Study on Krasnoselskii’s fixed point theorem for Caputo–Fabrizio fractional differential equations

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Abstract
This note is concerned with establishing existence theory of solutions to a class of implicit fractional differential equations (FODEs) involving nonsingular derivative. By using usual classical fixed point theorems of Banach and Krasnoselskii, we develop sufficient conditions for the existence of at least one solution and its uniqueness. Further, some results about Ulam–Hyers stability and its generalization are also discussed. Two suitable examples are given to demonstrate the results.

Keywords: Krasnoselskii’s fixed point theorem; Caputo–Fabrizio fractional differential equations; Hyers–Ulam stability

1 Introduction
FODEs have many applications in real world problems; see [1–3]. The concerned area has been investigated from different aspects in the last several years. These investigations include the existence theory of solutions by the fixed point theory, numerical analysis and stability theory by taking Hadamard, Riemann–Liouville, Caputo, etc., type fractional derivatives (for details, see [4–7]). But recently another form of derivative, called nonsingular type, has attracted much attention from the researchers. The existence theory, together with stability results, has been very well investigated for other FODEs; for details, see [8–10]. The considered differential operator has been introduced in 2015 by Caputo and Fabrizio [11] (in short, we write it as (CFFD)), which replaces the singular kernel by a nonsingular kernel of exponential type. In this research work, we establish the existence theory for the following class of fractional differential equations involving the CFFD:

\[
\begin{bmatrix}
\text{CF}_0^\theta D_t^\alpha u(x) = f(x, u(x), \text{CF}_0^\theta D_t^\alpha u(x)) \quad x \in [0, T] = J, \\
u(0) = u_0, \quad u_0 \in \mathbb{R},
\end{bmatrix}
\]

where \( \theta \in (0, 1) \), \( f : J \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \). The considered differential operator replaces the singular kernel by a nonsingular kernel of exponential type in (1). The mentioned operator has been observed to be more practical than the usual Caputo operator in some problems; for details, see [12–15].
So in this paper, we are using the fixed point theory to obtain some results for the existence and uniqueness of a solution to the considered problem (1). Also the stability theory of Ulam–Hyers type has been properly investigated for ordinary FODEs. Some results in this regards can be traced back in [16–19]. In recent years some remarkable work has been carried out about the mentioned FODEs; see [20–24]. Therefore in this article, we also developed some results about the stability for the proposed problem. Two proper examples are also given in the end.

2 Background materials

Some basic notions and results are provided below.

**Definition 1** ([25, 26]) Letting \( u \in H^1(\mathcal{J}) \), where \( H^1(0, T) \) is a Hilbert space, we define the nonsingular derivatives for \( \theta \in (0, 1] \) as

\[
\mathcal{C}_0^\theta D_x^\theta u(x) = \frac{M(\theta)}{1-\theta} \int_0^x u'(\eta) \exp \left( -\frac{\theta(x-\eta)}{1-\theta} \right) d\eta,
\]

(2)

provided the integral on the right-hand side of (2) converges on \((0, \infty)\), where \( M(\theta) \) is a normalization function with \( M(0) = M(1) = 1 \). Further, if \( u \) does not exist in \( H^1(\mathcal{J}) \), then the listed derivative of fractional order is defined as

\[
\mathcal{C}_0^\theta D_x^\theta u(x) = \frac{M(\theta)}{1-\theta} \int_0^x (u(x) - u(\eta)) \exp \left( -\frac{\theta(x-\eta)}{1-\theta} \right) d\eta,
\]

(3)

provided that the integral on the right-hand side of (3) converges on \((0, \infty)\). Further, let \( \lambda = \frac{1+\xi}{\theta} \), \( \theta \in [0, 1] \), \( \lambda \in [0, \infty] \), and then

\[
\lim_{\lambda \to 0} \frac{1}{\lambda} \exp \left( -\frac{x-\eta}{\lambda} \right) = \delta(x-\eta).
\]

Further,

\[
\lim_{\theta \to 1} \left[ \mathcal{C}_0^\theta D_x^\theta u(x) \right] = \lim_{\theta \to 1} \frac{M(\theta)}{1-\theta} \int_0^x u'(\eta) \exp \left( -\frac{\theta(x-\eta)}{1-\theta} \right) d\eta
\]

\[
= \lim_{\lambda \to 0} \frac{\theta N(\theta)}{\lambda} \int_0^x u'(\eta) \exp \left( -\frac{x-\eta}{\lambda} \right) d\eta
\]

\[
= u(x),
\]

where \( N(\theta) \) is the corresponding normalization term of \( M(\theta) \) with the property \( N(0) = N(\infty) = 1 \).

**Definition 2** ([25, 26]) The nonsingular kernel type fractional integral is given by

\[
\mathcal{C}_0^\theta I_x^\theta u(x) = \frac{(1-\theta)}{M(\theta)} u(x) + \frac{\theta}{M(\theta)} \int_0^x u(\eta) d\eta,
\]

(4)

provided that the integral on right-hand side converges on \((0, \infty)\). Further, if we set \( \theta = 1 \), then \( M(\theta) = 1 \) in (4), and we get the following classical integral:

\[
\lim_{\theta \to 1} \left[ \mathcal{C}_0^\theta I_x^\theta u(x) \right] = \lim_{\theta \to 1} \left[ \frac{(1-\theta)}{M(\theta)} u(x) + \frac{\theta}{M(\theta)} \int_0^x u(\eta) d\eta \right] = \int_0^x u(\eta) d\eta.
\]
Lemma 1 \cite{(11)} Let \( y \in C[0, T] \), then the solution of FODE (5)

\[
\left. \begin{array}{l}
C_0^\theta D_0^\theta u(x) = y(x), \quad x \in [0, T], \ 0 < \theta \leq 1,
\end{array} \right\} \tag{5}
\]

is given as

\[
u(x) = u_0 + D_0[y(x) - y_0] + D_0 \int_0^x y(\eta) \, d\eta,
\]

where \( D_0 = \frac{(1-\theta)}{\Gamma(\theta)} \), \( \tilde{D}_0 = \frac{\theta}{\Gamma(\theta)} \).

Proof Using the definition of \( C_0^\theta I_0^\theta \), (5) implies that

\[
u(x) = c + D_0y(x) + \tilde{D}_0 \int_0^x y(\eta) \, d\eta, \quad c \in \mathbb{R}.
\]

Using the initial condition \( u(0) = u_0 \) and \( y(0) = y_0 \in \mathbb{R} \), from (7), we get \( c = u_0 - D_0y_0 \).

Hence by plugging the value of \( c \) in (7), we get (6). \( \square \)

Remark 1 Henceforth, for simplicity, we use \( C_0^\theta D_0^\theta u(x) = h_u(x) \) for the implicit term in our analysis. Further, for simplicity, we use \( f(0, u(0), h_u(0)) = f_0 \).

3 Main work

Lemma 2 Under the conditions of Lemma 1, the solution of (1) is given by

\[
u(x) = u_0 + D_0[f(x, u(x), h_u(x)) - f_0] + D_0 \int_0^x f(\eta, u(\eta), h_u(\eta)) \, d\eta.
\]

To proceed further, we assume that

\( (C_1) \) There exist \( L_f > 0 \) and \( 0 < M_f < 1 \) such that

\[
|f(x, u, h_u) - f(x, \bar{u}, \bar{h}_u)| \leq L_f|u - \bar{u}| + M_f|h_u - \bar{h}_u|
\]

for all \( u, \bar{u}, h_u, \bar{h}_u \in \mathbb{R} \).

Let \( X = C(J) \) be a Banach space with norm \( \|x\| = \max_{x \in J} |u(x)| \).

Theorem 1 Under the assumption \( (C_1) \), if the condition \( (D_0 + \tilde{D}_0) \frac{L_f}{1 - M_f} < 1 \) holds, then the considered problem (1) has a unique solution.

Proof Define an operator \( S : X \rightarrow X \) by using (8) as

\[
Su(x) = u_0 + D_0[f(x, u(x), h_u(x)) - f_0] + D_0 \int_0^x f(\eta, u(\eta), h_u(\eta)) \, d\eta.
\]

Then for any \( u, \bar{u} \in X \), from (9), we have

\[
\|Su - S\bar{u}\| = \max_{x \in J}|Su(x) - S\bar{u}(x)|
\]

\[
= \max_{x \in J} \left| D_0[f(x, u(x), h_u(x)) - f(x, \bar{u}(x), h_{\bar{u}}(x))] \right|
\]
Hence $S$ is a contraction, therefore $S$ has a unique fixed point. Hence the corresponding problem (1) has a unique solution. □

**Theorem 2** ([27]) Let $E \subset X$ be a closed, convex, and nonempty subset of $X$, and suppose there exist two operators $S_1, S_2$ such that
1. $S_1u_1 + S_2u_2 \in E$ for all $u_1, u_2 \in E$;
2. $S_1$ is a contraction and $S_2$ is compact and continuous.
Then there exists at least one solution $u \in E$ to the operator equation $S_1u + S_2u = u$.

For further analysis, let the given assumption hold:

(C2) There exist constants $a_f, b_f, c_f > 0$ with $0 < c_f < 1$ such that

$$|f(x, u, v)| \leq a_f + b_f |u| + c_f |v|.$$ 

**Theorem 3** Under the assumption (C2), if $0 < D_0 \frac{L_f}{1 - M_f} < 1$ holds, then the considered problem (1) has at least one solution.

**Proof** Let us define two operators from (8) as

$$S_1u(x) = u_0 + D_0 \left[ f(x, u(x), h_u(x)) - f_0 \right]$$

and

$$S_2u(x) = D_0 \int_0^x f(\eta, u(\eta), h_u(\eta)) d\eta.$$ 

Let us define a set $E = \{ u \in X : \| u \| \leq r \}$. Since $f$ is continuous, so is $S_1$, and letting $u, \bar{u} \in E$, from (10), we have

$$\| S_1u - S_1\bar{u} \| = \max_{x \in J} |D_0 \left[ f(x, u(x), h_u(x)) - f(x, \bar{u}(x), h_{\bar{u}}(x)) \right] |$$

$$\leq D_0 L_f \frac{1}{1 - M_f} \| u - \bar{u} \|.$$ 

Hence $S_1$ is a contraction. Next to prove that $S_2$ is compact and continuous, for any $u \in E$, we have from (11)

$$\| S_2u \| = \max_{x \in J} |S_2u(x)| = \max_{x \in J} \left| D_0 \int_0^x f(\eta, u(\eta), h_u(\eta)) d\eta \right|$$

$$\leq D_0 (a_f + b_f r) \frac{1}{1 - c_f} = A,$$
which implies that \( \|S_2u\| \leq A \). Thus \( S_2 \) is bounded. Next, letting \( x_1 < x_2 \) in \( J \), we have

\[
\|S_2u(x_2) - S_2u(x_1)\| = \left| \tilde{D}_0 \int_0^{x_2} f(\eta, u(\eta), h_u(\eta)) \, d\eta - \tilde{D}_0 \int_0^{x_1} f(\eta, u(\eta), h_u(\eta)) \, d\eta \right|
\leq \tilde{D}_0 \int_0^{x_2} |f(\eta, u(\eta), h_u(\eta))| \, d\eta + \tilde{D}_0 \int_0^{x_1} |f(\eta, u(\eta), h_u(\eta))| \, d\eta
\leq \tilde{D}_0 \int_0^{x_2} \left( \frac{a_f + b_f r}{1 - c_f} \right) \, d\eta + \tilde{D}_0 \int_0^{x_1} \left( \frac{a_f + b_f r}{1 - c_f} \right) \, d\eta,
\]

which implies that

\[
\|S_2u(x_2) - S_2u(x_1)\| \leq \tilde{D}_0 \left( \frac{a_f + b_f r}{1 - c_f} \right)(x_2 - x_1).
\] (12)

From (12), we see that if \( x_1 \to x_2 \), then the right-hand side of (12) goes to zero, so \( |S_2u(x_2) - S_2u(x_1)| \to 0 \) as \( x_1 \to x_2 \). Thus the operator defined in (11), \( S_2 \), is continuous. Also \( S_2(E) \subset E \), therefore \( S_2 \) is compact and, due to Arzelá–Ascoli theorem, \( S \) has at least one fixed point. Hence the corresponding problem has at least one solution.

**4 Stability theory**

In this section, we establish some results regarding stability of Ulam type. Before proceeding further, we give some notion and a definition:

**Definition 3** The considered problem (1) is Ulam–Hyers stable if for any \( \epsilon > 0 \) such that

\[
|\text{CF}_0 \text{D}_x^\theta u(x) - f(x, u(x), \text{CF}_0 \text{D}_x^\theta u(x))| < \epsilon, \quad \forall x \in J,
\]

holds, there exists a unique solution \( \bar{u} \) with a constant \( C_f \) such that

\[
|u(x) - \bar{u}(x)| \leq C_f \epsilon, \quad \forall x \in J.
\]

Further the mentioned problem will be generalized Ulam–Hyers stable if there exists a nondecreasing function \( \vartheta : (0, 1) \to (0, \infty) \) such that

\[
|u(x) - \bar{u}(x)| \leq C_f \vartheta(\epsilon), \quad \forall x \in J
\]

with \( \vartheta(0) = 0 \).

Also we state an important remark.

**Remark 2** There exists a function \( \ell(x) \) depending on \( u \in X \) with \( \ell(0) = 0 \) and such that

1. \( |\ell(x)| \leq \epsilon, \forall x \in J \);
2. \( \text{CF}_0 \text{D}_x^\theta u(x) = f(x, u(x), h_u(x)) + \ell(x), \forall x \in J \).

**Lemma 3** The solution of the given perturbed problem

\[
\text{CF}_0 \text{D}_x^\theta u(t) = f(x, u(x), h_u(x)) + \ell(x), \quad \forall x \in J,
\]

\[u(0) = u_0\]
is given as

\[
\begin{aligned}
    u(x) &= u_0 + D_\theta [f(x, u(x), h_u(x)) - f_0] + \bar{D}_\theta \int_0^x f(\eta, u(\eta), h_u(\eta)) \, d\eta \\
    &\quad + D_\theta \ell(x) + \bar{D}_\theta \int_0^x \ell(\eta) \, d\eta, \quad \forall x \in J.
\end{aligned}
\]

(13)

Moreover, the solution satisfies the following inequality:

\[
\begin{aligned}
    \left| u(x) - \left[ u_0 + D_\theta [f(x, u(x), h_u(x)) - f_0] + \bar{D}_\theta \int_0^x f(\eta, u(\eta), h_u(\eta)) \, d\eta \right] \right| \\
    \leq \Omega \varepsilon, \quad \forall x \in J,
\end{aligned}
\]

(14)

where \( \Omega = D_\theta + \bar{D}_\theta T \).

**Proof** The solution (13) can be obtained easily by using Lemma 2. From it, it is obvious how to get result (14) using Remark 2.

**Theorem 4** Under the assumptions of Lemma 3, the solution of the considered problem (1) is Ulam–Hyers stable and also generalized Ulam–Hyers stable if

\[
\frac{L_f \Omega}{1 - M_f} < 1.
\]

**Proof** Let \( u \in X \) be any solution of problem (1) and \( \bar{u} \in X \) be the unique solution of the considered problem. Then take

\[
\begin{aligned}
    \| u - \bar{u} \| &= \max_{x \in J} \left| u - \left[ u_0 + D_\theta [f(x, \bar{u}(x), h_{\bar{u}}(x)) - f_0] + \bar{D}_\theta \int_0^x f(\eta, \bar{u}(\eta), h_{\bar{u}}(\eta)) \, d\eta \right] \right| \\
    &\leq \max_{x \in J} \left| u - \left[ u_0 + D_\theta [f(x, u(x), h_u(x)) - f_0] + \bar{D}_\theta \int_0^x f(\eta, u(\eta), h_u(\eta)) \, d\eta \right] \right| \\
    &\quad + \max_{x \in J} \left| D_\theta \left[ f(x, u(x), h_u(x)) - f(x, \bar{u}(x), h_{\bar{u}}(x)) \right] \right| \\
    &\quad + \max_{x \in J} \bar{D}_\theta \int_0^x \left| f(\eta, u(\eta), h_u(\eta)) - f(\eta, \bar{u}(\eta), h_{\bar{u}}(\eta)) \right| \, d\eta \\
    &\leq \Omega \varepsilon + \frac{\Omega L_f}{1 - M_f} \| u - \bar{u} \|.
\end{aligned}
\]

(15)

Hence from (15), we have

\[
\| u - \bar{u} \| \leq \frac{\Omega}{1 - \frac{L_f \Omega}{1 - M_f}} \varepsilon.
\]

(16)

Hence (16) yields that the solution is Ulam–Hyers stable. Further let \( \mathcal{C}_f = \frac{\Omega}{1 - \frac{L_f \Omega}{1 - M_f}} \) and suppose there exists a nondecreasing function \( \vartheta \in C((0,1),(0,\infty)) \). Then from (16) we can write

\[
\begin{aligned}
\| u - \bar{u} \| &\leq \mathcal{C}_f \vartheta(\varepsilon), \quad \text{with } \vartheta(0) = 0.
\end{aligned}
\]

(17)

Therefore (17) implies that the solution is also generalized Ulam–Hyers stable. \( \square \)
5 Application of our analysis

In this part of the paper, we test our obtained results on some problems given bellow.

Example 1 Take an implicit-type problem

\[
\begin{cases}
  \text{CF}_0 D_{x}^{\frac{1}{2}} u(x) = \frac{x^2}{10} + \frac{\sin(u(x)) + \sin([\text{CF}_0 D_{x}^{\frac{1}{2}} u(x)])}{50 + x^2}, & x \in [0,1], \\
  u(0) = 0.
\end{cases}
\] (18)

Clearly, from (18), \( T = 1 \) and

\[ f(x,u,h_u) = \frac{x^2}{10} + \frac{\sin |u(x)| + \sin |\text{CF}_0 D_{x}^{\frac{1}{2}} u(x)|}{50 + x^2} \]

is continuous for all \( x \in [0,1] \). Further, let \( u, \tilde{u}, h_u, \tilde{h}_u \in \mathbb{R} \), then one has

\[ |f(x,u,h_u) - f(x,\tilde{u},\tilde{h}_u)| \leq \frac{1}{50} |u - \tilde{u}| + \frac{1}{50} |h_u - \tilde{h}_u|. \] (19)

From (19), one has \( L_f = \frac{1}{50}, M_f = \frac{1}{50}, \theta = \frac{1}{2} \). Also

\[ |f'(x,u(x),h_u(x))| \leq \frac{1}{10} + \frac{1}{50} |u(x)| + \frac{1}{50} |h_u(x)|. \]

Thus \( a_f = \frac{1}{10}, b_f = c_f = \frac{1}{50} \), and then \( D_0 = \frac{1}{2}, \tilde{D}_0 = \frac{1}{2}, T = 1 \), and \( (D_0 + \tilde{D}_0) \frac{L_f}{1-M_f} = \frac{1}{50} < 1 \). Hence the conditions of Theorem 1 are satisfied, so (18) has a unique solution. Further, \( \frac{D_0 L_f}{1-M_f} = \frac{1}{50} < 1 \), therefore the conditions of Theorem 3 also hold. Thus the results of Theorem 3 hold. Further, to verify Theorem 4, we see that \( \Omega = 1, \Omega \frac{L_f}{1-M_f} = 0.0204 < 1 \). Hence the solution of the given problem is Ulam– Hyers stable and, consequently, generalized Ulam–Hyers stable.

Example 2 Here to strengthen our analysis, we investigate another problem:

\[
\begin{cases}
  \text{CF}_0 D_{x}^{\frac{99}{100}} u(x) = \frac{1}{80 + x^4} \frac{u(x)}{1 + |\text{CF}_0 D_{x}^{\frac{99}{100}} u(x)|^2} + \frac{\exp(-3x) \cos |\text{CF}_0 D_{x}^{\frac{99}{100}} u(x)|}{200 + 4x^2}, & x \in [0,1], \\
  u(0) = 1.
\end{cases}
\] (20)

Clearly, from (20), we have \( T = 1 \) and

\[ f(x,u(x),h_u(x)) = \frac{1}{80 + x^4} \frac{u(x)}{1 + |\text{CF}_0 D_{x}^{\frac{99}{100}} u(x)|^2} + \frac{\exp(-3x) \cos |\text{CF}_0 D_{x}^{\frac{99}{100}} u(x)|}{200 + 4x^2} \]

is continuous for all \( x \in [0,1] \). Further, for \( u, \tilde{u}, h_u, \tilde{h}_u \in \mathbb{R} \), one has

\[ |f(x,u,h_u) - f(x,\tilde{u},\tilde{h}_u)| \leq \frac{1}{120} |u - \tilde{u}| + \frac{1}{200} |h_u - \tilde{h}_u|. \] (21)

From (21), we take \( L_f = \frac{1}{120}, M_f = \frac{1}{200}, \theta = \frac{99}{100} \). Also

\[ |f'(x,u(x),h_u(x))| \leq \frac{1}{80} + \frac{1}{120} |u(x)| + \frac{1}{200} |h_u(x)|. \]
Thus $a_f = \frac{1}{80}$, $b_f = \frac{1}{120}$, $c_f = \frac{1}{200}$, and then $D_0 = \frac{1}{100}$, $\bar{D}_0 = \frac{99}{100}$ with $T = 1$, and $(D_0 + \frac{\bar{D}_0}{T}) L_f = 1.11 < 1$. Hence the conditions of Theorem 1 are satisfied, so (20) has a unique solution. Further, $\frac{D_0 L_f}{1 - M_f} = \frac{1}{1.11940} < 1$. Therefore the conditions of Theorem 3 also hold. Thus the results of Theorem 3 hold. Further, to verify Theorem 4, we see that $\Omega = 1$, $\Omega L_f = 0.0083752 < 1$. Hence the solution of the given problem is Ulam–Hyers stable and, consequently, generalized Ulam–Hyers stable.

6 Conclusion

The existence theory of solutions to nonsingular kernel-type FODEs has been framed. For the said theory, we have applied the usual Banach and Krasnoselskii fixed point theorems. Also some appropriate results about Ulam–Hyers and generalized Ulam–Hyers stability have been established by using the tools of nonlinear analysis. The obtained results have been testified by two interesting examples. To the best of our knowledge, the said results are new for FODEs involving CFFD. In the future, the above theory and analysis can be extended to more complicated and applicable problems of FODEs involving CFFD.

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Competing interests

The authors declare that they have no competing interests.

Authors’ contributions

All authors equally contributed to this manuscript, read and approved the final version.

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References

1. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies, vol. 204. Elsevier, Amsterdam (2006)
2. Podlubny, I.: Fractional Differential Equations, Mathematics in Science and Engineering. Academic Press, New York (1999)
3. Samko, S.G., Kilbas, A.A., Marichev, O.I.: Fractional Integrals and Derivatives: Theory and Applications. Gordon & Breach, New York (1993)
4. Shah, K.: Multipoint boundary value problems for systems of fractional differential equations: existence theory and numerical simulations. Ph.D. dissertation, University of Malakand, Pakistan (2016)
5. Wang, J.R., Xuezhu, L.: A uniform method to Ulam–Hyers stability for some linear fractional equations. Mediterr. J. Math. 13, 625–635 (2016)
6. Lazarevic, P.M., Aleksandar, M.S.: Finite-time stability analysis of fractional order time-delay systems: Gronwall’s approach. Math. Comput. Model. 49(3–4), 475–481 (2009)
7. Gara, R., Orsingher, E., Polito, F.: A note on Hadamard fractional differential equations with varying coefficients and their applications in probability. Mathematics 6, Article ID 4 (2018). https://doi.org/10.3390/math6010004
8. Borisut, P., Kumam, P., Ahmed, J., Sitthithakerngkiet, K.: Nonlinear Caputo fractional derivative with nonlocal Riemann–Liouville fractional integral condition via fixed point theorems. Symmetry 11(6), Article ID 829 (2019)
9. Ahmed, I., Kumam, P., Shah, K., Borisut, P., Sitthithakerngkiet, K., Dembba, M.A.: Stability results for implicit fractional pantograph differential equations via $\psi$-Hilfer fractional derivative with a nonlocal Riemann–Liouville fractional integral condition. Mathematics 8(1), Article ID 94 (2020)
10. Borisut, P., Kumam, P., Ahmed, I., Jirakitpuwapat, W.: Existence and uniqueness for $\psi$-Hilfer fractional differential equation with nonlocal multi-point condition. Math. Methods Appl. Sci. (2020). https://doi.org/10.1002/mma.6092

11. Toufik, M., Atangana, A.: New numerical approximation of fractional derivative with non-local and non-singular kernel: application to chaotic models. Eur. Phys. J. Plus 132, Article ID 444 (2017)

12. Algahtani, O.J.J.: Comparing the Atangana–Baleanu and Caputo–Fabrizio derivative with fractional order. Allen Cahn model. Chaos Solitons Fractals 89, 552–559 (2016)

13. Atanackovic, T.M., Pilipovic, S., Zorica, D.: Properties of the Caputo–Fabrizio fractional derivative and its distributional settings. Fract. Calc. Appl. Anal. 21(1), 29–44 (2018)

14. Ali, F., et al.: Application of Caputo–Fabrizio derivatives to MHD free convection flow of generalized Walters-B fluid model. Eur. Phys. J. Plus 131(10), Article ID 377 (2016)

15. Francisco, G., Torres, L., Escobar, R.F.: Fractional Derivatives with Mittag-Leffler Kernel. Springer, Berlin (2019)

16. Wang, C.: Stability of some fractional systems and Laplace transform. Acta Math. Sci. Ser. A 39(1), 49–58 (2019)

17. Sher, M., Shah, K., Fečkan, M., Khan, R.A.: Qualitative analysis of multi-terms fractional order delay differential equations via the topological degree theory. Mathematics 8(2), Article ID 218 (2020)

18. Abdeljawad, T.: Fractional operators with exponential kernels and a Lyapunov type inequality. Adv. Differ. Equ. 2017, Article ID 313 (2017)

19. Benchohra, M., Bouriah, S.: Existence and stability results for nonlinear boundary value problem for implicit differential equations of fractional order. Moroccan J. Pure Appl. Anal. 1, 22–36 (2015)

20. Alderremy, A.A., et al.: Certain new models of the multi space-fractional Gardner equation. Phys. A, Stat. Mech. Appl. 545, Article ID 123806 (2020)

21. Agarwal, P., Singh, R.: Modelling of transmission dynamics of Nipah virus (Niv): a fractional order approach. Phys. A, Stat. Mech. Appl. 547, Article ID 124243 (2020)

22. Agarwal, P., Bessem, M.S.: Fixed Point Theory in Metric Spaces: Recent Advances and Applications. Springer, Berlin (2019)

23. Morales-Delgado, V.F., Gómez-Aguilar, J.F., Saad, K.M., Khan, M.A., Agarwal, P.: Analytic solution for oxygen diffusion from capillary to tissues involving external force effects: a fractional calculus approach. Phys. A, Stat. Mech. Appl. 523, 48–65 (2019)

24. Choi, J., Agarwal, P.: A note on fractional integral operator associated with multiindex Mittag-Leffler functions. Filomat 30(7), 1931–1939 (2016)

25. Caputo, M.: A new definition of fractional derivative without singular kernel. Prog. Fract. Differ. Appl. 1(2), 73–85 (2015)

26. Abdeljawad, T., Baleanu, D.: On fractional derivatives with exponential kernel and their discrete versions. Rep. Math. Phys. 80(1), 11–27 (2017)

27. Burton, T.A., Furumochi, T.: Krasnoselskii’s fixed point theorem and stability. Nonlinear Anal., Theory Methods Appl. 49(4), 445–454 (2002)