Further restrictions on the topology of stationary black holes in five dimensions

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Abstract

We place further restriction on the possible topology of stationary asymptotically flat vacuum black holes in 5 spacetime dimensions. We prove that the horizon manifold can be either a connected sum of Lens spaces and “handles” $S^1 \times S^2$, or the quotient of $S^3$ by certain finite groups of isometries (with no “handles”). The resulting horizon topologies include Prism manifolds and quotients of the Poincare homology sphere. We also show that the topology of the domain of outer communication is a cartesian product of the time direction with a finite connected sum of $\mathbb{R}^4$, $S^2 \times S^2$‘s and $CP^2$‘s, minus the black hole itself. We do not assume the existence of any Killing vector beside the asymptotically timelike one required by definition for stationarity.

1 Introduction

In this paper, we derive further restrictions on the possible topologies of 5-dimensional, asymptotically flat, analytic, non-extremal, vacuum black hole spacetimes $(M, g)$ with compact horizon. The known solutions in this class to date are the Myers-Perry black holes [1]
(with horizon topology $S^3$), and the Emparan-Reall black rings and their generalizations [2, 3] (with horizon topology $S^2 \times S^1$). Whether these are all possible solutions is unclear at present, but it has been conjectured by Reall that there could be solutions with only precisely one extra $U(1)$-Killing field, $\psi$. This conjecture has recently received some support by the investigations of [4]. Our aim is to place some limits on the possible topologies of such, as yet conjectural, solutions, both in as far as the horizon is concerned, but also in as far as that of the entire domain of outer communication (exterior of the black hole) in $M$ is concerned.

Several general theorems are already known in this direction, and our analysis is to a large extent a combination of these: Firstly, it has been shown by [5] (see also [6, 7]) that the horizon cross section, $H$, can carry a metric of strictly positive scalar curvature. This result, which holds in $D$ dimensions under the assumption that matter satisfies the dominant energy condition, was proved originally in 4 dimensions by Hawking [8], where it implies that the horizon has topology $S^2$. It implies strong restrictions on the horizon topology also in higher, especially 5-, dimensions. Secondly, the topological censorship theorem [9, 10, 11] states that any curve in the domain of outer communication with endpoints in the asymptotic region can be deformed to a curve entirely within that region. Therefore, if the spacetime is asymptotically flat in the standard sense, then the domain of outer communication is simply connected. Thirdly, it is known that if the horizon is rotating, i.e. if the original asymptotically timelike Killing field $t$ is not tangent to the null generators of the horizon, then the rigidity theorem implies that there is at least one further Killing field $\psi$ generating an action of $U(1)$ on spacetime which commutes with $t$ [12, 13, 14]. If there is precisely one such $U(1)$ Killing field, then we have on the horizon,

$$\xi = t + \Omega \psi,$$

(1.1)

where $\xi$ is tangent to the null generators of the horizon, and where the constant $\Omega$ is the angular velocity of the horizon. In the known exact black hole solutions, we have in fact two further extra Killing fields generating an action of $U(1) \times U(1)$ instead of just $U(1)$. In that case, a complete classification of the possible solutions is available [15, 16]. In particular, the possible topologies of $H$ are $L(p, q), S^2 \times S^1, S^3$. Furthermore, the domain of outer communication then has topology $\mathbb{R} \times \Sigma$, where $\Sigma$ can be shown using results of [17] to be a direct sum of $\mathbb{R}^4$, and copies of $S^2 \times S^2, \pm \mathbb{C}P^2$‘s, minus the black hole $B$ itself. Fourthly, if the horizon is non-rotating, then the solution is isometric to the Schwarzschild spacetime [18, 19, 20, 21], with horizon topology $S^3$, and $\Sigma \approx \mathbb{R}^4 \setminus B$. Finally, some papers have also appeared concerning the nature of the past endpoint set of an event horizon in $D$-dimensions when the spacetime is dynamically evolving, see e.g. [22]. However, these do not appear to give further constraints on the final topology of the black hole beyond the ones that we have already mentioned.
2 Results

In this paper, we will consider the generic case of an analytic, stationary, rotating, vacuum black hole with compact, connected horizon cross sections, which, as we have explained, might only have isometry group $\mathbb{R} \times U(1)$. As we will see, the statement about the topology of $\Sigma$ remains true in that case, but the possibilities for the horizon topology that we derive are more than just $L(p, q), S^2 \times S^1, S^3$. More precisely, we will prove the following two results:

**Result 1.** The topology of $H$ can be one of the following:

1. If $\psi$ has a zero on $H$, then the topology of $H$ must be
   \[ H \cong \# l \cdot (S^2 \times S^1) \# L(p_1, q_1) \# \cdots \# L(p_k, q_k). \] (2.2)
   Here, $k$ is the number of exceptional orbits of the action of $U(1)$ on $H$ that is generated by $\psi$, and $l$ is the number of connected components of the zero set of $\psi$.

2. If $\psi$ does not have a zero on $H$, then $H \cong S^3/\Gamma$, where $\Gamma$ can be certain finite subgroups of $SO(4)$, or $H \cong S^2 \times S^1$. This class of manifolds includes again the Lens-spaces, but also Prism manifolds, the Poincare homology sphere, and various other quotients. All manifolds in this class are certain Seifert fibred spaces over $S^2$. The precise classification of the possibilities is given below in table 1.

Thus, in summary, our first result is that $H$ is either a connected sum of handles and Lens-spaces, or a certain kind of other spherical manifold with no handles. Our result is somewhat stronger than what is implied by merely knowing that $H$ carries a metric of positive scalar curvature, because it rules out the possibility that $H$ could be a connected sum of spherical manifolds. Our result is definitely much stronger than what can be concluded from merely knowing that it carries a $U(1)$-action.

If the topology of $M$ is such that it allows for an action of $U(1) \times U(1)$—as would be the case if the black hole solution could be connected continuously to a solution with two commuting axial Killing fields rather than just one—then the possibilities for the topology of $H$ are cut down to $L(p, q), S^2 \times S^1, S^3$, in particular, the other spherical manifolds, such as Prism manifolds, cannot appear.

Our second result concerns the topology of the domain of outer communication.

**Result 2.** There is a compact manifold $B$ with boundary $\partial B = H$ (the “black hole”) such that the domain of outer communication has topology $M \cong \Sigma \times \mathbb{R}$, where

\[ \Sigma \cong \left( \mathbb{R}^4 \# n \cdot (S^2 \times S^2) \# n' \cdot (\pm CP^2) \right) \setminus B. \] (2.3)

for some $n, n' \in \mathbb{N}$.

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1Our results can easily be generalized to multiple horizons.
2Here we allow that the Lens space be $L(0,1) := S^3$. 

Thus, from the topological viewpoint, the condition that there be an isometry group \( \mathbb{R} \times U(1) \times U(1) \) gives essentially the same restrictions on the topology of \( \Sigma \) as just assuming stationarity. For the known black hole solutions we have \( n = 0 = n' \), and this may well be in fact the only possibility, although we cannot prove this. If we assume, as is very reasonable, that \( M \) carries a spin structure, then \( n' = 0 \).

Because \( \Sigma \) is a manifold \textit{together} with an action of \( U(1) \), we can associate further invariants with the solution that specify the action. These invariants (considered first in [23, 24]) constitute the “decorated orbit space” \( \hat{\Sigma} = \Sigma/U(1) \), which consists of a manifold \( \hat{\Sigma} \) with boundary together with a collection of polyhedral arcs, the components of which are decorated by certain integers, \( (p_i, q_i) \), and Euler numbers, \( e_i \), as in eq. (2.23). These data specify both the topology of \( \Sigma \), as well as the precise nature of the \( U(1) \)-action, see figure 1. (Note that the same spaces can carry different inequivalent \( U(1) \) actions.) Below in sec. 2.2, we also give conditions on the decorated orbit space \( \hat{\Sigma} \), under which the action of \( U(1) \) on \( M \) can be extended, topologically, to an action of \( U(1) \times U(1) \). Of course, this is only a topological statement, the second factor of \( U(1) \) need not act by isometries. The point of this statement is that, if a solution is continuously connected to a solution with the higher symmetry \( U(1) \times U(1) \), i.e. a finite perturbation, then one obtains some additional information about the invariants in this way.

2.1 Result 1

As we have explained, the rigidity theorem implies the existence of a second Killing field \( \psi \) commuting with \( t \). The Killing field \( \psi \) generates an action of \( U(1) \) on the spacetime. The proof of this theorem [12, 13, 14] implies that we can choose a horizon cross section \( H \) such that \( \psi \) is tangent to \( H \), and such that \( \psi \) is a Killing vector of the induced metric \( \gamma \) on \( H \). Thus, \( (H, \gamma) \) is a compact Riemannian 3-manifold with an isometric action of \( U(1) \). Let

\[
\hat{H} = H/U(1) \tag{2.4}
\]

be the quotient space. As is well-known [25, 26], \( \hat{H} \) is a 2-dimensional orbifold with boundary. The boundary

\[
\partial \hat{H} \cong \bigcup_{i=1}^{l} S^1_i \tag{2.5}
\]

consists of \( l \) disjoint circles. Points in this boundary correspond to the fixed points of the \( U(1) \)-action, i.e. the places in \( H \) where \( \psi = 0 \). Furthermore, each orbifold point \( x_i \in \hat{H} \) is labelled by a pair \( (p_i, q_i) \), where \( i = 1, \ldots, k \), of relatively prime integers satisfying \( 0 < q_i < p_i \). Near such a point \( H \) is modelled upon the quotient \( D^2_i/\mathbb{Z}_{p_i} \) of a disk by the cyclic group of order \( p_i \) acting within the disk \( D^2_i \) by phases \( \exp(2\pi \sqrt{-1}/p_i) \). The points \( x_i \) correspond to the singular \( U(1) \)-fibres in \( H \). Each such fibre is surrounded by a solid 3-dimensional torus \( D^2_i \times S^1 \), and in such a solid torus, each \( U(1) \)-orbit winds around the disk \( n_i \)-times as
Figure 1: The figure shows how the “decorated orbit space” $\hat{\Sigma} = \Sigma/U(1)$ can look like. The weighted polyhedral arcs correspond to points in $\Sigma$ having a non trivial isotropy group $\mathbb{Z}_{p_i} \subset U(1)$. The boundary components correspond to points where the isotropy group is $U(1)$ (i.e. where $\psi = 0$), or the horizon.

it goes $p_i$-times around the $S^1$-direction, see figures 2 and 3 ($q_i n_i \equiv 1 \mod p_i$). We are going to distinguish the cases $\partial \hat{H} = \emptyset$ and $\partial \hat{H} \neq \emptyset$.

2.1.1 Case (i): $\partial \hat{H} = \emptyset$

This is the more interesting case and happens when $\psi \neq 0$ everywhere on $H$. Then, topologically, since $\hat{H}$ inherits an orientation from the 3-form $\epsilon$ on $H$ given by $i_\psi \epsilon$, both $\hat{H}$ and the fibres are oriented\(^3\). Thus, $H$ is a Seifert fibred space [25, 27] characterized by the decoration data

$$H : \{g; b; (p_1, q_1), \ldots, (p_k, q_k)\},$$

where $g$ is the genus of the oriented compact Riemann surface $\hat{H}$ without boundary. $b$ is an integer that characterizes the topology of the principal fibre bundle obtained after drilling out the exceptional fibres and replacing them with regular ones. The manifold $H$ is determined, as a manifold with $U(1)$-action, by the symbol (2.6), but different symbols obtained from a given one by certain operations give rise to the same $U(1)$-manifold [25, 26].

\(^3\)In the terminology of Seifert manifolds, our manifolds are of type “Oo”.
Symbols that may not be transformed into each other by such manipulations may still give rise to manifolds of the same topology of $H$, but the corresponding spaces will then have inequivalent actions of $U(1)$.

Our aim is to show that the orbifold Euler characteristic\(^\text{4}\) of $\hat{H}$ is positive, i.e. that

$$\chi_{\text{orbifold}}(\hat{H}) := 2 - 2g - \sum_{i=1}^{k} \left(1 - \frac{1}{p_i}\right) > 0.$$  \hfill (2.7)

By standard results on Seifert 3-manifolds with positive orbifold Euler characteristic [see e.g. table 4 of the review [28]; original refs. include [27, 29]], this restricts the possible decoration data and topologies to the following ones [excluding $S^2 \times S^1$ which is included in case (ii)]: In all cases we have $g = 0, k \leq 3$, and the possible fibrations (2.6), their corresponding 3-manifold $H$, and fundamental groups are summarized in table 1, where $D^*_n$ is the binary dihedral group of order $4n$, $T^*$ is the binary tetrahedral group of order 24, $O^*$ is the octahedral group of order 48, $I^*$ the icosahedral group of order 120 and $D^*_{2k(2n+1)}, T^*_{8,3,k}$ are groups of the indicated order that are given e.g. in [28]. In all cases, the fundamental

\(^4\)This is an invariant of any Seifert manifold, i.e. it is unchanged under the manipulations mentioned above.
group \( \pi_1(H) = \Gamma \) is finite, and by the Thurston elliptization theorem for 3-manifolds (see e.g. [30]), it follows that \( H = S^3/\Gamma \) in all cases.

Thus, what remains to be shown is that the orbifold Euler characteristic is positive. As above, let \( \gamma \) be the Riemannian metric on \( H \) induced by the spacetime metric. Then, as shown by [5], there exists a positive function \( \phi > 0 \) on \( H \) such that conformally transformed metric \( \tilde{\gamma} = \phi^{-3/2} \gamma \) is a metric with non-negative scalar curvature \( \tilde{S} \geq 0 \). The metric \( \gamma \) is invariant under \( U(1) \), so if we could show that also \( \phi \) can be chosen \( U(1) \)-invariant, then also \( \tilde{\gamma} \) is \( U(1) \)-invariant and furthermore has positive scalar curvature. It is easily seen that the argument of [5] can be adapted in a straightforward way to prove this. For completeness, we indicate how this is done following [5]. Let us introduce Gaussian null-coordinates (see e.g. [14]) near the horizon as

\[
g = 2du(dv + r \beta_a dx^a + r \alpha du) + \gamma_{ab} dx^a dx^b ,
\]

where the indices \( a, b, \ldots \) indicate tensor components tangent to \( H \). The Killing field \( \xi \), see eq. (1.1), is given in these coordinates by \( \xi = \partial/\partial u \), and the horizon is at \( r = 0 \). The function \( \alpha \) is constant on \( H \) and given by the surface gravity of the black hole. By considering variations of \( H \) along an outward directed spatial normal vector field, it is demonstrated in [5] that there holds the inequality

\[
\int_H \left( (\nabla^a f) \nabla_a f + \frac{1}{2} \{ S - (L_{\xi} \gamma)^{ab} (L_{\xi} \gamma)_{ab} \} f^2 \right) \sqrt{\gamma} d^3x \geq 0 ,
\]

for any smooth function \( f \) on \( H \), where indices are raised with \( \gamma^{ab} \). Since \( \xi \) is a Killing field, one can show that the Lie-derivative in fact vanishes in our situation. In view of the inequality, one knows that the spectrum \{\( \lambda_1, \lambda_2, \ldots \)\} of the differential operator \(-\nabla^a \nabla_a + \frac{1}{2} S\) is non-negative. It is then possible, by standard results [31], to choose a strictly positive eigenfunction, \( \phi > 0 \), for the first eigenvalue \( \lambda_1 \geq 0 \). The only additional new observation necessary for us is that, since the differential operator commutes with the flow of \( \psi \), we may...
| Topology of $H$ | $U(1)$-Fibration | Fundamental Group |
|-----------------|------------------|------------------|
| $S^3$           | $\{b\}$         | $\mathbb{Z}_{|b|}$ |
| $L(b, 1)$       | $\{b\}$         | $\mathbb{Z}_{|b|}$ |
| $L(bp_1 + q_1, n)$ | $\{b; (p_1, q_1)\}$ with $|bp_1 + q_1| > 1$ and $n = p_1 \mod bp_1 + q_1$ with $0 < n < bp_1 + q_1$ | $\mathbb{Z}_{|bp_1 + q_1|}$ |
| $L(bp_1p_2 + p_1q_2 + q_2p_1, mp_2 - nq_2)$ | $\{b; (p_1, q_1, (p_2, q_2)\}$ | $\mathbb{Z}_{bp_1p_2 + p_1q_2 + q_2p_1}$ |
| $P(r)$ | $\{-1; (2, 1), (2, 1), (r, 1)\}$ | $D^*_r$ |
| generalized Prism space | $\{b; (2, 1), (2, 1), (p_3, q_3)\}$ | $\mathbb{Z}_{(b+1)p_3 + q_3} \times D^*_r$ if g.c.d.$((b+1)p_3 + q_3, 2q_3) = 1$ $\mathbb{Z}_{|m|} \times D^*_r$ for $(b+1)p_3 + q_3 = 2^k m$ |
| $S^3/T^*$ | $\{-1; (2, 1), (3, 1), (3, 1)\}$ | $D^*_r$ |
| generalized octahedral space | $\{b; (2, 1), (3, q_2), (3, q_3)\}$ | $\mathbb{Z}_{b+3+2q_2+2q_3} \times T^*$ if g.c.d.$(12, 6b + 3 + 2q_2 + 2q_3) = 1$ $\mathbb{Z}_{|m|} \times T^*$ for $6b + 3 + 2q_2 + 2q_3 = 3^k m$ |
| $S^3/O^*$ | $\{-1; (2, 1), (3, 1), (4, 1)\}$ | $O^*$ |
| generalized cube space | $\{b; (2, 1), (3, q_2), (4, q_3)\}$ | $\mathbb{Z}_{12b+6+4q_2+3q_3} \times O^*$ |
| $S^4/T^*$ | $\{-1; (2, 1), (3, 1), (5, 1)\}$ | $T^*$ |
| generalized dodecahedral space | $\{b; (2, 1), (3, q_2)\}$ | $\mathbb{Z}_{30b+15+10q_2+6q_3} \times T^*$ |

Table 1: The orientable Seifert manifolds with positive orbifold Euler characteristic

choose $\phi$ to be invariant as well. If it is not initially, we simply make it $U(1)$ invariant by taking instead the average

$$\phi(x) \to \frac{1}{2\pi} \int_0^{2\pi} \phi \circ \theta_\tau(x) \, d\tau$$

along the flow $\theta_\tau$ of $\psi$, which is again strictly positive everywhere on $H$ and an eigenfunction of $-\nabla^a \nabla_a + \frac{1}{2} S$. The metric $\tilde{\gamma} = \phi^{-3/2} \gamma$ has non-negative scalar curvature $\tilde{S}$, because

$$\tilde{S} = \phi^{-1} \left( 2\lambda_1 + \frac{3}{2\phi^2} (\nabla^a \phi) \nabla_a \phi \right) \geq 0,$$

by the standard conformal transformation formula for the scalar curvature. This still leaves the possibility that $\tilde{S} = 0$ everywhere on $H$. To rule out this case, one can argue as follows. Let $\tilde{S}_{ab}$ be the Ricci tensor of $\tilde{\gamma}_{ab}$. Then, following Bourguignon (see [31]), by considering deformations of $\tilde{\gamma}_{ab}$ in the direction of $\tilde{S}_{ab}$, one could find a metric on $H$ which is Ricci flat, and since $H$ is a 3-manifold, flat. The only possibility is then $H \cong T^3$, but this case has been ruled out by [6].
Thus, we can assume that $\tilde{\gamma}$ is $U(1)$-invariant and has non-negative scalar curvature $\tilde{S} \geq 0$ which is non-zero somewhere on $H$. To continue, we recall that $H$ is a fibred space over $\hat{H}$, with fibres $S^1$, but it is not a principal fibre bundle in the open neighborhoods of the exceptional fibres. We drill out a neighborhood (solid 3-torus) of the form $D^2_r \times S^1$ around each exceptional fibre in $H$, where $D^2_r$ is a disk of radius $r$ in Riemannian normal coordinates centered on the fibre. The resulting compact manifold with boundary is denoted by $H_r$; its orbit space has the form

$$\hat{H}_r = \hat{H} \setminus \bigcup_{i=1}^k D^2_i,$$

(2.12)

i.e. it is a closed 2-manifold of genus $g$ with $k$ disks cut out, and hence has a boundary given by a union of $k$ circles $S^1_i$, $i = 1, \ldots, k$. The 3-manifold $H_r$ now has only regular fibres, so it has the structure of a principal fibre bundle over the 2-manifold $\hat{H}_r$ with boundary. We can then perform a “Kaluza-Klein” reduction of the metric $\tilde{\gamma}$ in the usual way, i.e., we can write

$$\tilde{\gamma} = e^\nu (d\varphi + \omega_i dx^i)^2 + e^{-\nu} h_{ij} dx^i dx^j,$$

(2.13)

where $\varphi$ is a $2\pi$-periodic coordinate on the fibres and $x^i$ are local coordinates of $\hat{H}_r$, so that $\psi = \partial/\partial \varphi$, where $\nu$ is a scalar field, $\omega$ a $U(1)$-connection, and $h$ a metric on $\hat{H}_r$. Furthermore, as a standard calculation shows, the scalar curvature $\tilde{S}$ of $\tilde{\gamma}$ can be decomposed as

$$e^{-\nu} \tilde{S} = \mathcal{R} - \frac{1}{4} e^{2\nu} \mathcal{F}_{ij} \mathcal{F}^{ij} - \frac{1}{2} (\partial_i \nu) \partial^i \nu,$$

(2.14)

where $\mathcal{F}$ is the curvature of $\omega$, $\mathcal{R}$ is the scalar curvature of $h$, and all indices are raised with $h$. We multiply this equation with $\sqrt{h} d^2 x$, the invariant integration element on $\hat{H}_r$, and integrate, taking $r > 0$ so small that $\tilde{S} > 0$ somewhere on $\hat{H}_r$. Then we get:

$$0 < \int_{\hat{H}_r} \left( e^{-\nu} \tilde{S} + \frac{1}{4} e^{2\nu} \mathcal{F}_{ij} \mathcal{F}^{ij} + \frac{1}{2} (\partial_i \nu) \partial^i \nu \right) \sqrt{h} d^2 x = \int_{\hat{H}_r} \mathcal{R} \sqrt{h} d^2 x.$$

(2.15)

On the right side, we now apply the Gauss-Bonnet theorem for the manifold with boundary $\hat{H}_r$. Letting $K$ be the extrinsic curvature of the boundary components $S^1_i$, $i = 1, \ldots, k$ oriented by the outward pointing normal, and $ds$ be the corresponding invariant line element, we get

$$0 < 2 - 2g - \sum_{i=1}^k \left( 1 - \frac{1}{2\pi} \int_{S^1_i} K ds \right).$$

(2.16)

The remaining task is to evaluate the boundary integrals in the limit as $r \to 0$. For small $r$ and within the $i$-th removed disk $D^2_i$, the metric $h$ takes the form $h \sim dr^2 + r^2 dy^2$ up to higher orders of $r$, where $y$ is a coordinate which is $2\pi/p_i$-periodic. This immediately gives the desired result for the orbifold Euler characteristic, since the boundary integrals then evaluate to $1/p_i$ in the limit as $r \to 0$. 
2.1.2 Case (ii): \( \partial \hat{H} \neq \emptyset \)

In this case, the orbit space \( \hat{H} \) is a 2-dimensional oriented, compact orbifold with \( l \) boundaries \( S^1_i, i = 1, \ldots, l \), and \( k \) orbifold points labelled by \((p_i, q_i), i = 1, \ldots, k\). Our first aim is to prove that topologically

\[
\hat{H} = D^2_0 \setminus \bigcup_{i=1}^l D^2_i,
\]

i.e. \( \hat{H} \) is a large disk \( D^2_0 \), with \( l \) small disks removed, see figure 4. On this large disk, there are \( k \) orbifold points. To see this, we make use of the topological censorship theorem already mentioned above. The point is that \( H \) is the boundary of a spatial slice \( \Sigma \). As shown in [32], we may choose this slice in such a way that it is invariant under the \( U(1) \)-action (i.e., \( \psi \) is tangent to \( \Sigma \)), and of course we have \( \partial \Sigma = -H \cup S^3_\infty \), where we mean a 3-sphere at infinity.

Then, by standard arguments, the quotient \( \hat{H} \) must lie on the boundary of the corresponding quotient \( \hat{\Sigma} = \Sigma / U(1) \). The quotient of the 4-manifold \( \Sigma \) by \( U(1) \) is discussed in more detail in the next subsection. Here we only need to know that \( \hat{\Sigma} \) is a space which locally is a manifold with boundaries, up to certain singularities that are localized along 1-dimensional curve segments. One of the boundaries (that reaching out to infinity) has topology \( \mathbb{R}^2 \) outside a compact set, and \( \hat{H} \) is a subset of this, see figure 5. Now let us assume that instead of a disk \( D^2_0 \) we would have a disk \( D^2_0 \) with \( h \) additional handles attached. Then it is quite obvious, see figure 6 for an example, that we could then find in \( \hat{\Sigma} \) a curve that slings through one of these handles and is hence not contractible. However, by the topological censorship theorem, in the domain of outer communication, any curve is contractible, and therefore by standard topological arguments, so is any curve in \( \hat{\Sigma} \). Hence, we have a contradiction unless \( h = 0 \).

The statement (2.17) now implies the desired decomposition eq. (2.2) by standard arguments of [25, 26]. For completeness, we briefly outline how these arguments are made. First, we cut out the removed disks \( D^2_i \), as illustrated in figure 7. Each of these operations corresponds, on the level of \( H \), to removing a handle \( S^2 \times S^1 \) and gluing back in a sphere. After removing \( l \) such handles, we are left with a disk and \( k \) orbifold points. These are now removed one by one, as illustrated in figure 8. Each of these operations corresponds, on the level of \( H \), to removing a Lens space \( L(p_i, q_i) \) and gluing in a 3-sphere. Thus, we arrive at the desired decomposition (2.2). The condition that there exist a metric of positive scalar curvature on \( H \) does not give any further restrictions, since such decompositions are known to admit such metrics.

2.2 Result 2

We are going to prove result 2 by considering the orbit space \( \hat{\Sigma} = \Sigma / U(1) \). For compact, simply connected 4-manifolds \( X \) with a \( U(1) \) action, the orbit space \( \hat{X} = X / U(1) \) has been
\( \hat{H} = H/U(1) \)

Figure 4: We claim that the orbit space \( \hat{H} \) is topologically a disk with some disks removed. The crosses represent the orbifold points.

analyzed by [23]. He shows that the orbit space is a singular space which at generic points is a 3-manifold. This 3-manifold has boundaries corresponding to certain fixed points of the action of \( U(1) \), together with certain piecewise smooth polygonal curves in \( \hat{X} \), which correspond to exceptional orbits where the isotropy group (i.e. the subgroup of \( U(1) \) leaving a point invariant) is discrete. More precisely, the nature of the orbit space is as follows: 
\( \hat{X} = \hat{L} \cup \hat{E} \cup \hat{F} \), where a hat always means the quotient by \( U(1) \), and where \( F \) is the space of fixed points in \( X \) (where the isotropy subgroup is \( U(1) \)), \( E \) is the space of exceptional orbits (where the isotropy group is \( \mathbb{Z}_p \subset U(1) \) for some \( p \)), and where \( L \) is the set of regular orbits (where the isotropy subgroup is trivial).

1. The set \( \hat{L} \) is open in \( \hat{X} \), and forms a smooth open manifold of dimension 3.

2. The set \( \hat{F} \) of fixed points is closed in \( \hat{X} \). It consists of isolated points \( x_i \), or boundary components \( \partial \hat{X} \cong S^2 \). Near an isolated point, we can find a coordinate system \( (y_1, \ldots, y_4) \) such that \( x_i \) corresponds to the origin of the coordinate system, and such
Figure 5: This figure shows how the orbit space $\hat{\Sigma}$ looks like. The orbit space $\hat{H}$ of the horizon forms part of the boundary of this space. In fact, $\hat{H}$ should be connected, and we argue that there cannot be any handles as suggested in this figure.

that the action of an element $e^{\sqrt{-1}t} \in U(1)$ is given by the matrix

$$
\begin{pmatrix}
\cos p_i t & \sin p_i t & 0 & 0 \\
-\sin p_i t & \cos p_i t & 0 & 0 \\
0 & 0 & \cos p_{i+1} t & \sin p_{i+1} t \\
0 & 0 & -\sin p_{i+1} t & \cos p_{i+1} t
\end{pmatrix}, \quad p_i, p_{i+1} \in \mathbb{Z}, \quad \gcd(p_i, p_{i+1}) = 1.
$$

(2.18)

Near each point of a boundary component, we can find a coordinate system $(y_1, \ldots, y_4)$ such that the action of an element $e^{\sqrt{-1}t} \in U(1)$ is given by the matrix

$$
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \cos t & \sin t \\
0 & 0 & -\sin t & \cos t
\end{pmatrix},
$$

(2.19)

where $\{0 = r = \sqrt{y_1^2 + y_2^2}\}$ corresponds to the points in the boundary component of $\hat{X}$, i.e. $(r, y_3, y_4), r > 0$ provide coordinates for $\hat{X}$ near that boundary point.

3. The set $\hat{E}$ of exceptional orbits consists of smooth arcs $\gamma_i$ in $\hat{X}$. Each such arc is labelled by a pair $(p_i, q_i)$ of integers. A point in $E$ has a neighborhood with coordinates
Figure 6: If $\hat{H}$ contained a handle, then we could sling through it a curve as shown, and this contradicts the topological censorship theorem.

$$(y_1, \ldots, y_4)$$ such that the action of an element $e^{\sqrt{-1} t} \in U(1)$ is given by the matrix

$$
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \cos p_i t & \sin p_i t \\
0 & 0 & -\sin p_i t & \cos p_i t
\end{pmatrix},
$$

near such a point, the Killing field $\psi$ generating $U(1)$ is given locally by $\partial/\partial y_1$, and the orbit space in the neighborhood of the arc is parametrized by $y_2$, which runs along the arc, and $y_3, y_4$, running transverse to the arc, located at $y_3 = y_4 = 0$ locally. Two arcs can intersect at an isolated fixed point (see previous item), and the numbers $q_i \in \mathbb{Z}$ are then assigned in such a way that if the adjacent arcs carry $p_i$ and $p_{i+1}$, then we have

$$\begin{vmatrix}
p_i & p_{i+1} \\
q_i & q_{i+1}
\end{vmatrix} = \pm 1. \quad (2.21)$$

The final invariant associated with the $U(1)$ manifold $X$ comes from the Euler numbers of the boundary components $\partial \hat{X} \cong S^2$ (if any). Let us shift the boundary slightly inwards in $\hat{X}$. Then we obtain a surface denoted $S^2_i$ inside $\hat{L}$, which is the base of a sub-$U(1)$ bundle in $X$ with base $S^2_i$ and fibres $U(1)$. We let $e_i$ be the Euler (= first Chern-) class of this bundle, i.e.

$$e_i = \frac{1}{2\pi} \int_{S^2_i} F \in \mathbb{Z}, \quad (2.22)$$
where $\mathcal{F}$ is the curvature of a connection in the $U(1)$-bundle over $S^2_1$ that can be obtained by decomposing a metric on $\hat{X}$ similar to eq. (2.13).

It was shown in [23] that the above invariants

$$X : \{ \hat{X}; e_1, \ldots, e_b; \gamma_1, (p_1, q_1), \ldots, \gamma_k, (p_k, q_k) \},$$

subject to the above constraint (2.21) are in one-to-one correspondence with the compact, oriented, simply connected $U(1)$-manifolds $X$, i.e. for each set of invariants there is precisely one such manifold, and vice-versa. Furthermore, it was shown in [23, 24] that the quadratic form of $X$ (i.e., the pairing $Q_X : H_2(X) \times H_2(X) \to \mathbb{Z}$) is congruent over $\mathbb{Z}$ to the matrix

$$Q_X = \bigoplus m \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus I_{m'} \oplus (-I_{m''})$$

for some $m, m', m'' \in \mathbb{N}$. Such a quadratic form is obtained also for connected sums of copies of $\pm \mathbb{C}P^2$ and copies of $S^2 \times S^2$, and therefore, since the topology of $X$ is uniquely determined by the invariant $Q_X$ according to [33, 34], $X$ has to be topologically a connected sum of copies of $S^2 \times S^2$, and $\pm \mathbb{C}P^2$'s. The projective spaces are forbidden if we assume, as appears to be reasonable from the physical viewpoint, that $X$ can carry a spin structure.

In our case, we would like to take $X = \Sigma$, where $\Sigma$ is a spatial slice. We know that $\Sigma$ is simply connected by the topological censorship theorem, but it is not compact. Its compactification is a manifold with boundary $\partial \Sigma = -H \cup S^3_\infty$. Nevertheless, it is not difficult to generalize the classification to this case. First, we glue a 4-dimensional ball $D^4_\infty$ into $\hat{\Sigma}$ along the boundary $\partial D^4_\infty = S^3_\infty$ at infinity, in such a way that the $U(1)$-actions match up. We call the resulting manifold with boundary $\Sigma_0 = \Sigma \cup D^4_\infty$. Then, if we take the
Figure 8: Removing an orbifold point corresponds to removing a Lens space $L(p_i, q_i)$. Here, $p_i = 0$ is allowed (this corresponds to an $S^3$).

quotient $\hat{\Sigma}_0 = \Sigma_0/U(1)$, the quotient $\hat{H}$ will correspond to a (new) part of the boundary $\Sigma_0$ that does not correspond to an axis as described in item 1. above. The quotient of the horizon $H$ might additionally contain points in $\hat{F}$, i.e. points corresponding to fixed points. Those correspond to the following situation, see figure 1.

1. Let $x_i$ be a point in $\hat{H} \cap \hat{F}$ corresponding to an isolated fixed point. Then at the corresponding points of $\Sigma_0$, we can choose coordinates $(y_1, y_2, y_3, y_4), y_1 > 0$ such that the action of an element $e^{\sqrt{-1}t} \in U(1)$ is given by the matrix

$$
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \cos t & \sin t \\
0 & 0 & -\sin t & \cos t
\end{pmatrix},
$$

where $y_1 = 0$ locally corresponds to the boundary $H$. The quotient space $\hat{\Sigma}_0$ is locally parametrized by the coordinates $r = \sqrt{y_3^2 + y_4^2} > 0, y_1 > 0$, and $y_2$, i.e. it locally has the structure of a corner.

2. Let $x_i$ be a point in $\hat{H} \cap \hat{E}$ corresponding to an exceptional orbit. It has a neighborhood with coordinates $(y_1, \ldots, y_4), y_1 > 0$ such that the action of an element $e^{\sqrt{-1}t} \in U(1)$ is given by the matrix

$$
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \cos p_t t & \sin p_t t \\
0 & 0 & -\sin p_t t & \cos p_t t
\end{pmatrix},
$$
Near such a point, the Killing field \( \psi \) generating \( U(1) \) is given locally by \( \partial/\partial y_2 \), and the horizon is locally located at \( y_1 = 0 \). The point \( x_i \) corresponds to a singular fibre with labels \((p_i, q_i)\) in \( H \) (see above), where \( \text{g.c.d.}(p_i, q_i) = 1 \). It is the intersection point of \( \hat{H} \) with one of the arcs in item 3. above.

By a straightforward application of van Kampen’s theorem, and using that \( \hat{\Sigma}_0 \) is simply connected, the fundamental group of \( \Sigma_0 \) is then found to be given by

\[
\pi_1(\Sigma_0) = \mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_I} \tag{2.27}
\]

where \( i = 1, \ldots, I \) runs through the set of isolated points on \( \hat{H} \) which are connected to another such point by an arc as in item 3. that is decorated by a pair \((p_i, q_i)\). Since we know that the fundamental group is in fact trivial, and since by definition \( p_i > 0 \), we can conclude that no such arc can exist in \( \hat{\Sigma}_0 \).

It is clear from this description that one can glue a decorated 3-manifold \( \hat{B} \) with boundary \( \partial \hat{B} = \hat{H} \) into \( \hat{\Sigma}_0 \) so as to give a decorated 3-manifold \( \hat{X} = \hat{\Sigma}_0 \cup \hat{B} \) (see eq. \((2.23)\)), and this will correspond, by the results of [23, 24], to a simply connected four manifold \( X \) with quadratic form \( Q_X \) as above. Then the decomposition \((2.3)\) follows, because for some \( n, n' \in \mathbb{N} \)

\[
\Sigma = X \setminus (B \cup D^4_{\infty}) \cong \left( \mathbb{R}^4 \# n \cdot (S^2 \times S^2) \# n' \cdot (\pm CP^2) \right) \setminus B. \tag{2.28}
\]

An illustration of the weighted orbit space \( \hat{\Sigma} \) is given above in figure 1. This space has as boundary components both points corresponding to the horizon \( (\hat{H}) \) as well as points corresponding to axes of the Killing field \( (\partial \hat{\Sigma} \setminus \hat{H}) \).

Finally, let us suppose that \((M, g)\) in fact has the isometry group \( U(1) \times U(1) \) acting on the spatial slice \( \Sigma \). Then the decorated orbit space is in fact more restricted, as we shall now explain. Let \( \hat{Y} = \Sigma/[U(1) \times U(1)] \). Then, as shown in [17, 15], the space \( \hat{Y} \) is homeomorphic to an upper half-plane, whose boundary is divided into several intervals \( I_i, i = 1, \ldots, r \), labeled by relatively prime integers \((p_i, q_i)\), except for a single special interval \( I_h \sim L(r, s) \), with \( \left| p_h - 1 \right| q_h = q_{h+1} + nr, \quad np_h - 1 = 1 \mod q_{h-1} \) \tag{2.29}.

Now let \( \hat{\Sigma} = \Sigma/[U(1) \times \{1\}] \) be the quotient by one \( U(1) \)-factor only. This gives a weighted orbit space as described above consisting of a 3-manifold with boundaries and weighted
arcs, see figure 1. The precise correspondence of this to $\hat{Y}$ is as follows. As a space, $\Sigma \cong \mathbb{R}^+ \times \mathbb{R}^2 \setminus \bigcup_i D_i^3$, where the boundary $S_i^2$ of the $i$-th ball corresponds to an interval $I_i$ that is labeled by $(1,0)$. If $(p_{i-1}, q_{i-1})$ resp. $(p_{i+1}, q_{i+1})$ are the labels of the preceding resp. following intervals, then it is possible to see that the $i$-th Euler class $e_i$ associated with $S_i^2$ [see eq. (2.22)] is given by

$$e_i = \frac{1}{2\pi} \int_{S_i^2} F \left| \begin{array}{cc} p_{i-1} & p_{i+1} \\ q_{i-1} & q_{i+1} \end{array} \right|.$$  \hspace{1cm} (2.30)

The last interval $(1,0)$ corresponds to the boundary component $\{0\} \times \mathbb{R}^2$ of $\hat{Y}$. The corresponding Euler class is found setting $p_{i+1} = 0, q_{i+1} = 1$. The horizon $H$ corresponds to either a separate boundary sphere $S_h^2$, or a part of $\{0\} \times \mathbb{R}^2$. The polyhedral arcs are obtained as follows. Take the boundary $\partial \hat{Y}$ (a line), and delete any interval that is labeled by $(1,0), (0,1)$ together with its nearest neighbors, and take away the interval $I_h$ corresponding to $H$. This cuts the line into several connected pieces which are labeled each by a pair of relatively prime integers. These connected pieces correspond to the polyhedral arcs. The procedure is explained in figure 9.

In summary, if the black hole has the isometry group $\mathbb{R} \times U(1) \times U(1)$, then the weighted orbit space of a slice is given by the simpler symbol

$$\Sigma : \{ \mathbb{R}^+ \times \mathbb{R}^2 \setminus \bigcup_i D_i^3; e_1, \ldots, e_b; \gamma_1, (p_1, q_1), \ldots, \gamma_k, (p_k, q_k) \},$$  \hspace{1cm} (2.31)

where none of the polyhedral arcs close, and where the Euler classes are as above in eq. (2.30).

3 Outlook

Let us finally discuss generalizations of our results. The most obvious question is whether one can obtain not only a classification of the topology, but in fact of the metrics of stationary black holes. In the case of symmetry group $\mathbb{R} \times U(1) \times U(1)$, this was achieved in [15, 16]. There it was found that what characterizes the solution uniquely (if it exists) are its conserved charges as well as the data of the weighted orbit space $\hat{Y}$ (see previous section and figure 9). The decoration data include the collection of winding numbers $(p_i, q_i)$, as well as the lengths $l_i$ of the intervals. In the situation considered in this paper, one can only assume the existence of one $U(1)$, and the decorated orbit space is now 3-dimensional (see eq. (2.23) and figure 1), and has labelled polyhedral curves and certain areas on the boundary that correspond to the horizon. In this case, one would expect that one needs further data to uniquely specify the spacetime metric, such as the length and angles between the arc segments, and/or e.g. the areas of the horizon domain (the shaded area in figure 1). The latter has been suggested by [35], but he seems to ignore the polyhedral arcs.

The second, easier, question is what happens when matter fields are included. Our results rely on the rigidity theorem [12, 13, 14] the topology theorem [5], and the topological
censorship theorem [10, 11]. For present proof of the rigidity theorem, the essential requirements are that the null-energy condition holds, that the theory possess a well-posed (characteristic) initial value formulation, and that the domain of outer communication be simply connected and analytic. These requirements hold e.g. for a cosmological constant of either sign, Maxwell fields, or the (bosonic sector of) minimal supergravity in 5 dimensions. The requirements of analyticity, initial value formulation, and simply connectedness (which would fail e.g. for asymptotically Kaluza-Klein theories) are actually only needed in order to define the extra $U(1)$-symmetry globally, but they are not required in order to merely conclude that the horizon metric is $U(1)$-invariant. Similarly, the proof of [5] only requires the dominant energy condition. As a consequence, our result 1 will continue to hold for any Einstein-matter theory satisfying the dominant energy condition. Result 2 on the other hand relies in an essential way on the fact that the domain of outer communication is simply connected, which in turn is a consequence of the topological censorship theorem. This theorem requires the null energy condition and allows one to conclude that the simply connectedness properties of the spacetime are essentially the same as those of the asymptotic region. Therefore, this result will not generalize in the present form if e.g. the spacetime is asymptotically Kaluza-Klein, hence not simply connected. Also, result 2 relies in an essential way on the global existence of a further $U(1)$ symmetry as guaranteed by the rigidity theorem. If the spacetime is not real analytic, the present proofs do not work, and result 2 again does not seem to follow.

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Figure 9: This figure shows the weighted 2-dimensional orbit space $\hat{Y} = \Sigma/[U(1) \times U(1)]$ and its relation to the corresponding 3-dimensional one $\hat{\Sigma} = \Sigma/[U(1) \times \{1\}]$. Thus, if there are two $U(1)$-factors, the corresponding 3-dimensional weighted orbit space is more restricted. In particular, in that case, there cannot be any knotted polyhedral arcs as suggested in figure 1 which represents the general situation.