

\textbf{BINARY HERMITIAN FORMS AND OPTIMAL EMBEDDINGS}

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\textbf{Abstract.} Fix a quadratic order over the ring of integers. An embedding of the quadratic order into a quaternionic order naturally gives an integral binary hermitian form over the quadratic order. We show that, in certain cases, this correspondence is a discriminant preserving bijection between the isomorphism classes of embeddings and integral binary hermitian forms.

1. Introduction

Let $K$ be a field of any characteristic and fix $L$, a separable quadratic extension of $K$. Let $n_L : L \to K$, $n_L(l) = \bar{l}$, be the norm map. In this paper we study relationship between embeddings of $L$ into a quaternion algebra and vector spaces over $L$ with a hermitian form, as well as an integral version of this problem. Let $V$ be a vector space over $L$. A hermitian form on $V$ is a quadratic form $h : V \to K$ such that $h(lv) = n_L(l)h(v)$ for all $l \in L$ and $v \in V$. This definition is different than what is customary in the literature, where a hermitian symmetric form is a function $s : V \times V \to L$, linear in the first variable, and $s(y, x) = s(x, y)$ for all $x, y \in V$. However, given $h$, we show that there exists a unique $s$ such that $h(v) = s(v, v)$ for all $v \in V$, even if the characteristic of $K$ is 2.

Let $i : L \to H$ be an embedding of $L$ into a quaternion algebra $H$. Let $n : H \to K$ be the (reduced) norm. Then $H$ is naturally a vector space over $L$ of dimension 2, with a hermitian form $h := n$ representing 1, since $n(1) = 1$. Conversely, let $(V, v, h)$ be a triple consisting of a 2-dimensional vector space $V$ over $L$, a non-degenerate hermitian form $h$ on $V$, and a point $v \in V$ such that $h(v) = 1$. Then one can define a quaternion algebra $H_V$ such that $H_V = V$, as vector spaces over $K$, $v$ is the identity element of $H_V$, and $l \mapsto lv$ is an embedding of $L$ into $H_V$, where $lv$ is the scalar multiplication inherited from $V$. The reduced norm is, of course, equal to the hermitian form $h$. This discussion can be placed in a categorical context; the following two categories are isomorphic: one whose objects are pairs $(i, L)$ where $i$ is an embedding of $L$ into a quaternion algebra, the other whose objects are triples $(V, v, h)$ of pointed, non-degenerate, binary hermitian spaces. We now observe that different choices of the point $v$ representing 1 give isomorphic objects. Indeed, if $u \in V$ is another element in $V$ representing 1, then $u$ is a unit element in the quaternion algebra $H_V$ arising from the triple $(V, v, h)$. Right multiplication by $u$ gives an isomorphism of the triples $(V, v, h)$ and $(V, u, h)$. As a consequence, there is a bijection between isomorphism classes of embeddings of $L$ into quaternion algebras and non-degenerate binary hermitian spaces $(V, h)$ representing 1.

As the next result, we establish an integral version of this bijection. More precisely, assume that $K$ is the field of fractions of a Dedekind domain $A$, and $B$ the integral closure of $A$ in $L$. An order $O$ in $H$ is an $A$-lattice, containing the identity element and closed under the multiplication in $H$. An embedding $i : L \to H$ is called optimal if $i(L) \cap O = i(B)$. If this is the case, then $O$ is naturally a projective $B$-of rank 2, such that the norm $n$ takes values in...

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A and represents 1. Conversely, let $\Lambda \subseteq V$ be a lattice, i.e. a projective $B$-module of rank 2, such that $h$ is integral on $\Lambda$, i.e. $h(\Lambda) \subseteq A$. Such a pair $(\Lambda, h)$ is called an integral binary hermitian form. Assume that there exists $v \in \Lambda$ such that $h(v) = 1$. We show that $\Lambda$ is an order (denoted by $O_\Lambda$) in $H_V$, and thus we have constructed a quaternionic order and an embedding of $B$ into it, given by $b \mapsto bv$. Just as in the field case, different choices of $v$ give isomorphic objects and, as a consequence, there is a bijection between isomorphism classes of embeddings of $B$ into quaternionic orders and non-degenerate integral binary hermitian forms $(\Lambda, h)$ representing 1.

We now turn to the question when a hermitian form represents 1, which is especially interesting in the integral case. To that end we establish a relationship between invariants of the objects studied. The determinant $d(\Lambda, h)$ of an integral binary hermitian form $(\Lambda, h)$ is a fractional ideal in $K$ generated by $\det(s(v_i, v_j))$ for all pairs $(v_1, v_2)$ of elements of $\Lambda$. We note that $d(\Lambda, h)$ is not necessarily an ideal. Integrality of $h$ implies only that $s$ takes values in the different ideal of $B$ over $A$ in $L$. Since the discriminant $D_{B/A}$ is the norm of the inverse different, the product $D_{B/A} \cdot d(\Lambda, h)$ is an integral ideal. This was previously observed in [1] where the discriminant of an integral binary hermitian form is defined as this product. Thus we follow the same convention and define the discriminant of $(\Lambda, h)$ to be the integral ideal $\Delta(\Lambda, h) = D_{B/A} \cdot d(\Lambda, h)$. If $h$ represents 1, so $\Lambda$ is the order $O_\Lambda$, then we have an identity

$$\Delta(O_\Lambda) = \Delta(\Lambda, h)$$

of integral ideals where $\Delta(O_\Lambda)$ is the discriminant of the order $O_\Lambda$. If $K = \mathbb{Q}$ and $L$ an imaginary quadratic extension, then these invariants can be refined to be integers, and this is an equality of two integers. The following is the most interesting result in this paper:

**Theorem 1.1.** Assume $L$ is a tamely ramified, imaginary quadratic extension of $\mathbb{Q}$. Let $B$ be the maximal order in $L$. Let $\Delta < 0$ be a square free integer. There is a bijection between the isomorphism classes of embeddings of $B$ into quaternionic orders of discriminant $\Delta$ and integral binary $B$-hermitian forms of discriminant $\Delta$.

The key here is to show that an integral binary hermitian form $(\Lambda, h)$ of discriminant $\Delta$ represents 1. The condition $\Delta$ is square free assures that $h$ represents 1 on $\Lambda \otimes \mathbb{Z}_p$ for all $p$. The condition $\Delta < 0$ implies that $h$ is indefinite on $\Lambda \otimes \mathbb{R}$. The theorem follows from the integral version of the Hasse-Minkowski theorem for indefinite quadratic forms in 4 variables.

## 2. Hermitian Forms

Let $K$ be a commutative field of any characteristic. Let $L$ be a separable quadratic $K$-algebra. In particular, there exists an involution $l \mapsto \bar{l}$ on $L$ such that the set of fixed points is $K$. A vector space over $L$ is a free $L$-module. We go over some basic results about hermitian forms on vector spaces.

**Definition 2.1.** Let $V$ be a vector space over $L$. A **hermitian symmetric form** is a function $s: V \times V \to L$ which is $L$-linear in the first argument and $s(x, y) = s(y, x)$.

Let $h: V \to K$ be the function defined by $h(x) = s(x, x)$ for all $x \in V$. This is a quadratic form, if we consider $V$ a vector space over $K$. In particular, the function $b(x, y) = h(x + y) - h(x) - h(y)$ is $K$-bilinear. If we let $n_L(l) = \bar{l}$, for $l \in L$, then $h(lx) = n_L(l)h(x)$. These two facts are a sufficient characterization of hermitian forms for our purposes, due to the
Proposition 2.2. Let $V$ be a vector space over $L$. Let $h : V \to K$ be a quadratic form, where $V$ is considered a vector space over $K$, satisfying $h(lx) = n_L(l)h(x)$ for all $l \in K$. Then there is a unique hermitian symmetric form $s(x, y)$ such that $h(x) = s(x, x)$.

Proof. Assume that $s$ exists. Then, for all $x, y \in V$ and $l \in L$,

$$s(x, y) + s(y, x) = b(x, y)$$
$$ls(x, y) + \overline{l}s(y, x) = b(lx, y).$$

We view this as a $2 \times 2$ system in unknowns $s(x, y)$ and $s(y, x)$. If $l \neq \overline{l}$ then this system has a unique solution given by

$$s_l(x, y) = \frac{\overline{l}b(x, y) - b(lx, y)}{\overline{l} - l}$$

where the subscript $l$ indicates that the solution, so far, depends on the choice of $l$. (Note that $\overline{l} - l$ is invertible, even when $L \cong K^2$.) We claim that $s_l(x, y)$ is independent of $l$. Indeed, let $l = c + d\pi$, where $L = K[\pi]$, and $c, d \in K$. Then

$$s_l(x, y) = \frac{\overline{l}b(x, y) - b(lx, y)}{\overline{l} - l}$$
$$= \frac{cb(x, y) + d\pi b(x, y) - b(cx + d\pi x, y)}{c + d\pi - c - d\pi}$$
$$= \frac{d\pi b(x, y) - db(\pi x, y)}{d\pi - d\pi}$$
$$= s_{\pi}(x, y).$$

Hence $s_l = s_{\pi}$ and we can write $s$ for any of the $s_l$. It is then easy to check that $s(x, x) = s_l(x, x) = h_l(x)$.

Now we check that $s(x, y)$ is linear in the first variable. It is clear that $s(x_1 + x_2, y) = s_l(x_1 + x_2, y)$. Let $l \in L \setminus K$. Then

$$s(lx, y) = s_{\overline{l}}(lx, y)$$
$$= \frac{lb(lx, y) - b(n_L(l)x, y)}{l - \overline{l}}$$
$$= \frac{lb(lx, y) - n_L(l)b(x, y)}{l - \overline{l}}$$
$$= \frac{\overline{l}b(x, y) - b(lx, y)}{\overline{l} - l}$$
$$= ls_l(x, y) = ls(x, y),$$

which shows $s$ is linear in the first argument. It remains to check that $s(x, y) = s(y, x)$. Note that $b(\overline{l}x, \overline{l}y) = n_L(l)b(x, y)$. Replacing $x$ with $lx$, we find that $b(n_L(l)x, \overline{l}y) = n_L(l)b(lx, y)$,
hence \( b(lx, y) = b(x, ly) \). Then

\[
s(y, x) = \mathcal{s}(y, x) = \frac{lb(y, x) - b(\bar{y}, x)}{l - \overline{\mathcal{I}}}
\]

\[
= \frac{lb(x, y) - b(lx, y)}{l - \overline{\mathcal{I}}}
\]

\[
= s_l(x, y) = s(x, y),
\]

or \( s(x, y) = \mathcal{s}(y, x) \), as desired. Uniqueness is clear. \( \square \)

Thus, we can make the following definition.

**Definition 2.3.** Let \( V \) be a vector space over \( L \). A hermitian form is a quadratic form \( h : V \to K \), where \( V \) is considered a vector space over \( K \), such that \( h(lv) = n_L(l)h(v) \) for all \( l \in L \), and for all \( v \in V \).

## 3. Quaternion Algebras

Let \( H \) be a quaternion algebra over \( K \), i.e. a central simple algebra over \( K \) of dimension 4. We know that by general theory (e.g. [5, §9a]), every element \( x \in H \) satisfies a reduced characteristic polynomial of degree two, associated to its action on the algebra via left-multiplication. The reduced trace \( \text{tr}(x) \) is the sum of the roots of the reduced characteristic polynomial of \( x \). Let the reduced norm \( n(x) \) be the product of the roots. The trace furnishes \( H \) with an involution \( \overline{x} := \text{tr}(x) - x \) such that \( n(x) = x\overline{x} \). Let \( L \) be a separable quadratic field extension of \( K \) and \( i : L \to H \) an embedding. Since \( L \) is separable, the restriction of the involution of \( H \) to \( L \) is non-trivial. The norm \( n \) is multiplicative, i.e. \( n(xy) = n(x)n(y) \) for all \( x, y \in H \). Hence the norm is a hermitian form on \( H \). This hints at a correspondence between embeddings of \( L \) into quaternion algebras and hermitian forms. The next section will formalize this idea. We finish this section with the following structural result.

**Proposition 3.1.** Let \( L \) be a separable quadratic field extension of \( K \). If \( i : L \to H \) is an embedding, then there exists \( u \in H \) such that \( \overline{u} = -u \) and \( \theta \in K^\times \) satisfying \( H = L + Lu, u^2 = \theta, \) and \( um = \overline{m}u \) for all \( m \in L \) (cf. [6, Definition 1.1, Chapter 1]). For every \( v = x + yu \in L + Lu, \)

\[
n(v) = n_L(x) - n_L(y)\theta,
\]

where \( n_L : L \to K \) is the norm.

**Proof.** By Proposition 2.2, there exists a unique, \( L \)-hermitian bilinear form \( s(v, w) : V \times V \to L \) such that \( s(v, v) = n(v) \) for all \( v \in H \). Take \( u \neq 0 \) with \( s(u, 1) = 0 \). Then \( H = L + Lu \). For any \( l \in L \setminus K, \)

\[
0 = s(u, 1) = s_l(u, 1) = \frac{\overline{l}b(u, 1) - b(lu, 1)}{\overline{l} - l}.
\]

Since \( b(u, 1) = n(u + 1) - n(u) - n(1) = \text{tr}(u) \) and \( b(lu, 1) = \text{tr}(lu) \), hence \( \overline{l}\text{tr}(u) = \text{tr}(lu) \).

If \( \text{tr}(u) \neq 0 \), then \( \overline{l} \in K \), a contradiction. Hence \( \text{tr}(u) = 0 \) and \( \text{tr}(lu) = 0 \). The first of these implies that \( \overline{u} = -u \), and so \( 0 = \text{tr}(lu) = lu + \overline{u} = lu - ul \), i.e. that \( lu = ul \) or \( ul = \overline{lu} \).

Finally, since \( \text{tr}(u) = 0, u^2 = -n(u) \in K \). Simplicity of \( H \) forces \( \theta \) to be non-zero. The formula for \( n(v) \) follows from orthogonality of 1 and \( u \). \( \square \)
Note that the hermitian symmetric form $s$, attached to $n$, in the basis $1, u$ is given by a diagonal matrix with $1$ and $-\theta$ on the diagonal. In particular, invertibility of $\theta$ implies that the hermitian form is non-degenerate.

4. Correspondence Over a Field

Here $L$ continues to be a separable quadratic extension of $K$. The correspondence alluded to before Proposition 3.1 can be formalized as an isomorphism of the categories $\text{PtdHerm}_L$ of non-degenerate pointed hermitian spaces and $\text{Quat}_L$ of embeddings of $L$ into quaternion algebras over $K$. We will define these terms more precisely, and give a proof of the isomorphism.

**Definition 4.1.** Let $\text{Quat}_L$ be the category whose objects are pairs $(i, H)$ where $H$ is a quaternion algebra over $K$, and $i : L \to H$ is an embedding of $L$ into $H$. The morphisms $f : (i, H) \to (i', H')$ are morphisms $f : H \to H'$ of $K$-algebras satisfying $f \circ i = i'$. Note that any $f$ must be an isomorphism of $H$ with $H'$.

**Definition 4.2.** Let $\text{PtdHerm}_L$ be the category whose objects are triples $(V, v, h)$, called pointed hermitian spaces, where $V$ is a two-dimensional $L$-vector space, $h : V \to K$ a non-degenerate hermitian form, and $v \in V$ such that $h(v) = 1$. The morphisms from $(V, v, h)$ to $(V', v', h')$ are isomorphisms $f : V \to V'$ with $f(v) = v'$ and $h(x) = h'(f(x))$.

We can now define a functor $F : \text{Quat}_L \to \text{PtdHerm}_L$, due to the following easy lemma:

**Lemma 4.3.** Let $F : \text{Quat}_L \to \text{PtdHerm}_L$ be given by $F(i, H) = (H, 1, n)$, where $n$ is the norm form of $H$, and $1$ is the identity element of $H$. Given a morphism $f : (i, H) \to (i', H')$, define $F(f) := f$. Then $F$ is a functor.

Now we construct a functor $G : \text{PtdHerm}_L \to \text{Quat}_L$ by defining a multiplication $\cdot$ on a given $(V, v, h)$, motivated by Proposition 3.1. Recall that there is a unique hermitian symmetric form $s(v, w)$ on $V$ with $h(x) = s(x, x)$ for all $x \in V$. Embed $L$ into $V$ via $l \mapsto lv$ where $lv$ denotes the scalar multiplication.

1. $v$ is the identity element,
2. for $x \in V$, and $l \in L$, $l \cdot x := lx$, the scalar multiplication,
3. for $y \in L^\perp = \{ w \mid s(v, w) = 0 \}$, $y \cdot (lv) := \overline{ly}$, for $l \in L$.
4. for $y \in L^\perp$, $y \cdot y := -h(y)v$.
5. conjugation on $V$ is $\overline{lv} + y = \overline{lv} - y$, for $l \in L$ and $y \in L^\perp$.

This gives $V = L \oplus L^\perp$ a structure of quaternion algebra, as in Proposition 3.1, since we can choose $u$ to be any element of $L^\perp$, and $\theta = -h(u)$. Non-degeneracy of $s$ implies that $\theta \neq 0$. The norm form is $h$. Let $H_V$ denote this algebra. Then we have

**Lemma 4.4.** Define $G(V, v, h) := (i_v, H_V)$, where $i_v(l) = lv$ for $l \in L$, and for a morphism $f : (V, v, h) \to (V', v', h')$, define $G(f) := f$. Then $G$ is a functor from $\text{PtdHerm}_L$ to $\text{Quat}_L$.

It is clear that the functors $F$ and $G$ are inverses of each other. Thus, we have

**Theorem 4.5.** The categories $\text{Quat}_L$ and $\text{PtdHerm}_L$ are isomorphic.
that the isomorphism classes of objects in the category $\text{PtdHerm}_L$ do not depend on the choice of the point representing 1. Thus we have the following:

**Corollary 4.6.** The functors $F$ and $G$ give a bijection between the isomorphism classes of embeddings of $L$ into quaternion algebras and the isomorphism classes of non-degenerate binary hermitian spaces $(V, h)$ such that $h$ represents 1.

### 5. Correspondence Over a Dedekind Domain

Assume that $K$ is a field of fractions of a Dedekind domain $A$. Let $B$ be the integral closure of $A$ in $L$. It turns out an integral version of the above isomorphism can be established, replacing the field with a Dedekind domain, quaternion algebras with orders, and binary hermitian forms on a vector space with integral binary hermitian forms on a $B$-projective module of rank 2. Let $i : L \to H$, be an embedding of $L$ into $H$. Recall that a lattice $\Lambda$ in the quaternion algebra $H$ is a finitely generated $A$-submodule containing a $K$-basis of $H$. If the lattice $\Lambda$ is a $B$-module, then it is also projective of rank 2. An order in $H$ is a lattice containing the identity element and closed under the multiplication.

**Definition 5.1.** Let $\text{Ord}_B$ be the category whose objects are pairs $(i, \mathcal{O})$ where $i : B \to \mathcal{O}$ is an embedding into an order $\mathcal{O}$ of quaternion algebra. The morphisms from $(i, \mathcal{O})$ to $(i', \mathcal{O}')$ are isomorphisms $f : \mathcal{O} \to \mathcal{O}'$ of $A$-algebras satisfying $f \circ i = i'$.

**Definition 5.2.** A binary $B$-hermitian form is a pair $(\Lambda, h)$ where $\Lambda$ is a projective $B$-module of rank two, and $h : \Lambda \otimes_A K \to \mathbb{C}$ is an $L$-hermitian form. The form is integral if $h(\Lambda) \subset A$.

In the above definition we have identified $\Lambda$ with $\Lambda \otimes_A 1 \subset V = \Lambda \otimes_A K$.

**Definition 5.3.** Let $\text{PtdProjHerm}_B$ be the category whose objects are triples $(\Lambda, v, h)$, called **pointed integral hermitian spaces**, where $(\Lambda, h)$ is an integral, binary $B$-hermitian form, with $h$ non-degenerate, and $v \in \Lambda$ with $h(v) = 1$. The morphisms from $(\Lambda, v, h)$ to $(\Lambda', v', h')$ are $B$-module isomorphisms $f : \Lambda \to \Lambda'$ preserving the pointed hermitian structure, i.e. for $x \in \Lambda$, $h(x) = h'(f(x))$ and $f(v) = v'$.

We will prove that the two categories are isomorphic. In one direction we have a functor given by the following:

**Lemma 5.4.** Let $P : \text{Ord}_B \to \text{PtdProjHerm}_B$ be defined by $P(i, \mathcal{O}) := (\mathcal{O}, 1, n)$, where $n$ is the norm of the quaternion algebra $\mathcal{O} \otimes_A K$, and for a morphism $f : (i, \mathcal{O}) \to (i', \mathcal{O}')$, $P(f) := f$. Then $P$ is a functor between these two categories.

In order to define a functor $Q : \text{PtdProjHerm}_B \to \text{Ord}_B$, we need to define a multiplication on a pointed $B$-hermitian module $(\Lambda, v, h)$. On $V = \Lambda \otimes_A K$, as in the previous section, we have a quaternion algebra structure $H_V$ arising from $h$ and the associated $L$-hermitian bilinear form $s : V \times V \to L$, such that $v$ is the identity. We need the following:

**Lemma 5.5.** The $B$-module $\Lambda$, regarded as a subset of $H_V$ (recall $V = H_V$ as sets), is closed under the multiplication on $H_V$ i.e. it is an order in $H_V$ denoted by $\mathcal{O}_\Lambda$.

**Proof.** We first reduce to the case of $A$ and $B$ principal ideal domains. Let $p$ be any prime of $A$, $S_p = A - p$, and $A_p = S_p^{-1}A$, $B_p = S_p^{-1}B$ the localizations of $A$ and $B$ respectively. Then $B_p$ is a semi-local Dedekind domain, hence it is a principal ideal domain. Let $\Lambda_p = S_p^{-1}\Lambda$. If we show, for $x, y \in \Lambda_p$, that $x \cdot y \in \Lambda_p$, for all $p$, then using $\Lambda = \bigcap_p \Lambda_p$, we conclude that $x \cdot y \in \Lambda$. 

Hence we can assume that $B$ is a principal ideal domain. Then $\Lambda$ is a free $B$-module of rank 2. Note that $v$ is a primitive element of the lattice $\Lambda$ since $h(v) = 1$. Hence $v$ is member of a $B$-basis $(v, w)$ of $\Lambda$ \cite[Ch. 7, Theorem 3.1, page 106]{2}. Let $v^\perp = w - s(w, v)v$, then $s(v, v^\perp) = 0$. Let $\gamma = s(w, v)$ and $\theta = v^\perp \cdot v^\perp = -h(v^\perp)$. Then to check $x \cdot y \in \Lambda$, we just need to check that

$$w \cdot w = (v^\perp + \gamma v) \cdot (v^\perp + \gamma v)$$

is an element of $\Lambda_p$. But we can compute

$$w \cdot w = (v^\perp + \gamma \cdot v) \cdot (v^\perp + \gamma \cdot v)$$

$$= v^\perp \cdot v^\perp + v^\perp \cdot \gamma \cdot v + \gamma \cdot v \cdot v^\perp + \gamma \cdot v \cdot \gamma \cdot v$$

$$= \theta \cdot v + \gamma^2 \cdot v^\perp$$

$$= (\theta - n(\gamma))v + \theta \cdot v + \gamma^2 \cdot v^\perp$$

Now note that $n(\gamma) = \gamma^2 = h(\gamma \cdot v)$. Hence,

$$-\theta = s(v^\perp, v^\perp) = s(w, w) + s(-\gamma v, w) + s(w, -\gamma v) + s(-\gamma v, -\gamma v)$$

$$= h(w) + h(\gamma v) - \gamma s(v, w) - \gamma s(w, v)$$

$$= h(w) + n(\gamma) - 2n(\gamma) = h(w) - n(\gamma),$$

since $s(v, w) = \gamma$. Hence $\theta - n(\gamma) = -h(w)$, which is in $A$. Finally,

$$\text{tr}(\gamma) = \gamma + \gamma^2 = s(w, v) + s(v, w)$$

$$= s(v + w, v + w) - s(v, v) - s(w, w)$$

$$= h(v + w) - h(v) - h(w),$$

which is also in $A$. Thus, the product from $V$ preserves $\Lambda$. \qed

Then we have the lemma

**Lemma 5.6.** Let $Q : \text{PtdProjHerm}_B \to \text{Ord}_B$ be defined by $Q(\Lambda, v, h) = (i_v, O_\Lambda)$, $i_v(b) = bv$ for $b \in B$, and for a morphism $f : (\Lambda, v, h) \to (\Lambda', v', h')$, define $Q(f) := f$. Then $Q$ is a functor.

Again, it is easy to check that the functors $P$ and $Q$ are inverses of each other. In particular:

**Theorem 5.7.** The categories $\text{Ord}_B$ and $\text{PtdProjHerm}_B$ are isomorphic.

Arguing in the same way as in the case of fields, Corollary 4.6, two pointed binary integral $B$-hermitian modules $(\Lambda, v, h)$ and $(\Lambda, u, h)$ are isomorphic. Thus the isomorphism classes in $\text{PtdProjHerm}_B$ are the same as the isomorphism classes of binary integral hermitian spaces $(\Lambda, h)$ such that $h$ represents 1.

**Corollary 5.8.** The functors $P$ and $Q$ give a bijection between the isomorphism classes of embeddings of $B$ into quaternionic orders of and the isomorphism classes of binary integral $B$-hermitian spaces $(\Lambda, h)$ such that $h$ represents 1.
All that remains is to examine in which situations there is existence of a point with \( h(v) = 1 \). This can be done, in the field and domain cases, by using local-global principles.

6. Discriminant Relation

We are in the setting of the previous section.

**Lemma 6.1.** Let \( \Lambda \) be a lattice in \( H \). The fractional ideal \( d(\Lambda) \) in \( K \) generated by \( \det(\text{tr}(u_iu_j)) \), for all quadruples \((u_1, \ldots, u_4)\) of elements in \( \Lambda \) is a square.

**Proof.** Assume firstly that \( A \) is a principal ideal domain. Then \( d(\Lambda) \) is a principal ideal generated by \( \det(\text{tr}(u_iu_j)) \) where \((u_1, \ldots, u_4)\) is any \( A \)-basis of \( \Lambda \). Let \( \Lambda' \) be another lattice and \((u'_1, \ldots, u'_4)\) its basis. Let \( T : H \to H \) be the linear transformation such that \( T(u_i) = u'_i \) for all \( i \). Then

\[
d(\Lambda') = \det(T)^2 d(\Lambda).
\]

Hence, in order to show that \( d(\Lambda) \) is a square, it suffices to do so for one lattice. Write \( H = L + Lu \), as in Proposition 3.1. Let \( \Lambda = B + Bu \). Assume that \( B = A[\pi] \) where \( \pi \) is a root of the polynomial \( x^2 + ax + b \). Pick 1, \( \pi, u, \pi u \) as a basis of \( \Lambda \). Then the matrix of traces is

\[
\begin{pmatrix}
2 & -a & 0 & 0 \\
-a & a^2 - 2b & 0 & 0 \\
0 & 0 & -2\theta & a\theta \\
0 & 0 & a\theta & -2b\theta
\end{pmatrix},
\]

and its determinant is \( (a^2 - 4b)(4b - a^2)\theta^2 = -(a^2 - 4b)^2\theta^2 \). Then \( d(\Lambda) = (D_{B/A}\theta)^2 \) (as ideals) where \( D_{B/A} \) is the discriminant of \( B \) over \( A \). The general case is now treated using localization. Let \( p \) be a maximal ideal in \( A \). Then, one easily checks, \( d(\Lambda)_p = d(\Lambda_p) \). Since \( A_p \) is a local Dedekind ring, hence a principal domain, \( d(\Lambda_p) \) must be an even power of \( p \). \( \square \)

**Definition 6.2.** The discriminant of an \( A \)-lattice \( \Lambda \) in \( H \) is the fractional ideal \( \Delta(\Lambda) \) in \( K \) such that \( \Delta(\Lambda)^2 = d(\Lambda) \).

If \( K = \mathbb{Q} \) then we can refine the notion of \( \Delta(\Lambda) \) to be a rational number, the generator of this ideal. We pick \( \Delta(\Lambda) \) to be the positive generator if \( H \otimes_{\mathbb{Q}} \mathbb{R} \) is the matrix algebra, and negative otherwise.

**Definition 6.3.** Let \( \Lambda \) be a projective \( B \)-module of rank 2 and \( h \) a hermitian form on \( \Lambda \), not necessarily integral. Let \( s \) be the bilinear hermitian form on \( \Lambda \) such that \( s(x, x) = h(x) \), for all \( x \in \Lambda \). Let \( d(\Lambda, h) \) be the fractional ideal in \( K \) generated by \( \det(s(v_i, v_j)) \) for all pairs \((v_1, v_2)\) of elements in \( \Lambda \).

If \( K = \mathbb{Q} \) and \( L \) a complex quadratic extension we refine \( d(\Lambda, h) \) to be the positive generator of this ideal if \( h \) is a positive definite hermitian form on \( \Lambda \otimes_{\mathbb{Z}} \mathbb{R} \) and negative otherwise. In other words this sign is the sign of \( \det(s(v_i, v_j)) \), for any basis \((v_1, v_2)\) of \( \Lambda \otimes_{\mathbb{Z}} \mathbb{R} \).

**Theorem 6.4.** Let \( \Lambda \) be a lattice in \( H \) that is also a \( B \)-module. Then we have the following identity of fractional ideals in \( K \):

\[
\Delta(\Lambda) = D_{B/A} \cdot d(\Lambda, n).
\]

If \( K = \mathbb{Q} \) and \( L \) a complex quadratic extension, this is an identity of rational numbers.
Proof. This is an identity of two fractional ideals, so it can be checked by localization at every prime ideal \( p \). The localizations \( A_p \) and \( B_p \) are a local and a semi-local, respectively, Dedekind domains. Thus they are principal ideal domains [3]. Hence, by localization, the proof can be reduced to the case of principal ideal domains. So we shall assume that \( A \) and \( B \) are both principal ideal domains. In that case \( d(\Lambda, n) \) is a principal ideal generated by \( \det(s(v_i, v_j)) \) where \( v_1, v_2 \) is a \( B \)-basis of \( \Lambda \). Let \( \Lambda' \) be another lattice in \( H \) that is also a \( B \)-module. Let \( v'_1, v'_2 \) be a \( B \)-basis of \( \Lambda' \). Let \( T : H \to H \) be the \( L \)-linear map defined by \( T(v_i) = v'_i \). Then 

\[
    d(\Lambda', n) = n_L(\det_L(T))d(\Lambda, n).
\]

Now note that if we view \( T \) as a \( K \)-linear map, then \( \det_K(T) = n_L(\det_L(T)) \). Since \( \Delta(\Lambda') = \det_K(T)\Delta(\Lambda) \), it follows that both sides of the proposed identity change in the same way, if we change the lattice. Hence it suffices to check the identity for one lattice. Take \( \Lambda = B + Bu \), then both sides are ideals generated by \( DB/A \theta \), where \( -\theta = u^2 \), so the identity of ideals holds.

The last statement is an easy check of signs, since the discriminant of complex quadratic fields is negative.

We finish this section with the following proposition which defines the discriminant of an integral binary hermitian form, and establishes its integrality.

**Proposition 6.5.** Let \( \Lambda \) be a projective \( B \)-module of rank 2 and \( h \) an integral hermitian form on \( \Lambda \). Then \( \Delta(\Lambda, h) = DB/A \cdot d(\Lambda, h) \) is an integral ideal, called the discriminant of \( (\Lambda, h) \).

Proof. By localization, we can assume that \( A \) is local. Let \( v_1, v_2 \in \Lambda \) and consider the entries of the matrix \( (s(v_i, v_i)) \). Since \( s(v, v) = h(v) \) the diagonal entries are integral. However, the off-diagonal entries need not be integral. Indeed, for every \( l \in B \), such that \( l \neq \overline{l} \), we have the formula

\[
    s(v_1, v_2) = \frac{l b(v_1, v_2) - b(lv_1, v_2)}{\overline{l} - l}.
\]

If \( p \) splits or is inert in \( B \), then there exists \( l \) such that \( \overline{l} - l \) is a unit. Hence \( \det(s(v_i, v_j)) \) is in \( A \). If \( p \) ramifies, then we can take \( l \) to be the uniformizing element in \( B \). In this case \( B = A[l] \) and the discriminant \( DB/A \) is precisely the norm of \( \overline{l} - l \). It follows that \( DB/A \cdot \det(s(v_i, v_j)) \) is integral.

7. Representing one

Here we address the question when a binary hermitian form represents 1, integral or otherwise. We first address the field case. The case of integral representations is significantly more complicated; a local-global principle is provided by [4, Theorem 104:3], which will reduce the problem of an integral representation of 1 to the problem of integral representations over \( p \)-adic rings.

7.1. Over a Field. Assume that \( K \) is a number field. Any binary \( L \)-hermitian form \( h \) is a quaternary quadratic form over \( K \). Recall that \( h \) is positive semi-definite at a real place of \( K \) (i.e. an embedding of \( K \) into \( \mathbb{R} \)) if \( h \) is always greater than or equal to 0 on \( V \otimes_K \mathbb{R} \). We say that \( h \) is positive semi-definite or indefinite if it is positive semi-definite or indefinite at all real places of \( K \). Then we have the following
Lemma 7.1. The quadratic form \( h \) is positive semi-definite or indefinite if and only if \( h \) represents 1.

Proof. Consider the 5-dimensional space \( V \oplus K \) with the quadratic form \( f(v, x) = h(v) - x^2 \). Since \( V \oplus K \) is 5-dimensional, by the Hasse-Minkowski theorem, it represents 0 if and only if it represents 0 at all real places. If \( h \) is positive semi-definite or indefinite, the form \( f \) is semi-define at all real places, so it represents 0 at all real places. Hence \( f \) represents 0, and if \( x \neq 0 \) in the representation, then \( h \) represents 1 by dividing by \( x \). Otherwise, \( h \) represents 0, and so it represents any element of \( K \). Conversely, if it represents 1, then it does so at every real place. Hence it cannot be negative semi-definite, and thus, it must be positive semi-definite or indefinite. \( \square \)

7.2. Over integers. We assume that \( K = \mathbb{Q} \) and \( L \) a quadratic imaginary field. Thus, \( A = \mathbb{Z} \) and \( B \) is the ring of integers in \( L \). Let \( D = D_B/A \) be the discriminant of \( L \). We shall assume that \( L \) is unramified at 2, in order to avoid discussing the difficult 2-adic theory of integral hermitian and quadratic forms.

Lemma 7.2. Let \( \Delta < 0 \) be a square-free integer. Let \( \Lambda \) be a projective \( B \)-module of rank 2 and \( h \) an integral hermitian form on \( \Lambda \) of the discriminant \( \Delta \). Then there exists \( \mathbf{v} \in \Lambda \) such that \( h(\mathbf{v}) = 1 \).

Proof. Let \( \mathbb{Z}_p \) be the \( p \)-adic completion of \( \mathbb{Z} \). Let \( B_p = B \otimes \mathbb{Z}_p \) and \( \Lambda_p = \Lambda \otimes \mathbb{Z}_p \). Note that the subscript \( p \) denotes the completion and not localization in this proof. Since \( h \) is assumed to be indefinite, by [2, Ch. 9, Theorem 1.5, page 131], it suffices to prove that for every \( p \) there exists \( \mathbf{v} \in \Lambda_p \) such that \( h(\mathbf{v}) = 1 \). Assume that \( p \) does not divide \( D \) i.e. \( p \) does not ramify in \( L \). Recall that \( \text{val}_p(\Delta) = 0 \) or 1, by the assumption on \( \Delta \). We claim that there exists \( \mathbf{v} \in \Lambda \) such that \( h(\mathbf{v}) = x \) where \( \text{val}_p(x) = 0 \). Indeed, if not, then \( v \mapsto h(v)/p \) is an integral form. But the discriminant of \( h/p \) is \( \Delta/p^2 \), not an integer, contradicting Proposition 6.5. Furthermore, since \( p \) does not ramify in \( L \), the norm map \( n_L : B_p^\times \to \mathbb{Z}_p^\times \) is surjective. Hence we can rescale \( u \) by an element in \( B_p^\times \) to get \( v \) such that \( h(v) = 1 \).

Now assume \( p \) divides \( D \), so \( p \) is tamely ramified in \( L \). In particular, \( p \) is odd, and we can write \( B_p = \mathbb{Z}[\pi] \) where \( \pi \) is a uniformizer of \( B_p \). We can assume that \( \pi = -\pi \) and, for the price of changing the uniformizer in \( \mathbb{Z}_p \), that \( \pi \pi = p \). Pick a \( B_p \)-basis \( (v_1, v_2) \) of \( \Lambda_p \) and write any \( \mathbf{v} \in \Lambda_p \) as a sum \( \mathbf{v} = \mathbf{x} v_1 + \mathbf{y} v_2 \) where \( x, y \in B_p \). Then \( h(\mathbf{v}) = (x \quad y) \begin{pmatrix} \alpha & \gamma \\ \overline{\gamma} & \beta \end{pmatrix} \begin{pmatrix} \pi \\ \overline{\pi} \end{pmatrix} \)

where \( \alpha, \beta \in \mathbb{Z}_p \). By the assumption, we have \( \text{val}_p(\Delta) = 0 \) or 1. If \( \text{val}_p(\Delta) = 1 \) then \( \text{val}_p(\gamma) \geq 0 \), and it is easy to see that the matrix can be diagonalized, by a change of \( B_p \)-basis in \( \Lambda_p \). Hence, in this case, the quadratic form \( h \) on \( \Lambda_p \) is equivalent to a diagonal form \( a_1 x_1^2 + a_2 x_2^2 + a_3 p x_3^2 + a_4 p x_4^2 \) for some \( a_i \) in \( \mathbb{Z}_p^\times \). Assume now that \( \text{val}_p(\Delta) = 0 \). Then \( \gamma = (a + b\pi)/\pi \), for some \( a \in \mathbb{Z}_p^\times \) and \( b \in \mathbb{Z}_p \). Writing \( x = x_1 + x_2 \pi \) and \( y = y_1 + y_2 \pi \), the \( 4 \times 4 \) matrix representing the
quadratic form \( h \) in the variables \((x_1, x_2, y_1, y_2)\) is
\[
\begin{pmatrix}
\alpha & 0 & b/2 & a/2 \\
0 & \alpha p & a/2 & pb/2 \\
b/2 & a/2 & \beta & 0 \\
a/2 & pb/2 & 0 & \beta p
\end{pmatrix}.
\]
The determinant of this matrix modulo \( p \) is \( a^4/16 \in \mathbb{Z}_p^\times \). Recall that, since \( p \) is odd, every \( \mathbb{Z}_p \)-integral form can be diagonalized. Since the determinant is \( p \)-adic unit, the form is integrally equivalent to a diagonal form
\[
a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2
\]
for some \( a_i \) in \( \mathbb{Z}_p^\times \). But in each of the two cases the diagonal quartic, quadratic form represents 1 by [2, Ch. 8, Theorem 3.1, page 115]. This completes the proof of lemma. \( \square \)

The following is a combination of the previous lemma, Corollary 5.8 and Theorem 6.4. We remind the reader that an integral binary \( B \)-hermitian form is a \( B \)-hermitian form \( h \) on a projective \( B \)-module \( \Lambda \) of rank 2, such that \( h(\Lambda) \subseteq \mathbb{Z} \).

**Theorem 7.3.** Let \( L \) be a quadratic imaginary extension of \( \mathbb{Q} \) of odd discriminant \( D < 0 \). Let \( B \) be its maximal order. Let \( \Delta < 0 \) be a square free integer. Then there is a natural bijection between isomorphism classes of embeddings of \( B \) into quaternionic orders of the discriminant \( \Delta \) and isomorphism classes of integral binary \( B \)-hermitian forms of the discriminant \( \Delta \).

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