Some remarks on dissipative Ermakov systems and damping in Bose–Einstein condensates

Dieter Schuch
Institut für Theoretische Physik, Goethe-Universität Frankfurt am Main, Max-von-Laue-Str. 1, D-60438 Frankfurt am Main, Germany
E-mail: schuch@em.uni-frankfurt.de

Radhey Shyam Kaushal
Department for Physics and Astrophysics, University of Dehli, Dehli, 110 007, India
E-mail: rkaushal@physics.du.ac.in

Abstract. For physical systems where the Hamiltonian is no longer a constant of motion, like oscillators with time-dependent frequency or dissipative systems, a dynamical invariant of the Ermakov type might still exist. This invariant depends on the classical position and velocity and an auxiliary variable that is proportional to position uncertainty. It is possible to construct this invariant via an algebraic method based on the classical Poisson bracket. A modified version for dissipative systems that also employs anti-Poisson brackets and is related to a description in an expanding coordinate system will be presented. The resulting invariant is identical to that obtained from a logarithmic nonlinear Schrödinger equation. The uncertainties of position and momentum, expressed in terms of the auxiliary variable, can be compared with the so-called moment-method for the description of the time-evolution of Bose–Einstein condensates, corresponding to a description in terms of the cubic nonlinear Gross–Pitaevskii equation. It will be shown how this method can be extended to also include dissipative damping effects into the Bose–Einstein dynamics.

1. Introduction

For physical systems where the Hamiltonian is no longer a constant of motion, like the harmonic oscillator (HO) with time-dependent (TD) frequency or dissipative systems with energy loss due to interaction with an environment, there might still be another constant of motion that does not have the dimension of an energy, like the Hamiltonian, but that of an action.

In a first step it will be shown that such an invariant, first established by Vassili Petrovich Ermakov in 1880(!) for the HO with TD frequency [1] can also be obtained via an algebraic method based on the description of the time-evolution in terms of Poisson brackets. In section 3, it will be shown that the same invariant and the equations of motion leading to it, can also be received from the time-evolution of a Gaussian wave packet (WP) that is a solution of the corresponding TD Schrödinger equation (SE). In this context, the dynamics of quantum uncertainties will play an important role. The coupled closed set of differential equations describing this dynamics is closely related to the algebraic method mentioned above and will be crucial for the further analysis.
Section 4 will show how the interaction with a dissipative environment, expressed in the form of a linear velocity dependent friction force can be included into a formal Hamiltonian description, involving exponentially expanding coordinates and a non-canonical transformation, but supplying a Hamiltonian that is a constant of motion. It will be shown that also for this system a dynamical Ermakov invariant with the dimension of an action can be obtained via a modified algebraic method, now also including anti-commutator terms.

In section 5, it will be shown that the same invariant can also be obtained from a quantum mechanical description of the problem, taking into account the environmental effect via a complex logarithmic nonlinear term, turning the SE into a nonlinear (NL) SE.

The time-evolution of a Bose–Einstein condensate (BEC) can be described in the mean field approximation in terms of the cubic-nonlinear Gross–Pitaevskii equation for the corresponding WP. This equation cannot usually be solved analytically but, in certain cases, it is possible to describe the dynamics of this system entirely in terms of so-called moments. Their definition and the closed set of coupled evolution equations will be given in section 6 and the relation to the equations of motion for the quantum uncertainties will be established.

Finally, it will be shown in section 7 how the results from the NLSE can be used to include dissipative effects into the dynamics of the BEC, based on the dynamical equations leading to the corresponding Ermakov invariant.

The last Section will summarize the results.

2. Classical Ermakov systems and dynamical algebra

In conservative classical Hamiltonian mechanics, the Hamiltonian function

- determines the dynamics of the system, and
- is a constant of motion that represents the conserved energy of the system.

In cases where the Hamiltonian is no longer a constant of motion, e.g., for the HO with TD frequency \( \omega = \omega(t) \), there still may exist another constant of motion, the Ermakov invariant, that does not have the dimension of an energy, but essentially that of an action.

This invariant can be obtained from the corresponding classical Newtonian equation of motion for the (one-dimensional) position coordinate \( q \),

\[
\ddot{q} + \omega^2(t) q = 0
\]

and the equation for an auxiliary variable \( \alpha(t) \),

\[
\ddot{\alpha} + \omega^2(t) \alpha = \frac{1}{\alpha^3}
\]

by eliminating \( \omega^2(t) \) from these two equations (see Ermakov [1]), leading to the constant of motion

\[
I = \frac{1}{2} \left[ (\dot{q} \alpha - q \dot{\alpha})^2 + \left( \frac{q}{\alpha} \right)^2 \right] = \text{const.}
\]

However, the same invariant can also be obtained in a different way via an algebraic approach (see, e.g., [2]). In this case, the Hamiltonian is rewritten in terms of (not explicitly TD) phase space functions \( \Gamma_n \) as

\[
H = \sum_n h_n(t) \Gamma_n
\]

where the dynamical algebra is the Lie algebra of the functions \( \Gamma_n \), which is closed with relation to the Poisson brackets \( \{ , \} \) as

\[
\{ \Gamma_n, \Gamma_m \} = \sum_r C_{nm}^r \Gamma_r
\]
with the $C^r_{nm}$ being the structure constants of the algebra.

The time-evolution of any phase space function $F(q,p,t)$ is given by

$$\frac{d}{dt} F = \{ F, H \}_- + \frac{\partial}{\partial t} F .$$

In particular, a **dynamical invariant** is characterized by $\frac{d}{dt} I = 0$, i.e.,

$$\frac{\partial}{\partial t} I = \{ H, I \}_- .$$

Looking for an invariant that is also a member of the dynamical algebra, i.e.,

$$I = \sum_n \kappa_n(t) \Gamma_n ,$$

Eq. (7) leads to a coupled set of evolution equations for the expansion coefficients $\kappa_n$,

$$\dot{\kappa}_r + \sum_n \left( \sum_m C^r_{nm} h_m(t) \right) \kappa_n = 0 .$$

In the following, we consider the dynamical algebra of

$$\Gamma_1 = \frac{p^2}{2m} , \quad \Gamma_2 = p q , \quad \Gamma_3 = \frac{m q^2}{2}$$

with the Poisson brackets $\{ \Gamma_1, \Gamma_3 \}_- = - \Gamma_2$, $\{ \Gamma_1, \Gamma_2 \}_- = - 2 \Gamma_1$, $\{ \Gamma_3, \Gamma_2 \}_- = 2 \Gamma_3$.

For the Hamiltonian of the HO with TD frequency,

$$H = \frac{1}{2m} p^2 + \frac{m}{2} \omega^2(t) q^2$$

with $h_1 = 1, h_2 = 0$ and $h_3 = \omega^2(t)$, the set of equations for the $\kappa_n$ can be written as

$$\dot{\kappa}_1 = - 2 \kappa_2$$
$$\dot{\kappa}_2 = \omega^2 \kappa_1 - \kappa_3$$
$$\dot{\kappa}_3 = 2 \omega^2 \kappa_2 .$$

This coupled set of equations can be reduced to a single second order differential equation by introducing a new variable $\alpha(t)$ via $\kappa_1 = \frac{1}{m} \alpha^2$, leading to

$$\ddot{\alpha} + \omega^2(t) \alpha = \frac{k}{\alpha^3}$$

with constant $k$.

With the $\kappa_n$, expressed in terms of $\alpha$ and $\dot{\alpha}$, the invariant (8) can be written as

$$I = \frac{1}{2} \left[ \alpha^2 \frac{p^2}{m^2} - 2 \dot{\alpha} \alpha \frac{p}{m} q + \left( \dot{\alpha}^2 + \frac{k^2}{\alpha^4} \right) q^2 \right] ,$$

which is identical to (3) for $k = 1$ and $p = m \dot{q}$. 
3. Quantum mechanical wave packets and Ermakov invariant

Now the (one-dimensional) TDSE for the HO with possibly TD frequency $\omega = \omega(t)$ shall be considered,

$$i\hbar \frac{\partial}{\partial t} \Psi(x,t) = \left\{ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{m}{2} \omega^2 x^2 \right\} \Psi(x,t), \tag{15}$$

with exact analytic Gaussian WP solutions, written as [3]

$$\Psi_{WP}(x,t) = N(t) \exp \left\{ i \left[ y(t) \tilde{x}^2 + \frac{1}{\hbar} < p > \tilde{x} + K(t) \right] \right\}, \tag{16}$$

where $\tilde{x} = x - < x >$ with $< x > = \int_{-\infty}^{+\infty} \Psi^* x \Psi dx = q(t) =$ classical trajectory. The complex TD quantity $y(t) = y_R(t) + i y_I(t)$ is connected with the WP width, or the position uncertainty, via $y_I = \frac{1}{\hbar} \tilde{x}$ with $< \tilde{x}^2 > = < x^2 > - < x >^2$.

Inserting WP (16) into the TDSE (15) yields the Newtonian equation

$$\ddot{q} + \omega^2(t) q = 0 \tag{17}$$

for the WP maximum and the complex Ricatti equation

$$\frac{2\hbar}{m} \ddot{y} + \left( \frac{2\hbar}{m} \dot{y} \right)^2 + \omega^2 = 0 \tag{18}$$

for the WP width.

Introducing a new variable $\alpha_L(t)$ via $\frac{2\hbar}{m} y_I = \frac{1}{\alpha_L^2(t)}$ with $\alpha_L = \sqrt{\frac{2m<\tilde{x}^2>}{\hbar}}$, the Ricatti equation (18) leads to an Ermakov equation for $\alpha_L$ which is identical to the auxiliary equation (2) for $\alpha$, so $\alpha_L \equiv \alpha$. Together with Eq. (17), this again yields the invariant $I_L = I = (3)$.

It can be shown that position and momentum uncertainties are related to $\alpha_L$ via

$$< \tilde{x}^2 >_L = \frac{\hbar}{2m} \alpha_L^2,$$

$$< \tilde{p}^2 >_L = \frac{\hbar m}{2} \left( \alpha_L^2 + \frac{1}{\alpha_L^2} \right),$$

$$< [\tilde{x}, \tilde{p}]_+ >_L = \hbar \dot{\alpha}_L \alpha_L,$$

$$U_L = < \tilde{x}^2 >_L < \tilde{p}^2 >_L = \frac{\hbar^2}{4} \left[ 1 + (\dot{\alpha}_L \alpha_L)^2 \right] \tag{19}$$

where $[ , ]_+$ denotes the anti-commutator.

This allows one to write the invariant $I_L$ also in the form

$$I_L = \frac{1}{2} \left[ \frac{\alpha_L^2}{m^2} \frac{\tilde{p}^2}{m^2} - 2\alpha_L \alpha_L \frac{\tilde{p}}{m} q + \left( \alpha_L^2 + \frac{k^2}{\alpha_L^2} \right) q^2 \right]$$

$$= \frac{1}{\hbar m} \left[ < \tilde{x}^2 >_L \tilde{p}^2 - < [\tilde{x}, \tilde{p}]_+ >_L q p + < \tilde{p}^2 >_L q^2 \right]. \tag{20}$$

Comparison with $I$ defined via the expansion coefficients $\kappa_n$, $I = \sum_n \kappa_n(t) \Gamma_n$, shows how these coefficients are related to the quantum mechanical uncertainties,

$$\kappa_1 = \frac{1}{m} \alpha_L = \frac{2}{\hbar} < \tilde{x}^2 >_L,$$

$$\kappa_2 = -\frac{1}{m} \dot{\alpha}_L \alpha_L = -\frac{1}{\hbar m} < [\tilde{x}, \tilde{p}]_+ >_L,$$

$$\kappa_3 = \frac{1}{m} \left( \alpha_L^2 + \frac{1}{\alpha_L^2} \right) = \frac{2}{\hbar m^2} < \tilde{p}^2 >_L. \tag{21}$$

Therefore, the coupled set of equations for the $\kappa_n$, Eqs. (12), represents a set of equations for the time-dependence of the quantum uncertainties.
4. Classical dissipative Hamiltonian in expanding coordinates

In the following a classical dissipative system with (linear) velocity dependent friction force shall be considered. It will be described in an exponentially-expanding coordinate system \cite{4} with the variables

\[ Q = e^{\gamma t/2} q , \quad P = m \dot{Q} = me^{\gamma t/2} (\dot{q} + \frac{\gamma}{2} q) , \]

where the transition \((q, p) \rightarrow (Q, P)\) is a non-canonical transformation.

The corresponding Hamiltonian function

\[ \hat{H}_{\text{exp}} = \frac{1}{2m} P^2 + \frac{m}{2} \Omega^2 Q^2 = \text{const.} \] (23)

with \( \Omega^2 = \left( \omega^2 - \frac{\gamma^2}{4} \right) \) can be expressed in terms of the physical variables \( q \) and \( p \) as

\[ \hat{H}_{\text{exp}} = \left[ \frac{1}{2m} p^2 + \frac{\gamma}{2} p q + \frac{m}{2} \omega^2 q^2 \right] e^{\gamma t} = \frac{1}{2m} p_0^2 + \frac{m}{2} \omega^2 q_0^2 = E_0 . \] (24)

The corresponding Hamiltonian equations of motion are

\[ \frac{\partial \hat{H}_{\text{exp}}}{\partial P} = \frac{1}{m} P = \dot{Q} = e^{\gamma t/2} \left( \dot{q} + \frac{\gamma}{2} q \right) \]
\[ -\frac{\partial \hat{H}_{\text{exp}}}{\partial Q} = -\Omega^2 Q = -\left( \omega^2 - \frac{\gamma^2}{4} \right) e^{\gamma t/2} q = \dot{P} = e^{\gamma t/2} \left( \dot{q} + \gamma \dot{q} + \frac{\gamma^2}{4} q \right) , \] (25)

equivalent to the Newtonian equation

\[ \ddot{Q} + \left( \omega^2 - \frac{\gamma^2}{4} \right) Q = 0 \] (26)

in the expanding coordinates, or, to a Newtonian equation including a friction term \( \gamma \dot{q} \) with friction coefficient \( \gamma \),

\[ \ddot{q} + \gamma \dot{q} + \omega^2(t) q = 0 \] (27)

in the physical coordinate system.

Although the dissipative system in the expanding canonical variables \((Q, P)\) can be described via the usual canonical formalism, this is obviously not possible in terms of the physical variables \((q, p)\).

However, since the relations between the physical and the canonical variables are known, it is possible to transform the canonical results into the ones on the physical level.

In particular, if the time-evolution of any phase space function \( F(Q, P, t) \) on the canonical level is given by

\[ \frac{d}{dt} F = \{ F, \hat{H}_{\text{exp}} \}_{(Q, P)} + \frac{\partial}{\partial t} F , \] (28)

on the physical level, additional Poisson brackets \{ , \} - and anti-Poisson brackets \{ , \} + occur,

\[ \frac{d}{dt} F(q, p, t) = \{ F, H \} - + \{ F, \frac{\gamma}{2} p q \} - - \{ F, \frac{\gamma}{2} p q \} + + \frac{\partial}{\partial t} F , \] (29)

where \( H \) is given by Eq. (11) (for further details see \cite{4}).

In this dissipative case, the dynamical invariant

\[ I_{\text{exp}} = \sum_n t_{n, \text{exp}}(t) \Gamma_n \] (30)
must now fulfil the relation
\[
\frac{d}{dt} I_{\text{exp}} = \{I_{\text{exp}}, H\}_- + \{I_{\text{exp}}, \frac{\gamma}{2} p \cdot q\}_- - \{I_{\text{exp}}, \frac{\gamma}{2} p \cdot q\}_+ + \frac{\partial}{\partial t} I_{\text{exp}}
\]
\[
= \{I_{\text{exp}}, H\}_- + \frac{\gamma}{2} \{I_{\text{exp}}, \Gamma_2\}_- - \frac{\gamma}{2} \{I_{\text{exp}}, \Gamma_2\}_+ + \frac{\partial}{\partial t} I_{\text{exp}} = 0 .
\]

In comparison with the conservative case, there are additional terms from \(\{\Gamma_n, \Gamma_2\}_- - \{\Gamma_n, \Gamma_2\}_+\). This leads to a modified set of coupled equations for \(\kappa_{n,\text{exp}} = e^{-\gamma t} \dot{\kappa}_{n,\text{exp}}\),
\[
\begin{align*}
\dot{k}_{1,\text{exp}} &= \gamma \kappa_{1,\text{exp}} - 2 \kappa_{2,\text{exp}} \\
\dot{k}_{2,\text{exp}} &= \omega^2 \kappa_{1,\text{exp}} - \kappa_{3,\text{exp}} \\
\dot{k}_{3,\text{exp}} &= \gamma \kappa_{3,\text{exp}} + 2 \omega^2 \kappa_{2,\text{exp}} .
\end{align*}
\]
With \(\kappa_{1,\text{exp}} = \frac{1}{m} \alpha^2_{NL}\), this set of coupled equations can be condensed into the modified Ermakov equation
\[
\ddot{\alpha}_{NL} + \left(\omega^2(t) - \frac{\gamma^2}{4}\right) \alpha_{NL} = \frac{1}{\alpha^2_{NL}} .
\]
The corresponding Ermakov invariant, written in terms of \(q\) and \(\alpha^2_{NL}\), reads
\[
I_{\text{exp}} = \frac{1}{2} e^{\gamma t} \left[ \dot{\eta} \alpha_{NL} - \left( \dot{\alpha}_{NL} - \frac{\gamma}{2} \alpha_{NL} \right) \eta \right]^2 + \left( \frac{\eta}{\alpha_{NL}} \right)^2 = \text{const.}
\]
Expressing \(q\) and \(\dot{q}\) in terms of the expanding coordinate \(Q\) and velocity \(\dot{Q}\), this invariant can be rewritten as
\[
I_{\text{exp}} = \frac{1}{2} \left[ \left( \dot{Q} \alpha_{NL} - Q \dot{\alpha}_{NL} \right)^2 + \left( \frac{Q}{\alpha_{NL}} \right)^2 \right] = \text{const.}
\]
In this form, it becomes obvious that the invariant is not only independent of \(\omega\), i.e., also existing for \(\omega = \omega(t)\), but also independent of \(\gamma\), i.e., also existing for \(\gamma = \gamma(t)\)! This form is identical to the one for the conservative case, only \(q\) being replaced by \(Q\) and \(\alpha\) by \(\alpha_{NL}\). The meaning of the replacement \(q \rightarrow Q\) has been explained above; the meaning of the replacement \(\alpha \rightarrow \alpha_{NL}\) will become clear from the following description of an effective quantum mechanical approach for dissipative systems.

5. Nonlinear Schrödinger equation for dissipative systems
It has been shown that quantum systems interacting with some kind of dissipative environment may not only be described in terms of a reduced density operator coupled to a large number of environmental degrees of freedom (drastically increasing computational effort and thus severely limiting the systems that can be treated), but also in terms of pure state wave functions (or WPs) obeying an effective NLSE [5] where the NL Hamiltonian can even be non-Hermitian and still provide normalizable wave functions [6].

Our approach in this direction starts from the reversible continuity equation for the probability density \(\rho(x,t) = \Psi^*(x,t) \Psi(x,t)\) by adding a time-reversal symmetry-breaking diffusion term, leading to an irreversible (Fokker–Planck-type) Smoluchowski equation [6]
\[
\frac{\partial}{\partial t} \rho + \frac{\partial}{\partial x} (\rho v_\perp) - D \frac{\partial^2 \rho}{\partial x^2} = 0
\]
with the convection velocity field \(v_\perp = \frac{h}{\gamma m} \left( \frac{\partial}{\partial x} \Psi - \frac{\partial}{\partial x} \Psi^* \right)\) and the (possibly TD) diffusion coefficient \(D\).
To obtain the corresponding SE, Eq. (36) must be separated into two equations for \( \Psi \) or \( \Psi^* \), respectively. Due to the diffusion term, this is not possible in general, only in particular cases. So, separation can be achieved with the ansatz

\[
-D \frac{\partial^2 \theta}{\partial x^2} = \gamma (\ln \varrho - < \ln \varrho >)
\]

This leads to the NLSE

\[
\begin{align*}
 i\hbar \frac{\partial}{\partial t} \Psi_{NL}(x, t) &= \left\{ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) + \frac{\hbar}{i} (\ln \Psi_{NL} - < \ln \Psi_{NL} >) \right\} \Psi_{NL}(x, t) \\
&= \{H_L + W\} \Psi_{NL}(x, t) = H_{NL} \Psi_{NL}(x, t)
\end{align*}
\]

with the complex logarithmic nonlinearity \( W = \frac{\hbar^2}{i} (\ln \Psi_{NL} - < \ln \Psi_{NL} >) \).

For this log NLSE, also exact Gaussian WP solutions exist [6, 7]. In particular, the motion of the WP maximum is determined by the Newtonian equation

\[
\ddot{q} + \gamma \dot{q} + \omega^2(t) q = 0 ,
\]

whereas the dynamics of the WP width, also in this case, follows from a modified complex Riccati equation,

\[
\frac{2\hbar}{m} \dot{y} + \gamma \left( \frac{2\hbar}{m} y \right) + \left( \frac{2\hbar}{m} y \right)^2 + \omega^2(t) = 0 .
\]

With the same definition for the imaginary part in terms of a new variable \( \alpha_{NL}(t) \) as in the case of the linear TDSE, i.e., \( \frac{2\hbar}{m} \dot{y} = \frac{\hbar}{\alpha_{NL}(t)} \), the Riccati equation (40) turns into the modified Ermakov equation (33) for the auxiliary variable leading to the Ermakov invariant (34) in the expanding coordinate system. Therefore, the Newtonian equation (39) for \( q(t) \) and this equation for \( \alpha_{NL}(t) \) provide the same Ermakov invariant, i.e., \( I_{NL} \equiv I_{exp} \).

As in the conservative case, also in this dissipative case the quantity \( \alpha_{NL} \) is related to the position uncertainty via \( \alpha_{NL} = \sqrt{\frac{2m<\dot{x}^2>_{NL}}{\hbar}} \).

All the uncertainties for the dissipative system, expressed in terms of \( \alpha_{NL} \) and \( \alpha_{NL} \), now have the modified form

\[
\begin{align*}
<\dot{x}^2>_{NL} &= \frac{\hbar}{2m} \alpha_{NL}^2 = \frac{\hbar}{2} \kappa_{1, exp} , \\
<\dot{p}^2>_{NL} &= \frac{\hbar m}{2} \left[ (\dot{\alpha}_{NL} + \frac{\gamma}{2} \alpha_{NL})^2 + \frac{1}{\alpha_{NL}^2} \right] = \frac{\hbar m^2}{2} \kappa_{3, exp} , \\
<\dot{x}, \dot{p}>_{NL} &= \hbar \left( \dot{\alpha}_{NL} \alpha_{NL} - \frac{\gamma}{2} \alpha_{NL}^2 \right) = - \hbar m \kappa_{2, exp} , \\
U_{NL} &= <\dot{x}^2>_{NL} <\dot{p}^2>_{NL} = \frac{\hbar}{4} \left[ 1 + \left( \dot{\alpha}_{NL} \alpha_{NL} - \frac{\gamma}{2} \alpha_{NL}^2 \right) \right]
\end{align*}
\]

where the relations between the uncertainties and the expansion coefficients \( \kappa_{n, exp} \) correspond to the ones in the conservative case. Therefore, the set of equations (32) for the time-evolution of the expansion coefficients \( \kappa_{n, exp} \) now corresponds to a set of equations for the time-evolution of the uncertainties of the dissipative system.

Important for the following is that:

- in the conservative case the quantum uncertainties, given by \( <\dot{x}^2>_L, <\dot{p}^2>_L \), etc. are directly related to
- the \( \kappa_n \) from the classical algebraic approach and
• the $\alpha_L, \dot{\alpha}_L$ from the WP solution of the TDSE;

in the **dissipative case** the **quantum uncertainties** are given by $< \tilde{x}^2 >_{NL}, < \tilde{p}^2 >_{NL}$, etc., derived from the log NLSE and uniquely related to

• the $\kappa_{n,exp}$ from the classical expanding coordinate system and

• the $\alpha_{NL}, \dot{\alpha}_{NL}$ from the WP solution of the log NLSE.

6. Bose–Einstein condensates and moment method

In the mean field approximation, a BEC can be described by a macroscopic WP for the condensate, $\Psi$, which obeys the **Gross–Pitaevskii equation**

\[
i\hbar \frac{\partial}{\partial t} \Psi = \left\{ -\frac{\hbar^2}{2m} \Delta + V(r,t) + g|\Psi|^2 \right\} \Psi.
\] (42)

The trapping potential $V(r,t)$ shall be given by $V(r,t) = \frac{m}{2} \omega^2(t)r^2$ with TD frequency (e.g., like in a Paul trap).

Although Eq. (42) cannot be solved analytically (even in the case of constant frequency $\omega$), the **dynamics** of the BEC characterized by this equation can be described by a set of coupled differential equations for the so-called **moments** $M_n (n = 1 - 4)$ (for details see, e.g., [8]).

These moments are defined by the following integral parameters (with $\hbar = m = 1$)

\[
M_1 = \int d^2x |u|^2 \sim \text{norm}
\]

\[
M_2 = \int d^2x r^2 |u|^2 \sim \text{width}
\]

\[
M_3 = i \int d^2x \left( \frac{\partial u^*}{\partial r} - \frac{\partial u}{\partial r} \right) |u|^2 \sim \text{radial momentum}
\]

\[
M_4 = \int d^2x \left( |\nabla u|^2 + \frac{k^2}{\rho^2} |u|^2 + |u|^4 \right) \sim \text{energy of WP}.
\] (43)

It can be shown that these moments $M_n$ satisfy a set of coupled first order differential equations (where $\frac{d}{dt} M_1 = 0$, corresponding to conservation of probability or particle number),

\[
\frac{d}{dt} M_2 = M_3,
\]

\[
\frac{d}{dt} M_3 = -\omega^2(t) M_2 + 4M_4,
\]

\[
\frac{d}{dt} M_4 = -\frac{1}{2} \omega^2(t) M_3.
\] (44)

With the help of the invariant $2M_1M_2 = M_3^2 = 1$, this set of equations can be reduced to a single equation for $M_2$,

\[
\frac{d^2}{dt^2} M_2 - \frac{1}{2M_2} \left( \frac{d}{dt} M_2 \right)^2 + \omega^2(t)M_2 = \frac{2}{M_2}.
\] (45)

Introducing a new variable $X(t)$ via $X(t) = \sqrt{M_2} = \text{WP width}$, changes this equation into

\[
\ddot{X} + \omega^2(t) X = \frac{k}{X^3}.
\] (46)
In comparison with the algebraic method for the conservative case and the WP solution of the corresponding TDSE it immediately follows (with $k = 1$) that

$$
M_2 = \frac{2m}{\hbar} < \hat{x}^2 >_L = \alpha_L^2 = m \kappa_1 \\
M_3 = \frac{2}{\hbar} < [\hat{x}, \hat{p}]_+ >_L = 2\alpha_L \alpha = -2 m \kappa_2 \\
M_4 = \frac{1}{m\hbar} < \hat{p}^2 >_L = \frac{1}{2} \left( \alpha_L^2 + \frac{1}{\alpha_L^2} \right) = \frac{1}{2} m \kappa_3 .
$$

(47)

Since $X = M_2^{1/2}$ is identical to $\alpha_L$, the equation for $X(t)$ is just the Ermakov equation from the TDSE.

7. Bose–Einstein condensates with dissipation

To include dissipative energy loss effects, attempts at different modifications of the Gross–Pitaevskii equation have been made, e.g., by using imaginary (non-Hermitian) loss terms of the kind $i\gamma(x)\Psi$, etc. [9].

Considering the quantum mechanical results discussed in Section 5 for the inclusion of dissipative friction effects, a modification of the Gross–Pitaevskii equation by adding the complex logarithmic term should be used, i.e.,

$$
\hbar \frac{\partial}{\partial t} \Psi = \left\{ - \frac{\hbar^2}{2m} \Delta + V(r,t) + g|\Psi|^2 + \frac{\hbar}{i} (\ln \Psi - < \ln \Psi >) \right\} \Psi .
$$

(48)

Although this equation also cannot be solved analytically, the moment method can still be applied.

Since the moments $M_n$ can be directly identified with the expansion coefficients $\kappa_n$ (or $\kappa_{n,exp}$, respectively) of the Ermakov invariant which are uniquely connected with the quantum uncertainties, the set of equations for $\kappa_{n,exp}$, including the dissipative effects from the log NLSE, can be directly translated into the corresponding equations for the moments $M_n$, including dissipation.

Therefore, all results concerning the log NLSE can be transferred to the dissipative BEC problem. The moments can be identified with the uncertainties of the NLSE in the same way as given in Eqs. (47), only replacing $< ... >_L$ by $< ... >_{NL}$.

The set of equations of motion for the moments changes into

$$
\frac{d}{dt} M_2 = M_3 + \gamma M_2 \\
\frac{d}{dt} M_3 = - \omega^2(t) M_2 + 4M_4 \\
\frac{d}{dt} M_4 = - \frac{1}{2} \omega^2(t) M_3 - \gamma M_4 ,
$$

(49)

corresponding to the set of evolution equations (32) for the $\kappa_{n,exp}$ or the uncertainties of the log NLSE, respectively, which again, can be condensed into the Ermakov equation for $X = \sqrt{M_2} = \sqrt{\frac{2m < \hat{x}^2 >_{NL}}{\kappa}}$,

$$
\ddot{X} + \left( \omega^2(t) - \frac{\gamma^2}{4} \right) X = \frac{1}{X^3} .
$$

(50)
8. Conclusions
For the HO with TD frequency, a model that can describe an ion in a Paul trap [10] or a confined BEC [8], the Hamiltonian (with the dimension of an energy) is no longer a constant of motion but there still exists the Ermakov invariant (with the dimension of an action) for this system. As an alternative to the original derivation [1], this invariant can also be obtained via a classical dynamical algebra. The (conservative) dynamics of quantum mechanical WPs fulfilling the corresponding TDSE shows the connection between this invariant, the classical position ($q$) and the quantum mechanical position uncertainty ($\alpha L$).

Another system where the energy is no longer a constant of motion and, in addition, the usual Hamiltonian formalism for the physical position and momentum cannot be applied to provide correct equations of motion, is the HO with linear velocity dependent friction force. However, a description in terms of the canonical formalism becomes feasible via a non-canonical transformation to a new set of exponentially-expanding variables. In this case, the formal canonical Hamiltonian function $H_{\text{exp}}$ is a constant of motion and corresponds to the initial energy of the system. It can be shown that for this system also an Ermakov invariant exists. The (non-canonical) backtransformation shows which changes in the conservative Hamiltonian formalism must be made in order to include the friction effect also into the equations that are expressed in the physical variables. This also leads to a modification of the algebraic method for the determination of the Ermakov invariant, particularly the set of coupled equations for the determination of the expansion coefficients $\kappa_n$ is affected.

It can be shown that the same dissipative system can also be treated analytically in a quantum mechanical context if a description in terms of an effective NLSE with complex logarithmic nonlinearity is used. The coupled equations of motion for the quantum uncertainties correspond, like already in the conservative case, to the coupled set of equations for the $\kappa_{n,\text{exp}}$ in the dissipative case. The Ermakov invariant of the NLSE is identical to the one obtained in the expanding coordinate system.

The dynamics of a conservative BEC can also be described by a (different) NLSE, the Gross–Pitaevskii equation with cubic nonlinearity. Although no analytical solutions to this equation exist, the dynamics can still be obtained using the so-called moment method. The relation between these moments, the quantum uncertainties of the TDSE and the expansion coefficients for the conservative Ermakov invariant have been clarified. The structural relationship is still valid if the dissipative effect is taken into account, thus allowing one to obtain the dynamics of a dissipative BEC just by comparing it with the results from the log NLSE.

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