Discreteness in deSitter Space and Quantization of Kähler Manifolds

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Abstract

Recently, it has been proposed that the dimension of the Hilbert space of quantum gravity in deSitter space is finite and moreover it is expressed in terms of the coupling constants by using the entropy formula. A weaker conjecture would be that the coupling constant in deSitter space should take only discrete values not necessarily given by the entropy formula. We discuss quantization of the horizon in deSitter space by using Berezin’s functorial quantization of Kähler manifolds and argue that the weak conjecture is valid for Euclidean deSitter space. Moreover it can be valid for a class of bounded complex symmetric spaces.

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1. Introduction

An important problem is to extend the microscopic derivation \cite{1} of the Bekenstein-Hawking formula for the black hole entropy to the case of the deSitter horizon. In three dimensions this problem has been solved by Maldacena and Strominger \cite{2} by using the Chern-Simons approach, see also \cite{3}.

Recently, it has been proposed by Banks \cite{4} and has been discussed by Witten \cite{5} that the dimension of the Hilbert space of quantum gravity in deSitter space might be finite and moreover it is expressed in terms of the coupling constants by using the formula for the entropy. A weaker conjecture would be that the coupling constant in deSitter space should take only discrete values not necessarily given by the entropy formula.

To explore the weak conjecture we discuss quantum mechanics of the four dimensional deSitter ($dS_4$) space. The horizon in the Euclidean $dS_4$ space is the two dimensional sphere. By using Berezin’s functorial quantization of Kähler manifolds \cite{6}, \cite{7}, \cite{8} we argue that the weak conjecture might be valid for Euclidean deSitter space. Moreover it can be valid for a class of bounded complex symmetric spaces.

2. DeSitter Entropy and Quantum Theory

DeSitter spacetime is the maximally symmetric solution of the Einstein equations with positive cosmological constant $\Lambda$. In spherically symmetric coordinates the metric has the form

$$ ds^2 = -(1 - \Lambda r^2/3)dt^2 + dr^2/(1 - \Lambda r^2/3) + r^2d\Omega^2. $$ (2.1)

The surface $r^2 = 3/\Lambda$ is the horizon, which has entropy \cite{4}

$$ S = 3\pi/\hbar G_N \Lambda. $$ (2.2)

We will keep the Planck constant and set $G_N = 1$, $\Lambda = 3$. Euclidean de Sitter space is the sphere $S^4$ and the entropy is

$$ S = \pi/\hbar. $$ (2.3)

Banks \cite{4} has argued that the Hilbert space of quantum gravity in asymptotically deSitter spacetime has a finite dimension $N$ and

$$ S = \ln N. $$ (2.4)
Recently, this proposal has been discussed by Witten \[5\] and Balasubramanian, Horava, Minic \[10\]. A bound on matter entropy in asymptotically deSitter spaces supporting the proposal was obtained by Bousso \[11\]. If this proposal is valid then from (2.3) and (2.4) we obtain that the Planck constant should take only a discrete set of values. If the coupling constant (the Planck constant in our setting) takes only a discrete set of values not necessarily consistent with (2.3) and (2.4) then we will talk about the discreteness of dS in the weak sense.

According to the membrane paradigm \[12\] all the properties of black holes can be derived from the consideration of the horizon as a physical membrane. Therefore if one considers quantum gravity then it seems natural to quantize the horizon.

In the Euclidean formulation the spacetime is $S^4$ and the horizon is the two dimensional sphere $S^2$ embedded into $S^4$. To understand the proposal we consider quantum mechanics of the horizon, i.e. quantization of $S^2$. Quantization of $S^2$ has been performed by Berezin (see next section) and it is quite remarkable that he has found that the Hilbert space of the corresponding quantum mechanics has a finite dimension and that the Planck constant should take only discrete set of values. Therefore one has a confirmation of the conjecture on the discreteness of dS in the weak sense.

3. Berezin’s Quantization of Kähler Manifolds

Berezin \[6\] has developed a general approach to quantize Kähler manifolds. Berezin’s quantization possesses the natural functorial properties. A remarkable property of this quantization is that the Hilbert space of quantum mechanics of a bounded complex symmetric space has a finite dimension and the Planck constant takes only discrete values. If one makes a rescaling to set the Planck constant equal to 1 then an appropriate combination of the coupling constants will take discrete values. These properties support the weak conjecture.

Let $(M, \omega)$ be a Poisson manifold, where $M$ is a manifold and $\omega$ is a skew-symmetric tensor field. Let us denote by $A(M)$ the Lie algebra of differentiable functions on $M$ with the Poisson bracket

$$\{f, g\} = \omega^{ik} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^k}. \quad (3.1)$$
Quantization of $A(M)$ is defined as a family $\{A_\hbar\}$ of associative algebras where the index $\hbar$ runs through a subset of positive real numbers. The algebra $A_\hbar$ consists of the functions on $M$ with the multiplication $f \ast g$, with the following properties

$$
\lim_{\hbar \to 0} f \ast g = fg, \quad \lim_{\hbar \to 0} \frac{1}{\hbar} (f \ast g - g \ast f) = i\{f, g\}.
$$

(3.2)

See [6] for further details. Such quantization has been performed for Kähler manifolds. Let $M$ be an $n$ dimensional Kähler with metric $ds^2 = g_{i\bar{k}} dz^i dz^k$ and symplectic form $\omega = g_{i\bar{k}} dz^i \wedge dz^k$, $d\omega = 0$. Let $F(z, \bar{z})$ be the Kähler potential, $g_{i\bar{k}} = -\frac{\partial^2}{\partial z^i \partial \bar{z}^k} \ln F$. Now let us consider the Hilbert space $H_\hbar$ of analytic functions on a submanifold $\tilde{M}$ of $M$ with the scalar product

$$
(f, g) = c(\hbar) \int f(z)\bar{g}(z) F^{1/\hbar}(z, \bar{z}) d\mu(z, \bar{z}),
$$

(3.3)

where $d\mu$ is the measure $d\mu = \omega^n$ and

$$
c^{-1}(\hbar) = \int F^{1/\hbar}(z, \bar{z}) d\mu(z, \bar{z}).
$$

(3.4)

Take an orthonormal basis $\{f_r(z)\}$ in $H_\hbar$ and consider the kernel

$$
L_\hbar(z, \bar{z}) = \sum_r f_r(z)\bar{f}_r(z).
$$

(3.5)

Functions in $A_\hbar$ are interpreted as symbols of operators in $H_\hbar$. Multiplication in $A_\hbar$ is defined as

$$
(f \ast g)(z, \bar{z}) = \int f(z, \bar{v}) g(v, \bar{z}) G_\hbar(z, \bar{z}|v, \bar{v}) d\mu(v, \bar{v}),
$$

(3.6)

where the kernel $G_\hbar$ is

$$
G_\hbar(z, \bar{z}|v, \bar{v}) = c(\hbar) \frac{L_\hbar(z, \bar{v}) L_\hbar(v, \bar{z})}{L_\hbar(z, \bar{z}) L_\hbar(v, \bar{v})}.
$$

(3.7)

It was shown in [6] that the correspondence principle is valid if the following condition is satisfied

$$
L_\hbar(z, \bar{z}) = \lambda F^{-1/\hbar}(z, \bar{z}),
$$

(3.8)

where $\lambda$ is a constant. This condition restricts the Planck constant $\hbar$. For compact complex symmetric spaces it leads to a discreteness of $\hbar$.  

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3 Here $f \ast g$ should be a function but not just a formal series in $\hbar$ as in deformation quantization [13].
There are four types of compact complex symmetric spaces $M_{p,q}^I$, $M_{p}^{II}$, $M_{p}^{III}$ and $M_{n}^{IV}$. There are special global coordinates covering submanifolds $\tilde{M}$ in these spaces. They are given by complex $p \times q$ matrices and also by symmetric $p \times p$, anti-symmetric $p \times p$ matrices and by $n$-dimensional vectors respectively. In particular $M_{1,1}^I = M_{1}^{II} = S^2$ and $M_{1,q}^I$ is the complex projective space $CP^q$.

The Kähler potential for the first three types is

$$F = \det(I + zz^*)^{-\nu}$$

where $\nu = p + q, p + 1, p - 1$ for $M_{p,q}^I$, $M_{p}^{II}$, $M_{p}^{III}$ respectively. Therefore the Hilbert spaces $\bar{H}_\hbar$ are finite dimensional and they consist of polynomials in $z$. The dimension of the Hilbert space $\bar{H}_\hbar$ is

$$\dim \bar{H}_\hbar = c(\hbar) \int d\mu.$$  \hfill (3.10)

For the correspondence principle as $\hbar \to 0$ be valid the condition (3.8) should be satisfied and as a result the Planck constant takes only discrete values. For the space of types I one has

$$\frac{1}{\hbar} = \frac{n}{(p + q)}.$$  \hfill (3.11)

For the space of type II

$$\frac{1}{\hbar} = \frac{n}{(p + 1)}$$  \hfill (3.12)

and for the space of type III

$$\frac{1}{\hbar} = \frac{n}{2(p - 1)},$$  \hfill (3.13)

where $n = 1, 2, ...$

In particular for the sphere $S^2$ one has

$$\frac{1}{\hbar} = \frac{n}{2}.$$  \hfill (3.14)

The Kähler potential in this case is $F = (1 + zz^*)^{-2}$ and

$$\dim \bar{H}_\hbar = 1 + 1/\hbar.$$  \hfill (3.15)

It is important to note that the discreteness of $1/\hbar$ is not related to the specific quantization described above. It is proved [7] that under some natural conditions this quantization is the unique maximum effective irreducible $w^*$ quantization up to natural equivalence.

Note also that if one applies the same method of quantization to the $n$-dimensional complex space $C^n$ which is a noncompact Kähler manifold then one obtains the standard quantum mechanics with the infinite dimensional Hilbert space and without any restrictions to the Planck constant $\hbar$.

It would be interesting to use Berezin’s quantization to extend noncommutative gauge theory [14] to the case of Kähler manifolds.
4. Discussions

In this note we argued that discreteness in quantum theory on deSitter space arises because quantum mechanics on the horizon has a finite dimensional Hilbert space and the coupling constant is quantized as (3.14).

By using the AdS/CFT correspondence it was suggested in [10] that $E_{dS_4}$ entropy scales as $N^2$, where $N$ is the number of quantum degrees of freedom. Such a behavior in principle can be consistent with the expression for the entropy (2.3) where the Planck constant is quantized as (3.14) if $n \sim N^2$. We have discussed the discreteness only in dS in 4 dimensions. It would be interesting to consider the discreteness in various dimensions. To this end quantization of bounded complex symmetric spaces might be suitable. Generalization of the AdS/CFT correspondence for the complex homogeneous domains was discussed in [15].

There is also another type of discreteness in the deSitter space related to the Chern-Simons approach to $dS_3$ case [2]. There one has the quantization of the level in the Chern-Simons lagrangian and it would be interesting to consider whether this discreteness can be combined with the discreteness discussed in this note into a unified picture.

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