A Global Linear and Local Superlinear (Quadratic) Inexact Non-Interior Continuation Method for Variational Inequalities Over General Closed Convex Sets

Le Thi Khanh Hien1 · Chek Beng Chua2

Received: 22 March 2019 / Accepted: 10 March 2020 / Published online: 18 March 2020 © Springer Nature B.V. 2020

Abstract
We use the concept of barrier-based smoothing approximations to extend the non-interior continuation method, which was proposed by B. Chen and N. Xiu for nonlinear complementarity problems based on Chen-Mangasarian smoothing functions, to an inexact non-interior continuation method for variational inequalities over general closed convex sets. Newton equations involved in the method are solved inexactly to deal with high dimension problems. The method is proved to have global linear and local superlinear/quadratic convergence under suitable assumptions. We apply the method to non-negative orthants, positive semidefinite cones, polyhedral sets, epigraphs of matrix operator norm cone and epigraphs of matrix nuclear norm cone.

Keywords Inexact non-interior continuation method · Variational inequality · Smoothing approximation · Polyhedral set · Epigraph of matrix operator norm · Epigraph of matrix nuclear norm · Strict complementarity

Mathematics Subject Classification (2010) 65K15 · 90C25 · 90C30

1 Introduction
Let $X$ be a given closed convex subset of a finite dimensional real vector space $E$ equipped with an inner product $(\cdot, \cdot)$, and $F : E \rightarrow E$ be a continuously differentiable map. We
consider the following variational inequality \( VI(X, F) \) over general closed convex sets: find \( x \in X \) such that
\[
(F(x), y - x) \geq 0 \quad \text{for all } y \in X. \tag{1}
\]
Many well-known optimization problems can be cast as VIs. For examples, when \( X \) is a cone in \( \mathbb{E} \), the variational inequality (VI) is equivalent to a nonlinear complementarity problem (NCP) of finding \( x \in X \) such that \( F(x) \in X^2 \) and \( (x, F(x)) = 0 \), where \( X^2 \) is the dual cone of \( X \); convex optimization problems and fixed point problems can also be cast as VIs, see [22, Chapter 1]. We let \( \| \cdot \| \) be the norm induced by the inner product \((\cdot, \cdot)\), and denote the Euclidean projection of \( z \) onto \( X \) by \( \Pi_X(z) \), i.e.,
\[
\Pi_X(z) = \arg \min_{x \in X} \frac{1}{2} \| x - z \|^2.
\]
The map
\[
(x, y) \in \mathbb{E} \times \mathbb{E} \mapsto (x - \Pi_X(x - y), F(x) - y)
\]
is called the natural map \( G_{\text{nat}} \). It was proved in [22, Proposition 1.5.8] that \( x \) is a solution of \( VI(X, F) \) if and only if \( x \) satisfies \( G_{\text{nat}}(x, y) = 0 \) for some \( y \in \mathbb{E} \), and this equation is called the natural map equation. In this paper, we are interested in solving \( VI(X, F) \) via the natural map equation.

The natural map equation, in general, is nonsmooth since the Euclidean projection is not always smooth; and as such it can not be solved by typical Newton-based methods. To remedy this restriction, one can use nonsmooth Newton-based methods, see e.g., [23, Chapter 7, Chapter 8], [29, 30, 34, 35], or use smoothing approximations of the Euclidean projection to solve the natural map equation numerically by a smoothing Newton continuation method. A continuous map \( p : \mathbb{E} \times \mathbb{R}_+ \to \mathbb{E} \), parameterized by \( \mu \in \mathbb{R}_+ \), is called a smoothing approximation (SA) of the Euclidean projection \( \Pi_X \) over a closed convex set \( X \) if \( p \) converges point-wise to \( \Pi_X \) as \( \mu \to 0 \), i.e., \( p(z, 0) = \Pi_X(z) \), and for each \( \mu \in \mathbb{R}_+ \), \( p(\cdot, \mu) \) is differentiable. When the convergence is uniform, \( p \) is called a uniform SA. A very well-known example of the SA is the Chen-Harker-Kanzow-Smale (CHKS) function (see [7, 25])
\[
p(z, \mu) = \frac{1}{2} (\sqrt{z^2 + 4\mu^2} + z),
\]
which approximates the projection onto the set of non-negative real numbers \( \mathbb{R}_+ \). The CHKS function belongs to the class of smoothing functions introduced by C. Chen and O. L. Mangasarian, see [10], which is computed as
\[
p(z, \mu) = \int_{-\infty}^{z} \int_{-\infty}^{t} \frac{1}{\mu} d \left( \frac{x}{\mu} \right) dx dt,
\]
where \( d(\cdot) \) is a certain probability density function. When \( d(x) = \frac{2}{(x^2 + 4\mu^2)^{3/2}} \) the double integral equals to the CHKS function. L. Qi and D. Sun [33] developed this type of convolution-based SAs to approximate general nonsmooth functions. However, it is not computable in most cases since it contains a multivariate integral. Recently, C. B. Chua and Z. Li [16] introduced barrier-based smoothing functions, which only approximate the Euclidean projection onto convex cone with nonempty interior. This type of SAs has been extended to general closed convex sets with non-empty interior in [15]. Note that the barrier-based SA can be computed via proximal mappings of smooth maximal monotone maps. We denote \( p_{\mu}(z) = p(z, \mu) \) to emphasize that \( \mu \) will be used as a parameter, and define a SA of the natural map \( G_{\text{nat}} \)
\[
H_{\mu}(x, y) = \left( x - p_{\mu}(x - y) \right) \cdot \frac{F(x) - y}{F(x) - y}.
\]
It is worth noting that, in the literature, there are many approaches to solve the VI in its general form Eq. 1 or when \( X \) or \( F \) have more specific structures, for examples, KKT
conditions based methods, merit function based algorithms, interior point methods, projection methods, to name a few. We refer the readers to [23] for a comprehensive review of algorithms for solving VIs. In this paper, we are specifically interested in the SA approach which we briefly review in the next paragraph.

Chen and Mangasarian are the pioneers in using smoothing methods for solving VIs. Based on the smoothing of the plus function, Newton-based algorithms are proposed in [10] for solving NCPs and box-constrained VIs. The authors in [24] proposed a class of smoothing functions and use it to approximate the mixed NCP by a smooth system of nonlinear equations. They then study the limiting behaviour of the path generated by the approximate solutions of a sequence of least squares problems. Chen et al in [12] define an important property for the sequence of the derivatives of the SA which is called the Jacobian consistency property; by using this property together with some mild assumptions, for the first time in the literature, a local superlinear convergence rate is established for a smoothing Newton method that solves a nonsmooth equation. Note that the global convergence rate is not established for this smoothing Newton method. The method is then applied to solve a box constrained VI. Li and Fukushima [27] derive the CHKS smoothing function (which satisfies the Jacobian consistency property) for the mixed complementarity problem; and under suitable conditions, they establish local quadratic convergence properties for a smoothing Newton method. In another series of works, the SA approach is incorporated with the idea of path-following methods to yield the non-interior continuation methods, which are also known as non-interior path-following methods, for solving NCPs, see [7, 8]. Burke and Xu [6], for the first time, establish the global linear convergence of the non-interior continuation method for linear complementarity problem, and extended the result to NCPs with uniform $P$-functions [38]. The method are further extended and well-documented in the literature, see e.g., [9, 14, 25] and references therein.

We notice that although SA algorithms have been deeply studied for NCPs and VIs with specific structures of $X$, studying the algorithms for solving the VI over a general convex set $X$ is still an interesting and challenging topic. The main difficulty of designing smoothing approximation algorithms for VIs over general convex sets lies in how we develop the smoothing approximation of the Euclidean projection and the properties of its derivative sequence (such as the Jacobian consistency property). To the best of our knowledge, there exists only two approaches of smoothing approximation of Euclidean projections onto general convex sets, which are mentioned above – the convolution-based and the barrier-based approximation. While the former approximation has been widely applied and employed to develop smoothing approximation algorithms for VIs with strong convergence rate guar- anty, the later still has potential developments to explore. In this paper, we will employ the barrier-based smoothing functions to approximately solve the natural map equation.

Solving smooth equations by classical Newton method was proved to have local quadratic convergence. However, it was also verified that if a large size problem is considered then solving a system of Newton equations at each iterate would be very expensive. Inexact Newton methods [19], which calculate an appropriate solution to the Newton equations satisfying some level of precision, are more practical and suitable for large scale problems. In particular, to solve the system of nonlinear equation $G(x) = 0$, instead of using the Newton’s step $x_{k+1} = x_k + s_k$, where $s_k$ is calculated by $G'(x_k)s_k = -G(x_k)$, the authors in [19] propose a class of inexact Newton methods which solves the Newton equation inexactly $G'(x_k)s_k = -G(x_k) + r_k$, where the relative residual $\|r_k\| / \|G(x_k)\|$ does not exceed some predetermined precision $\eta_k$. Many Newton-like methods that do not require to compute the closed forms of $G'(x_k)$ or its inverse belong to this class of inexact Newton methods. For example, Dennis [20] proposes a Newton-like method that
updates \( x_{k+1} = x_k - M(x_k)G(x_k) \), where \( M(x_k) \) is a linear operator satisfying some conditions; under these conditions we can prove that the relative residual, with \( r_k = G'(x_k)(-M(x_k)G(x_k)) + G(x_k) \), satisfies some precision, see [19, Section 4]. Some other examples include Newton-Krylov methods [5, 26], truncated Newton methods [32]. The inexact Newton step \( x_k \) can also be found by applying efficient iteration solvers for the system of linear equations such as the GMRES method [3, 36], the splitting methods [1, 2], conjugate gradient methods [37]. The inexact Newton methods also converge locally super-linearly/quadratically under some natural assumptions of the relative residuals, see [19]. Taking into account these advantages, we use inexact Newton methods to solve Newton equations in our path-finding continuation method.

1.1 Contribution

Our main contribution is that, using the barrier-based smoothing approximation, we improve the noninterior continuation method in [9] to solve VI over general closed convex sets. The method employs centering steps, which give its global linear convergence, together with approximate Newton steps, which help to achieve local superlinear/quadratic convergence. Newton equations involved in the method are solved inexactly to handle large scale problems. We provide the application of our method to non-negative orthants, positive semidefinite cones, polyhedral sets, epigraphs of matrix operator norm and epigraphs of matrix nuclear norm.

To achieve global linear convergence rate, we assume that the derivative \( \nabla H_{\mu_k}(x(k), y(k)) \), where \( \{(x(k), y(k))\}_{k \geq 0} \) is the sequence generated by our algorithm, is nonsingular and \( \{\|\nabla H_{\mu_k}(x(k), y(k))^{-1}\|\}_{k \geq 0} \) is bounded. We prove that the monotonicity of \( F \) is sufficient for the non-singularity of \( \nabla H_{\mu_k}(x(k), y(k)) \). It was proved in [13] that, when \( X \) is a positive semidefinite cone, the strong monotonicity of \( F \) together with the uniform boundedness of \( \{\|\nabla F(x(k))\|\}_{k \geq 0} \) is sufficient for the boundedness of the sequence \( \{\|\nabla H_{\mu_k}(x(k), y(k))^{-1}\|\}_{k \geq 0} \). We will extend this result to the case when \( X \) is a general convex set, see Section 3.3. It is worth mentioning that, for more specific cases of \( X \), the strong monotonicity condition of \( F \) can be relaxed. We refer the readers to [9] for the case when \( X \) is a non-negative orthant, to [15] for the case when \( X \) is an epigraph of operator norm or an epigraph of nuclear norm.

To obtain local superlinear convergence rate, we develop an important property of the derivative sequence \( \nabla P_{\mu}(z) \) for the barrier-based SA. In particular, we prove that if \( \nabla P_{\mu}(z) \) converges to a linear operator \( T^* \) when \( (z, \mu) \) converges to \( (z^*, \mu) \), where \( z^* = x^* - y^* \) and \( (x^*, y^*) \) is any limit point of \( \{(x(k), y(k))\}_{k \geq 0} \), then the operator \( T^* \) must be \( \Pi_X(z^*) \), which also implies that the projector onto \( X \) is differentiable at \( z^* \). To achieve \( \xi \)-order convergence rate with \( \xi > 1 \), we further need \( \|\nabla P_{\mu}(z) - T^*\| = O\left(\|(z-z^*, \mu)\|^{-\xi\xi-1}\right) \).

We verify the property with \( \xi = 2 \) (i.e., the local convergence rate will be quadratic) for non-negative orthants, positive semidefinite cones, polyhedral sets, epigraphs of matrix operator norm and epigraphs of matrix nuclear norm. We further show that for non-negative orthants, positive semidefinite cones, epigraphs of matrix operator norm and epigraphs of matrix nuclear norm, differentiability of \( \Pi_X(\cdot) \) at \( z^* \) is equivalent to strict complementarity of \( (x^*, y^*) \).
1.2 Organization and notation

The paper is organized as follows. In the next section, we give some preliminaries on barrier-based smoothing approximation. In Section 3, we describe the inexact non-interior continuation method and prove its global linear, and local superlinear/quadratic convergence. We present application of the algorithm to specific convex sets in Section 4. Finally, we conclude our paper in Section 5.

We end this section by explaining some notations that will be used in the paper. For a Fréchet-differentiable map $F$, we use $DF$ to denote the derivative of $F$ and $JF$ to denote its Jacobian. We use $\nabla f(x)$ and $\nabla^2 f(x)$ to denote the gradient and the Hessian of a twice Fréchet-differentiable function $f$. For a vector $x \in \mathbb{R}^m$, $[x]_+$ denotes the vector whose components are $[x_i]_+ = \max\{0, x_i\}$, $i = 1, \ldots, m$, and $\text{Diag}(x)$ denotes the $m \times m$ diagonal matrix with $(\text{Diag}(x))_{ii} = x_i$, $i = 1, \ldots, m$. For a matrix $z \in \mathbb{R}^{m \times m}$ we define two linear matrix operators

$$G(z) = \frac{1}{2}(z + z^T), \quad \Xi(z) = \frac{1}{2}(z - z^T).$$

We denote the following norms of a matrix $z \in \mathbb{R}^{m \times n}$

- $\|z\|_\infty$: the $l_\infty$ norm, i.e., $\|z\|_\infty = \max\{|z|_{ij}, 1 \leq i \leq m, 1 \leq j \leq n\}$,
- $\|z\|_1$: the $l_1$ norm, i.e., $\|z\|_1 = \sum_{1 \leq i \leq m, 1 \leq j \leq n} |z_{ij}|$,
- $\|z\|_F$: the Frobenius norm,
- $\|z\|_k$: the operator norm, i.e., the largest singular value of $z$,
- $\|z\|_*$: the nuclear norm, i.e., the sum of all singular values of $z$.

We use $I$ to denote the identity matrix, whose dimension is clear in the context, and $\mathbf{I}$ to denote the identity operator. For operators $P_1, P_2 : \mathbb{E} \rightarrow \mathbb{E}'$, we use $P_1 \prec P_2$ to mean $((P_2 - P_1)[u], u) > 0$ for all $u \neq 0$. We denote the set of $n \times n$ symmetric matrices by $\mathbb{S}^n$, the set of $n \times n$ orthogonal matrices by $\mathcal{O}^n$. For $z \in \mathbb{S}^n$, we use $\lambda_1(z) \geq \lambda_2(z) \geq \cdots \geq \lambda_m(z)$ to denote its eigenvalues with multiplicity, $\lambda_f(z) = (\lambda_1(z), \ldots, \lambda_m(z))^T$, and $\mathcal{O}(z)$ to denote the set of orthogonal matrices $u$ such that $z = u \text{Diag}(\lambda_f(z)) u^T$. For a given matrix $z \in \mathbb{R}^{m \times n}$ with $m \leq n$, we denote $\sigma_1(z) \geq \sigma_2(z) \geq \cdots \geq \sigma_m(z)$ to be singular values of $z$ with multiplicity and $\sigma_f(z) = (\sigma_1(z), \ldots, \sigma_m(z))^T$. We use $\circ$ to denote the Hadamard product (or the entry-wise product), i.e., $(z \circ w)_{ij} = z_{ij} w_{ij}$ for $z, w \in \mathbb{R}^{m \times n}$. For a $n \times n$ matrix $z$ we use $\text{Tr}(z)$ to denote its trace. The norm of a multilinear operator $L : \mathbb{E} \times \cdots \times \mathbb{E} \rightarrow \mathbb{E}'$ is given by

$$\|L\| = \sup_{v = (v_1, \ldots, v_k) \in \mathbb{E} \times \cdots \times \mathbb{E}} \left\{ \frac{\|Lv\|}{\|v\|} : v \neq 0 \right\},$$

where the norm of $v = (v_1, \ldots, v_k) \in \mathbb{E} \times \cdots \times \mathbb{E}$ is $\|v\| = \sqrt{\|v_1\|^2 + \cdots + \|v_k\|^2}$. We use $\text{int}(X)$ and $\text{relint}(X)$ to denote the interior and the relative interior of $X$.

2 Preliminaries

In this section, we give some background knowledge and establish some necessary results to be used in the sequel.

Definition 1 (a) A function $f : \text{int}(X) \rightarrow \mathbb{R}$ is called a barrier of $X$ if $f(x_k) \rightarrow \infty$ for any sequence $\{x_k\} \subset \text{int}(X)$ converging to a boundary point of $X$. 
(b) A function $f$ is called $\vartheta$-self-concordant barrier for $X$ if it is three times continuously differentiable, and satisfies the following conditions

(\alpha) the following value, called barrier parameter, is finite

$$\vartheta = \inf \left\{ t \geq 0 : \inf_{x \in \text{int}(X), h \in \mathbb{E}} t(h, \nabla^2 f(x)h) - \langle \nabla f(x), h \rangle^2 \geq 0 \right\};$$

(\beta) $|D^3 f(x)[h, h, h]| \leq 2(D^2 f(x)[h, h])^{3/2}$ $\forall x \in \text{int}(X), h \in \mathbb{E}$.

Proposition 1 If $f$ is a $\vartheta$-self-concordant barrier of $X$ then

$$|D^3 f(x)[h_1, h_2, h_3]| \leq 2\left(D^2 f(x)[h_1, h_1]D^2 f(x)[h_2, h_2]D^2 f(x)[h_3, h_3]\right)^{1/2}.$$

We refer to [28, Appendix 1] for the proof.

For a given closed convex set $X$ with a differentiable barrier $f$, we define the map $p : E \times \mathbb{R}_+ \rightarrow E$ as

$$p(z, \mu) + \mu^2 \nabla f(p(z, \mu)) = z \text{ when } \mu > 0$$

$$p(z, \mu) = \Pi_X(z) \text{ when } \mu = 0. \quad (3)$$

Condition (\alpha) in Definition 1 is used to prove the following theorem, which leads to Definition 2 – the definition of barrier-based smoothing approximation.

Theorem 1 [15, Theorem 3.1, Theorem 3.2] If $f$ is a twice continuously differentiable barrier on $X$, then the map $p$ defined via (3) is a smoothing approximation of the Euclidean projector $\Pi_X$. In addition, if $f$ is a $\vartheta$-barrier then the map $p$ is Lipschitz continuous with modulus $\sqrt{\vartheta}$ in the smoothing parameter; consequently, $p$ is a uniform smoothing approximation.

Definition 2 The barrier based smoothing approximation of Euclidean projection $\Pi_X$ defined by a given twice continuously differentiable barrier $f$ on $X$ is the map $p : E \times \mathbb{R} \rightarrow E$ that satisfies (3).

We remind that $p_\mu(z) = p(z, \mu)$. The following proposition provides an upper bound for $\|D^2 p_\mu(z)\|$ which is of paramount importance in achieving the global linear convergence of our inexact non-interior continuation method proposed in Section 3.

Proposition 2 If the barrier $f$ of $X$ is $\vartheta$-self-concordant, then for all $z \in E$ and $\mu > 0$ we have

$$\left\|D^2 p_\mu(z)\right\| \leq \frac{1}{4\mu}.$$

Proof Denote $g(z) = \nabla f(z)$. From the definition of barrier-based smoothing approximation, we have

$$p_\mu(z) + \mu^2 g(p_\mu(z)) = z.$$

Taking derivative on $z$ both sides, we get

$$D p_\mu(z)[u] + \mu^2 Dg(p_\mu(z))[D p_\mu(z)[u]] = u, \quad (4)$$

for all $u \in E$.

Taking derivative again on both sides of Eq. 4 gives us

$$D^2 p_\mu(z)[u, v] + \mu^2 D^2 g(p_\mu(z))[D p_\mu(z)u, D p_\mu(z)v] + \mu^2 Dg(p_\mu(z))D^2 p_\mu(z)u, v] = 0,$$
for all $u, v \in \mathcal{E}$. Therefore, we get
\[
\left\| D^2p_\mu(z)[u, v] \right\| = \left\| \mu^2 Dp_\mu(z)D^2g(p_\mu(z))[Dp_\mu(z)u, Dp_\mu(z)v] \right\|. \tag{5}
\]
Denote $\rho = \mu^2 Dp_\mu(z)D^2g(p_\mu(z))[Dp_\mu(z)u, Dp_\mu(z)v]$. Now we prove that $\|\rho\| \leq \frac{1}{4\mu}$.

We remind that $g = \nabla f$. For all $w \in \mathcal{E}$ with $\|w\| = 1$, we have
\[
\langle \rho, w \rangle = \mu^2 D^3f(p_\mu(z))[Jp_\mu(z)u, Jp_\mu(z)v, Jp_\mu(z)w].
\]
Since $f$ is a $\theta$-self-concordant barrier, we get
\[
\mu^2 D^3f(p_\mu(z))[Jp_\mu(z)u, Jp_\mu(z)v, Jp_\mu(z)w] \leq 2\mu^2 \prod_{d \in \{u, v, w\}} \left| D^2f(p_\mu(z))[Jp_\mu(z)d, Jp_\mu(z)d] \right|^{1/2}. \tag{6}
\]
Let $\nabla^2f(p_\mu(z)) = Q\text{Diag} \left( \lambda_i \right)_{1 \leq i \leq n} Q^T$ be eigenvalue decomposition of $\nabla^2f(p_\mu(z))$, whose eigenvalues are $\lambda_i$, $1 \leq i \leq n$. From (4), we have $Jp_\mu(z) = Q\text{Diag} \left( \frac{1}{1 + \mu^2 \lambda_i} \right)_{1 \leq i \leq n} Q^T$. By noting that $\frac{\mu^2 \lambda_i}{(1 + \mu^2 \lambda_i)^2} \leq \frac{1}{4}$, we imply that for $d \in \{u, v, w\}$ the following satisfies
\[
\mu^2 \left| D^2f(p_\mu(z))[Jp_\mu(z)d, Jp_\mu(z)d] \right| = d^T Q\text{Diag} \left( \frac{\mu^2 \lambda_i}{(1 + \mu^2 \lambda_i)^2} \right)_{1 \leq i \leq n} Q^T d \leq \frac{1}{4} \|d\|^2 \|Q\|^2 \leq \frac{1}{4}.
\]
Together with (6), we get $\langle \rho, w \rangle \leq \frac{1}{4\mu}, \forall w \in \mathcal{E}, \|w\| = 1$. Hence $\|\rho\| = \max_{w: \|w\| = 1} \langle \rho, w \rangle \leq \frac{1}{4\mu}$. We deduce from (5) that for all $u, v \in \mathcal{E}$ with $\|(u, v)\| = 1$ we have $\|D^2p_\mu(z)[u, v]\| \leq \frac{1}{4\mu}$. \hfill \Box

The following theorem connects some properties of the projection $\Pi_X(\cdot)$ with the behaviour of $Dp_\mu(z)$ at its limit point (which is supposed to exist). Theorem 2 serves as a cornerstone to prove the local convergence of our algorithm.

**Theorem 2** If $\lim_{(z, \mu) \to (z^*, 0)} Dp_\mu(z) = T^*$, then the projector $\Pi_X$ is strictly differentiable at $z^*$ and $D\Pi_X(z^*) = T^*$. Furthermore, if $\|Dp_\mu(z) - T^*\| = O(\|(z - z^*, \mu)\|^{\xi - 1})$ with $\xi > 1$ then $\Pi_X$ satisfies
\[
\|\Pi_X(z) - D\Pi_X(z^*) - \Pi_X(z^*)\| = O(\|z - z^*\|^\xi).
\]

**Proof** Since $\lim_{(z, \mu) \to (z^*, 0)} Dp_\mu(z) = T^*$, we have
\[
\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } 0 < \| (z - z^*, \mu) \| < \delta \Rightarrow \|Dp_\mu(z) - T^*\| < \varepsilon.
\]
Hence, \( \forall \varepsilon > 0, \ \exists \delta > 0, \) for any \( z_1, z_2 \) and \( \mu \) such that \( 0 < \|z_1 - z^*\| < \frac{\delta}{\sqrt{2}}, \)
\[
0 < \|z_2 - z^*\| < \frac{\delta}{\sqrt{2}} \quad \text{and} \quad \mu < \frac{\delta}{\sqrt{2}},
\]
the following holds
\[
\left\| p_\mu(z_2) - p_\mu(z_1) - T^*[z_2 - z_1] \right\| = \left\| \int_0^1 (Dp_\mu(tz_2 + (1-t)z_1) - T^*)dt[z_2 - z_1] \right\| < \varepsilon \|z_2 - z_1\|,
\]
(7)
since \( \|tz_2 + (1-t)z_1 - z^*\| \leq t \|z_2 - z^*\| + (1-t) \|z_1 - z^*\| < \frac{\delta}{\sqrt{2}} \) for all \( 0 \leq t \leq 1. \)
Moreover, if \( \|Dp_\mu(z) - T^*\| = O(\|(z - z^*, \mu)\|^{\xi-1}) \) then \( \forall \varepsilon > 0, \exists \delta > 0, \) for \( 0 < \|z - z^*\| < \frac{\delta}{\sqrt{2}} \) and \( \mu < \frac{\delta}{\sqrt{2}}, \) we get
\[
\left\| p_\mu(z^* + h) - p_\mu(z^*) - T^*[h] \right\| = \left\| \int_0^1 (Dp_\mu(z^* + th) - T^*)dt[h] \right\| = O(\|h\|^\xi) + O(\mu^{\xi-1})\|h\|.
\]
(8)
As \( \mu \) in (7) goes to 0, accompanied with \( p_\mu(z_2) \rightarrow \Pi_X(z_2), \) \( p_\mu(z_1) \rightarrow \Pi_X(z_1), \) it yields that
\[
\left\| \Pi_X(z_2) - \Pi_X(z_1) - T^*[z_2 - z_1] \right\| \leq \varepsilon \|z_2 - z_1\|.
\]
This expression shows that the projection \( \Pi_X \) is strictly differentiable at \( z^* \) and \( D\Pi_X(z^*) = T^*. \) Similarly, as \( \mu \) in (8) goes to 0, the second result follows
\[
\left\| \Pi_X(z^* + h) - \Pi_X(z^*) - T^*[h] \right\| = O(\|h\|^\xi).
\]

**Remark 1** The inverse direction of Theorem 2 does not always hold. If we use a wrong barrier function for the set \( X, \) the limit may not exist even when \( \Pi_X \) is differentiable at \( z^*; \) see an example in Appendix B. However, the question about the existence of a suitable barrier-based smoothing approximation which may guarantee the limit for a given general convex set \( X \) is still open to us.

### 3 An Inexact Non-Interior Continuation Method

In this section, we describe the inexact non-interior continuation for solving Problem (1) and prove its local superlinear (\( \xi \)- order) and global linear convergence. We use a \( \vartheta \)-self-concordant barrier \( f \) to formulate \( p_\mu(\cdot). \) Denote \( \phi_\mu(x, y) = x - p_\mu(x - y) \) and remind that
\[
H_\mu(x, y) = \left( \frac{x - p_\mu(x - y)}{F(x) - y} \right). \quad \text{We use the merit function}
\]
\[
\Psi_\mu(x, y) = \|F(x) - y\| + \|\phi_\mu(x, y)\|,
\]
and define the neighbourhood \( \mathcal{N}(\beta, \mu) = \{(x, y) : \Psi_\mu(x, y) \leq \beta \mu \}. \) Algorithm 1 fully describes our algorithm.

The algorithm starts with Step 0 which can be easily initialized by choosing arbitrary \( w^{(0)} \in \mathbb{E} \times \mathbb{E}, \mu_0 > 0 \) and \( \beta > \max\{\sqrt{\vartheta}, \Psi_{\mu_0}(w^{(0)})/\mu_0\}. \) Centering steps 1–3 are crucial to obtain the global convergence rate, while approximate Newton steps 4–5 are necessary.
Algorithm 1 Inexact non-interior continuation method.

Given \( \sigma, \alpha_1, \alpha_2, \alpha_3 \in (0, 1) \), two sequences \( \{\theta_1^{(k)}\}_{k \geq 0} \subset (0, 1) \) and \( \{\theta_2^{(k)}\}_{k \geq 0} \subset (0, 1) \) such that \( \sigma + \sqrt{2} \sup_k \{\theta_1^{(k)}\} < 1 \). Denote \( w^{(k)} = (x^{(k)}, y^{(k)}) \).

**Step 0** Set \( k = 0 \). Choose \( \mu_0 > 0 \), \( w^{(0)} \in \mathbb{E} \times \mathbb{E} \), \( \beta > \sqrt{\sigma} \) such that \( w^{(0)} \in \mathcal{N}(\beta, \mu_0) \).

**Step 1** (Calculate centering step) 

if \( H_0(w^{(k)}) = 0 \) then 
  terminate, \( w^{(k)} \) is a solution of the VI;
else if \( \Psi_{\mu_k}(w^{(k)}) = 0 \) then 
  set \( \tilde{w}^{(k+1)} = w^{(k)} \) and go to step 3;
else 
  let \( \Delta \tilde{w}^{(k)} \) solve the equation
  \[ -H_{\mu_k}(w^{(k)}) + r_1^{(k)} = DH_{\mu_k}(w^{(k)})[\Delta \tilde{w}^{(k)}], \]  
  where \( \|r_1^{(k)}\| \leq \theta_1^{(k)} \|H_{\mu_k}(w^{(k)})\| \).

end if

**Step 2** (Line search for centering step) Let \( \lambda_k \) be the maximum of \( 1, \alpha_1, \alpha_1^2, \ldots \) such that
\[ \Psi_{\mu_k}(w^{(k)} + \lambda_k \Delta \tilde{w}^{(k)}) \leq (1 - \sigma \lambda_k) \Psi_{\mu_k}(w^{(k)}). \]  
Set \( \tilde{w}^{(k+1)} = w^{(k)} + \lambda_k \Delta \tilde{w}^{(k)} \).

**Step 3** (Reduction based on centering step) Let \( \gamma_k \) be the maximum of the values \( 1, \alpha_2, \alpha_2^2, \ldots \) such that
\[ \tilde{w}^{(k+1)} \in \mathcal{N}(\beta, (1 - \gamma_k)\mu_k). \]  
Set \( \tilde{\mu}_{k+1} = (1 - \gamma_k)\mu_k \).

**Step 4** (Calculate approximate Newton step) Let \( \Delta \tilde{w}^{(k)} \) solve the equation
\[ -H_0(w^{(k)}) + r_2^{(k)} = DH_{\mu_k}(w^{(k)})[\Delta \tilde{w}^{(k)}], \]  
where \( \|r_2^{(k)}\| \leq \theta_2^{(k)} \|H_0(w^{(k)})\| \).
Set \( \tilde{w}^{(k+1)} = w^{(k)} + \Delta \tilde{w}^{(k)} \).

**Step 5** (Reduction based on approximate Newton step) 

if \( \tilde{w}^{(k+1)} \notin \mathcal{N}(\beta, \tilde{\mu}_{k+1}) \) then 
  set \( \mu_{k+1} = \tilde{\mu}_{k+1}, w^{(k+1)} = \tilde{w}^{(k+1)}, k = k + 1 \) and return to step 1;
else if \( H_0(\tilde{w}^{(k+1)}) = 0 \) then 
  terminate, \( \tilde{w}^{(k+1)} \) is a solution of the VI;
else 
  let \( \eta_k \) be the greatest value of \( 1, \alpha_3, \alpha_3^2, \ldots \) such that \( \tilde{w}^{(k+1)} \notin \mathcal{N}(\beta, \eta_k \alpha_3 \tilde{\mu}_{k+1}) \).
  Set \( \mu_{k+1} = \eta_k \tilde{\mu}_{k+1}, w^{(k+1)} = \tilde{w}^{(k+1)}, k = k + 1, \) and return to step 1.

end if

to obtain the local convergence rate. As proved in Theorem 4, when \( k \) is sufficiently large, Algorithm 1 updates \( w^{(k+1)} = \tilde{w}^{(k+1)} \) eventually. Newton (9) and (11) of centering steps and approximate Newton steps respectively are solved inexactly. Parameters \( \theta_1^{(k)} \) and \( \theta_2^{(k)} \) are to control the level of accuracy in solving the Newton equations. As mentioned in the introduction, we remark that finding the inexact solutions satisfying (9) and (11) does not require computing the explicit form of the Hessian matrix \( DH_{\mu_k} \) and its inverse; instead we
can use some Hessian-free Newton type methods or apply some iteration solvers to find the
inexact Newton directions.

3.1 Global linear convergence

We first list assumptions that will be used in sequel.

**Assumption 1** The derivative $D H_{\mu_k}(w^{(k)})$ is nonsingular and there exists a constant $C$ such that $\|D H_{\mu_k}(w^{(k)})^{-1}\| \leq C$ for all $k$.

**Assumption 2** We have

$$\|F(y) - F(x) - DF(x)[y - x]\| = o(\|y - x\|) \text{ for all } x, y \in \mathbb{E}. $$

**Assumption 3** There exist constant $\xi > 1$ and $L > 0$ such that

$$\|F(y) - F(x) - DF(x)[y - x]\| \leq L\|y - x\|^\xi \text{ for all } x, y \in \mathbb{E}.$$  

As mentioned in introduction section, we will prove in Section 3.3 that if $F$ is mono-
tone then $D H_{\mu_k}(w^{(k)})$ is nonsingular. Furthermore, we will extend the sufficient condition obtained in [13] for the uniform boundedness of $\|D H_{\mu_k}(w^{(k)})^{-1}\|$ to general convex set, and refer the readers to [9, 15] for more relaxed conditions that guarantee the boundedness when $X$ is a non-negative orthant, an epigraph of operator norm or an epigraph of nuclear norm. Assumption 3 is typical in global convergence analysis of non-interior continuation methods, see e.g., [13, Proposition 1], [9, Assumption 2]. Assumption 2 will be used to prove the local superlinear convergence, and Assumption 3 will be used to prove the local $\xi$-order convergence of our proposed method. Assumption 2 is satisfied if Assumption 3 is satisfied.

Proposition 3 provides bounds for Newton directions; see its proof in Appendix A.1.

**Proposition 3** Let $w^{(k)}$ be the $k$-th iterate of the Algorithm 1, $\Delta \tilde{w}^{(k)}$ be the solution of (9) and $\Delta \tilde{w}^{(k)}$ be the solution of (11). If Assumption 1 holds true, then

(i) $\|\Delta \tilde{w}^{(k)}\| \leq C(1 + \theta_1)\Psi_{\mu_k}(w^{(k)}) \leq C(1 + \theta_1)\beta_{\mu_k}$, where $\theta_1 = \sup \theta_1^{(k)}$,

(ii) $\|\Delta \tilde{w}^{(k)}\| \leq C(1 + \theta_2)(\beta + \sqrt{\theta})\mu_k$, where $\theta_2 = \sup \theta_2^{(k)}$.

We now apply Proposition 2 together with Proposition 3 to bound the value of $H_{\mu}(x + \lambda \Delta \tilde{x}, y + \lambda \Delta \tilde{y})$.

**Proposition 4** Let $0 \leq \lambda \leq 1$, $r_1 = \left(\frac{r_{1x}}{r_{1y}}\right)$, $(\Delta \tilde{x}, \Delta \tilde{y})$ be the solution of (9).

(i) If the barrier $f(x)$ is $\theta$-self-concordant, then

$$\|\phi_{\mu}(x + \lambda \Delta \tilde{x}, y + \lambda \Delta \tilde{y})\| \leq (1 - \lambda)\|\phi_{\mu}(x, y)\| + \frac{1}{4\mu}\|\|\Delta \tilde{x}, \Delta \tilde{y}\|^2 + \lambda \|r_{1x}\|^2.$$  

(ii) If Assumption 1 holds and the map $F$ satisfies Assumption 2 then

$$\|F(x + \lambda \Delta \tilde{x}) - (y + \lambda \Delta \tilde{y})\| \leq (1 - \lambda)\|F(x) - y\| + o(\lambda\|\Delta \tilde{x}, \Delta \tilde{y}\|) + \lambda \|r_{1y}\|.$$  


Proof (i) From (9), we get
\[ x - p_\mu(x - y) = -(I - Dp_\mu(x - y))(\Delta \tilde{x}) - Dp_\mu(x - y)[\Delta \tilde{y}] + r_{1x} \]
\[ = - (\Delta \tilde{x} - Dp_\mu(x - y)[\Delta \tilde{x} - \Delta \tilde{y}] - r_{1x}). \]
Therefore,
\[ \phi_\mu(x + \lambda \Delta \tilde{x}, y + \lambda \Delta \tilde{y}) - (1 - \lambda)\phi_\mu(x, y) \]
\[ = x + \lambda \Delta \tilde{x} - p_\mu(x - y + \lambda(\Delta \tilde{x} - \Delta \tilde{y})) - (x - p_\mu(x - y)) - \]
\[ \lambda(\Delta \tilde{x} - Dp_\mu(x - y)[\Delta \tilde{x} - \Delta \tilde{y}] - r_{1x}) \]
\[ = - p_\mu(x - y + \lambda(\Delta \tilde{x} - \Delta \tilde{y})) + p_\mu(x - y) + \lambda Dp_\mu(x - y)[\Delta \tilde{x} - \Delta \tilde{y}] + \lambda r_{1x}. \]

Denote \( \hat{\theta} = p_\mu(x - y + \lambda(\Delta \tilde{x} - \Delta \tilde{y})) - p_\mu(x - y) - \lambda Dp_\mu(x - y)[\Delta \tilde{x} - \Delta \tilde{y}] \). Using Lagrange’s remainder for first order Taylor polynomial, for all \( v \in \mathbb{E} \) we have \( \epsilon \in (0, 1) \) such that
\[ \left\{ v, \hat{\theta} \right\} = \left\{ v, \frac{1}{2}D^2 p_\mu(x - y + \epsilon \lambda(\Delta \tilde{x} - \Delta \tilde{y})) [\lambda(\Delta \tilde{x} - \Delta \tilde{y}), \lambda(\Delta \tilde{x} - \Delta \tilde{y})] \right\} \leq \frac{1}{8\mu} \lambda^2 \|v\| \|\Delta \tilde{x} - \Delta \tilde{y}\|^2, \]
where we have applied Proposition 2 for the last inequality. Hence
\[ \|\hat{\theta}\|^2 = \left\langle \hat{\theta}, \hat{\theta} \right\rangle \leq \frac{1}{8\mu} \lambda^2 \|\Delta \tilde{x} - \Delta \tilde{y}\|^2, \]
which implies \( \|\hat{\theta}\| \leq \frac{1}{8\mu} \lambda^2 \|\Delta \tilde{x} - \Delta \tilde{y}\|^2 \leq \frac{1}{4\mu} \lambda^2 \|\Delta \tilde{x}, \Delta \tilde{y}\|^2. \) Together with (12), we get (i).

(ii) From (9), we have \( F(x) - y = -DF(x)[\Delta \tilde{x}] + \Delta \tilde{y} + r_{1y}. \) This equality together with Assumption 2 yields
\[ F(x + \lambda \Delta \tilde{x}) - (y + \lambda \Delta \tilde{y}) - (1 - \lambda)(F(x) - y) \]
\[ = F(x + \lambda \Delta \tilde{x}) - (y + \lambda \Delta \tilde{y}) - (F(x) - y) + \lambda(-DF(x)[\Delta \tilde{x}] + \Delta \tilde{y} + r_{1y}) \]
\[ = F(x + \lambda \Delta \tilde{x}) - D\lambda F(x)[\lambda \Delta \tilde{x}] - F(x) + \lambda r_{1y} \]
\[ = \sigma(\lambda \|\Delta \tilde{x}\|) + \lambda r_{1y} \]
\[ = \sigma(\lambda \|\Delta \tilde{x}, \Delta \tilde{y}\|) + \lambda r_{1y}. \]

Finally, we establish the positive lower bounds for \( \lambda_k \) and \( y_k \) that are necessary to prove the global convergence in Theorem 3.

**Proposition 5** Let \( 0 \leq \lambda \leq 1. \) We consider \( \lambda_k \) and \( y_k \) in Step 2 and Step 3 of Algorithm 1 respectively. If Assumptions 1 and 2 are satisfied, then

(i) There exists \( \tilde{\lambda} \) such that \( \lambda_k \geq \alpha_1 \tilde{\lambda}, \) and

(ii) \( y_k \geq \alpha_2 \tilde{y} \) where \( \tilde{y} = \min \left\{ 1, \frac{\beta \sigma \lambda}{\beta + \sqrt{\delta}} \right\}. \)
Proof (i) By Proposition 4, we get
\[ \Psi_{\mu_k}(w^{(k)} + \lambda \Delta \tilde{w}^{(k)}) \leq (1 - \lambda) \Psi_{\mu_k}(w^{(k)}) + o(\lambda \| \Delta \tilde{w}^{(k)} \|) + \frac{1}{4\mu_k} \lambda^2 \| \Delta \tilde{w}^{(k)} \|^2 + \lambda \sqrt{2} \| r_1^{(k)} \|. \]

Using Proposition 3(i) and remind that \( \| r_1^{(k)} \| \leq \theta_1^{(k)} \| H_{\mu_k}(w^{(k)}) \| \), we deduce that there exists a function \( \sigma(\cdot) \) such that \( \lim_{x \to 0} \frac{\sigma(x)}{x} = 0 \) and
\[ \Psi_{\mu_k}(w^{(k)} + \lambda \Delta \tilde{w}^{(k)}) \leq (1 - \lambda) \Psi_{\mu_k}(w^{(k)}) + \sigma(\lambda \Psi_{\mu_k}(w^{(k)})) + \frac{1}{4} \lambda^2 C^2 (1 + \theta_1)^2 \beta \Psi_{\mu_k}(w^{(k)}) + \lambda \sqrt{2} \theta_1 \Psi_{\mu_k}(w^{(k)}). \] (13)

Note that \( \Psi_{\mu_k}(w^{(k)}) \leq \beta \mu_k \leq \beta \mu_0 \), i.e., \( \Psi_{\mu_k}(w^{(k)}) \) is bounded by \( \beta \mu_0 \). Hence, there exists \( \bar{\lambda} \geq 0 \) such that for all \( 0 \leq \lambda \leq \bar{\lambda} \), we have
\[ \frac{\sigma(\lambda \Psi_{\mu_k}(w^{(k)}))}{\lambda \Psi_{\mu_k}(w^{(k)})} + \frac{1}{4} \lambda^2 C^2 (1 + \theta_1)^2 \beta \lambda \leq 1 - \sigma - \sqrt{2} \theta_1, \]
which together with (13) leads to \( \Psi_{\mu_k}(w^{(k)} + \lambda \Delta \tilde{w}^{(k)}) \leq (1 - \sigma \lambda) \Psi_{\mu_k}(w^{(k)}) \). Therefore, for all \( 0 \leq \lambda \leq \bar{\lambda} \) the line search criteria (10) for centering step holds true. Then we get (i).

(ii) We note that
\[
\Psi((1-\gamma)\mu_k)(\tilde{w}^{(k+1)}) - \Psi_{\mu_k}(\tilde{w}^{(k+1)})
= \| \phi((1-\gamma)\mu_k)(\tilde{w}^{(k+1)}) - \phi_{\mu_k}(\tilde{w}^{(k+1)}) \|
\leq \| \phi((1-\gamma)\mu_k)(\tilde{w}^{(k+1)}) - \phi_{\mu_k}(\tilde{w}^{(k+1)}) \|
= \| p((1-\gamma)\mu_k)(\tilde{x}^{(k+1)} - \tilde{y}^{(k+1)}) - p_{\mu_k}(\tilde{x}^{(k+1)} - \tilde{y}^{(k+1)}) \|. \]

Using the Lipschitz continuity of \( p_{\mu} \) (see Theorem 1), we have
\[
\Psi((1-\gamma)\mu_k)(\tilde{w}^{(k+1)}) \leq \Psi_{\mu_k}(\tilde{w}^{(k+1)}) + \sqrt{\sigma} \gamma \mu_k. \] (14)

On the other hand, as proved above, we have \( \Psi_{\mu_k}(\tilde{w}^{(k+1)}) \leq (1 - \sigma \tilde{\lambda}) \Psi_{\mu_k}(w^{(k)}) \). Hence, using the fact \( \Psi_{\mu_k}(w^{(k)}) \leq \beta \mu_k \), we derive from (14) that
\[ \Psi((1-\gamma)\mu_k)(\tilde{w}^{(k+1)}) \leq (1 - \sigma \tilde{\lambda}) \Psi_{\mu_k}(w^{(k)}) + \sqrt{\sigma} \gamma \mu_k \leq \beta (1 - \sigma \tilde{\lambda} + \sqrt{\sigma} \gamma) \mu_k. \]

Finally, we get \( \Psi((1-\gamma)\mu_k)(\tilde{w}^{(k+1)}) \leq \beta (1 - \gamma) \mu_k \), i.e., \( \tilde{w}^{(k+1)} \in \mathcal{N}(\beta, (1 - \gamma) \mu_k) \) for all \( 0 \leq \gamma \leq \tilde{\gamma} \). The result (ii) follows then.

Remark 2 If we assume Assumption 3 holds, then we get the following inequality, which is stronger than the inequality in Proposition 4(ii),
\[
\| F(x + \lambda \Delta \tilde{x}) - (y + \lambda \Delta \tilde{y}) \| \leq (1 - \lambda) \| F(x) - y \| + L \lambda \xi \| \Delta \tilde{x} \| \xi + \lambda \| r_{1y} \|
\leq (1 - \lambda) \| F(x) - y \| + L \lambda \xi \| (\Delta \tilde{x}, \Delta \tilde{y}) \| \xi + \lambda \| r_{1y} \|. \]
Consequently, we obtain the following inequality which is stronger than Eq. 13

\[
\Psi_{\mu_k}(w^{(k)} + \lambda \Delta \tilde{w}^{(k)}) \\
\leq (1 - \lambda) \Psi_{\mu_k}(w^{(k)}) + \left( L \lambda \xi C(1 + \theta_1)\beta \mu_k \right)^{\xi-1} \\
+ \frac{1}{4} \lambda^2 C(1 + \theta_1)\beta C(1 + \theta_1) \Psi_{\mu_k}(w^{(k)}) + \lambda \sqrt{2} \theta_1 \Psi_{\mu_k}(w^{(k)}) \\
\leq (1 - \lambda) \Psi_{\mu_k}(w^{(k)}) \\
+ \lambda \left\{ \lambda^a (L C(1 + \theta_1)\beta \mu_0)^{\xi-1} + \frac{1}{4} C(1 + \theta_1)\beta C(1 + \theta_1) + \sqrt{2} \theta_1 \right\} \Psi_{\mu_k}(w^{(k)}),
\]

where \( a = \min \{ \xi - 1, 1 \} \). Hence the value of \( \tilde{\lambda} \) in Proposition 5 (i) is chosen to be

\[
\tilde{\lambda} = \min \left\{ 1, a \frac{1 - \sigma - \sqrt{2} \theta_1}{C(1 + \theta_1) \left( L C(1 + \theta_1)\beta \mu_0 \right)^{\xi-1} + \frac{1}{4} C(1 + \theta_1)\beta} \right\}.
\]

The following theorem states the global linear convergence of our algorithm. Its proof is similar to that of [9, Theorem 1]; we hence omit the details here.

**Theorem 3** Assuming Assumption 1 and Assumption 2 are satisfied, then we have

(i) For all \( k \geq 0 \), \( \mu_{k+1} \leq (1 - \alpha_2 \tilde{\gamma}) \mu_k \) and \( \mu_k \leq \mu_0 (1 - \alpha_2 \tilde{\gamma})^k \), where \( \tilde{\gamma} \) is defined in Proposition 5.

(ii) The sequence \( \{ H_0(w^{(k)}) \} \) converges to 0 R-linearly.

(iii) The sequence \( \{ w^{(k)} \} \) is a Cauchy sequence converging to a solution \( w^* = (x^*, y^*) \) of the VI (1).

### 3.2 Local superlinear/\( \xi \)-order convergence

We denote \( \varepsilon^{(k)} = x^{(k)} - y^{(k)} \) and \( z^* = x^* - y^* \), where \( (x^*, y^*) \) is any limit point of the sequence \( \{ (x^{(k)}, y^{(k)}) \} \). We now use Theorem 2 to establish the local convergence of Algorithm 1.

We will see in Section 4 that the condition \( \lim_{(z, \mu) \to (z^*, 0)} Dp_\mu(z) = T^* \) is satisfied in many specific convex sets.

**Theorem 4** Suppose Assumption 1 and Assumption 2 hold. If the derivative \( Dp_\mu(z) \) converges to a linear operator \( T^* \) when \( (z, \mu) \) goes to \( (z^*, 0) \), Algorithm 1 generates infinite sequence \( \{ w^{(k)} \} \) and \( r_2^{(k)} = o \left( \| H_0(w^{(k)}) \| \right) \), then the sequence \( \{ w^{(k)} \} \) converges superlinearly to \( (x^*, y^*) \).

Moreover, if \( \| Dp_\mu(z) - T^* \| = O \left( \| z - z^*, \mu \|^{\xi-1} \right) \) and \( r_2^{(k)} = O \left( \| H_0(w^{(k)}) \|^{\xi} \right) \) with \( \xi > 1 \) as given in Assumption 3, then the convergence is of \( \xi \)-order.

**Proof** By Theorem 2, we can deduce that \( T^* = D\Pi_X(z^*) \).
First, we prove that $\|\hat{w}^{(k+1)} - w^*\| = o\left(\|w^{(k)} - w^*\|\right)$. By Eq. 11,
\[
\|w^{(k)} + \Delta \hat{w}^{(k)} - w^*\|
= \left\| w^{(k)} - DH_{\mu_k}(w^{(k)})^{-1}(H_0(w^{(k)}) - r_2^{(k)}) - w^* \right\|
\leq \left\| DH_{\mu_k}(w^{(k)})^{-1}\right\| \left\| DH_{\mu_k}(w^{(k)})(w^{(k)} - w^*) - H_0(w^{(k)}) + r_2^{(k)} \right\|
\leq C \left( q_1 + q_2 + \left\| r_2^{(k)} \right\| \right),
\]
where
\[
q_1 = \left\| DF(x^{(k)})[x^{(k)} - x^*] - [y^{(k)} - y^*] - (F(x^{(k)}) - y^{(k)}) \right\|
= \left\| F(x^{(k)}) - DF(x^{(k)})[x^{(k)} - x^*] - F(x^*) \right\|
= o \left( \left\| x^{(k)} - x^* \right\| \right) \quad \text{by Assumption 2}
= o \left( \left\| w^{(k)} - w^* \right\| \right),
\]
and
\[
q_2 = \left\| (I - DP_{\mu_k}z^{(k)})[x^{(k)} - x^*] + DP_{\mu_k}z^{(k)}[y^{(k)} - y^*] - (x^{(k)} - \Pi_K(z^{(k)})) \right\|
\leq \left\| \Pi_K(z^{(k)}) - \Pi_K(z^*) \right\|
+ \left\| DP_{\mu_k}(z^{(k)})[y^{(k)} - y^*] \right\|
+ \left\| (DP_{\mu_k}(z^{(k)}))[z^{(k)} - z^*] \right\|. \quad (16)
\]
By Theorem 2, we get $q_2 = o \left( \left\| z^{(k)} - z^* \right\| \right) = o \left( \left\| w^{(k)} - w^* \right\| \right)$. Together with $r_2^{(k)} = o \left( \left\| H_0(w^{(k)}) \right\| \right)$, we deduce that
\[
\left\| w^{(k)} + \hat{w} - w^* \right\| = o \left( \left\| w^{(k)} - w^* \right\| \right). \quad (17)
\]
where $\tau_k$ is a sequence converging to 0. Furthermore,
\[
\left\| w^{(k)} - w^* \right\| = \left\| w^{(k)} + \hat{w} - w^* - \hat{w} \right\| \leq \left\| w^{(k)} + \hat{w} - w^* \right\| + \left\| \Delta \hat{w} \right\|.
\]
Hence,
\[
\left\| w^{(k)} - w^* \right\| \leq \tau_k \left\| w^{(k)} - w^* \right\| + \left\| \Delta \hat{w} \right\|.
\]
Applying Proposition 3 (ii), we have
\[
\left\| w^{(k)} - w^* \right\| \leq \tau_k \left\| w^{(k)} - w^* \right\| + C(1 + \theta_2)(\beta + \sqrt{\bar{\theta}})\mu_k
\leq \frac{1}{2} \left\| w^{(k)} - w^* \right\| + C(1 + \theta_2)(\beta + \sqrt{\bar{\theta}})\mu_k
\]
for sufficiently large $k$. Hence for sufficiently large $k$ the following inequality is satisfied
\[
\left\| w^{(k)} - w^* \right\| \leq 2C(1 + \theta_2)(\beta + \sqrt{\bar{\theta}})\mu_k. \quad (18)
\]
Now we prove that $\hat{w}^{(k+1)} \in N(\beta, (1 - \gamma_k)\mu_k)$ for sufficiently large $k$, then $w^{(k)} = \hat{w}^{(k)}$ eventually. Using the property $\sqrt{2} \| (a, b) \| \geq \| a \| + \| b \|$, similarly to (48) we can prove
\[
\Psi(1 - \gamma_k)\mu_k \left( \hat{w}^{(k+1)} \right) \leq \sqrt{2} \| H_0(\hat{w}^{(k+1)}) \| + (1 - \gamma_k)\mu_k \sqrt{\bar{\theta}}. \quad (19)
\]
Using the Lipschitz continuity of $H_0(w)$ near $w^*$ with Lipschitz constant $L_1$, Expression (17) and Inequality (18) we then get
\[
\Psi(1 - \gamma_k)\mu_k \left( \hat{w}^{(k+1)} \right)
\leq \sqrt{2}L_1 \left\| \hat{w}^{(k+1)} - w^* \right\| + (1 - \gamma_k)\mu_k \sqrt{\bar{\theta}}
\leq \sqrt{2}L_1 \tau_k \left\| w^{(k)} - w^* \right\| + (1 - \gamma_k)\mu_k \sqrt{\bar{\theta}}
\leq \sqrt{2}L_1 \tau_k \left[ C(1 + \theta_2)(\beta + \sqrt{\bar{\theta}})\mu_k (1 - \gamma_k)\mu_k \beta - (1 - \gamma_k)\mu_k (\beta - \sqrt{\bar{\theta}}) \right]
\leq (1 - \gamma_k)\mu_k \beta + \mu_k (2\sqrt{2}L_1 \tau_k C(1 + \theta_2)(\beta + \sqrt{\bar{\theta}}) - (1 - \alpha_2)(\beta - \sqrt{\bar{\theta}}))
where we have used the fact \( y_k \leq \alpha_2 \) from Step 3 of Algorithm 1 (if \( y_k = 1 \) then the algorithm terminates finitely). Finally, for sufficiently large \( k \) the value \( 2\sqrt{L_1 \tau_k C(1 + \theta_2)(\beta + \sqrt{\delta}) - (1 - \alpha_2)(\beta - \sqrt{\delta}) \) is negative number since \( \tau_k \to 0 \) and \( \beta > \sqrt{\delta} \) was chosen; we hence get
\[
\Psi((1-y_k)\mu_k(\hat{w}^{(k+1)}) \leq B(1-y_k)\mu_k.
\]

For locally \( \xi \)-order convergence, we use the result of [9, Lemma 7] which states that if the algorithm generates the \( k \)-th iterate by the approximate Newton step then \( \mu_k = O(\|w^{(k)} - w^*\|) \). Indeed, based on the approximate Newton step, we have \( w^{(k)} \notin N(\beta, \alpha_3 \mu_k) \). Moreover, similarly to (19), we get \( \Psi(\alpha_3 \mu_k(w^{(k)}) \leq \sqrt{2} \|H_0(w^{(k)})\| + \alpha_3 \mu_k \sqrt{\delta} \). Hence
\[
\beta \alpha_3 \mu_k < \Psi(\alpha_3 \mu_k(w^{(k)}) \leq \sqrt{2} \|H_0(w^{(k)})\| + \alpha_3 \mu_k \sqrt{\delta}.
\]

This implies
\[
\mu_k \leq \frac{\sqrt{2} \|H_0(w^{(k)})\|}{\alpha_3 (\beta - \sqrt{\delta})} \leq \frac{\sqrt{2} L_1 \|w^{(k)} - w^*\|}{\alpha_3 (\beta - \sqrt{\delta})}.
\]

Appealing to this fact, if \( \|Dp_\mu(z) - T^*\| = O(\|z - z^*\|, \mu)\) then from Theorem 2 and Expression (16) we yield \( q_2 = O(\|w^{(k)} - w^*\|) \). From Expression (15) and Assumption 3, we have \( q_1 = O(\|w^{(k)} - w^*\|) \). All together with \( q_2 = O(\|w^{(k)} - w^*\|) \), we deduce \( \|\hat{w}^{(k+1)} - w^*\| = O(\|w^{(k)} - w^*\|) \).

### 3.3 Sufficient conditions for Assumption 1

Assumption 1 is crucial in obtaining the global linear convergence of our algorithm. The following proposition provides some sufficient conditions that make Assumption 1 satisfied for general closed convex sets.

**Proposition 6**

(i) If \( F \) is monotone then \( DH_\mu(x, y) \) is nonsingular for all \( \mu > 0 \) and \( (x, y) \in \mathbb{E} \times \mathbb{E} \).

(ii) If \( F \) is strongly monotone and assume that \( \{DF(x^{(k)})\}_{k \geq 0} \) is bounded, then Assumption 1 holds.

**Proof**

(i) We remind that \( J_{p_\mu}(z) = [I + \mu^2 \nabla^2 f(p(z))]^{-1} \) (see Formula (4)). We also note that \( Dp_\mu(z)u = J_{p_\mu}(z)u, \forall u \in \mathbb{E} \). Then it is easy to see that \( 0 < Dp_\mu(z) < I \) for all \( z \in \mathbb{E} \). Denote \( Dp_\mu(x - y) = \tilde{D} \). We have
\[
DH_\mu(x, y) = \begin{pmatrix}
I - \tilde{D} & \tilde{D} \\
DF(x) & -I
\end{pmatrix}.
\]

The system of linear equations \( DH_\mu(x, y)((u, v)) = 0 \) is equivalent to
\[
\begin{align*}
(I - \tilde{D})[u] + \tilde{D}[v] &= 0, \\
DF(x)[u] - v &= 0,
\end{align*}
\]

which can be rewritten as
\[
\begin{align*}
(I - \tilde{D} + \tilde{D}(DF(x))[u] = 0, \\
v &= DF(x)[u].
\end{align*}
\]

It follows from \( 0 < \tilde{D} < I \) and monotonicity of \( F \) that \( (I - \tilde{D})\tilde{D} > 0 \) and \( \tilde{D}(DF(x))\tilde{D} > 0 \). Therefore,
\[
(I - \tilde{D} + \tilde{D}(DF(x))[u] = 0 \Leftrightarrow (I - \tilde{D} + \tilde{D}(DF(x))\tilde{D}\tilde{D}^{-1}[u] = 0
\]

\[
\Leftrightarrow ((I - \tilde{D})\tilde{D} + \tilde{D}(DF(x))\tilde{D})\tilde{D}^{-1}[u] = 0
\]

\[
\Leftrightarrow \tilde{D}^{-1}[u] = 0
\]

\[
\Leftrightarrow u = 0.
\]
We deduce that $D H_{\mu}(x, y)[(u, v)] = 0$ has the unique solution $(u, v) = (0, 0)$. The result (i) follows then.

(ii) The following proof is inspired by the proof of [17, Proposition 4.4].

Denote $M = D F(x^{(k)}), \tilde{D}^{(k)} = D p_{\mu}(x^{(k)} - y^{(k)})$ and $\tilde{D}^{(k)}[u] = \tilde{u}$. Since $F$ is strongly monotone, then there exists a constant $\varrho$ such that $\langle M d, d \rangle = \langle M^T d, d \rangle \geq \varrho \|d\|^2$ for all $d \in \mathbb{R}$. Furthermore, we are considering the barrier $f$ with positive semidefinite $\nabla^2 f$, hence $\langle M^T d + \mu \nabla^2 f(x^{(k)} - y^{(k)}) d, d \rangle \geq \varrho \|d\|^2, \forall d \in \mathbb{R}$. By Cauchy-Schwarz inequality,

$$\left\langle M^T d + \mu \nabla^2 f(x^{(k)} - y^{(k)}) d, d \right\rangle \leq \left\| M^T d + \mu \nabla^2 f(x^{(k)} - y^{(k)}) d \right\| \|d\|,$$

we then deduce $\|M^T d + \mu \nabla^2 f(x^{(k)} - y^{(k)}) d\| \geq \varrho \|d\|$ for all $d \in \mathbb{R}$.

Let $m_F$ be the constant such that $\|D F(x^{(k)})\| \leq m_F$ for all $k \geq 0$. For arbitrary $u \in \mathbb{R}$, we consider 2 cases

Case 1: $\|\tilde{u}\| = 1 + \frac{1}{m_F + \varrho} \|u\|$. We note that

$$\tilde{D}^{(k)}[u] = \tilde{u} \iff \left(I + \mu \nabla^2 f(x^{(k)} - y^{(k)})\right) [\tilde{u}] = u.$$

Then we have

$$\left\| \left(I - \tilde{D}^{(k)} + M^T \tilde{D}^{(k)} \right)[u] \right\| = \left\| \tilde{u} + \mu \nabla^2 f(x^{(k)} - y^{(k)})\tilde{u} - \tilde{u} + M^T \tilde{u} \right\|
\geq \varrho \|\tilde{u}\| \geq \frac{\varrho}{1 + m_F + \varrho} \|u\|.$$

Case 2: $\|\tilde{u}\| < 1 + \frac{1}{m_F + \varrho} \|u\|$. We have

$$\left\| u - \tilde{D}^{(k)}[u] + M^T \tilde{D}^{(k)}[u] \right\| \geq \|u\| - \|\tilde{u}\| - \left\| M^T \tilde{u} \right\|
\geq \|u\| - (1 + m_F) \|\tilde{u}\|
\geq \frac{\varrho}{1 + m_F + \varrho} \|u\|.$$

We have proved that $\left\| u - \tilde{D}^{(k)} u + M^T \tilde{D}^{(k)} u \right\| \geq \frac{\varrho}{1 + m_F + \varrho} \|u\|$ for all $u \in \mathbb{R}$. On the other hand, we note that the smallest singular value of arbitrary linear operator $L$, which equals to $\min_{u \neq 0} \left\{ \frac{\|L u\|}{\|u\|} \right\}$, is invariant under taking transpose. Hence, for all $u \in \mathbb{R}$ we have

$$\left\| u - \tilde{D}^{(k)} u + \tilde{D}^{(k)} Mu \right\| \geq \frac{\varrho}{1 + m_F + \varrho} \|u\|.$$

For fixed $(r, s) \in \mathbb{R} \times \mathbb{R}$, let $D H_{\mu}(x^{(k)}, y^{(k)})^{-1}[(r, s)] = (u, v)$. It follows from $D H_{\mu}(x^{(k)}, y^{(k)})(u, v)] = (r, s)$ that $(I - \tilde{D}^{(k)})[u] + \tilde{D}^{(k)} v = r, \quad DF(x^{(k)})[u] - v = s$. Therefore,

$$\begin{cases} (I - \tilde{D}^{(k)} + \tilde{D}^{(k)} DF(x^{(k)})) [u] = r + \tilde{D}^{(k)} [s], \\ v = DF(x^{(k)})[u] - s \end{cases}$$

(21)
From the first equation of (21) and Inequality (20), we deduce
\[ \|u\| \leq \frac{1 + m_F + \varrho}{\varrho} \|r + 5_s\| = O(\|(r, s)\|). \]
And the second equation of (21) implies that
\[ \|v\| = \|DF(x^{(k)})u - s\| \leq m_F \|u\| + \|s\| = O(\|(r, s)\|). \]
Consequently, \(\|DH_\mu(x^{(k)}, y^{(k)})^{-1}\|\) is uniformly bounded.

4 Application to Specific Convex Sets

In this section, we use notation \((x^*, y^*)\) and \(z^*\) as in Section 3.2. We now apply our result in Theorem 4 to specific convex sets. In particular, we choose appropriate barrier functions to formulate the corresponding barrier-based smoothing approximations \(p_\mu(\cdot)\) of the projection onto the specific convex sets, and we then verify the condition \(\|DP_\mu(z) - T^*\| = O(\|(z - z^*, \mu)\|)\) in Theorem 4 so that the local quadratic convergence of Algorithm 1 is assured by Theorem 4. We also prove in this section that when \(X\) is a non-negative orthant, a positive semidefinite cone, an epigraph of matrix operator norm or an epigraph of matrix nuclear norm, then differentiability of the projector \(\Pi_X(\cdot)\) at \(z^*\) is equivalent to strict complementarity of \((x^*, y^*)\).

To construct the smoothing approximation, throughout this section, we use the following \(\vartheta\)-self-concordant barriers \(f\) for \(X\).

1. When \(X\) is nonnegative orthant \(\mathbb{R}^n_+\), we use \(f(x) = -\sum_{i=1}^n \log x_i\).
2. When \(X\) is positive semidefinite cone \(\mathbb{S}^n_+\), we use \(f(x) = -\log \det x\).
3. When \(X\) is polyhedral set \(P(A, b) = \{x \in \mathbb{R}^n : Ax \geq b\}\) for some matrix \(A \in \mathbb{R}^{m \times n}\) and vector \(b \in \mathbb{R}^n\), we use \(f(x) = -\sum_{i=1}^m \log(A_i x - b_i)\).
4. When \(X\) is epigraph of matrix operator norm cone
   \[ K_{m,n} = \{(t, x) \in \mathbb{R}_+ \times \mathbb{R}^{m \times n} : t \geq \|x\|, \quad \text{with} \quad m \leq n, \] we use (see [28, Part 5.4.6]) \(f(t, x) = -\log \det \begin{pmatrix} tI_n & x^T \\ x & tI_m \end{pmatrix}\).
5. When \(X\) is epigraph of matrix nuclear norm
   \[ K_{m,n}^\| = \{(t, x) \in \mathbb{R}_+ \times \mathbb{R}^{m \times n} : t \geq \|x\|, \quad \text{with} \quad m \leq n, \] we use the modified Fenchel barrier function
   \[ f^\| (s_\alpha, s) = -\inf \{s_\alpha x_\alpha + \text{Tr}(s^T x) + f(x_\alpha, x) : (x_\alpha, x) \in K_{m,n} \}. \]

We use [31, Definition 2] for the definition of strict complementary solutions of VI. Specifically, considering the VI (1) over a convex cone \(K\), a pair of feasible primal-dual solution \((x, y)\) of (1) is strictly complementary if \(x \in \text{relint}(\mathcal{F})\) and \(y \in \text{relint}(\mathcal{F}^\Delta)\) for some face \(\mathcal{F}\) of \(K\). If \((0, y)\) is feasible for \(y \in \text{int}(K^\|)\) or \((x, 0)\) is feasible for \(x \in \text{int}(K)\) then the corresponding pair is also called strictly complementary. Here \(\mathcal{F}^\Delta = \{v \in K^\| : \forall u \in \mathcal{F}, \langle v, u \rangle = 0\}\) is the complementary face of \(\mathcal{F}\). The following theorem states our main result.
Theorem 5 When $X$ is a non-negative orthant, a positive semidefinite cone, a polyhedral set, an epigraph of matrix operator norm or an epigraph of matrix nuclear norm, then the following statements are equivalent.

(i) The projector $\Pi_X(\cdot)$ is differentiable at $z^*$.
(ii) The derivative $Dp_\mu(z)$ converges to $D\Pi(z^*)$ when $(z, \mu)$ converges to $(z^*, 0)$, and we then have

$$\|Dp_\mu(z) - D\Pi_X(z^*)\| = O(\|(z - z^*, \mu)\|).$$

When $X$ is a non-negative orthant, a positive semidefinite cone, an epigraph of matrix operator norm or an epigraph of matrix nuclear norm, then the above statements are further equivalent to strict complementarity of $(x^*, y^*)$.

Although Theorem 5 recovers the local quadratic convergence of non-interior continuation method for non-negative orthant and positive semidefinite cone, it is worth noting that Theorem 5 provides a new technique in proving these local convergence rate. We leave the proof of Theorem 5 for these cases (non-negative orthant and positive semidefinite cone) to Appendix A.2. We now provide the proofs for the remaining cases of $X$. We see that Theorem 2 already shows that Statement (i) is a consequence of Statement (ii). We now prove the inverse direction and prove the equivalence to strict complementarity of $(x^*, y^*)$ case by case.

4.1 Polyhedral Set $P(A, b)$

Before going to the detailed proof for the case of polyhedral sets, we provide some properties that will be used later.

4.1.1 Some preliminaries for polyhedral set

Denote

$$\mathcal{I}_0 = \{\mathcal{I} \subset \{1, \ldots, m\} : \exists x \in \mathbb{R}^n \text{ such that } A_i x = b_i \forall i \in \mathcal{I} \text{ and } A_i x > b_i \forall i \not\in \mathcal{I}\}.$$ 

Proposition 7 summarizes some results from [22, Part 4.1] and [29, Lemma 5].

Proposition 7 (i) Each nonempty face $\mathcal{F}_I$ of $P(A, b)$ defines an index set $\mathcal{I} \in \mathcal{I}_0$, and vice versa:

$$\mathcal{F}_I = \{x \in P(A, b) : A_i x = b_i, \forall i \in \mathcal{I}\}.$$

(ii) For each $x \in \text{relint}(\mathcal{F}_I)$, the normal cone of $P(A, b)$ at $x$ is defined by

$$\mathcal{N}_I = \text{cone}\{-A_i^T, i \in \mathcal{I}\},$$

which is independent of $x$ and depends only on the face $\mathcal{F}_I$.

(iii) It holds that

$$\bigcup_{\mathcal{I} \in \mathcal{I}_0} \mathcal{F}_I + \mathcal{N}_I = \mathbb{R}^n.$$

Moreover, if $\mathcal{I}, \mathcal{J}$ are distinct index sets in $\mathcal{I}_0$ such that

$$P_{\mathcal{I}, \mathcal{J}} = (\mathcal{F}_I + \mathcal{N}_I) \cap (\mathcal{F}_J + \mathcal{N}_J) \neq \emptyset,$$

then

(a) $P_{\mathcal{I}, \mathcal{J}} = (\mathcal{F}_I \cap \mathcal{F}_J) + (\mathcal{N}_I + \mathcal{N}_J)$; and

(b) $P_{\mathcal{I}, \mathcal{J}}$ is a common face of $\mathcal{F}_I + \mathcal{N}_I$ and $\mathcal{F}_J + \mathcal{N}_J$. 

© Springer
(iv) For each $x \in \mathcal{F}_I + \mathcal{N}_I$, we have $\Pi_{P(A,b)}(x) = \Pi_{S_I}(x)$, where

$$S_I = \text{aff}(\mathcal{F}_I) = \{ x \in \mathbb{R}^n : A_i x = b_i, \forall i \in I \}.$$  

The projector is directionally differentiable everywhere

$$\Pi'_{P(A,b)}(x; d) = \Pi_C(d)$$

where $C = C(x; P(A,b)) = T(\bar{x}; P(A,b)) \cap (\bar{x} - x)^\perp$, with $\bar{x} = \Pi_{P(A,b)}(x)$, is the critical cone of $P(A,b)$ at $x$. And the projector is Fréchet-differentiable at $x$ if and only if $x \in \text{int}(\mathcal{F}_I + \mathcal{N}_I)$.

Gradient and Hessian of the barrier $f(x) = -\sum_{i=1}^{m} \log(A_i x - b_i)$ is

$$\nabla f(x) = -\sum_{i=1}^{m} \frac{1}{A_i x - b_i} A_i^T, \quad \nabla^2 f(x) = \sum_{i=1}^{m} \frac{1}{(A_i x - b_i)^2} A_i^T A_i.\$$

The barrier-based smoothing approximation $p_\mu(z) = x$ is then defined by

$$x - \mu^2 \sum_{i=1}^{m} \frac{1}{A_i x - b_i} A_i^T = z$$

(22)

**Proposition 8** Let $z^*$ be a differentiable point of the projector $\Pi_{P(A,b)}$ and $\mathcal{F}_{I^*}$ be its neighbor face, i.e., $z^* \in \mathcal{F}_{I^*} + \mathcal{N}_{I^*}$. Let $(z, \mu)$ converge to $(z^*, 0)$ and $x = p_\mu(z)$. Then for each $i \in I^*$, there exist a positive constant $\kappa_i$ such that

$$\frac{\mu}{A_i x - b_i} > \frac{\kappa_i}{\mu}.\$$

See proof of Proposition 8 in Appendix A.3. We are ready to prove that Statement (i) implies Statement (ii) in Theorem 5 for polyhedral set.

**Proof** Let $A_{I^*}$ be the matrix containing the rows $A_i$, $i \in I^*$, $N$ be its null space, i.e., $N = \{ x : A_i x = 0, \forall i \in I^* \}$, and $N^\perp = \text{span}(A_i^T, i \in I^*)$. We can verify that $\Pi_N(w) = D \Pi_{P(A,b)}(Z^*) w$ for $w \in \mathbb{R}^n$ (see Statement (iv) of Proposition 7). Let $w$ with $\|w\| = 1$ be fixed and $Jp_\mu(z) w = u$. We recall that $Jp_\mu(z) = [I + \mu^2 \nabla^2 f(x)]^{-1}$, then we have

$$w = u + \mu^2 \nabla^2 f(x) u \Rightarrow u + \sum_{i=1}^{m} \frac{\mu^2}{(A_i x - b_i)^2} (A_i u) A_i^T.\$$

(23)

This implies

$$\|w\|^2 = \|u\|^2 + 2 \sum_{i=1}^{m} \frac{\mu^2}{(A_i x - b_i)^2} (A_i u)^2 + \sum_{i=1}^{m} \frac{\mu^2}{(A_i x - b_i)^2} (A_i u)^2 \geq \max \left\{ \|u\|^2, \sum_{i=1}^{m} \frac{\mu^2}{(A_i x - b_i)^2} (A_i u)^2 \right\}.$$  

(24)

Inequality (24) shows that $u$ is bounded. Furthermore, we remind that $x \rightarrow \bar{z}^*$ when $(z, \mu) \rightarrow (z^*, 0)$ and $A_i \bar{z}^* > b_i, \forall i \notin I^*$. Therefore, from (23) we deduce

$$\left\| w - u - \sum_{i \in I^*} \frac{\mu^2}{(A_i x - b_i)^2} (A_i u) A_i^T \right\| = \left\| \sum_{i \notin I^*} \frac{\mu^2}{(A_i x - b_i)^2} (A_i u) A_i^T \right\| = O(\mu).$$
Moreover, \( w = \Pi_N(w) + \Pi_{N^\perp}(w) \) and 
\[
\sum_{i \in \mathcal{I}^*} \frac{\mu^2}{(A_i x - b_i)^2} (A_i u) A_i^T \in N^\perp,
\]
we hence get
\[
\text{dist}(\Pi_N(w) - u, N^\perp) 
\leq \text{dist}(w - u - \sum_{i \in \mathcal{I}^*} \frac{\mu^2}{(A_i x - b_i)^2} (A_i u) A_i^T, N^\perp) + \text{dist}(\Pi_{N^\perp}(w), N^\perp) 
+ \text{dist}(\sum_{i \in \mathcal{I}^*} \frac{\mu^2}{(A_i x - b_i)^2} (A_i u) A_i^T, N^\perp)
\]
\[
= O(\mu).
\]
(25)

We have \( \Pi_N(u) = u - A_{\mathcal{I}^*}^+ A_{\mathcal{I}^*} u \), where \( A_{\mathcal{I}^*}^+ \) is the Moore-Penrose pseudo-inverse of \( A_{\mathcal{I}^*} \). Thus, \( \text{dist}(u, N) = \left\| A_{\mathcal{I}^*}^+ A_{\mathcal{I}^*} u \right\| = O(A_{\mathcal{I}^*} u) \). Moreover, Inequality (24) together with Proposition 8 yields that \( A_i u = O(\mu) \) for \( i \in \mathcal{I}^* \). Therefore,
\[
\text{dist}(u - \Pi_N(w), N) \leq \text{dist}(u, N) + \text{dist}(\Pi_{N^\perp}(w), N) = O(\mu).
\]
In company with (25), we imply
\[
\|\Pi_N(w) - u\| \leq \|\Pi_N(\Pi_N(w) - u)\| + \|\Pi_{N^\perp}(\Pi_N(w) - u)\| = O(\mu) = O\left(\|z-z^*,\mu\|\right).
\]
The result follows then.

\[\square\]

4.1.2 Epigraph of \( l_\infty \) norm \( C_n \)

The \( l_\infty \) norm cone, which is defined by \( C_n = \{(t, x) \in \mathbb{R} \times \mathbb{R}^n : t \geq \|x\|_\infty\} \), is a special case of polyhedral set since we can rewrite it as \( C_n = \{(t, x) \in \mathbb{R} \times \mathbb{R}^n : t - x_i \geq 0, t + x_i \geq 0, \forall i = 1, \ldots, n\} \). Then we use the following barrier for the \( l_\infty \) norm cone
\[
f(t, x) = -\sum_{i=1}^{n} \log(t - x_i) - \sum_{i=1}^{n} \log(t + x_i).
\]
Its gradient and Hessian are
\[
\nabla f(t, x) = -\sum_{i=1}^{n} \frac{a_i}{t - x_i} - \sum_{i=1}^{n} \frac{b_i}{t + x_i},
\]
\[
\nabla^2 f(t, x) = \sum_{i=1}^{n} \frac{a_i a_i^T}{(t - x_i)^2} + \sum_{i=1}^{n} \frac{b_i b_i^T}{(t + x_i)^2},
\]
where \( a_i = e_0 - e_i, b_i = e_0 + e_i \) and \( e_i, i = 0, \ldots, n \), are unit vectors of \( \mathbb{R}^{n+1} \). The barrier-based smoothing approximation \( p_\mu(z_o, z) = (t, x) \) is then defined by
\[
\begin{align*}
t - \sum_{i=1}^{n} \frac{2\mu^2}{t^2 - x_i^2} &= z_o, \\
x_i + \frac{2\mu^2}{t^2 - x_i^2} x_i &= z_i, \\
t &> |x_i|, 1 \leq i \leq n.
\end{align*}
\]
(26)

Remark 3 We note that the smoothing approximation in (26) coincides with the smoothing approximation proposed by Chen in his thesis [11]. However, Chen uses a different
approach to derive (26). In particular, he shows that the SA is the unique solution of the logarithmic penalty problem associated with the constrained optimization problem that finds the projection onto the epigraph of $l_\infty$ norm. We derive the barrier-based SA for general polyhedral sets in (22), and (26) is just a special case of (22).

The following proposition gives another approach other than that of [21, Proposition 3.2] to find the projection onto $C_\mu$ which is used in the next section for epigraph of matrix operator norm. We give its proof in Appendix A.4.

**Proposition 9** When $(z_\ast, z, \mu) \rightarrow (z_\ast^\ast, z^\ast, 0)$, the limit of the smoothing approximation defined by (26) is the pair $(t^\ast, x^\ast)$ given by

\[
\begin{align*}
t^\ast(z_\ast^\ast, z^\ast) &= \max \left\{ \frac{1}{k^* + 1} (z_\ast + \sum_{i=1}^{k^*} |z^\ast_{\pi(i)}|) , 0 \right\}, \\
x_i^\ast &= \begin{cases} 
\sgn(z_i^\ast + 1) t^\ast, & \text{for } i = \pi(1), \ldots, \pi(k^*), \\
\frac{z_i^\ast}{\pi(i)} + 1, & \text{for } i = \pi(k^* + 1), \ldots, \pi(n), 
\end{cases}
\end{align*}
\]

where $\pi$ is a permutation of $\{1, \ldots, n\}$ such that $|z_{\pi(1)}^\ast| \geq \ldots \geq |z_{\pi(n)}^\ast|$, and $k^*$ is the unique nonnegative integer satisfying

\[
|z_{\pi(k^*)}^\ast| > \max \left\{ \frac{1}{k^* + 1} (z_\ast + \sum_{i=1}^{k^*} |z^\ast_{\pi(i)}|) , 0 \right\} \geq |z_{\pi(k^*+1)}^\ast|,
\]

where we let $z_{\pi(0)}^\ast = \infty$ and $z_{\pi(n+1)}^\ast = 0$. Consequently, $\Pi_{C_\mu}(z_\ast^\ast, z^\ast) = (t^\ast, x^\ast)$.

### 4.2 Epigraph of matrix operator norm and epigraph of matrix nuclear norm

We recall the self-concordant barrier function used for $K_{m,n}$ is

\[
f(t, x) = -\logdet \left( \begin{pmatrix} tI_n & x^T \\ x & tI_m \end{pmatrix} \right).
\]

Its first derivative is

\[
\nabla f(t, x) = \left( -\text{Tr}(tI_n - \frac{1}{t} x^T x)^{-1} - \text{Tr}(tI_m - \frac{1}{t} xx^T)^{-1} \right) \frac{2}{t}(tI_m - \frac{1}{t} xx^T)^{-1} x.
\]

Denote $\Sigma = [\text{Diag}(\sigma_f(x)) \ 0]$, and let $x = u \Sigma v^T$. It follows from

\[
(tI_n - \frac{1}{t} x^T x) = v(tI_n - \frac{1}{t} \Sigma^T \Sigma) v^T \quad \text{and} \quad (tI_m - \frac{1}{t} xx^T) = u(tI_m - \frac{1}{t} \Sigma \Sigma^T) u^T
\]

that $\nabla f(t, x) = \left( -\sum_{i=1}^{m} \frac{2t}{t^2 - \sigma_i(x)^2} - \frac{m-n}{t} \right) \frac{2u(t^2 I_m - \Sigma \Sigma^T)^{-1} \Sigma}$. The equation that defines the corresponding barrier-based smoothing approximation $P_{\mu}(z_\ast, z)$ of the
projection onto $K_{m,n}$, $(t, x) + \mu^2 \nabla f(t, x) = (z_o, z)$, is rewritten as

\[
\begin{cases}
  t - \mu^2 \left( \sum_{i=1}^{m} \frac{2t}{t^2 - \sigma_i^2} + \frac{n-m}{t} \right) = z_o, \\
  \sigma_i + 2\mu^2 \frac{\sigma_i}{t^2 - \sigma_i^2} = \sigma_i^o, \\
  t > |\sigma_i|,
\end{cases}
\]

(29)

where $\sigma = \sigma(t, x), \sigma^o = \sigma(t, z) = u[\text{Diag}(\sigma^o)]v^T$.

For $K_{m,n}$, using the modified Fenchel barrier function gives us the corresponding smoothing approximation $P_{t, \mu}(z_o, z)$. From [15, Part 6.2], we have

\[
P_{t, \mu}(z_o, z) = P_{t, \mu}(-z_o, -z) + (z_o, z).
\]

(30)

Now we give characteristic of the projector onto $K_{m,n}$. For $(t, x) \in \mathbb{R} \times \mathbb{R}^{m \times n}$, we let $x = u[\text{Diag}(\sigma_f(x))]v^T$ be a singular value decomposition of $x$, and denote $a = \{i : \sigma_i(x) > 0\}, b = \{i : \sigma_i(x) = 0\}, c = \{m+1, \ldots, n\}, c' = \{1, \ldots, n-m\},$ $(q_0(t, \sigma(x)), q(t, \sigma(x))) = \Pi_{C_n}(t, \sigma(x))$.

and $\Omega_1, \Omega_2 \in \mathbb{R}^{m \times m}$, $\Omega_3 \in \mathbb{R}^{m \times (n-m)}$ as follows

\[
(\Omega_1)_{ij} = \begin{cases} q_i(t, \sigma(x)) - q_j(t, \sigma(x)) & \text{if } \sigma_i(x) \neq \sigma_j(x), \\
\sigma_i(x) - \sigma_j(x) & \text{if } \sigma_i(x) = \sigma_j(x), \\
0 & \text{otherwise},
\end{cases}
\]

\[
(\Omega_2)_{ij} = \begin{cases} q_i(t, \sigma(x)) + q_j(t, \sigma(x)) & \text{if } \sigma_i(x) + \sigma_j(x) \neq 0, \\
\sigma_i(x) + \sigma_j(x) & \text{if } \sigma_i(x) + \sigma_j(x) = 0, \\
0 & \text{otherwise},
\end{cases}
\]

\[
(\Omega_3)_{ij} = \begin{cases} q_i(t, \sigma(x)) & \text{if } \sigma_i(x) \neq 0, \\
\sigma_i(x) & \text{if } \sigma_i(x) = 0, \\
0 & \text{otherwise},
\end{cases}
\]

We can rewrite these matrices as follows

\[
\Omega_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (\Omega_1)_{a \alpha} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (\Omega_1)_{a \alpha} E_{\alpha \beta}, \\
\Omega_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} (\Omega_2)_{\gamma \gamma} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (\Omega_2)_{\gamma \gamma} E_{\gamma \gamma}, \\
\Omega_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (\Omega_3)_{a \gamma} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (\Omega_3)_{a \gamma} E_{\gamma \gamma},
\]

where $E_{\alpha \beta}, E_{\gamma \gamma}$ are two matrices whose entries are all ones, and

\[
\alpha = \{i : \sigma_i > q_0(t, \sigma(x))\}, \beta = \{i : \sigma_i = q_0(t, \sigma(x))\}, \gamma = \{i : \sigma_i < q_0(t, \sigma(x))\}.
\]

Let $K$ be number of $q_i(t, \sigma(x))$ such that $q_i(t, \sigma(x)) = q_0(t, \sigma(x))$. Denote

\[
\delta = \sqrt{1 + K}, \quad \rho(w_o, w) = \begin{cases} \delta^{-1}(w_o + Tr(\hat{S}(u_o^T w_o w_o))) & \text{if } t \geq \|x\|, \\
0 & \text{otherwise}.
\end{cases}
\]

The following theorem characterizes the differentiable property of the projection onto $K_{m,n}$.

**Theorem 6** [21, Theorem 3] The metric projector $\Pi_{K_{m,n}}$ is differentiable at $(t, x)$ if and only if $(t, x)$ satisfies one of the following conditions

(i) $t > \|x\|_2$,

(ii) $\|x\|_2 > t > -\|x\|_*$ but $\bar{B} = \emptyset$. 

\[\square\] Springer
(iii) \[ t < -\|x\|_. \]

Under condition (ii), we have \[ D \Pi_{K_{m,n}}(t, x)(w_o, w) = (w'_o, w'), \] where

\[ (w_o, w) \in \mathbb{R} \times \mathbb{R}^{m \times n}, \quad w'_o = \delta^{-1}\rho(w_o, w) \]

and

\[
w' = u \left[ \begin{array}{c}
d^{-1}\rho(w_o, w)I_{|\alpha|} (\Omega_1)_{\gamma\gamma} \circ \mathcal{S}(A)_{\gamma\gamma} \\
(\Omega_2)_{aa} \circ \mathcal{S}(A)_{aa} \\
(\Omega_2)_{ba} \circ \mathcal{S}(A)_{ba}
\end{array} \right] v_1^T
+ u \left[ \begin{array}{c}
(\Omega_3)_{ac} \circ B_{ac} \\
B_{bc'}
\end{array} \right] v_2^T
\]

with \( v = [v_1 | v_2] \), \( A = u^T w v_1 \), and \( B = u^T w v_2 \).

We need the following lemmas to prove the convergence of the derivative \( D p_{\mu}(z_o, z) \).

**Lemma 1** [13, Lemma 3] For any symmetric matrix \( x \in \mathbb{S}^n \), there exist \( \eta > 0 \) and \( \varepsilon > 0 \) such that \( \min_{p \in O_x} \|p - q\| \leq \eta \|x - y\|, \forall y \), and \( \|y - x\| \leq \varepsilon, \forall q \in O_y \).

From \( \sigma_k(z) = \min_{\text{rank}(y) < k} \|z - y\| \) (see [4, Chapter III]), we can derive the following lemma.

**Lemma 2** The \( k \)-th singular value \( \sigma_k(\cdot) \) of a matrix in \( \mathbb{R}^{m \times n} \) satisfied

\[ |\sigma_k(z_1) - \sigma_k(z_2)| \leq \|z_1 - z_2\| \text{ for all matrices } z_1, z_2 \text{ in } \mathbb{R}^{m \times n}. \]

### 4.2.1 Verifying the requirement of Theorem 5 (ii)

Let \( (z^*_o, z^*) \) be a differentiable point of the projector onto \( K_{m,n} \) (or \( K_{m,n}^\circ \)), and let \( (z_o, z, \mu) \) go to \( (z^*_o, z^*, 0) \). We now verify the following expression which is the requirement of Theorem 5(ii)

\[
\| D p_{\mu}(z_o, z) - D \Pi_K(z^*_o, z^*) \| = O(\|z_o - z^*_o, z - z^*, \mu\|).
\]

**Proof** Let \( (t, x) = p_{\mu}(z_o, z) \) with \( x = u[\text{Diag}(\sigma) \ 0]v^T, z = u[\text{Diag}(\sigma^o) \ 0]v^T \) and \( (t, \sigma), (z_o, \sigma^o) \) satisfying (29). Let \( u^*, v^* \) be the limit points of \( u, v \). When \( \mu \to 0 \), we have \( \sigma^o \to \sigma(t^*) = \zeta \) and \( z \to z^* = u^*[\text{Diag}(\zeta) \ 0](v^*)^T \). Totally similar to Proposition 9, we can prove \( (t, \sigma) \to (t^*, \sigma^*) = \Pi_{C_n}(z^*_o, \zeta) \), where

\[
\sigma^*_i = \begin{cases} 
t^* > 0 & \text{if } i \leq k^*, \\
\zeta_i < t^* & \text{if } i > k^*.
\end{cases}
\]

Then \( x \to x^* = u^*[\text{Diag}(\sigma^*) \ 0](v^*)^T \).

The second derivative of the barrier (28) is

\[
D^2 f(t, x): ((h_o, h); (k_o, k)) \rightarrow \text{Tr} \left( \begin{array}{c}
I_n \ x^T \\
(\ h_oI_n \ h) \\
(\ h_oI_m \ k)
\end{array} \right) \left( \begin{array}{c}
I_n \ x^T \\
(\ h_oI_n \ h) \\
(\ h_oI_m \ k)
\end{array} \right)^{-1}
\]
We note that
\[
(\begin{pmatrix} tI_n & x^T \\ x & tI_m \end{pmatrix})^{-1} = \begin{pmatrix} v & 0 \\ 0 & u \end{pmatrix} \begin{pmatrix} t^2I_n - \Sigma^T \Sigma \Sigma^T(t^2I_n - \Sigma^T \Sigma)^{-1} - \Sigma^T(t^2I_n - \Sigma^T \Sigma)^{-1} & \Sigma^T(t^2I_n - \Sigma^T \Sigma)^{-1} \\ -(t^2I_m - \Sigma^T \Sigma)^{-1} \Sigma & t^2I_m - \Sigma^T \Sigma)^{-1} \end{pmatrix} \begin{pmatrix} v & 0 \\ 0 & u \end{pmatrix}^{-1} = \begin{pmatrix} v_1 & v_2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix} \begin{pmatrix} v_1 & 0 \\ v_2 & 0 \end{pmatrix},
\]
where \(D_1 = \text{Diag}\left(\frac{t}{t^2 - \sigma_i^2} \right)_{i=1, \ldots, m}\), \(D_2 = \frac{1}{t}I_{n-m}, \Delta = \left(\frac{-\sigma_i}{t^2 - \sigma_i^2} \right)_{i=1, \ldots, m}\), \(v_1\) contains the first \(m\) columns of \(v\) and \(v_2\) contains the remaining \(n - m\) columns of \(v\). Then,
\[
D^2 f(t, x)[(h_o, h); (k_o, k)] = \text{Tr} \left( \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix} \begin{pmatrix} h_oI_m & 0 \\ 0 & h_oI_{n-m} \end{pmatrix} \begin{pmatrix} 0 & h_oI_{n-m} \\ 0 & h_oI_m \end{pmatrix} \begin{pmatrix} h_o(D_1^2 + \Delta^2) + \Delta \tilde{h}_1D_1 + \Delta \tilde{h}_2D_2 \\ \Delta \tilde{h}_1D_1 + \Delta \tilde{h}_2D_2 \end{pmatrix} \begin{pmatrix} 0 & h_oI_{n-m} \\ 0 & h_oI_m \end{pmatrix} \begin{pmatrix} 0 & h_oI_{n-m} \\ 0 & h_oI_m \end{pmatrix} \begin{pmatrix} k_oI_m & 0 \\ 0 & k_oI_{n-m} \end{pmatrix} \end{pmatrix}
\times \begin{pmatrix} k_oI_m & 0 \\ 0 & k_oI_{n-m} \end{pmatrix}.
\]
Hence we get
\[
D^2 f(t, x)[(h_o, h); (k_o, k)] = k_o \left( h_o(2\text{Tr}(D_1^2 + \Delta^2) + \|D_2\|^2_F) + 4\text{Tr}(\Delta \tilde{h}_1D_1) \right) + 2(\tilde{k}_1, 2h_o \Delta D_1 + D_1\tilde{h}_1D_1 + \Delta \tilde{h}_1^T \Delta) + 2(\tilde{k}_2, D_1\tilde{h}_2D_2),
\]
where \(\tilde{h}_i = u^T h_i v_i\) and \(\tilde{k}_i = u^T k_i v_i, i = 1, 2\).
Let \((w_o, w) \in \mathbb{R} \times \mathbb{R}^m\times n\) be fixed. If \(DP_\mu(z_o, z)[w_o, w] = (h_o, h),\) then
\[
(w_o, w) = (h_o, h) + \mu^2 D^2 f(t, x)[h_o, h],
\]
and hence for any \((k_o, k) \in \mathbb{R} \times \mathbb{R}^m\times n\), by formula (32), we have
\[
k_o(w_o - h_o) = (\tilde{k}_1, \tilde{w}_1 - \tilde{h}_1) + (\tilde{k}_2, \tilde{w}_2 - \tilde{h}_2)
= \langle (w_o - h_o, w - h), (k_o, k) \rangle = \mu^2 D^2 f(t, x)[(h_o, h); (k_o, k)]
= \mu^2 k_o \left( h_o(2\text{Tr}(D_1^2 + \Delta^2) + \|D_2\|^2_F) + 4\text{Tr}(\Delta \tilde{h}_1D_1) \right)
+ 2\mu^2(\tilde{k}_1, 2h_o \Delta D_1 + D_1\tilde{h}_1D_1 + \Delta \tilde{h}_1^T \Delta) + 2\mu^2(\tilde{k}_2, D_1\tilde{h}_2D_2),
\]
where \(\tilde{w}_i = u^T w_i, i = 1, 2\). Thus for every fixed \((w_o, w),\) we get
\[
(w_o - h_o) = \mu^2 \left( h_o(2\text{Tr}(D_1^2 + \Delta^2) + \|D_2\|^2_F) + 4\text{Tr}(\Delta \tilde{h}_1D_1) \right),
\]
\[
\tilde{w}_1 - \tilde{h}_1 = 2\mu^2(2h_o \Delta D_1 + D_1\tilde{h}_1D_1 + \Delta \tilde{h}_1^T \Delta),
\]
\[
\tilde{w}_2 - \tilde{h}_2 = 2\mu^2 D_1\tilde{h}_2D_2.
\]
By Theorem 6, the Euclidean projector \(\Pi_{K_m,n}\) is differentiable at \((z_o^*, z^*)\) if and only if
\begin{enumerate}
\item \(z_o^* > \|z^*\|,\)
\item \(\|z^*\|_2 > z_o^* > -\|z^*\|_4\) but \(t^*(z_o^*, \sigma_f(z^*))\), which is defined in (27), is not a singular value of \(z^*,\)
\item \(z_o^* < -\|z^*\|_4.\)
\end{enumerate}
We consider the first case $z_o^* > \|z^*\|$, i.e., $(z_o^*, z^*)$ lies in the interior of $K_{m,n}$. In this case $(t, x) \rightarrow \Pi_{K_{m,n}}(z_o^*, z^*) = (z_o^*, z^*)$ and $\text{D}\Pi_{K_{m,n}}(z_o^*, z^*) = I$. Thus,

$$
\| (h_o, h) - (w_o, w) \| = \| \mu^2 \nabla^2 f(t, x) (h_o, h) \| = O(\mu) = O(\| (z_o - z_o^*, z - z^*, \mu) \|).
$$

We now consider the second case. Denote $\tilde{w}^* = (u^*)^T w v^*$. From the third equation of (33), we imply that for $i = 1, \ldots, m$, $j = 1, \ldots, n - m$, we have $\tilde{w}_{i,m+j} - \tilde{h}_{i,m+j} = 2 \frac{\mu^2}{t^2 - \sigma_i^2} \tilde{h}_{i,m+j}$. Therefore, we get $\tilde{h}_{i,m+j} = \frac{1}{1 + 2 \frac{\mu^2}{t^2 - \sigma_i^2}} \tilde{w}_{i,m+j}$. Furthermore, from (29), we deduce

$$
\frac{t^2 - \sigma_i^2}{\mu^2} \rightarrow \frac{2\sigma_i}{t^2 - \sigma_i^2} - \frac{2t^*}{\zeta_i - t^*} \quad \text{for}\quad i = 1, \ldots, k^*, \quad \text{and}
$$

$$
\frac{t^2 - \sigma_i^2}{\mu^2} \rightarrow \frac{(t^*)^2 - (\zeta_i)^2}{(t^*)^2 - (\zeta_i)^2} = 0 \quad \text{for}\quad i = k^* + 1, \ldots, m.
$$

Hence, for $i = 1, \ldots, m$, $j = 1, \ldots, n - m$, we have

$$
\tilde{h}_{i,m+j} \rightarrow \begin{cases} 
\frac{t^*}{\xi_i} \tilde{w}_{i,m+j} & \text{if}\quad i = 1, \ldots, k^* \\
\tilde{w}_{i,m+j} & \text{if}\quad i = k^* + 1, \ldots, m.
\end{cases}
$$

For $i, j = 1, \ldots, m$, $i \neq j$, denote $\rho_{ij} = \frac{t^2 - \sigma_i^2}{\mu^2} \times (t^2 - \sigma_j^2)$. The $(i, j)$-th, $(j, i)$-th entries of the second equation of (33) give $\tilde{w}_{ij} - \tilde{h}_{ij} = \rho_{ij}^{-1} (\tilde{h}_{ij} t^2 + \tilde{h}_{ij} \sigma_i \sigma_j)$ and $\tilde{w}_{ji} - \tilde{h}_{ji} = \rho_{ij}^{-1} (\tilde{h}_{ji} t^2 + \tilde{h}_{ji} \sigma_i \sigma_j)$. Solving these equations imply that for $i, j = 1, \ldots, m$, $i \neq j$, we have

$$
\tilde{h}_{ij} = \frac{\rho_{ij}}{(\rho_{ij} + t^2)^2 - \sigma_i^2 \sigma_j^2} (\tilde{w}_{ij} (\rho_{ij} + t^2) - \tilde{w}_{ij} \sigma_i \sigma_j), \tilde{h}_{ji}
$$

$$
= \frac{\rho_{ij}}{(\rho_{ij} + t^2)^2 - \sigma_i^2 \sigma_j^2} (\tilde{w}_{ji} (\rho_{ij} + t^2) - \tilde{w}_{ij} \sigma_i \sigma_j).
$$

For $i = 1, \ldots, k^*$, $j = k^*$, $m$, we have

$$
\rho_{ij} \rightarrow \frac{t^* ((t^*)^2 - (\zeta_j)^2)}{\zeta_i - t^*} = \rho_{ij}^*;
$$

hence

$$
\tilde{h}_{ij} \rightarrow \frac{\rho_{ij}^*}{(\rho_{ij}^* + (t^*)^2)^2} (\tilde{w}_{ij} (\rho_{ij}^* + (t^*)^2) - \tilde{w}_{ij} t^* \zeta_j) = \frac{t^* \zeta_i - \zeta_j^2}{\xi_i - \xi_j} \tilde{w}_{ij}^* - \frac{\xi_j (\zeta_i - t^*)}{\xi_i - \xi_j} \tilde{w}_{ij}, \quad (37)
$$

$$
\tilde{h}_{ji} \rightarrow \frac{t^* \zeta_i - \zeta_j^2}{\xi_i - \xi_j} \tilde{w}_{ji}^* - \frac{\xi_j (\zeta_i - t^*)}{\xi_i - \xi_j} \tilde{w}_{ij}^*.
$$

For $i, j = 1, \ldots, k^*$, $i \neq j$, we have

$$
\rho_{ij} \rightarrow 0, \quad \rho_{ij} \rightarrow \frac{t^2 - \sigma_i^2}{2\mu^2} \times \frac{(t^2 - \sigma_j^2)}{2\mu^2} \rightarrow \frac{(t^*)^2}{(\zeta_i - t^*)(\zeta_j - t^*)};
$$

(38)
\[ \tilde{h}_{ij} = \frac{\rho_{ji}}{2\mu^2} + 2t^2 \frac{\rho_{ji}}{2\mu^2} + \frac{t^2 - \sigma_j^2}{2\mu^2} \left( \tilde{w}_{ij} \left( \rho_{ij} + t^2 \right) - \tilde{w}_{ji} \sigma_j \right) \to \frac{t^2}{t_i + t_j} (\tilde{w}_{ij}^* - \tilde{w}_{ji}^*) \]  \hspace{1cm} (40)

For \( i = k^* + 1, \ldots, m \), the \( i \)-th diagonal entry of the second equation in (33) is

\[ \tilde{w}_{ii} - \tilde{h}_{ii} = 2 \left( \frac{\mu}{t^2 - \sigma_i^2} \right)^2 \left( -2h_o t \sigma_i + \tilde{h}_{ii} (t^2 + \sigma_i^2) \right) \to 0, \]  \hspace{1cm} (41)

which shows that \( \tilde{h}_{ii} \to \tilde{w}_{ii}^* \). Similarly,

\[ \tilde{h}_{ij} \to \tilde{w}_{ij}^* \text{ for } i, j = k^* + 1, \ldots, m, i \neq j. \]  \hspace{1cm} (42)

For \( i = 1, \ldots, k^* \), the \( i \)-th diagonal entry of the second equation in (33) is

\[ \tilde{w}_{ii} - \tilde{h}_{ii} = 2 \left( \frac{\mu}{t^2 - \sigma_i^2} \right)^2 \left( -2h_o t \sigma_i + \tilde{h}_{ii} (t^2 + \sigma_i^2) \right), \]

which implies

\[ 2t^2 (\tilde{h}_{ii} - h_o) = \frac{1}{2} \left( \frac{t^2 - \sigma_i^2}{\mu^2} \right)^2 \mu^2 (\tilde{w}_{ii} - \tilde{h}_{ii}) + 2h_o t (\sigma_i - t) + \tilde{h}_{ii} (t^2 - \sigma_j^2) \to 0. \]  \hspace{1cm} (43)

Therefore,

\[ \tilde{h}_{ii} - h_o \to 0 \text{ for } i = 1, \ldots, k^*. \]  \hspace{1cm} (44)

Adding the first equation to the sum of the diagonal entries in the second equation of (33),

\[ w_o - h_o + \sum_{i=1}^{m} (\tilde{w}_{ii} - \tilde{h}_{ii}) \]

\[ = \mu^2 h_o \frac{n - m}{t} + 2 \sum_{i=1}^{m} \left( \frac{\mu}{t^2 - \sigma_i^2} \right)^2 \left( h_o (t^2 + \sigma_i^2 - 2t \sigma_i) - \tilde{h}_{ii} (2 t \sigma_i - t^2 - \sigma_i^2) \right) \]

\[ = \mu^2 h_o \frac{n - m}{t^2} + 2 \sum_{i=1}^{m} \left( \frac{\mu}{t + \sigma_i} \right) (h_o - \tilde{h}_{ii}) \to 0. \]  \hspace{1cm} (45)

Thus \( h_o + \sum_{i=1}^{m} \tilde{h}_{ii} \to w_o + \sum_{i=1}^{m} \tilde{w}_{ii}^* \). Together with \( \tilde{h}_{ii} \to \tilde{w}_{ii}^* \) for \( i = k^* + 1, \ldots, m \) and \( \tilde{h}_{ii} - h_o \to 0 \) for \( i = 1, \ldots, k^* \), we conclude that for \( i = 1, \ldots, k^* \),

\[ (k^* + 1)h_o \to w_o + \sum_{i=1}^{k^*} \tilde{w}_{ii}^*, \quad \tilde{h}_{ii} \to \frac{1}{k^* + 1} \left( w_o + \sum_{i=1}^{k^*} \tilde{w}_{ii}^* \right). \]  \hspace{1cm} (46)
In summary, \( h_0 \to h^*_0 = \frac{1}{k^*} \left( w_0 + \sum_{i=1}^{k^*} \tilde{w}_{ii}^* \right) \), and

\[
\frac{1}{k^* + 1} \left( w_0 + \sum_{i=1}^{k^*} \tilde{w}_{ii}^* \right) \quad \text{if } i, j = 1, \ldots, k^*, \ i = j,
\]

\[
\frac{t^*}{t^* + \xi_j} \left( \tilde{w}_{ij}^* - \tilde{w}_{ji}^* \right) \quad \text{if } i, j = 1, \ldots, k^*, \ i \neq j,
\]

\[
\frac{t^*}{t^* + \xi_j} \left( \tilde{w}_{ij}^* - \tilde{w}_{ji}^* \right) \quad \text{if } i = 1, \ldots, k^*, \ j = k^* + 1, \ldots, m
\]

\[
\frac{t^*}{t^* + \xi_j} \left( \tilde{w}_{ij}^* - \tilde{w}_{ji}^* \right) \quad \text{if } i = k^* + 1, \ldots, m, \ j = 1, \ldots, k^*
\]

\[
\frac{t^*}{t^* + \xi_j} \left( \tilde{w}_{ij}^* - \tilde{w}_{ji}^* \right) \quad \text{if } i = k^* + 1, \ldots, m, \ j = k^* + 1, \ldots, n
\]

\[
\frac{t^*}{t^* + \xi_j} \left( \tilde{w}_{ij}^* - \tilde{w}_{ji}^* \right) \quad \text{if } i = 1, \ldots, k^*, \ j = m + 1, \ldots, n.
\]

Therefore, a limit of \( D P_\mu(z_0, z) \) has the form \( T^*(w_0, w) = (h^*_0, w^* h^* (v^*)^T) \), which can be verified to be the derivative of the projector onto \( K_{m,n} \) at \((z_0^*, z^*)\) by Theorem 6.

Lipschitz continuity of a smoothing approximation with respect to \( \mu \) (see Theorem 1) and that of the projector imply

\[
\| (t, \sigma) - (t^*, \sigma^*) \| \leq \| p_\mu(z_0, \sigma^o) - \Pi_{K_{m,n}} (z_0^*, \xi) \| \leq \| p_\mu(z_0, \sigma^o) - p_0(z_0, \sigma^o) \| + \| \Pi_{K_{m,n}} (z_0, \sigma^o) - \Pi_{K_{m,n}} (z_0^*, \xi) \| = O(\| z - z^* \|).\]

Furthermore, by Lemma 2 we have \( \| \sigma^o - \xi \| = O(\| z - z^* \|). \) Therefore, the limits in (34) satisfy

\[
\frac{\sigma_i - \sigma_i^o}{\sigma_i^o - \sigma_i} - \frac{t^*}{t^* - t^*} = O(\| z - z^* \|), \ i = 1, \ldots, k^*, \mu.
\]

Moreover, using Lemma 1 we deduce there exist \( \eta_1, \eta_2, \varepsilon_1, \varepsilon_2 > 0 \) such that

\[
\forall z, \| z - z^* \| < \varepsilon_1, \forall u \in O(zz^T), \exists u^* \in O(z^*(z^*)^T) \text{ such that } \| u - u^* \| \leq \| z - z^* \|,
\]

\[
\forall z, \| z - z^* \| < \varepsilon_2, \forall v \in O(z^T z), \exists v^* \in O(z^*(z^*)^T) \text{ such that } \| v - v^* \| \leq \| z - z^* \|.
\]

In company with the fact \( T^* \) is independent of the choice \( u^*, v^* \), we can choose \( u^*, v^* \) such that \( \| \tilde{w} - \tilde{w}^* \| = O(\| z - z^* \|) \). Therefore, the limit in (35) satisfy

\[
\left\{ \begin{array}{ll}
\| \tilde{h}_{i,m+j} - \frac{t^*}{\tilde{x}_i} \tilde{w}_{i,m+j}^* \| = O(\| z - z^* \|) & \text{if } i = 1, \ldots, k^*
\\
\| \tilde{h}_{i,m+j} - \tilde{w}_{i,m+j}^* \| = O(\| z - z^* \|) & \text{if } i = k^* + 1, \ldots, m.
\end{array} \right.
\]

Totally similarly, we can prove that all of the involving limits to finding limits of \( h_0, \tilde{h} \) in (36)–(46), which have the form \( l h s \to r h s \), satisfy \( \| l h s - r h s \| = O(\| z - z^* \|) \). This leads to \( \| \tilde{h} - \tilde{h}^* \| = O(\| z - z^* \|) \). We then get Expression (31).
Now we consider case \( z_0^* < -\|z^*\|_v \) i.e., \(-z_0^*, z^*\) \(\in\) \(\text{int}(K_{(m,n)}^\mu)\). We deduce from (30) that \( \text{DP}_\mu:(z_0, z) = I - \text{DP}_\mu:(-z_0, -z) \). Furthermore, it follows from \((-z_0, -z) \to (-z_0^*, -z^*)\) \(\in\) \(\text{int}(K^\mu)\) that \( P_\mu:(-z_0, -z) \to \Pi_{K^\mu}:(-z_0^*, -z^*) = (-z_0^*, -z^*) \) and
\[
\left\| I - \text{DP}_\mu:(-z_0, -z) \right\| = \left\| I - (I + \mu^2\nabla^2 f^\mu(P_\mu:(-z_0, -z)))^{-1} \right\| = \left\| (I + \mu^2\nabla^2 f^\mu(P_\mu:(-z_0, -z)))^{-1} (\mu^2\nabla^2 f^\mu(P_\mu:(-z_0, -z))) \right\| = O(\mu).
\]
Hence \( \| \text{DP}_\mu:(z_0, z) - 0 \| = \| I - \text{DP}_\mu:(-z_0, -z) \| = O(\mu) = O(\|(z_0 - z_0^*, z - z^*)\|) \). We now verify expression (31) for \( K_{(m,n)}^\mu \). By Moreau decomposition \((-z_0^*, z^*) = \Pi_{K^\mu}:(z_0^*, z^*) - \Pi_K:(-z_0^*, -z^*)\), we imply that the projector onto \( K_{(m,n)}^\mu \) is differentiable at \((-z_0^*, z^*)\) if and only if projector onto \( K_{(m,n)}^\mu \) is differentiable at \((-z_0^*, -z^*)\).

On the other hand, by the result for \( K_{(m,n)}^\mu \), we have
\[
\| D\text{P}_{\mu}:(-z_0, -z) - \text{D}\Pi_{K_{(m,n)}}:(-z_0^*, -z^*) \| = O(\|(z_0 - z_0^*, z - z^*, \mu)\|).
\]
Therefore, the result follows from \( D\text{P}_{\mu}:(z_0, z) = I - \text{D}P_{\mu}:(-z_0, -z) \).

4.2.2 The equivalence of the differentiability of the projection and the strict complementarity

We now prove the remaining part of Theorem 5 that is the equivalence of the differentiability of the projection at \((-z_0, z^*) = (x_0^*, x^*) - \Pi^\mu_{K^\mu}:(y_0^*, y^*)\) and the strict complementarity of \((x_0^*, x^*), (y_0^*, y^*)\). Here \((x_0^*, x^*), (y_0^*, y^*)\) is a pair of the solutions of the VI.

Proof The cases \((x_0^*, x^*) = 0\) or \((x_0^*, x^*) \neq 0\) \(\in\) \(\text{int}(K_{(m,n)})\) are trivial. We consider non-trivial case, i.e., \((x_0^*, x^*) \neq 0\) \(\in\) \(\text{int}(K_{(m,n)})\). From [15, Section 6.3] we have
\[
x^* = u^*\text{[Diag} \sigma_1^*, \ldots, \sigma_m^*) \text{[Diag} \tau_1^*, \ldots, \tau_m^*) \text{[Diag} v^*)^T, y^* = u^*\text{[Diag} \tau_1^*, \ldots, \tau_m^*) \text{[Diag} v^*)^T,
\]
where \(x_0^* = \sigma_1^* = \ldots = \sigma_r^* \geq \sigma_{r+1}^* \geq \ldots \geq \sigma_m^* + 1 = 0\) and \(\tau_1^* \leq \ldots \leq \tau_r^* < \tau_{r+1}^* \ldots \tau_{m+1}^* = 0\), \(y_0^* = -\sum \tau_i^*\) for some \(r, \tau^r \in \{1, \ldots, m\}, r \geq \tau^r\).

By [18, Example 5.7], we have
\[
F_K = \left\{ (x_0, x) : x = u^* \begin{pmatrix} x_0 & 0 \\ 0 & M \end{pmatrix} (v^*)^T, M \in R^{(m-r)\times(n-r)}, \|M\| \leq x_0 \right\}
\]
is a face of \(K_{(m,n)}\) containing \((x_0^*, x^*)\). This face is with respect to the standard face
\[
S_r^\infty = \{ (x_0, \bar{x}) \in C_n : \bar{x}_i = x_{0i} \text{ for } 1 \leq i \leq r \}.
\]
By [18, Theorem 6.2], \((x_0^*, x^*) \in \text{relint}(F_K)\). Similarly, by [18, Example 5.6], we have
\[
F_{K^\mu} = \left\{ (y_0, y) : y = u^* \begin{pmatrix} -N & 0 \\ 0 & 0 \end{pmatrix} (v^*)^T, N \in S_r^\mu, TrN = y_0 \right\}
\]
is a face of \(K_{(m,n)}^\mu\) containing \((y_0^*, y^*)\). This face is with respect to the standard face
\[
S_r^{\mu} = \left\{ (y_0, \bar{y}) \in C_n^\mu : \sum_{i=1}^{\bar{r}} \bar{y}_i = y_0 \right\}.
\]

\(\copyright\) Springer
Furthermore, by [18, Theorem 6.2], we have \((\bar{x}^*, \bar{y}^*) = \text{relint}(\mathcal{F}_{K^2})\). Therefore, if \((x^*_o, y^*)\) and \((\bar{x}^*, \bar{y}^*)\) are strictly complementary then \(\mathcal{F}_{K^2} = \mathcal{F}_{K^\Delta}^\Delta\). Moreover, we note that
\[
\mathcal{F}_{K^\Delta} = \left\{ (y_0, y) : y = u^* \begin{pmatrix} -N & A \\ C & 0 \end{pmatrix} (v^*)^T, N \in R^{r \times r}, \text{Tr}(N) = y_0 \geq \left\| \begin{pmatrix} -N \\ -C \end{pmatrix} \right\| \right\}.
\]
This implies that \(r = r^2\); otherwise, the point \((y_0, \bar{y})\), which is defined by
\[
y_0 > 0, \quad \bar{y} = u^* \begin{pmatrix} -\text{Diag}(\bar{y}_1, \ldots, \bar{y}_r) & 0 \\ 0 & 0 \end{pmatrix} (v^*)^T, \quad \sum_{i=1}^r \bar{y}_i = y_0, \quad \bar{y}_i > 0,
\]
belongs to \(\mathcal{F}_{K^\Delta}\) but does not belong to \(\mathcal{F}_{K^2}\), this gives a contradiction. Then we deduce that
\[
z^* = x^* - y^* = u^*[\text{Diag}(x^*_o - \tau^*_1, \ldots, x^*_o - \tau^*_r, \sigma^*_r, \ldots, \sigma^*_m)](v^*)^T.
\]
Therefore the projector of \(z^*\) onto \(K_{m,n}\) is differentiable by Theorem 6.

Conversely, suppose that the projector onto \(K_{m,n}\) is differentiable at \((z^*_o, z^*)\), then we have \(r = r^2\). We know that each face of \(K_{m,n}^\Delta\) unique determines a standard face of \(C_{n}^{\Delta}\). Suppose that \(\mathcal{F}_{K^2} \neq \mathcal{F}_{K}^\Delta\), i.e., \(S^1_t\) is not the standard face of \(\mathcal{F}_{K}^\Delta\). Then the standard face of \(\mathcal{F}_{K}^\Delta\) has the form \(S^1_t = \{ (y_0, \bar{y}) \in C_{n}^{\Delta} : \sum_{i=1}^r \bar{y}_i = y_0 \} \) with \(\bar{r} \neq r\). If \(\bar{r} < r\) then the point \((y_0, \bar{y})\) defined in (47) belongs to \(\mathcal{F}_{K}^\Delta\) but definitely does not belong to the face of \(K_{m,n}^\Delta\) generated by \(S^1_t\). This is a contradiction. If \(\bar{r} > r\) then the point \((\bar{y}_0, \bar{y})\) with \(\bar{y}_0 > 0\) and
\[
\bar{y} = u^* \begin{pmatrix} -\text{Diag}(\bar{y}_1, \ldots, \bar{y}_r) & 0 \\ 0 & 0 \end{pmatrix} (v^*)^T, \quad \sum_{i=1}^r \bar{y}_i = y_0, \quad \bar{y}_i > 0,
\]
belongs to the face of \(K_{m,n}^\Delta\) generated by \(S^1_t\) but definitely does not belong to \(\mathcal{F}_{K}^\Delta\). We again get a contradiction. Therefore, \(\mathcal{F}_{K^2} = \mathcal{F}_{K}^\Delta\), i.e., \((x^*_o, x^*), (y^*_o, y^*)\) is strict complementary.

\section{Conclusion}

We analyse an inexact non-interior continuation method for variational inequalities over general closed convex sets. The method can deal with large scale problems by solving involving Newton equations inexactly. Proposition 2 is the key to achieving the global linear convergence of the algorithm. A \(\vartheta\)-self-concordant barrier of \(X\) is the sufficient condition to get the inequality \(\left\| D^2 p_{\mu}(z) \right\| \leq \frac{1}{4\mu} \) in this proposition. For local convergence, Theorem 2 serves as a cornerstone to establish the local quadratic convergence of the algorithm. Therefore, in Section 4, we always choose self-concordant barriers in application the algorithm to concrete closed convex sets, and verify the condition \(\left\| Dp_{\mu}(z) - DP_X(z^*) \right\| = O(||(z - z^*, \mu)||)\) for these sets. We further prove that differentiability of \(\Pi_X\) at \(z^*\) is equivalent to strict complementarity of \((x^*, y^*)\) when \(X\) is a non-negative orthant, a semidefinite cone, an epigraph of matrix operator norm or an epigraph of matrix nuclear norm.

\section*{Acknowledgments}

We thank the anonymous reviewers for their meticulous and insightful comments, which help us improve the paper. LTKH gives special thanks to Prof. Nicolas Gillis for his support.
Appendix A: Technical proofs

A.1 Proof of Proposition 3

(i) From Eq. 9 we get
\[ \| \Delta \tilde{w}^{(k)} \| \leq C \left( \| H_{\mu_k} (w^{(k)}) \| + \| r_1^{(k)} \| \right) \leq C (1 + \theta_1) \Psi_{\mu_k} (w^{(k)}) \leq C (1 + \theta_1) \beta \mu_k. \]

(ii) We have
\[ \| H_0 (w^{(k)}) \| - \Psi_{\mu_k} (w^{(k)}) \leq \left\| \phi_0 (w^{(k)}) \right\| - \left\| \phi_{\mu_k} (w^{(k)}) \right\| \leq \left\| \phi_0 (w^{(k)}) - \phi_{\mu_k} (w^{(k)}) \right\| \leq \sqrt{\theta} \mu_k, \] (48)

where we have used the property \( \| (a, b) \| \leq \| a \| + \| b \| \) for the first inequality and Theorem 1 for the last inequality. Inequality (48) with (11) give us
\[ \| \Delta \tilde{w} \| \leq C \left( \| H_0 (w^{(k)}) \| + \| r_2^{(k)} \| \right) \leq C (1 + \theta_2) (\Psi_{\mu_k} (w^{(k)}) + \sqrt{\theta} \mu_k) \leq C (1 + \theta_2) (\beta + \sqrt{\theta}) \mu_k. \]

A.2 Proof of Theorem 5

A.2.1 Non-negative orthant \( \mathbb{R}_+^n \)

Gradient and Hessian of the barrier function are
\[ \nabla f (x) = - \sum_{i=1}^n \frac{1}{x_i} e_i, \quad \nabla^2 f (x) = \sum_{i=1}^n \frac{1}{x_i^2} e_i e_i^T, \]
where \( e_i \) denote the \( i \)-th standard unit vector of \( \mathbb{R}^n \). The corresponding barrier-based smoothing approximation is \( p_\mu (z) = \frac{1}{2} \sum_{i=1}^n \left( z_i + \sqrt{z_i^2 + 4 \mu^2} \right) e_i \). Its Jacobian is
\[ J p_\mu (z) = \frac{1}{2} \text{Diag} \left( 1 + \frac{z_i}{\sqrt{z_i^2 + 4 \mu^2}} \right) e_i \bigg|_{i=1, \ldots, n}. \]
The projection of \( z \) onto \( \mathbb{R}_+^n \) is \( \Pi_{\mathbb{R}_+^n} (z) = [z]_+ \). We observe that the projector is differentiable at \( z^* \) if and only if \( z_i^* \neq 0, \forall i = 1, \ldots, n \). On the other hand, a pair \( (x^*, y^*) \) is strictly complimentary if and only if \( x_i^* + y_i^* > 0 \) for all \( i = 1, \ldots, n \), see [9]. Furthermore, \( x^* + y^* = \Pi_{\mathbb{R}_+^n} (z^*) + \Pi_{\mathbb{R}_-^n} (-z^*) \). Hence, it is easy to see that differentiability of the projector at \( z^* \) is equivalent to strict complimentarity of \( (x^*, y^*) \).

Now let \( z_i^* \neq 0 \) for \( i = 1, \ldots, n \), then we observe that the Jacobian \( J p_\mu (z) \) converges to
\[ J \Pi_{\mathbb{R}_+^n} (z^*) = \frac{1}{2} \text{Diag} \left( 1 + \frac{z_i^*}{|z_i^*|} \right) e_i \bigg|_{i=1, \ldots, n} \]
when \( (z, \mu) \rightarrow (z^*, 0) \). Since the map \( (z, \mu) \mapsto J p_\mu (z) \) is continuously differentiable at \( (z^*, 0) \), thus is locally Lipschitz at this point. Consequently, \( \| J p_\mu (z) - T^* \| = \)
\[ O(\|z - z^*\|, \mu). \] On the other hand, \[ \|DP_{\mu}(z) - D\Pi_{\mathbb{R}_+^n}(z^*)\| = \|JP_{\mu}(z) - J\Pi_{\mathbb{R}_+^n}(z^*)\|. \]

Thus, we get the result.

### A.2.2 Positive semidefinite cone \( \mathbb{S}_+^n \)

We have \( \nabla f(x) = -x^{-1}. \)

From the equation \( x + \mu^2 \nabla f(x) = z, \) we deduce that the corresponding barrier-based smoothing approximation is

\[ p_{\mu}(z) = \frac{1}{2} \left( z + (z^2 + 4\mu^2 I)^{1/2} \right). \]

Denote \( g : u \in \mathbb{R} \mapsto g(u) = u + \sqrt{u^2 + 4\mu^2}, \) \( g'(u) = 1 + \frac{u}{\sqrt{u^2 + 4\mu^2}} \) and \( g^{(1)} \) is a matrix whose \((i, j)\)-th entry with respect to a vector \( d \) is

\[
(g^{(1)}(d))_{ij} = \left\{ \begin{array}{ll}
g(d_i) - g(d_j) & \text{if } d_i \neq d_j \\
g'(d_i) & \text{if } d_i = d_j
\end{array} \right.
\]

\[
= \left\{ \begin{array}{ll}
1 + \frac{\sqrt{d_i^2 + 4\mu^2} - \sqrt{d_j^2 + 4\mu^2}}{d_i - d_j} & \text{if } d_i \neq d_j \\
1 + \frac{d_i}{\sqrt{d_i^2 + 4\mu^2}} & \text{if } d_i = d_j
\end{array} \right.
\]

Let \( z = q \text{Diag}(\lambda_f(z))q^T, \) then \( DP_{\mu}(z)[h] = \frac{1}{2}q \left[ g^{(1)}(\lambda_f(z)) \circ (q^T h q) \right] q^T. \) Projection of \( z \) onto \( \mathbb{S}_+^n \) is \( \Pi_{\mathbb{S}_+^n}(z) = q \text{Diag}([\lambda_f(z)]_+)q^T. \) We see that \( \Pi_{\mathbb{S}_+^n}(\cdot) \) is differentiable at \( z^* \) if and only if all eigenvalues \( \lambda_i^*, \) for \( i = 1, \ldots, n, \) of \( z^* \) are non-zeroes. Furthermore, strict complementarity of \( (x^*, y^*) \) is equivalent to the condition that all eigenvalues of \( x^* + y^* \) is positive. We now let \( z^* = q \text{Diag}(\lambda_f(z^*))q^T \) be the eigenvalue decomposition of \( z^*. \) Then,

\[ x^* + y^* = \Pi_{\mathbb{S}_+^n}(z^*) + \Pi_{\mathbb{S}_+^n}(-z^*) = q \text{Diag}([\lambda_f(z^*)]_+)q^T + q \text{Diag}([-\lambda_f(z^*)]_+)q^T = q \text{Diag}([\lambda_f(z^*)]_+ + [-\lambda_f(z^*)]_+)q^T. \]

Hence differentiability of \( \Pi_{\mathbb{S}_+^n}(\cdot) \) at \( z^* \) is equivalent to strict complementarity of \( (x^*, y^*). \)

Now we consider \( z^* \) whose eigenvalues are non-zeroes. Let \( (z, \mu) \) go to \((z^*, 0), \) then \( \lambda_f(z) \) converges to \( \lambda^*. \) Let \( \tilde{q} \) be a limit point of \( q. \) We then have \( z^* = \tilde{q} \text{Diag}(\lambda^*)\tilde{q}^T \) with \( \tilde{q} \in \mathcal{O}^n(z^*). \) We deduce from \( \lambda_i^* \neq 0, i = 1, \ldots, n \) that

\[
(g^{(1)}(\lambda_f(z)))_{ij} = 1 + \frac{\lambda_i + \lambda_j}{\sqrt{\lambda_i^2 + 4\mu^2} + \sqrt{\lambda_j^2 + 4\mu^2}} \rightarrow 1 + \frac{\lambda_i^* + \lambda_j^*}{|\lambda_i^*| + |\lambda_j^*|}.
\]
Therefore, by Theorem 2, when \((z, \mu) \to (z^*, 0)\), where \(z^*\) are differential points of \(\Pi_{\mathbb{S}^2}(\cdot)\), \(Dp_\mu(z)\) converges to \(D\Pi_{\mathbb{S}^2}(z^*)\) with

\[
D\Pi_{\mathbb{S}^2}(z^*)[h] = \frac{1}{2}q \left[ \tilde{g}^{(1)}(\lambda^*) \circ (\bar{q}^T h \bar{q}) \right] \bar{q}^T,
\]

(49)

where \(\tilde{g}^{(1)}(\lambda^*)_{ij} = 1 + \frac{\lambda^*_{ij} + \lambda^*_{ji}}{|\lambda^*_{ij}| + |\lambda^*_{ji}|}\). Note that Formula (49) is independent of the choice \(\bar{q}\).

Finally, similarly to the case \(\mathbb{R}^n_+\), we have \(\|Dp_\mu(z) - D\Pi_{\mathbb{S}^2}(z^*)\| = O(\|z - z^*, \mu\|)\) since \((z, \mu) \to Dp_\mu(z)\) is locally Lipschitz around \((z^*, 0)\).

### A.3 Proof of Proposition 8

We note that \(\mathcal{F}_{I^*}\) is the unique neighbour face of \(z^*\) and \(z^* \in \text{int}(\mathcal{F}_I + \mathcal{N}_I)\) as the projector is differentiable at \(z^*\) (see Proposition 7). When \((z, \mu) \to (z^*, 0)\), we have \(x \to \Pi_K(z^*) = \tilde{z}^*,\) satisfying \(A_i \tilde{z}^* = b_i, \forall i \in I^*\) and \(A_i \tilde{z}^* > b_i, \forall i \notin I^*.\) From Eq. 22 we have

\[
z^* - \tilde{z}^* = \lim_{(z, \mu) \to (z^*, 0)} (x - z) = \lim_{(z, \mu) \to (z^*, 0)} \sum_{i \in I^*} \frac{\mu^2}{A_{ij}x - b_i}(-A_i^T)
\]

(50)

If there exists \(j \in I^*\) and a subsequence \((z, \mu)_k \to (z^*, 0)\) such that \(\frac{\mu^2_k}{A_{jk}x_k - b_j} \to 0\) then we take the limit of this subsequence in (50) to get

\[
z^* - \tilde{z}^* = \lim_{k \to \infty} \sum_{i \in I^* \setminus \{j\}} \frac{\mu^2_k}{A_{ij}x_k - b_i}(-A_i^T) \in \text{cone}\{-A_i^T : i \in I^* \setminus \{j\}\} = \mathcal{N}_{I^* \setminus \{j\}}\]

On the other hand, \(\tilde{z}^* \in \mathcal{F}_{I^* \setminus \{j\}}\) as \(A_i \tilde{z}^* = b_i, \forall i \in I^* \setminus \{j\}\). Hence \(z^* = \tilde{z}^* + z^* - \tilde{z}^* \in \mathcal{F}_{I^* \setminus \{j\}} + \mathcal{N}_{I^* \setminus \{j\}}\), which implies \(I^* \setminus \{j\}\) is a neighbour face of \(z^*\). This contradicts to the fact \(I^*\) is the unique neighbour face of \(z^*\). Therefore, for all \(i \in I^*, \frac{\mu^2}{A_{ij}x - b_i}\) only have nonzero limit points. The result follows then.

### A.4 Proof of Proposition 9

By re-indexing \(z^*\) if necessary, we can assume that \(\pi\) is the identity permutation. For each \(i \in \{1, \ldots, n\}\), if \(|x_i^*| < t^*\) then

\[
x_i^* = \lim_{k \to \infty} x_i = \lim (z_i^* - \frac{2\mu^2}{t^2 - x_i^2}) = z_i^*.
\]

Together with \(\text{sgn}(x_i) = \text{sgn}(z_i)\) and \(t > |x_i|\), we deduce that \(x_i^* = \text{sgn}(z_i^*)t^* + z_i^*\) for \(i = 1, \ldots, n\). Moreover, \(|x_i| < \min\{t, |z_i|\}\) further implies that

\[
x_i^* = \begin{cases} \text{sgn}(z_i^*)t^* & \text{if } t^* < |z_i^*| \\ z_i^* & \text{if } t^* > |z_i^*|. \end{cases}
\]

Thus, there exists a unique positive integer \(k^*\) such that

\[
x_i^* = \begin{cases} \text{sgn}(z_i^*)t^* & \text{for } i = 1, \ldots, k^*, \\ z_i^* & \text{for } i = k^* + 1, \ldots, n, \end{cases}
\]
and \( |z_{k^*}^*| > t^* \geq |z_{k^*+1}^*| \). Summing up \((n + 1)\) equations in (26) gives
\[
z_o + \sum_{i=1}^{n} |z_i| = t + \sum_{i=1}^{n} |x_i| - \mu^2 \sum_{i=1}^{n} \frac{2(t - |x_i|)}{t^2 - x_i^2} = t + \sum_{i=1}^{n} |x_i| - \mu^2 \sum_{i=1}^{n} \frac{2}{t + |x_i|}.
\]
If \( t^* > 0 \) then taking limit gives, \( z_o^* + \sum_{i=1}^{n} |z_i^*| = t^* + \sum_{i=1}^{n} |x_i^*| \), and hence
\[
(k^* + 1)t^* = t^* + \sum_{i=1}^{n} |x_i^*| = z_o^* + \sum_{i=1}^{n} |z_i^*| - \sum_{i=1}^{n} |x_i^*| = z_o^* + \sum_{i=1}^{n} |z_i^*|.
\]
If \( t^* = 0 \) then \( |x_i| < t \) implies that \( x_i^* = 0 \) for \( i = 1, \ldots, n \). Thus
\[
z_o^* + \sum_{i=1}^{n} |z_i^*| = \lim(z_o + \sum_{i=1}^{n} |z_i|) = \lim(t + \sum_{i=1}^{n} |x_i| - \mu^2 \sum_{i=1}^{n} \frac{2}{t + |x_i|})
\]
\[
\leq \lim(t + \sum_{i=1}^{n} |x_i|) = t^* + \sum_{i=1}^{n} |x_i^*| = 0.
\]
Subsequently, \( z_o^* + \sum_{i=1}^{n} |z_i^*| \leq z_o^* + \sum_{i=1}^{n} |z_i^*| \leq 0 \). Hence,
\[
t^* = \max \left\{ \frac{1}{k^* + 1} (z_o^* + \sum_{i=1}^{n} |z_i^*|), 0 \right\}.
\]

**Appendix B: Example**

We consider the second order cone in \( \mathbb{R}^3 \)
\[ K_2 = \{(t, z) : z \in \mathbb{R}^2, t \in \mathbb{R}_+, \|z\| \leq t \}. \]
Firstly, we use the barrier \( f^{(1)}(t, z) = -\log(t^2 - \|z\|^2) \). Denote \( M = t^2 - z_1^2 - z_2^2 \). The gradient of \( f^{(1)} \) is
\[
\nabla f^{(1)}(t, z) = \left( -2tM^{-1}, 2z_1M^{-1}, 2z_2M^{-1} \right).
\]
The smoothing approximation \( p^{(1)}_\mu(t^o, z^o) = (t, z) \) regarding to \( f^{(1)}(t, z) \) is computed by
\[
\begin{align*}
    t - \mu^2 (2tM^{-1}) &= t^o \\
    z_1 + \mu^2 (2z_1M^{-1}) &= z_1^o \\
    z_2 + \mu^2 (2z_2M^{-1}) &= z_2^o.
\end{align*}
\]
The unique solution of (51) is
\[
\begin{align*}
    t &= \frac{1}{4} \left( 2t^o + \sqrt{(t^o - \|z^o\|)^2 + 8\mu^2} + \sqrt{(t^o + \|z^o\|)^2 + 8\mu^2} \right) \\
    z_1 &= \frac{1}{4\|z^o\|} \left( 2\|z^o\| + \sqrt{(t^o + \|z^o\|)^2 + 8\mu^2} - \sqrt{(t^o - \|z^o\|)^2 + 8\mu^2} \right) \\
    z_2 &= \frac{1}{4\|z^o\|} \left( 2\|z^o\| + \sqrt{(t^o + \|z^o\|)^2 + 8\mu^2} - \sqrt{(t^o - \|z^o\|)^2 + 8\mu^2} \right).
\end{align*}
\]
Denote $s_1 = \sqrt{(t^o - \|z^o\|)^2 + 8\mu^2}$, $s_2 = \sqrt{(t^o + \|z^o\|)^2 + 8\mu^2}$. We have
\[
J_{P_\mu}^{(1)}(t^o, z_1^o, z_2^o) = \begin{pmatrix}
d_{11} & d_{12} & d_{13} \\
d_{21} & d_{22} & d_{23} \\
d_{31} & d_{32} & d_{33}
\end{pmatrix}
\]
where
\[
d_{11} = \frac{1}{2} + \frac{t^o(t^o - \|z^o\|)}{4s_1} + \frac{t^o(t^o + \|z^o\|)}{4s_2},
\]
\[
d_{12} = d_{21} = -\frac{(t^o - \|z^o\|)z_1^o}{4\|z^o\|s_1} + \frac{(t^o + \|z^o\|)z_2^o}{4\|z^o\|s_2},
\]
\[
d_{13} = d_{31} = -\frac{(t^o - \|z^o\|)z_2^o}{4\|z^o\|s_1} + \frac{(t^o + \|z^o\|)z_2^o}{4\|z^o\|s_2},
\]
\[
d_{22} = \frac{1}{2} + \frac{(z_2^o)^2}{\|z^o\|^2} \left( \frac{t^o + \|z^o\|}{s_2} + \frac{t^o - \|z^o\|}{s_2} \right) + \frac{(z_2^o)^2(s_2 - s_1)}{\|z^o\|^2},
\]
\[
d_{23} = d_{32} = \frac{z_2^o}{\|z^o\|} \left( \frac{t^o + \|z^o\|}{s_2} + \frac{t^o - \|z^o\|}{s_2} \right) + \frac{(z_2^o)^2(s_2 - s_1)}{\|z^o\|^2},
\]
\[
d_{33} = \frac{1}{2} + \frac{(z_2^o)^2}{\|z^o\|^2} \left( \frac{t^o + \|z^o\|}{s_2} + \frac{t^o - \|z^o\|}{s_2} \right) + \frac{(z_2^o)^2(s_2 - s_1)}{\|z^o\|^2}.
\]

We choose $(t^o, z_1^o, z_2^o)$ such that $(t^o, z_1^o, z_2^o) \rightarrow (t^*, z_1^*, z_2^*) = (0, 1, 0)$; then $s_1 \rightarrow 1, s_2 \rightarrow 1, \|z^o\| \rightarrow 1$. We imply
\[
\lim_{(t^o, z_1^o, z_2^o, \mu) \rightarrow (0, 1, 0, 0)} J_{P_\mu}^{(1)}(t^o, z_1^o, z_2^o) = \begin{pmatrix}
1/2 & 1/2 & 0 \\
1/2 & 1/2 & 0 \\
0 & 0 & 1/2
\end{pmatrix},
\]
which equals to $J_{\Pi K_2}(0, 1, 0)$ by Theorem 2. Now we use another barrier
\[
f^{(2)}(u, v_1, v_2) = -\log(u^2 - \|v\|^2) - \log(u - v_1) - \log(u + v_1).
\]

Denote $M_\mu = u^2 - v^2 - w^2$. Gradient $\nabla f^{(2)}(u, v_1, v_2)$ of $f^{(2)}$ is
\[
\begin{pmatrix}
-2uM_\mu^{-1} - \frac{1}{u - v_1} - \frac{1}{u + v_1}, 2v_1M_\mu^{-1} + \frac{1}{u - v_1} - \frac{1}{u + v_1}, 2v_2M_\mu^{-1}
\end{pmatrix}.
\]

Let $p^{(2)}_\mu(t^*, z_1^*, z_2^*) = p^{(2)}_\mu(0, 1, 0) = (u, v_1, v_2)$, which is defined by
\[
\begin{cases}
u_1 + \mu^2 \left(2v_1M_\mu^{-1} + \frac{1}{u - v_1} - \frac{1}{u + v_1} \right) = 0 \\
u_2 + \mu^2 \left(2v_2M_\mu^{-1} \right) = 0.
\end{cases}
\]

The third equation implies $v_2 = 0$, hence $M_\mu = u^2 - v_1^2 = (u - v_1)(u + v_1)$. Thus, the first
and the second equation imply
\[
\begin{cases}
u_1 + \mu^2 \left(2uM_\mu^{-1} \right) = 1 \\
u_2 + \mu^2 \left(2v_1M_\mu^{-1} \right) = 0.
\end{cases}
\]

\[\varepsilon\] Springer
which give $u = \frac{1}{2}\sqrt{1 + 16\mu^2}$, $v_1 = \frac{1}{2}M\mu = 4\mu^2$. Denote $a = \sqrt{1 + 16\mu^2}$. Hessian matrix $\nabla^2 f^{(2)}(u, v_1, v_2)$ of the barrier $f^{(2)}$ at $(u, v_1, v_2)$ is

$$
\begin{pmatrix}
\frac{1+8\mu^2}{16\mu^4} + \frac{4}{(a-1)^2} + \frac{4}{(a+1)^2} - \frac{\alpha}{16\mu^2} - \frac{4}{(a-1)^2} + \frac{4}{(a+1)^2} & 0 \\
-\frac{\alpha}{16\mu^2} - \frac{4}{(a-1)^2} + \frac{4}{(a+1)^2} + \frac{1+8\mu^2}{16\mu^4} + \frac{4}{(a-1)^2} + \frac{4}{(a+1)^2} & 0 \\
0 & \frac{1}{2\mu^2}
\end{pmatrix}
= \begin{pmatrix}
\frac{1+8\mu^2}{8\mu^4} - \frac{a}{8\mu^4} & 0 \\
-\frac{a}{8\mu^4} & \frac{1+8\mu^2}{8\mu^4} & 0 \\
0 & 0 & \frac{1}{2\mu^2}
\end{pmatrix}.
$$

We remind that $J_{P_{\mu}^{(2)}}(0, 1, 0) = [I + \mu^2\nabla^2 f^{(2)}(u, v_1, v_2)]^{-1} = \begin{pmatrix}
1/2 & 1/2 & 0 \\
1/2 & 1/2 & 0 \\
0 & 0 & 2/3
\end{pmatrix}$. Thus, $\lim_{\mu \to 0} J_{P_{\mu}^{(2)}}(0, 1, 0)$ equals $\begin{pmatrix}
1/2 & 1/2 & 0 \\
1/2 & 1/2 & 0 \\
0 & 0 & 2/3
\end{pmatrix}$. It does not coincide with the Jacobian of $\Pi_{K_2}$ at $(0, 1, 0)$. This shows that the limit $\lim_{(x^0, s^0, t^0, \mu) \to (0, 1, 0, 0)} D_{P_{\mu}^{(2)}}(t^0, z^0)$ does not exist.

**References**

1. Bai, Z.-Z., Golub, G.H., Lu, L.-Zh., Yin, J.-F.: Block triangular and skew-hermitian splitting methods for positive-definite linear systems. SIAM J. Sci. Comput. 26(3), 844–863 (2005)
2. Bai, Z.-Z., Golub, G.H., Ng, M.K.: Hermitian and skew-hermitian splitting methods for non-hermitian positive definite linear systems. SIAM J. Matrix Anal. Appl. 24(3), 603–626 (2003)
3. Bellavia, S., Morini, B.: A globally convergent newton-gmres subspace method for systems of nonlinear equations. SIAM J. Sci. Comput. 23(3), 940–960 (2001)
4. Bhatia, R.: Matrix analysis. Springer, Berlin (1997)
5. Brown, P.N., Saad, Y.: Hybrid krylov methods for nonlinear systems of equations. SIAM J. Sci. Stat. Comput. 11(3), 450–481 (1990)
6. Burke, J., Xu, S.: The global linear convergence of a noninterior path-following algorithm for linear complementarity problem. Math. Oper. Res. 23, 719–734 (1998)
7. Chen, B., Harker, P.T.: A non-interior-point continuation method for linear complementarity problems. SIAM J. Matrix Anal. Appl. 14(4), 1168–1190 (1993)
8. Chen, B., Harker, P.T.: A continuation method for monotone variational inequalities. Math. Program. 69(1), 237–253 (1995)
9. Chen, B., Xiu, N.: A global linear and local quadratic noninterior continuation method for nonlinear complementarity problems based on Chen-Mangasarian smoothing functions. SIAM J. Optim. 9(3), 605–623 (1999)
10. Chen, C., Mangasarian, O.L.: A class of smoothing functions for nonlinear and mixed complementarity problems. Comput. Optim. Appl. 5(2), 97–138 (1996)
11. Chen, C.H.: Numerical Algorithms for a Class of Matrix Norm Approximation Problems. PhD thesis, Nanjing University (2012)
12. Chen, X., Qi, L., Sun, D.: Global and superlinear convergence of the smoothing newton method and its application to general box constrained variational inequalities. Math. Comput. 67(222), 519–540 (1998)
13. Chen, X., Tseng, P.: Non-interior continuation methods for solving semidefinite complementarity problems. Math. Program. 95(3), 431–474 (2003)
14. Chen, X.J., Ye, Y.: On homotopy-smoothing methods for box-constrained variational inequalities. SIAM J. Control Optim. 37(2), 589–616 (1999)
15. Chua, C.B., Hien, L.T.K.: A superlinearly convergent smoothing newton continuation algorithm for variational inequalities over definable sets. SIAM J. Optim. 25(2), 1034–1063 (2015)
16. Chua, C.B., Li, Z.: A barrier-based smoothing proximal point algorithm for NCPs over closed convex cones. SIAM J. Optim. 23(2), 745–769 (2013)
17. Chua, C.B., Yi, P.: A continuation method for nonlinear complementarity problems over symmetric cones. SIAM J. Optim. 20, 2560–2583 (2010)
18. de Sá, E.: Faces of the unit ball of a unitarily invariant norm. Linear Algebra Appl. 197, 451–493 (1994)
19. Dembo, R.S., Eisenstat, S.C., Steihaug, T.: On newton-like methods. SIAM J. Numer. Anal. 19, 400–408 (1982)
20. Dennis, J.E. Jr.: On newton-like methods. Numer. Math. 11(4), 324–330 (1968)
21. Ding, C., Sun, D., Toh, K.-Ch.: An introduction to a class of matrix cone programming. Math. Program. 144(1-2), 141–179 (2014)
22. Facchinei, F., Pang, J.-S.: Finite-Dimensional Variational Inequalities and Complementarity Problems, vol. I. Springer, New York (2003)
23. Facchinei, F., Pang, J.-S.: Finite-Dimensional Variational Inequalities and Complementarity Problems, vol. II. Springer, New York (2003)
24. Gabriel, S.A., Moré, J.J.: Smoothing of mixed complementarity problems (1995)
25. Kanzow, C.: Some noninterior continuation methods for linear complementarity problems. SIAM J. Matrix Anal. Appl. 17(4), 851–868 (1996)
26. Knoll, D.A., Keyes, D.E.: Jacobian-free newton–krylov methods: a survey of approaches and applications. J. Comput. Phys. 193(2), 357–397 (2004)
27. Li, D., Fukushima, M.: Smoothing newton and quasi-newton methods for mixed complementarity problems. Comput. Optim. Appl. 17(2), 203–230 (2000)
28. Nesterov, Y.E., Nemirovski, A.S.: Interior Point Polynomial Algorithms in Convex Programming. SIAM Stud. Appl. Math. SIAM Publication, Philadelphia (1994)
29. Pataki, G., Tunçel, L.: On the generic properties of convex optimization problems in conic form. Math. Program. 89(3), 449–457 (2000)
30. Qi, L., Sun, D.: Smoothing functions and smoothing Newton method for complementarity and variational inequality problems. J. Optim. Theory Appl. 113(1), 121–147 (2002)
31. Ralph, D.: Global convergence of damped Newton’s method for nonsmooth equations, via the path search. Math. Oper. Res. 19(1), 302–318 (1994)
32. Saad, Y., Schultz, M.H.: Gmres: A generalized minimal residual algorithm for solving nonsymmetric linear systems. SIAM J. Sci. Stat. Comput. 7(3), 856–869 (1986)
33. Stoer, J.: Solution of Large Linear Systems of Equations by Conjugate Gradient Type Methods, pp. 540–565. Springer, Berlin (1983)
34. Xu, S.: The global linear convergence of an infeasible non-interior path-following algorithm for complementarity problems with uniform p-functions. Math. Program. 87(3), 501–517 (2000)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.