On the non-Markovian Enskog equation for granular gases

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Abstract

We develop a rigorous formalism for the description of the kinetic evolution of many-particle systems with dissipative interaction. The links of the evolution of a hard sphere system with inelastic collisions described within the framework of marginal observables governed by the dual Bogolyubov–Born–Green–Kirkwood–Yvon (BBGKY) hierarchy and the evolution of states described by the Cauchy problem of the Enskog kinetic equation for granular gases are established. Moreover, we consider the Boltzmann–Grad asymptotic behavior of the constructed non-Markovian Enskog kinetic equation for granular gases in a one-dimensional space.

Keywords: granular gas, inelastic collision, dual BBGKY hierarchy, Enskog equation, kinetic evolution

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1. Introduction

A granular gas is a dynamical system of significant interest not only in view of its applications but also as a many-particle system displaying a collective behavior that differs from the statistical behavior of other gases; for example, it is related to typical macroscopic properties [1–9].

As is known, the collective behavior of many-particle systems can be effectively described within the framework of a one-particle marginal distribution function governed by the kinetic equation in a suitable scaling limit of underlying dynamics [10–14]. At present, considerable advances are being made in the rigorous derivation of the Boltzmann kinetic equation of a system of hard spheres in the Boltzmann–Grad scaling limit [15–17]. At the same time, many
recent papers [18–20] (and see references therein) are considering the Boltzmann-type and
Enskog-type kinetic equations for inelastically interacting hard spheres, modelling granular
gases, as the original evolution equations and the rigorous derivation of such kinetic equations
remain a problem [21–26].

The goal of this paper is to develop an approach based on the dynamics of particles
with dissipative interaction to properly justify the kinetic equations that previous works have
already applied a priori to the description of granular gases. In the paper, we consider the
problem of potentialities inherent in the description of the evolution of states of a hard sphere
system with inelastic collisions in terms of a one-particle distribution function. We establish
that, in fact, if the initial state is completely specified by a one-particle marginal distribution
function, then all possible states at an arbitrary moment of time can be described within the
framework of a one-particle distribution function without any approximations.

To outline the structure of the paper and the main results, in section 2, we develop
an approach to the description of the kinetic evolution of many hard spheres with inelastic
collisions within the framework of the evolution of marginal observables. Then, in section 3,
the main results related to the origin of the kinetic evolution of granular gases are stated. We
prove that underlying dynamics governed by the dual Bogolyubov–Born–Green–Kirkwood–
Yvon (BBGKY) hierarchy for marginal observables can be completely described within the
framework of the one-particle marginal distribution function governed by the non-Markovian
Enskog kinetic equation with inelastic collisions. In this case, we prove that all possible
correlations, created by hard sphere dynamics, are described by the explicitly defined marginal
functionals with respect to the solution of the established kinetic equation. In section 4,
we consider a one-dimensional granular gas. The Boltzmann–Grad asymptotic behavior of
the constructed non-Markovian Enskog kinetic equation with inelastic collisions in a one-
dimensional space is outlined. Finally, in section 5, we conclude with some observations and
perspectives for future research.

2. Hierarchies of evolution equations for granular gases

It is well known that many-particle systems are described in terms of two sets of objects:
observables and states. The functional of the mean value of observables defines a duality
between observables and states, and as a consequence two approaches to the description of
the evolution exist. Usually, the evolution of many-particle systems is described within the
framework of the evolution of states by the BBGKY hierarchy for marginal distribution
functions. An equivalent approach to this description is given in terms of the marginal
observables governed by the dual BBGKY hierarchy. In the same framework, the systems
of particles with dissipative interaction, namely hard spheres with inelastic collisions, can be
described.

2.1. Dual BBGKY hierarchy and semigroups of operators of hard spheres with inelastic
collisions

We consider a system of a non-fixed, i.e. arbitrary, but finite average number of identical par-
ticles of a unit mass with the diameter \( \sigma > 0 \), interacting as hard spheres with inelastic collisions.
Every particle is characterized by the phase coordinates \( (q_i, p_i) \equiv x_i \in \mathbb{R}^3 \times \mathbb{R}^3, \ i \geq 1 \).

Let \( C_\gamma \) be the space of sequences \( b = (b_0, b_1, \ldots, b_n, \ldots) \) of bounded continuous
functions \( b_n \in C_n \) defined on the phase space of \( n \) hard spheres that are symmetrical
with respect to the permutations of the arguments \( x_1, \ldots, x_n \), equal to zero on the set of
forbidden configurations \( \mathcal{W}_n \equiv \{(q_1, \ldots, q_n) \in \mathbb{R}^{3n} | |q_i - q_j| < \sigma \ for \ at \ least \ one \ pair \)
respectively. In (3), (4), the following notations are used:

\[ L \] and for \( t \) functions with compact supports by \( \varepsilon \)

where on the set \( B_0 \) but finite average number of hard spheres is described by the sequences \( B_s \)

\[ \delta \] means a scalar product,

and the post-collision momenta are determined by

\[ B_s(t, x_1, \ldots, x_n) \mid_{s=0} = B^0_0(x_1, \ldots, x_n), \quad s \geq 1, \]

where on the set \( C_{x,0} \subset C \), the free motion operator \( \mathcal{L}(j) \) and the operator of inelastic collisions

\[ \mathcal{L}_{\text{int}}(j_1, j_2) \]

are defined by the formulas

\[ \mathcal{L}(j) \equiv \left\{ p_j, \frac{\partial}{\partial q_j} \right\}, \]

and

\[ \mathcal{L}_{\text{int}}(j_1, j_2) b_j \equiv \sigma^2 \int_{\mathbb{R}^3} \eta(\eta, (p_{j_1} - p_{j_2}))(b_j(x_1, \ldots, x^*_j, \ldots, x_s)) \]

\[ - b_j(x_1, \ldots, x_s) \delta(q_{j_1} - q_{j_2} + \sigma \eta), \]

respectively. In (3), (4), the following notations are used: \( x^*_j \equiv (q_j, p^*_j) \), the symbol \( \langle \cdot, \cdot \rangle \)

means a scalar product, \( \delta \) is the Dirac measure, \( S^0_1 \equiv \{ \eta \in \mathbb{R}^3 | |\eta| = 1, \langle \eta, (p_{j_1} - p_{j_2}) \rangle \geq 0 \} \)

and the post-collision momenta are determined by

\[ p^*_{j_1} = p_{j_1} - (1 - \epsilon) \eta(\eta, (p_{j_1} - p_{j_2})), \]

\[ p^*_{j_2} = p_{j_2} + (1 - \epsilon) \eta(\eta, (p_{j_1} - p_{j_2})), \]

where \( \epsilon = \frac{1 - \sigma^2}{2} \in [0, \frac{1}{2}] \) and \( \sigma \in [0, 1] \) is a restitution coefficient [7].

We give explicit examples of recurrence evolution equations (1):

\[ \frac{\partial}{\partial t} B_1(t, q_1, p_1) = \left\{ p_1, \frac{\partial}{\partial q_1} \right\} B_1(t, q_1, p_1), \]

\[ \frac{\partial}{\partial t} B_2(t, x_1, x_2) = \sum_{j=1}^2 \left\{ p_j, \frac{\partial}{\partial q_j} \right\} B_2(t, x_1, x_2) \]

\[ + \sigma^2 \int_{S^0_1} \eta(\eta, (p_1 - p_2))(B_2(t, x^*_1, x^*_2) - B_2(t, x_1, x_2)) \delta(q_1 - q_2 + \sigma \eta) \]

\[ + \sigma^2 \int_{S^0_1} \eta(\eta, (p_1 - p_2))(B_1(t, x^*_1) - B_1(t, x_1)) \delta(q_1 - q_2 + \sigma \eta) \]

\[ + \sigma^2 \int_{S^0_1} \eta(\eta, (p_1 - p_2))(B_1(t, x^*_2) - B_2(t, x_2)) \delta(q_1 - q_2 + \sigma \eta). \]
We refer to recurrence evolution equations (1) as the dual BBGKY hierarchy for hard spheres with inelastic collisions or for granular gases.

The first term of a generator of the dual BBGKY hierarchy (1) is the Liouville operator

\[ \mathcal{L}_g = \sum_{j=1}^s \mathcal{L}(j) + \sum_{j_1 < j_2 = 1} \mathcal{L}_{\text{int}}(j_1, j_2), \]

which is an infinitesimal generator of the semigroup of operators \( S_g(t), t \geq 0 \), of a system of \( s \) inelastically interacting hard spheres. The semigroup of operators \( S_g(t), t \geq 0 \), is defined almost everywhere on the phase space \( \mathbb{R}^{3s} \setminus \mathcal{W}_s \) \( \times \mathbb{R}^{3s} \), namely, outside the set \( \mathcal{M}_s^0 \) of the Lebesgue measure zero, as a shift operator of the arguments of functions from the space \( \mathcal{C}_s \) along the phase trajectories of hard spheres with inelastic collisions

\[ (S_g(t)b_j)(x_1, \ldots, x_s) = S_g(t, 1, \ldots, s)b_j(x_1, \ldots, x_s) \]

where the function \( X_j(t) = X_j(t, x_1, \ldots, x_s) \) is a phase trajectory of the \( j \)th particle constructed in [26], and the set \( \mathcal{M}_s^0 \) consists of phase space points specified in the initial data \( x_1, \ldots, x_s \) that generate multiple collisions during the evolution [14].

On the space \( \mathcal{C}_s \), one-parameter mapping (6) is a bounded \( \ast \)-weak continuous semigroup of operators [28], and \( \left\| S_g(t) \right\|_{\mathcal{C}_s} \leq 1 \).

Let \( L_\alpha = \bigoplus_{n=0}^\infty \alpha^n L_\alpha^n \) be the space of sequences \( f = (f_0, f_1, \ldots, f_n, \ldots) \) of integrable functions \( f_n(x_1, \ldots, x_n) \) defined on the phase space of \( n \) hard spheres that are symmetrical with respect to the permutations of the arguments \( x_1, \ldots, x_n \), equal to zero on the set of forbidden configurations \( \mathcal{W}_n \), and equipped with the norm \( \left\| f \right\|_{L_\alpha^n} = \sum_{n=0}^\infty \alpha^n \int dx_1 \ldots dx_n f_n(x_1, \ldots, x_n) \), where \( \alpha > 1 \) is a real number. We denote by \( L_\alpha^1 \subset L_\alpha \) the everywhere dense set in \( L_\alpha^1 \) of finite sequences of continuously differentiable functions with compact supports.

On the space of integrable functions, the semigroup of operators \( S_g^s(t), t \geq 0 \), is defined adjoint to the semigroup of operators (6) in the sense of the continuous linear functional (the functional of mean values of observables)

\[ \langle b | f \rangle = \sum_{i=0}^\infty \frac{1}{i!} \int_{\mathbb{R}^{3s} \times \mathbb{R}^{3s}} dx_1 \ldots dx_i b_j(x_1, \ldots, x_i) f_n(x_1, \ldots, x_i). \]

The adjoint semigroup of operators is defined by the Duhamel equation

\[ S_g^s(t, 1, \ldots, s) = \int_0^t \sum_{i=1}^s S_g^s(t, i) \int_0^t \sum_{j_1 < j_2 = 1} \mathcal{L}_{\text{int}}^s(j_1, j_2)S_g^s(t, 1, \ldots, s), \quad (7) \]

where for \( t \geq 0 \) the operator \( \mathcal{L}_{\text{int}}^s(j_1, j_2) = \sum_{\tau=0}^s \mathcal{L}_{\text{int}}^s(j_1, j_2) \mathcal{L}_g^s(\tau, 1, \ldots, s) \)

\[ \mathcal{L}_{\text{int}}^s(j_1, j_2)f = \alpha^2 \int_{\mathbb{R}^{3s}} d\eta \eta \left( p_{j_1} - p_{j_2} \right) \left( \frac{1}{1 - 2\varepsilon} f_n(x_1, \ldots, x_{j_1}, \ldots, x_{j_2}, \ldots, x_s) \right) \]

\[ \times \delta(q_{j_1} - q_{j_2} + \sigma \eta) - f_n(x_1, \ldots, x_s) \delta(q_{j_1} - q_{j_2} - \sigma \eta). \quad (8) \]

In (8), notations similar to formula (4) are used, \( x_{j_1}^\varepsilon = (q_{j_1}, p_{j_1}^\varepsilon) \), and the pre-collision momenta (solutions of equations (5)) are determined as follows:

\[ p_{j_1}^\varepsilon = p_{j_1} - \frac{1 - \varepsilon}{1 - 2\varepsilon} \eta \left( \eta, (p_{j_1} - p_{j_2}) \right), \]

\[ p_{j_2}^\varepsilon = p_{j_2} + \frac{1 - \varepsilon}{1 - 2\varepsilon} \eta \left( \eta, (p_{j_1} - p_{j_2}) \right). \quad (9) \]
Hence, an infinitesimal generator of the adjoint semigroup of operators \( S_t^\gamma \) is defined on \( L^1_{0,s} \) as the operator: \( L^\gamma = \sum_{j=1}^s L^\gamma (j) + \sum_{j_1 < j_2=1}^s L^\gamma_{int} (j_1, j_2) \), where we introduced the operator adjoint to free motion operator (3): \( L^\gamma (j) = -(p_j, \frac{\partial}{\partial p_j}) \).

On the space \( L^1_{0,s} \), the one-parameter mapping defined by equation (7) is a bounded strong continuous semigroup of operators.

We remark that, according to the definition of the marginal observables \([29]\) the dual BBGKY hierarchy (1) for hard spheres with inelastic collisions can be rigorously derived due to the properties of semigroups (6) of hard spheres with inelastic collisions.

### 2.2. A solution expansion of the dual BBGKY hierarchy with hard sphere inelastic collisions

On the space \( C_y \) for abstract initial-value problem (1), (2), the following statement is true.

A solution \( B(t) = (B_0, B_1(t, x_1), \ldots, B_s(t, x_1, \ldots, x_s), \ldots) \) of the Cauchy problem (1), (2) is determined by the expansions \([29]\)

\[
B_s(t, x_1, \ldots, x_s) = \sum_{n=0}^{s-1} \frac{1}{n!} \sum_{j_1 \neq \ldots \neq j_n=1}^s A_{1+n}^0(t, [Y \setminus Z], Z) B^s_{n-1}(x_1, \ldots, x_{j_1-1}, x_{j_1+1}, \ldots, x_s),
\]

(10)

where the \((1 + n)\)th-order cumulant of semigroups of operators \([30]\) of hard spheres with inelastic collisions (6) is defined by the formula

\[
A_{1+n}^0(t, [Y \setminus X], X) = \sum_{P: (|Y \setminus X|, X) \in |Y \setminus X|} (-1)^{|P| - 1} |[P] - 1|! \prod_{X \in \theta(X)} S_{\phi(X)}(t, \theta(X)),
\]

(11)

and \( Y = (1, \ldots, s), Z = (j_1, \ldots, j_n) \subset Y, [Y \setminus Z] \) is the set consisting of one element \( Y \setminus Z = (1, \ldots, j_1 - 1, j_1 + 1, \ldots, j_n - 1, j_n + 1, \ldots, s) \); i.e., this set is a connected subset of the partition \( P \), such that \( |P| = 1 \), the mapping \( \theta(\cdot) \) is a declusterization operator defined by the formula \( \theta([Y \setminus Z]) = Y \setminus Z \). Under the condition \( \gamma < e^{-1} \), for a sequence of marginal observables (10) the estimate holds

\[
\|B(t)\|_{C_y} \leq e^2 (1 - \gamma e)^{-1} \|B(0)\|_{C_y}.
\]

(12)

The simplest examples of marginal observables (10) are given by the following expansions:

\[
B_1(t, x_1) = A_1(t, 1) B^1_0 (x_1),
\]

\[
B_2(t, x_1, x_2) = A_1(t, 1, 2) B^2_0 (x_1, x_2) + A_2(t, 1, 2) (B^1_0 (x_1) + B^1_0 (x_2)).
\]

We note that one-component sequences of marginal observables correspond to observables of a certain structure; namely, the marginal observable \( B^{(1)} = (0, b_1(x_1), 0, \ldots) \) corresponds to the additive-type observable, and the marginal observable \( B^{(k)} = (0, \ldots, 0, b_k(x_1, \ldots, x_s), 0, \ldots) \) corresponds to the \( k \)-ary-type observable \([27]\). If in capacity of initial data (2) we consider the additive-type marginal observable, then the structure of solution expansion (10) is simplified and attains the form

\[
B^{(1)}_s (t, x_1, \ldots, x_s) = A_s (t, 1, \ldots, s) \sum_{j=1}^s b_t (x_j), \quad s \geq 1.
\]

(13)
For $B(0) = (B_0, B_1, \ldots, B_s, \ldots) \in C^0_\gamma \subset C_\gamma$ of finite sequences of infinitely differentiable functions with compact supports, the sequence of functions (10) is a classical solution and for arbitrary initial data $B(0) \in C_\gamma$ it is a generalized solution.

We note that expansion (10) can be also represented in the form of the weak formulation of the perturbation (iteration) series [27] as a result of applying analogues of the Duhamel equation to cumulants of semigroups of operators (11).

2.3. A functional of mean values of marginal observables

The mean value of the marginal observable $B(t) = (B_0, B_1(t), \ldots, B_s(t), \ldots) \in C^0_\gamma$ at $t \geq 0$ in the initial state described by the sequence of marginal distribution functions $F(0) = (1, F_0^0, \ldots, F_s^0, \ldots) \in L^1_0$ is determined by the functional

$$
\langle B(t) | F(0) \rangle = \sum_{s=0}^{\infty} \frac{1}{s!} \int_{(\mathbb{R}^3)^s} dx_1 \ldots dx_s B_s(t, x_1, \ldots, x_s) F^0_s(x_1, \ldots, x_s).
$$

(14)

Owing to estimate (12), functional (14) exists under the condition that $\gamma < e^{-1}$.

In particular, functional (14) of mean values of the additive-type marginal observables $B^{(1)}(0) = (0, B^{(1)}_1(0, x_1), 0, \ldots)$ takes the form

$$
\langle B^{(1)}(t) | F(0) \rangle = \langle B^{(1)}(0) | F(t) \rangle
$$

$$
= \int_{(\mathbb{R}^3)^2} dx_1 B^{(1)}_1(0, x_1) F_{1}(t, x_1),
$$

where the one-particle marginal distribution function $F_{1}(t, x_1)$ is determined by the series expansion [30]

$$
F_{1}(t, x_1) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^3)^s} dx_2 \ldots dx_{n+1} \mathcal{B}^{(n+1)}_1(t, x_1, \ldots, x_{n+1}),
$$

and the generating operator $\mathcal{B}^{(n+1)}_1(t)$ of this series is the $(1+n)$-th order cumulant of adjoint semigroups of hard spheres with inelastic collisions. In the general case for mean values of marginal observables, the following equality is true:

$$
\langle B(t) | F(0) \rangle = \langle B(0) | F(t) \rangle,
$$

where the sequence $F(t) = (1, F_1(t), \ldots, F_s(t), \ldots)$ is a solution of the Cauchy problem of the BBGKY hierarchy of hard spheres with inelastic collisions [26]. This equality signifies the equivalence of two pictures of the description of the evolution of hard spheres by means of the BBGKY hierarchy and the dual BBGKY hierarchy (1).

Furthermore, we consider the initial states of hard spheres specified by a one-particle marginal distribution function, namely

$$
F^{(1)}_s(x_1, \ldots, x_s) = \prod_{i=1}^{s} F^0_s(x_i) \chi_{\mathbb{R}^3 \setminus \mathcal{W}_s}, \quad s \geq 1,
$$

(15)

where $\chi_{\mathbb{R}^3 \setminus \mathcal{W}_s} \equiv \chi(q_1, \ldots, q_s)$ is a characteristic function of allowed configurations $\mathbb{R}^{3s} \setminus \mathcal{W}_s$ of $s$ hard spheres and $F^0_s \in L^1(\mathbb{R}^3 \times \mathbb{R}^3)$. Initial data (15) is intrinsic for the kinetic description of many-particle systems because in this case all possible states are characterized by means of a one-particle marginal distribution function.

3. The non-Markovian Enskog kinetic equation for granular gases

In view of the fact that initial state is completely specified by a one-particle marginal distribution function on allowed configurations (15), the evolution of states can be described
within the framework of the sequence \( F(t \mid F_1(t)) = (1, F_1(t), F_2(t \mid F_1(t)), \ldots, F_s(t \mid F_1(t)), \ldots) \) of the marginal functionals of the state \( F_s(t, x_1, \ldots, x_s \mid F_1(t)) \), \( s \geq 2 \), which are explicitly defined with respect to the solution \( F_1(t, x_1) \) of the kinetic equation. We refer to such a kinetic equation of inelastically interacting hard spheres as the non-Markovian Enskog kinetic equation for granular gases.

3.1. A description of the collective behavior of granular gases by means of kinetic equations

In the case of initial states (15), the dual picture to the Heisenberg picture of the evolution of a system of hard spheres with inelastic collisions described in terms of the dual BBGKY hierarchy (1) for marginal observables is the evolution of states described within the framework of a system of hard spheres with inelastic collisions described in terms of the dual BBGKY hierarchy (1) for marginal observables is the evolution of states described within the framework of the non-Markovian Enskog kinetic equation and a sequence of explicitly defined functionals of a solution of this kinetic equation.

In fact, for mean value functional (14), the following representation holds:

\[
\langle B(t) \mid F^c \rangle = \langle B(0) \mid F(t \mid F_1(t)) \rangle ,
\]

where \( F^c = (1, F_1^{(c)}, \ldots, F_s^{(c)}, \ldots) \) is the sequence of initial marginal distribution functions (15), and the sequence \( F(t \mid F_1(t)) = (1, F_1(t), F_2(t \mid F_1(t)), \ldots, F_s(t \mid F_1(t))) \) is a sequence of the marginal functionals of the state \( F_s(t, x_1, \ldots, x_s \mid F_1(t)) \) represented by the series expansions over the products with respect to the one-particle marginal distribution function \( F_1(t) \):

\[
F_s(t, x_1, \ldots, x_s \mid F_1(t)) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^3)^n} dx_{s+1} \ldots dx_{s+n} \mathfrak{A}_{1+s}^n(t, [Y], X \setminus Y) \prod_{i=1}^{s+n} F_i(t, x_i), \quad s \geq 2.
\]

In series (17) we used the notations \( Y \equiv (1, \ldots, s), X \equiv (1, \ldots, s+n) \); and the \((n+1)\)th-order generating operator \( \mathfrak{A}_{1+n}(t) \), \( n \geq 0 \), is defined as follows [31]:

\[
\mathfrak{A}_{1+n}(t, [Y], X \setminus Y) \equiv \sum_{k=0}^{n} (-1)^k \sum_{m_1=1}^{s} \ldots \sum_{m_k=1}^{s+n-m_1-\ldots-m_k} \frac{n!}{(n-m_1-\ldots-m_k)!} \times \hat{\mathfrak{A}}_{1+n-m_1-\ldots-m_k}^k(t, [Y], s+1, \ldots, s+n-m_1-\ldots-m_k) \prod_{j=1}^{k} \sum_{k_j=0}^{m_j} \sum_{i_1=1}^{\prod_{j=1}^{k} k_j} \sum_{k_{i_1} \cdots k_{i_k} = 0}^{k_{i_1} \ldots k_{i_k}} \frac{1}{(k_{n-m_1-\ldots-m_1+i_j-1})!} \times \hat{\mathfrak{A}}_{1+k_{i_1}+k_{i_2}+\ldots+k_{i_k}}^k(t, i_j, s+n-m_1-\ldots-m_j+1, \ldots, s+n-m_1-\ldots-m_j) \cdots \times k_j \equiv m_j, \quad k_{n-m_1-\ldots-m_j+1} = 0, \quad \text{and we denote the (1 + n)th-order scattering cumulant by the operator } \hat{\mathfrak{A}}_{1+n}(t): \]

\[
\hat{\mathfrak{A}}_{1+n}(t, [Y], X \setminus Y) \equiv \mathfrak{A}_{1+n}^*(t, [Y], X \setminus Y) \mathfrak{A}_{\mathbb{R}^{3(s+n)} \setminus \{\mathbb{R}^3\}} \prod_{i=1}^{s+n} \mathfrak{A}_i(t, i)^{-1}.
\]
\[ \mathcal{V}_1(t, \{Y\}) = \hat{A}_1(t, \{Y\}) = S^s(t, 1, \ldots, s)X_{\mathbb{R}^s} \prod_{i=1}^{s} S^i(t, i)^{-1}, \]

\[ \mathcal{V}_2(t, \{Y\}, s + 1) = \hat{A}_2(t, \{Y\}, s + 1) = \hat{A}_1(t, \{Y\}) \sum_{i_1 = 1}^{s} \hat{A}_2(t, i_1, s + 1). \]

We emphasize that, in fact, functionals (17) characterize the correlations generated by dynamics of a hard sphere system with inelastic collisions. If \( \|F_1(t)\|_{L^2(\mathbb{R}^3 \times \mathbb{R}^3)} < e^{-(3t+2)} \), for arbitrary \( t \geq 0 \) series (17) converges in the norm of the space \( L^2 \).

The second element of the sequence \( F(t \mid F_1(t)) \), i.e. the one-particle marginal distribution function \( F_1(t) \), is determined by the following series expansion:

\[ F_1(t, x_1) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^3 \times \mathbb{R}^3)^n} dx_2 \ldots dx_{n+1} \hat{A}^n_{1+n}(t)X_{\mathbb{R}^{(1+n) \times \mathbb{R}^{1+n}}} \prod_{i=1}^{n+1} F_i^0(x_i), \tag{20} \]

where the generating operator \( \hat{A}^n_{1+n}(t) \equiv \hat{A}^n_{1+n}(t, 1, \ldots, n+1) \) is the \( (1+n) \)-th-order cumulant of adjoint semigroups of hard spheres with inelastic collisions.

For \( t \geq 0 \), the one-particle marginal distribution function (20) is a solution of the following Cauchy problem of the non-Markovian Enskog kinetic equation [31]

\[ \frac{\partial}{\partial t} F_1(t, q_1, p_1) = -\left(p_1, \frac{\partial}{\partial q_1}\right) F_1(t, q_1, p_1) + \sigma^2 \int_{\mathbb{R}^3 \times \mathbb{R}^3} dp_2 \, d\eta(\eta, (p_1 - p_2)) \]

\[ \times \left( \frac{1}{(1-2\epsilon)^2} F_2(t, q_1, p_1, q_1 - \sigma \eta, p_2 \mid F_1(t)) \right. \]

\[ - F_2(t, q_1, p_1, q_1 + \sigma \eta, p_2 \mid F_1(t)) \bigg). \tag{21} \]

\[ F_1(t)_{t=0} = F_1^0, \tag{22} \]

where the collision integral is determined by the marginal functional of the state (17) in the case of \( s = 2 \), and the expressions \( p_i^1 \) and \( p_i^2 \) are the pre-collision momenta of hard spheres with inelastic collisions (9), i.e. solutions of equations (5).

Thus, if initial states are specified by a one-particle marginal distribution function on allowed configurations, then the evolution of marginal observables governed by the dual BBGKY hierarchy (1) can also be described within the framework of the non-Markovian kinetic equation (21) and a sequence of marginal functionals of the state (17). In other words, for the mentioned initial states, the evolution of all possible states of a hard sphere system with inelastic collisions at an arbitrary moment of time can be described within the framework of a one-particle distribution function without any approximations.

3.2. The proof of main results

We establish the validity of equality (16) for mean value functional (14).

In the particular case of initial data (2) specified by the \( s \)-ary marginal observable \( s \geq 2 \), i.e. \( B^{(s)}(0) = (0, \ldots, 0, b_s, 0, \ldots) \), equality (16) takes the form

\[ \langle B^{(s)}(t) \mid F_1(t) \rangle = \langle B^{(s)}(0) | F_1(t) \rangle \]

\[ = \frac{1}{s!} \int_{(\mathbb{R}^3 \times \mathbb{R}^3)^s} dx_1 \ldots dx_s b_s(x_1, \ldots, x_s) F_1(t, x_1, \ldots, x_s \mid F_1(t)), \tag{23} \]

where the marginal functional of the state \( F(t, x_1, \ldots, x_s \mid F_1(t)) \) is determined by series expansion (17).
To verify the validity of equality (23), we transform the functional $\langle B^{(i)}(t) | F^c \rangle$ to the form

$$\langle B^{(i)}(t) | F^c \rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{(R^3 \times R^3)^n} dx_1 \ldots dx_n \frac{1}{(n - s)!}$$

$$\times \sum_{j_1 \neq \ldots \neq j_{n-s}=1} A_{1+n-s}(t, \{1, \ldots, j_1 = 1, j_1 + 1, \ldots, j_{n-s} = 1, j_{n-s} + 1, \ldots, s\}) \prod_{j=1}^{n} F_0^0(i) \mathcal{X}_{R^{2(n-1)|W_{n+1}}},$$

$$= \frac{1}{s!} \int_{(R^3 \times R^3)^s} dx_1 \ldots dx_s \sum_{n=0}^{\infty} \frac{1}{n!} \int_{(R^3 \times R^3)^n} dx_{n+1} \ldots dx_{s+n} \mathcal{X}_{1+n}(t, \{Y\}, X \setminus \{Y\}) \prod_{j=1}^{s+n} F_0^0(i) \mathcal{X}_{R^{2(s+n)|W_{s+n}}},$$

where we used the notations accepted in formula (17), and the operator $\mathcal{X}_{1+n}(t, \{Y\}, X \setminus \{Y\})$ is the $(1 + n)$th-order cumulant of adjoint semigroups of hard spheres with inelastic collisions. For $F_0^0 \in L^1(R \times R)$ and $b_i \in C^s$ obtained functional (24) exists under the condition that $\|F_0^0\|_{L^1(R \times R)} < \epsilon^{-1}$.

Then we expand the cumulants $\mathcal{X}_{1+n}(t, \{Y\}, s + 1, \ldots, s + n) \mathcal{J}_{s+n}(1, \ldots, s + n)$

$$= \sum_{k_1=0}^{n} \frac{n!}{(n - k_1)!k_1!} \mathcal{J}_{1+n-k_1}(t, \{Y\}, s + 1, \ldots, s + n - k_1)$$

$$\times \sum_{k_2=1}^{k_1} \frac{k_1!}{k_2! (k_1 - k_2)!} \ldots \sum_{k_{n-k_1}=1}^{k_{n-k_1-1}} \frac{k_{n-k_1-1}!}{k_{n-k_1}! (k_{n-k_1} + s - 1 - k_{n-k_1})!}$$

$$\times \prod_{i=1}^{s+n-k_1} \mathcal{X}_{k_{i-1}+s+1-\ldots-k_{i-1}-\ldots-1}(t, i, s + n - k_1 + 1 + k_{s+n-k_1+2-i}, \ldots, s + n - k_1 + 1 + k_{s+n-k_1+2-i}, \ldots, s + n - k_1 + 1 + k_{s+n-k_1+2-i}, n \geq 0).$$

In expansion (25), it is assumed that $k_{s+n} \equiv 0$, and the operator $\mathcal{J}_{s+n}$ is determined by the formula

$$\mathcal{J}_{s+n}(1, \ldots, s + n) f_{s+n} \subset X_{R^{2(s+n)|W_{s+n}}} f_{s+n},$$

where $X_{R^{2(s+n)|W_{s+n}}}$ is a characteristic function of allowed configurations $\mathcal{R}^{3(s+n)} \setminus \mathcal{W}_{s+n}$ of a system of $s + n$ hard spheres.

We give several examples of recurrence relations (25) in terms of scattering cumulants (19). Acting on both sides of equality (25) by the evolution operators $\prod_{i=1}^{s} \mathcal{X}_{i}(t, i)^{-1}$, we obtain

$$\mathcal{H}_1(t, \{Y\}) = \mathcal{V}_1(t, \{Y\}),$$

$$\mathcal{H}_2(t, \{Y\}, s + 1) = \mathcal{V}_2(t, \{Y\}, s + 1) + \mathcal{V}_1(t, \{Y\}) \sum_{i_1=1}^{s} \mathcal{H}_2(t, i_1, s + 1),$$

where the operator $\mathcal{H}_{1+n}(t)$ is the $(1 + n)$th-order scattering cumulant (19).
We note that solutions of recurrence relations (25) are given by expansions (18). As a result of the application of kinetic cluster expansions (25) to the series expansion on the right-hand side of equality (24), the following equality is true (see appendix)

\[
\sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathbb{R}^3 \times \mathbb{R}^3} dx_{x+1} \cdots dx_{x+n} \mathcal{A}_{1+n}^* (t, \{Y\}, X \setminus Y) \mathcal{X}_{\mathbb{R}^{n+1+\nu}} \prod_{i=1}^{s+n} F_{1}^{0} (x_i)
\]

\[
= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathbb{R}^3 \times \mathbb{R}^3} dx_{x+1} \cdots dx_{x+n} \mathcal{B}_{1+n} (t, \{Y\}, X \setminus Y) \prod_{i=1}^{s+n} F_{1} (t, x_i),
\]

(27)

where the \((n + 1)\)th-order generating evolution operator \(\mathcal{B}_{1+n} (t)\) is a solution of recurrence relations (25); i.e., it is determined by formula (18) and the function \(F_1 (t)\) is represented by series expansion (20).

Thus, equality (23) is valid.

In the case of initial data (2) specified by the additive-type marginal observables, i.e. \(B^{(1)} (0) = (0, b_1, 0, \ldots)\), according to solution expansion (13), equality (16) takes the form

\[
\langle B^{(1)} (t) | F^c \rangle = \int_{\mathbb{R}^3 \times \mathbb{R}^3} dx_1 b_1 (x_1) F_1 (t, x_1),
\]

(28)

where the one-particle marginal distribution function \(F_1 (t)\) is determined by series (20). This equality is proven in a similar manner to the first step (24) of the proof of equality (24).

The validity of equality (16) in the case of the general type of marginal observables is proven in much the same way as the validity of equalities (23) and (28).

3.3. The derivation of the non-Markovian Enskog kinetic equation with inelastic collisions

Next, we establish that the one-particle marginal distribution function defined by series (20) is governed by the non-Markovian Enskog kinetic equation (21).

In view of the validity of the following equalities for cumulants of adjoint semigroups of hard spheres for \(f \in L^1_0 (\mathbb{R}^3 \times \mathbb{R}^3)\) in the sense of the norm convergence on the space \(L^1 (\mathbb{R}^3 \times \mathbb{R}^3)\)

\[
\lim_{t \to 0} \frac{1}{t} \mathcal{A}_{1}^* (t, 1) f_1 (x_1) = \mathcal{L}^*_1 (1) f_1 (x_1),
\]

\[
\lim_{t \to 0} \frac{1}{t} \int_{\mathbb{R}^3 \times \mathbb{R}^3} dx_2 \mathcal{A}_{2}^* (t, 1, 2) f_2 (x_1, x_2) = \int_{\mathbb{R}^3 \times \mathbb{R}^3} dx_2 \mathcal{L}^*_2 \mathcal{L}^*_1 (1, 2) f_2 (x_1, x_2),
\]

\[
\lim_{t \to 0} \frac{1}{t} \int_{\mathbb{R}^3 \times \mathbb{R}^3} dx_2 \cdots dx_{n+1} \mathcal{A}_{n+1+\nu}^* (t, 1, \ldots, n+1) f_{n+1} = 0, \quad n \geq 2,
\]

where the operators \(\mathcal{L}^*_1 (1)\) and \(\mathcal{L}^*_2 \mathcal{L}^*_1 (1, 2)\) are defined by formulas (3) and (8) respectively, as a result of the differentiation over the time variable of function (20) in the sense of pointwise convergence on the space \(L^1 (\mathbb{R}^3 \times \mathbb{R}^3)\) we obtain

\[
\frac{\partial}{\partial t} F_1 (t, x_1) = - \langle p_1, \frac{\partial}{\partial q_1} \rangle F_1 (t, x_1) + \int_{\mathbb{R}^3 \times \mathbb{R}^3} dx_2 \mathcal{L}^*_2 \mathcal{L}^*_1 (1, 2) \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathbb{R}^3 \times \mathbb{R}^3} dx_3 \cdots dx_{n+2} \mathcal{A}_{n+1+\nu}^* (t, 1, 2, 3, \ldots, n+2) \prod_{i=1}^{n+2} F_{1}^{0} (x_i) \mathcal{X}_{\mathbb{R}^{n+1+\nu}} \prod_{i=1}^{n+2} F_{1}^{0} (x_i)
\]

(29)

To represent the second term on the right-hand side in this equality in terms of one-particle marginal distribution function (20), we expand cumulants \(\mathcal{A}_{n+1+\nu}^* (t, 1, 2, 3, \ldots, n+2)\) into
kinetic cluster expansions (25) in the case of \( s = 2 \). Then we transform the series over the summation index \( n \) and the sum over the index \( k_l \) to the two-fold series. As a result, it holds

\[
\sum_{n=0}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^3 \times \mathbb{R}^3)^n} dx_3 \ldots dx_{n+2} \mathcal{A}_{1+n}^{(t)}(t, \{1, 2\}, 3, \ldots, n + 2) \mathcal{X}_{(2^{n+2})-n2} \prod_{i=1}^{n+2} F_i^{(0)}(x_i)
\]

\[
= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k_1=0}^{\infty} \int_{(\mathbb{R}^3 \times \mathbb{R}^3)^{n+k_1}} dx_3 \ldots dx_{n+2+k_1} \mathcal{U}_{1+n}^{(t)}(t, \{1, 2\}, 3, \ldots, n + 2) \sum_{k_2=0}^{k_1} \frac{1}{k_1! k_2! (k_1 - k_2)!} \prod_{j=1}^{n+2} \mathcal{A}_{1+k_{1,2}}^{(t)}(t, l, n + 3 + k_n + 4 - i, \ldots, n + 2 + k_n + 3 - i) \prod_{j=1}^{n+2+k_1} F_i^{(0)}(x_j).
\]

According to equality (27) in the case of \( s = 2 \), the last series in equality (30) can be expressed in terms of series expansion (20) for a one-particle marginal distribution function.

We treat the constructed identity for a one-particle distribution function as the kinetic equation for a hard sphere system with inelastic collisions. We refer to this evolution equation as the non-Markovian Enskog kinetic equation.

Hence, for additive-type marginal observables, the non-Markovian Enskog kinetic equation (21) is dual to the dual BBGKY hierarchy for hard spheres with inelastic collisions (1) with respect to bilinear form (14).

Thus, we establish the validity of statement (16).

### 3.4. Some properties of the collision integral

We represent the collision integral \( \mathcal{I}_E \) of the non-Markovian Enskog equation (21) in the form of an expansion with respect to the Boltzmann–Enskog collision integral

\[
\mathcal{I}_E = \sigma^2 \int_{\mathbb{R}^3 \times \mathbb{R}^3} dp_2 dp_1 \mathcal{E}(\eta, (p_1 - p_2)) \left( \frac{1}{(1 - 2s)^2} F_1(t, q_1, p_1^s) F_1(t, q_1 - \sigma \eta, p_2^s) - F_1(t, q_1, p_1) F_1(q_1 + \sigma \eta, p_2) \right),
\]

where the momenta \( p_1^s, p_2^s \) are defined by equalities (9). We observe that such expansion of the collision integral \( \mathcal{I}_E \) is given in terms of the marginal correlation functionals

\[
G_2(t, x_1, x_2 \mid F_1(t)) = F_2(t, x_1, x_2 \mid F_1(t)) - F_1(t, x_1) F_1(t, x_2).
\]

Indeed, in view of the validity of the equality

\[
(\mathcal{A}_1(t, 1^2, 2^2) - I) F_1(t, q_1, p_1^s) F_1(t, q_1 \pm \sigma \eta, p_2^s) = 0,
\]

where we use notations adopted to the conventional notation of the Enskog collision integral, we have

\[
\mathcal{A}_1(t, 1^2, 2^2) = \mathcal{A}_1(t, 1^2, 2^2) - I) F_1(t, q_1, p_1^s) F_1(t, q_1 \pm \sigma \eta, p_2^s) = 0,
\]
\[ \mathcal{I}_E = \mathcal{I}_{BE} + \sum_{n=1}^{\infty} \mathcal{I}_E^{(n)} = \mathcal{I}_{BE} + \sigma^2 \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\mathbb{R}^3 \times \mathbb{R}^3} dp_1 dq_1 \cdots dq_{n+2} d\rho_{n+2}(\eta, (p_1 - p_2)) \times \left( \frac{1}{1 - 2\epsilon^2} \mathcal{V}_{1+\sigma}(t, \{1^0, 2^0\}, 3, \ldots, n + 2) F_1(t, q_1, p_{1}) \right) \times F_1(t, q_1 - \sigma \eta, p_2) \prod_{i=3}^{n+2} F_1(t, q_i, p_i) - \mathcal{V}_{1+\sigma}(t, \{1, 2^+\}, 3, \ldots, n + 2) \times F_1(t, q_1 + \sigma \eta, p_2) \prod_{i=3}^{n+2} F_1(t, q_i, p_i) \right). \] (31)

We note that the properties of a solution of kinetic equation (21) for hard spheres with inelastic collisions are determined by the properties of generating operators of collision integral expansion (31).

For the abstract Cauchy problem of the non-Markovian Enskog kinetic equation (21) in the space of integrable functions \(L^1(\mathbb{R}^3 \times \mathbb{R}^3)\), the following statement is true.

A global in-time solution of the Cauchy problem of the non-Markovian Enskog equation (21) is determined by function (20). If \(\|F_1^0\|_{L^1(\mathbb{R}^3 \times \mathbb{R}^3)} < e^{-10}(1 + e^{-9})^{-1}\), then for \(F_1^0 \in L^1(\mathbb{R}^3 \times \mathbb{R}^3)\) function (20) is a strong solution and for arbitrary initial data \(F_1^0 \in L^1(\mathbb{R}^3 \times \mathbb{R}^3)\) it is a weak solution.

We also note that on the basis of the derived non-Markovian Enskog equation (21) we can formulate the Markovian Enskog equation with inelastic collisions. The Markovian approximation means that the generating operators of the collision integral \(\mathcal{I}_E\) must be replaced on the limit operators \(\mathcal{V}_{1+\sigma}(\{1, 2\}, 3, \ldots, n + 2) = \lim_{\epsilon \to 0} \mathcal{V}_{1+\sigma}(\epsilon^{-1}t, \{1, 2\}, 3, \ldots, n + 2)\), where \(\epsilon > 0\) is the suitable scale parameter [14].

4. On the Boltzmann–Grad asymptotic behavior of one-dimensional granular gases

It is well known that in the Boltzmann–Grad scaling limit [10, 11], the dynamics of a one-dimensional system of hard spheres with elastic collisions are trivial (a free motion or the Knudsen flow) [15]. However, as observed in [32], with this scaling limit, the dynamics of inelastically interacting hard rods are not trivial and are governed by the Boltzmann kinetic equation for granular gases [23, 33]. In this section, the approach to the rigorous derivation of Boltzmann-type equation for one-dimensional granular gases is outlined. We remark that a one-dimensional system of hard spheres, i.e. hard rods, with inelastic collisions exhibits the essential properties of granular gases in view that in a multidimensional case under inelastic collisions, only the normal component of relative velocities dissipates [23].

4.1. The Boltzmann–Grad limit of the non-Markovian Enskog equation with inelastic collisions

We consider the Boltzmann–Grad asymptotic behavior of the non-Markovian Enskog equation solution for a one-dimensional granular gas. In this case, for \(t \geq 0\) in dimensionless form, the
Cauchy problem (21), (22) has the form
\[
\frac{\partial}{\partial t} F_1(t, q_1, p_1) = -p_1 \frac{\partial}{\partial q_1} F_1(t, q_1, p_1) + \int_0^\infty dP P \left( \frac{1}{(1 - 2\epsilon)^2} F_2(t, q_1, \tilde{p}_1^*(p_1, P), q_1 - \epsilon, p_2^*(p_1, P) \mid F_1(t)) \right) \]
\[
+ \int_0^\infty dP P \left( \frac{1}{(1 - 2\epsilon)^2} F_2(t, q_1, \tilde{p}_1^*(p_1, P), q_1 + \epsilon, p_2^*(p_1, P) \mid F_1(t)) \right)
\]
where \( \epsilon > 0 \) is a scaling parameter (the ratio of a hard sphere diameter (the length) \( \sigma > 0 \) to the mean free path), the collision integral is determined by the marginal functional of the state (17) in the case of \( s = 2 \), and the expressions
\[
p_1^*(p_1, P) = p_1 - P + \frac{\epsilon}{2\epsilon - 1} P,
\]
\[
p_2^*(p_1, P) = p_1 - \frac{\epsilon}{2\epsilon - 1} P
\]
are transformed pre-collision momenta in a one-dimensional space.

If initial one-particle marginal distribution functions satisfy the following condition:
\[
|F_1^{x,0}(x_1)| \leq C e^{-\frac{\beta}{\epsilon} p_1^2},
\]
where \( \beta > 0 \) is a parameter and \( C < \infty \) is a constant, then every term of the series
\[
F_1^x(t, x_1) = \sum_{n=0}^\infty \frac{1}{n!} \int_{(\mathbb{R} \times \mathbb{R})^n} dx_2 \ldots dx_{n+1} \mathfrak{A}_{1+n}^*(t) \prod_{i=1}^{n+1} F_1^{x,0}(x_i) \mathfrak{X}_{\mathbb{R} \times \mathbb{R}^n \setminus \mathfrak{W}_{1+n}},
\]
exists, for finite time interval function (35) is the uniformly convergent series with respect to \( x_1 \) from arbitrary compact, and it is determined a weak solution of the Cauchy problem of the non-Markovian Enskog equation (32), (33).

The proof of this statement is based on analogues of the Duhamel equations for cumulants of semigroups of operators \( \mathfrak{A}_{1+n}^*(t) \), \( n \geq 1 \), and estimates established for the iteration series of the BBGKY hierarchy for hard spheres [15, 34, 35].

The following Boltzmann–Grad limit theorem holds. If initial marginal one-particle distribution function \( F_1^{x,0} \) satisfies condition (34) and in the sense of a weak convergence there exists the limit
\[
w- \lim_{\epsilon \to 0} \left( F_1^{x,0}(x_1) - f_1^0(x_1) \right) = 0,
\]
then for a finite time interval there exists the Boltzmann–Grad limit of solution (35) of the Cauchy problem of the non-Markovian Enskog equation for granular gas (32) in the sense of a weak convergence:
\[
w- \lim_{\epsilon \to 0} \left( F_1^x(t, x_1) - f_1(t, x_1) \right) = 0,
\]
where the limit one-particle marginal distribution function is defined by the series which is uniformly convergent on arbitrary compact set series:

\[
f_1(t, x_1) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathbb{R}^n} dx_n \cdots dx_1 \frac{Q^0_{1+n}(t)}{n+1} \prod_{i=1}^{n+1} f_i^0(x_i),
\]

and the generating operator \(Q^0_{1+n}(t)\) is the \((n+1)\)th-order cumulant of adjoint semigroups \(S_n^0(t)\) of point particles with inelastic collisions. An infinitesimal generator of the semigroup of operators \(S_n^0(t)\) is defined as the operator:

\[
(Q^0_{n+1} f_n)(x_1, \ldots, x_n) = -\sum_{j=1}^{n} p_j \frac{\partial}{\partial q_j} f_n(x_1, \ldots, x_n)
\]

\[
+ \sum_{j=1}^{n} \sum_{j=0}^{n} [p_{j} - p_{j}^\circ] \left( \frac{1}{(1 - 2\varepsilon)^2} f_n(x_1, \ldots, x_j^\circ, \ldots, x_{j+1}^\circ, \ldots, x_n) \right)
\]

\[- f_n(x_1, \ldots, x_n) \delta(q_{j} - q_{j}),
\]

where \(x_j^\circ \equiv (q_j, p_j^\circ)\), and the pre-collision momenta \(p_j^\circ\) are determined by the following expressions:

\[
p_j^\circ = p_j + \frac{\varepsilon}{2\varepsilon - 1} (p_j - p_j).
\]

\[
p_j^\circ = p_j - \frac{\varepsilon}{2\varepsilon - 1} (p_j - p_j).
\]

If \(f_1^0\) satisfies condition (34), then for \(t \geq 0\) the limit one-particle distribution function represented by series (38) is a weak solution of the Cauchy problem of the Boltzmann-type kinetic equation of point particles with inelastic collisions

\[
\frac{\partial}{\partial t} f_1(t, q, p) = -p \frac{\partial}{\partial q} f_1(t, q, p) + \int_{-\infty}^{+\infty} dp_1 |p - p_1|
\]

\[
\times \left( \frac{1}{(1 - 2\varepsilon)^2} f_1(t, q, p^\circ) f_1(t, q, p_1^\circ) - f_1(t, q, p) f_1(t, q, p_1) \right) + \sum_{n=1}^{\infty} T_n^{(n)}.
\]

In kinetic equation (39), the remainder \(\sum_{n=1}^{\infty} T_n^{(n)}\) of the collision integral is determined by the expressions \(T_0^{(n)}\), which have a similar structure to the expressions \(T_E^{(n)}\) from series expansion (31)}
where the generating operators $\mathcal{V}_{1+n}(t) \equiv \mathcal{V}_{1+n}(t, [1, 2], 3, \ldots, n+2), n \geq 0$, are represented by expansions (18) with respect to the cumulants of semigroups of scattering operators of point hard rods with inelastic collisions in a one-dimensional space

$$\hat{S}_{n}^{0}(t, 1, \ldots, n) \equiv S_{n}^{\alpha, 0}(t, 1, \ldots, s) \prod_{i=1}^{n} S_{1}^{\alpha, 0}(t, i)^{-1}. \quad (40)$$

In fact, the series expansions for the collision integral or solution (35) of the non-Markovian Enskog equation for a granular gas are represented as the power series over the density, so that the terms $I_{n}^{0}, n \geq 1$, of the collision integral of the Boltzmann kinetic equation (39) are corrections with respect to the density to the Boltzmann collision integral for a one-dimensional granular gas, as claimed previously [23, 33].

We remark that since the scattering operator of point hard rods is an identity operator in the limit of elastic collisions, i.e. in the limit $\varepsilon \to 0$, the collision integral of the Boltzmann kinetic equation (39) in a one-dimensional space is equal to zero. In the quasi-elastic limit [33], the limit one-particle marginal distribution function satisfies the nonlinear friction kinetic equation for one-dimensional granular gases [32, 33].

### 4.2. On the propagation of a chaos in granular gases

Taking into consideration the results (37) on the Boltzmann–Grad asymptotic behavior of the non-Markovian Enskog equation (21), for marginal functionals of the state (17) in a one-dimensional space, the following statement is true.

For a finite time interval in the sense of a weak convergence of the space of bounded functions for marginal functionals of the state (17), it holds

$$w-\lim_{\varepsilon \to 0} (F_{s}(t, x_{1}, \ldots, x_{s} \mid F_{1}(t)) - f_{s}(t, x_{1}, \ldots, x_{s} \mid f_{1}(t))) = 0, \quad s \geq 2,$$

where the limit marginal functionals $f_{s}(t \mid f_{1}(t)), s \geq 2$, with respect to limit one-particle distribution function (38) are determined by the series expansions with a structure similar to series (17) and the generating operators represented by expansions (18) over the cumulants of semigroups of scattering operators (40) of point hard rods with inelastic collisions in a one-dimensional space.

In the case of a hard rod system with elastic collisions, the limit marginal functionals are products of the limit one-particle distribution function, which describes the free motion of point hard rods, which means the propagation of initial chaos.

Thus, in the Boltzmann–Grad scaling limit, solution (35) of the non-Markovian Enskog equation (32) is governed by the Boltzmann equation for a one-dimensional granular gas (equation (39)). The limit marginal functionals of the state are represented by the corresponding series expansions with respect to the limit one-particle distribution function (38), which describes how the initial chaos propagates in one-dimensional granular gases.

We also point out that in a multidimensional space, the Boltzmann–Grad asymptotic behavior of the non-Markovian Enskog equation (21) is similar to the Boltzmann–Grad asymptotic behavior of a hard sphere system with elastic collisions [15]; namely it is governed by the Boltzmann equation for a granular gas [7] and the limit marginal functionals of the state are represented by the products of its solution, which means the propagation of initial chaos.

### 5. Conclusion

Within the framework of the nonequilibrium grand canonical ensemble, the origin of the microscopic description of the evolution of observables of a hard sphere system with inelastic
collisions was considered. In case of initial states (15) specified by a one-particle distribution function solution (10), the Cauchy problem of the dual BBGKY hierarchy with hard sphere inelastic collisions (1), (2) and a solution of the Cauchy problem of the non-Markovian Enskog equation for granular gases (21), (22) together with marginal functionals of the state (17), give two equivalent approaches to the description of the evolution of a hard sphere system with inelastic collisions. In fact, the rigorous justification of the Enskog kinetic equation for granular gases is a consequence of the validity of equality (16).

We note that the structure of collision integral expansion (31) of the non-Markovian Enskog equation for granular gases (21) is such that the first term of this expansion is the Boltzmann–Enskog collision integral and the next terms describe all possible correlations connected with the forbidden configurations.

One of the advantages of the developed approach is the possibility to construct kinetic equations in scaling limits, involving correlations at initial time which can characterize the condensed states of a hard sphere system with inelastic collisions.

We emphasize that the approach to the derivation of the Boltzmann equation with inelastic collisions over the new evolution operators (18) and the function (19) is represented by series expansion (20).

Finally, we remark that the developed approach is also related to the problem of a rigorous derivation of the non-Markovian kinetic-type equations from underlying many-particle dynamics, which make it possible to describe the memory effects of granular gases.

Appendix

We give a sketch of the proof of equality (27), i.e. the validity of the equality

\[ \sum_{n=0}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^3 \times \mathbb{R}^3)^n} dx_{t+1} \ldots dx_{t+n} \mathcal{A}_{t+n}^{*+n}(t, \{Y\}, X \setminus Y) \mathcal{X}_{t+n}^{n-1}(\{Y\}) i! \prod_{i=1}^{x+n} F_i^0(x_i) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^3 \times \mathbb{R}^3)^n} dx_{t+1} \ldots dx_{t+n} \mathcal{Y}_{t+n}(t, \{Y\}, X \setminus Y) i! \prod_{i=1}^{x+n} F_i(t, x_i), \]

where the \((n+1)\)th-order generating evolution operator \(\mathcal{Y}_{t+n}(t)\) is determined by formula (18) and the function \(F_i(t)\) is represented by series expansion (20).

We expand the cumulants \(\mathcal{A}_{t+n}^*(t), n \geq 0\), of adjoint semigroups of hard spheres with inelastic collisions over the new evolution operators \(\mathcal{Y}_{t+n}(t)\), \(n \geq 0\), into kinetic cluster expansions (25), then in consequence of the representation of series expansion over the summation index \(n\) and the sum over the summation index \(k_1\) as a two-fold series, we derive

\[ \sum_{n=0}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^3 \times \mathbb{R}^3)^n} dx_{t+1} \ldots dx_{t+n} \mathcal{A}_{t+n}^{*+n}(t, \{Y\}, X \setminus Y) \mathcal{X}_{t+n}^{n-1}(\{Y\}) i! \prod_{i=1}^{x+n} F_i^0(x_i) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^3 \times \mathbb{R}^3)^n} dx_{t+1} \ldots dx_{t+n} \mathcal{Y}_{t+n}(t, \{Y\}, X \setminus Y) \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \sum_{k_3=0}^{\infty} \ldots \sum_{k_{n+1}=0}^{\infty} \frac{1}{k_1! (k_1 + k_2)! \ldots (k_1 + k_2)!} \int_{(\mathbb{R}^3 \times \mathbb{R}^3)^{k_1}} dx_{t+k_1+1} \]

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... \sum_{i=1}^{n+s} \prod_{k=0}^{n+s+k_i} \mathcal{X}_1^{*+k_{i+1}+1-k_{i+2}+2-}(t, i, n + s + 1 + k_{n+s+2}+\ldots, n + s + k_{n+s+k_{i+1}-i})\) \\
\sum_{q_{n+s+k_{i+1}-i}}^{n+s+k_{i+1}+1} F_1^0(x_j, \mathcal{X}_1^{*+k_{i+1}+1-k_{i+2}+2-}(q_i, q_{n+s+k_{i+1}-i}, \ldots, q_{n+s+k_{i+1}-i}), \ldots) \\
F_1(t, x_i) = \sum_{k_1=0}^{k_4} \sum_{k_2=0}^{k_3} \sum_{k_3=0}^{k_2} \sum_{k_4=0}^{k_1} \frac{1}{k_1! k_2! k_3! k_4!} (k_{n+s+1} + k_{n+s+2} + \ldots + k_{n+s+k_{i+1}-i})! \ldots (k_4 - k_3)!
\int_{(\mathbb{R}^3)^4} dx_{n+s+k_{i+1}+1} \ldots dx_{n+s+k_{i}} \prod_{j=1}^{n+s+k_{i+1}+1} \mathcal{X}_1^{*+k_{i+1}+1-k_{i+2}+2-}(t, i, n + s + 1 + k_{n+s+2}+\ldots, n + s + k_{n+s+k_{i+1}-i})\) \\
\sum_{q_{n+s+k_{i+1}-i}}^{n+s+k_{i+1}+1} F_1^0(x_j, \mathcal{X}_1^{*+k_{i+1}+1-k_{i+2}+2-}(q_i, q_{n+s+k_{i+1}-i}, \ldots, q_{n+s+k_{i+1}-i}), \ldots) \\
(A.1)

in obtained expansion the series over the index $k_1$ can be expressed in terms of series expansion (20) for a one-particle marginal distribution function.
Thus, equality (27) is true.

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