Exceptional sets in dynamical systems and Diophantine approximation

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Abstract. The nature and origin of exceptional sets associated with the rotation number of circle maps, Kolmogorov-Arnol’d-Moser theory on the existence of invariant tori and the linearisation of complex diffeomorphisms are explained. The metrical properties of these exceptional sets are closely related to fundamental results in the metrical theory of Diophantine approximation. The counterpart of Diophantine approximation in hyperbolic space and a dynamical interpretation which led to the very general notion of ‘shrinking targets’ are sketched and the recent use of flows in homogeneous spaces of lattices in the proof of the Baker-Sprindžuk conjecture is described briefly.

1 Introduction

An important consequence of the natural connection between resonance and Diophantine equations is the frequent occurrence of Diophantine conditions in the study of dynamical systems. A set in a space is called exceptional if its members fail to satisfy some property satisfied by ‘almost all’ points in the space. More precisely, let \((X, \mu)\) be a measure space. The set \(E \subset X\) is exceptional if \(\mu(E) = 0\). From this point of view, the rationals \(\mathbb{Q}\) form an exceptional set in \(\mathbb{R}\) and the Cantor set in \([0, 1]\) is one with respect to Lebesgue measure. The sets to be considered are general ‘limsup’ sets; the standard way of showing them to be null is to use the easy half (convergence case) of the Borel-Cantelli theorem.

As is well known, Hausdorff measure and dimension provide a means of distinguishing sets of Lebesgue measure zero (or null) sets. In this article, the Hausdorff measure and dimension of some exceptional sets of importance in dynamical systems will be discussed. The purpose of this paper is to show how these sets arise from certain Diophantine conditions failing to hold. The paper is not intended to cover all recent developments which can be found in the references cited. We begin with a simple example; further details on this and dynamical systems in general can be found in the comprehensive introduction to dynamical systems of Katok and Hasselblatt [17] or the more elementary text of Arrowsmith and Place [7].

2 Rotation number

The rotation number is a measure of how far on average a continuous orientation preserving homeomorphism \(f: \mathbb{S}^1 \to \mathbb{S}^1\) moves a point round the circle
and illustrates nicely the way Diophantine properties can arise in analysis (see [47, Chapter 11] or [59, Chapter 1] for accounts of this topic). When $f$ is continuous it has a continuous lift $F : \mathbb{R} \to \mathbb{R}$ which satisfies $e \circ F = f \circ e$, where $e : \mathbb{R} \to S^1$ is the exponential map given by

$$z = e(t) = e^{2\pi it}.$$ 

The lift $F$ is unique up to translation by an integer. Moreover when the homeomorphism $f$ is orientation preserving, then $f$ is of degree 1, $F$ is strictly increasing and the function $F - \text{id}$, given by $F(t) - t$, is 1-periodic. Thus $F(t+1) - t - 1 = F(t) - t$ and $F(t+1) = F(t) + 1$. When $f$ has a fixed point at $z_0$ say, so that $f(z_0) = z_0$, then $F(t_0) = t_0 + k_0$ for some $k_0 \in \mathbb{Z}$, whence for each $N \in \mathbb{N}$, the set of positive ($\geq 1$) integers,

$$F^N(t_0) = t_0 + Nk_0,$$

and

$$\lim_{N \to \infty} \frac{F^N(t_0)}{N} = \lim_{N \to \infty} \frac{t_0 + Nk_0}{N} = k_0 \in \mathbb{Z},$$

where $F^N$ is the $N$-fold iteration $F \circ \cdots \circ F$. A similar argument shows that the limit $\rho(F)$ for a $q$-periodic point is a rational with denominator $q$. The limit

$$\rho(F) = \lim_{N \to \infty} \frac{F^N(t)}{N} = \lim_{N \to \infty} \frac{F^N(t) - t}{N}$$

turns out to be independent of $t$ (for proofs see [47, Prop. 11.1.1] and [59, p 33]). In fact it can be shown that $\rho(F)$ is irrational if and only if $f$ has no periodic points [47,59].

Two rotation numbers $\rho(F)$ and $\rho(F')$ for lifts $F, F'$ for $f$ differ by an integer and so the rotation number $\rho(f)$ for $f$ can defined as the fractional part $\{\rho(F)\}$ of $\rho(F)$. As a simple example, the rotation number of a rotation $r_\alpha$ by an angle $2\pi \alpha$, where $0 \leq \alpha < 1$, given by

$$r_\alpha(z) = z e(\alpha) = z e^{2\pi i \alpha}$$

is, naturally enough, $\alpha$. It follows immediately that $\rho(f)$ is irrational if and only if $f$ has no periodic points. If $\rho(f)$ is irrational then for $z \in S^1$, the closure $A$ of the orbit

$$\omega(z) = \{f^n(z) : n \in \mathbb{N}\}$$

does not depend on $z$ and $A$ is either perfect and nowhere dense or $A = S^1$. In the latter case $f$ is transitive and is topologically conjugate to the rotation $r_{\rho(f)}$, i.e., there exists an orientation preserving homeomorphism $\varphi : S^1 \to S^1$ such that

$$f = \varphi^{-1} \circ r_{\rho(f)} \circ \varphi \quad \text{or} \quad \varphi \circ f = r_{\rho(f)} \circ \varphi,$$

(2)
written \( f \sim r_{\rho(f)} \).

Denjoy showed that when \( f \) is \( C^2 \) rather than just \( C^1 \), the irrationality of \( \rho(f) \) implies that \( f \) is transitive so that \( f \sim r_{\rho(f)} \), i.e., \( f \) is topologically conjugate to a rotation. This partial converse to the rotation \( r_{\rho} \) having rotation number \( \rho \) is best possible as there are counterexamples when \( f \) is \( C^{2-\varepsilon} \) [7].

Two kinds of Diophantine approximation enter the picture when the differentiability of the conjugacy function \( \varphi \) is considered. When \( \varphi \) is \( C^r \), where \( r \) is finite or infinite (but \( \varphi \) is not analytic), the existence or otherwise of a smooth conjugacy is determined by the Diophantine type of \( \rho \) (see §2.1 below). If \( f \) is \( C^r \) and \( \rho(f) \) is of Diophantine type then \( f \) is smoothly conjugate to \( r_{\rho(f)} \). The precise details of the differentiability conditions are rather complicated; further details and references are in [47] and [78,79].

When \( \varphi \) is analytic and \( \rho \) is a Bruno number, i.e., \( \rho \in \mathbb{R} \setminus \mathbb{Q} \) and the denominator \( q_n \) of the \( n \)-th convergent in the continued fraction expansion of \( \rho \in \mathbb{R} \setminus \mathbb{Q} \) satisfies

\[
\sum_{n=1}^{\infty} \frac{\log q_{n+1}}{q_n} < \infty,
\]

then any analytic \( f \) with rotation number \( \rho \) is analytically conjugate to \( r_{\rho} \) (continued fractions and convergents are explained in [17,37]). This condition is also optimal since if \( \rho \) is not a Bruno number, there is a Blaschke product with rotation number \( \rho \) not analytically conjugate to \( r_{\rho} \) [78].

To illustrate the ideas, we shall consider analytic conjugation when \( f \) is close to a rotation and see how Diophantine conditions emerge. Suppose the diffeomorphism \( f \) is analytic with (analytic) lift \( F \) given by

\[
F(x) = x + \rho + a(x),
\]

where \( a \) is a 1-periodic analytic function, real on the real axis and bounded on the strip \( \{ z \in \mathbb{C} : |\Im z| < \varepsilon \} \) for some \( \varepsilon > 0 \). Consider the lifts \( F, \varphi \) and \( R_{\rho} \) of \( f, \varphi \) and \( r_{\rho} \) respectively and the lift

\[
F \circ \Phi(t) = \Phi \circ R_{\rho}(t)
\]

of equation (3), where \( R_{\rho} \) is translation by \( \rho \) given by \( R_{\rho}(t) = t + \rho \) and where \( \Phi \) is assumed to be close to the identity, i.e., \( \Phi(t) = t + h(t) \), where \( h(t) \) is small. Then (3) reduces to the functional equation

\[
t + h(t) + \rho + a(t + h(t)) = t + \rho + h(t + \rho),
\]

i.e.,

\[
h(t + \rho) - h(t) = a(t + h(t)).
\]

One then seeks a solution to this equation by successive approximation. Since \( a \) and \( h \) are small, consider the first order approximation

\[
h^0(t + \rho) - h^0(t) = \tilde{a}(t) = a(t) - a_0,
\]
where \( \alpha_0 = \int_0^1 a(t) dt \), the mean of \( a(t) \). Since \( a \) and \( h \) are periodic, they have Fourier series expansions

\[
a(t) = \sum_{k \in \mathbb{Z}} a_k e^{2\pi i k t} \quad \text{and} \quad h^0(t) = \sum_{k \in \mathbb{Z}} h^0_k e^{2\pi i k t}.
\]

On substituting in (5) and comparing the coefficients of \( e^{2\pi i k t} \), the coefficients of the linear approximation \( h^0 \) to the unknown function \( h \) can be determined in terms of those of the function \( a \), as follows: \( h^0_0 = 0 \) and for \( k \neq 0 \),

\[
h^0_k = \frac{a_k}{e^{2\pi i k \rho} - 1}.
\]

It is evident that for \( h^0 \) to exist, the denominator and numerator of \( h^0_k \) must vanish together (this is why the function \( \tilde{a}(t) = a(t) - a_0 \) is introduced). Since \( a \) is analytic, its Fourier coefficients \( a_k \) decay rapidly and

\[
|a_k| < \delta e^{-|k|\varepsilon},
\]

where \( \sup \{|a(z)|: |\Im z| < \varepsilon \} < \delta \). Now as \( 2\theta/\pi \leq \sin \theta \leq \theta \) when \( 0 \leq \theta \leq \pi/2 \), it is readily verified that for any real \( \theta \),

\[
|e^{i\theta} - 1| = 2|\sin \theta/2| \geq 2\|\theta\|/\pi,
\]

where \( \|\theta\| = \theta \) when \( 0 \leq \{\theta\} \leq 1/2 \) and \( 1 - \{\theta\} \) otherwise (\( \{\theta\} \) is the fractional part of the real number \( \theta \)). Hence when \( k \neq 0 \),

\[
|e^{2\pi i k \langle \rho \rangle} - 1| = |e^{2\pi i k \rho} - 1| \geq 2\|k\rho\|/\pi.
\]

Note that by (6), the denominator of \( h^0_k \) is \( e^{2\pi i k \rho} - 1 \).

Dirichlet’s theorem [37, §11.2] states that given any real number \( \rho \) and positive integer \( N \), there exists a positive integer \( k \leq N \) and an integer \( j \) such that

\[
\left| \rho - \frac{j}{k} \right| < \frac{1}{kN} \leq \frac{1}{k^2}.
\]

It follows that there exist infinitely many \( k \in \mathbb{N} \) such that

\[
\|k\rho\| \leq k^{-1}
\]

(recall that \( \|k\rho\| = \min\{|k\rho - p|: p \in \mathbb{Z}\} \)). Indeed for rational \( \rho \), the denominator will vanish for infinitely many \( k \), so destroying the convergence of the Fourier series for \( h \). Even for irrational \( \rho \), the denominator will get arbitrarily small and so make convergence problematic. However, the very rapid decay in the numerator \( a_k \) can be set against this to obtain convergence for those numbers \( \rho \) for which \( \|k\rho\| \) does not get too small. To be more precise, comparing the moduli of the denominator and numerator of \( h^0_k \) in (6), we see that the Fourier series for \( h^0 \) will converge if \( \rho \) satisfies

\[
\|k\rho\| \geq K|k|^{-v}
\]

for some \( K, v > 0 \). By Dirichlet’s theorem, we must take \( v > 1 \).
2.1 Diophantine type

A number $\rho$ satisfying

$$\left| \rho - \frac{j}{k} \right| \geq \frac{K}{k^{\sigma+2}},$$

for some $\sigma > 0$ and for all $k \in \mathbb{N}$, is said to be of Diophantine type $(K, \sigma)$ [4] and the set of such numbers is denoted by $\mathcal{D}(K, \sigma)$; the set of numbers of Diophantine type $(K, \sigma)$ for some positive $K, \sigma$ is denoted $\mathcal{D} = \bigcup_{K, \sigma} \mathcal{D}(K, \sigma)$.

Similar conditions turn up in other settings, as we will see below. The set of Bruno numbers includes $\mathcal{D}$ since if $\xi \in \mathcal{D}(K, \sigma)$, then $q_{n+1} = O(q_n^\sigma)$ and $\sum_n (\log q_{n+1})/q_n < \infty$. Note that $\mathcal{D}(0) = \bigcup_{K>0} \mathcal{D}(K, 0) = \mathfrak{B}$, the set of badly approximable numbers.

Using a version of Newton’s tangent method, a sequence of successive approximations $\Phi_n$ is constructed which converges to an analytic function $\Phi$ satisfying (3), again providing $\rho$ is of type $(K, \sigma)$ for some $K, \sigma$. Projecting from the lift, the desired conjugacy exists, providing $\rho$ is of type $(K, \sigma)$, i.e., providing $\rho \in \mathcal{D}$. More details are in [5, §12].

The size of the complementary set $E = \mathbb{R} \setminus \mathcal{D}$ of points for which the iterative method does not guarantee convergence – and so conjugacy – should be as small as possible. The exponent $\sigma$ in the Diophantine type affects the range of validity of the result significantly, so we consider the set $E_\sigma$ given by

$$E_\sigma = \bigcup_{K>0} \{ \xi \in \mathbb{R} : |\xi - p/q| < Kq^{-2-\sigma} \text{ for some } p/q \in \mathbb{Q} \}$$

$$= \bigcup_{K>0} \{ \xi \in \mathbb{R} : \|q\xi\| < Kq^{-1-\sigma} \text{ for some } q \in \mathbb{N} \}.$$

The set

$$E = \bigcap_{\sigma>0} E_\sigma = \bigcap_{\sigma \in \mathbb{Q}^+} E_\sigma = \lim_{\sigma \to \infty} E_\sigma$$

consists of points not of Diophantine type for any $K, \sigma > 0$. Given $\sigma > 0$, almost all real numbers are of Diophantine type $(K, \sigma)$ for some positive $K$, i.e., the set

$$\mathcal{D}(\sigma) = \bigcup_{K>0} \{ \xi \in \mathbb{R} : |\xi - p/q| \geq Kq^{-2-\sigma} \text{ for each } p/q \in \mathbb{Q} \} \quad (8)$$

is of full measure, since the complementary set

$$E_\sigma = \bigcap_{K>0} \{ \xi \in \mathbb{R} : |\xi - p/q| < Kq^{-2-\sigma} \text{ for some } p/q \in \mathbb{Q} \}$$
is null. This can be proved directly but we will consider a related set \( K_v \) and use Khintchine’s theorem. This theorem also implies that the set \( B \) of badly approximable numbers is null. Note that badly approximable numbers are good for conjugacy.

### 2.2 Very well approximable numbers and Khintchine’s theorem

The set of \( v \text{-approximable} \) numbers is defined as

\[
K_v = \{ \xi \in \mathbb{R} : \| q\xi \| < q^{-v} \text{ for infinitely many } q \in \mathbb{N} \}; \tag{9}
\]

when \( v > 1 \), the points are often called very well approximable (VWA). The set \( K_v \) can be usefully expressed as a ‘limsup’ set, i.e.,

\[
K_v = \bigcap_{N=1}^{\infty} \bigcup_{q=N}^{\infty} B_{\psi(q)}(q) = \limsup_{N \to \infty} B_{\psi(N)}(N), \tag{10}
\]

where \( B_q(q) = \{ \xi \in \mathbb{R} : \| q\xi \| < \delta \} \). It is not difficult to verify that for any \( \varepsilon > 0 \),

\[
K_{\sigma+1+\varepsilon} \subset E_{\sigma} \subset K_{\sigma+1}, \tag{11}
\]

see [12, §7.2]. The set \( D(\sigma) \) (defined in [8]) is complementary to the sets of points approximable to exponent \( \sigma + 2 \) and consists of points which are, in a sense, not close to the rationals or very well approximable points. Points in \( D(\sigma) \) have desirable approximation properties for certain applications.

Khintchine’s theorem relates the Lebesgue measure of sets more general than \( K_v \) to the convergence or divergence of a sum. The Lebesgue measure of a subset \( A \) of Euclidean space will be denoted by \( |A| \). A set \( A \) is full if the Lebesgue measure of its complement is null. First let \( \psi : \mathbb{N} \to \mathbb{R}^+ \) be a function and let

\[
K(\psi) = \{ \xi \in \mathbb{R} : \| q\xi \| < \psi(q) \text{ for infinitely many } q \in \mathbb{N} \} \tag{12}
\]

be the set of \( \psi \text{-approximable} \) points.

**Theorem 1.** If the sum \( \sum_{q \in \mathbb{N}} \psi(q) \) converges, then \( K(\psi) \) is null, while if the sum diverges and \( \psi \) is decreasing, \( K(\psi) \) is full.

Khintchine’s theorem is reminiscent of the Borel-Cantelli lemma and indeed the proof when the sum converges is the same, as it suffices to show that

\[
\sum_q |\{ \xi \in \mathbb{R} : \| q\xi \| < \psi(q) \}| < \infty.
\]

The divergent case is, however, much more difficult and relies on pairwise (quasi-)independence and deeper ergodic ideas. Proofs of Khintchine’s theorem and generalisations are in [17, Chapter VII], [38] and [72]; the last two references include discussions of ‘zero-one’ laws.
Exceptional sets

Since $K(\psi) = K_v$ when $\psi(x) = x^{-v}$ and since the sum $\sum_q q^{-v}$ converges for $v > 1$, Khintchine’s theorem implies that $|K_{\sigma+1}| = 0$ for $\sigma > 0$. It follows from (1) that $|E_\sigma| \leq |K_{\sigma+1}| = 0$ for $\sigma > 0$. Since $D := \bigcup_{\sigma>0} D(\sigma)$, $D$ is of full measure and its complement, the set of of numbers not of Diophantine type $D(K,\sigma)$ for any $K > 0$, is an exceptional set.

There is a very general higher dimensional form of the theorem due to Groshev which is useful to us. Let $K^{(m,n)}(\psi)$ be the set of $m \times n$ real matrices $X$ such that

$$\|q X\| := \max\{|q \cdot X_1|, \ldots, |q \cdot X_n|\} < \psi(|q|),$$

where $X_i$ is the $i$-th column of $X$ and $|q| = \max\{|q_1|, \ldots, |q_m|\}$ is the height of $q$, for infinitely many $q \in \mathbb{Z}^m$. The set of $m \times n$ real matrices is identified in the natural way with $\mathbb{R}^{mn}$.

**Theorem 2.** If the sum $\sum_{q \neq 0} |q|^{m-1} \psi(|q|)^n$ converges, then $K^{(m,n)}(\psi)$ is null, while if the sum diverges and $\psi$ is decreasing, $K^{(m,n)}(\psi)$ is full.

### 2.3 Hausdorff dimension and the Jarník-Besicovitch theorem

The dependence of the exceptional sets $E_\sigma$ and $K_{\sigma+1}$ on $\sigma$, where $\sigma > 0$, is revealed through their Hausdorff dimension (definitions and expositions are in [12,34,54]). In fact the dimension $\dim K_{\sigma+1}$ of the latter is given by another famous result in number theory, namely the Jarník-Besicovitch theorem [14,45] (see [12] for more details).

**Theorem 3.** When $v > 1$,

$$\dim K_v = \frac{2}{v+1}.$$  

By Dirichlet’s theorem, when $v \leq 1$, $K_v = \mathbb{R}$, whence $\dim K_v = 1$ [34,54]. The proof that $2/(v+1)$ is an upper bound for the dimension follows straightforwardly from $K_v$ being a limsup set and a natural covering argument. To prove it a lower bound is much harder. Jarník’s original proof and his more general Hausdorff measure result for simultaneous Diophantine approximation relied heavily on arithmetic arguments. Besicovitch’s later independent proof was more geometric and was a basis for the widely used regular systems [30] and ubiquitous systems [30].

It follows from the inclusions (1) and the Jarník-Besicovitch theorem that when $\sigma \geq 0$,

$$\frac{2}{\sigma+2+\varepsilon} \leq \dim E_\sigma \leq \frac{2}{\sigma+2}.$$  

Since $\varepsilon > 0$ is arbitrary, $\dim E_\sigma = 2/(\sigma + 2)$. Now let $E = \bigcap_{\sigma>0} E_\sigma = \lim_{\sigma \to \infty} E_\sigma$ since $E_\sigma$ decreases as $\sigma$ increases. Then

$$\dim E = \lim_{\sigma \to \infty} \frac{2}{\sigma+2} = 0,$$

(14)
and the complement of $D$ has Hausdorff dimension 0. Thus when $f$ is analytic, $f$ is analytically conjugate to a rotation unless the rotation number of $f$ lies in a set of Hausdorff dimension 0. Rational points lie in this set and they play a special role in this problem. Because of the connection with the physical phenomenon of resonance, the rationals are called resonant. The set $E_\sigma$ can be thought of as consisting of families of regularly spaced points and associated ‘fractal dust’. More details are given in [5, 59, 78] and there is a simplified account in [54]. The phenomenon is widespread, see for example §4 below on linearisation and in normal forms [27, 31].

In higher dimensions, the Hausdorff dimension of the set

$$K^{(1,n)}_v = \{ \xi \in \mathbb{R}^n : ||q\xi|| < q^{-v} \text{ for infinitely many } q \in \mathbb{N} \}$$

(15)

is $(n + 1)/(v + 1)$ when $v > 1/n$ [10]; and that of the set

$$K^{(n,1)}_v = \{ \xi \in \mathbb{R}^n : ||q \cdot \xi|| < |q|^{-v} \text{ for infinitely many } q \in \mathbb{Z}^n \}$$

(16)

is $n - 1 + (n + 1)/(v + 1)$ when $v > 1/n$. For the general systems of linear forms which combines both results, see [28]. Levesley has proved a general inhomogeneous form [52] which was used by Dickinson in normal forms for pseudo-elliptic operators [27].

Jarník also showed that the exceptional set of badly approximable numbers has full Hausdorff dimension 1 [44]. This interesting result was extended greatly by Schmidt [67] who showed that the set was ‘thick’ and by Dani [13, 21] who extended the ideas to dynamical systems. Kleinbock obtained a general inhomogeneous version of Jarník’s theorem by exploiting these ideas [48, 49], for an expository account of dimension and dynamical systems, see [62].

3 Kolmogorov-Arnol’d-Moser (KAM) theory

The next example comes from the study of the stability of the solar system [57]. This is one of the oldest problems in mechanics and is of course a special case of understanding the motions of $N$ bodies of point mass subject to Newtonian attraction. The solution is well known for $N = 2$ and periodic solutions exist, with the bodies moving in elliptic orbits about their centre of mass. In the absence of any effects such as friction, the solution persists for all time. For $N \geq 3$, however, the situation is extraordinarily complicated and far from understood. This is the case for solar systems, in which one of the bodies is a sun with mass $m_N$ very much larger than the other masses $m_1, \ldots, m_{N-1}$ of the planets. If as a first approximation, the centre of mass of the system is assumed to coincide with that of the sun and if the gravitational interactions between the planets and other effects are neglected, then the system decouples into $N - 1$ two-body problems, in which each planet describes an elliptical orbit around the sun, with period $T_j$ say and frequency $\omega_j = 2\pi/T_j$, $j = 1, \ldots, N - 1$. 
For each vector \( \omega = (\omega_1, \ldots, \omega_n) \) of frequencies in the \( n \)-dimensional torus \( T^n = S^1 \times \cdots \times S^1 \), the map \( \varphi_\omega : \mathbb{R} \to T^n \) given by

\[
\varphi_\omega(t) = \varphi_\omega(0) + t\omega
\]

is a quasi-periodic flow on the torus. The case \( n = 1 \) corresponds to uniform motion around a circle and so is periodic. When the frequencies are all rational, the flow is a closed path on the torus and returns to its starting point after a time equal to the lowest common multiple of the denominators of the frequencies. If the frequencies are not all rational, then by Kronecker’s theorem \([37]\), the flow winds round the torus, densely filling a subspace of dimension given by the number of rationally independent frequencies. Thus when the frequencies are independent over the rationals, the closure of \( \varphi_\omega(\mathbb{R}) \) is the torus \( T^n \). Solutions given by quasi-periodic flow \( \varphi_\omega \) on the torus \( T^n \), where \( n = N - 1 \) will persist for all time.

The planetary interactions are represented by a small perturbation of the original Hamiltonian. The stability of the solar system then reduces to the question of whether the solutions of the perturbed Hamiltonian system will continue to wind round a perturbed invariant torus, so that motion of the planets remains quasi-periodic and persists for all time. This model is of course idealised and takes no account of the long term fate of the universe. Weierstrass constructed a formal series solution but was unable to establish convergence because the denominators in the terms of the series contained factors of the form \( q \cdot \omega \) which can become very small for certain integer vectors \( q \) \([56, \text{Chap. 1}, \S 2]\). Investigating the three body problem led Poincaré to speculate that quasi-periodic solutions need not exist. However, Siegel overcame a related ‘small denominator’ problem arising in the linearisation of complex diffeomorphisms \([56,68,69]\), see \( \S 4 \) below) and a little later Kolmogorov formulated the theorem that quasi-periodic solutions for a perturbed analytic Hamiltonian system not only existed but were relatively abundant in the sense that they formed a complicated Cantor type set of positive Lebesgue measure \([51]\). This theorem was proved completely in 1962 by Arnol’d \([3]\); independently Moser proved an analogous result for sufficiently smooth, area-preserving planar maps (‘twist’ maps) \([4,20]\), the degree of differentiability and the Diophantine conditions being relaxed substantially by Rüssmann \([65,66]\) (see also \([3,\S 6.3], [4, \text{Chap. 1}, [63]\)). It follows that for planets with mass very much less than the sun and for the majority of initial conditions in which the orbits are close to co-planar circles, distances between the bodies will remain perpetually bounded, \( i.e., \) the planets will never collide, escape or fall into the sun. Further details can be found in \([4,22]\).

As in \( \S 2 \), the Diophantine condition \([18]\) in the theorem below guarantees convergence of the Fourier series and of an infinite dimensional extension of Newton’s tangent method. The theorem is expressed in terms of general analytic Hamiltonians and has been taken from \([56]\), p. 44.
Theorem 4 (Kolmogorov-Arnol’d). Let $U$ be an open bounded subset of $\mathbb{R}^n$ and let the function $H(x,y,\mu)$ be a real analytic function of $x,y,\mu$ for all $x,y \in U$ and near $\mu = 0$. Moreover, $H$ is assumed to have period $2\pi$ in $x_1, \ldots, x_n$ and $H_0 = H(x,y,0)$ to be independent of $x$. Let $c \in \mathbb{R}^n$ be chosen so that the frequencies
$$\omega_k = \frac{\partial H_0(c)}{\partial x_k}, \quad 1 \leq k \leq n,$$
in $\omega = (\omega_1, \ldots, \omega_n)$ satisfy the Diophantine condition
$$|q \cdot \omega| = |q_1 \omega_1 + \cdots + q_n \omega_n| \geq K|q|_1^{-v},$$
where $|q|_1 = |q_1| + \cdots + |q_n|$, for some positive constants $K = K(\omega)$ and $v = v(\omega)$ for all non-zero $q = (q_1, \ldots, q_n) \in \mathbb{Z}^n$. Suppose further that the Hessian
$$\det \left( \frac{\partial^2 H_0(c)}{\partial x_j \partial x_k} \right) \neq 0.$$Then for sufficiently small $|\mu|$, there exists an invariant torus described by
$$(x,y) = (\theta + f(\theta,\mu), c + g(\theta,\mu)),$$where $\theta = (\theta_1, \ldots, \theta_n)$ have period $2\pi$ in each component and $f(\theta,\mu), g(\theta,\mu)$ are analytic functions of $\mu$ which vanish for $\mu = 0$. Moreover the flow on this torus is given by
$$\dot{\theta} = \omega.$$The exponent $v$ is subject to two conflicting requirements. It should be large enough ($v > n$) to ensure that the Diophantine condition above is not too restrictive but small enough to ensure that the perturbation has physical significance and that the stability is robust. The proof breaks down when the frequencies lie in the complementary exceptional set $E_v$, say of frequencies which are close to resonance in the sense that given any $C > 0$, there exists a $q \in \mathbb{Z}^n$ such that
$$|q \cdot x| < C|q|_1^{-v}.$$This set is closely related to the set
$$\mathcal{L}_v(\mathbb{R}^n) = \{ x \in \mathbb{R}^n : |q \cdot x| < |q|_1^{-v} \text{ for infinitely many } q \in \mathbb{Z}^n \}$$and in fact by an argument similar to that giving [11], for any $\varepsilon > 0$,
$$\mathcal{L}_{v+\varepsilon}(\mathbb{R}^n) \subset E_v \subset \mathcal{L}_v(\mathbb{R}^n)$$(see [12, §7.5.2]). This inclusion implies that the two sets $\mathcal{L}_v(\mathbb{R}^n)$ and $E_v$ have the same Hausdorff dimension. The set $\mathcal{L}_v(\mathbb{R}^n)$ is related to $K_v^{(n,1)}$ and
the Hausdorff dimension of \( \hat{L}_v(\mathbb{R}^n) \) is a special case of an analogue, proved by Dickinson [26], of the general form of the Jarník-Besicovitch theorem:

\[
\dim \hat{L}_v(\mathbb{R}^n) = \dim E_v = n - 1 + \frac{n}{v + 1}.
\]

(20)

when \( v > n - 1 \) (see also [32]); note that \( \hat{L}_v(\mathbb{R}^n) = \mathbb{R}^n \) otherwise.

The resonance sets \( R_q = \{ x \in \mathbb{R}^n : q \cdot x = 0 \} \) have dimension \( n - 1 \) and \( \hat{L}_v(\mathbb{R}^n) \) consists of families of hyperplanes through the origin with integer vector normals plus ‘fractal dust’. Another approach is via ‘averaging’ [3]; this can involve Diophantine approximation on manifolds [29].

4 Linearisation

The linearisation of a complex analytic diffeomorphism \( f : \mathbb{C}^n \to \mathbb{C}^n \) in a neighbourhood of a fixed point is related to the rotation number. The fixed point can be taken to be the origin and the linearisation means that near the origin \( f \) is analytically conjugate to its linear part (Jacobian) \( Df|_0 = A \) say. The linearising transformation \( \phi \) is given by the solution to the functional equation

\[
f \circ \phi = \phi \circ A
\]

and when \( n = 1 \) is known as Schröder’s equation. Linearisation can be regarded as obtaining a normal form and is analogous to diagonalising a matrix. Problems similar to those discussed above arise when the eigenvalues \( \alpha_1, \ldots, \alpha_n \) of \( A \) are close to being resonant in the sense that they are close to satisfying the equation

\[
\alpha_k = \prod_{r=1}^{n} \alpha_r^{j_r}
\]

for some \( k \) and all \( j = (j_1, \ldots, j_n) \) with \( |j|_1 \geq 2 \) and \( j_r \geq 0, r = 1, \ldots, n \). Linearisation is well understood when \( n = 1 \) and the diffeomorphism \( f : \mathbb{C} \to \mathbb{C} \) with \( f(0) = 0 \) can be linearised when \( |(Df)|_0 = |f'(0)| \neq 1 \). The interesting case when \( |f'(0)| = 1 \) is closely related (via lifts) to the conjugacy of a circle map to a rotation, discussed in §2 above, and necessary and sufficient conditions for the linearisation of a diffeomorphism \( f : \mathbb{C} \to \mathbb{C} \) in terms of Bruno numbers are known [3, 28, 73]. Less is known when \( n \geq 2 \) but Siegel established sufficient conditions on \( Df|_0 \) which guarantee the existence of a linearising transformation \( \phi \). By analogy with the terminology Diophantine type, the point \( (\alpha_1, \ldots, \alpha_n) \) in \( \mathbb{C}^n \) is said to be of multiplicative type \((K, v)\) [3, p. 191] if

\[
|\alpha_k - \prod_{r=1}^{n} \alpha_r^{j_r}| \geq K|j|_1^{-v}
\]

(21)
for all $j \in (\mathbb{N} \cup \{0\})^n$ with $|j|_1 \geq 2$ and all $k = 1, \ldots, n$. Siegel showed that if the vector $(\alpha_1, \ldots, \alpha_n)$ of eigenvalues of $Df|_0$ is of multiplicative type $(K, v)$ for some $K, v > 0$, then $f$ can be linearised locally \[68,69\]. Siegel’s condition is not seriously restrictive when $v > (n - 1)/2$ as then for any $K > 0$ the set of points of multiplicative type $(K, v)$ has full measure. However, the size of the neighbourhood of linearisation depends on $v$ (it also depends on $K$ but less significantly) and so it is desirable not to make $v$ too large.

Given an exponent $v$, the complementary set of points $\mathcal{E}_v$ say in $\mathbb{C}^n$ (regarded as $\mathbb{R}^{2n}$) consists of points which fail to be of multiplicative type $(K, v)$ for any $K > 0$ and so fail to satisfy the conditions of Siegel’s theorem. The multiplicative Diophantine condition (21) is hard to handle and the question is simplified by means of the map

$$(z_1, \ldots, z_n) \mapsto (e^{2\pi i z_1}, \ldots, e^{2\pi i z_n})$$

which preserves the Hausdorff dimension and reduces the problem to sets with a simpler structure \[31\]. Multiplicative type is replaced by the more amenable notion of a point $z = x + iy$ in $\mathbb{C}^n$ being of mixed additive type $(K, v)$, i.e.,

$$\max \left( |x_k - \sum_{r=1}^n j_r x_r|, \|y_k - \sum_{r=1}^n j_r y_r\| \right) \geq K|j|_1^{-v}$$

for each $j \in (\mathbb{N} \cup \{0\})^n$ with $|j|_1 \geq 2$ and each $k = 1, \ldots, n$.

Given $v > 0$, the complementary set of points which are not of mixed additive type for any $K > 0$ is null. It is closely related to and has the same Hausdorff dimension as the simpler set $F_v$ say of points $(x, y) \in \mathbb{R}^{2n}$ such that

$$\max \{|j \cdot x|, \|j \cdot y\|\} < |j|^{-v}$$

for infinitely many $j \in \mathbb{Z}^n$. This set is roughly speaking made up of a ‘distance from 0’ part and a ‘distance from $\mathbb{Z}$’ part. The resonant sets are a system of cartesian products of a hyperplane normal to $q$ through the origin in $\mathbb{R}^n$ and a family of parallel hyperplanes. The lower bound for the dimension is obtained by showing that the resonant sets form a ubiquitous system, the upper by using the natural limsup cover for $F_v$ \[30,31\]. If $v \geq (n - 1)/2$, then $F_v$ and $\mathcal{E}_v$ are null with Hausdorff dimension

$$\dim F_v = \dim \mathcal{E}_v = 2(n - 1) + \frac{n + 1}{v + 1}. \quad (22)$$

Mixed additive type can also be used in the analysis of linearising periodic differential equations or, in other words, for finding the normal form of a vector field on $\mathbb{C}^n \times S^1$ near a singular point. The periodic differential equation

$$\dot{z} = Bz + Q(z, t),$$

Exceptional sets where $B$ is an $n \times n$ complex matrix and $Q : \mathbb{C}^n \times S^1 \to \mathbb{C}^n$ is analytic and has period $2\pi$ in $t$ and satisfies

$$Q(0, t) = 0, \quad \frac{\partial Q_j(0, t)}{\partial z_k} = 0 \quad \text{for} \quad 1 \leq j, k \leq n,$$

can be linearised to the form $\dot{\zeta} = B\zeta$ in a neighbourhood of 0 if the real and imaginary parts of the eigenvalues of $B$ form a vector of mixed additive type (see [29, Theorem B], [69]). For each $v > 0$, the exceptional set of such vectors has the same Hausdorff dimension as $\mathcal{E}_v$ in (22). Similar results hold for autonomous differential equations.

5 Diophantine approximation in hyperbolic space

Diophantine approximation has a fertile interpretation in hyperbolic space in which many results can be interpreted dynamically, for example geodesic orbits on manifolds play an important role. The action of a discrete subgroup of the group of orientation preserving Möbius transformations of the upper half plane to itself allows a fertile generalisation of the classical theory of Diophantine approximation. The rationals $\mathbb{Q}$ can be interpreted as the orbit of the point at infinity under the linear fractional or Möbius transformation of the extended complex plane $\mathbb{C} \cup \{\infty\}$ in which

$$z \mapsto \frac{az + b}{cz + d}, \quad a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1,$$

i.e., as the orbit of $\infty$ under the modular group $SL(2, \mathbb{Z})$ acting on points in the upper half plane. These observations allow a very nice translation of classical Diophantine approximation into the hyperbolic space setting by considering the action of a Kleinian group $G$ and the (Euclidean) distance of the orbit of a special point in the limit set $\Lambda(G)$ from the other points in the set. Here $\Lambda(G)$ is the set of accumulation points of the orbit $G(x)$ of the point $x$ in hyperbolic space and turns out to be independent of $x$. Details of the basic properties of Kleinian groups acting on hyperbolic space can be found in [2,10,58]. From now on, the groups considered will be non-elementary and geometrically finite. Approximation of real numbers by rationals is replaced by approximating points in $\Lambda(G)$ by points in the orbit of the special point $p$ which is either a parabolic fixed point of $G$ when $G$ has them or a hyperbolic fixed point otherwise. More precisely, when the unit ball $B^{n+1}$ endowed with the hyperbolic metric $\rho$ (given by $d\rho = |dx|_2/(1 - |x|^2)$) is the model for hyperbolic space, given a point $\xi \in \Lambda(G)$, we consider the quantity

$$|\xi - g(p)|_2 < \lambda_g^{-v} \quad (23)$$

as $g$ runs through $G$. The ‘denominator’ in this setting is the reciprocal of the modulus of the Jacobian $g'(0)$ of each $g \in G$ at 0,

$$\lambda_g = |g'(0)|^{-1}$$
and is comparable to $e^{\rho(g(0))}$ (i.e., $\lambda_g e^{-\rho(g(0))}$ is bounded above and below by positive constants). In the upper half plane model $\lambda_g = |g'(i)|^{-1}$. Thus $\lambda_g \to \infty$ as $|g(0)| \to 1$, i.e., as the orbit of the origin moves out towards the boundary of hyperbolic space \( \mathbb{H} \).

It turns out that not only is $\lambda_g$ an appropriate analogue for the modulus of the denominator of a rational $p/q$ in the classical setting but also properties of the Dirichlet series $\sum_{g \in G} \lambda_g^{-s}$ are connected with the Hausdorff dimension of $\Lambda(G)$. In fact the \textit{exponent of convergence}

$$\delta(G) = \inf\{s > 0 : \sum_{g \in G} \lambda_g^{-s} < \infty\}$$

of $G$ is equal to $\dim A(G)$ \[4,5\]. The appropriateness of $\lambda_g$ as a ‘denominator’ of $g$ can be seen clearly by considering the case $G = \text{SL}(2, \mathbb{Z})$ and $p = \infty$. Each group element $g$ satisfies $g(\infty) = p/q$ for some $p/q \in \mathbb{Q}$ and a straightforward calculation yields that $\lambda_g$ is comparable to $q^2$. By analogy with the classical case, a point $\xi \in A(G)$ is called $v$-\textit{approximable} if the inequality

$$|\xi - g(p)|_2 < \lambda_g^{-v}$$

holds for infinitely many $g \in G$.

When $G$ has parabolic fixed points (the most interesting case), Dirichlet’s theorem has the following analogue: There exists a constant $C > 0$ depending only on $G$, such that for each $\xi \in A(G)$ and each integer $N > 1$,

$$|\xi - g(p)|_2 < C/\sqrt{\lambda_g N} \leq C/\lambda_g$$

for some parabolic $p$ and $g \in G$ with $\lambda_g \leq N$ \[4,5\]. The analogy can be pursued further, in the first place to Khintchine’s theorem \[4,5\]. We suppose $G$ has a parabolic element $p$ of rank $r_p$. Let $\mu$ be Patterson measure \[11\] and $\psi : [1/2, \infty) \to \mathbb{R}^+$ be decreasing and satisfy $\psi(2x) > c\psi(x)$ for some constant $c > 0$. Then the set $W(G, p; \psi)$ of points $\xi$ satisfying

$$|\xi - g(p)|_2 < \psi(\lambda_g)$$

infinitely often has Patterson measure 0 or 1 according as the sum

$$\sum_{k=1}^{\infty} \psi(K^k)^{2\delta(G)-r_p}$$

converges or diverges for some $K > 1$ (when $G$ has no parabolic elements, the sum is simpler).

The set $W_v(\text{SL}(2, \mathbb{Z}), \infty)$ of $v$-approximable points in the limit set corresponds to $K_v$, the set of $v$-approximable points in $\mathbb{R}$, although the exponent $v$ is normalised differently. For any $\varepsilon > 0$,

$$W_v(\text{SL}(2, \mathbb{Z}), \infty) \subset K_{2v+1} \subset W_{v-\varepsilon}(\text{SL}(2, \mathbb{Z}), \infty),$$

so that $\dim W_v(\text{SL}(2, \mathbb{Z}), \infty) = 1/(v+1)$ by the Jarník-Besicovitch theorem. Moreover a more general counterpart of the theorem holds \[12\].
Theorem 5. Let $G$ be a non-elementary geometrically finite group and suppose $v \geq 0$. If $G$ has a parabolic element $p$ say, then

$$\dim W_v(G, p) = \min \left\{ \frac{\delta(G) + r_p}{2v + 1}, \frac{\delta(G)}{v + 1} \right\},$$

where $r_p$ is the rank of $p$. Otherwise $G$ is convex co-compact and

$$\dim W_v(G, p) = \frac{\delta(G)}{v + 1}.$$ 

This setting provides a beautiful dynamical interpretation of approximation. When $G$ is geometrically finite, the quotient space $M = \mathbb{H}^{n+1}/G$ obtained by identifying equivalent points in $\mathbb{H}^{n+1}$ under the action of $G$ is an $n$-dimensional manifold. When $G$ is a Fuchsian group, the manifold is a Riemann surface of constant negative curvature, when $n = 1$ and $G = \text{SL}(2, \mathbb{Z}), \mathbb{H}^2/\text{SL}(2, \mathbb{Z})$, where $\mathbb{H}^2$ is the upper half plane, is the modular surface. The manifold $M$ can be decomposed into a disjoint union of a compact part with a finite number of exponentially ‘narrowing’ cuspidal ends (corresponding to a set of inequivalent parabolic fixed points of $G$) and ‘exploding’ ends or funnels (corresponding to the free faces of a convex fundamental polyhedron for $G$). The set of points $\xi$ in $A(G)$ for which there exists a constant $c(\xi)$ such that

$$|\xi - g(p)| \geq c(\xi)/\lambda_g$$

holds for all $g \in G$ is the set of badly approximable points, denoted by $\mathcal{B}(G, p)$. When $p$ is a parabolic fixed point of $G$, the set $\mathcal{B}(G, p)$ corresponds to ‘bounded’ geodesics on the manifold; the dimension results for $\mathcal{B}(G, p)$ imply that $M$ is of full Hausdorff dimension $\delta(G)$ [15,36]. On the other hand, the set $W(G, p; \psi)$ corresponds to excursions by geodesics into the cuspidal ends of the manifold (the rate being governed by the function $\psi$); i.e., to ‘divergent’ geodesics which persistently enter a ‘shrinking’ neighbourhood of the cusp associated with $p$. The work of Dani and and Margulis on the structure of bounded and divergent trajectories in homogeneous spaces has had a profound influence on Diophantine approximation [18,19,20,21,22,23,24,53].

The notion of a shrinking target introduced in [40] arose from a general consideration of Diophantine approximation and has proved fruitful; it is exploited in [13] and [41]. Let $X$ be a metric space with a Borel probability measure $\mu$ and $T: X \to X$ measure preserving and ergodic. Then given the function $\varrho: \mathbb{R}^+ \to \mathbb{R}^+$, the set

$$W(a, \varrho) = \{x \in X: T^q(x) \in B(a, \varrho(q)) \text{ for infinitely many } q \in \mathbb{N} \},$$

where $B(a, \varepsilon)$ is a ball of radius $\varepsilon$ in $X$, is an analogue of the sets $W(G, p; \psi)$ and $\mathcal{K}^{1, n}(\psi)$. By the Birkhoff ergodic theorem [15], if $\varrho(q) = \varrho_0$, a constant, for $q$ sufficiently large and $\mu(B(a, \varrho)) > 0$, the set

$$\{x \in X: T^q(x) \in B(a, \varrho_0) \text{ for infinitely many } q \in \mathbb{N} \}$$
has full $\mu$-measure, in other words the forward orbit of almost all points falls into the ball infinitely often. It follows that the complement of $W(a, g_0)$ in $X$, consisting of points whose forward orbits land in the ball only a finite number of times, is of zero $\mu$-measure. This set corresponds to the set of badly approximable points in a dynamical system which has been studied in \([23,76]\).

On the other hand, when $g(q) \to 0$ as $q \to \infty$, points in $W(a, g)$ have trajectories which hit a shrinking ball or target infinitely often and are called $g$-approximable. Points in the backward orbit of $a$ in $X$ are resonant points corresponding to the rationals in $K_v^{(1,n)}$ and orbit points in $W(G, p; \psi)$. In this framework, the theory of Diophantine approximation in the hyperbolic space setting can be used with ideas from ergodic theory to analyse the structure of a variety of apparently quite unrelated sets in complex dynamics and other dynamical systems. For further details see \([41]\).

Expanding Markov maps $T$ of the unit interval (such as the continued fraction map) have been analysed along these lines \([1]\) and the theory has been extended to higher dimensional tori and to maps which are multiplication by integer matrices modulo $\mathbb{Z}^n$ \([43]\). The very well approximable sets associated with a given dynamical system have unexpected links with exceptional sets arising from points in the phase space which have ‘badly behaved’ ergodic averages and with multifractal spectra \([35,41]\).

### 6 Extremal manifolds and flows

So far, we have been looking at sets which arise in dynamical systems through Diophantine conditions and using number theory to show that these sets are exceptional (i.e., null) and determining their Hausdorff dimension. However, the tables have been turned with dynamical systems ideas being used to great effect, first to prove Oppenheim’s conjecture \([53]\) and more recently in Diophantine approximation, including the important topic of approximation on manifolds \([58]\). The latter stemmed from Mahler’s conjecture work in transcendence theory during the 1930’s. He conjectured that the set of points $(x, x^2, \ldots, x^n)$ on the Veronese curve which are $v$-simultaneously approximable is relatively null for $v > 1/n$, i.e.,

$$|\{x \in \mathbb{R} : \max_{j=1,\ldots,n} \{\|qx^j\|\} < q^{-v} \text{ for infinitely many } q \in \mathbb{N}\}| = 0$$

when $v > 1/n$. This property, which is a special case of Khintchine’s theorem and says that the exponent in Dirichlet’s theorem on simultaneous Diophantine approximation cannot be improved for almost all points, is called extremality. By Khintchine’s transference principle, the set of simultaneously VWA points is also dually VWA, in the sense that the set of points $\xi$ satisfying

$$|g_0 + g_1\xi + \cdots + g_n\xi^n| < |q|^{-v}$$
for infinitely many \( q \in \mathbb{Z}^n \) is null for \( v > n \). Mahler’s conjecture was proved by Sprindžuk \[71\] who then conjectured that non-degenerate analytic manifolds were extremal. Extremality was later strengthened to a stronger ‘multiplicative’ form by Baker \[8\] and Sprindžuk \[73\] in which given any \( v > 1 \), the inequality
\[
\|q \cdot y\| \leq \prod_{j=1}^{n} \max\{1, |q_j|\}^{-v}
\]
holds for infinitely many \( q \in \mathbb{N} \) for almost all points \( y = (y_1, \ldots, y_n) \in M \). Such points are called very well multiplicatively approximable or VWMA. The ‘strong extremality’ conjecture was proved by Kleinbock and Margulis \[50\] for smooth manifolds which are non-degenerate (or ‘non-flat’ locally) almost everywhere. Their proof uses actions on the lattice
\[
A_y = \begin{pmatrix} 1 & y^T \\ 0 & I_n \end{pmatrix} \mathbb{Z}^{n+1},
\]
where \( y \in \mathbb{R}^n \), in the homogeneous space \( SL_{n+1}(\mathbb{R})/SL_{n+1}(\mathbb{Z}) \) of unimodular lattices by the semigroup \( \{g_t : t \in \mathbb{N}^n\} \), where
\[
g_t = \text{diag}(e^{\sum_{i=1}^{n} t_i}, e^{-t_1}, \ldots, e^{-t_n}).
\]
Elements in \( A_y \) are of the form \( (q_0 + q \cdot y, q) \), \( q \in \mathbb{Z}^n, q_0 \in \mathbb{Z} \) and the distance of the lattice \( A \) from the origin is
\[
\delta(A) = \inf_{a \in A \setminus 0} |a|
\]
(recall \( |a| \) is the height of \( a \)). By Mahler’s Compactness Criterion, an infinite sequence \( A^{(k)}, k = 1, 2, \ldots \), diverges to infinity in \( SL_{n+1}(\mathbb{R})/SL_{n+1}(\mathbb{Z}) \) if and only if \( \delta(A^{(k)}) \to 0 \) as \( k \to \infty \). It can be shown that if a point \( y \) is very well multiplicatively approximable, then there exists a \( \gamma > 0 \) such that
\[
\delta(g_t A_y) \leq e^{-\gamma |t|_1}
\]
for infinitely many \( t \in \mathbb{N}^n \). By the Borel-Cantelli Lemma, the set of VWMA points on the manifold is exceptional (\( i.e. \), the manifold is strongly extremal) if the sum
\[
\sum_{t \in \mathbb{N}^n} |\{x \in B : \delta(g_t A_{f(x)}) \leq e^{-\gamma |t|_1} \}| < \infty,
\]
where \( B \) is a neighbourhood of a non-degenerate point \( x_0 \) and \( y = f(x) \), where \( f \) is the local parametrisation function. This is established by modifying the arguments of Dani and Margulis \[21\].

7 Conclusion

Exceptional sets in dynamical systems can arise from Diophantine conditions and can be analysed using number theory. The number theory has in turn
served as a model for an analogous theory in groups actions in hyperbolic space. The interpretation of the orbit structure in terms of Diophantine approximation has been very fruitful, leading to a general notion of shrinking targets and in another dramatic development to the recent solution of the Baker-Sprindžuk conjecture and the proof of multiplicative Khintchine-type theorems for $C^n$ manifolds in $\mathbb{R}^n$ [13]. However, Beresnevich’s proof [11], which uses ideas based on those of Sprindžuk, of a Khintchine-type theorem in the case of convergence for manifolds requires weaker differentiability conditions, showing that classical methods remain effective. The complementary case of divergence is much more difficult but progress is being made.

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References

1. A. G. Abercrombie and R. Nair, An exceptional set in the ergodic theory of Markov maps on the interval, Proc. Lond. Math. Soc. 75 (1997), 221–240.
2. L. V. Ahlfors, Möbius transformations in several dimensions, Lecture Notes, School of Mathematics, University of Minnesota, 1954.
3. V. I. Arnol’d, Small denominators and problems of stability of motion in classical and celestial mechanics, Usp. Mat. Nauk 18 (1963), 91–192, English transl. in Russian Math. Surveys, 18 (1963), 85–191.
4. _____, Mathematical methods of classical mechanics, Springer-Verlag, 1978, Translated by K. Vogtmann and A. Weinstein.
5. _____, Geometrical methods in ordinary differential equations, Springer-Verlag, 1983, Translated by J. Szücs.
6. V. I. Arnol’d, V. V. Kozlov, and A. I. Neishtadt, Mathematical aspects of classical and celestial mechanics, Encyclopaedia of Mathematical Sciences, vol. 3, Dynamical Systems III, Springer-Verlag, 1980, Translated by A. Jacob.
7. D. K. Arrowsmith and C. M. Place, An introduction to dynamical systems, Cambridge University Press, 1990.
8. A. Baker, Transcendental number theory, second ed., Cambridge University Press, 1979.
9. A. Baker and W. M. Schmidt, Diophantine approximation and Hausdorff dimension, Proc. Lond. Math. Soc. 21 (1970), 1–11.
10. A. F. Beardon, The geometry of discrete groups, Springer-Verlag, 1983.
11. V. V. Beresnevich, A Groshev type theorem for convergence on manifolds, Acta Math. Hung. (2001), to appear.
12. V. I. Bernik and M. M. Dodson, *Metric Diophantine approximation on manifolds*, Cambridge Tracts in Mathematics, No. 137, Cambridge University Press, 1999.

13. V. I. Bernik, D. Y. Kleinbock, and G. A. Margulis, *Khintchine-type theorems for manifolds: the convergence case for standard and multiplicative versions*, submitted to Inter. Math. Res. Notices.

14. A. S. Besicovitch, *Sets of fractional dimensions (IV): on rational approximation to real numbers*, J. Lond. Math. Soc. 9 (1934), 126–131.

15. C. J. Bishop and P. W. Jones, *Hausdorff dimension and Kleinian groups*, Acta Math. 111 (1997), 1–39.

16. J. D. Bovey and M. M. Dodson, *The Hausdorff dimension of systems of linear forms*, Acta Arith. 45 (1986), 337–358.

17. J. W. S. Cassels, *An introduction to Diophantine approximation*, Cambridge Tracts in Math. and Math. Phys., No. 45, Cambridge University Press, 1957.

18. S. G. Dani, *On orbits of unipotent flows on homogeneous spaces*, Ergod. Th. Dyn. Sys. 4 (1984), 25–34.

19. , *Divergent trajectories of flows on homogeneous spaces and homogeneous Diophantine approximation*, J. reine angew. Math. 359 (1985), 55–89.

20. , *Bounded orbits of flows on homogeneous spaces*, Comm. Math. Helv. 61 (1986), 636–660.

21. , *On orbits of unipotent flows on homogeneous spaces, II*, Ergod. Th. Dyn. Sys. 6 (1986), 167–182.

22. , *Orbits of horospherical flows*, Duke Math. J. 53 (1986), 177–188.

23. , *On badly approximable numbers, Schmidt games and bounded orbits of flows*, Number theory and dynamical systems (M. M. Dodson and J. A. G. Vickers, eds.), LMS Lecture Note Series, vol. 134, Cambridge University Press, 1987, pp. 69–86.

24. S. G. Dani and G. A. Margulis, *Limit distributions of orbits of unipotent flows and values of quadratic forms*, Adv. Soviet Math. 16, Amer. Math. Soc., RI 16 (1993), 91–137.

25. R. de la Llave, *A tutorial on KAM theory*, Smooth Ergodic Theory and Applications, Proceedings of Symposia in Pure Mathematics, Summer Research Institute, Seattle WA, Amer. Math. Soc., 1999, to appear 2001.

26. H. Dickinson, *The Hausdorff dimension of systems of simultaneously small linear forms*, Mathematika 40 (1993), 367–374.

27. H. Dickinson, T. Gramchev, and M. Yoshino, *First order pseudodifferential operators on the torus: normal forms, Diophantine approximation and global hypoellipticity*, Ann. Univ. Ferrara, Sez. VII – Sc. Mat. 61 (1995), 51–64.

28. M. M. Dodson, *Hausdorff dimension, lower order and Khintchine’s theorem in metric Diophantine approximation*, J. reine angew. Math. 432 (1992), 69–76.

29. M. M. Dodson, B. P. Rynne, and J. A. G. Vickers, *Averaging in multi-frequency systems*, Nonlinearity 2 (1989), 137–148.

30. , *Diophantine approximation and a lower bound for Hausdorff dimension*, Mathematika 37 (1990), 59–73.

31. , *The Hausdorff dimension of exceptional sets associated with normal forms*, J. Lond. Math. Soc. 49 (1994), 614–624.

32. M. M. Dodson and J. A. G. Vickers, *Exceptional sets in Kolmogorov-Arnol’d-Moser theory*, J. Phys. A 19 (1986), 349–374.
33. M. M. Dodson and J. A. G. Vickers (eds.), *Number theory and dynamical systems*, Lond. Mathematical Society Lecture Note Series, vol. 134, Cambridge University Press, 1989.
34. K. Falconer, *The geometry of fractal sets*, Cambridge Tracts in Mathematics, No. 85, Cambridge University Press, 1985.
35. , *Representation of families of sets, dimension spectra and Diophantine approximation*, Math. Proc. Camb. Philos. Soc. 128 (2000), 111–121.
36. J.-L. Fernandez and M. V. Melián, *Bounded geodesics of Riemann surfaces and hyperbolic manifolds*, Trans. Amer. Math. Soc. 9 (1995), 3533–3549.
37. G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, 4th ed., Clarendon Press, 1960.
38. G. Harman, *Metric number theory*, LMS Monographs New Series, vol. 18, Clarendon Press, 1998.
39. M. R. Herman, *Recent results and some open questions on Siegel’s linearisation theorem on germs of complex analytic diffeomorphisms of $\mathbb{C}^n$ near a fixed point*, Proceedings of the VIIIth International Congress of Mathematical Physics, Marseilles 1986 (M. Mebkhout and R. Seneor, eds.), World Scientific, 1987, pp. 138–184.
40. R. Hill and S. L. Velani, *Ergodic theory of shrinking targets*, Invent. Math. 119 (1995), 175–198.
41. , *Metric Diophantine approximation in Julia sets of expanding rational maps*, Publ. Math. I.H.E.S. (1997), no. 85, 193–216.
42. , *The Jarník-Besicovitch theorem for geometrically finite Kleinian groups*, Proc. Lond. Math. Soc. 77 (1998), 524–550.
43. , *The shrinking target problem for matrix transformations of tori*, J. Lond. Math. Soc. 60 (1999), 381–398.
44. V. Jarník, *Zur metrischen Theorie der diophantischen Approximationen*, Prace Mat.-Fiz. (1928-9), 91–106.
45. , *Diophantischen Approximationen und Hausdorffsches Mass*, Mat. Sbornik 36 (1929), 371–382.
46. , *Über die simultanen diophantischen Approximationen*, Math. Z. 33 (1931), 503–543.
47. A Katok and B. Hasselblatt, *Introduction to the modern theory of dynamical systems*, Cambridge University Press, 1995.
48. D. Y. Kleinbock, *Applications of ergodic theory to metric Diophantine approximation*, Smooth Ergodic Theory and Applications, Proceedings of Symposia in Pure Mathematics, Summer Research Institute, Seattle WA, American Mathematical Society, 1999, to appear 2001.
49. , *Badly approximable systems of affine forms*, J. Number Th. 79 (1999), 83–102.
50. D. Y. Kleinbock and G. A. Margulis, *Flows on homogeneous spaces and Diophantine approximation on manifolds*, Ann. Math. 148 (1998), 339–360.
51. A. N. Kolmogorov, *On the conservation of conditionally periodic motions under small perturbations of the Hamiltonian*, Dokl. Akad. Nauk SSSR 98 (1954), 527–530, in Russian.
52. J. Levesley, *A general inhomogeneous Jarník-Besicovitch theorem*, J. Number Th. 71 (1998), 65–80.
53. G. A. Margulis, *Formes quadratiques indéfinies et flats unipotents sur l’espaces homogènes*, C. R. Acad. Sci., Paris, Ser. I 304 (1987), 249–253.
54. P. Mattila, *Geometry of sets and measures in Euclidean space*, Cambridge University Press, 1995.
55. J. Moser, *On invariant curves of area-preserving maps of an annulus*, Nachr. Akad. Wiss. Gött., Math. Phys. Kl. (1962), 1–20.
56. , *Stable and random motions in dynamical systems*, Princeton University Press, 1973.
57. , *Is the solar system stable?*, Math. Intelligencer 1 (1978), 65–71.
58. P. J. Nicholls, *The ergodic theory of discrete groups*, LMS Lecture Notes, vol. 143. Cambridge University Press, 1989.
59. Z. Nitecki, *Differentiable dynamics*, MIT Press, 1971.
60. S. J. Patterson, *Diophantine approximation in Fuchsian groups*, Phil. Trans. Roy. Soc. Lond. A 262 (1976), 527–563.
61. , *The limit set of a Fuchsian group*, Acta Math. 136 (1976), 241–273.
62. Ya. B. Pesin, *Dimension theory in dynamical systems. Contemporary views and applications*, Chicago Lectures in Math., University of Chicago Press, 1997.
63. J. Pöschel, *Integrability of Hamiltonian systems on Cantor sets*, Comm. Pure Appl. Math. 35 (1982), 653–696.
64. C. A. Rogers, *Hausdorff measure*, Cambridge University Press, 1970.
65. H. R. Rüssmann, *On the existence of invariant curves of twist mappings of the annulus*, Geometric Dynamics, Lecture Notes in Mathematics, vol. 1007, Springer-Verlag, 1983, pp. 677–712.
66. , *On the frequencies of quasi-periodic solutions of analytic nearly integrable Hamiltonian systems*, Progress in Nonlinear Differential Equations and their applications, 12 (V. Lazutkin S. Kuksin and J. Pöschel, eds.), vol. 12, Birkhäuser Verlag, Basel, 1994, pp. 51–58.
67. W. M. Schmidt, *Badly approximable systems of linear forms*, J. Number Th. 1 (1969), 139–154.
68. C. L. Siegel, *Iteration of analytic functions*, Ann. Math. 43 (1942), 607–612.
69. , *Über die Normalform analytischer Differentialgleichungen in der Nähe einer Gleichgewichtslösung*, Nachr. Akad. Wiss. Gött. Math-Phys. Kl (1952), 21–30.
70. C. L. Siegel and J. K. Moser, *Lectures on celestial mechanics*, Springer-Verlag, 1971.
71. V. G. Sprindžuk, *A proof of Mahler’s conjecture on the measure of the set of S-numbers*, Amer. Math. Soc. Transl. Ser. 2 51 (1966), 215–272.
72. , *Metric theory of Diophantine approximations*, John Wiley, 1979, Translated by R. A. Silverman.
73. , *Achievements and problems in Diophantine approximation theory*, Usp. Mat. Nauk 35 (1980), 3–68, English transl. in Russian Math. Surveys, 35 (1980), 1–80.
74. B. Stratmann and S. L. Velani, *The Patterson measure for geometrically finite groups with parabolic elements, new and old*, Proc. Lond. Math. Soc. 71 (1995), 197–220.
75. D. Sullivan, *Entropy, Hausdorff measures old and new, and the limit set of geometrically finite Kleinian groups*, Acta Math. 153 (1984), 259–277.
76. M. Urbánski, *The Hausdorff dimension of the set of points with non-dense orbit under a hyperbolic dynamical system*, Nonlinearity 4 (1991), 385–397.
77. S. L. Velani, *An application of metric Diophantine approximation in hyperbolic space to quadratic forms*, Publ. Math. 38 (1994), 175–185.
78. J. C. Yoccoz, *An introduction to small divisors problems*, From number theory to physics, Springer-Verlag, 1992, Les Houches, 1989, pp. 659–679.
79. ———, *Petits diviseurs en dimension 1*, Astérisque 231 (1995).