EXOTIC CLUSTER STRUCTURES ON $SL_n$ WITH BELAVIN–DRINFIELD DATA OF MINIMAL SIZE, II.
CORRESPONDENCE BETWEEN CLUSTER STRUCTURES AND BD TRIPLES

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Abstract. Using the notion of compatibility between Poisson brackets and cluster structures in the coordinate rings of simple Lie groups, Gekhtman Shapiro and Vainshtein conjectured a correspondence between the two. Poisson Lie groups are classified by the Belavin–Drinfeld classification of solutions to the classical Yang Baxter equation. For any non trivial Belavin–Drinfeld data of minimal size for $SL_n$, the companion paper constructed a cluster structure with a locally regular initial seed, which was proved to be compatible with the Poisson bracket associated with that Belavin–Drinfeld data.

This paper proves the rest of the conjecture: the corresponding upper cluster algebra $\mathcal{A}_C(C)$ is naturally isomorphic to $\mathcal{O}(SL_n)$, the torus determined by the BD triple generates the action of $(\mathbb{C}^\ast)^{2k_T}$ on $\mathbb{C}(SL_n)$, and the correspondence between Belavin–Drinfeld classes and cluster structures is one to one.

1. Introduction

Since cluster algebras were introduced in [7], a natural question was the existence of cluster structures in the coordinate rings of a given algebraic variety $V$. Partial answers were given for Grassmannians $V = Gr_k(n)$ [14] and double Bruhat cells [2]. If $V = G$ is a simple Lie group, one can extend the cluster structure found in the double Bruhat cell to one in $\mathcal{O}(G)$. The compatibility of cluster structures and Poisson brackets, as characterized in [9] suggested a connection between the two: given a Poisson bracket, does a compatible cluster structure exist? Is there a way to find it?

In the case that $V = G$ is a simple complex Lie group, R-matrix Poisson brackets on $G$ are classified by the Belavin–Drinfeld classification of solutions to the classical Yang Baxter equation [1]. Given a solution of that kind, a Poisson bracket can be defined on $G$, making it a Poisson–Lie group.

The Belavin–Drinfeld (BD) classification is based on pairs of isometric subsets of simple roots of the Lie algebra $\mathfrak{g}$ of $G$. The trivial case when the subsets are empty corresponds to the standard Poisson bracket on $\mathfrak{g}$ . It has been shown in [11] that extending the cluster structure introduced in [2] from the double Bruhat cell to the whole Lie group yields a cluster structure that is compatible with the standard Poisson bracket. This led to naming this cluster structure “standard”, and trying to find other cluster structures, compatible with brackets associated with non trivial Poisson–Lie group, cluster algebra, Belavin–Drinfeld triple.

Date: November, 2015.
2000 Mathematics Subject Classification. 53D17,13F60.

Key words and phrases. Poisson–Lie group, cluster algebra, Belavin–Drinfeld triple.
BD subsets. The term “exotic” was suggested for these non standard structures [12].

Gekhtman, Shapiro and Vainshtein conjectured a bijection between BD classes and cluster structures on simple Lie groups [11,13]. According to the conjecture, for a given BD class for \( G \) and cluster structures on simple Lie groups [11, 13]. The conjecture states that the structure is regular, and that the upper cluster algebra coincides with the ring of regular functions on \( G \). The conjecture was proved for the standard case and for \( G = SL_n \) with \( n < 5 \) in [11]. The Cremmer–Gervais case, which in some sense is the “furthest” from the standard one, was proved in [13]. It was also found to be true for all possible BD classes for \( SL_5 \) [5].

This paper considers the conjecture for \( SL_n \) when the BD data is of minimal size, i.e., the two subsets contain exactly one simple root. In the companion paper [6] the first part of the conjecture was proved: starting with two such subsets \( \{ \alpha \} \) and \( \{ \beta \} \), an algorithm was given constructing a set \( B_{\alpha\beta} \) of functions that served as an initial cluster. Defining an appropriate quiver \( Q_{\alpha\beta} \) (or an exchange matrix \( B_{\alpha\beta} \)), one can obtain an initial seed for a cluster structure on \( SL_n \). This structure is locally regular and it is compatible with the Poisson bracket associated with the BD data \( \{ \alpha \} \rightarrow \{ \beta \} \).

This paper completes the proof of the conjecture: the bijection between cluster structures and BD classes of this type is established, the upper cluster algebra is proved to be naturally isomorphic to the ring of regular functions on \( SL_n \), and a description of a global toric action is given.

2. Background

2.1. Cluster structures. Let \( \{ z_1, \ldots, z_m \} \) be a set of independent variables, and let \( S \) denote the ring of Laurent polynomials generated by \( z_1, \ldots, z_m \):

\[
S = \mathbb{Z}[z_1^{\pm 1}, \ldots, z_m^{\pm 1}].
\]

(Here and in what follows \( z^{\pm 1} \) stands for \( z, z^{-1} \).) The ambient field \( F \) is the field of rational functions in \( n \) independent variables (distinct from \( z_1, \ldots, z_m \)), with coefficients in the field of fractions of \( S \).

A seed (of geometric type) is a pair \((\mathbf{x}, \tilde{B})\), where \( \mathbf{x} = (x_1, \ldots, x_n) \) is a transcendence basis of \( F \) over the field of fractions of \( S \), and \( \tilde{B} \) is an \( n \times (n + m) \) integer matrix whose principal part \( B \) (that is, the \( n \times n \) matrix formed by columns \( 1, \ldots, n \)) is skew-symmetric. The set \( \mathbf{x} \) is called a cluster, and its elements \( (x_1, \ldots, x_n) \) are called cluster variables. Set \( x_{n+i} = z_i \) for \( i \in [1, m] \) (we use the notation \([a, b]\) for the set of integers \( \{a, a+1, \ldots, b\} \). When \( a = 1 \) we write just \([m]\) for the set \([1, m]\)). The elements \( x_{n+1}, \ldots, x_{n+m} \) are called stable variables (or frozen variables). The set \( \tilde{\mathbf{x}} = (x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m}) \) is called an extended cluster. The square matrix \( B \) is called the exchange matrix, and \( \tilde{B} \) is called the extended exchange matrix.

We sometimes denote the entries of \( \tilde{B} \) by \( b_{ij} \), or say that \( \tilde{B} \) is skew-symmetric when the matrix \( B \) has this property.

Let \( \Sigma = (\tilde{\mathbf{x}}, \tilde{B}) \) be a seed. The set \( \tilde{\mathbf{x}}_k = (\tilde{\mathbf{x}} \setminus \{x_k\}) \cup \{x'_k\} \) is called the adjacent cluster in direction \( k \in [n] \), where \( x'_k \) is defined by the exchange relation

\[
(2.1) \quad x_k \cdot x'_k = \prod_{b_{kj} > 0} x_j^{b_{kj}} + \prod_{b_{kj} < 0} x_j^{-b_{kj}}.
\]
A matrix mutation $\mu_k \left( \tilde{B} \right)$ of $\tilde{B}$ in direction $k$ is defined by

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k \\ b_{ij} + \frac{1}{2} \left( |b_{ik}| b_{kj} + b_{ik} |b_{kj}| \right) & \text{otherwise.} \end{cases}$$

Seed mutation in direction $k$ is then defined $\mu_k \left( \Sigma \right) = \left( \tilde{x}_k, \mu_k \left( \tilde{B} \right) \right)$.

Two seeds are said to be mutation equivalent if they can be connected by a sequence of seed mutations.

Given a seed $\Sigma = (x, \tilde{B})$, the cluster structure $C(\Sigma)$ (sometimes denoted $C(\tilde{B})$), if $x$ is understood from the context) is the set of all seeds that are mutation equivalent to $\Sigma$. The number $n$ of rows in the matrix $\tilde{B}$ is called the rank of $C$.

Let $\Sigma$ be a seed as above, and $A = \mathbb{Z}[x_{n+1}, \ldots, x_{n+m}]$. The cluster algebra $A = A(\Sigma) = A(\tilde{B})$ associated with the seed $\Sigma$ is the $A$-subalgebra of $F$ generated by all cluster variables in all seeds in $C(\tilde{B})$. The upper cluster algebra $\overline{A} = \overline{A}(\Sigma) = \overline{A}(\tilde{B})$ is the intersection of the rings of Laurent polynomials over $\tilde{A}$ in cluster variables taken over all seeds in $C(\tilde{B})$. The famous Laurent phenomenon \cite{fz} claims the inclusion $A(\Sigma) \subseteq \overline{A}(\Sigma)$.

A useful tool to prove that some function belongs to the upper cluster algebra is the following Lemma \cite{fz} Lemma 8.3:

Lemma 2.1. Let $C$ be a cluster structure of geometric type and $\overline{A}$ be the corresponding upper cluster algebra. Suppose $M_{\frac{m}{j_1}} = M_{\frac{m}{j_2}} = M$ for $M_1, M_2 \in \overline{A}$, $m_1, m_2 \in \mathbb{N}$ and coprime cluster variables $f_1 \neq f_2$. Then $M \in \overline{A}$.

It is sometimes convenient to describe a cluster structure $C(\tilde{B})$ in terms of its quiver $Q(\tilde{B})$: it is a directed graph with $n+m$ nodes labeled $x_1, \ldots, x_{n+m}$ (or just $1, \ldots, n+m$), and an arrow pointing from $x_i$ to $x_j$ with weight $b_{ij}$ if $b_{ij} > 0$.

Let $V$ be a quasi-affine variety over $\mathbb{C}$, $C(V)$ be the field of rational functions on $V$, and $O(V)$ be the ring of regular functions on $V$. Let $C$ be a cluster structure in $F$ as above, and assume that $\{f_1, \ldots, f_{n+m}\}$ is a transcendence basis of $C(V)$. Then the map $\varphi : x_i \to f_i$, $1 \leq i \leq n+m$, can be extended to a field isomorphism $\varphi : F_C \to \mathbb{C}(V)$, with $F_C = F \otimes \mathbb{C}$ obtained from $F$ by extension of scalars. The pair $(\mathbb{C}, \varphi)$ is then called a cluster structure in $\mathbb{C}(V)$ (or just a cluster structure on $V$), and the set $\{f_1, \ldots, f_{n+m}\}$ is called an extended cluster in $(\mathbb{C}, \varphi)$. Sometimes we omit direct indication of $\varphi$ and just say that $C$ is a cluster structure on $V$. A cluster structure $(\mathbb{C}, \varphi)$ is called regular if $\varphi(x)$ is a regular function for any cluster variable $x$, and a seed $\Sigma$ is called locally regular if all the cluster variables in $\Sigma$ and in all the adjacent seeds are regular functions. The two algebras defined above have their counterparts in $F_C$ obtained by extension of scalars; they are denoted $A_C$ and $\overline{A}_C$. If, moreover, the field isomorphism $\varphi$ can be restricted to an isomorphism of $A_C$ (or $\overline{A}_C$) and $O(V)$, we say that $A_C$ (or $\overline{A}_C$) is naturally isomorphic to $O(V)$.

The following statement is a weaker analogue of Proposition 3.37 in \cite{fz}:

Proposition 2.2. Let $V$ be a Zariski open subset in $\mathbb{C}^{n+m}$ and $\left( C = C \left( \tilde{B} \right) ; \varphi \right)$ be a cluster structure in $\mathbb{C}(V)$ with $n$ cluster and $m$ stable variables such that

1. $\text{rank} \tilde{B} = n$;
2. there exists an extended cluster $\tilde{x} = (x_1, \ldots, x_{n+m})$ in $C$ such that $\varphi(x_i)$ is regular on $V$ for $i \in [n+m]$;
(3) for any cluster variable \( x'_k, k \in [n] \), obtained by applying the exchange relation \((2.1)\) to \( \varphi (x'_k) \) is regular on \( V \);

(4) for any stable variable \( x_{n+i}, i \in [m] \), the function \( \varphi (x_{n+i}) \) vanishes at some point of \( V \);

(5) each regular function on \( V \) belongs to \( \varphi (\mathcal{A}_C (\mathcal{C})) \).

Then \( \mathcal{C} \) is a regular cluster structure and \( \mathcal{A}_C (\mathcal{C}) \) is naturally isomorphic to \( O (V) \).

2.2. Compatible Poisson brackets. Let \( \{ \cdot, \cdot \} \) be a Poisson bracket on the ambient field \( \mathcal{F} \). Two elements \( f_1, f_2 \in \mathcal{F} \) are log canonical if there exists a rational number \( \omega_{f_1, f_2} \) such that \( \{ f_1, f_2 \} = \omega_{f_1, f_2} f_1 f_2 \). A set \( F \subseteq \mathcal{F} \) is called a log canonical set if every pair \( f_1, f_2 \in F \) is log canonical.

A cluster structure \( \mathcal{C} \) in \( \mathcal{F} \) is said to be compatible with the Poisson bracket \( \{ \cdot, \cdot \} \) if every cluster is a log canonical set with respect to \( \{ \cdot, \cdot \} \). In other words, for every cluster \( x \) and every two cluster variables \( x_i, x_j \in x \) there exists \( \omega_{ij} \) s.t.

\[
\{ x_i, x_j \} = \omega_{ij} x_i x_j
\]

The skew symmetric matrix \( \Omega^x = (\omega_{ij}) \) is called the coefficient matrix of \( \{ \cdot, \cdot \} \) (in the basis \( x \)).

If \( C(\tilde{B}) \) is a cluster structure of maximal rank (i.e., \( \text{rank} \tilde{B} = n \)), a complete characterization of all Poisson brackets compatible with \( C(\tilde{B}) \) is known (see [9], and also [10] Ch. 4). In particular, an immediate corollary of Theorem 1.4 in [9] is:

**Proposition 2.3.** If \( \text{rank} \tilde{B} = n \) then a Poisson bracket is compatible with \( C(\tilde{B}) \) if and only if its coefficient matrix \( \Omega^x \) satisfies \( B \Omega^x = [D \ 0] \), where \( D \) is a diagonal matrix.

A Lie group \( \mathcal{G} \) with a Poisson bracket \( \{ \cdot, \cdot \} \) is called a Poisson–Lie group if the multiplication map \( \mu : \mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G}, \mu : (x, y) \mapsto xy \) is Poisson. That is, \( \mathcal{G} \) with a Poisson bracket \( \{ \cdot, \cdot \} \) is a Poisson–Lie group if

\[
\{ f_1, f_2 \}(xy) = \{ \rho_y f_1, \rho_y f_2 \}(x) + \{ \lambda_x f_1, \lambda_x f_2 \}(y),
\]

where \( \rho_y \) and \( \lambda_x \) are, respectively, right and left translation operators on \( \mathcal{G} \).

A Poisson–Lie bracket on \( SL_n \) can be extended to one on \( GL_n \), with the determinant being a Casimir function. It will sometimes be easier to discuss \( GL_n \), and any statement can be restricted to \( SL_n \) by removing the determinant function.

Given a Lie group \( \mathcal{G} \) with a Lie algebra \( \mathfrak{g} \), let \( (\cdot, \cdot) \) be a nondegenerate bilinear form on \( \mathfrak{g} \), and \( t \in \mathfrak{g} \otimes \mathfrak{g} \) be the corresponding Casimir element. For an element

\[
r = \sum_i a_i \otimes b_i \in \mathfrak{g} \otimes \mathfrak{g}
\]

denote

\[
[r, r] = \sum_{i,j} [a_i, a_j] \otimes b_i \otimes b_j + \sum_{i,j} a_i \otimes [b_i, a_j] \otimes b_j + \sum_{i,j} a_i \otimes a_j \otimes [b_i, b_j]
\]

and

\[
r^{21} = \sum_i b_i \otimes a_i.
\]

The Classical Yang–Baxter equation (CYBE) is

\[
[r, r] = 0,
\]

an element \( r \in \mathfrak{g} \otimes \mathfrak{g} \) that satisfies \((2.3)\) together with the condition

\[
r + r^{21} = t
\]

is called a classical R-matrix.
A classical R-matrix $r$ induces a Poisson-Lie structure on $G$: choose a basis $\{I_\alpha\}$ in $\mathfrak{g}$, and denote by $\partial_\alpha$ (resp., $\partial'_\alpha$) the left (resp., right) invariant vector field whose value at the unit element is $I_\alpha$. Let $r = \sum_{\alpha,\beta} I_\alpha \otimes I_\beta$, then

\begin{equation}
\{f_1, f_2\}_r = \sum_{\alpha,\beta} r_{\alpha,\beta} (\partial_\alpha f_1 \partial_\beta f_2 - \partial'_\alpha f_1 \partial'_\beta f_2)
\end{equation}

defines a Poisson bracket on $G$. This is called the Sklyanin bracket corresponding to $r$.

In [1] Belavin and Drinfeld give a classification of classical R-matrices for simple complex Lie groups: let $\mathfrak{g}$ be a simple complex Lie algebra with a fixed nondegenerate invariant symmetric bilinear form $(\ ,\ )$. Fix a Cartan subalgebra $\mathfrak{h}$, a root system $\Phi$ of $\mathfrak{g}$, and a set of positive roots $\Phi^+$. Let $\Delta \subseteq \Phi^+$ be a set of positive simple roots.

A Belavin–Drinfeld (BD) triple is two subsets $\Gamma_1, \Gamma_2 \subset \Delta$ and an isometry $\gamma : \Gamma_1 \to \Gamma_2$ with the following property: for every $\alpha \in \Gamma_1$ there exists $m \in \mathbb{N}$ such that $\gamma^j(\alpha) \notin \Gamma_1$ for $j = 0, \ldots, m - 1$, but $\gamma^m(\alpha) \in \Gamma_1$. The isometry $\gamma$ extends in a natural way to a map between root systems generated by $\Gamma_1, \Gamma_2$. This allows one to define a partial ordering on the root system: $\alpha \prec \beta$ if $\beta = \gamma^j(\alpha)$ for some $j \in \mathbb{N}$.

Select now root vectors $E_\alpha \in \mathfrak{g}$ that satisfy $(E_\alpha, E_{-\alpha}) = 1$. According to the Belavin–Drinfeld classification, the following is true (see, e.g., [2] Ch. 3).

**Proposition 2.4.** (i) Every classical R-matrix is equivalent (up to an action of $\sigma \otimes \sigma$ where $\sigma$ is an automorphism of $\mathfrak{g}$) to

\begin{equation}
r = r_0 + \sum_{\alpha \in \Phi^+} E_{-\alpha} \otimes E_\alpha + \sum_{\alpha, \beta \in \Phi^+, \alpha \prec \beta} E_{-\alpha} \wedge E_\beta
\end{equation}

(ii) $r_0 \in \mathfrak{h} \otimes \mathfrak{h}$ in (2.6) satisfies

\begin{equation}
(\gamma (\alpha) \otimes 1) r_0 + (1 \otimes \alpha) r_0 = 0
\end{equation}

for any $\alpha \in \Gamma_1$, and

\begin{equation}
r_0 + r_0^{21} = t_0,
\end{equation}

where $t_0$ is the $\mathfrak{h} \otimes \mathfrak{h}$ component of $t$.

(iii) Solutions $r_0$ to (2.7), (2.8) form a linear space of dimension $k_T = |\Delta \setminus \Gamma_1|$.

Two classical R-matrices of the form (2.6) that are associated with the same BD triple are said to belong to the same Belavin–Drinfeld class. The corresponding bracket defined in (2.5) by an R-matrix $r$ associated with a triple $T$ will be denoted by $\{\cdot, \cdot\}_r$.

**Remark 2.5.** The BD data for $SL_n$ will be considered up to the following two isomorphisms: the first reverses the direction of $\gamma$ and transposes $\Gamma_1$ and $\Gamma_2$, while the second one takes each root $\alpha_j$ to $\alpha_{\omega(\alpha)}$, where $\omega_0$ is the longest element in the Weyl group (which in $SL_n$ is naturally identified with the symmetric group $S_{n-1}$). These two isomorphisms correspond to the automorphisms of $SL_n$ given by $X \mapsto -X^t$ and $X \mapsto \omega_0 X \omega_0$, respectively. Since R-matrices are considered up to an action of $\sigma \otimes \sigma$, from here on we do not distinguish between BD triples obtained one from the other via these isomorphisms. We will also assume that in the map $\gamma : \alpha_i \mapsto \alpha_j$ we always have $i < j$. 


Slightly abusing the notation, we sometimes refer to a root \( \alpha_i \in \Delta \) just as \( i \), and write \( \gamma : i \mapsto j \) instead of \( \gamma : \alpha_i \mapsto \alpha_j \). For shorter notation, denote the BD triple \((\{\alpha\},\{\beta\},\gamma : \alpha \mapsto \beta)\) by \( T_{\alpha\beta} \), and naturally the corresponding Sklyanin bracket will be \( \{\cdot,\cdot\}_{\alpha\beta} \).

Given a BD triple \( T \) for \( G \), write
\[
\mathfrak{h}_T = \{ h \in \mathfrak{h} : \alpha(h) = \beta(h) \text{ if } \alpha \prec \beta \},
\]
and define the torus \( \mathcal{H}_T = \text{exp} \mathfrak{h}_T \subset G \).

The following conjecture was given by Gekhtman, Shapiro and Vainshtein in [11]:

**Conjecture 2.6.** Let \( G \) be a simple complex Lie group. For any Belavin–Drinfeld triple \( T = (\Gamma_1,\Gamma_2,\gamma) \) there exists a cluster structure \( \mathcal{C}_T \) on \( G \) such that

1. the number of stable variables is \( 2k_T \), and the corresponding extended exchange matrix has a full rank.
2. \( \mathcal{C}_T \) is regular.
3. the corresponding upper cluster algebra \( \mathcal{A}_C(\mathcal{C}_T) \) is naturally isomorphic to \( \mathcal{O}(G) \);
4. the global toric action of \( (\mathbb{C}^*)^{2k_T} \) on \( \mathcal{C}(G) \) is generated by the action of \( \mathcal{H}_T \otimes \mathcal{H}_T \) on \( G \) given by \( (H_1,H_2)(X) = H_1XH_2 \);
5. for any solution of CYBE that belongs to the Belavin–Drinfeld class specified by \( T \), the corresponding Sklyanin bracket is compatible with \( \mathcal{C}_T \);
6. a Poisson–Lie bracket on \( G \) is compatible with \( \mathcal{C}_T \) only if it is a scalar multiple of the Sklyanin bracket associated with a solution of CYBE that belongs to the Belavin–Drinfeld class specified by \( T \).

The conjecture was proved for the Belavin–Drinfeld class \( \Gamma_1 = \Gamma_2 = \emptyset \). This trivial triple corresponds to the standard Poisson–Lie bracket. We call the cluster structures associated with the non-trivial Belavin–Drinfeld data exotic.

In the Cremmer–Gervais case \( \Gamma_1 = \{\alpha_1,\ldots,\alpha_{n-2}\}, \Gamma_2 = \{\alpha_2,\ldots,\alpha_{n-1}\} \) and \( \gamma : \alpha_i \mapsto \alpha_{i+1} \). This case is, in some sense, “the furthest” from the standard case, because here \( |\Gamma_1| \) is maximal. Conjecture 2.6 was proved for the Cremmer–Gervais case in [13]. The conjecture is also true for all exotic cluster structures on \( SL_n \) with \( n \leq 4 \) [11] and for \( SL_5 \) [6].

One step towards a proof of the conjecture was this Proposition, from [11]:

**Proposition 2.7.** Let \( T = (\Gamma_1,\Gamma_2,\gamma) \) be a BD triple. Suppose that assertions 1 and 4 of Conjecture 2.6 are valid and that assertion 5 is valid for one particular \( R \)-matrix in the BD class specified by \( T \). Then 5 and 6 are valid for the whole BD class specified by \( T \).

3. Results

For \( SL_n \) with any \( n \geq 3 \) and BD data \( \{\alpha\} \mapsto \{\beta\} \), part 1 of conjecture 2.6 was proved in [5]. This paper completes the proof by showing that parts 2–6 are also true.

**Theorem 3.1.** For any \( n \geq 3 \) and BD data \( T = (\{\alpha\},\{\beta\},\alpha \mapsto \beta) \), the upper cluster algebra \( \mathcal{A}_C(\mathcal{C}_T) \) on \( SL_n \) is naturally isomorphic to \( \mathcal{O}(SL_n) \).

**Proof.** The proof relies on a Proposition 2.2. Conditions 1, 2 and 3 of the proposition hold, according to [6]. To satisfy Condition 4 note that the Poisson bracket
Theorem 3.2. The global toric action of \( \cdot \cdot \cdot \cdot \) on \( SL_n \) can be extended to \( \text{Mat}(n) \) by requiring that the determinant is a Casimir function. Clearly, on \( \text{Mat}(n) \) Condition 4 holds. Last, we need that for any \( n \geq 3 \) and BD data \( T = \{(\alpha), (\beta), \alpha \rightarrow \beta\} \), all regular functions on \( SL_n \) are in \( \overline{\mathcal{A}}_C(C_T) \). To prove this, it suffices to show that for all \((i,j) \in [n] \times [n]\) the function \( x_{ij} \) belongs to \( \overline{\mathcal{A}}_C(C_T) \). This is true according to Theorem 5.1 and therefore \( \overline{\mathcal{A}}_C(C_T) \) is naturally isomorphic to \( \mathcal{O}(SL_n) \).

The second result is statement 4 of conjecture 2.6.

**Theorem 3.2.** The global toric action of \((\mathbb{C}^*)^{2kT} \) on \( \mathbb{C}(SL_n) \) is generated by the action of \( \mathcal{H}_T \otimes \mathcal{H}_T \) on \( SL_n \) given by \((H_1, H_2)(X) = H_1 X H_2\). which will be proved in Section 6.

Now we can use Proposition 3.1 which implies that assertions 5 and 6 of Conjecture 2.6 are true.

4. Construction of an initial seed

For a given BD data \( T_{\alpha \beta} = \{(\alpha), (\beta), \gamma \rightarrow \beta\} \), an initial seed for the cluster structure \( C_{\alpha \beta} \) is described in [6]. The construction defines a set of matrices \( M \) such that the set of all determinants of these matrices forms the initial cluster. We repeat the main part of it here:

For a matrix \( X \) let \( M_{ij}(X) \) be the maximal contiguous submatrix of \( X \) with \( x_{ij} \) at the upper left hand corner. That is,

\[
M_{ij}(X) = \begin{cases} 
  \begin{bmatrix} 
    x_{ij} & \cdots & x_{in} \\
    \vdots & \ddots & \vdots \\
    x_{n-j+i,j} & \cdots & x_{n-j+i,n} 
  \end{bmatrix} & \text{if } j > i \\
  \begin{bmatrix} 
    x_{ij} & \cdots & x_{i,n-i+j} \\
    \vdots & \ddots & \vdots \\
    x_{nj} & \cdots & x_{n,n-i+j} 
  \end{bmatrix} & \text{otherwise.}
\end{cases}
\]

Now define for every \((i,j) \in [n] \times [n]\),

\[
f_{ij}(X) = \det M_{ij}(X).
\]

Let \( X^C_R \) denote the submatrix of \( X \) with rows in the set \( R \) and columns in \( C \) (with \( R, C \subseteq [n] \)). Then define two special families of matrices: for \( 1 \leq j \leq \alpha \) and \( i = n + j - \alpha \) set

\[
\tilde{M}_{ij}(X) = \begin{bmatrix} 
  X_{[j,\alpha+1]}^{[i,n]} & 0_{(n-i) \times (\mu-1)} \\
  0_{\mu \times (n-i)} & X_{[\beta,n]}^{[\mu]} 
\end{bmatrix}
\]

with \( \mu = n - \beta \), and for \( 1 \leq i \leq \beta \) and \( j = n + i - \beta \), set

\[
\tilde{M}_{ij}(X) = \begin{bmatrix} 
  X_{[j,\beta+1]}^{[i,\mu]} & 0_{(n-j) \times \mu} \\
  0_{(\mu-1) \times (n-j+1)} & X_{[\alpha,n]}^{[\mu]} 
\end{bmatrix},
\]

and here \( \mu = n - \alpha \). Note that these matrices are not block diagonal: in the first case the number of columns in each of the two blocks is greater than the number of rows by one, while in the second case the number of rows in each block is greater than the number of columns by one.
Take the set of matrices \( \{ M_{ij} \}_{i,j=1}^n \) and whenever \( i = n + j - \alpha \) or \( j = n + i - \beta \) replace \( M_{ij}(X) \) with \( \tilde{M}_{ij}(X) \). This assures that for a fixed pair \((i,j) \in [n] \times [n] \) there is still a unique matrix in the set with \( x_{ij} \) at the upper left corner. Denote this matrix (either \( M_{ij}(X) \) or \( \tilde{M}_{ij}(X) \)) by \( M_{ij} \), and set \( \varphi_{ij} = \det M_{ij} \).

Since \( \varphi_{11} = \det X \) is constant on \( SL_n \) we ignore it and set the initial cluster to be \( B_{\alpha \beta} = \{ \varphi_{ij} | i,j \in [n] \} \setminus \{ \varphi_{11} \} \).

To describe the initial quiver \( Q_{\alpha \beta} \) start with the initial quiver of the standard case on \( SL_n \): the vertices are placed on an \( n \times n \) grid, with rows numbered from top to bottom and columns numbered left to right. A vertex \((i,j) \) corresponds to the cluster variable \( f_{ij} \). The vertices in the first row and first column are frozen.

There are arrows from a vertex \((i,j) \) to vertices

- \((i+1,j) \) if \( i \neq n \)
- \((i,j+1) \) if \( j \neq n \)
- \((i-1,j-1) \) if \( i \neq 1 \) and \( j \neq 1 \)

The standard quiver for \( SL_5 \) is shown in Fig. 4.1 with squares representing frozen vertices and circles representing mutable ones.

The quiver \( Q_{\alpha \beta}(n) \) for the BD triple \( T_{\alpha \beta} \) on \( SL_n \) has similar form. It also has \( n^2 - 1 \) vertices on an \( n \times n \) grid, with same arrows as the standard one and the following changes:

1. Vertices \((\alpha + 1,1) \) and \((1,\beta + 1) \) are not frozen.
2. The arrows \((\alpha,1) \rightarrow (\alpha + 1,1) \) and \((1,\beta) \rightarrow (1,\beta + 1) \) are added.
3. The arrows \((n,\alpha + 1) \rightarrow (1,\beta + 1), (1,\beta + 1) \rightarrow (n,\alpha) \) are added.
4. The arrows \((\beta + 1,n) \rightarrow (\alpha + 1,1), (\alpha + 1,1) \rightarrow (\beta,n) \) are added.
The example of $Q_{2→3}(5)$ on $SL_5$ is given in Figure 4.1. The dashed arrows are the arrows that were added to the standard quiver.

Associating every vertex $(i,j)$ of the quiver $Q_{αβ}$ with the cluster variable $ϕ_{ij}$ turns the pair $(B_{αβ}, Q_{αβ})$ into the initial seed for a cluster structure $C_{αβ}$ on $SL_n$. This cluster structure is compatible with the bracket $\{·,·\}_{αβ}$, and the initial seed is locally regular. More details can be found in [6].

In Section 5 we will use the following notations: for a function $f_{ij} = det X_{\ell}[j,k]\left[i,ℓ\right]$ we write

\begin{align*}
    f_{ij}^→ &= det X_{\ell}[j,ℓ-1,ℓ+1]\left[i,k\right] \\
    f_{ij}^← &= det X_{\ell}[j-1,j+1,ℓ]\left[i,k\right] \\
    f_{ij}^↑ &= det X_{\ell}[j,ℓ]\left[i−1,i+1,...,k\right] \\
    f_{ij}^↓ &= det X_{\ell}[j-1,ℓ]\left[i,...,k−1,k+1\right].
\end{align*}

5. THE NATURAL ISOMORPHISM

**Theorem 5.1.** $x_{ij} \in \mathcal{T}_{αβ}$ for every $(i,j) \in [n] \times [n]$

**Proof.** According to Lemma 5.2 all $x_{in}$ and $x_{nj}$ are in $\mathcal{T}_{αβ}$. If $β ≠ n − 1$ we use induction on $n$: according to Proposition 5.3 $\mathcal{T}_{αβ}(n−1) \subset \mathcal{T}_{αβ}(n)$. By the induction hypothesis, $x_{ij} \in \mathcal{T}_{αβ}(n−1)$ for all $(i,j) \in [n−1] \times [n−1]$.

If $β = n − 1$ and $α ≠ 1$ then $\mathcal{T}_{αβ}$ is isomorphic to $\mathcal{T}_{1,n−α}$ (see Remark 2.5) and the argument above holds. Last, the case $α = 1, β = n − 1$ is covered by Lemma 5.10. □

**Lemma 5.2.** $x_{in} \in \mathcal{T}_{αβ}$ and $x_{nj} \in \mathcal{T}_{αβ}$ for all $i, j \in [n]$. 

**Figure 4.2.** The quiver $Q_{2→3}$ on $SL_5$
Proof. $x_{nj}$ is a cluster variable in the initial cluster for every $j \neq \alpha$. To see that $x_{na} \in \mathcal{A}$, note that

$$\varphi_{na} = x_{na} f_{1,\beta+1} - x_{n,\alpha+1} f_{1,\beta+1}^{-1},$$

and therefore

$$x_{na} = \frac{\varphi_{na} + x_{n,\alpha+1} f_{1,\beta+1}^{-1}}{f_{1,\beta+1}}. \quad (5.1)$$

Mutating at $(1, \beta + 1)$ it is not hard to see that

$$\varphi_{1,\beta+1}' = x_{na} f_{2,\beta+1} - x_{n,\alpha+1} f_{2,\beta+1}^{-1},$$

and so

$$x_{na} = \frac{\varphi_{1,\beta+1}' + x_{n,\alpha+1} f_{2,\beta+1}^{-1}}{f_{2,\beta+1}}. \quad (5.2)$$

We can now combine (5.1) and (5.2) into:

$$x_{na} = \frac{\varphi_{na} + x_{n,\alpha+1} f_{1,\beta+1}^{-1}}{f_{1,\beta+1}} = \frac{\varphi_{1,\beta+1}' + x_{n,\alpha+1} f_{2,\beta+1}^{-1}}{f_{2,\beta+1}}.$$ 

Clearly, $f_{1,\beta+1}$ and $f_{2,\beta+1}$ are cluster variables. The functions $f_{1,\beta+1}^{-1}$ and $f_{2,\beta+1}^{-1}$ can be obtained from the initial cluster through the mutation sequence

$$((n, \beta + 1), (n - 1, \beta + 1), \ldots (3, \beta + 1), (2, \beta + 1)),$$ 

as shown in [6] Lemma 2.1 now proves $x_{na} \in \mathcal{A}$.

In a similar way, for all $i \neq \beta$ the function $x_{in}$ is a cluster variable in the initial seed, and therefore $x_{in} \in \mathcal{A}$. Symmetric arguments to those above yield

$$x_{\beta n} = \frac{\varphi_{\beta n} + x_{\beta + 1, n} f_{\alpha + 1, 1}^{-1}}{f_{\alpha + 1, 1}}$$

and

$$x_{\beta n} = \frac{\varphi_{\beta n} + x_{\beta + 1, n} f_{\alpha + 1, 2}^{-1}}{f_{\alpha + 1, 2}}.$$ 

The functions $f_{\alpha + 1, 1}^{-1}$ and $f_{\alpha + 1, 2}^{-1}$ are in $\mathcal{A}$ (see [6] again), and $f_{\alpha + 1, 1}$ and $f_{\alpha + 1, 2}$ are cluster variables. According to Lemma 2.1 $x_{\beta n} \in \mathcal{A}$.

5.1. Mutation Sequence $S$.

**Proposition 5.3.** For $\beta \neq n - 1$ there is a cluster mutation sequence $S$ such that $S(Q_{\alpha \beta}(n))$ contains a subquiver isomorphic to $Q_{\alpha \beta}(n - 1)$. The vertices of this subquiver can be rearranged on an $(n - 1) \times (n - 1)$ grid such that the vertex $(i, j)$ now corresponds to the cluster variables $\varphi_{ij}$ as defined on $SL_{n-1}$.

**Proof.** Define $S$ as a composition $S = S_c \circ S_h$ of two sequences $S_c, S_h$ that are introduced hereinafter. Let $p_{h}^{(k)}$ be the sequence of quiver vertices starting at $(n, k)$ and moving diagonally up and to the left. That is, $p_{h}^{(k)} = ((n, k), (n - k, k - 1), \ldots)$. If the vertex $(n - k + 1, 1)$ is frozen, then it is the last vertex of $p_{h}^{(k)}$. If this is not the case, which means $k = n - \alpha$, proceed with

$$p_{h}^{(k)} = ((n, k), \ldots, (\alpha + 1, 1), (\beta, n), (\beta - 1, n - 1), \ldots)$$

1In [6] it is shown that $f_{1,\beta+1}^{-1}$ and $f_{2,\beta+1}^{-1}$ are cluster variables of the standard structure on $SL_n$. However, it is not difficult to see that this is also true for our case, because locally the quiver looks the same around the mutated vertices, and they correspond to the same cluster variables.
until hitting a frozen vertex (so the last vertex of $p^{(k)}_h$ must be frozen). Now let $S^{(k)}_h$ be the sequence of mutations along the vertices of the path $p^{(n+1-k)}_h$, excluding the last (frozen) vertex.

Next, define a family of quivers $\left\{ Q^{(k)}_h \right\}_{k=0}^{n-1}$: the first quiver is $Q^{(0)}_h = Q_{\alpha\beta}(n)$. The quiver $Q^{(k+1)}_h$ is obtained from $Q^{(k)}_h$ by deleting the vertex $(n, n-k)$ and all the arrows incident to it. If in $Q^{(k)}_h$ there is an arrow connecting $(n, n-k)$ to a vertex $(1, j)$, there are two more changes:

- an arrow $(n-1, n-k) \to (1, j)$ is added;
- the arrow $(1, j) \to (n, n-k-1)$ is replaced by $(1, j) \to (n-1, n-k-1)$.

The quivers $Q^{(3)}_h$ and $Q^{(4)}_h$ for BD triple $T_{24}$ (that is, $2 \mapsto 4$) on $SL_6$ are shown in Fig. 5.1 and Fig. 5.2, respectively.

Denote by $\sigma^{(k)}_h$ the following manipulations on a quiver $Q$:

1. Apply the mutation sequence $S^{(k)}_h$.
2. Freeze the last vertex of $S^{(k)}_h$.
3. Remove the last vertex of $p^{(n+1-k)}_h$ from the quiver.
4. Shift each vertex in $p^{(n+1-k)}_h$ to the position of its successor (so $(i, j)$ is shifted to $(i-1, j-1)$ if $j \neq 1$. If $(\alpha+1, 1)$ is in $p^{(n+1-k)}_h$, it is shifted to $(\beta, n)$).
We will show that after applying $S_h(k)$ to $Q_h(k-1)$, the last vertex of $p_h^{(n+1-k)}$ has only one neighbor: the last vertex of $S_h(k)$. When the latter is frozen, the last vertex of $p_h^{(n+1-k)}$ becomes isolated (as arrows connecting frozen vertices are ignored), and can be removed from the quiver.

We will assign a function $\psi_{ij}^k$ to every vertex $(i,j)$ of $Q_h(k)$. For a function $f_{ij} = \det X_{[i,k]}^{[j,\ell]}$ write $f_{ij}^{(m)} = \det X_{[i,k-m]}^{[j,\ell-m]}$. Any function $\varphi_{ij}$ can be written as determinant of some matrix $M$, as defined in Section 4. In this case, $\varphi_{ij}^{(m)}$ will stand for the determinant of the submatrix obtained from $M$ by deleting the last $m$ rows and last $m$ columns. Now let

$$\psi_{ij}^h = \begin{cases} f_{ij}^{(1)} f_{1,\beta+1}^{(1)\rightarrow} f_{1,\beta+1}^{(1)\rightarrow} & i = n - \alpha + 1 \\ \varphi_{ij}^{(1)} & \text{otherwise} \end{cases}$$

set $\psi_{ij}^0 = \varphi_{ij}$, and define

$$\psi_{ij}^{k+1} = \begin{cases} \psi_{ij}^h & (i,j) \in p_h^{(n+1-k)} \\ \psi_{ij}^k & \text{otherwise} \end{cases}$$

so the pair $(Q_h^{(k)}, \{\psi_{ij}^k\})$ defines a seed that will be denoted $\Sigma_h^{(k)}$.

In a similar way, define $p_v^{(k)}$, $\sigma_v^{(k)}$, $\sigma_v(k)$ and $Q_v^{(k)}$: let $p_v^{(k)}$ denote the path $((k,n),(k-1,n-1),\ldots,(1,n-k+1))$ if $(1,n-k+1)$ is a frozen vertex (i.e., if $k \neq n - \beta$). Otherwise the path continues:
$((n - \beta, n), (n - \beta - 1, n - 1), \ldots, (1, \beta + 1), (n, \alpha), (n - 1, \alpha - 1) \ldots)$ until hitting a frozen vertex. The sequence $S_p^{(k)}$ is the sequence of mutations at the vertices of $p_v^{(n-k)}$, excluding the last (frozen) one. The quivers $Q_v^{(k)}$ are defined recursively: set $Q_v^{(0)} = Q_h^{(n-1)}$ and let $Q_v^{(k+1)}$ be the quiver obtained from $Q_v^{(k)}$ by removing the vertex $(n - k, n)$ and all the arrows incident to it. If there is an arrow in $Q_v^{(k)}$ connecting $(n - k, n)$ to a vertex $(i, 1)$ then in addition

- an arrow $(n - k, n - 1) \rightarrow (\alpha + 1, 1)$ is added;
- the arrow $(\alpha + 1, 1) \rightarrow (n - k - 1, n)$ is replaced by $(\alpha + 1, 1) \rightarrow (n - k - 1, n - 1)$.

We now define $\sigma_v^{(k)}$ on a quiver $Q$:

1. Apply the mutation sequence $S_v^{(k)}$.
2. Freeze the last vertex of $S_v^{(k)}$.
3. Remove the last vertex of $p_v^{(n-k)}$ from the quiver.
4. Shift each vertex in $p_v^{(n-k)}$ to the position of its successor (so $(i, j)$ is shifted to $(i - 1, j - 1)$ if $j \neq 1$. If $(1, \beta + 1)$ is in $p_v^{(n-k)}$, it is shifted to $(n, \alpha)$).

Let $\sigma_h = \sigma_{h(1)}^{(n-1)} \circ \cdots \circ \sigma_{h(n)}^{(1)}$ and $\sigma_v = \sigma_{v(1)}^{(n-2)} \circ \cdots \circ \sigma_{v(2)}^{(2)} \circ \sigma_{v(1)}^{(1)}$ (and correspondingly, let $S_h = S_{h(1)}^{(1)} \circ \cdots \circ S_{h(n)}^{(2)} \circ S_h^{(1)}$ and $S_v = S_v^{(n-2)} \circ \cdots \circ S_v^{(2)} \circ S_v^{(1)}$). According to Lemma 5.4, $\sigma_h(\Sigma_{\alpha\beta}(n)) = \Sigma_{h(\alpha\beta)}^{(n-1)}$, and according to Lemma 5.5, $\sigma_v(\Sigma_h^{(n-1)}) = \Sigma_{\alpha\beta}(n - 1)$. Therefore

$$\sigma_v \circ \sigma_h(\Sigma_{\alpha\beta}(n)) = \Sigma_{h(\alpha\beta)}^{(n-1)}(n - 1),$$

which completes the proof.

The following lemma uses the Desnanot–Jacobi identity (see [3] Th. 3.12): for a square matrix $A$, denote by “hatted” subscripts and superscripts deleted rows and columns, respectively. Then

\begin{equation}
\det A \cdot \det A_{\hat{c_1}\hat{c_2}} = \det A_{\hat{c_1}} \cdot \det A_{\hat{r_2}} - \det A_{\hat{r_1}} \cdot \det A_{\hat{c_2}}.
\end{equation}

By adding an appropriate row, we get a similar result for a non square matrix $B$ with number of rows greater by one than the number of columns:

\begin{equation}
\det B_{\hat{r_1}} \det B_{\hat{r_2}r_3} = \det B_{\hat{c_1}} \det B_{\hat{c_1}\hat{r_3}} - \det B_{\hat{r_2}} \det B_{\hat{c_2}},
\end{equation}

and naturally, a similar identity can be obtained for a matrix with number of columns greater by one than the number of rows.

**Lemma 5.4.** For any $\beta \neq n - 1$, the seed $\Sigma_h^{(n-1)}$ is obtained from the initial seed $\Sigma_{\alpha\beta}(n)$ through the sequence $\sigma_h = \sigma_{h(1)}^{(n-1)} \circ \cdots \circ \sigma_{h(n)}^{(2)} \circ \sigma_{h(1)}^{(1)}$.

**Proof.** Apply the sequence $S_h^{(1)}$ to the quiver $Q_h^{(0)}$: mutating at $(n, n)$ removes the arrows $(n - 1, n - 1) \rightarrow (n - 1, n)$ and $(n - 1, n - 1) \rightarrow (n, n - 1)$, and reverses the arrows incident to $(n, n)$ so now they are $(n, n) \rightarrow (n - 1, n)$, $(n, n) \rightarrow (n, n - 1)$ and $(n - 1, n - 1) \rightarrow (n, n)$. The next mutation at $(n - 1, n - 1)$ (shown in Fig. 5.3) then removes the arrows $(n - 2, n - 2) \rightarrow (n - 2, n - 1)$ and $(n - 2, n - 2) \rightarrow (n - 1, n - 2)$, and reverses arrows so now we have $(n - 1, n - 1) \rightarrow (n - 2, n - 1)$, $(n - 1, n - 1) \rightarrow (n - 1, n - 2)$, $(n - 2, n - 2) \rightarrow (n - 1, n - 1)$, and $(n, n) \rightarrow (n - 1, n - 1)$. Use induction now to see that this mutation looks the same at each vertex of the sequence. As a result, after the last mutation at $(2, 2)$ the vertex $(1, 1)$
has only \((2, 2)\) as a neighbor. Consequently, after freezing \((2, 2)\) the vertex \((1, 1)\) becomes isolated, and can not take part in any future mutations. It is therefore removed from the quiver. It is not hard to see that 
\[
\sigma^{(1)}_h \left( Q^{(0)}_h \right) = Q^{(1)}_h.
\]

At each step of this sequence the exchange relation at \((i, i)\) is

\[
\varphi_{ii} \varphi'_{ii} = \varphi_{i-1,i-1} \varphi'_{i+1,i+1} + \varphi_{i-1,i} \varphi'_{i,i-1}.
\]

Assume (by induction) that \(\varphi'_{i+1,i+1} = \varphi^{(1)}_{ii}\) and write \(A = X^{\left[i-1,n\right]}_{\left[i-1,n\right]}\), with \(\ell\) for the last row (and column) of \(A\). The exchange rule \((5.5)\) then becomes

\[
\varphi_{ii} \varphi'_{ii} = \det A \det A^{\hat{i}\ell} + \det A^{\hat{i}1} \det A^{\hat{i}\ell}
\]

and according to \((5.3)\)

\[
\varphi_{ii} \varphi'_{ii} = \det A^{\hat{i}1} \det A^{\hat{i}\ell} = \varphi_{ii} \varphi^{(1)}_{i-1,i-1},
\]

which means

\[
\varphi'_{ii} = \varphi^{(1)}_{i-1,i-1},
\]

and this proves that 
\[
\sigma_1 \left( \Sigma_{\alpha,\beta}(n) \right) = \Sigma^{(1)}_h.
\]

We will now show that \(\sigma^{(k)}_h \left( Q^{(k-1)}_h \right) = Q^{(k)}_h\) for every \(k \in [n-1]\). If \(k \notin \{\alpha + 1, n - \alpha\}\) then we use induction again to assume that locally at each step the path \(p^{(n+1-k)}_h\) is isomorphic to \(p^{(n)}_h\), so the quiver mutations are (locally) identical, and after the mutations and the removal of the last vertex we end up with \(Q^{(k)}_h\).

The mutation rule at \((n-m, n+1-k-m)\) of the sequence \(S^{(k)}_h\) looks like the one at \((n-m, n-m)\) in the sequence \(S^{(1)}_h\). Put \(i = n-m\) and \(j = n+1-k-m\) so the exchange relation is

\[
\varphi_{ij} \varphi'_{ij} = \varphi_{i-1,j-1} \varphi'_{i+1,j+1} + \varphi^{(1)}_{i-1,j} \varphi_{i,j-1}.
\]
Assume by induction $\varphi'_{i+1,i+1} = \varphi^{(1)}_{ij}$, and write $A = X^{[j,\mu]}_{[i,n]}$ with $\mu = n - i + j$. We can now use (5.3) to obtain

$$\varphi'_{ij} = \varphi^{(1)}_{ij}.$$  

Hence, for $k \notin \{\alpha + 1, n - \alpha\}$ we have

$$(5.6)\quad \sigma^{(k)}_h \left( \Sigma^{(k-1)}_h \right) = \Sigma^{(k)}_h.$$  

If $k = \alpha + 1$, then the above holds for the first part of the sequence $S^{(k)}_h$, which is $((n, n - \alpha), \ldots, (\alpha + 2, 2))$. The sequence then continues with a mutation at $(\alpha + 1, 1)$ as illustrated in Fig. 5.4.

The exchange relation is

$$(5.7)\quad \varphi_{\alpha+1,1,\alpha+1,1} = \varphi_{\alpha+2,1} \varphi_{\alpha+1,1} \varphi_{\beta+1,n} + \varphi'_{\alpha+2,2} \varphi_{\beta,n},$$

and writing

$$A = \begin{bmatrix}
x_{\beta n} & x_{\alpha 1} & \cdots & \cdots & x_{\alpha \mu} \\
x_{\beta+1,n} & x_{\alpha+1,1} \\
0 & x_{\alpha+2,1} & \ddots \\
\vdots & \vdots & \ddots \\
0 & x_{n 1} & \cdots & x_{n \mu}
\end{bmatrix}$$
with $\mu = n - \alpha$, one can use (5.3) to obtain

$$
\varphi'_{\alpha+1,1} = \det \begin{bmatrix}
    x_\beta n & x_{\alpha 1} & \cdots & \cdots & x_{\alpha,\mu - 1} \\
    x_{\beta+1,n} & x_{\alpha+1,1} & & & \\
    0 & x_{\alpha+2,1} & \ddots & & \\
    \vdots & \vdots & \ddots & \ddots & \\
    0 & x_{n-1,1} & \cdots & x_{n-1,\mu - 1}
\end{bmatrix} = \varphi^{(1)}_{\beta n}.
$$

The next mutations are at $(\beta - k, n - k)$ with $k = 0, 1, \ldots, \beta + 2$ (for $k = 0$ we label $(\alpha + 1, 1)$ as $(\beta + 1, n + 1)$). The relevant part of the quiver is shown in Fig. 5.5.

Four arrows - $(\beta - k - 1, n - k) \rightarrow (\beta - k - 1, n - k - 1)$, $(\beta - k - 1, n - k - 1) \rightarrow (\beta - k, n - k - 1)$, $(\beta - k - 1, n - k - 1) \rightarrow (\alpha, 1)$ and $(\beta - k - 1, n - k - 1) \rightarrow (\alpha + 2, 1)$ are removed, and four new arrows are added: $(\alpha, 1) \rightarrow (\beta - k - 1, n - k - 1)$, $(\alpha + 2, 1) \rightarrow (\beta - k - 1, n - k - 1)$, $(\beta - k, n - k) \rightarrow (\beta - k + 1, n - k + 1)$ and $(\beta - k, n - k - 1) \rightarrow (\beta - k + 1, n - k + 1)$.

In addition, all the arrows incident to $(\beta - k, n - k)$ are inverted.

The exchange rule here (with $i = \beta - k$, and $j = n - k$) is

$$
\varphi_{ij}\varphi'_{ij} = \varphi_{i-1,j-1}\varphi'_{i+1,i+1} + \varphi_{i-1,j}\varphi'_{i,j-1}\varphi_{\alpha 1}\varphi_{\alpha 2,1}.
$$
Assume by induction $\varphi_{i+1,j+1}^{(1)} = \varphi_{ij}^{(1)}$, and as in the previous cases, write

$$A = \begin{bmatrix}
 x_{i-1,j-1} & \cdots & x_{i-1,n} & 0 & \cdots & 0 \\
 \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
 \vdots & & x_{\beta \alpha} & x_{\alpha 1} & \cdots & x_{\alpha \mu} \\
 x_{\beta+1,j-1} & \cdots & x_{\beta+1,n} & x_{\alpha+1,1} & \vdots & \vdots \\
 0 & \cdots & 0 & \vdots & \ddots & \vdots \\
 \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
 0 & \cdots & 0 & x_{n 1} & \cdots & x_{n \mu}
\end{bmatrix}$$

and let $\ell$ denote the last row (and column) of $A$, and now [5.8] turns to

$$\varphi_{ij}^{(1)} = \det A \det A_{1\ell}^{(1)} + \det A_{i}^{(1)} \det A_{1\ell}^{(1)} = \det A_{i}^{(1)} \det A_{1\ell}^{(1)},$$

so that $\varphi_{ij}^{(1)} = \varphi_{i-1,j-1}^{(1)}$. This proves [5.6] for $k = \alpha + 1$.

The last case is $k = n - \alpha$, with $S_h^{(k)} = ((n, \alpha + 1), (n - 1, \alpha), \ldots)$. Fig. 5.6 shows the mutation at $(n, \alpha + 1)$: The arrows $(n-1, \alpha+1) \rightarrow (n-1, \alpha), (n-1, \alpha) \rightarrow (n, \alpha)$ and $(1, \beta + 1) \rightarrow (n-1, \alpha + 1)$ is added, while an arrow $(1, \beta + 1) \rightarrow (n-1, \alpha + 1)$ is inverted. The four arrows incident to $(n, \alpha + 1)$ are inverted.

The exchange relation here is

$$\varphi_{n, \alpha + 1} \varphi_{n, \alpha + 1}^{(1)} = \varphi_{n \alpha} \varphi_{n-1, \alpha + 1}^{(1)} + \varphi_{n-1, \alpha} \varphi_{1, \beta + 1}^{(1)}.$$
because the function associated with the vertex \((n - 1, \alpha + 1)\) is now \(\varphi^{(1)}_{n-1,\alpha+1}\) (in the quiver \(Q^{(k-1)}_{\alpha\beta}\)). Recall that \(\varphi_{n\alpha} = x_{n\alpha}f_{1,\beta+1} - x_{n,\alpha+1}f_{1,\beta+1}^{\tau}\), so we can write
\[
\varphi_{n\alpha+1}\varphi'_{n,\alpha+1} = \left( x_{n\alpha}f_{1,\beta+1} - x_{n,\alpha+1}f_{1,\beta+1}^{\tau} \right) x_{n-1,\alpha+1} + \left| \begin{array}{c} x_{n-1,\alpha} \ \ x_{n-1,\alpha+1} \\ x_{n\alpha} \ \ x_{n,\alpha+1} \end{array} \right| f_{1,\beta+1} \]
and therefore
\[
\varphi'_{n,\alpha+1} = x_{n-1,\alpha}f_{1,\beta+1} - x_{n,\alpha+1}f_{1,\beta+1}^{\tau},
\]
Proceeding along the path \(S_h^{(k)} = ((n, \alpha + 1), (n - 1, \alpha), \ldots)\), the result is
\[
\varphi'_{ij} = f_{i-1,j-1}^{(1)}f_{1,\beta+1} - f_{i-1,j-1}^{(\tau)}f_{1,\beta+1}^{\tau},
\]
for all \(i = n - \alpha - 1 + j\). This shows that \(5.6\) holds for \(k = n - \alpha\) as well.

We have shown that \(\sigma_h^{(k)}(\Sigma_h^{(k-1)}) = \Sigma_h^{(k)}\) for \(k \in [1, n]\). Thus, writing \(\sigma_n = \sigma_h^{(n-1)} \circ \sigma_h^{(2)} \circ \sigma_h^{(1)}\) we have proved
\[
\sigma_h(\Sigma_{\alpha\beta}) = \Sigma_{h(\alpha\beta)}.
\]

**Lemma 5.5.** The seed \(\Sigma_{\alpha\beta}(n-1)\) is obtained from the initial seed \(\Sigma_h^{(n-1)}\) through the sequence \(\sigma_v = \sigma_v^{(n-1)} \circ \cdots \circ \sigma_v^{(2)} \circ \sigma_v^{(1)}\).

**Proof.** The proof is almost identical to the proof of Lemma [5.4] \(\square\)

### 5.2. Mutation sequences \(T_h\).

Start with the mutation sequence \(T_1\), mutating along columns of the quiver \(Q_{1\rightarrow n-1}\), bottom to top, from the right column to the left. That is, define
\[
T_1 = (\langle n, n \rangle, \langle n - 1, n \rangle, \ldots, (1, n), (n, n - 1), (n - 1, n - 1), \ldots, \\
(2, n - 1), \ldots, (n, 2), (n - 1, 2), \ldots, (2, 2),)
\]
and let
\[
T_m = (\langle n, n \rangle, \ldots, (m + 1, n), (n, n - 1), \ldots (m + 1, n - 1), \ldots, \\
(n, m + 1), \ldots, (m + 1, m + 1)).
\]
This sequence has only columns \(n, n - 1, \ldots, m + 1\) and each column only contains rows \(n, n - 1, \ldots, m + 1\). Note that \(T_m \subset T_{m-1}\). Set \(\Sigma_0 = (B_{1,n-1}, Q_{1,n-1}(n))\) (the initial seed) and define \(\Sigma_m = (B_m, Q^m)\) as the seed obtained from \(\Sigma_{m-1}\) through the sequence of mutations \(T_m\). Let \(\varphi_{ij}^m\) denote the function associated with the vertex \((i, j)\) in the seed \(\Sigma_m\), and let
\[
\tilde{f}_{ij} = f_{ij}f_{21} - f_{ij}^\dagger f_{21}^\dagger,
\]
and
\[
\tilde{f}_{ij}^{[1]} = f_{ij}f_{21} - f_{ij}^{(1)}f_{21}^\dagger.
\]
Note that if \(\varphi = \tilde{f}_{ij}\) then \(\varphi^{(1)} \neq \tilde{f}_{ij}^{[1]}\).
**Proposition 5.6.** In the seed $\Sigma_1$,

1. for $i,j \in \{2, \ldots , n\}$ the cluster variables take the form

$$\varphi_{ij}^1 = \begin{cases} f_{i-1,j-1}^{(1)} & j > i, \\ \tilde{f}_{i-1,j-1}^{[1]} & j \leq i; \end{cases}$$

2. after freezing all the vertices $(2,j)$ and $(i,2)$, and vertices $(1,n)$ and $(2,1)$, all the vertices $(1,j)$ and $(i,1)$ become isolated (connected only to frozen vertices) and can therefore be ignored. The subquiver on vertices $i,j \in \{2, \ldots , n\}$ is isomorphic to the standard quiver $Q_{\text{std}}(n-1)$ on $SL_{n-1}$.

**Proof.** We look at the sequence $T_1$ step by step: start with mutating the initial cluster at $(n,n)$. The relevant part of the quiver is given in Fig. 5.7. So the exchange relation is

$$\varphi_{nn} \varphi_{nn}^1 = \varphi_{n-1,n-1} \cdot \varphi_{21} + \varphi_{n-1,n} \cdot \varphi_{n,n-1}$$

and since $f_{n-1,n} = x_{n-1,n}$, it comes down to

$$\varphi_{nn} \varphi_{nn}^1 = x_{nn} \left( x_{n-1,n-1} f_{21} - x_{n,n-1} f_{21}^{[1]} \right)$$

which means

$$\varphi_{nn}^1 = x_{n-1,n-1} f_{21} - x_{n,n-1} f_{21}^{[1]} = \tilde{f}_{n-1,n-1}^{[1]}.$$ 

The arrows $(n-1,n-1) \rightarrow (n,n-1), (n-1,n) \rightarrow (n-1,n)$ and $(2,1) \rightarrow (n-1,n)$ are removed, and an arrow $(n,n-1) \rightarrow (2,1)$ is added. All the arrows that touch $(n,n)$ are inverted (as shown in Fig. 5.7).

Now mutate at $(n-1,n)$. The exchange relation is determined by the partial quiver in Fig. 5.8

$$\varphi_{n-1,n} \varphi_{n-1,n}^1 = \varphi_{n-2,n} \varphi_{n,n}^1 + \varphi_{n-2,n-2}$$

and since we had $\varphi_{n,n} = \tilde{f}_{n-1,n}$, we get

$$\varphi_{n-1,n}^1 = x_{n-2,n-1} = f_{n-2,n-1}^{(1)}.$$
The arrow \( (k-1, n-1) \rightarrow (k-1, n) \) was removed, and an arrow \( (k+1, n) \rightarrow (k-1, n-1) \) was added. All the arrows touching \( \varphi_{kn} \) were inverted.

The sequence continues mutating at \((k,n)\) with \( k = n - 2, n - 3, \ldots, 2 \): locally the quiver looks like in Fig. 5.8 and so

\[
\varphi_{kn} = \varphi_{k+1,n} \varphi_{k-1,n} + \varphi_{k-1,n-1}
\]

and so

\[
\varphi_{kn}^{1} = x_{k-1,n-1} = f_{k-1,n-1}^{(1)}.
\]

The last mutation on column \( n \) is at the vertex \((1,n)\). The exchange relation is determined by Fig. 5.8 with \( k = 1 \), so now \( k - 1 \) should be replaced with \( n \), and so

\[
\varphi_{1n}^{1} = \varphi_{2n}^{1} \varphi_{n} + \varphi_{n1}
\]

and so

\[
\varphi_{1n}^{1} = x_{n1} x_{n2} + \begin{vmatrix} x_{n1} & x_{n2} \\ x_{n1} & x_{n2} \end{vmatrix}
\]

Therefore

\[
(5.11) \quad \varphi_{1n}^{1} = x_{n1}.
\]

Assume now that columns \( n, \ldots, j+1 \) were mutated, and look at the sequence \( T_{1} \) on the column \( j \). Start at \((n,j)\) where the exchange relation is shown in Fig. 5.9

\[
\varphi_{n,j}^{1} = \varphi_{n,j-1}^{1} \varphi_{n,j+1} + \varphi_{n-1,j-1}^{1} \varphi_{n+1,j+1}
\]

and so

\[
\varphi_{1n}^{1} = x_{n+1,j+1}.
\]

The arrows \((n-1, j-1) \rightarrow (n, j-1)\) and \((2,1) \rightarrow (n, j+1)\) were removed, and two arrows were added: \((n, j-1) \rightarrow (n-1, j-1)\) and \((n, j-1) \rightarrow (2,1)\). The arrows that touch \((n,j)\) were inverted.
We move on, mutating at \((k, j)\) with \(k > j\). The exchange relation is now given in Fig. 5.10

\[
\varphi_{kj} \varphi_{kj}^1 = \varphi_{k,j-1} \varphi_{k,j+1}^1 + \varphi_{k-1,j-1} \varphi_{k+1,j+1}^1
\]

\[
= f_{k,j-1} f_{k,j+1}^1 + f_{k-1,j-1} f_{k+1,j+1}^1
\]

\[
= f_{21} \left( f_{k,j-1} f_{k,j+1}^{(1)} + f_{k-1,j-1} f_{k,j}^{(1)} \right)
\]

\[
- f_{21}^* \left( f_{k,j-1} f_{k,j+1}^{(1)\dagger} + f_{k-1,j-1} f_{k,j}^{(1)\dagger} \right)
\]

and we can use \(5.10\) with \(A = X_{[k-1:0]}^{[1]}[0:n]\) (with \(\ell = n-k+j\)), so the first parenthesis is just \(f_{kj} f_{k-1,j-1}^{(1)}\) and the second one is \(f_{kj} f_{k-1,j-1}^{(1)\dagger}\), and so

\[
\varphi_{kj}^1 = f_{k-1,j-1}^1.
\]

After this mutation the arrows \((k-1, j-1) \to (k, j-1)\) and \((k+1, j+1) \to (k, j+1)\) were removed, and two arrows were added: \((k, j-1) \to (k+1, j+1)\) and \((k, j+1) \to (k-1, j-1)\). The arrows that touch \((k, j)\) were inverted.
Four arrows were removed:

\[
\text{parenthesis and the exchange relation: arrows that touch (5.12)}
\]

relation here is and there are two cases here:

Next look at the mutation on the main diagonal, at \((j, j)\): Fig. 5.11 describes the exchange relation:

\[
\varphi_{jj} = \varphi_{j-1,j-1} + \varphi_{j+1,j+1} + \varphi_{j-1,j} + \varphi_{j+1,j}
\]

and we use (5.3) again, now with \(A = X_{[j-1...n]}\) to get \(f_{jj}f_{j-1,j-1}^{(1)}\) in the first parenthesis and \(f_{jj}f_{j-1,j-1}^{(1)}\) in the second one. Thus the exchanged variable is \(\varphi_{jj}^{(1)} = f_{j-1,j-1}\).

Four arrows were removed: \((j - 1, j - 1) \rightarrow (j, j - 1), (j - 1, j - 1) \rightarrow (j - 1, j), (j - 1, j + 1) \rightarrow (j, j - 1))\) and \((j + 1, j + 1) \rightarrow (j - 1, j)\). The four arrows that touch \((j, j)\) were inverted.

proceeding along column \(j\) we mutate now at \((k, j)\) where \(k < j\). The exchange relation here is

\[ \varphi_{kj}\varphi_{kj} = \varphi_{k-1,j-1}\varphi_{k+1,j+1} + \varphi_{k-1,j}\varphi_{k+1,j}, \quad (5.12) \]

and there are two cases here:

First, if \(k = j - 1\) then (5.12) reads

\[
\varphi_{kj}\varphi_{kj} = \varphi_{j-2,j-1}\varphi_{j+1,j+1} + \varphi_{j-2,j}\varphi_{j+1,j}
\]

and using (5.3) with \(A = X_{[j-2...n]}\) yields \(f_{j-1,j}f_{j-2,j-1}^{(1)}\) in the first parenthesis, and with \(A = X_{[j-2...n-2]}\) gives \(f_{j-1,j}f_{j-2,j-1}^{(1)}\) in the second one. Recall that

\[ \mu_x \]

Figure 5.11. Mutation \(T_1\) at vertex \((j, j)\)
$\varphi_{j-1,j} = \tilde{f}_{j-1,j}$ and therefore

$$\varphi_{j-1,j}^1 = f_{j-2,j-1}^{(1)}.$$  

The second case is when $k < j - 1$ and then (5.12) reads

$$\varphi_{kj} \varphi_{k,j}^1 = f_{k-1,j-1}^{(1)} f_{k,j} + f_{k-1,j} f_{k,j-1}^{(1)}$$

and with (5.3) on the matrix $A = X_{[j-1\ldots n]}_{[k-1\ldots \ell]}$ (here $\ell = n + k - j$) it comes down to

$$\varphi_{kj}^1 = f_{k-1,j-1}^{(1)}.$$  

In both cases the quiver changes are: the arrows $(k-1,j-1) \to (k-1,j)$ and $(k+1,j+1) \to (k-1,j)$ were removed and $(k+1,j) \to (k-1,j-1)$ and $(k+1,j) \to (k+1,j+1)$ were added. □

**Proposition 5.7.** In the seed $\Sigma_2$,

1. for all $i,j \in \{3, \ldots, n\}$ the cluster variables take the form

$$\varphi_{ij}^2 = f_{i-2,j-2}^{(2)};$$

2. after freezing all the vertices $(3,j)$ and $(i,3)$, all vertices $(2,j)$ and $(i,2)$ become isolated and can be ignored. The subquiver on vertices $i,j \in \{3, \ldots, n\}$ is isomorphic to the standard quiver $Q_{n-2}$ on $SL_{n-2}$.

**Proof.** Following the pattern of the proof of Proposition 5.6 we can look at the mutations step by step. All exchange relations can be resolved using the Desnanot–Jacobi identity (5.3). If all the functions in the exchange relation are proper minors of the matrix $X$, this is pretty much straightforward. If the exchange relation involves a function of the form $f_{ij}$, it is still not too hard: the structure of the quiver assures there must be two such functions (this can be easily proved by induction), each of which has the form $g \cdot f_{21} + g^2 f_{21}^2$. The arrows that connect the corresponding two vertices to the vertex associated with the exchanged variable point in opposite directions (one towards this vertex and the other away from it. This is also not hard to see). Therefore, the exchange relation can be broken into two parts: the first has only determinants of dense submatrices of $X$ multiplied by $f_{21}$. The second one has determinants of same submatrices with just one row
replaced by another (recall that if $g = \det X_{[i,k]}^{[j,\ell]}$ then $g^\perp = \det X_{[i,n-1,k+1]}^{[j,\ell]}$). Using (5.3) on each part separately yields the result. \hfill \Box

**Proposition 5.8.** For $m \geq 3$,
1. The subquiver of $Q^m$ of rows $m + 1, \ldots, n$ and columns $m + 1, \ldots, n$ is isomorphic to the standard quiver $Q_{\text{std}}(n - m)$ (on $SL_{n-m}$).
2. The functions $\varphi_{ij}^m \in \mathcal{B}_m$ with $j > i$ are
$$\varphi_{ij}^m = f_{i-m,j-m}. $$

**Proof.** By induction, based on Propositions 5.6 and 5.7. All the relevant exchange relations look like the standard ones. \hfill \Box

**Corollary 5.9.** If $i < j < n$, then $x_{ij}$ is a cluster variable.

**Proof.** Set $m = n - j$, and then $\varphi_{i+m,n}^m = f_{i,n-m} = x_{ij}$ is a cluster variable. \hfill \Box

**Lemma 5.10.** For every $(i, j) \in [n-1] \times [n-1]$ the function $x_{ij}$ belongs to the upper cluster algebra $\mathcal{A}_{1\to n-1}$.  

**Proof.** First, by Corollary 5.9, $x_{ij} \in \mathcal{A}_{1\to n-1}$ for all $j > i$.

To see that $x_{n-1,n-1}$ belongs to $\mathcal{A}_{1\to n-1}$ note that
$$\varphi_{n-1,n-1} = \begin{vmatrix} x_{n-1,n-1} & x_{n-1,n} \\ x_{n,n-1} & x_{nn} \end{vmatrix},$$
and we have
$$\varphi_{nn}^1 = x_{n-1,n-1}f_{21} - x_{n,n-1}f_{21}^\perp,$$
so
$$x_{n-1,n-1} = \frac{\varphi_{n-1,n-1} + x_{n-1,n}x_{nn} - 1}{\varphi_{nn}^1 + x_{n,n-1}f_{21}^\perp}. $$

So according to Lemma 2.1, $x_{n-1,n-1} \in \mathcal{A}$.

If $i = n - 1$ and $j < i$ then
$$\varphi_{i,j} = \begin{vmatrix} x_{n-1,j} & x_{n-1,j+1} \\ x_{nj} & x_{n,j+1} \end{vmatrix},$$
and
$$\varphi_{n,j+1}^1 = x_{n-1,j}f_{21} - x_{nj}f_{21}^\perp.$$

So
$$x_{n-1,j} = \frac{\varphi_{n,j+1} + x_{nj}x_{n-1,j+1}}{\varphi_{n,j+1}^1 + x_{nj}f_{21}^\perp}. $$

Inductively assuming $x_{n-1,j+1} \in \mathcal{A}$, this satisfy the conditions of Lemma 2.1 and therefore $x_{ij} \in \mathcal{A}_{1\to n-1}$.

If $i < n - 1$ we use induction: Let $D_{ij}$ be the set of all $x_{k\ell}$ with $k \geq i$ and $\ell \geq j$ without $x_{ij}$. For a pair $(i, j)$ assume that $D_{ij} \subseteq \mathcal{A}_{1\to n-1}$ (that is, all $x_{k\ell}$ with $k \geq i$ and $\ell \geq j$ are in $\mathcal{A}_{1\to n-1}$, except maybe $x_{ij}$ itself). We have
$$\varphi_{ij} = \det X_{[i,n]}^{[j,i]} = x_{ij} \cdot f_{i+1,j+1} - p_1$$
where $p_1$ is a polynomial in the variables $x_{k\ell} \in D_{ij}$ and therefore $p_1 \in \mathcal{A}_{1\to n-1}$.

Similarly,
$$\varphi_{ij}^2 = \det X_{[i,n-2]}^{[j,j-2]} = x_{ij}f_{i+1,j+1} - p_2$$
with \( p_2 \in \mathcal{A}_{1\rightarrow n-1} \) again. Hence,

\[
x_{ij} = \frac{\varphi_{ij} + p_1}{f_{i+1,j+1}} = \frac{\varphi_{ij}^{(2)} + p_2}{f_{i+1,j+1}}
\]

and so by Lemma 2.1 we get \( x_{ij} \in \mathcal{A}_{1\rightarrow n-1} \).

6. The Toric Action

Proving Theorem 3.2 is equivalent to proving the following (see [11]):

1. for any \( H_1, H_2 \in \mathcal{H}_T \) and any \( X \in SL_n \),

\[
y_i (H_1 X H_2) = H_1^{\eta_i} H_2^{\zeta_i} y_i (X)
\]

for some weights \( \eta_i, \zeta_i \in h^*_T \) (\( i \in [n+m] \));

2. \( \text{span} \{ \eta_i \}_{i=1}^{\dim \mathcal{G}} = \text{span} \{ \zeta_i \}_{i=1}^{\dim \mathcal{G}} = h^*_T \); 

3. for every \( i \in [\dim SL_n - 2k_T] \),

\[
\sum_{j=1}^{\dim SL_n} b_{ij} \eta_j = \sum_{j=1}^{\dim SL_n} b_{ij} \zeta_j = 0.
\]

For a seed \((\tilde{x}, \tilde{B})\) in \( C_T \), and \( y_i = \varphi (x_i) \) for \( i \in [n+m] \).

**Proposition 6.1.** For any BD triple \( T = (\{\alpha\}, \{\beta\}, \alpha \mapsto \beta) \) on \( SL_n \) statements 1,2 and 3 above hold true.

**Proof.** Recall that

\[
\mathfrak{h}_T = \{ h \in \mathfrak{h} : \alpha (h) = \beta (h) \}.
\]

Therefore, taking the basis \( \{ h_i = e_{ii} - e_{i+1,i+1} \} \) we can parametrize \( \mathfrak{h}_T \) with linear combinations \( h = \sum c_i h_i \) with coefficient vectors \((c_1, \ldots, c_{n-1})\) subject to a restriction derived from (6.1). To understand this restriction look at the Cartan matrix \( C \) of \( sl_n \):

\[
C_{ij} = \begin{cases} 2 & i = j \\ -1 & |i - j| = 1 \\ 0 & \text{otherwise} \end{cases}
\]

let \([C]_i\) denote the \( i \)-th row of \( C \), and let \( A \) be the one row matrix \( A = [C]_\alpha - [C]_\beta \). Then the coefficient vector must be in the null space of \( A \) (to satisfy \( \alpha (h) = \beta (h) \)). Since \( C \) is a symmetric \((n-1) \times (n-1)\) matrix, \( A^T = C (e_\alpha - e_\beta) \) and clearly its null space is \( n-2 \) dimensional.

Assume \( y_m \) is a function of the form \( f_{ij} = \det X_{[i,j]}^{[p,q]} \). Take a diagonal matrix \( H_1 \in \mathcal{H}_T \) where \( H_1 = \exp h \) for some \( h = \text{diag}(d_1, \ldots, d_n) \). Now set \( \eta_m (h) = d_i + \ldots + d_k \). It is easy to verify that

\[
y_m (H_1 X) = H_1^{\eta_m} y_m (X) .
\]
If $y_m$ is a function of the form

$$
\theta_j = \det \begin{bmatrix}
    x_{ij} & \cdots & x_{i,\alpha+1} & 0 & \cdots \\
    \vdots & \ddots & \vdots & \vdots & \ddots \\
    x_{n1} & \cdots & x_{n,\alpha} & x_{n,\alpha+1} & 0 & \cdots \\
    \cdots & 0 & x_{1,\beta} & x_{1,\beta+1} & \cdots & x_{1n} \\
    \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
    \vdots & \vdots & x_{\beta j} & \cdots & x_{\beta n} \\
    0 & x_{n-\beta} & x_{n-\beta, j} & \cdots & x_{n-\beta, n}
\end{bmatrix}
$$

it is not hard to see that setting $\eta_m(h) = d_1 + \cdots + d_{n-\beta} + d_i + \cdots + d_n$ yields (6.2) again.

The last case is when $y_m$ is of type $\psi$ - ,

$$
\psi_i = \det \begin{bmatrix}
    x_{ij} & \cdots & x_{jn} & 0 & \cdots & 0 \\
    \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
    x_{\beta j} & \cdots & x_{\beta n} & x_{\alpha 1} & \cdots & x_{\alpha,n-\alpha} \\
    x_{\beta+1,j} & \cdots & x_{\beta+1,n} & x_{\alpha+1,1} & \cdots & \vdots \\
    \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
    \vdots & \vdots & x_{n1} & \cdots & x_{n,n-\alpha}
\end{bmatrix}
$$

with $j = n + i - \beta$. We can write $\psi_i = f_{ij}f_{\alpha+1,1} - f_{ij}f_{\alpha+1,1}^\dagger$, or using determinants of submatrices

$$
\psi_i = \det X^{[j,\alpha]}_{[i,\beta]} \det X^{[1,n-\alpha]}_{[\alpha+1,\alpha]} - \det X^{[j,\alpha]}_{[1,\beta-1,\beta+1]} \det X^{[1,n-\alpha]}_{[\alpha,\alpha+2,\ldots,n]}.
$$

Therefore, (6.2) holds if

$$
d_\beta + d_{\alpha+1} = d_{\beta+1} + d_{\alpha},
$$

i.e., the sum of weights on rows $\beta, \alpha+1$ is equal to the sum of those on rows $\beta+1, \alpha$.

In this case the weight is $\eta_m(h) = d_i + \cdots + d_\beta + d_{\alpha+1} + \cdots + d_n$. This is equivalent to the condition that $h = \text{diag}(d_1, \ldots, d_n)$ is in the null space of the row matrix $B = e_\alpha - e_{\alpha+1} - e_\beta + e_{\beta+1}$ (here $e_k$ is a row vector with 1 in the $k$-th entry and 0 elsewhere). Let $T$ be the transformation matrix from the basis $\{h_i\}_{i=1}^{n-1}$ to the standard one (so the $j$-th column of $T$ is $e_j - e_{j+1}$), and let $v^T = (c_1, \ldots, c_{n-1})$ be the coefficient vector of $h$ in the basis $\{h_i\}_{i=1}^{n-1}$ (so that $h = Tv$). It is not hard to verify that $BT = [C]\alpha - [C]\beta$, hence

$$
Bh = BTv = Av = 0,
$$

because $v$ is in the null space of $A$ by its definition. This establishes (6.3), and therefore (6.2) holds in this case as well.

Repeating the above with right multiplication by $H_2$, now with weights $\zeta_m$, yields statement 2.

Proving statement 2 is easy: as described above, the weight associated with the function $f_i$ is $\eta_i(h) = d_i + \cdots + d_n$. Therefore the set $\{\eta_2, \ldots, \eta_n\}$ spans $h^\ast$, because this is a set of $n-1$ linearly independent vectors in an $n-1$ dimensional space. Since $h_T^\ast$ is a subspace of $h^\ast$ (because $h_T$ is a subspace of $h$), this set also spans $h_T^\ast$. The same holds for the weights $\zeta_i$. 
Figure 6.1. The neighbors of \((i,j)\) when \(1 < i,j < n\)

To prove statement \([3]\) we rephrase it as follows: for every mutable vertex \(v = (i,j)\) of the quiver, the sum of weights over the neighbors with arrows pointing towards \(v\) is equal to the sum of weights over the neighbors with arrows pointing out of \(v\). This is true because \(b_{ij} \in \{0, 1, -1\}\). It will be proved for the weights \(\eta_m\) corresponding to the left multiplication \(y_m(HX) = H^{\eta_m}y_m(X)\) but symmetric arguments will hold for \(\zeta_m\) with right multiplication to show \(y_m(XH) = H^{\zeta_m}y_m(X)\).

As was explained above, for \(h = \text{diag}(d_1, \ldots, d_n)\) the weights \(\eta_i\) can be defined as

\[
\eta_i(h) = \begin{cases} 
    d_i + \cdots + d_k & \text{if } y_m = f_{ij} = \det X^{[j,k]}_{[i,k]} \\
    d_i + \cdots + d_n + d_1 + \cdots + d_{n-\beta} & \text{if } y_m = \theta_j = f_{ij}f_{1,\beta+1} - f_{ij}^\top f_{1,\beta+1} \\
    d_i + \cdots + d_\beta + d_{\alpha+1} + \cdots + d_n & \text{if } y_m = \psi_i = f_{ij}f_{\alpha+1,1} - f_{ij}^\top f_{\alpha+1,1}. 
\end{cases}
\]

Assume \(y_m\) and \(y_{m'}\) are two cluster variables that correspond to two vertices on the same diagonal of the quiver, that is, \(y_m\) corresponds to the vertex \((i,j)\) and \(y_{m'}\) corresponds to \((i+k, j+k)\). Then

\[
(6.4) \quad \eta_{m'}(h) = \eta_m(h) + d_i + \cdots + d_{k-1}.
\]

With this fact in mind, look at the sum of weights over all the vertices adjacent to a mutable vertex \((i,j)\) and consider the following cases:

1. If \(1 < i,j < n\) then there are three arrows pointing to \((i,j)\) from the vertices \(A = (i-1,j)\), \(B = (i,j-1)\) and \(C = (i+1,j+1)\) and three arrows from \((i,j)\) to the vertices \(D = (i,j+1)\), \(E = (i+1,j)\) and \(F = (i-1,j-1)\) (see Fig. 6.1). According to \((6.4)\),

\[
\begin{align*}
\eta_A &= \eta_D + d_{i-1} \\
\eta_B &= \eta_E + d_i \\
\eta_C &= \eta_F - d_i - d_{i-1}
\end{align*}
\]

and therefore \(\eta_A + \eta_B + \eta_C = \eta_D + \eta_E + \eta_F\).
2. If $i = j = n$ then $(i, j)$ has only three neighbors (Assuming $\beta \neq n - 1$) as shown in Fig. 6.2: $A = (n - 1, n)$ and $B = (n, n - 1)$ with arrows to $(n, n)$ and $C = (n - 1, n - 1)$ with an arrow pointing to it from $(n, n)$. In this case it is clear that

\[
\begin{align*}
\eta_A &= d_{n-1} \\
\eta_B &= d_n \\
\eta_C &= d_{n-1} + d_n
\end{align*}
\]

and so $\eta_A + \eta_B = \eta_C$.

3. If $i = n$ and $j \notin \{\alpha, \alpha + 1, n\}$, then there are four neighboring vertices: $A = (n, j - 1)$ and $B = (n - 1, j)$ with arrows pointing to $(n, j)$, and $C = (n, j + 1)$ and $D = (n - 1, j - 1)$ with arrows pointing to them. Here $y_A = x_{n,j-1}$ and

\[
y_D = \begin{vmatrix} x_{n-1,j-1} & x_{n-1,j} \\ x_{n,j-1} & x_{nj} \end{vmatrix}
\]

so $\eta_A = d_n$ and $\eta_D = d_{n-1} + d_n$. According to (6.4)

\[
\eta_B = \eta_C + d_{n-1},
\]

so again, $\eta_A + \eta_B = \eta_C + \eta_D$.

4. The vertex $(n, \alpha)$ has three neighbors with arrows pointing at it: $A = (n - 1, \alpha)$, $B = (1, \beta + 1)$ and $C = (n, \alpha - 1)$, and two neighbors with arrows from $(n, \alpha)$ to them: $D = (n, \alpha + 1)$ and $E = (n - 1, \alpha - 1)$ (see Fig. 6.3). We have

\[
\begin{align*}
\eta_A &= \eta_D + d_{n-1} \\
\eta_B &= \eta_E - d_{n-1} - d_n \\
\eta_C &= d_n
\end{align*}
\]

and $\eta_A + \eta_B + \eta_C = \eta_D + \eta_E$.

5. The vertex $(n, \alpha + 1)$ has two neighbors with arrows pointing at it: $A = (n - 1, \alpha + 1)$ and $B = (n, \alpha)$. There are three neighbors with arrows from $(n, \alpha + 1)$ to them: $C = (n, \alpha + 2)$, $D = (1, \beta + 1)$ and $E = (n - 1, \alpha)$. Fig. 6.4 shows $(n, \alpha + 1)$ and its neighbors. So here

\[
\begin{align*}
\eta_A &= \eta_C + d_{n-1} \\
\eta_B &= \eta_D + d_n \\
\eta_E &= d_{n-1} + d_n
\end{align*}
\]

\footnote{If $\beta = n - 1$ then $D = (\alpha + 1, 1)$ is also a neighbor and it is easy to verify that the result holds.}
and $\eta_A + \eta_B = \eta_C + \eta_D + \eta_E$.

6. The vertex $(i, n)$ with $i \notin \{\beta, \beta + 1, n\}$. There are four neighbors as Fig. 6.3 shows: $A = (i - 1, n)$ and $B = (i, n - 1)$ with arrows pointing to $(i, n)$, and $C = (i + 1, n)$ and $D = (i - 1, n - 1)$ with arrows pointing to them. Here $y_A = x_{i-1,n}$ and $y_D = \left[ \begin{array}{cc} x_{i-1,n-1} & x_{i-1,n} \\ x_{i,n-1} & x_{in} \end{array} \right]$ so $\eta_A = d_{i-1}$ and $\eta_D = d_{i-1} + d_i$. According to (6.4) $\eta_B = \eta_C + d_{n-1}$, so again, $\eta_A + \eta_B = \eta_C + \eta_D$. 

---

**Figure 6.3.** The neighbors of $(n, \alpha)$

**Figure 6.4.** The neighbors of $(n, \alpha + 1)$
7. The vertex \((\beta, n)\) has three neighbors with arrows pointing at it: \(A = (\beta - 1, n)\), \(B = (\alpha + 1, 1)\) and \(C = (\beta, n - 1)\), and two neighbors with arrows from \((\beta, n)\) to them: \(D = (\beta + 1, n)\) and \(E = (\beta - 1, n - 1)\) (see Fig. 6.6). We have

\[
\begin{align*}
\eta_A &= d_{\beta - 1} \\
\eta_B &= \eta_E - d_{\beta - 1} - d_\beta \\
\eta_C &= \eta_D + d_\beta
\end{align*}
\]

and therefore \(\eta_A + \eta_B + \eta_C = \eta_D + \eta_E\).
Figure 6.7. The neighbors of \((\beta + 1, n)\)

8. The vertex \((\beta + 1, n)\) has two neighbors with arrows pointing at it: \(A = (\beta, n)\) and \(B = (\beta + 1, n - 1)\). There are three neighbors with arrows from \((\beta + 1, n)\) to them: \(C = (\alpha + 1, 1)\), \(D = (\beta + 2, n)\) and \(E = (\beta, n - 1)\) (see Fig 6.7). So here

\[
\begin{align*}
\eta_A &= \eta_C + d_\beta \\
\eta_B &= \eta_D + d_{\beta+1} \\
\eta_E &= d_\beta + d_{\beta+1}
\end{align*}
\]

and \(\eta_A + \eta_B = \eta_C + \eta_D + \eta_E\).

9. The vertex \((1, \beta + 1)\) has three neighbors with arrows pointing at it: \(A = (n, \alpha + 1)\), \(B = (2, \beta + 2)\) and \(C = (1, \beta)\). There are two neighbors with arrows from \((1, \beta + 1)\) to them: \(D = (2, \beta + 1)\) and \(E = (n, \alpha)\) (see Fig 6.8). So here

\[
\begin{align*}
\eta_A &= d_n \\
\eta_B &= \eta_E - d_n - d_1 \\
\eta_C &= \eta_D + d_1
\end{align*}
\]

and again we have \(\eta_A + \eta_B + \eta_C = \eta_D + \eta_E\).

10. Last is the vertex \((\alpha + 1, 1)\), with three neighbors with arrows pointing at it: \(A = (1, \alpha)\), \(B = (\alpha + 2, 2)\) and \(C = (\beta + 1, n)\) and two neighbors with arrows from \((\alpha + 1, 1)\) to them: \(D = (\alpha + 1, 2)\) and \(E = (\beta, n)\) (see Fig 6.9). So now

\[
\begin{align*}
\eta_A &= \eta_D + d_\alpha \\
\eta_B &= \eta_E - d_{\alpha+1} - d_\beta \\
\eta_C &= d_{\beta+1}
\end{align*}
\]

and with \((6.3)\), the result is \(\eta_A + \eta_B + \eta_C = \eta_D + \eta_E\). 

\qed
Acknowledgments

The author was supported by ISF grant #162/12. The author thanks Michael Gekhtman for his helping comments and answers. Special thanks to Alek Vainshtein for his support and encouragement, as well as his mathematical, technical and editorial advices.
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