Complex structure of a DT surface with $T^2$ topology *

H.Kawai $^a$, N.Tsuda $^a$ and T.Yukawa $^{a,b}$

$^a$ National Laboratory for High Energy Physics (KEK), Tsukuba 305, Japan

$^b$ Coordination Center for Research and Education, The Graduate University for Advanced Studies, Hayama-cho, Miura-gun, Kanagawa 240-01, Japan

A method of defining the complex structure (i.e. moduli) for dynamically triangulated (DT) surfaces with torus topology is proposed. Distribution of the moduli parameter is measured numerically and compared with the Liouville theory for the surface coupled to $c = 0, 1$ and $2$ matter. Equivalence between the dynamical triangulation and the Liouville theory is established in terms of the complex structure.

1. Introduction

Two-dimensional quantum gravity (2DQG) is interesting from the point of view of string theory in non-critical or critical dimensions and the statistical mechanics of random surfaces. As has been known the matrix model and the continuous theory, i.e. the Liouville field theory, exhibit the same critical exponent and Green’s function. However, the mutual relation between DT and the Liouville theory has not been so evident, because the DT surfaces are known to be fractal and its typical fractal dimension is considered to be four. On the other hand, we will show in this paper, the complex structure is well defined even for the DT surfaces, and we can check a precise equivalence of the DT and the Liouville theory.

It is known that 2DQG can be interpreted as a special case of the critical string with the conformal mode having a linear dilaton background. The critical string is defined to have two local symmetries, i.e. the local scale (Weyl) invariance and the reparametrization (Diffeo) invariance. After imposing these two symmetries on the world-sheet, no degree of freedom is left for the metric $g_{\mu\nu}$ except for a set of parameters $\tau$, which specify the moduli space $\mathcal{M}$ of the complex structure: $\mathcal{M} = \{g_{\mu\nu}\}/\text{Diffeo} \otimes \text{Weyl} = \{\tau\}$. Therefore, if we find a way to impose the local scale invariance on DT, we achieve a step closer to the study of the numerical simulations of the critical string, although it is not very easy at the present stage. However, considering the complex structure is very useful for obtaining clear signals in various measurements because the complex structure can be extracted independently to the rather complicated fluctuations of the conformal mode. In our previous work [1], we have established how to define and measure the complex structure and the conformal mode on the DT surfaces with $S^2$ topology. To be concrete we focus on the case of torus $T^2$ in this study, but the generalization would be straightforward.

2. Determination of a moduli on the torus

In the continuous formulation the period $\tau$ is obtained by the following procedure. First we introduce harmonic 1-form $\tilde{j}_\mu dx^\mu$ where $\tilde{j}_\mu$ satisfies the divergence and rotation free conditions $\partial_\mu \tilde{j}_\mu = 0$ and $\partial_\mu \tilde{j}_\nu - \partial_\nu \tilde{j}_\mu = 0$ respectively, with $j^\mu = \sqrt{g} g^{\mu\nu} \tilde{j}_\nu$. Since there are two linearly independent solutions, we can impose two conditions such as

$$\oint_\alpha j_\mu dx^\mu = 0, \quad \oint_\beta j_\mu dx^\mu = \frac{1}{r},$$

where $\alpha$ and $\beta$ represent two independent paths on the torus which intersect each other only once and $r$ denotes the resistivity of the surface. Under these conditions the period $\tau$ is given by

$$\tau = \oint_\alpha \tilde{j}_\mu dx^\mu + i \oint_\beta \tilde{j}_\mu dx^\mu = \frac{ir \oint_\alpha \tilde{j}_\mu dx^\mu}{\oint_\beta \tilde{j}_\mu dx^\mu + i \oint_\beta \tilde{j}_\mu dx^\mu},$$

where $\tilde{j}_\mu$ is the complex conjugate of $j_\mu$.
where $\tilde{j}_\mu$ is the dual of $j_\mu$ defined by $\tilde{j}_\mu = \epsilon_{\mu\nu}g^{\nu\lambda}j_\lambda$. This procedure can be easily translated to the case of triangulated surfaces by identifying $j_\mu$ with the current on the resistor network of the dual graph. Then $r\int_\alpha j_\mu dx^\mu$ and $\sum_\beta j_\mu dx^\mu$ correspond to the potential drop along $\alpha$-cycle and the total current flowing across $\alpha$-cycle respectively. We can easily impose two conditions eq.(1) by inserting electric batteries in the dual links crossing the $\alpha$-cycle and apply constant voltages(1V), see Fig.1. Writing the electric potential at the vertex $v$ as $V(v)$ and assuming that each bond has resistance 1Ω, the current conservation reads $V(v) = \frac{1}{2}(\sum_{i=1,2,3}V(v_i) + \delta_{\alpha\text{-}cycle})$, where $\delta_{\alpha\text{-}cycle}$ represents the voltages of the batteries placed along the $\alpha$-cycle and $v_{1,2,3}$ are the three neighboring vertices of $v$. We solve these set of equations iteratively by the successive over-relaxation method, and estimate the total currents flowing across the $\alpha$- and $\beta$-cycles as

$$\int_\alpha \tilde{j}_\mu dx^\mu = \sum_{\alpha\text{-}cycle}(V(v_{\text{right}}) - V(v_{\text{left}}) + 1),$$

$$\int_\beta \tilde{j}_\mu dx^\mu = \sum_{\beta\text{-}cycle}(V(v_{\text{right}}) - V(v_{\text{left}})),$$

where $V(v_{\text{left}})$ and $V(v_{\text{right}})$ denote the potentials of vertices placed at either side of the $\alpha$- or $\beta$-cycle. In the case of $T^2$ topology the resistivity is not easily determined because of the lack of the $SL(2,C)$-invariance as we have made use of in the $S^2$ topology case. Here, we borrow the resistivities obtained in case of the $S^2$ topology in order to determine the moduli of the torus.

Here, we present the theoretical predictions of the distribution function of the moduli. The genus-one partition function with $c$ scalar fields is given by \(Z\) by

$$Z \simeq \int_{\mathcal{F}} \frac{d^2\tau}{(\tau_2)^2} \{C(\tau)\}^{c-1},$$

$$C(\tau) = \left(\frac{1}{2}\tau_2\right)^{-\frac{c}{2}} e^{\frac{i}{\tau_2} \int_\tau \tau_1} \prod_{n=1}^{\infty} (1 - e^{2\pi i \tau_n})^{-2},$$

where $\tau$ is a moduli parameter $\tau = \tau_1 + i\tau_2$, and $\mathcal{F}$ denotes the fundamental region. According to the integrand of eq. (3), we find that the density distribution function of $\tau$ in the fundamental region is given by up to an overall numerical factor

$$\tau_2^{-\frac{c}{2}} e^{\frac{i}{\tau_2} \int_\tau \tau_1} \prod_{n=1}^{\infty} (1 - e^{2\pi i \tau_n})^{2(1-c)}.$$ (5)

The distribution of $\tau_2$ for $c > 1$ indicates the instability of the vacuum due to the tachyon of the bosonic string theory. It may become a clear evidence for the branched polymers.

3. Numerical results and discussions

Fig.2 shows the distribution of the period $\tau$ for a surface with 16K triangles in the case of the pure gravity with about $1.5 \times 10^4$ independent configurations. Roughly speaking larger values of $\tau_2$ in the fundamental region represent tori deformed like a long thin tube, while smaller values of $\tau_2$ represent almost regular tori. In order to compare numerical results with the predictions of the Liouville theory eq.(4), we consider the distribution functions integrated over $\tau_1$. Fig.3 shows the distribution of $\tau_2$ in the case of surfaces coupled with a scalar field($c = 1$) with 2K, 4K, 8K and 16K triangles. Fig.4 shows the distributions of $\tau_2$ in the case of the pure-gravity with 2K, 4K, 8K and 16K triangles. It is clear that the numerical results agree fairly well with the predictions of the Liouville theory for sufficiently

\(1\) In the case of the sphere, for example, the anomalous dimensions and the string susceptibility turn to be complex for $c > 1$. 
Figure 2. Plot of the moduli(\(\tau\)) on the complex-plane with a total number of triangles of 16K. A dot denotes \(\tau\) which is mapped into the fundamental region of each configuration.

Figure 3. Density-distributions of \(\tau_2\) in the case of pure-gravity.

Figure 4. Density-distributions of \(\tau_2\) in the case of \(c = 1\).

Figure 5. Density-distributions of \(\tau_2\) in the case of \(c = 2\).

large number of triangles. Fig. shows the distributions of \(\tau_2\) in the case of two scalar fields(\(c = 2\)) with 4K, 8K and 16K triangles. In this case, we cannot detect the divergence of the distribution of \(\tau_2\). It would be hard to obtain a large value of \(\tau_2\) for relatively small number of triangles, because we need many triangles to form a long narrow shape to the torus. We conclude that the DT surfaces have the same complex structure as the Liouville theory in the thermodynamic limit for \(c \leq 1\) cases.

-Acknowledgements-

We are grateful to J.Kamoshita for useful discussions and comments. One of the authors (N.T) was supported by a Research Fellowships of the Japan Society for the Promotion of Science for Young Scientists.

REFERENCES

1. H. Kawai, N. Tsuda and T. Yukawa, Phys.Lett.B351 (1995) 162, \texttt{hep-lat/9512014}, \texttt{hep-lat/9604019}.
2. See for examples: N. Seiberg, Prog.Theo.Phys.Suppl.102 (1990) 319; A. Gupta, S. P. Trivedi and B. Wise, Nucl.Phys.B340 (1990) 475; M. Bershadsky and I. R. Klebanov, Phys.Rev.Lett.65 (1990) 3088.