Quasi adiabatic dynamics of energy eigenstates for a solvable quantum system at finite temperature

Takaaki Monnai

Department of Materials and Life Sciences, Seikei University, Tokyo 180-8633, Japan
E-mail: monnai@st.seikei.ac.jp

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Abstract. It is a fundamental problem to characterize nonequilibrium processes. For a moving one-dimensional potential, we explore the nonequilibrium dynamics of the initial energy eigenstates for a confined quantum system interacting with a large reservoir. For concreteness, we investigate a dragged harmonic oscillator linearly interacting with an assembly of harmonic oscillators, and explore the deviation from adiabatic processes by rigorously calculating the so-called persistent amplitude. In particular, we show that the phase of the persistent amplitude is considered to be common both for the ground and excited states. Also, we can define the quasi adiabatic processes in a well-defined double limit of small perturbation and a sufficiently long time in terms of the phase and absolute value of the persistent amplitude.

Keywords: rigorous results in statistical mechanics
Quasi adiabatic dynamics of energy eigenstates for a solvable quantum system at finite temperature

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1. Introduction

Recently, considerable attention has been paid to the characterization of the nonequilibrium processes of many-body systems. Remarkable progress includes some thermodynamic relations such as the fluctuation theorems [1–7], energetics of mesoscopic systems [8–11, 13], relaxation of thermally isolated quantum systems [14–16], to name but a few. By calculating the fidelity or so-called persistent amplitude, we can evaluate the relaxation time of isolated quantum many-body systems [17–19]. On the other hand, the energetics of mesoscopic systems concerns the fluctuation of entropy production or work done [9, 11, 12]. For example, the work distribution characterizes how far the system is out of equilibrium for nonadiabatic processes in terms of the spectral fluctuation. There are many other measures of the sort of distance from equilibrium such as the relative entropy, fidelity, deviation from adiabatic theorem for finite external perturbation, and so on. The deviation from adiabatic theorem is considered to be relevant for characterizing the slowly varying processes. In this article, we explore how the time evolution of the initial eigenstates deviates from those of adiabatic processes for an externally perturbed quantum system interacting with a large reservoir by rigorously calculating the persistent amplitude. For concreteness, we consider a uniformly dragged harmonic potential interacting with a reservoir which is an assembly of an infinite number of harmonic oscillators [20–24]. Such a model is useful to discuss a quantum system coupled to an environment: quantum Langevin equation [22, 23], atoms interacting with an electric field [24], exact case studies of the work distribution and quantum fluctuation theorem [7, 25].

This paper is organized as follows. In section 2, we describe our model. In section 3, we calculate the persistent amplitude of the ground state in terms of the Wick’s theorem. In section 4, we explore the excited states by considering the case of finite temperature. Section 5 is devoted to a summary.
2. Model

In this section, we describe our model, and diagonalize the Hamiltonian. We consider a harmonic potential linearly interacting with an assembly of harmonic oscillators [7, 23–25] as a genuine model of a system interacting with a reservoir. We externally control the center of the potential \( f(t) \). Then, the total Hamiltonian is

\[
\hat{H}(t) = \frac{\hat{p}^2}{2m} + \frac{k}{2} \left( (\hat{q} - f(t))^2 + \frac{1}{2} \int d\lambda (\hat{p}_\lambda^2 + \omega_\lambda^2 (\hat{q}_\lambda - \kappa_\lambda \hat{q}))^2 \right).
\]

Here, \( m \) is the mass and \( k \) stands for the spring constant. Also, \( \hat{q} \) and \( \hat{p} \) are the position and momentum of a particle, and \( \hat{q}_\lambda \) and \( \hat{p}_\lambda \) are those of the reservoir degrees of freedom. These operators satisfy the canonical commutation relations \([\hat{q}, \hat{p}] = i\hbar\), \([\hat{q}_\lambda, \hat{p}_\lambda] = i\hbar\delta(\lambda - \lambda')\), and the system variables commute with those of the reservoir. We assume that the coupling strength \( \kappa_\lambda \) between the system and the reservoir is weak so that the Hamiltonian does not admit any bound states. It is convenient to define the normal mode for the reservoir, \( \hat{a}_\lambda = \frac{1}{\sqrt{2\omega_\lambda}} (\omega_\lambda \hat{q}_\lambda + i\hat{p}_\lambda) \). Then, the total Hamiltonian is diagonalized as

\[
\hat{H}(t) = \hat{H}_0 + \hat{V}_I(t) = \int d\lambda \hbar \omega_\lambda \hat{A}^\dagger_\lambda \hat{A}_\lambda + \int \frac{\hbar}{2} k f(t) \int d\lambda \left( \frac{\kappa_\lambda \omega_\lambda \sqrt{\omega_\lambda}}{\eta_-(\omega_\lambda)} \hat{A}_\lambda + \frac{\kappa_\lambda \omega_\lambda \sqrt{\omega_\lambda}}{\eta_+(\omega_\lambda)} \hat{A}^\dagger_\lambda \right) + \frac{k}{2} f(t)^2.
\]

Here, the normal mode is a linear combination of canonical operators \( \hat{A}_\lambda = \hat{a}_\lambda - \frac{\kappa_\lambda \omega_\lambda \sqrt{\omega_\lambda}}{\eta_+(\omega_\lambda)} \left\{ \frac{m \omega_\lambda \hat{q} + i\hbar}{\sqrt{2m}} - \int d\lambda' \left( \frac{\kappa_{\lambda'} \omega_{\lambda'} \sqrt{\omega_{\lambda'}}}{\omega_{\lambda'} - \omega_\lambda' + \frac{\pi k^2}{\hbar^2}} + \frac{\kappa_{\lambda'} \omega_{\lambda'} \sqrt{\omega_{\lambda'}}}{\omega_{\lambda'} + \omega_\lambda'} \right) \right\} \), and we introduced the dispersion function \( \eta_{\pm}(z) = mz^2 - k - \int d\lambda' \kappa_{\lambda'}^2 \omega_{\lambda'}^2 - \int d\lambda' \frac{\kappa_{\lambda'}^2 \omega_{\lambda'}^2}{\omega_{\lambda'}^2 - \omega_\lambda' \pm \frac{\pi k^2}{\hbar^2}} \). Note that we omitted the vacuum energy, which is time-independent. The normal mode is obtained by diagonalizing the Hamiltonian \( \hat{H} \) in the absence of \( f(t) \) in (1) as \( \hat{H} = \int d\lambda \hbar \omega_\lambda \hat{A}_\lambda^\dagger \hat{A}_\lambda \). We remark that the frequency \( \omega_\lambda \) of the normal mode is the same as that of the reservoir [26]. This point is in contrast to the case of discrete spectrum where the Hamiltonian is diagonalized by normal modes whose frequencies are zeros of the dispersion function [20, 21]. It is straightforward to verify that the normal mode satisfies the canonical commutation relation \([\hat{A}_\lambda, \hat{A}^\dagger_{\lambda'}] = \delta(\lambda - \lambda') \). To derive equations (2) from (1), we also used the inversion \( \hat{q} = -\sqrt{\frac{\hbar}{2}} \int d\lambda (\frac{\kappa_\lambda \omega_\lambda \sqrt{\omega_\lambda}}{\eta_-(\omega_\lambda)} \hat{A}_\lambda + \frac{\kappa_\lambda \omega_\lambda \sqrt{\omega_\lambda}}{\eta_+(\omega_\lambda)} \hat{A}^\dagger_\lambda)^{\dagger} \). To calculate the persistent amplitude in the following sections, it is convenient to use the interaction picture. The interaction Hamiltonian in the interaction picture is given as

\[
\hat{H}_I(t) = \sqrt{\frac{\hbar}{2}} k f(t) \int d\lambda \left( \frac{\kappa_\lambda \omega_\lambda \sqrt{\omega_\lambda}}{\eta_-(\omega_\lambda)} \hat{A}_\lambda e^{-i\omega_\lambda t} + \frac{\kappa_\lambda \omega_\lambda \sqrt{\omega_\lambda}}{\eta_+(\omega_\lambda)} \hat{A}^\dagger_\lambda e^{i\omega_\lambda t} \right) + \frac{k}{2} f(t)^2.
\]

The dispersion functions \( \eta_\pm(z) \) contain a term \( \pi \int d\lambda \kappa_\lambda^2 \delta(z - \omega_\lambda) \), and in the absence of the interaction between the system and reservoir the weak coupling limit \( \kappa_\lambda \rightarrow 0 \) should be taken after the contour integration as shown in appendix.

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3. Persistent amplitude

In this section, we recapitulate the way to calculate the vacuum persistent amplitude for the ground state $|0\rangle$. At $t = 0$, we assume that the interaction and Heisenberg pictures coincide, and their vacuum states are common. Instead of the adiabatic switching-off of the interaction in the scattering theory, we move the center of the trap in a cyclic way $f(T) = f(0)$ for a long time $T$ as shown in figure 1. The sign of the velocity $\dot{f}(t)$ is changed at $t = \frac{T}{2}$, however, thermodynamic properties of the stationary state are considered to depend only on the absolute value $|\dot{f}(t)|$. For sufficiently slow processes, the initial vacuum eigenstate $|0\rangle$ is considered to evolve toward a state $U(t)|0\rangle$ which is close to the vacuum eigenstate of $\hat{H}(t)$ according to the adiabatic theorem [28]. Note that the adiabatic theorem requires non-degenerated eigenstates, while our calculation similarly holds for the case of a discrete spectrum by replacing the reservoir Hamiltonian which measures the difference between the initial and final states. To illustrate the significance of the vacuum persistent amplitude, first we consider a general adiabatic process described by a cyclic Hamiltonian $\hat{H}_I(t)$, where (4) is equal to

$$e^{-\frac{1}{\hbar}\Delta ET},$$

with the use of (3). The absolute value of (5) is unity. Here, $\Delta E$ is the difference between the eigenenergies of the ground states at the initial time $t = 0$ and intermediate time $t$ ($0 \ll t \leq T$). Indeed, we have from (3) $\langle 0|e^{\frac{1}{\hbar}\int_0^T dt \hat{H}(t)}|0\rangle = e^{\frac{1}{\hbar}\int_0^T dt \hat{H}(t)}|0\rangle$ with the eigenenergy $E_0$ of $\hat{H}(0)$, and the adiabatic evolution makes the initial state $|0\rangle$ to the corresponding ground state of $\hat{H}(t)$ for $0 \ll t \leq T$. For quasi adiabatic non-equilibrium processes, however, the state at time $t$ ($0 < t \leq T$) is out of equilibrium and not an energy eigenstate. By quasi adiabatic, we consider small but finite $v$ which yields deviation from the adiabatic theorem. Quantitative characterization of the quasi adiabatic process is described in section 5. The absolute value of the persistent amplitude (4) is close to unity for quasi adiabatic processes. We note that the eigenenergies of $\hat{H}(t)$ and $\hat{H}(0)$ are the same, since they are related by a unitary operator $\hat{H}(t) = D(t)\hat{H}(0)D(t)^\dagger$ with $D(t) = e^{-\frac{1}{\hbar}\int \frac{d\lambda}{\pi \lambda^2} \int_0^T f(t)A_{\lambda} - \frac{\Delta \lambda}{2}\int_0^T f(t)A_{\lambda}^2}$. This invariance of the eigenenergies is specific to our model, however, it is compatible with the adiabatic theorem, i.e. $\Delta E \to 0$ for $v \to 0$. The phase is nonzero for nonadiabatic processes.
The vacuum persistent amplitude is further calculated as

$$
\langle 0 | T \{ e^{-\frac{i}{\hbar} \int_0^T \hat{V}(t) dt} \} | 0 \rangle e^{-\frac{i}{2\hbar} \int_0^T dt f(t)^2},
$$

where we defined the interaction Hamiltonian minus the energy stored in the harmonic potential

$$
\hat{V}(t) = \hat{H}_I(t) - \frac{k}{2} f(t)^2.
$$

We use the Wick’s theorem \[27\]

$$
T \{ e^{-\frac{i}{\hbar} \int_0^T \hat{V}(t) dt} \} = N \{ e^{-\frac{i}{\pi} \int_0^T \hat{V}(t) dt} \} e^{-\frac{1}{2\hbar^2} \int_0^T dt_1 \int_0^T dt_2 \langle 0 | T \{ \hat{V}(t_1) \hat{V}(t_2) \} | 0 \rangle}.
$$

Here, \( N \) stands for the normal ordering. Then, the phase and the absolute value of the persistent amplitude \( \Theta \) and \( e^{-\Gamma} \) are related to the propagator \( \langle 0 | T \{ \hat{V}(t_1) \hat{V}(t_2) \} | 0 \rangle \) as

$$
e^{i\Theta} e^{-\Gamma} = e^{-\frac{i}{\pi} \int_0^T dt_1 \int_0^T dt_2 \langle 0 | T \{ \hat{V}(t_1) \hat{V}(t_2) \} | 0 \rangle} e^{-\frac{k}{2\hbar} \int_0^T dt f(t)^2},
$$

since the vacuum expectation value of the normal ordered product is unity. In \(8\), the propagator is calculated as

$$
\langle 0 | T \{ \hat{V}(t_1) \hat{V}(t_2) \} | 0 \rangle = \frac{\hbar k^2}{2} f(t_1) f(t_2) \int d\lambda \frac{\kappa^2 \omega^3}{|\eta_+(\omega\lambda)|^2} e^{-i\omega|t_1-t_2|}.
$$

From equations \(8\) and \(9\), the phase \( \Theta \) is given as

$$
\Theta = \left( \frac{\hbar k^2}{4\hbar} \int_0^T dt_1 \int_0^T dt_2 f(t_1) f(t_2) \int d\lambda \frac{\kappa^2 \omega^3}{|\eta_+(\omega\lambda)|^2} e^{-i\omega|t_1-t_2|} \right) T + \mathcal{O}(1),
$$

where the \( \mathcal{O}(1) \) contribution is negligible for long \( T \) and calculation of the phase shift per unit time. Note that the first term of the right hand side of \(10\) is actually real as shown for our model in the following section.

Figure 1. Cyclic manipulation of the center of the potential. The blue line shows the uniform dragging case \( f(t) = vt \) for \( 0 \leq t \leq \frac{T}{2} \) and \( f(t) = v(T - \frac{t}{2}) \) for \( \frac{T}{2} \leq t \leq T \).
4. Uniform dragging

In this section, we calculate the phase (10) for the uniform dragging case, which is shown in figure 1.

For the case of uniform dragging \( f(t) = vt \) for \( 0 \leq t \leq \frac{T}{2} \) and \( f(t) = v(T - t) \) for \( \frac{T}{2} \leq t \leq T \), the phase is then evaluated as

\[
\Theta = -\frac{k}{24h}v^2T^3 + \frac{k^2}{24h}v^2\int d\lambda \frac{\kappa^2_\lambda \omega^2_\lambda}{|\rho_+(\omega_\lambda)|^2}T^3 + \frac{k^2v^2}{2h}\int d\lambda \frac{\kappa^2_\lambda}{|\rho_+(\omega_\lambda)|^2}T + \mathcal{O}(v^2).
\]

(11)

Remarkably, the absolute value of the persistent amplitude (4)

\[
e^{-\Gamma} = e^{-\frac{2k^2v^2}{\hbar} \int d\lambda \frac{\kappa^2_\lambda}{|\rho_+(\omega_\lambda)|^2} \sin^2 \frac{\omega_\lambda T}{4}}
\]

converges to unity in the quasi static limit \( v \to 0 \), which is consistent with the adiabatic theorem.

With the use of the lemma \( \int d\lambda \frac{\kappa^2_\lambda \omega^2_\lambda}{|\rho_+(\omega_\lambda)|^2} = \frac{1}{k} \) detailed in the appendix, the first and second terms of (11) cancel each other, and we can further calculate the phase as

\[
\Theta = \frac{k^2v^2}{2h} \left( \int d\lambda \frac{\kappa^2_\lambda}{|\rho_+(\omega_\lambda)|^2} T + \mathcal{O}(1) \right).
\]

(13)

5. Finite temperature

In this section, we explore the case of excited states. We show that the phase of the excited states are the same as that of the ground state. For this purpose, let us consider the initial canonical state \( \hat{\rho}_c = \frac{1}{Z}e^{-\beta \hat{H}(0)} \) at an inverse temperature \( \beta \). Here, \( Z = \text{Tr}e^{-\beta \hat{H}(0)} \) is the partition function. We calculate the persistent amplitude for the canonical state \( \langle T(e^{-\frac{i}{\hbar} \int_0^T \hat{H}_c(t)dt})c \rangle_c = \text{Tr}\hat{\rho}_c T\{e^{-\frac{i}{\hbar} \int_0^T \hat{H}_c(t)dt}\} \). With the use of the initial energy eigenstate \( |E_\mu\rangle \), the persistent amplitude of \( \hat{\rho}_c \) is equal to

\[
\langle T(e^{-\frac{i}{\hbar} \int_0^T \hat{H}_c(t)dt})c \rangle_c = \int d\mu \frac{1}{Z}e^{-\beta E_\mu} \langle E_\mu | T\{e^{-\frac{i}{\hbar} \int_0^T \hat{H}_c(t)dt}\}|E_\mu\rangle,
\]

(14)

where \( \mu \) labels the excited states. In particular, we calculate the phase \( \Theta_\mu \) for \( \langle E_\mu | T\{e^{-\frac{i}{\hbar} \int_0^T \hat{H}_c(t)dt}\}|E_\mu\rangle \).

We note that the normal ordering can be decomposed as

\[
N\{e^{-\frac{i}{\hbar} \int_0^T \hat{V}(t)dt}\} = e^{-\frac{i}{\hbar} \int_0^T \hat{V}^+(t)dt} e^{-\frac{i}{\hbar} \int_0^T \hat{V}^-(t)dt},
\]

(15)

which is achieved by expanding both sides. Here, we introduced field operators \( \hat{V}^+(t) = -\sqrt{\frac{2}{i}}k f(t) \int d\lambda \frac{\kappa^2_\lambda \omega_\lambda}{\rho_+(\omega_\lambda)} \hat{A}_\lambda^1 e^{-i\omega_\lambda t} \) and \( \hat{V}^-(t) = -\sqrt{\frac{2}{i}}k f(t) \int d\lambda \frac{\kappa^2_\lambda \omega_\lambda}{\rho_+(\omega_\lambda)} \hat{A}_\lambda^1 e^{i\omega_\lambda t} \).

We also use a lemma [25, 28, 29]
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\[ \langle e^{\int \lambda T \hat{A}_\lambda} \rangle_c = e^{\int \lambda \frac{\hbar^2}{2m} \coth \left( \frac{\hbar \omega_\lambda}{2} \right)} \] \hspace{0.5cm} (16)

Then, we can calculate the persistent amplitude by applying (16) to \( \hat{V}(t) = \int \lambda T \hat{A}_\lambda \)
with \( \alpha_\lambda = \int_0^T dt \sqrt{\frac{\hbar}{2}} k f(t) \frac{\lambda \omega_\lambda}{\eta_\lambda(\omega_\lambda)} e^{-i \omega_\lambda t} \)

\[ \left\{ \begin{array}{l}
T \{ e^{-\frac{i}{\hbar} \int_0^T \hat{H}_1(t) dt} \} \\
= \langle N \{ e^{-\frac{i}{\hbar} \int_0^T \hat{V}(t) dt} \} \rangle_c e^{-\frac{1}{2 \hbar} \int_0^T dt_1 \int_0^T dt_2 \langle 0|\hat{V}(t_1)\hat{V}(t_2)|0 \rangle e^{-\frac{1}{4 \hbar} \int_0^T dt f(t)^2} \\
= \langle e^{\int \lambda T \hat{A}_\lambda} \rangle_c e^{-\frac{1}{2 \hbar} \int_0^T dt_1 \int_0^T dt_2 \langle 0|\hat{V}(t_1)\hat{V}(t_2)|0 \rangle e^{-\frac{1}{4 \hbar} \int_0^T dt f(t)^2}} \\
= e^{-\frac{1}{2 \hbar} \int f \lambda |\alpha_\lambda|^2 (\coth \frac{\hbar \omega_\lambda}{2} - 1) e^{-\frac{1}{2 \hbar} \int_0^T dt f(t)^2}. \hspace{0.5cm} (17)
\end{array} \right. \]

Here, we used the definition of \( \hat{V} \) below (6), the property of normal ordering (15) and (16). In the last line, the first exponential factor is equal to

\[ e^{\frac{\hbar^2 \omega_\lambda^2}{2} \int d\lambda \frac{\eta_\lambda^2(\omega_\lambda)}{|\eta_\lambda(\omega_\lambda)|^2} \sin^4 \frac{\omega_\lambda T}{4} \left( \coth \frac{\hbar \omega_\lambda}{2} - 1 \right), \hspace{0.5cm} (18) \]

which is real describing the decay of the persistent amplitude for the canonical state and does not contribute to the phase shift. The absolute value is proportional to (18) and is an increasing function of \( \beta \), and a higher temperature requires smaller \( v \) to achieve the adiabatic process. On the other hand, the remaining exponential factors are the same as (8). Therefore, the persistent amplitude of the canonical state is

\[ \left\{ \begin{array}{l}
\langle e^{\int \lambda T \hat{A}_\lambda} \rangle_c \\
= e^{\int \lambda T \hat{A}_\lambda} \frac{\eta_\lambda^2(\omega_\lambda)}{|\eta_\lambda(\omega_\lambda)|^2} T + \mathcal{O}(1)} \\
= e^{-\frac{1}{2 \hbar} \int f \lambda |\alpha_\lambda|^2 (\coth \frac{\hbar \omega_\lambda}{2} - 1) e^{-\frac{1}{4 \hbar} \int_0^T dt f(t)^2}. \hspace{0.5cm} (19)
\end{array} \right. \]

Here, it is remarkable that the phase is independent from \( \beta \) and only the absolute value depends on the inverse temperature. From (14) and (19) and the \( \beta \) independence of the phase for arbitrary \( \beta \), the phase of the individual \( \langle E_\mu | T \{ e^{-\frac{i}{\hbar} \int_0^T \hat{H}_1(t) dt} \} | E_\mu \rangle \) is considered to be equal to that of (13). Therefore, for arbitrary \( v \) the phase \( \Theta_\mu \) is identical for all the excited states \( |E_\mu\rangle \),

\[ \Theta_\mu = \frac{k^2 v^2}{\hbar} \left( \int d\lambda \frac{\eta_\lambda^2(\omega_\lambda)}{|\eta_\lambda(\omega_\lambda)|^2} T + \mathcal{O}(1) \right), \hspace{0.5cm} (20) \]

which is our main result. It is non-trivial that the phase is identical for all the excited states for finite \( v \). For example, this is in marked contrast to the velocity dependence of the absolute value \( e^{-1} \). Moreover, (19) is almost independent from the inverse temperature \( \beta \) in a nontrivial double limit of small perturbation \( v^2 \ll \frac{\hbar^2}{k^2} \) and long time \( T \to \infty \), where \( v^2 T \) is kept constant, and the persistent amplitude is well-approximated by \( e^{i \Theta} \) with (13). Hence, the persistent amplitude \( \langle E_\mu | T \{ e^{-\frac{i}{\hbar} \int_0^T \hat{H}_1(t) dt} \} | E_\mu \rangle \) is independent from the energy scale specified by the inverse temperature \( \beta \) from \( E_\mu \cong \langle \hat{H} \rangle_c \) in this double limit.

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6. Summary

We have explored the dynamics of the energy eigenstates for nonequilibrium processes. In particular, we rigorously calculated the persistent amplitude, which measures a sort of distance between the initial and final states for an arbitrary external perturbation. On the other hand, the absolute value of the persistent amplitude $e^{-\Gamma}$ is an exponentially decaying function of $v$.

In particular, the phase is common for all the excited states, while the absolute value is an increasing function of the inverse temperature. Then, the quasi adiabatic processes qualitatively argued in section 3 are characterized by the persistent amplitude (18) in a well-defined double limit of small perturbation $v \to 0$ and long time $T \to \infty$ with $v^2 T^2$ kept finite. To explore the persistent amplitude for other models is an interesting task for the future.

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Appendix. Proof of the lemma

Let us show the lemma $ \int d\lambda \frac{\kappa^2 \omega^2}{\eta_+(\omega \lambda)} = \frac{1}{k}$. A slightly different calculation is shown in [25]. We note that the dispersion functions satisfy $\eta_-(x) - \eta_+(x) = -\pi i \int d\lambda (x - \omega \lambda) \kappa^2 \omega^2 \lambda$. Then, we can calculate the coefficient as

$$\int d\lambda \frac{\kappa^2 \omega^2}{\eta_+(\omega \lambda)} = \int d\lambda \int_0^\infty dx \delta(x - \omega \lambda) \frac{\kappa^2 \omega^2}{\eta_-(x) \eta_+(x) x}$$

$$= \int d\lambda \int_0^\infty dx \delta(x - \omega \lambda) \frac{\kappa^2 \omega^2}{\eta_-(x) \eta_+(x) x}$$

$$= \int_0^\infty \frac{1}{-\pi i} \left( \frac{1}{\eta_+(x)} - \frac{1}{\eta_-(x)} \right) dx$$

$$= \int_{-\infty}^\infty \frac{1}{-\pi i} \frac{1}{\eta_+(x)} dx$$

$$= \int_C \frac{1}{-\pi i} \frac{1}{\eta_+(z)} dz - \lim_{r \to 0} \int_0^\pi \frac{r e^{i\theta}}{\eta_+(re^{i\theta})} d\theta$$

$$- \lim_{R \to \infty} \int_0^\pi \frac{R e^{i\theta}}{\eta_+(Re^{i\theta})} d\theta$$

$$= \frac{1}{k}.$$  \hfill (A.1)
As shown in figure A.1, the contour $C$ consists of the real axis $[-\infty, -r]$, $[r, \infty]$, and semi circles on the upper-half plane.