Index Theory and Adiabatic Limit in QFT

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To my wife

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Abstract

The paper has the form of a proposal concerned with the relationship between the three mathematically rigorous approaches to quantum field theory: 1) local algebraic formulation of Haag, 2) Wightman formulation and 3) the perturbative formulation based on the microlocal renormalization method. In this project we investigate the relationship between 1) and 3) and utilize the known relationships between 1) and 2). The main goal of the proposal lies in obtaining obstructions for the existence of the adiabatic limit (confinement problem in the phenomenological standard model approach). We extend the method of deformation of Dütsch and Fredenhagen (in the Bordemann-Waldmann sense) and apply Fedosov construction of the formal index—an analog of the index for deformed symplectic manifolds, generalizing the Atiyah-Singer index. We present some first steps in realization of the proposal.

1 Introduction

The paper has the form of a proposal concerned with the relationship between the three mathematically rigorous approaches to quantum field theory. Namely they are: 1) local algebraic formulation of Haag, 2) Wightman formulation and 3) the approach based on the microlocal causal renormalization method going back to Bogolubov and Stückelberg, promoted mostly by Fredenhagen and his co-workers. In this project we investigate the relationship between 1) and 3) and utilize the known relationships between 1) and 2). The weakness of the 3)
approach lies in its dependence on the existence of the adiabatic limit, otherwise the formal power series are physically meaningless. With few exceptions only (e.g., QED) the existence problem for the adiabatic limit is open, even no obstructions for its existence are known. The main goal of the proposal lies in obtaining obstructions for the existence of the adiabatic limit. The problem corresponds to the confinement problem in the phenomenological standard model approach. We extend the method of deformation in the Bordeman-Waldmann sense as worked out by Dütsch and Fredenhagen by noticing the parallelism between deformation applied by Dütsch and Fredenhagen and the existence of adiabatic limit on the one side and the deformations of symplectic manifolds and the existence of the asymptotic representation of Fedosov on the other. It was suggested by Bordemann and Waldmann [1]. We extend their suggestion here. Fedosov constructed a formal analog of the index for deformed symplectic manifolds, generalizing the Atiyah-Singer index, and has shown that the existence of the asymptotic representation is equivalent to the integrality of the index. We notice further that the construction of his index may be applied to the Dütsch and Fredenhagen deformations and that his construction of necessity and sufficiency constraints may be carried to the Dütsch and Fredenhagen deformations provided we could utilize a Fredholm module over a fixed subalgebra of free fields, which is canonically connected to free fields. Quite independently we notice that in the local algebraic theory the charges cannot superpose by principle, as they determine the selection sectors. Moreover in 1) there are two diverse kinds of non-superposing quantities, namely A) such as generalized charges and B) such as spacetime coordinates, i.e. classical parameters with direct physical meaning, allowing the theory to have physical interpretation. We propose to treat them both more symmetrically in that the reason for the lack of coherent superpositions for B) should in principle be the same as for the lack of superpositions for A). Therefore B) should also be represented by the elements of the algebra of fields which do not mix the coherent selection sectors (of the Hilbert space acted on by all fields, also the charged fields) and thus by elements which determine selection sectors. This leads us to the concept of spacetime which is classical, i.e. whose points cannot superpose, but with noncommutative algebra of coordinates. In order to keep the geometric particle interpretation of Haag we identify the algebra with the Haag’s algebra of detectors. Spacetime structure should determine its (pseudo-riemannian) spectral triple and (after Wick rotation) the corresponding Fredholm module. We identify the last module with the Fredholm module necessary for the construction (of the sufficiency condition) of the adiabatic limit. The first profit of this assumption is that it allows us to keep the particle interpretation even on curved spacetime without any time-like Killing vector field, a long standing problem in quantum field theory on curved spacetimes. Another profit: we expect nontrivial limitations put on allowed values of coupling constants, which are deformation parameters in the Dütsch-Fredenhagen approach (integrality of Fedosov index assuring the existence of asymptotic representation puts strong restrictions on possible values of deformation parameter). Last but not least we get the time’s arrow for non-superposing quantities for free, as an immediate
consequence of non-commutativity of multiplication in the algebra of spacetime coordinates.

The proposal is divided into five tasks: I. To provide details of the proof of the stability theorem under deformation of D"utsch and Fredenhagen with the modification in definition of the algebra of observables meaning that we restrict ourselves to ghost-free fields in the construction of the algebra and explain the relationship between the two definitions (see section 3 for details). II. To reconstruct the asymptotic behavior of the analog of Fedosov asymptotic representation for QED utilizing the Blanchard-Seneor analysis and relationships between 1) and 2). III. To formulate necessary conditions for the existence of the asymptotic representation in QED in terms of the formal index. IV. Having given a compact spectral triple to construct a formal deformation of the triple in the sense of Bordemann-Waldmann, and examine stability of the compact spectral triple structure under the deformation. V. Having given a completely integrable Faddeev model to investigate more deeply analytic properties of the linear representation of the quantum monodromy matrix on a dense subset of the Fock space, given in the Korepin, Bogoliubov and Izergin monograph (section 4). Then incorporating the relationship between point-like fields and local algebras try to carry the quantum group structure and their action on the corresponding spacetime algebra of bounded operators.

2 A Tentative Hypothesis

In 1957 at the conference in Chapel Hill, Richard Feynman presented his famous Gedanken Experiment supporting the claim, that the gravitational field has a quantum mechanical character in more or less the same sense as the electromagnetic field, and thus should be be quantized in more or less he same way as the electromagnetic field and other matter fields. The postulate that all physical processes (all the more quantum mechanical processes) should be described by amplitudes (and not probabilities themselves) was very natural at that time, i.e. only three decades after the discovery of matrix mechanics. Thus naively speaking: one can confine oneself to observables acting in a fixed Hilbert space and moreover there was no reason visible at that time for taking into consideration other collections (algebras) of observables than those which act irreducibly in the Hilbert space (Hermann Weyl in his famous book\textsuperscript{1} even referred to the Aristotelian nihil frustra principle, in order to support the restriction to irreducible representations only). The mentioned postulate together with the "natural" assumption that, say, an electron is a spacetime object (in more or less the same sense as, say, a grain of sand) indeed gave a solid argument speaking for the quantum character of gravitational filed, i.e. in the sense that it should undergo the superposition principle, and should be quantized similarly to matter fields. The warning, which Feynman gave on that occasion, that quantum mechanics may not be correct for macroscopic objects, suggesting some possibilities for

\textsuperscript{1}See [2], 238; moreover, he derives the Schr"{o}dinger equation from the irreducibility, compare \textit{Ibidem}, Chap. IV.D.
other alternatives has apparently at least been ignored, universally recognized
as his scientific honesty at most. Nowadays, over half a century after the con-
ference, the principal arguments of Feynman speaking for the quantum character
of gravity, get lost much of their cogency. First of all the simplified scheme: ob-
servables + Hilbert space in which they act irreducibly, had to be substantially
subtilized. According to the subsequent investigations in QFT and quantum
statistical mechanics, we have all the grounds to expect, that the Hilbert space
has to be divided into subspaces, called superselection sectors, and the super-
position of amplitudes cannot take place freely in the whole Hilbert space but
only within one and the same sector, whenever the system in question is more
complex. In particular it seems hardly possible that two states with different
(generalized) charge numbers (e.g. different hadron numbers, or state with even
half spin with a state with an odd half spin) can superpose. Beyond doubt
the assumption that such states cannot superpose is a "good approximation" in
the light of nowadays knowledge, independently of the possible disputes on how
is in "reality". Similarly we have all the grounds to suppose, that states with
different electric charge numbers do not superpose (although there exists in this
case an alternative theory of A. Staruszkiewicz, compare discussion below). From
the investigations of Haag and his school (algebraic quantum field the-
ory) it follows that the charge structure (global gauge symmetry groups) can be
obtained from the structure of the equivalence classes of representations of the
algebra of observables, where all the representations in question come out of a
special natural class fulfilling the so called superselection condition. The alge-
bra of (quasi-local) observables has the property (among other properties) that
transforms sector into the same sector (of the Hilbert space of point-like fields in
the sense of Wightman, corresponding to the algebra of quasi-local observables,
in which all the point-like fields – also the charge carrying fields with nontrivial
gauge – act, whenever such corresponding fields do exist). Thus no element of
the observable algebra leads us out of a coherent subspace (selection sector).

\[\text{It should be emphasized here that this is an assumption (or hypothesis) of experimental}
\text{tentative character and it cannot be mathematically inferred from the ordinary quantum}
\text{mechanics contrary to what is sometimes misstated. All } \text{"proofs" of the theorems "that}
\text{such and such quantity is classical in the sense that it does not undergo the superposition}
\text{principle" turned up ultimately to be ineffective and contained serious gaps, e. g. that the}
\text{argument of falls short of the claim was subsequently shown in }.\]

\[\text{Essentially on the same}
\text{footing the "proof" of Landau and Lifshitz, that the Coulomb field is classical as well as other}
\text{similar "proofs" fall short of their goals because the superselection structure goes beyond the}
\text{competencies of the ordinary quantum mechanics.}\]

\[\text{One can reconstruct in this way e. g. the isospin group. However some subtle difficulties}
\text{arise in case of the electric charge in choosing the suitable selection rule and the suitable class}
\text{of representations in this case. They are caused by the unlimited range of the electromagnetic}
\text{interactions (zero rest mass of the photon) and with the construction of the "Hilbert space"}
\text{with the indefinite product within the algebraic formalism it is difficult to construct such}
\text{space and to distinguish the Hilbert space of } \text{"physical states" in it. Below we return to this}
\text{problem.}\]

\[\text{Compare discussion below; the relation between quasi-local algebra in the sense of Haag}
\text{and pointlike fields in the sense of Wightman is essential for the whole proposal.}\]

\[\text{More precisely: no Wigtman point-like filed smeared out over a compact domain, corre-
\text{sponding to an element of the local algebra of observables leads us out of the superselection}\]
Roughly speaking the superselection condition allowing us to select the natural class of representations tells that in space-like infinity each representation of the class behaves like the vacuum representation (there are some important troubles just with this condition for the electric charge). This suggests that the quantities which do not undergo the superposition principle, such as charges, are characterized by a decomposition parameters of representations of the observable algebra (or some subalgebra of the corresponding algebra of smeared out fields, whenever they exist) into irreducible representations. Roughly but suggestively speaking: non-superposing quantities are decomposition parameters of the representation of the smeared out fields corresponding to observables or some other distinguished subalgebra of fields into irreducible representations. Similarly in the quantum statistical mechanics the quantities which do not superpose shows up as decomposition parameters of representations of the same algebra into irreducible representations, but this time for representations of the statistical and not the vacuum sector. Haag’s approach and his school partially based on observable algebra understood in the classical sense (introduced by Dirac in his famous handbook on quantum mechanics) but taking into account division into selection sectors, thus based on representation theory of one and the same algebraic structure, was not able as yet to explain in the same manner the structure of local gauge group symmetries; not to mention the difficulties with electric charge and indefinite product. In my opinion the fundamental reason for the lack of success here lies in this: The algebraic theory introduces two kinds of non-superposing quantities with no deeper interrelation between them: 1) such as charges and 2) such as spacetime coordinates, which are classical quantities with direct physical interpretation, enabling the algebraic theory to have a physical interpretation, yet the the local gauge symmetries connect the two kinds of quantities. Here I propose the

**POSTULATE.** Not only generalized charges, but all non-superposing quantities, including classical directly observable parameters, should be decomposition parameters of representations of some fixed subalgebra of the algebra of smeared out point-like fields into irreducible representations.

This is of course a hypothesis of tentative character. In order to keep the physical interpretation and in order to enable concrete computations, we have to supply the postulate and have to point out the subalgebra which corresponds to the algebra of spacetime coordinates. Namely we supply the postulate with the hypothesis that the subalgebra is given by the so called *-algebra of detectors $\mathcal{A}$ (not unital; roughly speaking it is generated by the elements of the

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6From time to time opinions arise claiming that the troubles have only technical character and are not fundamental, compare e.g. [6]. Some knew perspectives was presented in: [7]. But this status quo lasts since the early seventies of the previous century without any breakthrough visible in the solution. It seems that it will be difficult to avoid the analysis of the relation of the algebraic (Haag’s) and Wightman formulation with the perturbative formulation of QFT, compare [7, 23, 10].
observable algebra of the form $L^*L$, where $L$ are quasi local annihilators, which differ from the Doplicher annihilators only by the property that the ideal which they form is not norm closed\(^7\). Therefore we admit the classical quantities to possess their own subalgebra, which determines their own superselection sectors. At the preliminary stage at least it seems reasonable to assume that the algebra of spacetime coordinates determines the universal structure of superselection sectors for all macroscopic quantities (compare the geometric physical interpretation proposed by Haag \[^5\]). The physical motivation for this definition follows from the geometric particle interpretation of the algebraic theory proposed by Haag as well as the role of classical spacetime coordinates and the algebra of detectors (Doplicher annihilators) in this interpretation \[^5\]. Actually a similar postulate was put foreword by Haag himself, when expressing the conjecture that local gauge groups can be explained within the algebraic formulation similarly as the global groups, allowing a wider class of representations of observable algebra, in particular going out of all the sectors of the Hilbert space in which the algebra of corresponding (smeared out) point-like fields (including charge carrying fields) acts. Haag’s postulate, however, does not give any explicit computational hints (which, among other things, is confirmed by the lack of its realization); in particular it is not clear how to look for the additional representations. Such additional representations would be necessary if no other subalgebra besides the observable algebra would be allowed to determine the superselection sectors. Although some restriction of this kind has to be put on the allowed subalgebra in order to give an objective sense to a non-superposing quantity, the algebra of observables is too big and we have to seek a smaller one. If no other algebra fixing the superselection sectors than the algebra of observables were allowed, no local superselection sectors would be left, by the local normality principle\(^8\). However there is no indication (within the algebraic theory) that the algebra of observables determines all relevant selection sectors, e. g. all sectors sufficient to define all relevant non-superposing macroscopic quantities, sufficient for the physical interpretation of the theory. Even contrary: for the geometric particle interpretation at least all information comes from the use of the subalgebra of detectors and coincidence arrangements of detector\(^9\), indicating that the subalgebra of detectors is sufficient to pick up all relevant sectors, thus suggesting that the whole algebra of observables mix too many sectors of objective physical meaning. Moreover the assumption that non-superposing quantities (including macroscopic quantities) should be construable via the selection sectors inside the Hilbert space acted on by the corresponding (smeared out) point-like fields (exactly as for the the charges and the algebra of observables) finds a justification in the fact that the quantum theory of fields is in agreement at least with the phenomenological theory\[^1\] of quantum

\(^7\)Compare \[^5\], p. 283, algebra of detectors is denoted there by $C$.

\(^8\)\[^5\], p. 131.

\(^9\)\[^5\], p. 272.

\(^10\)Whenever such fields do exists and the correspondence mentioned in the footnote 4 is meaningful, compare discussion below.

\(^11\)I mean the well known FAPP-type methods of H. Žurek and his school.
measurement, assuming that the detectors determine their own selection sectors (assumption which cannot be derived from the ordinary quantum mechanics, as was emphasized e. g. by Penrose).

Perhaps we should explain that the classical character of spacetime (in the physical sense used here) and the non-commutative character of the algebra of spacetime coordinates ($\mathcal{A}$) are not a priori inconsistent. The term classical used here means that the superposition takes place only within one and the same sector (of the sector structure in the Hilbert space acted on by the fields, determined by the smeared out fields corresponding to $\mathcal{A}$). Therefore no superposition exist between (elements of spaces of) in-equivalent irreducible representations of $\mathcal{A}$; thus no ”superposition” of two different spacetime ”points” can exist; as the ”points” of $\mathcal{A}$ correspond to equivalence classes of irreducible representations of $\mathcal{A}$. Parameters numbering the irreducible representations are in a one-to-one correspondence with the spectrum of a commutative subalgebra $\mathcal{A}_{cl}$. Assume for a moment (only for heuristic aims) that $\mathcal{A}_{cl}$ is a subalgebra of $\mathcal{A}$ and therefore it is equal to the center of $\mathcal{A}$. We can therefore localize $\mathcal{A}$ with respect to $\mathcal{A}_{cl}$. Heuristically the elements $\mathcal{A}(x)$ of the localization with $x$ ranging over open subsets $\mathcal{U}$ of the spectrum of $\mathcal{A}_{cl}$ are the elements of the algebra of detectors $\mathcal{A}(\mathcal{U})$ of $\mathcal{U}$. This ”approximation” is, however, too coarse and unrealistic. In the geometric particle interpretation at least, we consider detectors (asymptotically) localized within compact subsets. Although the subsets are small in comparison to distances between localization centers and two detectors with different localization centers (asymptotically) commute, in all relevant coincidence arrangements of detectors, see [5] p. 272, they cannot be shrunk to points. Here points are used in ordinary commutative sense, and have immediate physical meaning of spacetime points used in algebraic quantum field theory (which we intend to identify with elements of the spectrum of $\mathcal{A}_{cl}$). Therefore we are forced to use a non-commutative localization, say of Ore type, with respect to a commutative subalgebra $\mathcal{A}_{cl}$ of $\mathcal{A}$ not contained in the center of $\mathcal{A}$. Of course these are heuristic remarks only, motivated on the geometric interpretation of Haag, suggesting that in general realistic situation there should exist a commutative subalgebra $\mathcal{A}_{cl}$ in the algebra of detectors $\mathcal{A}$ whose spectrum elements are parameters with immediate physical interpretation.

12 Therefore the ”parameters” numbering irreducible representations of the spacetime algebra cannot superpose. In passing: also the classical manifold (in the sense: commutative) can be described by a non-commutative algebra Morita equivalent to the commutative algebra of smooth functions on the manifold, compare e. g. [8]. Of course this case is trivial from the physical point of view, and by this, it is not very interesting for physicists.

13 In general for decompositions into irreducible representations $\mathcal{A}_{cl}$ is a maximal commutative subalgebra in the commutant of $\mathcal{A}$. Here we assume that the algebra $\mathcal{A}$ acts in a fixed Hilbert space, the same in which the corresponding Wightman fields act, and assume that the action defines a faithful representation of $\mathcal{A}$ which is to be decomposed.

14 This is only heuristic, as detectors are localizable only asymptotically.

15 In general we cannot, however, expect that the spectrum of $\mathcal{A}_{cl}$ will be sufficient to designate all points of $\mathcal{A}$, for example the representation of $\mathcal{A}_{cl}$ induced by an irreducible representation of $\mathcal{A}$ is not irreducible in general if $\mathcal{A}_{cl}$ does not lie in the center of $\mathcal{A}$. Algebraically speaking: possibly many different localizations are needed to reconstruct the algebra $\mathcal{A}$ and its relevant spectrum, giving different types of coincidence arrangements of detectors.
sequences of the above postulate (suitably supplemented) one have to introduce (natural) analytic structures allowing concrete computations. We shall describe only some first steps towards this goal, based on the (rigorous) micro-local perturbative approach of Brunetti and Fredenhagen and formulate its connection to local algebraic approach of Haag in terms of formal index theorem of Fedosov and asymptotic representations (generalizing the asymptotic representations of Fedosov). This allows us to introduce spectral triple formalism of Connes, via its construction for free fields.

3 Spacetime and QFT

Here we formulate the hypothesis of the previous section in more concrete mathematical terms. We use the tools of non-commutative geometry, and introduce a natural structure of spacetime in terms of this geometry, which may be adopted to this operator-algebraic situation and explain its natural connection to structures which one finds in quantum field theory. We will use the local perturbative construction of the algebra of observables in gauge theories as proposed by Dütsch, Fredenhagen and Brunetti [9,10]. But first we remind that the analogue \((\mathcal{A}, D, \mathcal{H})\) of the Connes’ spectral triple for pseudo-riemannian manifold, as proposed by Strohmaier [11], is given by a pre-C*-algebra \(\mathcal{A}\) with involution \(*\) acting as an algebra of bounded operators not in the ordinary Hilbert space but in a Krein space \(\mathcal{H}\). The involution is represented by taking the Krein adjoint, the Dirac operator \(D\) is self-adjoint in the Krein sense. Important role is played by the so called fundamental symmetries of the Krein space \(\mathcal{H}\). These are operators: \(J: \mathcal{H} \to \mathcal{H}\), such that: \(J^2 = 1\) and \((\cdot, \cdot) = (J\cdot, \cdot)\), where \((\cdot, \cdot)\) is the indefinite inner product in the Krein space \(\mathcal{H}\). With the help of \(J\), one can obtain ordinary (riemannian) spectral triples from pseudo-riemannian spectral triples in a similar way as this is done in quantum field theory by ”Wick rotation”, when passing to riemannian signature. After this

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Most of all we should be interested in the coincidence arrangements of detectors encountered in particle physics, of course.

16 There are several competitive proposals for this analogue, some of them propose to include (the smooth) fundamental symmetries into the construction of the operator \(D\) acting in ordinary Hilbert space (for example Connes and Marcolli [12] propose to construct a spectral triple in ordinary Hilbert space abandoning, however, (ordinary) self-adjointness of \(D\), but keeping the self-adjointness of \(D^2\)). We rejected those propositions whose construction is based on foliations into Cauchy hyper-surfaces, which seem to be less general. The non-compact riemannian case (non-unital \(\mathcal{A}\)) is worked out in: [13]. Actually first steps has been prepared only in this non-compact direction, but no fundamental difficulties are expected here. An extension of spectral triple formalism to type III algebras has been proposed in: [15].

17 Let us remind briefly that the Krein space \(\mathcal{H}\) is a linear space with indefinite non-degenerate inner product \((\cdot, \cdot)\) which admits a direct sum decomposition \(\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-\) into subspaces \(\mathcal{H}_+ \) and \(\mathcal{H}_-\) on which \((\cdot, \cdot)\) is positive definite and respectively negative definite and such that \(\mathcal{H}_+\) and \(\mathcal{H}_-\) are closed in norm topology induced on them by the inner product \((\cdot, \cdot)\). Thus \((\cdot, \cdot)\) induces on \(\mathcal{H}_+\) and \(\mathcal{H}_-\) the structure of ordinary Hilbert spaces. For any such decomposition \(\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-\), one defines the operator of fundamental symmetry \(J\) putting it equal to \(+1\) on \(\mathcal{H}_+\) and \(-1\) on \(\mathcal{H}_-\). Moreover \((\cdot, \cdot)_J = (\cdot, \cdot)\) is an ordinary positive definite inner product inducing on \(\mathcal{H}\) an ordinarily Hilbert space structure. Norms induced by these inner products defined by any two symmetries \(J\) are equivalent.
digression we go back to the perturbative construction of the algebra of observables as proposed in [9, 10]. We start from free fields in a theory with gauge symmetry. Afterwards we construct the algebra of fields (and algebras of observables and detectors) without performing the adiabatic limit, noticing that their construction depends locally on interaction. It is based on the old ideas of Bogoliubov and Stückelberg, developed by Epstein and Glaser, and then by Dütsch, Brunetti and Fredenhagen, who applied to it the Hörmander’s microlocal analysis of wave fronts for hyperbolic operators. The price we pay for the clear separation of local aspects (renormalization) from the global (adiabatic limit) lies in this: algebras thus constructed are formal power series algebras only, with mathematically well defined coefficients built of free fields, and are deformations of the free field algebras in the sense of Bordemann-Waldmann [1]. Therefore only a halfway is thus reached: the existence of adiabatic limit remains to be examined. We return to the existence problem below, but first we give some details of the construction of Dütsch and Fredenhagen. The local algebra $\mathcal{F}(\mathcal{U})$ of free fields with gauge symmetry (as well as interacting fields, if one assumes the existence of adiabatic limit) does not act in ordinary Hilbert space, but in a space with indefinite inner product, compare the Gupta-Bleuler formalism. In order to give a mathematical sense to some operator manipulations performed by physicists some assumptions of topology-analytic character are necessary (to make the various kinds of taking adjoint of an operator more precise, etc.). We assume in particular that $\mathcal{H}$ is a Krein space (indefinite inner product is non-degenerate and the subspaces $\mathcal{H}_+$ and $\mathcal{H}_-$ of the footnote 17 are closed in norms induced by the indefinite inner product). Thus the Gupta-Bleuler operator $\eta$ is a fundamental symmetry of the Krein space $\mathcal{H}$ (one of many such, and which was denoted above by $\eta$). It is clear that also in this situation we can repeat the general argument of Haag, that the elements of the algebra of fields which represent observables cannot lead us out of coherent subspaces of $\mathcal{H}$. This time, however, situation is more complicated, as we identify two vectors of $\mathcal{H}$ which differ by the so called ”admixture”, a vector on which the indefinite inner products is zero and, moreover, not all vectors of $\mathcal{H}$ are regarded as physical (in particular the indefinite inner product must be positive on them).

In order to reconstruct the so called physical Hilbert space $\mathcal{H}$ we have to use the full BRST formalism (or its equivalent, Dütsch and Fredenhagen use the Kugo-Ojima operator $Q$). In particular the net $\mathcal{U} \to \mathcal{F}(\mathcal{U})$ of local fields is such that every local algebra $\mathcal{F}(\mathcal{U})$ is a *-algebra with a $\mathbb{Z}_2$-gradation. A graded derivation $s$ acts on the algebra $\mathcal{F} = \bigcup_{\mathcal{U}} \mathcal{F}(\mathcal{U})$ of quasi-local fields, such that $s^2 = 0$, $s(\mathcal{F}(\mathcal{U})) \subseteq \mathcal{F}(\mathcal{U})$, $s(F^*) = -(-1)^{\delta(F)} s(F)^*$, $s(AB) = s(A)B + (-1)^{\delta(A)} As(B)$, where the $\mathbb{Z}_2$-gradation is defined by $F \to (-1)^{\delta(F)} F$ ($\delta(F)$ is the ghost number of the field $F$, and $s$ is the BRST transformation). From the properties of $s$ it follows that the kernel $\ker s = \mathfrak{A}_0$ as well as the image $s(\mathcal{F}) = \mathfrak{A}_{00}$ of derivation $s$ are *-sub-algebras of $\mathcal{F}$. Dütsch and Fredenhagen define then the algebra of

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18This operator was denoted by $\eta$ in the Polish translation of the book: W. Heitler, ”The Quantum Theory of Radiation”, Clarendon Press, Oxford, 1954, II.10.
quasi-local observables and the net of local observables as follows:

\[ \mathfrak{A}_0 \mod \mathfrak{A}_{00} \quad \text{and} \quad \mathcal{U} \rightarrow \mathfrak{A}_0 \cap \mathcal{F}(\mathcal{U}) \mod \mathfrak{A}_{00} \cap \mathcal{F}(\mathcal{U}), \]

which makes sense because \( s^2 = 0 \) and \( \mathfrak{A}_0 \supseteq \mathfrak{A}_{00} \). The action of field operators on \( \mathcal{H} \) is such that the involution is represented by the Krein adjoint. We assume additionally that the gradation on \( \mathcal{F} \) can be represented by a \( \mathbb{Z}_2 \)-gradation on \( \mathcal{H} \), such that \( \mathfrak{A}_0^{(0)}, \mathfrak{A}_{00}^{(0)} \) and \( \mathfrak{A}_0^{(1)}, \mathfrak{A}_{00}^{(1)} \) are subspaces in \( \mathfrak{A}_0 \) and \( \mathfrak{A}_{00} \) of grade 0 and 1 respectively. We adopt this gradation as the gradation of the (even) pseudo-riemannian spectral triple \((\mathfrak{A}, D, \mathcal{H})\) mentioned above. We propose also a slight modification in the above definition of the algebra of observables (quasi-local and local) and we put instead:

\[ \mathfrak{A} = \mathfrak{A}_0^{(0)} \mod \mathfrak{A}_{00}^{(0)} \quad \text{and} \quad \mathcal{U} \rightarrow \mathfrak{A}(\mathcal{U}) = \mathfrak{A}_0^{(0)} \cap \mathcal{F}(\mathcal{U}) \mod \mathfrak{A}_{00}^{(0)} \cap \mathcal{F}(\mathcal{U}), \]

thus confining ourselves in their definition to ghost-free fields. The algebra \( \mathcal{A} \) of spacetime coordinates is not directly identified with the algebra of detectors, but with the sub-algebra \( \mathcal{A} \) of \( \mathfrak{A}_0^{(0)} \) for which \( \mathfrak{A} \mod \mathfrak{A}_{00}^{(0)} \) is the algebra of detectors (this is the identification proposed above with the necessary modification caused by the fact that not all vectors of the Krein space \( \mathcal{H} \) are physical and by the identification of vectors differing by an "admixture"). We construct the representation of the algebra of observables in the ordinary (physical) Hilbert space \( \mathcal{H} \) exactly as D"utsch and Fredenhagen. If the graded commutator with an operator \( Q \) represents \( s \) (in short: if \( Q \) represents \( s \), i. e.

\[ s(F) = QF - (-1)^{\delta(A)} FQ, \]

then \( Q \) has to be self-adjoint in the sense of Krein and \( Q^2 = 0 \) (in order to ensure fulfillment of the following conditions \( s(F^*) = -(-1)^{\delta(A)} s(F)^* \) and \( s^2 = 0 \)). Because the physical vectors should be \( s \)-invariant D"utsch and Fredenhagen introduce the following definitions: \( \mathcal{H}_0 = \ker Q \) and \( \mathcal{H}_{00} = \text{Im} Q \). Then they assume:

(i) \( (\varphi, \varphi) \geq 0 \) for every \( \varphi \in \mathcal{H}_0 \) (Positivity),

(ii) \( [\varphi \in \mathcal{H}_0 \land (\varphi, \varphi) = 0] \implies \varphi \in \mathcal{H}_{00}; \)

and put

\( \mathfrak{H} = \mathcal{H}_0 \mod \mathcal{H}_{00}, \)

with the following inner product on \( \mathfrak{H} : \)

\( \langle [\varphi_1], [\varphi_2] \rangle_{\mathfrak{H}} = (\psi_1, \psi_2), \quad \psi_j \in [\varphi_j] = \varphi_j + \mathcal{H}_{00}. \)

\( \mathfrak{H} \) with so defined inner product is a pre-Hilbert space (the closure turns \( \mathfrak{H} \) into a Hilbert space). Next, the formula

\[ \pi([A])[\varphi] = [A\varphi], \quad \text{where} \quad [A] = A + \mathfrak{A}_{00}^{(0)} \quad \text{and} \quad A \in \mathfrak{A}_0^{(0)}, \]

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indicated above. Thus if one starts from free fields acting on the Krein space $\mathcal{H}$, and then construct the deformation of the algebra of fields, i.e. build the formal power series of free fields via the microlocal method of Brunetti-Fredenhagen \[ \text{[10]}, \] then one can extend naturally the above construction of representation of observables (and detectors) for free fields to a formal Bordeman-Waldmann-type representation of deformed algebras of observables and detectors. To formulate strictly the “stability” theorem we need to introduce some further definitions. Namely in order to construct the deformation we replace every element $F \in \mathcal{F}$ with a formal power series $\tilde{\mathcal{F}} = \sum_n g^n F_n$, in which $F_0 = F$, $F_1 , \ldots , F_n \in \mathcal{F}$, $\delta(F_n) = \text{const}$. We replace $s$ and $Q$ with a similar power series $\tilde{s} = \sum_n g^n s_n$ (every $s_n$ is a graded derivation), $\tilde{Q} = \sum_n g^n Q_n$, $Q_n \in \mathcal{L}(\mathcal{H})$, $s_0 = s$, $Q_0 = Q$, thus

$$\tilde{s}^2 = 0, \quad \tilde{Q}^2 = 0, \quad (\tilde{Q}\phi, \psi) = (\phi, \tilde{Q}\psi), \quad \tilde{s}(\tilde{F}) = \tilde{Q}\tilde{F} - (-1)^{\delta(\tilde{F})}\tilde{F}\tilde{Q}. \quad \text{(19)}$$

Next we define the formal algebra of observables \[ \mathcal{H}_0 \] by $(\ker \tilde{s})^{(0)} \bmod (\text{Im} \tilde{s})^{(0)}$, and then replace $\mathcal{H}_0$ and $\mathcal{H}_00$ with $\mathcal{H}_0 = \ker \tilde{Q}$ and $\mathcal{H}_00 = \text{Im} \tilde{Q}$ and define $\hat{\mathcal{H}} = \ker \tilde{Q} \bmod \text{Im} \tilde{Q}$. The inner product in $\hat{\mathcal{H}}$ induces an inner product in $\tilde{\mathcal{H}}$ which assumes values in a formal power series field over $\mathbb{C}$. It follows from the above construction that the formal algebra of observables has a natural formal representation $\check{\pi}$ on $\hat{\mathcal{H}}$. D"utsch and Fredenhagen adopt the definition that a formal power series $\tilde{b} = \sum_n g^n b_n$, $b_n \in \mathbb{C}$, is positive iff there exists another power series $\tilde{c} = \sum_n g^n c_n$, $c_n \in \mathbb{C}$, such that $\tilde{c}^\ast \tilde{c} = \tilde{b}$, i.e. such that $b_n = \sum_{k=0}^n c_k c_n - k$.

In this situation D"utsch and Fredenhagen proved the following stability theorem under deformation. If the positivity assumption is fulfilled, then

(i) $(\tilde{\varphi}, \tilde{\varphi}) \geq 0$ for every $\tilde{\varphi} \in \mathcal{H}_0$

(ii) $[\tilde{\varphi} \in \mathcal{H}_0 \land (\tilde{\varphi}, \tilde{\varphi}) = 0] \implies \tilde{\varphi} \in \mathcal{H}_00$;

(iii) For every $\varphi \in \mathcal{H}_0$ there exists $\tilde{\varphi} \in \mathcal{H}_0$, such that $(\tilde{\varphi})_0 = \varphi$.

(iv) Let $\pi$ and $\check{\pi}$ be the representation of free field algebra constructed above and the formal representation of its deformation in $\mathcal{H}$ or $\tilde{\mathcal{H}}$ respectively. Then

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\[ \text{[19]} \text{It is not important here but in computational practice, i.e. in the formal power series of the microlocal renormalization method of Brunetti-Fredenhagen, } g \text{ is a smooth function on the spacetime manifold (understood in the ordinarily sense) with compact support -- local coupling "constant".} \]

\[ \text{[20]} \text{M. D"utsch and K. Fredenhagen define here: } (\ker \tilde{s}) \bmod (\text{Im} \tilde{s}). \]

\[ \text{[21]} \text{With a slightly different definition of the algebra of observables, as has already been indicated above.} \]
\(\tilde{\pi}(\tilde{A}) \neq 0 \text{ if } \pi((\tilde{A})_0) \neq 0.\)

A state \(\omega\) on the algebra of observables \(\tilde{\mathfrak{A}}(\mathcal{U})\) is defined by the following conditions (compare \([9, 1]\))

(i) \(\omega: \tilde{\mathfrak{A}}(\mathcal{U}) \rightarrow \tilde{\mathbb{C}}\) is linear, i.e. \(\omega(\tilde{a}[\tilde{A}] + [\tilde{B}]) = \tilde{a}\omega([\tilde{A}]) + \omega([\tilde{B}])\),

(ii) \(\omega([\tilde{A}]^*) = \omega([\tilde{A}])^*\) for all \([\tilde{A}] \in \tilde{\mathfrak{A}}(\mathcal{U})\),

(iii) \(\omega([\tilde{A}]^*[\tilde{A}]) \geq 0\) for all \([\tilde{A}] \in \tilde{\mathfrak{A}}(\mathcal{U})\) and

(iv) \(\omega(\tilde{1}) = 1\).

The physical vector-states constructed in \([9]\) define naturally states in the Bordemann-Waldmann sense \([1]\):

\[\omega_{\tilde{\phi}}([\tilde{A}]) = \langle [\tilde{\phi}], [\tilde{A}] [\tilde{\phi}] \rangle_{\tilde{\mathcal{H}}}, \quad [\tilde{\phi}] \in \tilde{\mathcal{H}},\]

where positivity follows from the positivity of Wightman distributions of gauge invariant fields, see \([9]\).

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We have thus arrived at the first preliminary task of our proposal: to provide details of the proof of the \textit{stability theorem under deformation} formulated above with the modifications indicated (i.e. with the modification in definition of observable algebra and explain relationship between the two definitions).

Actually the first part of this task follows from the proof of the \textit{stability theorem} as presented in \([9]\), because \(s\) preserves gradation. Only the comparison of the two definitions of the algebra of observables needs a closer inspection, but again, the relation between the two definitions for QED may essentially be read of from \([9]\). In this case the representation \(\pi\) of our algebra of observables constructed above, in contrary to the algebra of observables of Dütsch and Fredenhagen, \textit{is faithful}, and it is generated by \([F^{\mu\nu}], [\psi], [\bar{\psi}]\) and Wick monomials thereof (of course here \([\cdot]\) are understood as classes modulo elements of the ideal \(\mathfrak{A}_{(0)}\)), whereas the "canonical" representatives of \(\tilde{\mathcal{H}}\) are vectors (of \(\mathcal{H}\)) containing transversal photons, electrons and positrons only, as follows from \([9]\). Our definition of the algebra of observables is therefore justified in QED at least and we can in this case confine ourselves to ghost-free fields when constructing observables. What remains to be investigated in the first task is the relation between the two definitions of the algebra of observables for theories with more involved gauge freedom.

Now we pass to the existence problem for the adiabatic limit, which in the formulation of Dütsch and Fredenhagen is equivalent to the following question: under what (accessible) conditions the formal series are convergent, and thus
when the formal representation of the deformed algebra turns into an actual representation of an actual (C*-)-algebra in an ordinarily Hilbert space? But on the other hand Fedosov [16] proved an interesting theorem in the theory of deformations of symplectic (or even Poisson) manifolds. Namely he showed that the deformed algebra admits a so called asymptotic operator representation in ordinary Hilbert space iff his (Fedosov’) formal index fulfills some integrality conditions. His formal index is a formal analog of the Atiyah-Singer index (better: it is a generalization of the Atiyah-Singer index adopted to deformed algebras and their formal representations), in particular it is a topological invariant of the symplectic manifold, so it is an invariant of the algebra (of smooth functions on the manifold), which is subject to deformation as well as of the deformed algebra (the latter in the general non-commutative sense: it is a formal K-theory invariant). The formal (Fedosov’) index can be carried on deformations considered here. The algebra of free fields (or rather the non-commutative algebra $A$ of spacetime defined above, corresponding to free fields) plays the role of the algebra of smooth functions on the symplectic manifold subject to deformation. Next we confine ourselves to the QED case (in the above deformation formulation, compare e. g. [9]). We know that in this case the adiabatic limit does exist, i.e. Wightman distributions do exist (or Green functions), according to the Blachard-Seneor [17] theorem. We may therefore reconstruct the action of smeared out fields (a construction by now rather well known formally analogous to the Gelfand-Naimark-Segal construction of representation from a state, firstly applied by Wightman). Having given this and the machinery of constructing local algebras (of bounded operators) from fields [15] we intend to read of the asymptotic conditions fulfilled by the representation so constructed which are induced by the asymptotic conditions of Blanchard and Seneor’s paper, fulfilled by Green functions. We may thus construct the analog of Fedosov’ asymptotic representation with the explicit asymptotic conditions fulfilled by power series of which we a priori know that they admit an actual representation.

Thus we arrive at the second task of our proposal: to formulate necessary conditions for the existence of the asymptotic representation of deformation in QED in terms of the formal index.

To this end we intend to mimic the argument of Fedosov which he applies in the construction of the analogous necessary condition [23]: just as in the case of Fedosov’s necessary conditions we expect that they will ultimately depend on (ordinary) K-theory invariant of the algebra subject to deformation (in our case this is the algebra $A$ for free fields and its representation constructed as above). We expect to obtain in this way integrality-type conditions for the index on $A$ (for free fields) which we propose to compare with the index map induced by the ordinary spectral triple $(A, D_3, H_3)$ corresponding to $(A, D, H)$ via the ”Wick rotation” induced by an admissible [23] fundamental symmetry $J$:

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22 Compare theorem 7.1.2 and its proof in ref. [16].
23 Compare [11]
i.e. we propose the Dirac operator $D$ to be so chosen that the index map induced by $D_3$ on $A$ coincides with the index map in the construction of the necessary conditions. However this topological-type condition embracing only the global aspect of the theory may be insufficient for reconstruction of $D$ (even in this undeformed, i.e. free-field case). One may hope to reconstruct in this way only the sign of $F = D_3 |D_3|^{-1}$ of $D_3$. We expect, however, that the full reconstruction of $D$ in the undeformed (i.e. free field) case will be difficult. The local information which shows up in the microlocal renormalization is useless for the reconstruction of "undeformed" $D$. But if the undeformed $D$ was unknown, then any effort to proceed the other way round after Fedosov and investigate the sufficiency condition for the existence of the asymptotic representation would be hopeless (still in QED). We propose to make only first steps towards this goal. We assume that we have undeformed ordinary (riemannian) spectral triple $(A, D_3, \mathcal{H}_3)$ and that it is compact (i.e. unital). Now we could incorporate the microlocal renormalization of Brunetti and Fredenhagen [10] to utilize the local information.

Thus we have arrived at the third task of our proposal: to construct a formal deformation $(\tilde{A}, \tilde{D}_3, \tilde{\mathcal{H}}_3)$ of $(A, D_3, \mathcal{H}_3)$ along the lines of Dütsch and Fredenhagen (or Bordeman and Waldmann), thus to investigate stability of the spectral triple structure $(A, D_3, \mathcal{H}_3)$ under deformation, i.e. try to prove the analog of the above stability theorem for compact spectral triple.

If the stability is preserved, then we can expect to have the full analog of the Fedosov theorem (in compact case only) and imitate the main steps of Fedosov having the full abstract calculus of symbols worked out by Connes and Moscovici [20] for the undeformed $(A, D_3, \mathcal{H}_3)$. Again we expect that even in this simplified case (QED: existence of Green functions assured) the full analog of Fedosov theorem will be difficult to work out, as the non-compact triples involve much more technicalities. Yet the full version (necessity and sufficiency) would be very desirable as we expect in this case that the integrality of the index (necessity and sufficiency condition) puts strong limitations on the allowable values of the deformation parameter, i.e. the coupling constant $g$. This goes outside our proposal, but we expect that in general situation (not only for QED) an analog of Fedosov theorem holds: namely that the actual

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24 Of course modulo a trivial modification on the kernel, but preserving the index, so that $F = \text{index } D_3$, compare e.g. [5].

25 Independently of this many examples of Fredholm modules — bounded versions of ordinary riemannian spectral triples, connected to free (quantum) fields has been constructed, at least for fields without any gauge freedom. Compare e.g. [19], where it is shown how the free fermion charged fields give rise to natural constructions of Fredholm modules. In the same book [19], Chap. IV.13, connection of the adiabatic limit and the Bogoliubov-Epstein-Glaser local renormalization with the local index formula is noticed and emphasized.

26 In fact Fedosov proved his theorem on sufficiency for the existence of asymptotic representation for compact manifolds only. But an analogue theorem is certainly true for the non-compact case as well (after some reasonable assumptions of course).
asymptotic representation does exist (and so the adiabatic limit exist) whenever the index induced by \((A, D, \mathcal{H})\) fulfills some integrality conditions. By what we already know of the charge structure from the algebraic quantum field theory we expect that such index describes charge structure of the theory. Because on the other hand the properties of the index reflect universal properties of the (non-commutative but classical) spacetime, the charge structure would come out of (non-commutative) spacetime properties. At this place I quote a problem posed by Staruszkiewicz: How is it possible at all that the electric charges in general, and the electric charges of particles so much diverse as leptons and hadrons in particular, are all equal to the multiples of one and the same universal unit charge? How is it possible that the electric charge of electron and the electric charge of proton are equal with an an unusually small experimental error, such that their ratio is equal to 1 with the experimental error less than \(10^{-21}\)? We agree with A. Staruszkiewicz that the simplest explanation of this problem is to assume that the electric charges of proton and electron are mathematically equal and that the charge structure (in particular the property of the electric charge cited above) reflects a property of spacetime and not properties of particles themselves, just as for spin, whose properties reflect the rotation symmetries – a subgroup of spacetime symmetries, and result from the properties of irreducible unitary representations of the subgroup. The problem of Staruszkiewicz is an important motivation for this proposal. However the hypothesis presented here differs significantly from the theory proposed by Staruszkiewicz. Here we intend to reconcile the puzzle of Staruszkiewicz with the observed fact that the electric charge (and generalized charges, such as baryon number, lepton number, generalized isospin, ...) do not superpose similarly as macroscopic immediately observable quantities, and propose a tentative hypotheses that the generalized charges do not superpose. Staruszkiewicz adopts the initial assumption that the electric charge can in principle at least superpose, and consequently, that the phase of wave function – a degree of freedom canonically conjugate to the charge emerging from the U(1)-gauge, is subject to quantization. Thus he lives open the question: why we do not observe any coherent superpositions of electric charges? These assumptions (of this proposal and that of Staruszkiewicz) lead to different conceptions of spacetime.

What are the conceptual and computational gains of the hypothesis proposed here and of the conception of non-commutative spacetime adopted here? Perhaps it is worth to emphasize that the inclusion of the algebra of spacetime coordinates as a structural ingredient of the theory along the lines proposed here allows in principle to keep the particle interpretation on curved spacetime, even if the spacetime does not posses any time-like Killing vector fields, considering

\footnote{More exactly: by the corresponding \((A, D_3, \mathcal{H}_3)\).}

\footnote{Compare the Doplicher, Haag and Roberts analysis in ref. [5].}

\footnote{As far as reflected by the index.}

\footnote{This is rather an artifact of the (possibly an oversimplifying) assumption that the regime of validity of the ordinary quantum mechanics is unrestricted then of the Staruszkiewicz’s theory itself (one has to assume at least that the Coulomb field falls within the regime). In this approach an \textit{ad hoc} "vector reduction mechanism" is needed.}
the relationship between the algebra of spacetime coordinates and the algebra of detectors. This allows (potentially) to make a practical use of the renormalization theory of Brunetti and Fredenhagen. Indeed we can, in principle at least, pick up the vacuum-like states by incorporating the relationship between annihilators and detectors. This would give a solution to the well known problem set for e. g. by Buchholz in section 8 of his review article [23]. We should emphasize that the geometric method proposed here introduces a whole variety of non-commutative geometry tools and connects them with the existence problem for the adiabatic limit, a problem which is still open (to the author’s knowledge) for theories with non-abelian gauge symmetry (confinement). Last but not least: we get for free the time’s arrow for non-superposing quantities, as a consequence of the non-commutative character of the algebra of spacetime coordinates.

4 Time’s Arrow for Non-superposing Quantities

Vector fields (e.g. the vector field corresponding to time evolution) on an ordinary manifold correspond canonically to one-parameter groups of automorphisms of the algebra of smooth functions on the manifold (e.g. the one parameter group of time automorphisms). The non-commutative multiplication in the algebra of space-time coordinates has the mathematical consequence that the "non-commutative transformations" corresponding to a vector field are not automorphisms of the algebra (a phenomenon connected to Morita equivalence) and do not form any group in the ordinary sense in general. There are several competitive structures which have to replace the ordinary group (the so called quantum group is one of the main candidates \(^{31}\)) but it is beyond doubt that in general the group property ensuring the existence of the inverse transformation among the "non-commutative transformations" for any "non-commutative transformation" (e.g. ensuring the existence of the backward time evolution "\(-t\)" for every time evolution "\(+t\)"") is not fulfilled in general. This is the case for example for quantum groups. However the possibility that some classical parameters corresponding to spectra of some commutative sub-algebras of spacetime coordinates are acted on by the quantum group (determining say the time evolution) as by an ordinary one-parameter group in not \textit{a priori} excluded; in other words: besides the classical parameters evolving non-deterministically, there could in principle exist parameters evolving deterministically. To explain this let us consider a model. Because the full theory involves extremely complicated computational machinery, and moreover one of its most fundamental ingredients is not explicitly constructed, i.e. the operator \(D\), we are forced to consider a very simplified (even oversimplified) model. Namely we consider quantum fields in two-dimensional spacetime, which are completely integrable, constructed by Faddeev and his school, such e.g. as the quantized nonlinear

\(^{31}\)Until recently it was widely believed that quantum groups do not fit into the spectral triple format. Quite recent works show that the two formalisms may be reconciled. Let us cite the breakthrough papers only: \([24, 25, 26, 27]\)
Schrödinger or sine-Gordon equation. They are constructed from the classical inverse scattering transform, just by replacement of the "classical" fields in the monodromy matrix with point-like operator valued distributions, thus obtaining the quantum monodromy matrix $\tau(\lambda)$, compare the monograph of Korepin, Bogoliubov and Izergin [28], and utilizing the normal ordering (Wick theorem). Let us remind that in such models (two dimensional spacetime) renormalization is finite [29] (no Haag’s theorem) so that the interacting fields may be represented in the Fock space along with free fields, and there is no necessity in smearing them out over open sets of full dimension. The distributions in the monodromy matrix $\tau(\lambda)$ so obtained, which in general are only sesquilinear forms on a dense subset $H_0$ of the Hilbert space (here the Fock space), can moreover be multiplied (Wick theorem applicable) on the dense subset in this simplified situation. Here the dense subset $H_0$ is obtained when acting on the Fock vacuum state by all the polynomials in elements of the second column of the monodromy matrix. Thus we obtain a linear representation (quite singular from the analytic point of view) of the set of linear operators, i.e. the monodromy matrix elements, on the linear subspace $H_0$. As shown in [28] the construction of the monodromy matrix $\tau(\lambda)$ is equivalent with determination of the time evolution (e.g. in the case of "second-quantized" nonlinear Schrödinger equation, it is equivalent to the Bethe Ansatz). Let us stop for a moment at the pure linear-algebraic level of the mentioned representation in the linear space $H_0$ without any care for analytic subtleties in assuring a strict mathematically well defined relationship to the Fock space, keeping in mind only the formal analogy to the Fock space inscribed in the construction of the representation. This is what mathematicians actually did when inventing quantum group. Namely the algebra generated by the elements of the monodromy matrix is from the pure algebraic point of view an algebraic quantum group in the sense of Manin [30]. Thus in the analysis of the algebra (quantum group) we follow mathematicians for a while in order to make clear our motivation for the last task of the proposal. From the commutation relations of the algebra (quantum group) it follows that it coacts on the algebra generated by the first column of the monodromy matrix. The later corresponds formally to the algebra of annihilators with adjoined unit, and thus correspond to our spacetime algebra, via the correspondence between fields and local algebras, which is assured in this completely integrable case. In general the quantum group so constructed is a Yangian, whose structure is still quite

32In fact quantum groups were invited by Drinfeld [32], who placed the algebra into the category of specific bi-algebras with adequately defined structure embracing the algebras of smooth functions on Lie groups with the fully fledged adequately rigid topological structures, generalizing the properties of algebras of smooth functions on Lie groups, asserting non-triviality of the theory of representations of the object. Prof. S. L. Woronowicz introduced the topological structure along the $C^*$-algebra format and extended the Peter-Weyl theory on the quantum compact groups. Further analytic structures, as e.g. differential structure along the spectral triple format was invited in the papers cited in 31 footnote. However the topological and analytical structures invited thereafter have no clear connection to the whole analytic structure of the initial physical situation (Faddeev models).

33I. e. commutation relations of the monodromy matrix elements.

34Compare [28], p. 47.
complicated. This is the case for the nonlinear Schrödinger and sine-Gordon
textbooks of Schrödinger and sine-Gordon
models at least. In particular the Yang-Baxter matrix with parameter \( R(q_1, q_2) \)
corresponding to it has a pole at \( q_1 = q_2 \). Therefore we go further in our mathe-
matical simplifications and assume that we have such a model (if there exits such
and is still reasonable) whose Yang-Baxter matrix is not singular at \( q_1 = q_2 \), so
that we can assume that \( R \) is a function of one parameter \( q \) only. Then we may
use the root of unity phenomenon, investigated and generally described mostly
by Lusztig [33, 34]. Namely, if \( q \) is a primitive root of unity of odd degree, then
quantum groups corresponding to Yang-Baxter matrices with one parameter \( q \)
contain a "big" commutative sub-algebras and the structures of the quantum
groups generate natural ordinary group structures on these sub-algebras and the
actions of the quantum groups on their uniform spaces induce ordinary group
actions on the spectra of the commutative sub-algebras. Perhaps the Manin
problem of GL\(_q\)(2) co-acting on the algebra of the Manin plane is the simplest il-
lustration of the phenomenon, compare e. g. [31], pp. 151-153. In this case
the mentioned sub-algebras lie in the center (of the corresponding algebras).
Because on the other hand the algebras "of functions" of these quantum groups
and of their uniform spaces are not in general Morita equivalent to commutative
algebras, even for \( q \) equal to a primitive root of unity, then their actions on the
spectra of commutative sub-algebras is not in general equivalent to ordinary
group actions. In particular neither the algebra "of functions" of \( GL_q(2) \) nor
the algebra of the Manin plane are Morita equivalent to commutative algebras,
even if \( q \) is equal to a primitive root of unity [35].

Thus we arrive at the fourth task of our proposal: to investigate more
deeply the analytic properties of the linear representation of the quantum
monodromy matrix \( \tau(\lambda) \) on the dense subset \( H_0 \) of the Fock space, given
in [28]. Then incorporating the relationship between point-like fields and
local algebras (as developed in the following papers: [36, 18, 37]) try to
carry the quantum group structure and their action on the corresponding
spacetime algebra of bounded operators. The goal is to convert the formal
argument demonstrated above into an actual.

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