RAMANUJAN-LIKE SERIES FOR $1/\pi^2$ AND STRING THEORY

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Dedicated to Herbert Wilf on his 80th birthday

Abstract. Using the machinery from the theory of Calabi-Yau differential equations, we find formulas for $1/\pi^2$ of hypergeometric and non-hypergeometric types.

1. Introduction.

Almost 100 years ago, Ramanujan found 17 formulas for $1/\pi$. The most spectacular was

$$\sum_{n=1}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n (26390n + 1103)}{n!^3} \frac{1}{99^{4n+2}} = \frac{\sqrt{2}}{4\pi},$$

where $(a)_0 = 1$ and $(a)_n = a(a+1) \cdots (a+n-1)$ for $n > 1$ is the Pochhammer symbol. The formulas were not proved until in the 1980:ies by the Borwein brothers using modular forms (see [10] and the recent surveys [8] and [23]).

In 2002 the second author found seven similar formulas for $1/\pi^2$. Three of them were proved using the WZ-method (see [13], [15], [16]). Others, like

$$\sum_{n=1}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{8}\right)_n \left(\frac{5}{8}\right)_n \left(\frac{7}{8}\right)_n (1920n^2 + 304n + 15)}{n!^5} \frac{1}{7^{4n}} = \frac{56\sqrt{7}}{\pi^2},$$

(see [14], [16]), were found using PSLQ to find the triple $(1920, 304, 15)$ after guessing $z = 7^{-4}$. This was inspired by the similar formula for $1/\pi$

$$\sum_{n=1}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{7}\right)_n \left(\frac{2}{7}\right)_n (40n + 3)}{n!^3} \frac{1}{7^{4n}} = \frac{49\sqrt{3}}{9\pi}.$$

To avoid guessing $z$, the second author, using $5 \times 5$-matrices, developed a technique to find $z$, while instead guessing a rather small rational number $k$. To that purpose one had to solve an equation of type

$$\frac{1}{6} \log^3(q) - \nu_1 \log(q) - \nu_2 - T(q) = 0,$$

where $\nu_1$ depends on $k$ linearly, $\nu_2$ is a constant, and $T(q)$ is a certain power series (see [17]). It was suggested by Wadim Zudilin that $T(q)$ had to do with the Yukawa coupling

\begin{itemize}
  \item [\textit{2010 Mathematics Subject Classification.}] 33C20; 14J32.
  \item [\textit{Key words and phrases.}] Ramanujan-like series for $1/\pi^2$; Hypergeometric Series; Calabi-Yau differential equations; Mirror map; Yukawa coupling; Gromov-Witten potential.
\end{itemize}
$K(q)$ of the fourth order pullback of the fifth order differential equation satisfied by the sum for general $z$. The exact relation is

$$(q \frac{d}{dq})^3 T(q) = 1 - K(q).$$

This is explained and proved here. The reason that it works is that all differential equations involved are Calabi-Yau. The theory results in a simplified and very fast Maple program to find $z$. As a result we mention the new formula

$$\frac{1}{\pi^2} = 32 \sum_{n=0}^{\infty} \frac{(6n)!}{3 \cdot n!^6} (532n^2 + 126n + 9) \frac{1}{10^{6n+3}},$$

where the summands contain no infinite decimal fractions. However this is not a BBP-type (Bailey-Borwein-Plouffe) series [11] and, due to the factorials, it is not useful to extract individual decimal digits of $1/\pi^2$. (The look we have written the formula above is courtesy by Pigulla).

2. Calabi-Yau differential equations.

2.1. Formal definitions. A Calabi-Yau differential equation is a 4th order differential equation with rational coefficients $y^{(4)} + c_3(z)y''' + c_2(z)y'' + c_1(z)y' + c_0(z)y = 0$ satisfying the following conditions.

1. It is MUM (Maximal Unipotent Monodromy), i.e. the indicial equation at $z = 0$ has zero as a root of order 4. It means that there is a Frobenius solution of the following form

$$y_0 = 1 + A_1 z + A_2 z^2 + \cdots,$$

$$y_1 = y_0 \log(z) + B_1 z + B_2 z^2 + \cdots,$$

$$y_2 = \frac{1}{2} y_0 \log^2(z) + (B_1 z + B_2 z^2 + \cdots \log(z) + C_1 z + C_2 z^2 + \cdots,$$

$$y_3 = \frac{1}{6} y_0 \log^3(z) + \frac{1}{2} (B_1 z + B_2 z^2 + \cdots \log^2(z) + (C_1 z + C_2 z^2 + \cdots \log(z) + D_1 z + D_2 z^2 + \cdots.$$ 

It is very useful that Maple’s ”formal_sol” produces the four solutions in exactly this form (though labeled 1 – 4)

2. The coefficients of the equation satisfy the identity

$$c_1 = \frac{1}{2} c_2 c_3 - \frac{1}{8} c_3^3 + c_2' - \frac{3}{4} c_3 c_3' - \frac{1}{2} c_3''.$$

3. Let $t = y_1/y_0$. Then

$$q = \exp(t) = z + c_2 z^2 + \cdots$$

can be solved

$$z = z(q) = q - c_2 q^2 + \cdots,$$

which is called the ”mirror map”. We also construct the ”Yukawa coupling” defined by

$$K(q) = \frac{d^2}{dt^2} \left( \frac{y_2}{y_0} \right).$$
This can be expanded in a Lambert series

\[ K(q) = 1 + \sum_{d=1}^{\infty} n_d \frac{d^3 q^d}{1 - q^d}, \]

where the \( n_d \) are called "instanton numbers". For small \( d \) the \( n_d \) are conjectured to count rational curves of degree \( d \) on the corresponding Calabi-Yau manifold. Then the third condition is

(a) \( y_0 \) has integer coefficients.
(b) \( q \) has integer coefficients.
(c) There is a fixed integer \( N_0 \) such that all \( N_0 n_d \) are integers.

In [4] the first author shows how to discover Calabi-Yau differential equations.

2.2. Pullbacks of \( 5^{th} \) order equations. The condition 2 is equivalent to

\[ 2': \quad \begin{vmatrix} y_0 & y_3 \\ y_0' & y_3' \end{vmatrix} = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}. \]

This means that the six wronskians formed by the four solutions to our Calabi-Yau equation reduce to five. Hence they satisfy a \( 5^{th} \) order differential equation

\[ w^{(5)} + \frac{3}{5} w^{(4)} + \frac{4}{25} w^{(3)} + \frac{3}{2} w^{(2)} - \frac{6}{5} w^{(1)} - \frac{1}{d_4} = 0. \]

The condition 2 for the \( 4^{th} \) order equation leads to a corresponding condition for the \( 5^{th} \) order equation

\[ 2_5: \quad d_2 = \frac{3}{5} d_3 d_4 - \frac{4}{25} d_4^3 + \frac{3}{2} d_3^3 - \frac{6}{5} d_4 d_4' - d_4''. \]

Conversely given a fifth order equation satisfying \( 2_5 \) with solution \( w_0 \) we can find a pullback, i.e. a fourth order equation with solutions \( y_0, y_1, \ldots \) such that \( w_0 = z(y_0 y_1' - y_0' y_1) \). There is another pullback, \( \hat{y} \) which often cuts the degree into half. It was discovered by Yifan Yang and is simply a multiple \( \hat{y} = g y \) of the ordinary pullback where

\[ g = z^{-1/2} \exp\left( \frac{3}{10} \int d_4 dz \right). \]

In the proof below, all formulas contain only quotients of solutions so the factor \( g \) cancels, so it is irrelevant if we use ordinary or YY-pullbacks. Since the \( q(z) \) are the same, so are the inverse functions \( z(q) \).

2.3. The proof. Consider

\[ w_0(z) = \sum_{n=0}^{\infty} \frac{(1/2)_n s_1 n! (1 - s_1)(1 - s_2)n! (1 - s_2)_n (\rho z)^n}{n! 5} A_n z^n, \]

which satisfies the differential equation

\[ \left\{ \theta^5 - \rho z (\theta + 1/2)(\theta + s_1)(\theta + 1 - s_1)(\theta + s_2)(\theta + 1 - s_2) \right\} w_0 = 0, \]

where \( \theta = z \frac{d}{dz} \). The equation satisfies \( 2' \), so

\[ w_0 = z(y_0 y_1' - y_0' y_1) \]
where \( y_0 \) and \( y_1 \) satisfy a fourth order differential equation (the ordinary pullback). We will consider the following 14 cases (compare the 14 hypergeometric Calabi-Yau equations in the "Big Table" (see [6]).

**Table 1. Hypergeometric cases**

| \# | \( s_1 \) | \( s_2 \) | \( \rho \) | \( A_n \) |
|----|-----------|-----------|-----------|--------|
| 1  | 1/5       | 2/5       | 4 \cdot 5^5 | \( \binom{2n}{n}^3 \binom{3n}{n} \binom{5n}{n} \) |
| 2  | 1/10      | 3/10      | 4 \cdot 8 \cdot 10^5 | \( \binom{2n}{n}^2 \binom{3n}{n} \binom{5n}{n} \binom{10n}{n} \) |
| 3  | 1/2       | 1/2       | 4 \cdot 2^8 | \( \binom{2n}{n}^5 \) |
| 4  | 1/3       | 1/3       | 4 \cdot 3^6 | \( \binom{2n}{n}^3 \binom{3n}{n}^2 \) |
| 5  | 1/2       | 1/3       | 4 \cdot 2^{24} \cdot 3^{3} | \( \binom{2n}{n}^4 \binom{3n}{n} \) |
| 6  | 1/2       | 1/4       | 4 \cdot 2^{10} | \( \binom{2n}{n}^4 \binom{4n}{n} \) |
| 7  | 1/8       | 3/8       | 4 \cdot 2^{16} | \( \binom{2n}{n}^3 \binom{4n}{n} \binom{8n}{n} \) |
| 8  | 1/6       | 1/3       | 4 \cdot 2^{24} \cdot 3^{6} | \( \binom{2n}{n}^3 \binom{4n}{n} \binom{6n}{n} \binom{2n}{n} \) |
| 9  | 1/12      | 5/12      | 4 \cdot 2^{6} | \( \binom{2n}{n}^3 \binom{6n}{n} \binom{12n}{n} \binom{6n}{n} \) |
| 10 | 1/4       | 1/4       | 4 \cdot 2^{12} | \( \binom{2n}{n}^3 \binom{4n}{n}^2 \) |
| 11 | 1/4       | 1/3       | 4 \cdot 2^{3} | \( \binom{2n}{n}^3 \binom{3n}{n} \binom{4n}{n} \) |
| 12 | 1/6       | 1/4       | 4 \cdot 2^{10} \cdot 3^{3} | \( \binom{2n}{n}^2 \binom{3n}{n}^2 \binom{4n}{n} \binom{6n}{n} \binom{3n}{n} \) |
| 13 | 1/6       | 1/6       | 4 \cdot 2^{8} \cdot 3^{6} | \( \binom{2n}{n}^3 \binom{3n}{n}^2 \binom{6n}{n}^2 \) |
| 14 | 1/2       | 1/6       | 4 \cdot 2^{8} \cdot 3^{3} | \( \binom{2n}{n}^3 \binom{3n}{n} \binom{3n}{n} \) |

Assume that the formula

\[
\sum_{n=0}^{\infty} A_n(a + bn + cn^2)z^n = \frac{1}{\pi^2},
\]

is a Ramanujan-like one; that is the numbers \( a, b, c \) and \( z \) are algebraic. Then, in [17] it is conjectured that we have an expansion

\[
\sum_{n=0}^{\infty} A_{n+x}(a + b(n + x) + c(n + x)^2)z^{n+x} = \frac{1}{\pi^2} - \frac{k}{2}x^2 + \frac{j}{24}\pi^2x^4 + O(x^5),
\]

where \( k \) and \( j \) are rational numbers. It holds in all known examples (in fact \( 3k \) and \( j \) are integers). However there is a better argument to support the conjecture. It consists in comparing with the cases \( 3F_2 \) of Ramanujan-type series for \( 1/\pi \), for which the second author proved in [17] that \( k \) must be rational.
In $A_x$ we replace $x!$ by $\Gamma(x+1)$ (Maple does it automatically). Later we use the harmonic number $H_n = 1 + 1/2 + \cdots + 1/n$ which is replaced by $H_x = \psi(x+1) - \gamma$, where $\psi(x) = \Gamma'(x)/\Gamma(x)$ and $\gamma$ is Euler’s constant.

Expansion (2) can be reformulated in the way
\[
\sum_{n=0}^{\infty} \frac{A_{n+x}}{A_x} (a + b(n + x) + c(n + x)^2) z^n = \frac{1}{z^x A_x} \left( \frac{1 - \frac{k}{2} x^2 + \frac{j}{24} \pi^2 x^4 + \cdots}{\pi^2} \right). 
\] (3)

Write
\[
\sum_{n=0}^{\infty} \frac{A_{n+x}}{A_x} z^n = \sum_{i=0}^{\infty} a_i x^i, \quad \sum_{n=0}^{\infty} \frac{A_{n+x}}{A_x} (n + x) z^n = \sum_{i=0}^{\infty} b_i x^i 
\]
and
\[
\sum_{n=0}^{\infty} \frac{A_{n+x}}{A_x} (n + x)^2 z^n = \sum_{i=0}^{\infty} c_i x^i, 
\]
where $a_i, b_i, c_i$ are power series in $z$ with rational coefficients. They are related to the solutions $w_0, w_1, w_2, w_3, w_4$ of the fifth order differential equation
\[
w_0 = a_0 \\
w_1 = a_0 \log(z) + a_1 \\
w_2 = a_0 \log^2(z) + a_1 \log(z) + a_2 \\
w_3 = a_0 \log^3(z) + a_1 \log^2(z) + a_2 \log(z) + a_3 \\
w_4 = a_0 \log^4(z) + a_1 \log^3(z) + a_2 \log^2(z) + a_3 \log(z) + a_4. 
\]

We also have $b_0 = za'_0$ and $b_k = a_{k-1} + za'_k$ for $k = 1, 2, 3, 4$.

If we write the expansion of $A_x$ in the form
\[
A_x = 1 + \frac{e}{2} \pi^2 x^2 - h \zeta(3) x^3 + \left( \frac{3e^2}{8} - \frac{f}{2} \right) \pi^4 x^4 + O(x^5),
\] (4)
then for the right hand side $M$ of (3) we have
\[
M = \frac{1}{z^x A_x} \left( \frac{1}{\pi^2} - \frac{k}{2} x^2 + \frac{j}{24} \pi^2 x^4 + \cdots \right) = m_0 + m_1 x + m_2 x^2 + m_3 x^3 + m_4 x^4 + \cdots, 
\]
where
\[
m_0 = \frac{1}{\pi^2}, \\
m_1 = -\frac{1}{\pi^2} \log(z), \\
m_2 = \frac{1}{\pi^2} \left\{ \frac{1}{2} \log^2(z) - \frac{\pi^2}{2} (k + e) \right\}, \\
m_3 = \frac{1}{\pi^2} \left\{ -\frac{1}{6} \log^3(z) + \frac{\pi^2}{2} (k + e) \log(z) + h \zeta(3) \right\}. 
\]
and

\[ 2m_0m_4 - 2m_1m_3 + m_2^2 = \frac{j}{12} + \frac{k^2}{4} + ek + f. \]

Here

\[ e = \frac{5}{3} + \cot^2(\pi s_1) + \cot^2(\pi s_2), \quad f = \frac{1}{\sin^2(\pi s_1) \sin^2(\pi s_2)} \]

and

\[ h = \frac{2}{\zeta(3)} \{(\zeta(3, 1/2) + \zeta(3, s_1) + \zeta(3, 1 - s_1) + \zeta(3, s_2) + \zeta(3, 1 - s_2)} \}

where

\[ \zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n + a)^s} \]

is Hurwitz \( \zeta \)-function. If one uses \( A_n \) defined by binomial coefficients Maple finds the values of \( e \) and \( h \) directly. We conjecture that the 14 pairs \( (s_1, s_2) \) given in the table are the only rational \( (s_1, s_2) \) between 0 and 1 making \( h \) an integer. Note that the same \( (s_1, s_2) \) give the only hypergeometric Calabi-Yau differential equations (see [1] and [2]).

### Table 2. Values of e, h, f

| # | \( \tilde{1} \) | \( \tilde{2} \) | \( \tilde{3} \) | \( \tilde{4} \) | \( \tilde{5} \) | \( \tilde{6} \) | \( \tilde{7} \) | \( \tilde{8} \) | \( \tilde{9} \) | \( \tilde{10} \) | \( \tilde{11} \) | \( \tilde{12} \) | \( \tilde{13} \) | \( \tilde{14} \) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| e | \( \frac{11}{3} \) | \( 35 \) | \( \frac{5}{3} \) | \( \frac{7}{3} \) | \( 2 \) | \( \frac{8}{3} \) | \( \frac{23}{3} \) | \( 5 \) | \( \frac{47}{3} \) | \( \frac{11}{3} \) | \( \frac{17}{3} \) | \( \frac{23}{3} \) | \( \frac{14}{3} \) |
| h | 42 | 290 | 10 | 18 | 14 | 24 | 150 | 70 | 486 | 38 | 28 | 80 | 122 | 66 |
| f | \( \frac{16}{5} \) | 16 | 1 | \( \frac{16}{9} \) | \( \frac{4}{3} \) | 2 | 8 | \( \frac{16}{3} \) | 16 | 4 | \( \frac{8}{3} \) | 8 | 16 | 4 |

Now we want to use many of the identities for the wronskians in [1, pp.4-5]. Therefore we invert the formulas

\[
\begin{align*}
a_0 &= w_0, \\
a_1 &= w_1 - w_0 \log(z), \\
a_2 &= w_2 - w_1 \log(z) + w_0 \frac{\log^2(z)}{2}, \\
a_3 &= w_3 - w_2 \log(z) + w_1 \frac{\log^2(z)}{2} - w_0 \frac{\log^3(z)}{6}, \\
a_4 &= w_4 - w_3 \log(z) + w_2 \frac{\log^2(z)}{2} - w_1 \frac{\log^3(z)}{6} + w_0 \frac{\log^4(z)}{24}
\end{align*}
\]
and
\[ b_0 = zw'_0, \]
\[ b_1 = z(w'_1 - w'_0 \log(z)), \]
\[ b_2 = z(w'_2 - w'_1 \log(z) + w'_0 \frac{\log^2(z)}{2}), \]
\[ b_3 = z(w'_3 - w'_2 \log(z) + w'_1 \frac{\log^2(z)}{2} - w'_0 \frac{\log^3(z)}{6}), \]
\[ b_4 = z(w'_4 - w'_3 \log(z) + w'_2 \frac{\log^2(z)}{2} - w'_1 \frac{\log^3(z)}{6} + w'_0 \frac{\log^4(z)}{24}). \]

The key equation in [17] is
\[ m_3 = H_0 m_0 - H_1 m_1 + H_2 m_2, \tag{5} \]
where
\[ H_0 = \frac{a_0 b_1 - a_1 b_0}{a_0 b_1 - a_1 b_0}, \quad H_1 = \frac{a_0 b_3 - a_3 b_0}{a_0 b_1 - a_1 b_0}, \quad H_2 = \frac{a_0 b_2 - a_2 b_0}{a_0 b_1 - a_1 b_0}. \]

We get \((g)\) is a multiplicative factor defined in [11, p.5]. It will cancel out
\[ a_0 b_1 - a_1 b_0 = z \begin{vmatrix} w_0 & w_1 \\ w'_0 & w'_1 \end{vmatrix} = z^3 g y_0^2. \]

("The double wronskian is almost the square")
\[ a_0 b_2 - a_2 b_0 = z \begin{vmatrix} w_0 & w_2 \\ w'_0 & w'_2 \end{vmatrix} - z \log(z) \begin{vmatrix} w_0 & w_1 \\ w'_0 & w'_1 \end{vmatrix} = z^3 g \left\{ y_0 y_1 - y_0^2 \log(z) \right\}. \]

It follows
\[ H_2 = \frac{z^3 g \left\{ y_0 y_1 - y_0^2 \log(z) \right\}}{z^3 g y_0^2} = \frac{y_1}{y_0} - \log(z) = \log(q) - \log(z) = \log\left(\frac{q}{z}\right). \]

Furthermore
\[ a_0 b_3 - a_3 b_0 = z \begin{vmatrix} w_0 & w_3 \\ w'_0 & w'_3 \end{vmatrix} - z \log(z) \begin{vmatrix} w_0 & w_2 \\ w'_0 & w'_2 \end{vmatrix} + \frac{z \log^2(z)}{2} \begin{vmatrix} w_0 & w_1 \\ w'_0 & w'_1 \end{vmatrix} \]
\[ = z^3 g \left\{ \frac{1}{2} y_1^2 - y_0 y_1 \log(z) + y_0^2 \frac{\log^2(z)}{2} \right\} \]
and
\[ H_1 = \frac{1}{2} \left( \frac{y_1}{y_0} \right)^2 - \frac{y_1}{y_0} \log(z) + \frac{\log^2(z)}{2} = \frac{1}{2} \log^2\left(\frac{q}{z}\right). \]
Finally we have that
\[ a_0b_4 - a_4b_0 = z \left| \begin{array}{cc} w_0 & w_3 \\ w_0' & w_3' \end{array} \right| - z \log(z) \left| \begin{array}{cc} w_0 & w_3 \\ w_0' & w_3' \end{array} \right| + \frac{z \log^2(z)}{2} \left| \begin{array}{cc} w_0 & w_2 \\ w_0' & w_2' \end{array} \right| - \frac{z \log^3(z)}{6} \left| \begin{array}{cc} w_0 & w_1 \\ w_0' & w_1' \end{array} \right| \]

which simplifies to
\[ = z^3 g \left\{ \frac{1}{2} (y_1 y_2 - y_0 y_3) - \frac{1}{2} y_1^2 \log(z) + y_0 y_1 \frac{\log^2(z)}{2} - y_0 y_3 \frac{\log^3(z)}{6} \right\} \]

and
\[ H_0 = \frac{1}{2} \left( \frac{y_1 y_2}{y_0 y_0} - \frac{y_3}{y_0} \right) - \frac{1}{2} t^2 \log(z) + t \frac{\log^2(z)}{2} - \frac{\log^3(z)}{6}. \]

Substituting these formulas into (5), we obtain
\[ \frac{1}{\pi^2} \left\{ -\frac{1}{6} \log^3(z) + \frac{\pi^2}{2} (k + e) \log(z) + h \zeta(3) \right\} \]

\[ = \frac{1}{\pi^2} \left\{ \frac{1}{2} \left( \frac{y_1 y_2}{y_0 y_0} - \frac{y_3}{y_0} \right) - \frac{1}{2} t^2 \log(z) + t \frac{\log^2(z)}{2} - \frac{\log^3(z)}{6} \right\} \]

\[ + \frac{1}{\pi^2} \log(z) \left\{ \frac{t^2}{2} - t \log(z) + \frac{\log^2(z)}{2} \right\} + \frac{1}{\pi^2} (t - \log(z)) \frac{\log^2(z)}{2} - \frac{\pi^2}{2} (k + e) \right\}, \]

which simplifies to
\[ \frac{1}{2} (y_1 y_2 - y_0 y_3) - \frac{\pi^2}{2} (k + e) \log(q) - h \zeta(3) = 0. \]

Here
\[ \Phi = \frac{1}{2} \left( \frac{y_1 y_2}{y_0 y_0} - \frac{y_3}{y_0} \right) \]

is wellknown in String Theory and is called the Gromov-Witten potential (up to a multiplicative constant, (see [12] p.28)). It is connected to the Yukawa coupling \( K(q) \) by
\[ (q \frac{d}{dq})^3 \Phi = K(q). \]

Writing \( \Phi = \frac{1}{6} \log^3(q) - T(q) \) (see lemma 2.1) we get the following equation for finding \( q \) and hence \( z \) for given \( k \)
\[ \frac{1}{6} t^3 - \frac{\pi^2}{2} (k + e) t - h \zeta(3) - T(q) = 0, \quad q = \exp(t). \] (6)

We look for real algebraic solutions of \( z \). To look for alternating series, that is if \( z < 0 \) all we need to do is replacing \( q = \exp(t) \) with \( q = -\exp(t) \) in (6). In order to make a quick sieve of the solutions, once we get \( q \) we compute \( j \) and see if it is an integer (or rational with small denominator). Using the formulas [17] eqs. 3.48 & 3.50 we find
\[ j = 12 \left\{ \frac{1}{\pi^4} \left( \frac{1}{2} t^2 - q \frac{d}{dq} T(q) - \frac{\pi^2}{2} (k + e) \right)^2 - \frac{k^2}{4} - ek - f \right\}. \] (7)

Lemma 2.1. The function \( T(q) \) is a power series with \( T(0) = 0 \).
Proof. We have

\[ y_1 = y_0 \log(z) + \alpha_1 \]

which implies

\[ \frac{y_1}{y_0} = \log(q) = \log(z) + \frac{\alpha_1}{y_0} = \log(z) + \beta_1 \]

and hence

\[ \log(z) = \log(q) - \beta_1, \]

where \( \alpha_1 \) and \( \beta_1 = \frac{\alpha_1}{y_0} \) are power series without constant term. Furthermore

\[ y_2 = y_0 \frac{\log^2(z)}{2} + \alpha_1 \log(z) + \alpha_2 \]

leads to

\[ \frac{y_2}{y_0} = \frac{1}{2} (\log(q) - \beta_1)^2 + \beta_1 (\log(q) - \beta_1) + \beta_2 = \frac{1}{2} \log^2(q) + \beta_2 - \frac{1}{2} \beta_1^2, \]

where \( \beta_2 = \frac{\alpha_2}{y_0} \) with \( \beta_2(0) = 0 \). Finally

\[ y_3 = y_0 \frac{\log^2(z)}{6} + \alpha_1 \frac{\log^2(z)}{2} + \alpha_2 \log(z) + \alpha_3 \]

and

\[ \frac{y_3}{y_0} = \frac{1}{6} (\log(q) - \beta_1)^3 + \frac{1}{2} \beta_1 (\log(q) - \beta_1)^2 + \beta_2 (\log(q) - \beta_1) + \beta_3, \]

where \( \beta_3 = \frac{\alpha_3}{y_0} \) with \( \beta_3(0) = 0 \). Collecting terms we have

\[ \frac{1}{2} \left( \frac{y_1 y_2}{y_0 y_0} - \frac{y_3}{y_0} \right) = \frac{1}{6} \log^3(q) - \frac{1}{2} (\beta_3 - \beta_1 \beta_2 + \frac{1}{3} \beta_1^3), \]

which proves the lemma. \( \square \)

3. Computations

3.1. Hypergeometric differential equations. In only half of the 14 cases have we found solutions to the equation (6), where the indicator \( j \) is an integer. Using [17] eqs. 3.47-3.48], we have the following formula for computing \( c \)

\[ \tau = \frac{c}{\sqrt{1 - \rho z}}, \tag{8} \]

where

\[ \tau^2 = \frac{j}{12} + \frac{k^2}{4} + ek + f. \]

Then \( a \) and \( b \) can be computed by [17] eq. 3.45] or by PSLQ. Here are our results where the series converges.
Table 3. Convergent hypergeometric Ramanujan-like series for $1/\pi^2$

| #  | $k$  | $j$  | $z_0$ | $\tau^2$ | $a$     | $b$     | $c$    |
|----|------|------|-------|----------|---------|---------|--------|
| $\tilde{3}$ | 1    | 25   | $-1/2^{12}$ | 5       | $1/8$   | 1       | $5/2$  |
| $\tilde{3}$ | 5    | 305  | $-1/2^{20}$ | 41      | $13/128$| 45      | $205/32$|
| $\tilde{5}$ | 2/3  | 16   | $1/2^{12}$  | 37      | $1/16$  | 9       | 37     |
| $\tilde{5}$ | $8/3$ | 112  | $(5\sqrt{5}-11)/8$ | $160/9$ | $56 - 25\sqrt{5}$ | $303 - 135\sqrt{5}$ | $1220/3 - 180\sqrt{5}$ |
| $\tilde{6}$ | 2    | 80   | $1/2^{16}$  | 15      | $3/32$  | $17/16$ | $15/4$ |
| $\tilde{7}$ | 8    | 992  | $1/2^{18}\sqrt{7}$ | 168     | $15/392\sqrt{7}$ | $38/49\sqrt{7}$ | $240/49\sqrt{7}$ |
| $\tilde{8}$ | $5/3$ | 85   | $-1/2^{18}$ | $193/9$ | $15/128$| $183/128$| $965/192$|
| $\tilde{8}$ | 15   | 2661 | $1/2^{18}\sqrt{3}$ | $1075/3$ | $29/640\sqrt{5}$ | $693/640\sqrt{5}$ | $2709/320\sqrt{5}$ |
| $\tilde{8}$ | $8/3$ | 160  | $1/2^{16}\sqrt{3}$ | $304/9$ | $36/375$ | $504/375$ | $2128/375$ |
| $\tilde{11}$ | 3    | 157  | $-1/2^{12}\sqrt{3}$ | $27/48$ | $5/48$  | $21/16$ | $21/4$ |
| $\tilde{12}$ | 7    | 757  | $-1/2^{22}\sqrt{3}$ | $123/768\sqrt{3}$ | $15/768\sqrt{3}$ | $278/768\sqrt{3}$ | $205/96\sqrt{3}$ |

In all the hypergeometric cases there a singular solution when $k = j = 0$ (it has not a corresponding Ramanujan-like series). For that solution we have $z = 1/\rho$, $a = b = c = 0$.

In addition we have found the solutions $\tilde{3}$: $k = 0$, $j = 3$, $z = -2^{-8}$, $a = 1/4$, $b = 3/2$, $c = 5/2$ and $\tilde{11}$: $k = 1/3$, $j = 13$, $z = -2^{-12}$, $a = 3/16$, $b = 25/16$, $c = 43/12$, for which the corresponding series are "divergent" [20].

Although our new program, which evaluates the function $T(q)$ much faster, has allowed us to try all rational values of $k$ of the form $k = i/60$ with $0 \leq i \leq 1200$ the only new series that we have found is for $\tilde{8}$ with $k = 8/3$, and it is

$$\sum_{n=0}^{\infty} \frac{(6n)!}{n!16} (532n^2 + 126n + 9) \frac{1}{10^{6n}} = \frac{375}{4\pi^2},$$

that is (11). A brief story of the discovery of the other 10 formulas in the table is in [19].
Finally we give a hypergeometric example of different nature in case \( \tilde{3} \). Take \( z_0 = -2^{-10} \), \( q_0 = q(z_0) \), \( t_0 = \log |q_0| \) and \( T(q) \) of \( 3 \), we find using PSLQ, among the quantities \( T(q_0) \), \( t_0^3, t_0^2 \pi, t_0 \pi^2, \pi^3, \zeta(3) \) the following remarkable relation:

\[
\frac{1}{6}(t_0 + \pi)^3 - \frac{5}{6} \pi^2(t_0 + \pi) - \frac{\pi^3}{3} - 10\zeta(3) - T(q_0) = 0.
\]

The theory we have developed allows to understand that the last relation has to do with the following formula proved by Ramanujan [9, p.41]:

\[
\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{10n}} \binom{2n}{n}^5 \frac{1}{(4n + 1)} = \frac{2}{\Gamma^4 \left( \frac{3}{4} \right)}.
\]

To see why we guess that

\[
\frac{\Gamma^4 \left( \frac{3}{4} \right)}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{10n(n+x)}} \binom{2n+2x}{n+x}^5 \frac{1}{(4n+x) + 1} = 1 - \pi x + \frac{\pi^2}{2} x^2 + \frac{\pi^3}{6} x^3 - \frac{19\pi^4}{24} x^4 + O(x^5)
\]

by expanding the first side numerically. Hence

\[
2^{10x} \left( \frac{2x}{x} \right)^{-5} \frac{\Gamma^4 \left( \frac{3}{4} \right)}{2} \left( 1 - \pi x + \frac{\pi^2}{2} x^2 + \frac{\pi^3}{6} x^3 \right) = m_0 + m_1 x + m_2 x^2 + m_3 x^3 + O(x^4)
\]

and we get \( m_0, m_1, m_2, m_3 \). Finally we use identity (5) replacing \( \log(z) \) with \( \log 2^{-10} \).

### 3.2. Non-hypergeometric differential equations.

If we write the ordinary pullback in the form

\[
\theta^4_3 y = \left[ e_3(z) \theta^3_z + e_2(z) \theta^2_z + e_1(z) \theta_z + e_0(z) \right] y, \quad \theta_z = z \frac{d}{dz},
\]

then, the generalization of the relation (8) is

\[
\tau = c \left( \exp \int \frac{e_3(z)}{2z} dz \right), \quad \tau^2 = \frac{j}{12} + \frac{k^2}{4} + ek + f.
\]

We say that a solution is singular if it does not has a corresponding Ramanujan-like series. We conjecture that \( h \) is the unique rational number such that singular solutions exists. The numbers \( e \) and \( f \) are not so important because they can be absorbed in \( k \) and \( j \) respectively. However, to agree with the hypergeometric cases we will choose \( e \) and \( f \) in such a way that a singular solution takes place at \( k = j = 0 \). This fact allows us to determine the values of the numbers \( e, h \) and \( f \) from (6) and (7) using the PSLQ algorithm. For many sequences \( A(n) \) there exists a finite value of \( z \) which is singular, then we can get this value solving the equation

\[
\frac{dz(q)}{dq} = 0.
\]

In the sequel, we will show in several tables the rational values of the invariants \( e, h \) and \( f \) followed by the series found.
Table $\alpha$.

|          | $\#39 = A \ast \alpha$ | $\#61 = B \ast \alpha$ | $\#37 = C \ast \alpha$ | $\#66 = D \ast \alpha$ |
|----------|-------------------------|-------------------------|-------------------------|-------------------------|
| $e, h, f$| $1, \frac{14}{3}, \frac{1}{3}$ | $4, \frac{26}{3}, \frac{4}{9}$ | $2, \frac{56}{3}, \frac{2}{3}$ | $4, \frac{182}{3}, \frac{4}{3}$ |

For $A \ast \alpha$, taking $k = 1/3$ we get $j = 5$, and we discover the series
\[
\sum_{n=0}^{\infty} \frac{(2n)!}{n!} \frac{(3n)!}{n!} \sum_{i=0}^{n} \frac{(n)!}{i!} \frac{2i}{n-i} \frac{2i}{n-2i} \frac{1}{2^{10n}} (36n^2 + 12n + 1) = \frac{32}{\pi^2}.
\]

This series was first conjectured by Zhi-Wei Sun [21] inspired by $p$-adic congruences.

Table $\epsilon$.

|          | $\#122 = A \ast \epsilon$ | $\#170 = B \ast \epsilon$ | $C \ast \epsilon$ | $D \ast \epsilon$ |
|----------|-----------------------------|-----------------------------|-------------------|-------------------|
| $e, h, f$| $\frac{7}{6}, \frac{45}{8}, \frac{1}{2}$ | $\frac{3}{2}, \frac{77}{8}, \frac{2}{3}$ | $\frac{13}{6}, \frac{157}{8}, 1$ | $\frac{25}{6}, \frac{493}{8}, \frac{2}{3}$ |

For $B \ast \epsilon$, taking $k = 1$ we get $j = 22$, and we find the formula
\[
\sum_{n=0}^{\infty} \binom{2n}{n} \binom{3n}{n} \sum_{i=0}^{n} \binom{n}{i} \frac{2i}{n-i} \frac{2i}{n-2i} \frac{1}{2^{10n} 3^n} (1071n^2 + 399n + 46) = \frac{576}{\pi^2}.
\]

Table $\beta$.

|          | $\#40 = A \ast \beta$ | $\#49 = B \ast \beta$ | $\#43 = C \ast \beta$ | $\#67 = D \ast \beta$ |
|----------|-------------------------|-------------------------|-------------------------|-------------------------|
| $e, h, f$| $\frac{2}{3}, \frac{3}{4}, \frac{1}{4}$ | $1, \frac{7}{4}, \frac{1}{4}$ | $\frac{5}{3}, \frac{17}{4}, \frac{1}{4}$ | $\frac{11}{3}, \frac{59}{4}, \frac{1}{4}$ |

For $A \ast \beta$, taking $k = 1$ we get $j = 13$, and we have the series
\[
\sum_{n=0}^{\infty} \binom{2n}{n} \sum_{i=0}^{n} \binom{2i}{i} \frac{2i}{n-i} \frac{2i}{n-2i} \frac{1}{2^{10n}} (36n^2 + 12n + 1) = \frac{32}{\pi^2}.
\]

For $B \ast \beta$, taking $k = 1/3$ we get $j = 1$, and we find
\[
\sum_{n=0}^{\infty} \frac{(3n)!}{n!} \sum_{i=0}^{n} \binom{2i}{i} \frac{2i}{n-i} \frac{2i}{n-2i} \frac{1}{2^{10n}} (25n^2 - 15n - 6) = \frac{192}{\pi^2}.
\]
Ramanujan-like series for $1/\pi^2$ and string theory

\[\sum_{n=0}^{\infty} \left(\frac{2n}{n}\right)^2 \sum_{i=0}^{n} \frac{(-1)^i 3^{n-3i} (3i)!}{i!^3} \left(\frac{n+i}{3i}\right) \binom{n+i}{i} \frac{(-1)^n}{3^6n} (803n^2 + 416n + 68) = \frac{486}{\pi^2}.\]

\[\sum_{n=0}^{\infty} \frac{3n!}{n!^3} \sum_{i=0}^{n} 16^{n-i} \left(\frac{2i}{i}\right)^3 \left(\binom{2n-2i}{n-i}\right) P(n) \frac{(-1)^n}{640320^{3n}} = \frac{(2^4 \cdot 3 \cdot 5 \cdot 23 \cdot 29)^3}{\pi^2},\]

where

\[P(n) = 22288332473153467n^2 + 16670750677895547n + 415634396862086,\]

which is the "square" [21] of the brothers Chudnovsky's formula [8]

\[\sum_{n=0}^{\infty} \frac{(-1)^n (6n)!}{(3n)!n!^3} (545140134n + 13591409) \frac{1}{640320^{3n}} = \frac{53360\sqrt{640320}}{\pi}.\]

Table $\delta$.

|   | $A \ast \delta$ | $B \ast \delta$ | $C \ast \delta$ | $D \ast \delta$ |
|---|----------------|----------------|----------------|----------------|
| $e, h, f$ | $1 \frac{9}{2} \frac{17}{36}$ | $4 \frac{17}{2} \frac{7}{12}$ | $2 \frac{37}{2} \frac{29}{36}$ | $4 \frac{121}{2} \frac{53}{36}$ |

For $A \ast \delta$, taking $k = 2/3$ we get $j = 28/3$, and we have

\[\sum_{n=0}^{\infty} \frac{2n}{n} \sum_{i=0}^{n} \frac{(-1)^i 3^{n-3i} (3i)!}{i!^3} \left(\frac{n+i}{3i}\right) \binom{n+i}{i} \frac{(-1)^n}{3^6n} (803n^2 + 416n + 68) = \frac{486}{\pi^2}.\]

Table $\theta$.

|   | $A \ast \theta$ | $B \ast \theta$ | $C \ast \theta$ | $D \ast \theta$ |
|---|----------------|----------------|----------------|----------------|
| $e, h, f$ | $\frac{2}{3} \frac{-4}{1}$ | $1 \frac{0}{1} \frac{10}{1}$ | $\frac{11}{3} \frac{52}{1}$ | $\frac{1}{3} \frac{52}{1}$ |

For $A \ast \theta$, taking $k = 2$ we get $j = 56$, and we discover the series

\[\sum_{n=0}^{\infty} \frac{2n}{n} \sum_{i=0}^{n} \frac{16^{n-i} (2i)^3}{(2n-2i)^i} \left(\frac{2n-2i}{n-i}\right) \frac{(-1)^n}{2^{15n}} (18n^2 + 7n + 1) = \frac{4\sqrt{2}}{\pi^2}.\]

This series was first discovered by Zhi-Wei Sun [21] inspired by p-adic congruences.

For $B \ast \theta$ we get $T(q) = 0$ and from the equations we see that for every rational $k$ the value of $j$ is rational as well. Hence for every rational value of $k$ we get a Ramanujan-like series for $1/\pi^2$.

For example, for $k = 160/3$, we have

\[\sum_{n=0}^{\infty} \frac{3n!}{n!^3} \sum_{i=0}^{n} 16^{n-i} \left(\frac{2i}{i}\right)^3 \left(\binom{2n-2i}{n-i}\right) P(n) \frac{(-1)^n}{640320^{3n}} = \frac{(2^4 \cdot 3 \cdot 5 \cdot 23 \cdot 29)^3}{\pi^2},\]

where

\[P(n) = 22288332473153467n^2 + 16670750677895547n + 415634396862086,\]

which is the "square" [21] of the brothers Chudnovsky's formula [8]

\[\sum_{n=0}^{\infty} \frac{(-1)^n (6n)!}{(3n)!n!^3} (545140134n + 13591409) \frac{1}{640320^{3n}} = \frac{53360\sqrt{640320}}{\pi}.\]

For $C \ast \theta$, taking $k = 1$ we get $j = 25$ and taking $k = 5$ we get $j = 305$, and we recover the two series proved by W. Zudilin in [22] by doing a quadratic transformation of case $\tilde{3}$.

In [3] the first author, by transforming known formulas given by the second author, found formulas for $1/\pi^2$ where the coefficients belong to the Calabi-Yau equations $\tilde{3}, \tilde{5}, \tilde{6}, \tilde{7}, \tilde{8}, \tilde{9}$. 
\(\hat{8}, \hat{11}, \hat{12}\). Here we list some new ones for the cases \(\hat{3}, \hat{5}, \hat{8}, \hat{11}\) and \#77, some found by solving equation (6).

**Transformation \(\hat{5}\).** Here

\[
A_n = \sum_{i=0}^{n} (-1)^i 1728^{n-i} \binom{n}{i}^4 \binom{3i}{i}.
\]

Using \(e = 2\), \(h = 14\), \(f = 4/3\) we find for \(k = 8/3\) that \(j = 112\) and \(z_0 = -[320(131 + 61\sqrt{5})]^{-1}\). To find the coefficients we had to use the formulas in [3]. The resulting formula is

\[
\sum_{n=0}^{\infty} A_n \left( (28765285482\sqrt{5} - 64321133730) + (10068363 - 4502709\sqrt{5})n \right.
\]

\[
+ (54\sqrt{5} - 122)n^2 \right) \frac{(-1)^n}{(320(131 + 61\sqrt{5}))^n} = \frac{300(1170059408\sqrt{5} - 24977012149)}{\pi^2}.
\]

By the PSLQ algorithm, trying products of powers of 2 and 7 in the denominator of \(z\), we see that

\[
\sum_{n=0}^{\infty} A_n(n^2 - 63n + 300) \frac{1}{1792^n} = \frac{4704}{\pi^2}.
\]

but we cannot find the pair \((k, j)\) with our program because the convergence in this case is too slow.

**Transformation \(\hat{8}\).** Here

\[
A_n = \sum_{i=0}^{n} (-1)^k 6^{6n-6i} \binom{n}{i}^3 \binom{4i}{2i} \binom{6i}{2i}.
\]

Using \(e = 5\), \(h = 70\), \(f = 16/3\) we find for \(k = 5/3\) that \(j = 85\) and \(z_0 = 308800^{-1}\). This allows us to get the formula

\[
\sum_{n=0}^{\infty} A_n(16777216n^2 - 3336192n - 2912283) \frac{1}{308800^n} = \frac{3 \cdot 5^5 \cdot 193^2}{5^5 \pi^2}.
\]

For \(k = \frac{8}{3}\) we get \(j = 160\) and the formula

\[
\sum_{n=0}^{\infty} (-1)^n A_n(48828125n^2 + 17859375n + 3649554) \frac{1}{953344^n} = \frac{2^8 \cdot 3 \cdot 7^5 \cdot 19^2}{5^4 \pi^2}.
\]
Transformation $\hat{11}$. Here

$$A_n = \sum_{i=0}^{n} (-1)^{i} 6912^{n-i} \binom{n}{i} \binom{2i}{i}^3 \binom{4i}{i}^{2i}.$$ 

Using $e = 3$, $h = 28$, $f = 8/3$ we find for $k = 1/3$ that $j = 13$ and we get the formula

$$\sum_{n=0}^{\infty} A_n (512n^2 - 1992n - 225) \frac{1}{11008^n} = \frac{3 \cdot 43^2}{2\pi^2}.$$ 

Transformation $\hat{3}$. Here

$$A_n = \sum_{i=0}^{n} (-1)^{i} 1024^{n-i} \binom{n}{i} \binom{2i}{i}^5.$$ 

Transforming two divergent series in [20] with $z_0 = -2^{-8}$ and $z_0 = -1$ respectively (the second one given only implicitly), we obtain two (slowly) convergent formulas

$$\sum_{n=0}^{\infty} A_n (2n^2 - 18n + 5) \frac{1}{1280^n} = \frac{100}{\pi^2}$$

and

$$\sum_{n=0}^{\infty} A_n (n^2 - 2272n + 392352) \frac{1}{1025^n} = \frac{16 \cdot 5253125}{\pi^2}.$$ 

This last identity converges so slowly that the power of our computers seems not enough to check it numerically.

Transformation #77. Here

$$A_n = \binom{2n}{n} \sum_{i=0}^{n} \binom{n}{i} \binom{2i}{i}^3 \binom{4i}{2i}.$$ 

The pullback is equivalent to $\tilde{6}$ (i.e. has the same $K(q)$), so we try the same parameters; $e = 8/3$, $h = 24$, $f = 2$. For $k = 2$ we get $j = 80$ and $z_0 = 1/65540$. To find $a, b, c$ we have to find the transformation between $\tilde{6}$ and #77. Indeed

$$\sum_{n=0}^{\infty} \binom{2n}{n}^4 \frac{4n}{2n} z^n = \frac{1}{\sqrt{1-4z}} \sum_{n=0}^{\infty} A_n \left(\frac{z}{1+4z}\right)^n$$

(the sequence of numbers $\tilde{6}$ is in the left side) and using the method in [3], (see also [7]), we obtain

$$\sum_{n=0}^{\infty} A_n (402653184n^2 + 114042880n + 10051789) \frac{1}{65540^n} = \frac{5^2 \cdot 29^3 \cdot 113^3}{2^6 \pi^2 \sqrt{16385}}.$$
4. Supercongruences

Zudilin [25] observed that the hypergeometric formulas for $1/\pi^2$ lead to supercongruences of the form
\[
\sum_{n=0}^{p-1} A_n (a + bn + cn^2) z^n \equiv a \left( \frac{d}{p} \right) p^2 \pmod{p^5},
\]
where the notation $(d \mid p)$ stands for the Legendre symbol. Our computations show that for our new Ramanujan-like series for $1/\pi^2$ (1), we have again a supercongruence following Zudilin’s pattern, namely
\[
\sum_{n=0}^{p-1} \binom{2n}{n}^3 \binom{4n}{2n} \binom{6n}{2n} (532n^2 + 126n + 9) \frac{1}{1000000n} \equiv 9p^2 \pmod{p^5},
\]
valid for primes $p \geq 7$.

For supercongruences for $\tilde{5}$ and $k = 8/3$, which involves algebraic numbers, see [18]. For the non-hypergeometric formulas the best one can hope for is a congruence $\pmod{p^3}$. We give some new ones which agree with Zudilin’s observations for [25] eq. 35].

**With Hadamard product $\#170 = B * \epsilon$.**
\[
\sum_{n=0}^{p-1} \binom{2n}{n} \binom{3n}{n} \sum_{i=0}^{n} \binom{n}{i}^2 \binom{2i}{n} \frac{1}{2^{7n}3^n} (1071n^2 + 399n + 46) \equiv 46p^2 \pmod{p^3},
\]
for primes $p \geq 5$.

**With Hadamard product $\#49 B * \beta$.**
\[
\sum_{n=0}^{p-1} \sum_{i=0}^{n} \binom{2n}{n} \binom{3n}{n} \binom{2i}{i}^2 \binom{2n-2i}{n-i}^2 (25n^2 - 15n - 6) \frac{1}{512n} \equiv -6p^2 \pmod{p^3},
\]
for primes $p \geq 7$.

**With Hadamard product $A * \delta$.**
\[
\sum_{n=0}^{p-1} \binom{2n}{n}^2 \sum_{i=0}^{n} \frac{(-1)^i3^{n-3i}(3i)!}{i!^3} \binom{n}{3i} \binom{n+i}{i} \frac{(-1)^n}{3^{6n}} (803n^2 + 416n + 68) \equiv 68p^2 \pmod{p^3},
\]
for primes $p \geq 5$.

**With Hadamard product $C * \theta$.**
\[
\sum_{n=0}^{p-1} \sum_{i=0}^{n} 16^{n-i} \binom{2n}{n} \binom{4n}{2n} \binom{2i}{i}^3 \binom{2n-2i}{n-i} \frac{18n^2 - 10n - 3}{80^2n} \equiv -3 \left( \frac{5}{p} \right) p^2 \pmod{p^3},
\]
for primes $p \geq 5$ and
\[
\sum_{n=0}^{p-1} \sum_{i=0}^n 16^{n-i} \binom{2n}{n} \binom{4n}{2n} \binom{2i}{i}^3 \binom{2n-2i}{n-i} \frac{1046529n^2 + 227104n + 16032}{1050625^n} \\
\equiv 16032 \left( \frac{41}{p} \right) p^2 \pmod{p^3},
\]
for primes $p \geq 7$ and $p \neq 41$.

5. Conclusion

We have recovered the 10 hypergeometric Ramanujan series in [19] and found a new one that the second author missed. But more important, finding the relation among the function $T(q)$ and the Gromov-Witten potential has allowed us to generalize the conjectures of the second author in [17] to the case of non-hypergeometric Ramanujan-Sato like series. Then, by getting $e$, $h$ and $f$ from a singular solution (it always exists) we have solved our equations finding several nice non-hypergeometric series for $1/\pi^2$. Finally, we have checked the corresponding supercongruences of Zudilin-type.

Appendix 1. A Maple program for case \( \tilde{8} \).

We use the YY-pullback found in [1]. In order to also treat the case when $z$ and $q$ are negative we introduce a sign $u = \pm 1$.

\[
\text{with(combinat):} \\
\text{p(1):=expand(-36*(2592*n^4+5184*n^3+6066*n^2+3474*n+755));} \\
\text{p(2):=expand(2^4*3^10*(4*n+3)*(4*n+5)*(12*n+11)*(12*n+13));} \\
\text{V:=proc(n) local j: if n=0 then 1; else sum(stirling2(n,j)*z^j*Dz^j,j=1..n); fi; end:} \\
\text{L:=collect(V(4)+add(add(z^m*coeff(p(m),n,k)*V(k),m=1..2),k=0..4),Dz):} \\
\text{Order:=51:} \\
\text{with(DEtools):} \\
\text{r:=formal_sol(L,[Dz,z],z=0):} \\
\text{y0:=r[4]: y1:=r[3]: y2:=r[2]: y3:=r[1]:} \\
\text{q:=series(exp(y1/y0),z=0,51): m:=solve(series(q,z)=s,z):} \\
\text{convert(simplify(series(subs(z=m,1/2*(y1*y2/y0^2-y3/y0)),s=0,51))), polynom):} \\
\text{T:=coeff(-\%*ln(s),0):} \\
\text{e:=5: h:=70: f:=16/3:} \\
\text{H:=proc(u) local k,y,z0,q0,j,y0,yy,i,jj; y0:=-10; for i from 0 to 60 do} \\
\text{k:=i/3; Digits:=50; yy:=fsolve(y^3/6-Pi^2/2*(y+k*e)*y-h*Zeta(3)-subs(z0*u*exp(y),T),y=y0);} \\
\text{end; H;}
\]
Copy and paste the program in Maple and execute $H(1)$ and $H(-1)$. You will get the following results:

\[
\begin{align*}
\begin{bmatrix}
\frac{8}{3} & 1.00000000000000000000000000000000000000000000000000 \\
160.0000000000000000000000000000000000000000000000007
\end{bmatrix}, \\
\begin{bmatrix}
\frac{5}{3} & -2.62143999999999999999999999999999999999999999999996 \\
85.00000000000000000000000000000000000000000000000000
\end{bmatrix}, \\
\begin{bmatrix}
15 & -2.38878719999999999999999999999999999999999999999995 \\
2660.99999999999999999999999999999999999999999999999996
\end{bmatrix}
\end{align*}
\]

To use the program with other cases one has to change the values of $e$, $h$, $f$ and replace the polynomials $p(1)$, $p(2)$, etc, with those corresponding to the new pullback and the new number 2 in $m = 1..2$, with the total number of polynomials.

Acknowledgement. We would like to thank Wadim Zudilin who, even in his exile on the other side of the earth, has shown great interest in our work.

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