A Proof to the Riemann Hypothesis
Using a Simplified Xi-Function

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Abstract: The Riemann hypothesis, proposed by Bernhard Riemann in 1859, has been of great interest to the mathematics community due to its implications for the distribution of prime numbers. In this paper, we propose a proof to the Riemann hypothesis. First, we provide a brief review of the simplified Riemann $\xi(s)$ function and its important properties. Then, we follow Bombieri’s idea as stated in the Official Problem Description, and show that on the critical line, all zeros of the $\xi(s)$ function are simple, all local maxima are positive, and all local minima are negative. Based on this, we break up the complex domain into several sub-domains based on the zero counters of the imaginary part of $\xi(s)$ function, where it takes different signs. Using the Cauchy-Riemann equations, we demonstrate that the zeros of the $\xi(s)$ function are not located outside the critical line, thus proving the Riemann hypothesis. Finally, we provide some comments on Pólya’s counterexample.

Key Words: Riemann hypothesis.

1 Introduction

In 1859, Riemann [1] used an analytical continuation method to extend the zeta function $\zeta(s)$ into the following form:

$$\xi(s) = \frac{1}{2} s(s - 1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s),$$

(1)
where, the function $\xi(s)$ can be analytically expressed as

$$\xi(s) = 1 + \frac{1}{2} s(s - 1) \int_1^\infty \Psi(\tau) \left( \tau^{s/2 - 1} + \tau^{-(1+s)/2} \right) d\tau,$$

$$\Psi(\tau) = \sum_{n=1}^\infty e^{-\pi n^2 \tau} = \frac{1}{2} \left( \theta(\tau) - 1 \right), \quad (2)$$

$\theta(\tau)$ is the third Jacobian theta function. Riemann stated that every zero of the function $\xi(s)$, which is also the non-trivial zero of zeta function $\zeta(s)$, is located in the critical strip $0 \leq \text{Re}(s) \leq 1$. Then, he asserted that all zeros are most likely to lie on the line $\text{Re}(s) = 1/2$. This conjecture later became the so-called Riemann hypothesis which is very fundamental in the study of analytic number theory [2-3]. Due to its importance, in 1900, the Riemann hypothesis was proposed as one of the famous Hilbert problems - number eight of twenty-three. And in 2000, it was selected again as one of the seven Clay Millennium Prize Problems [4-5].

In this paper, we will prove the Riemann hypothesis using a simplified $\xi(s)$ function.

## 2 Formula transformation

First, we briefly review the simplified formula of Riemann $\xi(s)$ function and its important properties. The formula was originally made by Riemann in his paper [1] before he put forth the hypothesis. Later, the work was continued by many authors such as Jensen [6], Landau [7], Hardy [8], Wilton [9], Ramanujan [10], Titchmarsh [11], Ingham [12] and others. Here we need to change some notations in the formula in order to conveniently apply the Cauchy-Riemann equations in our work. Starting with $\Psi(\tau) = \sum_{n=1}^\infty e^{-\pi n^2 \tau} = \frac{1}{2} \left( \theta(\tau) - 1 \right), \quad (2)$, we introduce a new complex variable $z \in \mathbb{C}$, $z = x + iy$ to make a transformation

$$s = \frac{1}{2}(z + 1). \quad (3)$$

Then, $\xi(s)$ is rewritten as follows

$$\xi(s) = \Xi(z) = 1 + \frac{1}{8} (z^2 - 1) \int_1^\infty \Psi(\tau) \tau^{-3/4}(\tau^{z/4} + \tau^{-z/4}) d\tau. \quad (4)$$

By $\Xi(z)$, the concerned strip $0 \leq \text{Re}(s) \leq 1$ to search for the zeros of the function $\xi(s)$ in the $s$-plane has been changed to $-1 \leq \text{Re}(z) \leq 1$ in the $z$-plane for the function $\Xi(z)$, and the
line \( \text{Re}(s) = 1/2 \) becomes \( \text{Re}(z) = x = 0 \). Furthermore, we change the integral argument by

\[
\tau = e^{4t},
\]

thus,

\[
\Xi(z) = \frac{1}{2} + \frac{1}{2}(z^2 - 1) \int_0^\infty \Psi(e^{4t})e^{(e^{zt} + e^{-zt})}dt,
\]

or, using a hyperbolic function \( \cosh(w) = (e^w + e^{-w})/2 \), we have

\[
\Xi(z) = \frac{1}{2} + (z^2 - 1) \int_0^\infty F(t)\cosh(zt)dt,
\]

where,

\[
F(t) = \Psi(e^{4t})e^t = \sum_{n=1}^\infty \exp(t - \pi n^2 e^{4t}).
\]

Now we will continue working on the function \( F(t) \). Taking the first and second order differentials, we obtain:

\[
F'(t) = \sum_{n=1}^\infty \exp(t - \pi n^2 e^{4t}) (1 - 4\pi n^2 e^{4t}),
\]

\[
F''(t) = \sum_{n=1}^\infty \exp(t - \pi n^2 e^{4t}) \left[(1 - 4\pi n^2 e^{4t})^2 - 16\pi n^2 e^{4t}\right]
\]

\[
= F(t) + G(t),
\]

where,

\[
G(t) = \sum_{n=1}^\infty \exp(t - \pi n^2 e^{4t}) \left[16\pi^2 n^4 e^{8t} - 24\pi n^2 e^{4t}\right]
\]

is called Jensen function [6]. At \( t = 0 \), we have

\[
F(0) = \sum_{n=1}^\infty \exp(-\pi n^2) = \frac{1}{2}(\theta(1) - 1),
\]

\[
F'(0) = \sum_{n=1}^\infty \exp(-\pi n^2) (1 - 4\pi n^2) = -\frac{1}{2},
\]

and at \( t \to \infty \),

\[
\lim_{t \to \infty} F(t) \sinh(zt) = 0, \quad \lim_{t \to \infty} F'(t) \cosh(zt) = 0,
\]
where, the theta function $\theta(1)$ in (12) is a finite value that can be obtained from Whittaker and Watson [13], while (13) and (14) should have been originally found by Riemann [1] because they are indispensable for working with the integral representation of the $\xi(s)$ function. With this understanding, we can simplify equation (7). Using integration by parts,

$$
\int_0^\infty F(t) \cosh(zt) \, dt = \int_0^\infty \frac{1}{z} F(t) \, d \sinh(zt)
$$

$$
= \frac{1}{z} F(t) \sinh(zt) \bigg|_0^\infty - \frac{1}{z} \int_0^\infty F'(t) \, \sinh(zt) \, dt
$$

$$
= -\frac{1}{z^2} F'(t) \cosh(zt) \bigg|_0^\infty + \frac{1}{z^2} \int_0^\infty F''(t) \, \cosh(zt) \, dt
$$

$$
= \frac{1}{z^2} F'(0) + \frac{1}{z^2} \int_0^\infty [F(t) + G(t)] \, \cosh(zt) \, dt.
$$

(15)

Thus,

$$
(z^2 - 1) \int_0^\infty F(t) \cosh(zt) \, dt = F'(0) + \int_0^\infty G(t) \, \cosh(zt) \, dt.
$$

(16)

Substituting (16) into (7), we obtain an integral representation form of the Riemann’s $\xi(s)$ function, which we call the simplified $\xi(s)$ function in this paper.

$$
\Xi(z) = \int_0^\infty G(t) \cosh(zt) \, dt.
$$

(17)

It is easy to check that the Jensen function $G(t) > 0$ on $0 < t < \infty$, and $G(t) \to 0$ very fast when $t \to \infty$. In a two-page note, Wintner [14] proved that $G(t)$ is strictly decreasing, i.e., $G'(t) < 0$. This important property had also been proved independently by Spira [15] in 1971. In his note, Wintner pointed out a property crediting to Jensen, Hurwitz and others: if the definition domain of $G(t)$ is extended to the negative $t$-axis, it is also an even function, i.e., $G(-t) = G(t)$, and $G'(0) = 0$. Since it is not easy to see this property from (11), we briefly repeat the proof as follows. Recall (8),

$$
F(t) = \Psi(\tau)e^t, \quad \tau(t) = e^{4t}, \quad \tau'(t) = 4\tau(t), \quad \tau(-t) = \frac{1}{\tau(t)},
$$

(18)
we can calculate
\[ G(t) = F''(t) - F(t) = \left[ 16\tau^2\Psi''(\tau) + 24\tau\Psi'(\tau) \right] e^t. \] (19)

Applying derivatives to the identity of Jacobi that Riemann gave in his paper [1]
\[ 2\Psi(\tau) + 1 = \tau^{-1/2}\left[ 2\Psi\left(\frac{1}{\tau}\right) + 1 \right], \] (20)
we obtain
\[ \left[ 16\tau^2\Psi''(\tau) + 24\tau\Psi'(\tau) \right] e^t = \left[ \frac{16}{\tau^2}\Psi''\left(\frac{1}{\tau}\right) + \frac{24}{\tau}\Psi'\left(\frac{1}{\tau}\right) \right] e^{-t} = G(-t). \] (21)
Thus, \( G(t) \) is analogous to a Gaussian function. Its bell-shaped curve is shown in Figure 1. This feature has motivated many researchers to create other similar \( G(t) \) functions to study the Riemann hypothesis, e.g., see [16].

Since \( G(t) \) is a smooth (infinitely differentiable) even function, Jensen [17] showed that its derivatives of any order at \( t = 0 \) and \( \infty \) are given by
\[ G^{(2k-1)}(0) = 0, \quad G^{(2k)}(0) = \text{exists}, \quad \lim_{t \to \infty} G^{(k)}(t) = 0. \] (22)
Although Jensen did not give a formula for even order derivatives, it is shown from [2] and [19] that \( G^{(2k)}(0) \) is related to \( \theta^{(2k+2)}(1) \), \( \theta^{(2k+1)}(1) \), \( \cdots \), \( \theta'(1) \). In recent years, Romik [18]
found a formula to calculate all derivatives of the $\theta(\tau)$ function at $\tau = 1$. Using Romik’s formula, we have calculated

$$G(0) \approx 3.5736, \quad G''(0) \approx -267.6880, \quad G^{(4)}(0) \approx 51978.4213, \ldots$$  \hspace{1cm} (23)

Jensen [17] also discovered many other properties of $G(t)$ and its derivatives, which have been well summarized by Gélinas [19]. In his reading notes, Gélinas has also extended more interesting equations for the $G(t)$ function, e.g., for $t \geq 0$,

$$(2\pi\tau - 3)\pi\tau e^{t-\pi\tau} < \frac{1}{8} G(t) < (2\pi\tau - 3)\pi\tau e^{t-\pi\tau} + 32\pi^2 \tau^3 e^{t-4\pi\tau},$$  \hspace{1cm} (24)

and

$$4G(t) < \frac{-G'(t)}{\pi\tau - \pi} < 4G(t) + 8(8\pi - 18)\pi\tau e^{t-\pi\tau}.$$  \hspace{1cm} (25)

These two inequalities will play important roles in the following work.

### 3 The Proof

We will use [17] to study the Riemann hypothesis. Let $z = x + iy$. The function $\Xi(z)$ can be divided into its real and imaginary parts as follows:

$$\Xi(z) = u(x, y) + i v(x, y)$$

$$= \int_0^\infty G(t) \cosh(xt) \cos(yt) \, dt + i \int_0^\infty G(t) \sinh(xt) \sin(yt) \, dt,$$  \hspace{1cm} (26)

Therefore, $\Xi(z) = 0$ is equivalent to $u(x, y) = v(x, y) = 0$.

The idea for this work came from Bombieri’s assertion in the Official Problem Description [4], which states that “The Riemann Hypothesis is equivalent to the statement that all local maxima of $\xi(t)$ (i.e., $u(0, y)$ in the above equation) are positive and all local minima are negative.” Thus, in the first step, we propose three interconnected theorems to prove that $\Xi(z)$ satisfies Bombieri’s equivalence condition on the critical line $x = 0$, where

$$u(0, y) = \Xi(iy) = \int_0^\infty G(t) \cos(yt) \, dt =: U(y),$$

$$v(0, y) = 0.$$  \hspace{1cm} (27)
**Theorem 1:** For sufficient large \( y \), the \( k \)-th derivative of \( U(y) \) satisfies the following asymptotic equation

\[
U^{(k)}(y) = (-1)^k \left( \frac{\pi}{8} - \frac{1}{4y} \right)^k U(y) \left( 1 + \mathcal{O}(y^{-1}) \right).
\] (28)

Proof: According to Titchmarsh [20], when the modulus of \( s \) is sufficient large, the \( \xi(s) \) function behaves asymptotically as

\[
\log \xi(s) = \frac{1}{2} s \log s + \mathcal{O}(1),
\] (29)

Thus, we set \( s = (1 + iy)/2 \) into (29), where \( y \) is sufficient large,

\[
\log U(y) = \log \xi \left( \frac{1 + iy}{2} \right) = -\frac{y}{4} \arctan \left( \frac{y}{1} \right) + \frac{1}{4} \log \sqrt{1 + y^2} + \mathcal{O}(1)
\]

\[
= -\frac{\pi}{8} y + \frac{1}{4} \log y + \mathcal{O}(y^{-1}).
\] (30)

Differentiating \( U(y) \), we have

\[
U'(y) = -\left( \frac{\pi}{8} - \frac{1}{4y} \right) U(y) \left( 1 + \mathcal{O}(y^{-1}) \right).
\] (31)

Continuing to differentiate this equation for several times, we get \( \square \) (28).

**Remark:** Titchmarsh gave the order estimation (29) through an analysis of (1), in which he ignored the term \( s(s - 1) \). If this term is added back to (29), we get

\[
\log \xi(s) = \frac{1}{2} s \log s + 2 \log s + \mathcal{O}(1).
\] (32)

Then, by a similar work (28) will be adjusted as

\[
U^{(k)}(y) = (-1)^k \left( \frac{\pi}{8} - \frac{9}{4y} \right)^k U(y) \left( 1 + \mathcal{O}(y^{-1}) \right).
\] (33)

As will be seen soon, this adjustment has no effect to the proof of the following Theorem 2.

**Theorem 2:** The function \( U(y) \) in (27) has only simple zeros.

Proof: Assume \( U(y) \) has a zero \( y_0 \) of order \( n \). We can express \( U(y) \) as a Taylor series expansion around \( y_0 \) and get \( U(y) \approx U^{(n)}(y_0)(y - y_0)^n/n! \), where \( U^{(n)}(y_0) \neq 0 \). Thus, in the interval \( y_0 < y < y_0 + \delta \), for sufficiently small \( \delta > 0 \), if \( U(y) > 0 \), then all derivatives \( U^{(k)}(y) > 0 \) for \( 1 \leq k \leq n \). We need to show that this is not possible for \( n \geq 2 \).
We differentiate $U(y)$ in (27), and then integrate by parts to get

$$yU'(y) + U(y) = -\int_0^\infty G'(t)t \cos(yt) \, dt. \quad (34)$$

We differentiate $U(y)$ again to obtain the second-order derivative,

$$U''(y) = -\int_0^\infty G(t)t^2 \cos(yt) \, dt. \quad (35)$$

These two equations motivate us to establish a relationship between $G'(t)$ and $G(t)t^2$. Using the left-hand inequality in (24), we have

$$\pi \tau e^{-\pi \tau} < \frac{1}{8(2\pi \tau - 3)}G(t), \quad (36)$$

which is substituted into the right-hand inequality in (25) to obtain

$$4G(t) < \frac{-G'(t)}{\pi \tau - \pi} < 4G(t) + \frac{8\pi - 18}{2\pi \tau - 3}G(t). \quad (37)$$

Since $\tau = e^{4t} = 1 + 4t + \cdots$, we rearrange (37) as

$$16\pi G(t) < -\frac{G'(t)}{t} < 16\pi \frac{(e^{4t} - 1)}{4t} \left[ 1 + \frac{4\pi - 9}{4\pi e^{4t} - 6} \right] G(t). \quad (38)$$

This equation implies that $-G'(t)/t$ is at least $16\pi$ times greater than $G(t)$. Figure 2 shows a comparison of the three curves, which also validates (24) and (25).

We assume that the ratio of $-G'(t)/t$ to $G(t)$ can be expressed as a function of $a(t)$. As both $G(t)$ and $-G'(t)/t$ are even functions, $a(t)$ is also even. We can express $a(t)$ as a series:

$$a(t) = \frac{-G'(t)}{tG(t)} = a_0 + a_2t^2 + a_4t^4 + a_6t^6 + \cdots. \quad (39)$$

By setting $t = 0$ in (39), we can find the value of $a_0$:

$$a_0 = \lim_{t \to 0} \frac{-G'(t)}{tG(t)} = -\frac{G''(0)}{G(0)} \approx 1.4902 \times 16\pi \approx 74.9076 > 0. \quad (40)$$

Using $G(0)$, $G''(0)$, and $G^{(4)}(0)$ in (23), we can obtain the value of $a_2$:

$$a_2 = \lim_{t \to 0} \frac{1}{t^2} \left( -\frac{G'(t)}{tG(t)} - a_0 \right) = \frac{3[G''(0)]^2 - G(0)G^{(4)}(0)}{6[G(0)]^2} \approx 381.3732 = 5.0912a_0. \quad (41)$$
Although we can continue this process to determine other coefficients, the calculation becomes very tedious. However, we can use an approximate set of coefficients \( \{a_{2k}\} \) for the relation between \( G'(t)t \) and \( G(t)^2 \), as long as the resulting series (39) is close enough to the \( a(t) \) function.

From (37) or Figure 2, we know that there exists a smooth function \( b(t) \approx 1 \) such that
\[
a(t) = \frac{-G'(t)}{tG(t)} = 16\pi \left[ 1 + b(t) \frac{4\pi - 9}{4\pi e^t - 6} \right] \frac{e^t - 1}{4t},
\]
where \( b(0) < b(t) < 1 \), and \( b(0) \approx 0.9026 \) can be calculated from \( a_0 \) by
\[
a_0 = 16\pi \left[ 1 + b(0) \frac{4\pi - 9}{4\pi - 6} \right]. \tag{43}
\]
From (42), \( a(t) \) is dominated by \( (e^t - 1)/(4t) \), which has two even asymptotic functions: \( \sinh(4t)/(4t) \) and \( \cosh(4t) \). We can then easily verify that
\[
a_0 \frac{\sinh(4t)}{4t} \leq a(t) \leq a_0 \cosh(4t), \tag{44}
\]
as shown by the curve plots in Figure 3. Thus, taking Taylor expansions for \( \sinh(4t)/(4t) \) and \( \cosh(4t) \), respectively, and using (39) for \( a(t) \), we have
\[
a_0 \sum_{k=0}^{\infty} \frac{4^{2k}}{(2k+1)!} t^{2k} \leq \sum_{k=0}^{\infty} a_{2k} t^{2k} \leq a_0 \sum_{k=0}^{\infty} \frac{4^{2k}}{(2k)!} t^{2k}. \tag{45}
\]
The three series in (45) have a common starting value $a_0$ at $t = 0$, and the curvature of $a(t)$ (i.e., from (41), $a_2 = 5.0912a_0$) is between the other two: $(8/3)a_0 < a_2 < 8a_0$. When $t \to \infty$, according to (42),

$$
\lim_{t \to \infty} a(t) \frac{a_0 \sinh(4t)/(4t)}{a_0 \cosh(4t)} = 32\pi a_0, \quad \lim_{t \to \infty} a(t) \frac{a_0 \cosh(4t)}{a_0 \sinh(4t)/4t} = 0,
$$

which means that $a_{2k}$ approaches $a_04^{2k}/(2k+1)!$ as $k$ becomes large. These properties enable $a_{2k}$ to be well approximated by a weighted average of the other two sequences

$$
a_{2k} = (1 - c_{2k})a_0 \frac{4^{2k}}{(2k+1)!} + c_{2k}a_0 \frac{4^{2k}}{(2k)!}, \quad (k = 0, 1, 2, \cdots),
$$

where we restrict $-1/(2k) < c_{2k} \leq 1$ such that $0 < a_{2k} \leq a_04^{2k}/(2k)!$. For example, we have manually created a set of $a_{2k}$ as shown in Table 1. The resulting fitted curve is also plotted in Figure 3 for comparison. The fitting error is denoted by

$$
\Delta = \max_{0 \leq t < \infty} \left| a(t) \left( \sum_{k=0}^{\infty} a_{2k} t^{2k} \right)^{-1} - 1 \right|.
$$

According to (46), $a_{2k}$ will become more accurate as $k$ becomes large, thus the fitting error occurs in the small $t$ region. Therefore, for our example, we get $\Delta \leq 0.008$. Certainly, one can obtain a smaller $\Delta$ through some optimization method.
Using the function $a(t)$ and the equation $-G'(t) \cdot t = G(t) \cdot t^2 \cdot a(t)$, we can combine $U(y)$, $U'(y)$, and $U''(y)$ together. Referring to equations (34) and (48), we have

$$yU'(y) + U(y) = -\int_0^\infty G'(t) t \cos(yt) \, dt = \int_0^\infty G(t) t^2 a(t) \cos(yt) \, dt$$

$$= \int_0^\infty G(t) t^2 a(t) \left( \sum_{k=0}^{\infty} a_{2k} t^{2k} \right) \frac{1}{(2k+1)!} \cos(yt) \, dt$$

$$= A \int_0^\infty G(t) \left( a_0 t^2 + a_2 t^4 + a_4 t^6 + a_6 t^8 + \cdots \right) \cos(yt) \, dt,$$

(49)

where $A$ denotes a parameter that varies in a very narrow region $(1 - \Delta < A < 1 + \Delta)$, and has been treated as a constant and taken out from the integral. By differentiating further for the higher order derivatives, we obtain

$$U^{(2k)}(y) = (-1)^k \int_0^\infty G(t) t^{2k} \cos(yt) \, dt,$$

(50)

which we can use to rewrite the previous equation as

$$yU'(y) + U(y) = -A \left[ a_0 U''(y) - a_2 U^{(4)}(y) + a_4 U^{(6)}(y) - a_6 U^{(8)}(y) + \cdots \right].$$

(51)

According to Hardy [8], $U(y)$ has infinitely many zeros on $0 < y < \infty$. Let $y_0$ be one such zero. We assume $y_0$ is sufficiently large because Platt and Trudgian [21] have already shown that the zeros up to $y \leq 6 \cdot 10^{12}$ are simple. Let’s assume that $y_0$ is a zero of order 2. Then, in the neighborhood of $y_0$, $U(y)$ can be expressed as

$$U(y) = \frac{1}{2} U''(y_0) (y - y_0)^2 \left( 1 + O(y - y_0) \right),$$

(52)
where $U''(y_0) \neq 0$. Thus, in this neighborhood, $U''(y) \neq 0$. Since $y$ is large, according to Theorem 1, we have

$$U^{(2k+2)}(y) = \left(\frac{\pi}{8} - \frac{1}{4y}\right)^{2k} U''(y) \left(1 + \mathcal{O}(y^{-1})\right), \quad (k = 1, 2, 3, \cdots). \quad (53)$$

We substitute (53) into (51) and get

$$yU'(y) + U(y) = -Ac U''(y) + \mathcal{O}\left(\frac{U''(y)}{y}\right), \quad (54)$$

where

$$c = a_0 - a_2 \left(\frac{\pi}{8} - \frac{1}{4y}\right)^2 + a_4 \left(\frac{\pi}{8} - \frac{1}{4y}\right)^4 - a_6 \left(\frac{\pi}{8} - \frac{1}{4y}\right)^6 + \cdots. \quad (55)$$

It is seen that (55) is an alternating series. According to (47), if $0 \leq c_{2k} \leq 1$, we have

$$a_0 \frac{4^{2k}}{(2k + 1)!} \leq a_{2k} \leq a_0 \frac{4^{2k}}{(2k)!}, \quad (k = 0, 1, 2, \cdots). \quad (56)$$

Thus, the series satisfies the conditions of Leibniz’ theorem:

$$a_2 \left(\frac{\pi}{8} - \frac{1}{4y}\right)^2 \leq 4.5387 a_0 \left(\frac{\pi}{8}\right)^2 < a_0, \quad (57)$$

$$a_{2k+2} \left(\frac{\pi}{8} - \frac{1}{4y}\right)^{2k+2} \leq a_0 \frac{4^{2k+2}}{(2k + 2)!} \left(\frac{\pi}{8} - \frac{1}{4y}\right)^{2k+2} \leq a_0 \frac{4^{2k}}{(2k + 1)!} \left(\frac{\pi}{8} - \frac{1}{4y}\right)^{2k} \leq a_0 \frac{4^{2k}}{(2k)!} \left(\frac{\pi}{8} - \frac{1}{4y}\right)^{2k}, \quad (k = 1, 2, 3, \cdots), \quad (58)$$

and

$$\lim_{k \to \infty} a_{2k} \left(\frac{\pi}{8} - \frac{1}{4y}\right)^{2k} \leq \lim_{k \to \infty} a_0 \frac{4^{2k}}{(2k)!} \left(\frac{\pi}{8}\right)^{2k} = 0. \quad (59)$$

If $-1/(2k) < c_{2k} \leq 0$, such as $c_4$ in the previous example, one can check from Table 1 that $a_{2k}$ still satisfies the Leibniz conditions. Thus, the series (55) converges and the value

$$a_0 - a_2 \left(\frac{\pi}{8} - \frac{1}{4y}\right)^2 \leq c \leq a_0. \quad (60)$$

From (57), we have $c > 0$, consequently, $Ac > 0$. Therefore, for $y = y_0 + \delta$, where $\delta > 0$ is sufficiently small, according to (52), if $U(y) > 0$, we have $U'(y) > 0$ and $U''(y) > 0$.
Otherwise, if \( U(y) < 0 \), we have \( U'(y) < 0 \) and \( U''(y) < 0 \). However, both of these cases contradict (54).

Let’s assume that \( y_0 \) is a zero of order 3. We differentiate equation (51) with respect to \( y \),

\[
yU''(y) + 2U'(y) = -A\left[a_0U'''(y) - a_2U^{(5)}(y) + a_4U^{(7)}(y) - \cdots\right].
\]  

(61)

In the neighborhood of \( y_0 \), \( U(y) \) can be expressed as

\[
U(y) = \frac{1}{3!}U'''(y_0)(y - y_0)^3\left(1 + O(y - y_0)\right),
\]  

(62)

where \( U'''(y_0) \neq 0 \), and thus \( U'''(y) \neq 0 \) too. According to Theorem 1, we have

\[
U^{(2k+3)}(y) = \left(\frac{\pi}{8} - \frac{1}{4y}\right)^{2k}U'''(y)\left(1 + O(y^{-1})\right), \quad (k = 1, 2, 3, \cdots).
\]  

(63)

When (63) is substituted into (61), we get

\[
yU''(y) + 2U'(y) = -AcU'''(y) + O\left(\frac{U'''(y)}{y}\right),
\]  

(64)

where \( c > 0 \) is the same as in (55). For \( y = y_0 + \delta \), according to (62), if \( U(y) > 0 \), we will have \( U'(y) > 0 \), \( U''(y) > 0 \), and \( U'''(y) > 0 \); otherwise, if \( U(y) < 0 \), we have \( U'(y) < 0 \), \( U''(y) < 0 \), and \( U'''(y) < 0 \). Then again, both these two cases contradict (64).

If \( U(y) \) has a zero \( y_0 \) of higher order \( n \geq 4 \), we will differentiate (51) several times,

\[
yU^{(n-1)}(y) + (n - 1)U^{(n-2)}(y) = -A\left[a_0U^{(n)}(y) - a_2U^{(n+2)}(y) + \cdots\right],
\]  

(65)

and use the same process to get contradictions. Thus, \( U(y) \) has only simple zeros. \( \square \)

Remark: Theorem 2 can be interpreted by Figure 4. The figure shows a section of the \( U(y) \) curve around the 5-7th zeros. The curve has a geometry of increasing and concave-downward (or, near the neighboring zeros, decreasing and concave-upward). This indicates that the existence of zeros of any high orders is not possible.

**Theorem 3**: The function \( U(y) := u(0, y) \) in (27) has only one stationary point \( y_m \) between every two consecutive zeros, where \( U'(y_m) = 0 \). This makes \( U(y_m) \) either a positive local maximum or a negative local minimum.
Figure 4: A section of $U(y)$ curve from the Riemann’s $\xi(s)$ function.

Proof: According to Theorem 2, all zeros of $U(y)$ are simple. By Rolle’s theorem, there exists at least one $y_m$ between every two consecutive zeros such that $U'(y_m) = 0$. Then, in a neighbourhood region $|y - y_m| < \delta$, $U'(y)$ can be approximated as

$$U''(y) = \frac{1}{n!} U^{(n+1)}(y_m)(y - y_m)^n \left(1 + \mathcal{O}(y - y_m)\right),$$

where $n \geq 1$ and $U^{(n+1)}(y_m) \neq 0$. If $n = 2$, then $U''(y_m) \neq 0$. Let’s assume $U''(y_m) > 0$. Then, in the region $y_m < y < y_m + \delta$, $U''(y) > 0$, $U''(y) > 0$, and $U''(y) > 0$, which contradicts (64). The same cases occur when $n \geq 3$. Therefore, $n = 1$. In a neighbourhood of $y_m$, (51) becomes

$$yU''(y) + U(y) = -AcU''(y) + \mathcal{O}\left(\frac{U''(y)}{y}\right),$$

which is the same as (54), but $U(y)$ and $U'(y)$ are not related to each other from (66). Since $U''(y_m) = 0$, if $U(y_m) > 0$, then $U''(y_m) < 0$, showing that $U(y_m)$ is a positive local maximum. Otherwise, if $U(y_m) < 0$, then $U''(y_m) > 0$, indicating that $U(y_m)$ is a negative local minimum. This also implies that both positive local minimum and negative local maximum cannot occur, i.e., the count of stationary point $y_m$ between two consecutive zeros is 1. \Box
According to Theorem 3, the function $\Xi(z)$ satisfies Bombieri's equivalence condition. By Bombieri's claim, the Riemann hypothesis holds, meaning that the $\Xi(z)$ function will have no zeros in the whole $z$-plane except for the critical line. Therefore, the work is completed as long as Bombieri's claim is true. Although we trust that Bombieri has a proof for his theorem, it has not yet been found in the public literature. Therefore, we provide an independent proof to complete this paper. Of course, Bombieri should be credited if his original proof is later found.

Our next task is to search for the zeros of the $\Xi(z)$ function in the critical strip. The traditional method for this, as described by Pólya [16], involves finding a function $A(z)$ such that $|A(z)\Xi(z) - 1| = B(z)$ is bounded to be smaller than 1, which would then show that $\Xi(z)$ has no zeros. However, finding such an $A(z)$ has never been easy. Our method was motivated by a numerical study of the zero-counters of the $\Xi(z)$ function (either its real part or imaginary part) in the critical strip. Figure 5 shows some zero-counters of the $u(x, y)$ function, which was produced by Gélinas [22] using the Pari/GP and Gnuplot system in 2017. The picture shows that the strip can be divided into several sub-domains by these counters. Since $u(x, y) \neq 0$ in each sub-domain, if we can show that the function $v(x, y) \neq 0$ on the counters, then $\Xi(z)$ is guaranteed to be nonzero in the entire strip.

Likewise, instead of using $u(x, y)$-counters, the strip can be divided by zero counters of the $v(x, y)$ function. By doing so, we discovered that, based on Theorem 3, the Cauchy-Riemann equations can be used perfectly to finish the job, and the Bombieri's equivalence theorem will also be verified.

**Theorem 4:** For any given $y$, there exists an $\varepsilon > 0$ such that in the two neighborhoods of the critical line $0 < |x| < \varepsilon$, the function $\Xi(z)$ has no zero.

**Proof:** We first mention a similar work. In 1914, Bohr and Landau [23] had shown that almost all zeros are located inside a small domain $|x| < \varepsilon$. In comparison, this theorem excludes the $\varepsilon$-domain except the line $x = 0$.

From [26], we have $v(0, y) = 0$, and $v(x, y)$ is an odd function of $x$, i.e., $v(-x, y) = -v(x, y)$. Thus, for any $(x, y)$, if $v_x(0, y) \neq 0$, there exists an $\varepsilon_1 > 0$, such that when $0 < |x| < \varepsilon_1$, we have $v(x, y) = v_x(0, y)x + \mathcal{O}(x^2) \neq 0$. 

15
Figure 5: The zero contours of \( \Re(\xi(s)) \) in the critical strip, calculated by Pari/GP and Gnuplot system in 2017. Picture permission from Gélinas.
If \((x, y)\) is near a point \((0, y_m)\) where \(v_x(0, y_m) = 0\), from the Cauchy-Riemann equation,

\[
\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},
\]

we have \(u_y(0, y_m) = -v_x(0, y_m) = 0\). According to Theorem 3, the curve \(u(0, y)\) reaches
a local maximum or minimum at the point \((0, y_m)\), such that \(u(0, y_m) \neq 0\). Since \(u(x, y)\) is continuous, there exists an \(\varepsilon_2 > 0\), such that when \(\sqrt{x^2 + (y - y_m)^2} < \varepsilon_2\), we have \(u(x, y) \neq 0\).

Thus, for any \(z = x + iy\), assuming that \(z_m = 0 + iy_m\) is the closest point to \(z\) where \(v_x(0, y_m) = 0\), we take \(\varepsilon = \max(\varepsilon_1, \varepsilon_2)\). When \(0 < |x| < \varepsilon\), \(u(x, y)\) and \(v(x, y)\) cannot be 0 at the same time. Therefore, \(\Xi(z) \neq 0\).

\[\square\]

**Theorem 5**: The function \(\Xi(z)\) has no zero in the region \(|x| > 0\).

**Proof**: Due to the symmetry of the function \(\Xi(z)\), we only need to consider the half-plane where \(x > 0\). From Theorem 4, there exists an \(\varepsilon\)-domain \(\omega = \{(x, y) : 0 < x < \varepsilon\}\), which can be divided into several sub-domains \(\omega_k\) \((k = 0, \pm 1, \pm 2, \cdots)\) depending on whether \(v_x(0, y)\) is greater than 0 or less than 0. If \(v(x, y)\) is positive in \(\omega_k\), then it is negative in \(\omega_{k+1}\), and the boundary between \(\omega_k\) and \(\omega_{k+1}\) is a zero curve \(y = \varphi_k(x)\), where \(v(x, \varphi_k(x)) = 0\). Specifically, we have

\[
\omega_k = \{(x, y) : 0 < x < \varepsilon, \ \varphi_{k-1}(x) < y < \varphi_k(x), \ (-1)^k v(x, y) > 0\}. \quad (69)
\]

Since \(v(x, y)\) is analytic (continuous and differentiable) in \(x > 0\), we can extend each sub-domain \(\omega_k\) to fill the entire half-plane:

\[
\Omega_k = \{(x, y) : x > 0, \ \varphi_{k-1}(x) < y < \varphi_k(x), \ (-1)^k v(x, y) > 0\}. \quad (70)
\]

The boundary between \(\Omega_k\) and \(\Omega_{k+1}\) is still denoted by \(y = \varphi_k(x)\), where \(v(x, \varphi_k(x)) = 0\).

The boundary curves are zero contours of the \(v(x, y)\) function. Although they may have complicated geometries, we can first assume that all curves \(y = \varphi_k(x)\) neither intersect nor bifurcate in the half-plane \(x > 0\). A general scenario can be expressed by Figure 6. In the sub-domain \(\Omega_{2k}\), \(v(x, y) > 0\). Therefore, on the left boundary \(x = 0\), we have \(-u_y = v_x > 0\), and \(u(0, y)\) reaches a minimum \(u_{\min} < 0\) at the corner of \(\Omega_{2k}\) and \(\Omega_{2k+1}\), which is the starting
point of the curve $y = \varphi_{2k}(x)$. We denote two directional vectors along and perpendicular to the curve by

$$
e_1 = (1, \varphi'_{2k}(x)), \quad e_2 = (-\varphi'_{2k}(x), 1). \tag{71}$$

Since $v(x, y) < 0$ in the next sub-domain $\Omega_{2k+1}$, using the Cauchy-Riemann equations,

$$0 > \nabla v \cdot e_2 = \left( \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \right) \cdot \left( -\varphi'_{2k}(x), 1 \right) = -\varphi'_{2k}(x) \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}
= \varphi'_{2k}(x) \frac{\partial u}{\partial y} + \frac{\partial u}{\partial x} = \nabla u \cdot e_1. \tag{72}$$

Thus, along the curve, $u(x, \varphi_{2k}(x)) < u_{\min} < 0$. In the same way, it can be shown that along the next curve, $u(x, \varphi_{2k+1}(x)) > u_{\max} > 0$. In conclusion, for any $z = x + iy$ where $x > 0$, if $(x, y)$ is in one of $\Omega_k$, then $v(x, y) \neq 0$. Otherwise, $(x, y)$ must be on a curve $y = \varphi_k(x)$ where $u(x, y) \neq 0$. Therefore, $\Xi(z) \neq 0$.

Suppose that two curves, namely $y = \varphi_{2k}(x)$ and $y = \varphi_{2k+1}(x)$, intersect at a point $P$, as shown in Figure 7. From previous work, we know that along the curve $y = \varphi_{2k}(x)$, $u(x, y) < u_{\min} < 0$, and along the curve $y = \varphi_{2k+1}(x)$, $u(x, y) > u_{\max} > 0$. Thus, the intersection point $P$ becomes a singular point of $u(x, y)$. This contradicts the fact that $\Xi(z)$ is analytic; therefore, any two curves will not intersect.
Finally, suppose that one curve, \( y = \varphi_{2k}(x) \), bifurcates at a point \( P \) and gives rise to a new curve, \( y = \hat{\varphi}_{2k}(x) \), and two new sub-domains in which \( v(x, y) \) takes different signs, as shown in Figure 8. We already know that on the lower boundary \( u(x, \varphi_{2k-1}(x)) > u_{\text{max}} > 0 \), and on the upper boundary \( u(x, \varphi_{2k}(x)) < u_{\text{min}} < 0 \). Thus, in \( \Omega_{2k} \), \( u(x, y) \) is decreasing in the direction perpendicular to the boundary curve, i.e., \( \nabla u \cdot e_2 < 0 \). According to the Cauchy-Riemann equations, \( \nabla v \cdot e_1 = -\nabla u \cdot e_2 > 0 \), i.e., \( v(x, y) \) is increasing in the direction along the boundary curve. Since \( v(x, y) > 0 \) in \( \Omega_{2k} \), from the \( \Omega_{2k} \) side to approach the new curve \( y = \hat{\varphi}_{2k}(x) \), there is

\[
v(x, \hat{\varphi}_{2k}(x)) > 0. \tag{73}
\]

But across the new curve \( y = \hat{\varphi}_{2k}(x) \), we immediately have \( v(x, y) < 0 \). This means the function \( v(x, y) \) is discontinuous on the curve \( y = \hat{\varphi}_{2k}(x) \), which contradicts \( \Xi(z) \) being analytic. Thus, \( y = \varphi_{2k}(x) \) cannot bifurcate, and we complete the proof. \( \square \)

Based on the five theorems presented above, all zeros of the \( \Xi(z) \) function are located on the critical line \( x = 0 \). Thus, the Riemann hypothesis is true. Moreover, we have also verified Bombieri’s equivalence theorem.

Figure 7: Sub-domains of \( \Omega_k \) showing two curves intersecting at point \( P \).
4 A Counterexample

After the death of J.L.W.V. Jensen, Pólya was given access to his Nachlass and published a fundamental article [17] in German in a Danish journal, which unfortunately was not well known. There, he gave detailed proofs of all the interesting properties of the Riemann $\xi(s)$ function that were found by Jensen. At the end, Pólya was left with deciding whether Jensen had ever found a proof of the Riemann hypothesis, or if Jensen’s contributions could lead to a successful proof of the hypothesis. To settle this issue, he produced a devastating example of an entire function that had almost all the properties of the Riemann $\xi(s)$ function investigated by Jensen, but had zeros off the critical line. Since then, any proofs of the Riemann hypothesis using the Jensen function would have been considered invalid if they apply equally well to Pólya’s counterexample, where the analogue of the Riemann hypothesis fails.

Compared to (17), Pólya’s counterexample is a linear combination of two equalities,

$$\Xi(z) = e^{\frac{1}{2}z^2} \left( \cosh(z) + \alpha \right) = \frac{2}{\sqrt{2\pi e}} \int_0^\infty e^{-\frac{1}{2}t^2} \left( \cosh(t) + \alpha \sqrt{e} \right) \cosh(zt) dt, \quad (74)$$

where $\alpha$ is a parameter. When $-1 < \alpha < 1$, the function $e^{\frac{1}{2}z^2} (\cosh(z) + \alpha)$ has infinite simple
zeros on the $y$-axis, which behaves the same way as the Riemann $\xi(s)$ function. However, when $\alpha = 1$, since

$$\cosh(iy) + 1 = \cos(y) + 1 = 2 \cos^2\left(\frac{y}{2}\right),$$

(75)

it has infinite zeros of order 2 on the $y$-axis, i.e., $y_k = (2k + 1)\pi$, for $k = 0, \pm 1, \pm 2, \cdots$. And when $\alpha > 1$, the function $\cosh(z) + \alpha$ has no zeros on the $y$-axis any more. Instead, the zeros appear in the region $x \neq 0$, as $z_k = \ln(\alpha \pm \sqrt{\alpha^2 - 1}) + i2k\pi$. Thus, it is necessary to verify whether Pólya’s example could invalidate the results presented in this paper.

The major dispute with Jensen’s work came from the integral kernel, i.e., Jensen’s function $G(t)$ in (11), which is a Gaussian-type function with a bell-shaped curve as shown in Figure 1. In Pólya’s example, the integral kernel

$$G(t) = \frac{2}{\sqrt{2\pi e}} e^{-\frac{1}{2}t^2} \left( \cosh(t) + \alpha \sqrt{e} \right)$$

(76)

is also a Gaussian-type function, but its $\Xi(z)$ function has zeros outside the critical line $x = 0$. This fact indicates that owning an integral kernel of a Gaussian-type function is not a sufficient condition for the $\Xi(z)$ function to be zero-free outside the critical line. According to Theorems 3-5, i.e., Bombieri’s equivalence theorem, if $\Xi(z)$ has a zero outside the critical line, then it must have a positive local minimum or negative local maximum in the critical line. This is indeed the case for Pólya’s example. From (74),

$$\Xi(iy) = U(y) = e^{-\frac{1}{2}y^2} (\cos(y) + \alpha)$$

(77)

does have an infinite number of positive local minima when $\alpha > 1$. Thus, in the case of $\alpha > 1$, Pólya’s counterexample does not contradict the theorems of this paper.

The next question is why the zeros of order 2 occur on the critical line in Pólya’s counterexample when $\alpha = 1$. This actually comes from a different property of the $G(t)$ function. From (76), we can calculate:

$$G'(t) = tG(t) - \frac{2}{\sqrt{2\pi e}} e^{-\frac{1}{2}t^2} \sinh(t)$$

$$= tG(t) - \frac{\sinh(t)}{\cosh(t) + \alpha \sqrt{e}} G(t).$$

(78)
This equation shows an apparent gap compared to [38], in which \(-G'(t)/t\) is even lower than \(G(t)\). Thus, Pólya’s \(G(t)\) decreases to 0 much slower than Jensen’s. After taking the Fourier cosine transform of the two \(G(t)\)s in [27], we get two different \(U(y)\) functions. In Pólya’s case, as shown in (77), \(U(y)\) has a bound function of \(e^{-\frac{1}{2}y^2}\), so \(U'(y)/U(y) \approx -y\). In Jensen’s case, from Theorem 1, the bound is \(U(y) = e^{-(\pi/8)y}\) and then \(U'(y)/U(y) \approx -\pi/8\). Thus, for large \(y\), Pólya’s \(U(y)\) may create a very high slope \(U'(y)\) near a zero, which could break the simple zero condition.

We can show the curve property for Pólya’s \(U(y)\) directly. Differentiating (77), we get:

\[
U'(y) = -yU(y) - e^{-\frac{1}{2}y^2}\sin(y),
\]

\[
U''(y) = (y^2 - 1)U(y) + e^{-\frac{1}{2}y^2}(2y\sin(y) + \alpha). \tag{79}
\]

When \(\alpha\) approaches 1, a critical value of \(\alpha_c\) that makes \(U(y) \to 0^+\), \(U'(y) > 0\), \(U''(y) < 0\) can be found by solving the following equations:

\[
\cos(y) + \alpha = 0, \\
-\sin(y) \geq 0, \\
2y\sin(y) + \alpha \leq 0. \tag{80}
\]

It turns out that \(\alpha_c \approx 0.98617\). This indicates that when \(\alpha \geq \alpha_c\), a curve geometry of increasing and concave-upward, i.e., \(U(y) > 0\), \(U'(y) > 0\), \(U''(y) > 0\), has already formed around a zero, which will eventually change the zero from order 1 to order 2 when \(\alpha \to 1\). Figure 9 shows the curve geometry around this zero, which can be compared to Figure 4.

In conclusion, Pólya’s \(G(t)\) function has a special property that results in \(U(y) > 0\), \(U'(y) > 0\), and \(U''(y) > 0\) near a zero when \(\alpha \to \alpha_c\). This causes the zero to change from order 1 to order 2. In contrast, Jensen’s \(G(t)\) function does not possess this property. Therefore, in the case of \(\alpha = 1\), Pólya’s counterexample does not contradict Theorem 2.

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**Important reminder:** All calculations in this article are written in an Excel file, which can be downloaded from https://github.com/linxiao19570629/rh. We highly recommend that you download the Excel file and refer to it while reading this article. If you have any questions or comments, feel free to contact us.