Hydrodynamics of non-integrable systems from relaxation-time approximation

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We develop a general kinetic theory framework to describe the hydrodynamics of strongly interacting, nonequilibrium quantum systems in which integrability is weakly broken, leaving a few residual conserved quantities. This framework is based on a generalized relaxation-time approximation; it gives a simple, but surprisingly accurate, prescription for computing nonequilibrium transport even in strongly interacting systems. This approximation reproduces the crossover from generalized to conventional hydrodynamics in interacting one-dimensional Bose gases with integrability-breaking perturbations, both with and without momentum conservation. It also predicts the hydrodynamics of chaotic quantum spin chains, in good agreement with matrix product operator calculations.

**Introduction.**— Hydrodynamics has experienced a revival in the past decade, as an effective theory of strongly interacting quantum matter far from equilibrium [1–10]. A major factor in this revival has been the advent of new experimental platforms, from quark-gluon plasmas [11] to strongly interacting ultracold gases [12, 13] and pristine solid-state systems that feature strong interactions and long mean free times [14–17]. Hydrodynamics is particularly rich for low-dimensional fluids, featuring transport anomalies such as long-time tails [18–23]; in one dimension, hydrodynamics is further enriched by the proximity of many realistic systems to integrability. In the integrable limit, conventional hydrodynamics breaks down, and a new framework, called “generalized hydrodynamics” (GHD), has been developed [24–49]. GHD incorporates the distinctive features of integrable dynamics: namely, the presence of infinitely many conservation laws, at least the “delta functions” conserving momentum and energy are nontrivial to evaluate.

Realistic systems, however, are only approximately integrable. On short timescales they obey GHD, but on the longest timescales they cross over to conventional hydrodynamics. A general theory of this crossover has remained elusive, despite recent progress [55, 66–85]. In principle one can write a collisional Boltzmann equation for weak integrability breaking [82, 85]. However, in general the collision integral is intractable, as it depends on all the matrix elements of the integrability-breaking perturbation. In special cases, such as long-range interactions, slowly fluctuating noise, or weakly interacting systems, the integrability-breaking perturbation can itself be expressed in terms of GHD data [82, 85]. More generally, however, integrability-breaking perturbations lie outside GHD: for example, umklapp scattering involves large momentum transfer, and thus cannot be captured by a long-wavelength theory such as GHD. In the absence of the GHD framework, evaluating the collision integral is an intractable task.

In the present work, we propose and explore a simple approximation to the collision integral, which we call the “generalized relaxation time approximation” (GRTA), by analogy with the conventional relaxation time approximation (RTA) for weakly interacting electrons [86]. Although our approach resembles the RTA in spirit, its actual implementation is completely different, and its consequences are correspondingly more nontrivial. The RTA deals with nearly free particles, so their scattering kinematics is simple. By contrast, in an interacting integrable system, the momentum carried by each quasiparticle is a nonlinear functional of the full quasiparticle distribution function. Thus, when one describes a scattering process in an integrable system, not only the matrix elements but also the “delta functions” conserving momentum and energy are nontrivial to evaluate.

Instead we implement the GRTA as follows. In GHD, one regards a system as locally being in a generalized Gibbs ensemble (GGE) [87–89], with chemical potentials for each conservation law [89]. The key step in our approach is to replace the local GGE with a local thermal Gibbs state, subject to the residual conservation laws, at some finite rate \(1/\tau\) (where \(\tau\) is the generalized relaxation time). The assumption is that there is a unique local relaxation rate for the quasiparticle distribution function. This assumption is justified under certain assumptions, and (as we discuss below) fails sometimes; however, we find that it is remarkably accurate at reproducing numerical time evolution, even when the integrability-breaking perturbations are not especially small. For initial states far from equilibrium, the GRTA (unlike the RTA) gives rise to nontrivial relaxation dynamics, as the local equilibrium state is a nontrivial functional of the local quasiparticle distribution. Moreover, contrary to the simplest implementation of the RTA, GRTA preserves conservation laws and is suitable to study hydrodynamics. Thus, we argue the GRTA captures the “generic” crossover from generalized to conventional hydrodynamics.
\textit{Boltzmann equation} — GHD describes the dynamics of integrable systems in terms of their quasiparticles. We characterize quasiparticles with given quantum number ("rapidity") \( \lambda \) by their density \( \rho_\lambda(x,t) \). Note that \( \lambda \) is a shorthand for both continuous and discrete labels. The distribution of quasiparticles \( \rho_\lambda(x,t) \) is in one-to-one correspondence with a local equilibrium macrostate \([90]\). In an integrable system with conserved charges \( \{Q_n\} \), local equilibrium can be equivalently characterized by a generalized Gibbs ensemble (GGE) density matrix \( \hat{\rho}_{\text{GGE}} = Z^{-1} e^{-\sum \beta_n Q_n} \).

In integrable systems, quasiparticles scatter elastically with phase shifts leading to Wigner time delays \([29,33]\): the effective velocity \( v_\lambda^{\text{eff}}[\rho] \) of a quasiparticle with rapidity \( \lambda \) depends on the density of all the other quasiparticles \([24,25,46,91]\). Transport properties can be inferred from the fact that quasiparticles carry some charge \( h_i(\lambda) \), where \( i \) labels the conserved charges of the integrable system. The density of charge \( i \) reads \( q_i(x,t) = \int d\lambda h_i(\lambda) \rho_\lambda(x,t) \), with the associated Euler current \( j_i(x,t) = \int d\lambda h_i(\lambda) \partial_t \rho_\lambda(x,t) v_\lambda^{\text{eff}}[\rho] + \ldots \) where the dots represent higher order (diffusive) corrections \([39-41,51]\) that will be negligible for our purposes. The conservation laws \( \partial_t q_i + \partial_x j_i = 0 \) form the basis of GHD \([24,25]\).

We now imagine perturbing such an integrable system with Hamiltonian \( H_0 \) by a small, non-integrable perturbation \( g\hat{V} \) that destroys all but a few conservation laws. We assume that the expressions for charges and currents are unchanged – neglecting \( \mathcal{O}(g) \) corrections to these quantities, and force terms that are treated elsewhere \([92]\). The leading effect of the non-integrable perturbation is to thermalize quasiparticle distributions at long times \( t \gg \mathcal{O}(g^{-2}) \). Integrability breaking endows the GHD equation with a collision integral

\[
\partial_t \rho_\lambda + \partial_x \left( v_\lambda^{\text{eff}}[\rho] \rho_\lambda \right) = \mathcal{I}_\lambda[\rho].
\]

that mixes quasiparticle sectors. This collision integral preserves only a few of the original conserved charges \( q_\alpha \), \( \alpha = 1, \ldots, N \), \( N = 1,2 \) or 3 in most physical situations of interest; with \( \int d\lambda Z_\lambda[\rho] h_\alpha(\lambda) = 0 \). This collision integral term can be derived perturbatively using Fermi’s Golden Rule (FGR), and is of order \( \mathcal{O}(g^2) \). It involves the matrix elements (form factors) of the integrability breaking perturbations, which can be expressed in terms of hydrodynamical data only for non-interacting systems, and for perturbations involving low-momentum transfer such as slowly varying noisy potentials or long-range interactions \([82]\). This equation (1) was analyzed within linear response in Ref. \([82]\), and was shown to lead to diffusive hydrodynamics in general.

\textit{Numerical solution} — First, we develop a general numerical scheme to solve (1), which can be used both near and far from equilibrium. Following the numerical methods of Refs. \([28,49,92]\) in the integrable case, we find it convenient work with the “normal modes” of GHD, which are given by the occupation ratios (Fermi factors) \( n_\lambda = \rho_\lambda/\rho^{\text{tot}}_\lambda \), where \( \rho^{\text{tot}}_\lambda = \rho_\lambda + \rho_\lambda^{\text{h}} \) is the total density of states at rapidity \( \lambda \) and \( \rho_\lambda^{\text{h}} \) the density of holes. There is a one-to-one correspondence between the density of quasiparticles \( \rho_\lambda \) and the occupation ratios \( n_\lambda \), provided by the Bethe equations. In terms of \( n_\lambda \), the Boltzmann equation (1) takes the advection form

\[
\partial_t n_\lambda + v_\lambda^{\text{eff}}[\rho] \partial_x n_\lambda = I_\lambda[\rho],
\]

where \( I_\lambda \) is simply related to \( \mathcal{I}_\lambda[\rho] \) \([93]\). We then solve this equation by finite elements, discretizing space, time, and rapidity space. We use a backward first order scheme \( n_\lambda(x,t) = n_\lambda(x-v_\lambda^{\text{eff}}[\rho(x,t)] \Delta t, t-\Delta t) + \Delta t I_\lambda[n(x,t)] \), where crucially, the velocity and collision integrals in the right-hand side are evaluated at time \( t \) to improve stability. We solve this equation by iterations, and check convergence with respect to the small parameters \( \Delta t, \Delta x \) and \( \Delta \lambda \).

We can implement this method for a general collision integral, e.g., for the case of slow noise \([93]\) where the collision integral can be written explicitly. However, as noted above, collision integrals are generally intractable; in what follows, therefore, we adopt a different perspective, and explain how to approximate the collision integral via the GRTA.

\textit{Generalized relaxation-time approximation} — For most physical integrability-breaking perturbations, the matrix elements of the integrability breaking perturbation cannot be expressed in terms of hydrodynamic data. Even in the few cases where the collision integrals can be written down explicitly, they are impractical to implement numerically. For context, we remark that even for weakly-interacting fermions, collision integrals are often approximated by using the relaxation-time approximation (RTA), which suffices to capture most of the relaxation physics and to describe experiments. Here, we introduce a generalized relaxation-time approximation (GRTA), which amounts to choosing a simple form for the collision integral:

\[
\partial_t \rho_\lambda + \partial_x \left( v_\lambda^{\text{eff}}[\rho] \rho_\lambda \right) = -\left( \rho_\lambda - \rho^{\text{Gibbs}}_\lambda[\rho] \right)/\tau.
\]

This right-hand side enforces local thermalization on a typical relaxation timescale \( \tau \) as follows: \( \rho^{\text{Gibbs}}_\lambda[\rho] \) is a non-linear functional of the state \( \rho_\lambda \), defined as the distribution of quasiparticles of a Gibbs state with the same value of the conserved quantities \( q_\alpha \) (\( \alpha = 1, \ldots, N \) corresponding to the charges preserved by the integrability breaking perturbation) as the state \( \rho_\lambda \). For example, consider a Bose gas where the integrability breaking perturbation preserves energy \( E \), particle number \( N \) and momentum \( P \). Then the distribution \( \rho^{\text{Gibbs}}_\lambda[\rho] \) corresponds to the (boosted) Gibbs ensemble density matrix \( \hat{\rho}^{\text{Gibbs}} = \frac{1}{Z} e^{-\beta(\hat{H} - \mu N - \nu P)} \) where \( \beta, \mu \) and \( \nu \) are chosen so that the average particle number, energy and momentum

\[
\hat{\rho}_{\text{GGE}} = \frac{1}{Z} e^{-\beta(\hat{H} - \mu N - \nu P)}
\]

are the same as in the state $\rho_{s(L)}$. By definition, we have
\[\int d\lambda (\rho_{s(L)} - \rho_{\text{Gibbs}}(\lambda)) I_{s(L)}(\lambda) = 0,\]
ensuring the conservation of the charges $\tilde{Q}_{s(L)}$.

We evaluate the right-hand side of eq. (3) as follows. We compute the (density of) conserved charges $q_{\alpha}$ (say particle number, momentum and energy) in the state $\rho_{s(I)}(x, t)$, and invert the equation of states of the model – known from equilibrium thermodynamic Bethe ansatz (TBA) [94] – to find the Lagrange multipliers (in our example, $\beta, \mu$ and $\nu$) of the Gibbs state corresponding to those values. Using TBA, we then compute the density of quasiparticles $\rho_{\alpha}^{\text{Gibbs}}[\rho]$ corresponding to those Lagrange multipliers and thus $I_{s(L)}$ [93]. We take $\tau$ to be an unknown constant, a single phenomenological parameter to be determined by comparing the solution of eq. (3) to numerics or experiments.

Physically, the GRTA assumes that local relaxation is controlled by a single relaxation rate. Of course, realistic FGR collision integrals have a lot more structure, involving a hierarchy of relaxation rates. However, we expect this approximation to capture the key physics of integrability breaking. One can formalize this intuition as follows. The relaxation of charges in the presence of weak integrability-breaking is captured by the equation
\[\partial_t Q_{s(L)} = -\sum_{j} \Gamma_{s(L)j} Q_{j},\]
where $\Gamma$ is a matrix that itself is a function of the equilibrium state. One can write the conductivity tensor for the residual conserved charges in terms of the regularized inverse $\Gamma^{-1}$ and of properties of the integrable dynamics [82, 85]. The spectrum of the matrix $\Gamma$ contains zero modes corresponding to the residual conserved charges, as well as other eigenmodes that capture the characteristic decay rates. If there is a gap between the zero modes and the decaying modes, one can identify this gap with $1/\tau$, and replace the matrix $\Gamma$ with a projector onto modes that decay at rate $\sim 1/\tau$. The GRTA corresponds to replacing $\Gamma^{-1} = \tau$ for all decaying charges, which approximately coincides with the projection approach, provided that all residual conserved currents have approximately similar overlaps with the slowest-decaying modes of $\Gamma$. (This construction indicates that the GRTA will fail whenever there are arbitrarily slowly relaxing modes, as we expect on physical grounds, and also when the currents of residual charges have very different overlaps with the slowest-relaxing modes of $\Gamma$.)

Hydrodynamics of Bose gases — To illustrate the GRTA approach, we study the crossover from generalized to conventional hydrodynamics in one-dimensional Bose gases, governed by the Lieb-Liniger Hamiltonian
\[\hat{H}_0 = \int dx \left( -\frac{\nabla^2}{2m} - \mu \right) \hat{\Psi} \hat{\Psi} + c \hat{\Psi} \hat{\Psi} \hat{\Psi} \hat{\Psi}, \]
with $m = 1/2$ and $c = 1$ in what follows. We first consider integrability-breaking perturbations that relax momentum: in this case, the conserved quantities in the Gibbs state of eq. (3) are particle number and energy. We implemented both far from equilibrium free expansions into vacuum of a cloud of atoms which model experiments on ultracold Bose gases (Fig. 1a) [43, 95–105], and linear response setups where the initial state is a small local perturbation on top of an equilibrium Gibbs state (Fig. 1b). We checked that the conservation of both energy and particle number are satisfied to a very good accuracy for all time scales that are plotted ($< 0.5\%$). We find that while the variance of the profiles of the local perturbation in both energy and particle density grow quadratically (indicating ballistic transport) in the integrable case, they crossover to a linear (diffusive) behavior at times $t \gg \tau$. Diffusive hydrodynamics is expected as momentum is not conserved, and we see that energy and particle number have different diffusion constants, inher-
ited from the different Drude weights of the integrable limit.

We have also solved eq. (3) for a Bose gas (4) with a perturbation preserving particle number, energy and momentum. Our scheme fully preserves Galilean invariance, so the particle current is momentum and is therefore conserved: we observe “sound modes” propagating ballistically in the non-integrable case, which broaden diffusively on the time scales we simulated. We also observe a small heat mode near the origin. This is consistent with what is expected from conventional, Navier-Stokes hydrodynamics in one dimension. We note that conventional hydrodynamics is generically anomalous in one dimension, and adding noise to our equations is expected to broaden the sound peaks in a superdiffusive way (dynamical exponent $z = 3/2$) – instead of diffusive – as predicted by the theory of non-linear fluctuating hydrodynamics [106–108]. It will be interesting to include noise in our framework to check this.

Energy transport in spin chains — The GRTA approach has the advantage of being very general, and can also be applied to chaotic spin chains near integrability. To illustrate this, we consider the spin-$\frac{1}{2}$ XXZ spin chain with staggered transverse fields

$$\hat{H} = \sum_i (\hat{S}_{i}^x \hat{S}_{i+1}^x + \hat{S}_{i}^y \hat{S}_{i+1}^y + \Delta \hat{S}_{i}^z \hat{S}_{i+1}^z) + h_x (-1)^i \hat{S}_{i}^x,$$

with anisotropy $\Delta = 1/2$. When $g \equiv h_x = 0$, this model is integrable, and energy transport is purely ballistic as the total energy current is a conserved quantity. As higher-order corrections vanish exactly, energy transport can be captured extremely well by GHD [28]. The staggered perturbation $h_x \neq 0$ breaks integrability and the $U(1)$ symmetry of the XXZ model.

We consider energy transport in the Hamiltonian (5) by preparing a local region with temperature $T = 10$ embedded in a uniform equilibrium background with temperature $T = 2$ [109]. We simulate the dynamics of this system up to time $t = 20$ by evolving the density matrix using time-evolving block decimation (TEBD) [110–112], and compare with the GRTA (3) for various values of $\tau$. We compare the local temperature profiles $T(x,t)$ between the two approaches, using the equilibrium equation of state of Eq. (5) to convert energy density to temperature. (This accounts for the shift in the equilibrium energy density due to the perturbation $h_x$, which can readily be captured using perturbation theory). We find a best fit for the single parameter $\tau$ by matching the full temperature profiles from the TEBD simulations and the GRTA.

We find that GRTA is able to describe the non-integrable dynamics of (5) remarkably well with a single parameter $\tau$ for each $h_x$, for various values of $h_x$ ranging from 0.05 to 0.6 corresponding to almost two decades in $\tau$. Moreover, the fitted values of $\tau$ vs $h_x$, taking account the uncertainties of the fits, all agree perfectly with the simple FGR scaling

$$\tau \simeq C h_x^{-2},$$

with $C \approx 0.32(5)$. This is remarkable as in general we expect that relaxation times should depend on temperature, and the initial state considered has a wide range of temperatures. Allowing for limited dependence of $\tau$ on the state $\rho$ — such as through the local temperature — might be necessary to capture strongly nonequilibrium setups with even wider temperature ranges.

This scaling implies that the whole time evolution for all values of $h_x$ we have considered can be described very accurately using a single free parameter $C$. While we obviously expect corrections to this GRTA approach, combined with the expected FGR scaling (6), it clearly captures most of the physics of integrability breaking. Surprisingly GRTA is able to describe energy transport even for strongly chaotic chains with $h_x = 0.6$, for which the relaxation time $\tau$ is of order $O(1)$.

Discussion — In this work we have introduced the GRTA as a numerically efficient approximation to study the nonequilibrium dynamics of systems with weak integrability breaking. The GRTA treats integrability-breaking in a rather drastic approximation, where all but the residual conserved charges decay on a single timescale $\tau$. Nevertheless, this approximation works surprisingly well to capture the hydrodynamics of physically relevant integrability-breaking perturbations $gV$ (such as a staggered transverse field in the XXZ model) at the cost of introducing $\tau(g) = C g^{-2}$ with a single fit parame-
ter C. Many natural extensions of this method suggest themselves. For instance, in cases where some charges relax much slower than others, we can treat the dynamics of the fast charges within GRTA (treating the slow modes as conserved) and then relax the slow charges separately. This could be relevant, for example, in ultracold atomic experiments, where integrability breaking due to collisions can be much faster than atom loss or momentum relaxation due to the trap. Another natural extension would be to add noise to the GRTA equations (of strength given by the fluctuation-dissipation theorem) and explore whether this allows one to recover the predicted anomalous scaling of the sound peaks in Fig. 1. Finally, our implementation of the integrable dynamics itself has been restricted to Euler scale hydrodynamics. An important open question is to develop an efficient scheme for numerically solving the GHD equations beyond the Euler scale [39]; incorporating the GRTA into this scheme would allow us to answer currently open questions about the fate of anomalous diffusion in non-integrable spin chains [83].

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I. BOLTZMANN EQUATION IN NORMAL MODE BASIS

The form of the Boltzmann equation written in eq (1) of the main text is not the most convenient when it comes to solving it numerically. Instead it is simpler to work with the occupation ratios (Fermi factors) or normal modes of the theory which are related to the density of quasiparticles via the local Bethe equation

\[ \rho^\text{tot}_\lambda(x,t) = \frac{p'_\lambda}{2\pi} + \int d\lambda' T_{\lambda,\lambda'} \rho_{\lambda'}(x,t), \]  

with the kernel \( T_{\lambda,\lambda'} \) the scattering kernel of the theory and \( p'_\lambda = dp/d\lambda \), with \( p_\lambda \) the bare momentum. We can omit the space-time dependence and also write all equations abstractly working directly with operators and vectors acting on rapidity space, see Ref.\(^1\) for a recent review. In this language, the kernel \( \hat{T} \) of the integrable model in question is an operator that acts as the convolution when applied to a vector, i.e. \( \hat{T} h|_\lambda = (T \ast h)|_\lambda = \int d\lambda' T_{\lambda,\lambda'} h_{\lambda'} \). All the operators we will deal with will be diagonal, e.g. \( \hat{n} h|_\lambda = n_\lambda h_\lambda \). This way the Bethe equation reads simply

\[ \hat{n} \rho = \frac{1}{2\pi} \hat{n} e' \text{dr}, \]  

where \( e(\lambda) \) denotes the bare energy. Using the Bethe equations, we have \( \hat{v} \rho = \frac{1}{2\pi} \hat{n} e' \text{dr} \), so the derivatives read

\[ \partial_t \rho + \partial_x (\hat{v} \rho) = \hat{I}[\rho] \]  

with the effective velocity given by\(^2\text{-}^4\)

\[ v^\text{eff}_\lambda = \frac{(e')^{\text{dr}}_\lambda}{(p')^{\text{dr}}_\lambda}, \]  

where \( e(\lambda) \) denotes the bare energy. Using the Bethe equations, we have \( \hat{v} \rho = \frac{1}{2\pi} \hat{n} e' \text{dr} \), so the derivatives read

\[ \partial_t \rho = \frac{1}{2\pi} \left( \hat{1} + \hat{n} \hat{T} \text{dr} \right) \partial_t \hat{n} \rho' \text{dr}, \]

\[ \partial_x (\hat{v} \rho) = \frac{1}{2\pi} \left( \hat{1} + \hat{n} \hat{T} \text{dr} \right) \partial_x \hat{n} e' \text{dr}, \]  

where we have defined the dressed kernel \( \hat{T} \text{dr} = (\hat{1} - \hat{T} \hat{n})^{-1} \hat{T} \). Introducing the operator \( \hat{R} \) given by \( \hat{R} = \hat{1} - \hat{n} \hat{T} \), we have \( \hat{1} + \hat{n} \hat{T} \text{dr} = \hat{R}^{-1} \), so that equation (2) becomes

\[ \frac{1}{2\pi} \hat{R}^{-1} (\partial_t \hat{n} + \hat{v} \partial_x \hat{n}) \rho' \text{dr} = \hat{I}[\rho] \]  

Using \( \rho^\text{tot}_\lambda = \frac{1}{2\pi} p\rho' \text{dr}_\lambda \), we have

\[ \partial_t n_\lambda + v^\text{eff}_\lambda \partial_x n_\lambda = I_\lambda[n], \]  

with the modified collision integral \( I_\lambda[n] \equiv \frac{1}{\rho^\text{tot}_\lambda} \left( \hat{R} \hat{I}[\rho] \right)|_\lambda \).
FIG. 1. **Bose gas with slowly varying noise:** Time evolution of particle density \( n(x,t) \) in a Lieb-Liniger Bose gas with mass \( m = 1/2 \) and interaction parameter \( c = 1 \), subject to slowly varying noise. The initial state is a thermal state with temperature \( T = 1 \), with a local excess temperature \( T_e = 4 \) at the origin. Time profiles are shown both with and without the integrability-breaking noise perturbation. *Inset:* Evolution of the densities of energy \( \epsilon(t) \), particle \( n(t) \), and momentum \( p(t) \) as a function of time, starting from a uniform boosted Gibbs state with \( \nu = 1.5, \mu = 0 \) and \( \beta = 1 \).

**II. NUMERICAL SOLUTION OF THE BOLTZMANN EQUATION**

Motivated by the method of characteristics widely used in the context of PDEs and by Ref.\(^5\) in the integrable case, we propose a solution of Eq (2) based on a finite element, backward (implicit) first order scheme solution given by

\[
n_\lambda(x,t) = n_\lambda(x - v^{\text{eff}}[n(x,t)]\Delta t, t - \Delta t) + \Delta t I_\lambda[n(x,t)],
\]

which is obviously correct up to order \( O(\Delta t^2) \) and was found to be numerically stable in the integrable limit \( I_\lambda = 0 \) in Ref.\(^5\). A solution to Eq. (6) is found upon convergence when reducing the size of \( \Delta t, \Delta x \) and \( \Delta \lambda \). In practice we find quick convergence in most problems considered, and no further refinement of the method is needed. We remark however that going beyond first order in \( \Delta t \) would not provide further insights into the phenomenology considered in this work, though it might be useful for future applications of our approach to access longer times. Solving (7) at time step \( N_t \) requires an interpolation scheme to map out all possible arguments of the solution (characteristic curves) at time step \( N_t - 1 \), which we do by means of cubic splines.

In this implicit scheme, the right-hand side of (7) depends on the state at time \( t \), and as such the whole equation must be solved iteratively, which in practice does not pose a problem reaching convergence quickly. Finding the effective velocity \( v^{\text{eff}} \), as well as the collision term, which in principle is arbitrary, requires solving integral equations for the dressed quantities and the quasiparticle densities using (1) and (3), only bounded by the discretization of \( x \)-space, the upper rapidity cutoff \( \Lambda \), and the number of species of quasiparticle excitations. It is thus imperative to find an efficient way to carry out all (improper) integrals. Using a Legendre-Gauss quadrature scheme allows us to find convergence in all these for up to the times considered in this work using as little as \( N_\lambda = 100 \) points in rapidity space in the simplest instances with \( \Lambda = 20 \), and up \( N_\lambda = 350 \) in the most numerically demanding cases.

**A. Solution to the Boltzmann equation in the presence of slowly varying noise**

To illustrate the versatility of our numerical approach let us consider the case of a Bose gas with slowly varying noise. This is modeled by the Lieb-Liniger Hamiltonian with an extra smoothly varying time-dependent potential coupled to particle density, \( V(x)\eta(t)\hat{q}_0 \). In this case, the collision integral can be evaluated using Fermi’s Golden Rule (FGR) directly in terms of GHD data\(^6\)

\[
I_\lambda = \rho_{\lambda}^{\text{tot}} 1_\lambda^{\text{dr}} \int d\varphi |\hat{V}(k_{\lambda+\varphi} - k_\lambda)|^2 |\hat{\eta}(\epsilon_{\lambda+\varphi} - \epsilon_\lambda)|^2 \rho^{\text{tot}}_{\lambda+\varphi} 1_{\lambda+\varphi}^{\text{dr}} |n_{\lambda+\varphi} - n_\lambda|,
\]

where the functions \( \hat{V} \) and \( \hat{\eta} \) characterize correlations of the noise in frequency and momentum space. This expression is valid for slowly varying noise so the collision integral is dominated by low-momentum, single particle-hole excitations.
created by the integrability breaking perturbation. Here $k$ and $\epsilon$ are the physical momentum and energy of the excitations, which are given by $k' = (\rho')^{1/2}$ and $\epsilon' = (E')^{1/2}$. Here, we ignore forces due to the varying potentials.

In Fig. 1 we show the results of preparing an initial state with an homogeneous background temperature profile of $T = 1$ and $\mu = 0$ perturbed with a temperature bump with temperature $T + \Delta T = 4$, and letting it evolve under the unitary dynamics given by the perturbed Lieb-Liniger Hamiltonian. While the approach is only controlled for slowly varying noise so that energy and momentum are almost conserved, we show here the solution for very broad functions $V$ and $\hat{q}$ to accentuate the effects of the perturbation at relatively short times, by taking those functions to be Gaussians with large variance. Convergence of the results shown is attained choosing time steps $\Delta t \approx 0.05$ (for the integrable case) or $\Delta t \approx 0.005$ (for the non-integrable case), $\Delta x \approx 0.5$ and $N_\lambda \approx 200$. We observe that in stark contrast with the integrable case, the non-integrable dynamics does not feature ballistically moving peaks. The main effects of the integrability breaking perturbation are already visible from the time evolution of a uniform state, as showcased in the inset of Fig. 1: whereas particle number is conserved, the system is heating up (energy increasing), while momentum relaxes.

### B. Solution of the Boltzmann equation with GRTA

In principle the aforementioned scheme works generically for any collision integral. In this work we are mostly interested in the specific choice given by the GRTA, Eq (3) of the main text $\tilde{Z}_\lambda[\rho] = - (\rho_\lambda - \rho^{\text{Gibbs}}_\lambda) / \tau$ (with the counterpart version in normal modes written as in (6)). As explained in the main text, $\rho^{\text{Gibbs}}_\lambda$ is the Gibbs state with the same value of the charge densities $q_\alpha(x, t)$, $\alpha = 1,...,N$ as the state $\rho_\lambda$, so that $\frac{d}{dx} \int dx q_\alpha = 0$. As such, it is a non-linear functional of $\rho_\lambda$ and thus makes the right-hand side far from trivial. Its numerical implementation is straightforward: the first step is to discretize the Lagrange multipliers involved in the problem (e.g. in the case where the GRTA conserves particle number $Q_0 = N$ and energy $Q_2 = E$, these will be $\mu$ and $\beta$). For each set of Lagrange multipliers $\{\beta_\alpha\}$ we compute the associated conserved charge densities $\{q_\alpha\}$ via the equation of states known from TBA, $\{q_\alpha\} = \{f(\beta_\alpha)\}^{1/2}$. This yields a multidimensional grid for each $q_\alpha$. Since in all physical cases of interest $N \leq 3$, this first step can be solved very efficiently. In practice, it helps to adapt the range of this grid to the initial state under investigation. The second step is to compute at each time step and position $x$ the various charge densities of the conserved charges $q_\alpha = \int d\lambda \rho_\lambda h_\alpha(\lambda)$, and find the corresponding Lagrange multipliers that are closest to the ones used when computing the charge density grids. This can be further improved by interpolation of the grid, and we have also tried steepest-descent schemes. This gives us $\rho^{\text{Gibbs}}_\lambda$ at each space-time coordinate and thus the GRTA collision integral $I_\lambda$. We then follow the same steps indicated above to solve (7).

We have illustrated this approach in the main text for both interacting Bose gases and spin chains. In the case of the Lieb-Liniger model our simulations were obtained setting the coupling constant $c = 1$ and $m = 1/2$. For the far-from-equilibrium results we prepared the initial state to a $T = 2/3$ temperature background with a temperature excess of value $T = 20$ at the origin. We obtained converged results using $N_\lambda = 200$ and $\Delta T = 0.02$, $\Delta x = 0.75$. For the linear response regime we prepared the initial state with a local excess of temperature $T = 1.1$ on top of a background at temperature $T = 1$. We find convergence using $N_\lambda = 200$, $\Delta t = 0.05$ and $\Delta x = 0.5$. For the XXZ
model (see below for more details) the parameters used were $N_\lambda = 300$, $\Delta t = 0.2$ and $\Delta x = 0.075$.

### III. ENERGY TRANSPORT IN NON-INTEGRABLE QUANTUM SPIN CHAINS

We apply GRTA to the XXZ model with a staggered magnetic field along the transverse direction. We simulate the dynamics from first principles using the TEBD algorithm, as well as using the Boltzmann equation (6) with the GRTA approximation. All TEBD simulations are carried out by evolving the homogeneous infinite temperature density matrix in imaginary time with the inhomogenous Hamiltonian

$$\hat{H} = \sum_i \frac{\beta(i + \frac{1}{2})}{\beta_0} (\hat{S}_i^x \hat{S}_{i+1}^x + \hat{S}_i^y \hat{S}_{i+1}^y + \Delta \hat{S}_i^z \hat{S}_{i+1}^z) + \frac{\beta(i)}{\beta_0} h_x (-1)^i \hat{S}_i^x.$$  \hspace{1cm} (9)

We evolve using a 2nd order Trotter decomposition of the imaginary time evolution operator using Trotter steps of size $\Delta \beta = 0.001$ and number $\beta_0 / \Delta \beta$, effectively attaining the far-from-equilibrium initial (inverse) temperature profile given by $\beta(x)$ in the homogenous Hamiltonian. We take the temperature profile to be a Gaussian centered on the middle bond of a length 200 chain:

$$\beta(x) = \beta_0 - (\beta_0 - \beta_M) e^{-(x-x_0)^2/L^2},$$  \hspace{1cm} (10)

with $\beta_0 = 0.5$, $\beta_M = 0.1$, and $L = 8$. We then time evolve the resulting initial state using a 4th order Trotter decomposition of the time evolution operator (of the homogeneous Hamiltonian) with Trotter step $\delta t = 0.4$. We set the maximum bond dimension $\chi_{\text{max}} = 512$, and a truncation cutoff of $\epsilon = 10^{-9}$, which suffice to obtain converged results for all our simulations.

We find that GHD+GRTA treating $\tau$ as a free parameter agrees remarkably well with TEBD time evolution. The results for $h_x = 0.1$ and $h_x = 0.6$ are shown in Fig. 2. Surprisingly, we find very good agreement even far away from the integrable point $h_x = 0$: for $h_x = 0.6$, $\tau \sim 1$ indicating a very quick crossover to diffusion. As shown in the main text, the fitted values of $\tau$ agree very well with the FGR scaling $\tau = Ch_x^{-2}$, meaning that ultimately, non-equilibrium energy transport for any (reasonably small) value of $h_x$ can be inferred from fitting $C$ from a single instance of $h_x$.

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