with exponential kernels (see Section 2.1.2), where the conditional intensity can be evaluated recursively, which explains the predominance of exponential MHP in the literature.

The goal of this paper is to overcome these limitations. We develop a stochastic optimization algorithm for parametric MHP estimation that does not directly evaluate the conditional intensity of the MHP; we call this the ASLSD algorithm (Adaptively Stratified Least Squares Descent). As an accompaniment to our derivation and analysis, an implementation in python is available at https://github.com/saadilabyad/aslsd. This algorithm is computationally efficient, accurate for a wide range of sample sizes, and flexible enough to allow for regularization and sparsity terms to be easily included.

1.1 MHP Estimation methods

We briefly review the state of the art for MHP estimation. The time complexity, assumptions, objective function, and regularization type of the algorithms discussed here are summarized in Table 1. Estimation procedures fall into three main categories:

- **Method of moments.** These procedures are typically based on spectral properties of the MHP, and usually aim to convert the estimation problem into solving a system of equations. Most of these methods are non-parametric, and require mild assumptions beyond stationarity of the MHP as they use second order properties of the process.

- **Maximum likelihood estimation (MLE).** As in other statistical problems, MLE benefits from sound theoretical guarantees. However, for general MHP, the evaluation of the log-likelihood of the MHP has a quadratic time complexity in the number of jumps observed. The application of the expectation-maximization (EM) algorithm to MHP estimation usually improves the convergence of optimization algorithms, but the high computational cost of the E step in EM methods does not allow for efficient algorithms.

- **Least squares estimation.** This class of methods is rarely used in the context of MHP. The cost of evaluating the least squares objective function is roughly as expensive as that of evaluating the log-likelihood. Nonetheless, in this paper we show that, unlike the log-likelihood, the least squares error (LSE) has an additive decomposition that is particularly suitable for efficient stochastic approximation.

**Method of moments** Hawkes [21,22] applies methods developed for the general analysis of the spectra of point processes by Bartlett [8]. Hawkes shows a link between the Laplace transform of the autocovariance function $\nu$ of the increments of stationary MHP and the mean conditional intensity and kernels of the MHP. Bacry, Dayri, and Muzy [3] use this link to propose a non-parametric estimation method in the specific case of stationary MHP with symmetric kernels and Laplace transforms diagonalizable in the same orthogonal basis. All these assumptions (except stationarity) are relaxed by Bacry and Muzy [2], who show that the MHP parameters solve a system of Wiener–Hopf equations; we use this algorithm as a nonparametric baseline in our numerical examples. Achab et al. [1] use the first three cumulants of the MHP to propose a non-parametric estimation method for the adjacency matrix of the MHP (the matrix of $L_1$ norms of the kernels). This algorithm is fast, as it depends linearly on the number of jumps of the MHP; however, this method is not meant for the estimation of the kernels themselves. Finally, the work of Gao, Zhou, and Zhu [17] relies on the spectrum of the cumulative number of jumps of different types instead of the autocovariance property.

While algorithms based on the method of moments apply to a wide range of models, they are particularly inefficient when the number of observations is small. These moment-based methods are also
particularly prone to the curse of dimensionality (with respect to the number of dimensions $d$ of the MHP), and regularization for the sake of dimensionality reduction seems difficult for these models.

**Maximum likelihood estimation** A different paradigm consists in maximising the log-likelihood of the sample path, see Daley and Vere-Jones [14]. To the best of our knowledge, the fastest parametric approach, in the case where the kernels are a sum of exponentials with fixed decay rates, is that in Bompaire, Bacry, and Gaïffas [9]; we use this algorithm as a parametric baseline in our numerical examples. Lemonnier and Vayatis [27] use Bernstein polynomials to give a density argument to justify the choice of a linear combination of exponential decays. In the case of linear combinations of non-exponential kernels, Bacry et al. [5] propose a mean field approximation of the log-likelihood to speed up standard parametric estimation. Despite the speed of this method, it is difficult to generalize it due to the mean-field and linearity assumptions.

Another limitation of log-likelihood methods for MHP estimation is the flatness of the log-likelihood. A classic approach to solve this issue is the EM procedure introduced by Veen and Schoenberg [42] and Lewis and Mohler [28], which is based on Hawkes and Oakes’s [24] cluster representation of the MHP. In the general case, the complexity of an EM iteration remains quadratic, but significantly smooths the objective. The ADM4 algorithm of Zhou, Zha, and Song [46] also builds on the EM approach, with the assumption that the kernels of the MHP are of a fixed form with a single scale coefficient. This method uses sparsity and low rank penalties to estimate high-dimensional MHP. Finally, Zhou, Zha, and Song [47] show that the kernels satisfy an Euler–Lagrange equation and use a Seidel method to solve it numerically. Again, these methods are not applicable to general kernels without a significant computational burden.

**Least squares estimation** Among M-estimation methods for MHP, the log-likelihood is significantly more popular than the least squares functional. To the best of our knowledge, the work of Reynaud-Bouret and Schbath [36] is the first to introduce this objective for MHP. Their estimation method is meant for piece-wise constant kernels with finite support, with a view towards applications to genomics. Bacry et al. [7] develop an approach for more conventional kernels; namely, linear combinations of exponential kernels with fixed decays. They are interested in dimensionality reduction via sparsity inducing penalties, as the number of kernels in the MHP is quadratic in the number of event types. However, their method is not applicable to general kernels. We review least-square estimation methods in Section 2.2.

### 1.2 Work outline

Section 2 recalls the definitions of MHP and the least squares estimation framework. In Section 3, Theorem 3.1 provides a decomposition of the LSE as a sum of functions. The rationale behind this decomposition is that, if these functions and their partial derivatives can be evaluated quickly, then a Monte Carlo estimator, of the gradient of the LSE is inexpensive to evaluate. We construct this Monte Carlo estimator using adaptive stratified sampling for variance reduction purposes, allowing for general kernels for the MHP. We combine this estimator with numerical schemes from the stochastic gradient descent literature to propose a new fast estimation method for MHP with general kernels and large datasets. We evaluate our procedure on synthetic data in Section 4 and benchmark it against state of the art algorithms. Finally, we give example applications of our method in Section 5 using the MemeTracker dataset, and a dataset of malaria infections in China.

## 2 Background on MHP estimation

Before defining MHP, we recall the definition of the larger class of point processes and their conditional intensity. The objective function we use to estimate MHP in this work is the LSE; we recall its definition and discuss the challenges of LSE minimization with standard exact or stochastic first order methods in Section 2.2.
2.1 Hawkes processes

For any positive integer \( n \), we denote by \([n]\) the set of integers from 1 to \( n \). We fix a dimension \( d \in \mathbb{N}^* \) for our process, where \( \mathbb{N}^* \) denotes the set of strictly positive integers.

2.1.1 Model definition

Following the notation and terminology in Daley and Vere-Jones [13], we briefly introduce point processes.

Definition 2.1 (Point process). A \( d \)-dimensional orderly point process is a random sequence of times \( T = \{ t_m^i : m \in \mathbb{N}^*, i \in [d], t_m^i < t_m^{i+1} \} \). The associated counting process \( N \) is defined for \( t \geq 0 \) by

\[
N_t := (N^i_t)_{i \in [d]}, \quad \text{where} \quad N^i_t := \sum_{i=1}^{+\infty} \mathbb{1}_{\{t \geq t^i_m\}}.
\]

We denote the total number of jumps up to time \( t \) by \( N_t := \sum_{i=1}^d N^i_t \).

Consider a \( d \)-dimensional point process \( T \) and the associated counting process \( N \). Let \( \mathcal{F} = (\mathcal{F}_t)_{t \geq 0} \) be the natural filtration of \( N \). The counting process \( N \) is characterized by its conditional intensity, the infinitesimal rate of events per unit of time given the history of the process.

Definition 2.2 (Conditional Intensity). For \( i \in [d] \), the conditional intensity of \( N^i \) is defined by

\[
\lambda^i(t) := \lim_{h \to 0^+} \frac{\mathbb{P}(N^i_{t+h} - N^i_t = 1 | \mathcal{F}_t)}{h},
\]

and we write \( \lambda := (\lambda_1, \ldots, \lambda_d)^\top \).

Definition 2.3 (Compensator). For \( t \geq 0 \), the compensator of the counting process \( N^i \) is defined by

\[
\Lambda^i(t) := \int_0^t \lambda^i(s)ds.
\]

The Doob–Meyer decomposition of \( N^i \) is \( N^i = \Lambda^i + M^i \), that is, \( M^i_t := N^i_t - \Lambda^i(t) \) defines a local martingale.

We define the MHP model as in Liniger [30].

Definition 2.4 (MHP). Let \( N \) be a \( d \)-dimensional counting process with conditional intensity \( \lambda \). We say that \( N \) is a (linear) MHP if, for all \( i \in [d] \) and for all \( t \geq 0 \), we have

\[
\lambda^i(t) = \mu^i + \sum_{j=1}^d \sum_{m.t_m^i < t} \phi^i_{ij}(t - t_m^i),
\]

where

- For all \( i, j \in [d] \), \( \phi^i_{ij} : [0, +\infty) \to [0, +\infty) \) is in \( L_1 \). The functions \( \phi^i_{ij} \) are called the kernels of the MHP, and we write in matrix notation
  \( \Phi(t) = (\phi^i_{ij}(t))_{ij} \).

- For all \( i \in [d] \), \( \mu^i > 0 \). The scalars \( \mu^i \) are called baseline intensities, and we write in vector notation \( \mu = (\mu_1, \ldots, \mu_d)^\top \).

We refer to such a process as a \((\mu, \Phi)\)-MHP.
2.1.2 Examples of kernels

This paper is aimed at general parametric classes of kernel models. In both theoretical and numerical applications of MHP, monotonically decaying kernels predominate. In particular, exponential kernels are commonly used in the literature (see the discussion in Section 2.2). We now provide some examples which we use to test our method.

**Definition 2.5** (Exponential kernel). Let \( r \in \mathbb{N}^* \). For \( x \geq 0 \), the exponential kernel \( \phi^E_{(\omega, \beta)} \) is

\[
\phi^E_{(\omega, \beta)}(x) := \sum_{l=1}^{r} \omega_l \beta_l e^{-\beta_l x},
\]

where the parameters are the vector of weights \( \omega := (\omega_l)_{l \in [1,r]} \in [0, +\infty)^r \) and the vector of decays \( \beta := (\beta_l)_{l \in [1,r]} \in (0, +\infty)^r \). We have \( \|\phi^E_{(\omega, \beta)}\|_1 = \sum_{l=1}^{r} \omega_l \).

In nature, there are systems generating streams of data with non-monotonic kernels; for example, when events might trigger each other with some delay. A famous example in the seismology literature is the *linkin* model in Hawkes and Oakes [23], which uses Gamma mixture kernels. For computational reasons, we will consider a model built from truncated Gaussian mixture distributions. While the method we propose is designed as a parametric estimation algorithm, non-parametric variations of our algorithm with large Gaussian kernel bases (which allows for general kernels through a standard density argument) are, in principle, possible.

**Definition 2.6** (Gaussian kernel). Let \( r \in \mathbb{N}^* \). For \( x \geq 0 \), the Gaussian kernel \( \phi^N_{(\omega, \beta, \delta)} \) is

\[
\phi^N_{(\omega, \beta, \delta)}(x) = \sum_{l=1}^{r} \omega_l \frac{1}{\beta_l \sqrt{2\pi}} \exp \left( -\frac{(x-\delta_l)^2}{2\beta_l^2} \right),
\]

where the parameters are the vector of weights \( \omega := (\omega_l)_{l \in [1,r]} \in [0, +\infty)^r \), the vector of means \( \delta := (\delta_l)_{l \in [1,r]} \in \mathbb{R}^r \), and the vector of standard deviations \( \beta := (\beta_l)_{l \in [1,r]} \in (0, +\infty)^r \). We have \( \|\phi^N_{(\omega, \beta, \delta)}\|_1 = \sum_{l=1}^{r} \omega_l \left( 1 - F_N(\delta_l / \beta_l) \right) \), with \( F_N \) the standard normal distribution function.

To demonstrate the effect of non-monotonicity of the kernel on inter-arrival times, we simulate a univariate Hawkes process with a Gaussian kernel. Figure 1 shows a multi-modal empirical distribution of inter-arrival times, with modes near the component means and their harmonics.

Further examples of kernels suitable for our method can be found in Appendix D.

2.2 Least squares estimation

We are interested in estimation of the parameters of a \((\mu^\circ, \Phi^\circ)\)-MHP. In what follows, we fix a horizon \( T > 0 \). We observe a single sample path of jump times \( \mathcal{T}_T \) of \( N \) on the interval \([0, T] \) and we would like to estimate \((\mu^\circ, \Phi^\circ)\) given these observations. For all \( i \in [d] \) and \( t \leq T \), we consider the intensity model for \( \lambda_i \) as in [1], with candidate baseline and kernel model \((\mu, \Phi)\). The idea behind least squares estimation is to choose \((\mu, \Phi)\) such that the conditional intensity model \( \lambda \) is close to the ground truth \( \lambda^\circ \) in the \( L_2 \) sense.

**2.2.1 The LSE**

**Definition 2.7** (Least squares error). The LSE of the linear MHP model \((\mu, \Phi)\) is

\[
\mathcal{R}_T(\mu, \Phi) := \frac{1}{T} \sum_{k=1}^{d} \int_0^T \lambda_k(t)^2 dt - \frac{2}{T} \sum_{k=1}^{d} \sum_{m=1}^{N^k_T} \lambda_k(t^k_m),
\]

(2)
Using the Doob–Meyer decomposition of $N$, one can show that
\[
\mathbb{E} [R_T(\mu, \Phi)] = \frac{1}{T} \sum_{k=1}^{d} \int_{0}^{T} \mathbb{E} [(\lambda_k(t) - \lambda^0_k(t))^2] \, dt - \frac{1}{T} \sum_{k=1}^{d} \int_{0}^{T} \mathbb{E} [\lambda^0_k(t)^2] \, dt,
\]
where $\lambda^0$ is the intensity of the MHP that generates the observations. As highlighted in Reynaud-Bouret and Schbath [36, Lemma 3], the main rationale behind the definition of the LSE is that $R_T$ is a statistical contrast functional. From (3) one can see that $\mathbb{E} [R_T(\mu, \Phi)]$ is minimized if and only if
\[
\lambda_k(t) = \lambda^0_k(t) \quad \text{a.s., for almost all} \quad t \leq T, k \in [d].
\]
In this work, we consider parametric models $(\mu^\theta, \Phi^\theta)$ and denote by $\Theta$ the parameter space. To simplify notation, we drop the superscript $\theta$ when the dependence on $\theta$ is clear and write $\mu, \Phi, \lambda$. By abuse of notation, we see the LSE in (2) as a function on $\Theta$, and so write $R_T(\theta)$ and consider the estimator
\[
\theta^*_T \in \arg \min_{\theta \in \Theta} R_T(\theta).
\]
It is not clear whether the estimator $\theta^*_T$ is unbiased or consistent and, if so, at which rate $\theta^*_T$ converges. Nevertheless, this estimator gives satisfactory results in practice, as seen in Reynaud-Bouret and Schbath [36], Gaïffas and Guilloux [16], Hansen, Reynaud-Bouret, and Rivoirard [20], Bacry et al. [7], and in our numerical experiments (see Section 4).

2.2.2 Minimizing the LSE

Directly minimizing the LSE, for example with exact first order methods, is particularly inefficient without further assumptions because of the auto-regressive structure of the conditional intensity.

Evaluating the LSE is expensive Evaluating the conditional intensity model $\lambda_i(t)$ based on $[1]$ has a complexity in time which is roughly linear in the number of jumps of the MHP up to time $t$. Therefore, the time complexity to evaluate the second term
\[
\frac{2}{T} \sum_{k=1}^{d} \sum_{m=1}^{N_k^m} \lambda_k(t_m^k)
\]

Figure 1: Simulated path of a Gaussian MHP

Kernel plot and histogram of inter-arrival times of a univariate Hawkes process with a Gaussian kernel, $r = 3$. Orange vertical dashed lines correspond to the means of the Gaussian mixture. The apparent mode near $t = 2$ corresponds to the difference $3 - 1$. 
in the LSE in (2), is roughly $O(N_T^2)$ for a general kernel matrix $\Phi$. Now, consider the evaluation of the first term

$$\frac{1}{T} \sum_{k=1}^{d} \int_0^T \lambda_k(t)^2 dt$$

in the LSE. If the integral in (6) is approximated numerically using a quadrature rule, the cost of such approximation is at least linear since the conditional intensity has a discontinuity at each jump time. If we re-use the evaluations of the conditional intensity at jump times, which we computed for the evaluation of (5), we still face the problem that the conditional intensity varies between jump times. For general kernels, it is not clear how to interpolate the conditional intensity accurately between jump times, therefore, we might also have to evaluate $\lambda$ between consecutive jump times. Another drawback of numerical integration is that the evaluation of the gradient would typically require a finite difference method or operator overloading, increasing the overall complexity.

In a nutshell, evaluating the LSE at a given $\theta$ has complexity $O(N_T^2)$ in the general case, and the evaluation of the gradient in closed-form is roughly as expensive (if we ignore for now the curse of dimensionality coming from the number of parameters of the model). This makes exact first order methods impractical for general kernels, unless the kernels considered are sum-of-basis exponential functions (see Section 3.3.2).

Towards a SGD approach A classic strategy to accelerate first order methods is the use of Stochastic Gradient Descent (SGD), which relies on an approximation of the gradient of the objective function; see Robbins and Monro [37]. The application of SGD to the minimization of the LSE in (2) faces some difficulties. For example, for $d = 1$, write

$$R_T(\theta) = \frac{1}{T}(\mu^2 - 2\mu) + \frac{1}{T} \int_{t_N}^T \lambda(t)^2 dt + \frac{1}{T} \sum_{m=1}^{N_T-1} \left( \int_{t_m}^{t_{m+1}} \lambda(t)^2 dt - 2\lambda(t_{m+1}) \right),$$

and define

$$f_m(\theta) := \begin{cases} \int_{t_m}^{t_{m+1}} \lambda(t)^2 dt - 2\lambda(t_{m+1}) & \text{if } m \in [1, N_T - 1], \\ \int_{t_N}^T \lambda(t)^2 dt & \text{if } m = N_T, \end{cases}$$

to yield the decomposition

$$R_T(\theta) = \frac{1}{T}(\mu^2 - 2\mu) + \frac{1}{T} \sum_{m=1}^{N_T} f_m(\theta).$$

This decomposition preserves the chronological order of the data because, for all $m$, the function $f_m$ only depends on jump times up to $t_{m+1}$. However, computing $f_m(\theta)$ and its derivatives has complexity roughly of order $O(m)$. Due to this linear cost, we propose a new additive decomposition of the LSE and build a fast, yet accurate, Monte Carlo estimator of large finite sums.

3 SGD for MHP estimation

In this section, we state our decomposition of the LSE in Theorem 3.1, which is at the heart of the method we propose. In Section 3.4, we build an adaptive stratified Monte Carlo sampling estimator of the gradient of the LSE based on this decomposition. Finally, Section 3.5 presents our numerical scheme.

3.1 Notation and definitions

We start with some notation and definitions, noticing that in dimension $d > 1$, the chronological ordering of jump times requires some care.

Definition 3.1. For all $i \in [d]$ and for all $t > 0$, let

$$\kappa(i, t) := N_t^i.$$
or equivalently, \( \kappa(i,t) := \sup\{m : t^i_m < t\} \). We define, for all \( n \in \mathbb{N}^* \), and \( i,j \in [d] \),
\[
\kappa(i,j,n) := \kappa(i,t^i_n);
\]
that is, the index of the last jump of type \( i \) before the \( n \)-th jump of type \( j \).

For \( i = j \), it is clear that \( \kappa(i,j,n) = n - 1 \).

**Definition 3.2.** For all \( h \in \mathbb{N}^* \) and \( i,j \in [d] \), the upper-inverse of \( \kappa(j,i,) \) is given by
\[
\varpi(i,j,h) := \inf\{p \in \mathbb{N}^* : \kappa(j,i,p) \geq h\};
\]
that is, the index of the earliest jump of type \( i \) preceded by at least \( h \) jumps of type \( j \). For simplicity, we write
\[
\varpi(i,j,h) := \varpi(i,j,1).\]

For \( i = j \), it is clear that \( \varpi(i,j,h) = h + 1 \).

Next, we introduce some quantities of interest, which arise in our computations. For all \( i \in [d] \), define the global event rate
\[
\eta^i_T := \frac{N^i_T}{T}.
\]
This is both the maximum likelihood estimator and the least squares estimator of the intensity under a homogeneous Poisson model. For all \( i,j \in [d] \), and \( t \geq 0 \), define
\[
\psi_{ij}(t) := \int_0^t \phi_{ij}(u)du.
\]

**Remark 3.1.** Since \( \phi_{ij} \geq 0 \), we have \( \lim_{t \to +\infty} \psi_{ij}(t) = \|\phi_{ij}\|_1 \).

For all \( i,j,k \in [d] \), and \( t,s \geq 0 \), define
\[
\Upsilon_{ijk}(t,s) := \int_0^t \phi_{ik}(u)\phi_{kj}(u+s)du.
\]

**Remark 3.2.** Note that \( \lim_{t \to +\infty} \Upsilon_{ik}(t,0) = \|\phi_{ki}\|_2^2 \). The function \( \Upsilon_{ijk} \) quantifies the correlation (or auto-correlation) between \( \phi_{ki} \) and \( \phi_{kj} \).

### 3.2 Decomposition of the LSE

Assume that the observed sample path of \( N \) is non-trivial, in particular, we observe at least one event of each type (i.e. for all \( i \in [d] \), \( N^i_T > 1 \)) and, for each event type, the last observed event of that type is preceded by at least one event of every type (i.e. for all \( i,j \in [d] \), \( \varpi(i,j) < N^j_T \)). We state our decomposition of the LSE as a sum involving the functions \( (\phi_{ij}), (\psi_{ij}), \) and \( (\Upsilon_{ijk}) \), as well as the background rates \( (\mu_i) \).

**Theorem 3.1 (Least squares error).** The LSE \( R_T \) satisfies
\[
R_T(\theta) = \frac{2}{T} \sum_{k=1}^{d} \sum_{i=1}^{d} \sum_{j=1}^{d} \sum_{m=\varpi(i,j)}^{N^i_T} \kappa(j,i,m) \sum_{n=1}^{N^j_T} \Upsilon_{ijk}(T-t^i_m, t^i_m - t^j_n)
\]
\[- \frac{2}{T} \sum_{k=1}^{d} \sum_{i=1}^{d} \sum_{m=\varpi(k,j)}^{N^i_T} \kappa(j,k,m) \sum_{n=1}^{N^j_T} \phi_{kj}(t^k_m - t^j_n) + \sum_{k=1}^{d} (\mu^2_k - 2\eta^k_T \mu_k)
\]
\[+ \frac{2}{T} \sum_{k=1}^{d} \sum_{m=1}^{N^j_T} \psi_{ki}(T-t^i_m) + \frac{1}{T} \sum_{k=1}^{d} \sum_{m=1}^{N^j_T} \Upsilon_{ik}(T-t^i_m, 0).\]
We give a proof of this expansion in Appendix A. Note that the coupling between background rates \( \mu \) and kernels \( \phi \) is only through the term
\[
\frac{2}{T} \sum_{k=1}^{d} \mu_k \sum_{i=1}^{d} \sum_{m=1}^{N_k} \psi_{ki}(T - t_m^i),
\]
while the coupling between kernels \( \phi \) is only through the term
\[
\frac{2}{T} \sum_{k=1}^{d} \sum_{i=1}^{d} \sum_{j=1}^{d} \sum_{m=1}^{N_k} \kappa(j,i,m) \sum_{n=1}^{N_i} \Upsilon_{ijk}(T - t_m^i, t_m^j - t_n^i).
\]
This decomposition guides our choice of parametric families for our algorithm, as we need the functions \( \psi_i \) and \( \Upsilon_{ijk} \) and their derivatives to be computable efficiently. In Appendix D we give a selection of families where these functions are available in closed form.

3.3 Model parameterization

For all \( i,j \in [d] \), denote the vector of parameters of the kernel \( \phi_{ij} \) by \( \vartheta_{ij} \), the dimension of \( \vartheta_{ij} \) by \( \rho_{ij} \), and the total number of parameters of the model by
\[
n_{\text{param}} := d + \sum_{i=1}^{d} \sum_{j=1}^{d} \rho_{ij}.
\]
For all \( k \in [d] \), concatenate these vectors as \( \vartheta_{k}^\top = (\vartheta_{k1}^\top, \ldots, \vartheta_{kd}^\top) \), and for each \( k \in [d] \), define \( \theta_{k}^\top = (\mu_k, \vartheta_{k}^\top) \).

Finally, define the vector of parameters \( \theta \) of the \((\mu, \Phi)\)MHP by
\[
\theta^\top = (\theta_1^\top, \ldots, \theta_d^\top).
\]

3.3.1 Parallelization

For all \( k \in [d] \), the intensity \( \lambda_k \) only depends on \( \theta_k \) and the observed jumps. Define the partial LSE
\[
R_T^{(k)}(\theta_k) := \frac{1}{T} \int_0^T \lambda_k(t)^2 dt - \frac{2}{T} \sum_{m=1}^{N_k} \lambda_k(t_m^k). \tag{14}
\]
It is clear from the definition of the LSE that \( R_T(\theta) = \sum_{k=1}^{d} R_T^{(k)}(\theta_k) \).

Therefore, minimizing \( R_T \) is equivalent to \( d \) independent minimization programs that can be solved in parallel. In the remainder of Section 3, we fix \( k \in [d] \) and focus on minimizing \( R_T^{(k)} \), which we re-write using Theorem 3.1.

Corollary 3.1. For all \( \theta_k \) we have
\[
R_T^{(k)}(\theta_k) = \sum_{i=1}^{d} \sum_{j=1}^{d} \sum_{m=1}^{N_{k,i}} \sum_{n=1}^{N_{k,j}} \frac{2}{T} \Upsilon_{ijk}(T - t_m^i, t_m^j - t_n^i) - \left( \frac{2}{T} \sum_{j=1}^{d} \sum_{m=1}^{N_{k,j}} \sum_{n=1}^{N_{k,i}} \phi_{kj}(t_n^i - t_m^j) \right) + \mu_k^2 - 2 \vartheta_{ij} \mu_k \tag{15}
\]
3.3.2 A quadratic program: SBF MHP

We highlight those MHP models where the kernels are a sum of basis functions (SBF); these are widely used in the literature. Our method applies to general linear MHP, and includes SBF MHP as a special case. In our numerical examples, we observe that SBF models often display better stability than more general parametric families, which may experience a form of ‘mode collapse’ (see Appendix C).

Consider a \((\mu, \Phi)\)-MHP and assume that for all \(i, j \in [d]\)

\[
\phi_{ij} = \sum_{l=1}^{r_{ij}} \omega_{ijl} \tilde{\phi}_{ijl},
\]

(16)

where \(\tilde{\phi}_{ijl}\) are fixed (known) functions. The parameters of the model are the background intensities \(\mu = \{\mu_i > 0 : i \in [d]\}\), and the weights \(\omega = \{\omega_{ijl} > 0 : i, j \in [d], l \in [r_{ij}]\}\).

We refer to this model as a \((\mu, \Phi)\)-SBF. In this case, the minimization of \(R_T^{(k)}\) is a quadratic program (QP). One might be tempted to first compute the matrices in this QP formulation, then use standard solvers to estimate the model parameters (for example, dual-primal methods, see Vandenberghe [41]).

The major difficulty is the pre-computation of the QP formulation, which in general, takes quadratic time. This complexity can be reduced by assuming \(r_{ij} = r\), \(\tilde{\phi}_{ijl} = \tilde{\phi}_l\), for all \(i, j, l\), and by choosing \(\tilde{\phi}_l\) to be an exponential decay. In this case, the pre-computation of the QP formulation will still have linear complexity in time, which is slower than the method we develop in this paper.

3.4 Gradient approximation

Next, for all \(\theta_k\), we construct an unbiased estimator of the gradient \(\nabla R_T^{(k)}(\theta_k)\) of the LSE [15], which we use as an input to our optimization. To do this, we use variance reduction techniques to construct an efficient unbiased estimator using the additive structure of [15].

3.4.1 Monte Carlo approximation problem

We first provide some notation to deal with the different types of sums involved in Corollary 3.1. For the rest of this section, fix \(i, j \in [d]\). Let

\[
A_T^j := \{T - t_m^i : m \in [N_T^j]\},
\]

\[
B_T^j := \{t_m^i - t_n^i : m \in [\varpi(i, j), N_T^j], n \in [\kappa(i, j, m)]\},
\]

\[
\tilde{B}_T^j := \{(T - t_m^i, t_m^i - t_n^i) : m \in [\varpi(i, j), N_T^j], n \in [\kappa(i, j, m)]\}.
\]

For some domain \(E\) consider a function \(f_{\theta_k} : E \to [0, +\infty)\) parameterized by \(\theta_k\). Let \(E_S \subseteq E\) be finite, with \(N_S := |E_S|\), and consider the generic problem of estimating

\[
S_T(\theta_k) := \sum_{x \in E_S} f_{\theta_k}(x).
\]

In our problem, we need to address the following configurations:

i) The single sums

- \(\sum_{m=1}^{N_T^j} \psi_{ki}(T - t_m^i)\); corresponds to \(f_{\theta_k} = \psi_{ki}, \mathcal{E} = \mathbb{R}, \mathcal{E}_S = A_T^j\).
- \(\sum_{m=1}^{N_T^j} \Upsilon_{ijk}(T - t_m^i, 0)\); corresponds to \(f_{\theta_k} = \Upsilon_{ijk}(\cdot, 0), \mathcal{E} = \mathbb{R}, \mathcal{E}_S = A_T^j\).

ii) The double sums

- \(\sum_{m=\varpi(i, j)}^{N_T^j} \sum_{n=1}^{\kappa(i,j,m)} \phi_{ij}(t_m^i - t_n^i)\); corresponds to \(f_{\theta_k} = \phi_{ij}, \mathcal{E} = \mathbb{R}, \mathcal{E}_S = B_T^j\).
- \(\sum_{m=\varpi(i, j)}^{N_T^j} \sum_{n=1}^{\kappa(i,j,m)} \Upsilon_{ijk}(T - t_m^i, t_m^j - t_n^j)\); corresponds to \(f_{\theta_k} = \Upsilon_{ijk}, \mathcal{E} = \mathbb{R}^2, \mathcal{E}_S = \tilde{B}_T^j\).
The vanilla Monte Carlo approach to this problem is to uniformly sample $N_{MC} < N_S$ elements of $E_S$, denoted by $E_{MC}$, and to consider the unbiased estimator

$$\hat{S}_T(\theta_k) := \frac{N_S}{N_{MC}} \sum_{x \in E_{MC}} f_{\theta_k}(x).$$

In practice, only mild variations of this approach are needed to achieve satisfactory Monte Carlo estimation of the single sums, even in the case of a nearly unstable MHP. However, the estimation of the double sums is significantly more challenging and, in practice, vanilla Monte Carlo is too imprecise for our problem because it does not capture the variations of $f_{\theta_k}$ on the domain $E_S$. For this reason, we develop a stratified sampling approach.

### 3.4.2 Estimating the single sums

Let $f_{\theta_k}$ denote $\psi_{ki}$ or $\Upsilon_{ik}(\cdot, 0)$. In this case, we want to estimate

$$S_T(\theta_k) = \sum_{m=1}^{N^+_I} f_{\theta_k}(T - t'_m).$$

Fix $n_{max} \in \mathbb{N}$ and consider a fixed increasing sequence of integers $b_0 < b_1 < \cdots < b_{n_{max}}$, with $b_0 := 1$ and $b_{n_{max}} < N^+_I$. The integer intervals $[b_p, b_{p+1} - 1]$ are the strata of a stratified Monte Carlo estimator of

$$\sum_{m=1}^{b_{n_{max}}-1} f_{\theta_k}(T - t'_m).$$

Define an unbiased estimator of $S_T(\theta_k)$ by

$$\hat{S}_T(\theta_k) := \sum_{p=1}^{n_{max}-1} \frac{b_{p+1} - b_p}{q_p} Z_p + \sum_{m=b_{n_{max}}}^{N^+_I} f_{\theta_k}(T - t'_m), \quad (17)$$

where for each $p$, we sample uniformly $q_p$ integers $(m_1^{(p)}, \ldots, m_{q_p}^{(p)})$ in the integer interval $[b_p, b_{p+1} - 1]$ without replacement and define

$$Z_p := \sum_{l=1}^{q_p} f_{\theta_k}(T - t'_{m_l^{(p)}}). \quad (18)$$

We do not set $b_{n_{max}} = N^+_I$ because, even for a stable MHP, there are values of $m$ such that $t'_m \sim T$, and these values may contribute significantly to the sum. We summarize this procedure in Algorithm 1.

**Algorithm 1:** Estimation of a single sum

**Result:** Estimator $\hat{S}_T(\theta_k)$

Initialize $\hat{S}_T(\theta_k) = \sum_{m=b_{n_{max}}}^{N^+_I} f_{\theta_k}(T - t'_m)$;

for $p$ in $[n_{max}-1]$ do

- Sample $q_p$ integers $(m_1^{(p)}, \ldots, m_{q_p}^{(p)})$;
- Use (18) to compute $Z_p$;
- Increment $\hat{S}_T(\theta_k)$ by $\frac{b_{p+1} - b_p}{q_p} Z_p$;
end
3.4.3 Estimating the double sums

We now consider the estimation of the terms

\[ \sum_{m=\pi(i,j)}^{N_j^k} \kappa(j,i,m) \sum_{n=1}^{N_i^k} \Phi_{ijk}(T - t_m^i, t_m^i - t_n^j), \]

and

\[ \sum_{m=\pi(k,j)}^{N_j^k} \kappa(j,k,m) \sum_{n=1}^{N_k^j} \phi_{kj}(t_m^k - t_n^j), \]

which appear in [15]. We briefly outline our method, and give a detailed description in Appendix B.2. Approaches similar to those used to estimate the single sums fail because of the larger number of terms in these sums, many of which make very small contributions, leading to a high variance in a simple estimation. Ideally, one could use stratified sampling of the data by time differences, which is computationally expensive, both in time and memory. Instead, we use stratified sampling by index differences as a proxy for time differences, which is significantly faster and memory efficient. Indeed, this approach does not store any additional data.

For event indices \( m, n \) such that \( t_n^j < t_m^i \), the difference \( \kappa(j,i,m) - n \) is the number of events of type \( j \) in the interval \((t_n^j, t_m^i)\). We refer to the quantity \( m = \kappa(j, i, m) - n + 1 \) as the lag between indices \( m, n \).

For all \( h \in [\kappa(j, i, N_j^k)] \), define the sets of event times with a given lag in their indices

\[ B_{ij,h}^T := \{ t_m^i - t_n^j : m \in [\pi(i,j), N_i^k], n = \kappa(j,i,m) - h + 1 \}, \]

\[ B_{ik,h}^T := \{ (T - t_m^i, t_m^i - t_n^j) : m \in [\pi(i,j), N_i^k], n = \kappa(j,i,m) - h + 1 \}. \]

For each element of the sets \( B_{ij,h}^T \) and \( B_{ik,h}^T \), the indices \( (m, n) \) are such that \( t_n^j \) is the \( h \)-th to last jump of type \( j \) before \( t_m^i \). Let \( E_{ij,h}^T \) denote \( B_{ij,h}^T \) or \( B_{ik,h}^T \) as appropriate for the sum under consideration. Define

\[ S_{ij,h}^T(\theta) := \sum_{x \in E_{ij,h}^T} f_{\theta}(x), \quad \text{hence} \quad S_T(\theta) := \sum_{h=1}^{\kappa(j,i,N_j^k)} S_{ij,h}^T(\theta). \]

The sets \( (E_{ij,h}^T) \) form a partition of \( E_{ij}^T \) of size \( \kappa(j,i,N_j^k) \), which is typically still too large to use as a stratification\(^1\). Our heuristic for this estimator is to first note that

\[ \lim_{t \to \infty} \phi_{ij}(t) = 0 \quad \text{and} \quad \lim_{s \to \infty} Y_{ijk}(t, s) = 0. \]

We expect the contribution of \( S_{ij,h}^T(\theta) \) to \( S_T(\theta) \) to decrease after a certain index \( h_{\text{max}} \), as the sequence \( (\min E_{ij,h}^T) \) is strictly decreasing. Hence, we focus on the estimation of

\[ S_{T_{\text{max}}}(\theta) := \sum_{h=1}^{h_{\text{max}}} S_{ij,h}^T(\theta) \]

separately from the estimation of the remainder

\[ S_{T_{\text{rest}}}(\theta) := \sum_{h=h_{\text{max}}+1}^{\kappa(j,i,N_j^k)} S_{ij,h}^T(\theta). \]

\(^1\)For example, one can think of the case \( j = i \), where \( \kappa(j,i,N_j^k) = N_j^k - 1. \)
We group several index lag sets to reduce the number of strata. Formally, let \( n_B \in \mathbb{N}^* \). Consider a partition \( B = (b_1, \ldots, b_{n_B}) \) of \([h_{\text{max}}]\). By abuse of notation, for all \( b \in B \) take the disjoint union and sum

\[
\mathcal{E}^{ij,b} := \bigcup_{h \in b} \mathcal{E}^{ij,h}, \quad S^{b}_T(\theta_k) := \sum_{h \in b} S^{h}_T(\theta_k).
\]

Figure 2 illustrates the construction of this stratification in the case \( i = j \).

For all \( p \in [n_B] \), sample \( q^p \) points \((x^{b_p}_{1}, \ldots, x^{b_p}_{q^p})\) uniformly and without replacement from \( \mathcal{E}^{ij,b_p} \), and define an unbiased estimator of \( S^{b}_T(\theta_k) \) by

\[
\hat{S}^{b_p,q^p}_T(\theta_k) = \frac{|\mathcal{E}^{ij,b}|}{q^p} \sum_{m=1}^{q^p} f_{\theta_k}(x^{b_p}_{m}).
\]

(19)

Fix in advance the total number \( Q \) of points we want to sample

\[
Q := \sum_{p=1}^{n_B} q^p.
\]

Let \( q := (q^1, \ldots, q^p) \) denote the absolute allocation. For all \( p \in [n_B] \), define the relative allocation \( q^p := q^p/Q \), and write \( \tilde{q} = (\tilde{q}^1, \ldots, \tilde{q}^{n_B}) \). An unbiased estimator of \( S^{\text{max}}_T(\theta_k) \) is

\[
\hat{S}^{\tilde{q}}_T(\theta_k) = \sum_{p=1}^{n_B} \hat{S}^{b_p,q^p}_T(\theta_k).
\]

(20)

In practice, one could fix \( q \) a priori, and choose \( h_{\text{max}} \) of order 30 to 50, for example. We see in Section 4 that this leads to satisfactory results in a variety of cases, particularly for monotonically decaying kernels. However, for general kernels, this approach is not sufficiently robust; we instead adaptively determine
the allocation of sampled points per stratum in Appendix B.2. As in the single sum case, use a standard stratified Monte Carlo approach with a fixed allocation to estimate the remainder

\[ S^\text{rest}_T(\theta_k) := \sum_{h=h_{\text{max}}+1}^{\kappa_{(j,j,N)^T}} S^h_T(\theta_k). \]

We denote the estimator of the remainder by \( \hat{S}^\text{rest}_T(\theta_k) \), and the estimator of the sum \( S_T(\theta_k) \) by

\[ \hat{S}_T(\theta_k) = \hat{S}_q^T(\theta_k) + \hat{S}^\text{rest}_T(\theta_k). \] (21)

This procedure is summarized in Algorithm 2.

---

**Algorithm 2:** Estimation of a double sum

**Result:** Estimator \( \hat{S}_T(\theta_k) \)

Initialize \( \tilde{q}_s^{(0)}(\theta_k) \);

for \( s \) in \([n_K]\) do

  for \( p \) in \([n_B]\) do

    Set \( \Delta q^p(s) = \tilde{q}_s^{p,(s-1)}(\theta_k)(n_B + \Delta Q_s); \)

    Sample without replacement \( \Delta q^p(s) \) points \( (x^m_{i,b,p}) \) in \( E_{ij},b \);

    Use (31) to compute \( \hat{S}_q^p(\Delta s)(\theta_k) \) and \( \hat{\sigma}_q^p(\Delta s)(\theta_k)^2 \);

    Use (33) to compute \( \hat{S}_q^p(s)(\theta_k) \) and \( \hat{\sigma}_q^p(s)(\theta_k)^2 \);

  end

Use (35) to compute \( \hat{S}^\text{rest}_T(\theta_k) \);

Use (21) to compute \( \hat{S}_T(\theta_k) \);

---

### 3.5 The procedure

#### 3.5.1 The gradient estimator

We use the sum estimators constructed above to define the gradient estimate \( G^{(k)}_T(\theta_k) \). For all \( i \in [d] \), denote the sum estimates by

- \( \hat{S}_{\psi,ki,T}(\theta_k) \) the estimator of \( \sum_{m=1}^{N_i^T} \psi_{ki}(T - t^i_m) \),
- \( \hat{S}_{\phi,kj,T}(\theta_k) \) the estimator of \( \sum_{m=1}^{N_j^T} \phi_{kj}(t^k_m - t^j_n) \),
- \( \hat{S}_{\psi,ki,T}(\theta_k) \) the estimator of \( \sum_{m=1}^{N_i^T} \psi_{ki}(T - t^i_m) \),
- \( \hat{S}_{\phi,kj,T}(\theta_k) \) the estimator of \( \sum_{m=1}^{N_j^T} \phi_{kj}(t^k_m - t^j_n) \).

We denote by \( G^{(k)}_T|_{\mu_k} \) the \( \mu_k \) component of \( G^{(k)}_T \) (the estimator of the partial derivative of the partial LSE \( R^{(k)}_T \) with respect to \( \mu_k \)), defined by

\[ G^{(k)}_T|_{\mu_k}(\theta_k) = 2 \left( \mu_k - \eta^{(k)}_T + \frac{1}{T} \sum_{i=1}^{d} \hat{S}_{\psi,ki,T}(\theta_k) \right). \] (22)
For \( p \in [d] \) and \( l \in [\rho_{ij}] \), let \( \vartheta_{kpl} \) be the \( l \)-th parameter of \( \phi_{kp} \). Denote by \( \mathcal{G}_{T}^{(k)}_{\vartheta_{kpl}} \) the \( \vartheta_{kpl} \) component of \( \mathcal{G}_{T}^{(k)} \) (the estimator of the partial derivative of the partial LSE \( \mathcal{R}_{T}^{(k)} \) with respect to \( \vartheta_{kpl} \)) defined by

\[
\mathcal{G}_{T}^{(k)}_{\vartheta_{kpl}}(\theta_{k}) = \frac{2}{T} \sum_{i=1, i \neq p}^{d} \frac{\partial \hat{S}_{\varphi, kp, T}}{\partial \vartheta_{kpl}} + \frac{\partial \hat{S}_{\varphi, pp, T}}{\partial \vartheta_{kpl}} + 2 \frac{\partial \hat{S}_{\varphi, kk, T}}{\partial \vartheta_{kpl}} - 2 \frac{\partial \hat{S}_{\varphi, \Phi, kp, T}}{\partial \vartheta_{kpl}} + \frac{2}{T} \frac{\partial \hat{S}_{\varphi, kp, T}}{\partial \vartheta_{kpl}} + \frac{1}{T} \frac{\partial \hat{S}_{\varphi, pp, T}}{\partial \vartheta_{kpl}}.
\]

**Remark 3.3.** To give a concise expression for the complexity in time of the computation of the estimate \( \mathcal{G}_{T}^{(k)}(\theta_{k}) \), suppose that

- the number of parameters per kernel is constant; i.e. there exists \( \rho \in \mathbb{N} \) such that for all \( i, j \in [d] \), we have \( \rho_{ij} = \rho \),
- the total sample size for the estimation of each single sum is constant, denoted \( Q^{(1)} \),
- the total sample size for the estimation of each double sum is constant, denoted \( Q^{(2)} \).

Under these assumptions, the complexity in time of the computation of a gradient estimate is roughly of order \( O(\rho d Q^{(2)} + \rho d Q^{(1)}) \).

### 3.5.2 Numerical scheme

Let \( n_{\text{iter}} \in \mathbb{N}^* \) denote the number of iterations in the optimization procedure. We initialize the parameters at a value \( \theta_{k}^{(0)} \), which can be chosen in a deterministic way or randomly sampled.

We use classic stochastic gradient methods to build a sequence \( (\theta_{k}^{(t)})_{t \in [n_{\text{iter}}]} \). Consider methods of the form

\[
\theta_{k}^{(t+1)} = \text{proj}_{\Theta_{k}} \left( \theta_{k}^{(t)} + \Delta \theta_{k}^{(t+1)} \right),
\]

where \( \Delta \theta_{k}^{(0)} = 0 \) and \( (\Delta \theta_{k}^{(t)})_{t \in [n_{\text{iter}}]} \) is a sequence that depends on the gradient estimate \( \mathcal{G}_{T}^{(k)}(\theta_{k}^{(t)}) \).

In the SGD literature, there exists a variety of constructions of the sequence \( (\Delta \theta_{k}^{(t)})_{t \in [n_{\text{iter}}]} \), leading to numerical schemes with very different properties. In this work, we consider the ADAM algorithm, proposed by Kingma and Ba [25], which is an adaptive learning rate method known to outperform most other state of the art algorithms on several machine learning tasks; see [Appendix B] for more details.

### 3.6 Complexity comparisons with other methods

Table 1 summarizes some features of our algorithm in comparison with existing techniques. The **Algorithm** column contains the reference of the algorithm, and its name if it has been named in the publication. **Parametric** specifies whether this is a parametric or non-parametric method. **Complexity** is the time complexity of a method, using the notation of this paper. Some algorithms consider a discretization of the kernels, we denote by \( n_{\text{dis}} \) the resolution of this discretization. In case the algorithms contain an inner loop, we denote its number of iterations by \( n_{\text{iter}} \). We denote by \( n_{\text{samples}} \) the number of observed sample paths of the MHP in case the method considers several sample paths. **Assumptions** refers to the additional assumptions made on the MHP (SBF \( \text{exp.} \) for an SBF MHP with exponential kernels, SBF \( \text{uni.} \) for an SBF MHP with \( r = 1 \) and a unique basis function \( \phi_{ij} = \phi \), \( \text{exp.} \) for an MHP with exponential kernels). **Type** is either the type of objective function used (LSE for Least Squares Error, \( \text{LL} \) for log-likelihood, \( \text{LL-EM} \) for marginal likelihood in an EM framework) or \( \text{MM} \) in the case of the method of moments. **Regularization** refers to the type of penalty used in the algorithm, if any. In case no penalty is used in the paper, but could be incorporated to the method with mild modifications, we write an asterisk.
Table 1: Comparison of the computational complexity of our algorithm ASLSD with state of the art estimation of MHP. Our two baseline cases are denoted by SumExp and WH, both here and in subsequent sections.

| Algorithm         | Parametric | Complexity                                      | Assumptions                  | Type   | Regularization |
|-------------------|------------|-------------------------------------------------|------------------------------|--------|----------------|
| ASLSD (This paper) | param.     | $O(n_{iter} \rho d^2 (Q^{(2)} + n_{max} \rho (Q^{(1)})$) | -                            | LSE    | *              |
| [7]               | param.     | $O(N_T \rho^2 d + n_{max} \rho^2 d^3)$          | SBF exp.                     | LSE    | Sparsity, Low Rank |
| MF [5]            | param.     | $O(d^2 \cdot AT \times \text{max}(AT, d \rho + d^3 \rho^3)$ | SBF, Stable, Mean field      | LL     | *              |
| (SumExp) [2]      | param.     | $O(n_{iter} N^2)$                               | SBF exp.                     | LL     | *              |
| MF [5]            | non-param. | $O(n_{iter} N^2)$                               | exp.                         | LL-EM  |                |
| MPL [33]          | non-param. | $O(n_{iter} N^2)$                               | -                            | LL-EM  | Good’s penalty |
| ADM4 [5]          | param.     | $O(d^3 n_{iter} + d^3 N_T n_{max} + n_{max} \rho d^2 N^2 + n_{max} \rho^2 N^2)$ | SBF uni.                     | LL-EM  | Sparsity, Low rank |
| MMEL [7]          | non-param. | $O(n_{iter} d^2 n_{iter} + n_{iter} n_{max} (dN_T + N^2 N^2))$ | -                            | LL-EM  | Kernel smoothing |
| MLE-SGLP [44]     | param.     | $O(n_{iter} d^2 n_{iter})$                      | SBF                           | LL-EM  | Sparse, Lasso, pairwise similarity |
| (WH) [2]          | non-param. | $O(n_{iter} d^3 n_{iter} + d^3 n_{iter}^3)$     | Stable                        | Autocovariance | -          |
| NPHC [1]          | param.     | $O(N_T d^2 + n_{iter} d^2)$                     | Stable                        | Integrated | cumulants |
4 Numerical experiments

In this section, we evaluate our estimation procedure on data simulated with an exact cluster based algorithm. To reproduce the results in this section, see the code in the `Experiments` folder of [https://github.com/saadlabyad/aslsd](https://github.com/saadlabyad/aslsd).

**Evaluation metrics** Consider a \((\mu^\circ, \Phi^\circ)\)-MHP observed on a window \([0, T]\), and a model \((\mu, \Phi)\). We define metrics to evaluate the performance of our algorithms:

- **L2RelErr** : We define this metric by
  \[
  \text{L2RelErr} := \frac{\|\mu^\circ - \mu\|_2^2}{\|\mu^\circ\|_2^2} + \frac{\|\Phi^\circ - \Phi\|_2^2}{\|\Phi^\circ\|_2^2},
  \] where \(\|\Phi\|_2 := \int_0^{+\infty} \phi^2_{ij}(t)dt\). \((24)\)

- **WassErr** : The (first) Wasserstein distance between probability measures \(f\) and \(g\) is given by
  \[
  \mathcal{W}_1(f, g) := \inf_{\pi \in \Gamma(f, g)} \int_{[0, +\infty) \times [0, +\infty)} |x - y|d\pi(x, y),
  \] where \(\Gamma(f, g)\) is the space of measures on \([0, +\infty) \times [0, +\infty)\) with marginals \(f, g\). Define
  \[
  \text{WassErr} := \sum_{i=1}^{d} |\mu^\circ_i - \mu_i| + \sum_{i=1}^{d} \sum_{j=1}^{d} \mathcal{W}_1 \left( \frac{\phi^\circ_{ij}}{||\phi^\circ||_1} , \frac{\phi_{ij}}{||\phi||_1} \right) + \sum_{i=1}^{d} \sum_{j=1}^{d} ||\phi^\circ_{ij}||_1 - ||\phi_{ij}||_1. \tag{25}\]

**Goodness-of-fit** Residual analysis is the state of the art goodness-of-fit test for MHP. This analysis relies on the time transformation property given by Ogata [33], Section 3.3. For all \(m \in \mathbb{N}^+, k \in [d]\), let
  \[
  s^k_m = \Lambda_k(1^k_m).
  \] For each \(k \in [d]\), define the point process \(S^k := \{s^k_m : m \in \mathbb{N}^+\}\); then \((S^k)_{k \in [d]}\) are independent standard Poisson processes. The inter-arrival times of \(S^k\) (‘residuals’), for a model that fits the data well must therefore be close to a standard exponential distribution. To assess if fitted residuals satisfy this property, we display the Q-Q plots of the residuals against a standard exponential distribution. As visual comparison of the fit of different models can be difficult, we also use the probability plot of residuals, which are defined by
  \[
  z^k_m = 1 - \exp(-s^k_m), \text{ and } Z^k := \{z^k_m : m \in \mathbb{N}^+\}.
  \] If the residuals \(S^k\) are exponentially distributed, then \((Z^k)_{k \in [d]}\) are independent random variables, uniformly distributed on \([0, 1]\). For improved clarity, we subtract the \(y = x\) line from the probability plots and rescale \(z^k_m\) with a multiplicative factor \(\sqrt{N^k_T - 1}\); for large \(N^k_T\), Donsker’s theorem indicates that this results in a process which is approximately a Brownian bridge. In each probability plot, we plot dashed lines corresponding to the 99% critical value for the Kolmogorov–Smirnov test.

**Benchmarks** We compare the performance of our algorithm to the following state of the art methods:

- **SumExp** : an SBF exponential MHP model, fitted using the algorithm in Bonnaffe et al. [2]. This is an interesting benchmark to evaluate the quality of the fit for the decay parameter \(\beta\) in a non SBF exponential using our method.

- **WH** : the algorithm proposed in Bacry and Muzy [2], a non-parametric estimation method which solves a Wiener–Hopf system derived from the autocovariance of the MHP. This method applies to any stable MHP.

These two algorithms are implemented in the python `tick` package [6], which we use for comparison. In Section 4.1, we specify the parameter values for the various MHP that we consider, and we plot paths of the solver to illustrate the performance of our estimation algorithm. In Section 4.2 we compare the results of our algorithm with those of our benchmarks for each evaluation metric.
4.1 Data

For each kernel type, we simulate one path of each MHP and fit the corresponding models, to illustrate the trajectory of our solver.

4.1.1 Exponential kernels

Univariate case  Consider a univariate MHP with exponential kernel, with true parameters

\[ \mu^o = 1.5, \quad \omega^o = 0.2, \quad \beta^o = 1. \]

For this ground truth MHP, we consider two estimation models:

- **SbfExp1D**: SBF exponential model as in Section 3.3.2 with only one basis function \( r = 1 \), and fix \( \beta = \beta^o \). We only estimate \( \mu^o, \omega^o \).

- **Exp1D**: exponential model as in Section 2.1.2 and estimate \( \mu^o, \omega^o, \beta^o \).

We simulate one path of this process up to \( T = 10^7 \), which results in 18,754,765 jumps. We fit the SbfExp1D model and plot the path of our solver in Figure 3. We fit the model Exp1D and plot the

![Figure 3: Contour plot of the LSE and SGD updates in the SbfExp1D model.](image)

White circle is the randomly chosen initial point. Green circle is the true value of the parameters used for simulation. White line is the trajectory of parameter estimates using our algorithm.

updates of the parameters and the estimates of the partial derivatives of the LSE in Figure 4.

Multivariate case  Consider a bivariate MHP with exponential kernels, and true parameters

\[ \mu^o = \begin{pmatrix} 1.5 \\ 1 \end{pmatrix}, \quad \omega^o = \begin{pmatrix} 0.2 & 0.6 \\ 0.7 & 0.1 \end{pmatrix}, \quad \beta^o = \begin{pmatrix} 1 & 1.5 \\ 2 & 1.3 \end{pmatrix}. \]

For this ground truth MHP, we consider an estimation model Exp2D which is an exponential model as in Section 2.1.2. We simulate one path of this process up to \( T = 10^5 \), which results in 1,098,456 jumps. We fit the Exp2D model; Figure 5 plots the estimated kernels.

4.1.2 Gaussian kernels

Univariate unimodal case  Consider a univariate MHP with Gaussian kernel, and with true parameters

\[ \mu^o = 1.5, \quad \omega^o = 0.5, \quad \beta^o = 0.5, \quad \delta^o = 3. \]
For this ground truth MHP, we consider the Gauss1D estimation model, which is a Gaussian model as in Section 2.1.2. We simulate one path of this process up to $T = 10^6$, which results in 3,005,742 jumps, and use our procedure to estimate $\mu^\circ$, $\omega^\circ$, $\beta^\circ$, $\delta^\circ$. We fit the Gauss1D model and provide more insight on the estimation of each parameter in Figure 6.

**Univariate multimodal case** Consider a univariate MHP with a basis of three Gaussian kernels, with true background rate $\mu^\circ = 0.01$ and with true kernel parameters

- $\omega_1^\circ = 0.2$
- $\omega_2^\circ = 0.3$
- $\omega_3^\circ = 0.4$
- $\beta_1^\circ = 0.4$
- $\beta_2^\circ = 0.6$
- $\beta_3^\circ = 0.8$
- $\delta_1^\circ = 1$
- $\delta_2^\circ = 3$
- $\delta_3^\circ = 8$.

For this ground truth MHP, we consider the SbfGauss1D10R estimation model, which is a SBF Gaussian model as in Section 3.3.2, with ten basis function ($r = 10$), where for all $l \in [10]$ set $\beta_l = 0.5$ and $\delta_l = l - 1$. To reflect typical usage, the model is somewhat misspecified, as the true kernel mixture cannot be expressed in terms of the SBF family.

We simulate one path of this process up to $T = 10^7$, which results in 986,996 jumps, and use our procedure to estimate the background rate $\mu^\circ$ and the kernel parameters $(\omega^\circ_l)_{l \in [3]}$. We fit the SbfGauss1D10R model and provide more insight on the fitted kernel and residuals in Figure 8.

We have also considered fitting this model using a non-SBF approach. This exhibited a persistent problem of mode collapse — after sufficient training, all the components of the mixture coalesce, leading to a poorly fitting model. We discuss this in more detail in Appendix C.

### 4.2 Performance for various sample sizes

For each ground truth MHP $(\mu^\circ, \Phi^\circ)$ specified earlier, we simulate $n_{\text{paths}}$ paths $(\mathcal{T}^{(p)})_{p \in [n_{\text{paths}}]}$ of the process up to a terminal horizon $T$. We consider $(T_q)_{q \in [n_{\text{times}}]}$, an increasing sequence of times. For each $(\mu^\circ, \Phi^\circ)$, for each simulated path $\mathcal{T}^{(p)}$, and for each integer $q \in [n_{\text{times}}]$, we define $\mathcal{T}^{(p,q)} := \mathcal{T}^{(p)} \cap [0, T_q]$, i.e., the path of the process truncated at $T_q$, containing $N^{(p)}_{T_q}$ jumps. Finally, for each ground truth MHP, for each evaluation metric, and for each of the three algorithms considered (our method and the two benchmarks), we fit a given MHP model to the observations $\mathcal{T}^{(p,q)}$ and compute the error $\epsilon^{(p,q)}$. For each time discretization step $q$, we compute the empirical mean (resp. 25th percentile and 75th percentile) of $(\epsilon^{(p,q)})_{p \in [n_{\text{paths}}]}$ and denote it by $M_q^{(q)}$ (resp. $Q_{0.25}^{(q)}$ and $Q_{0.75}^{(q)}$). Define the mean number of jumps per path $
abla_q := \frac{1}{n_{\text{paths}}} \sum_{p=1}^{n_{\text{paths}}} N^{(p)}_{T_q}$. 

---

Figure 4: SGD updates in the Exp1D model. Orange lines correspond to the true parameter values in the parameter updates column (left), and to zero in the gradient updates column (right).

Figure 5: SGD updates in the Gauss1D model. Orange lines correspond to the true parameter values in the parameter updates column (left), and to zero in the gradient updates column (right).
Figure 5: Fitted kernels for the Exp2D model.

Figure 9 plots the mean errors \( M(q)_{q \in \mathbb{N}_{\text{times}}} \) against the mean number of jumps \( \tilde{N}_q_{q \in \mathbb{N}_{\text{times}}} \), and the shaded area between the lower quartiles \( Q_{0.25}^{(q)}_{q \in \mathbb{N}_{\text{times}}} \) and the upper quartiles \( Q_{0.75}^{(q)}_{q \in \mathbb{N}_{\text{times}}} \). We see that our algorithm outperforms the WH benchmark with respect to each evaluation metric in the two exponential cases, Exp1D and Exp2D. In the unimodal Gaussian case Gauss1D, ASLSD outperforms WH for smaller datasets (under \( 10^6 \) jumps) in the Wasserstein metric \( \text{WassErr} \). For the multimodal Gaussian example SbGauss1D10R, WH typically outperforms ASLSD, however this is due to the (deliberate) misspecification of the SBF Gaussian model which is seen to approach its lower \( L^2 \) error bound. Our procedure consistently outperforms the SumExp benchmark for all ground truths.

5 Applications

In this section, we apply our estimation procedure to real world data. To reproduce the results in this section, see code in the Applications folder of https://github.com/saadlabyad/aslsd.

5.1 News propagation

Data In this application, we are interested in the diffusion of information across different media platforms. Gomez Rodriguez, Leskovec, and Schölkopf [19] compiled news articles from several websites that mention a selection of keywords into the MemeTracker dataset. There exist different versions of the MemeTracker dataset, with different data and different structures; we are interested in the one proposed by Gomez Rodriguez et al. [18]. Each file lists the times of occurrences of a given keyword in a variety of media outlets. The primary focus of the MemeTracker dataset is the analysis of event cascades: the occurrences of keywords happen through generic sentences shared in websites (referred to as memes). In this application, we do not take an event cascade viewpoint on the data in MemeTracker, but rather model the excitation in the occurrences of a given keyword irrespective of the meme to which it belongs to.
Figure 6: SGD updates in the **Gauss1D** model. Orange lines correspond to the true parameter values.

Figure 7: Fit results for the **Gauss1D** model.
Figure 8: Fit results for the SbfGauss1D10R model.

Left: blue line corresponds to the fitted SbfGauss1D10R model using ASLSD, orange lines to the ground truth, and green lines to the $L^2$ projection of the ground truth MHP on the parametric space of kernels of SbfGauss1D10R. Right: Q-Q plot of fitted residuals.

Figure 9: Performance of ASLSD and benchmarks.

Error plots for the various ground truth MHP. Blue lines correspond to our algorithm, orange lines to the WH benchmark, and green lines to the SumExp benchmark. In the lower-rightmost plot, dashed blue line is the lower bound of the $L2RelErr$ between the ground truth MHP and a SbfGauss1D10R MHP. This lower bound corresponds to the $L2RelErr$ between the ground truth MHP and the $L^2$ projection of the ground truth MHP on the parametric space of kernels of SbfGauss1D10R.
We model mentions of the keyword related to the British Royal family, a few months after the wedding of Prince William and Catherine Middleton on 29 April 2011. We limit ourselves to data between 1 November 2011 at midnight UTC and 1 March 2012 at midnight UTC. The data is timestamped in Unix time in hours with a second resolution. When two or more posts have the same timestamp, we only keep the event that appears first in the dataset.

**Univariate** First, we aggregate all publication timestamps into a path of a one-dimensional point process. We use our method to fit two exponential models: with one exponential (Exp1D1R) and with six exponentials (Exp1D6R). For benchmark purposes, we fit SumExp (with 6 fixed exponential decays) and WH. Figure 10 displays the probability plots of the residuals for each model, showing that our method is competitive with state of the art algorithms. Figure 11 plots the fitted kernels.

Distinction by media nationality Diffusion dynamics of news related to the British Royal family might significantly differ between British news outlets, North American and Australian media, and those from other nationalities. In the 5,000 news websites that appear in the MemeTracker dataset, it is not sufficient to use the top-level domain of the website to deduce its country. We manually verify the

---

2The exact keyword is *prince william-william-kate middleton-kate-middleton-westminster-watch-marriage-queen-king-elizabeth-charles.*

3For example, several major news websites had a .com top-level domain at the time of data collection of MemeTracker, such as *The Economist* (British), *El País* (Spanish) or *Globo* (Brazilian).
nationality of the media sources; the list of media nationalities is available as a csv file in the Applications folder of our repository. We model publication times of US and UK articles related to this keyword as a bi-dimensional MHP (dimension $i = 1$ corresponds to US articles and $i = 2$ to UK articles). We use our method to fit two exponential models: with one exponential ($\text{Exp2D1R}$) and three exponentials ($\text{Exp2D3R}$); as well as one Gaussian model ($\text{Gauss2D1R}$). For benchmark purposes, we fit $\text{SumExp}$ (with 6 fixed exponential decays) and WH. Figure 12 displays the Q-Q plots and probability plots for these models, showing that our algorithms compete with those of the benchmarks. Figure 13 plots the fitted kernels.

In this case, the data seems well modeled by an Exponential MHP. The hyperparameters for the WH algorithm were selected by hand to give reasonable performance, however further tuning may be possible, and may reduce the oscillatory behaviour we observe, particularly in the cross-excitation kernels $\phi_{12}$ and $\phi_{21}$.

5.2 Epidemic propagation

Data In this second application, we model the propagation of Malaria in China. Unwin, Routledge, Flaxman, Rizoiu, Lai, Cohen, Weiss, Mishra, and Bhatt [39] use a slightly modified univariate linear Hawkes model with a delayed Rayleigh kernel to study the transmission in this context. To fit their model, given the typically small number of observations in the applications they consider, Unwin et al. [39] compute exactly the log-likelihood of their observations and input it to a standard optimization solver. We use their data for the propagation of malaria in the Yunan province between 1 January 2011 and 24 September 2013, which is available in Unwin et al. [40].

Figure 12: Residual plots.
Q-Q plots (left column) and rescaled probability plots (right column) for the US (upper row) and UK publications (lower row). WH displays very poor goodness of fit and is omitted for clarity.
**Model** We fit two models using ASLSD: $\text{SbfGauss1D10R}$ is a SBF Gaussian model with ten Gaussians (with uniformly spaced means in $[0, 20]$ and standard deviations equal to 1.9), and $\text{Gauss1D1R}$ is a non-SBF Gaussian model. In addition to this, we consider two benchmarks: $\text{SumExp}$, and $\text{Poisson}$, which is a naive homogeneous Poisson model. (Given the small number of observations, the standard implementation of the $\text{WH}$ method would not run on this dataset.)

For this data, the $\text{SumExp}$ model gives residuals which pass a KS-test, with a p-value of 7.8%. The $\text{Gauss1D1R}$ gives a better fit, passing the KS test with a p-value of 14.6%. Unwin et al. [39] propose that the self-excitation of malaria infection is better modelled by a non-monotonic kernel, because of the delay in contagions due to the mosquito phase of the disease. They force this by using a delayed Rayleigh kernel, with a delay of 15 days. It is therefore not surprising that $\text{Gauss1D1R}$ outperforms $\text{SumExp}$. However, the fitted mean of the Gaussian kernel in $\text{Gauss1D1R}$ is around 5 days, well below the 15 days delay in Unwin et al. [39]. The $\text{SbfGauss1D10R}$ model gives a significantly better p-value, at 40.5%. In addition to the mode at 5, we see a second mode appearing slightly below 15.

Given the delays in contagion caused by the malaria life cycle (in particular during the mosquito phase, as discussed, for example, in Stopard, Churcher, and Lambert [38] or Unwin et al. [39]), we might expect the true kernel to have negligible mass below 10 days. However, we speculate that this dataset may not be based on full observation of malaria cases (for example, due to some cases remaining unreported). This can introduce significant biases in the estimated self-excitation structure, which may explain the observed primary mode with a delay of 5 days.

6 Conclusion

In this work, we proposed a new estimation method for linear multivariate Hawkes processes, which applies to large datasets and general kernels. Our method is a fast stochastic method that can be used
Model selection In the procedure we proposed, we did not consider robustness to model error. Therefore, model selection is still an unexplored area in the literature of MHP. Cross validation methods are one of the major model selection frameworks, but it is not clear how to apply them to MHP due to the auto-regressive form of the conditional intensity. Our methods can easily be extended to include regularization terms, for example to encourage sparsity, opening the way for a wide variety of modelling approaches to be considered.

Non-linear Hawkes The conditional intensity of a point process has to remain positive, and the fact that the kernels of a linear MHP are positive is sufficient to satisfy this condition. However, because
their kernels are positive, linear MHP are unable to model inhibition between event types. In high-frequency limit order book data for example, inhibitory behaviours are well known: for example, Lu and Abergel [31] observe empirically that order cancellations on one side that change the spread may inhibit submission of contraside market orders. Non-linear Hawkes processes (Brémaud and Massoulié [10]) can model inhibitory interactions, but the estimation of non-linear Hawkes processes is a hard problem and the literature is scarce, with the notable exceptions of the work of Wang, Xie, Du, and Song [43] and Menon and Lee [32]. To the best of our knowledge, there exists no fast method to calibrate non-linear Hawkes processes with general kernels. The additive decomposition we propose in Theorem 3.1 cannot be directly transposed to the LSE of a non-linear Hawkes process; in fact it seems improbable to obtain any useful additive decomposition of the LSE in this case. Following the approach of Menon and Lee [32], it would be interesting to investigate the use of other contrast functions than the LSE for the estimation of non-linear Hawkes processes.
A Proof of Theorem 1

In this appendix, we prove [Theorem 3.1]

**Lemma A.1.** For all \( i, j \in [d] \) and all \( t \geq 0 \), define

\[
\varphi_{ij}(t) := \sum_{n=1}^{\kappa(j,t)} \phi_{ij}(t - t_n^j).
\]

**Note that for all** \( t \leq t_1^j \), we have \( \varphi_{ij}(t) = 0 \). **For all** \( i, j \in [d] \) and all \( t \geq 0 \), define

\[
C_{ij}(t) := \int_0^t \varphi_{ij}(u)du.
\]

**Then,**

\[
C_{ij}(T) = \sum_{n=1}^{N_T^j} \psi_{ij}(T - t_n^j).
\]  \hspace{1cm} \text{(26)}

**Proof.** Fix \( i, j \in [d] \). By definition

\[
C_{ij}(T) = \int_0^T \sum_{n=1}^{\kappa(j,t)} \phi_{ij}(t - t_n^j)dt = \int_{t_1^j}^T \sum_{n=1}^{\kappa(j,t)} \phi_{ij}(t - t_n^j)dt.
\]

Split the integral

\[
C_{ij}(T) = \int_{t_1^j}^T \sum_{n=1}^{\kappa(j,t)} \phi_{ij}(t - t_n^j)dt + \sum_{m=1}^{N_T^j - 1} \int_{t_m^j}^{t_{m+1}^j} \sum_{n=1}^{\kappa(j,t)} \phi_{ij}(t - t_n^j)dt.
\]

For all \( t \in \left[ t_{N_T^j}^j, T \right] \) we have \( \kappa(j, t) = N_T^j \). For all \( m \in \left[ N_T^j - 1 \right] \), for all \( t \in \left[ t_m^j, t_{m+1}^j \right] \) we have \( \kappa(j, t) = m \). Hence,

\[
C_{ij}(T) = \int_{t_1^j}^{t_{N_T^j}} \sum_{n=1}^{N_T^j} \phi_{ij}(t - t_n^j)dt + \sum_{m=1}^{N_T^j - 1} \int_{t_m^j}^{t_{m+1}^j} \sum_{n=1}^{m} \phi_{ij}(t - t_n^j)dt.
\]

Use Fubini’s Theorem to write

\[
C_{ij}(T) = \sum_{n=1}^{N_T^j} \int_{t_n^j}^{t_{n+1}^j} \phi_{ij}(t - t_n^j)dt + \sum_{m=1}^{N_T^j - 1} \sum_{n=1}^{m} \int_{t_n^j}^{t_{n+1}^j} \phi_{ij}(t - t_n^j)dt.
\]

Re-index the sums of the second term and use a change of variable to conclude

\[
C_{ij}(T) = \sum_{n=1}^{N_T^j} \int_{t_n^j}^{t_{n+1}^j} \phi_{ij}(u)du + \sum_{n=1}^{N_T^j - 1} \int_{t_n^j}^{t_{n+1}^j} \phi_{ij}(u)du,
\]

\[
= \sum_{n=1}^{N_T^j} \int_{t_n^j}^{t_{n+1}^j} \phi_{ij}(u)du + \sum_{n=1}^{N_T^j - 1} \int_{t_n^j - t_n^j}^{t_{n+1}^j - t_n^j} \phi_{ij}(u)du,
\]

\[
= \sum_{n=1}^{N_T^j} \int_0^{t_{n+1}^j - t_n^j} \phi_{ij}(u)du.
\]
Lemma A.2. For all \( i,j,k \in [d] \) define
\[
I_{ijk}(T) := \int_0^T \varphi_{ki}(t) \varphi_{kj}(t) dt.
\]
Then,
\[
I_{iik}(T) = \sum_{m=1}^{N_T^k} \sum_{n=1}^{N_T^k} \gamma_{i,m,n}(T-t_{nm}^i, t_{nm}^i - t_{nm}^i) + 2 \sum_{m=1}^{N_T^k} \sum_{n=1}^{N_T^k} \gamma_{i,m,n}(T-t_{nm}^i, t_{nm}^i - t_{nm}^i),
\]
and if \( i \neq j \)
\[
I_{ijk}(T) = \sum_{m=1}^{N_T^k} \sum_{n=1}^{N_T^k} \gamma_{i,m,n}(T-t_{nm}^i, t_{nm}^j - t_{nm}^i) + \sum_{m=1}^{N_T^k} \sum_{n=1}^{N_T^k} \gamma_{i,m,n}(T-t_{nm}^i, t_{nm}^j - t_{nm}^i).
\]

Proof. Fix \( j,k,m \in [d] \). Let \( N = N_T^k + N_T^m \). Define
\[
(t_{q,p})_{q \in [1,N]} = \{t_{q,p}^i, p \in \{k, m\}, i \in [1,N_T^q]\},
\]
such that for all \( q \in [1, N-1] \), \( q < t_{q+1} \). For all \( q \in [N] \), define \( \zeta(q) \in \{k, m\} \) and \( \iota(q) \in [N_T^{\zeta(q)}] \) such that \( t_q = t_{\iota(q)}^{\zeta(q)} \). Define for all \( s \geq 0 \)
\[
I_{jkm}(s) = \int_0^s \varphi_{jk}(t) \varphi_{jm}(t) dt.
\]
We prove by induction that for all \( q \in [N] \),
\[
I_{jkm}(t_q) = \sum_{i=1}^{N_T^k} \sum_{n=1}^{N_T^k} \int_{t_q}^{t_{q+1}} \varphi_{jk}(t) \varphi_{jm}(t) dt + \sum_{i=1}^{N_T^m} \sum_{n=1}^{N_T^m} \int_{t_q}^{t_{q+1}} \varphi_{jk}(t) \varphi_{jm}(t) dt.
\]
For \( q = 1 \), the sum is empty so the result is true. Let \( q \in [N-1] \), assume the property is true for \( q \). Define
\[
\kappa_k := \kappa(k, \zeta(q+1), \iota(q+1)), \quad \kappa_m := \kappa(m, \zeta(q+1), \iota(q+1)).
\]
The proof that the property is true for \( q+1 \) results directly from the decomposition
\[
I_{jkm}(t_{q+1}) - I_{jkm}(t_q) = \sum_{i=1}^{N_T^k} \sum_{n=1}^{N_T^k} \int_{t_q}^{t_{q+1}} \varphi_{jk}(t) \varphi_{jm}(t) dt = \sum_{i=1}^{N_T^k} \sum_{n=1}^{N_T^k} \int_{t_q}^{t_{q+1}} \varphi_{jk}(t) \varphi_{jm}(t) dt.
\]
Therefore, we have
\[
I_{jkm}(T) = \sum_{i=1}^{N_T^k} \sum_{n=1}^{N_T^k} \int_0^T \varphi_{jk}(t) \varphi_{jm}(t) dt + \sum_{i=1}^{N_T^m} \sum_{n=1}^{N_T^m} \int_0^T \varphi_{jk}(t) \varphi_{jm}(t) dt.
\]
Use the change of variables \( u := t-t_k^i \) and \( u := t-t_m^m \), to write
\[
I_{jkm}(T) = \sum_{i=1}^{N_T^k} \sum_{n=1}^{N_T^k} \int_0^{T-t_k^i} \varphi_{jk}(u) \varphi_{jm}(u + t_k^i - t_m^m) du + \sum_{i=1}^{N_T^m} \sum_{n=1}^{N_T^m} \int_0^{T-t_m^m} \varphi_{jm}(u) \varphi_{jk}(u + t_m^m - t_k^i) du.
\]
The result then follows from the definitions of \( Y_{jkm} \) and \( Y_{jmk} \). \( \square \)
Theorem 3.1 (Least squares error). The LSE $R_T$ satisfies

$$R_T(\theta) = \frac{2}{T} \sum_{k=1}^{d} \sum_{i=1}^{d} \sum_{j=1}^{d} \sum_{m=\varpi(i,j)}^{N_{r}^{k}} \sum_{n=1}^{m} \gamma_{ij}(T - t_{m}^{i},t_{m}^{i} - t_{m}^{n})$$

$$- \frac{2}{T} \sum_{k=1}^{d} \sum_{i=1}^{d} \sum_{m=\varpi(k,j)}^{N_{r}^{k}} \sum_{n=1}^{m} \phi_{kj}(v_{m}^{k} - t_{m}^{n}) + \frac{d}{T} \sum_{k=1}^{d} \sum_{m=1}^{N_{r}^{k}} \gamma_{iik}(T - t_{m}^{i},0).$$

Proof of Theorem 3.1. Fix $\theta \in \Theta$. By definition

$$R_T(\theta) := \frac{1}{T} \sum_{k=1}^{d} \int_{0}^{T} \lambda_{k}(t)^{2} dt - \frac{2}{T} \sum_{k=1}^{d} \sum_{m=1}^{N_{r}^{k}} \lambda_{k}(t_{m}^{k}).$$

It is clear that

$$\sum_{k=1}^{d} \sum_{m=1}^{N_{r}^{k}} \lambda_{k}(t_{m}^{k}) = \sum_{k=1}^{d} N_{r}^{k} \mu_{k} + \sum_{k=1}^{d} \sum_{j=1}^{d} \sum_{m=1}^{N_{r}^{k}} \sum_{n=1}^{m} \phi_{kj}(v_{m}^{k} - t_{m}^{n}).$$

Let $j \in [d]$. By definition

$$\int_{0}^{T} \lambda_{j}(t)^{2} dt = \int_{0}^{T} \left( \mu_{j}^{2} + 2 \mu_{j} \sum_{l=1}^{d} \varphi_{jk}(t) + \left( \sum_{k=1}^{d} \varphi_{jk}(t) \right)^{2} \right) dt,$$

$$= \mu_{j}^{2} T + 2 \mu_{j} \sum_{k=1}^{d} C_{jk}(T) + \int_{0}^{T} \left( \sum_{k=1}^{d} \varphi_{jk}(t) \right)^{2} dt.$$

We note that

$$\int_{0}^{T} \left( \sum_{k=1}^{d} \varphi_{jk}(t) \right)^{2} dt = \sum_{k=1}^{d} \sum_{m=1}^{d} \int_{0}^{T} \varphi_{jk}(t) \varphi_{jm}(t) dt.$$ 

The result follows directly from the application of Lemma A.1 and Lemma A.2.

B Insights on the gradient estimator

In this appendix, we bring additional insight to the construction of the estimators of single sums from Section 3.4.2 and the estimators of double sums from Section 3.4.3.

B.1 Estimating the single sums

Our heuristic for the construction of this estimator is first to note that the sequence $m \mapsto f_{\theta_{m}}(T - t_{m}^{i})$ is decreasing, because the functions $\psi_{ki}(\cdot)$ and $\gamma_{iik}(\cdot,0)$ are increasing. Second, qualitatively, we expect $t_{m}^{i} \ll T$ for a stationary process, except for the largest values of index $m$. Finally, we expect the variance of $f_{\theta_{m}}(T - t_{m}^{i})$ to increase with $m$. The hyper-parameters of this estimator are the bounds of the strata, $b_{p}$, and the number of points sampled in the strata, $q_{m}$. In our numerical experiments, we chose the index $b_{\text{max}}$ such that $N_{r}^{T} - b_{\text{max}} \sim 10^{p}$. One can imagine schemes where $b_{\text{max}}$ is chosen adaptively, but our experiments suggested this was not useful in practice. Given $b_{\text{max}}$, we choose $b_{\text{max}} - 1 = b_{\text{max}} - \delta$ where $\delta \in \mathbb{N}$ is a hyper-parameter, and chose the other bounds ($b_{p}$), such that

$$b_{p+1} - b_{p} \sim (b_{p+2} - b_{p+1})^{2}.$$
B.2 Estimating the double sums

Constructing a stratification For a given number of sample points $Q$, we build an adaptive strategy to allocate our points in $q$. A naive criterion is to minimize the total variance of the estimator. Define the total standard deviation in bucket $b$ by

$$
\sigma_T^b(\theta_k) := \sqrt{\frac{1}{|E_T^{ij,b}|} \sum_{x \in E_T^{ij,b}} f_{\theta_k}^2(x) - \frac{1}{|E_T^{ij,b}|^2} \left( \sum_{x \in E_T^{ij,b}} f_{\theta_k}(x) \right)^2}.
$$

For all $p \in [n_B]$, define

$$
\hat{q}_p^T(\theta_k) := \sqrt{\frac{|E_T^{ij,b}|}{|E_T^{ij,b}| - 1} \sigma_T^b(\theta_k) - \sum_{p'=1}^{n_B} \frac{|E_T^{ij,b}|}{|E_T^{ij,b'}| - 1} \sigma_T^{b'}(\theta_k)},
$$

and let

$$
\hat{q}_*(\theta_k) = (\hat{q}_1^T(\theta_k), \ldots, \hat{q}_N^T(\theta_k)).
$$

The variances of the estimators, accounting for sampling without replacement, are

$$
\text{Var} \left[ \hat{S}_{s,T}^{b}(q^T) (\theta_k) \right] = \frac{|E_T^{ij,b}|}{q^T - 1} \left( 1 - \frac{|E_T^{ij,b}|}{q^T - 1} \right) \sigma_T^b(\theta_k)^2,
$$

$$
\text{Var} \left[ \hat{S}_{s,T}^T(\theta_k) \right] = \sum_{p=1}^{n_B} \frac{|E_T^{ij,b}|}{q^T - 1} \left( 1 - \frac{|E_T^{ij,b}|}{q^T - 1} \right) \sigma_T^b(\theta_k)^2.
$$

The variance of the estimator, as a function of $\hat{q}$, is

$$
\text{Var} \left[ \hat{S}_{s,T}^T(\theta_k) \right] = \frac{1}{Q} \sum_{p=1}^{n_B} \frac{|E_T^{ij,b}|}{|E_T^{ij,b}| - 1} \sigma_T^{b^p}(\theta_k)^2 - \sum_{p=1}^{n_B} \frac{|E_T^{ij,b}|}{|E_T^{ij,b'}| - 1} \sigma_T^{b^p}(\theta_k)^2.
$$

Use Jensen’s inequality to write

$$
\text{Var} \left[ \hat{S}_{s,T}^T(\theta_k) \right] \geq \frac{1}{Q} \sum_{p=1}^{n_B} \frac{|E_T^{ij,b}|}{|E_T^{ij,b}| - 1} \sigma_T^{b^p}(\theta_k)^2 - \sum_{p=1}^{n_B} \frac{|E_T^{ij,b}|}{|E_T^{ij,b'}| - 1} \sigma_T^{b^p}(\theta_k)^2.
$$

The inequality in (30) is tight, and this lower bound is attained for $\hat{q} = \hat{q}_*(\theta_k)$. For a given number of sample points $Q$, the allocation $q = \hat{q}_*(\theta_k)$ is the allocation that minimizes the variance of the estimator $\hat{S}_{s,T}^T(\theta_k)$. However, the computation of the optimal allocation $\hat{q}_*(\theta_k)$ is expensive, because the computation of the vector of variances $\sigma_T^b(\theta_k)$ has quadratic worst-case complexity.

In Section 4, we discuss cases where it is not necessary to choose $\hat{q} = \hat{q}_*(\theta_k)$ for the estimation procedure to converge, notably for some decreasing kernels. But for general kernels, setting an arbitrary allocation $\hat{q}$ does not lead to satisfactory estimates in practice, this is why we propose an adaptive estimator of the optimal allocation $\hat{q}_*(\theta_k)$ below.

Efficient adaptive estimation of $\hat{q}_*(\theta_k)$ We slightly modify the work of Etoré and Jourdain [15] on adaptive stratified Monte Carlo sampling to the case of simple random sampling without replacement. Fix $\theta_k$ and an initial allocation guess $\hat{q}_0^T(\theta_k)$. Let $n_K$ denote the number of iterations used to estimate $\hat{q}_*(\theta_k)$, and $Q$ the total number of points that we sample in this procedure. Fix $(\Delta Q_s)_{s \in [n_K]}$ such that at each step $s \in [n_K]$ we sample $Q_s := n_B + \Delta Q_s$ points. Denote the points sampled in stratum $b$ at step $s$ by

$$
\Delta q_s^b := b + \delta_d_s^b.
$$
so that we sample at least one point in each stratum at each step $k$. Denote by $q^b_s := \sum_{s=1}^s \Delta q^b_s$ the total number of points sampled in stratum $E_T^{ij,b}$ up to and including step $s$. Begin the procedure with an initial guess of the optimal allocation $\tilde{q}^{(0)}(\theta_k)$. For all $p \in [n_B]$, denote by $\tilde{S}_T^{b_p,(s)}(\theta_k)$, $\tilde{q}_T^{b_p,(s)}(\theta_k)$, and $\tilde{q}_T^{1}(\theta_k)$, our estimates constructed inductively, based on all the samples up to step $s$ inclusive. At step $s$, for all $p$, compute
\[
\Delta q^b_s = \tilde{q}_T^{b_p,(s-1)}(\theta_k)(n_B + \Delta Q_s).
\]
Next, for all $p$, sample without replacement $\Delta q^b_s$ points $(x^b_m)_m$ in $E_T^{ij,b}$, and obtain
\[
\tilde{S}_T^{b_p}(\theta_k) = \tilde{S}_T^{b_p,(s-1)}(\theta_k),
\tilde{q}_T^{b_p}(\theta_k) = \tilde{q}_T^{b_p,(s-1)}(\theta_k),
\tilde{q}_T^{b_p}(\theta_k)^2 = \tilde{q}_T^{b_p,(s-1)}(\theta_k)^2.
\]

In some cases, using a fixed, preset allocation $\tilde{q}$ is sufficient to get satisfactory numerical results. In particular, for decaying kernels, we note that the sequence $(S_T^b(\theta_k))_h$ is decreasing with the lag $h$. For a sufficiently fine stratification, a monotonically decaying allocation leads to good results. This is not the case for more general kernels, hence the importance of constructing an adaptive estimator $\tilde{q}_*(\theta_k)$ of the optimal allocation $\tilde{q}_*(\theta_k)$ without increasing significantly the complexity of the procedure.

At each step $s$, compute the sample mean and sample variance of the corresponding batches
\[
S_T^{b_p,(s-1)}(\theta_k) = |E_T^{ij,b}| \sum_{m=1}^{q^b_s} f_{th}(x^b_m),
\]
\[
\sigma_T^{b_p,(s-1)}(\theta_k)^2 = \frac{|E_T^{ij,b}|}{q^b_s - 1} \sum_{m=1}^{q^b_s} \left( f_{th}(x^b_m) - S_T^{b_p,(s-1)}(\theta_k) \right)^2.
\]
To avoid unnecessary computations, update the running estimates in batches with
\[
\tilde{S}_T^{b_p,(s)}(\theta_k) = \frac{q^b_s}{q^b_s - 1} \tilde{S}_T^{b_p,(s-1)}(\theta_k) + \frac{\Delta q^b_s}{q^b_s - 1} \tilde{S}_T^{b_p,(s-1)}(\theta_k),
\]
\[
\sigma_T^{b_p,(s)}(\theta_k)^2 = \frac{q^b_s}{q^b_s - 1} \sigma_T^{b_p,(s-1)}(\theta_k)^2 + \frac{\Delta q^b_s}{q^b_s - 1} \sigma_T^{b_p,(s-1)}(\theta_k)^2
\]
\[
\frac{|E_T^{ij,b}|}{q^b_s - 1} \frac{\Delta q^b_s}{q^b_s - 1} \left( S_T^{b_p,(s-1)}(\theta_k) - \tilde{S}_T^{b_p,(s-1)}(\theta_k) \right)^2.
\]

**Sensitivity of $\tilde{q}_*(\theta_k)$ to $\theta_k$** Note that for a SBF MHP such that each kernel $\phi_{ij}$ is represented by only one basis function; i.e. only one corresponding parameter $\omega_{ij} > 0$, we have $\nabla \tilde{q}_*(\theta_k) = 0$. In the general case though, the optimal allocation $\tilde{q}_*(\theta_k)$ depends on $\theta_k$. In the context of a gradient-based optimization procedure, consider a sequence of parameters
\[
(\theta_k^{(1)}, \ldots, \theta_k^{(t)}).
\]
Use an exponential moving average (EMA) of the estimates
\[
\tilde{q}_*(\theta_k^{(1)}), \ldots, \tilde{q}_*(\theta_k^{(t)})
\]
as an initial guess for this procedure in step $t+1$. Denote by $w$ the EMA weight, fixed at the beginning of the procedure, and write
\[
\tilde{q}_*(\theta_k^{(0)}) = w \cdot \tilde{q}_*(\theta_k^{(1)}) + (1 - w) \cdot \tilde{q}_*(\theta_k^{(t-1)}).
\]
The role of the EMA step in our procedure is to try to use the corresponding sequence of optimal share estimates
\[
\left( \hat{q}_* \left( \theta_{k}^{(t)} \right), \ldots, \hat{q}_* \left( \theta_{k}^{(t)} \right) \right),
\]
to get a heuristic for \( \hat{q}_* \left( \theta_{k}^{(t+1)} \right) \).

**Estimation of the remainder** \( S_{T}^{\text{rest}}(\theta_k) \) As in the single sum case, use a standard stratified Monte Carlo approach with a fixed allocation to estimate the remainder
\[
S_{T}^{\text{rest}}(\theta_k) := \sum_{h=h_{\text{max}}+1}^{\kappa(j,i,N_i)} \sum_{n=1}^{q_p} f_{\theta_k}(x_{b'_p}^n).
\]

Let \( n_R \in \mathbb{N}^* \) denote the number of strata we use for this estimation, and consider a partition \( B' = (b'_1, \ldots, b'_{n_R}) \) of \([h_{\text{max}} + 1, \kappa(j,i,N_i)]\). Sample \( q_p \) points \((x_{1}^1, \ldots, x_{q_p}^1)\) uniformly and without replacement from \( E_{ij} \) for all \( p \in [n_R] \). Fix in advance the number of sample points per stratum, which we denote by
\[
q = (q_1, \ldots, q_p).
\]
Next, use the unbiased estimator of \( S_{T}^{\text{rest}}(\theta_k) \) defined by
\[
\hat{S}_{T}^{\text{rest}}(\theta_k) = \sum_{p=1}^{n_R} \sum_{q_p} q_{p} f_{\theta_k}(x_{b'_p}^n).
\]

**Adaptive learning rate** Denote the element-wise product between vectors or matrices by \( \odot \); the learning rate hyper-parameter by \( t \mapsto a_{\text{rate}}(t) > 0 \); and two moment hyper-parameters by \( a_{M1} > 0 \) and \( a_{M2} > 0 \). Denote a last hyper-parameter \( a_{E} > 0 \), which is used to avoid division by zero. The scheme we consider is defined by
\[
g_1^{(t)} = \frac{a_{M1} \cdot g_1^{(t-1)} + (1 - a_{M1}) \cdot \mathcal{G}_T^{(k)}(\theta_{k}^{(t)})}{1 - a_{M1}^2},
\]
\[
g_2^{(t)} = \frac{a_{M2} \cdot g_2^{(t-1)} + (1 - a_{M2}) \cdot \mathcal{G}_T^{(k)} \odot \mathcal{G}_T^{(k)}(\theta_{k}^{(t)})}{1 - a_{M2}^2},
\]
\[
\Delta \theta_{k}^{(t)} = -a_{\text{rate}}(t) \cdot \frac{g_1^{(t)}}{\sqrt{g_2^{(t)}} + a_{E}}.
\]

This scheme differs from the standard ADAM scheme by the non-constant learning rate \( a_{\text{rate}}(t) \). We start this rate at a fixed value \( a_{\text{rate}}(0) > 0 \) and then use
\[
a_{\text{rate}}(t) := \frac{a_{\text{rate}}(0)}{2^{[t/200]}},
\]

**C Mode collapse**

In our numerical experiments, we observe that when a given kernel is modelled as a multimodal mixture, and the calibration fits all the parameters of the mixture model (rather than being a sum-of-basis-functions model), the estimates are prone to numerical instability.

For example, consider the Gaussian MHP in the univariate multimodal case from Section 4.1.2. Consider fitting a Gauss4 be a univariate Gaussian model as in Section 2.1.2 with four Gaussian functions
(r = 4), where for all l ∈ [10] set β_l = 0.5. Figure 16 plots the updates of model parameters per gradient iterations, we see a clear mode collapse towards the lowest of the true means. The SBF Gaussian model helps circumvent this issue, which may be explained through the fact the LSE is a quadratic function of the weights ω. We leave the detailed study of this problem for future work.

D Kernel-specific results

In this appendix, we derive the formulas of the associated ψ and Υ functions and their partial derivatives for some parametric classes of kernels; as we need them in closed-form in our algorithm.

Consider a (µ, Φ)−linear MHP. Fix i, j, k ∈ [d], and denote by θ_{ki}, θ_{kj} the vectors of parameters of the kernels φ_{ki} and φ_{kj}. 

**D.1 Monotonic kernels**

**D.1.1 Exponential**

We assume that the kernel

\[ \phi_{ki} := \phi^{E}_{(\omega_{ki}, \beta_{ki})} \]

is an exponential kernel with parameters \( \theta_{ki} = (\omega_{ki}, \beta_{ki}) \), but do not make any assumptions on \( \phi_{kj}',(k',i') \neq (k,i) \).

**Proposition D.1.** For all \( x \geq 0 \), we have

\[ \psi_{ki}(x) = \sum_{l=1}^{r_{ki}} \omega_{kil} (1 - e^{-\beta_{kil} x}). \]

Fix \( p \in [r_{ki}] \). The partial derivatives of \( \phi_{ki} \) and \( \psi_{ki} \) with respect to model parameters are

\[ \frac{\partial \phi_{ki}}{\partial \omega_{kip}}(x) = - \beta_{kip} e^{-\beta_{kip} x}, \]
\[ \frac{\partial \phi_{ki}}{\partial \beta_{kip}}(x) = \omega_{kip} (1 - \beta_{kip} x) e^{-\beta_{kip} x}, \]
\[ \frac{\partial \psi_{ki}}{\partial \omega_{kip}}(x) = 1 - e^{-\beta_{kip} x}, \]
\[ \frac{\partial \psi_{ki}}{\partial \beta_{kip}}(x) = \omega_{kip} x e^{-\beta_{kip} x}. \]

We give closed-form formulas for Υ_{ijk} and its partial derivatives given various parametric classes for kernel \( \phi_{kj} \).

**Proposition D.2** (Exponential with exponential). We assume that the kernel

\[ \phi_{kj} := \phi^{E}_{(\omega_{kj}, \beta_{kj})} \]
is an exponential kernel with parameters \( \theta_{kj} = (\omega_{kj}, \beta_{kj}) \). Then for all \( t, s \geq 0 \)

\[
T_{ijk}(t, s) = \sum_{l=1}^{r_k} \sum_{i'=1}^{r_j} \sum_{j'=1}^{r_l} \omega_{ki'} \omega_{kj'} \frac{\beta_{ki'} \beta_{kj'}}{\beta_{ki'} + \beta_{kj'}} e^{-\beta_{kj'} t} (1 - e^{-\beta_{kj'} (t + s)}),
\]

where \( \delta > 0 \) controls the delay. We define the delayed exponential kernel using this basic approach.

Let \( p \in [r_k] \) and \( q \in [r_j] \). If \( i \neq j \), the partial derivatives of \( T_{ijk} \) with respect to the parameters are

\[
\frac{\partial T_{ijk}}{\partial \theta_{kip}}(t, s) = \sum_{l=1}^{r_l} \omega_{ki} \beta_{kl} e^{-\beta_{kl} t} \left( 1 - e^{-\beta_{kl} (t + s)} \right) \frac{\beta_{kl}^2}{\beta_{kl}^2 + \beta_{ki}^2},
\]

\[
\frac{\partial T_{ijk}}{\partial \theta_{kjq}}(t, s) = \sum_{l=1}^{r_l} \omega_{kj} \beta_{kl} e^{-\beta_{kl} t} \left( 1 - e^{-\beta_{kl} (t + s)} \right) \frac{\beta_{kl}^2}{\beta_{kl}^2 + \beta_{kj}^2},
\]

\[
\frac{\partial T_{ijk}}{\partial \theta_{kip}}(t, s) = \omega_{ki} \beta_{ki} e^{-\beta_{ki} (t + s)} \left( 1 - e^{-\beta_{ki} (t + s)} \right) \frac{\beta_{ki}^2}{\beta_{ki}^2 + \beta_{ki}^2},
\]

\[
\frac{\partial T_{ijk}}{\partial \theta_{kjq}}(t, s) = \omega_{kj} \beta_{kj} e^{-\beta_{kj} (t + s)} \left( 1 - e^{-\beta_{kj} (t + s)} \right) \frac{\beta_{kj}^2}{\beta_{kj}^2 + \beta_{kj}^2}.
\]

If \( i = j \), the partial derivatives of \( T_{ijk} \) with respect to the parameters are given by

\[
\frac{\partial T_{ijk}}{\partial \theta_{kip}}(t, s) = \omega_{ki} \beta_{ki} e^{-\beta_{ki} (t + s)} \left( 1 - e^{-2 \beta_{ki} (t + s)} \right) \frac{\beta_{ki}^2}{\beta_{ki}^2 + \beta_{ki}^2},
\]

\[
\frac{\partial T_{ijk}}{\partial \theta_{kjp}}(t, s) = \omega_{kj} \beta_{kj} e^{-\beta_{kj} (t + s)} \left( 1 - e^{-2 \beta_{kj} (t + s)} \right) \frac{\beta_{kj}^2}{\beta_{kj}^2 + \beta_{kj}^2}.
\]

D.2 Non monotonic kernels

D.2.1 Delayed exponential

Given a monotonically decaying function \( \phi \), a naive way to model a delayed response is to consider kernels of the form

\[
x \mapsto \mathbf{1}_{(x > \delta)} \phi(x - \delta),
\]

where \( \delta > 0 \) controls the delay. We define the delayed exponential kernel using this basic approach.

Definition D.1 (Delayed exponential kernel). Let \( r \in \mathbb{N}^* \). For \( x \geq 0 \), the delayed exponential kernel \( \phi^{DE}_{(\omega, \beta, \delta)} \) is

\[
\phi^{DE}_{(\omega, \beta, \delta)}(x) := \sum_{i=1}^{r} \omega_i e^{-\beta_i (x - \delta)} \mathbf{1}_{(x > \delta)},
\]

where the parameters are the vector of weights \( \omega := (\omega_i)_{i \in [1, r]} \in [0, +\infty)^r \), the vector of decays \( \beta := (\beta_i)_{i \in [1, r]} \in (0, +\infty)^r \), and the vector of delays \( \delta := (\delta_i)_{i \in [1, r]} \in (0, +\infty)^r \). We have

\[
\| \phi^{DE}_{(\omega, \beta, \delta)} \|_1 = \sum_{i=1}^{r} \omega_i.
\]
Remark D.1. Given the discontinuity of the kernel when the delay \( \delta \) is varied, gradient methods do not seem appropriate for calibrating \( \delta \). If \( \delta \) is unknown, an alternative model family (e.g. a mixture of Rayleigh or Gaussian kernels) may be more appropriate.

Proposition D.3. For all \( x \geq 0 \), we have
\[
\psi_{ki}(x) = \sum_{i=1}^{r_{ki}} \omega_{ki}(1 - \exp(-\beta_{ki}(x - \delta_{ki}))) \tag{52}
\]

Fix \( p \in \{r_{ki}\} \). The partial derivatives of \( \psi_{ki} \) and \( \phi_{ki} \) with respect to model parameters are given by
\[
\frac{\partial \psi_{ki}}{\partial \kappa_{ki}}(x) = \beta_{ki} e^{-\beta_{ki}(x - \delta_{ki})} \tag{53}
\]
\[
\frac{\partial \psi_{ki}}{\partial \delta_{ki}}(x) = \omega_{ki}(1 - \beta_{ki}(x - \delta_{ki})) e^{-\beta_{ki}(x - \delta_{ki})} \tag{54}
\]
\[
\frac{\partial \psi_{ki}}{\partial \delta_{ki}}(x) = 1 - e^{-\beta_{ki}(x - \delta_{ki})} \tag{55}
\]
\[
\frac{\partial \phi_{ki}}{\partial \delta_{ki}}(x) = \omega_{ki}(x - \delta_{ki}) e^{-\beta_{ki}(x - \delta_{ki})} \tag{56}
\]

Remark D.2. In terms of the intensity \( \Lambda_k \) and the compensator \( \Lambda_k \), the fact that \( \phi_{ki} \) is a delayed exponential kernel is similar to having \( r_{ki} \) additional event types with jump times \( t^{(0)}_i \) and standard exponential decay kernels. Nonetheless, since these jump times \( t^{(0)}_i \) are obtained by translation of the jump times \( t^{(1)}_i \), the delayed exponential kernel cannot be replicated by any linear MHP with exponential kernels only because the \( r_{ki} \) fictional jump types do not self or cross excite with the other jump types.

We give closed-form formulas for \( \Upsilon_{ijk} \) and its partial derivatives given various parametric classes for kernel \( \phi_{ki} \).

Proposition D.4 (Delayed exponential with delayed exponential)

\[
\phi_{kj} := \phi_{kj}^{DE, DE, DE} \tag{63}
\]
is a delayed exponential kernel with parameters \( \theta_{kj} = (\omega_{kj}, \delta_{kj}, \beta_{kj}) \). Define
\[
b_{ijk,lt}^j = \begin{cases} 
\beta_{kjt} & \text{if } \delta_{kjt} - \delta_{klt} < s, \\
\beta_{klt} & \text{if } \delta_{kjt} - \delta_{klt} > s, \\
0 & \text{if } \delta_{kjt} - \delta_{klt} = s.
\end{cases} \tag{57}
\]

Then for all \( t, s \geq 0 \)
\[
\Upsilon_{ijk}(t, s) = \sum_{i=1}^{r_{ki}} \sum_{l=1}^{r_{ki}} \omega_{ki}\omega_{kj} \beta_{kjt} \beta_{klt} e^{-b_{ijk,lt}^j(t - s)(t - s)} \tag{58}
\]

Let \( p \in \{r_{ki}\} \) and \( q \in \{r_{kj}\} \). If \( i \neq j \), the partial derivatives of \( \Upsilon_{ijk} \) with respect to the parameters are given by
\[
\frac{\partial \Upsilon_{ijk}}{\partial \kappa_{kj}}(t, s) = \sum_{i=1}^{r_{ki}} \sum_{l=1}^{r_{ki}} \omega_{ki}\omega_{kj} \beta_{kjt} \beta_{klt} e^{-b_{ijk,lt}^j(t - s)(t - s)} \tag{59}
\]
\[
\frac{\partial \Upsilon_{ijk}}{\partial \delta_{kj}}(t, s) = \sum_{i=1}^{r_{ki}} \sum_{l=1}^{r_{ki}} \omega_{ki}\omega_{kj} \beta_{kjt} \beta_{klt} e^{-b_{ijk,lt}^j(t - s)(t - s)} \tag{60}
\]
\[
\frac{\partial \Upsilon_{ijk}}{\partial \delta_{kj}}(t, s) = \sum_{i=1}^{r_{ki}} \sum_{l=1}^{r_{ki}} \omega_{ki}\omega_{kj} \beta_{kjt} \beta_{klt} \left( \frac{\beta_{kjt}}{\beta_{kj} + \beta_{kj}} e^{-b_{ijk,lt}^j(t - s)(t - s)} \right) \tag{61}
\]
\[
\frac{\partial \Upsilon_{ijk}}{\partial \beta_{kj}}(t, s) = \sum_{i=1}^{r_{ki}} \sum_{l=1}^{r_{ki}} \omega_{ki}\omega_{kj} \beta_{kjt} \beta_{klt} \left( \frac{\beta_{kjt}}{\beta_{kj} + \beta_{kj}} e^{-b_{ijk,lt}^j(t - s)(t - s)} \right) \tag{62}
\]

If \( i = j \), the partial derivatives of \( \Upsilon_{ijk} \) with respect to the parameters are given by
\[
\frac{\partial \Upsilon_{ijk}}{\partial \kappa_{kj}}(t, s) = \omega_{ki}\beta_{kjt} e^{-\beta_{kjt}^j} (1 - e^{-\beta_{kjt}^j}) \tag{63}
\]
\[
+ \sum_{t' \in \{r_{ki}\}, t' \neq t} \omega_{ki}\omega_{kj} \beta_{kjt} \beta_{klt} \left( e^{-b_{ijk,lt}^j(t - s)(t - s)} - e^{-\beta_{kjt}(t - s)(t - s)} - e^{-\beta_{kjt}(t - s)(t - s)} \right) \tag{64}
\]
\[
+ \sum_{i \in \{r_{kj}\}, i \neq i} \omega_{ki}\omega_{kj} \beta_{kjt} \beta_{klt} \left( e^{-b_{ijk,lt}^j(t - s)(t - s)} - e^{-\beta_{kjt}(t - s)(t - s)} - e^{-\beta_{kjt}(t - s)(t - s)} \right) \tag{65}
\]
\[
+ \sum_{i \in \{r_{kj}\}, i \neq i} \omega_{ki}\omega_{kj} \beta_{kjt} \beta_{klt} \left( e^{-b_{ijk,lt}^j(t - s)(t - s)} - e^{-\beta_{kjt}(t - s)(t - s)} - e^{-\beta_{kjt}(t - s)(t - s)} \right) \tag{66}
\]
We assume that the kernel $D.2.2$ Gaussian

Then for all $t,s$

For all $f$

write $(Gaussian \text{ with } Gaussian)$ Proposition D.6 is a Gaussian kernel with parameters $\phi_{k,i} := \phi^N_{(\omega_{k,i}, \beta_{k,i}, \delta_{k,i})}$ is a Gaussian kernel with parameters $\theta_{k,i} = (\omega_{k,i}, \beta_{k,i}, \delta_{k,i})$, but do not make any assumptions on $\phi_{k',i'}$, for $(k', i') \neq (k, i)$. We write $f_N$ and $F_N$ for the standard Gaussian density and distribution function respectively.

**Proposition D.5.** For all $x \geq 0$, we have

$$\psi_{k,i}(x) = \sum_{l=1}^{r_{k,i}} \omega_{k,il} \left( F_N \left( x - \frac{\delta_{k,il}}{\beta_{k,il}} \right) - F_N \left( \frac{\delta_{k,il}}{\beta_{k,il}} \right) \right). \tag{65}$$

Fix $p \in \{r_{k,i}\}$. The partial derivatives of $\phi_{k,i}$ and $\psi_{k,i}$ with respect to model parameters are given by

$$\frac{\partial \phi_{k,i}}{\partial \omega_{k,ip}}(x) = \frac{1}{\beta_{k,ip} \sqrt{2\pi}} \exp \left( -\frac{(x - \delta_{k,ip})^2}{2\beta_{k,ip}^2} \right), \tag{66}$$

$$\frac{\partial \phi_{k,i}}{\partial \beta_{k,ip}}(x) = \omega_{k,ip} \frac{\beta_{k,ip}^2}{\beta_{k,ip}^2 \sqrt{2\pi}} \exp \left( -\frac{(x - \delta_{k,ip})^2}{2\beta_{k,ip}^2} \right) \left( \left( \frac{x - \delta_{k,ip}}{\beta_{k,ip}} \right)^2 - 1 \right), \tag{67}$$

$$\frac{\partial \phi_{k,i}}{\partial \delta_{k,ip}}(x) = \frac{\omega_{k,ip}}{\beta_{k,ip}^2 \sqrt{2\pi}} \exp \left( -\frac{(x - \delta_{k,ip})^2}{2\beta_{k,ip}^2} \right), \tag{68}$$

$$\frac{\partial \psi_{k,i}}{\partial \omega_{k,ip}}(x) = F_N \left( x - \frac{\delta_{k,ip}}{\beta_{k,ip}} \right) - F_N \left( \frac{\delta_{k,ip}}{\beta_{k,ip}} \right), \tag{69}$$

$$\frac{\partial \psi_{k,i}}{\partial \beta_{k,ip}}(x) = F_N \left( x - \frac{\delta_{k,ip}}{\beta_{k,ip}} \right) \left( \frac{\delta_{k,ip} - x}{\beta_{k,ip}} \right) f_N \left( x - \frac{\delta_{k,ip}}{\beta_{k,ip}} \right) - \delta_{k,ip} f_N \left( \frac{\delta_{k,ip}}{\beta_{k,ip}} \right), \tag{70}$$

$$\frac{\partial \psi_{k,i}}{\partial \delta_{k,ip}}(x) = -\omega_{k,ip} \frac{\beta_{k,ip}^2}{\beta_{k,ip}^2} \left( f_N \left( x - \frac{\delta_{k,ip}}{\beta_{k,ip}} \right) - F_N \left( \frac{\delta_{k,ip}}{\beta_{k,ip}} \right) \right). \tag{71}$$

We give closed-form formulas for $\gamma_{t,ik}$ and its partial derivatives given various parametric classes for kernel $\phi_{k,i}$.

**Proposition D.6** (Gaussian with Gaussian). We assume that the kernel $\phi_{k,i} := \phi^N_{(\omega_{k,i}, \beta_{k,i}, \delta_{k,i})}$ is a Gaussian kernel with parameters $\theta_{k,i} = (\omega_{k,i}, \delta_{k,i}, \delta_{k,i})$. Define

$$b_{ijk,i't'} = \frac{\beta_{k,ij} \beta_{k,i'j'}}{\sqrt{\beta_{k,ii'} + \beta_{k,ii'}^2}}, \tag{72}$$

$$d_{ijk,i't'}^* = \frac{\beta_{k,ij}^2}{\beta_{k,ii'} + \beta_{k,ii'}^2} \frac{\beta_{k,ij}^2}{\beta_{k,ii'}^2 + \beta_{k,ii'}^2} \delta_{k,ij} + \frac{\beta_{k,ij}^2}{\beta_{k,ii'} + \beta_{k,ii'}^2} (\delta_{k,ij} - s). \tag{73}$$

Then for all $t, s \geq 0$

$$\gamma_{t,ik}(t, s) = \sum_{l=1}^{r_{k,i}} \sum_{l'=1}^{r_{k,i'}} \omega_{i'l'} \omega_{ik} \omega_{i'k} \frac{\exp \left( \frac{-1}{2} \frac{(x - \delta_{k,il} - \delta_{k,il'})^2}{\beta_{k,il}^2 + \beta_{k,il'}^2} \right)}{\sqrt{2\pi(\beta_{k,ili'}^2 + \beta_{k,ili'}^2)}} \left[ 2 \pi \left( \sqrt{\beta_{k,ili'}^2 + \beta_{k,ili'}^2} \right) \right] F_N \left( \frac{x - d_{ijk,ii'}^*}{b_{ijk,ii'}} \right) - F_N \left( \frac{d_{ijk,ii'}^*}{b_{ijk,ii'}} \right). \tag{74}$$
Let $p \in [r_i]$ and $q \in [r_j]$. If $i \neq j$, the partial derivatives of $Y_{ijk}$ with respect to the parameters are given by

\[
\frac{\partial Y_{ijk}}{\partial \omega_{kp}}(t, s) = \frac{r_{ki}}{\omega_{kp}} \sum_{j' = 1}^{r_{ki}} \exp \left( -\frac{1}{2} \frac{(t - \delta_{ijk,p}t')^2}{\omega_{kp}^2 + \beta_{kp}^2} \right) \left[ F_N \left( t - d_{i,jk,p}t' \right) - F_N \left( -d_{i,jk,p}t' \right) \right].
\]

(75)

\[
\frac{\partial Y_{ijk}}{\partial \omega_{kj}}(t, s) = \frac{r_{ki}}{\omega_{kj}} \sum_{l' = 1}^{r_{kj}} \exp \left( -\frac{1}{2} \frac{(t - \delta_{ijk,l}t')^2}{\omega_{kj}^2 + \beta_{kj}^2} \right) \left[ F_N \left( t - d_{i,j,k,p}t' \right) - F_N \left( -d_{i,j,k,p}t' \right) \right],
\]

(76)

\[
\frac{\partial Y_{ijk}}{\partial \omega_{kp}}(t, s) = \frac{r_{ki}}{\omega_{kp}} \sum_{j' = 1}^{r_{ki}} \exp \left( -\frac{1}{2} \frac{(t - \delta_{ijk,p}t')^2}{\omega_{kp}^2 + \beta_{kp}^2} \right) \left[ F_N \left( t - d_{i,jk,p}t' \right) - F_N \left( -d_{i,jk,p}t' \right) \right].
\]

(77)

\[
\frac{\partial Y_{ijk}}{\partial \omega_{kj}}(t, s) = \frac{r_{ki}}{\omega_{kj}} \sum_{l' = 1}^{r_{kj}} \exp \left( -\frac{1}{2} \frac{(t - \delta_{ijk,l}t')^2}{\omega_{kj}^2 + \beta_{kj}^2} \right) \left[ F_N \left( t - d_{i,j,k,p}t' \right) - F_N \left( -d_{i,j,k,p}t' \right) \right],
\]

(78)

\[
\frac{\partial Y_{ijk}}{\partial \omega_{kp}}(t, s) = \frac{r_{ki}}{\omega_{kp}} \sum_{j' = 1}^{r_{ki}} \exp \left( -\frac{1}{2} \frac{(t - \delta_{ijk,p}t')^2}{\omega_{kp}^2 + \beta_{kp}^2} \right) \left[ F_N \left( t - d_{i,jk,p}t' \right) - F_N \left( -d_{i,jk,p}t' \right) \right].
\]

(79)

\[
\frac{\partial Y_{ijk}}{\partial \omega_{kj}}(t, s) = \frac{r_{ki}}{\omega_{kj}} \sum_{l' = 1}^{r_{kj}} \exp \left( -\frac{1}{2} \frac{(t - \delta_{ijk,l}t')^2}{\omega_{kj}^2 + \beta_{kj}^2} \right) \left[ F_N \left( t - d_{i,j,k,p}t' \right) - F_N \left( -d_{i,j,k,p}t' \right) \right].
\]

(80)

If $i = j$, the partial derivatives of $Y_{ijk}$ with respect to the parameters are given by

\[
\frac{\partial Y_{ijk}}{\partial \omega_{kp}}(t, s) = 2\omega_{kp} \exp \left( -\frac{1}{2} \frac{(t - \delta_{ijk,p}t')^2}{\beta_{kp}^2} \right) \left[ F_N \left( \sqrt{2}(t - \delta_{ijk,p}t') \beta_{kp} \right) - F_N \left( -\sqrt{2}(t - \delta_{ijk,p}t') \beta_{kp} \right) \right].
\]

(81)
Proposition D.7 (Gaussian with Exponential). We assume that the kernel

\[ \phi_{k,j} := \phi_{(\omega_{k,j}, \beta_{k,j})} \]

is an exponential kernel with parameters \( \theta_{k,j} = (\omega_{k,j}, \beta_{k,j}) \). Then for all \( t, s \geq 0 \)

\[
Y_{i,tk}(t, s) = \sum_{i' \in [r_{k,t}]} \sum_{i'' \in [r_{k,s}]} \omega_{k,ti} \beta_{k,ti} \exp \left( -\frac{\beta_{k,ti}^2}{2} \frac{(t - \delta_{k,ti} + s - \delta_{k,ti})^2}{\beta_{k,ti}} \right) \times \left( F_N \left( \frac{t - (\delta_{k,ti} - \beta_{k,ti} r_{k,t})}{\beta_{k,ti}} \right) - F_N \left( \frac{t - (\delta_{k,ti} - \beta_{k,ti} r_{k,t})}{\beta_{k,ti}} \right) \right)
\]

We can also obtain \( Y \) for mixtures including both Gaussians and Exponentials (but see Remark D.3).
Let $p \in \{r_k\}$ and $q \in \{r_{kj}\}$. The partial derivatives of $Y_{ijk}$ with respect to the parameters are given by

\[
\frac{\partial Y_{ijk}}{\partial \omega_{kip}}(t, s) = \sum_{t' = 1}^{r_{ki}} \frac{\omega_{kij} \beta_{kji} \exp \left( -\beta_{kji} t' + s - \frac{\beta_{kji} \beta_{kji}^2}{2} \right)}{\beta_{kji}} \times \left( F_{N}(t - (\delta_{kji} - \beta_{kji} \beta_{kji}^2) - F_{N} \left( -\frac{\delta_{kji} - \beta_{kji} \beta_{kji}^2}{\beta_{kji}} \right) \right),
\]

(86)

\[
\frac{\partial Y_{ijk}}{\partial \omega_{kjq}}(t, s) = \sum_{t' = 1}^{r_{kj}} \omega_{kij} \beta_{kji} \exp \left( -\beta_{kji} t' + s - \frac{\beta_{kji} \beta_{kji}^2}{2} \right) \times \left( F_{N}(t - (\delta_{kji} - \beta_{kji} \beta_{kji}^2) - F_{N} \left( -\frac{\delta_{kji} - \beta_{kji} \beta_{kji}^2}{\beta_{kji}} \right) \right),
\]

(87)

\[
\frac{\partial Y_{ijk}}{\partial \beta_{kip}}(t, s) = \sum_{t' = 1}^{r_{ki}} \omega_{kij} \beta_{kji} \exp \left( -\beta_{kji} t' + s - \frac{\beta_{kji} \beta_{kji}^2}{2} \right) \times \left( \beta_{kji} \beta_{kji}^2 \left( F_{N}(t - (\delta_{kji} - \beta_{kji} \beta_{kji}^2) - F_{N} \left( -\frac{\delta_{kji} - \beta_{kji} \beta_{kji}^2}{\beta_{kji}} \right) \right) + (\beta_{kji} \beta_{kji}^2) f_{N}(t - (\delta_{kji} - \beta_{kji} \beta_{kji}^2) - F_{N} \left( -\frac{\delta_{kji} - \beta_{kji} \beta_{kji}^2}{\beta_{kji}} \right) \right) - (\beta_{kji} \beta_{kji}^2) \right),
\]

(88)

\[
\frac{\partial Y_{ijk}}{\partial \beta_{kjq}}(t, s) = \sum_{t' = 1}^{r_{kj}} \omega_{kij} \beta_{kji} \exp \left( -\beta_{kji} t' + s - \frac{\beta_{kji} \beta_{kji}^2}{2} \right) \times \left( -\beta_{kji} \left( F_{N}(t - (\delta_{kji} - \beta_{kji} \beta_{kji}^2) - F_{N} \left( -\frac{\delta_{kji} - \beta_{kji} \beta_{kji}^2}{\beta_{kji}} \right) \right) \right),
\]

(89)

\[
\frac{\partial Y_{ijk}}{\partial \beta_{kjl}}(t, s) = \sum_{t' = 1}^{r_{kj}} \omega_{kij} \beta_{kji} \exp \left( -\beta_{kji} t' + s - \frac{\beta_{kji} \beta_{kji}^2}{2} \right) \times \left( \frac{1}{\beta_{kjl}} - \frac{\delta_{kji} - s - \frac{\beta_{kji} \beta_{kji}^2}{2}}{\beta_{kji}} \right) \times \left( F_{N}(t - (\delta_{kji} - \beta_{kji} \beta_{kji}^2) - F_{N} \left( -\frac{\delta_{kji} - \beta_{kji} \beta_{kji}^2}{\beta_{kji}} \right) \right) + \beta_{kji} \left( f_{N}(t - (\delta_{kji} - \beta_{kji} \beta_{kji}^2) - F_{N} \left( -\frac{\delta_{kji} - \beta_{kji} \beta_{kji}^2}{\beta_{kji}} \right) \right)),
\]

(90)

Proposition D.8 (Exponential with Gaussian). We assume that the kernel

\[
\phi_{kj} := \phi_{\omega_{kq}, \beta_{kj}, \delta_{kj}}
\]

is a Gaussian kernel with parameters $\theta_{kj} = (\omega_{kq}, \beta_{kj}, \delta_{kj})$. Then for all $t, t' \geq 0$

\[
Y_{ijk}(t, s) = \sum_{t' = 1}^{r_{ki}} \sum_{t' = 1}^{r_{kj}} \omega_{kij} \omega_{kji} \beta_{kii} \exp \left( -\beta_{kii} t' + s - \frac{\beta_{kii} \beta_{kii}^2}{2} \right) \times \left( F_{N}(t - (\delta_{kii} - \beta_{kii} \beta_{kii}^2) - F_{N} \left( -\frac{\delta_{kii} - \beta_{kii} \beta_{kii}^2}{\beta_{kii}} \right) \right),
\]

(92)

Let $p \in \{r_k\}$ and $q \in \{r_{kj}\}$. The partial derivatives of $Y_{ijk}$ with respect to the parameters are given by

\[
\frac{\partial Y_{ijk}}{\partial \omega_{kip}}(t, s) = \sum_{t' = 1}^{r_{ki}} \omega_{kij} \beta_{kji} \exp \left( -\beta_{kji} t' + s - \frac{\beta_{kji} \beta_{kji}^2}{2} \right) \times \left( F_{N}(t - (\delta_{kji} - \beta_{kji} \beta_{kji}^2) - F_{N} \left( -\frac{\delta_{kji} - \beta_{kji} \beta_{kji}^2}{\beta_{kji}} \right) \right),
\]

(94)

\[
\frac{\partial Y_{ijk}}{\partial \omega_{kjq}}(t, s) = \sum_{t' = 1}^{r_{kj}} \omega_{kij} \beta_{kji} \exp \left( -\beta_{kji} t' + s - \frac{\beta_{kji} \beta_{kji}^2}{2} \right) \times \left( F_{N}(t - (\delta_{kji} - \beta_{kji} \beta_{kji}^2) - F_{N} \left( -\frac{\delta_{kji} - \beta_{kji} \beta_{kji}^2}{\beta_{kji}} \right) \right),
\]

(95)
\[ \frac{\partial Y_{kl}}{\partial \beta_{kl}}(t, s) = \omega_{k1p} \beta_{k1p} \sum_{l=1}^{r_k} \omega_{kji} e^{-\beta_{k1p}(\delta_{kji} - \beta_{k1p} \delta_{kji}^2)} \]

\[ \times \left( \frac{1}{2} \delta_{kji} - s + \frac{\beta_{k1p} \delta_{kji}^2}{2} \right) \left( F_N \left( \frac{t - (\delta_{kji} - s - \beta_{k1p} \delta_{kji}^2)}{\beta_{kji}} \right) \right. \]

\[ - F_N \left( \frac{t - \beta_{k1p} \delta_{kji}^2}{\beta_{kji}} \right) \]

\[ + F_N \left( \frac{t - \beta_{k1p} \delta_{kji}^2}{\beta_{kji}} \right) \left( \frac{t - (\delta_{kji} - s - \beta_{k1p} \delta_{kji}^2)}{\beta_{kji}} \right) \]

\[ = 2F_N \left( \frac{\delta_{kji} - s - \beta_{k1p} \delta_{kji}^2}{\beta_{kji}^2} \right) \]  

\[ \frac{\partial Y_{jk}}{\partial \beta_{jk}}(t, s) = -\omega_{k1p} \beta_{k1p} \sum_{l=1}^{r_k} \omega_{kji} e^{-\beta_{k1p}(\delta_{kji} - \beta_{k1p} \delta_{kji}^2)} \]

\[ \times \left( \frac{1}{2} \delta_{kji} - s + \frac{\beta_{k1p} \delta_{kji}^2}{2} \right) \left( F_N \left( \frac{t - (\delta_{kji} - s - \beta_{k1p} \delta_{kji}^2)}{\beta_{kji}} \right) \right. \]

\[ - F_N \left( \frac{t - \beta_{k1p} \delta_{kji}^2}{\beta_{kji}} \right) \]

\[ + F_N \left( \frac{t - \beta_{k1p} \delta_{kji}^2}{\beta_{kji}} \right) \left( \frac{t - (\delta_{kji} - s - \beta_{k1p} \delta_{kji}^2)}{\beta_{kji}} \right) \]

\[ - \beta_{kji} \left( \frac{t - (\delta_{kji} - s - \beta_{k1p} \delta_{kji}^2)}{\beta_{kji}} \right) \]

\[ + \frac{1}{\beta_{kji}} \left( \frac{t - (\delta_{kji} - s - \beta_{k1p} \delta_{kji}^2)}{\beta_{kji}} \right) \]

\[ \left( - \beta_{k1p} \delta_{kji}^2 \right) \]  

**Remark D.3.** The term \( \exp \left( - \beta_{k1p} (\delta_{kji} - s - \beta_{k1p} \delta_{kji}^2) \right) \) in (91) can lead to overflow. Therefore, implementing directly the formulas in this last proposition leads to numerical instabilities, and is of limited practical use.

**D.2.3 Rayleigh kernel**

**Definition D.2** (Rayleigh kernel). Let \( r \in \mathbb{N}^+ \). For \( x \geq 0 \), the Rayleigh kernel \( \phi^r_{(\omega, \beta)} \) is

\[ \phi^r_{(\omega, \beta)}(x) = \sum_{i=1}^{r} \omega_i x_i \beta_i \exp \left( - \frac{x^2}{2 \beta_i^2} \right), \]

where the parameters are the vector of weights \( \omega := (\omega_i)_{i \in \{1, r\} \in [0, \infty)^r} \), and the vector of scale parameters \( \beta := (\beta_i)_{i \in \{1, r\} \in (0, \infty)^r} \). We have

\[ \| \phi^r_{(\omega, \beta)} \|_1 = \sum_{i=1}^{r} \omega_i. \]

We assume that the kernel

\[ \phi_{k1} := \phi^r_{(\omega_{k1}, \beta_{k1})} \]

is a Rayleigh kernel with parameters \( \theta_{k1} = (\omega_{k1}, \beta_{k1}) \), but do not make any assumptions on \( \phi_{k'i'} \) for \( (k', i') \neq (k, i) \).

**Proposition D.9.** For all \( x \geq 0 \), we have

\[ \psi_{k1}(x) = \sum_{i=1}^{r} \omega_{k1i} \left( 1 - \exp \left( - \frac{x^2}{2 \beta_i^2} \right) \right) \]

Fix \( p \in [r_k] \). The partial derivatives of \( \phi_{k1} \) and \( \psi_{k1} \) with respect to model parameters are given by

\[ \frac{\partial \phi_{k1}}{\partial \omega_{k1p}}(x) = \frac{x}{\beta_{k1p}} \exp \left( - \frac{x^2}{2 \beta_{k1p}^2} \right), \]

\[ \frac{\partial \psi_{k1}}{\partial \omega_{k1p}}(x) = - \frac{x^2}{2 \beta_{k1p}^2} \exp \left( - \frac{x^2}{2 \beta_{k1p}^2} \right), \]

\[ \frac{\partial \psi_{k1}}{\partial \beta_{k1p}}(x) = - \omega_{k1p} x^2 \exp \left( - \frac{x^2}{2 \beta_{k1p}^2} \right). \]
We give closed-form formulas for $\Upsilon_{i,j,k}$ and its partial derivatives given various parametric classes for kernel $f_{k,j}$. 

**Proposition D.10** (Rayleigh with Rayleigh). We assume that the kernel $\phi_{k,j} := \phi_{\omega_{k,j}, \beta_{k,j}}$ is a Rayleigh kernel with parameters $\theta_{k,j} = (\omega_{k,j}, \beta_{k,j})$. Define 

$$b_{i,j,k,t} = \frac{\beta_{i,k,t}^2}{\sqrt{\beta_{i,k,t}^2 + \beta_{j,k,t}^2}},$$  

$$b_{(i,j),k,t} = \frac{\beta_{i,k,t}^2}{\beta_{i,k,t}^2 + \beta_{j,k,t}^2},$$  

$$b_{(i,k),j,t} = \frac{\beta_{i,k,t}^2}{\beta_{i,k,t}^2 + \beta_{j,k,t}^2}.$$  

Then for all $t, s \geq 0$ 

$$\Upsilon_{i,j,k}(t, s) = \sum_{\ell=1}^{r_k} \omega_{k,\ell} \omega_{k,\ell} \sum_{\ell=1}^{r_k} \omega_{k,\ell} \omega_{k,\ell} \exp \left( -\frac{s^2}{2(\beta_{i,k,\ell}^2 + \beta_{j,k,\ell}^2)} \right) \times \left[ \frac{\beta_{i,k,\ell}^2}{\beta_{i,k,\ell}^2 + \beta_{j,k,\ell}^2} \exp \left( -\frac{(\beta_{i,k,\ell}^2)^2}{2(\beta_{i,k,\ell}^2 + \beta_{j,k,\ell}^2)} \right) - t + sb_{(i,j),k,t}^2 \beta_{i,k,\ell}^2 \exp \left( -\frac{(t + sb_{(i,j),k,t}^2)^2}{2(\beta_{i,k,\ell}^2 + \beta_{j,k,\ell}^2)} \right) \right],$$  

let $p \in [r_k]$ and $q \in [r_k]$. If $i \neq j$, the partial derivatives of $\Upsilon_{i,j,k}$ with respect to the parameters are given by 

$$\frac{\partial \Upsilon_{i,j,k}(t, s)}{\partial \omega_{k,p}} = \sum_{\ell=1}^{r_k} \omega_{k,\ell} \omega_{k,\ell} \sum_{\ell=1}^{r_k} \omega_{k,\ell} \omega_{k,\ell} \exp \left( -\frac{s^2}{2(\beta_{i,k,\ell}^2 + \beta_{j,k,\ell}^2)} \right) \times \left[ \frac{\beta_{i,k,\ell}^2}{\beta_{i,k,\ell}^2 + \beta_{j,k,\ell}^2} \exp \left( -\frac{(\beta_{i,k,\ell}^2)^2}{2(\beta_{i,k,\ell}^2 + \beta_{j,k,\ell}^2)} \right) - t + sb_{(i,j),k,t}^2 \beta_{i,k,\ell}^2 \exp \left( -\frac{(t + sb_{(i,j),k,t}^2)^2}{2(\beta_{i,k,\ell}^2 + \beta_{j,k,\ell}^2)} \right) \right],$$  

$$\frac{\partial \Upsilon_{i,j,k}(t, s)}{\partial \beta_{k,q}} = \sum_{\ell=1}^{r_k} \omega_{k,\ell} \omega_{k,\ell} \sum_{\ell=1}^{r_k} \omega_{k,\ell} \omega_{k,\ell} \exp \left( -\frac{s^2}{2(\beta_{i,k,\ell}^2 + \beta_{j,k,\ell}^2)} \right) \times \left[ \frac{\beta_{i,k,\ell}^2}{\beta_{i,k,\ell}^2 + \beta_{j,k,\ell}^2} \exp \left( -\frac{(\beta_{i,k,\ell}^2)^2}{2(\beta_{i,k,\ell}^2 + \beta_{j,k,\ell}^2)} \right) - t + sb_{(i,j),k,t}^2 \beta_{i,k,\ell}^2 \exp \left( -\frac{(t + sb_{(i,j),k,t}^2)^2}{2(\beta_{i,k,\ell}^2 + \beta_{j,k,\ell}^2)} \right) \right],$$  

$$\frac{\partial \Upsilon_{i,j,k}(t, s)}{\partial \beta_{k,p}} = \sum_{\ell=1}^{r_k} \omega_{k,\ell} \omega_{k,\ell} \sum_{\ell=1}^{r_k} \omega_{k,\ell} \omega_{k,\ell} \exp \left( -\frac{s^2}{2(\beta_{i,k,\ell}^2 + \beta_{j,k,\ell}^2)} \right) \times \left[ \frac{2b_{(i,j),k,t}^2}{\beta_{i,k,\ell}^2 + \beta_{j,k,\ell}^2} \exp \left( -\frac{(2b_{(i,j),k,t}^2)^2}{2(\beta_{i,k,\ell}^2 + \beta_{j,k,\ell}^2)} \right) - t + sb_{(i,j),k,t}^2 \beta_{i,k,\ell}^2 \exp \left( -\frac{(t + sb_{(i,j),k,t}^2)^2}{2(\beta_{i,k,\ell}^2 + \beta_{j,k,\ell}^2)} \right) \right],$$  

where $F_N$ is the Rayleigh distribution function.
\[ \frac{\partial \Omega_{ijkl}(t,s)}{\partial \omega_{kij}} = \omega_{kij} \sum_{l=1}^{r_{kl}} \omega_{kl} \exp \left[ \frac{-s^2}{2(\beta_{kij}^2 + \beta_{kij}^4)} \right] \times \left[ s^2 + 2\frac{a_s}{b_{ijkl}} (k^{ijkl}_{ijkl} - b^{ijkl}_{ijkl}) s \exp \left( \frac{(s b^{ijkl}_{ijkl})^2}{2\beta_{ijkl}^2} \right) \right] \]

\[ - \frac{1}{\beta_{kij}^2} \left[ \frac{(k^{ijkl}_{ijkl})^2}{\beta_{kij}^4 + \beta_{kij}^4 t^2} + \left( t + b^{ijkl}_{ijkl} s \right) \left( t + (1 + b^{ijkl}_{ijkl}) s \right) \exp \left( \frac{(t + b^{ijkl}_{ijkl})^2}{2\beta_{ijkl}^2} \right) \right] \]

\[ + \sqrt{2} \frac{a_s}{b_{ijkl}} \left[ \frac{(k^{ijkl}_{ijkl})^2}{\beta_{kij}^4 + \beta_{kij}^4 t^2} + \left( t + b^{ijkl}_{ijkl} s \right) \left( t + (1 + b^{ijkl}_{ijkl}) s \right) \exp \left( \frac{(t + b^{ijkl}_{ijkl})^2}{2\beta_{ijkl}^2} \right) \right] \]

\[ - \sqrt{2} \frac{a_s}{b_{ijkl}} \left[ \frac{(k^{ijkl}_{ijkl})^2}{\beta_{kij}^4 + \beta_{kij}^4 t^2} + \left( t + b^{ijkl}_{ijkl} s \right) \left( t + (1 + b^{ijkl}_{ijkl}) s \right) \exp \left( \frac{(t + b^{ijkl}_{ijkl})^2}{2\beta_{ijkl}^2} \right) \right] \]

(111)

If \( i = j \), the partial derivatives of \( \Omega_{ijkl} \) with respect to the parameters are given by

\[ \frac{\partial \Omega_{ikk}(t,s)}{\partial \omega_{ikk}} = 2\omega_{ikk} \exp \left[ \frac{-s^2}{4\beta_{ikk}^2} \right] \times \left[ \frac{s}{4\beta_{ikk}^2} \exp \left( \frac{-s^2}{4\beta_{ikk}^2} \right) - \frac{2s + s^3 \beta_{ikk}}{4\beta_{ikk}^2} \exp \left( \frac{(2s + s^3 \beta_{ikk})^2}{4\beta_{ikk}^4} \right) \right] \]

\[ + \frac{\sqrt{2}}{2} \frac{\beta_{ikk}}{\omega_{ikk}} \left[ 1 - \frac{s^2}{4\beta_{ikk}^2} \right] \exp \left( \frac{(2s + s^3 \beta_{ikk})^2}{4\beta_{ikk}^4} \right) \]

\[ + \sum_{l \neq k} \omega_{il} \exp \left( \frac{-s^2}{2(\beta_{ikk}^2 + \beta_{ikk}^4 t^2)} \right) \times \left[ \frac{s \beta_{il}}{2(\beta_{ikk}^2 + \beta_{ikk}^4 t^2)} \exp \left( \frac{(s \beta_{il})^2}{2(\beta_{ikk}^2 + \beta_{ikk}^4 t^2)^2} \right) \right] \]

\[ - \left( \frac{\beta_{il}^2 + \beta_{ikk}^2 t + s \beta_{il}}{(\beta_{ikk}^2 + \beta_{ikk}^4 t^2)^2} \right) \exp \left( \frac{-s^2}{2(\beta_{ikk}^2 + \beta_{ikk}^4 t^2)} \right) \]

\[ + \sqrt{2} \frac{\beta_{ikk}}{(\beta_{ikk}^2 + \beta_{ikk}^4 t^2)^{\frac{3}{2}}} \left[ 1 - \frac{s^2}{4\beta_{ikk}^2} \right] \exp \left( \frac{(2s + s^3 \beta_{ikk})^2}{4\beta_{ikk}^4} \right) \]

\[ + \frac{\sqrt{2}}{2} \frac{\beta_{ikk}}{(\beta_{ikk}^2 + \beta_{ikk}^4 t^2)^{\frac{3}{2}}} \left[ 1 - \frac{s^2}{4\beta_{ikk}^2} \right] \exp \left( \frac{(2s + s^3 \beta_{ikk})^2}{4\beta_{ikk}^4} \right) \]

\[ + \sum_{l \neq k} \omega_{ikl} \exp \left( \frac{-s^2}{2(\beta_{ikk}^2 + \beta_{ikk}^4 t^2)} \right) \times \left[ \frac{s \beta_{ikl}}{2(\beta_{ikk}^2 + \beta_{ikk}^4 t^2)} \exp \left( \frac{(s \beta_{ikl})^2}{2(\beta_{ikk}^2 + \beta_{ikk}^4 t^2)^2} \right) \right] \]

\[ - \left( \frac{\beta_{ikl}^2 + \beta_{ikk}^2 t + s \beta_{ikl}}{(\beta_{ikk}^2 + \beta_{ikk}^4 t^2)^2} \right) \exp \left( \frac{-s^2}{2(\beta_{ikk}^2 + \beta_{ikk}^4 t^2)} \right) \]

\[ + \sqrt{2} \frac{\beta_{ikk}}{(\beta_{ikk}^2 + \beta_{ikk}^4 t^2)^{\frac{3}{2}}} \left[ 1 - \frac{s^2}{4\beta_{ikk}^2} \right] \exp \left( \frac{(2s + s^3 \beta_{ikk})^2}{4\beta_{ikk}^4} \right) \]

\[ + \frac{\sqrt{2}}{2} \frac{\beta_{ikk}}{(\beta_{ikk}^2 + \beta_{ikk}^4 t^2)^{\frac{3}{2}}} \left[ 1 - \frac{s^2}{4\beta_{ikk}^2} \right] \exp \left( \frac{(2s + s^3 \beta_{ikk})^2}{4\beta_{ikk}^4} \right) \]

(112)
\[
\frac{\partial T_{ikp}(t,s)}{\partial s_{kp}} = \omega_{kp} \exp \left( \frac{s^2}{4\sigma^2_{kp}} \right) \times \left[ \frac{1}{2\sigma^2_{kp}} \left( \frac{s^3}{2\sigma^2_{kp}} - s \right) \exp \left( -\frac{s^2}{4\sigma^2_{kp}} \right) - \frac{1}{2\sigma^2_{kp}} \left( 2t + s \right)^2 \left( \frac{s^2}{4\sigma^2_{kp}} - 1 \right) \exp \left( -\frac{(2t + s)^2}{4\sigma^2_{kp}} \right) \right]
\]

\[
+ \frac{\sqrt{\pi}}{2\sigma^2} \left( 1 - \frac{s^2}{2\sigma^2} \right)^2 \left( F_{\sigma^2} \left( \frac{2t + s}{\sqrt{2\sigma^2}} \right) - F_{\sigma^2} \left( \frac{s}{\sqrt{2\sigma^2}} \right) \right)
\]

\[
- \frac{\sqrt{\pi}}{8 \sigma^2_{kp}} \left( 1 - \frac{s^2}{2\sigma^2_{kp}} \right)^2 \left( 2t + s \right) \exp \left( -\frac{s}{\sqrt{2\sigma^2_{kp}}} \right)
\]

\[
+ \omega_{kp} \sum_{t' \in \{ \tilde{t}_{ikl} \}, t' \neq t} \omega_{klt'} \exp \left( -\frac{s^2}{2(\sigma^2_{kp} + \sigma^2_{klt'})} \right) \times \left[ -2s(\hat{b}_{iklp})^2 \left( 1 + 2(\hat{b}_{iklp})^2 \right) \exp \left( -\frac{s(\hat{b}_{iklp})^2}{2\sigma_{klt'}^2} \right) \right]
\]

\[
- \frac{t + s(\hat{b}_{iklp})^2}{\beta_{kp}} \left( \frac{s^2}{\beta_{kp}^2 + \beta_{klt'}^2} - 2(\hat{b}_{iklp})^2 \right) \left( F_{\beta_{klt'}} \left( \frac{t + s(\hat{b}_{iklp})^2}{\beta_{klt'}} \right) - \frac{s}{\beta_{klt'}} \cdot F_{\beta_{klt'}} \left( \frac{s}{\beta_{klt'}} \right) \right)
\]

\[
+ \frac{\sqrt{\pi} \sigma_{klt'}}{2 \beta_{klt'}} \beta_{klt'} \left( \frac{s^2}{\beta_{klt'}^2 + \beta_{klt'}^2} \right) \left( 1 - \frac{s^2}{\beta_{klt'}^2} \right) \exp \left( -\frac{s^2}{\beta_{klt'}^2} \right) \frac{F_{\beta_{klt'}} \left( \frac{s^2}{\beta_{klt'}^2} \right) - \frac{s}{\beta_{klt'}} \cdot F_{\beta_{klt'}} \left( \frac{s^2}{\beta_{klt'}^2} \right) \right)
\]

\[
\left( 1 - \frac{s^2}{\beta_{klt'}^2} \right) \frac{F_{\beta_{klt'}} \left( \frac{s^2}{\beta_{klt'}^2} \right) - \frac{s}{\beta_{klt'}} \cdot F_{\beta_{klt'}} \left( \frac{s^2}{\beta_{klt'}^2} \right) \right)
\]

D.2.4 Triangular

The triangular kernel is a simple example of a finite support non-monotonic kernel.

**Definition D.3 (Triangular kernel).** Let \( t \in \mathbb{N}^* \). For \( s \geq 0 \), the triangular kernel \( \phi_{\omega, \alpha, \beta, \delta} \) is

\[
\phi_{\omega, \alpha, \beta, \delta}(t,s) = \sum_{i=1}^{r} \omega_{i} \left( \frac{t - \alpha_{i}}{\beta_{i}} \cdot I_{s \leq \alpha_{i} \leq \beta_{i}} - \frac{t - \alpha_{i} - \delta_{i}}{\beta_{i}} \cdot I_{0 \leq s \leq \beta_{i} - \alpha_{i} - \delta_{i}} \right)
\]

where the parameters are the vector of weights \( \omega := (\omega_{i})_{i \in [1,r]} \in [0, +\infty)^{r} \), the vector of left corners \( \alpha := (\alpha_{i})_{i \in [1,r]} \in (0, +\infty)^{r} \), the vector of distances to the altitude feet \( \beta := (\beta_{i})_{i \in [1,r]} \in (0, +\infty)^{r} \), and the vector of distances between altitude feet and
Proposition D.11. For all \( \theta \) is a triangular kernel with parameters \( \theta_{ki} = (\omega_{ki}, \alpha_{ki}, \beta_{ki}, \delta_{ki}) \), we have

\[
\psi_{ki}(x) = \sum_{l=1}^{r} \omega_{kl} \left( \frac{(x - \alpha_{kl})^2}{2\delta_{kl}} \right) + \frac{\beta_{kl}}{2} + (x - \alpha_{kl} - \beta_{kl}) - \frac{(x - \alpha_{kl} - \beta_{kl})^2}{2\delta_{kl}}.
\]

Figure 17: Plot of a triangular kernel against time, \( r = 1 \).

We assume that the kernel

\[
\phi_{ki} := \phi_{[\omega_{ki} - \alpha_{ki}, \beta_{ki}, \delta_{ki}]}^T
\]

is a triangular kernel with parameters \( \theta_{ki} = (\omega_{ki}, \alpha_{ki}, \beta_{ki}, \delta_{ki}) \), but do not make any assumptions on \( \phi_{k'i'} \) for \((k', i') \neq (k, i)\).

The partial derivatives of \( \phi_{ki} \) and \( \psi_{ki} \) with respect to model parameters are given by

\[
\frac{\partial \phi_{ki}}{\partial \omega_{kl}}(x) = \frac{x - \alpha_{kl}}{\beta_{kl}} 1_{\{0 \leq x - \alpha_{kl} \leq \beta_{kl}\}} \frac{1}{\beta_{kl}} (x - \alpha_{kl} - \beta_{kl} - \delta_{kl}) 1_{\{0 \leq x - \alpha_{kl} - \beta_{kl} \leq \delta_{kl}\}},
\]

\[
\frac{\partial \phi_{ki}}{\partial \alpha_{kl}}(x) = -\frac{\omega_{kl}}{\beta_{kl}} 1_{\{0 < x - \alpha_{kl} \leq \beta_{kl}\}} + \frac{\omega_{kl}}{\delta_{kl}} 1_{\{0 < x - \alpha_{kl} < \delta_{kl}\}},
\]

\[
\frac{\partial \phi_{ki}}{\partial \beta_{kl}}(x) = -\frac{\omega_{kl}}{\delta_{kl}} 1_{\{0 < x - \alpha_{kl} < \beta_{kl}\}} + \frac{\omega_{kl}}{\delta_{kl}} 1_{\{0 < x - \alpha_{kl} - \beta_{kl} \leq \delta_{kl}\}},
\]

\[
\frac{\partial \phi_{ki}}{\partial \delta_{kl}}(x) = \omega_{kl} - \frac{x - \alpha_{kl} - \beta_{kl}}{\beta_{kl} 1_{\{0 < x - \alpha_{kl} < \beta_{kl}\}}},
\]

\[
\frac{\partial \psi_{ki}}{\partial \omega_{kl}}(x) = \frac{(x - \alpha_{kl})^2}{2\delta_{kl}} 1_{\{0 \leq \alpha_{kl} \leq \beta_{kl}\}} + \frac{\beta_{kl}}{2} + (x - \alpha_{kl} - \beta_{kl}) - \frac{(x - \alpha_{kl} - \beta_{kl})^2}{2\delta_{kl}} 1_{\{0 \leq \alpha_{kl} - \beta_{kl} \leq \delta_{kl}\}},
\]

\[
\frac{\partial \psi_{ki}}{\partial \alpha_{kl}}(x) = -\frac{\omega_{kl}}{\beta_{kl}} 1_{\{0 < x - \alpha_{kl} \leq \beta_{kl}\}} - \frac{\omega_{kl}}{\delta_{kl}} 1_{\{0 < x - \alpha_{kl} - \beta_{kl} \leq \delta_{kl}\}},
\]

\[
\frac{\partial \psi_{ki}}{\partial \beta_{kl}}(x) = -\frac{\omega_{kl}}{\delta_{kl}} 1_{\{0 < x - \alpha_{kl} < \beta_{kl}\}} + \frac{\omega_{kl}}{\delta_{kl}} 1_{\{0 < x - \alpha_{kl} - \beta_{kl} \leq \delta_{kl}\}},
\]

\[
\frac{\partial \psi_{ki}}{\partial \delta_{kl}}(x) = \omega_{kl} - \frac{x - \alpha_{kl} - \beta_{kl}}{\beta_{kl} 1_{\{0 < x - \alpha_{kl} < \beta_{kl}\}}},
\]

\[
\frac{\partial \psi_{ki}}{\partial \omega_{kl,f}}(x) = -\omega_{kl} - \frac{x - \alpha_{kl}}{\beta_{kl}} 1_{\{0 \leq x - \alpha_{kl} \leq \beta_{kl}\}} + \omega_{kl} \left( \frac{x - \alpha_{kl} - \beta_{kl}}{\delta_{kl}} \right) 1_{\{0 \leq x - \alpha_{kl} - \beta_{kl} \leq \delta_{kl}\}}.
\]
\[
\frac{\partial \psi}{\partial k_{ijp}}(x) = \frac{\omega_{k_{ijp}}}{2} \left( \frac{x - a_{k_{ijp}} - \delta_{k_{ijp}}}{\delta_{k_{ijp}}} \right)^2 1_{(0 \leq x - a_{k_{ijp}} - \delta_{k_{ijp}} \leq \delta_{k_{ijp}})} + \frac{\omega_{k_{ijp}}}{2} \left( \frac{x - a_{k_{ijp}} + \delta_{k_{ijp}}}{\delta_{k_{ijp}}} \right)^2 1_{(x > a_{k_{ijp}} + \delta_{k_{ijp}})}.
\] (122)

We give closed-form formulas for \( T_{ijk} \) and its partial derivatives given various parametric classes for kernel \( \phi_{kj} \). For all \( x, y, a, b \in \mathbb{R} \), define

\[ F_r(x, y, a, b) := \frac{(y - x)^3}{3} - (a + b) \frac{(y - x)^2}{2} + ab(y - x). \]

The partial derivatives of \( F_r \) are given by

\[ \frac{\partial F_r}{\partial x}(x, y, a, b) = -(y - x)^2 + (a + b)(y - x) - ab, \]
\[ \frac{\partial F_r}{\partial y}(x, y, a, b) = (y - x)^2 - (a + b)(y - x) + ab, \]
\[ \frac{\partial F_r}{\partial a}(x, y, a, b) = -\frac{(y - x)^2}{2} + b(y - x), \]
\[ \frac{\partial F_r}{\partial b}(x, y, a, b) = -\frac{(y - x)^2}{2} + a(y - x). \]
\[ \text{For } z := (x, y, a, b), \text{ denote the gradient of } F_r \text{ in } z \text{ by } \nabla F_r(z). \]

**Proposition D.12** (Triangular with triangular). We assume that the kernel

\[ \phi_{kj} := \phi_{\omega_{kj}, \alpha_{kj}, \beta_{kj}, \delta_{kj}} \]

is a triangular kernel with parameters \( \theta_{kj} = (\omega_{kj}, \alpha_{kj}, \beta_{kj}, \delta_{kj}) \). Define the integration lower bounds

\[ z_{(1)}_{ij't'} := \max(0, \alpha_{ki}, \alpha_{kj'}, \alpha_{ki}, \alpha_{kj'}) - s, \]
\[ z_{(2)}_{ij't'} := \max(0, \alpha_{ki}, \alpha_{kj'}, \alpha_{ki}, \alpha_{kj'}) - s + \beta_{kj'}, \]
\[ z_{(3)}_{ij't'} := \max(0, \alpha_{ki} + \beta_{ki} - \delta_{kj'}, \alpha_{kj'} - s), \]
\[ z_{(4)}_{ij't'} := \max(0, \alpha_{ki} + \beta_{ki} - \delta_{kj'}, \alpha_{kj'} - s + \beta_{kj'}). \]

and the integration upper bounds

\[ y_{(1)}_{ij't'} := \min(t, \alpha_{ki} + \beta_{ki}, \alpha_{kj'}, \alpha_{ki}, \alpha_{kj'}) - s, \]
\[ y_{(2)}_{ij't'} := \min(t, \alpha_{ki} + \beta_{ki}, \alpha_{kj'}, \alpha_{ki}, \alpha_{kj'}) - s + \beta_{kj'}, \]
\[ y_{(3)}_{ij't'} := \min(t, \alpha_{ki} + \beta_{ki} - \delta_{kj'}, \alpha_{kj'} - s), \]
\[ y_{(4)}_{ij't'} := \min(t, \alpha_{ki} + \beta_{ki} - \delta_{kj'}, \alpha_{kj'} - s + \beta_{kj'}). \]

Define the coefficients

\[ a_{(1)} = \alpha_{ki}, \]
\[ a_{(2)} = \alpha_{ki} + \beta_{ki}, \]
\[ b_{(1)} = \alpha_{kj'} - s, \]
\[ b_{(2)} = \alpha_{kj'} - s + \beta_{kj'}. \]

To simplify the expression of \( T_{ijk} \), denote

\[ a_{(3)} = a_{(1)} + b_{(1)} - \delta_{kj'}, \]
\[ b_{(3)} = b_{(1)} + \delta_{kj'}. \]

Define the weights

\[ c_{(1)} := \frac{1}{\delta_{kj'}}, \]
\[ c_{(2)} := \frac{1}{\beta_{kj'}}, \]
\[ c_{(3)} := -\frac{1}{\delta_{kj'}}, \]
\[ c_{(4)} := \frac{1}{\beta_{kj'}}. \]

Then for all \( t, s \geq 0 \)

\[ T_{ijk}(t, s) = \sum_{i=1}^{r_k} \sum_{F=1}^{s_k} \omega_{ki} \omega_{kj} \sum_{m=1}^{4} c_{(m)} \chi_{(m)}(t, s) 1_{t < y_{(m)}_{ij't'}}, \]
\[ F_r(t, s) 1_{t < y_{(m)}_{ij't'}}, a_{(m)}, b_{(m)} \]
\[ \text{Define } \]
\[ z_{(m)}_{ij't'} := (x_{(m)}_{ij't'}, y_{(m)}_{ij't'}, a_{(m)}, b_{(m)})^T. \]
Let $p \in \{r_k\}$ and $q \in \{r_k\}$. If $i \neq j$, the partial derivatives of $\Gamma_{ijk}$ with respect to the parameters are given by

\[
\frac{\partial \Gamma_{ijk}}{\partial \omega_{kip}}(t, s) = \sum_{l' \in \{r_k\} \setminus \{r_k\}}^r \omega_{kl'} \sum_{m=1}^4 c^{(m)}_{lp} 1_{s_p \leq s_{lp}, t_{ip} \leq t_{lp}} F_e \left(z^{(m)}_{lp, ts} \right),
\]

(140)

\[
\frac{\partial \Gamma_{ijk}}{\partial \omega_{kjq}}(t, s) = \sum_{l' \in \{r_k\} \setminus \{r_k\}}^r \omega_{kl'} \sum_{m=1}^4 c^{(m)}_{lp} 1_{s_p \leq s_{lp}, t_{ip} \leq t_{lp}} \left( \frac{\partial z^{(m)}_{lp, ts}}{\partial \omega_{kjq}} \nabla F_r \left(z^{(m)}_{lp, ts} \right) \right),
\]

(141)

\[
\frac{\partial \Gamma_{ijk}}{\partial \beta_{kjp}}(t, s) = \omega_{kip} \sum_{l' \in \{r_k\} \setminus \{r_k\}}^r \omega_{kl'} \sum_{m=1}^4 c^{(m)}_{lp} 1_{s_p \leq s_{lp}, t_{ip} \leq t_{lp}} \left( \frac{\partial z^{(m)}_{lp, ts}}{\partial \beta_{kjp}} \nabla F_r \left(z^{(m)}_{lp, ts} \right) \right),
\]

(142)

\[
\frac{\partial \Gamma_{ijk}}{\partial \beta_{kjq}}(t, s) = \omega_{kjq} \sum_{l' \in \{r_k\} \setminus \{r_k\}}^r \omega_{kl'} \sum_{m=1}^4 c^{(m)}_{lp} 1_{s_p \leq s_{lp}, t_{ip} \leq t_{lp}} \left( \frac{\partial z^{(m)}_{lp, ts}}{\partial \beta_{kjq}} \nabla F_r \left(z^{(m)}_{lp, ts} \right) \right),
\]

(143)

\[
\frac{\partial \Gamma_{ijk}}{\partial \beta_{kjp}}(t, s) = \omega_{kip} \sum_{l' \in \{r_k\} \setminus \{r_k\}}^r \omega_{kl'} \sum_{m=1}^4 c^{(m)}_{lp} 1_{s_p \leq s_{lp}, t_{ip} \leq t_{lp}} \left( \frac{\partial z^{(m)}_{lp, ts}}{\partial \beta_{kjp}} \nabla F_r \left(z^{(m)}_{lp, ts} \right) \right),
\]

(144)

\[
\frac{\partial \Gamma_{ijk}}{\partial \beta_{kjq}}(t, s) = \omega_{kjq} \sum_{l' \in \{r_k\} \setminus \{r_k\}}^r \omega_{kl'} \sum_{m=1}^4 c^{(m)}_{lp} 1_{s_p \leq s_{lp}, t_{ip} \leq t_{lp}} \left( \frac{\partial z^{(m)}_{lp, ts}}{\partial \beta_{kjq}} \nabla F_r \left(z^{(m)}_{lp, ts} \right) \right),
\]

(145)

\[
\frac{\partial \Gamma_{ijk}}{\partial \beta_{kjp}}(t, s) = \omega_{kip} \sum_{l' \in \{r_k\} \setminus \{r_k\}}^r \omega_{kl'} \sum_{m=1}^4 c^{(m)}_{lp} 1_{s_p \leq s_{lp}, t_{ip} \leq t_{lp}} \left( \frac{\partial z^{(m)}_{lp, ts}}{\partial \beta_{kjp}} \nabla F_r \left(z^{(m)}_{lp, ts} \right) \right),
\]

(146)

\[
\frac{\partial \Gamma_{ijk}}{\partial \beta_{kjq}}(t, s) = \omega_{kjq} \sum_{l' \in \{r_k\} \setminus \{r_k\}}^r \omega_{kl'} \sum_{m=1}^4 c^{(m)}_{lp} 1_{s_p \leq s_{lp}, t_{ip} \leq t_{lp}} \left( \frac{\partial z^{(m)}_{lp, ts}}{\partial \beta_{kjq}} \nabla F_r \left(z^{(m)}_{lp, ts} \right) \right),
\]

(147)

If $i = j$, the partial derivatives of $\Gamma_{ijk}$ with respect to the parameters are given by

\[
\frac{\partial \Gamma_{ijk}}{\partial \omega_{kip}}(t, s) = 2 \omega_{kip} \sum_{m=1}^4 c^{(m)}_{lp} 1_{s_p \leq s_{lp}, t_{ip} \leq t_{lp}} F_e \left(z^{(m)}_{lp, ts} \right),
\]

(148)

\[
\frac{\partial \Gamma_{ijk}}{\partial \omega_{kjp}}(t, s) = \sum_{l' \in \{r_k\} \setminus \{r_k\}}^r \omega_{kl'} \sum_{m=1}^4 c^{(m)}_{lp} 1_{s_p \leq s_{lp}, t_{ip} \leq t_{lp}} F_e \left(z^{(m)}_{lp, ts} \right),
\]

(149)
\[
\frac{\partial Y_{ik}}{\partial \beta_{kip}}(t, s) = \omega_{kip}^2 \left[ \sum_{m=1}^{4} c_{pp}^{(m)} 1_{p \neq p', s \neq s'} \left( \frac{\partial z_{pp,ts}^{(m)}}{\partial \beta_{kip}} \nabla F_r \left( z_{pp,ts}^{(m)} \right) \right) \right.
\]
\[
+ 1_{p \neq p', s \neq s'} \frac{\partial z_{pp,ts}^{(m)}}{\partial \beta_{kip}} F_r \left( z_{pp,ts}^{(m)} \right) \right]
\]
\[
+ \omega_{kip} \sum_{l \in \{ r_{kl} \}, t \neq p} \omega_{kl} \left[ \sum_{m=1}^{4} c_{pp}^{(m)} 1_{p \neq p', s \neq s'} \left( \frac{\partial z_{pp,ts}^{(m)}}{\partial \beta_{kip}} \nabla F_r \left( z_{pp,ts}^{(m)} \right) \right) \right.
\]
\[
+ 1_{p \neq p', s \neq s'} \frac{\partial z_{pp,ts}^{(m)}}{\partial \beta_{kip}} F_r \left( z_{pp,ts}^{(m)} \right) \right]
\]
\[
+ \omega_{kip} \sum_{l \in \{ r_{kl} \}, t \neq p} \omega_{kl} \left[ \sum_{m=1}^{4} c_{pp}^{(m)} 1_{p \neq p', s \neq s'} \left( \frac{\partial z_{pp,ts}^{(m)}}{\partial \beta_{kip}} \nabla F_r \left( z_{pp,ts}^{(m)} \right) \right) \right.
\]
\[
+ 1_{p \neq p', s \neq s'} \frac{\partial z_{pp,ts}^{(m)}}{\partial \beta_{kip}} F_r \left( z_{pp,ts}^{(m)} \right) \right]
\]

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