RATIONAL HOMOLOGY MANIFOLDS AND HYPERSURFACE NORMALIZATIONS

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ABSTRACT. We prove a criterion for determining whether the normalization of a local complete intersection is a rational homology manifold, using a perverse sheaf known as the multiple-point complex. This perverse sheaf is naturally associated to any “parameterized space”, and has several interesting connections with the Milnor monodromy and mixed Hodge Modules.

1. Introduction

Let $U$ be an open neighborhood of the origin in $\mathbb{C}^{n+1}$, let $X \subseteq U$ be a local complete intersection (LCI) containing 0, and let $\pi : Y \to X$ be the normalization of $X$.

As $X$ an LCI, the shifted constant sheaf $\mathbb{Q}^\bullet X[n]$ is perverse, and there is a surjection of perverse sheaves $\mathbb{Q}^\bullet X[n] \to I_X^n \to 0$, where $I_X^n$ is the intersection cohomology complex on $X$ with constant $\mathbb{Q}$ coefficients. Since the category of perverse sheaves on $X$ is Abelian, there is a perverse sheaf $N_X^\bullet$ on $X$ such that

$$0 \to N_X^\bullet \to \mathbb{Q}^\bullet X[n] \to I_X^n \to 0 \quad (\dagger)$$

is a short exact sequence of perverse sheaves.

Thus, if $I_Y^\bullet$ is intersection cohomology on $Y$ with constant $\mathbb{Q}$ coefficients, we have $\pi_! I_Y^\bullet \cong I_X^n$ ( $\pi$ is a small resolution, in the sense of Goersky and Macpherson [2]), and we obtain the short exact sequence of perverse sheaves

$$0 \to N_X^\bullet \to \mathbb{Q}^\bullet X[n] \to \pi_! I_Y^\bullet \to 0$$

on $X$. We refer to this exact sequence as the fundamental short exact sequence of the normalization. This short exact sequence, and the perverse sheaf $N_X^\bullet$ in particular, have been examined recently in several papers by the author and D. Massey in the case where the normalization $Y$ is smooth ([4], [3]), where $N_X^\bullet$ is called the multiple-point complex of the normalization (see Section 4).

Disregarding the normalization, if one just examines the short exact sequence (\dagger), D.Massey has recently shown in [6] that, in the case where $X = V(f)$ is a hypersurface,

$$N_X^\bullet \cong \ker\{\text{id} - \widetilde{T}_f\},$$

where $\widetilde{T}_f$ is the monodromy action on the vanishing cycles $\phi_f[-1]\mathbb{Q}_U[n + 1]$, and the kernel takes place in the category of perverse sheaves on $X = V(f)$. In this context, Massey refers to $N_X^\bullet$ as the comparison complex on $X$.

Looking at (\dagger), one notices immediately that $\mathbb{Q}^\bullet X[n] \cong I_X^n$ if and only if $N_X^\bullet = 0$; that is, the LCI $X$ is a rational homology manifold (or, a $\mathbb{Q}$-homology manifold) precisely when the complex $N_X^\bullet$ vanishes (for this criterion, see for example [1], [7]). We will recall $\mathbb{Q}$-homology manifolds and their properties in Section 2. It is then
natural to ask that, given the normalization $Y$ of $X$ and the resulting fundamental short exact sequence, is there a similar result relating $N_X^*$ to whether or not $Y$ is a $Q$-homology manifold?

We answer this question in our main result:

**Main Theorem 1** (Theorem 2.3). $Y$ is a $Q$-homology manifold if and only if $N_X^*$ has stalk cohomology concentrated in degree $-n+1$; i.e., for all $p \in X$, $H^k(N_X^*)_p$ is non-zero only possibly when $k = -n+1$.

In general, it is quite difficult to compute these stalk cohomology groups, even in the “next simplest” case where the normalization of a hypersurface has an isolated singularity, e.g., the normalization of a surface with a curve singularity, which we will work out in detail in Section 5.

**Remark 1.1.** M. Saito has recently drawn interesting connections with the multiple-point complex $N_X^*$ to the setting of mixed Hodge modules in a recent preprint [10]. In particular, Saito shows, for an arbitrary reduced complex algebraic variety $X$ of pure dimension $n$, that the weight zero part of the cohomology group $H^1(X;Q)$ is given by

$$W_0H^1(X;Q) \cong \ker\{H^0(Y;Q) \to H^0(X;F_X)\},$$

where $\pi : Y \to X$ is the normalization of $X$, and $F_X$ is a certain constructible sheaf on $X$, given by the cokernel of the natural morphism of sheaves $Q_X \to \pi_*Q_Y$.

This constructible sheaf is none other than the cohomology sheaf $H^{-n+1}(N_X^*)$; this follows immediately from taking the long exact sequence in cohomology of the fundamental short exact sequence of the normalization. Consequently, we can interpret Saito’s result as an isomorphism

$$W_0H^1(X;Q) \cong \ker\{H^0(Y;Q) \to H^{-n+1}(X;N_X^*)\},$$

since $H^0(X;H^{-n+1}(N_X^*)) \cong H^{-n+1}(X;N_X^*)$.

Furthermore, the bounded complex of Mixed Hodge Modules $\mathcal{M} = D^b(MHM(X))$ underlying the sheaf $F_X$ considered by Saito is easily seen to be $N_X^*[1]$. In our setting, the shifted constant sheaf $Q_X[n]$ is perverse, and our choice of shift ensures $N_X^*$ is a perverse sheaf on $X$. This is not true in the general setting Saito considers, so no shift is necessary.

Finally, in the hypersurface case, Saito’s calculation of $H^0(Y;F_X)$ via invariant cycles of the monodromy follows from Massey’s isomorphism of perverse sheaves $N_X^* \cong \ker\{\text{id} - \widetilde{T}_X\}$ [6]. Saito’s calculation of this cohomology group for a general reduced complex algebraic variety then seems to allude to a similar isomorphism holding in $D^b_c(X)$.

2. Main result

Before we prove our main result, we first recall a theorem of Borho and MacPherson [1] giving us several equivalent characterizations of rational homology manifolds:

**Theorem 2.1.** ([B-M]) The following are equivalent:

1. $X$ is a $Q$-homology manifold (i.e., $I_X^* \cong Q^*_X[n]$);
2. $D(Q^*_X[n]) \cong Q^*_X[n]$;
3. For all $p \in X$, for all $k$, $H^k(X,X\setminus\{p\};Q) = 0$ unless $k = 2n$, and $H^{2n}(X,X\setminus\{p\};Q) \cong Q$. 


The proof of Theorem 2.3 relies on the following well-known lemma.

**Lemma 2.2.** Let $X$ be a complex analytic space of pure dimension $n$. Then, for $p \in X$, the rank of $H^{-n}(I^*_X)_p$ is equal to the number of irreducible components of $X$ at $p$.

**Proof.** This result is well-known to experts, see e.g. Theorem 1G (pg. 74) of [11], or Theorem 4 (pg. 217) [5] □

Note that taking stalk cohomology at $p \in X$ of the fundamental short exact sequence yields the short exact sequence

$$0 \to \mathbb{Q} \to H^{-n}(\pi_*I^*_Y)_p \to H^{-n+1}(N^*_X)_p \to 0,$$

and isomorphisms $H^k(\pi_*I^*_Y)_p \cong H^{k+1}(N^*_X)_p$ for $-n + 1 \leq k \leq -1$. With this in mind, we claim that:

**Theorem 2.3.** $Y$ is a $\mathbb{Q}$-homology manifold if and only if $N^*_X$ has stalk cohomology concentrated in degree $-n + 1$.

**Proof.** ($\Rightarrow$) Suppose that $Y$ is a $\mathbb{Q}$-homology manifold, and let $p \in X$ be arbitrary. Since $Y$ is a $\mathbb{Q}$-homology manifold, $\mathbb{Q}_Y[n] \cong I^*_Y$ in $D^b_c(Y)$, from which it follows $H^k(N^*_X)_p = 0$ for $k \neq -n + 1$ by the above isomorphisms.

($\Leftarrow$) Suppose that, for all $p \in X$, $H^k(N^*_X)_p \neq 0$ only possibly when $k = -n + 1$. We wish to show that the natural morphism $\mathbb{Q}_Y[n] \to I^*_Y$ is an isomorphism in $D^b_c(Y)$.

There is still the short exact sequence

$$0 \to \mathbb{Q} \to H^{-n}(\pi_*I^*_Y)_p \to H^{-n+1}(N^*_X)_p \to 0$$

and $H^k(\pi_*I^*_Y)_p = 0$ for $k \neq -n$, since $H^k(\pi_*I^*_Y)_p \cong H^{k+1}(N^*_X)_p$ for all $p \in X$ and $-n + 1 \leq k \leq -1$. In degree $-n$, we have

$$H^{-n}(\pi_*I^*_Y)_p \cong \bigoplus_{q \in \pi^{-1}(p)} H^{-n}(I^*_Y)_q.$$ 

This then implies that, for all $q \in Y$, $H^k(I^*_Y)_q = 0$ for $k \neq -n$. Our goal is to calculate this stalk cohomology in degree $-n$. Since $Y$ is normal, and thus locally irreducible, it follows by Lemma 2.2 that $H^{-n}(I^*_Y)_q \cong \mathbb{Q}$ for all $q \in Y$.

Finally, we claim that the natural morphism $\mathbb{Q}_Y[n] \to I^*_Y$ is an isomorphism in $D^b_c(Y)$. In stalk cohomology at any point $q \in Y$, both $H^k(\mathbb{Q}_Y[n])_q$ and $H^k(I^*_Y)_q$ are non-zero only in degree $k = -n$, with stalks isomorphic to $\mathbb{Q}$. Consequently, the natural morphism is an isomorphism in $D^b_c(Y)$ provided that the morphism

$$\mathbb{Q} \cong H^{-n}(\mathbb{Q}_Y[n])_q \to H^{-n}(I^*_Y)_q \cong \mathbb{Q}$$

is not the zero morphism. But this is just the “diagonal” morphism from a single copy of $\mathbb{Z}$ to the number of connected components of $Y \setminus \{p\}$, which is clearly non-zero. Thus, $Y$ is a $\mathbb{Q}$-homology manifold. □
Corollary 2.4. Suppose that \( N^*_X \) has stalk cohomology concentrated in degree \(-n+1\). Then, for all \( p \in X \), if \( j_p : \{ p \} \to X \) is the inclusion map, we have
\[
H^k(j^*_p N^*_X) \cong \begin{cases} \tilde{H}^{n+k-1}(K_{X,p}; \mathbb{Q}), & \text{for } 0 \leq k \leq n-1; \\
0, & \text{else.} \end{cases}
\]
This follows by applying \( j^*_p \) to the fundamental short exact sequence of the normalization, and taking stalk cohomology.

3. INTERPRETATION IN TERMS OF COMPARISON COMPLEX

Recall that, by D. Massey, if \( X = V(f) \) is a hypersurface, \( N^*_X = \ker \{ \text{id} - \tilde{T}_f \} \) is the perverse eigenspace of the eigenvalue 1 of the monodromy action on \( \phi_f[-1]Q_0^*[n+1] \), where \( U \) is an open neighborhood of the origin in \( \mathbb{C}^{n+1} \).

Since the content of this paper is interesting only in the case where \( \dim_\mathbb{Q} \Sigma f = n-1 \) (otherwise, \( X \) is its own normalization), we will assume throughout that this is the case; consequently, the stalk cohomology \( H^k(\phi_f[-1]Q_0^*[n+1])_p \) is possibly non-zero only for \(-n+1 \leq k \leq 0\).

In general, it is not the case that, given a morphism of perverse sheaves, the cohomology of the stalk of the kernel of \( G \) is isomorphic to the kernel of the cohomology on the stalks; that is, there may exist points \( p \in \Sigma f \) such that
\[
H^k(\ker \{ \text{id} - \tilde{T}_f \})_p \not\cong \ker \{ \text{id} - (\tilde{T}_f)_p^k \}.
\]
However, this isomorphism does hold in degree \(-n+1\) for all \( p \in \Sigma f \) (See Lemma 5.1 of [6]):

Proposition 3.1. Let \( \pi : Y \to V(f) \) be the normalization of \( V(f) \), and suppose \( Y \) is a \( \mathbb{Q} \)-homology manifold. Then, the following isomorphisms hold for all \( p \in \Sigma f \):
\[
H^k(\ker \{ \text{id} - \tilde{T}_f \})_p \cong \begin{cases} \ker \{ \text{id} - (\tilde{T}_f)_p^{-n+1} \}, & \text{if } k = -n+1; \\
0, & \text{if } k \neq -n+1. \end{cases}
\]
\[
H^{-n+1}(\ker \{ \text{id} - (\tilde{T}_f)_p^{-n+1} \})_p \cong \ker \{ \text{id} - (\tilde{T}_f)_p^{-n+1} \},
\]
\[
H^{-n+1}(\coker \{ \text{id} - (\tilde{T}_f)_p^{-n+1} \})_p \cong \ker \{ \text{id} - (\tilde{T}_f)_p^{-n+1} \},
\]
where \( \text{id} - (\tilde{T}_f)_p^{-n+1} \) is the Milnor monodromy action on \( H^1(F_{f,p}; \mathbb{Q}) \).

Proof. Since \( H^k(\ker \{ \text{id} - \tilde{T}_f \})_p = 0 \) for \( k \neq -n+1 \), the result follows from the short exact sequences
\[
0 \to \ker \{ \text{id} - (\tilde{T}_f)_p^{-n+1} \} \to H^1(F_{f,p}; \mathbb{Q}) \to H^{-n+1}(\ker \{ \text{id} - (\tilde{T}_f) \})_p \to 0,
\]
and
\[
0 \to H^{-n+1}(\ker \{ \text{id} - (\tilde{T}_f) \})_p \to H^1(F_{f,p}; \mathbb{Q}) \to H^{-n+1}(\coker \{ \text{id} - (\tilde{T}_f) \})_p \to 0.
\]

By taking stalk cohomology of the fundamental short exact sequence, we have
\[
0 \to H^{-n}(\tilde{Q}_X)_p \to H^{-n}(\tilde{I}_X)_p \to \ker \{ \text{id} - (\tilde{T}_f)_p^{-n+1} \} \to 0.
\]
Since \( \pi_* \tilde{I}_Y \cong \tilde{I}_X \), and \( H^{-n}(\pi_* \tilde{I}_Y)_p \cong \mathbb{Q}^{\pi^{-1}(p)} \),
\[
\ker \{ \text{id} - (\tilde{T}_f)_p^{-n+1} \} \cong \mathbb{Q}^{\pi^{-1}(p) - 1}
\]
for all $p \in X$, yielding the following nice lower-bound:

**Corollary 3.2.**

$$\dim \mathbb{Q} H^1(F_{f,p}; \mathbb{Q}) \geq |\pi^{-1}(p)| - 1.$$ 

4. $N^*_X$ as the Multiple-Point Complex

In the case where the normalization $Y \xrightarrow{\pi} X$ is a $\mathbb{Q}$-homology manifold, the short exact sequence

$$0 \to \mathbb{Q} \to H^{-n}(\pi, I^*_p) \to H^{-n+1}(N^*_X)_p \to 0$$

allows us to identify, given Lemma 2.2, that

$$m(p) := \dim \mathbb{Q} H^{-n+1}(N^*_X)_p = |\pi^{-1}(p)| - 1.$$ 

Consequently, we conclude that the support of $N^*_X$ is none other than the **image multiple-point set** of the morphism $\pi$, which we denote by $D$; precisely, we have

$$D := \{p \in X \mid |\pi^{-1}(p)| > 1\}.$$ 

For this reason, we refer to the perverse sheaf $N^*_X$ as the **multiple-point complex** of $X$ (or, of the morphism $\pi$, as we do in [3] and [4]).

In such cases (see Section 5), it is useful to partition $X$ into subsets $X_k = m^{-1}(k)$ for $k \geq 1$; clearly, one has

$$D = \bigcup_{k \geq 1} X_k.$$ 

In the case where $X = V(f)$ is a hypersurface in some open neighborhood $U$ of the origin in $\mathbb{C}^{n+1}$, we prove in [3] that a strong relationship holds between the **characteristic polar multiplicities** of $N^*_X$ and the Lê numbers of the function $f$. This same result holds for hypersurface normalizations that are $\mathbb{Q}$-homology manifolds. More precisely, the exact same proof yields:

**Theorem 4.1.** Suppose that $\tilde{X}$ is a $\mathbb{Q}$-homology manifold, and $\pi : (\tilde{X} \times \mathbb{C}, \{0\} \times S) \to (U, 0)$ is a one-parameter unfolding with parameter $t$, with $\text{im} \pi = X = V(f)$ for some $f \in \mathcal{O}_{U,0}$. Suppose further that $z = (z_1, \cdots, z_n)$ is chosen such that $z$ is an IPA-tuple for $f_0 = f|_{V(t)}$ at $0$. Then, if $N^*_{X,t_0} = N^*_X|_{V(t-t_0)} [-1]$, the following formulas hold for the Lê numbers of $f_0$ with respect to $z$ at $0$: for $0 < |t_0| \ll \epsilon \ll 1$,

$$\lambda^{0}_{f_0,z}(0) = -\lambda^{0}_{N^{*}_{X,t_0},z}(0) + \sum_{p \in B_i \cap V(t-t_0)} \left( \lambda^{0}_{f_0,z}(p) + \lambda^{0}_{N^{*}_{X,t_0},z}(p) \right),$$

and, for $1 \leq i \leq n-2$,

$$\lambda^{i}_{f_0,z}(0) = \sum_{q \in B_i \cap V(t-t_0,z_1,z_2,\cdots,z_i)} \lambda^{i}_{f_0,z}(q).$$

In particular, the following relationship holds for $0 \leq i \leq n-2$:

$$\lambda^{i}_{f_0,z}(0) + \lambda^{i}_{N^{*}_{X,t_0},z}(0) = \sum_{p \in B_i \cap V(t-t_0,z_1,z_2,\cdots,z_i)} \left( \lambda^{i}_{f_0,z}(p) + \lambda^{i}_{N^{*}_{X,t_0},z}(p) \right)$$
For a precise definition of characteristic polar multiplicities, see [9]; for deformations with isolated polar activity (IPA-deformations and IPA-tuples), see [8].

In [3], such an unfolding \( \pi \) considered above is equivalent to \( X \) having a smooth normalization. Given Theorem 2.3, the exact same proof for this smooth case from Theorem 5.2 of [3] works for the case where the normalization of \( X = V(f) \) is a \( \mathbb{Q} \)-homology manifold and one has a one-parameter unfolding of \( \pi \).

5. Example

We consider the following "trivial, non-trivial" example of the normalization of a surface \( X \) with one-dimensional singularity in \( \mathbb{C}^3 \), which nicely illustrates the content of Theorem 2.3.

Let \( f(x, y, z) = xz^2 - y^2(y + x^3) \), so that \( X = V(f) \subseteq \mathbb{C}^3 \) has critical locus \( \Sigma f = V(y, z) \). Then, if we let \( Y = V(u^2 - x(y + x^3), uy - xz, uz - y(y + x^3)) \subseteq \mathbb{C}^4 \), the projection map \( \pi : Y \to X \) is the normalization of \( X \).

It is easy to check that \( \Sigma Y = V(x, y, z, u) \), and
\[
\pi^{-1}(\Sigma f) = V(u^2 - x^4, y, z).
\]
It then follows that \( X_k = \emptyset \) if \( k > 2 \), and \( X_2 = V(y, z) \setminus \{0\} \), so that
\[
\text{supp} N_X^* = V(y, z) = \Sigma f.
\]

For \( p \in X \),
\[
H^{-2}(\pi_* I_Y^*)_p \cong \bigoplus_{q \in \pi^{-1}(p)} H^{-2}(I_Y^*)_q \quad (\dagger 6.1)
\]

But \( \pi^{-1}(p) \subseteq Y \setminus \Sigma Y \), and \( (I_Y^*)_\pi^{-1}(p) \cong (\mathbb{Q}^*_Y[2])_{\pi^{-1}(p)} \), so from (\dagger 6.1), it follows that
\[
H^{-2}(\pi_* I_Y^*)_p \cong \mathbb{Q}^2.
\]

Similarly, since \( (I_Y^*)_{Y \setminus \Sigma Y} \cong \mathbb{Q}^*_Y \setminus \Sigma Y [2] \), it follows that
\[
H^0(N_X^*)_p \cong H^{-1}(\pi_* I_Y^*)_p = 0.
\]

When \( p = 0 \), we find
\[
H^k(I_Y^*)_0 \cong \begin{cases} \mathbb{H}^k(K_{Y,0}; I_Y^*), & \text{if } k \leq -1 \\ 0, & \text{if } k > -1 \end{cases}
\]

Since \( Y \) has an isolated singularity at the origin in \( \mathbb{C}^4 \), we further have
\[
\mathbb{H}^k(K_{Y,0}; I_Y^*) \cong H^{k+2}(K_{Y,0}; \mathbb{Q}).
\]

For \( 0 < \epsilon \ll 1 \), the sphere \( S_\epsilon \) transversely intersects \( Y \) near \( 0 \), so the real link \( K_{Y,0} = Y \cap S_\epsilon \) is compact, orientable, smooth manifold of (real) dimension 3. We are interested in computing the two integral cohomology groups \( H^0(K_{Y,0}; \mathbb{Q}) \) and \( H^1(K_{Y,0}; \mathbb{Q}) \).

Because \( K_{Y,0} \) is a compact, connected, orientable manifold, we can apply Poincaré duality to find \( H^0(K_{Y,0}; \mathbb{Q}) \cong \mathbb{Q} \).

Consider the standard parameterization of the twisted cubic \( \nu : \mathbb{P}^1 \to \mathbb{P}^3 \) via
\[
\nu([s : t]) = [s^3 : st^2 : t^3 : s^2t] = [x : y : z : u]
\]
which lifts to a map \( \nu : \mathbb{C}^2 \to \mathbb{C}^4 \) which parameterizes the affine cone over the twisted cubic, i.e., the normalization \( Y = V(u^2 - xy, uy - xz, uz - y^2) \). Then, we claim that \( \nu \) is a 3-to-1 covering map away from the origin. Clearly, since \( \nu \) parameterizes \( Y \), we see that \( \nu \) is a surjective local diffeomorphism onto \( \nu(\mathbb{C}^2) = Y \).

Suppose that \( \nu(s, t) = \nu(s', t') \). Then, we must have \( s^3 = (s')^3 \) and \( t^3 = (t')^3 \), so that there are cube roots of unity \( \eta \) and \( \omega \) for which \( s = \eta s' \) and \( t = \omega t' \). But then,

\[
s^2 t = (s')^2(t') = \eta^2 \omega s^2 t,
\]

so either \( \eta^2 \omega = 1 \), or \( st = 0 \). Since \( \eta \) and \( \omega \) are both cube roots of unity, if \( \eta^2 \omega = 1 \), then \( \eta = \omega \). Additionally, note that \( st = 0 \) implies \( (s, t) = 0 \). It then follows that \( \nu \) is 3-to-1 away from the origin.

Consider then the (real analytic) function

\[ r(x, y, z, u) = |x|^2 + 3|y|^2 + |z|^2 + 3|u|^2 \]

on \( \mathbb{C}^4 \); \( r \) is proper, transversally intersects \( Y \) away from \( 0 \), and \( Y \cap r^{-1}(0, \epsilon) \) gives a fundamental system of neighborhoods of the origin in \( Y \). Consequently, \( Y \cap r^{-1}(\epsilon) \) gives, up to homotopy, the real link \( K_{Y,0} \). The composition \( \nu(r(s, t)) \) then gives:

\[
\nu(r(s, t)) = |s|^3 + 3|s|^2 t + |t|^3 + 3|s|^2 t^2
\]

\[
= |s|^6 + 3|s|^4 |t|^2 + 3|s|^2 |t|^2 + |t|^6
\]

\[
= (|s|^2 + |t|^2)^3 = \epsilon,
\]

provided that \( |s|^2 + |t|^2 = \sqrt[3]{\epsilon} \); that is, \( \nu \) maps the 3-sphere in \( \mathbb{C}^2 \) 3-to-1 onto the real link \( K_{Y,0} \). Since the 3-sphere is simply-connected, it is the universal cover of \( K_{Y,0} \). The group of deck transformations given by multiplying \( (s, t) \) by a cube root of unity then yields the isomorphism \( \pi_1(K_{Y,0}) \cong \mathbb{Z}/3\mathbb{Z} \). Thus, \( H_1(K_{Y,0}; \mathbb{Z}) = \mathbb{Z}/3\mathbb{Z} \).

Since \( K_{Y,0} \) is a compact, connected, and orientable manifold, we can apply Poincaré duality. Consequently, \( H^2(K_{Y,0}; \mathbb{Z}) = \mathbb{Z}/3\mathbb{Z} \) as well. By the Universal Coefficient theorem for cohomology, we then have \( H_2(K_{Y,0}; \mathbb{Z}) = 0 \) so that \( H^1(K_{Y,0}; \mathbb{Z}) = 0 \) by Poincaré duality. Using \( \mathbb{Q} \) coefficients, this implies:

\[
H^k(K_{Y,p}; \mathbb{Q}) \cong \begin{cases} \mathbb{Q}, & \text{if } k = 0, 3 \\ 0, & \text{else} \end{cases}
\]

for all \( p \in Y \), so that \( Y \) is a \( \mathbb{Q} \)-homology manifold.

Equivalently, we find:

\[
H^k(N_X^*; \mathbb{Q}) \cong \begin{cases} \mathbb{Q}, & \text{if } k = -1 \text{ and } p \in \Sigma f \setminus \{0\} \\ 0, & \text{if } k \neq -1, p \in \Sigma f \end{cases}
\]

i.e., \( N_X^* \) has stalk cohomology concentrated in degree \(-1\).

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