System for solving multicriteria problems with fuzzy preferences

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Abstract. The new version of the software system DASS designed to solve multicriteria problems using methods of the criteria importance theory is described. A new approach has been developed by the authors, allowing to take into account inaccurate and fuzzy information about preferences of decision makers.

1. Introduction
The article presents an approach to solving multicriteria problems, based on the methods of criteria importance theory [1-4]. The process of solving such problems is considered as an iterative procedure in which a decision maker (DM) interacts with a computer system in order to analyze multicriteria decision options and make the best choice according to his/her preferences [4-5]. Often, in practice, it is not possible to obtain accurate estimates of quantitative parameters that actually reflect the preferences of decision makers. Therefore, the most promising are methods of multicriteria analysis using incomplete, inaccurate and fuzzy information about preferences.

The criteria importance theory makes it possible to correctly take into account qualitative (non-numerical) information about the DM’s preferences, in particular information about the relative importance of criteria. Within this theory, special methods have been developed that allow conclusions to be drawn on decision-making based on incomplete information on the preferences of the DM [6]. In addition, methods for obtaining and using fuzzy information about preferences are currently being developed [7–8].

The article presents a description of the new version of the computer system DASS [4-5], intended for the analysis of multicriteria problems of choice by the methods of the criteria importance theory. The latest version of the system implements analysis methods that use incomplete and fuzzy information about the DM’s preferences.

2. Mathematical model and definitions from the criteria importance theory
The further discussion is based on the following mathematical model of the situation of multicriteria choice under uncertainty, which is used in the criteria importance theory:

\[ M = \langle X, K_1, ..., K_m, Z_0, \Re \rangle, \]

here \( X \) is the set of alternatives (variants), \( K_1, ..., K_m \) are the criteria \( (m \geq 2) \), i.e. the functions \( K_i : X \to Z_0 \), with \( Z_0 = \{1, ..., q\} \) is the range of values of the criteria (the set of scale estimates) \( (q \geq 2) \), and \( \Re \)
is the DM's preferences model. It is assumed that each of the criteria is independent in preference from the others and its larger values are preferable to the smaller ones. Each alternative \( x \) from the set \( X \) is characterized by its vector estimate \( y(x) = (K_1(x), ..., K_m(x)) \). The set of all vector estimates, both attainable (corresponding to alternatives from \( X \)) and hypothetical, is \( Z = Z^n \). So, comparison of alternatives by preference is equivalent to a comparison of their vector estimates.

The DM’s preferences are modeled on \( Z \) using the non-strict preference relation \( \mathcal{R} \): \( yRz \) means that the vector estimate \( y \) is no less preferable than \( z \). The relation \( \mathcal{R} \) is a (partial) quasi-order (that is, it is reflexive and transitive) and produces on \( Z \) the relations of indifference \( I \) and (strict) preference \( P \):

\[
 yIz \iff yRz \land zRy; \quad yPz \iff yRz \land \neg zRy.
\]

An alternative is called non-dominated, if there is no other alternative from the set \( X \), which is more preferable with respect to \( P \). Otherwise, the alternative is called dominated. The best alternative should be chosen among the set of non-dominated alternatives [1].

Since the preferences of the DM increase along the criterion scale \( Z_0 \), the Pareto relation \( \mathcal{R}^p \) is defined on the set \( Z \) of vector estimates:

\[
 yR^p z \iff y_i \geq z_i, \quad i = 1, ..., m.
\]

As a rule, it is not possible to obtain a solution to the multicriteria choice problem only with the help of the Pareto relation. Therefore, it needs to be expanded, involving additional information about the preferences of the DM.

The criteria importance theory uses information about the relative importance of criteria for the DM, which is formally introduced as follows [1-2]. Denote by \( y^\phi \) the vector estimate obtained from the vector estimate \( y = (y_1, ..., y_m) \) by permuting its components \( y_i \) and \( y_j \).

**Definition 1.** The criteria \( K_i \) and \( K_j \) are equally important, when any vector estimate \( y \) from \( Z \) is the same in preference with \( y^\phi \). The message is denoted by \( i \sim j \) and determines the indifference relation on set \( Z \):

\[
 y^{i \sim j} \iff z = y^j, \quad y_i \neq y_j.
\]  \hspace{1cm} (1)

**Definition 2.** The criterion \( K_i \) is more important than the criterion \( K_j \), when any vector estimate \( y \) from \( Z \), in which \( y_i > y_j \), is more preferable than \( y^\phi \). The message is denoted by \( i > j \) and determines the preference relation on set \( Z \):

\[
 y^{i > j} \iff z = y^j, \quad y_i > y_j.
\]  \hspace{1cm} (2)

Qualitative information \( \Omega \) about the criteria importance is a set of messages of the form \( i \sim j \) and \( i > j \).

Definitions 1 and 2 imply that a comparison by preference must be made for all vector estimates \( y \) from \( Z \). However, due to the large number of them, this is usually impossible. Therefore, in practice, only a limited set of specially selected vector estimates are used [1]. In addition to a large number of comparisons, another practical problem may be the inconsistency of the DM’s answers when comparing different vector estimates. To take into account conflicting, inaccurate and poorly reliable information on the preferences of DMs, we proposed to use fuzzy preference relations [9] as a model \( R \) of preferences [7-8].

Let us introduce the following fuzzy relations on the set of criteria: equalness \( \mu^\sim \); superiority in importance \( \mu^\succ \); non-strict superiority in importance \( \mu^\succeq = \mu^\sim \cup \mu^\succ \). To obtain fuzzy information about the relative importance of the criteria we propose to use the following approach. Consider the frequency of responses of the DM in favor of each conclusion about the relative importance of the criteria. For example, during the determination of the relative importance of the first and second criteria, the DM compares by preference the following pairs of vector estimates: (2, 1, 1)/(1, 2, 1), (3, 1, 1)/(1, 2, 1), (2, 1, 3)/(1, 2, 3), (3, 1, 3)/(1, 2, 3). Then we believe that with a degree of certainty \( \mu^\sim (1, 2) = 0.25 \) the criteria are equally important, with a degree of certainty \( \mu^\succ (1, 2) = 0.5 \) the first
If Pareto relation: indifference can be used to solve a multicriteria problem. Let us now consider how the obtained fuzzy information about the relative importance of criteria can be used to solve a multicriteria problem.

**Definition 3.** The following fuzzy relations are introduced on the set \( Z \) of vector estimates: indifference \( \mu^I \); preference \( \mu^P \); non-strict preference \( \mu^R = \mu^I \cup \mu^P \). Fuzzy relations for transpositions \( y^i \) of vector estimates \( y \) from \( Z \) are assumed to be equal to fuzzy relations of importance for the corresponding criteria:

\[
\mu^I(y, y^{ij}) = \mu^- (i, j), \tag{3}
\]

\[
\mu^P(y, y^{ij}) = \mu^>(i, j), \ y_i > y_j. \tag{4}
\]

Given the symmetry of the fuzzy relation \( \mu^- \), it implies

\[
\mu^R(y, y^{ij}) = \mu^\sim (i, j), \ y_i > y_j. \tag{5}
\]

To construct fuzzy indifference and preference relations on the set \( Z \) of all vector estimates, we introduce the following notation.

\( \Pi(y) \) is the set of all vector estimates, including \( y \), obtained from \( y \) by the permutation of components.

\( D(y, z) \) is the set of all vector estimates \( w \in \Pi(y) \), which are preferable to \( z \) in accordance with the Pareto relation: \( w \in P^\sim z \).

\( T(y, w) \) is the set of transpositions \( (i, j) \), applying which successively to the vector estimate \( y \) it is possible to obtain a vector estimate \( w \in \Pi(y) \). The transposition \( (i, j) \) is the transformation \( y \rightarrow y^i \) considered above. Denote by \( [i, j] \) the transposition \( (i, j) \) in which the numbers of the components are ordered in such a way that \( y_i > y_j \) is satisfied in the transformed vector estimate \( y \).

**Definition 4.** If \( z \in \Pi(y) \), then

\[
\mu^I(y, z) = \max_{T(y, z)} \min_{(i, j) \in T(y, z)} \mu^- (i, j). \tag{6}
\]

\[
\mu^R(y, z) = \max_{T(y, z)} \min_{(i, j) \in T(y, z)} \mu^> (i, j), \tag{7}
\]

\[
\mu^P(y, z) = \max_{T(y, z)} \max_{(p, r) \in T(y, z)} \min_{[i, j] \in T(y, z)} \{\mu^\sim (i, j), \mu^>(p, r)\}. \tag{8}
\]

If \( z \notin \Pi(y) \) and \( D(y, z) \neq \emptyset \), then \( \mu^I(y, z) = 0 \) and

\[
\mu^R(y, z) = \mu^R(y, w) = \max_{w \in D(y, z)} \mu^R(y, w). \tag{9}
\]

If \( z \notin \Pi(y) \) and \( D(y, z) = \emptyset \), then \( \mu^R(y, z) = \mu^I(y, z) = \mu^P(y, z) = 0 \).

As a result, the best alternative \( x^* \) should be chosen among the alternatives with the greatest degree of non-domination, which can be estimated by the following expression:

\[
1 - \max_{x \in X} \mu^P(y(x), y(x^*)). \tag{10}
\]

3. **Description of the computer system DASS**

The first version of the computer system DASS (Decision Analysis Support System) was created in 2006 [4-5]. It included a basic interface for entering information on preferences of DMs and available at the time methods of building a preference relations within the framework of the criteria importance theory approach. Subsequent versions of the system were created during 2012–2017 and continue to evolve at present [3–4]. They implemented new methods for comparing vector estimates by preference. In addition, in the current version 2.4 of the DASS, procedures for obtaining conciliation decisions are implemented [6]. All current versions of the DASS are available on the website [http://www.mcodm.ru/soft/dass](http://www.mcodm.ru/soft/dass).
The DASS version 2.4 includes the following components:
- framework for processing and storing data about the problem and preferences;
- library of mathematical algorithms;
- user interface in English and Russian.

Data about the problem and preferences can be entered and edited in the user interface, as well as export/import in XML format.

General data on a multicriteria choice problem consists of a textual description of the problem, a list of $m$ criteria, the number of gradations $q$ of the general ordinal scale of criteria and a list of $n$ alternatives with vectors of their estimates for each criterion – integers from 1 to $q$ (see figure 1).

Data on preferences of the DM is obtained step by step in a dialogue mode. First, you need to select the type of information about the importance of criteria (no information, qualitative, quantitative interval, quantitative accurate) and the type of the criteria scale (ordinal, first ordered metric, scale with interval constraints, interval scale) (see figure 2).

When choosing quality information about the importance of criteria, it is necessary to order the list of criteria in accordance with their relative importance. When choosing quantitative information, it is required to enter estimates of the degree of superiority in the importance of criteria, exact or in the form of intervals of possible values.

In accordance with the selected combination of types of information about the importance of criteria and type of their scale and the introduced estimates of preference parameters, the system will
generate a list of non-dominated and dominated alternatives using the developed methods for comparing vector estimates by preference (see figure 3). When you select any dominated alternative with the left mouse button, a text explanation will be displayed, which of the alternatives is more preferable than the selected one (see figure 3). Figure 3. The results of the comparison of alternatives in terms of preference in the form of a list of non-dominated and dominated alternatives.

The DASS version 2.4 allows you to switch to any of the described types of preference information at any time, resulting in a list of non-dominated and dominated alternatives. However, it is recommended to use the iterative-fragmentary approach [5], which implies a successive and consistent refinement of the DM’s preference model on the basis of gradually increasing information about preferences (moving from qualitative to quantitative information about the criteria importance and improving their scale from ordinal to quantitative). With this approach, the original set of alternatives non-dominated by the Pareto relation is successively narrowed, so that at one stage of solving the problem there can be one non-dominated alternative, or several non-dominated alternatives equivalent by the relation \( I \). In this case, a solution to the multicriteria choice problem is found, and further refinement of preferences not required. It is also possible that further refinement of preferences is difficult or impossible, and the set of non-dominated alternatives still contains alternatives that are incomparable by preference. Then you should use additional methods to choose the best alternative among the set of non-dominated ones. The DASS version 2.4 implements two such methods [6]:

1) choose the alternative with the highest value of additive value function calculated using centroid values of the preferences parameters;

2) determine the maximum likelihood optimal (ML-optimal) alternative, i.e. the alternative with the highest probability of being optimal for a uniform distribution of parameter value probabilities.

4. Conclusion
The described approach to the use of fuzzy information about the preferences of DMs in the framework of the criteria importance theory seems to be promising for the interactive solution of multicriteria choice problems. At the same time, the development of effective methods for calculating fuzzy preference relations in accordance with the definition 4 remains relevant.

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References

[1] Podinovski V V 2007 *Introduction into the Criteria Importance Theory in Multicriteria Problems of Decision Making* (Moscow: Fizmatlit) p 64
[2] Podinovski V V 1976 Multicriterial problems with importance-ordered criteria *Automation and remote control* 37 (11) 1728–36
[3] Podinovski V V, Potapov M A, Nelyubin A P and Podinovskaya O V 2014 Criteria importance theory: state of the art and directions of further development *The 12th All-Russia Conf. on Control Problems VSPU-2014* 7697–702
[4] Nelyubin A P, Podinovski V V and Potapov M A 2018 Methods of criteria importance theory and their software implementation *Computational Aspects and Applications in Large-Scale Networks. Springer Proc. in Mathematics & Statistics* 247 189–196
[5] Podinovski V V 2008 Analysis of multicriteria choice problems by methods of the theory of criteria importance, based on computer systems of decision making support *J. of Computer and System Sciences International* 47 (2) 221–225
[6] Nelyubin A P and Podinovski V V 2017 Multicriteria choice based on criteria importance methods with uncertain preference information *Computational Mathematics and Mathematical Physics* 57 (9) 1475–83
[7] Potapov M A, Nelyubin A P and Solovyev I S 2018 Methods for analyzing fuzzy information about the relative importance of criteria *Information Technologies in Science, Education, and Control* 3 (7) 7–14
[8] Podinovski V V, Potapov M A, Nelyubin A P 2018 The solution to the choice problem with fuzzy information about preferences *Proc. Int. Conf. KROMSH-2018 (Simferopol)* 147–149
[9] Orlovsky S A 1978 Decision-making with a fuzzy preference relations *Fuzzy Sets and Systems* 1 155–167