Spontaneous dynamics of two-dimensional Leidenfrost wheels

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It has been recently discovered that liquid Leidenfrost drops levitated by their vapor above a flat hot surface undergo symmetry breaking leading to spontaneous motion (A. Bouillant et al., Nature Physics, 14 1188–1192, 2018). Motivated by these experiments, we theoretically investigate the spontaneous dynamics of Leidenfrost drops on the basis of a simplified two-dimensional model, focusing on near-circular drops small in comparison to the capillary length. The model couples the equations of motion of the drop, which flows as a rigid wheel, and a thin-film model governing the vapor flow, the profile of the deformable vapor-liquid interface and thus the hydrodynamic forces and torques on the drop. We find that the symmetric Leidenfrost state is unstable above a critical drop radius: $R_1$ for a free drop and $R_2 > R_1$ for an immobilized drop. In these respective cases, symmetry breaking is manifested in supercritical-pitchfork bifurcations into steady states of pure rolling and constant angular-velocity. In qualitative agreement with the experiments, when an immobilized drop is suddenly released it initially moves at constant acceleration $a g$, where $\alpha$ is an angle characterizing the slope of the liquid-vapor profile and $g$ is the gravitational acceleration; furthermore, $\alpha$ exhibits a maximum with respect to the drop radius, at a radius increasing with the temperature difference between the surface and the drop. At long times, the translation and rotational velocities become comparable and the drop tends to the steady pure-rolling state.

Introduction.—A liquid drop can levitate above a hot surface if the temperature of the surface sufficiently exceeds the boiling temperature of the liquid [1]. This effect was first studied by J. G. Leidenfrost in 1756 and ever since then has been a source of great scientific curiosity. The Leidenfrost effect is associated with a sharp transition from nucleate to film boiling as the surface temperature is increased past the so-called Leidenfrost point. The vapor film formed by evaporation at the bottom of the drop prevents direct contact. As a consequence, Leidenfrost drops exhibit increased lifetimes and spectacular mobility, attributes that either hold promise or are concerning for many contemporary applications. Experimental and theoretical studies over the last decade have unraveled the wealth of interesting dynamics exhibited by Leidenfrost drops, including oscillations, bouncing, take-off and directed propulsion using asymmetrically structured surfaces and thermal gradients [2].

Remarkably, it has been discovered only very recently that Leidenfrost drops can also undergo symmetry breaking, leading to spontaneous motion in the absence of any external gradients or asymmetries. Specifically, it has been shown experimentally in [3] that a rotational flow can be spontaneously established within an immobilized millimeter-scale Leidenfrost drop — not too small or too large; moreover, once such a drop is released, it begins to move horizontally at an acceleration approximately equal to the product of the gravitational acceleration $g$ and the angle measured between the surface and the virtual inclined plane defined by the narrow necks of the asymmetrically deformed liquid-vapor interface. These observations have received significant attention as they may help explain the extreme mobility of Leidenfrost drops as well as open new avenues for practical exploitation. Symmetry breaking and spontaneous motion have also been recently observed for so-called inverse-Leidenfrost drops [4].

Theoretical analyses of Leidenfrost statics and dynamics [5–7], as well as related configurations of levitated drops [8, 9], have traditionally employed a lubrication approximation to model the vapor film, which is either patched or matched to a hydrostatic model of the top side of the drop. These studies, however, which neglect the liquid flow within the drop on account of the high liquid-vapor viscosity contrast, show no hint of symmetry breaking.

In this Letter, we theoretically illuminate the mechanics enabling symmetry breaking and spontaneous motion of Leidenfrost drops based on a simplified two-dimensional model. We shall focus on the case where the drop is small compared to the capillary length, in which case it turns out to be straightforward to account for the internal liquid flow.

Two-dimensional model.—Consider a two-dimensional liquid drop (area $\pi R^2$, density $\bar{\rho}$, viscosity $\bar{\mu}$) levitated by a thin vapor layer (thermal conductivity $k$, density $\rho$, viscosity $\mu$) above a flat solid surface of temperature $\Delta T$ relative to the drop. Following standard modeling of Leidenfrost drops, we assume an isothermal drop at boiling temperature and that area variations due to the evaporation are negligible on the time scale of interest.

The cornerstone assumption underlying our theory is that $R \ll \ell_c = \sqrt{\gamma/\bar{\rho} g}$, the capillary length, wherein $\gamma$ is the liquid-vapor interfacial tension; equivalently, this implies the small-Bond-number limit $B = (R/\ell_c)^2 \ll 1$. Similarly to the case of a sessile nonwetting drop, in that limit the shape of a two-dimensional Leidenfrost drop is approximately a circle except close to a small “flat spot” at the bottom of the drop. The liquid pressure is therefore approximately uniform and equal to the capillary...
pressure $\gamma/R$. Assuming that is also the scaling of the pressure in the vapor film below the flat spot, a balance with gravity gives the scaling $L = BR$ for the flat-spot length. This standard argument assumes that the dynamical stresses associated with the internal liquid flow are $\ll \gamma/R$.

Consider now the internal liquid flow, which is usually thought to be shear-driven by the vapor being squeezed symmetrically outwards from beneath the flat spot. For $B \ll 1$, and given the high liquid-vapor viscosity contrast, such a shear-driven flow would be confined to the $O(L)$ vicinity of the flat spot, symmetric and weak relative to the vapor flow [2]. We shall see, however, that when this symmetric state is unstable, the net forces and torques exerted on the drop by the vapor film can inertially drive an asymmetric drop-scale flow. The latter flow, in turn, entrains the vapor flow, which is no longer driven solely by evaporation.

An asymmetric drop-scale flow is indeed evident in the experiments [3]. Theoretically, given the circular shape of the drop boundary, we hypothesize that the drop-scale flow is a superposition of a rigid-body translation, at velocity $U(t)\hat{e}_x$, and rigid-body rotation, at angular velocity $\Omega(t)\hat{e}_y$ (see Fig. 1a). Here $t$ is time and we introduce unit vectors $\hat{e}_x$ and $\hat{e}_z$ pointing horizontally and vertically upwards, respectively, with $\hat{e}_y = \hat{e}_z \times \hat{e}_x$. This ansatz is inspired by the analysis of a nonwetting drop rolling down a gently inclined plane [10, 11], where a similar leading-order drop-scale solution holds. In the present problem, where we allow for a time dependence of the rigid-body motion, substitution into the time-dependent Navier–Stokes equations and the interfacial conditions on the circular boundary furnishes the condition that the inertial time scale $T$, on which $U(t)$ and $\Omega(t)$ vary, must be much larger than $1/\Omega$; this is in addition to said condition that the drop-scale dynamical stresses are $\ll \gamma/R$. These assumptions appear to hold in the experiments [3] and can be tested a posteriori for the two-dimensional theory [12].

**Vapor film.**—With this drop-scale physical picture, we now turn our attention to modeling the vapor film. We shall employ the lubrication approximation on the premise, to be verified, that the film is thin relative to its $O(L)$ width. It is convenient to work in a frame co-moving with the drop center-of-mass, with the corresponding Cartesian coordinate system defined in Fig. 1.

We denote by $u = u\hat{e}_x + w\hat{e}_z$ the vapor velocity field in the co-moving frame, by $p$ the corresponding vapor pressure field and by $h$ the film-thickness profile. In the lubrication approximation, $p$ is independent of $z$ and the momentum and continuity equations respectively become

$$-\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial z^2} = 0, \quad \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0. \quad (1a, b)$$

At the solid substrate, the flow satisfies a no-slip condition,

$$u = -U, \quad w = 0 \quad \text{at} \quad z = 0. \quad (2a, b)$$

To formulate the boundary conditions on the liquid-vapor interface, we assume that (i) the liquid velocity and pressure at the bottom of the drop are approximately uniform and equal to $-\Omega(t)R\hat{e}_x$ and $\gamma/R$, respectively [12]; (ii) evaporation at the interface contributes a normal vapor speed $-\lambda/h$, where $\lambda = k\Delta T/l$, $l$ being the latent heat of evaporation [3]; and (iii) the interface velocity $\partial h/\partial t$ is negligible compared to the normal vapor speed $\dot{h}$. Accordingly, the kinematic boundary conditions and continuity of tangential velocity across the interface give

$$u = -\Omega R, \quad w = -\frac{\lambda}{h} - \Omega R \frac{\partial h}{\partial x} \quad \text{at} \quad z = h, \quad (3a, b)$$

while the dynamic boundary condition gives

$$\frac{\gamma}{R} - p = \frac{\partial^2 h}{\partial x^2}. \quad (4)$$
Integrating the momentum equation (11) together with the boundary conditions (2a) and (3a) gives

$$u + U = \frac{1}{2\mu} \frac{\partial p}{\partial x} (z - h) + \frac{z}{h} (U - \Omega R).$$

(5)

Next, integrating the continuity equation (1) using the boundary conditions (2a) and (3a) gives

$$\frac{\partial}{\partial x} \left( \frac{h^3}{12\mu} \frac{\partial p}{\partial x} \right) + \frac{U + \Omega R}{2} \frac{\partial h}{\partial x} = -\lambda. \quad (6)$$

To close the thin-film problem governing $p$ and $h$ we require four boundary conditions. These can be represented in terms of coefficients in the large-$|x|$ expansions of $h$. Thus, since $p \to 0$ in that limit, (4) implies

$$h \sim \frac{x^2}{2R} + a_\pm x + O(1) \quad \text{as} \quad x \to \pm \infty. \quad (7)$$

One relation between the coefficients $a_\pm$ can be deduced from the vertical force balance

$$\mathcal{L} - mg = 0, \quad \mathcal{L} = \int_{-\infty}^{\infty} p \, dx, \quad (8)$$

where $m = \pi \rho R^2$ is the drop mass per unit length and $\mathcal{L}$ the “lift” force per unit length: substituting (4) into (8) gives $a_- - a_+ = \pi B$. To obtain a second relation, we argue that the linear terms in (7) match with $O(\beta R)$ drop-scale deviations from a circular shape; such deviations are hydrostatic [12] and therefore even in $x$. It follows that $a_+ = -a_- = -\pi B/2$.

The lubrication model is now closed and provides $p$ and $h$ for an instantaneous value of the sum $U + \Omega R$. Then $u$ follows from (3). It is insightful to rewrite the far-field expansions (7) in the expanded form

$$h \sim \frac{1}{2R} \left( x + \frac{\pi BR}{2} \right)^2 + h_\pm + o(1) \quad \text{as} \quad x \to \pm \infty, \quad (9)$$

where the constants $h_\pm$ are outputs of the lubrication problem. The geometric interpretation of these constants can be understood from Fig. 2. In particular, the slope

$$\alpha = \frac{h^- - h^+}{\pi BR} \quad (10)$$

is small and represents an angle characteristic of the profile asymmetry.

**Drop dynamics.**—We turn to the dynamics of the drop as a rigid body. To this end, we consider an integral linear-momentum balance in the $x$ direction and an integral angular-momentum balance in the $y$ direction, with the hydrodynamic forces and torques on the drop calculated based on the lubrication approximation of the vapor film. We thereby find the equations of motion

$$m \frac{d\mathcal{L}}{dt} = \mathcal{P} - \mathcal{F}, \quad I \frac{d\Omega}{dt} = T - R\mathcal{P} + RF, \quad (11a,b)$$

where we define the moment of inertia $I = mR^2/2$, the “propulsion” and friction forces per unit length

$$\mathcal{P} = -\frac{1}{2} \int_{-\infty}^{\infty} p \, dx, \quad \mathcal{F} = \mu (U - \Omega R) \int_{-\infty}^{\infty} \frac{dx}{h}, \quad (12a,b)$$

respectively, and the torque per unit length

$$T = -\int_{-\infty}^{\infty} xp \, dx. \quad (13)$$

Note that the propulsion $\mathcal{P}$ and torque $T$ are nonlinear functions of $U + \Omega R$; using (4) and (3), these can be written

$$\mathcal{P} = \frac{\gamma}{2R} (h_- - h_+), \quad T = \gamma (h_- - h_+). \quad (14a,b)$$

The friction $\mathcal{F}$ is linear in $U - \Omega R$, the constant of proportionality being a positive nonlinear function of $U + \Omega R$.

**Scalings.**—Let $H$ be the characteristic thickness of the vapor film. Assuming the vertical velocity scaling $W = \lambda/H$ based on the evaporation term in (3b), (11) implies the scaling $U = LW/H$ for the horizontal velocity. Comparing the corresponding lubrication pressure scaling [cf. (11)] with the capillary pressure $\gamma/R$ provides the estimate $H = \Gamma B^2R$, where we define the parameter

$$\Gamma = B^{-3/2} (\mu \lambda/\gamma R)^{1/4} \propto \Delta T^{1/4} R^{-13/4}. \quad (15)$$

It then follows from (12) that the horizontal force on the drop is of order $F = \gamma H/R$ and hence that the inertial time scale is of order $T = mU/F$.

We shall see that our model predicts spontaneous dynamics for small values of $\Gamma$. In that case, the vapor film decomposes into a “bubble” bounded by two narrow necks, with the scalings corresponding to the symmetric case being those indicated in Fig. 1. These scalings can be derived following [6]. The bubble pressure is $\approx \gamma/R$ and hence from the vertical-force balance (8) the distance between the necks is $\approx \pi L$.

**Dimensionless model.**—Let $\tilde{x} = x/L$, $\tilde{h} = h/H$, $\tilde{p} = pR/\gamma$, $\tilde{U} = U/U$, $\tilde{\Omega} = \Omega \nu R/U$, $\tilde{s} = \tilde{U} \pm \tilde{\Omega}$ and $\tilde{t} = t/T$. The lubrication problem (11), (13) and (14) becomes

$$\frac{\partial}{\partial \tilde{x}} \left( \frac{h^3}{12\mu} \frac{\partial \tilde{p}}{\partial \tilde{x}} \right) + \frac{s^+ \partial \tilde{h}}{2 \partial \tilde{x}} = -\frac{1}{\tilde{h}}, \quad 1 - \tilde{p} = \Gamma \frac{\partial^2 \tilde{h}}{\partial \tilde{x}^2}, \quad (16a,b)$$
\[
\tilde{h} \sim \frac{1}{2T} \left( \tilde{x} \pm \frac{\pi}{2} \right) + \tilde{h}_\pm + o(1) \quad \text{as} \quad \tilde{x} \to \pm \infty. \quad (17)
\]

The constants \( \tilde{h}_\pm = h_\pm / H \) are outputs of the lubrication problem \((10)-(17)\), which depends only on the dynamic variable \( s^\pm \) and the parameter \( \Gamma \). The equations of motion \((11)\) become

\[
\begin{align*}
\dot{\tilde{U}} &= \frac{1}{2} A(s^+; \Gamma) - s^- B(s^+; \Gamma), \quad (18a) \\
\dot{\tilde{\Omega}} &= \frac{1}{2} A(s^+; \Gamma) + s^- B(s^+; \Gamma), \quad (18b)
\end{align*}
\]

where the dot stands for \( d/d\tilde{t} \) and [cf. \((12)-(14)\)]

\[
A(s^+; \Gamma) = \tilde{h}_- - \tilde{h}_+ , \quad B(s^+; \Gamma) = \int_{-\infty}^\infty \frac{d\tilde{x}}{h + \tilde{x}}. \quad (19a, b)
\]

Note that \( \tilde{h} \) is invariant under the transformation \( \tilde{x} \to -\tilde{x} \) and \( s^+ \to -s^+ \). It follows that \( A \) and \( B \) are odd and even in \( s^+ \), respectively. Also, while \( A \) can be either positive or negative, \( B \) is always positive. Given that \( A \) is odd in \( s^+ \), it is obvious that \( (s^+, s^-) = (0, 0) \) is a fixed point of the dynamical system \((18a, b)\) for all \( \Gamma \). This trivial fixed point corresponds to the usual symmetric state.

**Free drops.**—A straightforward analysis of \((18a, b)\), wherein \( A \) and \( B \) are calculated by solving the lubrication problem \((10)-(17)\) numerically, shows that the symmetric state is unstable for \( \Gamma < \Gamma_F \approx 0.0235 \). In that regime, there are stable asymmetric “pure-rolling” steady states for which \( s^- = 0 \) and \( s^+ \) satisfies \( A(s^+; \Gamma) = 0 \). The resulting supercritical pitchfork bifurcation is shown in Fig. 3, the insets showing the profiles \( \tilde{h}(\tilde{x}) \) for the symmetric and asymmetric steady states at \( \Gamma = 10^{-3} \).

**Immobilized drops.**—It is clear that the steady pure-rolling states do not conform to the constant-acceleration motion observed in the experiments \([3]\). It appears important that in the experiments the drop is initially immobilized, by a needle, and then released; importantly, while the drop is immobilized, a rotational flow within the drop builds up.

To model the first stage in the experiment where the drop is immobilized, we assume a horizontal restoring force passing through the drop center of mass. Then, the angular-momentum balance \((18a, b)\) gives, with \( s^\pm = \pm \Omega, \) the reduced one-dimensional dynamical system

\[
\dot{\tilde{\Omega}} = A(\tilde{\Omega}; \Gamma) - 2\tilde{\Omega} B(\tilde{\Omega}; \Gamma). \quad (20)
\]

Analogously to the free-drop case, the symmetric steady state \( \tilde{\Omega} = 0 \) is unstable for \( \Gamma < \Gamma_I \approx 0.00253 \). In that regime, there are stable steady states of angular velocity \( \tilde{\Omega} = \tilde{\Omega}^* \) satisfying

\[
A(\tilde{\Omega}^*; \Gamma) - 2\tilde{\Omega}^* B(\tilde{\Omega}^*; \Gamma) = 0. \quad (21)
\]

The resulting supercritical pitchfork bifurcation is shown in Fig. 4, the inset depicting the profile \( \tilde{h}(\tilde{x}) \) for the rotational steady state at \( \Gamma = 10^{-3} \).

**Released drops.**—Now assume that an immobilized drop rotating at the angular velocity \( \tilde{\Omega}^* \) is suddenly released at \( \tilde{t} = 0 \) with \( \tilde{U} = 0 \). From \((18a, b)\) and \((20)\), the drop starts to move with initial acceleration

\[
\dot{\tilde{U}} = A(\tilde{\Omega}^*; \Gamma), \quad (22)
\]

which is plotted in Fig. 4 as a function of \( \Gamma \). A simulation of the free-drop equations \((18a, b)\), starting from this initial state gives, is shown in Fig. 5. For small \( \tilde{t} \), the acceleration and angular velocity are approximately constant. We argue that this initial period corresponds to the regime observed in the experiments \([15]\). For large \( \tilde{t} \), the drop approaches the steady pure-rolling state.

Using \((10)\) and \((19)\), the dimensional acceleration corresponding to \((22)\) is

\[
\frac{dU}{dt} = \alpha g. \quad (23)
\]
This expression agrees with that observed in the three-dimensional experiments \cite{3} if we identify the characteristic slope $\alpha$ with the measured “inclination angle”, namely the geometric angle between the surface and a line passing through the local neck minima (numerically, $\alpha$ is within 14\% of that geometric angle for $\Gamma < \Gamma_1$).
Our prediction that $\alpha$ exhibits a maximum as a function of $R$, at a value of $R$ increasing with $\Delta T$ \cite{15}, provides further qualitative agreement with the experiments. Because our model is two-dimensional, however, any quantitative agreement would be fortuitous. In particular, for the physical parameters in the experiments, the value $\Gamma_1$ suggests a bifurcation radius $\approx 4 mm$ which widely overestimates the observed value $\approx 0.6 mm$ and appears to contradict our assumption $B \ll 1$ (the capillary length is $\approx 2.5 mm$ for the experimental physical parameters); the fact that the measured bifurcation radius satisfies the small-$B$ condition, however, suggests that that assumption is physically representative.

Concluding remarks.—We have presented a simplified two-dimensional model of a Leidenfrost drop which predicts symmetry breaking leading to spontaneous dynamical regimes, including “pure-rolling” steady states, not yet observed experimentally.

Generalizing our model to three dimensions would allow to check quantitative agreement with the experiments. It would also be desirable to generalize the theory to moderate $B$, where the experiments show a second bifurcation from the rotational state to a symmetric state characterized by a two-cell liquid drop-scale flow. We suspect this latter transition occurs as the liquid flow shear-driven by the vapor film, which for $B \ll 1$ is localized and negligible in magnitude relative to the vapor and inertial drop-scale flows, extends to the whole drop and grows comparable in magnitude to the vapor flow.

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\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{Fig5}
\caption{Inertial dynamics of a drop released at time $\tilde{t} = 0$ from an immobilized steady state of constant angular velocity $\tilde{\Omega}$, for $\Gamma = 10^{-4}$.
}
\end{figure}

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\bibitem{11} E. Yariv and O. Schnitzer, Phys. Rev. Fluids \textbf{4}, 093602 (2019).
\bibitem{12} Relative to the capillary stress $\gamma/R$, the inertial and viscous dynamic stresses respectively scale like the Weber number $We = \bar{\rho}U^2R/\gamma$ and $BCa$, wherein $Ca = \mu U/\gamma$ is the capillary number and $U$ the characteristic velocity defined later. (The $B$ factor is because the viscous stresses vanishes for a rigid-body motion.) We require these dynamic stresses to be $o(B\gamma/R)$, rather than $o(\gamma/R)$, for the reason mentioned below \cite{5}. Furthermore, the condition $1/\Omega \ll T$, or $R/U \ll T$, is equivalent to $\Gamma B^2 \ll We$, wherein $\Gamma$ is defined in \cite{15} and we use the estimates for $U$ and $T$ derived in “Scalings”. We accordingly assume that $\Gamma B^2 \approx We \ll B$ and $Ca \ll 1$.
\bibitem{13} The deviation of the drop shape from a circular shape, due to gravity, implies a negligible $O(B\Omega R)$ velocity perturbation \cite{11}. Meanwhile, a tangential-stress balance at the liquid-vapor interface implies a velocity perturbation $O(\mu L/\bar{\mu} H)$ relative to the characteristic vapor velocity $U$, where $H = \Gamma B^2 R$ and $\Gamma$ is defined in \cite{15}. For $U = O(\Omega R)$ this implies the condition $\mu/\bar{\mu} \ll B^4 \Omega$. Flow measurements at the drop base confirm an approximately uniform flow \cite{3}.
\bibitem{14} In the co-moving frame, $h$ varies on the inertial time scale $T$. Using the estimates derived in “Scalings”, the ratio between $dh/dt$ and the normal vapor speed is $BR/UT$, which is small since $B \ll 1$ and $T \gg R/U$.
\bibitem{15} From private communications with the authors of \cite{3}, $U < \Omega R$ throughout the measurements; moreover, the experimental data suggests an inertial time scale $\approx 10$ times larger than the observation time.
\end{thebibliography}