Escape, bound and capture geodesics in local static coordinates in Schwarzschild spacetime

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The classical geodesics of timelike particles in Schwarzschild spacetime is analyzed according to the particle starting radius \( r \), velocity \( v \) and angle \( \alpha \) against the radial outward direction in the reference system of an local static observer. The region of escape, bound and capture orbits in the parameter space of \((r, v, \alpha)\) are solved using the three cases of the effective potential. It is found that generally for radius smaller than \( 4M \) or velocity larger than \( c/\sqrt{2} \) there will be no bound orbits. While for fixed radius larger than \( 4M \) (velocity smaller than \( c/\sqrt{2} \)), as velocity (radius) increase from zero (the black hole surface), the particle is always captured until a critical value \( r_{\text{crit1}} \) \((v_{\text{crit1}})\) when the bound orbit start to appear around \( \alpha = \pi/2 \) between a double-napped cone structure. As the radius (velocity) increases to another critical value \( r_{\text{crit2}} \) \((v_{\text{crit2}})\) then the bound directions and the outward cone becomes escape direction, leaving only the inward cone separating the capture and bound directions. The angle of this cone will increase to its asymptotic value as radius (velocity) increases to its asymptotic value. The implication of the results in shadow of black holes formed by massive particles and black hole accretion is briefly mentioned.

Keywords: Schwarzschild spacetime, escape cone, bound orbits, shadow

I. INTRODUCTION

The study of test particle geodesics in a spacetime in General Relativity is important because the knowledge of the spacetime can be completely deduced from the geodesic motions in it and vice versa. Among known spacetimes, Schwarzschild spacetime is one of the simplest and consequently numerous studies on geodesic motions in it exist in literature, emphasizing different aspects of theoretical considerations or astrophysical applications. A very incomplete collection of these lies in Ref. [1–9].

Among motion of different kinds of particles in Schwarzschild and related spacetimes, the motion of photons is the most intensively studied (see [10–16] for early works). In particular, Synge found that photons can only escape to infinity in the Schwarzschild spacetime if their escaping direction was within a cone region [16] (see Fig. 1). This cone was then called the escape cone in later literature and is the complement of the cone of avoidance in Chandrasekhar’s book [17]. This is now considered as an pioneering work in the study of shadow of black holes [18–22]. The timelike geodesics for massive test particles in Schwarzschild spacetime was studied in Ref. [10–13] and classified in Ref. [17, 23] in term of conserved specific orbital angular momentum \( L \) and specific energy \( E \) of the test particle. However, \( L \) and \( E \) of a test particle can not be conveniently determined locally, i.e., they are more easily measured by an external observer but not local static observer. The quantities naturally connected with \( E \) and \( L \) in the local static reference system are the local radial coordinate \( r \), the particle velocity \( v \) and the velocity direction. For massive particles, the question what initial velocity and direction will lead to bound orbits in the Schwarzschild spacetime, has not been studied. In this work, we would like to focus on the following questions: suppose that a test particle is located in Schwarzschild spacetime at radius \( r \), with local velocity \( v \) and velocity direction angle \( \alpha \) against the outward radial
direction, then what kind of geodesic will this test particle has, i.e., will the particle fall into the black hole, move along a bound orbit or escape to infinity.

We will show that for massive particles starting from a fixed radius \( r \) and velocity \( v \), there will be no bound orbit but either escape or capture ones if the radius is larger than \( 4M \) or the velocity is smaller than \( \frac{c}{\sqrt{2}} \). If the fixed radius is larger than \( 4M \), then the bound orbit will appear when the velocity is larger than a critical value and all these bound orbits should have initial velocities pointing outside a double-napped cone structure (see Fig. 8 and 10). Directions inside the cones leads to capture orbits. The angle of the inward cone against the radial outward direction will increase as the velocity further increase until a second critical value beyond which the bound directions and outward capture directions become escape directions. As the velocity further increases, the angle of the inward cone separating the capture and escape directions will further increase until its asymptotic value. The process is similar if one fixes the velocity \( v \) to a value smaller than \( \frac{c}{\sqrt{2}} \) but increase the starting radius \( r \). We will solve the exact formula for the angle of the cones \( \alpha_e(r, v) \) of massive test particle as a function of radius \( r \) and velocity \( v \). Moreover, all the critical value of velocity or radius will also be obtained.

The work is organized as follows. In section II we setup the radial geodesic equation in the Schwarzschild metric and summarize pervious result in the null case. In section III we do the case study of the massive particle motion and classify the types of orbits in the parameter space of \((r, v, \alpha)\). Lastly in section IV we discuss the implication of this work and its possible extensions.

II. THE SETUP AND MOTION OF MASSLESS PARTICLES

The Schwarzschild metric takes the form

\[
ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2),
\]

in which \((t, r, \theta, \phi)\) are the coordinates and \(M\) is the mass of the black hole. The geodesic equation together with initial conditions completely determines the orbits of particles. Since we are interested only in the final state of particle’s motion, i.e. being captured, bounded or escape to infinity, but not its detail shape, we merely need to focus on the radial geodesic equation, which is

\[
\frac{1}{2} \left(\frac{dr}{d\tau}\right)^2 + V = \frac{1}{2}E^2,
\]

where \(\tau\) is the affine parameter and \(V\) is the effective potential given by

\[
V = \frac{1}{2} \left(1 - \frac{2M}{r}\right) \left(\frac{L^2}{r^2} + \kappa\right).
\]

Here \(\kappa = 0, 1\) for massless and massive particle respectively. \(E\) and \(L\) are two constants of motion introduced in the first integration of the time and angular geodesics equations

\[
E = \left(1 - \frac{2}{r}\right) \frac{dt}{d\tau},
\]

\[
L = r^2 \frac{d\phi}{d\tau}.
\]

Note that in Schwarzschild spacetime, the geodesic motions are planar and therefore we can always set \(\theta(\tau) = \pi/2\). \(E\) and \(L\) are interpreted respectively as the specific energy at infinity radius and orbital angular momentum of the test particle around the \(\theta = 0\) axis.

Eq. (2) is mathematically equivalent to the equation describing a one-dimensional classical mechanical motion of a unit mass particle with total energy \(E^2/2\) in effective potential \(V\). Consequently, the relationship between \(E^2/2\) and \(V\) will determine the final state of particles. Because we are using the natural units \(G = c = 1\), the length and time will have the same dimension as mass, therefore to simplify the analysis we will perform and use in the rest of the work the following change of variables

\[
\frac{r}{M} \rightarrow r, \quad \frac{\tau}{M} \rightarrow \tau, \quad \frac{L}{M} \rightarrow L.
\]
With this change, the effective potential becomes

$$V = \frac{1}{2} \left( 1 - \frac{2}{r} \right) \left( \frac{L^2}{r^2} + \kappa \right).$$

(7)

The final states for the motion of massless particles starting from radius $r$ with local velocity $c$ and velocity angle $\alpha$ against the radial outward direction has been studied in Ref. 16, 17. For completeness, we summarize the finding here. The most essential result in this case is a critical angle given by

$$\alpha_e(r) = \arccos \left( -\frac{(r - 3)\sqrt{r + 6}}{r^{3/2}} \right)$$

(8)

in the two dimensional parameter space spanned by $(r, \alpha)$ (see Fig. 1). This angle is half of the local opening of the escape cone only within which the particle can escape. In other words, it divides the parameter space into two regions: the escape region and the capture region. Massless particles starting in the escape (capture) region will escape from (be captured by) the hole to infinity (its singularity), as illustrated in Fig. 1.

III. MOTION OF MASSIVE PARTICLES

For massive particle, the parameter space has one more dimension, the particle velocity $v$, and becomes $(r, v, \alpha)$. Depending on the location of the initial point in this parameter space, the effective potential $V(r, L)$ that the test particle experience will be different. Consequently, unlike the case of photons, there is a possibility of a bound orbit besides the escape and capture geodesics.

In order to clarify which part of the parameter space will lead to respectively the escape, bound and capture orbits, we first analyze the effective potential (3) and then connect its properties to different regions in the parameter space. Using $\kappa = 1$ in Eq. (3), the potential, now denoted by $V_1$, becomes

$$V_1 = \frac{1}{2} \left( 1 - \frac{2}{r} \right) \left( \frac{L^2}{r^2} + 1 \right).$$

(9)

This potential equals 0 at the surface of the black hole and $\frac{1}{2}$ at infinity. In between, the potential can have local extrema if the extremal condition

$$\frac{\partial V_1(r, L)}{\partial r} = 0$$

(10)

have real solutions of $r$ that are larger than 2.
Solving Eq. (10), we find
\[ r_{\pm} = \frac{1}{2} \left( L \pm \sqrt{L^2 - 12} \right) L. \] (11)

Thus if \( L < \sqrt{12} \), there is no real solution and no local extreme in \( r \in (2, \infty) \). In this case, the potential \( V_1 \) increases monotonically from 0 to its asymptotic value \( \frac{1}{2} \). When \( L = 12 \), these two roots are degenerate
\[ r_+ = r_- = 6 \] (12)
and this radius is a flat but non-extremal point of the potential. For \( L > \sqrt{12} \), it can be readily verified that \( r_- \) will be a local maximum and \( r_+ (> r_-) \) is a local minimum of the potential. Moreover, it is also easy to show that in this case \( 3 < r_- \leq 6 \) and \( r_+ \geq 6 \).

Substituting \( r = r_{\pm} \) into (9) we obtain respectively the maximum and minimum values of the potential
\[ V_{1,\text{min}} = \frac{(L^2 + L\sqrt{L^2 - 12} - 4)^2}{L(L + \sqrt{L^2 - 12})^3}, \quad V_{1,\text{max}} = \frac{(L^2 - L\sqrt{L^2 - 12} - 4)^2}{L(L - \sqrt{L^2 - 12})^3}. \] (13)

In particular, one can also verify that the potential minimum \( V_{1,\text{min}} \) increases monotonically as \( L \) increases but is always smaller than the asymptotic value \( \frac{1}{2} \). The potential maximum \( V_{1,\text{max}} \) also increases with \( L \) and will be equal to the asymptotic value when \( L = 4 \). The above cases of the potential are summarized in Fig. 2 and has appeared in e.g. Ref. [17].

In the following subsections, we then discuss the possible motion types in each case of the potential in Fig. 2, the connection of the quantities \( L \) and \( E \) to local observables \( v \) and \( \alpha \), and most importantly how the physical variables \( (r, v, \alpha) \) affects the outcome of the orbits.

![FIG. 2: The effective potential as a function of radius for some fixed \( L \). \( L^2 = 11.9 \) (a), \( L^2 = 13.5 \) (b) and \( L^2 = 18 \) (c). In each case, typical escape (dashed green), bound (dotted brown) and capture (solid red) orbits are indicated.](image)

### A. Case (1): \( L^2 < 12 \)

In this case, it is seen from Fig. 2 (a) that there are only two possibilities for the final state of the geodesics. The particle will escape if and only if the energy is equal or higher than the asymptotic value
\[ \frac{1}{2} E^2 \geq \frac{1}{2} \] (14)
and the initial moving direction is not inward
\[ \frac{d \tau}{d r} \geq 0. \] (15)
Other particles not satisfying these two conditions in Case (1) will be captured by the black hole.

In order to translate the case condition $L^2 < 12$ and escape conditions (14) and (15) to requirements on parameters $r$, $v$ and $\alpha$, we now consider a tetrad $e_\mu^a$ associated with a static observer at coordinates $(r, \theta, \phi)$

$$e_0^\mu = \left( \left( 1 - \frac{2}{r} \right)^{-1/2}, 0, 0, 0 \right),$$  \hfill (16)

$$e_1^\mu = \left( 0, \left( 1 - \frac{2}{r} \right)^{1/2}, 0, 0 \right),$$  \hfill (17)

$$e_2^\mu = \left( 0, 0, \frac{1}{r}, 0 \right),$$  \hfill (18)

$$e_3^\mu = \left( 0, 0, 0, \frac{1}{r \sin \theta} \right).$$  \hfill (19)

This tetrad links the Schwarzschild metric (1) to the Minkowski metric

$$ds^2 = -d\tilde{x}_0^2 + d\tilde{x}_1^2 + d\tilde{x}_2^2 + d\tilde{x}_3^2$$  \hfill (20)

with coordinates $(\tilde{x}_0, \tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$. Note that the spacial direction $\tilde{x}_1$ points to the radial outward direction. In this local coordinate system, the energy $E$ in Eq. (4), angular momentum $L$ in Eq. (5) and the $dr/d\tau$ in Eq. (15) become respectively

$$E = \left( 1 - \frac{2}{r} \right) \frac{dt}{d\tau} = \left( 1 - \frac{2}{r} \right) \frac{\partial t}{\partial \tilde{x}_a} \frac{d\tilde{x}_a}{d\tau} = \sqrt{1 - \frac{2}{r}} \frac{d\tilde{x}_0}{d\tau},$$  \hfill (21)

$$L = r^2 \frac{d\phi}{d\tau} = r^2 \frac{\partial \phi}{\partial \tilde{x}_a} \frac{d\tilde{x}_a}{d\tau} = r \frac{d\tilde{x}_3}{d\tilde{x}_0} \frac{d\tilde{x}_0}{d\tau} = \frac{d\tilde{x}_3}{d\tau},$$  \hfill (22)

$$\frac{dr}{d\tau} = \frac{\partial r}{\partial \tilde{x}_a} \frac{d\tilde{x}_a}{d\tau} = \sqrt{1 - \frac{2}{r}} \frac{d\tilde{x}_1}{d\tau},$$  \hfill (23)

Here we used the fact that $\theta(\tau) = \pi/2$. In the local system, the above quantities are conveniently expressed in terms of the local velocity $v$ of the particle and its angle $\alpha$ with the radial direction

$$\frac{d\tilde{x}_0}{d\tau} = \frac{1}{\sqrt{1 - v^2}}, \quad \frac{d\tilde{x}_1}{d\tilde{x}_0} = v \cos \alpha, \quad \frac{d\tilde{x}_3}{d\tilde{x}_0} = v \sin \alpha.$$  \hfill (24)

Using these relations in Eqs. (21) and (22), we get

$$E = \frac{1}{\sqrt{1 - v^2}},$$  \hfill (25)

$$L = \frac{r v \sin \alpha}{\sqrt{1 - v^2}},$$  \hfill (26)

$$\frac{dr}{d\tau} = \frac{1}{\sqrt{1 - v^2}} v \cos \alpha.$$  \hfill (27)

Substituting Eqs. (25)–(27) into the case condition $L^2 < 12$, and the escape conditions (14) and (15), we have

$$L^2 = \frac{r^2 v^2 \sin^2 \alpha}{1 - v^2} < 12,$$  \hfill (28)

$$\frac{1}{2} E^2 = \frac{1}{2} - \frac{2}{1 - v^2} \geq \frac{1}{2},$$  \hfill (29)

$$\frac{dr}{d\tau} = \sqrt{1 - \frac{2}{r}} v \cos \alpha \geq 0.$$  \hfill (30)

The case condition (28) bound the velocity direction to the range described by

$$\sin \alpha < \frac{2 \sqrt{3(1 - v^2)}}{rv}.$$  \hfill (31)
Condition (29) further bounded the escape orbits to the part in parameter space satisfying
\[ v \geq \sqrt{\frac{2}{r}}. \]  
(32)

One can show that with \( v \) satisfying Eq. (32), the right hand side of (31) is less than one, and therefore its inverse sine function can be taken and its solution becomes
\[ \alpha < \arcsin \left( \frac{2\sqrt{3(1-v^2)}}{rv} \right) \equiv \alpha_1 \text{ or } \alpha > \pi - \alpha_1. \]  
(33)

Together with Eq. (30) which requires \( \alpha \leq \pi/2 \), then the total requirement of escape on the velocity direction becomes
\[ \alpha < \alpha_1. \]  
(34)

The rest of the parameter space in Case (1) then will lead to the capture of the particle.

Both the escape and capture regions in Case (1) are plotted in Fig. 3. The top surface of the escape region and the non-trivial part of the inner boundary of the capture region are plotted using \( \alpha_1 \) in Eq. (34). The vertical boundary on the back of the escape region as well as the vertical boundary on the lower front part of the capture region are plotted using Eq. (32).

![FIG. 3: Regions in parameter space \((r, v, \alpha)\) that allows escape (green) and capture (red) orbits when \(L^2 < 12\).](image)

**B. Case (2):** \(12 \leq L^2 < 16\)

As stated in the beginning of this section, in this case there exists one local maximum and one local minimum in the effective potential, as seen in Fig. 2 (b) and therefore exists possible outcomes of the orbits. In this case, the necessary and sufficient conditions for escape geodesics are formally the same as Eqs. (14) and (15)
\[ \frac{1}{2} E^2 \geq \frac{1}{2}, \quad \frac{dE}{d\tau} \geq 0. \]  
(35)

While the necessary and sufficient condition for the orbit to be bounded is that the energy is smaller than the local maximum of the potential
\[ \frac{1}{2} E^2 < V_{1,\text{max}} \]  
(36)

and the starting radius \( r \) is on the outside of the potential maximum radius
\[ r > r_. \]  
(37)
Besides the above escape and bound orbits, the particles in other region of Case (2) will always be captured by the hole. The case condition \(12 \leq L^2 < 16\) indeed contain two inequalities
\[
L^2 \geq 12, \quad \text{and} \quad L^2 < 16.
\] (38) (39)
The first of these, Eq. (38), is solved similar to the case condition of subsection III A and the result is similar to Eq. but with the direction of the inequality changed
\[
\alpha_1 \leq \alpha \leq \pi - \alpha_1, \ r > \frac{2\sqrt{3(1-v^2)}}{v}.
\] (40)
The second inequality, Eq. (39), will produce a different constraint on \(\alpha\)
\[
\sin \alpha < \frac{4\sqrt{1-v^2}}{rv}.
\] (41)
Now for the escape orbit, the conditions (35) should have solutions of the same form as Eq. (32) and \(\alpha < \pi\). These solutions guarantee the right hand side of Eq. (41) is equal or less than 1 and an inverse sine function can be taken. Therefore, combining with the case condition (40) and (41), the escape condition in Case (2) becomes
\[
\alpha_1 \leq \alpha < \alpha_2 \equiv \arcsin \left(\frac{4\sqrt{1-v^2}}{rv}\right), \ v \geq \frac{\sqrt{2}}{r}.
\] (42)
For the bound orbit condition (36), after substituting Eqs. (13), (25) and (26) and taking account into Eq. (38), the angle \(\alpha\) can be solved from it to be
\[
\alpha_1 < \alpha < \pi - \alpha - \alpha_1 \text{ when } 2\sqrt{3(1-v^2)} < r < \frac{18}{1+8v^2}, \ v > \frac{1}{2} \text{ and}
\]
\[
\alpha_e < \alpha < \pi - \alpha_e \text{ when } \left(r > \frac{18}{1+8v^2}, \ v > \frac{1}{2}\right) \text{ or } \left(r > \frac{4}{v} - 2, \ v < \frac{1}{2}\right),
\] (43)
where \(\alpha_e\) is the angle
\[
\alpha_e = \arcsin \left(\sqrt{\frac{8r^2v^4 + 20r^2v^2 - 72rv^2 - r^2 - 36r + 108 + \sqrt{r^2 - 2(8rv^2 + r - 18)^3}}{2r^3v^2(rv^2 - 2)}}\right).
\] (44)
When \(v = 1\), this angle reduces to that of the photon in Eq. (8). The condition (37) when combined with Eq. (38) has a solution of \(\alpha\) as
\[
\alpha_3 < \alpha < \pi - \alpha_3 \equiv \arcsin \left(\frac{\sqrt{1-v^2}}{v\sqrt{r-3}}\right) \text{ when } r > \frac{1+2v^2}{v^2}, \ v > \frac{1}{2} \text{ and}
\]
\[
\alpha_1 < \alpha < \pi - \alpha_1 \text{ when } r > \frac{2\sqrt{3(1-v^2)}}{v}, \ v < \frac{1}{2}.
\] (45)
One can show that the combination of Eqs. (43) to (45) yields
\[
\alpha_e < \alpha < \pi - \alpha_e, \text{ when } \left(r > \frac{1+2v^2}{v^2}, \ v > \frac{1}{2}\right) \text{ or } \left(r > \frac{4}{v} - 2, \ v < \frac{1}{2}\right)
\] (46)
and further combination with the case conditions Eqs. (40) and (41) produces the final region of bound orbits in the parameter space as
\[
\sin \alpha_e < \sin \alpha < \sin \alpha_2, \frac{4}{v} - 2 < r < \frac{2}{v^2} \text{ when } v < \frac{1}{2}, \text{ and } \frac{1+2v^2}{v^2} < r < \frac{2}{v^2} \text{ when } \frac{1}{2} < v < \frac{1}{\sqrt{2}}.
\] (47)
Other regions in the parameter space in Case (2) leads to capture orbits.
In Fig. 4 we show all three kinds of regions in the parameter space in Case (2). The top and bottom surface of the escape region are plotted using \(\alpha = \alpha_1\) and \(\alpha = \alpha_2\) in Eq. (42) respectively. The front and back surface of the bound region are plotted using the \(\sin \alpha = \sin \alpha_e\) and \(\sin \alpha = \sin \alpha_2\) in Eq. (47). The capture region is also bounded by \(\sin \alpha = \sin \alpha_1\) and \(\sin \alpha = \sin \alpha_2\) but with proper portions of escape and bound regions removed.
The potential and types of geodesics in this case are illustrated in Fig. 2 (c). From the derivation of Eq. (41), one see that the case condition $L^2 \geq 16$ implies

$$\alpha_2 \leq \alpha \leq \pi - \alpha_2.$$  \hfill (48)

In this potential, the particle will escape if

$$\frac{1}{2} E^2 > V_{1,\text{max}}, \quad \frac{dr}{d\tau} \geq 0$$ \hfill (49)

or

$$\frac{1}{2} \leq \frac{1}{2} E^2 \leq V_{1,\text{max}},$$ \hfill (50)

$$r > r_-.$$ \hfill (51)

and will be bounded if

$$\frac{1}{2} E^2 < \frac{1}{2},$$ \hfill (52)

$$r > r_-.$$ \hfill (53)

In other regions of the parameter space in case (3), the particle will be captured.

For the escape orbits, from Eq. (36) and its result Eq. (43), one know that Eq. (49) should lead to

$$\alpha_1 < \alpha < \alpha_e \text{ when } \left( r > \frac{18}{1 + 8v^2}, \ v > \frac{1}{2} \right) \text{ or } \left( r > \frac{4}{v} - 2, \ v < \frac{1}{2} \right).$$ \hfill (54)

Taking account into Eq. (48), this region of escape orbits is finally constrained by

$$\alpha_2 < \alpha < \alpha_e \text{ when } r > \frac{2}{v^2}.$$ \hfill (55)

For Eq. (50), noticing Eqs. (14), (36) and their solutions Eq. (32), (43), its solution is found as

$$\alpha_e < \alpha < \pi - \alpha_e, \ r > \frac{2}{v^2}.$$ \hfill (56)

As in Eq. (37), the condition $r > r_-$ still yields Eq. (48) which puts on top of Eq. (56) an extra condition and then for this escape orbit we have finally

$$\alpha_e < \alpha < \pi - \alpha_e \text{ when } \left( r \geq \frac{1 + 2v^2}{v^2}, \ v > \frac{1}{\sqrt{2}} \right) \text{ or } \left( r > \frac{2}{v^2}, \ v < \frac{1}{\sqrt{2}} \right).$$ \hfill (57)
The regions in the parameter space bounded by the Eqs. (55) and (57) therefore correspond to escape orbits in Case (3).

Now for the region of bound orbits, Eqs. (52) and (53) together with Eq. (48) produce

$$\alpha_2 < \alpha < \pi - \alpha_2, \quad \frac{4\sqrt{1 - v^2}}{v} < r < \frac{2}{v^2}, \quad v < \frac{1}{\sqrt{2}}.$$  

The rest of the regions in the parameter space in Case (3) will all lead to capture orbits. The regions of escape, bound and capture orbits in the parameter space \((r, v, \alpha)\) in this case is shown in Fig. 5. As seen in Eqs. (55) and (57), the top, bottom and back surfaces of the region of escape orbits are described by \(\alpha = \alpha_2\), \(\alpha = \pi - \alpha_e\) and \(r = \frac{2}{v^2}\) respectively. The back and front surface of the region of bound orbits are \(\sin \alpha = \sin \alpha_2\) and \(r = \frac{2}{v^2}\) respectively, as dictated by Eq (58). Finally, the \(\alpha = \pi - \alpha_e\) and \(\alpha = \pi - \alpha_2\) are respectively the lower and upper surface of the region of capture orbits.

FIG. 5: The parameter space that allows escape (green), bound (brown) and capture (red) orbits when \(L^2 > 16\).

D. Combined escape, bound and capture regions

FIG. 6: The combined regions of escape (green), bound (brown) and capture (red) orbits. These regions completely partition the entire parameter space.

With all cases solved, we can now combine all regions of escape orbits in Cases (1), (2) and (3) into one region in the parameter space. The result is plotted in Fig. 6. Similarly, the combination of all regions of bound orbits also forms one region, as well as all regions of capture orbits. It is seen that the escape region is enclosed by the vertical surfaces \(r = \frac{2}{v^2}\), the surface \(\alpha = \pi - \alpha_e\) on the top, as well as the parameter space limits \(v = 1\) and \(\alpha = 0\). The bound
region is enclosed by the two surfaces \( \sin \alpha = \sin \alpha_e \) and \( r = \frac{2}{v^2} \), while the capture region is bounded by \( \sin \alpha = \sin \alpha_e \), \( r = \frac{2}{v^2} \) and the parameter space limits \( v = 0, v = 1 \) and \( \alpha = 0, \alpha = \pi \).

![Graph](image)

FIG. 7: The regions of escape (dashed green), bound (solid brown) and capture (dot-dashed red) orbits for \( v = 0.80c \) (a), \( v = 0.71c \) (b), \( v = 0.67c \) (c) and \( v = 0.40c \) (d) and different \( r \) and \( \alpha \).

![Graph](image)

FIG. 8: The capture (dashed red arrows), bound (dot-dashed brown arrows) and escape (dotted green arrows) directions for \( v = 0.4c \) (see Fig. 7 (d)) but different radius. From (a) to (g) the radii are respectively 8.1\( M \), 8.4\( M \), 10.0\( M \), 12.2\( M \), 12.6\( M \), 28.0\( M \) and 108.0\( M \). The critical radius for the appearance of the bound cone is 8.0\( M \) and that of the escape cone is 12.5\( M \). The angle of the inward cone of the double-napped cones or that of the single cone, denoted by the blue dashed lines, against the radial outward direction are respectively 0.534\( \pi \), 0.571\( \pi \), 0.664\( \pi \), 0.731\( \pi \), 0.74\( \pi \), 0.883\( \pi \) and 0.969\( \pi \) rad.

In Fig. 7, we show the the escape, bound and capture regions in the two dimensional parameter space of \((r, \alpha)\) for some typical velocities. The upper boundary separating the capture and escape regions, and the entire boundary separating the bound and capture regions are given by \( \sin \alpha = \sin \alpha_e \). The lower boundary separating the capture and escape regions as well as that between the bound and escape regions, is given by \( r = \frac{2}{v^2} \). It is seen that as anticipated, for particles with velocity larger than \( \frac{c}{\sqrt{2}} \) whose asymptotic energy will be larger than 1, they either escape to infinity or enter the black hole but no bound orbit can be formed. While for particles with velocity lower than \( \frac{c}{\sqrt{2}} \), there can always exist a region in the \((r, \alpha)\) space in which this particle follow a bound orbit. Moreover, for fixed velocity and radius but varying angles that allow bound orbits, the region of bound orbits is between the angles \( \alpha = \alpha_e \) and \( \alpha = \pi - \alpha_e \), which indeed form a double-napped cone structure in the real local coordinate space (see Fig. 7 (c), (d)). The two opposite cone have the same opening angle \( \alpha_e \) and orbits with directions in the cone would all be captured. For fixed velocity \( v < \frac{c}{\sqrt{2}} \), the effect of increase \( r \) is shown in Fig. 8. As \( r \) increases from 2, the double-napped cone starts to appear and the two cones split from \( \alpha = \pi/2 \) only after \( r = r_{\text{crit}1} \) where

\[
r_{\text{crit}1} = \begin{cases} 
\frac{4 - 2v}{1 + 2v^2} & \text{if } v < \frac{c}{2}, \\
\frac{c^2}{2} & \text{if } \frac{c}{2} \leq v < \frac{c}{\sqrt{2}},
\end{cases}
\]

(see Fig. 8 (a)), and as the radius further increases the angles between the two cones increases until another critical radius

\[
r = r_{\text{crit}2} = \frac{2}{v^2}
\]

(60)
At this point the cone angle $\alpha = \alpha_v$ and the directions between the two cones and within the outward cone suddenly become escape directions and only one cone $\alpha = \pi - \alpha_v$ is left, separating the escape region and the capture region (see Fig. 8 (e)). As $r$ further increases, the escape cone angle $\pi - \alpha_v$ will also increase, approaching $\pi/2$ near infinite $r$ (see Fig. 8 (f), (g)). Moreover, comparing Fig. 7 (c) and (d) we see that with the decrease of $v$, the portion of bound orbit in this two-dimension parameter space also become larger.

FIG. 9: The regions of escape (dashed green), bound (solid brown) and capture (dotdashed red) orbits for $r = 3$ (a), $r = 4$ (b), $r = 4.5$ (c) and $r = 10$ (d) and different $\alpha$ and $v$.

FIG. 10: The capture (dashed red arrows), bound (dotdashed brown arrows) and escape (dotted green arrows) directions for $r = 10$ (see Fig. 9 (d)) but different velocities. From (a) to (g) the velocities are respectively 0.335$c$, 0.344$c$, 0.374$c$, 0.444$c$, 0.454$c$, 0.734$c$ and 0.999$c$. The critical velocity for the appearance of the bound cone is 0.333$c$ and that of the escape cone is 0.447$c$. The angle of the inward cone of the double-napped cones or that of the single cone, denoted by the blue dashed lines, against the radial outward direction are respectively $0.528\pi$, $0.569\pi$, $0.631\pi$, $0.703\pi$, $0.710\pi$, $0.810\pi$ and $0.846\pi$ rad.

In Fig. 9, the two dimensional parameter space of $(v, \alpha)$ for fixed radius is plotted. Similar to the case of fixed $r$, the boundaries separating the escape, bound and capture regions are also the curve $\sin \alpha = \sin \alpha_v$ and $r = \frac{2}{\sqrt{\pi v}}$. For any radius, the escape cone angle will increase as $v$ increases. Moreover, the bound state only starts to appear when $r > 4$. This radius agrees with particle sphere radius, separating the escape and capture region if the incoming particle has almost zero velocity at infinity [24]. For a fixed starting radius $r > 4$, the effect of changing $v$ to the regions and their angles are illustrated in Fig. 10. As the velocity increase from zero there were no escape or bound orbits in any direction until a critical value

$$v = v_{\text{crit}1} = \begin{cases} 
1 & \text{if } 4 < r < 6, \\
\frac{\sqrt{r} - 2}{4} & \text{if } r > 6,
\end{cases}$$

when again the double-napped cone structure starts to appear and the two cones split from $\alpha = \pi/2$ (see Fig. 10 (a)-(c)). The angle between the two opposite cones increases as velocity further increase until a second critical value

$$v = v_{\text{crit}2} = \sqrt{\frac{2}{r}}$$
Beyond which the bound directions and the outward cone directions become escape directions (see Fig. 10(d), (e)). As \( v \) further increases to the speed of light, the escape cone opening angle further increase until its limit value determined by \( \alpha_e(r) \) of photon (see Fig. 10(f), (g)).

IV. IMPLICATIONS AND DISCUSSION

Because the geodesics we considered are reversible, the smaller escape cone, i.e., a larger capture cone, of massive particle comparing to photons starting from the same radius implies that massive particles will form a larger shadow of the same black hole comparing to the shadow formed by light rays. For both photons and massive particles, the Eq. (44) gives a universal expression of the shadow angular size. Assuming that the galactic central black hole mass is \( 4.1 \times 10^6 \) solar mass and distance is 8.122 kpc, its photon shadow size was estimated around 25.89 microarcseconds [19]. On the other hand, besides electromagnetic signals, with the discovery of supernova neutrino from SN1987A [26, 20] and blazar TXS 0506+056 [27, 28], and the recent observation of gravitational wave (GW) signal [29, 30] and the observation of the binary neutron star merger GW170817 and GW170817A [33, 34], it is now well known that neutrinos and GWs can also be used as astrophysical messengers. Because (at least two of the three) neutrinos are massive, in principle the shadow formed by neutrino signal should be slightly larger than that of the photons. For neutrinos of the supernova origin, their typical energy is at the order of 10 MeV and using the mass square difference of the same black hole comparing to the shadow formed by lightrays. For both photons and massive particles, the particle comparing to photons starting from the same radius implies that massive particles will form a larger shadow.

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Another circumstance where results in this work can be relevant is the accretion of black holes. In these models, usually the falling of materials and the escape of photons and particles play crucial rules in the dynamics of the accretion. However, many of the accretion theories simply assume a simple radius of escape or velocity of escape for particles but did not considered a detailed dependance of escape on the particle velocity, escape radius and escape angle [37-39]. Although the background spacetime in accretion models are usually more complicated than a Schwarzschild spacetime, the results here regarding this detailed dependance of the escape region on the initial parameters should provide a primary step towards a better understanding of escape of particles with arbitrary velocity/angles in slightly more complicated spacetimes, e.g., Schwarzschild spacetime with (weak) magnetic field [40], Reissner-Nordstrom black holes [41, 42] and Kerr spacetime with small orbital angular momentum (with/without magnetic field) [43, 44].

On a somewhat different direction that the results here might be useful is for the navigation of spacecraft near massive compact objects. Suppose that the spacecraft can only accelerate locally to a maximum velocity \( V \) but its velocity direction can be chosen arbitrarily, and follow geodesics afterwards, then the question is can the spacecraft escape to infinity or enter the desired bound orbit along the chosen direction and avoid being captured by the black hole. In this case, the result Eq. (44) and Figs. 3 to 9 especially the partition of the parameter space in Fig. 6 will provide the spacecraft necessary values of needed velocity and its direction at the given radius. Finally, let us point out that it is straightforward to extend the current work to other spherically symmetric static spacetimes and equatorial motions in axially symmetric spacetimes. A more dramatic improvement would be to use field approach rather than the optical geodesic one in studying the final state of particles. This is particularly important for the study of GWs because in that case a GW with frequency 1 Hz has wavelength comparable to size of massive black holes of \( 10^5 M_{\odot} \). If the GW frequency is 0.1 Hz or slightly below, which is well in the reach of near future GW detectors [47, 48], then its wavelength will be larger than the the millon sun mass black hole sizes and consequently its wave nature will enhance wave effect such as absorption and interference of the GW. In these situation, it would be necessary to use a field treatment to properly solve the outcome of the propagating GWs.

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