THE STRATIFIED GRASSMANNIAN

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Abstract. We study a quasicategory variant of the stratified Grassmannian of Ayala, Francis and Rozenblyum, which bypasses the theory of conically-smooth stratified spaces. We prove that its maximal sub-$\infty$-groupoid is equivalent, non-canonically, to $\ast \amalg \mathbb{Z}_+ \times \coprod_{k \geq 1} BO(k)$.

Introduction and Summary

For the purpose of constructing a theory of factorisation homology that can take as input any $(\infty,n)$-category and evaluate it on appropriate variframed stratified spaces, Ayala–Francis–Rozenblyum defined in [AFR18b] the ‘fibrewise constructible tangent bundle’ $T^{\text{fib}}$ of a (conically-smooth) stratified space which intrinsically depended on their earlier work, in part with Tanaka, on the general theory of conically-smooth stratified spaces ([AFT17]; [AFR18a]). The functor $T^{\text{fib}}$ on a stratified space $X$ (i.e., its nonrelative special case to which we restrict ourselves) is given in the form of a classifying map $\text{Exit}(X) \to V^{\text{inj}}$, whose domain is their version of the exit path $\infty$-category of Lurie–MacPherson ([Lur]; the two notions are equivalent by a result of [AFR18a]), and with target the ‘stratified Grassmannian’. By exodromy, such a functor classifies a constructible sheaf on $X$, which may be interpreted as the sheaf of sections of the ‘tangent bundle’ although, to our knowledge, no étalé space of this sheaf has been discussed in the literature.

There are numerous senses in which $T^{\text{fib}}$ is the ‘correct’ notion of tangent bundle in stratified space theory. One is the ease with which one may define and work with tangential structures, and spaces of tangential structures, on stratified spaces, using $T^{\text{fib}}$; this has been already exploited in op.s cit. Another is our work in preparation that employs it in functorial field theory, in order to generalise a suggestion by Lurie from [Lur08] (taken up in [Sch14]) as to the construction of a fully-extended functorial field theory using a given $E_n$-algebra, which makes no use of $T^{\text{fib}}$, and does not immediately generalise. Our work in preparation shows that this the ‘framed base case’ of a more general construction that yields field theories given disk-algebras with any tangential structure. Scrutinising bordisms and Lurie’s suggestion using $T^{\text{fib}}$ is the starting point of this construction where, immediately, the need for a hands-on definition of $V^{\text{inj}}$ arises.

Indeed, this note was intended to be a section in an upcoming paper, but we separated it as it might be of independent interest. Our purpose is to give a direct construction of a variant of $V^{\text{inj}}$ that is suggested by [AFR18b]. In a future work we will in particular give a corresponding variant of $T^{\text{fib}}$ (in the nonrelative case) that circumvents the theory of conically-smooth stratified spaces, though it is directly informed by it.
We now summarise the contents. All notation is defined in the main text.

Section 1 constructs a topological monoid whose operation is given by direct-summing of vector spaces. To this end, we circumvent the more systematic treatment of spectra as $\mathbb{E}_\infty$-rings by adding some redundancy that achieves on-the-nose associativity. In order to develop a real $K$-theory (spectrum) for stratified spaces, it might be better to pursue a different treatment.

Definition 2.3, suggested and inspired by [AFR18b, Remark 2.7], gives a full definition of the stratified Grassmannian as an $\infty$-category, in a way that bypasses the theory of conically-smooth stratified spaces and vector bundles. In order to distinguish it from $V_{\text{inj}}$, we denote it by $V_{\rightarrow}$. Proposition 2.7 proves that the maximal sub-$\infty$-groupoid of $V_{\rightarrow}$ is equivalent to countably many copies of the disjoint union of the ordinary positive infinite Grassmannians together a point, i.e., $\ast \amalg \bigoplus_{k \geq 1} BO(k)$. That that of $V_{\text{inj}}$ is equivalent to $\bigoplus_{k \geq 0} BO(k)$ was claimed in [AFR18b]. The factor $\mathbb{Z}_+$ is due to the redundancy from Section 1 mentioned above, and so, unlike the factor in $\mathbb{Z} \times BO$, does not arise in order to handle different connected components of spaces whose vector bundles one wishes to classify. The proof is a direct comparison; it in fact yields an isomorphism, contrary, we believe, to the situation with $V_{\text{inj}}$. This follows an explicit study of low-dimensional simplices.

We should note that the proof applies mutatis mutandis to the delooping of any topological monoid $M$ such that $m : M \to M$ being invertible implies $m = 1$. That is, the proof shows in this case that $(\ast / N\Delta(BM))_{\sim} \simeq M$ as $\infty$-groupoids.

Some standard constructions are gathered in Appendix A.

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**Conventions.** We say $\infty$-category to mean a quasicategory, that is, a simplicial set that satisfies the weak Kan condition ([BV73]; [Joy08]; [Lur08]; [JT07]). We reserve the term $\infty$-groupoid to mean a Kan complex. When a topological space $X$ appears in place of an $\infty$-category, we mean the $\infty$-groupoid $\text{Sing}_\ast(X)$ of its singular chains. By a $\text{Kan-enriched category}$ we mean a locally Kan category, i.e., a simplicially-enriched category whose hom-spaces are Kan complexes. The set $\mathbb{N}$ of natural numbers includes 0. By $\mathbb{Z}_+$ we denote the discrete space of positive integers.

1. **Direct sums on infinite Grassmannians**

Let $H$ be a separable real Hilbert space of countably-infinite dimension, so, up to isometric isomorphism, the real sequence space $\ell^2$.

**Definition.** For $k \in \mathbb{N}$, $BO(k) := \text{Gr}_k(H)$ denotes the Grassmannian of $k$-dimensional subspaces of $H$. 
BO(k) is an infinite-dimensional (Hilbert) manifold modelled on H, and, thus topologised, is homotopy-equivalent (e.g. [Pal66, 4 ff.] combined with Whitehead’s theorem) to the colimit infinite Grassmannian $Gr_k(\mathbb{R}^\infty) = \text{colim} Gr_k(\mathbb{R}^n)$ along the closed embeddings $Gr_k(\mathbb{R}^n) \hookrightarrow Gr_k(\mathbb{R}^{n+1})$ induced by the first-coordinate inclusions $\mathbb{R}^n \hookrightarrow \mathbb{R}^{n-1}$. For our purposes, $\mathbb{R}^\infty$, $H$, and $\ell^2$ are interchangeable.

**Notation.** $BO_\| := \bigsqcup_{k \geq 0} BO(k)$, $BO_\|^+ := \bigsqcup_{k \geq 1} BO(k)$.

**Remark.** The purpose of the notation is to distinguish it from the (connected component of the zeroeth space of the real $K$-theory) spectrum $BO$, where a different, non-discrete colimit is taken.

The aim of this section is to define a monoidal structure based on direct-summing of vector spaces, in the spirit of the direct-summing maps

$$\oplus : Gr_k(\mathbb{R}^n) \times Gr_l(\mathbb{R}^m) \to Gr_{k+l}(\mathbb{R}^{n+m})$$

Passing to infinite Grassmannians, these give maps

$$BO(k) \times BO(l) \to Gr_{k+l}(H \oplus H).$$

Choosing an isomorphism $H \oplus H \cong H$ yields a map

$$BO(k) \times BO(l) \to BO(k + l),$$

which defines a map

$$\oplus : BO_\| \times BO_\| \to BO_\|$$

connected-componentwise.

The problem with this map is that there is no choice of an isomorphism $H \oplus H \cong H$ that would make the map above associative, so it does not promote $BO_\|$ to a topological monoid. The canonical associativity of direct-summing of vector bundles on (paracompact Hausdorff) spaces translates to a monoidal structure on $BO_\|$ (or its stable version $BO$) only up to coherent homotopy. A systematic treatment in this direction, i.e., the theory of $E_\infty$-rings and its application to spectra, is laid out in [Lur].

For our purposes, it will suffice to point out that an isomorphism $H \oplus H \cong H$, or equivalently a pairing function (bijection) $\mathbb{N} \times \mathbb{N} \cong \mathbb{N}$ cannot be associative, as this would contradict injectivity. In order to attain hands-on access to the stratified Grassmannian, we have chosen to strictify $(BO_\|, \oplus)$ in a certain way instead. This involves a trade-off: it does give a topological monoid, but also introduces some redundancy.

**Notation.** $BO^N(k) := Gr_k(\mathbb{R}^{\oplus N})$.

**Notation.** $BO^\infty_{\|} := \{0\} \amalg \bigsqcup_{N \geq 1} \bigsqcup_{k \geq 1} BO^N(k)$.

**Remark 1.1.** Of course, each $BO^N(k)$ is equivalent (even homeomorphic) to $BO(k) = BO^1(k)$, but this is non-canonical. Thus, with some choice of a pairing function and some choice of parenthesisation for large exponents, we have

$$BO^\infty_{\|} \simeq \ast \amalg \mathbb{Z}_+ \times BO^+_{\|}.$$
We separated the zero vector space singleton \( \{0\} = BO(0) = BO^N(0) \) from the disjoint union so as not to count it separately for each \( N \geq 1 \).

**Construction 1.2.** Direct-summing of vector spaces gives maps

\[ BO^N(k) \times BO^M(l) \to BO^{N+M}(k+l), \]

which define a map

\[ \boxplus : BO_{\infty}^N \times BO_{\infty}^M \to BO_{\infty}^{N+M} \]

connected-componentwise. The zero vector space acts as the identity. This is easily seen to be associative.

The notation is meant to distinguish this addition map from (1) which depended on the choice of a pairing function.

We now deloop the topological monoid \( (BO_{\infty}^\ast, \boxplus) \).

**Definition 1.3.** By \( B_{\boxplus}^\ast \) we denote the Kan-enriched category with a single object \( \ast \), endomorphism space \( BO_{\infty}^\ast \), and composition \( \boxplus \).

**Definition 1.4.** \( B_{\boxplus}^\ast := N^\Delta(BO_{\infty}^\ast) \).

2. The stratified Grassmannian

2.1. The delooping in low dimensions. Using notation from Appendix A.1, we will discuss explicitly the 1-, 2- and 3-simplices of \( B_{\boxplus}^\ast \) for future reference, and leave higher simplices to the interested reader.

**Warning 2.1.** By ‘vector space’ we will mean a point of \( BO_{\infty}^\ast \).

2.1.1. 1-morphisms. Let

\[ F : \mathcal{C}[1] \to B_{\boxplus}^\ast \]

be a map of simplicial categories, i.e., a 1-simplex of \( B_{\boxplus}^\ast \). Both objects \( 0, 1 \in [1] \) are sent to \( \ast \). The mapping poset \( P_{0,1} \) has the sole nontrivial element \( 01 \in N_0(P_{0,1}) \), the image of which determines \( F \). Write \( F(01) = V_{01} \in \text{Sing}_0 = \text{Sing}_0 BO_{\infty}^\ast \); so \( V_{01} \) is a vector space.

2.1.2. 2-morphisms. Let

\[ F : \mathcal{C}[2] \to B_{\boxplus}^\ast \]

be a 2-simplex of \( B_{\boxplus}^\ast \). Let \( \iota_{ab}^*F : \mathcal{C}[1] \to B_{\boxplus}^\ast \) be the three nontrivial faces, \( \iota_{ab} : [1] \hookrightarrow [2] \) given by \( 0 \mapsto a, 1 \mapsto b \) for \( a < b \) in \([2]\), so that they are determined by vector spaces \( V_{ab} = F(ab) = \text{Sing}_0 = \text{Sing}_0 BO_{\infty}^\ast \); so \( V_{01} \) is a vector space.

includes two new pieces of information: a vector space \( V_{012} = F(012) \in \text{Sing}_0 \), and, seeing \( \preceq \in N_1(P_{0,2}) \), a path \( \gamma = F(\preceq) \in \text{Sing}_1 \) with source \( V_{02} \) and target \( V_{012} \). Notice now that \( 012 \) is in the image of

\[ P_{1,2} \times P_{0,1} \to P_{0,2}, \]
namely $012 = 12 \cup 01$. As $F$ is functorial, we have $V_{012} = V_{12} \boxplus V_{01}$. Thus, $F$ is determined by three spaces $V_{01}$, $V_{12}$ and $V_{02}$, together with a path $\gamma: V_{02} \to V_{12} \boxplus V_{01}$ in $BO_\infty^\boxplus$. Pictorially:

$$
\begin{tikzpicture}
  \node (v1) {$V_{01}$};
  \node (v2) [above right of=v1] {$V_{12}$};
  \node (v3) [below right of=v1] {$V_{12} \boxplus V_{01}$};
  \node (v0) [below right of=v2] {$*$};
  \node (v4) [right of=v0] {$*$};
  \draw[->] (v1) to (v2);
  \draw[->] (v1) to node [sloped, above] {$F(\prec)$} (v3);
  \draw[->] (v2) to (v4);
  \draw[->] (v3) to (v4);
\end{tikzpicture}
$$

If $V_{12}$ is the identity, i.e. $V_{12} = 0$, then this is just a path from $V_{02}$ to $V_{12}$.

2.1.3. 3-morphisms. Let

$$F: \mathfrak{c}[3] \to B^\boxplus O$$

be a 3-simplex of $B^\boxplus O$. The six non-degenerate edges $V_{ab} = F(ab)$, $0 \leq a < b \leq 3$, are vector spaces. The four non-degenerate faces

$$\iota_{abc}: \mathfrak{c}[2] \hookrightarrow \mathfrak{c}[3] \to B^\boxplus O$$

are of the form of (2), which specifically is the face $(a, b, c) = (0, 1, 2)$. Explicitly, we have four paths $V_{ac} \overset{\sim}{\to} V_{abc} = V_{bc} \boxplus V_{ab}$:

$$V_{02} \simeq V_{12} \boxplus V_{01}, \quad V_{03} \simeq V_{13} \boxplus V_{01}, \quad V_{13} \simeq V_{23} \boxplus V_{12}, \quad V_{03} \simeq V_{23} \boxplus V_{02}.$$ 

The mapping poset $P_{0,3}$ is as follows:

$$
\begin{tikzpicture}
  \node (03) at (0,0) {$03$};
  \node (013) at (-1,-1) {$013$};
  \node (023) at (1,-1) {$023$};
  \node (0123) at (0,-2) {$0123$};
  \draw[->] (03) to node [sloped, below] {$\prec_1$} (013);
  \draw[->] (03) to node [sloped, above] {$\prec_2$} (023);
  \draw[->] (013) to node [sloped, below] {$\prec_2$} (0123);
  \draw[->] (023) to node [sloped, above] {$\prec_1$} (0123);
\end{tikzpicture}
$$

The left and right triangles therein depict the two non-degenerate elements of $N_2(P_{0,3})$, which $F$ maps to $\text{Sing}_2(BO_\infty^\boxplus)$. That is, writing $\gamma = F(\prec)$, $F$ gives homotopies filling the triangles in

$$
\begin{tikzpicture}
  \node (v0) at (0,0) {$V_{0123}$};
  \node (v1) at (1,1) {$V_{03}$};
  \node (v2) at (1,-1) {$V_{23} \boxplus V_{02}$};
  \node (v3) at (0,1) {$V_{13} \boxplus V_{01}$};
  \draw[->] (v0) to node [sloped, above] {$\gamma_1$} (v1);
  \draw[->] (v0) to node [sloped, below] {$\gamma_2$} (v2);
  \draw[->] (v1) to node [sloped, above] {$\gamma_1$} (v3);
  \draw[->] (v2) to node [sloped, below] {$\gamma_2$} (v3);
\end{tikzpicture}
$$

The first and third paths of (3) give further decompositions of the sums on the left and right. For future reference, note the decomposition

$$V_{0123} = V_{23} \boxplus V_{12} \boxplus V_{01}.$$
Remark 2.2. If all but \( V_{01}, V_{02}, V_{03} \) are non-zero, then the right triangle of (4) reduces to

\[
\begin{array}{c}
V_{03} \\
\gamma_2 \\
\gamma_1 \downarrow \\
\gamma_{12} \\
V_{02} \\
\gamma_1 \downarrow \\
V_{01}
\end{array}
\]

In globular terms, this describes vertical composition.

2.2. The coslice of the delooping. The advantage of the following definition is two-fold: it bypasses the theory of conically-smooth stratified spaces, and it is more amenable to related bundle-theoretic constructions than the original definition. See Appendix A.2 for the meaning of the notation.

Definition 2.3. \( V \rightarrow := */\mathfrak{B}^\circ\).

We call this the stratified Grassmannian, and elucidate now its morphisms up to dimension 2.

Remark 2.4. Note that \( \mathfrak{B}^\circ \) is far from being an \( \infty \)-groupoid: only the zero vector space is invertible.

Via the identification \( \Delta^0 \times \Delta^n \simeq \Delta^{n+1} \) as in Remark A.9, \( n \)-simplicies of \( V \rightarrow \) are \( (n+1) \)-simplices \( \mathfrak{B}^\circ \) with no qualification, since \( \mathfrak{B}^\circ \) has a unique 0-simplex. Thus:

2.2.1. 0-simplices. A 0-simplex is a vector space \( V \) (see Warning 2.1).

2.2.2. 1-morphisms. A 1-simplex is as in (2), with source \( V_{01} \) and target \( V_{02} \) in the sense, or convention, of Remark A.10, together with a map \( V_{01} \hookrightarrow V_{12} \oplus V_{01} \xrightarrow{\gamma^{-1}} V_{02} \).

In this sense, morphisms of \( V \rightarrow \) can be said to be ‘injections of vector spaces’.

2.2.3. 2-morphisms. A 2-simplex is a map

\[
\gamma : \Delta^0 \times \Delta^2 \rightarrow \mathfrak{B}^\circ
\]

whose edges may be described as follows:

\[
\begin{array}{c}
W_{12} \\
1 \\
W_{13} \\
V_{01} \\
W_{23} \\
2 \\
V_{02} \\
V_{03} \\
3 \\
0
\end{array}
\]

(6)
We have the following three induced faces of $\gamma$:

$$
\begin{array}{c}
\Delta^0 \star \Delta^1 \xrightarrow{id \times f_i} \Delta^0 \star \Delta^2 \xrightarrow{\gamma} B \boxdot O \\
\downarrow \sim \\
\Delta^2 \xrightarrow{id \times f_i} \Delta^1
\end{array}
$$

where $i = 0, 1, 2$ is skipped by $f_i : [1] \hookrightarrow [2]$. We call the $\{0,1\}$-edge of $\Delta^2$ its source edge, and the $\{0,2\}$-edge of $\Delta^2$ its target edge. In globular terms, this is justified by the special case where $V_{12} = 0$ is the identity. Say, therefore, the induced $f_2$-face of $\gamma$ is its source face, and the $f_1$-face its target face. These two faces share their respective source edges:

$$
\begin{array}{c}
\Delta^0 \star \Delta^0 \xrightarrow{id \times f_i} \Delta^0 \star \Delta^1 \simeq \Delta^2 \xrightarrow{\gamma_2 \gamma_1} B \boxdot O \\
\downarrow \sim \\
\Delta^1 \xrightarrow{\gamma_0}
\end{array}
$$

where $f_1 : [0] \hookrightarrow [1]$ skips 1.

This common source edge $\gamma_{01}$ is determined by the vector space $V_{01}$ ((6)). Then the source face of $\gamma$ is of type $\gamma_2 = (V_{01} \hookrightarrow W_{12} \boxdot V_{01} \simeq V_{02})$, and its target face is of type $\gamma_1 = (V_{01} \hookrightarrow W_{13} \boxdot V_{01} \simeq V_{03})$. The intermediate face is of type $\gamma_0 = (V_{02} \hookrightarrow W_{23} \boxdot V_{02} \simeq V_{03})$, yielding the diagram

$$
\begin{array}{c}
V_{01} \xrightarrow{\sim} W_{12} \boxdot V_{01} \xrightarrow{\gamma_2} V_{02} \xrightarrow{\sim} W_{23} \boxdot V_{02} \xrightarrow{\gamma_0} V_{03} \\
\sim \xrightarrow{\gamma_1} W_{13} \boxdot V_{01}
\end{array}
$$

Consider now the final face of $\gamma$, given as follows:

$$
\begin{array}{c}
\Delta^2 \xleftarrow{(A.5)} \xrightarrow{\gamma} \Delta^0 \star \Delta^2 \xrightarrow{\gamma} B \boxdot O \\
\sim \xrightarrow{\Gamma}
\end{array}
$$

It is of type $\Gamma = (W_{12} \hookrightarrow W_{23} \boxdot W_{12} \simeq W_{13})$. Composing (concatenating) the upper maps in (7) and inserting $\Gamma$ gives

$$
\begin{array}{c}
V_{01} \xrightarrow{\sim} W_{23} \boxdot W_{12} \boxdot V_{01} \xrightarrow{(id \boxdot \gamma_2) \star \gamma_0} V_{03} \\
\sim \xrightarrow{\Gamma \boxdot id} W_{13} \boxdot V_{01} \xrightarrow{\gamma_1 \sim} V_{03}
\end{array}
$$

The left triangle clearly commutes. The homotopy in the left triangle of (4) (with all paths inverted) commutes the right triangle of (8) in view of (5).

**Remark 2.5.** If all but $V_{01}, V_{02}, V_{03}$ are non-zero, then (8) reduces to

$$
\begin{array}{c}
V_{01} \xrightarrow{\sim} V_{02} \xrightarrow{\gamma_0} V_{03} \\
\sim \xrightarrow{\gamma_1}
\end{array}
$$
together with a homotopy commuting it, i.e., a homotopy \( \gamma_1 \simeq \gamma_2 \ast \gamma_0 \) in \( BO^\infty_H \) between paths from \( V_{01} \) to \( V_{03} \).

**Notation 2.6.** For \( C \) an \( \infty \)-category, let \( C^- \) denote its maximal sub-\( \infty \)-groupoid, i.e. the \( \infty \)-groupoid obtained by discarding the non-invertible morphisms of \( C \). That of \( \mathcal{V}^- \) is denoted by \( \mathcal{V}^- \).

**Proposition 2.7.** \( \mathcal{V}^- \simeq BO^\infty_H \simeq * \amalg BO^+_{H} \).

We give the correspondences at the levels of vertices and morphisms whence the stated categorical (in Lurie’s terminology), i.e., weak (in Joyal’s terminology) equivalence will be clear. We should note again that the second equivalence, from Remark 1.1, is not canonical.

**Proof.** The one-to-one correspondence at the level of vertices is given in Section 2.2.1. We first consider the identity morphism of a point of \( \mathcal{V}^- \). Let \( d : [1] \to [0] \) be the trivial degeneracy map. The identity 1-morphism of \( V_{01} \) in \( \mathcal{V}^- \) is by definition the pullback

\[
\text{id}_{V_{01}} = \left( \Delta^0 \ast \Delta^1 \xrightarrow{id \ast d} \Delta^0 \ast \Delta^0 \xrightarrow{V} BO^\infty \right).
\]

In terms of (2), where now \( V_{01} = V \), the edge \( W_{12} = V_{12} \) is the further pullback

\[
\begin{align*}
\Delta^1 \xrightarrow{id_1} & \Delta^0 \ast \Delta^1 \xrightarrow{id \ast d} \Delta^0 \ast \Delta^0 \xrightarrow{V} BO^\infty \\
\downarrow & \downarrow \\
\Delta^0 \xrightarrow{f_0} & \Delta^1
\end{align*}
\]

which factors as the diagram indicates. (Here, \( f_0 \) skips \( 0 \in [1] \).) The corresponding map \( C[1] \to BO^\infty \) of simplicial categories factors then as

\[
C[1] \to C[0] \to C[1] \to BO^\infty,
\]

which picks out the identity morphism of \( BO^\infty \), i.e., \( W_{12} = 0 \), and, moreover, the path \( V_{01} \simeq V_{01} \) is constant.

Now, a 1-morphism \( \gamma_2 = \left( V_{01} \xrightarrow{W_{12} \boxplus V_{01}} V_{02} \right) \) in the sense of Section 2.2.2 is invertible if there is a 2-morphism \( \gamma \) with source face \( \gamma_2 \) and target face \( \gamma_1 = \text{id}_{V_{01}} \).

In terms of (6), this means \( V_{03} = V_{01}, W_{13} = 0, \) and so \( W_{12} = 0 = W_{23} \), whence we are in the situation of Remark 2.5 with \( \gamma_1 \) constant. The statement follows. \( \square \)

**Appendix A. Nerves, slices**

Let \( \Delta \) denote the simplex category, and \( sSet = \text{pSh}(\Delta) \) that of simplicial sets, i.e. set-valued presheaves on \( \Delta \), and \( \mathcal{C}_{\Delta} \) that of simplicial categories, that is, the category of categories enriched in \( sSet \). We assume the reader is familiar with the nerve \( N(C) = N_{\bullet}(C) \in sSet \) of an ordinary category \( C \).

In this section, \( \Delta^k = \text{Hom}_{\Delta}(\cdot, [k]) \in sSet \) denotes the standard \( k \)-simplex. For \( X = X_{\bullet} \) a simplicial set, the Yoneda lemma identifies elements of \( X_k \) with maps \( \Delta^k \to X \); a fact we will use without mention.
A.1. Simplicial nerves. We recall the simplicial nerve construction ([Cor82], though see [Lur22, 00KT]) that featured crucially in the delooping \( B^\infty O \) of \( (BO^\infty_\infty, \boxplus) \), following [Lur09, §1.1.5]. The simplicial nerve is also called the homotopy-coherent nerve.

Similarly to the Yoneda embedding \( \Delta \hookrightarrow sSet, [k] \mapsto \Delta^k \), which gives a simplicial set for each \( k \in \mathbb{N} \), there exists functor

\[
\mathcal{C} : \Delta \to \mathbf{Cat}_\Delta.
\]

**Definition A.1.** We first define \( \mathcal{C} \) on objects, then on morphisms.

1. The simplicial category \( \mathcal{C}[k] \) has the same objects as those of \([k]\), and the simplicial sets of morphisms in each \( \mathcal{C}[k] \) are given by

\[
\text{Hom}_{\mathcal{C}[k]}(i, j) = \mathbb{N}(P_{i, j}),
\]

where \( P_{i, j}, 0 \leq i, j \leq k \) is empty if \( i > j \), and

\[
P_{i, j} = \{ I \subseteq \{ i \leq a + 1 \leq \cdots \leq j \} \subseteq [k] : a, b \in I \}
\]

if \( i \leq j \). Its nerve is taken with respect to its poset structure, with partial order \( \leq \) given by inclusions of the \( I \). For each triple \( i \leq j \leq p \) in \([k]\), there is a map \( P_{i, p} \times P_{i, j} \to P_{i, p} \) defined by taking unions. The nerve functor applied to these maps defines maps \( \text{Hom}_{\mathcal{C}[k]}(j, p) \times \text{Hom}_{\mathcal{C}[k]}(i, j) \to \text{Hom}_{\mathcal{C}[k]}(i, p) \) of simplicial sets, which is associative since so is taking unions.

2. A map \( f : [l] \to [k] \) in \( \Delta \) induces a map \( \mathcal{C}[l] \to \mathcal{C}[k] \) as follows: on objects, it is given by \([l] \ni i \mapsto f(i) \in [k]\), and on the mapping posets it is given by \( P_{i, j} \ni I \mapsto f(I) \in P_{f(i), f(j)} \), applying \( \mathbb{N} \) to which defines the map \( f = \mathcal{C}f : \mathcal{C}[l] \to \mathcal{C}[k] \).

**Definition A.2.** We call the \( P_{i, j} \) mapping posets, and their nerves mapping spaces.

**Definition A.3.** The simplicial nerve \( N^\Delta(D) = N^\Delta_\Delta(D) \) of a simplicial category \( D \) is the simplicial set whose set of \( k \)-simplices is defined by

\[
N^\Delta_k(D) = \text{Hom}_{\mathbf{Cat}_\Delta}(\mathcal{C}[k], D).
\]

This is contravariant in \([k]\) via the covariance of \( \mathcal{C} \).

In other words, \( N^\Delta \) is the restriction of the Yoneda embedding \( \mathbf{Cat}_\Delta \to \mathbf{pSh}(\mathbf{Cat}_\Delta) \) along \( \mathcal{C} : \Delta \to \mathbf{Cat}_\Delta \).

If \( D \) is Kan-enriched, then \( N^\Delta(D) \) is an \( \infty \)-category ([CP86], [Lur09, Proposition 1.1.5.10], [Lur22, 00LJ]).

A.2. Joins and (co)slices. For \( f : K \to \mathcal{C} \) a functor from a simplicial set to an \( \infty \)-category, there is ([Joyal02], [Lur22, 01GP]) a right fibration \( \mathcal{C}/f \to \mathcal{C} \) and a left fibration \( f/\mathcal{C} \to \mathcal{C} \), whose domains are respectively called the slice and coslice of \( \mathcal{C} \) at \( f \). We will recall their definitions, but refer the reader to the op. cit. for the named lifting properties.

In the following, our convention is that \( X_{-1} = \emptyset \) for \( X \bullet \), a simplicial set, and a product is empty if one of its factors is \( \emptyset \). The following is equivalent to the more
standard definition; see [Lur22, 0234]. It is a simplicial version of Milnor’s general
topological construction from [Mil56].

**Definition A.4.** The join \( X \star Y = (X \star Y)_\bullet \) of two simplicial sets \( X = X_\bullet, Y = Y_\bullet \) is defined by
\[
(X \star Y)_k = \{ \langle \pi, f_-, f_+ \rangle : \pi : \Delta^k \to \Delta^1, f_- : \Delta^k|_0 \to X, f_+ : \Delta^k|_1 \to Y \},
\]
where \( \pi, f_-, f_+ \) are maps of simplicial sets, and \( \Delta^k|_i = \{ i \} \times_{\Delta^1} \Delta^k, i = 0, 1, \) is
defined using \( \pi \). Given \( \phi : [l] \to [k] \) in \( \Delta^1 \), the corresponding \( \phi : \Delta^l \to \Delta^k \) defines a
map \((X \star Y)_k \to (X \star Y)_l \) by restrictions.

**Remark A.5.** We have injections \( \iota_0 : X \hookrightarrow X \star Y, \iota_1 : Y \hookrightarrow X \star Y \). For the former,
let \( f : \Delta^k \to X \) be a \( k \)-simplex of \( X \). Defining \( \pi : \Delta^k \to \{ 0 \} \hookrightarrow \Delta^1 \) and setting
\( f_- = f \), and necessarily \( f_+ : \emptyset \to Y \), gives a map \( X_k \to (X \star Y)_k \). In the inclusion
of \( Y \) into \( X \star Y \), \( \pi \) is defined by factoring through the projection to 1 and setting
\( f_- \) empty instead.

**Remark A.6.** The join construction is functorial in both arguments. Given \( \phi : X \to X' \), \( \psi : Y \to Y' \), we write \( \phi \star \psi \) for the induced map \( X \star Y \to X' \star Y' \).

**Definition A.7.** Let \( K \) be a simplicial set, \( \mathcal{C} \) an \( \infty \)-category, and \( f : K \to \mathcal{C} \) a map. The slice \( \mathcal{C}/f \) of \( \mathcal{C} \) at \( f \) is the simplicial set defined by
\[
(\mathcal{C}/f)_n = (\text{Hom}_{\mathcal{C}}(\Delta^n \star K, \mathcal{C})),
\]
where the subscript \( K \) indicates that the set in question consists of maps \( \phi : \Delta^n \star K \to \mathcal{C} \) whose precomposition \( K \xrightarrow{\phi} \Delta^n \star K \xrightarrow{f} \mathcal{C} \) is \( f \).

The face and degeneracy maps are given by precomposition and functoriality: a
map \( \psi : \Delta^m \to \Delta^n \) induces a map \( \Delta^m \star K \xrightarrow{\psi \star \text{id}} \Delta^n \star K \xrightarrow{f} \mathcal{C} \), which is clearly in
\((\mathcal{C}/f)_m\), i.e., \( (\phi \circ (\psi \star \text{id}))|_K = f \). The slice is again an \( \infty \)-category.

The projection \( \mathcal{C}/f \to \mathcal{C} \) is given by precomposing \( \phi : \Delta^n \star K \to \mathcal{C} \) with \( \Delta^n \xrightarrow{\iota_0} \Delta^n \star K \).

The coslice \( f/\mathcal{C} \) is defined analogously, with \( \Delta^n \star K \) replaced by \( K \star \Delta^n \), \( \iota_1 \) by \( \iota_0 \) and
vice versa, throughout. It is again an \( \infty \)-category.

**Notation A.8.** Let \( \iota_x : \Delta^0 \to \mathcal{C} \) be given by a vertex \( x \in \mathcal{C}_0 \). We write \( \mathcal{C}/x \rightleftharpoons \mathcal{C}/\iota_x, \ x/\mathcal{C} : \iota_x/\mathcal{C} \). They are respectively called the over- and under-\( \infty \)-category at \( x \).

**Remark A.9.** There are canonical isomorphisms \( \Delta^k \star \Delta^l \simeq \Delta^{k+l+1} \), such that the composition \( \Delta^k \xrightarrow{\iota_k} \Delta^k \star \Delta^l \xrightarrow{\iota_{k+l+1}} \Delta^{k+l+1} \) is given by \( [k] \mapsto [k+1+l], i \mapsto i \), and such that the composition \( \Delta^l \xrightarrow{\iota_l} \Delta^k \star \Delta^l \xrightarrow{\iota_{k+l+1}} \Delta^{k+l+1} \) is given by \( [l] \mapsto [k+1+l], i \mapsto k+1+i \).

**Remark A.10.** We should explicate the degeneracies in an under-\( \infty \)-category \( x/\mathcal{C} \). Via Remark A.9, a 0-simplex of \( x/\mathcal{C} \) is a 1-simplex of \( \mathcal{C} \) with source \( x \). Given a 1-simplex of \( x/\mathcal{C} \), written \( \sigma : \Delta^0 \star \Delta^1 \to \mathcal{C} \), the source and target \( \sigma_0, \sigma_1 \), are given,
according to Definition A.7, by \( \sigma_0 : \Delta^0 \star \Delta^0 \xrightarrow{\text{id}} \Delta^0 \star \Delta^1 \xrightarrow{\sigma} \mathcal{C} \), and similarly with \( \text{id} \star 1 \) for \( \sigma_1 \). The faces of higher simplices are to be understood analogously.
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