A DESCRIPTION OF n-ARY SEMIGROUPS POLYNOMIAL-DERIVED FROM INTEGRAL DOMAINS

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Abstract. We provide a complete classification of the n-ary semigroup structures defined by polynomial functions over infinite commutative integral domains with identity, thus generalizing Glazek and Gleichgewicht’s classification of the corresponding ternary semigroups.

1. Introduction

Let $R$ be an infinite commutative integral domain with identity and let $n \geq 2$ be an integer. In this note we provide a complete description of all the $n$-ary semigroup structures defined by polynomial functions over $R$ (i.e., the $n$-ary semigroup structures polynomial-derived from $R$).

For any integer $k \geq 1$, let $\left[ k \right] = \{1, \ldots, k\}$. Recall that a function $f: R^n \rightarrow R$ is said to be associative if it solves the following system of $n-1$ functional equations:

$$f(x_1, \ldots, f(x_{i+n-1}), \ldots, x_{2n-1}) = f(x_1, \ldots, f(x_{i+1}, \ldots, x_{i+n}), \ldots, x_{2n-1}), \quad i \in [n-1].$$

In this case, the pair $(R, f)$ is called an $n$-ary semigroup.

The introduction of $n$-ary semigroups goes back to Dörnte [1] and led to the generalization of groups to $n$-ary groups (polyadic groups). The next extensive study on polyadic groups was due to Post [10] and was followed by several contributions towards the description of $n$-ary groups and similar structures. To mention a few, see [2–6, 8, 9, 11].

We now state our main result, which provides a description of the possible associative polynomial functions from $R^n$ to $R$. Let $\text{Frac}(R)$ denote the fraction field of $R$ and let $\mathbf{x} = (x_1, \ldots, x_n)$.

**Main Theorem.** A polynomial function $p: R^n \rightarrow R$ is associative if and only if it is one of the following functions:

1. $p(\mathbf{x}) = c$, where $c \in R$,
2. $p(\mathbf{x}) = x_1$,
3. $p(\mathbf{x}) = x_n$,
4. $p(\mathbf{x}) = c + \sum_{i=1}^{n} x_i$, where $c \in R$,
5. $p(\mathbf{x}) = \sum_{i=1}^{n} \omega^{i-1} x_i$ (if $n \geq 3$), where $\omega \in R \setminus \{1\}$ satisfies $\omega^{n-1} = 1$,
6. $p(\mathbf{x}) = -b + a \prod_{i=1}^{n} (x_i + b)$, where $a \in R \setminus \{0\}$ and $b \in \text{Frac}(R)$ satisfy $ab^{n-1} - b \in R$ and $ab^k \in R$ for every $k \in [n-1]$.

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The Main Theorem shows that the associative polynomial functions of degree greater than 1 are symmetric, i.e., invariant under any permutation of the variables.

**Example 1.** The third-degree polynomial function \( p: \mathbb{Z}^3 \to \mathbb{Z} \) defined on the ring \( \mathbb{Z} \) of integers by

\[
p(x_1, x_2, x_3) = 9x_1x_2x_3 + 3(x_1x_2 + x_2x_3 + x_3x_1) + x_1 + x_2 + x_3
\]

is associative since it is the restriction to \( \mathbb{Z} \) of the associative polynomial function \( q: \mathbb{Q}^3 \to \mathbb{Q} \) defined on the field \( \mathbb{Q} \) of rationals by

\[
q(x_1, x_2, x_3) = \frac{1}{3} + 9 \prod_{i=1}^{3} \left( x_i + \frac{1}{3} \right)
\]

The classification given in the Main Theorem was already obtained for ternary semigroups (i.e., when \( n = 3 \)) by Glazek and Gleichgewicht [7]. Surprisingly, the classification for arbitrary \( n \) remains essentially the same except for certain solutions of type \((v)\) (already mentioned in [1]), whose existence is subordinate to that of nontrivial roots of unity. Note that, when \( n \) is odd, \((v)\) always provides the solution

\[
p(x) = \sum_{i=1}^{n} (-1)^{i-1} x_i,
\]

which was the unique solution of type \((v)\) found in [7] for \( n = 3 \).

In Section 2 we give the proof of the Main Theorem. In Section 3 we analyze some properties of these \( n \)-ary semigroup structures: we show that they are medial, determine the \( n \)-ary groups defined by polynomial functions, and discuss irreducibility issues for these \( n \)-ary semigroups.

2. Technicalities and Proof of the Main Theorem

Throughout this section, with every function \( f: \mathbb{R}^n \to \mathbb{R} \) we associate \( n \) functions \( f_i: \mathbb{R}^{2n-1} \to \mathbb{R} \), \( i \in [n] \), defined by

\[
f_i(x_1, \ldots, x_{2n-1}) = f(x_1, \ldots, f(x_i, \ldots, x_{i+n-1}), \ldots, x_{2n-1}).
\]

It follows that \( f \) is associative if and only if \( f_1 = \ldots = f_n \).

It is clear that the definition of \( R \) enables us to identify the ring \( R[x_1, \ldots, x_n] \) of polynomials of \( n \) indeterminates over \( R \) with the ring of polynomial functions of \( n \) variables from \( \mathbb{R}^n \) to \( R \). Recall that, for any integer \( d \geq 0 \), a polynomial function \( p: \mathbb{R}^n \to \mathbb{R} \) of degree \( \leq d \) can be written as

\[
p(x) = \sum_{j_1, \ldots, j_n \leq d} c_{j_1, \ldots, j_n} x_1^{j_1} \cdots x_n^{j_n}, \quad c_{j_1, \ldots, j_n} \in R.
\]

For every \( i \in [n] \), we denote the degree of \( p \) in \( x_i \) by \( \deg(p, x_i) \). We also denote the degree of \( p \) by \( \deg(p) \).

**Proposition 2.** For every associative polynomial function \( p: \mathbb{R}^n \to \mathbb{R} \), we have \( \deg(p, x_i) \leq 1 \) for every \( i \in [n] \). Moreover, if \( \deg(p, x_n) = 0 \) (resp. \( \deg(p, x_1) = 0 \)), then \( p \) is either a constant or the projection on the first (resp. the last) coordinate.

**Proof.** Let \( p: \mathbb{R}^n \to \mathbb{R} \) be an associative polynomial function and let \( p_1, \ldots, p_n \) be the polynomial functions associated with \( p \) as defined in [2]. For every \( i \in [n] \), we let \( d_i = \deg(p, x_i) \). By associativity, we have

\[
d_1 = \deg(p_1, x_1) = \deg(p_1, x_1) = d_1^2, \quad i \in [n] \setminus \{1\},
\]

\[
d_n = \deg(p_n, x_{2n-1}) = \deg(p_n, x_{2n-1}) = d_n^2, \quad i \in [n] \setminus \{n\},
\]

and

\[
d_i = \deg(p_i, x_{2n-1}) = \deg(p_i, x_{2n-1}) = d_i^2, \quad i \in [n] \setminus \{1, n\}.
\]
which shows that $d_1 \leq 1$ and $d_n \leq 1$.

Again by associativity, we have
\[
d_id_{i+1} = \deg(p_i, x_n) = \deg(p_1, x_n) = d_1d_n, \quad i \in [n],
\]
\[
d_i = \deg(p_{i+1}, x_i) = \deg(p_i, x_i) = d_id_i, \quad i \in [n-1],
\]
\[
d_i = \deg(p_{i-1}, x_{n-i+1}) = \deg(p_i, x_{n+1}) = d_id_n, \quad i \in [n] \setminus \{1\}.
\]
If $d_1 = d_n = 1$, then the first set of equations shows that $d_i = 1$ for every $i \in [n]$. If $d_n = 0$, then the third set of equations shows that $p$ is of the form $p(x) = c_1x_1 + c_0$ and hence we can conclude immediately. We proceed similarly if $d_1 = 0$. \(\square\)

By Proposition 2, an associative polynomial function $p: R^n \to R$ can always be written in the form
\[
p(x) = \sum_{j_1, \ldots, j_n \in \{0, 1\}} c_{j_1, \ldots, j_n} x_1^{j_1} \cdots x_n^{j_n}, \quad c_{j_1, \ldots, j_n} \in R.
\]

Using subsets of $[n]$ instead of Boolean indexes, we obtain
\[
p(x) = \sum_{J \subseteq [n]} c_J \prod_{j \in J} x_j, \quad c_J \in R.
\]

In order to prove the Main Theorem, we only need to describe the class of associative polynomial functions of the form (3).

To avoid cumbersome notation, for every subset $S = \{j_1, \ldots, j_k\}$ of integers and every integer $m$, we set $S + m = \{j_1 + m, \ldots, j_k + m\}$. Also, for every $i \in [n]$, we let
\[
A_i = \{1, \ldots, i-1\} = [i-1],
\]
\[
B_i = \{i, \ldots, i+n-1\} = [n] + i - 1,
\]
\[
C_i = \{i+n, \ldots, 2n-1\} = [n-i] + n + i - 1,
\]
with the convention that $A_1 = C_n = \emptyset$.

**Lemma 3.** If $p: R^n \to R$ is of the form (3), then for every $i \in [n]$ the associated function $p_i: R^{2n-1} \to R$ is of the form
\[
p_i(x_1, \ldots, x_{2n-1}) = \sum_{S \subseteq [2n-1]} r_S^i \prod_{j \in S} x_j
\]
and its coefficients are given in terms of those of $p$ by
\[
r_S^i = \begin{cases} c_{J_S} \cup (i) & \text{if } S \cap B_i \neq \emptyset, \\
                      c_{J_S} \cap (i) + c_{J_S} & \text{otherwise},
\end{cases}
\]
where $J_S = (S \cap A_i) \cup ((S \cap C_i) - n + 1)$ and $K_S = (S \cap B_i) - i + 1$.

**Proof.** We first note that
\[
p(x_1, \ldots, x_{n+i-1}) = \sum_{K \subseteq [n]} c_K \prod_{k \in K} x_{k+i-1} = \sum_{K \subseteq [n]} c_K \prod_{k \in K} x_k.
\]

Then, partitioning $J \subseteq [n]$ into $J \cap A_i$, $J \cap \{i\}$, and $J \cap (C_i - n + 1)$, we obtain
\[
p_i(x_1, \ldots, x_{2n-1}) = \sum_{J \subseteq [n]} c_J \prod_{j \in J \cap A_i} x_j \prod_{j \in (J \cap \{i\}) \cap C_i} x_j \prod_{j \in (J \cap (C_i - n + 1)) \cap C_i} x_j \left( \sum_{K \subseteq [n]} c_K \prod_{k \in K} x_k \right)
\]
\[
= \sum_{J \subseteq [n]} c_J c_K \prod_{j \in J \cap A_i} x_j \prod_{j \in (J \cap \{i\}) \cap C_i} x_j \prod_{j \in (J \cap (C_i - n + 1)) \cap C_i} x_j \prod_{k \in K} x_k
\]
\[
+ \sum_{J \subseteq [n]} c_J \prod_{j \in J \cap A_i} x_j \prod_{j \in (J \cap \{i\}) \cap C_i} x_j \prod_{j \in (J \cap (C_i - n + 1)) \cap C_i} x_j.
\]
Similarly, we show that unless $c_x$ Comparing the coefficients of \( \ell \) polynomial functions.

Proof. We assume that $c_{[n]} = 0$ and prove by induction that $c_J = 0$ for every $J \subseteq [n]$ such that $|J| > 2$. Suppose that $c_J = 0$ for every $J \subseteq [n]$ such that $|J| > k$ for some $k \geq 3$. Fix $J_0 \subseteq [n]$ such that $|J_0| = k - 1$. We only need to show that $c_{J_0} = 0$.

Assume first that $\ell = \min(J_0) \leq (n + 1)/2$.

(i) Case $\ell = 1$. Let

$$S = J_0 \cup ((J_0 + n - 1) \setminus \{n\}) \subseteq [2n - 1].$$

We have $S \cap A_1 = \emptyset$, $S \cap B_1 = J_0$, and $(S \cap C_1) - n + 1 = J_0 \setminus \{1\}$. By Lemma 3, we have $r_S^1 = c_J^{2}$. Setting $m = \min((n) \setminus J_0)$ we also have $|S \cap A_m| = |A_m| = m - 1$ and $|(S \cap C_m) - n + 1| = |J_0| - (m - 1)$. Moreover, $S \cap B_m \neq \emptyset$ for otherwise we would have $J_0 = \{1\}$, which contradicts $|J_0| \geq 2$.

Thus, using Lemma 3 associativity, and the induction hypothesis, we have $r_S^m = 0$ and therefore

$$c_{J_0}^2 = r_S^1 = r_S^m = 0.$$

(ii) Case $1 < \ell \leq (n + 1)/2$. Let

$$S = (J_0 + \ell - 1) \cup ((J_0 + n - 1) \setminus \{n + \ell - 1\}) \subseteq [2n - 1].$$

We proceed as above to obtain $r_S^\ell = c_J^{2\ell}$. By associativity it is then sufficient to show that $r_S^{2\ell - 1} = 0$. Using the notation of Lemma 3 we can readily see that $|K_S^{2\ell - 1}| \geq |J_0|$. Hence by Lemma 3 we only need to show that $c_K^{2\ell - 1} = 0$. If $|K_S^{2\ell - 1}| > |J_0|$, then we conclude by using the induction hypothesis. If $|K_S^{2\ell - 1}| = |J_0|$, then we can apply case (i) since $\min(K_S^{2\ell - 1}) = 1$.

If $\ell > (n + 1)/2$, we proceed symmetrically by setting $\ell' = \max(J)$ and considering the cases $\ell' = n$ and $(n + 1)/2 \leq \ell' < n$ separately. \qed

Proposition 5. A polynomial function $p : \mathbb{R}^n \to \mathbb{R}$ of the form $\sum_{i=1}^n c_i x_i$ with $c_{[n]} = 0$ is associative if and only if it is one of the functions of types (i)–(v).

Proof. It is straightforward to see that the functions of types (i)–(v) are associative polynomial functions.

Now, by Proposition 3 the polynomial function $p$ has the form

$$p(x) = c_0 + \sum_{i=1}^n c_i x_i, \quad c_0, \ldots, c_n \in \mathbb{R}.$$

Comparing the coefficients of $x_1$ in $p_1$ and $p_2$, we obtain the equation $c_1^2 = c_1$. Similarly, we show that $c_2^2 = c_n$. If $c_1 = 0$ or $c_n = 0$, we conclude by Proposition 2. Thus we can assume that $c_1 = c_n = 1$. Comparing the coefficients of $x_i$ in $p_i$ and $p_{i-1}$ for $2 \leq i \leq n$, we obtain the equations $c_1 c_i = c_2 c_{i-1}$, or equivalently, $c_1 = c_i^2$ and $c_2 c^{-1} = 1$. Finally, since the constant term in $p_i$ is $c_0 + c_i c_0$, we must have $c_0 = 0$ unless $c_1 = \cdots = c_n$. \qed

1In fact, $m = \text{mex}_{[n]}(J_0)$, where ‘mex’ stands for the minimal excluded number, well known in combinatorial game theory.
Lemma 6. Let \( p : \mathbb{R}^n \to \mathbb{R} \) be an associative polynomial function of the form (5). If \( c_{[n]} \neq 0 \), then \( p \) is a symmetric function.

Proof. Let us first prove that \( c_J = c_{J'} \) for every \( J, J' \in [n] \) such that \( |J| = |J'| = n-1 \). Setting \( S = [2n-1] \setminus \{n\} \), we see by Lemma 3 that \( r_S^i = c_{[n]} c_{\{n\}} \setminus \{i+1\} \) for \( i \in [n] \) and we conclude by associativity.

We now proceed by induction. Suppose that \( c_J = c_{J'} \) for every \( J, J' \in [n] \) such that \( |J| = |J'| > k \) for some \( 2 \leq k \leq n - 1 \) and set \( c_{[j]} = c_J \) for every \( J \subseteq [n] \) such that \( |J| = k \). Fix \( J_0 \) such that \( |J_0| = k - 1 \) and set \( S = J_0 \cup C_1 \) and \( m = \min((n \setminus J_0) \setminus n - 1) \). By Lemma 3 and associativity we have \( c_{[m]} c_{[J_0]} r_S^r = r_S^{m+1} = c_{n-1} c_{[J_0] + 1} \). □

The interest of Lemma 6 is shown by the following obvious result.

Lemma 7. A symmetric function \( f : \mathbb{R}^n \to \mathbb{R} \) is associative if and only if the associated functions \( f_1, \ldots, f_n \) satisfy the condition \( f_1 = f_2 \).

Recall that the \( n \)-variable elementary symmetric polynomial functions of degree \( k \leq n \) are defined by

\[
P_k(x) = \sum_{K \subseteq [n], |K| = k} \prod_{i \in K} x_i.
\]

Proposition 8. A polynomial function \( p : \mathbb{R}^n \to \mathbb{R} \) such that \( \deg(p) > 1 \) is associative if and only if it is of the form

\[
p(x) = \sum_{k=0}^n c_k P_k(x),
\]

where the coefficients \( c_k \in \mathbb{R} \) satisfy the conditions

\[
c_{j+1} c_k + c_j \delta_{k,0} = c_j c_{k+1}, \quad j \in [n-1], \quad k \in [n] - 1.
\]

Proof. By Proposition 3 and Lemma 6 any associative polynomial function \( p : \mathbb{R}^n \to \mathbb{R} \) such that \( \deg(p) > 1 \) is of the form (5). By Lemma 7 such a polynomial function is associative if and only if \( p_1 = p_2 \), that is, with the notation of Lemma 6 \( r_S^r = r_S^{r+1} \) for every \( S \subseteq [2n-1] \).

Set \( j = |S \cap C_1|, k = |S \cap B_1|, j' = |S \cap A_2| + |S \cap C_2|, \) and \( k' = |S \cap B_2| \). We have either \( j' = j + 1 \) and \( k' = k + 1 \), or \( j' = j + 1 \) and \( k' = k + 1 \), or \( j' = j \) and \( k' = k \). Therefore we get the equations

\[
c_{j+1} c_k + c_j \delta_{k,0} = c_j c_{k+1}, \quad j \in [n-1], \quad k \in [n] - 1,
\]

\[
c_{j+2} c_{k-1} + c_{j+1} \delta_{k-1,0} = c_{j+1} c_k, \quad j \in [n-1], \quad k \in [n] - 1.
\]

We conclude by observing that both sets of conditions are equivalent. □

Let us now consider the special case where \( R \) is a field.

Proposition 9. Assume that \( R \) is a field. The associative polynomial functions from \( \mathbb{R}^n \) to \( \mathbb{R} \) of degree \( > 1 \) are of the form

\[
p_{a,b}(x) = -b + a \prod_{i=1}^n (x_i + b),
\]

where \( a \in R \setminus \{0\} \) and \( b \in R \).

Remark 1. The functions \( p_{a,b} \) defined in (7) can be written in several equivalent forms. It is easy to see that they are associative since so are \( p_{0,0} \) and \( p_{a,b} = \varphi \circ p_{a,0} \circ (\varphi^{-1}, \ldots, \varphi^{-1}) \) where \( \varphi(x) = x - b \).
Proof. Since the coefficient $c_n$ in (5) is nonzero by Proposition 5 we can set $a = c_n$ and $b = c_{n-1}/a$. Using equation (5) for $j = n - 1$ and $k \geq 1$, we obtain $c_k = b c_{k+1}$, that is, $c_k = ab^{n-k}$. Using again equation (6) for $j = n - 1$ and $k = 0$, we obtain $c_0 = -b + a b^n$. We conclude by observing that the function $p_{a,b}$ is associative (see remark above).

We see from the proof of Proposition 9 that the system of equations (6) has a unique solution in $\text{Frac}(R)$. Therefore we can characterize the associative polynomial functions of degree $> 1$ as the restrictions to $R$ of nonzero multiples of the product function, up to an affine transformation in $\text{Frac}(R)$.

**Proposition 10.** Any associative polynomial function $p: R^n \to R$ such that $\deg(p) > 1$ is of type (vi).

3. Further properties

We now investigate a few properties of the semigroup structures that we have determined.

3.1. $n$-ary groups. After classifying the ternary semigroups defined by polynomial functions, Glazek and Gleichgewicht [7] determined the corresponding ternary groups. Using the Main Theorem, we can also derive a description of the $n$-ary groups defined by polynomial functions. Recall that an $n$-ary quasigroup is given by a nonempty set $G$ and an $n$-ary operation $f: G^n \to G$ such that for every $a_1, \ldots, a_n, b \in G$ and every $i \in [n]$ the equation

\[ f(a_1, \ldots, a_{i-1}, z, a_{i+1}, \ldots, a_n) = b, \]

has a unique solution in $G$. An $n$-ary group is then an $n$-ary quasigroup $(G, f)$ that is also an $n$-ary semigroup. Recall also that in an $n$-ary group, with any element $x$ is associated the element $\overline{x}$, called skew to $x$, defined by the equation $f(x, \ldots, x, \overline{x}) = x$.

**Proposition 11.** The $n$-ary groups $(R, p)$ defined by polynomial functions $p : R^n \to R$ of degree $\leq 1$ are of type (iv) with $\overline{x} = (2 - n) x - c$ and type (v) with $\overline{x} = x$.

Proof. We immediately see that the polynomials of types (i)–(iii) do not define $n$-ary groups. It is well known that the polynomials of types (iv) and (v) define $n$-ary groups.\footnote{Polynomial functions of type (v) were already considered by Dörnte [11] p. 5] in the special case of complex numbers.}

In general, the $n$-ary semigroups $(R, p)$ defined by type (vi) are not $n$-ary groups. In the special case where $R$ is a field, we have the following immediate result.

**Proposition 12.** If $R$ is a field, the $n$-ary semigroup $(R \setminus \{-b\}, p_{a,b})$, where $p_{a,b}$ is defined in [7], is an $n$-ary group. It is isomorphic to $(R \setminus \{0\}, p_{a,0})$.

3.2. Medial $n$-ary semigroup structures. We observe that all the $n$-ary semigroup structures given in the Main Theorem are medial. This is a general fact for functions of degree $\leq 1$ on a commutative ring. This is also immediate for the function $p_{a,b}$ defined in [7] because it is the restriction to $R$ of an $n$-ary operation that is isomorphic to a nonzero multiple of the $n$-ary product operation on $\text{Frac}(R)$. From this observation it follows that, for the $n$-ary groups given in Proposition 11 the map $x \mapsto \overline{x}$ is an endomorphism.
3.3. (Ir)reducibility of $n$-ary semigroup structures. Recall that if $(G, \circ)$ is a semigroup, then there is an obvious way to define an $n$-ary semigroup by $f(x_1, \ldots, x_n) = x_1 \circ \cdots \circ x_n$. In this case, the $n$-ary semigroup $(G, f)$ is said to be derived from $(G, \circ)$ or reducible to $(G, \circ)$, otherwise it is said to be irreducible. It is clear that the $n$-ary semigroups defined in types (i)–(iii) are derived from the corresponding semigroups. However, the $n$-ary semigroups defined in type (v) are not reducible. Indeed, otherwise we would have $y = y \circ 0 \circ \cdots \circ 0$ for all $y \in R$, and therefore

$$x \circ y = x \circ (y \circ 0 \circ \cdots \circ 0) = x \circ (y \circ 0) \circ 0 \circ \cdots \circ 0 = x + \omega(y \circ 0),$$

for $x, y \in R$, which leads to $x \circ y = x + \omega y + c$, where $c = \omega^2(0 \circ 0)$. We know from the Main Theorem that this function does not define a semigroup. We can prove similarly that the $n$-ary semigroups defined in type (iv) are reducible if and only if $c = (n - 1)c_0$ for $c_0 \in R$ and, when $R$ is a field, that the semigroup $(R \setminus \{0\}, p_{a,0})$ is derived from a semigroup if and only if $a = a_{0}^{n-1}$ for $a_0 \in R$.

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