Fast-forward scaling of atom-molecule conversion in Bose-Einstein condensates

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Robust stimulated Raman exact passages are requisite for controlling nonlinear quantum systems, with the wide applications ranging from ultracold molecules, non-linear optics to superchemistry. Inspired by shortcuts to adiabaticity, we propose the fast-forward scaling of stimulated Raman adiabatic processes with the nonlinearity involved, describing the transfer from an atomic Bose-Einstein condensate to a molecular one by controllable external fields. The fidelity and robustness of atom-molecule conversion are shown to surpass those of conventional adiabatic passages, assisted by fast-forward driving field. Finally, our results are extended to the fractional stimulated Raman adiabatic processes for the coherent superposition of atomic and molecular states.

I. INTRODUCTION

Over the past few decades, coherent control has been considered a strategic cross-sectional field of research for atomic, molecular, optical physics, and photochemistry, providing a quantum mechanics-based method for the manipulation of populations and pathways, typically by laser pulses [1–7]. For example, the coherent control chemistry interaction exemplifies the fascinating application in chemistry, thus manipulating and enhancing the product yield [8]. Also, it can be applied directly to so-called superchemistry, in which the coherent Raman transition generates a molecular Bose-Einstein condensate (BEC) from an atomic BEC [9, 10]. Currently, with the availability of quantum technologies, quantum control has also emerged as an essential physical basis for state preparation and transfer in quantum information science and quantum sensing [11, 12].

In this context, there exist several promising tools for controlling quantum states, developed in terms of adiabatic passages [1, 4, 6], composite pulses [13, 14], optimal control theory [12], and single-shot shaped pulses [15, 16]. Besides, the concept of “shortcuts to adiabaticity” (STA), provides an alternative control paradigm, sharing the speed and robust advantages, see the review [17, 18]. In the scenario of population transfer, counter-diabatic (CD) driving [19–21] (or quantum transitionless algorithm [22–24]), invariant-based inverse engineering (IE) [25, 26, 37], fast-forward (FF) scaling [28–30] and dressed state method [31] are capable of speeding up the conventional rapid adiabatic passage (RAP) and stimulated Raman adiabatic passage (STIRAP), respectively, in two- and three-level quantum systems. Despite the experimental demonstrations in the quantum platforms with nitrogen-vacancy (NV) center spins [32–34], cold atoms [35], and superconducting circuits [36], we still have to choose or combine these approaches, which are best suited for specific systems and objectives, by clarifying the features, overlaps, and relations among all these approaches [37, 38].

Quite naturally, STIRAP and its variants [5, 6] have been applied directly to magneto- and photo-association of a BEC, by using partially overlapping pulses (from pump and Stokes lasers) to produce complete population transfer between two quantum states of an atom or molecule [39–45]. However, the nonlinearity induced from three-body collision leads to the dynamical instability and inefficiency with the breakdown of adiabaticity [46–48]. In recent years, the technique of STA [49, 50], in addition to optimal control [53, 54] and adiabatic tracking [51, 52], is considered as a preferable option to tackle this issue, by improving the stability and efficiency of nonlinear STIRAP.

In this article, we shall explore the FF scaling of atom-molecule conversion in BECs with the second-order nonlinearities involved, by extending the FF assisted STIRAP [55, 56]. By using the dark state in nonlinear A-type STIRAP as an ansatz with the magnification factor, we construct the FF driving field, coupling between atomic and molecular BECs. It is concluded that the combination between FF field and nonlinear STIRAP overcomes the instability and inefficiency of photo- and magneto-association of atomic BEC, by averting the unwanted diabatic transition. Furthermore, the FF driving field can be similarly designed in fractional STIRAP (f-STIRAP), generating the coherent superposition of atomic and molecular BECs. Conceptually, the FF scaling approach in nonlinear STIRAP is different from the CD driving in linear STIRAP, though they have similar forms and physical implementations. The instantaneous eigenstates are degenerate and non-orthogonal in the nonlinear systems, resulting in obscure calculation of CD field. By comparing the original CD field, the derived FF field is also more general and efficient with extra control parameters. In addition, the state evolution is always along the dark state by using the FF scaling approach in nonlinear STIRAP, thus without populating the excited state. This provides the advantage over the IE method [49], when the irreversible losses is inevitable. Finally, we emphase our results can be applicable to other nonlinear systems, i.e., nonlinear optics and BEC in an accelerated optical lattice, in the presence of the third-order Kerr-type nonlinearities.

Our paper is organized as follows. In Sec. II, we briefly review the model and Hamiltonian, and present the original nonlinear STIRAP and its variant with pump and Stokes Gaussian pulses. In Sec. III, we derive the formula of FF driving field accordingly. In Sec. IV, the stability and efficiency of FF assisted STIRAP and f-STIRAP are featured. Finally, we draw the conclusion in Sec. V.

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where the second-order nonlinearities are still involved, which may lead to the dynamical instability [49, 50]. In principle, there may exist more non-orthogonal eigenstates than the dimension of the Hilbert space in nonlinear systems [57]. Nevertheless, in analogy to the linear counterpart [1, 6], the nonlinear \( \Lambda \)-type STIRAP still supports a dark state (or so-called the coherent population trapping state) with zero eigenvalue [40, 44], which is decoupled from the excited state. Therefore, by setting \( c_2^0 = 0 \), and using \( |c_1^0|^2 + |c_3^0|^2 = 1 \), we obtain the instantaneous population,

\[
|c_1^0|^2 = 1 - |c_3^0|^2 = \frac{2\Omega_p}{\Omega_s + \Omega_c},
\]

from which the dark state, corresponding to the eigenvector \( |\Psi_0(t)\rangle = [c_1^0, c_2^0, c_3^0]^T \), is calculated as \( |\Psi_0(t)\rangle = N(\Omega_c,0) - c_0^0\Omega_p|3\rangle \), with \( \Omega_c = (\Omega_s^2 + 4\Omega_p^2)^{1/2} \) and the normalization constant \( N \). As used in conventional linear STIRAP [1, 6], the dark state is further reformulated into

\[
|\Psi_0(t)\rangle = \cos \Theta(t)|1\rangle - \sin \Theta(t)|3\rangle,
\]

with the mixing angle,

\[
\Theta(t) = \arctan \left( \frac{c_1^0\Omega_p}{\Omega_s} \right) = \frac{\sqrt{\Omega_p}}{\sqrt{\Omega_s + \Omega_c}}.
\]

This dark state has been experimentally verified for instance, through the superposition state of atomic and molecular BECs [45]. Apart from it, we have the other two eigenstates: \([0, \pm 1/\sqrt{2}, 1/\sqrt{2}]^T \), with the eigenvalues being \( \pm \Omega_s/2 \). When \( \Omega_c/\Omega_p < 1/\sqrt{2} \), two more eigenstates exist \([1/(2 - \Omega_s^2/\Omega_p^2)^{1/2}, \pm 1/\sqrt{2}, \Omega_s/\Omega_p]^T \) with the eigenvalues being \( \pm \Omega_s/\sqrt{2} \). Due to the lack of orthogonality between the dark state and any of the other eigenstates, the usual adiabatic condition in linear STIRAP are not valid at all. Thus, one can apply the linear stability analysis around the fixed stable point [43], corresponding to the dark state, to calculate three orthogonal eigenstates, \( w_0 = N_0[-\Omega_s/2, 0, c_1^0\Omega_p]^T \), and \( w_\pm = N_\pm[c_1^0\Omega_p, \epsilon_\pm, \Omega_s]^T \) with the corresponding eigenvalues, \( \epsilon_0 = 0 \) and \( \epsilon_\pm = \pm \sqrt{\Omega_s\Omega_c} \), respectively, \( N_0, N_\pm \) being the normalization constant. In this manner, the adiabatic condition suitable for the nonlinear STIRAP is derived as [43],

\[
A_{\text{ad}} \approx \frac{N_+ + N_-}{w_+ + w_-}^{1/2} \left( \frac{\bar{\Omega}_s \bar{\Omega}_c - \bar{\Omega}_s \Omega_p}{\Omega_s + \Omega_c} \right) \ll 1,
\]

where \( \cdots \) means the derivative with respect to time. As a consequence, within the adiabatic approximation (6), the pump and Stokes Gaussian pulses [1, 6],

\[
\Omega_p(t) = \Omega_0 e^{-(t-\tau_1)^2/T_p^2},
\]

\[
\Omega_s(t) = \Omega_0 e^{-(t+\tau_1)^2/T_p^2},
\]

transfer from state \(|1\rangle\) at initial time \( t = t_i \) to \(|3\rangle\) at final one \( t = t_f \) along the dark state (4). Noting that \( \tau, T, \Omega_0 \) are the center, width and amplitude of the Gaussian pulses. More
general, one pump and two Stokes Gaussian pulses [58],
\[
\begin{align*}
\Omega_p(t) &= \Omega_0 \sin \beta e^{-(t-\tau_f)/T}^2, \\
\Omega_s(t) &= \Omega_0 e^{-(t-\tau_f)/T}^2 + \Omega_0 \cos \beta e^{-(t-\tau_f)/T}^2.
\end{align*}
\]
which becomes asymptotically
\[
0 \to \infty \quad \frac{\Omega_p(t)}{\Omega_s(t)} \to \tan \beta,
\]
are assumed to create the coherent superposition of \(|1\rangle\) and \(|3\rangle\) adiabatically by using dark state. In this circumstance, the constant \(\beta\) can be determined by the final target state, by combining with Eqs. (5) and (9). More specially, for the nonlinear f-STIRAP [59], one has to set \(\beta = \arctan \sqrt{2}\), when the conditions \(\Theta(t_f) = \pi/4\) and \(c_0^i(t_f) = 1/\sqrt{2}\) are stipulated to guarantee the state superposition of \(|1\rangle\) and \(|3\rangle\) with equal amplitude. As we all know, to fulfill the adiabatic condition (6), it usually takes long time, thus spoiling the state or repeating operations [55,56]. In this section, we shall develop the FF scaling approach to speed up the nonlinear STIRAP and f-STIRAP.

\section*{III. FAST-FORWARD SCALING APPROACH}

In this section, we shall generalize the FF scaling approach, with the motivation to accelerate the nonlinear STIRAP or f-STIRAP subjected to a slow variation of pulses. Inspired by some earlier results [28], the FF field for accelerating adiabatic processes with several examples has been worked out for discrete multi-level quantum system [55,56] and the nonlinear Gross-Pitaevskii or the corresponding Schrödinger equations [60,61]. In order to recapitulate the FF scaling for our proposal, we choose the dark state (4) as an ansatz
\[
|\Psi_{FF}(t)\rangle = \cos \Theta(R(t)) e^{i f(t)} |1\rangle - \sin \Theta(R(t)) e^{i f(t)} |3\rangle,
\]
with the “magnification factor” \(R(t)\) for the re-scaled time and a phase factor \(f_{1,3}(t)\) to satisfying the time-dependent Schrödinger equation, \(i \partial_t |\Psi_{FF}(t)\rangle = H_{FF}(t) |\Psi_{FF}(t)\rangle\), where the Hamiltonian has the form
\[
H_{FF}(t) = \begin{pmatrix}
0 & \Omega_{pF}^e c_1 & \Omega_{pF}^c c_1 \\
\Omega_{pF}^e c_1 & 0 & \Omega_{pF}^F \\
\Omega_{pF}^c c_1 & \Omega_{pF}^F & 0
\end{pmatrix},
\]
with the modified Rabi frequencies of pump, Stokes and an additional FF fields being \(\Omega_{pF}^F = \Omega_p^F(t), \Omega_{sF}^F = \Omega_s^F(t), \Omega_{cF}^F = \Omega_c^F(t)\). Here the conditions \(f_{1}(t_f) = f_{1,3}(t_f) = 0\) are required to connecting to adiabatic references. By inserting this ansatz (10) into the dynamical equation, we have the following equations:
\[
\begin{align*}
\frac{\Omega_p^F(t)}{\Omega_s^F(t)} &= \sin \Theta(R(t)) e^{i \Delta f(t)} e^{i f(t)}, \\
\frac{d \Delta f(t)}{dt} &= \frac{2 \cos 2 \Theta(R(t))}{\sin \Theta(R(t))} \text{Re}[\Omega_{cF}^F e^{i \Delta f(t)}], \\
\frac{\partial \Theta}{\partial R} \frac{\partial R}{dt} &= \text{Im}[\Omega_{cF}^F e^{i \Delta f(t)}],
\end{align*}
\]
from which the Rabi frequencies in the FF scaling approach are finally obtained as
\[
\begin{align*}
\frac{\Omega_p^F(t)}{\Omega_s^F(t)} &= \frac{\Omega_p(R(t))}{\Omega_s(R(t))} e^{i \Delta f(t) - f(t)}, \\
\Omega_{cF}^F(t) &= e^{-i \Delta f(t)} \left[ \frac{2 \cos 2 \Theta(R(t))}{2 \cos 2 \Theta(R(t))} \frac{d \Delta f(t)}{dt} + i \frac{\partial \Theta}{\partial R} \frac{\partial R}{dt} \right],
\end{align*}
\]
with \(\Delta f(t) = f_3(t) - f_1(t)\). When \(f_{1,3}(t) = 0\) is further assumed, the Rabi frequencies can be thus simplified as
\[
\begin{align*}
\frac{\Omega_p^F(t)}{\Omega_s^F(t)} &= \frac{\Omega_p(R(t))}{\Omega_s(R(t))}, \\
\Omega_{cF}^F(t) &= i \frac{\partial \Theta}{\partial R} \frac{\partial R}{dt}.
\end{align*}
\]
Obviously, the additional FF driving field \(\Omega_{cF}^F(t)\), depending on the magnification factor, is necessarily required for the acceleration of adiabatic passages, while the pump and Stokes fields keep the same shapes but with the re-scaled time. When \(R(t) = \eta t\), and the rate of change in \(R(t)\) meets \(\eta \ll 1\), the adiabatic process is recovered, and thus the FF field, i.e. \(\Omega_{cF}^F(t) = 0\), is vanished. In the case of \(R(t) = t\), the FF driving field can be written as \(\Omega_{cF}^F(t) = i \Theta\), which is in reminiscence of CD driving in linear STIRAP [23]. However, the mixing angle with \(c_1^i\) involved in Eq. (5) results in different auxiliary interaction coupling between \(|1\rangle\) and \(|2\rangle\), from the linear counterpart.

Now we have to emphasize the similarity and difference among the FF scaling, CD driving and IE methods. Though the additional coupling between \(|1\rangle\) and \(|3\rangle\) are required for both FF scaling and CD driving methods, the FF scaling approach is more general in the sense that when \(f_{1,3}(t) = 0\), the FF field has real and imaginary parts, whose pulse area becomes larger than \(\pi\), that of CD driving in linear STIRAP [55]. Moreover, one can always adopt various function of \(R(t)\) for accelerating the adiabatic passage, which provides more flexibility as well [28,60,61]. More importantly, the concept of FF scaling approach is different from CD driving. In order to obtain the CD term, one has to calculate the diabatic transition after diagonalizing Hamiltonian in linear STIRAP. However, in nonlinear STIRAP, we can’t use eigenstates and their orthogonality to obtain the CD driving directly. Instead, we assume the dark state as an ansatz for constructing the FF driving field, such that the state transfer is always along the dark state without populating the excited state. This also provides the advantage over the IE method used in Ref. [49], in which the intermediate state \(|2\rangle\) is populated, thus leading to the inevitable losses, although there is no need for the direct coupling between the states \(|1\rangle\) and \(|3\rangle\).

\section*{IV. EFFICIENCY AND STABILITY}

In this section, we shall provide insights into the features on the efficiency and stability of FF scaling approach in nonlinear A-type system, by exemplifying the FF assisted nonlinear STIRAP and f-STIRAP, respectively.
(a,b) we recover the original nonlinear system. By using Gaussian shapes of pump and Stokes pulses, see Eq. (7), the state can be adiabatically transferred from an atomic BEC at initial time \( t_i = -5T \) to a molecular one at \( t_f = 5T \), as shown in Fig. 2 (a,b), where \( \Omega_0 = 100 \) (in the units of \( 1/T_0 \)), \( \tau = 0.64T \), \( T = 1 \) (in the units of \( T_0 \)) and \( T_0 = 3.1 \times 10^{-3} \) s. The final population \(|c|^2 = 0.9914\) is achieved with total time \( 10T \). For comparison, FF assisted STIRAP (c) for state transfer (d) is also presented, where \( \Omega_{p,FF} \) (solid red) and \( \Omega_{s,FF} \) (dashed blue) with \( \Omega_0 = 5 \) (in the units of \( 1/T_0 \)) and other parameters are the same. Assisted by the FF driving field \( \Omega_{FF} \) (dotted black), the final population \(|c|^2 = 0.9999\) is achieved with total time \( 5T \).

We first check the conventional STIRAP in such \( \Lambda \)-type nonlinear system. By using Gaussian shapes of pump and Stokes pulses, see Eq. (7), the state can be adiabatically transferred from an atomic BEC at initial time \( t_i = -5T \) to a molecular one at \( t_f = 5T \), as shown in Fig. 2 (a,b), where \( \Omega_0 = 100 \) (in the units of \( 1/T_0 \)), \( \tau = 0.64T \), \( T = 1 \) (in the units of \( T_0 \)). Normally, \( T = 3.1 \times 10^{-3} \) s can be chosen in the practical experiment [39], corresponding to \( \Omega_0 = 3.226 \) MHz. The final population \(|c(t_f)|^2 = 0.9914\) is achieved without exciting state \( |2\rangle \), when the total time is \( 10T \) with the fixed \( \Omega_0 = 100 \). Remarkably, the modified pump/Stokes fields and FF driving field are designed to accelerate the nonlinear STIRAP, as shown in Fig. 2 (c,d), where \( \Omega_0 = 5 \) (in the units of \( 1/T_0 \)), and other parameters are the same as those in Fig. 2 (a,b). By introducing \( R = 2t \), the total time is decreased up to \( 5T \), assisted by FF field. From the comparison in Fig. 2, we demonstrate that the assisted FF driving field really speed up the original nonlinear STIRAP, following the dark state, as seen in population evolution. The final population reaches \(|c|^2 = 0.9999\) in Fig. 2 (d), above the previous result of STIRAP, when the parameters don’t fulfill the adiabatic condition (6). In order to quantify the acceleration, we calculate the adiabatic condition (6), see Fig. 3, where \( A_{ad} \ll 1 \) under the parameters used in conventional nonlinear STIRAP, while \( A_{ad} \) becomes much larger for the parameters when the transfer time is shorten with lower intensity \( \Omega_0 \). Of course, one can try \( R = t \), while keeping the total time \( 10T \), which is not shown here for simplicity. But we can still show that the auxiliary FF field assists to achieve the high-fidelity state transfer with \( \Omega_0 = 5 \), since the amplitude of Gaussian pulses is not large enough to fulfill the adiabatic condition. In this sense, the FF driving field speeds up nonlinear STIRAP when \( R = t \), because the system evolution requires a larger transfer time with fixed small \( \Omega_0 \).

Next, we shall also apply the FF driving field for accelerating nonlinear f-STIRAP. In Fig. 4 (a,b) we recover the original adiabatic process to generate the coherent superposition of an atomic BEC and a molecular BEC with equal amplitude, where Gaussian pulses of pump and Stokes fields are used, see Eq. (8), with \( \Omega_0 = 20 \) (in the units of \( T_0 \)), \( \beta = \sqrt{2} \), and other parameters are the same as those in Fig. 2. Similar to nonlinear STIRAP, we apply the FF driving field, modified pump and Stokes fields to speed up the nonlinear f-STIRAP as well, see in Fig. 4 (c,d). By selecting \( R = 2t \), we can shorten...
the total time from $10T$ to $5T$, with small amplitude $\Omega_0 = 5$. With the assisted FF driving field, the state evolution follows exactly the adiabatic reference in Fig. 4 (b,d) to achieve the perfect coherent superposition with the ratio $1:1$, but without populating excited state [2]. Profiting from the example in Fig. 4, the FF scaling approach should work well for the state superposition, thus generalizing to other ratio of amplitudes by changing the parameter $\beta$ in Eq. (8) through the mixing angle (5).

Finally, we shall address the issue on efficiency and stability of nonlinear STIRAP and f-STIRAP assisted by the FF driving field. First of all, we demonstrate that the fidelity $F$ depends strongly on the amplitude $\Omega_0$ of STIRAP and f-STIRAP, as shown in Fig. 5 (a), where the fidelity can be defined as $F = |\langle \Psi_0(t_f) | \psi(t_f) \rangle |^2$, with $\Psi_0(t_f)$ being the target state following the dark state and $| \psi(t_f) \rangle$ being the final solution of time-dependent Schrödinger equation with their own pump, Stokes and FF driving fields for STIRAP and f-STIRAP, respectively. It is clearly that the adiabatic passages don’t work for small value of $\Omega_0$, due to the breakdown of adiabatic criteria. For instance, when $\Omega_0 = 10$ is much less $\Omega_0 = 100$ used in Fig. 2, the fidelity of STIRAP is far away from 1. Remarkably, the designed FF driving field helps accelerate the adiabatic passage with perfect fidelity, $F \approx 1$, for arbitrary value of $\Omega_0$. However, one has to keep in mind that the energetic cost of STA, that is, the physical constraint on the FF driving field sets the limitation to shorten the time, relevant to quantum speed limits [62]. Worthy to mention, we also confirm that FF scaling approach improves the stability with respect to the fluctuation of $T$, as shown in Fig. 5 (b), where $T$ is the width of Gaussian pulses in Eqs. (7) and (8) and $T_0 = 3.1 \times 10^{-5} \text{s}$ is used as before. The fidelity decreases dramatically when $T$ is shortened, while it always keep $F \approx 1$ when the assisted FF driving field is complemented. Furthermore, the stability with respect to $\tau$, affecting the sequence of Gaussian pulses, is improved by the FF driving field as well. For instance, the fidelities are decreased up to $F = 0.9790$ and 0.9953, respectively, for original STIRAP and f-STIRAP, when $\tau = 0.5T$. However, with the assisted FF field, the fidelities keep $F \approx 1$ in both protocols, where $T$ is re-scaled by magnification factor $R(t)$ with the assisted FF field. Actually, the improvement of robustness makes sense that the area of whole pump and Stokes pulses is increased by $\Omega_0 = 20$ in the units of $1/T_0$ and other parameters are the same. Assisted by the FF driving field $\Omega^\text{FF}_F$ (dotted black), the superposition of atomic and molecular BECs with equal amplitudes is achieved with total time $5T$.

\[
\begin{align*}
\text{FIG. 4.} & \quad \text{f-STIRAP (a) for generating the coherent state superposition (b) is presented, where $\Omega_c$ (solid red) and $\Omega_p$ (dashed blue) sequences are Gaussian type with the amplitude $\Omega_0 = 20$ (in the units of $1/T_0$), $\tau = 0.64T$, $T = 1$ (in the units of $T_0$) and $T_0 = 3.1 \times 10^{-5} \text{s}$. The superposition of an atomic BEC and a molecular BEC with equal amplitudes is provided with total time $10T$. For comparison, FF assisted f-STIRAP (c) for generating such superposition (d) is also presented, where $\Omega^\text{FF}_c$ (solid red) and $\Omega^\text{FF}_p$ (dashed blue) with $\Omega_0 = 20$ (in the units of $1/T_0$) and other parameters are the same. Assisted by the FF driving field $\Omega^\text{FF}_F$ (dotted black), the superposition of atomic and molecular BECs with equal amplitudes is achieved with total time $5T$.
\end{align*}
\]

\[
\begin{align*}
\text{FIG. 5.} & \quad \text{FF scaling approach improves the stability with respect to the fluctuation of $T$, as shown in Fig. 5 (b), where $T$ is the width of Gaussian pulses in Eqs. (7) and (8) and $T_0 = 3.1 \times 10^{-5} \text{s}$ is used as before. The fidelity decreases dramatically when $T$ is shortened, while it always keep $F \approx 1$ when the assisted FF driving field is complemented. Furthermore, the stability with respect to $\tau$, affecting the sequence of Gaussian pulses, is improved by the FF driving field as well. For instance, the fidelities are decreased up to $F = 0.9790$ and 0.9953, respectively, for original STIRAP and f-STIRAP, when $\tau = 0.5T$. However, with the assisted FF field, the fidelities keep $F \approx 1$ in both protocols, where $T$ is re-scaled by magnification factor $R(t)$ with the assisted FF field. Actually, the improvement of robustness makes sense that the area of whole pump and Stokes pulses is increased by $\Omega_0 = 20$ in the units of $1/T_0$ and other parameters are the same. Assisted by the FF driving field $\Omega^\text{FF}_F$ (dotted black), the superposition of atomic and molecular BECs with equal amplitudes is achieved with total time $5T$.
\end{align*}
\]
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In summary, we have worked out the FF scaling of coherent control for atom and molecular BECs, with the second-order nonlinearity involved. By using the dark state in nonlinear A-type system, we can derive the FF driving field, combing with the modified pump and Stokes fields, to achieve high-fidelity state conversion from an atomic BEC to a molecular one, when the adiabatic condition is not fulfilled. Moreover, the result can be directly generalized to f-STIRAP for the coherent superposition of an atomic and a molecular BECs. The FF assisted STIRAP and f-STIRAP have a higher tolerance to the parameter fluctuations, for instance, in the intensity and width of Gaussian pulses. Besides, the original adiabatic passages are speeded up, but without populating the intermediate excited state, which prevents from the losses due to inevitable dissipation [44] or dephasing effect [63].

What we should emphasize that FF scaling approach in nonlinear system are different from CD driving and IE methods of STA. In nonlinear system, there exist non-orthogonal degenerate eigenstates, which hinders the calculation of CD driving, even though the formula of FF fields is similar. With extra parameters in the phase of dark state (4), the FF driving provides more flexibility for the atom-molecular conversion, with large area of pulses. We realize that the intermediate state (2) is populated in an alternative IE method [49]. However, the price paid in FF scaling approach is the supplement of the auxiliary coupling between states (1) and (3). Therefore, when it comes to experimental realization, one can pick up the suitable recipes or protocols as seen above, taking into account the physical feasibility and limitation.

Last but not least, the FF scaling approach is interestingly extended to fast and robust control of unstable nonlinear systems, for instance, BEC in optical lattices with the third-order Kerr-type nonlinearity [64], and other applications of coupled waveguides [65, 66] and frequency conversion in nonlinear optics [67] in analogous fashion.

V. CONCLUSION

FIG. 5. Fidelity, $F = |\langle \Psi(t_f) | \Psi(t) \rangle|^2$, versus the amplitude $\Omega_0$ (a) and the width $T$ (b) of Gaussian pulses, where $\Psi(t_f)$ is target state and $\Psi(t)$ is the final solution of time-dependent Schrödinger equation with its own corresponding pump and Stokes fields, and FF driving field for STIRAP and f-STIRAP, respectively. Here the fidelities of STIRAP, f-STIRAP and FF assisted STIRAP are denoted by solid black, dashed blue and dotted red lines, and that of FF assisted f-STIRAP (dash-dotted orange) are undistinguishable. Other parameters are the same as those in Fig. 2 and Fig. 4.

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