New currents with Killing–Yano tensors

Ulf Lindström¹,² and Özgür Sarıoğlu¹,*

¹ Department of Physics, Faculty of Arts and Sciences, Middle East Technical University, 06800, Ankara, Turkey
² Department of Physics and Astronomy, Theoretical Physics, Uppsala University, SE-751 20 Uppsala, Sweden

E-mail: ulf.lindstrom@physics.uu.se and sariglu@metu.edu.tr

Received 10 May 2021, revised 13 July 2021
Accepted for publication 28 July 2021
Published 27 August 2021

Abstract

New relations involving the Riemann, Ricci and Einstein tensors that have to hold for a given geometry to admit Killing–Yano tensors (K YT s) are described. These relations are then used to introduce novel conserved ‘currents’ involving such KYTs. For a particular current based on the Einstein tensor, we discuss the issue of conserved charges and consider implications for matter coupled to gravity. The condition on the background geometry to allow asymptotic conserved charges for a current introduced by Kastor and Traschen is found and a number of other new aspects of this current are commented on. In particular we show that it vanishes for rank \((D - 1)\) KYTs in \(D\) dimensions.

Keywords: Killing, Yano, tensor, conserved, current, geometry, symmetry

1. Introduction

Killing–Yano tensors (KYTs) have long been studied in various settings. They can be thought of as square roots of Killing tensors with which they share some properties. In particular they are relevant to gravity, supergravity and string theory for finding hidden symmetries for particles and backgrounds, for separating variables in Hamilton–Jacobi equations and for finding the symmetries of the Dirac equation and its super extensions. The literature is vast and this is not a review, so we shall just mention some references that we have found useful in our present endeavor.

A general background to Killing tensors and KYTs is the nice paper [1]. A classical application to finding new supersymmetries is contained in ‘Susy in the sky’ [2]. Relevant for string theory are the more recent paper [3] and the extensive treatise [4]. There are further applications in general relativity (GR) [5, 6] to G-structures [7, 8], to WZW models [9], to classical mechanics [10] and to symmetries of the Dirac operator [11]. A comprehensive survey of these topics,
together with many more references, can be found in [12]. Finally, supersymmetric conformal KYTs are discussed in [13], and partly in [14].

We are interested in the effect of KYTs on the geometry. Part of our motivation is purely mathematical, investigating the interplay between the properties of a generic rank $n$ KYT and the rest of the geometry. As a consequence, we are also able to construct conserved antisymmetric contravariant tensors that we refer to as conserved currents. Not being Noether currents, these tensors correspond to conserved integrals that are not in general flux integrals. They can nevertheless in some cases lead to conserved charges along the lines of the Abbott–Deser (AD) construction for a Killing vector contracted with the energy momentum tensor. Our setting is GR in $D$ dimensions coupled to matter. Assuming that this admits a KYT of rank $n$, we derive two new identities for such KYTs and use them to find new conserved currents. We apply our identities to several known solutions of GR and discuss possible conserved charges for the new currents as well as other constraints on the matter content.

Our discussion is inspired by a result of Kastor and Traschen [15, 16], who constructed a conserved current for an arbitrary rank KYT. We show how this KT-current\(^3\) in general splits into sums of conserved currents and how special gravitational backgrounds allow particular such splittings. In [15, 16], it is stated that any spacetime that allows asymptotic KYTs will give rise to conserved charges using the KT-current. We find that in general there are obstructions to this, and derive a relation that the perturbed background geometry has to satisfy for these charges to exist. These obstructions can be traced back to the linearized Bianchi identities needed for the conservation of the KT-current.

After the definition of KYTs in section 2, we describe the new identities in section 3 and the currents in section 4. In section 5, we discuss some of the consequences of the existence of a KYT on the matter fields coupled to GR. Section 6 contains a reformulation of the KT-current in terms of the Weyl and Schouten tensors. This rewrite allows us to show that the KT-current identically vanishes in $D = 3$ for all KYTs and for rank $n = D - 1$ KYTs in $D \geq 4$. It also helps us to identify new constituent currents for special dimensions and KYT ranks. Moreover, it contains the derivation of a condition on the geometry for a general KT-current to give rise to asymptotic AD charges [17]. Sections 5 and 6 also contain gratifying checks on our identities for the FLRW geometry, the Kantowski–Sachs metric and the Kerr–Newman black hole. Section 7 deals with various special cases of the $n = 2$ KT-current. In appendix A, we discuss and exclude AD charges for one of our new currents based on the Einstein tensor. Appendix B contains the proof that another of our currents is conserved for conformally flat geometries.

2. Killing–Yano tensors

The KYTs generalise Killing vectors and Killing tensors to rank $n$ antisymmetric tensor fields with analogous properties. They can be thought of as being the components of an $n$-form\(^4\) $f_{a_1 \ldots a_n} = f_{[a_1 \ldots a_n]}$ satisfying

$$\nabla_{(a_1} f_{a_2 \ldots a_n]} = 0,$$

which implies the further properties

$$\nabla_{a_1} f_{a_2 \ldots a_{n+1}} = \nabla_{[a_1} f_{a_2 \ldots a_{n+1}]} \quad \text{and} \quad \nabla_{a_1} f^{a_1 \ldots a_n} = 0.$$  

\(^3\) See section 6 for the definition of a KT-current.

\(^4\) So in $D$ dimensions, one has $n \leq D$. 
These can be used to derive the nontrivial identity\(^5\) [15]
\[
\nabla_a \nabla_b f_{c_1...c_n} = (-1)^{n+1} \frac{(n + 1)}{2} R^d_{abc_1...c_n} f_{c_1...c_n},
\]
which generalises the analogous formula for a Killing vector \(\nabla_a \nabla_b f_c = R^d_{abc} f_d\) when \(n = 1\).

3. KYT identities

Let us rewrite (2.3) explicitly for \(n = 2\):
\[
\nabla_a \nabla_b f_{cd} = -\frac{3}{2} R^e_{abcde} f_{de} + \frac{1}{2} R^e_{abe} f_{cd} + \frac{1}{2} R^e_{acd} f_{eb} + \frac{1}{2} R^e_{adb} f_{ec}.
\]
We contract the \((a, c)\) indices in (3.1). Since \(\nabla_a f^{ab} = 0\), we find that
\[
\nabla_a \nabla_b f^{ac} = [\nabla_a, \nabla_b] f^{ac} = R_{ab} f^{ac} + R_{abc} f^{ad} = \frac{1}{2} R_{ab} f^{ac} - \frac{1}{2} R^{ac} f_{ab} + \frac{1}{2} R_{dab} f^{da},
\]
where the first line follows from the definition of the commutator of covariant derivatives and the second line from the contraction of indices on the right-hand side of (3.1). From the equality (3.2) we find
\[
\frac{1}{2} (R_{ab} f^{ac} + R^{ac} f_{ab}) = \frac{1}{2} R_{dab} f^{da} + R_{bda} f^{da},
\]
using \(R_{abcd} = 0\). We split the last term into two halves using the dummy index pair \((a, d)\) and employ \(R_{abcd} = 0\) again to arrive at
\[
R_{ab} f^{ac} + R^{ac} f_{ab} = 0.
\]
To our knowledge the identity (3.3) was first reported in [18], but does not seem to be widely known (see however [19–21]). It can alternatively be derived by acting on the defining property (2.1) with a second covariant derivative, considering various index combinations and applying the Ricci identity. This also leads to an identity between the Weyl tensor and \(f\) which we omit. See [20, 21] for details.

3.1. Generalisation of (3.3) for arbitrary rank \(n\) KYT

We repeat the steps above for the generic rank \(n\) case starting from (2.3). Contracting the \((a, c_n)\) indices gives
\[
g^{ac_n} \nabla_a \nabla_b f_{c_1...c_n} = \nabla^{c_n} \nabla_b f_{c_1...c_n} = \left[\nabla_{c_n}, \nabla_b\right] f_{c_1...c_n},
\]
\(^5\) We use ‘identity’ in the less strict sense where the properties of \(f\) have to be taken into account.
which yields
\[ R^d_b f_{[c_1 \ldots c_n-1]d} + (-1)^n(n-1) R^d_{bde[c_1} f_{e_2 \ldots e_{n-1}]d} = R^d_b f_{[c_1 \ldots c_n-1]d} + (-1)^n(n-1) \frac{1}{2} R_{ad}[^b_c f_{e_2 \ldots e_{n-1}]ad}. \] (3.5)

Since
\[ R^d_b f_{[c_1 \ldots c_n-1]d} = \frac{1}{n}\left[R^d_b f_{[c_1 \ldots c_n-1]d} + (-1)^n(n-1) R^d_{c_1 f_{e_2 \ldots e_{n-1}]d}\right], \]
\[ R_{ad}[^b_c f_{e_2 \ldots e_{n-1}]ad} = \frac{1}{n}\left(4 R^d_{bde[c_1} f_{e_2 \ldots e_{n-1}]d} + (-1)^n(n-2) R^d_{e_1 c_1 f_{e_2 \ldots e_{n-1}]b} + \frac{1}{2} R^d_{e_1 c_1 f_{e_2 \ldots e_{n-1}[ad]} \right) \]
(3.5) can be recast as
\[ R^d_b f_{[c_1 \ldots c_n-1]d} + (-1)^n R^d_{[c_1 f_{e_2 \ldots e_{n-1}]d} + (n-2) \left((-1)^n R^d_{bde[c_1} f_{e_2 \ldots e_{n-1}]d} + \frac{1}{2} R^d_{e_1 c_1 f_{e_2 \ldots e_{n-1}[ad]} \right) = 0. \] (3.6)

This is the generalisation of (3.3) for a rank $n$ KYT, and to our knowledge, has not been reported elsewhere in the literature.\(^6\)

As a quick check, it identically reduces to (3.3) when $n = 2$. Note that when any pair of free indices are contracted in (3.6), one gets identically zero on the left-hand side and there is nothing to infer from such contractions.

### 3.2. A new identity

Let us go back to (3.1) for a rank $n = 2$ KYT. This time we differentiate, i.e. consider
\[ \nabla_a (\nabla_b \nabla_c f_{de}) - \nabla_b (\nabla_a \nabla_c f_{de}) = [\nabla_a, \nabla_b] \nabla_c f_{de}, \] (3.7)
and use (3.1) on the left-hand side of (3.7). Using the Bianchi identity $\nabla^a R_{bcda} = 0$ and multiplying by an overall factor of 2, one gets
\[ f_{[a} \nabla^i R_{de]ab} = R^i_{b[cd} \nabla_e f_{ia} + R^i_{d[ea} \nabla_c f_{bi} + 2 R^i_{ab]} e f_{ej}. \] (3.8)

Contracting the $(a,e)$ indices in the latter and multiplying by an overall factor of 3 then gives
\[ 2 f_{[a} \nabla^a R_{e]b} + f^a_i \nabla_i R_{abcd} = 2 R_{ab]e} \nabla_d f_{ia} + 2 R^a_{[d} \nabla_i f_{ba} + R_{acde} \nabla_b f_{ia} + 4 R_{ab]e} \nabla_c f_{ia} + 2 R^a_{b} \nabla_c f_{da} \]
\[ = 3 R_{ab[e} \nabla_d f_{ia} + 2 R_{ab]} e \nabla_d f_{ia} + 3 R^a_{d} \nabla_c f_{ba} + R^a_{b} \nabla_c f_{da} + R_{acde} \nabla_b f_{ia}. \]

\(^6\) It has been pointed out to us by one of the referees that it might be related to the material in subsection 3.4 of [22]. Indeed the integrability condition in [22] gives the relation (3.3) for a $n = 2$ KYT when traced over one set of indices. For a $n = 3$ KYT tracing and anti-symmetrising two different index pairs recovers out (3.6).\(^7\)

\(^7\) The analogs of the steps taken here for the case of a Killing vector $f$, i.e. $n = 1$ case, gives the well-known result that the Lie derivative of the Riemann tensor along the Killing vector vanishes, i.e. $\mathcal{L}_f R_{ab} = 0$, which leads to $\mathcal{L}_f R_{ab} = 0$, $\mathcal{L}_f R = 0$ and hence to $\mathcal{L}_f G_{ab} = 0$.\(^8\)

---

\(^8\) Class. Quantum Grav. 38 (2021) 195011 1 U Lindström and Ö Sarıoğlu
Finally contracting the \((b, c)\) indices in the last equality gives

\[
f_{ab} \nabla^a R - f^{ab} \nabla_a R_{db} - f^{bd} \nabla_b R_{da} = 0,
\]

which is equivalent to

\[
f^{ab} \nabla_a G_{bd} = 0, \tag{3.9}
\]

where \(G_{ab}\) denotes the Einstein tensor. As far as we know, this identity has not been reported elsewhere.

### 3.3. Generalisation of (3.9) for arbitrary rank \(n\) KYT

It is again worthwhile repeating the steps taken from (3.7)–(3.9) for a generic rank \(n\) KYT. Starting from (2.3), we have, in analogy to (3.7),

\[
\nabla_a (\nabla_b \nabla_c f_{c_1...c_n}) - \nabla_b (\nabla_a \nabla_c f_{c_1...c_n}) = [\nabla_a, \nabla_b] \nabla_c f_{c_1...c_n}. \tag{3.10}
\]

Using (2.3), the Bianchi identity \(\nabla_{[a} R_{bc]de} = 0\) and some algebra, one finds

\[
(\nabla_d R_{ab} (c_1) f_{c_2...c_n})^d = 2 R_{ab [c_1} \nabla_{c_2...c_n]} + R_{bd [c_1} \nabla_{c_2...c_n]} + R_{da [c_1} \nabla_{c_2...c_n]} \tag{3.11}
\]

analogous to (3.8). On both sides of (3.11), if one contracts first the index pair \((a, c_n)\) and then the pair \((b, c)\), one finds that the right-hand side vanishes identically. However the left-hand side yields

\[
(n - 1) (\nabla^b R_{[c_1} f_{c_2...c_n-1]} ab + \frac{1}{2} (\nabla^a R) f_{a[c_1...c_n-1]} = 0. \tag{3.12}
\]

This is the generalisation of (3.9) for a rank \(n\) KYT, and reduces to (3.9) when \(n = 2\). To our knowledge, this identity is also new.

### 4. New currents

Let us return to the \(n = 2\) case, and the associated identities (3.3) and (3.9). The antisymmetry of the KYT and (3.3) immediately give

\[
G_{ab} f^{ac} + G^{ac} f_{ab} = 0, \tag{4.1}
\]

i.e. the analogous identity for the Einstein tensor. This suggests defining the ‘current’

\[
K_{ab} \equiv 2 G_{c[a} f^b c] = G^c f^{bc} - G^b f^{ac} = 2 G^c f^{bc}, \tag{4.2}
\]

where the last equality follows from (4.1). It is easy to see that this antisymmetric tensor is covariantly conserved

\[
\nabla_a K_{ab} = 0. \tag{4.3}
\]

\* Apart from [23], the relation (4.1) appears neither to have been considered nor used.
This can be shown in at least two separate ways. The easier one starts by using the last equality in (4.2), and employing (2.2) and the property $\nabla_a G^{ab} = 0$. Alternatively, one can use the penultimate equality in (4.2). This results in a total of four terms for $\nabla_a K^{ab}$, three of which cancel out by (2.2) and $\nabla_a G^{ab} = 0$ as before. The remaining piece, $(\nabla_a G^{bc}) f^{ac}$, does vanish due to (3.9).

A question that comes to mind is whether the current $K^{ab}$ (4.2) can be used for finding new conserved Killing charges, in the sense of e.g. [15, 17]. The stakes are high because of the presence of the Einstein tensor, which through the Einstein field equations, naturally relates to the matter sources. It seems unlikely, since the current $K^{ab}$ (4.2) does not have a Noether origin, i.e. conservation is not modulo field equations and it is not derived as a Noether current for a symmetry, but that fact does not exclude asymptotic charges in the sense of [17], (AD charges)\(^9\) for the KT-current. In appendix A we explicitly show and explain the absence of an asymptotic AD-charge for maximally symmetric spacetimes.

Perhaps naively but naturally, one is also tempted to generalise the expression (4.2) for $K^{ab}$ and define

$$J^{c_1 \cdots c_n}_E = G^{d[c_1} f^{c_2 \cdots c_n]}_{\cdots d},$$

as a possible new current. It should be noted that the covariant conservation of this expression requires

$$\nabla_a J^{c_2 \cdots c_n}_E = \frac{1}{n} \left( \nabla_a G^{d[c_2} f^{c_3 \cdots c_n]}_{\cdots d} + (-1)^{n+1}(n-1)\nabla_a G^{d[c_2} f^{c_3 \cdots c_n]}_{\cdots d} \right) + G^{d_a} \nabla_a f^{c_2 \cdots c_n}_{\cdots d} + (-1)^{n+1}(n-1)G^{d[c_2} \nabla_a f^{c_3 \cdots c_n]}_{\cdots d} = 0.$$  (4.5)

Using (2.2) and $\nabla_a G^{ab} = 0$, the latter becomes

$$\nabla_a J^{c_2 \cdots c_n}_E = (-1)^{n+1}(n-1) \frac{1}{n} \nabla_a G^{d[c_2} f^{c_3 \cdots c_n]}_{\cdots d} = 0.$$  (4.6)

We first observe that this expression vanishes for $n = 1$. This reproduces the well-known covariant conservation of the Killing vector current, e.g. in [15]. Secondly, we note that (4.6) vanishes if

$$G^{d[c_2} f^{c_3 \cdots c_n]}_{\cdots d} \sim G^{d_a} f^{c_2 c_3 \cdots c_n}_{\cdots d},$$  (4.7)

which is true for $n = 2$ according to (3.9). Nevertheless, it does not vanish for general $n$, which can be seen from the $n$-dependent coefficient in (3.12). However, it does vanish for special cases, such as conformally flat geometries (see appendix B).

Closer scrutiny of (3.12) reveals that one can in fact generalise (4.2) for a generic rank $n$ K YT by defining

$$K^{a_1 \cdots a_n} \equiv R_{[a_1} f^{a_2 \cdots a_n]}^{c_1} + \frac{(-1)^n}{n} R f^{a_1 \cdots a_n},$$  (4.8)

that is covariantly conserved, $\nabla_{a_1} K^{a_1 \cdots a_n} = 0$, and reduces to (4.2) for $n = 2$.

\(^9\) See subsection 6.1 for a detailed discussion.
5. Constraints on matter sources from (4.2) and (4.3)

In this section we restrict our attention to the consequences of (4.2) and (4.3) on continuous matter distributions that are described by a stress–energy tensor \( T_{ab} \), which acts as a source in Einstein’s field equations. To keep the discussion concise, we only consider the stress tensors of a perfect fluid and of an electromagnetic field.

5.1. The perfect fluid

The stress tensor of a perfect fluid is given by

\[
T_{ab} = \rho u_a u_b + p(g_{ab} + u_a u_b),
\]

where \( u^a \) is a unit timelike four-velocity of the fluid with \( u^a u_a = -1 \) and the functions \( p \) and \( \rho \), respectively, denote the pressure and the mass-density of the fluid. The stress tensor satisfies the equations of motion

\[
\nabla^a T_{ab} = 0,
\]

which yields

\[
u^a \nabla_a \rho + (\rho + p) \nabla^a u_a = 0,
\]

\[
(p + \rho) u^a \nabla_a u_b + (g_{ab} + u_a u_b) \nabla^a p = 0.
\]

If the spacetime of interest admits a KYT of rank \( n = 2 \), then (4.3), or equivalently (3.9) which becomes

\[
f^a_{bc} \nabla_a T_{bc} = 0,
\]

imposes yet another set of conditions in analogy to (5.3) and (5.4) above. These read

\[
f^a_{ab} u_b \nabla_a \rho + (\rho + p) f^a_{ab} \nabla_a u_b = 0,
\]

\[
(p + \rho) f^a_{ab} u_b \nabla_a u_c + (g_{bc} + u_b u_c) f^a_{ab} \nabla_a p = 0.
\]

The new identities (5.5) and (5.6) can be checked by using e.g. the Robertson–Walker metric written as

\[
\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t) \left( \mathrm{d}r^2 + b^2(r)(\mathrm{d}\theta^2 + \sin^2 \theta \, \mathrm{d}\phi^2) \right),
\]

where \( b(r) \equiv \sin r, r, \sinh r \) corresponding to the three spatial—spherical, Euclidean, hyperboloidal, respectively—geometries. This metric admits four independent rank \( n = 2 \) KYTs [21], the components of which read

\[
f^{(1)}_{\theta \phi} = 2a^3 b \sin \varphi, \quad f^{(1)}_{\varphi \phi} = a^3 b \cos \varphi \sin 2\theta, \quad f^{(1)}_{\theta \phi} = 2a^3 b^2 b' \cos \varphi \sin^2 \theta;
\]

\[
f^{(2)}_{\theta \phi} = 2a^3 b \cos \varphi, \quad f^{(2)}_{\varphi \phi} = a^3 b \sin \varphi \sin 2\theta, \quad f^{(2)}_{\theta \phi} = 2a^3 b^2 b' \sin \varphi \sin^2 \theta;
\]

\[
f^{(3)}_{r \phi} = 2a^3 b \sin^2 \theta, \quad f^{(3)}_{\theta \phi} = a^3 b^2 b' \sin 2\theta;
\]

\[
f^{(4)}_{\theta \phi} = 2a^3 b^3 \sin \theta.
\]

Here we have omitted the arguments of the metric functions \( a \) and \( b \), and used a prime over \( b \) to indicate differentiation with respect to \( r \). One can show separately for each KYT (5.8) that (5.5) and (5.6) (as well as (5.3) and (5.4), of course) are satisfied for the Robertson–Walker metric.
As for another example, one can consider the Kantowski–Sachs metric in $D = 4$: 

$$d s^2 = -d t^2 + X^2(t) d r^2 + Y^2(t) (d \theta^2 + \sin^2 \theta \, d \phi^2). \quad (5.9)$$

This is a solution of the Einstein field equations for dust and admits the rank $n = 2$ KYT [21] with a single component

$$f_{\theta \phi} = 2 Y^3(t) \sin \theta. \quad (5.10)$$

It follows easily that (5.5) and (5.6) (as well as (5.3) and (5.4), of course) are satisfied for the Kantowski–Sachs metric.

5.2. The electromagnetic field

The electromagnetic stress tensor is given by

$$T_{ab} = F_{ac} F^c_b - \frac{1}{4} g_{ab} F_{cd} F^{cd}. \quad (5.11)$$

From the Einstein field equations, one must again have that (5.2) is satisfied. Using $\nabla_{\{a} F_{b\}c} = 0$ carefully, this yields

$$\nabla^a T_{ab} = (\nabla^a F_{ac}) F^c_b = 0. \quad (5.12)$$

If Maxwell’s equations admit a current, then they read

$$\nabla^a F_{ab} = j_a, \quad (5.13)$$

and (5.12) can be thought of as $F^{bc} j_c = 0$, a non-trivial requirement to be satisfied by the components of the current. For a nontrivial solution for the current $j_c$, the ‘coefficients’ $F^{bc}$ must be such that $\det(F^{bc}) = 0$.\footnote{In $D = 4$, one has $\det(F^{ab}) \sim (F_{ab}^* F^{ab})^2$, of course.} Put in another way, one must have $\nabla^a F_{ab} = 0$ provided $\det(F^{bc}) \neq 0$.

If the spacetime of interest admits a KYT of rank $n = 2$, then (4.3), or equivalently (3.9) which becomes $f^{ab} \nabla_a T_{bc} = 0$, imposes

$$(f^{ab} \nabla_a F_{bd}) F^d_c + \frac{3}{2} F^{bd} \nabla_a \left( f_{\{a}^b F_{c\}d} \right) = 0. \quad (5.14)$$

The celebrated Kerr–Newman solution in $D = 4$ is an example for which the new identities put forward can be checked. The metric and the vector potential are given by

$$d s^2 = - \left( \frac{\Delta - a^2 \sin^2 \theta}{\Sigma} \right) d t^2 - \frac{2 a \sin^2 \theta \left( r^2 + a^2 - \Delta \right)}{\Sigma} d r \, d \phi \right) + \left( \frac{(r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta}{\Sigma} \right) \sin^2 \theta \, d \phi^2 + \frac{\Sigma}{\Delta} d r^2 + \Sigma \, d \theta^2, \quad (5.16)$$

$$A_a d x^a = - \frac{q r}{\Sigma} \left( d r - a \sin^2 \theta \, d \phi \right),$$

where

$$\Sigma = r^2 + a^2 \cos^2 \theta \quad \text{and} \quad \Delta = r^2 + a^2 + q^2 - 2Mr. \quad (5.17)$$
One has \( G_{ab} = 2T_{ab} \) and \( \nabla^b F_{ab} = 0 \) here, with \( F_{ab} = 2\partial_i (u A_b) \) as usual. Kerr–Newman metric shares the same rank \( n = 2 \) KYT with the Kerr metric, i.e. (5.15) with \( q = 0 \). Its components explicitly read
\[

f_r = a \cos \theta \quad f_\theta = a r \sin \theta \quad f_\varphi = a^2 \cos \theta \sin^2 \theta \quad f_{\varphi \theta} = r (r^2 + a^2) \sin \theta. \tag{5.18}
\]

One can show explicitly that the identities (3.3), (3.9) and (4.3) (together with (4.2)) and (5.14) are all nontrivially satisfied for the Kerr–Newman metric.

### 6. The KT-current

In this section we discuss under what condition conservation of a general rank \( n \) KT-current gives rise to asymptotically conserved charges, rewrite the KT-current in terms of the Weyl and Schouten tensors and show that this current vanishes for rank \( n = D - 1 \) KYTs in \( D \) dimensions.

In [15], a covariantly conserved current\(^{11}\) was constructed
\[
j^{a_1 \ldots a_n} = -\frac{(n-1)}{4} R^{[a_1 a_2} f^{a_3 \ldots a_n]bc} + (-1)^{n+1} R_c [a_1 f^{a_2 \ldots a_n}c] - \frac{1}{2n} R f^{a_1 \ldots a_n}, \tag{6.1}\]
with \( \nabla_a j^{a_1 \ldots a_n} = 0 \), for a spacetime that admits a rank \( n \) KYT. To show the conservation of \( j^{a_1 \ldots a_n} \) the following Bianchi identities are needed:
\[
\nabla_{[a} R_{bc]}{}^{de} = 0, \quad \nabla_a R_{bcde} + 2 \nabla_{[b} R_{c]de} = 0, \quad \nabla_a R^a_b - \frac{1}{2} \nabla_b R = 0. \tag{6.2}\]

A look at the newly found current (4.8) shows that one can in fact split the KT-current into two separately covariantly conserved pieces. To see this, introduce
\[
\bar{K}^{a_1 \ldots a_n} = -\frac{(n-1)}{4} R^{[a_1 a_2} f^{a_3 \ldots a_n]bc} + \frac{1}{2n} R f^{a_1 \ldots a_n}, \tag{6.3}\]
with \( \nabla_a \bar{K}^{a_1 \ldots a_n} = 0 \) and write the KT-current as [24]
\[
j^{a_1 \ldots a_n} = \bar{K}^{a_1 \ldots a_n} + (-1)^{n-1} K^{a_1 \ldots a_n}. \tag{6.4}\]

### 6.1. AD charges for the KT-current

A covariantly conserved antisymmetric rank \( n \) tensor field is equivalent to a co-closed \( n \)-form. By the extension of the Poincaré lemma to the exterior co-derivative this means that it is equal to the co-derivative of an \((n + 1)\)-form in an open set, under quite general conditions on this set. In what follows we apply this fact to the background geometry to construct conserved charges for linearized currents.

In [15, 25], the existence of asymptotic charges based on the KT-current was shown for asymptotically flat and asymptotically AdS geometries. The method is a generalisation of the idea of employing asymptotic Killing vectors [17] to define the corresponding conserved charges. We first treat the current based on a rank-2 KYT. So consider a \( D \)-dimensional

---

\(^{11}\) We shall refer to (6.1) as the KT-current henceforth.
spacetime $\bar{g}_{ab}$, which is often referred to as ‘the background spacetime’ with a completely antisymmetric rank-2 K YT $f_{ab}$ satisfying
\[ \bar{\nabla}_a \bar{f}_{bc} + \bar{\nabla}_b \bar{f}_{ac} = 0. \] (6.5)

Now the spacetime $g_{ab}$ whose new Killing–Yano charge(s) we are after does not necessarily have to admit exact K YT$s. We assume that the metric $g_{ab}$ can be asymptotically split into a background plus a deviation as
\[ g_{ab} \equiv \bar{g}_{ab} + h_{ab} \quad \text{so that} \quad \bar{g}^{ab} = \bar{g}^{ab} - h^{ab} + O(h^2), \] (6.6)

where $h^{ab} = g^{ac} h_{cd} g^{db}$. In what follows, all indices are raised and lowered with the generic background metric $\bar{g}_{ab}$, e.g. $h \equiv \bar{g}^{ab} h_{bc}$ and $\Box \equiv \nabla^a \nabla_a$. To $O(h)$ his leads to the following linearized curvature, Ricci tensor and curvature scalar:
\[
\begin{align*}
(R_{abc})_L &= \bar{R}_{abc} + 2 \Box_{[a} \bar{\nabla}^b \bar{h}_{bc]} \cdot c, \\
(R^a_{b})_L &= \frac{1}{2} \left( \bar{\nabla}^c \bar{\nabla}^d h_{bc} + \bar{\nabla}^c \bar{\nabla}_b h^{ac} - \bar{\nabla}^a \bar{\nabla}_b h - \Box_{\bar{h}}^a \right) - h^{ac} \bar{R}_{bc}, \\
R_L &= \bar{\nabla}_a \bar{\nabla}^a h^{bc} - \Box_{\bar{h}} - h^{ab} \bar{R}_{ab}.
\end{align*}
\]

(6.7) (6.8) (6.9)

To see if the linearised KT-current is conserved, we shall need the following versions of the identities (6.2) that hold modulo terms of $O(h^2)$ and higher:
\[
\begin{align*}
\Box_{[a} (R_{bc]} d)_{L} + (\Gamma_{[a} h)_{L} \cdot (\bar{R}_{bc]} d) &= 0, \\
\bar{\nabla}_a (R_{bc} d)_{L} + 2\Box_{[a} (R_{c]d})_{L} + (\Gamma_{a} h)_{L} \cdot (\bar{R}_{bc} d) + 2(\Gamma_{[a} h)_{L} \cdot (\bar{R}_{c]d}) &= 0, \\
\bar{\nabla}_a (R^b_{c})_{L} + \frac{1}{2} \bar{R} h_{bc} + (\Gamma_{a} h)_{L} \cdot (\bar{R}^b_{c}) &= 0.
\end{align*}
\] (6.10)

Here $(\Gamma_{a})_L$ denotes the usual action of a connection on a tensor as exemplified by
\[ (\Gamma_{a})_L \cdot (T^b_{\quad c}) = \Gamma^{b}_{\quad ae} T^e_{\quad c} - \Gamma^{e}_{\quad ae} T^b_{\quad c}. \] (6.11)

The linearized connection is
\[ (\Gamma^a_{ab} h)_{L} = \frac{1}{2} R^{b}_{\quad c e} \left( \bar{\nabla}_a h_{bc} + \bar{\nabla}_b h_{ac} - \bar{\nabla}_c h_{ab} \right). \] (6.12)

Note that for flat or maximally symmetric backgrounds, the relations (6.10) become the same as (6.2) with all curvature objects replaced by their linearized counterparts. It is this form that is needed for background conservation of the linearised current. We shall also need the assumption that $\bar{g}_{ab}$ asymptotically admits K YT$s due to this splitting and that $h_{ab}$ vanishes sufficiently fast at the hypersurface of interest $\Sigma$ (see (6.15) below) which is used for defining the charges. When the linearized connection terms in (6.10) vanish, the current $p^b$ is background covariantly conserved, i.e. $\bar{\nabla}_{a} (j^{ac})_{L} = 0$. Since the current is antisymmetric, the covariant conservation is expected to give rise to an ordinary conservation law via
\[ \bar{\nabla}_a (j^{ac})_{L} = \frac{1}{\sqrt{|g|}} \partial_a \left( \sqrt{|g|} (j^{ac})_{L} \right) = 0. \] (6.13)

From this we infer as usual that the integral
\[ \int d^{D-1} x \sqrt{|g|} (j^{ab})_{L} \]
(6.14)
is constant. In [15, 25] the latter is turned into a flux integral over a $(D - 3)$-dimensional hypersurface\(^{12}\) by further invoking the Stokes’ theorem: the crucial step is the determination of the potential for the current, as described in the beginning of this section. We thus need to express $(j^{ac})_L$ as the divergence of a completely antisymmetric rank-3 tensor $(j^{ac})_L = \nabla_d \tilde{\rho}^{acd}$, where $\tilde{\rho}^{acd} = \tilde{\rho}^{[ac]d}$. Then, up to a trivial normalization, the conserved ‘charge’ can be obtained by

$$Q^{ac}_i \sim \int_\Sigma dS_i \sqrt{\gamma} \tilde{\rho}^{aci}, \quad (6.15)$$

where $i$ ranges over the $(D - 3)$-dimensional hypersurface $\Sigma$ and $\gamma$ is the induced metric on $\Sigma$.

The asymptotic charges for the KT-current were given in [15] for an arbitrary rank $n$ KT in an asymptotically flat background and in [25] for an arbitrary rank $n$ KT in a maximally symmetric background. Their existence again rests on the KT-current being expressible as the covariant divergence of an $(n + 1)$-form. Since the construction of such a potential is nontrivial, here we complement this discussion by deriving a condition that the background has to satisfy for such an $(n + 1)$-form to exist.

Following [15], the general rank KT-current can be written as

$$j^{a_1 \ldots a_n} = N_n \delta^{a_1 \ldots a_n}_{b_1 \ldots b_{n+1}c_1 c_2} f^{b_1 \ldots b_n} R_{d_1 d_2} c_1 c_2, \quad (6.16)$$

where $\delta^{a_1 \ldots a_n}_{b_1 \ldots b_{n+1}c_1 c_2}$ is totally antisymmetric in all up and down indices, and

$$N_n = \frac{(n + 1)(n + 2)}{4n}. \quad (6.17)$$

As explained above, we are only interested in the linearized part of (6.16) and find

$$(j^{a_1 \ldots a_n})_L = N_n \delta^{a_1 \ldots a_n}_{b_1 \ldots b_{n+1}c_1 c_2} f^{b_1 \ldots b_n} (R_{d_1 d_2} c_1 c_2)_L. \quad (6.18)$$

In terms of the linearized Riemann tensor in (6.7), the current may be written

$$(j^{a_1 \ldots a_n})_L = N_n \delta^{a_1 \ldots a_n}_{b_1 \ldots b_{n+1}c_1 c_2} f^{b_1 \ldots b_n} \left( \bar{R}_{d_1 d_2} c_1 h_{c_2} c_1 + 2 \nabla_{d_1} \nabla_{c_2} h_{d_2} c_1 \right). \quad (6.19)$$

In [15] it is shown that, for a flat background, this may be written as

$$(j^{a_1 \ldots a_n})_L = \nabla_e \bar{\rho}^{a_1 \ldots a_n} \quad (6.20)$$

where the $(n + 1)$-form $\bar{\rho}^{a_1 \ldots a_n} = \bar{\rho}^{[a_1 \ldots a_n]}$ is

$$\bar{\rho}^{a_1 \ldots a_n} = 2N_n \delta^{a_1 \ldots a_n}_{b_1 \ldots b_{n+1}c_1} f^{b_1 \ldots b_n} \nabla_{c_2} h_{d_2} c_1 - \frac{1}{2n} (h \nabla^e f^{a_1 \ldots a_n} - (n + 1) h^{d_1} \nabla_{d_1} f^{a_1 \ldots a_n}). \quad (6.21)$$

Similar manipulations as in [15] give the following result for the general case\(^{13}\)

$$(j^{a_1 \ldots a_n})_L = \nabla_e \bar{\rho}^{a_1 \ldots a_n} + N_n \left( f^{[a_1 \ldots a_n} R_{c_1 c_2] e} c_1 h_{c_2} c_1 + 2 h_{e_2} c_1 \nabla_{c_2} \nabla_{c_1} f^{a_1 \ldots a_n} \right). \quad (6.22)$$

\(^{12}\) For a rank $n$ KT, the analogous step involves an integral over a hypersurface of dimension $D - 1 - n$.

\(^{13}\) Note that there are no additional curvature terms generated in the process.
with $\ell$ as in (6.21).

Using (2.3) and the explicit antisymmetrisation, vanishing of the terms in parenthesis (6.22) can be expressed in terms of the background curvature as

$$J^{[a_1 \ldots a_m]} C_{e_1 \epsilon_1}^{e_2 \epsilon_2} h^{e_2} + 2(-1)^m \delta_{e_2}^{e_1} C_{e_1 \epsilon_1}^{e_2 \epsilon_2} c_1 f^{a_2 \ldots a_m} e^c = 0.$$  (6.23)

For the KT construction of asymptotic charges, the condition (6.23) has to hold. It is clearly fulfilled for the flat case which leads to the results in [15]. For a maximally symmetric background

$$\bar{R}_{abcd} = \frac{2\Lambda}{(D-1)(D-2)} (\bar{g}_{ae} \bar{g}_{bd} - \bar{g}_{ad} \bar{g}_{bc}),$$
$$\bar{R}_{ab} = \frac{2\Lambda}{(D-2)} \bar{g}_{ab},$$
$$\bar{R} = \frac{2\Lambda D}{(D-2)},$$
$$\bar{G}_{ab} = -\Lambda \bar{g}_{ab}.$$ (6.24)

(6.23) is also fulfilled and leads to the results in [25]. This agrees with the known cases where the linearized Bianchi identities (6.10) ensure conservation of the KT-current.

Using the expansion of the Riemann tensor in terms of the Weyl and Schouten tensors, given in (6.25) below, (6.23) may alternatively be written as

$$J^{[a_1 \ldots a_n]} C_{e_1 \epsilon_1}^{e_2 \epsilon_2} h^{e_2} + 2(-1)^m \delta_{e_2}^{e_1} C_{e_1 \epsilon_1}^{e_2 \epsilon_2} c_1 f^{a_2 \ldots a_m} e^c = 0.$$  (6.24)

This expression may be further simplified using the traceless property of the Weyl tensor.

### 6.2. KT-current in terms of the Weyl and Schouten tensors

It is also instructive to rewrite the KT-current (6.1) using the decomposition of the Riemann tensor in terms of the Weyl tensor $C$ and the Schouten tensor $S$

$$S_{ab} \equiv \frac{1}{(D-2)} \left( R_{ab} - \frac{1}{2(D-1)} R g_{ab} \right),$$
$$R_{ab} = (D-2)S_{ab} + S g_{ab} \quad \text{with} \quad S \equiv g^{cd} S_{cd},$$
$$R_{abcd} = C_{abcd} + 4\delta^{[a} S_{c]d} \quad \text{and} \quad G_{ab} = (D-2)(S_{ab} - S g_{ab}).$$  (6.25)

These let one express (6.1) alternatively as

$$J^{[a_1 \ldots a_n]} = \frac{(n-1)}{4} C^{[a_1 a_2} f^{a_3 \ldots a_n] e} + (-1)^{n-1} \left( \frac{D - (n+1)}{D - 2} \right) C_{e_1}^{e_2 \ldots a_n} f^{e_1 \ldots e_{n-1}} c_1 R^{[a_1 \ldots a_n] e} + \left( \frac{n-1}{2(D-1)(D-2)} - \frac{1}{2n} \right) R f^{a_1 \ldots a_n}.$$  (6.26)

14 When $n = 1$, (6.23) simply reads $h R^{a} f_{a} - h^{a} R_{a} f^{a} = 0$. 

\begin{align}
= \frac{(n - 1)}{4} C^{(a_1 a_2 \ldots a_n)} f_{a_1 \ldots a_n} f_{a_1 \ldots a_n} + (D - (n + 1)) \left( (-1)^{n-1} S_c^{[a_1 f_{a_2 \ldots a_n} b_c} - \frac{1}{n} S f^{a_1 \ldots a_n} \right). \tag{6.26}
\end{align}

The latter equality shows that when the rank \( n = D - 1 \), the KT-current (6.1) reduces to
\begin{align}
j^{a_1 \ldots a_n} \big|_{n=D-1} = -\frac{(D - 2)}{4} C^{(a_1 a_2 \ldots a_n)} f_{a_1 \ldots a_n}. \tag{6.27}
\end{align}

Note also that since the Weyl tensor vanishes identically in \( D = 3 \), so does the whole KT-current \( j^a \) for \( n = 2 \). Moreover, when \( D = 4 \) one has a special current for a rank \( n = 3 \) KYT from (6.27)
\begin{align}
j^{a_1 a_2 a_3} = -\frac{1}{2} C^{(a_1 a_2 \ldots a_n)} f_{a_1 \ldots a_n}. \tag{6.28}
\end{align}

In fact one can show that this also vanishes and thus the KT-current does not exist in this case either. The Hodge dual of a KYT is a closed conformal Killing tensor (KT) [26]. In particular this means that a rank \( n = D - 1 \) KYT is dual to a closed conformal Killing vector, (defined in (7.9) below), as discussed in [27]. We thus first dualize the \( n = 3 \) KYT to a closed conformal Killing vector \( \tilde{f}_a \) (defined in (7.9) below) to write
\begin{align}
\tilde{f}_a = \sqrt{|g|} \epsilon_a^{bcd} f_{bcd} \Rightarrow f_{bcd} = \tilde{f}_a \epsilon^a_{bcd} \tag{6.29}
\end{align}

satisfying
\begin{align}
\nabla_a \tilde{f}_b = \frac{1}{4} (\nabla_c \tilde{f}_e) g_{ab}. \tag{6.30}
\end{align}

Dualizing also \( j^{a_1 a_2 a_3} \), we may then write the relation (6.28) up to some signs and factors as
\begin{align}
\epsilon_{da_1 a_2 a_3} j^{a_1 a_2 a_3} \sim \epsilon_{da_1 a_2 a_3} C^{a_1 a_2} \epsilon_a^{bcd} \tilde{f}_e. \tag{6.31}
\end{align}

Using the formula for contracting one index on the Levi-Civita symbol and the traceless property of the Weyl tensor then shows that the right-hand side vanishes, and thus that \( j^{a_1 a_2 a_3} = 0 \). This can also be seen, perhaps more directly, from the fact that
\begin{align}
C_{abcd} \tilde{f}_d = 0 \tag{6.32}
\end{align}
in \( D = 4 \) when \( \tilde{f}_d \) satisfies (6.29), see e.g. [20]. We have
\begin{align}
C^{a_1 a_2} \epsilon_a^{bcd} \tilde{f}_e = 2 \tilde{f}_e C^{a_1 a_2} \epsilon^* \epsilon_2^* \epsilon^{a_1 a_2} \epsilon^* \tilde{f}_e = 0, \tag{6.33}
\end{align}
where we used a relation between the right and left duals of the Weyl tensor (see, e.g. [28]) and the last equality follows by (6.32).

The condition (6.32) implies that either \( \tilde{f}_e \) is a null vector or the space is conformally flat. It is gratifying to see that for the conformally flat case the existence of the charge condition (6.24) also vanishes.

Clearly the argument leading from (6.28) to (6.33) holds equally well for a KT-current based on a rank \( n = D - 1 \) KYT in \( D \) dimensions, so that such a KT-current also has to vanish.
7. Comments on the KT and related currents

The KT current has many interesting special cases for particular geometries. We also found that a number of conserved ‘currents’ related to the KT current can be defined. In this section we summarize these cases for completeness and collect their interrelations in a table. We start by reproducing the \( n = 2 \) KT-current (6.1), for convenience:

\[
-4f^{ab} = R^{abcd} f_{cd} + 4R_{c}^{[a} f^{b]c} + R f^{ab}. \tag{7.1}
\]

It is interesting to note that the expression multiplying \((D - 3)\) in (7.2)

\[
J_{(1)}^{ab} \equiv 2 f^{c[a} S_{b]c} + f^{ab} S \tag{7.3}
\]

is conserved for certain geometries. We have

\[
\nabla_a J_{(1)}^{ab} = f^{ca} \nabla_a S_{b} \tag{7.4}
\]

which vanishes when \( S_{ab} \) is a Codazzi tensor, i.e. a symmetric two-tensor whose covariant derivative is also symmetric

\[
\nabla_a S_{b} = \nabla_b S_{a}. \tag{7.5}
\]

The Weyl–Schouten theorem [29, 30] states that:

A Riemannian manifold of dimension \( D \) with \( D \geq 3 \) is conformally flat if and only if the Schouten tensor is a Codazzi tensor for \( D = 3 \), or the Weyl tensor vanishes for \( D > 3 \).

Hence we need a conformally flat metric in \( D = 3 \) for \( J_{(1)}^{ab} \) to be conserved. For higher dimensions, we note that a metric \( g \) has a harmonic Weyl tensor

\[
\nabla_a C_{cd}^{ab} = 0, \tag{7.6}
\]

if and only if its Schouten tensor is a Codazzi tensor. In this case we see from (7.2) that \( f^{ab} \) is the sum of two independently conserved currents, one proportional to \( J_{(1)}^{ab} \) and a new current

\[
J_{(2)}^{ab} \equiv f^{cd} C_{cd}^{ab}, \tag{7.7}
\]

according to \((D > 3)\)

\[
-4f^{ab} = J_{(2)}^{ab} + 2(D - 3) J_{(1)}^{ab}. \tag{7.8}
\]

A rank \( n \) conformal Killing–Yano tensor (CKYT) \( \tilde{f} \) obeys

\[
\nabla_b \tilde{f}_{a_1...a_n} = \nabla_{\{b} \tilde{f}_{a_1...a_n\}} + n g_{b[a_1} \tilde{f}_{a_2...a_n]} \tag{7.9}
\]

with

\[
\tilde{f}_{a_2...a_n} \equiv \frac{1}{D - n + 1} \nabla_b f_{a_2...a_n}^{b}. \tag{7.10}
\]

When the first term in (7.9) vanishes, the tensor is called a closed conformal Killing–Yano tensor (CCKYT). A differential form is a KYT if, and only if, its Hodge dual is a CCKYT.
Table 1. Relations between various currents in section 7.

| Current | Cons. conditions | Relation to the KT-current $f^{ab}$ |
|---------|-----------------|---------------------------------|
| $J^{ab}_{(1)}$ | $2 f^{cd} S_{cd} + f^{ab} S$ | $-4 j^{ab} = C^{abcd} f_{cd} + 2(D - 3) j_{(1)}^{ab}$ |
| $J^{ab}_{(2)}$ | $f^{cd} C_{cd}^{ab}$ | Weyl harmonic |
| $J^{ab}_{(3)}$ | $f^{cd} R_{cd}^{ab}$ | Riemann harmonic |
| $J^{ab}_{(4)}$ | $4R^{1}{}_{ab} f^{bc} + R f^{ab}$ | $R_{ab}$ Codazzi |

The current $J^{ab}_{(2)}$ in (7.7) can be extended to involve a conformal Yano two-form $\hat{f}$. When acting on by the covariant derivative

$$\nabla_a (\hat{f} c^{ab} c_{cd} f^{cd}) = (\nabla_a (\hat{f} c^{ab} c_{cd} f^{cd}) + 2 g_{ab} (\hat{f} c_{cd} f^{cd}) + \hat{f} c_{cd} c^{cd} + \hat{f} c_{cd} f^{cd},$$

(7.11)

the first term vanishes due to the anti-symmetrization of $\nabla \hat{f}$ which imposes the first Bianchi identity on $C$, the second vanishes since $C$ is trace-free and the third since the Weyl tensor is harmonic.

It may also be of interest to consider a metric $g$ with a harmonic Riemann tensor

$$\nabla_a R^{ab}_{cd} = 0.$$  

(7.12)

This requires the Ricci tensor to be a Codazzi tensor, instead of the Schouten tensor:

$$\nabla_a R_{bc} = \nabla_b R_{ac}.$$  

(7.13)

Returning to the form (7.1) for the current $J^{ab}$ we note that

$$J^{ab}_{(3)} \equiv f^{cd} R_{cd}^{ab}$$

(7.14)

satisfies

$$\nabla_a J^{ab}_{(3)} = g_{ac} \nabla^c (f^{cd} R_{cd}^{ab}) + f^{cd} \nabla_a R_{cd}^{ab} = 0,$$

(7.15)

where the first term vanishes due to the first Bianchi identity and the second due to (7.12). Since the full current $f^{ab}$ is conserved, we realize that writing

$$J^{ab}_{(4)} \equiv j^{ab} - p^{ab}_{(3)}$$

(7.16)

yields, in analogy to the harmonic Weyl tensor case, a third current, which must be conserved,

$$\nabla_a J^{ab}_{(4)} = 0,$$

(7.17)

due to (7.13), which may also be explicitly verified (table 1).

8. Conclusions and comments

In this paper we have presented new identities for KYTs and shown how they may be used to find new conserved currents. These currents are all of the Kastor–Traschen type, i.e. not Noether currents. As shown in [15, 25], such currents may nevertheless lead to asymptotically conserved charges of AD type. We found a condition for such conserved charges to exist for the KT-current. We also displayed the linearized form of the Bianchi identities and pointed out that only for certain backgrounds do they directly lead to background conserved linearized
KT-currents. An interesting question is if there are other backgrounds and/or modifications of the current that allow such conservation using these linearized identities.

For our current $K^{ab}$, based on the Einstein tensor, we investigated this possibility too and showed that it does not give an AD charge for a maximally symmetric space time (see appendix A). There are however many more cases, both currents and backgrounds, that should be studied.

It is particularly interesting to note that we were able to find new conserved currents for $n > 2$ KY forms. These should be relevant for higher dimensional solutions to Einstein’s equation.

There are several directions into which the present efforts may be extended: treating conformal KYTs as we touched upon in the text. Extending the geometry to allow for torsion which will introduce modified Killing–Yano equations as in e.g. [8]. This opens up for supersymmetric extensions, such as discussed in [13].

Acknowledgments

ÖS would like to thank D O Devecioğlu for help with xAct in the early stages of this work. The research of UL is supported in part by the 2236 Co-Funded Scheme2 (CoCirculation2) of TÜBİTAK (Project No: 120C067)\textsuperscript{15}. We are grateful to Dr. Vojtech Witzany for asking us a question that led to the clarification around (6.10) and to an anonymous referee for helping us to further refine the related arguments.

Data availability statement

No new data were created or analysed in this study.

Appendix A. No AD charge for $K^{ab}$ in maximally symmetric spacetimes

This appendix serves as an illustration for the method of deriving AD charges and its limitations. We adapt and apply the arguments given in subsection (6.1) to the currents $K^a = G^a_c f^c$ and $K^{ab} = G^{[a}_c f^{b]c}$ (4.2) for the maximally symmetric and flat backgrounds. We show explicitly that only the first can be used in defining new conserved quantities as done in [16, 25] for the KT-current.

So one starts with a $D$-dimensional background $\bar{g}_{ab}$ admitting a rank-2 KYT $\bar{f}_{ab}$ satisfying (6.5). For such a maximally symmetric spacetime, one has

\[
\bar{R}_{abcd} = \frac{2\Lambda}{(D-1)(D-2)} (\bar{g}_{ac} \bar{g}_{bd} - \bar{g}_{ad} \bar{g}_{bc}),
\]

\[
\bar{R}_{ab} = \frac{2\Lambda}{(D-2)} \bar{g}_{ab},
\]

\[
\bar{R} = \frac{2\Lambda D}{(D-2)},
\]

\[
\bar{G}_{ab} = -\Lambda \bar{g}_{ab}.
\]

\textsuperscript{15} However the entire responsibility for the publication is ours. The financial support received from TÜBİTAK does not mean that the content of the publication is approved in a scientific sense by TÜBİTAK.
Then one finds the following which are frequently used in the ensuing calculations:

\[ \nabla_a f^{ab} = 0, \quad \nabla_a \bar{f}_{bc} = \nabla_b \bar{f}_{ca} = \nabla_c \bar{f}_{ab}. \]  
(A.1)

\[ \nabla_a \nabla_b \bar{f}_{cd} = \frac{2\Lambda}{(D-1)(D-2)} (\bar{g}_{ab} \bar{f}_{dc} + \bar{g}_{ac} \bar{f}_{bd} + \bar{g}_{ad} \bar{f}_{cb}), \]  
(A.2)

\[ \Box \bar{f}_{ab} = \frac{2\Lambda}{(D-1)} \bar{f}_{ba}, \quad \bar{\nabla}^a \nabla_b \bar{f}_{ac} = \frac{2\Lambda}{(D-1)} \bar{f}_{ca}. \]  
(A.3)

The ‘linearized’ version of \( K^a = G_i^a f^c \) is background covariantly conserved, i.e. \( \nabla_a (K^a)_{\mu} = 0 \). It should therefore have a potential \( (K^a)_{\mu} = \bar{\nabla}_\ell (d\bar{a}) \) according to the general argument.

Keeping in mind that all indices are raised and lowered with the maximally symmetric \( \bar{g}_{ab} \), one finds that the linearized Ricci tensor and the Ricci scalar read\(^{16}\)

\[ (R_{ab})_{\mu} = \frac{1}{2} \left( \nabla^c \nabla_b h_{ac} + \nabla^c \nabla_a h_{bc} - \Box h_{ab} - \nabla_a \nabla_b h \right), \]  
(A.4)

\[ R_{\mu} = \nabla^c \nabla_d h_{cd} - \Box h - \frac{2\Lambda}{(D-2)} h. \]  
(A.5)

These further give

\[ (G^a)_{\mu} = \frac{1}{2} \left( \nabla^c \nabla^d h_{bc} + \nabla^c \nabla_b h^{ac} - \Box h^b - \nabla^a \nabla_b h \right) \]
\[ - \frac{1}{2} \delta^a_b \left( \nabla_c \nabla_d h^{cd} - \Box h - \frac{2\Lambda}{(D-2)} h \right) - \frac{2\Lambda}{(D-2)} h^b. \]  
(A.6)

With this linearized Einstein tensor the current can be rearranged to

\[ (K^a)_\mu = (G^a)_{\mu} \bar{f}^b + (G^b)_{\mu} \bar{f}^a. \]  
(A.7)

Arguments analogous to those given in the discussion surrounding (6.13) can now be repeated, replacing \((j^a)_\mu \) with \((K^a)_\mu \) and lead to a conserved charge as in (6.15).

The ‘linearized’ version of the antisymmetric ‘current’\(^{17}\) \( K^a = 2 G_i^a f^c \),

\[ (K^a)_{\mu} = -(G^a)_{\mu} \bar{f}^{bc} + (G^b)_{\mu} \bar{f}^{ca} \]  
(A.8)

can be similarly treated leading to

\[ (K^a)_{\mu} = 3 \nabla_d \left\{ \bar{f}^{bd} \nabla^c \nabla^e \bar{h}^{\mu}_{b} + h^{bd} \nabla^c \nabla^e \bar{f}^{\mu}_{b} + \frac{1}{2} \bar{f}^{bde} \nabla^\mu e \bar{h} \right\} \]
\[ + \nabla_d \left\{ \bar{f}^{bd} \nabla^c \nabla^e \bar{h}^{\mu}_{b} + h^{bd} \nabla^c \nabla^e \bar{f}^{\mu}_{b} + \frac{1}{2} \bar{f}^{bde} \nabla^\mu e \bar{h} \right\} \]
\[ + \bar{f}^{bd} \nabla^c \nabla^e h^{\mu}_{b} + \bar{f}^{ca} \nabla^e h^{\mu}_{b} - h^{bd} \bar{f}^{ca} \nabla^e \bar{h} + h^{cd} \bar{f}^{ca} \nabla^e \bar{h} \]
\[ + \frac{4\Lambda}{(D-1)(D-2)} \left( h \bar{f}^{ca} + 2h_{b} \bar{f}^{cd} \right). \]  
(A.9)

---

\(^{16}\) These easily follow by adapting (6.9) accordingly to a maximally symmetric background.

\(^{17}\) As shown in section 4, \( \nabla_a K^a = 0 \) if the spacetime \( g_{\alpha\beta} \) admits a KKY \( f_{\alpha\beta} \) itself.
The first line is in the desired structure but the remaining parts of (A.9) do not fulfill the requirements of a proper $\overline{\ell}$. This is so even when one takes $\Lambda \to 0$, the same choice as in [15], to work in an asymptotically flat background. This shows that the current $K^{ab}$ (4.2) does not admit the construction of an $AD$-charge.

In retrospect the reason for this is clear. When defining $K^{ac} = 2 G_b^{[a} f^{c]b}$, we needed to use (3.3):

$$R_{ab} f^{ac} + R^{ac} f_{ab} = 0 .$$

(A.10)

When $f^{ac} = \overline{f}^{ac}$ is a background KYT, as in (A.8), this holds with the background Ricci tensor $\overline{R}_{ab}$, and the background current $K^{ac} = 2 G_b^{[a} \overline{f}^{c]b}$ is background conserved. However this will not in general be the case for the linearized current $(K^{ac})_L = 2 (G^{[a})_L \overline{f}^{c]b}$ since

$$(\overline{R}_{ab})_L \overline{f}^{ac} + (R^{ac})_L \overline{f}_{ab} \neq 0 .$$

(A.11)

Indeed the background covariant divergence of (A.9) is easily seen to be nonvanishing.

**Appendix B. Conservation of $J_E$ in conformally flat geometries**

In this section, we show that conformal flatness in fact guarantees the conservation of the current $J_E$ in (4.4) for an arbitrary rank $n$ KYT. Using

$$R_{abcd} f_{c1 \ldots c_n}^{ad} = - \frac{1}{2} R_{adbc} f_{c1 \ldots c_n}^{ad} ,$$

(B.1)

we rewrite (3.5) as

$$\frac{(-1)^{n+1}}{2} (2 R_{[d} f_{c2 \ldots c_n]}^{bc} + (-1)^{n+1}(n - 1) R_{[bc}^{da} f_{c3 \ldots c_n]}^{ad} ) = (-1)^{n+1} R_{d}^{bc} f_{c2 \ldots c_n]}^{bc} - \frac{(n - 1)}{2} R_{c2}^{[da} f_{c3 \ldots c_n]}^{bc}^{ad} .$$

(B.2)

Using (6.25) gives

$$\frac{(-1)^{n+1}}{2} (2 R_{[d}^{bc} f_{c2 \ldots c_n]}^{bc} + (-1)^{n+1}(n - 1) C_{[bc}^{da} f_{c3 \ldots c_n]}^{ad} - 4(-1)^{n+1} \times (n - 1) S_{[d}^{bc} f_{c3 \ldots c_n]}^{bc} ) = (-1)^{n+1} R_{d}^{bc} f_{c2 \ldots c_n]}^{bc} - \frac{(n - 1)}{2} C_{bc}^{[da} f_{c3 \ldots c_n]}^{bc} - (n - 1) \times (S_{[d}^{bc} f_{c3 \ldots c_n]}^{bc} - S_{bc}^{[d} f_{c3 \ldots c_n]}^{bc} ) .$$

(B.3)

For vanishing Weyl tensor and ignoring the metric terms in $S_{ab}$ (B.3) becomes

$$AR_{[d}^{bc} f_{c2 \ldots c_n]}^{bc} = BR_{d}^{bc} f_{c2 \ldots c_n]}^{bc} - CR_{[d}^{bc} f_{c3 \ldots c_n]}^{bc} ,$$

(B.4)

where

$$A \equiv (-1)^{n+1} (1 + 2\alpha(1 - n)) ,$$
$$B \equiv (-1)^{n+1} (1 + \alpha(1 - n)) ,$$
$$C \equiv -\alpha(1 - n) ,$$

18
with \( \alpha = 1/(D-2) \) depending on the dimension \( D \) of the spacetime according to (6.25). Observing that

\[
n R^d_{\{b \alpha} f_{\ldots \ldots \alpha_d} = R^d_{\{b} f_{\ldots \ldots} \alpha_d \} \cdot (n-1) \alpha \frac{(n-1)(n-1)}{D-2} \]

(B.5) can be rewritten as

\[
(Bn - A)R^d_{\{b f_{\ldots} \alpha d} = (Cn + (-1)^{n+1} A)R^d_{\{b f_{\ldots} \alpha d} \cdot \]

(B.6)\( \Leftrightarrow \)

\[
(-1)^n (\alpha n - 1)n - n + 1 + 2(1 - n)) R^d_{\{b f_{\ldots} \alpha d} \]

\( \Leftrightarrow \)

\[
(-1)^{n+1} R^d_{\{b f_{\ldots} \alpha d} = R^d_{\{b f_{\ldots} \alpha d} \]

\( \Leftrightarrow \)

\[
R^d_{\{b f_{\ldots} \alpha d} = R^d_{\{b f_{\ldots} \alpha d} \cdot \]

(B.7)

This leads to (4.7) which guarantees the conservation of \( J_E \), provided that the metric terms in the Schouten tensor also work out. However, from (B.3) this requires

\[
-2\delta^d_{\{b f_{\ldots} \alpha d} = -\delta^d_{\{b f_{\ldots} \alpha d} + \delta^d_{b f_{\ldots} \alpha d} \cdot \]

which indeed holds. So this proves that conformal flatness guarantees the conservation of the current \( J_E \) for an arbitrary rank \( n \) KYT.

References

[1] Hansen D 2014 Killing–Yano tensors Independent Project Report (supervisor: N Obers) Niels Bohr Institute
[2] Gibbons G W, Rietdijk R H and van Holten J W 1993 Susy in the sky Nucl. Phys. B 404 42–64
[3] Jonghe F D, Peeters K and Sfetsos K 1997 Killing–Yano supersymmetry in string theory Class. Quantum. Grav. 14 35–46
[4] Chervonyi Y and Lunin O 2015 Killing–Yano tensors in string theory: J. High Energy Phys. JHEP09(2015)182
[5] Carter B 1968 Global structure of the Kerr family of gravitational fields Phys. Rev. 174 1559
[6] Walker M and Penrose R 1970 On quadratic first integrals of the geodesic equations for type \{22\} spacetimes Commun. Math. Phys. 18 265–74
[7] Papadopoulos G 2008 Killing–Yano equations and G structures Class. Quantum. Grav. 25 105016
[8] Papadopoulos G 2012 Killing–Yano equations with torsion, worldline actions and G-structures Class. Quantum. Grav. 29 115008
[9] Lunin O and Tian J 2020 Separation of variables in the WZW models (arXiv:2012.15083 [hep-th])
[10] Cariglia M, Gibbons G W, van Holten J-W, Horvathy P A, Kosiński P and Zhang P-M 2014 Killing tensors and canonical geometry in classical dynamics Class. Quantum. Grav. 31 125001
[11] Cariglia M 2014 Hidden symmetries of the Dirac equation in curved space-time Relativity and Gravitation (Springer Proceedings in Physics vol 157) ed J Bičák and T Ledvinka (Berlin: Springer) p 25
[12] Santillan O P 2012 Hidden symmetries and supergravity solutions J. Math. Phys. 53 043509
[13] Howe P S and Lindström U 2018 Some remarks on (super)-conformal Killing–Yano tensors J. High Energy Phys. JHEP11(2018)049
[14] Kuzenko S M, Lindström U, Raptakis E S N and Tartaglino-Mazzucchelli G 2021 Symmetries of \( N = (1, 0) \) supergravity backgrounds in six dimensions J. High Energy Phys. JHEP03(2021)157
[15] Kastor D and Traschen J 2004 Conserved gravitational charges from Yano tensors J. High Energy Phys. JHEP08(2004)045
[16] Kastor D, Ray S and Traschen J 2007 Do Killing–Yano tensors form a Lie algebra? Class. Quantum Grav. 24 3759–68
[17] Abbott L F and Deser S 1982 Stability of gravity with a cosmological constant Nucl. Phys. B 195 76–96
[18] Collinson C D 1974 The existence of Killing tensors in empty space-times Tensor 28 173–6
[19] Collinson C D 1976 On the relationship between Killing tensors and Killing–Yano tensors Int. J. Theor. Phys. 15 311–4
[20] Stephani H 1978 A note on Killing tensors Gen. Relativ. Gravit. 9 789–92
[21] Ibohal N 1997 On the relationship between Killing–Yano tensors and electromagnetic fields in curved spaces Astrophys. Space Sci. 249 73–93
[22] Batista C 2015 Integrability conditions for Killing–Yano tensors and conformal Killing–Yano tensors Phys. Rev. D 91 024013
[23] Menekay Ç 2013 Killing family of tensors in classical gravitational theories MS Thesis Middle East Technical University
[24] Acik Ö, Ertem U, Önder M and Vercin A 2010 Basic gravitational currents and Killing–Yano forms Gen. Relativ. Gravit. 42 2543
[25] Cebeci H, Sarıoğlu Ö and Tekin B 2006 Gravitational charges of transverse asymptotically AdS spacetimes Phys. Rev. D 74 124021
[26] Frolov V P and Kubizňák D 2008 Higher-dimensional black holes: hidden symmetries and separation of variables Class. Quantum Grav. 25 154005
[27] Batista C 2014 Killing–Yano tensors of order $n-1$ Class. Quantum Grav. 31 165019
[28] Martínez i Portillo J 2016 Classification of Weyl and Ricci tensors Bachelor’s Degree Project (Universitat Politècnica de Catalunya)
[29] Weyl H 1918 Reine Infinitesimalgeometrie Math. Z. 2 384–411
[30] Schouten J A 1921 Über die konforme abbildung n-dimensionaler mannigfaltigkeiten mit quadratischer massbestimmung auf eine mannigfaltigkeit mit euklidischer massbestimmung Math. Z. 11 58–88