MICROLOCAL BRANES ARE CONSTRUCTIBLE SHEAVES

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Dedicated to Paul S. Nadler

Abstract. Let $X$ be a compact real analytic manifold, and let $T^*X$ be its cotangent bundle. In a recent paper with E. Zaslow [28], we showed that the dg category $\text{Sh}_c(X)$ of constructible sheaves on $X$ quasi-embeds into the triangulated envelope $F(T^*X)$ of the Fukaya category of $T^*X$. We prove here that the quasi-embedding is in fact a quasi-equivalence. When $X$ is a complex manifold, one may interpret this as a topological analogue of the identification of Lagrangian branes in $T^*X$ and regular holonomic $D_X$-modules developed by Kapustin [15] and Kapustin-Witten [16] from a physical perspective.

As a concrete application, we show that compact connected exact Lagrangians in $T^*X$ (with some modest homological assumptions) are equivalent in the Fukaya category to the zero section. In particular, this determines their (complex) cohomology ring and homology class in $T^*X$, and provides a homological bound on their number of intersection points. An independent characterization of compact branes in $T^*X$ has recently been obtained by Fukaya-Seidel-Smith [9].

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1. Introduction

1.1. Summary. Let $X$ be a compact real analytic manifold, and let $T^*X$ be its cotan-
gent bundle. Let $Sh_c(X)$ be the differential graded (dg) category of constructible complexes of sheaves on $X$. Objects of $Sh_c(X)$ are complexes of sheaves of $C$-vector spaces on $X$ with bounded constructible cohomology; morphisms are obtained from the naive morphism complexes by inverting quasi-isomorphisms. The (ungraded) cohomology category of $Sh_c(X)$ is the usual bounded derived category $D_c(X)$ of cohomologically constructible complexes on $X$. (See Section 2 below for more details.)

In a joint paper with Eric Zaslow [28] (reviewed in Section 3 below), we developed a Fukaya $A_\infty$-category of exact Lagrangian branes in $T^*X$. Objects are (not necessarily compact) exact Lagrangian submanifolds of $T^*X$ equipped with brane structures and perturbations; morphisms are given by transverse intersections, and composition maps by counts of pseudoholomorphic polygons. A key assumption on the objects is that they have reasonable compactifications so that we can make sense of “intersections at infinity”. Perturbations are organized so that morphisms always propagate a small amount “forward in time”. The construction is a close relative of the category of vanishing cycles proposed by Kontsevich [22] and Hori-Iqbal-Vafa [13], and developed by Seidel [30], [31], [33].

This version of the Fukaya category of the cotangent bundle encodes the geometry of infinitesimal paths in the base. Many other theories (such as cyclic homology [24], and the chiral de Rham complex [14]) also relate small loops in the base to the de Rham complex (and hence $D$-modules), while other versions of Floer theory on the cotangent bundle are closely related to the full path space of the base (for example, in the work of Viterbo [37]).

Now let us pass to a stable setting and consider the dg category of right modules over the above Fukaya category. We write $F(T^*X)$ for the full subcategory of twisted complexes of representable modules, and refer to it as the triangulated envelope of the Fukaya category. We write $DF(T^*X)$ for the cohomology category of $F(T^*X)$, and refer to it as the derived Fukaya category.

The main result of [28] was the construction of an $A_\infty$-quasi-embedding

$$\mu_X : Sh_c(X) \rightarrow F(T^*X)$$

which we will refer to as microlocalization. This is an $A_\infty$-functor such that when we pass to cohomology, we obtain a fully faithful embedding of triangulated categories

$$H(\mu_X) : D_c(X) \rightarrow DF(T^*X).$$

Our aim in this paper is to show that $\mu_X$ is in fact a quasi-equivalence. This is the assertion that the embedding $H(\mu_X)$ of triangulated categories is an equivalence. In other words, we must see that it is essentially surjective, or in plain words that

\textit{every object of }$DF(T^*X)$\textit{ is isomorphic to an object coming from }$D_c(X).$
This can be viewed as a categorification of the fact [17, Theorem 9.7.10] that the characteristic cycle homomorphism is an isomorphism from constructible functions to conical Lagrangian cycles. To simplify the statement, let us assume for the moment that $X$ is oriented. Then as a consequence of our main result, one can deduce a commutative diagram relating the microlocalization $\mu_X$ to the characteristic cycle homomorphism

$$
\begin{align*}
K_0(D_c(X)) & \xrightarrow{K_0(H(\mu_X))} K_0(DF(T^*X)) \\
\chi \downarrow & \downarrow \\
\mathcal{F}_c(X) & \xrightarrow{CC} \mathcal{L}_{\text{con}}(T^*X)
\end{align*}
$$

Here $K_0$ denotes the underlying Grothendieck group, $\mathcal{F}_c(X)$ the group of constructible functions on $X$, and $\mathcal{L}_{\text{con}}(T^*X)$ the group of conical Lagrangian cycles. The homomorphism $\chi$ is the local Euler-Poincaré index, and the homomorphism $\xi$ simply dilates a Lagrangian brane down to a conical cycle. The reader could consult [17] for a comprehensive treatment of the characteristic cycle homomorphism $CC$, local Euler-Poincaré index $\chi$ and related topics. The fact that the diagram is commutative follows from the compatibility of the definition of $\mu_X$ with the functoriality formula [29, Theorem 4.2] (see also the results of [10] in the complex algebraic setting).

1.2. Sketch of arguments. To establish our main assertion, we use the following variation of a standard argument. Consider the toy problem of showing a set of vectors $\{v_\alpha\}$ span a finite dimensional vector space $V$. We can reformulate this in terms of the identity map $id_V : V \to V$. Namely, we can ask if $id_V$ can be written in the form

$$
id_V = \sum_\alpha \lambda_\alpha \otimes v_\alpha$$

where $\{\lambda_\alpha\}$ are some set of functionals. If so, then by applying $id_V$ to any $v \in V$, we see that $v$ is in the span of the set $\{v_\alpha\}$. We may interpret this argument as expressing the identity map $id_V$ as a sum of projections $\lambda_\alpha \otimes v_\alpha$ onto the span of our collection. More generally, if $V$ is not necessarily finite dimensional, it still suffices to show that for any finite dimensional subspace $V' \subset V$, the restriction of the identity $id_V|_{V'}$ can be expressed in the above form.

The same proof can be implemented in the less elementary setting of a triangulated $A_{\infty}$-category $\mathcal{C}$. Suppose we want to show that a collection of objects $\{c_\alpha\}$ classically generates all of $\mathcal{C}$. In other words, we want to see that any object $c$ can be realized as a finite sequence of iterated cones beginning with maps among the objects $\{c_\alpha\}$. Consider the identity functor $id_{\mathcal{C}}$ in the triangulated $A_{\infty}$-category of $A_{\infty}$-functors from $\mathcal{C}$ to $\mathcal{C}$. Then it suffices to show that $id_{\mathcal{C}}$ itself can be realized as a finite sequence of iterated cones of functors of the form $f_\alpha \otimes c_\alpha$, where $f_\alpha$ is a bounded object of the dg category of left $A_{\infty}$-modules. In some situations such as the one we will consider below, the above is stronger than what can actually be shown. But it still suffices to verify the weaker assertion that for any full subcategory $\mathcal{C}'$ of $\mathcal{C}$ generated by finitely many objects, the restriction $id_{\mathcal{C}}|_{\mathcal{C}'}$ is in the full subcategory generated by functors of the form $f_\alpha \otimes c_\alpha$. 
As an example, consider the problem of representing coherent sheaves on projective space \( \mathbb{P}^n \) by complexes of vector bundles. In [2], Beilinson introduced a resolution of the structure sheaf \( \mathcal{O}_{\Delta_{\mathbb{P}^n}} \) of the diagonal \( \Delta_{\mathbb{P}^n} \subset \mathbb{P}^n \times \mathbb{P}^n \) by external products of vector bundles. For any coherent sheaf on \( \mathbb{P}^n \), convolution with this resolution produces the desired complex of vector bundles.

We will apply the above strategy in the context of \( F(T^*X) \) to see that every object comes from \( Shc(X) \) via the microlocalization functor \( \mu_X \).

To put this plan in action, we must have a way to get our hands on functors between categories of branes, in particular the identity functor. For this, we take advantage of the symmetry of cotangent bundles and introduce a duality equivalence

\[ \alpha_X : F(T^*X) \to F(T^*X). \]

By definition, it acts on the underlying Lagrangians of our branes by the antipodal anti-symplectomorphism

\[ a : T^*X \to T^*X \quad a(x,\xi) = (a, -\xi). \]

As a consequence of our main result, we will see that the brane duality \( \alpha_X \) corresponds to Verdier duality \( D_X \) under the microlocalization \( \mu_X \).

Now given compact real analytic manifolds \( X_0, X_1 \), we will construct functors

\[ F(T^*X_0) \to F(T^*X_1) \]

by thinking of branes in the product \( T^*X_0 \times T^*X_1 \) as integral kernels. Consider the dg category \( \text{mod}_r(F(T^*X_1)) \) of right \( A_\infty \)-modules and the corresponding Yoneda embedding

\[ \mathcal{Y}_r : F(T^*X_1) \to \text{mod}_r(F(T^*X_1)) \]

\[ \mathcal{Y}_r(P_1) : P'_1 \mapsto \hom_{F(T^*X_1)}(P'_1, P_1). \]

For each object \( L \) of \( F(T^*X_0 \times T^*X_1) \), we define an \( A_\infty \)-functor by considering the mapping functional

\[ \hat{\Psi}_{L^*} : F(T^*X_0) \to \text{mod}_r(F(T^*X_1)) \]

\[ \hat{\Psi}_{L^*}(P_0) : P_1 \mapsto \hom_{F(T^*X_0 \times T^*X_1)}(L, P_0 \times \alpha_{X_1}(P_1)). \]

Note that \( \hat{\Psi}_{L^*} \) is functorial in \( L \) in the contravariant sense. As a consequence of our main result, we will see that there is a functor

\[ \Psi_{L^*} : F(T^*X_0) \to F(T^*X_1) \]

that represents \( \hat{\Psi}_{L^*} \) in the sense that we have a quasi-isomorphism of functors

\[ \hat{\Psi}_{L^*} \simeq \mathcal{Y}_r \circ \Psi_{L^*} : F(T^*X_0) \to \text{mod}_r(F(T^*X_1)). \]

Two special cases of the above construction merit special mention. First, for the microlocalization \( L_\Delta = \mu_X(\mathbb{C}_{\Delta_X}) \) of the constant sheaf \( \mathbb{C}_{\Delta_X} \) along the diagonal \( \Delta_X \subset X \times X \), we check directly that the functor \( \hat{\Psi}_{L_\Delta^*} \) is represented by the identity functor \( \text{id}_{F(T^*X)} \) in the sense that there is a quasi-isomorphism of functors

\[ \hat{\Psi}_{L_\Delta^*} \simeq \mathcal{Y}_r : F(T^*X) \to \text{mod}_r(F(T^*X)). \]
Second, for an external product \( L_0 \times L_1 \), we check directly that the functor \( \tilde{\Psi}_{L_0 \times L_1} \) plays the expected role of a projection in the sense that there is a quasi-isomorphism of functors

\[
\tilde{\Psi}_{L_0 \times L_1^*} \simeq \mathcal{Y}_t(L_0) \otimes \mathcal{Y}_t(\alpha_{X_1}(L_1)) : F(T^*X_0) \to \text{mod}_t(F(T^*X_1)).
\]

Here \( \mathcal{Y}_t \) denotes the Yoneda embedding

\[
\mathcal{Y}_t : F(T^*X_0) \to \text{mod}_t(F(T^*X_0))
\]

\[
\mathcal{Y}_t(P_0) : P_0 \mapsto \text{hom}_{F(T^*X_0)}(P_0, P_0^t)
\]

for left \( A_\infty \)-modules over \( F(T^*X_0) \). Note that the above formula says that \( \tilde{\Psi}_{L_0 \times L_1^*} \) is in fact a projection onto the span of the dual brane \( \alpha_{X_1}(L_1) \). Although one might prefer simpler formulas, our conventions are set up to agree with the pushforward of sheaves.

An alternative, more geometric framework for constructing functors between categories of branes is provided by the beautiful formalism of world-sheet foam introduced by Khovanov-Rozansky from a physical viewpoint [25], or that of quilted Riemann surfaces developed by Wehrheim-Woodward in a mathematical context [39]. The latter was an original inspiration for the strategy of proof undertaken here. To keep this paper as self-contained as possible, we have opted for the above homological approach, though we have included a brief discussion explaining its compatibility with that of [39].

Now to see that \( F(T^*X) \) is classically generated by objects coming from \( Sh_c(X) \), we would like to realize the identity functor \( id_{F(T^*X)} \) as an iterated cone of external products \( f_\alpha \otimes \mu_X(\mathcal{F}_\alpha) \), for some objects \( \mathcal{F}_\alpha \) of \( Sh_c(X) \), and some bounded objects \( f_\alpha \) of \( \text{mod}_t(F(T^*X_0)) \). It is not difficult to see that this strong an assertion can not be true. Instead, we fix a conical Lagrangian \( \Lambda \subset T^*X \) and consider the full \( A_\infty \)-subcategory

\[
F(T^*X)_\Lambda \subset F(T^*X)
\]

consisting of branes whose boundaries at infinity lie in the boundary of \( \Lambda \). By [28], every finite collection of objects of \( F(T^*X) \) lies in such a subcategory \( F(T^*X)_\Lambda \) for some \( \Lambda \). Thus to arrive at our desired conclusion, it suffices to realize the identity functor of \( F(T^*X)_\Lambda \) as an iterated cone of external products \( f_\alpha \otimes \mu_X(\mathcal{F}_\alpha) \).

Translating this into the above setting of branes in the product \( T^*X \times T^*X \), we seek to express the restriction of the functor \( \tilde{\Psi}_{L_\Delta^*} \) to the subcategory \( F(T^*X)_\Lambda \) as an iterated cone of the restrictions of functors of the form \( \tilde{\Psi}_{L_\alpha \times \alpha_X(\mu_X(\mathcal{F}_\alpha))} \). By the functoriality of our constructions, this would follow immediately if the brane \( L_{\Delta_X} \) could be realized as an iterated cone of branes of the form \( L_\alpha \times \alpha_X(\mu_X(\mathcal{F}_\alpha)) \). Of course, this is not true (for example, it would imply the identity functor \( id_{F(T^*X)} \) in fact could be written in terms of external products), but we can achieve the following: there is a collection of \([0,1] \)-families of objects \( \mathcal{L}_{\Delta_X, \alpha, t} \) of \( F(T^*X \times T^*X) \) satisfying the following properties:

1. \( L_{\Delta_X} \) can be realized as an iterated cone of the branes \( \mathcal{L}_{\Delta_X, \alpha, 0} \).
2. \( \mathcal{L}_{\Delta_X, \alpha, 1} \) is an external product of the form \( L_\alpha \times \alpha_X(\mu_X(\mathcal{F}_\alpha)) \).
3. For all \( t \in [0,1] \), the functors

\[
\tilde{\Psi}_{\mathcal{L}_{\Delta_X, \alpha, t}} : F(T^*X)_\Lambda \to \text{mod}_t(F(T^*X))
\]
are quasi-isomorphic.

The key point in explaining the third property is that the families $\mathcal{L}_{\Delta X_{\alpha},t}$ are non-characteristic with respect to the conical Lagrangian $\Lambda \subset T^*X$ of the first factor. Roughly speaking, for a given conical Lagrangian $\Lambda' \subset T^*X$, we can arrange (after appropriate perturbations) so that the boundaries at infinity of the families $f_{\Delta X_{\alpha},t}$ do not intersect the boundary of $\Lambda \times \Lambda'$.

With the preceding in hand, by applying the identity functor to any object of $F(T^*X)_{\Lambda}$, we immediately conclude that it is quasi-isomorphic to an object coming from $\mathcal{S}h_c(X)$.

Independently of the above arguments, one can construct an explicit $A_\infty$-functor

$$\pi_X : F(T^*X) \to \mathcal{S}h_c(X)$$

which is a quasi-inverse to $\mu_X$ (see Section 4.6 for details). To define $\pi_X$, for each open subset $U \hookrightarrow X$, consider the corresponding costandard brane $L_U$ (see Section 3.6).

The underlying Lagrangian of $L_U$ can be taken to be the graph $\Gamma_{-\frac{d}{d\log m}}$ for any non-negative function $m : X \to \mathbb{R}$ that vanishes precisely on the complement $X \setminus U$.

Given an object $L$ of $F(T^*X)$, the assignment

$$U \mapsto \text{hom}_{F(T^*X)}(L_U \otimes \text{or}_X[-\text{dim} X], L)$$

defines a contravariant $A_\infty$-functor from the category of open sets of $X$ to the dg category of chain complexes. Without much difficulty, it is possible to reinterpret this as a constructible complex of sheaves on $X$ which we take to be $\pi_X(L)$. It follows quickly from the definitions that we have a canonical quasi-isomorphism of functors

$$\pi_X \circ \mu_X \simeq \text{id}_{\mathcal{S}h_c(X)}$$

confirming that $\pi_X$ and $\mu_X$ are quasi-inverses. The construction of $\pi_X$ is very similar to some results of [18, 19] though presented in the language of constructible sheaves rather than Fáy functors.

Finally, it is simple to understand basic properties of the functors $\pi_X$ and $\mu_X$ such as their constructibility. First, fix a stratification $\mathcal{S} = \{S_\alpha\}$ of $X$, and consider the full subcategory

$$\mathcal{S}h_\mathcal{S}(X) \subset \mathcal{S}h_c(X)$$

of complexes constructible with respect to $\mathcal{S}$. Then by construction, we have

$$\mu_X : \mathcal{S}h_\mathcal{S}(X) \to F(T^*X)_{\Lambda_\mathcal{S}}$$

where $\Lambda_\mathcal{S} = \bigcup_\alpha T_{S_\alpha}X$. Conversely, given a conical Lagrangian $\Lambda \subset T^*X$, as part of the construction of $\pi_X$, we verify that we have

$$\pi_X : F(T^*X)_\Lambda \to \mathcal{S}h_\mathcal{S}(X)$$

for any stratification $\mathcal{S}$ such that $\Lambda \subset \Lambda_\mathcal{S}$. It is also simple to see that the functors $\pi_X$ and $\mu_X$ interchange the brane duality $\alpha_X$ with Verdier duality $D_X$ and are also compatible with the basic operations on sheaves and branes.
1.3. Applications. We will postpone most applications to future papers and restrict ourselves here to one immediate application to symplectic topology.

The question of the possible structure of compact Lagrangian submanifolds of $T^*X$ has seen some progress in recent years. For a recent example of the subject, we refer the reader to the paper of Seidel [32]. It contains a brief summary of other relevant works of Lalonde-Sikarov [26], Viterbo [38], and Buhovsky [1], and is the paper in the subject which is closest to this one in methods. Namely, the main point is that we may reinterpret properties of objects of the Fukaya category of $T^*X$ in terms of the structure of their underlying Lagrangian submanifolds.

Consider a compact connected Lagrangian submanifold $L \subset T^*X$. To ensure that we may lift $L$ to an object of the Fukaya category of $T^*X$, we assume first that $L$ is exact and has trivial Maslov class. For simplicity in the following statement, we will also assume $\pi_1(X)$ is trivial. Thus in particular, the second Stiefel-Whitney class $w_2(T^*X)$, which is the square of the pullback of $w_1(X)$, must vanish so $T^*X$ is spin. This also implies that $L$ is orientable since the difference between $w_1(L)$ and the restriction of the pullback of $w_1(X)$ is the $\mathbb{Z}/2\mathbb{Z}$-reduction of the Maslov class. Further, we will assume that $w_2(L)$ is the restriction of the pullback of $w_2(X)$, so that $L$ is relatively spin with respect to the background class given by the pullback of $w_2(X)$.

The following application of the equivalence of $F(T^*X)$ and $\mathcal{Sh}_{c}(X)$ generalizes part of the main statement of [32] from the case when $X$ is a sphere. During the preparation of this paper, we learned that a similar characterization of compact exact branes has recently been obtained by Fukaya-Seidel-Smith [9, 35] by a variety of different methods.

**Theorem 1.3.1.** Assume $X$ and $L$ are as above. Then $L$ is equivalent in the Fukaya category of $T^*X$ to (a shift of) the zero section. From this, we conclude:

1. $[L] = \pm [X] \in H_{\dim X}(T^*X, \mathbb{C})$.
2. $H^*(L, \mathbb{C}) \simeq H^*(X, \mathbb{C})$.
3. If $L' \subset T^*X$ is another Lagrangian submanifold satisfying the same conditions, then we have a lower bound on the (possibly infinite) number of intersection points

$$\#(L \cap L') \geq \sum_k \dim H^k(L, \mathbb{C}).$$

**Proof.** By assumption, we may equip $L$ with a brane structure so that it becomes an object of $F(T^*X)$. Applying $\pi_X$ to this brane produces an object $\mathcal{F}$ of $\mathcal{Sh}_{c}(X)$. Since $L$ is compact, $\mathcal{F}$ is constructible with respect to the trivial stratification of $X$. In other words, the cohomology sheaves of $\mathcal{F}$ are local systems. By assumption, $X$ is simply-connected, so these local systems are all trivial.

Applying $\pi_X$ to standard calculations in $F(T^*X)$, we see that

$$\text{Ext}^{*}_{\mathcal{Sh}_{c}(X)}(\mathcal{F}, \mathcal{F}) \simeq H^*(L).$$

In particular, $H^m(L) = 0$, for $m > \dim X$, implies $\text{Ext}^m_{\mathcal{Sh}_{c}(X)}(\mathcal{F}, \mathcal{F}) = 0$, for $m > \dim X$. By writing $\mathcal{F}$ as a successive extension of (shifts of) the constant sheaf $\mathbb{C}_X$, we see that this bound forces $\mathcal{F}$ to reduce to (a shift of) $\mathbb{C}_X^{\otimes k}$ for some $k > 0$. But then since $L$ is connected, $H^0(L) \simeq \mathbb{C}$, and so $\mathcal{F}$ is isomorphic to (a shift of) $\mathbb{C}_X$ itself.
Thus applying $\mu_X$, we conclude that $L$ is equivalent to (a shift of) the zero section in $F(T^*X)$. This implies assertions (2) and (3) immediately.

For assertion (1), by applying $\mu_X$ to the skyscraper sheaf $C_{\{x\}}$, we can consider the conormal to a point $T^*_x X$ as an object of $F(T^*X)$. Since $\text{Ext}_{\mathcal{S}_X}^n(C_X, C_{\{x\}})$ is isomorphic to $\mathbb{C}$ concentrated in degree zero, (after a possible shift) the cohomology of $\text{hom}_{F(T^*X)}(L, T^*_x X)$ is as well. In particular, the Euler characteristic of the complex of intersection points is equal to $\pm 1$. This implies assertion (1). □

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2. Constructible sheaves

In this section, we first review background material on the constructible derived category, then recall a differential graded model of it. Finally, we explain how one can construct constructible sheaves out of certain $A_\infty$-functors.

2.1. Derived category. In this section, we briefly recall the construction of the constructible derived category of a real analytic manifold. For a comprehensive treatment of this topic, the reader could consult the book of Kashiwara-Schapira [17].

Let $X$ be a topological space. Let $\text{Top}(X)$ be the category whose objects are open sets $U \hookrightarrow X$, and morphisms are inclusions $U_0 \hookrightarrow U_1$ of open sets:

$$\text{hom}_{\text{Top}(X)}(U_0, U_1) = \begin{cases} \text{pt} & \text{when } U_0 \hookrightarrow U_1, \\ \emptyset & \text{when } U_0 \not\hookrightarrow U_1. \end{cases}$$

Let $\text{Vect}$ be the abelian category of complex vector spaces.

The derived category of sheaves of complex vector spaces on $X$ is traditionally defined via the following sequence of constructions:

1. Presheaves. Presheaves on $X$ are functors $\mathcal{F} : \text{Top}(X)^{\circ} \to \text{Vect}$ where $\text{Top}(X)^{\circ}$ denotes the opposite category. Given an open set $U \hookrightarrow X$, one writes $\mathcal{F}(U)$ for the sections of $\mathcal{F}$ over $U$, and given an inclusion $U_0 \hookrightarrow U_1$ of open sets, one writes $\rho_{i_0}^{U_1} : \mathcal{F}(U_1) \to \mathcal{F}(U_0)$ for the corresponding restriction map.

2. Sheaves. Sheaves on $X$ are presheaves $\mathcal{F} : \text{Top}(X)^{\circ} \to \text{Vect}$ which are locally determined in the following sense. For any open set $U \hookrightarrow X$, and covering $\bigcup = \{U_i\}$ of $U$ by open subsets $U_i \hookrightarrow U$, there is a complex of vector spaces

$$0 \longrightarrow \mathcal{F}(U) \longrightarrow \prod_i \mathcal{F}(U_i) \longrightarrow \prod_{i,j} \mathcal{F}(U_i \cap U_j),$$

where $\delta = \prod_i \rho_{i_0}^{U_i}$ and $\delta_0 = \prod_{i,j} \left( \rho_{i_0}^{U_i \cap U_j} - \rho_{j_0}^{U_i \cap U_j} \right)$. A sheaf is a presheaf for which $\ker(\delta) = \ker(\delta_0)/\text{im}(\delta) = 0$ for all open sets and coverings of open sets.

Sheaves on $X$ form an abelian category and thus one can continue with the following sequence of general homological constructions:
3. Complexes. Let $C(X)$ be the abelian category of complexes of sheaves on $X$ with morphisms the degree zero chain maps. Given a complex of sheaves $\mathcal{F}$, one writes $H(\mathcal{F})$ for the (graded) cohomology sheaf of $\mathcal{F}$.

4. Homotopy category. Let $K(X)$ be the homotopy category of sheaves on $X$ with objects complexes of sheaves and morphisms homotopy classes of maps. This is a triangulated category whose distinguished triangles are isomorphic to the standard mapping cones.

5. Derived category. The derived category $D(X)$ of sheaves on $X$ is defined to be the localization of $K(X)$ with respect to homotopy classes of quasi-isomorphisms (maps inducing isomorphisms on cohomology). Acyclic objects form a null system in $K(X)$, and thus $D(X)$ inherits the structure of triangulated category.

With the derived category $D(X)$ in hand, one can define many variants by imposing topological and homological conditions on objects.

6. Bounded derived category. The bounded derived category $D^b(X)$ is defined to be the full subcategory of $D(X)$ of bounded complexes.

Two standard equivalent descriptions are worth keeping in mind: first, there is the more flexible description of $D^b(X)$ as the full subcategory of $D(X)$ of complexes with bounded cohomology; second, there is the computationally useful description of $D^b(X)$ as the homotopy category of complexes of injective sheaves with bounded cohomology.

7. Constructibility. Assume $X$ is a real analytic manifold. Fix an analytic-geometric category $\mathcal{C}$ in the sense of [36]. For example, one could take $\mathcal{C}(X)$ to be the subanalytic subsets of $X$ as described in [3].

Let $\mathcal{S} = \{S_\alpha\}$ be a Whitney stratification of $X$ by $\mathcal{C}$-submanifolds $i_\alpha : S_\alpha \hookrightarrow X$. An object $\mathcal{F}$ of $D(X)$ is said to be $\mathcal{S}$-constructible if the restrictions $i_\alpha^* H(\mathcal{F})$ of its cohomology sheaf to the strata of $\mathcal{S}$ are finite-rank and locally constant.

The $\mathcal{S}$-constructible derived category $D_\mathcal{S}(X)$ is the full subcategory of $D(X)$ of $\mathcal{S}$-constructible objects. The constructible derived category $D_\mathcal{C}(X)$ is the full subcategory of $D(X)$ of objects which are $\mathcal{S}$-constructible for some Whitney stratification $\mathcal{S}$.

Note that if the stratification $\mathcal{S}$ is finite (for example, if $X$ is compact), then the finite-rank condition implies that all $\mathcal{S}$-constructible objects have bounded cohomology. In other words, within $D(X)$, every object of $D_\mathcal{S}(X)$ is isomorphic to an object of $D^b(X)$.

2.2. Differential graded category. The derived category $D(X)$ is naturally the cohomology category of a differential graded (dg) category $Sh(X)$. To define it, we will return to the sequence of homological constructions listed above and perform some modest changes. Two principles guide such definitions: (1) structures (such as morphisms and higher exts) should be defined at the level of complexes not their cohomologies; and (2) properties (such as constructibility) should be imposed at the level of cohomologies rather than complexes. The first principle ensures we will not lose important information, while the second ensures we will have sufficient flexibility. As an example of the latter, we prefer the realization of the bounded derived category $D^b(X)$ as the full subcategory of $D(X)$ of complexes with bounded cohomologies rather than of strictly bounded complexes.
The reader could consult [20, 6, 21] for background on dg categories, in particular, a discussion of the construction of dg quotients.

Recall that sheaves on \( X \) form an abelian category. The following sequence of homological constructions can be performed on any abelian category:

1. **Dg category of complexes.** Let \( C_{dg}(X) \) be the dg category with objects complexes of sheaves and morphisms the usual complexes of maps between complexes. In particular, the degree zero cycles in such a morphism complex are the usual degree zero chain maps which are the morphisms of the ordinary category \( C(X) \).

2. **Dg derived category.** The dg derived category \( Sh(X) \) is defined to be the dg quotient of \( C_{dg}(X) \) by the full subcategory of acyclic objects. This is a triangulated dg category whose cohomology category \( H(Sh(X)) \) is canonically equivalent (as a triangulated category) to the usual derived category \( D(X) \).

One can cut out full triangulated dg subcategories of \( Sh(X) \) by specifying full triangulated subcategories of its cohomology category \( H(Sh(X)) \simeq D(X) \).

3. **Bounded dg derived category.** The bounded dg derived category \( Sh^b(X) \) is defined to be the full dg subcategory of \( Sh(X) \) of objects projecting to \( D^b(X) \).

4. **Constructibility.** Assume \( X \) is a real analytic manifold, and fix an analytic-geometric category \( C \). The constructible dg derived category \( Sh_c(X) \) is the full dg subcategory of \( Sh(X) \) of objects projecting to \( D_c(X) \). For a Whitney stratification \( S \) of \( X \), the \( S \)-constructible dg derived category \( Sh_S(X) \) is the full dg subcategory of \( Sh(X) \) of objects projecting to \( D_S(X) \).

The formalism of Grothendieck’s six (derived) operations \( f^*, f_*, f!, f^!, \mathcal{H}om, \otimes \) can be lifted to the constructible dg derived category \( Sh_c(X) \) (see for example [6] for a general discussion of deriving functors in the dg setting). In our case, one concrete approach is to recognize that the natural map \( C_{dg,c}(\mathcal{J}\text{inj}(X)) \to Sh_c(X) \) from the dg category \( C_{dg,c}(\mathcal{J}\text{inj}(X)) \) of complexes of injective sheaves with constructible cohomology is a quasi-equivalence. With this in hand, one can define derived functors by evaluating their naive versions on \( C_{dg,c}(\mathcal{J}\text{inj}(X)) \). Since we will only consider derived functors, we will denote them by the above unadorned symbols.

Throughout the remainder of this paper, we assume that \( X \) is a real analytic manifold, and we fix an analytic-geometric category \( C \). All subsets will be \( C \)-subsets unless otherwise stated.

2.3. **Standard bases.** We recall here several standard bases for the \( T \)-constructible dg derived category \( Sh_T(X) \) for a triangulation \( T = \{ \tau_a \} \) of \( X \) by simplices \( j_a : \tau_a \hookrightarrow X \). (They are also well-known as basic examples in the theory of exceptional collections.)

Define \( C_*(T) \) to be the full dg category of \( Sh_T(X) \) of standard objects \( j_a C_{\tau_a} \). The morphisms between standard objects are quasi-isomorphic to complexes concentrated in degree zero

\[
\text{hom}_{Sh_T(X)}(j_b C_{\tau_b}, j_a C_{\tau_a}) \simeq \begin{cases} 
\mathbb{C} & \text{when } \tau_a \subset \tau_b, \\
0 & \text{when } \tau_a \not\subset \tau_b.
\end{cases}
\]
The composition maps are given by the linearization of the obvious poset relations.

**Lemma 2.3.1.** $\text{Sh}_T(X)$ is the triangulated envelope of $\mathcal{C}_*(T)$.

**Proof.** Let $i_{\geq k} : T_{\geq k} \hookrightarrow X$ denote the union of the simplices of $T$ of dimension greater than or equal to $k$, and let $j_{\leq k} : T_{\leq k} \hookrightarrow X$ denote the union of the simplices of $T$ of dimension less than $k$.

Let $\text{Sh}_{T_{\geq k}*}(X)$ denote the full dg subcategory of $\text{Sh}_T(X)$ of objects of the form $F \simeq i_{\geq k*}F_{\geq k}$. By the standard triangle

$$j_{\leq k}j_{\leq k+1}^!F \to F \to F \to i_{\geq k+1*}i_{\geq k+1}^*F_{\geq k+1},$$

this is equivalent to $j_{\leq k}^!F \simeq 0$. Let us show by induction that $\text{Sh}_{T_{\geq k}*}(X)$ is generated by the standard objects $j_{\leq k}^!C_{\tau_\alpha}$ associated to simplices $\tau_\alpha$ with dim $\tau_\alpha \geq k$. In particular, the assertion of the lemma is the case $k = 0$.

For $k = \text{dim } X$, $\text{Sh}_{T_{\geq k}*}(X)$ consists of complexes of standard objects on the top-dimensional simplices and nothing else.

Suppose we know the assertion for all $\ell > k$. For any object $F$ of $\text{Sh}_{T_{\geq k}*}(X)$, we have a distinguished triangle

$$j_{\leq k}j_{\leq k+1}^!j_{\leq k+1}^*F \to F \to i_{\geq k+1*}i_{\geq k+1}^*F \to,$$

By induction, we can express the object $i_{\geq k+1*}i_{\geq k+1}^*F$ in terms of the standard objects associated to the simplices of $T_{\geq k+1}$. Applying $j_{\leq k}^!$ to the above triangle, we obtain

$$j_{\leq k}^!j_{\leq k+1}^!j_{\leq k+1}^*F \simeq 0,$$

and so we can express the object $j_{\leq k+1}^!j_{\leq k+1}^*F$ in terms of the standard objects associated to the simplices of dimension $k$. □

We can obtain a costandard basis by applying Verdier duality to the standard basis as follows.

Define $\mathcal{C}_*(T)$ to be the full dg category of $\text{Sh}_T(X)$ of costandard objects $j_{\geq k}^!\omega_{\tau_\alpha} \simeq D(j_{\geq k}C_{\tau_\alpha})$. Here $\omega_{\tau_\alpha} \simeq D(C_{\tau_\alpha})$ denotes the Verdier dualizing complex of $\tau_\alpha$; it is canonically isomorphic to the shifted orientation sheaf $or_{\tau_\alpha}[\text{dim } \tau_\alpha]$, and so, given a choice of orientation of $\tau_\alpha$, isomorphic to the shifted constant sheaf $C_{\tau_\alpha}[\text{dim } \tau_\alpha]$. The morphisms between costandard objects are quasi-isomorphic to complexes concentrated in degree zero

$$\text{hom}_{\text{Sh}_T(X)}(j_{\geq k}^!\omega_{\tau_\alpha}, j_{\geq k}^!\omega_{\tau_\beta}) \simeq \begin{cases} \mathbb{C} & \text{when } \tau_\alpha \subseteq \tau_\beta, \\ 0 & \text{when } \tau_\alpha \not\subseteq \tau_\beta. \end{cases}$$

The composition maps are given by the linearization of the obvious poset relations.

Since Verdier duality is an anti-equivalence, Lemma 2.3.1 implies the following.

**Lemma 2.3.2.** $\text{Sh}_T(X)$ is the triangulated envelope of $\mathcal{C}_*(T)$.

Here is an alternative basis of standard objects associated to open sets. For each simplex $\tau_\alpha$ of $T$, let $i_{\alpha} : st_\alpha \hookrightarrow X$ be its star

$$st_\alpha = \bigcup_{\tau_\beta \subset \tau_\alpha} \tau_\beta.$$
Note that $st_a$ is an open contractible submanifold of $X$, and we have $\tau_a \subset \tau_b$ if and only if $st_b \subset st_a$.

Define $\mathcal{C}_{sts}(\mathcal{T})$ to be the full dg category of $Sh_{\tau}(X)$ of standard objects $i_{as}C_{st_a}$. The morphisms between standard objects are quasi-isomorphic to complexes concentrated in degree zero

$$\text{hom}_{Sh_{\tau}(X)}(i_{as}C_{st_a}, i_{bs}C_{st_b}) \simeq \begin{cases} \mathbb{C} & \text{when } \tau_a \subset \tau_b, \\ 0 & \text{when } \tau_a \not\subset \tau_b. \end{cases}$$

The composition maps are given by the linearization of the obvious poset relations.

**Lemma 2.3.3.** $Sh_{\tau}(X)$ is the triangulated envelope of $\mathcal{C}_{sts}(\mathcal{T})$.

**Proof.** We continue with the notation of the proof of Lemma 2.3.1.

Let us show by induction that $Sh_{\tau_a}(X)$ is generated by the standard objects $i_{as}C_{st_a}$ associated to the stars of simplices with $\dim \tau_a \geq k$. Recall that the proof of Lemma 2.3.1 shows that $Sh_{\tau_a}(X)$ is generated by the standard objects $j_{as}C_{\tau_a}$ associated to simplices with $\dim \tau_a \geq k$.

For $k = \dim X$, if $\dim \tau_a = \dim X$, then $st_a = \tau_a$ and so $j_{as}C_{\tau_a} = i_{as}C_{st_a}$.

Suppose we know the assertion for all $k > \dim \tau_a$. Let $i'_{a} : st'_{a} \to X$ be the punctured star $st'_{a} = st_{a} \setminus \tau_{a}$, and consider the distinguished triangle

$$j_{as}C_{\tau_a} \to i_{as}C_{st_a} \to i'_{as}C_{st'_{a}} \to [1].$$

Here $\nu_{\tau_a} \simeq j_{a}C_{X}$ is canonically isomorphic to the shifted normal orientation sheaf $or_{X/\tau_a}[-\dim \tau_a]$, and so, given a choice of normal orientation, isomorphic to the shifted constant sheaf $C_{\tau_a}[-\dim \tau_a]$.

By induction, we can express $i'_{as}C_{st'_{a}}$ in terms of the standard objects $i_{bs}C_{st_b}$ associated to the stars of simplices with $\dim \tau_b \geq k$. Therefore we can express the standard object $j_{as}C_{\tau_a}$ as well. \qed

We can obtain a costandard basis by applying Verdier duality to the above standard basis as follows.

Define $\mathcal{C}_{st!}(\mathcal{T})$ to be the full dg category of $Sh_{\tau}(X)$ of costandard objects $i_{at}C_{st_a} \simeq D(i_{as}C_{st_a})$. The morphisms between costandard objects are quasi-isomorphic to complexes concentrated in degree zero

$$\text{hom}_{Sh_{\tau}(X)}(i_{bt}C_{st_b}, i_{at}C_{st_a}) \simeq \begin{cases} \mathbb{C} & \text{when } \tau_a \subset \tau_b, \\ 0 & \text{when } \tau_a \not\subset \tau_b. \end{cases}$$

The composition maps are given by the linearization of the obvious poset relations.

Since Verdier duality is an anti-equivalence, Lemma 2.3.3 implies the following.

**Lemma 2.3.4.** $Sh_{\tau}(X)$ is the triangulated envelope of $\mathcal{C}_{st!}(\mathcal{T})$.

2.4. **Quasi-representable modules.** This section is not logically needed in what follows. We include it to give a feeling for the information contained in an $A_{\infty}$-module over the constructible dg derived category $Sh_{c}(X)$. By restricting a quasi-representable $A_{\infty}$-module over $Sh_{c}(X)$ to the costandard sheaves of open subsets, one obtains what could be called an $A_{\infty}$-sheaf. For background on $A_{\infty}$-categories, the reader could


consult [33] and the references therein. Note that any dg category can be viewed as an
an $A_\infty$-category with vanishing higher compositions.

Let $Ch = Ch_{dg}(pt)$ denote the dg category of chain complexes of complex vector
spaces, and let $mod_r(Sh_c(X))$ denote the $A_\infty$-category of $A_\infty$-functors
$$\mathcal{M} : Sh_c(X)^\circ \to Ch.$$ In keeping with usual nomenclature (inspired by considering categories with a single
object), we will also refer to such functors as right $Sh$-functor $A$-category $\mathcal{C}(\mathcal{T})$ of $\mathcal{C}(\mathcal{T})$-modules satisfying $(f-r)$ and $(S-lc)$ is quasi-represented by an object of $Sh_S(X)$.

Lemma 2.4.1. For any right $Sh_c(X)$-module $\mathcal{M}$ satisfying the finite-rank condition
$(f-r)$, and any triangulation $\mathcal{T}$, there is an object $\mathcal{F}_\mathcal{T}$ of $Sh_{\mathcal{T}}(X)$ that quasi-represents
the restriction $\mathcal{M}|_{Sh_{\mathcal{T}}(X)}$.

Proof. Recall the full dg category $\mathcal{C}(\mathcal{T}) \subset Sh_c(X)$ of costandard objects introduced
in the preceding section. By Lemma 2.3.2 the embedding realizes $Sh_{\mathcal{T}}(X)$ as the
triangulated envelope of $\mathcal{C}(\mathcal{T})$, and hence the corresponding restriction of modules is
a quasi-embedding as well
$$mod_r(Sh_{\mathcal{T}}(X)) \hookrightarrow mod_r(\mathcal{C}(\mathcal{T})).$$ Thus to prove the lemma, it suffices to define an object $\mathcal{F}_\mathcal{T}$ of $Sh_{\mathcal{T}}(X)$ along with a
quasi-isomorphism of modules
$$\mathcal{Y}_r(\mathcal{F}_\mathcal{T})|_{\mathcal{C}(\mathcal{T})} \simeq \mathcal{M}|_{\mathcal{C}(\mathcal{T})}$$
Consider the (oriented) standard objects \(j_{\alpha^*}\omega_{\tau_a}\) associated to the simplices \(j_{\alpha} : \tau_a \hookrightarrow X\). They provide the simplest right \(\mathcal{C}(\mathcal{T})\)-modules in the sense that

\[
\text{hom}_{\text{Sh}(X)}(j_{b^*}\omega_{\tau_b}, j_{a^*}\omega_{\tau_a}) \simeq \text{hom}_{\text{Sh}(X)}(\omega_{\tau_b}, j^!_{b^*}j_{a^*}\omega_{\tau_a}) \simeq \begin{cases} \mathbb{C} & \text{ when } a = b, \\ 0 & \text{ when } a \neq b. \end{cases}
\]

We will use this to show that \(\mathcal{M}|_{\mathcal{C}(\mathcal{T})}\) can be expressed as an iterated cone of shifts of the standard modules \(\mathcal{Y}_r(j_{\alpha^*}\omega_{\tau_a})|_{\mathcal{C}(\mathcal{T})}\).

The argument, similar to that of Lemma 2.3.1, is an induction on the dimension of the simplices of \(\mathcal{T}\), beginning with the open simplices. Let \(n = \dim X\), and for \(0 \leq k \leq n\), let \(\mathcal{I}_k\) be those indices labelling simplices of \(\mathcal{T}\) of dimension \(k\).

(Step \(n\)) For \(a_n \in \mathcal{I}_n\), consider the evaluation \(\mathcal{M}(a_n) = \mathcal{M}(j_{\tau_{a_n}}\omega_{\tau_{a_n}})\).

We claim that there is a canonical map of right \(\mathcal{C}(\mathcal{T})\)-modules

\[
q_n : \mathcal{M} \to \sum_{a_n \in \mathcal{I}_n} \mathcal{Y}_r(j_{a_n^*}(\mathcal{M}(a_n) \otimes \omega_{\tau_{a_n}})).
\]

To see this, note that for any \(\tau_b\) the right hand side evaluates to be

\[
\left(\sum_{a_n \in \mathcal{I}_n} \mathcal{Y}_r(j_{a_n^*}(\mathcal{M}(a_n) \otimes \omega_{\tau_{a_n}}))\right)(j_{b^*}\omega_{\tau_b}) \simeq \begin{cases} \mathcal{M}(a_n) & \text{ when } b = a_n, \\ 0 & \text{ when } b \neq a_n. \end{cases}
\]

Thus we can take the first-order part of \(q_n\) to be the identity when \(b = a_n\) and zero otherwise. Furthermore, there are no higher-order terms to define.

Let \(\mathcal{M}_{<n}\) denote the cone of \(q_n\). By construction, since \(q_n\) evaluated at any \(j_{a^*}\omega_{\tau_a}\), for any \(a \in \mathcal{I}_n\), is an isomorphism at the chain level, we can arrange so that \(\mathcal{M}_{<n}(j_{a^*}\omega_{\tau_a}) = 0\), for all \(a \in \mathcal{I}_n\).

(Step \(n-1\)) For \(a_{n-1} \in \mathcal{I}_{n-1}\), consider the evaluation \(\mathcal{M}_{<n}(a_{n-1}) = \mathcal{M}_{<n}(j_{a_{n-1}^*}\omega_{\tau_{a_{n-1}}}\).

We claim that there is a canonical map of right \(\mathcal{C}(\mathcal{T})\)-modules

\[
q_{n-1} : \mathcal{M}_{<n} \to \sum_{a_{n-1} \in \mathcal{I}_{n-1}} \mathcal{Y}_r(j_{a_{n-1}^*}(\mathcal{M}_{<n}(a_{n-1}) \otimes \omega_{\tau_{a_{n-1}}})).
\]

To see this, note that for any \(\tau_b\), the left hand side evaluates to be

\[
\left(\sum_{a_{n-1} \in \mathcal{I}_{n-1}} \mathcal{Y}_r(j_{a_{n-1}^*}(\mathcal{M}_{<n}(a_{n-1}) \otimes \omega_{\tau_{a_{n-1}}}))\right)(j_{b^*}\omega_{\tau_b}) \simeq \begin{cases} \mathcal{M}_{<n}(a_{n-1}) & \text{ when } b = a_{n-1}, \\ 0 & \text{ when } b \neq a_{n-1}. \end{cases}
\]

Thus we can take the first-order part of \(q_{n-1}\) to be the identity when \(b = a_{n-1}\) and zero otherwise. Furthermore, there are no higher-order terms to define.

Let \(\mathcal{M}_{<n-1}\) denote the cone of \(q_{n-1}\). By construction, since \(q_{n-1}\) evaluated at \(j_{a^*}\omega_{\tau_a}\), for any \(a \in \mathcal{I}_{n-1} \cup \mathcal{I}_n\), is an isomorphism at the chain level, we can arrange so that \(\mathcal{M}_{<n-1}(j_{a^*}\omega_{\tau_a}) = 0\), for all \(a \in \mathcal{I}_{n-1} \cup \mathcal{I}_n\).

And so on. In the end, we see that \(\mathcal{M}\) can be expressed by a finite sequence of cones of Yoneda modules of shifted standard objects.

Consider a second triangulation \(\mathcal{T}'\) of \(X\) that refines \(\mathcal{T}\) in the sense that each simplex of \(\mathcal{T}\) is a union of simplices of \(\mathcal{T}'\). So we have fully faithful dg embeddings

\[
\text{Sh}_\mathcal{T}(X) \hookrightarrow \text{Sh}_{\mathcal{T}'}(X) \hookrightarrow \text{Sh}_\mathcal{C}(X),
\]
and the corresponding restriction of modules

\[ \text{mod}_r(Sh_c(X)) \to \text{mod}_r(Sh_T(X)) \to \text{mod}_r(Sh_T(X)). \]

**Lemma 2.4.2.** Suppose the right \( Sh_c(X) \)-module \( M \) also satisfies the \( S \)-locally constant property (\( S \)-lc) (in addition to (f-r)), and that the triangulation \( T \) refines the stratification \( S \). Then for any triangulation \( T' \) that refines \( T \), there is a canonical isomorphism \( F_T \simeq F_{T'} \) of objects of \( Sh_T(X) \).

**Proof.** Recall that for any right \( Sh_c(X) \)-module \( M \) satisfying the finite-rank condition (f-r), and any triangulation \( T \), Lemma 2.4.1 provides an object \( F_T \) along with a quasi-isomorphism of right \( Sh_T(X) \)-modules

\[ Y_r(F_T) \simeq M|_{Sh_T(X)}. \]

Applying Lemma 2.4.1 to \( T' \) and restricting to \( Sh_T(X) \) provides a quasi-isomorphism of right \( Sh_T(X) \)-modules

\[ Y_r(F_{T'})|_{Sh_T(X)} \simeq M|_{Sh_T(X)}. \]

Finally, composing the above quasi-isomorphisms gives a quasi-isomorphism of right \( Sh_T(X) \)-modules

\[ Y_r(F_{T'})|_{Sh_T(X)} \simeq Y_r(F_T). \]

With the above quasi-isomorphism in hand, to prove the lemma, it suffices to show that \( F_{T'} \) is in fact an object of \( Sh_T(X) \). For each pair \( \tau'_a \subset \tau'_b \) of simplices of \( T' \) such that \( \tau'_a, \tau'_b \) both lie in a single stratum of \( S \), there is a diagram which commutes at the level of cohomology

\[
\begin{array}{ccc}
\mathcal{M}(i_b! \omega_{\tau_b}) & \to & \mathcal{M}(i_a! \omega_{\tau_a}) \\
\downarrow & & \downarrow \\
\text{hom}_{Sh_c(X)}(i_b! \omega_{\tau_b}, F_{T'}) & \to & \text{hom}_{Sh_c(X)}(i_a! \omega_{\tau_a}, F_{T'})
\end{array}
\]

The assumption (\( S \)-lc) on \( \mathcal{M} \) implies that the upper horizontal arrow is a quasi-isomorphism, and so the lower horizontal arrow is as well. Thus the object \( F_{T'} \) is in fact \( T \)-constructible. \( \square \)

We summarize the preceding development in the following statement.

**Proposition 2.4.3.** Let \( \mathcal{M} \) be an object of \( \text{mod}_r(Sh_c(X)) \) satisfying the properties (f-r) and (\( S \)-lc). Then \( \mathcal{M} \) is quasi-represented by an object of \( Sh_S(X) \).

**Proof.** Fix a triangulation \( T \) refining the stratification \( S \). By Lemma 2.4.1 there is an object \( F_T \) quasi-representing the restriction \( \mathcal{M}|_{Sh_T(X)} \). Given any stratification \( S' \), we can find a triangulation \( T' \) that simultaneously refines \( T \) and \( S' \). Thus by Lemma 2.4.2, the object \( F_T \) quasi-represents \( \mathcal{M} \) on all of \( Sh_c(X) \). Finally, \( \mathcal{M} \) satisfies property (\( S \)-lc) and hence the Yoneda \( Sh_c(X) \)-module \( Y_r(F_T) \) does as well. This clearly implies that \( F_T \) is \( S \)-constructible. \( \square \)
3. Microlocal branes

In the sections below, we review some basic aspects of the Fukaya category of the cotangent bundle $T^*X$ from [28]. In particular, we discuss how constructible sheaves on $X$ embed into its triangulated envelope. We also collect some technical results on isotopies and standard branes needed in what follows.

3.1. Preliminaries. In what follows, we work with a fixed compact real analytic manifold $X$ with cotangent bundle $\pi : T^*X \to X$. We often denote points of $T^*X$ by pairs $(x, \xi)$ where $x \in X$ and $\xi \in T^*_xX$. The material of this section is a condensed version of the discussion of [28].

Let $\theta \in \Omega^1(T^*X)$ denote the canonical one-form $\theta(v) = \xi(\pi_*(v))$, for $v \in T_{(x,\xi)}(T^*X)$, and let $\omega = d\theta \in \Omega^2(T^*X)$ denote the canonical symplectic structure. For a fixed Riemannian metric on $X$, let $|\xi| : T^*X \to \mathbb{R}$ denote the corresponding fiberwise linear length function.

3.1.1. Compactification. To better control noncompact Lagrangians in $T^*X$, it is useful to work with the cospherical compactification $\overline{T^*X} : \pi : \overline{T^*X} \to X$ of the projection $\pi : T^*X \to X$ obtained by attaching the cosphere bundle at infinity $\pi^\infty : T^\infty X \to X$.

Concretely, we can realize the compactification $\overline{T^*X}$ as the quotient

$$\overline{T^*X} = ((T^*X \times \mathbb{R}_{\geq 0}) \setminus (X \times \{0\})) / \mathbb{R}_+$$

where $\mathbb{R}_+$ acts by dilations on both factors. The canonical inclusion $T^*X \hookrightarrow \overline{T^*X}$ sends a covector $\xi$ to the class of $[\xi, 1]$. The boundary at infinity $T^\infty X = \overline{T^*X} \setminus T^*X$ consists of classes of the form $[\xi, 0]$ with $\xi$ a non-zero covector. Given a Riemannian metric on $X$, one can identify $\overline{T^*X}$ with the closed unit disk bundle $D^*X$, and $T^\infty X$ with the unit cosphere bundle $S^*X$, via the map

$$[\xi, r] \mapsto (\xi, \hat{r}), \text{ where } |\hat{\xi}|^2 + \hat{r}^2 = 1.$$

The boundary at infinity $T^\infty X$ carries a canonical contact distribution $\kappa \subset T(T^\infty X)$ with a well-defined notion of positive normal direction. Given a Riemannian metric on $X$, under the induced identification of $T^\infty X$ with the unit cosphere bundle $S^*X$, the distribution $\kappa$ is the kernel of the restriction of $\theta$.

3.1.2. Conical almost complex structure. To better control holomorphic disks in $T^*X$, it is useful to work with an almost complex structure $J_{con} \in \text{End}(T(T^*X))$ which near infinity is invariant under dilations.

A fixed Riemannian metric on $X$ provides a canonical splitting $T(T^*X) \simeq T_b \oplus T_f$, where $T_b$ denotes the horizontal base directions and $T_f$ the vertical fiber directions, along with a canonical isomorphism $j_0 : T_b \to T_f$ of vector bundles over $T^*X$. We refer to the resulting almost complex structure

$$J_{Sas} = \begin{pmatrix} 0 & j_0^{-1} \\ -j_0 & 0 \end{pmatrix} \in \text{End}(T_b \oplus T_f).$$

as the Sasaki almost complex structure, since by construction, the Sasaki metric on $T^*X$ is given by $g_{Sas}(v, v) = \omega(u, J_{Sas}v)$.
Fix positive constants \( r_0, r_1 > 0 \), a bump function \( b : \mathbb{R} \to \mathbb{R} \) such that \( b(r) = 0 \) for \( r < r_0 \), and \( b(r) = 1 \), for \( r > r_1 \), and set \( w(x, \xi) = |\xi|^{|b(\xi)|} \), where as usual \( |\xi| \) denotes the length of a covector with respect to the original metric on \( X \). We refer to the compatible almost complex structure
\[
J_w = \begin{pmatrix} 0 & w^{-1}j_0^{-1} \\ -w_j_0 & 0 \end{pmatrix} \in \text{End}(T_b \oplus T_f)
\]
as a(n asymptotically) conical almost complex structure since near infinity \( J_{\text{con}} \) is invariant under dilations. The corresponding metric \( g_{\text{con}}(u, v) = \omega(v, J_{\text{con}}v) \) presents \( T^*X \) near infinity as a metric cone over the unit cosphere bundle \( S^*X \) equipped with the Sasaki metric.

One can view the conical metric \( g_{\text{con}} \) as being compatible with the compactification \( \overline{T}^*X \) in the sense that near infinity it treats base and angular fiber directions on equal footing. Near infinity the metrics on the level sets of \( |\xi| \) are given by scaling the Sasaki metric on the unit cophere bundle by the factor \( |\xi|^{1/2} \).

3.2. Brane structures. By a Lagrangian \( j : L \hookrightarrow T^*X \), we mean a closed (but not necessarily compact) half-dimensional submanifold such that \( TL \) is isotropic for the symplectic form \( \omega \). One says that \( L \) is exact if the pullback of the one-form \( j^* \theta \) is cohomologous to zero.

By a brane structure on a Lagrangian \( L \hookrightarrow T^*X \), we mean a three-tuple \( (\mathcal{E}, \bar{\alpha}, b) \) consisting of a flat (finite-dimensional) vector bundle \( \mathcal{E} \to L \), along with a grading \( \bar{\alpha} : L \to \mathbb{R} \) (with respect to the canonical bicanonical trivialization) and a relative pin structure \( b \) (with respect to the background class \( \pi^*(w_2(X)) \)). To remind the interested reader, we include below a short summary of what the latter two structures entail.

3.2.1. Gradings. The almost complex structure \( J_{\text{con}} \in \text{End}(T(T^*X)) \) provides a holomorphic canonical bundle \( \kappa = (\wedge^\dim X T^\text{hol}(T^*X))^{-1} \). According to [28], there is a canonical trivialization \( \eta^2 \) of the bicanonical bundle \( \kappa^{\otimes 2} \) (and a canonical trivialization of \( \kappa \) itself if \( X \) is assumed oriented). Consider the bundle of Lagrangian planes \( \text{Lag}_{T^*X} \to T^*X \), and the squared phase map
\[
\alpha : \text{Lag}_{T^*X} \to U(1)
\]
\[
\alpha(\mathcal{L}) = \eta(\wedge^\dim X \mathcal{L})^2/|\eta(\wedge^\dim X \mathcal{L})|^2.
\]

For a Lagrangian \( L \hookrightarrow T^*X \) and a point \( x \in L \), we obtain a map \( \alpha : L \to U(1) \) by setting \( \alpha(x) = \alpha(T_x L) \). The Maslov class \( \mu(L) \in H^1(L, \mathbb{Z}) \) is the obstruction class \( \mu = \alpha^*(dt) \), where \( dt \) denotes the standard one-form on \( U(1) \). Thus \( \alpha \) has a lift to a map \( \bar{\alpha} : L \to \mathbb{R} \) if and only if \( \mu = 0 \), and choices of a lift form a torsor over the group \( H^0(L, \mathbb{Z}) \). Such a lift \( \bar{\alpha} : L \to \mathbb{R} \) is called a grading of the Lagrangian \( L \hookrightarrow T^*X \).

3.2.2. Relative pin structures. Recall that the group \( \text{Pin}^+(n) \) is the double cover of \( O(n) \) with center \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \). A pin structure on a Riemannian manifold \( L \) is a lift of the structure group of \( TL \) to \( \text{Pin}^+(n) \). The obstruction to a pin structure is the second Stiefel-Whitney class \( w_2(L) \in H^2(L, \mathbb{Z}/2\mathbb{Z}) \), and choices of pin structures form a torsor over the group \( H^1(L, \mathbb{Z}/2\mathbb{Z}) \).
A relative pin structure on a submanifold \( L \hookrightarrow M \) with background class \( [w] \in H^2(M, \mathbb{Z}/2\mathbb{Z}) \) can be defined as follows. Fix a Čech cocycle \( w \) representing \( [w] \), and let \( w|_L \) be its restriction to \( L \). Then a pin structure on \( L \) relative to \( [w] \) can be defined to be an \( \omega|_L \)-twisted pin structure on \( TL \). Concretely, this can be represented by a \( \text{Pin}^+(n) \)-valued Čech 1-cochain on \( L \) whose coboundary is \( w|_L \). Such structures are canonically independent of the choice of Čech representatives.

For Lagrangians \( L \hookrightarrow T^*X \), we will always consider relative pin structures \( \flat \) on \( L \) with respect to the fixed background class \( \pi^*(w_2(X)) \in H^2(T^*X, \mathbb{Z}/2\mathbb{Z}) \).

3.3. **Fukaya category.** We recall here the construction of the Fukaya \( A_\infty \)-category of the cotangent bundle \( T^*X \). Our aim is not to review all of the details, but only those relevant to our later proofs. For more details, the reader could consult [28] and the references therein. In technical terms, the construction is a close relative of the category of vanishing cycles proposed by Kontsevich [22] and Hori-Iqbal-Vafa [13], and developed by Seidel [30], [31], [33].

3.3.1. **Objects.** An object of the Fukaya category of \( T^*X \) is a four-tuple \( (L, \mathcal{E}, \tilde{\alpha}, \flat) \) consisting of an exact (not necessarily compact) closed Lagrangian submanifold \( L \hookrightarrow T^*X \) equipped with a brane structure: this includes a flat vector bundle \( \mathcal{E} \rightarrow L \), along with a grading \( \tilde{\alpha} : L \rightarrow \mathbb{R} \) (with respect to the canonical bicanonical trivialization) and a relative pin structure \( \flat \) (with respect to the background class \( \pi^*(w_2(X)) \)).

To ensure reasonable behavior near infinity, we place two assumptions on the Lagrangian \( L \). First, consider the compactification \( \overline{T^*X} \) obtained by adding to \( T^*X \) the cosphere bundle at infinity \( T^\infty X \). Then we fix an analytic-geometric category \( \mathcal{C} \) once and for all, and assume that the closure \( \overline{L} \hookrightarrow \overline{T^*X} \) is a \( \mathcal{C} \)-subset. Along with other nice properties, this implies the following two key facts:

1. The boundary at infinity \( L^\infty = \overline{L} \cap T^\infty X \) is an isotropic subset of \( T^\infty X \) with respect to the induced contact structure.
2. There is a real number \( r > 0 \) such that the restriction of the length function

   \[ |\xi| : L \cap \{ |\xi| > r \} \rightarrow \mathbb{R} \]

   has no critical points.

As discussed below, the above properties guarantee we can make sense of “intersections at infinity”.

Second, to have a manageable theory of pseudoholomorphic maps with boundary on such Lagrangians, we also assume the existence of a perturbation \( \psi \) that moves the initial Lagrangian \( L \) to a nearby Lagrangian tame (in the sense of [34]) with respect to the conical metric \( g_{\text{con}} \). As confirmed in the Appendix, all such perturbations lead to equivalent calculations.

We use the term Lagrangian brane to refer to objects of the Fukaya category. When there is no chance for confusion, we often write \( L \) alone to signify the Lagrangian brane.
3.3.2. **Morphisms.** To define the morphisms between two branes, we must perturb Lagrangians so that their intersections occur in some bounded domain. To organize the perturbations, we recall the inductive notion of a fringed set \( R_{d+1} \subset \mathbb{R}^{d+1} \). A fringed set \( R_1 \subset \mathbb{R}_+ \) is any interval of the form \((0,r)\) for some \( r > 0 \). A fringed set \( R_{d+1} \subset \mathbb{R}^{d+1} \) is a subset satisfying the following:

1. \( R_{d+1} \) is open in \( \mathbb{R}^{d+1} \).
2. Under the projection \( \pi : \mathbb{R}^{d+1} \to \mathbb{R}^d \) forgetting the last coordinate, the image \( \pi(R_{d+1}) \) is a fringed set.
3. If \((r_1, \ldots, r_d, r_{d+1}) \in R_{d+1}\), then \((r_1, \ldots, r_d, r'_{d+1}) \in R_{d+1} \) for \( 0 < r'_{d+1} < r_{d+1} \).

A Hamiltonian function \( H : T^*X \to \mathbb{R} \) is said to be controlled if there is a real number \( r > 0 \) such that in the region \( |\xi| > r \) we have \( H(x, \xi) = |\xi| \). The corresponding Hamiltonian isotopy \( \varphi_{H,t} : T^*X \to T^*X \) equals the normalized geodesic flow \( \gamma_t \) in the region \( |\xi| > r \).

As explained in [23], given Lagrangians branes \( L_0, \ldots, L_d \subset T^*X \), and controlled Hamiltonian functions \( H_0, \ldots, H_d \), we may choose a fringed set \( R \subset \mathbb{R}^{d+1} \) such that for \((\delta_d, \ldots, \delta_0) \in R\), there is a real number \( r > 0 \) such that for any \( i \neq j \), we have

\[ \varphi_{H_i,\delta_i(L_i)} \cap \varphi_{H_j,\delta_j(L_j)} \text{ lies in the region } |\xi| < r. \]

By a further compactly supported Hamiltonian perturbation, we may also arrange so that the intersections are transverse.

We consider finite collections of Lagrangian branes \( L_0, \ldots, L_d \subset T^*X \) to come equipped with such perturbation data, with the brane structures \((\mathcal{E}_i, \bar{\alpha}_i, b_i)\) and taming perturbations \( \psi_t \) transported via the perturbations. Note that the latter makes sense since the normalized geodesic flow \( \gamma_t \) is an isometry of the metric \( g_{con} \). Then for branes \( L_i, L_j \) with \( i < j \), the graded vector space of morphisms between them is defined to be

\[ \text{hom}_{F(T^*X)}(L_i, L_j) = \bigoplus_{p \in \psi_t(\varphi_{H_i,\delta_i(L_i)}) \cap \varphi_{H_j,\delta_j(L_j)}} \text{Hom}(E_i[p], E_j[p])[-\deg(p)]. \]

where the integer \( \deg(p) \) denotes the Maslov grading of the linear Lagrangian subspaces at the intersection.

It is worth emphasizing that near infinity the salient aspect of the above perturbation procedure is the relative position of the perturbed branes rather than their absolute position. The following informal viewpoint can be a useful mnemonic to keep the conventions straight. In general, we always think of morphisms as “propagating forward in time”. Thus to calculate the morphisms \( \text{hom}_{F(T^*X)}(L_0, L_1) \), we have required that \( L_0, L_1 \) are perturbed near infinity by normalized geodesic flow so that \( L_1 \) is further in the future than \( L_0 \). But what is important is not that they are both perturbed forward in time, only that \( L_1 \) is further along the timeline than \( L_0 \). So for example, we could perturb \( L_0, L_1 \) near infinity by normalized anti-geodesic flow as long as \( L_0 \) is further in the past than \( L_1 \).

3.3.3. **Compositions.** Signed counts of pseudoholomorphic polygons provide the differential and higher composition maps of the \( A_\infty \)-structure. We use the following approach of Sikorav [33] (or equivalently, Audin-Lalonde-Polterovich [1]) to ensure that the relevant moduli spaces are compact, and hence the corresponding counts are finite.
First, as explained in [28], the cotangent bundle $T^*X$ equipped with the canonical symplectic form $\omega$, conical almost complex structure $J_{\text{con}}$, and conical metric $g_{\text{con}}$ is tame in the sense of [34]. To see this, one can verify that $g_{\text{con}}$ is conical near infinity, and so it is easy to derive an upper bound on its curvature and a positive lower bound on its injectivity radius.

Next, given a finite collection of branes $L_0, \ldots, L_d$, denote by $L$ the union of their perturbations $\psi_i(\varphi_{H_i, \delta_i}(L_i))$ as described above. By construction, the intersection of $L$ with the region $|\xi| > r$ is a tame submanifold (in the sense of [34]) with respect to the structures $\omega, J_{\text{con}}$, and $g_{\text{con}}$. Namely, there exists $\rho_L > 0$ such that for every $x \in L$, the set of points $y \in L$ of distance $d(x, y) \leq \rho_L$ is contractible, and there exists $C_L$ giving a two-point distance condition $d_L(x, y) \leq C_L d(x, y)$ whenever $x, y \in L$ with $d(x, y) < \rho_L$.

Now, consider a fixed topological type of pseudoholomorphic map

$$u : (D, \partial D) \to (T^*X, L).$$

Assume that all $u(D)$ intersect a fixed compact region, and there is an a priori area bound $\text{Area}(u(D)) < A$. Then as proven in [34], one has compactness of the moduli space of such maps $u$. In fact, one has a diameter bound (depending only on the given constants) constraining how far the image $u(D)$ can stretch from the compact set.

In the situation at hand, for a given $A_\infty$-structure constant, we must consider pseudoholomorphic maps $u$ from polygons with labeled boundary edges. In particular, all such maps $u$ have image intersecting the compact set given by a single intersection point. The area of the image $u(D)$ can be expressed as the contour integral

$$\text{Area}(u(D)) = \int_{u(\partial D)} \theta.$$

Since each of the individual Lagrangian branes making up $L$ is exact, the contour integral only depends upon the integral of $\theta$ along minimal paths between intersection points. Thus such maps $u$ satisfy an a priori area bound. We conclude that for each $A_\infty$-structure constant, the moduli space defining the structure constant is compact, and its points are represented by maps $u$ with image bounded by a fixed distance from any of the intersection points.

Finally, as usual, the composition map

$$m^d : \text{hom}_{F(T^*X)}(L_0, L_1) \otimes \cdots \otimes \text{hom}_{F(T^*X)}(L_d, L_d) \to \text{hom}_{F(T^*X)}(L_0, L_d)[2-d]$$

is defined as follows. Consider elements $p_i \in \text{hom}(L_i, L_{i+1})$, for $i = 0, \ldots, d - 1$, and $p_d \in \text{hom}(L_0, L_d)$. Then the coefficient of $p_d$ in $m^d(p_0, \ldots, p_{d-1})$ is defined to be the signed sum over pseudoholomorphic maps from a disk with $d + 1$ counterclockwise cyclically ordered marked points mapping to the $p_i$ and corresponding boundary arcs mapping to the perturbations of $L_{i+1}$. Each map contributes according to the holonomy of its boundary, where adjacent perturbed components $L_i$ and $L_{i+1}$ are glued with $p_i$.

Continuation maps with respect to families of perturbed branes ensure the consistency of all of our definitions. While the details of this were not elaborated on in [28], Section 3.7 and the Appendix contain a discussion about continuation maps which contains what is needed here as a special case.
Consider the dg category of right modules over the Fukaya category of $T^*X$. Throughout this paper, we write $F(T^*X)$ for the full subcategory of twisted complexes of representable modules, and refer to it as the triangulated envelope of the Fukaya category. We use the term Lagrangian brane to refer to an object of the Fukaya category, and brane to refer to an object of its triangulated envelope $F(T^*X)$.

Before continuing, it is worth remarking about the status of units for Lagrangian branes, and thus the precise relation between the Fukaya category and its triangulated envelope $F(T^*X)$. Thanks to standard arguments, all of the Lagrangian branes considered in this paper (in particular, all those arising from sheaves via microlocalization as explained in Section 3.5) are cohomologically unital. Furthermore, we expect every Lagrangian brane to be cohomologically unital, though we have not attempted to show this. Rather, our definition of the triangulated envelope $F(T^*X)$ as a category of modules automatically provides (strict) units. The Yoneda embedding from the Fukaya category to $F(T^*X)$ is cohomologically fully faithful on cohomologically unital Lagrangian branes. Likewise, since $F(T^*X)$ is unital, the Yoneda embedding from $F(T^*X)$ to (left or right) modules over $F(T^*X)$ is cohomologically fully faithful. This does not rule out the possibility of exotic Lagrangian branes that are for instance orthogonal to all other branes including themselves. While potentially interesting, exploring such phenomena is beyond the aims of this paper.

It is also worth remarking that since $Sh_\omega(X)$ is split-closed, as a consequence of our main result, it follows that $F(T^*X)$ is split-closed as well.

### 3.4. Duality and time reversal.

#### 3.4.1. Duality. We introduce here the duality on branes that corresponds to Verdier duality on sheaves. In Section 5.4, as a consequence of our main result, we will confirm this compatibility.

Consider the antipodal anti-symplectomorphism
$$a : T^*X \to T^*X \quad a(x, \xi) = (x, -\xi).$$
It induces a duality equivalence
$$\alpha_X : F(T^*X)^\circ \sim F(T^*X).$$
On Lagrangian branes, $\alpha_X$ is given by the map
$$(L, \mathcal{E}, \bar{a}, \bar{b}) \mapsto (a(L), a^*(\mathcal{E}^\vee) \otimes or_X, - \dim X - a^*(\bar{a}), a^*(\bar{b})).$$
Here $or_X$ denotes the pullback to $T*X$ of the orientation local system of $X$. Note as well that given a taming perturbation $\psi$ for $L$, one can take the composition $\psi \circ a$ as a taming perturbation for $a(L)$. On morphisms and pseudoholomorphic disks, $\alpha_X$ is given by transport of structure via the antipodal map $a$.

#### 3.4.2. Time reversal. Although our proofs will not require the material of this section, we include it as a prelude to the informal discussion of Section 5.4.

Let $(T^*X)^{-}$ denote the cotangent bundle with its opposite symplectic structure. So except for the symplectic form being negated, no other aspect of the geometry is changed. In particular, we continue to work with a Riemannian metric on $X$ for which the notion of normalized geodesic flow is unchanged.
We can repeat the construction of \( F(T^*X) \) word for word in order to construct \( F((T^*X)^-) \). So when perturbing branes, we continue to work with Hamiltonian functions \( H : (T^*X)^- \to \mathbb{R} \) which are controlled in the sense that there is a real number \( r > 0 \) such that in the region \(|\xi| > r\) we have \( H(x, \xi) = |\xi| \). Here the corresponding Hamiltonian isotopy \( \varphi_{H,t} : (T^*X)^- \to (T^*X)^- \) equals normalized anti-geodesic flow \( \gamma_{-t} \) in the region \(|\xi| > r\) because we are dealing with the opposite symplectic structure.

To calculate the \( A_c \)-structure among an ordered collection of Lagrangians branes \( L_0, \ldots, L_d \subset (T^*X)^- \), we continue to repeat our previous definition and choose controlled Hamiltonian functions \( H_0, \ldots, H_d \), and a fringed set \( R \subset \mathbb{R}^{d+1} \) such that for \((\delta_d, \ldots, \delta_0) \in R\), there is a real number \( r > 0 \) such that for any \( i \neq j \), we have

\[
\varphi_{H_i, \delta_i}(L_i) \cap \varphi_{H_j, \delta_j}(L_j) \text{ lies in the region } |\xi| < r.
\]

By a further compactly supported Hamiltonian perturbation, we may also arrange so that the intersections are transverse.

Similarly, the rest of the definition of \( F((T^*X)^-) \) continues to follow that of \( F(T^*X) \) word for word.

In case of confusion concerning the above perturbation procedure near infinity, it is useful to return to the mnemonic that morphisms propagate forward in time. Thus in word by word.

In the case of \( F(T^*X) \), to calculate the morphisms \( \text{hom}_{F(T^*X)}(L_0, L_1) \), we have required that \( L_0, L_1 \) are perturbed near infinity by normalized geodesic flow so that \( L_1 \) is further in the future than \( L_0 \). In the case of \( F((T^*X)^-) \), we think of the opposite symplectic structure as reversing the timeline. To calculate the morphisms \( \text{hom}_{F((T^*X)^-)}(L_0, L_1) \), we again perturb \( L_0, L_1 \) near infinity so that \( L_1 \) is further in the future than \( L_0 \). But now this implies that near infinity we must perturb \( L_1 \) by normalized anti-geodesic flow a greater amount than we perturb \( L_0 \). Since we use the opposite symplectic structure here, this implies that we continue to use controlled Hamiltonian functions.

Finally, there is a time reversal equivalence

\[
\rho_X : F(T^*X)^\circ \xrightarrow{\sim} F((T^*X)^-).
\]

On Lagrangian branes, \( \rho_X \) is given by the map

\[
(L, \mathcal{E}, \tilde{\alpha}, b) \mapsto (L, \mathcal{E}^\vee, -\tilde{\alpha}, b).
\]

On morphisms and pseudoholomorphic disks, \( \rho_X \) is induced by the identity map.

### 3.5. Microlocalization

We review here the microlocalization quasi-embedding

\[
\mu_X : Sh_c(X) \xrightarrow{\sim} F(T^*X)
\]

constructed in \([28]\). Some useful notation: for a function \( m : X \to \mathbb{R} \) and number \( r \in \mathbb{R} \), we write \( X_{m=r} \) for the subset \( \{x \in X | m(x) = r\} \) and similarly for inequalities.

Let \( i : U \hookrightarrow X \) be an open submanifold that is a \( \mathcal{C} \)-subset of \( X \). Since the complement \( X \setminus U \) is a closed \( \mathcal{C} \)-subset of \( X \), we can find a non-negative function \( m : X \to \mathbb{R}_{\geq 0} \) such that \( X \setminus U \) is precisely the zero-set of \( m \). Since the complement of the critical values of \( m \) form an open \( \mathcal{C} \)-subset of \( \mathbb{R} \), the subset \( X_{m=\eta} \) is an open submanifold with smooth hypersurface boundary \( X_{m=\eta} \), for any sufficiently small \( \eta > 0 \).
Now let \( i_\alpha : U_\alpha \hookrightarrow X \), for \( \alpha = 0, \ldots, d \), be a finite collection of open submanifolds that are \( C \)-subsets of \( X \). Fix non-negative function \( m_\alpha : X \to \mathbb{R}_{\geq 0} \), for \( \alpha = 0, \ldots, d \), such that \( X \setminus U_\alpha \) is precisely the zero-set of \( m_\alpha \). There is a fringed set \( R \subset \mathbb{R}^{d+1} \) such that for any \((\eta_d, \ldots, \eta_0) \in R\), the following holds. First the hypersurfaces \( X_{m_\alpha=\eta_\alpha} \) are all transverse. Second, for \( \alpha < \beta \), there is a quasi-isomorphism of complexes

\[
\text{hom}_{\text{Sh}_c(X)}(i_{\alpha*}C_{U_\alpha}, i_{\beta*}C_{U_\beta}) \simeq (\Omega(X_{m_\alpha \geq \eta_\alpha} \cap X_{m_\beta > \eta_\beta}, X_{m_\alpha = \eta_\alpha} \cap X_{m_\beta > \eta_\beta}, d)
\]

where \( (\Omega, d) \) denotes the relative de Rham complex which calculates the cohomology of the pair. Furthermore, the composition of morphisms in \( \text{Sh}_c(X) \) corresponds to the wedge product of forms.

Next let \( f_\alpha : X_{m_\alpha > \eta_\alpha} \to \mathbb{R} \), for \( \alpha = 0, \ldots, d \), be the logarithm \( f_\alpha = \log m_\alpha \). While choosing the sequence of parameters \((\eta_d, \ldots, \eta_0)\), we can also choose a sequence of small positive parameters \((\epsilon_d, \ldots, \epsilon_0)\) such that the following holds. For any \( \alpha < \beta \), consider the open submanifold \( X_{m_\alpha > \eta_\alpha, m_\beta > \eta_\beta} = X_{m_\alpha > \eta_\alpha} \cap X_{m_\beta > \eta_\beta} \) with corners equipped with the function \( f_{\alpha, \beta} = \epsilon_\beta f_\beta - \epsilon_\alpha f_\alpha \). Then there is an open set of Riemannian metrics on \( X \) such that for all \( \alpha < \beta \), it makes sense to consider the Morse complex \( \mathcal{M}(X_{m_\alpha > \eta_\alpha, m_\beta > \eta_\beta}; f_{\alpha, \beta}) \), and there is a quasi-isomorphism

\[
(\Omega(X_{m_\alpha \geq \eta_\alpha} \cap X_{m_\beta > \eta_\beta}, X_{m_\alpha = \eta_\alpha} \cap X_{m_\beta > \eta_\beta}, d) \simeq \mathcal{M}(X_{m_\alpha > \eta_\alpha, m_\beta > \eta_\beta}; f_{\alpha, \beta})
\]

Furthermore, following arguments of \([12], [23]\), homological perturbation theory provides a quasi-equivalence between the \( A_\infty \)-composition structure on the collection of Morse complexes and the dg structure given by the wedge product of forms.

Finally, we define the microlocalization quasi-embedding

\[
\mu_X : \text{Sh}_c(X) \to F(T^*X)
\]

as follows. Recall by Lemma 2.3.3 that the standard objects \( i_*C_U \) associated to open submanifolds \( i : U \hookrightarrow X \) generate the constructible dg derived category \( \text{Sh}_c(X) \). Thus to construct \( \mu_X \), it suffices to find a parallel collection of standard objects of \( F(T^*X) \).

Given an open submanifold \( i : U \hookrightarrow X \) and function \( m : X \to \mathbb{R}_{\geq 0} \) with zero-set the complement \( X \setminus U \), define the standard Lagrangian \( L_{U; f_*} \to T^*X|_U \) to be the graph

\[
L_{U; f_*} = \Gamma_{d|f},
\]

where \( df \) denotes the differential of the logarithm \( f = \log m \).

The standard Lagrangian \( L_{U; f_*} \) comes equipped with a canonical brane structure \((E, \tilde{a}, b)\) and taming perturbation \( \psi \). Its flat vector bundle \( E \) is trivial, and its grading \( \tilde{a} \) and relative pin structure \( b \) are the canonical structures on a graph. Its taming perturbation \( \psi \) is given by the family of standard Lagrangians

\[
L_{X_{m=\eta}, f_{\eta_*}} = \Gamma_{d\eta}, \quad \text{for sufficiently small } \eta > 0,
\]

where \( f_\eta = \log m_\eta \) is the logarithm of the shifted function \( m_\eta = m - \eta \).

Now one can extend the fundamental result of Fukaya-Oh \([5]\) identifying Morse moduli spaces and Fukaya moduli spaces to the current setting. Namely, one can show that for any finite ordered collection of open submanifolds \( i_\alpha : U_\alpha \hookrightarrow X \), for \( \alpha = 0, \ldots, d \), and any finite collection of \( A_\infty \)-compositions respecting the order, there is a fringed set \( R \subset \mathbb{R}^{d+1} \) such that for any parameters \((\eta_d, \ldots, \eta_0) \in R \), the Morse moduli spaces of
the ordered collection of functions $f_{\eta}$ are isomorphic to the Fukaya moduli spaces of the ordered collection of standard branes $L_{X,m=\eta,f_{\eta}}$ (after further variable dilations of the functions and branes).

Thus we can define a quasi-embedding $\mu_X$ so that on objects we have

$$\mu_X(i_*C_U) = L_{U,f_*}$$

where $f = \log m$ for any choice of non-negative function $m : X \to \mathbb{R}_{\geq 0}$ such that the complement $X \setminus U$ is the zero-set of $m$. In particular, the standard branes for different choices of $m$ are all isomorphic, and so we will choose one and simply denote it by $L_{U_*}$.

In what follows, it will also be useful to recall the result of [28] describing where $\mu_X$ takes other standard objects.

Consider the standard sheaf $i_*C_Y$ associated to an arbitrary submanifold $i : Y \hookrightarrow X$. Given a non-negative function $m : X \to \mathbb{R}_{\geq 0}$ with zero-set the boundary $\partial Y = \overline{Y} \setminus Y$, define the standard Lagrangian $L_{Y,f_*} : T^*X|_Y$ to be the fiberwise sum

$$L_{Y,f_*} = T^*_YX + \Gamma_{df_*}$$

where $T^*_YX \hookrightarrow T^*X$ denotes the conormal bundle to $Y$, and $\Gamma_{df_*} \hookrightarrow T^*X|_Y$ the graph of the differential of the logarithm $f = \log m$. By construction, $L_{Y,f_*}$ depends only on the restriction of $m$ to $Y$.

The standard Lagrangian $L_{Y,f_*}$ comes equipped with a canonical brane structure $(\mathcal{E}, \bar{\alpha}, b)$ and taming perturbation $\psi$. Its flat vector bundle $\mathcal{E}$ is the pullback of the normal orientation bundle $\pi^*(or_X \otimes or_Y^{-1})$, where $or_X, or_Y$ denote the orientation bundles of $X, Y$ respectively. Its grading $\bar{\alpha}$ is characterized by the following property. Suppose we perturb $L_{Y,f_*}$ so that it becomes a graph over an open set. Then if the perturbation is in the direction of anti-geodesic flow near infinity, the transported grading coincides with the canonical grading carried by the graph. If the perturbation is in the direction of geodesic flow near infinity, then the transported grading is equal to the shift by $\text{codim} Y$ of the canonical grading on the graph. Under either such perturbation, its relative pin structure $b$ coincides with the canonical such structure on the graph. Finally, its taming perturbation $\psi$ is given by the family of standard Lagrangians

$$L_{Y,m=\eta,f_{\eta}*} = \Gamma_{df_{\eta}},$$

where $f_{\eta} = \log m_{\eta}$ is the logarithm of the shifted function $m_{\eta} = m - \eta$.

Then by [28], the microlocalization $\mu_X(i_*C_Y)$ is isomorphic to the standard brane $L_{Y,f_*}$. In particular, the standard branes $L_{Y,f_*}$ for different choices of $f$ are all isomorphic, and so we will choose one and simply denote it by $L_{Y_*}$.

In the next section, it will be helpful to have in mind the following aspect of the construction of $\mu_X$. Our perturbation conventions allow for all of the standard branes to be perturbed near infinity in the direction of normalized geodesic flow. In particular, this implies that the perturbations contract the boundaries of standard branes towards the interior of the corresponding submanifolds. Thus all calculations can be understood in terms of submanifolds with smooth, transversely intersecting boundaries.
3.6. Costandard branes. The material of this section is needed in Section 5.1 to confirm that the microlocalization $\mu_X$ intertwines Verdier duality $D_X$ and the brane duality $\alpha_X$. It does not play a role in the proof in Section 4 that $\mu_X$ is a quasi-equivalence.

Let $i : U \hookrightarrow X$ be an open submanifold, and $i_! \omega_U$ be the corresponding costandard object. Let us first study the left Yoneda $\mathcal{Sh}(X)$-module $\mathcal{Y}(i_! \omega_U)$ applied to a finite collection of standard objects associated to open submanifolds.

For $\alpha = 0, \ldots, d$, consider an open submanifold $i_\alpha : U_\alpha \hookrightarrow X$, and the corresponding standard object $i_\alpha^! \mathcal{C}_{U_\alpha}$. Consider the problem of calculating the directed dg structure among the ordered collection

$$i_\alpha^! \omega_U, i_{0,0}^! \mathcal{C}_{U_0}, \ldots, i_{d,d}^! \mathcal{C}_{U_d}.$$  

For $\alpha = 0, \ldots, d$, fix a non-negative function $m_\alpha : X \to \mathbb{R}_{\geq 0}$ such that the complement $X \setminus U_\alpha$ is the zero-set of $m_\alpha$. Then the techniques of [28] allow us to fix $i_\alpha^! \omega_U$ but simplify the other objects. To be precise, there is a fringed set $R \subset \mathbb{R}^{d+1}$ such that for any $(\eta_0, \ldots, \eta_d) \in R$, the above directed dg structure is quasi-equivalent to that of the ordered collection

$$i_\alpha^! \omega_U, i_{0,\eta_0}^! \mathcal{C}_{X_{m_0 > 0}}, \ldots, i_{d,\eta_d}^! \mathcal{C}_{X_{m_d > 0}}$$

where $i_{\alpha,\eta_\alpha} : X_{m_\alpha > \eta_\alpha} \hookrightarrow X$ denotes the inclusion.

Furthermore, another application of the techniques of [28], along with the adjunction identity

$$\text{hom}_{\mathcal{Sh}(X)}(i_\alpha^! \omega_U, i_{\alpha,\eta_\alpha}^! \mathcal{C}_{X_{m_\alpha > \eta_\alpha}}) \simeq \text{hom}_{\mathcal{Sh}(X)}(\omega_U, i_! i_{\alpha,\eta_\alpha}^! \mathcal{C}_{X_{m_\alpha > \eta_\alpha}}),$$

shows that for sufficiently small $\eta > 0$, we can replace the above ordered collection by the ordered collection

$$i_{\eta}^! \omega_{X_{m > \eta}}, i_{0,\eta_0}^! \mathcal{C}_{X_{m_0 > 0}}, \ldots, i_{d,\eta_d}^! \mathcal{C}_{X_{m_d > 0}},$$

where as above $i_{\eta} : X_{m > \eta} \hookrightarrow X$ denotes the inclusion.

Finally, we can calculate the above morphism complexes via de Rham complexes. Namely, there are quasi-isomorphisms of complexes

$$\text{hom}_{\mathcal{Sh}(X)}(i_\alpha^! \omega_U, i_{\alpha}^! \mathcal{C}_{U_\alpha}) \simeq (\Omega(X_{m > \eta} \cap X_{m_\alpha > \eta_\alpha} ; \omega_X^\vee), d),$$

and for $\alpha < \beta$, there are quasi-isomorphisms of complexes

$$\text{hom}_{\mathcal{Sh}(X)}(i_{\alpha}^! \mathcal{C}_{U_\alpha}, i_{\beta}^! \mathcal{C}_{U_\beta}) \simeq (\Omega(X_{m_\alpha \geq \eta_\alpha} \cap X_{m_\beta > \eta_\beta}, X_{m_\alpha = \eta_\alpha} \cap X_{m_\beta > \eta_\beta}), d).$$

Furthermore, the composition of morphisms corresponds to the wedge product of forms.

Now in parallel with standard branes, we define costandard branes as follows. For simplicity, and since it suffices for our later needs, we restrict to the case of open submanifolds.

Given an open submanifold $i : U \hookrightarrow X$ and function $m : X \to \mathbb{R}_{\geq 0}$ with zero-set the complement $X \setminus U$, define the costandard Lagrangian $L_{U,f} : [-df, -\Gamma_{df}] \to T^* X|_U$ to be the graph

$$L_{U,f} = -\Gamma_{df},$$

where $df$ denotes the differential of the logarithm $f = \log m$.  

The costandard Lagrangian $L_{U,f!}$ comes equipped with a canonical brane structure $(\mathcal{E}, \tilde{\alpha}, b)$ and taming perturbation $\psi$. Its flat vector bundle $\mathcal{E}$ is the pullback of the orientation bundle $\pi^*(or_X)$, its grading $\tilde{\alpha}$ is the shift by $\dim X$ of the canonical grading on a graph, and its relative pin structure $b$ is the canonical structure on a graph. Finally, its taming perturbation $\psi$ is given by the family of costandard Lagrangians

$$L_{X,m=\eta,f!} = -\Gamma_{df_\eta}, \quad \text{for sufficiently small } \eta > 0,$$

where $f_\eta = \log m_\eta$ is the logarithm of the shifted function $m_\eta = m - \eta$.

Alternatively, we could use the brane duality $\alpha_X$, and take as definition the motivating identity

$$L_{U,f!} \simeq \alpha_X(L_{U,f*}).$$

In particular, the costandard branes $L_{U,f!}$ for different choices of $f$ are all isomorphic, and so we will choose one and denote it by $L_{U!}$.

The construction of the microlocalization $\mu_X$ favors standard objects over costandard objects. For example, without further arguments, it does not follow immediately that $\mu_X$ takes the costandard sheaf $i_!\omega_U$ to a costandard brane $L_{U!}$. Equivalently, without further arguments, it does not immediately follow that $\mu_X$ intertwines Verdier duality $\mathcal{D}_X$ with brane duality $\alpha_X$. To eventually see this (cf. Proposition 5.1.1), we will use the following partial identification of costandard branes.

Consider the microlocalization $\mu_X$, and the resulting pullback functor on left modules

$$\mu_X^* : \text{mod}_f(F(T^*X)) \rightarrow \text{mod}_f(Sh_c(X))$$

$$\mu_X^*(\mathcal{M}) = \mathcal{M} \circ \mu_X.$$ 

In particular, for an object $L$ of $F(T^*X)$, composing $\mu_X^*$ with the Yoneda embedding $\mathcal{Y}_L$ for left modules provides a left $Sh_c(X)$-module

$$\mu_X^*(\mathcal{Y}_L(F)) = \text{hom}_{F(T^*X)}(L, \mu_X(F)).$$

**Proposition 3.6.1.** For any open submanifold $i : U \hookrightarrow X$, there is a quasi-isomorphism of left $Sh_c(X)$-modules

$$\mu_X^*(\mathcal{Y}_L(U!)) \simeq \mathcal{Y}_L(i_!\omega_U) : Sh_c(X) \rightarrow \text{Ch}.$$ 

**Proof.** Let us understand the left $Sh_c(X)$-module $\mu_X^*(\mathcal{Y}_L(U!))$ applied to a finite collection of standard objects associated to open submanifolds.

Fix a representative $L_{U,f!}$ where as usual $f = \log m$ for a non-negative function $m : X \rightarrow \mathbb{R}_{\geq 0}$ such that the complement $X \setminus U$ is the zero-set of $m$. For $\alpha = 0, \ldots, d$, consider an open submanifold $i_\alpha : U_\alpha \hookrightarrow X$, a non-negative function $m_\alpha : X \rightarrow \mathbb{R}_{\geq 0}$ such that the complement $X \setminus U_\alpha$ is the zero-set of $m_\alpha$, and the corresponding standard brane $L_{U_\alpha,f_\alpha*}$, where as usual $f_\alpha = \log m_\alpha$.

Consider the problem of calculating the directed $A_\infty$-composition maps among the ordered collection

$$L_{U,f!}, L_{U_0,f_0*}, \ldots, L_{U_d,f_d*}.$$ 

Then our perturbation conventions allow us to fix $L_{U,f!}$ but perturb the other branes. To be precise, there is a fringed set $R \subset \mathbb{R}^{d+1}$ such that for any $(\eta_0, \ldots, \eta_0) \in R$, we
must calculate the directed $A_\infty$-composition maps among the ordered collection

$$L_{U,f_1}, L_{X_{m_0 > \eta_0}, f_0, \eta^*}, \ldots, L_{X_{m_d > \eta_d}, f_d, \eta^*}.$$  

Here as earlier the underlying Lagrangian of the standard brane $L_{X_{m>\eta}, f_0, \eta^*}$ is the graph of the differential of $f_\alpha = \log(m_\alpha - \eta_\alpha)$ over the open set $X_{m_\alpha > \eta_\alpha}$.

By definition, the taming perturbation of the costandard brane $L_{U,f}$ moves it to the costandard brane $L_{X_{m>\eta}, f_0^!}$, for sufficiently small $\eta > 0$. Here as above the underlying Lagrangian of the costandard brane $L_{X_{m>\eta}, f_0^!}$ is the negative of the graph of the differential of $f_\eta = \log(m - \eta)$ over the open set $X_{m>\eta}$. Thus we are left to calculate the directed $A_\infty$-composition maps among the ordered collection

$$L_{X_{m>\eta}, f_0^!}, L_{X_{m_0 > \eta_0}, f_0, \eta^*}, \ldots, L_{X_{m_d > \eta_d}, f_d, \eta^*}.$$  

The techniques of [28] extend directly to this situation: the relevant Fukaya moduli spaces can be identified with the corresponding Morse moduli spaces. In turn, following arguments of [12], [23], homological perturbation theory provides a quasi-equivalence between the Morse $A_\infty$-composition structure and the dg structure given by the wedge product of forms. Finally, as discussed above, the dg structure on differential forms calculates the dg structure on the corresponding constructible sheaves. □

3.7. Non-characteristic isotopies. We discuss here the invariance of calculations among microlocal branes under a very specific class of non-characteristic Hamiltonian isotopies. What we explain is the minimum technical result needed to establish our main theorem. Further generalizations are discussed in the Appendix.

3.7.1. Motivation: sheaf calculations. This section is intended as motivation for the Floer calculations to follow, but is not logically needed in what follows.

Let’s consider a family of stratifications of $X$ parametrized by the real line $\mathbb{R}$. More precisely, by a one-parameter family of stratifications of $X$, we mean a single Whitney stratification $\mathcal{G} = \{\mathcal{G}_\beta\}$ of $\mathbb{R} \times X$ satisfying the following:

1. The restrictions of the projection $p_\mathbb{R} : \mathbb{R} \times X \to \mathbb{R}$ to each stratum $\mathcal{G}_\beta$ of $\mathcal{G}$ is nonsingular.

2. There is a compact interval $[a, b] \subset \mathbb{R}$ such that the induced stratification of $p_\mathbb{R}^{-1}(\mathbb{R} \setminus [a, b])$ obtained by restricting $\mathcal{G}$ is locally constant.

For each $s \in \mathbb{R}$, we will denote by $\mathcal{G}(s) = p_\mathbb{R}^{-1}(s) \cap \mathcal{G}$ the fiber of $\mathcal{G}$. Note that condition (1) above implies that the topological type of the stratification $\mathcal{G}(s)$ is constant with respect to $s \in \mathbb{R}$. More precisely, by the Thom Isotopy Lemma, one can construct a homeomorphism $\psi : \mathbb{R} \times X \to \mathbb{R} \times X$ such that $p_\mathbb{R} \circ \psi = p_\mathbb{R}$, $\psi(\mathcal{G}) = \mathbb{R} \times \mathcal{G}(0)$.

Suppose two one-parameter family of stratifications $\mathcal{G} = \{\mathcal{G}_\beta\}$, $\mathcal{G}' = \{\mathcal{G}'_\alpha\}$ of $X$ are transverse. Note that this is equivalent to the fibers $\mathcal{G}(s)$ and $\mathcal{G}'(s)$ being transverse for all $s \in \mathbb{R}$. Then the Whitney stratification $\mathcal{G} \cap \mathcal{G}'$ of $\mathbb{R} \times X$ with strata the intersections of strata $\{\mathcal{G}_\beta \cap \mathcal{G}'_\alpha\}$ is again a one-parameter family of stratifications of $X$.

In particular, fix a Whitney stratification $\mathcal{G} = \{\mathcal{G}_\alpha\}$ of $X$, and let $\mathcal{S}_\mathbb{R} = \{\mathbb{R} \times \mathcal{G}_\alpha\}$ be the constant one-parameter family of stratifications of $\mathbb{R} \times X$. We will say that $\mathcal{G}$ is $\mathcal{S}$-non-characteristic if $\mathcal{G}$ is transverse to $\mathcal{S}_\mathbb{R}$.
Now consider the $\mathcal{S}$-constructible dg derived category $\text{Sh}_S(X)$. Consider as well any object $\mathcal{F}$ of the $\mathcal{S}$-constructible dg derived category $\text{Sh}_S(\mathbb{R} \times X)$, and denote by $\mathcal{F}_s$ its restriction to the fiber $X = p_\mathbb{R}^{-1}(s)$. In general, $\mathcal{F}_s$ is not an object of $\text{Sh}_S(X)$, but can be paired with objects of $\text{Sh}_S(X)$ in the ambient constructible dg derived category $\text{Sh}_c(X)$.

**Lemma 3.7.1.** Suppose $\mathcal{S}$ is an $\mathcal{S}$-non-characteristic one-parameter family of stratifications of $X$. Then for any object $\mathcal{F}$ of $\text{Sh}_S(\mathbb{R} \times X)$, and any test object $\mathcal{P}$ of $\text{Sh}_S(X)$, there are functorial quasi-isomorphisms among the complexes

$$\text{hom}_{\text{Sh}_c(X)}(\mathcal{P}, \mathcal{F}_s), \quad \text{for all } s \in \mathbb{R}.$$ 

**Proof.** Consider the object of $\text{Sh}_c(\mathbb{R})$ given by the pushforward

$$p_\mathbb{R}! \text{Hom}_{\text{Sh}_c(\mathbb{R} \times X)}(\mathbb{C}_\mathbb{R} \otimes \mathcal{P}, \mathcal{F}).$$

By base change, its stalk at $s \in \mathbb{R}$ is the complex $\text{hom}_{\text{Sh}_c(X)}(\mathcal{P}, \mathcal{F}_s)$. As mentioned above, the Thom Isotopy Lemma provides a stratum-preserving homeomorphism

$$\psi : \mathbb{R} \times X \to \mathbb{R} \times X \quad \psi(\mathcal{S} \cap \mathcal{S}_{\mathbb{R}}) = \mathbb{R} \times (\mathcal{S}(0) \cap \mathcal{S})$$

such that $p_\mathbb{R} \circ \psi = p_\mathbb{R}$. Thus the projection via $p_\mathbb{R}! \simeq p_\mathbb{R}! \circ \psi!$ of any object of $\text{Sh}_{\mathcal{S} \cap \mathcal{S}_{\mathbb{R}}}(\mathbb{R} \times X)$ is constant. □

Our aim in the next two sections is to produce an analogue of Lemma 3.7.1 with sheaves on $X$ replaced by branes in $T^*X$. In particular, we will be interested in a brane version of the following specific example.

**Example 3.7.2.** Let $\mathcal{S}$ be a one-parameter family of stratifications of $X$. Suppose further that $\mathcal{S}$ consists of three strata: a smooth submanifold $i : \mathcal{Y} \hookrightarrow \mathbb{R} \times X$, its smooth boundary $\partial \mathcal{Y} = \overline{\mathcal{Y}} \setminus \mathcal{Y} \hookrightarrow \mathbb{R} \times X$, and their complement $(\mathbb{R} \times X) \setminus \overline{\mathcal{Y}}$. For each $s \in \mathbb{R}$, we will denote by

$$i_s : \mathcal{Y}_s = p_\mathbb{R}^{-1}(s) \cap \mathcal{Y} \hookrightarrow X$$

the fiber of $\mathcal{Y}$.

For a given stratification $\mathcal{S}$ of $X$, we will say that $\mathcal{Y}$ is an $\mathcal{S}$-non-characteristic one-parameter family of submanifolds with smooth boundaries if the corresponding three stratum stratification $\mathcal{S}$ is $\mathcal{S}$-non-characteristic.

If $\mathcal{Y}$ is an $\mathcal{S}$-non-characteristic one-parameter family of submanifolds with smooth boundaries, then for any test object $\mathcal{P}$ of $\text{Sh}_S(X)$, there are functorial quasi-isomorphisms among the complexes

$$\text{hom}_{\text{Sh}_c(X)}(\mathcal{P}, i_{s*} \mathcal{C}_{\mathcal{Y}_s}), \quad \text{for all } s \in \mathbb{R}.$$ 

3.7.2. **Continuation of Floer calculations.** Let $L \hookrightarrow T^*X$ be a tame exact Lagrangian brane. Consider a time-dependent Hamiltonian function $H_s : X \times \mathbb{R} \to \mathbb{R}$ such that its differential $dH_s$ is compactly supported in $X$ and $\mathbb{R}$. Let $\varphi_s : T^*X \to T^*X$ be the associated Hamiltonian flow. Acting on the initial brane $L$, we obtain a family of tame exact Lagrangian branes $\mathcal{L}_s = \varphi_s(L)$ satisfying the following:

1. $\mathcal{L}_0 = L$. 

Near infinity in $\mathbb{R}$, the family $L_s$ is locally constant: there is a compact interval $[a, b] \rightarrow \mathbb{R}$ such that for $s \in \mathbb{R} \setminus [a, b]$, the family $L_s$ is locally constant.

Near infinity in $T^*X$, the family $L_s$ is constant: there exists $r > 0$ such that

$$\{||\xi|| > r\} \cap L_s = \{||\xi|| > r\} \cap L_0, \quad \text{for all } s \in \mathbb{R}.$$ 

The family $L_s$ is the most general possible compactly supported motion of the exact brane $L$. We will consider more general families in the next section.

The following is a straightforward generalization of by now standard techniques in Floer theory. Namely, once we confirm the necessary a priori estimates on the possible diameters of pseudoholomorphic disks, continuation maps provide the sought after natural transformations.

**Proposition 3.7.3.** For any test $P$ of $F(T^*X)$, there are functorial quasi-isomorphisms among the Floer complexes

$$\text{hom}_{F(T^*X)}(P, L_s), \quad \text{for all } s \in \mathbb{R}.$$ 

**Proof.** Fix parameters $a \ll 0 \ll b \in \mathbb{R}$. Following Seidel [33, Section 10c], given test objects $P_1, \ldots, P_d$ of $F(T^*X)$, we would like to define maps

$$T_d : \text{hom}_{F(T^*X)}(P_1, L_a) \otimes \left( \bigotimes_{k=1}^{d-1} \text{hom}_{F(T^*X)}(P_k+1, P_k) \right) \rightarrow \text{hom}_{F(T^*X)}(P_d, L_b)[1 - d]$$

assembling into an $A_\infty$-transformation of right Yoneda modules. For compact branes, one immediately has the sought after maps: a signed count of pseudoholomorphic disks with moving boundary conditions given by the family $L_s$, for $s \in \mathbb{R}$, and static conditions given by the test branes $P_1, \ldots, P_d$ provides the structure constants of the maps. Furthermore, the transformations satisfy a compatibility with respect to the concatenation of families. In particular, the transformations are quasi-isomorphisms since the constant family gives the identity functor.

In our current setting, to implement this approach, we need to be careful to make sure the relevant moduli spaces remain compact. This is the content of the rest of the proof of the proposition.

First, following [33], it is convenient to recast the moduli problem of pseudoholomorphic polygons with moving boundary conditions in terms of pseudoholomorphic sections of varying almost complex targets over holomorphic polygons.

Given a non-negative integer $d$, consider the manifold with boundary $D_d \hookrightarrow \mathbb{C}$ obtained from the closed unit disk $D \hookrightarrow \mathbb{C}$ by removing the $(d + 1)$st roots of unity from the boundary $\partial D \hookrightarrow \mathbb{C}$. Thus the boundary $\partial D_d \hookrightarrow D_d$ is a disjoint union of $d + 1$ cyclically ordered open intervals $I_k$ indexed by $k = 0, 1, \ldots, d$. We equip the interior $D^*_d = D_d \setminus \partial D_d$ with the standard symplectic structure $\omega_d$ obtained by restricting the standard exact symplectic structure of $\mathbb{C}$. (The interaction of $\omega_d$ with the boundary $\partial D_d$ will not play a significant role.) We will consider $D_d$ together with $\omega_d$ as a fixed symplectic manifold with boundary.

Consider the product manifold $D_d \times T^*X$ with the obvious projection onto the first factor $\pi : D_d \times T^*X \rightarrow D_d$. Identify the boundary component $I_0 \hookrightarrow \partial D_d$ with the real
line \( \mathbb{R} \), and consider the embedded submanifold

\[ \mathfrak{L}_{\text{move}} = I_0 \times_{D_d} \mathfrak{L}_s \hookrightarrow D_d \times T^*X. \]

Similarly, given a collection of static test objects \( P_1, \ldots, P_d \hookrightarrow T^*X \), consider the embedded submanifolds

\[ P_{k, \text{stat}} = I_k \times P_k \hookrightarrow D_d \times T^*X, \quad \text{for } k = 1, \ldots, d. \]

Consider a one-form \( \kappa \) on the base \( D_d \) with values in functions on the fiber \( T^*X \). Given a vector field \( v \) on \( D_d \), the evaluation \( \kappa(v) \) provides a family of functions on \( T^*X \) parametrized by \( D_d \). By passing to the corresponding Hamiltonian vector fields, we may think of \( \kappa \) as a section of the vector bundle \( \text{Hom}(TD_d, \pi_* TT^*X) \). In other words, \( \kappa \) provides a connection \( \nabla_\kappa \) on the trivial family of symplectic manifolds defined by \( \pi \).

Now, given the submanifold \( \mathfrak{L}_{\text{move}} \hookrightarrow D_d \times T^*X \), we will choose a compactly supported one-form \( \kappa \) so that \( \mathfrak{L}_{\text{move}} \) is preserved by the parallel transport of the connection \( \nabla_\kappa \). To this end, it is convenient to choose a collared neighborhood \( \mathcal{N}_0 \hookrightarrow D_d \) of the boundary component \( I_0 \hookrightarrow \partial D_d \). Thus we have an identification \( \mathcal{N}_0 \simeq I_0 \times [0,1) \) which we think of as being given by coordinates \((s,t)\). Then it is straightforward to construct a one-form \( \kappa \) as above satisfying the following:

1. There is a compact subset of \( \mathcal{N}_0 \times T^*X \) outside of which \( \kappa \) vanishes.
2. In local coordinates, \( \kappa \) can be expressed as \( f(s,t,x,\xi)ds \), where \((x,\xi)\) are local coordinates on \( T^*X \).
3. Along the boundary component \( I_0 \), the submanifold \( \mathfrak{L}_{\text{move}} \) is preserved by the parallel transport of the connection \( \nabla_\kappa \).

Next, let \( j \) be a compatible complex structure on the symplectic manifold \( D_d \) (for a surface, any correctly oriented complex structure is compatible), and let \( J_{\text{con}} \) be the conical almost complex structure on \( T^*X \) discussed in the previous section. Together with \( \kappa \), the two structures provide an almost complex structure \( J_\kappa \) on the family \( D_d \times T^*X \) such that the projection \( \pi \) is pseudoholomorphic. Namely, one takes

\[ J_\kappa(v,w) = (j(v), J_{\text{con}}(w) + J_{\text{con}}(\sigma_\kappa(j(v))) + \sigma_\kappa(v)) \]

where \( \sigma_\kappa \in \text{Hom}(TD_d, \pi_* TT^*X) \) is the section associated to \( \kappa \). In what follows, we will always consider \( D_d \times T^*X \) equipped with the almost complex structure \( J_\kappa \). Note that while we will think of \( \kappa \) and \( J_{\text{con}} \) as fixed, we will allow the choice of \( j \) to vary.

The moduli problem of \( J_{\text{con}} \)-holomorphic polygons in \( T^*X \) with moving boundary condition \( \mathfrak{L}_s \) and static boundary conditions \( P_1, \ldots, P_d \), coincides with that of \( J_\kappa \)-holomorphic sections of \( \pi \) with boundary conditions \( \mathfrak{L}_{\text{move}}, P_{1,\text{stat}}, \ldots, P_{d,\text{stat}} \). To get bounds for solutions, we introduce a symplectic structure on \( D_d \times T^*X \) as follows. Let \( \theta \) be the canonical one-form on \( T^*X \), and consider the two-form

\[ \omega_\kappa = d\theta + d\kappa + c\omega_d \]

where \( c > 0 \) is some fixed constant. Using the explicit form of \( \kappa \), it is simple to check that we can choose \( c \) large enough so that \( \omega_\kappa \) will be non-degenerate. Furthermore, it is simple to check that the complex structure \( J_\kappa \) is compatible with \( \omega_\kappa \), and the boundary conditions \( \mathfrak{L}_{\text{move}}, P_{1,\text{stat}}, \ldots, P_{d,\text{stat}} \) are Lagrangian.
With the preceding setup in hand, we are now in a context where we can appeal to standard results to verify the proposition. By construction, all of the branes under consideration are tame, and so we have a priori bounds on the diameters of the relevant pseudoholomorphic disks. Thus standard techniques as outlined in [33, Section 10c] provide continuation maps giving a functorial quasi-isomorphism.

3.7.3. Non-characteristic isotopies of branes. In this section, we will consider more general families of exact branes in $T^*X$ parametrized by the real line $\mathbb{R}$.

By a one-parameter family of closed (but not necessarily compact) submanifolds (without boundary) in $T^*X$, we mean a closed submanifold

$$L \hookrightarrow \mathbb{R} \times T^*X$$

satisfying the following:

1. The restriction of the projection $p_\mathbb{R} : \mathbb{R} \times X \to \mathbb{R}$ to the submanifold $L$ is nonsingular.
2. There is a real number $r > 0$, such that the restriction of the product $p_\mathbb{R} \times |\xi| : T^*X \to \mathbb{R} \times (r, \infty)$ to the subset $\{ |\xi| > r \} \cap L$ is proper and nonsingular.
3. There is a compact interval $[a, b] \hookrightarrow \mathbb{R}$ such that the restriction of the projection $p_X : \mathbb{R} \times T^*X \to T^*X$ to the submanifold $p_\mathbb{R}^{-1}([\mathbb{R} \setminus [a, b]] \cap L$ is locally constant.

Note that conditions (1) and (2) will be satisfied if the restriction of the projection $p_\mathbb{R} : \mathbb{R} \times T^*X \to \mathbb{R}$ to the closure $\overline{L} \hookrightarrow T^*X$ is nonsingular as a stratified map, but the weaker condition stated is a useful generalization. It implies in particular that the fibers $L_s = p_\mathbb{R}^{-1}(s) \cap L \hookrightarrow T^*X$ are all diffeomorphic, but imposes no requirement that their boundaries at infinity should all be homeomorphic as well.

By a one-parameter family of tame Lagrangian branes in $T^*X$, we mean a one-parameter family of closed submanifolds $L \hookrightarrow \mathbb{R} \times X$ in the above sense such that the fibers $L_s = p_\mathbb{R}^{-1}(s) \cap L \hookrightarrow T^*X$ also satisfy:

1. The fibers $L_s$ are exact tame Lagrangians with respect to the usual symplectic structure and any almost complex structure conical near infinity.
2. The fibers $L_s$ are equipped with a locally constant brane structure $(E_s, \tilde{\alpha}_s, \flat_s)$ with respect to the usual background classes.

Note that if we assume that $L_0$ is an exact Lagrangian, then $L_s$ being an exact Lagrangian is equivalent to the family $L$ being given by the flow $\varphi_{H_s}$ of the vector field of a time-dependent Hamiltonian $H_s : T^*X \to \mathbb{R}$. Note as well that a brane structure consists of topological data, so can be transported unambiguously along the fibers of such a family.

Remark 3.7.4. In fact, it is possible to prove continuation of Floer homology for even more general families. A motivation for this level of generality is that it allows one to check that all taming perturbations for a brane lead to equivalent calculations. See the Appendix for a discussion in this direction.

Fix a conical Lagrangian $\Lambda \subset T^*X$, and let $\Lambda^\infty = \overline{\Lambda} \cap T^\infty X$ be its boundary at infinity. Let $F(T^*X)_\Lambda$ be the full subcategory of $F(T^*X)$ generated by Lagrangian branes $L$ whose boundary at infinity $L^\infty = \overline{L} \cap T^\infty X$ lies in $\Lambda^\infty$. 
Suppose \( \mathcal{L} \hookrightarrow \mathbb{R} \times T^*X \) is a one-parameter family of tame Lagrangian branes. We will say that \( \mathcal{L} \) is \( \Lambda \)-non-characteristic if

\[
\mathfrak{T}_s \cap \Lambda^\infty = \emptyset, \quad \text{for all } s \in \mathbb{R}.
\]

**Proposition 3.7.5.** Suppose \( \mathcal{L} \hookrightarrow \mathbb{R} \times T^*X \) is a \( \Lambda \)-non-characteristic one-parameter family of tame Lagrangian branes. For any test object \( P \) of \( F_\Lambda(T^*X) \), there are functorial quasi-isomorphisms among the Floer complexes

\[
\text{hom}_{F(T^*X)}(P, L_s), \quad \text{for all } s \in \mathbb{R}.
\]

**Proof.** Our strategy will be to “factor” the family \( \mathcal{L} \) into many small steps which each fall into a broad class of manageable moving boundary conditions. Over each step, we will be able to establish quasi-isomorphisms of right Yoneda modules. To prove the proposition, we will then take compositions of these quasi-isomorphisms.

To begin, since we assume that \( \mathcal{L} \) is locally constant away from a compact interval \( [a, b] \hookrightarrow \mathbb{R} \), it suffices to show that for each fixed parameter \( s_0 \in \mathbb{R} \), there is a neighborhood \( I_{s_0} = [a_0, b_0] \subset \mathbb{R} \) with \( a_0 < s_0 < b_0 \), such that there are functorial quasi-isomorphisms among the Floer complexes

\[
\text{hom}_{F(T^*X)}(P, L_s), \quad \text{for all } s \in I_{s_0}.
\]

Fix a parameter \( s_0 \in \mathbb{R} \). Consider a finite number of \( A_\infty \)-operations among \( L_{s_0} \) and a fixed collection of test objects \( P_1, \ldots, P_d \) of \( F_\Lambda(T^*X) \). Consider the moduli problem of pseudoholomorphic disks that calculate the structure constants of the operations. Recall that we have an a priori diameter bound on such pseudoholomorphic disks: there is a large constant \( r_{s_0} > 0 \) such that none of the relevant disks enter the region of \( T^*X \) given by \( |\xi| > r_{s_0}/2 \). Moreover, by the continuity of the diameter bound, we can find an open interval \( K_{s_0} \subset \mathbb{R} \) containing \( s_0 \) such that the pseudoholomorphic disks that calculate the same structure constants for \( L_s \), for all \( s \in K_{s_0} \), do not enter the region of \( T^*X \) given by \( |\xi| > r_{s_0} \).

Now for fixed constants \( r_2 > r_1 > r_0 > r_{s_0} \), there exists a sufficiently small closed subinterval \( I_{s_0} = [a_0, b_0] \subset K_{s_0} \), with \( a_0 < s_0 < b_0 \), such that we can “factor” the family \( \mathcal{L} \) over the parameters \( I_{s_0} \) into two families of the following form.

First, we can define a family \( \mathcal{L}' \) over \( I_{s_0} \) satisfying the following. Let \( L' \hookrightarrow I_{s_0} \times T^*X \) be the union of the moving brane \( \mathcal{L}' \) and the static branes \( I_{s_0} \times P_1, \ldots, I_{s_0} \times P_d \). Then by construction, we can arrange that

1. \( \mathcal{L}'_{a_0} = \mathcal{L}_{a_0} \);
2. \( \mathcal{L}' \) is constant in the region \( |\xi| < r_1 \);
3. \( \mathcal{L}' \) coincides with \( \mathcal{L} \) in the region \( |\xi| > r_2 \);
4. in the region \( |\xi| > r_0 \), the product function

\[
p \times |\xi| : L' \to \mathbb{R} \times (r_0, \infty)
\]

is nonsingular.

Second, we can define a family \( \mathcal{L}'' \) over \( I_{s_0} \) satisfying the following

1. \( \mathcal{L}_{a_0}'' = \mathcal{L}_{b_0}' \);
2. \( \mathcal{L}'' \) is constant in the region \( |\xi| > r_2 \);
3. \( \mathcal{L}'' \) coincides with \( \mathcal{L} \) in the region \( |\xi| < r_1 \).
(4) $\mathcal{L}''_{b_0} = \mathcal{L}_{b_0}$.

It is worth commenting that our choice to work locally near a fixed parameter $s_0$ is due to the fact that it is easy to find such a factorization locally in $s$.

Now if we can establish individually that the Yoneda modules associated to the fibers of such families $\mathcal{L}', \mathcal{L}''$ are quasi-isomorphic, then it will follow by composition that the Yoneda modules associated to the fibers of the family $\mathcal{L}$ itself over the interval $I_{s_0}$ are quasi-isomorphic.

**Case 1 ($\mathcal{L}'$ constant away from infinity).** The first case is particularly easy in that there is in fact a strict identification of the $A_{\infty}$-operations under consideration. By construction, for the parameter $s_0 \in \mathbb{R}$, we have arranged so that the pseudoholomorphic disks that calculate the corresponding structure constants do not leave the region $|\xi| < r_1$. Furthermore, we have arranged so that for any $s \in I_{s_0}$, the intersection of $\mathcal{L}_s$ and the region $|\xi| < r_1$ is constant, and in particular, the diameter bound constraining the relevant disks holds independently of $s$. Thus the identity map on intersection points gives an identification of the $A_{\infty}$-operations.

**Case 2 ($\mathcal{L}''$ constant near infinity).** With the assumptions of the second case, we must check that a compactly supported family leads to a quasi-isomorphism of Yoneda modules. This was the content of Proposition 3.7.3.

□

A further generalization of Proposition 3.7.5 will be presented in the Appendix. For now, we will simply mention the single application of Proposition 3.7.5 that will be used in what follows.

**Example 3.7.6.** Fix a Whitney stratification $\mathcal{S} = \{S_\alpha\}$ of $X$, and let $\Lambda_{\mathcal{S}} = \bigcup_{\alpha} T^*_{S_\alpha} X \hookrightarrow T^*X$ be the associated conical Lagrangian. Let $\mathcal{Y} \hookrightarrow \mathbb{R} \times X$ be an $\mathcal{S}$-non-characteristic one-parameter family of submanifolds with smooth boundaries in $X$.

Fix a non-negative function $m : \mathbb{R} \times X \to \mathbb{R}$ that vanishes precisely on the boundary $\partial \mathcal{Y} \hookrightarrow \mathbb{R} \times X$, and consider the function $f : \mathcal{Y} \to \mathbb{R}$ given by $f = \log m$. For each $s \in \mathbb{R}$, consider the function $f_s : \mathcal{Y}_s \to \mathbb{R}$ obtained by restricting $f$ to the fiber $\mathcal{Y}_s = p_{\mathbb{R}}^{-1}(s) \hookrightarrow X$.

Define the one-parameter family of closed submanifolds $\mathcal{L}_{\mathcal{Y}, f_s} \hookrightarrow \mathbb{R} \times T^*X$ to be the union of the fiberwise sums

$$\mathcal{L}_{\mathcal{Y}, f_s} = \bigsqcup_{s \in \mathbb{R}} (s, T^*_{\mathcal{Y}_s} X + \Gamma_{df_s}) \hookrightarrow \mathbb{R} \times T^*X$$

where $T^*_{\mathcal{Y}_s} X$ denotes the conormal bundle to $\mathcal{Y}_s \hookrightarrow X$, and $\Gamma_{df_s}$ the graph of the differential of $f_s$ over $\mathcal{Y}_s$.

By construction, $\mathcal{L}_{\mathcal{Y}, f_s}$ is a $\Lambda_{\mathcal{S}}$-non-characteristic one-parameter family of tame Lagrangian branes. Thus by Proposition 3.7.5, for any object $P$ of $F_{\Lambda_{\mathcal{S}}}(T^*X)$, there are functorial quasi-isomorphisms among the Floer complexes

$$\text{hom}_{F(T^*X)}(P, \mathcal{L}_{\mathcal{Y}, f_s}), \quad \text{for all } s \in \mathbb{R}.$$
4. MICROLOCALIZATION IS A QUASI-EQUIVALENCE

In this section, we prove that the constructible dg derived category $Sh_c(X)$ is quasi-equivalent to the derived Fukaya category $F(T^*X)$. We show that every object of $F(T^*X)$ is quasi-isomorphic to the microlocalization of an object of $Sh_c(X)$.

4.1. Statement of results. Consider the microlocalization quasi-embedding

$$\mu_X : Sh_c(X) \hookrightarrow F(T^*X).$$

By definition, the fact that it is a quasi-embedding means that it induces a fully faithful functor on cohomology categories

$$H(\mu_X) : D_c(X) \hookrightarrow DF(T^*X).$$

Thus to show that $\mu_X$ is a quasi-equivalence, we must show that $H(\mu_X)$ is essentially surjective, or in other words, that every object of $DF(T^*X)$ is isomorphic to an object coming from $D_c(X)$.

By construction, the image of $\mu_X$ is generated by the standard branes $L_{Y*} \hookrightarrow T^*X$ associated to submanifolds $Y \hookrightarrow X$. Recall the Yoneda embedding into right modules

$$Y_r : F(T^*X) \rightarrow \text{mod}_r(F(T^*X)) \quad Y_r(L) : L' \mapsto \text{hom}_{F(T^*X)}(L', L).$$

The main technical result of this section is the following.

**Theorem 4.1.1.** Let $L$ be an object of $F(T^*X)$. The Yoneda module $Y_r(L)$ can be expressed as a twisted complex of the Yoneda modules of standard branes $Y_r(L_{Y*})$.

**Remark 4.1.2.** Along the way, we will also directly establish the (weaker) statement that if $\text{hom}_{F(T^*X)}(L_{\{x\}}, L)$ is acyclic for the standard branes $L_{\{x\}} \hookrightarrow T^*X$ associated to all points $x \in X$, then $\text{hom}_{F(T^*X)}(P, L)$ is acyclic for all test objects $P$.

Complexes of the form $\text{hom}_{F(T^*X)}(L_{\{x\}}, L)$ will appear as the coefficients of the modules $Y_r(L_{Y*})$ in the decomposition of the module $Y_r(L)$. See Remark 4.5.1 for a more precise statement on the structure of the coefficients.

Theorem 4.1.1 immediately implies the following.

**Theorem 4.1.3.** Microlocalization is a quasi-equivalence

$$\mu_X : Sh_c(X) \sim F(T^*X)$$

The next four sections are devoted to the proof of Theorem 4.1.1. In the final section, we discuss constructibility properties of the quasi-equivalence.

4.2. Two Floer calculations. Let $X_0, X_1$ be compact real analytic manifolds.

We consider here the product manifold $X_0 \times X_1$, and the Fukaya category of its cotangent bundle $T^*(X_0 \times X_1)$ with respect to the usual background structures (cf. Section 3.2 or [28]). Our aim is to identify the Yoneda modules associated to some simple but important examples of branes on the product.
4.2.1. **Product branes.** Consider the product map on the set of branes

\[ \text{Ob}(F(T^* X_0)) \times \text{Ob}(F(T^* X_1)) \to \text{Ob}(F(T^* X_0 \times T^* X_1)) \]

\[ ((L_0, E_0, \alpha_0, b_0), (L_1, E_1, \alpha_1, b_1)) \mapsto (L_0 \times L_1, E_0 \otimes E_1, \alpha_0 + \alpha_1, b_{0,1}). \]

Here $b_{0,1}$ denotes the canonical induced relative pin structure on $L_0 \times L_1$ with respect to the usual background class

\[ (\pi_0 \times \pi_1)^* w_2(X_0 \times X_1) = \pi_0^* w_2(X_0) + \pi_1^* w_1(X_0) \cdot \pi_1^* w_1(X_1) + \pi_1^* w_2(X_1). \]

To construct $b_{0,1}$, observe that the relative pin structures $b_0, b_1$ with respect to the usual background classes $\pi_0^* w_2(X_0), \pi_1^* w_2(X_1)$ respectively together provide a twisted lift of $TL_0 \times TL_1$ to the pin group with respect to the background class

\[ \pi_0^* w_2(X_0) + w_1(L_0) \cdot w_1(L_1) + \pi_1^* w_2(X_1). \]

By assumption, the Maslov classes of $L_0, L_1$ vanish, and so $w_1(L_0), w_1(L_1)$ are the restrictions of $\pi_0^* w_1(X_0), \pi_1^* w_1(X_1)$ respectively.

We use the term product branes to refer to objects of $F(T^* X_0 \times T^* X_1)$ that arise via the preceding construction. When there is no chance of confusion, we will denote a product brane by the product of the underlying Lagrangians.

**Lemma 4.2.1.** For any test objects $L_0, P_0$ of $F(T^* X_0)$, and $L_1, P_1$ of $F(T^* X_1)$, there is a functorial (in each object) quasi-isomorphism of Floer complexes

\[ \text{hom}_{F(T^* X_0 \times T^* X_1)}(L_0 \times L_1, P_0 \times P_1) \simeq \text{hom}_{F(T^* X_0)}(L_0, P_0) \otimes \text{hom}_{F(T^* X_1)}(L_1, P_1). \]

**Proof.** One can choose all further necessary structures such as perturbations to be a product of the corresponding structures on the factors. In this way, one obtains in fact a strict isomorphism of complexes. \(\square\)

4.2.2. **Diagonal brane.** Consider the smooth, closed submanifold given by the diagonal $\Delta_X \subset X \times X$, and let $C_{\Delta_X}$ denote the constant sheaf along $\Delta_X$.

Let $L_{\Delta_X}$ be the standard object of $F(T^* X \times T^* X)$ obtained as the microlocalization

\[ L_{\Delta_X} \simeq \mu_{X \times X}(C_{\Delta_X}). \]

We will refer to $L_{\Delta_X}$ as the diagonal brane though its underlying Lagrangian is the conormal bundle

\[ T^*_{\Delta_X} (X \times X) = \{(x, \xi; x, -\xi) \in T^* X \times T^* X\} \]

(so strictly speaking, not truly the diagonal – the true diagonal is not Lagrangian). Its flat vector bundle is the pullback of the normal orientation bundle $\mathcal{O}_{X \times X} \otimes \mathcal{O}_{\Delta_X}^{-1}$.

**Proposition 4.2.2.** For any test objects $P_0, P_1$ of $F(T^* X)$, there is a functorial quasi-isomorphism of Floer complexes

\[ \text{hom}_{F(T^* X)}(P_1, P_0) \simeq \text{hom}_{F(T^* X \times T^* X)}(L_{\Delta_X}, P_0 \times \alpha_X(P_1)). \]

**Proof.** Consider the intersection points and pseudoholomorphic disks involved in calculating the right hand side as a function of the branes $P_0, P_1$. Observe that to make any calculation involved, we can fix the brane $L_{\Delta_X}$ and work with perturbations that only move the other branes.
Now apply the product $id \times a_1$ of the identity map $id$ of the first factor and the antipodal map of the second factor

$$a_1 : T^*X \to T^*X \quad a_1(x_1, \xi_1) = (x_1, -\xi_1)$$

to the objects under consideration. Observe that $id \times a_1$ takes the conormal Lagrangian $T^*_\Delta_X(X \times X)$ to the diagonal submanifold $\Delta_{T^*X}$, the Lagrangian $P_0$ to itself, and the dualized Lagrangian $\alpha_X(P_1)$ back to $P_1$.

Standard gluing arguments imply that $id \times a_1$ takes a pseudoholomorphic map

$$(u_0, u_1) : D \to T^*X \times T^*X$$

with a $L_\Delta_X$-labelled boundary component

$$(u_0, u_1)|_C : C \to L_\Delta_X$$

to a pseudoholomorphic map

$$u_0 \cup a_1(u_1) : D \cup_C \overline{D} \to T^*X$$

where $\overline{D}$ denotes the conjugate disk, and $D \cup_C \overline{D}$ the gluing of $D, \overline{D}$ along $C$.

Tracing through brane structures, we see that the above identification of moduli spaces provides the sought-after functorial quasi-isomorphism

$$\text{hom}_{F(T^*X)}(L_{\Delta_X}, P_0 \times \alpha_X(P_1)) \simeq \text{hom}_{F(T^*X)}(P_1, P_0)$$

Note that the appearance of the orientation bundle $or_X$ on the dualized brane $\alpha_X(P_1)$ matches up with the appearance of the normal orientation bundle $or_{X \times X} \otimes or_{-\Delta_X}^{\Delta_X}$ on the standard brane $L_{\Delta_X}$.

4.3. Triangulation of diagonal. We explain here how the choice of a triangulation of $X$ allows us to express the diagonal brane $L_{\Delta_X}$ in terms of costandard branes. The primary content of the section is in developing notation and collecting preliminaries for the arguments of subsequent sections.

Fix a triangulation $\mathcal{T} = \{\tau_\alpha\}$ of $X$, and consider the $\mathcal{T}$-constructible dg derived category $Sh_\mathcal{T}(X)$. Consider the inclusions $j_\alpha : \tau_\alpha \hookrightarrow X$, and the corresponding standard sheaves $j_\alpha^* C_{\tau_\alpha}$. By Lemma 2.3.1, we can express any object of $Sh_\mathcal{T}(X)$, and in particular the constant sheaf $C_X$, as an iterated cone of maps among the standard sheaves $j_\alpha^* C_{\tau_\alpha}$.

Identify $X$ with the diagonal $\Delta_X \subset X \times X$ (via either projection), and consider the induced triangulation $\Delta_\mathcal{T} = \{\Delta_\tau_\alpha\}$ of $\Delta_X$. Consider the inclusions $d_\alpha : \Delta_\tau_\alpha \hookrightarrow \Delta_X$, and the corresponding standard sheaves $d_\alpha^* C_{\Delta_\tau_\alpha}$. Again, by Lemma 2.3.1, we can express the constant sheaf $C_{\Delta_X}$ as an iterated cone of maps among the standard sheaves $d_\alpha^* C_{\Delta_\tau_\alpha}$.

Next, recall that the diagonal brane $L_{\Delta_X}$ is the standard object of $F(T^*X \times T^*X)$ obtained as the microlocalization

$$L_{\Delta_X} \simeq \mu_{X \times X}(C_{\Delta_X})$$

By construction, its underlying Lagrangian is the conormal bundle $T^*_\Delta_X(X \times X)$. 
For each simplex of $T$ consider the standard object $L_{\Delta_\tau_a^*}$ of $F(T^*X \times T^*X)$ obtained as the microlocalization
\[
L_{\Delta_\tau_a^*} \simeq \mu_X \times \Delta (d_a! C_{\Delta_\tau_a}).
\]
By construction, we can take its underlying Lagrangian to be in the following form. Fix a non-negative function $m_a : X \rightarrow \mathbb{R}$ that vanishes precisely on the boundary $\partial \tau_a \subset X$, and consider the function $f_a : \tau_a \rightarrow \mathbb{R}$ given by the logarithm $f_a = \log m_a$. Then the underlying Lagrangian of $L_{\Delta_\tau_a}$ can be identified with the fiberwise sum
\[
T^*_\tau_a (X \times X) + \Gamma p^*_2 df_a \subset T^*(X \times X),
\]
where $\Gamma p^*_2 df_a$ denotes the graph of the pullback $p^*_2 df_a$ via projection to the second factor $p_2 : X \times X \rightarrow X$.

Since we can express the constant sheaf $C_{\Delta_\tau_a}$ as a twisted complex of standard sheaves $d_a! C_{\Delta_\tau_a}$, we can express the brane $L_{\Delta_\tau_a}$ as a twisted complex of standard branes $L_{\Delta_\tau_a^*}$. It will be convenient to have a slightly modified version of the preceding as recorded in the following.

**Lemma 4.3.1.** The dual brane $\alpha_X (L_{\Delta_X})$ can be expressed as a twisted complex of the standard branes $L_{\Delta_\tau_a}$.

**Proof.** Tracing through the definitions, we have the elementary identity $\alpha_X (L_{\Delta_X}) = \mu_X (D_X (C_{\Delta_X}))$. Thus we can repeat the preceding discussion replacing the constant sheaf $C_{\Delta_X}$ by its Verdier dual $D_X (C_{\Delta_X})$.

**□**

4.4. Moving the diagonal. We continue with the notations of the preceding section.

Fix a point $x_a \in \tau_a$, and consider the standard sheaf $C_{\{x_a\}} \times j_{\tau_a} C_{\tau_a}$ as an object of $Sh_c (X \times X)$.

Consider the standard object $L_{\{x_a\} \times \tau_a^*}$ of $F(T^*X \times T^*X)$ obtained as the microlocalization
\[
L_{\{x_a\} \times \tau_a^*} \simeq \mu_X \times \{x_a\} \times j_{\tau_a} C_{\tau_a}.
\]
By construction, we can take its underlying Lagrangian to be the fiberwise sum
\[
T^*_\{x_a\} (X \times X) + \Gamma p^*_2 df_a.
\]
Observe that $L_{\{x_a\} \times \tau_a^*}$ is the external product
\[
L_{\{x_a\} \times \tau_a^*} \simeq L_{\{x_a\}_a!} \times L_{\tau_a^*}
\]
of the factors
\[
L_{\{x_a\}} \simeq \mu_X (C_{\{x_a\}}) \quad L_{\tau_a^*} \simeq \mu_X (j_{\tau_a} C_{\tau_a}).
\]

Consider the conical Lagrangian $\Lambda_T \subset T^*X$ given by the union of the conormal bundles of the simplices of the triangulation
\[
\Lambda_T = \bigsqcup a T^*_a X.
\]
We will use the results of Section 3.7 to verify the following.
By construction, we can take its underlying Lagrangian to be the fiberwise sum of complexes $P_0 \times L_{\Delta a}$. There are obvious identifications $\tau_a = \{x \in P_0 \times \Delta a| a(x) > \eta\}$, and the family of submanifolds:

$$\tau_a = \{(x_0, x_1) \in X \times X| x_0 = \psi_t(x_1)\}$$

satisfying the obvious identifications

$$\tau_{a,0} = \Delta a \quad \tau_{a,1} = \{x_a\} \times \tau_a.$$  

Consider the inclusion $d_{a,t} : \tau_{a,t} \hookrightarrow X \times X$, the corresponding standard sheaf $d_{a,t}(\mathcal{C}_{\tau_{a,t}})$, and its microlocalization

$$L_{\tau_{a,t}} = \mu_{X \times X}(d_{a,t}(\mathcal{C}_{\tau_{a,t}})).$$

By construction, we can take its underlying Lagrangian to be the fiberwise sum

$$T_{a,t}^\ast(X \times X) \ast \Gamma_{p \ast \mu}.$$  

We have the obvious identifications

$$L_{\tau_{a,0}} = L_{\Delta a} \quad L_{\tau_{a,1}} = L_{\{x_0\} \times \tau_a} = L_{\{x_a\} \times \tau_a}.$$  

Now fix test objects $P_0, P_1$ of $F(T^\ast X)$, with $P_0 \subset L_{\Delta a}$, and let $\Lambda \subset T^\ast X$ be a conical Lagrangian with $P_1 \subset L_{\Lambda}$. We would like to show the family of Floer complexes

$$\text{hom}_{F(T^\ast X \times T^\ast X)}(P_0 \times P_1, L_{\Delta a} \ast)$$

has constant cohomology with respect to $t$. Choose a small $\eta > 0$, and consider the submanifold $\tau_{a,\eta} = \{x \in \tau_a| a(x) > \eta\}$, and the family of submanifolds

$$\tau_{a,\eta, t} = \{(x_0, x_1) \in X \times X| x_1 \in \tau_{a,\eta}, x_0 = \psi_t(x_1)\}.$$  

Consider the inclusion $d_{a,\eta,t} : \tau_{a,\eta, t} \hookrightarrow X \times X$, the corresponding costandard sheaf $d_{a,\eta,t}(\mathcal{C}_{\tau_{a,\eta, t}})$, and its microlocalization

$$L_{\tau_{a,\eta,t}} = \mu_{X \times X}(d_{a,\eta,t}(\mathcal{C}_{\tau_{a,\eta, t}})).$$

By construction, we can take its underlying Lagrangian to be the fiberwise sum

$$T_{a,\eta,t}^\ast(X \times X) \ast \Gamma_{p \ast \mu}.$$  

where $f_{a,\eta} : \tau_{a,\eta} \rightarrow \mathbb{R}$ is the function given by the logarithm $f_{a,\eta} = \log(m_a - \eta)$. Then for sufficiently small $\eta > 0$, and all $t$, we have a quasi-isomorphism

$$\text{hom}_{F(T^\ast X \times T^\ast X)}(P_0 \times P_1, L_{\Delta a,\eta,t} \ast) \simeq \text{hom}_{F(T^\ast X \times T^\ast X)}(P_0 \times P_1, L_{\Delta a,\eta, t} \ast).$$

Finally, by construction, the family of branes $L_{\tau_{a,\eta,t}}$ is non-characteristic with respect to the product conical Lagrangian $\Lambda_{\tau} \times \Lambda$. Thus by Proposition 3.7.6 (see in particular Example 3.7.6), the family of complexes

$$\text{hom}_{F(T^\ast X \times T^\ast X)}(P_0 \times P_1, L_{\Delta a,\eta,t} \ast)$$

is a functorial quasi-isomorphism of complexes.
has constant cohomology. Furthermore, at \( t = 0 \), the family calculates the left hand side of the proposition, and at \( t = 1 \), it calculates the right hand side (cf. Lemma 4.2.1).

4.5. **Proof of Theorem 4.1.1.** Now let us wrap up the proof of Theorem 4.1.1. We continue with the notation of the preceding sections.

Fix once and for all an object \( L \) of \( F(\mathbb{T}^*X) \). Our aim is to show that for any test object \( P \) of \( F(\mathbb{T}^*X) \), we can functorially express the Floer complex \( \text{hom}_{F(\mathbb{T}^*X)}(P,L) \) as a twisted complex of the Floer complexes \( \text{hom}_{\mathbb{F}(\mathbb{T}^*X)}(P,Y_*\mathbb{C}) \) of standard branes \( Y_* \hookrightarrow \mathbb{T}^*X \) associated to submanifolds \( Y \hookrightarrow X \).

First, by Proposition 4.2.2 there is a functorial quasi-isomorphism of Floer complexes

\[
\text{hom}_{F(\mathbb{T}^*X)}(P,L) \simeq \text{hom}_{F(\mathbb{T}^*X \times \mathbb{T}^*X)}(L\Delta_X,L \times \alpha_X(P)).
\]

It will be convenient to rewrite the preceding in a slightly modified form. Since the brane duality \( \alpha_X \) is an anti-equivalence, we have a functorial quasi-isomorphism

\[
\text{hom}_{F(\mathbb{T}^*X)}(P,L) \simeq \text{hom}_{F(\mathbb{T}^*X \times \mathbb{T}^*X)}(\alpha_X(L) \times P,\alpha_X(L\Delta_X)).
\]

Next, fix a triangulation \( \mathcal{T} = \{ \tau_{a} \} \) of \( X \) along with the induced triangulation \( \Delta_{\mathcal{T}} = \{ \Delta_{\tau_{a}} \} \) of \( \Delta_X \subset X \times X \). By Lemma 4.3.1 we can express \( \alpha_X(L\Delta_X) \) as a twisted complex of the standard branes \( L_{\Delta_{\tau_{a}}\mathbb{C}} \).

Suppose further that \( \mathcal{T} \) is chosen fine enough so that \( L_{\infty} \subset \Delta_{X}^{\infty} \). Then by Proposition 4.4.1 there is a functorial quasi-isomorphism of complexes

\[
\text{hom}_{F(\mathbb{T}^*X \times \mathbb{T}^*X)}(\alpha_X(L) \times P,L_{\Delta_{\tau_{a}}\mathbb{C}}) \simeq \text{hom}_{F(\mathbb{T}^*X)}(\alpha_X(L),L_{\tau_{a}}) \otimes \text{hom}_{F(\mathbb{T}^*X)}(P,L_{\tau_{a}}).
\]

Putting together the preceding identifications, we have functorially expressed the Floer complex \( \text{hom}_{F(\mathbb{T}^*X)}(P,L) \) as a twisted complex with terms the Floer complexes \( \text{hom}_{F(\mathbb{T}^*X)}(\alpha_X(P),L_{\tau_{a}}) \). This completes the proof of Theorem 4.1.1. \( \square \)

**Remark 4.5.1.** In the expression of the brane \( L \) as a twisted complex of the standard branes \( L_{\tau_{a}}\mathbb{C} \), the coefficients appearing are the functionals

\[
\text{hom}_{F(\mathbb{T}^*X)}(\alpha_X(L),L_{\tau_{a}}) \simeq \text{hom}_{F(\mathbb{T}^*X)}(L_{\tau_{a}}),L).
\]

Although we will not use it, the proof of Theorem 4.1.1 leads to the following precise form of \( L \) as a twisted complex. First, we can identify the coefficients \( \text{hom}_{\mathbb{F}(\mathbb{T}^*X)}(L_{\tau_{a}}),L) \) with the shifted Floer complexes \( \text{hom}_{\mathbb{F}(\mathbb{T}^*X)}(L_{\tau_{a}}),L) \). Next, the triangulation provides the structure of dual cell complexes on the collection of standard branes \( L_{\tau_{a}}\mathbb{C} \) and costandard branes \( L_{\tau_{a}}\mathbb{C} \) associated to the simplices \( \tau_{a} \). Finally, this induces the structure of a twisted complex on the branes \( \text{hom}_{\mathbb{F}(\mathbb{T}^*X)}(L_{\tau_{a}}),L) \leftrightarrow \) appearing in the decomposition of \( L \).

4.6. **From branes to sheaves.** Since microlocalization is a quasi-equivalence

\[
\mu_X : Sh_c(X) \rightarrow F(\mathbb{T}^*X),
\]

the corresponding pullback of right \( A_{\infty}\)-modules is also a quasi-equivalence

\[
\mu_X^* : \text{mod}_{X}(F(\mathbb{T}^*X)) \rightarrow \text{mod}_r(Sh_c(X)).
\]

In particular, given an object \( L \) of \( F(\mathbb{T}^*X) \), we can take its Yoneda module \( \mathcal{Y}_r(L) \), and ask what object \( \mathcal{F} \) of \( Sh_c(X) \) quasi-represents \( \mu_X^*\mathcal{Y}_r(L) \).
Here is an informal way to think about an object $F$ that quasi-represents $\mu_X^*\mathcal{Y}_r(L)$. Given an open submanifold $i : U \hookrightarrow X$, we have quasi-isomorphisms of complexes

$$F(U) \simeq \text{hom}_{Sh_*(X)}(i_!\mathcal{C}_U, F) \simeq \text{hom}_{F(T^*X)}(U_U \otimes \omega_X[- \dim X], L).$$

The first quasi-isomorphism is by adjunction, and the second is by the fact that $F$ quasi-represents $\mu_X^*\mathcal{Y}_r(L)$. Here we have taken $L_U \otimes \omega_X[- \dim X]$ since $\omega_X \simeq \omega_X[\dim X]$.

Similarly, given an inclusion of open submanifolds $i^!_0 : U_0 \hookrightarrow U$, we have a diagram that commutes at the level of cohomology

$$
\begin{array}{ccc}
F(U_1) & \xrightarrow{\rho^1_0} & F(U_0) \\
\downarrow & & \downarrow \\
\text{hom}_{F(T^*X)}(L_{U_1} \otimes \omega_X[- \dim X], L) & \xrightarrow{i^!_0} & \text{hom}_{F(T^*X)}(L_{U_0} \otimes \omega_X[- \dim X], L).
\end{array}
$$

Here $\rho^1_0$ denotes the sheaf restriction map, and $i^!_0$ denotes the map induced by the canonical degree zero morphism in the complex

$$\text{hom}_{F(T^*X)}(L_{U_0}, L_{U_1}) \simeq \text{hom}_{Sh_*(X)}(i^!_0\mathcal{C}_{U_0}, i^!_0\mathcal{C}_{U_1}) \simeq (\Omega(U_0), d).$$

Finally, we record the following consequence of the proof of Theorem 4.1.1. Fix a conical Lagrangian $\Lambda \hookrightarrow T^*X$, and let $F(T^*X)_\Lambda$ be the full subcategory of $F(T^*X)$ of twisted complexes of Lagrangian branes $L$ such that $L^\infty \subseteq \Lambda^\infty$.

**Proposition 4.6.1.** Given an object $L$ of $F(T^*X)_\Lambda$, consider its Yoneda module $\mathcal{Y}_r(L)$, and an object $F$ of $Sh_*(X)$ that quasi-represents the pullback $\mu_X^*\mathcal{Y}_r(L)$.

Then for any stratification $S = \{S_\alpha\}$ of $X$ such that $\Lambda \subset S = \cup_\alpha T^*_\alpha X$, the object $F$ lies in $Sh_*(X)$.

**Proof.** In the proof of Theorem 4.1.1 we showed that $L$ can be expressed as a twisted complex of the standard branes $L_{\tau_b}$, for any triangulation $\mathcal{T} = \{\tau_b\}$ refining $S$. In other words, $F$ can be expressed as a twisted complex of the standard sheaves $j_{b *}\mathcal{C}_{\tau_b}$.

In fact, in place of the triangulation $\mathcal{T}$, we can take any disjoint cell decomposition $C = \{c_b\}$ of $X$ refining $S$ in the sense that each cell $j_b : c_b \hookrightarrow X$ lies in some stratum $S_\alpha$. To see this level of generality, observe that all we need for the proof of Theorem 4.1.1 is that the dualizing complex $\omega_X = D_X(\mathbb{C}_X)$ can be expressed as a twisted complex of the standard sheaves $j_{b *}\mathcal{C}_{c_b}$, and that each cell $c_b$ can be deformation contracted within the stratum $S_\alpha$ containing $c_b$ to a point $x_b \in c_b$.

Thus $F$ can be expressed as a twisted complex of the standard sheaves $j_{b}\mathcal{C}_{c_b}$ on the cells of any such cell decomposition $C = \{c_b\}$. The proposition now follows from Lemma 4.6.2 immediately below. \qed

**Lemma 4.6.2.** Let $F$ be an object of $Sh_*(X)$. Suppose that for any cell decomposition $C = \{c_b\}$ of $X$ refining the stratification $S = \{S_\alpha\}$, we can express $F$ as a twisted complex of the standard sheaves $j_{b}\mathcal{C}_{c_b}$ on the cells. Then $F$ is $S$-constructible.

**Proof.** Fix any point $p \in X$, and let $S_\alpha$ be the stratum of $S$ containing $p$. Choose an open ball $B_{\alpha,p} \subset S_\alpha$ containing $p$, and a normal slice $N_{\alpha,p} \subset X$ to $S_\alpha$ at $p$. We will consider $B_{\alpha,p}$ as a smooth manifold, and equip $N_{\alpha,p}$ with the stratification induced by restricting $S$. 

By the Thom Isotopy Lemma, there is a stratum preserving homeomorphism from the product \( B_{\alpha,p} \times N_{\alpha,p} \), equipped with the product stratification, to a neighborhood \( U_p \subset X \) of the point \( p \), with the stratification induced by restricting \( \mathcal{S} \). Furthermore, the restriction of the homeomorphism to each stratum is a diffeomorphism.

Choose any triangulation \( \mathcal{T}_{N_{\alpha,p}} \) of the normal slice \( N_{\alpha,p} \) refining the stratification induced by \( \mathcal{S} \). Consider the induced product stratification of the neighborhood \( U_p \cong B_{\alpha,p} \times N_{\alpha,p} \). Extend this to any cell decomposition \( \mathcal{C} = \{ c_b \} \) of all of \( X \) by cutting up the complement \( X \setminus U_p \) into cells.

By construction, the restriction of the standard sheaf \( j_{c_b}^* C_{c_b} \) of any cell \( c_b \) of \( \mathcal{C} \) to the cell \( B_{\alpha,p} \) is constant. Hence by assumption, the restriction of \( F \) to the cell \( B_{\alpha,p} \) is constant as well. Thus the restriction of \( F \) to the stratum \( S_{\alpha} \) is locally constant. \( \square \)

5. Functoriality

Now that we have established that microlocalization is a quasi-equivalence

\[
\mu_X : Sh_c(X) \xrightarrow{\sim} F(T^*X),
\]

we can collect some formal consequences for future applications.

5.1. Duality revisited. Recall the brane duality equivalence

\[
\alpha_X : F(T^*X)^\circ \xrightarrow{\sim} F(T^*X)
\]

introduced in Section 3.4. Our aim in this section is to confirm that microlocalization \( \mu_X \) intertwines brane duality with Verdier duality

\[
\mathcal{D}_X : Sh_c(X)^\circ \xrightarrow{\sim} Sh_c(X).
\]

For a submanifold \( i_Y : Y \hookrightarrow X \), Verdier duality exchanges the associated standard and costandard objects

\[
i_Y^* \mathbb{C}_Y \simeq \mathcal{D}(i_Y^! \omega_Y).
\]

Likewise, by construction, brane duality exchanges the associated standard and costandard branes

\[
\alpha_X(L_{Y^*}) \simeq L_Y?.
\]

**Proposition 5.1.1.** There is a quasi-isomorphism

\[
\mu_X \circ \mathcal{D}_X \simeq \alpha_X \circ \mu_X : Sh_c(X)^\circ \xrightarrow{} F(T^*X).
\]

**Proof.** By Proposition 3.6.1, for any open submanifold \( i_U : U \hookrightarrow X \), there is a quasi-isomorphism of left \( Sh_c(X) \)-modules

\[
\mu_X^*(\mathcal{Y}_U(L_U^!)) \simeq \mathcal{Y}_U(i_U^! \omega_U) : Sh_c(X) \rightarrow Ch.
\]

In other words, for any test object \( F \) of \( Sh_c(X) \), there is a functorial quasi-isomorphism

\[
\text{hom}_{F(T^*X)}(L_U^!, \mu_X(F)) \simeq \text{hom}_{Sh_c(X)}(i_U^! \omega_U, F)
\]

Since microlocalization \( \mu_X \) is a quasi-embedding, there is a functorial quasi-isomorphism

\[
\text{hom}_{Sh_c(X)}(i_U^! \omega_U, F) \simeq \text{hom}_{F(T^*X)}(\mu_X(i_U^! \omega_U), \mu_X(F)).
\]
Finally, since \( \mu_X \) is a quasi-equivalence (so objects of the form \( \mu_X(F) \) generate \( F(T^*X) \)), and the Yoneda functor \( \mathcal{Y}_I \) is a quasi-embedding, there is an isomorphism
\[
\mu_X(i_U!)\omega_U) \cong L_U!.
\]

We conclude that there are isomorphisms
\[
\mu_X(D_X(i_U, \mathbb{C}_U)) \cong \mu_X(i_U!\omega_U) \cong L_U! \cong \alpha_X(L(U^*_s)) \cong \alpha_X(\mu_X(i_U^!\mathbb{C}_U)).
\]

By Lemma 2.3.3, standard objects \( i_U^!\mathbb{C}_U \) generate \( Sh_c(X) \), and so we have the asserted quasi-isomorphism of functors. \( \square \)

5.2. Integral transforms for sheaves. In this and the following section, we describe standard functors for sheaves and branes. At this stage, their compatibility is completely formal: it depends only on the fact that microlocalization is a quasi-equivalence intertwining Verdier duality with brane duality.

Given two real analytic manifolds \( X_0, X_1 \), consider the standard projections
\[
P_0 : X_0 \times X_1 \to X_0 \quad P_1 : X_0 \times X_1 \to X_1.
\]

Following [17] Section 3.6, for an object \( \mathcal{K} \) of \( Sh_c(X_0 \times X_1) \), we have the integral transforms
\[
\Phi_K : Sh_c(X_0) \to Sh_c(X_0) \quad \Phi_{K*} : Sh_c(X_0) \to Sh_c(X_1)
\]
\[
\Phi_K^*(\mathcal{F}) = p_0!(\mathcal{K} \otimes p_1^*(\mathcal{F})) \quad \Phi_{K*}(\mathcal{F}) = p_1!(\mathcal{K} \otimes p_0^*(\mathcal{F}))
\]

Similarly, reversing the roles of \( X_0 \) and \( X_1 \), we have the integral transforms
\[
\Phi_{K!} : Sh_c(X_0) \to Sh_c(X_1) \quad \Phi_K^! : Sh_c(X_1) \to Sh_c(X_0)
\]
\[
\Phi_{K!}(\mathcal{F}) = p_1!(\mathcal{K} \otimes p_0^*(\mathcal{F})) \quad \Phi_K^!(\mathcal{F}) = p_0!(\mathcal{K} \otimes p_1^*(\mathcal{F}))
\]

Standard identities imply the following: (1) \( (\Phi_{K*}, \Phi_{K*}) \) and \( (\Phi_{K!}, \Phi_K^!) \) are each adjoint pairs. (2) The construction is functorial in \( \mathcal{K} \) in the sense that we have functionals
\[
\Phi^* \simeq \Phi_* : Sh_c(X_0) \otimes Sh_c(X_0 \times X_1)^0 \otimes Sh_c(X_1)^0 \to Ch
\]
\[
(\mathcal{F}_1, \mathcal{K}, \mathcal{F}_0) \mapsto hom_{Sh_c(X_0)}(\Phi_K^*(\mathcal{F}_1), \mathcal{F}_0) \simeq hom_{Sh_c(X_1)}(\mathcal{F}_1, \Phi_{K*}(\mathcal{F}_0))
\]
\[
\Phi \simeq \Phi^! : Sh_c(X_0)^0 \otimes Sh_c(X_0 \times X_1)^0 \otimes Sh_c(X_1) \to Ch
\]
\[
(\mathcal{F}_0, \mathcal{K}, \mathcal{F}_1) \mapsto hom_{Sh_c(X_1)}(\Phi_{K!}(\mathcal{F}_0), \mathcal{F}_1) \simeq hom_{Sh_c(X_0)}(\mathcal{F}_0, \Phi_K^!(\mathcal{F}_1))
\]

(3) We have quasi-isomorphisms of functors
\[
\Phi_K^* \simeq D_{X_0} \circ \Phi_K^! \circ D_{X_1} \quad \Phi_{K*} \simeq D_{X_1} \circ \Phi_{K!} \circ D_{X_0}
\]
\[
\Phi_{K!} \simeq D_{X_1} \circ \Phi_{K*} \circ D_{X_0} \quad \Phi_K^! \simeq D_{X_0} \circ \Phi_K^* \circ D_{X_1}
\]

Example 5.2.1. Our notation is motivated by the following example. Fix a \( C \)-map \( f : X_0 \to X_1 \), consider the graph
\[
\Gamma_f = \{(x_0, x_1) \in X_0 \times X_1 | x_1 = f(x_0)\},
\]
and let \( \mathbb{C}_{\Gamma_f} \) denote the constant sheaf along \( \Gamma_f \). Then we have canonical identifications of functors
\[
(f^*, f_*) \simeq (\Phi_{\mathbb{C}_{\Gamma_f}}^*, \Phi_{\mathbb{C}_{\Gamma_f}}) \quad (f!, f^!) \simeq (\Phi_{\mathbb{C}_{\Gamma_f}}, \Phi_{\mathbb{C}_{\Gamma_f}}^!)
\]
5.3. Integral transforms for branes. We discuss here the analogous integral transforms associated to objects of $F(T^*X_0 \times T^*X_1)$.

For any object $L$ of $F(T^*X_0 \times T^*X_1)$, we have functors

$$
\tilde{\Psi}_L^*: F(T^*X_1) \to \text{mod}_r(F(T^*X_0)),
$$
$$
\tilde{\Psi}_{L\ast}(P_1): P_0 \mapsto \text{hom}_{F(T^*X_0 \times T^*X_1)}(L, P_0 \times \alpha_{X_1}(P_1)) \tag{1.1} \nonumber
$$
$$
\tilde{\Psi}_{L!}(P_1): P_1 \mapsto \text{hom}_{F(T^*X_0 \times T^*X_1)}(L, \alpha_{X_0}(P_0) \times P_1) \tag{2.2} \nonumber
$$
$$
\tilde{\Psi}_L^*(P_1): P_0 \mapsto \text{hom}_{F(T^*X_0 \times T^*X_1)}(L, \alpha_{X_0}(P_0) \times P_1) \tag{3.3} \nonumber
$$
Note that the constructions are functorial in $L$ in the contravariant sense.

**Proposition 5.3.1.** Consider an object $K$ of $\text{Sh}_c(X_0 \times X_1)$, and its microlocalization $L = \mu_{X_0 \times X_1}(K)$. Then there are functorial quasi-isomorphisms

$$
\mathcal{Y}_t \circ \mu_{X_0} \circ \Phi_K^* \simeq \tilde{\Psi}_L^* \circ \mu_{X_1}
$$
$$
\mathcal{Y}_r \circ \mu_{X_1} \circ \Phi_K \simeq \tilde{\Psi}_{L!}^* \circ \mu_{X_0}
$$
$$
\mathcal{Y}_t \circ \mu_{X_1} \circ \Phi_K! \simeq \tilde{\Psi}_{L\ast}^* \circ \mu_{X_0}
$$

**Proof.** We establish the second quasi-isomorphism (the case of the usual pushforward); the arguments for the others are similar. It suffices to consider test objects $L_0$ of $F(T^*X_0)$ and $L_1$ of $F(T^*X_1)$ of the form $L_0 = \mu_{X_1}(\mathcal{F}_0)$ and $L_1 = \mu_{X_1}(\mathcal{F}_1)$, and to establish a functorial quasi-isomorphism

$$
\text{hom}_{\text{Sh}_c(X_0 \times X_1)}(\mathcal{F}_1, p_{1*}(\text{hom}(K, p_0^1(\mathcal{F})))) \simeq \text{hom}_{F(T^*X_0 \times T^*X_1)}(L_0 \times \alpha_{X_1}(L_1)).
$$

By standard identities, this is nothing more than a functorial quasi-isomorphism

$$
\text{hom}_{\text{Sh}_c(X_0 \times X_1)}(\mathcal{K}, p_0^1(\mathcal{F})) \circ p_1^1(\mathcal{D}_{\text{X}_1}(\mathcal{F}_1)) \simeq \text{hom}_{F(T^*X_0 \times T^*X_1)}(L_0 \times \alpha_{X_1}(L_1)).
$$

Now the assertion follows immediately from [Proposition 5.1.1](#).

We see from the proposition that the modules arising from the functors $\tilde{\Psi}_L^*, \tilde{\Psi}_{L\ast}, \tilde{\Psi}_{L!}, \tilde{\Psi}_L^！$ are representable. Namely, for $L \simeq \mu_{X_0 \times X_1}(K)$, we can take the representing functors to be the compositions

$$
\Psi_L^* = \mu_{X_0} \circ \Phi_K^* \circ \pi_{X_1}
$$
$$
\Psi_{L\ast} = \mu_{X_1} \circ \Phi_K \circ \pi_{X_0}
$$
$$
\Psi_{L!} = \mu_{X_1} \circ \Phi_K! \circ \pi_{X_0}
$$
$$
\Psi_L^！ = \mu_{X_0} \circ \Phi_K^！ \circ \pi_{X_1}
$$

Their basic properties are immediate from the definitions: (1) $(\Psi_L^*, \Psi_{L\ast})$ and $(\Psi_{L!}, \Psi_L^！)$ are each adjoint pairs. (2) The construction is functorial in $L$ in the sense that we have functors

$$
\Phi^* \simeq \Phi_* : F(T^*X_0 \times T^*X_1) \to \text{fun}_{A_{\infty}}(F(T^*X_0), \text{mod}_r(F(T^*X_1)))
$$
$$
\Psi^! \simeq \Psi^! : F(T^*X_0 \times T^*X_1) \to \text{fun}_{A_{\infty}}(F(T^*X_1), \text{mod}_r(F(T^*X_0)))
$$

where $\text{fun}_{A_{\infty}}$ denotes the $A_{\infty}$-category of $A_{\infty}$-functors. (We have written the functors in this form rather than as functionals since the notion of internal hom for $A_{\infty}$-categories is
straightforward, while that of tensor product is more delicate.) (3) There are functorial quasi-isomorphisms

\[ \Psi_L^* \simeq \alpha_{X_0} \circ \Psi_L^1 \circ \alpha_{X_1}, \quad \Psi_{L^*} \simeq \alpha_{X_1} \circ \Psi_{L^*} \circ \alpha_{X_0} \]

\[ \Psi_{L^*} \simeq \alpha_{X_1} \circ \Psi_{L^*} \circ \alpha_{X_0}, \quad \Psi_{L^*} \simeq \alpha_{X_0} \circ \Psi_{L^*} \circ \alpha_{X_1} \]

intertwining the functors with brane duality.

**Example 5.3.2.** Let \( \mathcal{F} : X_0 \to X_1 \) be a \( \mathcal{C} \)-map. Consider the graph

\[ \Gamma_{\mathcal{F}} = \{(x_0, x_1) \in X_0 \times X_1 | x_1 = \mathcal{F}(x_0)\}, \]

and let \( \mathbb{C}_{\Gamma_{\mathcal{F}}} \) denote the constant sheaf along \( \Gamma_{\mathcal{F}} \).

Let \( L_{\mathcal{F}} \) be the standard object of \( F(T^*X_0 \times T^*X_1) \) obtained as the microlocalization

\[ L_{\mathcal{F}} \simeq \mu_{X_0 \times X_1}(\mathbb{C}_{\Gamma_{\mathcal{F}}}). \]

By construction, when \( \mathcal{F} \) is smooth, we can take the Lagrangian underlying \( L_{\mathcal{F}} \) to be the conormal bundle \( T_{\Gamma_{\mathcal{F}}}^*(X_0 \times X_1) \).

Applying the above constructions, we obtain functors \( \Psi_{L^1}, \Psi_{L^*}, \Psi_{L^*}, \Psi_{L^1} \).

**Corollary 5.3.3.** For any \( \mathcal{C} \)-map \( \mathcal{F} : X_0 \to X_1 \), there are quasi-isomorphisms

\[ \Psi_{L^1} \circ \mu_{X_1} \simeq \mu_{X_0} \circ \mathcal{F}^* \]

\[ \Psi_{L^*} \circ \mu_{X_0} \simeq \mu_{X_1} \circ \mathcal{F}^* \]

\[ \Psi_{L^1} \circ \mu_{X_0} \simeq \mu_{X_1} \circ \mathcal{F} \]

\[ \Psi_{L^*} \circ \mu_{X_0} \simeq \mu_{X_1} \circ \mathcal{F} \]

5.4. **Correspondence interpretation.** In this informal section, we sketch how the integral transforms of the preceding sections are related to the beautiful theory of quilted Riemann surfaces and generalized branes mathematically developed by Wehrheim- Woodward [39] (and pioneered from a physical perspective by Khovanov-Rozansky [25] under the name world-sheet foam). We do not use the discussion of this section and include it for the interested reader already familiar with the constructions of [39].

In what follows, we assume that all of our manifolds are orientable, so that their cotangent bundles are spin. The main reason for imposing this condition will be that for a product \( T^*X_0 \times T^*X_1 \), the canonical background class will then be the product of the canonical background classes

\[ (\pi_0 \times \pi_1)^*w_2(X_0 \times X_1) = \pi_0^*w_2(X_0) + \pi_1^*w_2(X_1). \]

5.4.1. **Generalized branes.** By a generalized Lagrangian submanifold of \( T^*X \), we mean a sequence of compact real analytic manifolds \( pt = X_{-m}, X_{-m+1}, \ldots, X_{-1}, X_0 = X \), for some \( m > 0 \), and a sequence of Lagrangian submanifolds \( L = (L_{(-m,-m+1)}, \ldots, L_{(-1,0)}) \) in the successive products

\[ L_{(-k,-k+1)} \subset (T^*X_{-k})^+ \times T^*X_{-k+1}, \quad \text{for } k = 1, \ldots, m. \]

As usual, to control the behavior of \( L_{(-k,-k+1)} \) near infinity, we require that its closure in the product compactification is a \( \mathcal{C} \)-subset, and that there is a perturbation to a tame Lagrangian.

A brane structure on a generalized Lagrangian submanifold \( L \) consists of a sequence \( \mathcal{E} = (\mathcal{E}_{(-m,-m+1)}, \ldots, \mathcal{E}_{(-1,0)}) \) of flat vector bundles

\[ \mathcal{E}_{(-k,-k+1)} \to L_{(-k,-k+1)}, \quad \text{for } k = 1, \ldots, m. \]
and a sequence of gradings and relative pin structures. For simplicity, we will take the gradings and relative pin structures to be defined with respect to the canonical product bicanonical trivializations and background forms respectively.

5.4.2. Composition of correspondences. Following Wehrheim-Woodward [39], there is a triangulated category $DF^\#(T^*X)$ whose objects are twisted complexes of generalized Lagrangian branes. Work in progress of Mau-Wehrheim-Woodward [27] will provide an $A_\infty$-enhancement of this story but we content ourselves here with discussing things at the cohomological level.

A primary motivation for introducing generalized Lagrangian branes is that Lagrangian correspondences act on them: there is a triangulated functor

$$DF^\#(T^*X_0) \otimes DF((T^*X_0)^- \times T^*X_1) \rightarrow DF^\#(T^*X_1)$$

given on objects by concatenation

$$(L = (L_{(-m,-m+1)}) \cdots , L_{(1,0)}, L_{(0,1)}) \mapsto L^\#L_{(0,1)} = (L_{(-m,-m+1)}), \cdots , L_{(1,0)}, L_{(0,1)}).$$

The structure of the categories $DF^\#(T^*X_0), DF^\#(T^*X_1)$ and the composition functor are given by counting quilted Riemann surfaces. In particular, there is a Floer functional to chain complexes

$$DF^\#(T^*X_0)^\circ \otimes DF((T^*X_0)^- \otimes T^*X_1)^\circ \otimes DF^\#(T^*X_1) \rightarrow D(Ch)$$

$$(L_0, L_{(0,1)}, L_1) \mapsto \text{hom}_{DF^\#(T^*X_1)}(L_0^\#L_{(0,1)}, L_1).$$

Observe that there is an obvious functor $DF(T^*X) \rightarrow DF^\#(T^*X)$, and thus given an object $L$ of $DF^\#(T^*X)$, we can think of it as defining a left $DF(T^*X)$-module via the Yoneda map

$$\mathcal{Y}_{\ell,\text{ord}} : DF^\#(T^*X) \rightarrow \text{mod}_\ell(DF(T^*X))$$

$$\mathcal{Y}_{\ell,\text{ord}}(L) : P \mapsto \text{hom}_{DF^\#(T^*X)}(L, P)$$

5.4.3. Compatibility. Now consider the product symplectomorphism

$$a_0 \times id_1 : T^*X_0 \times T^*X_1 \sim (T^*X_0)^- \times T^*X_1$$

$$(a_0 \times id_1)(x_0, \xi_0; x_1, \xi_1) = (x_0, -\xi_0; x_1, \xi_1).$$

It induces an equivalence by transport of structure

$$(a_0 \times id_1)_* : DF(T^*X_0 \times T^*X_1) \sim DF((T^*X_0)^- \times T^*X_1).$$

To ensure the compatibility of the following proposition, we introduce a twisted version of the above equivalence. Namely, we define the equivalence

$$(a_0 \times id_1)^\natural_* : DF(T^*X_0 \times T^*X_1) \sim DF((T^*X_0)^- \times T^*X_1)$$

to be the composition of $(a_0 \times id_1)_*$ with the twist by the pullback $p_0^\natural(or_{X_0})$ of the orientation bundle from the first factor.

Given an object $L$ of $DF(T^*X_0 \times T^*X_1)$, we write

$$L_{(0,1)} = (a_0 \times id_1)^\natural(L)$$

for the corresponding object of $DF((T^*X_0)^- \times T^*X_1)$.

We leave the proof of the following to the interested reader; it is not used in other parts of the paper. Similar identities exist for the other “integral transforms”.
Proposition 5.4.1. Given objects $P_0$ of $\mathcal{D}(T^* X_0)$ and $L$ of $\mathcal{D}(T^* X_0 \times T^* X_1)$, there is a functorial isomorphism of left $\mathcal{D}(T^* X_1)$-modules

$$\mathcal{Y}_{\text{ord}}(P_0 \# L_{(0,1)}) \simeq \tilde{\Psi}_L(P_0).$$

6. Appendix: invariance of calculations

We discuss here some aspects of the invariance of calculations among microlocal branes. We assume the standard (though highly intricate) theory for compact exact branes (in the form explained by Seidel [33]), and comment about the modest modifications needed to treat the noncompact branes we consider. We do not attempt anything approximating a comprehensive discussion, but rather specifically argue for the independence of the taming perturbation in the definition of a microlocal brane (see Section 3.3).

6.1. Almost complex structures. Recall that in the definition of $F(T^* X)$, we work with an asymptotically conical almost complex structures $J_{\text{con}} \in \text{End}(T(T^* X))$ (see Section 3.1). Then we require that every microlocal brane $L$ comes equipped with a taming perturbation $\psi$ that moves it to a brane $\psi(L)$ that is tame with respect to the induced metric $g_{\text{con}}(v,v) = \omega(v, J_{\text{con}} v)$ (see Section 3.3). These requirements ensure that the moduli spaces defining the structure constants of the $A_\infty$-operations of $F(T^* X)$ are compact (see again Section 3.3).

Our aim here is to show that in fact calculations among microlocal branes are independent of the class of asymptotically conical almost complex structures. For any finite calculation (finite number of objects, finite number of $A_\infty$-operations), we will show that as long as we choose a compatible almost complex structure $J$ such that $T^* X$ and the branes under consideration are tame, the resulting $A_\infty$-operations will be compatible with those defined with respect to any other such almost complex structure $J'$ (in particular, an asymptotically conical almost complex structure). Furthermore, our arguments can be made compatibly for increasing unions of finite calculations.

To isolate the role of the almost complex structure, let us fix a finite collection of $A_\infty$-operations. Consider the corresponding moduli problems of $J$-holomorphic disks that calculate the structure constants of the operations. Recall that we have an a priori diameter bound on the relevant $J$-holomorphic disks: there is a large $r > 0$ such that none of the disks enter the region of $T^* X$ given by $|\xi| > r/2$.

Now replace $J$ by a compatible almost complex structure $J_{\text{cut}}$ of the following form:

1. $J_{\text{cut}} = J$ in the region $|\xi| < r/2$. 
(2) \( J_{\text{cut}} = J' \) in the region \( |\xi| > r \).

So \( J_{\text{cut}} \) equals \( J \) in a compact region, \( J' \) near infinity, and whatever one chooses in between. By construction, for the fixed collection of \( A_\infty \)-operations, we have the same a priori diameter bound on the relevant \( J_{\text{cut}} \)-holomorphic disks. This follows from the property (1) above and the local derivation of the diameter bound. Thus the corresponding moduli spaces for \( J \) and for \( J_{\text{cut}} \) are in fact equal.

Finally, choose a \([0, 1]\)-family of compatible almost complex structure \( J_t \) satisfying:

1. \( J_0 = J_{\text{cut}} \).
2. \( J_1 = J' \).
3. \( J_t = J_{\text{cut}} = J' \) in the region \( |\xi| > r \) for all \( t \).

Since \( J_t \) is constant near infinity, we can apply standard homotopy arguments \[33\] to compare the \( A_\infty \)-operations for \( J_{\text{cut}} \) and for \( J' \).

6.2. Taming perturbations. We apply here Lemma 6.1.1 to see that the choice of taming perturbation \( \psi \) in the definition (see Section \[33\]) of a microlocal brane \( L \) does not affect calculations.

To isolate the role of the taming perturbation, let us fix a finite collection of test branes \( P_1, \ldots, P_d \subset T^*X \), and without loss of generality, assume that they are already mutually transverse and do not intersect each other or \( L \) at infinity.

**Lemma 6.2.1.** Suppose the brane \( L \) is equipped with two taming perturbations \( \psi \) and \( \psi' \). Then for any finite collection of test objects \( P_1, \ldots, P_d \subset F(T^*X) \), there is a quasi-isomorphism intertwining the \( A_\infty \)-operations within the collection \( \psi(L), P_1, \ldots, P_d \) and the collection \( \psi'(L), P_1, \ldots, P_d \).

**Proof.** By assumption, we have compatible almost complex structures \( J \) and \( J' \) such that the respective collections of branes \( \psi(L), P_0, \ldots, P_d \) and \( \psi'(L), P_0, \ldots, P_d \) are tame with respect to the respective induced metrics \( g \) and \( g' \).

Pulling back the almost complex structures \( J \) and \( J' \), we obtain tame almost complex structures \( J_0 = \psi^*(J) \) and \( J'_0 = \psi'^*(J') \) such that the branes \( L, P_0, \ldots, P_d \) are tame with respect to both \( J_0 \) and \( J'_0 \). Thus we are in the setting of Lemma 6.1.1 and can conclude that the \( A_\infty \)-operations calculated with respect to \( J_0 \) and \( J'_0 \) are quasi-isomorphic. By construction, this is the same as a quasi-isomorphism intertwining the \( A_\infty \)-operations within the collection \( \psi(L), P_1, \ldots, P_d \) and the collection \( \psi'(L), P_1, \ldots, P_d \). \( \square \)

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