Classical and quantum Cosmology of the Sáez-Ballester theory

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We study the generalization of the Sáez-Ballester theory applied to a flat FRW cosmological model. Classical exact solutions up to quadratures are easily obtained using the Hamilton-Jacobi approach. Contrary to claims in the specialized literature, it is shown that the Sáez-Ballester theory cannot provide a realistic solution to the dark matter problem of Cosmology. Furthermore the quantization procedure of the theory can be simplified by reinterpreting the theory in the Einstein frame, where the scalar field can be interpreted as part of the matter content of the theory, in this approach, exact solutions are also found for the Wheeler-DeWitt equation in the quantum regime.

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I. INTRODUCTION

The inclusion of scalar fields into homogeneous cosmologies is a typical practice to study different scenarios, such as inflation, dark matter, and dark energy[1]. However, since the early seventies, the problem exists of finding the appropriate sources of matter and its corresponding Lagrangian to solve an specific scenario[2, 3].

In this respect, Saez and Ballester (SB)[4] formulated a scalar-tensor theory of gravitation in which the metric is coupled to a dimensionless scalar field in order to solve the so-called missing matter problem in Cosmology. Some works about the classical regime are already present in the literature[5, 6, 7, 8]. In particular, in ref. 8 the authors consider the coupling parameter time-dependent and take a particular ansatz for mathematical convenience for...
solving the field equations.

In spite of a the dimensionless character of the scalar field, an antigravity regime appears, and this fact has been used to suggest a new possible way to solve the missing matter problem in non-flat FRW cosmologies. On the other hand, the quantization program of the theory has yet to be made.

In this paper, we shall study a generalization of the SB theory and transform it into a conventional tensor theory, where the dimensionless scalar field is interpreted as an exotic matter. We found the general behaviour for the kinetic scalar field dependent to the scale factor of the universe, but the behaviour corresponds to stiff matter and not for a dust universe, then the missing matter problem is not solved.

With respect to the quantization program, in this approach we can construct the quantization program of the theory using the usual ADM formalism\[2\]. Also, we can in principle quantize the theory following the Loop Quantum Cosmology program.

In this work, we shall use this formulation to obtain classical and quantum solutions in quadratures, for the flat barotropic FRW cosmology, including a cosmological term $\lambda$.

The paper is arranged as follows, In section II we write the generalization Sáez-Ballester formalism in the usual manner, that is, we calculate the corresponding energy-momentum tensor to the scalar field and give the equivalent lagrangian density. Next, we proceed to obtain the corresponding canonical lagrangian $L_{\text{can}}$ to a flat FRW universe through the lagrange transformation, we calculate the classical hamiltonian, we also present solutions to some models. In section III, using the transformation and the Hamiltonian constraint $H$, we find the Wheeler-DeWitt (WDW) equation of the corresponding cosmological model under study. Section IV is devoted to conclusions and outlook.

II. GENERALIZED SAEZ-BALLESTER THEORY

The simplest generalization of the Sáez-Ballester theory\[4\] with a cosmological term is

$$L_{\text{geo}} = (R - 2\lambda - F(\phi)\phi_{,\gamma}\phi^{,\gamma}), \quad (1)$$

where $R$ the scalar curvature, $\phi^{,\gamma} = g^{\gamma\alpha}\phi_{,\alpha}$, and $F(\phi)$ is a dimensionless and arbitrary functional of the scalar field. According to common wisdom, the Lagrangian $\boxed{(1)}$ would correspond to a scalar field theory without scalar potential but with an exotic kinetic term.
The complete action is then

$$ I = \int_{\Sigma} \sqrt{-g} (\mathcal{L}_{geo} + \mathcal{L}_{mat}) \, d^4x , $$

(2)

where we have included a matter Lagrangian $\mathcal{L}_{mat}$, and $g$ is the determinant of metric tensor. The field equations derived from the above action are

$$ G_{\alpha\beta} + g_{\alpha\beta} \lambda - F(\phi) \left( \phi_{,\alpha} \phi_{,\beta} - \frac{1}{2} g_{\alpha\beta} \phi_{,\gamma} \phi_{,\gamma} \right) = 8\pi GT_{\alpha\beta} , $$

(3a)

$$ 2F(\phi) \phi_{,\alpha} + \frac{dF}{d\phi} \phi_{,\alpha} \phi_{,\gamma} = 0 , $$

(3b)

in which $G$ is the gravitational constant, and a semicolon means covariant derivative.

The same set of equations (3a,3b) is obtained if we consider the scalar field $\phi$ as part of the matter budget, i.e. say $\mathcal{L}_\phi = F(\phi) g^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta}$. In this new line of reasoning, action (2) can be rewritten as a geometrical part (Hilbert-Einstein with $\Lambda$) and matter content (usual matter plus a term that corresponds to the scalar field component of Sáez-Ballester theory),

$$ I = \int_{\Sigma} \sqrt{-g} (R - 2\lambda + \mathcal{L}_{mat} + \mathcal{L}_\phi) \, d^4x . $$

(4)

Even though the philosophy is different to that of the original SB theory, the similarity of the latter to a standard scalar field theory at the classical level will help us to infer the correspondence quantum formulation. We expect the quantum picture will also be the correct one for the SB theory, as all the formulation is based upon the same (classical) Hamiltonian constraint.

Using this action we obtain the classical Hamiltonian of the generalized SB theory for a Friedmann-Robertson-Walker background. Let us start with the line element for a homogeneous and isotropic universe,

$$ ds^2 = -N^2(t)dt^2 + a^2(t) \left[ \frac{dr^2}{1 - \kappa r^2} + r^2 d\Omega^2 \right] , $$

(5)

where $a(t)$ is the scale factor, $N(t)$ is the lapse function, and $\kappa$ is the curvature constant that can take the values 0, 1 and $-1$, for flat, closed and open universe, respectively. The total Lagrangian density then reads

$$ \mathcal{L} = \frac{6\dot{a}^2 a}{N} - 6\kappa Na + \frac{F(\phi)a^3}{N} \dot{\phi}^2 + 16\pi G N a^3 \rho - 2Na^3\lambda , $$

(6)

where $\rho$ is the matter energy density; we will assume that it complies with a barotropic equation of state of the form $p = \gamma \rho$, where $\gamma$ is a constant. The conjugate momenta are
obtained from
\[
\Pi_a = \frac{\partial L}{\partial \dot{a}} = 12\dot{a}, \quad \rightarrow \quad \dot{a} = \frac{N\Pi_a}{12a},
\]
\[
\Pi_\phi = \frac{\partial L}{\partial \dot{\phi}} = \frac{2Fa^3\phi}{N}, \quad \rightarrow \quad \dot{\phi} = \frac{N\Pi_\phi}{2Fa^3}.
\] (7)

From the canonical form of the Lagrangian density (6) and the solution for the barotropic fluid equation of motion we find the Hamiltonian density for this theory
\[
\mathcal{H} = \frac{a^{-3}}{24} \left[ a^2\Pi_a^2 + \frac{6}{F(\phi)}\Pi_\phi^2 + 144\kappa a^4 + 48a^6\lambda - 384\pi G\rho_\gamma a^{3(1-\gamma)} \right],
\] (8)
where \(\rho_\gamma\) is an integration constant.

A. Classical solutions for flat FRW

Using the transformation \(\Pi_q = \frac{dS_q}{dq}\), the Einstein-Hamilton-Jacobi corresponding to Eq. (8) is
\[
a^2 \left( \frac{dS_a}{da} \right)^2 + \frac{6}{F(\phi)} \left( \frac{dS_\phi}{d\phi} \right)^2 + 48a^6\lambda - 384\pi G\rho_\gamma a^{3(1-\gamma)} = 0 ,.
\] (9)

The EHJ equation can be further separated in the equations
\[
\frac{6}{F(\phi)} \left( \frac{dS_\phi}{d\phi} \right)^2 = \mu^2 ,
\] (10)
\[
a^2 \left( \frac{dS_a}{da} \right)^2 + 48a^6\lambda - 384\pi G\rho_\gamma a^{3(1-\gamma)} = -\mu^2 ,
\] (11)
where \(\mu\) is a separation constant. With the help of Eqs. (7), we can obtain the solution up to quadratures of Eqs. (10) and (11),
\[
\int \sqrt{F(\phi)} d\phi = \frac{\mu}{2\sqrt{6}} \int a^{-3}(\tau) d\tau ,
\] (12a)
\[
\Delta \tau = \int \frac{a^2 da}{\sqrt{\frac{2}{3}\pi G\rho_\gamma a^{3(1-\gamma)} - \frac{2}{3}a^6 - \nu^2}} ,
\] (12b)
with \(\nu = \frac{\mu}{12}\)

Eq. (12a) readily indicates that
\[
F(\phi)\dot{\phi}^2 = 6\nu^2 a^{-6}(\tau) ,
\] (13)
despite of the particular form of the functional $F(\phi)$. Also, this structure is directly obtained for this model solving the equation (3b). Moreover, the matter contribution of the SB scalar field to the rhs of the Einstein equations would be

$$\rho_{\phi} = \frac{1}{2} F(\phi) \dot{\phi}^2 \propto a^{-6}. \quad (14)$$

That is, the contribution of the scalar field is the same as that of stiff matter with a barotropic equation of state $\gamma = 1$.

This is an interesting result, since the original SB theory was thought of as a form to solve the missing matter problem of Cosmology, now generically called the dark matter problem; to solve the latter, one needs a fluid behaving as dust with $\gamma = 0$. It is surprising that such a general result remain unnoticed until now in the literature about SB.

Also, that we have identified the general evolution of the scalar field with that of a stiff fluid means that the Eq. (12b) can be integrated separately without a complete solution for the scalar field. For completeness, we give below a compilation of exact solutions in the case of the original SB theory.

If $F(\phi) = \omega \phi^m$, then we have two cases that correspond to $m = -2$ and $m \neq -2$; the general solution for the scalar field is

$$\phi = \begin{cases} 
\exp \left[ \frac{6\nu}{6\omega} \int a^{-3}(\tau) d\tau \right] & m = -2 \\
\left[ \frac{2\nu(m+2)}{\sqrt{6\omega}} \int a^{-3}(\tau) d\tau \right]^{\frac{2}{m+2}} & m \neq -2
\end{cases} \quad (15)$$

which can be completely integrated once the time dependence of the scale factor $a$ has been resolved.

- Stiff plus a cosmological constant, $\gamma = -1$. The master equation become

$$\Delta \tau = \int \frac{a^2 da}{\sqrt{b_{-1}a^6 - \nu^2}}, \quad (16)$$

where $b_{-1} = \frac{8}{3} \pi G \rho_{-1} - \frac{\lambda}{3}$, whose solution is

$$\Delta \tau = \frac{1}{3 \sqrt{b_{-1}}} \ln \left[ b_{-1}a^3 + \sqrt{b_{-1}b_{-1}a^6 - \nu^2} \right]. \quad (17)$$

The volume function is then

$$a^3 = \frac{1}{2b_{-1}} \left( e^{3 \sqrt{b_{-1}} \Delta \tau} + b_{-1} \nu^2 e^{-3 \sqrt{b_{-1}} \Delta \tau} \right), \quad (18)$$
whereas that of the scalar field is
\[ \phi = \begin{cases} 
E^{\frac{4}{\sqrt{6\omega}}} \arctan \left( \frac{E^{\frac{3}{\sqrt{b_0}} \Delta \tau}}{\nu \sqrt{b_0}} \right) & m = -2 \ , \\
\frac{2(m+2)}{\sqrt{6\omega}} \arctan \left( \frac{E^{\frac{3}{\sqrt{b_0}} \Delta \tau}}{\nu \sqrt{b_0}} \right)^{\frac{m+2}{m+2}} & m \neq -2 \ . 
\end{cases} \]
\[ (19) \]

For the case \( \gamma = 1 \) the same solutions are found and only a redefinition of the constants is needed.

- Stiff plus a cosmological constant plus dust, \( \gamma = 0 \). In this case the master equation becomes
\[ \Delta \tau = \int \frac{a^2 da}{\sqrt{\frac{8}{3} \pi G \rho_0 a^{3} - \frac{\lambda}{3} a^{6} - \nu^2}} \]
\[ (20) \]
whose solution is
\[ \Delta \tau = \frac{1}{\sqrt{3|\lambda|}} Ln \left[ b_0 + \frac{2|\lambda| a^3}{\sqrt{|\lambda|}} + 2 \frac{b_0 a^3 + |\lambda|}{3} a^6 - \nu^2 \right] . \]
\[ (21) \]
with \( |\lambda| > 0 \) and \( b_0 = \frac{8}{3} \pi G \rho_0 \). The volume function is now
\[ a^3 = \frac{3}{4 \sqrt{3|\lambda|}} e^{-\sqrt{3|\lambda|} \tau} \left[ 4 \nu^2 + \left( e^{\sqrt{3|\lambda|} \tau} - \frac{3 b_0}{\sqrt{3|\lambda|}} \right)^2 \right] . \]
\[ (22) \]
In this way, the solution for the field \( \phi \) is
\[ \phi = \begin{cases} 
E^{\frac{4}{\sqrt{6\omega}}} \arctan \left( \frac{\sqrt{3|\lambda| E^{\sqrt{3|\lambda|} \Delta \tau}}}{2 \nu \sqrt{3|\lambda|}} \right) & m = -2 \ , \\
\frac{4(m+2)}{\sqrt{6\omega}} \arctan \left( \frac{\sqrt{3|\lambda| E^{\sqrt{3|\lambda|} \Delta \tau}}}{2 \nu \sqrt{3|\lambda|}} \right)^{\frac{m+2}{m+2}} & m \neq -2 \ . 
\end{cases} \]
\[ (23) \]

The classical solution when \( F(\phi) = w e^{m \phi} \) have the following structure
\[ \phi(\tau) = \frac{2}{m} Ln \left[ \frac{m}{2 \sqrt{\frac{6\nu^2}{w}}} \int a^{-3}(\tau) d\tau + e^{\frac{m \phi_0}{w}} \right] , \]
\[ (24) \]
where the integration value must be consider the last calculations over the scale factor.

The solutions above were checked to comply with the Einstein field equations encoded in equations \((3b)\), using the REDUCE 3.8 package.
III. QUANTUM FRW COSMOLOGICAL MODEL

One of the open problem of SB is the lack of a quantum model, in this section using the generalization of the ideas presented in the previous sections we use canonical quantization. By the usual representation for the momenta operators $\Pi_q = -i \frac{\partial}{\partial q}$, $(\hbar = 1)$, including the factor ordering problem in the $a$ and $\phi$ variables, we obtain the Wheeler-DeWitt equation

$$
\left[ -a^2 \frac{\partial^2}{\partial a^2} - qa \frac{\partial}{\partial a} - \frac{6}{F(\phi)} \frac{\partial^2}{\partial \phi^2} - \frac{6s}{F(\phi)} \frac{\partial^2}{\partial \phi^2} + 144\kappa a^4 + 48a^6 \lambda - 384\pi G \rho a^{3(1-\gamma)} \right] \Psi = 0,
$$

(25)

where $q$ and $s$ are real constants that measures the ambiguity in the factor ordering in the operators $\Pi_a$ and $\Pi_\phi$, $\Psi$ is the wave function for this cosmological model. Employing the variables separation method, $\Psi(a, \phi) = A(a)B(\phi)$, (25) gives the set of equations

$$
-a^2 \frac{d^2 A}{da^2} - qa \frac{dA}{da} + \left( 144\kappa a^4 + 48a^6 \lambda - 384\pi G \rho a^{3(1-\gamma)} - \mu^2 \right) A = 0,
$$

(26)

$$
\phi \frac{d^2 B}{d\phi^2} + s \frac{dB}{d\phi} - \frac{\mu^2}{6} \phi F(\phi) B = 0.
$$

(27)

The equation (26) does have not a general solution for any $\kappa$, then we solve for flat case and the particular values in the $\gamma$ parameter. When $\gamma = -1$, the exact solution is

$$
A(a) = a^{\frac{1-s}{2}} \left( \frac{\sqrt{b}}{3} a^3 \right),
$$

(28)

where $\nu = \frac{1}{6} \sqrt{(1-q)^2 - 4\mu^2}$ and $b = 384\pi G \rho_{-1} - 48\lambda$. We can see that when $b > 0$, the generic Bessel function $Z_\nu \rightarrow J_\nu$, and when $b < 0$, $Z_\nu \rightarrow (K_\nu, I_\nu)$.

Other soluble case is when $\gamma = 1$, the solution is the same, and the changes appear in the constants $\mu^2 \rightarrow 384\pi G \rho_1 + \mu^2$ and $b = -48\lambda$. In this form, we obtain the exact solution to the wave function $\Psi(a, \phi)$ in this theory.

For solve the equation (27), we apply this approach at Sáez-Ballester theory. The case when $m \neq -2$ [9] is written in term of generic Bessel function $Z_\eta$ as

$$
B(\phi) = c \phi^{\frac{1-s}{2}} Z_{\eta} \left( \frac{2\sqrt{-\xi}}{m+2} \phi \frac{m+2}{n+2} \right),
$$

(29)

where $c$ is a integration constants, and $\eta = \frac{1-s}{m+2}$, $\xi = \frac{\mu^2 \omega}{6}$. Also, we can see that the generic Bessel function $Z_\eta \rightarrow J_\eta$ when $\omega < 0$, or $(K_\eta, I_\eta)$ when $\omega > 0$.

We can build the wave packet, introducing the continuum parameters $\eta$ and $\nu$ as

$$
\Psi_{\eta\nu} = \int_\eta \int_\nu F(\eta) G(\nu) \phi^{\frac{1-s}{2}} Z_{\eta} \left( \frac{2\sqrt{-\xi}}{n+2} \phi \frac{n+2}{m+2} \right) a^{\frac{1-s}{2}} Z_\nu \left( \frac{\sqrt{b}}{3} a^3 \right) d\eta d\nu
$$

(30)
For particular values in the constant $m$, the exact solutions are very simple. For instance, when $m = -2$, we have the Euler equation whose solution is

$$
B(\phi) = \phi^{\frac{1}{2}} \begin{cases} 
\frac{[c_1\phi^\alpha + c_2\phi^{-\alpha}]}{(1 - s)^2 > 4b} \\
\frac{[c_1 + c_2\ln\phi]}{(1 - s)^2 = 4b} \\
\frac{[c_1\sin(\alpha\ln\phi) + c_2\cos(\alpha\ln(\phi))]}{(1 - s)^2 < 4b}
\end{cases}
$$

(31)

with $\alpha = \frac{1}{2}\sqrt{(1 - s)^2 - 4b}$ and $b = -\frac{\omega^2}{6}$.

When $m = -6$ and $s = -1$, making the transformations $z = \phi^{-2}$ and $B = \frac{u}{z}$, leads to a constant coefficient linear equation, (27) is transformed to $4\frac{d^2u}{dz^2} - \frac{\omega^2}{6}u = 0$ who exact solutions becomes

$$
u(z) = \begin{cases} 
c_1 \sinh \left( \sqrt{\frac{\omega^2}{24}}z \right) + c_2 \cosh \left( \sqrt{\frac{\omega^2}{24}}z \right) & \omega > 0 \\
c_1 \sin \left( \sqrt{\frac{\omega^2}{24}}z \right) + c_2 \cos \left( \sqrt{\frac{\omega^2}{24}}z \right) & \omega < 0
\end{cases}
$$

(32)

in the original variables

$$
B(\phi) = \phi^2 \begin{cases} 
c_1 \sinh \left( \sqrt{\frac{\omega^2}{24}}\frac{1}{\phi^2} \right) + c_2 \cosh \left( \sqrt{\frac{\omega^2}{24}}\frac{1}{\phi^2} \right) & \omega > 0 \\
c_1 \sin \left( \sqrt{\frac{\omega^2}{24}}\frac{1}{\phi^2} \right) + c_2 \cos \left( \sqrt{\frac{\omega^2}{24}}\frac{1}{\phi^2} \right) & \omega < 0
\end{cases}
$$

(33)

**IV. CONCLUSIONS**

We studied the generalization of the Sáez-Ballester theory by including a dimensionless functional of the scalar field $F(\phi)$. The classical dynamics of the theory were obtained from the corresponding classical Lagrangian and Hamiltonian densities; the solutions were in turn given up to quadratures.

One general result here is that the evolution of the scale factor of the Universe does not depend upon the particular form of the functional $F(\phi)$; actually, the contribution of the scalar field in the SB theory is that of perfect fluid with a stiff (barotropic) equation of state. If any, its contribution to the matter budget of the Universe is only relevant at early times.

A separate conclusion is that the SB, whether in its original form as given in Ref. [4] or in the generalized case studied here, cannot be an answer to the dark matter riddle of Cosmology.

In the quantum regime was necessary to build one equivalent density lagrangian in order to apply this, and does not possible to write this solution in closed form. In this sense,
we check this approach using the original Sáez-Ballester formalism, obtaining the exact solutions in both regimes, classical and quantum for particular values in the $\gamma$ parameter. This formalism will be used with anisotropic cosmological models, which will be reported in other work.

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