Higgsed antisymmetric tensors and topological defects

JAN TROOST
Theoretische Natuurkunde, Vrije Universiteit Brussel
Pleinlaan 2, B-1050 Brussel, Belgium

ABSTRACT

We find topological defect solutions to the equations of motion of a generalised Higgs model with antisymmetric tensor fields. These solutions are direct higher dimensional analogues of the Nielsen-Olesen vortex solution for a gauge field in four dimensions.

1. Introduction

It is well known that the Higgs-mechanism involving a complex scalar and a vector particle can be extended to antisymmetric tensor fields. The formal extension consists in a model containing a (h-1)-form, an h-form and a scalar field with a Higgs type Lagrangian in a spacetime of arbitrary dimension \( D = d + 1 \). When the scalar field gets a vacuum expectation value, the h-form eats the degrees of freedom of the (h-1)-form and acquires a mass.

In this paper we will look for solutions to the equations of motion of the generalised Higgs model. We will work in close analogy to the paper of Nielsen and Olesen on the vortex-solution in four dimensions (also of use in superconductivity). In the same approximation as in \( [3] \), we will find topological defect solutions that extend over \( d - h - 1 \) dimensions.

These solutions have their importance in the study of the different phases in antisymmetric tensor field theories and they play a role in determining the physical content of a brane-antibrane system after tachyon condensation.

2. The antisymmetric tensor Higgs model

We study a model in \( D = d + 1 \) dimensions with the following field content: an antisymmetric tensor of degree \( h, \omega_h \), an antisymmetric tensor of degree \( h - 1, C_{h-1} \) and a scalar field \( f \). We consider the following action:

\[
S = \int d^{d+1}x \left( d\omega_h \ast d\omega_h + df \ast df - m(f)^2 (dC_{h-1} + q \omega_h) \ast (dC_{h-1} + q \omega_h) - U(f) \right),
\]

where \( m(f) \) and \( U(f) \) are general functions of the scalar field. The gauge symmetries of this action are:

\[
\begin{align*}
\delta \omega_h &= d\epsilon_{h-1} \\
\delta C_{h-1} &= -q \epsilon_{h-1} + d\xi_{h-2}.
\end{align*}
\]

1 troost@tena4.vub.ac.be; Aspirant F.W.O.
2 When using form-notation we will be sloppy with numerical factors, but in component form we believe to have every factor straight.
If the scalar field $f$ acquires a vacuum expectation value, it is appropriate to use the gauge freedom to gauge away the $(h-1)$-form $C_{h-1}$ completely. A massive $h$-form $\omega_h$ and a real scalar will be left as physical fields, as in the ordinary Higgs-mechanism. Later on we will make use of a specific form of the potential:

$$U(f) = -c_2 f^2 + c_4 f^4,$$

where we took over some of the conventions of [3] for easy comparison. Note that for $D = 4$ and $h = 1$ and the quartic potential, the model matches up with the Higgs-model, where $\omega_1$ is the gauge field, $C_0$ represents the phase of the complex scalar, and $f$ its modulus. The function $m(f)$ is then given by $m(f)^2 = f^2$.

The Lagrangian is expressed in component form as follows:

$$\mathcal{L} = -\frac{1}{2(h+1)!}((h+1)\partial_{(M_{h+1}\omega_{M_1\ldots M_h})}^2$$

$$-\frac{1}{2}(\partial_M f)^2 - m(f)^2 \frac{1}{2h!} (h \partial_{[M_h} C_{M_1\ldots M_{h-1}]} + q \omega_{M_1\ldots M_h})^2 - U(f)$$

The equations of motion corresponding to this Lagrangian are:

$$0 = \frac{1}{\sqrt{|g|}} \partial_{M_{h+1}} \sqrt{|g|}(d\omega)^{M_{h+1}M_1 \ldots M_h}$$

$$-m(f)^2 q (h \partial^{[M_h} C^{M_1\ldots M_{h-1}]} + q \omega_{M_1\ldots M_h})$$

$$0 = \frac{1}{\sqrt{|g|}} \partial_M (\sqrt{|g|} \partial^M f)$$

$$-\frac{1}{h!} m(f) m(f)' (h \partial_{[M_h} C_{M_1\ldots M_{h-1}]} + q \omega_{M_1\ldots M_h})^2 - U(f)'$$

$$0 = \partial_{M_1} (\sqrt{|g|} m(f)^2 (h \partial^{[M_h} C^{M_1\ldots M_{h-1}]} + q \omega^{M_1\ldots M_h})), $$

where a prime denotes differentiation with respect to $f$. The last equation is the equation of motion corresponding to the $C_{h-1}$ form. It merely states that the $h$-form $\omega_h$ couples to a conserved current

$$j^{M_1 \ldots M_h} = q \sqrt{|g|} m(f)^2 (h \partial^{[M_h} C^{M_1\ldots M_{h-1}]} + q \omega^{M_1\ldots M_h}).$$

### 3. Ansatz and solution

In close analogy to [3], we will look for a topological defect solution of dimension $d - h - 1$. The $(h+1)$-form field strength will measure the number of topological defects passing through a $h + 1$ dimensional plane perpendicular to the defects. We can define a magnetic flux $\Phi$ flowing through a $(h+1)$-dimensional ball and calculate it in terms of the $(h-1)$-form field strength

$$\Phi \equiv \int_{B^{h+1}} d\omega_h$$

$$= \int_{S^h} \omega_h$$

$$= -\frac{1}{q} \int_{S^h} dC_{h-1},$$
where we have used the fact that there is no current over the h-sphere that is the boundary of the (h+1)-ball.

We consider an ansatz with $SO(h+1) \times \text{Poincaré} (d-h-1, 1)$ symmetry. We use the following coordinates adapted to the symmetry: $(r, \phi, \theta_1, \ldots, \theta_h, t, z_1, \ldots, z_{d-h})$

The ansatz reads in these coordinates:

$$\omega_h = |\omega(r)| r^h d\Omega_h$$

$$C_{h-1}^\pm = k_{h-1}(\pm |\ell_{h-1} + f_{h-1}(\theta_{h-1})|) d\Omega_{h-1}$$

where $d\Omega_h$ denotes the volume form of the h-sphere with volume $s_h$. Moreover we take the constants $k_{h-1}$ and $\ell_{h-1}$ to be

$$k_{h-1} = \frac{2\pi}{s_h} = \frac{\pi^{(h-1)/2}}{\Gamma(h/2)}$$

$$\ell_{h-1} = \frac{\sqrt{\pi}}{2} \frac{\Gamma(h/2)}{\Gamma((h+1)/2)},$$

and the function $f_{h-1}$ satisfies

$$\frac{d}{d\theta_{h-1}} f(\theta_{h-1}) = \sin^{h-1} \theta_{h-1},$$

following [6]. From these formulae we easily derive:

$$d\omega_h = \partial_r (|\omega|r^h) dr \wedge d\Omega_h$$

$$dC_{h-1} = k_{h-1} \sin^{h-1} \theta_{h-1} d\ell_{h-1} \wedge d\Omega_{h-1} = k_{h-1} d\Omega_h$$

The (h-1)-form ansatz is chosen such that from the formula for the flux (3.1), we can conclude that there will be a topological defect in the $(t, z_1, \ldots, z_{d-h-1})$ direction. Filling in the ansatz in the equations of motion results in the following set of differential equations:

$$0 = \frac{1}{r^h} \partial_r (r^h \partial_r f) - m(f)m(f)'(\frac{k_{h-1}}{r^h} + q|\omega|)^2 - U(f)'$$

$$0 = \partial_r (\frac{1}{r^h} \partial_r (|\omega|r^h)) - m(f)^2 q(q|\omega| + \frac{k_{h-1}}{r^h})$$

Following [3], we consider the situation in which $f$ tends to a constant value at infinity (transverse to the topological defect). In that approximation, we can solve (3.10) for $|\omega|$ in terms of modified Bessel functions:

$$|\omega| = -\frac{1}{q} \frac{k_{h-1}}{r^h} + \frac{c}{q} (qmr)^{-\nu+1} K_{\nu}(qmr)$$

3 For the case $h = 1$ it is clear that the magnetic flux is quantized [3]. For $h = 2$ see [4]. To us it seems that in the case $h \geq 2$ you could suppose the existence of an electric charge for the (h-1)-form to have quantization of the magnetic flux for the h-form, reasoning along the lines of [6].

4 Details of the standard manipulations are in the first appendix.
where the index of the modified Bessel function is given by $\nu = \frac{h+1}{2}$ and $c$ is an integration constant. The magnetic field strength becomes:

$$ |H| \equiv \frac{1}{\rho h} \partial_r (r^h |\omega|) $$

$$ = cm(qmr)^{-\nu+1} K_{\nu-1}(qmr) $$

(3.12)

where we have used a property of the modified Bessel function given in the second appendix (B.3). The asymptotic behavior of the magnetic field is then (B.2):

$$ |H|_{r \to \infty} = c \sqrt{\frac{\pi m}{2q}} (qmr)^{-\nu+1} e^{-qmr} + \ldots $$

(3.13)

We can define a characteristic length measuring the distance over which the magnetic field differs appreciably from zero:

$$ \lambda \equiv \frac{1}{qm} $$

(3.14)

This is the analog of the penetration depth in superconductivity.

We turn now to the other equation of motion (3.9), the one corresponding to the scalar field $f$. We will restrict from this point on to the special case of a quartic Higgs potential (2.3). If we assume that the deviation of the $h$-form from $-\frac{k_0}{qr}$ is negligible compared to the effect of a steep potential, in other words, if the coefficients $c_2$ and $c_4$ are large, then (3.9) is approximately satisfied if $f$ takes the constant value that minimizes the potential:

$$ < f > \equiv v = \sqrt{\frac{c_2}{2c_4}} $$

(3.15)

To get an idea of the position dependence of $f$ we consider fluctuations around its vacuum expectation value $v$:

$$ f = v + \rho(r) $$

(3.16)

At infinity we find an approximate solution

$$ \rho(r) \approx e^{-\sqrt{c_2}r} $$

(3.17)

giving rise to a new characteristic length,

$$ \xi = \frac{1}{\sqrt{4c_2}} $$

(3.18)

the generalisation of the correlation length in superconductivity. It is nothing but the inverse of the mass of the Higgs-particle, and it measures the distance over which $f$ differs appreciably form its vacuum expectation value. All of this to spell out that the behavior of the fields is analogous to the well known special case of the Nielsen-Olesen vortex-solution, due to the general properties of the modified Bessel functions. We further remark that to have a clear corelike topological defect of dimension $d-h-1$, we
need $\xi$ and $\lambda$ both small. For $r << \lambda$ we require that the magnetic flux $\Phi = V(S^h)|\omega|$ vanish, fixing the integration constant to:

$$c = (2\pi)^{-\nu+1}(qm)^h,$$

(3.19)

where we made use of formula (B.1) in the appendix, (3.4) and (3.11). This finishes the discussion of the approximate solution.

4. A remark

For some applications it is useful to have an estimate of the energy density of the $d-h-1$-dimensional topological defect. In [3] the idea was to match up the energy density of the vortex with the string tension of the dual string model. The reasoning was that the string model would be a good effective description of the field theory in a regime where the vortex solution becomes the most important classical solution to the action [3]. Then you can link the field theory parameters to the string tension, $\frac{1}{2\pi \alpha'}$. Following [4] we can study the special case of the membrane-like solution in $D = 5 + 1$ and with $h = 2$ and try to match its energy density to the M2-brane tension. We only make a rough analysis, indicating that this is less straightforward.

Treating the mass $m$ as a constant, the magnetic contribution to the energy density is in the general case [3]:

$$E_m \approx \frac{sh}{2} \int_0^\infty |H|^2 r^h dr \approx \frac{sh}{2(2\pi)^{2\nu-2}} q^{h-1} m^{h+1} \int_0^\infty K_{\nu-1}(z)^2 z dz$$

(4.1)

From the asymptotic behavior of the modified Bessel functions (B.2), it is clear that the integrand converges fast enough at infinity, but near $z = 0$ (B.1), the integrand behaves as $z^{-h+2}$. For a finite result we thus need $h < 3$. Then the integral is of order unity and the magnetic energy density of order

$$E_m \approx q^{h-1} m^{h+1}$$

(4.2)

The contribution to the energy density of the topological defect due to the scalar field $f$ can roughly be approximated by [3]:

$$E_f \approx \xi^{h+1} c_2 v^2 \approx \frac{c_2^h}{c_4}$$

(4.3)

We note that for $h = 1$ and $m = v = \sqrt{\frac{c_2}{c_4}}$ the two estimates match [3]. For other values of $h$ like $h = 2$ the naive analysis gives an ambiguous result. We will not pursue this here. Notice though, that our sketchy analysis of the asymptotic behavior of the magnetic field and the energy density is finer than the one in [4].

5. Conclusions

We studied Higgsed antisymmetric tensor field theories. Specifically we searched for and found topological defect solutions, generalizing the work by Nielsen and Olesen.
on vortex solutions in four dimensions [3]. We wrote down the approximate behavior of the fields at infinity in terms of modified Bessel functions, making use of the assumption that the scalar field there reaches a constant vacuum expectation value. We briefly indicated the possibility of identifying the energy density of the topological defect solutions in terms of field theory parameters, with possible other descriptions of the same objects. We hope that these explicit solutions may be of benefit to a study of the phases of antisymmetric tensor theories and to the study of brane anti-brane systems in string theory.

Acknowledgments: Thanks are due to Fernando Quevedo for advice and to Alex Sevrin and Walter Troost for useful discussions.
APPENDIX

A. Solution of the differential equation

We solve the differential equation (3.10):

\[ 0 = \partial_r \left( \frac{1}{r^h} \partial_r (|\omega| r^h) \right) - m(f)^2 q (|\omega| + \frac{k_{h-1}}{r^h}) \]

for constant f by standard techniques. First, we define a new unknown function, \( \Omega \), and a new variable \( z \):

\[ \Omega \equiv |\omega| + \frac{k_{h-1}}{q} \frac{1}{r^h} \]
\[ z \equiv qmr \]

In terms of the new variables the differential equation takes the form:

\[ \partial_z \partial_{\Omega} + \frac{h}{z} \partial_{\Omega} = \Omega + \frac{h}{z^2} \Omega \]

(A.3)

To bring this to a well known form, we define still another function \( X = \Omega z^{\frac{h-1}{2}} \), in terms of which the differential equation reads:

\[ \partial_z \partial_X + \frac{1}{z} \partial_X = (1 + \frac{\nu^2}{z^2})X \]

(A.4)

where \( \nu = \frac{h+1}{2} \). Excluding the solutions which blow up at infinity, we find the following standard solution for \( X \) in terms of modified Bessel functions, including an integration constant:

\[ X = \frac{c}{q} K_{\nu}(z) \]

(A.5)

Returning to the original variables gives:

\[ \Omega = (qmr)^{-\nu+1} \frac{c}{q} K_{\nu}(qmr) \]

(A.6)

\[ |\omega| = -\frac{k_{h-1}}{q} \frac{1}{r^h} + \frac{c}{q} (qmr)^{-\nu+1} K_{\nu}(qmr) \]

(A.7)

B. Some properties of modified Bessel functions

For easy reference we list here the asymptotic behavior of the modified Bessel functions \( K_{\nu} \) and a property of the derivative that we will need in the body of the text:

\[ z \to 0 \]

\[ K_0(z) \approx -\log z \]

\[ K_{\nu}(z) \approx \frac{1}{2} \Gamma(\nu) \left( \frac{1}{2} z \right)^{-\nu} \]

(B.1)
\[ z \to \infty \]

\[ K_\nu(z) \approx \sqrt{\frac{\pi}{2z}} e^{-z} \]  \hspace{1cm} (B.2)

\[ K_0(z)' = -K_1(z) \]

\[ \frac{1}{z} \frac{d}{dz} (z^\nu K_\nu(z)) = z^{\nu-1} K_{\nu-1}(z) \]  \hspace{1cm} (B.3)
References

1. F. Englert and R. Brout, Phys Rev. Lett. 13 (1964) 321; P. Higgs, Phys. Lett. 12 (1964) 132; G. Guralnik, C. Hagen and T. Kibble, Phys. Rev. Lett. 13 (1964) 585; P. Higgs, Phys. Rev. 145 (1966) 1156; T. Kibble, Phys. Rev. 155 (1967) 1554

2. E.g. F. Quevedo and C. Trugenberger Nucl. Phys. B501 (1997) 143, hep-th/9604196 and references therein.

3. H. Nielsen and P. Olesen, Nucl. Phys. B61 (1974) 45

4. Piljin Yi, hep-th/9901159

5. A. Sen JHEP 9808 (1998) 010, hep-th/9805013; JHEP 9808 (1998) 012, hep-th/9805170; JHEP 9809 (1998) 023, hep-th/9808141

6. R. Nepomechie, Phys. Rev. D31 (1985) 1921 C. Teitelboim, Phys. Lett. 167B (1986) 63; Phys. Lett. 167B (1986) 69