Reduced-State Synchronization of Quantum Networks: Convergence, Graphical Information Hierarchy, and the Missing Symmetry

Guodong Shi, Shuangshuang Fu, and Ian R. Petersen

Abstract

We establish a thorough treatment of reduced-state synchronization for qubit networks with the aim of driving the qubits’ reduced states to a common trajectory. The evolution of the quantum network’s state is described by a master equation, where the network Hamiltonian is either a direct sum or a tensor product of identical qubit Hamiltonians, and the coupling terms are given by a set of permutation operators over the network. The permutations introduce naturally quantum directed interactions. We show that reduced-state synchronization is achieved if and only if the quantum interaction graphs corresponding to the permutation operators form a strongly connected union graph. The proof is based on an algebraic analysis making use of the Perron-Frobenius theorem for non-negative matrices. The convergence rate and the limiting orbit are explicitly characterized. Numerical examples are provided illustrating the obtained results. Further, we investigate the missing symmetry in the reduced-state synchronization from a graphical point of view. The information-flow hierarchy in quantum permutation operators is characterized by different layers of information-induced graphs. We show that the quantum synchronization equation is by nature equivalent to several parallel cut-balanced consensus processes, and a necessary and sufficient condition is obtained for quantum reduced-state synchronization under switching interactions applying recent work of Hendrickx and Tsitsiklis.

Keywords: Quantum networks, Reduced-state synchronization, Master equations
1 Introduction

1.1 Motivation and Background

Synchronization of a network of coupled dynamics is ubiquitous in nature, where examples from a variety of scientific disciplines arise such as power grids [1], social networks [2], brain cells [3], and animal groups [4]. As early as the 17th century, Huygens discovered that two pendulum clocks hung together tend to synchronize, which he referred to as “an odd kind of sympathy” [5]. In the past decades, tremendous research efforts have been made in establishing rigorous mathematical modeling and treatment that can provide definitions and explanations to such synchronization phenomena [6].

Frequency synchronization is often considered for coupled phase oscillators (a typical example is the Kuramoto oscillator [7, 8]), where the interactions of the network components (nodes) are described by nonlinear diffusive couplings and the inherent free motion of each node is relatively simple. A nice survey of the developments along this line is provided in [9]. Phase synchronization, which requires the trajectories of the nodes to asymptotically agree, applies to the cases when the node interaction is linear with possibly more complex and even nonlinear free motion [10, 11]. Initiated by [12, 13] (though the basic idea can be traced back to [14]), distributed consensus control, where synchronization is reduced to the condition that all nodes’s state converge to the same limit, recently has attracted much attention in the area of control theory, and a particular advantage is that general switching interactions can be well handled. The strong involvement of a graph-theoretic approach has indeed reshaped the study of networked control systems in the past decade [15].

Scientific interest on synchronization subject to the laws of quantum mechanics [16] has also been noticed. Taking advantage of quantum algorithms to enhance classical clock synchronization was considered from the point of view of quantum information processing in [17, 18]. Quantum synchronization driven by external periodic signals was studied numerically in [19, 20]. Recent work [21] introduced a quantum analogue of frequency and phase synchronization concepts.
1.2 Related Works

Sepulchre et al. [22] generalized consensus algorithms to non-commutative spaces and presented convergence results for quantum stochastic maps to a fully mixed state. Mazzarella et al. [23] made a systematic study of consensus-seeking in quantum networks, where four classes of consensus quantum states based on invariance and symmetry properties were introduced and a quantum gossip algorithm [24] was proposed for reaching a symmetric consensus state over a quantum network. The class of quantum gossip algorithms can be further extended to symmetrization problems in a group-theoretic framework and be applied to consensus on probability distributions and quantum dynamical decoupling [25].

Developments in continuous-time quantum consensus seeking were made in [26, 27, 28] for Markovian quantum dynamics governed by master equations [29]. In [26], using a group-theoretic analysis, the authors proposed a consensus master equation involving permutation operators and showed that a symmetric state consensus can be achieved under such evolutions. In [28], the same master equation approach was proposed for swapping operators with nontrivial network Hamiltonians. A type of quantum synchronization is shown in the sense that the network trajectory tends to an orbit determined by the network Hamiltonian and the symmetrization of the initial state, which implies that all quantum nodes asymptotically reach the same orbit [28]. For research on quantum network control and information processing we refer to [30, 31, 32]; for a survey for quantum control theory we refer to [33].

1.3 Main Results

In this paper, we aim to develop a thorough investigation of reduced-state synchronization for qubit (i.e., quantum bit) networks. Note that the reduced state of a qubit, given by tracing out the remaining qubits’ information, fully captures the information that qubit holds [16]. As a result, reduced-state synchronization is analogous to phase synchronization in the classical sense. Our discussions rely heavily on the concepts of quantum consensus which originated in [23, 26], and the main contribution of the current paper is to push the graphical theoretic analysis in [28] to its limit and therefore establish a thorough understanding of quantum reduced-state synchronization.

The evolution of the quantum network state is given by a Lindblad master equation where the Lindblad operators are in a set of qubit permutations [34]. Every permutation operator
defines a directed quantum interaction graph, and the aim of the qubits is to reach reduced-state synchronization in the sense that their reduced states tend to a common orbit. We show that such reduced-state synchronization is reached if and only if the quantum interaction graphs corresponding to the permutations form a strongly connected union graph. The proof is built on an algebraic argument using the Perron-Frobenius theorem. The limiting orbit is explicitly given and the convergence speed is also characterized.

We also study the missing symmetry in the reduced-state synchronization, compared to the symmetric consensus state proposed in [23, 26]. We make a graph-theoretic study of the information-flow hierarchy in quantum permutation operators, and present detailed characterizations of the different layers of information-induced graphs. Such a characterization helps us to establish a clear bridge between quantum and classical consensus dynamics, based on which the full details in the quantum state evolution can be visualized. As a by-product of this graphical analysis, we show that the quantum synchronization equation is by nature equivalent to a parallel cut-balanced classical consensus processes, and then a necessary and sufficient condition is obtained for reaching quantum reduced-state synchronization under switching interactions in light of a recent result by Hendrickx and Tsitsiklis [39].

1.4 Paper Organization

Section 2 presents some preliminaries including relevant concepts in linear algebra, graph theory and quantum systems. Section 3 introduces the n-qubits network model, its state evolution and the problem of interest. Section 4 establishes convergence conditions for quantum reduced-state synchronization on the considered quantum network, presents detailed characterizations of the limit orbit, and provides numerical examples illustrating the obtained results. Section 5 turns to a graph-theoretic description of the information hierarchy in the quantum permutation operators, based on which we obtain a full interpretation of the missing symmetry between reduced-state consensus and symmetric-state consensus. Finally, Section 6 concludes the paper with a few remarks.
2 Preliminaries

In this section, we introduce some concepts and theories in linear algebra [35], graph theory [37], and quantum systems [16].

2.1 Directed Graphs

A (simple) directed graph \( G = (V, E) \), or in short, a digraph, consists of a finite set \( V = \{1, \ldots, N\} \) of nodes and an arc set \( E \), where an element \( e = (i, j) \in E \) denotes an arc from node \( i \in V \) to \( j \in V \) with \( i \neq j \). A directed path between two vertices \( v_1 \) and \( v_k \) in \( G \) is a sequence of distinct nodes \( v_1, v_2, \ldots, v_k \) such that for any \( m = 1, \ldots, k - 1 \), there is an arc from \( v_m \) to \( v_{m+1} \); \( v_1v_2\ldots v_k \) is called a semi-path if for any \( m = 1, \ldots, k - 1 \), either \((v_m, v_{m+1}) \in E\) or \((v_{m+1}, v_m) \in E\). We call graph \( G \) to be fully connected if \((i, j) \in E \) for all \( i \neq j \in V \); strongly connected if, for every pair of distinct nodes in \( V \), there is a path from one to the other; quasi-strongly connected if there exists a node \( v \in V \) such that there is a path from \( v \) to all other nodes; weakly connected if there is a semi-path between any two distinct nodes. The in-degree of \( v \in V \), denoted \( \text{deg}^{-}(v) \), is the number of nodes from which there is an arc entering \( v \). The out-degree \( \text{deg}^{+}(v) \) can be correspondingly defined. The directed graph \( G \) is called balanced if \( \text{deg}^{+}(v) = \text{deg}^{-}(v) \) for any \( v \in V \).

A subgraph of \( G \) associated with \( V^* \subseteq V \), denoted \( G|_{V^*} \), is the graph \((V^*, E^*)\) with \((i, j) \in E^* \) if and only if \((i, j) \in E \) for \( i, j \in V^* \). A weakly connected component (or, simply component) of \( G \) is a weakly connected subgraph associated with some \( V^* \subseteq V \), with no arc between \( V^* \) and \( V \setminus V^* \). The following lemma is well-known characterizing the connectivity of balanced digraphs. We provide a simple proof in Appendix A.1.

**Lemma 1** A balanced digraph \( G = (V, E) \) is weakly connected if and only if it is strongly connected.

The Laplacian of \( G \), denoted \( L(G) \), is defined as

\[
L(G) = D(G) - A(G),
\]

where \( A(G) \) is the \( N \times N \) matrix given by \( [A(G)]_{kj} = 1 \) if \((j, k) \in E \) and \([A(G)]_{kj} = 0 \) otherwise, and \( D(G) = \text{diag}(d_1, \ldots, d_N) \) with \( d_k = \sum_{j=1, j \neq k}^{N} [A(G)]_{kj} \). By definition it is self-evident that zero is always an eigenvalue of \( L(G) \) for any directed graph \( G \).
2.2 Linear Algebra

Given a matrix $M \in \mathbb{C}^{m \times n}$, the vectorization of $M$, denoted by $\text{vec}(M)$, is the $mn \times 1$ column vector $(\lfloor M \rfloor_{11}, \ldots, \lfloor M \rfloor_{m1}, \lfloor M \rfloor_{12}, \ldots, \lfloor M \rfloor_{m2}, \ldots, \lfloor M \rfloor_{1n}, \ldots, \lfloor M \rfloor_{mn})^T$. We have $\text{vec}(ABC) = (C^T \otimes A)\text{vec}(B)$ for all matrices $A, B, C$ with $ABC$ well defined, where $\otimes$ stands for the Kronecker product. We always use $I_\ell$ to denote the $\ell \times \ell$ identity matrix, and $1_\ell$ for the all one vector in $\mathbb{R}^\ell$.

The following lemma is known as the Geršgorin disc Theorem.

**Lemma 2** (pp. 344, [35]) Let $A \in \mathbb{C}^{m \times m}$. Then all eigenvalues of $A$ are located in the union of $m$ discs

$$
\bigcup_{i=1}^{m} \left\{ z \in \mathbb{C} : |z - \lfloor A \rfloor_{ii}| \leq \sum_{j=1,j\neq i}^{m} |\lfloor A \rfloor_{ij}| \right\}.
$$

A matrix $A \in \mathbb{R}^{m \times m}$ is called a nonnegative matrix if all its elements are nonnegative real numbers. We call $A$ a *stochastic matrix* if $A1_m = 1_m$, i.e., all the row sums of $A$ are equal to one. We call $A$ to be doubly stochastic if both $A$ and $A^T$ are stochastic. A matrix $A$ is called to be *irreducible* if $A$ cannot be conjugated into block upper triangular form by a permutation matrix $P$, i.e.,

$$
PAP^{-1} = \begin{pmatrix} E & F \\ 0 & G \end{pmatrix},
$$

where $E$ and $G$ are square matrices of sizes greater than zero.

For any nonnegative matrix $A \in \mathbb{R}^{N \times N}$, we can define its induced graph $G_A = (V, E_A)$ by that $V = \{1, \ldots, N\}$ and $(i, j) \in E_A$ if and only if $i \neq j$ and $[A]_{ji} > 0$. A nonnegative matrix $A$ is irreducible if and only if $G_A$ is strongly connected. The following lemma is the famous Perron-Frobenius Theorem [36] for irreducible nonnegative matrices.

**Lemma 3** Let $A$ be an irreducible nonnegative matrix. Then its spectral radius $\lambda(A) > 0$ is a simple eigenvalue of $A$ which corresponds to a positive eigenvector.
2.3 Quantum Mechanics

2.3.1 Quantum System and Master Equation

The state space associated with any isolated quantum system is a Hilbert space which is a complex vector space with inner product \[ |16| \]. The state of a quantum system is a unit vector in the system’s state space. For any Hilbert space \( \mathcal{H}_* \), it is convenient to use \(|\cdot\rangle\), known as the Dirac notion, to denote a unit (column) vector in \( \mathcal{H}_* \). The complex conjugate of \(|\xi\rangle\) is denoted as \( \langle\xi| \). The state space of a composite quantum system is the tensor product, denoted \( \otimes \), of the state space of each component system. For a quantum system, its state can also be described by a density operator \( \rho \), which is Hermitian, positive in the sense that all its eigenvalues are non-negative, and \( \text{tr}(\rho) = 1 \). For any \(|p\rangle, |q\rangle \in \mathcal{H}_* \), we use the notion \(|p\rangle\langle q|\) to denote the operator over \( \mathcal{H}_* \) defined by

\[
(|p\rangle\langle q|)|\eta\rangle = \langle q, |\eta\rangle|p\rangle, \quad \forall|\eta\rangle \in \mathcal{H}_*,
\]

where \( \langle \cdot, \cdot \rangle \) represents the inner product on the Hilbert space \( \mathcal{H}_* \). In standard quantum mechanical notation the inner product \( \langle p|q\rangle \) is denoted as \( \langle p|q\rangle \).

The evolution of the state \(|\xi\rangle\) of a closed quantum system is described by the Schrödinger equation. Equivalently these dynamics can be also written in forms of the evolution of the density operator \( \rho \), called the von Neumann equation. When a quantum system interacts with the environment, a Markovian approximation can be applied under the assumption of a short environmental correlation time permitting the neglect of memory effects \[29\]. Markovian master equations have been widely used to model quantum systems with external inputs in quantum control, especially for Markovian quantum feedback \[40\]. The so-called Lindblad master equation is described as \[34\]

\[
\frac{d}{dt}\rho(t) = -\frac{i}{\hbar}[H, \rho(t)] + \sum_{k=1}^{K} \gamma_k \mathcal{D}[L_k]\rho(t),
\]

where \( H \) is a Hermitian operator on the underlying Hilbert space known as the system Hamiltonian, \( i^2 = -1 \), \( \hbar \) is the reduced Planck constant, the non-negative coefficient \( \gamma_k \)'s specify the relevant relaxation rates, and

\[
\mathcal{D}[L_k]\rho = L_k\rho L_k^\dagger - \frac{1}{2} L_k^\dagger L_k\rho - \frac{1}{2} \rho L_k^\dagger L_k.
\]

Here the \( L_k \)'s are the Lindblad operators representing the coupling of the system to the environment.
2.3.2 Partial Trace

Let $H_A$ and $H_B$ be the state spaces of two quantum systems $A$ and $B$, respectively. Their composite system is described as a density operator $\rho^{AB}$. Let $L_A$, $L_B$, and $L_{AB}$ be the spaces of (linear) operators over $H_A$, $H_B$, and $H_A \otimes H_B$, respectively. Then the partial trace over system $B$, denoted by $\text{Tr}_{H_B}$, is an operator mapping from $L_{AB}$ to $L_A$ defined by

$$\text{Tr}_{H_B}(|p_A\rangle\langle q_A| \otimes |p_B\rangle\langle q_B|) = |p_A\rangle\langle q_A| \text{Tr}(|p_B\rangle\langle q_B|), \quad \forall |p_A\rangle, |q_A\rangle \in H_A, |p_B\rangle, |q_B\rangle \in H_B.$$  

The reduced density operator (state) for system $A$, when the composite system is in the state $\rho^{AB}$, is defined as $\rho^A = \text{Tr}_{H_B}(\rho^{AB})$. The physical interpretation of $\rho^A$ is that $\rho^A$ holds the full information of system $A$ in $\rho^{AB}$. For a detailed description we hereby refer to [16].

3 Problem Definition

3.1 Qubit Network and Permutations

In quantum mechanical systems, a two-dimensional Hilbert space forms the most basic quantum system, called a qubit system. Let $H$ be a qubit space with a basis denoted by $|0\rangle$ and $|1\rangle$. In this paper, we consider a quantum network with $n$ qubits indexed in the set $V = \{1, \ldots, n\}$ and the state space of this $n$-qubit quantum network is denoted as the Hilbert space $H^\otimes n = H \otimes \cdots \otimes H$. The density operator of this $n$-qubit network is denoted as $\rho$.

Interactions among the qubits are introduced by permutations. An $n$’th permutation is a bijection over $V$, denoted by $\pi$. Denote the set of all $n$’th permutations as $\mathfrak{P}$. There are $n!$ elements in $\mathfrak{P}$. We can define the product of two permutations $\pi_1, \pi_2 \in \mathfrak{P}$ as their composition, denoted $\pi_1\pi_2 \in \mathfrak{P}$, in that

$$\pi_1\pi_2 : \pi_1\pi_2(i) = \pi_1(\pi_2(i)), \quad i \in V.$$  

In this way the set $\mathfrak{P}$ equipped with this product operation defines a group known as the permutation group. Now associated with any $\pi \in \mathfrak{P}$, we define the corresponding operator $U_\pi$ over $H^\otimes n$, by

$$U_\pi(q_1 \otimes \cdots \otimes q_n) = q_{\pi(1)} \otimes \cdots \otimes q_{\pi(n)}, \quad q_i \in H, i = 1, \ldots, n.$$  

In this way, the operator $U_\pi$ permutes the states of the qubits. Particularly, a permutation $\pi$ is called a swapping between $j$ and $k$, if $\pi(j) = k$, $\pi(k) = j$, and $\pi(s) = s, s \in V \setminus \{j, k\}$. 

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The following lemma holds whose proof has been put in Appendix A.2.

**Lemma 4** For any $\pi \in \mathfrak{P}$, we have $U_\pi U_\pi^\dagger = U_\pi^\dagger U_\pi = I$.

The following definition provides a graphical interpretation to $U_\pi$.

**Definition 1** The quantum interaction graph associated with $U_\pi$, denoted $G_\pi = (V, E_\pi)$, is the directed graph over $V$ with

$$E_\pi := \left\{ (i, \pi(i)) : i \neq \pi(i), i \in V \right\}.$$ 

The quantum interaction graph $G_\pi$ indicates the information flow along the permutation operator $U_\pi$ (see Figure 1 for an illustration).

### 3.2 State Evolution

Let $H$ be the (time-invariant) Hamiltonian of the $n$-qubit quantum network. Let $\mathfrak{P}_* \subseteq \mathfrak{P}$ be a subset of the permutation group. We investigate the state evolution of the quantum network described by the following master equation

$$\frac{d\rho}{dt} = -\frac{i}{\hbar} [H, \rho] + \sum_{\pi \in \mathfrak{P}_*} \left( U_\pi \rho U_\pi^\dagger - \rho \right).$$  \hspace{1cm} (3)

Let $H_0$ be a Hermitian operator over $\mathcal{H}$. Denote the direct sum $H_0^{\oplus n} = \sum_{i=1}^n I^{\otimes (i-1)} \otimes H_0 \otimes I^{\otimes (n-i)}$. We make the following assumption as our standing assumption throughout the whole paper.

**Assumption** There exists a Hermitian operator $H_0$ over $\mathcal{H}$ such that either $H = H_0^{\oplus n}$ or $H = H_0^{\oplus n}$. 

Figure 1: The quantum interaction graph $G_\pi$ over a three-qubit network for a given $\pi$ with $\pi(1) = 2, \pi(2) = 3, \pi(3) = 1$. 

\[
\begin{pmatrix}
1 & 2 & 3 \\
2 & 3 & 1
\end{pmatrix}
\]
Remark 1 From Lemma 4 we see that Eq. (3) defines a quantum master equation in the Lindblad form (2).

Remark 2 If we choose a positive number \( w_\pi > 0 \) as the weight of the permutation \( \pi \in \mathcal{P}_* \), the system (3) becomes

\[
\frac{d\rho}{dt} = -\frac{i}{\hbar} [H, \rho] + \sum_{\pi \in \mathcal{P}_*} w_\pi \left( U_\pi \rho U_\pi^\dagger - \rho \right).
\] (4)

Extension of the results established for (3) in the current paper to the weighted dynamics (4), (even for the case with \( w_\pi \) being time-dependent), is straightforward. The paper therefore focuses on the discussion of (3) for the ease of presentation.

Remark 3 The system (4) with the constraint that every element in \( \mathcal{P}_* \) is a swapping operator was investigated in [27] (and a more detailed discussion later in [28]). The system (4) with \( H = 0 \) was studied in [26]. The aim of [27, 26] is to drive the quantum system to a symmetric state as introduced in [23]. As a result, \( \mathcal{P}_* \) must be a generating subset of the entire permutation group \( \mathcal{P} \) in [27, 26].

Remark 4 The assumption that either \( H = H_0^\otimes n \) or \( H = H_0^\boxtimes n \) with \( H_0 \) being a Hermitian operator over \( \mathcal{H} \) was introduced in [28]. The reason is that the network Hamiltonian must be compatible with the permutation operators in that \( [H, U_\pi] = 0 \) for all \( \pi \in \mathcal{P} \) if the assumption holds [28]. From a physical point of view, \( H = H_0^\otimes n \) means that there are no internal interactions among the qubits, while \( H = H_0^\boxtimes n \) means that the internal interactions spread homogeneously across the network.

3.3 Objectives

Let \( \mathcal{H}_i \) denote the two-dimensional Hilbert space corresponding to qubit \( i, i \in V \). We denote by

\[
\rho^k(t) := \text{Tr}_{\otimes_{j \neq k} \mathcal{H}_j} (\rho(t))
\]

the reduced state of qubit \( k \) at time \( t \) for each \( k = 1, \ldots, n \), where \( \otimes_{j \neq k} \mathcal{H}_j \) stands for the remaining \( n - 1 \) qubits’ space \( \otimes_{j \neq k} \mathcal{H}_j \) and \( \text{Tr}_{\otimes_{j \neq k} \mathcal{H}_j} \) is the partial trace. Note that \( \rho^k(t) \) contains all the information that qubit \( k \) holds in the composite state \( \rho(t) \). We introduce the following synchronization condition to the considered qubit network.
Definition 2 The system (3) achieves global reduced-state synchronization if
\[
\lim_{t \to \infty} \left\| \rho^j(t) - \rho^k(t) \right\| = 0 \tag{5}
\]
for all \(j, k \in V\) and for any initial state \(\rho_0 = \rho(0)\).

4 Reduced-State Synchronization

In this section, we study the asymptotic behavior of the quantum state for the system (3). We show that synchronization is achieved over the qubit networks in the sense that the reduced states at each qubit tend a common trajectory, as long as the qubit interactions given by permutations in \(\mathcal{P}\) provide suitable connectivity.

This section is organized as follows. In Subsection 4.1, we present the main results and provide some remarks on the essence of these results. Next, we show some numerical examples verifying the conclusions drawn as well as illustrating the intuition behind in Subsection 4.2. Finally, Subsection 4.3 provides all the technical proofs of the statements.

4.1 Main Results

Note that for any two digraphs sharing the same node set \(G_1 = (V, E_1)\) and \(G_2 = (V, E_2)\), we can define their union as \(G_1 \cup G_2 = (V, E_1 \cup E_2)\). Recall that the quantum interaction graph associated with \(U_\pi\) for \(\pi \in \mathcal{P}\) is denoted \(G_\pi\). Note also that for any given subset of a group, we can define its generated subgroup as the smallest subset of the group containing all of the elements of the set and by itself being a group. Let \(\mathcal{C}_\mathcal{P}\) be the subgroup generated by \(\mathcal{P}\).

For the cases with \(H = H_0^\otimes n\) and \(H = H_0^\oplus n\), our main results regarding the synchronization condition for the system (3) are as follows, respectively.

Theorem 1 Suppose \(H = H_0^\otimes n\). The system (3) achieves global reduced-state synchronization if and only if
\[
\mathcal{G}_\mathcal{P}_* := \bigcup_{\pi \in \mathcal{P}_*} G_\pi
\]
is strongly connected. In this case when reduced-state synchronization is achieved we have
\[
\lim_{t \to \infty} \left\| \rho^k(t) - e^{-iH_0 t/\hbar} \left[ \text{Tr}_{\otimes_{j=1}^{n-1} H_j} \left( \frac{1}{|\mathcal{C}_\mathcal{P}_*|} \sum_{\pi \in \mathcal{C}_\mathcal{P}_*} U_\pi \rho_0 U_\pi^\dagger \right) \right] e^{iH_0 t/\hbar} \right\| = 0
\]
for all \(k \in V\).
Theorem 2 Suppose $H = H_0^\otimes n$. The system (3) achieves global reduced-state synchronization if and in general only if $G_{\mathcal{P}_*}$ is strongly connected. In this case when reduced-state synchronization is achieved we have

$$\lim_{t \to \infty} \| \rho(t) - e^{-iHt/\hbar} \left( \frac{1}{|\mathcal{C}_{\mathcal{P}_*}|} \sum_{\pi \in \mathcal{C}_{\mathcal{P}_*}} U_\pi \rho_0 U_\pi^\dagger \right) e^{iHt/\hbar} \| = 0.$$  \hspace{1cm} (6)

Remark 5 Theorems 1 and 2 improved the results in [27, 28, 26] in the following aspects. First of all, trajectories’ convergence is established for arbitrary $\mathcal{P}_*$ without any additional assumptions (cf. the upcoming Proposition 3). Moreover, a tight graphical condition is obtained on $\mathcal{P}_*$ for guaranteeing reduced-state synchronization, which only requires $G_{\mathcal{P}_*}$ to be strongly connected. Finally, the results fill the gap between the two quantum consensus concepts, i.e., reduced-state consensus and symmetric-state consensus, introduced in [23], by clear distinctions of $\mathcal{P}_*$.

As will be shown in the following discussions, $G_{\mathcal{P}_*}$ is also not even weakly connected if it is not strongly connected. This means that the qubits in the set $V$ contain two disjoint parts which never interact with each other under permutation operators given by $\pi \in \mathcal{P}_*$. We however cannot fully rule out the internal qubit interactions enforced by $H_0^\otimes n$. Therefore, here by saying “in general only if” in Theorem 2 we mean that we can always construct examples of $H_0$ and qubit networks, under which strong connectivity of $G_{\mathcal{P}_*}$ becomes essentially necessary for reduced-state synchronization.

The proof of Theorems 1 and 2 are built step by step via the following intermediate conclusions, which may be of independent interest. We define

$$\rho_{\mathcal{P}_*} = \frac{1}{|\mathcal{C}_{\mathcal{P}_*}|} \sum_{\pi \in \mathcal{C}_{\mathcal{P}_*}} U_\pi \rho U_\pi^\dagger$$

as the $\mathcal{P}_*$-average of a given density operator $\rho \in \mathcal{L}_H^\otimes n$. Clearly this $\mathcal{P}_*$-average of the initial value $\rho_0$ plays a crucial role in above results. The $\mathcal{P}_*$-average is the arithmetic mean of the images of the permutation operators $U_\pi$ over $\rho$ for $\pi$ in the generated subgroup $\mathcal{C}_{\mathcal{P}_*}$. When $\mathcal{C}_{\mathcal{P}_*} = \mathcal{P}_*$, the $\mathcal{P}_*$-average of $\rho$ becomes

$$\rho_{\mathcal{P}_*} = \frac{1}{n!} \sum_{\pi \in \mathcal{P}_*} U_\pi \rho U_\pi^\dagger,$$

which corresponds to the quantum average consensus state introduced in [23]. As has been shown in [23], for any $\rho$, its $\mathcal{P}$-average $\rho_{\mathcal{P}_*}$ is symmetric in the sense that it is invariant under
any permutation operation, which immediately implies that the reduced states at each qubit are identical in $\rho_{\mathcal{P}^*}$.

The following proposition provides a tight criterion regarding when a $\mathcal{P}^*$-average generates identical reduced states.

**Proposition 1** Let $\rho_{\mathcal{P}^*}$ be the $\mathcal{P}^*$-average of $\rho \in \mathcal{L}_{\mathcal{H}^\otimes n}$ and denote $\rho^k_{\mathcal{P}^*} = \text{Tr}_{\otimes j \neq k} \mathcal{H}_j (\rho_{\mathcal{P}^*})$. Then $\rho^i_{\mathcal{P}^*} = \rho^j_{\mathcal{P}^*}$, $i, j \in \mathcal{V}$ for all $\rho \in \mathcal{L}_{\mathcal{H}^\otimes n}$ if and only if $G_{\mathcal{P}^*}$ is strongly connected.

In fact, we can even show the following relation between $\rho^k_{\mathcal{P}^*}$ and $\rho^k_{\mathcal{P}^*}$.

**Proposition 2** Suppose $G_{\mathcal{P}^*}$ is strongly connected. Then there holds

$$\rho^k_{\mathcal{P}^*} = \rho^k_{\mathcal{P}^*} = \frac{1}{n} \sum_{m=1}^{n} \rho^m$$

for all $k \in \mathcal{V}$, where $\rho^m = \text{Tr}_{\otimes j \neq m} \mathcal{H}_j (\rho)$ is the reduced state of qubit $m$ of $\rho$.

Consider the state evolution of the system (3) with $H = 0$. We call

$$\frac{d\rho}{dt} = \sum_{\pi \in \mathcal{P}^*} (U_\pi \rho U_\pi^\dagger - \rho)$$

(7)

the quantum consensus master equation. The following proposition holds.

**Proposition 3** For the system (7) we have

$$\lim_{t \to \infty} \rho(t) = \sum_{\pi \in \mathcal{P}^*} U_\pi \rho_0 U_\pi^\dagger / |\mathcal{C}_{\mathcal{P}^*}|$$

for all initial state $\rho_0 = \rho(0)$.

Note that Proposition 3 indicates that for (7), the network’s density operator converges to the $\mathcal{P}^*$-average of the initial density operator $\rho_0$. This convergence holds true for any choice of $\mathcal{P}^*$. With Proposition 2 we further see that the synchronization orbit of the qubits’ reduced states established in Theorem 1 under $\mathcal{P}^*$ is exactly the same as the synchronization orbit obtained under the entire permutation group $\mathcal{P}$: the quantum consensus master equation averages the initial reduced states of the qubits in the network.

From the proof of Proposition 3 we can even show that the convergence is exponential, with the exact convergence rate given by

$$\min_{\lambda_i \neq 0} \text{Re}(\lambda_i (L_\mathcal{P}^*))$$.
Here
\[ L_* := \sum_{\pi \in \Psi_\pi} \left( I_{2^n} \otimes I_{2^n} - U_{\pi} \otimes U_{\pi} \right) \]

with \( U_{\pi} \) being the matrix representation of \( U_{\pi} \) and \( \otimes \) representing the Kronecker product, and \( \lambda_i(L_*) \) stands for an eigenvalue. From the proof it is also clear that this convergence rate also characterizes the rate of mixing of the limiting trajectories for both \( H = H_0^{\oplus n} \) and \( H = H_0^{\otimes n} \) in Theorems 1 and 2.

4.2 Numerical Examples

Consider three qubits indexed in the set \( \mathcal{V} = \{1, 2, 3\} \). We take \( \Psi_* = \{\pi_*\} \) with \( \pi_*(1) = 2, \pi_*(2) = 3, \pi_*(3) = 1 \) as shown in Figure 1. The corresponding \( G_{\pi_*} \) is a directed cycle which is obviously strongly connected. The initial network state is chosen to be \( \rho_0 = |10+\rangle \langle 10+| \)

with \( |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \). The network Hamiltonian is chosen to be \( H = \sigma_z \oplus \sigma_z \oplus \sigma_z \) or \( H = \sigma_z \otimes \sigma_z \otimes \sigma_z \), where
\[
\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

is one of the Pauli matrices.

4.2.1 Synchronization in Reduced States

We plot the evolution of the reduced states of the three qubits for the system on one Bloch sphere with initial value \( \rho_0 = |10+\rangle \langle 10+| \) for \( H = \sigma_z \oplus \sigma_z \oplus \sigma_z \) and \( H = \sigma_z \otimes \sigma_z \otimes \sigma_z \), respectively, in Figure 2. The qubits’ orbits asymptotically tend to the same trajectory for both of the two cases. However due to the internal interactions raised by the tensor products in the network Hamiltonian, the evolution of the qubits’ states gives different orbits for the two choices of \( H \).

The trace distance between two density operator \( \rho_1, \rho_2 \) over the same Hilbert space, is defined as
\[
D(\rho_1, \rho_2) = \frac{1}{2} \text{Tr} \sqrt{(\rho_1 - \rho_2)^\dagger (\rho_1 - \rho_2)}.
\]

We plot the trace distance function
\[
D(\rho^1(t), \rho^2(t)) + D(\rho^1(t), \rho^3(t)) + D(\rho^2(t), \rho^3(t))
\]
Figure 2: The evolution of the reduced states of the three qubits for initial value \( \rho_0 = |10+\rangle\langle10+| \) with \( H = \sigma_z \oplus \sigma_z \oplus \sigma_z \) (left), and \( H = \sigma_z \otimes \sigma_z \otimes \sigma_z \) (right), respectively.

Figure 3: The trace distance function \( D(\rho^1(t), \rho^2(t)) + D(\rho^1(t), \rho^3(t)) + D(\rho^2(t), \rho^3(t)) \) with \( H = \sigma_z \oplus \sigma_z \oplus \sigma_z \) (left), and \( H = \sigma_z \otimes \sigma_z \otimes \sigma_z \) (right), respectively.
Figure 4: The evolution of the reduced states of the three qubits for a different initial value with $H = \sigma_z \otimes \sigma_z \otimes \sigma_z$ (left), and $H = \sigma_z \otimes \sigma_z \otimes \sigma_z$ (right), respectively. Clearly when $H = \sigma_z \otimes \sigma_z \otimes \sigma_z$, drastic change appears for the shape of the limiting orbit compared to Figure 2.

for the system (3) with initial value $\rho_0 = |10+\rangle \langle 10 + |$, again for $H = \sigma_z \otimes \sigma_z \otimes \sigma_z$, and $H = \sigma_z \otimes \sigma_z \otimes \sigma_z$, respectively, in Figure 3. Clearly they all converge to zero with an exponential rate and they show exactly the same convergence speed since the speed only depends on $P^*$, as discussed in the previous subsection.

On the other hand, from Theorem 1 we know that when $H = \sigma_z \otimes \sigma_z \otimes \sigma_z$, the limiting orbit of each qubit’s reduced state is always parallel to the $x - y$ plane of the bloch sphere, no matter how the initial density operator is selected. In fact, we also know from Theorem 1 that in this case the $z$-axis position of the limiting orbit is determined uniquely by the $P^*$-average of the initial network state. However, when $H = \sigma_z \otimes \sigma_z \otimes \sigma_z$, there are internal interactions among the qubits, and as a result, the shape of the limiting orbit under $H = \sigma_z \otimes \sigma_z \otimes \sigma_z$ is no longer predictable with respect to the choice of initial density operators. We illustrate this point in Figure 4.

4.2.2 Partial Symmetrization

We now investigate the difference between the $P^*$-average and the $P$-average

$$\rho_{P^*} - \rho_P = \frac{1}{|\mathcal{P}_x|} \sum_{\pi \in \mathcal{P}_x} U_{\pi} \rho U^\dagger_{\pi} - \frac{1}{n!} \sum_{\pi \in \mathcal{P}} U_{\pi} \rho U^\dagger_{\pi}. $$
Figure 5: The zero-pattern of the average difference $\rho_{\pi^*} - \rho_{\pi}$. Potential nonzero entries are shadowed.

Under the standard computational basis of $\mathcal{L}_{H^\otimes 3}$

$$\{|p_1 p_2 p_3\rangle\langle q_1 q_2 q_3| : |p_i\rangle, |q_i\rangle \in \{|0\rangle, |1\rangle\}$$

we plot the zero-pattern for the entries of $\rho_{\pi^*} - \rho_{\pi}$ with the given $\mathcal{P}_* = \{\pi_*\}$ in Figure 5. The zero-pattern of $\rho_{\pi^*} - \rho_{\pi}$ is obtained as follows: we randomly select $\rho$, and shadow every entry that can be nonzero among the selections. From Figure 5 we clearly see the missing symmetry in $\rho_{\pi^*}$ indicated by the zero-pattern, which by itself shows certain symmetry.

4.3 Proofs of Statements

We now provide the detailed proofs of the various conclusions stated previously.

4.3.1 Proof of Proposition 1

The proof relies on some technical lemmas.

First of all we establish a lemma characterizing the property of the quantum interaction graph induced by permutation operators.

**Lemma 5** For any $\pi \in \mathcal{P}$, $G_\pi$ is a union of some disjoint directed cycles.
Next, the following lemma provides an operator form of the quantum operation $U_{\pi}$ induced by the permutation $\pi \in \mathcal{P}$.

**Lemma 6** For any $\pi \in \mathcal{P}$ and $A_i \in \mathcal{L}_H$, we have $U_{\pi}(A_1 \otimes \cdots \otimes A_n)U_{\pi}^\dagger = A_{\pi(1)} \otimes \cdots \otimes A_{\pi(n)}$.

The last lemma establishes the relation between the two digraphs $\bigcup_{\pi \in \mathcal{P}} \mathcal{G}_\pi$ and $\bigcup_{\pi \in \mathcal{C}_\mathcal{P}} \mathcal{G}_\pi$.

**Lemma 7** The digraph $\bigcup_{\pi \in \mathcal{P}} \mathcal{G}_\pi$ is strongly connected if and only if $\bigcup_{\pi \in \mathcal{C}_\mathcal{P}} \mathcal{G}_\pi$ is fully connected.

The proofs of Lemmas 5, 6, 7 have been put in Appendix A.3, A.4, A.5, respectively. We are now in a place to prove Proposition 1.

(Sufficiency.) Take $\rho \in \mathcal{L}_H \otimes n$. We denote by $\Theta_H := \{\theta_1, \ldots, \theta_4\}$ a basis of $\mathcal{L}_H$ and write

$$\rho = \sum_{i_1, \ldots, i_n} C_{i_1 \ldots i_n} \left(\theta_{i_1} \otimes \cdots \otimes \theta_{i_n}\right),$$

where $C_{i_1 \ldots i_n} \in \mathbb{C}$ and $\theta_{i_s} \in \Theta_H, s = 1, \ldots, n$. We now have

$$\rho_{\pi_s}^k = \text{Tr}_{i \neq k} \text{Tr}_{H_i} \left[ \frac{1}{|\mathcal{C}_\mathcal{P}|} \sum_{\pi \in \mathcal{C}_\mathcal{P}} U_{\pi} \left( \sum_{i_1, \ldots, i_n} C_{i_1 \ldots i_n} \left(\theta_{i_1} \otimes \cdots \otimes \theta_{i_n}\right) \right) U_{\pi}^\dagger \right]$$

$$= a) \text{Tr}_{i \neq k} \text{Tr}_{H_i} \left[ \frac{1}{|\mathcal{C}_\mathcal{P}|} \sum_{\pi \in \mathcal{C}_\mathcal{P}} \sum_{i_{1}, \ldots, i_n} C_{i_1 \ldots i_n} \left(\theta_{i_{\pi(1)}} \otimes \cdots \otimes \theta_{i_{\pi(n)}}\right) \right]$$

$$= b) \frac{1}{|\mathcal{C}_\mathcal{P}|} \sum_{\pi \in \mathcal{C}_\mathcal{P}} \sum_{i_{1}, \ldots, i_n} C_{i_1 \ldots i_n} \left[\theta_{i_{\pi(k)}} \prod_{j \neq k} \text{Tr}(\theta_{i_{\pi(j)}})\right],$$

(9)

where $a)$ holds from Lemma 6 and $b)$ follows from the definition of partial trace.

Take $m \neq k \in \mathcal{V}$. Since by assumption $\bigcup_{\pi \in \mathcal{P}_{\pi}} \mathcal{G}_\pi$ is strongly connected, $\bigcup_{\pi \in \mathcal{C}_{\mathcal{P}_{\mathcal{P}}}} \mathcal{G}_\pi$ is fully connected according to Lemma 7. As a result, there exists $\pi_s \in \mathcal{C}_{\mathcal{P}}$ such that $\pi_s(m) = k$. We
thus conclude
\[
\rho^m_{\pi_*} = \frac{1}{|\mathcal{C}_p_*|} \sum_{\pi \in \mathcal{C}_p_*} \sum_{i_1, \ldots, i_n} C_{i_1 \ldots i_n} \left[ \theta_{i_m(m)} \prod_{j \neq m} \text{Tr}(\theta_{i_x(j)}) \right]
\]
\[
eq \frac{1}{|\mathcal{C}_p_*|} \sum_{\pi \in \mathcal{C}_p_*} \sum_{i_1, \ldots, i_n} C_{i_1 \ldots i_n} \left[ \theta_{i_{\pi_*}(m)} \prod_{j \neq \pi_*} \text{Tr}(\theta_{i_x(j)}) \right]
\]
\[
eq \frac{1}{|\mathcal{C}_p_*|} \sum_{\pi \in \mathcal{C}_p_*} \sum_{i_1, \ldots, i_n} C_{i_1 \ldots i_n} \left[ \theta_{i_{\pi_*}(k)} \prod_{j \neq \pi_*} \text{Tr}(\theta_{i_x(j)}) \right]
\]
\[
eq \frac{1}{|\mathcal{C}_p_*|} \sum_{\pi \in \mathcal{C}_p_*} \sum_{i_1, \ldots, i_n} C_{i_1 \ldots i_n} \left[ \theta_{i_{\pi_*}(k)} \prod_{j \neq k} \text{Tr}(\theta_{i_x(j)}) \right]
\]
\[= \rho^k_{\pi_*}, \quad (10)
\]
where c) follows from the fact that \{\pi : \pi_{\pi_*} \in \mathcal{C}_p_*\} = \mathcal{C}_p_* for any \pi_* \in \mathcal{C}_p_* since \mathcal{C}_p_* is by itself a group; d) is from the selection of \pi_* which satisfies \pi_*(m) = k; e) holds again from \{\pi : \pi_{\pi_*} \in \mathcal{C}_p_*\} = \mathcal{C}_p_* . This proves the sufficiency part of the proposition.

(Necessity.) If \( \mathcal{G}_{\pi_*} := \bigcup_{\pi \in \mathcal{C}_p_*} \mathcal{G}_\pi \) is not strongly connected, then \( \mathcal{G}_{\pi_*} \) is also not weakly connected by Lemmas 1 and 7. This means that \( \mathcal{V} \) can be divided into two disjoint subsets \( \mathcal{V}_1 \) and \( \mathcal{V}_2 \) such that \( \rho_{\pi_*} \) never mixes the information of \( \rho^k \)'s in \( \mathcal{V}_1 \) and \( \mathcal{V}_2 \). We can easily construct examples of \( \rho_0 \) based on this understanding under which \( \rho^i_{\pi_*} \neq \rho^j_{\pi_*} \) for \( i \in \mathcal{V}_1 \) and \( j \in \mathcal{V}_2 \). This concludes the proof of Proposition 1.

\[\square\]

4.3.2 Proof of Proposition 2

We continue to make use of the notations introduced in the proof of Proposition 1. By (9) we have
\[
\rho^k_{\pi_*} = \frac{1}{|\mathcal{C}_p_*|} \sum_{\pi \in \mathcal{C}_p_*} \sum_{i_1, \ldots, i_n} C_{i_1 \ldots i_n} \left[ \theta_{i_{\pi_*}(k)} \prod_{j \neq k} \text{Tr}(\theta_{i_x(j)}) \right]
\]
\[= \sum_{i_1, \ldots, i_n} C_{i_1 \ldots i_n} \left[ \frac{1}{|\mathcal{C}_p_*|} \sum_{\pi \in \mathcal{C}_p_*} \theta_{i_{\pi_*}(k)} \prod_{j \neq k} \text{Tr}(\theta_{i_x(j)}) \right]. \quad (11)
\]
Furthermore, from Lemma 7, \( \bigcup_{\pi \in \mathcal{C}_p_*} \mathcal{G}_\pi \) is fully connected when \( \bigcup_{\pi \in \mathcal{C}_p_*} \mathcal{G}_\pi \) is strongly connected. As a result, introducing
\[
\mathcal{I}_k(m) := \left\{ \pi \in \mathcal{C}_p_* : \pi(k) = m \right\},
\]
we have \( |\mathcal{I}_k(l)| = |\mathcal{I}_k(m)| \) for all \( m, l \in \mathcal{V} \). Consequently, we conclude that
\[
\frac{1}{|\mathcal{C}_p_*|} \sum_{\pi \in \mathcal{C}_p_*} \theta_{i_{\pi_*}(k)} \prod_{j \neq k} \text{Tr}(\theta_{i_x(j)}) = \frac{1}{n} \sum_{s=1}^{n} \theta_i \prod_{j \neq s} \text{Tr}(\theta_{i_j}), \quad (12)
\]
which does not depend on the choice of $P_*$.

Finally, plugging in (11) with (12) we obtain

$$\rho_{P_*}^k = \frac{1}{n} \sum_{s=1}^{n} \sum_{i_1,...,i_n} C_{i_1...i_n} \left[ \theta_{i_s} \prod_{j \neq s} \text{Tr}(\theta_{i_j}) \right]$$

$$= \frac{1}{n} \sum_{m=1}^{n} \rho^m$$

for all $k \in V$. The proof is complete. \[\square\]

4.3.3 Proof of Proposition 3

We need a few preliminary lemmas.

The following lemma characterizes the fixed points of the so-called complete positive map. Similar conclusion was drawn in [41] and was later adapted to the following statement in [42] (Lemma 5.2).

**Lemma 8** Let $\mathcal{H}$ be a Hilbert space and denote by $\mathcal{L}_\mathcal{H}$ the space of linear operators over $\mathcal{H}$. Define $T : \mathcal{L}_\mathcal{H} \mapsto \mathcal{L}_\mathcal{H}$ by

$$T(X) = \sum_{j=1}^{K} M_j^\dagger X M_j, \quad X \in \mathcal{L}_\mathcal{H}$$

where $M_j \in \mathcal{L}_\mathcal{H}$ for all $j$, $\sum_{j=1}^{K} M_j^\dagger M_j = \sum_{j=1}^{K} M_j M_j^\dagger = I$. Then, for any given $X_0 \in \mathcal{L}_\mathcal{H}$, $\sum_{j=1}^{K} M_j^\dagger X_0 M_j = X_0$ if and only if $X_0 M_j = M_j X_0, j = 1,\ldots,K$.

Define $K_{P_*} : \mathcal{L}_\mathcal{H}^\otimes n \mapsto \mathcal{L}_\mathcal{H}^\otimes n$ by $K_{P_*}(\rho) = \sum_{\pi \in P_*} (U_{\pi} \rho U_{\pi}^\dagger - \rho)$. The following two lemmas hold.

**Lemma 9** $\text{Null}(K_{P_*}) := \{ \rho : K_{P_*}(\rho) = 0 \} = \{ \rho : \rho = \sum_{\pi \in P_*} U_{\pi} \rho U_{\pi}^\dagger / |C_{P_*}| \}$.

**Proof.** From Lemma 8, we know $U_{\pi} \rho_0 = \rho_0 U_{\pi}, \pi \in P_*$, if $K_{P_*}(\rho_0) = 0$. Obviously $U_{\pi} \rho_0 = \rho_0 U_{\pi}, \pi \in P_*$ implies $U_{\pi} \rho_0 = \rho_0 U_{\pi}, \pi \in P_*$, which in turn leads to $\rho_0 = \{ \rho = \sum_{\pi \in P_*} U_{\pi} \rho U_{\pi}^\dagger / |C_{P_*}| \}$.

On the other hand, if $\rho_0 = \sum_{\pi \in P_*} U_{\pi} \rho_0 U_{\pi}^\dagger / |C_{P_*}|$, then $U_{\pi} \rho_0 U_{\pi}^\dagger = \rho_0, \pi \in P_*$. This leads to that $\rho_0 = \{ \rho : K_{P_*}(\rho) = 0 \}$. The proof is now complete. \[\square\]

**Lemma 10** The zero eigenvalue of $K_{P_*}$’s algebraic multiplicity is equal to its geometric multiplicity.
Proof. We make use of the Perron-Frobenius theorem to prove the desired conclusion. Let $U_\pi$ denote the matrix representation of $U_\pi$ under the following basis of $H^\otimes n$:

$$B := \{ |p_1 \cdots p_n \rangle : |p_i \rangle \in \{ |0\rangle, |1\rangle \} \}.$$

From the definition of $U_\pi$ it is clear that the following claim holds.

Claim (i). $U_\pi$ is a doubly stochastic matrix for any $\pi \in \mathcal{P}$.

We further consider the standard computational basis of $L_{H^\otimes n}$:

$$B := \{ |p_1 \cdots p_n \rangle \langle q_1 \cdots q_n | : |p_i \rangle, |q_i \rangle \in \{ |0\rangle, |1\rangle \} \}.$$

We make another claim.

Claim (ii). $U_\pi (|p_1 \cdots p_n \rangle \langle q_1 \cdots q_n |) U_\pi^\dagger = |p_\pi(1) \cdots p_\pi(n) \rangle \langle q_\pi(1) \cdots q_\pi(n) |$.

This claim can be easily proved by verifying the images of the above two operators are the same for all $|\xi \rangle \in B$. As a result, there is an order of the elements in $B$ under which the matrix representation of $K_{\mathcal{P}_*}$ is

$$K_* := \sum_{\pi \in \mathcal{P}_*} (U_\pi \otimes U_\pi - 1_{2n} \otimes I_{2n}),$$

where $\otimes$ stands for the Kronecker product. Making use of Claim (i) we know that

$$\left( 1^T_{2n} \otimes 1^T_{2n} \right) U_\pi \otimes U_\pi = \left( 1^T_{2n} U_\pi \right) \otimes \left( 1^T_{2n} U_\pi \right) = 1^T_{4n}$$

and

$$U_\pi \otimes U_\pi \left( 1_{2n} \otimes 1_{2n} \right) = \left( U_\pi 1_{2n} \right) \otimes \left( U_\pi 1_{2n} \right) = 1_{4n},$$

i.e., each $U_\pi \otimes U_\pi$ is a doubly stochastic matrix.

We now focus on the matrix $H_* := \sum_{\pi \in \mathcal{P}_*} U_\pi \otimes U_\pi$ and its induced graph $G_{H_*}$. Since every $U_\pi \otimes U_\pi$ is doubly stochastic, $G_{H_*}$ is a balanced digraph. This further implies that every weakly connected component of $G_{H_*}$ is balanced, and thus strongly connected by Lemma 1. In other words, there exists a permutation matrix $P_* \in \mathbb{R}^{4^n \times 4^n}$ such that

$$P_* H_* P_*^{-1} = \text{diag}(P_1, \ldots, P_{c_0}),$$

where each $P_i$ is the adjacency matrix of each weakly connected component and $c_0$ stands for the number of those weakly connected components of $G_{H_*}$. Consequently, each $P_i$ is irreducible.
Finally, applying the Geršgorin disc Theorem, i.e., Lemma 2, we conclude that \( \lambda(P_i) \leq |\mathcal{P}_i| \). We also know \( |\mathcal{P}_i| \) is an eigenvalue of \( P_i \) due to the stochasticity of each \( U_{\pi} \otimes U_{\pi} \). Imposing the Perron-Frobenius theorem, i.e., Lemma 3, we further conclude that \( |\mathcal{P}_i| \) is a simple eigenvalue of every \( P_i \). This immediately yields that the zero eigenvalue of \( K_\ast \)'s algebraic multiplicity is equal to its geometric multiplicity since \( K_\ast = H_\ast - |\mathcal{P}_i| \cdot I_4n \). The proof is complete. \( \square \)

We are now in a position to prove Proposition 3. Lemma 10 ensures that the zero eigenvalue of \( K_\ast \)'s algebraic multiplicity is equal to its geometric multiplicity. Lemma 2 ensures that all non-zero eigenvalues of \( K_\ast \) have negative real parts. These two facts imply that, for the system (7), \( \rho(t) \) converges to a limit, and the limit is a fixed point, say \( \rho_\ast \), in the null space of \( K_\ast\mathcal{P}_i \).

From Lemma 9 we know that

\[
\rho_\ast = \sum_{\pi \in \mathcal{C}_{\mathcal{P}_i}} U_{\pi} \rho_\ast U_{\pi}^\dagger / |\mathcal{C}_{\mathcal{P}_i}|. \tag{13}
\]

From Lemma 8 we also see that

\[
\frac{d}{dt} \sum_{\pi \in \mathcal{C}_{\mathcal{P}_i}} U_{\pi} \rho(t) U_{\pi}^\dagger / |\mathcal{C}_{\mathcal{P}_i}| = \sum_{\pi \in \mathcal{C}_{\mathcal{P}_i}} U_{\pi} \left( \frac{d}{dt} \rho(t) \right) U_{\pi}^\dagger / |\mathcal{C}_{\mathcal{P}_i}| = \sum_{\pi \in \mathcal{C}_{\mathcal{P}_i}} U_{\pi} \sum_{\pi \in \mathcal{P}_i} \left( U_{\pi} \rho(t) U_{\pi}^\dagger - \rho \right) U_{\pi}^\dagger / |\mathcal{C}_{\mathcal{P}_i}| \]

\[
= \sum_{\pi \in \mathcal{C}_{\mathcal{P}_i}} U_{\pi} \rho(t) U_{\pi}^\dagger / |\mathcal{C}_{\mathcal{P}_i}| - \sum_{\pi \in \mathcal{C}_{\mathcal{P}_i}} U_{\pi} \rho(t) U_{\pi}^\dagger / |\mathcal{C}_{\mathcal{P}_i}| \equiv 0, \tag{14}
\]

where we have used the fact that \( \mathcal{C}_{\mathcal{P}_i} \) is a subgroup so that \( \pi \mathcal{C}_{\mathcal{P}_i} = \mathcal{C}_{\mathcal{P}_i} \) for any \( \pi \in \mathcal{C}_{\mathcal{P}_i} \).

Therefore, combining (13) and (14) we know that

\[
\rho_\ast = \sum_{\pi \in \mathcal{C}_{\mathcal{P}_i}} U_{\pi} \rho_\ast U_{\pi}^\dagger / |\mathcal{C}_{\mathcal{P}_i}| = \sum_{\pi \in \mathcal{C}_{\mathcal{P}_i}} U_{\pi} \rho_0 U_{\pi}^\dagger / |\mathcal{C}_{\mathcal{P}_i}|. \tag{15}
\]

We have now completed the proof of Proposition 3.

4.3.4 Proof of Theorems 1 and 2

We introduce \( \tilde{\rho}(t) = e^{iHt/\hbar} \rho(t) e^{-iHt/\hbar} \). By our standing assumption, \( H = H_0^{\otimes n} \) or \( H = H_0^{\odot n} \) for some \( H_0 \in \mathcal{L}_H \). In either case the form of \( H \) yields that \( [H, U_{\pi}] = 0 \) for all \( \pi \in \mathcal{P}_i \). As a result,
the evolution of $\tilde{\rho}(t)$ satisfies
\[
\frac{d\tilde{\rho}}{dt} = \sum_{\pi \in \mathcal{P}^*} (U_\pi \tilde{\rho} U_\pi^\dagger - \tilde{\rho}).
\] (16)

In light of Proposition 3, we see that in both cases we have
\[
\lim_{t \to \infty} \left\| \rho(t) - e^{-iHt/\hbar} \rho_* e^{iHt/\hbar} \right\| = 0,
\] (17)
where
\[
\rho_* = \frac{1}{|\mathfrak{C}_\rho|} \sum_{\pi \in \mathfrak{C}_\rho} U_\pi \rho_0 U_\pi^\dagger.
\]

(Sufficiency.) Note that $[e^{iH}, U_\pi] = 0$ if $[H, U_\pi] = 0$. As a result, either $H = H_0^\otimes n$ or $H = H_0^\oplus n$ implies that
\[
e^{-iHt/\hbar} \rho_* e^{iHt/\hbar} = \frac{1}{|\mathfrak{C}_\rho|} \sum_{\pi \in \mathfrak{C}_\rho} U_\pi \left( e^{-iHt/\hbar} \rho_0 e^{iHt/\hbar} \right) U_\pi^\dagger.
\]
As a result, applying Proposition 1 for the $\mathfrak{C}_\rho$-average of $e^{-iHt/\hbar} \rho_0 e^{iHt/\hbar}$, we immediately know that global reduced-state synchronization is guaranteed by (17). This proves the sufficiency statements in Theorems 1 and 2. Particularly, we know that
\[
e^{iHt/\hbar} = e^{iH_0 t} \otimes \cdots \otimes e^{iH_0 t}
\]
for all $t \in \mathbb{R}$ when $H = H_0^\otimes n$. With some straightforward calculations we further obtain
\[
\text{Tr}_{\otimes_{j=1}^{n-1} \mathcal{H}_j} \left[ e^{-iHt/\hbar} \rho_* e^{iHt/\hbar} \right] = e^{-iH_0 t/\hbar} \left[ \text{Tr}_{\otimes_{j=1}^{n-1} \mathcal{H}_j} \rho_* \right] e^{iH_0 t/\hbar}.
\]

(Necessity (Theorem 1).) Suppose global reduced-state synchronization is achieved. Then by definition of reduced-state synchronization we have (note that Proposition 3 holds without any connectivity requirement on $G_\rho$)
\[
\lim_{t \to \infty} \left\| e^{-iH_0 t/\hbar} \left[ \text{Tr}_{\otimes_{j \neq k} \mathcal{H}_j} \rho_* - \text{Tr}_{\otimes_{j \neq m} \mathcal{H}_j} \rho_* \right] e^{iH_0 t/\hbar} \right\| = 0
\]
for all $\rho_0$ and $k, m \in \mathcal{V}$. This implies that for all initial value $\rho_0$,
\[
\text{Tr}_{\otimes_{j \neq k} \mathcal{H}_j} \rho_* = \text{Tr}_{\otimes_{j \neq m} \mathcal{H}_j} \rho_* , \quad k, m \in \mathcal{V}.
\]
Then Proposition 1 concludes the necessity proof.

We have now completed the proof. □
5 The Missing Symmetry: A Graphical Look

In this section, we take a further look at the incomplete symmetry in the $\mathcal{P}_\pi$-average, as reflected by the zero-pattern of the difference between $\mathcal{P}_\pi$-average and $\mathcal{P}$-average (cf., Figure 5). We do this by investigating the dynamics of every element of the density operator along the master equation, from a graphical point of view. We show that this graphical approach not only provides a full characterization of the missing symmetry, but also naturally leads to a much deeper understanding of the original quantum dynamics.

5.1 The Information-Flow Hierarchy

The quantum interaction graph $G_\pi = (V, E_\pi)$ provides a characterization of the information flow among the qubit network under $U_\pi$. The “resolution” of this characterization is however considerably low since merely the directions of the information flow are indicated in $G_\pi$. In order to provide some more accurate characterizations of the information flow under $U_\pi$, we introduce the following definition by identifying the elements in the basis of $\mathcal{H}^\otimes n$ and $\mathcal{LH}^\otimes n$ as classical nodes.

**Definition 3** Let $\pi \in \mathcal{P}$ and denote $V = \{|p_1 \cdots p_n \rangle : p_i \in \{0, 1\}\}$ and $V' = \{|p_1 \cdots p_n \rangle\langle q_1 \cdots q_n| : p_i, q_i \in \{0, 1\}\}$. Associated with the permutation $\pi$, we define

(i) the state-space graph $G_\pi = (V, E_\pi)$ so that $E_\pi$ consists of all non-self-loop arcs in

$$\left\{ |p_1 \cdots p_n \rangle, |p_{\pi(1)} \cdots p_{\pi(n)} \rangle | : p_i \in \{0, 1\}\right\};$$

(ii) the operator-space graph $\mathcal{G}_\pi = (V', E_\pi)$ so that $E_\pi$ consists of all non-self-loop arcs in

$$\left\{ |p_1 \cdots p_n \rangle\langle q_1 \cdots q_n|, |p_{\pi(1)} \cdots p_{\pi(n)} \rangle\langle q_{\pi(1)} \cdots q_{\pi(n)}| | : p_i, q_i \in \{0, 1\}\right\}.$$

From their definitions we see that both $G_\pi$ and $\mathcal{G}_\pi$ are simple digraphs, and in fact they are always balanced from the nature of permutation indicated in Lemma 5. Note that $V$ and $V'$ are basis of $\mathcal{H}^\otimes n$ and $\mathcal{LH}^\otimes n$, respectively. Clearly $G_\pi$ and $\mathcal{G}_\pi$ provide not only the directions of the information flow, but also the information itself in the flow of $U_\pi$. For the permutation $\pi(1) = 2, \pi(2) = 3, \pi(3) = 1$ over a three-qubit network, its interaction graph $G_\pi$, state-space graph $G_\pi$, and operator-space graph $\mathcal{G}_\pi$, are respectively illustrated in Figure 6.

For the state-space graph, we have the following result. Let $G_{\mathcal{P}} := \bigcup_{\pi \in \mathcal{P}} G_\pi$. 

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Figure 6: The interaction graph $G_\pi$, the state-space graph $G_\pi$, and the operator-space graph $\mathcal{H}_\pi$. Various properties of these graphs can be verified from Propositions 4 and 5, e.g., $|E_\pi| = 6$, and $|E_\pi| = 2^4 \times |E_\pi| - |E_\pi|^2 = 60$.

**Proposition 4** (i) Suppose $G_\pi$ is a directed cycle. Then $G_\pi$ has $2^n - 2$ arcs for all $n \in \mathbb{N}^+$. Moreover, if $n \in \mathbb{N}^+$ is odd, then $G_\pi$ has exactly $2 + (2^n - 2)/n$ strongly connected components, among which two are singletons and the remaining $(2^n - 2)/n$ are directed cycles with size $n$.

(ii) $G_\pi$ has $n+1$ strongly connected components with their sizes ranging from $\binom{n}{0}$ to $\binom{n}{n}$, and each of its strongly connected components is fully connected. Consequently there are

$$E_n := \sum_{k=1}^{n-1} \binom{n}{k} \left[ \binom{n}{k} - 1 \right]$$

arcs in $G_\pi$.

**Remark 6** From Lemma 5, $G_\pi$ is a union of disjoint directed cycles for any $\pi$. This means that Proposition 4 (i) for $\pi$ whose interaction graph $G_\pi$ is a directed cycle, can be easily generalized to arbitrary permutations.

For the operator-space graph, the following proposition holds. Denote $\mathcal{G}_\pi := \bigcup_{\pi \in \mathcal{P}_*} \mathcal{G}_\pi$ and recall that $K_{\mathcal{H}_\pi} := \mathcal{L}_{\mathcal{H}_\pi^n} \rightarrow \mathcal{L}_{\mathcal{H}_\pi^n}$ with $K_{\mathcal{P}_*} (\rho) = \sum_{\pi \in \mathcal{P}_*} (U\pi \rho U_\pi^\dagger - \rho)$, and $\mathcal{C}_{\mathcal{P}_*}$ is the generated subgroup by $\mathcal{P}_*$.
Proposition 5
(i) $|\mathcal{E}_\pi| = 2^{n+1}|E_\pi| - |E_\pi|^2$.

(ii) Suppose $G_\pi$ is a directed cycle and $n \in \mathbb{N}^+$ is odd. Then $G_\pi$ has exactly $4 + (2^{2n} - 4)/n$ strongly connected components, among which four are singletons and the rest $(2^{2n} - 4)/n$ are directed cycles with size $n$.

(iii) There are a total of $\dim(\text{Null}(K_{P*}))$ strongly connected components in $G_{P*}$.

(iv) The components of $G_{P*}$ and $G_{P^*}$ give the same partition of $\mathcal{V}$, i.e., they agree on the same subsets of nodes.

Propositions 4 and 5 provide some detailed descriptions of the information hierarchy for the quantum permutation operators, which can be quite useful in understanding the evolution of the quantum synchronization master equation. They are established via combinatorial analysis approach applied to $E_\pi$ and $\mathcal{E}_\pi$, whose detailed proofs are put in Appendix A.6 and A.7, respectively.

5.2 Quantum vs. Classical Consensus Equation

We denote by

$$[ho(t)]_{p_1\cdots p_n \langle q_1 \cdots q_n |}$$

the $|p_1 \cdots p_n \langle q_1 \cdots q_n |$-entry of the matrix representation of $\rho(t)$ under the basis of $\mathcal{V}$, where $p_i, q_i \in \{0,1\}$ for all $i$. From Lemma 6, the quantum consensus master equation (7) can be written as

$$\frac{d}{dt} [\rho(t)]_{p_1\cdots p_n \langle q_1 \cdots q_n |} = \sum_{\pi \in \mathcal{P}^*} \left([\rho(t)]_{p_{\pi-1(1)} \cdots p_{\pi-1(n)} \langle q_{\pi-1(1)} \cdots q_{\pi-1(n)}} - [\rho(t)]_{p_1\cdots p_n \langle q_1 \cdots q_n |} \right)$$

under this matrix representation $[\rho(t)]$ of $\rho(t)$.

Note that (18) exactly defines a classical consensus dynamics (see [13]) over the digraph $G_{P*}$, where each node $|p_1 \cdots p_n \langle q_1 \cdots q_n |$’s dynamics are influenced by its in-neighbors in $G_{P*}$:

$$\mathcal{N}_{p_1\cdots p_n \langle q_1 \cdots q_n |}(\mathcal{P}^*) := \{|p_{\pi-1(1)} \cdots p_{\pi-1(n)} \langle q_{\pi-1(1)} \cdots q_{\pi-1(n)} | : \pi \in \mathcal{P}^* \}.$$ 

The following understanding becomes immediate.

**Proposition 6** Convergence to a $\mathcal{P}^*$-average along the quantum master equation (7) is equivalent to componentwise convergence to a classical average consensus along (18).
Here by componentwise classical average consensus, we mean that average consensus is achieved, i.e., every node’s state converges to the average of the subnetwork nodes’ initial values, over each strongly connected component of $\mathcal{G}_\pi$. Proposition 6 together with the understandings we established for $\mathcal{G}_\pi$ provides a deep characterization of the original synchronization master equation.

**Remark 7** From Propositions 5 and 6, we immediately know that whenever $n$ is odd, the convergence rate to a synchronization trajectory under any permutation $\pi$ is equal to the rate of convergence to a classical consensus over an $n$-node directed cycle. This rate can thus be explicitly given as

$$1 - \cos\left(\frac{2\pi}{n}\right),$$

following the spectral analysis to graphs (cf., Section 1.4.3, [38]).

### 5.3 The Zero Pattern

We now investigate the zero-pattern of the difference between $\mathcal{P}_\ast$-average and $\mathcal{P}$-average. Let

$$[\rho_{\mathcal{P}_\ast} - \rho_{\mathcal{P}}]_{|p_1\cdots p_n\rangle\langle q_1\cdots q_n|}$$

be the $|p_1\cdots p_n\rangle\langle q_1\cdots q_n|$-entry of $\rho_{\mathcal{P}_\ast} - \rho_{\mathcal{P}}$ under the basis $V$. The following result holds.

**Theorem 3** (i) There exists $\rho \in \mathcal{L}_{H^n}$ for which $[\rho_{\mathcal{P}_\ast} - \rho_{\mathcal{P}}]_{|p_1\cdots p_n\rangle\langle q_1\cdots q_n|} \neq 0$ if and only if the strongly connected components in $\mathcal{G}_\mathcal{P}$ and $\mathcal{G}_{\mathcal{P}_\ast}$ that contain $|p_1\cdots p_n\rangle\langle q_1\cdots q_n|$, have different sizes.

(ii) Suppose $G_{\mathcal{P}_\ast}$ is strongly connected. Then $[\rho_{\mathcal{P}_\ast} - \rho_{\mathcal{P}}]_{|p_1\cdots p_n\rangle\langle q_1\cdots q_n|} = 0$ for all $\rho \in \mathcal{L}_{H^n}$ if one of the following conditions holds:

a) $p_1 = \cdots = p_n$ and $q_1 = \cdots = q_n$;

b) $p_1 = \cdots = p_n$, and $\sum_{i=1}^{n} q_i \in \{1, n-1\}$;

c) $|p_1\cdots p_n\rangle = |q_1\cdots q_n\rangle$, and $\sum_{i=1}^{n} q_i \in \{1, n-1\}$;

d) $|p_1\cdots p_n\rangle = |\bar{q}_1\cdots \bar{q}_n\rangle$, where $\bar{q}_i = 1 - q_i$, and $\sum_{i=1}^{n} q_i \in \{1, n-1\}$.

**Proof.** (i). The conclusion follows directly from Proposition 6

(ii). We only need to make sure that the strongly components in $\mathcal{G}_\mathcal{P}$ and $\mathcal{G}_{\mathcal{P}_\ast}$ that contain $|p_1\cdots p_n\rangle\langle q_1\cdots q_n|$ have the same size.
If \( p_1 = \cdots = p_n \) and \( q_1 = \cdots = q_n \), then \(|p_1 \cdots p_n\rangle\langle q_1 \cdots q_n|\) is an isolated node in \( \mathcal{G}_p \). Thus a) always ensures the above same-size condition, actually for arbitrary \( \mathcal{P}_* \).

Now we move to Condition b) and suppose \( p_1 = \cdots = p_n \) with \( \sum_{i=1}^{n} q_i = 1 \). Without loss of generality we let \( q_1 = 1 \). Since \( \mathcal{G}_p \) is strongly connected, for any \( i_* \in V \), there exist \( \pi_1, \ldots, \pi_k \) such that \( \pi_k \cdots \pi_1(1) = i_* \). This implies that the component containing \(|p_1 \cdots p_n\rangle\langle q_1 \cdots q_n|\) in \( \mathcal{G}_p \) also has \( n \) nodes. We can thus invoke (i) to conclude that \( [\rho_{p_*} - \rho_\pi] |p_1 \cdots p_n\rangle\langle q_1 \cdots q_n| = 0 \) for all \( \rho \in \mathcal{L}_{\mathcal{H}^\otimes n} \). While the other case in Condition b) with \( \sum_{i=1}^{n} q_i = n - 1 \) holds from a symmetric argument.

Conditions c) and d) ensure the same-size condition in (i), via a similar analysis as we use to investigate Condition b). We thus omit their details. The proof is now complete. \( \square \)

Theorem 3(i) is a tight graphical characterization of the missing symmetry in the \( \mathcal{P}_* \)-average. Theorem 3(ii) further explicitly shows some symmetry kept in the \( \mathcal{P}_* \)-average when only reduced-state consensus is guaranteed (e.g., \( \mathcal{G}_p \) is strongly connected, cf., Proposition 1).

In this way, Theorem 3 provides a full explanation to the zero-pattern shown in Figure 5.

5.4 Robustness under Switching Permutations: Cut-Balance Consensus

In this subsection, we investigate the robustness of the quantum synchronization master equation (3) subject to switching of permutation operators. To this end, we introduce \( \mathcal{P}_* \) as the set containing all the subsets of \( \mathcal{P}_* \), and a piecewise constant switching signal \( \mu(\cdot) : \mathbb{R}_\geq 0 \mapsto \mathcal{P}_* \). We use \( \mathcal{P}_{\mu(t)} \) to denote the set of permutations selected at time \( t \). Consider the following dynamics

\[
\frac{d\rho}{dt} = -\frac{i}{\hbar} [H, \rho] + \sum_{\pi \in \mathcal{P}_{\mu(t)}} \left( U_\pi \rho U_\pi^\dagger - \rho \right),
\]

which is evidently a time-varying version of (3).

For the ease of presentation we assume that there is a constant \( \mu_D > 0 \) as a lower bound between any two consecutive switching instants of \( \mu(\cdot) \). We introduce the following definition.

**Definition 4** We call \( \pi \in \mathcal{P}_* \) a persistent permutation under \( \mu(\cdot) \) if the intervals for which \( \pi \) appears in \( \mathcal{P}_{\mu(t)} \) sum to an infinite length. Then we introduce

\[
\mathcal{P}_*^p : \left\{ \pi : \pi \text{ is a persistent permutation} \right\}
\]

and denote \( \mathcal{C}_{\mathcal{P}_*^p} \) as the generated subgroup by \( \mathcal{P}_*^p \).
The following result holds showing some fundamental robustness of the quantum synchronization equation with respect to switching of permutations (environments).

**Theorem 4** The system (19) achieves global reduced-state synchronization for all $t_0 \geq 0$ if and only if $G_{\pi p^*}$ is strongly connected.

Indeed, under our standing assumption, to obtain the asymptotical properties of (19) we only need to investigate the following quantum consensus equation:

$$\frac{d\rho}{dt} = \sum_{\pi \in \mathcal{P}} \mu(t) (U_\pi \rho U_\pi^* - \rho).$$

(20)

It is straightforward to see from our previous analysis that Theorem 4 is equivalent to the following result on the convergence of (20). The proof of the result is based on the relationship between quantum and classical consensus dynamics, and the results on classical consensus for the so-called “cut-balanced graphs” established recently in [39].

**Proposition 7** The system (20) ensures global $\mathcal{P}_*$-average consensus in the sense that $\lim_{t \to \infty} \rho(t) = \rho_0^{\mathcal{P}_*}$ for all $t_0 \geq 0$ and all $\rho_0 = \rho(t_0)$ if and only if $C_{\mathcal{P}_*} = C_{\mathcal{P}_*}^r$.

**Proof.** Recall that the system (20) has the form:

$$\frac{d}{dt} \rho(t)_{|p_1 \cdots p_n \rangle \langle q_1 \cdots q_n|} = \sum_{\pi \in \mathcal{P}_{\mu(t)}} \left( \rho(t)_{|p_{\pi^{-1}(1)} \cdots p_{\pi^{-1}(n)} \rangle \langle q_{\pi^{-1}(1)} \cdots q_{\pi^{-1}(n)}|} - \rho(t)_{|p_1 \cdots p_n \rangle \langle q_1 \cdots q_n|} \right)$$

(21)

under the basis $\mathcal{V}$. Note that (21) admits a classical consensus dynamics over the node set $\mathcal{V}$ with time-varying node interaction structures, where at time $t$ node $|p_1 \cdots p_n \rangle \langle q_1 \cdots q_n|$ is influenced by its in-neighbors in the set

$$\mathcal{N}_{|p_1 \cdots p_n \rangle \langle q_1 \cdots q_n|}(t) := \left\{|p_{\pi^{-1}(1)} \cdots p_{\pi^{-1}(n)} \rangle \langle q_{\pi^{-1}(1)} \cdots q_{\pi^{-1}(n)}| : \pi \in \mathcal{P}_{\mu(t)} \right\}.$$

Similarly, the node $|p_1 \cdots p_n \rangle \langle q_1 \cdots q_n|$ influences its out-neighbors in the set

$$\mathcal{N}^+_{|p_1 \cdots p_n \rangle \langle q_1 \cdots q_n|}(t) := \left\{|p_{\pi(1)} \cdots p_{\pi(n)} \rangle \langle q_{\pi(1)} \cdots q_{\pi(n)}| : \pi \in \mathcal{P}_{\mu(t)} \right\}.$$

Note that for any $\pi \in \mathcal{P}$, we know that $\mathcal{G}_\pi$ is balanced. This immediately leads to

$$|\mathcal{N}^+_{|p_1 \cdots p_n \rangle \langle q_1 \cdots q_n|}(t)| = |\mathcal{N}^-_{|p_1 \cdots p_n \rangle \langle q_1 \cdots q_n|}(t)|.$$
As a result, this guarantees that (21) defines a cut-balanced classical consensus process in the sense that $\deg^+_t(\mathcal{S}) = \deg^-_t(\mathcal{S})$ for any node set $\mathcal{S} \subseteq \mathcal{V}$ and for any $t \geq 0$, where by definition

$$\deg^+_t(\mathcal{S}) := \left| \{ z \in \mathcal{V} \setminus \mathcal{S} : \exists v \in \mathcal{S} \text{ and } \pi \in \mathcal{P}_{\mu(t)} \text{ s.t. } (v, z) \in E_\pi \} \right|$$

(22)

and

$$\deg^-_t(\mathcal{S}) := \left| \{ z \in \mathcal{V} \setminus \mathcal{S} : \exists v \in \mathcal{S} \text{ and } \pi \in \mathcal{P}_{\mu(t)} \text{ s.t. } (z, v) \in E_\pi \} \right|$$

(23)

Finally, by Proposition 5(iii)-(iv), there are $\dim(\text{Null}(K_{\mathcal{P}_\ast}))$ strongly connected components in $\mathcal{G}_{\mathcal{P}_\ast}$, and thus the nodes in those different components can never interact under the dynamics (21). Further we notice that Proposition 6 continues to hold for system (21) since the conclusion only characterizes the asymptotic trajectories which are independent of fixed or switching permutations in the dynamics. Thus, the desired result holds directly from Theorem 1 in [39], and this concludes the proof.

6 Conclusions

Reduced-state synchronization of qubit networks was considered with the aim of driving the qubits’ reduced states to a common trajectory. The evolution of the quantum network’s state is described by a Lindblad master equation, where the network Hamiltonian is either a direct sum or a tensor product of identical qubit Hamiltonians, and the coupling terms are given by a set of permutation operators over the network. We define quantum directed interaction graphs for the permutation operators. We obtained a necessary and sufficient condition for reduced-state synchronization with convergence rate and the limiting orbit explicitly characterized. Numerical examples were provided illustrating the obtained results. The missing symmetry in the reduced-state synchronization was also investigated from a graphical point of view. The information-flow hierarchy in quantum permutation operators is characterized by three layers of information-induced graphs. We also proved that the quantum synchronization equation is by nature equivalent to a cut-balanced consensus process, thus a necessary and sufficient condition is obtained for quantum reduced-state synchronization under switching interactions applying the work of Hendrickx and Tsitsiklis.
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Appendix

A.1 Proof of Lemma 1

If a digraph $G = (V, E)$ is balanced, then apparently it must hold that $\deg^+(S) = \deg^-(S)$ for any node set $S \subseteq V$, where by definition

$$\deg^+(S) := \left| \{ z \in V \setminus S : \exists v \in S \text{ s.t. } (v, z) \in E \} \right| \quad (24)$$

and

$$\deg^-(S) := \left| \{ z \in V \setminus S : \exists v \in S \text{ s.t. } (z, v) \in E \} \right|. \quad (25)$$

Only the necessity statement needs to be verified. Suppose $G = (V, E)$ is not strongly connected. Then there exists a partition of $V$ into two nonempty and disjoint subsets of nodes $V_1$ and $V_2$ such that $V_1 \cup V_2 = V$, for which there is no arc leaving from $V_1$ pointing to $V_2$. On the other hand the graph is weakly connected, so there exists at least one arc from $V_2$ to $V_1$. Taking $S = V_1$ in the above argument we reach a contradiction. This concludes the proof.

A.2 Proof of Lemma 4

It is well known that any permutation $\pi$ can be written as

$$\pi = \varsigma_1 \ldots \varsigma_k$$

for some $k \geq 1$ with each of the $\varsigma_k$ being a swapping permutation. From the definition of the permutation-induced quantum interaction it is easy to see that

$$U_{\pi} = U_{\varsigma_1} \ldots U_{\varsigma_k}.$$
Since $U_\varsigma U_\varsigma^\dagger = U_\varsigma^\dagger U_\varsigma = I$ whenever $\varsigma \in \mathcal{P}$ is a swapping, we further conclude

$$U_\varsigma U_\varsigma^\dagger = U_{\varsigma_1} \ldots U_{\varsigma_k} U_{\varsigma_k}^\dagger \ldots U_{\varsigma_1}^\dagger = I;$$
$$U_\varsigma^\dagger U_\varsigma = U_{\varsigma_k}^\dagger \ldots U_{\varsigma_1}^\dagger U_{\varsigma_1} \ldots U_{\varsigma_k} = I.$$ 

This concludes the proof.

A.3 Proof of Lemma 5

Take $\pi \in \mathcal{P}$ and $i \in V$. Since $\pi$ is a bijection over $V$ there must exist an integer $K \geq 1$ such that $\pi^K(i) = i$. Without loss of generality we can assume such $K$ has been taken as the smallest integer satisfying $\pi^K(i) = i$ and $K \geq 2$. Note that it is impossible that $\pi^{k_1}(i) = \pi^{k_2}(i)$ for some $0 \leq k_1 < k_2 \leq K$ since otherwise $\pi^{k_2-k_1}(i) = i$, which contradicts the choice of $K$. This means that

$$i, \pi(i), \ldots, \pi^K(i) = i$$

admits a directed cycle in $G_\pi$. Examining every $i \in V$ using the above argument concludes the lemma immediately.

A.4 Proof of Lemma 6

From Lemma 4 and the definition of $U_\pi$ we know that $U_\pi^\dagger = U_\pi^{-1} = U_{\pi^{-1}}$, where $\pi^{-1}$ is the inverse of $\pi$ in the permutation group $\mathcal{P}$.

The following equalities hold:

$$\langle p_1 \cdots p_n | U_\pi \left( A_1 \otimes \cdots \otimes A_n \right) U_\pi^\dagger | q_1 \cdots q_n \rangle$$

$$= \langle p_{\pi^{-1}(1)} \cdots p_{\pi^{-1}(n)} | \left( A_1 \otimes \cdots \otimes A_n \right) | q_{\pi^{-1}(1)} \cdots q_{\pi^{-1}(n)} \rangle$$

$$= \sum_{i=1}^{n} \langle p_{\pi^{-1}(i)} | A_i | q_{\pi^{-1}(i)} \rangle$$

$$= \sum_{\pi(i)=1}^{n} \langle p_i | A_{\pi(i)} | q_i \rangle$$

$$= \sum_{i=1}^{n} \langle p_i | A_{\pi(i)} | q_i \rangle$$

$$= \langle p_1 \cdots p_n | A_{\pi(1)} \otimes \cdots \otimes A_{\pi(n)} | q_1 \cdots q_n \rangle$$

(26)

for all $|p_i\rangle, |q_i\rangle \in H, i \in V$. This concludes the proof.
A.5 Proof of Lemma 7

Apparently only the sufficiency statement needs to be proved. From Lemma 5 we know that for any \( \pi \in \mathcal{P} \), there is an integer \( K \geq 1 \) such that \( \pi^K = I \). This means that \( \pi^{-1} = \pi^{K-1} \).

From the definition of \( G_\pi \), \( \bigcup_{\pi \in \mathcal{P}_*} G_\pi \) being fully connected is equivalent to that for any two nodes \( i \neq j \in V \), there exists a permutation \( \pi_s \in \mathcal{C}_\mathcal{P}_* \) such that
\[
\pi_s(i) = j.
\] (27)

Note that the above argument yields that any \( \pi_s \in \mathcal{C}_\mathcal{P}_* \) can be written as \( \pi_s = \pi_k \cdots \pi_1 \) with \( \pi_s \in \mathcal{P}_s, 1 \leq s \leq k \). In other words, (27) leads to
\[
\pi_k \cdots \pi_1(i) = j, \quad \pi_s \in \mathcal{P}_s, \ 1 \leq s \leq k.
\] (28)

We immediately conclude from (28) that \( \bigcup_{\pi \in \mathcal{P}_*} G_\pi \) is strongly connected.

A.6 Proof of Proposition 4

Recall that for a positive integer \( n \), two integers \( a \) and \( b \) are said to be congruent modulo \( n \), denoted \( a \equiv b \pmod{n} \), if \( n \) divides their difference \( a - b \).

(i). Since \( G_\pi \) is a directed cycle, without loss of generality we assume that \( \pi(i) = i + 1 \pmod{n} \). Suppose
\[
|p_1 \cdots p_n\rangle = |p_{\pi(1)} \cdots p_{\pi(n)}\rangle.
\]

Then we obtain \( p_1 = p_2 = \cdots = p_n \) from the definition of \( \pi \). This implies that \( (|p_1 \cdots p_n\rangle, |p_{\pi(1)} \cdots p_{\pi(n)}\rangle) \) defines an arc in \( E_\pi \) as long as \( p_1 = p_2 = \cdots = p_n \) does not hold. We immediately conclude that
\[
|E_\pi| = 2^n - 2.
\]

Now we investigate the property of the strongly connected components of \( G_\pi \). Note that as \( G_\pi \) is apparently balanced, each of its weakly connected components is strongly connected by Lemma 1. The following lemma holds.

Lemma 11 Suppose \( n \geq 3 \) is an odd integer and \( G_\pi \) associated with \( \pi \in \mathcal{P} \) is a directed cycle. Then \( G_{\pi k} \) is also a directed cycle for all \( k \neq 0 \pmod{n} \).

Proof. Again without loss of generality we assume that \( \pi(i) = i + 1 \pmod{n} \).
We first prove the conclusion for $k = 2$. By Lemma 5, we only need to show that $G_\pi^2$ is strongly connected. Take $i_* \neq j_* \in V$. The following modular equation (with respect to $x$)

$$i_* + 2x = j_* \pmod{n}$$

always has a solution since $n \geq 3$ is an odd integer. Let $x_0 \in \mathbb{N}$ be a solution of (29). Then

$$(\pi^2)^{x_0}(i_*) = j_*,$$

which yields a path from $i_*$ to $j_*$ in $G_\pi^2$ with length $x_0$. This proves that $G_\pi^2$ is strongly connected, which must be a directed cycle.

Now let $0 \leq k \leq n - 1$. Since $\pi^n = I$, for any $k \neq 0 \pmod{n}$ we can find a positive integer $\gamma$ satisfying $\pi^k = (\pi^\gamma)^2$. As a result, the overall conclusion follows from a straightforward induction argument. This completes the proof. □

**Lemma 12** Suppose $G_\pi$ is a directed cycle and let $n$ be an odd integer. Take $|p_1 \cdots p_n\rangle$ with $p_i \in \{0, 1\}$ for all $i$ and assume that at least two $p_i$’s take distinct values. Then the elements in

$$\left\{|p_{\pi^k(1)} \cdots p_{\pi^k(n)}\rangle, k = 0, 1, \ldots, n - 1\right\}$$

define the set of nodes to which there is a path from $|p_1 \cdots p_n\rangle$ in $G_\pi$. Consequently, the component where $|p_1 \cdots p_n\rangle$ locates contains exactly $n = \left|\left\{|p_{\pi^k(1)} \cdots p_{\pi^k(n)}\rangle, k = 0, 1, \ldots, n - 1\right\}\right|$ nodes.
nodes and $n$ directed arcs. Invoking the fact that there are a total of $2^n - 2$ arcs in $G_{\pi}$, such components with a size $n$ count $(2^n - 2)/n$. The total number of components are certainly $(2^n - 2)/n + 2$ (two singleton components corresponds to $p_1 = \cdots = p_n = 0$ and $p_1 = \cdots = p_n = 1$, respectively). The fact that each non-singleton component is a directed cycle simply follows from that $\pi^n = I$.

(ii) Suppose $|p_1 \cdots p_n \rangle, |q_1 \cdots q_n \rangle \in V$ satisfy $\sum_{i=1}^n p_i = \sum_{i=1}^n q_i = k$ with $0 \leq k \leq n$. Then obviously we can find a $\pi \in \mathcal{P}$ such that $|q_1 \cdots q_n \rangle = |p_{\pi(1)} \cdots p_{\pi(n)} \rangle$. This immediately leads to that the subset of nodes $$\left\{ |p_1 \cdots p_n \rangle, \sum_{i=1}^n p_i = k \right\}$$ induces a fully connected component of $G_\mathcal{P}$. The rest of the conclusions follows from direct computations.

The proof of Proposition 4 is now complete.

A.7 Proof of Proposition 5

(i). Let $U_{\pi}$ be the matrix representation of the operator $U_{\pi}$ under the basis $V$ of $\mathcal{H}^\otimes n$. From the definition of $U_{\pi}$ and $G_{\pi}$ we see that $U_{\pi}$ is exactly the adjacency matrix of $G_{\pi}$. Define $J_{\pi} : \mathcal{L}_{\mathcal{H}^\otimes n} \mapsto \mathcal{L}_{\mathcal{H}^\otimes n}$ by that $$J_{\pi}(\rho) = U_{\pi}\rho U_{\pi}^\dagger, \; \rho \in \mathcal{L}_{\mathcal{H}^\otimes n}.$$ From the correspondence of tensor product and Kronecker product we see that $U_{\pi} \otimes U_{\pi}$ is a matrix representation of $J_{\pi}$ under the basis $V$, as well as the adjacency matrix of $G_{\pi}$.

Suppose $|E_{\pi}| = m$. There is a permutation matrix $P \in \mathbb{R}^{2^n \times 2^n}$ such that $$PU_{\pi}P^{-1} = \tilde{U}_{\pi} = \begin{pmatrix} I_{2^n-m} & 0 \\ 0 & Q_{\pi} \end{pmatrix}$$ with $Q_{\pi}$ being a stochastic matrix with zero diagonals. It is therefore straightforward to directly compute that there are $$m^2 + 2m(2^n - m) = 2^{n+1}m - m^2$$ nonzero and non-diagonal entries in $U_{\pi} \otimes U_{\pi}$, which immediately yields the desired conclusion.

(ii). The conclusion follows immediately combining Proposition 4(i) and the structure of $\tilde{U}_{\pi}$ shown in (30).
(iii). By definition\[ L(\mathcal{G}_P^*) := \sum_{\pi \in \mathcal{P}^*} \left( I_{4^n} - U_{1} \otimes U_{\pi} \right) \]
is the Laplacian of $\mathcal{G}_P^*$. Recall that every weakly strongly connected component of $\mathcal{G}_P^*$ is strongly connected since it is balanced. From Lemma 10 we further know that the multiplicity of the zero eigenvalue of $L(\mathcal{G}_P^*)$ equals to the number of strongly connected components of $L(\mathcal{G}_P^*)$. On the other hand $-L(\mathcal{G}_P^*)$ is the matrix representation of $K_{\mathcal{G}_P^*}$ under the basis $\mathcal{V}$. The desired conclusion holds.

(iv). The conclusion follows from (iii) and the fact that $\dim(\text{Null}(K_{\mathcal{G}_P^*}))$ is fully determined by $\mathcal{C}_{\mathcal{G}_P^*}$ from Lemma 9.

We have now completed the proof of Proposition 5.

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GUODONG SHI
College of Engineering and Computer Science, The Australian National University,
Canberra, ACT 0200 Australia
Email: guodong.shi@anu.edu.au

SHUANGSHUANG FU
College of Engineering and Computer Science, The Australian National University,
Canberra, ACT 0200 Australia
Email: shuangshuang.fu@anu.edu.au

IAN R. PETERSEN
School of Engineering and Information Technology, University of New South Wales,
Canberra, ACT 2600 Australia
Email: i.r.petersen@gmail.com