GLOBAL EXISTENCE OF WEAK SOLUTION TO THE FREE BOUNDARY PROBLEM FOR COMPRESSIBLE NAVIER-STOKES

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Abstract. In this paper, the compressible Navier-Stokes system (CNS) with constant viscosity coefficients is considered in three space dimensions. We prove the global existence of spherically symmetric weak solutions to the free boundary problem for the CNS with vacuum and free boundary separating fluids and vacuum. In addition, the free boundary is shown to expand outward at an algebraic rate in time.

1. Introduction. The compressible isentropic Navier-Stokes equation in $\mathbb{R}^3$ is described by

$$
\begin{cases}
\rho_t + \text{div}(\rho U) = 0, \\
(\rho U)_t + \text{div}(\rho U \otimes U) + \nabla P - \mu \Delta U - (\lambda + \mu) \nabla \text{div} U = 0,
\end{cases}
$$

(1)

where $\rho$, $U$ and $P(\rho)$ are the density, the velocity and the pressure respectively. The pressure is given by

$$
P(\rho) = \rho^\gamma, (\gamma > 1)
$$

$\mu$ and $\lambda$ are the shear viscosity and bulk viscosity coefficients respectively. They satisfy the following physical restrictions:

$$
\mu > 0, \quad \lambda + \frac{2}{3}\mu \geq 0.
$$

There are huge amounts of literature on the studies about global existence and behaviors of solution to (1). The important progress on the global existence of strong or weak solutions in spatial one dimension has been made by many authors, refer to [8, 11, 13, 17, 22] and references therein. However, the regularity, uniqueness and dynamical behavior of the weak solutions for arbitrary initial data remain largely open for the compressible Navier-Stokes equations with constant viscosity coefficients. For multidimensional problem (1), The local existence and uniqueness of classical solutions are known in [21, 25] in the absence of vacuum and recently,

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for strong solutions also, in [1, 2, 3, 24] for the case that the initial density need not be positive and may vanish in open sets. Particularly, Choe and Kim in [3] showed that the radially symmetric strong solutions exist globally in annulus region $0 < a \leq r \leq b$. The global classical solutions were first obtained by Matsumura-Nishida [20] for initial data close to a non-vacuum equilibrium in some Sobolev space $H^s$. Later, Hoff [10] studied the problem for discontinuous initial data. For the existence of solutions for arbitrary data in three-dimension, the major breakthrough is due to Lions [18], where he established global existence of weak solutions for the whole space, periodic domains or bounded domains with Dirichlet boundary conditions provided $\gamma \geq \frac{9}{5}$. The restriction on $\gamma$ is improved to $\gamma > \frac{3}{2}$ by Feireis [4, 5, 6]. For solutions with spherical symmetry, Jiang and Zhang in [15] relaxed the restriction on $\gamma$ in [18] to the case $\gamma > 1$, and got the global existence of the weak solutions for $N = 2$ or $N = 3$. Huang, Li and Xin [14] established the global existence and uniqueness of classical solutions with small initial energy and large oscillations in 3D. Further understanding of the regularity and the asymptotic behavior of solutions near the interfaces between the gas and vacuum was given by Luo, Xin and Yang in [19, 26]. It is worth noting that Okada and Makino proved the existence of the global weak solutions to the free boundary problem of spherically symmetric motion of viscous gases in [23] in which they consider external region $1 \leq r \leq a(t)$ and the boundary condition $u(0, t) = 0, \rho(a(t), t) = 0$. However a natural question is whether it exists or not for the region $0 \leq r \leq a(t)$ and the discontinuous boundary condition $(u(0, t) = 0, P(a(t), t) = \nu(u_r + \frac{2n}{r}(a(t), t)))$?

In this paper, we prove global existence of spherically symmetric weak solutions to the free boundary value problem of three dimensions isentropic compressible Navier-Stokes equations. Compared with no vacuum initial ($\rho_0 > 0$) in [16], we obtain the result which contains vacuum ($\rho_0 \geq 0$). Further, we also obtain long time behavior of free boundary compared with [23]. However, compared with $\gamma = 1$ in pressure term in [9], where $\gamma > 1$ in pressure term which is difficult to take approximate limits due to strong nonlinear. An interesting new entropy estimate is established in this paper away from center, which provides some high regularity for the density.

The subsequent contents of the paper are organized as follows. In section 2 we will present the main result of this paper. In section 3 we construct an approximate solution sequence and derive a priori estimates for the approximate solutions. The key uniform estimates away from the symmetry center are established in section 4. In this section, these estimates do not depend on $\epsilon$. Based on these, in section 5, we take the limits to obtain the global existence of weak solutions of the original system.

*Notation used throughout this paper.* Let $\Omega$ be a domain in $\mathbb{R}^3$. Let $m$ be an integer and let $1 \leq p \leq \infty$. By $W^{m,p}(\Omega)$ ($W^{m,p}_0(\Omega)$) we denote the usual Sobolev space defined over $\Omega$. $W^{m,2}(\Omega) \equiv H^m(\Omega)(W^{m,2}_0(\Omega) \equiv H^m_0(\Omega))$. We define

$$\mathcal{L}^p(\Omega) := \left\{ f \in L^1_{loc}(\Omega) : \int_{\Omega} |f(r)|^p r^2 dr < \infty \right\},$$

with norm

$$\|f\|_{\mathcal{L}^p(\Omega)}^p := \int_{\Omega} |f(r)|^p r^2 dr.$$

For simplicity we also use the following abbreviations:

$$\|f\|_{L^p} \equiv \|f\|_{L^p}.$$
The same letter $C$ (sometimes used as $C(X)$ to emphasize the dependence of $C$ on $X$) will denote various positive constants.

2. **Main results.** In the present paper, we are mainly concerned with spherically symmetric solution to system (1). To this end, we are looking for a spherically symmetric solution as follows:

$$ r = |x|, \quad \rho(x,t) = \rho(r,t) \quad U(x,t) = u(r,t) \frac{2}{r} $$

and let $\nu = \lambda + 2\mu$, the system (1) are change to

$$
\begin{align*}
\rho_t + (\rho u)_r + \frac{2\rho u}{r} &= 0, \\
(\rho u)_t + (\rho u^2 + P(\rho))_r + \frac{2\rho u^2}{r} &= \nu(u_r + \frac{2u}{r})_r,
\end{align*}
$$

$(r,t) \in \Omega_T$ with

$$\Omega_T = \{(r,t)|0 \leq r \leq a(t), \quad 0 \leq t \leq T\}$$

with initial data

$$(\rho, \rho u)(r,0) = (\rho_0, m_0)(r) := (\rho_0, \rho_0 u_0)(r), \quad r \in (0, a_0),$$

and the free boundary condition

$$
\left( P(\rho) - \nu \left( u_r + \frac{2u}{r} \right) \right)(a(t), t) = 0, \quad t \geq 0.
$$

At the center of symmetry we impose the Dirichlet boundary condition

$$u(0, t) = 0, \quad t \geq 0.$$  

where $a(t)$ is the free boundary defined by

$$
\begin{align*}
\frac{da(t)}{dt} &= u(a(t), t), \quad t \geq 0, \\
a(0) &= a_0.
\end{align*}
$$

Which is the interface separating the gas from the vacuum. Before giving the main result, we introduce first the definition of the global weak solution to (2)-(5).

**Definition 2.1.** (Weak solution) $(\rho(t,r), u(t,r), a(t))$ with $\rho \geq 0$ a.e. is said to be a weak solution to the free surface problem (2)-(5) on $\Omega_T \times [0, T]$, provided that it holds that

1) $\rho \in L^\infty(0, T; L^1(\Omega_t)) \cap L^2(\Omega_t))$, $\sqrt{\rho} \in L^\infty(0, T; L^2(\Omega_t))$, $\rho(a(t), t) > 0, \quad t \in [0, T], \quad a(t) \in H^1([0, T]) \cap C^0([0, T]),$

2) For any $t_2 \geq t_1 \geq 0$ and $\phi \in C^1(\Omega_t \times [0, t])$ there holds

$$
\int_{\Omega_t} \rho \phi_r^2 dr|_{t_1}^{t_2} - \int_{t_1}^{t_2} \int_{\Omega_t} (\rho \phi_t + \rho u \phi_r) r^2 dr dt = 0;
$$

3) For $\psi \in C^1(\Omega_t \times [0, t])$ satisfying $\psi(r,t) = 0$ on $\partial\Omega_t$, there holds

$$
\int_{\Omega_t} \rho u \psi_r^2 dr|_{t_1}^{t_2} - \int_{t_1}^{t_2} \int_{\Omega_t} \left( \rho u \psi_t + \rho u^2 \psi_r + P(\rho)(\psi_r + \frac{2\psi}{r}) \right) r^2 dr dt
= -\nu \int_{t_1}^{t_2} \int_{\Omega_t} (u_r + \frac{2u}{r})(\psi_r + \frac{2\psi}{r}) r^2 dr dt.
$$

The free boundary condition (4) is satisfied in the sense of trace.
Suppose that the spherically symmetric initial values \((\rho_0, u_0)(r)\) satisfy
\[
\rho_0(r) \geq 0 \quad r \in \Omega_0, \quad \rho_0, u_0 \in \mathcal{H}^1(\Omega_0),
\] (6)
with \(\Omega_0 = [0, a_0]\).

The main results of the present paper can be stated in the following

**Theorem 2.2.** Assume that (6) holds and the initial data and boundary values are consistent in the sense
\[
\left( \rho_0' - \nu \left( u_0r + \frac{2u_0}{a_0} \right) \right)(a_0) = 0, \quad \rho_0(a_0) > 0.
\] (7)

Then, the FBVP (2)-(5) has a global spherically symmetric weak solution \(\rho(t, r), u(t, r), a(t)\) for which the support of \(\rho\) is bounded on the left by a curve \(\bar{r}(t)\), satisfying the following:

(a) \(c \leq a(t) \leq C_T, \quad a(t) \in H^1([0, T]) \cap C^{1/2}([0, T]), \quad \rho(a(t), t) > 0, \quad t \in [0, T], \)
\[
\|\rho^\gamma - \rho(u_r + \frac{2u}{r})\|_{L^2(\Omega_t)} + \int_0^T \|\rho^\gamma - \rho(u_r + \frac{2u}{r})\|_{\mathcal{H}^1(\Omega_t)} dt \leq C(T)
\]
with \(\Omega_\delta = (a(t) - \delta, a(t))\) for some small constant \(\delta > 0\), and the free boundary condition (4) is satisfied in the sense of trace.

(b) The function \(r : [0, \infty) \to [0, \infty)\) is a semicontinuous curve, so that if \(\mathcal{F}\) is the set
\[
\mathcal{F} := \{(t, r) | t \geq 0, \text{ and } \bar{r}(t) < r(a(t)) \text{,}\}
\]
then \(\mathcal{F} \cap \{t > 0\} \cap \{r < a(t)\}\) is open.

(c) The density \(\rho \in C([0, \infty); W^{1, \infty}[0, a(t)])^*\). Also, \(\rho(\cdot, t) \equiv 0\) in \(\mathcal{F}^c\), and if \(\rho u\) is taken to be zero in \(\mathcal{F}^c\), then the weak form of the mass equation holds for test function \(\phi \in C^1(\Omega_t \times [0, t]):\)
\[
\int_{\Omega_t} \rho \phi r^2 drdt - \int_{t_1}^{t_2} \int_{\Omega_t} (\rho \phi_t + \rho \phi_r) r^2 drdt = 0,
\]
\[
(\text{d)} \quad \text{For } \psi \in C^2(\Omega_t \times [0, t]) \text{ satisfying } \psi(r, t) = 0 \text{ on } \partial \Omega_t, \text{ there holds}
\]
\[
\int_{\Omega_t} \rho \psi r^2 drdt - \int_{t_1}^{t_2} \int_{\Omega_t} \left\{\rho \psi_t + \rho \psi_r + P(\rho)(\psi_r + \frac{2\psi}{r})\right\} r^2 drdt = -\nu \int_{t_1}^{t_2} \int_{\Omega_t} (u_r \psi_r + \frac{2u \psi}{r^2}) r^2 drdt.
\]

Finally, we investigate the long time behavior of global solutions and the motion of the interface of the free surface problem (2)-(5). We have.

**Theorem 2.3.** (Long Time Expanding and Decay Rate) Let the assumptions in Theorem 2.2 hold. Then
\[
\rho(a(t), t) \leq C(1 + t)^{1/3},
\] (8)
\[
\rho(a(t), t) \leq C(1 + t)^{-\frac{1}{\gamma}},
\] (9)
\[
a_M(t) = \max_{a \in [0, T]} a(s) \geq \begin{cases} C(1 + t)^{\frac{1}{\gamma}}, & \gamma > \frac{N+1}{N}, \\ C(1 + t)^{\frac{1}{N}}, & \gamma = \frac{N+1}{N}, \\ C(1 + t)^{-\frac{1}{\gamma}}, & \gamma \in (1, \frac{N+1}{N}). \end{cases}
\] (10)

Thus we obtain
\[
a_M(t) \to \infty, \quad \text{as } t \to \infty.
\]
3. Global existence of approximate FBVP. Consider a modified FBVP problem for Eq. (2) with the following initial data and boundary conditions for any fixed small enough $\epsilon > 0$:

$$
(\rho^\epsilon, u^\epsilon)|_{t=0} = (\rho_0, u_0)(r), \quad \epsilon \leq r \leq a_0, \quad (11)
$$

$$
u'(\epsilon, t) = 0, \quad \left( P(\rho^\epsilon) - \nu(u^\epsilon_r + \frac{2u^\epsilon}{r}) \right)(a^\epsilon(t), t) = 0, \quad t \geq 0, \quad (12)
$$

where

$$
\begin{cases}
\frac{da^\epsilon(t)}{dt} = u^\epsilon(a^\epsilon(t), t), \quad t \geq 0, \\
a(0) = a_0.
\end{cases} \quad (13)
$$

To simplify the presentation, we drop the superscript $\epsilon$.

**Proposition 1.** Let $T > 0$, and $\epsilon > 0$ be fixed, assume the initial data $(\rho_0, u_0)$ satisfies

$$\inf_{[\epsilon, a_0]} \rho_0 > 0, \quad (\rho_0, u_0) \in H^1([\epsilon, a_0]).$$

Then, there exists a unique global smooth solution $(\rho, u, a)$ of the FBVP (2) and (11)-(12) which satisfies

$$c \leq a(t) \leq C_T, \quad c, r \leq \rho \leq C_{\epsilon, T},$$

$$\|(\rho, u)\|_{H^1([\epsilon, a(t)])} + \int_0^T \|(\rho_r, u_r, u_{rr})\|_{L^2([\epsilon, a(t)])}^2 dt$$

$$+ \int_0^T \|\rho^\gamma - \rho(u_r + \frac{2u}{r})\|_{L^2([\epsilon, a(t)])}^2 dt + \int_0^T \|(a, a')^2(t)\|_{L^2([\epsilon, a(t)])} dt \leq C(\epsilon, T).$$

Furthermore, if $u_t(r, 0) \in L^2([\epsilon, a_0])$, then

$$\|u\|_{H^2([\epsilon, a(t)])} + \|u_t\|_{L^2([\epsilon, a(t)])} + \|\rho^\gamma - \rho(u_r + \frac{2u}{r})\|_{H^1([\epsilon, a(t)])}$$

$$+ \int_0^T \|u_t\|^2_{H^1([\epsilon, a(t)])} + |a''(t)|^2 dt \leq C(\epsilon, T).$$

In this section, we establish the a-priori estimates for any approximate solution $(\rho, u, a)$ with $\rho > 0$ to FBVP (2) and (11)-(12). We start with a basic energy estimate.

**Lemma 3.1.** Let $\gamma > 1$, $T > 0$, and $(\rho, u, a)$ with $\rho > 0$ be any regular solution to the FBVP (2) and (11)-(12) for $t \in [0, T]$ under the assumptions of Proposition 1. Then

$$\int_0^{a(t)} \rho r^2 dr = \int_0^{a_0} \rho_0 r^2 dr = M_0,$$

$$\int_0^{a(t)} \left( \frac{1}{2} \rho u^2 + \frac{1}{\gamma - 1} \rho^\gamma \right) r^2 dr + \int_0^t \int_0^{a(s)} \nu \left( u_r + \frac{2u}{r} \right)^2 r^2 dr ds = E_0, \quad (14)$$

The proof of Theorem 2.2 is based on a basic energy estimate, higher-order estimate and passing to the limit for the approximate solutions. To pass to the limit, we need further higher-order bounds which do not depend on $\epsilon$ and mainly use and adapt the idea of Hoff and Jenssen in the study of symmetric nonbarotropic flows with large data and forces [12].
or
\[
\int_0^{a(t)} \left( \frac{1}{2} \rho u^2 + \frac{1}{\gamma - 1} \rho^\gamma \right) r^2 \, dr + \nu \int_0^t \int_0^{a(s)} \left( u_r^2 + \frac{2u^2}{r^2} \right) r^2 \, dr \, ds \\
+ 2\nu \int_0^t u_2 (a(s), s) a(s) \, ds = E_0. \tag{15}
\]

**Proof.** Multiplying \((2)_2\) by \(r^2 u\), using \((2)_1, (13)\), integration by parts and \((6)\), one gets
\[
\int_0^{a(t)} \left( \frac{1}{2} \rho u^2 + \frac{1}{\gamma - 1} \rho^\gamma \right) r^2 \, dr + \nu \int_0^t \int_0^{a(s)} \nu \left( u_r + \frac{2u}{r} \right)^2 r^2 \, dr \, ds = E_0.
\]
Since
\[
\int_0^t \int_0^{a(s)} \nu \left( u_r + \frac{2u}{r} \right)^2 r^2 \, dr \, ds = \nu \int_0^t \int_0^{a(s)} \left( u_r^2 + 4u^2 + 2(u^2)r \right) dr \, ds,
\]
thus we obtain \((15)\). This completes the proof of the Lemma 3.1. \(\blacksquare\)

**Lemma 3.2.** Under the same assumptions as Lemma 3.1, Then
\[
c_0 \equiv E_0^{-\frac{1}{\gamma - 1}} M_0^{\frac{\gamma - 1}{\gamma}} \leq a(t) \leq C(1 + t)^{1/3}. \tag{16}
\]
\[
a(t) \in H^1(0, T). \tag{17}
\]

**Proof.** First, by the equation \((2)_1\) and \((14)\), one gets
\[
M_0 \equiv \int_0^{a_0} \rho_0 r^2 \, dr = \int_0^{a(t)} \rho r^2 \, dr \leq \left( \int_0^{a(t)} \rho^\gamma r^2 \, dr \right)^{\frac{1}{\gamma}} \left( \int_0^{a(t)} r^2 \, dr \right)^{\frac{\gamma - 1}{\gamma}} \leq (E_0)^{\frac{1}{2}} 3^{\frac{1 - \gamma}{\gamma}} (a(t))^{\frac{3(\gamma - 1)}{\gamma}},
\]
which yields
\[
c_0 \equiv E_0^{-\frac{1}{\gamma - 1}} M_0^{\frac{\gamma - 1}{\gamma}} \leq a(t). \tag{18}
\]
Next, \((15)\) yields
\[
a(t)^{\frac{3}{2}} = \int_0^t \frac{d}{ds} a(s)^{\frac{3}{2}} \, ds + a_0^{\frac{3}{2}}
\]
\[
= \int_0^t \frac{3}{2} a(s)^{\frac{3}{2}} u(a(s), s) \, ds + a_0^{\frac{3}{2}}
\]
\[
\leq \frac{3}{2} \left( \int_0^t a(s) u^2(a(s), s) \, ds \right)^{1/2} t^{1/2} + a_0^{\frac{3}{2}}
\]
\[
\leq \frac{3}{2} \frac{1}{2\nu} (E_0)^{1/2} t^{1/2} + a_0^{\frac{3}{2}}
\]
\[
\leq C(t^{1/2} + 1),
\]
where
\[
\rho_0 = \frac{\rho}{\rho_0}, \quad a(s) = a(s) \rho_0
\]
thus, we get
\[ a(t) \leq C(t^{1/2} + 1)^{\frac{3}{2}} \leq C(t + 1)^{\frac{3}{2}}. \]
It follows from (15) and (18) that
\[ 2\nu C_0 \int_0^t (a'(\tau))^2 d\tau = 2\nu C_0 \int_0^t [u(a(\tau), \tau)]^2 d\tau \leq 2\nu \int_0^t [u(a(\tau), \tau)]^2 a(\tau) d\tau \leq C. \]
This completes the proof of the Lemma 3.2.

**Lemma 3.3.** Under the same assumptions as Lemma 3.1, Then
\[
\rho(a(t), t) = \left( \frac{\nu}{\nu'} \right)^{-\frac{1}{2}} \left( \frac{\nu}{\gamma} \rho_0^{-\gamma}(a_0) + t \right)^{-\frac{1}{2}} \leq C(1 + t)^{-\frac{1}{2}},
\]
\[
\rho(a(t), t) \leq \rho_0(a_0).
\]

**Proof.** From (12) and Eq. (2), we get
\[
\frac{d}{dt} \rho(a(t), t) = -\frac{1}{\nu} \rho^{\gamma+1}(a(t), t),
\]

thus, we have
\[
\rho(a(t), t) = \rho_0(a_0)(1 + \frac{\gamma}{\nu'} \rho_0^{-\gamma}(a_0)t)^{-\frac{1}{2}} = \rho_0(a_0)(\frac{\gamma}{\nu'} \rho_0^{-\gamma}(a_0))^{-\frac{1}{2}} (\frac{\nu}{\gamma} \rho_0^{-\gamma}(a_0) + t)^{-\frac{1}{2}} = (\frac{\gamma}{\nu'})^{-\frac{1}{2}} (\frac{\nu}{\gamma} \rho_0^{-\gamma}(a_0) + t)^{-\frac{1}{2}},
\]
which gives Lemma 3.3 and \( \rho(a(t), t) \leq \rho_0(a_0) \). This completes the proof of the Lemma 3.3.

In order to estimate \( \rho \), we first establish the following technical result.

**Lemma 3.4.** Under the same assumptions as Lemma 3.1. Define
\[
\varphi(r, t) = \nu(u_r + \frac{2u}{r}) - \rho u^2 - \rho^\gamma - \int_r^\infty \frac{2\rho u^2}{y} dy,
\]
and
\[
\psi(r, t) = \int_0^t \varphi(r, s) ds + \int_r^\infty \rho_0 u_0 dy.
\]
Then there exists a constant \( C(\epsilon, T) \), such that
\[
|\psi(r, t)| \leq C(\epsilon, T) \quad \text{for} \quad r \in [\epsilon, a(t)], \quad t \in [0, T].
\]

**Proof.** We observe that
\[
(\rho u)_t - \varphi(r, t)_r = 0,
\]
and
\[
\psi_t = \varphi, \quad \psi_r = \rho u.\]
\[ |\psi(r, t)| = |\psi(a(t), t) + \int_{a(t)}^{r} \psi(y, t)dy| \]
\[ \leq |\int_{0}^{t} \rho u^2(a(s), s) + \int_{\epsilon}^{a(s)} \frac{2\rho u^2}{y} dy)ds| + |\int_{\epsilon}^{a(t)} \rho_0 u_0 dy| + |\int_{a(t)}^{r} \rho u(y, t)dy| \]
\[ \leq \rho_0 a_0 \frac{1}{\epsilon} \int_{0}^{t} u^2(a(s), s) a(s) ds + \frac{2}{\epsilon^3} \int_{0}^{t} \int_{\epsilon}^{a(s)} \rho u^2 y^2 dy ds| + |\int_{\epsilon}^{a_0} \rho_0 u_0 dy| + |\int_{a(t)}^{r} \rho u(y, t)dy| \]
\[ \leq \frac{C}{\epsilon^3} \]
This completes the proof of the Lemma 3.4.

**Lemma 3.5.** Under the same assumptions as Lemma 3.1, Then
\[ c(\epsilon) \leq \rho \leq C(\epsilon), \quad (r, \tau) \in [\epsilon, a(t)] \times [0, T]. \]

**Proof.** Define \( H(\psi) = \exp(\nu^{-1}\psi), \) so that
\[
D_t(\rho H(\psi)) = \partial_t(\rho H(\psi)) + u \cdot \partial_r(\rho H(\psi))
\[
= \rho H(\psi) + u \rho H(\psi) + \rho H(\psi) \nu^{-1} \psi_t + \rho u H(\psi) \nu^{-1} \psi_r
\]
\[ = -\nu^{-1} \rho H(\psi) \left( \rho \gamma + \int_{\epsilon}^{r} 2\rho u^2 dy \right), \]
thus, we get
\[ D_t \ln(\rho H(\psi)) = -\nu^{-1} \left( \rho \gamma + \int_{\epsilon}^{r} 2\rho u^2 dy \right). \]
Consequently
\[ \rho H(\psi) = \rho_0 H(\psi_0) \exp \left\{ -\int_{0}^{t} \nu^{-1} \left( \rho \gamma + \int_{\epsilon}^{r} 2\rho u^2 dy \right) ds \right\}. \tag{19} \]
First, from Lemma 3.4 and the definition of \( H(\psi), \) we have \( c(\epsilon, T) \leq H(\psi) \leq C(\epsilon, T). \) Consequently, we get
\[ \rho \leq C(\epsilon, T). \]
Next, by (19), we get
\[ \frac{1}{\rho} = \frac{H(\psi)}{\rho_0 H(\psi_0)} \exp \left\{ \int_{0}^{t} \nu^{-1} \left( \rho \gamma + \int_{\epsilon}^{r} 2\rho u^2 dy \right) ds \right\}. \]
Since \( H(\psi) \leq C(\epsilon, T) \) and \( \rho \) is bounded above, we conclude that
\[ \frac{1}{\rho} \leq C(\epsilon, T). \]
This completes the proof of the Lemma 3.5.

It is convenient to deal with the FBVP (2) in the Lagrangian coordinates. For simplicity we assume that \( \int_{\epsilon}^{a_0} \rho_0 r^2 dr = 1, \) which implies
\[ \int_{\epsilon}^{a(t)} \rho r^2 dr = \int_{\epsilon}^{a_0} \rho_0 r^2 dr = 1. \]
For \( r \in [\epsilon, a(t)], t \in [0, T] \), define the Lagrangian coordinates transformation
\[
x(r, t) = \int_{\epsilon}^{r} \rho(y, t)y^2dy, \quad t = \tau.
\]
which translates the domain \([0, T] \times [\epsilon, a(t)]\) into \([0, T] \times [0, 1]\) and satisfies
\[
\frac{\partial x}{\partial r} = \rho r^2, \quad \frac{\partial x}{\partial t} = -\rho u r^2, \quad \frac{\partial \tau}{\partial t} = 0, \quad \frac{\partial r}{\partial \tau} = 1 \quad \text{and} \quad \frac{\partial r}{\partial \tau} = u.
\]
The free boundary value problem (2) and (11)-(12) are changed to
\[
\begin{cases}
\rho \tau + \rho \frac{\partial}{\partial r} (\rho u^2) = 0, \\
u_r + P(\rho) - \psi u r^2 = \nu u^2 (\rho u^2)_x,
\end{cases}
\] (20)
for \((x, \tau) \in [0, 1] \times [0, T]\), with the initial data and boundary conditions given by
\[
(\rho, u)|_{t=0} = (\rho_0, u_0)(x), \quad x \in [0, 1],
\]
\[
u(0, \tau) = 0, \quad (P(\rho) - \nu \rho u^2)_x(1, \tau) = 0, \quad \tau \in [0, T],
\] (22)
where \( r = r(x, \tau) \) is defined by
\[
\frac{d r(x, \tau)}{d \tau} = u(x, \tau), \quad (x, \tau) \in [0, 1] \times [0, T],
\]
and the fixed boundary \( x = 1 \) corresponds to the free boundary \( a(\tau) = r(1, \tau) \) in Eulerian form determined by
\[
\frac{d a(\tau)}{d \tau} = u(1, \tau), \quad \tau \in [0, T], \quad a(0) = a_0.
\]
Energy estimation of Lemma 3.1 has follow form in Lagrangian coordinates.

**Lemma 3.6. (Energy estimation)** Under the same assumptions as Lemma 3.1, Then
\[
\int_{0}^{1} \left[ \frac{1}{2} u^2 + \frac{1}{\gamma - 1} \rho r^2 \right] dx + \int_{0}^{t} \int_{0}^{1} \nu \rho [(ur^2)_x]^2 dx ds = E_0,
\]
or
\[
\int_{0}^{1} \left[ \frac{1}{2} u^2 + \frac{1}{\gamma - 1} \rho r^2 \right] dx + \int_{0}^{t} \int_{0}^{1} \nu \rho (u_r^2)^2 dx ds + 2 \nu \int_{0}^{t} \int_{0}^{1} \frac{1}{\rho} u_r^2 dx ds + \int_{0}^{t} 2 \nu u^2 (1, s) ds = E_0.
\]

**Lemma 3.7. (Entropy estimation)** Under the same assumptions as Lemma 3.1, Then
\[
\int_{0}^{1} (u + \nu u^2 (\ln \rho)_x)^2 dx + \frac{4}{\gamma} \int_{0}^{\tau} \int_{0}^{1} \left( (\rho^{\gamma/2}u)^2 \right)_x^2 dx ds \leq C.
\]
*Proof.* From (20), one has
\[
u_r + P_x r^2 = -\nu u^2 \frac{\rho_x}{\rho} = -\nu u^2 (\ln \rho)_{x\tau},
\]
which yields
\[
(u + \nu u^2 (\ln \rho)_x)_\tau = 2 \nu u u (\ln \rho)_x - P_x r^2.
\]
Corollary 1. Under the same assumptions as Lemma 3.1, Then

\[ \int_0^1 (\rho_x)^2 dx \leq C(\epsilon, T). \]
Lemma 3.8. (Higher-order boundedness) Let $h > 0$ and $T > 0$ be given. Then there is a constant $C = C(T, \epsilon)$ such that

$$A(T) \leq C(T, \epsilon).$$

Proof. Multiplying (20) by $u_\tau$ and integrating, we have

$$\int_0^1 (u_\tau)^2 dx - \int_0^1 P(u_\tau^2) dx$$

Multiplying (20) by $u_\tau$ and integrating, we have

$$\int_0^1 P(u_\tau^2) dx = -\nu \int_0^1 \rho (u_\tau^2) dx$$

$$\int_0^1 P(u_\tau^2) dx = -\nu \int_0^1 \rho (u_\tau^2) dx$$

$$\int_0^1 P(u_\tau^2) dx = -\nu \int_0^1 \rho (u_\tau^2) dx$$

$$\frac{d}{dt} \int_0^1 \rho (u_\tau^2) dx$$

This leads to

$$\frac{d}{dt} \int_0^1 \rho (u_\tau^2) dx + \int_0^T \int_0^1 (u_\tau)^2 dx d\tau$$

$$= E_2 + \int_0^T \int_0^1 P(u_\tau^2) dx d\tau + \frac{\nu}{2} \int_0^T \int_0^1 \rho_x (u_\tau^2_x) dx d\tau$$

$$+ \nu \int_0^T \int_0^1 \rho (u_\tau^2_x (2u_\tau^2) dx d\tau$$

$$\leq E_2 + J_1 + J_2 + J_3,$$

(28)

where $E_2 = \frac{\nu}{2} \int_0^1 \rho_0 (u_0^2) dx$.

We first estimate $J_1$. (6) and Lemma 3.1 implies

$$J_1 = \int_0^T \int_0^1 P u_x r^2 dx d\tau + \int_0^T \int_0^1 P u_r \frac{2}{r} dx d\tau$$

$$\leq \int_0^T \int_0^1 P u_x r^2 dx - \int_0^1 P u_x r^2 (x, 0) dx - \int_0^T \int_0^1 u_x (P r^2 + 2ru) dx d\tau$$

$$+ \int_0^T \int_0^1 \rho^{-1} u_r \frac{2}{r} dx d\tau$$

$$\leq \int_0^T \int_0^1 P u_x r^2 dx + C(\epsilon, T) \int_0^T \int_0^1 \rho (u_\tau^2) dx d\tau + \delta \int_0^T \int_0^1 u_\tau^2 dx d\tau + C(\epsilon, T)$$

$$\leq \delta \int_0^1 \rho (u_\tau^2) dx + \delta \int_0^T \int_0^1 u_\tau^2 dx d\tau$$

$$+ C(\epsilon, T) \int_0^T \int_0^1 \rho (u_\tau^2) dx + C(\epsilon, T).$$

(29)

The equation (20) implies

$$J_2 \leq C(\epsilon, T) \int_0^T \int_0^1 (u_\tau^2) dx d\tau.$$
Young’s inequality and (27) give

\[ J_3 = \nu \int_0^T \int_0^1 \rho (uv^2 x) \left( \frac{2u^2}{r^2} + 4rux \right) dx \, d\tau \]
\[ \leq C(\epsilon, T) \int_0^T \|u\|_{L^\infty} \left\{ \int_0^1 \rho (uv^2 x)^2 \, dx + \int_0^1 \left( \frac{2u}{r^2} \right)^2 \, dx + \int_0^1 (rux)^2 \, dx \right\} \, d\tau \]
\[ \leq C(\epsilon, T) \int_0^T \|u\|_{L^\infty} \int_0^1 \rho (uv^2 x)^2 \, dx \, d\tau + C(\epsilon, T). \]  

Substituting (29) – (31) into (28), we obtain

\[ \int_0^1 \rho (uv^2 x)^2 \, dx + \int_0^T \int_0^1 (u_t)^2 \, dx \, d\tau \]
\[ \leq C(\epsilon, T) \int_0^T \int_0^1 |(uv^2 x)|^3 \, dx \, d\tau \]  

(32)

the first term in (32) can estimated as follows

\[ \int_0^T \int_0^1 |(r^2 u)_x|^{3} \, dx \, d\tau \]
\[ \leq \sup_{0 \leq \tau \leq T} \left( \int_0^1 |(r^2 u)_x|^2 \, dx \right)^{1/2} \times \left\{ \int_0^T \int_0^1 |(r^2 u)_x|^2 \, dx \, d\tau \right\}^{1/4} \]
\[ + \left( \int_0^T \int_0^1 |(r^2 u)_x|^2 \, dx \, d\tau \right)^{1/4} \left( \int_0^T \int_0^1 |(r^2 u)_x|^2 \, dx \, d\tau \right)^{3/4} \}
\[ \leq C(\epsilon, h) A(T)^{1/2} \left( 1 + \int_0^T \int_0^1 |(r^2 u)_x|^2 \, dx \, d\tau \right)^{1/4} \]  

(33)

Equation (20)2 can re-write as

\[ \rho^{-1} (u_t r^{-2} + P_x - \nu \rho_x (r^2 u)_x) = \nu (r^2 u)_x. \]

Which and the fact \(\|u\|_{L^\infty} \leq \|u\|_{L^2}^{1/2} \|u_x\|_{L^2}^{1/2}\) give

\[ \nu^2 \int_0^T \int_0^1 |(r^2 u)_x|^2 \, dx \, d\tau \]
\[ \leq C(T, \epsilon) \int_0^T \int_0^1 \rho^{-1} |u_t r^{-2} + P_x r^{-2} - \nu \rho_x (r^2 u)_x|^2 \, dx \, d\tau \]
\[ \leq C(T, \epsilon) \left( \int_0^T \int_0^1 |u_t|^2 \, dx \, dt + \int_0^T \int_0^1 |P_x|^2 \, dx \, dt \right) \]
\[ + C(T, \epsilon) \left( \int_0^T \| (r^2 u)_x \|_{L^\infty}^2 \int_0^1 |\rho_x|^2 \, dx \, dt \right) \]
\[ \leq C(T, \epsilon) \left( \int_0^T \int_0^1 |u_t|^2 \, dx \, dt + \int_0^T \| (r^2 u)_x \|_{L^2}^2 \, dt \right) + \delta \int_0^T \| (r^2 u)_x \|_{L^2}^2 \, dt \]
Consequently
\[ \nu^2 \int_0^T \int_0^1 |(r^2u)_{xx}|^2 dx d\tau \leq C(T, \epsilon) \left( \int_0^T \int_0^1 |u_\tau|^2 dx d\tau + \int_0^T \| (r^2u)_x \|^2_{L^2} dt \right) \leq C(T, \epsilon)(A(T) + 1), \]

thus, by (33) we have
\[ \int_0^T \int_0^1 |(r^2u)_x|^3 dx d\tau \leq C(T, \epsilon)A(T)^{1/2} \left( 1 + A(T) \right)^{1/4} \]

\[ \leq C(T, \epsilon) \left( 1 + A(T)^{3/4} \right). \]

Which together with (32) give
\[ A(T) \leq C(T, \epsilon) \left( 1 + A(T)^{3/4} + \int_0^T \| u \|_{L^\infty} A(\tau) d\tau \right), \]

Applying Cauchy’s inequality and Gronwall’s inequality we finally obtain
\[ A(T) \leq C(T, \epsilon). \]

This completes the proof of the Lemma 3.8.

From the Lemma 3.5 and Lemma 3.8, we get

**Corollary 2.** Under the same assumptions as Lemma 3.1, Then
\[ \int_0^1 (u_x)^2 dx + \int_0^T \int_0^1 (u_\tau)^2 dx d\tau \leq C(T, \epsilon). \]

**Lemma 3.9.** Under the same assumptions as Lemma 3.1, Then
\[ \int_0^T \int_0^1 (u_{xx})^2 + \left( |(P - \nu r^2 u_x)_x|^2 + \int_0^T (u_x)^2 + (u_\tau)^2 \right) \right) dx \leq C(T, \epsilon), \]

and
\[ \int_0^1 (u_\tau)^2 dx + \int_0^1 (u_{xx})^2 dx + \int_0^T \int_0^1 (u_{x\tau})^2 dx d\tau \]
\[ + \int_0^T (a''(\tau))^2 d\tau \leq C(\epsilon, T). \]

**Proof.** Equation (20) can re-write as
\[ \rho^{-1}(u_\tau r^{-2} + P_x - \nu r^2 (u_x) x) = \nu \left( r^2 u_{xx} + \frac{2u_x}{\rho r} + \frac{2u_x}{\rho^2 r} - \frac{2u}{(\rho r)^2} \right), \]

which gives
\[ \int_0^T \int_0^1 (u_{xx})^2 dx d\tau \]
\[ \leq \int_0^T \int_0^1 \left( (u_\tau)^2 + (\rho_x)^2 + (\rho_x u_x)^2 + (u_x)^2 + (u_\tau)^2 + u^2 \right) dx d\tau \]
\[ \leq \int_0^T \int_0^1 \left( (\rho_x u_x)^2 + (u_\tau)^2 \right) dx d\tau + C(T, \epsilon) \]
Thus
\[
\int_0^T \int_0^1 (u_{xx})^2 \text{d}x \text{d}\tau \leq C(T, \epsilon).
\]
By (15), (16) and (20)\_2
\[
\int_0^T ((a(\tau))^2 + (a'(\tau))^2) \text{d}\tau \leq C(T, \epsilon),
\]
and
\[
\int_0^T \int_0^1 \left((p - \nu \rho (u^2)_x)_x\right)^2 \text{d}x \text{d}\tau \leq \int_0^T \int_0^1 (r^{-2} u\_x)^2 \text{d}x \text{d}\tau \leq C(T, \epsilon).
\]
Differentiating (20)\_2 with respect to \( \tau \) gives
\[
r^{-2} u\_{\tau\tau} - 2r^{-3} u\_\tau + P\_\tau = \nu(u^2)_x\_\tau
\]
Taking the inner product of (34) with \( u\_\tau \) over \([0, 1]\]
\[
\begin{align*}
\frac{d}{dt} \int_0^1 \frac{1}{2} r^{-2}(u\_\tau)^2 \text{d}x + \nu \int_0^1 (u\_\tau)^2 r^2 \text{d}x + \nu \frac{u^2}{r}(1, \tau) \\
= \int_0^1 r^{-3} u(u\_\tau)^2 \text{d}x + \int_0^1 p\_\tau u\_\tau \text{d}x - \nu \int_0^1 \rho \rho (u^2)_x u\_\tau \text{d}x \\
- \nu \int_0^1 2\rho u\_\tau x u\_\tau \text{d}x + \nu \int_0^1 \frac{u^2}{\rho^4} \rho r^2 \text{d}x + \nu \int_0^1 \frac{2u\_\tau u\_\tau \rho}{\rho} \text{d}\tau x + \nu \int_0^1 \frac{2u^2 u\_\tau}{\rho^2} \text{d}x \\
\leq C \int_0^1 r^{-2}(u\_\tau)^2 \text{d}x + \delta \int_0^1 \rho (u\_\tau)^2 r^2 \text{d}x + C \int_0^1 (\rho r)^2 \text{d}x - \nu \int_0^1 \rho \rho (u^2)_x u\_\tau \text{d}x \\
+C \int_0^1 (u\_\tau)^2 \text{d}x + C \int_0^1 u^2 \text{d}x \\
\leq \int_0^1 r^{-2}(u\_\tau)^2 \text{d}x + \delta \int_0^1 \rho (u\_\tau)^2 r^2 \text{d}x + \int_0^1 (u\_\tau)^2 \text{d}x,
\end{align*}
\]
where we have used the follow estimate
\[
-\nu \int_0^1 \rho \rho (u^2)_x u\_\tau \text{d}x
\]
\[
= \nu \int_0^1 \rho^2 (u^2)_x^2 u\_\tau \text{d}x
\]
\[
\leq C(\epsilon, T) \| (u^2)_x \|_{L^\infty} \left( \int_0^1 \rho (u^2)_x^2 \text{d}x \right)^{1/2} \left( \int_0^1 \rho (u\_\tau)^2 r^2 \text{d}x \right)^{1/2}
\]
\[
\leq C(\epsilon, T) \| (u^2)_x \|_{L^\infty}^2 + \delta \int_0^1 \rho (u\_\tau)^2 r^2 \text{d}x
\]
\[
\leq C(\epsilon, T) \| (u^2)_xx \|_{L^2}^2 + \delta \int_0^1 \rho (u\_\tau)^2 r^2 \text{d}x
\]
\[ \leq C(\epsilon, T) (\|u_{xx}\|_{L^2} + \|u_x\|_{L^2} + \|\rho_x\|_{L^2} + \|u\|_{L^2}) + \delta \int_0^1 \rho(u_{xx})^2 r^2 dx \]
\[ \leq C(\epsilon, T) (\|u_{xx}\|_{L^2} + 1) + \delta \int_0^1 \rho(u_{xx})^2 r^2 dx. \]

Using Grönwall’s inequality to (35), we obtain
\[ \int_0^1 r^{-2} (u_x)^2 dx + \int_0^T \int_0^1 \rho(u_{xx})^2 r^2 dxd\tau + \int_0^T \frac{u_x^2}{r}(1, \tau) d\tau \leq C(\epsilon, T). \]

Consequently by (20), we have
\[ \int_0^1 ((p - \nu \rho(u_x^2)_{xx})_x)^2 dx \leq \int_0^1 (r^{-2} u_x)^2 dx \leq C(T, \epsilon), \]
\[ \int_0^T a''(\tau) d\tau \leq C(T, \epsilon), \]
and
\[ \int_0^1 (u_{xx})^2 dx \leq \int_0^1 ((u_x)^2 + (u_x)^2 + (\rho_x)^2) dx \leq C(\epsilon, T). \]

This completes the proof of the Lemma 3.9.

The proof of proposition 1. First, we can change Proposition 1 in the Lagrangian coordinates. Next, with the help of Lemmas 3.1-3.9 for \((\rho, u, a)\) and continuity argument, the global strong solution to the FBVP (2) and (11)-(12) under the assumptions of Proposition 1 can be shown by standard argument as in [22].

4. Uniform estimates away from symmetry center. In order to obtain limiting solutions as \(\epsilon \to 0\), we will need further higher order bounds which are uniform in \(\epsilon\). These will be obtained away from the origin of Lagrangian space in the following sense. Define curves \(r^{\epsilon}_h(t)\) for \(h \geq 0\) by
\[ h = \int_{r^{\epsilon}_h(t)} r^\epsilon (r, t) r^2 dr. \]
\[ \frac{\partial r^{\epsilon}_h}{\partial t} = u^\epsilon (r^{\epsilon}_h, t). \]

Thus \(r^{\epsilon}_h(t)\) is the position at time \(t\) of a fixed fluid particle. So, given \(h > 0\) there is a positive constant \(C\) independent of \(\epsilon\) and \(T\), such that
\[ Ch^\frac{\gamma}{\gamma-1} \leq r^{\epsilon}_h(t). \]

Thus the Lagrangian coordinates transformation translates the domain \([0, T] \times [r^{\epsilon}_h(t), a(t)]\) into \([0, T] \times [h, 1]\).

In this section, we derive some desired uniform estimates for \((\rho^\epsilon, u^\epsilon, a^\epsilon)\) to the modified FBVP (2) and (11)-(12). To simplify the presentation, we drop the superscript \(\epsilon\).

Lemma 4.1. The upper bound of the density:
\[ c(h) \leq \rho \leq C(h), \quad (r, \tau) \in [r^{\epsilon}_h(t), a(t)] \times [0, T]. \]

Proof. Thanks to the fact \(Ch^\frac{\gamma}{\gamma-1} \leq r^{\epsilon}_h(t)\), this lemma can be proved similarly as Lemma 3.5. So the details are omitted here.

Thanks to Lemma 3.6 and Lemma 4.1, we get the following corollary.
Corollary 3. Under the same assumptions as Lemma 4.1, Then
\[ \int_0^T \int_0^1 (P - \nu \rho(u^2)_x)_x^2 dx d\tau \leq C(h, T). \]

Similar to the proof of Lemma 3.7 we have the following Entropy estimation.

Lemma 4.2. (Entropy estimation) Under the same assumptions as Lemma 4.1, Then
\[ \int_1^h \frac{1}{h} \left( u + \nu r^2 (\ln \rho)_x \right)^2 dx + \frac{4}{7} \int_0^T \int_h^1 \left( \frac{\rho^{\gamma/2} x}{r^2} \right)^2 dxd\tau \leq C(h). \]

Thanks to Lemma 3.6 and Lemma 4.1, we get the following corollary.

Corollary 4. Under the same assumptions as Lemma 4.1, Then
\[ \int_1^h r^4 (\rho x)_x^2 dx \leq C(h). \]

We shall make repeated use of a cut-off function which is convected with the flow and which vanishes near the origin. So we construct a cut-off function \( \phi(x) \) satisfies
\[
\phi(x) = \begin{cases} 
0 & x \in [0, h], \\
\frac{1}{h} (x - h) & x \in (h, 2h), \\
1 & x \in [2h, 1].
\end{cases}
\]

Now we introduce the higher-order functional for a given solution and define
\[
\mathcal{B}(T) = \sup_{0 \leq t \leq T} \int_h^1 \phi^2 \rho [(r^2 u)_x]_x^2 dx + \int_0^T \int_h^1 \phi^2 u_r^2 dx d\tau. \tag{36}
\]

For which corresponding to the following form in Euler coordinates:
\[
\mathcal{B}(T) = \sup_{0 \leq t \leq T} \int_{\tau^a(t)}^{\tau^d(t)} \phi^2 u_r^2 r^2 dr + \int_0^T \int_{\tau^a(t)}^{\tau^d(t)} \phi^2 u_r^2 r^2 dr dt.
\]

Lemma 4.3. Let \( h > 0 \) and \( T > 0 \) be given. Then there exists a constant \( C = C(h, T) \) such that, if \( \mathcal{B} \) is as defined above in (36), then
\[ \mathcal{B}(t) \leq C(h, T), \]
and
\[ \int_0^T \int_{2h}^1 \left| (\nu \rho(u^2)_x - P(\rho))_x \right|^2 dxd\tau \leq C(h, T), \]

Proof. Multiplying (20) by \( \phi^2 u_r \), one has
\[
\frac{\nu}{2} \int_h^1 \phi^2 \rho [(r^2 u)_x]_x^2 dx + \int_0^T \int_h^1 \phi^2 u_r^2 dx d\tau
= \frac{\nu}{2} \int_h^1 \phi^2 \rho [(r^2 u)_x]_x^2 \big|_{t=0} dx + \int_0^T \int_h^1 (\phi^2 u_r^2)_x P dxd\tau
- \frac{\nu}{2} \int_0^T \int_h^1 \phi^2 [(r^2 u)_x]^3 \rho^2 dxd\tau - 2 \nu \int_0^T \int_h^1 \rho \phi(x)(u^2)_x (u^2)_x dxd\tau.
\]
In the sequel, we derive bounds for $I_i$ (1 ≤ $i$ ≤ 4) on the right-hand side of (37).

\[ I_1 = \int_0^T \left( \int_0^1 \phi^2 u_x r^2 \, dx \right) \, d\tau + \int_0^T \left( \int_0^1 \beta^2 P u_x r^2 \, dx \right) \, d\tau + 2 \int_0^T \left( \int_0^1 \phi \phi_x u_x r^2 \, dx \right) \, d\tau \]

\[ = \int_0^1 \phi^2 u_x r^2 \, dx - \int_0^1 \beta^2 P u_x r^2 \, dx - \int_0^T \phi \phi_x u_x r^2 \, d\tau \]

\[ + \int_0^T \left( \int_0^1 \phi^2 \rho^{-1} u_x r^2 \, dx \right) \, d\tau + 2 \int_0^T \int_0^1 \phi \phi_x u_x r^2 \, d\tau \]

\[ \leq \int_0^1 \phi^2 u_x r^2 \, dx + \delta \int_0^T \int_0^1 \phi^2 u_x r^2 \, d\tau + C(h,T) \]

\[ \leq \delta \int_0^1 \phi^2 \rho \left( (u^2)_{x} \right)^2 \, dx + \delta \int_0^T \int_0^1 u_x r^2 \, d\tau + C(h,T), \] (38)

\[ I_2 \leq C(h,T) \int_0^T \int_0^1 \rho^2 \phi^2 \left( (u^2)_{x} \right)^3 \, d\tau, \] (39)

\[ I_3 = -2\nu \int_0^T \int_0^1 \phi \phi_x \rho \left( (u^2)_{x} \right) \left( (u^2)_{x} \right) \, dx \, d\tau \]

\[ = -2\nu \int_0^T \int_0^1 \phi \phi_x u_x r^2 \left( (u^2)_{x} \right) \, dx \, d\tau - 4\nu \int_0^T \int_0^1 \phi \phi_x u_x^2 \left( (u^2)_{x} \right) \, dx \, d\tau \]

\[ \leq \delta \int_0^T \int_0^1 \phi^2 u_x \, d\tau + C(h,T) \int_0^T \int_0^1 \phi^2 \rho^2 \left( (u^2)_{x} \right)^2 \, d\tau \]

\[ + C(h,T) \int_0^T \int_0^1 \phi^2 \rho^2 u_x r^2 \, d\tau \]

\[ \leq \delta \int_0^T \int_0^1 \phi^2 u_x \, d\tau + C(h,T) \int_0^T \int_0^1 u_x \, d\tau + C(\delta, h, T), \] (40)

where we have used the fact

\[ \int_0^T \| u \|_{L^2} \, d\tau \leq \int_0^T \| u \|_{L^2} \| u_x \|_{L^2} \, d\tau \leq \int_0^T \| u_x \|_{L^2} \, d\tau \]

\[ \leq C(h,T) \int_0^T \int_0^1 \rho \left( (u^2)_{x} \right)^2 \, dr \leq C(h,T). \]

\[ I_4 = \nu \int_0^T \int_0^1 \left( \phi^2 u_x r^2 \right) \rho \left( (u^2)_{x} \right) \, dx \, d\tau \]

\[ = 2\nu \int_0^T \int_0^1 \phi \phi_x u_x r^2 \rho \left( (u^2)_{x} \right) \, dx \, d\tau + 2\nu \int_0^T \int_0^1 \phi^2 u_x r^2 \rho \left( (u^2)_{x} \right) \, dx \, d\tau \]

\[ + \nu \int_0^T \int_0^1 \phi^2 u_x^2 \frac{1}{r^2} \left( (u^2)_{x} \right) \, dx \, d\tau \]
For $\int_0^T \int_h^1 \phi^2 \rho[(r^2 u)_x]^2 dxd\tau$ term on the right here we apply the following Sobolev estimate:

\[
\int_0^T \int_h^1 \phi^2 \rho[(r^2 u)_x]^2 dxd\tau \\
\leq \int_0^T \| u \|_{2,\infty}^2 \int_h^1 \phi^2 \rho[(r^2 u)_x]^2 dxd\tau \\
\leq C(h, T) \int_0^T \int_h^1 \phi^2 \rho[(r^2 u)_x]^2 dxd\tau + C(h, T) \int_0^T \| u \|_{2,\infty}^2 \int_h^1 \phi^2 \rho[(r^2 u)_x]^2 dxd\tau + C.
\]
\[
\leq C(h, T) \sup_{0 \leq t \leq T} \left( \int_h^1 \phi^2 \rho (r^2 u)_x^2 \, dx \right)^{\frac{1}{2}} \times \left\{ 1 + \left( \int_0^T \int_h^1 \phi^2 [(\rho(r^2 u)_x)_x]^2 \, dx \, d\tau \right)^{\frac{1}{4}} \right\}. \tag{43}
\]

For \(\int_0^T \int_h^1 \phi^2 [(\rho(r^2 u)_x)_x]^2 \, dx \, d\tau\) term on the right here, by the equation (20), we have
\[
u u_\tau + r^2 P(\rho)_x = \nu r^2 (\rho(r^2 u)_x)_x.
\]
Which together with Corollary 4 gives
\[
\int_0^T \int_h^1 \phi^2 [(\rho(r^2 u)_x)_x]^2 \, dx \, d\tau \leq C(h, T) \int_0^T \int_h^1 \phi^2 u_x^2 \, dx \, d\tau + C(h, T), \tag{44}
\]
Substituting (44) into (43), we have
\[
\int_0^T \int_h^1 \phi^2 (r^2 u)_x^3 \, dx \, d\tau \leq C(h, T)B^{\frac{1}{4}} \times \left\{ 1 + B^{\frac{1}{4}} \right\},
\]
which together with (42) yield
\[
B(T) \leq C(h, T)B^{\frac{1}{4}} \left[ 1 + B(T)^{\frac{1}{4}} \right] + \int_0^T \|u\|^2_{L^\infty} B(\tau) \, d\tau \leq C(h, T) \left[ 1 + B(T)^{\frac{1}{4}} \right] + \int_0^T \|u\|^2_{L^\infty} B(\tau) \, d\tau.
\]
By Gronwall’s inequality, we get
\[
B(T) \leq C(h, T).
\]
Which together with the equation (20) and Lemma 4.1 give
\[
\int_0^T \int_{2h}^1 \left| (\nu \rho (ur^2)_x - P(\rho))_x \right|^2 \, dx \, d\tau = \int_0^T \int_{2h}^1 \nu^{-2} u_x^2 \, dx \, d\tau \leq C(h, T),
\]
and
\[
\int_{2h}^1 \left| (\nu (ur^2)_x - P(\rho)) \right|^2 \, dx \, d\tau \leq \nu \int_{2h}^1 (\rho (ur^2)_x)^2 \, dx \, d\tau + \int_{2h}^1 \rho \, dx \, d\tau \leq C(h, T).
\]
This completes the proof of the Lemma 4.3. \qed

5. **Proof of Theorem 2.2.** By virtue of the a priori estimates derived in Sections 4, we are now able to prove our main theorem by taking appropriate limits in a manner analogous to that in [12].

To begin with, we let \((\rho_0, u_0)\) be initial data satisfying the hypotheses of Theorem 2.2. Let \(\psi(x) \in C_0^\infty(\mathbb{R})\) satisfy \(\psi(x) = 1\) when \(|x| \leq \frac{a_0}{2}\) and \(\psi(x) = 0\) when \(|x| \geq a_0\),
and define $\psi_\delta(x) \doteq \psi(x)$. we denote by $H_\delta$ a standard mollifier (in $r$) of width $\delta$. for simplicity we still denote by $(\rho^\epsilon, u^\epsilon_0)$ the extension of $(\rho^\epsilon_0, u^\epsilon_0)$ in $\mathbb{R}$, i.e. 

$$
\rho^\epsilon_0(r) \doteq \begin{cases} 
\rho_0(\epsilon) + \epsilon, & r \in [0, \epsilon], \\
\rho_0(r) + \epsilon, & r \in [\epsilon, a_0], \\
\rho_0(a_0) + \epsilon, & r \in [a_0, \infty),
\end{cases}
$$

$$
u^\epsilon_0(r) \doteq \begin{cases} 
u_0(r), & r \in (2\delta + \epsilon, a_0), \\
0 & \text{otherwise}.
\end{cases}
$$

For $\epsilon < \min\{a_0(1-\delta), \delta\}$, we define the smooth approximate initial data $(\rho_{\epsilon, \delta}^0, u_{\epsilon, \delta}^0)$ as follows:

$$
\rho_{\epsilon, \delta}^0(r) \doteq (\rho_0^\epsilon * H_\delta)(r), \\
u_{\epsilon, \delta}^0(r) \doteq (u_0^\epsilon * H_\delta)(r) \{1 - \psi_\delta(a_0 - r)\} + (u_0^\epsilon * H_\delta)(a_0) \psi_\delta(a_0 - r) \\
+ \left\{ \frac{1}{\nu}(\rho_0^\epsilon(a_0))^{- \gamma} - \frac{2u_0^\epsilon(a_0)}{a_0} \right\} \int_\epsilon^r \psi_\delta(a_0 - x) dx.
$$

The resulting data $(\rho_{\epsilon, \delta}^0, u_{\epsilon, \delta}^0)$ then satisfy the hypotheses of Theorem 2.2 with the constants independent of $\epsilon$ and $\delta$. Thus, there is a global-in-time smooth solution $(\rho_{\epsilon, \delta}, u_{\epsilon, \delta})$ of the system (2)-(5)). This is a result of Okada’s work [22] in the annular domain $[\epsilon, a(t)]$. Next, we want to pass to the limit to get a global weak solution. we define the particle path $r_{\epsilon, \delta}^h(t)$ associated with the approximate solution $(\rho_{\epsilon, \delta}, u_{\epsilon, \delta})$ by 

$$
h = \int_\epsilon^{r_{\epsilon, \delta}^h(t)} \rho_{\epsilon, \delta}(t, r)r^2 dr,
$$

$$
\frac{da_{\epsilon, \delta}(t)}{dt} = u_{\epsilon, \delta}(a_{\epsilon, \delta}(t), t).
$$

5.1. Convergence of the approximate solution $(\epsilon, \delta) \to (0, 0)$. By the a priori estimates established in Section 4, we have the following four Propositions, which imply that there is a subsequence $(\epsilon, \delta) \to (0, 0)$, such that the approximate solutions and their associated particle paths are convergent.

Proposition 2. Let $(\rho_{\epsilon, \delta}, u_{\epsilon, \delta}, a_{\epsilon, \delta})$ and $r_{\epsilon, \delta}^h(t)$ be as described above.

Then there is a subsequence $(\epsilon, \delta) \to (0, 0)$ and function $a(t)$ such that $a_{\epsilon, \delta}(t) \to a(t)$, uniformly on $C^{1/2}([0, T])$ and $c \leq a(t) \leq C_T$ a(t) $\in H^1(0, T)$.

Proof. We have from the Definition (45) of $a_{\epsilon, \delta}(t)$

$$
\frac{da_{\epsilon, \delta}(t)}{dt} = u_{\epsilon, \delta}(a_{\epsilon, \delta}(t), t),
$$

thus for $0 \leq t_1 < t_2 \leq T$, we have

$$
|a_{\epsilon, \delta}(t_2) - a_{\epsilon, \delta}(t_1)| = \left| \int_{t_1}^{t_2} u_{\epsilon, \delta}(a_{\epsilon, \delta}(t), t) dt \right|
\leq \left( \int_{t_1}^{t_2} (u_{\epsilon, \delta})^2(a_{\epsilon, \delta}(t), t) dt \right)^{1/2} (t_2 - t_1)^{1/2}
\leq C(t_2 - t_1)^{1/2},
$$

and

$$
|a_{\epsilon, \delta}(t)| \leq (1 + t)^{1/2} \leq C(T).
$$

Using the Ascoli-Arzela Lemma we get

$$a_{\epsilon, \delta}(t) \to a(t), \text{ uniformly on } C^{1/2}([0, T]).$$
which together with Lemma 3.2, gives
\[ c \leq a(t) \leq C_T, a(t) \in H^1(0, T). \]
This completes the proof of the Proposition 2.

**Proposition 3.**
1. There is a subsequence \( \epsilon \to 0 \) such that \( r^{\epsilon, \delta}_h(t) \) converges uniformly for \((h, t)\) in compact subsets of \((0, M_0] \times [0, \infty)\) and the limit \( r_h(t) \) is Hölder continuous in \((h, t)\) on these compact sets.
2. \( r(t) = \lim_{h \to 0} r_h(t), \)
then
\[ \lim_{t \to 0} r(t) = 0. \]
3. If the “fluid region” defined by
\[ F : = \{(t, r)| t \geq 0, \text{and} \ r(t) < r(t) < a(t)\}, \]
then \( F \cap \{0 < t\} \cap \{r(t) < a(t)\} \) is an open set.

**Proposition 4.** There is a subsequence still denoted by \( u^{\epsilon, \delta}(r, t) \) and limit function \( u(r, t) \) such that
\[ u^{\epsilon, \delta}(r, t) \to u(r, t) \]
uniformly on compact subsets of \( F \cap \{0 < t\} \). Limit function \( u(r, t) \) is Hölder continuous on these compact subsets.
Moreover, for any \( \phi \in D'(\Omega_t \times [t_1, t_2]) \), then
\[ \int_0^t \int_0^{a(s)} u^{\epsilon, \delta}_r \phi r^2 dr ds \to \int_0^t \int_0^{a(s)} u_r \phi r^2 dr ds, \]
\[ \int_0^t \int_0^{a(s)} u^{\epsilon, \delta} \phi dr ds \to \int_0^t \int_0^{a(s)} u \phi dr ds. \]

By virtue of the a priori estimates established in Sections 3, one can show Proposition 3 and Proposition 4 in the same manner as Proposition 1 and Proposition 2 in the proof of [12].

**Proposition 5.** Then there is a subsequence \( (\epsilon, \delta) \to (0, 0) \) and function \( \rho(r, t) \) such that
\[ \rho^{\epsilon, \delta} \to \rho \quad \text{uniformly on compact subsets of} \ F \cap \{0 < t\} \quad \text{and} \quad \rho(a(t), t) > 0. \] (46)
In addition, if \( \rho^{\epsilon, \delta} \) is taken to be zero for \( r \leq \epsilon \), then
\[ \rho^{\epsilon, \delta}(\cdot, t) \to 0 \quad \text{in} \quad L^1([0, r(t)]). \] (47)
Also, for \( h > 0 \) and \( T > 0 \), there is a constant \( C(h, T) \), such that for \( 0 \leq t \leq T \) and \( r_h(t) \leq t \leq a(t) \)
\[ C(h, T)^{-1} \leq \rho \leq C(h, T). \]
Finally, for \( h > 0 \) and \( t > 0 \)
\[ h = \int_{z(t)}^{r_h(t)} \rho(t, r)r^2 dr. \] (48)
Proof. For $0 < \tau \leq t \leq T$ and $r_{h_0}^{\epsilon,\delta}(t) \leq r_1 \leq r_2 \leq a(t)$,

$$|\rho^{\epsilon,\delta}(r_2, t) - \rho^{\epsilon,\delta}(r_1, t)| \leq \int_{r_1}^{r_2} |\rho^{\epsilon,\delta}_r(r, t)| dr \leq \sqrt{r_2 - r_1} \int_{r_1}^{r_2} (|\rho^{\epsilon,\delta}_r(r, t)|^2 r^2 dr)^{1/2} \leq C(h_0, \tau, T) \sqrt{r_2 - r_1}.$$

To prove Hölder continuity in time, we let $0 < \tau \leq t_1 \leq t_2 \leq T$ and $r_{h_0}^{\epsilon,\delta}(t) \leq r_1$, for $t \in \left[ t_1, t_2 \right]$ then take for $k = \sqrt{t_2 - t_1}$

$$|\rho^{\epsilon,\delta}(r_1, t_2) - \rho^{\epsilon,\delta}(r_1, t_1)| \leq \frac{1}{k} \int_{r_1}^{r_1+k} |\rho^{\epsilon,\delta}(r, t_2) - \rho^{\epsilon,\delta}(r, t_1)| dr \leq \frac{1}{k} \int_{r_1}^{r_1+k} \left( |\rho^{\epsilon,\delta}(r, t_2) - \rho^{\epsilon,\delta}(r, t_1)| + |\rho^{\epsilon,\delta}(r_2, t_2) - \rho^{\epsilon,\delta}(r_2, t_1)| \right) dr$$

$$+ \frac{1}{k} \int_{r_1}^{r_1+k} |\rho^{\epsilon,\delta}(r_1, t_2) - \rho^{\epsilon,\delta}(r_1, t_1)| dr \leq \frac{1}{k} \int_{r_1}^{r_1+k} \int_{t_1}^{t_2} |\partial_t \rho^{\epsilon,\delta}(r, t)| dt dr + \frac{1}{k} \int_{r_1}^{r_1+k} \int_{r_1}^{r_1} |\partial_z \rho^{\epsilon,\delta}(z, t)| dz dr$$

$$+ \frac{1}{k} \int_{r_1}^{r_1+k} \int_{t_1}^{t_2} |\partial_t \rho^{\epsilon,\delta}(z, t_1)| dz dr \leq \frac{1}{k} \int_{r_1}^{r_1+k} \int_{t_1}^{t_2} |\partial_t \rho^{\epsilon,\delta}| dt dr + C(h_0, \tau, T) \sqrt{k} \leq \frac{1}{k} \int_{r_1}^{r_1+k} \int_{t_1}^{t_2} \left| \rho^{\epsilon,\delta}_r u^{\epsilon,\delta} + \rho^{\epsilon,\delta} u_r^{\epsilon,\delta} + \frac{2\rho^{\epsilon,\delta} u^{\epsilon,\delta}}{r} \right| dt dr + C(h_0, \tau, T) \sqrt{k} \leq C(h_0, \tau, T) k + C(h_0, \tau, T) \sqrt{k} \leq C(h_0, \tau, T) \sqrt{k} = C(h_0, \tau, T)(t_2 - t_1)^{\frac{1}{2}},$$

which together with $\rho^{\epsilon,\delta} \leq C(h_0, T)$, Ascoli-Arzela Lemma and Lemma 3.3 give the proof of (46). The proof of (47) and (48) are same to Proposition 3 in [12]. \qed

5.2. Weak forms of the Navier-Stokes equations. We now turn to the proof that the limiting functions are indeed a weak solution of the Navier-Stokes equations in $\Omega \times [0, T)$. First, the limiting functions $\rho$ and $u$ have been defined in the fluid region $\mathcal{F}$ but not elsewhere. We therefore define $\rho$, $\rho u$ to be identically zero in the vacuum region $\mathcal{F}^c$.

Next, in order to show the limits of these approximate solutions are indeed weak solutions of the original system, we prove the crucial uniform integrability of the approximate densities and energies. Where we define $|E| = \int_{\mathcal{F}} r^2 dr$.

Lemma 5.1. (1) If $a \geq 0$ and $\rho : [a, a(t)] \to R$ is positive, Then for any measurable set $E = E(t) \subset [a, a(t)]$,

$$\int_E \rho u^2 dr \leq E_0^{\frac{1}{2}} |E|^{\frac{(\gamma - 1)}{2}}.$$
(2) If \( b > 0 \), then there is a constant \( C(b, T) \) such that, if \( E = E(t) \subset [b, a(t)] \), then
\[
\int_0^T \int_E \rho u^2 r^2 \, dr \, dt \leq C(b, T) \left( \int_0^T |E|^{\frac{2(\gamma - 1)}{\gamma}} \, dt \right)^{1/2},
\]
and
\[
\int_0^T \int_E \rho^\gamma r^2 \, dr \, dt \leq C(b, T) \int_0^T |E|^{\frac{\gamma - 1}{\gamma}} \, dt.
\]

Proof. (1) Using the (14), one has
\[
\int_E \rho u^2 r^2 \, dr \leq \left( \int_E \rho^\gamma r^2 \, dr \right)^{1/2} \left( \int_E \rho^{\gamma - 1} r^2 \, dr \right)^{1/2} \leq \int_0^T |E|^{\frac{\gamma - 1}{\gamma}} \, dt.
\]
(2) Thanks to the above inequality, yields
\[
\int_0^T \int_E \rho u^2 r^2 \, dr \, dt \leq \left( \int_0^T \rho \right)^{1/2} \left( \int_0^T \rho^2 r^2 \, dr \right)^{1/2} \left( \int_0^T \rho^{\gamma - 1} r^2 \, dr \right)^{1/2} \int_0^T \left( \int_0^T \rho^2 r^2 \, dr \right)^{1/2} \, dt \leq C(b, T) \left( \int_0^T |E|^{\frac{\gamma - 1}{\gamma}} \, dt \right)^{1/2}.
\]
Where we have used as follows fact. From (15), we obtain
\[
\int_0^T |u(r, t)|^2 \, dt = \int_0^T |u(a(t), t) + \int_{a(t)}^r u_y(y, t) \, dy|^2 \, dt \leq \int_0^T |u(a(t), t)|^2 \, dt + \int_0^T \int_{a(t)}^r |u_y(y, t)|^2 \, dy \, dt \leq \int_0^T |u(a(t), t)|^2 \, dt + \int_0^T a(t) \int_b^a (u_y)^2 \, dy \, dt \leq C(b, T) \left( \int_0^T |u(a(t), t)|^2 \, dt + \int_0^T \int_b^a (u_y)^2 \, dy \, dt \right) \leq C(b, T).
\]
(3) By Lemma 3.5, we get
\[
\int_0^T \int_E \rho^\gamma r^2 \, dr \, dt \leq \|\rho\|_{L^\infty([0, T] \times E)}^{-1} \int_0^T \int_E \rho r^2 \, dr \, dt \leq C(b, T) \int_0^T |E|^{\frac{\gamma - 1}{\gamma}} \, dt.
\]
This completes the proof of the Lemma 5.1.

In the following proposition, we show that \((\rho, u)\) satisfies the weak form (2) of the mass equation.
Lemma 5.1, Proof. We first derive the weak form of the mass equation. Thus let $u$ be the difference between each of the above terms and the corresponding terms with $\rho^{c,\delta}$, $u^{c,\delta}$ replaced by the limits $\rho$, $u$. First, at time $t = t_1$ or $t_2$ by Lemma 5.1,
\[
\int_0^{a(t)} \rho^{c,\delta} \phi_r^2 dr|_{t_1}^{t_2} - \int_0^{a(t)} \rho \phi_r^2 dr = 0.
\] (49)
We consider the difference between each of the above terms and the corresponding terms with $\rho^{c,\delta}$, $u^{c,\delta}$ replaced by the limits $\rho$, $u$. First, at time $t = t_1$ or $t_2$ by Lemma 5.1,
\[
\int_0^{a(t)} \rho^{c,\delta} \phi_r^2 dr - \int_0^{a(t)} \rho \phi_r^2 dr \leq \int_0^{a(t)} (\rho^{c,\delta} - \rho) \phi_r^2 dr + \|\phi\|_\infty \left( \int_0^{r_h^{c,\delta}} \rho^{c,\delta} \phi_r^2 dr + \int_0^{r_h} \rho^{c,\delta} \phi_r^2 dr + \int_0^{\bar{r}} \rho \phi_r^2 dr \right)
\] (50)
letting $(c, \delta) \to 0$, then $\bar{r} \to 0$ and using the definitions of the curves $r_h^{c,\delta}$, $r_h$, $\bar{r}$ and Proposition 5, we have
\[
\int_0^{a(t)} \rho^{c,\delta} \phi_r^2 dr|_{t_1}^{t_2} \to \int_0^{a(t)} \rho \phi_r^2 dr|_{t_1}^{t_2}.
\]
The same argument applies to the second term in the equation (49), and for the last term in (49) we have
\[
\int_0^{t_2} \int_0^{a(t)} \rho^{c,\delta} u^{c,\delta} \phi_r^2 \phi_r^2 dr dt - \int_0^{t_1} \int_0^{a(t)} \rho u \phi_r^2 dr dt \leq \int_0^{t_2} \int_0^{a(t)} (\rho^{c,\delta} u^{c,\delta} - \rho u) \phi_r^2 dr dt + \|\phi_r\|_\infty \left( \int_0^{t_1} \int_0^{a(t)} |\rho^{c,\delta} u^{c,\delta}| \phi_r^2 dr dt + \int_0^{t_2} \int_0^{a(t)} |\rho u| \phi_r^2 dr dt \right)
\] (50)
For the first term on the right-hand side of (50), we have it goes to zero by the uniform convergence of $u^{c,\delta}$ to $u$, and $\rho^{c,\delta}$ to $\rho$. And the argument of the other terms are also same to [12]. We have thus proved that, for function $\phi$ which are $C_0^1$ on $\Omega_t \times [t_1, t_2]$,
\[
\int_0^{a(t)} \rho \phi_r^2 dr|_{t_1}^{t_2} = \int_0^{t_2} \int_0^{a(t)} (\rho \phi_r + \rho u \phi_r) \phi_r^2 dr dt.
\]
Then, the proof of (b) and (c) which omitted here are same to [12]. □

Next, We now turn to the formulation and proof of the weak form of the momentum equation.
Proposition 7. Let $\rho$, $u$ be the functions given in Propositions 6. Let $t_1 < t_2$ and $\psi$ be a $C^1$ function on $[t_1, t_2] \times \Omega$, such that $\psi(t, 0) = 0$ for $t \in [t_1, t_2]$. Then, the following identity holds.

\[
\int_0^{a(t)} \rho \psi r^2 dr \bigg|_{t_1}^{t_2} - \int_{t_1}^{t_2} \int_0^{a(t)} \left( \rho u \psi_t + \rho u^2 \psi_r + P \left( \psi_r + \frac{2 \psi}{r} \right) \right) r^2 dr dt = -\nu \int_{t_1}^{t_2} \int_0^{a(t)} \left( u_r^\epsilon + \frac{2 u}{r} \right) \left( \psi_r + \frac{2 \psi}{r} \right) r^2 dr dt.
\]

\[ (51) \]

Proof. We first consider a simpler case in which the test function vanishes in a neighborhood of the origin. Assume that $\psi$ is a $C^1$ function on $[t_1, t_2] \times \Omega$, satisfying $\psi = 0$ on $[0, b]$ for some $b > \epsilon$. The weak form of the momentum equation \ref{momentum} holds for the approximate solutions:

\[
\int_0^{a(t)} \rho^{\epsilon, \delta} u^{\epsilon, \delta} \psi r^2 dr \bigg|_{t_1}^{t_2} - \int_{t_1}^{t_2} \int_0^{a(t)} \left( \rho^{\epsilon, \delta} u^{\epsilon, \delta} \psi_t + \rho^{\epsilon, \delta} (u^{\epsilon, \delta})^2 \psi_r \right) r^2 dr dt \\
= -\nu \int_{t_1}^{t_2} \int_0^{a(t)} \left( u_r^{\epsilon, \delta} + \frac{2 u^{\epsilon, \delta}}{r} \right) \left( \psi_r + \frac{2 \psi}{r} \right) r^2 dr dt.
\]

\[ (52) \]

We proceed by showing that each term in \ref{momentum} converges to the corresponding term in \ref{momentum}. The convergence of the terms $\rho^{\epsilon, \delta} u^{\epsilon, \delta} \psi$ and $\rho^{\epsilon, \delta} u^{\epsilon, \delta} \psi_t$ is established just as for the term $\rho^{\epsilon, \delta} (u^{\epsilon, \delta})^2 \psi_r$ in the proof of Proposition 6. For the $\rho^{\epsilon, \delta} (u^{\epsilon, \delta})^2 \psi_r$ term we have

\[
\left| \int_{t_1}^{t_2} \int_0^{a(t)} \rho^{\epsilon, \delta} (u^{\epsilon, \delta})^2 \psi_r r^2 dr dt - \int_{t_1}^{t_2} \int_0^{a(t)} \rho u^2 \psi_r r^2 dr dt \right| \leq \|\psi_r\|_{L^\infty} \left( \int_{t_1}^{t_2} \int_b \rho^{\epsilon, \delta} (u^{\epsilon, \delta})^2 r^2 dr dt + \int_{t_1}^{t_2} \int_{E_h^{\epsilon, \delta}} \rho^{\epsilon, \delta} (u^{\epsilon, \delta})^2 r^2 dr dt \right)
\]

\[ + \|\psi_r\|_{L^\infty} \left( \int_{t_1}^{t_2} \int_{E_h} \rho^{\epsilon, \delta} u^2 r^2 dr dt \right) \]

\[ (53) \]

Where $E_h^{\epsilon, \delta} = [r_h^{\epsilon, \delta}, r_h] \cap [b, a(t)]$ (or $E_h^{\epsilon, \delta} = [r_h, r_h^{\epsilon, \delta}] \cap [b, a(t)]$ and $E_h = [r, r_h] \cap [b, a(t)]$ (or $E_h = [r_h, r] \cap [b, a(t)]$). The last term in \ref{momentum} tends to zero because Proposition 4-5. For the first term in \ref{momentum}:

\[
\int_{t_1}^{t_2} \int_b \rho^{\epsilon, \delta} (u^{\epsilon, \delta})^2 r^2 dr dt \leq \int_{t_1}^{t_2} \|u^{\epsilon, \delta}\|_{L^\infty([b, a(t)])} \int_b \rho^{\epsilon, \delta} u^{\epsilon, \delta} r^2 dr dt \\
\leq \int_{t_1}^{t_2} \|u^{\epsilon, \delta}\|_{L^\infty([b, a(t)])} \left( \int_b \rho^{\epsilon, \delta} r^2 dr \right)^{\frac{1}{2}} \left( \int_b \rho^{\epsilon, \delta} (u^{\epsilon, \delta})^2 r^2 dr \right)^{\frac{1}{2}} dt \\
\leq C \left( \int_{t_1}^{t_2} \|u^{\epsilon, \delta}\|_{L^\infty([b, a(t)])} dt \right)^{\frac{1}{2}} \left( \int_{t_1}^{t_2} \int_b \rho^{\epsilon, \delta} u^2 r^2 dr dt \right)^{\frac{1}{2}} \\
\leq C(b, T) (h(t_2 - t_1))^{\frac{1}{2}} \to 0, \quad as \quad h \to 0.
\]
For the second term in (53), by Lemma 5.1, we have
\[
\int_{t_1}^{t_2} \int_{E_h} \rho^\varepsilon,\delta \frac{(u^\varepsilon,\delta)^2}{2} r^2 \, dr \, dt \leq C(b,T) \left( \int_{t_1}^{t_2} |E_h^\varepsilon,\delta|^{\frac{2(\gamma-1)}{\gamma}} \, dt \right)^{1/2},
\]
which goes to zero as \((\varepsilon, \delta) \to (0,0)\) because \(r_h^\varepsilon,\delta \to r_h\) as \((\varepsilon, \delta) \to (0,0)\) with \(h\) fixed. Thanks to \(\sqrt{\rho^\varepsilon,\delta} u^{\varepsilon,\delta} \to \sqrt{\rho} u\) in \(L^2([0,a(t)] \times [t_1,t_2])\)
\[
\int_{t_1}^{t_2} \int_{E_h} \rho u^2 r^2 \, dr \, dt \leq \liminf_{(\varepsilon,\delta) \to (0,0)} \int_{t_1}^{t_2} \int_{E_h} \rho^\varepsilon,\delta (u^\varepsilon,\delta)^2 r^2 \, dr \, dt
\]
\[
\leq C(b,T) \liminf_{(\varepsilon,\delta) \to (0,0)} \left( \int_{t_1}^{t_2} |E_h^\varepsilon,\delta|^{\frac{2(\gamma-1)}{\gamma}} \, dt \right)^{1/2}
\]
\[
\leq C(b,T) \left( \int_{t_1}^{t_2} |E_h|^{\frac{2(\gamma-1)}{\gamma}} \, dt \right)^{1/2} \to 0, \quad \text{as} \quad h \to 0.
\]
This shows \(\rho^\varepsilon,\delta (u^\varepsilon,\delta)^2 \psi \) term in (52) converges to the corresponding term in (51). The pressure term in (52), Set \(\psi \equiv \psi_r + 2 \frac{\psi_r}{\rho} \) we have
\[
\left| \int_{t_1}^{t_2} \int_0^{a(t)} (\rho^{\varepsilon,\delta})^\gamma \psi_r r^2 \, dr \, dt - \int_{t_1}^{t_2} \int_0^{a(t)} \rho^{\varepsilon,\delta} \psi_r r^2 \, dr \, dt \right|
\]
\[
\leq \|\psi_r\|_{L^\infty} \left( \int_{t_1}^{t_2} \int_b^{a(t)} (\rho^{\varepsilon,\delta})^\gamma r^2 \, dr \, dt + \int_{t_1}^{t_2} \int_{E_h^{\varepsilon,\delta}} (\rho^{\varepsilon,\delta})^\gamma r^2 \, dr \, dt \right)
\]
\[
+ \|\psi_r\|_{L^\infty} \left( \int_{t_1}^{t_2} \int_{E_h} \rho^{\varepsilon,\delta} r^2 \, dr \, dt + \int_{t_1}^{t_2} \int_{r_h}^{a(t)} (\rho^{\varepsilon,\delta})^\gamma - \rho^\gamma \psi_r r^2 \, dr \, dt \right).
\]
Where \(E_h^{\varepsilon,\delta} = [r_h^{\varepsilon,\delta}, r_h] \cap [b, a(t)]\) (or \(E_h^{\varepsilon,\delta} = [r_h, r_h^{\varepsilon,\delta}] \cap [b, a(t)]\) and \(E_h = [r, r_h] \cap [b, a(t)]\) (or \(E_h = [r, r_h \cap [b, a(t)]\)). The last term in (54) tends to zero because Proposition 5. For the first term in (54):
\[
\int_{t_1}^{t_2} \int_b^{a(t)} (\rho^{\varepsilon,\delta})^\gamma r^2 \, dr \, dt \leq \int_{t_1}^{t_2} \| (\rho^{\varepsilon,\delta})^\gamma - 1 \|_{L^\infty([b, a(t)])} \int_b^{a(t)} \rho^{\varepsilon,\delta} r^2 \, dr \, dt
\]
\[
\leq C(b,T) h^\gamma (t_2 - t_1) \to 0, \quad \text{as} \quad h \to 0.
\]
For the second term in (54), by Lemma 5.1, we have
\[
\int_{t_1}^{t_2} \int_{E_h^{\varepsilon,\delta}} (\rho^{\varepsilon,\delta})^\gamma r^2 \, dr \, dt \leq C(b,T) \int_{t_1}^{t_2} |E_h^{\varepsilon,\delta}|^{\frac{\gamma-1}{\gamma}} \, dt,
\]
which goes to zero as \((\varepsilon, \delta) \to (0,0)\) because \(r_h^{\varepsilon,\delta} \to r_h\) as \((\varepsilon, \delta) \to (0,0)\) with \(h\) fixed. Thanks to \(\rho^{\varepsilon,\delta} \to \rho\) in \(L^\gamma([0,a(t)] \times [t_1,t_2])\)
\[
\int_{t_1}^{t_2} \int_{E_h} \rho^{\gamma} r^2 \, dr \, dt \leq \liminf_{(\varepsilon,\delta) \to (0,0)} \int_{t_1}^{t_2} \int_{E_h} (\rho^{\varepsilon,\delta})^\gamma r^2 \, dr \, dt
\]
\[
\leq C(b,T) \liminf_{(\varepsilon,\delta) \to (0,0)} \int_{t_1}^{t_2} |E_h|^{\frac{\gamma-1}{\gamma}} \, dt
\]
\[
\leq C(b,T) \int_{t_1}^{t_2} |E_h|^{\frac{\gamma-1}{\gamma}} \, dt.
\]
which goes to zero as \( h \to 0 \) because \( r_h \to r \) as \( h \to 0 \). This shows pressure term converges to the corresponding term in (51).

The last term in (52) converges to the corresponding term in (51) by Proposition 4.

To extend the identity (52) to the case that test functions are supported in \([0, a(t)]\), we fix an increasing \( C^1 \) function \( \chi : [0, \infty) \to [0, 1] \) with \( \chi \equiv 0 \) on \([0, 1]\) and \( \chi \equiv 1 \) on \([2, \infty)\), and define \( \chi^b(r) := \chi(r/b) \) for \( b > 0 \). Let \( \psi \) be a \( C^1 \) function on \([t_1, t_2] \times [0, a(t)]\) such that \( \psi(t, 0) = 0 \) for \( t \in [t_1, t_2] \), and define \( \psi^b := \chi^b \psi \). Then, the (52) applies to the test functions \( \psi^b = \chi^b \psi \). We obtain

\[
\int_0^{a(t)} \rho u \chi^b \psi r^2 dr dt - \int_{t_1}^{t_2} \int_0^{a(t)} \left( \rho u \chi^b \psi_t + \rho u^2 (\chi^b \psi)_r + P(\chi^b \psi)_r + \frac{2P \chi^b \psi}{r} \right) r^2 dr dt = -\nu \int_{t_1}^{t_2} \int_{\Omega} (u_r (\chi^b \psi)_r + \frac{2u \chi^b \psi}{r^2}) r^2 dr dt,
\]

(55)

The first, second and fifth terms converge to the corresponding terms in (51) as \( b \to 0 \) by the Dominated Convergence Theorem. For the third term we have

\[
\int_{t_1}^{t_2} \int_0^{a(t)} \rho u^2 (\chi^b \psi + \chi^b \psi_r) dr dt = \int_{t_1}^{t_2} \int_0^{a(t)} \rho u^2 (\chi^b \psi_r + \chi^b \psi) dr dt,
\]

(56)

and the second term on the right here clearly tends to the third term in (51) as \( b \to 0 \). Since \( \psi(t, 0) = 0 \), we can write \( \psi = r \phi \) for a smooth \( \phi \). Then since \( \chi^b \leq \frac{C}{b} \), we can bound the first term on the right-hand side of (56) by

\[
\int_{t_1}^{t_2} \int_b^{2b} \rho u^2 C_b \frac{r}{b} |\phi| r^2 dr dt \leq C \int_{t_1}^{t_2} \int_b^{2b} \rho u^2 r^2 dr dt,
\]

(57)

which tends to zero as \( b \to 0 \) by the bound on the limiting energy. The same argument applies to the the fourth terms in (55). For the last term

\[
\nu \int_{t_1}^{t_2} \int_0^{a(t)} \left( u_r + \frac{2u}{r} \right) \left( (\chi^b \phi)_r + \frac{2\chi^b \phi}{r} \right) r^2 dr dt = \nu \int_{t_1}^{t_2} \int_0^{a(t)} \left( u_r + \frac{2u}{r} \right) \left( \chi_r \phi + \chi \phi_r + \frac{2\chi \phi}{r} \right) r^2 dr dt
\]

(58)

Similarly to (57), we deduce that

\[
\left| \int_{t_1}^{t_2} \int_0^{a(t)} \left( u_r + \frac{2u}{r} \right) \chi^b r^2 dr dt \right| \leq \int_{t_1}^{t_2} \int_b^{2b} \left( u_r + \frac{2u}{r} \right) \frac{C_r}{b} |\phi| r^2 dr dt \leq \left( \int_{t_1}^{t_2} \int_b^{2b} \left( u_r + \frac{2u}{r} \right)^2 r^2 dr dt \right)^{\frac{1}{2}} \left( \int_{t_1}^{t_2} \int_b^{2b} \left( \frac{C_r}{b} |\phi| \right)^2 r^2 dr dt \right)^{\frac{1}{2}}
\]

which tends to zero as \( b \to 0 \). The other terms on the right-hand side of (58) converge to the corresponding terms in (51) as \( b \to 0 \) by the Dominated Convergence Theorem. This completes the proof of the proposition. \(\square\)
5.3. **Proof of theorem 2.2 and 2.3.** The proof of theorem 2.2: The (a) of theorem 2.2 is Proposition 2, Lemma 4.3, Proposition 4 and Proposition 5. The (b) of theorem 2.2 follows from Proposition 3. The weak forms of the mass (c) and momentum equations (d) are proved in Propositions 6 and 7. From Proposition 6, the regularity assertions in (c) of Theorem 2.2 follow immediately.

The proof of theorem 2.3: The upper bound on the expanding rate of the free boundary (8) follows similarly to (16). The point-wise decay of density (9) can be shown as for Lemma 3.3. We use the similar arguments as Lemma 6.1 in Refs. [7] to prove expanding rate of the domain occupied by the fluid (10), we omit the details.

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