On rational maps from a general surface in $\mathbb{P}^3$ to surfaces of general type

Lucio Guerra $^1$ and Gian Pietro Pirola $^2$

Abstract

We study dominant rational maps from a general surface in $\mathbb{P}^3$ to surfaces of general type. We prove restrictions on the target surfaces, and special properties of these rational maps. We show that for small degree the general surface has no such map. Moreover a slight improvement of a result of Catanese, on the number of moduli of a surface of general type, is also obtained.

1 Introduction

Let $X$ be a smooth complex projective variety of general type. Let $R(X)$ be the field of rational functions of $X$. Consider the set of the geometric subfields of $R(X)$, that is:

$$\mathcal{F}(X) = \{ K : \mathbb{C} \subset K \subset R(X) \}.$$ 

An element $K \in \mathcal{F}(X)$ corresponds to a dominant rational map $X \dashrightarrow Y$, where $Y$ is a smooth projective variety with $K \cong R(Y)$, up to birational isomorphisms of $Y$. Consider moreover the subset:

$$\mathcal{F}_0(X) = \{ K \in \mathcal{F}(X) : [R(X) : K] \text{ is finite} \}.$$ 

Elements of this subset correspond to generically finite dominant rational maps. We may then define various geometric subsets of $\mathcal{F}(X)$, such as:

$$\mathcal{I}S(X) = \{ R(Y) \in \mathcal{F}_0(X) : Y \text{ is of general type} \},$$

$$\mathcal{G}(X) = \{ R(Y) \in \mathcal{F}_0(X) : Y \text{ is not rationally connected} \}.$$ 

We call $\mathcal{I}S(X)$ the Iitaka-Severi set of $X$, we denote by $s(X)$ the cardinality of $\mathcal{I}S(X)$ and by $g(X)$ the cardinality of $\mathcal{G}(X)$.

The recent solution of the Iitaka-Severi conjecture [31] [14] [29] gives that $\mathcal{I}S(X)$ is a finite set. In general $\mathcal{G}(X)$ is not finite (for instance if $X$ dominates an abelian variety). The problem remains how to compute or at least to estimate the number $s(X)$. We call this the refined Iitaka-Severi problem.

When $X$ is a curve of genus $g \geq 2$ effective bounds on $s(X)$ in terms of $g$ are known [16] [30]. In higher dimensions not much is known, but see [12] [15] [25] where upper bounds are given under some geometric restrictions.

If $X$ is a curve, general in moduli, then $g(X) = s(X) = 1$. This may be proved by counting moduli of maps by means of the Hurwitz formula.

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The same in fact holds for the general smooth plane curve of degree bigger than 3, or for a general hyperplane section of a regular surface \( [3, 27] \). The proof of these facts may be based on a Hodge-Lefschetz theoretical argument (using monodromy, see \([32, \S3.2.3]\), which implies that the Jacobian of \( H \) is simple. So we have two basic methods: a moduli count and a Hodge theoretical argument.

In higher dimensions, we believe the following could be true:

**Conjecture.** Let \( X \) be a very general hypersurface of \( \mathbb{P}^n \) of degree \( d > n + 1 \). Then \( g(X) = s(X) = 1 \).

The case of curves and the results of Amerik \([1]\) give evidence to the conjecture. In this paper we can prove the following (see \( 4.2.1 \)):

**Theorem.** If \( X \) is a general surface of \( \mathbb{P}^3 \) of degree \( 5 \leq d \leq 11 \) then \( s(X) = 1 \).

The proof uses both methods described for the case of curves. Using the Hodge theoretic argument, we obtain restrictions for the target surfaces (see \( 3.5.2 \)):

**Proposition.** Let \( X \) be a general surface of \( \mathbb{P}^3 \) of degree \( d \geq 5 \). Let \( Y \) be a minimal surface of general type, and assume that \( f : X \to Y \) is a dominant rational map. If \( f \) is not birational then \( Y \) is simply connected of geometric genus \( p_g(Y) = 0 \).

Here we mention that simply connected surfaces of geometric genus \( p_g(Y) = 0 \) are known to exist \([2, 19]\), and moreover they are homeomorphic to rational surfaces, as follows from Freedman’s theorem \([10]\). However the moduli space of these surfaces is still largely unknown.

Then we approach the moduli of rational maps. First we consider the moduli of target surfaces. There is a well known result of Catanese \([5, 6]\) on the moduli of surfaces of general type, for which we propose a new approach based on the stability \([4, 9, 26, 28]\), which produces a slight improvement (see \( 2.5.1 \)):

**Theorem.** Let \( Y \) be a minimal surface of general type, \( M(Y) \) be the number of moduli of \( Y \). The following estimate holds: \( M(Y) \leq 11 \chi(\mathcal{O}_Y) + K_Y^2 \).

Then we study the moduli of maps in terms of their ramification. Roughly speaking we associate to a rational map \( f : X \to Y \) the piece of the ramification divisor that is seen on \( X \) (not in the exceptional divisor of the resolution of \( f \)). This is a complete intersection curve \( D \) on the surface \( X \subset \mathbb{P}^3 \).

As an outcome of the vanishing \( p_g(Y) = q(Y) = 0 \) we have that a property of Cayley-Bacharach type is enjoyed by the fibers of the rational map, and this in turn implies some estimate for the degree \( (3.4.1) \) and the ramification \( (3.4.3) \) of the map.

A lower bound for the moduli of the ramification divisor \( D \subset X \) follows from an argument \( (2.2.1) \) which combines the rigidity theorem for rational maps and the bend and break lemma of Mori theory. An upper bound for the moduli of the curve \( D \subset \mathbb{P}^3 \) is obtained in terms of the degree of this complete intersection. Finally all the constraints force the inequality \( d \geq 12 \).

The problem we studied was a good field for the interplay of methods that come from different areas: projective and birational geometry, moduli and
stability theory. It is possible that a different approach is necessary in order to settle the above conjecture. One possibility is to try a degeneration argument. The results of this paper will be useful (in the surface case), allowing for instance to consider a restricted class of target varieties.

2 Surfaces of general type

In this section we study the number of moduli of a curve which deforms in a surface of general type, and in the last subsection we study the number of moduli of a surface of general type.

2.1 Rigidity

We need the following rigidity theorem for rational maps:

**Theorem 2.1.1.** If $X$ and $Y$ are varieties of general type, of the same dimension, a dominant rational map $f : X \to Y$ admits no non-constant deformation.

This was first proved by Kobayashi-Ochiai [17], and also follows from the more general statement known as the Iitaka-Severi conjecture, nowadays a theorem in virtue of the recent work of Tsuji [31], Hacon-McKernan [14], and Takayama [29], and the original approach of Maehara [21]. An updated account will be presented in a forthcoming paper [13].

2.2 Bend and break

We prove a lemma which combines the rigidity theorem and the basic idea of Mori theory.

**Lemma 2.2.1.** Let $S$ and $B$ be smooth connected projective surfaces. Let $C$ be a smooth connected projective curve. Let $F : C \times B \to S$ be a rational map. Assume that the family $F(C_b)$ is a two-dimensional family of curves on $S$. Then $S$ is not of general type.

**Proof.** Assume by contradiction that $S$ is of general type. We first remark that $F$ is dominant. Take the general point $s \in S$ and the general point $(t, b) \in F^{-1}(s)$. We remark that $F^{-1}(s)$ is a curve on $C \times B$. Fix a general point $x \in C$ and consider the rational map $F_x : B \to S : F_x(b) = F(x, b)$.

We now show that $F_x$ cannot be dominant. Otherwise the rigidity theorem gives $F_t = F_x$ for $t$ belonging to a Zariski open subset of $B$. This implies then $F(t, b) = F(x, b)$, that is the curve $C_b$ is contracted to a point. This gives a contradiction.

We have then that $F_x(B)$ is a curve and hence there is a curve $D$ on $B$ such that $F(x, y) = s$ for all $y \in D$. The family of curves obtained by restriction of $F$ defines a map $G : C \times D \to S$ such that $G(x, y) = s$ for $y \in D$. Using Mori’s trick (see [23], thm. 5 and 6) it follows that there is a rational curve $R \subset S$ passing through $s$. Therefore $S$ cannot be of general type. 

\[ \square \]
2.3 Modular dimension

A family of curves parametrized by a nonsingular variety $U$ is a surjective proper morphism $q : X \to U$, with 1-dimensional fibers $X_t = q^{-1}(t)$. We assume that $X$ is reduced, in order to avoid multiple components in the general curve, we also assume that all components of $X$ dominate $U$. A smooth family of curves will be a family of curves for which $q$ is a smooth morphism. Let $\mathcal{M}_g$ be the moduli space of smooth connected curves of genus $g$. For any smooth family of connected curves of genus $g$, there is a modular map

$$\mu : U \to \mathcal{M}_g$$

defined by $\mu(t) = [X_t]$ (see e.g. [24], ch. 5). The dimension of the image of this map is the number of moduli of curves in the family. For an arbitrary family of curves we define the number of moduli as follows.

Assume first that $X$ is irreducible. There is a dominant map $k : W \to U$, which is a finite morphism of $W$ onto a Zariski open subset of $U$, and there is a smooth connected family of curves $\mathcal{C} \to W$, together with a morphism of families

$$\begin{array}{ccc}
\mathcal{C} & \xrightarrow{i} & X \\
\downarrow & & \downarrow \\
W & \xrightarrow{\pi} & U
\end{array}$$

such that for $t \in U$ the induced morphism

$$\prod_{k(z) = t} C_z \to X_t$$

is the normalization map of $X_t$. This family $\mathcal{C} \to W$ defines a modular map

$$\mu : W \to \mathcal{M}_g.$$  

**Definition 2.3.1.** Let $X \to U$ be a family of curves. If $X$ is irreducible, the dimension of the image of the modular map $W \to \mathcal{M}_g$ is by definition the modular dimension of the family:

$$M(X/U) = \dim \mu(W).$$

In general, if $X = \bigcup X_i$ is the irreducible decomposition, then by definition the modular dimension of the family is:

$$M(X/U) = \max_i M(X_i/U).$$

A family of curves in a variety $Y$ is a family of curves $X \to U$ such that $X \subset U \times Y$. Over a Zariski open subset $U' \subset U$ the family is flat, there is the natural map $U' \to \mathcal{H}(Y)$ to the Hilbert scheme of $Y$, sending $t \mapsto X_t$, and the dimension of the image of this map is the dimension of the family, we call it $f$. We remark that in general $M(X'/U) \leq f$. We can rewrite the proposition.

**Proposition 2.3.2.** Let $X/U$ be a family of curves on a surface of general type, let $f$ be the dimension of the family. Then:

$$f - 1 \leq M(X/U) \leq f.$$  

**Proof.** The fibers of the modular map $\mu : W \to \mathcal{M}_g$ define families of curves with constant moduli. In a surface of general type by [2.2.1] they have dimension $\leq 1$. \qed
2.4 Stability

Let $M$ be a smooth projective complex variety of dimension $n$. Let $A$ be a line bundle on $M$. We would like to recall the notion of Mumford-Takemoto semistability of a vector bundle with respect to $A$. Usually $A$ is ample or at least nef and big. We will often abuse notation and identify a line bundle $A$ with its first Chern class $c_1(A)$. Let $E$ be a vector bundle of rank $r$ on $M$. We say that $E$ is semistable with respect to $A$ if for any injective sheaf map $\phi : F \rightarrow E$, where $F$ is a coherent sheaf of rank $s$, then:

$$\frac{c_1(F) \cdot A^{n-1}}{s} \leq \frac{c_1(E) \cdot A^{n-1}}{r}.$$ 

Let $T_M$ and $\Omega^1_M$ be the tangent and the cotangent bundle of $M$. From the work of Yau \[33, 34\] it follows that if the canonical bundle is ample then $T_M$ is $K_M$-semistable. The following general version was proved by Enoki (see \[28\] and \[9\]):

**Theorem 2.4.1.** Let $M$ be a canonical projective variety (that is, $M$ has only canonical singularities and ample canonical divisor) of dimension $n$. Let $\mu : N \rightarrow M$ be a smooth resolution. Then the tangent bundle $T_N$ is $\mu^*K_M$-semistable.

Now let $Y$ be a minimal surface of general type and $\Omega_Y^1$ be its cotangent bundle. The canonical model of $Y$ has canonical singularities. Then 2.4.1 implies that every $\Omega_Y^1(mK_Y)$ is semistable with respect to $K_Y$. We obtain the following:

**Corollary 2.4.2.** Let $Y$ be a minimal surface of general type. If $L$ is a line bundle on $Y$ and $L \subset \Omega_Y^1(mK_Y)$ is a sheaf inclusion, then: $2K_Y L \leq (2m+1)K_Y^2$.

**Remark 2.4.3.** Bogomolov (see for instance \[4\] and \[26\]) introduced a slightly weaker notion of stability, called the $T$-(semi)stability in the book of Kobayashi \[18\], p. 184. Bogomolov was able to prove that for a minimal surface of general type $\Omega_Y^1$ is $T$-semistable. This is also a straightforward consequence of 2.4.2. It is likely possible that 2.4.2 follows from Yau’s work, or from the theory of Bogomolov, but presently the authors do not have a reference for that.

2.5 Surface moduli estimate

In this section we prove a bound for the number of moduli $M(Y)$ of a minimal surface $Y$ of general type, slightly improving a result of Catanese. The proof relies on 2.4.2

**Theorem 2.5.1.** Let $Y$ be a minimal surface of general type, and let $M(Y)$ be the number of moduli of $Y$. The following estimates hold:

1. $M(Y) \leq 11\chi(\mathcal{O}_Y) + K_Y^2$;
2. if $K_Y^2 = 1$ then $M(Y) \leq 10\chi(\mathcal{O}_Y) + 1$. 

Let $T_Y$ be the tangent bundle of $Y$. We have $h^2(T_Y) = h^0(\Omega^1_Y(K_Y))$, $h^0(T_Y) = 0$ and $h^1(T_Y) \geq M(Y)$, by deformation theory. From Riemann-Roch we have $\chi(T_Y) = 2K_Y^2 - 10\chi(\mathcal{O}_Y)$, hence:

$$M(Y) \leq h^0(\Omega^1_Y(K_Y)) - \chi(T_Y) = h^0(\Omega^1_Y(K_Y)) + 10\chi(\mathcal{O}_Y) - 2K_Y^2.$$ 

Therefore Proposition 2.5.1 is an immediate consequence of the following:

**Proposition 2.5.2.** In the present setting we have:

1. $h^0(\Omega^1_Y(K_Y)) \leq \chi(\mathcal{O}_Y) + 3K_Y^2$;
2. if $K_Y^2 = 1$ then $h^0(\Omega^1_Y(K_Y)) \leq 3$.

**Proof.** Assume that there is a line bundle $L \subset \Omega^1_Y(K_Y)$ with $h^0(L) = \dim H^0(Y, L) > 0$. Otherwise $h^0(\Omega^1_Y(K_Y)) = 0$ and the statement is trivially true since $\chi(\mathcal{O}_Y) + 3K_Y^2 \geq 4$. After saturation of $L$, we can define an exact sequence:

$$0 \rightarrow L \rightarrow \Omega^1_Y(K_Y) \rightarrow M \otimes I_\theta \rightarrow 0,$$

where $M$ is a line bundle and $I_\theta$ is the ideal of a zero dimensional scheme $\theta$.

We obtain

$$h^0(\Omega^1_Y(K_Y)) \leq h^0(L) + h^0(M \otimes I_\theta) \leq h^0(L) + h^0(M).$$

Note that $L + M = \det(\Omega^1_Y(K_Y)) = 3K_Y$.

(a) First assume $K_Y^2 \geq 2$. We have two subcases:

1. $h^0(M) \neq 0$. In this case using the multiplication

$$\mu : H^0(L) \otimes H^0(M) \rightarrow H^0(3K_Y)$$

we obtain by Hopf’s lemma

$$h^0(L) + h^0(M) - 1 \leq \dim \text{Im}(\mu) \leq h^0(3K_Y) = \chi(\mathcal{O}_Y) + 3K_Y^2.$$

We know from base point freeness and 1-connectedness ([31], Ch.7 §§5.6) that $|3K_Y|$ contains some smooth irreducible curve $D$, and therefore the strict inequality $h^0(L) + h^0(M) - 1 < h^0(3K_Y)$ holds. In conclusion:

$$h^0(\Omega^1_Y(K_Y)) \leq \chi(\mathcal{O}_Y) + 3K_Y^2.$$

(2) $h^0(L) = h^0(\Omega^1_Y(K_Y))$. Take a smooth irreducible curve $D$ in $|3K_Y|$, as before. We have $h^0(L) \leq h^0(L_D)$. In fact $L(-D)$ is contained in $T_Y(-K_Y)$, which has no sections. We can apply Clifford’s theorem and 2.4.2 to get:

$$2(h^0(L_D) - 1) \leq DL = 3K_Y L \leq \frac{9}{2} K_Y^2,$$

so finally

$$h^0(\Omega^1_Y(K_Y)) \leq h^0(L_D) \leq 1 + \frac{9}{4} K_Y^2 \leq \chi(\mathcal{O}_Y) + 3K_Y^2.$$
(b) Now consider the case $K_Y^2 = 1$. Let us prove that: if $L \subset \Omega_Y^1(K_Y)$ then $h^0(L) \leq 1$. From \[2.4.2\] we have $2K_Y L \leq 3$ that is

$$K_Y L \leq 1.$$ 

Assume by contradiction $h^0(L) \geq 2$. Write $|L| = F + |H|$ where $F$ is the fixed part of the system and $H$ is the free part. We have $1 \leq K_Y H \leq K_Y F + K_Y H = K_Y L = 1$ that is $K_Y H = 1$. It follows that $H^2$ is odd (since $H^2 - K_Y H$ is even by Riemann-Roch) and $\geq 0$. The Hodge index theorem gives $H^2 = 1$ and $H \equiv K_Y$ numerically. It follows that $h^0(2H) \geq 3$. But now Ramanujam vanishing gives $h^0(2H) = h^0(2K_Y) = h^0(-K_Y) = 0$ and $h^2(2H) = 0$. That would imply $h^0(2H) = h^0(2K_Y) = 1 + K_Y^2 = 2$, which gives a contradiction.

Now consider the determinant map $c : \wedge^2 H^0(\Omega_Y^1(K_Y)) \to H^0(3K_Y)$. From the assertion above, the kernel of $c$ does not contain any decomposable non trivial element. Otherwise, if $s_1 \wedge s_2 = 0$ then the two sections define a rank 1 subsheaf $L$ of $\Omega_Y^1(K_Y)$ with $h^0(L) \geq 2$. Since $h^0(3K_Y) = 4$ it follows that $h^0(\Omega_Y^1(K_Y)) \leq 3$. \hfill $\Box$

**Corollary 2.5.3.** Let $Y$ be a simply connected minimal surface of general type with $p_g(Y) = 0$. We have

1. $M(Y) \leq K_Y^2 + 11 \leq 19$;
2. if $K_Y^2 = 1$ then $M(Y) \leq 11$.

**Proof.** Under the present hypotheses, we have $\chi(O_Y) = 1$. Then $K_Y^2 \leq 9$ by the Miyaoka-Bogomolov inequality. Moreover by Yau’s theorem if $K_Y^2 = 9$ then $Y$ is not simply connected (and is rigid). Then the result follows from \[2.5.1\]. \hfill $\Box$

**Remark 2.5.4.** The following estimate of the number of moduli of minimal surfaces of general type was given by Fabrizio Catanese ([5] thm. B, and [6] thm. 20.6):

$$M(Y) \leq 10\chi(O_Y) + 3K_Y^2 + 18.$$ 

The estimate in \[2.5.1\] is a slight improvement, as is easily seen using the Noether inequality.

**Remark 2.5.5.** For a surface $X$ of degree $d \geq 5$ of $\mathbb{P}^3$

$$M(X) = M(d) = \binom{d + 3}{3} - 16 = \frac{(d + 1)(d + 2)(d + 3)}{6} - 16.$$ 

3 **Surfaces of projective space**

We study rational maps from a surface in $\mathbb{P}^3$ to a surface of general type. Under certain special assumptions, we obtain some estimate for the degree of the map and some control of the ramification. In the last subsection we prove that for the general surface in $\mathbb{P}^3$ the assumptions are indeed verified.
3.1 Hurwitz formula

In this section $X$ will be a smooth surface of $\mathbb{P}^3$ of degree $d > 4$, with Picard group generated by the hyperplane section, $Y$ will be a minimal surface of general type with $p_g(Y) = 0$, and $f : X \to Y$ will be a dominant rational map of degree $m$. Consider the diagram of maps

$$
\begin{array}{ccc}
Z & \xrightarrow{\phi} & X \\
\downarrow^{h} & \searrow & \downarrow^{f} \\
\phantom{Z} & \phantom{X} & \phantom{Y}
\end{array}
$$

(1)

where $\phi$ is the blowing up which resolves the singularity of $f$, and $h$ is the morphism which extends $f$, so that (as rational maps) $h \circ \phi^{-1} = f$.

Let $E$ be the ramification divisor of $\phi : Z \to X$. Every connected component of the support of $E$ is a connected tree of rational curves. Let $H$ be the hyperplane divisor of $X$. Set $L = \phi^*H$. Let $K_X$, $K_Z$ and $K_Y$ be the canonical divisors of $X$, $Z$ and $Y$. We have

$$
K_X = (d - 4)H.
$$

Since the Néron-Severi group of $X$ is generated by the hyperplane $H$ we have that the Néron-Severi group (=Picard group) of $Z$ is generated by $L$ and the irreducible components $E_i$ of the support of $E$. Let $R \subset Z$ be the ramification divisor of $h$. The Hurwitz formulae give (modulo linear equivalence):

$$
K_Z = h^*(K_Y) + R = \phi^*(K_X) + E = (d - 4)L + E
$$

(2)

Write:

$$
h^*(K_Y) = rL - W,
$$

(3)

$$
R = sL + W + E,
$$

(4)

where:

$$
W = \sum a_iE_i.
$$

The coefficients $a_i$ and $r, s$ are integers, with $r \geq 0, s \geq 0$ and

$$
r + s = d - 4.
$$

We prove the following:

**Lemma 3.1.1.** The divisor $W = \sum a_iE_i$ is effective, that is $a_i \geq 0$ for all $i$.

**Proof.** Write $W = A - B$ where $A$ and $B$ are effective divisors supported on $E'$ with disjoint irreducible components, in particular :

$$
A \cdot B \geq 0.
$$

Now $h^*(K_Y) = rL - A + B$ is a nef divisor since $Y$ is minimal surface of general type. Then, since $L \cdot B = 0$, we get:

$$
0 \leq B \cdot h^*(K_Y) = -B \cdot A + B^2 \leq B^2.
$$

This implies $B = 0$ since $B$ is contracted by $\phi$. □
Lemma 3.1.2. Using the notation established above, we have:

\[ mK_Y^2 \leq r^2 d. \]

Moreover \( mK_Y^2 = r^2 d \) holds if and only if \( W = 0 \). In particular we have \( r > 0 \).

Proof. Since \( L^2 = H^2 = d \) we obtain:

\[ mK_Y^2 = (h^*K_Y)^2 = r^2 H^2 + W^2 = r^2 d + W^2 \leq r^2 d. \]

Since \( K_Y^2 > 0 \) we get \( r > 0 \).

Remark 3.1.3. To show that \( r > 0 \) it is enough to assume that \( Y \) is simply connected with \( p_g(Y) = 0 \) and \( K_Y \) nef. This implies clearly \( r \geq 0 \). If we assume by contradiction \( K_Y^2 = 0 \) and \( r = 0 \), we would obtain \( W = 0 \), that is \( f^*(K_Y) = 0 \). But this would give \( f_*f^*(K_Y) = mK_Y = 0 \). Since \( Y \) is simply connected it would follow then \( K_Y = 0 \), and hence \( p_g(Y) = 1 \). This is a contradiction.

3.2 Cayley-Bacharach condition

We recall the classical notion of the Cayley-Bacharach condition (see [11]). A set of distinct points \( T = \{p_1, \ldots, p_{m-1}, p_m\} \subset \mathbb{P}^3 \) is in Cayley-Bacharach position with respect to \( \mathcal{O}(d) \) if any surface of degree \( d \) passing through any subset of \( T \) of cardinality \( m-1 \), must contain also the remaining point. There are many results on points in Cayley-Bacharach position (see [7]). We will use only the following elementary lemma:

Lemma 3.2.1. Assume \( T = \{p_1, \ldots, p_{m-1}, p_m\} \) in \( \mathbb{P}^3 \) is in Cayley-Bacharach position with respect to \( \mathcal{O}(n) \), \( n > 0 \). Then:

1. \( m \geq n + 2 \).
2. If \( n + 2 \leq m \leq 3n + 1 \) then \( T \) is contained in a plane.
3. If \( n + 2 \leq m \leq 2n + 1 \) then \( T \) is contained in a line.

3.3 Trace of holomorphic forms

Let \( X \) be a smooth surface of \( \mathbb{P}^3 \) of degree \( d \geq 5 \). Assume that \( f : X \rightarrow Y \) is a generically finite dominant rational map of degree \( m = \deg f \). We will use the method of [20]. Any rational correspondence between \( X \) and \( Y \), \( \Gamma \subset X \times Y \), defines a trace map \( tr(\Gamma) : H^{2,0}(X) \rightarrow H^{2,0}(Y) \) defined by the composition of pull-back and push-down \( \pi_Y \circ \pi_X \), where \( \pi_X : \Gamma \rightarrow X \) and \( \pi_Y : \Gamma \rightarrow Y \) are induced by the projections. When \( f : X \rightarrow Y \) is generically finite, the trace of \( f \)

\[ tr(f) : H^{2,0}(X) \rightarrow H^{2,0}(Y) \]

is associated to the graph of \( f \). Let \( y \) be a general point of \( Y \), and assume that \( f \) is étale at \( y \). Set \( T = \{p_1, \ldots, p_m\} = f^{-1}(y) \). Taking a local coordinate \( z \) around the point \( y \) we define then by pullback coordinates around any point \( p_i \in T = f^{-1}(y) \). Now if \( \omega \in H^{2,0}(X) \) using the parameters defined above as local identification we get the local trace formula:
\[ tr(f)(\omega)_y = \sum_{p_i \in T} \omega_{p_i}. \]

Assuming that \( tr(f)(\omega) = 0 \) and that \( \omega \) vanishes in \( m - 1 \) points of \( T \), it follows that \( \omega \) must vanish in the remaining one. If \( tr(f) = 0 \), and this certainly happens when \( p_g(Y) = \dim H^{2,0}(Y) = 0 \), then \( T \) is in Cayley-Bacharach position with respect to \( O(d - 4) \). In particular we have then:

**Proposition 3.3.1.** Let \( X \) be a smooth surface in \( \mathbb{P}^3 \) of degree \( d \), and \( Y \) be a smooth surface with \( p_g(Y) = 0 \). Let \( f : X \rightarrow Y \) be a generically finite rational map of degree \( m = \deg f \). Then the points of the general fiber of \( f \) are in Cayley-Bacharach position with respect to \( O(d - 4) \).

**3.4 Degree of maps**

We now prove the following:

**Proposition 3.4.1.** Let \( X \) be a smooth surface in \( \mathbb{P}^3 \) of degree \( d \), and \( Y \) be a non-rational smooth surface with \( p_g(Y) = q(Y) = 0 \). Let \( f : X \rightarrow Y \) be a generically finite rational map of degree \( m = \deg f \). We have

1. \( m \geq d - 1 \).
2. If \( d > 5 \) and \( X \) does not contain rational curves then \( m \geq d \).

**Proof.**

1. From 3.2.1 it follows that \( m > d - 3 \). Assume by contradiction \( m = d - 2 \). By 3.2.1 we still have that the points of a general fiber of \( f \) are on a line. Let \( S^k(X) \) be the \( k \)-symmetric product of \( X \) and define the rational map \( Y \twoheadrightarrow S^{d-2}(X), y \mapsto f^{-1}(y) \). Taking the two residual points of the line which contains \( f^{-1}(y) \) we define a rational map \( k : Y \twoheadrightarrow S^2(X) \). The map \( k \) is birational onto its image. This follows since generically the two points define the line and then the fiber of \( f \). The main point of [20] is that the image of \( k \) cannot define a correspondence between \( X \) and \( Y \), since otherwise this also should be a trace null correspondence. The analysis of [20] proves then that \( Y \) is birationally isomorphic either to the product of two curves of \( X \) or to the 2-symmetric product of a curve in \( X \) or else to a rational ruled surface over a curve of \( X \). Since \( Y \) is regular (it is dominated by the regular variety \( X \)) it would follow that \( k(Y) \) is covered by rational curves: \( Y \) is rational, and we obtain a contradiction.

2. Assume now by contradiction \( m = d - 1 \) and \( d > 5 \). We will show that \( X \) contains a rational curve. Since \( d > 5 \) the general fiber of \( f \) is contained in a line (see 3.2.1). Arguing as before we get a rational map \( Y \twoheadrightarrow S^{d-1}(X) \) and by taking the residue point on the line we get a map \( g : Y \twoheadrightarrow X \). This map cannot be dominant \( (p_g(X) > 0) \). Then either \( g(Y) \) is a point or a curve. If \( g(Y) = p \) is a point the general fibers of \( f \) are the general fibers of the projection \( \pi_p : X \twoheadrightarrow \mathbb{P}^2 \) from \( p \), then \( f = \pi_p \) (as rational maps) and hence \( Y \) is birational to \( \mathbb{P}^2 \). This gives a contradiction. It follows that \( g(Y) \) is a curve. Since \( Y \) is regular it follows that \( g(Y) \) is regular and hence a rational curve. \( \square \)

**Remark 3.4.2.** In her unpublished thesis Renza Cortini has classified all smooth surfaces \( X \) of degree \( d \) admitting a rational map \( X \twoheadrightarrow \mathbb{P}^2 \) of degree \( d - 2 \).
We can now improve Proposition 3.1.2.

**Proposition 3.4.3.** Let $X$ be a smooth surface in $\mathbb{P}^3$ of degree $d$, which contains no rational curves. Let $Y$ be a non-rational smooth surface with $p_g(Y) = q(Y) = 0$, and let $f : X \dasharrow Y$ be a generically finite dominant rational map of degree $m$. Assume moreover that $r = 1$, in the notation of Proposition 3.1.2. Then we have $K_Y^2 = 1$ and $d \leq 6$.

**Proof.** By Proposition 3.4.1 1) we have $m \geq d - 1$. Since $r = 1$, from the proof of Proposition 3.1.2 we have
\[(d - 1)K_Y^2 \leq mK_Y^2 = d + W^2 \leq d.\]
This forces $K_Y^2 = 1$ and $m \leq d$. Now assume that $d > 6$. By Proposition 3.4.1 2) it follows that $m = d$. Recall the maps $h : Z \to Y$ and $\phi : Z \to X$ in diagram (1). From Proposition 3.1.2 again we have $W = 0$, hence:
\[h^*K_Y = L = \phi^*H.\]
In particular we obtain $h_*L = dK_Y$. Since $d > 6$ the general fiber of the map $f : X \dasharrow Y$ is contained in a line. Thus we obtain a surface $S$ in the Grassmannian of lines in $\mathbb{P}^3$ and a rational map $k : Y \dasharrow S$ birational onto its image.

We are going to show that a general plane $\Pi$ in $\mathbb{P}^3$ can be chosen in such a way that a number of special conditions are satisfied. Let $S^0 \subset S$ be the Zariski open subset of $S$ consisting of lines $\ell$ such that the cardinality of $\ell \cap X$ is exactly $d = m$, and define:
\[U_0 = \{\Pi \in \mathbb{P}^3 : \Pi \supset \ell \text{ for some } \ell \in S^0\}.\]
Let $\Gamma \subset X$ be the set of points of indeterminacy of the map $f$, and define:
\[U_1 = \{\Pi \in \mathbb{P}^3 : \Pi \cap \Gamma = \emptyset\}.\]
Define moreover:
\[U_2 = \{\Pi \in \mathbb{P}^3 : \Pi \cap X \text{ is a connected smooth curve}\},
\[U_3 = \{\Pi \in \mathbb{P}^3 : \exists \ell \in S^0, \exists x \in X : \ell \cap \Pi = \{x\}\}.\]
We claim that the $U_i$ are dense Zariski constructible subsets of $\mathbb{P}^3$, so their intersection is non-empty. This requires some well known basic facts, except possibly the density of $U_0$.

This point is easily seen by means of projective duality. It is enough to prove that every plane must contain some line of $S$. There is a dual surface $S'$ in the Grassmannian of lines of $\mathbb{P}^3$, and the dual assertion is that every point of $\mathbb{P}^3$ belongs to some line of $S'$. Otherwise $S'$ covers a surface $X'$ in $\mathbb{P}^3$, and through any two points of $X'$ there is a line of $S'$, so $X'$ is a plane and $S'$ is the variety of lines in the plane. In this case the dual surface $S$ is the variety of lines through some fixed point in $\mathbb{P}^3$. Then $Y$ is a rational surface, and this is a contradiction.

It follows that we can take $\Pi \in U_0 \cap U_1 \cap U_2 \cap U_3$. Since $\Pi \in U_2$ we see that $C = \Pi \cap X$ is a smooth curve of genus $\frac{(d-1)(d-2)}{2}$. Since $\Pi \in U_1$ we get that the restriction map $f_C : C \to f(C)$ is everywhere defined, and since moreover $\Pi \in U_3$ then $f_C$ is birational onto its image. On the other hand since $\Pi \in U_0$
then $C$ contains $d$ distinct points, of some line $\ell$, which collapse on $f(C)$. Let $a$ be arithmetic genus of $f(C)$. We have

$$a \geq \frac{(d-1)(d-2)}{2} + \frac{d(d-1)}{2} = (d-1)^2.$$  

We remark in fact that $f(C)$ is contained in a surface and has a $d$–ple point: the above bound follows from the adjunction formula applied on the blow-up of $Y$. On the other hand since $f(C) = h_\ast L = dK_Y$ we have

$$2a - 2 = dK_Y \cdot (d+1)K_Y = d^2 + d.$$  

Hence

$$2(d-1)^2 \leq d^2 + d + 2.$$  

That is $d^2 \leq 5d$ and $d \leq 5$.

To outline the importance of the number $s$ we give the following

**Definition 3.4.4.** We call the number $s = d - 4 - r$ the birational index of the ramification of $f$.

### 3.5 General surfaces

For a general surface in $\mathbb{P}^3$ all the assumptions required for the results in the present section are indeed satisfied. We start by collecting some well known facts:

**Theorem 3.5.1.** Assume that $X$ is a general surface of $\mathbb{P}^3$ of degree $d \geq 5$. Then

i) (Noether-Lefschetz) The Néron-Severi group of $X$ is generated by the hyperplane section $H$.

ii) (Lefschetz) The Hodge substructure of $H^2(X)$ orthogonal to the hyperplane section is irreducible.

iii) (Xu) The surface $X$ does not contain any rational or elliptic curve.

iv) The only birational automorphism of $X$ is the identity.

**Proof.** (i) follows from (ii), and for (ii) see Voisin [32], §3.2.3. (iii) is proved in [33], and (iv) is well known.

We apply this to obtain:

**Proposition 3.5.2.** Let $X \subset \mathbb{P}^3$ be a general surface of degree $d \geq 5$ and $Y$ be a surface of general type. Let $f : X \dashrightarrow Y$ be a dominant rational map of degree $m > 1$. Then:

1. $p_g(Y) = 0$,

2. $Y$ is simply connected.
Proof. 1. Consider the Hodge structure map \( f^* : H^2(Y) \to H^2(X) \), which is defined by means of diagram (1) as the composition of the ordinary pullback \( h^* : H^2(Y) \to H^2(Z) \) followed by the Gysin map \( \phi_* : H^2(Z) \to H^2(X) \), and consider the injection \( f^* : H^{2,0}(Y) \to H^{2,0}(X) \). Let \( T_Y \supset H^{2,0}(Y) \) and \( T_X \supset H^{2,0}(X) \) be the Hodge substructures orthogonal to the Néron-Severi Hodge structures of \( Y \) and respectively of \( X \). We have \( H^{2,0}(Y) = T^{2,0}_Y \) and \( H^{2,0}(X) = T^{2,0}_X \). Then \( f^*T_Y \subset T_X \) is a Hodge substructure of \( T_X \).

Assume, by contradiction, that \( H^{2,0}(Y) \neq 0 \). Then \( f^*H^{2,0}(Y) \) is not trivial and hence by 3.5.1 ii) the inclusion \( f^*T_Y \subset T_X \) is an equality. In particular

\[
f^* : H^{2,0}(Y) \to H^{2,0}(X)
\]

is an isomorphism. It follows that the canonical map of \( X \) factors through \( f \) (as a rational map). When \( d > 4 \) the canonical map is an embedding. It would follow that \( f \) is a birational map, which is a contradiction.

2. Let \( \rho : W \to Y \) be the universal covering of \( Y \). Consider diagram (1), in which \( f \) is extended to a morphism \( h \) on the blow up \( Z \). Since \( Z \) is simply connected we may lift \( h \) to a holomorphic map \( g : Z \to W \). It follows that \( g \) is surjective, \( W \) is projective and the fundamental group of \( Y \) is finite. Since the deck transformations of \( \rho : W \to Y \) give automorphisms of \( W \), and \( X \) has only the trivial birational automorphism it follows \( \deg(g) > 1 \). This defines a dominant rational map \( g' : X \dashrightarrow W \) of degree \( > 1 \). Using the first part of the proposition we get \( p_g(W) = 0 \). Since \( q(Y) = q(W) = 0 \) we also obtain:

\[
\chi(O_Y) = \chi(O_W) = 1.
\]

The proportionality theorem for the holomorphic Euler characteristic gives

\[
\chi(O_W) = \deg(\rho)\chi(O_Y).
\]

Therefore \( \deg(\rho) = 1 \), and the fundamental group of \( Y \) is trivial. \( \Box \)

Remark 3.5.3. A similar proposition holds for the general hypersurface of \( \mathbb{P}^n \) of degree \( > n + 1 \).

4 Families of rational maps

We consider a family of rational maps from surfaces in \( \mathbb{P}^3 \) to surfaces of general type, and we study the number of moduli of the ramification divisors. In the final subsection we prove our main result, that for small degree the general surface in \( \mathbb{P}^3 \) has no such map.

4.1 Families and moduli

From now on \( X \) will be a general smooth surface of \( \mathbb{P}^3 \) of degree \( d \geq 5 \), \( Y \) will be a simply connected minimal surface of general type with \( p_g(Y) = 0 \). We also assume that \( f : X \dashrightarrow Y \) is a dominant rational map of degree \( m \). The resolution of indeterminacy of \( f \) is given in diagram (1). Moreover we would like to consider a family of such mappings.
Let $U$ be a smooth base variety and $p \in U$ be a general point. Assume we are given a smooth family $q_1 : \mathcal{X} \to U$ with $\mathcal{X} \subset U \times \mathbb{P}^3$, and smooth projective families $q_2 : Z \to U$ and $q_3 : Y \to U$, and a diagram of families

$$
\begin{array}{c}
\Phi \downarrow H \\
\mathcal{X} \longrightarrow Y
\end{array}
$$

such that $\Phi$ is birational and $H$ is generically finite and dominant. Moreover at the given point $p$ the diagram of families specializes to diagram (1) relative to the map $f$. Thus we have defined a family of rational maps $F_t = H_t \circ \Phi^{-1}_t$.

We have now the moduli space $M_d$ of smooth surfaces of degree $d$ in $\mathbb{P}^3$, and also the connected component $\mathcal{N}$ of the moduli space of minimal surfaces of general type with invariants $\chi(Y)$ and $K_2^2$, which contains all points $[Y_t = q_3^{-1}(t)]$ for $t \in U$. Define

$$N = \dim \mathcal{N}.$$ 

Consider the modular maps:

$$\mu : U \to M_d \quad \text{and} \quad \nu : U \to \mathcal{N}.$$ 

We assume that $\mu$ is generically finite and dominant,

$$\dim U = M(d) = \left(\frac{d+3}{3}\right) - 16.$$ 

For every rational map in the family there are several ramification loci. Let $R_t \subset Z_t$ be the ordinary ramification divisor of $H_t$, and denote by $\mathcal{R}_t$ the support of $R_t$. Define $B_t \subset Y_t$ to be the reduced branch divisor, the support of the divisor $H_t(\mathcal{R}_t)$, and define moreover $D_t \subset X_t$ to be the support of the divisor $\Phi_t(\mathcal{R}_t)$, the part of the ramification that appears in $X_t$. These divisors form algebraic families, and those which are reduced divisors form algebraic families of curves in the sense of [2.3.1] (shrinking the base $U$ if necessary). Let $\mathcal{R} \subset Z$ be the relative ramification divisor of $H$ over $U$, and denote by $\mathcal{R} \subset Z$ the total space of the family of curves $\mathcal{R}_t$. Let $B \subset Y$ be the total space of the family $B_t$, and let $\mathcal{D} \subset \mathcal{X}$ be the total space of the family $D_t$. Here $B$ and $\mathcal{D}$ are reduced. Since $X_t$ is a general surface of $\mathbb{P}^3$, then $D_t$ is a complete intersection curve, by [3.5.1]. Moreover generically any component of $D_t$ has geometric genus $> 1$, by [3.5.1 iii]).

We can write formulas (2) and (4)

$$H_t^*(K_{Y_t}) = rL_t - W_t,$$

$$R_t = sL_t + W_t + E_t.$$ 

Note that the birational index $s$ is invariant in the family. We have $s+r = d-4$.

It follows that $D_t$ is a complete intersection of type $(d, s')$ where $s' \leq s$.

We are going to study the modular dimension of this family $D_t$, as defined in [2.3.1]. An upper bound follows from considering this as a family of complete intersection curves in $\mathbb{P}^3$.

Define for $d > k$ and $d > 2$ the function:

$$M(d, k) = \begin{cases} 
\left(\frac{d+3}{3}\right) + \left(\frac{k+3}{3}\right) - \left(\frac{d-k+3}{3}\right) - 17 & \text{if } k > 1 \\
\left(\frac{d+2}{2}\right) - 9 & \text{if } k = 1 \\
-1 & \text{if } k = 0
\end{cases}.$$
Proposition 4.1.1. Using the notation established above, we have:

\[ M(\mathcal{D}/U) \leq M(d, s). \]

Proof. Let \( \mathcal{D}' \) be an irreducible component of \( \mathcal{D} \). After a base change \( W \to U \) (and replacing \( U \) by an appropriate open subset if necessary) we have a family \( \mathcal{C} \to W \) whose members are the normalizations \( C_z \) of irreducible components \( A_z \) of curves of the family \( \mathcal{D} \), see [2,3]. The curves in this family are complete intersections of type \( (d, s') \) with \( s' \leq s \), of geometric genus \( g \). By [3,5,iii) we have that \( g > 1 \). The case \( s' = 0 \) is trivial and \( s' = 1 \) is similar to the case \( s' > 1 \). So assume \( s' > 1 \).

Let \( \mathcal{H}(d, s') \) be the Hilbert scheme of complete intersection curves in \( \mathbb{P}^3 \) of type \( (d, s') \). By acting with the projective group \( G \) we find a family of curves \( hA_z \), \( h \in G \), of \( \mathbb{P}^3 \) and this defines a morphism \( W \times G \to \mathcal{H}(d, s') \), let \( \tilde{W} \) be the image of this morphism, and consider the modular map \( \mu : \tilde{W} \to M_g \). Since \( g > 1 \), the automorphisms group of \( C_z \) is finite and hence no continuous family of automorphisms of \( \mathbb{P}^3 \) can fix the curve \( A_z \). It follows that \( \dim \mu(W) \leq \dim \tilde{W} - \dim G = \dim \tilde{W} - 15 \). Therefore we have:

\[ M(\mathcal{D}'/U) = \dim \mu(\tilde{W}) \leq \dim \tilde{W} - 15 \leq \dim \mathcal{H}(d, s') - 15 = M(d, s'), \]

and clearly \( M(d, s') \leq M(d, s) \).

On the other hand, a lower bound for the modular dimension comes from the bend and break lemma.

Proposition 4.1.2. Using the notation established above, we have:

\[ M(d) - N \leq M(\mathcal{D}/U) + 1. \]

Proof. Let \( \Gamma \subset U \) be the general fiber of the modular map \( \nu : U \to \mathcal{N} \). We may assume that \( \Gamma \) contains \( p \). Define \( f = \dim \Gamma \), and we remark that

\[ f \geq M(d) - N. \]

Let \( \mathcal{X}_\Gamma = q_1^{-1}(\Gamma) \) and \( \mathcal{Z}_\Gamma = q_2^{-1}(\Gamma) \) be the restrictions of the families \( \mathcal{X} \) and \( \mathcal{Z} \) to \( \Gamma \). The maps parametrized by \( \Gamma \) are rational maps \( F_t : \mathcal{X}_t \to Y \) where \( Y \) is fixed. Let moreover \( \mathcal{R}_\Gamma \) and \( \mathcal{B}_\Gamma \), \( \mathcal{D}_\Gamma \) be the restricted families of ramification and branch divisors.

We recall from covering theory that covering maps of \( Y \) with given branch locus \( B \) and given degree \( m \) are classified in terms of homomorphisms from the fundamental group of \( Y \setminus B \) to the symmetric group in \( m \) elements (see [22] ch. III §4). It follows that the map \( t \mapsto B_t \) from \( \Gamma \) to the Hilbert scheme of \( Y \) is generically finite onto its image (since so is the map \( U \to M_d \) in our assumptions). So the branch loci \( B_t \), \( t \in \Gamma \), form a family of curves in \( Y \) of dimension \( f \). It follows from [2,3,2] that the modular dimension of this family is \( M(\mathcal{B}_\Gamma/\Gamma) \geq f - 1 \). A fortiori the family of ramification divisors \( R_t \), \( t \in \Gamma \), has modular dimension \( M(\mathcal{R}_\Gamma/\Gamma) \geq M(\mathcal{B}_\Gamma/\Gamma) \geq f - 1 \).

Assume now that \( s > 0 \). The family \( D_t \), \( t \in \Gamma \), has the same modular dimension \( M(\mathcal{D}_\Gamma/\Gamma) = M(\mathcal{R}_\Gamma/\Gamma) \). An irreducible component of \( \mathcal{R}_\Gamma \) whose fibers consist of rational curves has modular dimension 0. Any other irreducible
component arises from an irreducible component of $D_{\Gamma}$ with the same modular map. Summing up we have:

$$f - 1 \leq M(B_{\Gamma}/\Gamma) \leq M(\overline{D}_{\Gamma}/\Gamma) = M(D_{\Gamma}/\Gamma) \leq M(D/U).$$

Finally, if $s = 0$ then $D_t = 0$ and $B_t$ consists of rational curves, which do not move in $Y$, hence it follows that $f = 0$, and this gives the statement.

4.2 Main result

We are in a position to prove our main theorem, by combining the previous results. We keep the notation of the last section.

From 4.1.1 and 4.1.2 we obtain the inequality:

$$M(d) - N \leq M(d, s) + 1,$$

which is a necessary condition for the existence of a rational map $f$, with birationality index $s$, on a general surface $X$ of degree $d$ in $\mathbb{P}^3$. Moreover from 2.5.3 we have:

$$N \leq 19.$$

**Theorem 4.2.1.** If $5 \leq d \leq 11$ then the general surface of degree $d$ has no (non-trivial) rational map which dominates a surface of general type.

**Proof.** If $s = 0$ then inequality (5) gives $M(d) - N \leq 0$, that is:

$$\left(\frac{d+3}{3}\right) - 16 \leq N;$$

since $d > 4$ this gives $N \geq 40$, a contradiction. If $s = 1$ since $r > 0$ we have $d \geq 6$. From (5) we have the inequality:

$$\left(\frac{d+3}{3}\right) - 16 - \left(\frac{d+2}{2}\right) + 9 \leq N + 1;$$

for $d > 5$ this gives $N \geq 48$, again a contradiction. Assume $s > 1$, so that $d > 6$. By 3.4.3 we have $r > 1$. From (5) we have:

$$\left(\frac{d+3}{3}\right) - 16 - N \leq \left(\frac{d+3}{3}\right) + \left(\frac{s+3}{3}\right) - \left(\frac{d-s+3}{3}\right) - 17 + 1$$

whence we obtain:

$$19 \geq N \geq \left(\frac{d-s+3}{3}\right) - \left(\frac{s+3}{3}\right) = \left(\frac{r+7}{3}\right) - \left(\frac{s+3}{3}\right).$$

Now since $r \geq 2$ we have:

$$19 \geq N \geq \left(\frac{9}{3}\right) - \left(\frac{s+3}{3}\right) = 84 - \left(\frac{s+3}{3}\right),$$

and so:

$$\left(\frac{s+3}{3}\right) \geq 65.$$

This gives $s \geq 6$ and hence $d = s + r + 4 \geq 12$. 

\qed
Remark 4.2.2. The previous computation proves that the only possible case for \( d = 12 \) gives \( s = 6 \) and \( r = 2 \).

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**Lucio Guerra**  
Dipartimento di Matematica, Università di Perugia  
Via Vanvitelli 1, 06123 Perugia, Italia  
guerra@unipg.it

**Gian Pietro Pirola**  
Dipartimento di Matematica, Università di Pavia  
via Ferrata 1, 27100 Pavia, Italia  
gianpietro.pirola@unipv.it