Exact Witten Index in D=2 supersymmetric Yang-Mills quantum mechanics

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Abstract

A new, recursive method of calculating matrix elements of polynomial hamiltonians is proposed. It is particularly suitable for the recent algebraic studies of the supersymmetric Yang-Mills quantum mechanics in any dimensions. For the $D = 2$ system with the SU(2) gauge group, considered here, the technique gives exact, closed expressions for arbitrary matrix elements of the hamiltonian and of the supersymmetric charge, in the occupation number representation. Subsequent numerical diagonalization provides spectrum and restricted Witten index of the system with very high precision (taking into account up to $10^5$ quanta).

Independently, the exact value of the restricted Witten index is derived analytically for the first time.

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1 Introduction

Supersymmetric quantum mechanics provides a simple laboratory to study many properties of supersymmetric systems [1, 2, 3]. Recently supersymmetric Yang-Mills quantum mechanics (SYMQM) in $d$ space dimensions attracts a lot of attention because of its possible relation with M-theory [4] for $d = 9$ and large number of colours. Even though the three loop calculations [5, 6] question the exact equivalence between the two, it still remains a valuable model of the latter sharing many of its features (e.g., continuum spectrum of scattering states with the threshold bound state – a supergraviton). At the same time, studies of the lower-dimensional systems with various gauge groups provide the global understanding of the whole family of these models with many interesting limiting cases. A good example is the $D = 4$ SYMQM with the SU(2) gauge group whose spectrum, in the zero fermion sector, is identical with that of the well known 0-volume glueballs [7, 8, 9, 10]. Moreover, supersymmetry guarantees that, in addition to the continuum of the scattering states, there must also exist localized gluino-glueball bound states with the same masses. Indeed such solutions were found in Ref. [10] where a new algebraic approach to study these systems was proposed. Compact version of the $D = 4$ SYMQM was also studied recently by van Baal who pushed rather far analytical understanding of this system [11, 12].

The technique of [10] is the adaptation of the hamiltonian methods [13] to supersymmetric systems with local gauge invariance. It consists of two steps: first, a finite (e.g., cut off) basis of gauge-invariant eigenstates of the bosonic and fermionic occupation number operators $\hat{B} = a_b a_b^\dagger$ and $\hat{F} = f_b f_b^\dagger$, is generated (we omit all but colour indices for a moment). Second, the matrix representation of the hamiltonian (and any other relevant observable) is derived and numerically diagonalized. All this is automated by implementing standard rules of quantum mechanics in an algebraic language like Mathematica. A faster, compiler based, version is now available [14]. As the cutoff $N_{\text{cut}}$ we choose the number of bosons $B$, i.e., the cut off basis consists of all states with $B \leq N_{\text{cut}}$ and all allowed fermions. The cutoff is gauge and rotationally invariant and consequently the spectrum reveals the full SO($D-1$) symmetry for each $N_{\text{cut}}$. The technique has been applied to Wess-Zumino quantum mechanics, $D = 2$ and $D = 4$ SYMQM and to $D = 5-10$ YMQM, all based on the SU(2) gauge group [10, 15]. In all cases studied until now the spectrum of lower states converges with $N_{\text{cut}}$ before the number of states becomes unmanageable. Calculations become more and more time consuming.
with increasing $D$ and $N$ but the $D = 10$ case is within the reach of today’s computers for the first few values of $N$. Similar methods have been independently developed in Refs [16, 17] to study lower dimensional supersymmetric field theories.

In the first part of this letter we present a new method to calculate matrix representations and apply it to the $D = 2$ SYMQM in Section 3. We calculate recursively the matrix elements of the hamiltonian thereby eliminating lengthy and space consuming process of generation and storing of the basis. For the $D = 2$ system recursions can be solved resulting in closed expressions for any matrix element of the hamiltonian $H$. We then diagonalize $H$ numerically and calculate the restricted Witten index for this system to virtually arbitrary precision. The method can be generalized to higher $D$, allowing to reach values of the cutoff larger than those in Ref. [10].

The second part contains an exact calculation of the restricted index, exploiting the analytic properties of the densities of bosonic and fermionic states, suitably regularized in the infrared. To our knowledge this is the first analytic calculation of the restricted index for this system.

## 2 Supersymmetric Yang-Mills quantum mechanics in two dimensions

This model is exactly soluble [2] also for higher gauge groups [18]. It is however still interesting as it shares some of the complexity of higher dimensional models. For example it has a continuous spectrum which is a characteristic feature of many other supersymmetric models and poses some challenge for the hamiltonian methods. Moreover it has a nontrivial Witten index which was defined only recently [10].

The system, reduced from $D = 2$ to one (time) dimension [19], is described by the three real bosonic variables $x_a(t)$ and three complex, fermionic degrees of freedom $\psi_a(t)$, both in the adjoint representation of SU(2), $a = 1, 2, 3$.

The hamiltonian reads [2]

$$ H = \frac{1}{2} p_a p_a + i g \epsilon_{abc} \psi_a^\dagger x_b \psi_c, $$

where the quantum operators $x, p, \psi, \psi^\dagger$ satisfy the canonical (anti)commutation
rules
\[ [x_a, p_b] = i \delta_{ab}, \quad \{\psi_a, \psi_b^\dagger\} = \delta_{ab}, \quad (2) \]
and can be written in terms of the creation and annihilation operators
\[ x_a = \frac{1}{\sqrt{2}} (a_a + a_a^\dagger), \quad p_a = \frac{1}{i \sqrt{2}} (a_a - a_a^\dagger), \quad (3) \]
\[ \psi_a = f_a, \quad \psi_a^\dagger = f_a^\dagger. \quad (4) \]
The system has a gauge invariance with generators
\[ G_a = \epsilon_{abc} (x_b p_c - i \psi_a^\dagger \psi_c). \quad (5) \]
Therefore the physical Hilbert space consists only of the gauge-invariant states. This constraint is easily accommodated by constructing all possible combinations of creation operators (creators) invariant under SU(2), and using them to generate a complete gauge-invariant basis of states. There are four lower order creators:
\[ (aa) \equiv a_a^\dagger a_a^\dagger, \quad (af) \equiv a_a^\dagger f_a^\dagger, \quad (aff) \equiv \epsilon_{abc} a_a^\dagger f_b^\dagger f_c^\dagger, \quad (fff) \equiv \epsilon_{abc} f_a^\dagger f_b^\dagger f_c^\dagger. \quad (6) \]
Fermionic creators satisfy \((ff) = (af)^2 = (aff)^2 = (fff)^2 = 0\), therefore the whole basis can be conveniently organized into the four towers of states, each tower beginning with one of the following states
\[ |0_F\rangle = |0\rangle; \quad |1_F\rangle = (af)|0\rangle; \quad |2_F\rangle = (aff)|0\rangle; \quad |3_F\rangle = (fff)|0\rangle, \quad (7) \]
where we have labeled the states by the gauge-invariant fermionic number \(F = f_a f_a^\dagger\). To obtain the whole basis it is now sufficient to repeatedly act on the four vectors \((7)\) with the bosonic creator \((aa)\). Acting with other creators either gives zero, due to the Pauli principle, or produces a state from another tower, already obtained by application of \((aa)\). The basis with cutoff \(N_{\text{cut}}\) is then obtained by applying \((aa)\) up to \(N_{\text{cut}}\) times to each of the four “base” states of Eq. \((7)\). Obviously our cutoff is gauge-invariant, since it is defined in terms of the gauge-invariant creators.

The hamiltonian \((1)\) reduces in the physical basis to that of a free bosonic particle
\[ H = \frac{1}{2} p_a p_a + g x_a G_a, \quad (8) \]
therefore it preserves the fermionic number and can be diagonalized independently in each sector spanned by the four towers in Eq. \((7)\).
The spectrum is doubly degenerate even at finite $N_{\text{cut}}$, because of the particle-hole symmetry which is preserved by our cutoff. Particle-hole symmetry relates empty and filled fermionic states ($|0_F\rangle \leftrightarrow |3_F\rangle$) and their 1-particle 1-hole counterparts ($|1_F\rangle \leftrightarrow |2_F\rangle$). On the other hand, the supersymmetry generator

$$Q = \sum_a \psi_a p_a,$$

connects sectors which differ by 1 in the fermionic number, e.g., it connects $0_F-1_F$ sectors and $2_F-3_F$ sectors, but it does not connect $1_F$ and $2_F$ sectors. This can be easily seen in terms of the “$B$-parity”, $P_B = (-1)^B$: $Q$ not only changes $F$, but also $P_B$; however, the $F=1$ and $F=2$ sectors have the same $P_B$, hence matrix elements of $Q$ between these sectors vanish.

**Witten index vs restricted index.** Because the particle-hole symmetry interchanges odd and even fermionic numbers, the Witten index

$$I_W(T) = \sum_i (-1)^{F_i} \exp (-TE_i),$$

vanishes identically for this model\(^1\). Nevertheless one can obtain a nontrivial and interesting information by defining the index restricted to a one pair (e.g., $0_F-1_F$) of sectors

$$I(T)^{(0,1)} = \sum_{i,F_i=0,1} (-1)^{F_i} \exp (-TE_i).$$

Since $Q$ does not connect the $1_F$ sector with the $2_F$ sector, supersymmetry balances independently, and identically, fermionic and bosonic states within the $0_F-1_F$ and $2_F-3_F$ pairs, with the usual exception of the vacuum. Therefore the restricted index is a good and nontrivial measure of the amount of the violation of SUSY, even when the total Witten index vanishes. Obviously, $I^{(2,3)} = -I^{(0,1)}$. In principle, one could also consider $I^{(0,3)}$ and $I^{(1,2)}$, however, similarly to the global index, they vanish due to the particle-hole symmetry. Studying the restricted index is particularly interesting in this model since, due to the continuum spectrum, it does not have to be an integer.

### 3 Exact matrix elements and numerical diagonalization

In order to simplify numerical computations, it is useful to avoid dealing explicitly with the Fock space vectors. This can be achieved by writing every

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\(^1\)This is also true for any finite cutoff since $N_{\text{cut}}$ preserves the particle-hole symmetry.
quantity of interest as a vacuum expectation value (vev) of a gauge-invariant operator, and deriving recursive relations between such vev’s.

We shall deal here explicitly with the $F = 0$ and $F = 1$ sectors, the $F = 2$ and $F = 3$ sectors can be obtained by the particle-hole symmetry. To begin with, let us introduce the gauge-invariant operators

$$A = \sum_b a_b a_b, \quad A^\dagger = \sum_b a_b^\dagger a_b^\dagger = (aa), \quad B = B^\dagger = \sum_b a_b a_b^\dagger, \quad (12)$$

which satisfy the commutation relations

$$[A, A^\dagger] = 4B - 6, \quad [A, B] = 2A. \quad (13)$$

States in the $F = 0$ sector have the form

$$|2n, 0\rangle = \frac{1}{\sqrt{c_n}} (A^\dagger)^n |0\rangle, \quad (14)$$

and those with $F = 1$ can be written as

$$|2n+1, 1\rangle = \frac{1}{\sqrt{c_n}} F^\dagger (A^\dagger)^n |0\rangle, \quad F = \sum_b a_b f_b, \quad F^\dagger = \sum_b a_b^\dagger f_b^\dagger = (af). \quad (15)$$

Observe that in the $F = 0$ sector $f_a f_b^\dagger = \delta_{ab}$ and therefore $F F^\dagger = B$. We are interested in the scalar products

$$\langle 0 | A^{n'} (A^\dagger)^n |0\rangle = \delta_{n',n} c_n, \quad (16)$$

$$\langle 0 | A^{n'} F F^\dagger (A^\dagger)^n |0\rangle = \langle 0 | A^{n'} B (A^\dagger)^n |0\rangle = \delta_{n',n} c'_n, \quad (17)$$

and in the matrix elements of $H$ and $Q$. Since

$$H = -\frac{1}{4} (A + A^\dagger - 2B + 3), \quad (18)$$

the matrix elements of $H$ can be written in terms of the above vev’s and of $\langle 0 | A^{n'} B^2 (A^\dagger)^n |0\rangle$. Given Eqs. (14) and (15), the matrix elements of $Q$ are also expressible by the vev’s defined in Eqs. (16) and (17).

A general technique to compute the desired vev’s is to consider a generic matrix element

$$\langle 0 | A^{n_1} \Theta A^{n_2} \Xi (A^\dagger)^n |0\rangle \quad (19)$$

$$\langle 0 | A^{n_1-1} \Theta A^{n_2+1} \Xi (A^\dagger)^n |0\rangle + \langle 0 | A^{n_1-1} [A, \Theta] A^{n_2} \Xi (A^\dagger)^n |0\rangle, \quad (19)$$

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where \( \Theta = B, A^\dagger \) and \( \Xi = 1, B \). Eq. (19) allows to “shift to the left” \( \Theta \) until it is immediately to the right of \( \langle 0 | \) Then we use \( \langle 0 | A^\dagger = 0 \) or \( \langle 0 | B = \langle 0 | 3; \) the commutator terms are of lower order and therefore the iteration of Eq. (19) closes, giving the desired recursion.

This method can be applied to SYMQM in arbitrary dimension. For \( D = 2 \), recursions can be solved, providing explicit expressions for the matrix elements of \( H \) and \( Q \) between orthonormalized states. To this end we consider first the \( F = 0 \) sector and define

\[
\langle 0 | A^n A^\dagger = l_n \langle 0 | A^{n-1}, \tag{20}
\]

\( l_n \) can be computed recursively: \( l_1 = 6 \) and

\[
l_n \langle 0 | A^{n-1} = \langle 0 | A^{n-1} A A^\dagger = \langle 0 | A^{n-1} (A^\dagger A + 4B - 6) = \langle 0 | [(l_{n-1} - 6)A^{n-1} + 4([A^{n-1}, B] + BA^{n-1})] = (l_{n-1} + 8n - 2) \langle 0 |
\]

therefore \( l_n = 2n + 4n^2 \). Moreover, \( c_n = l_n c_{n-1} \) and \( c_0 = 1 \). We next compute

\[
\langle 2n, 0 | B | 2n, 0 \rangle = \frac{1}{c_n} \langle 0 | A^n B (A^\dagger)^n | 0 \rangle = \frac{1}{c_n} \langle 0 | (2n + B) A^n (A^\dagger)^n | 0 \rangle = 2n + 3 \tag{22}
\]

and

\[
\langle 2n-2, 0 | A | 2n, 0 \rangle = \langle 2n, 0 | A^\dagger | 2n-2, 0 \rangle = \sqrt{\frac{c_n}{c_{n-1}}} = \sqrt{2n + 4n^2}. \tag{23}
\]

Therefore, the nonzero matrix elements of \( H \) are

\[
\langle 2n, 0 | H | 2n-2, 0 \rangle = \langle 2n-2, 0 | H | 2n, 0 \rangle = -\frac{1}{4} \sqrt{2n + 4n^2}, \tag{24}
\]

\[
\langle 2n, 0 | H | 2n, 0 \rangle = n + \frac{3}{4}. \tag{25}
\]

The \( F = 1 \) sector is dealt with in the same way, and the result is

\[
\langle 2n+1, 1 | H | 2n-1, 1 \rangle = \langle 2n-1, 1 | H | 2n+1, 1 \rangle = -\frac{1}{4} \sqrt{6n + 4n^2} \tag{26}
\]

\[
\langle 2n+1, 1 | H | 2n+1, 1 \rangle = n + \frac{5}{4}. \tag{27}
\]

For the supersymmetry generator we obtain

\[
\langle 2m, 0 | Q | 2n+1, 1 \rangle = -i \sqrt{m + \frac{3}{2}} \delta_{m,n} + i \sqrt{m} \delta_{m,n+1}. \tag{28}
\]
In the cut Hilbert space, with states up to the \(|2N_{\text{cut}}+2,0\rangle\) and \(|2N_{\text{cut}}+1,1\rangle\),
\(Q\) is represented by the \((N_{\text{cut}}+2)\times(N_{\text{cut}}+1)\) matrix 
\[ \langle 2i, 0 | Q | 2j+1, 1 \rangle = \frac{1}{2} \sum_k Q_{ik} Q^\dagger_{kj}. \]
(29)
All matrix elements of \(H\) and \(H_{\text{cut}}\) are equal, apart from
\[ \langle 2N_{\text{cut}}+2, 0 | H_{\text{cut}} | 2N_{\text{cut}}+2, 0 \rangle = \frac{1}{2} (N_{\text{cut}} + 1), \]
(30)
cf. Eq.(25). Hence we confirm that we can define a cut off hamiltonian \(H_{\text{cut}}\) whose spectrum has exact supersymmetry at any finite \(N_{\text{cut}}\) \([10, 16, 17]\).

The hamiltonian matrix has a tridiagonal structure and it can be diagonalized numerically in a very efficient way. We use the \(O(N^2)\) algorithm implemented in the lapack library, which computes all eigenvalues for \(N_{\text{cut}} = 10^5\) in a few minutes on a PC.

A plot of the restricted index as a function of the Euclidean time is presented in Fig. 1 confirming results of \([10]\) to a much higher precision, and strongly suggesting the exact, time independent, value \(I(T) = 1/2\).

Note the intriguing multiple crossing at \(T = 1\) where the index seems to attain its asymptotic value for all \(N_{\text{cut}}\). Upon closer inspection it turns out that the curves do not cross at the same point. However the intersection points approaches \(I_W = 1/2, T = 1\) as \(\delta I_W = O(1/N_{\text{cut}}), \delta T = O(1/\sqrt{N_{\text{cut}}})\). This suggests existence of a “duality” transformation which relates, for finite \(N_{\text{cut}}\), SUSY violations above and below the “critical” energy \(E \sim 1\).

4 Exact calculations of the restricted index

Gauge-invariant eigenstates of the free hamiltonian, Eq. (8) are labeled by the absolute value of the momentum \(p\). In this representation the index reads \(\footnote{We consider the index restricted to the \(F = 0\) and \(F = 1\) sectors but will omit for the simplicity the \((0,1)\) superscript.}
\[ I(T) = \int_0^\infty (\rho_B(p) - \rho_F(p)) e^{-Tp^2/2} dp, \]
(31)
Since the spectrum is continuous, definition of the bosonic and fermionic densities, \(\rho_B\) and \(\rho_F\), requires an infrared regularization. Therefore we first
Figure 1: The restricted Witten index computed for values of $N_{\text{cut}}$ ranging from 125 to 128000.

solve the free problem in a spherical well of radius $R$ and then calculate the index as a limit:

$$ I(T) = \lim_{R \to \infty} I_R(T), $$

(32)

with the IR regularized index given by the discrete sum

$$ I_R(T) = \sum_n \left( e^{-T(p_n^B)^2/2} - e^{-T(p_n^F)^2/2} \right), $$

(33)

where $p_n^B(p_n^F)$ are the discrete momenta of the bosonic and fermionic states respectively.

The spectrum of a free particle in a three dimensional spherical well of radius $R$ is determined by the boundary condition $j_l(pR) = 0$, for the $l$-th spherical wave. However the local gauge symmetry limits allowed values of $l$ to $l = 0, 1$. Only these two angular momenta can be combined with the

\footnote{We thank Ken Konishi for the discussion on this point.}
fermionic colour spin, \( s = 1 \), into a scalar, \( J = 0 \), gauge-invariant state in the \( F = 0 \) and \( F = 1 \) sectors.\(^4\) It follows that the sum in Eq. (33) is over the positive zeroes \( z_n^{(l)} \) of the first two Bessel functions.

\[ p_n^B = \frac{z_n^{(0)}}{R}, \quad p_n^F = \frac{z_n^{(1)}}{R}, \quad (34) \]

A simple way to compute the \( R \to \infty \) limit of Eq. (33) is to replace \( z_n^{(1)} \) by its large-\( n \) asymptotic expansion

\[ z_n^{(0)} = \pi n, \quad z_n^{(1)} = \beta_n - 1/\beta_n + O(n^{-3}), \quad \beta_n = \pi (n + \frac{1}{2}); \quad (35) \]

we note that that, in the \( R \to \infty \) limit, higher-order terms do not contribute and the sum can be replaced by an integral:

\[
I_W(T) = \lim_{u \to 0} \int_0^\infty \frac{dn}{n} \left\{ \exp \left[-u^2 n^2\right] - \exp \left[-\frac{2u^2}{\pi^2} \right] \exp \left[-u^2(n + \frac{1}{2})^2\right] \right\} \]

\[
= \lim_{u \to 0} \frac{\sqrt{\pi}}{2u} \left\{ 1 - \exp \left[-\frac{2u^2}{\pi^2}\right] \text{erfc} \left[\frac{u}{2}\right]\right\} = \frac{1}{2}, \quad (36)\]

where

\[ u^2 = \frac{\pi^2 T}{2R^2}, \quad \text{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty \exp[-t^2] \, dt, \quad \text{erfc}(0) = 1. \quad (37)\]

A more illuminating strategy to compute \( I_W \) starts with the well known theorem on meromorphic functions to rewrite Eq. (33) as a contour integral

\[ I_R(T) = \frac{1}{2\pi i} \int_\Gamma e^{-T(z/R)^2/2} h(z) \, dz, \quad (38)\]

where \( \Gamma \) denotes any contour enclosing counterclockwise positive real axis and contained within the angular region \(-\pi/4 < \arg(z) < \pi/4\). The weight \( h \) is given by

\[ h(z) = j_0'(z)/j_0(z) - j_1'(z)/j_1(z) + 1/z. \quad (39)\]

The first two terms follow directly from the above theorem, the last one subtracts the pole at \( z = 0 \). Since \( j_1(z) \) has a second-order zero at \( z = 0 \) the theorem would not apply. However we can subtract by hand the singular

\[ \text{The “angular momentum” considered here is the colour angular momentum which generates rotations in the three dimensional colour space.}\]
terms of the Laurent expansion around $z = 0$ from both $j_0$ and $j_1$, since the $z_0 = 0$ contributions cancel in Eq. (33) anyway.

Now, choose for $\Gamma$ two straight lines\footnote{Contribution from the section of the big circle is negligible at large $R$.} at angles $\pm \vartheta$, $0 < \vartheta < \pi/4$, running to (from) the origin from (to) infinity, and rescale $z$. This gives

$$I_R(T) = \frac{R}{2\pi i} \int_{\Gamma} e^{-T u^2/2} h(uR) du,$$

with the same contour $\Gamma$ in the $u$ plane.

The whole $R$ dependence of the integrand is now in $h(uR)$. When $R \to \infty$, poles on the real axis condense into a cut. However above and below the cut the integrand simplifies considerably for large $R$. We exploit this by deforming the contour into two lines parallel to the real axis at distance $\epsilon$ and a vertical section along the imaginary axis. Due to the Schwarz reflection principle, contribution from the vertical section vanish. Hence $I_R(T) = I^+_R + I^-_R$ with

$$I^+_R(T) = \mp R \frac{1}{2\pi i} \int_0^\infty e^{-T(p\pm i)^2/2} h(R(p \pm i\epsilon)) dp,$$

At large $R$ and fixed $\epsilon$ trigonometric functions are dominated by the terms growing exponentially with $R$ and in fact cancel in the ratios in Eq. (39). Consequently we have simply (e.g., above the cut)

$$\lim_{R \to \infty, \epsilon > 0 \text{ fixed}} h(R(p + i\epsilon)) = \frac{z - 2i}{z(z - i)} = \frac{z}{z - R(p + i\epsilon)}.$$  

and symmetrically below the cut.

Now we can move toward the upper and lower ends of the cut by taking $\epsilon \to 0$. The imaginary contributions from upper and lower contours cancel and our main result reads

$$I(T) = \lim_{R \to \infty} \frac{R}{\pi} \int_0^\infty e^{-T s^2/2} \frac{dp}{R^2 p^2 + 1}.$$  

This can be easily calculated with the aid of the proper time integral representation

$$I(T) = \lim_{R \to \infty} \frac{R}{\sqrt{2\pi}} \int_0^\infty e^{-T s^2/2} \int_0^\infty e^{-s(R^2 p^2 + 1)} ds dp = \lim_{R \to \infty} \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-s} ds \sqrt{s} = \frac{1}{2}.$$
which finally proves our conjecture based on Fig. 1. The limit (43) singles out only the contribution from the $p = 0$ state. In fact Eq. (43) is nothing but

$$I(T) = \int_0^\infty \delta(p) e^{-T p^2/2} dp.$$  \hspace{1cm} (45)

This form clearly proves that supersymmetry is restored in the $R \to \infty$ limit. Densities of the bosonic and fermionic states are exactly equal for any nonzero energy, while the lowest state does not have a supersymmetric partner, which results in the point-like contribution from the zero momentum. This also confirms our earlier assertion that, in the $D = 2$ system, restricted Witten index is time independent even for the continuum spectrum [20].

Summarizing, supersymmetric quantum mechanical systems can now be solved with better precision in any dimensions. This provides a useful numerical tool for the nonperturbative solutions of matrix theories. Here it was applied to the $N = 2$ case. However, since the present recursive approach uses only gauge-invariant operators, it is even more useful for higher gauge groups. Finally, analytical calculation of the restricted Witten index for the $D = 2$ showed explicitly how the supersymmetry is restored while removing the infrared cutoff.

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