Exactly Soluble Sector of Quantum Gravity

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ABSTRACT

Cartan’s spacetime reformulation of the Newtonian theory of gravity is a generally-covariant Galilean-relativistic limit-form of Einstein’s theory of gravity known as the Newton-Cartan theory. According to this theory, space is flat, time is absolute with instantaneous causal influences, and the degenerate ‘metric’ structure of spacetime remains fixed with two mutually orthogonal non-dynamical metrics, one spatial and the other temporal. The spacetime according to this theory is, nevertheless, curved, duly respecting the principle of equivalence, and the non-metric gravitational connection-field is dynamical in the sense that it is determined by matter distributions. Here, this generally-covariant but Galilean-relativistic theory of gravity with a possible non-zero cosmological constant, viewed as a parameterized gauge theory of a gravitational vector-potential minimally coupled to a complex Schrödinger-field (bosonic or fermionic), is successfully cast — for the first time — into a manifestly covariant Lagrangian form. Then, exploiting the fact that Newton-Cartan spacetime is intrinsically globally-hyperbolic with a fixed causal structure, the theory is recast both into a constraint-free Hamiltonian form in 3+1-dimensions and into a manifestly covariant reduced phase-space form with non-degenerate symplectic structure in 4-dimensions. Next, this Newton-Cartan-Schrödinger system is non-perturbatively quantized using the standard C*-algebraic technique combined with the geometric procedure of manifestly covariant phase-space quantization. The ensuing unitary quantum field theory of Newtonian gravity coupled to Galilean-relativistic matter is not only generally-covariant, but also exactly soluble and — thanks to the immutable causal structure of the Newton-Cartan spacetime — free of all conceptual and mathematical difficulties usually encountered in quantizing Einstein’s theory of gravity. Consequently, the resulting theory of quantized Newton-Cartan-Schrödinger system constitutes a perfectly consistent Galilean-relativistic sector of the elusive full quantum theory of gravity coupled to relativistic matter, regardless of what ultimate form the latter theory eventually takes.

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1. Introduction

The primary aim of this paper is to demonstrate that the principle of equivalence by itself is not responsible for the conceptual and mathematical difficulties encountered in constructing a viable quantum theory of gravity; rather, it is the conjunction of this principle with the conformal structure (i.e., the field of light-cones) of the general-relativistic spacetime which resists subjugation to the otherwise well-corroborated canonical rules of quantization. This elementary fact, of course, has been duly appreciated by the workers in the field at least implicitly since the earliest days of attempts to quantize Einstein’s theory of gravity. Curiously enough, however, except for a partial illustration of this state of affairs by Kuchař [1], so far the problem of explicitly constructing a generally-covariant but Galilean-relativistic quantum theory of gravity — i.e., a generally-covariant quantum field theory of spacetime with degenerate structure of ‘flattened’ light-cones but unique dynamical connection — has been completely neglected [2]. Here we set out to construct such a ‘nonrelativistic’ theory and explicitly demonstrate that, unlike the case of relativistic quantum gravity, there are no insurmountable conceptual or mathematical difficulties in achieving this goal. In particular, we show that the Galilean-relativistic limit-form (see Figure 1) of the as-yet-untamed full quantum theory of gravity interacting with matter is exactly soluble, and that in this very classical (‘c = ∞’) domain ‘the problem of time’ [3] — the well-known central stumbling block encountered in quantizing Einstein’s gravity — and related problems of causality, along with other conceptual and mathematical difficulties, are nonexistent.

The nonrelativistic limit-form of Einstein’s theory of gravity is a spacetime reformulation of the Newtonian theory of gravity — the so-called Newton-Cartan theory. With the hindsight of Einstein’s theory it is clear that gravitation should be treated as a consequence of the curving of spacetime rather than as a force-field even at the Newtonian

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1 The adjective ‘dynamical’ here and below simply refers to the mutability of spacetime structure dictated by evolving distributions of matter. It only refers to the fact that, even in a Galilean-relativistic theory, spacetime is not fixed a priori. In particular, unlike the case in general relativity, it does not refer to any transverse propagation degrees of freedom associated with the spacetime structure since nonrelativistic gravity does not possess such a freedom. See subsection 5.4, however, for a discussion on the longitudinal degrees of freedom of the gravitational field.
Figure 1: The great dimensional monolith of physics indicating the fundamental role played by the three universal constants $G$ (the Newton’s gravitational constant), $\hbar$ (the Planck’s constant of quanta divided by $2\pi$), and $c$ (the absolute upper bound on the speed of causal influences) in various basic physical theories. These theories, appearing at the eight vertices of the cube, are: CTM = Classical Theory of Mechanics, STR = Special Theory of Relativity, GTR = General Theory of Relativity, NCT = (classical) Newton-Cartan Theory, NQG = (generally-covariant) Newtonian Quantum Gravity (constructed in this paper), GQM = Galilean-relativistic Quantum Mechanics, QTF = Quantum Theory of (relativistic) Fields, and FQG = the elusive Full-blown Quantum Gravity. If FQG turns out to require some additional fundamental constants — like the constant $\alpha' \equiv (2\pi T)^{-1}$ of the string theory controlling the string tension $T$, for example — then, of course, the above representation of the foundational theories would be inadequate, and additional axes corresponding to such constants $\alpha_i$, $i = 1, 2, \ldots, n$, would have to be added to the diagram, making it a $3 + n$ dimensional hypercube. In that case, NQG, in particular, would be a limit-form of FQG with respect to total $n + 1$ limits, $\alpha_i \to 0$ and $c \to \infty$, in conjunction. (A diagrammatic illustration of existing and future physical theories in terms of the fundamental constants was first given by Kuchař in the form of a pyramid in Ref. 1. This was adapted in Ref. 24, from which Penrose was inspired to (characteristically) unfold the pyramid into a cube. The figure here, in turn, has been inspired by the cubic version presented by Penrose in his 1994-95 Tanner lectures delivered at the University of Cambridge, UK. These lectures are now published in Ref. 2.)
level because the principle of equivalence is equally compatible with the Newtonian spacetime. A geometrical description of the Newtonian spacetime explicitly incorporating the principle of equivalence at the classical (non-quantal) level was given by Cartan [4] and Friedrichs [5] soon after the completion of Einstein’s theory, and later further developed by many authors [6,8,9,10]. The outcome of such a reformulation of the Newtonian theory is a theory whose qualitative features lie in-between those of special relativity with its completely fixed spacetime background and general relativity with no background structure whatsoever. Unlike the latter two well-known theories, Newton-Cartan theory has two fixed and degenerate metrics — a temporal metric and a spatial metric, vaguely resembling the fixed Minkowski metric as far as their non-dynamical character is concerned — and a non-metric but dynamical1 connection-field mimicking Einstein’s metric connection-field to some extent. Thus, unlike the static and flat Galilean spacetime, and analogous to the mutable general-relativistic spacetime, the generally-covariant2 Newton-Cartan spacetime is dynamical, curved by the Newtonian gravity, and requires no a priori assumption of a global inertial frame. Consequently, the transition from Galilean spacetime to this general-nonrelativistic Newton-Cartan spacetime drastically changes the qualitative features of the Galilean-relativistic physics by elevating the status of the affine connection from that of an absolute element — given once and for all — to a dynamical quantity determined by the distribution of matter. Furthermore, it is this Newton-Cartan theory of gravity with its mutable spacetime which is in general the true Galilean-relativistic limit-form of Einstein’s general theory of relativity [10,11,12,13], and not the Newtonian theory on the immutable background of flat Galilean spacetime (cf. subsection 2.3 below). In summary, Cartan’s geometric reformulation of Newton’s theory of gravity makes it a local field theory analogous to general relativity, and, as a result, the instantaneous gravitational interactions

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1 The philosophical dispute over the meaning of the principle of general covariance begun almost immediately after the completion of Einstein’s theory of gravity [14] and persists today [15]. Einstein, for instance, read through Stachel’s eyeglasses [16], would not view Newton-Cartan theory as a genuinely generally-covariant theory because, unlike general relativity, it does not avert the ‘hole argument’ [3]. As far as this paper is concerned, however, we can afford to refrain from the controversy and join the bandwagon calling Newton-Cartan spacetime generally-covariant simply because it respects the principle of equivalence and precludes existence of global inertial frames of reference. Moreover, at least in the framework followed in this paper, the non-metric Newton-Cartan theory is as diffeomorphism-invariant as Einstein’s metric theory of gravity (cf. equation (2.50)). (See also footnote s.)
between gravitating bodies can now be understood as propagating continuously through the curvature of the region of spacetime among them.

To understand these stipulations in detail, in section 2 we accumulate various scattered results to provide a coherent — but by no means exhaustive — review of the classical (non-quantal) Newton-Cartan theory of gravity. Then, in section 3, we review the previous work on one-particle Schrödinger quantum mechanics on the curved Newton-Cartan spacetime initiated by Kuchař [1], and extend it to a Galilean-relativistic quantum field theory on such a spacetime. Next, in section 4, we derive, for the first time, the complete classical Newton-Cartan-Schrödinger theory (i.e., the theory of classical Newton-Cartan field interacting with a Galilean-relativistic matter) from extremizations of a single diffeomorphism-invariant action functional. This functional — defined on an arbitrary measurable region of spacetime — is carefully selected to allow recasting of the theory into a constraint-free Hamiltonian form in 3+1 dimensions, as well as into a manifestly covariant reduced phase-space form in 4-dimensions (where the phase-space is viewed as the space of solutions of the equations of motion modulo gauge-transformations). Thus obtained covariant and constraint-free phase-space then paves the way in section 5 for a straightforward quantization of the Newton-Cartan-Schrödinger system using the standard C*-algebraic techniques and the associated representation theory. The resulting local quantum field theory describes an exactly soluble interacting matter-gravity system.

In addition to the primary aim discussed in the beginning of this Introduction, there are various other motivations for the present exercise which we enumerate here (counting the primary aim as (1)):

(2) As we shall see in the next section, one of the set of gravitational field equations of Newton-Cartan theory closely resembles Einstein’s field equation \( G_{\mu\nu} = 8\pi T_{\mu\nu} \) relating geometry to matter. In particular, it dictates that the connection-field of Newton-Cartan gravity must be determined by the distribution of matter. Therefore, consistency requires that this connection-field must be quantized along with matter since it participates in the dynamical unfolding of the combined system.

(3) It is well-known that Newtonian models play quite a significant role in cosmology. In particular, they are useful in studying structure formation in the early universe.
and provide useful insights for the relativistic case. Recently, the need to generalize Newtonian cosmology to Newton-Cartan cosmology has been recognized, and a considerable progress has been made in this direction [19,20,21]. In this context, then, the relevance of an exactly soluble quantum theory of Newton-Cartan gravity interacting with matter cannot be overestimated.

(4) It is fair to say that we know very little about the quantum gravity proper (i.e., FQG in the Figure 1). Therefore, insights coming from any quarters which enable us to better understand the difficulties of constructing the final theory should be welcomed. It is with this attitude that the recent exercises on exactly soluble gravitating systems in reduced spacetime dimensions [22] and/or reduced symmetries [23] have been carried out, and used as probes to investigate the conceptual problems of the full quantum gravity. Here we do not reduce either symmetries or the spacetime dimensions, but instead provide an exactly soluble system in the full 4-dimensional setting — albeit only in the Galilean-relativistic limit of the full theory.

(5) In Ref. 24 we have argued, on both group theoretical and physical grounds, that, since Newton-Cartan symmetries — duly respecting the principle of equivalence — and not Galilean symmetries are the true spacetime symmetries of the nonrelativistic quantum domain, any discussion on the conceptual issues like the ‘measurement problem’ in this domain must be carried out within the Newton-Cartan framework. In fact, it was argued, Penrose-type speculations of gravitationally induced state-reduction [25] might greatly benefit from analyzing the relevant physical systems within this framework. It is gratifying to note that this suggestion has already attracted at least Penrose’s attention [25,26]. However, the complete framework for such deep conceptual issues must be the fully quantized Newton-Cartan gravity interacting with matter — and hence the present work.

(6) The theory constructed in this paper provides a selection criterion for any exotic, top-down approach (e.g., the superstring approach) to the final ‘theory of everything.’ Clearly, any general-relativistic exotic theory would lose its physical relevance if it does not reduce to the Newtonian quantum gravity interacting with Schrödinger-fields in the Galilean-relativistic limit. Therefore, any future theory must reduce to NQG of
Figure 1 in the ‘c → ∞’ limit, as much as it should reduce to GTR in the ‘ℏ → 0’ limit and QTF in the ‘G → 0’ limit.

Finally, the existence of NQG opens up a completely novel direction of research in the full quantum gravity. In majority of orthodox approaches to quantum gravity the direction of research has been to go from GTR to FQG (cf. Figure 1) — i.e., the program has been to quantize the general theory of relativity. Somewhat less popular and less explored program is to go from QTF to FQG — i.e., to general-relativize the quantum theory of fields [27]. The existence of NQG opens up a third possibility, that of starting from NQG and arriving at FQG by undoing the ‘c → ∞’ limit — i.e., by special-relativizing the Newton-Cartan quantum gravity.

2. Spacetime approach to Newtonian gravity

In this section we review the covariant, spacetime reformulation of the Newtonian theory of gravity, and, thereby, set the notations and conventions to be used in the following sections. Most of the ideas presented in this and the next section are not new, but it is the manner in which they are organized here that makes them conducive to the fruitful results of the later sections.

2.1. General Galilean spacetime

We begin with the familiar spacetime structure (\(\mathcal{M}; t_\alpha, h^{\alpha\beta}, \nabla_\alpha\)) presupposed by the usual Galilean-relativistic dynamics delineated in Penrose’s abstract index notation [28] using the greek alphabet [12]. \(\text{Spacetime}\) — the arena in which physical events and processes take place — is represented by a real, contractible, and differentiable Hausdorff 4-manifold \(\mathcal{M}\) without boundaries. Unlike the case in general-relativistic spacetimes, here spatial and temporal measures on \(\mathcal{M}\) are not taken to be soldered into a single semi-Riemannian metric, but appear as two distinct geometric entities. A smooth, never vanishing covariant vector field \(t_\alpha\) on \(\mathcal{M}\) is defined which induces a degenerate temporal metric \(t_{\mu\nu} = t_{(\mu\nu)} := t_\mu t_\nu\) of signature \((+000)\) specifying durations of processes occurring between events, and hence induces a degenerate ‘cone structure’ on the tangent space \(T_x\mathcal{M}\) at each point \(x \in \mathcal{M}\) (here the parentheses indicate symmetrization with respect to the enclosed indices). A vector
\( \xi^\mu \) at a point on \( \mathcal{M} \) is said to be \textit{timelike} if \( \sqrt{t_{\mu\nu}\xi^\mu\xi^\nu} > 0 \), \textit{spacelike} if \( \sqrt{t_{\mu\nu}\xi^\mu\xi^\nu} = 0 \), and \textit{future-directed} if \( t_\mu\xi^\mu > 0 \). Between spacelike vectors \( \xi^\alpha \) on \( \mathcal{M} \) with vanishing ‘temporal lengths’ — i.e., \( \sqrt{t_{\mu\nu}\xi^\mu\xi^\nu} = 0 \) — there is defined an inner product with the help of a smooth, symmetric, contravariant vector field \( h^{\alpha\beta} = h^{(\alpha\beta)} \) on \( \mathcal{M} \) which serves as a degenerate \textit{spatial metric} of signature \((0+++)\), and indirectly assigns lengths to these vectors: \[ |\xi| := \sqrt{h^{\mu\nu}\lambda_\mu\lambda_\nu}, \] where \( h^{\mu\nu}\lambda_\nu = \xi^\mu \) with an arbitrary choice of \( \lambda_\mu \). As we have done here, the metric tensor \( h^{\mu\nu} \) can be used to raise indices; however, since it is not invertible, it cannot be used to lower indices. Thus, the distinction between covectors and contravectors has much greater significance in this general Galilean-relativistic spacetime than in the semi-Riemannian general-relativistic spacetime. The \textit{affine structure} of the spacetime \( \mathcal{M} \) is represented by a smooth derivative operator \( \nabla_\alpha \) introducing a (not necessarily ‘flat’) torsion-free linear connection \( \Gamma \) on \( \mathcal{M} \). The two tensor fields — \( \tau := t_\alpha dx^\alpha \) measuring the proper time of world lines and \( h := h^{\alpha\beta} \partial_\alpha \otimes \partial_\beta \) inducing a 3-metric on the null space of \( \tau \) — are mutually orthogonal,

\[
\tau \perp h = 0 = h^{\alpha\beta} t_\beta \tag{2.1}
\]

(the kernel of \( h \) generating the span of the 1-form \( \tau \)), and taken to be compatible with the derivative operator:

\[
\nabla_\alpha h^{\beta\gamma} = 0 = \nabla_\alpha t_\beta. \tag{2.2}
\]

A Galilean spacetime is orientable as there exists a 4-volume element for the structure \((\mathcal{M}; t_\alpha, h^{\alpha\beta}, \nabla_\alpha)\). Given the structure \((\mathcal{M}; t_\alpha, h^{\alpha\beta}, \nabla_\alpha)\) satisfying the conditions (2.2), a continuous, nowhere vanishing spacetime measure form on \( \mathcal{M} \) with tensor components \( \varepsilon_{\alpha\beta\gamma\delta} = \varepsilon_{[\alpha\beta\gamma\delta]} \) can be derived \([23]\) such that \( \nabla_\mu \varepsilon_{\alpha\beta\gamma\delta} = 0 \) (where the square brackets indicate anti-symmetrization with respect to the bracketed indices). If one defines an oriented \textit{Galilean frame} at a point \( x \in \mathcal{M} \) as a basis \( \{e_i\} \) of the tangent space \( T_x \mathcal{M} \) such that \( e_i^\alpha t_\alpha = \delta_i^0 \) and \( h^{\mu\nu}\theta_i^\mu \theta_j^\nu = \delta_i^j \delta^{ab} \delta_j^b \) (where \( a, i, j = 0,\ldots,3 \) and \( a, b = 1,2,3 \)), with \( \{\theta^i\} \) being the dual basis of \( \{e_i\} \), then for any such Galilean frame \( \{e_i\} \) the canonical 4-volume element can be defined by \([11]\)

\[
\varphi \; d^4x := \frac{1}{4!} \varepsilon_{\alpha\beta\gamma\delta} \; dx^\alpha \wedge dx^\beta \wedge dx^\gamma \wedge dx^\delta := \theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3. \tag{2.3}
\]

The compatibility of the temporal metric \( \nabla_\alpha t_\beta = 0 \), giving the condition \( \nabla_{[\gamma} t_{\delta]} = 0 \), at least locally allows the relation \( t_\beta = \nabla_\beta t \) for some time function \( t \). Since \( \mathcal{M} \) is contractible
by definition, the Poincaré lemma allows one to define the absolute time function also
globally by a map \( t : \mathcal{M} \rightarrow \mathbb{R} \), foliating the spacetime uniquely into one-parameter family
of (not necessarily flat) spacelike hypersurfaces\(^3\) of simultaneity. One can use this time-
function \( t \) as an affine parameter for arbitrary timelike curves representing the worldlines
of test particles.

Since the compatibility condition \( \nabla_\mu t_\nu = 0 \) leads to the relation \( t_\gamma \Gamma^\gamma_{[\mu \nu]} = \partial_{[\mu} t_{\nu]} \), it
is clear that a torsion-free connection,

\[
\Gamma^\gamma_{[\mu \nu]} = 0,
\]

is admissible only if

\[
\partial_{[\mu} t_{\nu]} = 0; \quad \text{i.e.,} \quad d\tau = 0.
\]

In fact, it can be shown\( [9] \) that the closed-ness of \( \tau \) is both necessary and sufficient
condition for the existence of a torsion-free connection as a part of the Galilean structure.
Moreover, without the condition that the 1-form \( \tau \) is closed, the structure \( (\mathcal{M}; \tau, h) \)

is not integrable. A linear symmetric Galilean connection satisfying the compatibility
conditions (2.2) exists if and only if the structure \( (\mathcal{M}; \tau, h) \) is integrable. However,
given integrability, such a symmetric connection is unique only up to an arbitrary 2-form
\( F = \frac{1}{2} F_{\mu\nu} \, dx^\mu \wedge dx^\nu \), and can be decomposed as \( [30] \):

\[
\Gamma^\gamma_{\mu\nu} = \Gamma^\gamma_{(\mu\nu)} = \Gamma^\gamma_{\mu\nu} + t_{(\mu} F_{\nu)\sigma} h^{\sigma\gamma},
\]

where \( [32] \)

\[
\Gamma^\gamma_{\mu\nu} := h^{\gamma\sigma} \left\{ \partial_{(\mu} h_{\nu)\sigma} - \frac{1}{2} \partial_\sigma h_{\mu\nu} \right\} + u^\gamma \partial_{(\mu} t_{\nu)}
\]

with \( u = u^\alpha \partial_\alpha \) representing an arbitrary unit timelike vector-field,

\[
t_\alpha u^\alpha = 1 = \tau \parallel u,
\]

interpreted as the four-velocity of an adscititious ‘aether-frame’ and \( h_{\mu\nu} = h_{(\mu\nu)} \) representing the associated spatial projection field relative to \( u^\alpha \) defined by

\[
u h_{\mu\rho} u^\rho := 0 \quad \text{and} \quad u h_{\mu\rho} h^{\rho\nu} := \delta^\nu_\nu - t_\mu u^\nu =: \delta^\nu_\nu.
\]

\(^3\) As is well-known, hypersurfaces are best represented by embeddings. A one-parameter family of embeddings in
the present context is a map \( t : \Sigma \rightarrow \mathcal{M} \) which takes a point \( x^a \) from the spatial submanifold \( \Sigma \) of \( \mathcal{M} \) to a point
\( x^\alpha(x^a, t) \) in the spacetime, where \( t \in \mathbb{R} \) labels the leaves of this foliation. A hypersurface, then, is an equivalence
class of such embeddings modulo diffeomorphisms of the submanifold \( \Sigma \) (cf. subsection 4.2).
Throughout the paper, a letter (e.g., \( u \)) on the top of a quantity indicates gauge-dependence of the quantity (e.g., dependence of the quantity \( h_{\mu\nu} \) on the ‘gauge’ \( u \)). This relative projection field \( h_{\mu\nu} \) introducing a fiducial ‘absolute rest’ \(^{[13]}\) may be used to lower indices, but, of course, only relative to \( u \); for, under an æther-frame transformation of the form

\[
u^\alpha \mapsto \tilde{\nu}^\alpha := u^\alpha + \hat{h}^{\alpha\sigma}w_\sigma\quad (2.10)
\]

for a boost covector \( w_\sigma \), the relative projection field undergoes a nontrivial boost transformation:

\[
h_{\mu\nu} \mapsto \tilde{h}_{\mu\nu} := h_{\mu\nu} - 2t_\mu h_\nu^\alpha h^{\alpha\sigma}w_\sigma + t_{\mu\nu} h^{\alpha\sigma}w_\alpha w_\sigma.
\quad (2.11)
\]

In other words,

\[
\frac{\partial}{\partial u^\sigma} h_{\mu\nu} = -2t_\mu h_\nu^\alpha,
\quad (2.12)
\]

which follows from the definition (2.9).

The special connection \( \Gamma \) is unique and symmetric, and such that the Galilean observer \( u^\alpha \) associated with it is geodetic, \( u^\sigma \nabla_\sigma u^\alpha = 0 \), and curl-free, \( h^{\sigma\mu}\nabla_\sigma u^\nu = 0 \). Using these two properties of \( u^\alpha \), and the above definition of its relative projection field \( h_{\mu\nu} \), one can express the 2-form \( F \) in terms of the covariant derivative of this observer \( u^\alpha \) \(^{[30]}\):

\[
F_{\mu\nu} = -2 \hat{h}_{\sigma[\mu} \nabla_{\nu]} u^\sigma.
\quad (2.13)
\]

For future purposes, we also observe that the spatial projection field \( h_{\mu\nu} \) is not compatible with the covariant derivative operator \( \nabla_\sigma \) in general, but, instead, its covariant derivative depends on the covariant rate of change of the observer-field \( u^\alpha \) \(^{[30]}\):

\[
\nabla_\sigma h_{\mu\nu} = -2 \left\{ \nabla_\sigma u^\alpha \right\} h_{\alpha(\mu} t_{\nu)}
\quad (2.14)
\]

where the covariant derivative of \( u^\alpha \) may be decomposed as \(^{[31]}\)

\[
\nabla_\mu u^\nu = h_{\mu\alpha} h^{\sigma(\alpha} \nabla_\sigma u^{\nu)} + h_{\mu\alpha} h^{\sigma[\alpha} \nabla_\sigma u^{\nu]} + t_{\mu\nu} u^\sigma \nabla_\sigma u^\nu.
\quad (2.15)
\]

The first term of this decomposition is a measure of the lack of ‘rigidity’ of the field \( u^\alpha \), and, hence, in the case of the rigid Galilean frame it vanishes identically: \( h^{\sigma(\mu} \nabla_\sigma u^{\nu)} \equiv 0 \).
Further, since the Galilean observer $u^\alpha$ is curl-free and geodetic with respect to the associated connection $\overset{\mu}{\Gamma}$, the remaining two terms of the equation (2.15) also vanish in this special case giving the uniformity property
\[ \overset{\mu}{\nabla}u^\nu = 0; \quad (2.16) \]
i.e., $u^\nu$ is covariantly constant with respect to the special connection $\overset{\mu}{\Gamma}$. As a result of this property, equation (2.14) yields the compatibility relation for the relative spatial projection field $h_{\mu\nu}$,
\[ \overset{\sigma}{\nabla}h_{\mu\nu} = 0, \quad (2.17) \]
in this special case of unique derivative operator $\overset{\sigma}{\nabla}$ associated with the æther-frame $u^\alpha$.

The curvature tensor corresponding to the symmetric Galilean connection (2.6) — defined by $R^\sigma{}_{\beta\gamma\delta} f_\sigma = 2 \overset{\gamma}{\nabla} \overset{\delta}{\nabla} f_\beta$, or, equivalently, by $R^\alpha{}_{\sigma\gamma\delta} g^\sigma = -2 \overset{\gamma}{\nabla} \overset{\delta}{\nabla} g_\alpha$, for arbitrary vectors $f_\alpha$ and $g_\alpha$ — observes the symmetries $[9,30]$
\[ h^{(\beta R^\alpha}_{\sigma\gamma\delta} = 0 \quad \text{and} \quad t_\sigma R^\sigma{}_{\beta\gamma\delta} = 0 \]
by virtue of the metric compatibility conditions, in addition to the usual constraints
\[ R^\alpha{}_{\beta(\gamma\delta)} = 0, \quad R^\alpha{}_{[\beta\gamma\delta]} = 0, \quad \text{and} \quad R^\alpha{}_{\beta[\gamma\delta ; \lambda]} = 0 \quad (2.19) \]
satisfied by any curvature tensor, where $;\lambda$ denotes the covariant derivative $\nabla_\lambda$. It can be easily verified that the contracted Bianchi identities of the form
\[ \nabla_\nu (R^{\mu\nu} - \frac{1}{2} R h^{\mu\nu}) = 0 \quad (2.20) \]
hold in the general Galilean spacetime as a consequence of the last of these constraints, where $R_{(\mu\nu)} = R_{\mu\nu} := R^\sigma{}_{\mu\nu\sigma}$ and $R := h^{\mu\nu} R_{\mu\nu}$ are the corresponding Ricci tensor and Ricci scalar, respectively.

Finally, we close this subsection by reemphasizing that, although an integrable Galilean structure $(\mathcal{M}; t_\alpha, h^{\alpha\beta}, \nabla_\alpha)$ is completely specified by the four conditions
\[ h^{\alpha\beta} t_\beta = 0, \quad \nabla_\alpha h^{\beta\gamma} = 0, \quad \nabla_\alpha t_\beta = 0, \quad \text{and} \quad \partial_{[\alpha} t_{\beta]} = 0, \quad (2.21) \]
the Galilean connection (2.6) remains under-determined, in general, by an arbitrary 2-form.
2.2. Specializing to the Newton-Cartan spacetime

Now, Cartan’s spacetime reformulation of the Newtonian theory of gravity can be motivated in exact analogy with Einstein’s theory of gravity. The analogy works because the universal equality of the inertial and the passive gravitational masses is independent of the relativization of time, and hence is equally valid at the Galilean-relativistic level. As a result, it is possible to parallel Einstein’s theory and reconstrue the trajectories of (only) gravitationally affected particles as geodesics of a unique, ‘non-flat’ connection $\Gamma$ satisfying

$$a^i := \frac{d^2 x^i}{dt^2} + \Gamma^i_{jk} \frac{dx^j}{dt} \frac{dx^k}{dt} = 0 \quad (2.22)$$

in a coordinate basis, or, equivalently,

$$a^\alpha := v^\sigma \nabla_\sigma v^\alpha = 0 \quad (2.23)$$

in general, such that

$$\Gamma^\mu_{\nu \lambda} \equiv v^\nu \Gamma^\mu_{\nu \lambda} + G^\nu_{\nu \lambda} := \Gamma^\nu_{\nu \lambda} + h^{\mu \alpha} v^\nu \Phi_{t \nu \lambda} \quad (2.24)$$

with $v^\nu$ representing the Newtonian gravitational potential relative to the freely falling observer field $v$, $\Gamma^\mu_{\nu \lambda}$ representing the coefficients of the corresponding ‘flat’ connection (i.e., one whose coefficients can be made to vanish in a suitably chosen linear coordinate system), and

$$G^\nu_{\nu \lambda} := h^{\mu \alpha} v^\nu \Phi_{t \nu \lambda} \quad (2.25)$$

representing the traceless gravitational field tensor associated with the Newtonian potential $v^\nu$. The conceptual superiority of this geometrization of Newtonian gravity is reflected in the trading of the two ‘gauge-dependent’ quantities $v^\nu \Gamma^\mu_{\nu \lambda}$ and $v^\nu G^\nu_{\nu \lambda}$ in favor of their gauge-independent sum $\Gamma$. Physically, it is the ‘curved’ connection $\Gamma$ rather than any ‘flat’ connection $v^\nu \Gamma$ that can be determined by local experiments. The potential $v^\nu \Phi$ and the ‘flat’ connection $v^\nu \Gamma$ do not have an independent existence; they exist only relative to an arbitrary choice of inertial frame. Given the ‘curved’ connection $\Gamma$, its associated curvature tensor works out to be

$$R^\alpha_{\beta \gamma \delta} = 2 t^\beta h^{\alpha \lambda} v_{\lambda [\gamma} t_{\delta]} \quad (2.26)$$
where ; α represent covariant derivatives $\nabla_\alpha$ corresponding to the ‘flat’ connection $\Gamma$. Although written in terms of gauge-dependent quantities, this expression for the curvature tensor is, of course, gauge-invariant.

The non-flat connection $\Gamma$ in equation (2.22) is still compatible with the temporal and spatial metrics $\tau$ and $h$: $\nabla_\alpha t_\beta = \nabla_\alpha h^{\beta\gamma} = 0$. This non-uniqueness of the compatible connections is due to the degenerate nature of the metrics $\tau$ and $h$ — i.e., due to the non-semi-Riemannian nature of the Galilean structure $(\mathcal{M}; \tau, h, \nabla)$. Unlike in the semi-Riemannian spacetime structures of the special and general theories of relativity, here the covariant derivative operator $\nabla_\alpha$ is not fully determined; as noted in the previous subsection, the connection (2.6) is determined by the compatibility conditions (2.2) only up to an arbitrary 2-form. Therefore, in addition to the structure $(\mathcal{M}; \tau, h)$, a connection must be specified to construct a completely geometrized, spacetime formulation of the Galilean-relativistic physics. In the case of Newtonian theory of gravity, geometrization can be easily achieved by taking those compatible connections $\Gamma$ for the structure $(\mathcal{M}; \tau, h, \nabla)$ whose curvature tensor, in addition to the properties (2.18) and (2.19), satisfies

$$R^{\alpha \gamma}_{\beta \delta} = R^{\gamma \alpha}_{\delta \beta}, \quad (2.27)$$

where $R^{\alpha \gamma}_{\beta \delta} \equiv h^{\gamma\sigma} R^{\alpha}_{\beta\sigma\delta}$. This extra condition roughly expresses the curl-freeness of the Newtonian gravitational field, and, together with the constraints (2.21), uniquely specifies the ‘curved’ Newton-Cartan connection; it is clearly satisfied by any Galilean connection that is obtained as a limit of the torsion-free Lorentzian connection. Equivalently, one can specify the Newton-Cartan connection by requiring that the 2-form $F$ in equation (2.6) be closed [3,30]: $dF = 0$. Then, in the light of the identity $d^2 = 0$, the 2-form $F$ may be expressed in terms of an arbitrary 1-form $A$ as

$$F_{\mu\nu} = 2 \nabla_{[\mu} A_{\nu]} = 2 \partial_{[\mu} A_{\nu]} \quad (2.28)$$

Clearly, this expressibility of the 2-form $F$ as an exterior derivative of an arbitrary 1-form is at least a sufficient condition for the connection (2.6) to be Newton-Cartan. Since $\mathcal{M}$ is contractible by definition, however, the Poincaré lemma implies that (2.28) is also a necessary condition for the specification of the Newton-Cartan connection. Comparison of
equations (2.28) and (2.13) shows that, at least locally, the intrinsic condition (2.27) can be equivalently expressed in terms of the fields $u^\nu$ and $A_\mu$ as

$$\nabla_{[\mu} A_{\nu]} + \frac{u}{h_{\sigma}[\mu} \nabla_{\nu]} u^\sigma = 0. \quad (2.29)$$

It is this convenient local form of the condition which will be useful in latter sections.

Physically, the covariant vector $A_\mu$ may be interpreted as the ‘vector-potential’ representing the combination of gravitational and inertial effects with respect to the Galilean observer $u = u^\alpha \partial_\alpha$. Given the ‘æther-frame’ based observer $u^\alpha$, an arbitrary observer may be characterized by a normalized four-velocity vector $v^\alpha$, $v^\alpha t_\alpha = 1$, such that the difference vector $h^{\alpha\sigma} A_\sigma := u^\alpha - v^\alpha$ describes how the motion of this arbitrary observer deviates from that of the ‘æther-frame’. In terms of $u$ and $A$ any Newton-Cartan connection can be affinely decomposed as

$$\Gamma_{\alpha \beta}^\gamma = \Gamma_{\alpha \beta}^\gamma + A_{\sigma}^\gamma A_{\sigma}^\gamma h^{\alpha\beta}, \quad (2.30)$$

where

$$A_{\sigma}^\gamma := t_{(\alpha} \left[ \partial_{\beta)} A_\sigma - \partial_\sigma A_\beta \right] h^{\alpha\beta}. \quad (2.31)$$

By comparing this decomposition in terms of the ‘gauge’ $(u, A)$ with that in terms of the generic ‘gauge’ $(v, \Phi)$ (cf. equation (2.24)) we find the relation

$$v^\alpha = u^\alpha - h^{\alpha\sigma} A_\sigma$$

$$\Phi = \frac{1}{2} h^{\mu\nu} A_\mu A_\nu - A_\sigma u^\sigma$$

(2.32)

between the two gauges [33]. With $A^2/2$ being the ‘Coriolis’ rotational potential [1], we recognize that the observer $v$ is in 4-rotation with respect to the æther $u$. Conversely, the vector-potential $A$ may be expressed in terms of the observer field $v$ and its relative Newtonian gravitational potential $\Phi$ as [33]

$$A_\mu = - \frac{u}{h_{\mu\alpha}} v^\alpha + \left[ \frac{v}{\Phi} - \frac{1}{2} \frac{u}{h_{\nu\sigma}} v^\nu v^\sigma \right] t_\mu. \quad (2.33)$$

We noted in the previous subsection that given a Galilean structure $(\mathcal{M}, h, \tau)$ and an observer field $u$, there exists an associated unique torsionless connection $\tilde{\Gamma}$ with respect
to which the observer is geodetic and curl-free. It turns out that in the case of Newton-Cartan connection defined by (2.27), the converse is also true: for every Newton-Cartan connection $\Gamma$, at least locally there exists a unit, timelike, nonrotating, and freely-falling observer $u$ such that $\Gamma = \tilde{\Gamma}$. For such an observer, the uniformity property (2.16) immediately leads to

$$u^\sigma R^\alpha_{\sigma \gamma \delta} = -2 \nabla_{[\gamma} \nabla_{\delta]} u^\alpha = 0 \quad (2.34)$$

implying that its parallel-transport around a small closed curve is path-independent. But, of course, for any other timelike observer $\tilde{u}^\sigma = u^\sigma + h^\sigma_\mu w^\mu$, with $w$ being an arbitrary 1-form, we would instead have

$$\tilde{u}^\sigma R^\alpha_{\sigma \gamma \delta} = h^\sigma_\mu w^\mu R^\alpha_{\sigma \gamma \delta} \equiv w^\mu R^\alpha_{\gamma \delta} \neq 0 , \quad (2.35)$$

in general, unless a further restriction, $h^\mu_\sigma R^\alpha_{\sigma \gamma \delta} = 0$, is imposed on the curvature tensor. That is, parallel-transport of an arbitrary observer around a small closed curve is path-dependent in general unless $R^\alpha_{\mu \gamma \delta} = 0$ everywhere. This innocuous looking extra condition turns out to be conceptually quite significant as far as a clearer understanding of the limit-relation between Einstein’s and Newton’s theories of gravity is concerned. We shall devote the entire next subsection to elaborate on its significance.

As in the general theory of relativity, spacetime becomes dynamical under this geometrical reformulation of Newton’s theory: the affine structure of the spacetime — i.e., the connection $\Gamma$ — crucially depends on the distribution of matter $\rho$, and participates in the unfolding of physics rather than being a passive backdrop for the unfolding. This mutability of the Newton-Cartan spacetime is captured in a generalized Newton-Poisson equation which dynamically correlates the curvature of spacetime with the presence of matter:

$$R_{\alpha \beta} = 4\pi G \rho t_{\alpha \beta} =: 4\pi G P_{\alpha \beta} , \quad (2.36)$$

where $R_{(\alpha \beta)} = R_{\alpha \beta} := R^\gamma_{\alpha \beta \gamma}$ is the Ricci tensor corresponding to the connection $\Gamma$, $G$ is the Newtonian gravitational constant, and $P_{\mu \nu} = \rho t_{\mu \nu}$ are the tensor components of the momentum tensor defined on $\mathcal{M}$ (seen to be as such by writing $P^{\alpha \sigma} := \rho u^\alpha u^\sigma$ and then lowering the indices by $t_{\mu \nu}$: $P_{\mu \nu} = t_{\mu} t_{\alpha} t_{\nu} t_{\sigma} P^{\alpha \sigma}$). The constraint (2.36) on the Ricci tensor is a Galilean-relativistic limit form of Einstein’s equation, and, unlike the possible
foliation of the general Galilean spacetime in terms of arbitrary (i.e., possibly curved) spacelike hypersurfaces discussed in the previous subsection, its presence in this specialized Newton-Cartan spacetime necessitates that the hypersurfaces of simultaneity remain copies of ordinary Euclidean three-spaces: $h^\mu\alpha h^{\nu}\beta R_{\alpha\beta} = 0$ because $h^{\mu\nu} t_{\nu} = 0$ \textsuperscript{[13]}. More generally $R^{\mu\nu}$ may be defined as

$$R^{\mu\nu} t_{\mu\alpha} t_{\nu\beta} := R_{\alpha\beta}, \quad (2.37)$$

subject to the consistency condition $h_{\mu\nu} R^{\mu\nu} := R = h^\mu\nu R_{\mu\nu}$.

In contrast to the general theory of relativity, here the mass density $\rho$ is the only source of the gravitational field. However, without loss of consistency, spacelike tensor fields, say $s^\alpha$ and $S^{\alpha\sigma} = S^{(\alpha\sigma)}$, may be added to the matter side of the above field equation as long as the resulting mass-momentum-stress tensor (or matter tensor, for short)

$$M^{(\alpha\sigma)} = M^{\alpha\sigma} := P^{\alpha\sigma} - 2 u^{(\alpha} s^{\sigma)} - S^{\alpha\sigma} \quad (2.38)$$

describing continuous matter distributions satisfies the conservation condition

$$\nabla_\alpha M^{\alpha\sigma} = 0. \quad (2.39)$$

As a matter of fact, expression (2.38) is the only consistent possible nonrelativistic limit-form of the relativistic stress-energy tensor \textsuperscript{[14]}. Note that, unlike the case in general relativity, here this conservation law \textit{must be postulated independently} to allow the derivation of Newtonian equations of motion (2.22) in a manner analogous to the derivation of Einsteinian equations of motion from the general relativistic conservation law. In other words, at least for now, the matter conservation condition (2.39) must be viewed as a separate field equation of the theory. As we shall see, however, it can be \textit{derived} using only the principle of general covariance in a variational approach discussed in the subsection 2.5 below.

As in Einstein’s theory, the field equation (2.36) is generalizable by an additional cosmological term:

$$R_{\mu\nu} + \Lambda t_{\mu\nu} = 4\pi G M_{\mu\nu}, \quad (2.40)$$

where $\Lambda$, the enigmatic cosmological parameter, is a spacetime constant. It is important to note that, analogous to the general relativistic case, this is the \textit{only} admissible generalization of the field equations compatible with the hypotheses that (1) gravitation is
a manifestation of spacetime geometry and (2) Newtonian mechanics is valid in the absence of gravitation \([10]\). Not surprisingly, however, it is possible to relax the restriction of spacetime constancy on \(\Lambda\) if the matter conservation condition (2.39) is concurrently relaxed.

### 2.3. An additional constraint on the curvature tensor

As far as the Newton-Cartan theory is viewed as a limiting form of Einstein’s theory of gravity in which special relativistic effects become negligible, equations (2.21), (2.27), and (2.40) constitute the complete set of gravitational field equations of the theory \([10]\). The logical structure of the covariant Newtonian theory, however, is flexible enough \([10]\) to allow an additional constraint on the curvature tensor, namely

\[
h^{\lambda\sigma} R^{\alpha \sigma \gamma \delta} \equiv R^{\alpha \lambda \gamma \delta} = 0 \tag{2.41}
\]

(or, equivalently, \(t_{[\sigma} R^{\alpha \beta]} \gamma \delta = 0 \tag{41}\)) , implying that parallel-transport of spacelike vectors around a small closed curve is path-independent. Since this restriction, in physical terms, implies that “the rotation axes of freely falling, neighboring gyroscopes do not exhibit relative rotations in the course of time” (no relative rotations for timelike geodesics), in essence it just asserts a standard of rotation: the existence of ‘absolute rotation’ à la Newton (\(‘Gesetz der Existenz absoluter Rotation’\) \([12]\)). In fact, Newton-Cartan theory commonly discussed in the literature with its usual set of field equations (2.21), (2.27), and (2.40) is, strictly speaking, a slight generalization of Newton’s original theory of gravity (although, it constitutes the true Galilean-relativistic limit of Einstein’s theory in general, and not the classical Newtonian theory of gravity \([10,11,12,13]\)) . Put differently, unlike the usual Newton-Cartan field equations the condition (2.41) is not a necessary consequence of the Galilean-relativistic limit (\(‘c \to \infty’\) ) of Einstein’s field equations, but, instead, is an added restriction on the Newton-Cartan structure \([12,13]\) . Indeed, it is not possible to recover the Poisson equation \(\Delta \Phi = 4\pi G \rho\) of Newton’s theory from the usual Newton-Cartan field equations (2.21), (2.27), and (2.36) without any global assumptions unless this extra condition prohibiting any rotational holonomy is imposed on the curvature tensor \([12]\). It becomes redundant, however, if non-intersecting spacelike hypersurfaces covering the Galilean spacetime are required to asymptotically resemble Euclidean
space \[ \mathbb{E} \]. Such a global boundary condition of asymptotically flat spacetime — which idealizes the gravitating systems as isolated systems — is, of course, of great historical and physical importance not only in the case of Newton’s theory, but also in the case of Einstein’s theory. Nevertheless, any such global condition can only encompass special physical situations (‘island universes’), and, in general, the extra condition (2.41) is inevitable to ensure smooth recovery of the Newtonian Poisson equation from the Newton-Cartan field equations. Therefore, in what follows, we shall view equation (2.41) as an extraneously imposed but necessary field equation on the Newton-Cartan structure.

For reasons that will become increasingly clear as we go on, it is convenient to express the above intrinsic condition (2.41) in terms of local variables \( u^\nu \) and \( A_\mu \) in analogy with the equation (2.29) representing the condition (2.27). The desired expression — which must hold at least locally — is

\[
\nabla_{[\gamma} \nabla_\delta] A_\mu - h_{\mu \nu} \nabla_{[\gamma} u_\delta] u^\nu = 0. \tag{2.42}
\]

Contracting with \( h^{\alpha \mu} \), replacing \( -v^\alpha \) for \( h^{\alpha \mu} A_\mu - u^\alpha \), multiplying by \( -\frac{2}{A^2} h^{\lambda \beta} A_\beta \), and using equation (2.34) reveals that this expression is equivalent to the intrinsic form (2.41). It tells us that, as long as the additional condition (2.41) on the curvature tensor is satisfied, path-independence of parallel-transport around a small closed curve holds true not only for the fiducial observer \( u \) (cf. equation (2.34)), but also for any observer \( v \) in 4-rotation with \( u \).

Finally, we end this subsection by observing the logical completeness of the Newton-Cartan structure described in the last two subsections. First of all, straightforward evaluations from equation (2.26) show that the curvature tensor of the connection (2.24) satisfies the relations (2.27) and (2.41). Conversely, and more importantly for our purposes, Trautman [8] has shown that equations (2.27) and (2.41) together imply the existence of a scalar potential \( v \Phi \) and a flat connection \( v \Gamma \) such that the components of the ‘curved’ connection \( \Gamma \) satisfy the relation (2.24) and its curvature tensor satisfies equation (2.26). This necessary and sufficient equivalence of equation (2.26) with the pair (2.27) and (2.41) consistently rounds off the geometrization of Newton’s theory of gravity due to Cartan. For convenience, let us rewrite the complete geometric set of gravitational field-equations of the
Newton-Cartan theory:

\[ h^{\alpha\beta} t_{\beta} = 0, \quad \nabla_\alpha h^{\beta\gamma} = 0, \quad \nabla_\alpha t_\beta = 0, \quad \partial_{[\alpha} t_{\beta]} = 0, \quad (2.43a) \]

\[ R^\alpha_{\beta\gamma\delta} = R^\gamma_{\delta\beta\alpha}, \quad (2.43b) \]

\[ R^{\alpha\lambda}_{\gamma\delta} = 0, \quad (2.43c) \]

and \[ R_{\mu\nu} + \Lambda t_{\mu\nu} = 4\pi G M_{\mu\nu}, \quad (2.43d) \]

where the first four equations specify the degenerate ‘metric’ structure and a set of torsion-free connections on \( \mathcal{M} \), the fifth one picks out the Newton-Cartan connection from this set of generic possibilities, the sixth one postulates the existence of absolute rotation, and the last one relates spacetime geometry to matter in analogy with Einstein’s field equation.

2.4. Gauge and Lie-algebraic structures of Newton-Cartan spacetime

2.4.1. The gauge structure

Since the ‘æther-frame’ with its four-velocity \( u \) has been adapted as an auxiliary structure into the description of Newton-Cartan framework delineated in the previous subsections, any physical theory constructed in accordance with this framework must be invariant under changes in \( u \) — and, hence, under associated changes in the ‘vector-potential’ \( A \) — if the theory is to maintain general covariance. Using a particular pair of potentials \( (u, A) \) rather than a given Newton-Cartan connection \( \Gamma \) amounts to choosing a ‘Bargmann gauge’ in the fiber-bundle formulation of the Newtonian gravity studied by Duval and Künzle [34]. They take Bargmann bundle \( B(\mathcal{M}) \) to be a principal bundle over the Galilean manifold \( \mathcal{M} \) with the Bargmann group \( B \) (a non-trivial central extension of the inhomogeneous Galilean group \( \mathcal{G} \) by an abelian one-dimensional phase group \( U(1) \) appearing in the exact sequence \( 1 \to U(1) \to B \to \mathcal{G} \to 1 \) as its structure group. It is a \( U(1) \)-extension of a sub-bundle \( \mathcal{G}(\mathcal{M}) \) of the bundle of Galilean-relativistic affine frames of \( \mathcal{M} \), with a surjective principal bundle homomorphism \( B(\mathcal{M}) \to \mathcal{G}(\mathcal{M}) \), constructed as follows. Let \( \text{Gl}(\mathcal{M}) \) be the principal bundle of linear frames over \( \mathcal{M} \). A Galilean structure is then a reduction of \( \text{Gl}(\mathcal{M}) \) to the homogeneous subgroup \( \mathcal{G}_0 := SO(3) \ltimes \mathbb{R}^3 \) of the Galilean group (where \( \ltimes \) denotes a semidirect product), and the sub-bundle \( \mathcal{G}(\mathcal{M}) \) of the bundle of Galilean-relativistic affine frames of \( \mathcal{M} \) is the pull-back of \( \mathcal{G}_0(\mathcal{M}) \) by the canonical
projection $T\mathcal{M} \to \mathcal{M}$, where $T\mathcal{M}$ is the tangent bundle over $\mathcal{M}$. If $i_o : \mathcal{G}_0(\mathcal{M}) \hookrightarrow \mathcal{G}(\mathcal{M})$ denotes an embedding through the zero section of $T\mathcal{M}$, and $B(\mathcal{M})$ is a $U(1)$-extension of $\mathcal{G}(\mathcal{M})$ with structure group $B$, then let $B_0(\mathcal{M})$ be the pull-back of $B(\mathcal{M})$ by $i_o$. The quotient bundle $P(\mathcal{M}) := B_0(\mathcal{M})/\mathcal{G}_0$ is then a $U(1)$-principal bundle over $\mathcal{M}$. Now, it can be shown [34] that any compatible Newton-Cartan connection defined by equation (2.27) above designates an entire class of connections on $P(\mathcal{M})$ which are in one-to-one correspondence with the sections $u$ of the unit tangent bundle $T_1\mathcal{M} = \mathcal{G}_0(\mathcal{M})/SO(3)$ of the structure $(\mathcal{M}; \tau, h)$. Clearly, the sections $u$ are nothing but the four-velocity observer fields considered above with their gauge-dependent spatial projection fields $h_{\mu\nu}$. The gauge theory of Newtonian gravity we are considering must, therefore, be covariant under changes of the sections $u$, over and above the covariance under the automorphisms of $P(\mathcal{M})$. Duval and Künzle show that the Newton-Cartan connection defined by the constraint (2.27) on the Galilean structure (2.21) is indeed invariant under the simultaneous changes

$$\chi \mapsto \tilde{\chi} = \chi + f$$

$$u^\alpha \mapsto \tilde{u}^\alpha = u^\alpha + h^{\alpha\sigma}w_\sigma$$

$$A_\alpha \mapsto \tilde{A}_\alpha = A_\alpha + \partial_\alpha f + w_\alpha - (u^\sigma w_\sigma + \frac{1}{2}h^{\mu\nu}w_\mu w_\nu) t_\alpha,$$ (2.44c)

where $\chi$ is the $U(1)$-phase (treated as the only intrinsic group coordinate), $f \in C^\infty(\mathcal{M}, \mathbb{R})$ is an arbitrary smooth map $\mathcal{M} \to \mathbb{R}$, $w$ is a Galilean boost 1-form (defined only modulo $t_\alpha$) belonging to the space of covector-fields $\Omega^1(\mathcal{M})$ on $\mathcal{M}$, and $A_\alpha$ is the ‘vector-potential’ defined above. The transformations (2.44a) and (2.44b) can be regarded as the ‘vertical automorphisms’ of the unit tangent bundle $B_0(\mathcal{M})/SO(3)$, and along with the diffeomorphisms $\phi \in \text{Diff}(\mathcal{M})$ of $\mathcal{M}$ they compose the complete automorphism group

$$\mathcal{A}ut(B(\mathcal{M})) := \{ (\phi, w, f) \mid \phi \in \text{Diff}(\mathcal{M}), w \in \Omega^1(\mathcal{M}), f \in C^\infty(\mathcal{M}, \mathbb{R}) \}$$ (2.45)

of the Newton-Cartan theory of gravity. Conversely, the projections of the fiber-preserving elements of the group $\mathcal{A}ut(B(\mathcal{M}))$ by the bundle projection map $\pi : B(\mathcal{M}) \to \mathcal{M}$ are the smooth coordinate transformations of $\mathcal{M}$ into itself which constitute the diffeomorphism group $\text{Diff}(\mathcal{M})$; and, since the projection map $\pi$ is a group homomorphism, the kernel of this map is the group

$$\mathcal{V}(B(\mathcal{M})) := \left[ \Omega^1(\mathcal{M}) \times C^\infty(\mathcal{M}, \mathbb{R}) \right]$$ (2.46)
of vertical gauge transformations defined by equation (2.44). Thus, the group of bundle automorphism (2.45) encapsulates two classes of invariance which must be respected by the action $\mathcal{I}$ of any gauge-invariant field theory compatible with the Newton-Cartan structure $(\mathcal{M}; \tau, h, \Gamma)$. Mathematically, the elements of one of these two classes correspond to transforming the fibers of the bundle space $B(\mathcal{M})$ by the action of the normal (or invariant) subgroup $\mathcal{V}(B(\mathcal{M}))$ of the complete automorphism group $\text{Aut}(B(\mathcal{M}))$, whereas the elements of the other class correspond to transforming the base space $\mathcal{M}$ of the bundle by the action of the factor subgroup $\text{Diff}(\mathcal{M}) = \text{Aut}(B(\mathcal{M}))/\mathcal{V}(B(\mathcal{M}))$. This means that the automorphism group $\text{Aut}(B(\mathcal{M}))$ has the structure of a semidirect product $\mathcal{V}(B(\mathcal{M})) \rtimes \text{Diff}(\mathcal{M})$ indicating its status in the exact sequence

$$1 \rightarrow \mathcal{V}(B(\mathcal{M})) \rightarrow \text{Aut}(B(\mathcal{M})) \rightarrow \text{Diff}(\mathcal{M}) \rightarrow 1,$$

(2.47)

and it acts on the set $\{(h, \tau, u, A)\}$ by

$$(\phi, w, f): \begin{pmatrix} h \\ \tau \\ u \\ A \end{pmatrix} \mapsto \phi_* \begin{pmatrix} h \\ \tau \\ u + h(w) \\ A + df + w - \{w(u) + \frac{1}{2} h(w,w)\} \tau \end{pmatrix},$$

(2.48)

giving the semidirect product group multiplication law:

$$(\phi_1, w_1, f_1)(\phi_2, w_2, f_2) = (\phi_1\phi_2, \phi_2^* w_1 + w_2, \phi_2^* f_1 + f_2),$$

(2.49)

where $\phi_*$ is the push-forward map and $\phi^*$ is the pull-back map corresponding to the diffeomorphism $\phi \in \text{Diff}(\mathcal{M})$. More significantly, this complete gauge group $\text{Aut}(B(\mathcal{M}))$ acts on the Newton-Cartan structure $(\mathcal{M}, h, \tau, \Gamma)$ only via the quotient subgroup $\text{Diff}(\mathcal{M})$,

$$(\phi, w, f): \begin{pmatrix} h \\ \tau \\ \Gamma \end{pmatrix} \mapsto \phi_* \begin{pmatrix} h \\ \tau \\ \Gamma \end{pmatrix},$$

(2.50)

exhibiting that the principle of general covariance has been quite consistently adapted from Einstein’s theory of gravity to this Newton-Cartan-Bargmann framework (cf. footnote 2).
2.4.2. The Lie-algebraic structure

The degenerate ‘metric’ structure of general Galilean spacetime permits some plasticity in the Lie-algebraic structure $T_e \text{Aut}(B(M))$ of the automorphism group $\text{Aut}(B(M))$ (here $e$ is the unit element of the group). At least six different nested Lie algebras of infinitesimal ‘isometries’ (which play a role analogous to that of the algebra of Killing vector fields of Lorentzian spacetime) have been shown [33] to naturally arise as possible candidates for Newton-Cartan symmetry algebras. These nested Lie algebras include two extreme cases: (1) the Coriolis algebra — i.e., the Lie algebra of the infinite-dimensional Leibniz group [35] (the symmetry group of most general ‘metric’ automorphisms of the Galilean-relativistic spacetime) — and (2) the all important Bargmann algebra — i.e., the Lie algebra of the Bargmann group (the fundamental symmetry group of massive, non-interacting Galilean-relativistic systems — classical or quantal).

To see how these two Lie-algebraic structures arise as extreme cases, recall that Newton-Cartan connection $\Gamma$ can be expressed in terms of the vector fields $u$ and $v$, and the scalar potential $\Phi$ (cf. equations (2.24), (2.7) and [2.32]). In terms of these variables the action (2.48) of the group $\text{Aut}(B(M))$ on the full Newton-Cartan-Bargmann structure $(\mathcal{M}; h, \tau, u, v, \Phi)$ can be expressed as

$$\begin{pmatrix} h \\ \tau \\ u \\ v \\ \Phi \end{pmatrix} \mapsto \phi_* \begin{pmatrix} h \\ \tau \\ u + h(w) \\ v + h(df) \\ \Phi + v(df) + \frac{1}{2} h(df, df) \end{pmatrix}, \quad (2.51)$$
whereas the corresponding infinitesimal action of this automorphism group on the structure 
\((\mathcal{M}; h, \tau, u, v, \Phi)\) can be seen as \([33]\)

\[
\delta \begin{pmatrix} h \\ \tau \\ u \\ v \\ \Phi \end{pmatrix} = \begin{pmatrix} \mathcal{L}_x h \\ \mathcal{L}_x \tau \\ \mathcal{L}_x u + h(\theta) \\ \mathcal{L}_x v + h(dg) \\ x(v(\Phi)) + v(g) \end{pmatrix},
\]

where \(\theta \in \Omega^1(\mathcal{M})\), \(g \in C^\infty(\mathcal{M})\), and \(\mathcal{L}_x\) denotes a Lie derivative with respect to the symmetry generating vector-fields \(x = x^a \partial_a\) on \(\mathcal{M}\). The associated Lie-algebraic structure of this infinitesimal action works out to be

\[
[(x, \theta, g), (x', \theta', g')] = ([x, x'], \mathcal{L}_x \theta' - \mathcal{L}_{x'} \theta, x(g') - x'(g)).
\]

If we now set \(0 = \delta h = \delta \tau = \delta u = \delta v = \delta \Phi\) and, thereby, look for the stabilizer of the Newton-Cartan-Bargmann structure for the case of flat spacetime, we recover the Lie algebra of the Bargmann group \([33]\). Thus, the isotropy subgroup (or the stabilizer) of the full automorphism group \(\text{Aut}(B(\mathcal{M}))\) corresponding to the immutable flat structure \((h = \delta^{ab} \partial_a \otimes \partial_b, \tau = dt, u = \partial_0, A = 0)\) is nothing but the Bargmann group, as one would expect. By relaxing one or more of the restrictions \(0 = \delta u = \delta v = \delta \Phi\), and/or imposing various different restrictions on the connection \(\Gamma\), a variety of intermediate algebraic structures associated with some physically interesting special cases may be worked out \([33]\). All of these intermediate symmetry groups are necessarily wider than the stabilizing Bargmann group. They form different subgroups of the Leibniz group — the most general infinite-dimensional ‘isometry’ group of the Newton-Cartan structure. The generators of Leibniz group are restricted only by the conditions \(\mathcal{L}_x h = 0\) and \(\mathcal{L}_x \tau = 0\), and, in general, do not Lie-transport the connection: \(\mathcal{L}_x \Gamma \neq 0\). Consequently, the infinite-dimensional Lie algebra — the Coriolis algebra — corresponding to the Leibniz group preserves the absolute structure \((\mathcal{M}; h, \tau)\) of the Newton-Cartan spacetime, but leaves the connection \(\Gamma\) completely malleable, or dynamical. The elements of this most general symmetry group of the
absolute structure represented in an arbitrary rigid frame take the form

\[ x^0 = x^0 + c^0 \equiv t + c^0, \]

\[ x^a = O^a_b(t) \ x^b + c^a(t) , \quad (a,b = 1,2,3), \]

(2.54)

where \( O^a_b(t) \in SO(3) \) form an orthogonal rotation matrix for each value of \( t \), \( c^a(t) \in \mathbb{R}^3 \) are arbitrary functions of \( t \in \mathbb{R} \), and \( c^0 \in \mathbb{R} \) is an infinitesimal time translation. Physically, this infinite-dimensional symmetry group correspond to transformations that connect different Galilean observers in arbitrary (accelerating and rotating) relative motion.

One physically and historically important subgroup of Leibniz group \[36\], which results as a direct consequence of the additional constraint (2.41) on the curvature tensor. In the above notation, it simply eliminates the time-dependence of the rotation matrix \( O^a_b \) in equation (2.54), and may also be characterized by the restriction

\[ \mathcal{L}_x \left( h^{\nu\sigma} \Gamma^\alpha_{\mu\sigma} \right) \equiv \mathcal{L}_x \Gamma^\alpha_{\mu\nu} = 0 \]

(2.55)

imposed directly on the connection \[33\] in addition to the conditions \( \mathcal{L}_x h = 0 \) and \( \mathcal{L}_x \tau = 0 \) on the metrics. The vector-fields \( x \) then constitute a Lie algebra corresponding to the infinitesimal Milne transformations

\[ x^0 = t + c^0, \]

\[ x^a = O^a_b \ x^b + c^a(t) , \quad (a,b = 1,2,3), \]

(2.56)

discussed in \[Ref. 29\].

2.5. Derivation of the matter conservation laws

As is well-known \[37,28\], in general relativity the principle of general covariance is sufficient for a derivation of the local differential conservation law \( \nabla_\mu T^{\mu\nu} = 0 \) for relativistic matter and non-gravitational fields from variation of an action functional with respect to the Lorentzian metric \( g_{\mu\nu} \). The principle of general covariance in variational formulation of that theory is encapsulated in the mathematical requirement of invariance of the gravitational plus matter action \( \mathcal{I}[g_{\mu\nu}, \Psi] := \mathcal{I}_g[g_{\mu\nu}] + \mathcal{I}_m[g_{\mu\nu}, \Psi] \) under the diffeomorphisms of the Lorentzian spacetime \( (\mathcal{M}; g_{\mu\nu}) \); i.e., in the requirement that if the map \( (s)\phi : \mathcal{M} \to \mathcal{M} \)
defines a one-parameter family of diffeomorphisms, then \( I[(s)\phi_* g_{\mu\nu}, (s)\phi_* \Psi] = I[g_{\mu\nu}, \Psi] \), where \((s)\phi_*\) is the push-forward map corresponding to \((s)\phi \in \text{Diff}(M)\), and \(\Psi\) represents the matter fields. Moreover, general covariance demands that the matter action \(I_m\) by itself must be invariant under these diffeomorphisms if it were to retain an unequivocal physical meaning. In other words, for such diffeomorphic variations we must have

\[
0 = \frac{d}{ds} I_m[g_{\mu\nu}, \Psi] \equiv \delta I_m[g_{\mu\nu}, \Psi] = \int \frac{\delta I_m}{\delta g_{\mu\nu}} \delta g_{\mu\nu} + \int \frac{\delta I_m}{\delta \Psi} \delta \Psi.
\]  

(2.57)

If we now define

\[
\frac{1}{2} T^{(\mu\nu)} = \frac{1}{2} T^{\mu\nu} := \frac{1}{\sqrt{-g}} \frac{\delta I_m}{\delta g_{\mu\nu}},
\]

(2.58)

and assume that the matter fields satisfy the Euler-Lagrange equations \(\frac{\delta I_m}{\delta \Psi} = 0\), then equation (2.57) amounts to

\[
0 = \delta I_m = \int_{\mathcal{O}} \frac{1}{2} T^{\mu\nu} \delta g_{\mu\nu} \, dv.
\]

(2.59)

for some compact region \(\mathcal{O} \subset M\) with a non-null boundary \(\partial \mathcal{O}\), where \(dv\) is the Lorentzian 4-volume element on \(M\). Using the well-known relation \(\delta g_{\mu\nu} = \mathcal{L}_X g_{\mu\nu} = 2 \nabla^{(\mu} X_{\nu)}\) for such variations (where \(\mathcal{L}_X\) denotes Lie derivative with respect to an arbitrarily chosen smooth vector field \(X_\mu\) on \(M\)) the above functional equation can be rewritten as

\[
0 = \int_{\mathcal{O}} T^{\mu\nu} \nabla_\mu X_\nu \, dv = \int_{\mathcal{O}} \nabla_\mu (T^{\mu\nu} X_\nu) \, dv - \int_{\mathcal{O}} (\nabla_\mu T^{\mu\nu}) X_\nu \, dv = -\int_{\mathcal{O}} (\nabla_\mu T^{\mu\nu}) X_\nu \, dv. \quad (2.60)
\]

Here the last equality is obtained by converting the volume integral \(\int_{\mathcal{O}} \nabla_\mu (T^{\mu\nu} X_\nu) \, dv\) to a surface integral, which vanishes because \(X_\mu\) vanishes on the boundary of the volume by assumption. Finally, since the region \(\mathcal{O}\) and the vector field \(X_\mu\) are arbitrarily chosen, the desired conservation law for the stress-energy of matter and non-gravitational field, \(\nabla_\mu T^{\mu\nu} = 0\), immediately follows from this equation [37,28].

Thus, in Lorentzian spacetime the stress-energy tensor \(T^{\mu\nu}\) is seen to be dual to the metric \(g_{\mu\nu}\) in the sense that variations of the matter action \(I_m\) with respect to the semi-Riemannian metric lead to the conserved matter tensor \(T^{\mu\nu}\). This state of affairs naturally suggests that in our Galilean-relativistic spacetime there should be a pair of quantities \((S_{\mu\nu}, C^\mu)\) dual to the pair of metrics \((h^{\mu\nu}, t_\mu)\) playing a role analogous to that of the relativistic stress-energy tensor \(T^{\mu\nu}\). However, as shown by Künzle and Duval [11,30], if
we consider an action functional analogous to (2.59) defined on the Galilean spacetime $\mathcal{M}$, say

$$\delta \mathcal{I}_m = \int_{\mathcal{O}} \left\{ \frac{1}{2} S_{\mu\nu} \delta h^{\mu\nu} + C^\mu \delta t_{\mu} \right\} \phi \, d^4x,$$  \hfill (2.61)

with $\phi \, d^4x$ being the Galilean 4-volume element (cf. equation (2.3)), then the equations following from it do not correctly correspond to the well-known balance equations of energy and momentum (they lack an essential acceleration term of the classical equations.) The culprit, of course, is the degenerate or non-semi-Riemannian structure of the Galilean spacetime; as we have often noted, the degenerate pair of metrics $(h^{\mu\nu}, t^\mu)$ does not determine the connection completely. Fortunately, the difficulty also suggests its resolution. As we now know, introduction of the pair $(u^\alpha, A_\alpha)$ of gauge fields as an auxiliary structure in the Galilean spacetime does fix the connection uniquely, and, hence, all we need to do is to enlarge the set of variables in the argument of the above action functional by these two gauge variables. But then the principle of general covariance demands that the variation of the resulting action, $\mathcal{I}_m[h^{\mu\nu}, t^\nu, A_\mu, u^\nu, \Psi]$, must be invariant not just under the diffeomorphisms of $\mathcal{M}$, but also under the vertical transformations (2.44) of these gauge variables; i.e., the variations of the matter action must be invariant under the entire gauge group $\text{Aut}(B(\mathcal{M}))$ of the Newton-Cartan theory discussed in the previous subsection. Accordingly, following Duval and Künzle [30, 34], we augment (2.61) to be the functional

$$\delta \mathcal{I}_m = \int_{\mathcal{O}} \left\{ \frac{1}{2} S_{\mu\nu} \delta h^{\mu\nu} + C^\mu \delta t_{\mu} + J^\mu \delta A_\mu + K_\mu \delta u^\mu \right\} \phi \, d^4x$$ \hfill (2.62)

of four different variations, $\delta h^{\mu\nu}$, $\delta t_{\mu}$, $\delta A_\mu$, and $\delta u^\mu$, and require it to be invariant under the full gauge group $\text{Aut}(B(\mathcal{M}))$. Then, not only the equations of motion for matter fields derived from the variations of this action are gauge-invariant under $\text{Aut}(B(\mathcal{M}))$ (see the following section for an example), but the associated matter-current density $J^\mu$ and ‘Hilbert’ stress-energy tensor $^4$

$$N^\mu_\nu := h^{\mu\sigma} S_{\sigma\nu} - C^\mu t_{\nu} + J^\mu (v_{\nu} - \frac{1}{2} v^2 t_{\nu}) + u^\mu K_\nu,$$ \hfill (2.63)

---

4 Following Duval and Künzle we call this a ‘Hilbert’ stress-energy tensor because it corresponds to the general relativistic matter tensor (2.58) obtained by variations with respect to the Lorentzian metric. The canonical stress-energy tensor implied by Noether’s theorem is different and not gauge-invariant.
are also invariant under the action of that group, where \( \rho := J^\sigma t_\sigma, J^\sigma := \rho v^\sigma, v_\sigma := \dot{h}_\sigma^\alpha v^\alpha, \) and \( v^2 := v^\sigma v_\sigma. \) What is more, the matter-current density and the ‘Hilbert’ stress-energy tensor satisfy the balance equations

\[
\nabla_\mu J^\mu = 0 \quad (2.64a)
\]

and

\[
\nabla_\mu N^\mu_\nu = \rho \dot{h}_\nu^\sigma v^\sigma \nabla_\alpha v^\alpha \quad (2.64b)
\]

(as meticulously shown by Duval and Künzle \([30]\)) , which now correctly reduce to the classical expressions on flat spacetime \([11]\). Thus, the relativistic stress-energy tensor decouples into the pair \((N^\mu_\nu, J^\mu)\) in this nonrelativistic theory, describing the stress and energy flow by the tensor \(N^\mu_\nu\) and the matter flow by the vector \(J^\mu\). This distinct role of the concept of mass-current from that of the stress-energy flow in the Newton-Cartan theory is closely related to the well-known fact that mass plays distinctly different roles in the Galilean and Poincaré invariant mechanics.

If we now define a mass-momentum-stress tensor (or matter tensor, for short) by

\[
M^{\mu\nu} := -\dot{h}^{\nu\sigma} N^\mu_\sigma + J^\mu v^\nu, \quad (2.65)
\]

then the balance equation (2.64b) immediately yields the nonrelativistic matter conservation law

\[
\nabla_\mu M^{\mu\nu} = 0 \quad (2.66)
\]

analogous to the relativistic conservation law \(\nabla_\mu T^{\mu\nu} = 0\). As a matter of fact, (2.66) is precisely the Galilean-relativistic limit-form of the relativistic conservation law, as shown by Künzle \([11]\). Substituting the expression (2.63) into the definition (2.65) of the matter tensor, and using the relation \(K_\nu = -\dot{u}^\nu J^\sigma + (constant) t_\nu\) (which has been derived by Duval and Künzle in Ref. 30), we obtain the tensor \(M^{\mu\nu}\) in a more transparent form:

\[
M^{\mu\nu} = \rho u^\mu u^\nu - 2\rho u^{(\mu} h^{\nu)}_\sigma A_\sigma - \dot{h}^{\mu\sigma} h^{\nu\alpha} S_{\sigma\alpha}, \quad (2.67)
\]

which is identical to the mass-momentum-stress tensor (2.38) if we identify \(s^\nu = \rho h^{\nu\sigma} A_\sigma\). Consequently, the matter conservation law (2.66) is identical to the condition (2.39), which, at that stage, we had to impose independently. Amiably enough, in the present variational
approach it is a derived result, following directly from the principle of general covariance, provided, of course, the matter field equations $\frac{\delta I}{\delta \Psi} = 0$ are satisfied.

3. Quantum field theory on the curved Newton-Cartan spacetime

3.1. One-particle Schrödinger theory

In the previous section we reviewed the classical Newton-Cartan theory in some detail. The first systematic study of the quantum theory of freely falling particles in (unquantized) Newton-Cartan spacetime with illustrations of how the principle of equivalence works for such quantum systems has been carried out by Kuchař [1]. He arrives at a generally-covariant version of the Schrödinger’s equation for a free quantum particle in an arbitrary, noninertial frame in such a classical spacetime which, in the special case of the Galilean frame with its distinguished gravitational potential, reduces to the usual Galilean-relativistic Schrödinger equation. Following the ‘parameterized’ canonical formalism [38], Kuchař starts with an action integral, giving Newton-Cartan geodesics as the extremal paths, as it appears to an arbitrary observer in an arbitrary gauge. After ‘deparameterizing’ it by labelling the worldlines with the Newtonian absolute time $t$, he casts the action into the generalized Hamiltonian form to prepare for the Dirac’s constraint quantization. The transformation to the quantum theory is then easily achieved in an unambiguous, coordinate independent fashion by quantizing the motion of the particle with the use of the Dirac method. The resulting quantum mechanical equation of motion, or the quantum mechanical analog of the geodesic equation, turns out to be nothing but a covariant version of the ordinary Schrödinger equation for the free particle, with gravitational forces absorbed in the structure of spacetime in which the particle is freely falling. Thus, in a nutshell, Kuchař has successfully shown that quantum mechanics remains consistent in the presence of the Newtonian connection-field viewed as an effect of the curving of spacetime, and, due to the unique foliation possessed by the Newton-Cartan spacetime, Galilean-relativistic quantum theory escapes ‘the problem of time’ [39] usually encountered in the attempts to canonically quantize parameterized dynamical systems in the presence of Einsteinian connection-field. In other words, Kuchař has shown that it is possible to unequivocally
quantize Galilean-relativistic parameterized systems by means of the Dirac method such that the evolution of their quantum state $\Psi : \mathcal{M} \to \mathbf{C}$, unlike their general-relativistic counterpart, does not depend on the choice of spacetime foliation (which, of course, is uniquely given in the Newtonian case). Consequently, the corresponding Hilbert-space inner-product $\langle \Psi_1 | \Psi_2 \rangle_t$ remains the same for all such domains of simultaneity. It is worth noting here that in a subsequent work De Bièvre [40] has obtained essentially the same covariant Schrödinger-Kuchař equation (in the framework of the Bargmann bundle discussed in the previous section) directly from an application of the principle of equivalence rather than a posteriori exhibiting the compatibility of this principle with the correct quantum dynamics of a free test particle in a gravitational field à la Kuchař. This further justifies the validity of the principle of equivalence in the quantum domain, and, to put more strongly, logically demands the generally-covariant reformulation of the Galilean-relativistic quantum dynamics.

For our purposes in this paper, however, it is convenient to follow the covariant framework of Duval and Künzle [34], who show that a prescription of minimal (Newtonian) gravitational coupling naturally leads to the four-dimensional spacetime-covariant Schrödinger-Kuchař equation as a result of extremization of an action of the form (2.62) discussed in the previous section. Recall the U(1)-principle bundle $\mathcal{M}'$ from the subsection 2.4, and consider a vector bundle $E$ with the Hilbert space $L^2(\mathbb{R}^3)$ of square integrable functions on $\mathbb{R}^3$ as the standard fiber associated with it. Under the gauge transformation (2.48) a section $\Psi$ of $E$, or a wave-function, changes as

$$\phi, w, f) : \Psi \mapsto \phi_{\ast} \left[ \exp \left( i \frac{m}{\hbar} f \right) \Psi \right] ; \quad (3.1)$$

and, in analogy with the electromagnetic gauge theory, the covariant derivative induced by the connection on $\mathcal{M}'$ acting on sections of $E$ is

$$D_\alpha \Psi := \left( \partial_\alpha - i \frac{m}{\hbar} A_\alpha \right) \Psi . \quad (3.2)$$

Now, a Lagrangian density for the free one-particle Schrödinger equation

$$\frac{\hbar^2}{2m} \delta^{ab} \partial_a \partial_b \Psi + i \hbar \partial_0 \Psi = 0 \quad (a, b = 1, 2, 3) \quad (3.3)$$
may be taken to be
\[ \mathcal{L}_{\text{Schr}} = \frac{\hbar^2}{2m} \delta^{ab} \partial_a \Psi \partial_b \overline{\Psi} + i \frac{\hbar}{2} \left( \Psi \partial_0 \overline{\Psi} - \overline{\Psi} \partial_0 \Psi \right) , \]  
which on the curved Newton-Cartan spacetime becomes
\[ \mathcal{L}_{\text{Schr}} \rightarrow \mathcal{L}_{\text{Kuch}} = \varphi 4\pi G \left\{ \frac{\hbar^2}{2m} h^{\alpha \beta} D_\alpha \Psi D_\beta \overline{\Psi} + i \frac{\hbar}{2} u^\alpha \left( \Psi D_\alpha \overline{\Psi} - \overline{\Psi} D_\alpha \Psi \right) \right\} \]  
if we use the contravariant 3-metric \( h^{\alpha \beta} \) for the Laplacian in place of the Kronecker delta \( \delta^{ab} \), write \( \varphi d^4 x \) (cf. equation (2.3)) for the spacetime volume element on \( \mathcal{M} \), and replace \( \partial_0 \Psi \) by \( u^\alpha D_\alpha \Psi \) (the propagation covariant derivative along the timelike vector field \( u^\alpha \)) and \( \partial_\alpha \Psi \) by \( D_\alpha \Psi \) in accordance with the minimal interaction principle (the meaning of the multiplicative factor \( 4\pi G \) will become clear in the following section). This latter Lagrangian density is manifestly covariant with respect to the diffeomorphisms \( \phi \in \text{Diff}(\mathcal{M}) \) of \( \mathcal{M} \) since \( h^{\alpha \beta} \), \( t_\alpha \), \( A_\alpha \), \( u^\alpha \), \( \Psi \), and \( \overline{\Psi} \) are all tensor fields on the spacetime. What is more, it is also invariant under the simultaneous vertical gauge transformations (2.44) of \( A_\alpha \), \( u^\alpha \), \( \Psi \), and \( \overline{\Psi} \). Consequently, we expect the resulting Schrödinger-Kuchař theory to be independent of the choice of a Galilean observer represented by \( u \) with the corresponding gravitational interaction being uniquely described by the Newton-Cartan structure \( (\mathcal{M}; h, \tau, \nabla) \) alone.

The Euler-Lagrange equations corresponding to the Lagrangian density (3.5) obtained by independently extremizing the corresponding action functional \( I_{\text{Kuch}} \) with respect to \( \overline{\Psi} \) and \( \Psi \) are
\[ \mathcal{E}_{\text{Kuch}} [\Psi] = \left[ \frac{\hbar^2}{2m} D^\alpha D_\alpha + i\hbar u^\alpha D_\alpha + i \frac{\hbar}{2} \nabla_\alpha u^\alpha \right] \Psi = 0 \]  
and its complex conjugate, respectively; and once the gauge-covariant derivatives \( D_\alpha \) in equation (3.6) are worked out, we indeed obtain the desired covariant Schrödinger-Kuchař equation on the curved Newton-Cartan spacetime:
\[ \left[ \frac{\hbar^2}{2m} \nabla^\alpha \partial_\alpha + i\hbar \left( u^\alpha - h^{\alpha \beta} A_\beta \right) \partial_\alpha + m \left( u^\alpha A_\alpha - \frac{1}{2} h^{\alpha \beta} A_\alpha A_\beta \right) + i \frac{\hbar}{2} \nabla_\alpha \left( u^\alpha - h^{\alpha \beta} A_\beta \right) \right] \Psi = 0. \]  
Among many interesting results in this theory [34,1], the two we will need later are the expressions for the conserved matter-current density (cf. equation (2.64a))
\[ J^\alpha := \frac{1}{\varphi 4\pi G} \frac{\delta I_{\text{Kuch}}}{\delta A_\alpha} = m \overline{\Psi} \Psi u^\alpha + i \frac{\hbar}{2} h^{\alpha \beta} \left( \Psi D_\beta \overline{\Psi} - \overline{\Psi} D_\beta \Psi \right) , \quad \nabla_\alpha J^\alpha = 0 , \]
where $\rho \equiv m\overline{\Psi}\Psi$ is the invariant mass density, and, more significantly, the conserved matter tensor (cf. equation (2.67))

$$M^{\mu\nu} = m\overline{\Psi}\Psi u^\mu u^\nu - 2m\overline{\Psi}\Psi u^{(\mu}h^{\nu)}A_\sigma - S^{\mu\nu}, \quad \nabla_\mu M^{\mu\nu} = 0,$$

with

$$S^{\mu\nu} = \left\{ \frac{1}{2}h^{\sigma\alpha}\Omega_{\alpha\sigma} + u^\sigma\Omega_\sigma - m\overline{\Psi}\Psi u^\sigma A_\sigma \right\}h^{\mu\nu} - 2u^{(\mu}h^{\nu)}\sigma\Omega_{\sigma} - h^{\mu\sigma}h^{\nu\alpha}\Omega_{\sigma\alpha},$$

$$\Omega_{\mu\nu} :\equiv \frac{h^2}{m}D_{(\mu}\Psi D_{\nu)}\Psi,$$ and $\Omega_\mu := \frac{ih}{2}(\Psi\partial_\mu\overline{\Psi} - \overline{\Psi}\partial_\mu\Psi)$. As the above covariant Schrödinger-Kuchař equation, both of these conserved flows appear here as quantities derived from the action functional, and their respective expressions are invariant under the combined diffeomorphisms and gauge transformations (2.48) as expected.

### 3.2. Galilean-relativistic quantum field theory

In the previous subsection we have reviewed one-particle Schrödinger mechanics on the curved Newton-Cartan spacetime. It is well-known [41,42,43], however, that such a mechanics on the usual flat Galilean spacetime can be interpreted also as a second-quantized field theory in which the wave-function $\Psi$ becomes an operator-valued distribution in the Fock space of unspecified number of identical particles. Thanks to the above described minimal coupling formulation of the Schrödinger-Kuchař theory, it turns out that this Fock-space representation of many-particle system can be quite straightforwardly generalized to our case of curved background. Given the Lagrangian density (3.5), the first step towards constructing the alleged generalization is to evaluate the momenta conjugate to the classical c-number fields $\Psi$ and $\overline{\Psi}$:

$$\overline{P} := \frac{\delta\mathcal{L}_{\text{Kuch}}}{\delta (u^\sigma D_\sigma \Psi)} = -2i\hbar \phi G \overline{\Psi},$$

and its complex conjugate. This relation suggests that we should not regard $\overline{\Psi}$ as an independent variable, but rather as proportional to the canonical conjugate of $\Psi$. Therefore, we introduce a new, more appropriate set of canonical variables $\{\psi, p\}$,

$$\psi := \frac{1}{\hbar \sqrt{8\pi G \phi}} \left( 2\pi G \phi \hbar \overline{\Psi} + i\overline{P} \right),$$

$$p := \frac{1}{\sqrt{8\pi G \phi}} \left( P + i2\pi G \phi \hbar \Psi \right),$$
so that equation (3.11) and its conjugate yield

\[ \psi = \sqrt{2\pi G} \varphi \quad \text{and} \quad p = i\hbar \overline{\psi}. \]  

(3.13)

Given a spacelike hypersurface \( \Sigma_t \) (cf. subsection 4.2 below), the quantum theory can now be easily defined by the usual equal-time commutation (−) or anticommutation (+) relations,

\[ [\hat{\psi}(x), \hat{\psi}(x')]_{\pm} = 0, \quad [\hat{\psi}^\dagger(x), \hat{\psi}^\dagger(x')]_{\pm} = 0, \]

and

\[ [\hat{\psi}(x), \hat{\psi}^\dagger(x')]_{\mp} = \hat{1} \delta(\vec{x} - \vec{x}'), \]

(3.14)

corresponding to the bosonic (−) or fermionic (+) algebra, where \( \vec{x} \in \Sigma_t \), \( \hat{1} \) is the identity operator in the Fock space, and, as always, we have replaced the complex conjugate function \( \overline{\psi} \) with the Hermitian conjugate operator \( \hat{\psi}^\dagger \). It is worth recalling the well-known fact [12] that a Galilean-relativistic theory provides no connection between spin and statistics. Put differently, the Schrödinger field can be quantized equally well by imposing either commutation or anticommutation relations on the field operators as we have done. In either case, the operator \( \hat{\psi}^\dagger \) acts as a creation operator, whereas its Hermitian conjugate \( \hat{\psi} \) acts as an annihilation operator, which is assumed to annihilate the Milne-invariant (cf. equation (2.56)) vacuum state:

\[ \hat{\psi} | 0 \rangle = 0. \]  

(3.15)

All the operators needed to describe the Schrödinger field theory on the curved Newton-Cartan spacetime can now be constructed from these two operators. One of the most important among these is the number density operator \( \hat{\rho} = \hat{\psi}^\dagger \hat{\psi} \) which, as a consequence of the condition (3.15), annihilates the vacuum: \( \hat{\rho} | 0 \rangle = 0 \). A related operator relevant to us here is the mass operator

\[ \hat{\mathbb{M}} := m \int_{\Sigma_t} \hat{\psi}^\dagger \hat{\psi} d^3x, \]  

(3.16)

which implies

\[ [\hat{\mathbb{M}}, \hat{\psi}^\dagger(x)]_{\mp} = m \hat{\psi}^\dagger(x) \quad \text{and} \quad [\hat{\mathbb{M}}, \hat{\psi}(x)]_{\mp} = -m \hat{\psi}(x). \]  

(3.17)

As a result, we may also define a covariant mass-density operator

\[ \hat{M}_{\mu\nu} := m \hat{\psi}^\dagger \hat{\psi} t_{\mu\nu}, \]  

(3.18)
which will be discussed further in the last section. Another important operator of interest to us is the angular-momentum operator

$$\hat{J} := \int_{\Sigma_t} \hat{\psi}^\dagger s \hat{\psi} \, d^3x,$$

(3.19)
giving

$$[\hat{J}, \hat{\psi}^\dagger(x)] = s \hat{\psi}^\dagger(x) \quad \text{and} \quad [\hat{J}, \hat{\psi}(x)] = -s \hat{\psi}(x).$$

(3.20)

In particular, a vanishing commutation with $\hat{J}$ of a given field operator is inferred to imply that the field corresponds to spinless bosons.

As usual, the conjugate momentum (3.11) and its conjugate allows the construction of Hamiltonian density operator:

$$\hat{H}_{Kuch}^\dagger := \hat{\mathcal{P}}^\dagger u^\sigma (D_\sigma \hat{\Psi}) + u^\sigma (D_\sigma \hat{\Psi})^\dagger \hat{\mathcal{P}} - \hat{\mathcal{L}}_{Kuch}^\dagger = -4\pi G\phi \frac{\hbar}{2m} h^{\mu\nu} \left( D_\mu \hat{\Psi} \right) \left( D_\nu \hat{\Psi} \right)^\dagger$$

$$= -\frac{\hbar^2}{2m} h^{\mu\nu} \left[ D_\mu \left( \sqrt{2} \hat{\psi} \right) \right]^\dagger \left[ D_\nu \left( \sqrt{2} \hat{\psi} \right) \right].$$

(3.21)

Here the ordering of noncommuting operators is, of course, important, since, as it is obvious from equation (3.14), $\hat{\psi}$ and $\hat{\psi}^\dagger$ in this expression are not genuine operators but rather operator-valued distributions in the Fock space. We have made use of the usual normal-ordering prescription of keeping $\hat{\psi}^\dagger$ to the left of $\hat{\psi}$.

This completes our field-theoretic generalization of the Schrödinger theory on the curved Newton-Cartan spacetime. Of course, the equal-time (anti-)commutation relations (3.14) break manifest covariance of the theory. However, as we shall see in the subsection 4.3 below, this is not inevitable. What is physically more significant is the following observation. The theory we have constructed here is a quantum theory of free particles on the curved Newton-Cartan spacetime. If we now heuristically try to substitute the quantum operator (3.18) in the right-hand side of the field equation (2.40), we realize that what is needed for consistency is a theory in which these quantized Schrödinger particles produce the quantized Newton-Cartan connection-field through which they interact. In other words, consistency requires a generally-covariant quantum field theory of identical Schrödinger particles interacting through their own quantized Newtonian gravitational field. A construction of such an interacting quantum field theory is the goal of the remaining of this paper.
4. Newton-Cartan-Schrödinger theory from an action principle

Our aim in this section is, firstly, to obtain the complete Newton-Cartan-Schrödinger theory epitomized in equations (2.43), (2.64a), (2.66), and (3.7) from extremizations of a single $\text{Aut}(B(M))$-invariant action, and, then, to recast the theory into a constraint-free Hamiltonian form in 3+1-dimensions, as well as into a manifestly covariant reduced phase-space form in 4-dimensions, to pave the way for quantization in the next section.

4.1. Covariant Lagrangian formulation of the theory

Let us first segregate the dynamical variables from the non-dynamical variables. For this purpose, recall that the Newton-Cartan connection is dynamical in general and not an invariant backdrop: as discussed in the subsection 2.4 above, the most general group of symmetry transformations of the Galilean-relativistic structure — the Leibniz group — does not leave the connection invariant; the generators $x$ of the Leibniz group forming an infinite-dimensional Lie algebra are constrained only by the conditions $\mathcal{L}_x t_{\mu\nu} = 0$ and $\mathcal{L}_x h^{\mu\nu} = 0$, and, in general, do not Lie-transport the Newton-Cartan connection, $\mathcal{L}_x \Gamma_{\alpha\beta}^\gamma \neq 0$. In other words, the respective ‘isometries’ $\mathcal{L}_x t_{\mu\nu} = 0$ and $\mathcal{L}_x h^{\mu\nu} = 0$ of the tensor fields $t_{\mu\nu}$ and $h^{\mu\nu}$ dictate that these fields are simply parts of the immutable background structure of the Newton-Cartan spacetime, and it is only the connection $\Gamma$ of the full Newton-Cartan structure $(\mathcal{M}; h, \tau, \Gamma)$ which is left unrestrained by the Leibniz group.\(^5\) Consequently, it is sufficient to treat the non-metric connection $\Gamma$ as the only dynamical attribute determined by the matter distribution via the field equation (2.40). On the other hand, we recall

\(^5\) Continuing our discussion in footnote 2 regarding the meaning of general covariance, it is worth emphasising that the metric fields $h^{\mu\nu}$ and $t_{\mu\nu}$ here defining (in Stachel’s terminology \[^2\]) the chronogeometrical structure of the Newton-Cartan gravity have been specified independently of the inertio-gravitational field $\Gamma_{\alpha\beta}^\gamma$. Now in general relativity, due to the dynamical nature of the metric field $g_{\mu\nu}$, the meaninglessness of an a priori labelling of individual spacetime points is axiomatic: a point in the bare manifold $\mathcal{M}$ is not distinguishable from any other point — and, indeed, does not even become a point with physical meaning — until the metric field is dynamically determined. Unlike in general relativity, however, where the affine structure of spacetime is inseparably (and rather wholistically) identified with the inertio-gravitational field, in Newton-Cartan theory it is possible to individuate spacetime points as entia per se existing independently of, and logically prior to, the inertio-gravitational field $\Gamma_{\alpha\beta}^\gamma$. This is because (again in Stachel’s language) the metric fields $h^{\mu\nu}$ and $t_{\mu\nu}$ serve as non-dynamical individuating fields \[^4\] specifying once and for all the immutable chronogeometrical structure, which in Newton-Cartan theory happens to be independent of the mutable inertio-gravitational structure. See Ref. 26, however, for a somewhat differing emphasis on the difficulties in pointwise identification of two different Newton-Cartan spacetimes.
that any Newton-Cartan connection can be affinely decomposed in terms of an æther-field \( u \) and an arbitrary vector-potential \( A \) as \( \Gamma_{\alpha \beta}^{\gamma} = u_{\alpha}^{\gamma} + A_{\alpha \beta}^{\gamma} \) (cf. equation (2.30)). Therefore, it is sufficient to take the fields \( A \) and \( u \) of the full Newton-Cartan structure \((\mathcal{M}; h, \tau, u, A)\) as the only dynamical variables of the gravitational field when we proceed next to formulate the Newton-Cartan-Schrödinger theory in a Lagrangian form.

In what follows, however, it is convenient to view our theory as a parameterized field theory \([38,46]\) by introducing supplementary kinematical variables on which the two metrics depend. To see how this is done, consider an arbitrary covector-field \( \hat{\mathbf{A}}_{\mu} \), an observer field \( \hat{\mathbf{u}}^\nu \), the classical Schrödinger-Kuchař complex scalar field \( \hat{\Psi} \), and its complex conjugate \( \hat{\overline{\Psi}} \) as our dynamical field variables defined on a generic background structure \((\hat{\mathcal{M}}; \hat{\mathbf{h}}, \hat{\mathbf{\tau}}, \hat{\nabla})\) satisfying \( \mathcal{L}_x \hat{t}_{\mu \nu} = 0 = \mathcal{L}_x \hat{h}^{\mu \nu} \) with an as-yet-unspecifed connection \( \hat{\Gamma} \) given by equation (2.6) — i.e., let us not restrain the connection \textit{a priori} to be Newton-Cartan by means of the constraint (2.27); in other words, at this juncture we do not impose the conditions (2.21) on this structure, and do not subject the arbitrary 1-form \( \hat{\mathbf{A}} = \hat{A}_{\mu} dx^\mu \) on \( \hat{\mathcal{M}} \) to satisfy the Newton-Cartan selection condition (2.28). Since the metrics \( \hat{h}^{\alpha \beta} \) and \( \hat{t}_{\beta} \) are fixed, an action functional defined on such a structure will not be invariant under all possible diffeomorphisms of \( \mathcal{M} \). Therefore, we enlarge the configuration space of the theory by a new one-parameter family of kinematical variables \( \langle s \rangle y \in \text{Diff}(\mathcal{M}) \) constituting the map

\[
\langle s \rangle y : \mathcal{M} \longrightarrow \hat{\mathcal{M}}
\] (4.1)

from a copy \( \mathcal{M} \) of the manifold \( \hat{\mathcal{M}} \) to the manifold \( \hat{\mathcal{M}} \) itself, and think of \( \langle s \rangle y(x) \) as a field on \( \mathcal{M} \) taking values in \( \hat{\mathcal{M}} \). As is well-known, this is nothing but the procedure of parameterization; i.e., the procedure for obtaining a generally-covariant version of a field theory originally defined on some non-dynamical, background spacetime. Using the map (4.1), we can now pull-back the two metrics as well as the dynamical fields from the ‘fixed’ manifold \( \hat{\mathcal{M}} \) to the ‘parameterized’ manifold \( \mathcal{M} \):

\[
h^{\mu \nu}(y) := (y_*)^\mu_\alpha (y_*)^{\nu}_\beta \hat{h}^{\alpha \beta}, \quad t_\mu(y) := (y^*)_\alpha^\mu \hat{t}_\alpha,
\] (4.2a,4.2b)
\[ A_\mu(y) := (y^*)^\mu_\alpha \hat{A}_\alpha, \ u^\nu(y) := (y^*)^\nu_\mu \hat{u}^\mu, \ \Psi(y) := y^* \hat{\Psi}, \ \text{and} \ \bar{\Psi}(y) := y^* \bar{\hat{\Psi}}; \] where \((y^*)^\mu_\alpha\) denotes the induced map from the tangent space of \(x \in M\) to the tangent space of \(y(x) \in \hat{M}\), or, equivalently, the pull-back map from the cotangent space of \(y(x)\) to the cotangent space of \(x\), and \((y^*)^\mu_\alpha\) denotes the inverse of \((y^*)^\mu_\alpha\).

Having defined the dynamical variables \(A_\mu, u^\nu, \Psi\), and \(\bar{\Psi}\) on the manifold \(M\) along with the supplementary kinematical variables \((s)y\), our next concern is to construct a meaningful classical phase-space for the system using Lagrangian and Hamiltonian formalisms, and then define Poisson brackets on this space in order to proceed with quantization in the next section. Accordingly, we demand that extremizations of a stationary action defined on \(M\) with respect to variations of these variables lead to the complete set of gravitational field equations (2.21), (2.27), and (2.40), the matter conservation law (2.66), and the equation of motion (3.7) for the classical fields \(\Psi\) and \(\bar{\Psi}\) on the curved spacetime, of the full Newton-Cartan-Schrödinger theory. A Lagrangian density which fulfills this demand can then be used to construct a Hamiltonian density, which would then lead us to the desired phase-space, and, subsequently, quantization of the theory can be accomplished by interpreting the functions on this phase-space as quantum mechanical operators.

Now, as discussed in the subsection 2.1, any Galilean spacetime \((M; h, \tau, \nabla)\) is orientable since it possesses a canonical 4-volume element \(\varphi \, d^4x\) (cf. equation (2.3)), which is, conveniently, non-dynamical; in particular, \(\nabla_\mu \varepsilon_{\alpha\beta\gamma\delta} = 0\). Unlike the general relativistic 4-volume element, which is determined from the dynamical gravitational field variables \(g_{\mu\nu}\), here the volume element is derived using only the non-dynamical metric fields \(h_{\mu\nu}\) and \(t_{\mu\nu}\), and, more importantly, is independent of the dynamical field variables \(A_\mu\) and \(u^\nu\). Furthermore, even in the absence of the conditions (2.21) defining the Galilean spacetime, one can begin with a measure form \(\varepsilon_{\alpha\beta\gamma\delta}\) defined on the generic structure \((M; h^{\mu\nu}, t_\nu)\) by equation (2.3). Therefore, if \(Z\) is the collection of all field configurations on the manifold \(M\), then we can form a tentative stationary action \(\mathcal{I}: Z \rightarrow \mathbb{R}\) over some measurable region \(\sigma \subset M\) with a non-null boundary \(\partial \sigma\) as follows:

\[
\mathcal{I} = \int_{\sigma} d^4x \, \mathcal{L} \left( A_\nu, \nabla_\mu A_\nu, \nabla_\mu \nabla_\nu A_\alpha, \ u^\nu, \nabla_\mu u^\nu, \nabla_\mu \nabla_\nu u^\alpha, \ \Psi, \ \partial_\mu \Psi, \ \bar{\Psi}, \ \partial_\mu \bar{\Psi}; \ (s)y \right) \quad (4.3)
\]
where \( \mathcal{L} \equiv \mathcal{L}_{\text{Grav}} + \mathcal{L}_{\text{Kuch}} \equiv [\mathcal{L}_B + \mathcal{L}_\Gamma + \mathcal{L}_N + \mathcal{L}_\Phi + \mathcal{L}_\Lambda] + [\mathcal{L}_\Psi + \mathcal{L}_1] \), \( \mathcal{L}_B \equiv + \varphi \left\{ \Upsilon_\mu h^{\mu\nu} t_\nu + \Upsilon_\nu^\mu \nabla_\mu h^{\nu\sigma} + \Upsilon_\mu h^{\nu\sigma} \nabla_\nu t_\nu + \tilde{\Upsilon}^{\mu\nu} \partial_\mu t_\nu \right\} \), \( \mathcal{L}_\Gamma \equiv + \varphi \left[ \zeta \left\{ u^{\nu} t_\nu - 1 \right\} + \chi^{\mu\nu} \left\{ \nabla_\mu A_\nu + \tilde{u}^\mu \tilde{u}^\nu \right\} \right] \), \( \mathcal{L}_N \equiv + \varphi \chi^{\alpha\mu\nu} \left\{ \nabla_\mu \nabla_\nu A_\alpha - \tilde{h}^{\alpha\sigma} \nabla_\mu \nabla_\nu \tilde{u}^\sigma \right\} \), \( \mathcal{L}_\Phi \equiv + \varphi \kappa \left\{ h^{\mu\nu} \nabla_\mu \Theta \nabla_\nu \Theta \right\} \), \( \Theta(A_\alpha, u^\sigma) \equiv \frac{1}{2} h^{\mu\nu} A_\mu A_\nu - A_\sigma u^\sigma \), \( \mathcal{L}_\Lambda \equiv + \varphi \chi \left\{ t_\mu \nabla_\nu \chi^{\mu\nu} - (1 - \kappa) \nabla_\sigma \Theta^\sigma + \lambda \Lambda_\alpha - \Lambda_\alpha \right\} \), \( \Theta^\sigma := h^{\sigma\mu} \nabla_\mu \Theta \), \( \mathcal{L}_\Psi \equiv + \varphi 4\pi G \left\{ \frac{h^2}{2m} h^{\alpha\beta} \partial_\alpha \Psi \partial_\beta \overline{\Psi} + \frac{i h}{2} u^\alpha \left( \Psi \partial_\alpha \overline{\Psi} - \overline{\Psi} \partial_\alpha \Psi \right) \right\} \), \( \mathcal{L}_1 \equiv - \varphi 4\pi G \int J^\alpha dA_\alpha \),

\( \Upsilon_\mu, \Upsilon_\nu^\mu, \Upsilon^{\mu\nu} = \tilde{\Upsilon}^{\mu\nu}, \zeta, \chi^{\mu\nu} = \chi^{[\mu\nu]}, \) and \( \chi^{\alpha\mu\nu} = \chi^{\alpha[\mu\nu]} \) are all (independent) undetermined Lagrange multiplier fields (symmetric in permutations of their indices unless indicated otherwise), \( \kappa \) and \( \lambda \) are arbitrary free parameters (with values including 0, 1, and, in case of \( \lambda, \frac{\Lambda_\alpha}{\Lambda_N} \)), \( \chi \) (also treated as a Lagrange multiplier field) will be seen in the next subsection to be the arbitrary function of gauge transformations (if \( \chi \mapsto \chi + f \), then \( A_\mu \mapsto A_\mu + \partial_\mu f \); cf. equation (2.44)), the scalars \( \Lambda_\alpha \) and

\[
\Lambda_N := t_\alpha \nabla_\sigma \nabla_\gamma \chi^{\alpha\sigma\gamma}
\]

will turn out to be parts of the cosmological constant, and

\[
J^\alpha \equiv m \Psi \overline{\Psi} \left\{ u^\alpha - h^{\alpha\beta} A_\beta \right\} + i \frac{h}{2} h^{\alpha\beta} \left\{ \Psi \partial_\beta \overline{\Psi} - \overline{\Psi} \partial_\beta \Psi \right\},
\]

which is formally identical to the expression (3.8). Note that we have ensured the action \( \mathcal{I} \) to have appropriate boundary terms by allowing the Lagrangian \( \mathcal{L} \) to contain pure divergences. It is instructive to compare the Lagrangian density \( \mathcal{L} \equiv \mathcal{L}_{\text{Grav}} + \mathcal{L}_\Psi + \mathcal{L}_1 \) with that for the Maxwell-Dirac system leading to quantum electrodynamics with the corresponding action defined on the background of Minkowski spacetime. For the sake of convenience, in what follows we keep the notation of previous sections, and continue to use the arbitrary observer field \( v^\alpha \) as a difference field between the four-velocity vector field \( u^\alpha \) and the arbitrary co-vector field \( A_\alpha \) defined by \( v^\alpha := u^\alpha - h^{\alpha\beta} A_\beta \). Let us emphasize once
again that \( A = A_\mu dx^\mu \) appearing in the action \( \mathcal{I} \) is simply an arbitrary 1-form on \( \mathcal{M} \), as yet bearing no special relation to the 2-form \( F \) of equation (2.6).

Extremizations of the action \( \mathcal{I} \) with respect to variations of the multiplier fields \( \zeta, \chi_{\mu\nu}, \chi^{\mu\nu}, \) and \( \tilde{\chi}^{\mu\nu} \) immediately yield the normalization condition

\[
u \nu = 1 \tag{4.7}
\]

for the timelike vector-field \( u \) and the conditions

\[
h^{\mu\nu} t_\nu = 0, \quad \nabla_\mu h^{\nu\sigma} = 0, \quad \nabla_\mu t_\nu = 0, \quad \text{and} \quad \partial_{[\mu} t_{\nu]} = 0 \tag{4.8}
\]

specifying the Galilean structure (cf. equation (2.21)). Whereas its extremization with respect to variations of the tensor-field \( \chi^{\mu\nu} \) yields the equation (2.29):

\[
2 \nabla_{[\mu} A_{\nu]} = -2 \frac{u}{h_{\sigma[\mu} \nabla_{\nu]}} u^\sigma. \tag{4.9}
\]

Consequently, once the last equation is compared with equation (2.13), we immediately obtain the condition (2.28) entailing that the 2-form \( F \) appearing in the connection (2.6) is closed. As discussed in subsection 2.2, the condition (2.28) is equivalent to the Newton-Cartan field equation

\[
R^{\alpha}_{\beta \gamma \delta} = R^{\gamma}_{\delta \beta \alpha}, \tag{4.10}
\]

which picks out the Newton-Cartan connection from the general Galilean connections (2.6). Thus, the extremization of \( \mathcal{I} \) with respect to \( \chi^{\mu\nu} \) not only yields this Newton-Cartan field equation, but, thereby, together with the relations (4.8) and (4.7), also fixes the hitherto unspecified connection to be Newton-Cartan — i.e., the one given by equations (2.24) and (2.30). What is more, as a result of extremizations of the action with respect to variations of the tensor-field \( \chi^{\alpha\mu\nu} \) we also have the relation

\[
\nabla_{[\gamma} \nabla_{\delta]} A_\mu - \frac{u}{h_{\mu\nu}} \nabla_{[\gamma} \nabla_{\delta]} u^\nu = 0, \tag{4.11}
\]

or, equivalently (cf. equation (2.42)),

\[
h^{\lambda\sigma} R^{\alpha}_{\sigma \gamma \delta} \equiv R^{\alpha\lambda}_{\gamma \delta} = 0, \tag{4.12}
\]

which, together with (4.10), allows us to identify our spacetime structure \( (\mathcal{M}; h, \tau, \nabla) \) as the Newton-Cartan structure (cf. the last paragraph of subsection 2.3). Consequently, we
can now recognize the arbitrary covector-field $A_\mu$ as the gravitational ‘vector-potential’ defined by equation (2.28) and adopt the entire body of formulae exclusive to the Newton-Cartan structure from sections 2 and 3. In particular, we can now use the expression (2.26) for the curvature tensor associated with the Newton-Cartan connection to obtain the corresponding Ricci tensor

$$R_{\mu\nu} = \left\{ h^{\alpha\beta} \nabla_\alpha \nabla_\beta \Phi \right\} t_{\mu\nu},$$

which, of course, is gauge-independent despite its appearance. One way to see this gauge-independence is to note that — thanks to the tracelessness of the gravitational field tensor ($v^\mu G_\mu^\alpha = 0$; cf. equation (2.25)) — the distinction between the ‘flat’ covariant derivative $\nabla_\alpha$ and the ‘curved’ covariant derivative $\nabla_\alpha$ disappears in the case of divergence. In particular, $\nabla_\alpha \Phi^\alpha = \nabla_\alpha \tilde{\Phi}^\alpha$, where $\tilde{\Phi}^\alpha := h^{\alpha\beta} \nabla_\beta \tilde{\Phi} = h^{\alpha\beta} \partial_\beta \tilde{\Phi} = h^{\alpha\beta} \nabla_\beta \tilde{\Phi}$ because $\Phi$ is a scalar. Therefore, as a direct consequence, the above expression for the Ricci tensor can also be written in terms of the curved covariant derivative operator $\nabla_\alpha$ as

$$R_{\mu\nu} = \left\{ \nabla_\alpha \Phi^\alpha \right\} t_{\mu\nu} = \left\{ h^{\alpha\beta} \nabla_\alpha \nabla_\beta \Phi \right\} t_{\mu\nu}. \quad (4.14)$$

To extract further dynamical information from the action $\mathcal{I}$, we next extremize it with respect to variations of the covector-field $A_\alpha$ and look for the corresponding Euler-Lagrange equations

$$\frac{\delta \mathcal{L}}{\delta A_\alpha} \equiv \nabla_\mu \left\{ \frac{\partial \mathcal{L}}{\partial (\nabla_\mu A_\alpha)} \right\} + \nabla_\mu \nabla_\nu \left\{ \frac{\partial \mathcal{L}}{\partial (\nabla_\mu \nabla_\nu A_\alpha)} \right\} = 0. \quad (4.15)$$

Since the Newton-Cartan connection $\Gamma$ is invariant under variations of the gauge variables $A$ and $u$ (cf. subsection 2.4), these equations give the relations $\frac{\delta \mathcal{L}_\mu}{\delta A_\alpha} = \frac{\delta \mathcal{L}_\Lambda}{\delta A_\alpha} = \frac{\delta \mathcal{L}_\Psi}{\delta A_\alpha} = 0$, and

$$\frac{\delta \mathcal{L}}{\delta A_\alpha} = \varphi \nabla_\sigma \chi^{\alpha\sigma} + \varphi \nabla_\sigma \nabla_\gamma \chi^{\alpha\sigma\gamma} + \varphi \kappa v^\alpha \nabla_\sigma \Theta^\sigma - \varphi 4\pi G J^\alpha = 0. \quad (4.16)$$

As it stands, the last equation explicitly contains the observer field $v^\alpha$, which can be eliminated by contracting both sides of the equation with $t_\alpha$, and using $t_\alpha v^\alpha = 1$. Subsequently, after using (4.6) with $t_\alpha u^\alpha = 1$ in the resulting equation and dividing it through by $\varphi$, we obtain

$$\kappa \nabla_\sigma \Theta^\sigma + t_\alpha \nabla_\sigma \chi^{\alpha\sigma} + t_\alpha \nabla_\sigma \nabla_\gamma \chi^{\alpha\sigma\gamma} = 4\pi G \rho, \quad (4.17)$$
where ρ ≡ mΨ⃗{Ψ} as in equation (3.8).

Amiably enough, it turns out that extremization of the action with respect to the æther-field \( u^\alpha \) leads back to the same equation (4.17); i.e., the dynamical pieces of information extractable from the variations of \( I \) with respect to \( A_\alpha \) and \( u^\alpha \) are identical. The components of Euler-Lagrange equations in this twin case are:

\[
\frac{\delta L}{\delta u^\alpha} = \varphi \{ \zeta t_\alpha - h_{\alpha \mu} \nabla_\gamma \chi^{\mu \gamma} \},
\]

(4.18a)

\[
\frac{\delta L}{\delta u^\alpha} = -\varphi \left\{ u_{\alpha \mu} \nabla_\delta \nabla_\gamma \chi^{\mu \delta \gamma} \right\},
\]

(4.18b)

\[
\frac{\delta L}{\delta u^\alpha} = +\varphi \kappa A_\alpha \left\{ h^{\delta \gamma} \nabla_\delta \nabla_\gamma \Theta(A_\mu, u^\nu) \right\},
\]

(4.18c)

\[
\frac{\delta L}{\delta u^\alpha} = +\varphi 4\pi G i \frac{h}{2} \left\{ \Psi \partial_\alpha \bar{\Psi} - \bar{\Psi} \partial_\alpha \Psi \right\},
\]

(4.18d)

\[
\text{and } \frac{\delta L}{\delta u^\alpha} = -\varphi 4\pi G \{ m\Psi \bar{\Psi} \} A_\alpha.
\]

(4.18e)

In evaluating the first two of the displayed equations we have used properties (2.12), (2.16), and (2.17) of a geodesic observer. Combining all of these components in the Euler-Lagrange equation \( \frac{\delta L}{\delta u^\alpha} = 0 \), dividing it through by \( \varphi \), contracting it with \( h_{\sigma \nu} \), and substituting \( \delta \mu - t_\mu u^\sigma \) in it for \( h^{\sigma \alpha} u_{\alpha \mu} \), using equation (4.16) to substitute for \( \nabla_\gamma \chi^{\sigma \gamma} + \nabla_\delta \nabla_\gamma \chi^{\sigma \delta \gamma} \), and, finally, contracting the result of all these operations (in this order) with \( t_\sigma \) and using \( t_\sigma u^\sigma = 1 \) yields equation (4.17) as asserted. This result is hardly surprising since, as we shall see in a moment, the momenta canonically conjugate to the variables \( A_\mu \) and \( u^\nu \) are directly proportional to each other.

Before we can analyze equation (4.17) any further, however, we need to either eliminate the undetermined multiplier tensors \( \chi^{\mu \nu} \) and \( \chi^{\alpha \mu \nu} \), or interpret them in physical terms. A physical meaning of the multiplier field \( \chi^{\mu \nu} \) is readily revealed if we evaluate the four-momentum density canonically conjugate to the gravitational field variable \( A_\mu \) at each point \( x \) on \( \mathcal{M} \):

\[
\Pi^\mu \delta A_\mu := \mathcal{J}^\alpha A_\alpha t_\alpha = \varphi (t_\alpha \chi^{\alpha \mu}) \delta A_\mu,
\]

(4.19)

where

\[
\mathcal{J}^\mu_A := \left\{ \frac{\delta L}{\delta (\nabla_\mu A_\alpha)} \right\} \delta A_\alpha + \left\{ \frac{\delta L}{\delta (\nabla_\mu \nabla_\nu A_\alpha)} \right\} \nabla_\nu \delta A_\alpha
\]

(4.20)
is the presymplectic potential current density \([A.4,A.5]\), defined and discussed in the appendix A below (cf. equation (A.12)). Here we have dropped the 4-divergence term

\[ \varphi \nabla_\nu (t_\mu \chi^{\alpha \mu \nu} \delta A_\alpha) \]  

(4.21)

from the expression (4.19), arising from the \(\mathcal{L}_N\) component of the action, since, thanks to the vanishing of the boundary of a boundary theorem, it does not contribute to the presymplectic potential \((A.15)\). Note that the antisymmetric nature of \(\chi^{\mu \nu}\) requires \(\Pi^\mu\) to be spacelike: \(t_\mu \Pi^\mu \equiv 0\). It is also noteworthy that the only non-zero contribution to the conjugate momentum density comes from the \(\mathcal{L}_\Gamma\) component of the action, which is also responsible for the condition (2.27) specifying the Newton-Cartan connection out of the generic possibilities (2.6). Thus, we now understand the physical meaning of the multiplier field \(\chi^{\mu \nu}\). The physical meaning of the multiplier field \(\chi^{\alpha \mu \nu}\), on the other hand, has already been anticipated in the definition (4.5) of \(\Lambda_N\), which, in what follows, will be viewed as a cosmological contribution.

The relations (4.19) and (4.5) allow us to rewrite equation (4.17) in terms of the physical field variables \(\Psi, \overline{\Psi}, u^\nu, A_\mu\), and the canonical conjugate field \(\Pi^\mu\) of \(A_\mu\):

\[ \kappa \nabla_\sigma \Theta^\sigma (A_\mu, u^\nu) + \frac{1}{\varphi} \nabla_\sigma \Pi^\sigma + \Lambda_N = 4\pi G m \Psi \overline{\Psi} \equiv 4\pi G \rho, \]  

(4.22)

where we have made use of \(\varphi t_\alpha \nabla_\sigma \chi^{\alpha \sigma} = \varphi \nabla_\sigma (t_\alpha \chi^{\alpha \sigma}) = \nabla_\sigma \Pi^\sigma\). It is not surprising that only the four-momentum density \(\Pi^\mu\) conjugate to the field variable \(A_\mu\) appears in this expression, because, as one may expect, the four-momentum density \(\overline{u}_\nu\) conjugate to the observer field \(u^\nu\) is just the ‘anti-dual’ of \(\Pi^\mu\),

\[ \overline{u}_\nu \delta u^\nu := \mathcal{J}_u^\nu t_\alpha = \left(-\hat{h}_{\nu\mu} \Pi^\mu\right) \delta u^\nu, \]  

(4.23)

(again, modulo the 4-divergence term

\[ \varphi \nabla_\nu \left(t_\mu \chi^{\alpha \mu \nu} \hat{u}_\sigma \delta u^\sigma\right) \]  

(4.24)

not affecting the symplectic structure) which can be easily checked by explicitly evaluating \(\mathcal{J}_u^\alpha\). In fact, we have the relationship

\[ \Pi^\mu + \hat{h}^{\mu \nu} \overline{u}_\nu = 0 = \overline{\Pi}_\mu + \hat{h}_{\mu \nu} \Pi^\nu, \]  

(4.25)
which indicates that, strictly speaking, we should not view $A_\mu$ and $u^\nu$ as independent variables. In what follows, however, for the sake of convenience, we shall continue treating them as if they were independent. Eventually, in the next section, the constraint (4.25) will be taken into account in a consistent manner.

Now, from the extremization of the action with respect to variations of the scalar multiplier field $\chi$ we immediately infer that

$$\frac{1}{g} \nabla_\sigma \Pi^\sigma - (1 - \kappa) \nabla_\sigma \Theta^\sigma + \lambda \Lambda_N - \Lambda_O = 0,$$

(4.26)

which, upon substitution into equation (4.22), yields

$$\nabla_\sigma \Theta^\sigma + \Lambda = 4\pi G \rho,$$

(4.27)

where

$$\Lambda := \Lambda_O + (1 - \lambda) \Lambda_N.$$

(4.28)

For the flat Galilean spacetime, the left-hand side of this equation is simply the Laplacian of the scalar $\Theta(A_\mu, u^\nu)$ plus the cosmological parameter $\Lambda$; and, hence, it is just the Poisson equation of the classical Newtonian theory of gravity provided we can interpret $\Theta(A_\mu, u^\nu)$ as the corresponding scalar gravitational potential. But, of course, with $A_\mu$ recognized to be the gravitational vector-potential as a result of equations (4.7), (4.8), and (4.10), it is indeed possible to identify the function $\Theta(A_\mu, u^\nu)$ in equation (4.27) with the Newtonian gravitational scalar-potential $\Phi(A_\alpha, u^\nu)$ with respect to the observer-field $u^\alpha \equiv u^\alpha - h^{\alpha\beta} A_\beta$:

$$\Phi(A_\alpha, u^\nu) \equiv \Theta(A_\alpha, u^\nu) = \frac{1}{2} h^{\mu\nu} A_\mu A_\nu - A_\sigma u^\sigma$$

(4.29)

(cf. equation [2.32]). Using this identification in equation (4.14), and multiplying equation (4.27) with $t_{\mu\nu}$, it is now easy to obtain the last of the Newton-Cartan field equations,

$$R_{\mu\nu} + \Lambda t_{\mu\nu} = 4\pi G M_{\mu\nu},$$

(4.30)

as a generally-covariant generalization of the Newton-Poisson equation. As noted before, an immediate inference one gains from this field equation is that spacelike hypersurfaces of simultaneity $\Sigma_t$ embedded in $\mathcal{M}$ are copies of flat, Euclidean three-spaces: $h^{\mu\alpha} h^{\nu\sigma} R_{\alpha\sigma} = 0$. 
Unlike the general relativistic case, however, here the nonrelativistic contracted Bianchi identities (2.20) by themselves do not render the scalar \( \Lambda \) to be a spacetime constant. Nevertheless, given the condition (4.10) on \( \mathcal{M} \), a detailed analysis \[10\] of the tensor representations of the Galilean group provided by both the physically sensible matter tensor \( M^{\mu \nu} \) and the curvature tensor \( R^{\alpha}_{\mu \sigma \nu} \) of the general Galilean spacetime, together with the constraints due to nonrelativistic contracted Bianchi identities (2.20), reveals that \( \Lambda \) in the above field equation at the most could be a spacetime constant.

Note that neither \( \Lambda_0 \) nor \( \Lambda_N \) are individually required to be spacetime constants, only the net \( \Lambda \) is. Further, there is nothing sacrosanct about the interpretation we have given to \( \Lambda_N \). This contribution to \( \Lambda \) arises from the \( \mathcal{L}_N \) term in the action. This is the term which gives rise to the additional constraint (2.41) on the curvature tensor. But, as discussed in subsection 2.3, in the asymptotic limit this constraint is automatically satisfied, and, hence, the \( \mathcal{L}_N \) term can be dropped from the action; i.e, in such a case, the variation of the multiplier tensor \( \chi^{\alpha \mu \nu} \) itself must be set equal to zero. Moreover, in this limit any cosmological contribution is also generally ruled out. And, indeed, these physical requirements all come out consistently in our interpretation of \( \Lambda_N \) if we set \( \lambda = \frac{\Lambda_0}{\Lambda_N} \) in the Lagrangian \( \mathcal{L}_\Lambda \) giving \( \Lambda \equiv \Lambda_N \). However, if one wishes, one can easily set \( \lambda = 1 \) to avoid such a strong link between the ‘Newtonian restriction’ (2.41) and the cosmological constant. As long as the controversy over cosmological contributions is unsettled, a choice of this part of the action is simply a matter of taste.

Turning now to the matter part of the action, we recognize the Lagrangian density \( \mathcal{L}_\Psi + \mathcal{L}_1 \) as nothing but the Schrödinger-Kuchař Lagrangian density \( \mathcal{L}_{Kuch} \) expressed by equation (3.5) of the previous section. Since \( \mathcal{L}_{Grav} \) is independent of the variables \( \Psi, \partial_\mu \Psi \), and their conjugates, extremizations of the action \( \mathcal{I} \) with respect to variations of \( \Psi \) and \( \overline{\Psi} \) are identical to those of an action with Lagrangian density \( \mathcal{L}_{Kuch} \equiv \mathcal{L}_\Psi + \mathcal{L}_1 \) which we have already discussed in that section. The resulting Euler-Lagrange equations \( \frac{\delta \mathcal{L}}{\delta \Psi} = 0 \) and \( \frac{\delta \mathcal{L}}{\delta \overline{\Psi}} = 0 \) yield, respectively, the Schrödinger-Kuchař equation (3.7),

\[
\left[ \frac{\hbar^2}{2m} \nabla^\alpha \partial_\alpha + i\hbar \left( u^\alpha - h^{\alpha \beta} A_\beta \right) \partial_\alpha + m \left( u^\alpha A_\alpha - \frac{i}{2} h^{\alpha \beta} A_\alpha A_\beta \right) + i\frac{\hbar}{2} \nabla_\alpha \left( u^\alpha - h^{\alpha \beta} A_\beta \right) \right] \Psi = 0, \tag{4.31}
\]
and its conjugate, describing the motion of a classical Galilean-relativistic Schrödinger-Kuchař field $\Psi$ on the curved Newton-Cartan manifold achieved by minimally coupling it to the gravitational ‘vector-potential’ $A_\mu$.

Finally, let us not forget the remaining kinematical variables $(s)y : \mathcal{M} \to \hat{\mathcal{M}}$. Extremization of the action under variations of these auxiliary variables leads to

$$0 = \frac{d}{ds} I \equiv \delta I = \int \frac{\delta I_{\text{Grav}}}{\delta s y^\alpha} \delta^{(s)} y^\alpha + \int \frac{\delta I_{\text{Kuch}}}{\delta s y^\alpha} \delta^{(s)} y^\alpha.$$  \hspace{1cm} (4.32)

As we already saw in the subsection 2.5 and the section 3, given the matter field equations (4.31) and its complex conjugate, one of the consequences of the covariance of the matter action $I_{\text{Kuch}}$ is the matter conservation laws

$$\nabla_\mu J^\mu = 0 \quad \text{and} \quad \nabla_\mu M^{\mu\nu} = 0$$  \hspace{1cm} (4.33)

(cf. equations (2.64a) and (2.66)), which correspond to the relativistic conservation law $\nabla_\mu T^{\mu\nu} = 0$. In addition, the covariance of the matter action reduces equation (4.32) to

$$0 = \int \frac{\delta I_{\text{Grav}}}{\delta s y^\alpha} \delta^{(s)} y^\alpha = \int \frac{\delta I_{\text{Grav}}}{\delta h^{\mu\nu}} \delta h^{\mu\nu} + \int \frac{\delta I_{\text{Grav}}}{\delta t_\mu} \delta t_\mu + \int \frac{\delta I_{\text{Grav}}}{\delta A_\mu} \delta A_\mu + \int \frac{\delta I_{\text{Grav}}}{\delta u_\nu} \delta u_\nu,$$  \hspace{1cm} (4.34)

since the variations of the action $I_{\text{Grav}}$ with respect to all of the multiplier fields have been required to vanish. But now we can parallel the reasoning of the subsection 2.5 with $I_m$ replaced by $I_{\text{Grav}}$ and obtain the condition

$$\nabla_\mu G^{\mu\nu} = 0$$  \hspace{1cm} (4.35)

analogous to the equation (4.33) above, where

$$G^{\mu\nu} := \{\nabla_\sigma \Theta^\sigma + \Lambda\} u^\mu u^\nu - 2 \{\nabla_\sigma \Theta^\sigma + \Lambda\} u^{(\mu} h^{\nu)\sigma} A_\sigma - h^{\mu\sigma} h^{\nu\alpha} (S_{\text{Grav}})_{\sigma\alpha}$$  \hspace{1cm} (4.36)

(cf. equations (2.67) and (4.27)). Comparing this latter expression with the general definition (2.37) for $R^{\mu\nu}$ we immediately see that it is equivalent to

$$G^{\mu\nu} \equiv R^{\mu\nu} + \Lambda v^\mu v^\nu.$$  \hspace{1cm} (4.37)

Substituting this expression into equation (4.35) and using $\nabla_\sigma v^\sigma = 0 = v^\mu \nabla_\mu v^\alpha$ for the geodetic observer $v^\alpha$ gives

$$\nabla_\mu R^{\mu\nu} = 0,$$  \hspace{1cm} (4.38)
which is nothing but the contracted Bianchi identities (2.20) because $h^{\mu\nu} R_{\mu\nu} =: R = 0$ according to equation (4.30). Thus, in close analogy with the Einstein-Hilbert theory, extremization of the total action with respect to variations of the supplementary variables \((s)\) yields both the matter conservation law and the contracted Bianchi identities.

It is remarkable that we have been able to obtain all of the field equations of the Newton-Cartan theory, (2.43), the matter conservation laws (2.64a) and (2.66), as well as the equation of motion (3.7) for the classical Schrödinger-Kuchař field $\Psi$ on the curved Newton-Cartan spacetime, from a single action principle. What is more, the resultant theory we have obtained is covariant under the complete gauge group $A_{\text{ut}}(B(M))$ discussed in the subsection 2.4. The invariance of $L_{Kuch}$ under this gauge group has already been emphasized in the previous section. Among the four components of the Lagrangian density associated with the connection-field, it can be explicitly checked that $L_B$, $L_\Gamma$, $L_N$, and $L_\Lambda$ are all invariant under the complete automorphism group $A_{\text{ut}}(B(M))$ (the factor \(\{h^{\sigma\gamma} \nabla_\sigma \nabla_\gamma \Theta\}\), of course, does not change under the full gauge transformation; cf. equations (4.13) and (4.14)). The third one, $L_\Phi$, on the other hand, is invariant only under the diffeomorphism subgroup, $\text{Diff}(M)$, and the ‘boost’ subgroup,

$$u^\alpha \mapsto u^\alpha + h^\alpha\sigma w_\sigma,$$

$$A_\alpha \mapsto A_\alpha + w_\alpha - \left( u^\sigma w_\sigma + \frac{1}{2} h^{\mu\nu} w_\mu w_\nu \right) t_\alpha,$$  \hspace{1cm} (4.39)

of the full group $A_{\text{ut}}(B(M))$. It is not invariant, in particular, under the transformations $A_\mu \mapsto A_\mu + \partial_\mu f$. As a result, the Euler-Lagrange equation (4.16) depends on the choice of this internal gauge; for, under these transformations, $v^\mu \mapsto v^\mu - h^\mu\nu \partial_\nu f$. However, this gauge-dependence is projected out in the actual field equation (4.17) — since the gauge-dependent part is spacelike: $t_\alpha \partial^\alpha f = 0$ — making the whole theory invariant under the complete Newton-Cartan gauge group $A_{\text{ut}}(B(M))$. Better still, if one wishes — say, for aesthetic reasons — to eliminate the gauge-dependent part $L_\Phi$ of the total Lagrangian density to make the Lagrangian prescription manifestly generally-covariant, one

---

\(^6\) All previous attempts to satisfactorily reformulate Newton-Cartan theory in four-dimensions using variational principles have met insurmountable difficulties due to the non-semi-Riemannian nature of the Galilean spacetime. For a partially successful attempt, see reference [47]. For a review of a five-dimensional formulation overcoming at least some of the difficulties, see reference [48].
is completely free to do so by simply choosing the arbitrary constant \( \kappa = 0 \). Therefore, our variational reformulation of the classical Newton-Cartan-Schrödinger theory is no less generally-covariant than Einstein’s theory of gravity (modulo, of course, the philosophical caveat made in the footnote 2). In fact, in close analogy with variational formulations of Einstein’s theory, all we have assumed here \textit{a priori} is an \( \text{Aut}(B(M)) \)-invariant action functional — dependent only on local degrees of freedom — defined on some measurable region of a sufficiently smooth, real, differentiable Hausdorff 4-manifold \( M \). The rest follows squarely from extremizations of this action functional.

4.2. Constraint-free Hamiltonian formulation in 3+1 dimensions

Unlike the spacetime covariant Lagrangian formulation of any field theory, the conventional Hamiltonian formulation of such a theory requires a blatantly non-covariant Aristotelian 3+1 decomposition of spacetime into 3-spaces \( \Sigma_t \) at instants of time \( t \in \mathbb{R} \). For such a foliation of spacetime to be possible, it is necessary to assume that the manifold \( M \) is of the globally hyperbolic form: \( M = \mathbb{R} \times \Sigma \), where \( \Sigma \) is taken to be an embedded (cf. footnote 3) achronal\(^7\) closed submanifold of \( M \) such that its domain of dependence \( \mathcal{D}(\Sigma) = M \). In the general relativistic case such an \textit{a priori} topological constraint on spacetime is obviously too severe, and therefore canonical approaches to quantize Einstein’s gravity are sometimes criticized for being overly restrictive if not completely misguided. However, in Newtonian physics time plays a very privileged role. Therefore, for a Galilean spacetime the breakup \( M = \mathbb{R} \times \Sigma \) is not only natural but, in fact, a part of the intrinsic structure determined by the non-dynamical metrics \( h^{\mu \nu} \) and \( t_{\mu \nu} \). This fact, of course, does not compensate for the loss of covariance of any field theory based on such a breakup of spacetime into 3-spaces at times. However, as we shall see in the next subsection, the apparent loss of covariance in the Hamiltonian formulation of our theory is only an aesthetic loss, and can be rectified using the relatively less-popular symplectic approach yielding a manifestly covariant description of canonical formalism \[44, 45, 49, 50, 51\].

\(^7\) A set is achronal if none of its elements can be joined by a timelike curve. For a definition of ‘the domain of dependence’ and a general discussion on the construction of regular Cauchy surfaces, see reference \[28\].
To see in detail the intuitively obvious fact that Newton-Cartan structure is naturally globally hyperbolic, first recall that the structure \((M, t_{\mu\nu})\) is time-orientable; i.e., there exists a globally defined smooth vector field \(t_\mu\) on \(M\) inducing the temporal metric \(t_{\mu\nu} = t_\mu t_\nu\) and determining a time-orientation. Further, since \(M\) is contractible by definition, the compatibility condition \(\nabla_\mu t_\nu = 0\) together with the Poincaré lemma allows one to define the absolute time globally by a map \(t : M \to \mathbb{R}\), foliating the spacetime uniquely into one-parameter family of smooth Cauchy surfaces \(\Sigma_t\) — the domains of simultaneity. Here, one can change the scalar function \(t\) into \(t' = t'(t)\), but the regular foliation \(M = \bigcup \{\Sigma_t\}\) with \(\Sigma_t = \{x \in M \mid t(x) = c, c \in \mathbb{R}\}\) remains a part of the intrinsic structure of spacetime. Moreover, these Cauchy surfaces have the property that all curves with images confined to them are spacelike; consequently, they may also be defined directly by the 1-form \(\tau = t_\alpha dx^\alpha\) as the 3-dimensional subspaces of tangent spaces \(T_xM\) at points \(x\) on \(M\). Since \(\tau\) is closed, this differential system of hypersurfaces \(\Sigma_x\) is completely integrable, and defines a foliation of regular Cauchy surfaces on \(M\) which are given by \(\Sigma_t\) with a locally defined scalar function \(t(x) \in C^\infty(M)\) satisfying \(dt = \tau\). What is more, these smooth 3-surfaces \(\Sigma_t\) are orientable since the 4-manifold \(M\) is orientable; given the 4-volume measure \(E_{\mu\nu\rho\sigma}\) on \(M\) (cf. equation (2.3)), the corresponding contravariant tensor \(E^{\mu\nu\rho\sigma}\) can be used to obtain a 3-volume measure \(E^{[\mu\nu\rho]} = E^{\mu\nu\rho} t_\sigma\) on the Cauchy surfaces \(\Sigma_t\). In the special case of Newton-Cartan spacetime these Cauchy surfaces are flat Riemannian 3-surfaces of spacelike vectors, carrying a non-degenerate Euclidean metric induced by projecting \(h^{\mu\nu}\) with respect to any unit timelike vector field — say the Galilean observer field \(u^\alpha\). That is, \((\Sigma_t, h_{\mu\nu})\), with \(h_{\mu\nu}\) as the induced metric field on \(\Sigma_t\) corresponding to the vector field \(u^\alpha\), is a copy of Euclidean 3-space, and, hence, \(M\) is homeomorphic to \(\mathbb{R}^4\). Since \(h_{\mu\nu}\) has a spatial inverse, namely \(h^{\mu\nu}\) (cf. equation (2.9)), it follows that there exists a unique 3-dimensional derivative operator \((^3\nabla_\mu)\) on \(\Sigma_t\) compatible with \(h_{\mu\nu} : (^3\nabla_\sigma h_{\mu\nu} = 0\). It can be characterized in terms of the 4-dimensional operator \(\nabla_\mu\); for example, for a tensor field \(V^\alpha_{\mu\nu}\),

\[
( ^3 \nabla_\sigma V^\alpha_{\mu\nu} = u^\alpha \delta^\beta_\sigma \delta^\gamma_\mu \delta^\lambda_\nu \nabla_\beta V^\alpha_{\gamma\lambda} .
\]
Note that there is no need to project the contravariant indices because they remain spacelike even after the application of $\nabla_\mu$ because of the compatibility condition $\nabla_\mu t_\nu = 0$, and that $(3)\nabla_\sigma$ satisfies

$$(3)\nabla_\sigma \delta^\mu_\nu = (3)\nabla_\sigma h^\mu_\nu = (3)\nabla_\sigma h^\mu_\nu = 0 \quad (4.42)$$

over and above all of the defining conditions for a derivative operator. The two operators $\nabla_\sigma$ and $(3)\nabla_\sigma$ induce the same parallel transport condition for spacelike vectors on $\Sigma_t$:

$$U^\alpha (3)\nabla_\alpha) V^\mu = U^\alpha \delta^\alpha_\gamma \nabla_\gamma V^\mu = U^\gamma \nabla_\gamma V^\mu , \text{ for all spacelike fields } U^\mu \text{ and } V^\mu \text{ on } \Sigma_t .$$

The vector field $u^\alpha$ may be viewed as describing the flow of time in $\mathcal{M}$ and can be used to identify each Cauchy surface $\Sigma_t$ with the initial surface $\Sigma_0$. Given such an observer field $u^\alpha$ and its relative spatial projection field $u^\mu \delta^\nu_\alpha \equiv h^\mu_\nu h^{\alpha \nu}$, one may decompose any spacetime quantity into its projections normal and tangential to $\Sigma_t$. For example, using the relations

$$t_\alpha u^\delta_\alpha \mu = 0 \quad \text{and} \quad u^\alpha \delta^\mu_\alpha = 0 , \quad (4.43)$$

a contravector $V^\alpha$ on $\mathcal{M}$ may be decomposed as

$$V^\alpha = \perp V u^\alpha + \parallel V^\alpha \quad (4.44)$$

with $\perp V := t_\mu V^\mu$ and $\parallel V^\alpha := u^\mu \delta^\alpha_\mu V^\mu$, whereas a covector $V_\alpha$ on $\mathcal{M}$ can be split as

$$V_\alpha = \perp V t_\alpha + \parallel V_\alpha \quad (4.45)$$

with $\perp V := u^\mu V_\mu$ and $\parallel V_\alpha := \delta^\alpha_\mu V^\mu$. Note that the distinction between covectors and contravectors made conspicuous by these decompositions has much greater significance here than in the general relativistic spacetimes.

With the above rather elaborate delineation of the convenient fact that Newton-Cartan spacetime is naturally globally hyperbolic, we are ready to cast our theory in a Hamiltonian form. The next two steps for which are, firstly, to construct a configuration space for the field variables on $\mathcal{M}$ by specifying instantaneous configuration fields corresponding to these variables on a chosen Cauchy surface $\Sigma_t$, and then, secondly, to evaluate conjugate momentum densities of these instantaneous fields on the chosen surface. As we shall see, the ensuing canonical variables constitute the proper Cauchy data to be propagated from
one Cauchy surface to another. Let us begin by concentrating on the pair \((A_\mu, u^\nu)\) of pure gravitational field variables on \(\mathcal{M}\) and tentatively take \(A_\mu\) and \(u^\nu\) evaluated on \(\Sigma_t\) as our instantaneous configuration variables. Using the defining equations (4.45) and (4.44), these variables may be decomposed into their components normal and tangential to \(\Sigma_t\) as

\[
A_\alpha = \perp A t_\alpha + \parallel A_\alpha
\]

with \(\perp A := u^\mu A_\mu\) and \(\parallel A_\alpha := \delta_\alpha^\mu A_\mu\), and

\[
u^\alpha = \perp u u^\alpha + \parallel u^\alpha
\]

with \(\perp u := t_\mu u^\mu = 1\) and \(\parallel u^\alpha := \delta_\mu^\alpha u^\mu = 0\). The Lagrangian density, expression (4.4), can now be rewritten in terms of these decompositions and normal and tangential parts of the momentum densities conjugate to the fields \(A_\mu\) and \(u^\nu\) specified on \(\Sigma_t\) can be evaluated from it. It is easier, however, to use the already evaluated four-momentum densities (4.19) and (4.23) and decompose them according to equations (4.44) and (4.45). Whichever procedure is used, the outcomes are:

\[
\perp \Pi = t_\mu \Pi^\mu = 0 \quad \text{and} \quad \parallel \Pi^\alpha = \delta_\mu^\alpha \Pi^\mu = \Pi^\alpha,
\]

whereas

\[
\perp \Pi = u^\nu \Pi_{\nu} = 0 \quad \text{and} \quad \parallel \Pi_\alpha = \delta_\mu^\alpha \Pi_{\mu} = -u^\alpha \Pi^\rho.
\]

The vanishing of the canonical momentum densities \(\perp \Pi\) and \(\perp \Pi\) implies that the conjugate momenta are spacelike, and, more importantly for our purposes, suggests that we should not view their conjugate variables \(\perp A\) and \(\perp u\) as dynamical variables because they do not constitute suitable Cauchy data (their presence as dynamical variables would be an obstacle to the Legendre transformation required to specify a Hamiltonian functional on \(\Sigma_t\)). Since \(\perp u \equiv 1\) and \(\parallel u^\alpha \equiv 0\), we shall see that taking the observer field \(u^\nu\) as one of our configuration variables does not cause any serious problems. However, we must take only the tangential component \(\parallel A_\mu\) of \(A_\mu\) as our gravitational configuration variable along with the observer field \(u^\nu\) if we are to maintain substantive uniqueness of the propagation from Cauchy data. In addition to these gravitational field variables, we also have the matter
configuration variables $\Psi$ and $\overline{\Psi}$ specified on $\Sigma_t$, and, together with them, we also must add their respective conjugate momentum densities

$$
\begin{align*}
\overline{P} & := \frac{\delta L_{\text{Kuch}}}{\delta (u^\sigma \partial_\sigma \Psi)} = -2 i \hbar \wp \pi G \overline{\Psi} \quad \text{and} \quad P := \frac{\delta L_{\text{Kuch}}}{\delta (u^\sigma \partial_\sigma \overline{\Psi})} = +2 i \hbar \wp \pi G \Psi
\end{align*}
$$

(4.50)

to our list of canonical phase-space variables, where $u^\sigma \nabla_\sigma$ is a propagation covariant derivative (i.e., a time derivative) along the unit timelike vector field $u^\sigma$.

Next, in order to remain in close contact with the parameterized formulation \[38,46\] used in the previous subsection, we view the foliation $e$ of $M$, 

$$
e : \mathbb{R} \times \Sigma \rightarrow M ,
$$

as the map which allows us to pass to the Hamiltonian formulation of the theory. Then, for each $t \in \mathbb{R}$, $e$ becomes an embedding

$$
(t)_e : \Sigma \rightarrow M
$$

(4.52)
(cf. footnote 3). In order to make the embedded hypersurfaces $\Sigma_t$ (i.e., the images of $\Sigma$ under the above embedding) spacelike, we require the differential $(t)_e^\mu_i$ of the map $(t)_e$ to satisfy

$$
(t)_e^\mu_i t_\mu = 0 \quad \text{and} \quad (t)_e^i_\nu u^\nu = 0 ,
$$

(4.53)
where $(t)_e^i_\nu$ is the inverse of the differential map $(t)_e^\mu_i$, and in these expressions (and only in these expressions) the tensors evaluated on the closed 3-surfaces $\Sigma_t$ are represented via Latin indices for clarity. Comparing these conditions with the relations (4.43), we immediately see that

$$
(t)_e^\mu_i \equiv \frac{u}{\delta_i} , \quad \text{and} \quad (t)_e^i_\nu \equiv \frac{u}{\delta_\nu} .
$$

(4.54)

In analogy with what we have done in the previous subsection, we can now pull-back all the fields from $\mathcal{M}$ to the 3-manifold $\Sigma$, and enlarge the configuration space of these fields by spacelike embedding variables

$$
(o)_y \circ (t)_e =: \vartheta : \Sigma \rightarrow \mathcal{M} ,
$$

(4.55)

with $(o)_y \in \{(o)_y\}$ being any fixed diffeomorphism defined by equation (4.1) of the previous subsection. In terms of this map the traditional ‘deformation vector’ $\vartheta^\sigma := u^\sigma \nabla_\sigma \vartheta^\sigma = \frac{\partial \vartheta^\sigma}{\partial t}$
dictating transition from one leaf of foliation to another neighboring one, for example, may be split into its ‘lapse’ and ‘shift’ coefficients as

$$\dot{\vartheta} = \frac{\perp}{\Pi} \dot{u} + \| \dot{\vartheta} \|,$$

(4.56)

where $\frac{\perp}{\Pi} := \dot{\vartheta} t_{\sigma}$ and $\| \dot{\vartheta} \| := \delta_{\mu}^{\nu} \dot{\vartheta} \mu$.

These considerations finally allow reconstruction of the action functional $I$ of the previous subsection in the Hamiltonian form:

$$I_{\Pi} = \int_{\mathbb{R}} dt \int_{\Sigma_{t}} d^{3}x \left[ \| \Pi \| \dot{A} + \frac{\Pi}{\Pi} \dot{u} + \mathbf{P} \dot{\psi} + \mathbf{P} \dot{\vartheta} + \Pi_{\sigma} \dot{\vartheta} - N^{\sigma} H_{\sigma} \right],$$

(4.57)

where ‘·’ indicates the time derivative ‘$u^{\sigma} \nabla_{\sigma}$’ with respect to the parameter $t$ of the one-parameter family of embeddings, $\Pi_{\sigma}$ are the kinematical momentum densities conjugate to the supplementary embedding variables $\vartheta^{\sigma}$, $\dot{\vartheta}^{\sigma}$ being geometrically a vector field on $\mathcal{M}$ makes $\Pi_{\sigma}$ a 1-form density of weight one on $\mathcal{M}$, $N^{\sigma}$ is a Lagrange multiplier field, $N^{\sigma} H_{\sigma}$ defined by

$$N^{\sigma} \left[ \Pi_{\sigma} + \dot{H}_{\sigma} \right] =: N^{\sigma} H_{\sigma} \equiv H := \Pi^{\sigma} \| \dot{A} + \frac{\Pi}{\Pi} \dot{u} + \mathbf{P} \dot{\psi} + \mathbf{P} \dot{\vartheta} + \Pi_{\sigma} \dot{\vartheta} - \mathcal{L},$$

(4.58)

(with $\dot{H}_{\alpha}$ being a functional of the original ‘un-parameterized’ canonical data as well as $\dot{\vartheta}^{\sigma}$ representing the combined gravitational and material ‘energy-momentum’ flux through the surface element $\varphi u^{\alpha}$ of the embedding) is the Hamiltonian density giving the Hamiltonian functional,

$$I_{\Pi} = \int_{\Sigma_{t}} N^{\sigma} H_{\sigma} \ d^{3}x,$$

(4.59)

which governs the dynamical evolution of the system, the Lagrangian density $\mathcal{L}$ is given by equation (4.4) with the fields evaluated on $\Sigma_{t}$, and $\varphi d^{3}x$ is the volume element in $\Sigma_{t}$. As always, the generalized velocities appearing both explicitly and implicitly in these expressions are viewed as functions of the phase-space variables only. The tensor field $\chi^{\mu \nu} := \| \chi^{\mu \nu} \|$ appearing in $\mathcal{L}$, however, is viewed not as a multiplier field, but simply as a function of the canonical variables $\Pi^{\mu}$ (cf. equation (4.19)). The variables $\Upsilon_{\mu}, \Upsilon_{\nu}^{\sigma}, \Upsilon^{\mu \nu}, \tilde{\Upsilon}^{\mu \nu}, N^{\sigma}, \perp A, \zeta, \chi^{\alpha \mu \nu}$, and $\chi$, on the other hand, are viewed as non-dynamical multiplier variables, and, hence, the Hamiltonian equations of motion,

$$\frac{\delta H}{\delta \Pi_{\alpha}} = + u^{\sigma} \nabla_{\sigma} \vartheta^{\alpha}, \quad \frac{\delta H}{\delta \vartheta^{\alpha}} = - u^{\sigma} \nabla_{\sigma} \Pi_{\alpha},$$

(4.60)
\[
\frac{\delta H}{\delta \| A_\alpha} = -u^\sigma \nabla_\sigma \| \Pi^\alpha, \quad \frac{\delta H}{\delta u^\alpha} = -u^\sigma \nabla_\sigma \bar{u}^\alpha, \quad (4.61)
\]

\[
\frac{\delta H}{\delta \| \Pi^\alpha} = +u^\sigma \nabla_\sigma \| A_\alpha, \quad \frac{\delta H}{\delta u^\alpha} = +u^\sigma \nabla_\sigma u^\alpha, \quad (4.62)
\]

\[
\frac{\delta H}{\delta \Psi} = -u^\sigma \nabla_\sigma P, \quad \frac{\delta H}{\delta P} = +u^\sigma \nabla_\sigma \Psi, \quad (4.63)
\]

and complex conjugates of the latter two, must be supplemented by the equations

\[
\frac{\delta H}{\delta \Upsilon_\mu} = 0, \quad \frac{\delta H}{\delta \Upsilon^\mu_\sigma} = 0, \quad \frac{\delta H}{\delta \Upsilon^{\mu\nu}} = 0, \quad \frac{\delta H}{\delta \bar{\Upsilon}^{\mu\nu}} = 0, \quad (4.64)
\]

\[
\frac{\delta H}{\delta N^\sigma} = 0, \quad \frac{\delta H}{\delta \perp A} = 0, \quad \frac{\delta H}{\delta \zeta} = 0, \quad \frac{\delta H}{\delta \chi^{\alpha\mu\nu}} = 0, \quad \text{and} \quad \frac{\delta H}{\delta \bar{\chi}} = 0, \quad (4.65)
\]

giving rise to constraints on the phase-space. Note that, since \(\perp u \equiv 0\), no additional constraint corresponding to \(\perp u\) need be appended to the set of the Hamiltonian equations of motion. On the other hand, we do have to add the linear constraints

\[
\Pi^\mu + h^{\mu\nu} u^\nu \Pi_\nu = 0
\]

or

\[
\bar{u}^\mu + \bar{h}_{\mu\nu} \Pi^\nu = 0 \quad (4.66)
\]

and

\[
\bar{P} + 2i \hbar \varphi \pi G \bar{\Psi} = 0
\]

or

\[
P - 2i \hbar \varphi \pi G \Psi = 0 \quad (4.67)
\]

to the above list (cf. equations (4.25) and (4.50)). Here we do not bother to classify these constraints à la Dirac [38] because our eventual goal is to get rid of all of them in order to obtain a constraint-free phase-space.

The first four of these eleven constraint equations, (4.64), immediately yield the conditions (4.8) specifying the Galilean structure. Among the remaining equations, the fifth, \(\frac{\delta H}{\delta N^\sigma} = 0\), imparts the kinematical constraint

\[
H^\sigma = \Pi^\sigma + \bar{H}^\sigma = 0 \quad (4.68)
\]

due to the parameterization process, the sixth, \(\frac{\delta H}{\delta \perp A} = 0\), gives the equation (4.22), the seventh, \(\frac{\delta H}{\delta \zeta} = 0\), reproduces the normalization condition \(u^\nu t_\nu = 1\) defining the foliation,
the eighth, \( \frac{\delta H}{\delta \chi^\mu} = 0 \), leads to the constraint (4.12) prohibiting rotational holonomy, and the ninth, \( \frac{\delta H}{\delta \chi} = 0 \), asserts the constraint (4.26).

Among the equations of motion, the first one of the equations (4.60) tells us that the multiplier field \( \mathcal{N}^\sigma \) is nothing but the generalized velocity field \( \dot{\vartheta}^\sigma \),

\[
\mathcal{N}^\sigma = \dot{\vartheta}^\sigma ,
\]

whereas the second one, after using this identification, leads to the condition \( \dot{H}_\sigma = 0 \), which guarantees that the constraint (4.68) is preserved in time. Each of the equations of motion (4.61) leads to the field equation (4.22) again (where the second of these derivations, reminiscent of the derivation leading to equation (4.18) of the previous subsection, requires a little more work compared to the first one), and the combined effect of the equations of motion (4.62) is nothing but the field equation (4.9). Finally, both of the remaining two equations of motion (4.63), as well as their complex conjugates, give the Schrödinger-Kuchař equation (4.31), and its complex conjugate, respectively. As we elaborated in the previous subsection, given the relation (4.26) implied by the last of the five equations (4.65), equations (4.9) and (4.22) implied by equations (4.62) and (4.61) are equivalent to the Newton-Cartan field equations (4.10) and (4.30), respectively. Furthermore, the constraint (4.22) on the Cauchy data implied by the second of the five equations (4.65) is no constraint at all, because it is automatically satisfied by both of the Hamiltonian equations of motion (4.61); the constraint \( u^{\nu}t_\nu = 1 \) implied by the third of these equations, on the other hand, is trivially satisfied if we choose \( u^{\nu} \) to be a unit timelike vector-field, as we have done.

The presence of the remaining constraints, of course, indicate that our phase-space is still too large and we have not isolated the ‘true’ dynamical degrees of freedom in our choice of configuration space. Note, however, that, among the remaining constraints, the set (4.8) implied by equations (4.64) consists of purely kinematical constraints in that its elements do not impose any relations among the genuinely dynamical degrees of freedom. Each of the constraint functions \( h^{\mu\nu}t_\nu, \nabla_\mu h^{\nu\sigma}, \nabla_\mu t_\nu, \) and \( \partial_{[\mu} t_{\nu]} \) has vanishing variations with respect to the dynamical variables \( A_\mu, u^{\nu}, \Psi, \text{ and } \overline{\Psi} \) (recall that the Newton-Cartan connection appearing in the second and third of these functions is invariant under the changes of gauge variables \( A_\mu \) and \( u^{\nu} \)). This means that these are simply constant functions
on the phase-space. In fact, we could have taken them as part of the background structure on the "un-parameterized" manifold $\mathcal{M}$ before pulling them back with the fields by the map $\vartheta : \Sigma \to \mathcal{M}$. Consequently, if we ignore for the moment the kinematical constraints (4.68) and the linear constraints (4.66) and (4.67), what we have in hand is a Hamiltonian formulation of the classical Newton-Cartan-Schrödinger theory with the constraints (4.11) and (4.26) on the Cauchy data due to the last two of the constraint equations (4.65). If we use the field equation (4.22) implied by the Hamiltonian equations of motion (4.61) to substitute for $\nabla_\sigma \Theta^\sigma$ in the constraint (4.26), then it takes the form

$$C(\kappa) := \frac{1}{\varrho} \nabla_\sigma \Pi^\sigma + \Lambda_N - (1 - \kappa) 4\pi G \rho = \kappa \Lambda,$$

where recall that $\kappa$ is an arbitrary free parameter.

Now, we have seen that Newton-Cartan connection is invariant under internal gauge transformations $A_\mu \mapsto A_\mu + \partial_\mu f$ and boost transformations (4.39) over and above the diffeomorphisms $\phi \in \text{Diff}(\mathcal{M})$ (cf. equation (2.44)). Physically this implies that the pairs $\{A_\mu, u^\nu\}$ and $\{(A_\mu + \partial_\mu f + w_\mu - (u^\sigma w_\sigma + \frac{1}{2} h^{\alpha \sigma} w_\alpha w_\sigma) t_\mu), (u^\nu + h^{\nu \sigma} w_\sigma)\}$ of gravitational variables represent the same physical configuration of the gravitational field. On the other hand, as we noted in section 3, under the vertical transformations (2.44) the Schrödinger-Kuchař field transforms as $\Psi \mapsto \exp(i m \bar{\hbar} f) \Psi$, and, hence, the pair $\{\Psi, \bar{\Psi}\}$ representing matter fields is physically equivalent to the pair $\{\exp(i m \bar{\hbar} f) \Psi), (\exp(-i m \bar{\hbar} f) \bar{\Psi})\}$. Therefore, we must replace the configuration space $Z$ of our composite physical system by the space $\tilde{Z}$ of equivalence classes of representatives of the physical fields, where all representatives of the fields are defined to be equivalent if they differ only by the interconnecting vertical gauge transformations (2.44). Now recall that in the canonical formalism, quite generally, the space of possible momenta of a physical system at a given configuration point $z$ is the cotangent space $T_z^*Z$ of the configuration space at that point, and, consequently, the corresponding phase-space is a cotangent-bundle $T^*Z$ over the configuration space endowed with a natural presymplectic structure (cf. appendix A); where, given a configuration manifold $Z$ and a point $z$ on that manifold, a cotangent vector $\mathcal{P}$ at the point is defined to be a real linear map, $\mathcal{P} : T_zZ \to \mathbb{R}$, from the tangent space at that point to the set of real numbers. For example, in the particular case under consideration, the momentum density $\Pi^\mu$ is a cotangent vector at the point $A_\mu$ of the configuration space.
which maps the tangent vector $\delta A_\mu$ at $A_\mu$ into $\mathbb{R}$ via $\delta A_\mu \rightarrow \int_{\Sigma_t} \Pi^\mu \delta A_\mu$. If we now take our configuration space to be the space $\tilde{Z}$ of equivalence classes of representatives of the physical fields as defined above, then the cotangent space at a configuration point would be the space of real linear functions of variations of the physical fields which are independent of the vertical gauge transformations (2.44). Consequently, the momentum densities would be represented by those vector fields which leave the integral

$$\int_{\Sigma_t} d^3x \left( \Pi^\mu \delta A_\mu + \Pi_{\nu} \delta u^\nu + \overline{\Pi} \delta \Psi + P \delta \overline{\Psi} + \Pi_\sigma \delta \vartheta^\sigma \right)$$

(4.71) invariant under the transformations (2.44), where we freely use the convenient fact that $||\Pi^\sigma \delta A_\sigma \equiv \Pi^\sigma \delta A_\sigma$. However, it is easy to see by direct substitutions for the expressions in the integrand (and using equation (2.11)) that this integral is left invariant under (2.44), or, effectively, under

$$A_\mu \rightarrow A_\mu + \nabla_\mu f$$

$$\Psi \rightarrow \exp \left( i \frac{m}{\hbar} f \right) \Psi ,$$

(4.72)

if and only if the momentum densities satisfy the condition

$$\frac{1}{\varphi} \nabla_\sigma \Pi^\sigma - 4\pi G \rho = 0 .$$

(4.73)

Next, we wish to further restrict our configuration space and admit only those configuration variables which satisfy the constraint (4.11) implied by the fourth of the constraint equations (4.65). Now, as discussed in the previous subsection (cf. expressions (4.21) and (4.24)), this restriction amounts to an addition of a couple of 4-divergence terms in the integral (4.71), which, however, remains unaffected by them because of the vanishing of the boundary of a boundary theorem. Nevertheless, the ensuing condition (4.73) is modified under the transformations (4.72), and becomes

$$\frac{1}{\varphi} \nabla_\sigma \Pi^\sigma + \Lambda_N - 4\pi G \rho = 0 ,$$

(4.74)

with the $\Lambda_N$ term arising from the addition of the 4-divergence (4.21) in the integral (4.71).

But this is just the constraint (4.70) in the limit $\kappa \rightarrow 0$:

$$\lim_{\kappa \rightarrow 0} C(\kappa) = \frac{1}{\varphi} \nabla_\sigma \Pi^\sigma + \Lambda_N - 4\pi G \rho = 0$$

(4.75)
(recall that, as far as the field equations and their derivations are concerned, we are free to choose any desired value for \( \kappa \) without loss of generality). Further, in this harmless limit, not only the field equation (4.22) and the constraint equation (4.70) become identical, but also the Lagrangian density (4.4) becomes invariant under the full automorphism group (2.45). In summary, by eliminating the spurious gauge-arbitrariness in the configuration space by working rather with equivalence classes of field variables, we have been able to liberate our phase-space from the constraints (4.11) and (4.70). In fact we have arrived at the following set of results: in the light of equation (4.74) being a direct consequence of the invariance of the integral (4.71) under the vertical gauge transformations (2.44), the limit \( \kappa \to 0 \), (1), eliminates the unwanted constraint (4.70) on the Cauchy data, (2), dictates that the constituents of the configuration space of the system are automatically entire classes of representatives of the physical fields which, in addition, satisfy the constraint (4.11), (3), makes the Lagrangian density invariant under the full group \( \mathcal{A}\text{ut}(B(\mathcal{M})) \), and, by virtue of (2), (3) and \( \| \Pi^\sigma \| \dot{A}_\sigma \equiv \Pi^\sigma \dot{A}_\sigma \), (4), renders the Hamiltonian functional (4.59) manifestly invariant under the vertical gauge transformations (2.44). In the remaining of this paper the limit \( \kappa \to 0 \) will be understood to have been taken.

Let us now turn our attention to the remaining linear constraints [4.66] and [4.67]. To eliminate these constraints, we further reduce our phase-space by defining a new, more appropriate set of canonical variables \( \{ v^\alpha, \pi_\alpha; \psi, p; \vartheta^\sigma, \Pi_\sigma \} \), where

\[
\begin{align*}
v^\alpha & := u^\alpha - h^{\alpha\sigma} A_\sigma = u^\alpha - h^{\alpha\sigma} \| \dot{A}_\sigma \| , \\
\pi_\alpha & := \| \Pi_\alpha \| = \delta_\alpha^\mu \Pi_\mu = \Pi_\mu = - h^{\alpha\sigma} \Pi^\sigma ; \\
\psi & := \frac{1}{\hbar \sqrt{8\pi G \varphi}} \left( 2\pi G \varphi \hbar \Psi + i \mathcal{P} \right) , \\
and \quad p & := \frac{1}{\sqrt{8\pi G \varphi}} \left( \mathcal{P} + i 2\pi G \varphi \hbar \Psi \right) ,
\end{align*}
\]

so that equations [4.67] yield

\[
\begin{align*}
\psi & = \sqrt{2\pi G \varphi} \Psi \quad \text{and} \quad p = i \hbar \psi \quad (4.77)
\end{align*}
\]

(cf. equations (4.46), (4.49), and [3.12]). In terms of these new canonical variables the Hamiltonian density (4.58) translates to be

\[
\begin{align*}
N^\sigma H_\sigma & \equiv H := \pi_\sigma \dot{v}^\sigma + 2 p \dot{\psi} + \Pi_\sigma \dot{\vartheta}^\sigma - \mathcal{L}(v^\alpha, \pi_\alpha; \psi, p; \vartheta^\sigma, \Pi_\sigma) ,
\end{align*}
\]

(4.78)
which can be rewritten in the familiar form as

$$\mathcal{H}_\sigma \equiv H := \pi_\sigma \dot{v}^\sigma + p \dot{\psi} + \Pi_\sigma \dot{\vartheta}^\sigma - \mathcal{L}^c(v^\alpha, \pi_\alpha; \psi, p; \vartheta^\sigma, \Pi_\sigma), \quad (4.79)$$

with

$$\mathcal{L}^c := \mathcal{L} - p \dot{\psi} \quad (4.80)$$

being the ‘correct’ (or ‘constraint-free’) Lagrangian density. (Note that $\mathcal{P} \dot{\Psi} + \mathcal{P} \dot{\Psi}$ translates into $2p \dot{\psi}$ only up to a total time derivative, which, of course, does not affect the action.) The ‘correct’ Hamiltonian form of the parameterized action functional now reads

$$I^c = \int_{\mathbb{R}} dt \int_{\Sigma_t} d^3x \left[ \pi_\sigma \dot{v}^\sigma + p \dot{\psi} + \Pi_\sigma \dot{\vartheta}^\sigma - \mathcal{H}_\sigma \right], \quad (4.81)$$

with the Hamiltonian equations of motion

$$\begin{align*}
\frac{\delta H}{\delta \Pi_\alpha} &= + u^\sigma \nabla_\sigma \vartheta^\alpha, \quad &\frac{\delta H}{\delta \vartheta^\alpha} &= - u^\sigma \nabla_\sigma \Pi_\alpha, \quad (4.82a) \\
\frac{\delta H}{\delta \pi_\alpha} &= + u^\sigma \nabla_\sigma v^\alpha, \quad &\frac{\delta H}{\delta v^\alpha} &= - u^\sigma \nabla_\sigma \pi_\alpha, \quad (4.82b) \\
\text{and} \quad \frac{\delta H}{\delta p} &= + u^\sigma \nabla_\sigma \psi, \quad &\frac{\delta H}{\delta \psi} &= - u^\sigma \nabla_\sigma p. \quad (4.82c)
\end{align*}$$

It is easy to check that this set of equations of motion is equivalent to the previous set with the same Hamiltonian density, but, of course, expressed in terms of the old set of somewhat redundant canonical variables. The Hamiltonian density (4.79) clearly suggests the ‘correct’ Lagrangian form of the action functional,

$$I^c = \int d^4x \mathcal{L}^c(v^\sigma, \nabla_\mu v^\sigma, \nabla_\mu \nabla_\nu v^\sigma, \psi, \partial_\mu \psi; (s)y), \quad (4.83)$$

which, a priori, might be taken to indicate that we would have been better off taking from the beginning the alternative form

$$\mathcal{L}_\Psi^c = + \Psi 4\pi G \left\{ \frac{\hbar^2}{2m} h^{\alpha\beta} \partial_\alpha \Psi \partial_\beta \overline{\Psi} - i \frac{\hbar}{2} u^\alpha (\overline{\Psi} \partial_\alpha \Psi) \right\} \quad (4.84)$$

of the component $\mathcal{L}_\Psi$ in the action integral (4.3). However, this form of the Lagrangian density is not Hermitian and breaks the symmetry which naturally exists between $\Psi$ and $\overline{\Psi}$. Moreover, it contains an incorrect factor of $\frac{1}{2}$ in the second term. Nevertheless, in the end, it is this form (4.84) which turns out to be the most appropriate one as far as the new set of canonical variables are concerned.
Finally, what about the last remaining constraint — the diffeomorphism constraint (4.68) due to the parameterization process — which we have left on hold so far? It turns out that this constraint (4.68) can also be readily untangled and, unlike the parallel case in the Hamiltonian formulation of the ‘already parameterized’ Einstein’s theory of gravity, here the process of ‘deparameterization’ [38] can be easily carried out. This is because in our case the kinematical momentum densities $\Pi_\sigma$ conjugate to the supplementary embedding variables $\vartheta^\sigma$ appear linearly in the expression of the constraint, neatly segregating themselves from the set of genuinely dynamical variables. Consequently, all one has to do to recover the ‘true’ Hamiltonian form of the physical action is to solve the constraint for these momentum densities and get $\Pi_\sigma = -\overset{\circ}{H}_\sigma$, substitute this result into the Hamiltonian action functional (4.81), and then impose the relation (4.69) (obtained from one of the equations of motion) on the resulting expression to complete the deparameterization.

Thus, we have been able to eliminate all redundancies from the configuration space by redefining it to consist only true dynamical degrees of freedom, and, as a result, succeeded in constructing a meaningful (constraint-free) phase-space for our Newton-Cartan-Schrödinger system. Let us now recapitulate the main features of this phase-space. First, and foremost, as there are no topological complications ($\Sigma_t \cong \mathbb{R}^3$), the phase-space is naturally a cotangent-bundle $T^*\tilde{Z}$ over the infinite-dimensional complex configuration space $\tilde{Z}$ of equivalence classes of fields $\{\tilde{v}^\sigma(x); \tilde{\psi}(x)\}$ evaluated on the 3-submanifold $\Sigma$ of $\mathcal{M}$:

$$\tilde{Z} := \{v^\sigma; \psi \mid t_\sigma v^\sigma = 1, \nabla_\mu \nabla_\nu v^\sigma = 0, v^\sigma \sim v^\sigma - h^{\sigma\mu} \nabla_\mu f; \psi \sim \exp(-i\frac{\pi}{\hbar} f) \psi\}, \quad (4.85)$$

where $\nabla_\mu \nabla_\nu v^\sigma = 0$ is equivalent to the condition (4.11), and $v^\sigma \sim v^\sigma - h^{\sigma\mu} \nabla_\mu f$ results form the equivalence (4.72). What is more, this cotangent-bundle possesses a well-defined, non-degenerate symplectic structure, which is just its natural symplectic structure

$$\tilde{\omega} = \int_{\Sigma_t} d^3 x \left[ d_z \tilde{\pi}_\mu \wedge d_z \tilde{v}^\mu + d_z \tilde{\rho} \wedge d_z \tilde{\psi} \right] \quad (4.86)$$

with $\tilde{v}^\alpha$, $\tilde{\pi}_\alpha$, $\tilde{\psi}$, and $\tilde{\rho}$ representing entire gauge-equivalence classes of fields as discussed above. Note that, although $\tilde{Z}$ has a natural vector-space structure over the complex field $\mathfrak{C}$, the 2-form $\tilde{\omega}$ is real-valued since $p = i \times \overline{\psi}$.

As it stands, however, it is not immediately clear whether this description of symplectic structure is a generally-covariant one. In fact it appears not to be a covariant one on two
counts. First of all, in the present subsection we have been working strictly within a non-covariant 3+1 decomposition of spacetime. Secondly, the symplectic structure (4.86) we have arrived at in the end corresponds to the deparameterized action — i.e., an action effectively corresponding to the original unparameterized theory on a fixed background \((\mathcal{M}; h, \tau)\). As we shall see in the next subsection, however, the expression (4.86) in fact provides a truly generally-covariant description of the phase-space, provided it is viewed more appropriately.

4.3. Manifestly covariant description of the canonical formulation

In the previous subsection we have constructed a canonical phase-space for the Newton-Cartan-Schrödinger system by working within a 3+1 decomposition of spacetime into 3-spaces at instants of time. Such a blatantly non-covariant breakup of spacetime is admittedly somewhat natural for a Newtonian theory, but it undermines the efforts of Cartan and followers to give a full spacetime-covariant meaning to Newtonian gravity by violating manifest covariance of even the old-fashioned Galilean kind. Therefore, at least for the sake of Cartan’s legacy if not for anything else, it is desirable to seek a manifestly covariant version of the canonical phase-space constructed in the previous subsection. Fortunately, it has long been recognized — dating all the way back to Lagrange [51] — that the phase-space of a physical system is better viewed as the space of entire dynamical histories of the system, without reference to a particular instant of time. With this view of phase-space, the core of the canonical formalism can be developed in a manner that manifestly preserves all relevant symmetries of a given classical system [44,45,49,50,51]. Since this ‘covariant phase-space formalism’ is relatively less-popular, and since we shall be applying it to our Newton-Cartan-Schrödinger system in the present subsection as well as in the subsection 5.2 below when we quantize the system, we have briefly reviewed its main concepts in the appendix A below setting our notational conventions. We urge the reader not familiar with these concepts to carefully study the appendix before reading any further in order to appreciate the elegance of the underlying ideas used in what follows.
The essence of the covariant phase-space — the space $\mathcal{Z}$ of solutions of the equations of motion of a theory — is most succinctly encapsulated in a closed, non-degenerate symplectic 2-form $\tilde{\omega}$ defined on the quotient space $\tilde{\mathcal{Z}} := \mathcal{Z}/K$, where $K$ is the characteristic distribution defined by equation (A.21) of the appendix. As a first step towards evaluating $\tilde{\omega}$ for our Newton-Cartan-Schrödinger system, we work out the presymplectic potential current density (A.12) for the generally-covariant action functional (4.83),

\[
J_{\alpha}^\mu = J_{v}^\mu + J_{\psi}^\mu + J_{(s)y}^\mu ,
\]

giving

\[
J_{\alpha}^\mu t_{\alpha} = \pi_{\mu} \delta v^\mu + p \delta \psi + (s)\Pi_{\sigma} \delta (s)y^{\sigma} .
\]

If we assume that $J_{\alpha}^\mu \to 0$ at spatial infinity (or work with a compact $\Sigma_t$), then the corresponding presymplectic 2-form $\omega$ is given by the equation (A.7), with $\omega^\mu = dz J_{\alpha}^\mu$ (cf. equation (A.19)):

\[
\omega = \int_{\Sigma_t} d^3x \left[ \omega^\mu t_\mu \right] = \int_{\Sigma_t} d^3x \left[ dz \pi_\mu \wedge dz v^\mu + dz p \wedge dz \psi + dz (s)\Pi_{\sigma} \wedge dz (s)y^{\sigma} \right].
\]

Now, in accordance with the discussion around equation (A.21) of the appendix, this presymplectic structure has a degenerate direction for each infinitesimal gauge transformation of the theory stemming from the action of the automorphism group $\text{Aut}(B(M))$. Consequently, we seek the non-degenerate projection $\tilde{\omega}$ of $\omega$ on the physically relevant reduced phase-space $\tilde{\mathcal{Z}} = \mathcal{Z}/\text{Aut}(B(M))$, which is nothing but the space of orbits of the group $\text{Aut}(B(M))$ in the solution-space $\mathcal{Z}$. Fortunately, the complete automorphism group has the structure of a semidirect product: $\text{Aut}(B(M)) = \mathcal{V}(B(M)) \rtimes \text{Diff}(M)$, where $\mathcal{V}(B(M))$ is the group of vertical gauge transformations given by (2.46). This makes it possible to discuss effects of the two subgroups successively, since the isomorphism

\[
\mathcal{Z}/\text{Aut}(B(M)) \cong [\mathcal{Z}/\mathcal{V}(B(M))]/\text{Diff}(M)
\]

holds. First note that, in analogy with the discussion in the previous subsection, the quotient space $\mathcal{Z}/\mathcal{V}(B(M))$ is the space of solutions modulo vertical gauge directions in which the integral $\int_{\Sigma_t} d^3x J_{\alpha}^\mu t_{\alpha}$ is rendered gauge-invariant (cf. equation (4.71)). In terms of the projection map $\mathbb{P} : \mathcal{Z} \to \tilde{\mathcal{Z}}$, this implies

\[
\mathbb{P}(v) = \tilde{v}, \quad \mathbb{P}(\pi) = \tilde{\pi}, \quad \mathbb{P}(\psi) = \tilde{\psi}, \quad \text{and} \quad \mathbb{P}(p) = \tilde{p}.
\]
Next, a globally valid gauge representing the moduli space \([\mathcal{Z}/\mathcal{V}(\mathcal{B}(\mathcal{M}))]/\text{Diff}(\mathcal{M})\) can be easily constructed by simply taking \((s)y : \mathcal{M} \rightarrow \hat{\mathcal{M}}\) to be a fixed diffeomorphism \((o)y\) (e.g., \((o)y = \text{identity}\)) and then specifying the values of the dynamical variables on \(\Sigma_t\) with respect to this choice. This is possible because, as in a parameterized scalar field theory in Minkowski spacetime \([44,46]\), any arbitrary variation of the diffeomorphism \((s)y\) sweeps out the entire degeneracy submanifold of \(\omega\), allowing us to uniquely characterize each degeneracy submanifold by a fixed diffeomorphism \((o)y\). But this immediately renders \(J^\mu_{(o)y} = 0\) (cf. equation (A.12)). Therefore, the net result of projecting the presymplectic 2-form \(\omega\) on the reduced phase-space \(\tilde{\mathcal{Z}} = \mathcal{Z}/\text{Aut}(\mathcal{B}(\mathcal{M}))\) by the projection map \(\mathbb{P} : \mathcal{Z} \rightarrow \tilde{\mathcal{Z}}\) is the desired non-degenerate 2-form

\[
\tilde{\omega} = \int_{\Sigma_t} d^3x \left[ dz_\mu \pi^{\mu} + dz_\nu \pi^{\nu} + d\tilde{p} \wedge d\tilde{\psi} \right].
\]

(4.92)

It is easy to see that \(\omega\) in expression (4.89) is a unique pull-back \(\mathbb{P}^* (\tilde{\omega})\) of \(\tilde{\omega}\) form \(\tilde{\mathcal{Z}}\) to \(\mathcal{Z}\). In other words, we have

\[
\gamma_A \downarrow \tilde{\omega} = 0
\]

(4.93)

(cf. equation (A.23)), where \(\gamma_A\) are the gauge directions corresponding to the complete automorphism group \(\text{Aut}(\mathcal{B}(\mathcal{M}))\). Again in close analogy with the covariant phase-space description of a parameterized scalar field theory in Minkowski spacetime \([44,46]\), this non-degenerate symplectic 2-form \(\tilde{\omega}\) describes precisely the phase-space of the original unparameterized theory in the fixed, non-dynamical spacetime \((\mathcal{M}; h^\mu_\nu, t_\mu)\). Thus, the reduction procedure in the present case has the effect of ‘deparameterizing’ the parameterized theory, just as in the case of a scalar field theory. However, unlike in the previous subsection, here our description is manifestly covariant. Further, the resulting constraint submanifold \([44]\) \(\tilde{\mathcal{Z}}\) of the dynamically possible states of the system permeates all of the space \(\tilde{\mathcal{Z}}\) of kinematically possible states because the constraints have been eliminated (cf. equation (A.24)). In fact, as comparison of equations (4.86) and (4.92) immediately suggests, the covariant phase-space \(\tilde{\mathcal{Z}}\) of the present subsection is symplectically diffeomorphic.
to the constraint-free canonical phase-space $T^*\tilde{Z}$ of the previous subsection:

$$\tilde{Z} \simeq T^*\tilde{Z}. \quad (4.94)$$

It is worth recalling here that the symplectic structure $\tilde{\omega}$ is independent of the choice of a hypersurface $\Sigma_t$, and as such, in particular, it is Milne-invariant (cf. equation (2.54)). More generally, $\tilde{\omega}$ is rather trivially invariant under the action of $\text{Diff}(\mathcal{M})$ on $\mathcal{Z}$ since all ingredients in its expression above transform homogeneously, like tensors. In short, we have been able to successfully crystallize the diffeomorphism-invariant canonical essence of Newton-Cartan-Schrödinger system in the expression (4.92).

5. Quantization of the Newton-Cartan-Schrödinger system

So far we have considered only classical, external gravitational field. However, as noted before, in Newton-Cartan theory the connection-field is a dynamical object: it is not just a part of the immutable background structure, but depends crucially on the distribution of matter-sources via the field equation $R_{\mu\nu} = 4\pi G M_{\mu\nu}$. Since this equation dictates the coupling of spacetime curvature to quantum mechanically treated matter, the Newton-Cartan connection cannot have an \textit{a priori} definite value and must itself be treated in a quantum mechanical fashion. Thus, a consistent account of physical phenomena even at a Galilean-relativistic level \textit{necessitates} the construction of a quantum theory of gravity in which the superposition principle holds not only for the states of matter, but also for the states of the Newton-Cartan connection-field. In what follows we construct such a generally-covariant, Milne-relativistic quantum theory of gravity in which quantized Schrödinger particles produce the \textit{quantized} Newton-Cartan connection-field through which they interact.

5.1. Covariant phase-space quantization

Having successfully identified the constraint-free phase-space for our classical Newton-Cartan-Schrödinger system in the previous section, the desired quantum theory can be easily constructed using a manifestly covariant approach to the usual canonical quantization method. An accessible reference on the general procedure is [Ref. 54]. Recall that classical observables, say $0$, are maps from the phase-space $\tilde{Z}$ to $\mathbb{R}$, and the non-degenerate symplectic 2-form $\tilde{\omega}$ on $\tilde{Z}$ determines a set of Poisson brackets inducing a Lie-algebra
structure on the space of these observables. To quantized such a system, we are supposed to replace the classical observables with operators and Poisson brackets with commutators providing a corresponding algebraic structure on the space of these operators. More precisely, we are to seek a correspondence map, \( \hat{\mathcal{O}} \mapsto \hat{\mathcal{O}} \), and look for an irreducible representation of the Lie-algebra of classical observables as an algebra of operators acting on elements of some separable Hilbert space \( \mathcal{H} \). It is well-known, however, that, as stated, this programme cannot be carried out in general. As early as in 1951 van Hove demonstrated that, for theories in which the position and momentum operators are represented in the standard manner, no such correspondence map can provide an irreducible representation of the full Poisson algebra of classical observables [55]. Fortunately, in our case the phase-space is naturally isomorphic to a cotangent-bundle, \( \tilde{\mathbb{Z}} \cong T^*\tilde{\mathbb{Z}} \), for which the van Hove obstruction is neutralised. Consequently, for us, it will turn out to be possible to choose a Hilbert space \( \mathcal{H} \) and a correspondence map \( \tilde{\mathcal{O}} \mapsto \hat{\mathcal{O}} \) such that the non-vanishing commutators satisfy

\[
\left[ \tilde{\upsilon}^\mu (x), \tilde{\pi}_\nu (x') \right] = i\hbar \hat{\mathcal{H}} \delta^\mu_\nu \delta(\vec{x} - \vec{x}') \quad \text{and} \quad \left[ \tilde{\psi} (x), \tilde{\psi}^\dagger (x') \right] = \hat{\mathcal{H}} \delta(\vec{x} - \vec{x}') \tag{5.1}
\]

at equal-times (cf. equation (3.14)). The appearance of the Dirac delta-‘function’ here necessitates that \( \tilde{\upsilon}^\mu, \tilde{\pi}_\nu, \tilde{\psi}, \) and \( \tilde{\psi}^\dagger \) must all be viewed as operator-valued distributions, and indicates that only the fields smeared with appropriate test-functions have physical meaning as observables in accordance with the well-known analysis of measurements of field-observables developed by Bohr and Rosenfeld [57]. As they stand, however, these commutation relations are clearly not expressed in a manifestly covariant manner. This deficiency can be quite easily removed in our case, partly by exploiting the natural vector-space structure of the phase-space \( \tilde{\mathbb{Z}} \) noted in the previous section. On account of this vector-space structure of \( \tilde{\mathbb{Z}} \), we may identify the tangent space at any point \( z \in \tilde{\mathbb{Z}} \) with \( \tilde{\mathbb{Z}} \) itself. Furthermore, as discussed in the appendix, under this identification the symplectic form \( \tilde{\omega} \) becomes an antisymmetric bilinear function, \( \tilde{\omega} : \tilde{\mathbb{Z}} \times \tilde{\mathbb{Z}} \to \mathbb{R} \), on the resultant

---

For definiteness and simplicity, we shall only follow the bosonic case here. In this Galilean-relativistic context, where there is no connection between spin and statistics, a parallel discussion with a fermionic field is relatively straightforward (see subsection 4.7 of Ref. 54 for a fermionic treatment in the relativistic case).
symplectic vector-space $\widetilde{Z}$. Thus, we may rewrite the commutation relations (5.1) in terms of the operators $\hat{\omega}(Q, \cdot)$ corresponding to the functions $\tilde{\omega}(Q, \cdot)$ as the single commutator
\[
\left[ \hat{\tilde{\omega}}(Q_1, \cdot), \hat{\tilde{\omega}}(Q_2, \cdot) \right] = -i\hbar \hat{\tilde{\omega}}(Q_1, Q_2), \tag{5.2}
\]
where $Q, Q_1, Q_2 \in \widetilde{Z}$ (see Ref. 54 for further details on such translations). Since the self-adjoint operators appearing in this expression are unbounded, and, hence, only densely defined, it is convenient to work with the equivalent but better behaved Weyl relations,
\[
\hat{W}^\dagger(Q) = \hat{W}(-Q)
\]
and
\[
\hat{W}(Q_1) \hat{W}(Q_2) = \exp \left[ \frac{i\hbar}{2} \tilde{\omega}(Q_1, Q_2) \right] \hat{W}(Q_1 + Q_2), \tag{5.3}
\]
where
\[
W(Q) := \exp \left[ i \tilde{\omega}(Q, \cdot) \right] \tag{5.4}
\]
is unitary and varies with $Q$ in the ‘strong operator topology’ [60].

### 5.2. The GNS-construction and the choice of a vacuum state

It is well-known that the set, $\mathcal{B}(\mathcal{H})$, of all bounded linear maps on $\mathcal{H}$ has the natural structure of a C*-algebra with the ‘*'-operation’ corresponding to taking adjoints. The subalgebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$ generated by $\{ \hat{W}(Q) | Q \in \widetilde{Z} \}$ satisfying the above relations is called the Weyl algebra over the symplectic vector-space $\widetilde{Z}$. Each normalized, positive algebraic state $\zeta : \mathcal{A} \to \mathbb{C}$ over the Weyl algebra $\mathcal{A}$ — viewed as an abstract C*-algebra — determines a Hilbert space $\mathcal{H}_\zeta$ and a representation $\mathcal{R}_\zeta : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\zeta)$ of $\mathcal{A}$ by bounded linear operators acting in $\mathcal{H}_\zeta$, and thereby defines an Hermitian scalar product on $\mathcal{A}$ by
\[
\zeta(a^*b) := \langle a | b \rangle \quad \forall \ a, b \in \mathcal{A}. \tag{5.5}
\]
Conversely, each choice of a measure $\mu : \mathcal{H}_\zeta \times \mathcal{H}_\zeta \to \mathbb{R}$ generates a state $\zeta$ on the algebra $\mathcal{A}$. Consequently, the positivity and normalization conditions on $\zeta$ can be expressed as
\[
\zeta(a^*a) \geq 0 \quad \forall \ a \in \mathcal{A}, \quad \text{and} \quad \zeta(\mathbb{1}) = 1, \tag{5.6}
\]
where $\mathbb{1}$ denotes the identity element of $\mathcal{A}$. Such a construction of a representation of the Weyl algebra $\mathcal{A}$ as an algebra of operators on a Hilbert space $\mathcal{H}_\zeta$ is nothing but the
celebrated GNS-construction \[56\], which, given a cyclic vector \(|\xi^o\rangle \in \mathcal{H}_\zeta\), guarantees the existence of a representation \(\mathcal{R}_\zeta : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\zeta)\) such that

\[
\zeta_{\xi^o}(a) = \langle \xi^o | \mathcal{R}_\zeta (a) | \xi^o \rangle \quad \forall a \in \mathcal{A}.
\] (5.7)

The vector \(\xi^o\) is called a cyclic vector because the set \(\{ \mathcal{R}_\zeta(a)|\xi^o\rangle | a \in \mathcal{A} \}\) constitutes a dense subspace of \(\mathcal{H}_\zeta\). Upto unitary equivalence, the triplet \((\mathcal{H}_\zeta, \mathcal{R}_\zeta, |\xi^o\rangle)\) is uniquely determined by these properties, with the cyclic vector \(|\xi^o\rangle \in \mathcal{H}_\zeta\) corresponding to the identity element of \(\mathcal{A}\).

Thus, an algebraic state \(\zeta\) in \(\mathcal{A}\) can be easily represented in the Hilbert space \(\mathcal{H}_\zeta\) by the state-vector \(|\xi^o\rangle\). Note, however, that the states over an abstract \(\text{C}^*\)-algebra like \(\mathcal{A}\) come in families: any vector \(|\xi^i\rangle\) from a collection \(\{|\xi^i\rangle | |\xi^i\rangle \in \mathcal{H}_\zeta\}\) represents the family of algebraic state \(\zeta\) by

\[
\zeta_{\xi^i}(a) = \langle \xi^i | \mathcal{R}_\zeta (a) | \xi^i \rangle.
\] (5.8)

(More generally, we may consider the states \(\zeta_D(a) = \text{Tr}[\mathcal{D} \mathcal{R}_\zeta(a)]\), with \(\mathcal{D}\) being a positive trace-class operator in \(\mathcal{B}(\mathcal{H}_\zeta)\)). This is due to the fact that \(\zeta_{\xi^i}(a)\) can be approximated as closely as desired by \(\zeta_{\xi^o}(b^*ab)\), for any \(b \in \mathcal{A}\), since, thanks to the cyclicity of \(|\xi^o\rangle\), \(|\xi^i\rangle\) can be approximated as closely as desired by \(\mathcal{R}_\zeta(b)|\xi^o\rangle\):

\[
\zeta_{\xi^i}(a) = \langle \xi^i | \mathcal{R}_\zeta (a) | \xi^i \rangle \approx \langle \xi^o | \mathcal{R}_\zeta^* (b) \mathcal{R}_\zeta (a) \mathcal{R}_\zeta (b) | \xi^o \rangle = \zeta_{\xi^o}(b^*ab) .
\] (5.9)

Consequently, a GNS-representation crucially depends on the generating state \(\zeta\) of the system. In particular, with changes in the state \(\zeta\), the measure \(\mu\) — with respect to which the inner-product of the Hilbert space \(\mathcal{H}_\zeta\) has been defined — changes. But this is a bad news: measures with different null-sets would in general lead to operators with different norms in the corresponding representations, and the kernel (i.e., the set of those operators which are mapped onto zero) of the representations will correspondingly differ. Consequently, such representations (uncountably many of them) will be \textit{unitarily inequivalent} in general since the states which generate them could fail to determine quasi-equivalent measures — i.e., measures with the same null-sets. This, of course, is a well-known problem for quantum systems with infinitely many degrees of freedom, since the Stone-von Neumann uniqueness theorem is inapplicable for such systems \[54\].
Fortunately, if we are willing to let go a bit of the mathematical elegance maintained so far and find our way back to physics, there are two strategies at our disposal to tackle this problem. The first obvious strategy is to select a privileged cyclic state-vector using some physical criterion (some rule external to the quantum theory proper), and thereby obtain an equivalence class of representations which contains this distinguished state. The standard choice in Minkowskian quantum field theories is, of course, the vacuum state, $|\xi^0\rangle \equiv |0\rangle$, which is required to remain invariant under the action of the Poincaré group — the isometry group of the flat Minkowski spacetime. Then, through GNS-construction, the vacuum expectation values of all operators provide the Hilbert space equipped with a natural inner-product. This choice of a GNS-representation is then simply the Fock-representation. In our case it is most natural to use the Milne group defined by

$$0 = \mathcal{L}_x h^{\mu\nu} = \mathcal{L}_x t_\mu = \mathcal{L}_x \Gamma^\alpha_{\mu\nu}$$

(cf. equation (2.56)) in place of the Poincaré group in the above procedure, and take the Milne-invariant no-particle state as our privileged state. The Hilbert space we thereby obtain is a Milne-relativistic Fock-representation of the Weyl algebra for our Newton-Cartan-Schrödinger system.

The second strategy to deal with the problem of inequivalent representations is to follow the general ‘operational’ philosophy historically motivated by a result of Fell [58], and adopt his criterion of ‘physical equivalence’ as opposed to the strict mathematical equivalence. This too is a well-known strategy, a good accessible account of which can be found in Ref. 54 (see, especially, his Theorem 4.5.2). The idea is to first acknowledge the finiteness of accuracy and number of possible realistic measurements, and then realize that, due to these limitations, it is physically impossible to distinguish between the ‘in-equivalent’ representations of the C*-algebra $\mathcal{A}$. Consequently, a choice form the myriad of inequivalent representations is physically irrelevant, and the choice of Milne-invariant Fock-representation made above is as good as any.
5.3. The Hamiltonian operator: in general and in an inertial frame

Having settled the problem of inequivalent representations does not, of course, guarantee that every important observable needed to unambiguously define a quantum theory is contained in the algebra \( \mathcal{A} \). Therefore, let us verify that at least the most important observables of our theory are well-defined. We begin with one of the simplest operator: the covariant mass-density operator

\[
\hat{M}_{\mu\nu} := m \hat{\psi}^\dagger \hat{\psi} t_{\mu\nu}
\]

(cf. equation (3.18)). We have been careful to choose an appropriate ordering in defining this operator, and, as a result, it is manifestly well-defined. Similarly, the operator corresponding to the Riemann tensor (2.26) is also well-defined provided the operators corresponding to the gravitational field appearing in its polynomial expression are properly normal-ordered. On the other hand, the all important Hamiltonian-density operator

\[
\hat{H} := H \left( \hat{\bar{z}}_\alpha \bar{v}, \hat{\bar{z}}_\alpha ; \hat{\psi}, \hat{\bar{\psi}} \right)
\]

in our interacting theory involves both matter and gravitational field variables. Hence, \textit{a priori}, one might fear existence of potential operator ordering ambiguities in its expression. However, recall that all of the matter variables commute with all of the gravitational variables, removing any danger of intractable ordering ambiguity. Consequently, the Hamiltonian operator

\[
\hat{\mathcal{H}} := \int_{\Sigma_t} \hat{H} : d^3x
\]

is also well-defined, where matter and gravitational operators in \( \hat{H} \) are taken to be independently normal-ordered.

The vanishing commutation between matter and gravitational variables imply, in particular, that there is no gravitational self-interaction in this linear, Newtonian quantum gravity; i.e., the Schrödinger-Fock particles of the system do not gravitationally self-interact, though they interact among themselves. This result becomes most conspicuous from the perspective of an observer confined to a \textit{local} inertial (i.e., Galilean) frame of reference equipped with a Cartesian coordinate system. In a local inertial frame the inertial and gravitational parts of the Newton-Cartan connection-field may be unambiguously
(but, of course, non-covariantly) separated as in the equation (2.24) above, and a linear coordinate system may be introduced such that the two metric fields assume their canonical forms: \( h = \delta^{ab} \partial_a \otimes \partial_b \) and \( \tau = dt \). Furthermore, the connection-field in such a frame corresponds to a gauge-choice \( u = \frac{\partial}{\partial t} \) and \( A = -\Phi \tau \) in equation (2.30), with \( \Phi \) viewed as the usual Newtonian gravitational potential. With a further gauge-choice of \( \chi = -\frac{1}{2} \Phi \), and setting the rest of the multiplier fields to zero (without any loss of information, of course), the action functional (4.3) in the inertial frame becomes

\[
\mathcal{I} = \int dt \int d\vec{x} \left[ \frac{1}{8\pi G} \Phi \Delta \Phi + \frac{\hbar^2}{2m} \delta^{ab} \partial_a \psi \partial_b \psi + i \frac{\hbar}{2} (\bar{\psi} \partial_t \psi - \psi \partial_t \bar{\psi}) - m \bar{\psi} \psi \Phi \right],
\]

where we recall that \( \kappa = 0 \), and for simplicity we also take \( \Lambda_0 = 0 \). Extremization of this functional with respect to variations of the scalar potential \( \Phi \) immediately yields the Newton-Poisson equation

\[
\Delta \Phi = \frac{4\pi G}{\langle \psi|\psi \rangle} m \bar{\psi} \psi,
\]

where \( \psi := \sqrt{2\pi G}\bar{\psi} \psi \), and we set \( \langle \psi|\psi \rangle := \int d\vec{x} \bar{\psi} \psi = 1 \). On the other hand, extremization of the action with respect to variations of the matter field \( \Psi \) leads to the familiar Schrödinger equation in the presence of an external gravitational field:

\[
i\hbar \frac{\partial}{\partial t} \psi = \left[ -\frac{\hbar^2}{2m} \Delta + m \Phi \right] \psi.
\]

The last two equations may be interpreted as describing a single Galilean-relativistic particle gravitationally interacting with its own Newtonian field. As such, the coupled equations (5.15) and (5.16) constitute a nonlinear system, which can be easily seen as such by first formally solving equation (5.15) for the gravitational potential giving

\[
\Phi (\vec{x}) = -Gm \int d\vec{x}' \frac{\bar{\psi} (\vec{x}') \psi (\vec{x}')}{|\vec{x}' - \vec{x}|},
\]

and then — by substituting this solution into equation (5.16) — obtaining the nonlinear integro-differential equation

\[
i\hbar \frac{\partial}{\partial t} \psi (\vec{x}, t) = -\frac{\hbar^2}{2m} \Delta \psi (\vec{x}, t) - Gm^2 \int d\vec{x}' \frac{\bar{\psi} (\vec{x}', t) \psi (\vec{x}', t)}{|\vec{x}' - \vec{x}|} \psi (\vec{x}, t).
\]

However, when \( \psi \) is promoted to a ‘second-quantized’ field operator \( \hat{\psi} \) satisfying

\[
\left[ \hat{\psi} (\vec{x}), \hat{\psi}^\dagger (\vec{x}') \right] = \mathbb{1} \delta (\vec{x} - \vec{x}'),
\]

\[
\int d\vec{x} \int d\vec{x}' \frac{\bar{\psi} (\vec{x}) \psi (\vec{x}')}{|\vec{x}' - \vec{x}|} = 1.
\]

This integral equation is similar to the Hellman-Feynman theorem in quantum mechanics, and it implies that the expectation value of the Hamiltonian is conserved. The solution of this integral equation is known as the Hellman-Feynman potential, and it provides a way to calculate the expectation value of the Hamiltonian without explicitly solving the Schrödinger equation. However, in the context of general relativity, the Hellman-Feynman potential is not a constant of motion, and it depends on the initial conditions of the system. Therefore, the Hellman-Feynman potential is not a good candidate for a conserved quantity in general relativity.
this equation describes a system of many identical particles in the Heisenberg picture, with \( \hat{\psi} \) acting as an annihilation operator in the corresponding Fock space, analogous to the covariantly described ‘free’ system discussed in the subsection 3.2 above. In particular, the normal-ordered Hamiltonian operator for the system now reads

\[
\hat{H} = \hat{H}_o + \hat{H}_i
\]

with \( \hat{H}_o := \int d\vec{x} \hat{\psi}^\dagger(\vec{x}) \left[ -\frac{\hbar^2}{2m} \frac{\Delta}{2} \right] \hat{\psi}(\vec{x}) \),

and \( \hat{H}_i := -\frac{1}{2} G m^2 \int d\vec{x} \int d\vec{x}' \frac{\hat{\psi}^\dagger(\vec{x}) \hat{\psi}^\dagger(\vec{x}') \hat{\psi}(\vec{x}) \hat{\psi}(\vec{x}')}{|\vec{x}' - \vec{x}|} \), \( (5.20) \)

which, upon substitution into the Heisenberg equation of motion

\[
i\hbar \frac{\partial}{\partial t} \hat{\psi}(\vec{x}, t) = \left[ \hat{\psi}(\vec{x}, t), \hat{H} \right]
\]

yields an operator equation corresponding to the equation (5.18). It is easy to show \([12]\) that the action of the Hamiltonian operator \( \hat{H} \) on a multi-particle state is given by

\[
\langle \vec{x}_1 \vec{x}_2 \ldots \vec{x}_n | \hat{H} | \xi \rangle = \left[ -\frac{\hbar^2}{2m} \sum_{a=1}^{n} \Delta_a - G m^2 \sum_{a<b} \frac{1}{|\vec{x}_a - \vec{x}_b|} \right] \langle \vec{x}_1 \vec{x}_2 \ldots \vec{x}_n | \xi \rangle , \quad (5.22)
\]

which is consistent with the classical multi-particle Hamiltonian with gravitational pair-interactions. Evidently, the interaction Hamiltonian annihilates a single particle state \( |\vec{x}\rangle := \hat{\psi}^\dagger(\vec{x})|0\rangle \),

\[
\hat{H}_i |\vec{x}\rangle = 0 , \quad (5.23)
\]

implying that, thanks to the appropriate normal-ordering of the operators in \( \hat{H} \), the matter particles do not gravitationally self-interact. Moreover, the number operator, defined by \( \hat{N} := \int d\vec{x} \hat{\psi}^\dagger(\vec{x}) \hat{\psi}(\vec{x}) \), commutes with the total Hamiltonian operator. In other words, the number of particles in the theory under consideration is a constant of motion: as expected, our Galilean-relativistic interaction does not lead to particle production.
5.4. The intuitive physical picture

It is well-known that Newtonian gravitational field does not possess any dynamical degrees of freedom of its own — they ‘remain frozen’ in the ‘$c \to \infty$’ limit. On the other hand, Einsteinian gravitational radiation propagates in vacuum with the speed of light $c$. Considering this in the light of the local quantum field theory of Newton-Cartan gravity we have constructed here, it is rather convenient to maintain that Newtonian gravitational field also possesses propagating degrees of freedom, but it so happens that such gravitational disturbances travel instantaneously — i.e., with the Galilean-relativistic speed of light ‘$c = \infty$’. Indeed, it is possible to view Newton-Poisson equation $\Delta \Phi = 4\pi G \rho$ as a wave-equation for the Newtonian gravitational waves propagating with infinite speed:

$$\lim_{c \to \infty} \left[ \frac{\Delta \Phi}{c^2} - \frac{\partial^2 \Phi}{\partial t^2} = 4\pi G \rho \right] \longrightarrow [\Delta \Phi = 4\pi G \rho].$$

Moreover, unlike the usual weak-field approach to Einstein’s gravity (‘$c \neq \infty$’) where such longitudinal degrees of freedom of the gravitational field are ‘gauged-away’ and ignored (transverse-traceless gauge), in the theory constructed here we have been able to avoid any gauge-fixing procedure. Therefore, as far as Newton-Cartan theory is viewed as a Galilean-relativistic limit-form of Einstein’s theory of gravity, the limit relation (5.24) is a better interpretation of the Newton-Poisson equation than the usual one in which one insists that there are no gravitational waves in a Newtonian theory.

Now, consider the quantized gravitational degrees of freedom. According to the above view, these also propagate with infinite speed. Moreover, if we take expressions (3.16) and (3.19) in our theory as the mass and angular-momentum operators, respectively (which are indeed the correct operators for an inertial observer), then

$$\left[ \hat{\mathbf{m}}, \hat{\bar{\mathbf{v}}}^\alpha \right] = 0 \quad \text{and} \quad \left[ \hat{\mathbf{j}}, \hat{\bar{\mathbf{v}}}^\alpha \right] = 0$$

(5.25)

imply that Newton-Cartan gravitons are the massless and spinless mediating particles between the matter particles. In other words, the theory constructed here may be viewed, in a particle interpretation, as a theory of non-self-interacting quantized Schrödinger particles producing the longitudinal Newton-Cartan gravitons — the massless, spin-0 exchange bosons propagating with infinite speed — and interacting through them.
6. Conclusion

If Einstein’s gravity is viewed as a result of two physical principles, (1) strong equivalence of gravitational and inertial masses, and (2) relativization of time, then it is the second principle which prevents it from any straightforward subjugation to the otherwise well-corroborated rules of quantization. As demonstrated here, the first one by itself is completely unproblematic as far as the canonical quantization of gravity is concerned. On the other hand, the invocation of relativization of time in addition to the first principle induces the so far intractable ‘problem of time’ via the Hamiltonian constraint in general relativity — as is quite well-known [3]. Because of the presence of preferred foliation — whose covariant normal constitutes the kernel of the spatial metric endowing spacetime with an absolute, observer-independent notion of distant simultaneity — no such intractable constraint arises in the Hamiltonian formulation of the classical Newton-Cartan-Schrödinger theory. Consequently, we have been able to successfully and unambiguously quantize this interacting Galilean-relativistic field theory as an unconstrained Hamiltonian system in a manifestly covariant manner. What is more, as discussed in the Introduction, this exercise opens up a completely novel direction of research in quantum gravity: the program of the special-relativization of the quantum theory of Newton-Cartan gravity.

Note Added to Proof

Several years after I published this paper I learned from Roger Penrose — who, in turn, had learned from John Stachel — that in the 1930s a remarkable Soviet physicist called Bronstein had already represented the fundamental theories of physics in a map very much like the one in my Fig. 1 above. His two-dimensional map, however, neglected the Newtonian theory of gravity, which was eventually included by Zelmanov in 1967, thereby transforming Bronstein’s two-dimensional picture into a full three-dimensional cube. Neither Bronstein nor Zelmanov, however, made the crucial distinction between Newton’s original theory of gravity and Cartan’s spacetime reformulation of it, let alone made any indication of quantization of the latter theory. In particular, Zelmanov’s version of the cube (followed up by Okun in 1991 [59]) wrongly represents Newton’s theory of gravity as a Galilean-relativistic limit-form of Einstein’s theory of gravity. In fact, as we
saw above, the true limit-form of Einstein’s theory is the Newton-Cartan theory with its mutable connection field. To appreciate the importance of this crucial distinction in the contemporary context, see, for example, section 14.4.2 of Ref. [60] and references therein. For an excellent historical account of Bronstein’s remarkable life and work in physics — including the Soviet history of “the cube of theories” — see Ref. [61] and references therein.

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Appendix: Review of the covariant phase-space formalism

Recall that the essence of the canonical formalism for a classical system with finitely many degrees of freedom is succinctly captured by the symplectic 2-form \[ \omega = dp_i \wedge dq^i , \] (A.1)

where \( q^i , i,j = 1,\ldots,N \), are the generalized coordinates describing the configuration of the system, and \( p_j \) are the corresponding conjugate momenta (here Einstein’s summation convention is understood). The collection of all possible values of coordinates and momenta, \( (q_1, \ldots, q_N; p_1, \ldots, p_N) \), is referred to as the phase-space of the system, on which the dynamical evolution is determined by a Hamiltonian function \( \mathcal{H} \) via the Hamiltonian equations of motion. If we combine the \( p_i \) and \( q^i \) in a single variable \( Q^i , i=1,\ldots,2N \), with \( Q^i := p_i \) for \( i \leq N \) and \( Q^i := q^{i-N} \) for \( i > N \), then we can think of \( \omega \) as an antisymmetric
2N×2N matrix \( \omega_{IJ} \), with inverse \( \omega^{IJ} \), whose nonzero matrix elements are

\[
\omega_{i,i+N} = -\omega_{i+N,i} = 1. \tag{A.2}
\]

With the help of this invertible matrix \( \omega^{IJ} \), the Hamiltonian equations of motion can be succinctly expressed as

\[
\frac{dQ^I}{dt} = \omega^{IJ} \frac{\partial H}{\partial Q^J}, \tag{A.3}
\]

and the Poisson bracket of any two functions \( U(Q^I) \) and \( V(Q^I) \) as

\[
\{ U, V \} = \omega^{IJ} \frac{\partial U}{\partial Q^I} \frac{\partial V}{\partial Q^J}. \tag{A.4}
\]

Since the essential features of the symplectic form \( \omega \) can be described in a coordinate independent manner \([52,53,19]\), it can be viewed as the invariant geometric structure that underlies the definitions of Hamiltonian equations and Poisson brackets of classical mechanics. If we now view our phase-space as the cotangent-bundle \( T^*Z \) on a configuration space \( Z \) with coordinates \( q^i \), and let \( p_j \) be the corresponding components of a covector \( p \) at \( q \in Z \), then the 2-form (A.1) is a closed 2-form on \( T^*Z \),

\[
d\omega = 0, \tag{A.5}
\]

because its components in this particular coordinate system are constant. Conversely, if \( \omega \) is any closed 2-form on a \( 2N \) dimensional manifold \( Z \) such that it is non-degenerate (i.e., the matrix \( \omega_{ij}(z) \) is invertible at each point \( z \in Z \)), then, according to Darboux’s theorem \([19]\), locally one can always introduce coordinates \( (p_i, q^j) \) on \( Z \), called the canonical coordinates, which put \( \omega \) in the standard form (A.1). This immediately suggests a generalization: the cotangent-bundle form of phase-space is only a special form of phase-space (but, of course, of a fundamentally important kind) since, in general, the ‘isomorphism’ \( Z \cong T^*Z \) holds only in a local neighborhood of a point \( z \in Z \). In other words, globally the phase-space may not be anything like a cotangent-bundle. In general phase-space is defined as a pair \( (Z, \omega) \) in which \( Z \) is a smooth manifold, called a symplectic manifold, and \( \omega \), defined everywhere on \( Z \), is a closed, non-degenerate and real-valued 2-form called the symplectic structure on \( Z \). The ‘observables’ are then functions of the form \( U : Z \to \mathbb{R} \) generating a one-parameter family of canonical transformations on \( Z \). In particular, the Hamiltonian of a
classical system generating the time evolution is a function \( \mathbf{H} : \mathcal{Z} \to \mathbb{R} \) on the phase-space \( \mathcal{Z} \) of the system, and the dynamical trajectories governed by the Hamiltonian equations of motion correspond to integral curves on \( \mathcal{Z} \) of the Hamiltonian vector-field, \( X_{\mathbf{H}} : \mathcal{Z} \to \mathcal{Z} \), defined by

\[
X_{\mathbf{H}} \omega + d\mathbf{H} := 0. \tag{A.6}
\]

It is easy to verify that the flow of \( X_{\mathbf{H}} \) preserves \( \omega \) in the sense that \( \mathcal{L}_{X_{\mathbf{H}}} \omega = 0 \). Moreover, for a curve \( Q(t) \) whose tangent vector at every point coincides with \( X_{\mathbf{H}} \), the component form of equation (A.6) in local coordinates \( (p_i, q^j) \) is precisely the equation (A.3).

The above abstract definition of phase-space quite elegantly encapsulates the important fact that it is the assignment of a symplectic structure \( \omega \) on the classical phase-space \( \mathcal{Z} \), rather than a particular choice of coordinates \( (p_i, q^j) \) on \( \mathcal{Z} \), that is more intrinsic to the canonical formulation of a classical theory. Yet, \( prima facie \), the very concept of phase-space appears to be non-covariant because phase-spaces are usually constructed by first decomposing spacetime into spacelike hypersurfaces at instants of time, and then specifying the initial data \( (p_i, q^j) \) on one of these hypersurfaces. However, thanks to the one-to-one correspondence between each dynamically allowed trajectory of a given classical system and its initial data, such a manifestly non-covariant route to phase-space is not indispensable.

It is clear from equation (A.6) that each point \( z \in \mathcal{Z} \) is an appropriate initial data for uniquely determining a complete Hamiltonian trajectory of the system. This establishes an isomorphism between the space of solutions to the dynamical equations and the canonical phase-space of the system, and, thereby, allows one to pull-back the symplectic structure from the phase-space to the space of solutions. As a result, we can identify our phase-space \( \mathcal{Z} \) with the manifold of solutions to the equations of motion, which now we may also denote by \( \mathcal{Z} \). This space of solutions \( \mathcal{Z} \) equipped with the symplectic structure \( \omega \) is then our desired manifestly covariant phase-space \( (\mathcal{Z}, \omega) \). Of course, as allowed by Darboux’s theorem, one can always choose a coordinate system on \( \mathcal{Z} \), at least locally, and identify the solutions to the dynamical equations with their Cauchy data in that coordinate system; but there is no fundamental necessity to violate covariance in this manner. Besides, globally such an identification may not be possible in general, because, as noted above, globally the manifold \( (\mathcal{Z}, \omega) \) may not be isomorphic to a cotangent-bundle. A cotangent-bundle can,
of course, be ‘polarized’ — i.e., foliated by the individual cotangent spaces of constant $q$ — and, thereby, the configuration and momentum variables can be distinguished. But a general symplectic manifold may not be polarizable in this manner, and, consequently, the global identification of solutions to the equations of motion with integral curves of the Hamiltonian vector-field becomes untenable in general. Instead, the ‘time-development’ of a classical system is understood within this framework by interpreting the Hamiltonian flow as a mapping between entire histories of the system. Given a notion of time, the mapping from one dynamical trajectory to another infinitesimally distant one, whose initial data at time $t + \epsilon$ are the same as the initial data of the first trajectory at time $t$, is interpreted as the ‘time-evolution’ from $t$ to $t + \epsilon$ generated by the Hamiltonian function $H : Z \to \mathbb{R}$.

The (pre)symplectic structure in the covariant description of an infinite-dimensional phase-space $(\mathcal{Z}, \omega)$ of a field theory is defined by a real valued function of the field variables,\[ \omega = \int_{\Sigma_t} \omega^\mu n_\mu \, d\Sigma_t, \tag{A.7} \]
where $\Sigma_t$ is some spacelike hypersurface, $n_\mu$ is the inverse of an ‘outward-pointing’ unit normal to this surface, and $\omega^\mu$ — called the presymplectic current density — is a conserved and closed 2-form (which may be degenerate, and hence the prefix ‘pre’ meaning ‘before the removal of degeneracy’; see below). As discussed above, if the manifold $\mathcal{Z}$ is polarizable so that we can distinguish between configuration and momentum variables, then we can express it as a cotangent-bundle of the configuration space. The above general expression for $\omega$ can then be reduced (at least in the case of a first-order action) to the standard canonical presymplectic structure,\[ \omega = \int_{\Sigma_t} dp_i \wedge dq^j \, d\Sigma_t, \tag{A.8} \]
once a choice of global coordinates $(p_i, q^j)$ is made. Despite its appearance in (A.7) and (A.8), $\omega$ is independent of the choice of hypersurface $\Sigma_t$ (provided that either $\Sigma_t$ is chosen to be compact or the field variables on it are subjected to satisfy suitable boundary conditions at spatial infinity).

This manifest covariance of $\omega$ may require some convincing. Let us look more closely at how it can be shown by considering a dynamical theory for a collection of smooth fields $Q^r(x)$ on a spacetime $\mathcal{M}$ equipped with a derivative operator $\nabla_\mu$; here $r$ is a collective
label for fields representing spacetime indices as well as internal and discrete indices. We assume that our spacetime $\mathcal{M}$ is globally hyperbolic, i.e., topologically $\mathbb{R} \times \Sigma$, where each image $\Sigma_t$ of $\Sigma$ for any $t \in \mathbb{R}$ is diffeomorphic to $\mathbb{R}^3$. Let $Z$ denote the infinite-dimensional manifold constituted by the fields $Q^r$ on $\mathcal{M}$. Since functions on $Z$ are functionals of the form $f[Q(x)]$, an action functional $I : Z \to \mathbb{R}$ may be constructed over some measurable region of $\mathcal{M}$ such that equations of motion for a field $Q^r$ can be obtained by extremizing the action under any variation $\delta Q^r \equiv Y^r$ of the field which vanishes on the boundary of the region. The variations $\delta Q$ here are tangent vectors $Y \in T_QZ$ at points in space $Z$ corresponding to the fields $Q$, the tangent spaces $T_QZ$ at $Q$ are vector spaces of the variations $Y$, and the first variation $\delta I$ of the action can be expressed as an exterior derivative of $I$ in $Z$ applied to $Y$:

$$\delta I = d_z I (Y) ;$$

i.e., the variational derivative ‘$\delta$’ is equivalent to the exterior derivative ‘$d_z$’ on the space $Z$. Then, for a given measurable region $\sigma \subset \mathcal{M}$ with a non-null boundary $\partial \sigma$ and a collection of fields $Q \in Z$, the variation of action

$$I [Q] = \int_{\sigma} dv \ L (Q^r, \nabla_\mu Q^r, \ldots, \nabla_\mu \nabla_\mu_2 \ldots \nabla_\mu_k Q^r)$$

(A.10)

is equal to

$$d_z I (Y) = \int_{\sigma} dv \left\{ \frac{\delta L}{\delta Q^r} \delta Q^r + \nabla_\mu J_\mu^\mu \right\} ,$$

(A.11)

where $dv$ is the volume element on $\mathcal{M}$ compatible with the derivative operator $\nabla_\mu$,

$$J_\mu^\mu := \left\{ \frac{\delta L}{\delta (\nabla_\mu Q^r)} \right\} \delta Q^r + \ldots + \left\{ \frac{\delta L}{\delta (\nabla_\mu_2 \ldots \nabla_\mu_k Q^r)} \right\} \nabla_\mu_2 \ldots \nabla_\mu_k \delta Q^r$$

(A.12)

is what is called the \textit{presymplectic potential current density}, and the variational derivative $\frac{\delta}{\delta l}$ of a local function is defined as

$$\frac{\delta}{\delta l} := \frac{\partial}{\partial l} - \nabla_\mu \left\{ \frac{\partial}{\partial (\nabla_\mu l)} \right\} + \nabla_\mu \nabla_\nu \left\{ \frac{\partial}{\partial (\nabla_\mu \nabla_\nu l)} \right\} - \ldots .$$

(A.13)

Under the requirement that the action $I$ remains stationary for any variation $Y$ of $Q$ which vanishes on the boundary, the subspace $Z \subset Z$ of solutions to the dynamical equations is
defined by the condition \( \frac{\delta L}{\delta Q} = 0 \). Conversely, on the space \( Z \) consisting of fields \( Q \) which extremize \( I \), the pull-back of equation (A.11) reduces to the boundary term

\[
i^* d_z I(Y) = \oint_{\partial \Sigma} J^\mu_Q(Y) n_\mu \, ds,
\]

where \( i : Z \hookrightarrow Z \) is the natural embedding of the submanifold \( Z \) into \( Z \), \( n_\mu \) is the inverse of an `outward-pointing` unit normal to the boundary \( \partial \Sigma \) as before, \( ds \) is the surface element on \( \partial \Sigma \), and now the tangent vector \( Y \in T_Q Z \) is a solution to the linearized equations of motion at \( Q \) (i.e., both \( Q \) and \( Q + \epsilon Y \) satisfy the equations of motion to lowest order in \( \epsilon \)). Clearly, the surface term (A.14) does not contribute to the variation of the action because \( J^\mu_Q(Y) \) vanishes when \( Y \equiv \delta Q = 0 \) is assumed on the boundary. Nevertheless, it is precisely this term that captures the manifestly covariant essence of the canonical structure of phase-space. In order to see this, consider a volume segment \( \partial (\Sigma_{t'} - \Sigma_t) \subset M \) which is bounded by two Cauchy surfaces \( \Sigma_{t'} \) and \( \Sigma_t \) connected by a timelike world tube \( T_\infty \) at spatial infinity, and define a 1-form \( \theta_{\Sigma_t}(Y) \) on \( Z \) for all tangent vectors \( Y \in T_Q Z \) by

\[
\theta_{\Sigma_t}(Y) := \int_{\Sigma_t} J^\mu_Q(Y) n_\mu \, d\Sigma_t,
\]

so that in terms of this 1-form equation (A.14) becomes

\[
i^* d_z I(Y) = \theta_{\Sigma_{t'}}(Y) - \theta_{\Sigma_t}(Y) + \theta_{T_\infty}(Y).
\]

Note that, in general, this *presymplectic potential* \( \theta_{\Sigma_t}(Y) \) (as it is sometimes called in suggestive analogy with the electromagnetic vector-potential) is not unique and *depends* on the choice of hypersurface \( \Sigma_t \). However, a presymplectic 2-form on \( Z \) can now be defined as the exterior derivative of \( \theta_{\Sigma_t}(Y) \),

\[
\omega_{\Sigma_t}(Y_1, Y_2) := d_z \theta_{\Sigma_t}(Y_1, Y_2),
\]

which in the light of equation (A.16) is immediately seen to behave as

\[
0 \equiv i^* d_z^2 I = \omega_{\Sigma_{t'}} - \omega_{\Sigma_t} + \omega_{T_\infty}
\]

under changes of the Cauchy surface \( \Sigma_t \). Thus, in case \( \Sigma_t \) is non-compact, simply by a suitable choice of boundary condition (e.g., \( J^\mu_Q \to 0 \) at spatial infinity ensuring the
vanishing of the term $\omega_{T_{\infty}}$ we obtain the desired covariant behavior of the presymplectic structure as claimed above.

In the view of equation (A.17), the closed-ness of the 2-form $\omega$ is immediate: $d_z \omega = 0$. Whereas comparisons of equations (A.7), (A.15), (A.17), and (A.18) show that the corresponding presymplectic current density

$$\omega^\mu (Y_1, Y_2) := d_z J^\mu_Q (Y_1, Y_2) ,$$

(A.19)

whose integral over $\Sigma_t$ gives the 2-form $\omega_{\Sigma_t}$ on $Z$, is conserved,

$$\nabla^\mu \omega^\mu = 0 ,$$

(A.20)

thanks to the equations of motion $\delta L \delta Q = 0$ and their linearized versions satisfied, respectively, by the fields $Q$ and their variations $Y$. Thus, at each spacetime point $x$, $\omega^\mu (x)$ is a vector-valued 2-form on the space $Z$ of classical solutions, but in its dependence on $x$, it is a conserved current density. Furthermore, in case $\Sigma_t$ is non-compact, if a 4-divergence term is added to the Lagrangian, $\mathcal{L} \rightarrow \mathcal{L} + \nabla_\mu \lambda^\mu$, then the potential current density $J^\mu_Q$ changes only by an exact form $d_z \lambda^\mu$ plus an identically conserved vector density, changing the 2-form $\omega$ only by a ‘surface term at infinity’ [44].

It is clear from our notation in the equation (A.17) that $\omega$ can also be viewed as a skew-symmetric bilinear function $\omega : T_z Z \times T_z Z \rightarrow \mathbb{R}$ on the tangent vector-space $T_z Z$ at some point $z$ on the phase-space $Z$. The phase-space of a linear (i.e., non-self-interacting) dynamical system is the prime example of such a symplectic vector-space with a bilinear form [54]. The field equations of linear dynamical systems are ‘already linearized’; i.e., the equations of motion for such systems are linear in the linear canonical coordinates because their Hamiltonians are at most quadratic functions on $Z$. Consequently, their phase-spaces, viewed as manifolds $Z$ of solutions, have a natural vector-space structure, and one may identify the tangent space $T_z Z$ at any point $z \in Z$ with the space $Z$ itself. The symplectic form $\omega (Q_1, Q_2)$, with $Q_1, Q_2 \in Z$, then becomes a bilinear map on the symplectic vector-space $Z$, $\omega : Z \times Z \rightarrow \mathbb{R}$, which is independent of the choice of point $z$ used to make this identification.

The presymplectic 2-form $\omega$ on $Z$ we have defined so far is necessarily degenerate if the action functional $I$ admits any gauge-arbitrariness. Conversely, a degenerate symplectic
structure would necessitate gauge-type symmetries in a theory (the issue, of course, is closely related to the question of whether or not the Cauchy problem for $\frac{\delta}{\delta Q} = 0$ is well posed). However, at least in simple cases, it is possible to obtain a genuine (i.e., non-degenerate) symplectic 2-form $\tilde{\omega}$ on a reduced phase-space $\tilde{Z}$ by the so-called reduction procedure [44,49]. A 2-form $\tilde{\omega}$ of the pair $(\tilde{Z}, \tilde{\omega})$ is said to be non-degenerate (or weakly non-degenerate, to be more precise) if $\tilde{\omega}(Y_1, Y_2) = 0$ for all $Y_2 \in \tilde{Z}$ implies $Y_1 = 0$, and it is said to be strongly non-degenerate if the map $T_z\tilde{Z} \to T^*_z\tilde{Z} : Y \mapsto Y \downharpoonright \omega$ is a linear isomorphism at each point $z \in \tilde{Z}$ (the criteria of weak and strong non-degeneracy are clearly equivalent when $\tilde{Z}$ is finite dimensional). The reduction procedure amounts to projecting down the presymplectic 2-form $\omega$ from the space $Z$ of solutions of the equations of motion to the physical phase-space $\tilde{Z}$ — the space of solutions to the equations of motion modulo gauge-transformations. The projection map $P : Z \to \tilde{Z}$ which accomplishes this, assigns each element of $Z$ to its gauge-equivalence class. It can be shown that every infinitesimal gauge transformation corresponds to a degenerate direction of the 2-form $\omega$ on $Z$ by showing that $\omega$ is actually a unique pull-back $P^*(\tilde{\omega})$ from $\tilde{Z}$ to $Z$ of a non-degenerate 2-form $\tilde{\omega}$ on the reduced phase-space $\tilde{Z}$. This reduced phase-space, in turn, is just the space of orbits in $Z$ of the group $K$ of gauge transformations; i.e., the reduced phase-space is the quotient space $\tilde{Z} := Z/K$, where $K$ is the characteristic distribution of $\omega$ with fibers $K_z$ at $z \in Z$ defined as [49]

$$K_z := \{Y \in T_zZ \mid Y \downharpoonright \omega = 0\} \subset T_zZ. \tag{A.21}$$

In other words, any nonvanishing vector field $Y \in K$ tangent to the $K$-orbits on $Z$ is a degenerate direction of $\omega$: $Y \in V_zK(\tilde{Z}) \Rightarrow \omega(Y, Y) = 0 \forall Y \in T_zZ$, where $V_zK(\tilde{Z})$ denotes the set of vector fields tangent to $K$. Then, for one thing, the Lie derivative of $\omega$ along the degenerate directions is always zero:

$$Y \in V_zK(\tilde{Z}) \Rightarrow \mathcal{L}_Y \omega = d_z(Y \downharpoonright \omega) + Y \downharpoonright d_z\omega = 0. \tag{A.22}$$

Agreeably, the quotient space $Z/K = \tilde{Z}$ is a Hausdorff manifold since the characteristic distribution $K$ on $Z$ is an integrable submanifold: if $Y_1$ and $Y_2$ are degeneracy vector fields with values in $K$, then so is $[Y_1, Y_2]$. Clearly, the gauge-directions are eliminated in passing from $Z$ to $\tilde{Z}$, rendering $\tilde{\omega}$ on $\tilde{Z}$ manifestly gauge-invariant. In practice, all one has to do
to ensure gauge-invariance is to make sure that a given presymplectic (i.e., degenerate) structure \( \omega \) on \( Z \) is a pull-back of the genuine (i.e., non-degenerate) symplectic structure \( \tilde{\omega} \) from the reduced phase-space \( \tilde{Z} \): \( \omega = \mathbb{P}^*(\tilde{\omega}) \). This is guaranteed if and only if \( \tilde{\omega} \) on \( \tilde{Z} \) has vanishing components in the gauge-directions:

\[
Y_k \downarrow \tilde{\omega} = 0. \tag{A.23}
\]

Finally, note that, although \( \omega \) is exact by definition (\( \omega = d\theta \); cf. equation (A.17)), \( \tilde{\omega} \) need not satisfy this property, because, in general, the quotient space \( \tilde{Z} \) could be topologically much more subtle compared to the space \( Z \). It is always possible, however, to find a local neighborhood on \( \tilde{Z} \) such that \( \tilde{\omega} = d\tilde{\theta} \) within it. If \( \tilde{Z} \) happens to be symplectically diffeomorphic to a cotangent-bundle, then, of course, \( \tilde{\omega} \) is its natural exact symplectic structure with \( \tilde{\theta} \) being the standard canonical 1-form \[49\]. For convenience, let us illustrate the relations between various spaces we have encountered by the following diagram:

\[
\begin{array}{ccc}
Z & \xrightarrow{i^*} & Z \\
\downarrow \mathbb{P} & & \downarrow \mathbb{P} \\
\tilde{Z} & \xrightarrow{i^*} & \tilde{Z}
\end{array} \tag{A.24}
\]

Here \( \tilde{Z} \) denotes collection of all kinematically possible field configurations modulo gauge-translations (cf. equation (4.85)), whereas \( \tilde{Z} \subseteq \tilde{Z} \) denotes collection of purely dynamically possible field configurations modulo gauge-translations. If constraints are present, however, then \( \tilde{Z} \subset \tilde{Z} \), and not all kinematically possible states are dynamically possible.
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