On massive higher spins in $d = 3$

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ABSTRACT: In this paper we consider a frame-like gauge invariant description of massive higher spin bosons and fermions in $d = 3$ and provide for the first time a proof that such formulation does describe just one massive physical degree of freedom with the appropriate helicity. For this purpose we completely fix all the gauge symmetries and show that all other auxiliary components vanish on-shell, while the only remaining highest component satisfies the correct equations. As a bonus, we show that the Lagrangians for the so-called self-dual massive spin-3 and spin-4 fields proposed by Aragone and Khoudeir (as well as their generalization to arbitrary integer and half-integer spins) can be obtained from the gauge invariant ones by the appropriate gauge fixing.

KEYWORDS: Field Theories in Lower Dimensions, Gauge Symmetry, Higher Spin Gravity, Higher Spin Symmetry

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1 Introduction

A classical (now called metric-like) formulation for massive higher spin bosons and fermions has been constructed by Singh-Hagen [1, 2]. It requires an introduction of a number of auxiliary fields which themselves vanish on-shell and provide all necessary constraints. Later on, a gauge invariant metric-like formulation for the higher spin massive bosons [3] and fermions [4] were proposed. The main idea was to combine an appropriate set of massless fields and join them together keeping all the (modified) gauge symmetries. In this, the description of the massless fields was based on the Fronsdal formulation [5, 6] which uses double traceless completely symmetric (spin-)tensors, while the gauge transformation parameters are traceless ones. Taking into account that a double traceless tensor is equivalent to two traceless ones one can completely fix the gauge setting half of these traceless components to zero. The result coincides with the Singh-Hagen formulation.

An alternative (now called frame-like) formulation for the massless higher spin bosons and fermions was proposed by Vasiliev [7–9]. It requires an introduction of the physical and auxiliary one-forms which enter the free Lagrangian as well as a set of extra fields which do not enter the free Lagrangian but (being equivalent to higher derivatives of the physical field) play a very important role in an interacting theory. A frame-like gauge invariant formulation for the massive higher spin fields was also constructed [10–12] along the same lines as in the metric-like case.

As is well known, in three dimensions all massless higher spin fields with $s \geq 3/2$ do not have any physical degrees of freedom being a pure gauge, while an irreducible representation for the massive case corresponds to just one physical degree of freedom with a helicity $+s$ or $-s$. A first formulation for the arbitrary spin massive bosons and fermions was proposed by Tyutin and Vasiliev [13]. It used a set of Lagrangian multipliers to achieve all necessary constraints. One more possibility (specific to three dimensions) is a so-called topologically massive formulation (see e.g. [14] and references therein). At last, a frame-like gauge invariant formulation for the massive higher spins has been constructed along the same line as in $d \geq 4$ spaces (see review [15] and references therein). In spite of the fact that massless higher spins in $d = 3$ do not have any physical degrees of freedom,
the construction works and already found some successful applications [15, 16]. However, a
strict proof that such formulation does describe just one degree of freedom with the correct
helicity was absent up to now. Our aim here is to fill this gap.

Our strategy here is to fix the gauge and show that all the auxiliary components
vanish on-shell, while the remaining main field does satisfy the correct equations. Note,
that starting with the spin-2 one faces a situation where one and the same field plays double
role being a gauge field for one transformations and a Stueckelberg field for another one.
So to correctly fix the gauge we decompose our one-forms into irreducible components
and use a metric-like multispinor formalism similar to the one used in [14]. Then we
show that the equations for the Stueckelberg components follow from the equations for the
other components and thus we can set them to zero directly in the Lagrangian (see e.g.
discussion on this theme in appendix A of [17]). After that we show that all other auxiliary
components indeed vanish on-shell leaving us with just one field $\omega^{(2s)}$ which satisfies

$$\frac{1}{s} D^\alpha_{\beta} \omega^{(2s-1)\beta} - M \omega^{(2s)} \approx 0, \quad D_{\beta(2)} \omega^{(2s-2)\beta(2)} \approx 0.$$ 

In the paper [18] Aragone and Khoudeir proposed the Lagrangians for the so-called
self-dual massive spin-3 and spin-4 fields generalizing the massive spin-2 case [19]. Such
formulation uses a one-form as the main gauge field as well as a set of zero-forms as the
auxiliary ones. Our second aim in this work is to show that these Lagrangians as well as
their generalization for the arbitrary spin bosons and fermions can be directly obtained
from our gauge invariant formulation by the appropriate gauge fixing.

**Notation and conventions.** For simplicity we work in flat three dimensional space. All
objects are one-forms or zero-forms having a number of completely symmetric local spinor
indices which we denote $\alpha(n) = (\alpha_1, \alpha_2, \ldots, \alpha_n)$. A coordinate independent description of
the background space is provided by the frame one-form $\omega^{(2)}$ and a Lorentz covariant
derivative $D$ such that $D \wedge D = 0$. Also we use two and three-forms defined as:

$$e^{\alpha(2)} \wedge e^{\beta(2)} = \epsilon^{\alpha\beta} E^{\alpha\beta}, \quad E^{\alpha(2)} \wedge e^{(2)} = \epsilon^{\alpha\beta} \epsilon^{\alpha\beta} E.$$ 

In the main text wedge product sign will be omitted.

## 2 Bosons

In three dimensions a massive higher spin boson describes in general two physical degrees
of freedom. But in the frame-like formalism it is possible to separate a complete system
into two independent subsystems each describing just one helicity (see [16] for details).
Such description uses a set of one-forms $\Omega^{\alpha(2k)}$, $1 \leq k \leq s - 1$ and one zero-form $B^{(2)}$.
The Lagrangian:

$$\mathcal{L} = \sum_{k=1}^{s-1} (-1)^{k+1} \left[ -a_k \Omega^{(2k)} D \Omega^{(2k)} + \frac{M}{2(k+1)} \Omega^{(2k-1)\beta(2)} \epsilon^{\beta(2)\gamma} \Omega^{(2k-1)\gamma} \right]$$

$$-B_{\alpha\beta} \epsilon^{\alpha\beta} \gamma B^{\alpha\beta} - M s \epsilon^{\alpha(2)} B^{(2)} - 2a_0 \Omega^{(2)} \epsilon^{\alpha(2)} B^{\alpha\gamma},$$

(2.1)
where
\[ a_k^2 = \frac{(s + k + 1)(s - k - 1)}{2(k + 1)(2k + 3)} M^2, \]  
(2.2)
is invariant under the following gauge transformations:

\[ \delta \Omega^{\alpha(2k)} = D\eta^{\alpha(2k)} + a_k e^{\alpha\beta(2)} \eta^{(2k)\beta(2)} + \frac{a_{k-1}}{k} e^{\alpha(2)} \eta^{(2k-2)} \]
\[ + \frac{Ms}{2k(k + 1)} e^{\alpha\beta} \eta^{\alpha(2k-1)\beta}, \]
\[ \delta \Omega^{\alpha(2)} = D\eta^{\alpha(2)} + a_1 e^{\alpha\beta(2)} \eta^{\alpha(2)\beta(2)} + \frac{Ms}{4} e^{\alpha\beta} \eta^{\alpha\beta}, \]
\[ \delta B^{\alpha(2)} = 2a_0 \eta^{\alpha(2)}. \]

Let us consider \( \eta^{\alpha(2)} \)-transformations:

\[ \delta \Omega^{\alpha(2)} = D\eta^{\alpha(2)} + \frac{Ms}{4} e^{\alpha\beta} \eta^{\alpha\beta}, \]
\[ \delta \Omega^{\alpha(4)} = \frac{a_1}{6} e^{\alpha(2)} \eta^{\alpha(2)}, \]
\[ \delta B^{\alpha(2)} = 2a_0 \eta^{\alpha(2)}. \]

(2.4)
The zero-form \( B^{\alpha(2)} \) is a Stueckelberg field for these transformations and so we can partially fix the gauge putting \( B^{\alpha(2)} = 0 \) directly in the Lagrangian. But for the sake of comparison with the more complicated case below, it is instructive to show that the equation for the field \( B^{\alpha(2)} \) follows from the equations of other fields. Indeed, from the gauge transformations it follows (as we have explicitly checked):

\[ D \frac{\delta L}{\delta \Omega^{\alpha(2)}} - \frac{Ms}{4} e^{\alpha\beta} \frac{\delta L}{\delta \Omega^{\alpha\beta}} - \frac{a_1}{6} e^{\alpha(2)} \frac{\delta L}{\delta \Omega^{\alpha(2)}} = 2a_0 \frac{\delta L}{\delta B^{\alpha(2)}}. \]

(2.5)
Now we put \( B^{\alpha(2)} = 0 \) and this gives us the Lagrangian:

\[ L = \sum_{k=1}^{s-1} (-1)^{k+1} \left[ \frac{1}{2} \Omega_{\alpha(2k)} D\Omega^{\alpha(2k)} + \frac{Ms}{2(k + 1)} \Omega_{\alpha(2k-1)\beta} e^{\beta \gamma} \Omega^{\alpha(2k-1)\gamma} - a_k \Omega_{\alpha(2k)\beta(2)} e^{\beta(2)} \Omega^{\alpha(2k)} \right], \]

(2.6)
which is still invariant under the remaining gauge transformations \( \eta^{\alpha(2k)}, k > 1 \). Each one-form \( \Omega^{\alpha(2k)} \) for \( k > 1 \) plays double role being gauge field for one transformation and a Stueckelberg field for another one. Thus to correctly fix all the gauge transformations we use a metric-like multipiner formalism similar to the one used e.g. in [14]. Namely, we decompose each one-form into three zero-forms as follows:

\[ \Omega^{\alpha(2k)} = e^{\beta(2)} \omega_+^{\alpha(2k)\beta(2)} + e^{\alpha\beta} \omega_0^{\alpha(2k-1)\beta} + e^{\alpha(2)} \omega_-^{\alpha(2k-2)}, \quad D = e^{\alpha(2)} D_{\alpha(2)}. \]

(2.7)
Then by a straightforward calculations we can rewrite the Lagrangian in terms of these new variables:

\[ L = \sum_{k=1}^{s-1} (-1)^{k+1} \left[ \omega_{+,\alpha(2k+1)} D^{\gamma} \omega_+^{\alpha(2k+1)\gamma} + 2k \omega_{+,\alpha(2k)\beta(2)} D^{\beta(2)} \omega_0^{\alpha(2k)} \right. \]
\[ + 2k \omega_{0,\alpha(2k-1)} D^{\gamma} \omega_-^{\alpha(2k-1)\gamma} + 2k(k + 1)(2k - 1) \omega_0,\alpha(2k-2) \beta(2) D^{\beta(2)} \omega_-^{\alpha(2k-2)} \]
\[ \omega_0,\alpha(2k-2) \beta(2) D^{\beta(2)} \omega_-^{\alpha(2k-2)} \]
\[ \omega_0,\alpha(2k-2) \beta(2) D^{\beta(2)} \omega_-^{\alpha(2k-2)} \]
\[-k(k - 1)(2k - 1)(2k + 1)\omega_{-\alpha(2k-3)\beta}D^\beta_\gamma \omega_\alpha^{(2k-3)\gamma} + \frac{Ms}{2(k + 1)} \left( -\omega_{+\alpha(2k+2)}\omega_\alpha^{(2k+2)} + 2(k + 1)\omega_\alpha^{(2k)} + k(k + 1)(2k - 1)\omega_{-\alpha(2k-2)}\omega_\alpha^{(2k-2)} \right) + a_k \left( 2(k + 2)\omega_0^{(2k+2)} + 2(k + 1)(2k + 3)\omega_{-\alpha(2k)}\omega_\alpha^{(2k)} \right) \right].

(2.8)

The Lagrangian equations for the \(\omega_{+\theta} \) components (which are gauge invariant hence notations):

\[
\mathcal{R}_+^{\alpha(2k+2)} = -kD^\beta_\gamma \omega_0^{(2k+1)\beta} + \frac{k}{(k + 1)(2k + 1)} D^\alpha_\beta \omega_\alpha^{(2k)} - \frac{Ms}{(k + 1)} \omega_+^{(2k+2)} + 2(k + 2)a_k \omega_0^{(2k+2)} + k(k + 1)(2k - 1)\omega_{-\alpha(2k)}\omega_\alpha^{(2k)}.
\]

(2.9)

\[
\mathcal{R}_0^{\alpha(2k)} = -2kD^\beta_\gamma \omega_+^{(2k)\beta(2)} + 2D^\alpha_\beta \omega_0^{(2k-1)\beta} + 2(k + 1)D^\gamma_\beta \omega_-^{(2k-2)} + 2Ms(2k) - 2(k + 1)a_k \omega_0^{(2k)} + 2k(k + 1)(2k + 3)a_k \omega_0^{(2k)} + k(k - 1)(2k + 1)Ms \omega_-^{(2k-2)} - 2k(k - 1)(2k + 1)a_k \omega_0^{(2k-2)}.
\]

(2.10)

Now let us consider a concrete gauge transformation \(\eta^{(2k)}\):

\[
\delta\omega_0^{(2k)} = \frac{a_k}{(k + 1)(2k + 1)} e^{(2)\eta} \eta^{(2k)},
\]

\[
\delta\omega_+^{(2k)} = D\eta^{(2k)} + \frac{Ms}{2k(k + 1)} e^{(2)\eta} \eta^{(2k-1)\beta} + \frac{k}{k + 1} D\eta^{(2)} \eta^{(2k)},
\]

(2.12)

\[
\delta\omega_-^{(2k-2)} = a_k \eta^{(2k-2)\beta(2)}.
\]

(2.13)

For the new variables we obtain:

\[
\delta\omega_+^{(2k)} = a_k \frac{k}{(k + 1)(2k + 1)} \eta^{(2k)},
\]

\[
\delta\omega_-^{(2k-2)} = \frac{1}{k(k + 1)} D^\beta_\gamma \eta^{(2k-2)\beta(2)} + \frac{Ms}{2k(k + 1)} \eta^{(2k)}.
\]
From these equations we obtain a relation (which we have explicitly checked):

\[-D_\beta R_0^{(2k-1)\beta} - \frac{2(k+1)}{k(2k-1)(2k+1)} D^{\alpha(2)} R_0^{\alpha(2k-2)} + Ms R_0^{\alpha(2k)} - \frac{2ka_k}{(2k+1)} R_+^{\alpha(2k)} = 2k(k+1) \left[ D_{\beta(2)} R_+^{\alpha(2k+2)} + a_{k-1} R_+^{\alpha(2k)} \right].\]  

(2.14)

Taking into account that the highest component $\omega_+^{\alpha(2s)}$ is not a Stueckelberg field, we find that the equation for the $\omega_+^{\alpha(2s-2)}$ component follows from the equations of other fields. Then, using (2.14) recursively we find that the equations for all $\omega_+$ components follow from the equations of other fields. Thus we can completely fix all the gauge symmetries setting all $\omega_+^{\alpha(2k)} = 0$, $2 \leq k \leq s-1$. Then the equations (2.9) reduce to

\[R_+^{\alpha(2k+2)} = 2(k+2)a_k \omega_0^{\alpha(2k+2)} + \frac{k}{k(2k+1)} D^{\alpha(2)} \omega_0^{\alpha(2k)} \approx 0, \quad k \geq 1.\]  

(2.15)

There is no such equation for $k = 0$, but we still can use the equation (2.14) for $k = 1$ and obtain:

\[-D_\beta R^{\alpha\beta} - \frac{4}{3} D^{\alpha(2)} R_+ - Ms R_0^{\alpha(2)} - \frac{2a_1}{3} R_0^{\alpha(2)} - 4D_{\beta(2)} R_+^{\alpha(2)\beta(2)} = 16a_0^2 \omega_0^{\alpha(2)} \approx 0.\]  

(2.16)

Thus we see that all components $\omega_0 \approx 0$. At last, from

\[R_- = 3Ms \omega_- \approx 0,\]  

(2.17)

\[R_0^{\alpha(2k)} = 2k(k+1)(2k+3)a_k \omega_-^{\alpha(2k)} + 2(k+1) D^{\alpha(2)} \omega_-^{\alpha(2k-2)} \approx 0.\]

we find that all components $\omega_- \approx 0$ as well. This leaves us with just one main field $\omega_+^{\alpha(2s)}$ satisfying the two equations:

\[\frac{1}{s} D_\beta R_+^{\alpha(2s-1)\beta} - Ms \omega_+^{\alpha(2s)} \approx 0, \quad D_{\beta(2)} \omega_+^{\alpha(2s-2)\beta(2)} \approx 0.\]  

(2.18)

3 Fermions

A frame-like description of the massive spin-$(s+1/2)$ fermions is similar to the one for massive bosons.\(^1\) It requires a set of one-forms $\Phi^{\alpha(2k+1)}$, $0 \leq k \leq s-1$ and one zero-form $\phi^\alpha$. The Lagrangian

\[\frac{1}{i} \mathcal{L} = \sum_{k=0}^{s-1} (-1)^{k+1} \left[ \frac{1}{2} \Phi_\alpha^{(2k+1)} D\Phi^{\alpha(2k+1)} + \frac{(2s+1)M}{2(2k+3)} \Phi_\alpha^{(2k+2)} e^{\beta(2)\gamma} \Phi^{\alpha(2k)} \right]

+ a_k \Phi_\alpha^{(2k-1)\beta(2)} e^{\beta(2)\gamma} \Phi^{\alpha(2k-1)} + \frac{1}{2} \phi_\alpha E_\alpha^\beta D\phi^\beta - \frac{(2s+1)M}{2} E_\alpha \phi^\alpha + a_0 \Phi_\alpha E_\beta^\alpha \phi^\beta,\]  

(3.1)

\(^1\)Let us stress that for all the formulas that follow it is important to consider all fermionic objects (both fields and gauge parameters) as anticommuting ones.
where
\[ a_k^2 = \frac{(s + k + 1)(s - k)}{2(k + 1)(2k + 1)} M^2, \quad a_0^2 = 2s(s + 1)M^2, \tag{3.2} \]
is invariant under the following gauge transformations:
\[
\delta_0 \Phi^{\alpha(2k+1)} = D\zeta^\alpha(2k+1) + \frac{(2s + 1)M}{(2k + 1)(2k + 3)} e^\alpha_\beta \zeta^{(2k)\beta} + \frac{a_k}{k(2k + 1)} e^{(2)}_\beta \zeta^{(2k-1)} + a_{k+1} e^{(2)}_\beta \zeta^{(2k+1)\beta}(2), \tag{3.3} \]
\[
\delta_0 \Phi^\alpha = a_0 \zeta^\alpha.
\]
First, let us consider the following gauge transformations:
\[
\delta \Phi^\alpha = D\zeta^\alpha + \frac{(2s + 1)M}{3} e^\alpha_\beta \zeta^\beta, \quad \delta \Phi^{(3)} = \frac{a_1}{3} e^{(2)}_\beta \zeta^\alpha, \quad \delta \phi^\alpha = a_0 \zeta^\alpha. \tag{3.4} \]
This produces a relation (which we explicitly checked):
\[
D \frac{\delta L}{\delta \Phi^\alpha} = \frac{(2s + 1)M}{3} e^\alpha_\beta \frac{\delta L}{\delta \Phi^\beta} - \frac{a_1}{3} e^{(2)}_\beta \frac{\delta L}{\delta \Phi^{(3)}} = a_0 \frac{\delta L}{\delta \phi^\alpha}. \tag{3.5} \]
Thus the equation for the field \( \phi^\alpha \) follows from the equations for other fields and we can put \( \phi^\alpha = 0 \) directly in Lagrangian. We obtain then the Lagrangian:
\[
\frac{1}{i} \mathcal{L} = \sum_{k=0}^{s-1} (-1)^{k+1} \left[ \frac{1}{2} \Phi_{\alpha(2k+1)} D\Phi^{\alpha(2k+1)} + \frac{(2s + 1)M}{2(2k + 3)} \Phi_{\alpha(2k)\beta} e^\beta_\gamma \Phi^{\alpha(2k)\gamma} \right.
\[
+ a_k \Phi_{\alpha(2k-1)\beta(2)} e^{(2)}_\beta \Phi^{\alpha(2k-1)} \right], \tag{3.6} \]
which is still invariant under the remaining gauge transformations. We proceed with the transition to the new variables:
\[
\Phi^{\alpha(2k+1)} = e^{(2)}_\beta \phi^\alpha_{+(2k+1)\beta(2)} + e^\alpha_\beta \phi^\alpha_{(2k)\beta} + e^{(2)}_\beta \phi^\alpha_{-(2k-1)} \tag{3.7} \]
The Lagrangian in terms of these new variables has the form:
\[
\frac{1}{i} \mathcal{L} = \sum_{k=0}^{s-1} (-1)^{k+1} \left[ \phi_{+(2k+1)\beta(2)} D^\beta_\gamma \phi^\alpha_{+(2k+2)\gamma} + (2k + 1) \phi_{+(2k+1)\beta(2)} D^\beta_\gamma \phi^\alpha_{-(2k-1)\gamma} \right.
\[
+ (2k + 1) \phi_{0,\alpha(2k)\beta} D^\beta_\gamma \phi^\alpha_{0(2k+2)\gamma} + k(2k + 1)(2k + 3) \phi_{0,\alpha(2k-1)\beta(2)} D^\beta_\gamma \phi^\alpha_{-(2k-1)\gamma} \right.
\[
- k(2k + 1)(2k + 1) \phi_{-\alpha(2k-2)\beta} D^\beta_\gamma \phi^\alpha_{-(2k-2)\gamma} \right.
\[
+ \frac{(2s + 1)M}{2(2k + 3)} \left( -\phi_{+(2k+3)\gamma} + (2k + 3) \phi_{0,\alpha(2k+1)} \phi^\alpha_{-(2k-1)} \right)
\[
+ k(2k + 1)(2k + 3) \phi_{-\alpha(2k-1)} \phi^\alpha_{-(2k-1)} \right]
\[
-a_k \left( (2k + 3) \phi_{0,\alpha(2k+1)} \phi^\alpha_{+(2k+1)} + (k + 1)(2k - 1)(2k + 1) \phi_{-\alpha(2k-1)} \phi^\alpha_{-(2k-1)} \right) \right]. \tag{3.8} \]
Thus we can completely fix all the gauge transformations setting $\phi^{(2k+1)} = 0$. Here also taking into account that the highest component $\phi^{(2k+1)}$ is not a Stueckelberg field we see that equation for the $\phi^{(2k-1)}$ component follows from the equations for other fields. Then, using relation (3.14) recursively we find the same property for all $\phi^{(2k+1)}$ components. Thus we can completely fix all the gauge transformations setting $\phi^{(2k+1)} = 0$, $1 \leq k \leq s-2$. 

Lagrangian equations for these variables look like:

\[
\mathcal{F}^{(2k+3)}_+ = \frac{2}{(2k+3)} D^{\alpha} \beta \phi^{(2k+2)\beta} + \frac{(2k+1)}{(k+1)(2k+3)} D^{(2)} \phi^{(2k+1)} + (2s+1)M \phi^{(2k+3)} + (2k+5) a_{k+1} \phi^{(2k+3)}, \\
\mathcal{F}^{(2k+1)}_0 = -(2k+1) D^{(2)} \phi^{(2k+1)\beta} + 2 D^{\alpha} \beta \phi^{(2k+1)} + (2k+3) D^{\alpha} \phi^{(2k-1)} + (2k+3) \psi^{(2k+1)} + (k+2)(2k+3) a_{k+1} \phi^{(2k+1)}, \\
\mathcal{F}^{(2k-1)}_- = -k(k+1)(2k+3) D^{(2)} \phi^{(2k-1)\beta} - 2k(k+1)(2k+1) D^{\alpha} \phi^{(2k-2)\beta} - k(k+1)(2k+1) M \phi^{(2k-1)} - (k+1)(2k+1) a_k \phi^{(2k-1)}.
\]

Let us consider $\zeta^{(2k+1)}$-transformations:

\[
\delta \Phi^{(2k+3)} = \frac{a_{k+1}}{(k+1)(2k+3)} \phi^{(2k+1)} + \frac{(2s+1)M}{(2k+1)(2k+3)} \phi^{(2k+1)} + (2k+5) a_{k+1} \phi^{(2k+3)}, \\
\delta \Phi^{(2k+1)} = D \zeta^{(2k+1)} + \frac{(2s+1)M}{(2k+1)(2k+3)} \phi^{(2k+1)} + (2k+5) a_{k+1} \phi^{(2k+3)}, \\
\delta \Phi^{(2k-1)} = a_k \epsilon \phi^{(2k-1)}.
\]

In terms of the new variables this gives:

\[
\delta \phi^{(2k+1)} = \frac{a_{k+1}}{(k+1)(2k+3)} \phi^{(2k+1)}, \\
\delta \phi^{(2k+3)} = \frac{1}{(k+1)(2k+3)} D^{\alpha} \phi^{(2k+1)}, \\
\delta \phi^{(2k+1)} = -\frac{2}{(k+1)(2k+3)} D^{\alpha} \phi^{(2k+1)} + \frac{(2s+1)M}{(2k+1)(2k+3)} \phi^{(2k+1)}, \\
\delta \omega^{(2k-1)} = \frac{1}{(k+1)(2k+1)} D^{(2)} \phi^{(2k-1)\beta} + \frac{(2s+1)M}{(2k+1)(2k+3)} \phi^{(2k+1)}, \\
\delta \phi^{(2k+1)} = a_k \phi^{(2k+1)}.
\]

Hence we obtain the following relation (which we also explicitly checked):

\[
-2D^{\alpha} \beta \mathcal{F}^{(2k+1)} - \frac{(2k+3)}{k(k+1)(2k+1)} D^{\alpha} \mathcal{F}^{(2k-1)} + (2s+1)M \mathcal{F}^{(2k+1)} + \frac{(2k+1)}{k+1} a_{k+1} \mathcal{F}^{(2k+1)} = (2k+1)(2k+3) \left[ D^{(2)} \mathcal{F}^{(2k-1)\beta} + a_k \mathcal{F}^{(2k+1)} \right].
\]

Here also taking into account that the highest component $\phi^{(2k+1)}$ is not a Stueckelberg field we see that equation for the $\phi^{(2k-1)}$ component follows from the equations for other fields. Then, using relation (3.14) recursively we find the same property for all $\phi^{(2k+1)}$ components. Thus we can completely fix all the gauge transformations setting $\phi^{(2k+1)} = 0$, $1 \leq k \leq s-2$. 

\[
\frac{2}{(2k+3)} D^{\alpha} \beta \phi^{(2k+2)\beta} + \frac{(2k+1)}{(k+1)(2k+3)} D^{(2)} \phi^{(2k+1)} + (2s+1)M \phi^{(2k+3)} + (2k+5) a_{k+1} \phi^{(2k+3)}.
\]
Then the relation (3.9) reduces to:

\[ F_{\alpha}^{(2k+3)} = -(2k + 5)a_{k+1}\phi_0^{(2k+3)} + \frac{(2k + 1)}{(k + 1)(2k + 3)} D_{\alpha}^{(2)} \phi_0^{(2k+1)} \approx 0. \]  

(3.15)

One more equation we obtain from (3.14) at \( k = 0 \):

\[ -2D_{\alpha}^{\beta} F_{0}^{\beta} + (2s + 1)M F_{0}^{\alpha} - a_{1} F_{\alpha}^{\alpha} - 3D_{\beta(2)} F_{0}^{\alpha(2)} = 9a_{0}^{2} \phi_{0}^{\alpha} \approx 0. \]  

(3.16)

Thus all the components \( \phi_{0}^{\alpha(2k+1)} \approx 0 \). At last, using

\[ F_{0}^{\alpha} = 6a_{1} \phi_{\alpha}^{\alpha} \approx 0, \]  

(3.17)

\[ F_{0}^{\alpha(2k+1)} = (k + 2)(2k + 1)(2k + 3)a_{k+1} \phi_{-}^{\alpha(2k+3)} + (2k + 3)D_{\alpha}^{(2)} \phi_{-}^{(2k-1)} \approx 0, \]

we find that all the components \( \phi_{-}^{\alpha(2k+1)} \approx 0 \) as well. This leaves us with just one highest field \( \phi_{+}^{\alpha(2s+1)} \) satisfying

\[ \frac{2}{(2s + 1)} D_{\alpha}^{\beta} \phi_{+}^{\alpha(2s)} - M \phi_{+}^{\alpha(2s+1)} \approx 0, \quad D_{\beta(2)} \phi_{+}^{\alpha(2s-1)(2)} \approx 0. \]  

(3.18)

4 Self-dual fields a la Aragone-Khoudeir

In this section we show that the self-dual Lagrangians for massive spin-3 and spin-4, proposed by Aragone and Khoudeir in [18], appear as the gauge fixed versions of our gauge invariant Lagrangians. For completeness, we begin with the self-dual massive spin-2 [19].

Spin 2. The Lagrangian in terms of the one-form \( \Omega_{\alpha}^{(2)} \) looks like:

\[ L = \frac{1}{2} \Omega_{\alpha}^{(2)} D \Omega_{\alpha}^{(2)} + \frac{M}{2} \Omega_{\alpha,\beta} e_{\gamma}^{\beta} \Omega_{\alpha}^{\gamma} \]  

(4.1)

and does not have any gauge symmetries. To show, that it indeed describes just one physical degree of freedom, we again use new variables:

\[ \Omega_{\alpha}^{(2)} = e_{\beta(2)} \omega_{+}^{\alpha(2),\beta(2)} + e_{\alpha,\beta} \omega_{0}^{\alpha,\beta} + e_{\alpha}^{(2)} \omega_{-}. \]  

(4.2)

Lagrangian in new variables has the form:

\[ L = \omega_{+,\alpha(2),\beta(2)} D_{\gamma}^{\beta} \gamma \omega_{+,\gamma}^{\alpha(3)} + 2\omega_{+,\alpha(2),\beta(2)} D_{\gamma(2)}^{\beta(2)} \omega_{0}^{\alpha(2)} + 2\omega_{0,\alpha,\beta} D_{\gamma}^{2} \gamma \omega_{0}^{\alpha(2)} + 4\omega_{0,\alpha(2)} D_{\alpha(2)}^{\omega_{-}} \]  

\[ -\frac{M}{2} \omega_{+,\alpha(4)} \omega_{+}^{(4)} + 2M \omega_{0,\alpha(2)} \omega_{0}^{(2)} + 3M \omega_{-}. \]  

(4.3)

Lagrangian equations for the new fields look like:

\[ \mathcal{R}_{+}^{\alpha(4)} = \frac{1}{2} D_{\beta}^{\alpha(3)\beta} + \frac{1}{3} D_{\alpha(2)}^{\omega_{0}} - M \Omega_{+}^{\alpha(4)}, \]  

\[ \mathcal{R}_{0}^{\alpha(2)} = -2D_{\beta(2)}^{\alpha(2)\beta(2)} + 2D_{\beta}^{\alpha\beta} \omega_{0}^{\alpha\beta} + 4D_{\alpha(2)}^{\omega_{-}} + 4M \omega_{0}^{\alpha(2)}, \]  

\[ \mathcal{R}_{-} = -4D_{\alpha(2)}^{\omega_{0}} + 6M \omega_{-}. \]  

(4.4)
Now by the straightforward calculations we obtain:
\[ D_{\beta(2)} R^{(2)}_+ + \frac{1}{4} D^\alpha R^{(2)}_0 + \frac{1}{3} D^{(2)} R^- \frac{M}{2} R_0 = -2M^2 \omega^{(2)} \approx 0. \]  
(4.5)
Moreover, it follows that
\[ D_{\alpha(2)} \omega^{(2)}_0 + \frac{1}{4} R^- = \frac{3M}{2} \omega_- \approx 0. \]  
(4.6)
This leaves us with just one field \( \omega^{(4)}_+ \) satisfying
\[ \frac{1}{2} D^\alpha_\beta \omega^{(3)\beta}_+ - M \omega^{(4)}_+ \approx 0, \quad D_{\beta(2)} \omega^{(2)\beta}_+ \approx 0. \]  
(4.7)
**Spin 3.** In this case our initial Lagrangian and gauge transformations have the form:
\[
\mathcal{L} = - \frac{1}{2} \Omega^{(4)} D \Omega^{(4)} - \frac{M}{2} \Omega^{(3)\beta} e^\beta \gamma \Omega^{(3)\gamma} - \frac{M}{2} \Omega_{\alpha(2)\beta(2)} e^{(2)\gamma} \Omega^{(2)}
\]
\[ + \frac{1}{2} \Omega^{(2)} D \Omega^{(2)} + \frac{3M}{4} \Omega^{(3)\beta} e^\beta \gamma \Omega^{(2)} ,
\]  
(4.8)
\[ \delta \Omega^{(4)} = D \eta^{(4)} + \frac{M}{4} e^\alpha_\beta \eta^{(3)\beta}, \quad \delta \Omega^{(2)} = \frac{M}{2} e^{(2)\beta} \eta^{(2)\beta}(2). \]  
(4.9)
Now we fix the gauge setting (i.e. setting \( \omega^{(4)}_+ = 0 \))
\[ \Omega^{(2)} = e^\alpha_\beta \omega^{(2)}_0 + e^{(2)\alpha(2)} \omega_-, \]  
(4.10)
leaving the highest one-form \( \Omega^{(4)} \) intact. We obtain
\[
\mathcal{L} = - \frac{1}{2} \Omega^{(4)} D \Omega^{(4)} - \frac{M}{2} \Omega^{(3)\beta} e^\beta \gamma \Omega^{(3)\gamma} + 2M \Omega_{\alpha(2)\beta(2)} e^{(2)\gamma} \omega^{(2)_0}
\]
\[ + 2E \left[ 4\omega^{(0)} \alpha \beta D^\beta \gamma \omega^{(0)\gamma} + 8\omega^{(0)} \alpha \beta D^{(2)\alpha(2)} \omega_- + 6M \omega^{(0)} \alpha(2) \omega^{(2)_0} + 9M \omega^2 \right]. \]  
(4.11)
The resulting Lagrangian has the same structure as the corresponding one in [18] (up to transition to multispinor formalism).

**Spin 4.** In this case we have three one-forms \( \Omega^{(0)} \), \( \Omega^{(4)} \), \( \Omega^{(2)} \) and two gauge transformations \( \eta^{(6)} \) and \( \eta^{(4)} \). We fix the gauge setting
\[ \Omega^{(4)} = e^\alpha_\beta \omega^{(3)\beta}_0 + e^{(2)\alpha(2)} \omega_- \]  
(4.12)
leaving the highest one-form \( \Omega^{(6)} \) intact. We obtain the Lagrangian in the form:
\[
\mathcal{L} = - \frac{1}{2} \Omega^{(6)} D \Omega^{(6)} + \frac{M}{2} \Omega^{(5)\beta} e^\beta \gamma \Omega^{(5)\gamma} - 8\omega^{(0)\alpha(3)} e^{(3)\gamma} \Omega^{(3)\gamma} \omega^{(2)}
\]
\[ + 4E \left[ -4\omega^{(0)\alpha(3)} D^\beta \gamma \omega^{(3)\gamma} + 18\omega^{(0)\alpha(2)\beta(2)} D^{(2)\alpha(2)} \omega^{(2)} + 30\omega_- \omega^{(2)\alpha(2)} D^\gamma \gamma \omega_- \right.
\]
\[ + 2\omega^{(0)\alpha(2)} D^\beta \gamma \omega^{(3)\gamma} + 2\omega^{(0)\alpha(2)} D^{(2)\alpha(2)} \omega_+ + 10\omega^{(0)\alpha(2)\beta(2)} \omega^{(2)_0} 
\]
\[ - 2M \left( 2\omega^{(0)\alpha(4)} \omega^{(4)} + 30\omega_- \omega^{(2)\alpha(2)} - 2\omega^{(0)\alpha(2) \omega^{(2)_0} - 3\omega_- \omega_-} \right), \]  
(4.13)
which also has the same general structure as the corresponding one in [18].

Similarly, it is possible to obtain the generalization of such construction to arbitrary integer spin, though it appears to be rather complicated. Let us turn to the half-integer spins instead. Again for completes we begin with the simplest spin-3/2 case.
Spin 3/2. The Lagrangian has a very simple form:

\[ L = \frac{1}{2} \Phi^{\alpha \beta} D \Phi_{\alpha \beta} + \frac{M}{2} \Phi^{\alpha} e^{\alpha \beta} \phi^\beta \]  

and does not have any gauge symmetries. To show that it indeed describes just one physical degree of freedom we introduce new variables:

\[ \Phi^{\alpha} = e_{\beta(2)}^{\alpha \beta(2)} + e^{\alpha \beta} \phi^\beta. \]  

Lagrangian in terms of the new variables looks like:

\[ L = \phi^{\alpha(2) \beta(2)} + \phi^{\alpha(2) \beta} + \phi^{\alpha} D \phi^{\beta} - \frac{M}{2} \phi^{\alpha(3)} \phi^{\beta(3)} + \frac{3M}{2} \phi^{\alpha} \phi^{\beta}, \] 

while Lagrangian equations have the form:

\[ R^{\alpha(3)} = 2D^{\alpha(2) \beta(2)} - D^{\alpha(2) \beta} + \frac{1}{3} \phi^{\alpha(3)}, \]

\[ R^{\alpha} = -D^{\beta(2) \phi^{\alpha(2)}} + 2D^{\alpha \beta(2)} \phi^{\beta} + 3\phi^{\alpha}. \]  

By direct and simple calculations we obtain:

\[ 2D^{\alpha(2) \beta(2)} + \frac{2}{3} D^{\alpha \beta} R^{\beta} - MR^{\alpha} = -3M^2 \phi^{\alpha} \approx 0, \] 

and as a result:

\[ 2D^{\alpha(2) \beta(2)} - M \phi^{\alpha(3)} \approx 0, \quad D^{\beta(2) \phi^{\alpha(2)}} \approx 0. \]  

Spin 5/2. For our last example the initial Lagrangian and gauge transformations look like:

\[ L = \frac{1}{2} \Phi^{\alpha \beta(3)} D \Phi_{\alpha \beta(3)} + \frac{M}{2} \Phi^{\alpha(2) \beta(2) \gamma(2) \phi^{\alpha(2) \gamma}} + a_1 \Phi^{\alpha(3)} \phi^{\beta(2) \phi^{\alpha}}, \]

\[ \phi^{\alpha(3)} = D \zeta^{\alpha(3)} + \frac{M}{3} \phi^{\alpha \beta(2) \phi^{\beta(2)}}, \quad \phi^{\alpha} = a_1 e^{\alpha(3)} \phi^{\beta(2) \phi^{\beta}}. \]  

Now we fix the gauge setting:

\[ \Phi^{\alpha} = e^{\alpha \beta} \phi^\beta. \] 

leaving the highest one-form \( \Phi^{\alpha(3)} \) intact. Then we obtain:

\[ L = \frac{1}{2} \Phi^{\alpha(3)} D \Phi^{\alpha(3)} + \frac{M}{2} \Phi^{\alpha(2) \beta(2) \phi^{\alpha(2) \gamma}} - 2a_1 \Phi^{\alpha(2) \phi^{\beta(2) \phi^{\alpha}} \phi^{\beta(2) \phi^{\alpha}} - E \left[ 4\phi_{\alpha(3)} D^{\alpha \beta(2) \phi^{\beta(2)} + 10M \phi_{\alpha(3)} \phi^{\beta(2)} \phi^{\beta}}. \right. \] 

Note that as in the bosonic case it is possible to obtain a generalization of such construction to the arbitrary half-integer spin.
Conclusion. In this note we have shown that the frame-like gauge invariant Lagrangians constructed previously do describe just one physical degree of freedom with the appropriate helicity. For this purpose we completely fix all the gauge symmetries and show that all other auxiliary components vanish on-shell, while the only remaining highest component satisfies the correct equations. As a bonus, we show that the Lagrangians for the so-called self-dual massive spin-3 and spin-4 fields proposed by Aragone and Khoudeir (as well as their generalization to arbitrary integer and half-integer spins) can be obtained from the gauge invariant ones by the appropriate gauge fixing. Note however that such description appears to be much more complicated as the initial gauge invariant one.

We hope that the (relatively) simple gauge invariant Lagrangians (2.6) and (3.6) can serve as a nice starting point for the investigation of the Lorentz covariant formulation for the massive higher spin interactions in $d = 3$ (for the light-cone formulation see [20–22]).

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