Gyromagnetic Ratios of Bound Particles

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(March, 1997)

Abstract

A new approach to calculation of the binding corrections to the magnetic moments of the constituents in a loosely bound system, based on the Bargmann-Michel-Telegdi equation, is suggested. Binding corrections are calculated in this framework, and the results confirm earlier calculations performed by other methods. Our method clearly demonstrates independence of the binding corrections on the magnitude of the spin of the constituents.

I. INTRODUCTION

It is a common practice to describe the Zeeman effect in atoms in terms of the gyromagnetic ratios of the constituents, and also to use when needed the Clebsch-Gordon coefficients. However, due to the binding effects, magnetic moments of the electrons and the nuclei do not coincide exactly with the magnetic moments of the free particles. Respective bound-state corrections to the gyromagnetic ratios of the constituents in the hydrogenic atoms have been calculated a long time ago in Refs. [1–3]. Explicit calculations in these works were done for the case of spin one-half constituents, and the methods of [1,2] do not admit straightforward generalization for the case of the nuclei with other spins. So, it was not clear from these works if the spin one-half formulae for the heavy particle gyromagnetic ratio may be

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applied without alteration for nuclei with other spins, for example in the case of deuterium. This problem was resolved by quite general and abstract considerations in [3] based on the representation theory of the Poincare group. It was shown there that respective expressions are equally valid for any spin of the nucleus. This problem is of more than academic interest since the high precision result for the ratio of the bound electron and bound deuteron gyromagnetic ratios was used in the 1986 adjustment of the fundamental constants [4] for extracting a high precision value of the free deuteron gyromagnetic ratio.

Recently there emerged a new interest in calculating binding corrections to the electron magnetic moment in high $Z$ atoms without expansion in $Z\alpha$. The first calculations produced results (see, e.g., review in [5]) which in the small $Z$ limit seemed to contradict the old results in [1–3] although the results of later work [6] are consistent with the [1–3]. It also turned out that the validity of the old results for arbitrary spin of the nuclei, which was proved in [3] was not widely known. Inspired by these questions we decided to reconsider the problem once more. We have studied a new approach to the problem which is different from all other methods used earlier [1–3]. This approach is physically quite transparent and immediately provides results valid for particles with any spin with accuracy of order $(Z\alpha)^2$. It also clearly demonstrates that the problem of the proper description of the lowest order binding corrections to the gyromagnetic ratios for particles of any spin may be completely resolved in the nonrelativistic framework, and is not logically connected with the problem of Lorentz invariant description of bound states (see especially comments on the recoil corrections below). We present below this new calculation of the gyromagnetic ratios for bound particles.

II. SPIN MOTION IN AN EXTERNAL FIELD

In order to establish notation we start with the standard nonrelativistic Hamiltonian for the interaction of the magnetic moment with an external magnetic field

$$
H_{\text{nonrel}} = -\mu H,
$$

where the magnetic moment for arbitrary spin in terms of its "native" magneton $\mu_0 = e/(2m)$ and gyromagnetic ratio $g_s$, has the form

$$
\mu = g_s\mu_0 S.
$$

Leading binding corrections of order $(Z\alpha)^2$ to the gyromagnetic ratio are of relativistic nature, and are induced by the relativistic corrections of order $v^2/c^2$ to the Hamiltonian. The observation that all terms of order $v^2/c^2$ in the spin Hamiltonian in an external field,  

1This question was raised in private communication by Profs. B. Taylor and P. Mohr. We have to admit that we ourselves have realized that Ref. [3] contains the result for arbitrary spin only after completion of this note.

2We use the system of units where $\hbar = c = 1$, and use the definition above even in the case of negatively charged electron in order to be able to present a derivation of the binding corrections to the gyromagnetic ratio which is equally valid both for positively and negatively charged particles.
may easily be restored with the help of the Bargmann-Michel-Telegdi (BMT) equation [7],
is of crucial importance for the considerations below.

Consider four-vector \(a_\mu\), which is a relativistic generalization of the average spin vector in the rest frame [8]. The classical relativistic BMT equation describes the behavior of this four-vector in an external electromagnetic field. This equation, which is a direct consequence of relativistic invariance, is equally valid for arbitrary magnitude of the average spin, and in terms of the particle gyromagnetic ratio has the form [8]

\[
\frac{da_\mu}{d\tau} = \mu_0 [g_s F^{\mu\nu} a_\nu - (g_s - 2) u^\mu F^{\nu\lambda} u_\nu a_\lambda],
\]

where \(\tau\) is the relativistic proper time for the particle.

We would like to mention in passing that this equation nicely demonstrates the special role of the "normal" gyromagnetic moment \(g_s = 2\), which for any spin has a simple kinematic origin and is connected with the form of the ordinary Lorentz force (the famous Thomas one-half). Technically the special role of \(g_s = 2\) is connected with the correlation between the magnitude of the coefficient \(e/m\) before the term with the magnetic field in the expression for the Lorentz force and the magnitude of the factor before the magnetic field in the nonrelativistic spin Hamiltonian above.

In terms of the ordinary time \(t\) and the average spin three-vector \(\zeta\) the BMT equation has a more complicated form [8]

\[
\frac{d\zeta}{dt} = \mu_0 \left\{ \frac{g_s m + (g_s - 2)(E_p - m)}{E_p} [\zeta \times H] + \frac{(g_s - 2) E_p}{E_p + m} (vH)[v \times \zeta] \right. \\
+ \left. \frac{g_s m + (g_s - 2) E_p}{E_p + m} [\zeta \times [E \times v]] \right\}.
\]

In order to obtain the quantum mechanical description of the spin motion in the external field we follow the idea of [8]. First, we rewrite the BMT equation in terms of the canonical variables substituting \(v = (p - eA)/E_p\). Then one may easily restore the quantum mechanical Hamiltonian which according to the canonical commutation relations leads to the BMT equation

\[
\mathcal{H}_{\text{spin}} = \mathcal{H}' - \mu_0 \frac{g_s m + (g_s - 2)(E_p - m)}{E_p} SH + \mu_0 \frac{(g_s - 2) E_p}{(E_p + m) E_p^2} ((p - eA)H)((p - eA)S)
\]

\[
-\mu_0 \frac{g_s m + (g_s - 2) E_p}{(E_p + m) E_p} (S[E \times (p - eA)]),
\]

where \(H'\) contains all terms of the Hamiltonian which are spin independent.

All relativistic corrections both to the normal and anomalous magnetic moments may be calculated with the help of the quantum mechanical Hamiltonian in eq.(5).
A. Nonrecoil Limit

Let us ignore first all recoil factors and calculate corrections to the gyromagnetic ratio of order \((Z\alpha)^2\). Physically this corresponds to a problem with a light particle in the field of an infinitely heavy Coulomb center. In this case canonical coordinates of the light particle coincide with the coordinates in the center of mass frame, and the problem simplifies. We expand all coefficients in eq.(5) up to order \(p^2/m^2\) and preserve only terms linear in the external magnetic field and spin

\[
\mathcal{H}_{\text{exp}} = \mathcal{H}' - \mu_0 \left\{ g_s (1 - \frac{p^2}{2m^2}) S H + (g_s - 2) \frac{p^2}{2m^2} S H - \frac{g_s - 2}{2m^2} (p H) (p S) \right\}
\]

\[
- \frac{e}{m} \left\{ \frac{g_s}{2} + \frac{g_s - 2}{2} \right\} |(S[E \times A])|.
\]

Next we simplify the perturbation, anticipating that the matrix elements will be calculated between spherically symmetric wave functions and that the external Coulomb electric field corresponds to the potential \(V = -4\pi Ze/r\),

\[
\mathcal{H}_{\text{exp}} = \mathcal{H}' - \mu_0 \left\{ g_s [1 - \frac{p^2}{2m^2}] S H + (g_s - 2) \frac{p^2}{3m^2} S H + \frac{Z\alpha}{6mr} S H \right\}.
\]

Matrix elements between the Coulomb-Schrodinger wave functions for the \(nS\) states are given by the relations

\[
\langle n | Z\alpha \frac{m}{mr} | n \rangle = \langle n | \frac{p^2}{m^2} | n \rangle = \frac{(Z\alpha)^2}{n^2}, \]

and we easily obtain

\[
\langle n | \mathcal{H}_{\text{exp}} | n \rangle = \langle n | \mathcal{H}' | n \rangle - \mu_0 \left\{ g_s (1 - \frac{(Z\alpha)^2}{3n^2}) + (g_s - 2) \frac{(Z\alpha)^2}{2n^2} \right\} S H,
\]

or

\[
g_{\text{bound}} = g_s \left[ 1 - \frac{(Z\alpha)^2}{3n^2} + \frac{g_s - 2}{g_s} \frac{(Z\alpha)^2}{2n^2} \right] \approx g_s \left[ 1 - \frac{(Z\alpha)^2}{3n^2} + \frac{\alpha(Z\alpha)^2}{4\pi n^2} \right]. \]

This result reproduces the old results \([1-3]\) in the nonrecoil approximation.

III. CENTER OF MASS MOTION IN EXTERNAL FIELD

Consider next the Coulomb bound system of two particles. We are seeking corrections to the magnetic moments of the constituents, and the first problem is to separate center of mass motion from the internal degrees of freedom. This is of course trivial in the absence of an external field due to translation invariance. But an external field breaks translation invariance and the usual variables in the respective Schrodinger equation do not separate any more, even in the nonrelativistic case. One might think that this is completely unimportant, since in any case we consider the external field as a small perturbation and are interested
only in the terms which are linear in the external field. However, as we will see below, even in the case of vanishingly small external field one cannot ignore its influence on the proper description of the center of mass motion if recoil corrections are to be accounted for.

Let us consider two charged nonrelativistic particles which interact via the Coulomb potential in an external homogeneous magnetic field. The Lagrangian of this system has the form

\[ L = \frac{m_i \dot{r}_i^2}{2} + e_i A_i \dot{r}_i - V(r_1 - r_2), \]  

where \( A_i = H \times r_i/2 \) and \( V(r_1 - r_2) = 4\pi e_1 e_2 / |r_1 - r_2| \). The center of mass and relative coordinates are defined by the standard relations

\[ r = r_1 - r_2, \]  

\[ R = \mu_1 r_1 + \mu_2 r_2, \]  

where \( \mu_1 = m_1/(m_1 + m_2), \mu_2 = m_2/(m_1 + m_2) \). In these variables the Lagrangian has the form

\[ L = \frac{(m_1 + m_2) \dot{R}^2}{2} + \frac{m_r \dot{r}^2}{2} + (e_1 + e_2)A(R) \dot{R} \]  

\[ + \frac{e_1 \mu_2 - e_2 \mu_1}{2} ([H \times r] \dot{R} + [H \times R] \dot{r}) + (e_1 \mu_2^2 + e_2 \mu_1^2) A(r) \dot{r} - V(r). \]

Respective canonical momenta are as follows

\[ P = \frac{\partial L}{\partial \dot{R}} = (m_1 + m_2) \dot{R} + (e_1 + e_2)A(R) + (e_1 \mu_2 - e_2 \mu_1)A(r) \]  

\[ = p_1 + p_2, \]  

\[ p = \frac{\partial L}{\partial \dot{r}} = m_r \dot{r} + (e_1 \mu_2^2 + e_2 \mu_1^2) A(r) + (e_1 \mu_2 - e_2 \mu_1)A(R) \]  

\[ = \frac{\mu_2 - \mu_1}{2} P + \frac{p_1 - p_2}{2}, \]

\[ p_1 = \mu_1 P + p, \]

\[ p_2 = \mu_2 P - p, \]

and the Hamiltonian is equal to
the commutation relation (see, e.g., [1])

The Hamiltonian in eq.(15) still admits a conserved vector $P + (e_1 + e_2)A(R) + (e_1 \mu_2 - e_2 \mu_1)A(r)$,

$[\mathcal{H}, P + (e_1 + e_2)A(R) + (e_1 \mu_2 - e_2 \mu_1)A(r)] = 0$. (18)

This vector is equal to the sum of the respective conserved vectors for the free constituents $P + (e_1 + e_2)A(R) + (e_1 \mu_2 - e_2 \mu_1)A(r) = p_1 + e_1 A(r_1) + p_2 + e_2 A(r_2)$, and it is conserved because addition to the free Hamiltonian of the Coulomb potential which depends only on the relative distance does not break conservation of this sum.

Now it is clear that the proper description of the center of mass motion may be achieved with the help of a unitary transformation which transforms the conserved vector into the one corresponding to a free charged particle $P + (e_1 + e_2)A(R) + (e_1 \mu_2 - e_2 \mu_1)A(r) \rightarrow P + \ldots$
\((e_1 + e_2)A(R)\). This recipe was suggested in \([\text{I}]\), and the respective unitary transformation has the form

\[
U = e^{-i(e_1\mu_2 - e_2\mu_1)A(r)R}.
\] (19)

The unitary transformed Hamiltonian

\[
H' = U^{-1}HU = \frac{(P - (e_1 + e_2)A(R))^2}{2(m_1 + m_2)} + \frac{(p - (e_1\mu_2^2 + e_2\mu_1^2)A(r))^2}{2m_r}
\] (20)

\[
+ 2(e_1\mu_2 - e_2\mu_1)^2A(r)\frac{P - (e_1 + e_2)A(R)}{m_1 + m_2} + 2(e_1\mu_2 - e_2\mu_1)^2\frac{A^2(r)}{(m_1 + m_2)},
\]

now satisfies the commutation relation

\[
[H', P + (e_1 + e_2)A(R)] = 0,
\] (21)

characteristic for a free particle of charge \(e_1 + e_2\) in an external field (compare eq.(16)). Hence, in all calculations of the properties of the bound system one should use the unitary transformed Hamiltonian. Any perturbations which we will add below to the Hamiltonian also should be unitary transformed.

IV. RECOIL CORRECTIONS

Unitary transformation of the spin interaction Hamiltonian corresponding to the BMT equation reduces to substitution of the transformed momenta instead of respective particle velocities in eq.(4)

\[
v_i \rightarrow p'_i - e_iA(r_i) = \mu_iP - (e_1 + e_2)A(R) \pm [p - (e_1 - (e_1 + e_2)\mu_i^2)A(r)],
\] (22)

and the spin Hamiltonian acquires (say, for the first particle) the form

\[
H_{\text{spin}} = -\mu_0 \frac{g_m m_1 + (g_s - 2)(E_{p'_1} - m_1)}{E_{p'_1}^2}SH
\] (23)

\[
+ \mu_0 \frac{(g_s - 2)E_{p'_1}}{E_{p'_1}^2 + m_1}(p'_1 - e_1A(r_1))H((p'_1 - e_1A(r_1))S
\]

\[
- \mu_0 \frac{g_s m_1 + (g_s - 2)E_{p'_1}}{E_{p'_1}^2 + m_1}(S \times (p'_1 - e_1A(r_1))).
\]

Thus far we completely ignored all interaction terms between the two particles except the Coulomb term. This is a valid approximation in the nonrecoil limit, but it is easy to realize that the Breit interaction (the transverse photon exchange) immediately generates terms linear and quadratic in the mass ratio. Hence, we have to amend the Hamiltonian
above by the Breit interaction term. This term may be easily calculated in the first order
perturbation theory (see, e.g., [1]) and has the form (again after the unitary transformation)
\[\mathcal{H}_{Br} = g_s \frac{4\pi e_1 e_2}{2m_1 m_2} S[\frac{r}{r^3} \times (p' - e_2 A(r_2))] = g_s \mu_0 \frac{4\pi e_2}{m_2} \left[J_1 \frac{r}{r^3} \times (p' - e_2 A(r_2))\right],\]  
(24)

which contains a term linear in \(\mathbf{H}\)
\[\mathcal{H}_{Br} \approx g_s \mu_0 \frac{[e_2 - (e_1 + e_2)\mu_2^2]}{m_2} \mathbf{S}[\mathbf{E} \times \mathbf{A}(r)] = g_s \mu_0 \frac{4\pi e_2 [e_2 - (e_1 + e_2)\mu_2^2]}{3m_2 r} \mathbf{S}\mathbf{H}.\]  
(25)

Now we add the Breit interaction term to the spin Hamiltonian above, substitute the
center of mass and relative coordinates, go to the center of mass system, preserve only the
terms of order \((Z\alpha)^2\) and obtain
\[\mathcal{H}_{tot} = -\mu_0 (\mathbf{S}\mathbf{H}) \{g_s (1 - \frac{p^2}{2m_1^2}) + (g_s - 2) \frac{p^2}{2m_1^2} - \frac{g_s - 2}{2} \frac{p^2}{3m_1^2} - \frac{4\pi e_2 e_1}{3m_1 r} \left[\frac{g_s}{2} + \frac{g_s - 2}{2}\right] - g_s \frac{4\pi e_2 [e_2 - (e_1 + e_2)\mu_2^2]}{3m_2 r}\}.\]  
(26)

Then
\[g_{1\text{bound}} = g_s \{(1 - \frac{<p^2>}{2m_1^2}) - \left[\frac{4\pi e_2 [e_1 - (e_1 + e_2)\mu_1^2]}{6m_1} + \frac{4\pi e_2 [e_2 - (e_1 + e_2)\mu_2^2]}{3m_2}\right] < \frac{1}{r} > \} \]  
(27)

\[+(g_s - 2)\left\{\frac{<p^2>}{2m_1^2} - \frac{<p^2>}{6m_1^2} - \frac{4\pi e_2 [e_1 - (e_1 + e_2)\mu_1^2]}{6m_1}\right\} < \frac{1}{r} > \} \].

Now we use matrix elements
\[<n|\frac{1}{r}|n> = -\frac{4\pi e_1 e_2}{n^2} m_r \approx -\frac{4\pi e_1 e_2}{n^2} (1 - \frac{m_1}{m_2} + \frac{(m_1}{m_2})^2),\]  
(28)

\[<n|\frac{p^2}{m_1^2}|n> = \frac{(4\pi e_1 e_2)^2}{n^2} \mu_2^2 \approx \frac{(4\pi e_1 e_2)^2}{n^2} (1 - \frac{2m_1}{m_2} + 3\frac{(m_1}{m_2})^2),\]

and obtain
\[g_{1\text{bound}} = g_s \{(1 - \frac{(4\pi e_1 e_2)^2}{2n^2} \mu_2^2) + \frac{4\pi e_2 \mu_2 [e_1 - (e_1 + e_2)\mu_1^2]}{6} + \frac{4\pi e_2 \mu_1 [e_2 - (e_1 + e_2)\mu_2^2]}{3} \frac{4\pi e_1 e_2}{n^2}\} \]  
(29)

\[+(g_s - 2) \left\{\frac{(4\pi e_1 e_2)^2}{3n^2} \mu_2^2 + \frac{4\pi e_2 \mu_2 [e_1 - (e_1 + e_2)\mu_1^2]}{6} \frac{4\pi e_1 e_2}{n^2}\} \}.\]
Our treatment was completely symmetric with respect to both constituents, so the gyromagnetic ratio for the second particle may be obtained from the expression above by a simple substitution of indices

\[
g_{2\text{bound}} = g_s \left\{ \left( 1 - \frac{4\pi e_1 e_2}{2n^2} \right) \mu_1^2 \right\} + \frac{4\pi e_1 \mu_1 [e_2 - (e_1 + e_2)\mu_2^2]}{6} + 4\pi e_1 \mu_2 [e_1 - (e_1 + e_2)\mu_1^2] \frac{4\pi e_1 e_2}{3n^2} + (g_s - 2) \frac{(4\pi e_1 e_2)^2}{3n^2} \mu_1^2 \\
+ \frac{4\pi e_1 \mu_1 [e_2 - (e_1 + e_2)\mu_2^2]}{6} \frac{4\pi e_1 e_2}{n^2} \}.
\]  

(30)

It is interesting to consider the formulae for the gyromagnetic ratios of the constituents in the case when one particle (say, the second one) is much heavier than the other. It is physically evident that in the linear approximation in the small mass ratio one may easily obtain the binding corrections to the gyromagnetic ratios from the nonrecoil treatment in the second part above, simply by substituting reduced mass instead of the particle mass in the expressions for the matrix elements and adding the Breit contribution which is linear in the small mass ratio from the very beginning. We have checked that this simple-minded approach reproduces the formulae above with linear accuracy in the mass ratio. On the other hand naive calculation of the terms linear in the mass ratio from the two particle Hamiltonian above, ignoring the unitary transformation, leads to a wrong result. This remark emphasizes once more the importance of the unitary transformation above for the proper separation of the center of mass motion. Let us emphasize that the need for the unitary transformation emerges already in a completely nonrelativistic framework, and ignoring it leads to incorrect results even in the linear approximation in the small mass ratio.

V. CONCLUSION

In this short note we proposed a new method for calculation of the leading binding corrections to the gyromagnetic ratios of bound particles. We have confirmed the validity of the old results by using the BMT equation to obtain the magnetic interactions in the two-particle Hamiltonian. Our new approach clearly demonstrates the purely kinematic nature of the distinction between the normal and anomalous parts of the particle magnetic moment, and the independence of the leading binding corrections on the magnitude of the particle spin.

ACKNOWLEDGMENTS

We would like to thank Barry Taylor and Peter Mohr who attracted our attention to the problem discussed in this note.

M. E. is deeply grateful for the kind hospitality of the Physics Department at Penn State University, where this work was performed. The authors appreciate the support of this work by the National Science Foundation under grant number PHY-9421408.
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