Guidance of Agents in Cyclic Pursuit

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Abstract

This report studies the emergent behavior of systems of agents performing cyclic pursuit controlled by an external broadcast signal detected by a random set of the agents. Two types of cyclic pursuit are analyzed: 1) linear cyclic pursuit, where each agent senses the relative position of its target or leading agent 2) non-linear cyclic pursuit, where the agents can sense only bearing to their leading agent and colliding agents merge and continue on the path of the pursued agent (a so-called "bugs" model). Cyclic pursuit is, in both cases, a gathering algorithm, which has been previously analyzed. The novelty of our work is the derivation of emergent behaviours, in both linear and non-linear cyclic pursuit, in the presence of an exogenous broadcast control detected by a random subset of agents. We show that the emergent behavior of the swarm depends on the type of cyclic pursuit. In the linear case, the agents asymptotically align in the desired direction and move with a common speed which is proportional to the ratio of the number of agents detecting the broadcast control to the total number of agents in the swarm, for any magnitude of input (velocity) signal. In the non-linear case, the agents gather and move with a shared velocity, which equals the input velocity signal, independently of the number of agents detecting the broadcast signal.

Keywords: cyclic pursuit, broadcast control, random leaders, emergent behavior
Symbols and Abbreviations

\( n \) - Number of agents
\( p_i \) - position of agent \( i \), \( p_i \in \mathbb{R}^2 \)
\( x_i \) - \( x \) coordinate of \( p_i \)
\( y_i \) - \( y \) coordinate of \( p_i \)
\( P \) - vector of stacked positions of all agents, \( P = (p_1, \ldots, p_n)^T \)
\( X \) - vector of \( x \) coordinates, \( X = (x_1, \ldots, x_n)^T \)
\( Y \) - vector of \( y \) coordinates, \( Y = (y_1, \ldots, y_n)^T \)
\( u_i \) - local gathering control applied by agent \( i \), \( u_i \in \mathbb{R}^2 \)
\( U_c \) - external broadcast control, \( U_c \in \mathbb{R}^2 \)
\( b_i \) - flag indicating whether agent \( i \) detected the broadcast control, 1/0
\( B \) - vector indicator of agents detecting the broadcast control, \( B(i) = b_i \)
\( N^l \) - the set of agents detecting the broadcast control
\( n^l \) - the number of agents detecting the broadcast control, \( n^l = |N^l| \)
\( \|v\| \) - Euclidean norm of vector \( v \)
\( |s| \) - absolute value of scalar \( s \)
\( d_i \) - distance of agent \( i \) from \( i + 1 \), \( d_i = \|p_i - p_{i+1}\| \)
\( (\cdot) \) - conjugate of \( (\cdot) \), scalar or vector
\( v^* \) - transpose conjugate of vector \( v \), \( v^* = (\overline{v})^T \)
\( 1_n \) - vector of ones, size \( n \)
\( 0_n \) - vector of zeroes, size \( n \)
1 Introduction

In this work we consider the behaviour of \( n \) agents performing cyclic pursuit, when an exogenous velocity control is broadcast by a controller and detected by a random set of agents. The cyclic pursuit problem is formulated as \( n \) agents chasing each other. The agents are ordered from 1 to \( n \), and agent \( i \) acquires information about its leading agent (prey) \( i+1 \). The agent indices are (modulo \( n \)) throughout this paper. The agents start from arbitrary positions on a plane.

In graph theoretic terms, cyclic pursuit can be represented by a directed cycle graph, whose nodes are the agents and the directed edges depict the information flow, as shown in Fig. 1.

![Information flow in cyclic pursuit](image)

Figure 1: Information flow in cyclic pursuit

In this work we consider two types of information possibly acquired by the agents

1. Relative position to the chased ”target” agent

2. Bearing only information, i.e. direction to the target

Let \( p_i(t) \) be the position of agent \( i \) at time \( t \); \( p_i(t) \in \mathbb{R}^2 \). We assume the agents to be identical, memory-less particles, modeled as single integrators.
1. In case of relative position information, the autonomous kinematics of agent $i$ can be expressed as

$$\dot{p}_i(t) = k(p_{i+1}(t) - p_i(t)); \quad k > 0 \quad i = (1, ... n)$$  \hspace{1cm} (1)

In the sequel we assume, without loss of generality, $k = 1$. It’s easy to see that in this case (1) can be decoupled into

$$\dot{x}_i(t) = x_{i+1}(t) - x_i(t)$$  \hspace{1cm} (2)

$$\dot{y}_i(t) = y_{i+1}(t) - y_i(t)$$  \hspace{1cm} (3)

Thus, we can consider only the $x$ coordinates and obtain results for the $y$ coordinates by similarity. Let $X(t) = (x_1(t), ..., x_n(t))^T$. Then, from (2), we have

$$\dot{X}(t) = MX(0)$$  \hspace{1cm} (4)

Eq. (4) is a linear system, where $M$ is the circulant matrix (5).

$$M = circ[-1, 1, 0, 0, ..., 0] = \begin{bmatrix} -1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ 1 & 0 & \cdots & 0 & -1 \end{bmatrix}$$  \hspace{1cm} (5)

This system shall be referred to in the sequel as linear cyclic pursuit.

2. Bearing only information flow generates a non-linear cyclic pursuit. Let $d_i(t)$ be the distance between agents $i$ and $i + 1$ at time $t$:

$$d_i(t) = \|p_{i+1}(t) - p_i(t)\|$$  \hspace{1cm} (6)

where $\|\|$ represents the Euclidean norm. Then, if $d_i(t) \neq 0$ the law of bearing-only autonomous motion can be written as

$$\dot{p}_i(t) = \frac{p_{i+1}(t) - p_i(t)}{d_i(t)}$$

where we assumed the speed of all agents to be 1. Moreover, we assume that if $d_i(\hat{t}) = 0$ at some time $\hat{t}$, then $p_i(t) = p_{i+1}(t)$ for all $t \geq \hat{t}$, i.e. when agents $i$ and $i + 1$ collide they merge and continue as one agent in the direction of $i + 1$. This model is known in the literature as the "bugs" model, see e.g. [25], [14] and references therein.
The novelty of our work is the derivation of emergent behaviours, in both linear and non-linear cyclic pursuit, in the presence of an exogenous broadcast velocity signal detected by a random subset of agents. The impact of the external velocity signal on the movement of agents in the linear case is shown in section 2 and for the non-linear case in section 3.

1.1 Literature survey and our contribution

1.1.1 Linear cyclic pursuit

Autonomous linear cyclic pursuit, belongs to the larger class of networks with directed, fixed topology graphs, denoted by $G$, where each node applies protocol (8)

$$\begin{align*}
\dot{x}_i(t) &= u_i(t) \\
u_i(t) &= \sum_{j \in N_i} a_{ij}(x_j(t) - x_i(t)); \quad i \in \{1, \ldots, n\}
\end{align*}$$

where $N_i$ is the neighborhood of $i$, defined as $N_i = \{j \in \{1, \ldots, n\}; a_{ij} > 0\}$. In our case $N_i = \{i + 1 \mod(n)\}$, $a_{ij} = 1\forall i$ and $G$ is as depicted by Fig. 1. Olfati-Saber, Fax and Murray show in [20], [21] that a network of integrators with directed information flow, $G$, that is strongly connected, using Protocol (8), yields the following results:

- It globally asymptotically solves an agreement problem, i.e. $x_i(t \to \infty) = x_j(t \to \infty) = \alpha; \quad \forall i, j$, [see Proposition 2 in [20]]

- A sufficient condition for $\alpha = \text{Avg}(x(0))$, i.e. the agreement to be the average agreement, is $\sum_{i=1}^{n} u_i = 0$.

Note that if $G$ is undirected and symmetric, i.e. $a_{ij} = a_{ji}$, then the condition $\sum_{i=1}^{n} u_i = 0$ automatically holds and $\text{Avg}(x(t))$ is an invariant quantity, see [22]

- $G$ globally asymptotically solves the average-consensus problem using Protocol (8) if and only if $G$ is balanced.

Recalling that

- A digraph is called strongly connected if for every pair of vertices there is a directed path between them.
• A node is called balanced if the total weight of edges entering the node and leaving the same node are equal.

• If all nodes in the digraph are balanced then the digraph is called balanced.

We observe that the circular flow graph, depicted in Fig. 1, is strongly connected, balanced and, using \( u_i(t) = x_{i+1}(t) - x_i(t) \), satisfies \( \sum_{i=1}^{n} u_i = 0 \), the system described by 2, 3 solves the average consensus problem.

Addressing specifically the problem of linear cyclic pursuit, other researchers derived similar results. Bruckstein et al., in [4], see Section "Linear Insects", showed that for every initial condition, the agents exponentially converge to a single point, computable from the initial conditions of the agents. Marshal, in [16] Section 2.2.6, offers an alternate proof for the same.

If the agents apply heterogeneous gains to the \( u_i \), i.e. \( u_i(t) = k_i(p_{i+1}(t) - p_i(t)) \); \( k_i > 0 \) \( \forall i \), then the point of convergence of the agents can be controlled by selecting these gains, as shown in [27]. Moreover, if convergence to a point is achievable, then other formations are achievable by a simple modification, where each agent pursues a displaced version of the next agent, as discussed in [15].

All of the above studies consider zero input or autonomous systems in cyclic pursuit. Our main contribution is in the addition of an external broadcast control, detected by a random set of agents, and the derivation of the asymptotic behaviour of the system in this case. Ren, Beard, and McLain, in [24], consider the problem of dynamic consensus, which at first glance is similar to ours but is only a simple, special case, of our paradigm and results. Applying their general graph case to the cyclic pursuit case, the update law they apply is

\[
\dot{x}_i(t) = (x_{i+1}(t) - x_i(t)) + U(t)
\]

i.e. a common input \( U(t) \) is applied at time \( t \) to all agents. They show (Theorem 3) that in this case \( \|x(t) - \zeta(t)\| \xrightarrow{t \to \infty} 0 \), where \( \zeta(t) \) is the integral of \( U(t) \) starting at the equilibrium point of the autonomous, zero input system. Moore and Lucarelli, [19] consider the case where a separate input enters each agent. However, they limit their analysis specifically the case where an input enters only one node (or agent), say \( k \). Moreover, the input is not a general velocity control, as in our paradigm, but attraction of agent \( k \) to a
goal position, say $x^{sp}$, i.e.

$$U_i(t) = \begin{cases} (x^{sp} - x_i(t)) & \text{if } i = k \\ 0 & \text{otherwise} \end{cases} \quad (9)$$

Dimarogonas, Gustavi et al. in [7], [9] also consider the global mission of converging to a known destination point, but allow for multiple leaders. Leaders are predesignated agents holding the information of the goal destination, thus the external input in this case becomes

$$U_i(t) = \begin{cases} (x^{sp} - x_i(t)) & \text{if } i \in K \\ 0 & \text{otherwise} \end{cases} \quad (10)$$

where $K$ is the predesignated set of leaders. In [23], W. Ren extends the problem of reaching a goal position to that of consensus to a time varying reference state, and shows necessary and sufficient conditions under which consensus is reached on the time-varying reference state. This is the problem of tracking a time dependent state and not of steering by an external velocity signal received by a random set of agents, as in our case.

In other works, the model is even further from our paradigm. Some assume that the (predesignated) leaders have a fixed state value and do not abide by the agreement protocol. For example, Jadbabaie et al. in [12] consider Vicsek’s discrete model [28], and introduce a leader that moves with a fixed heading. Yet others add special agents to the swarm with the purpose of controlling the collective behavior. In [10], [11] these special agents are referred to as "shills". The basic local rules of motion of the existing agents in the system are not changed, however the shill does not obey the same local rules but has a local control of its own, depending on the states of the ordinary agents and a secret goal function.

We recall that in our paradigm the position of the agents is not known to themselves and the leaders are are not special agents but regular agents, randomly selected from the swarm, obeying the same gathering rule of motion as the remaining agents. The external input is a velocity signal aimed at steering the swarm in a desired direction and not a goal position for the swarm. Thus our problem, as well as solution and results, is different from the above discussed cases, covered in the literature.

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1Shill is a decoy who acts as an enthusiastic, internally driven agent, that looks like an ordinary agent, having the goal to stimulate the participation of others
1.1.2  Cyclic pursuit with non-linear local control

Our paradigm for non-linear cyclic pursuit discussed herein, comprising sensing direction to prey, chasing along the line of sight with unit speed and capture (merge) upon collision, is commonly referred to in the existing literature as the bugs problem, also known as “ants”, see e.g. [4], [3]. In [3] the convergence of \( n \) ants in cyclic pursuit to an encounter point is proved. Richardson shows in [25] that the encounter occurs in finite time. In [4], Bruckstein et. al extend the model allowing each bug (ant) to move at different, time dependent speed, \( v_i(t) \). They show that integrability of the speed plays a central role in the emerging behaviour of the swarm. Speed \( v_i(t) \) is integrable iff the cumulative distance travelled by ant \( i \) at time \( t \), \( V_i(t) = \int_0^t v_i(s)\,ds \) holds \( V_i(\infty) < \infty \). Constant speeds are not integrable, hence, according to Theorem 1.ii in [4], the time of the last ant collision, i.e. termination time, is finite.

The question of simultaneous mutual capture, i.e. the existence of a time \( t_c \) such that the distances \( d_i(t_c) = \| p_{i+1}(t_c) - p_i(t_c) \| = 0 \) and \( d_i(t) > 0 \) \( \forall t \in [t_0, t_c) \) for all \( i \), was also investigated. In 1971 Klamkin and Newman, [14], showed that if \( n = 3 \), the 3 bugs travel at the same speed and the initial positions of the bugs are not collinear, then the meeting of the three bugs must be mutual, i.e. all bugs capture their prey simultaneously. Klamkin and Newman speculate that this result generalizes to more bugs. In [2], Behroozi and Gagnon prove that it does indeed generalize to \( n = 4 \) if the bugs initial positions form a convex polygon. Only non-convex configurations can give rise to a premature capture but a non-convex configuration cannot evolve from a convex configuration. Behroozi and Gagnon in [2] generalize some aspects of the proof for the \( n \)-bug systems but leave some open questions. Thus, conditions for mutual capture for \( n \)-bug systems, \( n > 4 \), remain conjectures supported by simulations, see e.g. [1]. Richardson shows in [25], that, in the general case of \( n \) bugs in \( k \) dimensions, it is possible for bugs to capture their prey without all bugs simultaneously doing so even for non-collinear initial positions, however the probability of a non-mutual capture occurrence is zero.

We note that the simple bugs model is not the only commonly used model for non-linear cyclic pursuit. Another frequently used model for systems in cyclic pursuit is based on higher order agent behaviour, like the unicycle model, describing wheeled vehicles, subject to a single non-holonomic con-
Our contribution is in deriving the emergent behaviour of the "bugs", when an external controller broadcasts a velocity signal which is detected by a random set of "bugs" in the group. This problem has not been previously investigated. [5] analyzes the problem of agents, modeled by single integrator dynamic, in bearing-only cyclic pursuit with a moving target. The moving target can be seen as a leader broadcasting velocity, but this leader is a special purpose agent which does not participate in the cyclic pursuit. Moreover, there is a basic assumption that all agents in cyclic pursuit detect target’s velocity and can sense the bearing to target. This model is entirely different from our paradigm where each agent can sense the bearing only to its leading agent and the desired velocity is available only to a (random) subset of agents.

1.2 Paper outline and main results

The paper is organized as follows:

- In section 2 we derive the emergent behaviour in case of linear cyclic pursuit, with a broadcast steering control, using properties of linear systems, and show that in this case the agents will asymptotically align in the direction of $U_c$ and move asymptotically as a time-independent linear formation with velocity $\frac{n_l}{n} U_c$, where $n_l$ is the number of agents detecting the steering control. In this case there is no restriction on the size of $\|U_c\|$.

- In section 3 we derive the emergent behaviour for the assumed non-linear model (the bugs model with external input detected by a random set of agents) and show that $\|U_c\|$ within an upper bound ensures convergence to a moving point. In this case, if at least one agent detected the broadcast control, the agents will all move, after the mutual capture time, as a single point with velocity $U_c$.

All the analytically derived results are illustrated by simulations.
2 Linear cyclic pursuit with broadcast control

In our paradigm, the equation of motion of agents performing linear cyclic pursuit, i.e. sensing relative position to the chased agent, in the presence of a broadcast velocity control, $U_c$, can be written as

$$ \dot{p}_i(t) = p_{i+1}(t) - p_i(t) + b_i(t)U_c(t) $$

where

$$ b_i(t) = \begin{cases} 1 & \text{if agent } i \text{ detected the external control} \\ 0 & \text{otherwise} \end{cases} $$

Since $p_i = (x_i, y_i)^T$, and $U_c(t) = (U_x(t), U_y(t))^T$, eq. (11) evolves independently in the $x$ and $y$ directions, thus we can consider only one component, say $x$. If we aggregate the $x$ position component of all the agents we can write

$$ \dot{X}(t) = MX(t) + B(t)U_x(t) $$

where

- $X = (x_1, x_2, ..., x_n)^T$
- $M$ is the circulant matrix (14) representing the interactions graph of agents in linear cyclic pursuit

$$ M = circ[-1, 1, 0, 0, ..., 0] = \begin{bmatrix} -1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 1 & 0 & \cdots & 0 \\ \vdots & & & & & \\ 1 & 0 & \cdots & 0 & -1 \end{bmatrix} $$

- $B(t)$ is the leaders indicator vector at time $t$, i.e. the $i$'th entry in the vector $B(t)$ is $b_i(t)$ and $b_i(t)$ is defined by (12)

In eq. (13), $M$ is time independent and if we assume $B(t)$ and $U_x(t)$ to be piecewise constant, i.e. we assume that the time-line can be divided into intervals, $t \in [t_k, t_{k+1})$, where $B(t) \equiv B(t_k)$, and $U_x(t) \equiv U_x(t_k)$, $t_k$ is the time of change of leaders or of the broadcast control. In the sequel we treat each time interval, $[t_k, t_{k+1})$, separately. Thus, it is convenient to suppress the subscript $k$. Moreover, it is convenient to denote by $t$ the relative time.
since the beginning of the interval \( t = 0 \) and by \( x(0) \) the state of the system at this time, by \( B \) the leaders indicator during the interval and by \( U_x \) the exogenous control during the interval. In each time interval eq. (13) evolves as a linear time independent system, which has the well known solution (cf. [13])

\[
x(t) = e^{Mt}x(0) + \int_0^t e^{M(t-\tau)}BU_x d\tau
\]

We note that \( M \) defined by (14) is a normal matrix, (Appendix A.5), therefore it is unitarily diagonizable, i.e. \( M = VAV^* \), where \( V \) is a unitary matrix of eigenvectors, \( V^* \) denotes the transpose of the complex conjugate of \( V \) and \( \Lambda \) is a diagonal matrix of eigenvalues.

**Remark**: Due to the circulant structure of \( M \), \( V \) is the DFT matrix

In particular, since \( M \) is the circulant matrix with \( c_0 = -1 \), \( c_1 = 1 \) and \( c_k = 0; \ k = 2, 3, ..., n - 1 \), we have (from Appendix C)

- the eigenvalues of \( M \) can be written as
  \[
  \lambda_k = \rho^k - 1; \ k = 0, ..., n - 1
  \]
  where \( \rho \triangleq e^{-2\pi j/n} \)

- with corresponding eigenvectors
  \[
  v_k = \frac{1}{\sqrt{n}} \left( 1, \rho^k, \rho^{2k}, ..., \rho^{k(n-1)} \right)^T ; \ k \in \{0, 1, ..., n - 1\}
  \]

It is easily seen that \( \lambda_0 = 0 \), with a corresponding eigenvector \( v_0 = \frac{1}{\sqrt{n}} 1_n \), while the remaining eigenvalues, \( \lambda_k; \ k = 1, ..., n - 1 \), have negative real part.

These features of \( M \) constitute the basis for the derivation of the emergent behaviour of swarms in linear cyclic pursuit with broadcast control.

Following the methodology in [26], we write eq. (15) as

\[
X(t) = x^{(h)}(t) + x^{(u)}(t)
\]

where

- \( x^{(h)}(t) = e^{Mt}X(0) \) represents the zero input, homogeneous, solution

- \( x^{(u)}(t) = \int_0^t e^{M(t-\tau)}BU_x d\tau \) represents the contribution of the exogenous input (broadcast control) to the group dynamics
2.1 Homogeneous linear cyclic pursuit

\[ x^{(h)}(t) = e^{Mt}x(0) = Ve^{\Lambda t}V^*x(0) = v_0v_0^TX(0) + \sum_{k=1}^{n-1} v_ke^{\lambda_k t}v_k^*X(0) \quad (19) \]

Since \( \Re(\lambda_k) < 0 \) for \( k = 1, \ldots, n - 1 \), we have

\[ \sum_{k=1}^{n-1} v_ke^{\lambda_k t}v_k^*X(0) \xrightarrow{t \to \infty} 0_n \]

\[ x^{(h)}(t) \xrightarrow{t \to \infty} v_0v_0^TX(0) = \frac{1}{n} \sum_{i=1}^{n} x_i(0) 1_n \]

Thus, a homogeneous system of \( n \) agents performing linear cyclic pursuit will asymptotically converge to a point, the centroid, determined by the initial conditions, which is a well known result, see e.g. [4]. However, the derivation in this section introduces a methodology which will be useful in the sequel.

2.2 The effect of the exogenous control

Recalling that we consider a time interval where \( B, U_x \) are constant, we obtain

\[ x^{(u)}(t) = \int_0^t e^{M(t-\tau)}BU_x d\tau = \int_0^t e^{M\nu}BU_x d\nu = \left[ \int_0^t e^{M\nu} d\nu \right] BU_x \quad (20) \]

Using the diagonalization of \( M \), as in subsection 2.1, we have

- \( M = V\Lambda V^* \), where \( V \) is a unitary matrix formed by the eigenvectors of \( M \), given by (17), and \( \Lambda \) is a diagonal matrix of the corresponding eigenvalues of \( M \), given by (16)

- \( e^{M\nu} = Ve^{\Lambda\nu}V^* = \sum_{k=0}^{n-1} e^{\lambda_k \nu} [v_kv_k^*] \)

Since \( M \) has a single zero eigenvalue and the remaining eigenvalues have negative real part, we can decompose \( x^{(u)}(t) \) in two parts:

\[ x^{(u)}(t) = x^{(a)}(t) + x^{(b)}(t) \quad (21) \]

where

- \( x^{(a)}(t) \) is the zero eigenvalue dependent term, representing the movement in the agreement space

- \( x^{(b)}(t) \) is the remainder, representing the deviation from the agreement space
2.2.1 Movement in the agreement space

\[ x^{(a)}(t) = \int_0^t e^{\lambda_0 \nu} v_0 v_0^* BU_x d\nu = v_0 v_0^* BU_x t = \frac{n_l}{n} U_x t \mathbf{1}_n \quad (22) \]

where \( \lambda_0 = 0 \), \( n_l \) is the number of leaders in the considered time interval and \( v_0 = \frac{1}{\sqrt{n}} \mathbf{1}_n \), \( v_0^* = \frac{1}{\sqrt{n}} \mathbf{1}_n^T \) and \( \mathbf{1}_n^T B = n_l \). Recalling that eq. (22) holds also for the \( y \) axis, with the corresponding component of \( U_c \), i.e.

\[ y^{(a)}(t) = \frac{n_l}{n} U_y t \mathbf{1}_n \quad (23) \]

we have

**Lemma 2.1.** A group of \( n \) agents performing linear cyclic pursuit, with \( n_l \) agents receiving an exogenous velocity vector control \( U_c \), will asymptotically align in the direction of the vector \( U_c \) and move with a common speed that is proportional to the ratio of \( n_l \) to \( n \), i.e. \( \frac{n_l}{n} U_c \).

2.2.2 Deviation from the agreement space

Consider now the remainder \( x^{(b)}(t) \) of the input-related part, i.e. the part of \( x^{(u)}(t) \) containing all eigenvalues of \( M \) other than the zero eigenvalue and representing the agents’ state deviation from the agreement subspace.

We have

\[ x^{(b)}(t) = \left[ \sum_{i=1}^{n-1} \int_0^t (e^{\lambda_i \nu}) v_i v_i^* d\nu \right] BU_x \quad (24) \]

\[ = \left[ \sum_{i=1}^{n-1} \frac{1}{\lambda_i} (1 - e^{\lambda_i t}) v_i v_i^* \right] BU_x \quad (25) \]

Since all eigenvalues \( \lambda_i \) for \( i = 1, \ldots, n-1 \) have strictly negative real parts, \( x^{(b)}(t) \) converges asymptotically to a time independent vector, denoted by \( \xi_x \), given by:

\[ \xi_x \triangleq x^{(b)}(t \to \infty) = \left[ \sum_{i=1}^{n-1} \frac{1}{\lambda_i} v_i v_i^* \right] BU_x \quad (26) \]

Similarly, for the \( y \) axis,

\[ \xi_y \triangleq y^{(b)}(t \to \infty) = \left[ \sum_{i=1}^{n-1} \frac{1}{\lambda_i} v_i v_i^* \right] BU_y \quad (27) \]
and all the following properties of $\xi_x$ hold also for $\xi_y$. Note that $M$ depends only on the number of agents, thus for a given number of agents and a constant broadcast control, $U_c$, the deviations depend only on $B$, i.e. on the agents detecting the broadcast control. Moreover, if all the agents receive the broadcast control then there are no deviations, i.e. the agents converge and move as a single point.

**Lemma 2.2.** A group of $n$ agents performing linear cyclic pursuit, \textit{with all agents receiving an exogenous velocity control $U_c$}, will asymptotically move as a single point with velocity $U_c$.

**Proof.** If all the agents receive the broadcast control then $n_l = n$ and $B = 1_n$. According to Lemma 2.1, all the agents will move with a velocity $U_c$. It remains to show that there is no deviation. Recalling that $v_i^*$ is a left eigenvector of $M$ with eigenvalue $\lambda_i$ and $B = 1_n$ we can rewrite eq. (26) as

$$
\xi_x = \left[ \sum_{i=1}^{n-1} \frac{1}{\lambda_i} v_i \frac{1}{\lambda_i} v_i^* M \right] 1_n U_x = 0_n
$$

and similarly $\xi_y = 0_n$

QED

### 2.3 Asymptotic trajectories of $n$ agents with an exogenous velocity control detected by $n_l$ agents

This section summarizes the results derived in sections 2.1 - 2.2.2.

The asymptotic position of agent $i$, chasing agent $i + 1$, in the two-dimensional space, when an external control $U_c = (U_x \ U_y)^T$ is detected by $n_l$ agents, will be

$$p_i(t \to \infty) = [\alpha + \beta U_c t + \gamma_i U_c]
$$

where

- $\alpha = (\alpha_x \ \alpha_y)^T = \frac{1}{n} \sum_{i=1}^{n} p_i(0)$ is the agreement, or gathering, point when there is no external input
- $\beta = \frac{n_l}{n}$ and $\beta U_c$ is the collective velocity.
- $\alpha + \beta U_c t$ is the position of the moving agreement point at time $t$
• $\gamma_i U_c$ is the deviation of agent $i$ from the moving agreement point, where

$$\gamma = \left[ \sum_{i=1}^{n-1} \frac{1}{\lambda_i} v_i v_i^* \right] B$$

and $\gamma_i$ is the $i^{th}$ element of $\gamma$

### 2.4 Illustration of linear cyclic pursuit - single interval

We illustrate the derived analytical results by 2 simulated examples. Both examples assume six agents, starting from the same random positions, shown in Fig. 2, but differing in the broadcast control and the set of agents detecting it. Example1 and Example2 were run for 50 secs (500000 points, $dt=0.0001$). This simulation time was long enough to obtain the analytically derived asymptotic behaviour.

![Initial nodes topology](image)

**Figure 2: Initial positions**

1. **Example1**
   - broadcast control, $U_c = (5, 1)$ (Slope=0.2)
   - set of ad-hoc leaders $\{0, 1, 0, 0, 0, 0\}$, thus $n_l = 1$ (out of 6)
Simulated results for Example 1:

Figure 3: Emergent trajectories

A solid line represents the leader while the trajectories of followers are shown by dotted lines.
In this figure a solid red line represents a leader while dotted lines represent followers. The velocities converge to $u|x_{T_{max}}$, $u|x_{T_{max}}$ which agree with the derived asymptotic velocities of $\frac{n_l}{n}U_c$, where $n_l = 1$, $n = 6$. 

Figure 4: Agents velocities in a long time interval
This figure shows the position of the agents at the end of the time interval. A star represents a leader while o represents followers. The slope of the line equals the slope of $U_c$, i.e. the agents align in the direction of $U_c$.

2. Example 2

- broadcast control, $U_c = (6, 3)$ (Slope=0.5)
- set of ad-hoc leaders $\{1, 1, 0, 1, 1, 1\}$, thus $n_l = 5$ (out of 6)

Simulated results for Example 2:
Displayed $u x_{T_{\max}}, u y_{T_{\max}}$ agree with the derived asymptotic velocities of $\frac{n_t}{n} U_c$, where $n_t = 5, n = 6$
Figure 8: Agents asymptotically align along the direction of $U_c$

We observe that in both examples the agents asymptotically align in the direction of $U_c$ and move as a linear formation with velocity $\frac{n_I}{n} U_c$, as expected.

We emphasize that this behaviour is indeed asymptotic, not obtained in a short time interval, as shown in figures 9 and 10, next.
Figure 9: Agents velocities in a short time interval (5sec)

Figure 10: Agents positions ("alignment") at the end of a short interval (5sec)
2.5 Illustration of linear cyclic pursuit - multiple intervals

We recall that we assumed \( B(t) \) and \( U_c(t) \) to be piecewise constant, i.e. \( B(t) \triangleq B(t_k) \), and \( U_c(t) \triangleq U_c(t_k) \), where \( t_k \) is the time of change of the set of ad-hoc leaders or of the broadcast control, and we treated separately each time interval, \([t_k, t_{k+1})\). In the above sections we considered a single time interval, where \( U_c \) and \( B \) are constant.

In this section we show, by simulation, the emergent behaviour over multiple time intervals. We consider as before 6 agents, starting at initial positions as illustrated in Fig. 2 but, starting with \( U_c(0) = (6, 3)^T \), we allow for discrete changes in \( U_c \) every 10 secs. \( U_c(t) \) is shown in Fig. 11. In this example the set of ad-hoc leaders is randomly selected at \( t = 0 \) and remains constant afterwards, i.e. \( B(t) = B(0) \forall t \).

The emergent trajectories are illustrated in Fig. 12.

\[ \text{Figure 11: } U_c(t) \text{ profile} \]
Figure 12: Emerging trajectories with $U_c(t)$ as in Fig. 11

Fig. 13 shows the agents velocities vs. analytically computed asymptotic velocity of the linear formation emerging from a broadcast velocity signal as shown in Fig. 11.

Figure 13: Emerging velocities vs computed formation velocity
We note that changes in the random set of ad-hoc leaders, within an interval where $U_c$ is constant, will affect the speed of the agents (if the *number* of ad-hoc leaders changes) and also the arrangement of the agents within the linear formation.
3 Non-linear cyclic pursuit

We assume the model for non-linear cyclic pursuit to be the ”bugs” model, where agent $i$ chases agent $i + 1$ (agent $n$ chases agent 1) with constant, common, speed along the line of sight, merges with it upon capture and the two agents continue with velocity $\dot{p}_{i+1}$. Upon capture, the number of agents is reduced. Agent $i$ is said to capture agent $i + 1$ if the distance between them, $d_i$, is zero

$$d_i(t) = \| p_{i+1}(t) - p_i(t) \|$$

(30)

where $p_i(t)$ is the position of agent at time $t$ and $\| . \|$ represents the Euclidean norm.

In case of an external broadcast velocity control, upon capture, the merged agent will be assumed to detect the broadcast velocity signal, if either one or both of the merged agents detected the broadcast velocity signal.

We first analyze the properties of the ”bugs” model, without external broadcast velocity signal, and then derive the impact of the broadcast velocity which is detected by a random set of agents.

3.1 ”Bugs” model

Let $d_i(t)$ be defined by eq. (30) and assume, without loss of generality, the speed of all agents to be 1. Then, the pursuit is formally defined as follows:

$$\begin{align*}
\text{if } d_i(t) \neq 0 & \quad \dot{p}_i(t) = \frac{p_{i+1}(t) - p_i(t)}{d_i(t)} \\
\text{if } d_i(\hat{t}) = 0 & \quad p_i(t) = p_{i+1}(t) \text{ for all } t \geq \hat{t}
\end{align*}$$

(31)

(32)

Lemma 3.1. If $d_i(t) \neq 0$ for all $i$ and the motion of each agent is represented by (31), then

(a) $d_i(t)$ is monotonically non-increasing for all $i$.

(b) There exists a finite time $T_m$ such that $d_i(T_m) = 0$ for all $i$

Proof. In the sequel, for simplicity of notation, we omit explicit reference to time $t$, whenever it is not confusing.
(a) : \( d_i(t) \) is non-increasing iff \( \dot{d}_i(t) \leq 0 \) for all \( t \).

From (30) we have

\[
\dot{d}_i = \frac{1}{d_i} (\dot{p}_{i+1} - \dot{p}_i, \dot{p}_{i+1} - \dot{p}_i) = (\dot{p}_{i+1} - \dot{p}_i, \dot{p}_i)
\]

where we used \( \|\dot{p}_i\| = \|\dot{p}_{i+1}\| = 1 \) and \( \theta \) is defined as in Fig. 14.

(b) : Richardson shows in [25], Lemma 1.1, that \( T_m \) exists and satisfies

\[
T_m \leq t_0 + 2n \sum_{i=1}^{n} d_i(t_0).
\]

The proof is repeated in Appendix D, for completeness.

QED
3.2 "Bugs" model with external input

Let $K_i$ denote the set of $k$ indices of $K_i = \{i - k, i - k + 1, \ldots, i - 1\}$. If $d_j = 0 \quad \forall j \in K_i$ and $d_i \neq 0$ we say that the $k$ agents in $K_i$ collapsed into agent $i$.

In order to analyze the motion of agent $i$ performing bearing-only cyclic pursuit when a velocity signal, $U_c$, is broadcast by an external controller and detected by a random set of agents, we have to consider three different scenarios:

1. $i$ is a free agent, i.e. $d_i \neq 0$ and $K_i = \{\}$

2. $k$ agents labeled $(i - k, i - k + 1, \ldots, i - 1)$ have merged into $i$, i.e. $d_i \neq 0$ and $d_j = 0 \quad \forall j \in K_i$

3. Agent $i$ collapsed into agent $j$, i.e. $d_j \neq 0$, $d_i = 0$ and $i \in K_j$

These cases are formalized by eqs. (37) - (38):

$$\dot{p}_i = \begin{cases} \frac{p_{i+1} - p_i}{d_i} + b_i U_c & \text{if } d_i \neq 0 \text{ and } K_i = \{\} \\ \frac{p_{i+1} - p_i}{d_i} + (\vee_{j \in K_i} b_j) U_c & \text{if } d_i \neq 0 \quad \forall j \in K_i \end{cases}$$

(37)

$$p_i = p_j \quad \forall i \in K_j$$

(38)

where $\vee$ stands for logical "or",

$$b_i = \begin{cases} 1; & \text{if } i \in N^l \\ 0; & \text{otherwise} \end{cases}$$

(39)

and $N^l$ is the set of agents that detected the broadcast control.

If $d_i = 0 \quad \forall i$ then

$$\dot{p}_i = (\vee_i) U_c \quad \forall i$$

Thus, once gathered, the agents will move as a single agent with velocity $U_c$, if at least one of the agents detected the external control $U_c$.

In section 3.2.1, we show the impact of the exogenous velocity signal, $U_c$ on a single distance, $d_i(t)$, defined by (30), when either $i$ or $i + 1$, or both, detect $U_c$. We derive the behaviour of $d_i$, in each case, as a function of $\|U_c\|$ and the instantaneous geometry. Since the agents are mobile, the geometry is time dependent. Therefore, we cannot deduce from $d_i(t)$ instantaneously
decreasing that it will decrease for all $t$ or that it will reach zero. In section 3.2.2 we derive an upper bound on $\|U_c\|$ that ensures convergence to a point in finite time, $T_m$, i.e. $\sum_{i=1}^n d_i(T_m) = 0$, independently of the instantaneous geometries.

### 3.2.1 Single $d_i$ behaviour

Let $d_i > 0 \ \forall i$. This is a valid assumption since subsequent to a collision (capture) the system evolves as a cyclic pursuit with fewer agents. Thus,

\begin{align*}
\dot{d}_i &= \frac{1}{d_i} (\hat{p}_{i+1} - \hat{p}_i, p_{i+1} - p_i) \\
\dot{p}_i &= \frac{p_{i+1} - p_i}{d_i} + b_i U_c
\end{align*}

(40)

The impact of $U_c$ on any distance $d_i > 0$ can be separated into four cases:

1. Neither $i$ nor $i + 1$ detected the signal $U_c \Rightarrow b_i = 0, \quad b_{i+1} = 0$
2. Both $i$ and $i + 1$ detected the signal $U_c \Rightarrow b_i = 1, \quad b_{i+1} = 1$
3. $i$ detected the signal $U_c$, but $i + 1$ did not $\Rightarrow b_i = 1, \quad b_{i+1} = 0$
4. $i + 1$ detected the signal $U_c$, but $i$ did not $\Rightarrow b_i = 0, \quad b_{i+1} = 1$

**Case 1:** $b_i = 0, \quad b_{i+1} = 0$

This case is identical to Lemma 3.1(a), Fig. 14, and therefore $\dot{d}_i = \cos(\theta) - 1 \leq 0$, independently of the broadcast $U_c$.

**Case 2:** $b_i = 1, \quad b_{i+1} = 1$

In this case $\frac{p_{i+1} - p_i}{d_i} = \hat{p}_i - U_c$ and (40) can be rewritten as

\begin{align*}
\dot{d}_i &= (\hat{p}_{i+1} - \hat{p}_i, \hat{p}_i - U_c) \\
&= ((\hat{p}_{i+1} - U_c) - (\hat{p}_i - U_c), \hat{p}_i - U_c)
\end{align*}

Recalling that $\frac{p_j - U_c}{d_j} = \frac{p_{j+1} - p_j}{d_j}; \quad j = i, i + 1$ we have $\|\hat{p}_j - U_c\| = 1; \quad j = i, i + 1$ and thus

\begin{equation}
\dot{d}_i = \cos(\theta) - 1
\end{equation}

(42)
where $\theta$ is now the angle between $\dot{p}_i - U_c$ and $\dot{p}_{i+1} - U_c$, see Fig. 15. Thus, in this case $\dot{d}_i \leq 0$, for any $U_c$, same as case 1.

Figure 15: $U_c$ detected by both $i$ and $i + 1$

**Case 3:** $b_i = 1, \quad b_{i+1} = 0$.

In this case $\dot{p}_{i+1}$ is not affected by $U_c$, as shown in Fig. 16, where we have $\|\dot{p}_i - U_c\| = 1$, and $\|\dot{p}_{i+1}\| = 1$. 

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32
In this case

\[
\dot{d}_i = (\dot{p}_{i+1} - \dot{p}_i, \dot{p}_i - U_c) \\
= ((\dot{p}_{i+1} - U_c) - (\dot{p}_i - U_c), \dot{p}_i - U_c) \\
= ((\dot{p}_{i+1} - U_c), (\dot{p}_i - U_c)) - 1 \\
= ((\dot{p}_{i+1}), (\dot{p}_i - U_c)) - (U_c, (\dot{p}_i - U_c)) - 1 \\
= \cos(\theta) - \|U_c\| \cos(\alpha) - 1
\]

where the angles $\theta, \alpha$ are defined as in Fig. 16. In order to have $\dot{d}_i \leq 0$ we must have $-\|U_c\| \cos(\alpha) \leq 1 - \cos(\theta)$. Therefore, $d_i$ will be non-increasing

- for any $\|U_c\|$ if $\cos(\alpha) \geq 0$, i.e. $\alpha$ is an acute angle
\[ \text{for } \|U_c\| < \frac{1 - \cos(\theta)}{-\cos(\alpha)} \text{ if } \cos(\alpha) < 0, \text{ i.e. } \alpha \text{ is an obtuse angle} \]

**Case 4:** \( b_i = 0, \quad b_{i+1} = 1 \).

In this case \( \dot{p}_i \) is not affected by \( U_c \), as shown in Fig. 17, where we have \( \|p_i\| = 1 \), and \( \|\dot{p}_{i+1} - U_c\| = 1 \).

\[ \dot{d}_i = \langle \dot{p}_{i+1} - \dot{p}_i, \dot{p}_i \rangle \]
\[ = \langle \dot{p}_{i+1}, \dot{p}_i \rangle - 1 \]
\[ = \left( \langle \dot{p}_{i+1} - U_c, \dot{p}_i \rangle + \langle U_c, \dot{p}_i \rangle \right) - 1 \]
\[ = \cos(\theta) + \|U_c\| \cos(\alpha) - 1 \]

Figure 17: \( U_c \) detected by \( i+1 \) but not by \( i \).
Therefore, in this case, $d_i$ will be non-increasing

- for any $\|U_c\|$ if $\cos(\alpha) < 0$, i.e. $\alpha$ is an obtuse angle

- for $\|U_c\| < \frac{1 - \cos(\theta)}{\cos(\alpha)}$ if $\cos(\alpha) > 0$, i.e. $\alpha$ is an acute angle

### 3.2.2 Gathering in finite time

In section 3.2.1 we derived the *instantaneous* behaviour of a single inter-agents distance when none or both or one of the limiting agents detected the broadcast control. Since the geometry of the agents is time dependent we cannot deduce the emergent behaviour of the system from the instantaneous behaviour. Given that the agents behave according to the "bugs" model with broadcast control, see (37), we want to show that there exist conditions that ensure the gathering of the agents to a moving point in finite time. These conditions are derived in Theorem 3.2 as an upper bound on the magnitude of $U_c$.

**Theorem 3.2.** If $d_i$ is defined by (30) and $d_i(t) \neq 0; i = 1, ..., n$, then

(a) if $\|U_c\| \leq \frac{1}{2n^2}$, there exists a finite time $T_m$ such that $\sum_{i=1}^{n} d_i(t \geq T_m) = 0$

(b) (a) holds for all times when $\sum d_i(t) \neq 0$ even if $d_i(t) = 0$ for some $i$, i.e. the capture is non-mutual

(c) $T_m \leq t_0 + \frac{2n \sum d_i(t_0)}{1 - 2n^2 \|U_c\|}$, where $t_0$ is the initial time.

**Proof.** (a):

Following the methodology in [25], we prove that for $\|U_c\| < \frac{1}{2n^2}$ we have $\sum_{i=1}^{n} \dot{d}_i \leq -c$, where $c = \frac{1}{2n} - n \|U_c\| > 0$, therefore there exists a time $T_m$ such that $\sum_{i=1}^{n} d_i(T_m) = 0$.

For $d_i(t) \neq 0$ we have

\[
\dot{p}_i = u_i + b_i U_c \tag{43}
\]

\[
u_i = \frac{p_{i+1} - p_i}{d_i} \tag{44}
\]
Using \( d_i^2 = \| p_{i+1} - p_i \|^2 \), (44) and (43), we have

\[
\dot{d}_i = u_i^T (\dot{p}_{i+1} - \dot{p}_i) \\
= u_i^T ((u_{i+1} + b_{i+1} U_c - u_i - b_i U_c) \\
= u_i^T u_{i+1} - 1 + (b_{i+1} - b_i) u_i^T U_c
\]

where we used \( \| u_i \| = 1 \). But

\[
u_i^T u_{i+1} - 1 = -\frac{1}{2} (u_{i+1} - u_i)^T (u_{i+1} - u_i) \\
= -\frac{1}{2} \| u_{i+1} - u_i \|^2
\]

Thus

\[
\dot{d}_i = -\frac{1}{2} \| u_{i+1} - u_i \|^2 + (b_{i+1} - b_i) u_i^T U_c
\] (45)

Let \( V_c = (b_{i+1} - b_i) U_c \)

\[
\dot{d}_i \leq -\frac{1}{2} \| u_{i+1} - u_i \|^2 + \| V_c \| = -\frac{1}{2} \| u_{i+1} - u_i \|^2 + |b_{i+1} - b_i| \| U_c \|
\] (46)

\[
\sum_{i=1}^n \dot{d}_i \leq -\frac{1}{2} \sum_{i=1}^n \| u_{i+1} - u_i \|^2 + \| U_c \| \sum_{i=1}^n |b_{i+1} - b_i|
\] (47)

\[
\leq -\frac{1}{2} \sum_{i=1}^n (\sum_{i=1}^n \| u_{i+1} - u_i \|)^2 + n_b \| U_c \|
\] (48)

where \( n_b \) is the number of pairs of agents \( \{i, i+1\} \) such that \( |b_{i+1} - b_i| = 1 \) and we used the Cauchy-Schwartz inequality, (69) in Appendix D. Since \( n_b \leq n \) we can write

\[
\sum_{i=1}^n \dot{d}_i \leq -\frac{1}{2n} \left( \sum_{i=1}^n \| u_{i+1} - u_i \| \right)^2 + n \| U_c \|
\] (49)

Repeating the reasoning in [25] and Appendix D we can show that \( \sum_{i=1}^n \| u_{i+1} - u_i \| > 1 \) and therefore

\[
\sum_{i=1}^n \dot{d}_i \leq -\frac{1}{2n} + n \| U_c \|
\]

Thus, if \( \| U_c \| < \frac{1}{2n^2} \) then \( \sum_{i=1}^n \dot{d}_i < 0 \)

(b): After any capture, say \( i \) captures \( i+1 \), the two agents merge and thus \( n \) is reduced. Let \( \hat{n} \) be the number of remaining agents at time \( t \) and
\( \hat{d}_i; i = 1, \ldots, \hat{n} \) the new distances. Then, by the same reasoning as for the proof of Lemma 3.2 we have

\[
\sum_{i=1}^{\hat{n}} \hat{d}_i \leq -\frac{1}{2\hat{n}} + \hat{n}\|U_c\| \tag{50}
\]

But, by definition, \( \sum_{i=1}^{n} d_i = \sum_{i=1}^{\hat{n}} \hat{d}_i \) and \( \hat{n} \leq n \), thus

\[
\sum_{i=1}^{n} \hat{d}_i = \sum_{i=1}^{\hat{n}} \hat{d}_i \leq -\frac{1}{2\hat{n}} + \hat{n}\|U_c\| \tag{51}
\]

\[
\leq -\frac{1}{2n} + n\|U_c\| \tag{52}
\]

\[
\leq -\frac{1}{2\hat{n}} + n\|U_c\| \tag{53}
\]

(c): The distances \( d_i(t) \) are continuous and there can be only a finite number of captures, thus by integrating (53) we obtain

\[
\sum_{i=1}^{\hat{n}} \hat{d}_i(T_m) = 0 \leq \sum_{i=1}^{\hat{n}} \hat{d}_i(t_0) + (-\frac{1}{2\hat{n}} + n\|U_c\|)(T_m - t_0)
\]

\[
T_m \leq t_0 + \frac{2n \sum_{i=1}^{\hat{n}} \hat{d}_i(t_0)}{1 - 2n^2\|U_c\|} \tag{54}
\]

QED

Remark 3.1. The condition \( \|U_c\| < \frac{1}{2n^2} \) is a very stringent bound for gathering and moving as a single point, since in general \( n_b < n \) and we do not require mutual capture, i.e \( n \) can decrease with time.

Remark 3.2. The bound (54) for gathering time holds only for \( \|U_c\| < \frac{1}{2n^2} \).

4 Illustration by simulation of emergent behaviour in bearing-only cyclic pursuit

We simulated the cyclic pursuit with broadcast control to test the theory developed above.
4.1 Simulation parameters

- User defined number of agents, \( n \), and external velocity control, \( U_c \).
  Since in our ”bugs” model the speed of agents is one we limited \( U_c \), to \( \|U_c\| \leq 1 \).
- Randomly selected initial positions and agents detecting the external control
- Bearing-only simulation description
  - \( i \) captures \( i + 1 \) (and merges with it) at time \( t_c \), if \( d_i(t_c) \leq \epsilon \) or \( i \) overtakes \( i + 1 \). In the presented simulations we used \( \epsilon = 0.001 \)
  - For \( t \geq t_c \)
    * \( p_i(t) = p_{i+1}(t) \)
    * \( \dot{b}_{i+1} = \dot{b}_i \vee b_{i+1} \)
    * \( \dot{p}_{i+1}(t) = \frac{p_{i+2}(t) - p_{i+1}(t)}{d_{i+1}} + \dot{b}_{i+1}U_c \)

4.2 Examples of simulations results

In this section we show sample simulation results for various initial topologies, various external inputs and sets of ad-hoc leaders. All the presented simulation results used \( n = 6 \).

4.2.1 Example1-bugs: Gathering property of the homogeneous ”bugs” model

We show an example of gathering of the ”bugs” model, without external input Let Example1-bugs denote the case of initial topology as in Fig. 18 and \( U_c = (0,0) \).
In Fig. 19, \((x_{T_{\text{max}}}, y_{T_{\text{max}}})\) denote the position of each agent, at the end of the time interval. We see that the agents indeed converged to a point, but this differs from the (displayed) initial centroid.
For comparison, we show the behaviour of the agents, starting from the same initial conditions, but performing linear cyclic pursuit. The gathering point in this case is the initial centroid.

Figure 20: Agents gathering in case of linear cyclic pursuit without external input

4.2.2 Example2-bugs: Impact of external input and incomplete sets of ad-hoc leaders

Next we show the impact of an external input, $U_c$, on the emergent behaviour, for various values of $U_c$, such that $\|U_c\| \leq 1$, and various incomplete sets of agents detecting it. We show for each example the behaviour of $d_i(t)$, the distance of agent $i$ to $i+1$, $\forall i$, as well as agents’ trajectories and velocities. We observe that in all considered cases

- There exists a time $t_c$ of mutual capture where $d_i = 0$ $\forall i$
- For $t \geq t_c$ all agents move as a single point with velocity $U_c$
- The value of $t_c$ and the behaviour of distances to prey, $d_i(t)$ for $t < t_c$, as well as of agents’ velocities for $t < t_c$ depend on the value of the external input and on the set of agents detecting the broadcast signal
All cases of Example2-bugs were run starting from the positions shown in Fig. 21.

![Initial nodes topology](image)

**Figure 21: Initial agents positions for Example2-bugs**

Cases considered were

- **Example2.1-bugs**: Same broadcast signal, different ad-hoc leaders
  - Example2.1.1-bugs: $U_c = (0.5, 0.3)^T$, $B = (111011)^T$
  - Example2.1.2-bugs: $U_c = (0.5, 0.3)^T$, $B = (001000)^T$
- **Example2.2-bugs**: Same set of ad-hoc leaders, increasing magnitude of broadcast signal
  - Example2.2.1-bugs: $B = (010010)^T$, $U_c = (0.013, 0)^T$
  - Example2.2.2-bugs: $B = (010010)^T$, $U_c = (0.5, 0.3)^T$
  - Example2.2.3-bugs: $B = (010010)^T$, $U_c = (-1, 0)^T$

where $B$ is a vector of pointers to the agents detecting the exogenous control.
Example 2.1-bugs: Same broadcast signal, different ad-hoc leaders
Chaser to prey distances - $d_i(t)$

Figure 22: Example 2.1.1-bugs: Distances behaviour
Figure 23: Example2.1.2-bugs: Distances behaviour
Velocities of agents

Figure 24: Example 2.1.1-bugs: Agents’ velocities

Figure 25: Example 2.1.2-bugs: Agents’ velocities

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Agents’ Trajectories

Figure 26: Example 2.1.1-bugs: Agents’ trajectories

Figure 27: Example 2.1.2-bugs: Agents’ trajectories
Example 2.2-bugs: Same set of ad-hoc leaders, increasing magnitude of broadcast signals
Chaser to prey distances $d_i(t)$

Figure 28: Example 2.2.1-bugs: Distances behaviour
Figure 29: Example2.2.2-bugs: Distances behaviour

Figure 30: Example2.2.3-bugs: Distances behaviour
Agents’ velocities

Figure 31: Example2.2.1-bugs: Velocities behaviour

Figure 32: Example2.2.2-bugs: Velocities behaviour
Figure 33: Example2.2.3-bugs: Velocities behaviour
Agents’ trajectories

Figure 34: Example2.2.1-bugs: Trajectories behaviour

Figure 35: Example2.2.2-bugs: Trajectories behaviour
Figure 36: Example2.2.3-bugs: Trajectories behaviour
4.2.3 Example3-bugs: Broadcast signal received by all

In this section we show that if the broadcast control is received by all then the gathering property of the cyclic pursuit is independent of the value of $\|U_c\|$. We show simulation results for

- Example3.1-bugs: $U_c = (5, 3)^T$
- Example3.2-bugs: $U_c = (-3, 2)^T$
- Example3.3-bugs: $U_c = (0, 0)^T$

From the presented results we observe that the time to convergence is identical in all cases, i.e. identical to the case $U_c = (0, 0)^T$. These results correspond to the theory in section 3.2.1, case 2, where we show that if $b_i = b_{i+1} = 1$ then $\dot{d}_i$ is independent of $U_c$.

**Distances behaviour - $d_i(t)$**

![Figure 37: Example3.1-bugs: Distances behaviour](image)

Figure 37: Example3.1-bugs: Distances behaviour
Figure 38: Example3.2-bugs: Distances behaviour

Figure 39: Example3.3-bugs: Distances behaviour in "bugs" cyclic pursuit, without external control
In all the examples shown above, we obtained

- Identical distances behaviour, i.e. independent of $U_c$

- The time to convergence was 64.5979
Agents’ velocities

Figure 40: Example3.1-bugs: Velocities behaviour

Figure 41: Example3.2-bugs: Velocities behaviour
Agents’ trajectories

Figure 42: Example 3.1-bugs: Trajectories behaviour

Figure 43: Example 3.2-bugs: Trajectories behaviour
Remark 4.1. The broadcast signal being received by all is a particular case of the broadcast signal being received by a random set of agents. Since we do not enforce it, in order to enable the general case of a random, not complete, set of agents receiving the broadcast signal we need to enforce \( \|Uc\| \leq 1 \).
A   About matrices

Following [?], let $M_n$ denote the class of all $n \times n$ matrices. Two matrices $A, B \in M_n$ are similar, denoted by $A \sim B$, if there exists an invertible (non-singular) matrix $S \in M_n$ s.t. $A = SBS^{-1}$. Similar matrices are just different basis representation of a single linear transformation. Similar matrices have the same characteristic polynomial, c.f. Theorem 1.3.3 in [?] and therefore the same eigenvalues.

A.1   Algebraic and geometric multiplicity of eigenvalues

Let $\lambda$ be an eigenvalue of an arbitrary matrix $A \in M_n$ with an associated eigenvector $v \in \mathbb{C}^n$.

Definitions:

- The spectrum of $A \in M_n$ is the set of all the eigenvalues of $A$, denoted by $\sigma(A)$.
- The spectral radius of $A$ is $\rho(A) = \max|\lambda|: \lambda \in \sigma(A)$.
- For a given $\lambda \in \sigma(A)$, the set of all vectors $v \in \mathbb{C}^n$ satisfying $Av = \lambda v$ is called the eigenspace of $A$ associated with the eigenvalue $\lambda$. Every nonzero element of this eigenspace is an eigenvector of $A$ associated with $\lambda$.
- The algebraic multiplicity of $\lambda$ is its multiplicity as a root of the characteristic polynomial $\det(\lambda I - A)$.
- The geometric multiplicity of $\lambda$ is the dimension of the eigenspace associated with $\lambda$, i.e. the number of linearly independent eigenvectors associated with that eigenvalue.
- We say that $\lambda$ is simple if its algebraic multiplicity is 1; it is semisimple if its algebraic and geometric multiplicities are equal.
- The algebraic multiplicity of an eigenvalue is larger than or equal to its geometric multiplicity.
- We say that $A$ is defective if the geometric multiplicity of some eigenvalue is less than its algebraic multiplicity.
A.2 Diagonizable matrices

**Definition:** If $A \in M_n$ is similar to a diagonal matrix, then $A$ is said to be diagonizable.

*Theorem 1.* See Theorem 1.3.7 in [?].

The matrix $A \in M_n$ is diagonizable iff there are $n$ linearly independent vectors, $v^{(1)}, v^{(2)}, \ldots, v^{(n)}$, each of which is an eigenvector of $A$. If $v^{(1)}, v^{(2)}, \ldots, v^{(n)}$ are linearly independent eigenvectors of $A$ and $S = [v^{(1)}, v^{(2)}, \ldots, v^{(n)}]$ then $S^{-1}AS$ is a diagonal matrix $\Lambda$ and the diagonal entries of $\Lambda$ are the eigenvalues of $A$.

*Lemma 1.* Let $\lambda_1, \ldots, \lambda_k; \ k \geq 2$ be distinct eigenvalues of $A \in M_n$ and suppose $v^{(i)}$ is an eigenvector associated with $\lambda_i; \ i = 1, \ldots, n$. Then the vectors $[v^{(1)}, v^{(2)}, \ldots, v^{(k)}]$ are linearly independent.

Proof of Lemma 1.3.8 in [?]

*Theorem 2.* If $A \in M_n$ has $n$ distinct eigenvalues, then $A$ is diagonizable.

*Proof.* Since all eigenvalues are distinct Lemma 1 ensures that the associated eigenvectors are linearly independent and thus, according to Theorem 1, $A$ is diagonizable. QED

**Notes:**

1. Having distinct eigenvalues is sufficient for diagonalizability, but not necessary.

2. A matrix is diagonizable iff it is non-defective, i.e. it has no eigenvalue with geometric multiplicity strictly less than its algebraic multiplicity.

A.3 Left eigenvectors

**Definition:** A non-zero vector $y \in \mathbb{C}^n$ is a left eigenvector of $A \in M_n$ associated with eigenvalue $\lambda$ of $A$ if $y^*A = \lambda y^*$. From [?], Theorem 1.4.12, we have the following relationship between left and right eigenvectors and the multiplicities of the corresponding eigenvalue:

*Theorem 3.* Let $\lambda \in \mathbb{C}$ be an eigenvalue of $A \in M_n$ associated with right eigenvector $x$ and left eigenvector $y^*$. Then the following hold:

- (a) If $\lambda$ has algebraic multiplicity 1, then $y^*x \neq 0$

- (b) If $\lambda$ has geometric multiplicity 1, then it has algebraic multiplicity 1 iff $y^*x \neq 0$
A.4  About non-symmetric real matrices

- the eigenvalues of non-symmetric real $n \times n$ matrix are real or come in complex conjugate pairs
- the eigenvectors are not orthonormal in general and may not even span an n-dimensional space
  - Incomplete eigenvectors can occur only when there are degenerate eigenvalues, i.e. eigenvalues with algebraic multiplicity greater than 1, but do not always occur in such cases
  - Incomplete eigenvectors never occur for the class of normal matrices
- Diagonalization theorem: an $n \times n$ matrix $A$ is diagonizable iff $A$ has $n$ linearly independent eigenvectors

A.5  Normal matrices

Definition 1. A matrix $A \in M_n$ is called normal if $A^* A = AA^*$

Definition 2. A matrix $A \in M_n$ is called Hermitian if $A^* = A$

Theorem 4. If $A \in M_n$ has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ the following statements are equivalent:

(a) $A$ is normal
(b) $A$ is unitarily diagonizable
(c) $\sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2 = \sum_{j=1}^{n} |\lambda_j|^2$
(d) There is an orthonormal set of $n$ eigenvectors of $A$

Remark A.1. All normal matrices are diagonizable but not all diagonizable matrices are normal.

A.6  Unitary matrices and unitary similarity

Unitary matrices, $U \in M_n$, are non-singular matrices such that $U^{-1} = U^*$, i.e $U^* U = U U^* = I$. A real matrix $U \in M_n(\mathbb{R})$ is real orthogonal if $U^T U = I$. The following are equivalent:
(a) $U$ is unitary
(b) $U$ is non-singular and $U^{-1} = U^*$
(c) $UU^* = I$
(d) $U^*$ is unitary
(e) The columns of $U$ are orthonormal
(f) The rows of $U$ are orthonormal
(g) For all $x \in \mathbb{C}^n$, $\|x\|_2 = \|Ux\|_2$

**Definition A.1.** :
- $A$ is unitarily similar to $B$ if there is a unitary matrix $U$ s.t. $A = UBU^*$
- $A$ is unitarily diagonizable if it is unitarily similar to a diagonal matrix

**B Laplacian representation of Graphs**

Graphs provide natural abstraction for how information is shared between agents in a network. Algebraic graph theory associate matrices, such as Adjacency and Laplacian, with graphs, cf. [?]. In this appendix some useful facts from algebraic graph theory are presented. Given a multi-agent system, the network can be represented by a directed or an undirected graph $G = (V, E)$, where $V$ is a finite set of vertices, labeled by $i$; $i = 1, ..., n$ representing the agents, and $E$ is the set of edges, $E \subseteq [V \times V]$, representing inter-agent information exchange links.\(^2\) A simple graph contains no self-loops, namely there is no edge from a node to itself. If the graph is undirected then the edge set $E$ contains unordered pairs of vertices. In directed graphs (digraphs) the edges are ordered pairs of vertices. We say that the graph is connected if for every pair of vertices in $V$ there is a path with those vertices as its end vertices. If this is not the case, the graph is called disconnected. We refer to a connected graph as having one connected component. A disconnected graph has more than one component.

\(^2\) Vertices are also referred to as nodes and the two terms will be used interchangeably
B.1 Directed graphs - digraphs

A directed graph (or digraph), denoted by \( D = (V, E) \), is a graph whose edges are ordered pairs of vertices. For the ordered pair \((i, j) \in E\), when vertices \(v_i, v_j\) are labelled \(i, j\), \(i\) is said to be the tail of the edge, while \(j\) is its head.

Definitions:

1. A digraph is called strongly connected if for every pair of vertices there is a directed path between them.

2. The digraph is called weakly connected if it is connected when viewed as a graph, that is, a disoriented digraph.

3. A digraph has a rooted out-branching, or spanning tree, if there exists a vertex \(r\) (the root) such that for every other vertex \(i \neq r \in N\) there is a directed path from \(r\) to \(i\). In this case, every \(i \neq r \in N\) is said to be reachable from \(r\). In strongly connected digraphs each node is a root.

4. A node is called balanced if the total weight of edges entering the node and leaving the same node are equal.

5. If all nodes in the digraph are balanced then the digraph is called balanced.

B.1.1 Properties of Laplacian matrices associated with digraphs

- The non-symmetric Laplacian, \( L \), associated with a digraph \( G \) of order \( n \) has the following properties:

  (a) \( L \) has at least one zero eigenvalue and all remaining eigenvalues have positive real part.

  (b) \( L \) has a simple zero eigenvalue and all other eigenvalues have positive real part if and only if \( G \) has a directed spanning tree.

  (c) \( L \) is real, therefore any complex eigenvalues must occur in conjugate pairs.\(^3\)

----

\(^3\)The eigenvalues of a real non-symmetric matrix may include real values, but may also include pairs of complex conjugate values.
(d) There is a right eigenvector of ones, \( \mathbf{1}_n \), associated with the zero eigenvalue, i.e. \( L \mathbf{1}_n = \mathbf{0}_n \).

(e) The left eigenvector of \( L \) corresponding to \( \lambda = 0 \), denoted by \( \mathbf{w}_l \) is positive and \( \sum_{i=1}^{n} w_l(i) = 1 \).

(f) \( w_l = \mathbf{1}_n^T \) if and only if the digraph is balanced.

- If the Laplacian \( L \) of the digraph is a normal, i.e. \( LL^T = L^T L \), then
  1. There exists an orthonormal set of \( n \) eigenvectors of \( L \).
  2. \( L \) is unitarily diagonizable, i.e. \( L = U \Lambda U^* \), where \( U \) is a unitary matrix of eigenvectors and \( \Lambda \) is a diagonal matrix of eigenvalues.
  3. The digraph must be balanced and thus \( w_l = \mathbf{1}_n^T \).

C Properties of circulant matrices

A circulant matrix is an \( n \times n \) matrix having the form

\[
C = \begin{bmatrix}
c_0 & c_1 & c_2 & \cdots & c_{n-1} \\
c_{n-1} & c_0 & c_1 & \cdots & c_{n-2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
c_1 & c_2 & \cdots & \cdots & c_0
\end{bmatrix}
\]  

(55)

which can also be characterized as an \( n \times n \) matrix \( C \) with entry \( (k,j) \) given by

\[
C_{k,l} = c_{(l-k) \mod (n)}
\]

Every \( n \times n \) circulant matrix \( C \) has eigenvectors (cf. [?], [?])

\[
v_k = \frac{1}{\sqrt{n}} \left( 1, e^{-2\pi jk/n}, e^{-4\pi jk/n}, \ldots, e^{-2\pi jk(n-1)/n} \right)^T; \quad k \in \{0, 1, \ldots, n-1\}
\]

(56)

where \( j = \sqrt{-1} \), with corresponding eigenvalues

\[
\lambda_k = \sum_{l=0}^{n-1} c_l e^{-2\pi jlk/n}
\]

(57)

From the definition of eigenvalues and eigenvectors we have

\[
Cv_k = \lambda_k v_k; \quad k = 0, 1, \ldots, n-1
\]

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which can be written as a single matrix equation

\[ CU = U\Lambda \]

where \( \Lambda = \text{diag}(\lambda_k); \quad k = 0, \ldots, n - 1 \) and

\[ U = \begin{bmatrix} v_0, v_1, \ldots, v_{n-1} \end{bmatrix} = \frac{1}{\sqrt{n}} \begin{bmatrix} e^{-2\pi jmk}; \quad m, k = 0, \ldots, n - 1 \end{bmatrix} \]

\( U \) is a unitary matrix, i.e. \( UU^* = U^*U = I \) (cf. [?], proof by direct computation) and

\[ C = U\Lambda U^* \quad (58) \]

Note that \( F_n = \sqrt{n}U^* \) is the known Fourier matrix.

### C.1 Cyclic pursuit

The Laplacian representing cyclic pursuit is a special case of circulant matrix

\[ L = \begin{bmatrix} 1 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ -1 & 0 & \cdots & 1 \end{bmatrix} \quad (59) \]

Thus the eigenvalues of the cyclic pursuit Laplacian are

\[ \lambda_k = 1 - e^{-2\pi jk/n}; \quad k = 0, \ldots, n - 1 \quad (60) \]

and the eigenvectors are given by eq. (56).

### D Proof of mutual capture existence in finite time, in non-linear cyclic pursuit without broadcast control - Lemma 1.1 in [25]

This proof, sketched in [25], is brought here for completeness. Let \( p_i(t) \) be the position of agent \( i \) at time \( t \) and let agent \( i \) chase \( i + 1 \), where \( i \) is mod \( n \). Denote by \( d_i(t) \) is the distance between \( i \) to \( i + 1 \) at time \( t \), i.e. \( d_i(t) = \| p_{i+1}(t) - p_i(t) \| \).
The dynamics of the agents are modeled by

$$\begin{align*}
\dot{p}_i &= \frac{p_{i+1}(t) - p_i(t)}{d_i(t)} \quad \text{if} \quad d_i(t) > 0 \\
p_i(t) &= p_{i+1}(t) \quad \forall \ t \geq \hat{t} \quad \text{s.t.} \quad d_i(\hat{t}) = 0
\end{align*}$$

(61)

In the sequel, for simplicity, we shall omit specific mention of $t$, whenever self explanatory.

**Lemma D.1.** $n$ agents in cyclic pursuit, modelled by (61), will collide (gather) within a finite time given by $T_m \leq t_0 + 2n \sum_{i=1}^{n} d_i(t_0)$, where $d_i(t_0)$ are the initial distances between agents and $T_m$ is the time of mutual capture (termination time).

**Proof.** We show that there exists a time $T_m$ such that $d_i(T_m) = 0$ for all $i$ or, since $d_i > 0$ for all $i$

$$\sum_{i=1}^{n} d_i(T_m) = 0 \quad (62)$$

To show (62) we will show that there exists a positive real number $c > 0$ such that $\sum_{i=1}^{n} d_i(t \leq T_m) \leq -c$.

We assume that no agents have collided at time $t < T_m$. Note that upon our model, when two agents collide they become one and $n$ is reduced, therefore this assumption holds. Given eq. (61) we have

$$\sum_{i=1}^{n} d_i \dot{p}_i = 0$$

Thus

$$\dot{p}_1^T \sum_{i=1}^{n} d_i \dot{p}_i = \sum_{i=1}^{n} d_i \dot{p}_1^T \dot{p}_i = 0 \quad (63)$$

In order for (63) to hold, there must exist an agent $j$ such that $\dot{p}_1^T \dot{p}_j < 0$.

$$\dot{p}_1^T \dot{p}_j = \|\dot{p}_1\| \|\dot{p}_j\| \cos(\alpha) < 0 \quad (64)$$

where $\alpha$ is the angle between $\dot{p}_1$ and $\dot{p}_j$. For eq. (64) to hold we must have $\pi/2 < \alpha < 3\pi/2$ and since $\|\dot{p}_i\| = 1; i = 1, \ldots, n$, we have $\|\dot{p}_j - \dot{p}_1\| > 1$

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From the definition of $d_i$ and $\hat{p}_i$ we have

\begin{align*}
  d_i^2 &= \|p_{i+1} - p_i\|^2 \\
  \dot{d}_i &= \dot{p}_i^T (p_{i+1} - \hat{p}_i) \\
  &= \dot{p}_i^T \hat{p}_{i+1} - 1 \\
  &= -\frac{1}{2} \|\hat{p}_{i+1} - \hat{p}_i\|^2
\end{align*}

(65) (66) (67) (68)

\[
\sum_{i=1}^{n} \dot{d}_i = -\frac{1}{2} \sum_{i=1}^{n} \|\hat{p}_{i+1} - \hat{p}_i\|^2 \leq -\frac{1}{2n} (\sum_{i=1}^{n} \|\hat{p}_{i+1} - \hat{p}_i\|)^2
\]

(69)

where we used the Cauchy-Schwartz inequality

\[
\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 \geq (\sum_{i=1}^{n} a_i b_i)^2
\]

(69)

with $a_i = \|\hat{p}_{i+1} - \hat{p}_i\|$ and $b_i = 1$ for $i = 1, \ldots, n$.

Applying to $\sum_{i=1}^{n} \|\hat{p}_{i+1} - \hat{p}_i\|$ the triangle inequality we have

\[
\sum_{i=1}^{j-1} \|\hat{p}_{i+1} - \hat{p}_i\| \geq \|\hat{p}_j - \hat{p}_1\|
\]

(70)
where \( j \) is some index such that \( \| \hat{\dot{p}}_j - \hat{\dot{p}}_1 \| > 1 \), which according to (63) must exist. Since all summands are \( \geq 0 \) we, have \( \sum_{i=1}^{n} \| \hat{\dot{p}}_{i+1} - \hat{\dot{p}}_i \| > 1 \) and therefore

\[
\sum_{i=1}^{n} d_i < -\frac{1}{2n} \tag{71}
\]

Integrating both sides of (71) from \( t_0 \) to \( T_m \) and recalling that at termination (mutual capture) time \( \sum_{i=1}^{n} d_i(T_m) = 0 \), we obtain

\[
T_m < t_0 + 2n \sum_{i=1}^{n} d_i(t_0)
\]

QED
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