Well-posedness, global existence and blow-up phenomena for an integrable multi-component Camassa-Holm system

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Abstract

This paper is concerned with a multi-component Camassa-Holm system, which has been proven to be integrable and has peakon solutions. This system includes many one-component and two-component Camassa-Holm type systems as special cases. In this paper, we first establish the local well-posedness and a continuation criterion for the system, then we present several global existence or blow-up results for two important integrable two-component subsystems. Our obtained results cover and improve recent results in [25, 36].

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1 Introduction

In this paper, we consider the following multi-component system proposed by Xia and Qiao in [34]:

\begin{equation}
\begin{aligned}
m_{jt} &= (m_j H)_x + m_j H + \frac{1}{(N+1)^2} \sum_{i=1}^{N}[m_i(u_j-u_{jx})(v_i+v_{ix}) + m_j(u_i-u_{ix})(v_i+v_{ix})], \\
n_{jt} &= (n_j H)_x - n_j H - \frac{1}{(N+1)^2} \sum_{i=1}^{N}[n_i(u_i-u_{ix})(v_j+v_{jx}) + n_j(u_i-u_{ix})(v_i+v_{ix})], \\
m_j &= u_j - u_{jxx}, n_j = v_j - v_{jxx}, 1 \leq j \leq N,
\end{aligned}
\end{equation}

where $H$ is an arbitrary function of $u_j, v_j, 1 \leq j \leq N$, and their derivatives. The above 2N-component Camassa-Holm system is proved to be integrable in the sense of Lax pair and infinitely many conservation laws in [34], where its peakon solutions for the case $N = 2$ are also obtained.

Since $H$ is an arbitrary function of $u_j, v_j, 1 \leq j \leq N$, and their derivatives, thus Eq. (1.1) is actually a large class of systems. As $N = 1$, $v_1 = 2$ and $H = -u_1$, Eq. (1.1) is reduced to the standard Camassa-Holm (CH) equation

\begin{equation}
m_t + um_x + 2u_xm = 0, \quad m = u - u_{xx},
\end{equation}

which was derived by Camassa and Holm [4] in 1993 as a model for the unidirectional propagation of shallow water waves over a flat bottom. The CH equation, also as a model for the propagation of axially symmetric waves in hyperelastic rods [17], has a bi-Hamiltonian structure [7, 22] and is completely integrable [4, 6]. One of the remarkable properties of the CH equation is the existence of peakons. One can refer to [4, 14, 15] for the existence of peakon solitons and multi-peakons. The Cauchy problem and initial boundary
problem of the CH equation has been studied extensively: local well-posedness \cite{8,11,18,26,31,19,20}, global strong solutions \cite{5,8,11,19,20}, blow-up solutions in finite time \cite{5,8,10,12,27,19,20} and global weak solutions \cite{3,9,13,35}.

As $N = 1$ and $H = -\frac{1}{2}(u_1 - u_1x)(v_1 + v_1x)$, Eq. (1.1) is reduced to the following system proposed by Song, Qu and Qiao in \cite{32}:

\begin{equation}
\begin{cases}
    m_t + \frac{1}{2}(u - u_x)(v + v_x)m_x = 0, \\
    n_t + \frac{1}{2}(u - u_x)(v + v_x)n_x = 0.
\end{cases}
\end{equation}

(1.3)

The above system is proved to be integrable not only in the sense of Lax-pair but also in the sense of geometry, namely, it describes pseudospherical surfaces \cite{32}. Besides, exact solutions to this system such as cuspons and W/M-shape solitons are also obtained in \cite{32}.

As $N = 1$ and $H = -\frac{1}{2}(u_1v_1 - u_1xv_1x)$, Eq. (1.1) is reduced to the following system proposed by Xia and Qiao in \cite{30,33}:

\begin{equation}
\begin{cases}
    m_t + \frac{1}{2}((uv - u_xv_x)m)_x - \frac{1}{2}(uv_x - vu_x)m = 0, \\
    n_t + \frac{1}{2}((uv - u_xv_x)n)_x + \frac{1}{2}(uv_x - vu_x)n = 0,
\end{cases}
\end{equation}

(1.4)

which describes a nontrivial one-parameter family of pseudo-spherical surfaces. In \cite{30,33}, the authors showed this system is integrable with Lax pair, bi-Hamiltonian structure, and infinitely many conservation laws. They also studied the peaked soliton and multi-peakon solutions to the system. Recently, Yan, Qiao and Yin \cite{36} studied the local well-posedness for the Cauchy problem of the system and derived a precise blow-up scenario and a blow-up result for the strong solutions to the system.

As $v = 2u$, both Eq. (1.3) and Eq. (1.4) are reduced to the following cubic Camassa-Holm equation

\begin{equation}
m_t + (m(u^2 - u_x^2))_x = 0, \quad m = u - u_{xx},
\end{equation}

(1.5)

which was proposed independently by Fokas \cite{21}, Fuchssteiner \cite{24}, Olver and Rosenau \cite{28}, and Qiao \cite{29} as an integrable peakon equations with cubic nonlinearity. Its Lax pair, peakon and soliton solutions, local well-posedness and blow-up phenomena have been studied in \cite{29,23,25}.

The aim of this paper is to establish the local well-posedness and a continuation criterion for the Cauchy problem of Eq. (1.1) in Besov spaces, and present several global existence or blow-up results for the two component subsystems: Eq. (1.3) and Eq. (1.4). Our obtained results cover and improve recent results in \cite{25,36}. Compared with the Camassa-Holm equation, one of the remarkable features of Eq. (1.1) is that it has higher-order nonlinearities. Thus, we have to estimate elaborately these higher-order nonlinear terms for the study of the local well-posedness and the continuation criterion of Eq. (1.1) in Besov spaces.
Besides, we derive that \( \|m(t)\|_{L^1} \) (\( \|n(t)\|_{L^1} \)) and \( \int_{\mathbb{R}} (mu_x)(t, x)dx = \int_{\mathbb{R}} (nu_x)(t, x)dx \) are conservation laws for Eq. (1.3) and Eq. (1.4), respectively. The above conservation laws, which have not been derived or used in the associated previous papers \[25, 36\], are useful and crucial in some blow-up results stated in the following fourth section.

The rest of our paper is then organized as follows. In Section 2, we recall the Littlewood-Paley decomposition and some basic properties of the Besov spaces. In Section 3, we establish the local well-posedness and provide a continuation criterion for Eq. (1.1). The last section is devoted to establishing several global existence or blow-up results for Eq. (1.3) and Eq. (1.4).

From now on we always assume that \( H = H(u_1, \cdots, u_N, v_1, \cdots, v_N, u_{1x}, \cdots, u_{1x}, v_{1x}, \cdots, v_{1x}) \) is a polynomial of degree \( l \), \( C > 0 \) stands for a generic constant, \( A \lesssim B \) denotes the relation \( A \leq CB \). Since all function spaces in this paper are over \( \mathbb{R} \), for simplicity, we drop \( \mathbb{R} \) in the notations of function spaces if there is no ambiguity.

## 2 Preliminaries

To begin with, we introduce the Littlewood-Paley decomposition.

**Lemma 2.1.** \[2\] Let \( \mathcal{C} = \{ \xi \in \mathbb{R}, \frac{1}{4} \leq |\xi| \leq \frac{5}{4} \} \) be an annulus. There exist radial functions \( \chi \) and \( \phi \) valued in the interval \([0, 1]\), belonging respectively to \( \mathcal{D}(B(0, \frac{1}{4})) \) and \( \mathcal{D}(\mathcal{C}) \), such that

\[
\forall \xi \in \mathbb{R}, \quad \chi(\xi) + \sum_{j \geq 0} \phi(2^{-j} \xi) = 1.
\]

The nonhomogeneous dyadic blocks \( \triangle_j \) and the nonhomogeneous low-frequency cut-off operator \( S_j \) are then defined as follows:

\[
\triangle_j u = 0 \quad \text{if} \quad j \leq -2, \quad \triangle_{-1} u = \chi(D) u,
\]

\[
\triangle_j u = \phi(2^{-j}D) u \quad \text{if} \quad j \geq 0, \quad S_j u = \sum_{j' \leq j-1} \triangle_{j'} u \quad \text{for} \quad j \in \mathbb{Z}.
\]

**Definition 2.1.** \[2\] Let \( s \in \mathbb{R} \) and \( (p, r) \in [1, \infty]^2 \). The nonhomogeneous Besov space \( B^s_{p,r} \) consists of all \( u \in \mathcal{S}'(\mathbb{R}) \) such that

\[
\|u\|_{B^s_{p,r}} \overset{\text{def}}{=} \left( \sum_{j \in \mathbb{Z}} \left( 2^{js} \|\triangle_j u\|_{L^p} \right)^r \right)^{1/r} < \infty.
\]

Let us give some classical properties of the Besov spaces.
Lemma 2.2. The set $B_{p,r}^s$ is a Banach space, and satisfies the Fatou property, namely, if $(u_n)_{n \in \mathbb{N}}$ is a bounded sequence of $B_{p,r}^s$, then an element $u$ of $B_{p,r}^s$ and a subsequence $u_{\psi(n)}$ exist such that

$$\lim_{n \to \infty} u_{\psi(n)} = u \text{ in } S' \text{ and } \|u\|_{B_{p,r}^s} \leq C \liminf_{n \to \infty} \|u_{\psi(n)}\|_{B_{p,r}^s}.$$ 

Lemma 2.3. Let $m \in \mathbb{R}$ and $f$ be an $S^m$-multiplier (i.e. $f : \mathbb{R} \to \mathbb{R}$ is smooth and satisfies that for each multi-index $\alpha$, there exists a constant $C_\alpha$ such that $|\partial^\alpha f(x)| \leq C_\alpha (1 + |x|)^{m-|\alpha|}, \forall x \in \mathbb{R}$). Then the operator $F(D)$ is continuous from $B_{p,r}^s$ to $B_{p,r}^{s-m}$.

Lemma 2.4. [18] (i) For $s > 0$ and $1 \leq p, r \leq \infty$, there exists $C = C(d, s)$ such that

$$\|uv\|_{B_{p,r}^s} \leq C(\|u\|_{L^\infty}\|v\|_{B_{p,r}^s} + \|v\|_{L^\infty}\|u\|_{B_{p,r}^s}).$$

(ii) If $1 \leq p, r \leq \infty$, $s_1 \leq \frac{1}{p}$, $s_2 > \frac{1}{p}$, $s_2 \geq \frac{1}{p}$, if $r = 1$ and $s_1 + s_2 > \max\{0, \frac{2}{p} - 1\}$, there exists $C = C(s_1, s_2, p, r)$ such that

$$\|uv\|_{B_{p,r}^{s_1}} \leq C\|u\|_{B_{p,r}^{s_1}}\|v\|_{B_{p,r}^{s_2}}.$$ 

Lemma 2.5. [18] Let $1 \leq p \leq p_1 \leq \infty$, $1 \leq r \leq \infty$, $s > -\min\{\frac{1}{p_1}, 1 - \frac{1}{p}\}$. Assume $f_0 \in B_{p,r}^s$, $F \in L^1(0, T; B_{p,r}^s)$, $v \in L^p(0, T; B_{\infty, \infty}^{-M})$ for some $\rho > 1$ and $M > 0$, and

$$\frac{\partial}{\partial t}v \in L^1(0, T; B_{p_1, \infty}^{\frac{1}{p_1}} \cap L^\infty), \quad \text{if } s < 1 + \frac{1}{p_1},$$

$$\frac{\partial}{\partial t}v \in L^1(0, T; B_{p_1, r}^{s-1}), \quad \text{if } s > 1 + \frac{1}{p_1}, \text{ or } s = 1 + \frac{1}{p_1} \text{ and } r = 1.$$ 

Then the following transport equation

$$\begin{cases} \frac{\partial}{\partial t}f + v \cdot \nabla f = F \\ f|_{t=0} = f_0, \end{cases}$$

has a unique solution $f \in C([0, T]; B_{p,r}^s)$, if $r < \infty$, or $f \in L^\infty(0, T; B_{p,r}^s) \cap \left( \bigcap_{s' < s} C([0, T]; B_{p,r}^{s'}) \right)$, if $r = \infty$.

Moreover, the following inequality holds true:

$$\|f(t)\|_{B_{p,r}^s} \leq \|f_0\|_{B_{p,r}^s} + \int_0^t \|F(\tau)\|_{B_{p,r}^s} \, d\tau + C \int_0^t V_{p_1}'(\tau) \|f(\tau)\|_{B_{p,r}^s} \, d\tau$$

with

$$V_{p_1}'(t) = \begin{cases} \|\partial_x v(t)\|_{B_{p_1, \infty}^{\frac{1}{p_1}} \cap L^\infty}, & \text{if } s < 1 + \frac{1}{p_1}, \\ \|\partial_x v(t)\|_{B_{p_1, r}^{s-1}}, & \text{if } s > 1 + \frac{1}{p_1} \text{ or } s = 1 + \frac{1}{p_1}, \text{ } r = 1. \end{cases}$$
3 Local well-posedness

In this section, we study the local well-posedness for Eq. (1.1).

To begin with, noticing \((1 - \partial_x^2)^{-1} = \frac{1}{2} e^{-|x|} \ast\), we have the following inequalities which will be frequently used in the sequel:

\[
\|u\|_{B^s_{p,r}} = \|(1 - \partial_x^2)^{-1} m\|_{B^s_{p,r}} \approx \|m\|_{B^s_{p,r}^2}, \quad \forall \ s \in \mathbb{R}, \ 1 \leq p, r \leq \infty.
\]
\[
\|u\|_{L^\infty} = \frac{1}{2} e^{-|x|} \ast m\|_{L^\infty} \leq \|m\|_{L^\infty},
\]
\[
\|u_x\|_{L^\infty} = \frac{1}{2} (-\text{sign}(x) e^{-|x|}) \ast m\|_{L^\infty} \leq \|m\|_{L^\infty},
\]
\[
\|u_{xx}\|_{L^\infty} = \|u - m\|_{L^\infty} \leq 2 \|m\|_{L^\infty},
\]

where \(m = u - u_{xx}\).

We now rewrite Eq. (1.1) as follows:

\[
M_t = H(U, U_x)M_x + A(H, H_x)M + B(U, U_x)M,
\]
\[
M_{t=0} = M_0,
\]

where \(M = (m_1, \ldots, m_N, n_1, \ldots, n_N)^T, \ M_0 = (m_{10}, \ldots, m_{N0}, n_{10}, \ldots, n_{N0})^T, U = (u_1, \ldots, u_N, v_1, \ldots, v_N)^T, \ H = H(U, U_x)\) is a polynomial of degree \(l\), and

\[
A(H, H_x) = \begin{pmatrix}
H_x I_{N \times N} + H I_{N \times N} & 0 \\
0 & H_x I_{N \times N} - H I_{N \times N}
\end{pmatrix},
\]

\[
B(U, U_x) = \begin{pmatrix}
B_{11} & 0 \\
0 & B_{22}
\end{pmatrix}
\]

with

\[
B_{11} = \frac{1}{(N+1)^2} \begin{pmatrix}
(u_1 - u_{1x})(v_1 + v_{1x}) & \cdots & (u_1 - u_{1x})(v_N + v_{Nx}) \\
\vdots & \ddots & \vdots \\
(u_N - u_{Nx})(v_1 + v_{1x}) & \cdots & (u_N - u_{Nx})(v_N + v_{Nx})
\end{pmatrix} + \sum_{i=1}^{N} [(u_i - u_{ix})(v_i + v_{ix})] I_{N \times N},
\]

and

\[
B_{11} = -\frac{1}{(N+1)^2} \begin{pmatrix}
(u_1 - u_{1x})(v_1 + v_{1x}) & \cdots & (u_N - u_{Nx})(v_1 + v_{1x}) \\
\vdots & \ddots & \vdots \\
(u_1 - u_{1x})(v_N + v_{Nx}) & \cdots & (u_N - u_{Nx})(v_N + v_{Nx})
\end{pmatrix} - \sum_{i=1}^{N} [(u_i - u_{ix})(v_i + v_{ix})] I_{N \times N}.
\]
3.1. Local existence and uniqueness

Theorem 3.1. Let \( 1 \leq p, r \leq \infty \), \( s > \max\{1 - \frac{1}{p}, \frac{1}{p}\} \), and \( M_0 \in B^s_{p,r} \). Then exists a time \( T > 0 \) such that Eq. (3.1) has a unique solution \( M \in L^\infty(0, T; B^s_{p,r}) \cap E^s_{p,r}(T) \) with

\[
E^s_{p,r}(T) \triangleq \begin{cases} \quad C([0, T]; B^s_{p,r}) \cap C^1([0, T]; B^{s-1}_{p,r}), & \text{if } r < \infty, \\ \quad \bigcap_{i < s} \left( C([0, T]; B^i_{p,r}) \cap C^1([0, T]; B^{i-1}_{p,r}) \right), & \text{if } r = \infty. \end{cases}
\]

The proof relies heavily on the following lemma.

Lemma 3.1. Let \( 1 \leq p, r \leq \infty \) and \( s > \max\{1 - \frac{1}{p}, \frac{1}{p}\} \). Suppose that \( M^1 \) and \( M^2 \) are two solutions of the Eq. (3.1) with the initial data \( M^1_0, M^2_0 \in L^\infty(0, T; B^s_{p,r}) \cap C([0, T]; S^r) \). Let \( M^{12} = M^1 - M^2 \), \( U^{12} = U^1 - U^2 \), and \( q = \max\{1, 2\} \) (where \( l \) is the polynomial order of \( H \)). Then, for all \( t \in [0, T] \), we have

1. if \( s > \max\{1 - \frac{1}{p}, \frac{1}{p}\} \), but \( s \neq 2 + \frac{1}{p} \), then

\[
\| M^{12}(t) \|_{B^{s}_{p,r}} \leq \| M^1_0 \|_{B^{s}_{p,r}} e^{C \int_0^t (\| M^1(\tau) \|_{B^{s}_{p,r}}^q + \| M^2(\tau) \|_{B^{s}_{p,r}}^q + 1) \, d\tau};
\]

2. if \( s = 2 + \frac{1}{p} \), then

\[
\| M^{12}(t) \|_{B^{s}_{p,r}} \leq \| M^1_0 \|_{B^{s}_{p,r}}^\theta (\| M^1(t) \|_{B^{s}_{p,r}} + \| M^2(t) \|_{B^{s}_{p,r}})^{1-\theta} e^{\theta C \int_0^t (\| M^1(\tau) \|_{B^{s}_{p,r}}^q + \| M^2(\tau) \|_{B^{s}_{p,r}}^q + 1) \, d\tau},
\]

where \( \theta \in (0, 1) \).

Proof. Let \( H^i = H(U^i, U^i_x) \), \( A^i = A(H^i, H^i_x) \), \( B^i = B(U^i, U^i_x) \), \( i = 1, 2 \), and \( H^{12} = H^1 - H^2 \), \( A^{12} = A^1 - A^2 \), \( B^{12} = B^1 - B^2 \). It is obvious that \( M^{12} \) solves the following transport equation

\[
M^{12}_{t} - H^1 M^{12}_x = F_1 + F_2 + F_3
\]

where \( F_1 = H^{12} M^2_x F_2 = (A^1 + B^1) M^{12} \) and \( F_3 = (A^{12} + B^{12}) M^2 \).

We claim that for all \( s > \max\{1 - \frac{1}{p}, \frac{1}{p}\} \), we have

\[
\| uv \|_{B^{s}_{p,r}} \lesssim \| u \|_{B^{s}_{p,r}} \| v \|_{B^{s}_{p,r}}.
\]

Indeed, if \( s > 1 + \frac{1}{p} \), then \( B^{s-1}_{p,r} \) is an algebra. Thus we have

\[
\| uv \|_{B^{s}_{p,r}} \lesssim \| u \|_{B^{s-1}_{p,r}} \| v \|_{B^{s-1}_{p,r}} \lesssim \| u \|_{B^{s-1}_{p,r}} \| v \|_{B^{s}_{p,r}}.
\]

On the other hand, if \( \max\{1 - \frac{1}{p}, \frac{1}{p}\} < s \leq 1 + \frac{1}{p} \), then applying Lemma (ii) with \( s_1 = s - 1 \) and \( s_2 = s \) yields (3.4).
Therefore, for all \( s > \max\{1 - \frac{1}{p}, \frac{1}{p}\} \), noticing the fact that \( B^s_{p,r} \) is an algebra, one may infer the following inequalities:

\[
\| F_i \|_{B^{s-1}_{p,r}} \lesssim \| H^{12}_{B^{s-1}_{p,r}} \|_{B^{s}_{p,r}} M^{12}_{B^{s}_{p,r}} \\
\lesssim \left( \| U_{1}^{12}_{B^{s}_{p,r}} + \| U_{12}^{12}_{B^{s}_{p,r}} \| \right) \left( \| U_{1}^{1} + \| U_{1}^{1} + \| U_{2}^{1} + \| U_{2}^{1} \| \right) M^{2}_{B^{s}_{p,r}} \\
\lesssim \| M^{12}_{B^{s-1}_{p,r}} \| \left( \| M^{1}_{B^{s}_{p,r}} \| + \| M^{2}_{B^{s}_{p,r}} \| \right) + 1,
\]

\[
\| F_2 \|_{B^{s-1}_{p,r}} \lesssim \left( \| A^{1}_{B^{s}_{p,r}} + \| B^{1}_{B^{s}_{p,r}} \| \right) \left( \| M^{12}_{B^{s}_{p,r}} \| + \| M^{2}_{B^{s}_{p,r}} \| \right) + 1,
\]

\[
\| F_3 \|_{B^{s-1}_{p,r}} \lesssim \left( \| A^{12}_{B^{s}_{p,r}} + \| B^{12}_{B^{s}_{p,r}} \| \right) \left( \| M^{1}_{B^{s}_{p,r}} \| + \| M^{2}_{B^{s}_{p,r}} \| \right) + 1,
\]

with \( q = \max\{l, 2\} \).

Thus, for the case (1) \( s > \max\{1 - \frac{1}{p}, \frac{1}{p}\} \) and \( s \neq 2 + \frac{1}{p} \), using Lemma 2.5 with the above three inequalities and \( p_1 = p \) and

\[
V_{p_1}^{s}(t) = \| \partial_x H^{1}_{B^{s-2}_{p,r}} + \| \partial_x H^{1}_{B^{s}_{p,r}} \|_{L^{\infty}} \leq \| \partial_x H^{1}_{B^{s}_{p,r}} \|
\]

we have

\[
\| M^{12}_{B^{s}_{p,r}} \| \leq \| M^{12}_{B^{s-1}_{p,r}} \| + \int_{0}^{t} (F_1 + F_2 + F_3)(\tau) \| M^{12}_{B^{s-1}_{p,r}} \| d\tau + C \int_{0}^{t} V_{p_1}^{s}(\tau) \| M^{12}_{B^{s-1}_{p,r}} \| d\tau
\]

\[
\leq \| M^{12}_{B^{s-1}_{p,r}} \| + C \int_{0}^{t} \| M^{12}_{B^{s-1}_{p,r}} \| \left( \| M^{1}_{B^{s}_{p,r}} \| \right) \| M^{2}_{B^{s}_{p,r}} \| d\tau
\]

\[
\leq \| M^{12}_{B^{s-1}_{p,r}} \| + C \int_{0}^{t} \| M^{12}_{B^{s-1}_{p,r}} \| \left( \| M^{1}_{B^{s}_{p,r}} \| \right) \| M^{2}_{B^{s}_{p,r}} \| d\tau
\]

\[
\text{Hence, the Gronwall lemma gives the inequality.}
\]

For the critical case (2) \( s = 2 + \frac{1}{p} \), let us choose \( s_1 \in (\max\{1 - \frac{1}{p}, \frac{1}{p}\} - 1, s - 1) \), \( s_2 \in (s - 1, s) \). Then \( s = s_1 + (1 - \theta)s_2 \) with \( \theta = \frac{s_2 - 1}{s_2 - s_1} \in (0, 1) \). By using the interpolation inequality and the consequence
of the case (1), we get
\[
\|M^{12}(t)\|_{B_{p,r}^{-1}} \leq \|M^{12}(t)\|_{B_{p,r}^{0}}^{\theta} \|M^{12}(t)\|_{B_{p,r}^{1-\theta}}^{1-\theta} \\
\leq \left(\|M_{0}^{12}\|_{B_{p,r}^{0}} + \int M^{12}(t) \, dt\right)^{\theta} \|M^{1}(t)\|_{B_{p,r}^{2}}^{\theta} \|M^{2}(t)\|_{B_{p,r}^{2}}^{1-\theta} \\
\leq \|M_{0}^{12}\|_{B_{p,r}^{0}} \|M^{1}(t)\|_{B_{p,r}^{1}} + \|M^{2}(t)\|_{B_{p,r}^{1}}^{1-\theta} e^{\theta C \int_{0}^{1} \lambda_{2}^{0} \lambda_{1} \, dt},
\]
which completes the proof of the lemma.

Proof of Theorem 3.1. Since uniqueness in Theorem 3.1 is a straightforward corollary of Lemma 3.1, we need only to prove the existence of a solution to Eq.(3.1). We shall proceed as follows.

First step: constructing approximate solutions.

Starting from \(M^{0} = M_{0}\) we define by induction a sequence \((M^{n})_{n \in \mathbb{N}}\) by solving the following linear transport equation
\[
(3.6) \\
\begin{align*}
M_{t}^{n+1} - H^{n}M_{x}^{n+1} &= A^{n} M^{n} + B^{n} M^{n}, \\
M_{t=0}^{n+1} &= M_{0},
\end{align*}
\]
where \(M^{n} = (m_{1}^{n}, \ldots, m_{N}^{n}, n_{1}^{n}, \ldots, n_{N}^{n})^{T}, U^{n} = (u_{1}^{n}, \ldots, u_{N}^{n}, v_{1}^{n}, \ldots, v_{N}^{n})^{T}, H^{n} = H(U^{n}, U_{x}^{n}), A^{n} = A(H^{n}, H_{x}^{n}), B^{n} = B(U^{n}, U_{x}^{n}).
\]

Second step: uniform bounds.

Let \(q = \max\{l, 2\}\). The condition \(s > \max\{1 - \frac{1}{p}, \frac{1}{p}\}\) yields that \(B_{p,r}^{s}\) is an algebra. Thus, we have
\[
\|A^{n} M^{n} + B^{n} M^{n}\|_{B_{p,r}^{s}} \leq \left(\|A^{n}\|_{B_{p,r}^{s}} + \|B^{n}\|_{B_{p,r}^{s}}\right)\|M^{n}\|_{B_{p,r}^{s}}
\]
we get

\[
\|M^{n+1}(t)\|_{B^s_{p,r}} \leq \|M_0\|_{B^s_{p,r}} + C \int_0^t \left(1 + \|M^n(\tau)\|_{B^s_{p,r}}^q\right) \|M^n(\tau)\|_{B^s_{p,r}} d\tau \\
+ C \int_0^t \|\partial_x H^n(\tau)\|_{B^s_{p,r}} \|M^{n+1}(\tau)\|_{B^s_{p,r}} d\tau \\
\leq \|M_0\|_{B^s_{p,r}} + C \int_0^t \left(1 + \|M^n(\tau)\|_{B^s_{p,r}}^q\right) \|M^n(\tau)\|_{B^s_{p,r}} \|M^{n+1}(\tau)\|_{B^s_{p,r}} d\tau \\
+ C \int_0^t \left(\|U^n\|_{B^s_{p,r}}^q + \|U^n_x\|_{B^s_{p,r}}^q + \|U^n_x\|_{B^s_{p,r}}^q\right) \|M^{n+1}(\tau)\|_{B^s_{p,r}} d\tau \\
\leq \|M_0\|_{B^s_{p,r}} + C \int_0^t \left(\|M^n(\tau)\|_{B^s_{p,r}}^q + 1\right) \|M^{n+1}(\tau)\|_{B^s_{p,r}} d\tau.
\]

The Gronwall lemma yields that

\[
\|M^{n+1}(t)\|_{B^s_{p,r}} \leq \|M_0\|_{B^s_{p,r}} e^{\int_0^t \left(1 + \|M^n(\tau)\|_{B^s_{p,r}}^q\right) d\tau} \left(1 + \|M^n(\tau)\|_{B^s_{p,r}}\right) \|M^n(\tau)\|_{B^s_{p,r}} d\tau).
\]

Notice that \( f(t) = \frac{f_{\text{sol}}}{(1+f_0^2-f_0 e^{2CT})} \) is the solution to the following equation:

\[
f'(t) = e^{\int_0^t \left(1 + f^n(\tau)\right) d\tau} \left(f_0 + C \int_0^t e^{-\int_0^t \left(1 + f^n(\tau)\right) d\tau} \left(1 + f^n(\tau)\right) d\tau\right).
\]

We fix a \( T > 0 \) such that \( 1 + \|M_0\|_{B^s_{p,r}}^q - \|M_0\|_{B^s_{p,r}}^q \ e^{2CT} > 0 \) and suppose that

\[
\forall \ t \in [0, T], \ \|M^n\|_{B^s_{p,r}} \leq \frac{\|M_0\|_{B^s_{p,r}} e^{2CT}}{(1 + \|M_0\|_{B^s_{p,r}}^q - \|M_0\|_{B^s_{p,r}}^q \ e^{2CT})}.
\]

Plugging the above inequality into (3.7) and using (3.8) yield

\[
\|M^{n+1}(t)\|_{B^s_{p,r}} \leq \|M_0\|_{B^s_{p,r}} e^{2CT}.
\]

Therefore, \( (M^n)_{n \in \mathbb{N}} \) is bounded in \( L^\infty(0, T; B^s_{p,r}) \).

Third step: convergence.

Similar to the proof of (3.5), we have, for \( s > \max\{1 - \frac{1}{p}, \frac{1}{p}\} \) and \( s \neq 2 + \frac{1}{p} \),

\[
\|(M^{n+m+1} - M^{n+1})(t)\|_{B^s_{p,r}}^{-1} \leq \int_0^t \|(M^{n+m} - M^n)(\tau)\|_{B^s_{p,r}}^{-1} \|M^n(\tau)\|_{B^s_{p,r}}^q + \|M^{n+1}(\tau)\|_{B^s_{p,r}}^q + \|M^{n+m}(\tau)\|_{B^s_{p,r}}^q + 1 d\tau \\
+ C \int_0^t \|(M^{n+m+1} - M^{n+1})(\tau)\|_{B^s_{p,r}}^{-1} (\|M^{n+m}(\tau)\|_{B^s_{p,r}}^q + 1) d\tau.
\]
Taking advantage of the Gronwall inequality gives
\[
\| (M^{n+m+1} - M^{n+1})(t) \|_{B^{s-1}_{p,r}}^p \\
\leq C e^{\int_0^t (\| M^{n+m} \|_{B_{p,r}^s} + 1) dt'} \int_0^{t'} e^{-C \int_0^{t'} (\| M^{n+m} \|_{B_{p,r}^s} + 1) dt'} \| (M^{n+m} - M^n)(\tau) \|_{B^{s-1}_{p,r}}^p d\tau.
\]

Since \((M^n)_{n\in\mathbb{N}}\) is bounded in \(L^\infty(0, T; B^{s-1}_{p,r})\), we finally get a constant \(C_T\), independent of \(n\) and \(m\), such that
\[
\| (M^{n+m+1} - M^{n+1})(t) \|_{B^{s-1}_{p,r}}^p \leq C_T \int_0^t \| (M^{n+m} - M^n)(\tau) \|_{B^{s-1}_{p,r}} d\tau.
\]

Finally, arguing by induction, we arrive at
\[
\| (M^{n+m+1} - M^{n+1})(t) \|_{B^{s-1}_{p,r}}^p \leq \frac{(TC_T)^{n+1}}{(n+1)!} \| M^m - M^0 \|_{L^\infty(B^{s-1}_{p,r})} \leq C_T \frac{(TC_T)^{n+1}}{(n+1)!},
\]
which implies that \((M^n)_{n\in\mathbb{N}}\) is a Cauchy sequence in \(L^\infty(0, T; B^{s-1}_{p,r})\).

For the critical case \(s = 2 + \frac{1}{p}\), from the above argument, we get that \((M^n)_{n\in\mathbb{N}}\) is a Cauchy sequence in \(L^\infty(0, T; B^{s-1-\varepsilon}_{p,r})\) with sufficiently small \(\varepsilon\). Then applying the interpolation method with uniform bounds in \(L^\infty(0, T; B^{s}_{p,r})\) obtained in the second step, we show that \((M^n)_{n\in\mathbb{N}}\) is also a Cauchy sequence in \(L^\infty(0, T; B^{s-1}_{p,r})\) for the critical case.

Final step: conclusion.

Let \(M\) be the limit of the sequence \((M^n)_{n\in\mathbb{N}}\) in \(L^\infty(0, T; B^{s-1}_{p,r})\). According to the Fatou lemma \([7,2]\), \(M\) also belongs to \(L^\infty(0, T; B^{s}_{p,r})\). It is then easy to pass to the limit in Eq. \([3.4]\) and to conclude that \(M\) is a solution of Eq. \([3.4]\). Note that \(A(U, U_x)M + B(H, H_x)M\) of Eq. \([3.4]\) also belongs to \(L^\infty(0, T; B^{s}_{p,r})\).

According to Lemma \([2,5]\) we have \(M \in C([0, T]; B^{s}_{p,r})\) if \(r < \infty\), or \(M \in \left( \bigcap_{s' < s} C([0, T]; B^{s'}_{p,r}) \right)\), if \(r = \infty\).

Again using the equation, we see that \(M_t \in C([0, T]; B^{s-1}_{p,r})\) if \(r < \infty\), or \(M_t \in \left( \bigcap_{s' < s} C([0, T]; B^{s'-1}_{p,r}) \right)\), if \(r = \infty\). This completes the proof of Theorem \([3.1]\). \(\square\)

### 3.2. A continuation criterion

In this subsection, we state a continuation criterion for Eq. \([3.1]\).

**Theorem 3.2.** Let \(M_0 \in B^{s}_{p,r}\) with \(1 \leq p, r \leq \infty\), \(s > \max\{1 - \frac{1}{p}, \frac{1}{p}\}\), and \(T > 0\) be the maximal existence time of the corresponding solution \(M\) to Eq. \([3.1]\). If \(T\) is finite, then we have
\[
\int_0^T \| M(\tau) \|_{L^\infty}^q d\tau = \infty,
\]
where \(q = \max\{1, 2\} \ (l\ is\ the\ polynomial\ order\ of\ H)\).
We now consider the case $1 < p < \infty$.

Step 1. If $\sigma > 1$, then we claim that

\begin{equation}
\|M\|_{L^p_T(B_{\sigma,r}^-)} < \infty, \quad \text{and} \quad \int_0^T \|M(\tau)\|_{L^\infty}^p \, d\tau < \infty \Rightarrow \|M\|_{L^p_T(B_{\sigma,r}^-)} < \infty.
\end{equation}

In fact, by using (3.9) and Lemma 2.5 with $p_1 = \infty$ and

\[
V'_{p_1}(t) = \|\partial_x H\|_{B_{\sigma,r}^-} \leq \|\partial_x H\|_{B_{\sigma,r}^-} \leq \|U\|_{B_{\sigma,r}^-}^t + \|U_x\|_{B_{\sigma,r}^-}^t + \|U_{xx}\|_{B_{\sigma,r}^-}^t \leq \|M\|_{B_{\sigma,r}^-}^t,
\]

we have

\[
\|M(t)\|_{B_{\sigma,r}^-} \leq \|M_0\|_{B_{\sigma,r}^-} + C \int_0^t (\|M(\tau)\|_{L^\infty}^p + 1) \|M(\tau)\|_{B_{\sigma,r}^-}^p \, d\tau.
\]

Hence, the Gronwall lemma gives

\[
\|M(t)\|_{B_{\sigma,r}^-} + 1 \leq (\|M_0\|_{B_{\sigma,r}^-} + 1) e^{C \int_0^t (\|M(\tau)\|_{L^\infty}^p + 1) \|M(\tau)\|_{B_{\sigma,r}^-}^p \, d\tau},
\]

which implies

\[
\|M\|_{L^p_T(B_{\sigma,r}^-)} < \infty, \quad \text{and} \quad \int_0^T \|M(\tau)\|_{L^\infty}^p \, d\tau < \infty \Rightarrow \|M\|_{L^p_T(B_{\sigma,r}^-)} < \infty.
\]

If $\sigma - 1 + 1/p > 1$, then repeat the above process. Clearly, this process stops within a finite number of steps.

Our claim (3.10) is guaranteed.

Step 2. If $\sigma = 1$, then by using (3.9) and Lemma 2.5 with $p_1 = p$ and

\[
V'_{p_1}(t) = \|\partial_x H\|_{B_{\sigma,r}^-} \leq \|U\|_{B_{1,\infty}}^t + \|U_x\|_{B_{1,\infty}}^t + \|U_{xx}\|_{B_{1,\infty}}^t,
\]

we have

\[
\|M(t)\|_{B_{1,r}^-} \leq \|M_0\|_{B_{1,r}^-} + C \int_0^t (\|M(\tau)\|_{L^\infty}^p + 1) \|M(\tau)\|_{B_{1,r}^-}^p \, d\tau.
\]

Hence, the Gronwall lemma gives

\[
\|M(t)\|_{B_{1,r}^-} + 1 \leq (\|M_0\|_{B_{1,r}^-} + 1) e^{C \int_0^t (\|M(\tau)\|_{L^\infty}^p + 1) \|M(\tau)\|_{B_{1,r}^-}^p \, d\tau},
\]

which implies

\[
\|M\|_{L^p_T(B_{1,r}^-)} < \infty, \quad \text{and} \quad \int_0^T \|M(\tau)\|_{L^\infty}^p \, d\tau < \infty \Rightarrow \|M\|_{L^p_T(B_{1,r}^-)} < \infty.
\]
\[ \|M\|_{L^p(B_{p,r})}^q \leq \|M\|_{B_{p,r}}^q + \|M\|_{L^\infty}^q \]

we have
\[ \|M(t)\|_{B_{p,r}} \leq \|M_0\|_{B_{p,r}} + C \int_0^t (\|M(\tau)\|_{L^\infty}^q + 1) \|M(\tau)\|_{B_{p,r}}^q + 1) d\tau \\
+ C \int_0^t (\|M\|_{B_{p,r}}^q + \|M\|_{L^\infty}^q + 1) \|M(\tau)\|_{B_{p,r}} d\tau. \]

Hence, the Gronwall lemma gives
\[ \|M(t)\|_{B_{p,r}} + 1 \leq (\|M_0\|_{B_{p,r}} + 1)e^{C \int_0^t (\|M(\tau)\|_{L^\infty}^q + 1) \|M(\tau)\|_{B_{p,r}}^q + 1) d\tau}, \]

which implies
\[ (3.11) \quad \|M\|_{L^p(B_{p,r})}^q < \infty \text{ and } \int_0^T \|M(\tau)\|_{L^\infty}^q d\tau < \infty \Rightarrow \|M\|_{L^p(B_{p,r})} < \infty. \]

Step 3. If \( \sigma \in (0,1) \), applying Lemma 2.3 with \( p_1 = \infty \) and
\[ V_{p_1}(t) = \|\partial_x H\|_{B_{\infty,\infty}} \leq \|\partial_x H\|_{L^\infty} \leq \|U\|_{L^\infty}^2 + \|U_x\|_{L^\infty}^2 + \|U_{xx}\|_{L^\infty} \leq \|M\|_{L^\infty}^2 + 1, \]

we have
\[ \|M(t)\|_{B_{p,r}} \leq \|M_0\|_{B_{p,r}} + C \int_0^t (\|M(\tau)\|_{L^\infty}^q + 1) \|M(\tau)\|_{B_{p,r}}^q + 1) d\tau \\
+ C \int_0^t (\|M\|_{L^\infty}^q + 1) \|M(\tau)\|_{B_{p,r}} d\tau. \]

Hence, the Gronwall lemma gives
\[ \|M(t)\|_{B_{p,r}} + 1 \leq (\|M_0\|_{B_{p,r}} + 1)e^{C \int_0^t (\|M(\tau)\|_{L^\infty}^q + 1) d\tau}, \]

which implies
\[ (3.12) \quad \int_0^T \|M(\tau)\|_{L^\infty}^q d\tau < \infty \Rightarrow \|M\|_{L^p(B_{p,r})} < \infty. \]

Therefore, for all \( s > \max\{1 - \frac{1}{p}, \frac{1}{p_1}\} \), if \( T < \infty \), and \( \int_0^T \|M(\tau)\|_{L^\infty}^q d\tau < \infty \), then we have \( \lim sup_{t \to T} \|M(t)\|_{B_{p,r}} < \infty. \)

The cases \( p = 1 \) and \( p = \infty \) can be treated similarly. We also have for \( s > 1 \), if \( T < \infty \), and \( \int_0^T \|M(\tau)\|_{L^\infty}^q d\tau < \infty \), then \( \lim sup_{t \to T} \|M(t)\|_{B_{p,r}} < \infty. \) For the sake of simplicity, we omit the details here.\(^1\)

\(^1\)We present a simple flow chart. For \( p = \infty \), \( \sigma > 1 \), let \( \varepsilon \in (0,1) \). \( \|M\|_{B_{\infty,\infty}}^q \) \( \geq \|\partial_x H\|_{B_{\infty,\infty}}^q \) \( \geq \|M\|_{B_{\infty,\infty}}^q \) \( \geq \|\partial_x H\|_{B_{\infty,\infty}}^q \) \( \geq \|M\|_{L^\infty}^q \) \( \geq \|M\|_{L^\infty}^q \).

For \( p = 1 \), \( \sigma > 1 \), choose \( p_1 \) such that \( 1 < p_1 < \infty \) and \( \sigma > 1 + \frac{1}{p_1} \). \( \|M\|_{B_{1,1}}^q \) \( \geq \|M\|_{B_{1,1}}^q \) \( \geq \|\partial_x H\|_{B_{1,1}}^q \) \( \geq \|M\|_{L^\infty}^q \) \( \geq \|M\|_{L^\infty}^q \).
Finally, if \( \limsup_{t \to T} \| M(t) \|_{B^p_{r,r}} < \infty \), then by Theorem 3.4 we can extend the solution \( M \) beyond \( T \), which is a contradiction with the assumption of \( T \). Then we must have \( \int_0^T \| M(\tau) \|^2_{L^\infty} d\tau = \infty \). This completes the proof of the theorem.

Combining Theorem 3.1 and Theorem 3.2 we readily obtain the following corollary.

**Corollary 3.1.** Let \( M_0 \in B^s_{p,r} \) with \( 1 \leq p, r \leq \infty \) and \( s > \max\{1 - \frac{1}{p}, \frac{1}{p}\} \) and \( T > 0 \) be the maximal existence time of the corresponding solution \( M \) to Eq. (3.1). Then the solution \( M \) blows up in finite time if and only if \( \limsup_{t \to T} \| M(t) \|_{L^\infty} = \infty \).

**Remark 3.1.** Apparently, for every \( s \in \mathbb{R} \), \( B^s_{2,2} = H^s \). Theorem 3.1, Theorem 3.2 and Corollary 3.1 hold true in the corresponding Sobolev spaces \( H^s \) with \( s > \frac{1}{2} \), which recovers the corresponding results in [36] and [25] as \( N = 1 \), \( H = -\frac{1}{2}(u_1v_1 - u_{1x}v_{1x}) \) and \( N = 1 \), \( H = -\frac{1}{2}(u_1v_1 - u_{1x}v_{1x}), v = 2u \), respectively.

**Remark 3.2.** We pointed out that if \( N = 1 \) and \( H = -\frac{1}{2}(u_1v_1 - u_{1x}v_{1x}) \), then Theorem 3.1 improves the corresponding result in [36], where \( s > \max\{1 - \frac{1}{p}, \frac{1}{p}, \frac{1}{2}\} \) but \( s \neq 1 + \frac{1}{p} \). Besides, Theorem 3.2, which works in Besov spaces, also improves the corresponding result in [36], where the corresponding blow-up scenario works only in Sobolev spaces.

## 4 Global existence and blow-up phenomena for the two-component subsystems

### 4.1 \( N = 1, H = -\frac{1}{2}(u - u_x)(v + v_x) \)

#### 4.1.1 A precise blow-up scenario

As mentioned in the Introduction, for \( N = 1 \) and \( H = -\frac{1}{2}(u - u_x)(v + v_x) \), Eq. (1.1) is reduced to the following system:

\[
\begin{align*}
  m_t + \frac{1}{2}((u - u_x)(v + v_x)m)_x &= 0, \\
  n_t + \frac{1}{2}((u - u_x)(v + v_x)n)_x &= 0,
\end{align*}
\]

(4.1)

where \( m = u - u_{xx} \) and \( n = v - v_{xx} \).

Consider the following initial value problem

\[
\begin{align*}
  q_t(t, x) &= \frac{1}{2}(u - u_x)(v + v_x)(t, q), \quad t \in [0, T), \\
  q(0, x) &= x, \quad x \in \mathbb{R}.
\end{align*}
\]

(4.2)
**Lemma 4.1.** Let $m_0, n_0 \in H^s$ ($s > \frac{1}{2}$), and let $T > 0$ be the maximal existence time of the corresponding solution $(m, n)$ to Eq. (4.1). Then Eq. (4.2) has a unique solution $q \in C^1([0, T] \times \mathbb{R}; \mathbb{R})$. Moreover, the mapping $q(t, \cdot)$ ($t \in [0, T]$) is an increasing diffeomorphism of $\mathbb{R}$, with

\begin{equation}
q_x(t, x) = \exp\left(\int_0^t \frac{1}{2}(m(v + v_x) - n(u - u_x))(\tau, q(\tau, x))d\tau\right).
\end{equation}

Proof. According to Remark 3.1, we get that $m, n \in C([0, T]; H^s) \cap C^1([0, T]; H^{s-1})$ with $s > \frac{1}{2}$, from which we deduce that $\frac{1}{2}(u - u_x)(v + v_x)$ is bounded and Lipschitz continuous in the space variable $x$ and of class $C^1$ in time variable $t$, then the classical ODE theory ensures that Eq. (4.2) has a unique solution $q \in C^1([0, T] \times \mathbb{R}; \mathbb{R})$. Differentiating Eq. (4.2) with respect to $x$ gives

\begin{equation}
\begin{cases}
q_{xt}(t, x) = \frac{1}{2}(m(v + v_x) - n(u - u_x))(t, q_x(t, x)), \quad t \in [0, T), \\
q_x(0, x) = 1, \quad x \in \mathbb{R},
\end{cases}
\end{equation}

which leads to (4.3). So, the mapping $q(t, \cdot)$ ($t \in [0, T]$) is an increasing diffeomorphism of $\mathbb{R}$. \hfill \Box

**Lemma 4.2.** Let $m_0, n_0 \in H^s$ ($s > \frac{1}{2}$), and let $T > 0$ be the maximal existence time of the corresponding solution $(m, n)$ to Eq. (4.1). Then, we have for all $t \in [0, T)$,

\begin{align}
m(t, q(t, x))q_x(t, x) &= m_0(x), \\
n(t, q(t, x))q_x(t, x) &= n_0(x).
\end{align}

Proof. Combining Eq. (4.2), Lemma 4.1 and Eq. (4.1), we have

\[
\frac{d}{dt}(m(t, q(t, x))q_x(t, x)) = \left(m(t, q) + m_x(t, q)q(t, x)\right)q_x(t, x) + m(t, q)q_{xt}(t, x) \\
= \left(m(t, q) + \frac{1}{2}(u - u_x)(v + v_x)m_x(t, q)\right)q_x(t, x) = 0.
\]

Therefore, the Gronwall inequality yields (4.5). Similar arguments lead to (4.6). This completes the proof of the lemma. \hfill \Box

The following theorem shows a precise blow-up scenario for Eq. (4.1).

**Theorem 4.1.** Let $m_0, n_0 \in H^s$ ($s > \frac{1}{2}$), and let $T > 0$ be the maximal existence time of the corresponding solution $(m, n)$ to Eq. (4.1). Then the solution $(m, n)$ blows up in finite time if and only if

\[
\liminf_{t \to T} \inf_{x \in \mathbb{R}} (m(v + v_x) - n(u - u_x))(t, x) = -\infty.
\]

Proof. Assume that the solution $(m, n)$ blows up in finite time $T$ and there exists a constant $C$ such that

\[(m(v + v_x) - n(u - u_x))(t, x) \geq -C, \quad \forall (t, x) \in [0, T) \times \mathbb{R}.\]
By (4.3) and Lemma 4.2, we have that
\[ \|m(t)\|_{L^\infty} + \|n(t)\|_{L^\infty} \leq (\|m_0\|_{L^\infty} + \|n_0\|_{L^\infty}) e^{Ct}, \quad \forall t \in [0, T), \]
which contradicts to Corollary 3.1.

On the other hand, if \( \lim \inf_{t \to T} \inf_{x \in \mathbb{R}} (m(v + v_x) - n(u - u_x))(t, x) = -\infty \), then we can get
\[ \lim_{t \to T} \|m(t)\|_{L^\infty} = \infty \quad \text{or} \quad \lim_{t \to T} \|n(t)\|_{L^\infty} = \infty. \]
Thus according to Corollary 3.1, the solution \((m, n)\) blows up. This completes the proof of the theorem.

4.1.2 Global existence

We now give a global existence result.

**Theorem 4.2.** Let \( m_0, n_0 \in H^s \) \((s > \frac{1}{2})\). Assume that \( \text{supp } m_0 \subset [b, \infty) \), \( \text{supp } n_0 \subset (-\infty, a] \), with \( a \leq b \). Then the corresponding solution \((m, n)\) to Eq.(4.1) exists globally in time.

Proof. Note that, according to Lemma 4.1, the function \( q(t, x) \) is an increasing diffeomorphism of \( \mathbb{R} \) with \( q_x(t, x) > 0 \) with respect to time \( t \). Thus \( a \leq b \) implies \( q(t, a) \leq q(t, b) \). We infer from Lemma 4.1 and Lemma 4.2 that for all \( t \in [0, T) \), we have
\[
\begin{align*}
\text{(4.7)} & \qquad \begin{cases} 
m(t, x) = 0, & \text{if } x < q(t, b), 
n(t, x) = 0, & \text{if } x > q(t, a). 
\end{cases}
\end{align*}
\]
Noticing
\[
\begin{align*}
u(t, x) & = e^{-x} \int_{-\infty}^{x} e^{y} m(t, y) dy, 
v(t, x) + v_x(t, x) & = e^{x} \int_{x}^{\infty} e^{-y} n(t, y) dy, 
\end{align*}
\]
we have
\[
\begin{align*}
\text{(4.8)} & \qquad \begin{cases} u(t, x) - u_x(t, x) = 0, & \text{if } x \leq q(t, b), 
v(t, x) + v_x(t, x) = 0, & \text{if } x \geq q(t, a). 
\end{cases}
\end{align*}
\]
Therefore, for Eq.(4.1), \( (m(v + v_x) - n(u - u_x))(t, x) = 0 \) on \( \mathbb{R} \) for all \( t \in [0, T) \). Then Theorem 4.1 implies \( T = \infty \). This proves the solution \((m, n)\) exists globally in time.

4.1.3 Blow-up phenomena

As a straight corollary of Lemma 4.1-4.2 we have the following lemma.
**Lemma 4.3.** Let $m_0, n_0 \in H^s$ ($s > \frac{1}{2}$), and let $T > 0$ be the maximal existence time of the corresponding solution $(m, n)$ to Eq. (4.7). Assume further $m_0, n_0 \in L^1$. Then we have for all $t \in [0, T)$,

$$
\|m(t)\|_{L^1} = \|m_0\|_{L^1}, \|n(t)\|_{L^1} = \|n_0\|_{L^1}.
$$

Now we derive two useful conservation laws for Eq. (4.1).

**Lemma 4.4.** Let $m_0, n_0 \in H^s$ with $s > \frac{1}{2}$, and let $T > 0$ be the maximal existence time of the corresponding solution $(m, n)$ to Eq. (4.7). Then we have that for all $t \in [0, T)$,

$$
\begin{align*}
\int_R m(v + v_x)(t, x)dx &= \int_R m_0(v_0 + v_{0x})dx, \\
\int_R n(u - u_x)(t, x)dx &= \int_R n_0(u_0 - u_{0x})dx.
\end{align*}
$$

Proof. By Eq. (4.1), we have

$$
\begin{align*}
\frac{d}{dt} \int_R m(v + v_x)(t, x)dx &= \frac{d}{dt} \int_R n(u - u_x)(t, x)dx \\
&= \int_R ((v + v_x)m_t + (u - u_x)n_t)(t, x)dx \\
&= \frac{1}{2} \int_R (u - u_x)(v + v_x)(m(v_x + v_{xx}) + n(u_x - u_{xx})(t, x)dx \\
&= \frac{1}{2} \int_R (u - u_x)(v + v_x)(m(v + v_x) - n(u - u_x))(t, x)dx \\
&= \frac{1}{2} \int_R (u - u_x)(v + v_x)\partial_x ((u - u_x)(v + v_x))(t, x)dx = 0.
\end{align*}
$$

This completes the proof of the lemma.

**Lemma 4.5.** Let $m_0, n_0 \in H^s$ ($s > \frac{1}{2}$), and let $T > 0$ be the maximal existence time of the corresponding solution $(m, n)$ to Eq. (4.7). Assume that $m_0$ and $n_0$ do not change sign. Then there exists a constant $C = C(\|v_0 + 2v_{0x}\|_{L^1}, \|(u_0 - 2u_{0x})n_0\|_{L^1}, \|u_0\|_{H^1}, \|v_0\|_{H^1})$ such that

$$
(4.9) \quad |u_x(t, x)| \leq |u(t, x)|, \quad |v_x(t, x)| \leq |v(t, x)|,
$$

$$
(4.10) \quad \|u(t)\|_{H^1} + \|v(t)\|_{H^1} \leq Ce^{Ct}, \forall t \in [0, T).
$$

Proof. One can assume without loss of generality that $m_0 \geq 0, n_0 \geq 0$ for all $x \in \mathbb{R}$. Since $m_0 \geq 0$, (4.3) and (4.5) imply that

$$
(4.11) \quad m(t, x) \geq 0, \forall (t, x) \in [0, T) \times \mathbb{R}.
$$

Noticing

$$
u(t, x) = (1 - \partial_x^2)^{-1}m(t, x) = \frac{1}{2} \int_{\mathbb{R}} e^{-|x-y|}m(t, y)dy,$$
we obtain
\[ u(t, x) = \frac{e^{-x}}{2} \int_{-\infty}^{x} e^{y} m(t, y)dy + \frac{e^{x}}{2} \int_{x}^{\infty} e^{-y} m(t, y)dy, \]
and
\[ u_x(t, x) = -\frac{e^{-x}}{2} \int_{-\infty}^{x} e^{y} m(t, y)dy + \frac{e^{x}}{2} \int_{x}^{\infty} e^{-y} m(t, y)dy, \]
which lead to
\begin{align*}
(4.12) & \quad u(t, x) + u_x(t, x) = e^{x} \int_{x}^{\infty} e^{-y} m(t, y)dy \geq 0, \\
(4.13) & \quad u(t, x) - u_x(t, x) = e^{-x} \int_{-\infty}^{x} e^{y} m(t, y)dy \geq 0.
\end{align*}
From the above two inequalities, we have
\begin{equation}
(4.14) \quad |u_x(t, x)| \leq u(t, x), \quad \forall (t, x) \in [0, T) \times \mathbb{R}.
\end{equation}
Similar arguments lead to
\begin{align*}
(4.15) & \quad n(t, x) \geq 0, \quad \forall (t, x) \in [0, T) \times \mathbb{R}, \\
(4.16) & \quad v(t, x) + v_x(t, x) = e^{x} \int_{x}^{\infty} e^{-y} n(t, y)dy \geq 0, \\
(4.17) & \quad v(t, x) - v_x(t, x) = e^{-x} \int_{-\infty}^{x} e^{y} n(t, y)dy \geq 0, \\
(4.18) & \quad |v_x(t, x)| \leq v(t, x), \quad \forall (t, x) \in [0, T) \times \mathbb{R}.
\end{align*}
Using Eq. (4.11), we get
\begin{align*}
\frac{1}{2} \frac{d}{dt} (\|u(t)\|_{H^1}^2 + \|v(t)\|_{H^1}^2) &= \int_{\mathbb{R}} (m_t u + n_t v)(t, x)dx \\
&= \frac{1}{2} \int_{\mathbb{R}} ((u - u_x)(v + v_x)mu_x + (u - u_x)(v + v_x)nv_x)(t, x)dx \\
&\leq \frac{1}{2} \|((u - u_x)u_x)(t)\|_{L^\infty} \|(v + v_x)m(t)\|_{L^1} + \|((v + v_x)v_x)(t)\|_{L^\infty} \|((u - u_x)m(t))\|_{L^1}.
\end{align*}
Using (4.14) and (4.18), it yields that
\begin{align*}
\|((u - u_x)u_x)(t)\|_{L^\infty} &\leq 2 \|u(t)\|_{L^\infty}^2 \leq \|u(t)\|_{H^1}^2, \\
\|((v + v_x)v_x)(t)\|_{L^\infty} &\leq 2 \|v(t)\|_{L^\infty}^2 \leq \|v(t)\|_{H^1}^2.
\end{align*}
Using Lemma 4.4 with the fact that \(m, n, u - u_x, v + v_x \geq 0\), we obtain
\[ \|((v + v_x)m)(t)\|_{L^1} = \|((v + v_x)m_0)\|_{L^1}. \]
\[ \| (u - u_x) n \|_{L^1} = \| (u_0 - u_{0x}) n_0 \|_{L^1}, \]

Combining the above three relations, we deduce that
\[ \frac{d}{dt} (\| u(t) \|_{L^1}^2 + \| v(t) \|_{H^1}^2) \leq \frac{1}{2} (\| (v_0 + v_{0x}) m_0 \|_{L^1} + \| (u_0 - u_{0x}) n_0 \|_{L^1}) (\| u(t) \|_{H^1}^2 + \| v(t) \|_{H^1}^2). \]

Gronwall’s inequality then yields the desired inequality (4.10). This completes the proof of the lemma. \( \square \)

**Lemma 4.6.** Let \( m_0, n_0 \in H^s (s > \frac{1}{2}) \), and let \( T > 0 \) be the maximal existence time of the corresponding solution \((m, n)\) to Eq. (4.1). Assume further \( m_0, n_0 \in L^1 \). Set \( Q(t, x) = \frac{1}{2} (u - u_x)(v + v_x)(t, x) \). Then there exists a constant \( C = C(\| m_0 \|_{L^1}, \| n_0 \|_{L^1}) \) such that for all \( t \in [0, T) \),
\[ Q(t, x) + (Q(Q_x)) (t, x) + Q_x^2 (t, x) \leq C(|m| + |n|)(t, x). \]

**Proof.** It is easy to deduce from Eq. (4.1) that
\[ Q_x^2 (t, x) + (Q(Q_x)) (t, x) + Q_x^2 (t, x) \]
\[ = (-1 - \partial_x^2)^{-1} (Q_x v + \partial_x (Q_x v_x)) m - (1 - \partial_x^2)^{-1} (\partial_x (Q_x v) + (Q_x v_x)) m \]
\[ + (1 - \partial_x^2)^{-1} (Q_x u + \partial_x (Q_x u_x)) n - (1 - \partial_x^2)^{-1} (\partial_x (Q_x u) + (Q_x u_x)) n](t, x), \]

where \( Q_x = \frac{1}{2} (m v + v_x) - n(u - u_x) \). Applying Lemma 4.3, we arrive at
\[ (1 - \partial_x^2)^{-1} (Q_x v + \partial_x (Q_x v_x)) (t, x) m(t, x) \]
\[ \leq \| (1 - \partial_x^2)^{-1} (Q_x v + \partial_x (Q_x v_x)) (t) \|_{L^\infty} \| m(t, x) \| \]
\[ \leq \frac{1}{2} e^{-|x|} \| (Q_x (v(t)) \|_{L^1} + \| Q_x (v_x (t)) \|_{L^1}) \| m(t, x) \| \]
\[ \leq C \| Q_x (t) \|_{L^1} \| v(t) \|_{L^\infty} + \| v_x (t) \|_{L^\infty} \| m(t, x) \| \]
\[ \leq C (\| m(t) \|_{L^1} + \| n(t) \|_{L^1}) (\| u(t) - u_x (t) \|_{L^\infty} + \| v(t) + v_x (t) \|_{L^\infty}) \| m(t) \|_{L^\infty} + \| v_x (t) \|_{L^\infty} \| m(t, x) \| \]
\[ \leq C (\| m(t) \|_{L^1} + \| n(t) \|_{L^1}) e^{-|x|} \| (m(t)) \|_{L^1} + \| n(t) \|_{L^1}) e^{-|x|} \| (m(t)) \|_{L^1} + \| n(t) \|_{L^1} \| m(t, x) \| \]
\[ \leq C \| m(t, x) \| \]

Following along almost the same lines as above yields
\[ \| - (1 - \partial_x^2)^{-1} (\partial_x (Q_x v) + (Q_x v_x)) (t, x) m(t, x) \|_{L^\infty} \leq C |m(t, x)|, \]
\[ \| (1 - \partial_x^2)^{-1} (Q_x u - \partial_x (Q_x u_x)) (t, x) n(t, x) - (1 - \partial_x^2)^{-1} (\partial_x (Q_x u) - (Q_x u_x)) (t, x) n(t, x) \|_{L^\infty} \leq C |n(t, x)|. \]

Combining the above inequalities completes the proof of the lemma. \( \square \)
Lemma 4.7. Let $m_0, n_0 \in H^s \ (s > \frac{1}{2})$, and let $T > 0$ be the maximal existence time of the corresponding solution $(m, n)$ to Eq. (4.17). Assume that $m_0$ and $n_0$ do not change sign. Set $Q(t, x) = \frac{1}{2}(u - u_x)(v + v_x)(t, x)$. Then there exists a constant $C = C(||(v_0 \ominus v_0x)m_0||_{L^1}, ||(u_0 - u_0x)n_0||_{L^1}, ||u_0||_{H^1}, ||v_0||_{H^1})$ such that

\[
Q_x(t, x) + (Q(Q_x))_x(t, x) + Q_x^2(t, x) \leq Ce^{Ct}(|m| + |n|)(t, x).
\]

Proof. Applying Lemma 4.5 to the first term on the right hand side of (4.20) yields

\[
(1 - \partial^2_x)^{-1}((Q_x v) + \partial_x(Q_x v_x))(t, x)m(t, x)
\]

\[
\leq\|(1 - \partial^2_x)^{-1}((Q_x v) + \partial_x(Q_x v_x))(t)\|_{L\infty}|m(t, x)|
\]

\[
=\left[\frac{1}{2}e^{-|x|} \star \left( (m(v + v_x) - n(u - u_x))v \right) \right]_{L\infty}
\]

\[
+ \left[\frac{1}{2}(\text{sign}(x)e^{-|x|}) \star \left( (m(v + v_x) - n(u - u_x))v \right) \right]_{L\infty}|m(t, x)|
\]

\[
\leq C(||u - u_x||_{L\infty} + ||v + v_x||_{L\infty})(||v||_{L\infty} + ||v_x||_{L\infty})(|e^{-|x|} \star m||_{L\infty} + ||e^{-|x|} \star n||_{L\infty})|m(t, x)|
\]

\[
= C(||u - u_x||_{L\infty} + ||v + v_x||_{L\infty})(||v||_{L\infty} + ||v_x||_{L\infty})(||u||_{L\infty} + ||v||_{L\infty})|m(t, x)|
\]

\[
\leq Ce^{Ct}|m|(t, x),
\]

where we have used the fact that $m, n$ do not change sign. The left three terms can be treated in the same way. We have

\[
-(1 - \partial^2_x)^{-1}(\partial_x(Q_x v) + (Q_x v_x)m) + (1 - \partial^2_x)^{-1}((Q_x u) - \partial_x(Q_x u_x))n
\]

\[
- (1 - \partial^2_x)^{-1}(\partial_x(Q_x u) - (Q_x u_x))n|t, x) \leq Ce^{Ct}(|m| + |n|)(t, x).
\]

Plunging the above two inequalities into (4.20) completes the proof of the lemma.

Next, we present two blow-up results.

Theorem 4.3. Let $m_0, n_0 \in H^s \ (s > \frac{1}{2})$, and let $T > 0$ be the maximal existence time of the corresponding solution $(m, n)$ to Eq. (4.17). Set $Q(t, x) = \frac{1}{2}(u - u_x)(v + v_x)(t, x)$. Assume that $m_0$ and $n_0$ do not change sign, and that there exists some $x_0 \in \mathbb{R}$ such that $N(0, x_0) = |m(0, x_0)| + |n(0, x_0)| > 0$ and $Q_x(0, x) = \frac{1}{2}(m_0(v_0 + v_0x) - n_0(u_0 - u_0x))(x_0) \leq a_0$, where $a_0$ is the unique negative solution to the following equation

\[
1 + ag\left( -\frac{a}{N(0, x_0)} \right) + N(0, x_0) \int_0^{g\left( -\frac{a}{N(0, x_0)} \right)} f(s)ds = 0,
\]

with $f(x) = e^{Cx} - 1, \ x \geq 0, g(x) = \frac{1}{b} \log(x + 1), \ x \geq 0$.

Then the solution $(m, n)$ blows up at a time $T_0 \leq g\left( -\frac{Q_x(0, x_0)}{N(0, x_0)} \right)$. 

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Proof. In view of Lemma 4.1, we obtain that

\[ Q_{x}(t, x_0) + (Q_x(x))(t, x_0) + Q_x(t, x_0) \leq C e^{C t} (|m| + |n|)(t, x_0). \]

By Lemma 4.1 and Lemma 4.2, we have

\[
\frac{d}{dt} Q_x(t, q(t, x_0)) + Q_x(t, q(t, x_0)) \leq C e^{C t} (|m| + |n|)(t, q(t, x_0)) = C e^{C t} (|m_0(x_0)| + |n_0(x_0)|) q_x^{-1}(t, x_0)
\]

which yields

\[
CN(0, x_0) e^{C t} \exp \left( \int_0^t -Q_x(\tau, q(\tau, x_0))d\tau \right),
\]

form which it follows that

\[
\frac{d}{dt} (Q_x(t, q(t, x_0)) e^{\int_0^t Q_x(\tau, q(\tau, x_0))d\tau}) \leq CN(0, x_0) e^{C t}.
\]

Integrating from 0 to \( t \) yields

\[
\frac{d}{dt} \exp \left( \int_0^t Q_x(\tau, q(\tau, x_0))d\tau \right) = Q_x(t, q(t, x_0)) \exp \left( \int_0^t Q_x(\tau, q(\tau, x_0))d\tau \right) \leq N(0, x_0)(e^{C t} - 1) + Q_x(0, x_0).
\]

Integrating again from 0 to \( t \) yields

\[
(4.22) \quad (e^{\int_0^t \inf \{Q_x(\tau, x_0)\}d\tau} \leq \exp \left( \int_0^t Q_x(\tau, q(\tau, x_0))d\tau \right) \leq N(0, x_0) \int_0^t (e^{C s} - 1)ds + Q_x(0, x_0)t + 1.
\]

Next, we consider the following function

\[ F(a, t) = 1 + at + N(0, x_0) \int_0^t f(s)ds, a \leq 0, \]

where \( f(x) = e^{C x} - 1, \ x \geq 0 \). It is easy to see that

\[
\min_{t \geq 0} F(a, t) = F(a, g(-\frac{a}{N(0, x_0)})) = 1 + ag(-\frac{a}{N(0, x_0)}) + N(0, x_0) \int_0^{g(-\frac{a}{N(0, x_0)})} f(s)ds \leq G(a),
\]

where \( g(x) = \frac{1}{C} \log(x + 1), \ x \geq 0, \) is the inverse function of \( f \). Differentiating \( G(a) \) with respect to \( a \), we obtain

\[
\frac{d}{da} G(a) = g(-\frac{a}{N(0, x_0)}) - g'(-\frac{a}{N(0, x_0)}) \frac{a}{N(0, x_0)} + g'(-\frac{a}{N(0, x_0)}) \frac{a}{N(0, x_0)} N(0, x_0) \frac{a}{N(0, x_0)}
\]

\[
= g(-\frac{a}{N(0, x_0)}) > 0, \ a < 0.
\]

Notice that

\[
\lim_{a \to -\infty} g(-\frac{a}{N(0, x_0)}) = +\infty.
\]

Thus, we deduce that

\[
\lim_{a \to -\infty} G(a) = -\infty,
\]

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which, together with that fact that $G(0) = 1$ and the continuity of $G$, yields that there exists a unique $a_0 < 0$ satisfies $G(a_0) = 0$. Therefore, $G(a) \leq 0$ if $a \leq a_0$. Combining this with (4.22), if $Q_x(0, x_0) \leq a_0$, we may find a time $0 < T_0 \leq g(-\frac{Q_x(0, x_0)}{N(0, x_0)})$ such that

$$e^{\int_0^t \inf_{x \in \mathbb{R}} Q_x(\tau, x)\,d\tau} \to 0, \text{ as } t \to T_0,$$

which, implies that

$$\lim_{t \to T_0} \inf_{x \in \mathbb{R}} Q_x(t, x) \to -\infty, \text{ as } t \to T_0.$$

Therefore, in view of Theorem 3.11 we conclude that the solution $(m, n)$ blows up at the time $T_0$.

**Theorem 4.4.** Let $m_0, n_0 \in H^s$ $(s > \frac{1}{2})$, and let $T > 0$ be the maximal existence time of the corresponding solution $(m, n)$ to Eq. (4.1). Set $Q(t, x) = \frac{1}{2}(u - u_x)(v + v_x)(t, x)$. Assume that $m_0, n_0 \in L^1$, and that there exists some $a_0 \in \mathbb{R}$ such that $N(0, x_0) = |m_0(x_0)| + |n_0(x_0)| > 0$ and $Q_x(0, x_0) = \frac{1}{2}(m_0(v_x + v_0) - n_0(u_x - u_0))(x_0) \leq -(2CN(0, x_0))^\frac{1}{2}$. Then there exists a constant $C = C(\|m_0\|_{L^1}, \|n_0\|_{L^1})$ such that the solution $(m, n)$ blows up at a time $T_0 \leq \frac{-Q_x(0, x_0)}{CN(0, x_0)}$.

Proof. In view of Lemma 4.6 we obtain that

$$Q_x(t, x) + (Q(Q_x)_x)(t, x) + Q_x^2(t, x) \leq C(|m| + |n|)(t, x).$$

By Lemma 4.1 and Lemma 4.2 we have

$$\frac{d}{dt}Q_x(t, q(t, x)) + Q_x^2(t, q(t, x)) \leq C(|m| + |n|)(t, q(t, x)) = C(|m_0(x)| + |n_0(x)|)q_x^{-1}(t, x)$$

$$= CN(0, x)\exp\left(\int_0^t -\frac{1}{2}(m(v - v_x) - n(u - u_x))(\tau, q(\tau, x))\,d\tau\right)$$

$$= CN(0, x)\exp\left(\int_0^t -Q_x(\tau, q(\tau, x))\,d\tau\right),$$

form which it follows that

$$\frac{d}{dt}(Q_x(t, q(t, x))\exp(\int_0^t Q_x(\tau, q(\tau, x))\,d\tau)) \leq CN(0, x).$$

Integrating from 0 to $t$ yields

$$\frac{d}{dt}\exp(\int_0^t Q_x(\tau, q(\tau, x))\,d\tau) = Q_x(t, q(t, x))\exp(\int_0^t Q_x(\tau, q(\tau, x))\,d\tau) \leq CN(0, x)t + Q_x(0, x).$$

Integrating again from 0 to $t$ yields

$$(e^{\int_0^t \inf_{x \in \mathbb{R}} Q_x(\tau, x)\,d\tau}) \exp(\int_0^t Q_x(\tau, q(\tau, x))\,d\tau) \leq \frac{1}{2}CN(0, x)t^2 + Q_x(0, x)t + 1.$$
Hence, if there exists some \( x_0 \in \mathbb{R} \) such that \( N(0, x_0) > 0 \) and \( Q_x(0, x_0) \leq -(2CN(0, x_0))^{1/2} \), then we may find a time \( 0 < T_0 \leq -\frac{Q_x(0, x_0)}{CN(0, x_0)} \) such that

\[
e^{-\int_{t_0}^t \inf_{x \in \mathbb{R}} Q_x(\tau, x) d\tau} \to 0, \quad as \ t \to T_0,
\]

which implies that

\[
\lim \inf \inf_{t \to T} \inf_{x \in \mathbb{R}} Q_x(t, x) \to -\infty, \quad as \ t \to T_0.
\]

Therefore, in view of Theorem 4.1, we conclude that the solution \((m, n)\) blows up at the time \(T_0\). \(\square\)

**Remark 4.1.** We mention that, if \(v = 2u\), Theorem 4.3 is same as Theorem 5.2 and Theorem 5.3 in [25], while Theorem 4.4 represents a new blow-up result for Eq. (1.5).

### 4.2 \(N = 1, H = -\frac{1}{2}(uv - u_xv_x)\)

#### 4.2.1 A precise blow-up scenario

For \(N = 1\) and \(H = -\frac{1}{2}(uv - u_xv_x)\), Eq. (4.23) is reduced to the following system:

\[
\begin{cases}
m_t + \frac{1}{2}((uv - u_xv_x)m)_x - \frac{1}{2}(uv_x - vu_x)m = 0, \\
n_t + \frac{1}{2}((uv - u_xv_x)n)_x + \frac{1}{2}(uv_x - vu_x)n = 0, \\
(m, n)|_{t=0} = (m_0, n_0),
\end{cases}
\]

(4.23)

where \(m = u - u_{xx}\) and \(n = v - v_{xx}\).

Along the same lines as the proof of Lemma 4.1-4.2 and Theorem 4.1, we can obtain the following results.

**Lemma 4.8.** Let \(m_{10}, m_{20} \in H^s (s > \frac{1}{2})\), and let \(T > 0\) be the maximal existence time of the corresponding solution \(M = (m_1, m_2)\) to Eq. (4.23). Then the following system

\[
q_t(t, x) = \frac{1}{2}(uv - u_xv_x)(t, q), \quad t \in [0, T),
\]

\[
q(0, x) = x, \quad x \in \mathbb{R}.
\]

(4.24)

has a unique solution \(q \in C^1([0, T] \times \mathbb{R}; \mathbb{R})\). Moreover, the mapping \(q(t, \cdot) \) \((t \in [0, T])\) is an increasing diffeomorphism of \(\mathbb{R}\), with

\[
q_x(t, x) = \exp\left(\int_0^t \frac{1}{2}(mv_x + nu_x)(\tau, q(\tau, x)) d\tau\right).
\]

(4.25)
Lemma 4.9. Let \( m_{10}, m_{20} \in H^s \) (s > \( \frac{1}{2} \)), and let \( T > 0 \) be the maximal existence time of the corresponding solution \( M = (m_1, m_2) \) to Eq. (4.17). Then, we have for all \( t \in [0, T) \),

\[
\begin{align*}
&4.2.2 \text{ Blow-up phenomena} \\
\text{Lemma 4.9.} \quad \text{Let} \quad m_{10}, m_{20} \in H^s \quad \text{(s > \( \frac{1}{2} \))}, \quad \text{and let} \quad T > 0 \quad \text{be the maximal existence time of the corresponding solution} \quad M = (m_1, m_2) \quad \text{to Eq. (4.17)}. \quad \text{Then, we have for all} \quad t \in [0, T),
\end{align*}
\]

\[
(4.26) \quad m(t, q(t, x))q_x(t, x) = m_0(x)\exp\left( \frac{1}{2} \int_0^t (uv_x - vu_x)(\tau, q(\tau, x))d\tau \right),
\]

\[
(4.27) \quad n(t, q(t, x))q_x(t, x) = n_0(x)\exp\left( -\frac{1}{2} \int_0^t (uv_x - vu_x)(\tau, q(\tau, x))d\tau \right).
\]

Theorem 4.5. Let \( m_0, n_0 \in H^s \) (s > \( \frac{1}{2} \)), and let \( T > 0 \) be the maximal existence time of the corresponding solution \( (m, n) \) to Eq. (4.23). Then, the solution \( (m, n) \) blows up in finite time if and only if

\[
\lim_{t \to T} \inf_{x \in \mathbb{R}} ((mv_x + nu_x))(t, x) = -\infty \quad \text{or} \quad \lim_{t \to T} \sup_{t \in \mathbb{R}} \|(uv_x - vu_x)(t, \cdot)\|_{L^\infty} = +\infty.
\]

4.2.2 Blow-up phenomena

Now we derive four useful conservation laws for Eq. (4.23).

Lemma 4.10. Let \( m_0, n_0 \in H^s \) with \( s > \frac{1}{2} \), and let \( T > 0 \) be the maximal existence time of the corresponding solution \( (m, n) \) to Eq. (4.23). Then, we have for all \( t \in [0, T), \)

\[
\int_{\mathbb{R}} (mv_x)(t, x)dx = \int_{\mathbb{R}} m_0v_0dx, \quad \int_{\mathbb{R}} (nu_x)(t, x)dx = \int_{\mathbb{R}} n_0u_0dx,
\]

\[
\int_{\mathbb{R}} (mv)(t, x)dx = \int_{\mathbb{R}} m_0v_0dx, \quad \int_{\mathbb{R}} (nu)(t, x)dx = \int_{\mathbb{R}} n_0u_0dx.
\]

Proof. By Eq. (4.17), we have

\[
\begin{align*}
&\frac{d}{dt} \int_{\mathbb{R}} (mv_x)(t, x)dx = \frac{d}{dt} \int_{\mathbb{R}} (-nu_x)(t, x)dx \\
&= \int_{\mathbb{R}} (v_xm_t - u_xn_t)(t, x)dx \\
&= \frac{1}{2} \int_{\mathbb{R}} (uv_x - u_xv_x)(mv_x - nu_x - uv_x + vu_x)(v_xm + u_xn)dx \\
&= \frac{1}{2} \int_{\mathbb{R}} \partial_x((uv - u_xv_x)(uv_x - vu_x))(t, x)dx = 0,
\end{align*}
\]

and

\[
\begin{align*}
&\frac{d}{dt} \int_{\mathbb{R}} (mv)(t, x)dx = \frac{d}{dt} \int_{\mathbb{R}} (nu)(t, x)dx = \int_{\mathbb{R}} (m_tv + n_tu)(t, x)dx \\
&= \frac{1}{2} \int_{\mathbb{R}} ((uv - u_xv_x)(mv_x + nu_x) + (uv_x - vu_x)(mv - nu))(t, x)dx \\
&= \frac{1}{2} \int_{\mathbb{R}} ((uv - u_xv_x)\partial_x(uv - u_xv_x) - (uv_x - vu_x)\partial_x(uv_x - vu_x))(t, x)dx = 0.
\end{align*}
\]

This completes the proof of the lemma.
Lemma 4.11. Let \( m_0, n_0 \in H^s \) \((s > \frac{1}{2})\), and let \( T > 0 \) be the maximal existence time of the corresponding solution \((m, n)\) to Eq. (4.23). Assume that \( m_0 \) and \( n_0 \) do not change sign. Then there exists a constant 

\[ C = C(\|v_0 x m_0\|_{L^1}, \|v_0 m_0\|_{L^1}, \|u_0 x n_0\|_{L^1}, \|u_0 n_0\|_{L^1}, \|u_0\|_{H^1}, \|v_0\|_{H^1}) \]

such that

\begin{align*}
\|u(t)\|_{H^1} + \|v(t)\|_{H^1} &\leq C e^{Ct}, \forall t \in [0, T). 
\end{align*}

Proof. Without loss of generality, we assume that \( m_0 \geq 0, n_0 \geq 0 \). Repeating the arguments that were used in Lemma 4.10, we get that the inequalities (4.11)-(4.18) still hold true here. Next, according to Lemma 4.10 with \( m, u + u x, n, v - v x \geq 0 \), we obtain

\begin{align*}
\| (mv_x)(t) \|_{L^1} &\leq \| (m(v - v x))(t) \|_{L^1} + \| (mv)(t) \|_{L^1} \\
&= \int_{\mathbb{R}} (m(v - v x))(t, x) dx + \int_{\mathbb{R}} (mv)(t, x) dx \\
&= 2 \int_{\mathbb{R}} (mv)(t, x) dx - \int_{\mathbb{R}} mv_v(t, x) dx \\
&\leq 2\| (m_0 v_0) \|_{L^1} + \| (m_0 v_0 x) \|_{L^1}.
\end{align*}

Finally, form Eq. (4.28), we have

\[ \frac{1}{2} \frac{d}{dt} (\|u(t)\|_{H^1}^2 + \|v(t)\|_{H^1}^2) = \int_{\mathbb{R}} (m u + u x v)(t, x) dx \\
= \frac{1}{2} \int_{\mathbb{R}} ((uv - u_x v_x) m u_x + (w_v - v u_x) m u) \\
+ (u v - u_x v_x) n u_x - (w v - v u_x) n v)(t, x) dx \\
= \frac{1}{2} \int_{\mathbb{R}} (u^2 - u_x^2) v u_x + (u^2 - v_x^2) u x x)(t, x) dx \\
\leq \frac{1}{2} (\| (u^2 - u_x^2)(t) \|_{L^\infty} \| (mv)(t) \|_{L^1} + \| (v^2 - v_x^2)(t) \|_{L^\infty} \| (nu)(t) \|_{L^1} \\
\leq C (\| u(t) \|_{H^1}^2 + \| v(t) \|_{H^1}^2).
\]

Then the Gronwall lemma yields the desired inequality (4.29). This completes the proof of the lemma. \( \square \)

Theorem 4.6. Let \( m_0, n_0 \in H^s \) \((s > \frac{1}{2})\), and let \( T > 0 \) be the maximal existence time of the corresponding solution \((m, n)\) to Eq. (4.23). Set \( Q(t, x) = \frac{1}{2} (uv - u x v_x)(t, x) \). Assume that \( m_0, n_0 \) do not change sign, and that there exists some \( x_0 \in \mathbb{R} \) such that \( N(0, x_0) = |m(0, x_0)| + |n(0, x_0)| > 0 \) and \( Q_x(0, x_0) = \frac{1}{2} (m_0 v_{0x} + n_0 u_{0x})(x_0) \leq a_0 \), where \( a_0 \) is the unique negative solution to the following equation

\[ 1 + a g(-\frac{a}{N(0, x_0)}) + N(0, x_0) \int_0^{g(-\frac{a}{N(0, x_0)})} f(s) ds = 0, \]

with \( f(x) = \exp(e^{C_x} - 1), x \geq 0, g(x) = \frac{1}{e} \log \left( \log(x + 1) + 1 \right), x \geq 0. \)

Then the solution \((m, n)\) blows up at a time \( T_0 \leq g(-\frac{Q_x(0, x_0)}{N(0, x_0)}) \).
Proof. It follows from Eq. (4.23) that

\[
Q_{xt} + Q(Q_x)_x + Q_x^2 \\
= -(1 - \partial_x^2)^{-1} (\partial_x (Q_x u) + (Q_x u_x - u_x v_x) m) n \\
- (1 - \partial_x^2)^{-1} (\partial_x (Q_x v) + (Q_x v_x + u_x v_x) m) n \\
+ \frac{1}{2} (uv_x - vu_x) (mv_x - nu_x).
\]

Using Lemma 4.11 and following along the same lines as the proof of Lemma 4.7, we obtain that

\[
Q_{xt}(t,x_0) + (Q(0)_{x})_x(t,x_0) + Q_x^2(t,x_0) \leq Ce^{Ct}(|m| + |n|)(x_0).
\]

By Lemma 4.9 we get

\[
\frac{d}{dt} Q_x(t,q(t,x_0)) + Q_x^2(t,q(t,x_0)) \leq Ce^{Ct}(|m| + |n|)(t,q(t,x_0)) \\
\leq Ce^{Ct} N(0,x_0) \exp \left( \int_0^t - \frac{1}{2} (m(v + v_x) - n(u - u_x)) (\tau, q(\tau, x_0)) d\tau \right) \exp \left( \frac{1}{2} \int_0^t \| (uv_x - vu_x)(\tau) \|_{L^\infty} d\tau \right) \\
= Ce^{Ct} N(0,x_0) \exp \left( \int_0^t - Q_x(\tau, q(\tau, x_0)) d\tau \right) \exp \left( \frac{1}{2} \int_0^t \| (uv_x - vu_x)(\tau) \|_{L^\infty} d\tau \right).
\]

Again using Lemma 4.11 we have

\[
\exp \left( \frac{1}{2} \int_0^t \| (uv_x - vu_x)(\tau) \|_{L^\infty} d\tau \right) \leq \exp (C \int_0^t e^{C\tau} d\tau) = \exp(e^{Ct} - 1),
\]

from which it follows that

\[
\frac{d}{dt} (Q_x(t,q(t,x_0))) \exp \left( \int_0^t Q_x(\tau, q(\tau, x_0)) d\tau \right) \leq Ce^{Ct} \exp(e^{Ct} - 1) N(0,x_0).
\]

Integrating from 0 to t yields

\[
\frac{d}{dt} e^{\int_0^t Q_x(\tau, q(\tau, x_0)) d\tau} = e^{\int_0^t Q_x(\tau, q(\tau, x_0)) d\tau} Q_x(t,q(t,x_0)) \leq Q_x(0,x_0) + N(0,x_0) \int_0^t \exp(e^{C\tau} - 1) Ce^{C\tau} d\tau \\
= Q_x(0,x_0) + N(0,x_0)(\exp(e^{Ct} - 1) - 1).
\]

Integrating again from 0 to t yields

\[
(4.32) \quad (e^{\int_0^t Q_x(\tau, x_0) d\tau} \leq) e^{\int_0^t Q_x(\tau, q(\tau, x_0)) d\tau} \leq 1 + Q_x(0,x_0) t + N(0,x_0) \int_0^t (\exp(e^{C\tau} - 1) - 1) d\tau.
\]

Next, following along almost the same lines as in the proof of Lemma 4.8 with \( f(x) = \exp(e^{Cx} - 1), \) \( x \geq 0 \) and \( g(x) = C \log (\log(x + 1) + 1), \) \( x \geq 0, \) completes the proof of the theorem. \( \square \)

**Remark 4.2.** We mention that Theorem 4.6 is an improvement of Theorem 4.3 in [36]. Firstly, in [36] the authors assumed that \( \| u \|_{L^\infty}, \| v \|_{L^\infty} \leq Ce^{Ct}, \) while in our paper, \( \| u \|_{L^\infty}, \| v \|_{L^\infty} \leq Ce^{Ct} \) is ensured by Lemma 4.11. Secondly, in [36] \( x_0 \) is required to satisfy an additional restriction: \( Q_x(0,x_0) = \inf_{x \in \mathbb{R}} Q_x(0,x). \) Finally, \( a_0 \) in our result is more explicit and accurate than that in [36].

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