HESSIAN OPERATORS ON CONSTRAINT MANIFOLDS

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Abstract

On a constraint manifold we give an explicit formula for the Hessian matrix of a cost function that involves the Hessian matrix of a prolonged function and the Hessian matrices of the constraint functions. We give an explicit formula for the case of the orthogonal group $O(n)$ by using only Euclidean coordinates on $\mathbb{R}^{n^2}$. An optimization problem on $SO(3)$ is completely carried out. Its applications to nonlinear stability problems are also analyzed.

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1 Introduction

The nature of critical points of a smooth cost function $G_S : S \to \mathbb{R}$, where $(S, \tau)$ is a smooth Riemannian manifold, can be often determined by analyzing the Hessian matrix of the function $G_S$ at the critical points. In order to compute the Riemannian Hessian one needs to have a good knowledge of the Riemannian geometry of the manifold $S$ such as the affine connection associated with the metric and geodesic lines, see [1] and [12]. Usually, to carry out explicit computations one has to introduce local coordinate systems on the manifold. These elements are often difficult to construct and manipulate on specific examples. A large class of examples have been studied in [1], [10], [11] in connection with optimization algorithms.

A method to bypass the computational difficulties associated with a Riemannian manifold is to embed it in a larger space, usually an Euclidean space, and transfer the computations into this more simpler space.

In this paper we give a formula for the Hessian matrix of the function $G_S$ that involves the Hessian matrix of an extended function $G$ and the Hessian matrices of the constraint functions. More precisely, let $G_S : S \to \mathbb{R}$ be a smooth map defined on the manifold $S$. Suppose that $S$ is a submanifold of a smooth manifold $M$ that is also the preimage of a regular value for a smooth function $F := (F_1, \ldots, F_k) : M \to \mathbb{R}^k$, i.e. $S = F^{-1}(c)$, where $c$ is a regular value of $F$. Let $G : M \to \mathbb{R}$ be a prolongation of the function $G_S$. In the case when $(M, g)$ is a finite dimensional Riemannian manifold we can endow the submanifold $S$ with a Riemannian metric $\tau_s$, constructed in [4], that is conformal with the induced metric on $S$ by the ambient metric $g$. The gradient of the restricted function $G_S$ with respect to the Riemannian metric $\tau_s$ can be computed using the gradients with respect to the ambient Riemannian metric $g$ of the prolongation function $G$ and the constraint functions $F_1, \ldots, F_k$.

In order to compute the Hessian operator of the cost function $G_S$ we need to take covariant derivatives of the gradient vector field $\text{grad}_{\tau_s} G_S$. The covariant derivative on the submanifold $S$ is related to the covariant derivative of the ambient space in the following way: take the covariant derivative in the ambient space of a prolongation of the vector field $\text{grad}_{\tau_{ind}} G_S$ and project this vector field on the tangent space of the submanifold $S$. The new problem is to find a vector field defined on the ambient space that prolongs $\text{grad}_{\tau_{ind}} G_S$. The solution to this problem is given by the standard control vector field introduced in [4].
In Section 3 we apply the formula found in Section 2 to cost functions defined on the orthogonal group \( \mathbf{O}(n) \). In Section 4 we specialize the formula from the previous section to the 2-power cost function considered in [4].

In the last section we relate our main result to a stability problem for equilibrium points of a dynamical system. We give a new interpretation of the stability results using the augmented function technique introduced in [13], see also [3] and [21].

In paper [2] a formula for the Riemannian Hessian has been proved using the orthogonal projection and the Weingarten map for a general submanifold embedded in an Euclidean space. Another construction of a Hessian operator using orthogonal coordinates on the tangent planes of a submanifold embedded in an Euclidean space has been presented in [9].

### 2 Construction of the Hessian operator on constraint manifolds

As discussed in the Introduction, we work in the following setting. Let \( G_S : S \to \mathbb{R} \) be a smooth map defined on a manifold \( S \). Suppose that \( S \) is a submanifold of a smooth manifold \( M \) that is also the preimage of a regular value for a smooth function \( F := (F_1, \ldots, F_k) : M \to \mathbb{R}^k \), i.e. \( S = F^{-1}(c) \), where \( c \) is a regular value of \( F \). Let \( G : M \to \mathbb{R} \) be a prolongation of the function \( G_S \).

We recall the construction and the geometry of the standard control vector field introduced in [4]. The \( r \times s \) Gramian matrix generated by the smooth functions \( f_1, \ldots, f_r, g_1, \ldots, g_s : (M, g) \to \mathbb{R} \) is defined by the formula

\[
\Sigma^{(f_1, \ldots, f_r)}_{(g_1, \ldots, g_s)} = \begin{bmatrix}
< \text{grad } g_1, \text{grad } f_1 > & \ldots & < \text{grad } g_s, \text{grad } f_1 > \\
\ldots & \ldots & \ldots \\
< \text{grad } g_1, \text{grad } f_r > & \ldots & < \text{grad } g_s, \text{grad } f_r >
\end{bmatrix}.
\]

(2.1)

The **standard control vector field** has the formula

\[
v_0 = \sum_{i=1}^{k} (-1)^{i+k+1} \det \Sigma^{(F_1, \ldots, F_k)}_{(F_1, \ldots, F_{i-1}, F_i, F_{i+1}, \ldots, F_k, G)} \text{grad } F_i + \det \Sigma^{(F_1, \ldots, F_k)}_{(F_1, \ldots, F_{i-1}, G, F_{i+1}, \ldots, F_k)} \text{grad } G = \det \Sigma^{(F_1, \ldots, F_k)} \text{grad } G - \sum_{i=1}^{k} \det \Sigma^{(F_1, \ldots, F_k)}_{(F_1, \ldots, F_{i-1}, G, F_{i+1}, \ldots, F_k)} \text{grad } F_i,
\]

(2.2)

where \( \hat{\cdot} \) represents the missing term. The vector field \( v_0 \) is tangent to the submanifold \( S \) and consequently, the restriction defines a vector field on \( S \), i.e. \( v_0|_S \in \mathcal{X}(S) \). On the submanifold \( S \) we can define a Riemannian metric \( \tau_c \) such that

\[
v_0|_S = \text{grad}_{\tau_c} G_S.
\]

We make the notation \( \Sigma := \det \Sigma^{(F_1, \ldots, F_k)}_{(F_1, \ldots, F_k)} \). On the submanifold \( S \) the restricted function \( \Sigma|_S \) is everywhere different from zero as the submanifold \( S \) is the preimage of a regular value. Also, it has been proved in [4] that the relation between the Riemannian metric \( \tau_c \) defined on \( S \) and the induced Riemannian metric \( g_{ind}^S \) from the ambient space \( (M, g) \) is given by

\[
\tau_c = \frac{1}{\Sigma|_S} g_{ind}^S.
\]

To determine the relation between the gradient of the function \( G_S \) with respect to the metric \( \tau_c \) and the gradient of the function \( G_S \) with respect to the induced metric \( g_{ind}^S \) we have the following computation:

\[
g_{ind}^S(\text{grad}_{g_{ind}^S} G_S, w) = dG_S(w) = \tau_c(\text{grad}_{\tau_c} G_S, w) = g_{ind}^S(\frac{1}{\Sigma|_S} \text{grad}_{\tau_c} G_S, w) = g_{ind}^S(\frac{1}{\Sigma|_S} \text{grad}_{\tau_c} G_S, w), \ \forall w \in \mathcal{X}(S).
\]
Consequently,

$$\text{grad}_{g^{\text{ind}}} G_S = \frac{1}{\sum_i \sigma_i} \text{v}_0 | S.$$  

The above equality implies that a prolongation of the vector field $\text{grad}_{g^{\text{ind}}} G_S$ to the open subset $\Omega = \{ x \in M \mid \Sigma(x) \neq 0 \}$ of the ambient space $M$ is given by the vector field

$$\frac{1}{\sum_i \sigma_i} \text{v}_0 = \text{grad} G - \sum_{i=1}^k \sigma_i \text{grad} F_i,
\quad \text{(2.3)}$$

where $\sigma_i : \Omega \to \mathbb{R}$ are defined by

$$\sigma_i(x) := \frac{\det \Sigma(F_1, \ldots, F_k)}{\Sigma(x)}.$$  

If $x_0 \in S$ is a critical point of the function $G_S$, then the numbers $\sigma_i(x_0)$ are the Lagrange multipliers of the extended function $G$ constraint to submanifold $S$. More precisely, a critical point $x_0$ of the constraint function $G_S = G_{|S}$ is an equilibrium point for the standard control vector field $v_0$ which implies the equality that gives the Lagrange multipliers

$$\text{grad} G(x_0) = \sum_{i=1}^k \sigma_i(x_0) \text{grad} F_i(x_0).$$

The above Lagrange multipliers are uniquely determined due to regular value condition which implies that $\text{grad} F_1(x_0), \ldots, \text{grad} F_k(x_0)$ are linearly independent vectors in $T_{x_0}M$.

In what follows we show how the Hessian operator associated to the cost function $G_S : S \to \mathbb{R}$, where $S$ is endowed with the induced metric $g^{\text{ind}}$, is related with the Hessian operator of the extended function $G$ and the Hessian operators of the functions $F_i$ that describe the constraint submanifold $S$. By definition, see [1], the Hessian operator $\mathcal{H}(G_S)(x) : T_xS \to T_xS$ is defined by the equality

$$\mathcal{H}(G_S)(x) \cdot \eta_x = \nabla^S \eta_x \text{grad}_{g^{\text{ind}}} G_S,$$

where $\nabla^S$ is the covariant derivative on the Riemannian manifold $(S, g^{\text{ind}})$ and $\eta_x \in T_xS$. The relation between the covariant derivative $\nabla^S$ and the covariant derivative $\nabla$ associated with the ambient Riemannian manifold $(M, g)$ is given by

$$\nabla^S \xi = \mathbf{P}_{T_xS} \nabla \tilde{\eta}_x \xi,$$

where $\mathbf{P}_{T_xS} : T_xM \to T_xS$ is the orthogonal projection onto $T_xS$ with respect to the scalar product on the tangent space $T_xM$ induced by the ambient metric $g$, $\xi \in \mathcal{X}(M)$ is a prolongation on the ambient space of the vector field $\xi^S \in \mathcal{X}(S)$, and $\tilde{\eta}_x \in T_xM$ is the vector $\eta_x \in T_xS$ regarded as a vector in the ambient tangent space $T_xM$.

For $x \in S$, using the prolongation given by (2.3) for the vector field $\text{grad}_{g^{\text{ind}}} G_S$ we obtain:

$$\mathcal{H}(G_S)(x) \cdot \eta_x = \mathbf{P}_{T_xS} \nabla \tilde{\eta}_x \frac{1}{\sum \sigma_i} \text{v}_0 = \mathbf{P}_{T_xS} \nabla \tilde{\eta}_x \left( \text{grad} G - \sum_{i=1}^k \sigma_i \text{grad} F_i \right)$$

$$= \mathbf{P}_{T_xS} \nabla \tilde{\eta}_x \text{grad} G - \sum_{i=1}^k d\sigma_i(\tilde{\eta}_x) \mathbf{P}_{T_xS} \text{grad} F_i(x) - \sum_{i=1}^k \sigma_i(x) \mathbf{P}_{T_xS} \nabla \tilde{\eta}_x \text{grad} F_i$$

$$= \mathbf{P}_{T_xS} \mathcal{H}^G(x) \cdot \eta_x - \sum_{i=1}^k \sigma_i(x) \mathbf{P}_{T_xS} \mathcal{H}^F_i(x) \cdot \eta_x = \mathbf{P}_{T_xS} \left( \mathcal{H}^G(x) - \sum_{i=1}^k \sigma_i(x) \mathcal{H}^F_i(x) \right) \cdot \eta_x.$$
Consequently, the bilinear form associated to the Hessian operator \( \mathcal{H}^{G_S} \) is given by

\[
\mathcal{H}^{G_S}(\eta_x, \xi_x) = \langle \mathcal{H}^{G_S}(x) \cdot \eta_x, \xi_x \rangle_{g(x)} = \langle P_{T_xS} \mathcal{H}^G(x) \cdot \hat{\eta}_x, \hat{\xi}_x \rangle_g - \sum_{i=1}^{k} \sigma_i(x) \langle P_{T_xS} \mathcal{H}^{F_i}(x) \cdot \hat{\eta}_x, \hat{\xi}_x \rangle_g.
\]

The above considerations lead us to the main result of the paper.

**Theorem 2.1.** For any \( x \in S \), the symmetric covariant tensor associated with the Hessian operator of the cost function \( G_S \) has the following formula:

\[
[\mathcal{H}^{G_S}(x)] = [\mathcal{H}^G(x)]|_{T_xS \times T_xS} - \sum_{i=1}^{k} \sigma_i(x) [\mathcal{H}^{F_i}(x)]|_{T_xS \times T_xS}.
\]  

(2.5)

The above formula is valid for all points \( x \in S \), not just for critical points of the cost function \( G_S \). Choosing a base for the tangent space \( T_xS \subset T_xM \), \( \{\tilde{e}_a \in T_xM\}_{a=1}^{\dim S} \), the components of the Hessian matrix \( [\mathcal{H}^{G_S}(x)] \) are given by the following relation between the components of the Hessian matrix of the prolonged function \( G \) and the components of the Hessian matrices of the functions \( F_i \) that describe the submanifold \( S \):

\[
[\mathcal{H}^{G_S}(x)]_{ab} = \langle \mathcal{H}^G(x) \cdot \tilde{e}_a, \tilde{e}_b \rangle_g - \sum_{i=1}^{k} \sigma_i(x) \langle \mathcal{H}^{F_i}(x) \cdot \tilde{e}_a, \tilde{e}_b \rangle_g.
\]  

(2.6)

Note that the base for the tangent space \( T_xS \) is computed in the local coordinates of the ambient manifold \( M \) and does not imply the knowledge of a local coordinate system on the submanifold \( S \). We recall that for a \( C^2 \)-function \( G : (M, g) \rightarrow \mathbb{R} \), in a local coordinate system on the ambient manifold \( M \) we have the formula

\[
\mathcal{H}^G_{uv}(x) = \frac{\partial^2 G}{\partial x^u \partial x^v}(x) - \Gamma^w_{uv}(x) \frac{\partial G}{\partial x^w}(x), \quad u, v, w = 1, \dim M,
\]

where \( \Gamma^w_{uv} \) are the Christoffel's symbols associated to the metric \( g \).

### 3 Hessian operator for cost functions on \( O(n) \)

We give an explicit formula for the Hessian operator associated to a \( C^2 \) cost function \( G_{O(n)} : O(n) \rightarrow \mathbb{R} \). The orthogonal group is defined by:

\[ O(n) = \{ X \in M_{n \times n}(\mathbb{R}) | XX^T = I_n = X^T X \} \]

A general element of \( O(n) \) is represented by an orthonormal frame \( \{x_i = (x_{i1}, ..., x_{in})\}_{i=\overline{1,n}} \) in \( \mathbb{R}^n \), namely,

\[
X = \begin{bmatrix}
x_1 \\
... \\
x_n
\end{bmatrix} = \begin{bmatrix}
x_{11} & ... & x_{1n} \\
... & ... & ... \\
x_{n1} & ... & x_{nn}
\end{bmatrix}.
\]

We identify an orthogonal matrix with a vector in \( \mathbb{R}^{n^2} \) using the linear map \( J : M_{n \times n}(\mathbb{R}) \rightarrow \mathbb{R}^{n^2} \),

\[
X \xrightarrow{J} \tilde{x} = (x_1, ..., x_n).
\]  

(3.1)
Regarded as a subset of $\mathbb{R}^{n^2}$, the orthogonal group $\mathbf{O}(n)$ can be seen as the preimage $\mathcal{O}(n) \subset \mathbb{R}^{n^2}$ of the regular value $(\frac{1}{2}, \ldots, \frac{1}{2}, 0, \ldots, 0) \in \mathbb{R}^n \times \mathbb{R}^{(n-1)n}$ for the constraint functions:

\[
F_s(\tilde{x}) = \frac{1}{2}|x_s|^2, \quad s \in \{1, \ldots, n\},
\]

\[
F_{pq}(\tilde{x}) = <x_p, x_q>, \quad 1 \leq p < q \leq n,
\]

where $< \cdot, \cdot >$ is the Euclidean product in $\mathbb{R}^n$. Using the above identification we obtain the cost function $G_{\mathcal{O}(n)} := G_{\mathbf{O}(n)} \circ \mathcal{O}(n) \to \mathbb{R}$.

Starting with the canonical base $e_1, \ldots, e_n$ in $\mathbb{R}^n$, we obtain the canonical base $\{\tilde{e}_{ij}\}_{i,j=1}^{n}$ in $\mathbb{R}^{n^2}$ by the identification,$
\]

where $e_j$ is on the $i$-th slot.

By direct computations, we have the following formulas for the gradients of the constraint functions:

\[
\text{grad } F_s(\tilde{x}) = \sum_{i=1}^n x_i \tilde{e}_{si},
\]

\[
\text{grad } F_{pq}(\tilde{x}) = \sum_{i=1}^n (x_{qi} \tilde{e}_{pi} + x_{pi} \tilde{e}_{qi}).
\]

The $n^2 \times n^2$ Hessian matrices of the constraint functions are given by:

\[
[H^F_s(\tilde{x})] = \sum_{j=1}^n \tilde{e}_{sj} \otimes \tilde{e}_{sj},
\]

\[
[H^F_{pq}(\tilde{x})] = \sum_{j=1}^n (\tilde{e}_{pj} \otimes \tilde{e}_{qj} + \tilde{e}_{qj} \otimes \tilde{e}_{pj}).
\]

For a $C^2$ prolongation $G : \mathbb{R}^{n^2} \to \mathbb{R}$ of the cost function $G_{\mathcal{O}(n)}$ we have the formula for the Hessian matrix

\[
[H^G(\tilde{x})] = \sum_{a,b,c,d=1}^n \frac{\partial^2 G}{\partial x_{ab} \partial x_{cd}} \tilde{e}_{ab} \otimes \tilde{e}_{cd}.
\]

Identifying a rotation $X \in \mathbf{O}(n)$ with the corresponding point $\tilde{x} \in \mathcal{O}(n)$ and substituting in the formulas $(6.1)$ (see Annexe), we have the following formula for the restricted Hessian matrix:

\[
[H^{G_{\mathcal{O}(n)}}(\tilde{x})] = \left[H^{G_{\mathbf{O}(n)}}(\tilde{x})\right]_{|T_{\tilde{x}} \mathcal{O}(n) \times T_{\tilde{x}} \mathcal{O}(n)} - \sum_{s=1}^n \frac{\partial G}{\partial x_s}(\tilde{x}), \quad x_s > \frac{\partial G}{\partial x_s}(\tilde{x}), \quad x_s > \frac{\partial G}{\partial x_q}(\tilde{x}), \quad x_p < \frac{\partial G}{\partial x_q}(\tilde{x}) \right]_{|T_{\tilde{x}} \mathcal{O}(n) \times T_{\tilde{x}} \mathcal{O}(n)}
\]

\[
- \frac{1}{2} \sum_{1 \leq p < q \leq n} \left( <\frac{\partial G}{\partial x_p}(\tilde{x}), x_q > + <\frac{\partial G}{\partial x_q}(\tilde{x}), x_p > \right) \frac{\partial G}{\partial x_{pq}}(\tilde{x}) \right]_{|T_{\tilde{x}} \mathcal{O}(n) \times T_{\tilde{x}} \mathcal{O}(n)},
\]

where we denote by $\frac{\partial G}{\partial x_j}(\tilde{x}) := \sum_{i=1}^n \frac{\partial G}{\partial x_{ij}}(\tilde{x})e_j$, which is a vector in $\mathbb{R}^n$.

Equivalently,

\[
[H^{G_{\mathcal{O}(n)}}(\tilde{x})] = \left[H^{G_{\mathbf{O}(n)}}(\tilde{x})\right]_{|T_{\tilde{x}} \mathcal{O}(n) \times T_{\tilde{x}} \mathcal{O}(n)} - \sum_{i,j,s=1}^n x_{si} \frac{\partial G}{\partial x_{sj}}(\tilde{x}) \tilde{e}_{sj} \otimes \tilde{e}_{sj} \right]_{|T_{\tilde{x}} \mathcal{O}(n) \times T_{\tilde{x}} \mathcal{O}(n)}
\]

\[
- \frac{1}{2} \sum_{1 \leq p < q \leq n} \left( x_{qi} \frac{\partial G}{\partial x_{pi}}(\tilde{x}) + x_{pi} \frac{\partial G}{\partial x_{qi}}(\tilde{x}) \right) \tilde{e}_{pj} \otimes \tilde{e}_{pj} \right]_{|T_{\tilde{x}} \mathcal{O}(n) \times T_{\tilde{x}} \mathcal{O}(n)},
\]
To write the above formula in a more explicit form we need to choose a base for the tangent space

\[ T_X \mathbb{O}(n) = \{ X \Omega \mid \Omega = -\Omega^T \} \]

It is equivalent to choose a base for \( n \times n \) skew-symmetric matrices. We consider the following base:

\[ \Omega_{\alpha\beta} = (-1)^{\alpha+\beta}(e_\alpha \otimes e_\beta - e_\beta \otimes e_\alpha), \quad 1 \leq \alpha < \beta \leq n. \]  
(3.5)

Consequently, a base for the tangent space \( T_X \mathbb{O}(n) \) is given by (see Annexe (6.2)):

\[ X\Omega_{\alpha\beta} = (-1)^{\alpha+\beta}\sum_{i=1}^{n}(x_{i\alpha}e_i \otimes e_\beta - x_{i\beta}e_i \otimes e_\alpha), \quad 1 \leq \alpha < \beta \leq n. \]

As \( \mathcal{J} \) is a linear map its differential \( d_X \mathcal{J} \) equals \( \mathcal{J} \), we obtain the following base for \( T_\mathcal{J} \mathbb{O}(n) \):

\[ \tilde{\omega}_{\alpha\beta}(x) = (-1)^{\alpha+\beta}\sum_{i=1}^{n}(x_{i\alpha}\tilde{e}_i \otimes \tilde{e}_\beta - x_{i\beta}\tilde{e}_i \otimes \tilde{e}_\alpha), \quad 1 \leq \alpha < \beta \leq n. \]

In the base for \( T_\mathcal{J} \mathbb{O}(n) \) chosen as above we have the following formula of the element \((\gamma\tau)(\alpha\beta), 1 \leq \gamma < \tau \leq n \) and \( 1 \leq \alpha < \beta \leq n \), of the Hessian matrix for the cost function \( G_\Sigma \):

\[
\begin{align*}
\left[ H^{G_\Omega(n)}(x) \right] \sum_{(\gamma\tau)(\alpha\beta)} & = \epsilon \sum_{a,b,c=1}^{n} \left( x_{a\gamma}x_{c\alpha} \frac{\partial^2 G}{\partial x_{a\tau}x_{c\beta}} - x_{a\gamma}x_{c\beta} \frac{\partial^2 G}{\partial x_{a\tau}x_{c\alpha}} - x_{a\tau}x_{c\alpha} \frac{\partial^2 G}{\partial x_{a\gamma}x_{c\beta}} + x_{a\tau}x_{c\beta} \frac{\partial^2 G}{\partial x_{a\gamma}x_{c\alpha}} \right) \\
&- \epsilon \sum_{i,s=1}^{n} x_{i\sigma} (x_{s\gamma}x_{\beta\delta} - x_{s\gamma}x_{\alpha\delta} - x_{s\alpha}x_{\gamma\beta} + x_{s\alpha}x_{\gamma\delta}) \frac{\partial G}{\partial x_{i\sigma}} \\
&- \frac{\epsilon}{2} \sum_{1 \leq p < q \leq n} \sum_{i=1}^{n} (x_{p\gamma} \frac{\partial G}{\partial x_{q\alpha}}(x) + x_{p\alpha} \frac{\partial G}{\partial x_{q\gamma}}(x))(x_{q\gamma}x_{p\beta} - x_{q\beta}x_{p\gamma}) \\
&- x_{p\gamma}x_{q\alpha} \delta_{\alpha\beta} + x_{p\alpha}x_{q\gamma} \delta_{\alpha\beta} + x_{q\gamma}x_{p\beta} \delta_{\alpha\beta} - x_{q\beta}x_{p\gamma} \delta_{\alpha\beta} \\
&= (1)^{\alpha+\beta+\gamma+\tau},
\end{align*}
\]

where \( \epsilon = (1)^{\alpha+\beta+\gamma+\tau} \).

4 Characterization of the critical points for 2-power cost functions defined on \( \mathbb{O}(3) \)

We will exemplify the formulas discovered in the previous section for the case of 2-power cost functions on \( \mathbb{O}(3) \). The orthogonal group \( \mathbb{O}(3) \) is given by:

\[ \mathbb{O}(3) = \{ X \in M_{3 \times 3}(\mathbb{R}) \mid XX^T = I_3 = X^T X \}. \]

A general element of \( \mathbb{O}(3) \) is represented by an orthonormal frame \( \{ x_i = (x_{i1}, x_{i2}, x_{i3}) \}_{i=1}^{M} \) in \( \mathbb{R}^3 \). We identify an orthogonal matrix with a vector in \( \mathbb{R}^9 \) using the linear map \( \mathcal{J} : M_{3 \times 3}(\mathbb{R}) \to \mathbb{R}^9 \),

\[ X \xrightarrow{\mathcal{J}} \tilde{x} = (x_1, x_2, x_3). \]

(4.1)

Regarded as a subset of \( \mathbb{R}^9 \), the orthogonal group \( \mathbb{O}(3) \) can be seen as the preimage \( \mathbb{O}(3) \subset \mathbb{R}^9 \) of the regular value \( (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0, 0) \in \mathbb{R}^3 \times \mathbb{R}^3 \) of the constraint functions:

\[ F_1(\tilde{x}) = \frac{1}{2}||x_1||^2, \quad F_2(\tilde{x}) = \frac{1}{2}||x_2||^2, \quad F_3(\tilde{x}) = \frac{1}{2}||x_3||^2, \]

\[ F_{12}(\tilde{x}) =< x_1, x_2 >, \quad F_{13}(\tilde{x}) =< x_1, x_3 >, \quad F_{23}(\tilde{x}) =< x_2, x_3 >, \]

\[ F_{123}(\tilde{x}) =< x_1, x_2, x_3 >, \]
where $<\cdot, \cdot>$ is the Euclidean product in $\mathbb{R}^3$. The gradients of the constraint functions are given by:

$$
\text{grad} \; F_1(\tilde{x}) = (x_1, 0, 0), \quad \text{grad} \; F_2(\tilde{x}) = (0, x_2, 0), \quad \text{grad} \; F_3(\tilde{x}) = (0, 0, x_3),
$$

$$
\text{grad} \; F_{12}(\tilde{x}) = (x_2, x_1, 0), \quad \text{grad} \; F_{13}(\tilde{x}) = (x_3, 0, x_1), \quad \text{grad} \; F_{23}(\tilde{x}) = (0, x_3, x_2).
$$

The Hessian matrices of the constraint functions are given by:

$$
[H^{F_1}(\tilde{x})] = \begin{bmatrix}
I_3 & 0_3 & 0_3 \\
0_3 & I_3 & 0_3 \\
0_3 & 0_3 & I_3
\end{bmatrix},
[H^{F_2}(\tilde{x})] = \begin{bmatrix}
0_3 & 0_3 & 0_3 \\
0_3 & I_3 & 0_3 \\
0_3 & 0_3 & I_3
\end{bmatrix},
[H^{F_3}(\tilde{x})] = \begin{bmatrix}
0_3 & 0_3 & 0_3 \\
0_3 & 0_3 & 0_3 \\
0_3 & 0_3 & I_3
\end{bmatrix},
$$

$$
[H^{F_{12}}(\tilde{x})] = \begin{bmatrix}
0_3 & 0_3 & 0_3 \\
I_3 & 0_3 & 0_3 \\
I_3 & 0_3 & 0_3
\end{bmatrix},
[H^{F_{13}}(\tilde{x})] = \begin{bmatrix}
0_3 & 0_3 & 0_3 \\
0_3 & I_3 & 0_3 \\
I_4 & I_3 & 0_3
\end{bmatrix},
[H^{F_{23}}(\tilde{x})] = \begin{bmatrix}
0_3 & 0_3 & 0_3 \\
0_3 & 0_3 & I_3 \\
0_3 & I_3 & 0_3
\end{bmatrix}.
$$

Considering a cost function $G_{O(3)} : O(3) \to \mathbb{R}$, we identify it with $G_{O(3)} := G_{O(3)} \circ J : O(3) \to \mathbb{R}$ and construct a prolongation function $G : \mathbb{R}^9 \to \mathbb{R}$. Formula (3.4) becomes:

$$
[H^{G_{O(3)}}(\tilde{x})] = [H^{G}(\tilde{x})]|_{T_{\tilde{x}}O(3) \times T_{\tilde{x}}O(3)} - \sum_{s=1}^{3} \left( \frac{\partial G}{\partial x_s}(\tilde{x}), x_s > 0 \right) [H^{F_s}(\tilde{x})]|_{T_{\tilde{x}}O(3) \times T_{\tilde{x}}O(3)} - \frac{1}{2} \sum_{1 \leq p < q \leq 3} \left( \frac{\partial G}{\partial x_p}(\tilde{x}), x_q > 0 \right) [H^{F_{pq}}(\tilde{x})]|_{T_{\tilde{x}}O(3) \times T_{\tilde{x}}O(3)}. \quad (4.2)
$$

We choose the base for $T_{\tilde{x}}O(3)$ as in the previous section:

$$
\tilde{\omega}_{12}(\tilde{x}) = (x_{12}, -x_{11}, 0, x_{22}, -x_{21}, 0, x_{32}, -x_{31}, 0),
$$

$$
\tilde{\omega}_{13}(\tilde{x}) = (-x_{13}, 0, x_{11}, -x_{23}, 0, x_{21}, -x_{33}, 0, x_{31}),
$$

$$
\tilde{\omega}_{23}(\tilde{x}) = (0, x_{13}, -x_{12}, 0, x_{23}, -x_{22}, 0, x_{33}, -x_{32}).
$$

The restricted Hessian matrices for the constraint functions are the following:

$$
[H^{F_1}(\tilde{x})]|_{T_{\tilde{x}}O(3) \times T_{\tilde{x}}O(3)} = \begin{bmatrix}
1 - x_{13}^2 & -x_{12} x_{13} & -x_{11} x_{13} \\
-x_{12} x_{13} & 1 - x_{12}^2 & -x_{11} x_{12} \\
-x_{11} x_{13} & -x_{11} x_{12} & 1 - x_{11}^2
\end{bmatrix},
$$

$$
[H^{F_2}(\tilde{x})]|_{T_{\tilde{x}}O(3) \times T_{\tilde{x}}O(3)} = \begin{bmatrix}
1 - x_{23}^2 & -x_{22} x_{23} & -x_{21} x_{23} \\
-x_{22} x_{23} & 1 - x_{22}^2 & -x_{21} x_{22} \\
-x_{21} x_{23} & -x_{21} x_{22} & 1 - x_{21}^2
\end{bmatrix},
$$

$$
[H^{F_3}(\tilde{x})]|_{T_{\tilde{x}}O(3) \times T_{\tilde{x}}O(3)} = \begin{bmatrix}
1 - x_{33}^2 & -x_{32} x_{33} & -x_{31} x_{33} \\
-x_{32} x_{33} & 1 - x_{32}^2 & -x_{31} x_{32} \\
-x_{31} x_{33} & -x_{31} x_{32} & 1 - x_{31}^2
\end{bmatrix},
$$

$$
[H^{F_{12}}(\tilde{x})]|_{T_{\tilde{x}}O(3) \times T_{\tilde{x}}O(3)} = \begin{bmatrix}
-2 x_{23} x_{13} & -x_{23} x_{12} - x_{13} x_{22} & -x_{23} x_{11} - x_{13} x_{21} \\
-x_{23} x_{12} - x_{13} x_{22} & -2 x_{22} x_{12} & -x_{22} x_{11} - x_{12} x_{21} \\
-x_{23} x_{11} - x_{13} x_{21} & -x_{22} x_{11} - x_{12} x_{21} & -2 x_{11} x_{21}
\end{bmatrix},
$$

$$
[H^{F_{13}}(\tilde{x})]|_{T_{\tilde{x}}O(3) \times T_{\tilde{x}}O(3)} = \begin{bmatrix}
-2 x_{33} x_{13} & -x_{33} x_{12} - x_{13} x_{32} & -x_{33} x_{11} - x_{13} x_{31} \\
-x_{33} x_{12} - x_{13} x_{32} & -2 x_{32} x_{12} & -x_{32} x_{11} - x_{12} x_{31} \\
-x_{33} x_{11} - x_{13} x_{31} & -x_{32} x_{11} - x_{12} x_{31} & -2 x_{31} x_{11}
\end{bmatrix},
$$

$$
[H^{F_{23}}(\tilde{x})]|_{T_{\tilde{x}}O(3) \times T_{\tilde{x}}O(3)} = \begin{bmatrix}
-2 x_{33} x_{23} & -x_{33} x_{22} - x_{23} x_{32} & -x_{33} x_{21} - x_{23} x_{31} \\
-x_{33} x_{22} - x_{23} x_{32} & -2 x_{32} x_{22} & -x_{32} x_{21} - x_{22} x_{31} \\
-x_{33} x_{21} - x_{23} x_{31} & -x_{32} x_{21} - x_{22} x_{31} & -2 x_{31} x_{21}
\end{bmatrix}.
$$

\(^1\text{In the formulas for the restricted Hessian matrices of the constraint functions we have used the fact that } \tilde{x} \in O(3).\)
We will characterize the critical points of the following 2-power cost function,

\[ G_{O(3)}(X) = \frac{1}{2} \sum_{i=1}^{k} ||X - R_i||_F^2, \]

where \( R_1, \ldots, R_k \) are sample rotations and \( || \cdot ||_F \) is the Frobenius norm. The critical points of the above cost function have been computed using The Embedding Algorithm in [5]. Using the identification map \( J \) we obtain the cost function,

\[ G_{O(3)}(\tilde{x}) = \frac{1}{2} \sum_{i=1}^{k} ||\tilde{x} - \tilde{r}_i||^2, \]

where \( || \cdot || \) is the Euclidean norm on \( \mathbb{R}^9 \). For the obvious prolongation \( G : \mathbb{R}^9 \rightarrow \mathbb{R} \) of \( G_{O(3)} \) we have,

\[ \nabla G(\tilde{x}) = k(\tilde{x} - \tilde{r}), \]  
\[ [\nabla^2 G(\tilde{x})]_{|T_{X}O(3) \times T_{X}O(3)} = \begin{bmatrix} 2k & 0 & 0 \\ 0 & 2k & 0 \\ 0 & 0 & 2k \end{bmatrix}, \]

where \( \tilde{r} = \frac{1}{k} \sum_{i=1}^{k} \tilde{r}_i \). Applying formula (3.4) for the case \( n = 3 \), we obtain the components of the Hessian matrix of the cost function \( G_{O(3)} \):

\[
\begin{align*}
h_{11}(\tilde{x}) &= k(x_{11}r_{11} + x_{21}r_{21} + x_{31}r_{31} + x_{12}r_{12} + x_{22}r_{22} + x_{32}r_{32}), \\
h_{12}(\tilde{x}) &= -\frac{k}{2}(x_{12}r_{13} + x_{22}r_{23} + x_{32}r_{33} + x_{13}r_{12} + x_{23}r_{22} + x_{33}r_{32}), \\
h_{13}(\tilde{x}) &= -\frac{k}{2}(x_{11}r_{13} + x_{21}r_{23} + x_{31}r_{33} + x_{12}r_{11} + x_{23}r_{21} + x_{33}r_{31}), \\
h_{22}(\tilde{x}) &= k(x_{11}r_{11} + x_{21}r_{21} + x_{31}r_{31} + x_{13}r_{13} + x_{23}r_{23} + x_{33}r_{33}), \\
h_{23}(\tilde{x}) &= -\frac{k}{2}(x_{12}r_{11} + x_{22}r_{21} + x_{32}r_{31} + x_{11}r_{12} + x_{21}r_{22} + x_{31}r_{32}), \\
h_{33}(\tilde{x}) &= k(x_{12}r_{12} + x_{22}r_{22} + x_{32}r_{32} + x_{13}r_{13} + x_{23}r_{23} + x_{33}r_{33}).
\end{align*}
\]

For the columns of the matrices \( X \), respectively \( R = \frac{1}{k} \sum_{i=1}^{k} R_i \), we make the notations \( y_i = (x_{1i}, x_{2i}, x_{3i}) \), and respectively \( s_i = (r_{1i}, r_{2i}, r_{3i}) \). The components of the of the Hessian matrix of the cost function \( G_{O(3)} \) can be written in the equivalent form:

\[
\begin{align*}
h_{11}(\tilde{x}) &= k(<y_1, s_1> + <y_2, s_2>), \\
h_{12}(\tilde{x}) &= -\frac{k}{2}(<y_2, s_3> + <y_3, s_2>), \\
h_{13}(\tilde{x}) &= -\frac{k}{2}(<y_1, s_3> + <y_3, s_1>), \\
h_{22}(\tilde{x}) &= k(<y_1, s_1> + <y_3, s_3>), \\
h_{23}(\tilde{x}) &= -\frac{k}{2}(<y_2, s_1> + <y_1, s_2>), \\
h_{33}(\tilde{x}) &= k(<y_2, s_2> + <y_3, s_3>).
\end{align*}
\]

**Remark 4.1.** The above expressions for the Hessian matrix depend on the chosen base for the tangent space \( T_{X}O(3) \). If we rename the base chosen above as follows:

\[
\nu_1(\tilde{x}) = \tilde{\omega}_{23}(\tilde{x}), \quad \nu_2(\tilde{x}) = \tilde{\omega}_{13}(\tilde{x}), \quad \nu_3(\tilde{x}) = \tilde{\omega}_{12}(\tilde{x}),
\]

the formulas for the components of the Hessian matrix of the cost function \( G_{O(3)} \) have more natural
expressions with respect to the symmetry of $O(3)$:

\[
\begin{align*}
\tilde{h}_{11}(\tilde{x}) &= k(\langle y_2, s_2 \rangle + \langle y_3, s_3 \rangle), \\
\tilde{h}_{12}(\tilde{x}) &= -\frac{k}{2}(\langle y_1, s_2 \rangle + \langle y_2, s_1 \rangle), \\
\tilde{h}_{13}(\tilde{x}) &= -\frac{k}{2}(\langle y_1, s_3 \rangle + \langle y_3, s_1 \rangle), \\
\tilde{h}_{22}(\tilde{x}) &= k(\langle y_1, s_1 \rangle + \langle y_3, s_3 \rangle), \\
\tilde{h}_{23}(\tilde{x}) &= -\frac{k}{2}(\langle y_2, s_3 \rangle + \langle y_3, s_2 \rangle), \\
\tilde{h}_{33}(\tilde{x}) &= k(\langle y_1, s_1 \rangle + \langle y_2, s_2 \rangle). \quad \square
\end{align*}
\]

Using the intrinsic Riemannian geometry of the Lie group $SO(3)$, another formula for the Hessian matrix of the 2-power cost function has been given in [14].

Now we apply the above formulas to determine the nature of the critical points of the following example of 2-power cost function defined on the connected component of the identity matrix of the orthogonal group $O(3)$ which is $SO(3)$:

\[
G_{SO(3)}^\alpha(X) = \frac{1}{2} (||X - R_1||_F^2 + ||X - R_2||_F^2 + ||X - R_3||_F^2),
\]

where

\[
R_1 = \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{array} \right), \quad R_2 = \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{array} \right), \quad R_3 = \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha \\ 0 & \sin \alpha & \cos \alpha \end{array} \right), \quad \alpha \in [-\pi, \pi],
\]

are rotations along the $x$-axis. Using the identification map 3, we obtain the cost function defined on the connected component $SO(3)$ of the point $(1, 0, 0, 1, 0, 0, 0, 1) \in O(3)$:

\[
G_{SO(3)}^\alpha(\tilde{x}) = \frac{1}{2} (||\tilde{x} - \tilde{r}_1||^2 + ||\tilde{x} - \tilde{r}_2||^2 + ||\tilde{x} - \tilde{r}_3||^2),
\]

where $|| \cdot ||$ is the Euclidean norm on $\mathbb{R}^9$. For the obvious prolongation $G^\alpha : \mathbb{R}^9 \to \mathbb{R}$ of $G_{SO(3)}^\alpha$ we have:

\[
\nabla G^\alpha(\tilde{x}) = 3(\tilde{x} - \tilde{r}), \quad \left[ \mathcal{H} G^\alpha(\tilde{x}) \right]_{T_{\tilde{x}}SO(3) \times T_{\tilde{x}}SO(3)} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix},
\]

where $\tilde{r} = \frac{1}{3} (\tilde{r}_1 + \tilde{r}_2 + \tilde{r}_3) = (1, 0, 0, 0, \frac{1 + \cos \alpha}{3}, \frac{-1 + \sin \alpha}{3}, \frac{1 + \sin \alpha}{3}, \frac{-1 + \cos \alpha}{3})$.

By using the base $\{\nu_1(\tilde{x}), \nu_2(\tilde{x}), \nu_3(\tilde{x})\}$ for the tangent space $T_{\tilde{x}}SO(3)$, we obtain the coefficients of the Hessian matrix of the cost function $G_{SO(3)}^\alpha$:

\[
\begin{align*}
\tilde{h}_{11}(\tilde{x}) &= (-1 + \cos \alpha)(x_{22} + x_{33}) + (1 + \sin \alpha)(x_{32} - x_{23}), \\
\tilde{h}_{12}(\tilde{x}) &= -\frac{1}{2} [(1 + \sin \alpha)x_{31} + (-1 + \cos \alpha)x_{21} + 3x_{12}], \\
\tilde{h}_{13}(\tilde{x}) &= -\frac{1}{2} [(-1 + \cos \alpha)x_{31} - (1 + \sin \alpha)x_{21} + 3x_{13}], \\
\tilde{h}_{22}(\tilde{x}) &= (-1 + \cos \alpha)x_{33} - (1 + \sin \alpha)x_{23} + 3x_{11}, \\
\tilde{h}_{23}(\tilde{x}) &= -\frac{1}{2} [(-1 + \cos \alpha)(x_{32} + x_{23}) + (1 + \sin \alpha)(x_{33} - x_{22})], \\
\tilde{h}_{33}(\tilde{x}) &= (1 + \sin \alpha)x_{32} + (-1 + \cos \alpha)x_{22} + 3x_{11}.
\end{align*}
\]
The critical points of the 2-power cost function $G_{SO(3)}^2$ have been computed in [5] using the Embedding Algorithm. We find five sets of critical points as follows:

Set $\text{Rot}_{\text{black}} = \{ R^q | q = (0,0,\pm \sqrt{1-t^2},t), t \in [-1,1] \}$,

Set $\text{Rot}_{\text{green}} = \{ R^q | q = (\sqrt{1-x_2^{\min}(\alpha)},x_2^{\min}(\alpha),0,0), \alpha \in [-\pi,\pi] \}$,

Set $\text{Rot}_{\text{pink}} = \{ R^q | q = (-\sqrt{1-x_2^{\min}(\alpha)},x_2^{\min}(\alpha),0,0), \alpha \in [-\pi,\pi] \}$,

Set $\text{Rot}_{\text{red}} = \{ R^q | q = (\sqrt{1-x_2^{\max}(\alpha)},x_2^{\max}(\alpha),0,0), \alpha \in [-\pi,\pi] \}$,

Set $\text{Rot}_{\text{blue}} = \{ R^q | q = (-\sqrt{1-x_2^{\max}(\alpha)},x_2^{\max}(\alpha),0,0), \alpha \in [-\pi,\pi] \}$,

where $x_2^{\min}(\alpha)$ and $x_2^{\max}(\alpha)$ are the smallest, respectively largest real positive solutions of the polynomial

\[
Q_{2,\alpha}(Z) = \left( 128 \sin^4 \frac{\alpha}{2} - 32 \sin^2 \frac{\alpha}{2} + 4 \right) Z^4 - \left( 128 \sin^4 \frac{\alpha}{2} - 32 \sin^2 \frac{\alpha}{2} + 4 \right) Z^2 \\
- 16 \sin^6 \frac{\alpha}{2} + 16 \sin^5 \frac{\alpha}{2} \cos \frac{\alpha}{2} + 28 \sin^4 \frac{\alpha}{2} - 8 \sin^2 \frac{\alpha}{2} + 1,
\]

and $R^q$ is the rotation corresponding to the unit quaternion $q$, see formula (6.3) in Annexe.

We study the nature of the above critical points using the Hessian characterization.

**Case black.** The critical points corresponding to rotations from Set $\text{Rot}_{\text{black}}$ are absolute maximum for the cost function $G_{SO(3)}^2$ as have been pointed out in [5]. Applying formula (4.3) at critical points in $I(\text{Set}_{\text{black}})$ we obtain the eigenvalues $\lambda_1 = 0, \lambda_2 = -3 + \sqrt{3 + 2\sin \alpha - 2\cos \alpha}, \lambda_3 = -3 - \sqrt{3 + 2\sin \alpha - 2\cos \alpha}$. Consequently, a critical point of this set is a degenerate absolute maximum.

![Figure 1: Eigenvalues of the Hessian matrix of $G_{SO(3)}^2$ computed at critical points from Set $\text{Rot}_{\text{black}}$.](image)

**Case green.** Applying formula (4.3) at critical points in $I(\text{Set}_{\text{green}})$ we obtain for the Hessian matrix of the cost function $G_{SO(3)}^2$ two equal eigenvalues that are represented by the thick line in the Figure 2 and one simple eigenvalue.
We note that for $\alpha = -\frac{\pi}{4}$ and $\alpha = \frac{3\pi}{4}$ we obtain a bifurcation phenomena and the critical points in $\text{Set } \text{Rot}_{\text{green}}$ corresponding to this values of the parameter $\alpha$ have a degenerate Hessian matrix. For $\alpha \in [-\pi, -\frac{\pi}{4}) \cup (\frac{3\pi}{4}, \pi]$ the corresponding critical points in $\text{Set } \text{Rot}_{\text{green}}$ are saddle critical points for the cost function $G_{\text{SO}(3)}^a$. For $\alpha \in (-\frac{\pi}{4}, \frac{3\pi}{4})$ the corresponding critical points are local minima.

**Case pink.** For the critical points in the set $\text{Set } \text{Rot}_{\text{pink}}$ the Hessian matrix of the cost function $G_{\text{SO}(3)}^a$ has one negative eigenvalue and two equal positive eigenvalues. Consequently, this critical points are all saddle points.

**Case red.** The critical points in the set $\text{Set } \text{Rot}_{\text{red}}$ are all local minima as eigenvalues of the Hessian matrix of the cost function $G_{\text{SO}(3)}^a$ are all positive.
Case blue. Again a bifurcation phenomena appears for this case at the values $\alpha = -\frac{\pi}{4}$ and $\alpha = \frac{3\pi}{4}$.

For $\alpha \in [-\pi, -\frac{\pi}{4}) \cup (\frac{3\pi}{4}, \pi]$ the corresponding critical points in Set $^{\text{Rot}}_{\text{blue}}$ are local minima for the cost function $G^\alpha_{SO(3)}$. For $\alpha \in (-\frac{\pi}{4}, \frac{3\pi}{4})$ the corresponding critical points are saddle points.

5 Stability of equilibrium points using restricted Hessian

We apply the results of Section 2 to the stability problem of an equilibrium point for a dynamical system generated by a vector field $X_S$ defined on a manifold $S$. Let $x_e \in S$ be an equilibrium point for the dynamics on the manifold $S$ generated by the vector field $X_S$. Stability behavior of the equilibrium point $x_e$ can be determined using the direct method of Lyapunov. This method requires the knowledge of a Lyapunov function $G_S : S \to \mathbb{R}$ which has the following properties:

(i) $\dot{G}_S := L_{X_S} G_S \leq 0$;

(ii) $G_S(x) > G_S(x_e)$, for all $x \neq x_e$ in a neighborhood of $x_e$. 


In order to verify the above conditions one needs to construct a local system of coordinates on $S$ around the equilibrium point $x_e$. If $G_S$ is a $C^2$ differentiable function and $x_e$ is a critical point for $G_S$ then a sufficient condition for (ii) to hold is given by the positive definiteness of the Hessian matrix $\mathcal{H}^{G_S}(x_e)$. Usually is very difficult to construct local coordinates on the submanifold $S$ and in these cases we will bypass this difficulty by embedding the problem in an ambient space $M$ (usually an Euclidean space) and use formula (2.5) given in Theorem 2.1.

Suppose that the manifold $S$ is a preimage of a regular value for a map $F = (F_1, \ldots, F_k) : M \to \mathbb{R}^k$, where $(M, g)$ is an ambient Riemannian manifold. Let $X \in \mathfrak{X}(M)$ be a prolongation of the vector field $X_S$, i.e. $X|_S = X_S \in \mathfrak{X}(S)$ and $G : M \to \mathbb{R}$ be a $C^2$ prolongation of the function $G_S$. The equilibrium point $x_e$ is also an equilibrium point for the dynamics on $M$ generated by the vector field $X$. The conditions of the next result guaranties the applicability of the direct method of Lyapunov stated above.

Theorem 5.1. The following conditions:

(i) $\dot{G} := L_X G \leq 0$,

(ii) $\text{grad } G(x_e) = \sum_{i=1}^k \sigma_i(x_e) \text{grad } F_i(x_e)$,

(iii) $\left[\mathcal{H}^G(x_e)\right]|_{T_{x_e}S \times T_{x_e}S} - \sum_{i=1}^k \sigma_i(x_e) \left[\mathcal{H}^{F_i}(x_e)\right]|_{T_{x_e}S \times T_{x_e}S}$ is positive definite,

implies that the equilibrium point $x_e$ is stable for the dynamics generated by the vector field $X_S$.

The condition $L_{X_S} G_S \leq 0$ is implied by the condition (i) in the above theorem. Condition (ii) is equivalent with $x_e$ being a critical point of the function $G_S : S \to \mathbb{R}$, where $\sigma_i(x_e)$ are the Lagrange multipliers, and condition (iii) is equivalent with positive definiteness of the Hessian matrix $\mathcal{H}^{G_S}(x_e)$. The advantage of the above theorem is that all the necessary computations for verifying conditions (i), (ii) and (iii) are made using the coordinates of the ambient space $M$ which usually is an Euclidean space. Note that the constraint functions $F_1, \ldots, F_k$ do not need to be conserved quantities for the prolonged vector field $X$.

Usually the above theorem is applied backwards, where the vector field $X_S$ is the restriction of a vector field $X \in \mathfrak{X}(M)$ to an invariant submanifold $S$ under the dynamics generated by the vector field $X$. In the case when $F_1, \ldots, F_k, G$ are conserved quantities for the vector field $X$ and the conditions (ii) and (iii) of the above theorem are satisfied, then the equilibrium point $x_e$ is also stable for the dynamics generated by the vector field $X$ according to the algebraic method, see [6], [7], [8].

We will apply the above result to the following Hamilton-Poisson situation. Let $(M, \{\cdot,\cdot\})$ be a finite dimensional Poisson manifold and $X_H$ a Hamilton-Poisson vector field. The paracompactness of the manifold $M$ ensures the existence of a Riemannian metric. The conserved quantities are the Casimir functions $F_1 = C_1, \ldots, F_k = C_k$. A regular symplectic leaf $S$ is an open dense set of the submanifold generated by a regular value of the Casimir functions (usually the two sets are equal). The restricted vector field $X_S = (X_H)|_S$ on the symplectic leaf is again a Hamiltonian vector field with respect to the symplectic form induced by the Poisson structure $\{\cdot,\cdot\}$ and the Hamiltonian function $G_S = H|_S$. If $x_e \in S$ is an equilibrium point for the restricted Hamiltonian vector field $X_S$, then it is also an equilibrium point for $X_H$ and also it is a critical point of the restricted Hamiltonian function $G_S = H|_S$.

We are in the hypotheses of the Theorem 5.1 and the algebraic method for stability.

Theorem 5.2. A sufficient condition for stability of the equilibrium point $x_e$ with respect to the dynamics $X_H$ is given by the following condition:

$$\left(\mathcal{H}^{H}(x_e) - \sum_{i=1}^k \sigma_i(x_e) \mathcal{H}^{C_i}(x_e)\right)|_{T_{x_e}S \times T_{x_e}S}$$

is positive definite.
The matrix in the above theorem represents the Hessian matrix of the restricted function \( H_\beta : S \rightarrow \mathbb{R} \) as has been shown in Theorem [21]. It is also the Hessian matrix restricted to the tangent space \( T_\gamma S \) of the augmented function \( F : M \rightarrow \mathbb{R} \), \( F(x) = H(x) - \sum_{i=1}^{k} \sigma_i(x)C_i(x) \) used in [13], [3], and [21].

In the case of symmetries the augmented function method for stability of relative equilibria has been studied extensively in [18], [16], [17], [19], [20], [15]. The tangent space to the invariant submanifold \( S \) can be further decomposed taking into account the symmetry of the dynamical system under study.

6 Annexe

1. The Gramian associated to the constraint functions \([3.2]-[3.3]\) is \( \Sigma(\tilde{x}) := \det \Sigma(F_1,f_2,...,F_{n-1,n})(\tilde{x}) \). The matrix \( \Sigma(F_1,f_2,...,F_{n-1,n})(\tilde{x}) \) has the form

\[
\Sigma(F_1,f_2,...,F_{n-1,n})(\tilde{x}) = \begin{bmatrix} A & C^T \\ C & B \end{bmatrix},
\]

where

\[
A = \begin{bmatrix} <\text{grad} F_1, \text{grad} F_1> & \ldots & <\text{grad} F_n, \text{grad} F_1> \\ \vdots & \ddots & \vdots \\ <\text{grad} F_1, \text{grad} F_n> & \ldots & <\text{grad} F_n, \text{grad} F_n> \end{bmatrix},
\]

\[
B = \begin{bmatrix} <\text{grad} F_{12}, \text{grad} F_{12}> & \ldots & <\text{grad} F_{n-1,n}, \text{grad} F_{12}> \\ \vdots & \ddots & \vdots \\ <\text{grad} F_{12}, \text{grad} F_{n-1,n}> & \ldots & <\text{grad} F_{n-1,n}, \text{grad} F_{n-1,n}> \end{bmatrix},
\]

\[
C = \begin{bmatrix} <\text{grad} F_1, \text{grad} F_{n-1,n}> & \ldots & <\text{grad} F_n, \text{grad} F_{n-1,n}> \end{bmatrix},
\]

We have the following computations:

\[
<\text{grad} F_s(\tilde{x}), \text{grad} F_r(\tilde{x})> = \sum_{i=1}^{n} x_{si} \tilde{e}_{si}, \sum_{j=1}^{n} x_{rj} \tilde{e}_{rj} > = \sum_{i,j=1}^{n} x_{si} x_{rj} < \tilde{e}_{si}, \tilde{e}_{rj} > = \sum_{i,j=1}^{n} x_{si} x_{rj} \delta_{ij} = \sum_{i=1}^{n} x_{si} x_{rj} \delta_{sr} = <x_s, x_r> \delta_{sr},
\]

\[
<\text{grad} F_s(\tilde{x}), \text{grad} F_{\alpha\beta}(\tilde{x})> = \sum_{i=1}^{n} x_{si} \tilde{e}_{si}, \sum_{j=1}^{n} (\delta_{\alpha j} \tilde{e}_{\alpha j} + \delta_{\beta j} \tilde{e}_{\beta j}) > = \sum_{i=1}^{n} (x_{si} \delta_{\alpha j} + x_{si} \delta_{\beta j}) = <x_s, x_\beta> \delta_{\alpha j} + <x_s, x_\alpha> \delta_{\beta j},
\]

\[
<\text{grad} F_{\gamma\tau}(\tilde{x}), \text{grad} F_{\alpha\beta}(\tilde{x})> = \sum_{i=1}^{n} (x_{\gamma i} \tilde{e}_{\gamma i} + x_{\tau i} \tilde{e}_{\tau i}), \sum_{j=1}^{n} (\delta_{\gamma j} \tilde{e}_{\gamma j} + \delta_{\tau j} \tilde{e}_{\beta j}) > = \sum_{i=1}^{n} (x_{\gamma i} \delta_{\gamma j} + x_{\tau i} \delta_{\beta j} + x_{\gamma i} \delta_{\tau j} + x_{\tau i} \delta_{\gamma j}) = <x_{\gamma}, x_{\beta}> \delta_{\gamma j} + <x_{\gamma}, x_{\alpha}> \delta_{\beta j} + <x_{\tau}, x_{\alpha}> \delta_{\gamma j} + <x_{\tau}, x_{\beta}> \delta_{\gamma j}.
\]
Consequently, using the identification (4.1) for $\tilde{x} \in O(n) = J(O(n))$ we have,

$$\Sigma(\tilde{x}) = \det \left[ \begin{array}{ccc} I_n & 0 & 0 \\ 0 & 2I_{n(n-1)}/n \\ \end{array} \right] = 2^{n(n-1)/2}.$$ 

For $\tilde{x} \in O(n)$ we have the following computations:

$$\Sigma_s(\tilde{x}) = \det \Sigma(F_1, \ldots, F_n, F_{n-1}, \ldots, F_1)(\tilde{x})$$

$$= \det \left[ \begin{array}{cccc} 1 \cdots < \text{grad } G(\tilde{x}), \text{grad } F_1(\tilde{x}) > & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 \cdots < \text{grad } G(\tilde{x}), \text{grad } F_n(\tilde{x}) > & \cdots & 1 \\ \end{array} \right]$$

$$= < \text{grad } G(\tilde{x}), \text{grad } F_s(\tilde{x}) > = 2^{n(n-1)/2} < \text{grad } G(\tilde{x}), \text{grad } F_s(\tilde{x}) >,$$

$$\Sigma_{pq}(\tilde{x}) = \det \Sigma(F_1, \ldots, F_n, F_{n-1}, \ldots, F_1)(\tilde{x})$$

$$= < \text{grad } G(\tilde{x}), \text{grad } F_{pq}(\tilde{x}) > = 2^{n(n-1)/2 - 1} < \text{grad } G(\tilde{x}), \text{grad } F_{pq}(\tilde{x}) >.$$

Consequently,

$$\sigma_s(\tilde{x}) = < \text{grad } G(\tilde{x}), \text{grad } F_s(\tilde{x}) >, \quad \sigma_{pq}(\tilde{x}) = \frac{1}{2} < \text{grad } G(\tilde{x}), \text{grad } F_{pq}(\tilde{x}) >. \quad (6.1)$$

II. We have the following formula for the multiplication of the two $n \times n$ matrices $e_i \otimes e_j$ and $e_\alpha \otimes e_\beta$:

$$e_i \otimes e_j \cdot e_\alpha \otimes e_\beta = \delta_{ij\alpha} e_i \otimes e_\beta.$$

Using (3.5), a base for $T_{\tilde{x}}O(n)$ is given by the following matrices,

$$X\Omega_{\alpha\beta} = \left( \sum_{i,j=1}^n x_{ij} e_i \otimes e_j \right) (-1)^{\alpha+\beta} (e_\alpha \otimes e_\beta - e_\beta \otimes e_\alpha)$$

$$= (-1)^{\alpha+\beta} \sum_{i,j=1}^n x_{ij} (e_i \otimes e_j \cdot e_\alpha \otimes e_\beta - e_\alpha \otimes e_\beta \cdot e_i \otimes e_\beta)$$

$$= (-1)^{\alpha+\beta} \sum_{i,j=1}^n x_{ij} (\delta_{j\alpha} e_i \otimes e_\beta - \delta_{j\beta} e_i \otimes e_\alpha)$$

$$= (-1)^{\alpha+\beta} \sum_{i=1}^n (x_{i\alpha} e_i \otimes e_\beta - x_{i\beta} e_i \otimes e_\alpha). \quad (6.2)$$

In order to compute the restricted Hessian to the tangent space $T_{\tilde{x}}O(n)$ we need the following
computation,
\[< \tilde{\omega}_{\gamma^T}(\tilde{x}), e_{ab} \otimes e_{cd} \cdot \tilde{\omega}_{\alpha \beta}(\tilde{x})>\]
\[= (-1)^{\gamma + \tau} \sum_{i=1}^{n} (x_{i\gamma} \tilde{e}_{j\gamma} - x_{j\gamma} \tilde{e}_{i\gamma}), e_{ab} \otimes e_{cd} \cdot \sum_{i=1}^{n} (-1)^{\alpha + \beta} (x_{i\alpha} \tilde{e}_{i\beta} - x_{i\beta} \tilde{e}_{i\alpha}) >\]
\[= (-1)^{\alpha + \beta + \gamma + \tau} \sum_{i,j=1}^{n} (x_{i\gamma} x_{i\alpha} \delta_{ij} \tilde{e}_{\beta} - x_{j\gamma} x_{i\beta} \delta_{ij} \tilde{e}_{\alpha}) - x_{i\gamma} x_{i\beta} \delta_{ij} \tilde{e}_{\alpha} - x_{i\alpha} x_{i\beta} \delta_{ij} \tilde{e}_{\gamma} + x_{i\gamma} x_{i\alpha} \delta_{ij} \tilde{e}_{\beta} - x_{i\beta} x_{i\alpha} \delta_{ij} \tilde{e}_{\gamma})\]
\[= (-1)^{\alpha + \beta + \gamma + \tau} (x_{i\gamma} x_{i\alpha} \delta_{ij} \tilde{e}_{\beta} - x_{i\beta} x_{i\alpha} \delta_{ij} \tilde{e}_{\gamma} + x_{i\gamma} x_{i\alpha} \delta_{ij} \tilde{e}_{\beta} - x_{i\beta} x_{i\alpha} \delta_{ij} \tilde{e}_{\gamma})\]
where \(\tilde{\omega}_{\alpha \beta}(\tilde{x}) = \mathcal{J}(X_{\alpha \beta}) = (-1)^{\alpha + \beta} \sum_{i=1}^{n} (x_{i\alpha} \tilde{e}_{i\beta} - x_{i\beta} \tilde{e}_{i\alpha}), \ 1 \leq \alpha < \beta \leq n.

Consequently,
\[< \tilde{\omega}_{\gamma^T}(\tilde{x}), e_{s\gamma} \otimes e_{s\alpha} \cdot \tilde{\omega}_{\alpha \beta}(\tilde{x})> = (-1)^{\alpha + \beta + \gamma + \tau} (x_{s\gamma} \delta_{s\alpha} - x_{s\alpha} \delta_{s\gamma}) (x_{s\gamma} \tilde{e}_{\beta} - x_{s\beta} \tilde{e}_{\gamma})\]
\[< \tilde{\omega}_{\gamma^T}(\tilde{x}), e_{p\gamma} \otimes e_{p\alpha} \cdot \tilde{\omega}_{\alpha \beta}(\tilde{x})> = (-1)^{\alpha + \beta + \gamma + \tau} (x_{p\gamma} \delta_{p\alpha} - x_{p\alpha} \delta_{p\gamma}) (x_{p\gamma} \tilde{e}_{\beta} - x_{p\beta} \tilde{e}_{\gamma})\]
\[< \tilde{\omega}_{\gamma^T}(\tilde{x}), e_{q\gamma} \otimes e_{q\alpha} \cdot \tilde{\omega}_{\alpha \beta}(\tilde{x})> = (-1)^{\alpha + \beta + \gamma + \tau} (x_{q\gamma} \delta_{q\alpha} - x_{q\alpha} \delta_{q\gamma}) (x_{q\gamma} \tilde{e}_{\beta} - x_{q\beta} \tilde{e}_{\gamma})\]

III. The unit quaternions \(q = (q_0, q_1, q_2, q_3) \in S^3 \subset \mathbb{R}^4\) and \(-q \in S^3 \subset \mathbb{R}^4\) correspond to the following rotation in \(SO(3)\):

\[
R^q = \begin{pmatrix}
(q_0^2 + q_1^2 - q_2^2 - q_3^2)^2 & 2(q_0 q_1 - q_0 q_3 + q_1^3) & 2(q_0 q_1 + q_0 q_3 - q_1^3) & 2(q_1 q_3 + q_0 q_2) \\
2(q_0 q_1 + q_1 q_3 - q_0 q_2) & (q_0^2 - (q_1^2 + q_2^2 - q_3^2)^2 & 2(q_1 q_3 - q_0 q_2) & 2(q_1 q_3 + q_0 q_2) \\
2(q_0 q_1 - q_0 q_3 - q_1^3) & 2(q_0 q_1 + q_0 q_3 + q_1^3) & (q_0^2 - (q_1^2 + q_2^2 + q_3^2)^2 & 2(q_1 q_3 - q_0 q_2) \\
\end{pmatrix}
\]

\[\text{(6.3)}\]

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References

[1] P.A. Absil, R. Mahony, R. Sepulchre, Optimization Algorithms on Matrix Manifolds, Princeton University Press, 2008.

[2] P.A. Absil, R. Mahony, J. Trumpf, An Extrinsic Look at the Riemannian Hessian, Geometric Science of Information, Lecture Notes in Computer Science, Volume 8085, (2013), pp 361-368.

[3] J.A. Beck, C.D. Hall, Relative equilibria of a rigid satellite in a circular Keplerian orbit, J. Astronaut. Sci., Vol. 40, Issue 3 (1998), pp. 2152-247.

[4] P. Birtea, D. Comănescu, Geometric Dissipation for dynamical systems, Comm. Math. Phys., Vol. 316, Issue 2 (2012), pp. 375-394.

[5] P. Birtea, D. Comănescu, C.A. Popa, Averaging on Manifolds by Embedding Algorithm, J. Math. Imaging Vis., DOI 10.1007/s10851-013-0478-8.

[6] D. Comănescu, The stability problem for the torque-free gyrostat investigated by using algebraic methods, Applied Mathematics Letters, Volume 25, Issue 9 (2012), pp. 1185-1190.

[7] D. Comănescu, Stability of equilibrium states in the Zhukovski case of heavy gyrostat using algebraic methods, Mathematical Methods in the Applied Sciences, Volume 36, Issue 4 (2013), pp. 373-382.
[8] D. Comănescu, A note on stability of the vertical uniform rotations of the heavy top, ZAMM, Volume 93, Issue 9 (2013), pp. 697-699.

[9] D.L. Donoho, C. Grimes, Hessian eigenmaps: Locally linear embedding techniques for high-dimensional data, Proceedings of the National Academy of Sciences, 100(10) (2003), pp. 5591-5596.

[10] A. Edelman, T.A. Arias, S.T. Smith, The geometry of algorithms with orthogonality constraints, SIAM J. Matrix Anal. Appl., Vol. 20, Issue 2 (1998), pp. 303-353.

[11] R. Ferreira, J. Xavier, J. P. Costeira, V. Barroso, Newton Algorithms for Riemannian Distance Related Problems on Connected Locally Symmetric Manifolds, IEEE Journal of Selected Topics in Signal Processing, Volume 7, Issue 4 (2013), pp. 634-645.

[12] S. Gallot, D. Hulin, J. Lafontaine, Riemannian Geometry, Universitext, Springer-Verlag, Berlin, 3rd edition, 2004.

[13] J.H. Maddocks, Stability of relative equilibria, IMA J. Appl. Math., Vol. 46 (1991), pp. 71-99.

[14] M. Moakher, Means and averaging in the group of rotations, SIAM J. Matrix Anal. Appl., Vol. 24, Issue 1 (2002), pp. 1-16.

[15] J. A. Montaldi, M. Rodriguez-Olmos, On the stability of Hamiltonian relative equilibria with non-trivial isotropy, Nonlinearity, Vol. 24, Issue 10 (2011), pp. 2777-2783.

[16] J.-P. Ortega, T. S. Ratiu, Stability of Hamiltonian relative equilibria, Nonlinearity, Volume 12, Issue 3 (1999), pp. 693-720.

[17] J.-P. Ortega, T. S. Ratiu, Non-linear stability of singular relative periodic orbits in Hamiltonian systems with symmetry, Journal of Geometry and Physics, Volume 32, Issue 2 (1999), pp. 160-188.

[18] G.W. Patrick, Relative equilibria in Hamiltonian systems: the dynamic interpretation of nonlinear stability on a reduced phase space, Journal of Geometry and Physics, Volume 9 (1992), pp. 111-119.

[19] G.W. Patrick, M. Roberts, C. Wulff, Stability of Poisson equilibria and Hamiltonian relative equilibria by energy methods, Archive for Rational Mechanics and Analysis, Volume 174, Issue 3 (2004), pp. 301-344.

[20] M. Rodriguez-Olmos, Stability of relative equilibria with singular momentum values in simple mechanical systems, Nonlinearity, Volume 19, Issue 4 (2006).

[21] Y. Wang, S. Xu, Equilibrium attitude and nonlinear attitude stability of a spacecraft on a stationary orbit around an asteroid, J. Adv. Space Res., Vol. 52, Issue 8 (2013), pp. 1497-1510.