Genus one correlation to multi-cut matrix model solutions

L. Chekhov

Steklov Mathematical Institute,
Gubkina 8, 117966, GSP–1, Moscow, Russia

We calculate genus one corrections to Hermitian one-matrix model solution with arbitrary number of cuts directly from the loop equation confirming the answer previously obtained from algebro-geometrical considerations and generalizing it to the case of arbitrary potentials.

1 Introduction

Solving matrix models using the loop equation has more than decade long history. An algorithm for finding genus expansion solutions to the Hermitian one-matrix model was elaborated (see [1] and references therein) in the case where the limiting eigenvalue distribution spans a single interval on the real line in the large $N$ limit (in what follows, $N$ is the dimension of the Hermitian matrix). Such a situation is commonly called the one-cut solution. Results for multi-cut solutions were few. The problem was that it is not too difficult to find a solution for the loop equation, i.e., the genus expansion of the so-called loop mean—this problem was solved by Akemann [2]. The universal critical behavior of the corresponding correlators was considered in [3]. However, a difficult problem is to integrate the obtained solution in order to produce higher genus contributions to the free energy. Akemann has succeeded in finding such a genus-one contribution in the two-cut case only.

Recently, it was realized that multi-cut solutions to matrix models is the powerful tool for studying broken vacua structures of $\mathcal{N} = 2$ super Yang–Mills theory of Seiberg and Witten [4]. Simultaneously, a new relation between the superpotentials of $\mathcal{N} = 1$ supersymmetric gauge theories in four dimensions and free energies of matrix models in the planar limit was proposed [5, 6]. The superpotentials in some $\mathcal{N} = 1$ four-dimensional Yang–Mills theories has turned out to be expressed through a single holomorphic function [5], which was later identified with the planar limit of the free energy of the multi-support solutions to matrix models [6].

On the other hand, there are strong indications that such an analogy goes beyond the tree level and just the large-$N$ limit of the matrix model partition function. The relation between gauge theories and matrix model solutions were checked on the one-loop level for the solution with two cuts and a cubic matrix model potential [7]. A comparison in general case must involve the structure of the general matrix model multi-cut solution in

\[1\] E-mail: chekhov@mi.ras.ru.
genus one approximation, which has been missed as yet. Kostov [8] had provided arguments in favor of using the conformal field theory (CFT) methods for calculating such the corrections in the general case following the way proposed by Moore [9]; this method has been improved by Dijkgraaf, Sinkovics, and Temürhan [10] who used algebro-geometrical and CFT arguments for proposing the formula for genus one correction and comparing it with the gauge theory answer. Note, however, that the very applicability of Kostov’s star operator (dressing) method needs to be justified by direct calculations using the loop equation alone and, second, the algebro-geometrical methods need to be improved in a way to allow calculating contributions from "non-geometrical" part of the action—from the so-called ”moments” of the matrix model potential (such contributions always arise if the power of the polynomial that is the derivative of the potential exceeds the number of cuts in play; then, double points arise and the genus of the arising hyperelliptic surface can be much lesser than the highest degree of the potential. In this situation, however, the superpotential satisfying the Whitham hierarchy equations and the WDVV equations is still attained by the planar (large-$N$) limit of the free energy of the corresponding matrix model [11], [12].

In this paper, we derive the subleading (genus one) expression for the free energy of the generalized matrix model multi-cut solution directly from the loop equation, without referring to geometry or to the CFT reasoning thus supporting the calculations in [8] and [10]. In Secs. 2, 3, we closely follows the method of [1], [2] reviewing the iterative procedure for finding the loop mean in the genus expansion; the answer for the genus-one contribution $W_1(p)$ to the loop mean is presented in Sec. 4. We integrate the obtained answer in Sec. 4 thus obtaining the complete expression for the genus-one part $F_1$ of the free energy. As a byproduct, we obtain interesting combinatorial identities on the quantities entering $W_1(p)$.

## 2 Definitions. Loop equation

The partition function of the Hermitian one-matrix model is

$$Z[N, \{t_i\}] \equiv e^{N^2 F} = e^{\sum_{g=0}^\infty N^{2-2g} F_g[\{t_i\}]} \equiv \int d\Phi \, e^{-N \text{tr} V(\Phi)}, \tag{2.1}$$

where the integration is over Hermitian $N \times N$ matrices $\Phi$. The matrix potential, which we assume to be polynomial of degree $(m+1)$, is given by the power series

$$V(x) = \sum_{j=1}^{m+1} \frac{t_j}{j} x^j. \tag{2.2}$$

Deriving the loop equation implies introducing all coupling constants $t_j$ into play; a specific potential of finite order can be inspected by setting the extra couplings to zero in the final result. The coupling constants $t_j$ are then the sources for polynomial expectation values with the means defined customarily as

$$\langle Q(\phi) \rangle = \frac{1}{Z} \int d\Phi \, Q(\Phi) \, e^{-N \text{tr} V(\Phi)}.$$
Introducing the loop insertion operator

$$\frac{d}{dV}(p) \equiv -\sum_{j=1}^{\infty} \frac{j}{p^{j+1}} \frac{d}{dt_j},$$

(2.3)

the one-loop mean $W(p)$ can thus be obtained from the free energy $F$

$$W(p) \equiv \frac{1}{N} \sum_{k=0}^{\infty} \langle \text{Tr} \Phi^k \rangle = \frac{1}{N} \left\langle \frac{1}{p - \phi} \right\rangle = \frac{d}{dV}(p)F + \frac{1}{p}.$$  

(2.4)

Analogously, all the multi-loop correlators can be derived by applying $\frac{d}{dV}(p)$ to $F$ (or to $W(p)$)

$$W(p_1, \ldots, p_n) \equiv N^{n-2} \left\langle \frac{\text{tr} 1}{p_1 - \phi} \cdots \frac{\text{tr} 1}{p_n - \phi} \right\rangle_{\text{conn}}$$

$$= \frac{d}{dV}(p_n)\frac{d}{dV}(p_{n-1})\cdots \frac{d}{dV}(p_1)F, \quad n \geq 2.$$  

(2.5)

Here conn refers to the connected part. Because the loop correlators and the free energy have the same genus expansion,

$$W(p_1, \ldots, p_n) = \sum_{g=0}^{\infty} \frac{1}{N^{2g}} W_g(p_1, \ldots, p_n),$$

(2.6)

Eq. (2.5) is valid for each genus $g \geq 0$ separately. We assume $W(p)$ to be analytic at infinity with the asymptotic behavior

$$\lim_{p \to \infty} W(p) \sim \frac{1}{p},$$  

(2.7)

i.e.,

$$W_g(p)|_{p \to \infty} = \frac{1}{p^g} \delta_{g,0} + O(1/p^2).$$  

(2.8)

This means that higher $W_g(p)$ must be total derivatives:

$$W_g(p) = \frac{d}{dV}(p)F_g, \quad g \geq 1.$$  

(2.9)

The loop equation is the exact equation satisfied by the loop mean $W(p)$. Deriving the loop equation is irrelevant to the multiple cut structure and relies only on the invariance of the partition function under a field redefinition $\Phi \to \Phi + \epsilon/(p - \Phi)$. The only effect is that the contour $C_D$ encircles now a number $n$ of disjoint intervals on the real line:

$$\oint_{C_D} \frac{d\omega}{2\pi i} V'(\omega) W(\omega) = (W(p))^2 + \frac{1}{N^2} \frac{d}{dV}(p)W(p), \quad p \notin D,$$

(2.10)

where $V'(\omega) = \sum_j t_j \omega^{j-1}$. We assume that the density $\rho_N(\lambda) \equiv \frac{1}{N} \langle \delta(\lambda - \lambda_i) \rangle$ of eigenvalues of the matrices $\Phi$ has a compact support $D$ as $N \to \infty$. We let $D$ to comprise $n$ disjoint intervals:

$$D \equiv \bigcup_{i=1}^{n} [\mu_{2i-1}, \mu_{2i}], \quad \mu_1 < \mu_2 < \ldots < \mu_{2n}.$$  

(2.11)

3
Presenting $W(p)$ in terms of $\rho_N(\lambda)$, $W(p) = \int d\lambda \frac{\rho_N(\lambda)}{p-\lambda}$, we find that as $N \to \infty$, $W(p)$ is analytic outside $n$ cuts on the real axis.

Inserting genus expansion (2.6) into loop equation (2.10), we obtain

$$\oint_{C_D} \frac{d\omega}{2\pi i} V'(\omega) W_0(\omega) = (W_0(p))^2$$

(2.12)

for genus zero and

$$(\hat{K} - 2W_0(p))W_g(p) = \sum_{g'=1}^{g-1} W_{g'}(p)W_{g-g'}(p) + \frac{d}{dV}(p)W_{g-1}(p), \quad g \geq 1,$$

(2.13)

for higher genera, where $\hat{K}$ is a linear integral operator

$$\hat{K}f(p) \equiv \oint_{C_D} \frac{d\omega}{2\pi i} \frac{V'(\omega)}{p-\omega}f(\omega).$$

(2.14)

Given $W_0(p)$, one can then determine $W_g(p)$ for $g \geq 1$ iteratively genus by genus provided the operator $(\hat{K} - 2W_0(p))$ can be inverted uniquely.

We can solve Eq. (2.12) for the planar solution $W_0(p)$ as follows. Deforming the contour in Eq. (2.12) to infinity, we obtain

$$(W_0(p))^2 = V''(p)W_0(p) + \oint_{C_\infty} \frac{d\omega}{2\pi i} \frac{V'(\omega)}{p-\omega}W_0(\omega).$$

(2.15)

The solution is then formally

$$W_0(p) = \frac{1}{2} V''(p) - \frac{1}{2} \sqrt{\left(V'(p)\right)^2 + 4P(p)},$$

$$P(p) = \oint_{C_\infty} \frac{d\omega}{2\pi i} \frac{V'(\omega)}{p-\omega}W_0(\omega),$$

(2.16)

where the minus sign is chosen in order to fulfill the asymptotic Eq. (2.8) and $P(p)$ is a polynomial of power $m - 1$. If $W_0(p)$ has $n$ cuts in the complex plane, we propose the ansatz for the square root,

$$W_0(p) = \frac{1}{2} (V'(p) - y(p)),$$

(2.17)

where

$$y(p) \equiv M(p)\tilde{y}(p), \quad \text{and} \quad \tilde{y}(p) \equiv \sqrt{\prod_{i=1}^{2n} (p - \mu_i)}$$

(2.18)

(the notation comes from [11]), and $M(p)$ is assumed to be a polynomial of degree $m - n$, which still has to be determined. By convention, we set $\tilde{y}(p)|_{p \to \infty} \sim p^n$, and $M(p)$ is then

$$M(p) = \oint_{C_\infty} \frac{d\omega}{2\pi i} \frac{V'(\omega)}{(\omega - p)\tilde{y}(\omega)}.$$

(2.19)
Inserting this solution in Eq. (2.17) and deforming the contour back, we obtain the planar one-loop correlator with an \( n \)-cut structure,

\[
W_0(p) = \frac{1}{2} \oint_{C_D} \frac{d\omega}{2\pi i} \frac{V'(\omega) \bar{y}(p)}{p - \omega \bar{y}(\omega)}, \quad p \notin \mathcal{D}.
\]  

(2.20)

Then, the planar eigenvalue density \( \rho(\lambda) \equiv \lim_{N \to \infty} \rho_N(\lambda) \) has the form

\[
\rho(\lambda) = \frac{1}{2\pi i} \lim_{\epsilon \to 0} \left( W_0(\lambda - i\epsilon) - W_0(\lambda + i\epsilon) \right) = \frac{1}{2\pi} \Im \bar{y}(\lambda), \quad \lambda \in \mathcal{D}.
\]  

(2.21)

We still must determine the positions of branching points \( \mu_\alpha, \alpha = 1, \ldots, 2n \). Here, we have two sorts of restrictions. The first set comes from asymptotic conditions (2.8) to be satisfied by ansatz (2.20), from which we have

\[
\delta_{k,s} = \frac{1}{2} \oint_{C_D} \frac{d\omega}{2\pi i} \frac{\omega^k V'(\omega)}{\bar{y}(\omega)}, \quad k = 0, \ldots, n.
\]  

(2.22)

These conditions provide \( n+1 \) equations for \( 2n \) constants \( \mu_\alpha \) (which suffices only if \( n = 1 \)).

Another bunch of relations cannot be read from the analytical structure alone. Introducing the occupation numbers

\[
S_i = \oint_{A_i} \frac{d\omega}{4\pi i} M(\omega) \bar{y}(\omega) \equiv \oint_{A_i} \frac{d\omega}{4\pi i} y(\omega),
\]  

(2.23)

where \( A_i, i = 1, \ldots, n - 1 \) is the basis of \( A \)-cycles on the hyperelliptic Riemann surface \( \bar{y}^2 = \prod_{\alpha=1}^{2n} (x - \mu_\alpha) \) (we may conveniently choose them to be the first \( n - 1 \) cuts), we can consider these numbers as independent parameters of the theory in the Dijkgraaf–Vafa setting, thus imposing the restrictions

\[
\frac{d}{dV}(p) S_i = 0, \quad i = 1, \ldots, n - 1.
\]  

(2.24)

(Adopting a quite opposite standpoint, which is intrinsic for an original approach to the matrix models, see [14], [2], we must look for the genuine minimum of the matrix model action with respect to variables \( S_i \) as well as with respect to the times \( t_j \). This means that partial derivatives of \( F_0 \) with respect to \( S_i \) must vanish. One can obtain (see [14], [6]), that these derivatives are differences of chemical potential on disjoint cuts and those are equal to integrals over dual \( B \)-cycles on the corresponding hyperelliptic Riemann surface, that is, this set of conditions reads

\[
\frac{d}{dV}(p) \oint_{B_i} d\omega y(\omega) = 0.
\]  

(2.25)

Note that, as we shall see below, from the technical viewpoint, it does not matter which cycle set, \( A \)- or \( B \)-cycles, we take as the basic set. That is, the higher genus corrections become scheme dependent: choosing determining conditions (2.24) or (2.25), we obtain different expressions for the free energy in the the genus one approximation. These expressions are however of the same type; the difference will be exactly in the choice of the corresponding basis cycles.)

5
3 The iterative procedure

3.1 Moments of the model

We can determine higher genus contributions iteratively by inverting genus expanded loop equation (2.13). From \( W_g(p) \), all multi-loop correlators of the same genus can be obtained then simply by applying the loop insertion operator \( \frac{d}{dp} \) to it (see Eqs. (2.5) and (2.4)). As in the one-cut solution [1], changing variables from coupling constants to moments \( M^{(k)}_\alpha \) allows expressing higher genus correlators nonperturbatively in the coupling constants \( t_j \), and these correlators turn out to depend only on a finite number of the moments

\[
M^{(k)}_\alpha \equiv \oint_{\mathcal{C}_D} \frac{d\omega}{2\pi i} V'(\omega)\phi^{(k)}_\alpha(\omega), \quad k = 1, 2, \ldots, \quad \alpha = 1, \ldots, 2n, \tag{3.1}
\]

with

\[
\phi^{(k)}_\alpha(\omega) \equiv \frac{1}{(\omega - \mu_\alpha)^k} \phi^{(0)}(\omega), \quad \phi^{(0)}(\omega) \equiv \frac{1}{\tilde{y}(\omega)}. \tag{3.2}
\]

Note that expression (2.19) implies

\[
M^{(1)}_\alpha = M(\mu_\alpha). \tag{3.3}
\]

3.2 Determination of the basis

We now introduce a basis for the operator acting on \( W_g(p) \) in Eq. (2.13),

\[
(\hat{K} - 2W_0(p)) \chi^{(k)}_\alpha(p) \equiv \frac{1}{(p - \mu_\alpha)^k}, \quad k = 1, 2, \ldots, \quad \alpha = 1, \ldots, 2n. \tag{3.4}
\]

Given the r.h.s. of Eq. (2.13) to be a fractional rational function of \( p \) having poles at the \( \mu_\alpha \) only, \( W_g(p) \) must have the following structure:

\[
W_g(p) = \sum_{k=1}^{3g-1} \sum_{\alpha=1}^{2n} A^{(k)}_{\alpha,g} \chi^{(k)}_\alpha(p), \quad g \geq 1. \tag{3.5}
\]

Here \( A^{(k)}_{\alpha,g} \) are complicated functions of \( \mu_\beta \) and the moments \( M^{(k)}_\beta \). As the order of the highest pole in \( W_g(p) \) is insensitive to a multi-cut structure, \( W_g(p) \) will depend on at most \( 2n(3g-1) \) moments, just like the one-cut solution case [1].

A set of basis functions fulfilling Eq. (3.4) is defined by

\[
\tilde{\chi}^{(k)}_\alpha(p) \equiv \frac{1}{M^{(1)}_\alpha} \left( \phi^{(k)}_\alpha(p) - \sum_{r=1}^{k-1} M^{(k-r+1)}_\alpha \tilde{\chi}^{(r)}_\alpha(p) \right), \quad k = 1, 2, \ldots, \quad \alpha = 1, \ldots, 2n, \tag{3.6}
\]

which can be proved by induction. However, this definition is not unique, as the kernel of \( (\hat{K} - 2W_0(p)) \) is not empty,

\[
\text{Ker}(\hat{K} - 2W_0(p)) = \text{Span}\{p^l \phi^{(0)}(p); \quad l = 0, \ldots, n\}. \tag{3.7}
\]
This follows from boundary conditions (2.22). Now, because of the asymptotic behavior (2.23), only terms \( \sim O \left( \frac{1}{p} \right) \) at large \( p \) may be added to \( W_g(p) \) for \( g \geq 1 \). This requirement reduces the set of zero modes to be

\[
\text{allowed zero modes : } \{ p^l \phi^{(0)}(p); \ l = 0, \ldots, n - 2 \} .
\]

In the general case, any linear combination of \( n - 1 \) functions can be added to \( W_g(p) \). But, besides its asymptotic behavior, \( W_g(p) \) must fulfill Eq. (2.24), which can be used to fix the basis uniquely. The \( p \)-dependence of \( W_g(p) \) must be completely absorbed into derivatives with respect to \( \frac{d}{dV} \). This is possible only if the basis functions \( \chi^{(k)}_\alpha(p) \) can be expressed completely in terms of \( \frac{d\mu_\alpha}{dV} \), \( \frac{dM^{(k)}_\alpha}{dV} \) as functions of \( p \). So, \( \tilde{\chi}^{(k)}_\alpha(p) \) must be redefined in Eq. (3.6) in accordance with this demand, which fixes completely their zero mode components. The derivatives of branching points \( \mu_\alpha \) and the moments \( M^{(k)}_\alpha \) can be obtained by applying the loop insertion operator \( \frac{d}{dV} \) to definitions (3.1) and using restrictions (2.24) taking into account that

\[
\frac{d}{dV}(p) = \frac{\partial}{\partial V}(p) + \sum_{\alpha=1}^{2n} \frac{d\mu_\alpha}{dV}(p) \frac{\partial}{\partial \mu_\alpha} + \sum_{\alpha=1}^{2n} \sum_{k=1}^{\infty} \frac{dM^{(k)}_\alpha}{dV}(p) \frac{\partial}{\partial M^{(k)}_\alpha},
\]

Using the identity

\[
\frac{\partial}{\partial V}(p)V'(\omega) = -\frac{1}{(p - \omega)^2},
\]

the result for the moments reads

\[
\frac{dM^{(k)}_\alpha}{dV}(p) = (k + 1/2) \left( M^{(k+1)}_\alpha \frac{d\mu_\alpha}{dV}(p) - \phi^{(k+1)}_\alpha(p) \right)
+ \frac{1}{2} \left( \sum_{\beta=1, \beta \neq \alpha}^{2n} \frac{1}{(\mu_\beta - \mu_\alpha)^{k-1+1}} (\phi^{(k+1)}_\alpha(p) - M^{(k)}_\alpha \frac{d\mu_\beta}{dV}(p)) 
+ \frac{1}{2} \sum_{\beta=1, \beta \neq \alpha}^{2n} \frac{1}{(\mu_\beta - \mu_\alpha)^{k}} (M^{(k)}_\beta \frac{d\mu_\beta}{dV}(p) - \phi^{(k)}_\beta(p)) \right),
\]

\[
\alpha = 1, \ldots, 2n , \ k = 1, 2, \ldots .
\]

The quantities \( \frac{d\mu_\alpha}{dV}(p) \) are given by the solution of the following set of linear equations,

\[
0 = \sum_{\alpha=1}^{2n} \left( \mu_\alpha^k M^{(1)}_\alpha \frac{d\mu_\alpha}{dV}(p) - p^k \phi^{(1)}_\alpha(p) \right) + 2k \ p^{k-1} \phi^{(0)}_\alpha(p), \ k = 0, \ldots, n ,
\]

\[
0 = \sum_{\alpha=1}^{2n} \left( M^{(1)}_\alpha \frac{d\mu_\alpha}{dV}(p) - \phi^{(1)}_\alpha(p) \right) K_{\alpha,j}, \ j = 1, \ldots, n - 1 ,
\]

\[
K_{\alpha,j} = \oint_{A_j} d\lambda \frac{\tilde{g}(\lambda)}{\lambda - \mu_\alpha} ,
\]

when applying \( \frac{d}{dV} \) to (2.22) and using (2.24). While the first relations can be attained easily, deriving the latter can be performed in a tricky way first proposed by Akemann.
Indeed,
\[
\frac{d}{dV}(p)\tilde{y}(\omega) = -\frac{1}{2} \sum_{\alpha=1}^{2n} M(\omega)\tilde{y}(\omega) \frac{1}{\omega - \mu_\alpha} \frac{d\mu_\alpha}{dV}(p) + \tilde{y}(\omega) \frac{d}{dV}(p) M(\omega)
\]  
\tag{3.13}

Calculating the second term, we find from (2.19) that
\[
\frac{d}{dV}(p)M(\omega) = \oint_{C_{\infty,p}} \frac{d\xi}{2\pi i} \left[ \frac{V'(\xi)}{(\xi - \omega)\tilde{y}(\xi)} \left( \frac{1}{2} \sum_{\alpha=1}^{2n} \frac{1}{\xi - \mu_\alpha} \frac{d\mu_\alpha}{dV}(p) \right) - \frac{1}{(p - \xi)^2(\xi - \omega)\tilde{y}(\xi)} \right],
\]
and the first term can be evaluated by the residue at infinity replacing \(V'(\xi)\) by \(M(\xi)\tilde{y}(\xi)\) in this limit, while the second term is given by the residue at \(\xi = p\) (as the integration contour must encircle the infinity and \(p\)). The first term then becomes
\[
\frac{1}{2} \sum_{\alpha=1}^{2n} \frac{M(\omega) - M(\mu_\alpha)}{\omega - \mu_\alpha} \frac{d\mu_\alpha}{dV}(p),
\]  
\tag{3.14}

and it partially cancels the first term in (3.13); the difference is just the term proportional to \(M(\mu_\alpha) = M^{(1)}_\alpha\), while evaluating the second term upon substituting it into the integral in (2.24) needs partial integration:
\[
\frac{\partial}{\partial p} \oint_{A_j} d\omega \frac{1}{p - \omega} \frac{\tilde{y}(\omega)}{\tilde{y}(p)} = \frac{1}{2} \oint_{A_j} d\omega \frac{1}{p - \omega} \sum_{\alpha=1}^{2n} \left( \frac{1}{\omega - \mu_\alpha} - \frac{1}{p - \mu_\alpha} \right) \frac{\tilde{y}(\omega)}{\tilde{y}(p)}
\]
\[
= \frac{1}{2} \sum_{\alpha=1}^{2n} K_{\alpha,j} \phi^{(1)}_{\alpha}(p),
\]  
\tag{3.15}

which together with (3.13) and (3.14) gives (3.12).

Linear system of equations (3.12) implies that the solution always takes the form
\[
M^{(1)}_\alpha \frac{d\mu_\alpha}{dV}(p) = \phi^{(1)}_{\alpha}(p) + \sum_{l=0}^{n-2} \mathcal{L}_{\alpha,l} p^l \phi^{(0)}_{\alpha}(p), \quad \alpha = 1, \ldots, 2n.
\]  
\tag{3.16}

The quantities \(\mathcal{L}_{\alpha,l}, \alpha = 1, \ldots, 2n, l = 0, \ldots, n - 2\), are then determined by the following set of equations:
\[
0 = \sum_{\alpha=1}^{2n} \sum_{l=0}^{n-2} \mathcal{L}_{\alpha,l} \mu_\alpha^k p^l - \sum_{\alpha=1}^{2n} \sum_{l=0}^{k-1} \mu_\alpha^{k-l-1} p^l + 2k p^{k-1}, \quad k = 0, \ldots, n,
\]
\[
0 = \sum_{\alpha=1}^{2n} \sum_{l=0}^{n-2} \mathcal{L}_{\alpha,l} K_{\alpha,j} p^j, \quad j = 1, \ldots, n-1.
\]  
\tag{3.17}

We can redefine basis (3.6) in terms of total derivatives. Solving Eq. (3.14) for \(\phi^{(k+1)}_{\alpha}(p)\) and expressing there the \(p\)-dependence through \(\frac{d}{dV}(p)\)-terms plus zero modes, we can obtain a unique basis inductively from Eq. (3.6) properly subtracting the corresponding zero modes. We then have
\[
\chi^{(k)}_{\alpha}(p) \equiv \frac{1}{M^{(1)}_\alpha} \left( \phi^{(k)}_{\alpha}(p) \bigg|_{\text{part}} - \sum_{r=1}^{k-1} M^{(k-r+1)}_\alpha \chi^{(r)}_{\alpha}(p) \right), \quad \alpha = 1, \ldots, 2n, \quad n = 1, 2, \ldots.
\]  
\tag{3.18}
Few first basis functions are
\[
\chi^{(1)}_{\alpha}(p) = \frac{d\mu_{\alpha}}{dV}(p), \quad \alpha = 1, \ldots, 2n, \\
\chi^{(2)}_{\alpha}(p) = -\frac{2}{3} \frac{d}{dV}(p) \log |M^{(1)}_{\alpha}| - \frac{1}{3} \sum_{\beta=1, \beta \neq \alpha}^{2n} \frac{d}{dV}(p) \log |\mu_{\alpha} - \mu_{\beta}|.
\] (3.19)

4 Calculations in genus one

4.1 Genus one partition function

Having determined the basis, the loop equation can now be inverted step by step in genus. For genus \( g = 1 \), Eq. (2.13) reads
\[
(\hat{K} - 2W_{0}(p))W_{1}(p) = \frac{d}{dV}(p)W_{0}(p).
\] (4.1)

Given \( W_{0}(p) \) and the loop insertion operator, the r.h.s. becomes
\[
\frac{d}{dV}(p)W_{0}(p) = -3 \sum_{\alpha=1}^{2n} \frac{1}{(p - \mu_{\alpha})^{2}} - \frac{1}{8} \sum_{\alpha, \beta=1}^{2n} \frac{1}{(p - \mu_{\alpha})(p - \mu_{\beta})} \\
+ \frac{1}{4} \phi^{(0)}(p) \sum_{\alpha=1}^{2n} \frac{1}{p - \mu_{\alpha}} M^{(1)}_{\alpha} \frac{d\mu_{\alpha}}{dV}(p) \\
= \frac{1}{16} \sum_{\alpha=1}^{2n} \frac{1}{(p - \mu_{\alpha})^{2}} - \frac{1}{8} \sum_{\alpha, \beta=1}^{2n} \frac{1}{\mu_{\alpha} - \mu_{\beta}} \left( \frac{1}{p - \mu_{\alpha}} - \frac{1}{p - \mu_{\beta}} \right) \\
+ \frac{1}{4} \sum_{\alpha=1}^{2n} \sum_{l=0}^{n-2} L_{\alpha,l} \mu_{\alpha}^{l}.
\] (4.2)

Here we took into account that regular parts coming from \( \frac{p}{p - \mu_{\alpha}}, l = 1, \ldots, n - 2, \) vanish due to Eq. (3.17) in order for \( W_{0}(p, p) = \frac{d}{dV}(p)W_{0}(p) \) to satisfy the correct asymptotic behavior, and we can just replace \( p^{l} \) by \( \mu_{\alpha}^{l} \) in numerators of such expressions. The result for the one-loop correlator of genus one with \( n \) cuts can now be easily obtained using Eq. (3.19),
\[
W_{1}(p) = \frac{1}{16} \sum_{\alpha=1}^{2n} \chi^{(2)}_{\alpha}(p) - \frac{1}{8} \sum_{1 \leq \alpha < \beta \leq 2n} \frac{1}{\mu_{\alpha} - \mu_{\beta}} \left( \chi^{(1)}_{\alpha}(p) - \chi^{(1)}_{\beta}(p) \right) \\
+ \frac{1}{4} \sum_{\alpha=1}^{2n} \sum_{l=0}^{n-2} L_{\alpha,l} \mu_{\alpha}^{l} \chi^{(1)}_{\alpha}(p) \\
= \frac{1}{16} \sum_{\alpha=1}^{2n} \chi^{(2)}_{\alpha}(p) - \frac{1}{8} \sum_{1 \leq \alpha < \beta \leq 2n} \frac{1}{\mu_{\alpha} - \mu_{\beta}} \left( \chi^{(1)}_{\alpha}(p) - \chi^{(1)}_{\beta}(p) \right) \\
+ \frac{1}{4} \sum_{\alpha=1}^{2n} \sum_{l=0}^{n-2} L_{\alpha,l} \mu_{\alpha}^{l} \chi^{(1)}_{\alpha}(p)
\]
\[
\begin{align*}
&= \frac{1}{16} \sum_{\alpha=1}^{2n} \left( -\frac{2}{3} \frac{d}{dV}(p) \log |M^{(1)}_\alpha| - \frac{1}{3} \sum_{\beta=1, \beta \neq \alpha}^{2n} \frac{d}{dV}(p) \log |\mu_\alpha - \mu_\beta| \right) \\
&\quad - \frac{1}{8} \sum_{\alpha=1}^{2n} \sum_{\beta=1, \beta \neq \alpha}^{n} \frac{1}{\mu_\alpha - \mu_\beta} \left( \frac{d\mu_\alpha}{dV}(p) - \frac{d\mu_\beta}{dV}(p) \right) \\
&\quad + \frac{1}{4} \sum_{\alpha=1}^{2n} \mathcal{L}_\alpha(\mu_\alpha) \frac{d\mu_\alpha}{dV}(p), \\
\end{align*}
\]  
(4.3)

where the polynomials $\mathcal{L}_\alpha(p)$ are, by definition, $\sum_{l=0}^{n-2} \mathcal{L}_{\alpha,l} p^l$.

We now integrate (4.3) in order to obtain $F_1$. While integrating first two terms is easy, the term with zero modes is difficult. Even before presenting the result, we need more notation. Let us introduce the quantities

\[
Q_{j,i} \equiv \oint_{A_j} \lambda^{i-1} \frac{\tilde{y}(\lambda)}{\gamma(\lambda)} d\lambda, \quad i = 1, 2, \ldots
\]  
(4.4)

(part of them constitute the $A$-matrix [8, 10]) and the (polynomial) basis of holomorphic differentials on the hyperelliptic Riemann surface

\[
\tilde{H}_k(\lambda) \equiv \sum_{l=1}^{n-1} \tilde{H}_{l,k} \lambda^{l-1}, \quad k = 1, \ldots, n - 1,
\]  
(4.5)

such that

\[
\oint_{A_j} \frac{\tilde{H}_j(\lambda)}{\tilde{y}(\lambda)} d\lambda = \delta_{k,j},
\]  
(4.6)

whence

\[
\sum_{l=1}^{n-1} Q_{j,l} \tilde{H}_{l,k} = \delta_{j,k} \quad \text{for} \quad j, k = 1, \ldots, n - 1.
\]  
(4.7)

The following lemma, to be proven in the next subsection, establishes the relation between zero modes $\mathcal{L}_{\alpha,l}$ and $Q_{j,i}$.

**Lemma 1** For quantities $\mathcal{L}_{\alpha,l}$ determined by system (3.17) and the quantities (4.4) and (4.5), we have the relation

\[
\sum_{l=0}^{n-2} \mathcal{L}_{\alpha,l} \mu_\alpha^l = - \sum_{j=1}^{n-1} \oint_{A_j} \frac{\tilde{H}_j(\lambda)}{\tilde{y}(\lambda)(\lambda - \mu_\alpha)} d\lambda, \quad \alpha = 1, \ldots, 2n.
\]  
(4.8)

It follows from Lemma 1 that the last term in (4.3) is just

\[
-\frac{1}{2} \sum_{\alpha=1}^{2n} \frac{\partial}{\partial \mu_\alpha} \left( \log \det_{i,j=1, \ldots, n-1} Q_{j,i} \right) \frac{d\mu_\alpha}{dV}(p),
\]

i.e.,

\[
F_1 = -\frac{1}{24} \log \left( \prod_{\alpha=1}^{2n} M^{(1)}_\alpha \cdot \Delta^4 \cdot \det_{i,j=1, \ldots, n-1} Q_{j,i} \right)^{12},
\]  
(4.9)

where $\Delta = \prod_{1 \leq \alpha < \beta \leq 2n} (\mu_\alpha - \mu_\beta)$ is the Vandermonde determinant. This is our final answer for the genus one partition function.
4.2 Proof of the lemma

Note first that all the quantities $Q_{j,i}$, although being dependent only on $2n$ moduli $\mu_\alpha$ are nevertheless algebraically independent for all $j$ and $i < 2n$ (that is, no rational functions of $\mu_\alpha$ and $Q_{j,i}$ with $i < 2n$ are identically zero at all $\mu_\alpha$). From the combinatorial point of view, we can therefore consider all such $Q_{j,i}$ as independent quantities. We also prove relation (4.8) not for $L_{\alpha,i}$ themselves but for special sums

$$R_{k,i} \equiv \sum_{\alpha=1}^{2n} \mu_\alpha^k L_{\alpha,i}. \quad (4.10)$$

If we calculate all $R_{k,i}$ with $k = 0, \ldots, 2n - 1$, we automatically find all $L_{\alpha,i}$ through the inverse Vandermonde transformation. We can prove (4.8) applying the summation with $\mu_\alpha^k$ to both its parts. Obtaining identities at all $k$ is equivalent to proving the lemma.

For shortening notations, it is convenient to introduce the quantities

$$T_{l,m} \equiv \sum_{j=1}^{n-1} Q_{l,j}^{-1} Q_{j,m}, \quad l = 1, \ldots, n - 1, \quad m = 1, 2, \ldots. \quad (4.11)$$

Obviously, $T_{l,m} = \delta_{l,m}$ for $1 \leq m \leq n - 1$.

In the l.h.s. of (4.8), we therefore have

$$\sum_{\alpha=1}^{2n} \mu_\alpha^k L_{\alpha,i} = \sum_{i=0}^{n-2} R_{k+i,i}, \quad (4.12)$$

while in the r.h.s., we find the expression

$$- \sum_{\alpha=1}^{2n} \sum_{j=1}^{n-1} \int_{A_j} \frac{\sum_{i=1}^{n-1} Q_{i,j}^{-1} x^{i-1} \mu_\alpha^k}{\tilde{y}(x) (x - \mu_\alpha)} dx =$$

$$= - \sum_{\alpha=1}^{2n} \sum_{j=1}^{n-1} \int_{A_j} \frac{\sum_{i=1}^{n-1} Q_{i,j}^{-1} x^{i-1} (\mu_\alpha^k - x^k)}{\tilde{y}(x) (x - \mu_\alpha)} - \sum_{\alpha=1}^{2n} \int_{A_j} \frac{\sum_{i=1}^{n-1} Q_{i,j}^{-1} x^{i+k-1}}{\tilde{y}(x) (x - \mu_\alpha)}$$

$$= \sum_{s=0}^{k-1} \sum_{i=1}^{n-1} T_{i,i+s} \left( \sum_{\alpha=1}^{2n} \mu_\alpha^{k-s-1} \right) - 2 \sum_{i=1}^{n-1} (i + k - 1) T_{i,i+k-1}, \quad (4.13)$$

where we have integrated by part in the second term. Thus, we must prove the coincidence of (4.12) and (4.13) for all $k$.

First, it is straightforward to find from the first relations in (3.17) the expressions for $R_{k,i}$ with $k = 0, \ldots, n$:

$$R_{k,i} = \theta(k - 1 - i) \sum_{\alpha} \mu_\alpha^{k-i-1} - 2k \delta_{k-i-1,0}, \quad k = 0, 1, \ldots, n, \quad (4.14)$$

where $\theta(n) = 1$ for $n \geq 0$ and zero otherwise. The complementary set of relations follows from the second conditions in (3.17). Let us introduce the standard symmetric functions

$$\text{Sym}_k \equiv \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq 2n} \mu_{i_1} \mu_{i_2} \cdots \mu_{i_k}, \quad k \leq 2n,$$

$$\text{Sym}(\mu_\alpha)_k \equiv \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq 2n} \mu_{i_1} \mu_{i_2} \cdots \mu_{i_k}, \quad k \leq 2n - 1.$$
with the obvious recurrence relations

\[ \text{Sym}(\beta_{\alpha})_k = \text{Sym}_k - \mu_{\alpha} \text{Sym}(\beta_{\alpha})_{k-1}, \]
\[ \text{Sym}_0 = \text{Sym}(\beta_{\alpha})_0 = 1, \quad \text{Sym}_{-1} = \text{Sym}(\beta_{\alpha})_{-1} = 0. \quad (4.15) \]

In particular, we have

\[ \text{Sym}(\beta_{\alpha})_k = \text{Sym}_k - \mu_{\alpha} \text{Sym}_{k-1} + \ldots + (-\mu_{\alpha})^s \text{Sym}_{k-s} + (-\mu_{\alpha})^{s+1} \text{Sym}(\beta_{\alpha})_{k-s-1} \]
for \( s = 0, \ldots, k, \quad (4.16) \)

and

\[ \sum_{\alpha=1}^{2n} \text{Sym}(\beta_{\alpha})_s = (2n - s) \text{Sym}_s, \quad 0 \leq s \leq 2n. \quad (4.17) \]

We use the transformation

\[ \frac{\bar{y}(x)}{x - \mu_{\alpha}} = \frac{1}{\bar{y}(x)} \prod_{\beta \neq \alpha}^{2n} (x - \mu_{\beta}) = \frac{1}{\bar{y}(x)} \sum_{l=0}^{2n-1} x^{2n-l-1} (-1)^l \text{Sym}(\beta_{\alpha})_l \]
\[ = \frac{1}{\bar{y}(x)} \sum_{l=0}^{2n-1} \sum_{s=0}^l x^{2n-l-1} \mu_{\alpha}^l (-1)^s \text{Sym}_s \quad (4.18) \]

to obtain the set of constraints

\[ \sum_{k=0}^{2n-2} \sum_{s=1}^{2n-k-1} Q_{j,2n-k-s} R_{s,i} \text{Sym}_k(-1)^k = 0. \quad (4.19) \]

Using the above formulas and contracting the result with \( Q_{p,j}^{-1}, \ p = 1, \ldots, n-1, \) we obtain after some algebra using (4.14), (4.16), and (4.17)

\[ 0 = \sum_{\alpha=1}^{2n} \mu_{\alpha}^{n-i} (-1)^{p+i+n} \text{Sym}(\beta_{\alpha})_{n-p-1} + (p - i - 1)(-1)^{p-i-1} \text{Sym}_{2n-1-p-i} \]
\[ + \sum_{s=0}^{n-p-1} R_{n+s+1,i} (-1)^{p+n+s+1} \text{Sym}_{n-1-p-s} \]
\[ + \sum_{r=n}^{2n-1} T_{p,r} (-1)^{r+i+1} (r - 1 - i) \text{Sym}_{2n-1-r-i}, \quad p = 1, \ldots, n-1. \quad (4.20) \]

We can now find all the remaining quantities \( R_{n+1+s,i} \) for \( s = 0, \ldots, n-2 \) by inverting the lower triangular matrix standing by these quantities in the second row of (4.20). Namely, we introduce the quantities \( q_r, \ r = 0, 1, \ldots, \) using the determining relation

\[ \sum_{r=0}^{p} q_r \text{Sym}_{p-r}(-1)^{p-r} = \delta_{p,0} \quad \text{for} \quad p = 0, 1, \ldots. \quad (4.21) \]

The first few \( q_r \) are \( q_0 = 1, \ q_1 = \text{Sym}_1, \ q_2 = (\text{Sym}_1)^2 - \text{Sym}_2, \) etc. Except (4.21), we need another combinatorial identity on \( q_r, \) which follows from (4.21) and (4.15). Let us
multiply (4.21) by \( \text{Sym}_1 \equiv \sum_{\alpha=1}^{2n} \mu_\alpha \):

\[
\delta_{p,0} = \sum_{s=0}^{p} (s + 1) \text{Sym}_{s+1}(-)^s q_{p-s} + \sum_{\alpha=1}^{2n} \mu_\alpha^2 \sum_{s=0}^{p} \text{Sym}(\mu_\alpha)_{s-1}(-)^s q_{p-s}
\]

\[
= \sum_{s=0}^{p} (s + 1) \text{Sym}_{s+1}(-)^s q_{p-s} - \sum_{\alpha=1}^{2n} \sum_{q=2}^{p+1} \mu_\alpha^2 \left( \sum_{s=0}^{p+1-q} \text{Sym}_s(-)^s q_{p+1-q-s} \right)
\]

\[
= \sum_{s=0}^{p} (s + 1) \text{Sym}_{s+1}(-)^s q_{p-s} - \frac{2n}{\alpha=1} \mu_\alpha^{p+1},
\]

that is, we obtain the formula

\[
\sum_{s=0}^{p+1} s \cdot \text{Sym}_s(-)^s q_{p+1-s} = - \frac{2n}{\alpha=1} \mu_\alpha^{p+1}, \quad p = 0, 1, \ldots,
\]

(4.22)

while this sum vanishes for \( p = -1 \).

From (4.20), (4.21), we obtain for \( R_{p,i} \) with \( p > n \):

\[
R_{n+1+k,i} = - \sum_{r=0}^{k} q_r \sum_{s=1+i}^{n} T_{2n-s,2n-1-k+r}(2n-s-i-1)(-1)^{s-i-1} \text{Sym}_{s-i-1}
\]

\[
+ \sum_{\alpha=1}^{2n} \mu_\alpha^{n+k-i} - \sum_{r=0}^{k} q_r (-1)^{i+k-r+n} \text{Sym}_{n+k-r-i}(n-2-k+r-1),
\]

\[
k = 0, \ldots, n-2.
\]

(4.23)

Two last terms in (4.23) pertain to a “regular” part of formulas (4.12) and (4.13), i.e., to their parts that do not contain \( T_{p,i} \) with \( p > n-1 \). It is straightforward to see that these regular parts matches. The only nontrivial thing to do is to check matching “nonregular” parts. To make presentation shorter, we use ellipses to denote omitted “regular” parts. We have for \( k = 0, \ldots, n-2 \):

\[
\sum_{i=0}^{n-2} R_{i+k,i} = \sum_{i=0}^{n-k} \ldots + \sum_{p=0}^{k-3} R_{n+1+p,n+1+p-k}
\]

\[
= (\ldots) + \sum_{r=0}^{k-3} \left( - \sum_{p=0}^{r} q_r \sum_{s=n+2+p-k}^{n} T_{2n-s,2n-1-p+r} \times
\]

\[
\times (2n-s-(n+1+p-k)-1)(-1)^{s-n+p+k} \text{Sym}_{s-n-1-p+k-1} \right)
\]

\[
= (\ldots) - \sum_{0 \leq r \leq s \leq k-2} T_{n-2+k-s,2n-1-r} \left( \sum_{p=r}^{s} (2k-p-s-4)(-)^{s-p} \text{Sym}_{s-p} q_{p-r} \right)
\]

\[
= (\ldots) - \sum_{0 \leq r \leq s \leq k-2} T_{n-2+k-s,2n-1-r} \left[ 2k-2s-4 \right] \delta_{r,s} + \left( - \sum_{\alpha=1}^{2n} \mu_\alpha^{s-r} + 2n \delta_{r,s} \right)
\]

\[
= (\ldots) + \sum_{0 \leq r \leq s \leq k-2} T_{n-2+k-s,2n-1-r} \sum_{\alpha=1}^{2n} \mu_\alpha^{s-r}
\]

\[
- \sum_{0 \leq r \leq k-2} 2[n+k-r-2] T_{n-2+k-r,2n-1-r},
\]

(4.24)
which exactly gives the “nonregular” part in expression (4.13). The lemma is therefore proved.

5 Conclusion

The obtained answer (4.9) has a nice geometrical interpretation in terms of $G$-functions and isomonodromic deformations (see [10], [15], [16]). Worth mentioning is the relation between this answer and the answer in planar limit, which is a superpotential. These geometrical relations deserve special studying. We hope to provide also a geometrical explanation for combinatorial identities of type (4.8).

Worth mentioning is what will be the answer for $F_1$ if we assume the “old” matrix model approach with equated chemical potentials on the intervals of eigenvalue distribution. It is easy to see that the corresponding answer has form (4.9) with the only difference that instead of integrating over $A$-cycles in the matrix $Q_{i,j}$, one should integrate over the $B$-cycles. In the particular case of two-cut solution, this reproduces the Akemann answer.

Another question pertains to the $S$-dependence of $F_1$. The loop equation by definition may fix only the part of $F_1$ that depends on the potential; in principle, there can be a part that depends only on the occupation numbers and which cannot be attained by using the Virasoro invariance of the matrix model integral; however, this part must be saturated by the planar limit of the matrix model integral. It is nevertheless interesting and important to calculate $S$-derivatives of the answer $F_1$.

The author thanks B. Dubrovin, T. Grava, D. Korotkin, A. Marshakov, A. Mironov, and A. Zabrodin for numerous discussions on matrix models and related geometry. The author is grateful to the SISSA for the hospitality during his visit when this paper was initiated.

The paper was partially financially supported by the Russian Foundation for Basic Research (Grants Nos. 02–01–00484 and NSh-2052.2003.1) and by the Program Nonlinear Dynamics and Solitons.

References

[1] J. Ambjørn, L. Chekhov, C. F. Kristjansen, and Yu. Makeenko, Matrix model calculation beyond the spherical limit, *Nucl. Phys.* B404 (1993) 127

[2] G. Akemann, Higher genus correlators for the Hermitian matrix model with multiple cuts, *Nucl. Phys.* B482 (1996) 403; [hep-th/9606004]

[3] J. Ambjørn and G. Akemann, New universal spectral correlators, *J. Phys.* A29 (1996) L555–L560; [cond-mat/9606129]

[4] N. Seiberg and E. Witten, Electric-magnetic duality, monopole condensation, and confinement in $N = 2$ supersymmetric Yang–Mills theory, *Nucl. Phys.* B426 (1994) 19; [Erratum: ibid. B430 (1994) 485]; [hep-th/9407087]
[5] F. Cachazo, K. Intriligator, and C. Vafa, A large $N$ duality via a geometric transition, Nucl. Phys. B603 (2001) 3-41; hep-th/0103067
F. Cachazo and C. Vafa, $N = 1$ and $N = 2$ geometry from fluxes, hep-th/0206017

[6] R. Dijkgraaf and C. Vafa, Matrix Models, Topological Strings, and Supersymmetric Gauge Theories, Nucl. Phys. B644 (2002) 3–20; hep-th/0206255. On Geometry and Matrix Models, Nucl. Phys. B644 (2002) 21–39; hep-th/0207106. A Perturbative Window into Non-Perturbative Physics, hep-th/0208048.

[7] A. Klemm, M. Marino, and S. Theisen, Gravitational corrections in supersymmetric gauge theory and matrix models, JHEP 0303 (2003) 051; hep-th/0211216

[8] I. K. Kostov, Conformal field theory techniques in random matrix models, hep-th/9907060

[9] G. W. Moore, Matrix models of 2D gravity and isomonodromic deformations, in: Proc. Cargese meeting on Random Surfaces, Quantum Gravity, and Strings, 1990, Prog. Theor. Phys. Suppl. 102 (1990) 255.

[10] R. Dijkgraaf, A. Sinkovics, and M. Temürhan, Matrix models and gravitational corrections, hep-th/0211241.

[11] L. Chekhov and A. Mironov Matrix models vs. Seiberg–Witten/Whitham theories, Phys. Lett. B552 (2003) 293–302; hep-th/0209085

[12] L. Chekhov, A. Marshakov, A. Mironov, and D. Vasilisev, DV and WDVV, Phys. Lett. B562 (2003) 323–338; hep-th/0301071

[13] Yu. Makeenko, Loop equations in matrix models and in 2D quantum gravity, Mod. Phys. Lett. A6 (1991) 1901–1913.

[14] J. Jurkiewicz, Regularization of one-matrix models, Phys. Lett. B245 (1990) 178.

[15] B. Dubrovin and Y. Zhang, Bi-Hamiltonian hierarchies in 2D topological field theory at one-loop approximation, Commun. Math. Phys. 198 (1998) 311–361; hep-th/9712232

[16] A. Kokotov and D. Korotkin, Bergmann tau-fuiction on Hurwitz spaces and its applications, math-ph/0310008