Effect of position-dependent mass on dynamical 
breaking of type B and type $X_2\,\mathcal{N}$-fold supersymmetry

Bikashkali Midya$^1$, Barnana Roy$^1$ and Toshiaki Tanaka$^{2,3,4}$

$^1$ Physics and Applied Mathematics Unit, Indian Statistical Institute, Kolkata 700108, India
$^2$ Department of Physics, National Cheng Kung University, Tainan 701, Taiwan, Republic of China
$^3$ National Center for Theoretical Sciences, Taiwan, Republic of China

E-mail: bikash.midya@gmail.com, barnana@isical.ac.in and toshiaki@post.kek.jp

Received 5 January 2012, in final form 2 April 2012
Published 2 May 2012
Online at stacks.iop.org/JPhysA/45/205303

Abstract
We investigate effect of position-dependent mass profiles on dynamical 
breaking of $\mathcal{N}$-fold supersymmetry in several type B and type $X_2$ models. 
We find that $\mathcal{N}$-fold supersymmetry in rational potentials in the constant-mass 
background is steady against the variation of mass profiles. On the other hand, 
some physically relevant mass profiles can change the pattern of dynamical 
$\mathcal{N}$-fold supersymmetry breaking in trigonometric, hyperbolic and exponential 
potentials of both type B and type $X_2$. The latter results open the possibility 
of detecting experimentally the phase transition of $\mathcal{N}$-fold as well as ordinary 
supersymmetry at a realistic energy scale.

PACS numbers: 03.65.Ca, 03.65.Ge, 11.30.Pb, 81.90.+c

1. Introduction

In recent years, the study of quantum mechanical systems with a position-dependent mass 
(PDM) has attracted a lot of interest due to their relevance in describing the physics of many 
microstructures of current interest, such as compositionally graded crystals [1], semiconductor 
heterostructure [2], quantum dots [3], He clusters [4], metal clusters [5], etc. The concept of 
PDM comes from the effective-mass approximation [6, 7], which is a useful tool for studying 
the motion of carrier electrons in pure crystals and also for the virtual-crystal approximation 
in the treatment of homogeneous alloys (where the actual potential is approximated by a 
periodic potential) as well as in graded mixed semiconductors (where the potential is not 
periodic). Recent interest in this field stems from extraordinary developments in crystal-growth 
techniques such as molecular beam epitaxy, which allows the production of nonuniform
semiconductor specimens with abrupt heterojunctions [8]. In these mesoscopic materials, the effective mass of the charge carrier is position dependent. Consequently, the study of the position-dependent mass Schrödinger equation (PDMSE) becomes relevant for deeper understanding of the non-trivial quantum effects observed on these nanostructures. It has also been found that such equations appear in many different areas. For example, it has been shown that constant-mass Schrödinger equations in curved space and those based on deformed commutation relations can be interpreted in terms of PDMSE [9]. The PDM also appears in the nonlinear oscillator [10, 11] and $\mathcal{PT}$-symmetric cubic anharmonic oscillator [12]. The most general form of the PDM Hamiltonian proposed by von Roos [13] is defined by

$$H = -\frac{1}{4} \left( m(q)^{\alpha} \frac{d}{dq} m(q)^{\beta} \frac{d}{dq} m(q)^{\gamma} + m(q)^{\gamma} \frac{d}{dq} m(q)^{\beta} \frac{d}{dq} m(q)^{\alpha} \right) + V(q),$$

where the ambiguity parameters $\alpha$, $\beta$ and $\gamma$ are related by $\alpha + \beta + \gamma = -1$. The above Hamiltonian always has the following form:

$$H = -\frac{1}{2m(q)} \frac{d^2}{dq^2} + \frac{m'(q)}{2m(q)^2} \frac{d}{dq} + U(q),$$

where the effective potential $U(q)$ is given by

$$U(q) = V(q) - (\alpha + \gamma) \frac{m'(q)}{4m(q)^2} + (\alpha \gamma + \alpha + \gamma) \frac{m'(q)^2}{2m(q)^3}.$$ 

It is quite natural that the physical interests described above have also enhanced the studies on exact solutions to PDMSE [14–30] by employing various methods, e.g., supersymmetric (SUSY) quantum mechanics [31] and point canonical transformation [32] to mention a couple. Later, PDM quantum systems were successfully formulated in the framework of $\mathcal{N}$-fold SUSY in [33], which has provided until now the most general tool for constructing a PDM system which admits exact solutions because of its equivalence to weak quasi-solvability. To avoid confusion, we note here that $\mathcal{N}$-fold SUSY is different from nonlinear SUSY, which has been long employed since the work by Samuel and Wess [34] in 1983 to indicate the nonlinearly realized SUSY originating from the work by Volkov and Akulov [35] in 1972. For a review of $\mathcal{N}$-fold SUSY, see [36], while for recent works on nonlinear SUSY, see, e.g., [37] and references therein.

Very recently, new classes of exactly solvable PDM quantum systems whose eigenfunctions are expressible in terms of so-called $X_1$ polynomials were constructed in [28]. The new findings of $X_n$ polynomials ($n \geq 1$) were associated with the more fundamental mathematical concept of exceptional polynomial subspaces of codimension $n$ introduced in [38–40], whose origin can be traced back to the pioneering work on the classification of monomial spaces preserved by second-order linear differential operators [41].

The purpose of this paper is twofold. The first is to bring the purely mathematical concept of exceptional polynomial subspaces into more physical settings by allowing the position dependence of mass (in a spirit similar to [28]) in the framework of $\mathcal{N}$-fold SUSY. In the constant-mass case, the form of potentials related to exceptional polynomial systems is very limited. Thus, we can enlarge the physical applicability of the mathematical concept by introducing PDM to quantum systems. On the other hand, the framework of $\mathcal{N}$-fold SUSY enables us to talk about the physical phenomenon of dynamical $\mathcal{N}$-fold SUSY breaking. The second purpose is actually to examine the effect of PDM profiles on dynamical breaking of $\mathcal{N}$-fold SUSY. In this respect, it is rather surprising that there have been few papers, such as [11], where broken as well as unbroken SUSY is described in PDM backgrounds depending on the mass profiles. One of the main reasons would be that SUSY has been mostly used just as a technique to obtain exact solutions. The true significance of Witten’s SUSY quantum
mechanics [31], however, rather resides in the nonperturbative aspects of dynamical SUSY breaking. Hence, one of our main purposes is, in other words, to examine a change of the nonperturbative nature of quantum systems caused by variations of mass profiles in view of dynamical $\mathcal{N}$-fold SUSY breaking.

This paper is organized as follows. In section 2, we provide a self-contained review of $\mathcal{N}$-fold SUSY in a PDM background, especially for those who are not familiar with the subject. We also summarize the mathematical structure of type B and type $X_2$ $\mathcal{N}$-fold SUSY. In section 3, we construct several $\mathcal{N}$-fold SUSY PDM quantum systems and examine dynamical $\mathcal{N}$-fold SUSY breaking in different PDM backgrounds. The first three models of type $B, X_2$ $\mathcal{N}$-fold SUSY have rational, trigonometric and exponential potentials in the constant-mass case. We show in particular that the models whose bound state eigenfunctions were shown to be expressed in terms of $X_1$ polynomials in [42] for the constant-mass case and in [28] for the PDM cases can be obtained as type $B$ systems. The last three models of type $X_2$ $\mathcal{N}$-fold SUSY have rational, hyperbolic and exponential potentials in the constant-mass case. For both types of $\mathcal{N}$-fold SUSY, we find that the rational potentials have steady $\mathcal{N}$-fold SUSY against variation of mass profile, while all the other types of potentials can receive the effect of PDM on their dynamical breaking of $\mathcal{N}$-fold SUSY. Finally, we summarize the results and discuss their implications and prospects in section 4.

2. Review of $\mathcal{N}$-fold SUSY in a PDM background

An $\mathcal{N}$-fold SUSY one-body quantum mechanical system with PDM is composed of a pair of PDM Hamiltonians

$$H^{\pm} = -\frac{1}{2m(q)} \frac{d^2}{dq^2} + \frac{m'(q)}{2m(q)^2} \frac{d}{dq} + U^{\pm}(q),$$  

(2.1)

and an $\mathcal{N}$th-order linear differential operator

$$P_N^\pm = m(q)^{-\mathcal{N}/2} \frac{d^\mathcal{N}}{dq^\mathcal{N}} + \sum_{k=0}^{\mathcal{N}-1} w_k^{[\mathcal{N}]}(q) \frac{d^k}{dq^k},$$  

(2.2)

which satisfy the following intertwining relations:

$$P_N^\pm H^{\mp} = H^{\mp} P_N^\pm, \quad P_N^\mp H^{\pm} = H^{\pm} P_N^\mp.$$  

(2.3)

In the above, $P_N^\pm$ is the transposition [43] of $P_N^\mp$ given by

$$P_N^\pm = (P_N^\mp)^T = \left(-\frac{d}{dq}\right)^\mathcal{N} m(q)^{-\mathcal{N}/2} + \sum_{k=0}^{\mathcal{N}-1} \left(-\frac{d}{dq}\right)^k w_k^{[\mathcal{N}]}(q).$$  

(2.4)

Actually, the two relations in (2.3) are not independent; the first implies the second and vice versa since the PDM Hamiltonians (2.1) are invariant under the transposition $(H^{\pm})^T = H^{\mp}$.

One of the significant consequences of the intertwining relations (2.3) is weak quasi-solvability, that is, $H^{\pm}$ preserves a finite-dimensional linear space $\mathcal{V}_N^{\pm}$ spanned by the kernel of the operator $P_N^{\pm}$

$$H^{\pm}\mathcal{V}_N^{\pm} \subset \mathcal{V}_N^{\pm}, \quad \mathcal{V}_N^{\pm} = \ker P_N^{\pm}.$$  

(2.5)

Each space $\mathcal{V}_N^{\pm}$ is called a solvable sector of $H^{\pm}$. Except for the $\mathcal{N} = 2$ case (cf [36, 44]), virtually all the $\mathcal{N}$-fold SUSY systems so far found admit analytic expression of $\mathcal{V}_N^{\pm}$ in the closed form, and are thus quasi-solvable. In addition, it sometimes happens when either $H^-$ or $H^+$ does not depend essentially on $\mathcal{N}$ and preserves an infinite flag of the solvable sectors

$$\mathcal{V}_{-1}^{\pm} \subset \mathcal{V}_0^{\pm} \subset \cdots \subset \mathcal{V}_N^{\pm} \subset \cdots.$$  

(2.6)
In this case, it is said to be solvable, which is a necessary condition for exact solvability. We note that $H^-$ and $H^+$ are usually simultaneously solvable due to the intertwining relations (2.3).

A set of an $\mathcal{N}$-fold SUSY system $H^\pm$ and $P_{\mathcal{N}}^\pm$ provides a representation of $\mathcal{N}$-fold superalgebra defined by
\[
[Q_{\mathcal{N}}^+, H] = [Q_{\mathcal{N}}^+, Q_{\mathcal{N}}^-] = 0,
\]
\[
\{Q_{\mathcal{N}}^-, Q_{\mathcal{N}}^\pm \} = 2^{\mathcal{N}} P_{\mathcal{N}}^N(H),
\]
(2.7)
where $P_{\mathcal{N}}^N(x)$ is a monic polynomial of degree $\mathcal{N}$ in $x$. Indeed, it is realized by defining $H$ and $Q_{\mathcal{N}}^\pm$ as
\[
H = H^- \psi^- + H^+ \psi^+, \quad Q_{\mathcal{N}}^+ = P_{\mathcal{N}}^N \psi^+, \quad Q_{\mathcal{N}}^- = P_{\mathcal{N}}^N \psi^-,
\]
(2.8)
where $\psi^\pm$ is a pair of fermionic variables satisfying $\{\psi^+, \psi^-\} = 0$ and $\{\psi^-, \psi^+\} = 1$. It is easy to check that the above $H$ and $Q_{\mathcal{N}}^\pm$ satisfy the first part of algebra (2.7). In particular, the intertwining relations in (2.3) guarantee the commutativity of $H$ and $Q_{\mathcal{N}}^\pm$. Regarding the second part of the algebra, the monic polynomial $P_{\mathcal{N}}^N$ is given, in the above representation, by [43, 33]
\[
P_{\mathcal{N}}^N(H) = \det \left( H - H^\pm |_{\mathcal{V}_{\mathcal{N}}^\pm} \right),
\]
(2.9)
namely the characteristic polynomial for $H^\pm$ restricted to the solvable sectors $\mathcal{V}_{\mathcal{N}}^\pm$.

Whether the $\mathcal{N}$-fold SUSY of the system under consideration is dynamically broken is determined by a property of the solvable sectors $\mathcal{V}_{\mathcal{N}}^\pm$ since they characterize $\mathcal{N}$-fold SUSY states, namely states annihilated by the pair of $\mathcal{N}$-fold supercharges $Q_{\mathcal{N}}^\pm$. Let $|0\rangle$ and $|1\rangle$ be the fermionic vacuum and the one-fermion state, respectively, which satisfy
\[
\psi^- |0\rangle = 0, \quad |1\rangle = \psi^+ |0\rangle.
\]
Then, superstates $|\Psi_0^-\rangle = \Psi_0^- (q) |0\rangle$ and $|\Psi_0^+\rangle = \Psi_0^+ (q) |1\rangle$, respectively, are annihilated by
\[
Q_{\mathcal{N}}^\pm |\Psi_0^-\rangle = 0, \quad Q_{\mathcal{N}}^\pm |\Psi_0^+\rangle = 0,
\]
(2.11)
if and only if $\Psi_0^- (q) \in \mathcal{V}_{\mathcal{N}}^-$ and $\Psi_0^+ (q) \in \mathcal{V}_{\mathcal{N}}^+$, respectively. However, such states do not necessarily satisfy physical requirements. Suppose $\mathbb{S} \subset \mathbb{C}$ is a domain where both Hamiltonians $H^\pm$ have no singularities and are thus naturally defined, and $\mathfrak{H} (\mathbb{S})$ is a linear space of complex functions in which both $H^\pm$ act. In a usual physical application, the domain $\mathbb{S}$ is the real line $\mathbb{R}$ or a real half-line $\mathbb{R}_+ = (0, \infty)$, and the linear space $\mathfrak{H}$ is a Hilbert space $L^2$, so that $\mathfrak{H}(\mathbb{S}) \subset L^2(\mathbb{R})$, or $L^2(\mathbb{R}_+)$. In the latter cases, the physical requirement is the normalizability (square integrability) on $\mathbb{S}$. Then, there exist physical (normalizable) $\mathcal{N}$-fold SUSY states $|\Psi_0^-\rangle$ and/or $|\Psi_0^+\rangle$ which satisfy (2.11) if $\mathcal{V}_{\mathcal{N}}^- \subset L^2(\mathbb{S})$ and/or $\mathcal{V}_{\mathcal{N}}^+ \subset L^2(\mathbb{S})$, in other words, if $H^-$ and/or $H^+$ is quasi-exactly solvable. If no such physical $\mathcal{N}$-fold SUSY states in the Hilbert space $L^2(\mathbb{S})$ exist, then the $\mathcal{N}$-fold SUSY of the system is said to be dynamically broken. It was first shown correctly in [45] that the generalized Witten index characterizes $\mathcal{N}$-fold SUSY breaking, which corrected the wrong statement made earlier in [46].

For $\mathcal{N} > 1$, we can have an intriguing situation where not the whole of, but a subspace of the solvable sectors $\mathcal{V}_{\mathcal{N}}^- (\mathbb{S})$ and/or $\mathcal{V}_{\mathcal{N}}^+ (\mathbb{S})$ belongs to the Hilbert space $L^2(\mathbb{S})$. In this case, the $\mathcal{N}$-fold SUSY of the system is said to be partially broken. Partial breaking of $\mathcal{N}$-fold SUSY was first discovered in [47]. We note that it is different in nature from the partial breaking of (nonlinear) SUSY [48, 49].

Construction of an $\mathcal{N}$-fold SUSY system is in general quite difficult, especially for a larger value of $\mathcal{N}$, since the intertwining relations (2.3) are composed of coupled nonlinear differential equations for $U^\pm (q)$ and $w_k^N (q)$ $(k = 0, \ldots, \mathcal{N} - 1)$. For the direct calculations of intertwining relations in a PDM background in the cases of $\mathcal{N} = 1$ and 2, see [24]. To circumvent the difficulty, a systematic algorithm for constructing an $\mathcal{N}$-fold SUSY system was
developed in [47] for constant-mass quantum mechanics and was later generalized to PDM systems in [33]. The significant feature which is common in both constant-mass and PDM systems is that an $\mathcal{N}$-dimensional linear space of functions
\begin{equation}
\hat{V}_N = \langle \hat{\psi}_1(z), \ldots, \hat{\psi}_N(z) \rangle
\end{equation}
(2.12)
preserved by a second-order linear differential operator $\hat{H}$ can determine the whole of an $\mathcal{N}$-fold SUSY system. Indeed, we can construct a pair of $\mathcal{N}$th-order linear differential operators $\tilde{P}_N^\pm$ and another $\mathcal{N}$-dimensional vector space $\tilde{V}_N^\pm$ such that $\mathcal{V}_N^\pm = \text{ker} \tilde{P}_N^\pm$. Then, we can show that a pair of second-order linear differential operators given by
\begin{equation}
\tilde{H}^\pm = -A(z) \frac{d^2}{dz^2} + \left[ \frac{\mathcal{N} - 2}{2} A'(z) \pm Q(z) \right] \frac{d}{dz} - C(z)
\end{equation}
(2.13)
is weakly quasi-solvable with respect to the spaces $\tilde{V}_N^\pm$, namely $\tilde{H} \tilde{V}_N^\pm \subset \tilde{V}_N^\pm$. With the choice of the change of variable $z = z(q)$ and the gauge potential $W_N$, determined by
\begin{equation}
z'(q)^2 = 2m(q)A(z), \quad W_N^\pm = -\frac{1}{4} \ln |m(q)| + \frac{\mathcal{N} - 1}{4} \ln |2A(z)| + \int dz \frac{m(q)Q(z)}{2A(z)},
\end{equation}
we can obtain an $\mathcal{N}$-fold SUSY system by
\begin{equation}
H^\pm = e^{-W_N^\pm} \tilde{H} e^{W_N^\pm} \bigg|_{z = z(q)}, \quad P_N^\pm = e^{-W_N^\pm} \tilde{P}_N^\pm e^{W_N^\pm} \bigg|_{z = z(q)}.
\end{equation}
(2.15)
With the change of variable and the gauge transformation, both $H^\pm$ obtain the form of PDM Hamiltonian (2.1) and their effective potentials $U^\pm(q)$ are given by
\begin{equation}
U^\pm(q) = \frac{1}{2m(q)} \left[ \left( \frac{dW_N^-}{dq} \right)^2 - \frac{dW_N^-}{dq} + \frac{m'(q) dW_N^-}{m(q)} \right] - C(z(q))
\end{equation}
\begin{equation}
- (1 \pm 1) \left[ \frac{\mathcal{N} - 1}{2} Q'(z) - \frac{1}{2} A'(z) \tilde{u}_{N^{-1}}^{[\mathcal{N}]}(z) - A(z) \tilde{u}_{N^{-1}}^{[\mathcal{N}]}(z) \right] \bigg|_{z = z(q)}.
\end{equation}
(2.16)
The solvable sectors $V_N^\pm$ of $H^\pm$ are evidently given by
\begin{equation}
V_N^\pm = \text{ker} P_N^\pm = e^{-W_N^\pm} \tilde{V}_N^\pm \bigg|_{z = z(q)}.
\end{equation}
(2.17)
In principle, we can construct a pair of $\mathcal{N}$-fold SUSY PDM Hamiltonians $H^\pm$ and its solvable sectors $\mathcal{V}_N^\pm$ by using formulas (2.16) and (2.17). However, there is an easier way to obtain such a system when we already have an ordinary $\mathcal{N}$-fold SUSY constant-mass quantum system at hand. Suppose the latter system is such that its pair of potentials $V^{(0)\pm}(q)$, gauge potentials $W_N^{(0)\pm}(q)$ and solvable sectors $V_N^{(0)\pm}[q]$ are all known. Then, an $\mathcal{N}$-fold SUSY PDM system having a pair of effective potentials $U^\pm(q)$, gauge potentials $W_N^\pm(q)$ and solvable sectors $V_N^\pm[q]$ can be constructed immediately via the following prescription:
\begin{align}
U^\pm(q) &= V^{(0)\pm}(u(q)) + \frac{m''(q)}{8m(q)^2} - \frac{7m'(q)^2}{32m(q)^3}, \quad (2.18a) \\
W_N^\pm(q) &= -\frac{1}{4} \ln |m(q)| + W_N^{(0)\pm}(u(q)), \quad (2.18b) \\
V_N^\pm[q] &= m(q)^{1/4} V_N^{(0)\pm}[u(q)]. \quad (2.18c)
\end{align}
where function $u(q)$ is given by

$$u(q) = \int dq \sqrt{m(q)}.$$  \hspace{1cm} (2.19)

Actually, the above relations are consistent with the formulas obtained by the point canonical transformation; see, e.g., equations (2.7) and (2.8) in [22], equation (7) in [21] and equations (10), (13) and (14) in [19]. The above relations (2.18a) and (2.18c) have also been verified in [33] where type A $\mathcal{N}$-fold SUSY has been constructed in a PDM background. One of the most salient features unveiled by the algorithmic construction is that both constant-mass and PDM quantum systems with $\mathcal{N}$-fold SUSY have totally the same structure in the gauged $z$-space. That is, the functional forms of the gauged operators such as $\bar{P}_N^+$ and $H_-$ given by (2.13) are identical in both cases. It means in particular that the starting vector space $\bar{\mathcal{N}}_N$ determines all in the algorithm regardless of whether mass is constant or not. Hence, different types of $\mathcal{N}$-fold SUSY are characterized by different types of vector spaces $\bar{\mathcal{N}}_N$ and vice versa. Until now, four different types have been discovered, namely type A [50, 44], type B [51], type C [47] and type $X_2$ [52]. We note that almost all the models having essentially the same symmetry as $\mathcal{N}$-fold SUSY but called by other names in the literature, such as Pöschl–Teller and Lamé potentials, are actually particular cases of type A $\mathcal{N}$-fold SUSY. In this paper, we focus on constructing PDM quantum systems with type B and type $X_2$ $\mathcal{N}$-fold SUSY since the other types (types A and C) are not related to exceptional polynomial subspaces. In what follows, we shall review the general structure of these two types of $\mathcal{N}$-fold SUSY.

### 2.1. Type B $\mathcal{N}$-fold SUSY

Type B $\mathcal{N}$-fold SUSY was first discovered in [51] and was found to be associated with the following monomial space:

$$\bar{\mathcal{N}}_N^B = \bar{\mathcal{N}}_N^B := (1, z, \ldots, z^{N-2}, z^N),$$  \hspace{1cm} (2.20)

called type B, which was considered in [41] in the context of the classification of monomial spaces preserved by second-order linear ordinary differential operators. Applying the algorithm to the type B monomial space, we obtain [36] the gauged $\mathcal{N}$-fold supercharge components

$$\bar{P}_N^+ = \zeta(q)^N \left( \frac{d}{dz} - \frac{1}{z} \right) \frac{d^{N-1}}{dz^{N-1}}, \quad \bar{P}_N^- = \zeta(q)^N \frac{d^{N-1}}{dz^{N-1}} \left( \frac{d}{dz} + \frac{1}{z} \right),$$  \hspace{1cm} (2.21)

d and the functions which characterize the gauged Hamiltonians (2.13) are given by

$$A(z) = a_1 z^2 + a_2 z + a_3,$$

$$2Q(z) = -N a_3 z^2 + 2b_1 z - N a_1,$$

$$C(z) = N(N-3)a_3 z + N(N-2)a_2 z + c_0,$$  \hspace{1cm} (2.23)

and $\bar{W}_{N-1}^B(z) = z^{-1}$. The other linear space $\bar{\mathcal{N}}_N^+$ preserved by $\bar{H}_+$ is given by

$$\bar{\mathcal{N}}_N^+ = \zeta^{-1}(1, \zeta, \ldots, \zeta^N).$$

We note that both the monomial spaces (2.20) and (2.25) are actually exceptional polynomial subspaces of codimension 1, see [40]. We can easily check that the type B Hamiltonian $\bar{H}_+$ preserves an infinite flag of the following spaces:

$$\bar{\mathcal{N}}_N^+ e^{-\bar{W}_N^+} \subset \bar{\mathcal{N}}_N^+ e^{-\bar{W}_N^+} \subset \cdots \subset \bar{\mathcal{N}}_N^+ e^{-\bar{W}_N^+} \subset \cdots,$$  \hspace{1cm} (2.26)

where $\bar{\mathcal{N}}_N^+$ and $\bar{W}_N^+$ are given by (2.25) and (2.14), respectively, and thus $\bar{H}_+$ is solvable if and only if $a_3 = a_4 = 0$. On the other hand, the partner type B Hamiltonian $\bar{H}_-$ does not
appear to be solvable for any parameter value since the type B monomial space (2.20) does not constitute an infinite flag due to the fact that \( \tilde{V}^{(B)}_N \not\subset \tilde{V}^{(B)}_{N+1} \) for all \( N = 1, 2, \ldots \). However, it turns out [36] that, when \( a_3 = a_4 = 0 \) and \( H^+ \) becomes solvable, the partner Hamiltonian \( H^- \) does preserve an infinite flag of linear spaces given by

\[
\tilde{V}^{(A)}_1 e^{-W_N} \subset \tilde{V}^{(A)}_2 e^{-W_N} \subset \cdots \subset \tilde{V}^{(A)}_N e^{-W_N} \subset \cdots ,
\]

where \( W_N \) is given by (2.14) and \( \tilde{V}^{(A)}_N \) is the type A monomial space defined by

\[
\tilde{V}^{(A)}_N = (1, z, \ldots, z^{N-1}).
\]

That is, \( H^- \) and \( H^+ \) can be solvable simultaneously. In this paper, all the type B models that will be considered later satisfy the solvability condition \( a_3 = a_4 = 0 \). Thus, all the pairs of type B Hamiltonians \( H^\pm \) preserve the infinite-dimensional solvable sectors \( \tilde{V}^\pm \) given by

\[
\tilde{V}^- = (1, z(q), z(q)^2, \ldots) e^{-W_N(q)},
\]

\[
\tilde{V}^+ = (1, z(q)^2, z(q)^3, \ldots) e^{-W_N(q)}.
\]

An interesting consequence of the fact that \( H^- \) and \( H^+ \) preserve different types of infinite flag of spaces in the solvable case is that the eigenfunctions of \( H^- \) are expressed in terms of a classical polynomial system while those of \( H^+ \) are in terms of an \( X_1 \) Laguerre or Jacobi polynomial. It is exactly the underlying reason why some of the Hamiltonians whose eigenfunctions are expressed in terms of the \( X_1 \) Laguerre or Jacobi polynomials using an intertwining or SUSY techniques in [53, 54, 27].

2.2. Type \( X_2 \) \( \mathcal{N} \)-fold SUSY

Type \( X_2 \) \( \mathcal{N} \)-fold SUSY constructed in [52] is associated with the following exceptional polynomial subspace of codimension 2:

\[
\tilde{V}^-_N = (\tilde{\psi}_1(z; \alpha), \ldots, \tilde{\psi}_N(z; \alpha)),
\]

where \( \tilde{\psi}_n(z; \alpha) \) is a polynomial of degree \( n + 1 \) in \( z \) with a parameter \( \alpha \neq 0, 1 \) defined by

\[
\tilde{\psi}_n(z; \alpha) = (\alpha + n - 2)z^{n+1} + 2(\alpha + n - 1)(\alpha - 1)z^n + (\alpha + n)(\alpha - 1)\alpha z^{n-1}.
\]

Applying the algorithm to the \( X_2 \) space (2.30), we obtain [52] the gauged \( \mathcal{N} \)-fold supercharge components

\[
\tilde{P}^-_N = \tilde{z}(q)^N f(z; \alpha) \prod_{k=0}^{N-1} \left[ f(z; \alpha + k + 1) \frac{d}{dz} - f(z; \alpha + k) \right] \frac{f(z; \alpha + k + 1)}{f(z; \alpha + k + 1)}
\]

\[
\tilde{P}^+_N = \tilde{z}(q)^N \prod_{k=0}^{N-1} \left[ f(z; \alpha + N - k) \frac{d}{dz} + f(z; \alpha + N - k) \right] \frac{f(z; \alpha + N - k)}{f(z; \alpha + N - k)}
\]

\[
\text{where } \prod_{k=0}^{N-1} A_k := A_{N-1} \cdots A_1 A_0, \text{ and the functions } f(z; \alpha) \text{ and } \tilde{u}^{(A)}_{N-1}(z) \text{ are given by}
\]

\[
f(z; \alpha) = z^2 + 2(\alpha - 1)z + (\alpha - 1)\alpha,
\]

\[
\tilde{u}^{(A)}_{N-1}(z) = -(N - 1) f'(z; \alpha) - f(z; \alpha + N) / f(z; \alpha + N).
\]

The most general forms of the functions \( A(z) \), \( Q(z) \), and \( C(z) \) appearing in \( \tilde{H}^\pm \) depend on four parameters \( a_i \) (\( i = 1, \ldots, 4 \)), but in this paper we only consider models with \( a_4 = a_3 = 0 \). In the latter case, they read as

\[
A(z) = a_2 z^3 + a_1 z + (\alpha - 1)(\alpha + N - 1) a_2.
\]
Here we shall consider three examples of type B models constructed below satisfy the solvability condition followed by the corresponding results in the PDM case. As we referred to before, all the type B choices of mass function in a few examples. Also it will be shown that the bound state wavefunctions of functions on symmetry breaking or restoration, it will be appropriate to consider more than one choices of physically interesting mass functions. In order to explore in detail the impact of mass PDM cases, we shall see below that the answer is in the affirmative in some cases for particular

$$Q(z) = -a_2 z^2 - (3a_2 + a_1)z - (\alpha - 1)(3\alpha + 3\mathcal{N} - 7)a_2$$
$$+ \frac{2\alpha + \mathcal{N} - 8}{2} a_1 + \frac{4(\alpha - 1)D(z)}{f(z; \alpha)}, \quad (2.36)$$

$$C(z) = a_2 z + c_0 = \frac{4(\alpha - 1)D(z)}{f(z; \alpha)}, \quad (2.37)$$

where $D(z)$ is given by

$$D(z) = -[(2\alpha + \mathcal{N} - 3)a_2 - a_1]z - (\alpha - 1)(2\alpha + \mathcal{N} - 1)a_2 + \alpha a_1. \quad (2.38)$$

For their most general forms, please refer to [52]. The other linear space $\tilde{V}^+_\mathcal{N}$ preserved by $\tilde{H}^+$ is given by

$$\tilde{V}^+_\mathcal{N} = (\tilde{\chi}_1(z; \alpha + \mathcal{N}), \ldots, \tilde{\chi}_\mathcal{N}(z; \alpha + \mathcal{N})) f(z; \alpha)^{-1} f(z; \alpha + \mathcal{N})^{-1}, \quad (2.39)$$

where $\tilde{\chi}_n(z; \alpha)$ is a polynomial of degree $n + 1$ in $z$ defined by

$$\tilde{\chi}_n(z; \alpha) = (\alpha - n)(\alpha - n + 1)z^n + 2(\alpha - n - 1)(\alpha - n + 1)(\alpha - n)z^n$$
$$+(\alpha - n - 1)(\alpha - n)(\alpha - n)z^n. \quad (2.40)$$

The solvable sectors $\mathcal{V}^\pm_{\mathcal{N}}$ of the constant-mass Hamiltonians $H^\pm$ are

$$\mathcal{V}^\pm_{\mathcal{N}} = \langle \tilde{\phi}_1(z(q); \alpha), \ldots, \tilde{\phi}_\mathcal{N}(z(q); \alpha) \rangle e^{-W_\mathcal{N}(q)}, \quad (2.41)$$

$$\mathcal{V}^\pm_{\mathcal{N}} = \langle \tilde{\chi}_1(z(q); \alpha + \mathcal{N}), \ldots, \tilde{\chi}_\mathcal{N}(z(q); \alpha + \mathcal{N}) \rangle f(z(q); \alpha)^{-1} f(z(q); \alpha + \mathcal{N})^{-1} e^{-W_\mathcal{N}(q)}. \quad (2.42)$$

Finally, type $X_2$ Hamiltonians $H^\pm$ preserve the infinite flag of the spaces $\mathcal{V}^\pm_{\mathcal{N}} (\mathcal{N} = 1, 2, \ldots)$ and are simultaneously solvable if and only if $a_2 = (a_3 = a_4 = 0)$.0.

### 3. Type B and type $X_2 \mathcal{N}$-fold SUSY for PDM

In this section, we shall consider some models which belong to type B and type $X_2 \mathcal{N}$-fold SUSY. In order to study the effect of PDM in these models, we need to consider simultaneously the corresponding constant-mass type B and type $X_2$ models as well. In particular, we shall apply ourselves to the following question: does PDM have any effect on dynamical breaking of type B and type $X_2 \mathcal{N}$-fold SUSY? By comparing the solvable sectors of both constant and PDM cases, we shall see below that the answer is in the affirmative in some cases for particular choices of physically interesting mass functions. In order to explore in detail the impact of mass functions on symmetry breaking or restoration, it will be appropriate to consider more than one mass function in a few examples. Also it will be shown that the bound state wavefunctions of one of the partner potentials obtained in type B $\mathcal{N}$-fold SUSY are associated with exceptional $X_1$ Laguerre and Jacobi polynomials, while those of the other partner are associated with classical Laguerre and Jacobi polynomials.

#### 3.1 Effects of PDM on dynamical symmetry breaking of type B $\mathcal{N}$-fold SUSY

Here we shall consider three examples of type B $\mathcal{N}$-fold SUSY corresponding to three different choices of $A(z)$. In each of the examples, we first show the results in the constant-mass case, followed by the corresponding results in the PDM case. As we referred to before, all the type B models constructed below satisfy the solvability condition $a_1 = a_4 = 0$ and thus their solvable sectors in the constant-mass case are given by (2.29).
Example 3.1. \( A(z) = k(z - z_0) \) \( (k \neq 0) \).

Potentials:

\[
V^{(0)-}(q) = \frac{b_1^2}{8} q^2 + \frac{4(z_0 b_1 - \mathcal{N} k)^2 - k^2}{8k^2 q^2} + \frac{\mathcal{N} b_1}{2} + V_0, \tag{3.1}
\]

\[
V^{(0)+}(q) = \frac{b_1^2}{8} q^2 + \frac{4z_0 b_1^2 - k^2}{8k^2 q^2} + \frac{2k}{k q^2 + 2z_0} - \frac{8k z_0}{(k q^2 + 2z_0)^2} + V_0, \tag{3.2}
\]

where \( V_0 \) is an irrelevant constant given by

\[ V_0 = \frac{(z_0 b_1 - \mathcal{N} k) b_1}{2k} + \frac{b_1}{\mathcal{N}} - R. \]

Solvable sectors:

\[
V^{(0)-} = \langle 1, z(q), z(q)^2, \ldots \rangle q^{(2z_0 b_1 - 2\mathcal{N} k + k)/2k} e^{b q^2/4}, \tag{3.3}
\]

\[
V^{(0)+} = \langle 1, z(q)^2, z(q)^3, \ldots \rangle z(q)^{-1} q^{-(2z_0 b_1 - k)/2k} e^{-b q^2/4}. \tag{3.4}
\]

We assume \( k > 0 \) and \( z_0 > 0 \) so that the pair of potentials \( V^{\pm}(q) \) has no singularities except for \( q = 0 \). Thus, the system is naturally defined in \( L^2(\mathbb{R}_+), \mathbb{R}_+ = (0, \infty) \). In the latter Hilbert space, \( V^{(0)-}(\mathbb{R}_+) \subset L^2(\mathbb{R}_+) \) if and only if

\[ b_1 < 0 \quad \text{and} \quad z_0 b_1 > (\mathcal{N} - 1)k, \tag{3.5} \]

which cannot be satisfied by any \( b_1 \in \mathbb{R} \). On the other hand, \( V^{(0)+}(\mathbb{R}_+) \subset L^2(\mathbb{R}_+) \) if and only if

\[ b_1 > 0 \quad \text{and} \quad k > z_0 b_1. \tag{3.6} \]

Hence, \( \mathcal{N} \)-fold SUSY of the system is unbroken if and only if \( 0 < b_1 < k/z_0 \) on the constant-mass background.

Now, the relevant expressions for partner potentials, gauge potentials and corresponding solvable sectors of type B PDM systems can be obtained using (3.1)–(3.4), (2.18a) and (2.18c).

Since our main objective in this section is to study the effect of mass function on dynamical breaking of \( \mathcal{N} \)-fold SUSY, we give below only the solvable sectors \( V^{\pm} \) for an arbitrary mass function \( m(q) \):

\[
V^{-} = \langle 1, z(u(q)), z(u(q))^2, \ldots \rangle m(q)^{1/4} u(q)^{(2z_0 b_1 - 2\mathcal{N} k + k)/2k} e^{b u(q)^2/4}, \tag{3.7}
\]

\[
V^{+} = \langle 1, z(u(q))^2, z(u(q))^3, \ldots \rangle z(u(q))^{-1} m(q)^{1/4} u(q)^{-(2z_0 b_1 - k)/2k} e^{-b u(q)^2/4}, \tag{3.8}
\]

where \( u(q) \) is given by (2.19). At this point, we are in a position to choose a particular mass function. Let the mass function be

\[ m(q) = e^{-b q}, \quad b > 0, \quad q \in (-\infty, \infty), \tag{3.9} \]

which was considered in [28] where the PDM potentials were associated with \( X_1 \) Laguerre polynomials. This exponentially behaved mass function has often been used in the study of confined energy states for carriers in semiconductor quantum wells [19, 28]. It has also been used to compute transmission probabilities for scattering in abrupt heterostructures [25] which may be useful in the design of semiconductor devices [55]. For the mass function, the change of variable is given by

\[ u(q) = -\frac{2}{b} e^{-b q/2}, \tag{3.10} \]
and the pair of potentials $U^\pm(q)$ reads from (2.18a) as
\begin{equation}
U^-(q) = \frac{b_1^2}{2b^2} e^{-bq} + \frac{b_1^2(z_0b_1 - \mathcal{N}k^2 - k^2)}{8k^2} \exp \left[ -\left( \frac{z_0b_1}{k} - \mathcal{N} + 1 \right) \frac{b}{2} b + \frac{b_1}{b^2} e^{-bq} \right],
\end{equation}
\begin{equation}
U^+(q) = \frac{b_1^2}{2b^2} e^{-bq} + \frac{b_1^2(z_0b_1^2 - k^2)}{8k^2} \exp \left[ -\frac{kb^2}{2k e^{-bq} + z_0b^2} - \frac{2kz_0b^4}{(2k e^{-bq} + z_0b^2)^2} + V_0, \right.
\end{equation}
respectively. It is worth mentioning here that the potential $U^+(q)$ given in (3.11) is identical to the potential $V_{\text{eff}}(q)$ associated with exceptional $X_1$ Laguerre polynomials (e.g., equation (12) in [28]), if one takes $k = 1/2, b_1 = b^2/2$ and $z_0 = \alpha/b^2$. On the other hand, for the same choices of parameters the other potential $U^-(q)$ coincides with the potential (after making a translation $\alpha \rightarrow \alpha - \mathcal{N}$) previously obtained in [27] corresponding to classical Laguerre polynomials.

The solvable sectors of the potentials (3.11) and (3.12) are, respectively, given by
\begin{equation}
V^- = (1, e^{-bq} + \bar{z}_0, (e^{-bq} + \bar{z}_0)^2, \ldots) \exp \left[ -\left( \frac{z_0b_1}{k} - \mathcal{N} + 1 \right) \frac{b}{2} b + \frac{b_1}{b^2} e^{-bq} \right],
\end{equation}
\begin{equation}
V^+ = (1, (e^{-bq} + \bar{z}_0)^2, (e^{-bq} + \bar{z}_0)^3, \ldots) (e^{-bq} + \bar{z}_0)^{-1} \exp \left[ -\left( \frac{z_0b_1}{k} - 1 \right) \frac{b}{2} b - \frac{b_1}{b^2} e^{-bq} \right],
\end{equation}
where $\bar{z}_0 = z_0b^2/(2k)$. Here the potentials have no singularities in the finite part of the real line, so the domain is $\mathbb{R}$. Since $b > 0$, so $V^-(\mathbb{R}) \subset L^2(\mathbb{R})$ if and only if $b_1 < 0$. On the other hand, $V^+(\mathbb{R}) \subset L^2(\mathbb{R})$ if and only if $b_1 > 0$. Hence, the $\mathcal{N}$-fold SUSY of the PDM system is unbroken unless $b_1 = 0$. Comparing the solvable sectors of both the constant and PDM scenarios, it can be observed that it is not possible to break $\mathcal{N}$-fold SUSY dynamically for the particular choice of mass function $m(q) = e^{-bq}$. In addition, we have checked that many physically interesting mass functions also have no effect on symmetry breaking.

Example 3.2. $A(z) = a^2[1 - (z - z_0)^2]/2 ~ (a > 0)$.

Potentials:
\begin{equation}
V^{(0)-}(q) = \frac{(4b_1^2 - \mathcal{N}a^2)^2z_0}{4a^2} \frac{\sin aq}{\cos^2 aq} + \frac{(2b_1 - \mathcal{N}a^2)^2z_0^2 + (2b_1 + \mathcal{N}a^2)^2 - a^4}{8a^2} \tan^2 aq + \frac{b_1\mathcal{N}}{2} + V_0,
\end{equation}
\begin{equation}
V^{(0)+}(q) = \frac{(2b_1 - \mathcal{N}a^2)^2z_0}{4a^2} \frac{\sin aq}{\cos^2 aq} + \frac{(2b_1 - \mathcal{N}a^2)^2(z_0^2 + 1) - a^4}{8a^2} \tan^2 aq + \frac{b_1\mathcal{N}}{2} + V_0,
\end{equation}
where $V_0$ is an irrelevant constant given by
\begin{equation}
V_0 = \frac{b_1}{\mathcal{N}} + \frac{a^2(N^2 - 7)}{12} + \frac{(2b_1z_0 - \mathcal{N}z_0a^2)^2}{8a^2} - R.
\end{equation}

Solvable sectors:
\begin{equation}
V^{(0)-} = (1, z(q), z(q)^2, \ldots) \exp \left[ \frac{b_1\mathcal{N}}{2} \left( 1 + \frac{\sin aq}{1 - \sin aq} \right) \right]^\frac{z_0^2 - \mathcal{N}z_0^2a^2}{a^2},
\end{equation}
\begin{equation}
V^{(0)+} = (1, z(q)^2, z(q)^3, \ldots) z(q)^{-1} \exp \left( -\frac{b_1\mathcal{N}}{2} \left( 1 + \frac{\sin aq}{1 - \sin aq} \right) \right)^\frac{z_0^2 - \mathcal{N}z_0^2a^2}{a^2}.
\end{equation}
It is worth mentioning here that the potential \( V^{(0) +}(q) \) coincides with the potential whose bound state wavefunctions are given in terms of the exceptional \( X_j \) Jacobi polynomial [42] for \( a = 1, b_1 = B + N/2, z_0 = -(2A - 1)/(2B) \) whereas potential \( V^{(0) -}(q) \) coincides with the Scarf I potential [54] (after making a change \( B \rightarrow B + N \)) whose bound state wavefunctions are given in terms of classical Jacobi polynomials.

We choose here a domain of the system as \( S = (-\frac{\pi}{2}, \frac{\pi}{2}) \) and assume \( z_0 > 1 \) so that the pair of potentials \( V^{(0) \pm}(q) \) has no singularities except at the boundary \( \partial S = \{-\frac{\pi}{2}, \frac{\pi}{2}\} \). Thus, the Hilbert space for the system is \( L^2(S) \). Then, \( V^{(0) -}(S) \subset L^2(S) \) if and only if

\[
\frac{b_1}{a^2} + \frac{N - 1}{2} \pm \frac{(2b_1 - Na^2)z_0}{2a^2} > -\frac{1}{2},
\]

that is,

\[
\frac{Na^2z_0 - 1}{2z_0 + 1} < b_1 < \frac{Na^2z_0 + 1}{2z_0 - 1} \quad \text{for} \quad z_0 > 1. \tag{3.19}
\]

Similarly, \( V^{(0) +}(S) \subset L^2(S) \) if and only if

\[
-\frac{b_1}{a^2} + \frac{N - 1}{2} \pm \frac{(2b_1 - Na^2)z_0}{2a^2} > -\frac{1}{2},
\]

that is,

\[
b_1 > \frac{Na^2}{2} \quad \text{and} \quad z_0 > 1. \tag{3.20}
\]

Hence, \( N \)-fold SUSY of the system is broken for the constant-mass case if and only if \( z_0 > 1 \) and

\[
b_1 \leq \frac{Na^2z_0 - 1}{2z_0 + 1} \quad \text{or} \quad b_1 \geq \frac{Na^2z_0 + 1}{2z_0 - 1}. \tag{3.21}
\]

In a PDM case, the solvable sectors \( V^{\pm} \) of the type B PDM \( N \)-fold SUSY partner Hamiltonians \( H^{\pm} \) for an arbitrary mass function \( m(q) \) are deformed according to (2.18c) as

\[
V^- = \{1, z(u(q)), z(u(q))^2, \ldots \}m(q)^{\frac{1}{4} + \frac{Na^2z_0}{4a^2} \left( \frac{1 + \sin au(q)}{1 - \sin au(q)} \right)^{(2b_1 - Na^2z_0)z_0}} \left( \frac{1 + \sin au(q)}{1 - \sin au(q)} \right)^{(\frac{2b_1 - Na^2z_0}{4a^2}}), \tag{3.22}
\]

\[
V^+ = \{1, z(u(q))^2, z(u(q))^3, \ldots \}m(q)^{\frac{1}{4} + \frac{Na^2z_0}{4a^2} \left( \frac{1 + \sin au(q)}{1 - \sin au(q)} \right)^{\frac{2b_1 - Na^2z_0}{4a^2}}}, \tag{3.23}
\]

where \( u(q) \) is given by (2.19). In this case, the choice of mass function and the corresponding change of variable are given by

\[
m(q) = \frac{2}{\pi} e^{-2q^2}, \quad u(q) = \text{Erf} q, \quad q \in (-\infty, \infty). \tag{3.24}
\]

Consequently, the partner potentials \( U^{\pm}(q) \) read as

\[
U^-(q) = \frac{(4b_1 - Na^2z_0) \sin(a \text{Erf} q)}{4a^2 \cos^2(a \text{Erf} q)} - \frac{(3q^2 + 1) \pi e^{2q^2}}{4} + \frac{b_1N}{2} + V_0 + \frac{b_1N}{a^2} \left( 2b_1 - Na^2z_0 \right)^2 + \frac{2b_1 + Na^2z_0}{8a^2} - a^2 \tan^2(a \text{Erf} q). \tag{3.25}
\]
The solvable sectors of the potentials (3.25) and (3.26) are given by

\[ V^\pm = \{1, z(u(q)), z(u(q))^2, \ldots\} \ e^{-q/4} \left[ \frac{1 + \sin(a\text{Erf}(q))}{1 - \sin(a\text{Erf}(q))} \right]^{\frac{(2b_1 - Na^2)z_0}{4a^2}} \].

(3.27)

The potentials \( U^\pm(q) \) as well as the mass function are well behaved in \( q \in (-\infty, \infty) \). So, we can take the domain as the whole real line \( \mathbb{R} \). Since Erf \( q \rightarrow \pm1 \) as \( q \rightarrow \pm\infty \), so both solvable sectors \( V^\pm(\mathbb{R}) \) belong to \( L^2(\mathbb{R}) \), irrespective of the parameter values of \( b_1 \) and \( z_0 \). Hence, it manifests unbroken SUSY. So, in this case PDM affects the symmetry breaking scenario. But the mass profile \( m(q) = \text{sech}^2aq \), \( q \in (-\infty, \infty) \) has no effect on dynamical breaking of \( N \)-fold SUSY, which can be observed by considering the leading behavior of the solvable sectors (3.22) and (3.23). We have found that the same is true for many other mass functions.

Also associated with this mass profile, one of the partner potentials given in equation (3.29) is identical to the \( V_{\text{eff}}(q) \) whose bound state wavefunctions are given by exceptional Xi Jacobi polynomials (e.g., equation (18) in [28]), for the choice of parameters \( b_1 = (\alpha - \beta + N)a^2/2, \ z_0 = (\alpha + \beta)/(\alpha - \beta) \). The simplified form of the other partner potential \( U^+(q) \) matches with the potential previously obtained in [27] corresponding to classical Jacobi polynomials. It is worth mentioning that this mass profile \( m(q) = \text{sech}^2aq \) has been previously used in PDM Hamiltonians of BenDaniel–Duke [56] and Zhu–Kroemer [57] type, and an interesting connection was shown [58] between the discrete eigenvalues of such Hamiltonians and the stationary 1-soliton and 2-soliton solutions of the Korteweg–de Vries equation.

For the latter choice of mass function, the change of variable is given by \( u(q) = \tan^{-1}(\sinh(aq))/a \) and the corresponding pair of potentials \( U^\pm(q) \) read as

\[ U^\pm(q) = \left[ 2b_1(z_0 + 1) - Na^2z_0 \mp (N - 2)a^2 \right] \left[ 2b_1(z_0 + 1) - Na^2z_0 \mp (N + 2)a^2 \right] e^{2aq} + \frac{1}{2} \frac{a^2}{z_0 + 1} \left[ 1 + \frac{2(z_0 - 2)}{z_0 - 1 + (z_0 + 1)e^{2aq}} - \frac{4(z_0 - 1)}{(z_0 - 1 + (z_0 + 1)e^{2aq})^2} \right] \].

(3.29)

**Example 3.3.** \( A(z) = (z - z_0)^2/2 \).

Potentials:

\[ V^{(0)}(q) = \frac{(2b_1 + N)^2z_0^2}{8} e^{-2q} + \frac{(4b_1^2 - N^2)z_0}{4} e^{-q} + V_0. \]

(3.30a)
\[ V^{(0)}(q) = \frac{(2b_1 + N)^2 z_0^2}{8} e^{-q} + \frac{(2b_1 + N)^2 z_0}{4} e^{-q} - \frac{z_0 e^{-q}}{(1 + z_0 e^{-q})^2} + V_0, \]  
where \(V_0\) is an irrelevant constant given by 
\[ V_0 = \frac{k_1^2}{2} + \frac{b_1}{N} + \frac{N^2 + 11}{24} - R. \]
Solvable sectors: 
\[ V^{(0)} = (1, z(q), z(q)^2, \ldots) \exp \left[ -\frac{(2b_1 + N)z_0}{2} e^{-q} - \frac{N - 1 - 2b_1}{2} q \right], \]  
\[ V^{(0)+} = \langle 1, z(q)^2, z(q)^3, \ldots \rangle z(q)^{-1} \exp \left[ \frac{(2b_1 + N)z_0}{2} e^{-q} - \frac{N - 1 + 2b_1}{2} q \right]. \]  

We assume \(z_0 > 0\) so that the pair of potentials \(V^{(0)\pm}(q)\) has no singularities in \((-\infty, \infty)\). As we will show in what follows, the \(N\)-fold SUSY in this case can be partially broken. To see this, we first introduce a pair of \(k\)-dimensional subspaces \(V_k^{(0)\pm}\) of the solvable sectors \(V^{(0)\pm}\) as 
\[ V_k^{(0)\pm} = (1, \ldots, z(q)^{k-1}) \exp \left[ -\frac{(2b_1 + N)z_0}{2} e^{-q} - \frac{N - 1 - 2b_1}{2} q \right], \]  
\[ V_k^{(0)+} = \langle 1, z(q)^2, \ldots, z(q)^k \rangle z(q)^{-1} \exp \left[ \frac{(2b_1 + N)z_0}{2} e^{-q} - \frac{N - 1 + 2b_1}{2} q \right]. \]

Then, for a fixed \(k \in \mathbb{N}\), we have 
\[ V_k^{(0)-} \subset L^2(\mathbb{R}) \quad \iff \quad -N < 2b_1 < N + 1 - 2k, \]  
\[ V_k^{(0)+} \subset L^2(\mathbb{R}) \quad \iff \quad 2k - N - 1 < 2b_1 < -N. \]  
(3.34)  
(3.35)
From these conditions, it is easy to observe that \(V_k^{(0)-} \subset L^2(\mathbb{R})\) if and only if \(-N/2 < b_1 < (N + 1 - 2k)/2\) for a \(k \in \mathbb{N}\) satisfying \(k < N + 1/2\), while there is no \(k \in \mathbb{N}\) which satisfies condition (3.35) and thus \(V_k^{(0)+} \not\subset L^2(\mathbb{R})\) \(\forall b_1 \in \mathbb{R}\). Hence, the \(N\)-fold SUSY in the constant-mass background is partially broken if there is a positive integer \(k \leq N\) for which the parameter \(b_1\) satisfies 
\[ -\frac{N}{2} < b_1 < -\frac{N + 1 - 2k}{2}, \]  
and fully broken otherwise.

The solvable sectors \(V^\pm\) of the corresponding PDM Hamiltonians \(H^\pm\) are written as 
\[ V^- = (1, z(u(q)), z(u(q))^2, \ldots) m(q)^{1/4} \exp \left[ -\frac{(2b_1 + N)z_0}{2} e^{-u(q)} - \frac{N - 1 - 2b_1}{2} u(q) \right], \]  
(3.36a)  
\[ V^+ = \langle 1, z(u(q))^2, z(u(q))^3, \ldots \rangle z(u(q))^{-1/4} \times \exp \left[ \frac{(2b_1 + N)z_0}{2} e^{-u(q)} - \frac{N - 1 + 2b_1}{2} u(q) \right], \]  
(3.36b)
and the potentials \(U^\pm(q)\) can be obtained using (2.18a), (3.30a) and (3.30b). We have checked the normalizability of the solvable sectors (3.36a) and (3.36b) with the following two mass functions.
(i) \( m(q) = (1 - q^2)^{-1}, q \in (-1, 1) \) for which the change of variable is \( u(q) = \sin^{-1} q \). This mass profile has been used in [11, 10] while considering the effective-mass quantum nonlinear oscillator. This mass function has effect on dynamical symmetry breaking because it manifests broken SUSY (i.e. neither \( \mathcal{V}^- \) nor \( \mathcal{V}^+ \) belongs to \( L^2(-1, 1) \)), which is clear from the following expressions of \( \mathcal{V}^- \) and \( \mathcal{V}^+ \):

\[
\mathcal{V}^- = \langle 1, z(u(q)), z(u(q))^2, \ldots \rangle \frac{1}{(1 - q^2)^{1/4}} \times \exp \left[ -\frac{(2b_1 + N)z_0}{2} e^{-\sin^{-1} q} - \frac{N - 1 - 2b_1}{2} \sin^{-1} q \right],
\]

\[
\mathcal{V}^+ = \langle 1, z(u(q))^2, z(u(q))^3, \ldots \rangle \frac{1}{(1 - q^2)^{1/4}(e^{\sin^{-1} q} + z_0)} \times \exp \left[ -\frac{(2b_1 + N)z_0}{2} e^{-\sin^{-1} q} - \frac{N - 1 + 2b_1}{2} \sin^{-1} q \right].
\]

(ii) \( m(q) = 2e^{-2q^2}/\pi \) for which the solvable sectors (3.36a) and (3.36b) reduce to

\[
\mathcal{V}^- = \langle 1, z(u(q)), z(u(q))^2, \ldots \rangle \times \exp \left[ -\frac{q^2}{4} - \frac{(2b_1 + N)z_0}{2} e^{-\text{Erf} q} - \frac{N - 1 - 2b_1}{2} \text{Erf} q \right],
\]

\[
\mathcal{V}^+ = \langle 1, z(u(q))^2, \ldots, z(u(q))^N z(u(q))^{-1} \rangle \times \exp \left[ -\frac{q^2}{4} + \frac{(2b_1 + N)z_0}{2} e^{-\text{Erf} q} - \frac{N - 1 + 2b_1}{2} \text{Erf} q \right].
\]

From the above solvable sectors, we observe that both \( \mathcal{V}^- (\mathbb{R}) \) and \( \mathcal{V}^+ (\mathbb{R}) \) belong to \( L^2(\mathbb{R}) \), irrespective of the parameter value \( b_1 \), which means unbroken \( \mathcal{N} \)-fold SUSY. Hence, the mass function \( m(q) = 2e^{-2q^2}/\pi \) affects dynamical breaking of the \( \mathcal{N} \)-fold SUSY.

Hence, comparing the normalizability conditions in both the constant and PDM cases, we conclude that both mass functions change the behavior of symmetry breaking.

3.2. **Effects of PDM on dynamical symmetry breaking of type \( X_2 \mathcal{N} \)-fold SUSY**

In this section, we examine three different models of type \( X_2 \mathcal{N} \)-fold SUSY characterized by different choices of the two parameters \( a_1 \) and \( a_2 \); \( a_1 \neq 0 \) and \( a_2 = 0 \) for the first model, \( a_1 = 0 \) and \( a_2 \neq 0 \) for the second and \( a_1 a_2 \neq 0 \) for the third. The first two choices lead to the rational- and hyperbolic-type potential pairs already shown in [52], while the last choice to an exponential-type potential pair which is new and has not been investigated in the literature.

**Example 3.4.** \( A(z) = 2z [a_1 = 2] \).

Potentials:

\[
V^{(0)-}(q) = \frac{q^2}{2} + \frac{4\alpha^2 - 1}{8q^2} + 4 \left[ \frac{q^2 - \alpha + 1}{f(q^2; \alpha)} - \frac{4(\alpha - 1)q^2}{f(q^2; \alpha)^2} \right] - \mathcal{N} + V_0,
\]

\[
V^{(0)+}(q) = \frac{q^2}{2} + \frac{4(\alpha + \mathcal{N})^2 - 1}{8q^2} + 4 \left[ \frac{q^2 - \alpha - \mathcal{N} + 1}{f(q^2; \alpha + \mathcal{N})} - \frac{4(\alpha + \mathcal{N} - 1)q^2}{f(q^2; \alpha + \mathcal{N})^2} \right] + V_0,
\]

where \( V_0 = \mathcal{N} - \alpha + 3 - \epsilon_0 \) is an irrelevant constant.
Solvable sectors:

\[
\begin{align*}
\mathcal{V}^{(0) -}_{N} & = \langle \tilde{\psi}_1(q^2; \alpha), \ldots, \tilde{\psi}_{N'}(q^2; \alpha) \rangle \frac{q^{\alpha+1/2} e^{-q^2/2}}{f(q^2; \alpha)}, \\
\mathcal{V}^{(0) +}_{N} & = \langle \tilde{\chi}_1(q^2; \alpha + N'), \ldots, \tilde{\chi}_{N'}(q^2; \alpha + N') \rangle \frac{q^{-\alpha-N'+1/2} e^{q^2/2}}{f(q^2; \alpha + N')}. \quad (3.42a)
\end{align*}
\]

In this case, the solvability condition \(a_2(\equiv a_3 = a_4) = 0\) for type \(X_2\) is satisfied and thus the corresponding constant-mass Hamiltonians \(H^{(0)\pm}\) are simultaneously solvable.

For \(\alpha > 1\), a natural choice for the domain of these potentials is a real half-line \(S = \mathbb{R}_+\). On this domain \(\mathbb{R}_+\), it is evident from (3.42a) and (3.42b) that \(\mathcal{V}^{(0) -}_{N'}(\mathbb{R}_+) \subset L^2(\mathbb{R}_+)\) and \(\mathcal{V}^{(0) -}_{N'}(\mathbb{R}_+) \not\subset L^2(\mathbb{R}_+)\). Therefore, it manifests unbroken \(\mathcal{N}\)-fold SUSY of the system in the constant-mass background.

According to (2.18c), the solvable sectors \(\mathcal{V}^{\pm}_{N'}\) of the corresponding PDM Hamiltonians \(H^{\pm}\) for an arbitrary mass function \(m(q)\) read as

\[
\begin{align*}
\mathcal{V}^{-(+)}_{N} & = \langle \tilde{\psi}_1(u(q)^2; \alpha), \ldots, \tilde{\psi}_{N'}(u(q)^2; \alpha) \rangle \frac{m(q)^{1/4} u(q)^{\alpha+1/2} e^{-u(q)^2/2}}{f(u(q)^2; \alpha)}, \\
\mathcal{V}^{-(+)}_{N} & = \langle \tilde{\chi}_1(u(q)^2; \alpha + N'), \ldots, \tilde{\chi}_{N'}(u(q)^2; \alpha + N') \rangle \frac{m(q)^{1/4} u(q)^{-\alpha-N'+1/2} e^{u(q)^2/2}}{f(u(q)^2; \alpha + N')}, \quad (3.43)
\end{align*}
\]

where \(u(q)\) is given by (2.19) and the PDM potentials \(U^\pm(q)\) can be obtained using (2.18a), (3.41a) and (3.41b). In this case, we have not been able to find any realistic mass function which could break the \(\mathcal{N}\)-fold SUSY. In other words, we can say that the \(\mathcal{N}\)-fold SUSY in this case is steady against many variations of mass functions (e.g., \(m(q) = e^{-q}, \text{sech}^2 q\)).

**Example 3.5.** \(A(z) = (z^2 + \xi^2)/2, [a_2 = 1/2, \xi^2 = (\alpha - 1)(\alpha + N - 1) > 0]\).

Potentials:

\[
\begin{align*}
V^{(0) -}(q) & = \frac{\xi^2}{8} \cosh^2 q + \frac{N - 1}{4} \xi \sinh q + V_0 \\
& + \frac{1}{8 \cosh^2 q} \left[ 4(N - 1) \xi \sinh q + 4 \alpha^2 + 4(N - 2) \alpha - N^2 - 2N + 4 \right] \\
& - 2(\alpha - 1) \left[ \frac{\xi \sinh q - \alpha - N + 3}{f(\xi \sinh q; \alpha)} - 2(\alpha - 1) \frac{\xi \sinh q - N + 1}{f(\xi \sinh q; \alpha)} \right], \quad (3.45)
\end{align*}
\]

\[
\begin{align*}
V^{(0) +}(q) & = \frac{\xi^2}{8} \cosh^2 q + \frac{3N - 1}{4} \xi \sinh q + V_0 \\
& - \frac{1}{8 \cosh^2 q} \left[ 4(N + 1) \xi \sinh q - 4 \alpha^2 - 4(N - 2) \alpha + N^2 + 6N - 4 \right] \\
& - 2(\alpha + N - 1) \left[ \frac{\xi \sinh q - \alpha + 3}{f(\xi \sinh q; \alpha + N)} - 2(\alpha + N - 1) \frac{\xi \sinh q + N + 1}{f(\xi \sinh q; \alpha + N)} \right], \quad (3.46)
\end{align*}
\]

where \(V_0\) is an irrelevant constant given by

\[
V_0 = \frac{4\alpha^2 + 4(N - 4) \alpha + N^2 + 16}{8} - c_0.
\]

Solvable sectors:

\[
\begin{align*}
\mathcal{V}^{(0) -}_{N'} & = \langle \tilde{\psi}_1(\xi \sinh q; \alpha), \ldots, \tilde{\psi}_{N'}(\xi \sinh q; \alpha) \rangle \frac{e^{-\xi(\sinh q)/2 - \xi g d q}}{(\cosh q)^{N'-1/2} f(\xi \sinh q; \alpha)}, \quad (3.47a)
\end{align*}
\]
\[ V_{N}^{(0)+} = (\bar{\chi}_{1}(\xi \sin \theta; \alpha + N), \ldots, \bar{\chi}_{N}(\xi \sin \theta; \alpha + N)) \frac{e^{\xi \sin \theta/2 + \xi \text{gd} q}}{(\cosh q)^{N/2} f(\xi \sin \theta; \alpha + N)}, \]

(3.47b)

where \( \text{gd} q = \tan^{-1}(\sin \theta) \) is the Gudermann function. The solvability condition is not satisfied in this case and both Hamiltonians are only quasi-solvable. For \( \alpha > 1 \), the potentials \( V^{\pm}(q) \) given in (3.45) are defined on the whole real line \( \mathbb{R} \). From the solvable sectors (3.47a) and (3.47b), it is clear that neither \( V_{N}^{(0)-}(\mathbb{R}) \) nor \( V_{N}^{(0)+}(\mathbb{R}) \) belong to \( L^{2}(\mathbb{R}) \), so the \( N \)-fold SUSY is dynamically broken in the constant-mass background.

Now, the PDM potentials \( U^{\pm}(q) \) can be obtained with help of (2.18a), (3.45) and (3.46), and the solvable sectors \( V_{N}^{\pm} \) of the corresponding PDM Hamiltonians \( H^{\pm} \) for an arbitrary mass function \( m(q) \) read from (2.18c) as

\[ V_{N}^{+} = (\bar{\psi}_{1}(\xi \sin \theta; \alpha), \ldots, \bar{\psi}_{N}(\xi \sin \theta; \alpha)) \frac{m(q)^{1/4} e^{-\xi \sin \theta/2 - \xi \text{gd} q}}{(\cosh \theta)^{N/2} f(\xi \sin \theta; \alpha + N)}, \]

(3.48a)

\[ V_{N}^{-} = (\bar{\psi}_{1}(\xi \sin \theta; \alpha), \ldots, \bar{\psi}_{N}(\xi \sin \theta; \alpha)) \frac{\sqrt{\text{sech} \theta} e^{-\xi \sin \theta/2 - \xi \text{gd} q}}{(\cosh \theta)^{N/2} f(\xi \sin \theta; \alpha)}, \]

(3.48b)

where \( u(q) \) is given by (2.19). Let us now consider two cases:

(i) \( m(q) = \text{sech} \theta q, q \in (-\infty, \infty) \), for which the change of variable is \( u(q) = \text{gd} q \). Then, the solvable sectors of \( U^{\pm}(q) \) are given by

\[ V_{N}^{+} = (\bar{\psi}_{1}(\xi \sin \theta; \alpha), \ldots, \bar{\psi}_{N}(\xi \sin \theta; \alpha)) \frac{\sqrt{\text{sech} \theta} e^{-\xi \sin \theta/2 - \xi \text{gd} q}}{(\cosh \theta)^{N/2} f(\xi \sin \theta; \alpha + N)} \]

(3.49a)

\[ V_{N}^{-} = (\bar{\psi}_{1}(\xi \sin \theta; \alpha), \ldots, \bar{\psi}_{N}(\xi \sin \theta; \alpha)) \frac{\sqrt{\text{sech} \theta} e^{-\xi \sin \theta/2 - \xi \text{gd} q}}{(\cosh \theta)^{N/2} f(\xi \sin \theta; \alpha + N)}. \]

(3.49b)

In this case, the mass function as well as the potentials \( U^{\pm}(q) \) are well behaved on \( (-\infty, \infty) \), so we can consider the whole real line \( \mathbb{R} \) as a domain of the potentials. From the solvable sectors (3.49a) and (3.49b), it is clear that both \( V_{N}^{\pm}(\mathbb{R}) \) belong to \( L^{2}(\mathbb{R}) \), which means unbroken \( N \)-fold SUSY, i.e. the mass profile affects symmetry restoration.

(ii) \( m(q) = 2 e^{-2q^{2}} / \pi \). In this case, the solvable sectors \( V_{N}^{\pm} \) reduce to

\[ V_{N}^{+} = (\bar{\psi}_{1}(\xi \sin \theta; \alpha), \ldots, \bar{\psi}_{N}(\xi \sin \theta; \alpha)) \frac{\exp[-q^{2}/4 - \xi \sin \text{Erf} \theta/2 - \xi \text{gd} \text{Erf} \theta]}{(\cosh \text{Erf} \theta)^{N/2} f(\xi \sin \theta; \alpha)} \]

(3.50a)

\[ V_{N}^{-} = (\bar{\psi}_{1}(\xi \sin \theta; \alpha), \ldots, \bar{\psi}_{N}(\xi \sin \theta; \alpha)) \frac{\exp[-q^{2}/4 + \xi \sin \text{Erf} \theta/2 + \xi \text{gd} \text{Erf} \theta]}{(\cosh \text{Erf} \theta)^{N/2} f(\xi \sin \theta; \alpha + N)}. \]

(3.50b)

From the above solvable sectors (3.50a) and (3.50b), it is clear that both \( V_{N}^{\pm}(\mathbb{R}) \) belong to \( L^{2}(\mathbb{R}) \). That is, in this case we again have unbroken \( N \)-fold SUSY.

We note that there are other mass functions, e.g., \( m(q) = (\beta + q^{2})^2/(1 + q^{2})^2 \), which have no effect on the dynamical breaking of \( N \)-fold SUSY, i.e. it is also possible to construct PDM systems which maintain broken \( N \)-fold SUSY.
Example 3.6. $A(z) = (z + \zeta)^2/2$, $[a_2 = 1/2, a_1 = \zeta = \sqrt{(\alpha - 1)/(\alpha + N - 1)}]$.

Potentials:
\[
V^{(0)-}(q) = \frac{1}{8} e^{2q} - \frac{N + 1}{4} e^q - \frac{(N - 1)(N + 2\alpha - 2\zeta - 1)}{4} e^{-q} + \frac{\zeta^2}{8} [N^2 + 2N(4\alpha - 2\zeta - 3) + 4\alpha(2\alpha - 2\zeta - 3) + 4\zeta + 5] e^{-2q} - 2 \left[ \frac{(\alpha - \zeta - 1)}{f(e^q - \zeta; \alpha)} + \frac{2(\alpha - 1)}{f(e^q - \zeta; \alpha)} \right] + V_0, \quad (3.51)
\]
\[
V^{(0)+}(q) = \frac{1}{8} e^{2q} + \frac{N - 1}{4} e^q + \frac{(N - 1)(N + 2\alpha - 2\zeta - 1)}{4} e^{-q} + \frac{\zeta^2}{8} [N^2 + 2N(4\alpha - 2\zeta - 3) + 4\alpha(2\alpha - 2\zeta - 3) + 4\zeta + 5] e^{-2q} - 2 \left[ \frac{(\alpha + N - \zeta - 1)}{f(e^q - \zeta; \alpha + N)} + \frac{2(\alpha + N - 1)}{f(e^q - \zeta; \alpha + N)} \right] + V_0, \quad (3.52)
\]
where $V_0$ is an irrelevant constant given by
\[
V_0 = \frac{(N + 2\alpha)^2 + 2\zeta(N - 2\alpha) + 2(7\zeta - 8\alpha + 8) - c_0}{8}.
\]

Solvable sectors:
\[
\mathcal{V}_N^{(0)-} = \langle \hat{\psi}_1(e^q - \zeta; \alpha), \ldots, \hat{\psi}_N(e^q - \zeta; \alpha) \rangle
\]
\[
\times \exp \left[ -\frac{e^q}{2} + \frac{2\zeta - 2\alpha - N + 1}{2} e^{-q} - \frac{N - 2}{2} q \right], \quad (3.53)
\]
\[
\mathcal{V}_N^{(0)+} = \langle \hat{\chi}_1(e^q - \zeta; \alpha + N), \ldots, \hat{\chi}_N(e^q - \zeta; \alpha + N) \rangle
\]
\[
\times \exp \left[ -\frac{e^q}{2} - \frac{2\zeta - 2\alpha - N + 1}{2} e^{-q} - \frac{N - 2}{2} q \right]. \quad (3.54)
\]

This system is new and presented in this paper for the first time. The exponential-type $V_N^\pm(q)$ are naturally defined on the whole real line $\mathbb{R}$ since they have no singularity on it, so the Hilbert space is $L^2(\mathbb{R})$. Noting that $2\zeta - 2\alpha - N + 1 < 0$ for $\alpha > 1$, since $4\zeta^2 - 2(2\alpha + N - 1)^2 = -4\alpha - (N - 1)(N + 3) < -(N + 1)^2 < 0$, we see that $\mathcal{V}_N^{(0)-}(\mathbb{R}) \subset L^2(\mathbb{R})$ and $\mathcal{V}_N^{(0)+}(\mathbb{R}) \not\subset L^2(\mathbb{R})$ for $\zeta > 0$. Hence, it manifests unbroken $N$-fold SUSY. For $\zeta < 0$, on the other hand, neither $\mathcal{V}_N^{(0)-}(\mathbb{R})$ nor $\mathcal{V}_N^{(0)+}(\mathbb{R})$ belong to $L^2(\mathbb{R})$, so the $N$-fold SUSY is broken in the constant-mass background.

In a PDM background, the solvable sectors $\mathcal{V}_N^\pm$ of the type $X_2$ PDM Hamiltonians $H^\pm$ are deformed as (cf (2.18c))
\[
\mathcal{V}_N^{-} = \langle \hat{\psi}_1(e^{u(q)} - \zeta; \alpha), \ldots, \hat{\psi}_N(e^{u(q)} - \zeta; \alpha) \rangle
\]
\[
\times m(q)^{1/4} \exp \left[ -\frac{e^{u(q)}}{2} + \frac{2\zeta - 2\alpha - N + 1}{2} e^{-u(q)} - \frac{N - 2}{2} u(q) \right], \quad (3.55)
\]
\[
\mathcal{V}_N^{+} = \langle \hat{\chi}_1(e^{u(q)} - \zeta; \alpha + N), \ldots, \hat{\chi}_N(e^{u(q)} - \zeta; \alpha + N) \rangle
\]
\[
\times m(q)^{1/4} \exp \left[ -\frac{e^{u(q)}}{2} - \frac{2\zeta - 2\alpha - N + 1}{2} e^{-u(q)} - \frac{N - 2}{2} u(q) \right]. \quad (3.56)
\]
and the potentials $U^{\pm}(q)$ can be obtained using (2.18a), (3.51) and (3.52). In this case, the choice of mass functions is as follows:

(i) $m(q) = (1-q^2)^{-1}, q \in (-1, 1)$, for which the solvable sectors of the PDM Hamiltonians $H^\pm$ are given by

$$
\mathcal{V}_N^+ = \left\langle \tilde{\phi}_1(e^{u(q)} - \zeta; \alpha), \ldots, \tilde{\phi}_N(e^{u(q)} - \zeta; \alpha) \right\rangle \\
\times \exp \left[ -\frac{q^2}{4} + \frac{e^{\text{erf} q}}{2} + \frac{2 \zeta - 2 \alpha - N' + 1}{2} \zeta e^{-\text{erf} q} - \frac{N' - 2}{2} e^{-\text{erf} q} \right].
$$

(ii) $m(q) = 2 e^{-\zeta q}/\pi, q \in (-\infty, \infty)$, for which the $\mathcal{N}$-fold SUSY remains unbroken, which is evident from the corresponding solvable sectors given by

$$
\mathcal{V}_N^+ = \left\langle \tilde{\phi}_1(e^{u(q)} - \zeta; \alpha), \ldots, \tilde{\phi}_N(e^{u(q)} - \zeta; \alpha) \right\rangle \\
\times \exp \left[ -\frac{q^2}{4} + \frac{e^{\text{erf} q}}{2} + \frac{2 \zeta - 2 \alpha - N' + 1}{2} \zeta e^{-\text{erf} q} - \frac{N' - 2}{2} e^{-\text{erf} q} \right].
$$

From the above solvable sectors, it is clear that both $\mathcal{V}_N^+(-1, 1)$ do not belong to $L^2(-1, 1)$, so it manifests broken $\mathcal{N}$-fold SUSY irrespective of the sign of $\zeta$. Hence, comparing the normalizability conditions in both the constant and PDM cases, we conclude that the mass function $m(q) = (1-q^2)^{-1}$ affects dynamical breaking of $\mathcal{N}$-fold SUSY for $\zeta > 0$. 

From the normalizability conditions in the constant and PDM cases, we see that the mass function $m(q) = 2 e^{-\zeta q}/\pi$ affects the dynamical breaking of $\mathcal{N}$-fold SUSY for $\zeta < 0$.

4. Summary and perspectives

In this paper, we have investigated the effect of a PDM background on dynamical breaking of type B and type $X_2\mathcal{N}$-fold SUSY. We have selected three different models in the constant-mass background for each type, and then examined whether some of the physically relevant effective-mass profiles can affect the pattern of $\mathcal{N}$-fold SUSY breaking in each model. We summarize the results in table 1. We can easily see from table 1 that, except for the rational potentials, some of the PDM profiles can actually affect and change the patterns of dynamical $\mathcal{N}$-fold SUSY breaking in all the trigonometric, hyperbolic and exponential potentials. Although we have selected the specific types of $\mathcal{N}$-fold SUSY to develop the physical applicability of the new mathematical concept of exceptional polynomial subspaces, we can of course make a similar analysis on other types of $\mathcal{N}$-fold SUSY such as types A and C to find the positive effect of PDM on SUSY breaking in some models.
Hence, it would be possible to observe the experimental transition between a broken and an unbroken phase if an effective mass can be controlled experimentally such that the constant-mass limit can be also realized in an experimental setting. The physical meanings of a PDM depend on each physical system under consideration, for instance, the curvature of the local band structure of an alloy in the momentum space for electrons in a crystal with graded composition [1], the particle densities of $^3$He and $^4$He in pure and mixed helium clusters with doping atoms or molecules [4], the effective electron mass for electrons confined in a quantum dot [3] and for dipole excitations of sodium clusters [5] and so on. Thus, if we can prepare such an atomic, molecular, or condensed matter system which is described by a certain PDM quantum model subjected to an $\mathcal{N}$-fold SUSY potential with mass profiles, e.g., $m(q) = e^{-v^2 q^2}$ or $(1 - v^2 q^2)^{-1}$ where $v$ is an experimentally adjustable parameter such that $v \to 0$ is realizable, then the spectral change of the system could be observed at $v = 0$ due to the phase transition. The essence and novelty of our idea rely on the observation that the physically controllable PDM can cause the phase transition by changing the normalizability of the solvable sector, although the latter is superficially a simple mathematical aspect. Hence, it is quite important to note that the normalizability of wavefunctions can play a bigger role than the quantization of energy spectrum, which is referred to by any standard textbook on quantum mechanics.

We note that this experimental observability might have an impact not only on some atomic, molecular and condensed matter problems from which PDM quantum theory originated, but also on high-energy physics. Until now many high-energy physicists have believed that SUSY is realized at the GUT or Planck scale as a resolution of the naturalness and the hierarchy problem but is broken at least at the electroweak scale. Unfortunately, however, theoretical analysis on dynamical SUSY breaking in field theoretical models is extremely difficult and it is virtually impossible to make a GUT scale experiment. The aforementioned experimental observability suggests that we might extract some clues to understanding dynamical SUSY breaking in high-energy physics from realistic eV scale experiments in atomic, molecular and condensed matter physics. This is because Witten’s work [31] has indicated that the mechanism of dynamical SUSY breaking in quantum field theory and quantum mechanics is essentially the same. We also note that the careful non-perturbative analyses in [59, 60] have shown that the mechanism of dynamical breaking of ordinary and $\mathcal{N}$-fold SUSY is also the same. Hence, the dynamical aspects of SUSY quantum field theoretical models would be mimicked in $\mathcal{N}$-fold SUSY quantum mechanical toy models, regardless of whether or not $\mathcal{N}$-fold SUSY can be realized in higher dimensions. Therefore,
we believe that further studies in this direction are worth pursuing both theoretically and experimentally. From a theoretical point of view it is a challenging issue to investigate both a perturbation theory and the non-renormalization theorem in PDM quantum systems.

Acknowledgments

This work (TT) was partially supported by the National Cheng Kung University under the grant no. HUA:98-03-02-227.

References

[1] Geller M R and Kohn W 1993 Quantum mechanics of electrons in crystals with graded composition Phys. Rev. Lett. 70 3103–6
[2] Morrow R A and Brownstein K R 1984 Model effective-mass Hamiltonians for abrupt heterojunctions and the associated wave-function-matching conditions Phys. Rev. B 30 678–80
[3] Serra L and Lipparini E 1997 Spin response of unpolarized quantum dots Europhys. Lett. 40 667–72
[4] Barranco M, Pi M, GaitãE S and Navarro J 1997 Structure and energetics of mixed 4He–3He drops Phys. Rev. B 56 8997–9003
[5] Puente A, Serra L, and Casas M 1994 Dipole excitation of Na clusters with a non-local energy density functional Z. Phys. D 31 283–8
[6] Wannier G H 1937 The structure of electronic excitation levels in insulating crystals Phys. Rev. 52 191–7
[7] Slater J C 1949 Electrons in perturbed periodic lattices Phys. Rev. 76 1592–601
[8] Bastard G 1988 Wave Mechanics Applied to Semiconductor Heterostructures (Les Ulis: Les Editions de Physique)
[9] Quesne C and Tkachuk V M 2004 Deformed algebras, position-dependent effective masses and curved spaces: an exactly solvable Coulomb problem J. Phys. A: Math. Gen. 37 4267–81 (arXiv:math-ph/0403047)
[10] Cariñena J F, Rañada M F and Santander M 2007 A quantum exactly solvable non-linear oscillator with quasi-harmonic behaviour Ann. Phys. 322 434–59 (arXiv:math-ph/0604008)
[11] Midya B and Roy B 2009 A generalized quantum nonlinear oscillator J. Phys. A: Math. Theor. 42 285301 (arXiv:0910.3179 [quant-ph])
[12] Mostafazadeh A 2005 PT-symmetric cubic anharmonic oscillator as a physical model J. Phys. A: Math. Gen. 38 6557–70 (arXiv:quant-ph/0411137)
[13] Mostafazadeh A 2005 J. Phys. A: Math. Gen. 38 8185 (erratum)
[14] von Roos O 1983 Position-dependent effective masses in semiconductor theory Phys. Rev. B 27 7547–52
[15] Dekar L, Chetouani L and Hammann T F 1998 An exactly soluble Schrödinger equation with smooth position-dependent mass J. Math. Phys. 39 2551–63
[16] Milanoviç V and Ikoníç Z 1999 Generation of isospectral combinations of the potential and the effective-mass variations by supersymmetric quantum mechanics J. Phys. A: Math. Gen. 32 7001–15
[17] Plastino A R, Rigo A, Casas M, Garcia S and Plastino A 1999 Supersymmetric approach to quantum systems with position-dependent effective mass Phys. Rev. A 60 4318–25
[18] de Souza Dutra A and Almeida C A S 2000 Exact solvability of potentials with spatially dependent effective masses Phys. Lett. A 275 25–30 (arXiv:quant-ph/0306065)
[19] Roy B and Roy P 2002 A Lie algebraic approach to effective mass Schrödinger equations J. Phys. A: Math. Gen. 35 3961–9
[20] GönlöB, Özer O, GönlöB and Üzgün F 2002 Exact solutions of effective-mass Schrödinger equations Mod. Phys. Lett. A 17 2453–65 (arXiv:quant-ph/0211113)
[21] Samani K A and Loram F 2003 Shape invariant potentials for effective mass Schrödinger equation arXiv:quant-ph/0302191
[22] Roy B and Roy P 2001 Exact solutions of effective mass Schrödinger equations arXiv:quant-ph/0106028
[23] Alhaidari A D 2002 Solutions of the nonrelativistic wave equation with position-dependent effective mass Phys. Rev. A 66 042116 (arXiv:quant-ph/0207061)
[24] Ganguly A and Nieto L M 2007 Shape-invariant quantum Hamiltonian with position-dependent effective mass through second-order supersymmetry J. Phys. A: Math. Theor. 40 7265–81 (arXiv:0707.3624 [quant-ph])
[25] Midya B, Roy B and Roychoudhury R 2010 Position dependent mass Schrödinger equation and isospectral potentials: intertwining operator approach J. Math. Phys. 51 022109 (arXiv:1001.1190 [quant-ph])
[25] Koç R, Koca M and Şahinoğlu E 2005 Scattering in abrupt heterostructures using a position dependent mass Hamiltonian *Eur. Phys. J. B* 48 583–6 (arXiv:quant-ph/0510172)

[26] Quesne C 2006 First-order intertwining operators and position-dependent mass Schrödinger equations in d dimensions *Ann. Phys.* 321 1221–39 (arXiv:quant-ph/0508216)

[27] Bagchi B, Gorain P, Quesne C and Roychoudhury R 2005 New approach to (quasi)-exactly solvable Schrödinger equations with a position-dependent effective mass *Europhys. Lett.* 72 155–61 (arXiv:quant-ph/0505171)

[28] Midya B and Roy P 2009 Exceptional orthogonal polynomials and exactly solvable potentials in position-dependent-mass Schrödinger Hamiltonians *Phys. Lett. A* 373 4117–22 (arXiv:0910.1209 [quant-ph])

[29] Chen G 2004 Approximate series solutions of the N-dimensional position-dependent mass Schrödinger equation *Phys. Lett. A* 329 22–27

[30] Roy B and Roy P 2005 Effective mass Schrödinger equation and nonlinear algebras *Phys. Lett. A* 340 70–3

[31] Witten E 1981 Dynamical breaking of supersymmetry *Nucl. Phys.* B 188 513–54

[32] Kamran N and Olver P J 1990 Lie algebras of differential operators and Lie-algebraic potentials *J. Math. Anal. Appl.* 145 342–56

[33] Tanaka T 2006 -fold supersymmetry in quantum systems with position-dependent mass *J. Phys. A: Math. Gen.* 39 219–34 (arXiv:quant-ph/0509132)

[34] Samuel S and Wess J 1983 A superfield formulation of the non-linear realization of supersymmetry and its coupling to supergravity *Nucl. Phys.* B 221 153–77

[35] Volkov D V and Akulov V P 1972 Possible universal neutrino interaction *JETP Lett.* 16 438–40

[36] Tanaka T 2009 -fold supersymmetry and quasi-solvability *Mathematical Physics Research Developments* ed M B Levy (New York: Nova Science Publishers) pp 621–79 chapter 18

[37] Antoniadis I, Dudas E, Ghilencea D M and Tziveloglou P 2010 Non-linear MSSM *Phys. Lett.* B 586 241–48

[38] Gómez-Ullate D, Kamran N and Milson R 2007 Quasi-exact solvability in a general polynomial setting *Inverse Problems* 23 1915–42 (arXiv:0610065 [nlin.SI])

[39] Gómez-Ullate D, Kamran N and Milson R 2009 An extended class of orthogonal polynomials defined by a Sturm–Liouville problem *J. Math. Anal. Appl.* 359 352–67 (arXiv:0807.3939 [math-ph])

[40] Gómez-Ullate D, Kamran N and Milson R 2010 An extension of Bogochn’s problem: exceptional invariant subspaces *J. Approx. Theory* 162 987–1006 (arXiv:0805.3576 [math-ph])

[41] Post G and Turbiner A 1995 Classification of linear differential operators with an invariant subspace of monomials *Russ. J. Math. Phys.* 3 113–22 (arXiv:funct-an/9307001)

[42] Quesne C 2008 Exceptional orthogonal polynomials, exactly solvable potentials and supersymmetry *J. Phys. A: Math. Theor.* 41 392001 (arXiv:0807.4087 [quant-ph])

[43] Andrianov A A and Sokolov A V 2003 Nonlinear supersymmetry in quantum mechanics: algebraic properties and differential representations *Nucl. Phys.* B 660 25–50 (arXiv:hep-th/0301062)

[44] Tanaka T 2003 Type A -fold supersymmetry and generalized Bender–Dunne polynomials *Nucl. Phys.* B 662 413–46 (arXiv:hep-th/0212276)

[45] Aoyama H, Sato M and Tanaka T 2001 -fold supersymmetry in quantum mechanics: general formalism *Nucl. Phys.* B 619 105–27 (arXiv:quant-ph/0106037)

[46] Andrianov A A, Ioffe M V and Spiridonov V P 1993 Higher-derivative supersymmetry and the Witten index *Phys. Lett. A* 174 273–79 (arXiv:hep-th/9303005)

[47] González-López A and Tanaka T 2005 A novel multi-parameter family of quantum systems with partially broken -fold supersymmetry *J. Phys. A: Math. Gen.* 38 5133–57 (arXiv:hep-th/0405079)

[48] Bagger J and Wess J 1984 Partial breaking of extended supersymmetry *Phys. Lett. B* 138 105–10

[49] Bagger J A 1985 Nonlinear realizations and the partial breaking of extended supersymmetry *Physica D* 15 198–207

[50] Aoyama H, Sato M and Tanaka T 2001 General forms of a -fold supersymmetric family *Phys. Lett. B* 503 423–9 (arXiv:quant-ph/0012065)

[51] González-López A and Tanaka T 2004 A new family of -fold supersymmetry: type B *Phys. Lett. B* 586 117–24 (arXiv:hep-th/0307094)

[52] Tanaka T 2010 -fold supersymmetry and quasi-solvability associated with -Laguerre polynomials *J. Math. Phys.* 51 032101 (arXiv:0910.0328 [math-ph])

[53] Bagchi B, Quesne C and Roychoudhury R 2009 Isospectrality of conventional and new extended potentials, second-order supersymmetry and role of PT symmetry *Pramana J. Phys.* 73 337–47 (arXiv:0812.1488 [quant-ph])

[54] Cooper F, Khare A and Sukhatme U 1995 Supersymmetry and quantum mechanics *Phys. Rep.* 251 267–385 (arXiv:hep-th/9405029)
[55] Cruz H, Hernández-Cabrera A and Aceituno P 1990 Quantum tunnelling of electron through a GaAs–Ga$_{1-x}$Al$_x$As superlattice in a transverse magnetic field: an analytical calculation of the transmission coefficient J. Phys.: Condens. Matter 2 8953–9

[56] BenDaniel D J and Duke C B 1966 Space-charge effects on electron tunneling Phys. Rev. 152 683–92

[57] Zhu Q G and Kroemer H 1983 Interface connection rules for effective-mass wave functions at an abrupt heterojunction between two different semiconductors Phys. Rev. B 27 3519–27

[58] Bagchi B 2007 Position-dependent mass models and their nonlinear characterization J. Phys. A: Math. Theor. 40 F1041–5 (arXiv:0706.0607 [quant-ph])

[59] Aoyama H, Kikuchi H, Okouchi I, Sato M and Wada S 1999 Valley views: instantons, large order behaviors, and supersymmetry Nucl. Phys. B 553 644–710 (arXiv:hep-th/9808034)

[60] Sato M and Tanaka T 2002 $N$-fold supersymmetry in quantum mechanics–analyses of particular models J. Math. Phys. 43 3484–510 (arXiv:hep-th/0109179)