Nonuniversality in quantum wires with off-diagonal disorder: a geometric point of view

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Abstract

It is shown that, in the scaling regime, transport properties of quantum wires with off-diagonal disorder are described by a family of scaling equations that depend on two parameters: the mean free path and an additional continuous parameter. The existing scaling equation for quantum wires with off-diagonal disorder [Brouwer et al., Phys. Rev. Lett. 81, 862 (1998)] is a special point in this family. Both parameters depend on the details of the microscopic model. Since there are two parameters involved, instead of only one, localization in a wire with off-diagonal disorder is not universal. We take a geometric point of view and show that this nonuniversality follows from the fact that the group of transfer matrices is not semi-simple. Our results are illustrated with numerical simulations for a tight-binding model with random hopping amplitudes.

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Universality is a key concept in any approach to study localization in disordered systems. In the scaling theory of localization, it is commonly believed that the statistical distributions of the conductance, or of energy levels and wave functions, are entirely determined by the fundamental symmetries and the dimensionality of the sample [1]. Once symmetry and dimensionality are taken into account, all microscopic details of a sample can be represented by a single length scale \( \ell \), the “mean free path”, such that on length scales \( L \) larger than \( \ell \), the sample is completely characterized by the ratio \( L/\ell \). The concept of universality is the cornerstone of various field-theoretic, diagrammatic, and random-matrix approaches to localization [1–4].

For a quasi-one-dimensional geometry, i.e., for quantum wires, and for weak disorder (mean free path \( \ell \) is much larger than the Fermi wavelength \( \lambda \)) the statistical distribution of the conductance can be described by the transfer matrix approach of Dorokhov [3], and Mello, Pereyra, and Kumar [3] (DMPK). In this approach, the transport properties of the quantum wire are described in terms of a scaling equation for its transmission eigenvalues, the so-called DMPK equation [2]. Hüffmann [7] and Caselle [8] have provided the DMPK equation with a geometric foundation by reformulating it in terms of a Brownian motion of the transfer matrix on a symmetric space, a certain curved manifold from the theory of Lie groups and Lie algebras [9]. The existence of a unique natural mathematical framework to describe Brownian motion on symmetric spaces provides the geometric counterpart of the observed universality of the localization properties of disordered quantum wires.

In a recent work, two of the authors, together with Simons and Altland [10], have proposed an extension of the scaling approach of DMPK to quantum wires with off-diagonal disorder (e.g. a lattice model with random hopping amplitudes and no on-site disorder). At the band center \( \varepsilon = 0 \) such a system has an extra chiral or sublattice symmetry that is not present in the standard case of a wire with diagonal (potential) disorder. Therefore they belong to a different symmetry class, which is referred to as the chiral symmetry class. Chiral symmetry also plays an important role for two-dimensional Dirac fermions in a random vector potential [11–13], the lattice random flux model [14,16], non-Hermitian quantum mechanics [17,18], supersymmetric quantum mechanics [19], diffusion in a random medium [20], and in certain problems in QCD [21].

In Ref. [10], the extension of the DMPK equation to the chiral symmetry class was derived from a simple microscopic model. Here we discuss its geometric origin. This proves to be an exercise with implications that reach far beyond the construction of a mere chiral parallel to the geometric foundation of the DMPK equation of Refs. [7,8]: Upon inspection of the geometric structure underlying the “chiral DMPK equation”, we find that the localization properties of a disordered wire with off-diagonal disorder are not universal; the geometric approach allows for a one-parameter family of scaling equations for the transmission eigenvalues. The scaling equation that was originally found in Ref. [10] is a special point in this family. Below we discuss the geometric approach in more detail, identify the origin of this non-universality, and illustrate the results with numerical simulations of different microscopic models.

We start with a brief summary of the ideas that lead to the standard DMPK equation and its geometric interpretation. The cornerstone of this approach are the symmetry properties of the transfer matrix \( M \) of a disordered wire. For definiteness, we focus on the lattice Anderson model [3] in a geometry of \( N \) coupled chains, see Fig. 1. In this model, the
Hamiltonian consists of nearest neighbor hopping terms and a random on-site potential. We restrict our attention to spinless particles. The wire consists of a disordered region and of two ideal leads consisting of \( N \) (uncoupled) chains. In the leads, the wave function is represented by an \( N \)-component vector \( \psi_{\pm} \) for the amplitudes of left (+) and right (−) moving waves. The wave functions to the left and right sides of the disordered sample are related by the \( 2N \times 2N \) transfer matrix \( M \) \cite{2},

\[
\begin{pmatrix}
\psi_+ \\
\psi_-
\end{pmatrix}_{\text{right}} = M \begin{pmatrix}
\psi_+ \\
\psi_-
\end{pmatrix}_{\text{left}}.
\]

In this basis of left and right movers, flux conservation implies that the transfer matrix \( M \) obeys

\[
M^\dagger \Sigma_3 M = \Sigma_3,
\]

where \( \Sigma_3 = \sigma_3 \otimes 1_N \), \( \sigma_3 \) being the Pauli matrix and \( 1_N \) the \( N \times N \) unit matrix. We distinguish between the presence and absence of time-reversal symmetry, labeled by the symmetry parameter \( \beta = 1, 2 \), respectively. In the presence of time-reversal symmetry, \( M \) further satisfies

\[
\Sigma_1 M^* \Sigma_1 = M,
\]

where \( \Sigma_1 = \sigma_1 \otimes 1_N \). The transfer matrix \( M \) can be parametrized as \cite{2,22}

\[
M = \begin{pmatrix}
U & 0 \\
0 & U'
\end{pmatrix} \begin{pmatrix}
\cosh x & \sinh x \\
\sinh x & \cosh x
\end{pmatrix} \begin{pmatrix}
V & 0 \\
0 & V'
\end{pmatrix},
\]

where \( U, U', V, \) and \( V' \) are unitary matrices, and \( x \) is a diagonal matrix with diagonal elements \( x_j \). For \( \beta = 1 \), \( V' = V^* \) and \( U' = U^* \). The unitary matrices \( U, U', V, \) and \( V' \) serve as “angular coordinates” for \( M \), the parameters \( x_j \) serve as “radial coordinates” [the eigenvalues of \( MM^\dagger \) are \( \exp(\pm 2x_j) \)]. The radial coordinates \( x_j \) are related to the transmission eigenvalues \( T_j \) by \( T_j = 1/\cosh^2 x_j \). They determine the dimensionless conductance \( g \) through the Landauer formula,
\[ g = \sum_{j=1}^{N} T_j = \sum_{j=1}^{N} \frac{1}{\cosh^2 x_j}. \]  

(5)

In the original derivations of the DMPK equation \[1,2\], the wire is divided into thin slices, and the transfer matrix \( M \) of the entire wire is found by multiplication of the transfer matrices of the individual slices. For weak disorder, the parameters \( x_j \) of the transfer matrix undergo only small changes \( x_j \rightarrow x_j + \delta x_j \) upon each such multiplication. One can view this process as a “Brownian motion” for the parameters \( x_j \), the length \( L \) of the wire serving as a fictitious time. With the choice of a maximum information entropy distribution for the transfer matrix of the slice \[3\], one can write down the corresponding Fokker-Planck equation, which has the form

\[
\frac{\partial P}{\partial L} = D \sum_j \frac{\partial}{\partial x_j} J \frac{\partial}{\partial x_j} J^{-1} P,
\]

\[ J = \prod_j \sinh(2x_j) \prod_{j<k} \sinh(\beta(x_j - x_k)) \sinh(\beta(x_j + x_k)). \]

(6)

The proportionality constant \( D \) is determined by the details of the microscopic model.\[4\]

Hüffmann pointed out that there is a beautiful geometric structure underlying the scaling equation (6). He observed that Eqs. (2)–(4) express that the transfer matrix \( M \) is a member of a Lie group \( G_\beta \), with

\[ G_1 \simeq Z_2 \times Sp(N, \mathbb{R}), \quad G_2 \simeq U(N, N) \simeq U(1) \times SU(N, N), \]

(7)

and proposed to describe the \( L \) evolution of \( M \) as a “Brownian motion” on the manifold \( G_\beta \). For a rigorous formulation of this Brownian motion process, two further steps have to be taken. We would like to repeat them here, as we need to reconsider them when we deal with the case of a quantum wire with off-diagonal disorder.

1. The Lie groups \( G_\beta \) are not semi-simple: they are the direct product of the two components \( Z_2 \) and \( Sp(N, \mathbb{R}) \), or \( U(1) \) and \( SU(N, N) \), for \( \beta = 1 \) or 2 respectively. For the product \( MM^\dagger \), and hence for the radial coordinates \( x_j \), only the non-compact components \( Sp(N, \mathbb{R}) \) and \( SU(N, N) \) are relevant, so that we may restrict our attention to the semi-simple Lie groups \( Sp(N, \mathbb{R}) \) and \( SU(N, N) \). The remaining components \( Z_2 \) and \( U(1) \) correspond to the sign or phase of \( \text{det} M \) and do not affect the \( x_j \).

2. The algebraic structure on the semi-simple Lie groups \( Sp(N, \mathbb{R}) \) and \( SU(N, N) \) gives rise to a natural metric [3]. However, since this natural metric is not positive definite, it cannot be used to define a Brownian motion process. The problem is

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1 In the DMPK equation, \( D^{-1} = 2(\beta N + 2 - \beta) \ell \), where \( \ell \) is the mean free path, see e.g. Ref. [2,7].

2 \( U(N, N) \) is the group of complex matrices \( M \) with \( M^\dagger \Sigma_3 M = \Sigma_3 \); \( SU(N, N) \) is its subgroup of matrices with unit determinant; \( Sp(N, \mathbb{R}) \) is the group of real \( 2N \times 2N \) matrices \( M \) that obey \( M^T \Sigma_2 M = \Sigma_2 \), and \( Z_2 = \{ \pm 1 \} \). To see that \( G_1 \simeq Z_2 \times Sp(N, \mathbb{R}) \), see e.g. Ref. [3].
solved by dividing out the maximal compact subgroups $U(N)$ or $S(U(N) \times U(N))$ for $\beta = 1$ or 2, respectively. The resulting coset spaces $S_1 = Sp(N, \mathbb{R})/U(N)$ and $S_2 = SU(N, N)/S(U(N) \times U(N))$ are called symmetric spaces and have a natural positive definite metric [9]. In the parametrization (4), the procedure of dividing out the subgroups $U(N)$ or $S(U(N) \times U(N))$ corresponds to the identification of all transfer matrices $M$, $M'$ for which the product $M^{-1}M'$ is of the form

$$M^{-1}M' = \begin{pmatrix} V & 0 \\ 0 & V' \end{pmatrix},$$

where $V$ and $V'$ are arbitrary unitary matrices ($V' = V^*$ for $\beta = 1$ and $\det V'V = 1$ for $\beta = 2$). In shifting to the symmetric spaces $S_\beta$ no information on the radial coordinates $x_j$ is lost, because they are well-defined in each equivalence class.

The symmetric spaces $S_\beta$ admit a spherical coordinate system whose radial coordinates $x_1, \ldots, x_N$ are equal to the radial coordinates $x_1, \ldots, x_N$ of the transfer matrix $M$, cf. Eq. (3). The Brownian motion of the $x_j$ is described by the radial part of the Laplace-Beltrami operator on $S_\beta$. This radial part is known from the literature [7–9], and is found to be identical to the differential operator in Eq. (3). As a result, we find that their probability distribution $P(x_1, \ldots, x_N; L)$ satisfies the Fokker-Planck equation (6), where the constant $D$ is the diffusion constant on $S_\beta$. The appearance of a single diffusion constant $D$ signifies the universality of the localization properties in disordered quantum wires.

Let us now consider a quantum wire with off-diagonal disorder (random hopping), at the band center $\varepsilon = 0$. We consider the same geometry as in Fig. 1, the only difference being that now the randomness is in the hopping amplitudes between neighboring lattice site, the on-site potential being zero everywhere. On a bipartite lattice (which is the case we consider here, cf. Fig. 1), and with only off-diagonal (hopping) randomness, localization is different from the standard case described above because of the existence of an additional symmetry, known as a sublattice or chiral symmetry [10],

$$\Sigma_1 M \Sigma_1 = M.$$  

Hence the unitary matrices in the parametrization (4) satisfy $U = U'$ and $V = V'$. Let us now consider the extension of the DMPK equation (4) that includes the chiral symmetry (9), following H"{u}ffmann’s geometric approach.

The Lie groups $G_\beta^{ch}$ of transfer matrices $M$ that obey the chiral symmetry (9) are

$$G_1^{ch} \simeq GL(N, \mathbb{R}) \simeq \mathbb{Z}_2 \times \mathbb{R}_+^+ \times SL(N, \mathbb{R}),$$

$$G_2^{ch} \simeq GL(N, \mathbb{C}) \simeq U(1) \times \mathbb{R}_+^+ \times SL(N, \mathbb{C}).$$

3 $GL(N, \mathbb{R})$ and $GL(N, \mathbb{C})$ are the multiplicative groups of $N \times N$ matrices with real and complex elements, respectively; $SL(N, \mathbb{R})$ and $SL(N, \mathbb{C})$ are their subgroups of matrices with unit determinant; $\mathbb{R}_+^+$ is the multiplicative group of the positive real numbers. The origin of these transfer matrix groups is explained below Eq. (17).
The construction of a Brownian motion process for $M$ on $G_{\beta}^{ch}$ proceeds with the same two steps as in the standard case of diagonal disorder. However, there is an important difference. Unlike the transfer matrix groups $G_{\beta}$ discussed above, the transfer matrix groups $G_{\beta}^{ch}$ for off-diagonal disorder contain two non-compact factors, $\mathbb{R}^+$ and $SL(N, \mathbb{R})$, or $\mathbb{R}^+$ and $SL(N, \mathbb{C})$, for $\beta = 1$ or 2, respectively. (This difference was noted by Zirnbauer in a field-theoretic context \[23\].) Both of them determine the radial coordinates $x_j$: The factor $\mathbb{R}^+$ describes the position of the average $\bar{x} = (x_1 + \ldots + x_N)/N$, while the special linear group $SL(N)$ is connected to the relative positions of the $x_j$. One therefore has to consider two different Brownian motion processes: One on $\mathbb{R}^+$, to describe the $L$ evolution of $\bar{x}$, and one on the symmetric space $S_{\beta}^{ch} = SL(N, \mathbb{R})/SO(N)$ or $S_{\beta}^{ch} = SL(N, \mathbb{C})/SU(N)$, to describe the $L$ evolution of the differences of the radial coordinates $x_j$. [The symmetric spaces $S_{\beta}^{ch}$ are obtained after dividing out the maximal compact subgroup of $SL(N, \mathbb{R})$ or $SL(N, \mathbb{C})$, as in the standard case.] Since these two Brownian motion processes have two different and a priori unrelated diffusion constants, one needs their ratio as an extra parameter to characterize the standard case.

In this most general case, there is a one-parameter family of Fokker-Planck equations for the distribution $P(x_1, \ldots, x_N; L)$ of the radial coordinates $x_j$. This family of Fokker-Planck equations is constructed in two steps. First, Brownian motion on $\mathbb{R}^+$ results in a simple diffusion equation for the distribution $P_\bar{x}(\bar{x}; L)$ of the average $\bar{x}$,

$$ \frac{\partial P_\bar{x}}{\partial L} = \frac{D_R}{N} \frac{\partial^2}{\partial \bar{x}^2} P_\bar{x}. \quad (10) $$

Here $D_R/N$ is the diffusion coefficient for the Brownian motion on $\mathbb{R}^+$. (We chose to write the diffusion coefficient as $D_R/N$ for later convenience; it is the diffusion coefficient that one would obtain for the average $\bar{x}$ if the $N$ variables $x_j$ would diffuse independently with diffusion coefficient $D_R$.) Second, Brownian motion on $S_{\beta}^{ch}$ is described in terms of radial coordinates $y_j$, $j = 1, \ldots, N$, with the constraint $y_1 + \ldots + y_N = 0 \[\Box\]$. They correspond to the radial coordinates $x_j$ of the transfer matrix $M$ via $y_j = x_j - \bar{x}$. Using the explicit form for the Laplace-Beltrami operator on the symmetric spaces $S_{\beta}^{ch}$, the Fokker-Planck equation for the probability distribution $P_y(y_1, \ldots, y_N; L)$ of the $y_j$ reads

$$ \frac{\partial P_y}{\partial L} = D_S \sum_{j=1}^{N} \frac{\partial}{\partial y_j} J \frac{\partial}{\partial y_j} J^{-1} P_y, \quad J = \prod_{j<k} \sinh^\beta(y_j - y_k), \quad (11) $$

where we have to restrict to the subspace $y_1 + \ldots + y_N = 0$. Hence, for the distribution $P(x_1, \ldots, x_N; L)$ of the radial coordinates $x_j = \bar{x} + y_j$ of the transfer matrix $M$ we find the Fokker-Planck equations

$$ \frac{\partial P}{\partial L} = D_{ch} \sum_{j,k=1}^{N} \frac{\partial}{\partial x_j} \left( \delta_{jk} - \frac{1 - \eta}{N} \right) J \frac{\partial}{\partial x_k} J^{-1} P, \quad J = \prod_{j<k} \sinh^\beta(x_j - x_k), \quad (12) $$

where $D_{ch} = D_S$ and $\eta = D_R/D_S$. Equation (12) contains the full one-parameter family of scaling equations that describes localization at the band center $\varepsilon = 0$ in quantum wires.
with off-diagonal disorder. The case $\eta = 1$ (i.e., equal diffusion constants $D_R$ and $D_S$) corresponds to the equation that was derived before in Ref. [10].

A solution of the scaling equation (12) in the localized regime $L \gg D\text{ch}$ can be obtained by standard methods [2]. The distribution of the radial coordinates is Gaussian, with average and variance given by

$$\langle x_j \rangle = (N + 1 - 2j) \beta L D\text{ch}, \quad \langle x_k x_j \rangle - \langle x_j \rangle \langle x_k \rangle = 2 \left( \delta_{jk} - \frac{1 - \eta}{N} \right) L D\text{ch}. \quad (13)$$

For even $N$, the distribution of the conductance $g$ is log-normal, with

$$\langle \ln g \rangle = -\beta s + 2 \left[ \frac{1}{\pi} \left( 1 - 2 \frac{1 - \eta}{N} \right) \right]^{1/2} s^{1/2} + O(1),$$

$$\text{var} \ln g = 2 \left[ 1 + \left( 1 - \frac{2}{\pi} \right) \left( 1 - 2 \frac{1 - \eta}{N} \right) \right] s + O(1), \quad (14)$$

where $s = 2L D\text{ch}$. For odd $N$, the fluctuations of $\ln G$ are of the same order as the average,

$$\langle \ln g \rangle = -2 \left[ \frac{2}{\pi} \left( 1 - \frac{1 - \eta}{N} \right) \right]^{1/2} s^{1/2} + O(1),$$

$$\text{var} \ln g = 4 \left( 1 - \frac{2}{\pi} \right) \left( 1 - \frac{1 - \eta}{N} \right) s + O(1). \quad (15)$$

For comparison, with diagonal disorder, $\langle \ln g \rangle = -4DL$ and $\text{var} \ln g = 8DL [2]$. Although for even $N$ the conductance distribution is log-normal both for diagonal and for off-diagonal disorder, there are some important differences: The presence of a term $\propto L^{1/2}$ in $\langle \ln g \rangle$, the $\beta$-dependence of $\langle \ln g \rangle$, and the absence of universal fluctuations of $\ln g$ (they depend on $\eta$) are special for off-diagonal disorder and do not appear in the standard case of diagonal disorder [2].

What is $\eta$ for a particular microscopic model? Although we cannot answer this question in general, we can discuss some examples of microscopic models of quantum wires with hopping disorder and compare theoretical predictions from Eq. (12) with numerical simulations. We start from the Schrödinger equation for a set of coupled chains with random hopping, which in general can be written as

$$\varepsilon \psi_n = -t_n \psi_{n+1} - t_{n-1}^\dagger \psi_{n-1}. \quad (16)$$

Here $n$ labels the position along the wire (in units of the lattice spacing), $\psi_n$ is an $N$-component wavevector, and $t_n$ is an $N \times N$ hopping matrix, see Fig. 2a. (This is a more general formulation than the one of Fig. [2]. Note that the lattice of Fig. [2] is in this class.) The hopping matrices $t_n$ are real (complex) for $\beta = 1$ (2). We connect the disordered region of length $L$ to two ideal leads for $n < 0$ and $n > L$, characterized by $t_n = 1$. For zero energy it is possible to solve for the transfer matrix explicitly,

$$M(L) = \frac{1}{2} \left( \begin{array}{cc} m^\dagger + m^{-1} & m^\dagger m^{-1} \\ m^\dagger m^{-1} & m^\dagger + m^{-1} \end{array} \right), \quad m = \prod_{n=1}^{L/2} (t_{2n-1} t_{2n}^\dagger - 1). \quad (17)$$
FIG. 2. (a) Most general random hopping chain corresponding to the Schrödinger Equation (16) for $N = 3$. (b) Random flux model: A square lattice where each plaquette is threaded by a random flux $\Phi$. 

(For convenience, we assumed that $L$ is even; recall that we use a basis of left and right movers.) The product $m^\dagger m$ has eigenvalues $\exp(2x_j)$, $j = 1, \ldots, N$; the product $m^{-1}m^{-1\dagger}$ has eigenvalues $\exp(-2x_j)$. Being an arbitrary $N \times N$ matrix with real or complex elements, the matrix $m$ is an element of the linear group $GL(N, \mathbb{R})$ or $GL(N, \mathbb{C})$, and, taking into account the considerations outlined above, the $L$ dependence of the transfer matrix $M$ can be described by the random trajectory of the matrix $m$ in $GL(N, \mathbb{R})$ or $GL(N, \mathbb{C})$.

We now consider two examples. First, we consider the microscopic model used in Ref. [10] (and also Ref. [24]). There, a distribution for the $t_n$ was assumed that was invariant under unitary transformations of the chains,

$$t_n = \exp(W_n), \quad \langle (W_n)_{\mu\nu}(W_n)_{\rho\sigma}^* \rangle = \frac{1}{2} w^2 \beta \delta_{\mu\rho} \delta_{\nu\sigma}.$$

Here $W_n$ is a real (complex) matrix with independently and identically Gaussian distributed elements for $\beta = 1$ (2). In this case, for small $w$, the distribution of the radial coordinates $x_j$ of the transfer matrix $M$ can be found explicitly. One obtains that the diffusion rates $D_R$ and $D_S$ are equal, and the parameters $x_j$ obey a Fokker-Planck equation of the type (12) with $D_{ch} = w^2$ and $\eta = 1$ [10,24]. We mention that the most general Fokker-Planck equation (12), including the dependence on $\eta$, can be derived from Eq. (16) using $t_n = \exp(W_n)$ and a Gaussian distribution for the matrix $W_n$ that involves correlations between the diagonal elements,

$$\langle (W_n)_{\mu\nu}(W_n)_{\rho\sigma}^* \rangle = \frac{1}{2} w^2 \beta \left[ \delta_{\mu\rho} \delta_{\nu\sigma} - (1 - \eta)N^{-1} \delta_{\mu\nu} \delta_{\rho\sigma} \right].$$

As a second example, we consider the random flux model [14,16]. This model corresponds to a square lattice, where each plaquette contains a random flux, see Fig. 2b. The hopping matrices $t_n$ have the form

$$t_n = \begin{pmatrix} 1 & e^{i\phi_{1,n}} & 0 & \ldots & 0 \\ 0 & 1 & e^{i\phi_{2,n}} & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 & e^{i\phi_{N-1,n}} \\ 0 & 0 & \ldots & 0 & 1 \end{pmatrix}.$$

The phases $\phi_{j,n}$ are distributed such that the fluxes $\Phi_{j,n} = \phi_{j,n+1} - \phi_{j,n}$ are independently distributed. In order to find the distribution of the radial coordinates $x_j$ for the random flux
model, we need to compute the matrix $m$, and find the eigenvalues $\exp(2x_j)$, $j = 1, \ldots, N$, of $mm^\dagger$, see Eq. (17). We do not need to solve this problem exactly: Each of the hopping matrices $t_n$ in the random flux model has $\det t_n = 1$, which implies $\exp(2N\bar{x}) = \det mm^\dagger = 1$ for all lengths. Hence, the average radial coordinate $\bar{x}$ does not diffuse, so that $D_R = 0$. We conclude that $\eta = 0$ for the random flux model.

The reason why $D_R = 0$ or $\eta = 0$ for the random flux model is that $\det t_n = 1$ for all $n$. More generally, all random hopping models which have $\det t_n = 1$ are described by Eq. (12) with $\eta = 0$. One example of such a model is a random hopping model on a square lattice with only randomness in the transverse hopping amplitudes. In the general case, however, there will be both randomness in the transverse and in the longitudinal hopping amplitudes, so that $\eta > 0$.

To compare the Fokker-Planck equation (12) to numerical simulations, we considered the quantity

$$c(N) \equiv -\lim_{L \to \infty} \frac{\langle \ln g \rangle}{\text{var } \ln g} = \begin{cases} \frac{\beta N/2}{N+(1-2/\pi)(N-2+2\eta)}, & \text{if } N \text{ is even,} \\ 0, & \text{if } N \text{ is odd.} \end{cases}$$

(21)

Here, we used Eqs. (14) and (15) for $\langle \ln g \rangle$ and $\text{var } \ln g$. In the standard case of on-site disorder, one has $c(N) = 1/2$ [2]. We have compared Eq. (21) to numerical simulations for a square lattice of width $N$ and length $L$, attached to perfect leads. For $\beta = 1$, the transverse hopping amplitudes are taken from a uniform distribution in $[-0.2, 0.2]$, whereas for $\beta = 2$, the transverse hopping amplitudes are complex numbers with amplitude uniformly distributed in $[0, 0.1]$ and a random phase. In both cases the longitudinal hopping amplitudes are real numbers taken uniform in $[1-w, 1+w]$. We have numerically computed the ratio $c(N)$ by taking an average over more than $2 \times 10^4$ samples. Results for $c(N)$ as a function of $w$ for $N = 2$ and $N = 4$ and for zero energy $\varepsilon = 0$ are shown in Fig. 3. In the presence of
the chiral symmetry, i.e., for zero energy, we expect that $c(N)$ depends on the details of the microscopic model (i.e., on the parameter $w$) through the parameter $\eta$. For $w = 0$, we have $\eta = 0$, and hence $c(2) = \beta/2$, $c(4) = \beta/(3 - 2/\pi)$. As the randomness $w$ in the longitudinal hopping amplitudes increases, we expect an increase of $\eta$, and hence a decrease of $c(N)$, which is confirmed by the numerical data shown in the figure. We have also shown data for energy $\epsilon = 0.2$ where the chiral symmetry is broken. In this case we find $c(N) = 1/2$ for all $w$, in agreement with the literature [2]. [The slight increase of $c(N)$ with $w$ is attributed to a breakdown of the weak disorder condition.]

Except for the case of the random-flux model and the random-hopping model with transverse random hopping, where $\eta = 0$, we do not know how to compute the parameter $\eta$ explicitly. Numerical simulations show that, typically, the parameter $\eta$ is of order unity. The effect of a nonzero $\eta$ on the conductance distribution is most pronounced for small $N$. For large $N$, the effect is small, because only the radial coordinates near zero contribute to transport and the additional correlations between these coordinates caused by a nonzero $\eta$ decrease as $1/N$. Therefore, in the limit of large $N$, we still expect a universal conductance distribution. Here, we would like to remark that in Ref. [24], the Fokker-Planck equation (12) with $\eta = 1$ was used to describe localization in the random-flux model. Despite the fact that the random flux model corresponds to $\eta = 0$, excellent agreement was found between the theory and numerical simulations performed for $N = 15$ and up.

Before concluding, we would like to make two remarks. First, it is known that universality of one-parameter scaling breaks down in (quasi-) one-dimensional disordered systems if there exist long-range correlations in the disorder, for instance in periodic-on-average systems [25] or for random-hopping chains in the presence of a staggering of the hopping parameter [10], as is relevant e.g., for narrow gap semiconductors [26] or charge-density wave materials [27]. (In the latter case, the staggering of the hopping parameter adds a drift term to the Brownian motion of $\vec{x}$, whereas the Brownian motion of the relative positions of the $x_j$ remains unaffected [10].) The nonuniversality that we discuss in this paper is of quite a different origin: It follows directly from the geometric structure of the transfer matrix group; no long-range correlations in the disorder are involved, all the hopping amplitudes in the examples we considered have independent distributions. It should be noted that the appearance of a one-parameter family of scaling equations also occurs in the case of random Dirac fermions in two dimensions, where one finds a line of fixed points, rather than a single fixed point [11,13].

Our second remark is of a more mathematical nature. Altland and Zirnbauer [28,29] and Caselle [8] have argued that Cartan’s classification of all symmetric spaces [9] offers a complete classification of all possible random-matrix theories. These random-matrix theories appear in triplets: distributions of eigenvalues of Hermitian matrices (i.e., energy levels), distributions of eigenphases of unitary matrices (i.e., scattering phase shifts), and a Fokker-Planck equation for the radial eigenvalues of a transfer matrix. It is the latter kind of random-

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4 We have not shown the data points for $w = 0$ and $N = 2$ at finite energy, because in that case there is an extra reflection symmetry that is not taken into account in the standard DMPK equation.
matrix theories that we have considered here. While Cartan’s classification is complete for all semi-simple transfer matrix groups, it does not take into account the phenomenon that we have presented in this paper: that for some disordered systems the transfer matrix group is not semi-simple, so that they cannot be represented by a single element from Cartan’s table.

To summarize, we have considered the Fokker-Planck equation for the transmission eigenvalues of a quantum wire with off-diagonal (hopping) disorder from a geometric point of view. Under the same assumption of weak disorder that leads to the universal DMPK equation in the standard case of diagonal (on-site) disorder [32], we have found that the Fokker-Planck equation for the transmission eigenvalues of a quantum wire with off-diagonal disorder contains an extra parameter that depends on the microscopic details of the disorder. The existence of this extra parameter leads to a non-universality of transport properties in random-hopping chains, which is most prominent if the number $N$ of coupled chains is small.

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REFERENCES

[1] P. A. Lee and T. V. Ramakrishnan, Rev. Mod. Phys. 57, 287 (1985).
[2] A. D. Stone, P. A. Mello, K. A. Muttalib, and J.-L. Pichard in Mesoscopic Phenomena in Solids, edited by B. L. Altshuler, P. A. Lee, and R. A. Webb (North Holland, Amsterdam, 1991); P. A. Mello in Mesoscopic Quantum Physics, edited by E. Akkermans, G. Montambaux, J.-L. Pichard and J. Zinn-Justin (North-Holland, Amsterdam, 1995); C. W. J. Beenakker, Rev. Mod. Phys. 69, 731 (1997).
[3] B. L. Altshuler and B. D. Simons in Mesoscopic Quantum Physics, edited by E. Akkermans, G. Montambaux, J.-L. Pichard and J. Zinn-Justin (North-Holland, Amsterdam, 1995).
[4] K. B. Efetov, Supersymmetry in Disorder and Chaos, Cambridge University Press, New York (1997).
[5] O. N. Dorokhov, Pis’ma Zh. Eksp. Teor. Fiz. 36, 259 (1982) [JETP Letters 36, 318].
[6] P. A. Mello, P. Pereyra, and N. Kumar, Ann. Phys. (NY) 181, 290 (1988).
[7] A. H"uffmann, J. Phys. A 23, 5733 (1990).
[8] M. Caselle, Phys. Rev. Lett. 74, 2776 (1995); preprint cond-mat/9610017.
[9] S. Helgason, Groups and Geometric Analysis (Academic, Orlando, 1984).
[10] P. W. Brouwer, C. Mudry, B. D. Simons, A. Altland, Phys. Rev. Lett. 81, 862 (1998).
[11] A. W. W. Ludwig, M. P. A. Fisher, R. Shankar, and G. Grinstein, Phys. Rev. B 50, 7526 (1994).
[12] A. A. Nersesyan, A. M. Tsvelik, and F. Wenger, Phys. Rev. Lett. 72, 2628 (1994); Nucl. Phys. B 438, 561 (1995).
[13] C. Mudry, C. Chamon, and X.-G. Wen, Nucl. Phys. B 466, 383 (1996).
[14] P. A. Lee and D. S. Fisher, Phys. Rev. Lett. 47, 882 (1981).
[15] J. Miller and J. Wang, Phys. Rev. Lett. 76, 1461 (1996).
[16] For an overview of recent developments see A. Furusaki, Phys. Rev. Lett. 82, 604 (1999) and references therein.
[17] H. J. Sommers, A. Crisanti, H. Sompolinsky, and Y. Stein, Phys. Rev. Lett. 60, 1895 (1988).
[18] N. Hatano and D. R. Nelson, Phys. Rev. Lett. 77, 570 (1996).
[19] A. Comtet, J. Desbois, and C. Monthus, Ann. Phys. (NY) 239, 312 (1995).
[20] J. P. Bouchaud and A. Georges, Phys. Rep. 195, 127 (1990).
[21] J. J. M. Verbaarschot, Nucl. Phys. B 53, 88 (1997).
[22] L. K. Hua, Harmonic Analysis of Functions of Several Complex Variables in the Classical Domains (Amer. Math. Soc., Providence, R.I., 1963).
[23] M. R. Zirnbauer, J. Math. Phys. 37, 4986 (1996).
[24] C. Mudry, P. W. Brouwer, and A. Furusaki, cond-mat/9903026 (to appear in Phys. Rev. B).
[25] L. I. Deych, D. Zaslavsky, and A. A. Lisyansky, Phys. Rev. Lett. 81, 5390 (1998).
[26] A. A. Ovchinnikov and N. S. Érikhman, Zh. Eksp. Teor. Fiz. 73, 650 (1977) [Sov. Phys. JETP 46, 340 (1977)].
[27] P. A. Lee, T. M. Rice, and P. W. Anderson, Phys. Rev. Lett. 31, 462 (1973).
[28] A. Altland and M. R. Zirnbauer, Phys. Rev. Lett. 76, 3420 (1996); Phys. Rev. B 55, 1142 (1997).