ON THE DIRICHLET PROBLEM FOR A CLASS OF AUGMENTED HESSIAN EQUATIONS

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Abstract. In this paper, we consider the Dirichlet problem for a new class of augmented Hessian equations. Under sharp assumptions that the matrix function in the augmented Hessian is regular and there exists a smooth subsolution, we establish global second order derivative estimates for the solutions to the Dirichlet problem in bounded domains. The results extend the corresponding results in the previous paper [11] from the Monge-Ampère type equations to the more general Hessian type equations.

1. Introduction

In this paper, we study a class of augmented Hessian equations with the following form

\begin{equation}
S_k[D^2u - A(x, Du)] = B(x, Du), \quad \text{in } \Omega,
\end{equation}

associated with the Dirichlet boundary condition

\begin{equation}
u = \varphi, \quad \text{on } \partial \Omega,
\end{equation}

where \( \Omega \) is a bounded domain in \( n \) dimensional Euclidean space \( \mathbb{R}^n \), \( A : \Omega \times \mathbb{R}^n \to \mathbb{R}^{n \times n} \) is a symmetric matrix function, \( B : \Omega \times \mathbb{R}^n \to \mathbb{R} \) is a scalar function, \( \varphi \) is a smooth function on \( \partial \Omega \), \( Du \) denotes the gradient vector of \( u \), \( D^2u \) denotes the Hessian matrix of the second derivatives of \( u \), and \( S_k \) is a \( k \)-Hessian operator defined by

\begin{equation}
S_k[W] := S_k(\lambda),
\end{equation}

with \( \lambda \) denoting the eigenvalues \( \lambda_1, \ldots, \lambda_n \) of the \( n \times n \) symmetric matrix \( W \), \( S_k(\lambda) \) denoting the \( k \)-th order elementary symmetric function given by

\begin{equation}
S_k(\lambda) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k}, \quad k \leq n \text{ is an integer}.
\end{equation}

In the equation (1.1), the symmetric matrix \( W \) under consideration is the augmented Hessian matrix \( \{D^2u - A(x, Du)\} \), which is given by functions of the Hessian minus a lower order symmetric matrix function. As usual, we shall use \( (x, p) \) to define the points in \( \Omega \times \mathbb{R}^n \). We adopt the terminology from [24] and call the matrix function \( A \) regular if \( A \) is co-dimension one convex with respect to \( p \), in the sense that

\begin{equation}
A_{ij,kl}(x, p)\xi_i \xi_j \eta_k \eta_l \geq 0,
\end{equation}

for all \( (x, p) \in \bar{\Omega} \times \mathbb{R}^n \), \( \xi, \eta \in \mathbb{R}^n \), \( \xi \perp \eta \), where \( A_{ij,kl} = D^2_{p_ip_k}A_{ij} \). As explained in [24], condition (1.5) is the natural condition for regularity of the solution \( u \).

Equations of the form (1.1) have attracted much research interest and have many applications. If \( A \equiv 0 \), the equation (1.1) reduces to the standard Hessian equation, and clearly satisfies the regular condition (1.5). In this case, it becomes the standard Monge-Ampère equation when \( k = n \). For
these well known standard Hessian equations and Monge-Ampère equations, classical solvability of the corresponding Dirichlet problems was studied extensively, see [2], [3], [4], [7], [13], [16], [22], [32] and [33] etc. If \( A \) depends only on \( x \), the corresponding equation (1.1) has applications in Riemannian geometry, see [16] and [30]. Recently, Guan studied the Dirichlet problem for a class of Hessian type equations with \( A \) only depending on \( x \) on Riemannian manifolds under some very general structure conditions in [9].

For general \( A \) related to \( x \) and \( p \), when \( k = n \), the equation (1.1) is the Monge-Ampère type equation and is an important model equation in optimal transportation, geometric optics, isometric embedding and etc. For example, in optimal transportation, the matrix function \( A(x,p) = D_x^2 c(x,Y(x,p)) \) and the scalar function \( B(x,p) = | \det D_x^2 Y(x,p) | \) where \( c(x,y) \) is the cost function, \( Y \) is the optimal map, \( f \) and \( g \) are densities of the original and target measures. The existence of smooth solutions to the Dirichlet problem for the optimal transportation equation in small balls under the strict version of (1.5), denoted A3, was used in [19]. For more detailed definitions in optimal transportation, one can refer to [19]. Classical solvability for Dirichlet problem of the Monge-Ampère type equations in bounded domains was studied under sharp assumptions that \( A \) is regular as well as the existence of subsolutions in [11]. While for general \( k \), the augmented Hessian equation (1.1) is related to the so-called \( k \)-Yamabe problem in conformal geometry. For instance, in the conformal geometry case, the matrix function \( A \) is given by \( A(x,Du) = -\frac{1}{4}|p|^2 I + p \otimes p \), see [24]. One can check that such \( A \) also satisfies the regular condition (1.5). The Dirichlet boundary problem for a class of Hessian type equations related to conformal deformations of metrics on Riemannian manifolds with boundary was studied in [8]. Under the existence of a smooth subsolution, Guan [8] derived various \textit{a priori} estimates and proved the classical solvability for the Dirichlet problem.

In the current paper, we deal with the augmented Hessian equations not only coming from conformal geometry but for the general equation (1.1) together with the Dirichlet boundary data (1.2). Under the assumption that there exists a subsolution, we show that the regular condition (1.5) is sufficient for the second order derivative estimates of the Dirichlet problem for the general Hessian type equation (1.1) in smooth bounded domains without any geometric restrictions. Our main issue is to deal with the dependence in lower order terms for both \( A \) and \( B \), which was not an issue in [9].

Before stating our theorems, we shall present some definitions and well known properties of the function \( f = (S_k)^{\frac{1}{n}} \) (see for example [8], [20], [29]). We define the positive cone \( \Gamma^+_k \) in \( \mathbb{R}^n \),

\[
\Gamma^+_k := \{ \lambda \in \mathbb{R}^n \mid S_j(\lambda) > 0, \ \forall \ 1 \leq j \leq k \},
\]

and its closure,

\[
\Gamma_k = \Gamma^+_k \cup \partial \Gamma^+_k.
\]

Then, we have

\[
(S_k)^{\frac{1}{n}}(\lambda) > 0, \ \lambda \in \Gamma^+_k, \ \text{and} \ \ (S_k)^{\frac{1}{n}}(\lambda) = 0, \ \lambda \in \partial \Gamma^+_k,
\]

(1.9)

\( (S_k)^{\frac{1}{n}} \) is a concave function in \( \Gamma^+_k \),

(1.10)

\[
f_i := \frac{\partial (S_k)^{\frac{1}{n}}}{\partial \lambda_i} > 0, \ \text{in} \ \Gamma^+_k, \ 1 \leq i \leq n,
\]

and for every \( C > 0 \) and every compact set \( E \subset \Gamma^+_k \), there exists \( R = R(E,C) > 0 \) such that

(1.11)

\[
(S_k)^{\frac{1}{n}}(\lambda_1, \cdots, \lambda_{n-1}, \lambda_n + R) \geq C, \ \forall \ \lambda = (\lambda_1, \cdots, \lambda_n) \in E.
\]
If \( \lambda \in \Gamma_k \) with \( \lambda_1 \geq \cdots \geq \lambda_n \), the concavity of \( f \) leads to \( f_1 \leq \cdots \leq f_n \), see (ii) of Lemma 2.1 in [29]. If we define \( \Gamma_{k;\mu_1,\mu_2} = \{ \lambda \in \Gamma_k : \mu_1 \leq f(\lambda) \leq \mu_2 \} \) for any \( 0 < \mu_1 \leq \mu_2 \), then

\[
\sum_{i=1}^{n} f_i \geq \sigma_0, \quad \text{on} \quad \Gamma_{k;\mu_1,\mu_2},
\]

and, for \( k > 1 \),

\[
\lim_{|\lambda| \to \infty} \sum_{i=1}^{n} f_i = +\infty, \quad \text{on} \quad \Gamma_{k;\mu_1,\mu_2},
\]

where \( \sigma_0 \) is a constant depending on \( \mu_1 \) and \( \mu_2 \).

The properties (1.8) and (1.9) imply the degenerate ellipticity condition \( f_i \geq 0 \) in \( \Gamma_k \) for \( i = 1, \cdots, n \).

With the definition of the cone \( \Gamma_k \), we can define the admissible solutions corresponding to the equation (1.1). A solution \( u \in C^2(\Omega) \) is called \( k \)-A-admissible if

\[
D^2 u - A(x, Du) \in \Gamma_k, \quad \text{in} \quad \Omega.
\]

For simplicity, we denote the \( k \)-A-admissible solutions as the admissible solutions. Under the assumption \( B(x, Du) > 0 \), the admissible condition (1.14) ensures that the equation (1.1) is elliptic with respect to a solution \( u \in C^2(\Omega) \). Denoting the right hand side function by \( \tilde{B}(x, p) = (B(x, p))^\frac{1}{k} \), we assume that \( \tilde{B} \) is convex with respect to the gradient variables \( p \), that is

\[
D_{\xi k \xi l} \tilde{B}(x, p) \xi_k \xi_l \geq 0,
\]

for all \( (x, p) \in \Omega \times \mathbb{R}^n, \xi \in \mathbb{R}^n \).

We now state our main theorems as follows:

**Theorem 1.1.** Let \( u \in C^4(\Omega) \cap C^2(\bar{\Omega}) \) be an admissible solution of Dirichlet problem (1.1)-(1.2), where \( A \in C^2(\bar{\Omega} \times \mathbb{R}^n) \) is regular, \( B \in C^2(\bar{\Omega} \times \mathbb{R}^n) \) satisfies (1.15) and \( B > 0 \) in \( \bar{\Omega} \times \mathbb{R}^n \). Suppose also there exists an admissible function \( \tilde{u} \in C^2(\bar{\Omega}) \) satisfying

\[
D^2 \tilde{u} - A(x, Du) \in \Gamma_k^+, \quad \text{for all} \quad x \in \bar{\Omega}.
\]

Then we have the estimate

\[
\sup_{\Omega} |D^2 u| \leq C(1 + \sup_{\partial \Omega} |D^2 u|),
\]

where the constant \( C \) depends on \( k, n, A, B, \Omega, \tilde{u} \) and \( \sup_{\Omega} (|u| + |Du|) \).

We remark that the function \( \tilde{u} \) in this theorem is only assumed to satisfy (1.16). Note that Theorem 1.1 certainly holds if \( \tilde{u} \) is an admissible subsolution satisfying

\[
S_k[D^2 \tilde{u} - A(x, Du)] \geq B(x, Du), \quad \text{in} \quad \Omega.
\]

The convexity condition (1.15) on \( \tilde{B} \) can be removed in the Monge-Ampère case when \( k = n \), see [11]. When \( k = 1 \), the conclusion (1.17) follows from the classical Schauder theory, [6].

From Theorem 1.1 we can infer a global second derivative bound for solutions of the Dirichlet problem (1.1)-(1.2) from boundary estimates, which we can further derive if \( u \) is an admissible subsolution satisfying (1.18) with the given boundary trace. Therefore, we have the following theorem.
Theorem 1.2. In addition to the assumptions in Theorem 1.1, suppose the admissible function \( u \) is a subsolution satisfying (1.18) in \( \Omega \) and \( u = \varphi \) on \( \partial \Omega \) with \( \varphi \in C^4(\partial \Omega), \partial \Omega \in C^4 \). Then any admissible solution \( u \in C^4(\Omega) \cap C^2(\bar{\Omega}) \) of the Dirichlet problem (1.1)-(1.2) satisfies the global a priori estimate

(1.19) \[ \sup_{\Omega} |D^2u| \leq C, \]

where the constant \( C \) depends on \( k, n, A, B, \Omega, \varphi, u \) and \( \sup_{\Omega} (|u| + |Du|) \).

In these theorems, the a priori estimates for second derivatives are derived under hypotheses including the regular condition for \( A \) and the existence of a subsolution. We remark that these assumptions are sharp for the second order derivative estimates to the Dirichlet problem (1.1)-(1.2) in general bounded domains. Moreover, if \( A \) and \( B \) depend also on \( u \), we have the more general augmented Hessian equations,

(1.20) \[ S_k[D^2u - A(x, u, Du)] = B(x, u, Du), \text{ in } \Omega. \]

We shall also discuss the corresponding theorems in Section 5 under some mild additional structure conditions for both \( A \) and \( B \) with respect to \( u \).

We focus on the second order derivative estimates in the current paper. These second derivative estimates together with the solution bounds and the gradient estimates will yield the regularity and the classical existence results of the Dirichlet problem (1.1)-(1.2) for the augmented Hessian equations. Once we obtain the derivative estimates up to second order, the Evans-Krylov theorems then yields \( C^{2,\alpha} \) bounds of the solutions. Then we obtain the existence and uniqueness of the classical solution of the Dirichlet problem by the method of continuity. It would be interesting to derive the lower order estimates especially the gradient estimates under some additional structure conditions analogous to the natural conditions of Ladyzhenskaya and Ural’tseva for quasilinear elliptic equations [15, 21, 6]. This will be taken up in a sequel.

The paper is organized as follows: In Section 2, we introduce some preliminary lemmas. The first lemma is to construct a global barrier function for the linearized operator of \( F \), which is fundamental in the second derivative estimates. In Section 4 we prove the global second derivative estimates for solutions to equation (1.1), which reduce the second derivative bound to the boundary. In Section 4 we first show that the regular condition of the matrix \( A \) is preserved when we translate and rotate the coordinates. Then we obtain the second derivative bound on the boundary for regular \( A \). In Section 5 we discuss more general equations (1.20), where both \( A \) and \( B \) depend also on \( u \). More structure conditions on \( A \) and \( B \) will be explained for this general case.

2. Preliminaries

In this section, we introduce some notation and present some preliminary results needed in later sections.

We denote the augmented Hessian matrix by \( W \), that is

(2.1) \[ W = \{w_{ij}\} = \{u_{ij} - A_{ij}(x, Du)\}. \]

Let

(2.2) \[ F[u] =: F(w_{ij}) = (S_k)^\frac{1}{2}(w_{ij}) = (S_k)^\frac{1}{2}[u_{ij} - A_{ij}(x, Du)], \]

it is known that \( F \) is a concave operator with respect to \( w_{ij} \) for admissible \( u \). Introducing the linearized operator of \( F \):

(2.3) \[ L = F^{ij}[D_{ij} - D_{pk}A_{ij}(x, Du)D_k], \]
where \( F^{ij} = \frac{\partial F}{\partial u_{ij}} \), it follows that \( \{F^{ij}\} \) is positive definite [3]. We also set
\[
\mathcal{L} = L - \tilde{B}_p D_i = F^{ij} [D_{ij} - D_{pk} A_{ij}(x,Du) D_k] - \tilde{B}_p D_i.
\]

We shall first prove the following fundamental lemma, which is a key barrier construction needed in the second derivative estimates. Although its proof is similar to Lemma 2.1 in [1], for completeness and for the convenience in later sections, we still present the detailed proof. Also we take the opportunity to make a small correction to our previous proof, (in connection with the choice of \( x_0 \)).

**Lemma 2.1.** Let \( u \in C^2(\bar{\Omega}) \) be an admissible solution of equation (1.1), \( v \in C^2(\bar{\Omega}) \) be an admissible strict subsolution of equation (1.1) satisfying
\[
S_k [D^2 u - A(x,Du)] \geq B(x,Du) + \delta_0,
\]
for some positive constant \( \delta_0 \). Assume the matrix function \( A \) is regular satisfying (1.5), \( A_{ij}(x,p) \in C^2(\bar{\Omega} \times \mathbb{R}^n) \), \( i,j = 1, \cdots, n \), \( B(x,p) \in C^2(\bar{\Omega} \times \mathbb{R}^n) \). Then
\[
\mathcal{L} \left( e^{K(x,u,v)} \right) \geq \epsilon_1 \sum_i F^{ii} - C,
\]
holds in \( \Omega \) for positive constants \( K, \epsilon_1 \) and \( C \), which depend on \( k, A, B, \Omega \), \( \sup_{\bar{\Omega}} |Du| \) and \( \sup_{\Omega} |Du| \).

**Proof.** Since \( u \) is a strict subsolution of (1.1), for any \( x_0 \in \Omega \), the perturbation function \( \tilde{u} = u - \frac{\tau}{2} |x - x_0|^2 \) is still a strict subsolution, for sufficiently small \( \epsilon > 0 \), and satisfies
\[
F_{\tilde{u}} = (S_k)^{\frac{1}{2}} [D^2 \tilde{u} - A(x,D\tilde{u})] \geq (B(x,D\tilde{u}) + \tau)^{\frac{1}{2}},
\]
for some positive constant \( \tau \).

Let \( v = \tilde{u} - u, v_\epsilon = \tilde{u}_\epsilon - u \). By a direct calculation, we have
\[
L v = L(v_\epsilon) + L(\frac{\tau}{2} |x - x_0|^2)
\]
\[
= \epsilon F^{ii} - \epsilon F^{ij} D_{pk} A_{ij}(x,Du)(x-x_0)_k + F^{ij} \{D_{ij}(\tilde{u}_\epsilon - u) - [A_{ij}(x,D\tilde{u}_\epsilon) - A_{ij}(x,Du)]\}
+ F^{ij} \{A_{ij}(x,D\tilde{u}_\epsilon) - A_{ij}(x,Du) - D_{pk} A_{ij}(x,Du) D_k v_\epsilon\}.
\]

Since \( F \) is concave with respect to \( w_{ij} \), we have
\[
F_{\tilde{u}} - F[u] \leq F^{ij} \{D_{ij}(\tilde{u}_\epsilon - u) - [A_{ij}(x,D\tilde{u}_\epsilon) - A_{ij}(x,Du)]\}.
\]

Using the Taylor expansion, we have for some \( \theta \in (0,1) \),
\[
A_{ij}(x,D\tilde{u}_\epsilon) - A_{ij}(x,Du) = D_{pk} A_{ij}(x,Du) D_k v_\epsilon
\]
\[
= D_{pk} A_{ij}(x,\bar{p}) D_k v_\epsilon - D_{pk} A_{ij}(x,Du) D_k v_\epsilon
= \frac{\bar{p}}{2} A_{ij,kl}(x,\bar{p}) D_k v_\epsilon D_l v_\epsilon,
\]
where \( \bar{p} = (1 - \theta) Du + \theta D\tilde{u}_\epsilon, \bar{p} = (1 - \theta) Du + \bar{\theta} D\tilde{u} \), and \( \bar{\theta} \in (0,\theta) \). Next we choose a finite family of balls \( B_\rho(x^i) \), with centres at \( x^i, i = 1 \cdots N \), covering \( \bar{\Omega} \) and with fixed radii \( \rho < 1/2 \max_{\Omega} |D_{pk} A_{ij}(x,Du)| \).

Then we select \( x_0 = x^i \) for some \( i \) such that \( x \in B_\rho(x^i) \). Accordingly, we have for a fixed positive \( \epsilon \),
\[
\mathcal{L} \geq \epsilon F^{ii} - \epsilon F^{ij} D_{pk} A_{ij}(x,Du)(x-x_0)_k + F[\tilde{u}] - F[u] + \frac{\tau}{2} F^{ij} A_{ij,kl}(x,\bar{p}) D_k v_\epsilon D_l v_\epsilon
\]
\[
\geq \epsilon F^{ii} - \frac{\epsilon}{2} F^{ij} + \frac{\tau}{2} F^{ij} A_{ij,kl}(x,\bar{p}) D_k v_\epsilon D_l v_\epsilon + (B(x,D\tilde{u}_\epsilon) + \tau)^{\frac{1}{2}} - (B(x,Du))^{\frac{1}{2}} - C_1,
\]
holds for \( x \in B_\rho(x^i) \), where \( C_1 \) is a constant depending on \( B, Du \), and \( D\tilde{u} \). We see that (2.9) holds in all balls \( B_\rho(x^i), i = 1 \cdots N \), with a fixed positive constant \( \epsilon \). Then by the finite covering, (2.9) holds in \( \Omega \) with a uniform positive constant \( \epsilon \).
Let $\phi = e^{Kv}$ with positive constant $K$ to be determined, we have
\[
L\phi = Ke^{Kv}Lv + K^2e^{Kv}F^{ij}D_i\nu D_j\nu \\
\geq Ke^{Kv}\left\{\frac{\epsilon}{2n}|F^{ij}| + \frac{\theta}{2} F^{ij}A_{ij,kl}(x,\bar{p})D_k\nu D_l\nu - C_1 + K F^{ij}D_i\nu D_j\nu \right\}.
\]
Without loss of generality, assume that $Du = (D_1\nu, 0, \cdots, 0)$, we get
\[
L\phi \geq Ke^{Kv}\left\{\frac{\epsilon}{2n}|F^{ij}| + \frac{\theta}{2} F^{ij}A_{ij,11}(x,\bar{p})(D_1\nu)^2 + K F^{11}(D_1\nu)^2 - C_1 \right\}
\geq Ke^{Kv}\left\{\frac{\epsilon}{2n}|F^{ij}| + \frac{\theta}{2} \sum_{i \text{ or } j = 1} F^{ij}A_{ij,11}(x,\bar{p})(D_1\nu)^2 + K F^{11}(D_1\nu)^2 - C_1 \right\},
\]
here we use the fact that $A$ is regular in the second inequality.

Since the matrix $\{F^{ij}\}$ is positive definite, any $2 \times 2$ diagonal minor has positive determinant. By the Cauchy’s inequality, we have
\[
|F^{ij}| \leq \sqrt{F^{ii}F^{jj}} \leq \frac{1}{2}(F^{ii} + F^{jj}), \quad \text{(which leads to)} \sum_{i,j} |F^{ij}| \leq n \sum_i F^{ii},
\]
and
\[
|F^{1i}| \leq \sqrt{F^{11}F^{ii}} \leq \eta F^{ii} + \frac{1}{4\eta} F^{11},
\]
for any positive constant $\eta$.

Thus, we have
\[
L\phi \geq Ke^{Kv}\left\{\frac{\epsilon}{2} F^{ii} - \theta \eta F^{ii}A_{1i,11}(x,\bar{p})|(D_1\nu)^2 - \frac{\theta}{4\eta} F^{11}A_{1i,11}(x,\bar{p})|(D_1\nu)^2 + K F^{11}(D_1\nu)^2 - C_1 \right\}.
\]
Choosing $\eta$ small such that $|\eta| \leq \theta \max\{|A_{11,11}(x,\bar{p})|(D_1\nu)^2\}$ and $K$ large such that $K \geq \frac{\theta \max\{|A_{11,11}(x,\bar{p})|}{4\eta}$, we obtain
\[
L\phi \geq Ke^{Kv}\left\{\frac{\epsilon}{4} F^{ii} - C_1 \right\}.
\]
Thus, we have
\[
L\phi = L\phi - \tilde{B}_{pi}D_i\phi \geq Ke^{Kv}\left\{\frac{\epsilon}{4} F^{ii} - C_1 \right\} - \tilde{B}_{pi}D_i\phi.
\]
If we choose $\epsilon_1 = \min_{\bar{\Omega}}\left\{\frac{\epsilon}{4} Ke^{Kv}\right\}$ and $C = \max_{\bar{\Omega}}\left\{C_1 Ke^{Kv} + \tilde{B}_{pi}D_i\phi\right\}$, the conclusion of this lemma is proved.

\[\square\]

**Remark 2.1.** The global barrier construction in this lemma is fundamental to derive the second order derivative estimates for solutions to (1.1)-(1.2). In Section 3 we shall use this barrier function to reduce the global estimates of second order derivatives to boundary estimates when the matrix function $A$ is regular. If $A$ is strictly regular satisfying (3.25) in the next section, we do not need such a barrier function, as further discussed in Remark 3.2. In Section 4, this barrier function will be modified a bit to fit the barrier argument so that we can get the second order derivative bounds on the boundary for both the mixed tangential-normal and the double normal directions.

**Remark 2.2.** In previous papers [24, 27], another global barrier condition called $A$-boundedness condition is assumed, namely that a domain $\Omega$ is $A$-bounded with respect to $u$, if there exists a function
\( \varphi \in C^2(\tilde{\Omega}) \) satisfying
\[
(D_{ij}\varphi - D_{pk}A_{ij}(x, Du)D_k\varphi)\xi_i\xi_j \geq \delta_0|\xi|^2,
\]
for some \( \delta_0 > 0 \) and for all \( x \in \Omega, \xi \in \mathbb{R}^n \). When the diameter of \( \Omega \) is sufficiently small, the function \( \varphi = |x|^2 \) satisfies condition (2.10) for bounded \( Du \). Also, condition (2.10) is trivial in the standard Monge-Ampère case as seen by still taking \( \varphi(x) = |x|^2 \). In the optimal transportation case, there are also various examples showing that condition (2.10) is satisfied by regular cost functions, see [17]. From the condition (2.10), we immediately have the following inequality
\[
L\varphi = F_{ij}[D_{ij}\varphi - D_{pk}A_{ij}(x, Du)D_k\varphi] - \tilde{B}_pD_i\varphi \geq \delta_0F_{ii} - C,
\]
which has a similar form of the inequality (2.6) in Lemma 2.1. In this sense, the A-boundedness condition (2.10) can provide us with an alternative global barrier function \( \varphi \).

**Remark 2.3.** By adding the perturbation function \( ae^{bx} \) for small positive constant \( a \) and large positive constant \( b \), a non-strict classical subsolution for an elliptic partial differential equation can be made strict using the linearized operator and the mean value theorem, (see [3], Chapter 3). This can also be done near the boundary, preserving the boundary condition by adding the perturbation \( ae^{bd(x)} \), where \( a \) is small positive constant, \( b \) is a large positive constant, and \( d(x) = \text{dist}(x, \partial \Omega) \) is the distance function. Hence we need only assume the existence of a non-strict subsolution in Lemma 2.1; the inequality (2.6) will still hold for the corresponding strict subsolution. Thus, the second order apriori estimates will also hold under the existence of a non-strict subsolution and we only need to assume a non-strict subsolution in the hypotheses for our theorems.

**Remark 2.4.** We observe from the proof of Lemma 2.1 that the function \( u \) does not need to be a subsolution. If \( u \) is only an admissible function satisfying \( D^2u - A(x, Du) \in \Gamma_k^+ \) in \( \tilde{\Omega} \), then we have
\[
F[u] = (S_k)^{\frac{1}{k}}(D^2u - A(x, Du)) \geq \delta^\frac{1}{k} > 0 \text{ for some } \delta > 0.
\]
The corresponding perturbation function \( u_\delta = u - \frac{\delta}{2}|x-x_0|^2 \) still satisfies \( F[u_\delta] = (S_k)^{\frac{1}{k}}(D^2u - A(x, Du)) \geq \tau \) for some positive constant \( \tau \), and the conclusion (2.6) of Lemma 2.1 still holds. Hence, the existence of a non-strict subsolution in Remark 2.3 can be further relaxed by the existence of an admissible function satisfying \( D^2u - A(x, Du) \in \Gamma_k^+ \) in \( \tilde{\Omega} \). Furthermore in the optimal transportation case such functions are constructed in [27], whence, in particular, the subsolution condition can be removed altogether from Theorems 1.1 and 2.1 in [11], for equations arising from optimal transportation; (see [10]). This also facilitates a more direct proof of Theorem 1.1 in [25].

As usual we denote the second partial derivatives of \( F \) with respect to \( w_{ij} \) by \( F^{ij,kl} \), that is \( F^{ij,kl} = \frac{\partial^2 F}{\partial w_{ij} \partial w_{kl}} \). We also need the following lemma of Andrews [11] [5], for our global estimates in the next section.
Lemma 2.2. For any $n \times n$ symmetric matrix $W = \{w_{ij}\}$, one has that

$$F^{ij,kl}w_{ij}w_{kl} = \sum_{i,j} \frac{\partial^2 f}{\partial \lambda_i \partial \lambda_j} w_{ii} w_{jj} + \sum_{i \neq j} \frac{f_i - f_j}{\lambda_i - \lambda_j} w_{ij}^2.$$  

The second term on the right hand side is non-positive if $f$ is concave, and is interpreted as a limit if $\lambda_i = \lambda_j$.

3. Global second derivative estimates

In this section, by using the preparatory lemmas in Section 2, we give the proof of Theorem 1.1, which shows that the global bounds for second derivative estimates of the equation (1.1) are reduced to their boundary estimates. Our arguments mimic those in [19] and [27], and are modifications of the arguments presented in Section 17.6 of [6]. In the following proof, the function $w$ can be regarded as an admissible function satisfying (1.16), as explained in Remark 2.3.

Proof of Theorem 1.1. Let $v$ be an auxiliary function given by

$$v(x, \xi) := \log w_{\xi \xi} + \eta \left(\frac{1}{2}|Du|^2\right) + b\phi,$$

where $w_{\xi \xi} = w_{ij} \xi_i \xi_j = (u_j - A_{ij}) \xi_i \xi_j$ with a vector $\xi \in \mathbb{R}^n$, $\phi = e^{K(u-u)}$ is the barrier function as in Lemma 2.1 with $u$ satisfying (1.16), $\eta$ is a function to be determined and $b$ is a positive constant to be determined.

By differentiating the following equation in the $\xi$ direction,

$$F[u] := F(u_{ij} - A_{ij}(x, Du)) = (S_k)^{\frac{1}{2}} [u_{ij} - A_{ij}(x, Du)] = B(x, Du),$$

we have

$$F^{ij} D_\xi w_{ij} = F^{ij} [D_{ij} u_{\xi} - D_{\xi} A_{ij} - (D_{pk} A_{ij}) D_k u_{\xi}] = D_\xi B + (D_{pk} B) D_k u_{\xi}.$$  

By a further differentiation, we obtain

$$F^{ij} D_\xi^2 w_{ij} = F^{ij} [D_{ij} u_{\xi \xi} - D_{\xi \xi} A_{ij} - 2(D_{\xi pk} A_{ij}) D_k u_{\xi} - (D_{pk}^2 A_{ij}) D_k u_{\xi} D_l u_{\xi} - (D_{pk} A_{ij}) D_k u_{\xi} + (D_{pk} B) D_k u_{\xi} D_l u_{\xi} + (D_{pk} B) D_k u_{\xi} D_l u_{\xi}].$$

Assume that $v$ takes its maximum at an interior point $x_0 \in \Omega$ and a unit vector $\xi_0$, without loss of generality we can choose an orthogonal coordinate system $e_1, \ldots, e_n$ at this point such that $e_1(x_0) = \xi_0$, $\{w_{ij}\}$ is diagonal and $w_{11} \geq \cdots \geq w_{nn}$. We immediately have $F^{ij}$ is diagonal and $F^{11} \leq \cdots \leq F^{nn}$. At the point $x_0$, the function $v(x, e_1) = \log w_{11} + \eta \left(\frac{1}{2}|Du|^2\right) + b\phi$ attains its maximum. By a direct calculation, we have at the maximum point $x_0$,

$$D_i v = D_i w_{11} \frac{w_{11}}{w_{11}} + \eta' D_k u D_{ik} u + b D_i \phi = 0,$$

$$D_i v = D_i w_{11} \frac{w_{11}}{w_{11}} + \eta' (D_{ik} u D_{jk} u + D_k u D_{ijk} u) + \eta'' D_k u D_{ik} u D_j u + b D_i \phi \leq 0,$$

and

$$0 \geq \mathcal{L} v = F^{ij} [D_{ij} v - (D_{pk} A_{ij}) D_k v] - (D_{pk} B) D_k v = \frac{1}{w_{11}} \mathcal{L} w_{11} + \eta' \sum_{k} u_k \mathcal{L} u_k + b \mathcal{L} \phi - \frac{1}{w_{11}^2} F^{ii} (D_i w_{11})^2 + \eta' F^{ii} (D_{ki} u)^2 + \eta'' F^{ii} (D_k u D_{ki} u)^2.$$

We shall estimate each term in (3.7), we first observe that the term \( \eta' \sum u_k \mathcal{L} u_k \) has a lower bound by using (3.3), that is
\[
\eta' \sum u_k \mathcal{L} u_k \geq -\eta_0' C(\sum F^{ii} + 1),
\]
where \( \eta_0' = |\eta'|_{C^0} \), the positive constant \( C \) depends on \( n, A, B, \Omega \) and sup \( (|u| + |Du|) \). Unless otherwise specified, we shall use \( C \) to denote a positive constant with such dependance in this section.

By the barrier construction in Lemma 2.1, we have an estimate for \( b\mathcal{L} \phi \), that is
\[
(3.9) \quad b\mathcal{L} \phi = b\mathcal{L} \left(e^{K(u-u)}\right) \geq b\epsilon_1 \sum F^{ii} - bC,
\]
where the positive constants \( \epsilon_1 \) and \( C \) also depend on \( u \).

By the twice differentiated equation (3.4), we have
\[
(3.10) \quad \mathcal{L} u_{11} = -F^{ij,kl} D_1 w_{ij} D_1 w_{kl} + F^{ij} (D_{11} A_{ij} + 2(D_{1p_k} A_{ij}) u_{k1}) + F^{ij} D^2_{p_k p_l} A_{ij} u_{k1} u_{l1} + D_{11} \ddot{B} + 2(D_{1p_k} \ddot{B}) u_{k1} + (D_{p_k p_l} \ddot{B}) u_{k1} u_{l1},
\]
where
\[
F^{ij} (D_{11} A_{ij} + 2(D_{1p_k} A_{ij}) u_{k1}) = F^{ij} (D_{11} A_{ij} + 2(D_{1p_k} A_{ij})(w_{k1} + A_{k1})) \geq -CF^{ii} (1 + w_{11}),
\]
and
\[
F^{ij} D^2_{p_k p_l} A_{ij} u_{k1} u_{l1} = F^{ij} A_{i,j,kl}(w_{11} + A_{k1})(w_{11} + A_{l1}) = F^{ij} A_{i,j,kl} w_{11} + 2A_{i,j,kl} w_{k1} A_{l1} + A_{i,j,kl} A_{k1} A_{l1} = F^{ij} (A_{i,j,11} w_{11}^2 + 2A_{i,j,11} w_{11} A_{kl} + A_{i,j,kl} A_{k1} A_{l1}) \geq CF^{ii} (1 + w_{11}) \geq -CF^{ii} (1 + w_{11}),
\]
here we use the regular condition (1.5) for the matrix \( A \) in the first inequality. So we obtain that
\[
(3.11) \quad \mathcal{L} u_{11} \geq -F^{ij,kl} D_1 w_{ij} D_1 w_{kl} + (D^2_{p_k p_l} \ddot{B}) u_{k1} u_{l1} - C(1 + w_{11} + F^{ii} + F^{ii} w_{11}) - CF^{ii} w_{11}^2,
\]
here we use property (1.12): \( \sum f_i \geq \sigma_0 \) on \( \Gamma_{k_{1+1,2}} \) and the convexity condition (1.15) to derive the second inequality.

To obtain the estimate \( \mathcal{L} w_{11} \), we also need to estimate the term \( \mathcal{L} A_{11} \). We evaluate this term in detail,
\[
\mathcal{L} A_{11} = F^{ij} [D_{i,j} A_{11} + D_{ip_k} A_{11}(w_{k1} + A_{k1}) + D_{jp_k} A_{11}(w_{k1} + A_{k1}) + D^2_{p_k p_l} A_{11}(w_{k1} + A_{k1})(w_{l1} + A_{l1}) + D_{p_k} A_{11} u_{k1} - D_{p_k} A_{11} d_{k1} A_{11} - D_{p_k} A_{11} u_{k1} + A_{k1}] - D_{p_k} \ddot{B}[D_{k1} A_{11} + D_{p_l} A_{11}(w_{kl} + A_{kl})] \leq CF^{ii} (1 + w_{11}) + F^{ij} D^2_{p_k p_l} A_{11} w_{k1} w_{l1} + F^{ij} D_{p_k} A_{11} u_{k1} \leq CF^{ii} (1 + w_{11}) + CF^{ii} w_{11}^2 + F^{ij} D_{p_k} A_{11} u_{k1}.
\]
The above third derivative term can be estimated by using the differentiated equation (3.3), that is
\[
F^{ij} D_{p_k} A_{11} u_{k1} = F^{ij} D_{p_k} A_{11} u_{jik} = D_{p_k} A_{11} [F^{ij} A_{i,j,11} w_{k1} + A_{i,k} + \ddot{B}_{k1} + D_{p_l} \ddot{B} u_{l1}] = D_{p_k} A_{11} \{F^{ij} A_{i,j,11} w_{k1} + A_{i,k} + \ddot{B}_{k1} + D_{p_l} \ddot{B} u_{l1}\} \leq CF^{ii} (1 + w_{11}) + CF^{ii} w_{11}^2,
\]
here the property (1.12): \( \sum f_i \geq \sigma_0 \) on \( \Gamma_{k_{1+1,2}} \) is used. So we get
\[
(3.12) \quad \mathcal{L} A_{11} \leq CF^{ii} (1 + w_{11}) + CF^{ii} w_{11}^2.
\]
Combining (3.11) and (3.12), we obtain the estimate for \( \mathcal{L} w_{11} \),
\[
(3.13) \quad \mathcal{L} w_{11} = \mathcal{L} u_{11} - \mathcal{L} A_{11} \geq -F^{ij,kl} D_1 w_{ij} D_1 w_{kl} - C[F^{ii} (1 + w_{11}) + w_{ii}] - CF^{ii} w_{11}^2 - CF^{11} w_{11}^2.
\]
With the above estimates in hand, (3.7) becomes

\[
0 \geq \mathcal{L}v \geq -\frac{1}{w_{11}} F_{ij,kl}^{ij} D_{1} w_{ij} D_{1} w_{kl} - \frac{1}{w_{11}} F_{ii}^{ii} (D_{1} w_{11})^2 \\
+ \eta' F_{ii}^{ii} (D_{ik} u)^2 + \eta'' F_{ii}^{ii} (D_{ik} u)^2 - C \frac{w_{11}^2}{w_{11}^2} F_{ii}^{ii} w_{11}^2 - CF_{11} w_{11} \\
- \frac{C}{w_{11}} [F_{ii}^{ii} (1 + w_{jj}) + w_{ii}] + (b_1 - \eta' h C)F_{ii}^{ii} - C(b + \eta' h). \tag{3.14}
\]

Firstly, we need to estimate the first two terms on the right hand side of (3.14). We recall that \(w_{11}\) is the largest eigenvalue of \(\{w_{ij}\}\), and set

\[
I = \{i : w_{ii} \leq -\omega w_{11}\}, \quad J = \{i > 1 : w_{ii} > -\omega w_{11}\},
\]

where \(\omega \in (0, 1)\) is a positive constant to be chosen later. We have \(1 \notin I, 1 \notin J, I \cap J = \emptyset, 1 \in I \cup J = \{1, 2, \ldots, n\}, \) and \(J_c = 1 \cup I,\) where \(J_c\) is the complementary set of \(J.\) By Lemma 2.2 and the concavity of the operator \(F = (S_k)_{\frac{1}{2}},\) we have

\[
-\frac{1}{w_{11}} F_{ij,kl}^{ij} D_{1} w_{ij} D_{1} w_{kl} \geq -\frac{1}{w_{11}} \sum_{i \neq j} F_{ii}^{ii} - F_{ij}^{ij} (D_{1} w_{ij})^2 \\
\geq -\frac{2}{w_{11}} \sum_{i \geq 2} F_{ii}^{ii} - F_{ii}^{11} (D_{1} w_{ii})^2 \\
\geq -\frac{2}{w_{11}} \sum_{i \in J} F_{ii}^{ii} - F_{ii}^{11} (D_{1} w_{ii})^2 \\
\geq \frac{2}{w_{11}} \sum_{i \in J} (F_{ii}^{ii} - F_{ii}^{11}) [D_{i} w_{11} + (D_{i} A_{111} - D_{i} A_{111})]^2 \\
\geq \frac{2}{w_{11}} \sum_{i \in J} (F_{ii}^{ii} - F_{ii}^{11}) [(D_{i} w_{11})^2 - \frac{1}{\omega} (D_{i} A_{111} - D_{i} A_{111})^2] \\
\geq \frac{1}{w_{11}} \sum_{i \in J} F_{ii}^{ii} (D_{i} w_{11})^2 - C \frac{w_{11}^2}{w_{11}^2} \sum_{i \in J} F_{ii}^{ii} - F_{ii}^{11} \sum_{i \in J} (D_{i} w_{11})^2. \tag{3.16}
\]

Here we use the Cauchy’s inequality in the last second inequality, and we chose \(\omega = 1/3\) to obtain the last inequality. Therefore, by the relationship between the set \(I\) and \(J,\) we have

\[
-\frac{1}{w_{11}} F_{ij,kl}^{ij} D_{1} w_{ij} D_{1} w_{kl} - \frac{1}{w_{11}} F_{ii}^{ii} (D_{1} w_{11})^2 \\
\geq \sum_{i \in I} F_{ii}^{ii} (D_{i} w_{11})^2 - C \frac{w_{11}^2}{w_{11}^2} \sum_{i \in J} F_{ii}^{ii} - 2F_{11} \sum_{i \notin I} (D_{i} w_{11})^2 \\
\geq \sum_{i \in I} F_{ii}^{ii} (D_{i} k u D_{i} k u + b D_{i} \phi)^2 - C \frac{w_{11}^2}{w_{11}^2} \sum_{i \in J} F_{ii}^{ii} - 2F_{11} \sum_{i \notin I} (D_{i} k u D_{i} k u + b D_{i} \phi)^2 \\
\geq (\eta')^2 \sum_{i \in I} F_{ii}^{ii} (D_{i} k u D_{i} k u)^2 - b^2 C \sum_{i \in I} F_{ii}^{ii} - C \frac{w_{11}^2}{w_{11}^2} \sum_{i \in J} F_{ii}^{ii} - CF_{11} [(\eta')^2 w_{11}^2 + (\eta')^2 + b^2]. \tag{3.17}
\]

Where the second inequality is from (3.5). Next, if we choose the function \(\eta(t) = \frac{a}{2} (1 + t)^2,\) we have \(\eta(t) = a(t + 1), \eta''(t) = a\) and \(\eta''(t) - (\eta')^2 = a - a^2(1 + t)^2.\) For any \(t \in [0, C],\) we choose the positive constant \(a\) sufficiently small such that \(\eta''(t) - (\eta')^2 \geq 0.\) Therefore, we have now determined the function

\[
\eta(t) = \frac{a}{2} (1 + t)^2, \tag{3.18}
\]

Where \(a\) is a small positive constant.
Consequently, by (3.14), (3.17) and (3.18), we have

\begin{equation}
0 \geq \mathcal{L}v \geq \eta'F^{ii}(D_{ik}u)^2 + [\eta'' - (\eta')^2] F^{ii}(D_kuD_{ik}u)^2 - \frac{C}{w_{11}} F^{ii}w_{ii}^2 - CF^{11}w_{11} \\
- \frac{C}{w_{11}} [F^{ii}(1 + w_{jj})] + (b\varepsilon_1 - \eta_0 C) F^{ii} - C(b + \eta_0') \nonumber
\end{equation}

\begin{equation}
(3.19)
- b^2C \sum_{i \in I} F^{ii} - \frac{C}{w_{11}} \sum_{i \in I} F^{ii} - CF^{11}[(\eta')^2w_{11}^2 + (\eta')^2 + b^2] \\
\geq (\eta' - \frac{w}{w_{11}}) F^{ii}w_{ii}^2 - b^2C \sum_{i \in I} F^{ii} + (b\varepsilon_1 - \eta_0 C - C) F^{ii} - C(b + \eta_0') \nonumber
\end{equation}

\begin{equation}
- CF^{11}[(\eta')^2w_{11}^2 + (\eta')^2 + b^2] - CF^{11}w_{11}. \nonumber
\end{equation}

Note that we can always suppose \( w_{11} \) as large as we want, otherwise \( w_{11} \) is bounded and the proof is finished. We first choose \( w_{11} \) large such that \( \eta'/2 \geq C/w_{11} \). By the property (1.13), when \( k > 1 \), we can choose the constant \( b \) sufficiently large such that \( (b\varepsilon_1 - \eta_0 C - C) F^{ii} - C(b + \eta_0') > 0 \). Thus, we obtain from (3.19),

\begin{equation}
\frac{\eta'}{2} F^{ii}w_{ii}^2 \leq b^2C \sum_{i \in I} F^{ii} + C(\eta')^2F^{11}w_{11}^2 + Cb^2F^{11} + CF^{11}w_{11}. \nonumber
\end{equation}

Since we have

\begin{equation}
\frac{\eta'}{2} F^{ii}w_{ii}^2 \geq \frac{\eta'}{2} F^{11}w_{11}^2 + \frac{\eta'}{2} \sum_{i \in I} F^{ii}w_{ii}^2 \nonumber
\end{equation}

\begin{equation}
(3.21)
\geq \frac{\eta'}{2} F^{11}w_{11}^2 + \frac{\eta'}{18} w_{11}^2 \sum_{i \in I} F^{ii}. \nonumber
\end{equation}

By choosing \( a > 0 \) sufficiently small and \( w_{11} \) sufficiently large, from (3.20) and (3.21), we obtain the estimate, at \( x_0 \)

\begin{equation}
\frac{a}{4} F^{11}w_{11}^2 \leq Cb^2F^{11} + CF^{11}w_{11}, \nonumber
\end{equation}

that is

\begin{equation}
(3.22)
\frac{a}{4} w_{11}^2 \leq Cb^2 + Cw_{11}, \nonumber
\end{equation}

which directly leads to

\begin{equation}
(3.23)
w_{11} \leq C. \nonumber
\end{equation}

This implies the conclusion (1.17) for \( 2 \leq k \leq n \). While \( k = 1 \), since we have \( F^{ii} = 1 \) for \( i = 1 \cdots n \), then conclusion (1.17) can be easily derived from (3.19). Note that in this case the conclusion (1.22) also follows from the classical Schauder theory in [6]. Therefore, we complete the proof of Theorem 1.1.

\[ \Box \]

**Remark 3.1.** We call the matrix \( A \) strictly regular if

\begin{equation}
A_{ij,k\ell}(x,p)\xi_i\xi_j\eta_k\eta_\ell > |\xi|^2|\eta|^2, \nonumber
\end{equation}

for all \((x,p) \in \tilde{\Omega} \times \mathbb{R}^n, \xi, \eta \in \mathbb{R}^n, \xi \perp \eta.\) The global second derivative estimate (1.17) for the solution \( u \) will become much simpler if \( A \) is strictly regular. In this case, we only need to consider the simpler auxiliary function \( v := w_{\xi \xi} \) and maximize it over \( \tilde{\Omega} \), since the strict regular condition directly implies

\begin{equation}
F^{ij}D_{p_kp_l}A_{ij}D_{k\xi}uD_{\ell\xi}u \geq C_1F^{ij}|D^2u|^2, \nonumber
\end{equation}

\( (3.26) \)
for some $C_1 > 0$. By differentiating the equation (3.2) twice, we then have
\begin{equation}
0 \geq L u \geq F^{ii}[C_1|D^2 u|^2 - C_2(|D^2 u| + 1)] + (D^2_{p,q}) \tilde{B} u_{k1} u_{l1}
\geq F^{ii}[C_1|D^2 u|^2 - C_2(|D^2 u| + 1)],
\end{equation}
holds at the maximum point $x_0$, for positive constants $C_1$, $C_1'$ and $C_2$, here the property (1.13) is used to obtain the second inequality. Therefore, one easily obtains the global second derivative estimate (1.17).

Thus, to obtain the global second derivative estimate, the barrier construction in Lemma 2.1 is not required when the matrix function $A$ is strictly regular. Moreover in this case, the convexity condition (1.15) on $B$ is also not needed and we obtain more general interior estimates; (see [24], Theorem 2.1).

**Remark 3.2.** Under the assumption that the matrix $A$ is regular satisfying (1.5) on the right hand side $\tilde{B}$ to obtain the global second derivative estimates for both the Hessian operator $(S_k)^2 \hat{x}$ here and the Hessian quotient operator $(S_k/S_l)^{1-\epsilon}$ $(l < k \leq n)$, see [31] for the special case $(S_k/S_l)^{1-\epsilon}$ $(l < n)$; while in the Monge-Ampère case, the special structure of the determinant function allows us to obtain the second derivative estimate even if $\tilde{B}$ does not satisfy the convexity condition, see [27, 17, 11]. If an operator $f$ satisfies a stronger property, namely $\sum_i f_i/|\lambda| \rightarrow \infty$ as $|\lambda| \rightarrow \infty$, we also do not need to impose the convexity condition on the right hand side function $\tilde{B}$, see [28, 29] for references.

4. **Boundary estimates for second derivatives**

In this section, we shall establish the second derivative estimate $|D^2 u| \leq C$ on the boundary $\partial \Omega$ and finish the proof of Theorem 1.2. First, we note from Remark 2.3 that a non-strict subsolution can be made strict near the boundary. Hence, we can assume the subsolution $\bar{u}$ is strict provided we restrict to a neighbourhood of $\partial \Omega$. Accordingly, Lemma 2.4 can be retained in a neighbourhood of $\partial \Omega$, which will suffice for the boundary estimates. Note that in this section, $\bar{u}$ denotes the subsolution rather than merely an admissible function satisfying (1.16).

Next, we need to check the invariance properties of the equation (1.1) under translation and rotation of coordinates. For a fixed point $x_0 \in \Omega$, let $\bar{x} = x - x_0$, the equation (1.1) becomes
\[ S_k[D^2_{\bar{x}}u - A(\bar{x} + x_0, D_{\bar{x}}u)] = B(\bar{x} + x_0, D_{\bar{x}}u). \]
Let $\bar{A}(\bar{x}, D_{\bar{x}}u) = A(\bar{x} + x_0, D_{\bar{x}}u)$ and $\bar{B}(\bar{x}, D_{\bar{x}}u) = B(\bar{x} + x_0, D_{\bar{x}}u)$, we have
\[ S_k[D^2_{\bar{x}}u - \bar{A}(\bar{x}, D_{\bar{x}}u)] = \bar{B}(\bar{x}, D_{\bar{x}}u). \]
Accordingly we see that both the form of the equation and the regular condition (1.5) are invariant under translation of coordinates. So, we may suppose any $x_0 \in \Omega$ as the origin if necessary.

Let $\hat{x} = Rx$, where $R$ is a rotation matrix. Since the rotation matrix $R$ is orthogonal satisfying $\det R = 1$, the equation (1.1) becomes
\[ S_k \left\{ R^T [D^2 u - RA(R^{-1}\hat{x}, R^{-1}D_{\hat{x}}u)R^{-1}] \right\} = B(R^{-1}\hat{x}, R^{-1}D_{\hat{x}}u). \]
Observing that $S_k$ is invariant under rotation of coordinates, we have
\[ S_k[D^2_{\hat{x}}u - \hat{A}(\hat{x}, D_{\hat{x}}u)] = \hat{B}(\hat{x}, D_{\hat{x}}u), \]
with
\[ \left\{ \begin{array}{l}
\hat{A}(\hat{x}, D_{\hat{x}}u) = RA(R^{-1}\hat{x}, R^{-1}D_{\hat{x}}u)R^{-1}, \\
\hat{B}(\hat{x}, D_{\hat{x}}u) = B(R^{-1}\hat{x}, R^{-1}D_{\hat{x}}u).
\end{array} \right. \]
Thus, we obtain the invariance of both the form of equation (1.1) and the regular condition (1.5) under rotations of coordinates.

Consequently, for any given boundary point \( x_0 \in \partial \Omega \), by a translation and a rotation of the coordinates, we may take \( x_0 \) as the origin and take the positive \( x_n \) axis to be the inner normal of \( \partial \Omega \) at the origin. Near the origin, \( \partial \Omega \) can be represented as a graph

\[
x_n = \rho(x'),
\]

such that \( D\rho(0) = 0 \), where \( x' = (x_1, \cdots, x_{n-1}) \). Since \( u - \overline{u} = 0 \) on \( \partial \Omega \), we then have by differentiation twice,

\[
D_{\alpha\beta}(u - \overline{u})(0) = -D_n(u - \overline{u})(0)\rho_{\alpha\beta}(0), \quad \alpha, \beta = 1, \cdots, n - 1,
\]

which leads to the double tangential derivative estimate \(|D_{\alpha\beta}u(0)| \leq C, \alpha, \beta = 1, \cdots, n - 1\).

We then estimate the mixed tangential-normal derivatives \( D_{\alpha n}u(0), \alpha = 1, \cdots, n - 1 \). To obtain this estimation, we shall apply the standard barrier argument by modifying the barrier function introduced in Lemma 2.1. Rewrite equation (1.1) in the following form,

\[
F[u] = F(u_{ij} - A_{ij}(x, Du)) = (S_k)^{\frac{1}{2}} [u_{ij} - A_{ij}(x, Du)] = \tilde{B}(x, Du).
\]

By differentiating (4.3) with respect to \( x_k \), we have

\[
F^{ij}_k(u_{kij} - D_k A_{ij}(x, Du) - D_{p_i} A_{ij}(x, Du) u_{kl}) = \tilde{B}_k(x, Du) + \tilde{B}_{p_i}(x, Du) u_{kl}, \quad k = 1, \cdots, n,
\]

which leads to

\[
\mathcal{L}D_k u = \tilde{B}_k(x, Du) + F^{ij}_k D_k A_{ij}(x, Du), \quad k = 1, \cdots, n,
\]

where \( \mathcal{L} \) is defined by (2.3). For fixed \( \alpha < n \), we consider the following operator

\[
T = \partial_\alpha + \sum_{\beta < n} \rho_{\alpha\beta}(0) (x_\beta \partial_n - x_n \partial_\beta).
\]

By calculation, we have

\[
\mathcal{L}T u = \mathcal{L}[D_\alpha u + \sum_{\beta < n} \rho_{\alpha\beta}(0) (x_\beta D_n u - x_n D_\beta u)]
\]

\[
= \mathcal{L}D_\alpha u + \sum_{\beta < n} \rho_{\alpha\beta}(0) (x_\beta \mathcal{L}D_n u - x_n \mathcal{L}D_\beta u)
\]

\[
+ \sum_{\beta < n} \rho_{\alpha\beta}(0) \left[ 2(F^{\beta j} u_{nj} - F^{nj} u_{\beta j}) - F^{ij}(D_{p_\beta} A_{ij} u_n - D_{p_n} A_{ij} u_\beta) - (\tilde{B}_{p_\beta} u_n - \tilde{B}_{p_n} u_\beta) \right].
\]

For fixed \( \beta < n \), we observe that

\[
F^{\beta j} u_{nj} = F^{\beta j} A_{nj} \quad \text{and} \quad F^{nj} u_{\beta j} = F^{nj} A_{\beta j}.
\]

Combining (4.7), (4.8), and the differentiated equation (4.5) for \( k = 1, \cdots, n - 1 \), we derive

\[
|\mathcal{L}T (u - \overline{u})| \leq C(1 + \sum_i F^{ii}).
\]

We also observe that, on \( \partial \Omega \) near the origin,

\[
|T (u - \overline{u})| \leq C|x|^2.
\]

Next, we shall modify the barrier function constructed in Lemma 2.1 to get through barrier argument near the boundary. Let \( d = d(x) \) be the distance function from \( \partial \Omega \), we may take \( \delta > 0 \) small enough so that \( d \) is a smooth function in \( \Omega_\delta = \Omega \cap B_\delta(0) \). The key ingredient is the following lemma:
Lemma 4.1. For any $M > 0$, there exist constants $K$, $N$ sufficiently large and $\mu$, $\delta$ sufficiently small, such that the function
\[
\psi = 1 - \exp \left\{ K \left[ (u - u) - \mu d + Nd^2 \right] \right\},
\]
satisfies
\[
(4.11) \quad L\psi \leq -\frac{\epsilon_1}{2} \sum_i F^{ii} - M, \quad \text{in } \Omega_\delta, \quad \text{and } \psi \geq 0, \quad \text{on } \partial \Omega_\delta,
\]
for some positive constant $\epsilon_1$.

Proof. We shall follow the proof of Lemma 2.1 and make some necessary changes. By calculation, we have
\[
(4.12) \quad |Ld| \leq C \sum_i F^{ii},
\]
and
\[
Ld^2 = 2dLd + 2F^{ij}d_{ij},
\]
where $L$ is the linearized operator defined by (2.3).

For any $x_0 \in \Omega_\delta$, the perturbation function \( \underline{u} = u - \frac{\epsilon_1}{2} |x - x_0|^2 \) is still a strict subsolution of the equation (1.1). Let $v = (u - u) - \mu d + Nd^2$, by a calculation as in (2.7), we have
\[
(4.13) \quad Lv = L([u - u] - \mu d + Nd^2) = \epsilon F^{ii} + \epsilon F^{ij}D_{p_k}A_{ij}(x, Du)(x - x_0) + \epsilon F^{ij}[D_{ij}\underline{u} - A_{ij}(x, D\underline{u}) + 2Nd_{ij}] - \epsilon F^{ij}[D_{ij}u - A_{ij}(x, Du)] + \epsilon F^{ij}[A_{ij}(x, D\underline{u}) - A_{ij}(x, Du) - D_{p_k}A_{ij}(x, Du) D_{k}v]
\]
\[+(2Nd - \mu)Ld.\]

By the concavity of $F$, we have
\[
(4.14) \quad F^{ij}[D_{ij}\underline{u} - A_{ij}(x, D\underline{u}) + 2Nd_{ij}] \geq F[D_{ij}u - A_{ij}(x, Du)] - F[D_{ij}u - A_{ij}(x, Du)].
\]
Since $d_n(0) = 1$, $d_{\beta}(0) = 0$ for all $\beta < n$, and $d$ is a smooth function in $\Omega_\delta$, we can choose $\delta$ sufficiently small such that $d_n \geq 1/2$ and $d_{\beta} \leq 1/2$ in $\Omega_\delta$. By the property (1.11), we observe that the term $F(D_{ij}\underline{u}) - A_{ij}(x, D\underline{u}) + 2Nd_{ij})$ can be made as large as we want by choosing $N$ large enough. If $N$ is chosen large enough such that
\[
(4.15) \quad F(D_{ij}\underline{u} - A_{ij}(x, D\underline{u}) + 2Nd_{ij} \geq \max_{\Omega} \beta(x, Du) + \bar{M},
\]
where $\bar{M}$ is a large positive constant to be determined. We fix the radius $\delta < 1/2n$ max $|D_{p_k}A_{ij}(x, Du)|$. By (2.8), (4.12) and (4.15), for $x \in \Omega_\delta$, we have from (4.13),
\[
(4.16) \quad Lv \geq \epsilon F^{ii} + \epsilon F^{ij}D_{p_k}A_{ij}(x, Du)(x - x_0) + \frac{\epsilon}{2} F^{ij}A_{ij,kl}(x, \bar{p})D_{kl}v - C(\mu + 2N\delta)F^{ii} + \bar{M}
\]
\[\geq \frac{\epsilon}{2} F^{ii} + \frac{\epsilon}{2} F^{ij}A_{ij,kl}(x, \bar{p})D_{kl}v - C(\mu + 2N\delta)F^{ii} + \bar{M}.
\]
Let $\phi = e^{Ku}$ with positive constant $K$ to be determined. Following the same way to estimate $L\phi$ in Lemma 2.1 (by using the regularity condition (1.5) of the matrix function $A$ and the Cauchy’s inequality), we have
\[
(4.17) \quad L\phi \geq Ke^{Ku} \left\{ \frac{\epsilon}{4} F^{ii} - C(\mu + 2N\delta)F^{ii} + \bar{M} \right\},
\]
holds for sufficiently large positive constant $K$, here $K$ is fixed now.

Thus, by choosing $\mu = \epsilon/16C$ and $\delta = \epsilon/32NC$ small enough, we have
\[
(4.18) \quad L\phi = L\phi - \bar{B}_p \bar{D}_1\phi \geq Ke^{Ku} \left\{ \frac{\epsilon}{8} F^{ii} + \bar{M} \right\} - \bar{B}_p \bar{D}_1\phi.
\]
We choose \( \tilde{M} = \max \left\{ \frac{M + \tilde{B} \pi D_i \phi}{K e^{K \nu}} \right\} \), and set \( \epsilon_1 = \min \left\{ \frac{\epsilon}{4} K e^{K \nu} \right\} \), (note that \( K e^{K \nu} \) is small when \( K \) is a large constant, \( \epsilon_1 \) is the same constant as in Lemma 2.1), then

\[
(4.19) \quad \mathcal{L} \phi \geq \frac{\epsilon_1}{2} \sum_i F^{ii} + M.
\]

Since the constant \( \tilde{M} \) is now fixed, the large enough constant \( N \) can be fixed now to guarantee the inequality (4.15). Therefore, we have

\[
(4.20) \quad \mathcal{L} \psi = \mathcal{L}(1 - \phi) \leq -\frac{\epsilon_1}{2} \sum_i F^{ii} - M, \quad \text{in } \Omega_{\delta}.
\]

On \( \partial \Omega \cap \Omega_{\delta} \), we have \( v = 0 \). Since \( \mu \) and \( \delta \) are given by \( \mu = \epsilon / 16C \) and \( \delta = \epsilon / 32NC \) respectively, we get

\[
(4.21) \quad v = (u - u) - \mu d + N d^2 \leq (-\mu + N \delta) d = -\frac{\epsilon d}{32C} \leq 0, \quad \text{on } \Omega \cap \partial \Omega_{\delta}.
\]

Consequently, we have \( \phi = e^{K \nu} \leq 1 \) on \( \Omega \cap \partial \Omega_{\delta} \), furthermore,

\[
(4.22) \quad \psi = 1 - \phi \geq 0, \quad \text{on } \Omega \cap \partial \Omega_{\delta}.
\]

Therefore, we get \( \psi \geq 0 \) on \( \partial \Omega_{\delta} \). Together with the inequality (4.20), the conclusion (4.11) of Lemma 4.1 is proved.

With the barrier function \( \psi \) in hand, we shall employ a new barrier function

\[
(4.23) \quad \tilde{\psi} = a \psi + b |x|^2,
\]

with positive \( a \) and \( b \) to be determined. We can choose \( a \gg b \gg 1 \), then

\[
(4.24) \quad \mathcal{L} \tilde{\psi} = a \mathcal{L} \psi + b \mathcal{L}|x|^2 \leq -\frac{a \epsilon_1}{2} (1 + \sum_i F^{ii}),
\]

and

\[
(4.25) \quad \tilde{\psi} \geq b |x|^2, \quad \text{on } \partial \Omega_{\delta}.
\]

Therefore, for \( a \gg b \gg 1 \) and \( \delta \ll 1 \), there holds

\[
(4.26) \quad \left\{ \begin{array}{l}
|\mathcal{L}T(u - u)| + \mathcal{L} \tilde{\psi} \leq 0, \quad \text{in } \Omega_{\delta}, \\
|T(u - u)| \leq \tilde{\psi}, \quad \text{on } \partial \Omega_{\delta}.
\end{array} \right.
\]

By the maximum principle, we obtain the mixed tangential-normal derivative estimate

\[
(4.27) \quad |D_{\alpha n} u(0)| \leq C, \quad \alpha = 1, \ldots, n - 1.
\]

Finally, we will adapt the technique in [22] to estimate the double normal derivative \( D_{nn} u \) on the boundary. Note that the regularity of the matrix function \( A \) is also critical for this estimate. We observe that equation (1.1) can be rewritten as

\[
(4.28) \quad (D_{nn} u - A_{nn}(x, Du)) S_{k-1} \left\{ [D^2 u - A(x, Du)]' \right\} + R = B(x, Du),
\]

where \( [D^2 u - A(x, Du)]' = \{D_{\alpha \beta} - A_{\alpha \beta}(x, Du)\}_{1 \leq \alpha, \beta \leq n-1} \), \( R \) denotes the remaining terms which do not involve \( D_{nn} u - A_{nn}(x, Du) \). From the above double tangential derivative and mixed tangential-normal derivative bounds, the term \( R \) is bounded. Since an upper bound

\[
(4.29) \quad D_{nn} u(x) \leq C,
\]

is equivalent to an upper bound

\[
(4.30) \quad D_{nn} u(x) - A_{nn}(x, Du) \leq C,
\]
in order to obtain an upper bound \( D_{nn}u(x) \leq C \), by (4.28) we only need to get a positive lower bound for \( S_{k-1}\{[D^2u - A(x, Du)]\} \).

For any boundary point \( x \in \partial \Omega \), let \( \xi^{(1)}, \ldots, \xi^{(n-1)} \) be an orthogonal vector field on \( \partial \Omega \). Writing
\[
\nabla_\alpha u = \xi^{(\alpha)}_m D_m u, \quad \nabla_{\alpha\beta} u = \xi^{(\alpha)}_m \xi^{(\beta)}_l D_{ml} u, \quad \xi^{(\alpha)}_m = \xi^{(\alpha)}_m \xi^{(\beta)} D_m \gamma_l, \quad 1 \leq \alpha, \beta \leq n - 1,
\]
\[
\nabla u = (\nabla_1 u, \ldots, \nabla_{n-1} u), \quad A_{\alpha\beta}(x, \nabla u, -D_\gamma u) = \xi^{(\alpha)}_m \xi^{(\beta)} \gamma_l A_{ml}(x, \nabla u, -D_\gamma u), \quad 1 \leq \alpha, \beta \leq n - 1,
\]
and
\[
\nabla^2 u = \{\nabla_{\alpha\beta} u\}_{1 \leq \alpha, \beta \leq n - 1}, \quad \xi = \{\xi_{\alpha\beta}\}_{1 \leq \alpha, \beta \leq n - 1},
\]
where \( \gamma \) is the unit outer normal of \( \partial \Omega \) at \( x \). From the boundary condition \( u = \varphi \) on \( \partial \Omega \), we have
\[
\nabla^2 (u - \varphi) = D_\gamma (u - \varphi) \xi,
\]
which agree with (4.2). Thus, we have, on the boundary \( \partial \Omega \),
\[
\nabla_{\alpha\beta} u - A_{\alpha\beta}(x, \nabla u, -D_\gamma u) = D_\gamma (u - \varphi) \xi_{\alpha\beta} + \nabla_{\alpha\beta} \varphi - A_{\alpha\beta}(x, \nabla \varphi, -D_\gamma u), \quad \alpha, \beta = 1, \ldots, n - 1,
\]
Denote
\[
G[u](x) = G(\nabla_{\alpha\beta} u - A_{\alpha\beta}(x, \nabla u, -D_\gamma u)) = G(D_\gamma (u - \varphi) \xi_{\alpha\beta} + \nabla_{\alpha\beta} \varphi - A_{\alpha\beta}(x, \nabla \varphi, -D_\gamma u)) = \{S_{k-1}[D_\gamma (u - \varphi) \xi_{\alpha\beta} + \nabla_{\alpha\beta} \varphi - A_{\alpha\beta}(x, \nabla \varphi, -D_\gamma u)]\}^{1/2} \text{ on } \partial \Omega,
\]
and
\[
G^\alpha_{\beta} = \frac{\partial G}{\partial r_{\alpha\beta}},
\]
where \( r_{\alpha\beta} = D_\gamma (u - \varphi) \xi_{\alpha\beta} + \nabla_{\alpha\beta} \varphi - A_{\alpha\beta}(x, \nabla \varphi, -D_\gamma u) \).

Assume that \( \inf_{x \in \partial \Omega} G \) is attained at \( x_0 \in \partial \Omega \), we have for all \( x \in \partial \Omega \),
\[
G(D_\gamma (u - \varphi)(x) \xi_{\alpha\beta}(x) + \nabla_{\alpha\beta} \varphi(x) - A_{\alpha\beta}(x, \nabla \varphi(x), -D_\gamma u(x))) \geq G(D_\gamma (u - \varphi)(x_0) \xi_{\alpha\beta}(x_0) + \nabla_{\alpha\beta} \varphi(x_0) - A_{\alpha\beta}(x_0, \nabla \varphi(x_0), -D_\gamma u(x_0))).
\]
By the concavity of \( G \), we have
\[
G^\alpha_{\beta} [D_\gamma (u - \varphi)(x) \xi_{\alpha\beta}(x) + \nabla_{\alpha\beta} \varphi(x) - A_{\alpha\beta}(x, \nabla \varphi(x), -D_\gamma u(x))] \geq G^\alpha_{\beta} [D_\gamma (u - \varphi)(x_0) \xi_{\alpha\beta}(x_0) + \nabla_{\alpha\beta} \varphi(x_0) - A_{\alpha\beta}(x_0, \nabla \varphi(x_0), -D_\gamma u(x_0))],
\]
where \( G_{\alpha\beta}^\gamma = G^\alpha_{\beta}(D_\gamma (u - \varphi)(x_0) \xi_{\alpha\beta}(x_0) + \nabla_{\alpha\beta} \varphi(x_0) - A_{\alpha\beta}(x_0, \nabla \varphi(x_0), -D_\gamma u(x_0))) \).

We consider two possible cases:
\[\text{Case 1.} G[u] \geq G[u]/2 \text{ at } x_0. \text{ An upper bound for } D_{nn}u(x_0) \text{ follows directly from equation (4.28).} \]
\[\text{Case 2.} G[u] < G[u]/2 \text{ at } x_0. \text{ Fixing a principle coordinate system at the point } x_0 \text{ and a corresponding neighbourhood } N \text{ of } x_0 \text{ with } \gamma_n < 0 \text{ on } T = N \cap \partial \Omega. \text{ Since the orthogonal vector field } \xi^{(1)}, \ldots, \xi^{(n-1)} \text{ agrees with the coordinate system at the point } x_0, \text{ we have } \xi^{(i)}_j(x_0) = \delta_{ij}, i, j = 1, \ldots, n - 1, -D_\gamma u(x_0) = D_n u(x_0), \xi_{\alpha\beta}(x_0) = D_{n}\gamma_{\alpha\beta}(x_0), \text{ and } A_{\alpha\beta}(x_0, \nabla \varphi(x_0), -D_\gamma u(x_0)) = A_{\alpha\beta}(x_0, D' \varphi(x_0), D_n u(x_0)), \text{ where } D' = (D_1, \ldots, D_{n-1}). \text{ By the concavity of } G, \text{ we have, at } x_0,
\]
\[
G[u](x_0) - G[u](x_0) \geq G_{\alpha\beta}^\gamma [D_\gamma (u - \varphi)(x_0) \xi_{\alpha\beta}(x_0) + \nabla_{\alpha\beta} \varphi(x_0) - A_{\alpha\beta}(x_0, \nabla \varphi(x_0), -D_\gamma u(x_0))],
\]
where
\[
G_{\alpha\beta}^\gamma = G^\alpha_{\beta}(D_\gamma (u - \varphi)(x_0) \xi_{\alpha\beta}(x_0) + \nabla_{\alpha\beta} \varphi(x_0) - A_{\alpha\beta}(x_0, \nabla \varphi(x_0), -D_\gamma u(x_0))).
\]
here the regular condition \( (1.5) \) of \( A \) is used. By the ellipticity of the strict subsolution \( u \), we can fix a positive constant \( \delta_0 \) for which

\[
G[u] \geq \delta_0, \quad \text{for all } x \in T. 
\]

We also have

\[
0 < D_n(u - u)(x_0) \leq \kappa, \quad \text{for a positive constant } \kappa. 
\]

Combining (4.37), (4.38) and (4.39), since \( G[u] < G[u]/2 \) at \( x_0 \), we have

\[
G^\alpha_\beta x_0 [D_{\alpha \gamma}(x_0) + D_{p_n}A_{\alpha \beta}(x_0, D'\varphi(x_0), D_nu(x_0))] \geq \frac{\delta_0}{2\kappa} > 0. 
\]

Let

\[
\vartheta(x) := G^\alpha_\beta x_0 [\varphi_{\alpha \beta}(x) + D_{p_n}A_{\alpha \beta}(x, \nabla \varphi(x), -D_{\gamma}u(x_0))],
\]

we have, at \( x_0 \),

\[
\vartheta(x_0) := G^\alpha_\beta x_0 [D_{\alpha \gamma}(x_0) + D_{p_n}A_{\alpha \beta}(x_0, D'\varphi(x_0), D_nu(x_0))] \geq \frac{\delta_0}{2\kappa} > 0. 
\]

Since \( \vartheta(x) \) is smooth near \( \partial \Omega \), we can have

\[
\vartheta(x) \geq c > 0, \quad \text{on } T,
\]

holds for some small positive constant \( c \). By the regular condition \( (1.5) \) of \( A \), we observe that \( A_{\alpha \beta} \) is convex with respect to \( p_n \). Therefore, we have

\[
A_{\alpha \beta}(x, \nabla \varphi(x), -D_{\gamma}u(x)) \leq D_{p_n}A_{\alpha \beta}(x, \nabla \varphi(x), -D_{\gamma}u(x))(D_{\gamma}u(x) - D_{\gamma}u(x_0)).
\]

Since \( G^\alpha_\beta x_0 \) is positive definite and \( \gamma_n(x) < 0 \), we have, by (4.36) and (4.44),

\[
G^\alpha_\beta x_0 [\varphi_{\alpha \beta}(x) + D_{p_n}A_{\alpha \beta}(x, \nabla \varphi(x), -D_{\gamma}u(x_0))] D_nu(x)
\]

\[
\leq \gamma_n(x) \left\{ G^\alpha_\beta x_0 [D_{\gamma}f(x)\varphi_{\alpha \beta} + D_{p_n}A_{\alpha \beta}(x, \nabla \varphi(x), -D_{\gamma}u(x_0))D_{\gamma}u(x_0) + \nabla_{\alpha \beta}(x) - \nabla_{\alpha \beta}(x_0) + D_{\gamma}(x - \varphi)(x_0)\varphi_{\alpha \beta}(x_0) + A_{\alpha \beta}(x, \nabla \varphi(x), -D_{\gamma}u(x_0)) - A_{\alpha \beta}(x, \nabla \varphi(x_0), -D_{\gamma}u(x_0)) \right\}.
\]

Observing that \( \vartheta(x) \) is the coefficient of \( D_nu(x) \) in (4.45), by (4.43), we have

\[
D_nu(x) \leq \Theta(x), \quad \text{on } T,
\]

where \( \Theta(x) \) is a smooth function defined by

\[
\Theta(x) := \gamma_n(x)(\vartheta(x))^{-1} \left\{ G^\alpha_\beta x_0 [D_{\gamma}f(x)\varphi_{\alpha \beta} + D_{p_n}A_{\alpha \beta}(x, \nabla \varphi(x), -D_{\gamma}u(x_0))D_{\gamma}u(x_0) + \nabla_{\alpha \beta}(x) - \nabla_{\alpha \beta}(x_0) + D_{\gamma}(x - \varphi)(x_0)\varphi_{\alpha \beta}(x_0) + A_{\alpha \beta}(x, \nabla \varphi(x), -D_{\gamma}u(x_0)) - A_{\alpha \beta}(x, \nabla \varphi(x_0), -D_{\gamma}u(x_0)) \right\}.
\]

We now define a function on \( T \),

\[
w(x) := D_nu(x) - \Theta(x).
\]

By extending \( \varphi, \gamma \) and \( \mathcal{C} \) smoothly to the interior near the boundary to be constant in the normal direction, the function \( w \) is extended to \( \Omega_\delta = \Omega \cap B_\delta(x_0) \) in the neighbourhood of \( x_0 \). By calculation, we have

\[
|\mathcal{L}w| = |\mathcal{L}D_nu(x) - \mathcal{L}\Theta(x)| \leq C(1 + \sum_i F^{ii}), \quad \text{in } \Omega_\delta,
\]

here the differentiated equation (4.35) for \( k = n \) is used. Since \( w \) is extended to be constant in the normal direction, we have, by (4.46),

\[
w \leq 0, \quad \text{on } \partial \Omega_\delta.
\]
Therefore, we obtain

\begin{equation}
\begin{cases}
\mathcal{L}(K'\psi \pm w) \leq 0, & \text{in } \Omega_\delta, \\
K'\psi - w \geq 0, & \text{on } \partial\Omega_\delta,
\end{cases}
\end{equation}

where $K'$ is a sufficiently large constant, $\psi$ is the barrier function constructed in Lemma 4.1. By the maximum principle, we have

\begin{equation}
K'\psi - w \geq 0, \quad \text{in } \Omega_\delta,
\end{equation}

which leads to

\begin{equation}
D_n(K'\psi - w) \geq 0, \quad \text{at } x_0,
\end{equation}

namely

\begin{equation}
D_{nn}u(x_0) \leq C.
\end{equation}

Utilizing equation (4.28) again, we conclude from the above two cases a positive lower bound from below for $G[u]$ at $x_0$. Recall that $G[u]$ attains its minimum at $x_0$. Hence, by (4.28) we finally obtain an upper bound $D_{\gamma\gamma}u(x) \leq C$ at any boundary point $x \in \partial\Omega$.

By the ellipticity, we have $tr(D^2u - A(x, Du)) \geq 0$, that is

\begin{equation}
D_{nn}u \geq \sum_{i}^{n} A_{ii}(x, Du) - \sum_{i}^{n-1} D_{ii}u,
\end{equation}

we now get an estimate from below for $D_{nn}u$ on $\partial\Omega$.

In conclusion, we have obtained the following desired second derivative bound on the boundary,

\begin{equation}
\sup_{\partial\Omega} |D_2u| \leq C,
\end{equation}

where the constant $C$ depends on $n, A, B, \Omega, \varphi, u$ and $\sup_\Omega (|u| + |Du|)$.

We see that the second derivative estimate (4.56) on the boundary holds for solutions of augmented Hessian equations (1.1) for all $1 \leq k \leq n$. In fact, if we only consider the semilinear case when $k=1$, the proof will be much simpler. In this case, the double tangential derivative bound can also be derived from (4.2), and the double normal derivative bound can be obtained directly from the equation $\Delta u - \sum_{i}^{n} A_{ii}(x, Du) = B(x, Du)$.

Now, we can easily give the proof of Theorem 1.2.

**Proof of Theorem 1.2.** By the global estimate (1.17) and the above boundary estimate (4.56), we obtain the a priori second derivative estimate (1.19) and Theorem 1.2 follows.

\[ \square \]

**Remark 4.1.** In order to obtain the second derivative estimates for admissible solutions of (1.1) on the boundary without the subsolution assumption, a kind of geometric condition should be imposed on the domain $\Omega$. Recall that in the case for Monge-Ampère type equation (11.24), the concept of domain $A$-convexity is introduced extending that of $c$-convexity in optimal transportation. If $\Omega$ is a connected domain in $\mathbb{R}^n$ with $\partial\Omega \in C^2$, and $A \in C^1(\Omega \times \mathbb{R}^n; \mathbb{S}^n)$, we say that $\Omega$ is uniformly $(k-1)$-$A$-convex with respect to $u$, if

\begin{equation}
S_{k-1}(\kappa) \geq \delta_0 > 0,
\end{equation}

where $S_{k-1}$ is the $(k-1)$-th fundamental solution of the Monge-Ampère equation.
for some $\delta_0 > 0$ and all $p \in Du(\Omega)$, where $\kappa = (\kappa_1, \ldots, \kappa_n)$ denote the eigenvalues of
\begin{equation}
(D_i \gamma_j(x) - A_{ij,pk}(x,p) \gamma_k(x))\},
\end{equation}
for all $x \in \partial \Omega$, and unit outer normal $\gamma$. In other words, a domain $\Omega$ is called uniformly $(k-1)$-A-convex if the eigenvalues of the matrix $\{D_i \gamma_j(x) - A_{ij,pk}(x,p) \gamma_k(x) - c_0 \delta_{ij}\}$ (as a vector in $\mathbb{R}^{n-1}$) lie in $\Gamma_{k-1}$, provided $c_0$ is a sufficiently small positive constant, where $\delta_{ij}$ is the usual Kronecker delta. When $k = n$, the $(n-1)$-A-convexity corresponds to the A-convexity in $[11, 24]$. When $A \equiv 0$, the $(k-1)$-A-convexity corresponds to the $(k-1)$-convexity as in $[23, 33]$. Particularly, when $k = n$ and $A \equiv 0$, the $(k-1)$-A-convexity reduces to the usual convexity. If we impose a uniformly $(k-1)$-A-convexity condition on the domain $\Omega$, we can still obtain the second derivative estimates on the boundary for admissible solutions of (1.1). We remark that the subsolution assumption can be replaced by the A-boundedness condition and the uniformly $(k-1)$-A-convexity condition of the domain $\Omega$ in the main theorems of this paper, but in some sense it is a bit weaker to assume the existence of a subsolution with the same boundary trace.

5. General type equations

In many applications, the matrix function $A$ and the right hand side term $B$ in the augmented Hessian equations may also depend on $u$. In this section, we consider some more general type augmented Hessian equations with the form (1.20) and formulate the corresponding theorems for the a priori $C^2$ estimates of the solutions. The general type equation (1.20) reduces to the Monge-Ampère type equation when $k = n$, which has applications in near field optics, see $[12, 24, 18, 26]$. With both the matrix function $A$ and the scalar function $B$ depending on $u$, we still use similar notation as in the previous sections. Here the augmented Hessian matrix is $\{w_{ij}\} = \{u_{ij} - A_{ij}(x,u,Du)\}$ and the right hand side term is $B(x,u,Du)$. We rewrite the equation (1.20) in the following form
\begin{equation}
F[u] = (S_k)^{\frac{1}{k}}[D^2 u - A(x,u,Du)] = \tilde{B}(x,u,Du), \quad \text{in } \Omega,
\end{equation}
where $\tilde{B} = B^{\frac{1}{k}} > 0$ in $\tilde{\Omega} \times \mathbb{R} \times \mathbb{R}^n$. The corresponding Dirichlet boundary condition is
\begin{equation}
u = \varphi, \quad \text{on } \partial \Omega,
\end{equation}
where $\varphi$ is a smooth function on $\partial \Omega$. The matrix function $A$ is regular if
\begin{equation}A_{ij,kl}(x,u,p)\xi_i \xi_j \eta_k \eta_l \geq 0,
\end{equation}
for all $(x,u,p) \in \tilde{\Omega} \times \mathbb{R} \times \mathbb{R}^n$, $\xi, \eta \in \mathbb{R}^n$, $\xi \perp \eta$, where $A_{ij,kl} = D^2_{p_k p_l} A_{ij}$.

In order to construct the global barrier function as in Lemma 2.1, we need to assume additional monotonicity conditions on both $A$ and $\tilde{B}$ with respect to $u$. We assume the matrix function $A(x,u,p)$ is monotone with respect to $u$, that is
\begin{equation}D_u A_{ij}(x,u,p)\xi_i \xi_j \geq 0,
\end{equation}
for all $(x,u,p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$, $\xi \in \mathbb{R}^n$. We also assume the scalar function $\tilde{B}(x,u,p)$ is monotone with respect to $u$, that is
\begin{equation}\tilde{B}_u(x,u,p) \geq 0,
\end{equation}
for all $(x,u,p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$. Note that these monotonicity conditions can guarantee the comparison principle and the uniqueness of the solution. To obtain the global second derivative bound, we assume
that \( \tilde{B}(x,u,p) \) is convex with respect to \( p \), that is
\[
(5.6) \quad D^2_{p_k p_l} \tilde{B}(x,u,p) \xi_k \xi_l \geq 0,
\]
for all \( (x,u,p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n, \xi \in \mathbb{R}^n \).

We define the linearized operators of \( F \) by
\[
L = F^{ij}[D_{ij} - D_{p_k A_{ij}}(x,u,Du)D_k],
\]
and
\[
\mathcal{L} = L - \tilde{B}_p D_i = F^{ij}[D_{ij} - D_{p_k A_{ij}}(x,u,Du)D_k] - \tilde{B}_p D_i,
\]
where \( F^{ij} = \frac{\partial F}{\partial w_{ij}} \). With the monotonicity conditions (5.4) and (5.5), we can construct the barrier function similar to Lemma 2.1 by assuming the existence of an admissible supersolution.

**Lemma 5.1.** Assume the function \( A(x,u,p) \in C^2(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n) \) is regular, \( \tilde{B}(x,u,p) \in C^2(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n) \), \( A \) and \( \tilde{B} \) satisfy the monotonicity conditions (5.4) and (5.5) respectively. Suppose \( u \in C^2(\bar{\Omega}) \) is an elliptic solution of equation (5.1) associated with Dirichlet the boundary condition (5.2), \( \bar{u} \in C^2(\bar{\Omega}) \) is an admissible strict supersolution of equation (5.1) satisfying
\[
(5.9) \quad (S_k) \frac{D^2 \bar{u} - A(x,\bar{u},D\bar{u})}{2} \leq \tilde{B}(x,\bar{u},D\bar{u}) - \delta, \quad \text{in } \Omega,
\]
for some positive constant \( \delta \), and \( \bar{u} \geq \varphi \) on \( \partial \Omega \). If \( A \) is regular satisfying (5.3), then
\[
(5.10) \quad \mathcal{L} \left( e^{K(\bar{u} - u)} \right) \geq \epsilon_1 \sum_i F^{ii} - C,
\]
holds in \( \Omega \) for positive constants \( K, \epsilon_1 \) and \( C \), which depend on \( A, \tilde{B}, \Omega, \|u\|_{C^1} \) and \( \|\bar{u}\|_{C^1} \).

**Proof.** Since \( \bar{u} \) is an admissible strict supersolution of (5.1), for any \( x_0 \in \Omega \), the perturbation function \( \bar{u}_\epsilon = \bar{u} - \frac{\epsilon_1}{2} |x - x_0|^2 \) is still an admissible strict supersolution satisfying
\[
(5.11) \quad F[\bar{u}_\epsilon] = (S_k) \frac{D^2 \bar{u}_\epsilon - A(x,\bar{u}_\epsilon,D\bar{u}_\epsilon)}{2} \leq \tilde{B}(x,\bar{u}_\epsilon,D\bar{u}_\epsilon) - \tau, \quad \text{in } \Omega,
\]
for some positive constant \( \tau \). Consequently, we have
\[
(5.12) \quad 0 \leq F[\bar{u}_\epsilon] < \tilde{B}(x,\bar{u}_\epsilon,D\bar{u}_\epsilon), \quad \text{in } \Omega.
\]
By the monotone assumptions (5.4) and (5.5), we can automatically have \( u < \bar{u}_\epsilon \) in \( \Omega \) from the comparison principle.

Let \( v = \bar{u} - u, \ v_\epsilon = \bar{u}_\epsilon - u \). By calculation, we get
\[
L v = L(v_\epsilon) + L\left( \frac{\epsilon_1}{2} |x - x_0|^2 \right) = \epsilon F^{ii} - \epsilon F^{ij} D_{p_k} A_{ij}(x,u,Du)(x - x_0)_k
\]
\[
+ F^{ij} \left[ D_{ij}(\bar{u}_\epsilon - u) - [A_{ij}(x,\bar{u}_\epsilon,D\bar{u}_\epsilon) - A_{ij}(x,u,Du)] \right]
\]
\[
+ F^{ij} \left[ A_{ij}(x,u,D\bar{u}_\epsilon) - A_{ij}(x,u,Du) - D_{p_k} A_{ij}(x,u,Du) D_k v_\epsilon \right]
\]
\[
+ F^{ij} \left[ A_{ij}(x,\bar{u}_\epsilon,D\bar{u}_\epsilon) - A_{ij}(x,u,D\bar{u}_\epsilon) \right].
\]
Since \( u \leq \bar{u}_\epsilon \) and \( A \) satisfies the monotonicity condition (5.4), we have
\[
(5.14) \quad F^{ij} \{ A_{ij}(x,\bar{u}_\epsilon,D\bar{u}_\epsilon) - A_{ij}(x,u,D\bar{u}_\epsilon) \} = F^{ij} D_a A_{ij}(x,\hat{u},D\bar{u}_\epsilon)(\bar{u}_\epsilon - u) \geq 0,
\]
where \( \hat{u} = \theta \bar{u}_\epsilon + (1 - \theta) u \) for some \( \theta \in (0,1) \).

By the concavity of \( F \), we have
\[
(5.15) \quad F^{ij} \{ D_{ij}(\bar{u}_\epsilon - u) - [A_{ij}(x,\bar{u}_\epsilon,D\bar{u}_\epsilon) - A_{ij}(x,u,Du)] \} \geq F[\bar{u}_\epsilon] - F[u] = F[\bar{u}_\epsilon] - \tilde{B}(x,u,Du) \geq -C_1,
\]
where the first inequality of (5.12) is used, and $C_1$ is a positive constant depending on $\hat{B}$ and $\sup_{\Omega} (|u| + |Du|)$.

By the regular condition (5.3), other terms in (5.13) can be estimated in the same manner as in the proof of Lemma 2.1. We only need to substitute $\bar{u}$ and $\bar{u}_\varepsilon$ in place of $u$ and $u_\varepsilon$ respectively in that proof.

Then by the finite covering, the conclusion of Lemma 5.1 can be completed by following the steps in the proof of Lemma 2.1.

□

Similar to Theorem 1.1 and Theorem 1.2, we can use the barrier function in Lemma 5.1 to derive the second order derivative estimates for solutions of (5.1)-(5.2). We formulate these results as follows:

**Theorem 5.1.** In addition to the assumptions in Lemma 5.1, suppose the condition (5.6) holds for $\hat{B}$. Then we have the estimate

$$(5.16) \quad \sup_{\Omega} |D^2 u| \leq C (1 + \sup_{\partial\Omega} |D^2 u|),$$

where the constant $C$ depends on $n, A, \hat{B}, \Omega, \bar{u}$ and $\sup_{\Omega} (|u| + |Du|)$.

**Remark 5.1.** In the paper [10] we obtain corresponding second derivative bounds for the special case of generated prescribed Jacobian equations under the conditions $G_1, G_2, G_1^*, G_3w, G_4w$ introduced in [26], (without convexity or monotonicity hypotheses on $B$). These extend the optimal transportation case as indicated in Remark 2.4 and depend on the construction of a suitable barrier $u$.

In order to obtain the second order derivative estimates on the boundary, we still need a subsolution $u$ with the boundary value $\varphi$ as the deduction in Section 4. The subsolution is necessary to derive the double normal derivative bound on the boundary. Combining the boundary second derivative estimates with Theorem 5.1, we have the following result for the second order a priori estimates for Dirichlet problem (5.1)-(5.2) of the general type equations.

**Theorem 5.2.** In addition to the assumptions in Theorem 5.1, suppose also there exists an elliptic subsolution $\underline{u} \in C^2(\Omega)$ of the equation (5.1) with $\underline{u} = \varphi$ on $\partial\Omega$. Then we have the global estimate,

$$(5.17) \quad \sup_{\Omega} |D^2 u| \leq C,$$

where the constant $C$ depends on $n, A, \hat{B}, \Omega, \varphi, \underline{u}$ and $\sup (|u| + |Du|)$.

We have obtained the global second derivative estimates for solutions of the Dirichlet problem (5.1)-(5.2). If the lower order derivative estimates are also obtained, we can obtain the $C^{2,\alpha}$ estimates of the solutions by the Evans-Krylov theorems, [6]. Using the method of continuity, we can have the existence and uniqueness of the classical solutions of the Dirichlet problem (5.1)-(5.2).

**Remark 5.2.** We remark that the monotonicity condition (5.5) for right hand side function $\hat{B}$ is to guarantee the comparison principle and the uniqueness of the solution. Since we use the subsolution $u$ and the supersolution $\bar{u}$ here, we need to compare them with the solution $u$. If we assume the geometric conditions (2.10) and (4.57) of the domain $\Omega$ rather than the existence of a subsolution, condition (5.5)
is not necessary in Theorem 5.2 and may only be used to guarantee the uniqueness by the continuity method. Furthermore, if the condition $\tilde{B}_u \geq 0$ is not satisfied, one may still have the existence theorem for the classical solutions by using Leray-Schauder fixed point theorem (see Theorem 11.6 in [6]) or a degree argument used in [3-8]. We refer the reader to the proof of Theorem 1.5 in [28] for the procedure of using the Leray-Schauder fixed point theorem for similar purpose.

References

[1] Andrews, B.: Contraction of convex hypersurfaces in Euclidean space. Calc. Var. PDE. 2, 151-171 (1994)
[2] Caffarelli, L., Nirenberg, L., Spruck J.: The Dirichlet problem for nonlinear second order elliptic equations I: Monge-Ampère equations. Comm. Pure Appl. Math. 37, 369-402 (1984)
[3] Caffarelli, L.A., Nirenberg, L., Spruck, J.: The Dirichlet problem for nonlinear second-order elliptic equations III: Functions of eigenvalues of the Hessians. Acta Math. 155, 261-301 (1985)
[4] Chou, K.-S., Wang, X.-J.: A variational theory of the Hessian equation. Comm. Pure Appl. Math. 54, 1029-1064 (2001)
[5] Gerhardt, G.: Closed Weingarten hypersurfaces in Riemannian manifolds. J. Differential Geom. 43, 612-641 (1996)
[6] Gilbarg, D., Trudinger, N.S.: Elliptic Partial Differential Equation of Second Order, Springer-Verlag, Berlin-New York, (2001)
[7] Guan, B.: The Dirichlet problem for Monge-Ampere equations in non-convex domains and spacelike hypersurfaces of constant Gauss curvature. Trans. Amer. Math. Soc. 350, 4955-4972 (1998)
[8] Guan, B.: Conformal metrics with prescribed curvature functions on manifolds with boundary. Amer. J. Math. 129, 915-942 (2007)
[9] Guan, B.: Second order estimates and regularity for fully nonlinear elliptic equations on Riemannian manifolds. To appear in Duke Math. J.
[10] Jiang, F., Trudinger, N.S.: On Pogorelov estimates in optimal transportation and geometric optics. (In preparation).
[11] Jiang, F., Trudinger, N.S., Yang, X.-P.: On the Dirichlet problem for Monge-Ampére type equations. Calc. Var. PDE., 49, 1223-1236 (2014)
[12] Karakhanyan, A., Wang X.-J.: On the reflector shape design. J. Diff. Geom., 84, 561-610 (2010)
[13] Krylov, N.V.: Lecture on fully nonlinear second order elliptic equations, Lipschitz Lectures, Bonn University, (1993)
[14] Kochengin, S.A., Oliker V.I.: Determination of reflector surfaces from near-field scattering data. Inverse Problems. 13, 363-373 (1997)
[15] Ladyzhenskaya, O., Ural'ceva, N.: Linear and quasilinear elliptic equations, Academic Press, New York, (1968)
[16] Li Y. Y.: Some existence results of fully nonlinear elliptic equations of Monge-Ampère type. Comm. Pure Appl. Math. 43, 233-271 (1990)
[17] Liu, J., Trudinger, N.S.: On Pogorelov estimates for Monge-Ampère type equation. Discrete Contin. Dyn. Syst. Ser. A. 28, 1121-1135 (2010)
[18] Liu, J., Trudinger, N.S.: On classical solutions of near field reflection problems. Preprint. (2013)
[19] Ma, X.-N., Trudinger N.S., Wang, X.-J.: Regularity of potential functions of the optimal transportation problem. Arch. Rat. Mech. Anal. 177, 151-183 (2005)
[20] Sheng, W., Urbas, J.I.E., Wang, X.-J.: Interior curvature bounds for a class of curvature equations. Duke Math. J. 123, 235-264 (2004)
[21] Trudinger, N.S.: Fully nonlinear, uniformly elliptic equations under natural structure conditions, Trans. Amer. Math. Soc. 278, 751-769 (1983)
[22] Trudinger, N.S.: On the Dirichlet problem for Hessian equations, Acta Math. 175, 151-164 (1995)
[23] Trudinger, N.S.: Weak solutions of Hessian equations, Comm. Partial Diff. Eqns. 22, 1251-1261 (1997)
[24] Trudinger, N.S.: Recent developments in elliptic partial differential equations of Monge-Ampère type. ICM. Madrid, 3, 291-302 (2006)
[25] Trudinger, N.S.: A note on global regularity in optimal transportation. Bull. Math.Sci. 3, 551-557 (2013)
[26] Trudinger, N.S.: On the local theory of prescribed Jacobian equations. Discrete Contin. Dyn. Syst. 34, 1663-1681 (2014)
[27] Trudinger, N.S., Wang, X.-J.: On the second boundary value problem for Monge-Ampère type equations and optimal transportation. Ann. Scuola Norm. Sup. Pisa Cl. Sci. VIII, 143-174 (2009)
[28] Urbas, J.: Nonlinear oblique boundary value problems for Hessian equations in two dimensions. Ann. Inst. Henri Poincaré-Analyse Non Linéaire. 12, 507-575 (1995)
[29] Urbas, J.: The second boundary value problem for a class of Hessian equations. Comm. Partial Diff. Eqns. 26, 859-882 (2001)
[30] Urbas, J.: Hessian equations on compact Riemannian manifolds. Nonlinear Problems in Mathematical Physics and Related Topics II. Kluwer/Plenum, New York, 367-377 (2002)
[31] von Nessi, G.T., On the second boundary value problem for a class of modified-Hessian equations, Comm. Partial Diff. Eqns. 35, 745-785 (2010)
[32] Wang X.-J.: A class of fully nonlinear elliptic equations and related functionals. Indiana Univ. Math. J. 43, 25-54 (1994)
[33] Wang X.-J.: The k-Hessian equation. Lecture Notes in Mathematics. Springer Berlin/Heidelberg. 1977, 177-252 (2009)

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