HUBBLE CONSTANT, LENSING, AND TIME DELAY IN RELATIVISTIC MODIFIED NEWTONIAN DYNAMICS

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Abstract

The time delay in galaxy gravitational lensing systems has been used to determine the value of the Hubble constant. As with other dynamical phenomena on the galaxy scale, dark matter is often invoked in gravitational lensing to account for the “missing mass” (the apparent discrepancy between the dynamical mass and the luminous mass). Alternatively, modified gravity can be used to explain the discrepancy. In this paper, we adopt the tensor–vector–scalar gravity (TeVeS), a relativistic version of Modified Newtonian Dynamics, to study gravitational lensing phenomena and derive the formulae needed to evaluate the Hubble constant. We test our method on quasar lensing by elliptical galaxies in the literature. We focus on double-image systems with time delay measurement. Three candidates are suitable for our study: HE 2149–2745, FBQ J0951+2635, and SBS 0909+532. The Hubble constant obtained is consistent with the value used to fit the cosmic microwave background result in a neutrino cosmological model.

Key words: dark matter – gravitation – gravitational lensing: strong – quasars: individual (HE 2149–2745, FBQ J0951+2635, SBS 0909+532)

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1. INTRODUCTION

The Hubble constant \(H_0\) has been a long-debated quantity in cosmology for more than half a century. Basically, it comes from the relation between the cosmological distance and the receding velocity of galaxies, \(v = H_0d\). Its inverse represents the age of the universe.

The value of \(H_0\) is sensitive to the way we estimate the distance. Its value has been estimated using many distance-determination methods, such as Cepheids, tip of the red giant branch, maser galaxies, surface brightness fluctuations, the Tully–Fisher relation, Type Ia supernovae, gravitational time delay, the Sunyaev–Zel’dovich effect, and the cosmic microwave background (CMB; for details see Jackson 2007; Freedman & Madore 2010). In this work, we focus on gravitational lensing and time delay. Obtaining \(H_0\) using gravitational time delay was introduced by Refsdal (1964). Bright variable sources are needed and Refsdal suggested supernovae. Subsequent works on \(H_0\), however, used quasar lensing. Now there are 18 systems of quasar lensing with time delay measurement (Paraficz & Hjorth 2010).

One advantage of using time delay to derive the Hubble constant is that it is less sensitive to cosmological models. Thus, it provides a more direct probe of the cosmological distance (Freedman & Madore 2010). However, when determining the mass distribution, there are some uncertainties due to image deflections and distortions from gravitational lensing. This is commonly known as “mass sheet degeneracy” (Gorenstein et al. 1988). Another source of uncertainty in mass is, of course, the missing mass problem. Missing mass has been a long-standing issue. Oort (1932) and Zwicky (1933) were the first to put forward the notion of missing mass in our Galaxy and the Coma cluster. The missing mass problem was neatly confirmed by the observed flat rotation curve in spiral galaxies (Rubin & Ford 1970; van Albada et al. 1985; Begman 1989). For various aspects of the history of missing mass, the reader is referred to Sanders (2010). Today, missing mass exists in nearly all types of galactic systems, clusters of galaxies, large-scale structure, and CMB. In fact, the problem should be interpreted in terms of excess acceleration or gravity, i.e., there are some accelerations that cannot be accounted for by luminous matter only. To compensate for excess acceleration one can, on the one hand, introduce dark matter into the system. On the other hand, one can modify Newton’s law of motion or the law of gravity. Milgrom (1983) proposed the MODified Newtonian Dynamics (MOND) to explain both the flat rotation curve and the Tully–Fisher relation (Tully & Fisher 1977). MOND asserts that when the acceleration of an object that is under the influence of gravity only is smaller than about \(a_0 = 1.21 \times 10^{-10}\,\text{m}\,\text{s}^{-2}\), Newton’s second law of motion no longer holds. The acceleration of the object is not proportional to the gravitational force exerted on it. The proposed modification is

\[
\tilde{\mu}(|\mathbf{a}|/a_0)\mathbf{a} = -\nabla \Phi_N = \mathbf{a}_N ,
\]

where \(\mathbf{a}\) is the acceleration of the object and \(\Phi_N\) is the Newtonian gravitational potential. The function \(\tilde{\mu}(x)\) is called the interpolation function. It is a monotonically increasing function that connects the Newtonian and deep MOND regimes. With \(x = |\mathbf{a}|/a_0\), \(\tilde{\mu}(x) \approx 1\) for \(x \gg 1\) (Newtonian regime) and \(\tilde{\mu}(x) \approx x\) for \(x \ll 1\) (deep MOND regime). In later calculations, we will consider a spherically symmetric model. It is useful to introduce the inverted interpolation function \(\tilde{\nu}\) such that

\[
\mathbf{a} = -\nabla \Phi = \tilde{\nu}(|\mathbf{a}_N|/a_0)\mathbf{a}_N .
\]

For convenience, we call \(\Phi_N\) the Newtonian potential and \(\Phi\) the MONDian potential.

MOND is very successful in explaining the dynamics of galactic systems (see a review by Sanders & McGaugh 2002). Recently, McGaugh (2011a, 2012) showed that MOND can...
perfectly explain the Tully–Fisher relation in gas-rich spiral galaxies without invoking uncertainty parameters such as the mass-to-light ratio of galaxies. As usual, the result created some debate (Foreman & Scott 2012; Gnedin 2012; McGaugh 2011b). Nevertheless, many consider MOND to not be quite successful on the cluster of galaxies scale (see, e.g., Aguirre et al. 2001; Clowe et al. 2006; Angus & McGaugh 2008). A recent study on the gravitational redshift of clusters of galaxies (Wojtak et al. 2011) has generated some debate as to whether MOND is applicable on a cluster scale, although it seems that MOND does not have difficulty in interpreting the data (Bekenstein & Sanders 2012). In any case, the original MOND is a non-relativistic theory and cannot be applied to relativistic phenomena such as a gravitational lens and cosmology. Two decades after the original proposal by Milgrom (1983), Bekenstein (2004) proposed the tensor–vector–scalar (TeVe S) covariant relativistic gravity theory with MONDian dynamics as its non-relativistic limit. Adopting TeVe S, Chiu et al. (2006) derived the corresponding strong lens equation. More recently, Milgrom (2009) proposed another relativistic version of MOND called BiMOND. It turns out that TeVe S and BiMOND have identical gravitational lensing equations.

The lens equation has been applied to some galaxy lensing data in which the mass of the galaxies has been calculated and compared with population synthesis (e.g., Zhao et al. 2006; Ferreras et al. 2008; Chiu et al. 2011). In a related work, Sanders & Land (2008) showed that the MONDian lensing mass is consistent with the dynamical mass deduced from the fundamental plane of elliptical galaxies.

In a recent study, Ferreras et al. (2012) claimed that strong lens data are in conflict with the MOND paradigm. They found that the mass deduced by gravitational lensing is larger than the value inferred by population synthesis. However, uncertainties still abound. In their paper, for more than half of the sample (five out of nine) the lens contains more than one galaxy. Evaluation of the MONDian acceleration of a non-spherical mass distribution is still in its infancy. It is not clear how they solved the problem of arbitrary mass distribution or accessed the uncertainty involved in their paper. In fact, population synthesis depends on a range of factors and physics of the lensing galaxy, where uncertainties are not easy to estimate. For instance, the estimated mass may differ a lot if a different initial mass function (IMF) is used, e.g., in Ferreras et al. (2012) the mass obtained by the Salpeter IMF can be a factor of two larger than the Chabrier IMF. How to reduce the uncertainties is not clear at this stage.

In this work, we turn our attention to the Hubble constant. If the derived Hubble constant is not consistent with other independent measurements, then the theory will be in trouble. If the time delay of the images of the lensing system can be measured, then in addition to the strong lens equation, we have another simple method to estimate the mass of the lens. Unlike dark matter theory in which the mass model is adjustable, no mass is non-luminous in MOND at the galaxy scale, and the mass density profile can be deduced solely from the brightness distribution. Equating the mass obtained by the lens equation and the time delay equation gives a relation between the acceleration constant \(a_0\) and the Hubble constant \(H_0\). Time delay provides another test for MOND or other theories.

In the following, we describe the lens equation and the time delay equation in MOND. A discussion on some limiting cases is given before applying our method to the data available in the literature. We find three candidates suitable for our study. They are HE 2149–275, FBQ J0951+2635, and SBS 0909+532. Some concluding remarks will be given at the end.

2. GRAVITATIONAL LENSING AND TIME DELAY IN RELATIVISTIC MOND

The discussion on light deflection due to gravitational force can be traced to Newton’s Opticks. The modern view of light deflection is a relativistic gravitational effect, in which both the time-like and space-like parts in the metric contribute to the deflection angle. Using the General theory of Relativity (GR), Einstein derived a deflection angle which is twice the Newtonian one. It turns out that the deflection angle derived from TeVe S or BiMOND is also twice the Newtonian one (i.e., the same as GR)—except that the Newtonian potential is replaced by the MONDian potential. The deflection angle by a spherical lens in the small-angle approximation can be written as (Chiu et al. 2006, 2011)

\[
\Delta \varphi = 2 \int \frac{dt}{c} \approx \frac{2 \varrho_0}{c^2} \int_{D_{LS}}^{D_S} \frac{1}{\varrho} \frac{\partial \Phi(\varrho)}{\partial \varrho} d\zeta, \tag{3}
\]

where \(c\) is the speed of light, \(\varrho\) is the image position, \(\varrho_0\) is the distance from the center of the spherical lens, \(\varrho_0 \approx D_0 \vartheta\) is the closest approach of the light path from the center of the lens, \(\zeta^2 = \varrho^2 - \varrho_0^2\), and \(\Phi(\varrho)\) is the MONDian potential. \(D_L\), \(D_L'\), and \(D_{LS}\) are the angular distances of the lens from the observer, the observer from the lens, and the source from the lens, respectively. The direction of the image is in the direction of the closest approach (projected on the sky). For a spherical lens, there are two images located at both sides of the source, and the governing equation (called the lens equation) is given by

\[
\beta = \vartheta_c - \alpha(\vartheta_c) = \alpha(\vartheta_\pm) - \vartheta_\pm, \quad \alpha(\varrho) = \frac{\Delta \varphi}{D_{LS} / D_S}, \tag{4}
\]

where \(\beta\) is the source position and \(\vartheta_\pm\) are the image positions. The upper sign denotes an image on the same side as the source and the lower sign on the opposite side of the source. \(\varrho(\vartheta)\) is commonly called the reduced deflection angle.

Time delay is defined as the difference between the time light takes to travel along the actual path and along the undeflected path. It can be derived from Fermat’s principle or from the geodesic equation in relativistic gravitation theory. As with the deflection angle, the form of the time delay is the same for MOND and BiMOND (with the Newtonian potential for GR and the MONDian potential for MOND),

\[
t(\vartheta) = \frac{(1 + z_L)}{c} \left[ \frac{D_L D_S}{2D_{LS}} \alpha(\varrho)^2 - \int_{D_{LS}}^{D_L} \frac{2 \Phi(\varrho)}{c^2} d\zeta \right]. \tag{5}
\]

The first and second terms in Equation (5) are referred to as the geometric and the potential time delay. In cases where the difference in the time delay of the two images is available, the value of \(H_0\) (and the mass of the lens as well) can be obtained by solving the time delay difference equation (time delay equation for short),

\[
\Delta t = t(\vartheta_-) - t(\vartheta_+), \tag{6}
\]

and the lens equation (Equation (4)).

To illustrate the ideas, we start with a simple example. We consider a point mass lens in the Bekenstein form. The
Bekenstein form is a frequently used interpolation function with a very simple inverted form $B_{\text{K}}$ (which is what MOND is designed for). Hence, the Bekenstein form is a frequently used interpolation function with a very simple inverted form $B_{\text{K}}$ (which is what MOND is designed for). Hence, the Bekenstein form has exactly the same form as shown in Equation (9), there is one subtle difference. In both GR and MOND, the geometric time delay $\Delta t_{\text{G}}$ can be written as

$$\frac{D_{\text{LS}}}{D_{L}D_{S}} \frac{H_0 \Delta t_{\text{G}}}{(1+z_{\text{L}})} = \frac{\theta_{\text{E}}^2}{2\theta_{+}\theta_{-}} \left(\theta_{+}^2 - \theta_{-}^2\right).$$

In GR, $\theta_{\text{E}}^2 = \theta_0^2$, and hence the first term of Equation (9) is the geometric time delay in GR. However, because of Equation (8), the geometric time delay in MOND is less than the first term in Equation (9). The interesting fact is that the potential time delay $\Delta t_{\text{P}}$ in MOND (Bekenstein form for point mass and Hernquist model) is

$$\frac{D_{\text{LS}}}{D_{L}D_{S}} \frac{H_0 \Delta t_{\text{P}}}{(1+z_{\text{L}})} = \theta_{\text{E}}^2 \left[\log \left(\frac{\theta_{+}}{\theta_{-}}\right) + \frac{\pi}{2\theta_{0}} \left(\theta_{+} - \theta_{-}\right)\right]$$

Thus, part of the potential time delay cancels the geometric time delay “exactly” and renders the total time delay to have the same form as in GR. Therefore, the terms in Equation (9) have a clear identification, namely, the first term is the geometric time delay and the second term is the potential time delay, but it is not so in MOND, where the first term of Equation (9) is partly geometric and partly potential, and the second term is part of the potential time delay. Moreover, when one approaches the deep MOND regime ($\theta/\theta_{\pm} \to 0$), $\theta_{\text{E}} \to 0$ and the MONDian geometric time delay and the second term in Equation (9) tend to zero. The time delay becomes solely potential time delay.

In the deep MOND regime, the interpolation function becomes $\mu(a/a_0) \approx a/a_0$. If the extent of the luminous matter is also much smaller than $\theta$ (i.e., can be modeled practically
by a point mass), then the time delay difference is solely determined by the potential time delay, because the deflection angle approaches a constant in the deep MOND regime (Chiu et al. 2006). In this case, the time delay (difference) equation is independent of the choice of interpolation function,

$$\frac{D_{LS}}{D_L D_S} \frac{H_0 \Delta t}{(1+z_L)} = \frac{1}{2} \left( \theta_2^2 - \theta_1^2 \right). \tag{13}$$

We emphasize that Equation (13) does not have any free parameters for the interpolation function or the mass of the lens, not even $a_0$ (the most important constant of MOND), $a_0$ and the mass of lens are hidden in the lens equation, Equation (8).

We point out that Equation (13) is identical to GR with an isothermal lens model (see Witt et al. 2000). This is expected since both potentials have the same form, namely, a logarithmic isothermal lens model (see Witt et al. 2000). This is expected, as both potentials have the same form, namely, a logarithmic isothermal lens model.

The MONDian gravitational potential and acceleration of the Hernquist model are

$$\Phi_N = -\frac{GM}{(\rho + \rho_0)^{1/2}}, \quad \alpha_N = \frac{GM}{(\rho + \rho_0)^{2}}. \tag{15}$$

where the Hernquist radius $\rho_0$ is 0.551 times the effective radius (or half-light radius). The gradient of the MONDian potential is given by Equation (2). The lens equation and the time delay equation for the Hernquist lens in the MOND form are described in the Appendix.

Different cosmological models give a different angular distance for the same redshift. A criticism of MOND has been that it cannot form a large-scale structure. Basically, the criticism originated from an argument in GR with baryons only. However, the nonlinear growth of structure in MOND with neutrinos can reproduce the power spectrum (Skordis et al. 2006; Skordis 2009; Angus 2009; Diaferio & Angus 2012). Although the contribution of TeVe S fields to the background FLRW equation is very small ($\sim 10^{-3}$ or less; Skordis 2009), that is enough to affect the nonlinear growth in large-scale structure formation.

More than a decade ago, while addressing CMB in MOND, McGaugh (1999) already commented that any relativistic MOND theory should contain GR in the appropriate strong-field limit in order to solve CMB problems. This is indeed the case. For instance, Angus (2009) estimated the gravity before recombination and found that it is around $570a_0$ and that the MOND effect is negligible. Skordis et al. (2006) showed that in order to comply with the CMB observations, TeVe S needed 2 eV massive neutrinos (treated as non-relativistic particles) and $(\Omega_B, \Omega_\gamma, \Omega_\Lambda) = (0.05, 0.17, 0.78)$. However, in this model, the predicted third acoustic peak of CMB is lower than what is observed. Then, Angus (2009) proposed a model with 11 eV sterile neutrinos and got a better fit to the third peak. (11 eV sterile neutrino is consistent with the analysis of the Miniboone experiment which gave the mass range of sterile neutrinos as $4 \sim 18$ eV; Giunti & Laveder 2008). The sterile neutrino cosmological model of Angus (2009) is $(\Omega_B, \Omega_\gamma, \Omega_\Lambda) = (0.05, 0.23, 0.72)$ and $m_\nu = 11$ eV. We use both the 2 eV neutrino model and the 11 eV sterile neutrino model as our cosmological model for angular distance calculation. In the three cases we studied, the difference between the two models is small. The difference in $D_{LS}/(D_L D_S)$ is less than 0.4%.

### 4. RESULT AND CONCLUSION

Basically, our model is comprised of three parts: (1) a galaxy, (2) lensing, and (3) a gravity theory. (1) We model the lensing

| Name         | $z_L$ | $z_S$ | $\theta_1''$ | $\theta_2''$ | $\theta_1''_{eff}$ | $\epsilon$ | $\Delta(t)_{\text{days}}$ |
|--------------|------|-------|--------------|--------------|-------------------|------------|--------------------------|
| HE 2149−2745 | 0.495| 2.030 | 1.354        | 0.344        | 0.501             | 0.5        | 103 ± 12                 |
| FBQ J0951+2635 | 0.240| 1.246 | 0.879        | 0.221        | 0.166             | 0.25       | 16 ± 2                   |
| SBS 0909+532 | 0.830| 1.376 | 0.756        | 0.415        | 1.580             | ...        | 45±11                    |

Notes. The unit of $\theta_1''$ is arcsecond. The effective radius in the $R$ band is used for objects HE 2149−2745, FBQ J0951+2635 (Kochanek et al. 2000), and SBS 0909+532 (Lehar et al. 2000). Ellipticity $\epsilon = 1 - b/a$ is measured in the $R$ band (Lopez et al. 1998; Jakobsson et al. 2005).
galaxy using Hernquist’s model. Photometric measurement of the galaxy could give the effective radius (or half-light radius) of the galaxy, and the Hernquist radius is related to the effective radius. The only unknown is the total mass of the lensing galaxy, \( M \). (2) On the part of gravitational lensing, we consider strong lensing. The angular position and the redshift (of the lens and the two images) and the time delay difference between the two images are measured. When a cosmological model is adopted, the only unknown is the Hubble constant \( H_0 \). (3) We consider MOND in the Bekenstein form. The only unknown is the acceleration constant \( a_0 \). These three unknowns are constrained by the lens equation and the time delay equation (Equations (A1) and (15)). In this paper, we would like to assume a gravity theory and find the mass and Hubble constant. The flat rotation curve of spiral galaxies and the Tully–Fisher relation in gas-rich galaxies give a consistent value of the acceleration constant \( a_0 = 1.21 \times 10^{-10} \text{ m s}^{-2} \) (Sanders & McGaugh 2002; Famaey & Binney 2005; McGaugh 2011a). We evaluate \( H_0 \) using Equation (A7) and \( M \) using Equation (A8). The results for the three selected systems (HE 2149−2745, FBQS J0951+263, and SBS 0909+523) are summarized in Table 2.

In the last two columns of Table 2, \( x = a/a_0 \) is the ratio of the acceleration at the closest approach to the MOND acceleration constant. Recall that \( x \gg 1 \) is the Newtonian regime and \( x \ll 1 \) is the deep MOND regime. In Table 2, we see that the deep MOND point mass model did not give a reasonable value of \( H_0 \). This is understandable because these three cases are not in the deep MOND regime. In addition, the closest approach distance of the three cases is comparable to the measured Hernquist radius, thus the point mass model is not a good approximation. Recall that the isothermal mass model in GR and the deep MOND point mass model have identical expressions (Equation (13)). The two would give the same result (give or take some slight difference due to the different cosmological models). Perhaps it is not a coincidence that these three cases are not as successful as other time delay cases by the simple isothermal model in GR, as shown in Witt et al. (2000).

In general, MOND gives a smaller mass than GR, and the excess mass in GR can be interpreted as missing mass or dark matter. In the Newtonian regime (\( x \gg 1 \)), MOND and GR should give a similar mass. For the three cases in Table 2, \( x \) is mostly of the order of 1−10 (intermediate MOND regime). As expected, the mass and Hubble constant from MOND are smaller than the value computed in GR without dark matter. Once again, as in other galactic scale dynamical phenomena, MOND provides a consistent picture explaining the observed excess acceleration in gravitational lensing including the time delay phenomenon.

A source of uncertainty is the choice of the interpolation function. Recall that either in the Newtonian regime (\( x \gg 1 \)) or in the deep MOND regime (\( x \ll 1 \)), different interpolation functions should give the same result. However, our sample lies in the intermediate MOND regime. Table 2 shows the result from the Bekenstein form. Other interpolation functions are expected to give a somewhat different result. In any case, the major uncertainty comes from observation, in particular, the time delay measurement. The Hubble constant obtained from lensing and time delay, of course, must be consistent with the values from other measurements.

The Hubble constant sets the scale of distance. Hubble’s law relates the distance of an object to its redshift. Recently, Riess et al. (2009, 2011) calibrated low-redshift Type Ia supernovae with Cepheids and obtained a Hubble constant of \( H_0 = 73.8 \pm 2.4 \text{ km s}^{-1} \text{ Mpc}^{-1} \). Neutrino cosmological models will not change this value since they also give the same linear Hubble law at low redshifts. The \( H_0 \) found by time delay in this work (see Table 2) is consistent with the value(s) from Type Ia supernova data.

In Angus (2009), neutrino cosmological models in TeVb S were considered to fit the CMD acoustic spectrum. An 11 eV sterile neutrino model can fit the data well. The parameters used in the model were \( H_0 = 71.5 \text{ km s}^{-1} \text{ Mpc}^{-1} \), \((\Omega_B, \Omega_{\nu}, \Omega_{\Lambda}) = (0.05, 0.23, 0.72)\). Our result for \( H_0 \) is consistent with this (see Table 2).

In summary, this work is a first attempt to use MOND to interpret data from gravitational time delay. The Hubble constant evaluated for the sample in this study is consistent with the value obtained from Hubble’s law and also with those in the literature (see Table 2). When compared with GR in the Hernquist model (without a dark matter component), the evaluated mass of the lens in MOND is 27%−43% smaller than those from GR, and the Hubble constant is 17%−25% smaller than GR.

Applying a gravity theory to a static distribution of mass gives the dynamics (or equation of motion) of a point mass or a photon in the mass distribution. It is expected that some of the parameters in the dynamics are related to the mass distribution and some of them are unique to the underlying gravity theory. In principle, the parameters can be fixed by measuring the dynamics. Similarly, when we apply the gravity theory to the dynamics of the universe, some parameters are related to the energy content of the universe and some to the theory itself.
In the case of strong gravitational lensing, if we can assume or measure the density profile of the mass distribution (in MOND, the profile is supposed to be given by the brightness distribution), then there remains only one unknown parameter for the mass distribution, namely, the mass scale (e.g., the total mass). If both the image positions and the time delay are measured, then we can get rid of the mass scale. The remaining parameters in the dynamics are related directly (or through a distance scale) to the gravity theory. For instance, in MOND, the remaining parameters are the acceleration constant and the Hubble constant . In this paper, we adopted from other measurements (Sanders & McGaugh 2002; Famaey & Binney 2005; McGaugh 2011a), and the derived is consistent with the value from Hubble’s law measured by Type Ia supernovae and the value needed for fitting the CMB acoustic spectrum by a neutrino cosmological model. Gravitational lensing promises to provide a test ground for modified gravity.

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APPENDIX

LENS AND TIME DELAY EQUATIONS FOR THE HERNQUIST MODEL IN THE BEKENSTEIN FORM

In this appendix, we provide the equations necessary for the analysis in Sections 3 and 4. The lens equation for a spherical Hernquist lens (see Equation (15)) in the Bekenstein form can be written as (cf. Equation (8))

\[ \frac{(\theta_+ + \theta_-)}{(f_+ + f_-)\theta_E^2} = 1 + \frac{(g_+ + g_-)}{(f_+ + f_-)\theta_0}, \]  

(A1)

where \( f_\pm = f(\theta_\pm, \theta_0), g_\pm = g(\theta_\pm, \theta_0) \)

\[ f(\theta, \theta_0) = \frac{\theta [1 - \theta h(\theta, \theta_0)]}{(\theta^2 - \theta_0^2)^2}, \quad g(\theta, \theta_0) = \theta h(\theta, \theta_0), \]  

(A2)

\[ c(h, \theta_0) = \begin{cases} \frac{1}{\sqrt{\theta^2 - \theta_0^2}} \left[ \frac{\pi}{2} - \sin^{-1} \left( \frac{\theta_0}{\theta} \right) \right], & \text{for } \theta_0 < \theta \\ 1, & \text{for } \theta_0 = \theta \\ \frac{1}{\sqrt{\theta^2 - \theta_0^2}} \log \left( \frac{\theta_0}{\theta} + \sqrt{\theta^2 - 1} \right), & \text{for } \theta_0 > \theta \end{cases} \]  

(A3)

and \( \theta_0 = r_0/D_L, \theta_E \) and \( \theta_0 \) are given by Equation (10) with \( M \) being the total mass of the lens.

The time delay (difference) equation is given by (cf. Equation (9))

\[ \frac{\tilde{D}_{LS}}{D_L D_S} \Delta t = \frac{(\theta_+ + \theta_-)}{(g_+ + g_-)} \left[ q - \frac{1}{2} (\theta_+ + \theta_-) (g_+ - g_-) \right] \]  

(A4)

\[ + \theta_E^2 p \left( \frac{q(f_+ + f_-)}{(g_+ + g_-)} + \frac{\theta_+ \theta_- (g_+ f_- - g_- f_+)}{(g_+ + g_-)} \right), \]  

where

\[ p(\theta_+, \theta_-, \theta_0) = \log \left( \frac{\theta_+}{\theta_-} \right) + \theta_0 \left[ h(\theta_+, \theta_0) - h(\theta_-, \theta_0) \right], \]  

(A5)

\[ q(\theta_+, \theta_-, \theta_0) = (\theta_+^2 - \theta_0^2) h(\theta_+, \theta_0) \]  

\[ - (\theta_-^2 - \theta_0^2) h(\theta_-, \theta_0) - \theta_0 \log \left( \frac{\theta_+}{\theta_-} \right). \]  

(A6)

Eliminating \( M \) from Equations (A1) and (A4), we obtain a relation between \( \alpha_0 \) and \( H_0 \) in terms of observed quantities \( \theta_\pm, \theta_0, \Delta t. \quad \text{Explicitly,} \]

\[ \frac{4\tilde{D}_{LS}}{D_L D_S} \frac{\alpha_0}{c H_0} = \frac{[\theta_+ + \theta_0] \sqrt{D - (f_+ + f_-)\sqrt{N}}}{(g_+ + g_-)^2 \Delta t}, \]  

(A7)

and \( M \) is given by

\[ \frac{4\tilde{D}_{LS}}{D_L D_S} \frac{G H_0 \alpha_0}{c^3} = \frac{N^2}{\Delta t}, \]  

(A8)

where

\[ N^2 = \frac{\tilde{D}_{LS}}{D_L D_S (1 + z_L)} \frac{(\theta_+ + \theta_-)}{(g_+ + g_-)} \left[ q - \frac{1}{2} (\theta_+ + \theta_-) (g_+ - g_-) \right], \]  

(A9)

\[ \Delta t = p \left( \frac{q(f_+ + f_-)}{(g_+ + g_-)} + \frac{(\theta_+ + \theta_-)(g_+ f_- - g_- f_+)}{(g_+ + g_-)} \right). \]  

(A10)

For a given \( H_0, \alpha_0 \) is given explicitly by Equation (A7). For a given \( \alpha_0 \), Equation (A7) is a third-order equation in \( H_0 \). It can be shown that only one solution is suitable.

For other interpolation functions, we still have the lens equation and the time delay equation, but the mass \( M, \alpha_0 \) and \( H_0 \) cannot be separated as nicely as for the Bekenstein form. In any case, there are two relations for three variables. If the mass can be estimated using other methods, say velocity dispersion of the lensing galaxy, then it is possible to get \( \alpha_0 \) and \( H_0 \) simultaneously.

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