Properly Coloured Cycles and Paths: Results and Open Problems

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Abstract

In this paper, we consider a number of results and seven conjectures on properly edge-coloured (PC) paths and cycles in edge-coloured multigraphs. We overview some known results and prove new ones. In particular, we consider a family of transformations of an edge-coloured multigraph \(G\) into an ordinary graph that allow us to check the existence PC cycles and PC \((s, t)\)-paths in \(G\) and, if they exist, to find shortest ones among them. We raise a problem of finding the optimal transformation and consider a possible solution to the problem.

1 Introduction

The class of edge-coloured multigraphs generalize directed graphs. There are several other generalizations of directed graphs such as arc-coloured digraphs, hypertournaments and star hypergraphs, but the class of edge-coloured multigraphs has been given the main attention in graph theory literature because many concepts and results on directed graphs can be extended to edge-coloured multigraphs and there are several important applications of edge-coloured multigraphs. For a more extensive treatment of this topic, see [6, 7].

In this paper we overview some known results on properly coloured (PC) cycles and paths in edge-coloured multigraphs, prove new ones and consider several open problems on the topic. In Section 2 we briefly consider a problem of whether an edge-coloured graph has a PC cycle. In Sections 3 and 4 we offer a useful tool to study edge-coloured multigraphs. In investigating problems on PC subgraphs of edge-coloured multigraphs, it is convenient to transform an edge-coloured graph into an ordinary graph. We suggest a new technique that somewhat automates this transformation. Moreover,

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by proving some new results, we illustrate how the proposed technique allows us to obtain more efficient algorithms for PC cycle and PC \((s, t)\)-path problems by reducing the order and size of the transformed graph. We raise a problem of determining the minimum order and size of the transformed graph, and describe the family of graphs that may be the solution to the problem.

In Section 5 we study long PC cycles and paths in arbitrary edge-coloured multigraphs and Section 6 is devoted to longest (mostly Hamilton) PC cycles in edge-coloured complete graphs.

An \(m\)-path-cycle subgraph \(F\) of a multigraph \(G\) is a vertex-disjoint union of \(m\) paths and a number of cycles in \(G\) (some cycles can be of length 2). If \(m = 0\), we call \(F\) a cycle subgraph of \(G\). For a vertex set \(X\) of a multigraph \(G\), \(G\langle X \rangle\) denotes the subgraph of \(G\) induced by \(X\).

For a pair \(s, t\) of distinct vertices of \(G\), a path between \(s\) and \(t\) is called an \((s, t)\)-path.

We consider edge-coloured multigraphs, i.e., undirected multigraphs in which each edge has a colour, but no parallel edges have the same colour. If an edge-coloured multigraph \(G\) has \(c\) colours, we assume that the colours are \(1, 2, \ldots, c\) and we call \(G\) a \(c\)-edge-coloured multigraph. We denote the colour of an edge \(e\) of an edge-coloured multigraph \(G\) by \(\chi(e)\). When \(G\) has no parallel edges, we call \(G\) an edge-coloured graph.

Let \(G\) be a \(c\)-edge-coloured multigraph and let \(v \in V(G)\). By \(N_i(v)\) we denote the set of neighbours of \(v\) adjacent to \(v\) by an edge of colour \(i\); let \(d_i(x) = |N_i(x)|\). The maximum (minimum) monochromatic degree of \(G = (V, E)\) is defined by

\[
\Delta_{\text{mon}}(G) = \max \{d_j(v) : v \in V, 1 \leq j \leq c\} \\
(\delta_{\text{mon}}(G) = \min \{d_j(v) : v \in V, 1 \leq j \leq c\}).
\]

Let \(\chi(v) = \{i : 1 \leq i \leq c, N_i(v) \neq \emptyset\}\). A path or cycle \(Q\) of \(G\) is properly coloured (PC) if every two adjacent edges of \(Q\) are of different colours.

2 Existence of PC Cycles

Since a pair of parallel edges in a \(c\)-edge-coloured multigraph \((c \geq 2)\) forms a PC cycle, in this section, we consider only \(c\)-edge-coloured graphs.

It is easy to see that the problem of checking whether a \(c\)-edge-coloured graph has a PC cycle is more general (even for \(c = 2\)) than the simple problem of verifying whether a digraph contains a directed cycle. Indeed, consider a digraph \(D\) and, to obtain a 2-edge-coloured graph \(G\) from \(D\), replace each arc \(xy\) of \(D\) with edges \(xz_{xy}\) and \(z_{xy}y\) of colours 1 and 2, where \(z_{xy}\) is a new vertex \((z_{xy} \neq z_{x'y'}\) provided \(xy \neq x'y')\). Observe that \(G\) has a PC cycle if and only if \(D\) has a directed cycle.
The following theorem by Yeo [19] provides a simple recursive way of checking whether a \(c\)-edge-coloured graph has a PC cycle. (For \(c = 2\), Theorem 2.1 was first proved by Grossman and Häggkvist [12].)

**Theorem 2.1.** Let \(G\) be a \(c\)-edge-coloured graph, \(c \geq 2\), with no PC cycle. Then, \(G\) has a vertex \(z \in V(G)\) such that no connected component of \(G - z\) is joined to \(z\) with edges of more than one colour.

Let us consider the following function introduced by Gutin [13]:

\[ d(n, c), \text{ the minimum number } k \text{ such that every } c\text{-edge-coloured graph of order } n \text{ and minimum monochromatic degree at least } k \text{ has a PC cycle. } \]

It was proved in [13] that

\[ d(n, c) \leq \frac{1}{\lfloor c/2 \rfloor} (\log_2 n - \frac{1}{3} \log_2 \log_2 n + \Theta(1)). \tag{1} \]

Abouelaoualim et al. [1] stated a conjecture which implies that \(d(n, c) = 1\) for each \(c \geq 2\). Using a recursive construction inspired by Theorem 2.1 of \(c\)-edge-coloured graphs with minimum monochromatic degree \(p\) and without PC cycles, Gutin [13] showed that

\[ d(n, c) \geq \frac{1}{c} (\log_c n - \log_c \log_c n) \tag{2} \]

and, thus, the conjecture does not hold. The bounds (1) and (2) imply that \(d(n, c) = \Theta(\log_2 n)\) for every fixed \(c \geq 2\).

**Conjecture 2.2.** [13] There is a function \(s(c)\) dependent only on \(c\) such that

\[ d(n, c) = s(c) \log_2 n (1 + o(1)). \]

In particular, it would be interesting to determine \(s(2)\).

### 3 P-Gadgets

We consider gadget constructions which generalize some known constructions mentioned below. The P-gadget graphs \(G^*\) and \(G^{**}\) of an edge-coloured multigraph \(G\) described in the next section allow one to transform several problems on properly coloured subgraphs of \(G\) into perfect matching problems in \(G^*\) or \(G^{**}\).

Let \(G\) be an edge-coloured multigraph and let \(G' = G - \{x \in V(G) : |\chi(x)| = 1\}\). For each \(x \in V(G')\) let \(G_x\) be an arbitrary (non-edge-coloured) graph with the following four properties:

- **P1** \(\{x_q : q \in \chi(x)\} \subseteq V(G_x)\);
- **P2** \(G_x\) has a perfect matching;
For each \( p \neq q \in \chi(x) \), if the graph \( G_x - \{x_p, x_q\} \) is not empty, it has a perfect matching;

For each set \( L \subseteq \chi(x) \) with at least 3 elements; if the graph \( G_x - \{x_l : l \in L\} \) is not empty, it has no perfect matching.

Each \( G_x \) with the properties P1-P4 is called a \textbf{P-gadget}. Let us consider the following three P-gadgets; the first two are known in the literature and the third one is new.

1. One P-gadget is due to Szeider [17]:
   \[
   V(G_x) = \{x_i, x'_i : i \in \chi(x)\} \cup \{x'_a, x''_b\} \quad \text{and} \quad E(G_x) = \{x''_a x''_b, x'_i x'_a, x'_i x''_b : i \in \chi(x)\} \cup \{x'_i : i \in \chi(x)\}.
   \]
   We will call this the \textbf{SP-gadget}.

2. Another gadget is due to Bang-Jensen and Gutin [4]:
   \[
   V(G_x) = \{x_j : j \in \chi(x)\} \cup \{y_j : j \in \chi(x) \setminus \{m, M\}\},
   \]
   where \( m = \min \chi(x) \), \( M = \max \chi(x) \), and
   \[
   E(G_x) = \{x_j y_k : j \in \chi(x), k \in \chi(x) \setminus \{m, M\}\} \cup \{x_j x_k : j \neq k \in \chi(x)\}.
   \]
   We will call this the \textbf{BJGP-gadget}.

3. The following new gadget is a sort of crossover of the above two and is called the \textbf{XP-gadget}:
   \[
   V(G_x) = \{x_j : j \in \chi(x)\} \cup \{y_j : j \in \chi(x) \setminus \{m, M\}\},
   \]
   where \( m \) and \( M \) are defined above, and
   \[
   E(G_x) = \{x_m x_M\} \cup \{x_j y_j, x_m y_j, x_M y_j : j \in \chi(x) \setminus \{m, M\}\}.
   \]
   It is not difficult to verify that the tree P-gadgets indeed satisfy P1-P4.

Let \( z = \chi(x) \). Observe that the SP-gadget has \( 2z + 2 \) vertices and \( 3z + 1 \) edges, the BJGP-gadget \( 2z - 2 \) vertices and \( z(3z - 5)/2 \) edges, the XP-gadget \( 2z - 2 \) vertices and \( 3z - 5 \) edges. Thus, the XP-gadget has the minimum number of vertices and edges among the three P-gadgets. It is not difficult to verify that the XP-gadget has the minimum number of vertices and edges among all possible P-gadgets for \( z = 2, 3, 4 \). Perhaps, this is true for any \( z \).

\textbf{Conjecture 3.1.} \textit{The XP-gadget has the minimum number of vertices and edges among all possible P-gadgets for every} \( z \geq 2 \).

We will see in the next section why minimizing the numbers of vertices and edges in P-gadgets is important for speeding up some algorithms on edge-coloured multigraphs.
4 P-gadget Graphs

Let $G$ be a $c$-edge-coloured multigraph and let $G_x$ be a $P$-gadget for $x \in V(G')$. The graph $G^*$ is defined as follows: $V(G^*) = \cup_{x \in V(G')} V(G_x)$ and $E(G^*) = E_1 \cup E_2$, where $E_1 = \cup_{x \in V(G')} E(G_x)$ and $E_2 = \{y_qz_q : y, z \in V(G'), yz \in E(G), \chi(yz) = q, 1 \leq q \leq c\}$.

Let $s, t$ be a pair of distinct vertices of $G$ and let $H = G - \{s, t\}$. Let $G^{**}$ be constructed from $H^*$ by adding $s$ and $t$ and edges $E_3 = \{sx_i : sx \in E(G), \chi(sx) = i\} \cup \{tx_i : tx \in E(G), \chi(tx) = i\}$.

We will denote the number of vertices and edges in multigraph $sG, G^*$ and $G^{**}$ by $n, m, n^*, m^*$, respectively.

The following result relates perfect matchings of $G^*$ with PC cycle subgraphs of $G$. PC cycle subgraphs are important in several problems on edge-coloured multigraphs (for example, for the PC Hamilton cycle problem), see [6]. Recall that $G' = G - \{x \in V(G) : |\chi(x)| = 1\}$.

**Theorem 4.1.** Let $G$ be a connected edge-coloured multigraph such that $G'$ is non-empty. Then $G$ has a PC cycle subgraph with $r$ edges if and only if $G^*$ has a perfect matching with exactly $r$ edges in $E_2$.

**Proof:** Let $M$ be a perfect matching of $G^*$ with exactly edges

$$x_{p_1}^1y_{q_1}, \ldots, x_{p_r}^r, y_{q_r}$$

in $E_2$. For a vertex $x$ of $G'$, let $Q_x$ be the set of edges in $E_2$ adjacent to $G_x$. By P2, each $G_x$ has even number of vertices ($x \in V(G')$) and since $M$ is a perfect matching in $G^*$, there is even number of edges in $Q_x$. By P4, $Q_x$ has either no edges or two edges for each $x \in V(G')$. Let $X$ be the set of all vertices $x \in V(G')$ such that $|Q_x| = 2$. Then, by the definition of $G^*$, $G(X)$ contains a PC cycle factor. It remains to observe that $|X| = r$.

Now let $F$ be a PC cycle subgraph of $G$ with $r$ edges. Observe that the edges of $F$ correspond to a set $Q$ of $r$ independent edges of $G^*$ and that either no edges or two edges of $Q$ are adjacent to $G_x$ for each $x \in V(G')$. Now delete the vertices adjacent with $Q$ from each $G_x$ and observe that each remaining non-empty gadget has a perfect matching by P2 and P3. Combining the perfect matchings of the non-empty gadgets with $Q$, we get a perfect matching of $G^*$ with exactly $r$ edges from $E_2$.\[\]

The first part of the next assertion generalizes a result from [4]. The second part is based on an approach which leads to a more efficient algorithm than in [2].

**Corollary 4.2.** One can check whether an edge-coloured multigrap $G$ has a PC cycle and, if it does, find a maximum PC cycle subgraph of $G$ in time $O(n^* \cdot (m^* + n^* \log n^*))$. Moreover one can find a shortest PC cycle in $G$ in time $O(n \cdot n^* \cdot (m^* + n^* \log n^*))$.\[\]
Proof: We may assume that $G$ is connected and that $G'$ is not empty. By Theorem 4.1, it is enough to find a perfect matching of $G^\ast$ containing the maximum number of edges from $E_2$. Assign weight 0 (1, respectively) to edges of $G^\ast$ in $E_1$ ($E_2$, respectively). Now we need to find a maximum weight perfect matching of $G^\ast$ which can be done in time $O(n^* \cdot (m^* + n^* \log n^*))$ by a matching algorithm in [11].

To find a shortest PC cycle in $G$, choose a vertex $x \in V(G')$. We will find a shortest PC cycle in $G$ traversing $x$. By Theorem 4.1, it is enough to find a perfect matching of $G^\ast$ containing the minimum number of edges from $E_2$ while containing at least one edge from $E_2$ so that the corresponding PC cycle in $G$ should be non-trivial. We define the weights on edges of $G^\ast$ as follows. Assign $M$, where $M$ is a sufficiently large number, to each edge in $E_2$ incident with $G_x$. For all other edges, assign weight 1 (0, respectively) to edges of $G^\ast$ in $E_1$ ($E_2$, respectively). A maximum weight perfect matching of $G^\ast$ contains exactly two edges of weight $M$ by P4, and contains the minimum number of edges in $E_2$. Finding a maximum weight perfect matching of $G^\ast$ can be done in time $O(n^* \cdot (m^* + n^* \log n^*))$ and we iterate the process for each $x \in V(G')$.

The proof of the following result is analogous to the proof of Theorem 4.1.

**Theorem 4.3.** Let $G$ be an edge-coloured multigraph and let $s, t$ be a pair of distinct vertices of $G$. If $G^{**}$ is non-empty, then $G$ has a PC 1-path-cycle subgraph with $r$ edges in which the path is between $s$ and $t$ if and only if $G^{**}$ has a perfect matching with exactly $r$ edges not in $E_1$.

The next assertion generalizes a result from [2].

**Corollary 4.4.** Let $G$ be an edge-coloured multigraph. One can check whether there is a PC $(s, t)$-path in $G$ in time $O(m^{**})$ and if $G$ has one, a shortest PC $(s, t)$-path can be found in time $O(n^{**} \cdot (m^{**} + n^{**} \log n^{**}))$.

**Proof:** Let $L$ be a graph. Given a matching $M$ in $L$, a path $P$ in $L$ is $M$–augmenting if, for any pair of adjacent edges in $P$, exactly one of them belongs to $M$ and the first and last edges of $P$ do not belong to $M$. Consider a perfect matching $M$ of $H^\ast$, where $H = G - \{s, t\}$, which is a collection of perfect matchings of $G_x$ for all $x \in V(G')$. The existence of a perfect matching in $G_x$ is guaranteed by P2. Observe that $G$ has a PC $(s, t)$-path if and only if there is an $M$–augmenting $(s, t)$-path $P$ in $G^{**}$. Since an $M$–augmenting path $P$ can be found in time $O(m^{**})$ (see [13]), we can find a PC $(s, t)$-path in $G$, if one exists, in time $O(m^{**})$.

To find a shortest PC $(s, t)$-path, we assign each edge in $\bigcup_{x \in V(G')} E(G_x)$ weight 0 and every other edge of $G^{**}$ weight 1. Observe that a minimum
weight perfect matching \( Q \) in the weighted \( G^{**} \) corresponds to a shortest \( PC(s,t) \)-path. Finding a minimum weight perfect matching can be done in time \( O(n^{**} \cdot (m^{**} + n^{**} \log n^{**})) \).

5 Long PC Cycles and Paths

The following interesting result and conjecture were obtained by Abouelaoualim, Das, Fernandez de la Vega, Karpinski, Manoussakis, Martinhon and Saad [1].

**Theorem 5.1.** [1] Let \( G \) be a \( c \)-edge-coloured multigraph \( G \) with \( n \) vertices and with \( \delta_{mon}(G) \geq \lceil \frac{n+1}{2} \rceil \). If \( c \geq 3 \) or \( c = 2 \) and \( n \) is even, then \( G \) has a Hamilton PC cycle. If \( c = 2 \) and \( n \) is odd, then \( G \) has a PC cycle of length \( n - 1 \).

**Conjecture 5.2.** [1] Theorem 5.1 holds if we replace \( \delta_{mon}(G) \geq \lceil \frac{n+1}{2} \rceil \) by \( \delta_{mon}(G) \geq \lceil \frac{n}{2} \rceil \).

We cannot replace \( \delta_{mon}(G) \geq \lceil \frac{n+1}{2} \rceil \) by \( \delta_{mon}(G) \geq \lceil \frac{n-1}{2} \rceil \) due to the following example. Let \( H_1 \) and \( H_2 \) be \( c \)-edge-coloured complete multigraphs (for each pair \( x, y \) of vertices and each \( i \in \{1,2,\ldots,c\} \) and \( j \in \{1,2\} \), \( H_j \) has an edge between \( x \) and \( y \) of colour \( i \)) of order \( p+1 \) that have precisely one vertex in common. Clearly, a longest PC cycle in \( H_1 \cup H_2 \) is of length \( p+1 \).

Since the longest PC path problem is \( \mathcal{NP} \)-hard, it makes sense to study lower bounds on the length of a longest PC path. The following result was proved by Abouelaoualim et al. [1].

**Theorem 5.3.** Let \( G \) be a \( c \)-edge-coloured graph of order \( n \) with \( \delta_{mon}(G) = d \geq 1 \). Then \( G \) has a PC path of length at least \( \min\{n-1,2\lfloor\frac{c}{2}\rfloor d\} \).

The authors of [1] raised the following two conjectures.

**Conjecture 5.4.** Let \( G \) be a \( c \)-edge-coloured graph of order \( n \) and let \( d = \delta_{mon}(G) \geq 1 \). Then \( G \) has a PC path of length at least \( \min\{n-1,2cd\} \).

They also conjectured the following analog of Theorem 5.3 for multigraphs:

**Conjecture 5.5.** Let \( G \) be a \( c \)-edge-coloured multigraph of order \( n \) with \( \delta_{mon}(G) = d \geq 1 \). Then \( G \) has a PC path of length at least \( \min\{n-1,2d\} \).

6 Longest PC Cycles and Paths in Edge-Coloured Complete Graphs

Let \( K^c_n \) denote a \( c \)-edge-coloured complete graph with \( n \) vertices.

Feng, Giesen, Guo, Gutin, Jensen and Rafiey [10] proved the following:
Theorem 6.1. A $K^c_n$ ($c \geq 2$) has a PC Hamilton path if and only if $K^c_n$ contains a PC spanning 1-path-cycle subgraph.

This theorem was first proved by Bang-Jensen and Gutin [4] for the case $c = 2$ and they conjectured that Theorem 6.1 holds for each $c \geq 2$. Theorem 6.1 implies that the maximum order of a PC path in $K^c_n$ equals the maximum order of a PC 1-path-cycle subgraph of $K^c_n$.

As a result, the problem of finding a longest PC path in $K^c_n$ is polynomial-time solvable for arbitrary $c \geq 2$. To see that a PC 1-path-cycle subgraph of $K^c_n$ can be found in polynomial time, add a pair $x, y$ of new vertices to $K^c_n$ together with all edges needed to have a complete multigraph on $n + 2$ vertices. Let the colour of all edges between $x$ and $y$ and $K^c_n$ be $c+1$ and let the colour of $xy$ be $c+2$. Observe that the maximum order of a PC 1-path-cycle subgraph of $K^c_n$ equals the maximum order of a PC cycle subgraph of the $c + 2$-edge-coloured complete graph described above. It remains to apply Corollary 4.2.

The problem of finding a longest PC cycle $K^c_n$ has not been solved yet for $c \geq 3$ as we will see below. For $c = 2$, Saad [15] found a characterization for longest PC cycles using the following notions. A pair of distinct vertices $x, y$ of $G$ are colour-connected if there exist PC $(x, y)$-paths $P$ and $Q$ such that $\chi(f_P) \neq \chi(f_Q)$ and $\chi(\ell_P) \neq \chi(\ell_Q)$, where $f_P$ and $f_Q$ are the first edges of $P$ and $Q$, respectively, and $\ell_P$ and $\ell_Q$ are the last edges of $P$ and $Q$, respectively. We say that $G$ is colour-connected if every pair of distinct vertices of $G$ is colour-connected. Saad’s characterization is as follows.

Theorem 6.2. The length of a longest PC cycle in a colour-connected $K^2_n$ is equal to the maximum order of a PC cycle subgraph of $K^2_n$.

Colour-connectivity for $K^2_n$ is an an equivalence relation (see [6]). Using Theorem 6.2, Saad [15] showed that the problem of finding a longest PC cycle in $K^2_n$ is random polynomial. Using a special case of Corollary 4.2, Bang-Jensen and Gutin [5] proved that the problem is, in fact, polynomial-time solvable. Theorem 6.2 implies the following:

Corollary 6.3. [15] A $K^2_n$ has a PC Hamilton cycle if and only if $K^2_n$ is colour-connected and contains a PC cycle factor.

There is another characterization of $K^2_n$ with a PC Hamilton cycle due to Bankfalvi and Bankfalvi, see [6]. The straightforward extension of Corollary 6.3 is not true for any $c \geq 3$ [9]. In fact, no characterization of $K^c_n$ with a PC Hamilton cycle is known for any fixed $c \geq 3$ and it is a very interesting problem to obtain such a characterization. Even the following problem by Benkouar, Manoussakis, Paschos and Saad [8] is still open.

Problem 6.4. Determine the complexity of the PC Hamilton cycle problem for $c$-edge-coloured complete graphs when $c \geq 3$. 

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We conjecture that the PC Hamilton cycle problem for \( c \)-edge-coloured complete graphs when \( c \geq 3 \) is polynomial-time solvable.

In absence of characterization of \( K_n^c \) with a PC Hamilton cycle, sufficient conditions are interest. Manoussakis, Spyropoulos, Tuza and Voigt [14] proved the next result.

**Proposition 6.5.** If \( c \geq \frac{1}{2}(n - 1)(n - 2) + 2 \), then every \( K_n^c \) has a PC Hamilton cycle.

Let \( \Delta_{\text{mon}}(K_n^c) \) denote the largest monochromatic degree of \( K_n^c \). Bollobás and Erdős [9] posed the following:

**Conjecture 6.6.** Every \( K_n^c \) with \( \Delta_{\text{mon}}(K_n^c) \leq \lfloor n/2 \rfloor - 1 \) has a PC Hamilton cycle.

Improving some previous results on this conjecture, Shearer [16] showed that if \( 7\Delta_{\text{mon}}(K_n^c) < n \), then \( K_n^c \) has a PC Hamilton cycle. So far, the best asymptotic estimate was obtained by Alon and Gutin [3].

**Theorem 6.7.** [3] For every \( \epsilon > 0 \) there exists an \( n_0 = n_0(\epsilon) \) so that for each \( n > n_0 \), every \( K_n^c \) satisfying \( \Delta_{\text{mon}}(K_n^c) \leq (1 - \frac{1}{\sqrt{2}} - \epsilon)n \) contains a PC Hamilton cycle.

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