Discrete analogues of the Liouville equation

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Abstract

The notion of Laplace invariants is transferred to the lattices and discrete equations which are difference analogs of hyperbolic PDE’s with two independent variables. The sequence of Laplace invariants satisfy the discrete analog of two-dimensional Toda lattice. The terminating of this sequence by zeroes is proved to be the necessary condition for existence of the integrals of the equation under consideration. The formulae are presented for the higher symmetries of the equations possessing integrals. The general theory is illustrated by examples of difference analogs of Liouville equation.

1 Introduction

Liouville equation and its generalizations are studied about one hundred fifty years and very many results, including classification ones, are obtained in this field [1] – [14]. Surprisingly, it seems that in the discrete version these investigations were not undertaken, although many results can be transferred from the continuous case without any efforts. The aim of our paper is to advance in this direction. The equations with explicit dependence on the discrete variable are most difficult and for the present we consider only shift-invariant equations although this condition is, probably, too restrictive. Overcoming of this difficulty as well as deduction of classification theorems require further researches.

Let us consider differential-difference and totally discrete equations of the form

\[ u_{i+1,x} = f(x, u_{i+1}, u_i, u_{i,x}) \]
\[ u_{i+1,j+1} = f(u_{i+1,j}, u_{i,j+1}, u_{i,j}) \]

Further on we assume that for the equation \( f_{u_{i,x}} \neq 0 \), and the right hand side of the equation \( f \) depends explicitly on each of the indicated variables.
Apparently these equations can be treated as difference approximations of the equations of the form

\[ u_{xy} = f(x, y, u, u_x, u_y). \]  

(3)

They arise also as Bäcklund transformations and nonlinear superposition principles, and of course this concern both generalizations of the Liouville equation and equations integrable by Inverse Scattering Transform (the well known examples are dressing chain [16] and difference KdV equation [17]).

One can divide the different generalizations of the Liouville equation into several (intersecting) subclasses, subject to what property of the Liouville equation is chosen as definition of integrability, for example: explicit formula for the general solution, differential substitution to linear equation, symmetries depending on arbitrary functions, or integrals along both characteristics. In the present paper we accept the later property as definition. As far as we know it was offered as independent definition in the paper [7] and the corresponding subclass was called 'Liouville type equations'. Since this term is suitable rather to join (or intersection) of all subclasses mentioned above, we would say just 'equations possessing integrals'. Other term, 'Darboux integrable equations' [11, 12, 13] is not exact, since Darboux had used rather the first of the listed properties.

The given definition means that the lattice (1) admits the \( x \)-integral \( X \) and \( i \)-integral \( I \), that is functions on \( x \) and finite set of dynamical variables \( \{u_{i+m}, u_i^{(n)} = \partial^n_x u_i\} \ m \in \mathbb{Z}, \ n \in \mathbb{Z}_+ \) (\( i \) is assumed to be fixed here), such that the identities hold in virtue of the lattice

\[ D_x(X) = 0, \ (T_i - 1)(I) = 0. \]

The sequence of the coupled Riccati equations

\[ u_{i+1,x} - u_{i,x} = u_{i+1}^2 - u_i^2 \]  

(4)

can be considered as the classical example, its integrals are of the form

\[ I = u_{i,x} - u_i^2, \ X = (u_i, u_{i+1}, u_{i+2}, u_{i+3}) = \left( \frac{u_i - u_{i+2}}{u_i - u_{i+3}} \right) \left( u_{i+1} - u_{i+2} \right) \]

and

\[ I = (u_{i,x} - 1)^2/u_{i,x}^2, \ X = \frac{(u_{i+3} - u_{i+1})(u_{i+2} - u_i)}{u_{i+2} + u_{i+1}}. \]

It is not difficult to prove that in general case \( I \) depends only on derivatives \( u_i^{(n)} \), while \( X \) depends only on shifts \( u_{i+m} \). We will call the order of the higher
derivative on which the \(i\)-integral depends the order of \(i\)-integral, and the order of \(x\)-integral is equal to the difference between the maximal and minimal shifts of the variable \(u_i\) which are involved in it. So, in the last example \(\text{ord } I = 2\), \(\text{ord } X = 3\).

To solve the lattice possessing integrals one has to solve the difference equation \(X = c_i\) where \(c_i\) is an arbitrary sequence, and consistent with it ordinary differential equation \(I = C(x)\) where \(C(x)\) is an arbitrary function. Usually this allows to obtain the explicit solution as the final result, for example, one has for the lattice (4)

\[
I = \frac{\psi_{xx}}{2\psi_x} + \frac{\psi_x}{z_i - \psi}
\]

where function \(\psi(x)\) and parameters \(z_i\) are arbitrary. Notice, that

\[
2I = S_\psi = \frac{\psi_{xxx}}{\psi_x} - \frac{3\psi_{xx}}{2\psi_x}, \quad X = (z_i, z_{i+1}, z_{i+2}, z_{i+3}).
\]

Nevertheless, the relation between existence of integrals and explicit formula for solution remains unclear till now even in the most studied continuous case.

For the discrete equation (3) our definition of integrability means existence of \(i\)- and \(j\)-integrals satisfying relations

\[(T_i - 1)(I) = 0, \quad (T_j - 1)(J) = 0.\]

Now the dynamical variables are \(u_{i+m,j}, u_{i,j+n}, m, n \in \mathbb{Z}\), since the rest can be eliminated in virtue of (4). It is easy to prove that \(i\)-integral depends on the shifts \(u_{i,j+n}\) only, and \(j\)-integral depends on the shifts \(u_{i+m,j}\). The order of integrals is defined in just the same way as order of \(x\)-integral for the equation (1). As an example, let us consider the equation

\[u_{i+1,j+1} - u_{i,j+1} + \frac{1}{u_{i+1,j} - u_{ij}} = 0 \quad (6)\]

which admits \(i\)-integral of the first order and \(j\)-integral of the third order:

\[
I = u_{i,j+1} + \frac{1}{u_{ij}}, \quad J = \frac{(u_{i+3,j} - u_{i+2,j})(u_{i+1,j} - u_{ij})}{(u_{i+3,j} - u_{i+1,j})(u_{i+2,j} - u_{ij})}
\]

Now we summarize our results. In the Section 2 the notion of the Laplace invariants is transferred to the discrete case (cf. (3)). The sequences of Laplace invariants for the above examples and discrete Liouville equations from the Section 3 are terminated by zero and this allows us to conjecture, by analogy with continuous case, that this property is characteristic for the equations possessing integrals (1) – (4). The main result of our paper is the proof of this conjecture in one direction (necessity), given in the Section 5. The outline of the proof is the same as in continuous case. In Section 6 the constructive formulae are presented for the symmetries of the lattices (4) possessing integrals.
Laplace invariants satisfy some recurrent relations analogous to the discrete Toda lattice \([18] - [23]\). In Section 4 these relations are identified as the well known formulae for the Bäcklund transformations and nonlinear superposition principle for the two-dimensional Toda lattice. The proof of sufficiency in the continuous case is based essentially on the results on the structure of the integrals of this lattice \([8, 15]\) and therefore it would be very important to obtain analogous results in the discrete case as well.

2 Discrete Laplace invariants

Laplace invariants of the linearized equation are very useful tool in the theory of the equations (3). In particular, it was proved that the equation possesses integrals iff the sequence of the Laplace invariants 

\[ h_j \]

is terminated by zero at the both ends \([11] - [14]\). The invariants 

\[ h_{-1}, h_0 \]

are calculated directly by the right hand side of the equation and the rest are found recurrently by the formula

\[
(\log h_j)_{xy} = h_{j+1} - 2h_j + h_{j-1}
\]

which is equivalent to the two-dimensional Toda lattice.

It turns out that analogous constructions can be defined in the discrete situation as well. Firstly let us consider differential-difference operator

\[
L = T_i D_x + a_i D_x + b_i T_i + c_i = (T_i + a_i)(D_x + b_{i-1}) + H_{i,0} = (D_x + b_i)(T_i + a_i) + K_{i,0}
\]

where

\[
H_{i,0} = c_i - a_i b_{i-1}, \quad K_{i,0} = c_i - a_i b_i - D_x(a_i).
\]

One can easily check that if \(H_{i,0} \neq 0\) then the formula holds

\[
(D_x + b_{i,1})L = L_1(D_x + b_{i-1})
\]

where

\[
L_1 = (D_x + b_{i,1})(T_i + a_i) + H_{i,0}, \quad b_{i,1} = b_{i-1} - D_x(\log H_{i,0}).
\]

It means that the operator \(D_x + b_{i-1}\) maps the kernel of \(L\) into the kernel of the operator \(L_1\). Moreover, operator \(\frac{1}{H_{i,0}}(T_i + a_i)\) define the inverse transformation which maps the kernel of \(L_1\) into the kernel of \(L\):

\[
L \frac{1}{H_{i,0}}(T_i + a_i) = (T_i + a_i) \frac{1}{H_{i,0}}L_1.
\]

We shall call the mapping \(L = L_0 \mapsto L_1\) the Laplace \(x\)-transformation. Its iterations yield the sequence of operators

\[
L_j = (T_i + a_i)(D_x + b_{i-1,j}) + H_{ij}, \quad j \geq 0
\]

\[ (\log h_j)_{xy} = h_{j+1} - 2h_j + h_{j-1} \]
where $H_{ij}, b_{ij}$ are defined by recurrent formulae
\begin{align}
H_{i,j+1} &= H_{ij} + D_x(a_i) + a_i(b_{i,j+1} - b_{i-1,j+1}), \\
b_{i,j+1} &= b_{i-1,j} - D_x(\log H_{ij}), \quad b_{i,0} = b_i.
\end{align}

(10)

Laplace $i$-transformation is defined in the similar manner. Analog of the formula (8) looks
\begin{align}
(T_i + a_{i,1})L = L_{-1}(T_i + a_i)
\end{align}
where
\begin{align}
L_{-1} = (T_i + a_{i,1})(D_x + b_i) + K_{i+1,0}, \quad a_{i,1} = a_iK_{i+1,0}/K_{i,0}
\end{align}
and it is assumed that $K_{i,0} \neq 0$. Operator $\frac{1}{K_{i,0}}(D_x + b_i)$ defines the inverse transformation:
\begin{align}
L_{-1} = (T_i + a_{i,1})(D_x + b_i) &= (D_x + b_i) \frac{1}{K_{i+1,0}}L_{-1}.
\end{align}

Iteration of the $i$-transformation generates the sequence of the operators
\begin{align}
L_{-j} = (D_x + b_{i+j})(T_i + a_{ij}) + K_{ij}, \quad j \geq 0
\end{align}
where $K_{ij}, a_{ij}$ are defined by recurrent relations
\begin{align}
K_{i,j+1} &= K_{i+1,j} - D_x(a_{i,j+1}) - a_{i,j+1}(b_{i,j+1} - b_{i+j}), \\
a_{i,j+1} &= a_{ij}K_{i+1,j}/K_{ij}, \quad a_{i,0} = a_i.
\end{align}

(13)

It is more convenient for calculations to consider, under assumption $a_i \neq 0$ (this guarantees $a_{ij} \neq 0$), the values
\begin{align}
h_{ij} = \begin{cases} 
H_{ij}/a_i, & j \geq 0, \\
K_{i,-j-1}/a_{i,-j-1}, & j < 0
\end{cases}
\end{align}
which we will call Laplace invariants. Actually, this exchange is necessary, since the operators $L$ should be considered up to the gauge $L = a_{i+1}^{-1}L\alpha_i$ and the values $h_{ij}$ are just the invariants of this transformation. In particular, comparison of the formulae (6), (11) and their inverses demonstrates that the Laplace $x$- and $i$-transformations are inverse to each other up to the gauge transformation. One can easily check that $h_{ij}$ satisfy the lattice
\begin{align}
\left(\log \frac{h_{ij}}{h_{i+1,j}}\right)_x = h_{i,j-1} - h_{ij} - h_{i+1,j} + h_{i+1,j+1}
\end{align}
while the values $H_{ij}, K_{ij}$ do not satisfy closed system of equations.

Laplace invariants of the lattice (11) are understood as Laplace invariants of the linearization operator
\begin{align}
L = T_iD_x - f_{u,x}D_x - f_{u+1}T_i - f_u.
\end{align}
Passage from the values \( H_{ij} \) and \( K_{ij} \) to \( h_{ij} \) is always possible in virtue of the nondegeneracy condition \( f_{u_i} \neq 0 \). Notice also, that the property
\[
h_{i+1,j} = T_i(h_{ij})
\]
holds due to the shift-invariance of the lattice and therefore the Laplace invariants can become zero only on the whole line \( j = j_0 \).

Let us return to the examples presented in Introduction. One can easily check by direct calculation that for the lattice (4) the invariants \( h_{i,-3} \) and \( h_{i,0} \) become zero, as well as invariants \( h_{i,-3} \) and \( h_{i,1} \) for the lattice (3). So the conjecture arises that the criterion of the existence of the integrals mentioned above is valid also in the discrete case.

Laplace transformations and invariants for the difference-difference operators are defined quite similarly. Moreover they were studied in the review [18] and we only bring few key formulae. Let us consider the operator
\[
L = T_iT_j + a_{ij}T_j + b_{ij}T_i + c_{ij} = (T_i + a_{ij})(T_j + b_{i-1,j}) + H_{ij,0} = (T_j + b_{ij})(T_i + a_{i,j-1}) + K_{ij,0}
\]
where
\[
H_{ij,0} = c_{ij} - a_{ij}b_{i-1,j}, \quad K_{ij,0} = c_{ij} - a_{i,j-1}b_{ij}.
\]
Operator \( T_j + b_{i-1,j} \) maps the kernel of \( L \) into the kernel of the operator
\[
L_1 = (T_j + b_{ij,1})(T_i + a_{ij}) + H_{i,j+1,0}, \quad b_{ij,1} = b_{i-1,j}H_{i,j+1,0}/H_{ij,0}
\]
and operator \( 1/H_{ij,0}(T_i + a_{ij}) \) defines inverse transformation. Iterations of this transformation, which we will call Laplace \( j \)-transformation, yield the sequence of operators
\[
L_k = (T_i + a_{i,j+k})(T_j + b_{i-1,jk}) + H_{ijk}, \quad k \geq 0
\]
where \( H_{ijk}, b_{ijk} \) are defined recurrently:
\[
H_{ij,k+1} = H_{i,j+1,k} + a_{i,j+k}b_{ij,k+1} - a_{i,j+k+1}b_{i-1,j,k+1},
\]
\[
b_{ij,k+1} = b_{i-1,jk}H_{i,j,k+1}/H_{ijk}, \quad b_{ij,0} = b_{ij}.
\]
Laplace \( i \)-transformation and values \( K_{ijk}, a_{ijk} \) are defined analogously.

When \( a_{ij}b_{ij} \neq 0 \) one can introduce Laplace invariants \( h_{ijk} = H_{ijk}/(a_{i,j+k}b_{i-1,jk}) \) for \( k \geq 0 \), \( h_{ijk} = K_{i,j-k-1}/(a_{i,j-1,k-1}b_{i-1,k-1,j}) \) for \( k < 0 \). As in the previous case these quantities do not change under the gauge transformations and satisfy closed system of equation
\[
(h_{i,j+1,k+1} + 1)(h_{i,j+1,k+1} + 1) = (h_{i,j+1,k+1} + 1)(h_{i,j+1,k+1} + 1)h_{ijk}h_{i,j+1,k+1}.
\]
It is easy to check that for the example (3) invariants \( h_{ij,-3} \) and \( h_{ij,0} \) of the linearized equation become zero in accordance with our conjecture.
3 Discrete Liouville equations

Remind that the Liouville equation (1) can be obtained from the Toda lattice (7) by imposing the simplest boundary conditions $h_1 = h_{-1} = 0$. The natural question arises, what is the result of the same reduction of the discrete Toda lattices (14) and (15). We will see that structure and properties of these three equations are very similar: each of them possesses second order integrals, has only two nonzero Laplace invariants, and admits linearizing substitutions of the first order. Moreover, the passage to the limit turns the discrete versions into the Liouville equation itself. Let us consider these examples in more details.

Boundary conditions $h_{i,1} = h_{i,-1} = 0$ turn the equation (14) into the lattice of the form (1) on the variables $u_i$: 

\[ u_{i+1,x}u_i - u_{i+1}u_{i,x} = u_{i+1}u_i(u_{i+1} + u_i). \]  

(16)

It is easy to check that assuming $u_i(x) = \varepsilon u(x,y)$, $y = \varepsilon i$ and passing to the limit $\varepsilon \to 0$ one obtains the Liouville equation $(\log u)_{xy} = 2u$. Notice also that the lattice (16) can be obtained from (4) by means of substitution $\tilde{u}_i = u_{i+1} - u_i$.

The linearization operator for (16) is

\[ L = D_x T_i - \frac{D_{i+1}}{u_i} - (u_{i,x}/u_i + 2u_{i+1} + u_i) + u_{i+1}(u_{i,x}/u_i^2 - 1) \]

and calculating of the Laplace invariants gives $h_{i,1} = h_{i,-2} = 0$.

Separating the variables $u_i$ and $u_{i+1}$ and differentiating yields

\[ \frac{u_{i+1,x}}{u_{i+1}} - u_{i+1} = \frac{u_{i,x}}{u_i} + u_i, \quad \frac{u_{i+1,x}}{u_{i+1}} - \frac{u_{i+1,x}}{u_{i+1}} - u_{i+1,x} = \frac{u_{i,x}}{u_i} - \frac{u_{i,x}}{u_i^2} + u_{i,x} \]

from which the $i$-integral can be easily found:

\[ I = 2\frac{u_{i,x}}{u_i} - 3\frac{u_{i,x}^2}{u_i^2} - u_i. \]

It is not difficult to find the $x$-integral as well:

\[ X = (1 + u_i/u_{i+1})(1 + u_i/u_{i-1}). \]

Equation (16) is obtained from the linear equation $v_{i+1,x} - v_{i,x} = 0$ by means of the substitution

\[ u_i = \frac{(v_{i+1} - v_i)v_{i,x}}{v_{i+1}v_i} \]

(cf. substitution $u = v_xv_y/v^2$ which linearize the Liouville equation) and this allows at once to write down the formula for the general solution:

\[ u_i(x) = \frac{(c_{i+1} - c_i)\psi_x}{(c_{i+1} + \psi)(c_i + \psi)} \]
where $c_i$ are arbitrary constants and $\psi(x)$ is arbitrary function.

In the totally discrete case the boundary conditions $h_{ij,1} = h_{ij,-1} = 0$ reduce the equation (15) to equation of the form (4) on the variables $u_{ij} = h_{ij,0}$:

$$u_{i+1,j+1}(1 + 1/u_{i+1,j})(1 + 1/u_{i,j+1})u_{ij} = 1. \quad (17)$$

It turns into the lattice (14) after the passage to limit $u_{ij} = -\varepsilon u_i(\varepsilon j), \varepsilon \to 0$.

The Laplace invariants of this equation $h_{ij,1}$ and $h_{ij,-2}$ are equal to zero just as in the previous example. The integrals are given by formulae

$$I = \left(1 + \frac{u_{ij}(u_{i,j-1} + 1)}{u_{i,j-1}}\right)\left(1 + \frac{u_{ij}}{u_{i,j+1}(u_{ij} + 1)}\right),$$

$$J = \left(1 + \frac{u_{ij}}{u_{i-1,j}}\right)\left(1 + \frac{u_{ij}}{u_{i+1,j}(u_{ij} + 1)}\right).$$

The substitution

$$u_{ij} = -\frac{(v_{i+1,j} - v_{ij})(v_{i,j+1} - v_{ij})}{v_{i+1,j}v_{i,j+1}}$$

brings (17) to the linear equation $v_{i+1,j+1} - v_{i+1,j} - v_{i,j+1} + v_{ij} = 0$ and therefore the formula for the general solution is valid

$$u_{ij} = -\frac{(c_i + 1 - c_i)(k_j + 1 - k_j)}{(c_i + 1 + k_j)(c_i + k_j + 1)},$$

where $c_i, k_j$ are arbitrary constants.

4 Discrete Toda lattices

Now we slightly digress from our main subject in order to discuss the equations (14) and (15) which are the natural discrete analogues of the Toda lattice (7). These equations have been already studied in literature from this point of view (possibly in somewhat different notation) see e.g. [18] – [23]. We should like to reproduce for completeness some well-known formulae which demonstrate that the links between these three lattices are more close than it seems at the first glance. The results of this Section will not be necessary in what follows.

It is more convenient to start from the lattice

$$r_{j,xy} = \exp(r_{j+1} - 2r_j + r_{j-1}), \quad (18)$$

rather than from (13) which is related with it by means of substitution $h_j = r_{j,xy}$.

The Bäcklund transformation for this lattice is of the form

$$(r_{ij} - r_{i+1,j})_x = \exp(r_{i,j-1} - r_{ij} - r_{i+1,j} + r_{i+1,j+1}), \quad (19)$$

$$(r_{i+1,j} - r_{i,j-1})_y = \exp(r_{ij} - r_{i+1,j} - r_{i,j-1} + r_{i+1,j-1}). \quad (20)$$
Indeed, the cross-differentiation yields the formula

\[ r_{ij,xy} = \exp(r_{i,j+1} - 2r_{ij} + r_{i,j-1}) + c(x,y) \]

where function \( c(x,y) \) does not depend on \( i,j \) and can be easily removed by shift of the form \( r \to r + C(x,y) \). The lattice (19) is reduced to (14) by means of substitution \( h_{ij} = (r_{ij} - r_{i+1,j})_x \).

We see that the second subscript in the formulae (19), (20) corresponds to the shift in the lattice (18) while the first one denotes the iteration of the Bäcklund transformation. Now introduce the third subscript and consider the Bäcklund transformation which changes it and leaves the first subscript fixed (vice versa, the fixed third subscript must be assigned to the variable \( r \) in the formulae (19), (20)):

\[
(r_{ijk} - r_{i,j,k+1})_x = \exp(r_{i,j-1,k} - r_{ijk} - r_{ij,k+1} + r_{i,j+1,k+1}), \\
(r_{ij,k+1} - r_{i,j-1,k})_y = \exp(r_{ijk} - r_{ij,k+1} - r_{i,j-1,k} + r_{i,j-1,k+1}).
\]

Let us consider the superposition of these Bäcklund transformations. It is sufficient to consider only \( x \)-part, since differentiating on \( y \) brings to the same result. Calculating \( (r_{ijk} - r_{i+1,j,k+1})_x \) in two different ways one obtains after some transformations the relation

\[
(T_j - 1)(\exp(r_{i+1,j-1,k} - r_{i+1,j,k+1} - r_{i,j-1,k} + r_{ij,k+1}) - \exp(r_{i,j-1,k+1} - r_{i+1,j,k+1} + r_{i,j-1,k} + r_{i,j+1,k})) = 0.
\]

The expression in the brackets is equal to some constant \( c_{ik} \) which can be set to 1 by shift of the form \( r_{ijk} \to r_{ijk} + C_{ik} \), resulting in desired difference equation

\[
\exp(r_{i+1,j-1,k} + r_{ij,k+1}) - \exp(r_{i,j-1,k+1} + r_{i+1,jk}) = \exp(r_{i+1,j,k+1} + r_{i,j-1,k}).
\]

(21)

It is equivalent, up to the linear transformation of the coordinate axes, to the well-known Hirota-Miwa equation [19, 20]

\[
a_{i+1,jk}f_{i-1,jk} + b_{i,j+1,k}f_{i,j-1,k} + c_{ij,k+1}f_{ij,k-1} = 0
\]

(it is clear that all coefficients can be removed by means of scaling \( f \to \alpha^{i(i-1)}\beta^{j(j-1)}f \)). Substitution

\[
h_{ijk} = \exp(r_{i+1,j,k+1} - r_{i,j-1,k+1} - r_{i+1,jk} + r_{ij,k+1})
\]

maps equation (21) into equation

\[
(h_{i+1,j,k+1} + 1)(h_{i,j-1,k} + 1) = (h_{i,j-1,k+1} + 1)(h_{i+1,j,k+1})h_{i+1,j-1,k}h_{ij,k+1}
\]

which coincides with (15) up to the linear change of \( i,j,k \).
It should be stressed that the changes from the variable \( r \) to the variable \( h \) are different in all three cases, and therefore one cannot say that the lattice (14) defines the Bäcklund transformation for (7) and the equation (15) is its nonlinear superposition principle. Nevertheless, we have demonstrated that these three equations live on the same three-dimensional lattice generated by the Bäcklund transformations of the lattice (18).

5 Necessary conditions for existing of integrals

In this Section we prove the main theorem which allows to check, rather effectively, if the given equation (1) or (2) possesses the integrals.

We consider in details only the lattices (1). Let \( g \) be function on \( x \) and dynamical variables \( u_{i+m}, u_i^{(n)} \). Denote \( g^* \) its linearization, that is operator

\[
g^* = \sum_{m=-\infty}^{\infty} g_{u_{i+m}} T_i^m + \sum_{n=1}^{\infty} g_{u_i^{(n)}} D_n^x.
\]

Intermediate calculation proves that the properties hold

\[
(D_x(g))^* = D_x g^* \mod [L], \quad (T_i(g))^* = T_i g^* \mod [L]
\] (22)

where \( L \) is linearization operator of the lattice (1) and \([L]\) denotes set of all operators of the form \( ML \) with arbitrary differential-difference operators \( M \).

**Theorem 1** If the lattice (1) possesses the \( x \)-integral \( X \) of the order \( n \) then its Laplace invariant \( h_{i,-j} \) vanishes for some \( j, 0 < j \leq n \). Analogously, if it admits the \( i \)-integral \( I \) of the order \( n \) then the invariant \( h_{ij} \) vanishes for some \( j, 0 \leq j < n \).

**Proof.** Let us prove by contradiction the first statement. Assume that \( h_{i,-j} \neq 0 \) for all \( j = 1, \ldots, n \). It is the same as the quantities \( K_{ij} \neq 0 \) for all \( j = 0, \ldots, n-1 \) and therefore the formulae (12), (13) define the operators \( L_{-j} \) for \( j = 0, \ldots, n \). The relation (11) for these operators takes form

\[
(T_i + a_{ij}) L_{-j+1} = L_{-j}(T_i + a_{ij-1}), \quad j = 1, \ldots, n.
\]

The repeated application of this formula brings to \( L_{-j} A_{j-1} = 0 \mod [L] \) where the difference operators \( A_j \) are defined as follows

\[
A_{-1} = 1, \quad A_j = (T_i + a_{ij}) A_{j-1}, \quad j = 0, \ldots, n.
\]

Therefore one obtains the relation

\[
D_x A_j = -b_{i+j} A_j - K_{ij} A_{j-1} \mod [L], \quad j = 1, \ldots, n.
\]
Due to the shift $T_i$ one can assume without loss of generality that $X$ depends on $u_i, \ldots, u_{i+n}$. Consider the expansion of $X_\ast$ over the operators $A_j : X_\ast = \sum_{j=0}^n \xi_j A_{j-1}$ and multiply it by $D_x$ from the left. Collecting the coefficients on $A_j$ and $D_x$ in the relation $D_x X_\ast = 0 \bmod [L]$ brings to the sequence of equations

\[
(D_x - b_{i+n-1})(\xi_n) = 0, \\
(D_x - b_{i+j-1})(\xi_j) = \xi_{j+1}K_{ij}, \quad j = 1, \ldots, n-1, \\
D_x(\xi_0) = \xi_1K_{i,0}, \\
\xi_0 = 0
\]

from which the contradiction follows: $\xi_0 = \ldots = \xi_n = 0 \Rightarrow X_\ast = 0$.

The proof of the second statement is quite analogous and we just outline it. Assume that $H_{ij} \neq 0$ for all $0 \leq j < n$ and define the operators $L_j$ accordingly to (9), (10) and operators $B_j$ by formulae

\[
B_{-1} = 1, \quad B_j = (D_x + b_{i-1,j})B_{j-1}.
\]

Applying the relations

\[
T_iB_j = -a_iB_j - H_{ij}B_{j-1} \bmod [L], \quad j = 1, \ldots, n
\]

to the equation $T_i I_\ast = I_\ast \bmod [L]$ where $I_\ast = \sum_{j=0}^n \xi_j B_{j-1}$ one obtains the sequence of equations

\[
(a_iT_i + 1)(\xi_n) = 0, \\
(a_iT_i + 1)(\xi_j) = -\xi_{j+1}H_{ij}, \quad j = 1, \ldots, n-1, \\
\xi_0 = -\xi_1H_{i,0}, \\
T_i(\xi_0) = 0
\]

which implies contradiction $I_\ast = 0$. \( \blacksquare \)

In the totally discrete case the following statement can be proved in similar manner.

**Theorem 2** Let equation (3) possesses the $i$-integral ($j$-integral) of the order $n$. Then the Laplace invariant $h_{ijk} = 0$ for some $k$, $0 \leq k < n$ ($h_{ij,-k} = 0$, $0 < k \leq n$).

### 6 Higher symmetries

The notion of symmetry for the equations (1), (2) and (3) is introduced uniformly as any function on the dynamical variables of the given equation which satisfies the characteristic equation $L(g) = 0$ where $L$ is linearization operator.
Less formally, symmetry can be understood as equation $u_t = g$ compatible with the given one. It is known [12] that equations (3) possessing integrals admit also the rich family of the symmetries of the form

$$g = P(Y) + Q(X)$$

where $P$ and $Q$ some explicitly computable differential operators on $D_x$ and $D_y$ respectively and $X, Y$ are arbitrary $x$- and $y$-integrals.

The aim of this Section is to obtain the analogous formula in the discrete case. Let us prove the following Lemma as a preliminary.

**Lemma 3** Let the lattice (4) possesses integrals then the kernels of operators $D_x - f_{u_{i+1}}$, $T_i - f_{u_{i,x}}$ are not empty in the space of the functions on dynamical variables. Analogously, if the equation (3) possesses integrals then the kernels of the operators $T_i - f_{u_{i,j}}$, $T_j - f_{i+1,j}$ are not empty.

**Proof.** It is sufficient to differentiate the relation defining the integral on the higher dynamical variable. For example, differentiating on $u_i^{(n)}$ of the relation $(T_i - 1)I(x, u_i, \ldots, u_i^{(n)}) = 0$ where $u_{i+1}^{(j+1)} = D_{x}^{j}(f)$ in virtue of the lattice (4) yields $(f_{u_{i,x}}T_i - 1)(I_i^{(n)}) = 0$, therefore $1/I_i^{(n)}$ belongs to the kernel of the operator $T_i - f_{u_{i,x}}$.

Now let us prove the theorem about the symmetries of the lattice (4) (we restrict ourselves by this case since the arguments for the equation (3) are quite analogous).

**Theorem 4** Let the lattice (4) admits an $x$-integral of the order $m$ and an $i$-integral of the order $n$. Then the differential operator $P$ and the difference operator $Q$ exist of orders not greater than $m - 1$ and $n - 1$ respectively such that the formula

$$g = P(I) + Q(X)$$

defines the symmetry of the given lattice for any $i$-integral $I$ and $x$-integral $X$.

**Proof.** Accordingly to the Theorem 1, the existence of the $x$-integral implies that $h_{i,-j} = 0$ for some $j$, $0 < j \leq m$. Applying the Laplace $i$-transformation (12), (13) for $j - 1$ times one obtains the general solution of the equation $L(g) = 0$ in the form

$$g = \frac{1}{K_{i,0}}(D_x + b_{i}) \cdot \ldots \cdot \frac{1}{K_{i,j-2}}(D_x + b_{i+j-2})(\psi)$$

where $\psi$ satisfies equation

$$(D_x + b_{i+j-1})(T_i + a_{i,j-1})(\psi) = 0.$$
Let \((T_i + a_{i,j-1}) (\psi) = 0\), then \(\psi\) can be represented as \(\psi = K_{i,j-2} \cdots K_{i,0} \varphi I\) where \(I\) is an arbitrary \(i\)-integral and \(\varphi\) satisfies equation \((T_i + a_i)(\varphi) = 0\). Solvability of this equation was proved in Lemma 3 (notice that it is provided by \(i\)-integral) resulting in particular solution of the form \(g = P(I)\) of the characteristic equation.

Now consider Laplace \(x\)-transformation. Let the invariant \(h_{ij}\) vanish, \(0 \leq j < n\), then the solution of the characteristic equation can be represented as

\[
g = \frac{1}{H_{i,0}}(T_i + a_i) \cdots \frac{1}{H_{i,j-1}}(T_i + a_i)(\psi)
\]

where \(\psi\) satisfies equation

\[
(T_i + a_i)(D_x + b_{i-1,j})(\psi) = 0.
\]

Moreover, accordingly to (10),

\[
b_{i-1,j} = b_{i-j-1,0} - D_x \log(H_{i-1,j-1} \cdots H_{i,j,0})
\]

and therefore the function \(\psi = H_{i-1,j-1} \cdots H_{i,j,0} \varphi X\) is the particular solution if \(X\) is arbitrary \(x\)-integral and \(\varphi\) satisfies equation \(\varphi_x + b_{i-j-1,0} \varphi = 0\). This equation is solvable according to Lemma 3, and one obtains the particular solution of the determining equation of the form \(g = Q(X)\).

As an example we present the formula for the symmetries of the discrete Liouville equation (16):

\[
u_{i,t} = D_x (u_i I) + (T_i - 1)((u_i + \frac{u_i^2}{u_{i+1}})X).
\]

The first term in this formula describes the family of the evolution equations for which the lattice (14) defines the Bäcklund transformation. Note for comparison that in the continuous case operators \(P\) and \(Q\) for the symmetries of the equation \((\log u)_{xy} = 2u\) are \(P = D_x u, Q = D_y u\).

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