Abstract

It is stated in many text books that the any metric appearing in general relativity should be locally Lorentzian i.e. of the type $\eta_{\mu\nu} = \text{diag} (1, -1, -1, -1)$ this is usually presented as an independent axiom of the theory, which can not be deduced from other assumptions. In this work we show that the above assertion is a consequence of a standard stability analysis of the Einstein equations and need not be assumed.

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1 Introduction

It is well known that our daily space-time is approximately of Lorentz (Minkowski) type that is, it possess the metric $\eta_{\mu\nu} = \text{diag} (1, -1, -1, -1)$. The above statement is taken as one of the central assumptions of the theory of special relativity.

Further more it is assumed in the general theory of relativity that any space-time is locally of the type $\eta_{\mu\nu} = \text{diag} (1, -1, -1, -1)$, although it can not be presented so globally due to the effect of matter. This is a part of the demands dictated by the well known equivalence principle. The above principle is taken to be one of the assumptions of general relativity in
addition to the Einstein equations:

\[ G_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu} \]  

(1)
in which \( G_{\mu\nu} \) is the Einstein tensor, \( T_{\mu\nu} \) is the stress-energy tensor, \( G \) is the gravitational constant and \( c \) is the velocity of light.

In what follows we will show that such assumption is not necessary, (contrary to what is argued in so many text books, see for example [1]) rather we will argue that this metric is the only possible stable solution to the Einstein equation (1) in vacuum, that is for the case \( T_{\mu\nu} = 0 \). And thus reduce the number of assumptions needed to obtain the celebrated results of general relativity. By making the theory more compact we enhance its predictive strength.

Eddington [2, page 25] has considered the possibility that the universe contains different domains in which some domains are locally Lorentzian and others have some other local metric of the type \( \eta_{\mu\nu} = \text{diag} (-1, -1, -1, -1) \) or the type \( \eta_{\mu\nu} = \text{diag} (+1, +1, -1, -1) \). For the first case he concluded that the transition will not be possible since one will have to go through a static universe with a metric \( \eta_{\mu\nu} = \text{diag} (0, -1, -1, -1) \). Going to the domain in which \( \eta_{\mu\nu} = \text{diag} (+1, +1, -1, -1) \) means that one will have to pass through \( \eta_{\mu\nu} = \text{diag} (+1, 0, -1, -1) \) in which space becomes two dimensional\(^2\). The stability of those domains was not discussed by Eddington.

Greensite [3] and Carlini & Greensite [4, 5] have studied the metric \( \eta_{\mu\nu} = \text{diag} (e^{i\theta}, -1, -1, -1) \) in which \( \theta \) the "wick angle" was treated as a quantum field dynamical variable. They have shown that the real part of the quantum field effective potential is minimized for the Lorentzian metric \( \theta = 0 \) and for the same case the imaginary part of the quantum field effective potential is stationary. Further more they have calculated the fluctuations around this minimal value and have shown them to be of the order \( (l_p R)^3 \) in which \( l_p \) is the Planck length and \( R \) is the scale of the universe. Elizalde & collaborators [6] have shown that the same arguments apply to a five dimensional Kaluza-Klein universe of the type \( R^4 \times T^1 \).

Itin & Hehl [7] have deduced that space time must have a Lorentzian metric in order to support classical electric/magnetic reciprocity.

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\(^1\)Prof. Lynden Bell has noticed that there may be another way going through the metric \( \eta_{\mu\nu} = \text{diag} (\infty, -1, -1, -1) \), the author thanks him for his remark.

\(^2\)Again there may be another way going through the metric \( \eta_{\mu\nu} = \text{diag} (+1, \infty, -1, -1) \).
H. van Dam & Y. Jack Ng [8] have argued that in the absence of a Lorentzian metric one can not obtain an appropriate finite representation of space-time and hence the various quantum wave equations can not be written.

What is common to the above approaches is that additional theoretical structures & assumptions are needed in order to justify what appears to be a fundamental property of space-time. In this paper we claim otherwise. We will show that General relativistic equations and some "old fashioned" stability analysis will lead to a unique choice of the Lorentzian metric being the only one which is stable.

The plan of this paper is as follows: in the first section we describe the possible constant metrics which are not equivalent to each other by trivial manipulations. The second section will be devoted to the review the classical linearized theory of gravity which we adapt to our more general case of a non Lorentzian background metric. In the third section we use the linearized theory to study the stability of the possible constant metrics and thus divide them into two classes: stable and unstable. The last section will discuss some possible implications of our results.

2 Possible Constant Metrics

In this section we study what are the possible constant metrics available in the general theory of relativity which are not equivalent to one another by a trivial transformation, that amounts to a simple change of coordinates.

Let us thus study the four-dimensional interval:

\[ d\tau^2 = \eta_{\mu\nu} dx^\mu dx^\nu \] (2)

in what follows Greek letters take the traditional values of 1 – 4, and summation convention is assumed. \( \eta_{\mu\nu} \) is any real constant matrix.

Since \( \eta_{\mu\nu} \) is symmetric we can diagonalize it using a unitary transformation in which both the transformation matrix and the eigenvalues obtained are real. Thus without loss of generality we can assume that in a proper coordinate basis:

\[ \eta = \text{diag} (\lambda_1, \lambda_2, \lambda_3, \lambda_4). \] (3)

Next, by changing the units of the coordinates, we can always obtain:

\[ \eta = \text{diag} (\pm 1, \pm 1, \pm 1, \pm 1) \] (4)
notice that a zero eigen-value is not possible due to our assumption that the space is four dimensional.

We conclude that the metrics \( \eta \) given in equation (4) are the most general constant metrics possible. In what follows we will study the stability of those solutions.

### 3 Stability of Constant Metrics

To study the stability of the metric \( \eta_{\mu\nu} \) given in equation (4) we make an arbitrary small perturbation of the constant metric and obtain the perturbed metric \( g_{\mu\nu} \).

\[
g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}. \tag{5}
\]

The evolution of the metric \( g \) is studied under the Einstein equation (1). In what follows we adapt the well known linearized theory of gravity given in many text books such as [1], to our more generalized needs. In order to obtain the linearized equations the following quantities should be linearized:

1. The connection coefficients deduced from the metric:

\[
\Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\nu}(g_{\alpha\nu,\beta} + g_{\beta\nu,\alpha} - g_{\alpha\beta,\nu}) \tag{6}
\]

in which \( \partial \) stand for partial derivative, and \( g^{\mu\nu} \) is the inverse matrix of \( g_{\mu\nu} \).

2. The Ricci tensor \( R_{\mu\nu} \) which is deduced from the connection coefficients:

\[
R_{\mu\nu} \equiv \Gamma^\alpha_{\mu\alpha,\nu} - \Gamma^\alpha_{\mu\beta,\nu} + \Gamma^\alpha_{\beta\alpha} \Gamma_{\mu\nu}^\alpha - \Gamma^\alpha_{\beta\nu} \Gamma_{\mu\alpha}^\beta \tag{7}
\]

3. The Einstein tensor \( G_{\mu\nu} \) which is deduced from the Ricci tensor and the Curvature Scalar:

\[
G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R, \quad R \equiv g^{\mu\nu} R_{\mu\nu} \tag{8}
\]

#### 3.1 Notations

First we introduce some notations: the inverse metric of \( g_{\mu\nu} \) is \( g^{\mu\nu} \) given by:

\[
g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \tag{9}
\]
in which $\eta^{\mu\nu}$ is the inverse matrix of $\eta_{\mu\nu}$, by virtue of equation (4) it is also identical to it i.e., $\eta^{\mu\nu} = \eta_{\mu\nu}$. An easy calculation will show that to first order in $h$ we obtain:

$$h^{\mu\nu} = \eta^{\sigma\nu} \eta_{\mu\rho} h_{\sigma\rho}. \quad (10)$$

Further more we introduce the following notations:

$$h_\sigma^\mu = \eta^{\mu\nu} h_{\sigma\nu} \quad h_\mu^\mu = \eta^{\mu\nu} h_{\mu\nu}. \quad (11)$$

Generally speaking we use the constant metric $\eta$ to raise and lower indices.

### 3.2 The Connection Coefficients

Let us now calculate the linearized form of the affine connection which is given by equation (6). Inserting equation (5) and equation (9) and keeping only the first order terms in $h$ we obtain:

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2} \eta^{\rho\sigma} (h_{\alpha\nu,\beta} + h_{\beta\nu,\alpha} - h_{\alpha\beta,\nu}) \quad (12)$$

which can also be written using equation (11) as:

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2} (h_{\alpha}^{\mu ,\beta} + h_{\beta}^{\mu ,\alpha} - h_{\alpha\beta}^{\mu}) \quad (13)$$

### 3.3 The Ricci Tensor

The Ricci Tensor given in equation (7) can be written in a linearized form using the result of equation (13) and the notation defined in equation (11)

$$R_{\mu\nu} = \frac{1}{2} (h_{\alpha}^{\mu ,\alpha\nu} + h_{\nu}^{\alpha ,\alpha\mu} - h_{\mu\nu,\alpha}^{\alpha} - h_{\mu\nu}) \quad (14)$$

The linearized Curvature Tensor can be calculated from equation (14)

$$R = \eta^{\mu\nu} R_{\mu\nu} = h_{\mu\nu}^{\alpha ,\mu\alpha} - h_{\nu}^{\alpha} \quad (15)$$

### 3.4 The Einstein Tensor

Finally we obtain the linearized form of the Einstein Tensor which can be calculated from equations (8,14,15):

$$G_{\mu\nu} = \frac{1}{2} (h_{\mu}^{\alpha ,\alpha\nu} + h_{\nu}^{\alpha ,\alpha\mu} - h_{\mu\nu,\alpha}^{\alpha} - h_{\mu\nu}) - \frac{1}{2} \eta_{\mu\nu} (h_{\alpha\beta}^{\alpha\beta} - h_{\alpha}^{\alpha}) \quad (16)$$
In order to simplify the above notation the following quantity is defined:

\[ \bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \hat{h} \]  

(17)

Using this definition, equation (16) can be written as:

\[ 2G_{\mu\nu} = -\bar{h}_{\mu\nu,\alpha} - \eta_{\mu\nu} \bar{h}_{\alpha\beta} + \bar{h}_{\mu\alpha,\nu} + \bar{h}_{\nu\alpha,\mu}. \]  

(18)

### 3.5 Gauge Transformation

The metric being a tensor can always be transformed to another coordinate system by the transformation:

\[ g'_{\sigma\rho}(x') = g_{\mu\nu} \frac{\partial x^\mu}{\partial x'^\sigma} \frac{\partial x^\nu}{\partial x'^\rho} \]  

(19)

Let us introduce the transformation:

\[ x^\mu = x'^\mu - \xi^\mu \]  

(20)

in which \( \xi \) is same order of magnitude as \( h \). Further more let us define:

\[ g'_{\mu\nu}(x') = \eta_{\mu\nu} + h'_{\mu\nu}(x). \]  

(21)

Inserting equations (5, 21, 20) into equation (19) we obtain to first order in \( h \):

\[ h'_{\mu\nu} = h_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} \quad (\xi_{\mu} = \eta^{\mu
u} \xi_{\nu}) \]  

(22)

this is denoted as the "gauge transformation", by changing the coordinates infinitesimally we can always obtain a new \( h' \) which is different from the old \( h \) and is related to it by equation (22). Our obvious choice of gauge will be one that simplifies equation (18). Let us calculate the expression \( \bar{h}'_{\mu\alpha,\alpha} \):

\[ \bar{h}'_{\mu\alpha,\alpha} = \bar{h}_{\mu\alpha,\alpha} - \xi_{\mu,\alpha}. \]  

(23)

We can always choose \( \xi \) such that:

\[ \xi_{\mu,\alpha} = \bar{h}_{\mu\alpha,\alpha} \]  

(24)

in which the equation above is a second order equation for the \( \xi_{\mu} \)'s. This does not "fix" the gauge since we can always introduce a new gauge: \( \xi_{\mu}^1 = \xi_{\mu} + \xi_{\mu}^0 \) in which: \( \xi_{\mu,\alpha}^0 = 0 \). Choosing the gauge according to equation (24) we obtain:

\[ \bar{h}'_{\mu\alpha,\alpha} = 0 \]  

(25)

writing equation (18) in terms of \( h' \) and dropping the prime we see that:

\[ 2G_{\mu\nu} = -\bar{h}_{\mu\nu,\alpha}. \]  

(26)
4 Stability Analysis

In the lack of matter Einstein equation (1) becomes $G_{\mu\nu} = 0$ that is through equation (26) we obtain the following equations for $\bar{h}_{\mu\nu}$:

$$\bar{h}_{\mu\nu,\alpha\alpha} = 0.$$  

(27)

Next we introduce the Fourier decomposition of $\bar{h}_{\mu\nu}$:

$$\bar{h}_{\mu\nu} = \frac{1}{(2\pi)^{\frac{3}{2}}} \int^{-\infty}_{\infty} A_{\mu\nu}(x_0, \vec{k}) e^{i\vec{k} \cdot \vec{x}} d^3 k,$$

(28)

Introducing the decomposition equation (28) into equation (27) leads to:

$$\eta^{00} \partial_0^2 A_{\mu\nu} - \eta^{ij} k_i k_j A_{\mu\nu} = 0$$

(29)

in which $i, j$ are integers between $1 - 3$. Choosing $\eta^{00} = 1$ we see that the only way to avoid exploding solutions is to choose $\eta^{ij} = \text{diag} (-1, -1, -1)$, thus one stable metric would be:

$$\eta^{(1)} = \text{diag} (1, -1, -1, -1)$$

(30)

alternatively we can choose $\eta^{00} = -1$ in this case the only way to avoid exploding solutions is to choose $\eta^{ij} = \text{diag} (1, 1, 1)$, thus a second stable metric would be:

$$\eta^{(2)} = \text{diag} (-1, 1, 1)$$

(31)

that is $\eta^{(1)} = -\eta^{(2)}$.

In the case that the universe has a spatial cyclic topology in one or more directions the Fourier integral in this direction can be replaced by a Fourier series such that we only have $k_i$’s of the type:

$$k_i = \frac{2\pi n_i}{L_i}$$

(32)

in which $n_i$ is an integer and $L_i$ is the dimension of the spatially cyclic universe in the $i$ direction.

5 Conclusions

We conclude from equations (30,31) that the only constant stable solution is of a Lorentz (Minkowski) type.
For other constant solutions we expect instabilities for $k_i \to \infty$ where $i$ depends on the unstable solution chosen. Thus the instabilities vary on very small length scale of which $\lambda = \frac{2\pi}{k} \to 0$, this length can be the smallest for which the general theory of relativity is applicable, perhaps the planck scale $\lambda = l_p = 1.61610^{-35}m$, in that case an unstable solution will last for about $t = \frac{\lambda}{c} = 5.3910^{-44}\text{sec}$. However, in the presence of matter this may take longer. This may explain why in QED an unstable Euclidean metric is used such that $\eta = \text{diag} (1, 1, 1, 1)$, this is referred to as ”wick’s rotation” [9].

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