M-Theory Exotic Scalar Glueball Decays to Mesons at Finite Coupling

Based on arXiv:1808.01182 (with A. Misra)

Vikas Yadav

Department of Physics, IIT ROORKEE

December 11, 2018
Outline

- A general description of the UV-complete holographic type IIB dual of large-N thermal QCD *K. Dasgupta et al*[2009]
- Glueballs from M-theory metric perturbation
- Meson sector in type IIA background
- Decay widths
K. Dasgupta et al [2009]'s set up

- **Brane Construction (type IIB)**
  1. A stack of $N$ $D3$ branes at a high temperature were considered at the tip of a conifold along with a stack of $M$ $D5$ branes wrapping vanishing two cycle of the conifold. To ensure the cancelation of UV logarithmic divergence by turning off all the three form fluxes, a stack of $M$ anti-$D5$ branes wrapping the vanishing $S^2$ located at the tip, were also placed around the antipodal point relative to the location of regular $D5$ branes, on the blown-up $S^2$ of the cone.
  2. To include fundamental quarks, a stack of $N_f$ $D7$ branes were introduced via Ouyang embedding P.Ouyang [2003] in the UV dipping all the way into the IR. Again to effect UV conformality by ensuring constancy of the gauge couplings in the UV, $N_f$ anti-$D7$ branes were also placed in the UV dipping into the UV-IR interpolating region.
  3. In the UV, the color gauge group is $SU(N + M) \times SU(N + M)$, which is reduced to $SU(N + M) \times SU(N)$ for energies $\lesssim \mathcal{R}_{D5/D5}$ ($D5 - \overline{D5}$ branes’ separation).
  4. In the IR, after a Seiberg Duality cascade, the color gauge group could reduce to a strongly coupled $SU(M)$ (at finite temperature).
The type IIB supergravity solution involving resolved warped deformed conifold is given as:

\[ ds^2 = \frac{1}{\sqrt{h}} \left( -g_1 dt^2 + dx_1^2 + dx_2^2 + dx_3^2 \right) + \sqrt{h} \left[ g_2^{-1} dr^2 + r^2 d\mathcal{M}_5^2 \right], \]

where the black hole functions \( g_i \) in the above limit are of the form:

\[ g_{1,2}(r, \theta_1, \theta_2) = 1 - \frac{r_h^4}{r^4} + \mathcal{O} \left( \frac{g_s M^2}{N} \right). \]

The warp factor that includes the back-reaction is given in IR as:

\[ h = \frac{L^4}{r^4} \left[ 1 + \frac{3g_s M^2}{2\pi N} \log r \left\{ 1 + \frac{3g_s N_f}{2\pi} \left( \log r + \frac{1}{2} \right) + \frac{g_s N_f}{4\pi} \log \left( \frac{\sin \theta_1}{2} \frac{\sin \theta_2}{2} \right) \right\} \right], \]
**The ‘MQGP Limit’ and M-theory uplift**

In M. Dhuria, A. Misra[2013], the authors had considered the following limit:

\[ \text{MQGP limit : } \frac{g_s M^2}{N} \ll 1, \quad g_s N \gg 1, \text{finite } g_s, M. \]

This is due to the fact that here we will be dealing with the strongly coupled thermal system such as sQGP. Hence we not only have to consider a large t'Hooft coupling but also a finite string coupling. The finiteness of the string coupling necessitates addressing the same from an M theory perspective. This was the reason for coining the name: ‘MQGP limit’.

In M. Dhuria, A. Misra[2013], the authors first constructed the type IIA mirror of the above type IIB background via ‘delocalized’ (i.e. around some fixed values of the angular coordinates as non-zero deformation [‘\(b(r)\)’] does not permit a \(T^3\)-worth of isometries \([\exists \phi_{1,2} \rightarrow \phi_{1,2} + \text{constant, } \hat{\theta} \psi \rightarrow \psi + \text{constant}]\) ‘Strominger-Yau-Zaslow’ prescription of constructing a mirror via three T dualities and its 11-dimensional M-theory uplift.

In the K. Sil, A. Misra[2015] they chose to work around particular values of \(\theta_1\) and \(\theta_2\), given by:

\[ \theta_1 \sim \frac{\alpha \theta_1}{N^{1/5}}, \quad \theta_2 \sim \frac{\alpha \theta_2}{N^{3/10}}, \]

and \(\psi \sim 2n\pi, \, n = 0, 1, 2\) whereat the five dimensional spacetime defined by \(\{t, x_{1,2,3, r}\}\) decouples from the six dimensional internal space defined by \(\{\theta_{1,2}, \phi_{1,2}, \psi, x_{11}\}\).
Glueball are color-neutral bound states of gluons (gg, ggg, etc.). They are represented by quantum numbers $J^{PC}$, where $J$, $P$ and $C$ corresponds to angular momentum, parity and charge conjugation respectively.

Compactifying the 11-dimensional uplift along the M-theory circle, we land up at the type IIA metric, performing a double wick rotation and then compactifying again along the periodic temporal circle (with radius given by the reciprocal of the temperature), one obtains $QCD_{2+1}$ corresponding to the three non-compact directions of the black M3-brane world volume. The Type IIB background of K. Dasgupta et al [2009], in principle, involves $M_4 \times RWDC(\equiv \text{Resolved Wraped Deformed Conifold})$; asymptotically the same becomes $AdS_5 \times T^{1,1}$. 
Motivated however by, e.g., (a) asymptotically the type IIB background of K. Dasgupta et al [2009] and its delocalized type IIA mirror of M. Dhuria et al [2013] consist of $AdS_5$
and (b) gauge theory operators corresponds to the solution to the Laplace equation on $T^{1,1}$ S. Gubser [1998] (the operator $Tr F^2$ which shares the quantum numbers of the $0^{++}$ glueball couples to the dilaton and $Tr F^4$ which also shares the quantum numbers of the $0^{++}$ glueball couples to trace of metric fluctuations and the four-form potential, both in the internal angular directions),

- type IIB dilaton fluctuations, which we refer to as $0^{++}$ glueball
- type IIB complexified two-form fluctuations that couple to $d^{abc} Tr (F^a_{\mu \rho} F^b_\lambda F^c_{\rho \lambda \nu})$, which we refer to as $0^{--}$ glueball
- type IIA one-form fluctuations that couple to $Tr (F \wedge F)$, which we refer to as $0^{--}$ glueball
- M-theory metric’s scalar fluctuations which we refer to as another (lighter) $0^{++}$ glueball
- M-theory metric’s vector fluctuations which we refer to as $1^{++}$ glueball,
and
- M-theory metric’s tensor fluctuations which we refer to as $2^{++}$ glueball.
Glueballs from M-theory metric perturbations

The M-theory metric for D=11

\[ ds_{11}^2 = e^{-\frac{2\phi^{IIA}}{3}} \left[ g_{tt} dt^2 + g_{rr} 3 \left( dx^2 + dy^2 + dZ^2 \right) + g_{uu} du^2 + ds_{IIA}^2(\theta_1,2, \phi_1,2, \psi) \right] + e^{\frac{4\phi^{IIA}}{3}} \left( dx_{11} + A^{F1} + A^{F3} + A^{F5} \right)^2. \]

M-theory metric fluctuations corresponding to scalar glueball \( G_E \) with \( J^{PC} = 0^{++} \), Hashimoto et al [2009] :

\[
\begin{align*}
    h_{tt} &= -q_1(r) G_{tt}^M G_E(x^1, x^2, x^3) \\
    h_{rr} &= -q_2(r) G_{rr}^M G_E(x^1, x^2, x^3) \\
    h_{ra} &= q_3(r) G_{aa}^M \frac{\partial_a G_E(x^1, x^2, x^3)}{M^2}, \ a = 1, 2, 3 \\
    h_{ab} &= G_{ab}^M \left( q_4(r) \eta_{ab} - q_5(r) \frac{\partial_a \partial_b}{M^2} \right) G_E(x^1, x^2, x^3), \ b = 1, 2, 3 \\
    h_{11,11} &= q_6(r) G_{11,11}^M G_E(x^1 x^2 x^3)
\end{align*}
\]
Solutions for the functions $q_{i=1,2,...,6}$ VY et al[2018]

- Solution for functions $q_i$'s can be obtained by solving the EOM's obtained from the 11-D action which can be written using $\int C_3 \wedge G_4 \wedge G_4 = 0$ M. Dhuria et al [2013]

$$S_{11} = \int d^{11}x \sqrt{-\text{det}g} \left( R - \frac{1}{2 \times 4!} |G_4|^2 \right)$$

- EOM upto linear order in perturbation is given as:

$$R^{(1)}_{\hat{M}\hat{N}} = \frac{1}{12} \left( -3 G_{\hat{M}\hat{N}} \hat{P}_2 \hat{Q} R_{\hat{P}_2\hat{P}_3} + \frac{1}{3} G_{\hat{M}\hat{N}} \hat{P}_2 \hat{Q} \hat{P}_3 \hat{Q} R_{\hat{P}_2\hat{P}_3} G_{\hat{M}\hat{N}} - \frac{G^2}{12} h_{\hat{M}\hat{N}} \right)$$

Here, $\hat{M}, \hat{N}, \hat{P}_2, \hat{P}_3$ takes value from 0 to 10 while, $R^{(1)}_{\hat{M}\hat{N}}$ is perturbed part of the Ricci tensor and $G^2 = G_{\hat{A}\hat{B}\hat{C}\hat{D}} G^{\hat{A}\hat{B}\hat{C}\hat{D}}$ where, $\hat{A}, \hat{B}, \hat{C}, \hat{D}$ takes value from 0 to 10.
\[ -\frac{1}{3} a_2^3 c_1 q_1 (r - r_h)^3 + \frac{3}{2} a_2^2 c_1 q_1 (r - r_h)^2 - 4 a_2 c_1 q_1 (r - r_h) + 4 c_1 + O(r - r_h)^4. \]

\[ q_1 \]

\[ q_2 \]

\[ \alpha_4 \]

\[ -\beta_3 \]

\[ q_3 \]

\[ q_4 \]

\[ \frac{1}{2} \left( 2 \sqrt{2} c_1 q_5 - \gamma_{51} r_h - \gamma_{55} r_h - \gamma_{32} r_h \right) + \frac{1}{4} (r - r_h)^2 \left( \frac{4 \sqrt{2} c_1 q_5}{r_h^2} - \gamma_{52} - \gamma_{56} - \gamma_{33} \right) - \frac{2 \sqrt{2} c_1 q_5 (r - r_h)}{r_h} \]

\[ q_5 \]

\[ q_6 \]

\[ 0 \]

**Table:** IR Solutions
The flavor $D7$-branes are holomorphically embedded in the (desingularized) conifold geometry of the type IIB gravity dual K. Dasgupta et al [2009], P. Ouyang [2003]; A. Misra’s talk. Following closely the ideas of K. Dasgupta et al [2015], we first discuss the type IIA mirror of the Ouyang embedding.

The delocalised type IIA mirror metric of the resolved warped deformed conifold metric as M. Dhuria and A. Misra [2013], for fixed $\theta_1 = \frac{\theta_1}{N^{5}}$ in the $(\theta_2, T^3(x, y, z))$-subspace near $\theta_2 = \frac{\theta_2}{N^{10}}$ can be written as:

$$ds^2_{IIA}(\theta_2, T^3(x, y, z)) = d\theta_2 N^{\frac{7}{10}} \left( \xi_{\theta_2} \frac{\alpha_{\theta_2}^2}{\alpha_{\theta_2}^2} \sqrt{g_s} d\theta_2 + \xi_{\theta_2} y N^{-\frac{7}{20}} g_s^\frac{1}{4} dy ight) + \xi_{\theta_2} z \frac{(\log r) MN_f}{\alpha_{\theta_2}} N^{-\frac{13}{20}} g_s^\frac{1}{4} dz \right) + ds^2(T^3(x, y, z))$$

$$\xrightarrow{N \gg 1} \xi_{\theta_2} \frac{\alpha_{\theta_2}^2}{\alpha_{\theta_2}^2} \sqrt{g_s} d\theta_2^2 + ds^2(T^3(x, y, z)),$$

where the $T^3$ is formed of $\left( x, y, z; \theta_1 = \frac{\alpha_{\theta_1}}{N^{5}}, \theta_2 \sim \frac{\alpha_{\theta_2}}{N^{10}} \right)$.
The Buscher triple-T-type-IIA dual of the NS-NS $B$:

$$B^{IIA} \left( \theta_1 = \frac{\alpha_1}{N^{1/5}}, \theta_2 \sim \frac{\alpha_2}{N^{3/10}} \right) = d\theta_2 \wedge dx \ B_1(g_s, N) + d\theta_2 \wedge dy \ B_2(g_s, N) + d\theta_2 \wedge dz \ B_3(g_s, N)$$

Considering the branch of Ouyang embedding where $(\theta_1, x) = (0, 0)$ and, taking $z = z(r)$ and defining $\Sigma_0(r; g_s, N_f, N, M)$ and $\Sigma_1(r; g_s, N_f, N, M)$ as the embedding functions, the pulled back metric + NS-NS $B$ appearing the DBI action for $N_f \ D6(x^{0,1,2,3}, r, \theta_2, y)$-branes becomes:

$$\det (i^*(g + B)) = \Sigma_0(r; g_s, N_f, N, M) + \Sigma_1(r; g_s, N_f, M, N) (z'(r))^2.$$

The Euler-Lagrange EOM is:

$$\frac{d}{dr} \left( \frac{z'(r)}{\sqrt{\Sigma_0(r; g_s, N_f, N, M) + \Sigma_1(r; g_s, N_f, N, M)(z')^2}} \right) = 0.$$
Analogous to Dasgupta et al [2015], \( z = \text{constant} \) is a solution for EOM. Choosing \( z = \pm C \frac{\pi}{2} \), one can choose the \( D6/\overline{D6} \) branes to be at ‘antipodal points’ (in the toroidal analog of the spherical \( \psi \) coordinate).
Vector meson spectrum is obtained by considering gauge fluctuations of a background gauge field along the world volume of the embedded flavor D6-brane.

Using a redefinition for coordinates \((r, z) \mapsto (Y, Z) : r = r_h e^{\sqrt{Y^2 + Z^2}}, \tilde{z} = \arctan \frac{Z}{Y}\), the constant embedding of \(D6(\overline{D6})\)-branes correspond to \(z = \frac{\pi}{2}\) for \(C = 1\) for \(D6\)-branes and \(z = -\frac{\pi}{2}\) for \(C = -1\) for \(\overline{D6}\)-branes, both corresponding to \(Y = 0\).

Turning on a gauge field fluctuation \(\tilde{F} \frac{\sigma^3}{2}\) about a small background gauge field \(F_0 \frac{\sigma^3}{2}\) and the background \(i^*(g + B)\):

\[
\text{Str} \sqrt{\det_{\mathbb{R}^1,3,Z,\theta_2,y} \left( i^*(G + B) + (F_0 + \tilde{F}) \frac{\sigma^3}{2} \right) } \bigg|_{Y=0} \delta \left( \theta_2 - \frac{\alpha \theta_2}{N^{3/10}} \right) \\
= \sqrt{\det_{\theta_2,y} (i^*(g + B))} \text{Str} \sqrt{\det_{\mathbb{R}^1,3,Z} \left( i^*(g + B) + (F_0 + \tilde{F}) \frac{\sigma^3}{2} \right) } \bigg|_{Y=0} \delta \left( \theta_2 - \frac{\alpha \theta_2}{N^{3/10}} \right) \\
\times \delta \left( \theta_2 - \frac{\alpha \theta_2}{N^{3/10}} \right) .
\]
Keeping terms quadratic in $\tilde{F}$ and using KK expansion for gauge fields

$A_\mu(x^\nu, Z) = \sum_{n=1} \rho_\mu^{(n)}(x^\nu) \psi_\mu^n(Z)$ and $A_Z(x^\nu, Z) = \sum_{n=0} \pi^{(n)}(x^\nu) \phi_n(Z)$. We obtain following EOMs:

$\psi_i^m : \frac{d}{dZ} (\nu_1(Z) \dot{\psi}_m^i) + 2\nu_2(Z) M^2_{(m)} \psi_m^i = 0$;

$a = r_h \left( 0.6 + 4 \frac{g_s M^2}{N} (1 + \log r_h) \right), m = \tilde{m} \frac{r_h}{\sqrt{4 \pi g_s N}}$

while the form for the $\phi_n(Z)$ can be obtained by imposing normalization condition. Which gives $\phi_n = m_n^{-1} \dot{\psi}_n$ for all $n \geq 1$ while, for $n=0$ we get $\phi_0 = \frac{C}{\nu_1(Z)}$

$V \int dZ \nu_2(Z) \psi_n \psi_m = \delta_{nm}$

$\frac{V}{2} \int dZ \nu_1(Z) \partial_Z \psi_n \partial_Z \psi_m = m_n^2 \delta_{nm}$.

$\frac{V}{2} \int dZ \nu_1(Z) \phi_n \phi_m = \delta_{nm}$.
radial profile function for $\rho$-meson and $\pi$ meson VY et al [2018]

$$\psi_1(Z) = Z^{1/2} \left[ c_1 \psi_1 M - \frac{i \omega_1}{2 \sqrt{\omega_2}} 0 (2i \sqrt{\omega_2} Z) + c_2 \psi_1 W - \frac{i \omega_1}{2 \sqrt{\omega_2}} 0 (2i \sqrt{\omega_2} Z) \right],$$

Further for well-behaved $\psi_1'(Z)$ near $Z = 0$ one requires to set $c_2 \psi_1 = 0$. Therefore:

$$\psi_1(Z) = -\frac{c_{\psi_1} \sqrt{1/\omega_2} \omega_2 Z^2}{\sqrt{2}} - \sqrt{2} c_{\psi_1} (i \sqrt{\omega_2})^{3/2} Z + \sqrt{2} c_{\psi_1} \sqrt{i \sqrt{\omega_2}}, \quad (1)$$

and

$$\psi_1'(Z) = -\sqrt{2} c_{\psi_1} \omega_2 \sqrt{i \sqrt{\omega_2} Z} - \sqrt{2} c_{\psi_1} (i \sqrt{\omega_2})^{3/2}.$$

To satisfy Neumann boundary condition at $Z = 0$, one will hence set: $c_1 \psi_1 = c_{\psi_1} = N^{-\Omega_\psi}$, $\Omega_\psi > 1$. Also, for $b = 0.57, |\omega_2| = \mathcal{O} \left( r_h \log r_h, \frac{g_s M^2}{N} r_h(\log r_h)^2 \right) \ll 1$.

Near $Z = 0$:

$$\phi_0(Z) = \frac{C_{\phi_0}}{V_1(Z)}$$

$$\phi_0(Z \sim 0) = -\frac{0.682249 C_{\phi_0} 5 \sqrt{N} Z^2 \alpha_1 \alpha_2^2}{g_s |\log(r_h)| MN_f^2 r_h^2}.$$
Glueball Mesons interaction lagrangian

- The DBI action for D6 branes is written in terms of the 10 dimensional type-IIA metric and dilaton field. The glueball modes and dilaton field for type-IIA background were obtained in terms of 11-D M theory metric perturbations using Witten’s relation. The perturbed type-IIA field components and dilaton are given as:

\[
\begin{align*}
g_{IIA}^{tt} &= \sqrt{G_{11,11}^M} \left[ 1 + \frac{h_{11,11}}{2G_{11,11}^M} \right] G_{tt}^M + h_{tt}, \\
g_{IIA}^{rr} &= \sqrt{G_{11,11}^M} \left[ 1 + \frac{h_{11,11}}{2G_{11,11}^M} \right] G_{rr}^M + h_{rr}, \\
g_{IIA}^{ab} &= \sqrt{G_{11,11}^M} \left[ 1 + \frac{h_{11,11}}{2G_{11,11}^M} \right] G_{ab}^M + h_{ab}, \\
g_{IIA}^{ra} &= \sqrt{G_{11,11}^M} \left[ 1 + \frac{h_{11,11}}{2G_{11,11}^M} \right] G_{ra}^M + h_{ra}, \\
g_{IIA}^{yy} &= \sqrt{G_{11,11}^M} \left[ 1 + \frac{h_{11,11}}{2G_{11,11}^M} \right] G_{yy}^M, \\
g_{IIA}^{\theta_2 \theta_2} &= G_{11,11}^M \sqrt{G_{11,11}^M} \left[ 1 + \frac{3h_{11,11}}{2G_{11,11}^M} \right] A_{\theta_2 \theta_2},
\end{align*}
\]

where a, b run from 1 to 3 corresponding to the spatial part of the metric.
Substituting all the expressions for the type IIA metric components $g_{MN}^{IIA}$ and the M-theory perturbations $h_{MN}$ into the D6-brane DBI action and, keeping terms only up to linear order we get three different type of terms as:

$$\mathcal{L} \mathcal{O}_d(h^0) \mathcal{O}_\phi(h^0) \mathcal{O}_F(h) + \mathcal{L} \mathcal{O}_\phi(h^0) \mathcal{O}_F(h^0) \mathcal{O}_d(h) + \mathcal{L} \mathcal{O}_d(h^0) \mathcal{O}_F(h^0) \mathcal{O}_\phi(h).$$

Here $\mathcal{O}(h^0)$ represents term without any perturbation while $\mathcal{O}(h)$ represents term with linear order in perturbation. In both the terms subscripts $d,F,\phi$ correspond to parts of the integrand of the DBI action from which they are obtained, $\mathcal{O}_d$ corresponds to term obtained from $\sqrt{-\det(\iota^*(g+B))}$, $\mathcal{O}_\phi$ corresponds to the term $e^{-\phi}$ and, $\mathcal{O}_F$ corresponds to the term of type $g^{-1}Fg^{-1}$.

Hence, one can write the glueball-meson interaction Lagrangian up to quadratic order in magnetic fields:

$$S_{int} = Tr \mathcal{S}_{int} \int \left( \frac{1}{T_h} \right) d^3 x \left[ c_1 (\partial_\mu \pi)^2 G_E + c_2 \partial_\mu \pi \partial_\nu \pi \frac{\partial^\mu \partial^\nu}{M^2} G_E 
+c_3 \rho_\mu \rho^\mu G_E + c_4 \rho^\mu \rho_\mu \frac{\partial^\mu \partial^\nu}{M^2} G_E 
+c_5 \tilde{F}_{\mu \nu} \tilde{F}^{\mu \nu} G_E + c_6 \tilde{F}_{\mu \rho} \tilde{F}^{\nu \rho} \frac{\partial^\mu \partial^\nu}{M^2} G_E 
+c_7 \partial_\mu \pi [\pi, \rho^\mu] G_E + c_8 \partial_\mu \pi [\pi, \rho_\nu] \frac{\partial^\mu \partial^\nu}{M^2} G_E 
+c_9 (Z) \rho_\mu \tilde{F}_\nu^\mu \frac{\partial^\nu G_E}{M^2} 
+c_{10} \tilde{F}_{\mu \nu} \tilde{F}^{\mu \nu} G_E + c_{11} \partial_\mu \pi \partial^\mu \pi G_E + c_{12} \rho_\mu \rho^\mu G_E + c_{13} \partial_\mu \pi [\pi, \rho^\mu] G_E \right],$$

The couplings appearing in the DBI action after ignoring the derivatives and possible indices can be written as:

$$G_E Tr(\pi^2), \; G_E Tr(\pi, [\pi, \rho]), \; G_E Tr([\pi, \rho]^2), \; G_E Tr(\rho^2), \; G_E Tr(\rho, [\rho, \rho]), \; G_E Tr([\rho, \rho]^2).$$
At quadratic order in field strength tensor these are the only interaction terms. Terms with higher order in $\rho_{\mu}$ and $\pi$ can be obtained in the same manner by keeping higher order terms of $F$ in the DBI action. Assuming that,

$$\int_{Z=0}^{\infty} dZ = \int_{Z=0}^{\log \sqrt{3}b} dZ + \int_{\log \sqrt{3}b}^{\infty} dZ,$$

the coefficients $c_i's$ setting $q_2q_6(r) = 0$, are given as under:

$$c_1 = \int dZ \left[ e^{-2Z} \phi_0(Z)^2 \sqrt{-A_{\theta_2\theta_2}G_{11,11}^M 2G_{yy}^M - B_{\theta_2y}^2 + G_{11,11}^M G_{\theta_2y}^M} \sqrt{G_{x1x1}^M 3G_{11,11}^M 5/2 G_{rr}^M G_{rr}^M G_{rr}^M r_h^2 e^{2Z}}
\right.
\frac{G_{x1x1}^M G_{11,11}^M 7/4 G_{rr}^M r_h^2}
{- q_5(Z) m^2 - q_1(Z) - q_2(Z) + 3q_4(Z)}
\left. + 2e^{-2Z} \phi_0(Z)^2(q_2(Z) - q_4(Z) - q_6(Z)) \sqrt{-A_{\theta_2\theta_2}G_{11,11}^M 2G_{yy}^M - B_{\theta_2y}^2 + G_{11,11}^M G_{\theta_2y}^M} \sqrt{G_{x1x1}^M G_{11,11}^M 5/2 G_{xx2}^M G_{x3x}^M} \right]
\frac{G_{x1x1}^M G_{11,11}^M 7/4 G_{rr}^M r_h^2}

\frac{1}{216 \pi^2 M_5^2 \alpha_1^3 \alpha_1^2}{\alpha_2^2}
\left\{ g_s M \sqrt{\frac{1}{N}} N_f^2 e^{-4Z} \left( e^{4Z} - 1 \right) \phi_0(Z)^2 \left( 2 \sqrt{\frac{1}{N}} \alpha_2^2 + 81 \alpha_1^2 \right) \right\}
\times (\log(e^Z r_h)) \left( 72 a_r^2 r_h e^Z \log(e^Z r_h) + 3a^2 + 2r_h^2 e^{2Z} \right) M_g^2 (q_5(Z) + (q_1(Z) - q_2(Z) - q_4(Z) + 2q_6(Z)))
$$

which for $b \sim 0.6$ yields:

$$\left[ - \frac{1.16 \times 10^{-7} C_{\phi_0}^2 N^6/5 \alpha_1 \alpha_2^2 c_{1q_4}}{M N_f^2 r_h^3} + \frac{15.9759 C_{\phi_0}^2 \log(N)^5 \sqrt{N} N_f U^2 V^2 \log(r_h) (2c_{q1}^{UV} - 3.01538c_{q4}^{UV})}{\sqrt{g_s M U^2 V^2 r_h^6 \alpha_1 \alpha_2^2}} \right]$$

e tc.
We calculated the decay widths for following processes assuming $f_0[1710]$ as glueball candidate with $M_G > 2M_\rho$.

- $G_E \rightarrow 2\pi$
- $G_E \rightarrow 2\rho$
- $\rho \rightarrow 2\pi$
- $G_E \rightarrow 4\pi^0$
- $G_E \rightarrow \rho + 2\pi$
- $G_E \rightarrow \rho + 2\pi \rightarrow 2\pi$
- $G_E \rightarrow 2\rho \rightarrow 4\pi$
The decay width summed over \( a = 1, 2, 3 \) is:

\[
\Gamma_{G_E \rightarrow \pi\pi} = \frac{|2c_1 + c_2|^2 M_g^2}{32} \frac{1}{T_h^2} \left( \frac{1}{T_h} \right)^2 \times 3 \times \frac{1}{2} \approx \frac{3}{64} c_2^2 m_0^2 \pi^2 T^2
\]

which for \( b \sim 0.6 \):

\[
0.003 m_0 \left( \frac{1.834 \times 10^{-4} C_{\phi_0}^2 N_f^7/5 \alpha_\theta^3 c_1 q_4}{MN_f^2 r_h^3} + \frac{15.379 C_{\phi_0}^2 \log(N)^5 N_f^{UV^2} \log(r_h)(c_{2q_1}^{UV} - 3.015 c_{2q_4}^{UV})}{\sqrt{g_s^{UV} M^{UV^2} r_h^6 \alpha_\theta^3}} \right)
\]

\[
\equiv 0.003 m_0 \times \Lambda_{G_E \rightarrow 2\pi}.
\]

In our paper, we have assumed \( | \log r_h | = \frac{f_r h}{3} \log N, 0 < f_r h < 1 \), or equivalently \( r_h = N^{- \frac{f_r h}{3}} \). From PDG-2018, the 2\( \pi \)-decay width per unit mass associated with \( f_0[1710] \) is \( \sim 10^{-2} \). Therefore by a convenient choice of \( C_{\phi_0}, c_1 q_4, C_{\phi_0}^{UV}, c_{2q_1}^{UV} - 3.015 c_{2q_4}^{UV} \) : \( \Lambda_{G_E \rightarrow 2\pi} \sim 10 \) - implying a constraint on a linear combination of \( c_2^{\phi_0} c_1 q_4 \) and \( c_{2q_1}^{UV} (c_{2q_1}^{UV} - 3.015 c_{2q_4}^{UV}) \) - one obtains:

\[
\frac{\Gamma_{G_E \rightarrow 2\pi}}{m_0} = 10^{-2} - \text{clearly an exact match with the PDG-2018 results is also similarly possible.}
\]
For onshell decay for $G_E \rightarrow \rho \rho$. The differential width is given by

$$d\Gamma = \frac{1}{16\pi} \sum_{pol} |\mathcal{M}|^2 \frac{S}{m^2} d\Omega_{k_1}$$

where

$$\mathcal{M} = T \frac{1}{T_h} \epsilon_\alpha (k_1) \epsilon_\beta (k_2) (A\eta^{\alpha\beta} + B^{\alpha\beta})$$

$$\sum_{pol} = T^2 \left( \frac{1}{T_h} \right)^2 \frac{m^4}{4M_\rho^4} \left( A^2 \lambda (M_\rho^2, M_\rho^2, m^2) + 8A^2 \frac{M_\rho^4}{m^4} + 4 \frac{M_\rho^4}{m^4} X(k_1, k_2, m, B) \right),$$

For $b \sim 0.6$, the expression will be dominated by the $c_6^2$, $Ac_6$ and $c_4c_6$ terms (if $M_g = 2M_\rho + \epsilon$, $0 < \epsilon \ll M_\rho$ then the $c_4^2$ term will be further suppressed). Demanding $\Gamma_{G_E\rightarrow 2\rho} = \Gamma_{G_E\rightarrow 4\pi}$ for $M_g > 2M_\rho$ Brunner et al[2015], would require $c_4 = \frac{3}{4} c_6 M_g^2$; so for $M_g = m_0$ MeV $\equiv m_0 \left( \frac{r_h}{\pi \sqrt{4\pi g_s N}} \right)$, $c_4 = \frac{3 m_0^2}{4} c_6 \left( \frac{r_h}{\pi \sqrt{4\pi g_s N}} \right)^2$. 

$G_E \rightarrow 2\rho$ VY et al[2018]
The relevant interaction term in the action and decay width is given by:

\[ c_{16} T \left( \frac{1}{T_h} \right) \int d^3x \partial_\mu \pi [\pi, \rho^\mu], \]

\[ \Gamma_{\rho \to 2\pi} = T^{-2} \left( \frac{1}{T_h} \right)^2 \frac{c_{16}^2}{2}. \]

where:

\[ c_{16} = \frac{5.61 \times 10^{-9} C_{\phi_0}^2 \frac{4}{\sqrt{2}} \omega_2 \alpha_1 \alpha_2^2 c_{\psi_1} N^{-\frac{3}{3} + \frac{5}{3}}}{g_s M N_f^2} \]

\[ - \frac{43017.7 C_{\phi_0}^{UV} 2 f_r g_s^{UV} M^{UV} \frac{5}{\sqrt{N} f_r} N^{UV^2} \log(N) C_{\psi_1}^{UV} N^{-\frac{2 f_r h}{3}}}{\alpha_1 \alpha_2^2}. \]

By demanding \( \Gamma_{\rho \to 2\pi} = 149 \text{MeV (PDG-2018)} \); replacing MeV by \( \frac{r_h}{\pi \sqrt{4 \pi g_s N}} \), this implies a constraint on \( C_{\phi_0}^2 \left( c_{\psi_1} \right) \) and \( C_{\phi_0}^{UV} 2 c_{\psi_1}^{UV} \):

\[ \left[ \frac{5.61 \times 10^{-9} C_{\phi_0}^2 \frac{4}{\sqrt{2}} \omega_2 \alpha_1 \alpha_2^2 c_{\psi_1} N^{-\frac{3}{3} + \frac{5}{3}}}{g_s M N_f^2} \right]^2 = \frac{298}{T^{-2} \left( \frac{r_h}{\sqrt{4 \pi g_s N}} \right)^3}. \]
Direct glueball decay to $4\pi^0$ s

VY et al[2108]

For coupling to four $\pi^0$ we need to expand the DBI action upto quartic order in $F_{\mu\nu}$. The action restricted to quartic order, reads

$$S = -T_{D6}(2\pi\alpha')^4 \text{Str} \int d^4x dZ d\theta_2 d\gamma \left( \theta_2 - \frac{\alpha \theta_2}{N^{3/10}} \right) \exp^{-\Phi} \sqrt{-\det(i^*(g + B))}$$

$$\times \left\{ \frac{1}{32} \text{Str} \left( g^{-1} F g^{-1} F \right) \text{Tr} \left( g^{-1} F g^{-1} F \right) - \frac{1}{8} \text{Str} \left( g^{-1} F g^{-1} F g^{-1} F \right) \right\}$$

Putting everything together and setting $q_2(Z) = q_6(Z) = 0$, one gets the following interaction Lagrangian corresponding to the direct $G_E \rightarrow 4\pi$ decay:

$$S_{int}^{G_E \rightarrow 4\pi} = T \left( \frac{1}{T_h} \right) \text{Str} \int d^2x \left( c_{14} \partial_\mu \pi \partial^\mu \pi \partial_\nu \pi \partial^\nu \pi G_E(x^{1,2,3}) + c_{15} \partial_\sigma \pi \partial^\sigma \pi \partial_\mu \pi \partial^\nu \pi \frac{\partial_\mu \partial^\nu}{M_g^2} G_E(x^{1,2,3}) \right),$$

where:

$$c_{14} = \int dZ \left[ -\frac{Ge^{4Z} \phi_0(Z)^4 \left( -\frac{q_1(Z)}{2} - \frac{q_2(Z)}{2} + \frac{3q_4(Z)}{2} - \frac{q_5(Z)}{2} \right)}{8G^M_{x^1x^1} 2G^M_{1111} 2G^M_{rr} 2r_h^4} - \frac{Ge^{4Z} \phi_0(Z)^4 (q_2(Z) - q_4(Z) - q_6(Z))}{4G^M_{x^1x^1} 2G^M_{1111} 2G^M_{rr} 2r_h^4} \right]$$

which for $b \sim 0.6$ yields:

$$= 6.219 \times 10^{-16} \frac{C_{\phi_0} 4 N^{8/5} \alpha_1^3 \alpha_2^6 c_1 q_4}{g_s^2 M^3 N_f^6 r_h^9 \log^2(r_h)}$$

$$c_{15} = \int dZ \left[ -\frac{Ge^{4Z} \phi_0(Z)^4 q_5(Z)}{4G^M_{x^1x^1} 2G^M_{1111} 2G^M_{rr} 2r_h^4} \right] = 1.35 \times 10^{-13} N^{21/20}$$

$$= \left( 4.72 \times 10^{18} \frac{C_{\phi_0} \log(N)^5 N_f UV^2 \log(r_h)(c_{2 q_1} UV - 3.01538 c_{2 q_4} UV)}{\sqrt{g_s^U N^17/20 M^U V^2 r_h^8 \alpha_1^2 \alpha_2^2}} \right) - \frac{8.31 \times 10^{13} C_{\phi_0} 4 N^{11/20} \alpha_1^3 \alpha_2^6 c_1 q_4}{g_s^2 M^3 N_f^6 r_h^9 \log^2(r_h)}. $$
For $f_0[1710] (M_g > 2M_\rho)$:

$$\frac{\Gamma_{GE \to 4\pi^0}}{m_0} \sim 10^{17} c_{15}^2 \sim 10^{-5} \left( \frac{4.72 \times 10^{18} C^{UV}\phi_0 \log(N)^5 N^{UV}_{f} \log(r_h)(c_{2q_1}^{UV} - 3.015c_{2q_4}^{UV})}{\sqrt{g_s^{UV} N^{17/20} M^{UV^2} r_h^8 \alpha \theta_1 \alpha \theta_2}} \right)$$

\[-8.31 \times 10^{13} C^{\phi_0} \frac{N^{11/20} \alpha \theta_1 \alpha \theta_2 c_{1q_4}}{g_s^2 M^3 N^{6} r_h^9 \log^2(r_h)} \right)^2.\]

Currently PDG-2018 does not have an entry against the experimental value of $\frac{\Gamma_{GE \to 4\pi^0}}{m_0}$. Let us say it is $\sim 10^{-5+\text{required}_1}$, $\text{required}_1$ could be positive or negative. This implies the following constraint:

$$\left( \frac{4.72 \times 10^{18} C^{UV}\phi_0 \log(N)^5 N^{UV}_{f} \log(r_h)(c_{2q_1}^{UV} - 3.015c_{2q_4}^{UV})}{\sqrt{g_s^{UV} N^{17/20} M^{UV^2} r_h^8 \alpha \theta_1 \alpha \theta_2}} \right)$$

\[-8.31 \times 10^{13} C^{\phi_0} \frac{N^{11/20} \alpha \theta_1 \alpha \theta_2 c_{1q_4}}{g_s^2 M^3 N^{6} r_h^9 \log^2(r_h)} \right)^2 \bigg|_{N=10^2} \sim 10^{\text{required}_1}.\]
\[
G_E \rightarrow \rho + 2\pi^{\text{VY et al}[2018]}
\]

(a) \hspace{10cm} (b)

\[
\frac{\Gamma_{G_E \rightarrow \rho + 2\pi}}{m_0} \sim \frac{\Gamma(b)}{m_0} \sim c_6^2 c_{16}^2
\]

- Assuming the experimental value for \(\frac{\Gamma_{G_E \rightarrow \rho + 2\pi}}{m_0}\) - not yet known in PDG-2018 - is \(10^{-5+\text{required}}\), (26) for \(N = 10^2\), implies the following constraint:

\[
\left( - \frac{2.23 \times 10^7 f_{\text{rh}}^2 g_s^3 M_{\text{fr}}^2 \sqrt{\omega_2 c_{\psi_1}^2 c_{\psi_4}^2 \log^2 \left( \frac{3}{N} \right)} N^{\frac{f_{\text{rh}}}{3}}}{\alpha_{\theta_1} \alpha_{\theta_2}^2} \right)
\]

\[
460099 f_{\text{rh}} \sqrt{g_s^{\text{UV}} \log(N)^5 N_f^{UV^2} \log \left( \frac{3}{N} \right) c_{2\psi_1}^{UV^2} (c_{2q_1}^{UV} - 3.0153 c_{2q_4}^{UV}) N^{\frac{8 f_{\text{rh}}}{3}} - \frac{1}{10}}
\]

\[
\left| \frac{N = 10^2}{M^{UV^2} \alpha_{\theta_1} \alpha_{\theta_2}^2} \right| = 10^{\text{required}}
\]
The relevant interaction Lagrangian is given by:

\[
S_{\text{int}} = \mathcal{T} \text{Str} \int \left( \frac{1}{T_h} \right) d^3 x \left[ c_3 \rho^2 G_E + c_4 \rho \rho \frac{\partial \rho}{\partial \rho} G + c_5 \bar{F}_{\mu \nu} \bar{F}^{\mu \nu} G_E + c_6 \bar{F}_{\rho \mu} \bar{F}_{\rho}^{\mu} \frac{\partial \mu \partial \nu}{M^2} G_E \\
+ \imath c_7 \partial_{\rho} \pi \partial_{\rho} \pi G_E + \imath c_8 \partial_{\rho} \pi \partial_{\rho} \rho \frac{\partial \rho}{\partial \rho} G_E + c_9 (Z) \rho \rho \frac{\partial \rho}{\partial \rho} G_E \\
+ c_{10} \bar{F}_{\mu \nu} \bar{F}^{\mu \nu} G_E + c_{11} \partial_{\mu} \pi \partial_{\mu} \pi G_E + c_{12} \rho \rho \frac{\partial \rho}{\partial \rho} G_E + \imath c_{13} \partial_{\rho} \pi \partial_{\rho} \pi G_E \right].
\]
For $f_0[1710] : M_g > 2M_\rho$, (b) dominates. 
So the total decay width can be approximated by

$$\frac{\Gamma_{G_{E\rightarrow 2\rho\rightarrow 4\pi}}}{m_0} \sim c_6^2 c_{16}^4$$

$$\sim 10^{-10} N^{13/5} \left( - \frac{2.23007 \times 10^7 f_{rh}^2 g_s^3 M_N^2 \sqrt{\omega_2} c_{\psi_1}^2 c_1 q_4 \log^2 \left( 3\sqrt{N} \right) N^{fr_h} \alpha_1^2 \alpha_2^2}{\alpha_1^2 \alpha_2^2} \right)$$

$$- \frac{46099.5 f_{rh} g_s^{UV} \log(N)^5 N_f^{UV} \log \left( 3\sqrt{N} \right) c_2^{UV} c_1^{UV} (c_2^{q_1}^{UV} - 3.015 c_2^{q_4}^{UV}) N^{8fr_h} \left( 3\sqrt{N} \right) N^{fr_h} \frac{1}{10} }{M^{UV} N^{8fr_h} \alpha_1^2 \alpha_2^2}$$

It is expected: $\frac{\Gamma_{G_{E\rightarrow \rho+2\pi}}}{m_0} \approx \frac{\Gamma_{G_{E\rightarrow 2\rho\rightarrow 4\pi}}}{m_0}$ F. Brunner et al[2015] we obtain the same
Exact Match with PDG for $M_G > 2M_{\rho}$

- The normalization condition for $\psi_1(Z)$ implies following quadratic constraint on $c_{\psi_1}$ and $c_{\psi_1}^{\text{UV}}$:

$$V \left( \frac{5 \times 10^{-5}}{\alpha_1 \alpha_2^2} g_s^2 MN^4/5 N^2 f \sqrt{\omega_2} (1433.3 + b^2 (-2067.37 + \omega_2))(c_{\psi_1})^2 \log r_h ight) + \frac{244.91 \log r_h g_c^{\text{UV}} M_{\text{UV}} N^4/5 N_f^{\text{UV}} c_{\psi_1}^{\text{UV}}}{\alpha_1 \alpha_2^2} = 1$$

- The normalization condition for $\phi_0(Z)$ implies following quadratic constraint on $C_{\phi_0}$ and $C_{\phi_0}^{\text{UV}}$:

$$V \left( \frac{5.51 \times 10^{-9} C_{\phi_0}^2 N^1/5 (0.03 + 0.042 b^2) \alpha_1 \alpha_2^2}{g s r_h^2 \log r_h MN_f^2} \right) + \frac{793.58 C_{\phi_0}^{\text{UV}} g_s^{\text{UV}} M_{\text{UV}} N_f^{\text{UV}} r_h^2 \log r_h}{N^1/5 \alpha_1 \alpha_2^2} = 1$$

- $c_{\phi_0}^2 c_{q_4}^2 (C_{\phi_0}^{\text{UV}} C_{\phi_0}^{\text{UV}})^2 (c_{2q_1}^{\text{UV}} - 3.015 c_{2q_4}^{\text{UV}})$ can be adjusted to reproduce the PDG value of $\Gamma_{G_E \rightarrow 2\pi}$ exactly.

- Requiring $\Gamma_{G_E \rightarrow 2\rho} = \Gamma_{G_E \rightarrow 4\pi}$ yields: $c_4 \approx \frac{3}{4} c_6 m_0^2 \left( \frac{r_h}{\pi \sqrt{4\pi g_s N}} \right)^2$
We note that the combination of constants of integration appearing in the solutions to the EOMs of \( \phi_0(Z) \), \( \psi_1(Z) \) and \( q_{1,2,3,4,5,6}(Z) \) in the IR and UV:

- involving \( C^4 \phi_0 c_1 q_4 \) and \((C \phi_0 \text{ UV} C \phi_0)^4 (c_2 q_1 \text{ UV} - 3.015 c_2 q_4 \text{ UV}) \) appearing in \( \Gamma_{\phi_0 \rightarrow 4\pi^0} \)

- involving \( C^4 \psi_1 c_1 q_4 \) and \((c_2 \psi_1 \text{ UV} c_1 \psi_1)^4 (c_2 q_1 \text{ UV} - 3.015 c_2 q_4 \text{ UV}) \) appearing in

\[ \Gamma_{\psi_1 \rightarrow \rho + 2\pi} \approx \Gamma_{\psi_1 \rightarrow 2\rho \rightarrow 4\pi} \]

can be tuned and equality of these two combinations can be effected such that one can reproduce the PDG value of

\[ \Gamma_{\phi_0 \rightarrow 4\pi^0} = \Gamma_{\psi_1 \rightarrow \rho + 2\pi} \approx \Gamma_{\psi_1 \rightarrow 2\rho \rightarrow 4\pi} \]
Thanks