Asymptotics of some quantum invariants of the Whitehead chains

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Abstract

In this paper, we study the generalized volume conjecture of the colored Jones polynomials of the Whitehead chains with one belt colored by $M_1$ and all the clasps colored by $M_2$, evaluated at $(N + \frac{1}{2})$-th root of unity, where $M_1$ and $M_2$ are sequences of integers in $N$. By considering the limiting ratios, $s_1$ and $s_2$, of $M_1$ and $M_2$ to $(N + \frac{1}{2})$, we show that the exponential growth rate of the invariants coincides with the hyperbolic volume of the link complement with certain incomplete hyperbolic structure parametrized by $s_1$ and $s_2$. In the proof we figure out the correspondence between the critical point equations of the potential function and the hyperbolic gluing equations of particular triangulations of the link complements. Furthermore, we discover some connection between the potential function, the theory of angle structure and covolume function. As a corollary, we prove the volume conjecture for the Turaev-Viro invariants for all Whitehead chains.

1 Introduction

1.1 Overview of volume conjectures

The volume conjecture of the colored Jones polynomials discovered by R. Kashaev, H. Murakami and J. Murakami suggests that the $N$-th colored Jones polynomials of any hyperbolic knot grow exponentially with growth rate the hyperbolic volume of the knot complement.

**Conjecture 1.** [8][13] Let $K$ be a hyperbolic knot and $J'_N(K; t)$ be the normalized $N$-th colored Jones polynomials of $K$ evaluated at $t$ such that $J'_N(\text{unknot}, t) = 1$ for all $N \in \mathbb{N}$. We have

$$\lim_{N \to \infty} 2\pi N \log |J'_N(K; e^{2\pi i/N})| = \text{Vol}(S^3 \setminus K),$$

where $\text{Vol}(S^3 \setminus K)$ is the hyperbolic volume of the knot complement.

Later, H. Murakami and J. Murakami extended the above volume conjecture to non-hyperbolic link and suggested that the exponential growth rate captures the simplicial volume of the link complement.

**Conjecture 2.** [13] Let $L$ be a link and $J'_{\vec{N}}(L; t)$ be the $\vec{N}$-th normalized colored Jones polynomials of $L$ evaluated at $t$, where $\vec{N} = (N, \ldots, N)$. We have

$$\lim_{N \to \infty} 2\pi N \log |J'_{\vec{N}}(L; e^{2\pi i/N})| = v_3||S^3 \setminus L||,$$

where $||S^3 \setminus L||$ is the simplicial volume of the link complement.

In 2015, Q. Chen and T. Yang discovered a version of volume conjecture of Turaev-Viro invariants at $2r$-th root of unity, where $r$ is odd. The conjecture can be stated as follows.

**Conjecture 3.** [2] For every hyperbolic 3-manifold $M$ with finite volume, we have

$$\lim_{r \to \infty, r \text{ odd}} \frac{2\pi}{r} \log \left( TV_r(M, e^{2\pi i/r}) \right) = \text{Vol}(M)$$

It turns out when the 3-manifold is a link complement, the Turaev-Viro invariants is related to the unnormalized colored Jones polynomials of the link evaluated at $(N + \frac{1}{2})$-th root of unity as follows.
Theorem 1. [5] Let $L$ be a link in $S^3$ with $n$ components. Then given an odd integer $r = 2N + 1 \geq 3$, we have

$$TV_r(S^3 \setminus L, e^{2\pi i / r}) = 2^{n-1} \left( \frac{2\sin(\frac{\pi}{r})}{\sqrt{r}} \right)^2 \sum_{1 \leq M \leq \frac{2r}{2}} \left| J_M(L, e^{\frac{2\pi i}{r}}) \right|^2,$$

Here, $J_M(L, t)$ is the unnormalized colored Jones polynomials of the link $L$, i.e.

$$J_N(U, t) = \frac{t^{N/2} - t^{-N/2}}{t^2 - t^{-2}}$$

for the unknot $U$ and any $N \in \mathbb{N}$.

Therefore, by studying the asymptotics of the colored Jones polynomials of links at $(N + \frac{1}{2})$-th root of unity, one can relate the volume conjectures of the colored Jones polynomials of a link to the volume conjecture of the Turaev-Viro invariants of its complement. In particular, in [5], the volume conjecture of the Turaev-Viro invariants is extended to non-hyperbolic link complement and this generalization is proved for all knots with zero simplicial volume.

Conjecture 4. [5] For every link $L$ in $S^3$, we have

$$\lim_{r \to \infty} \frac{2\pi}{r} \log(TV_r(S^3 \setminus L, e^{2\pi i / r})) = v_3 ||S^3 \setminus L||.$$

Besides, in [5], R. Detcherry, E. Kalfagianni and T. Yang ask the following question:

Conjecture 5. (Question 1.7 in [5]) Is it true that for any hyperbolic link $L$ in $S^3$, we have

$$\lim_{N \to \infty} \frac{2\pi}{N + \frac{1}{2}} \log |J_M(L, e^{\frac{2\pi i}{N + \frac{1}{2}}})| = \text{Vol}(S^3 \setminus L)$$

The answer is affirmative when $L$ is the figure eight knot. In fact, for the figure eight knot we can say more. In [21], Thomas Au and the author consider the $M$-th colored Jones polynomials of the figure eight knot evaluated at $(N + \frac{1}{2})$-th root of unity. Here we regard $M$ as a sequence in $N$ and define the limiting ratio $s = \lim_{N \to \infty} \frac{M}{N + \frac{1}{2}}$. As an easy consequence of Theorem 7(ii) in [21], we have the following result.

Theorem 2. For the figure eight knot $4_1$, there exists $\delta > 0$ such that for $1 - \delta < s \leq 1$, we have

$$\lim_{N \to \infty} \frac{2\pi}{N + \frac{1}{2}} \log |J_M(4_1, e^{\frac{2\pi i}{N + \frac{1}{2}}})| = \text{Vol}(S^3 \setminus 4_1, u = 2\pi i(1 - s)),$$

where $\text{Vol}(S^3 \setminus 4_1, u = 2\pi i(1 - s))$ is the volume of $S^3 \setminus 4_1$ equipped with the incomplete hyperbolic structure such that the logarithm of the holonomy of the meridian $u = 2\pi i(1 - s)$.

An important remark is that Theorem 2 is true for the root of unity $t = e^{\frac{2\pi i}{N + \frac{1}{2}}}$ but false for the original root $t = e^{\frac{2\pi i}{N}}$. As a generalization, we propose the generalized volume conjecture to hyperbolic link as follows.

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1 The generalization volume conjecture has been studied in several literatures (e.g. [2], [10], [12]). Unfortunately, in the original setting, i.e. $t = e^{\frac{2\pi i}{N}}$, this generalization fails even for knot (see Section 4 in [3]). Besides, the original volume conjecture is not true for split link due to the choice of normalization. Even if the normalization problem is fixed, in [20] R. van der Veen shows that the conjecture is false for the Whitehead chains with more than one belt.

On the other hand, if we consider the root $q = e^{\frac{2\pi i}{N + \frac{1}{2}}}$ rather than the original root, the above problems disappear and it gives a hope to have a generalized volume conjecture to hyperbolic link. Note that the difference between these two roots has also been discussed in [21].
which depends smoothly on the parameters $s_i$ and $z_i$.

The figure eight knot [12], the 5-bridge knot, and knots with 6 or 7 crossings [17, 18]. Moreover, the hyperbolic gluing equations (e.g. edges equations, surgery equations) of certain triangulation of the link complement. Recall that in order to find the critical point of the potential function, we need to solve the critical point equations

$$\frac{\partial}{\partial z_i} \Phi_L(z_1, z_2, \ldots, z_n) = 0 \quad \text{for } i = 1, 2, \ldots, n$$

It turns out that in many situations, the critical point equations of the potential function coincide with the hyperbolic gluing equations (e.g. edges equations, surgery equations) of certain triangulation of the link complement with shapes parameters parametrized by $z_1, z_2, \ldots, z_n$. Finally, we need to argue that the critical value of the potential function gives the hyperbolic volume of the link complement.

This approach has been used to study the volume conjecture of the colored Jones polynomials for the figure eight knot [12], the 5-bridge knot [16], and knots with 6 or 7 crossings [17, 18]. Moreover, the correspondence of the critical point equations of the potential functions to the edges equations of certain triangulation has been studied for the twist knots [4] and two bridge knots [13].

Once the original volume conjecture is proved in this way, the generalized volume conjecture can be proved by the ‘continuity argument’. The continuity argument has been used in [21].

1.2 Strategy to prove the volume conjecture

In this subsection, we briefly discuss our approach to study the volume conjecture. A standard way to prove volume conjecture involves three steps. Roughly speaking, first of all, given the explicit formula of the colored Jones polynomials of a link $L$, we can convert the sum into an integral of the form

$$\int_D f(z_1, z_2, \ldots, z_n) \exp \left( \frac{N}{2\pi} \Phi_L(z_1, z_2, \ldots, z_n) \right) dz_1 dz_2 \ldots dz_n,$$

where $(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$ for some $n$, $f(z_1, z_2, \ldots, z_n)$ is a holomorphic function and $\Phi_L(z_1, z_2, \ldots, z_n)$ is called the potential function of the link $L$. After that, we apply the saddle point approximation, which says that the large $N$ behavior is determined by the critical value of the potential function at certain non-degenerate critical point.

The last step is to show that the real part of the critical value gives the hyperbolic volume of the link complement. Recall that in order to find the critical point of the potential function, we need to solve the critical point equations

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Once the original volume conjecture is proved in this way, the generalized volume conjecture can be proved by the ‘continuity argument’. The continuity argument has been used in [21]. The key point is that all the technical assumptions in the above analysis depends continuously on the potential function $\Phi_L(z_1, z_2, \ldots, z_n)$. In particular, if we change the colors $M_i$, we obtain a family of the potential functions which depends smoothly on the parameters $s_i$. Therefore, as long as $s_i$‘s are sufficiently close to 1, the above analysis works and the result follows.

Remark 1. It turns out that for Whitehead link and Whitehead chains, we obtain two potential functions, both of which give the hyperbolic volume at certain critical points. See Section 2 for more details.

1.3 Main results

1.3.1 Asymptotics of the quantum invariants of the Whitehead link

Let $M_1$ and $M_2$ be sequences of integers in $N$ with limiting ratios

$$s_i = \lim_{N \to \infty} \frac{M_i}{N + \frac{1}{2}}$$

where $z_1, z_2, \ldots, z_n > 1$ for some $n$, $f(z_1, z_2, \ldots, z_n)$ is a holomorphic function and $\Phi_L(z_1, z_2, \ldots, z_n)$ is called the potential function of the link $L$. After that, we apply the saddle point approximation, which says that the large $N$ behavior is determined by the critical value of the potential function at certain non-degenerate critical point.

The last step is to show that the real part of the critical value gives the hyperbolic volume of the link complement. Recall that in order to find the critical point of the potential function, we need to solve the critical point equations

$$\frac{\partial}{\partial z_i} \Phi_L(z_1, z_2, \ldots, z_n) = 0 \quad \text{for } i = 1, 2, \ldots, n$$

It turns out that in many situations, the critical point equations of the potential function coincide with the hyperbolic gluing equations (e.g. edges equations, surgery equations) of certain triangulation of the link complement with shapes parameters parametrized by $z_1, z_2, \ldots, z_n$. Finally, we need to argue that the critical value of the potential function gives the hyperbolic volume of the link complement.

This approach has been used to study the volume conjecture of the colored Jones polynomials for the figure eight knot [12], the 5-bridge knot [16], and knots with 6 or 7 crossings [17, 18]. Moreover, the correspondence of the critical point equations of the potential functions to the edges equations of certain triangulation has been studied for the twist knots [4] and two bridge knots [13].

Once the original volume conjecture is proved in this way, the generalized volume conjecture can be proved by the ‘continuity argument’. The continuity argument has been used in [21]. The key point is that all the technical assumptions in the above analysis depends continuously on the potential function $\Phi_L(z_1, z_2, \ldots, z_n)$. In particular, if we change the colors $M_i$, we obtain a family of the potential functions which depends smoothly on the parameters $s_i$. Therefore, as long as $s_i$‘s are sufficiently close to 1, the above analysis works and the result follows.

Remark 1. It turns out that for Whitehead link and Whitehead chains, we obtain two potential functions, both of which give the hyperbolic volume at certain critical points. See Section 2 for more details.

1.3 Main results

1.3.1 Asymptotics of the quantum invariants of the Whitehead link

Let $M_1$ and $M_2$ be sequences of integers in $N$ with limiting ratios
We use $WL$ to denote the Whitehead link and $J_{M_1,M_2}(WL,t)$ to denote the colored Jones polynomials of the Whitehead link with the belt colored by $M_1$ and the clasp colored by $M_2$. In the first part of this paper, we study Conjecture 6 for the Whitehead link. In fact, we obtain the asymptotic expansion formula for $J_{M_1,M_2}(WL,t)$ (See Section 2 for details). Our result can be stated as follows

**Theorem 3.** Conjecture 6 is true for the Whitehead link. Furthermore,

1. There exists potential functions $\Phi^{k(z_1,z_2)}$, a triangulation of $S^3\backslash WL$ and an assignment of shape parameters satisfying that

   (a) the critical point equations $\frac{\partial}{\partial z_i} \Phi(z_1, z_2) = 0$ give the hyperbolic gluing equations for this particular triangulation.

   (b) there exists a critical point $(z^{(\pm z_1,z_2)}, u^{(\pm z_1,z_2)})$ such that

   $$\Re \Phi^{\pm z_1, z_2}(z^{(\pm z_1,z_2)}, u^{(\pm z_1,z_2)}) = \Vol(S^3\backslash WL, u_i = 2\pi i(1 - s_i)),$$

   where $\Vol(S^3\backslash WL, u_i = 2\pi i(1 - s_i))$ is the hyperbolic volume of the Whitehead link complement equipped with the incomplete hyperbolic structure such that the logarithm of the holonomy of the meridian of the belt and the clasp are $u_1 = 2\pi i(1 - s_1)$ and $u_2 = 2\pi i(1 - s_2)$ respectively.

2. We have

$$\lim_{N \to \infty} \frac{2\pi}{N + \frac{1}{2}} \log \left| J_{M_1,M_2}(WL, e^{\frac{2\pi i}{N}}) \right| = \Re \Phi^{\pm z_1, z_2}(z^{(\pm z_1,z_2)}, u^{(\pm z_1,z_2)}) = \Vol(S^3\backslash WL, u_i = 2\pi i(1 - s_i))$$

In [21], the authors describe a procedure to compute the asymptotic expansion formula (AEF) for the Turaev-Viro invariants of the link complement from that of the colored Jones polynomials of the link. Note that the complements of all $a$-twisted Whitehead links $WL(a)$ are homeomorphic. In particular, in order to study their Turaev-Viro invariants, it suffices to study that of the Whitehead link. Let

$$H(s_1, s_2) = \frac{1}{2\pi} \Vol(S^3\backslash WL, u_i = 2\pi i(1 - s_i))$$

By using Theorem 3, we have the following result.

**Theorem 4.** Conjecture 5 is true for all twisted Whitehead link complements. Furthermore, the AEF of $TV_r(S^3\backslash WL(a), e^{\frac{2\pi i}{r}})$ is given by

$$TV_r(S^3\backslash WL(a), e^{\frac{2\pi i}{r}}) \sim \frac{(N + \frac{1}{2})\pi}{\sqrt{2} \det \text{Hess } H(1,1)} \exp \left( \frac{r}{2\pi} \times \Vol(WL(a)) \right),$$

1.3.2 Asymptotic of the quantum invariants of the Whitehead chains

In the second part of this paper, we generalize the previous results on Whitehead link to Whitehead chains. Let $W_{a,b,c,d}$ be the Whitehead chains with $a$ twists, $b$ belts, $c$ clasps and $d$ mirror image of clasps [20].

**Theorem 5.** We have

$$\lim_{N \to \infty} \frac{2\pi}{N + \frac{1}{2}} \log |J_N(W_{a,b,c,d}, e^{\frac{2\pi i}{N}})| = \Vol(S^3\backslash W_{a_1,c,d})$$

As a consequence, we have

**Corollary 1.** For any $a \in \mathbb{Z}$, $c, d$ with $c + d \geq 1$, Conjectures 4 and 7 are true for $W_{a,b,c,d}$.

Next, we study a special case of Conjecture 6 for the Whitehead chains $W_{a_1,c,0}$. More precisely, we colored the belt by $M_1$ and all the clasp components by $M_2$. 


Theorem 6. There exists some $\delta > 0$ such that for any $1 - \delta < s_1, s_2 < 1$, we have

$$\lim_{N \to \infty} \frac{2\pi}{N + \frac{1}{2}} \log |J_{M_1, M_2}(W_{a,1,c,0}, e^{\frac{2\pi}{N+1/2}})|$$

$$= \text{Vol}(S^3 \setminus W_{a,1,c,0}, u_1 = 2\pi i(1 - s_1), u_2 = u_3 = \ldots u_{c+1} = 2\pi i(1 - s_2))$$

Remark 2. In fact, we obtain the AEF for $J_{M_1, M_2}(W_{a,1,c,0}, e^{\frac{2\pi}{N+1/2}})$ (See Section 4.4 for more details). Besides, similar argument can be applied to obtain the AEF for $J_{M_1, M_2}(W_{a,1,0,d}, e^{\frac{2\pi}{N+1/2}})$.

1.3.3 Differential formula for the potential function and relation to hyperbolic geometry

From the discussion in Subsection 1.2, we have already seen the importance of the potential function. Thus, it is necessary to have a better understanding of the potential function. In the study of the asymptotics of the colored Jones polynomials, we discover the following formula satisfied by the potential function.

Lemma 1. Write $z_k = x_k + iy_k$ for $k = 1, 2$. The real part of the potential function satisfies the following differential equation:

$$\text{Re } \Phi_{\pm(s_1,s_2)}(z_1, z_2) = \frac{1}{2\pi} V_{\pm(s_1,s_2)}(z_1, z_2) + \sum_{k=1}^{2} y_k \frac{\partial}{\partial y_k} \text{Re } \Phi_{\pm(s_1,s_2)}(z_1, z_2),$$

(1)

where

$$V_{\pm(s_1,s_2)}(z_1, z_2) = D(e^{2\pi i(s_2-1)} - 2\pi i z_1 - 2\pi i z_2) - D(e^{2\pi i(s_2-1)} - 2\pi i z_2) + D(e^{2\pi i z_1})$$

and $D(z)$ is the Bloch-Wigner function.

This formula has two immediate consequences. First, the derivatives of the potential function is equal to the product of the shape parameters around the vertices (see Section 2.4 for more details). From this it is easy to see that the critical value of the potential function is the sum of the volume of the ideal tetrahedra which satisfy the hyperbolic gluing equations, i.e. the hyperbolic volume of the link complement.

Next, the formula relates the potential function to the theory of angle structure. Note that the subset

$$\mathfrak{A} = \left\{ (z_1, z_2) \mid \text{Im } \frac{d}{dz_i} \Phi_{\pm(s_1,s_2)}(z_1, z_2) = \frac{\partial}{\partial y_k} \text{Re } \Phi_{\pm(s_1,s_2)}(z_1, z_2) = 0 \right\}$$

corresponds to the situation where the sum of the angles around the vertices are $2\pi$. Therefore, $\mathfrak{A}$ is a subset of the space of angle structures and the potential function restricted on $\mathfrak{A}$ is exactly the volume function. In particular, by the theorem of angle structure, we know that the maximum point on $\mathfrak{A}$ is exactly the hyperbolic volume of the link complement.

Moreover, it is interesting to compare the differential formula (1) and the covolume function discussed in [9]. Recall that every hyper-ideal tetrahedron is determined (up to isometry) by six dihedral angles $\{a_{ij}\}_{1 \leq i < j \leq 4}$, where $a_{ij}$ is the dihedral angle between face $i$ and face $j$. Let $l_{ij}$ be the length of the edge between face $i$ and face $j$. Then $a_{ij}$ and $l_{ij}$ uniquely determined each other. Moreover, the volume function $\text{vol}$ satisfies the Schläfli’s formula:

$$\frac{\partial \text{vol}}{\partial a_{ij}} = \frac{l_{ij}}{2},$$

The Legendre transform of $\text{vol}$ is called the co-volume function $\text{cov}$, which is given by

$$\text{cov}(l_{ij}) = \text{vol}(a_{ij}) + \sum_{i<j} l_{ij} \cdot \frac{a_{ij}}{2}$$

5
Note that by the Schäfli’s formula,

\[ \frac{\partial \text{cov}}{\partial l_{ij}} = \sum_{i<j} \frac{\partial \text{vol}}{\partial a_{ij}} \frac{\partial a_{ij}}{\partial l_{ij}} + \sum_{i<j} l_{ij} \frac{\partial a_{ij}}{\partial l_{ij}} + a_{ij}^2 = \frac{a_{ij}^2}{2} \]

Thus we have

\[ \text{cov}(l_{ij}) = \text{vol}(a_{ij}) + \sum_{i<j} l_{ij} \frac{\partial \text{cov}}{\partial l_{ij}} \] (2)

Now, given a triangulation \( \{ \Delta_1, \ldots, \Delta_n \} \) of a 3-manifold \( M \) with edge lengths \( \{ l_{ij}^1, \ldots, l_{ij}^n \} \), where we identify \( l_{pq}^k \) with \( l_{rs}^b \) whenever these two edges are glued together, we can define the co-volume function \( \text{cov}_M \) of \( M \) to be

\[ \text{cov}_M(l_{ij}^1, \ldots, l_{ij}^n) = \sum_{k=1}^n \text{cov}_k(l_{ij}) = \sum_{k=1}^n \text{vol}_k(a_{ij}) + \sum_{i,j} l_{ij} A_{ij} \] (3)

where \( A_{ij} = \frac{\partial \text{cov}_M}{\partial l_{ij}} \) is the sum of the dihedral angles around the edge \( l_{ij} \). Equations (1), (2) and (3) suggest that we should think of \( y_k \) as some sort of ‘length’. We hope that this formula will help us to understand the theory of potential function.

The final remark for this equation is that the proof of Lemma 1 is based on the basic properties of the dilogarithm function and the Lobachevsky function. In particular, by using the argument, one can show that the potential functions discussed in [16, 17, 18], and the potential functions in [15, 4] after the reparametrization \( z \mapsto e^{2\pi i z} \), also satisfy the same type of differential formula.

### 1.4 Outline

In Section 2, we compute the potential functions of \( J_{M_1, M_2}(WL, e^{\frac{2\pi i}{N+\frac{1}{2}}} ) \) and find its asymptotic expansion formula. Besides, we prove the differential formula for the potential function. In Section 3, by using the results in previous section, we compute the asymptotic expansion formula for \( TV_r(S^3 \setminus WL) \). In Section 4, we first compute the AEF for \( J_S(W_{a,1,c,d}, e^{\frac{2\pi i}{N+\frac{1}{2}}} ) \). After that, we generalize previous result to the Whitehead chains with 1 belt and c clasps and the Whitehead chains with 1 belt and d mirror clasps.

### 1.5 Acknowledgements

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### 2 Asymptotics of \( J_{M_1, M_2}(WL, e^{\frac{2\pi i}{N+\frac{1}{2}}} ) \)

#### 2.1 Potential functions of \( J_{M_1, M_2}(WL, e^{\frac{2\pi i}{N+\frac{1}{2}}} ) \)

In this section, we compute the potential function of the \((M_1, M_2)\)-th unnormalized colored Jones polynomials for the Whitehead link \( WL \), where \( M_1, M_2 \) are sequences of positive integers in \( N \). First of all, the colored Jones polynomials is given by [22]

\[ J_{M_1, M_2}(WL, t) = t^{\frac{M_2-1}{2}} (t^\frac{1}{2} - t^{-\frac{1}{2}})^{M_2-1} \sum_{n=0}^{M_2-1} \left( t^{\frac{M_2(2n+1)}{2}} - t^{-\frac{M_2(2n+1)}{2}} \right) \cdot \hat{C}(n, t; M_2), \]

where

\[ \hat{C}(n, t; M_2) = t^{\frac{M_2(M_2-1)}{2}} \sum_{l=0}^{M_2-1-n} t^{-M_2(l+n)} \prod_{j=1}^{n} \frac{(1 - t^{M_2-l-j})(1 - t^{l+j})}{1 - t^j}. \]
Put \( t = \exp \left( \frac{2\pi i}{N + \frac{1}{2}} \right) \). By direct computation, we have

\[
(t^\frac{1}{2} - t^{-\frac{1}{2}})^{-1} = \frac{1}{2i \sin \left( \frac{\pi}{N + \frac{1}{2}} \right)}
\]

(4)

\[
M_1(2n+1) - M_1(2n+1) = \frac{1}{2} \left( (M_1 - (N + \frac{1}{2})(2n+1)) + (M_1 - (N + \frac{1}{2})(2n+1)) \right) - t^{-\left( (N + \frac{1}{2})(2n+1) \right) - (M_1 - (N + \frac{1}{2})(2n+1)}.
\]

(5)

Besides,

\[
M_2(l) = e^{\frac{2\pi i}{N+\frac{1}{2}} \left( M_2(l) - \frac{1}{2} \right)}
\]

(6)

\[
t^{-\left( M_2(l+n) \right)} = e^{-2\pi i \left( \frac{M_2(l+n)}{N+\frac{1}{2}} \right)}
\]

(7)

Recall that the quantum dilogarithm function \( \phi^h(z) \) is defined by

\[
\phi^h(z) = \int_\Omega 4x \sinh(\pi x) \sinh(\pi xh) \, dx,
\]

where

\[
\Omega = (-\infty, -\epsilon] \cup \{ z \in \mathbb{C} \mid |z| = \epsilon, \text{Im} \, z > 0 \} \cup [\epsilon, \infty)
\]

and

\[
z \in \left\{ z \in \mathbb{C} \mid -\frac{\pi h}{2} < \text{Re} \, z < \pi + \frac{\pi h}{2} \right\}
\]

For any \( z \in \mathbb{C} \) with \( 0 < \text{Re} \, z < \pi \), the quantum dilogarithm function satisfies the functional equation that

\[
1 - e^{2iz} = \exp \left( \phi^h \left( z - \frac{\pi h}{2} \right) - \phi^h \left( z + \frac{\pi h}{2} \right) \right)
\]

Furthermore, it satisfies

\[
\lim_{h \to 0} 2\pi i h \phi^h(z) = \text{Li}_2(e^{2iz})
\]
Put \( h = \frac{1}{N + \frac{1}{2}} \). Then we have

\[
\prod_{j=1}^{n} (1 - e^{i\theta_{2j} - i\theta_j}) = \prod_{j=1}^{n} (1 - e^{2\pi i \left( \frac{M_2}{N + \frac{1}{2}} - \frac{j + \frac{1}{2}}{N + \frac{1}{2}} \right)})
\]

\[
= \exp \left( \sum_{j=1}^{n} \left[ \varphi^h \left( \frac{M_2}{N + \frac{1}{2}} - \frac{j + \frac{1}{2}}{N + \frac{1}{2}} \right) - \varphi^h \left( \frac{M_2}{N + \frac{1}{2}} - \frac{(j + \frac{1}{2})\pi}{N + \frac{1}{2}} \right) \right] \right)
\]

\[
= \exp \left( \varphi^h \left( \frac{M_2}{N + \frac{1}{2}} - \frac{\pi}{2N + 1} \right) - \varphi^h \left( \frac{M_2}{N + \frac{1}{2}} - \frac{\pi}{2N + 1} \right) \right) \right) \right)
\]

(8)

\[
\prod_{j=1}^{n} (1 - e^{i\theta_j}) = \prod_{j=1}^{n} (1 - e^{\frac{2\pi i (j + \frac{1}{2})}{2N + 1}})
\]

\[
= \exp \left( \sum_{j=1}^{n} \left[ \varphi^h \left( \frac{\pi}{2N + 1} + \frac{\pi}{2N + 1} \right) - \varphi^h \left( \frac{\pi}{2N + 1} + \frac{\pi}{2N + 1} \right) \right] \right)
\]

(9)

\[
\prod_{j=1}^{n} (1 - e^{i\theta_j}) = \prod_{j=1}^{n} (1 - e^{\frac{2\pi i}{N + \frac{1}{2}}})
\]

\[
= \exp \left( \sum_{j=1}^{n} \left[ \varphi^h \left( \frac{\pi}{2N + 1} + \frac{\pi}{2N + 1} \right) - \varphi^h \left( \frac{\pi}{2N + 1} + \frac{\pi}{2N + 1} \right) \right] \right)
\]

(10)
Altogether, by (4) - (10), we have

\[
J_{M_1, M_2}(WL, e^{\frac{2\pi i}{N + \frac{1}{2}}}) = \frac{(-1)^{(M_1 - N)} e^{\frac{2\pi i (M_2^2 - M_2^2 - \frac{1}{2})}{2(N + \frac{1}{2})}}}{2 \sin(N + \frac{1}{2})} \times \sum_{n=0}^{M_2 - 1} \left[ e^{\frac{N}{N + \frac{1}{2}}} \left( e^{\frac{2\pi i}{N + \frac{1}{2}} (2\pi i \left( \frac{M_1}{N + \frac{1}{2}} - 1 \right) ) \left( \frac{1}{2} \pi i \left( \frac{N - 1}{N + \frac{1}{2}} \right) \right) \right) \right] \times \sum_{n=0}^{M_2 - 1 - n} e^{-\frac{N}{N + \frac{1}{2}}} \left[ e^{\frac{2\pi i}{N + \frac{1}{2}} (2\pi i \left( \frac{M_2}{N + \frac{1}{2}} - 1 \right) ) \left( \frac{1}{2} \pi i \left( \frac{N - 1}{N + \frac{1}{2}} \right) \right) \right] \times \left\{ \exp \left[ \varphi^h \left( \frac{M_2 \pi}{N + \frac{1}{2}} - \frac{l \pi}{N + \frac{1}{2}} - \frac{n \pi}{N + \frac{1}{2}} - \frac{\pi}{2(N + 1)} \right) \right] - \varphi^h \left( \frac{l \pi}{N + \frac{1}{2}} + \frac{\pi}{2(N + 1)} \right) - \varphi^h \left( \frac{n \pi}{N + \frac{1}{2}} + \frac{\pi}{2(N + 1)} \right) - \varphi^h \left( \frac{\pi}{2(N + 1)} \right) + \varphi^h \left( \frac{n \pi}{N + \frac{1}{2}} + \frac{\pi}{2(N + 1)} \right) \right\}
\]

Define the limiting ratio \( s_1 \) and \( s_2 \) by

\[
s_1 = \lim_{N \to \infty} \frac{M_1}{N + \frac{1}{2}} \quad \text{and} \quad s_2 = \lim_{N \to \infty} \frac{M_2}{N + \frac{1}{2}}.
\]

Then we have

\[
J_{M_1, M_2}(WL, e^{\frac{2\pi i}{N + \frac{1}{2}}}) = \frac{(-1)^{(M_1 - N)} e^{\frac{2\pi i (M_2^2 - M_2^2 - \frac{1}{2})}{2(N + \frac{1}{2})}}}{2 \sin(N + \frac{1}{2})} \exp \left( -\varphi^h \left( \frac{\pi}{2(2N + \frac{1}{2})} \right) \right) (I_+ + I_-)
\]

where

\[
I_\pm = e^{\pm \pi \left( \frac{M_2}{N + \frac{1}{2}} - 1 \right)} \sum_{n=0}^{M_2 - 1} \sum_{l=0}^{M_2 - 1 - n} \exp \left( (N + \frac{1}{2}) \Phi_{M_1, M_2}^\pm(s_1, s_2) \left( \frac{1}{2} \pi i, \frac{1}{2} \pi i, \frac{1}{2} \pi i \right) \right)
\]

with

\[
\Phi_{M_1, M_2}^\pm(s_1, s_2) (z_1, z_2) = \frac{1}{2\pi i} \left\{ \pm \left( 2\pi i \left( \frac{M_1}{N + \frac{1}{2}} - 1 \right) \right) \left( 2\pi i \left( \frac{z_1 - 1}{2} \right) \right) \right.
- \left( 2\pi i \left( \frac{M_2}{N + \frac{1}{2}} - 1 \right) \right) \left( 2\pi i z_1 + 2\pi i z_2 \right)
+ \left( 2\pi i \left( \frac{M_2}{N + \frac{1}{2}} - \frac{\pi}{2(N + 1)} \right) \right) \varphi^h \left( \frac{\pi}{2(N + 1)} \right)
- \varphi^h \left( \frac{\pi}{2(N + 1)} \right)
+ \varphi^h \left( \frac{\pi}{2(N + 1)} \right) - \varphi^h \left( \pi z_1 + \frac{\pi}{2(N + 1)} \right)
+ \varphi^h \left( \pi z_1 + \frac{\pi}{2(N + 1)} \right) \left\}
\]
As \( N \to \infty \), we have
\[
\Phi_{\pm(s_1,s_2)}(z_1, z_2) = \lim_{N \to \infty} \Phi_{M_1, M_2}^{\pm(s_1,s_2)}(z_1, z_2)
\]
\[
= \frac{1}{2\pi i} \left[ (2\pi i) \left( z_1 - \frac{1}{2} \right) + (2\pi i) \left( z_2 - \frac{1}{2} \right) \right]
\]
\[
+ \text{Li}_2(e^{2\pi i(z_2-1)-2\pi iz_1-2\pi iz_2}) - \text{Li}_2(e^{2\pi i(s_2-1)-2\pi iz_1})
\]
\[
+ \text{Li}_2(e^{2\pi iz_1}) - \text{Li}_2(e^{2\pi iz_1+2\pi iz_2}) + \text{Li}_2(e^{2\pi iz_1})
\]

2.2 The AEF of \( J_{N,N}(WL, \exp(\frac{2\pi i}{N+\frac{s}{2}})) \)

Next, we consider the \((N,N)\)-th colored Jones polynomials for the Whitehead Link. Note that in this case, the two potential functions \( \Phi^{s(1,1)} \) are the same and are given by
\[
\Phi^{(1,1)}(z_1, z_2) = \lim_{N \to \infty} \Phi_{N,N}^{s(1,1)}(z_1, z_2)
\]
\[
= \frac{1}{2\pi i} \left[ \text{Li}_2(e^{-2\pi iz_1-2\pi iz_2}) - \text{Li}_2(e^{-2\pi iz_2}) + \text{Li}_2(e^{2\pi iz_1}) - \text{Li}_2(e^{2\pi iz_1+2\pi iz_2}) + \text{Li}_2(e^{2\pi iz_1}) \right]
\]

Recall that for \( \theta \in (0, \pi) \), the dilogarithm function \( \text{Li}_2 \) and the Lobachevsky function \( L \) are related by
\[
\text{Li}_2(e^{2\pi i\theta}) = \text{Li}_2(1) + \theta(\pi - \theta) + 2iL(\theta)
\]

Let
\[
\Delta = \{(x, y) \in [0, 1]^2 \mid x + y \leq 1\}
\]

Then for \((z_1, z_2) \in \Delta\), the real part of \( \Phi^{(1,1)}(x_1, x_2) \) is given by
\[
\text{Re} \Phi^{(1,1)}(x_1, x_2) = \frac{1}{\pi} \left| L(-\pi z_1 - \pi z_2) - L(-\pi z_2) + L(\pi z_1) - L(\pi z_1 + \pi z_2) + L(\pi z_1) \right|
\]
\[
= \frac{1}{\pi} \left[ -2L(\pi x_1 + \pi x_2) + 2L(\pi x_2) + L(\pi x_1) \right]
\]
\[
= \frac{1}{\pi} f(x_1, x_2)
\]

Note that this function \( f(x_1, x_2) \) is the function appeared in Equation (3.22) of [22]. In particular, the real part of \( \Phi^{(1,1)}(x_1, x_2) \) has a unique maximum at \((\frac{1}{2}, \frac{1}{2})\) in the \( \Delta \), with maximum value
\[
\text{Re} \Phi^{(1,1)}\left( \frac{1}{2}, \frac{1}{2} \right) = \frac{1}{\pi} f\left( \frac{1}{2}, \frac{1}{2} \right) = \frac{1}{2\pi} \left[ 8L\left( \frac{\pi}{4} \right) \right] = \frac{1}{2\pi} \text{Vol}(WL)
\]

Furthermore, the critical point equations of the potential function are given by
\[
\begin{cases}
\log(1 - e^{-2\pi iz_1-2\pi iz_2}) + \log(1 - e^{2\pi iz_1+2\pi iz_2}) - \log(1 - e^{2\pi iz_1}) = 0 \\
\log(1 - e^{-2\pi iz_1-2\pi iz_2}) + \log(1 - e^{2\pi iz_1+2\pi iz_2}) - \log(1 - e^{-2\pi iz_2}) - \log(1 - e^{2\pi iz_2}) = 0
\end{cases}
\]

(15) (16)

Note that the point \((z_1, z_2) = (\frac{1}{2}, \frac{1}{2})\) indeed solve the equations \[(15)\] and \[(16)\]. Moreover, at the critical point,
\[
\frac{\partial^2}{\partial x_1^2} \Phi^{(1,1)}\left( \frac{1}{2}, \frac{1}{2} \right) = (2\pi i) \left( \frac{i}{1-i} - \frac{-i}{1+i} + \frac{-1}{2} \right) = 2\pi i \left( \frac{2i-1}{2} \right)
\]
\[
\frac{\partial^2}{\partial x_2^2} \Phi^{(1,1)}\left( \frac{1}{2}, \frac{1}{2} \right) = (2\pi i) \left( \frac{i}{1-i} - \frac{-i}{1+i} + \frac{i}{1-i} \right) = 2\pi i (2i)
\]
\[
\frac{\partial^2}{\partial x_1 \partial x_2} \Phi^{(1,1)}\left( \frac{1}{2}, \frac{1}{2} \right) = (2\pi i) \left( \frac{i}{1-i} - \frac{-i}{1+i} \right) = 2\pi i (i)
\]
Thus, by the Poisson Summation Formula (Proposition 4.6 of [16]), we can change the sum into integral.

\[ \Phi^{(1,1)}(z_1, z_2) \text{ has a non-degenerate critical point at } (\frac{1}{2}, \frac{1}{4}) \text{ with critical value} \]

\[
\Phi^{(1,1)} \left( \frac{1}{2}, \frac{1}{4} \right) = \frac{1}{2\pi} \left[ 2 \operatorname{Li}_2(i) - 2 \operatorname{Li}_2(-i) + \operatorname{Li}_2(-1) \right] \\
= \frac{1}{2\pi} \left[ 2 \left( \frac{\pi}{4} - \frac{\pi}{4} + 2i\operatorname{Li}(\frac{\pi}{4}) - 2 \left( \frac{3\pi}{4} - \frac{3\pi}{4} + 2i\operatorname{Li}(\frac{3\pi}{4}) \right) - \frac{\pi^2}{12} \right) \right] \\
= \frac{1}{2\pi} \left[ 8 \operatorname{Li}(\frac{\pi}{4}) + \frac{\pi^2}{12} \right]
\]

Next, we are going to study the AEF for \( J_{(N, N)}(WL, \exp(\frac{2\pi i}{N+2})) \) and show that the exponential growth is given by the critical value of \( \Phi^{(1,1)} \left( \frac{1}{2}, \frac{1}{4} \right) \). From previous discussion, since the critical value is the unique maximum of \( \Re \Phi(z_1, z_2) \) on \( \Delta \), it sufficies to study the AEF near this critical point. Since \( (\frac{1}{2}, \frac{1}{4}) \) is a critical point, we can find a 4 real dimension ball \( B_r \subset \mathbb{C}^2 \cong \mathbb{R}^4 \) centred at the critical point of radius \( r \) such that whenever \( (x_1 + \eta i, x_2), (x_1, x_2 + \eta i) \in B_r \), we have

\[ \left| \frac{d}{d\eta} \Re \Phi(x_1 \pm \eta i, x_2) \right|, \left| \frac{d}{d\eta} \Re \Phi(x_1, x_2 \pm \eta i) \right| < \pi \]

In particular, we may choose \( r \) small enough such that the intersection of the \( \partial B \) and \( \Delta \) is a circle around the critical point. Let

\[ S_{1}^{\pm} = \{(x_1 \pm \eta i, x_2) \mid (x_1, x_2) \in \partial B \cap \Delta, \eta \in [0, r/2] \} \cup \{(x_1 \pm (r/2)i, x_2) \mid (x_1, x_2) \in B \cap \Delta \} \]
\[ S_{2}^{\pm} = \{(x_1, x_2 \pm \eta i) \mid (x_1, x_2) \in \partial B \cap \Delta, \eta \in [0, r/2] \} \cup \{(x_1, x_2 \pm (r/2)i) \mid (x_1, x_2) \in B \cap \Delta \} \]

For every points on \( S_{1}^{\pm} \), we have

\[ \Re \Phi^{(1,1)}(x_1 \pm \eta i, x_2) - \frac{1}{2\pi} \operatorname{Vol}(WL) \leq \left[ \Re \Phi^{(1,1)}(x_1, x_2) - \frac{1}{2\pi} \operatorname{Vol}(WL) \right] + \pi \eta < 2\pi \eta \]  \hspace{1cm} (19)

Similarly, for every points on \( S_{2}^{\pm} \), we have

\[ \Re \Phi^{(1,1)}(x_1, x_2 \pm \eta i) - \frac{1}{2\pi} \operatorname{Vol}(WL) \leq \left[ \Re \Phi^{(1,1)}(x_1, x_2) - \frac{1}{2\pi} \operatorname{Vol}(WL) \right] + \pi \eta < 2\pi \eta \]  \hspace{1cm} (20)

Thus, by the Poisson Summation Formula (Proposition 4.6 of [16]), we can change the sum into integral.

Next, we want to apply the Saddle point approximation. A key observation is that we can choose \( r \) as small as possible such that the assumption in Proposition 3.5 in [16] is automatically satisfied. Moreover, by Lemma A.3 of [16], we have

\[ \exp \left( -\varphi \left( \frac{\pi}{2N+1} \right) \right) = \exp \left( -\frac{N + \frac{1}{2} \pi^2}{6} - \frac{1}{2} \log(N + \frac{1}{2}) - \frac{\pi i}{4} + \frac{\pi i}{12(N + \frac{1}{2})} \right) \]
\[ \sim_{N \to \infty} e^{-\frac{\pi}{4}} (N + \frac{1}{2})^{-\frac{1}{2}} \exp \left( \frac{N + \frac{1}{2} \pi^2}{6} \right) \]  \hspace{1cm} (21)

As a result, since

\[ -\det(\text{Hess} \Phi^{(1,1)}) \left( \frac{1}{2}, \frac{1}{4} \right) = -4\pi^2(1 + i) \neq 0, \]
Altogether, by the saddle point approximation (Proposition 3.5 in [16]), we have

\[ J_{N,N}(WL, e^{\frac{2\pi i}{N + \frac{1}{4}}}) \]

\[ = \frac{e^{\frac{2\pi i}{2}(N^2 - \frac{N}{2} - \frac{1}{2})}}{2\sin\left(\frac{\pi}{N + \frac{1}{2}}\right)} \exp\left(-\varphi^h\left(\frac{\pi}{2N + 1}\right)\right) \]

\[ \left[ \pi i \left(\frac{N}{N + \frac{1}{2}} - 1\right) \sum_{n=0}^{M_2 - 1} \sum_{l=0}^{N_2 - 1-n} \exp\left((N + \frac{1}{2})\Phi^{(1,1)}_{N,N} \left(\frac{n}{N + \frac{1}{2}}, \frac{l}{N + \frac{1}{2}}\right)\right) \right] \]

\[ + e^{-\pi i \left(\frac{N}{N + \frac{1}{2}} - 1\right)} \sum_{n=0}^{M_2 - 1} \sum_{l=0}^{N_2 - 1-n} \exp\left((N + \frac{1}{2})\Phi^{(1,1)}_{N,N} \left(\frac{n}{N + \frac{1}{2}}, \frac{l}{N + \frac{1}{2}}\right)\right) \]

\[ \sim \frac{e^{-\frac{2\pi i}{4}} (N + \frac{1}{2})^4}{\pi} \exp\left(\frac{N + \frac{1}{2}}{2\pi i} \sum_{(\omega_1, \omega_2) \in \left[\frac{1}{2}, 1\right]^2} \pi \partial x \right) \]

\[ \sim \frac{e^{-\frac{2\pi i}{4}} (N + \frac{1}{2})^{3/2}}{-4\pi^2 (1 + i)} \times \exp\left(\frac{N + \frac{1}{2}}{2\pi i} \sum_{(\omega_1, \omega_2) \in \left[\frac{1}{2}, 1\right]^2} \pi \partial x \right) \]

\[ \left(1 + O\left(\frac{1}{N}\right)\right) \]

(22)

with

\[ \lim_{N \to \infty} \phi^{(1,1)}_{N,N}(z_1, z_2) = \frac{\pi^2}{6} \]

(23)

**Remark 3.** If we use the normalization in [22] and [20], then there will be an extra factor

\[ \frac{1}{\sqrt{N}} \]

In particular, the exponent of \((N + \frac{1}{2})\) is 3/2, which agrees with the result in [22] and [27].

### 2.3 The AEF for the \((M_1, M_2)\)-th Colored Jones polynomials of Whitehead Link at \(q = \exp(\frac{2\pi i}{N + \frac{1}{4}})\)

Now we generalize the previous argument to study the AEF of \(J_{(M_1, M_2)}(WL, \exp(\frac{2\pi i}{N + \frac{1}{4}}))\). Recall that the potential functions \(\Phi^{(s_1, s_2)}(z_1, z_2)\) are given by

\[ \Phi^{(s_1, s_2)}(z_1, z_2) = \frac{1}{2\pi i} \left[ (2\pi i)(s_1 - 1)\right] \left[ (2\pi i)z_2 - 1 \right] - (2\pi i)(s_2 - 1) \left[ (2\pi i z_1 + 2\pi i z_2) \right] \]

For simplicity we consider \(\Phi^{(s_1, s_2)}(z_1, z_2)\). The same argument works for \(\Phi^{-}(s_1, s_2)(z_1, z_2)\). First of all, we pick a \(\delta > 0\) small enough such that the function \(F : \left[1 - \delta, 1\right]^2 \times B_r \to \mathbb{C}\) defined by

\[ F(s_1, s_2, z_1, z_2) = \Phi^{(s_1, s_2)}(z_1, z_2) \]

is well-defined and continuous. Besides, by Implicit function theorem, there exists a smooth family of critical points \(\left(\frac{z^{(s_1, s_2)}}{z_1}, \frac{z^{(s_1, s_2)}}{z_2}\right)\) of the family of potential functions \(\Phi^{(s_1, s_2)}(z_1, z_2)\) such that \((z^{(1,1)}_1, z^{(1,1)}_2) = (\frac{1}{2}, \frac{3}{4})\). Note that on the surfaces \(S^1_t\), we have

\[ \text{Re} \Phi^{(1,1)}(x_1 + \eta_i, x_2) - \text{Re} \Phi^{(1,1)}(z^{(1,1)}_1, z^{(1,1)}_2) - 2\pi \eta = \text{Re} \Phi^{(1,1)}(x_1 + \eta_i, x_2) - \frac{1}{2\pi} \text{Vol}(WL) - 2\pi \eta < 0 \]
Since \( S_1^{+} \) is a compact set, by continuity, we may choose \( \delta \) small enough such that
\[
\text{Re} \Phi^{(s_1, s_2)}(x_1 + \eta i, x_2) - \text{Re} \Phi^{(s_1, s_2)}(z_1^{(s_1, s_2)}, z_2^{(s_1, s_2)}) - 2\pi \eta < 0
\]
for any \( 1 - \delta < s_1, s_2 \geq 1 \). The same argument works for \( S_1^{-}, S_2^{+} \) and \( S_2^{-} \).

As a result, the assumption of the Poisson Summation formula is satisfied. Finally, for the same reason, we can choose \( \eta \) small enough such that the condition for the saddle point approximation is automatically satisfied. Therefore, the AEF of \( J_{M_1, M_2}(WL, e^{S_{1/2}}) \) is given by

\[
J_{M_1, M_2}(WL, e^{S_{1/2}}) = \frac{(-1)^{(M_1-N)}e^{\frac{2\pi}{N+2}}(M^2 - \frac{M^2}{N^2} - \frac{1}{4})}{2\sin\left(\frac{\pi}{N+2}\right)} \exp\left(-\varphi^h\left(\frac{\pi}{2(N + \frac{1}{2})}\right)\right) \sum_{n=0}^{\pi_i(M_1 - \frac{1}{N+2})} \sum_{l=0}^{\pi_i(M_2 - \frac{1}{N+2})} \exp\left((N + \frac{1}{2})\Phi_{M_1, M_2}^{\pm\left(M_1 - \frac{1}{N+2}, M_2 - \frac{1}{N+2}\right)}\right) \left(\frac{n}{N + \frac{1}{2}}, \frac{l}{N + \frac{1}{2}}\right)\right) \sum_{n=0}^{\pi_i(M_1 - \frac{1}{N+2})} \sum_{l=0}^{\pi_i(M_2 - \frac{1}{N+2})} \exp\left((N + \frac{1}{2})\Phi_{M_1, M_2}^{\pm\left(M_1 - \frac{1}{N+2}, M_2 - \frac{1}{N+2}\right)}\right) \left(\frac{n}{N + \frac{1}{2}}, \frac{l}{N + \frac{1}{2}}\right)\right]
\]

\[
\sim e^{-\frac{\pi}{4}}(13) \exp\left((N + \frac{1}{2})\Phi_{M_1, M_2}^{\pm\left(M_1 - \frac{1}{N+2}, M_2 - \frac{1}{N+2}\right)}(z_1, z_2)dz_1dz_2\right) + e^{-\frac{\pi}{4}}(13) \exp\left((N + \frac{1}{2})\Phi_{M_1, M_2}^{\pm\left(M_1 - \frac{1}{N+2}, M_2 - \frac{1}{N+2}\right)}(z_1, z_2)dz_1dz_2\right)
\]

\[
\sim e^{-\frac{\pi}{4}}(13) \exp\left((N + \frac{1}{2})\Phi_{M_1, M_2}^{\pm\left(M_1 - \frac{1}{N+2}, M_2 - \frac{1}{N+2}\right)}(z_1, z_2)dz_1dz_2\right) + e^{-\frac{\pi}{4}}(13) \exp\left((N + \frac{1}{2})\Phi_{M_1, M_2}^{\pm\left(M_1 - \frac{1}{N+2}, M_2 - \frac{1}{N+2}\right)}(z_1, z_2)dz_1dz_2\right)
\]

At the end of Section 2.5, we will show that
\[
\Phi^{(s_1, s_2)}(z_1^{(s_1, s_2)}, z_2^{(z_1, z_2)}) = \Phi^{-(s_1, s_2)}(z_1^{-(s_1, s_2)}, z_2^{-(s_1, s_2)})
\]

and
\[
\text{Re} \Phi^{(s_1, s_2)}(z_1^{(s_1, s_2)}, z_2^{(s_1, s_2)}) = \text{Re} \Phi^{-(s_1, s_2)}(z_1^{-(s_1, s_2)}, z_2^{-(s_1, s_2)}) = \text{Vol}(WL; u_1 = 2\pi i (1 - s_1)),
\]

where \( \text{Vol}(WL; u_1 = 2\pi i (1 - s_1)) \) is the hyperbolic volume of the Whitehead link complement with logarithm of holonomies \( u_1 = 2\pi i (1 - s_1) \) at the belt and \( u_2 = 2\pi i (1 - s_2) \) at the clasp. As a result, we
have

\[ J_{M_1, M_2}(WL, e^{\frac{\pi i}{N+\frac{1}{2}}}) \]

\[ \sim e^{-\frac{\pi i}{2}} \frac{(-1)^{(M_1 - N)} e^{\frac{\pi i}{N+\frac{1}{2}}} (M_2 - M_2 \cdot \frac{1}{2})}{2} (N + \frac{1}{2})^{3/2} \exp \left( \frac{N + \frac{1}{2} \pi^2}{6} \right) \]

\[ \times \exp \left( N + \frac{1}{2} \Phi^+ (s_1, s_2)(z_1^+(s_1, s_2), z_2^+(z_1, z_2)) \right) \] (25)

2.4 Geometry for deformation of hyperbolic structure and assignment for the shape parameters

To understand the critical values of the potential functions, we need to associate a concrete triangulation to the link complement that corresponds to the potential function. First of all, we prepare an ideal octahedra (figure 1). The idea is to glue the two blue faces together and the two red faces together (figure 2) to form a cylinder.
The following pictures illustrate how to obtain a cylinder from an ideal octahedron.

Figure 3: the faces $B_1$ and $B_2$ are glued together.

Figure 4: deform the object so that it looks like a cylinder.

Figure 5: the faces $D_1$ and $D_2$ are glued together.

Figure 6: deform the object to obtain a cylinder with a clasp removed.

Next, we consider the curl face of the cylinder. The labels 1, 2, 3, 4 refer to the truncated faces of the ideal octahedron. If we glue the top of the cylinder to the bottom according to the rule $C_2 \rightarrow C_1$ and $A_2 \rightarrow A_1$, then we obtain a decomposition of the boundary torus into parallelograms. Note that this torus is exactly the boundary torus of a tubular neighborhood of the belt component in the Whitehead link complement.
Next, we decompose the ideal octahedron into 5 ideal tetrahedra and assign shape parameters to them (Figure 8). The parameters $U$, $Z$ and $W$ are related by $U = (ZW)^{-1}$. The decomposition of the truncated faces and the assignments of shape parameters to them are shown in Figure 9.
After that, we slide the left edge down (Figure 10) and then obtain the geometric triangulation of the curl surfaces for each cylinder.

From direct computation, for the boundary torus of the belt component, the edges equations are given by
• vertex (i):

\[
[W'Z''(B_2^{-1}W)'(B_2U)'Z''U'][(B_2^{-1}W)''Z'U'W''Z'(B_2U)'] = -1 -1 -1 -1 -1 -1 \frac{1}{W'Z' B_2^{-1}W B_2U} = 1
\]

(26)

• vertex (ii):

\[
(WU'W'')(B_2U)'(B_2^{-1}W)''(B_2U) = \left( \frac{U'}{W'} \right) \left( \frac{(B_2U)''}{(B_2^{-1}W)''} \right)^{-1} = \frac{B_2(1 - B_2W)(1 - W)}{(1 - B_2U) \left( 1 - \frac{1}{U} \right)}
\]

(27)

• vertex (iii): (the same as vertex (i))

\[
[W'Z''(B_2^{-1}W)'(B_2U)'Z''U'][(B_2^{-1}W)''Z'U'W''Z'(B_2U)'] = -1 -1 -1 -1 -1 -1 \frac{1}{W'Z' B_2^{-1}W B_2U} = 1
\]

(28)

• vertex (iv): (the inverse of vertex (ii))

\[
(UW'U'')(B_2^{-1}W)'(B_2U)'(B_2^{-1}W) = \left[ \left( \frac{U'}{W'} \right) \left( \frac{(B_2U)''}{(B_2^{-1}W)''} \right) \right]^{-1} = \frac{(1 - B_2U) \left( 1 - \frac{1}{U} \right)}{B_2(1 - B_2W)(1 - W)}
\]

(29)

For the top and bottom truncated faces, we have

![Diagram](image)

Figure 11: triangulation of the boundary torus of the belt component

From direct computation, for the boundary torus of the clasp component, the edges equations are given by

• vertex (v):

\[
(W)(BU)(B^{-1}W)'(B_2U)'(W)'(U) = \left( \frac{U'}{W'} \right) \left( \frac{(B_2U)''}{(B_2^{-1}W)''} \right)^{-1} = \frac{B_2(1 - B_2W)(1 - W)}{(1 - B_2U) \left( 1 - \frac{1}{U} \right)}
\]

(30)
• vertex (vi): (the inverse of vertex (i))

\[(B_2^{-1}W)(BU)''(W')(U)''(BW')(U) = \left[ \frac{U''}{W''} \right] \left[ \frac{(B_2U)''}{(B_2^{-1}W)''} \right]^{-1} = \left[ \frac{B_2(1 - B_2)}{W} (1 - W) }{(1 - B_2U) (1 - U)} \right]^{-1}
\] (31)

Now we can explore the correspondence between the critical point equations of the potential function and the edges equations of this triangulation. For \( \Phi^{(s_1, s_2)}(z_1, z_2) \), recall that the critical point equations are given by

\[
\begin{align*}
\log(1 - e^{2\pi i (s_2 - 1)} e^{-2\pi i z_1 - 2\pi i z_2}) + \log(1 - e^{2\pi i z_1 + 2\pi i z_2}) - \log(1 - e^{2\pi i z_1}) \\
= 2\pi i (- (s_1 - 1) + (s_2 - 1)) \\
\log(1 - e^{2\pi i (s_2 - 1)} e^{-2\pi i z_1 - 2\pi i z_2}) + \log(1 - e^{2\pi i z_1 + 2\pi i z_2}) - \log(1 - e^{2\pi i (s_2 - 1) - 2\pi i z_2}) \\
- \log(1 - e^{2\pi i z_2}) \\
= 2\pi i (s_2 - 1)= 2\pi i (s_2 - 1)
\end{align*}
\] (32) (33)

Put \( Z = e^{2\pi i z_1} \), \( W = e^{2\pi i z_2} \), \( U = \frac{1}{zw} \), \( B_1 = e^{2\pi i s_1} \) and \( B_2 = e^{2\pi i s_2} \). After taking exponential, we have

\[
\begin{align*}
\frac{(1 - B_2 U) (1 - U)}{(1 - Z)} &= B_1^{-1} B_2 \\
\frac{(1 - B_2 U) (1 - U)}{(1 - B_2 W)} &= B_2 \\
\frac{(1 - B_2 U) (1 - U)}{(1 - B_2 W) (1 - W)} &= B_2
\end{align*}
\] (34) (35)

In particular, from (35) we have

\[
\begin{align*}
\left[ \frac{U''}{W''} \right] \left[ \frac{(B_2U)''}{(B_2^{-1}W)''} \right]^{-1} = 1
\end{align*}
\]

i.e. the edge equations of the triangulation are satisfied. Furthermore, by (34) the holonomy of the meridian of the clasp component is given by

\[
m_1 = \frac{W''Z'(B_2U)''}{W''u''} = \frac{(1 - B_2 U) (1 - U)}{(1 - Z)} \times \frac{1}{B_2} = B_1^{-1}
\]

For the second component, consider the red curve \( \Gamma \) in Figure 12. By (34) we have

\[
\Gamma = \frac{(B_2U)'(B_2U)'}{(B_2^{-1}W)'} \cdot \frac{W''}{U''} = \frac{(1 - B_2 W) (1 - W)}{(1 - B_2 U) (1 - U)} \cdot \frac{1}{B_2} = B_2^{-2}
\] (36)

The meridian and longitude of the clasp are related by the following figure:

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As a result, we have \( m_2^2 = \sqrt{\Gamma} = B_2 \) for suitable orientation of \( m_2 \). Similar arguments can be applied to \( \Phi^{-(s_1,s_2)}(z_1, z_2) \) by replacing \( B_1 \) by \( B_1^{-1} \). Next, recall that the longitude of a knot is defined to be a parallel copy of the knot with zero linking number. From Figure 12, since \( \Gamma \) has linking number 2, the longitude of the clasp component is given by

\[
l_2 = \Gamma \cdot m_2^{-2} = B_2^{-4}
\]  

(37)

2.5 Differential formula for the potential function

After we associate a triangulation of the link complement to the potential function, we need to prove that the critical value is indeed the hyperbolic volume of all the tetrahedra. This can be done by using the following differential formula satisfied by the potential function.

**Proposition 1.** Write \( z_k = x_k + iy_k \) for \( k = 1, 2 \). The real part of the potential function satisfies the following differential equation:

\[
\text{Re } \Phi^{\pm(s_1,s_2)}(z_1, z_2) = \frac{1}{2\pi} V^{\pm(s_1,s_2)}(z_1, z_2) + \sum_{k=1}^{2} y_k \frac{\partial}{\partial y_k} \text{Re } \Phi^{\pm(s_1,s_2)}(z_1, z_2),
\]

(38)

where

\[
V^{\pm(s_1,s_2)}(z_1, z_2) = D(e^{2\pi i(s_2-1)-2\pi iz_1-2\pi iz_2}) - D(e^{2\pi i(s_2-1)-2\pi iz_1}) + D(e^{2\pi iz_2}) - \frac{D(e^{2\pi iz_1})}{2\pi} + y \frac{\partial}{\partial y} \text{Re } \Phi^{\pm(s_1,s_2)}(z_1, z_2),
\]

(39)

and \( D(z) \) is the Bloch-Wigner function.

**Proof.** First of all, for \( z = x + iy \),

\[
\frac{d}{dz} \text{Li}_2(e^{2\pi iz}) = \frac{-\log(1 - e^{2\pi iz})}{2\pi i}
\]

\[
\Rightarrow \quad \text{Im } \frac{d}{dz} \text{Li}_2(e^{2\pi iz}) = \frac{-\text{Arg}(1 - e^{2\pi iz})}{2\pi i}
\]

By the Cauchy Riemann equations, we have

\[
\frac{\partial}{\partial y} \text{Re } \frac{\text{Li}_2(e^{2\pi iz})}{2\pi i} = \text{Arg}(1 - e^{2\pi iz})
\]

Therefore,

\[
\text{Re } \frac{\text{Li}_2(e^{2\pi iz})}{2\pi i} = \frac{\text{Im } \text{Li}_2(e^{2\pi iz})}{2\pi} = \frac{D(e^{2\pi iz}) - \log |e^{2\pi iz}| \text{Arg}(1 - e^{2\pi iz})}{2\pi} = \frac{D(e^{2\pi iz})}{2\pi} + y \frac{\partial}{\partial y} \text{Re } \frac{\text{Li}_2(e^{2\pi iz})}{2\pi i}
\]
Following the same arguments, it is easy to show that

\[
\text{Re} \frac{\text{Li}_2(e^{-2\pi i})}{2\pi i} = \frac{D(e^{-2\pi i})}{2\pi} + y \frac{\partial}{\partial y} \text{Re} \frac{\text{Li}_2(e^{-2\pi i})}{2\pi i}
\]

\[
\text{Re} \frac{\text{Li}_2(e^{2\pi i(s_1-1)-2\pi i z_1 - 2\pi i z_2})}{2\pi i} = \frac{D(e^{2\pi i(s_1-1)-2\pi i z_1 - 2\pi i z_2})}{2\pi} + \sum_{k=1}^{2} y_k \frac{\partial}{\partial y_k} \text{Re} \frac{\text{Li}_2(e^{2\pi i(s_1-1)-2\pi i z_1 - 2\pi i z_2})}{2\pi i}
\]

\[
\text{Re} \frac{\text{Li}_2(e^{2\pi i(s_2-1)-2\pi i z_2})}{2\pi i} = \frac{D(e^{2\pi i(s_2-1)-2\pi i z_2})}{2\pi} + y_2 \frac{\partial}{\partial y_2} \text{Re} \frac{\text{Li}_2(e^{2\pi i(s_2-1)-2\pi i z_2})}{2\pi i}
\]

Besides, for the linear terms, it is straightforward to verify that

\[
\text{Re} \left[ \pm (2\pi i (s_1 - 1))(2\pi i (z_1 - \frac{1}{2})) \right] = y_1 \frac{\partial}{\partial y_1} \text{Re} \left[ \pm (2\pi i (s_1 - 1))(2\pi i (z_1 - \frac{1}{2})) \right]
\]

\[
\text{Re} \left[ -(2\pi i (s_2 - 1))(2\pi i z_1 + 2\pi i z_2) \right] = \sum_{k=1}^{2} y_k \frac{\partial}{\partial y_k} \text{Re} \left[ -(2\pi i (s_2 - 1))(2\pi i z_1 + 2\pi i z_2) \right]
\]

As a result, by direct computation, we get the differential formula.

Together with the discussion in previous subsection, we have

\[
\text{Vol}(\mathbb{S}^3 \setminus WL, u_i = 2\pi i(1 \pm s_i)) = \text{Vol}(\mathbb{S}^3 \setminus WL, u_i = 2\pi i(1 - s_i))
\]

Thus we have

\[
\text{Re} \Phi^{\pm(s_1,s_2)}(z_1^{\pm(s_1,s_2)}, z_2^{\pm(s_1,s_2)}) = \frac{1}{2\pi} \text{Vol}(\mathbb{S}^3 \setminus WL, u_i = 2\pi i(1 - s_i))
\]

Next, from the definitions of the functions \(\Phi^{\pm(s_1,s_2)}(z_1^{\pm(s_1,s_2)}, z_2^{\pm(s_1,s_2)})\), if we regard \(s_1, s_2\) as complex variables, then we get two holomorphic functions in \(s_1, s_2\). In particular, for each fixed real \(s_2 \sim 1\), we have two holomorphic functions

\[
N^{\pm}(s_1) = \Phi^{\pm(s_1,s_2)}(z_1^{\pm(s_1,s_2)}, z_2^{\pm(s_1,s_2)})
\]

with the same real part

\[
\text{Re} N^{\pm}(s_1) = \frac{1}{2\pi} \text{Vol}(\mathbb{S}^3 \setminus WL, u_i = 2\pi i(1 - s_i))
\]

As a result, \(N^+(s_1) = N^-(s_1)\) for all real \(s_1 \sim 1\). Since this is true for all \(s_2 \sim 1\), we have

\[
\Phi^{+(s_1,s_2)}(z_1^{+(s_1,s_2)}, z_2^{+(s_1,s_2)}) = \Phi^{-(s_1,s_2)}(z_1^{+(s_1,s_2)}, z_2^{+(s_1,s_2)})
\]

for all \(s_1, s_2 \sim 1\).

3 The AEF for \(TV_r(WL, e^{2\pi i/r})\)

Let \(r = 2N + 1\). Recall from Theorem 1 that the \(r\)-th Turaev-Viro invariants for the link complement \(\mathbb{S}^3 \setminus L\) is related to the Colored Jones polynomials of the link as follows:

\[
TV_r(\mathbb{S}^3 \setminus L, e^{2\pi i/r}) = 2^{n-1} \left( \frac{2\sin(\frac{2\pi}{r})}{\sqrt{r}} \right)^2 \sum_{1 \leq M \leq \frac{n}{2}} \left| J^+_M(L, e^{\frac{2\pi i}{r} + \frac{2\pi i}{r}}) \right|^2
\]

From the formula of \(J^+_M(WL, e^{\frac{2\pi i}{r} + \frac{2\pi i}{r}})\), it is easy to see that the colored Jones polynomials of the belt tangle grow at most polynomially. Thus, in order to find an upper bound for the exponential growth rate of \(J^+_M(WL, e^{\frac{2\pi i}{r} + \frac{2\pi i}{r}})\), we only need to find an upper bound for the clasp tangle \(\tilde{C}(n, e^{\frac{2\pi i}{r} + \frac{2\pi i}{r}}; M_2)\).
Lemma 2. For any \( n \in \{1, 2, \ldots, M - 1\}, l \in \{1, 2, \ldots, M - 1 - n\}, \) let

\[
c_M(n, l; t) = \prod_{j=1}^{n} \left| \frac{(1 - t^{M-l-j})(1 - t^{l+j})}{1 - t^{l+j}} \right|
\]

For each \( M, \) let \( n_M \in \{1, 2, \ldots, M - 1\} \) and \( l_M \in \{1, 2, \ldots, M - 1 - n\} \) such that \( c_M(n_M, l_M) \) achieves the maximum among all \( c_M(n, l) \). Assume that \( \frac{n_M}{N + \frac{1}{2}} \to s \in [0, 1], \frac{n_M}{N + \frac{1}{2}} \to n_s \) and \( \frac{l_M}{N + \frac{1}{2}} \to l_s \). Then we have

\[
\lim_{N \to \infty} \frac{1}{N + \frac{1}{2}} \log(c_M(n, l; e^{\frac{2\pi i}{N + \frac{1}{2}}})) \leq \frac{\text{Vol}(\mathbb{S}^3 \setminus WL)}{2\pi}
\]

Furthermore, the equality holds if and only if \( s = 1, n_s = \frac{1}{2} \) and \( l_s = \frac{1}{4} \).

Proof. Let \( f(x, y, s) = \frac{1}{\pi} [L(\pi s - \pi x - \pi y) - L(\pi s - \pi y) + L(\pi y) - L(\pi x + \pi y) + L(\pi x)] \), where \( \Delta_s = \{(x, y, s) \in \mathbb{R}^3 | 0 \leq x + y \leq s, s \in [0, 1] \} \). Note that \( f(x, y, s_2) = \text{Re} \Phi^C_{s_1, s_2}(x, y) \). In order to find out the maximum of \( f \) inside the region \( \Delta_s \), we will first find out the critical point of \( f \) and then estimate the value of \( f \) along the boundary. Note that

\[
\begin{align*}
  f_x &= \log|2\sin(\pi s - \pi x - \pi y)| + \log|2\sin(\pi x + \pi y)| - \log|2\sin(\pi x)| \quad (41) \\
  f_y &= \log|2\sin(\pi s - \pi x - \pi y)| - \log|2\sin(\pi s - \pi y)| - \log|2\sin(\pi y)| + \log|2\sin(\pi x + \pi y)| \quad (42) \\
  f_s &= -\log|2\sin(\pi s - \pi x - \pi y)| + \log|2\sin(\pi s - \pi y)| \quad (43)
\end{align*}
\]

1. To find out the critical point of \( f \), note that

\[
\begin{align*}
  f_s &= 0 \implies \log|2\sin(\pi s - \pi x - \pi y)| = \log|2\sin(\pi s - \pi y)| \\
  &\implies x = 0 \quad \text{or} \quad x + 2y = 2s - 1 \quad (44)
\end{align*}
\]

Put (44) into \( f_y = 0 \), we have \( x = 0 \) or \( x + 2y = 1 \). As a result, \( f_y = f_s = 0 \) imply \( x = 0 \) or \( s = 1 \).

In both cases, the critical point lies on the boundary.

2. To estimate the value of \( f \) along the boundary, note that

(a) on the boundary where \( s = 1 \), it is easy to show that the point \( (x, y, s) = \left( \frac{1}{2}, \frac{1}{4}, 1 \right) \) is a unique maximum point with maximum value \( f \left( \frac{1}{2}, \frac{1}{4}, 1 \right) = \frac{1}{2\pi} \text{Vol}(\mathbb{S}^3 \setminus WL) \).

(b) on the boundary where \( x = 0 \), we have \( f(0, y, s) = 0 \).

(c) when \( y = 0 \) or \( x + y = s \), the function \( f \) is given by

\[
\begin{align*}
  f(x, 0, s) &= \frac{1}{\pi} [L(\pi s - \pi x) - L(\pi s)] \\
  f(x, s - x, s) &= \frac{1}{\pi} [L(\pi y) - L(\pi s)]
\end{align*}
\]

In both cases, we have

\[
|f| \leq \frac{1}{\pi} [2L(\frac{\pi}{3})] \leq \frac{1}{\pi} [6L(\frac{\pi}{3})] = \frac{1}{2\pi} \text{Vol}(\mathbb{S}^3 \setminus A_1) < \frac{1}{2\pi} \text{Vol}(\mathbb{S}^3 \setminus WL)
\]

where the last inequality follows from the fact that the figure eight knot complement can be obtained by doing surgery along the belt component of the Whitehead link. This completes the proof.

From Lemma 2 we have

\[
TV_r \left( \mathbb{S}^3 \setminus WL, e^{\frac{2\pi i}{N + \frac{1}{2}}} \right) \sim 2 \left( \frac{2\sin(\frac{2\pi}{r})}{\sqrt{r}} \right)^2 \sum_{1 \leq M_1, M_2 \leq N, s \in \{1, \ldots, 1\}} |J_{M_1, M_2}(WL, e^{\frac{2\pi i}{N + \frac{1}{2}}})|^2
\]

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By Lemma 9 in [21], we can replace the colored Jones polynomials with the AEF obtained in previous section. Let

$$H(s_1, s_2) = \text{Vol}(S^3 \setminus W, u_i = 2\pi i(1 - s_i))$$

and

$$G(s_1, s_2) = \frac{e^{\pi i(s_2 - 1)}}{\sqrt{-\det \text{Hess} \Phi^{+(s_1, s_2)}(z_1^{+(s_1, s_2)}, z_2^{+(s_1, s_2)})}} + \frac{e^{-\pi i(s_2 - 1)}}{\sqrt{-\det \text{Hess} \Phi^{-+(s_1, s_2)}(z_1^{-(s_1, s_2)}, z_2^{-(s_1, s_2)})}}$$

Note that $H(s_1, s_2)$ attains its maximum at $(s_1, s_2) = (1, 1)$. Using the method described in [21],

$$TV_r \left( S^3 \setminus W, e^{\frac{2\pi i}{r}} \right) 
\sim \frac{N}{r^3} \sum_{1 \leq M_1, M_2 \leq N, \eta \in (1 - \eta, 1)} \left| \tilde{J}_M \left( L, e^{\frac{2\pi i}{r}} \right) \right|^2$$

$$\sim \frac{32\pi^2}{r^3} \sum_{1 \leq M_1, M_2 \leq N, \eta \in (1 - \eta, 1)} \left| (N + \frac{1}{2})^{3/2} G(s_1, s_2) \exp((N + \frac{1}{2}) H(s_1, s_2)) \right|^2$$

$$\sim \frac{4\pi^2 N + \frac{1}{2}}{\sqrt{\det \text{Hess} H(1, 1)}} G(1, 1)^2 \exp \left( \frac{2N + 1}{2\pi} F(1, 1) \right)$$

$$\sim \frac{(N + \frac{1}{2})\pi}{\sqrt{\det \text{Hess} H(1, 1)}} \exp \left( \frac{r}{2\pi} \times \text{Vol}(WL) \right)$$

4 Generalization to $J_{M_1, M_2}(W_{a,1,c,d}, e^{\frac{2\pi i}{r}})$

4.1 Potential functions for $J_{M_1, M_2}(W_{a,1,c,d}, e^{\frac{2\pi i}{r}})$

In this section, we compute the potential functions of the $\tilde{J}_{M_1, M_2}(W_{a,1,c,d}, e^{\frac{2\pi i}{r}})$ with the belt colored by $M_1$ and all the other components colored by $M_2$. First of all, recall from [20] that the unnormalized $\tilde{J}_{M_1, M_2}(W_{a,1,c,d})$ is given by

$$J_{M_1, M_2}(W_{a,1,c,d}) = \sum_{n=0}^{M_2-1} t^{\frac{M_2-1}{2}} t^{-\frac{n}{2}} \prod_{l=0}^{M_2-1-n} \frac{1 - t^{M_2-l-j} - (1 - t^{l+j})}{1 - t}.$$

Note that the contribution of the twists is given by

$$\tilde{C}(n, t; M_2) = \sum_{l=0}^{M_2-1-n} t^{M_2(l+n)} (1 - t^{M_2-l-j}) (1 - t^{l+j}).$$

$$\tilde{C}(n, t; M_2) = t^{M_2(M_2-1)} \sum_{l=0}^{M_2-1-n} t^{M_2(l+n)} \prod_{j=1}^{n} \frac{1 - t^{M_2-l-j} - (1 - t^{l+j})}{1 - t}.$$

$$\tilde{C}(n, t; M_2) = t^{M_2(M_2-1)} \sum_{l=0}^{M_2-1-n} t^{M_2(l+n)} \prod_{j=1}^{n} \frac{1 - t^{M_2-l-j} - (1 - t^{l+j})}{1 - t}.$$

$$\tilde{C}(n, t; M_2) = t^{M_2(M_2-1)} \sum_{l=0}^{M_2-1-n} t^{M_2(l+n)} \prod_{j=1}^{n} \frac{1 - t^{M_2-l-j} - (1 - t^{l+j})}{1 - t}.$$

$$\tilde{C}(n, t; M_2) = t^{M_2(M_2-1)} \sum_{l=0}^{M_2-1-n} t^{M_2(l+n)} \prod_{j=1}^{n} \frac{1 - t^{M_2-l-j} - (1 - t^{l+j})}{1 - t}.$$
Besides, for the colored Jones polynomials of the mirror clasp, note that

\[
\prod_{j=1}^{n} (1 - t^{-(M_j - l - j)}) = \prod_{j=1}^{n} (1 - t^{-M_j + (N + \frac{1}{2}) + l + j}) \\
= \prod_{j=1}^{n} (1 - e^{2\pi i \left( \frac{M_j}{N + \frac{1}{2}} - \frac{l}{N + \frac{1}{2}} + \frac{j}{N + \frac{1}{2}} \right)}) \\
= \exp \left( \sum_{j=1}^{n} \left[ \varphi^h \left( -\pi \left( \frac{M_j}{N + \frac{1}{2}} - 1 \right) + \frac{l\pi}{N + \frac{1}{2}} + \frac{(j - \frac{1}{2})\pi}{N + \frac{1}{2}} \right) \right] \\
- \varphi^h \left( -\pi \left( \frac{M_j}{N + \frac{1}{2}} - 1 \right) + \frac{l\pi}{N + \frac{1}{2}} + \frac{(j + \frac{1}{2})\pi}{N + \frac{1}{2}} \right) \right) \\
= \exp \left[ \varphi^h \left( -\pi \left( \frac{M_j}{N + \frac{1}{2}} - 1 \right) + \frac{l\pi}{N + \frac{1}{2}} + \frac{n\pi}{2N + 1} \right) \right] - \exp \left[ \varphi^h \left( -\pi \left( \frac{M_j}{N + \frac{1}{2}} - 1 \right) + \frac{l\pi}{N + \frac{1}{2}} + \frac{n\pi}{2N + 1} \right) \right] \\
= \exp \left[ \varphi^h \left( -\pi \left( \frac{M_j}{N + \frac{1}{2}} - 1 \right) + \frac{l\pi}{N + \frac{1}{2}} + \frac{n\pi}{2N + 1} \right) \right] \tag{47}
\]

\[
\prod_{j=1}^{n} (1 - t^{-(l+j)}) = \prod_{j=1}^{n} (1 - e^{2\pi i \left( \frac{1}{N + \frac{1}{2}} - \frac{j}{N + \frac{1}{2}} \right)}) \\
= \exp \left( \sum_{j=1}^{n} \left[ \varphi^h \left( \pi - \frac{l\pi}{N + \frac{1}{2}} - \frac{(j + \frac{1}{2})\pi}{N + \frac{1}{2}} \right) - \varphi^h \left( \pi - \frac{l\pi}{N + \frac{1}{2}} - \frac{(j - \frac{1}{2})\pi}{N + \frac{1}{2}} \right) \right] \\
= \exp \left[ \varphi^h \left( \pi - \frac{l\pi}{N + \frac{1}{2}} - \frac{n\pi}{2N + 1} \right) - \varphi^h \left( \pi - \frac{l\pi}{N + \frac{1}{2}} - \frac{n\pi}{2N + 1} \right) \right] \tag{48}
\]

\[
\prod_{j=1}^{n} (1 - t^{-l-j}) = \prod_{j=1}^{n} (1 - e^{2\pi i \left( \frac{1}{N + \frac{1}{2}} + \frac{j}{N + \frac{1}{2}} \right)}) \\
= \exp \left( \sum_{j=1}^{n} \left[ \varphi^h \left( \pi - \frac{(j + \frac{1}{2})\pi}{N + \frac{1}{2}} \right) - \varphi^h \left( \pi - \frac{(j - \frac{1}{2})\pi}{N + \frac{1}{2}} \right) \right] \\
= \exp \left[ \varphi^h \left( \pi - \frac{n\pi}{2N + 1} + \frac{\pi}{2N + 1} \right) - \varphi^h \left( \pi - \frac{n\pi}{2N + 1} + \frac{\pi}{2N + 1} \right) \right] \tag{49}
\]
Altogether, by (5) - (10) and (46) - (49), we have

\[
J_{M_1, M_2}(W_{a, c}, d; e^{\frac{2\pi i}{N^2 + \frac{1}{2}}})
= (-1)^{(M_1 - N + a)}e^{\frac{2\pi i}{N^2 + \frac{1}{2}} + \frac{M_2}{2}}e^{\frac{2\pi i}{N^2 + \frac{1}{2}} + \frac{M_2}{2} - \frac{e^{-d}M_2(M_2 - 1)}{4}}e^{-\frac{3}{2}(2\pi i)(N + \frac{1}{2})}
\]

\[
\times \sum_{n=0}^{M_2-1} e^{-\frac{\pi i n}{N^2 + \frac{1}{2}}} \left[ 2\pi i \left( \frac{M_1}{N + \frac{1}{2}} - 1 \right) \right] \left[ 2\pi i \left( \frac{n}{N + \frac{1}{2}} - \frac{1}{2} \right) \right] + e^{-\frac{\pi i n}{N^2 + \frac{1}{2}}} \left[ 2\pi i \left( \frac{M_1}{N + \frac{1}{2}} - 1 \right) \right] \left[ 2\pi i \left( \frac{n}{N + \frac{1}{2}} - \frac{1}{2} \right) \right]
\]

\[
\times e^{-\frac{\pi i n}{N^2 + \frac{1}{2}}} \left[ 2\pi i \left( \frac{n}{N + \frac{1}{2}} - \frac{1}{2} + \frac{1}{N + \frac{1}{2}} \right) \right]
\]

\[
\times \prod_{k=1}^{d} \left\{ \sum_{l_k=0}^{M_2 - 1} e^{-\frac{\pi i l_k}{N^2 + \frac{1}{2}}} \left[ 2\pi i \left( \frac{M_2}{N + \frac{1}{2}} - 1 \right) \right] \left[ 2\pi i \left( \frac{l_k}{N + \frac{1}{2}} + \frac{n}{N + \frac{1}{2}} + \frac{\pi}{N + \frac{1}{2}} \right) \right] \right\}
\]

\[
\times \left\{ \exp \left[ -\phi^h \left( \frac{\pi}{2N + 1} \right) + \phi^h \left( \frac{n}{N + \frac{1}{2}} + \frac{\pi}{N + \frac{1}{2}} \right) \right] \right\}
\]

(50)

For each \( i = 1, 2, \ldots, c \), define the function \( \psi_{i, M_1, M_2}^{(s_1, s_2)}(z_1, z_{i+1}) \) by

\[
\psi_{i, M_1, M_2}^{(s_1, s_2)}(z_1, z_{i+1}) = \frac{1}{2\pi i} \left\{ -\left( 2\pi i \left( \frac{M_2}{N + \frac{1}{2}} - 1 \right) \right) \left( 2\pi i z_1 + 2\pi i z_{i+1} \right)
\]

\[
+ \frac{2\pi i}{N + \frac{1}{2}} \left[ \phi^h \left( \frac{M_2}{N + \frac{1}{2}} - \pi z_1 - \pi z_{i+1} - \frac{\pi}{2N + 1} \right) + \phi^h \left( \pi z_{i+1} + \frac{\pi}{2N + 1} \right) - \phi^h \left( \pi z_1 + \frac{\pi}{2N + 1} \right) \right]
\]

(51)
For each $i = 1, 2, \ldots, d$, define the function $\kappa_{i, M_1, M_2}^{(s_1, s_2)}(z_1, z_{c+i+1})$ by

\[
\kappa_{i, M_1, M_2}^{(s_1, s_2)}(z_1, z_{c+i+1}) = \frac{1}{2\pi i} \left\{ \left( 2\pi i \left( \frac{M_2}{N + \frac{1}{2}} - 1 \right) \right) (2\pi i z_1 + 2\pi i z_{c+i+1}) \\
- \frac{2\pi i}{N + \frac{1}{2}} \left[ \varphi^h \left( -\pi \left( \frac{M_2}{N + \frac{1}{2}} - 1 \right) + \pi z_1 + \pi z_{c+i+1} + \frac{\pi}{2N + 1} \right) \right] \\
- \varphi^h \left( -\pi \left( \frac{M_2}{N + \frac{1}{2}} - 1 \right) + \pi z_{c+i+1} + \frac{\pi}{2N + 1} \right) \\
- \varphi^h \left( \pi - \pi z_{c+i+1} - \frac{\pi}{2N + 1} \right) + \varphi^h \left( \pi - \pi z_1 - \pi z_{c+i+1} - \frac{\pi}{2N + 1} \right)
\right\}
\]

Then we have

\[
J_{M_1, M_2}(W_{a,1,c,d}, e^{\frac{2N}{N + \frac{1}{2}}}) = \frac{(-1)^{(M_1 - N + a)} e^{\frac{2\pi}{N + \frac{1}{2}} (M_2^2 - \frac{M_2}{2} - \frac{1}{2})} e^{\frac{2\pi}{N + \frac{1}{2}} (N + \frac{1}{2}) e^{\frac{2\pi}{N + \frac{1}{2}} cM_2(M_2 - 1)}}}{2 \sin\left(\frac{\pi}{N + \frac{1}{2}}\right)} \\
\times \exp\left( -\varphi^h \left( \frac{\pi}{2(N + \frac{1}{2})} \right) \right)^c \exp\left( \varphi^h \left( \frac{\pi}{2(N + \frac{1}{2})} \right) \right)^d \times (I_+ + I_-)
\]

where

\[
I_\pm = \sum_{n=0}^{M_2-1} \sum_{l_1=0}^{M_2-1-n} \cdots \sum_{l_c=0}^{M_2-1-n} \sum_{l_i=0}^{M_2-1-n} \cdots \sum_{l_d=0}^{M_2-1-n} \\
\exp\left( (N + \frac{1}{2}) \Phi_{M_1, M_2}^{(s_1, s_2); a, c, d} \left( \frac{n}{N + \frac{1}{2}}, \frac{l_1}{N + \frac{1}{2}}, \cdots, \frac{l_c}{N + \frac{1}{2}}, \frac{l_i}{N + \frac{1}{2}}, \cdots, \frac{l_d}{N + \frac{1}{2}} \right) \right)
\]

with

\[
\Phi_{M_1, M_2}^{(s_1, s_2); a, c, d}(z_1, z_2, \ldots, z_{c+1}, z_{c+2}, \ldots, z_{c+d+1}) = \frac{1}{2\pi i} \left\{ \pm 2\pi i \left( \frac{M_1}{N + \frac{1}{2}} - 1 \right) \left( 2\pi i \left( z_1 - \frac{1}{2} \right) \right) + a \left[ 2\pi i \left( z_1 - \frac{1}{2} \right) \right] \left[ 2\pi i \left( z_1 - \frac{1}{2} + \frac{1}{N + \frac{1}{2}} \right) \right] \right\} \\
+ \sum_{i=1}^c \psi_{M_1, M_2, i}^{(s_1, s_2)}(z_1, z_{i+1}) + \sum_{i=1}^d \kappa_{M_1, M_2, i}^{(s_1, s_2)}(z_1, z_{c+i+1})
\]

Take $N \to \infty$, we have

\[
\psi_{i, M_1, M_2, i}^{(s_1, s_2)}(z_1, z_{i+1}) = \frac{1}{2\pi i} \left[ -2\pi i (s_2 - 2)(2\pi i z_1 + 2\pi i z_{i+1}) \right] \\
+ \text{Li}_2 \left( e^{2\pi i (s_1-1) - 2\pi i z_1 - 2\pi i z_{i+1}} \right) - \text{Li}_2 \left( e^{2\pi i (s_1-1) - 2\pi i z_{i+1}} \right) \\
+ \text{Li}_2 \left( e^{2\pi i z_{i+1}} \right) - \text{Li}_2 \left( e^{2\pi i z_1 + 2\pi i z_{i+1}} \right) + \text{Li}_2 \left( e^{2\pi i z_1} \right),
\]

\[
\kappa_{i, M_1, M_2, i}^{(s_1, s_2)}(z_1, z_{c+i+1}) = \frac{1}{2\pi i} \left( 2\pi i (s_2 - 2)(2\pi i z_1 + 2\pi i z_{c+i+1}) \right) \\
- \text{Li}_2 \left( e^{-2\pi i (s_2-1) + 2\pi i z_1 + 2\pi i z_{c+i+1}} \right) + \text{Li}_2 \left( e^{-2\pi i (s_2-1) + 2\pi i z_{c+i+1}} \right) \\
- \text{Li}_2 \left( e^{-2\pi i z_{c+i+1}} \right) + \text{Li}_2 \left( e^{-2\pi i z_1 - 2\pi i z_{c+i+1}} \right) - \text{Li}_2 \left( e^{-2\pi i z_1} \right)
\]

26
and

\[
\Phi_{\pm(s_1,s_2)}(z_1, z_2, z_3, \ldots, z_c, z_{c+1}; z_{c+d+1}) = \frac{1}{2\pi i} \left\{ \pm 2\pi i (s_2 - 1) \left( 2\pi i \left( z_1 - \frac{1}{2} \right) \right) + a \left[ 2\pi i \left( z_1 - \frac{1}{2} \right) \right]^2 \right\} \\
+ \sum_{i=1}^{c} \psi_{i}^{(s_1,s_2)}(z, z_{i+1}) + \sum_{i=1}^{d} \kappa_{i}^{(s_1,s_2)}(z_1, z_{c+i+1})
\]

Note that

\[
\frac{\partial}{\partial z_1} \psi_i(z_1, z_{i+1}) = \log(1 - e^{2\pi i(s_2-1)}e^{-2\pi iz_1-2\pi iz_{i+1}}) + \log(1 - e^{2\pi iz_1+2\pi iz_{i+1}}) - \log(1 - e^{2\pi iz_{i+1}})
- 2\pi i(s_2 - 1)
\]

(55)

\[
\frac{\partial}{\partial z_{i+1}} \psi_i(z_1, z_{i+1}) = \log(1 - e^{2\pi i(s_2-1)}e^{-2\pi iz_1-2\pi iz_{i+1}}) + \log(1 - e^{2\pi iz_1+2\pi iz_{i+1}}) - \log(1 - e^{2\pi iz_{i+1}}) - 2\pi i(s_2 - 1)
\]

(56)

\[
\frac{\partial}{\partial z_1} \kappa_i(z_1, z_{c+i+1}) = \log(1 - e^{-2\pi i(s_2-1)}e^{2\pi iz_1+2\pi iz_{c+i+1}}) + \log(1 - e^{-2\pi iz_1-2\pi iz_{c+i+1}})
- \log(1 - e^{-2\pi iz_{c+i+1}}) + 2\pi i(s_2 - 1)
\]

(57)

\[
\frac{\partial}{\partial z_{c+i+1}} \kappa_i(z_1, z_{c+i+1}) = \log(1 - e^{-2\pi i(s_2-1)}e^{2\pi iz_1+2\pi iz_{c+i+1}}) + \log(1 - e^{-2\pi iz_1-2\pi iz_{c+i+1}})
- \log(1 - e^{-2\pi iz_{c+i+1}}) - \log(1 - e^{-2\pi iz_{c+i+1}}) + 2\pi i(s_2 - 1)
\]

(58)

As a result, the critical point equations for the potential function \(\Phi_{\pm(s_1,s_2)}(z_1, z_2, z_3, \ldots, z_c)\) are given by

\[
\sum_{i=1}^{c} \log(1 - e^{2\pi i(s_2-1)}e^{-2\pi iz_1-2\pi iz_{i+1}}) + \log(1 - e^{2\pi iz_1+2\pi iz_{i+1}}) - \log(1 - e^{2\pi iz_1})
\]

\[
+ \sum_{i=1}^{d} \log(1 - e^{-2\pi i(s_2-1)}e^{2\pi iz_1+2\pi iz_{c+i+1}}) + \log(1 - e^{-2\pi iz_1-2\pi iz_{c+i+1}}) - \log(1 - e^{-2\pi iz_1})
\]

\[
= 2\pi i \left[ \mp (s_1 - 1) + (c - d)(s_2 - 1) - 2a \left( z_1 - \frac{1}{2} \right) \right]
\]

(59)

\[
\log(1 - e^{2\pi i(s_2-1)}e^{-2\pi iz_1-2\pi iz_{i+1}}) + \log(1 - e^{2\pi iz_1+2\pi iz_{i+1}}) - \log(1 - e^{2\pi iz_1})
\]

\[
- \log(1 - e^{-2\pi i(s_2-1)}e^{2\pi iz_1+2\pi iz_{i+1}}) + \log(1 - e^{-2\pi iz_1-2\pi iz_{i+1}})
\]

\[
= 2\pi i(s_2 - 1) \quad \text{for } i = 1, 2, \ldots, c
\]

(60)

\[
\log(1 - e^{-2\pi i(s_2-1)}e^{2\pi iz_1+2\pi iz_{c+i+1}}) + \log(1 - e^{-2\pi iz_1-2\pi iz_{c+i+1}})
- \log(1 - e^{-2\pi i(s_2-1)}e^{2\pi iz_1+2\pi iz_{c+i+1}}) - \log(1 - e^{-2\pi iz_1-2\pi iz_{c+i+1}})
\]

\[
= -2\pi i(s_2 - 1) \quad \text{for } i = 1, 2, \ldots, d
\]

(61)

Put \(Z_i = e^{2\pi iz_1}\), \(U_i = \frac{1}{Z_i^*Z_i+1}\) and \(B_l = e^{2\pi i\lambda_l}\) for \(i = 1, 2, \ldots, c + d + 1\) and \(l = 1, 2\). After taking exponential, we have
\[
\left\{ \prod_{i=1}^{c} \frac{(1 - B_2 U_i) \left( 1 - \frac{1}{U_i} \right)}{B_2 (1 - Z_i)} \right\} \left\{ \prod_{i=1}^{d} \frac{(1 - B_2^{-1} U_i^{-1}) \left( 1 - \frac{1}{U_i^{-1}} \right)}{B_2^{-1} (1 - Z_i^{-1})} \right\} (Z_i^2)^{\alpha} = ((\hat{z}_1)^2)^{-1} \]

(62)

\[
\left(1 - \hat{\sigma}_2 \hat{u}_i \right) \left( 1 - \frac{1}{\hat{u}_i} \right) = \hat{\sigma}_2 \quad \text{for } i = 1, 2, \ldots, c \quad (63)
\]

\[
\left(1 - \hat{\sigma}_2 \hat{u}_i \right) \left( 1 - \hat{z}_{i+1} \right) = \hat{\sigma}_2 \quad \text{for } i = 1, 2, \ldots, d \quad (64)
\]

4.2 AEF for \( J_{N,N}(W_{a,1,c,d}, e^{N+\frac{i}{2}}) \)

When \( M_1 = M_2 = N \), the potential function becomes

\[
\phi_{(1,1);a,c,d}(z_1, z_2, z_3, \ldots, z_{c+d+1}) = \frac{1}{2\pi i} \left[ a \left( \frac{1}{\frac{1}{2}} \right) \right]^2 + \sum_{i=1}^{c} \psi_i^{(1,1)}(z_1, z_{i+1}) + \sum_{i=1}^{d} \kappa_i^{(1,1)}(z_1, z_{c+i+1})
\]

Besides, its critical point equations become

\[
\sum_{i=1}^{c} \left[ \log(1 - e^{2\pi i z_1 - 2\pi i z_{i+1}}) + \log(1 - e^{2\pi i z_1 + 2\pi i z_{i+1}}) - \log(1 - e^{2\pi i z_1}) \right] - \sum_{i=1}^{d} \left[ \log(1 - e^{2\pi i z_1 + 2\pi i z_{c+i+1}}) + \log(1 - e^{-2\pi i z_1 - 2\pi i z_{c+i+1}}) - \log(1 - e^{-2\pi i z_1}) \right] = -2a(z_1 - \frac{1}{2})
\]

(65)

\[
\log(1 - e^{-2\pi i z_1 - 2\pi i z_{i+1}}) + \log(1 - e^{2\pi i z_1 + 2\pi i z_{i+1}}) - \log(1 - e^{-2\pi i z_1}) = 0 \quad \text{for } i = 1, 2, \ldots, c \quad (66)
\]

\[
\log(1 - e^{2\pi i z_1 + 2\pi i z_{c+i+1}}) + \log(1 - e^{-2\pi i z_1 - 2\pi i z_{c+i+1}}) - \log(1 - e^{2\pi i z_1}) = 0 \quad \text{for } i = 1, 2, \ldots, d \quad (67)
\]

It is straightforward to verify that \((z_1, z_2, \ldots, z_c, z_{c+1}, \ldots, z_{c+d+1}) = (\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})\) is a solution of equations (65) - (67). Besides, it is the unique maximum point on the region

\[
D = \{(z_1, \ldots, z_{c+d+1}) \in [0, 1]^{c+d+1} \mid z_1 + z_i \leq 1 \text{ for } i = 2, \ldots, c + d + 1\}
\]

Moreover, since

\[
\psi_i^{(1,1)} \left( \frac{1}{2}, \frac{1}{2} \right) = \frac{1}{2\pi i} \left[ 2 \operatorname{Li}_2(i) - 2 \operatorname{Li}_2(-i) + \operatorname{Li}_2(-1) \right] = \frac{1}{2\pi} \left[ 8L \left( \frac{\pi}{4} \right) + \frac{\pi^2 i}{12} \right]
\]

(68)

\[
\kappa_i^{(1,1)} \left( \frac{1}{2}, \frac{1}{2} \right) = \frac{1}{2\pi i} \left[ 2 \operatorname{Li}_2(i) - 2 \operatorname{Li}_2(-i) - \operatorname{Li}_2(-1) \right] = \frac{1}{2\pi} \left[ 8L \left( \frac{\pi}{4} \right) - \frac{\pi^2 i}{12} \right]
\]

(69)
the critical value is given by

$$\Phi^{(1,1,a,c,d)} \left( \frac{1}{2}, \frac{1}{4}, \cdots, \frac{1}{4} \right) = \frac{1}{2\pi i} \sum_{i=1}^{\epsilon} \psi_i \left( \frac{1}{2}, \frac{1}{4} \right) + \frac{1}{2\pi i} \sum_{i=1}^{d} \kappa_i \left( \frac{1}{2}, \frac{1}{4} \right) = \frac{1}{2\pi} \left[ 8(c + d)L \left( \frac{\pi}{4} \right) + \frac{(c - d)\pi^2 i}{12} \right]$$

Besides, by Lemma A.3 of [16], we have

$$\exp \left( -\phi^h \left( \frac{\pi}{2N + 1} \right) \right) \sim e^{-\frac{\pi^2 i}{2N + 1}} \frac{N + \frac{\pi^2 i}{2}}{6} \exp \left( \frac{N + \frac{1}{2} \pi^2 i}{2} \right) \tag{70}$$

$$\exp \left( \phi^h \left( \pi - \frac{\pi}{2N + 1} \right) \right) \sim e^{-\frac{\pi^2 i}{2N + 1}} \frac{N + \frac{1}{2} \pi^2 i}{6} \exp \left( \frac{N + \frac{1}{2} \pi^2 i}{2} \right) \tag{71}$$

Nevertheless, since the Hessian of $\Phi^{(1,1),c}(\frac{1}{2}, \frac{1}{4}, \cdots, \frac{1}{4})$ is given by

$$\text{Hess} \left( \Phi^{(1,1),a,c,d} \left( \frac{1}{2}, \frac{1}{4}, \cdots, \frac{1}{4} \right) \right) = 2\pi i \begin{pmatrix}
(c + d)i - \frac{e^{-d} + 2a}{i} & 2i & 0 & 0 & \cdots & 0 \\
2i & 0 & 0 & 0 & \cdots & 0 \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
\end{pmatrix},$$

we have

$$\det(-\text{Hess}(\Phi^{(1,1),a,c,d}(\frac{1}{2}, \frac{1}{4}, \cdots, \frac{1}{4}))) = (-1)^{c + d + 1}(2\pi i)^{c + d}(c + d)(-4\pi^2(1 + i))$$

$$= -(c + d)(2\pi i)^{c + d}(4\pi^2(1 + i))$$

$$\neq 0$$

By similar argument as in Section 2.2, we have

$$J_{N,N}(W_{a,1,c,d}, e^{\frac{2\pi i}{N + \frac{1}{2}}})$$

$$= \frac{(-1)^{a + c}e^{-\frac{\pi^2 i}{2}(N + \frac{1}{2})}}{2\sin(\frac{\pi}{N + \frac{1}{2}})} \left[ \left( e^{\frac{\pi}{N + \frac{1}{2}}} \right) ^{M_2 - 1} \left( e^{\frac{\pi}{N + \frac{1}{2}}} \right) ^{M_2 - 1} \cdots \left( e^{\frac{\pi}{N + \frac{1}{2}}} \right) ^{M_2 - 1} \right]$$

$$\exp \left( 2\pi i \Phi_{N,N} \left( \frac{n}{N + \frac{1}{2}}, \frac{l_1}{N + \frac{1}{2}}, \cdots, \frac{l_d}{N + \frac{1}{2}} \right) \right)$$

$$\sim \left( -1 \right)^{a + c - d} \frac{1}{2\pi} \exp \left( (c + d)^2 + (c + d) \frac{\pi^2 i}{6} \right) \exp \left( \frac{N + \frac{1}{2}(c + d)\pi^2 i}{6} \right)$$

$$\times \int_D \exp \left( \left( c + d \right) L \left( \frac{\pi}{4} \right) + \left( -4a + c - d \right) \frac{\pi^2 i}{4} \right) \left( 1 + O \left( \frac{1}{N} \right) \right)$$

$$\times \exp \left( \frac{N + \frac{1}{2}}{2\pi} \left( c + d \right) L \left( \frac{\pi}{4} \right) \right)$$

$$\sim \left( -1 \right)^{a + c} \frac{1}{2\pi} \exp \left( -\left( c + d \right) \frac{\pi^2 i}{4} \right) \left( \frac{1}{N} \right)$$

$$\times \exp \left( \frac{N + \frac{1}{2}}{2\pi} \left( (c + d)L \left( \frac{\pi}{4} \right) + \left( -4a + c - d \right) \frac{\pi^2 i}{4} \right) + O \left( \frac{1}{N} \right) \right)$$

This completes the proof of Theorem 5
4.3 Volume conjecture for the Turaev-Viro invariants for $S^3 \setminus W_{a,b,c,d}$

Next, we are going to prove Corollary 1. Note that $S^3 \setminus W_{a,b,c,d}$ is homeomorphic to $S^3 \setminus W_{0,b,c,d}$. Thus we only need to prove the volume conjecture for $TV_c(S^3 \setminus W_{0,b,c,d})$. By Theorem 1, it suffices to show that

1. $J_{N,...,N}(W_{a,1,c,d}, e^{\frac{2\pi i}{N+\frac{1}{2}}})$ grows exponentially with

$$
\lim_{N \to \infty} \frac{2\pi \log |J_{N,...,N}(W_{a,1,c,d}, e^{\frac{2\pi i}{N+\frac{1}{2}}})|^2}{2N+1} = \text{Vol}(S^3 \setminus W_{a,1,c,d}) = v_3 |S^3 \setminus W_{a,b,c,d}|
$$

2. The growth rates of $J_M(W_{a,b,c,d})$ are less than or equal to $\text{Vol}(S^3 \setminus W_{a,b,c,d}, e^{\frac{2\pi i}{N+\frac{1}{2}}})$ for all colors $M$.

The first condition is guaranteed by Theorem 5. Besides, by Fusion rule, we can fuse the belts together and express the colored Jones polynomials of $W_{a,1,c,d}$ as a sum of colored Jones polynomials of $W_{0,1,c,d}$ with certain colors. Then by Lemma 2 and triangle inequality, we prove condition 2. This completes the proof of Corollary 1.

4.4 AEF for $J_{M_1,M_2}(W_{a,1,c,0}, e^{\frac{2\pi i}{N+\frac{1}{2}}})$

Finally, we colored the belt component by $M_1$ and all the clasps components by $M_2$ respectively and study the AEF of $J_{M_1,M_2}(W_{a,1,c,0}, e^{\frac{2\pi i}{N+\frac{1}{2}}})$ and $J_{M_1,M_2}(W_{a,1,0,d}, e^{\frac{2\pi i}{N+\frac{1}{2}}})$. By the same computation, we have

$$
J_{M_1,M_2}(W_{a,1,c,0}, e^{\frac{2\pi i}{N+\frac{1}{2}}}) = (-1)^{(M_1-N-a)} e^{\frac{2\pi i}{N+\frac{1}{2}} (M_2^2 - M_2 - \frac{1}{2})} e^{-\frac{\pi}{2} (2\pi i)(N+\frac{1}{2})} e^{\frac{2\pi i}{N+\frac{1}{2}} \frac{M_2(M_2-1)}{2}} 
$$

$$
\exp \left( \frac{\pi}{2(N+\frac{1}{2})} \right) \left[ \sum_{n=0}^{\pi \left( \frac{M_1}{N+\frac{1}{2}} \right)} \sum_{l_1=0}^{n} \cdots \sum_{l_c=0}^{M_2-1-n} \exp \left( (N+\frac{1}{2}) \Phi_{M_1,M_2}^{(s_1,s_2)},a,1,0 \left( \begin{array}{c} n \\frac{l_1}{N+\frac{1}{2}} \\frac{l_2}{N+\frac{1}{2}} \cdots \frac{l_c}{N+\frac{1}{2}} \end{array} \right) \right) \right] 
$$

By the same argument as Section 2.3, there exists $\delta > 0$ such that for $1 - \delta < s_1, s_2 \leq 1$, we have
\[ J_{M_1, M_2}(W_{a, 1, c, 0}, e^{\frac{2\pi i}{N+\frac{1}{2}}}) \]

\[ \sim \frac{(-1)^{(M_1 - N + a)} e^{\frac{2\pi i}{N+\frac{1}{2}}}}{2 \sin \left( \frac{\pi}{N+\frac{1}{2}} \right)} \exp \left( \frac{\pi i}{N+\frac{1}{2}} \right) e^{-\frac{\pi i}{2} (2\pi i)(N+\frac{1}{2})} e^{\frac{2\pi i}{N+\frac{1}{2}} c M_2 (M_2 - 1)} \]

\[ (N + \frac{1}{2})^{3/2} \]

4.5 Triangulation of the \( S^3 \setminus W_{0, 1, c, 0} \)

The triangulation of \( S^3 \setminus W_{0, 1, c, 0} \) is similar to that for the Whitehead link complement. First of all, we prepare \( c \) ideal octahedra. Next, we glue the top of the cylinder to the bottom of the next cylinder by the rule \( C_2 \rightarrow C_1 \) and \( A_2 \rightarrow A_1 \) (figure 16 and 17), then we obtain a decomposition of the boundary torus into parallelograms. Note that this torus is exactly the boundary torus of a tubular neighborhood of the belt component of \( W_{0, 1, c, 0} \).

Figure 13: decomposition of the boundary torus into parallelograms
Originally, each tetrahedron has its own assignment of shape parameters \(z_i, w_i\) and \(u_i\). If we put \(z_1 = z_2 = \cdots = z_{c+1} = z\), \(w_1 = w_1 = \cdots = w_{c+1} = w\) and \(u_1 = u_2 = \cdots = u_{c+1} = u\), by similar calculation in Section 2.4 one can verify that the edges equations can be reduced to a single equation

\[
\begin{pmatrix}
U'
\end{pmatrix}
\begin{pmatrix}
(W')^{-1}
\end{pmatrix}
= \begin{pmatrix}
(B_2U)^{\eta}
\end{pmatrix}
\begin{pmatrix}
(B_2^{-1}W)^{\eta}
\end{pmatrix}
\]

On the other hand, the critical point equations for \(J_{M, \ldots, M}(W_{0,1,c,0})\) are given by

\[
\left\{
\sum_{i=1}^{c} \left[ \log(1 - e^{2\pi i (s_2-1)} e^{-2\pi i z_2 - 2\pi i z_{i+1}}) + \log(1 - e^{2\pi i z_2 + 2\pi i z_{i+1}}) - \log(1 - e^{2\pi i z_2}) \right]
\right.
\]

\[
= 2\pi i \left[ - (s_1 - 1) + c (s_2 - 1) \right]
\]

\[
\log(1 - e^{2\pi i (s_2-1)} e^{-2\pi i z_2 - 2\pi i z_{i+1}}) + \log(1 - e^{2\pi i z_2 + 2\pi i z_{i+1}})
\]

\[
- \log(1 - e^{2\pi i (s_2-1)} e^{-2\pi i z_2}) - \log(1 - e^{2\pi i z_2})
\]

\[
= 2\pi i (s_2 - 1) \text{ for } i = 1,2, \ldots, c
\]

By similar argument in Section 2.4 one can show that the holonomy of the meridian around the belt component and every clasps components are exactly \(B_1\) and \(B_2\) respectively. Therefore, by Proposition 1 we have

\[
\text{Re} \Phi^{\pm(s_1,s_2)}; 0,1,c,0 (z_1, z_2, \ldots, z_{c+1})
\]

\[
= \frac{1}{2\pi} \text{Vol} (S^3 \setminus W_{0,1,c,0}, u_1 = \pm 2\pi i (1 - s_1), u_2 = \cdots = u_{c+1} = \pm 2\pi i (1 - s_2))
\]

This completes the proof of Theorem 6

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