Abstract

The two-dimensional Green-Naghdi equations with uneven bottom topography are studied in this paper. The function defining the bottom topography can be dependent on time. Group classification of these equations with respect to the function describing the topography of the bottom is performed in the paper. The algebraic approach used for the analysis of the classifying equations.

Keywords: Equivalence group, admitted Lie group, Green-Naghdi equations

Subject Classification (MSC 2010): 35C99, 76W05

1. Introduction

Mathematical modeling of physical phenomena is one of the main streams in continuum mechanics. Phenomena such as hydraulic currents, coastal currents, currents in rivers and lakes, currents in water intakes, tsunami simulation, breakout wave propagation, distribution of heavy gases and scales of atmospheric movements used in weather forecasting require mathematical consideration.

Motion of an ideal fluid flow under the force of gravity can be modeled by means of the Euler equations. However, the full Euler equations, even under the assumption of incompressibility, barotropy and absence of rotation, are still rather complicated for describing waves on a surface. One of these difficulties is that the free surface is a part of the solution. This difficulty has motivated scientists to derive simpler equations. For this reason development of approximate models and their analysis by analytical and numerical methods is an actual problem.

The need to reduce the original equations to simpler equations led to the construction of asymptotic expansion models with respect to a small parameter determined by the ratio of the depth of the fluid to the characteristic linear size. One class of such equations is the class of shallow water equations.

The shallow water equations describe the motion of an incompressible fluid in the gravitational field if the depth of the fluid layer is sufficiently small. They are widely used in the description of processes in the atmosphere, water basins, modeling of tidal oscillations, tsunami waves and gravitational waves (e.g., see the classical books such as [1, 2, 3, 4] and [5, 6]).

There are many approaches for deriving shallow water models, a review of which can be found in [7, 8]. The classical approach of deriving the shallow water equations consists of approximating the Euler equations for irrotational flows. The hierarchy of the shallow water
approximations is considered with respect to the shallowness parameter $\delta = h_0/L$, where $h_0$ is the mean depth of the fluid, and $L$ is the typical length scale of the wave. In particular, the Green-Naghdi equations, derived for describing the two-dimensional fluid flow over an uneven bottom, are accurate up to the dispersive terms of order $\delta^2$. The Green-Naghdi system of equations is the generalization of the equations derived first by Serre and later by Su and Garden to describe the one-dimensional propagation of fully nonlinear and weakly dispersive surface gravity waves over a flat bottom topography.

In the classical Green-Naghdi equations, there are no equivalence transformations defined by Galilean invariance. To overcome this obstacle, the Green-Naghdi equations with bottom topography depending on time were derived in [12]. For deriving these equations the authors used Matsuno’s approach. It should be noted that the equations derived in this manner coincide with the equations obtained by a different approach. The dependence on time can take into account the bottom motion during, for example, an underwater earthquake or moving object located on the bottom. Some experimental and theoretical results for two specific time deformations of the bottom are presented in [13]. The authors of [14] considered motion of the bottom defined by the formula $\zeta(x, y)T(t)$. Using the Fourier transform with respect to $(x, y)$ and the Laplace transform in $t$ of the linearized problem, analytical formulas for the free-surface elevation was derived there.

It is well-known that symmetries of a mathematical model are intrinsic properties inherited from physical phenomena. One of the tools for studying symmetries is the Lie group analysis method [15, 16], which is a basic method for constructing exact solutions of ordinary and partial differential equations. Even in the case of the one-dimensional shallow water equations for flat bottom, one encounters certain difficulties to obtain nontrivial exact solutions. Symmetries of differential equations yield a guaranteed source of group-invariant solutions. Applications of Lie groups to differential equations is the subject of many books and review articles [15, 16, 17, 18, 19, 20].

Applications of the group analysis method to the Green-Naghdi equations with a horizontal bottom topography in Eulerian and Lagrangian coordinates were studied in [21, 22]. In [22, 23], Noether’s theorem is applied for finding conservation laws of the one-dimensional classical Green-Naghdi equations with horizontal and uneven bottom topography, respectively.

Group properties of the one-dimensional Green-Naghdi equations with uneven bottom depending on time were studied in [12]. One of the main focuses of [12] was the representation of one-dimensional equations in a variational form. For this purpose, the equations were rewritten in mass Lagrangian coordinates. The variational form allowed using Noether’s theorem for constructing conservation laws. Because the first step of application of Noether’s theorem is the group analysis of the Euler-Lagrange equation, the complete group classification of the studied equations was performed.

The present paper is devoted to the group classification of the two-dimensional Green-Naghdi equations with respect to the bottom topography depending on time. The equations are considered in Eulerian coordinates. For the group classification we use the algebraic approach applied earlier to different types of systems (see, e.g., [24, 25, 26, 27, 28] and references therein). The algebraic approach takes the algebraic properties of an admitted Lie algebra into account and allows one a significant simplification of the group classification. It was applied in the case

---

1See also the references therein.
2Private communication with S.L. Gavrilyuk.
if the admitted generators belong to subalgebras of some Lie algebra.

The paper is organized as follows. In Section 2 the Green-Naghdi equations with uneven bottom depending on time are given. Equivalence transformations are presented in Section 3. Section 4 is devoted to the group classification of the studied equations. The obtained results are discussed in the Conclusions.

2. Studied equations

In the dimensional form, the two-dimensional Green-Naghdi equations with a bottom topography depending on time have the form

\[
\frac{\partial h}{\partial t} + \nabla (hu) = 0, \\
\frac{d}{dt} u + g\nabla (h + H) = \frac{1}{h} (\nabla A + B\nabla H),
\]

where

\[
A = h^2 \frac{d}{dt} \left( \frac{h}{3} \text{div}(u) - \frac{d}{dt} H \right), \quad B = h \frac{d}{dt} \left( \frac{h}{2} \text{div}(u) - \frac{d}{dt} H \right),
\]

and

\[
\frac{d}{dt} = \frac{\partial}{\partial t} + (u \cdot \nabla).
\]

Here \(h\) is the total depth of the fluid, \(H\) is the height of the fluid column between the bottom and the undisturbed level of the fluid (see Figure ??). The function \(H(x, y, t)\) describes the variable bottom topography, e.g. the ocean bathymetry or a moving object located on the bottom. This function is assumed to be known. The first equation of the latter system is the conservation law of mass of the fluid column over which the averaging occurs, and the second equation is the Eulerian form of the Newton’s second law, with contributions to the fluid particle acceleration due to gravity and due to the surface and bottom boundary conditions.

Notice that for \(H_t = 0\) these equations coincide with the Green-Naghdi equations \([29, 30, 31]\).

3. Equivalence transformations

The class of equations (2.1) is parameterized by the arbitrary elements \(H(x, y, t)\). Equivalence transformations of this class preserve the structure of its equations, but are allowed to change the arbitrary elements. The first step of the group classification of the class of equations of the form (2.1) is to describe the equivalence among equations of this class to which the group classification is carried out.

Generators of one-parameter groups of equivalence transformations are assumed to be in the form \([15, 32]\)

\[
X^e = \xi^t \partial_t + \xi^x \partial_x + \xi^y \partial_y + \eta^h \partial_h + \eta^u \partial_u + \eta^v \partial_v + \zeta^H \partial_H,
\]

where all coefficients of the generator depend on \((t, x, y, h, u, v, H)\).

\[\text{It should be also noted that even } H_t = 0 \text{ in } [30], \text{ their Green-Naghdi equations are written in the form (2.1)–(2.2).}\]
The class of differential equations (2.1) is defined by auxiliary equations for the arbitrary elements \( H \), which are given by
\[
H_h = 0, \quad H_u = 0, \quad H_v = 0.
\]

For finding equivalence transformations the infinitesimal criterion \([15]\) is used. For this purpose the determining equations for the components of generators of one-parameter groups of equivalence transformations were derived. The solution of these determining equations gives the general form of elements of the equivalence group of the class (2.1). Because of the cumbersome calculations we extend the equivalence transformations, found in \([12]\) for one-dimensional case, to the two-dimensional case
\[
X^e_1 = \partial_x, \quad X^e_2 = \partial_y, \quad X^e_3 = t\partial_x + u\partial_u, \quad X^e_4 = t\partial_y + u\partial_u,
\]
\[
X^e_5 = -y\partial_x + x\partial_y - v\partial_v + u\partial_u, \quad X^e_6 = \partial_t,
\]
\[
X^e_7 = -y\partial_x + x\partial_y - v\partial_v + u\partial_u, \quad X^e_8 = t\partial_x + 2x\partial_x + 2y\partial_y + u\partial_u + v\partial_v + 2h\partial_h
\]
\[
X^e_9 = t\partial_t - 2g\partial_y, \quad X^e_{10} = h\partial_h + g\partial_g + H\partial_H, \quad X^e_{11} = \partial_H, \quad X^e_{12} = t\partial_H.
\]

The corresponding transformations changing \( H \) are
\[
X^e_{10} : \quad \tilde{H} = H + a,
\]
\[
X^e_{11} : \quad \tilde{H} = H + at,
\]
where \( a \) is the group parameter. Hence, because of the transformations related to \( X^e_{10} \) and \( X^e_{11} \), if \( H(x, y, t) = G(x, y, t) + w(t) \) and \( w'' = 0 \), then one can assume that \( w = 0 \).

There are also two obvious involutions which correspond to the change of the variables:
\[
E_1 : \quad \tilde{x} = -x, \quad \tilde{y} = -y, \quad \tilde{u} = -u, \quad \tilde{v} = -v,
\]
\[
E_2 : \quad \tilde{t} = -t, \quad \tilde{u} = -u, \quad \tilde{v} = -v,
\]
where only changeable variables are presented.

**Remark 3.1.** The transformation corresponding to the generator \( X^e_3 \) and \( X^e_4 \) are the Galilean transformations. Notice that the Green-Naghdi equations, when \( H \) does not depend on time \( t \), do not admit the Galilean transformations as equivalence transformations. The Galilean invariance principle states that all mechanical laws are the same in any inertial frame of reference. This property is of fundamental importance for any mathematical model.

4. Group classification

Group classification is carried out up to equivalence transformations. The group classification problem consists of finding all Lie algebras admitted by equation (2.1). A part of these Lie algebras, called the kernel of admitted Lie algebras, is admitted for all arbitrary element \( H \). Another part depends on the specification of the arbitrary elements. This part contains nonequivalent extensions of the kernel of admitted Lie algebras.

The admitted Lie algebra consists of the generators
\[
X^e = \xi^t \partial_t + \xi^x \partial_x + \xi^y \partial_y + \eta^h \partial_h + \eta^u \partial_u + \eta^v \partial_v + \zeta^H \partial_H,
\]
where all coefficients of the generator depend on \((t, x, y, h, u, v, H)\) and satisfy the determining equations \([15]\).

Calculations, performed in the symbolic manipulation system Reduce \([33]\), show that the classifying equations are

\[
\frac{\partial S}{\partial x} = 0, \quad \frac{\partial S}{\partial y} = 0, \quad \frac{\partial^2 S}{\partial t^2} = 0, \tag{4.1}
\]

where \(x_i, (i = 1, 2, \ldots, 7)\) are constant,

\[
S = (x_1 + x_3 t)H_x + (x_2 + x_4 t)H_y + x_5 (H_y x - H_x y) + x_6 H_t + x_7 (H_t t + 2H_y y + 2H_x x - 2H),
\]

and

\[
X = \sum_{i=1}^{7} x_i X_i,
\]

with the generators

\[
\begin{align*}
X_1 &= \partial_x, \quad X_2 = \partial_y, \quad X_3 = t \partial_x + \partial_u, \quad X_4 = t \partial_y + \partial_u, \\
X_5 &= -y \partial_x + x \partial_y - v \partial_u + u \partial_v, \quad X_6 = \partial_t, \\
X_7 &= t \partial_t + 2x \partial_x + 2y \partial_y + u \partial_u + v \partial_v + 2h \partial_h.
\end{align*}
\]

There are several ways to analyze the classifying equations \((4.1)\). One of the methods has been applied for group classification of the gas dynamics equations \([15]\). Unfortunately, implementation of this algorithm leads to cumbersome calculations. An alternative method for analyzing the classifying equations consists of an algebraic approach, taking the algebraic properties of an admitted Lie algebra into account, thus allowing for a significant simplification of the group classification. This algebraic approach for group classification has been applied in \([24, 25, 26, 27, 28]^{[4]}\).

As for the algebraic approach, one notes that the generators admitted by equation \((2.1)\) compose a Lie algebra which is a subalgebra of the Lie algebra \(L_7 = \{X_1, X_2, \ldots, X_7\}\). Another observation is that the automorphisms of the Lie algebra \(L_7\) act similar to the equivalence transformations corresponding to the generators \(X_i, (i = 1, 2, \ldots, 7)\). Hence, each of the Lie algebras admitted by equation \((2.1)\) belongs to one of the classes of an optimal system of subalgebras of the Lie algebra \(L_7\). Thus, for the group classification of equation \((2.1)\) it is sufficient to construct an optimal system of subalgebras of the Lie algebra \(L_7\). Each representative of a class from the optimal system of subalgebras provides a set of constants \(x_i, (i = 1, 2, \ldots, 7)\). Using these constants, and substituting them into classifying equations \((4.1)\), one obtains an overdetermined system of equations for the function \(H(x, y, t)\). The general solution of the overdetermined system of equations gives the group classification of equation \((2.1)\).

4.1. Optimal system of subalgebras of \(L_7\)

For low-dimensional Lie algebras calculation of the optimal system of subalgebras (also called the representative list of subalgebras) is relatively easy. For high-dimensional Lie algebras the problem becomes complicated because it requires extensive computations. The difficulties can be facilitated by a two-step algorithm suggested in \([34]\). This algorithm replaces the problem

\[\text{[4]}\text{See also references therein.}\]
of constructing the optimal system of high-dimensional subalgebras by a similar problem for lower dimensional subalgebras.

Shortly the algorithm [34] can be formulated as follows. Let \( L \) be a Lie algebra \( L \) with the basis \( \{X_1, X_2, \ldots, X_r\} \). Assume that the Lie algebra \( L \) is decomposed as \( L = I \oplus F \), where \( I \) is a proper ideal of the algebra \( L \) and \( F \) is a subalgebra. Then the set of the inner automorphisms \( A = \text{Int} \ L \) of the Lie algebra \( L \) is decomposed \( A = A_I A_F \), where

\[
AI \subset I, \quad A_F F \subset F, \quad (A_I X)_F = X, \quad \forall X \in F.
\]

This means the following [34]. Let \( x \in L \) be decomposed as \( x = x_I + x_F \), where \( x_I \in I \), and \( x_F \in F \). Any automorphism \( B \in A \) can be written as \( B = B_I B_F \), where \( B_I \in A_I, \ B_F \in A_F \).

The automorphisms \( B_I \) and \( B_F \) have the properties:

\[
B_I x_F = x_F, \quad \forall x_F \in F, \quad \forall B_I \in A_I;
\]
\[
B_F x_I \in I, \quad B_F x_F \in F, \quad \forall x_I \in I, \quad \forall x_F \in F, \quad \forall B_F \in A_F.
\]

At the first step, an optimal system of subalgebras \( \Theta_{A_F}(F) = \{F_0, F_1, F_2, \ldots, F_p, F_{p+1}\} \) of the algebra \( F \) is formed. Here \( F_0 = \{0\}, \ F_{p+1} = \{F\} \) and the optimal system of the algebra \( F \) is constructed with respect to the automorphisms \( A_F \). For each subalgebra \( F_j, j = 0, 1, 2, \ldots, p+1 \) one has to find its stabilizer \( \text{St}(F_j) \subset A \):

\[
\text{St}(F_j) = \{B \in A \mid B(F_j) = F_j\}.
\]

Note that \( \text{St}(F_{p+1}) = A \).

The second step consists of forming the optimal system of subalgebras \( \Theta_A(L) \) of the algebra \( L \) as a collection of \( \Theta_{\text{St}(F_j)}(I \oplus F_j), j = 0, 1, 2, \ldots, p+1 \).

If the subalgebra \( F \) can be decomposed, then the two-step algorithm can be used for construction of \( \Theta_{A_F}(F) \).

The structure of the Lie algebra is defined by its commutator table. The commutator table of \( L_7 \) is

|      | \( X_1 \) | \( X_2 \) | \( X_3 \) | \( X_4 \) | \( X_5 \) | \( X_6 \) | \( X_7 \) |
|------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| \( X_1 \) | 0         | 0         | 0         | 0         | \( X_2 \) | 0         | 2\( X_1 \) |
| \( X_2 \) | 0         | 0         | 0         | 0         | \( -X_1 \) | 0         | 2\( X_2 \) |
| \( X_3 \) | 0         | 0         | 0         | 0         | \( -X_4 \) | \( -X_1 \) | 0         |
| \( X_4 \) | 0         | 0         | 0         | 0         | \( -X_3 \) | \( -X_2 \) | 0         |
| \( X_5 \) | \( -X_2 \) | \( X_1 \) | \( -X_4 \) | \( X_3 \) | 0         | 0         | 0         |
| \( X_6 \) | 0         | 0         | \( X_1 \) | \( X_2 \) | 0         | 0         | \( X_6 \) |
| \( X_7 \) | \( -2X_1 \) | \( -2X_2 \) | 0         | 0         | 0         | \( -X_6 \) | 0         |

Directly from the table one derives that the composition series of ideals is

\[
O \subset \{X_1, X_2\} \subset \{X_1, X_2, X_3, X_4\} \subset \\
\{X_1, X_2, X_3, X_4, X_5\} \subset \{X_1, X_2, X_3, X_4, X_5, X_6\} \subset L_7,
\]

and inner automorphisms are:

where \( a_i, (i = 1, 2, \ldots, 7) \) are parameters of the automorphism \( A \),

\[
p_1 = (x_1, x_2), \ p_2 = (x_3, x_4), \ a_1 = (a_1, a_2), \ a_2 = (a_3, a_4),
\]
\[
\begin{array}{|c|c|c|c|c|}
\hline
A & Ap_1 & Ap_2 & Ax_5 & Ax_6 \\
\hline
T & p_1 + 2a_1x_7 + x_5(a_1, -a_2) & p_2 & x_5 & x_6 \\
\hline
S & p_1 - x_6a_2 & p_2 + x_5(a_3, -a_4) & x_5 & x_6 \\
\hline
A_6 & p_1 - a_6p_2 & p_2 & x_5 & x_6 + a_6x_7 \\
\hline
A_7 & \pm a_7p_1 & p_2 & x_5 & \pm a_7x_6 \\
\hline
E_1 & -p_1 & p_2 & x_5 & x_6 \\
\hline
E_2 & p_1 & -p_2 & x_5 & -x_6 \\
\hline
\end{array}
\]

Table 1: Inner automorphisms of the Lie algebra \( L_7 \).

\( S \) is the rotation matrix
\[
S = \begin{pmatrix}
\cos a_5 & \sin a_5 \\
-\sin a_5 & \cos a_5
\end{pmatrix}.
\]

For application of the two-step algorithm one notes that the Lie algebra \( L_7 \) can be decomposed as follows
\[
L_7 = \{\{\{X_1, X_2\} \oplus \{X_3, X_4\} \oplus \{X_5\}\} \oplus \{X_6\}\} \oplus \{X_7\}\}.
\]

**Remark 4.1.** The Lie algebra \( L_7 \) coincides with the Lie algebra admitted by the two-dimensional gas dynamics equations with the general form of the state equation. The optimal system of subalgebras of the Lie algebra \( L_7 \) was constructed by the author during the work on the SUBMODELS program [33], leading by L.V. Ovsiannikov. The two-step algorithm [34] was also proposed by L.V. Ovsiannikov in the framework of the SUBMODELS program [35].

The optimal system of subalgebras of the Lie algebra \( L_7 \) is presented in Table 2 where the subalgebra representatives are denoted by a pair of numbers \((r, i)\): \( r \) is the dimension and \( i \) is the serial number of a subalgebra of dimension \( r \). The numbers \( r \) are given in front of each block containing sub-algebras of dimensions \( r \). The serial numbers \( i \) are presented in the first column. The bases of the subalgebras \((r, i)\) are written out in the second column.

### 4.2. Solutions of classifying equations \((4.1)\)

As mentioned above, substituting the constants \( x_i \), \( (i = 1, 2, ..., 7) \), determined by the basis generators of a subalgebra of the optimal system of subalgebras of the Lie algebra \( L_7 \), into classifying equations \((1.1)\), one obtains an overdetermined system of equations for the function \( H(x, y, t) \). Solving the overdetermined system of equations, one finds the function \( H(x, y, t) \) such that equations \((2.1)\) admit the corresponding Lie algebra. Considering all subalgebras of the optimal system of subalgebras, one obtains the group classification presented in Table 3 where representation of the function \( H(x, y, t) \) is given in the second column, and the corresponding admitted Lie algebra is presented in the third column. The constants \( k, l, \alpha \) and \( \beta \) are arbitrary, the function
\[
H = lt^2 + Q,
\]
where \( l \) is an arbitrary constant and \( Q \) is an arbitrary function of its arguments. If \( Q'' = 0 \), then normally the model has more admitted generators: it is a particular case of the previous model.

Here are two typical examples.
| N | Basis                                                                 | N | Basis                                                                 |
|---|----------------------------------------------------------------------|---|----------------------------------------------------------------------|
| r=7                                      |                                            | r=3                                      |
| 1 | $X_1, X_2, X_3, X_4, X_5, X_6, X_7$                                  | 11 | $X_1, X_2, X_6$                                                      |
| r=6                                      |                                            | r=1                                      |
| 1 | $X_1, X_2, X_3, X_4, X_5 + \alpha X_7, X_6$                          | 12 | $X_1, X_2, X_3 + X_6$                                               |
| 2 | $X_1, X_2, X_3, X_4, X_5, X_7$                                       | 13 | $X_3, X_4, X_7$                                                      |
| 3 | $X_1, X_2, X_3, X_4, X_5, X_7$                                       | 14 | $X_1, X_3 + \alpha X_7, X_7 + \beta X_4$                            |
| r=5                                      |                                            | r=2                                      |
| 1 | $X_1, X_2, X_5, X_6, X_7$                                            | 16 | $X_1, X_2, X_7$                                                      |
| 2 | $X_1, X_2, X_3, X_4, X_5 + \alpha X_7$                               | 17 | $X_1, X_2, X_7 + \alpha X_3$                                        |
| 3 | $X_1, X_2, X_3, X_4, X_5 + X_6$                                      | 18 | $X_1, X_2 + X_3, X_4$                                               |
| 5 | $X_1, X_2, X_3, X_6, X_7 + \alpha X_4$                               | 19 | $X_1, X_3, X_4$                                                      |
| 6 | $X_1, X_2, X_3, X_4, X_6$                                            | 20 | $X_1, X_2, X_3$                                                      |
| r=4                                      |                                            | r=3                                      |
| 1 | $X_1, X_2, X_5 + \alpha X_7, X_6$                                    | 21 | $X_5, X_7$                                                           |
| 2 | $X_3, X_4, X_5, X_7$                                                 | 3  | $X_6, X_7$                                                           |
| 4 | $X_1, X_2, X_5, X_7$                                                 | 4  | $X_1, X_6$                                                           |
| 5 | $X_1, X_2, X_5, X_7$                                                 | 5  | $X_1, X_3 + X_6$                                                     |
| 6 | $X_1, X_2, X_5, X_7$                                                 | 6  | $X_2, X_3 + X_6 + \alpha X_4$                                       |
| 7 | $X_1, X_2, X_3, X_4 + X_6$                                           | 7  | $X_3, X_7 + \alpha X_4$                                             |
| 8 | $X_1, X_2, X_3, X_4$                                                 | 8  | $X_1, X_7 + \alpha X_3 + \beta X_4$                                 |
| 9 | $X_1, X_2, X_3, X_4$                                                 | 9  | $X_2, X_3 + X_4 + \alpha X_1$                                       |
| 10 |                                                                      | 10 | $X_2 + \alpha X_1, X_3$                                             |
| r=3                                      |                                            | r=1                                      |
| 1 | $X_5, X_6, X_7$                                                      | 15 | $X_1, X_2$                                                           |
| 2 | $X_3, X_4, X_5 + \alpha X_7$                                         |                                            | r=1                                      |
| 3 | $X_1, X_2, X_5 + \alpha X_7$                                         | 1  | $X_5 + \alpha X_7$                                                   |
| 4 | $X_1, X_2, X_5 + X_6$                                                | 2  | $X_5 + X_6$                                                          |
| 5 | $X_2 + X_3, X_1 - X_4, X_5$                                         | 4  | $X_3 + X_6$                                                          |
| 6 | $X_1, X_2, X_5$                                                      | 5  | $X_6$                                                                 |
| 7 | $X_1, X_6, X_7 + \beta X_3$                                         | 6  | $X_7 + \alpha X_3$                                                   |
| 8 | $X_2, X_3 + X_6, X_4 + \alpha X_1$                                   | 8  | $X_3$                                                                 |
| 9 | $X_1, X_2 + X_3, X_6$                                               | 9  | $X_2 + X_3$                                                          |
| 10 | $X_1, X_3, X_6$                                                       | 10 | $X_1$                                                                 |

Table 2: Optimal system of subalgebras of the Lie algebra $L_7$.

### 4.2.1. Example 1

Consider the subalgebra 2.9: \( \{X_2 + X_3, X_4 + \alpha X_1\} \). These generators gives the two sets of nonzero constants: for \( X_2 + X_3 \) they are \( x_2 = 1 \) and \( x_3 = 1 \); and for \( X_4 + \alpha X_1 \) they are \( x_1 = \alpha \) and \( x_4 = 1 \). The classifying equations in this case become

\[
4.2.1 \quad tH_x + H_y = a_1 t + b_1 \quad \alpha H_x + tH_y = a_2 t + b_2,
\]

8
where \( a_i \) and \( b_i \) \((i = 1, 2)\) are arbitrary constants. The general solution of these equations is

\[
H = y\left(\frac{-tl_2 + l_1\alpha}{t^2 - \alpha} + k_2\right) + x\left(\frac{-tl_1 + l_2}{t^2 - \alpha} + k_1\right) + Q(t), \quad \alpha^2 + l_2^2 \neq 0, \tag{4.2}
\]

where \( l_1 = a_2 - b_1, \ l_2 = a_1\alpha - b_2, \ k_1 = a_1, \) and \( k_2 = a_2. \) This result is presented in Table 3 at number 15.

4.2.2. Example 2

Consider the subalgebra 2.1: \(\{X_5 + \alpha X_7, X_6\}\). These generators gives the two sets of nonzero constants: for \(X_5 + \alpha X_7\) they are \(x_5 = 1\) and \(x_7 = \alpha\); and for \(X_6\) it is \(x_6 = 1.\) The classifying equations in this case become

\[
(2\alpha x - y)H_x + (2\alpha y + x)H_y + \alpha tH_t = 2\alpha H + a_1 t + b_1, \tag{4.3}
\]

\[
H_t = a_2 t + b_2.
\]

Solving the second equation of (4.3), one finds \(H = a_2 \frac{\alpha^2}{2} + b_2 t + G(x, y),\) where because of equivalence transformations one can assume that \(b_2 = 0.\) Substituting the found \(H\) into the first equation of (4.3), and differentiating it with respect to \(t,\) one derives that \(a_1 = 0.\) The first equation of (4.3) becomes

\[
(2\alpha x - y)G_x + (2\alpha y + x)G_y = 2\alpha G + b_1. \tag{4.4}
\]

In polar coordinate system

\[
x = r \cos \varphi, \quad y = r \sin \varphi,
\]

equation (4.4) has more convenient representation

\[
Q_\varphi + 2\alpha r Q_r = 2\alpha Q + b_1.
\]

The general solution of the latter equation depends on \(\alpha:\)

\[
\alpha \neq 0 : \quad G(r, \varphi) = e^{2\alpha \varphi}Q(re^{-2\alpha \varphi}) - \frac{b_1}{2\alpha},
\]

\[
\alpha = 0 : \quad G(r, \varphi) = b_1 \varphi + Q(r),
\]

where the function \(Q\) is an arbitrary function of its arguments. These results are presented in Table 3 at number 11 and 12.

5. Conclusions

The two-dimensional Green-Naghdi equations with uneven bottom topography dependent on time are studied in this paper. Group classification of these equations with respect to the function describing the topography of the bottom is performed in the paper. For the group classification we applied the algebraic approach. This approach simplifies the method for solving classifying equations (4.1). For its application one notes that the generators admitted by equations (2.1) compose a Lie algebra which is a subalgebra of the Lie algebra \(L_7 = \{X_1, X_2, ..., X_7\}.\) As the actions of the equivalence transformations corresponding to (3.1) coincides with the actions of the inner automorphisms of the Lie algebra \(L_7\) given in Table 1 for the group classification one can use the optimal system of subalgebras of the Lie algebra \(L_7.\) Using the generators
of a chosen subalgebra from Table 2 one determines the coefficients \( x_i, (i = 1, 2, \ldots, 7) \), and substituting them into the classifying equations (4.1), one obtains an overdetermined system of equations for the function \( H(x, y, t) \). The general solution of the latter system provides the bottom topography such that system (2.1) admits the chosen subalgebra. The final result of the group classification is presented in Table 3.

## Acknowledgments

The research was supported by the Russian Science Foundation Grant No. 18-11-00238 ‘Hydrodynamics-type equations: symmetries, conservation laws, invariant difference schemes’.
References

[1] G. B. Whitham. *Linear and Nonlinear Waves*. Wiley, New York, 1974.

[2] J. Pedlosky. *Geophysical Fluid Dynamics*. Springer, New York, 1987. 2nd Edition.

[3] L. V. Ovsiannikov, N. I. Makarenko, V. I. Nahimov, V. Yu. Liapidevskii, P. I. Plotnikov, I. V. Sturova, V. I. Bukreev, and V. A. Vladimirov. *Nonlinear Problems of the Theory of Surface and Internal Waves*. Nauka, Novosibirsk, 1985. In Russian.

[4] R. Salmon. *Lectures on Geophysical Fluid Dynamics*. Oxford University Press, New York, 1998.

[5] G. K. Vallis. *Atmospheric and oceanic fluid dynamics. Fundamentals and large-scale circulation*. Cambridge University Press, Cambridge, 2006.

[6] A. S. Petrosyan. *Additional chapters of heavy fluid hydrodynamics with a free boundary*. Space Research Institute of the Russian Academy of Sciences, Moscow, 2014. In Russian.

[7] P. Bonneton, E. Barthélémy, F. Chazel, R. Cienfuegos, D. Lannes, F. Marche, and M. Tissier. Recent advances in Serre-Green-Naghdi modelling for wave transformation, breaking and runup processes. *Euro. J. Mech. B/Fluids*, 30:589–597, 2011.

[8] G. S. Khakimzyanov, D. Dutykh, Z. I. Fedotova, and D. E. Mitsotakis. Dispersive shallow water wave modelling. Part I: Model derivation on a globally flat space. *Commun. Comput. Phys.*, 23(1):1–29, 2018.

[9] Y. Matsuno. Hamiltonian formulation of the extended Green-Naghdi equations. *Phys D*, 301-302:1–7, 2015.

[10] F. Serre. Contribution à l’étude des écoulements permanents et variables dans les canaux. *Houille Blanche*, 3:374–388, 1953.

[11] C. H. Su and C. S. Gardner. Korteweg-de Vries equation and generalizations. III. Derivation of the Korteweg-de Vries equation and Burgers equation. *Journal of Mathematical Physics*, 10(3):10–23, 1969.

[12] E. I. Kaptsov, S. V. Meleshko, and N. F. Samatova. The one-dimensional green-naghdi equations with a time dependent bottom topography and their conservation laws. *Physics of Fluids*, 32(12):123607, 2020.

[13] J. L. Hammack. A note on tsunamis: their generation and propagation in an ocean of uniform depth. *J . Fluid Mech.*, 60:769–799, 1973. part 4.

[14] D. Dutykh and F. Dias. Water waves generated by a moving bottom. In A. Kundu, editor, *Tsunami and Nonlinear Waves*, pages 65–96. Springer-Verlag (Geo Sc.), Berlin, Heidelberg, 2007.

[15] L. V. Ovsiannikov. *Group Analysis of Differential Equations*. Nauka, Moscow, 1978. English translation, Ames, W.F., Ed., published by Academic Press, New York, 1982.

[16] P. J. Olver. *Applications of Lie Groups to Differential Equations*. Springer-Verlag, New York, 1986.
[17] N. H. Ibragimov. *Transformation Groups Applied to Mathematical Physics*. Nauka, Moscow, 1983. English translation, Reidel, D., Ed., Dordrecht, 1985.

[18] G. W. Bluman and S. Kumei. *Symmetries and Differential Equations*. Springer-Verlag, New York, 1989.

[19] N. H. Ibragimov, editor. *CRC Handbook of Lie Group Analysis of Differential Equations*, volume 1, 2, 3. CRC Press, Boca Raton, 1994, 1995, 1996.

[20] G. Gaeta. *Nonlinear Symmetries and Nonlinear Equations*. Kluwer, Dordrecht, 1994.

[21] Yu. Yu. Bagderina and A. P. Chupakhin. Invariant and partially invariant solutions of the Green-Naghdi equations. *Journal of Applied Mechanics and Technical Physics*, 46(6):791–799, 2005.

[22] P. Siriwat, C. Kaewmanee, and S. V. Meleshko. Symmetries of the hyperbolic shallow water equations and the Green-Naghdi model in Lagrangian coordinates. *International Journal of Non-Linear Mechanics*, 86:185–195, 2016.

[23] V. A. Dorodnitsyn, E. I. Kaptsov, and S. V. Meleshko. Symmetries and difference schemes of the 1D Green-Naghdi equations. *Journal of Nonlinear Mathematical Physics*. in press.

[24] Yu. N. Grigoriev, S. V. Meleshko, and A. Suriyawichitseranee. On the equation for the power moment generating function of the Boltzmann equation. group classification with respect to a source function. In O.O. Vaneeva, C. Sophocleous, R.O. Popovych, P.G.L. Leach, V.M. Boyko, and P.A. Damianou, editors, *Group Analysis of Differential Equations & Integrable Systems*, pages 98–110. University of Cyprus, Nicosia, 2012.

[25] Yu. A. Chirkunov. Generalized equivalence transformations and group classification of systems of differential equations. *Journal of Applied Mechanics and Technical Physics*, 53(2):147–155, 2012.

[26] A. A. Kasatkin. Symmetry properties for systems of two ordinary fractional differential equations. *Ufa Mathematical Journal*, 4(1):71–81, 2012.

[27] T. G. Mkhize, S. Moyo, and S. V. Meleshko. Complete group classification of systems of two linear second-order ordinary differential equations. Algebraic approach. *Mathematical Methods in the Applied Sciences*, 38:1824–1837, 2015.

[28] S. Opanasenko, V. Boyko, and R. O. Popovych. Enhanced group classification of nonlinear diffusion-reaction equations with gradient-dependent diffusivity. *Journal of Mathematical Analysis and Applications*, 484(1):123739, 2020.

[29] S. V. Bazdenkov, N. N. Morozov, and O. P. Pogutse. Dispersive effects in two-dimensional hydrodynamics. *Sov. Phys. Dokl.*, 32:262–264, 1987.

[30] V. Yu. Liapidevskii and K. N. Gavrilova. Dispersion and blockage effects in the flow over a sill. *J Appl Mech Tech Phys*, 49(7):34–45, 2008.

[31] Y. Matsuno. Hamiltonian structure for two-dimensional extended Green-Naghdi equations. *Proceedings of the Royal Society. Mathematical, physical and engineering sciences*, 472(2190), 2016.
[32] S. V. Meleshko. *Methods for Constructing Exact Solutions of Partial Differential Equations.* Mathematical and Analytical Techniques with Applications to Engineering. Springer, New York, 2005.

[33] A. C. Hearn. *REDUCE Users Manual, ver. 3.3.* The Rand Corporation CP 78, Santa Monica, 1987.

[34] L. V. Ovsiannikov. On optimal system of subalgebras. *Dokl. RAS,* 333(6):702–704, 1993.

[35] L. V. Ovsiannikov. Program SUBMODELS. Gas dynamics. *J. Appl. Maths Mechs,* 58(4):30–55, 1994.