COARSE SPACES, ULTRAFILTERS AND DYNAMICAL SYSTEMS

IGOR PROTASOV

Abstract. For a coarse space \((X, \mathcal{E})\), \(X^\sharp\) denotes the set of all unbounded ultrafilters on \(X\) endowed with the parallelity relation: \(p||q\) if there exists \(E \in \mathcal{E}\) such that \(E[P] \in q\) for each \(P \in p\). If \((X, \mathcal{E})\) is finitary then there exists a group \(G\) of permutations of \(X\) such that the coarse structure \(\mathcal{E}\) has the base \(\{(x, gx) : x \in X, g \in F\} : F \in [G]^{<\omega}, id \in F\}.\) We survey and analyze interplays between \((X, \mathcal{E})\), \(X^\sharp\) and the dynamical system \((G, X^\sharp)\).

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The dynamical Švarc-Milnor Theorem and Gromov Theorem arose at the dawn of Geometric Group Theory. In both cases, a group or a pair of groups act on some locally compact spaces, see [22, Chapter 1]. The Gromov coupling criterion was transformed into the powerful tool in coarse equivalences (see references in [23]), however some natural questions on the coarse equivalence of groups need more delicate combinatorial technique, see [4].

In this paper, we describe and survey the dynamical approach to coarse spaces originated in the algebra of the Stone-Čech compactification. We identify the Stone-Čech compactification \(\beta G\) of a discrete group \(G\) with the set of all ultrafilters on \(G\). The left regular action \(G\) on \(G\) gives rise to the action of \(G\) on \(\beta G\) by \((g, p) \mapsto gp\), \(gp = \{gP : P \in p\}\). In turn, the dynamical system \((G, \beta G)\) induces on \(\beta G\) the structure of a right topological semigroup. The product \(pq\) of ultrafilters \(p, q\) is defined by \(A \in pq\) if and only if \(\{g \in G : g^{-1}A \in q\} \in p\). The semigroup \(\beta G\) has very rich algebraic structure and the plenty of combinatorial applications, see nice paper [5], capital book [6] or booklet [9].

Let \((X, \mathcal{E})\) be a coarse space. We denote by \(X^\sharp\) the set of all ultrafilters \(p\) on \(X\) such that each member \(P \in p\) is unbounded in \((X, \mathcal{E})\). Then we define the parallelity equivalence \(||\) on \(X^\sharp\) by \(p||q\) if and only if there exists \(E \in \mathcal{E}\) such that \(E[P] \in q\) for each \(P \in p\). For \(p \in X^\sharp\), the orbit \(\overline{p} = \{q \in X^\sharp : q||p\}\) looks like a smile apart of some hidden cat. This cat appears if \((X, \mathcal{E})\) is finitary. By Theorem 3.1, there exists a group \(G\) of permutations of \(X\) such that \(\mathcal{E}\) has the base \(\{(x, gx) : x \in F\} : F \in [G]^{<\omega}, id \in F\}.\) In this case, \(X^\sharp = X^*, X^* = \beta X \setminus X\)
and $\overline{p} = Gp$. But even $(X, \mathcal{E})$ is not finitary, $X^\sharp$ contains some counterpart of the kernel of a dynamical system, see Theorem 2.3.

Our goal is to clarify interplays between $(X, \mathcal{E})$, $X^\sharp$, and the dynamical system $(G, X^\ast)$ in order to understand the dynamical nature of some extremal coarse spaces, in particular, tight, discrete and indiscrete.

1. Coarse spaces

Given a set $X$, a family $\mathcal{E}$ of subsets of $X \times X$ is called a coarse structure on $X$ if

- each $E \in \mathcal{E}$ contains the diagonal $\triangle_X$, $\triangle_X = \{(x, x) \in X : x \in X\}$;
- if $E, E' \in \mathcal{E}$ then $E \circ E' \in \mathcal{E}$ and $E^{-1} \in \mathcal{E}$, where $E \circ E' = \{(x, y) : \exists z ((x, z) \in E, (z, y) \in E')\}$, $E^{-1} = \{(y, x) : (x, y) \in E\}$;
- if $E \in \mathcal{E}$ and $\triangle_X \subseteq E' \subseteq E$ then $E' \in \mathcal{E}$;
- $\bigcup \mathcal{E} = X \times X$.

A subfamily $\mathcal{E}' \subseteq \mathcal{E}$ is called a base for $\mathcal{E}$ if, for every $E \in \mathcal{E}$, there exists $E' \in \mathcal{E}'$ such that $E \subseteq E'$. For $x \in X$, $A \subseteq X$ and $E \in \mathcal{E}$, we denote $E[x] = \{y \in X : (x, y) \in E\}$, $E[A] = \bigcup_{a \in A} E[a]$, $E_A[x] = E[x] \cap A$ and say that $E[x]$ and $E[A]$ are balls of radius $E$ around $x$ and $A$.

The pair $(X, \mathcal{E})$ is called a coarse space [22] or a ballean [10], [21].

For a coarse space $(X, \mathcal{E})$, a subset $B \subseteq X$ is called bounded if $B \subseteq E[x]$ for some $E \in \mathcal{E}$ and $x \in X$. The family $\mathcal{B}_{(X, \mathcal{E})}$ of all bounded subsets of $(X, \mathcal{E})$ is called the bornology of $(X, \mathcal{E})$.

A coarse space $(X, \mathcal{E})$ is called finitary, if for each $E \in \mathcal{E}$ there exists a natural number $n$ such that $|E[x]| < n$ for each $x \in X$.

We classify subsets of a coarse space $(X, \mathcal{E})$ by their size. A subset $A$ of $X$ is called

- large if $E[A] = X$ for some $E \in \mathcal{E}$;
- small if $L \setminus A$ is large for each large subset $L$;
- thick if, for each $E \in \mathcal{E}$, there exists $a \in A$ such that $E[a] \subseteq A$;
- prethick if $E[A]$ is thick for some $E \in \mathcal{E}$;
- thin (or discrete) if, for each $E \in \mathcal{E}$, there exists a bounded subset $B$ of $X$ such that $E_A[a] = \{a\}$ for each $a \in A \setminus B$. 

For finitary coarse spaces, the dynamical unification of above definitions will be given in Section 3.

Following [17], we say that two subsets $A, B$ of $X$ are

- **close** (write $A\delta B$) if there exists $E \in \mathcal{E}$ such that, $A \subseteq E[B]$, $B \subseteq E[B]$;
- **linked** (write $A\lambda B$) if either $A, B$ are bounded or there exist unbounded subsets $A' \subseteq A$, $B' \subseteq B$ such that $A'\delta B'$.

We say that a coarse space $E \in \mathcal{E}$ is

- **$\delta$-tight** if any two unbounded subsets of $X$ are close;
- **$\lambda$-tight** if any two unbounded subsets of $X$ are linked;
- **indiscrete** if $E \in \mathcal{E}$ has no unbounded discrete subsets;
- **ultradiscrete** if $\{X \setminus B : B \in \mathcal{B}(X, E)\}$ is an unltrafilter.

We note that $\lambda$-tight spaces appeared in [2] under the name *utranormal*, $\delta$-tight subsets are called *extremely normal* in [14] and *hypernormal* in [1].

An unbounded coarse space is called *maximal* if it is bounded in every stronger coarse structure. By [18, Theorem 3.1], every maximal coarse space is $\delta$-tight. A ballean $(X, \mathcal{E})$ is $\delta$-tight if and only if every subset of $X$ is large. If a $\lambda$-tight space is not indiscrete then it contains an ultradiscrete subspace [14, Theorem 2.2], so every finitary $\lambda$-tight space is indiscrete.

## 2. Ultrafilters

Let $X$ be a discrete space and let $\beta X$ denotes the Stone–Čech compactification of $X$. We take the points of $\beta X$ to be the ultrafilter on $X$, with the points of $X$ identified with the principal ultrafilters, so $X^* = \beta X \setminus X$ is the set of all free ultrafilters. The topology of $\beta X$ can be defined by stating that the sets of the form $A = \{p \in \beta X : A \in p\}$, where $A$ is a subset of $X$, are base for the open sets. The universal property of $\beta X$ states that every mapping $f : X \rightarrow Y$, where $Y$ is a compact Hausdorff space, can be extended to the continuous mapping $f^\beta : \beta X \rightarrow Y$.

Given a coarse space $(X, \mathcal{E})$, we endow $X$ with the discrete topology and denote by $X^\sharp$ the set of all ultrafilters $p$ on $X$ such that each member $P \in p$ is unbounded. Clearly, $X^\sharp$ is the closed subset of $X^*$ and $X^\sharp = X^*$ if $(X, \mathcal{E})$ is finitary.

Following [10], we say that two ultrafilters $p, q \in X^\sharp$ are *parallel* (and write $p \parallel q$) if there exists $E \in \mathcal{E}$ such that $E[P] \in q$ for each $P \in p$. By [10, Lemma 4.1, 1], $\parallel$ is an equivalence on $X^\sharp$. We denote by $\sim$ the minimal (by inclusion) closed (in $X^\sharp \times X^\sharp$) equivalence on $X^\sharp$ such that $\parallel \subseteq \sim$. The quotient $\nu(X, \mathcal{E})$ of $X^\sharp$ by $\sim$ is called the Higson corona of $(X, \mathcal{E})$. For $p \in X^\sharp$, we denote

$$\overline{p} = \{q \in X^\sharp : q \parallel p\}, \quad \bar{p} = \{q \in p : q \sim p\}.$$
A function $f : (X, \mathcal{E}) \to \mathbb{R}$ is called \textit{slowly oscillating} if, for every $E \in \mathcal{E}$ and $\epsilon > 0$, there exists a bounded subset $B$ of $X$ such that $\text{diam}f(E[x]) < \epsilon$ for each $x \in X \setminus B$.

We recall \cite{10} that a coarse space $(X, \mathcal{E})$ is \textit{normal} if any two asymptotically disjoint subsets $A, B$ of $X$ have disjoint asymptotic neighbourhoods. Two subsets $A, B$ of $X$ are called \textit{asymptotically disjoint} if $E[A] \cap E[B]$ is bounded for each $E \in \mathcal{E}$. A subset $U$ of $X$ is called an \textit{asymptotic neighbourhood} of a subset $A$ if $E[A \setminus U]$ is bounded for each $E \in \mathcal{E}$. A subset $U$ of $X$ is called an \textit{asymptotic neighbourhood} of a subset $A$ if $E[A \setminus U]$ is bounded for each $E \in \mathcal{E}$.

By \cite[Theorem 2.2]{10}, $(X, \mathcal{E})$ is normal if and only if, for any two disjoint and asymptotically disjoint subsets $A, B$ of $X$, there exists a slowly oscillating function $f : (X, \mathcal{E}) \to [0, 1]$ such that $f|_A = 0, f|_B = 1$.

By \cite[Proposition 1]{11}, $p \sim q$ if and only if $h^\beta(p) = h^\beta(q)$ for every slowly oscillating function $h : (X, \mathcal{E}) \to [0, 1]$.

By \cite[Theorem 7]{4}, $(X, \mathcal{E})$ is normal if and only if $\sim = \text{cl}||$.

By \cite[Theorem 9 and Corollary 10]{17}, if $\lambda_{(X, \mathcal{E})} = \lambda_{(X, \mathcal{E}')}$, then Higson coronas of $(X, \mathcal{E})$ and $(X, \mathcal{E}')$ coincide and if $(X, \mathcal{E})$ is normal then $(X, \mathcal{E}')$ is normal.

By \cite[Theorem 2.1.1]{21} a coarse space $(X, \mathcal{E})$ is metrizable if $\mathcal{E}$ has a countable base. If $\lambda_{(X, \mathcal{E})} = \lambda_{(X, \mathcal{E}')}$, then $(X, \mathcal{E}')$ needs not to be metrizable \cite[Theorem 3]{17}.

**Question 2.1** \cite{17}. Let $\delta_{(X, \mathcal{E})} = \delta_{(X, \mathcal{E}')}$. Is $(X, \mathcal{E})$ metrizable? If the answer to Question 2.1 would be positive then $\mathcal{E} = \mathcal{E}'$.

Let $(X, \mathcal{E})$ be a coarse space. We say that a subset $S$ of $X^\sharp$ is \textit{invariant} if $\overline{P} \subseteq S$ for each $P \in S$. Every non-empty closed invariant subset of $X^\sharp$ contains a minimal by inclusion closed invariant subset. We denote $K(X^\sharp) = \bigcup\{M : M$ is minimal closed invariant subset of $X^\sharp\}$.

**Theorem 2.2.** For $p \in X^\sharp$, $\text{cl}\overline{\mathcal{P}}$ is a minimal closed invariant subset if and only if, for every $P \in p$, there exists $E \in \mathcal{E}$ such that $\overline{P} \subseteq (E[P])^\sharp$.

**Proof.** Apply arguments proving this statement for metric spaces \cite[Theorem 3.1]{13}. \qed

**Theorem 2.3.** For $q \in X^\sharp$, $q \in \text{cl}K(X^\sharp)$ if and only if each subset $Q \in q$ is prethick.

**Proof.** Apply arguments proving Theorem 3.2 in \cite{13}. \qed

**Theorem 2.4.** Let $p, q$ be ultrafilters from $X^\sharp$ such that $\overline{P}, \overline{Q}$ are countable and $\text{cl}P \cap \text{cl}Q \neq \emptyset$. Then either $\text{cl}P \subseteq \text{cl}Q$ or $\text{cl}Q \subseteq \text{cl}P$.

**Proof.** Apply arguments proving Theorem 3.4 in \cite{13}. \qed
3. Dynamical systems

By a dynamical system we mean a pair \((G, T)\), where \(T\) is a compact space, \(G\) is a group of homeomorphisms of \(G\).

The following two theorems make a bridge between coarse spaces and dynamical systems. For usage of Theorem 3.1 in corona constructions see [3].

Let \(G\) be a transitive group of permutations of a set \(X\). We denote by \(X^G\) the set \(X\) endowed with the coarse structure with the base.

\[
\{(x, gx) : g \in F \} : F \in [G]^\omega, \; \text{id} \in F\}
\]

**Theorem 3.1.** For every finitary coarse space \((X, E)\), there exists a group \(G\) of permutations of \(X\) such that \((X, E) = X^G\).

**Proof.** Theorem 1 in [12], for more general results see [8], [15]. \(\square\)

**Theorem 3.2.** If \((X, E), (X, E')\) are finitary coarse spaces and \(|||_{(X, E)} = |||_{(X, E')}\) then \(E = E'\).

**Proof.** Theorem 15 in [17]. \(\square\)

If \((X, E) = X^G\), we say that \(X^G\) is the \(G\)-realization of \((X, E)\). Each \(G\)-realization of \((X, E)\) defines the dynamical system \((G, X^*)\) with the action \((g, p) = gp, gp = \{gP : P \in p\}\). We note that \(\overline{p} = Gp\) for each \(p \in X^*\). By Theorem 3.2, the partition of \(X^*\) into \(G\)-orbits does not depend on \(G\)-realizations of \((X, E)\). It follows that if some property formulated in terms of \(G\)-orbits of \((G, X^*)\) is proved for some \(G\)-realization of \((X, E)\) then it holds for any \(G\)-realization.

Given a finitary coarse space \((X, E)\), its \(G\)-realization of \(X^G\), a subset \(A \subseteq X\) and \(p \in X^*\), we define the \(p\)-companion of \(A\) by

\[
\triangle_p(A) = A^* \cap Gp.
\]

**Theorem 3.3.** For a subset \(A\) of \((X, E)\), the following statements hold

1. \(A\) is large iff \(\triangle_p(A) \neq \emptyset\) for each \(p \in X^*\);
2. \(A\) is thick iff \(\triangle_p(A) = Gp\) for some \(p \in X^*\);
3. \(A\) is thin iff \(|\triangle_p(A)| \leq 1\) for each \(p \in X^*\);

**Proof.** Theorem 3.1 and 3.2 in [20]. \(\square\)

We recall that a dynamical system \((G, T)\) is

- **minimal** if each orbit \(Gx\) is dense in \(T\);
- **topologically transitive** if some orbit \(Gx\) is dense in \(T\).
For a dynamical system \((G, T)\), \(\ker (G, T)\) denotes the closure of the union of all minimal closed \(G\)-invariant subsets of \(T\). Theorem 2.3 describes explicitly the kernel of the dynamical system \((G, X^*)\) of \(X_G\).

**Theorem 3.4.** Let \((X, \mathcal{E})\) be a finitary coarse space and \((X, \mathcal{E}) = X_G\). Then \((X, \mathcal{E})\) is \(\delta\)-tight if and only if the dynamical system \((G, X^*)\) is minimal.

*Proof.* Apply Theorem 3.3(1). \(\square\)

**Theorem 3.5.** Let \((X, \mathcal{E})\) be a finitary coarse space and \((X, \mathcal{E}) = X_G\). Then the following statements are equivalent

1. \((X, \mathcal{E})\) is \(\lambda\)-tight;
2. for any infinite subset \(A, B\) of \(X\), there exist \(p \in X^*, g \in G\) such that \(A \in p, B \in gp\);
3. for any family \(\{A_n : n \in \omega\}\) of infinite subsets of \(X\), there exists \(p \in X^*\) such that \(A_n^* \cap Gp \neq \emptyset\) for each \(n \in \omega\).

*Proof.* (1) \(\Rightarrow\) (2). Since \(A, B\) are linked, there exist \(A' \subseteq A, B' \subseteq B\) and \(H \in [G]^{<\omega}\) such that \(A' \subseteq HB'\). We take \(p \in X^*\) such \(A' \in p\). Then \(B' \in h^{-1}p\) for some \(h \in H\).

(2) \(\Rightarrow\) (3). We choose inductively a sequence \((g_n)_{n \in \omega}\) in \(G\) and a sequence \((C_n)_{n \in \omega}\) of subsets of \(G\) such that \(C_n \subseteq A_n, g_nC_n \subseteq A_{n+1}, C_{n+1} \subseteq g_nC_n\). Let \(h_n = g_ng_{n-1} \ldots g_0\). Then

\[
A_0 \cap h_0^{-1}A_1 \cap \cdots \cap h_{n+1}^{-1}A_{n+1} \neq \emptyset
\]

for each \(n \in \omega\). We take an arbitrary ultrafilter \(p \in X^\sharp\) such that \(A_0 \cap h_0^{-1} \cap \cdots \cap h_{n+1}^{-1}A_{n+1} \in p\) for each \(n \in \omega\). Then \(Gp \cap A_n^* \neq \emptyset\) for each \(n \in \omega\).

(3) \(\Rightarrow\) (1). Evident. \(\square\)

**Corollary 3.6.** If \((X, \mathcal{E}) = X_G\) and the dynamical system \((G, X^*)\) is topologically transitive then \((X, \mathcal{E})\) is \(\lambda\)-tight.

Under some set theoretical assumptions, there exists a group \(G\) of permutations of \(\omega\) such that \(\omega_G\) is \(\lambda\)-tight but \((G, \omega^*)\) is not topologically transitive, see [1, Corollary 4.24(5)].

**Question 3.7.** In ZFC, does there exist a group \(G\) of permutations of \(\omega\) such that \(\omega_G\) is \(\lambda\)-tight and \((G, \omega^*)\) is not topologically transitive?
**Theorem 3.8.** Let $K$ be a closed nowhere dense subset of $\omega^*$. Then there exists a transitive group $G$ of permutations of $\omega$ such that $\ker(G, \omega^*) = K$ and the orbit $Gp$ is dense in $\omega^*$ for each $p \notin K$.

**Proof.** We take a filter $\phi$ on $\omega$ such that $K = \overline{\phi}$ and $\overline{\phi}$ has the base $\{A : A \in \phi\}$. We denote by $G$ the group of all permutations $g$ of $\omega$ such that there exists $A_g$ such that $g(x) = x$ for each $x \in A_g$. Clearly, $G$ is transitive.

If $q \in K$ then $g(q) = q$ for each $g \in G$ so $K \subseteq \ker(G, \omega^*)$.

We fix $p \in \omega^* \setminus K$ and take an arbitrary $q \in \omega^*$, $p \neq q$. Let $P \in p$, $Q \in q$ and $P \cap Q = \emptyset$. Since $K$ is nowhere dense, there exists $A \in \phi$ such that $P \setminus A \in p$ and $Q \setminus A$ is infinite. By the definition of $G$, there exists $g \in G$ such that $g(P \setminus A) = Q \setminus A$. Hence, $Gp$ is dense in $\omega^*$ and $K = \ker(G, \omega^*)$. $\blacksquare$

**Corollary 3.9.** There are $2^c \lambda$-tight finitary coarse spaces on $\omega$ which are not $\delta$-tight.

**Proof.** In light of Corollary 3.6, it suffices to notice that there are $2^c$ free ultrafilters on $\omega$. $\blacksquare$

Each orbit of a dynamical system from the proof of Theorem 3.8 is either dense or a singleton. We construct a topologically transitive $(G, \omega^*)$ having an infinite discrete orbit.

**Example 3.10.** We partition $\omega$ into infinite subsets $\{W_n : n \in \mathbb{Z}\}$, fix a bijection $f_n : W_n \to W_{n+1}$ and denote by $f$ a bijection of $\omega$ such that $f|_{W_n} = f_n$.

For each $n \in \mathbb{Z}$, we pick $p_n \in \omega^*$ such that $W_n \in p_n$ and denote by $S$ the set of all permutations $g$ such that, for each $g(x) = x$, $x \in W_n$.

We take the group $G$ of permutations generated by $S \cup \{f\}$. Then $Gp_0 = \{p_n : n \in \mathbb{Z}\}$ and $Gp_0$ is discrete.

If $p \in W_0^*$ and $p \neq p_0$ then $Gp$ is dense in $\omega^*$ so $(G, \omega^*)$ is topologically transitive.

**Remark 3.11.** If $(X, \mathcal{E})$ is a finitary coarse space, $(X, \mathcal{E}) = X_G$ then, by Theorem 2.3, every infinite subset of $(X, \mathcal{E})$ is prethick if and only if $\ker(G, X^*) = X^*$.

If every infinite subset of a finitary coarse space $(X, \mathcal{E})$ is thick then $(X, \mathcal{E})$ is discrete. Indeed, if $(X, \mathcal{E})$ is not discrete then, by Theorem 3.3(3), there exists $q \in X^*$ and $g \in G$ such that $gq \neq q$. We take $Q \in q$ such that $gQ \cap Q = \emptyset$. It follows that $Q$ is not thick.

We say that a subset $A$ of $X_G$ is

- **sparse** if $\triangle_p(A)$ is finite for each $p \in G^*$;

- **scattered** if, for each infinite subset $Y$ of $A$ there exists $p \in Y^*$ such that $\triangle_p(Y)$ is finite.
**Theorem 3.12.** Let \((X, \mathcal{E})\) be a finitary coarse space and \((X, \mathcal{E}) = X_G\). Then the following statements are equivalent:

1. \((X, \mathcal{E})\) is indiscrete;
2. for every infinite subset \(A\) of \(X\), there exist \(p \in X^*\) and \(g \in G\) such that \(A \in p\), \(A \in gp\) and \(p \neq gp\);
3. every infinite subset \(A\) of \(X\) is not sparse.

*Proof.* The equivalence of (1) and (2) follows from Theorem 3.3(3), \((3) \implies (1)\) is evident.

To show \((2) \implies (3)\), we choose a sequence \((g_n)_{n \in \omega}\) in \(G\) and sequence \((A_n)_{n \in \omega}\) of subsets of \(A\) such that

\[
A_{n+1} \subset A_n, \quad g_nA_{n+1} \cap A_{n+1} = \emptyset, \quad n \in \omega
\]

and choose \(p \in X^*\) such that \(A_n \in p\) for each \(n \in \omega\). Then \(Gp \cap A^*\) is infinite and \(A\) is not sparse. \(\square\)

**Theorem 3.13.** A subset \(A\) of \(X_G\) is scattered if and only if \(Gp\) is discrete for each \(p \in A^*\).

*Proof.* Theorem 5.4 in [20]. \(\square\)

Every infinite sparse space \(X_G\) contains an infinite discrete subset, see the proof of \((2) \implies (3)\) in Theorem 3.12.

**Question 3.14.** Does every infinite scattered space \(X_G\) contain an infinite discrete subset?

We say that a finitary coarse space \(X_G\) is inscattered if it has no infinite scattered subsets.

By Theorem 3.13, \(X_G\) is inscattered if and only if the set \(\{p \in X^* : Gp\text{ is not discrete} \}\) is dense in \(X^*\).

**Example 3.15.** We show that inscattered (in particular, indiscrete) space needs not to be \(\lambda\)-tight. To this end, we take the set \(X\) of all rational number on \([0, 1]\), denote by \(G\) the group of all homeomorphisms of \(X\) and consider the finitary space \(X_G\). Let \(A = \{a_n : n \in \omega\}\), \(B = \{b_n : n \in \omega\}\) be subsets of \(X\) such that \((a_n)_{n \in \omega}\) converges to 0 and \((b_n)_{n \in \omega}\) converges to some irrational number. Then \(A, B\) are not linked, so \(X_G\) is not \(\lambda\)-tight. On the other hand, if a sequence \((c_n)_{n \in \omega}\) converges in \([0, 1]\), \(C = \{c_n : n \in \omega\}, p \in X^*, C \in p\) then \(\triangle_p(C)\) is infinite, so \(X_G\) is inscattered.
Every incattered space is indiscrete. Question 3.14 asks whether the converse statement hold. If the answer to this question would be positive then the answer to the following question is positive.

**Question 3.16.** Is every \( \lambda \)-tight finitary space inscattered?

As the results, we have got two lines

\[
\delta \text{-tight} \implies \text{topologically transitive} \implies \lambda \text{-tight} \implies \text{indiscrete},
\]

\[
\text{topologically transitive} \implies \text{inscattered} \implies \text{indiscrete}
\]

with open questions

\[
\lambda \text{-tight} \implies \text{inscattered}, \quad \text{indiscrete} \implies \text{inscattered}?
\]

**Question 3.17.** Is a dynamical system \((G, \omega^*)\) minimal provided that \(\ker(G, \omega^*) = \omega^*\) and \((G, \omega^*)\) is topologically transitive?

The Higson corona of every \( \lambda \)-tight space is a singleton.

The following example suggested by Taras Banakh shows that the Higson corona of finitary indiscrete space needs not to be a singleton.

**Example 3.18.** Let \((X_1, \mathcal{E}_1), (X_2, \mathcal{E}_2)\) be infinite indiscrete finitary spaces. We endow the union \(X\) of \(X_1\) and \(X_2\) with the smallest coarse structure \(\mathcal{E}\) such that the restrictions of \(\mathcal{E}\) to \(X_1\) and \(X_2\) coincide with \(\mathcal{E}_1\) and \(\mathcal{E}_2\). Then the Higson corona of \((X, \mathcal{E})\) is not a singleton.

A subset \(A\) of a coarse space \((X, \mathcal{E})\) is called \(n\)-thin (or \(n\)-discrete), \(n \in \mathbb{N}\) if for each \(E \in \mathcal{E}\) there exists a bounded subset \(B\) of \((X, \mathcal{E})\) such that \(|E_A[a]| \leq n\) for each \(a \in A \setminus B\). Every \(n\)-thin metrizable coarse space can be partitioned into \(\leq n\) thin subsets \([7]\), but the Bergman’s construction from \([19]\) gives a finitary \(n\)-thin space which can not be partitioned into \(\leq n\) thin subsets because the function \(f\) defined by \(f(x) = 0, \, x \in X_1\) and \(f(x) = 1, \, x \in X_2\) is slowly oscillating.

**Question 3.19.** Let \(G\) be a group of permutations of \(\omega\). Can every \(n\)-thin subset of \(\omega_G\) be partitioned into \(\leq n\) thin subsets?

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**CONTACT INFORMATION**

I. Protasov:
Faculty of Computer Science and Cybernetics
Kyiv University
Academic Glushkov pr. 4d
03680 Kyiv, Ukraine
i.v.protasov@gmail.com