Anticipating Reflected Stochastic Differential Equations

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Abstract: In this paper, we establish the existence of the solutions \((X, L)\) of reflected stochastic differential equations with possible anticipating initial random variables. The key is to obtain some substitution formula for Stratonovich integrals via a uniform convergence of the corresponding Riemann sums.

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1 Introduction and main results

Let $\sigma : \mathbb{R} \to \mathbb{R}$ be a continuous function and $B$ be an $\mathbb{R}$-valued standard Brownian motion on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0,1]}, P)$ satisfying the usual conditions. For $x \geq 0$, we consider the following stochastic differential equation on $\mathbb{R}_+ = [0, +\infty)$ with reflecting boundary condition:

$$X_t(x) = x + \int_0^t \sigma(X_s(x)) \circ dB_s + L^x_t, \quad \forall \ t \in [0, 1], \quad (1.1)$$

where $\circ$ denotes the Stratonovich integral. A pair $(X_t(x), L^x_t, t \in [0, 1])$ is called a solution to equation (1.1) if

(i) $X_0(x) = x, \ X_t(x) \geq 0$ for $t \in [0, 1]$,

(ii) $X_t(x), \ L^x_t$ are continuous and adapted to $\{\mathcal{F}_t\}_{t \in [0,1]}$,

(iii) $L^x_t$ is non-decreasing with $L^x_0 = 0$ and

$$\int_0^t \chi_{\{X_s(x) = 0\}} dL^x_s = L^x_t, \quad (1.2)$$

(iv) $(X_t(x), L^x_t)$ satisfies Eq.(1.1) almost surely for every $t \geq 0$.

There now exists a considerable body of literature devoted to the study of reflected stochastic differential equations (see [6, 9, 11, 2, 13, 5, 12] and references therein). It is well-known that Eq.(1.1) has a unique solution for any given initial value $x \geq 0$ if $\sigma$ and its derivative are Lipschitz continuous functions.

Now consider the following question: Does there still exist a pair $(X_t, L^x_t, t \in [0, 1])$ to solve Eq.(1.1) if the initial value is an arbitrary non-negative random variable $Z$ which may depend on the whole Brownian paths?

The answer is not immediately clear because one needs to deal with anticipating stochastic integration. The main purpose of the present paper is to give an affirmative answer to the above question. More precisely, our main result is the following

**Theorem 1.1.** Assume that the function $\sigma$, its derivatives $\sigma'$ and $\sigma''$ are Lipschitz continuous, and $Z$ is a nonnegative random variable. Then there is a pair $(X_t(Z), L^Z_t, t \in [0, 1])$ that solves the following anticipating reflected SDE,

$$X_t(Z) = Z + \int_0^t \sigma(X_s(Z)) \circ dB_s + L^Z_t, \quad \forall \ t \in [0, 1], \quad (1.3)$$
and satisfies

(i) $X_0(Z) = Z$, $X_t(Z) \geq 0$ for $t \in [0, 1]$,

(ii) $X_t(Z)$, $L^Z_t$ are continuous,

(iii) $L^Z_t$ is non-decreasing with $L^Z_0 = 0$ and

$$
\int_0^t \chi_{\{X_s(Z) = 0\}} dL^Z_s = L^Z_t. \tag{1.4}
$$

Where the stochastic integral in (1.3) is interpreted as anticipating Stratonovich integral. Let us now recall the definition. For any $t \in [0, 1]$, let $\pi$ denote an arbitrary partition of the interval $[0, t]$ of the form: $\pi = \{0 = t_0 < t_1 < \cdots < t_n = t\}$. Let $||\pi|| = \sup_{0 \leq k \leq n-1} \{(t_{k+1} - t_k)\}$ denote the norm of $\pi$. For a stochastic process $f = \{f_s, s \in [0, 1]\}$, we define its Riemann sums $S_\pi(f, t)$ by

$$
S_\pi(f, t) = \sum_{k=0}^{n-1} \frac{1}{t_{k+1} - t_k} \left( \int_{t_k}^{t_{k+1}} f_s ds \right) (B_{t_{k+1}} - B_{t_k}). \tag{1.5}
$$

We have the following

**Definition 1.1.** We say that a stochastic process $f = \{f_s, s \in [0, 1]\}$ such that $\int_0^1 \chi_{\{s \leq 0\}} \left| f_s \right| ds < +\infty$ a.s. is Stratonovich integrable if the family $S_\pi(f, t)$ converges in probability as $||\pi|| \to 0$. In this case the limit will be called the Stratonovich integral of the process $f$ on $[0, 1]$ and will be denoted by $\int_0^t f_s \circ dB_s$.

Let us now describe our approach. To prove Theorem 1.1, the natural idea is to replace $x$ in (1.1) by the initial random variable $Z$ and prove that the pair $(X_t(Z), L^Z_t)$ satisfies the anticipating SDE. To achieve this, the key is to establish the following substitution formula

$$
\int_0^t \sigma(X_s(x)) \circ dB_s|_{x=Z} = \int_0^t \sigma(X_s(Z)) \circ dB_s \tag{1.6}
$$

for all $t \in [0, 1]$.

To obtain (1.6), it seems that we can not apply the existing substitution formula in the literature (see [9], [10]) because the regularity of the solution $X_t(x)$ of (1.1) with respect to the initial value $x$ is not good enough to satisfy the required hypothesis. Instead, we prove (1.6) by showing the uniform convergence (w.r.t. $x$) of the corresponding Riemann Sums $S_\pi(t, x)$ (Theorem 5.1 in section 5 below ). The Garsia, Rodemich and Rumsey’s Lemma and moments estimates for one-point and two-point motions(Theorem 4.1, 4.2 in section 4
below) will play an important role. For the proof of (1.4), we need to study the continuity of the random field \( \int_0^t l(X_s(x))dL_s^x \) for any continuous function \( l(y) \) on \((0, \infty)\) with compact support.

This paper is organized as follows. Section 2 is to study the regularity of the solution \((X_t(x), L_t^x)\) of Eq.(1.1). In Section 3 we prove the continuity of functional of local times. Section 4 is to study moments estimates for one-point and two-point motions. In Section 5 we prove the uniform convergence (w.r.t.x) of the Riemann Sums \( S_\pi(t, x) \). The proof of Theorem 1.1 will be completed in Section 6.

2 Regularity of the solution \((X_t(x), L_t^x)\) of Eq.(1.1)

We first recall the deterministic Skorohod problem (see [10]).

**Definition 2.1.** Let \( y \in \{ f \in C([0, 1] \rightarrow \mathbb{R}), \ f(0) \geq 0 \} \). We will say that a pair \((x, k)\) of functions on \([0, 1]\) is a solution of the Skorohod problem associated with \( y \) if

(i) \( x_t = y_t + k_t, \ t \in [0, 1] \),

(ii) \( x_t \geq 0, \ t \in [0, 1] \),

(iii) \( k \) is increasing, continuous, \( k(0) = 0 \) and satisfies

\[
  k_t = \int_0^t \chi_{\{x_s = 0\}} dk_s.
\]

(2.1)

In this case, the function \( k \) is given by

\[
  k_t = -\inf_{s \leq t} \{(y_s \wedge 0)\}.
\]

(2.2)

**Proposition 2.1.** Assume that the function \( \sigma \) satisfies the same conditions as in Theorem 1.1, and \((X_t(x), L_t^x)\) is a solution of Eq.(1.1). Then there is a constant \( c \) such that

\[
  \mathbb{E}\{ \sup_{t \leq 1} |X_t(x) - X_t(y)|^p \} \leq \exp\{cp^2 + cp\} |x - y|^p
\]

(2.3)

for any \( x, y \in \mathbb{R}_+ \) and \( p \geq 1 \).
Proof. By Hölder inequality, we need only to prove Proposition 2.1 for $p \geq 4$. Let $a(x) := \frac{1}{2}(\sigma\sigma')(x)$ for any $x \in \mathbb{R}$, we have

$$X_t(x) = x + \int_0^t \sigma(X_s(x))dB_s + \int_0^t a(X_s(x))ds + L^x_t, \quad t \in [0, 1],$$

(2.4)

$$X_t(y) = y + \int_0^t \sigma(X_s(y))dB_s + \int_0^t a(X_s(y))ds + L^y_t, \quad t \in [0, 1].$$

(2.5)

By the reflection principle (2.2),

$$L^x_t = -\inf_{s \leq t} \{(x + \int_0^t \sigma(X_s(x))dB_s + \int_0^t a(X_s(x))ds) \land 0\}, \quad t \in [0, 1],$$

(2.6)

$$L^y_t = -\inf_{s \leq t} \{(y + \int_0^t \sigma(X_s(y))dB_s + \int_0^t a(X_s(y))ds) \land 0\}, \quad t \in [0, 1].$$

(2.7)

Thus,

$$|X_t(x) - X_t(y)| \leq |x - y| + \sup_{s \leq t} \left| \int_0^t [\sigma(X_s(x)) - \sigma(X_s(y))]dB_s \right| + |L^x_t - L^y_t|$$

$$\leq 2|x - y| + 2 \sup_{s \leq t} \left| \int_0^t [\sigma(X_s(x)) - \sigma(X_s(y))]dB_s \right|$$

$$+ 2 \int_0^t |a(X_s(x)) - a(X_s(y))|ds.$$  

(2.8)

Set $Y_t(x, y) = \sup_{s \leq t} \{|X_s(x) - X_s(y)|\}$, $\psi_t(x, y) = \|Y_t(x, y)\|_p$ for any $x, y \in \mathbb{R}_+$ and $p \geq 4$. By Burkholder (see [1]) and Hölder inequalities,

$$\| \sup_{s \leq t} \left| \int_0^t [\sigma(X_s(x)) - \sigma(X_s(y))]dB_s \right| \|_p$$

$$\leq cp^{\frac{1}{2}} \| \left( \int_0^t [\sigma(X_s(x)) - \sigma(X_s(y))]^2 ds \right)^{\frac{1}{2}} \|_p$$

$$\leq cp^{\frac{1}{2}} (\int_0^t \psi_s(x, y)^2 ds)^{\frac{1}{2}},$$

(2.9)

where we have used Lipschitz continuity of $\sigma$.

Similarly,

$$\| \int_0^t |a(X_s(x)) - a(X_s(y))|ds \|_p \leq c (\int_0^t \psi_s(x, y)^2 ds)^{\frac{1}{2}}.$$  

(2.10)
Using (2.8), (2.9) and (2.10), we get that
\[
\psi_t(x, y)^2 \leq 12|x - y|^2 + (12c^2p + 3c^2) \int_0^t \psi_s(x, y)^2 ds.
\] (2.11)

It follows from Gronwall’s lemma and (2.11) that
\[
E\{ \sup_{t \leq 1} |X_t(x) - X_t(y)|^p \} \leq \exp\{12cp^2 + 3c^2p + 3p\}|x - y|^p.
\]

Thus we complete the proof. □

In view of (2.6) and (2.7), the following is a direct consequence of Proposition 2.1.

**Proposition 2.2.** Assume that the function \(\sigma\) satisfies the same conditions as in Theorem 1.1, and \((X_t(x), L^x_t)\) is a solution of Eq. (1.1). Then there is a constant \(c\) such that
\[
E\{ \sup_{t \leq 1} |L^x_t - L^y_t|^p \} \leq \exp\{cp^2 + cp\}|x - y|^p
\] (2.12)

for any \(x, y \in \mathbb{R}_+\) and \(p \geq 1\).

Similar arguments lead to the following result.

**Proposition 2.3.** Assume that the function \(\sigma\) satisfies the same conditions as in Theorem 1.1, \((X_t(x), L^x_t)\) is a solution of Eq. (1.1). Then there is a constant \(c\) such that
\[
E\{ \sup_{0 \leq t \leq 1} |X_t(x)|^p \} \leq \exp\{cp^2 + cp\}(1 + x)^p,
\] (2.13)
\[
E\{ \sup_{0 \leq t \leq 1} |L^x_t|^p \} \leq \exp\{cp^2 + cp\}(1 + x)^p
\] (2.14)

for any \(x \in \mathbb{R}_+\) and \(p \geq 1\).

Next, we study the regularity of the solution \((X_t(x), L^x_t)\) of Eq. (1.1) w.r.t. \(t\).

**Proposition 2.4.** Assume that the function \(\sigma\) satisfies the same conditions as in Theorem 1.1, \((X_t(x), L^x_t)\) is a solution of Eq. (1.1). Then, for any \(R > 0\) and \(p \geq 1\), there exist constants \(C(p, R)\) such that, for \(s, t \in [0, 1]\),
\[
\sup_{0 \leq x \leq R} E\{ |X_t(x) - X_s(x)|^{2p} \} \leq C(p, R)|t - s|^p;
\] (2.15)
\[
\sup_{0 \leq x \leq R} E\{ |L^x_t - L^x_s|^{2p} \} \leq C(p, R)|t - s|^p.
\] (2.16)
Proof. For $0 \leq s \leq t \leq 1$, let $f(t) \equiv -x - \int_0^t \sigma(X_s(x))dB_s - \int_0^t a(X_s(x))ds$. By reflection principle,

$$L^x_t = \sup_{0 \leq u \leq t} \{f(u) \vee 0\}.$$ 

Noting that

$$L^x_t - L^x_s \leq \sup_{s \leq u \leq t} \{|f(u) - f(s)|\},$$

by Burkholder (see [1]) and Hölder inequalities, we have

$$E\{|L^x_t - L^x_s|^{2p}\} \leq c(p)E\{\sup_{s \leq u \leq t} |\int_s^u \sigma(X_v(x))dB_v|^{2p}\}$$

$$+ c(p)E\{\sup_{s \leq u \leq t} |\int_s^u a(X_v(x))dv|^{2p}\}$$

$$\leq c(p)E\left(\int_s^t \sigma(X_v(x))^2dv\right)^p$$

$$+ c(p)E\left(|\int_s^t a(X_v(x))dv|\right)^{2p}$$

$$\leq c(p)(1 + R^{2p})(|t - s|^p + |t - s|^{2p})$$

$$\leq c(p,R)|t - s|^p,$$ 

(2.18)

and this implies (2.16). Since

$$|X_t(x) - X_s(x)| \leq |\int_s^u \sigma(X_v(x))dB_v| + |\int_s^t a(X_v(x))dv| + |L^x_t - L^x_s|,$$

the inequality (2.15) follows from (2.16), Burkholder (see [1]) and Hölder inequalities. Thus we complete the proof of Proposition 2.4. \(\square\)

3 Continuity of functionals of local times

Let $(X_t(x), L^x_t)$ be a solution of Eq.(1.1). Because of Propositions 2.1-2.2 and Proposition 2.4, we may assume that $X_t(x), L^x_t$ are jointly continuous in $(t, x)$. Let $l(y)$ be a continuous function on $(0, \infty)$ with compact support. Put $F(t, x) = \int_0^t l(X_s(x))dL^x_s$ for $x \in \mathbb{R}_+$. We have the following

Proposition 3.1. Assume that the function $\sigma$ satisfies the same conditions as in Theorem 1.1, and $(X_t(x), L^x_t)$ is a solution of Eq.(1.1). Then the function $F(t, x)$ is jointly continuous in $(t, x)$. 

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Proof. Since
\[ |F(t, x) - F(s, x)| \leq \sup_{y>0} \{|l(y)|\} |L^x_t - L^x_s|, \]  
(3.1)
the function \( F(t, x) \) is continuous in \( t \) uniformly with respect to \( x \) in any compact set by Proposition 2.4 and Kolmogorov’s continuity criterion (see Theorem 1.4.1 in \[3\]). Thus, it suffices to show the continuity of \( F(t, x) \) w.r.t. \( x \) for any fixed \( t \). Let \( x_n, x \in \mathbb{R}_+ \) with \( x_n \to x \) as \( n \to +\infty \). By Propositions 2.1-2.2, and Kolmogorov’s continuity criterion (see Theorem 1.4.1 in \[3\]), we have
\[ L^x_{t_n} \to L^x_t, \quad X_t(x_n) \to X_t(x), \]  
(3.2)
uniformly in \( t \), as \( n \to +\infty \). Therefore, there exists a constant \( C \geq 1 \) such that for all \( n \geq 1 \)
\[ L^x_{t_n} \leq C + L^x_t. \]  
(3.3)
Since the function \( l(x) \) is bounded and continuous, by (3.2) and (3.3),
\[ |\int_0^t [l(X_s(x_n)) - l(X_s(x))] dL^x_{s_n}| \to 0 \]  
(3.4)
as \( n \to +\infty \). Because \( L^x_{t_n} \) and \( L^x_t \) are increasing and continuous, by (3.2), the sequence of finite measures \( dL^x_{t_n} \) on \([0, 1]\) converges weakly to the finite measure \( dL^x_t \) on \([0, 1]\). Therefore, for bounded continuous function \( l(X_s(x)) \) on \([0, 1]\), we have
\[ \lim_{n \to \infty} \int_0^t l(X_s(x)) dL^x_{s_n} = \int_0^t l(X_s(x)) dL^x_s. \]  
(3.5)
The proof of Proposition 3.1 follows from (3.4) and (3.5). \( \square \)

4 Moments estimates for one-point and two-point motions

For any \( R > 0 \) and \( x \in [0, R] \), let \( (X_t(x), L^x_t) \) be a solution of Eq.(1.1). We define \( S_\pi(t, x) \) and \( I(t, x) \) by
\[ S_\pi(t, x) := S_\pi(\sigma(X_t(x)), t), \]
\[ I(t, x) := \int_0^t \sigma'(X_s(x)) \circ dB_s = \int_0^t \sigma(X_s(x)) dB_s + \frac{1}{2} \int_0^t (\sigma')^2(X_s(x)) ds. \]
Write
\[
S_\pi(t, x) = n^{-1} \sum_{k=0}^{n-1} \frac{1}{t_{k+1} - t_k} \left( \int_{t_k}^{t_{k+1}} \sigma(X_s(x)) ds \right) (B_{t_{k+1}} - B_{t_k})
\]
\[
= \sum_{k=0}^{n-1} \sigma(X_{tk}(x)) (B_{t_{k+1}} - B_{t_k})
\]
\[
+ \sum_{k=0}^{n-1} \frac{1}{t_{k+1} - t_k} \left( \int_{t_k}^{t_{k+1}} (\sigma(X_s(x)) - \sigma(X_{tk}(x))) ds \right) (B_{t_{k+1}} - B_{t_k}).
\]
(4.1)

By Ito's formula, for \( s \geq t_k \),
\[
\sigma(X_s(x)) - \sigma(X_{tk}(x)) = \int_{t_k}^{s} \sigma'(X_u(x)) \sigma(X_u(x)) du + \frac{1}{2} \int_{t_k}^{s} \sigma''(X_u(x)) \sigma^2(X_u(x)) du.
\]
(4.2)

Thus we can write \( S_\pi(t, x) - I(t, x) \) as follows:
\[
S_\pi(t, x) - I(t, x) = A_{1\pi} + A_{2\pi} + A_{3\pi} + A_{4\pi},
\]
(4.3)

where
\[
A_{1\pi}(x) := \sum_{i=0}^{n-1} \sigma(X_{ti}(x))(B_{t_{i+1}} - B_{t_i}) - \int_0^{t_i} \sigma(X_s(x)) dB_s,
\]
\[
A_{2\pi}(x) := \sum_{i=0}^{n-1} \frac{1}{t_{i+1} - t_i} \int_{t_i}^{t_{i+1}} ds \left( \int_{t_i}^{s} \sigma'(X_u(x)) \sigma(X_u(x)) dB_u \right) (B_{t_{i+1}} - B_{t_i})
\]
\[
- \frac{1}{2} \int_0^{t_i} \sigma'(X_s(x)) \sigma(X_s(x)) ds,
\]
\[
A_{3\pi}(x) := \sum_{i=0}^{n-1} \frac{1}{t_{i+1} - t_i} \int_{t_i}^{t_{i+1}} ds \left( \int_{t_i}^{s} [(\sigma')^2(X_u(x)) \sigma(X_u(x))]ight.
\]
\[
+ \sigma''(X_u(x)) \sigma^2(X_u(x))] du \times (B_{t_{i+1}} - B_{t_i}),
\]
\[
A_{4\pi}(x) := \sum_{i=0}^{n-1} \frac{1}{t_{i+1} - t_i} \int_{t_i}^{t_{i+1}} ds \left( \int_{t_i}^{s} \sigma'(X_u(x)) dL_u \right) \times (B_{t_{i+1}} - B_{t_i}).
\]
Proposition 4.1. Assume that the function $\sigma$ satisfies the same conditions as in Theorem 1.1, $(X_t(x), L^x_t)$ is a solution of Eq.(1.1). Then for any $p \geq 2$ and $R > 0$ there exist constants $C(p, R)$ such that

$$\sup_{x \in [0, R]} E\{|S_\pi(t, x) - I(t, x)|^{2p}\} \leq C(p, R) \|\pi\|^{\frac{1}{2}p}. \quad (4.4)$$

**Proof.** In the sequel, we will use $c(p)$ to denote a generic constant which depends only on $p$ and whose value may be different from line to line. By Burkholder-Davis-Gundy inequalities, we have

$$\{E\{|A_{1\pi}(x)|^{2p}\}\}^{\frac{1}{p}} \leq c(p) \left[ E\left(\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} (\sigma(X_s(x)) - \sigma(X_{t_i}(x))|^2 \right) \right]^{\frac{1}{p}}$$

$$\leq c(p) \sum_{i=0}^{n-1} \left\{ E\left(\int_{t_i}^{t_{i+1}} (\sigma(X_s(x)) - \sigma(X_{t_i}(x))|^2 \right) \right\}^{\frac{1}{p}}$$

$$\leq c(p) \sum_{i=0}^{n-1} \left\{ \left( E\left((\sigma(X_s(x)) - \sigma(X_{t_i}(x))^{2p}\right) \right)^{\frac{1}{p}} \right. \right.$$}

where we have used Proposition 2.4. For $p \geq 1$, by Hölder inequality,

$$E\{|A_{3\pi}(x)|^{2p}\} \leq E\left(\sum_{i=0}^{n-1} \frac{1}{t_{i+1} - t_i} \int_{t_i}^{t_{i+1}} \int_{t_i}^{s} [(\sigma')^2(X_u(x))\sigma(X_u(x)) \right.$$

$$+ \sigma''(X_u(x))\sigma^2(X_u(x)) \right) \right)^p \times \left( \sum_{i=0}^{n-1} (B_{t_{i+1}} - B_{t_i})^2 \right)^p \right.$$}

$$\leq c(p) \|\pi\|^p \sup_{\pi} E\left\{ \left( \sum_{i=0}^{n-1} \sum_{i=0}^{n-1} (B_{t_{i+1}} - B_{t_i})^2 \right)^p \right\}$$

$$\leq c(p) \|\pi\|^p. \quad (4.6)$$
\[ E\{|A_{4\pi}(x)|^{2p}\} \leq E\left\{ \left( \sum_{i=0}^{n-1} \frac{1}{t_{i+1} - t_i} \int_{t_i}^{t_{i+1}} ds \int_s^{t_i} \sigma'(X_u(x))dL^x_u \right)^2 \right\}^{p} \]

\[ \times \left\{ \sum_{i=0}^{n-1} (B_{t_{i+1}} - B_{t_i})^2 \right\}^p \]

\[ \leq c(p) E\left\{ \left( \sum_{i=0}^{n-1} (L^x_{t_{i+1}} - L^x_{t_i})^2 \right)^p \times \left( \sum_{i=0}^{n-1} (B_{t_{i+1}} - B_{t_i})^2 \right)^p \right\} \]

\[ \leq c(p) E\left\{ \left( \sup_i (L^x_{t_{i+1}} - L^x_{t_i}) \right)^p \times \left( \sum_{i=0}^{n-1} (B_{t_{i+1}} - B_{t_i})^2 \right)^p \right\} \]

\[ \leq c(p) \left( E\left\{ (L^x_{t_{i+1}} - L^x_{t_i})^3 \right\} \right)^{\frac{p}{3}} \left( E\left\{ (L^x_{t_i})^3 \right\} \right)^{\frac{1}{3}} \]

\[ \times E\left\{ \left( \sum_{i=0}^{n-1} (B_{t_{i+1}} - B_{t_i})^2 \right)^p \right\} \]

\[ \leq c(p) \|\pi\|_{\mathcal{F}}^p, \]  \hspace{1cm} (4.7)

where Proposition 2.4 and Kolmogorov’s continuity criterion (see Theorem 1.4.1 in [3]) were used in the last inequality. Using Fubini Theorem, \( A_{2\pi} \) can be further written as

\[ A_{2\pi}(x) = A_{2\pi}^{(1)}(x) + A_{2\pi}^{(2)}(x) + A_{2\pi}^{(3)}(x), \]  \hspace{1cm} (4.8)

where

\[ A_{2\pi}^{(1)}(x) := \sum_{i=0}^{n-1} \frac{1}{t_{i+1} - t_i} \int_{t_i}^{t_{i+1}} (t_{i+1} - u) \left( \sigma'(X_u(x))\sigma(X_u(x)) - \sigma'(X_{t_i}(x))\sigma(X_{t_i}(x)) \right) du, \]

\[ A_{2\pi}^{(2)}(x) := -\frac{1}{2} \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \left( \sigma'(X_u(x))\sigma(X_u(x)) - \sigma'(X_{t_i}(x))\sigma(X_{t_i}(x)) \right) du, \]

\[ A_{2\pi}^{(3)}(x) := \sum_{i=0}^{n-1} \frac{1}{t_{i+1} - t_i} \left( \int_{t_i}^{t_{i+1}} (t_{i+1} - u)\sigma'(X_u(x))\sigma(X_u(x))dB_u \right) (B_{t_{i+1}} - B_{t_i}) \]

\[ - \frac{1}{t_{i+1} - t_i} \left( \int_{t_i}^{t_{i+1}} (t_{i+1} - u)\sigma'(X_u(x))\sigma(X_u(x))du \right). \]
Since $\sigma'\sigma$ is Lipschitz continuous, it follows that
\[
\{\mathbb{E}\{|A_{2\pi}^{(1)}(x)|^p\}\}^{\frac{1}{p}} \leq \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \{\mathbb{E}\{|\sigma'(X_u(x))\sigma(X_u(x)) - \sigma'(X_{t_i}(x))\sigma(X_{t_i}(x))|^p\}\}^{\frac{1}{p}} du
\]
\[
\leq c \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \{\mathbb{E}\{|X_u(x) - X_{t_i}(x)|^p\}\}^{\frac{1}{p}} du
\]
\[
\leq c \|\pi\|^{\frac{1}{p}}. \tag{4.9}
\]
Similar arguments lead to
\[
\{\mathbb{E}\{|A_{2\pi}^{(2)}(x)|^p\}\}^{\frac{1}{p}} \leq c \|\pi\|^{\frac{1}{p}}. \tag{4.10}
\]
Noting that $A_{2\pi}^{(3)}$ is a martingale. Using Burkholder-Davis-Gundy inequalities, we obtain that
\[
\{\mathbb{E}\{|A_{2\pi}^{(3)}(x)|^{2p}\}\}^{\frac{1}{p}}
\]
\[
\leq c(p) \left[ \mathbb{E}\left( \sum_{i=0}^{n-1} \left( \int_{t_i}^{t_{i+1}} (t_{i+1} - u)\sigma'(X_u(x))\sigma(X_u(x))dB_u \right) (B_{t_{i+1}} - B_{t_i}) \right) \right]^{\frac{1}{2}}
\]
\[
\leq c(p) \sum_{i=0}^{n-1} \mathbb{E}\left( \int_{t_i}^{t_{i+1}} (t_{i+1} - u)\sigma'(X_u(x))\sigma(X_u(x))dB_u \right) \|B_{t_{i+1}} - B_{t_i}\|^{2p} \tag{4.11}
\]
where Hölder inequality was used for the last inequality. Combining the estimates for $A_{2\pi}^{(1)}$, $A_{2\pi}^{(2)}$ and $A_{2\pi}^{(3)}$ together, we deduce that
\[
\{\mathbb{E}\{|A_{2\pi}(x)|^{2p}\}\}^{\frac{1}{p}} \leq c \|\pi\|^{\frac{1}{p}}. \tag{4.12}
\]
Now (4.4) follows from (4.5), (4.6), (4.7) and (4.12). The proof is complete. □
Next result is the moment estimates for the two point motions.

**Proposition 4.2.** Assume that the function \( \sigma \) satisfies the same conditions as in Theorem 1.1, \((X_t(x), L_t)\) is a solution of Eq.(1.1). Then for any \( p \geq 2 \) and \( R > 0 \) there exists constants \( C(p, R) \) and \( \beta \), independent of the partition \( \pi \), such that

\[
E\left\{ \sup_{t \in [0,1]} |S_\pi(t, x) - S_\pi(t, y)|^p \right\} \leq C(p, R)|x - y|^{2p},
\]

(4.13)

for all \( x, y \in [0, R] \).

**Proof.** Similarly as (4.3), write

\[
S_\pi(t, x) - S_\pi(t, y) = A_{1\pi}(x, y) + A_{2\pi}(x, y) + A_{3\pi}(x, y) + A_{4\pi}(x, y),
\]

(4.14)

where

\[
A_{1\pi}(x, y) := \sum_{i=0}^{n-1} (\sigma(X_{t_i}(x)) - \sigma(X_{t_i}(y)))(B_{t_{i+1}} - B_{t_i}),
\]

\[
A_{2\pi}(x, y) := \sum_{i=0}^{n-1} \left\{ \int_{t_i}^{t_{i+1}} ds \int_{t_i}^{s} (\sigma'(X_u(x))\sigma(X_u(x))
\right.
\]

\[
- \sigma'(X_u(y))\sigma(X_u(y))dB_u \left\} \times (B_{t_{i+1}} - B_{t_i}),
\]

\[
A_{3\pi}(x, y) := \sum_{i=0}^{n-1} \left\{ \frac{1}{t_{i+1} - t_i} \int_{t_i}^{t_{i+1}} ds \left\{ \int_{t_i}^{s} (\sigma'(X_u(x))\sigma(X_u(x))
\right.
\]

\[
+ \sigma''(X_u(x))\sigma^2(X_u(x)) - (\sigma')^2(X_u(y))\sigma(X_u(y))
\]

\[
- \sigma''(X_u(y))\sigma^2(X_u(y))|du \} \times (B_{t_{i+1}} - B_{t_i}) \right\},
\]

\[
A_{4\pi}(x, y) := \sum_{i=0}^{n-1} \frac{1}{t_{i+1} - t_i} \int_{t_i}^{t_{i+1}} ds \left\{ \int_{t_i}^{s} \sigma'(X_u(x))dL_u^x
\right.
\]

\[
- \int_{t_i}^{s} \sigma'(X_u(y))dL_u^y \right\} (B_{t_{i+1}} - B_{t_i})
\]

\[
= \sigma'(0) \sum_{i=0}^{n-1} \frac{1}{t_{i+1} - t_i} \int_{t_i}^{t_{i+1}} ds \left\{ \int_{t_i}^{s} dL_u^x - \int_{t_i}^{s} dL_u^y \right\} (B_{t_{i+1}} - B_{t_i}).
\]

By Burkholder-Davis-Gundy inequalities, the Lipschitz continuity of \( \sigma \) and (2.3), it follows easily that

\[
E\left\{ \sup_{t \in [0,1]} |A_{1\pi}(x, y)|^p \right\} \leq C(p, R)|x - y|^p,
\]

(4.15)
By virtue of (2.3) and the Lipschitz continuity of $\sigma'$, 

\[
\left( \mathbb{E}\left\{ \sup_{t \in [0,1]} |A_{2\pi}(x, y)|^p \right\} \right)^{\frac{1}{p}} \leq \sum_{i=0}^{n-1} \left\{ \frac{1}{t_{i+1} - t_i} \int_{t_i}^{t_{i+1}} ds \left( \mathbb{E}\left\{ \int_{t_i}^{s} (\sigma'(X_u(x))\sigma(X_u(x))) - \sigma'(X_u(y))\sigma(X_u(y)))dB_u \right) \right\} \times (B_{t_{i+1}} - B_{t_i})^2 \right\} \right)^{\frac{1}{p}} \leq \sum_{i=0}^{n-1} \left\{ \frac{1}{t_{i+1} - t_i} \int_{t_i}^{t_{i+1}} ds \left( \int_{t_i}^{s} \mathbb{E}[|X_u(x) - X_u(y)|^2] \right)^{\frac{1}{p}} du \right\} \leq c(p)|x - y|. \quad (4.16)
\]

By a similar, but simpler argument, we also found that 

\[
\left( \mathbb{E}\left\{ \sup_{t \in [0,1]} |A_{3\pi}(x, y)|^p \right\} \right)^{\frac{1}{p}} \leq c(p)|x - y|. \quad (4.17)
\]

Observe that 

\[
(A_{4\pi}(x, y))^2 \leq c \sum_{i=0}^{n-1} \frac{1}{t_{i+1} - t_i} \int_{t_i}^{t_{i+1}} ds \left( L_s^x - L_t^x \right)^2 \times \sum_{i=0}^{n-1} (B_{t_{i+1}} - B_{t_i})^2 \leq c\left( \sup_{0 \leq s \leq 1} |L_s^x - L_t^y| \right) \sum_{i=0}^{n-1} \frac{1}{t_{i+1} - t_i} \int_{t_i}^{t_{i+1}} (|L_s^x - L_t^x| + |L_s^y - L_t^x|) ds \times \sum_{i=0}^{n-1} (B_{t_{i+1}} - B_{t_i})^2 \leq c\left( \sup_{0 \leq s \leq 1} |L_s^x - L_t^y| \right) \sum_{i=0}^{n-1} \left( |L^x_{t_{i+1}} - L^x_{t_i}| + |L^y_{t_{i+1}} - L^x_{t_i}| \right) \times \sum_{i=0}^{n-1} (B_{t_{i+1}} - B_{t_i})^2. \quad (4.18)
\]
It follows that
\[
E \left\{ \sup_{t \in [0,1]} |A_{4\pi}(x,y)|^{2p} \right\} \
\leq c \left( E \left\{ \left( \sup_{0 \leq s \leq 1} |L_s^x - L_s^y|^{3p} \right) \right\} \right)^{\frac{1}{3}} \times \left( E \left\{ \left( \sum_{i=0}^{n-1} (B_{t_{i+1}} - B_{t_i})^2 \right)^{3p} \right) \right)^{\frac{1}{3}} \
\leq c(p,R) |x - y|^{\frac{1}{2}p}.
\] (4.19)

Putting together above estimates (4.15)-(4.17) and (4.19), we arrive at (4.13).
\[\square\]

The following result can be proved similarly as Proposition 4.2.

**Proposition 4.3.** Assume that the function \(\sigma\) satisfies the same conditions as in Theorem 1.1, \((X_t(x), L_t^x)\) is a solution of Eq.(1.1). Then for any \(p \geq 2\) and \(R > 0\) there exists a constant \(C(p,R)\), such that
\[
E \left\{ \sup_{t \in [0,1]} |I(t,x) - I(t,y)|^p \right\} \leq C(p,R) |x - y|^p
\] (4.20)
for all \(x, y \in [0, R]\).

## 5 Uniform convergence of the Riemann sums

**Theorem 5.1.** Assume that the function \(\sigma\) satisfies the same conditions as in Theorem 1.1, \((X_t(x), L_t^x)\) is a solution of Eq.(1.1). Then for any \(p \geq 2\) and \(R > 0\),
\[
\lim_{\|\pi\| \to 0} E \left\{ \sup_{x \in [0,R]} |S_\pi(t,x) - I(t,x)|^{2p} \right\} = 0.
\] (5.1)

**Proof.** By Propositions 4.2 and 4.3, it follows from Garsia-Rodemich and Rumsey’s Lemma (cf.[4]) that there exists a constant \(\beta_0 > 0\), independent of \(\pi\), such that for \(x, y \in [0, R]\),
\[
|S_\pi(t,x) - S_\pi(t,y)| \leq K_\pi(\omega) |x - y|^{\beta_0},
\] (5.2)
\[
|I(t,x) - I(t,y)| \leq K(\omega) |x - y|^{\beta_0},
\] (5.3)
where \(K_\pi(\omega), K(\omega)\) are random variables that satisfy
\[
\sup_{\pi} E\{|K_\pi|^p\} < \infty, \quad E\{|K|^p\} < \infty,
\] (5.4)
for all $p \geq 1$. This is possible because the constant in Propositions 4.2 is independent of $\pi$. Thus, given any $\varepsilon > 0$, there is $\delta > 0$ such that

$$
E\{ \sup_{x \in [0, R]} \sup_{|y-x| \leq \delta} |S_\pi(t, y) - S_\pi(t, x)|^p \} \leq \varepsilon, \quad (5.5)
$$

$$
E\{ \sup_{x \in [0, R]} |I(t, y) - I(t, x)|^p \} \leq \varepsilon. \quad (5.6)
$$

On the other hand, for any $\delta > 0$, we can find $x_1, \ldots, x_m$ such that $[0, R] \subset \bigcup_{i=1}^{m} B(x_i, \delta)$.

Consequently,

$$
\sup_{x \in [0, R]} |S_\pi(t, x) - I(t, x)|^p 
\leq \sup_{x_i} \sup_{y \in B(x_i, \delta)} |S_\pi(t, y) - I(t, y)|^p 
\leq c(p) \sup_{x_i} \sup_{y \in B(x_i, \delta)} |S_\pi(t, y) - S_\pi(t, x_i)|^p 
+c(p) \sum_{i=1}^{m} |S_\pi(t, x_i) - I(t, x_i)|^p 
+c(p) \sup_{x_i} \sup_{y \in B(x_i, \delta)} |I(t, y) - I(t, x_i)|^{2p}. \quad (5.7)
$$

By virtue of (5.5) and (5.6), this implies that

$$
E\{ \sup_{x \in [0, R]} |S_\pi(t, x) - I(t, x)|^p \} 
\leq 2c(p)\varepsilon + c(p)E\{ \sum_{i=1}^{m} |S_\pi(t, x_i) - I(t, x_i)|^p \}. \quad (5.8)
$$

Let first $\|\pi\| \to 0$ and then $\varepsilon \to 0$ to get (5.1) by Proposition 4.1 and (5.8). Thus we complete the proof. \quad \square

6 Proof of Theorem 1.1

We will prove that $(X_t := X_t(Z), L_t^Z)$ solves the anticipating reflected SDE (1.3). Let $l(y)$ be any given continuous function on $(0, \infty)$ with compact support. Since $(X_t(x), L_t^x)$ is a solution of Eq.(1.1), $F_t(x) = \int_{0}^{t} l(X_s(x))dL_s^x = 0$ for all $\omega$ in some measurable set $\Omega_{t,x}$ with $P(\Omega_{t,x}) = 1$. By continuity of $F_t(x)$ proved in Proposition 3.1, we have

$$
P(\omega : F_t(x, \omega) = 0 \text{ for all } (t, x) \in [0, 1] \times [0, +\infty)) = 1.
$$
Hence
\[ P(\omega : F_t(Z, \omega) = 0 \text{ for all } t \in [0, 1]) = 1, \] (6.1)
for any non-negative random variable \( Z \). This implies that
\[ L_t^Z = \int_0^t \chi_{\{X_s(Z) = 0\}} dL_s^Z \]

Next we prove (1.6). By Theorem 5.1, the following holds almost surely on
\( \{\omega; Z(\omega) \leq M\} \)
\[ \int_0^t \sigma(X_s(x)) \circ dB_s \Big|_{x=Z} \chi_{\{\omega; Z(\omega) \leq M\}} 
= \lim_{\|\pi\| \to 0} S_{\pi}(t, x) \Big|_{x=Z} \chi_{\{\omega; Z(\omega) \leq M\}} 
= \int_0^t \sigma(X_s(Z)) \circ dB_s \chi_{\{\omega; Z(\omega) \leq M\}}. \]

Letting \( M \to \infty \) we obtain the substitution formula (1.6), and therefore prove the Theorem. \( \square \)

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