NAVIER-STOKES’ EQUATIONS FOR RADIAL AND TANGENTIAL ACCELERATIONS

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Abstract

The Navier-Stokes equations are considered by the use of the method of Lagrangians with covariant derivatives (MLCD) over spaces with affine connections and metrics. It is shown that the Euler-Lagrange equations appear as sufficient conditions for the existence of solutions of the Navier-Stokes equations over (pseudo) Euclidean and (pseudo) Riemannian spaces without torsion. By means of the corresponding (n - 1) + 1 projective formalism the Navier-Stokes equations for radial and tangential accelerations are found.

1 Introduction

By the use of the method of Lagrangians with covariant derivatives (MLCD) [1, 2] the different energy-momentum tensors and the covariant Noether’s identities for a field theory as well as for a theory of continuous media can be found. On the basis of the (n - 1) + 1 projective formalism and by the use of the notion of covariant divergency of a tensor of second rank the corresponding covariant divergencies of the energy-momentum tensors could be found. They lead to Navier-Stokes’ identity and to the corresponding generalized Navier-Stokes’ equations.

The general scheme for obtaining the Navier-Stokes equations for radial and tangential acceleration could be given in the form
The structure of a Lagrangian theory of tensor fields over a differentiable manifold $M$ (dim $M = n$) could be represented in the form

Let the following structure

$$(M, V, g, \Gamma, P)$$

be given, where

(i) $M$ is a differentiable manifold with dim $M = n$,
(ii) $V = V^A_B \cdot e_A \otimes e^B \in \otimes^k l(M)$ are tensor fields with contravariant rank $k$ and covariant rank $l$ over $M$, $A$ and $B$ are collective indices,
(iii) $g \in \otimes_{Sym}^2 (M)$ is a covariant symmetric metric tensor field over $M$,
(iv) $\Gamma$ is a contravariant affine connection, $P$ is a covariant affine connection related to the covariant differential operator along a basis vector field $\partial_i$ or $e_i$. 

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in a co-ordinate or non-co-ordinate basis respectively

\[ \nabla \delta V = V^A_{B;i} \cdot \partial_A \otimes dx^B , \]

\[ V^A_{B;i} = (V^A_{B,i} + \Gamma^A_{C;i} \cdot V^C_{B} + P^D_{Bi} \cdot V^A_{D} , \]

\[ V^A_{B;i} = \frac{\partial V^A_{B}}{\partial x^i} . \]

A Lagrangian density \( L \) can be considered in two different ways as a tensor density of rank 0 with the weight \( q = 1/2 \), depending on tensor field’s components and their first and second covariant derivatives

(i) As a tensor density \( L \) of type 1, depending on tensor field’s components, their first (and second) partial derivatives, (and the components of contravariant and covariant affine connections), i.e.

\[ L = \sqrt{-d_g} \cdot L(g_{ij}, g_{ij,k}, V^A_{B}, V^A_{B;i}, V^A_{B;i,j}, \Gamma^{i}_{jk}, ..., P^{i}_{jk}, ...) , \]

where \( L \) is a Lagrangian invariant,

\[ d_g = \det(g_{ij}) < 0 , \quad g = g_{ij} \cdot dx^i dx^j , \]

\[ dx^i dx^j = \frac{1}{2} \cdot (dx^i \otimes dx^j + dx^j \otimes dx^i) , \]

\[ V^A_{B;i,j} = \frac{\partial V^A_{B}}{\partial x^i \partial x^j} . \]

The method using a Lagrangian density of type 1 is called Method of Lagrangians with partial derivatives (MLPD).

(ii) As a tensor density \( L \) of type 2, depending on tensor field’s components and their first (and second) covariant derivatives, i.e.

\[ L = \sqrt{-d_g} \cdot L(g_{ij}, g_{ij;k}, V^A_{B}, V^A_{B;i}, V^A_{B;i,j}) . \]

By the use of the variation operator \( \delta \), commuting with the covariant differential operator

\[ \delta \circ \nabla_\xi = \nabla_\xi \circ \delta + \nabla_\delta \xi , \quad \xi \in T(M) , \quad T(M) = \cup_{x \in M} T_x(M) , \]

we could find the Euler-Lagrange equations.

By the use of the Lie variation operator (identical with the Lie differential operator) \( \mathcal{L}_\xi \), we could find the corresponding energy-momentum tensors.

The method using a Lagrangian density of type 2 is called Method of Lagrangians with covariant derivatives (MLCD).

1.1 Euler-Lagrange’s equations

The Euler-Lagrange equations follow from the variation of the Lagrangian density of type 2 in the form \( \mathcal{R} \).
(i) for the tensor fields \( V \)
\[
\frac{\delta_v L}{\delta V^A} B + P^A_B = 0 ,
\]
(ii) for the metric tensor field \( g \)
\[
\frac{\delta_g L}{\delta g_{kl}} + \frac{1}{2} \cdot L \cdot g^{kl} + P^{kl} = 0 .
\]

**Special cases:** (Pseudo) Euclidean and (pseudo) Riemannian spaces without torsion.
\[
\frac{\delta_v L}{\delta V^A} B = 0 , \quad \frac{\delta_g L}{\delta g_{kl}} + \frac{1}{2} \cdot L \cdot g^{kl} = 0 .
\]

### 1.2 Energy-momentum tensors

By the use of the Lie variation operator the energy-momentum tensors follow:

(i) Generalized canonical energy-momentum tensor \( \theta = \overline{F}_i^j \cdot \partial_j \otimes dx^i \),
(ii) Symmetric energy-momentum tensor of Belinfante \( sT = T_i^j \cdot \partial_j \otimes dx^i \),
(iii) Variational energy-momentum tensor of Euler-Lagrange \( Q = Q_i^j \cdot \partial_j \otimes dx^i \).

The energy-momentum tensors obey the covariant Noether identities
\[
\begin{align*}
F_i + \theta_i^j \equiv 0 , \\
F + \delta \theta \equiv 0 , \\
\text{(first covariant Noether’s identity)} \\
\overline{F}_i \cdot j - sT_i \equiv \overline{Q}_i \cdot j , \\
\theta - sT \equiv Q . \\
\text{(second covariant Noether’s identity)}
\end{align*}
\]

Now we can draw a rough scheme of the main structure of a Lagrangian theory:

```
⌜ ←− L −→ ⌝
↓       ↓       ↓
↓       sT     δL/δV^A_B
↓       ↓       ↓
θ       ↓       Q
↓       ↓       ↓
ζ       → θ − sT ≡ Q ←−
↓       ↓       ↓
ζ       → F + δθ ≡ 0 ←−
```

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2 Invariant projections of the energy-momentum tensors

By the use of the \((n - 1) + 1\) projective formalism we can find the invariant projections of the energy-momentum tensors corresponding to a Lagrangian field theory or to a theory of continuous media. The idea of the projective formalism is the representation of the dynamic characteristics of a Lagrangian system by means of their projections along the world line of an observer and to local neighborhoods orthogonal to this world line. The tangent vector to the world line of the observer and its local neighborhoods determine the notion of frame of reference \(Fr(u, \tau, \xi_\perp)\), where \(u\) is the tangent vector of the world line, \(\tau\) is the parameter of the world line, interpreted as the proper time of the observer \(\xi_\perp\) is a contravariant vector field, orthogonal to \(u\). The variation of \(\xi_\perp\) determines the relative velocity and the relative acceleration between the points at the world line and the points in the neighborhoods lying in the subspace orthogonal to the vector \(u\).

Let the contravariant vector field \(u \in T(M)\), \(g(u, u) := e \neq 0\), and its corresponding projective metrics \(h_u\) and \(h^u\)

\[
h_u = g - \frac{1}{e} \cdot g(u) \otimes g(u) , \quad h^u = \overline{g} - \frac{1}{e} \cdot u \otimes u ,
\]

\[
\overline{g} = g^{ij} \cdot \partial_i \otimes \partial_j , \quad \partial_i \otimes \partial_j = \frac{1}{2} \cdot (\partial_i \otimes \partial_j + \partial_j \otimes \partial_i)
\]

be given. Then the following proposition can be proved:

**Proposition 1.** Every energy-momentum tensor \(G \sim (\theta, sT, Q)\) could be represented in the form \[8\]

\[
G = (\rho_G + \frac{1}{e} \cdot L \cdot k) \cdot u \otimes g(u) - L \cdot Kr + u \otimes g(k\pi) + k_s \otimes g(u) + (kS)g ,
\]

where

\[
k\pi = g^{\pi} , \quad k_S = g^\pi , \quad k_S = g^\pi ,
\]

\(\rho_G\) is the rest mass density, \(k = (1/e) \cdot [g(u)](Kr)u\), \(L\) is the pressure of the system, \(Kr = g^i_j \cdot \partial_i \otimes dx^j\) is the Kronecker tensor, \(k\pi\) is the conductive momentum density, \(k_s\) is the conductive energy flux density, \(kS\) is the stress tensor \[4\].

3 Covariant divergency of the energy-momentum tensors and the rest mass density

The covariant divergency \(\delta G\) of the energy-momentum tensor \(\delta G \sim (\theta, sT, Q)\) can be represented by the use of the projective metrics \(h^u, h_u\) of the contravariant vector field \(u\) and the rest mass density for the corresponding energy-momentum tensor \(\rho_G\).
\[ \delta G = (\rho G + \frac{1}{c} \cdot L \cdot k) \cdot g(a) + \\
+ [u(\rho G + \frac{1}{c} \cdot L \cdot k) + (\rho G + \frac{1}{c} \cdot L \cdot k) \cdot \delta u + \delta G] \cdot g(u) - \\
- KrL - L \cdot \delta Kr + \delta u \cdot g(G\pi) + g(\nabla_u G\pi) + g(\nabla \omega_u u) + \\
+ (\rho G + \frac{1}{c} \cdot L \cdot k) \cdot (\nabla_u g)(u) + (\nabla_u g)(G\pi) + (\nabla \omega_u g)(u) + \\
+ \delta(S)g, \quad a = \nabla_u u, \]

\[ \bar{\Psi}(\delta G) = (\rho G + \frac{1}{c} \cdot L \cdot k) \cdot a + \\
+ [u(\rho G + \frac{1}{c} \cdot L \cdot k) + (\rho G + \frac{1}{c} \cdot L \cdot k) \cdot \delta u + \delta G] \cdot u - \\
- \bar{g}(KrL) - L \cdot \bar{g}(\delta Kr) + \delta u \cdot G\pi + \nabla_u G\pi + \nabla \omega_u u + \\
+ (\rho G + \frac{1}{c} \cdot L \cdot k) \cdot (\nabla_u g)(u) + (\nabla_u g)(G\pi) + (\nabla \omega_u g)(u) + \\
+ \bar{g}(\delta(S))g) \]

In a co-ordinate basis \( \delta G \) and \( \bar{\Psi}(\delta G) \) will have the forms

\[ G_i^j, G_k^j = (\rho G + \frac{1}{c} \cdot L \cdot k) \cdot a_i + \\
+ [((\rho G + \frac{1}{c} \cdot L \cdot k) \cdot u^j + (\rho G + \frac{1}{c} \cdot L \cdot k) \cdot w^j, + G\pi^j, u^j, + G\pi^j, w^j, + G\pi^j, k) + \\
- L_i - L \cdot g_i^j, + w^j, + G\pi_i + g_{i, k} ((G\pi^j, k, u^j, k, + G\pi^j, k)) + \\
+ g_{i, k} ((G\pi^j, k, u^j, k, + G\pi^j, w^j) + \\
+ (g_{ik}, G\pi^j, k))_{ij} 
\]

\[ G_i^j, G_k^j = (\rho G + \frac{1}{c} \cdot L \cdot k) \cdot a_i + \\
+ [((\rho G + \frac{1}{c} \cdot L \cdot k) \cdot u^j + (\rho G + \frac{1}{c} \cdot L \cdot k) \cdot w^j, + G\pi^j, u^j, + G\pi^j, w^j, + G\pi^j, k) + \\
- L_j - g_t^j, - L \cdot g_t^j, g_t^j, + w^j, + G\pi_i + g_{i, k} ((G\pi^j, k, u^j, k, + G\pi^j, k)) + \\
+ g_{i, k} ((G\pi^j, k, u^j, k, + G\pi^j, w^j) + \\
+ (g_{ik}, G\pi^j, k))_{ij} 
\]

4 Navier-Stokes’ identities and Navier-Stokes’ equations

If we consider the projections of the first Noether identity along a non-null (non-isotropic) vector field \( u \) and its corresponding contravariant and covariant projective metrics \( h^a \) and \( h_u \) we will find the first and second Navier-Stokes identities.

From the Noether identities in the form

\[ \bar{\Psi}(F) + \bar{\Psi}(\delta \theta) \equiv 0, \quad \text{(first covariant Noether’s identity)} \]
\[ (\theta \bar{\Psi}) - (\iota T) \bar{\Psi} \equiv (Q) \bar{\Psi}, \quad \text{(second covariant Noether’s identity)} \]

we can find the projections of the first Noether identity along a contravariant non-null vector field \( u = u^i \cdot \partial_i \) and orthogonal to \( u \).

Since

\[ g(\bar{\Psi}(F), u) = g_{ik} \cdot g^i \cdot F_i \cdot u^i = g_{ik} \cdot F_i \cdot u^i = F(u), \]
\[ g(\bar{\Psi}(\delta \theta), u) = (\delta \theta)(u), \quad F = F_k \cdot dx^k, \]
we obtain the first Navier-Stokes identity in the form

\[ F(u) + (\delta \theta)(u) \equiv 0 \ . \] (7)

By the use of the relation

\[ \mathcal{g}[h_u(\mathcal{g}(F)) = \mathcal{g}(h_u[\mathcal{g}(F)]) = h^u(F) \ , \ \mathcal{g}(h_u) = h^u \ , \] (8)

\[ \mathcal{g}[h_u(\mathcal{g}(\delta \theta)) = \mathcal{g}(h_u[\mathcal{g}(\delta \theta)]) = h^u(\delta \theta) \ , \] (9)

the first Noether identity could be written in the forms

\[ h_u[\mathcal{g}(F)] + h_u[\mathcal{g}(\delta \theta)] \equiv 0 \ , \] (10)
\[ h^u(F) + h^u(\delta \theta) \equiv 0 \ . \] (11)

The last two forms of the first Noether identity represent the second Navier-Stokes identity.

If the projection \( h^u(F) \), orthogonal to \( u \), of the volume force \( F \) is equal to zero, we obtain the generalized Navier-Stokes equation in the form

\[ h^u(\delta \theta) = 0 \ , \] (12)
or in the form

\[ h_u[\mathcal{g}(\delta \theta)] = 0 \ . \] (13)

Let us now find the explicit form of the first and second Navier-Stokes identities and the explicit form of the generalized Navier-Stokes equation. For this purpose we can use the explicit form of the covariant divergence \( \delta \theta \) of the generalized canonical energy-momentum tensor \( \theta \).

(a) The first Navier-Stokes identity follows in the form

\[
F(u) + (\rho \rho + \frac{1}{c} \cdot L \cdot k) \cdot g(a, u) + \\
+ c \cdot [u] \cdot (\rho \rho + \frac{1}{c} \cdot L \cdot k) + (\rho \rho + \frac{1}{c} \cdot L \cdot k) \cdot \delta u + \delta^\theta \pi - \\
- (K r L)(u) - L \cdot (\delta K r)(u) + g(\nabla u \theta \pi, u) + g(\nabla \pi \theta, u) + \\
+ (\rho \rho + \frac{1}{c} \cdot L \cdot k) \cdot (\nabla u g)(u, u) + (\nabla u g)(\theta \pi, u) + (\nabla \pi g)(u, u) + \\
\] 

\[ + [\delta((\theta \pi) g)](u) \equiv 0 \ . \] (14)

Since

\[ g(u, a) = \pm l_u \cdot \frac{d l_u}{d \tau} - \frac{1}{2} \cdot (\nabla u g)(u, u) = \]

\[ = \frac{1}{2} \cdot \left[ \frac{d}{d \tau} (\pm l_u^2) - (\nabla u g)(u, u) \right] \ , \]

the first Navier-Stokes identity could be interpreted as a definition for the change of \( l_u^2 \) along the world line of the observer. The length of the non-isotropic contravariant vector \( u \) is interpreted as the velocity of a signal emitted or received
by the observer \[6\]. On this basis, the first Navier-Stokes identity is related to the change of the velocity of signals emitted or received by an observer moving in a continuous media or in a fluid.

(b) The second Navier-Stokes identity can be found in the form

\[
\left[ g \left( \nabla_u \theta \right) \right] + \left[ g \left( \delta \theta \right) \right] = \left( \rho \theta + \frac{1}{e} \cdot L \cdot k \right) \cdot h_u(a) - \left[ g \left( KrL \right) \right] - \left[ g \left( \delta Kr \right) \right] + \left[ g \left( \nabla_u g \right) \right] + \left[ g \left( \theta L \cdot k \right) \right] + \left( \rho \theta + \frac{1}{e} \cdot L \cdot k \right) \cdot h_u(a) - h_u \left( \delta \theta \right) = 0 . \tag{15}
\]

(c) The generalized Navier-Stokes equation \[ h_u \left[ g \left( \delta \theta \right) \right] = 0 \] follows from the second Navier-Stokes identity under the condition \[ h_u \left[ g \left( F \right) \right] = 0 \] or under the condition \[ F = 0 \]

\[
\left( \rho \theta + \frac{1}{e} \cdot L \cdot k \right) \cdot h_u(a) - \left[ g \left( KrL \right) \right] - \left[ g \left( \delta Kr \right) \right] + \left[ g \left( \nabla_u g \right) \right] + \left[ g \left( \theta L \cdot k \right) \right] + \left( \rho \theta + \frac{1}{e} \cdot L \cdot k \right) \cdot h_u(a) - h_u \left( \delta \theta \right) = 0 , \tag{16}
\]

\[ h_u(a) = g(a) - \frac{1}{e} \cdot g(u, a) \cdot g(u) . \tag{17} \]

The second Navier-Stokes identity could be considered as a definition for the density of the inner force. If the density of the inner force is equal to zero, i.e. if \( F = g(F) = 0 \), then the covariant divergency, \( \delta \theta = g(\delta \theta) \) of the generalized canonical energy-momentum tensor \( \theta \) is also equal to zero, i.e. \( \delta \theta = g(\delta \theta) = 0 \). Then the orthogonal to the contravariant vector field \( u \) projection of the second Navier-Stokes identity lead to the equations

\[
g \left[ h_u(F) \right] = 0 \quad \Leftrightarrow \quad g \left[ h_u(\delta \theta) \right] = 0 . \tag{18}
\]

The last equation is the **Navier-Stokes equation in spaces with affine connections and metrics**. Now, we can prove the following proposition:

**Proposition 2.** The necessary and sufficient condition for the existence of the Navier-Stokes equation in a space with affine connections and metrics is the
condition for the vanishing of the density of the inner force in a dynamic system described by the use of a Lagrangian invariant $L$, interpreted as the pressure $p$ of the system, i.e. the necessary and sufficient condition for

$$\mathcal{G}[h_u(\delta \theta)] = 0$$

is the condition

$$\mathcal{G}[h_u(F)] = 0 .$$

The proof follows directly from the projective second Navier-Stokes identity $\mathcal{G}[h_u(F)] + \mathcal{G}[h_u(\delta \theta)] \equiv 0$.

Special case: $(L_n, g)$-spaces: $S = C$, $f^i_j = g^j_i$, $g(u, u) = e = \text{const.} \neq 0$, $k = 1$.

$$\delta Kr = 0 ,$$

(a) First Navier-Stokes' identity

\[ F(u) + (\rho \theta + \frac{1}{e} \cdot L) \cdot g(a, u) + \]
\[ + e \cdot [u(\rho \theta + \frac{1}{e} \cdot L) + (\rho \theta + \frac{1}{e} \cdot L) \cdot \delta u + \delta^0 \theta] - \]
\[ - (Kr L)(u) + g(\nabla_u \theta \pi, u) + g(\nabla \theta \pi u, u) + \]
\[ + (\rho \theta + \frac{1}{e} \cdot L) \cdot (\nabla_u g)(u, u) + (\nabla_u g)(\theta \pi, u) + (\nabla \theta \pi g)(u, u) + \]
\[ + [\delta((\theta \pi g))](u) \equiv 0 . \] (19)

(b) Second Navier-Stokes' identity

\[ (\rho \theta + \frac{1}{e} \cdot L) \cdot h_u(a) - \]
\[ - h_u[\mathcal{G}(Kr L)] + \delta u \cdot h_u(\theta \pi) + \]
\[ + h_u(\nabla_u \theta \pi) + h_u(\nabla \theta \pi u) + \]
\[ + (\rho \theta + \frac{1}{e} \cdot L) \cdot h_u[\mathcal{G}(\nabla_u g)](u) + h_u[\mathcal{G}(\nabla_u g)(\theta \pi)] + \]
\[ + h_u[\mathcal{G}(\nabla \theta \pi g)(u)] + h_u[\mathcal{G}(\delta((\theta \pi g)))] \]
\[ + h_u[\mathcal{G}(F)] \equiv 0 . \] (20)

(c) Generalized Navier-Stokes' equation $h_u[\mathcal{G}(\delta \theta)] = 0$
\[(\rho_0 + \frac{1}{e} \cdot L) \cdot h_u(a) - h_u[\mathcal{G}(KrL)] + \delta u \cdot h_u(\theta^{\pi}) + h_u(\nabla_u \theta^{\pi}) + h_u(\nabla_u \pi u) + (\rho_0 + \frac{1}{e} \cdot L) \cdot h_u[\mathcal{G}(\nabla g)(u)] + h_u[\mathcal{G}(\nabla g)(\theta^{\pi})] + h_u[\mathcal{G}(\nabla g)(\theta^{\pi})] + h_u[\mathcal{G}(\nabla g)(\theta^{\pi})] + h_u[\mathcal{G}(\nabla g)(\theta^{\pi})] = 0 . \] (21)

Special case: \( V_n \)-spaces: \( S = C, f_i j = g_i j, \nabla \xi g = 0 \) for \( \forall \xi \in T(M), g(u, u) = e = \text{const.} \neq 0, k = 1, g(a, u) = 0. \)

(a) First Navier-Stokes’ identity

\[ F(u) + e \cdot [u(\rho_0 + \frac{1}{e} \cdot L) + (\rho_0 + \frac{1}{e} \cdot L) \cdot \delta u + \delta \theta^{\pi}] + \nabla \theta^{\pi}) = 0 . \] (22)

(b) Second Navier-Stokes’ identity

\[ (\rho_0 + \frac{1}{e} \cdot L) \cdot h_u(\theta^{\pi}) - h_u[\mathcal{G}(KrL)] + \delta u \cdot h_u(\theta^{\pi}) + h_u[\mathcal{G}(\theta^{\pi})] + h_u[\mathcal{G}(\theta^{\pi})] = 0 . \] (23)

(c) Generalized Navier-Stokes’ equation \( h_u[\mathcal{G}(\delta \theta)] = 0 \)

\[ (\rho_0 + \frac{1}{e} \cdot L) \cdot h_u(a) - h_u[\mathcal{G}(KrL)] + \delta u \cdot h_u(\theta^{\pi}) + h_u[\mathcal{G}(\theta^{\pi})] = 0 . \] (24)

If we express the stress (tension) tensor \( \theta^{\pi} \) by the use of the shear stress tensor \( k_u \partial \), rotation (vortex) stress tensor \( k_u \partial \), and the expansion stress invariant \( k_u \partial \), then the covariant divergency of the corresponding tensors could be found and at the end we will have the explicit form of the Navier-Stokes identities and the generalized Navier-Stokes’ equation including all necessary tensors for further applications. The way of obtaining the Navier-Stokes equations could be given in the following rough scheme
5 Invariant projections of Navier-Stokes’ equations

5.1 Navier-Stokes’ equations and Euler-Lagrange’s equations

Let us now consider the second Navier-Stokes identity in the form

\[ \mathcal{F}[h_u[h(F)]] + \mathcal{F}[h_u[\mathcal{F}(\delta \theta)]] \equiv 0 \]

or in the form

\[ F_\perp + \delta \theta_\perp \equiv 0 \, , \quad F_\perp = \mathcal{F}[h_u[h(F)]] \, , \quad \delta \theta_\perp = \mathcal{F}[h_u[\mathcal{F}(\delta \theta)]] \, , \]

\[ g(u, F_\perp) = 0 \, , \quad g(u, \delta \theta_\perp) = 0 \, . \]

The explicit form of the density \( F \) of the inner force could be given as

\[ F_i = F_i \cdot dx^i \, , \]

\[ F_i = \frac{\delta L}{\delta V^A} \cdot V^A_{B;i} + W_i \, , \]

\[ W_i = W_i(Tk^j, g_{jk;i}) \, . \]
where $T_{kl}^j$ are the components of the torsion tensor (in a co-ordinate basis $T_{kl}^j = \Gamma^i_{lk} - \Gamma^i_{kl}$).

For (pseudo) Euclidean and (pseudo) Riemannian spaces without torsion ($T_{kl}^i = 0$) the quantity $W$ is equal to zero ($W_i = 0$) and the density of the inner force $F$ has the form

$$\overline{F}_i = \frac{\delta L}{\delta V^A B} \cdot V^A_{B;i}$$

If the Euler-Lagrange equations are fulfilled in (pseudo) Euclidean and (pseudo) Riemannian spaces without torsion, i.e. if

$$\frac{\delta L}{\delta V^A B} = 0 ,$$

then $F = 0$ and

$$\overline{F}_i = \frac{\delta L}{\delta V^A B} \cdot V^A_{B;i} = 0 ,$$

and the following propositions could be proved:

**Proposition 3.** Sufficient conditions for the existence of the Navier-Stokes equation in (pseudo) Euclidean and (pseudo) Riemannian spaces without torsion are the Euler-Lagrange equations.

**Proposition 4.** Every contravariant vector field $u \in T(M)$ in (pseudo) Euclidean and (pseudo) Riemannian spaces without torsion is a solution of the Navier-Stokes equation if the Euler-Lagrange equations are fulfilled for the dynamic system, described by a given Lagrangian invariant $L = p$ interpreted as the pressure of the system.

**Corollary.** If $L = p = p(u^i, u^i_{\cdot j}, u^i_{\cdot j;k}, g_{ij}, g_{ij;k}, g_{ij;k;i}, V^A B, V^A B;i, V^A B;ij)$ is a Lagrangian density fulfilling the Euler-Lagrange equations for $u^i$ and $V^A B$ in (pseudo) Euclidean and (pseudo) Riemannian spaces without torsion, then the contravariant non-isotropic vector field $u$ is also a solution of the Navier-Stokes equation.

### 5.2 Representation of $F_\perp$ and $\delta \theta_\perp$

Now, we can use the corresponding to a vector field $\xi_\perp, g(u, \xi_\perp) = 0$ (orthogonal to the vector field $u$) projective metrics $h_{\xi_\perp}$ and $h^{\xi_\perp}$

$$h_{\xi_\perp} = g - \frac{1}{g(\xi_\perp, \xi_\perp)} \cdot g(\xi_\perp) \otimes g(\xi_\perp) ,$$

$$h^{\xi_\perp} = \overline{g} - \frac{1}{g(\xi_\perp, \xi_\perp)} \cdot \xi_\perp \otimes \xi_\perp .$$

The vector field $F_\perp$ could be written in the form [4], [5]

$$F_\perp = \frac{g(F_\perp, \xi_\perp)}{g(\xi_\perp, \xi_\perp)} \cdot \xi_\perp + \overline{g}[h_{\xi_\perp}(F_\perp)] = \mp g(F_\perp, n_\perp) \cdot n_\perp + \overline{g}[h_{\xi_\perp}(F_\perp)] =$$

$$= F_{\perp z} + F_{\perp c} , \quad \xi_\perp = l_{\xi_\perp} \cdot n_\perp , \quad g(n_\perp, n_\perp) = \mp 1 .$$
\( F_{\perp} \) is the radial inner force density and \( F_{\perp c} \) is the tangential (Coriolis) inner force density

\[
F_{\perp} = \mp g(F_{\perp z}, n_\perp) \cdot n_\perp, \quad F_{\perp c} = \mathcal{F}[h_{\xi z}(F_{\perp})],
\]

\[
g(F_{\perp z}, u) = 0, \quad g(F_{\perp c}, \xi_\perp) = 0, \quad g(F_{\perp c}, u) = 0.
\]

The Navier-Stokes equation could now be written in the form

\[
\delta \theta_\perp = \mp g(\delta \theta_\perp, n_\perp) \cdot n_\perp + \mathcal{F}[h_{\xi z}(\delta \theta_\perp)] = 0,
\]
or in the forms

\[
\delta \theta_\perp := \mp g(\delta \theta_\perp, n_\perp) \cdot n_\perp = 0,
\]

Navier-Stokes’ equation for radial accelerations,

\[
\delta \theta_\perp := \mathcal{F}[h_{\xi z}(\delta \theta_\perp)] = 0,
\]

Navier-Stokes’ equation for tangential accelerations.

### 5.3 Radial projections of Navier-Stokes’ equation. Navier-Stokes’ equation for radial accelerations

If we use the explicit form of the Navier-Stokes equation

\[
(\rho \theta + \frac{1}{c} \cdot L \cdot k) \cdot a_{\perp} -
- [\mathcal{F}(KrL)]_{\perp} - L \cdot [\mathcal{F}(\delta Kr)]_{\perp} + \delta u \cdot [g\pi]_{\perp} +
+ (\nabla u [g\pi])_{\perp} + (\nabla u g\pi)_{\perp} +
+ (\rho \theta + \frac{1}{c} \cdot L \cdot k) \cdot [\mathcal{F}(\nabla u g)(u)]_{\perp} + [\mathcal{F}(\nabla u g)(\delta g)]_{\perp} +
+ [\mathcal{F}(\nabla u g)(\pi)]_{\perp} + [\mathcal{F}(\delta (\nabla g))]_{\perp} = 0,
\]

\( L = p \),

and apply the projection of the Navier-Stokes equation along and orthogonal to the vector field \( \xi_{\perp} \), by the use of the representation of the acceleration \( a_{\perp} \) in the form

\[
a_{\perp} = g(a_{\perp}, n_\perp) \cdot n_\perp + [h_{\xi z}(a_{\perp})] = a_z + a_c,
\]

\[
a_z = g(a_{\perp}, n_\perp) \cdot n_\perp, \quad a_c = [h_{\xi z}(a_{\perp})],
\]

where \( a_z = g(a_{\perp}, n_\perp) \cdot n_\perp = \mp l_{a_z} \cdot n_\perp \) is the radial (centrifugal, centripetal) acceleration and \( a_c = [h_{\xi z}(a_{\perp})] = \mp l_{a_c} \cdot m_\perp, g(n_\perp, m_\perp) = 0 \), is the tangential (Coriolis) acceleration, we could find the explicit form of the Navier-Stokes equation for radial (centrifugal, centripetal) accelerations in the form
\((\rho \theta + \frac{1}{c} \cdot L \cdot k) \cdot a_z - \left[ \frac{\partial}{\partial t} (KrL) \right]_{\perp z} - L \cdot \left[ \frac{\partial}{\partial t} (\delta Kr) \right]_{\perp z} + \delta u \cdot \theta \pi_{\perp z} + \right) \\
\left( \nabla_{\perp z} u \theta \right)_{\perp z} + \left( \nabla_{\perp z} u \right)_{\perp z} + \\
\left( \rho \theta + \frac{1}{c} \cdot L \cdot k \right) \cdot \left[ \frac{\partial}{\partial t} (\nabla_u g) (u) \right]_{\perp z} + \left[ \frac{\partial}{\partial t} (\delta (\theta \pi g)) \right]_{\perp z} = 0 \right), \\
\left( \frac{\partial}{\partial t} (\partial \pi) \right)_{\perp z} + \left( \nabla_{\perp z} \left( \frac{\partial}{\partial t} \pi \right) \right)_{\perp z} + \\
\left( \rho \theta + \frac{1}{c} \cdot L \cdot k \right) \cdot \left[ \frac{\partial}{\partial t} (\nabla_u g) (u) \right]_{\perp z} + \left[ \frac{\partial}{\partial t} (\delta (\theta \pi g)) \right]_{\perp z} = 0 \right), \\
L = p \right) . \\
\left( \rho \theta + \frac{1}{c} \cdot L \cdot k \right) \cdot a_z - \left[ \frac{\partial}{\partial t} (KrL) \right]_{\perp z} - L \cdot \left[ \frac{\partial}{\partial t} (\delta Kr) \right]_{\perp z} + \\
\left( \nabla_{\perp z} u \theta \right)_{\perp z} + \left( \nabla_{\perp z} u \right)_{\perp z} + \\
\left( \rho \theta + \frac{1}{c} \cdot L \cdot k \right) \cdot \left[ \frac{\partial}{\partial t} (\nabla_u g) (u) \right]_{\perp z} + \left[ \frac{\partial}{\partial t} (\delta (\theta \pi g)) \right]_{\perp z} = 0 \right), \\
\left( \frac{\partial}{\partial t} (\partial \pi) \right)_{\perp z} + \left( \nabla_{\perp z} \left( \frac{\partial}{\partial t} \pi \right) \right)_{\perp z} + \\
\left( \rho \theta + \frac{1}{c} \cdot L \cdot k \right) \cdot \left[ \frac{\partial}{\partial t} (\nabla_u g) (u) \right]_{\perp z} + \left[ \frac{\partial}{\partial t} (\delta (\theta \pi g)) \right]_{\perp z} = 0 \right), \\
\left( \frac{\partial}{\partial t} (\partial \pi) \right)_{\perp z} + \left( \nabla_{\perp z} \left( \frac{\partial}{\partial t} \pi \right) \right)_{\perp z} + \\
\left( \rho \theta + \frac{1}{c} \cdot L \cdot k \right) \cdot \left[ \frac{\partial}{\partial t} (\nabla_u g) (u) \right]_{\perp z} + \left[ \frac{\partial}{\partial t} (\delta (\theta \pi g)) \right]_{\perp z} = 0 \right), \\
\left( \frac{\partial}{\partial t} (\partial \pi) \right)_{\perp z} + \left( \nabla_{\perp z} \left( \frac{\partial}{\partial t} \pi \right) \right)_{\perp z} +
**Special case:** Perfect fluids: $\theta \in 0, \theta_s = 0, \theta_S = 0, L = p$.

\[(\rho \theta + \frac{1}{e} \cdot L \cdot k) \cdot a_c - \left[ g(K_r L) \right]_c - L \cdot \left[ g(\delta K_r) \right]_c + (\rho \theta + \frac{1}{e} \cdot L \cdot k) \cdot [g(\nabla u g)]_c = 0 . \quad (32)\]

**Special case:** Perfect fluids in (pseudo) Euclidean and (pseudo) Riemannian spaces without torsion: $\theta \in 0, \theta_s = 0, \theta_S = 0, L = p, \nabla u g = 0, \delta K_r = 0$.

\[ (\rho \theta + \frac{1}{e} \cdot p) \cdot a_c = \left[ g(K r p) \right]_c , \quad (33) \]

\[ a_c = \frac{1}{(\rho \theta + \frac{1}{e} \cdot p)} \cdot [g(K r p)]_c . \quad (34) \]

### 6 Conclusions

The representations of the Navier-Stokes equation in its forms for radial (centrifugal, centripetal) and tangential (Coriolis') accelerations could be used for description of different motions of fluids and continuous media in continuous media mechanics, in hydrodynamics and in astrophysics. The method of Lagrangians with covariant derivatives (MLCD) appears to be a fruitful tool for working out the theory of continuous media mechanics and the theory of fluids in spaces with affine connections and metrics, considered as mathematical models of space-time.

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